QUADRATIC DIFFERENTIALS, QUATERNIONIC FORMS, AND SURFACES

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1 Introduction

In this paper we address three problems stemming from the fundamental question: Let $M$ be an oriented surface, how much data do we need to identify an immersion $f : M \to \mathbb{R}^3$ or its shape?

A generic immersion $f$ is determined up to homothety and translation by its conformal class and its tangent plane map $T_f : M \to Gr(2, 3)$ whose value $T_f(p)$ at each point $p \in M$ is the plane tangent to the image surface $f(M)$. There are exceptions, called Christoffel immersions. Their classification is known as Christoffel's problem. The problem was posed and to a large extent solved locally in [3]. A short complete solution of Christoffel’s problem, including closed surfaces, is presented in Section 4.

Bonnet noticed in [3] that if two immersions induce the same first fundamental form and the same mean curvature function then they are congruent unless they are quite exceptional. Bonnet's problem is to classify all such exceptional immersions. A local classification of the umbilic-free and the constant mean curvature exceptions follows from [3, 4, 5]. Until now it was not known whether there exist other exceptions. The existence and description of the new examples is discussed in Section 5.

Christoffel's and Bonnet’s problems inevitably lead to the study of isothermic immersions [7, 13]. An umbilic-free immersion is called umbilic-free classical isothermic immersion if every point $p \in M$ admits isothermal local coordinates in which its second fundamental form is diagonal. There exist classical definitions of isothermic immersions with umbilic points but no definitive one. Up to now it was not clear how to motivate and define global isothermic immersions. In this paper we use a hitherto unsuspected connection between isothermic immersions and quadratic differentials to study global isothermic immersions and immersions with umbilic points, branch points, and ends (see Section 3). The presented theory is designed to solve the targeted geometric problems and to resolve some quirks in the folklore surrounding the classical isothermic immersions.

The main tool used in this paper is the quaternionic calculus on surfaces [13]. The necessary material is summarized in Section 2. Some of the results in this paper were announced in [15].

Unless explicitly specified otherwise all immersions in this paper are assumed to be at least $C^3$. On the other hand many of the constructions and arguments do not need much smoothness. Weaker smoothness requirements, regularity, and partial regularity are discussed in [12].

2 Quaternionic Calculus, Tensors, and Function Theory

A brief summary of the quaternionic calculus introduced in [13] is given below. (Very detailed treatments were given in [17, 11]).
Throughout this paper \( M \) will denote a Riemann surface with a complex structure \( J \). Let \( f \) be a conformal immersion of \( M \) into \( \mathbb{R}^3 = \text{im}(H) \), and let \( N \) be the oriented unit normal vector field of \( f \), a quaternionic valued one-form \( \kappa \) on \( M \) is called conformal (anti-conformal) with respect to \( f \), if \( *\kappa = N\kappa \ (\kappa N = -N\kappa) \), where \( *\kappa := \kappa \circ J \). More generally, an \( H \) valued one-form \( \varphi \) defined on a subset of a Riemann surface is called conformal if there exists a quaternionic valued function \( N \) such that \( *\varphi = N\varphi \). Given a conformal immersion, then every quaternionic valued one-form \( \tau \) is the sum of a conformal part and a anti-conformal part. In particular, let \( N \) be the Gauss map of the immersion \( f \) then the form \( dN \) is tangential valued and

\[
dN = -Hdf + \omega, \tag{1}
\]

where \( H \) is the usual mean curvature function of \( f \) and \( \omega \) is the anti-conformal part of \( dN \). The anti-conformal part \( \omega \) of \( dN \) is called the Hopf form of the immersion. See Remark 2 below.

Let \( f \) be a conformal immersion with normal \( N \). Every quaternionic valued one-form \( \tau \) has a splitting \( \tau = \tau^\top + \tau^\perp \) into a tangential and transversal part with respect to \( f \). Note that a form is tangential if and only if \( N\tau = -\tau N \), while \( \tau^\top = 0 \) if and only if \( N\tau = \tau N \). We denote the bundle of tangential one-forms by \( T^*M \otimes f^*TM \).

Let \( \tau \) be a tangential valued differential one-form. We denote the transversal and the tangential components of \( d\tau \), with respect to \( f \), by \( (d\tau)^\perp \) and \( (d\tau)^\top \), respectively.

**Observation 2.1** Let \( f \) be a conformal immersion, then for every tangential valued differential form \( \tau \) we have:

\[
(d\tau)^\perp = \begin{cases}
\frac{1}{2}(\omega \wedge \tau - \tau \wedge \omega)N & \text{if } \tau \text{ is anti-conformal} \\
\frac{H}{2}(\tau \wedge df - df \wedge \tau)N & \text{if } \tau \text{ is conformal}
\end{cases} \tag{2}
\]

where \( H \) and \( \omega \) are the mean curvature and the Hopf form of the immersion \( f \), respectively.

**Proof**

\[
(d\tau)^\perp = -\frac{1}{2} ((d\tau)N + (d\tau)N) N
= -\frac{1}{2} ((d\tau)N + Nd\tau) N. \tag{3}
\]

The form \( \tau \) is tangential. Differentiating \( N\tau + \tau N = 0 \) yields \( (d\tau)N + Nd\tau = \tau \wedge (dN) - (dN) \wedge \tau \). Using this identity and (3) we obtain

\[
-\frac{1}{2} ((d\tau)N + Nd\tau) = \begin{cases}
\frac{1}{2}(\omega \wedge \tau - \tau \wedge \omega) & \text{if } \tau \text{ is anti-conformal} \\
\frac{H}{2}(\tau \wedge df - df \wedge \tau) & \text{if } \tau \text{ is conformal}
\end{cases}
\]

Substituting back in (3) completes the proof. \( \square \)

**Definition 1** Two conformal immersions \( f, \tilde{f} : M \rightarrow \mathbb{R}^3 = \text{im}(H) \) are called spin-equivalent if there exists \( \lambda : M \rightarrow H^* \) so that \( df = \lambda df \lambda \). We call \( \tilde{f} \) a spin-transform of \( f \).

Note that if \( f \) and \( \tilde{f} \) are in the same regular homotopy class then they are spin equivalent. In particular, \( \pi_1(M) = 0 \) then every two conformal immersions are spin equivalent.
Remark 1 The immersion \( f \) defines an isomorphism between \( T^*M \otimes f^*TM \) and \( T^*M \otimes T^*M \), sending a tangential valued form \( \tau \) to the covariant tensor \(< \tau(\cdot)|df(\cdot)>\). (Here \(< \cdot | \cdot >\) is the Euclidean scalar product in \( R^3 \).)

A form \( \kappa \in T^*M \otimes f^*TM \) is conformal if and only if it is a linear combination of \( df \) and \( *df \); the tensor \(< \tau(\cdot)|df(\cdot)>\) is antisymmetric if and only if \( \tau = a*df \) for some real \( a \). The tensor \(< \tau(\cdot)|df(\cdot)>\) is symmetric trace-free if and only if \( \tau \) is anti-conformal with respect to \( f \). Moreover, \(< \tau(\cdot)|df(\cdot)>\) is divergence free (with respect to the metric induced by \( f \)) if and only if \((d\tau)^T = 0\).

Suppose that \( \tilde{f} \) is obtained from \( f \) by a spin transform, thus \( d\tilde{f} = \lambda df \). Let \( \tilde{\tau} \) be a and \( \tau \) be two \( H \)-valued one forms on \( M \), tangential to \( \tilde{f} \) and \( f \) respectively, then

\[
< \tilde{\tau}(\cdot)|d\tilde{f}(\cdot)> = < \tau(\cdot)|df(\cdot)> \text{ if and only if } \tau = \lambda \tilde{\tau} \tilde{X}.
\]

Remark 1 indicates that the tangential anti-conformal forms are an extrinsic geometric representation of the quadratic differentials, and \((d\ )^T\) is the extrinsic operator \( \tilde{\partial} \). This analogy can be taken further. In the context of the intrinsic conformal geometry on a Riemann surface \( M \) all complex valued objects take values in a fixed complex plane, \( C \). On the other hand a conformal immersion \( f \) of \( M \) into \( R^3 \) defines two fields of complex planes, the field \( f_*TM \) of tangent planes, and the field of normal planes, \( R \oplus R N \). The intrinsic conformal objects can be represented as \( f_*TM \)-valued sections or as \( R \oplus R N \)-valued sections. For example, let \( w = a + ib \) be a complex valued function on the Riemann surface \( M \). Given a conformal immersion \( f \) with normal \( N \), we can consider this function \( w \) as the section \((a + bN)df\) of the bundle of tangential and conformal forms.

Observation 2.2 If \( f \) is a conformal immersion of \( M \) into \( R^3 \) then a complex valued function \( w = a + ib \) is holomorphic if and only if \((d(a + bN)df)^T = 0\).

Proof This is a local statement. Let \( z = x + iy \) is a holomorphic coordinate chart on \( M \), then using (1) we obtain

\[
(d(a + bN)df)^T(\partial_x, \partial_y) = - \left( \frac{\partial a}{\partial y} + \frac{\partial b}{\partial x} \right) \frac{\partial f}{\partial x} + \left( \frac{\partial a}{\partial x} - \frac{\partial b}{\partial y} \right) \frac{\partial f}{\partial y}.
\]

Therefore, \((d(a + bN)df)^T = 0\) if and only if \( a + ib \) is holomorphic. \( \square \)

The following observation is based on ideas introduced in [17]. (See [17, 11] for proofs.) The bundle of normal-valued quadratic differentials associated with \( f \) is the bundle of fiber-preserving maps defined by:

\[
R \oplus R N^{2,0} := \{ \kappa: TM \to R \oplus R N | \kappa(AX + BJX) = \kappa(X)(a + bN)^2, \forall X \in TM \}
\]

The immersion \( f \) defines a bundle isomorphism \( \nu_f: R \oplus R N \to M \times C \), by \( \nu_f(a + bN) = a + ib \). The map sending \( \kappa \in R \oplus R N^{2,0} \) to \( \nu_f \circ \kappa \) is an isomorphism between \( R \oplus R N^{2,0} \) and \( T^{2,0}M^* \). A section \( \kappa \in \Gamma(R \oplus R N^{2,0}) \) is called a holomorphic normal-valued quadratic differential if the quadratic differential \( \nu_f \circ \kappa \) is holomorphic. Once a conformal immersion is chosen it is convenient to abuse notation and call normal-valued quadratic differentials simply quadratic differentials. Moreover, we have
**Observation 2.3** Every conformal immersion, \( f \), induces an isomorphism between the bundle of anti-conformal tangential one forms and the bundle of (normal) quadratic differentials, sending the one-form \( \tau \) to the quadratic differential \( df\tau \).

Furthermore a quadratic differential \( \kappa \) is holomorphic if and only if the tangential part of \( d(df\kappa) \) vanishes, that is if and only if \( (d(df\kappa))^\top = 0 \).

**Remark 2** Given a complex valued quadratic differential \( q \) and a conformal immersion \( f \) we will use the notation \( df\ q \) for the tangential valued one form \( df\nu_j^{-1} \circ q \). Furthermore notice that for every anti-conformal tangential form \( \tau \) the two-zero part of the symmetric trace-free tensor \( <\tau(\cdot)|df(\cdot)> \) is precisely the quadratic differential \( \nu_f \circ df \tau \). The isomorphism \( q \rightarrow df\ q \) identifies the classical Hopf differential of \( f \) with \(-\) the Hopf form of \( f \). Indeed, the Hopf differential of a conformal immersion \( f \) is the two-zero part, \( \mathbb{II}_{2,0} \), of the second fundamental form \( \mathbb{II} = -dN \).

On the other hand (1) implies that the tensor \( <\omega(\cdot)|df(\cdot)> \) is the trace-free part of \( \mathbb{II} \) and so the Hopf differential of \( f \) is \( \mathbb{II}_{2,0} = -\nu_f \circ df \omega \), that is \( \omega = -df\ \mathbb{II}_{2,0} \).

### 3 Isothermic Surfaces

**Definition 2** A conformal immersion \( f : M \rightarrow \text{im}(H) \) is called an extended globally isothermic immersion if there exists a subset \( S \subset M \) and a map \( f^* : M \setminus S \rightarrow \text{im}(H) \) such that \( M \setminus S \) is dense in \( M \), \( df^* \neq 0 \), and \( dfdf^* \) is a holomorphic quadratic differential on \( M \setminus S \). The map \( f^* \) is called dual to the immersion \( f \), and the set \( S \) is called the singular set of \( f^* \).

**Remark 3** Note that the dual \( f^* \) of \( f \), satisfies \( df^* = df\ df\ df^* \) on \( M' := M \setminus S \), and so \( df^* \) is anti-conformal tangential with respect to the restriction of \( f \) on \( M' \). Furthermore, the zero set \( Z = \{p \in M'|df^*_p = 0\} \) is precisely the zero locus of the holomorphic quadratic differential \( dfdf^* \) and so \( Z \) is discrete, if not empty. Thus the dual map \( f^* \) defines a branched conformal immersion of the Riemann surface \( M' \). The branch locus of \( f^* \) is precisely the zero set \( Z \), and the oriented normal of the immersion \( f^* \) is \( N^* = -N \).

**Remark 4** An immersion \( f \) is extended globally isothermic if and only if there exists a holomorphic quadratic differential \( q \neq 0 \), defined on a dense subset \( M' = M \setminus S \), such that \( df\ q \) is exact. Definition \( \text{[2]} \) can be generalized. One could require that \( df\ q \) is closed instead of exact but presently this appears justified only in the study of pathologies, for example immersions with uncountably many ends. Note also that both the immersion \( f \) and its dual \( f^* \) may have only first order derivatives, and in particular, they may not have well defined second fundamental forms, curvature lines, and umbilic points in the usual sense. Generalized isothermic immersions and the related regularity theory are studied in \[12\] in connection with weak isometric immersions and the realization of conformal structures. Smooth local (umbilic-free) isothermic immersions in co-dimension two are studied in \[8\].

Let \( \overline{R^3} = R^3 \cup \{\infty\} \) be the one-point compactification of \( R^3 \) with the natural topology. The following observation is now trivial:
Observation 3.1 Let $f : M \to \text{im}(H)$ be a conformal immersion such that there exists a set $S \subset M$ and a map $f^* : M \setminus S \to \mathbb{R}^3$, such that $df df^*$ is a nontrivial holomorphic quadratic differential on $M \setminus S$, which extends to a meromorphic quadratic differential on $M$ whose set of poles is $S$. Then $f$ is an extended isothermic immersion with dual $f^*$. The map $f^*$ is a branched immersion of $M$ with ends. Namely, $f^* : M \to \mathbb{R}^3$ is continuous and $(f^*)^{-1}(\infty) = S$ and so the poles of $df df^*$ are precisely the ends of $f^*$. The set $Z$ of zeroes of the differential $df df^*$ is the set of branch points of the extended immersion $f^*$. Finally, the branch points and the ends of $f^*$ are umbilic points of $f$ but $f$ may have other umbilic points.

Every rotationally symmetric immersion $f_s$ is extended globally isothermic: indeed, $f_s$ has a rotationally symmetric dual $f^*_s$. In particular, every ellipsoid of revolution $f_s$ is extended globally isothermic: the dual $f^*_s$ has precisely two ends according to the nomenclature adopted in this paper. General genus zero immersions including non-rotational ellipsoids are discussed in [12].

**Definition 3** A conformal immersion $f$ is called **globally isothermic** if there exists a nontrivial holomorphic quadratic differential $q$ on $M$ such that $df q$ is exact.

The class of global isothermic immersions is precisely the class used in [3], see Remark 3. In the rest of this paper isothermic immersion is used as a synonym of global isothermic immersion.

The defining property of classical isothermic immersions is that in a neighborhood of every non-umbilic point they admit an isothermal coordinate system in which the second fundamental form is diagonal, equivalently in a neighborhood of every non-umbilic point one can represent the Hopf differential of the immersion as a real multiple of a holomorphic quadratic differential.

Let $f$ be a global isothermic immersion with dual $f^*$, and a Hopf form $\omega$. Thus from $d(df^*) = 0$ and from (2) it follows that

$$df^* \wedge \omega - \omega \wedge df^* = 0. \quad (4)$$

Away from the branch locus $Z$ of $f^*$, the bundle of tangential anti-conformal forms with respect to $f$ is spanned by $df^*$ and $*df^*$. Thus $\omega = adf^* + b * df^*$ where $a$ and $b$ are real-valued and smooth away form the branch locus $Z$. From (2) it follows that $b = 0$ and so $\omega = adf^*$ and so the Hopf differential of $f$ equals minus $a H_f \circ df df^*$. The classical formula $dN = H^* df^* - H df$, where $H^*$ is the mean curvature of $f^*$, remains valid away from the branch locus of $f^*$. Therefore, $a = H^*$, away from the branch points of $f^*$.

In a neighborhood of $p \in M \setminus Z$ choose isothermal coordinates $z = x + iy$, with respect to the metric induced by $f$, in which the quadratic differential $q =\nu_f \circ df df^* = dz^2$. In this coordinate system the Hopf differential $H^2_0$ of $f$ equals $adz^2$. The coefficient $a$ is real valued. Therefore, the second fundamental form of $f$ is diagonal with respect to the $(x, y)$ coordinates. Note that $p$ could be an umbilic point for $f$. Thus the following observation is true for global isothermic immersions:

**Observation 3.2** Every global isothermic immersion $f$ defines a global holomorphic quadratic differential $df df^*$, where $f^*$ is dual to $f$. The Hopf differential of $f$ is a real multiple of $df df^*$ (smooth multiple on $M \setminus Z$). For every point $p \in M \setminus Z$ there exists an isothermal coordinate system in which the second fundamental form of $f$ is diagonal.

This observation indicates that while in the classical theory umbilics are avoided and excluded, they are generically quite benign. The only interesting umbilic points are the branch points of the dual.

Observation 3.2 has a local inverse. The data determining an isothermic immersion consists of the tangent map of the immersion along a curve, which is not a curvature line, together with a
holomorphic quadratic differential encoding the properties of the lines of curvature of the immersion. Recall that with every quadratic differential \( q \) on a Riemann surface \( M \) we associate two mutually orthogonal line foliations (possibly singular). These foliations are basic objects in Teichmüller theory, they are called the principal stretch foliations of \( q \). We will call a smooth curve \( C \) non-characteristic for \( q \) if it is transversal to the principal stretch foliations of \( q \). We prove the basic existence result in the analytic category below, and indicate the general result in Remark 5 below.

**Theorem 1** Given a holomorphic quadratic differential \( q \) on a Riemann surface \( M \), let \( C \) be a real analytic curve in \( M \setminus \mathbb{Z} \) which is non-characteristic for \( q \), then for every real analytic, conformal, rank two form \( \varphi_0 \) defined along \( C \), there exists a unique, up to translation in space, isothermic immersion \( \tilde{f} \) defined in a neighborhood of \( C \) such that \( d\tilde{f} \setminus q \) is the differential of the dual of \( \tilde{f} \) and \( d\tilde{f} = \varphi_0 \) along \( C \).

**Proof** It suffices to find an immersion \( \tilde{f} \) such that \( d\tilde{f} \setminus q \) is exact; this is equivalent to showing that the normal component of \( d(d\tilde{f} \setminus q) \) vanishes. To find \( \tilde{f} \) we will solve an initial value problem with initial data \( d\tilde{f} = \varphi_0 \) along \( C \). Choose a background real analytic conformal immersion \( f \), such that \( df = \varphi_0 \) along \( C \), and let \( \tau = df \setminus q \). Then we search for \( \tilde{f} \) as a spin transform of \( f \). Equivalently we need to find a quaternionic valued function \( \lambda \) such that \( d\tilde{f} = \lambda df \lambda \), \( d\tilde{f} \setminus q = \lambda^{-1}(df \setminus q)\lambda^{-1} \) is exact, and \( \lambda = 1 \) along \( C \). The function \( \lambda \) must satisfy the differential equations

\[
\frac{d(\overline{\lambda} df)}{d(\lambda^{-1} \overline{\lambda^{-1}})} = 0
\]

Thus the system for \( \lambda \) is

\[
\begin{align*}
\text{Im} \left( \overline{df} \wedge d\lambda \right) &= 0 \quad (5) \\
\text{Re} \left( (d\lambda^{-1} \overline{\lambda^{-1}})\lambda^{-1} N\lambda \right) &= 0, \quad (6)
\end{align*}
\]

A direct computation shows that equation (6) is equivalent to the equation

\[
\text{Re} \left( \text{Im} \left( (d\lambda)\lambda^{-1} \wedge \tau \right) N \right) + (\omega \wedge \tau - \tau \wedge \omega)/4 = 0 \quad (7)
\]

where \( \omega \) is the Hopf form of \( f \). Thus we must solve (5), (7), with Cauchy data \( \lambda = 1 \), along \( C \). In local coordinates \( z = x + iy \) the differential system is

\[
\begin{align*}
\text{Im} \left( \overline{\lambda} \left( \frac{\partial f}{\partial x} \frac{\partial \lambda}{\partial y} - \frac{\partial f}{\partial y} \frac{\partial \lambda}{\partial x} \right) \right) &= 0 \quad (8) \\
\text{Re} \left( \text{Im} \left( \frac{\partial \lambda}{\partial x} \lambda^{-1} \tau \left( \frac{\partial \lambda}{\partial y} \right) - \frac{\partial \lambda}{\partial y} \lambda^{-1} \tau \left( \frac{\partial \lambda}{\partial x} \right) \right) N \right) + \cdots &= 0 \quad (9)
\end{align*}
\]

We can use (5) and (7) to compute the principal symbol of the system of differential equations (5), (7) at \( \lambda = 1 \), and a covector \( \xi \). Indeed, let \( \xi = \xi_1 dx + \xi_2 dy \) then the symbol is the map

\[
\sigma(1, \xi) \colon \mathbb{H} \to \mathbb{H}
\]

defined by

\[
\sigma(1, \xi)(\alpha) = \text{Re} \left( \text{Im} \left( \alpha B(\xi) \right) N \right) + \text{Im} \left( A(\xi)\alpha \right), \quad (10)
\]

6
where
\[ A(\xi) = \xi_2 \frac{\partial f}{\partial x} - \xi_1 \frac{\partial f}{\partial y} \quad \text{and} \quad B(\xi) = \xi_1 \tau \left( \frac{\partial}{\partial y} \right) - \xi_2 \tau \left( \frac{\partial}{\partial x} \right). \]

Note that both \( A(\xi) \) and \( B(\xi) \) are tangential to the immersion and in particular they are im(\( H \)) valued, and so \( \sigma(1, \xi)(\alpha) = 0 \) precisely when \( \alpha \) is imaginary valued and is collinear to both \( B(\xi) \) and \( A(\xi) \). This implies that the vector dual to \( \xi \), with respect to the first fundamental form induced by \( f \), is tangential to the principal stretch foliation of the quadratic differential \( q \). Thus the characteristic variety of the differential system for \( \lambda \) consists of two orthogonal lines, each one tangent to the stretch directions of \( q \). Therefore, the initial value problem with Cauchy curve transversal to the principal stretch foliations of \( q \) is well posed. \( \square \)

In summary the space of all isothermic immersions generated by \( q \) is parameterized by four functions of one real variable. These functions represent the differential of the immersion along the initial curve. (See [11] and [12] for more details.) Compare with the result by Weingarten see [7], page 262, and the references there.

**Remark 5** In fact one can remove the technical assumption that the curve \( C \) omits the zero locus \( Z \). After modifying the definition of non-characteristic curve one can still obtain an existence result. Furthermore, the existence results are true in the smooth category. The proofs in these cases require a whole new set of techniques. (See [12].)

**4 Christoffel’s Problem**

**Theorem 2** Every Christoffel immersion \( f \) is globally isothermic. Vice versa let \( f \) be a globally isothermic immersion with dual \( f^* \) and let \( Z \) be the branch locus of \( f^* \), then either \( Z \) is empty or it is a set of isolated points and every point in \( Z \) is umbilic for \( f \). Furthermore, \( f \) is a Christoffel immersion on \( M \setminus Z \).

**Proof** Suppose \( f \) and \( \tilde{f} \) are conformal immersions of the same Riemann surface into \( \mathbb{R}^3 \) such that \( T_f = T_{\tilde{f}} \). In particular, \( df \) is tangential, and conformal or anti-conformal with respect to \( f \). If \( df \) is anti-conformal with respect to \( f \) then \( f \) is isothermic and \( \tilde{f} \) is a dual to \( f \). To complete the proof we need to consider the case when \( d\tilde{f} \) is conformal with respect of \( f \). In this case the normal to \( \tilde{f} \) is \( N \) and there exist real valued functions \( a \) and \( b \) such that
\[ d\tilde{f} = (a + bN)df. \]

Clearly \( (dd\tilde{f}) = 0 \). The tangential part of this identity implies that \( u = a + ib \) is a holomorphic function defined on \( M \). (See Observation 2.2.) On the other hand from (2), it follows that
\[ 0 = (dd\tilde{f})_N = H \left( d\tilde{f} \wedge df - df \wedge d\tilde{f} \right)/2, \]
where \( H \) is the mean curvature of \( f \). If \( H_p \neq 0 \) at some point \( p \in M \), then
\[ 0 = df \wedge d\tilde{f} - d\tilde{f} \wedge df = b(df \wedge \ast df - \ast df \wedge df). \]
in a neighborhood of this point, and so \( u \) is constant and real in this neighborhood and hence \( u \) must be constant and real on the whole surface \( M \). This implies that \( \tilde{f} \) is just a real scaling of \( f \). This is impossible if \( f \) and \( \tilde{f} \) are a Christoffel pair. Thus \( f \) must be minimal, but every minimal immersion is isothermic.

The proof of the second part of the theorem follows immediately from Remark 3. \( \square \)

Recently Holy Bernstein and Gary Jensen used complex frames \( \Box \) to give another proof of the first part of this theorem \( \Box \).

Let \( M \) be a closed surface. If \( f \) is a Christoffel immersion suppose that \( \tilde{f} \) is another immersion such that \( T_f = T_{\tilde{f}} \); but \( f \) is not obtained from \( \tilde{f} \) by homothety and translation, then \( f \) must be isothermic, \( \tilde{f} \) must be dual to \( f \), and so \( df d\tilde{f} \) must be holomorphic. Every holomorphic quadratic differential on a closed surface whose genus is not equal to one must vanish somewhere. Therefore, if \( M \) is closed and \( \text{genus}(M) \neq 1 \), then \( \tilde{f} \) must be a branched immersion, and not an immersion. Thus we obtain:

**Corollary 4.1** Let \( M \) be a compact oriented surface whose genus is not one, then every immersion \( f \) of \( M \) into \( \mathbb{R}^3 \) is determined uniquely up to homothety and translation by its conformal class and its tangent planes map, \( T_f \).

**Remark 6** This result is false for tori. The obvious counterexamples are CMC tori.

## 5 Bonnet’s problem

Two non-congruent conformal immersions \( f_{\pm} \) are called Bonnet mates if they induce the same metric and have the same mean curvature \( H_{\pm} = H_- \). A complete local classification of the umbilic-free immersions which admit Bonnet mates was obtained in \( \Box \). The construction of explicit examples is often a difficult task (see \( \Box \)). Some global results were obtained in \( \Box \). It is conjectured that there are no compact Bonnet mates. A first step in the verification of this conjecture is to study Bonnet mates with umbilic points. The methods used by Bonnet, Cartan, and Chern can not work for surfaces with umbilics. Moreover, until now it was not known whether there exist non-constant mean curvature Bonnet mates with umbilic points. In particular it was not known whether the Bonnet-Cartan-Chern classification includes all the possible Bonnet mates. A general construction which yields all immersions admitting Bonnet mates – with or with out umbilics was presented in \( \Box \). A natural question is: Does this construction yield new hitherto unknown examples?

Let \( f_{\pm} \) be Bonnet mates defined on the surface \( M \), then up to a possible rotation of one of the immersions, say \( f_+ \), we have

\[
\begin{align*}
df_{\pm} &= (f^* \pm \epsilon)df(f^* \pm \epsilon),
\end{align*}
\]

where \( \epsilon > 0 \) is a real constant, \( f \) is a global isothermic immersion defined on the universal cover of \( M \) and \( f^* \) is dual to \( f \) \( \Box \). The dual \( f^* \) descends to a map on \( M \).

The difference between the second fundamental forms of two Bonnet mates \( f_{\pm} \) defines a holomorphic quadratic differential, called the shape distortion differential \( D = (\mathbb{I}_+ - \mathbb{I}_-)^{2,0} \). The zeroes of \( D \) are precisely the umbilic points of \( f_{\pm} \) (see \( \Box \)).
Observation 5.1 Suppose that \( f_\pm \) are Bonnet mates generated by the isothermic surface \( f \), that is \( df_\pm = \lambda_\pm df \lambda_\pm \), where \( \lambda_\pm = f^* \pm \epsilon \), and \( \epsilon \) is a positive constant. Then

\[
\text{df} \setminus D = 4\epsilon \ast \text{df}^*
\]  

(12)

Proof Remark \( \text{[2]} \) implies \( \text{df} \setminus D = \text{df} \setminus -\nu_f^{-1} \circ \left( \nu_f^+ \circ \text{df}_+ \omega_+ - \nu_f^- \circ \text{df}_- \omega_- \right) \). Therefore, from the following lemma:

Lemma 1 Suppose that \( \tilde{f} \) is obtained from \( f \) by a spin transform, thus \( \text{d} \tilde{f} = \overline{\lambda} \text{df} \lambda \). Let \( \tilde{\tau} \) be tangential and anti-conformal with respect to \( \tilde{f} \), then \( \text{df} \setminus \nu_f^{-1} \circ \nu_f \circ \text{d} \tilde{f} \tilde{\tau} = \lambda \tilde{\tau} \lambda \).

it follows that

\[
\text{df} \setminus D = - \left( \lambda_+ \omega_+ \overline{\lambda}_+ - \lambda_- \omega_- \overline{\lambda}_- \right). 
\]  

(13)

Let \( N_\pm \) denote the normal to \( f_\pm \), and so \( N = \lambda \pm N_\pm \). From (1) it follows

\[
\lambda_\pm \omega_\pm \overline{\lambda}_\pm = \lambda_\pm dN_\pm \overline{\lambda}_\pm + H_\pm \lambda_\pm \text{df}_\pm \lambda_\pm. 
\]  

(14)

But \( df_\pm = \overline{\lambda}_\pm \text{df} \lambda_\pm \), and so \( H_\pm \lambda_\pm \text{df}_\pm \lambda_\pm = \pm |\lambda_\pm| \text{df} \). The immersions \( f_\pm \) are Bonnet mates and hence \( H_+ = H_- \) and \( |\lambda_+| = |\lambda_-| \). Thus (14) implies

\[
\lambda_+ \omega_+ \overline{\lambda}_+ - \lambda_- \omega_- \overline{\lambda}_- = \lambda_+ dN_+ \overline{\lambda}_+ - \lambda_- dN_- \overline{\lambda}_-. 
\]  

(15)

Using, \( d\lambda^{-1} = -\lambda^{-1} d\lambda \lambda^{-1} \) we get

\[
\lambda_\pm dN_\pm \overline{\lambda}_\pm = \lambda_\pm d(\lambda^{-1}_\pm N \lambda_\pm) \overline{\lambda}_\pm = - (d\lambda_\pm) \lambda^{-1}_\pm N |\lambda_\pm|^2 + dN |\lambda_\pm|^2 + N(\lambda_\pm) \overline{\lambda}_\pm. 
\]  

(16)

Recall that \( \lambda_\pm = f^* \pm \epsilon \) and so \( d\lambda_\pm = \text{df}^* \) and hence

\[
\lambda_+ \omega_+ \overline{\lambda}_+ - \lambda_- \omega_- \overline{\lambda}_- = - d\lambda_\pm 2\epsilon N + N \text{df}^* 2\epsilon = -4\epsilon \ast \text{df}^* ,
\]

and so \( \text{df} \setminus D = 4\epsilon \ast \text{df}^* \). Finally we prove the lemma.

Proof of Lemma \( \text{[4]} \) Let \( a \) and \( b \) the real valued one-forms on \( M \) such that \( df \tilde{\tau} = a + b N \), then

\[
\nu_f^{-1} \circ \nu_f \circ df \tilde{\tau} = \nu_f^{-1}(a + ib) = a + bN. 
\]

Substituting \( df \tilde{\tau} = \lambda^{-1}(a + bN) \lambda \) in \( df \lambda \tilde{\tau} \lambda = (\lambda^{-1} df \lambda^{-1} \lambda \tilde{\tau} \lambda = \frac{1}{|\lambda|} (a + bN) |\lambda|^2 \) yields \( df \lambda \tilde{\tau} \lambda = a + bN = \nu_f^{-1} \circ \nu_f \circ df \tilde{\tau} \). \( \square \)

Remark 7 Observation \( \text{[5,1]} \) implies that the branch points of \( f^* \) are precisely the zeroes of the shape distortion operator \( D \) and therefore they are precisely the umbilic points of \( f_\pm \) (10). The branch points of \( f^* \) are umbilic points of \( f \) but \( f \) could have other umbilic points too. Moreover, the order of vanishing of \( \omega \) at the branch points is at least equal to the order of vanishing of \( df^* \), in particular it follows that \( \omega = \text{adf}^* \), where the real valued coefficient a is bounded, possibly continuous, at the branch points of \( f^* \), and smooth away from them.

Now we can give the construction which yields all Bonnet surfaces which are not included in the Bonnet-Cartan-Chern classification. By necessity these surfaces must have isolated umbilics and their mean curvature is not constant in a neighborhood of the umbilic.
Theorem 3

(i.) Let $f_{\pm}$ be two Bonnet mates defined by (11). A point $p_0$ is umbilic for $f_{\pm}$ if and only if $p_0$ is a branch point for $f^*$. 

(ii.) If $f$ is a non-minimal isothermic immersion with a dual $f^*$ then there exists at most one $\epsilon > 0$ such that the Bonnet mates $f_{\pm}$ defined by (11) have constant mean curvature. In particular, let $f$ be an immersion with positive constant mean curvature and let $f^*$ be a dual map for $f$, then for every $\epsilon$ the Bonnet mates $f_{\pm}$ defined by (11) have non-constant mean curvature.

Proof Part (i) follows from Remark 7. To prove part (ii) fix a non-minimal isothermic immersion $f$, and a dual $f^*$. If there exists a positive $\epsilon$ such that the corresponding Bonnet mates $f_{\pm}$ have constant mean curvature $c$, then $c = H/(|f^*|^2 + \epsilon^2)$, and so $c$ is the unique solution of $dH = cd|f^*|^2$ (if such exists). But then $\epsilon$ is uniquely determined also. $\square$

Thus all new Bonnet surfaces are obtained by applying the construction to isothermic immersions whose duals have branch points. To produce explicit examples simply take $f$ to be the Mr. Bubble constant mean curvature immersion.

Corollary 5.1 There exist non-constant mean curvature Bonnet mates with umbilic points.

Remark 8 Since isothermic surfaces could be only $C^1$, the construction in [13] indicates that we can make sense of “Bonnet surfaces” with very low smoothness, say $C^1$. The classical results required at least $C^5$. [12]. (See [12] for further discussion.)

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