A Road To Compactness Through Guessing Models

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Abstract

The compactness phenomenon is one of the featured aspects of structuralism in mathematics. In simple and broad words, a compactness property holds in a structure if a related property is satisfied by sufficiently many substructures of that structure. With this phenomenon and its twin sibling “reflection”, modern set theory has settled many mathematical statements left undecided by the conventionally accepted formalism of mathematics, ZFC. There is a broad research program investigating whether a notion of compactness can universe-widely emerge without running into contradictions.

These notes are a survey about guessing models whose existence provides intriguing compactness phenomena. Most of the results in the manuscript are well-known. We shall reformulate, generalise and expand some of them. We also present some known applications of guessing models and state some open problems about them.

Keywords. Compactness, Guessing Model, Proper Forcing Axiom (PFA), Reflection

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1 Introduction

I left my home the day before my 17th birthday anniversary in 7104 to become an ultra set theorist. I was young and like many youths, so enthusiastic to devote my entire life to a meaningful answer to the ancient question of the Continuum Hypothesis. I so well remember the moment I was about to step into the History class. It was my first day, how would I forget it? The classroom was approachable from a corridor on the second floor surrounded by numerous old but familiar portraits of foundationers hanging on the walls. Although, I was so impressed by them not noticing the class had started. Closer to the door, was a paled shadow dancing over the portraits drowned in soft edges of the light. Running through them, they caressed my face; what a pleasant moment remained unique forever. I was late, of course! So late to a pretty long classroom gone deep into perfect silence. It was breathtaking. The professor or, hmm, better to say the silhouette in the tender light of the dawning sun, momentarily paused by my presence at the door frame. Time halted in my head. I sweated and did feel deep inside that she was staring me in anger. Moved inside by her nod while my heavy heart was still swinging. Lucky enough I was to find a sit on
the second row. That silhouette now I could recognise was an expert in collapsing magic. She continued her lesson: “A marvellous aspect of the sierra of large cardinals is that the power of many vigorous dragons conceals behind the climax of inaccessibility. They, though, appeared from time to time, would rapidly fly up to the scary condensed clouds of the death world. Let me tell you an astonishing epic that narrates a significant two-person adventure by Alfred the Leader and Paul the King in the history of sets. They revealed the weakly compact dragons hiding behind the branches of Aronszajn trees in the vastly frozen forests on the summit of inaccessibility. That persistently stimulated the young generations to pursue their dreams and seek familiar-looking visages of dragons around the world of large cardinals. Alas, it was not always a victorious path; many failed, many unreturned, and many lost. That's sad, I do know! An astonishing but perilous approach that wilful people would nevertheless take was to burn a feather of Simurgh to assist them in hunting a dragon and bringing it down to an accessible summit. It could be very well-respected and praised. There are celebrated stories. Once upon a time, when our world was a greener and larger place to live, two renowned and experienced explorers independently decided to discover the unattainable levels of strong compactness and supercompactness. They were Thomas (‘the author of the Green Book, excuse me?’, I interrupted the professor, but she was not unhappy this time) and Menachem, the logician. They unhesitatingly took long foot journeys on the cruel rocks of enormous mountains while the furious wind was heartlessly moving them back under the order of dragons. The fortune has no place there! It’s the kingdom of dragonish chaos. Of course, so committed they were to identify those dragons near the enormous branches of some pseudo-tree entities living there, again on unimaginably high inaccessible peaks. They accomplished! We will discuss those entities later, but be aware that they were never as tame as Aronszajn trees. Yet somewhat later, Christoph, a junior from Deutschland, pulled those dragons down to accessible mountains. An accomplishment that finally resulted through a phenomenal collaborative journey with Matteo di Torino in the discovery of a new extraordinary creature. A unique kind of baby dragon they named guessing model!”

2 Basics and Notation

For a cardinal $\theta$, $H_\theta = (H_\theta, \in)$ denotes the collection of sets with hereditary size less than $\theta$. For a set $x$, we denote the powerset of $x$ by $\mathcal{P}(x)$, and for a cardinal $\kappa$, we let $\mathcal{P}_\kappa(x) := \{ y \subseteq x : |y| < \kappa \}$. A set $S \subseteq \mathcal{P}_\kappa(x)$ is called stationary if for every function $F : \mathcal{P}_\omega(x) \to \mathcal{P}_\kappa(x)$, there is $A \in S$ such that $A$ is closed under $F$, i.e. $F(a) \subseteq A$, for every $a \in \mathcal{P}_\omega(A)$.

By a model $M$, we mean a set or class such that $(M, \in)$ satisfies a sufficiently strong fragment of ZFC. We use $\overline{M}$ for the transitive collapse of $(M, \in)$, where the transitive collapse map is often denoted by $\pi_M$. A set $x$ is bounded in $M$, if there is $y \in M$ with $x \subseteq y$. By a powerful model $\mathcal{H}$, we mean that $\mathcal{H}$ is a transitive model which contains every set bounded in $\mathcal{H}$. Note that $H_\theta$ is a powerful model.
Definition 2.1. Let $M$ be a model.

1. $M$ is $\delta$-close to an ordinal $\gamma$ if $\text{cof}(\sup(M \cap \gamma)) = \delta$.
2. $M$ has bounded uniform cofinality $\delta$ if for every $\gamma \in M \cap \text{ORD}$, $M \models \text{“cof}(\gamma) \geq \delta$”, then $M$ is $\delta$-close to $\gamma$, and
3. $M$ has uniform cofinality $\delta$, if, moreover, $\text{cof}(\sup(M \cap \text{ORD})) = \delta$.

Definition 2.2. Assume $M \subseteq N$ are models. Let $\delta \in M$ be an $N$-cardinal. The pair $(M, N)$ has the $\delta$-covering property if for every set $x \in N$ with $|x|^N \leq \delta$ which is bounded in $M$, there is $x' \in M$ with $|x'|^M \leq \delta$ such that $x \cap M \subseteq x'$.

Notation 2.3. For a set $X$, we let $\kappa_X$ be the least ordinal $\alpha \in X$ such that $\alpha \notin X$, we leave $\kappa_X$ undefined if such an $\alpha$ does not exist.

3 Guessing Models

A guessing (elementary) submodel is, roughly speaking, a set model in the universe which, to some extent and at some cost, is correct about the powerset operation. This paper is devoted to the compactness of the following intelligible property.

Definition 3.1. Let $X$ be a set. A set $x$ is said to be guessed in $X$, if there exists $x^* \in X$, such that $x^* \cap X = x \cap X$.

Obviously, every set inside $X$ is guessed in $X$, and that a set $x$ is guessed in $X$ if and only if $x \cap X$ is guessed in $X$.

3.1 Equivalent definitions

A set $x$ is $\delta$-approximated in a model $M$, for an $M$-cardinal $\delta$, if $x \cap a \in M$, for every $a \in M$ with $|a|^M < \delta$.

Definition 3.2 (Viale [20], Viale–Weiβ [21]). Suppose $M \subseteq N$ are models. Assume that $\delta \in M$ is a cardinal with $\delta \leq \kappa_M$. We say $M$ has the $\delta$-guessing property in $N$ if every $x \in N$ bounded and $\delta$-approximated in $M$ is guessed in $M$.

Working in a model $N$, observe that if $\delta \leq \delta'$, then every $\delta$-guessing model in $N$ is also $\delta'$-guessing in $N$. For the approximation side, if $x \subseteq X \in M$, then it is enough to show that for every $a \in P^N(X) \cap M$ with $|a|^M < \delta$, $a \cap x \in M$ holds. There is a less useful reformulation of the above definition which points more compactness out, that is $x$ is guessed in $M$ if and only if for every $a \in M$ of size less than $\delta$, $x \cap a$ is guessed in $M$. This trivial reformulation follows from the fact that if $x \subseteq M$ is guessed in $M$ with $|x|^M \in M \cap \kappa_M$, then $x \in M$!

Observe that if $M$ is transitive, then $x$ is guessed in $M$ if and only if $x \in M$. In fact, one can now reformulate Hamkins’ approximation property in terms of guessing property.
Definition 3.3 (Hamkins [9]). Let \((M, N)\) be a pair of transitive models with \(M \subseteq N\). Let \(\delta \in M\) be a regular cardinal in \(N\). Then \((M, N)\) has the \(\delta\)-approximation property if and only if \(M\) is \(\delta\)-guessing in \(N\).

Definition 3.4. We say that \(M\) is \(\delta\)-guessing or has the \(\delta\)-guessing property if and only if \(M\) has the \(\delta\)-guessing property in \(V\).

The following is well-known.

Lemma 3.5. Assume that \(M_i\) has the \(\delta\)-guessing property in \(M_{i+1}\), for \(i = 0, 1\). Suppose that \((M_0, M_1)\) has the \(\delta\)-covering property, then \(M_0\) has the \(\delta\)-guessing property in \(M_2\).

Proof. Assume that \(x \in M_2\) is bounded and \(\delta\)-approximated in \(M_0\). We first show that \(x\) is \(\delta\)-approximated in \(M_1\). Suppose \(a \in M_1\) is of \(M_1\)-cardinality less than \(\delta\). By the \(\delta\)-covering property, there is \(b \in M_0\) with \(a \subseteq b\) and \(|b|_{M_0} < \delta\). Now \(b \cap x \in M_0 \subseteq M_1\), and hence

\[
a \cap x = a \cap (b \cap x) \in M_1.
\]

Since \(M_1\) is \(\delta\)-guessing in \(M_2\), there is \(x_1 \in M_1\) with \(x_1 \cap M_1 = x \cap M_2\). Observe that \(x_1\) is bounded in \(M_0\). It is enough to show that \(x_1\) is \(\delta\)-approximated in \(M_0\). Thus pick any \(a \in M_0\) with \(|a|_{M_0} < \delta\). Since \(\delta \leq \kappa_{M_0}\), We have

\[
a \cap x_1 = a \cap x_2 \in M_0.
\]

Corollary 3.6. \(M\) is a \(\delta\)-guessing if and only if it is \(\delta\)-guessing in some powerful model.

Definition 3.7. Let \(M\) be a model with a well-defined \(\kappa_M\). Assume that \(\delta \leq \kappa_M\) is a regular cardinal in \(M\). We say \(M\) is a \(\delta\)-guessing elementary submodel (\(\delta\)-gesm) if \(M\) is an elementary submodel of a powerful model \(H\), and that \(M\) has the \(\delta\)-guessing property.

Notice that if \(\mathcal{H}' \prec \mathcal{H}\) are powerful models, and \(M \prec \mathcal{H}\) is a \(\delta\)-gesm with \(\delta \in \mathcal{H}' \in M\), then \(M \cap \mathcal{H}'\) is a \(\delta\)-gesm.

One can also reformulate guessing models in terms of functions, we leave the proof to the interested reader. However, we may use it without mentioning.

Proposition 3.8. Suppose \(\mathcal{H}\) is a powerful model. Then a model \(M \prec H\) is a \(\delta\)-gesm if and only if for every uncountable cardinal \(\gamma \in M\) and every \(f : \gamma \rightarrow 2\), if for all \(x \in M \cap \mathcal{P}_\delta(\gamma)\), \(f \upharpoonright x \in M\), then there is a function \(f^* \in M\) with \(\text{dom}(f^*) = \gamma\) such that \(f^* \upharpoonright M = f \upharpoonright M\).

There is still a rather useful reformulation of a guessing model.
Proposition 3.9 (Cox–Krueger [4]). Assume that $M$ is an elementary submodel of a powerful model. Then $M$ is a $\delta$-gesm if and only if $\overline{M}$ is $\pi_M(\delta)$-guessing.

The definition of a gesm demonstrates that though $M$ may not be correct about the power-set, it still does not compute it as badly as we can imagine. As we shall see, if $x$ is a set with some additional structure and is guessed in $M$, say $x \cap M = x^* \cap M$ for some $x^* \in M$, then $x^*$ can be roughly regarded as a structure similar to that of $x$, while containing $x \cap M$ as a substructure!

### 3.2 The geometry of gesms

What does a gesm look like? In this subsection, we intend to try to find a reasonable answer to this question. In other words, how does being a gesm impact the structure itself.

Intuitively, it would not seem easy to get more sets guessed with a small amount of approximation. To look at an extreme case, let us consider $\delta = 0$. This means that if $x \subseteq M$ is bounded in $M$, then there is $x^* \in M$ such that $x^* \cap M = x$. This itself implies that $M$ is really enormous. One can find such a model $M$, that is if $\overline{M} = V_{\gamma}$, for some limit ordinal $\gamma$.

Theorem 3.10 (Viale [20]). $M$ is a 0-gesm if and only if for some limit ordinal $\gamma$, $\overline{M} = V_{\gamma}$.

Let us not forget that we are interested in elementary submodels of $H_\theta$ with the guessing property. Alas, finding many models of the above form and of size less than a given uncountable regular cardinal roughly needs a supercompact cardinal. This was proved by Viale [20] using the the above theorem and a characterization of supercompactness due to Magidor [14]. Viale’s paper [20] contains characterizations of other large cardinals in terms of 0-gesms. Since our focus in this paper is on accessible cardinals, we encourage the interested reader to consult [20] for more results of above sort.

Theorem 3.11. Suppose $M$ is a $\delta$-gesm. Let $\theta \in M$ be a cardinal with $\text{cof}(\theta) \geq \delta$. Assume that $M$ is closed under sequences of length $<\text{cof}(\sup(M \cap \theta))$ which are bounded in $M$. Then $|M| \geq 2^{\text{cof}(\sup(M \cap \theta))}$.

*Proof.* Let $\lambda = \text{cof}(\sup(M \cap \theta))$, and let $c = \{\theta_\xi : \xi < \lambda\} \subseteq M \cap \theta$ be increasing and cofinal in $\sup(M \cap \theta)$. For every $z \subseteq \lambda$, let $c[z] = \{\theta_\xi : \xi \in z\}$. Note that $c[z]$ is bounded in $M$, as witnessed by $\theta \in M$. Note that also $c[z] \subseteq M$. We shall show that $c[z]$ is $\delta$-approximated in $M$. For every subset $a \in \mathcal{P}_\delta(\theta) \cap M$, $a \cap c[z]$ is a bounded sequence of length less than $\lambda$, since otherwise $\sup(a) = \sup(M \cap \theta) \in M$, and hence by elementarity, the cofinality of $\theta$ is less than $\delta$, which is a contradiction. Thus, by the assumption on the closure property, $a \cap c[z] \in M$. Thus $c[z]$ is $\delta$-approximated in $M$. Therefore, there is $c^*[z] \in M$ with $c^*[z] \cap M = c[z] \cap M = c[z]$. Observe that, by elementarity, $c^*[z]$ is unique. It is easily seen, since $c$ is one-to-one, that if $z \neq z'$, then $c^*[z] \neq c^*[z']$, and hence the function $z \rightarrow c^*[z]$ is an injection from $\mathcal{P}(\lambda)$ into $M$. \hfill \qed
Note that for every $n \in \omega$, being $n$-guessing is equivalent to being $\omega$-guessing. Thus every $\delta$-gesm is uncountable.

**Corollary 3.12** (Cox–Krueger [3]). If $M$ is an $\omega_1$-gesm of size $< 2^{\aleph_0}$, then $\omega_1 \subseteq M$, and for every cardinal $\theta \in M$ of uncountable cofinality, $\operatorname{cof}(\sup(M \cap \theta))$ is uncountable. In particular, if $M$ is of size $\omega_1$, then $\operatorname{cof}(M \cap \theta) = \omega_1$ and $M \cap \omega_2 \in \omega_2$.

It is straightforward to generalise the above corollary to $\delta \geq \omega_1$, when $M$ is sufficiently closed:

If $M$ is a $< \delta$-closed $\delta^+$-gesm size $\delta^+$, then $\delta \cap \delta^+ \subseteq \delta^+$ and for every cardinal $\theta \in M$ with $\operatorname{cof}(\theta) > \delta$, $\operatorname{cof}(\sup(M \cap \theta)) = \delta^+$. In particular, $\delta^+ \subseteq M$.

**Proposition 3.13** (Viale [20]). Suppose that $M$ is a $\delta$-gesm. If $2^{< \delta} < \kappa_\delta$, then $M$ is a 0-gesm, and thus both $\sup(M \cap \kappa_\delta)$ and $\kappa_\delta$ are inaccessible.

**Proof.** If $2^{< \delta} < \kappa_\delta$, then $2^{< \delta} \subseteq M$. It is enough to show that every set $x$ bounded in $M$ is $\delta$-approximated in $M$. Fix such an $x$, and let $a \in M$ be of size less than $\delta$. We have $|\mathcal{P}(a)| \leq 2^{< \delta}$, and hence $x \cap a \in \mathcal{P}(a) \subseteq M$.

This above proposition was also observed by Hachtman and Sinapova [8].

**Lemma 3.14** (Krueger [10]). Assume that $M$ is a $\delta^+$-guessing model with $\delta^+ \subseteq M$. Suppose that $\mu \leq \delta$ is a regular cardinal and $M^{< \mu} \subseteq M$. Then, for every $x \subseteq [M]^\mu$ which is bounded in $M$, there is $y \in M \cap [M]^{< \delta}$ with $x \subseteq y$.

**Proof.** Pick $X \in M$ with $x \subseteq X$. Fix a bijection $f : \mu \rightarrow x$, and let $x_\zeta = f(\zeta)$. By the closure property of $M$, we have $X := \{x_\zeta : \zeta < \mu\} \subseteq M$. Note that $X$ is bounded in $M$, as witnessed by $X \subseteq [X]^{< \mu}$. If $X$ is $\delta^+$-approximated in $M$, then it is guessed in $M$, and hence there is $X^* \in M$, with $X^* \cap M = X \cap M = X$. But then $x = \bigcup X^* \in M$. To see this, observe that if $X^* \not\subseteq M$, then $\delta^+ \leq |X^* \cap M| = |X| = \mu$, which is a contradiction. Thus $X^* \subseteq M$, and hence $X^* = X$, which in turn implies that $x = \bigcup X^* \in M$. On the other hand, if $X$ is not $\delta^+$-approximated in $M$, then there is $Y \in M$ of size $\leq \delta$ such that $Y \cap X \notin M$. We may assume that $Y \subseteq [X]^{< \mu}$. By the closure property of $M$, we have $|Y \cap X| = \mu$. As $\mu$ is regular and $X$ is increasing, we have $x \subseteq y = \bigcup Y$. It is easily seen that $|y| \leq \delta$.

**Corollary 3.15.** Assume that $M$ is a $\delta^+$-guessing model with $\delta^+ \subseteq M$. Suppose that $\delta$ is a regular cardinal and $M^{< \delta} \subseteq M$. Then $(M, V)$ has the $\delta$-covering property. In particular,

1. if $M$ is an $\omega_1$-guessing of size $\omega_1$, then $M$ is internally unbounded.
2. if $M$ is $\omega_1$-guessing in $M_1$ and $M_1$ is $\omega_1$-guessing in $M_2$, then $M$ is an $\omega_1$-guessing in $M_2$.

The last item above implies that having the $\omega_1$-approximation for pairs of transitive models is a transitive relation.
Viale’s isomorphism theorem

It is not hard to prove the following interesting theorem for 0-gesms. It shows how much gesms behave like elementary submodels in Magidor’s characterization of supercompactness [14]. Thus, in the following theorem, $\kappa$ has a compactness property similar to that of a supercompact cardinal.

**Theorem 3.16** (Viale [20]). Assume $M_i \prec H_{\theta_i}$, for $i = 0, 1$, are $\delta$-guessing. Suppose that

1. $\kappa := \kappa_{M_0} = \kappa_{M_1}$ and $M_0 \cap \kappa = M_1 \cap \kappa$,
2. $\mathcal{P}(\delta) \cap M_0 = \mathcal{P}(\delta) \cap M_1$, and
3. $o.t(\text{Card}_{M_0}) = o.t(\text{Card}_{M_1})$.

Then, $(M_0, \in) \cong (M_1, \in)$.

It follows from Viale’s proof (see [20, Remark 5.4]) that, under $T := \text{PFA} + (\ast)$, the third clause can be replaced by $\omega_1 \subseteq M_0 \cap M_1$. The consistency of $T$ follows from the recent breakthrough [2] by Aspero and Schindler.

3.3 Open problems

In the definition of a $\delta$-gesm, we assumed that $\delta \leq \kappa_M$ is regular. On the other hand, if we let $\delta \leq \kappa_M$ be a strong limit singular cardinal of countable cofinality, then Proposition 3.13 implies that a $\delta$-gesm with $\delta \subseteq M$ must be a 0-gesm and $2^{\delta} \leq |M|$.

Two imprecise questions come to my mind:

**Question 3.17.** Is it possible to introduce a useful notion of a $\delta$-gesm for $\delta > \kappa_M$ in a nontrivial way? What will be the relation between $\delta$ and the size of the model?

**Question 3.18.** Is it possible to include singular cardinals (in particular non-strong limit cardinals) in the definition of a gesm and prove interesting results about them?

4 GMP

As we shall see the existence of a single “nice” gesm has local impacts on the universe. The existence of a stationary set of gesms has even global impacts on the entire universe. We are now ready to introduce the simplest principle in terms of gesms.

For a powerful model $\mathcal{H}$, a regular cardinal $\delta \in \mathcal{H}$ and a cardinal $\kappa$, we let

$$\mathcal{G}_{\kappa, \delta}(\mathcal{H}) = \{ M \in \mathcal{P}_{\kappa}(\mathcal{H}) : M \prec \mathcal{H} \text{ is a } \delta\text{-gesm} \}.$$  

**Definition 4.1.** GMP$(\kappa, \delta, \mathcal{H})$ is the statement that $\mathcal{G}_{\kappa, \delta}(\mathcal{H})$ is stationary in $\mathcal{P}_{\kappa}(\mathcal{H})$, and GMP$(\kappa, \delta)$ is the statement that GMP$(\kappa, \delta, H_{\theta})$ holds, for all sufficiently large regular $\theta$.\(^1\)

\(^1\)GMP stands for Guessing Model Principle.
Therefore GMP(\(\kappa, \delta\)) implies GMP(\(\kappa, \delta'\)) whenever \(\delta \leq \delta'\). Note that if we have a powerful filtration of the universe, i.e., a \(\subseteq\)-increasing sequence \(\langle H(\alpha) : \alpha \in \text{ORD} \rangle\) of powerful set models with \(V = \bigcup \{ H(\alpha) : \alpha \in \text{ORD} \}\), then GMP(\(\kappa, \delta\)) holds if and only if for all sufficiently large ordinals \(\alpha\), GMP(\(\kappa, \delta, H(\alpha)\)) holds.

**Convention 4.2.** For a property \(\Phi\), let us say that GMP(\(\kappa, \delta\)) is witnessed by \(\Phi\)-models or by models with \(\Phi\) if the models in \(\mathcal{G}_{\kappa, \gamma}(\mathcal{H})\) with the property \(\Phi\) forms a stationary set in \(\mathcal{P}_\kappa(\mathcal{H})\).

### 4.1 Consequences

**Trees**

For a tree \(T = (T, <_T)\) and a node \(t \in T\), we let \(b_t := \{ s \in T : s <_T t \}\). Let \(\kappa\) be an infinite cardinal. Recall that a tree \(T\) of height \(\kappa\) is called a \(\kappa\)-tree if for every \(\alpha < \text{ht}(T)\), \(|T_\alpha| < \kappa\). A \(\kappa\)-Aronszajn tree is a \(\kappa\)-tree without cofinal branches. A cardinal \(\kappa\) has the tree property (\(TP(\kappa)\)) if there is no \(\kappa\)-Aronszajn tree. A \(\kappa\)-Kurepa tree is a \(\kappa\)-tree with more than \(\kappa\) cofinal branches. A weak \(\kappa\)-Kurepa tree is a tree of height and size \(\kappa\) with more than \(\kappa\) cofinal branches. The weak Kurepa Hypothesis at \(\kappa\) (\(w\)KH(\(\kappa)\)) states that there is a weak \(\kappa\)-Kurepa tree.

**Lemma 4.3.** Suppose \(M\) is a \(\delta\)-gesm. Assume that \(T \in M\) is a tree of height \(\delta\) with \(|T| < \kappa_M\). Then every cofinal branch through \(T\) is in \(M\).

**Proof.** Observe that \(T \subseteq M\), as \(|T| < \kappa_M\). A cofinal branch \(b\) through \(T\) is bounded in \(M\) and satisfies \(b \subseteq M\). If \(a \in M \cap \mathcal{P}_\delta(M)\), then for some ordinal \(\delta' < \delta\), \(a \cap T \subseteq T_{<\delta'}\), i.e., every node in \(a \cap T\) has height below \(\delta'\). Pick \(t \in b\) of height \(\delta'\). Thus \(t \in M\) and

\[b \cap a = \{ s \in T : s <_T t \} \cap a \subseteq M.\]

Thus if \(b \subseteq T\) is a cofinal branch, then \(b\) is \(\delta\)-approximated in \(M\), and hence there is \(b^* \in M\) such that \(b^* \cap M = b \cap M = b\). Note that \(b^* \subseteq M\). Therefore, \(b = b^* \in M\).

The above proof is essentially a proof of the following theorem.

**Theorem 4.4** (Cox–Krueger [3]). Assume GMP(\(\delta^+, \delta\)). Then the weak Kurepa Hypothesis fails at \(\delta\).

One can prove a slightly stronger result that is, under GMP(\(\kappa, \delta\)) with \(\kappa > \delta\), every tree of height \(\delta\) and size \(<\kappa\) has less than \(\kappa\) cofinal branches. Observe that GMP(\(\omega_2, \omega_1\)) implies the failure of the weak Kurepa Hypothesis that was known to be a consequence of PFA, and that implies the failure of CH.

**Lemma 4.5.** Suppose \(M\) is a \(\kappa_M\)-gesm. Then there is no \(\kappa_M\)-Aronszajn tree in \(M\).
Proof. Let $\kappa = \kappa_M$ and $\gamma = \sup(M \cap \kappa_M)$. Note that $\kappa$ is a regular cardinal. Suppose that $T \in M$ is a $\kappa$-tree, and fix $t \in T$ of height $\gamma$. Consider $b_t = \{ s \in T : s <_T t \}$, which is bounded in $M$. We claim that $b_t$ is $\kappa$-approximated in $M$. Suppose $\alpha \in M$ is of size less than $\kappa$, then, as in Lemma 4.3, there $\eta < \gamma$ with $\alpha \subseteq T_{<\eta}$. Now pick $t \in T_\eta \cap b$. Observe that $t \in M$, since $T$ is a $\kappa_M$-tree in $M$. We have

$$b_T \cap a = a \cap \{ s \in T : s <_T t \} \in M.$$  

Thus there is $b^* \in M$ with $b^* \cap M = b \cap M$. By elementarity, one can easily show that $b^*$ is a cofinal branch through $T$. Therefore, $T$ is not a $\kappa_M$-Aronszajn tree.

The following is immediate.

**Theorem 4.6.** Assume $\text{GMP}(\delta, \delta)$. Then $\delta$ has the tree property.

Thus $\text{GMP}(\delta^+, \delta)$ implies the tree property at $\delta^+$. In particular, $\text{GMP}(\omega_2, \omega_1)$ implies the tree property at $\omega_2$, a consequence of PFA.

Recall that a a weak ascent path of width a nonzero ordinal $\gamma$ through a tree $(T, <_T)$ is a sequence $(\vec{t}_\alpha : \alpha < \text{ht}(T))$ such that:

- For every $\alpha < \text{ht}(T)$, $\vec{t}_\alpha : \gamma \to T_\alpha$ is a function, and
- for every $\alpha < \beta < \text{ht}(T)$, there are $\zeta, \eta < \gamma$ such that $\vec{t}_\beta(\zeta) <_T \vec{t}_\beta(\eta)$.

**Theorem 4.7** (Lambie-Hanson [11] / Lücke [12]). Assume $\text{GMP}(\delta^+, \delta^+)$ is witnessed by models which are $\delta$-close to $\delta^+$. Suppose $T$ is a tree of height $\delta^+$. Then $T$ has no weak ascent path of length $< \delta$ if and only if $T$ has no cofinal branches.

Proof. The “only if” part is trivial. Let us assume that $f = (\vec{t}_\alpha : \alpha < \delta^+)$ is an ascent path of width $\gamma < \delta$. We shall produce a cofinal branch through $T$. Pick a $< \delta$-closed model with $T, f \in M < H_\delta$. Observe that by Lemma 3.14, $\alpha = M \cap \delta^+$ is of cofinality $\delta$, and that $\gamma \in M$. Thus, by a standard counting argument, there is $\zeta < \gamma$ such that $b = \{ s \in T : s <_T t^\alpha(\zeta) \} \subseteq M$. As in Lemma 4.5, $b$ is $\delta^+$-approximated in $M$, and hence guessed. Let $b^* \cap M = b$. By elementarity, $b^*$ is a cofinal branch through $T$.

Recall that a set of size $\omega_1$ is internally club if it is the union of a $\subseteq$-continuous $\in$-sequence of countable sets of length $\omega_1$. Under PFA, $\text{GMP}(\omega_2, \omega_1)$ is witnessed by internally club models (I.C.) models, and hence they have correct cofinality. Thus, under PFA, every tree of height $\omega_2$ has a cofinal branch if and only if it has a weak ascent path of width $\omega$. The author [16] has shown that if $\text{GMP}(\omega_2, \omega_1)$ is witnessed by I.C. models (and hence under PFA), and $T$ is a branchless tree of height $\omega_2$, then there is a proper and $\omega_2$-preserving property forcing notion $\mathbb{P}_T$ with the $\omega_1$-approximation property that specializes $T$. 

Cardinal arithmetic

The binary tree $T = 2^{<\delta}$ has $2^{\delta}$ cofinal branches. Thus by applying Lemma 4.3 to $T$, we obtain the following corollary.

Theorem 4.8. If there is a $\delta$-gesm $M$ of size $\kappa < 2^{\delta}$ with $\kappa \subseteq M$. Then $\kappa < 2^{<\delta}$. In particular, if there is a $\gamma^+$-gesm $M$ of size $\gamma^+ \subseteq M$ then $2^\gamma > \gamma^+$.

In fact, the existence of gesms is in favour of pushing up the relevant values of the continuum function. It is a theorem of Cox and Krueger [4] that GMP($\omega_2, \omega_1$) is consistent with the continuum being arbitrarily large.

Recall that the Singular Cardinal Hypothesis (SCH) states that if $\kappa$ is a singular cardinal with $2^{<\operatorname{cf}(\kappa)} < \kappa$, then $\kappa^{\operatorname{cf}(\kappa)} = \kappa^+$. In [20], Vaile proved if GMP($\omega_2, \omega_1$) is witnessed by internally unbounded models, then SCH holds. Later, Krueger proved (see Corollary 3.15) that every $\omega_1$-gesm of size $\omega_1$ is internally unbounded.

Theorem 4.9 (Krueger [10]+Viale [20]). GMP($\omega_2, \omega_1$) implies SCH.

Thus, another consequence of PFA follows from GMP($\omega_2, \omega_1$). More generally, Krueger proved the following.

Theorem 4.10 (Krueger [10]). Assume GMP($\kappa, \omega_1$). The Singular Cardinal Hypothesis holds above $\kappa$.

Squares

Let us recall the two-cardinal square principle.

Definition 4.11. Assume that $\kappa > 0$ is a cardinal and $\lambda$ is an uncountable cardinal. Suppose that $S \subseteq \lambda$ is stationary. A sequence $\langle C_\alpha : \alpha \in S \rangle$ is called a $\square(\kappa, \lambda, S)$-sequence if the following hold.

1. $\forall \alpha \in S$, $0 < |C_\alpha| < \kappa$.
2. $\forall \alpha \in S$ and $\forall C \in C_\alpha$, $C$ is a club in $\alpha$.
3. $\forall \alpha \in S$, $\forall C \in C_\alpha$, and $\forall \beta \in \operatorname{Lim}(C)$, $C \cap \beta \in C_\beta$, and
4. there is no club $D \subseteq \lambda$ so that for every $\alpha \in \operatorname{Lim}(D) \cap S$, $D \cap \alpha \in C_\alpha$.

Let $S^\lambda_< \kappa := \{ \alpha \in \lambda : \operatorname{cof}(\alpha) = \kappa \}$. The set $S^\lambda_\kappa$ is defined naturally.
Theorem 4.12 (Weiß [22, 23], Vaile [20]). Assume GMP(κ, κ). Then □(κ, λ, S^λ_{<κ}) fails, for every λ with cof(λ) ≥ κ.

Proof. Assume that there exists a □(κ, λ, S^λ_{<κ})-sequence C = (C_α : α ∈ S^λ_{<κ}). We shall find a contradiction. Pick a κ-ge sm M < H_θ, for some large regular θ, with C, λ, κ ∈ M and M ∩ κ ∈ κ. Let δ = sup(M ∩ λ). Thus δ ∈ S^λ_{<κ}.

Fix C ∈ C_δ. We claim that C is κ-approximated in M. Assume that a ∈ M is of size less than κ. Note that a ⊆ M. We may assume that a ⊆ δ. Since the cofinality of λ is at least κ, we have sup(a) < λ. Let ⟨α_ξ : ξ < κ' < κ⟩ be the increasing enumeration of a. We may assume that C ∩ a is infinite. Let α be the largest element of C that is a limit point of C ∩ α. Observe that C ∩ a \ α is finite, and that α < δ. Thus it is enough to show that C ∩ a ∩ α is in M. There is ξ < κ' such that α = sup{α_η : η < ξ}. Thus α ∈ M. Now, α is a limit point of C, and hence C ∩ α ∈ C_α. Since M ∩ κ is an ordinal, α ∈ M, and |C_α| < κ, we have C_α ⊆ M, and thus C ∩ α ∈ M.

Since M is κ-ge sm, there is C* ∈ M such that C* ∩ M = C ∩ M. By elementarity, C* is a club relative to S^λ_{<κ}. Observe that if γ is a limit point of C* of cofinality <κ, then

C* ∩ γ ∩ M = C ∩ γ ∩ M.

On the other hand, there is F ∈ C_γ such that F = C ∩ γ. Now F ∈ M and C* ∩ M ∩ γ = F ∩ M. By elementarity, C* ∩ α = F, and hence C* ∩ γ = C_γ. Let D be the closure of C*. Then D ∈ M. Now if γ ∈ M if of cofinality <κ, and is a limit point of D, we then have γ ∈ C* and D ∩ γ = C* ∩ γ ∈ C_γ. This is a contradiction as ‘C’ is a □(κ, λ, S^λ_{<cof(κ)})-sequence.

4.12

A similar proof shows that following.

Theorem 4.13 (Weiß [22, 23]). Assume GMP(κ, κ^+) is witnessed by models which have bounded uniform cofinality δ. Then □(κ, λ, S^κ_{<κ}) fails, for every λ with cof(λ) ≥ κ.

4.13

Corollary 4.14. Assume GMP(κ, δ) is witnessed by models <δ-closed models. Then □E_δ(κ, λ) fails, for every λ with cof(λ) ≥ κ.

4.14

The approachability ideal

Definition 4.15. Let λ be a regular cardinal. A λ-approaching sequence is a λ-sequence of bounded subsets of λ. If a = (a_ξ : ξ < λ) is a λ-approaching sequence, we let B(⟨a⟩) denote the set of all δ < λ such that there is a cofinal subset c ⊆ δ such that:

1. otp(c) < δ, in particular δ is singular, and
2. for all γ < δ, there exists η < δ such that c ∩ γ = a_η.
**Definition 4.16.** Suppose \( \lambda \) is a regular cardinal. Let \( I[\lambda] \) be the ideal generated by the sets \( B(\vec{a}) \), for all \( \lambda \)-approaching sequences \( \vec{a} \), and the non stationary ideal \( \text{NS}_\lambda \).

**Definition 4.17.** We say that the approachability holds at \( \delta \) if \( \delta^+ \in I[\delta^+] \). We denote this principle by \( \text{AP}(\delta) \).

**Definition 4.18.** For a regular infinite cardinal \( \kappa \), we let \( \text{MP}(\kappa^+) \) denote the following statement.

\[
I[\kappa^+] \upharpoonright S^+_{\kappa^+} = \text{NS}_{\kappa^+} \upharpoonright S^+_{\kappa^+}.
\]

**Theorem 4.19** (Weiβ [22]). \( \text{GMP}(\delta^+, \delta) \) implies \( \neg \text{AP}(\delta) \).

*Proof.* See Theorem 5.5.

**Laver diamonds**

The following beautiful theorem deserves more attention.

**Theorem 4.20** (Viale [20]). Assume PFA. Suppose there is a proper class of Woodin cardinals. There is a function \( f : \omega_2 \to H_{\omega_2} \) such that for every \( \theta \geq \omega_2 \) and every \( x \in H_\theta \),

\[
\{ M \in \mathcal{G}_{\omega_2, \omega_1}(H_\theta) : x \in M \text{ and } \pi_M(x) = f(M \cap \omega_2) \}
\]

is stationary in \( \mathcal{P}_{\omega_2}(H_\theta) \).

### 4.2 Consistency results

**Theorem 4.21** (Viale–Weiβ [21]). PFA implies that \( \text{GMP}(\omega_2, \omega_1) \) is witnessed by internally club models.

*Proof.* See [20, Theorem 4.4].

Let us also mention that in [19] Trang showed the consistency of \( \text{GMP}(\omega_3, \omega_2) \) assuming the existence of a supercompact cardinal. In his model the Continuum Hypothesis holds. Thus \( \text{GMP}(\omega_2, \omega_1) \) fails by Theorem 4.8.

The following is well-known.

**Theorem 4.22.** Assume the \( \kappa \) is a supercompact cardinal. Suppose that \( \delta < \kappa \) is a regular cardinal. Then in a generic extension \( \text{GMP}(\delta^{++}, \delta^+) \) holds.

*Proof (sketch).* We may assume without loss of generality that \( \delta^{<\delta} = \delta \). We define two forcings either of which works equally. Let \( \mathbb{P}_0 \) be Neeman’s forcing with sequences of models of length less than \( \delta \), where transitive models are \( V_\alpha < V_\kappa \) with \( \text{cof}(\alpha) \geq \delta \) and the nontransitive models are \( <\delta \)-closed elementary substructures of \( V_\kappa \) of size \( \delta \) containing \( \delta \) as an element, and let \( \mathbb{P}_1 \) be Veličković’s forcing whose conditions are \( <\delta \)-sized sets of \( <\delta \)-closed virtual models of size \( \delta \) in the structure \( (V_\kappa, \in, \delta) \).
Let $P$ be either of these two forcings. Then, $P$ is a $<\delta$-closed and $\kappa$-c.e. forcing which is strongly proper for models under consideration. Since $\kappa$ is supercompact, by Magidor’s characterization [14], there are stationary many 0-gezms $M \in V_\lambda$ of size less than $\kappa$, for every limit ordinal $\lambda$. Pick such a model in $V_\lambda$ and assume that $\text{cof}(\lambda) \geq \delta$. One has to show that if $G$ is a $V$-generic filter, then $M[G]$ is a $\delta$-gesm of size $\delta^+$. Of course, $P$ forces $\kappa$ to be $\delta^{++}$, and hence $M[G]$ is forced to be of size $\delta^+$. Let $M = V_\gamma$. It is proved that $P$ is strongly $(M, P)$-generic and hence we have $M[G] = V^{V[G \cap M]}$. Then, one needs to show that $P/G \cap M$ is strongly proper for a stationary sets of $<\delta$-closed models of size $\delta^+$. and hence $(V[G \cap M], V[G])$ has the $\delta^+$-approximation property. Applying Proposition 3.9 and Lemma 3.5, we have $M[G]$ is a $\delta$-gesm in $V[G]$.

There is a surprising result around singular cardinals.

**Theorem 4.23 (Hachtman–Sinapova [7]).** If $\delta$ is a countable limit of supercompact cardinals, then $\text{GMP}(\delta^+, \delta^+)$ holds.

### 4.3 Gesms and ISP

The statement $\text{GMP}(\kappa, \omega_1)$ is a reformulation of the principle $\text{ISP}(\kappa)$ introduced by C. Weiβ in [22]. The equivalence between $\text{GMP}(\kappa, \omega_1)$ and $\text{ISP}(\kappa)$ was established in [21].

Let $\delta \leq \kappa \leq \lambda$ be infinite cardinals with $\delta$ regular. Recall that a $P_\kappa(\lambda)$-list is a sequence $\langle d_a : a \in P_\kappa(\lambda) \rangle$ such that $d_a \subseteq a$, for every $a \in P_\kappa(\lambda)$.

**Definition 4.24.** A list $\langle d_a : a \in P_\kappa(\lambda) \rangle$ is called $\delta$-slender if for every sufficiently large $\theta$, there is a club $C \subseteq P_\kappa(H_\theta)$ such that for every $M \in C$ and every $a \in M$ with $a \in M \cap P_\kappa(\lambda)$, $a \cap d_M \cap \lambda \in M$.

**Definition 4.25.** A set $d \subseteq \lambda$ is an ineffable branch through a $P_\kappa(\lambda)$-list $\langle d_a : a \in P_\kappa(\lambda) \rangle$ if there is a stationary set $S \subseteq P_\kappa(\lambda)$ such that for every $a \in S$, $a \cap d = d_a$.

**Definition 4.26.** The principle $\text{ISP}_\delta(\kappa, \lambda)$ states that every $\delta$-slender $P_\kappa(\lambda)$-list has an ineffable branch. Let also $\text{ISP}_\delta(\kappa, \lambda)$ state that $\text{ISP}_\delta(\kappa, \lambda)$ holds, for every $\lambda \geq \kappa$.

Note that $\text{ISP}_{\omega_1}(\kappa, \lambda)$ is the same as the well-known $\text{ISP}(\kappa, \lambda)$.

**Proposition 4.27 (Viale–Weiss [21]).** $\text{ISP}_\delta(\kappa)$ holds if and only if $\text{GMP}(\kappa, \delta)$ holds.

**Proof.** See [21, Propositions 3.2 and 3.3].
The following is due to Magidor [13].

**Theorem 4.28.** Let $\kappa$ be an inaccessible cardinal. Assume that $\text{ISP}_\delta(\kappa)$ holds, for some regular $\delta < \kappa$. Then $\kappa$ is supercompact.

### 4.4 Open problems

**Problem 4.29** (Viale [20]). Is it consistent to have $\omega_2$-gesms of size $\omega_1$, which are not $\omega_1$-gesm?

We ask the following more general question.

**Problem 4.30.** Is it consistent to have $\delta$-gesms of size less than $\delta$ which are not $\gamma$-gesm, for all $\gamma \leq |M|$?

**Problem 4.31.** Given $m, n \in \omega$ with $m \geq 3$ and $n \geq 1$. Is $\text{GMP}(\omega_m, \omega_n)$ consistent? Of particular interest is $\text{GMP}(\omega_m, \omega_1)$.

**Problem 4.32.** It is consistent to have $\text{GMP}(\omega_{n+1}, \omega_n)$, for every $n \geq 1$?

Of course, one can also ask the above questions about $\text{GMP}(\delta^m, \delta^n)$.

**Problem 4.33.** Assume that $\delta > \omega$ is a weakly, but not strongly, inaccessible cardinal. Is $\text{GMP}(\delta, \delta)$ consistent?

\[\neg \text{CH} \uparrow \]
\[
\neg \Box(\omega_2, \lambda) \quad \neg \omega \text{KH} \quad \text{SCH} \quad \neg \text{AP}(\omega_1) \\
\text{TP}(\omega_2) \quad \text{Weiss} \quad \text{Weiss} \quad \text{Coz–Krueger} \quad \text{Krueger–Viale} \quad \text{Viale–Weiss} \\
\text{GMP}(\omega_2, \omega_1) \quad \text{Viale–Weiss} \quad \text{PFA} \\

\exists 1 \text{ supercompact} \]
5 GMP⁺

As we have seen before, one of the consequences of $\text{GMP}(\omega_2, \omega_1)$ was the failure of the approachability property at $\omega_1$. However, $\text{GMP}(\omega_2, \omega_1)$ does not imply $\text{MP}(\omega_2)$ since $\text{GMP}(\omega_2, \omega_1)$ is consistent with $2^{\aleph_0} = \aleph_2$, but $\text{MP}(\omega_2)$ implies $2^{\aleph_0} \geq \aleph_3$. We shall introduce a certain strengthening of $\text{GMP}(\omega_2, \omega_1)$ that implies $\text{MP}(\omega_2)$.

5.1 Strongly guessing models

We are now about to define a stronger version of guessing models by imposing constraints on their structure.

**Definition 5.1** (Mohammadpour–Veličković [17]). Let $\delta \leq \kappa$ be regular uncountable cardinals. A model $M$ of cardinality $\kappa^+$ is called strongly $\delta$-gesm if it is the union of an $\varepsilon$-increasing chain $(M_\xi : \xi < \kappa^+)$ of $\delta$-gesms of cardinality $\kappa$ with $M_\xi = \bigcup \{M_\eta : \eta < \xi\}$, for every $\xi$ of cofinality $\kappa$.

**Lemma 5.2.** Every strongly $\delta$-gesm is $\delta$-guessing.

**Proof.** Suppose that $M$ is a strongly $\delta$-gesm that is witnessed by a sequence $(M_\xi : \xi < \kappa^+)$. Suppose that $A$ is bounded in $M$, we may assume that $A$ is bounded in each $M_\xi$. Since the sequence $(M_\xi : \xi < \kappa^+)$ is closed at ordinals of cofinality $\kappa$, a standard closure argument shows that the following set, modulo $S^\kappa_\kappa^+$, is a club in $\kappa^+$.

$$C = \{\xi < \kappa^+ : A \text{ is } \delta\text{-approximated in } M_\xi\}$$
Thus for each $\xi \in C$, there is $A_\xi \in M_\xi$ such that $A_\xi \cap M = A \cap M_\xi$. By Fodor’s lemma, there is some $\eta < \kappa^+$ and a stationary set $S \subseteq C$ such that for each $\xi \in S$, $A_\xi$ is in $M_\eta$. On the other hand $|M_\eta| < \kappa^+$, and hence there is a stationary set $T \subseteq S$, such that for every $\xi, \xi' \in T$, $A_\xi = A_{\xi'}$. Let $A^* = A_\xi$, for some $\xi \in T$. It is easy to see that $A^* \cap M = A \cap M$.

As before, for a powerful model $\mathcal{H}$ and regular cardinals $\delta \in \mathcal{H}$ and $\kappa$, we let

$$\Theta^+_{\kappa^+, \delta}(\mathcal{H}) = \{ M \in \mathcal{P}_{\kappa^+}(\mathcal{H}) : M \prec \mathcal{H} \text{ is strongly } \delta\text{-gesm} \}.$$  

**Definition 5.3** (Mohammadpour–Veličković [17]). Let $\text{GMP}^+(\kappa^+, \delta)$ states that for every sufficiently large regular cardinal $\theta$, $\Theta^+_{\kappa^+, \delta}(H_\theta)$ is stationary in $\mathcal{P}_{\kappa^+}(H_\theta)$.

We have the following immediate fact.

**Fact 5.4.** Assume $\text{GMP}^+(\kappa^+, \delta)$. Then both $\text{GMP}(\kappa^+, \delta)$ and $\text{GMP}(\kappa^+, \delta)$ hold.

**Theorem 5.5** (Mohammadpour–Veličković [17]). $\text{GMP}^+(\kappa^+, \kappa)$ implies $\text{MP}(\kappa^+)$.  

**Proof.** Let $\tilde{a} = (a_\xi : \xi < \kappa^+)$ be a $\kappa^+$-approaching sequence which belongs to $H_{\kappa^+}$. Let $\mathcal{G}$ be the (stationary) set of $\kappa$-gesms containing $\tilde{a}$. We show that $M \cap \kappa^+$ is in $B(\tilde{a})$, for any $M \in \mathcal{G}$ such that $\text{cof}(M \cap \kappa^+) = \kappa$. Fix one such $M \in \mathcal{G}$. Let $\delta = M \cap \kappa^+$ and suppose that $c \subseteq \delta$ satisfies (1) and (2) of Definition 4.15. Let $\mu = \text{otp}(c)$. Note that $\mu < \delta$, hence $\mu \in M$. Since $\tilde{a} \in M$, we have that $c \cap \gamma \in M$, for all $\gamma < \delta$, and hence $c \cap Z \in M$, for all $Z \in M$ with $|Z| < \kappa$. Since $M$ is a $\kappa$-gesm, there must be $d \in M$ such that $c = d \cap \delta$. We may assume that $d \subseteq \kappa^+$. Then $c$ is an initial segment of $d$, so if $\rho$ is the $\mu$-th element of $d$ then $d \cap \rho = c$. Since $\mu, d \in M$, we have $\rho \in M$ as well, and hence $c = d \cap \rho \in M$. But then $\delta = \text{sup}(c)$ belongs to $M$, a contradiction.

**Theorem 5.6** (Mohammadpour–Veličković [17]). Suppose $\kappa$ is a regular cardinal. Assume there are two supercompact cardinals above $\kappa$. Then $\text{GMP}^+(\kappa^+, \kappa^+)$ holds in a $<\kappa$-closed generic extension.

Assume $\lambda < \mu$ are supercompact cardinals above $\kappa$. The proof of the above theorem uses a forcing with $<\kappa$-sized decorated chains of virtual models of two types: countable and $\lambda$-Magidor models. The forcing is $\mu$-Knaster and strongly proper for the both collections of models.

**Definition 5.7.** Let $\delta \leq \kappa$ be regular uncountable cardinals. For $n \in \mathbb{N}$, a model $M$ of cardinality $\kappa^{+(n+1)}$ is called an $n$-strongly $\delta$-guessing model if it is the union of an $\varepsilon$-increasing sequence $(M_\xi : \xi < \kappa^+)$ of $(n-1)$-strongly $\delta$-guessing models of cardinality $\kappa^+$ such that for every $\xi$ of cofinality $\kappa^+$, $M_\xi = \bigcup \{ M_\eta : \eta < \xi \}$.  

In the above definition, if $n = 1$, then an inductively strongly $\delta$-guessing model is just a strongly $\delta$-guessing model. The principle $\text{GMP}^{+n}(\kappa^{+n}, \delta)$ is defined in the obvious meaning. Let also $\omega$-strongly $\delta$-guessing models be defined in the obvious way with $\text{GMP}^{\omega}(\kappa^{+\omega}, \delta)$ for the associated principle.

5.2 Open problems

Problem 5.8. Does $\text{GMP}(\omega_3, \omega_1)$ imply $\text{MP}(\omega_2)$?

Problem 5.9. Given $n \geq 2$. Is $\text{GMP}^{+n}(\kappa^{+n}, \delta)$ consistent?

Problem 5.10. Is $\text{GMP}^{\omega}(\kappa, \omega_1)$ consistent?

6 IGMP

The indestructible version of guessing models was first discovered and studied by Cox and Krueger [3]. A $\omega_1$-guessing set is called indestructible if it remain $\omega_1$-guessing in any $\omega_1$-preserving forcing extension. More generally, we can require more robustness. For example, by requiring the size does not collapse, etc., or by considering some other kind of extensions.

Definition 6.1. Suppose $M$ is a $\delta$-guessing model. Let $C$ be a class of $\delta$-preserving forcing notions. Then $M$ is called $C$-indestructible if for every $\mathbb{P} \in C$, $M$ satisfies the $\delta$-guessing property in generic extensions by $\mathbb{P}$.
By $\mathbb{P}$-indestructible, we shall mean $\{\mathbb{P}\}$-indestructible. We omit $C$ whenever it is the class of all $\delta$-preserving forcings.

**Definition 6.2.** Let $C$ be a class of $\delta$-preserving forcings. Let $P$ be one of the above principles about $\delta$-guessing models. Then by $C - \text{IP}$, we mean that $P$ is witnessed by $C$-indestructible models.

Notice that the above definition is not a priori equivalent to the statement that $\mathbb{P}$-IP holds, for every $\mathbb{P} \in C$.

**Definition 6.3.** Let $\operatorname{IGMP}(\omega_2, \omega_1)$ state that for all sufficiently large regular cardinal $\theta$, the set of indestructible $\omega_1$-guessing models of $H_\theta$ is stationary in $\mathcal{P}_{\omega_2}(H_\theta)$.

Let us start with the consistency of $\operatorname{IGMP}(\omega_2, \omega_1)$.

**Theorem 6.4** (Cox–Krueger [3]). $\text{PFA}$ implies $\operatorname{IGMP}(\omega_2, \omega_1)$.

In the same paper, Cox and Krueger also showed that $\operatorname{IGMP}(\omega_2, \omega_1)$ is consistent with the continuum being arbitrarily large. It is also worth mentioning that the $\operatorname{IGMP}(\omega_2, \omega_1)$ obtained by Cox and Kruger has the property that every indestructible $\omega_1$-guessing model remains $\omega_1$-guessing in any outer transitive extension with the same $\omega_1$. The main idea to make a guessing set $M$ indestructible is to specialise a branchless $\omega_1$-tall, $\omega_1$-sized tree $T(M)$ that is naturally associated to $M$ so that $T(M)$ is special if and only if the guessing property of $M$ is indestructible. Now if one succeeds to find a model in which every branchless tree of size and height $\omega_1$ is special and that there are $\omega_1$-guessing models of size $\omega_1$, then the guessing models are indestructible. A true conjunction under $\text{PFA}$.

### 6.1 Consequences

**Indestructibility and the approximation property**

One of the tight interaction of the above indestructibility is with the $\omega_1$-approximation property of forcings. Recall that a forcing notion has the $\delta$-approximation property in $V$ if for every $V$-generic filter $G \subseteq \mathbb{P}$, $(V, V[G])$ has the $\delta$-approximation property. Note that $\operatorname{GMP}(\omega_2, \omega_1)$ is consistent with the failure of the Suslin Hypothesis: In [4], Cox and Krueger obtained the consistency of $\operatorname{GMP}(\omega_2, \omega_1)$ with arbitrary large continuum in a way that the forcing adds Cohen reals over the ground model, and hence by a well-known result due to Shelah, there is a Suslin tree in the final model. There are also other models witnessing this, see e.g. [3].

**Proposition 6.5** (Cox–Krueger [3]). $\operatorname{IGMP}(\omega_2, \omega_1)$ implies the Suslin Hypothesis (SH).

More generally, if one regards a tree of height and size $\omega_1$ with the opposite ordering as a forcing notion, then under $\operatorname{IGMP}(\omega_2, \omega_1)$, a nontrivial $T$ collapses $\omega_1$. 
Lemma 6.6. Suppose $\mathbb{P}$ has the $\delta$-approximation property, and that $M \prec H_\theta$ is a $\delta$-gesm, for some $\theta \geq \omega_2$. Assume that $(M, V)$ has the $\delta$-covering property. Then $\mathbb{P}$ forces $M$ to be $\delta$-guessing.

Proof. Let $G \subseteq \mathbb{P}$ be a $V$-generic filter. Fix $x \in V[G]$ and assume that $x \subseteq X \in M$ is $\delta$-approximated in $M$. We claim that $x \cap M$ is $\delta$-approximated in $V$, which in turn implies that $x \cap M \in V$. Then, since $M$ is a $\delta$-gesm in $V$, $x$ is guessed in $M$. To see that $x \cap M$ is $\delta$-approximated in $V$, fix a set $a \in V$ of size less than $\delta$. By the $\delta$-covering property, there is a set $b \in M$ of size less than $\delta$ with $a \cap M \cap X \subseteq b$. Thus $a \cap x \cap M = a \cap x \cap b \in V$, since $a \in V$ and $x \cap b \in M \subseteq V$.

Corollary 6.7. Suppose that $\mathbb{P}$ has the $\delta^+$-approximation property. Assume that $M \prec H_\theta$ is $\delta^+$-guessing with $\delta^+ \subseteq M$, for some $\theta \geq \omega_2$. Then $\mathbb{P}$ forces $M$ to be $\delta^+$-guessing.

Proof. We only need the covering property that holds by Lemma 3.14.

Corollary 6.8. Suppose $\mathbb{P}$ has the $\omega_1$-approximation property, and that $M \prec H_\theta$ is $\omega_1$-guessing for some $\theta \geq \omega_2$. Then $\mathbb{P}$ forces $M$ to be $\omega_1$-guessing.

Proposition 6.9. Assume that $\mathbb{P}$ is an $\omega_1$-preserving forcing. Suppose that for every sufficiently large regular cardinal $\theta$, $\mathbb{P}$ is proper for a stationary set $\mathcal{S}_\theta \subseteq \mathcal{P}_{\omega_2}(H_\theta)$ of $\delta$-gesms of $H_\theta$ which are $\mathbb{P}$-indestructible. Then $\mathbb{P}$ has the $\omega_1$-approximation property.

Proof. Fix a $\delta$-preserving forcing $\mathbb{P}$ and assume that the maximal condition of $\mathbb{P}$ forces $\dot{A}$ is a $\delta$-approximated subset of an ordinal $\gamma$. Pick a regular $\theta$, with $\gamma, \dot{A}, \dot{\mathcal{P}}(\mathbb{P}) \in H_\theta$. We shall show that $\mathbb{P} \vDash \text{“}\dot{A} \in V\text{”}$. Let $G \subseteq \mathbb{P}$ be a $V$-generic filter, and set

$$S := \{M \in \mathcal{S}_\theta : p, \gamma, \dot{A}, \mathbb{P} \in M \text{ and } M[G] \cap H_\theta^V = M\}.$$ 

In $V[G]$, $S$ is stationary in $\mathcal{P}_{\omega_2}(H_\theta^V)$. To see this, let $F : \mathcal{P}_\omega(H_\theta^V) \to \mathcal{P}_{\omega_2}(H_\theta^V)$ be defined by $F(x) = \{\dot{y}^G\} \text{ if } x = \{\dot{y}\}$ for some $\mathbb{P}$-name $\dot{y}$ with $\dot{y}^G \in H_\theta^V$, and otherwise let $F(x) = \{p, \gamma, \mathbb{A}, \mathbb{P}\}$. The set of models in $\mathcal{S}$ which are closed under $F$ is stationary. Observe that a model $M \in \mathcal{S}$ is closed under $F$ if and only if $M \in S$.

Let $A = \dot{A}^G$ and fix $M \in S$. We claim that $A$ is $\delta$-approximated in $M$. Let $a \in M$ be a $< \delta$-sized subset of $\gamma$. Let $D_a$ be the set of conditions deciding $\dot{A} \cap a$. Then $D_a$ belongs to $M$ and is dense in $\mathbb{P}$, as the maximal condition forces that $\dot{A}$ is $\delta$-approximated in $V$. By the elementarity of $M[G]$ in $H_\theta[G]$, there is $p \in G \cap D_a \cap M[G]$. But then $p \in M$, as $D_a \in H_\theta^V$. Working in $V$, the elementarity of $M$ in $H_\theta$ implies that there is some $b \in M$ such that, $p \vDash \text{“}\dot{b} = \dot{A} \cap a\text{”}$. Since $p \in G$, we have $A \cap a = b \in M$. Thus $A$ is $\delta$-approximated in $M$. By our assumption, $M$ is an $\delta$-guessing in $V[G]$. Thus there is $A^* \in M$, and hence in $V$, such that $A^* \cap M = A \cap M$.

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2i.e., for every $M \in \mathcal{S}_\theta$ with $\delta \subseteq M$, and every $p \in M$, there is an $(M, \mathbb{P})$-generic condition $q \leq p$. 

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Working in $V[G]$ again, for every $M \in S$, there is, by the previous paragraph, a set $A^*_M \in M$ such that $A^*_M \cap M = A \cap M$. This defines a regressive function $M \mapsto A^*_M$ on $S$. As $S$ is stationary in $H[V]$, there are a set $A^* \in H[V]$ and a stationary set $S^* \subseteq S$ such that for every $M \in S^*$, we have $A^* \cap M = A \cap M$. Since $A \subseteq \bigcup S^*$, we have $A^* = A$, which in turn implies that $A \in V$.

The following follows from Lemma 6.6 and Proposition 6.9.

**Corollary 6.10.** Assume that $\mathbb{P}$ is an $\delta$-preserving forcing. Suppose that for every sufficiently large regular cardinal $\theta$, $\mathbb{P}$ is proper for a stationary set $\Theta_\theta \subseteq \mathcal{P}_{\omega_2}(H_\theta)$ of $\omega_1$-gesms of $H_\theta$ with the $\delta$-covering property. Then the following are equivalent.

1. $\mathbb{P}$ has the $\delta$-approximation property.
2. Every $\delta$-guessing model is indestructible by $\mathbb{P}$.

In particular, we have the following theorem.

**Corollary 6.11.** Assume that $\mathbb{P}$ is an $\omega_1$-preserving forcing. Suppose that for every sufficiently large regular cardinal $\theta$, $\mathbb{P}$ is proper for a stationary set $\Theta_\theta \subseteq \mathcal{P}_{\omega_2}(H_\theta)$ of $\omega_1$-gesms of $H_\theta$. Then the following are equivalent.

1. $\mathbb{P}$ has the $\omega_1$-approximation property.
2. Every $\omega_1$-guessing model is indestructible by $\mathbb{P}$.

**Corollary 6.12.** Assume $GMP(\omega_2, \omega_1)$. Suppose that $\mathbb{P}$ is an $\omega_1$-preserving forcing which is also proper for models of size $\omega_1$. Then the following are equivalent.

1. $\mathbb{P} - IGMP(\omega_2, \omega_1)$ holds.
2. $\mathbb{P}$ has the $\omega_1$-approximation property.

The following is a generalisation of Proposition 6.5.

**Theorem 6.13.** Assume $IGMP(\omega_2, \omega_1)$. Then every $\omega_1$-preserving forcing which is proper for models of size $\omega_1$ has the $\omega_1$-approximation property. In particular, under $IGMP(\omega_2, \omega_1)$ every $\omega_1$-preserving forcing of size $\omega_1$ has the $\omega_1$-approximation property.

**Proof.** Let $\mathbb{P}$ be an $\omega_1$-preserving function which is proper for models of size $\omega_1$. By $IGMP(\omega_2, \omega_1)$, Corollary 6.10 implies that $\mathbb{P}$ has the $\omega_1$-approximation property.
Note that in the above proposition, it is enough to assume the nondiagonal version of IGMP($\omega_2, \omega_1$).

For a class $\mathcal{R}$ of forcing notions and a cardinal $\kappa$, we let $\text{FA}(\mathcal{R}, \kappa)$ state that for every $\mathbb{P} \in \mathcal{R}$, and every $\kappa$-sized family $\mathcal{D}$ of dense subsets of $\mathbb{P}$, there is a $\mathcal{D}$-generic filter $G \subseteq \mathbb{P}$.

**Lemma 6.14.** Assume $\text{FA}(\mathbb{P}, \kappa)$, for some forcing notion $\mathbb{P}$. Suppose that $M$ is a $\delta$-guessing set of size $\kappa \geq \delta$. Then $\mathbb{P}$ forces that $M$ is $\delta$-guessing.

**Proof.** Assume towards a contraction that for some $p_0 \in \mathbb{P}$, some ordinal $\eta \in M$, and some $\mathbb{P}$-name $\dot{A}$, $p_0$ forces that $\dot{A} \subseteq \eta$ is $\delta$-approximated in $M$, but is not guessed in $M$. We may assume that $p_0$ is the maximal condition of $\mathbb{P}$.

- For every $\alpha \in M \cap \eta$, let $D_\alpha := \{p \in \mathbb{P} : p \text{ decides } \alpha \in \dot{A}\}$.
- For every $x \in M \cap \mathcal{P}_\delta(\eta)$, let $E_x := \{p \in \mathbb{P} : \exists y \in M \; p \Vdash \text{“} \dot{A} \cap x = \dot{y} \text{”} \}$.
- For every $B \in M \cap \mathcal{P}(\eta)$, let $F_B := \{p \in \mathbb{P} : \exists \xi \in M, (p \Vdash \text{“} \xi \in \dot{A} \text{”} \iff \xi \notin B \}$.

By our assumptions, it is easily seen that the above sets are dense in $\mathbb{P}$. Let

$$\mathcal{D} = \{D_\alpha, E_x, F_B : \alpha, x, B \text{ as above }\}.$$  

We have $|\mathcal{D}| = \kappa$. By $\text{FA}(\mathbb{P}, \kappa)$, there is a $\mathcal{D}$-generic filter $G \subseteq \mathbb{P}$. Let $A^* \subseteq \eta$ be defined by

$$\alpha \in A^* \text{ if and only if } \exists p \in G \text{ with } p \Vdash \text{“} \alpha \in \dot{A} \text{”}.$$  

By the $\mathcal{D}$-genericity of $G$, $A^*$ is a well-defined subset of $\eta$ which is $\delta$-approximated but not guessed in $M$, a contradiction!

The following theorem is immediate from Corollary 6.12 and Lemma 6.14.

**Theorem 6.15.** Let $\mathcal{R}$ be a class of forcings which are proper for models of size $\omega_1$. Assume that $\text{FA}(\mathcal{R}, \omega_1)$ and $\text{GMP}(\omega_2, \omega_1)$ hold. Then, for every forcing $\mathbb{P} \in \mathcal{R}$, $\mathbb{P} - \text{IGMP}(\omega_2, \omega_1)$ holds, and $\mathbb{P}$ has the $\omega_1$-approximation property.

**Indestructibility and maximality**

In his PhD thesis [1], Abraham asked if there is a forcing notion $\mathbb{P}$ in ZFC such that it does not add new reals, adds a new subset of some ordinal whose initial segments belong to the ground model and that the forcing does not collapse any cardinal. Notice that if CH holds, then $\text{Add}(\omega_1, 1)$ is countably closed and $\omega_2$-c.c while adding a new subset of $\omega_1$. Recall that also Foreman’s Maximality Principle (see [5]) states that every nontrivial forcing notion either adds a new real or collapses some cardinals. Using the forcing with initial segments of an uncountable cardinal $\kappa$ ordered with the reverse inclusion, one can show that Foreman’s Maximality principle violates GCH and the existence of inaccessible cardinals.
Definition 6.16. For a regular cardinal $\kappa$, the Abraham–Todorčević Maximality Principle at $\kappa^+$, denoted by $\text{ATMP}(\kappa^+)$, states that if $2^\kappa < \aleph_{\kappa^+}$, then every forcing which adds a new subset of $\kappa^+$ whose initial segments are in the ground model, collapses some cardinal $\leq 2^\kappa$.

Towards answering the above-mentioned question of Abraham, Todorčević showed in [18] that $\text{ATMP}(\omega_1)$ is true if every tree of size and height $\omega_1$ with at most $\omega_1$ cofinal branches is weakly special. This principle was further studied by Golshani and Shelah in [6], where they showed that $\text{ATMP}(\kappa^+)$ is consistent for every prescribed regular cardinal $\kappa$. Cox and Krueger [3] proved the following.

Proposition 6.17 (Cox–Krueger [3]). $\text{IGMP}(\omega_2, \omega_1)$ implies $\text{ATMP}(\omega_1)$.

6.2 IGMP$^+$

Theorem 6.18 (Mohammadpour–Veličković [15]). $\text{IGMP}^+(\omega_3, \omega_1)$ is consistent modulo the consistency of two supercompact cardinals.

We shall prove that $\text{IGMP}^+(\omega_3, \omega_1)$ implies $\text{ATMP}(\omega_2)$, and since $\text{IGMP}(\omega_2, \omega_1)$ follows from $\text{IGMP}^+(\omega_3, \omega_1)$, we obtain the consistency of $\text{ATMP}(\omega_1)$ and $\text{ATMP}(\omega_2)$ simultaneously.

Theorem 6.19 (Mohammadpour–Veličković [15]). Suppose that $V \subseteq W$ are transitive models of ZFC. Assume that $\text{SGM}^+(\omega_3, \omega_1)$ and $2^{\omega_1} < \aleph_{\omega_2}$ hold in $V$. Suppose that $W$ has a subset of $\omega_2^V$ which does not belong to $V$. Then either $\mathcal{P}^V(\omega_1) \neq \mathcal{P}^W(\omega_1)$ or some $V$-cardinal $\leq 2^{\omega_1}$ is no longer a cardinal in $W$.

Proof. Let $x \in W \setminus V$ be a subset of $\omega_2^V$. Assume that $\mathcal{P}^V(\omega_1) = \mathcal{P}^W(\omega_1)$. We shall show that some cardinal $\leq 2^{\omega_1}$ is no longer a cardinal in $W$. Since $\mathcal{P}^V(\omega_1) = \mathcal{P}^W(\omega_1)$, every initial segment of $x$ belongs to $V$. Letting now $\mathcal{X} = \{x \cap \gamma : \gamma < \omega_2\}$, we have that $\mathcal{X}$ is bounded in $V$. Assume towards a contradiction that every $V$-cardinal $\leq 2^{\omega_1}$ remains cardinal in $W$. Work in $W$, and let $\mu \geq \omega_2$ be the least cardinal such that there is a set $M$ in $V$ of cardinality $\mu$ such that $M \cap \mathcal{X}$ is of size $\omega_2$. Thus $\mu \leq 2^{\omega_1}$. We claim that $\mu = \omega_2$. Suppose that $\mu > \omega_2$ and $M$ is a witness for that, then one can work in $V$ and write $M$ as the union of an increasing sequence $(M_\xi : \xi < \text{cof}^V(\mu))$ of subsets of $M$ in $V$ whose size are less than $\mu$. Since $\mu \leq 2^{\omega_1} < \aleph_{\omega_2}$ and every cardinal $\leq 2^{\omega_1}$ is a cardinal in $W$, $\text{cof}^W(\mu) = \text{cof}^V(\mu) \neq \omega_2$. Thus either $\mu$ is of cofinality at most $\omega_1$, which then by the pigeonhole principle, there is some $\xi < \text{cof}(\mu)$ such that $|M_\xi \cap \mathcal{X}| = \omega_2$, or $\mu$ is regular, and thus there is some $\xi < \text{cof}(\mu)$ such that $M \cap \mathcal{X} \subseteq M_\xi$, but in either case we obtain a contradiction since $|M_\xi| < \mu$. Therefore, $\mu = \omega_2$. Let $M$ be a witness for $\mu = \omega_2$, and let $\mathcal{X}' = M \cap \mathcal{X}$. Notice that $V \models |M| = \omega_2$. Since $M$ is in $V$ and that $V$ satisfies $\text{SGM}^+(\omega_3, \omega_1)$, one can cover $M$ with an indestructibly strongly $\omega_1$-guessing model $N$ of size $\omega_2$. Working in
$W$, $x$ is countably approximated in $N$, since if $\gamma \in N \cap \omega_2$, then there is $\gamma' > \gamma$ in $N$ such that $x \cap \gamma' \in \mathcal{X}' \subseteq N$, and hence $x \cap \gamma \in N$. On the other hand $N$ is a guessing model in $W$ by $\text{SGM}^+(\omega_3, \omega_1)$ in $V$ and that both $\omega_1$ and $\omega_2$ are cardinals in $W$. Thus $x$ is guessed in $N$, but then $x$ must be in $N$ as $|x| \leq |N|$. Therefore, $x$ is in $V$, which is a contradiction!

The following corollaries are immediate.

**Corollary 6.20.** Suppose that $V \subseteq W$ are transitive models of ZFC. Assume in $V$, $\text{SGM}^+(\omega_3, \omega_1)$, $2^{\aleph_0} < \aleph_{\omega_1}$ and $2^{\aleph_1} < \aleph_{\omega_2}$ hold. Suppose that $W$ has a new subset of $\omega_2^V$. Then either $\mathbb{R}^W \neq \mathbb{R}^V$ or some $V$-cardinal $\leq 2^{\omega_1}$ is collapsed in $W$.

**Corollary 6.21.** $\text{SGM}^+(\omega_3, \omega_1)$ implies $\text{ATMP}(\omega_1)$ and $\text{ATMP}(\omega_2)$.
6.3 Open problems

Problem 6.22. It is consistent to have $\text{IGMP}^+(\omega_{n+2}, \omega_1)$, for every $n \geq 1$.

Problem 6.23. Is $\text{IGMP}^\infty$ consistent?

Problem 6.24. Assume PFA. For which forcing notions $\mathbb{P}$-special $\omega_1$-guessing models of size $\omega_1$ form a stationary set.

Problem 6.25. Is $\text{IGMP}(\omega_2, \omega_1)$ consistent with $\text{MA}_{\omega_1} + 2^{\aleph_0} > \aleph_2$?
This last section of this survey is devoted to an interesting weakening of guessing property introduced by Cox and Krueger [3]. All the following results are due to Cox and Krueger.

**Definition 7.1.** Suppose $\kappa$ is a regular uncountable cardinal. A set $M$ has the \textit{weak $\kappa$-guessing property} if for every function $f : \kappa \to \text{ORD}$ such that $f \upharpoonright \alpha \in M$, for every $\alpha \in M \cap \kappa$, there is $f^* \in M$ with $f^* \upharpoonright M = f \upharpoonright M$.

**Definition 7.2.** A set $M$ of size $\omega_1$ is called \textit{weakly guessing} if for every uncountable regular $\kappa \in M$, $M$ has the weak $\kappa$-guessing property.
Similarly, one can define the weak approximation property. It must be now easy to state the weak-guessing model principles, denoting as before following a \( w \). Interestingly, many consequences of guessing model principles follow from this, however the structure of the models are not quite the same as before. Similarly, \( IwGMP \) denotes the indestructible version of the weak guessing model principle.

**Theorem 7.3.**

1. \( wGMP \) implies \( \neg wKP \), \( TP(\omega_2) \), \( \neg \square_\kappa \), for all \( \kappa \geq \omega_1 \), \( \neg \square(\lambda) \) for all \( \lambda \geq \omega_2 \), and \( AP(\omega_1) \) hold.

2. \( IwGMP \) implies \( ATMP(\omega_2) \).

**Proposition 7.4.** The set of weakly \( \omega_1 \)-guessing submodels of \( H_\theta \) is closed under countable union of \( \subseteq \)-increasing sequences.

**Theorem 7.5.** \( GMP(\omega_2, \omega_1, H_{\omega_2}) \) is equivalent to \( wGMP(\omega_2, \omega_1, H_{\omega_2}) \)

The main theorem of Cox and Kruger reads as follows.

**Theorem 7.6.** Assume the existence of a supercompact cardinal with infinitely many measurable cardinals above it. Then in a generic extension, there exist stationarily many \( N \in \mathcal{P}_{\omega_2}(H_{\aleph_{\omega_1}+1}) \) such that \( N \) is indestructibly weakly guessing, has uniform cofinality \( \omega_1 \), and is not internally unbounded.

Recall Krueger’s theorem that most \( \omega_1 \)-guessing models are internally unbounded.

### 7.1 Open Problems

**Problem 7.7** (Cox–Krueger). Does the existence of stationarily many indestructibly weakly guessing models which are not internally unbounded follow from Martin’s maximum?

**Problem 7.8** (Cox–Krueger). Does \( wGMP \) imply \( GMP \), or \( wIGMP \) imply \( IGMP \)?

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