Abstract

Subatomic systems were recently introduced to identify the structural principles underpinning the normalization of proofs. “Subatomic” means that we can reformulate logical systems in accordance with two principles. The atomic formulas become instances of sub-atoms, i.e. of non-commutative self-dual relations among logical constants, and their rules are derivable by means of a unique deductive scheme, the medial shape. One of the neat results is that the cut-elimination of subatomic systems implies the cut-elimination of every standard system we can represent sub-atomically.

We here introduce Subatomic systems-1.1. They relax and widen the properties that the sub-atoms of Subatomic systems can satisfy while maintaining the use of the medial shape as their only inference principle. Since sub-atoms can operate directly on variables we introduce \( P \). The cut-elimination of \( P \) is a corollary of the cut-elimination that we prove for Subatomic systems-1.1. Moreover, \( P \) is sound and complete with respect to the clone at the top of Post’s Lattice. I.e. \( P \) proves all and only the tautologies that contain conjunctions, disjunctions and projections. So, \( P \) extends Propositional logic without any encoding of its atoms as sub-atoms of \( P \).

This shows that the logical principles underpinning Subatomic systems also apply outside the sub-atomic level which they are conceived to work at. We reinforce this point of view by introducing the set of medial shapes \( R_d \). The formulas that the rules in \( R_d \) deal with belong to the union of two disjoint clones of Post’s Lattice. The SAT-problem of the first clone is in \( P\)-Time. The SAT-problem of the other is \( NP\)-Time complete. So, \( R_d \) and the proof technology of Subatomic systems could help to identify proof-theoretical properties that highlight the phase transition from \( P\)-Time to \( NP\)-Time complete satisfiability.

1 Introduction

Subatomic systems were recently introduced to identify the structural principles underpinning the normalization of proofs. “Subatomic” means that we can reformulate logical systems in accordance with two principles. The atomic constituents of the formulas become instances of sub-atoms, i.e. of non-commutative self-dual relations among logical constants, and the rules are derivable by means of a unique deductive scheme, the medial shape. In its not full, but general enough, form it is:

\[
\frac{(A \land B) \land (C \lor D) \land (A \lor C) \lor (B \land D)}{A \land C \lor (B \land D)}
\]

where \( A, B, C, D \) are formulas and \( \alpha, \beta, \gamma, \delta \) relations. For example, let us focus on propositional logic. The sub-atomic rule

\[
\frac{(A \lor B) \land (C \lor D)}{(A \lor C) \lor (B \land D)}
\]

stands for the introduction to the right of the conjunction. It is a rule in deep inference which we can read as follows. Let \( A \lor B \) and \( C \lor D \) be two given disjunctions where \( B \) is the premise that allows to derive \( A \) and \( D \) the one for deriving \( C \). Then, the rule derives \( A \land C \) from the premise \( B \lor D \). The sub-atomic rule \( \frac{(f \lor t) a (t \lor f)}{(f a t) \lor (t a f)} \) represents the excluded-middle \( \frac{t}{a \lor \overline{a}} \). The sub-atoms \( f a t \) and \( t a f \) stand for the atoms \( a \) and \( \overline{a} \), respectively, where \( a \) is a self-dual non commutative relation which obeys the equivalence \( (f \lor t) a (t \lor f) = t a t = t \). Instead, the rule \( \frac{(f a t) \lor (f a t)}{(f \lor f) a (t \lor t)} \) corresponds to the contraction \( \frac{a \lor a}{a} \). Under the same representation of \( a \) as before, the conclusion represents \( a \) up to the standard equivalences \( f \lor f = f \) and \( t \lor t = t \).

One reason why Subatomic systems are a deep inference formalism is that they target the representation of a class of logical systems as wide as possible which may well include self-dual non-commutative logical operators and we know that there cannot be analytic and complete Gentzen (linear) proof systems with self-dual non-commutative connectives in...
them [10]. Another reason is that, by means of the uniform representation they allow, Subatomic systems help to identify sufficient conditions to characterize proof systems that enjoy decomposition, i.e. the reorganization of contractions inside a proof, and cut-elimination. This is possible because Subatomic systems abstract at the right level the proofs of decomposition and of cut-elimination that the literature contains in relation to deep inference logical systems for classical, modal, linear and sub-structural logics.

Very briefly, deep inference looks at deductive processes as rewriting procedures where rules apply to an arbitrary depth in the syntax tree of formulas. This is equivalent to saying that deep inference logical systems compose derivations and formulas exactly with the same set of logical connectives. Subatomic systems witness how effective the reduction can be of syntactic bureaucracy that follows from the deep inference approach to proof theory to get closer to the semantic nature of proof and proof normalization. An informative survey about deep inference is [3]. An up-to-date information about its literature is [4].

This paper introduces Subatomic systems-1.1 (Section 2), a slight generalization of the original Subatomic system in [11, 12] that we dub as version 1.0, for easiness of reference. Version 1.1 relaxes and widen the properties that the sub-atoms of version 1.0 can satisfy while maintaining the use of the medial shape as the only inference principle. As effect of the generalization, the formulas of Subatomic systems 1.1 build also on variables. Hence, we can introduce P (Section 3.) We show that P is sound and complete with respect to the clone at the top of Post’s Lattice (Section 6.) I.e. P proves all and only the tautologies that contain conjunctions, disjunctions and the self-dual projections π₀ and π₁. So, P extends Propositional logic without any encoding of its atoms as sub-atoms of P. We also prove that the cut and other rules are admissible for a specific fragment of P (Section 5.) The proof is a corollary of the same property that we prove for version 1.1 and which extends the one for version 1.0 (Section 4.)

The existence of P shows that the logical principles underpinning Subatomic systems also apply outside the subatomic level which they are conceived to work at. We reinforce this idea by introducing the set R₂₃ of medial shapes (Section 7.) The formulas that occur in the rules of R₂₃ belong to the union of the two clones C₂ and C₃ of Post’s Lattice [7]. Both C₂ and C₃ are two of the five maximal clones strictly contained in C₁. The logical operators that build the formulas of C₂ and C₃ are strongly interrelated but the satisfiability problem for C₂ is in P-Time while the one for C₃ is NP-Time complete. That R₂₃ can be a Subatomic system-1.1 is still an open question. The conjecture is that we need a further extension of Subatomic systems to prove a cut-elimination for a system with R₂₃ as its core. The relevance of R₂₃ is twofold. On one side, it can help focusing on proof-theoretical properties that highlight how and when the phase transition from the satisfiability in P-Time to the satisfiability in the class of NP-Time complete problems occurs. On the other, the way we obtain R₂₃ strongly suggests that Subatomic systems can be viewed as a framework where looking for grammars that follow a very regular pattern able to generate possibly interesting logical systems, so contributing to the so called systematic proof theory [1]. The side effect would be that the larger will be the class of interesting logical systems that we can generate by means of Subatomic system, the clearer the reason could be why the medial scheme rule is so pervasive, something that, so far, has no a priory convincing explanation.

2 Subatomic systems-1.1

We generalize Subatomic systems-1.0 [11, 12] to Subatomic systems-1.1.

Definition 1 (Subatomic systems-1.1). Let U be a denumerable set of constants t,u,v,... Let V be a denumerable set of variables x,y,w,... Let R be a denumerable set of symbol relations α,β,... and let < ≤ R² be a partial order among the symbols in R. Let F := U | V | R' R F generate formulas A, B, C,... Let InV := (U → U) ∪ (V → V) ∪ (R → R) be an involutive negation:

\[ \overline{A} = \begin{cases} \pi & \text{if } A = u \text{ and } u \in U \\ x & \text{if } A = x \text{ and } x \in V \\ \overline{B} \pi \overline{C} & \text{if } A = B \alpha C \text{ and } B \alpha C \in F \end{cases} \]

Fixed n ∈ N, let = ≤ F² be the least congruence on F generated by any subset E₁ = F₁,..., E₀ = F₀ of axioms taken
A Subatomic system-1.1 $S$ on $\mathcal{F}$, $\mathcal{R}$, $\prec$ and $=$ has all and only the instances of the following schemes:

| Down-rules | Up-rules |
|------------|----------|
| **Splitting** | $(A \beta B) \rightarrow (C \beta D)$ | $(A \alpha C) \rightarrow (B \overline{D})$  |
| | $(A \alpha C) \rightarrow (B \overline{D})$ | $(A \beta B) \rightarrow (C \beta D)$  |
| **Contractive** | $(A \alpha B) \rightarrow (C \alpha D)$ | $(A \beta C) \rightarrow (B \alpha D)$  |
| | $(A \beta C) \rightarrow (B \alpha D)$ | $(A \alpha B) \rightarrow (C \alpha D)$  |
| **Equational** | $E_i \overline{F_i}$ | $E_i \overline{F_i}$  |

Let $\mathcal{F}$ contain variables of $\mathcal{V}$ and the set of axioms is extended in two directions. Axioms (4), (5) admit the existence of relations that erase structure. Axioms (7) and (8) allow the existence of relations among constants and variables. This extends the proof theoretical technology of Subatomic systems-1.0 outside its intrinsic sub-atomic nature.

**Notation and terminology.** Let $S$ be a Subatomic system-1.1 with formulas $\mathcal{F}$ built on the symbols in $\mathcal{R}$. Let $\prec$ be the \textbf{order relation} on $\mathcal{R}$. A context $S\{}$ is a formula $A \in \mathcal{F}$ with any of its sub-formulas, possibly $A$ itself, erased. In the last case $S\{}$ is $\{\}$. A relation $\alpha$ of $S$ is \textbf{unitary} if it enjoys axiom (3). A relation $\alpha$ is a \textbf{right weakening} if it enjoys (4) and is a left weakening if (5) holds for it. A relation $\alpha \in \mathcal{R}$ is strong if no $\beta \in \mathcal{R}$ exists such that $\beta \prec \alpha$. A relation $\alpha \in \mathcal{R}$ is weak if no $\beta \in \mathcal{R}$ exists such that $\alpha \prec \beta$. The map $\overline{\alpha}$ is $\prec$-consistent if a strong $\alpha \in \mathcal{R}$ implies that $\overline{\alpha}$ is weak, and vice versa. A derivation $\frac{A}{B}$ of $S$ from $A$ to $B$ is any obvious concatenation of rules instances of $S$.

**Remark 1.** Strong relations are defined as minimal elements of the partial order $\prec \subseteq \mathcal{R}$. Dually, weak relations are maximal elements. We share this terminological choice with [12]. The justification is semantical. A relation is strong if its truth implies the truth of a weaker one. For example, the classical conjunction is strong and the classical disjunction weak.

**Proposition 1 (Excluded middle).** Let $S$ be a Subatomic system-1.1 with $=$ as its equational theory. Let $\alpha \in \mathcal{R}$ be strong. Let the following instances of (6) and (8) hold in $S$:

- $\forall v \overline{v} \overline{v} = u_o$  \hfill (10)
- $\forall u \gamma \overline{u} = u_o$  \hfill (11)
- $\forall x \overline{\gamma} \overline{x} = u_o$  \hfill (12)

where $u_o$ is a single and distinguished element of $\mathcal{U}$. The rule $\frac{u_o}{A \overline{A}}$ is derivable, for every $A \in \mathcal{F}$. 

Like in [11, 12], the role of Definition 1 is to delineate the formal framework we are going to work in. The constraints on the framework are very lax. It should not surprise how simple is to think of semantically meaningless instances of Subatomic systems where, for example, the two propositional constants $T$ (true) and $F$ (false) exist and are equated by an instance of (9).
Proof. The proof is by induction on the structure of \( A \). The two base cases with \( A = x \) or \( A = v \) hold because (12) and (10) hold in the given \( S \). Let \( A \in A_0 \gamma A_1 \) where, we underline, \( \gamma \) can also be \( \alpha \) itself. Moreover, \( \gamma \) strong implies that \( \overline{\gamma} \) weak.

\[
\begin{array}{c}
\alpha \quad (\text{inductive hypothesis}) \\
A_0 \gamma A_1 \\
\end{array}
\]

Then \( (A_0 \overline{A_0}) \gamma (A_1 \overline{A_1}) \). \( \gamma < \overline{\alpha} \).

\[
\begin{array}{c}
(A_0 \gamma A_1) \overline{A} \quad (\text{inductive hypothesis}) \\
(A_0 \gamma A_1) \overline{A} \quad (\text{inductive hypothesis}) \\
\end{array}
\]

Proposition 1 justifies the following:

Definition 2 (Unit). The constant \( u_0 \in \mathcal{U} \) is a unit if it enjoys axioms (10), (11) and (12).

Proposition 2 (Contraction). Let \( S \) be a Subatomic system-1.1 with = as its equational theory. Let \( \beta \in \mathcal{R} \) be weak. Let the following instances of (1), (2), (6) and (7) hold in \( S \):

\[
(A \beta B) \beta C = A \beta (B \beta C) \\
A \beta B = B \beta A \\
v \beta v = v \\
x \beta x = x
\]

The rule \( \frac{A \beta A}{A} \) is derivable, for every \( A \in \mathcal{F} \).

Proof. The proof is by induction on the structure of \( A \). The base cases \( A = x \) and \( A = v \) holds because (15) and (16) hold in the given \( S \). Let \( A \in A_0 \gamma A_1 \), for any \( \gamma < \beta \). Then the following derivation \( (A_0 \beta A_0) \gamma (A_1 \beta A_1) \) exists.

\[
\begin{array}{c}
(A_0 \gamma A_1) \overline{A} \\
\end{array}
\]

Finally, let \( A \in A_0 \beta A_1 \). Then

\[
\begin{array}{c}
(A_0 \beta A_1) \beta (A_0 \beta A_1) \\
\end{array}
\]

Propositions 1 and 2 say that the medial shape is an invariant of two inference mechanisms. One is “Splitting” or, dually, “annihilation”. It distributes negation. So, the proofs of a Subatomic system-1.1 can start from units which split into a pair of structures that annihilate each other. The other is “Contraction” or, dually, “sharing”. It distributes sub-formulas with the goal of identifying two occurrences of the same formula into a single one. This is a consequence of a step-wise deductive process that reduces the global identification to the identification on constants or variables only.

Fact 1 (Equation derivations). Let \( \mathcal{D} \) be a derivation that only contains equation rules of a given Subatomic system-1.1 \( S \). We can obtain derivations of \( S \) from \( \mathcal{D} \) in two steps: (i) negating every of formula of \( \mathcal{D} \), (ii) flipping \( \mathcal{D} \) up-side down.

3 The Subatomic system-1.1 \( \mathcal{P} \)

We introduce the instance \( \mathcal{P} \) of Subatomic systems-1.1 which we could not see how to obtain as an instance of Subatomic systems-1.0 [11, 12].

Definition 3 (Formulas of \( \mathcal{P} \)). Let \( \mathcal{F}_p \) be the language of formulas generated by:

\[
A, B ::= T \mid F \mid \forall v \mid \overline{\forall v} \mid A \land B \mid A \pi_0 B \mid A \pi_1 B \mid A \lor B
\]

The set \( \mathcal{V}_p \) contains the variables \( x, y, z, \ldots \) and \( \overline{\mathcal{V}_p} \) their negations. Both \( \pi_0 \) and \( \pi_1 \) stand for the self-dual projections on first or second argument, respectively.

Definition 4 (Order relation among the relations of \( \mathcal{P} \)). The operator \( \land \) is strong, \( \lor \) is weak and every \( \pi_i \) is in between. i.e. \( A \land B \prec_{\mathcal{P}} A \pi_0 B, A \pi_1 B \prec_{\mathcal{P}} A \lor B \).

The order relation of Definition 4 originates from the following lattice which pointwise sorts the boolean functions it contains under the assumption that \( F \) is smaller than \( T \).
Definition 5 (Negation among formulas of $P$). For every $x, A, B \in \mathcal{T}_P$, let $\overline{\cdot}$ be the following involutive and $\preceq_{P}$-consistent negation:

\[
\begin{align*}
\overline{T} &= F \\
\overline{F} &= T \\
\overline{A} &= x \\
\overline{A \lor B} &= \overline{A} \land \overline{B} \\
\overline{A \land B} &= \overline{A} \lor \overline{B} \\
\overline{A \pi_i B} &= \overline{A} \pi_i \overline{B}
\end{align*}
\]  

(\forall x \in \mathcal{V})

(17)

(18)

(19)

Axiom (19) sets $\pi_0$ and $\pi_1$ to be self-dual operators like the boolean functions they represent.

Definition 6 (Congruence on formulas of $P$). Let $F$ be the unit $u_\lor$ of $\lor$ and $T$ the unit $u_\land$ of $\land$. For every $A, B$ and $C$ in $\mathcal{T}_P$, let $\preceq_{P}$ be the congruence that the following axioms induce:

\[
\begin{align*}
(A \lor B) \alpha C &= A \alpha (B \lor C) & (\forall A, B, C \in \mathcal{F} \text{ and } \alpha \in \{\pi_0, \pi_1, \lor, \land\}) \\
A \alpha B &= B \alpha A & (\forall A, B \in \mathcal{F} \text{ and } \alpha \in \{\lor, \land\}) \\
A \lor F &= A & (\forall A \in \mathcal{F}) \\
A \pi_0 B &= A & (\forall A, B \in \mathcal{F}) \\
A \pi_1 B &= B & (\forall A, B \in \mathcal{F}) \\
u \land F &= F & (u \in \{F, T\}) \\
u \lor \overline{u} &= T & (u \in \{F, T\}) \\
x \lor \overline{x} &= T & (\forall x \in \mathcal{V}_P) \\
u \lor u &= u & (u \in \{F, T\}) \\
x \land x &= x & (\forall x \in \mathcal{V}_P)
\end{align*}
\]  

(20)

(21)

(22)

(23)

(24)

(25)

(26)

(27)

(28)

(29)

Definition 6 gives the least set of axioms. The missing ones can be derived by negation.

Definition 7 (System $P$). $P$ contains the rules:

\[
\begin{align*}
ai_1\downarrow (A \lor B) \pi_j (C \lor D) &= (A \pi_j C) \lor (B \pi_j D) & j \in \{0, 1\} \\
ai_1\uparrow (A \pi_j C) \land (B \pi_j D) &= (A \lor B) \pi_j (C \land D) & j \in \{0, 1\} \\
ni_1\downarrow (A \lor B) \land (C \land D) &= (A \land C) \lor (B \lor D) \\
ni_1\uparrow (A \land C) \lor (B \lor D) &= (A \lor B) \land (C \land D) \\
m_1\downarrow (A \land B) \lor (C \land D) &= (A \lor C) \land (B \lor D) \\
m_1\uparrow (A \lor C) \land (B \lor D) &= (A \land B) \lor (C \land D) \\
c_1\downarrow (A \pi_j B) \lor (C \pi_j D) &= (A \lor C) \pi_j (B \lor D) & j \in \{0, 1\} \\
c_1\uparrow (A \pi_j B) \land (C \pi_j D) &= (A \lor C) \pi_j (B \land D) & j \in \{0, 1\}
\end{align*}
\]
with formulas of $\mathcal{T}_P$ (Definition 3) taken up to both $\Rightarrow_p$ (Definition 6) and the negation in Definition 5, with $\land, \pi_0, \pi_1$ and $\lor$ ordered under $\lhd_p$ (Definition 4).

So, $P$ is a Subatomic system-1.1 because its formalization fits in the framework of Definition 1. Hence, Proposition 1, axioms (20), (21), (28) and (29), and the rules $a_i\downarrow, a_i\downarrow, s_i\downarrow$ imply:

**Corollary 1** (Excluded middle in $P$). For every $A \in \mathcal{T}_P$, the rule $\dfrac{T}{A \lor A}$ is derivable.

Moreover, Proposition 2, axiom (22), (23), (24), (26) and (27), and the rules $m\downarrow, c_0\downarrow, c_1\downarrow$ imply:

**Corollary 2** (Idempotence in $P$). For every $A \in \mathcal{T}_P$, the rule $\dfrac{A \lor A}{A}$ is derivable.

**Remark 2.** As far as we can see, $P$ cannot be a Subatomic system-1.0, in accordance with Definition 2.5 in [11, page 10] and [12, page 6]. The axiom scheme (3) of those two works classifies every unitary relation $\alpha$ as one for which we have:

$$A \alpha u = A = u \alpha A.$$  \hfill (30)

However, the natural behavior of the relations $\pi_0$ and $\pi_1$ of $P$ is given by (23) and (24), instances of (4) and (5). So, they cannot satisfy (30). We will see that the weaker behavior of $\pi_0$ and $\pi_1$ as compared to the one of a unitary relation requires to generalize the Splitting theorem (Section 4). 

**Remark 3.** There is an aspect of $P$ for which we have no convincing a priori justification. For every $j$, the rule $c_j\downarrow$ is $a_j\downarrow$ flipped up side down. Currently, we limit to observe that this is harmless. Both rules are semantically sound, i.e. the truth of the premise implies the one of the conclusion.

## 4 Cut-elimination in Subatomic systems-1.1

We here adapt the proof of the cut-elimination for Subatomic systems-1.0 [11, 12] to version 1.1.

**Definition 8** (Splittable down-fragment). Let $S$ be a Subatomic system-1.1. Then, $S\downarrow$ is the Splittable down-fragment of $S$ if:

1. $S\downarrow$ contains at least one weak relation;
2. For every weak relation $\beta$ in $S\downarrow$ with unit $u_\beta \in U$ the following axioms hold:

$$\overline{u_\beta} \alpha \overline{u_\beta} = \overline{u_\beta}$$ \hfill (31)

$$(A \beta B) \beta C = A \beta (B \beta C)$$ \hfill (32)

$$A \beta B = B \beta A$$ \hfill (33)

$$u \beta \overline{u} = \overline{u_\beta}$$ \hfill (34)

$$A \beta u_\beta = A$$ \hfill (35)

$$x \beta \overline{x} = \overline{u_\beta}$$ \hfill (36)

3. $S\downarrow$ contains all and only the splitting and equational down-rules, as in Definition 1. So, it does not contain any contractive down-rule.

Like in [11, 12], axioms (31), (32) and (33) are strongly linked to the way that splitting works. Once decomposed a proof into independent subproofs, they can be composed back into a new proof exactly because the here above axioms hold. Also (34) is in [11, 12]. Instead, both (35) and (36) are new.

Symmetrically to Definition 8, we can identify the splittable down-fragment.

**Definition 9** (Splittable up-fragment). Let $S$ be a Subatomic system-1.1 with a Splittable down-fragment $S\downarrow$ as in Definition 8. The Splittable up-fragment $S\uparrow$ contains all and only the splitting and equational up-rules of $S$ that correspond to the rules in $S\downarrow$.

**Definition 10** (Length of a derivation). Let $S\downarrow$ be a Splittable Subatomic system-1.1. The length $|D|$ of a derivation $D$ in $S\downarrow$ counts the number of rules of $D$ which do not correspond to the application of any axiom among (32), (33), (35), (34) and (36) at point 2 of Definition 8. I.e. axioms that involve weak relations do not contribute to the growth of the dimension of a derivation, while (31) does.

**Lemma 1** (Atomic deduction for Splittable Subatomic systems-1.1). Let $\beta$ be a weak relation with unit $u_\beta$ of a Splittable Subatomic system-1.1 $S\downarrow$. For every $u \in U$ and $B \in \mathcal{T}$, if $\vdash_{S\downarrow} u_\beta B$, then $\vdash_{S\downarrow} u \beta B$. Analogously, for every $x \in V$, if $\vdash_{S\downarrow} x_\beta B$, then $\vdash_{S\downarrow} x \beta B$. 

then $\vdash_{S\downarrow} x \beta B$ exists.
Theorem 1 (Shallow splitting). Let $\beta$ be a weak relation with unit $u_\beta$ in a Splittable Subatomic system $\text{I.1}$. For every $\alpha < \beta$, let $\frac{u_\beta}{B}$ be given.

1. If $\alpha$ is a right weakening, then $\frac{K_0 \parallel K_1}{B}$ exists such that $\frac{u_\beta}{A_0 \beta K_0}$ exists as well and $|Q_0| \leq |P|$. If $\alpha$ is a left weakening, replace 1 for 0.

2. If $\alpha$ is unitary, then $\frac{K_0 \parallel K_1}{B}$ exists such that, for every $i \in \{0, 1\}$, $\frac{u_\beta}{A_i \beta K_i}$ exists as well and $|Q_0| + |Q_1| \leq |P|$.

Proof. We prove both points simultaneously, proceeding by induction on $|P|$. The value of $|P|$ is at least 1 because $\alpha < \beta$ and $\alpha$ is not weak. Necessarily, an occurrence of (31) exists in $P$ which generates a formula out of $\frac{u_\beta}{\beta}$ with $\alpha$ in it.

- The base case is with $|P| = 1$ and (31) occurs in $P$. So, $P$ is composed by the three derivations $\frac{u_\beta}{B}$, $\frac{u_\beta}{B'}$, and $\frac{u_\beta}{B''}$ for every $i \in \{0, 1\}$, where $|P'| = |P''| = |P_0| = |P_1| = 0$. Lemma 1 holds on $P'$. So, $\frac{u_\beta}{B'}$ implies the existence of $D$ which is $\frac{u_\beta}{B'}$.

  Two cases are now possible.

  - Let $\alpha$ be unitary. For every $i \in \{0, 1\}$, the proof $Q_i$ is $\frac{u_\beta}{A_i \beta u_\beta}$. Moreover, $|Q_0| + |Q_1| = |P_0| + |P_1| < |P| = 1$ because none among $Q_0, Q_1, P_0$ and $P_1$ contain axioms that count 1.

  - If $\alpha$ is a right or a left weakening we proceed as here above, but focusing only on one of the two proofs $Q_0$ and $Q_1$.

- The inductive case has $|P| > 1$. We only develop the details of the relevant cases. The first relevant case is a refinement of point (3) in the original proof of Shallow splitting of [11, 12]. The refinement requires to consider the possibilities that we introduce a constant by distinguishing among unitary relations, right weakening and left weakening.

  - Let $\alpha$ and $\gamma$ be right weakening such that $P$ is $\frac{u_\beta}{(A_0 \alpha A_1) \beta B_0 \gamma C B_1}$. Because $|P'| < |P|$, by the inductive hypothesis $\frac{u_\beta}{A_0 \alpha A_1 \beta B_0 \gamma C B_1}$ exists such that $\frac{u_\beta}{(A_0 \alpha A_1) \beta B_0 \beta K_1}$ exists as well with $|Q'| \leq |P'| < |P|$. So, the inductive hypothesis holds on $Q'$ and a derivation $\frac{u_\beta}{(A_0 \alpha A_1) \beta B_0 \beta K_1}$ exists as well with $A_0 \beta K_0$. 

Proof. The derivation $D$ is $\frac{\alpha \parallel \beta u_\beta}{(\alpha \beta u_\beta) \beta u_\beta}$, while $D'$ is $\frac{\alpha \parallel \beta u_\beta}{(x \beta u_\beta) \beta u_\beta}$.
\[ |Q'| \leq |Q'| \leq |P'| < |P|. \] The proof we are looking for is \( Q' \). The derivation is (4) \( \frac{K_0 \alpha K_1}{B_0 \beta K_1} \), \( \frac{\alpha \not\vdash S_1}{B_0 \beta B_1} \), \( \frac{\varphi' \not\vdash S_1}{B_0 \beta B_1} \), \( \frac{s'}{B_0 \beta B_1} \).

Let \( \alpha \) be unitary and let \( \gamma \) be right weakening such that \( \mathcal{P} \) is (4) \( ((A_0 \alpha A_1) \beta B_0) \gamma C) \beta B_1 \). Because \( |P'| < |P| \), \( K_1 \not\vdash K_r \), by the inductive hypothesis \( \varphi' \not\vdash S_1 \), \( B_1 \), \( (A_0 \alpha A_1) \beta B_0 \beta K_2 \), \( K_0 \not\vdash K_1 \), exists such that \( \alpha \not\vdash S_1 \), \( (A_0 \alpha A_1) \beta B_0 \beta K_2 \), \( K_0 \not\vdash K_1 \), exists as well with \( |Q'| \leq |P'| < |P| \).

The inductive hypothesis holds on \( Q' \) and a derivation \( \varphi' \not\vdash S_1 \), exists such that, for every \( i \in [0, 1] \), the \( A_i \beta K_i \), \( K_0 \not\vdash K_1 \), \( \varphi' \not\vdash S_1 \), \( B_0 \beta B_1 \), \( \varphi' \not\vdash S_1 \), \( B_0 \beta B_1 \).

So, the inductive hypothesis holds on \( Q' \) and a derivation \( \alpha \not\vdash S_1 \), \( A_i \beta K_i \), \( K_0 \not\vdash K_1 \), \( \varphi' \not\vdash S_1 \), \( B_0 \beta B_1 \), \( \varphi' \not\vdash S_1 \), \( B_0 \beta B_1 \).

The cases with both \( \alpha \) and \( \gamma \) left weakening or with \( \alpha \) unitary and \( \gamma \) left weakening are symmetric.

The further relevant cases come from points (13) and (14) in the original proof of Shallow splitting of \([11, 12]\). In our case, point (13) requires to focus also on a right weakening \( \alpha \) in a proof \( \mathcal{P} \) with form \( \frac{\varphi' \not\vdash S_1}{A_0 \beta B} \). From (4) we get that \( \mathcal{D} \) is \( \frac{B \not\vdash K_r}{B} \). So, the proof \( Q \) is simply \( Q' \). For the analogous of (14) with a left weakening it is enough to proceed as just done her above.

**Definition 11 (Provable context).** Let \( \beta \) be a weak relation with unit \( u_\beta \) in some Subatomic system-1.1. A context \( H \) is provable if \( H[K] = u_\beta \).

Theorem 1 implies that Context reduction holds exactly as formulated and proved in \([11, 12]\):

**Theorem 2 (Context reduction).** Let \( \mathcal{S} \) be a Subatomic system-1.1 whose fragment \( \mathcal{S}_1 \) is splittable. Let \( \beta \) a weak relation in \( \mathcal{S} \) with unit \( u_\beta \). For every \( A \in \mathcal{F} \) and context \( S \), if \( \varphi' \not\vdash S_1 \), then there is \( K \in \mathcal{F} \) and a provable context \( H \) such that \( H[A] \).

**Theorem 3 (Splittable up-fragment is admissible).** Let \( \mathcal{S} \) be a Subatomic system-1.1 with splittable \( \mathcal{S}_1 \) and \( \mathcal{S}_1 \) in it. Let \( A, B, C, D \in \mathcal{F} \) and \( S \) be a context. Let \( \alpha \in \mathcal{R} \) be strong. For every \( \gamma \in \mathcal{R} \) such that \( \alpha < \gamma \), if \( \varphi' \not\vdash S_1 \), \( A \beta B \), \( S[A] \), then \( \alpha \not\vdash S_1 \), \( A \beta B \), \( S[A] \), which means that \( \rho' \) is admissible in \( \mathcal{S}_1 \).

**Proof.** We develop a case specific to version 1.1 where \( \gamma \) and, hence \( \overline{\gamma} \), is a right weakening. Theorem 2 on \( \mathcal{P} \) implies \( \varphi' \not\vdash S_1 \), \( \alpha \not\vdash S_1 \), \( (\alpha \beta C) \gamma (B \alpha C) \), \( \gamma Q_1 \beta Q_2 \), \( \alpha \not\vdash S_1 \), \( (A \gamma C) \beta Q_3 \), \( \alpha \not\vdash S_1 \), \( (C \gamma D) \beta Q_2 \), \( \alpha \not\vdash S_1 \), \( \alpha \not\vdash S_1 \), \( \alpha \not\vdash S_1 \), \( \alpha \not\vdash S_1 \), \( \gamma Q_1 \beta Q_2 \). Theorem 1 on \( Q_1 \) implies \( \alpha \not\vdash S_1 \), \( \alpha \not\vdash S_1 \), \( \alpha \not\vdash S_1 \), \( \alpha \not\vdash S_1 \), \( \gamma Q_1 \beta Q_2 \). Theorem 1 on \( Q_2 \) implies \( \alpha \not\vdash S_1 \), \( \alpha \not\vdash S_1 \), \( \alpha \not\vdash S_1 \), \( \alpha \not\vdash S_1 \), \( \gamma Q_1 \beta Q_2 \).
\[
\frac{H(\pi_\gamma)}{H(\pi_\gamma \alpha \pi_\beta)} \quad H((\alpha \beta Q_\lambda) \alpha (C \beta Q_C)) \\
H((\alpha \beta Q_\lambda) \alpha (C \beta Q_C)) \\
\square
\]

5 The splittable fragment \( P \downarrow \) in \( P \)

In this section we take advantage of having identified the properties that a Subatomic system-1.1 must meet to enjoy the cut-elimination property. From Definitions 8, 9 and 7 it follows:

**Fact 2.** The Splittable down-fragment \( P \downarrow \) of \( P \) contains the down-rules:

\[
\begin{align*}
\text{ai}_{\downarrow} & \quad (A \lor B) \pi_0 (C \lor D) \\
& \quad (A \pi_0 C) \lor (B \pi_0 D) \\
\text{ai}_{\downarrow} & \quad (A \lor B) \pi_1 (C \lor D) \\
& \quad (A \pi_1 C) \lor (B \pi_1 D) \\
\text{s}_{\downarrow} & \quad (A \lor B) \land (C \lor D) \\
& \quad (A \land C) \lor (B \land D)
\end{align*}
\]

while the Splittable up-fragment \( P \uparrow \) of \( P \) contains the up-rules:

\[
\begin{align*}
\text{ai}_{\uparrow} & \quad (A \pi_0 B) \land (C \pi_0 D) \\
& \quad (A \land C) \pi_0 (B \land D) \\
\text{ai}_{\uparrow} & \quad (A \pi_1 B) \land (C \pi_1 D) \\
& \quad (A \land C) \pi_1 (B \land D) \\
\text{s}_{\uparrow} & \quad (A \lor B) \land (C \land D) \\
& \quad (A \land C) \lor (B \land D)
\end{align*}
\]

Theorem 3 holds on \( P \), hence on the subset of rules of \( P \downarrow \) and \( P \uparrow \). So, we get:

**Corollary 3.** Every up-rule of \( P \uparrow \) is admissible in \( P \downarrow \).

6 The system \( P \) and Post’s Lattice

We show that \( P \) is related to Post’s Lattice [7]. It follows that \( P \) extends Propositional logic without relying on any representations of the atoms of \( P \) in terms of sub-atoms, i.e. in terms of some encoding which is based on self-dual non-commutative relations.

**Definition 12 (Clones [7]).** Let \( B \) be a set of boolean operators. A clone \([B]\) is the least set of boolean operators of any arity, closed under composition that contains: (i) propositional variables \( x, y, z, \ldots \); (ii) projections of every finite arity, \( \pi_1(x) = x \) included; (iii) \( f \in B \) applied to propositional variables.

The class of all clones is Post’s Lattice which is infinite and complete [7]. The top of the lattice is \( C_1 = [\lor, T, -] \) which strictly contains five pairwise incomparable maximal clones:

\[
\begin{align*}
C_2 &= [\land, \rightarrow, \leftrightarrow, \lor, T] \\
C_3 &= [F, \land, \lor, \leftrightarrow, \lor] \\
L_1 &= [F, \lor, \land, T] \\
A_1 &= [F, \leftrightarrow, \land, T, -] \\
D_3 &= [(x \land y) \lor (y \land z) \lor (x \land z), (x \land y) \lor (y \land z) \lor (x \land z)]
\end{align*}
\]

whose names come from [7, 6].
Proposition 3 (Soundness of $\mathcal{P}$). The Subatomic system-1.1 $\mathcal{P}$ is sound for $\mathcal{C}_1$. I.e., let $A, B_1, \ldots, B_n \in \mathcal{F}_\mathcal{P}$ be such that $B \in \mathcal{C}_1$ exists and $A$ is equivalent to $B$, up to De Morgan equivalences. Given $\not\vdash\mathcal{P} \tau$, if $B_1 \land \ldots \land B_n$ is true, then $A$ is true.

Proof. $\mathcal{P}$ only contains rules of $\mathcal{P}$. By definition, the conclusion of every rule in $\mathcal{P}$ is true whenever its premise is true. Since the formula on top of $\mathcal{P}$ is $T$ also $A \lor B_1 \lor \ldots \lor B_n = B_1 \land \ldots \land B_n \lor A = B_1 \land \ldots \land B_n \Rightarrow A$ must be true. Forcefully, the truth of $B_1 \land \ldots \land B_n$ implies the one of $A$.

The proof of completeness follows a standard technique.

Definition 13. Let $A[x_1, \ldots, x_n]$ denote any formula of $\mathcal{F}_\mathcal{P}$ such that $x_1, \ldots, x_n$ are all and only its variables. Let $T_A$ be the following truth table of $A[x_1, \ldots, x_n]$:

| $x_1$ | $x_2$ | $\ldots$ | $x_n$ | $A[x_1, \ldots, x_n]$ |
|-------|-------|----------|-------|----------------------|
| $F$   | $F$   | $\ldots$ | $F$   | $\chi_0$             |
| $F$   | $F$   | $\ldots$ | $T$   | $\chi^1$             |
| $\vdots$ | $\vdots$ | $\ldots$ | $\vdots$ | $\chi^{2^n-1}$ |

where $\chi_1 \in \{F, T\}$, for every $0 \leq l \leq 2^n - 1$. For every $0 \leq l \leq 2^n - 1$ and every $B \in \{x_1, \ldots, x, A[x_1, \ldots, x_n]\}$ let $T_A(l, B)$ the entry of $T_A$ at line $l$ and column $B$. By definition, let $\tau$ be the following map:

$\tau(l, x_i) = x_i$ if $T_A(l, x_i) = T$ $\tau(l, x_i) = \overline{x_i}$ if $T_A(l, x_i) = F$

$\tau(l, A[x_1, \ldots, x_n]) = A[x_1, \ldots, x_n]$ if $T_A(l, A[x_1, \ldots, x_n]) = T$ $\tau(l, A[x_1, \ldots, x_n])$ if $T_A(l, A[x_1, \ldots, x_n]) = F$.

Fact 3 (Arbitrary projections in $\mathcal{P}$). For every $x_i \in \forall \mathcal{P}$, the derivation $\vdash \mathcal{P} \exists x_i \exists \pi_i(x_i, \ldots, \ldots)$ exists by applying suitable combinations of the axioms (23) and (24). The same holds for every $\forall x_i \in \forall \mathcal{P}$.

Proposition 4 (Compactness of $\mathcal{P}$). Let $A[x_1, \ldots, x_n] \in \mathcal{F}_\mathcal{P}$ be given. Then, for every $0 \leq l \leq 2^n - 1$, the proof $\vdash \mathcal{P} \tau(l, A[x_1, \ldots, x_n]) \lor \tau(l, x_0) \lor \ldots \lor \tau(l, x_n)$ exists.

Proof. We proceed by induction on the structure of $A[x_1, \ldots, x_n]$.

Let $A[x_1, \ldots, x_n]$ be $x_i$. If $\tau(l, x_i) = x_i$, then $\mathcal{P}$ is (27) $\overline{x_i \lor x_i}$. If $\tau(l, x_i) = \overline{x_i}$, then $\mathcal{P}$ is (27) $\overline{x_i \lor x_i}$. Let $A[x_1, \ldots, x_n]$ be $\pi_i(x_1, \ldots, x_n)$. If $\tau(l, \pi_i(x_1, \ldots, x_n)) = \pi_i(A[x_1, \ldots, x_n])$, then $\mathcal{P}$ is (19) $\overline{\pi_i(x_1, \ldots, x_n) \lor \pi_i(\ldots, x_i, \ldots, \ldots)}$.

Fact 3 $\tau(l, \pi_i(x_1, \ldots, x_n)) \lor \tau(l, \pi_i(x_1, \ldots, x_n))$

If $A \in \{A_0 \land A_1, A_0 \land A_1\}$ it is enough to standardly apply the inductive hypothesis.

Theorem 4 (Completeness of $\mathcal{P}$). The Subatomic system-1.1 $\mathcal{P}$ is complete for $\mathcal{C}_1$. I.e., let $A, B_1, \ldots, B_n \in \mathcal{F}_\mathcal{P}$ be such that $B \in \mathcal{C}_1$ exists and $A$ is equivalent to $B$, up to De Morgan equivalences. Let us also assume that the truth of $B_1 \land \ldots \land B_n$ implies the truth of $A$. Then $\vdash \mathcal{P} A \lor B_1 \lor \ldots \lor B_n$ exists.

Proof. The assumption saying that the truth of $B_1 \land \ldots \land B_n$ implies the truth of $A$ is equivalent to saying that $A \lor B_1 \lor \ldots \lor B_n$ is a tautology. To keep the proof readable we assume that $x, y$ are all and only the free variables of $A \lor B_1 \lor \ldots \lor B_n$.

Of course, what we are going to do, works for any finite set of variables in $A \lor B_1 \lor \ldots \lor B_n$. Proposition 4 assures the existence of $\vdash \mathcal{P} \tau(l, x) \lor \tau(l, x) \lor \tau(l, y)$ for every $1 \leq l \leq 2^2$, where $X$ shortens $(A \lor B_1 \lor \ldots \lor B_n)[x, y]$ and $2^2$ is the number of $\tau(l, X) \lor \tau(l, x) \lor \tau(l, y)$.
of lines that all the combinations of the literals $x, \overline{x}, y$ and $\overline{y}$ generate in the truth table of $\tau(l, X)$. In fact, for every $1 \leq l \leq 4$, the proof $\mathcal{P}_l$ has form $\mathcal{P}_l = \vdash_{\mathcal{P}_l} \mathcal{P}$ because $X$, i.e. $A \lor \overline{B}_1 \lor \ldots \lor \overline{B}_n$, is a tautology. So $\mathcal{D}$ we are looking for is $X \lor \tau(l, x) \lor \tau(l, y)$

\[
\begin{align*}
\vdash_T & \quad (X \lor y) \land (X \lor \overline{y}) \\
\vdash_1 & \quad (X \lor y) \lor (X \lor \overline{y}) \\
\vdash_2 & \quad (X \lor y) \lor (X \lor \overline{y}) \\
\vdash_3 & \quad (X \lor y) \lor (X \lor \overline{y}) \\
\vdash_4 & \quad (X \lor y) \lor (X \lor \overline{y}) \\
\vdash_5 & \quad (X \lor y) \lor (X \lor \overline{y}) \\
\vdash_6 & \quad (X \lor y) \lor (X \lor \overline{y}) \\
\vdash_7 & \quad (X \lor y) \lor (X \lor \overline{y}) \\
\vdash_8 & \quad (X \lor y) \lor (X \lor \overline{y}) \\
\vdash_9 & \quad (X \lor y) \lor (X \lor \overline{y}) \\
\vdash_{10} & \quad (X \lor y) \lor (X \lor \overline{y}) \\
\end{align*}
\]

\[\square\]

### 7 Conclusion and developments

This work highlights how much effective the work in [11, 12] that aims at identifying the core mechanism of cut-elimination is. The notion of Subatomic system allows to prove modularly, generally and once the cut-elimination of interesting deep inference systems. We show that the original notion of subatomic system can be slightly generalized. This allows to identify new logical systems without any need to encode their constants sub-atomically and without loosing splitting, i.e. cut-elimination. The Subatomic system-1.1 P, sound and complete for the tautologies of Post’s clone $\mathcal{C}_1$, is a witness of how that is possible. Of course, the introduction of $\mathcal{P}$ is not breathtaking, but logical systems that smoothly incorporate self-dual operators — in the case of $\mathcal{P}$ they are operators as natural as projections — and which keep maintaining good logical properties are not so common [5, 8, 9].

On going work aims at using the framework of subatomic systems, may be upgraded to some version $x, y$ — this work introduces release 1.1 —, for systematically identifying logical systems with good properties and, possibly, of some relevance. Saying it in another way, the idea is to use the pattern that subatomic systems suggest for contributing to systematic proof theory [1]. The following list of medial shapes should show to what extent this idea can be concrete and potentially interesting:

\[
\begin{align*}
\text{ai} & \quad (A \rightarrow B) \overline{\alpha}_0 (C \rightarrow D) \quad (A \overline{\alpha}_0 C) \rightarrow (B \overline{\alpha}_0 D) \\
\text{ai} & \quad (A \overline{\alpha}_0 C) \rightarrow (B \overline{\alpha}_0 D) \\
\text{ai} & \quad (A \overline{\alpha}_0 C) \rightarrow (B \overline{\alpha}_0 D) \\
\text{ai} & \quad (A \overline{\alpha}_0 C) \rightarrow (B \overline{\alpha}_0 D) \\
\text{ai} & \quad (A \overline{\alpha}_0 C) \rightarrow (B \overline{\alpha}_0 D) \\
\text{ai} & \quad (A \overline{\alpha}_0 C) \rightarrow (B \overline{\alpha}_0 D) \\
\text{ai} & \quad (A \overline{\alpha}_0 C) \rightarrow (B \overline{\alpha}_0 D) \\
\end{align*}
\]

The whole list is candidate to become a subatomic system $R_{23}$. Endowed it with the right equational theory among the propositional logic formulas that the rules infer, $R_{23}$ should derive tautologies of $\mathcal{C}_2 \cup \mathcal{C}_3$ we recall in Section 6. Lewis shows that the satisfiability of formulas of $\mathcal{C}_2$ belongs to P-Time problems while the satisfiability of formulas in $\mathcal{C}_3$ is NP-Time complete [6]. So, $R_{23}$ would be a logical system where looking for proof-theoretical properties that highlight the phase transition from P-Time to NP-Time complete satisfiability.

The rules of $R_{23}$ come from the following complete sub-lattice:
which is inside the complete lattice of binary boolean functions pointwise ordered in accordance with the convention that F is smaller than T. The lattice shows that it is natural to work with more than one weak relation in the same system. Both \( \rightarrow \) and \( \leftarrow \) are weak and play the same role as that played by \( \land \) in the lattice that drives the definition of \( P \) in Section 3. Two weak relations are required because the negation \( \leftarrow \) of \( \rightarrow \) is the least upper bound of \( \pi_0 \) and \( \pi_1 \) and not of \( \pi_0 \) and \( \pi_1 \) of which \( \rightarrow \) is greatest lower bound. Of course, symmetrically, the same observation holds for \( \leftarrow \).

The lattice here above should immediately suggests that the search of subatomic systems need not be confined to the set of sixteen two-valued boolean functions. For any \( k \geq 3 \), the use of \( k \)-valued operators as relations for subatomic systems is perfectly viable. For example, the subatomic system that corresponds to the paradigmatic deep inference system BV [5] can be seen as a system that uses 3-valued operators that define Coherence Spaces [2]. Considered the huge number of \( k \)-valued operators, as \( k \) grows, subatomic systems look like grammars that generate specific languages, i.e. logical systems, with good proof theoretical properties, of possible unexpected interest, as consequence of the consistent use of non standard logical operators. This should definitely make it evident the contribution that the introduction of Subatomic systems-1.1 can give to Systematic Proof Theory.

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