On triviality of the reduced Whitehead group over Henselian fields

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Abstract. Let $F$ be a Henselian field of $q$-cohomological dimension 3, where $q$ is a prime. Let $\Gamma_F$ be the totally ordered Abelian value group of $F$ and let $D$ be a central division algebra over $F$ of index a power of $q$ such that the characteristic of the residue field $F$ is coprime to $q$. We show that when $1 \leq \dim_{F_q}(\Gamma_F/q\Gamma_F) \leq 3$, the reduced Whitehead group of $D$ is trivial.

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1. Introduction. Let $E$ be an arbitrary field and $A$ be a finite-dimensional central simple algebra over $E$. We denote the group of units of $A$ by $A^*$. The reduced Whitehead group of $A$ is given by

$$SK_1(A) = \{a \in A^* : \text{Nrd}_A(a) = 1\}/[A^*, A^*],$$

where $[A^*, A^*]$ is the commutator subgroup of $A^*$ and $\text{Nrd}_A$ is the reduced norm map, $\text{Nrd}_A : A^* \to E^*$. If $A \cong M_n(D)$, then there is an isomorphism $SK_1(A) \cong SK_1(D)$ [1, cf. §23, Corollary 1]. Moreover, if $D_i$ ($1 \leq i \leq r$) are central division algebras of $p_i$-power degrees, where $p_i$ are distinct primes, then $SK_1(D_1 \otimes D_2 \otimes \cdots \otimes D_r) \cong SK_1(D_1) \times SK_1(D_2) \times \cdots \times SK_1(D_r)$ [1, cf. §23, Lemma 6]. Thus, to study $SK_1$, it is enough to consider central division algebras of prime power degrees.

Let $G_E$ denote the absolute Galois group of a field $E$, i.e., $G_E = \text{Gal}(E^*/E)$, where $E^*$ is the separable closure of $E$. The $q$-cohomological dimension of $E$ is the least positive integer $d$ such that for all discrete $G_E$-modules $A$, which are

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q-primary torsion groups, the Galois cohomology groups $H^i(G_E, A)$ are trivial for $i \geq d + 1$. The $q$-cohomological dimension of $E$ is denoted by $cd_q(E)$.

For a torsion free Abelian group $\Gamma_F$ and a prime $q$, let the $q$-rank of $\Gamma_F$ be $r_q := \dim_{\mathbb{F}_q}(\Gamma_F/q\Gamma_F)$. In this article, our aim is to prove the following theorem.

**Theorem 1.1.** Let $(F, v)$ be a Henselian field with totally ordered Abelian value group $\Gamma_F$ and characteristic of the residue field $\overline{F}$, $\text{char}(\overline{F}) = \overline{p}$. Let $q \neq \overline{p}$ be a prime and $D$ be a finite-dimensional central division algebra over $F$ of index a power of $q$. If $cd_q(F) = 3$ and $1 \leq r_q \leq 3$, then $SK_1(D) = (1)$.

In [8], Suslin conjectured that $SK_1(D) = (1)$ for any central division algebra $D$ of index $q^2$, where $q$ is a prime, over fields of cohomological dimension 3. The above theorem provides further evidence for the validity of the conjecture.

In order to prove Theorem 1.1, we first obtain a relation between the $q$-cohomological dimension of a Henselian field and its residue field. More precisely, we prove the following proposition.

**Proposition 1.2.** Let $(F, v)$ be a Henselian field with totally ordered Abelian value group $\Gamma_F$ and characteristic of the residue field $\overline{F}$, $\text{char}(\overline{F}) = \overline{p}$. For a prime $q \neq \overline{p}$, assume that $r_q$ is finite. Then we have

$$cd_q(F) = cd_q(\overline{F}) + r_q.$$  

Along with the above Proposition 1.2, the proof of Theorem 1.1 uses valuation theory on division algebras over Henselian fields as developed in [3,9].

2. Preliminaries.

2.1. Notations and basic definitions. In this section, we recall some notions on division algebras over Henselian fields. We refer the reader to [9] for more details.

Let $(F, v)$ be a Henselian valued field with the value group $v(F^*) = \Gamma_F$. We assume that $\Gamma_F$ is a totally ordered additive Abelian group. All division algebras in this article are assumed to be finite-dimensional over its center. Let $D$ be a central division algebra over $F$. The valuation $v$ extends uniquely to the valuation $v_D$ on $D$ [9, cf. Corollary 1.7] and by [9, Theorem 1.4], it is given by

$$v_D(d) := \frac{1}{\text{ind}D}v(\text{Nrd}_D(d)) \text{ for } d \in D^*.$$  

We denote the value group of $v_D$ by $\Gamma_D = v_D(D^*)$. We denote the residue division algebra by $\overline{D} = \{x \in D : v_D(x) \geq 0\}/\{x \in D : v_D(x) > 0\}$, and its center by $Z(\overline{D})$. There is a well-defined group homomorphism [9, cf. §1.1.1]

$$\theta_D : \Gamma_D \to \text{Aut}(Z(\overline{D})/F),$$

where $F$ is the residue field of $F$ with respect to the valuation $v$. We call $\theta_D$ the canonical homomorphism of the valuation $v_D$.

If $D$ satisfies the following equality

$$[D : F] = [\overline{D} : F] |\Gamma_D : \Gamma_F|,$$
then $D$ is said to be defectless over $F$. The division algebra $D$ is said to be tame over $F$ if $D$ is defectless over $F$, the extension $Z(D)/F$ is separable, and $\text{char}(F) \nmid |\ker(\theta_D) : \Gamma_F|$. Recall that $D$ is said to be totally ramified over $F$ if $|\Gamma_D : \Gamma_F| = [D : F]$ or equivalently, $\overline{D} = \overline{F}$ and $D$ is defectless [9, cf. §7.4.1].

For a tame central division algebra $D$, we denote by $G$ the Abelian Galois group $\text{Gal}(Z(\overline{D})/\overline{F})$ [9, cf. Proposition 1.5]. Let

$$\widetilde{N} = N_{Z(\overline{D})/\overline{F}} \circ \text{Nrd}_{\overline{D}} : \overline{D} \to \overline{F}, \quad \text{and} \quad \zeta = \text{ind} D/(\text{ind} D) [Z(\overline{D}) : F].$$

If we further assume that $\overline{D}$ is a field, then $\zeta^2 = [D : F]/[\overline{D} : \overline{F}]^2$. By [9, Proposition 1.5], $\overline{D}/\overline{F}$ is an Abelian Galois extension, and there is an isomorphism of Abelian groups $\text{Gal}(\overline{D}/\overline{F}) \simeq \Gamma_D/\ker \theta_D$. Furthermore, as $D$ is defectless

$$\zeta^2 = |\Gamma_D : \ker \theta_D||\ker \theta_D : \Gamma_F|/|\overline{D} : \overline{F}| = |\ker \theta_D : \Gamma_F|. \quad (2.1)$$

We associate the following groups with the valuation $v_D$ (cf. Ershov [3]).

- $SL(D) = \{x \in D^* : \text{Nrd}_D(x) = 1\};$
- $U = \{x \in D^* : v_D(x) = 0\};$
- $SL^{v_D}(D) = \{x \in SL(D) : \widetilde{N}(x) = 1\};$
- $SK^{v_D}(D) = SL^{v_D}(D)/[U, D^*];$
- $\mathcal{K} = \left(\left[\overline{U}, \overline{D}^*\right] \cap SL(\overline{D})\right)/[\overline{D}^*, \overline{D}^*];$
- $C_{v_D} = \mu(\overline{F}) \cap \widetilde{N}(\overline{D}^*).$

We recall the definition of the $-1$-Tate cohomology group. Let $G$ be a finite Abelian group and $M$ be a $G$-module, with operation written multiplicatively. Let $N_G : M \to M$ be the $G$-norm map given by $m \mapsto \prod_{\sigma \in G} \sigma(m)$. Let $I_G(M) = \langle m\sigma(m)^{-1} : m \in M, \sigma \in G \rangle$ be a subgroup of $M$. Then the $-1$-Tate cohomology group of $G$ with respect to $M$ is

$$\hat{H}^{-1}(G, M) = \ker N_G/I_G(M).$$

2.2. Some known results. We recall some known results which are used in the proof of the main theorem. We refer to the earlier subsection for notations and terminology. We have the following important result due to Ershov.

**Theorem 2.1** ([3, page 68]). Let $F$ be a Henselian field and let $D$ be a tame central division algebra over $F$. With notations as in the above subsection, we have the following diagram with exact rows and column.
Now we state a result on the triviality of the reduced Whitehead group of division algebras of $q$-prime power indices over a field $K$ of $q$-cohomological dimension at most 2. More precisely,

**Theorem 2.2** ([4, Theorem 1.1]). Let $K$ be a field of characteristic $p$ (which can be zero) and let $q$ be a prime number different from $p$. Suppose that $cd_q(K) \leq 2$ and that $A$ is a central simple algebra over $K$ whose index is a power of $q$. Then the reduced Whitehead group of $A$ is trivial.

### 3. Computation of cohomological dimension.

Let $F$ be a Henselian field with valuation $v$ and char$(F) = \overline{p}$. We denote by $F^t$ the inertia field of $F$ and $G^t = \text{Gal}(F^s/F^t)$ its corresponding Galois group (see [2, §5.2]). We denote by $G_F$ the absolute Galois group of the field $F$. By [2, Theorem 5.2.7], the absolute Galois group of the residue field $\overline{F}$, $G_{\overline{F}} \simeq \text{Gal}(F^t/F) \simeq G_F/G^t$. Consider the inflation map induced by the above isomorphism

$\inf: H^i(G_{\overline{F}}, \mathbb{Z}/n) \to H^i(G_F, \mathbb{Z}/n)$, for $(n, \overline{p}) = 1$.

By [2, Theorem 5.3.3 (1)], the ramification Galois group $G^v = \text{Gal}(F^s/F^v)$ is the unique $\overline{p}$-Sylow subgroup of $G^t$. For $\overline{p} \nmid n$, we thus have $H^i(G^v, \mathbb{Z}/n) = 0$ for $i \geq 1$ [6, cf. I.§3.3, Corollary 2]. Thus, by the inflation-restriction sequence [5, VII.§6, Proposition 5], we get the following isomorphism

$H^i(G^t/G^v, \mathbb{Z}/n) \simeq H^i(G^t, \mathbb{Z}/n)$, for $i \geq 0$ and $(n, \overline{p}) = 1$.

Let $q$ be a prime distinct from $\overline{p}$. By [2, Theorem 5.3.3 (3)], there is an isomorphism of profinite groups

$G^t/G^v \simeq \prod_{q \neq \overline{p}} \mathbb{Z}_{q^r}$, (3.1)

where for each prime $q \neq \overline{p}$, $r_q$ is the $q$-rank of $\Gamma_F$, i.e., $r_q$ is the $\mathbb{F}_q$-dimension of $\Gamma_F/q\Gamma_F$. Thus, for $i \geq 0$ and $\text{gcd}(n, \overline{p}) = 1$, we have

$H^i(G^t, \mathbb{Z}/n) \simeq H^i(G^t/G^v, \mathbb{Z}/n) \simeq H^i\left(\prod_{q \neq \overline{p}} \mathbb{Z}_{q^r}, \mathbb{Z}/n\right)$. (3.2)

We first prove the following lemma which appears as a part of an exercise in [6].
Lemma 3.1 ([6, I.§4.5, Exercise (1)]). For a prime number $q$ and a natural number $r_q$, we have

$$\text{cd}_q(\mathbb{Z}_q^{r_q}) = r_q \text{ and } H^{r_q}(\mathbb{Z}_q^{r_q}, \mathbb{Z}/q) \text{ is a group of order } q.$$ 

Proof. We prove this lemma by induction on $r_q$. Suppose $r_q = 1$, then $\text{cd}_q(\mathbb{Z}_q) = 1$ and $H^1(\mathbb{Z}_q, \mathbb{Z}/q)$ is a group of order $q$. We assume the result for $r_q - 1 < \infty$, i.e., $\text{cd}_q(\mathbb{Z}_q^{r_q-1}) = r_q - 1$ and $H^{r_q-1}(\mathbb{Z}_q^{r_q-1}, \mathbb{Z}/q)$ is a group of order $q$. Now to prove the result for $r_q$, consider the short exact sequence

$$1 \to \mathbb{Z}_q \to \mathbb{Z}_q^{r_q} \to \mathbb{Z}_q^{r_q-1} \to 1.$$ 

By the induction hypothesis, $\text{cd}_q(\mathbb{Z}_q^{r_q-1}) = \text{cd}_q(\mathbb{Z}_q^{r_q}/\mathbb{Z}_q) = r_q - 1 < \infty$ and $\text{cd}_q(\mathbb{Z}_q) = 1$. Moreover, $\mathbb{Z}_q$ is a pro-$q$-group and $H^1(\mathbb{Z}_q, \mathbb{Z}/q)$ is a group of order $q$. Therefore, by [6, I.§4.1, Proposition 22]

$$\text{cd}_q(\mathbb{Z}_q^{r_q}) = r_q.$$ 

By the spectral sequence [6, I.§3.3, Remark],

$$H^{r_q}(\mathbb{Z}_q^{r_q}, \mathbb{Z}/q) = H^{r_q-1}(\mathbb{Z}_q^{r_q-1}, H^1(\mathbb{Z}_q, \mathbb{Z}/q)) = H^{r_q-1}(\mathbb{Z}_q^{r_q-1}, \mathbb{Z}/q),$$

which is a group of order $q$ by the induction hypothesis. Hence, the lemma is proved. \qed

We now prove the Proposition 1.2. The proof is on similar lines as the discrete valued case [6, cf. II.§4.3, Proposition 12] and we present it here for completeness.

Proof of Proposition 1.2. Consider the following exact sequence

$$1 \to H \to \prod_{q' \neq p} \mathbb{Z}_q^{r_{q'}} \to \mathbb{Z}_q^{r_q} \to 1,$$

where the product is taken over all primes $q' \neq p$ and the last map is the projection onto the $q$th component. By [6, I.§3.3, Corollary 2], $\text{cd}_q(H) = 0$ and by Lemma 3.1, $\text{cd}_q(\mathbb{Z}_q^{r_q}) = r_q < \infty$. Since the group $\prod_{q' \neq p} \mathbb{Z}_q^{r_{q'}}$ is Abelian, using [6, I.§4.1, Proposition 22], we get

$$\text{cd}_q \left( \prod_{q' \neq p} \mathbb{Z}_q^{r_{q'}} \right) = r_q.$$ 

(3.3)

In fact, using the inflation-restriction sequence [5, VII.§6, Proposition 5], we get an isomorphism

$$H^{r_q}(\mathbb{Z}_q^{r_q}, \mathbb{Z}/q) \simeq H^{r_q} \left( \prod_{q' \neq p} \mathbb{Z}_q^{r_{q'}}, \mathbb{Z}/q \right).$$

(3.4)

In particular, $H^{r_q}(\prod_{q' \neq p} \mathbb{Z}_q^{r_{q'}}, \mathbb{Z}/q)$ is a group of order $q$ (see Lemma 3.1). Hence, $H^{r_q}(G^t, \mathbb{Z}/q)$ is also a group of order $q$ (see Equation 3.2), and applying [6, I.§3.3, Proposition 15] to the groups $G^t, G^\varphi$, we also have $\text{cd}_q(G^t) \leq r_q$. 


As a result, \( cd_q(G_t) = r_q \). Moreover, using Equation (3.3) above and [6, I.§3.3, Proposition 15], we have
\[
\text{cd}_q(G_F) \leq \text{cd}_q(G_{\Gamma}) + r_q.
\]
Assume \( \text{cd}_q(G_F/G_t)(= \text{cd}_q(G_{\Gamma})) = n < \infty \), using the spectral sequence [6, I.§3.3, Remark], we get
\[
H^{n+r_q}(G_F, \mathbb{Z}/q) = H^n(G_F/G_t, H^{r_q}(G_t, \mathbb{Z}/q)).
\]
Since by our assumption \( \text{cd}_q(G_F/G_t) = n \) and as observed above, \( H^{r_q}(G_t, \mathbb{Z}/q) \) is a group of order \( q \), we get \( H^n(G_F/G_t, H^{r_q}(G_t, \mathbb{Z}/q)) \neq 0 \). Hence, \( \text{cd}_q(G_F) = \text{cd}_q(G_{\Gamma}) + r_q \).

4. Triviality of the reduced Whitehead group. In this section, we give a proof of Theorem 1.1. We start with the following lemma. We denote the absolute Galois group of a field \( E \) by \( G_E \).

**Lemma 4.1.** Let \( L/E \) be a Galois field extension of degree \( q^r \) for some \( r \in \mathbb{N} \) and a prime number \( q \). Further assume that the \( q \)-cohomological dimension of \( E, \text{cd}_q(G_E) \leq 1 \). Then the multiplicative \( \text{Gal}(L/E) \)-module \( L^* \) is cohomologically trivial.

**Proof.** We show that \( H^1(\text{Gal}(L/E), L^*) \) and \( H^2(\text{Gal}(L/E), L^*) \) are trivial. Indeed, by Hilbert’s theorem 90, \( H^1(\text{Gal}(L/E), L^*) = 0 \) [5, cf. X.§1, Proposition 2]. By [5, X.§4, Corollary], the \( q \)-group \( H^2(\text{Gal}(L/E), L^*) \) injects into the Brauer group of \( E \). Since \( \text{cd}_q(G_E) \leq 1 \), the \( q \)-primary torsion part of the Brauer group is trivial. Hence, \( H^2(\text{Gal}(L/E), L^*) = 0 \). Thus, the Tate cohomology groups with positive exponents 1 and 2 are trivial. Using [5, IX.§5, Theorem 8], we get that the \( \text{Gal}(L/E) \)-module \( L^* \) is cohomologically trivial, i.e., for every subgroup \( H \leq \text{Gal}(L/E) \) and every integer \( i, \widehat{H}^i(H, L^*) = 0 \). □

A.R. Wadsworth suggested the proof of the following statement.

**Lemma 4.2.** Let \( \Gamma_F \subset \Gamma_D \) be totally ordered Abelian groups with \( \Gamma_F \) as a subgroup of \( \Gamma_D \). Suppose that the cardinality of \( \Gamma_D/\Gamma_F, |\Gamma_D/\Gamma_F| = q^n \), where \( q \) is a prime number, and \( n \) is a natural number. Assume that \( r_q \) is finite. Then we have
1. If \( r_q = 0 \), then \( \Gamma_D = \Gamma_F \).
2. If \( r_q = 1 \), then \( \Gamma_D/\Gamma_F \) is cyclic.

In general, the number of invariant factors of \( \Gamma_D/\Gamma_F \) is at most \( r_q \).

**Proof.** We have \( \Gamma_D/\Gamma_F \approx q^n\Gamma_D/q^n\Gamma_F \subset \Gamma_F/q^n\Gamma_F \). Furthermore, as \( \Gamma_F \) is a torsion-free Abelian group, the multiplication by \( q \) map from \( q^{i-1}\Gamma_F/q^i\Gamma_F \) to \( q^i\Gamma_F/q^{i+1}\Gamma_F \) is a group isomorphism for every \( i \geq 1 \). Hence, we have\[
\Gamma_F/q^n\Gamma_F \cong q\Gamma_F/q^2\Gamma_F \cong \ldots \cong q^i\Gamma_F/q^{i+1}\Gamma_F \cong \ldots \cong q^{n-1}\Gamma_F/q^n\Gamma_F.
\]
Thus, if \( r_q = 0 \), then \( \Gamma_F = q\Gamma_F = \ldots = q^n\Gamma_F \); hence, \( \Gamma_D = \Gamma_F \). If \( r_q = 1 \), the above isomorphism shows \( |\Gamma_F/q^n\Gamma_F| = q^n \). The finite Abelian \( q \)-group \( \Gamma_F/q^n\Gamma_F \) is a direct product of cyclic groups of \( q \)-power order; it is therefore
cyclic since \((\Gamma_F/q^n\Gamma_F)/q(\Gamma_F/q^n\Gamma_F) \simeq \Gamma_F/q\Gamma_F\), which is cyclic. Therefore, 
\(\Gamma_D/\Gamma_F\) is cyclic.

As \(\dim_{\mathbb{F}_q}(\Gamma_F/q\Gamma_F) = r_q\), the isomorphism \((\Gamma_F/q^n\Gamma_F)/q(\Gamma_F/q^n\Gamma_F) \simeq \Gamma_F/q\Gamma_F\) shows that the number of invariant factors of \(\Gamma_D/\Gamma_F\), which is a

subgroup of \(\Gamma_F/q^n\Gamma_F\), is at most \(r_q\).

\[\square\]

We recall the definition of a symbol algebra appearing in the next lemma. Let \(E\) be a field and \(n \geq 2\) an integer such that \(E\) contains a primitive \(n\)-th root of unity \(\omega\). For any \(a, b \in E\), consider the \(E\)-algebra \((a, b)_n\) generated by two elements \(i, j\) subject to the relations \(i^n = a, j^n = b, ij = \omega ji\). This algebra is called a symbol algebra.

**Lemma 4.3.** We keep the notations of Theorem 1.1. If \(r_q = 2\) or 3, then every nonsplit tame totally ramified central division algebra \(T\) over \(F\) of index a power of \(q\) is a symbol algebra. Moreover, \(SK_1(T) = (1)\).

**Proof.** Suppose that the degree of \(T\) is \(q^t\). By \[9\], Proposition 7.72 and Corollary 7.76], \(\overline{F}\) contains a primitive root of unity of order \(\exp(\Gamma_T/\Gamma_F)\) and the invariant factors of \(\Gamma_T/\Gamma_F\) occur in pairs, and \(\exp(T) = \exp(\Gamma_T/\Gamma_F)\).

Since \(T\) is a nonsplit tame totally ramified division algebra, \(\Gamma_T \neq \Gamma_F\). Hence, \(\Gamma_T/\Gamma_F \simeq \mathbb{Z}/q^i\mathbb{Z} \times \mathbb{Z}/q^j\mathbb{Z}\) by Lemma 4.2. Therefore, \(\exp(T) = q^t = \exp(T)\).

By \[9\], Theorem 11.23 (ii)], \(SK_1(T) \simeq \mu_{\exp(T)}(\overline{F})/\mu_{\exp(T)}(\overline{F}) = (1)\). Furthermore, \(T\) is a symbol algebra by \[9\], Proposition 7.74].

\[\square\]

We now proceed to prove the main Theorem 1.1 stated in the introduction.

**Proof of Theorem 1.1.** By [Proposition 1.2], \(cd_q(F) = cd_q(\overline{F}) + r_q\). Note that the division algebra \(D\) over the Henselian field \(F\) is tame. Indeed, we have

\[\left[ D : F \right] = \partial_{D/F} \cdot [D : \overline{F}] \cdot [\Gamma_D : \Gamma_F],\]

where \(\partial_{D/F}\) is the defect and \(\partial_{D/F} = \bar{p}^l\) for some \(l \in \mathbb{N}\) [9, cf. Theorem 4.3]. Since \(q \neq \bar{p}\) and \(D\) has the degree a power of \(q\), \(\bar{p} \nmid |\ker(\theta_D) : \Gamma_F|\), and the defect, \(\partial_{D/F} = 1\), i.e., in particular, \(D\) is defectless. Similarly, \(\bar{p} \nmid [Z(\overline{D}) : \overline{F}]\), and hence \(Z(\overline{D})/\overline{F}\) is a separable field extension. Moreover, by \[9\], Proposition 1.5 (iii)], the field extension \(Z(\overline{D})/\overline{F}\) is normal. Hence, \(Z(\overline{D})/\overline{F}\) is a Galois extension. Put \(G = \text{Gal}(Z(\overline{D})/\overline{F})\).

We first consider the case \(r_q = 1\). In this case, \(cd_q(\overline{F}) = 2\). As \(Z(\overline{D})/\overline{F}\) is a finite extension of fields, we have \(cd_q(Z(\overline{D})) = 2\) (see [5, Proposition 10]).

Thus, by Theorem 2.2, \(SK_1(\overline{D}) = (1)\). The canonical homomorphism \(\theta_D : \Gamma_D \to \text{Gal}(Z(\overline{D})/\overline{F})\) is surjective by \[9\], Proposition 1.5]. Clearly we have \(\Gamma_F \subseteq \ker(\theta_D) \subseteq \Gamma_D\). By Lemma 4.2, \(\Gamma_D/\Gamma_F\) is a cyclic group, and hence \(\Gamma_D/\ker(\theta_D) \simeq \text{Gal}(Z(\overline{D})/\overline{F})\) is a cyclic group as well. By Hilbert’s theorem 90 [5, X, §1, Corollary], \(H^{-1}(G, Z(\overline{D})^*) = (1)\). As \(\text{deg}(\overline{D})\) is a power of \(q\) and \(cd_q(Z(\overline{D})) \leq 2\), by Merkurjev-Suslin Theorem [7, Theorem 24.8], \(\text{Nrd}_{\overline{D}}(\overline{D}^*) = Z(\overline{D})^*\). Hence,

\[\hat{H}^{-1}(G, \text{Nrd}_{\overline{D}}(\overline{D}^*)) = \hat{H}^{-1}(G, Z(\overline{D})^*) = (1).\]
Since \( \ker \theta_D/\Gamma_F \) is a cyclic group, it can not carry a nondegenerate alternating pairing if it is not trivial. In view of [9, Proposition 8.17 (iv)], we have \( \ker \theta_D = \Gamma_F \). Hence, \( \zeta = 1 \) and \( SK_1(D) = (1) \) by Theorem 2.1.

Now assume that \( r_q = 2 \). We require the following result [3, cf. Corollary (c), page 69]

\[
\exp(\Gamma_D/\Gamma_F) = \exp([D^*, D^*]/[U, D^*]).
\]

By [9, Proposition 8.59], \( D \sim S \otimes_F T \in Br(F) \), where \( S \) is an inertially split central division algebra and \( T \) is a tame totally ramified central division algebra over \( F \). Since \( cd_q(F) \leq 1 \), \( Z(D) = \overline{D} \), and thus \( SK_1(D) = (1) \). By Lemma 4.1, \( \hat{H}^0(G, D^*) := F^*/N^0_{D/D}(D^*) = (1) \) and \( \hat{H}^{-1}(G, \text{Nrd}_D(D^*)) = \hat{H}^{-1}(G, \overline{D}^*) = (1) \). Therefore, \( \overline{N}(D^*) = F^* \), and \( \mu_\zeta(\overline{F}) \cap \overline{N}(D^*) = \mu_\zeta(\overline{F}) \).

Furthermore, exactness of the first row in the diagram of Theorem 2.1 implies that

\[
SK_{v,p}(D) = (1) \quad \text{and} \quad SL(D)/[U, D^*] \simeq \mu_\zeta(\overline{F}).
\]

If \( T = F \), then \( D \sim S \in Br(F) \). Hence, \( SK_1(D) \simeq SK_1(S) \). Thus, we may assume that \( D \) is an inertially split central division algebra. In this case, \( \zeta = 1 \) [9, cf. Proposition 8.64]. So by Theorem 2.1 and Equation 4.2, \( SK_1(D) = (1) \). Now assume that \( T \neq F \). By Equation 4.2, \( \zeta^2 = |\ker \theta_D : \Gamma_F| \). By [9, Theorem 8.60], \( \exp(\ker \theta_D/\Gamma_F) = \exp(\Gamma_T/\Gamma_F) \), and \( |\ker \theta_D : \Gamma_F| = |\Gamma_T : \Gamma_F| = [T : F] = q^{2t} \), where \( \deg T = q^t \). Therefore, by Lemma 4.3

\[
\zeta = q^t = \exp(\ker \theta_D/\Gamma_F) \leq \exp(\Gamma_D/\Gamma_F),
\]

and by [9, Proposition 8.17(v)], the \( \zeta \)-th primitive root of unity belongs to \( \overline{F} \). The isomorphism \( SL(D)/[U, D^*] \simeq \mu_\zeta(\overline{F}) \) implies that \( SL(D)/[U, D^*] \) is a cyclic group of exponent \( q^t \). Therefore, \( \exp([D^*, D^*]/[U, D^*]) \leq q^t \). By Equation 4.1 and Equation 4.3,

\[
\exp(\Gamma_D/\Gamma_F) = \exp([D^*, D^*]/[U, D^*]) = q^t.
\]

Therefore, \( SL(D)/[U, D^*] = [D^*, D^*]/[U, D^*] \). In view of the following exact sequence (cf. Theorem 2.1)

\[
1 \rightarrow [D^*, D^*]/[U, D^*] \rightarrow SL(D)/[U, D^*] \rightarrow SK_1(D) \rightarrow 1,
\]

we get that, \( SK_1(D) = (1) \).

If \( r_q = 3 \), then \( cd_q(\overline{F}) = 0 \) and the \( q \)-Sylow subgroup of the absolute Galois group of \( \overline{F} \) is trivial [6, cf. I.§3.3, Corollary 2]. Thus, \( \overline{F} \) does not have a proper separable field extension of degree a power of \( q \). For a defectless central division algebra \( D \) over \( F \) whose index is a power of \( q \), we thus have \( \overline{D} = Z(\overline{D}) = \overline{F} \), and \( [D : F] = [\Gamma_D : \Gamma_F] \), i.e., \( D \) is totally ramified. Hence, by Lemma 4.3, \( SK_1(D) = (1) \).

\[ \square \]

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