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ABSTRACT
In this paper, we propose a novel primal–dual inexact gradient projection method for nonlinear optimization problems with convex-set constraint. This method only needs inexact computation of the projections onto the convex set for each iteration, consequently reducing the computational cost for projections per iteration. This feature is attractive especially for solving problems where the projections are computationally not easy to calculate. Global convergence guarantee and $O(1/k)$ ergodic convergence rate of the optimality residual are provided under loose assumptions. We apply our proposed strategy to $\ell_1$-ball constrained problems. Numerical results exhibit that our inexact gradient projection methods for solving $\ell_1$-ball constrained problems are more efficient than the exact methods.

1. Introduction

The primary focus of this paper is on designing, analysing and testing numerical methods for nonlinear optimization problems with convex-set constraint, which have wide applications in diverse disciplines, such as machine learning, statistics, signal processing, and control [1–6]. For example, maximum likelihood estimation with sparse constraints [7,8] often yields norm ball constrained problems. Another example is principal component analysis [9–11] in statistical analysis and engineering to find simpler or low-dimensional representations for data, and a common approach in this case is to require each of the generated coordinate to be a weighted combination of only a small subset of the original dimension. This often leads to constraints of different kinds of norm balls.

Gradient Projection Method (GPM) [12–14] is one of the most common approaches for solving convex-set constrained problems. At each iteration, it...
moves along the direction of the negative gradient, and then projects iterates onto the constraint set if it is outside of the set. GPM only needs to use the first-order derivatives, so it is often considered as a scalable solver for large-scale optimization problems. The convergence of GPM is analysed in [15,16], and a linear convergence rate is shown in [17] under the assumption of smoothness and strong convexity of the objective. Many variants have appeared to improve the original GPM, and the Spectral Projected Gradient (SPG) method proposed in [18,19] is one of the most popular variants where the spectral stepsize [20] and non-monotone line search [21] are incorporated for purposes of acceleration and reducing the function evaluations spent on line search. It is deemed that GPM is an effective solver for large-scale settings as long as the projection operations at each iteration can be carried out efficiently. However, this may not be the case in many situations as the constraint set could be so complex that the projection can not be easily computed. Therefore, this limits the applicability of GPM and also other projection-based algorithms including Nesterov’s optimal first-order method [22,23] and projected Quasi-Newton [24], which are efficient for problems involving simple constraint sets such as affine subspace and norm balls. One way to circumvent this obstacle is to avoid calculating the projections by using Frank-Wolfe method (conditional gradient method) [25–27], but the convergence is shown to be generally slower than GPM [28].

To reduce the computational effort spent on the projections, inexact gradient projection methods [29,30] have been proposed to accept an approximate projection for each iteration. In this way, the subproblem solver can early terminate once a sufficiently accurate solution to the projection subproblem is reached. In Inexact Spectral Projected Gradient (ISPG) method proposed by Birgin and Necoara [29], the subproblem solver is terminated if the subproblem objective reduction induced by the (feasible) inexact solution is at least a fraction of that induced by the exact projection. Another inexact GPM was recently proposed by Patrascu and Necoara, which requires the subproblem solution to approach the constraint set over the iteration by imposing a sequence of monotonically decreasing feasibility tolerance for the subproblems. It is shown to converge sublinearly by assuming Lipschitz differentiability and convexity of the objective, and to converge linearly by further assuming strong convexity.

The key technique in such methods is the termination criterion for the subproblem solver which has significant effect on the overall performance of GPM. If the termination criterion is set to be very strict, meaning the subproblem needs to be solved relatively accurately, then the computational burden of projections may not be alleviated. On the other hand, terminating the subproblem too early might slow down the overall convergence. Such termination criterion should be practical to implement, and can be easily manipulated to control the progress (namely, the ‘inexactness’) achieved by the subproblem solver. The inexact GPM in [29] and [30] represents two typical kinds of contemporary inexact termination criteria. The subproblem termination criterion in [29] requires prior knowledge
or at least an estimate of the optimal objective. This can be used as a reliable standard to show ‘how far’ the current iterate is from being optimal. However, it is impractical in most cases since the optimal objective is generally unknown before finishing the solve. Therefore, this method relies on Dykstra’s projection algorithm for solving the subproblems as the inexact termination criterion will be eventually satisfied during the algorithm with some (uncontrollable) tolerance. The subproblem termination criterion in [30] represents a stark contrast to [29]. It depends on driving the tolerance for subproblem optimality residual gradually to zero. Therefore, the performance of this variant of GPM highly depends on how fast the tolerance converges to zero. And the appropriate tolerance value may vary much for different problems due to different problem sizes or constraint set structures.

In this paper, we propose a novel primal–dual Inexact Gradient Projection Method (IGPM), which allows inexact solution of the projection subproblem. Unlike other inexact methods, we use the duality gap to evaluate the ‘inexactness’ of a subproblem solution. For each projection subproblem solve, we require the gap between the initial objective and current primal value is at least a fraction of that between the initial objective and the dual objective. This condition is further relaxed to be satisfied only within some prescribed tolerance monotonically driven to zero. Compared with existing methods, this inexact termination criterion does not involve the optimal objective. Moreover, the two gaps used in this criterion coincide at optimal primal–dual solution by strong duality, and that the latter gap is an upper bound for the gap between the initial and optimal primal values. Therefore, this termination criterion clearly reflects how close the inexact solution is to the optimal solution compared with the starting point. This intuition can help the user to easily manipulate the inexactness of the subproblem solve. We analyse global convergence and worst-case complexity of the proposed inexact method for general nonlinear Lipschitz-differentiable objectives (not necessarily convex). To demonstrate the effectiveness of our proposed strategy, we apply our inexact method to the $\ell_1$ ball constrained optimization problems. We decompose the $\ell_1$ ball projection into a finite number of projections onto hyperplanes, and then use the inexact termination criterion to reduce this computational effort to only a few hyperplane projections.

We summarize the novelties of our work below.

- Our proposed IGPM is especially designed for the cases that the projection can not be computed efficiently. By reducing the computational effort for each iteration of the IGPM, the overall computational cost can be alleviated. The key technique of this strategy is a novel termination criterion for the projection subproblem using the duality gap. This criterion need no use of the optimal objective, and clearly reflects how close the current objective is to the optimal objective. We emphasize that we do not have any requirement for the selection of the projection solver. In fact, the proposed strategy can be incorporated
into any projection solver that generates a sequence of convergent primal–dual iterates.

- We derive global convergence and worst-case complexity analysis under loose assumptions without requiring the convexity of the objective. We show that the optimality residual has a $O(1/k)$ ergodic convergence rate.

1.1. Organization

In the remainder of this section, we outline our notation and introduce various concepts that will be employed throughout the paper. We describe the inexact termination criterion and our proposed inexact gradient projection method in Section 2. The analysis of global convergence and worst-case complexity are provided in Section 3. We apply the proposed strategy to solve the $\ell_1$ ball constrained problem in Section 4. The results of numerical experiments are presented in Section 5. Concluding remarks are provided in Section 6.

1.2. Notation

Let $\mathbb{R}^n$ be the space of real $n$-vectors and $\mathbb{R}_+$ be the non-negative orthant of $\mathbb{R}^n$, i.e., $\mathbb{R}_+^n := \{ x \in \mathbb{R}^n : x \geq 0 \}$. Define the positive natural numbers set as $\mathbb{N} = \{ 1, 2, 3, \ldots \}$.

The $\ell_1$ norm is indicated as $\| \cdot \|_1$ with the $n$-dimensional $\ell_1$ ball with radius $\gamma$ denoted as $\mathbb{B}_n(\gamma) := \{ x \in \mathbb{R}^n : \| x \|_1 \leq \gamma \}$. Given a set $\Omega \subset \mathbb{R}^n$, we define the convex indicator for $\Omega$ by

$$
\delta_\Omega(x) = \begin{cases} 
0, & \text{if } x \in \Omega, \\
+\infty, & \text{if } x \notin \Omega,
\end{cases}
$$

and its support function by

$$
\delta^*_\Omega(u) = \sup_{x \in \Omega} \langle x, u \rangle.
$$

If $f$ is convex, then the subdifferential of $f$ at $\tilde{x}$ is given by

$$
\partial f(\tilde{x}) := \{ z \mid f(\tilde{x}) + \langle z, x - \tilde{x} \rangle \leq f(x), \forall x \in \mathbb{R}^n \}.
$$

The subdifferential of the indicator function of $\Omega$ at $x$ is known as the normal cone,

$$
\partial \delta_\Omega(x) = N(x|\Omega) = \{ v \mid \langle v, y - x \rangle, \forall x \in \Omega \}.
$$

The Euclidean projection of $w$ onto $\Omega$ is given by

$$
\mathcal{P}_\Omega(w) = \arg\min_{x \in \Omega} \| x - w \|_2^2.
$$

The proximal operator associated with a convex function $h(x)$ is defined as

$$
prox_h(w) := \arg\min_x h(x) + \frac{1}{2} \| x - w \|_2^2.
$$
It is easily seen that $P_{\Omega}(w) = \text{prox}_{\delta_{\Omega}}(w)$. Let $\text{sign}(\cdot)$ represent the signum function of a real number $t$, i.e.

$$\text{sign}(t) := \begin{cases} 
1, & \text{if } t > 0, \\
0, & \text{if } t = 0, \\
-1, & \text{if } t < 0.
\end{cases}$$

For two vectors $a$ and $b$ of the same dimension, the Hadamard product $a \circ b$ is defined as $(a \circ b)_i = a_i b_i$. The absolute value of vector $a \in \mathbb{R}^n$ is defined as $|a| = (|a_1|, \ldots, |a_n|)^T$, and the sign of $a$ is denoted as $\text{sign}(a) = (\text{sign}(a_1), \ldots, \text{sign}(a_n))^T$. We use $e$ to represent the vector filled with all 1s of appropriate dimension, and $0_n$ to denote the $n$-dimensional vector of all zeros.

2. Algorithm description

In this section, we formulate our problem of interest and outline our proposed inexact methods. We present our algorithm in the context of the generic nonlinear optimization problem with convex set constraint

$$\min_{x \in \mathbb{R}^n} f(x) \quad \text{(NLO)}$$

s.t. $x \in \Omega,$

where $\Omega \subset \mathbb{R}^n$ is a closed convex set and $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is differentiable over $\Omega$.

At each iteration, gradient projection method starts from a feasible point, and moves along the negative gradient direction with stepsize $\beta_k > 0$. If the resulted point is not in the convex set, it is projected onto $\Omega$. The $k$th iteration can be expressed as

$$x_{k+1} = P_{\Omega}(x_k - \beta_k \nabla f(x_k)). \quad (1)$$

Many options for selecting $\beta_k$ in (1) have appeared. Goldstein [31], and Levitin and Polyak [32] show that for $L$-Lipschitz differentiable $f$, global convergence can be guaranteed with $\beta \in (0, 2/L)$. McCormick and Tapia [33] suggest exact line search to enforce sufficient decrease in the objective. To avoid the minimization of a piecewise continuously differentiable function, Bertsekas [28] suggests a backtracking line search for finding an appropriate $\beta_k$. Another well-known approach is the spectral projected gradient (SPG) method [18,19], where the stepsize $\beta_k$ is selected based on the Barzilai–Borwein (BB) formula [20]

$$\beta_k^{BB1} = \frac{s_{k-1}^T s_{k-1}}{s_{k-1}^T y_{k-1}} \quad \text{or} \quad \beta_k^{BB2} = \frac{s_{k-1}^T y_{k-1}}{y_{k-1}^T y_{k-1}} \quad (2)$$

with $s_{k-1} = x_k - x_{k-1}$ and $y_{k-1} = \nabla f(x_k) - \nabla f(x_{k-1})$. After obtaining the projection

$$z_k = P_{\Omega}(x_k - \beta_k \nabla f(x_k)), \quad (3)$$
the SPG method further requires sufficient decrease in the objective by (possibly non-monotone) line search between $x_k$ and $z_k$ for stepsize $\alpha_k$

$$x_{k+1} = x_k + \alpha_k(z_k - x_k).$$

In this paper, our inexact strategy is presented in the framework of SPG methods. For the sake of simplicity, we use fixed $\beta_k \equiv \beta$ for the description and analysis. However, we are well aware that our proposed strategy can be easily generalized to other variants where $\beta_k$ is determined by either line search or the truncated BB stepsizes.

We start the description of our inexact method by analysing the primal and dual of projection subproblems. At the $k$th iteration, we use the following shorthand for simplicity

$$g_k := \nabla f(x_k) \quad \text{and} \quad v_k := x_k - \beta \nabla f(x_k).$$

At $x_k$, the problem of projecting $v_k$ onto $\Omega$ can be formulated as

$$\min_z p(z; x_k) := \frac{1}{2} \|z - v_k\|^2 + \delta_\Omega(z), \quad (4)$$

with the optimal solution being denoted as $z_k^* := \arg\min_z p(z; x_k)$. For a feasible $z$, let $d = z - x_k$ and define the objective reduction induced by $z$ as

$$\Delta p(z; x_k) := p(x_k; x_k) - p(z; x_k). \quad (5)$$

Notice that a successful direction should give sufficient decrease in the objective, and in particular, $\Delta p(x_k; x_k) = 0$. The associated Fenchel–Rockafellar dual [34] of (4) is given by

$$\max_u q(u; x_k) := -\frac{1}{2} \|u - v_k\|^2 - \delta_\Omega^*(u) + \frac{1}{2} \|v_k\|^2. \quad (6)$$

By weak duality, we have $p(z; x_k) - q(u; x_k) \geq 0$ and the equality holds at primal–dual optimal solution $(z^*, u^*)$ by strong duality.

Given an iterative subproblem solver for computing $z_k^*$, we use superscript $(j)$ to denote the $j$th iteration of the $k$th subproblem solve. Assume this solver is able to generate a sequence of primal–dual pairs $\{(z_k^{(j)}, u_k^{(j)})\}$ with $\{z_k^{(j)}\} \in \Omega$. We assume the primal iterates generated by the solver are ‘no worse than’ the trivial iterate $x_k$ in the sense

$$p(x_k; x_k) \geq p(z_k^{(j)}; x_k). \quad (7)$$

In other words, it always holds true that $\Delta p(z_k; x_k) \geq 0$. This is a reasonable requirement, since any iterate violating this condition can be replaced with $x_k$. It is fixed during the solve of the $k$th subproblem, but will be sequentially cut down to zero during the entire IGPM framework (which will be discussed later). With
these ingredients, we introduce the following important ratio corresponding to the $j$th iterate of the subproblem solver:

$$
\gamma_k^{(j)} := \frac{p(x_k; x_k) - p(z_k^{(j)}; x_k) + \omega_k}{p(x_k; x_k) - q(u_k^{(j)}; x_k) + \omega_k}.
$$  \hspace{1cm} (8)

We define this ratio to evaluate the progress made by the $j$th iterate of the subproblem solver, since it can reflect how close the current objective is from the optimal value. We make the following observations

- First, the denominator is always greater than or equal to the numerator since $p(z_k^{(j)}; x_k) \geq q(u_k^{(j)}; x_k)$ by weak duality. On the other hand, the numerator is guaranteed to be non-negative due to (7). Therefore, this ratio always satisfies $\gamma_k^{(j)} \in [0, 1]$.

- For optimal primal–dual $(z^*_k, u^*_k)$, we have

$$
\frac{p(x_k; x_k) - p(z^*_k; x_k) + \omega_k}{p(x_k; x_k) - q(u^*_k; x_k) + \omega_k} = 1
$$

by strong duality $p(z^*_k; x_k) = q(u^*_k; x_k)$, meaning an exact projection is found. Therefore, as the primal–dual iterates $(z_k^{(j)}, u_k^{(j)})$ approaches $(z^*_k, u^*_k)$, we have $\gamma_k^{(j)}$ converges to 1.

- The inexact method in [29] uses the objective reduction induced by the current iterate $z_k^{(j)}$ over that caused by the minimizer (the exact projection) to evaluate the progress, i.e.

$$
\frac{\Delta p(z_k^{(j)})}{\Delta p(z^*_k)}.
$$  \hspace{1cm} (9)

To see the relationship between ratio (9) and $\gamma_k^{(j)}$, note that weak duality implies $p(z^*_k; x_k) \geq q(u_k^{(j)}; x_k)$ yielding

$$
\Delta p(z^*_k) \leq p(x_k; x_k) - q(u_k^{(j)}; x_k).
$$

Therefore, we know that the proposed ratio $\gamma_k^{(j)}$ satisfies

$$
\frac{\Delta p(z_k^{(j)}; x_k) + \omega_k}{\Delta p(z^*_k; x_k) + \omega_k} \geq \gamma_k^{(j)}.
$$  \hspace{1cm} (10)

If we set $\omega_k = 0$, the ratio on the left-hand side of (10) coincides with (9).

- The exactness of the subproblem solution is controlled by the threshold $\gamma$. The primary purpose of adding the relaxation parameter $\omega$ is mainly to prevent numerical issues when the denominator of $\gamma_k^{(j)}$ becomes tiny around the optimal solution, and this parameter can be removed by setting $\omega = 0$ in our
theoretical analysis and implementation. We introduce this parameter since it can allow the acceptance of a ‘more inexact’ subproblem solution and driving this relaxation $\omega_k$ to 0.

Observe that a large value of $\gamma_k^{(j)}$ close to 1 implies a relatively accurate solution, whereas a small value of $\gamma_k^{(j)}$ indicates the iterate is far from being optimal. Therefore, we can set a threshold $\gamma \in (0,1]$ for this ratio to let the subproblem be solved inexact once $\gamma_k^{(j)}$ exceeds the prescribed threshold. Specifically, given $(z_k^{(j)}, u_k^{(j)})$, we terminate the projection algorithm if

$$
\gamma_k^{(j)} = \frac{p(x_k; x_k) - p(z_k^{(j)}; x_k) + \omega_k}{p(x_k; x_k) - q(u_k^{(j)}; x_k) + \omega_k} \geq \gamma. \quad (11)
$$

- By weak duality, if (11) is satisfied, then

$$
\frac{p(x_k; x_k) - p(z_k^{(j)}; x_k) + \omega_k}{p(x_k; x_k) - p(z_k^*; x_k) + \omega_k} \geq \gamma. \quad (12)
$$

- It should be noticed that the relaxation parameter $\omega_k \geq 0$ brings more flexibility to our algorithm, making an inaccurate solution more easily accepted. In fact, it can be set $\omega_k = 0$, because $p(x_k; x_k) = q(u_k^{(j)}; x_k)$ implies $x_k$ is optimal for the $k$-th subproblem. In this case, it can be shown later that $x_k$ is also optimal for problem (NLO).
- It should be clear that we do not have any requirement for the subproblem solver, as long as it can generate a sequence of primal–dual iterates converging to the optimal solution so that (11) will be eventually satisfied.

We are now ready to introduce our inexact gradient projection method. Algorithm 1 presents a version using constant stepsize $\beta > 0$, which is often required to be smaller than $1/L$ as elaborated in later sessions.

**Algorithm 1** IGPM without line search

1. Given $\varepsilon > 0$, choose $0 < \gamma < 1$ and $0 < \beta \leq 1/L$.
2. Initialize $x_0$, $\omega_0$ and set $k = 0$.
3. repeat
4. Solve (4) approximately for $z_k \in \Omega$ satisfying (11).
5. Update $x_{k+1} \leftarrow z_k$ and $\omega_{k+1}$.
6. Set $k \leftarrow k + 1$.
7. until $\|z_k - x_k\| \leq \varepsilon$. 
If the Lipschitz constant $L$ of $f$ is impractical to estimate, we impose a backtracking line search method between the current point and the projected point to enforce a sufficient decrease in the objective. In this case, the stepsize $\beta$ actually can be allowed to vary in a prescribed interval [35], such as the spectral stepsize where $\beta$ is computed using the BB formulation (2). The iteration of the IGPM with backtracking line search is outlined in Algorithm 2.

**Algorithm 2 IGPM with line search**

1: Given $\varepsilon > 0$, choose $0 < \eta < 1$, $0 < \gamma < 1$, $0 < \alpha \leq 1$, $\beta > 0$ and $0 < \theta < 1$.
2: Initialize $x_0$, $\omega_0$, and set $k = 0$.
3: repeat
4: Solve (4) approximately for $z_k \in \Omega_1$ satisfying (11) and let $d_k = z_k - x_k$.
5: Let $\alpha_k$ be the largest value in $\{\theta^0 \alpha, \theta^1 \alpha, \theta^2 \alpha, \cdots \}$ such that

$$f(x_k + \alpha_k d_k) \leq f(x_k) + \eta \alpha_k g_k^T d_k.$$  

(13)

6: Update $x_{k+1} \leftarrow x_k + \alpha_k d_k$ and $\omega_{k+1}$.
7: Set $k \leftarrow k + 1$.
8: until $\|z_k - x_k\| \leq \varepsilon$.

3. Convergence analysis

In this section, we provide global convergence and worst-complexity analysis of our proposed methods. Throughout our analysis, we make the following assumptions about (NLO).

**Assumption 3.1:**

(i) $f$ is bounded below on $\Omega$ by $f$, i.e. $f(x) \geq f$ for any $x \in \Omega$.

(ii) $f$ is $L$-Lipschitz differentiable, meaning there exists a constant $L > 0$ such that

$$\|\nabla f(x) - \nabla f(y)\|_2 \leq L\|x - y\|_2,$$

for any $x, y \in \Omega$.

(iii) The relaxation sequence $\{\omega_k\}$ is summable, i.e. $\hat{\omega} := \sum_{k=0}^{+\infty} \omega_k < +\infty$.

3.1. Global convergence

In this subsection, we analyse the global convergence of our proposed two versions of inexact methods. We first provide the following straightforward results about the subproblem (4) in the following lemma, which can be easily found in many places such as [36].
**Lemma 3.2:** For any \( x_k \in \Omega \), let \( z_k^* \) be the optimal solution of the subproblem (4). If \( \Delta p(z; x_k) \geq 0 \) for \( z \in \Omega \), then \( d = z - x_k \) is a descent direction for \( f \) at \( x_k \).

Furthermore, \( x_k \) is a first-order optimal solution to (NLO) if and only if \( z_k^* = x_k \).

It is obviously true that \( z_k^* = x_k \) is equivalent to
\[
\Delta p(z_k^*; x_k) = 0, \tag{14}
\]
by the strictly convexity of the projection subproblem. Therefore, if a point is optimal to the (NLO), it must satisfy condition (14). The first result of our analysis is to show that every limit point of the iterates generated by our algorithms must satisfy this condition.

Also notice that Lemma 3.2 implies that \( x_k \) satisfies the first-order optimality condition of (NLO) if and only if
\[
\|z_k^* - x_k\|^2 = \|x_k - P_{\Omega}(x_k - \beta g_k)\|^2 = 0. \tag{15}
\]

Therefore, we can define \( E(x_k; \beta) := \|x_k - P_{\Omega}(x_k - \beta g_k)\|^2 \) as the optimality residual, as is used by Burke [36]. The second result of our analysis shows that this optimality residual converges to zero for our proposed methods.

Before proceed to our main result, we show that the line search stepsize \( \{\alpha_k\} \) for Algorithm 2 is bounded away from 0 in the following lemma.

**Lemma 3.3:** Suppose \( \{x_k\} \) is generated by Algorithm 2. It holds true that
\[
\alpha_k \geq \frac{\theta(1 - \eta)}{\beta L}.
\]

**Proof:** First of all, by (7), we require the subproblem solution always satisfies \( \Delta p(z_k; x_k) \geq 0 \), or equivalently, \( -\frac{1}{2}\|d_k\|^2 - \beta g_k^T d_k \geq 0 \). Hence, we have
\[
-g_k^T d_k / \|d_k\|^2 \geq \frac{1}{2\beta}. \tag{16}
\]

To find a lower bound for \( \alpha_k \), it follows from Assumption 3.1(ii) that
\[
f(x_k + \alpha d_k) \leq f(x_k) + \alpha g_k^T d_k + \frac{L}{2} \alpha^2 \|d_k\|^2,
\]
Therefore, for any \( \alpha \in (0, -((2(1 - \eta)g_k^T d_k)/(L\|d_k\|^2))] \), the Armijo condition (13) is satisfied. Hence, the backtracking procedure must end up with
\[
\alpha_k \geq -\frac{2\theta(1 - \eta)g_k^T d_k}{L\|d_k\|^2},
\]
yielding \( \alpha_k \geq ((\theta(1 - \eta))/((\beta L))) \) by Nesterov (16).

Next we show that for both algorithms, the subproblem objective reductions induced by either the inexact projection \( z_k \) or the exact projection \( z^* \) vanish.
Lemma 3.4: Suppose \( \{x_k\} \) is generated by Algorithm 1 or 2. It holds true that
\[
\lim_{k \to \infty} \Delta p(z_k; x_k) = 0 \quad \text{and} \quad \lim_{k \to \infty} \Delta p(z^*_k; x_k) = 0.
\]

Proof: We first prove this for Algorithm 1 and suppose \( \{x_k\} \) is generated by Algorithm 2. Notice that in this case we set \( x_{k+1} = z_k \). It follows from Assumption 3.1(ii) and \( \beta \leq 1/L \) that
\[
f(x_{k+1}) \leq f(x_k) + g_k^T (x_{k+1} - x_k) + \frac{L}{2} \|x_{k+1} - x_k\|^2_2
\]
\[
\leq f(x_k) + g_k^T (x_{k+1} - x_k) + \frac{1}{2\beta} \|x_{k+1} - x_k\|^2_2 \tag{17}
\]
\[
= f(x_k) - \frac{1}{\beta} \Delta p(x_{k+1}; x_k),
\]
then we have
\[
\Delta p(x_{k+1}; x_k) \leq \beta(f(x_k) - f(x_{k+1})).
\]
Summing up both sides of this inequation from 0 to \( t \) gives
\[
\sum_{k=0}^{t} \Delta p(x_{k+1}; x_k) \leq \beta \sum_{k=0}^{t} (f(x_k) - f(x_{k+1})) = \beta(f(x_0) - f(x_{t+1})).
\]
Letting \( t \to \infty \) and from Assumption 3.1(i), we have \( \lim_{k \to \infty} \Delta p(x_{k+1}; x_k) = 0 \). This proves the first limit.

To prove the second limit, we have from (12) that
\[
\Delta p(x_{k+1}; x_k) + \omega_k \geq \gamma(\Delta p(z^*_k; x_k) + \omega_k). \tag{18}
\]
Therefore, \( \Delta p(z^*_k; x_k) \leq \gamma(\Delta p(z_k; x_k) + (1-\gamma)\omega_k) \), which, combined with the first limit and \( \lim_{k \to \infty} \omega_k = 0 \) by Assumption 3.1(iii), yields \( \lim_{k \to \infty} \Delta p(z^*_k; x_k) = 0 \). This completes the proof for Algorithm 1.

Now suppose \( \{x_k\} \) is generated by Algorithm 2. We start with the Armijo condition (13) and sum up its both sides from 0 to \( t \) to obtain
\[
-\sum_{k=0}^{t} \eta \alpha_k g_k^T d_k \leq \sum_{k=0}^{t} (f(x_k) - f(x_k + \alpha_k d_k)) = f(x_0) - f(x_{t+1}).
\]
Letting \( t \to \infty \), we have \( -\sum_{k=0}^{\infty} \eta \alpha_k g_k^T d_k \leq f(x_0) - f \) by Assumption 3.1(i). It follows from Lemma 3.3 and \( g_k^T d_k \leq 0 \) that \( g_k^T d_k \to 0 \). The fact
\[
\Delta p(z_k; x_k) = -\beta g_k^T d_k - \frac{1}{2} \|d_k\|^2_2 \leq -\beta g_k^T d_k, \tag{19}
\]
indicates \( \Delta p(z_k; x_k) \to 0 \). It then follows from (18) that \( \Delta p(z^*_k; x_k) \to 0 \) is also true. \( \blacksquare \)
We are now ready to provide the first global convergence result for both Algorithms.

**Theorem 3.5:** Suppose \( \{x_k\} \) is generated by Algorithm 1 or 2. It holds true that every limit point \( x^* \) of \( \{x_k\} \) must be a first-order optimal solution for \( (NLO) \).

**Proof:** Based on (14), it suffices to show that the subproblem at \( x^* \) has a stationary point \( z^* = x^* \), or equivalently, \( \Delta p(z^*; x^*) = 0 \). We prove this by contradiction by assuming that there exists a limit point \( x^* \) of \( \{x_k\} \) such that

\[
\Delta p(z^*; x^*) = p(x^*; x^*) - p(z^*; x^*) \geq \epsilon > 0. \tag{20}
\]

Notice by Lemma 3.4, there exists \( k_0 \in \mathbb{N} \) such that

\[
\Delta p(z^*_k; x_k) = p(x_k; x_k) - p(z^*_k; x_k) < \epsilon / 2, \tag{21}
\]

for all \( k > k_0 \).

To derive a conclusion contradicting (21), first of all, notice that

\[
|p(x_k; x_k) - p(x^*; x^*)| = \left| \frac{1}{2} \beta^2 \|g_k\|^2 - \frac{1}{2} \beta^2 \|g_*\|^2 \right|
\]

\[
= \frac{1}{2} \beta^2 \|g_k\|^2 - \|g_*\|^2
\]

\[
= \frac{1}{2} \beta^2 \|g_k - g_*\|^2 (g_k + g_*)\n\]

\[
\leq \frac{1}{2} \beta^2 \|g_k - g_*\|^2 (\|g_k\| + \|g_*\|)
\]

\[
\leq \frac{1}{2} \beta^2 \|g_k - g_*\|^2 (\|g_k\|^2 + \|g_*\|^2). \tag{22}
\]

By Assumption 3.1(ii), we know there exists \( \delta > 0 \) such that \( \|g_k\| \leq \delta \). It then follows from (22) that

\[
|p(x_k; x_k) - p(x^*; x^*)| \leq \frac{1}{2} \beta^2 \|g_k - g_*\|^2 (\delta + \|g_*\|^2). \tag{23}
\]

Now consider a subsequence \( \mathcal{K} \subseteq \mathbb{N} \) such that \( \{x_k\}_{k \in \mathcal{K}} \to x^* \). By the continuity of \( g \), there exists \( k_1 \in \mathbb{N} \) such that \( \|g_k - g_*\|^2 \leq (\epsilon / (2\beta^2(\delta + \|g_*\|^2))) \) for any \( k > k_1, k \in \mathcal{K} \). Combining (23), we have

\[
|p(x_k; x_k) - p(x^*; x^*)| \leq \frac{1}{2} \beta^2 \frac{\epsilon}{\delta + \|g_*\|^2} (\delta + \|g_*\|^2) = \epsilon / 4. \tag{24}
\]

On the other hand, notice that \( p(z^*; x) \) is continuous with respect to \( x \). Therefore, there exists \( k_2 \in \mathbb{N} \) such that

\[
|p(z^*; x^*) - p(z^*; x_k)| < \epsilon / 4 \tag{25}
\]

for any \( k > k_2, k \in \mathcal{K} \).
Now let $k_3 = \max\{k_0, k_1, k_2\}$. It following from (20), (24) with (25) that
\[
p(x_k; x_k) - p(z^*; x_k)
= (p(x_k; x_k) - p(x^*; x_k)) + (p(x^*; x^*) - p(z^*; x^*)) + (p(z^*; x^*) - p(z^*; x_k))
> -\varepsilon/4 + \varepsilon - \varepsilon/4 = \varepsilon/2
\]
for any $k > k_3$, $k \in K$. This is equivalent to
\[
p(z^*; x_k) < p(x_k; x_k) - \varepsilon/2
\]
for any $k > k_3 \geq k_0$, $k \in K$, contradicting (21) since $z_k^*$ is the unique optimal solution of the $k$th subproblem. □

We next investigate the convergence results of the optimality residual $E(x_k; \beta)$. The key technique of this proof relies on showing $E(x_k; \beta)$ is bounded above by the subproblem objective reduction caused by the exact projection, which has been shown to converge to 0. Before proceeding, we have the following lemma about the relationship between primal optimal solution $z_k^*$ and dual optimal solution $u_k^*$.

**Lemma 3.6:** Let $z_k^*$ be the optimal solution of subproblem (4) and $u_k^*$ be the optimal solution of the dual subproblem (6). It then holds true that
\[
v_k = z_k^* + u_k^* \quad \text{and} \quad u_k^* \in N(z_k^*|\Omega).
\]

**Proof:** By the definition of $z_k^*$ and $u_k^*$, we can rewrite
\[
z_k^* = \arg\min_z \delta_{\Omega}(z) + \frac{1}{2}\|z - v_k\|_2^2,
\]
\[
u_k^* = \arg\min_u \delta_{\Omega}^*(u) + \frac{1}{2}\|u - v_k\|_2^2,
\]
or, equivalently
\[
z_k^* = \text{prox}_{\delta_{\Omega}}(v_k) \quad \text{and} \quad u_k^* = \text{prox}_{\delta_{\Omega}^*}(v_k).
\]
Using the Moreau decomposition [37], we have
\[
v_k = \text{prox}_{\delta_{\Omega}}(v_k) + \text{prox}_{\delta_{\Omega}^*}(v_k) = z_k^* + u_k^*.
\]
The optimality condition of the primal subproblem (4) gives
\[
0 \in z_k^* - v_k + N(z_k^*|\Omega).
\]
Combining this with (27), we have $u_k^* \in N(z_k^*|\Omega)$. □
We are now ready to analyse the convergence of the optimality residual $E(x_k; \beta)$.

**Theorem 3.7:** Suppose $\{x_k\}$ is generated by Algorithm 1 or 2. It holds that

$$\lim_{k \to \infty} E(x_k; \beta) = 0. \quad (28)$$

**Proof:** Consider the dual optimal solution $u^*_k$. We have from Lemma 3.6 that $u^*_k \in N(z_k^*|\Omega)$, meaning

$$(x - z_k^*)^T u^*_k \leq 0, \quad \forall x \in \Omega. \quad (29)$$

It follows that

\begin{align*}
E(x_k; \beta) &= \|x_k - z_k^*\|^2 \\
&\leq (\|x_k - z_k^*\|^2 - 2(x_k - z_k^*)^T u_k^*) \\
&= (\|x_k - z_k^* - u_k^*\|^2 - \|u_k^*\|^2) \\
&= (\|x_k - v_k\|^2 - \|v_k - z_k^*\|^2) \\
&= 2[p(x_k; x_k) - p(z_k^*; x_k)] \\
&= 2\Delta p(z_k^*; x_k),
\end{align*}

where the inequality follows from (29), the third equality is by Lemma 3.6 and the fourth equality is from the definition of $p(\cdot; x_k)$. Combing this with Lemma 3.4, we have $\lim_{k \to \infty} E(x_k; \beta) = 0$, completing the proof. \hfill \blacksquare

**3.2. Worst-case complexity analysis**

In this subsection, we provide the worst-case complexity analysis of our proposed IGPM. It should be noticed that we do not necessarily require the objective $f$ to be convex. Therefore, we are interested in showing the convergence rate for the ergodic average of the optimality residual $E(\cdot; \beta)$, where the ergodic average of $E(x_t; \beta)$ can be denoted as $\frac{1}{k} \sum_{t=0}^{k-1} E(x_t; \beta)$.

We first show the complexity result for Algorithm 1 in the following theorem.

**Theorem 3.8:** Suppose $\{x_k\}$ is generated by Algorithm 1. Then the ergodic average $(1/k) \sum_{t=0}^{k-1} E(x_t; \beta)$ has convergence rate $O(1/k)$. In particular,

$$\frac{1}{k} \sum_{t=0}^{k-1} E(x_t; \beta) \leq \frac{1}{k} \left( \frac{2}{\gamma} (\beta(f(x_0) - \hat{f}) + (1 - \gamma)\hat{\omega}) \right). \quad (31)$$
Proof: From (17), we have for $\beta \leq 1/L$ that

$$f(x_{t+1}) - f(x_t) \leq -\frac{1}{\beta} \Delta p(x_{t+1}; x_t)$$

$$\leq -\frac{\gamma}{\beta} \Delta p(z^*_t; x_t) - \frac{(\gamma - 1)}{\beta} \omega_t$$

$$\leq -\frac{\gamma}{2\beta} E(x_t; \beta) - \frac{(\gamma - 1)}{\beta} \omega_t,$$

where the second inequality follows by (18) and last inequality follows by (30). Summing up both sides of the above inequality from 0 to $k-1$, we have

$$\sum_{t=0}^{k-1} (\gamma E(x_t; \beta) + 2(\gamma - 1)\omega_t) \leq 2\beta \sum_{t=0}^{k-1} (f(x_t) - f(x_{t+1}))$$

$$= 2\beta (f(x_0) - f(x_k))$$

$$\leq 2\beta (f(x_0) - f),$$

by Assumption 3.1(i). It follows that

$$\sum_{t=0}^{k-1} E(x_t; \beta) \leq \frac{2}{\gamma} \left( \beta (f(x_0) - f) + (1 - \gamma) \sum_{t=0}^{k-1} \omega_t \right)$$

$$\leq \frac{2}{\gamma} \left( \beta (f(x_0) - f) + (1 - \gamma) \hat{\omega} \right),$$

where the second inequality follows by Assumption 3.1(iii). Dividing both sides by $k$, we know (31) is true.  

The complexity analysis of IGPM with backtracking line search is shown in the following theorem.

**Theorem 3.9:** Suppose $\{x_k\}$ is generated by Algorithm 2. Then the ergodic average $(1/k) \sum_{t=0}^{k-1} E(x_t; \beta)$ has convergence rate $O(1/k)$. In particular,

$$\frac{1}{k} \sum_{t=0}^{k-1} E(x_t; \beta) \leq \frac{1}{k} \left( \frac{2(1 - \gamma)}{\gamma} \hat{\omega} + \frac{2\beta^2 L}{\eta(1 - \eta)\theta \gamma} (f(x_0) - f) \right). \quad (32)$$

**Proof:** It holds true that

$$E(x_t; \beta) + 2\omega_t \leq 2(\Delta p(z^*_t; x_t) + \omega_t) \leq \frac{2}{\gamma} (\Delta p(z_t; x_t) + \omega_t)$$

$$\leq \frac{2}{\gamma} (-\beta g_t^T d_t + \omega_t), \quad (33)$$

where the first inequality is from (30), the second inequality is from (12) and the last inequality is by (19). Therefore, we have

$$-g_t^T d_t \geq \frac{\gamma}{2\beta} E(x_t; \beta) + \frac{\gamma - 1}{\beta} \omega_t. \quad (34)$$
On the other side, by (13) and Lemma 3.3, we have
\[ f(x_t) - f(x_{t+1}) \geq -\eta \alpha_t g_t^T d_t \geq -\frac{\eta(1 - \eta)\theta g_t^T d_t}{\beta L}, \]
which, together with (34), leads to
\[ f(x_t) - f(x_{t+1}) \geq \frac{\eta(1 - \eta)\theta \gamma}{2\beta^2 L} E(x_t; \beta) + \frac{\eta(1 - \eta)\theta(\gamma - 1)}{\beta^2 L} \omega_t. \]
Summing up both sides from 0 to \( k - 1 \), we have
\[
\frac{\eta(1 - \eta)\theta \gamma}{2\beta^2 L} \sum_{t=0}^{k-1} E(x_t; \beta) + \frac{\eta(1 - \eta)\theta(\gamma - 1)}{\beta^2 L} \sum_{t=0}^{k-1} \omega_t \\
\leq f(x_0) - f(x_k) \leq f(x_0) - f
\]
by Assumption 3.1(i). Therefore,
\[
\sum_{t=0}^{k-1} E(x_t; \beta) \leq \frac{2(1 - \gamma)}{\gamma} \hat{\omega} + \frac{2\beta^2 L}{\eta(1 - \eta)\theta \gamma} (f(x_0) - f)
\]
Dividing both sides by \( k \), we obtain (32).

4. Inexact projections onto \( \ell_1 \) ball

In this section, we apply our inexact gradient projection methods for solving the \( \ell_1 \) ball constrained problem, where the set \( \Omega \) in (NLO) is an \( \ell_1 \) ball. Many recent problems in machine learning \([4,38]\), statistics \([7,39]\), signal processing \([40,41]\) and compressed sensing \([42,43]\) can be formulated into the minimization of a nonlinear objective on an \( \ell_1 \)-norm ball,
\[
\min_x f(x) \quad \text{s.t. } x \in \mathbb{B}^n(\tau),
\]
where \( \tau \) is the radius of the \( \ell_1 \) ball \( \mathbb{B}^n(\tau) \).

We design an active-set method for solving the \( \ell_1 \)-ball projection subproblems, which can be shown to terminate in finite number of iterations for finding the exact projection. For each iteration, this method only needs to compute the projection onto a hyperplane in a space with lower dimension. We then incorporate this method into our proposed inexact gradient projection methods, so that the computational cost for each \( \ell_1 \) ball projection subproblem reduces to only several projections onto the hyperplanes. For simplicity, we remove the subscript \( k \) from our description as we focus on the projection subproblem, so that \( p(\cdot) \) and \( q(\cdot) \) denote the primal and dual value of the projection problem.
The problem of projecting $\mathbf{v}$ onto the $\ell_1$-ball can be formulated as
\[
\min_{\mathbf{z}} \frac{1}{2} \| \mathbf{z} - \mathbf{v} \|_2^2 \quad \text{s.t.} \quad \mathbf{z} \in \mathbb{B}^n(\tau). \tag{36}
\]
Most existing $\ell_1$-ball projection algorithms take advantage of the symmetry of the $\ell_1$-ball, and transform this projection problem into the projection onto the simplex defined as
\[
\mathcal{S}(\tau, n) := \{ \mathbf{x} \in \mathbb{R}^n | \sum_{i=1}^n x_i = \tau \text{ and } x_i \geq 0, \forall i = 1, 2, \ldots, n \}.
\]
The following lemma, which is presented in [44,45], shows the relationship between the projection onto the $\ell_1$-ball and the simplex.

**Lemma 4.1:** Let $\mathbf{w} = \mathcal{P}_{\mathcal{S}(\tau, n)}(|\mathbf{v}|)$, then
\[
\mathcal{P}_{\mathbb{B}^n}(\mathbf{v}) = \begin{cases} 
\mathbf{v}, & \text{if } \mathbf{v} \in \mathbb{B}^n(\tau), \\
\text{sign}(\mathbf{v}) \circ \mathbf{w}, & \text{otherwise}.
\end{cases}
\]

This lemma allows us to only focus on computing the projection of a vector in the non-negative orthant $\mathbb{R}^n_+$. Therefore, we consider the projection of $\mathbf{v} \in \mathbb{R}^n_+$ onto $\mathcal{S}(\tau, n)$, which is formulated as
\[
\min_{\mathbf{w}} \frac{1}{2} \| \mathbf{w} - \mathbf{v} \|_2^2 \\
\text{s.t.} \quad \sum_{i=1}^n w_i = \tau, \tag{37}
\]
\[
\quad w_i \geq 0, \quad i = 1, \ldots, n.
\]
The following lemma [46] is often used to determine the projection onto the simplex.

**Lemma 4.2:** There exists a unique $\mathbf{v} \in \mathbb{R}$ such that the optimal solution of (37) can be given by
\[
(\mathcal{P}_{\mathcal{S}(\tau, n)}(\mathbf{v}))_i = \max(v_i - \mathbf{v}, 0), \quad i = 1, \ldots, n.
\]

Using Lemma 4.2, many existing algorithms focus on solving the piecewise linear equation
\[
\sum_{i=1}^n \max(v_i - \mathbf{v}, 0) = \tau \tag{38}
\]
for $\mathbf{v}$, and then compute the projection accordingly. We now show the simplex projection problem (37) can be further transformed into several projections onto
the hyperplanes of the form
\[ H(\tau, n) := \{ x \in \mathbb{R}^n \mid e^T x = \tau \}. \]

Note that the feasible set of (37) is the intersection of two convex sets, the hyperplane \( H(\tau, n) \) and the non-negative orthant \( \mathbb{R}^n_+ \). Now consider projecting \( v \) onto the hyperplane \( H(\tau, n) \). This will result in two cases.

Case (i) All components of
\[ P_{H(\tau, n)}(v) = v - \frac{e^T v - \tau}{n} e \]
are all non-negative, then we know \( P_{H(\tau, n)}(v) \in \mathbb{B}(\tau)^n \) and that
\[ P_{H(\tau, n)}(v)_i = \max(v_i - (e^T v - \tau)/n, 0). \]

By Lemma 4.2, we know
\[ P_{S(\tau, n)}(v) = P_{H(\tau, n)}(v). \]
An exact projection onto the simplex is found.

Case (ii) There exists at least one component \( P_{H(\tau, n)}(v)_j \) of \( P_{H(\tau, n)}(v) \) is negative, i.e. \( v_j - (e^T v - \tau)/n < 0 \). Since
\[ \sum_{i=1}^n P_{H(\tau, n)}(v) = \sum_{i=1}^n (v_i - (e^T v - \tau)/n) = \tau, \]
we know
\[ \sum_{i=1}^n \max(v_i - (e^T v - \tau)/n, 0) \geq \tau - (v_j - (e^T v - \tau)/n) > \tau. \]
Hence for the unique root \( v \) of (38), it must be true that \( (e^T v - \tau)/n < v \). A key observation is that in the exact projection onto \( S(\tau, n) \), the corresponding component must be zero since we use \( v > (e^T v - \tau)/n \) to determine the projection by Lemma 4.2. Moreover, if we have \( v_l - (e^T v - \tau)/n = 0 \), it must be true that the corresponding \( l \)th component in the exact projection onto \( S(\tau, n) \) must also be zero.

Our projection algorithms are constructed based on these two cases.

4.1. Exact \( \ell_1 \)-ball projection algorithm

Based on the two cases discussed above, we can project \( v \) onto \( H(\tau, n) \) and check whether there are negative complements in the projection. Given vector
If \( w \), denote
\[
I_+(w) = \{ i \mid w_i > 0 \}, \quad I_0(w) = \{ i \mid w_i = 0 \} \quad \text{and} \quad I_-(w) = \{ i \mid w_i < 0 \}.
\]

If case (i) happens, meaning \( I_-(P_{\mathcal{H}(\tau,n)}(v)) = \emptyset \), we should terminate with an exact projection. If case (ii) happens, meaning \( I_-(P_{\mathcal{H}(\tau,n)}(v)) = \{ i \mid w_i < 0 \} \neq \emptyset \), then we know
\[
P_{S(\tau,n)}(v)_i = 0, \quad i \in I_-(P_{\mathcal{H}(\tau,n)}(v)) \cup I_0(P_{\mathcal{H}(\tau,n)}(v)).
\]

This property makes it possible for us to only focus on the components in \( I_+(P_{\mathcal{H}(\tau,n)}(v)) \), and eliminate the non-positive components. After eliminating these components, the simplex projection problem reduces to a simplex projection problem in a lower-dimensional space, and the same argument will also apply. If we repeat this procedure to obtain the next iterate, then eventually we must encounter case (i) where all the components of the projection onto the hyperplane are non-negative since \( v \) is finite dimensional. This procedure is stated in Algorithm 3, where we use superscript \( (j) \) to the \( j \)th iteration.

### Algorithm 3 Exact \( \ell_1 \)-ball projection method

1. Given \( \tau > 0 \) and \( v \notin \mathbb{R}^n(\tau) \).
2. Initialize \( y^{(0)} = w^{(0)} = |v|, w^* = 0_n \) and set \( j = 0 \).
3. repeat
   4. Calculate \( w^{(j+1)} = y^{(j)} - \frac{1}{|I_+|}(e^T y^{(j)} - \tau)e \).
   5. Update \( I_+(w^{(j+1)}), I_0(w^{(j+1)}) \) and \( I_-(w^{(j+1)}) \).
   6. Set \( y^{(j+1)} = w^{(j+1)} \).
   7. Set \( j \leftarrow j + 1 \).
4. until \( I_-(w^{(j)}) = \emptyset \).
5. Set \( w^*_{I_+(w^{(j)})} = y^{(j+1)} \).
6. Output \( z^* = \text{sign}(v) \circ w^* \).

In this algorithm, at each iteration, we compute the projection onto a hyperplane in a reduced space of smaller dimension, which needs \( 2|I_+(w^{(j+1)})| \) operations. This can dramatically reduce the computational cost per iteration if a lot of zero components are detected for each iteration. For the worst case, we have \( |I_-(w^{(j+1)})| = 1 \) and \( |I_0(w^{(j+1)})| = 0 \) for each iteration. In this case, this procedure needs \( n \) iterations to terminate, and the computational cost at the \( j \)th iteration is \( 2(n - j) \). Therefore, the worse-case complexity of this algorithm is given by
\[
\sum_{j=0}^{n-1} 2(n - j) = n(n + 1).
\]
In [46], they consider the similar algorithm but without reducing the working space for each iteration, and therefore has worst-case complexity of \(2n^2\). However, it is reported in [46] that \(\mathcal{O}(n \log n)\) complexity is generally observed in practice.

### 4.2. Inexact \(\ell_1\)-ball projection algorithm

As shown in the previous subsection, Algorithm 3 can be used to find the exact projection onto the \(\ell_1\) ball. Therefore, the \(\ell_1\) ball constraint problem (35) can be solved by incorporating Algorithm 3 within the gradient projection methods. Next we design an inexact version of this projection algorithm to use our inexact gradient projection methods.

The design of such an inexact algorithm needs to address two issues. First, since our IGPM only accepts feasible iterates, we need to guarantee that the subproblem solver returns a point within the \(\ell_1\) ball. To achieve this, we add a scaling phase in the projection algorithm to obtain a feasible iterate

\[
\hat{w}^{(j+1)} = \tau \frac{w^{(j+1)}}{\|w^{(j+1)}\|_1}.
\]

It should be noticed that this feasible iterate is only used for computing the ratio \(\gamma^{(j)}\), but not for updating the next iterates. Since the algorithm terminates in finite number of iterations, \(\hat{w}^{(j+1)}\) eventually converges to the primal optimal solution (in the reduced space).

On the other hand, for \(\Omega = \mathbb{B}_n(\tau)\), the support function can be shown as \(\delta^*_\Omega = \tau \|u^{(j)}\|_{\infty}\). Therefore, the dual of the \(\ell_1\)-ball projection problem is given by

\[
\max_u -\frac{1}{2} \|u - v\|_2^2 - \tau \|u\|_{\infty} + \frac{1}{2} \|v\|_2^2.
\]

We also need a dual estimate \(u^{(j)}\) to calculate \(q(u^{(j)})\) in \(\gamma^{(j)}\) without solving the dual problem. Note that by (27), we have \(u^* = v - z^*\). This motivates us to use

\[
u^{(j+1)} = v^{(j)} - w^{(j+1)}
\]

as our dual estimate. Therefore, \(u^{(j)}\) will eventually converges to the optimal dual solution (in the reduced space). Overall, based on our selection of \(\hat{w}^{(j+1)}\) and \(u^{(j+1)}\), we know the corresponding ratio \(\gamma^{(j)}\) will eventually exceed \(\gamma\), and we can terminate the subproblem solver with final point \(\hat{w}^{(j+1)}\).

Applying the same argument of Algorithm 3, we know this algorithm terminates in at most \(n\) iterations. If we did not terminate this solver prematurely, the ratio \(\gamma_k^{(j)}\) would converge to 1, which coincides with the exact solver. Therefore, we can terminate this solver prematurely when the ratio \(\gamma_k^{(j)}\) exceeds the prescribed threshold \(\gamma\).
Algorithm 4 Inexact $\ell_1$-ball projection method

1: Given $\tau > 0$ and $v \notin \mathbb{R}^n(\tau)$.
2: Initialize $y^{(0)} = w^{(0)} = |v|$, $w^{(*)} = 0_n$ and set $j = 0$.
3: repeat
4: Calculate $w^{(j+1)} = y^{(j)} - \frac{1}{|I_+|} (e^T y^{(j)} - \tau)e$.
5: Update $I_+(w^{(j+1)})$, $I_0(w^{(j+1)})$ and $I_-(w^{(j+1)})$.
6: Calculate $\hat{w}^{(j+1)} = w^{(j+1)} / \|w^{(j+1)}\|_1$ and $u^{(j+1)} = y^{(j)} - \hat{w}^{(j+1)}$.
7: Calculate $\gamma^{(j+1)}$ according to (8).
8: Set $y^{(j+1)} = w^{(j+1)}_{I_+(w^{(j+1)})}$.
9: Set $j \leftarrow j + 1$.
10: until $I_-(w^{(j)}) = \emptyset$ or $\gamma^{(j)} \geq \gamma$.
11: Set $w^{*}_{I_+(w^{(j)})} = y^{(j)}$.
12: Output $z^* = \text{sign}(v)w^*$.

5. Numerical experiments

In this section, we apply our proposed methods to sparse signal recovery problem [41], which aims to recover sparse signal from linear measurements. The sparse signal recovery problem can be formulated as a least squared problem with an $\ell_1$-ball constraint

$$
\min_{x \in \mathbb{R}^n} \frac{1}{2} \|Ax - b\|_2^2
$$

s.t. $\|x\|_1 \leq \tau,$

(39)

where $A \in \mathbb{R}^{m \times n}$ is the measurement matrix and $b \in \mathbb{R}^m$ is the observation vector.

We test both dense and sparse measurement matrix $A \in \mathbb{R}^{m \times n}$ cases. Entries of $A \in \mathbb{R}^{m \times n}$ is sampled from a Gaussian distribution. Denote $s$ as the number of non-zeros of $\bar{x}$, where $\bar{x}$ is the being recovered signal. We set up experiment as

(a) Construct $\bar{x} \in \mathbb{R}^{n \times 1}$ with randomly choosing $n-s$ components to be zero.

Each non-zero entry equals $\pm 1$ with equal probability.

(b) Form $b = A\bar{x}$.

(c) Solve (39) for $\hat{x}$.

Combining our proposed inexact projection framework with the $\ell_1$ projection methods, we have four approaches for solving the $\ell_1$-ball constrained problem (35). They are

- GPM1: exact projection method without line search using Algorithm 3.
- GPM2: exact projection method with line search using Algorithm 3.
- IGPM1: inexact projection method without line search using Algorithm 4.
Table 1. Column names.

| Alg | Algorithm used for solving (39) |
|-----|----------------------------------|
| γ   | Different choices of $\gamma = 0.6, 0.7, 0.8, 0.9$ |
| Time| CPU time |
| Outer $k$ | Number of iterations of gradient projection algorithm |
| Inner $j$ | Total number of iterations of projection subproblem solver |
| #backtracking | Total number of backtracking |

Table 2. Dense matrix with $n = 2 \times 10^3$, $m = 10^4$, $s = 10^2$.

| Alg | $\gamma$ | Time | Outer $k$ | Inner $j$ | #backtracking |
|-----|---------|------|----------|----------|--------------|
| GPM1 | 1       | 1.12 | 44.65    | 189.10   | 0            |
| GPM2 | 1       | 3.85 | 70.95    | 311.40   | 331.15       |
| IGPM1 | 0.6     | 0.81 | 44.60    | 117.70   | 0            |
| IGPM2 | 0.6     | 3.12 | 62.50    | 185.30   | 294.00       |
| IGPM1 | 0.7     | 0.82 | 44.70    | 122.60   | 0            |
| IGPM2 | 0.7     | 3.00 | 61.05    | 193.45   | 289.50       |
| IGPM1 | 0.8     | 0.81 | 44.70    | 125.45   | 0            |
| IGPM2 | 0.8     | 2.74 | 59.85    | 223.50   | 286.45       |
| IGPM1 | 0.9     | 0.80 | 44.65    | 127.40   | 0            |
| IGPM2 | 0.9     | 2.99 | 67.95    | 274.40   | 317.95       |

- IGPM2: inexact projection method with line search using Algorithm 4.

In the experiments, we initialize $x_0 = 0_n$. The initial value of sequence $\omega_k$ is chosen as $\omega_0 = 10^{-3}$. The radius $\tau$ of $\ell_1$ norm ball equals $n-s$. For algorithms without line search, the Lipschitz constant of the objective of problem (39) is the largest eigenvalue of the positive semidefinite matrix $A^T A$, denoted as $\lambda_{\max}$. We calculate $\lambda_{\max}$ approximately by using the power method [47], and then we set $\beta = \frac{0.8}{\lambda_{\max}}$. For algorithms with backtracking line search, we use the parameters $\beta = 0.01$, $\eta = 0.01$, $\theta = 0.7$, $\alpha_0 = 1$.

The algorithms are terminated whenever

$$\|z_k - x_k\|_\infty \leq 10^{-4}.$$ 

Our code is a Python implementation and run on a Dell Precision Tower 7810 Workstation with Intel Xeon processor at 2.40 GHz, 64 Gb of main memory, and CPU: E5-2630 v3.

The results are taking average of 20 runs, which are shown in Tables 2–7. The column names are explained in Table 1.

As shown in Tables 2–4, for dense measurement matrix, IGPM1 and IGPM2 outperform GPM1 and GPM2, respectively, in terms of CPU time. The inexact version in most cases does not need more outer iterations to terminate, and it always takes fewer inner iterations, so that the overall computational time is shorter. In particular, the inexact version without line search method in the overdetermined cases ($n \leq m$) exhibits a remarkable superiority for solving (39).
Table 3. Dense matrix with $n = 10^4$, $m = 10^4$, $s = 10^2$.

| Alg  | $\gamma$ | Time | Outer $k$ | Inner $j$ | #backtracking |
|------|----------|------|-----------|-----------|---------------|
| GPM1 | 1        | 10.80| 90.45     | 520.95    | 0             |
| GPM2 | 1        | 15.30| 69.80     | 393.35    | 326.35        |
| IGPM1| 0.6      | 8.33 | 90.85     | 390.75    | 0             |
| IGPM2| 0.6      | 11.33| 58.25     | 283.00    | 275.75        |
| IGPM1| 0.7      | 7.96 | 90.55     | 406.65    | 0             |
| IGPM2| 0.7      | 11.52| 90.55     | 297.70    | 285.65        |
| IGPM1| 0.8      | 7.77 | 90.50     | 417.00    | 0             |
| IGPM2| 0.8      | 11.44| 60.50     | 304.95    | 289.55        |
| IGPM1| 0.9      | 7.69 | 90.50     | 426.40    | 0             |
| IGPM2| 0.9      | 13.09| 70.75     | 367.65    | 330.70        |

Table 4. Dense matrix with $n = 10^4$, $m = 2 \times 10^3$, $s = 10^2$.

| Alg  | $\gamma$ | Time | Outer $k$ | Inner $j$ | #backtracking |
|------|----------|------|-----------|-----------|---------------|
| GPM1 | 1        | 35.57| 479.50    | 2874.95   | 0             |
| GPM2 | 1        | 5.17 | 59.00     | 421.45    | 89.70         |
| IGPM1| 0.6      | 24.75| 479.80    | 2471.30   | 0             |
| IGPM2| 0.6      | 3.95 | 69.95     | 282.60    | 111.45        |
| IGPM1| 0.7      | 25.19| 479.65    | 2581.75   | 0             |
| IGPM2| 0.7      | 3.88 | 68.00     | 285.90    | 107.80        |
| IGPM1| 0.8      | 25.48| 479.55    | 2643.35   | 0             |
| IGPM2| 0.8      | 3.83 | 62.95     | 300.90    | 96.55         |
| IGPM1| 0.9      | 25.67| 479.55    | 2699.80   | 0             |
| IGPM2| 0.9      | 3.67 | 59.30     | 299.70    | 90.35         |

Table 5. Sparse matrix with $n = 10^5$, $m = 10^4$, $s = 10^4$.

| Alg  | $\gamma$ | Time | Outer $k$ | Inner $j$ | #backtracking |
|------|----------|------|-----------|-----------|---------------|
| GPM2 | 1        | 28.19| 29.25     | 37.3      | 173.8         |
| IGPM2| 0.6      | 22.92| 28.3      | 14.8      | 169.95        |
| IGPM2| 0.7      | 24.52| 29.3      | 20.15     | 174.00        |
| IGPM2| 0.8      | 24.41| 29.2      | 21.3      | 173.45        |
| IGPM2| 0.9      | 24.91| 29.2      | 24.2      | 173.45        |

Table 6. Sparse matrix with $n = 10^5$, $m = 10^5$, $s = 10^4$.

| Alg  | $\gamma$ | Time | Outer $k$ | Inner $j$ | #backtracking |
|------|----------|------|-----------|-----------|---------------|
| GPM2 | 1        | 81.70| 83.55     | 412.75    | 8.35          |
| IGPM2| 0.6      | 20.84| 32.05     | 56.80     | 10.10         |
| IGPM2| 0.7      | 21.60| 32.60     | 62.90     | 10.10         |
| IGPM2| 0.8      | 38.90| 57.70     | 136.10    | 9.65          |
| IGPM2| 0.9      | 42.77| 61.15     | 174.15    | 9.35          |

For the cases of sparse matrix, we increase the dimension of matrix $A$, and restrict the density of the matrix, where density means the number of non-zero-valued elements divided by the total number of elements. We set the density to be $\frac{n}{1000m}$ for all sparse measurement matrix experiments. Due to the high-dimension of matrix $A$, calculating the largest eigenvalue of matrix $A^TA$ becomes impractical. Therefore, we only compare exact and inexact versions with backtracking. Simulation results exhibit that our IGPM2 has a better performance than the GPM2 in saving computational time.
Table 7. Sparse matrix with $n = 10^5$, $m = 10^6$, $s = 10^4$.

| Alg   | $\gamma$ | Time | Outer $k$ | Inner $j$ | #backtracking |
|-------|----------|------|-----------|-----------|---------------|
| GPM2  | 1        | 102.97 | 96.65     | 392.75    | 122.20        |
| IGPM2 | 0.6      | 71.83  | 99.90     | 259.30    | 126.00        |
| IGPM2 | 0.7      | 72.20  | 98.30     | 275.55    | 124.10        |
| IGPM2 | 0.8      | 72.02  | 96.65     | 283.70    | 121.95        |
| IGPM2 | 0.9      | 73.00  | 96.75     | 294.40    | 122.25        |

A key observation in all these experiments is that our algorithms are generally insensitive to the selection of the threshold $\gamma$ when it is varying on a wide range. In fact, without tuning this parameter carefully, we merely show the performance on these values to avoid the potential slow tail convergence for the subproblem. It indicates that our proposed inexact strategy could achieve stable superior performance for problems with different sizes and sparsity with the same values of $\gamma$. To accelerate the local behaviour, one may also use dynamically changing $\gamma$ and drive it to 1 over the iteration.

6. Conclusions

We have proposed a framework of inexact primal–dual gradient projection methods for solving nonlinear problems with convex-set constraint. The key technique of such methods is a novel criterion to terminate the projection subproblem prematurely, making the methods suitable for situations where the projections onto the convex set are not easy to compute. We presented the methods in two cases with or without backtracking line search. The global convergence and $O(1/k)$ ergodic convergence rate of the optimality residual in worst-case have been provided under loose assumptions. The proposed methods are applied to $\ell_1$-ball constrained problems, and their performance is exhibited through numerical test on sparse recovery problems.

Our primary focus here is to reduce the computational cost per iteration of the gradient projection methods so that an inexact projection can be accepted while maintaining the convergence and efficiency. We are very aware that there exist more efficient stepsizes for $\beta$, such as the well-known BB stepsizes. It should be noticed that our proposed strategy can be easily applied when using truncated BB stepsizes, and the theoretical analysis can be easily generalized to such cases.

Disclosure statement

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