A DISTRIBUTED CONTROL PROBLEM FOR A THREE-DIMENSIONAL LAGRANGIAN AVERAGED NAVIER-STOKES-α MODEL

E.J. VILLAMIZAR-ROA AND E. ORTEGA-TORRES

Abstract. A distributed optimal control problem with final observation for a three-dimensional Lagrange averaged Navier-Stokes-α model is studied. The solvability of the optimal control problem is proved and the first-order optimality conditions are established. Moreover, by using the Lagrange multipliers method, an optimality system in a weak and a strong form is derived.

1. Introduction

We are interested in the study of an optimal control problem for a Lagrange averaged Navier-Stokes-α model (also known as LANS-α or the viscous Camassa-Holm system). The LANS-α model, introduced by S. Chen, C. Foias, D.D. Holm, E. Oslon, E.S. Titi, and S. Wynne in [10], is the first one to use Lagrangian averaging to address the turbulence closure problem, that is, the problem of capturing the physical phenomenon of turbulence at computably low resolution. This model provides closure by modifying the nonlinearity in the Navier-Stokes equations to stop the cascading of turbulence at scales smaller than a certain length, but without introducing any extra dissipation (c.f. [6, 10, 11, 12, 23, 26]). The mathematical model is obtained by regularizing the 3D Navier-Stokes equations through a filtration of the fluid motion that occurs below a certain length scale α^1/2; the length scale is the filter width derived from inverting the Helmholtz operator I − αΔ. Explicitly, LANS-α model can be written in the following form:

\[
\begin{cases}
(I - \alpha \Delta) u_t + \nu (I - \alpha \Delta) A u + (u \cdot \nabla) (I - \alpha \Delta) u - \alpha (\nabla u)^* \cdot \Delta u + \nabla p = f & \text{in } Q, \\
\nabla \cdot u = 0 & \text{in } Q, \\
\n\nabla \cdot u = 0 & \text{on } \Gamma \times (0, T), \\
\nu(x, 0) = u_0(x) & \text{in } \Omega,
\end{cases}
\]

where \( u \) and \( p \) are unknown, representing respectively, the large-scale (or averaged) velocity and the pressure, in each point of \( Q = \Omega \times (0, T), 0 < T < \infty \). Here, \( \Omega \) is a bounded domain in \( \mathbb{R}^3 \) (with boundary \( \Gamma \) of class \( C^2 \)) where the fluid is occurring, and \( (0, T) \) is a time interval. The operator \( A \) denotes the known Stokes operator. Moreover, the right hand side \( f \) is a fixed external force and \( u_0 \) a given initial velocity field. The positive constant \( \nu \) represents the kinematic viscosity of the fluid.

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The interest of studying the LANS-α models arises principally in the approximation of many problems relating to turbulent flows because it preserves the properties of transport for circulation and vorticity dynamics of the Navier-Stokes equations. One of the main reasons justifying its use is the high-computational cost that the Navier-Stokes model requires \cite{6}. Notice that when $\alpha = 0$ the LANS-α model reduces to the classical Navier-Stokes system. We refer \cite{6,10,11,12,13,15,22,23,26} and references therein, for a complete description of the development of the LANS-α model, as well as, a discussion about the physical significance, namely, in turbulence theory.

From a mathematical point of view, several advances related the well-posedness, long time behavior, decay rates of the velocity and the vorticity, the connection between the solutions of the LANS-α model and the 3D Navier-Stokes system and Leray-α model, the existence and uniqueness of solutions for stochastic versions, have been developed in last years, see for instance \cite{4,6,7,8,13,14,15,16,22,26,30} and references therein. In particular, opposed to three dimensional Navier-Stokes equations, for LANS-α model, the existence and uniqueness of weak solutions is known (see for instance \cite{16}). This point is relevant in control problems because it permits to guarantee that the reaction of the flow produced by the action of a control is unique.

In this paper we are interested in an optimal control problem for the LANS-α model \eqref{1.1} where the body force is regarded as the control and a final observation is considered; in this sense we say that it is an optimal control problem for a distributed parameter system with final observation. More precisely, we wish to minimize the functional

$$J(u, v) = \frac{\gamma_1}{2} \int_0^T \|u(t) - u_d(t)\|^2_{D(A)}dt + \frac{\gamma_2}{2} \int_{\Omega} |u(x, T) - u_T(x)|^2dx + \frac{\gamma_3}{2} \int_0^T \|v(t)\|^2_2 dt,$$

where the velocity field is subject to state system \eqref{1.1} where $f$ is now replaced by the distributed control field $v$. The functions $u_d, u_T$ are given and denote the desired state, and the parameters $\gamma_1, \gamma_2, \gamma_3 > 0$ stand the cost coefficients for the control. The exact mathematical formulation will be given in Section 3. We will prove the solvability of the optimal control problem and state the first-order optimality conditions. By using the Lagrange multipliers method we derive an optimality system in a weak and strong formulation. To the best of our knowledge, this paper is the first work dealing with optimal control problems where the state variable satisfies the 3D LANS-α model \eqref{1.1}. However, in the recent papers \cite{25,27} the authors studied the problem of optimal control of the viscous Camassa-Holm equation in one dimension. The models treated in \cite{25,27} can be viewed as one dimensional versions of the three dimensional LANS-α model.

Related to the nonstationary Navier-Stokes system, there are many results available in the literature concerned with the study of optimal control problems (see \cite{17} and references therein). In particular, necessary conditions for optimal control of 2D-Navier-Stokes model can be found in \cite{11,18,19,21}. Necessary conditions for optimal control of 3D Navier-Stokes were obtained in \cite{9}.

The paper is organized as follows. In section 2 we establish the notation to be used and recall some known results for the LANS-α model. In section 3 we setting the precise optimal control problem and prove the existence of optimal solutions. In section 4 we
derive the first-order optimality conditions, and by using the Lagrange multipliers method we derive an optimality system.

2. Preliminaries

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^3 \) with boundary \( \Gamma \) of class \( C^2 \). We denote by \( D(\Omega) \) and \( D'(\Omega) \) the space of functions of class \( C^\infty(\Omega) \) with compact support, and the space of distributions on \( \Omega \), respectively. Throughout this paper we use standard notations for Lebesgue and Sobolev spaces. In particular, the \( L^2(\Omega) \)-norm and the \( L^2(\Omega) \)-inner product, will be represented by \( \| \cdot \| \) and \( \langle \cdot, \cdot \rangle \), respectively. We consider the solenoidal Banach spaces \( H \) and \( V \) defined, respectively, as the closure in \( (L^2(\Omega))^3 \) and \( (H^1(\Omega))^3 \) of

\[
V = \{ u \in (D(\Omega))^3 : \nabla \cdot u = 0 \text{ in } \Omega \}.
\]

Here, \( \nabla \cdot u \) denotes the divergence of the field \( u \). The norm and the inner product in \( V \) will be denoted by \( \| u \|_V \) and \( \langle \nabla u, \nabla v \rangle \), respectively. Throughout this paper, if \( X \) is a Banach space with dual space \( X' \), the duality paring between \( X' \) and \( X \) will be denoted by \( \langle \cdot, \cdot \rangle_{X',X} \). To simplify the notation, we will use the same notation for vectorial valued and scalar valued spaces.

For \( X \) Banach space, \( \| \cdot \|_X \) denotes its norm and \( L^p(0,T;X) \) denotes the standard space of functions from \( [0,T] \) to \( X \), endowed with the norm

\[
\| u \|_{L^p(0,T;X)} = \left( \int_0^T \| u(t) \|_X^p \, dt \right)^{1/p}, \quad 1 \leq p < \infty, \quad \| u \|_{L^\infty(0,T;X)} = \sup_{t \in (0,T)} \| u(t) \|_X.
\]

In the sequel we will identify the spaces \( L^p(0,T;X) := L^p(X) \) and \( L^p(0,T;L^p(\Omega)) := L^p(Q) \).

We recall the following compactness result:

**Lemma 2.1.** ([23]) Let \( B_0,B \) and \( B_1 \) be Banach spaces with \( B_0 \hookrightarrow B \hookrightarrow B_1 \) continuously and \( B_0 \hookrightarrow B \) compact. For \( 1 \leq p \leq \infty \) and \( T < \infty \) consider the Banach space

\[
W = \{ u \in L^p(0,T;B_0), \ u_t \in L^1(0,T;B_1) \}.
\]

Then \( W \hookrightarrow L^p(0,T;B) \) compactly.

Let \( P : L^2(\Omega) \to H \) be the Leray projector, and denote by \( A = -P\Delta \) the Stokes operator with domain \( D(A) = H^2(\Omega) \cap V \). It is known that \( A \) is a self-adjoint positive operator with compact inverse. Since \( \Gamma \) is of class \( C^2 \), the norms \( \| Au \| \) and \( \| u \|_{H^2} \) are equivalent. For \( u \in D(A) \) and \( v \in L^2(\Omega) \) we define the element of \( H^{-1}(\Omega) \equiv (H_0^1(\Omega))' \) by

\[
\langle (u \cdot \nabla)v, w \rangle_{H^{-1},H_0^1} = \sum_{i,j=1}^3 \langle \partial_i v_j, u_i w_j \rangle_{H^{-1},H_0^1}, \quad \forall w \in H_0^1(\Omega).
\]

In particular, if \( v \in H^1(\Omega) \), the definition of \( \langle (u \cdot \nabla)v, w \rangle_{H^{-1},H_0^1} \) coincides with the definition of

\[
\langle (u \cdot \nabla)v, w \rangle = \sum_{i,j=1}^3 \int_\Omega (u_i \partial_i v_j) w_j \, dx.
\]
Let us denote by \((\nabla u)^*\) the transpose of \(\nabla u\). Thus, if \(u \in D(A)\) then \((\nabla u)^* \in H^1(\Omega) \subset L^6(\Omega)\). Consequently, for \(v \in L^2(\Omega)\) we have that \((\nabla u)^* \cdot v \in L^{3/2}(\Omega) \subset H^{-1}(\Omega)\) with

\[
\langle (\nabla u)^* \cdot v, w \rangle_{H^{-1}, H^1_0} = \sum_{i,j=1}^3 \int_\Omega (\partial_j u_i)v_iw_j \, dx, \quad \forall w \in H^1_0(\Omega).
\]

One can check that for \(u, w \in D(A)\), \(v \in L^2(\Omega)\), the following equality holds

\[
\langle (u \cdot \nabla)v, w \rangle_{H^{-1}, H^1_0} = -\langle (\nabla u)^* \cdot v, u \rangle_{H^{-1}, H^1_0}.
\]

(2.2)

We consider the nonlinear operator \(B : D(A) \times D(A) \to D(A)'\) defined by

\[
\langle B(u, v), w \rangle_{D(A)', D(A)} = \langle (u \cdot \nabla)(v - \alpha \Delta v), w \rangle_{V', V} + \langle (\nabla u)^* \cdot (v - \alpha \Delta v), w \rangle_{V', V}.
\]

(2.3)

Thus, from (2.2) we have

\[
\langle B(u, v), u \rangle_{D(A)', D(A)} = 0, \quad \forall u, v \in D(A).
\]

(2.4)

Also, we have that

\[
|\langle B(u, v), w \rangle_{D(A)', D(A)}| \leq C \|u\|_V \|\nabla u\|_V \|Aw\| + C \alpha(\|u\|_L^6 \|\nabla u\|_L^3 + \|\nabla u\|_V \|\Delta u\|_V) \|\Delta v\|
\]

\[
\leq C \|\nabla u\|_V \|\Delta u\|_V + C \alpha \|\nabla u\|_V \|\Delta u\|_V \|\Delta w\|_V \|\Delta v\|
\]

\[
\leq C \|\nabla u\|_V \|\Delta u\|_V \|\Delta w\|_V \|\Delta v\|.
\]

(2.5)

Therefore,

\[
\|B(u, v)\|_{D(A)'} \leq C \|\nabla u\|_V \|\Delta u\|_V \leq C \|u\|_V \|v\|_{D(A)}, \quad \forall u, v \in D(A),
\]

and thus, for all \(u, v \in L^\infty(V) \cap L^2(D(A))\) it holds \(B(u, v) \in L^2(D(A)')\). Denoting by \(\Delta_\alpha = I - \alpha \Delta\), one gets

\[
\Delta_\alpha u \in L^\infty(V') \cap L^2(H) \quad \text{and} \quad \Delta_\alpha A u \in L^2(D(A)') \quad \forall u \in L^2(D(A)) \cap L^\infty(V).
\]

With the above notations, the system (1.1) can be rewritten as

\[
\begin{cases}
\Delta_\alpha u_t + \nu \Delta_\alpha A u + B(u, u) + \nabla p = f \quad \text{in} \; Q, \\
\nabla \cdot u = 0 \quad \text{in} \; Q, \\
u = 0, \quad A u = 0 \; \text{on} \; \Gamma \times (0, T), \\
\end{cases}
\]

(2.6)

Now we are in position to establish the definition of weak solution of Problem (1.1) (equivalently (2.6)).

**Definition 2.2.** (Weak solution) For \(f \in L^2(Q)\) and \(u_0 \in V\), a weak solution of the problem (2.6) is a field \(u \in L^2(D(A)) \cap L^\infty(V)\) with \(u_t \in L^2(H)\) satisfying

\[
\begin{cases}
\frac{d}{dt}((u, w) + \alpha(\nabla u, \nabla w)) + \nu(Au, w + \alpha Aw) \\
+ \langle B(u, u), w \rangle_{D(A)', D(A)} = (f, w), \quad \forall w \in D(A),
\end{cases}
\]

(2.7)

or equivalently,

\[
\begin{cases}
\Delta_\alpha u_t + \nu \Delta_\alpha A u + B(u, u) = f \quad \text{in} \; D(A)', \\
A u = 0 \; \text{in} \; \Gamma \times (0, T), \\
u = 0 \; \text{in} \; \Omega, \\
\end{cases}
\]

(2.8)
Theorem 2.3. (Existence and uniqueness of weak solution) Assuming that $f \in L^2(Q)$ and $u_0 \in V$, there exists a unique weak solution of (2.6).

Proof. The existence of weak solutions follows from the classical Galerkin approximations and energy estimates; the uniqueness follows from a standard Gronwall argument (see e.g. \([2, 5, 6, 8, 16, 30]\)).

3. A distributed control problem: Existence of optimal solution

We start by establishing the control in the system. We denote the control by $v \in L^2(Q)$ which will be use as a source term in (2.6)1; thus the right-hand side of equality (2.8)1 will be defined as:

$$(f, w) = (v, w), \quad \forall w \in D(A).$$

In order to specify exactly the problem, we make some considerations. We define the Banach space

$$W = \{u \in L^2(D(A)) \cap L^\infty(V) : u_t \in L^2(H)\},$$

with norm given by

$$\|w\|_W := \max\{\|u\|_{L^2(D(A))}, \|u\|_{L^\infty(V)}, \|u_t\|_{L^2(H)}\}.$$

Since $D(A) \hookrightarrow V \hookrightarrow H$, and $D(A) \hookrightarrow V$ compactly, from Lemma 2.1 we have $W \hookrightarrow L^2(V)$ compactly; furthermore, and as $D(A), V, H$ are Hilbert spaces, $W \hookrightarrow C([0, T]; V)$ (cf. \[29\]). We also consider the subspace $W_0$ of $W$ defined by

$$W_0 := \{w \in W : Au = 0 \text{ on } \Gamma \times (0, T)\}.$$

In order to establish the control problem, we assume the following general hypotheses:

(H1) The regularization parameters $\gamma_1, \gamma_2$ and $\gamma_3$, which measures the cost of the control, are fixed positive numbers.

(H2) The initial data $u_0 \in V$, the desired velocity $u_d \in L^2(D(A))$ and the function $u_T \in H$.

(H3) The set of admissible controls $U_{ad}$ is defined by:

$$U_{ad} = \{v \in L^2(Q) : v_{a,i}(x, t) \leq v_i(x, t) \leq v_{b,i}(x, t) \text{ a.e. on } Q, \ i = 1, 2, 3\},$$

where the control constraints $v_a, v_b$ are required to be in $L^2(Q)$ with $v_{a,i}(x, t) \leq v_{b,i}(x, t)$ a.e. on $Q$. Notice that $U_{ad}$ is a non-empty, convex and closed set in $L^2(Q)$.

Under the hypotheses (H1)-(H3), for $(u, v) \in W_0 \times U_{ad}$ we define the following objective functional

$$J(u, v) = \frac{\gamma_1}{2} \int_0^T \|u(t) - u_d(t)\|^2_{D(A)} dt + \frac{\gamma_2}{2} \int_{\Omega} |u(x, T) - u_T(x)|^2 dx + \frac{\gamma_3}{2} \int_0^T \|v(t)\|^2 dt.$$

Thus, we consider the following distributed optimal control problem:

$$\begin{cases}
\text{Minimize} & J(u, v) \\
(u, v) \in W_0 \times U_{ad},
\end{cases}$$

(3.3)
subject to the state equation

\[
\begin{aligned}
\Delta_{\alpha}u_t + \nu \Delta_{\alpha}Au + B(u, u) &= v \text{ in } L^2(D(A)', u(0, x) = u_0(x) \text{ in } V.
\end{aligned}
\]  

(3.4)

Then, the set of admissible solutions to (3.3)-(3.4) is defined as:

\[
\mathcal{S}_{ad} = \{(u, v) \in \mathbb{W}_0 \times U_{ad} : J(u, v) < \infty \text{ and } (u, v) \text{ satisfies (3.4)}\}.
\]

(3.5)

3.1. Existence of solution. We will show that the optimal control problem (3.3)-(3.4) has a solution.

Theorem 3.1. Under the assumptions (H1)-(H4), there exists a solution \((\hat{u}, \hat{v}) \in \mathcal{S}_{ad}\) to the optimal control problem (3.3)-(3.4).

Proof. By Theorem 2.3, the pair \((u, v_a) \in \mathcal{S}_{ad}\); thus the set \(\mathcal{S}_{ad} \neq \emptyset\). Now, since \(J\) is bounded from below \((J(u, v) \geq 0)\), there exists a minimizing sequence \((u^m, v^m)\) in \(\mathcal{S}_{ad}\) such that

\[
\lim_{m \to \infty} J(u^m, v^m) = \inf \{J(u, v) : (u, v) \in \mathcal{S}_{ad}\},
\]

and for all \(w \in D(A)\) it holds:

\[
\langle \Delta_{\alpha}u_t^m, w \rangle_{D(A)', D(A)} + \nu \langle \Delta_{\alpha}Au^m, w \rangle_{D(A)', D(A)} + \langle B(u^m, u^m), w \rangle_{D(A)', D(A)} = \langle v^m, w \rangle.
\]

(3.6)

From (3.6), using integration by parts on \(\Omega\), we have

\[
\langle u_t^m, w \rangle + \alpha \langle \nabla u_t^m, \nabla w \rangle + \nu \langle \nabla u^m, \nabla w \rangle + \nu \alpha \langle Au^m, w \rangle + \langle B(u^m, u^m), w \rangle = \langle v^m, w \rangle.
\]

(3.7)

Then, setting \(w = u^m(t)\) in (3.7) and taking into account (2.4), we get

\[
\frac{1}{2} \frac{d}{dt} (\|u^m\|^2 + \alpha \|\nabla u^m\|^2) + \nu \|\nabla u^m\|^2 + \nu \alpha \|Au^m\|^2 = \langle v^m, u^m \rangle.
\]

Using the Hölder and Young inequalities it holds

\[
\|v^m, u^m\| \leq \|v^m\|\|u^m\| \leq C \|v^m\|^2 + \nu \|\nabla u^m\|^2.
\]

Therefore,

\[
\frac{d}{dt} (\|u^m\|^2 + \alpha \|\nabla u^m\|^2) + \nu \|\nabla u^m\|^2 + 2 \nu \alpha \|Au^m\|^2 \leq C \|v^m\|^2.
\]

(3.8)

Integrating (3.8) from 0 to \(t \in [0, T]\), we obtain

\[
\|u^m(t)\|^2 + \alpha \|\nabla u^m(t)\|^2 + \nu \int_0^t (\|\nabla u^m(s)\|^2 + \alpha \|Au^m(s)\|^2)ds \leq C \|v^m\|^2 + \alpha \|\nabla u_0\|^2.
\]

(3.9)

Since \(v^m \in U_{ad}\) and \(u_0 \in V\), from (3.9) we conclude that

\[
\{u^m\}_{m \geq 1} \text{ is uniformly bounded in } L^\infty(V) \cap L^2(D(A)) \quad \text{ and } \quad \{v^m\}_{m \geq 1} \text{ is uniformly bounded in } L^2(Q).
\]

(3.10)  

(3.11)
On the other hand, from (3.6) and by applying integration by parts on \( \Omega \) we get

\[
(D_\alpha u^m_t, w)_{D(A)'}, D(A) = -\nu(\nabla u^m, \nabla w) - \nu \alpha (Au^m, Aw) - (B(u^m, u^m), w)_{D(A)', D(A)} + (v^m, w),
\]

and then, by using the Hölder inequality together inequalities (2.5) and (3.10), we obtain

\[
|\langle D_\alpha u^m_t, w \rangle_{D(A)', D(A)}| \leq C_{\nu, \alpha} (\|\nabla u^m\| + \|Au^m\| + \|B(u^m, u^m)\|_{D(A)'} + \|v^m\|) \|w\|_{D(A)}
\]

\[
\leq C_{\nu, \alpha} (\|\nabla u^m\| + \|Au^m\| + \|v^m\|) \|w\|_{D(A)}.
\]

Since \( D_\alpha u^m_t, w \rangle_{D(A)', D(A)} = \langle u_t + \alpha Au_t, w \rangle_{D(A)', D(A)} \) for all \( w \in D(A) \), the last inequality implies

\[
\|u^m_t + \alpha Au^m_t\|_{D(A)'} \leq C (\|\nabla u^m\| + \|Au^m\| + \|v^m\|),
\]

and by using the Young inequality

\[
\|u^m_t + \alpha Au^m_t\|_{D(A)'}^2 \leq C (\|\nabla u^m\|^2 + \|Au^m\|^2 + \|v^m\|^2).
\]

By integrating (3.12) from 0 to \( t \in [0, T] \) and taking into account (3.10)-(3.11) we have

\[
\int_0^t \|u^m_t(s) + \alpha Au^m_t(s)\|_{D(A)'}^2 ds \leq C.
\]

Since the operator \( A \) is self adjoint and positive, the following inequality holds (see [30])

\[
\|v\|_{D(A)'} \leq \|v + \alpha Av\|_{D(A)'} \text{ for each } v \in D(A)'.
\]

Then, by using triangular inequality and (3.14), we get

\[
\|\alpha Au^m_t\|_{D(A)'}^2 \leq \|u^m_t + \alpha Au^m_t\|_{D(A)'}^2 + \|u^m_t\|_{D(A)'}^2 \leq C \|u^m_t + \alpha Au^m_t\|_{D(A)'}^2.
\]

Thus, from (3.13) and (3.15), we conclude that

\[
\{Au^m_t\}_{m \geq 1} \text{ is uniformly bounded in } L^2(D(A)'),
\]

\[
\{u^m_t\}_{m \geq 1} \text{ is uniformly bounded in } L^2(Q).
\]

Moreover, from (3.10) and (3.17) we have

\[
\{u^m\}_{m \geq 1} \text{ is uniformly bounded in } W,
\]

with \( W \) compactly imbedded in \( L^2(V) \) (cf. Lemma 2.1).

Then, from (3.10), (3.11), (3.16), (3.17) and (3.18), there exists a subsequence, which again we denote by \( (u^m, v^m) \), converging to some limit \( (\hat{u}, \hat{v}) \in W \times U_{ad} \) such that as \( m \to \infty \),

\[
u^m \to \hat{u} \text{ weakly in } L^2(D(A)) \text{ and strongly in } L^2(V),
\]

\[
u^m_t \to \hat{u}_t \text{ weakly in } L^2(Q),
\]

\[
Au^m_t \to A\hat{u}_t \text{ weakly in } L^2((D(A)'),
\]

\[
v^m \to \hat{v} \text{ weakly in } L^2(Q).
\]

Passing to the limit as \( m \to \infty \) in (3.6), we can obtain that \( (\hat{u}, \hat{v}) \) satisfies (3.4), with \( A\hat{u} = 0 \) on \( \Gamma \times (0, T) \), and \( J(\hat{u}, \hat{v}) < \infty \). Since \( u_0 = u^m(0) \) in \( V \) for all \( m \), and \( u^m \in W \), it
holds $u_0 = \hat{u}(0)$ in $V$. Consequently $\hat{u} \in \mathbb{W}_0$ and thus we get that $(\hat{u}, \hat{v}) \in \mathcal{S}_{ad}$. Then we obtain
\[
\lim_{m \to \infty} J(u^m, v^m) = \inf \{ J(u, v) : (u, v) \in \mathcal{S}_{ad} \} \leq J(\hat{u}, \hat{v}). \tag{3.19}
\]
As the functional $J : \mathcal{S}_{ad} \to \mathbb{R}$ is weakly lower semicontinuous, we have that (cf. \[20\])
\[
J(\hat{u}, \hat{v}) \leq \lim_{m \to \infty} \inf J(u^m, v^m). \tag{3.20}
\]
Finally, from (3.19) and (3.20) we conclude that $J(\hat{u}, \hat{v}) = \inf \{ J(u, v) : (u, v) \in \mathcal{S}_{ad} \}$. \hfill \ensuremath{\square}

4. First order optimality conditions

In order to derive an optimal system by using the Lagrange multiplier method, we formulate an abstract Lagrange multiplier principle. Let $X$ and $Y$ be two Banach spaces, $J : X \to \mathbb{R}$ and $G : X \to Y$. Consider the problem
\[
\min_{z \in X} J(z) \quad \text{subject to } G(z) = 0. \tag{4.1}
\]
The Lagrange function corresponding to the problem (4.1) is defined by
\[
\mathcal{L}(z, \lambda_0, \lambda) = \lambda_0 J(z) - \langle \lambda, G(z) \rangle_{Y', Y},
\]
where $\lambda_0 \in \mathbb{R}$ and $\lambda \in Y'$ are called Lagrange multipliers. Then the following result is known (see e.g. \[20\,24\]).

**Theorem 4.1.** \[20\,24\] (The Lagrange multiplier rule). Let $\hat{z}$ be a solution of (4.1). Assume that the functional $J$ and the mapping $G$ are continuously differentiable at the point $\hat{z}$ and that the range of the mapping $G(\hat{z}) : X \to Y$ is closed. Then there exists a nonzero Lagrange multiplier $(\lambda_0, \lambda) \in \mathbb{R}^+ \times Y'$, such that
\[
\mathcal{L}_z(\hat{z}, \lambda_0, \lambda) h = \lambda_0 J_z(\hat{z}) h - \langle \lambda, G_z(\hat{z}) h \rangle_{Y', Y} = 0 \quad \forall h \in X,
\]
\[
\mathcal{L}(\hat{z}, \lambda_0, \lambda) \leq \mathcal{L}(z, \lambda_0, \lambda), \forall z \in X,
\]
where $\mathcal{L}_z(\cdot, \cdot, \cdot)$ denotes the Fréchet derivative of $\mathcal{L}$. Furthermore, if $G_z(\hat{z}) : X \to Y$ is an epimorphism, then $\lambda_0 \neq 0$ and $\lambda_0$ can be taken as 1.

In order to derive the first-order optimality conditions for the problem (3.3)-(3.4), we will apply Theorem 4.1.

Observing (3.3), we define the operator
\[
F : \mathbb{W}_0 \times L^2(D(A)') \to L^2(D(A)')
\]
\[
(u, v) \mapsto F(u, v) := \Delta_\alpha u + \nu \Delta_\alpha A u + B(u, u) - v.
\]

**Lemma 4.2.** The operator $F$ is Fréchet differentiable with respect to $u$.

**Proof.** Using (2.3), $B(u + w, u + w) = B(u, u) + B(u, w) + B(w, u) + B(w, w)$. Then
\[
F(u + w, v) - F(u, v) = \Delta_\alpha w + \nu \Delta_\alpha A w + B(u, w) + B(w, u) + B(w, w). \tag{4.2}
\]
Denoting by $Lw = \Delta_\alpha w + \nu \Delta_\alpha A w + B(u, w) + B(w, u)$, from (4.2) we get
\[
\| F(u + w, v) - F(u, v) - Lw \|_{L^2(D(A)')} = \| B(w, w) \|_{L^2(D(A)')}. \tag{4.3}
\]
Since \( w \in \mathbb{W}_0 \), from (2.5) we obtain
\[
\|B(w, w)\|_{L^2(D(A)')} \leq C \|w\|_{L^\infty(V)} \|w\|_{L^2(D(A))} \leq C \|w\|^2_{\mathbb{W}_0},
\]
and then from (4.3) we have
\[
\|F(u + v, v) - F(u, v) - Lw\|_{L^2(D(A)')} \leq C \|w\|^2_{\mathbb{W}_0}.
\]
Thus,
\[
\lim_{\|w\|_{\mathbb{W}_0} \to 0} \frac{\|F(u + v, v) - F(u, v) - (\Delta_\alpha w_t + \nu \Delta_\alpha Aw + B(u, w) + B(w, u))\|_{L^2(D(A)')}}{\|w\|_{\mathbb{W}_0}} = 0.
\]

Therefore, the Fréchet derivative of \( F \) with respect to \( u \) in an arbitrary \((u, v)\) is given by the operator \( F_u(u, v) : \mathbb{W}_0 \to L^2(D(A)') \) such that for each \( w \in \mathbb{W}_0 \),
\[
F_u(u, v)w = \Delta_\alpha w_t + \nu \Delta_\alpha Aw + B_u(u, u)w,
\]
where \( B_u(u, u)w = B(u, w) + B(w, u) \) is the Fréchet derivative of \( B \) with respect to \( u \) in an arbitrary point \((u, u)\). \( \square \)

Now let us consider the closed linear subspace \( \mathbb{V}_0 \) of \( \mathbb{W}_0 \) defined by

\[
\mathbb{V}_0 = \{ w \in \mathbb{W}_0 : w(x, 0) = 0 \ \forall x \in \Omega \}.
\]

The following preliminary result holds:

**Lemma 4.3.** Let \((u, v) \in \mathbb{W}_0 \times L^2(Q)\) and \( g \in L^2(D(A)') \) be given. Then there exists a unique solution \( w \in \mathbb{V}_0 \) of the linear problem
\[
F_u(u, v)w = g.
\]

**Proof.** The proof follows by using the classical Galerkin approximations and energy estimates (see [2, 5]). \( \square \)

**Lemma 4.4.** The functional \( J \) is Fréchet differentiable with respect to \( u \).

**Proof.** From definition of the functional \( J \) we get:
\[
J(u + v, v) - J(u, v) = \gamma_1 \int_0^T (Aw, Au - Au_d) dt + \frac{\gamma_1}{2} \int_0^T \|Aw\|^2 dt + \gamma_2(w(T), u(T) - u_T) + \frac{\gamma_2}{2} \|w(T)\|^2.
\]

Then
\[
|J(u + v, v) - J(u, v) - \gamma_1 \int_0^T (Aw, Au - Au_d) dt - \gamma_2(w(T), u(T) - u_T)| \leq C(\|w\|^2_{L^2(D(A))} + \|w(T)\|^2) \leq C \|w\|^2_{\mathbb{W}_0},
\]

which implies that
\[
\lim_{\|w\|_{\mathbb{W}_0} \to 0} \frac{|J(u + v, v) - J(u, v) - \gamma_1 \int_0^T (Aw, Au - Au_d) dt - \gamma_2(w(T), u(T) - u_T)|}{\|w\|_{\mathbb{W}_0}} = 0.
\]
Thus, the Fréchet derivative of $J$ with respect to $u$ in an arbitrary $(u, v)$ is the operator $J_u(u, v) : \mathbb{W}_0 \to \mathbb{R}$ defined by:

$$J_u(u, v)w = \gamma_1 \int_0^T (Aw, Au - A_d)dt + \gamma_2(w(T), u(T) - u_T), \; w \in \mathbb{W}_0. \tag{4.6}$$

With the above notations, let us define the Lagrange function $L$ for the control problem (3.3)-(3.4). Then, there exists a $\lambda \in \mathbb{R}$ such that

$$\mathcal{L}(u, v, \lambda) = J(u, v) - \langle F(u, v), \lambda \rangle_{L^2(D(A)'}, L^2(D(A))), \tag{4.7}$$

where $\lambda \in L^2(D(A))$. Thus, the Fréchet derivative of $L$ with respect to $u$ is

$$L_u(u, v, \lambda)w = J_u(u, v)w - \langle F_u(u, v)w, \lambda \rangle_{L^2(D(A)'}, L^2(D(A))), \; \forall w \in \mathbb{W}_0. \tag{4.8}$$

Now we will state and prove the necessary first-order optimality conditions.

**Theorem 4.5. (Necessary conditions)** Let $(\hat{u}, \hat{v}) \in S_{ad}$ be a solution of the optimal control problem (3.3)-(3.4). Then, there exists a $\lambda \in L^2(D(A))$ such that

$$\mathcal{L}(\hat{u}, \hat{v}, \lambda)h = J_u(\hat{u}, \hat{v})h - \langle F_u(\hat{u}, \hat{v})h, \lambda \rangle_{L^2(D(A)'}, L^2(D(A))) = 0, \; \forall h \in \mathbb{Y}_0. \tag{4.9}$$

Moreover, the minimum principle holds

$$\mathcal{L}(\hat{u}, \hat{v}, \lambda) \leq \mathcal{L}(u, v, \lambda) \; \forall v \in \mathcal{U}_{ad}. \tag{4.10}$$

**Proof.** We will apply Theorem 4.1 for that, in particular, we need to prove the surjectivity of the operator $F_u(\hat{u}, \hat{v})$. Thus, in order to simplify the calculations, we rewrite the problem (3.3)-(3.4) in an equivalent optimal control problem. For this purpose, by considering $z \in \mathbb{Y}_0$, we use the change of variable $u = \hat{u} + z$. Thus, by replacing $u$ in (3.3) we get

$$\Delta_\alpha z_t + \nu \Delta_\alpha Az + B(\hat{u}, z) + B(z, \hat{u}) + B(z, z) = 0. \tag{4.11}$$

Therefore we obtain the following equivalent optimal control problem:

$$\min_{z \in \mathbb{Y}_0} \tilde{J}(z) := \min_{z \in \mathbb{Y}_0} J(\hat{u} + z, \hat{v}), \tag{4.12}$$

subject to the state equation

$$G(z) = \Delta_\alpha z_t + \nu \Delta_\alpha Az + B(\hat{u}, z) + B(z, \hat{u}) + B(z, z) = 0. \tag{4.13}$$

Observe that $z = 0$ is the optimal solution of the control problem (4.12)-(4.13) provided $(\hat{u}, \hat{v})$ minimizes $J$.

Thus, we will apply Theorem 4.1 for the problem (4.12)-(4.13). For that, we will verify all its conditions.

- **Step one.** The operator $G$ is continuously differentiable with respect to $z$.

Following the proof of Lemma 4.2, it is not difficult to obtain that the derivative of the operator $G : \mathbb{Y}_0 \to L^2(D(A)'$ at a point $\hat{z}$ is given by the linear and continuous operator $G_z(\hat{z}) : \mathbb{Y}_0 \to L^2(D(A)'$ defined by

$$G_z(\hat{z})h = \Delta_\alpha h_t + \nu \Delta_\alpha Ah + B(\hat{u}, h) + B(h, \hat{u}) + B(h, \hat{z}) + B(\hat{z}, h). \tag{4.14}$$

**Remark 4.1.** Notice that at the optimal solution $\hat{z} = 0$ of $\tilde{J}$ it holds $G_z(0)h = F_u(\hat{u}, \hat{v})h$ for all $h \in \mathbb{Y}_0$, being $(\hat{u}, \hat{v}) \in S_{ad}$ the optimal solution of the control problem (3.3)-(3.4).
The functional $\tilde{J}$ is continuously differentiable with respect to $z$. Notice that from definition of $J$ we have

$$
J(\hat{u} + z + \epsilon h, \hat{v}) - J(\hat{u} + z, \hat{v}) = \gamma_1 \epsilon \int_0^T (Ah, A(\hat{u} + z) - Au_d) dt + \frac{\epsilon^2 \gamma_1}{2} \int_0^T \|Ah\|^2 dt + \epsilon \gamma_2 (h(T), (\hat{u} + z)(T) - u_T) + \frac{\epsilon^2 \gamma_2}{2} \|h(T)\|^2. \tag{4.15}
$$

Since

$$
\tilde{J}_z(z) h = \lim_{\epsilon \to 0} \frac{\tilde{J}(z + \epsilon h) - \tilde{J}(z)}{\epsilon} = \lim_{\epsilon \to 0} \frac{J(\hat{u} + z + \epsilon h, \hat{v}) - J(\hat{u} + z, \hat{v})}{\epsilon},
$$

then, from (4.15), the derivative of $\tilde{J}$ is given by the linear and continuous operator $\tilde{J}_z(z) : \mathbb{Y}_0 \to \mathbb{R}$ defined by

$$
\tilde{J}_z(z) h = \gamma_1 \int_0^T (Ah, A\hat{u} + Az - Au_d) dt + \gamma_2 (h(T), \hat{u}(T) + z(T) - u_T), \forall h \in \mathbb{Y}_0. \tag{4.16}
$$

**Remark 4.2.** Notice that at the optimal solution $\hat{z} = 0$ of $\tilde{J}$ it holds $\tilde{J}_z(0) h = J_u(\hat{u}, \hat{v}) h$, for all $h \in \mathbb{Y}_0$, being $(\hat{u}, \hat{v}) \in S_{ad}$ the optimal solution of the control problem (3.3)-(3.4).

**Step three.** $G_z(0) : \mathbb{Y}_0 \to L^2(D(A)')$ is surjective.

Observing (4.14), the derivative of $G$ at the optimal solution $\hat{z} = 0$ of $\tilde{J}$ is

$$
G_z(0) h = \Delta_\alpha h_t + \nu \Delta_\alpha Ah + B(\hat{u}, h) + B(h, \hat{u}). \tag{4.17}
$$

Then, by using Lemma 4.3 together Remark 4.1, for each $g \in L^2(D(A)')$ there exists a unique $h \in \mathbb{Y}_0$ such that $G_z(0) h = g$. Notice that the range of the mapping $G_z(0) : \mathbb{Y}_0 \to L^2(D(A)')$ is a closed set. Thus, the conditions of Theorem 4.1 are verified. Consequently, by defining the Lagrange functional

$$
\tilde{L}(z, \lambda) = \tilde{J}(z) - \langle G(z), \lambda \rangle_{L^2(D(A)', L^2(D(A))}, \tag{4.18}
$$

Theorem 4.1 guarantees the existence of a $\lambda \in L^2(D(A))$ such that

$$
\tilde{L}_z(0, \lambda) h = \tilde{J}_z(0) h - \langle G_z(0) h, \lambda \rangle_{L^2(D(A)', L^2(D(A))} = 0, \forall h \in \mathbb{Y}_0. \tag{4.19}
$$

From Remark 4.1 Remark 4.2 (4.8) and (4.19), we obtain

$$
\tilde{L}_z(0, \lambda) h = \mathcal{L}_u(\hat{u}, \hat{v}) h \ \forall h \in \mathbb{Y}_0, \tag{4.20}
$$

with $(\hat{u}, \hat{v}) \in S_{ad}$ the optimal solution of $J$. Therefore, from (4.19) and (4.20), the equality (4.9) is verified. The inequality (4.10) follows directly from Theorem 4.1. 

**Remark 4.3.** Following Theorem 1.5 in [17] (see also [24]), since $U_{ad}$ is a convex set, the minimum principle (4.10) implies

$$
\mathcal{L}_v(\hat{u}, \hat{v}, \lambda)(v - \hat{v}) \geq 0 \ \forall v \in U_{ad}. \tag{4.21}
$$
4.1. The weak formulation of an optimality system. The optimality system will be obtained from the necessary optimality conditions given in Theorem 4.5.

From (4.4), (4.6) and (4.9), we obtain the adjoint equation in a weak formulation

$$\int_0^T \langle \Delta \alpha h_t + \nu \Delta \alpha Ah + B_u(\hat{u}, \hat{u})h, \lambda \rangle_{D(A)^t, D(A)} dt = \gamma_1 \int_0^T (Ah, A\hat{u} - Au_d) dt$$

$$+ \gamma_2 (h(T), \hat{u}(T) - u_T), \forall h \in \mathbb{Y}_0.$$ (4.22)

From the minimum principle (4.10) we have

$$0 \leq J(\hat{u}, v) - J(\hat{u}, \hat{v}) + \langle F(\hat{u}, \hat{v}) - F(\hat{u}, v), \lambda \rangle_{L^2(D(A)^t), L^2(D(A))},$$

which implies that

$$0 \leq \frac{\gamma_3}{2} \int_0^T (\|v\|^2 - \|\hat{v}\|^2) dt + (v - \hat{v}, \lambda).$$ (4.23)

Using the equality $\|v\| - \|\hat{v}\|^2 = 2(v - \hat{v}, \hat{v}) + \|v - \hat{v}\|^2$, from (4.23) we get

$$0 \leq \int_0^T \gamma_3 (v - \hat{v}, \hat{v}) dt + \int_0^T (v - \hat{v}, \lambda) dt + \frac{\gamma_3}{2} \int_0^T \|v - \hat{v}\|^2 dt.$$ (4.24)

From (4.24) we can extract the following optimality condition

$$\int_0^T (\gamma_3 \hat{v} + \lambda, v - \hat{v}) dt \geq 0 \quad \forall v \in \mathcal{U}_{ad}.$$ (4.25)

Thus we have the variational inequality

$$(\hat{v} + \frac{1}{\gamma_3} \lambda, v - \hat{v}) \geq 0 \quad \text{a.e. in } Q, \forall v \in \mathcal{U}_{ad}.$$ (4.26)

Moreover, since $\mathcal{U}_{ad}$ is a convex and closed set in $L^2(Q)$, by the theorem of the projection onto a closed convex set (see [3]), the control $\hat{v}$ in the inequality (4.26) can be characterized as a projection; thus we have the optimality condition

$$\hat{v} = \text{Proj}_{\mathcal{U}_{ad}}(\cdot - \frac{1}{\gamma_3} \lambda) \quad \text{a.e. in } Q.$$ (4.27)

Consequently, the equations (4.4), (4.22) and the condition (4.27) form an optimality system in a weak formulation for the optimal control problem considered.

Remark 4.4. Taking into account the definition of $\mathcal{U}_{ad}$, the projection representation (4.27) for $\hat{v} = (\hat{v}_1, \hat{v}_2, \hat{v}_3)$ is in each component

$$\hat{v}_i = \text{Proj}_{[v_a, v_b]}(\cdot - \frac{1}{\gamma_i} \lambda_i) \quad \text{a.e. in } Q, \quad i = 1, 2, 3.$$
4.2. The strong form of the optimality system. We wish to represent the optimality system as a system of partial differential equations with boundary, initial and terminal conditions. Since we do not know at this moment whether the \( \lambda_t \) exists, we need to analyze the regularity of \( \lambda \).

Using integration by parts, for \( \lambda \in L^2(D(A)) \) and \( h \in \mathbb{V}_0 \) we get

\[
(\Delta_\alpha h_t, \lambda)_{D(A)'D(A)} = (h_t, \lambda) + \alpha(\nabla h_t, \nabla \lambda) - (h_t, \Delta \lambda) = (\Delta_\alpha \lambda, h_t),
\]

and then

\[
(\Delta_\alpha h_t, \lambda)_{L^2(D(A)'L^2(D(A))} = (\Delta_\alpha \lambda, h_t)_{L^2(Q), L^2(Q)}.
\]

(4.28)

Also, by using integration by parts in (4.22), for \( h \in \mathbb{V}_0 \) and \( \lambda \in L^2(D(A)) \), we have

\[
(\nu \Delta_\alpha Ah, \lambda)_{D(A)'D(A)} = \nu(Ah, \lambda) - \alpha \nu(\Delta Ah, \lambda)_{D(A)'D(A)} = \nu(Ah, \lambda) - \alpha(\Delta \lambda)
\]

\[
(\nu \Delta_\alpha Ah, \lambda)_{D(A)'D(A)} = \nu(A \Delta_\alpha \lambda, h)_{D(A)'D(A)}.
\]

(4.29)

Then for all \( h \in \mathbb{V}_0 \) we can write

\[
(\nu \Delta_\alpha Ah, \lambda)_{L^2(D(A)'L^2(D(A))} = \nu(A \Delta_\alpha \lambda, h)_{L^2(D(A)'L^2(D(A)))}.
\]

(4.30)

On the other hand, since \( A = - P \Delta \), we can obtain

\[
\gamma_1 \int_0^T \langle A \hat{u} - A u_d, h \rangle_{D(A)'D(A)} dt = \gamma_1 \int_0^T (\Delta h, A(\hat{u} - u_d)) dt
\]

\[
= \gamma_2 \int_0^T \langle \Delta A(\hat{u} - u_d), h \rangle_{D(A)'D(A)} dt.
\]

Thus, from (4.22), (4.28), (4.29), (4.30) and (4.31), we get

\[
(\Delta_\alpha h_t, \lambda)_{L^2(Q), L^2(Q)} = - (\nu A \Delta_\alpha \lambda, h)_{L^2(D(A)'L^2(D(A))} - (B^*_u(\hat{u}, \hat{u})\lambda, h)_{\mathbb{V}_0'\mathbb{V}_0} - \gamma_1 \langle \Delta A(\hat{u} - u_d), h \rangle_{L^2(D(A)'L^2(D(A))} + \gamma_2 (h(T), \hat{u}(T) - u_T).
\]

(4.32)

In order to obtain a representation of the weak time derivative of \( \Delta_\alpha \lambda \) we analyze the regularity of \( B^*_u(\hat{u}, \hat{u})\lambda \) in (4.32).

Notice that from (2.3) and (2.2) we get

\[
(\Delta_\alpha h_t, \lambda)_{D(A)'D(A)} = \langle \hat{u} \cdot \Delta_\alpha h, \lambda \rangle_{V',V} - \alpha(\nabla \hat{u}^* \cdot \Delta h, \lambda)
\]

\[
+ \langle h \cdot \nabla \Delta_\alpha \hat{u}, \lambda \rangle_{V',V} - \alpha(\nabla h^* \cdot \Delta \hat{u}, \lambda)
\]

\[
= - \langle \hat{u} \cdot \Delta \lambda, \Delta_\alpha h \rangle - \alpha(\lambda \cdot \nabla \hat{u}, \Delta h)
\]

\[
- \langle h \cdot \nabla \lambda, \Delta_\alpha \hat{u} \rangle - \alpha(\lambda \cdot \nabla h, \Delta \hat{u}).
\]

(4.33)

We bound the terms in (4.33). From Hölder and Sobolev inequalities we obtain

\[
|\langle \hat{u} \cdot \nabla \lambda, \Delta_\alpha h \rangle| \leq C \Vert \hat{u} \Vert_{L^6} \Vert \nabla \lambda \Vert_{L^3} \Vert \Delta_\alpha h \Vert \leq C \Vert \hat{u} \Vert V \Vert \lambda \Vert_{D(A)} \Vert h \Vert_{D(A)},
\]

(4.34)

\[
|\langle \lambda \cdot \nabla \hat{u}, \Delta h \rangle| \leq C \Vert \lambda \Vert_{L^\infty} \Vert \nabla \hat{u} \Vert \Vert \Delta h \Vert \leq C \Vert \lambda \Vert_{D(A)} \Vert \hat{u} \Vert V \Vert h \Vert_{D(A)}.
\]

(4.35)
By observing that \( w \cdot \nabla v = 0 \) on \( \Gamma \) if \( w, v \in D(A) \), and using integration by parts on \( \Omega \), for \( w, v, z \in D(A) \) we have
\[
(w \cdot \nabla v, \Delta z) = (\nabla (w \cdot \nabla v), \nabla z) = (\nabla v \nabla w, \nabla z) + (w \nabla (\nabla v), \nabla z),
\]
where \( w \nabla (\nabla v) = \sum_{i=1}^{3} w_i \frac{\partial}{\partial x_i} \nabla v. \) Then, by using (4.36), the fact that \( \| \nabla v \|_{L^1} \leq C \| v \|_{D(A)} \) and \( D(A) \subset L^\infty(\Omega) \), we obtain
\[
|(h \cdot \nabla \lambda, \Delta \hat{u})| = |(h \cdot \nabla \lambda, \hat{u}) - \alpha(h \cdot \nabla \lambda, \Delta \hat{u})| \\
\leq |(h \cdot \nabla \lambda, \hat{u})| + \alpha|\nabla \lambda \nabla h, \nabla \hat{u})| + \alpha|h \nabla (\nabla \lambda), \nabla \hat{u})| \\
\leq C \| h \|_{L^3} \| \nabla \lambda \| \| \hat{u} \|_{L^6} + C(\| \nabla \lambda \|_{L^4} \| \nabla h \|_{L^4} + \| h \|_{L^\infty} \| \nabla (\nabla \lambda) \|) \| \nabla \hat{u} \| \\
\leq C \| h \|_{D(A)} \| \lambda \|_{D(A)} \| \hat{u} \|_{V},
\]
(4.37)
\[
|(\lambda \cdot \nabla h, \Delta \hat{u})| \\
\leq |(\nabla h \nabla \lambda, \nabla \hat{u})| + |(\nabla (\nabla h), \nabla \hat{u})| \\
\leq C(\| \nabla h \|_{L^4} \| \nabla \lambda \|_{L^4} + \| \lambda \|_{L^\infty} \| \nabla (\nabla h) \|) \| \nabla \hat{u} \| \\
\leq C \| h \|_{D(A)} \| \lambda \|_{D(A)} \| \hat{u} \|_{V}.
\]
(4.38)
From (4.30), (4.33)-(4.35), (4.37) and (4.38), and by using the Hölder inequality, for \( \lambda \in L^2(D(A)), \hat{u} \in \mathcal{W}_0 \) and \( h \in \mathcal{Y}_0 \), we have
\[
|(B^*_u(\hat{u}, \hat{u})\lambda, h_{\mathcal{V}_0, \mathcal{Y}_0})| \leq C \| \hat{u} \|_{L^\infty(\mathcal{V})} \| \lambda \|_{L^2(D(A))} \| h \|_{L^2(D(A))},
\]
which implies
\[
B^*_u(\hat{u}, \hat{u})\lambda \in L^2(D(A)'),
\]
(4.39)
Then, for all \( h \in \mathcal{Y}_0 \) we can rewrite (4.32) as the following equality
\[
(\Delta_{\alpha} \lambda, h)_{L^2(Q)} = \langle -\nu A \Delta_{\alpha} \lambda - B^*_u(\hat{u}, \hat{u}) \lambda - \gamma_1 \Delta A(\hat{u} - u_d), h \rangle_{L^2(D(A)'),L^2(D(A))} \\
+ \gamma_2(h(T), \hat{u}(T) - u_T).
\]
Since \( h(T) \) can be arbitrary, when \( \Delta_{\alpha} \lambda(T) = \gamma_2(\hat{u}(T) - u_T) \), we have the existence of a representation of \( \Delta_{\alpha} \lambda_T \) in distributional sense as being
\[
\Delta_{\alpha} \lambda_T = \nu A \Delta_{\alpha} \lambda + B^*_u(\hat{u}, \hat{u}) \lambda + \gamma_1 \Delta A(\hat{u} - u_d).
\]
Thus we obtain that \( \lambda \in L^2(D(A)) \) is the solution of
\[
\left\{ \begin{array}{l}
\Delta_{\alpha} \lambda_T - \nu A \Delta_{\alpha} \lambda - B^*_u(\hat{u}, \hat{u}) \lambda = \gamma_1 \Delta A(\hat{u} - u_d) \text{ in } L^2(D(A)'), \\
\Delta_{\alpha} \lambda(T) = \gamma_2(\hat{u}(T) - u_T).
\end{array} \right.
\]
(4.40)
From (4.33), (4.37) and (4.39) we have
\[
\langle B^*_u(\hat{u}, \hat{u})\lambda, h \rangle_{D(A)' \cap D(A)} = -\langle \hat{u} \cdot \nabla \lambda, h \rangle + \alpha(\hat{u} \cdot \nabla \lambda, \Delta h) - \alpha(\lambda \cdot \nabla \hat{u}, \Delta h) \\
- \langle h \cdot \nabla \lambda, \Delta \hat{u} \rangle + \alpha(\lambda \cdot \nabla \hat{u}, \Delta \hat{u}).
\]
(4.41)
Observing that \( v \cdot \nabla w = 0 \) on \( \Gamma \) if \( v, w \in D(A) \), and using integration by parts on \( \Omega \), for \( \lambda, h \in D(A) \) we get
\[
\alpha(\hat{u} \cdot \nabla \lambda, \Delta h) - \alpha(\lambda \cdot \nabla \hat{u}, \Delta h) = -\alpha(\nabla(\hat{u} \cdot \nabla \lambda), \nabla h) + \alpha(\nabla(\lambda \cdot \nabla \hat{u}), \nabla h) \\
= \alpha(\Delta(\hat{u} \cdot \nabla \lambda), h) - \alpha(\Delta(\lambda \cdot \nabla \hat{u}), h).
\]
(4.42)
Taking into account (222), we get
\[
- \langle h \cdot \nabla \lambda, \Delta \hat{u} \rangle = \langle h \cdot \nabla \Delta_{\alpha} \hat{u}, \lambda \rangle_{V' \cap V} = -((\nabla \lambda)^* \cdot \Delta_{\alpha} \hat{u}, h).
\]
(4.43)
Thus, from \((4.41) - (4.43)\) we obtain
\[
\langle B_\alpha^*(\hat{u}, \hat{u}), h \rangle_{D(A)' \cdot D(A)} = \langle -\hat{u} \cdot \nabla \lambda + \alpha \Delta (\hat{u} \cdot \nabla \lambda) - \alpha \Delta (\lambda \cdot \nabla \hat{u}), h \rangle_{D(A)' \cdot D(A)} - \langle (\nabla \lambda)^* \cdot \Delta \alpha \hat{u} + \alpha \lambda \cdot \nabla \Delta \hat{u}, h \rangle_{D(A)' \cdot D(A)},
\]
which implies the following equality in \(L^2(D(A))'\):
\[
B_\alpha^*(\hat{u}, \hat{u}) = -\hat{u} \cdot \nabla \lambda + \alpha \Delta (\hat{u} \cdot \nabla \lambda) - (\nabla \lambda)^* \cdot \Delta \alpha \hat{u} + \alpha \lambda \cdot \nabla \Delta \hat{u}. \tag{4.44}
\]
Therefore, from \((4.40)\) and \((4.44)\), we have
\[
\begin{cases}
\Delta \alpha \lambda_t - \nu \Delta \alpha A \lambda + \hat{u} \cdot \nabla \lambda + \alpha \Delta (\hat{u} \cdot \nabla \lambda + \lambda \cdot \nabla \hat{u}) + (\nabla \lambda)^* \cdot \Delta \alpha \hat{u} \\
- \alpha \lambda \cdot \nabla \Delta \hat{u} = \gamma_1 \alpha A (\hat{u} - u_d), \quad \text{in} \quad L^2(D(A)'), \\
\nabla \cdot \lambda = 0 \quad \text{in} \quad Q, \\
\lambda = 0 \quad \text{on} \quad \Gamma \times (0, T), \\
\Delta \alpha \lambda(T) = \gamma_2 (\hat{u}(T) - u_T) \quad \text{in} \quad \Omega.
\end{cases} \tag{4.45}
\]

Summarizing the state equation \((3.4)\), the adjoint equation \((4.45)\) and the optimality condition \((4.27)\) we get the optimality system, in the strong form, as desired.

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(E.J. Villamizar-Roa) (Corresponding Author) Universidad Industrial de Santander, Escuela de Matemáticas, A.A. 678, Bucaramanga, Colombia.

E-mail address: jvillami@uis.edu.co

(E. Ortega-Torres) Departamento de Matemáticas, Universidad Católica del Norte, Casilla 1280, Antofagasta-Chile.

E-mail address: eortega@ucn.cl