Coefficient estimates for some subclasses of bi-univalent functions related to m-fold symmetry

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Abstract:
The purpose of present paper is to introduce and investigate two new subclasses \( \mathcal{N}_{\Sigma m}(\tau, \gamma, \alpha) \) and \( \mathcal{N}_{\Sigma m}(\tau, \gamma, \beta) \) of analytic and m-fold symmetric bi-univalent functions in the open unit disk. Among other results belonging to these subclasses upper coefficients bounds \( |a_{m+1}| \) and \( |a_{2m+1}| \) are obtained in this study. Certain special cases are also indicated.

Keywords: m-fold symmetry, bi-univalent functions, coefficient estimates.

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1. Introduction

Let $S$ denote the family of functions analytic in the open unit disk $U = \{ z : z \in \mathbb{C}, |z| < 1 \}$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$ and having the form

$$ f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \quad (1) $$

Also let $\mathcal{A}$ denote the subclass of functions in $S$ which are univalent in $U$.

The Koebe One Quarter Theorem (e.g., see [6]) ensures that the image of $f(z) \in \mathcal{A}$ contains the disk of radius $1/4$. Thus every univalent function $f$ has an inverse $f^{-1}$ satisfying

$$ f^{-1}(f(z)) = z, \quad (z \in U) $$

and

$$ f(f^{-1}(w)) = w, \quad (|w| < r(f), r(f) \geq \frac{1}{4}) $$

where

$$ g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots. \quad (2) $$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $U$ if both $f$ and $f^{-1}$ are univalent in $U$.

Let $\Sigma$ denotes the class of analytic and bi-univalent functions in $U$. Some examples of functions in class $\Sigma$ are

$$ h_1(z) = \frac{z}{1-z}, \quad h_2(z) = -\log(1-z), \quad h_3(z) = \frac{1}{2} \log \left( \frac{1+z}{1-z} \right), \quad z \in U. $$

For each function $f \in \mathcal{A}$, the function $h(f(z)) = (f(z)^m)^{\frac{1}{m}}$, $(z \in U, m \in \mathbb{N})$ is univalent and maps the unit disk $U$ into a region with m-fold symmetry. A function is said to be m-fold symmetric (see [9,10]) if it has the following normalized form:

$$ f(z) = z + \sum_{k=2}^{\infty} a_{mk+1} z^{mk+1}, \quad (z \in U, m \in \mathbb{N}). \quad (3) $$

We denote $S_m$ the class of m-fold symmetric univalent functions in $U$, which are normalized by the series expansion (3). In fact, the functions in the class $\mathcal{A}$ are one-fold symmetric. Analogous to the concept of m-fold symmetric univalent functions, we here introduced the concept of m-fold symmetric bi-univalent functions. Each function $f \in \Sigma$ generates an m-fold symmetric bi-univalent function for each integer $m \in \mathbb{N}$. Furthermore, for the normalized form of $f$ is given by (3), they obtained the series expansion for $f^{-1}$ as follows:

$$ g(w) = w - a_{m+1} w^{m+1} + \left( (m+1)a_{m+1}^2 - a_{2m+1} \right) w^{2m+1} - \frac{1}{2} (m+1)(3m+2)a_{m+1}^2 - (3m+2)a_{m+1} a_{2m+1} + a_{3m+1} w^{3m+1} + \cdots, \quad (4) $$

where $f^{-1} = g$. We denote by $\Sigma_m$ the class of m-fold symmetric bi-univalent functions in $U$. It is easily seen that for $m=1$, the formula (4) coincides with the formula (2) of the class $\Sigma$. Some examples of m-fold symmetric bi-univalent functions are given as follows:

$$ \left( \frac{z^m}{1-z^m}, \frac{1}{2} \log \left( \frac{1+z^m}{1-z^m} \right) \right) $$

with the corresponding inverse functions

$$ \left( \frac{w^m}{1+w^m}, \frac{e^{2w^m} - 1}{e^{2w^m} + 1} \right) \quad \text{and} \quad \left( \frac{w^m - 1}{e^w w^m} \right), $$

respectively.

Recently, many authors investigated bounds for various subclass of m-fold bi-univalent functions (see [1,2,3,4,5,7,9,12,13,15]). The aim of the present paper is to introduce the new subclass $N_{\Sigma_m}(\tau, \gamma; \alpha)$ and $N_{\Sigma_m}(\tau, \gamma; \beta)$ of $\Sigma_m$ and find estimates on the coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ for functions in each of these new subclass.

In order to prove our main results, we require the following lemma.

Lemma 1. ([6]). If $h \in P$, then $|c_k| \leq 2$ for each $k \in \mathbb{N}$, where $P$ is the family of all functions $h$ analytic in $U$ for which $Re\left(h(z)\right) > 0$, $(z \in U)$

where

$$ h(z) = 1 + c_1 z + c_2 z^2 + \cdots. $$

Definition 1. A function $f(z) \in \Sigma_m$ given by (3) is said to be in the class $N_{\Sigma_m}(\tau, \gamma; \alpha)$ if the following condition are satisfied:

$$ \left| arg \left( 1 + \frac{1}{\tau} \left[ (1+\gamma) z f''(z) + \gamma f'(z) - 1 \right] \right) \right| < \alpha \frac{\pi}{2}, \quad (z \in U) $$

and

$$ \left| \frac{1}{\tau} \left[ (1+\gamma) z f''(z) + \gamma f'(z) - 1 \right] \right| < a \frac{\pi}{2}, \quad (z \in U) $$

where $a = a_{m+1}$.
where the function $g = f^{-1}$ is given by (4).

**Definition 2.** A function $f(z) \in \mathcal{N}_m$ given by (3) is said to be in the class $\mathcal{N}_m(\tau, \gamma; \alpha)$ if the following conditions are satisfied:

$$\text{Re}\left(1 + \frac{1}{\tau}\left((1 + \gamma)z^2 f''(z) + zf'(z) - 1\right)\right) > \beta,$$

$(z \in U)$

(7)

and

$$\text{Re}\left(1 + \frac{1}{\tau}\left((1 + \gamma)w^2 g''(w) + wg'(w) - 1\right)\right) > \beta,$$

$(w \in U)$

(8)

where the function $g = f^{-1}$ is given by (4).

### 2. Coefficient Estimates for the Functions Class $\mathcal{N}_m(\tau, \gamma; \alpha)$

We begin this section by finding the estimates on the coefficients $|a_m|_1$ and $|a_{2m+1}|$ for functions in the class $\mathcal{N}_m(\tau, \gamma; \alpha)$.

**Theorem 2.1** Let $f(z) \in \mathcal{N}_m(\tau, \gamma; \alpha)$ $(0 < \alpha \leq 1; \tau \in \mathbb{C} \setminus \{0\}; 0 \leq \gamma < 1)$ be of the form (3). Then

$$|a_m|_1 \leq \frac{2\alpha|\tau|}{\sqrt{2m[(2m+2m+1)(m+1)-(m+m+1)(m+1)]}}$$

(9)

and

$$|a_{2m+1}| \leq \frac{2\alpha^2|\tau|^2(m+1)}{m^2(m+m+1)^2} + \frac{\alpha|\tau|}{m^2(2m+2m+1)}$$

(10)

**Proof.** It follows from (5) and (6) that

$$1 + \frac{1}{\tau}\left((1+\gamma)z^2 f''(z) + zf'(z) - 1\right) = [p(z)]^\alpha$$

(11)

and

$$1 + \frac{1}{\tau}\left((1+\gamma)w^2 g''(w) + wg'(w) - 1\right) = [q(w)]^\alpha$$

(12)

where the functions $p(z)$ and $q(w)$ are in $\mathcal{P}$ and have the following series representations:

$$p(z) = 1 + p_m z^m + p_{2m} z^{2m} + p_{3m} z^{3m} + \ldots$$

and

$$q(w) = 1 + q_m w^m + q_{2m} w^{2m} + q_{3m} w^{3m} + \ldots$$

(13)

Now, equating the coefficients in (11) and (12), we obtain

$$\frac{m(m+m+1)a_{m+1}}{\tau} = \alpha p_m,$$

(15)

and

$$\frac{(2m(2m+2m+1)a_{2m+1} - m(m+m+1)^2 a_{m+1})}{\tau} = \alpha q_m$$

(16)

and

$$\frac{-m(m+m+1)a_{m+1}}{\tau} = \alpha q_m - \frac{\alpha(a-1)}{2} p_m^2$$

(17)

From (15) and (17), we find

$$p_m = -q_m$$

(19)

and

$$2m^2(m+m+1)^2 a_{m+1} \tau = \alpha^2(p_m^2 + q_m^2)$$

(20)

From (16), (18) and (20), we get

$$(2m + 2m + 1)(m+1)-(m+m+1)^2 2ma_{m+1} \tau = \alpha(p_{2m} + q_{2m}) + \frac{(a-1)m^2(m+m+1)}{\tau^2} a_{m+1}$$

(21)

Therefore, we have

$$a_{m+1} \leq \frac{2a^2\tau^2(p_{2m} q_{2m})}{2m[(2m+2m+1)(m+1)-(m+m+1)^2-(a-1)m^2(m+m+1)]}$$

(22)

Applying Lemma 1 for the coefficients $p_{2m}$ and $q_{2m}$, we have

$$|a_{m+1}| \leq \frac{2a^2|\tau|^2(m+1)}{m^2(m+m+1)^2} + \frac{\alpha|\tau|}{m^2(2m)}$$

(23)

This gives the desired bound for $|a_{m+1}|$ as asserted in (9).

In order to find the bound on $|a_{2m+1}|$, by subtracting (18) from (16), we get

$$2m[(2m+2m+1)a_{2m+1} - (2m+2m+1)(m+1)a_{m+1}] = 4m^2 m[(2m+2m+1)(m+1)-(a-1)m^2(m+m+1)]$$

(24)

It follows from (19) and (24) that

$$a_{2m+1} \leq \frac{a^2\tau^2(p_{2m} + q_{2m})(m+1)}{4m(m+m+1)^2} + \frac{\alpha(p_{2m} - q_{2m})}{4m(2m+2m+1)}$$

(25)
Applying Lemma 1 once again for the coefficients \( p_m, p_{2m}, q_m, \) and \( q_{2m} \), we readily obtain
\[
|a_{2m+1}| \leq \frac{2a^2|\tau|^2(m+1)}{m^2(m+m+1)^2} + \sqrt{\frac{a|\tau|}{m(m+2m+1)}}.
\] (26)

3. Coefficient Bounds for the Functions Class \( \mathcal{N}_{\Sigma m}(\tau, \gamma; \beta) \)

This section is devoted to find the estimates on the coefficients \( |a_{m+1}| \) and \( |a_{2m+1}| \) for functions in the class \( \mathcal{N}_{\Sigma m}(\tau, \gamma; \beta) \).

**Theorem 3.1** Let \( f(z) \in \mathcal{N}_{\Sigma m}(\tau, \gamma; \beta) \) \((0 \leq \beta < 1; \tau \in \mathbb{C} \setminus \{0\}, 0 \leq \gamma < 1) \) be of the form (3).

Then
\[
|a_{m+1}| \leq \frac{2|\tau|(1-\beta)}{m((1+2m^2+2m+1)(m+1) - (m+1)^2)}
\] (27)
and
\[
|a_{2m+1}| \leq \frac{4|\tau|^2(1-\beta)^2(m+1)}{m^2(m+2m+1)^2} + \frac{2|\tau|(1-\beta)}{m(2m+2m+1)}
\] (28)

**Proof.** It follows from (7) and (8) that there exist \( p, q \in \mathbb{P} \) such that
\[
1 + \frac{1}{\tau} \left[ (1+\gamma^2)f''(z) + f'(z)(1+\gamma) - \gamma f(z) \right] = \beta + (1-\beta)p(z)
\] (29)
and
\[
1 + \frac{1}{\tau} \left[ (1+\gamma^2)g''(w) + g'(w)(1+\gamma) - \gamma g(w) \right] = \beta + (1-\beta)q(w),
\] (30)
where \( p(z) \) and \( q(z) \) have the forms (13) and (14), respectively. By suitably comparing coefficients in (29) and (30), we get
\[
\frac{m(m+m+1)a_{m+1}}{\tau} = (1-\beta)p_m,
\] (31)
\[
(2m(2m+2m+1)a_{2m+1} - (m+m+1)^2a_{m+1}^2) = (1-\beta)p_{2m},
\] (32)
\[
\frac{-m(m+m+1)a_{m+1}}{\tau} = (1-\beta)q_m,
\] (33)
\[
(2m(2m+2m+1)[(m+1)a_{m+1}^2 - a_{2m+1}^2] - (m+m+1)^2a_{m+1}^2) = (1-\beta)q_{2m}.
\] (34)

Adding (32) and (34), we have
\[
\frac{2m^2(m+m+1)^2a_{m+1}^2}{\tau} = (1-\beta)^2(p_m^2 + q_m^2).
\] (36)

Applying Lemma 1 once again for the coefficients \( p_m, p_{2m}, q_m, q_{2m} \), we readily obtain
\[
\frac{((2m+2m+1)(m+1) - (m+m+1)^2)2ma_{m+1}^2}{\tau} = (1-\beta)(p_{2m} + q_{2m}).
\] (37)

Applying Lemma 1, we obtain
\[
|a_{m+1}| \leq \frac{2|\tau|(1-\beta)}{m((2m+2m+1)(m+1) - (m+m+1)^2)}
\] (27)

This is the bound on \( |a_{m+1}| \) asserted in (27).

In order to find the bound on \( |a_{2m+1}| \), by subtracting (34) from (32), we get
\[
2m(2m+2m+1)a_{2m+1} - (2m+2m+1)(m+1)a_{m+1}^2 = (1-\beta)(p_{2m} - q_{2m})
\] (38)

Or, equivalently,
\[
a_{2m+1} = \frac{2m(2m+2m+1)(m+1)a_{m+1}^2}{2m(2m+2m+1)} + \frac{2|\tau|(1-\beta)}{m(2m+2m+1)}.
\] (39)

It follows from (35) and (36) that
\[
a_{2m+1} = \frac{\tau(1-\beta)^2(m+1)(p_m^2 + q_m^2) + \tau(1-\beta)(p_{2m} - q_{2m})}{2m^2(m+m+1)^2}
\] (40)

Applying Lemma 1 once again for the coefficients \( p_m, p_{2m}, q_m, q_{2m} \), we easily obtain
\[
|a_{2m+1}| \leq \frac{4|\tau|^2(1-\beta)^2(m+1)}{m^2(m+m+1)^2} + \frac{2|\tau|(1-\beta)}{m(2m+2m+1)}
\] (41)

4. Corollaries and Consequences

For one-fold symmetric bi-univalent functions and \( \tau = 1 \), Theorem 2.1 and Theorem 3.1 reduce to Corollary 1 and Corollary 2, respectively, which were proven very recently by Frasin [8] (see also [11]).

**Corollary 4.** Let \( f(z) \in \mathcal{N}_{\Sigma}(\alpha, \gamma)(0 < \alpha \leq 1; 0 \leq \gamma < 1) \) be of the form (1).

Then
\[
|a_2| \leq \frac{2a}{\sqrt{2(1-\alpha) - \gamma(\alpha + 1)}}
\] (42)

and
\[
|a_3| \leq \frac{4a^2}{(2+\gamma)^2} + \frac{a}{(3-2\gamma)}.
\] (43)

**Corollary 5.** Let \( f(z) \in \mathcal{N}_{\Sigma}(\beta, \gamma)(0 < \alpha \leq 1; 0 \leq \gamma < 1) \) be of the form (1).

Then
\[ |a_2| \leq \sqrt{\frac{2(1-\beta)}{(2+2\gamma+y^2)}} \] (44)

and

\[ |a_3| \leq \frac{8(1-\beta)^2}{(2+y)^2} + \frac{2(1-\beta)}{3(2y)} \] (45)

The classes \( \mathcal{N}_a(\alpha, \gamma) \) and \( \mathcal{N}_b(\beta, \gamma) \) are defined in the following way:

**Definition 3.** A function \( f(z) \in \Sigma \) given by (1) is said to be in the class \( \mathcal{N}_a \) if the following conditions are satisfied:

\[ \left| \arctan \left( \frac{(1+y)z f''(z)+zf'(z)}{(1+y)zf'(z)-zg'(z)} \right) \right| < \frac{\pi}{2} \quad (z \in U) \] (46)

And

\[ \left| \arctan \left( \frac{(1+y)w^2 f''(w)+wzf'(w)}{(1+y)wzg'(w)-wg'(w)} \right) \right| < \frac{\pi}{2} \quad (w \in U) \] (47)

\( (0 < \alpha \leq 1 ; 0 \leq \gamma < 1) \),

where the function \( g = f^{-1} \) is given by (2).

**Definition 4.** A function \( f(z) \in \Sigma \) given by (1) is said to be in the class \( \mathcal{N}_b(\beta, \gamma) \) if the following conditions are satisfied:

\[ \text{Re} \left( \frac{(1+y)z f''(z)+zf'(z)}{(1+y)zf'(z)-zg'(z)} \right) > \beta \quad (z \in U) \] (48)

And

\[ \text{Re} \left( \frac{(1+y)w^2 f''(w)+wzf'(w)}{(1+y)wzg'(w)-wg'(w)} \right) > \beta \quad (w \in U) \] (49)

\( (0 \leq \beta < 1 ; 0 \leq \gamma < 1) \),

where the function \( g = f^{-1} \) is given by (2).

If we set \( \gamma = 0 \) and \( \tau = 1 \) in Theorem 2.1 and Theorem 3.1, then the classes \( \mathcal{N}_a(\tau, \gamma; \alpha) \) and \( \mathcal{N}_b(\tau, \gamma; \beta) \) reduce to the classes \( \mathcal{N}_a^{(m)} \) and \( \mathcal{N}_b^{(m)} \) investigated recently by Srivastava et al. [11] and thus, we obtain the following corollaries:

**Corollary 6.** Let \( f(z) \in \mathcal{N}_a^{(m)} \) \( (0 < \alpha \leq 1) \) be of the form (3).

Then

\[ |a_{m+1}| \leq \frac{2\alpha}{\sqrt{m(2m+1)(m+1)^2+2m^2(m+1)^2(\alpha-1)}} \] (50)

and

\[ |a_{2m+1}| \leq \frac{2\alpha(m+1)}{m(2m+1)^2} \] (51)

**Corollary 7.** Let \( f(z) \in \mathcal{N}_b^{(m)} \) \( (0 \leq \beta \leq 1) \) be of the form (4).

Then

\[ |a_{m+1}| \leq \frac{2(1-\beta)}{\sqrt{m(2m+1)(m+1)^3-m(m+1)^2}} \] (52)

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مختصرات المعاليم لبعض الأصناف الجزئية لدوال ثنائية التكافؤ المرتبطة بالنظرية المتناظرة

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المستخلص:

الغرض من البحث الحالي هو أن نقدم ونتخريج عن صنفين جزئيين جديدين $N_{Y_{m}}(r, \gamma, \beta)$ و $N_{Y_{m}}(r, \gamma, \alpha)$ من الدوال ثنائية التكافؤ المستمرة ذات الطولية $m$، والتحليلة في نجوم الوحدة المنتج ومن بين النتائج الأخرى لهذه الأصناف الجزئية حدود المعاليم العليا ($|a_{2m+1}|, |a_{m+1}|$) تم الحصول عليها في هذه الدراسة.