Complete and vertical lifts of Poisson vector fields and infinitesimal deformations of Poisson tensor

Alina Dobrogowska, Grzegorz Jakimowicz, Karolina Wojciechowicz

Institute of Mathematics, University of Białystok, Ciolkowskiego 1M, 15-245 Białystok, Poland

E-mail: alina.dobrogowska@uwb.edu.pl, g.jakimowicz@uwb.edu.pl, kzukowska@math.uwb.edu.pl

Abstract

In this paper we prove that both complete and vertical lifts of a Poisson vector field from a Poisson manifold \((M, \pi)\) to its tangent bundle \((TM, \pi_{TM})\) are also Poisson. We use this fact to describe the infinitesimal deformations of Poisson tensor \(\pi_{TM}\). We study some of their properties and present a extensive set of examples in a low dimensional case.

Keywords: Lie algebroid, linear Poisson structure, tangent and vertical lifts of vector fields, bi-Hamiltonian structure, Lie algebra, tangent lift of Poisson structure

1 Introduction

The aim of this paper is to present the extension and generalization of results obtained in the articles [8, 9]. We want to describe explicitly how to build some infinitesimal deformations of Poisson tensors generated by the algebroid structure of differential forms using complete and vertical lifts of Poisson vector fields.

The paper is organized as follow. In the Section 2 we recall such concepts, definitions and well known results from the Poisson geometry as the Poisson vector field, the Schouten–Nijenhuis bracket of vector fields, the bi-Hamiltonian structure, the Poisson cohomology, the infinitesimal deformations of Poisson tensors, see [3, 4, 7, 10, 15, 23, 27, 28, 29]. We also introduce
some notions and results of the theory of Lie algebroids including related Poisson structures on the dual bundle to the Lie algebroid. As reference to this material we recommend [4, 5, 14, 16, 19, 30].

Section 3 contains main results of the paper. Using the concept of lifting multivectors from $M$ to $TM$, see [12, 20, 21, 31], we lift Poisson vector fields from $M$ to Poisson vector fields on $TM$ considered as a dual bundle to the algebroid $T^*M$. The complete and vertical lifts of Poisson vector fields will be the building blocks for generating new Poisson structures on $TM$. These structures are compatible with the original algebroid structure and can be considered as infinitesimal deformations. Some of these structures belong to deformation of cohomology of Lie algebroid (the fiber-wise linear Poisson cohomology), see also [6]. The next part of the paper is devoted to the examples of main theorems.

2 Preliminaries and notations

We give here a short review of some basic notions and facts in Poisson geometry which will be used in the main part of the paper.

Let $(M, \pi)$ be $N$-dimensional Poisson manifold, where $\pi \in \Gamma^\infty (\wedge^2 TM)$ is a Poisson tensor. The Poisson bracket on $M$ is given by $\{f, g\} = \pi(df, dg)$ and it is a skew-symmetric bilinear mapping satisfying the Jacobi identity and the Leibniz rule. In a system of local coordinates $x = (x^1, \ldots, x^N)$ on $M$ it can be written in the form

$$\{f, g\}(x) = \sum_{i,j=1}^N \pi^{ij}(x) \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j},$$  \hfill (1)

where $\pi^{ij}(x) = -\pi^{ji}(x) = \{x^i, x^j\}$.

We denote by $\mathcal{X}(M)$ the space of smooth vector fields on a manifold $M$. The Leibniz identity means that the map $f \mapsto \{f, h\}$ is a derivation for all $h \in C^\infty(M)$. Thus, there is a unique vector field on $M$ called the Hamiltonian vector field of $h$, such that $C^\infty(M) \ni h \mapsto X_h = \{\cdot, h\} \in \mathcal{X}(M)$. It means that a Poisson bracket defines an antihomomorphism $[X_{h_1}, X_{h_2}] = -X_{\{h_1, h_2\}}$. The functions $c_i \in C^\infty(M)$ for which the Hamiltonian vector field $X_{c_i}$ vanishes identically are called Casimir functions.

Generally a vector field $X \in \mathcal{X}(M)$ on a Poisson manifold $M$ such that

$$\mathcal{L}_X \pi = 0$$  \hfill (2)
is called a Poisson vector field. It can be viewed as an infinitesimal automorphism of the Poisson structure. The Lie derivative of the bi-vector $\pi$ along $X$ is given by the formula

$$\mathcal{L}_X \pi (df, dg) = \mathcal{L}_X (\pi (df, dg)) - \pi (\mathcal{L}_X (df), dg) - \pi (df, \mathcal{L}_X (dg)), \quad (3)$$

or equivalently in the terms of the Poisson bracket

$$\mathcal{L}_X \pi (df, dg) = X (\{f, g\}) - \{X (f), g\} - \{f, X (g)\}, \quad (4)$$

where $f, g \in C^\infty (M)$. Then the condition on the vector field $X$ to be Poisson is

$$X (\{f, g\}) - \{X (f), g\} - \{f, X (g)\} = 0. \quad (5)$$

In local coordinates when $X = \sum_{i=1}^N v^i \frac{\partial}{\partial x^i}$ we rewrite it in the form

$$\sum_{s=1}^N \left( \frac{\partial \pi^{ij}}{\partial x^s} v^i - \pi^{is} \frac{\partial v^j}{\partial x^s} - \pi^{js} \frac{\partial v^i}{\partial x^s} \right) = 0. \quad (6)$$

It can be seen that Hamiltonian vector fields are always Poisson as the equation (5) reduces to the Jacobi identity. Thus it is a property of the Poisson tensor that Lie derivative of $\pi$ with respect to $X_h$ vanishes $\mathcal{L}_{X_h} \pi = 0$. The Hamiltonian vector fields are a Lie algebra ideal in the Lie algebra of Poisson vector fields.

Next, we denote by $\mathcal{X}^k (M) = \Gamma^\infty (\Lambda^k TM)$ the space of $k$-vector fields on $M$. In addition, we recall the definition of the Schouten–Nijenhuis bracket, see [22, 26], which is a bilinear map $\mathcal{X}^k (M) \times \mathcal{X}^l (M) \ni (X, Y) \mapsto [X, Y] \in \mathcal{X}^{k+l-1} (M)$ determined by the properties

$$[X, Y] = -(-1)^{(k-1)(l-1)} [Y, X], \quad X \in \mathcal{X}^k (M), Y \in \mathcal{X}^l (M), \quad (7)$$

$$[X, Y \wedge Z] = [X, Y] \wedge Z + (-1)^{(k-1)} Y \wedge [X, Z], \quad X \in \mathcal{X}^k (M),$$

$$Y \in \mathcal{X}^l (M), Z \in \mathcal{X}^p (M),$$

$$[X, Y] = \mathcal{L}_X Y, \quad X \in \mathcal{X} (M), Y \in \mathcal{X}^l (M),$$

where $[X, Y]$ is the commutator bracket of vector fields, $[X, f] = X (f)$ and $[f, g] = 0$ for $X, Y \in \mathcal{X} (M), f, g \in \mathcal{X}^0 (M) = C^\infty (M)$. The Schouten bracket satisfies the graded Jacobi identity

$$[X, [Y, Z]] = [[X, Y], Z] + (-1)^{(k-1)(l-1)} [Y, [X, Z]] \quad (8)$$
for $X \in \mathcal{X}^k(M), Y \in \mathcal{X}^l(M), Z \in \mathcal{X}^p(M)$. Thus $\bigoplus_{k=0}^{\infty} \mathcal{X}^k(M), [\cdot, \cdot]$ is a graded Lie algebra.

We say that two Poisson tensors $\pi_1$ and $\pi_2$ are compatible if the linear combination $\pi_1 + \lambda \pi_2, \lambda \in \mathbb{R}$, is again a Poisson structure. It is equivalent to the vanishing of Schouten–Nijenhuis bracket $[\pi_1, \pi_2] = 0$. In particular $[\pi, \pi] = 0$ is the Jacobi identity. The manifold $M$ equipped with two compatible Poisson structures $\pi_1, \pi_2$ is called a bi-Hamiltonian manifold, see \cite{2, 11, 17}.

The concept of a Poisson cohomology was first introduced by Li chnerowicz \cite{15}. For a Poisson tensor $\pi$ the $\mathbb{R}$-linear map $\delta_\pi : \mathcal{X}^k(M) \longrightarrow \mathcal{X}^{k+1}(M)$ defined by the Schouten–Nijenhuis bracket

$$\delta_\pi(X) = [\pi, X]$$

is a coboundary, i.e. $\delta_\pi \circ \delta_\pi = 0$. This is the consequence of the property $[\pi, [\pi, \cdot]] = 0$. The operator $\delta_\pi$ is called the Lichnerowicz–Poisson differential. This mapping generates the Lichnerowicz complex of multivector fields

$$\cdots \xrightarrow{\delta_\pi} \mathcal{X}^k(M) \xrightarrow{\delta_\pi} \mathcal{X}^{k+1}(M) \xrightarrow{\delta_\pi} \mathcal{X}^{k+2}(M) \xrightarrow{\delta_\pi} \cdots$$

The cohomology of this complex $(\mathcal{X}^*(M), \delta_\pi)$ is called Poisson cohomology. For symplectic manifolds, the Poisson cohomology is isomorphic to the de Rham cohomology. In the special case when we have linear Poisson structures the Poisson cohomology is related to Lie algebra cohomology (Chevalley–Eilenberg cohomology). The Poisson cohomology groups are denoted by $H^*_\pi(M) = \bigoplus_{k=0}^{\infty} H^k_\pi(M)$, where

$$H^k_\pi(M) = \frac{\ker (\delta_\pi : \mathcal{X}^k(M) \longrightarrow \mathcal{X}^{k+1}(M))}{\text{Im} (\delta_\pi : \mathcal{X}^{k-1}(M) \longrightarrow \mathcal{X}^k(M))}.$$  

In particular, the zero Poisson cohomology group $H^0_\pi(M)$ coincides with the ring of Casimir functions of $\pi$, i.e. $[\pi, c] = \{\cdot, c\} = 0$. The first Poisson cohomology group $H^1_\pi(M)$ is the quotient of the Lie algebra of infinitesimal symmetries (space of Poisson vector fields), i.e. $[\pi, X] = -L_X\pi = 0$, over the space of Hamiltonian vector fields, i.e. $[\pi, h] = X_h$. It is called the space of outer automorphisms of Poisson manifold. Next the Poisson cohomology group $H^2_\pi(M)$ is the quotient of the space of bi-vector fields $X$ which satisfy the condition $[\pi, X] = 0$ over the space of bi-vector fields which can be
presented in the form \( X = [\pi, Y] \), where \( Y \in \mathcal{X}(M) \). As reference to this material, we recommend [1, 6, 10, 13].

The Poisson cohomology is a useful tool in Poisson geometry, it plays an important role in deformation theory and gives some information about the geometry of the manifold. If we consider a formal one-parameter deformation of a Poisson structure \( \pi \) given by

\[
\pi_\lambda = \pi + \lambda X, \tag{12}
\]

where \( X \in \mathcal{X}^2(M) \), \( \lambda \in \mathbb{R} \), then the condition for \( \pi_\lambda \) to be a Poisson tensor gives

\[
[\pi_\lambda, \pi_\lambda] = [\pi + \lambda X, \pi + \lambda X] = 2\lambda[\pi, X] + \lambda^2[X, X] = 0. \tag{13}
\]

If \( X \in H^2_\pi(M) \) then

\[
[\pi + \lambda X, \pi + \lambda X] = \lambda^2[X, X] \tag{14}
\]

satisfies the Jacobi identity up to terms of order \( \lambda^2 \). So \( \pi + \lambda X^2 \) is called an infinitesimal deformation of Poisson tensor \( \pi \). Thus \( H^2_\pi(M) \) is interpreted as nontrivial infinitesimal deformations.

Next we recall that a Lie algebroid \((A, [, ], a)\) over manifold \( M \) is a vector bundle \( q_M : A \rightarrow M \) together with a vector bundle map \( a : A \rightarrow TM \), called the anchor, and a Lie bracket \([, ]_A\) on the space of sections \( \Gamma(A) \). A Lie bracket \([, ]_A : \Gamma(A) \times \Gamma(A) \rightarrow \Gamma(A)\) satisfies the following Leibniz rule

\[
[X_A, fY_A] = f[X_A, Y_A] + a(X_A)(f)Y_A \tag{15}
\]

for any sections \( X_A, Y_A \in \Gamma(A) \) and function \( f \in C^\infty(M) \). The anchor meets the condition

\[
a([X_A, Y_A]) = [a(X_A), a(Y_A)]. \tag{16}
\]

This notion was introduced by Pradines [24], see also [16, 19]. It is well-known that a Poisson structure on \( M \) induces the algebroid structure on the cotangent bundle \( A = T^*(M) \) of \( M \) by the property

\[
[df, dg] = d\{f, g\}, \tag{17}
\]

\[
a(df)(g) = \{f, g\} \tag{18}
\]

for \( f, g \in C^\infty(M) \), see [18].
There is a natural correspondence between Lie algebroids $A$ and Poisson structures which are linear on the fibers on the total space of the dual bundle $A^*$ of $A$. For instance, if $A = T^*(M)$

$$A = T^*M \xrightarrow{a} TM$$

then the Poisson bracket on $A^* = TM$ is given by the properties

$$\{ f \circ q_M, g \circ q_M \}_{TM} = 0,$$
$$\{ f \circ q_M, l_{dg} \}_{TM} = -a(dg)(f) \circ q_M,$$
$$\{ l_{df}, l_{dg} \}_{TM} = l_{[df, dg]}$$

for functions $f, g \in \mathcal{C}^\infty(M)$ on the base and fiber-wise linear functions $l_{df}, l_{dg} \in \mathcal{C}^\infty(TM)$, where $l_{df}$ is given by pairing

$$l_{df}(X) = \langle X, df(q_M(X)) \rangle, \quad \forall X \in TM.$$  

We denote by $y^i = l_{dx^i}$. This type of Poisson tensors is called a fiber-wise linear Poisson structure, see [10]. The Poisson tensor can be rewritten in the form

$$\pi_{TM}(x, y) = \sum_{i,j=1}^{N} \pi^{ij}(x) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial y^j} + \frac{1}{2} \sum_{i,j,s=1}^{N} \frac{\partial \pi^{ij}}{\partial x^s}(x)y^s \frac{\partial}{\partial y^i} \wedge \frac{\partial}{\partial y^j}$$

or presented graphically

$$\pi_{TM}(x, y) = \begin{pmatrix} 0 & \pi(x) \\ \pi(x) & \sum_{s=1}^{N} \frac{\partial \pi}{\partial x^s}(x)y^s \end{pmatrix},$$

where $(x, y) = (x^1, \ldots, x^N, y^1, \ldots, y^N)$ is a system of local coordinates on $TM$. Some of the properties of this Poisson structure are well known, see [8, 12]. If $c_1, \ldots, c_r$ are Casimir functions for the Poisson structure $\pi$, then the functions

$$c_i \circ q_M \quad \text{and} \quad l_{dc_i} = \sum_{s=1}^{N} \frac{\partial c_i}{\partial x^s}(x)y^s, \quad i = 1, \ldots r,$$

(26)
are Casimir functions for the Poisson tensor $\pi_{TM}$. Subsequently if the functions $\{H_i\}_{i=1}^k$ are in involution with respect to the Poisson tensor $\pi$, then the functions

$$H_i \circ q_M \quad \text{and} \quad l_{dH_i} = \sum_{s=1}^N \frac{\partial H_i}{\partial x^s} (x) y^s, \quad i = 1, \ldots, k,$$

are in involution with respect to the Poisson tensor $\pi_{TM}$ given by (25).

Cranic and Moerdijk [6] introduced the deformation cohomology for Lie algebroids. In the next section we show how to lift the Poisson cohomology on $M$ to a fiber-wise linear Poisson cohomology on $A^* = TM$.

## 3 Lift of Poisson vector fields and some infinitesimal deformations of Poisson tensors

In the beginning we recall the known facts about lift of multivector fields from manifold $M$ to $TM$, see [12, 20, 31]. Given a $k$-vector field in local coordinates

$$X = \sum_{i_1, \ldots, i_k=1}^N v^{i_1 \ldots i_k} (x) \frac{\partial}{\partial x^{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{i_k}} \in \mathcal{X}^k(M)$$

we have the following complete lift to $\mathcal{X}^k(TM)$

$$X_C = \sum_{i_1, \ldots, i_k=1}^N v^{i_1 \ldots i_k} (x) \frac{\partial}{\partial y^{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial y^{i_{i-1}}} \wedge \frac{\partial}{\partial y^{i_{i-1}}} \wedge \frac{\partial}{\partial y^{i_k}} +$$

$$\quad \quad + \sum_{s=1}^N \frac{\partial v^{i_1 \ldots i_k}}{\partial x^s} (x) y^s \frac{\partial}{\partial y^{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial y^{i_k}}.$$ (29)

The vertical lift of $X$ from $M$ to $TM$ we denote by $X_V$ and it is defined by

$$X_V = \sum_{i_1, \ldots, i_k=1}^N v^{i_1 \ldots i_k} (x) \frac{\partial}{\partial y^{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial y^{i_k}} \in \mathcal{X}^k(TM).$$ (30)
For $X \in \mathcal{X}^k(M)$ and $Y \in \mathcal{X}^l(M)$ we obtain the following commutator relations for $X_C, X_V \in \mathcal{X}^k(TM)$, $Y_C, Y_V \in \mathcal{X}^l(TM)$

\[
[X_C, Y_C] = [X, Y]_C, \quad [X_C, Y_V] = [X, Y]_V, \quad [X_V, Y_C] = 0.
\]

Using the above formulas we lift Poisson vector fields on $M$ to Poisson vector fields on $TM$.

**Theorem 1.** If $X = \sum_{i=1}^N v^i(x) \frac{\partial}{\partial x^i}$ is a Poisson vector field on a Poisson manifold $(M, \pi)$ then

\[
X_C = \sum_{i=1}^N v^i(x) \frac{\partial}{\partial x^i} + \sum_{i,s=1}^N \frac{\partial v^i}{\partial x^s}(x)y^s \frac{\partial}{\partial y^i},
\]

\[
X_V = \sum_{i=1}^N v^i(x) \frac{\partial}{\partial y^i}
\]

are Poisson vector fields on the Poisson manifold $(TM, \pi_{TM})$.

**Proof.** To check this, it is enough to check that it holds on local system of coordinates $(x^1, \ldots, x^N, y^1, \ldots, y^N)$. By direct calculation we obtain

\[
X_C \left( \{x^i, x^j\}_{TM} \right) - \{X_C(x^i), x^j\}_{TM} - \{x^i, X_C(x^j)\}_{TM} = \quad (36)
\]

\[
= -\{v^i(x), x^j\}_{TM} - \{x^i, v^j(x)\}_{TM} = 0,
\]

\[
X_C \left( \{x^i, y^j\}_{TM} \right) - \{X_C(x^i), y^j\}_{TM} - \{x^i, X_C(y^j)\}_{TM} = \quad (37)
\]

\[
= X_C \left( \pi^{ij}(x) \right) - \{v^i(x), y^j\}_{TM} - \{x^i, \sum_{s=1}^N \frac{\partial v^j}{\partial x^s}(x)y^s\}_{TM} = \quad (37)
\]

\[
= \sum_{s=1}^N \left( v^s(x) \frac{\partial \pi^{ij}}{\partial x^s}(x) - \pi^{sj}(x) \frac{\partial v^i}{\partial x^s}(x) - \pi^{is}(x) \frac{\partial v^j}{\partial x^s}(x) \right) = \quad (37)
\]

\[
= (\mathcal{L}_X \pi) (dx^i, dx^j) = 0,
\]
\[ X_C \left( \{ y^i, y^j \}_{TM} \right) - \{ X_C(y^i), y^j \}_{TM} - \{ y^i, X_C(y^j) \}_{TM} = \] 
\[ = X_C \left( \sum_{s=1}^{N} \frac{\partial \pi^{ij}}{\partial x^s}(x)y^s \right) - \left\{ \sum_{s=1}^{N} \frac{\partial v^i}{\partial x^s}(x)y^s, y^j \right\}_{TM} - \left\{ y^i, \sum_{s=1}^{N} \frac{\partial v^j}{\partial x^s}(x)y^s \right\}_{TM} = \] 
\[ = \sum_{m,s=1}^{N} y^m \frac{\partial}{\partial x^m} \left( v^s(x) \frac{\partial \pi^{ij}}{\partial x^s}(x) - \pi^{sj}(x) \frac{\partial v^i}{\partial x^s}(x) - \pi^{is}(x) \frac{\partial v^j}{\partial x^s}(x) \right) = \] 
\[ = \sum_{m,s=1}^{N} y^m \frac{\partial}{\partial x^m} \left( (L_{X\pi}) (dx^i, dx^j) \right) = 0. \] 

Performing the similar calculation for \( X_V \)
\[ X_V \left( \{ x^i, x^j \}_{TM} \right) - \{ X_V(x^i), x^j \}_{TM} - \{ x^i, X_V(x^j) \}_{TM} = 0 \] 
\[ = X_V \left( \pi^{ij}(x) \right) - \{ x^i, v^j(x) \}_{TM} = 0, \]
\[ X_V \left( \{ y^i, y^j \}_{TM} \right) - \{ X_V(y^i), y^j \}_{TM} - \{ y^i, X_V(y^j) \}_{TM} = \] 
\[ = X_V \left( \sum_{s=1}^{N} \frac{\partial \pi^{ij}}{\partial x^s}(x)y^s \right) - \left\{ v^i(x), y^j \right\}_{TM} - \left\{ y^i, v^j(x) \right\}_{TM} = \] 
\[ = \sum_{s=1}^{N} \left( v^s(x) \frac{\partial \pi^{ij}}{\partial x^s}(x) - \pi^{sj}(x) \frac{\partial v^i}{\partial x^s}(x) - \pi^{is}(x) \frac{\partial v^j}{\partial x^s}(x) \right) = \] 
\[ = \sum_{s=1}^{N} (L_{X\pi}) (dx^i, dx^j) = 0. \] 

We see that these are also Poisson vector fields on \((TM, \pi_{TM})\). \( \square \)

Above, the first vector field, given by (34), is a fiber–wise linear vector field and the second, given by (35), is a fiber–wise constant vertical vector field.

**Theorem 2.** If \( X_C, X_V \) are complete and vertical lifts of vector field \( X \in \mathcal{X}(M) \), then the bi-vector
\[ \pi_{X_C, X_V} = X_C \wedge X_V \] 
is a Poisson tensor on \( TM \).
Proof. It is easy to see, from definition, that (42) is antisymmetric. If so, it is enough to check if the Jacobi identity holds. Direct calculation, using properties of the Schouten–Nijenhuis bracket given by (7) yields

$$
[\pi_{X_C,X_V},\pi_{X_C,X_V}] = [X_C \wedge X_V, X_C \wedge X_V] =
[X_C \wedge X_V, X_C] \wedge X_V - X_C \wedge [X_C \wedge X_V, X_V] =
- [X_C, X_C \wedge X_V] \wedge X_V + X_C \wedge [X_V, X_C \wedge X_V] =
- [X_C, X_C] \wedge X_V \wedge X_V - X_C \wedge [X_C, X_V] \wedge X_V +
[2X_C \wedge [X_C, X_V] \wedge X_V = 2 [X, X]_V \wedge X_C \wedge X_V = 0.
$$

Then $[\pi_{X_C,X_V},\pi_{X_C,X_V}] = 0$. It means that $\pi_{X_C,X_V}$ is a Poisson tensor. □

In a local system of coordinates $(x^1, \ldots, x^N, y^1, \ldots, y^N)$ the matrix of the Poisson tensor has the form

$$
\pi_{X_C,X_V}(x,y) = \begin{pmatrix} 0 & v(x)v^\top(x) \\ -v(x)v^\top(x) & \sum_{s=1}^N \left( \frac{\partial v}{\partial x^s}(x)v^\top(x) - v(x) \left( \frac{\partial v}{\partial x^s}(x) \right)^\top \right) y^s \end{pmatrix}, \quad (44)
$$

where $v^\top = (v^1, \ldots, v^N)$.

Note also that if $X$ is a Poisson vector field on a Poisson manifold $(M, \pi)$, then the Lie derivative of the Casimir function $c_i$ is again a Casimir function, i.e.

$$
\mathcal{L}_X c_i = X(c_i) = c_j. \quad (45)
$$

In the case when we get zero, we have the following theorem.

**Theorem 3.** Let $X$ be a vector field on $M$ and let $f$ be smooth function on $M$. If $X(f) = 0$ then for complete and vertical lifts of vector field $X$ we have

$$
X_{C}(f \circ q_M) = 0, \quad X_{C}(l_{df}) = 0, \quad (46)
$$

$$
X_{V}(f \circ q_M) = 0, \quad X_{V}(l_{df}) = 0. \quad (47)
$$

Proof. In the beginning we assume that for a initial vector field $X = \sum_{i=1}^N v^i \frac{\partial}{\partial x^i}$
we have $X(f) = \sum_{i=1}^{N} v^i \frac{\partial f}{\partial x^i} = 0$. Next, a direct verification shows that

$$X_C(f \circ q_M) = X(f) = 0, \quad (48)$$

$$X_V(f \circ q_M) = \sum_{s=1}^{N} y^s \frac{\partial}{\partial x^s} X(f) = 0, \quad (49)$$

$$X_V(f \circ q_M) = 0, \quad (50)$$

$$X_V(l_{df}) = X(f) = 0. \quad (51)$$

Note that from the above theorem follows that if $X(f) = 0$ then $f$ is a Casimir function for a Poisson tensor $\pi_{X_C, X_V}$.

**Theorem 4.** Let $(M, \pi)$ be a Poisson manifold and let $c \in H^0_\pi(M)$ be a Casimir function for $\pi$. For each Poisson vector field $X$ on $M$

$$\pi_{TM, X_C, X_V, c} = \pi_{TM} + \lambda c(x) \pi_{X_C, X_V} \quad (52)$$

is a Poisson tensor on $TM$.

**Proof.** From (7) we know that

$$[[\pi_{TM, X_C, X_V, c}, \pi_{TM, X_C, X_V, c}], \pi_{TM, X_C, X_V, c}] = [[\pi_{TM, X_C, X_V, c}, \pi_{TM}], \pi_{TM}] + \lambda [c(x)X_C \wedge X_V, \pi_{TM}] + \lambda^2 [c^2(x) [X_C \wedge X_V, X_C \wedge X_V] =$$

$$= 2\lambda c(x) [\pi_{TM, X_C} \wedge X_V - 2\lambda c(x) X_C \wedge [\pi_{TM}, X_V] = 0. \quad (53)$$

Above we use that $\pi_{TM}$ and $X_C \wedge X_V$ are Poisson tensors and $X_C, X_V$ are Poisson vector fields on $(TM, \pi_{TM})$.

This is a consequence of the facts that both bi-vector fields are Poisson tensors and $\pi_{X_C, X_V} \in H^2_{\pi_{TM}}(TM)$. In local coordinates $(x^1, \ldots, x^N, y^1, \ldots, y^N)$ we get the following infinitesimal deformation of the Poisson tensor $\pi_{TM}$

$$\pi_{TM, X_C, X_V, c}(x, y) = \left(\begin{array}{c}
\pi(x) + \lambda c(x)v(x)v(x) \\
\pi(x) - \lambda c(x)v(x)v(x) \\
\sum_{s=1}^{N} \left( \frac{\partial \pi}{\partial x^s}(x) + \lambda c(x) \left( \frac{\partial v}{\partial x^s}(x)v(x) - v(x) \left( \frac{\partial v}{\partial x^s}(x) \right) \right) \right) y^s \end{array}\right). \quad (54)$$

This type of Poisson structures has already appeared in our article [9], where we described in detail the algebroid structure associated with it. The next statement describes the case that Casimir functions for a certain class of tensors do not change.
Theorem 5. Let $c_1, \ldots, c_r$ be Casimir functions for the Poisson structure $\pi$ such that $\mathcal{L}_{X}c_i = 0$. Then the functions

$$c_i \circ q_M \quad \text{and} \quad l_{dc_i} = \sum_{s=1}^{N} \frac{\partial c_i}{\partial x_s}(x)y_s, \quad i = 1, \ldots, r,$$

(55)

are the Casimir functions for the Poisson tensor $\pi_{TM, X, Y, V}$.  

Proof. This is the consequence of Theorem 3 and formula (45). \qed

In the next step, we will consider a situation a little more general. We will assume that we have two non-proportional Poisson vector fields $X, Y \in \mathcal{X}(M)$ on a Poisson manifold $(M, \pi)$ expressed in local coordinates as

$$X = \sum_{i=1}^{N} v^i(x) \frac{\partial}{\partial x^i}, \quad Y = \sum_{i=1}^{N} w^i(x) \frac{\partial}{\partial x^i},$$

(56)

where $v^i, w^i \in C^\infty(M)$. Then we obtain four Poisson vector fields on a Poisson manifold $(TM, \pi_{TM})$

$$X_C = \sum_{i=1}^{N} v^i(x) \frac{\partial}{\partial x^i} + \sum_{i,s=1}^{N} \frac{\partial v^i}{\partial x^s}(x)y^s \frac{\partial}{\partial y^i}, \quad Y_C = \sum_{i=1}^{N} w^i(x) \frac{\partial}{\partial x^i} + \sum_{i,s=1}^{N} \frac{\partial w^i}{\partial x^s}(x)y^s \frac{\partial}{\partial y^i},$$

$$X_V = \sum_{i=1}^{N} v^i(x) \frac{\partial}{\partial y^i}, \quad Y_V = \sum_{i=1}^{N} w^i(x) \frac{\partial}{\partial y^i},$$

(57)

In this case, we can build three different types of bi-vector fields. A detailed analysis of these cases will be presented in the following statements.

Theorem 6. If $X_V$ and $Y_V$ are given by (57) then

$$\pi_{X_V, Y_V} = X_V \wedge Y_V = (X \wedge Y)_V$$

(58)

is a Poisson tensor on $TM$.

Proof. A direct calculation gives us

$$[\pi_{X_V, Y_V}, \pi_{X_V, Y_V}] = 2[Y_V, X_V] \wedge X_V \wedge Y_V = 0$$

(59)

from (7) and (33). \qed
This is a consequence of simple observation that any structure of bi-vector field with a matrix form
\[
\begin{pmatrix}
0 & 0 \\
0 & A(x)
\end{pmatrix},
\] (60)
where \(A(x) \in \mathfrak{so}(N)\), is a Poisson tensor on \(TM\). Moreover, the Poisson tensor (58) is compatible with the Poisson tensor \(\pi_{TM}\).

**Theorem 7.** Let \((M, \pi)\) be a Poisson manifold and let \(c \in H^0_{\pi}(M)\) be a Casimir function for \(\pi\). For any Poisson vector fields \(X, Y\) on \(M\)
\[
\pi_{TM,XV,YV,c} = \pi_{TM} + \lambda c(x) X \wedge Y
\] (61)
is a Poisson tensor on \(TM\).

**Proof.** The statement follows from the facts that \(\pi_{TM}, \pi_{XV,YV}\) are the Poisson tensors and the bi-vector field (58) belongs to the second Poisson cohomology group \(\pi_{XV,YV} \in H^2_{\pi_{TM}}(TM)\), i.e. \([\pi_{TM}, X \wedge Y] = 0\). □

The local expression of (61) is given by
\[
\pi_{TXV,YV,c}(x, y) = \begin{pmatrix}
0 \\
\frac{\pi(x)}{\pi(x)} \sum_{s=1}^N \frac{\partial \pi}{\partial x^s}(x) y^s + \lambda c(x) \left( v(x) w^\top(x) - w(x) v^\top(x) \right)
\end{pmatrix} \quad (62)
\]
where \(v^\top = (v^1, \ldots, v^N)\) and \(w^\top = (w^1, \ldots, w^N)\).

Additionally if \(X \wedge Y\) is also a Poisson tensor on \(M\) then this is the lift of bi-Hamiltonian structure \((M, \pi, X \wedge Y)\) to \(TM\), see [8]. This is the case if the following condition is satisfied
\[
[X \wedge Y, X \wedge Y] = 2[X, Y] \wedge X \wedge Y = (63)
\]
\[
= \sum_{i,j,k,n=1}^N v^i(x) w^k(x) \left( v^n(x) \frac{\partial w^i}{\partial x^n}(x) - w^n(x) \frac{\partial v^i}{\partial x^n}(x) \right) \frac{\partial}{\partial x^j} \wedge \frac{\partial}{\partial x^j} \wedge \frac{\partial}{\partial x^k} = 0.
\]

From the equality for Lichnerowicz–Poisson differential \(\delta_{\pi_{TM}} ((X \wedge Y)_V) = (\delta_{\pi} (X \wedge Y))_V\) the map \(H^2_\pi(M) \ni X \wedge Y \mapsto (X \wedge Y)_V \in H^2_{\pi_{TM}}(TM)\) is a homomorphism of Poisson cohomology space, see [20].

**Theorem 8.** 1. Let \(X_i \in \mathcal{X}(M)\) for \(i = 1, 2, 3, 4\) and let \(X_{iV}\) be the vertical lifts of the vector fields \(X_i\) on \(TM\). Then
\[
\pi_{X_{1V},X_{2V},X_{3V},X_{4V}} = X_{1V} \wedge X_{2V} + X_{3V} \wedge X_{4V}
\] (64)
is the Poisson tensor on \(TM\).
2. Let \((M, \pi)\) be a Poisson manifold and let \(c \in H^0_\pi(M)\) be a Casimir function for \(\pi\). For any Poisson vector fields \(X_i\) for \(i = 1, 2, 3, 4\) on \(M\)

\[
\pi_{TM,X_1,V,X_2,V,X_3,V,X_4,V,c} = \pi_{TM} + \lambda c(x) (X_1,V \wedge X_2,V + X_3,V \wedge X_4,V)
\]

is a Poisson tensor on \(TM\).

**Proof.** Proof is obtained by direct calculation.

**Remark:** The above procedure can be repeated many times.

**Theorem 9.** Let \(X, Y \in \mathcal{X}(M)\) be such that \([X, Y] = 0\) and let \(X_C, Y_C, Y_V\) be the complete and vertical lifts of the vectors \(X, Y\) on \(TM\), respectively. Then

\[
\pi_{X_C,Y_V} = X_C \wedge Y_V,
\]

\[
\pi_{X_C,Y_C} = X_C \wedge Y_C
\]

are Poisson tensors on \(TM\).

**Proof.** After direct calculation of the Jacobi identity using the Schouten–Nijenhuis bracket we obtain

\[
[\pi_{X_C,Y_V}, \pi_{X_C,Y_V}] = 2[X_C, Y_V] \wedge X_C \wedge Y_V = 2[X, Y]_V \wedge X_C \wedge Y_V = 0.
\]

Similarly for the second construction

\[
[\pi_{X_C,Y_C}, \pi_{X_C,Y_C}] = 2[X_C, Y_C] \wedge X_C \wedge Y_C = 2[X, Y]_C \wedge X_C \wedge Y_C = 0.
\]

**Theorem 10.** Let \((M, \pi)\) be a Poisson manifold and let \(c \in H^0_\pi(M)\) be a Casimir function for \(\pi\). For any Poisson vector fields \(X, Y\) on \(M\) such that \([X, Y] = 0\)

\[
\pi_{TM,X_C,Y_V,c} = \pi_{TM} + \lambda c(x) X_C \wedge Y_V,
\]

\[
\pi_{TM,X_C,Y_C,c} = \pi_{TM} + \lambda c(x) X_C \wedge Y_C
\]

are the Poisson tensors on \(TM\).
Proof. By calculation of the Schouten–Nijenhuis bracket we obtain
\[ [\pi_{TM,X_C,Y_V,c}, \pi_{TM,X_C,Y_V,c}] = 2\lambda_c(x)[\pi_{TM}, X_C \wedge Y_V] = 0, \]
(72)
because \( X_C \) and \( Y_V \) are Poisson vector fields for \( \pi_{TM} \). The proof for the second bi-vector is completely analogous. \( \square \)

In the local coordinates expressions of the Poisson structures introduced in Theorem 10 are the following

\[ \pi_{TM,X_C,Y_V,c}(x, y) = \]
(73)

\[
\begin{pmatrix}
0 & \pi(x) + \lambda_c(x)v(x)w^\top(x) \\
\pi(x) - \lambda_c(x)v(x)w^\top(x) & \sum_{s=1}^N \left( \frac{\partial \pi}{\partial x^s}(x) + \lambda_c(x) \left( \frac{\partial v}{\partial x^s}(x) w^\top(x) - w(x) \left( \frac{\partial v}{\partial x^s}(x) \right)^\top \right) \right) y^s
\end{pmatrix}
\]

\[ \pi_{TM,X_C,Y_V,c}(x, y) = \]
(74)

\[
\begin{pmatrix}
\pi(x) - \lambda_c(x) \sum_{s=1}^N \left( \frac{\partial \pi}{\partial x^s}(x) w^\top(x) - w(x) \left( \frac{\partial v}{\partial x^s}(x) \right)^\top \right) y^s & \pi(x) + \lambda_c(x) \sum_{s=1}^N \left( \frac{\partial \pi}{\partial x^s}(x) w^\top(x) - w(x) \left( \frac{\partial v}{\partial x^s}(x) \right)^\top \right) y^s \\
\pi(x) - \lambda_c(x) \sum_{s=1}^N \left( \frac{\partial \pi}{\partial x^s}(x) w^\top(x) - w(x) \left( \frac{\partial v}{\partial x^s}(x) \right)^\top \right) y^s & \pi(x) + \lambda_c(x) \sum_{s=1}^N \left( \frac{\partial \pi}{\partial x^s}(x) w^\top(x) - w(x) \left( \frac{\partial v}{\partial x^s}(x) \right)^\top \right) y^s
\end{pmatrix}
\]

Theorem 11. Let \( c_1, \ldots, c_r \) be Casimir functions for the Poisson structure \( \pi \) such that \( \mathcal{L}_{X_c} = 0, \mathcal{L}_{Y_c} = 0 \). Then the functions

\[ c_i \circ \varphi_M \quad \text{and} \quad l_{de_i} = \sum_{s=1}^N \frac{\partial c_i}{\partial x_s}(x) y_s, \quad i = 1, \ldots, r, \]
(75)

are the Casimir functions for the Poisson tensors \( \pi_{TM,X,V,c}, \pi_{TM,X,C,Y_V,c} \) and \( \pi_{TM,X,C,Y_V,c} \).

Proof. This is the consequence of Theorem 10 and formula (75). \( \square \)

In addition, \( c_i \circ \varphi_M \) is always a Casimir function for the Poisson tensor \( \pi_{TM,X,Y_V,c} \).

Those procedures can be repeated many times, with certain assumptions, which gives the following theorems.

Theorem 12. Let \( X_i \in \mathcal{X}(M) \) for \( i = 1, 2, 3, 4 \) and let \( X_{i,C}, X_{i,V} \) be the complete and vertical lifts of the vectors \( X_i \) on \( TM \).

1. If \( [X_1, X_4] = 0, [X_2, X_3] = 0, [X_3, X_4] = 0, \) then

\[ \pi_{X_{1,C},X_{2,V},X_{3,C},X_{4,V}} = X_{1,C} \wedge X_{2,V} + X_{3,C} \wedge X_{4,V} \]
(76)
is the Poisson tensor on \( TM \).
2. If \([X_i, X_j] = 0\) for \(i, j = 1, 2, 3, 4\), then
\[
\pi_{X_1,C,X_2,C,X_3,C,X_4,V} = X_1,C \wedge X_2,C + X_3,C \wedge X_4,V \tag{77}
\]
\[
\pi_{X_1,C,X_2,C,X_3,C,X_4,C} = X_1,C \wedge X_2,C + X_3,C \wedge X_4,C \tag{78}
\]
are the Poisson tensors on \(T M\).

3. If \([X_1, X_i] = 0, [X_2, X_i] = 0\), then
\[
\pi_{X_1,C,X_2,C,X_3,V,X_4,V} = X_1,C \wedge X_2,C + X_3,V \wedge X_4,V \tag{79}
\]
is the Poisson tensor on \(T M\).

4. If \([X_1, X_i] = 0\), then
\[
\pi_{X_1,C,X_2,V,X_3,V,X_4,V} = X_1,C \wedge X_2,V + X_3,V \wedge X_4,V \tag{80}
\]
is the Poisson tensor on \(T M\).

**Proof.** Our proof starts with observation that
\[
[X \wedge Y, Z \wedge W] = [X, Z] \wedge Y \wedge W + [Z, Y] \wedge X \wedge W + [X, W] \wedge Z \wedge Y + [W, Y] \wedge Z \wedge X \tag{81}
\]
for \(X, Y, Z, W \in \mathcal{X}(TM)\). Applying this equality for all above bi-vector field cases, we get our conclusions.  

The above theorem gives the restrictive conditions for these structures to be bi–Hamiltonian. Moreover we have the following corollary.

**Corollary 1.** If the bi-vectors \([76, 80]\) are Poisson tensors and \(X_i\) for \(i = 1, 2, 3, 4\) are Poisson vector fields on \(M\) then Poisson tensor \(\pi_{TM}\) is compatible with them.

### 4 Examples

Let us take \(\mathbb{R}^3\) with local coordinates \(x = (x^1, x^2, x^3)\) and let us consider the linear Poisson structure given by the Poisson tensor
\[
\pi(x) = x^1 \frac{\partial}{\partial x^2} \wedge \frac{\partial}{\partial x^3}, \tag{82}
\]
which equivalently can be written in the following form

\[
\pi(x) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & x^1 \\ 0 & -x^1 & 0 \end{pmatrix}.
\] (83)

Linear Poisson structure given above is related to the Lie algebra \(A_{3,1}\). The commutation rule for this Lie algebra is \([e_2, e_3] = e_1\) and it has only one invariant which is \(e_1\), see [25]. In this case the Casimir function for \(\pi\) assumes following form

\[c_1(x) = x^1.\]

It is easy to see that a Poisson vector field in this case is given by

\[X = x^1 \left( \frac{\partial v^2}{\partial x^2}(x) + \frac{\partial v^3}{\partial x^3}(x) \right) \frac{\partial}{\partial x^1} + v^2(x) \frac{\partial}{\partial x^2} + v^3(x) \frac{\partial}{\partial x^3},\] (84)

where \(\frac{\partial v^2}{\partial x^2}(x) + \frac{\partial v^3}{\partial x^3}(x) = f(x^1)\) and \(f\) is an arbitrary function of one variable.

We can lift the Poisson tensor on \(\mathbb{R}^3\) to \(T\mathbb{R}^3\), then

\[
\pi_{TM}(x, y) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x^1 \\ 0 & 0 & 0 & 0 & -x^1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x^1 & 0 & 0 & y^1 \\ 0 & -x^1 & 0 & 0 & -y^1 & 0 \end{pmatrix},
\] (85)

where \((x, y) = (x^1, x^2, x^3, y^1, y^2, y^3)\). The Casimir functions for this structure are given by \(c_1(x, y) = x^1\), \(c_2(x, y) = y^1\). We can also see that this is a Lie–Poisson structure associated with Lie algebra \(A_{6,4}\), for which commutation rules are \([e_1, e_2] = e_5, [e_1, e_3] = e_4, [e_2, e_4] = e_6\) and \((x^1, x^2, x^3, y^1, y^2, y^3) \mapsto (e_6, -e_4, e_5, e_1, e_2)\), see [8, 25].

Now we present the list of some infinitesimal deformations of the Poisson tensor \(\pi_{TM}\) given in (85) through the choice of different Poisson vector fields.

1. Let us now take as a Poisson vector field

\[X = \sqrt{x^3} \frac{\partial}{\partial x^2}\] (86)

and put \(\lambda c(x) = 1\). Then the complete and vertical lifts are given by

\[X_C = \sqrt{x^3} \frac{\partial}{\partial x^2} + \frac{y^3}{2 \sqrt{x^3}} \frac{\partial}{\partial y^2}, \quad X_V = \sqrt{x^3} \frac{\partial}{\partial y^2}\] (87)

17
The Poisson tensor described by Theorem 4, which in local coordinates can be written as in the \((54)\), is given by

\[
\pi_{TM,x,c,x,v,c}(x,y) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & x^3 & x^1 \\
0 & 0 & 0 & 0 & -x^1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & -x^3 & x^1 & 0 & 0 & y^1 \\
0 & -x^1 & 0 & 0 & -y^1 & 0
\end{pmatrix},
\]

for \(\lambda = 1\). By direct calculation and changing the variables we can prove that this is a tensor for the Lie–Poisson structure associated with the Lie algebra \(A_{6,6}\) from the classification given in [25]. Commutation relations for this Lie algebra are

\[
\begin{align*}
[e_1, e_2] &= e_6, \\
[e_1, e_3] &= e_4, \\
[e_1, e_4] &= e_5,
\end{align*}
\]

and

\[
\begin{align*}
[e_2, e_3] &= e_5,
\end{align*}
\]

Moreover the Casimir functions for structure \(\pi_{TM,x,c,x,v,c}\) are \(c_1(x) = x^1\) and \(c_2(x,y) = l_{dx^1} = y^1\) from Theorem 5.

2. Let us now take Poisson vector fields

\[
X = x^1 \frac{\partial}{\partial x^1} + x^2 \frac{\partial}{\partial x^2}, \quad Y = \frac{\partial}{\partial x^3},
\]

Then their vertical lifts are of the form

\[
X_V = x^1 \frac{\partial}{\partial y^1} + x^2 \frac{\partial}{\partial y^2}, \quad Y_V = \frac{\partial}{\partial y^3}
\]

and bi-vector can be expressed as

\[
X_V \wedge Y_V = x^1 \frac{\partial}{\partial y^1} \wedge \frac{\partial}{\partial y^2} + x^2 \frac{\partial}{\partial y^2} \wedge \frac{\partial}{\partial y^3}.
\]

Then by taking as above \(\lambda c(x) = 1\) and considering Poisson tensor \(\pi\) we get, from Theorem 7, that \(\pi_{TM} + X_V \wedge Y_V\) is a Poisson tensor and it is given by following matrix

\[
\pi_{TM,x_v,y_v,c}(x,y) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & x^1 & 0 \\
0 & 0 & 0 & 0 & -x^1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & x^1 & 0 & 0 & 0 & y^1 + x^2 \\
0 & -x^1 & 0 & -y^1 - x^2 & 0 & 0
\end{pmatrix}.
\]
If we take mapping \((x^1, x^2, x^3, y^1, y^2, y^3) \mapsto (2x^1, y^1 + x^2, 2x^3, y^1 - x^2, y^2, y^3) \mapsto (e_1, e_2, e_4, e_6, e_3, e_5)\) then we can recognize that above tensor is a Poisson tensor for Lie–Poisson structure related to direct sum \(A_{5,5} \oplus \langle e_6 \rangle\) for which commutation rules are given by \([e_3, e_4] = e_1, [e_2, e_5] = e_1, [e_3, e_5] = e_2\). Furthermore the Casimir functions are \(c_1(x) = x^1, c_2(x, y) = y^1 - x^2\).

3. Let us take now four Poisson vector fields

\[
X_1 = x^1 \frac{\partial}{\partial x^1} + x^3 \frac{\partial}{\partial x^3}, \quad Y_1 = -\frac{x^2}{x^1} \frac{\partial}{\partial x^3},
\]

\[
X_2 = x^1 \frac{\partial}{\partial x^2}, \quad Y_2 = \frac{\partial}{\partial x^3}.
\]

Then we can lift them vertically to Poisson vector fields on \(T\mathbb{R}^3\) and get

\[
X_{1,V} = x^1 \frac{\partial}{\partial y^1} + x^3 \frac{\partial}{\partial y^3}, \quad Y_{1,V} = -\frac{x^2}{x^1} \frac{\partial}{\partial y^3},
\]

\[
X_{2,V} = x^1 \frac{\partial}{\partial y^2}, \quad Y_{2,V} = \frac{\partial}{\partial y^3}.
\]

Then by taking as above \(\lambda c(x) = 1\) and considering Poisson tensor \(\pi\) we get, from Theorem [3] that \(\pi_{TM} + X_{1,V} \wedge Y_{1,V} + X_{2,V} \wedge Y_{2,V}\) is a Poisson tensor and it is of the form

\[
\pi_{TM,X_{1,V},Y_{1,V},X_{2,V},Y_{2,V},c}(x, y) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & x^1 \\
0 & 0 & 0 & 0 & -x^1 & 0 \\
0 & 0 & 0 & 0 & -x^2 & 0 \\
0 & 0 & -x^1 & 0 & 0 & y^1 + x^1 \\
0 & -x^1 & 0 & x^2 & -y^1 - x^1 & 0
\end{pmatrix}.
\]

It is easy to see that it is a Poisson tensor for Lie–Poisson structure related to the Lie algebra \(A_{6,17}\) by taking the mapping \((x^1, x^2, x^3, y^1, y^2, y^3) \mapsto (x^1, -x^2, x^3, -y^1, y^2 - \frac{1}{2}x^2, y^3 - \frac{1}{2}x^3) \mapsto (e_6, e_4, e_5, e_3, e_2, e_1)\). Commutation relation for Lie algebra \(A_{6,17}\) are \([e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = e_6\) and \([e_2, e_5] = e_6\) and Casimir functions are \(c_1(x) = x^1, c_2(x, y) = (x^2)^2 + 2y^1x^1\).
4. Let us now consider the Poisson vector fields

\[ X = x^3 \frac{\partial}{\partial x^2} + x^1 \frac{\partial}{\partial x^3}, \quad Y = \frac{\partial}{\partial x^2} \]  

and let us put \( \lambda c(x) = 1 \). Then from (57) we get complete and vertical lifts of the vector fields on \( T\mathbb{R}^3 \), given by

\[ X_C = x^3 \frac{\partial}{\partial x^2} + x^1 \frac{\partial}{\partial x^3} + y^3 \frac{\partial}{\partial y^2} + y^1 \frac{\partial}{\partial y^3}, \quad Y_V = \frac{\partial}{\partial y^2}. \]  

It is easy to see that \([X, Y] = 0\) is a Poisson tensor. Then from Theorem 10 we get that \( \pi_T M + X_C \wedge Y_V \) is also a Poisson tensor and it is of the form

\[ \pi_{T M, X_C, Y_V, c}(x, y) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x^3 & x^1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -x^3 & 0 & 0 & 0 & 0 \\ 0 & -x^1 & 0 & 0 & 0 & 0 \end{pmatrix}. \]  

By direct calculation and changing the variables we can prove that this is a tensor for a Lie–Poisson structure related to direct sum \( A_{5,1} \oplus \langle e_6 \rangle \). Commutation rules for Lie algebra \( A_{5,1} \) are \([e_3, e_5] = e_1\) and \([e_4, e_5] = e_2\) where \((x_1, x_2, x_3, y_1, y_2, y_3) \mapsto (e_1, e_5, e_2, e_6, -e_4, -e_3)\). Moreover the Casimir functions for structure \( \pi_{T M, X_1, X_2, Y_2, X_3, Y_3, c} \) are \( c_1(x) = x_1, c_2(x) = x_3 \) and \( c_3(x, y) = x_1 y_2 - x_3 y_3 \).

5. Let us take now four Poisson vector fields

\[ X_1 = \frac{\partial}{\partial x^3}, \quad Y_1 = x^1 \frac{\partial}{\partial x^2}, \]  

\[ X_3 = \frac{\partial}{\partial x^2}, \quad Y_2 = x^3 \frac{\partial}{\partial x^2}. \]  

Then from (57) we can lift them to Poisson vector fields on \( T\mathbb{R}^3 \) and get

\[ X_{1,C} = \frac{\partial}{\partial x^3}, \quad Y_{1,V} = x^1 \frac{\partial}{\partial y^2}; \]  

\[ X_{2,C} = \frac{\partial}{\partial x^2}, \quad Y_{2,V} = x^3 \frac{\partial}{\partial y^2}. \]
Then by taking as above $\lambda c(x) = 1$ and considering Poisson tensor $\pi$ we get, from Corollary 1, that $\pi_{TM} + X_{1,C} \wedge Y_{1,V} + X_{2,C} \wedge Y_{2,V}$ is a Poisson tensor and it is of the form

$$
\pi_{TM,X_{1,C},Y_{1,V},X_{2,C},Y_{2,V},c}(x, y) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & x^3 & x^1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & -x^3 & 0 & 0 & y^1 & 0 \\
0 & -x^1 & 0 & 0 & -y^1 & 0 \\
\end{pmatrix}.
$$

By changing the variables and direct calculation we can recognize that this is a Poisson tensor for a Lie–Poisson structure associated with the Lie algebra $A_{6,3}$. From [25], commutation rules for this Lie algebra are $[e_1, e_2] = e_6, [e_1, e_3] = e_4$ and $[e_2, e_3] = e_5$, where $(x^1, x^2, x^3, y^1, y^2, y^3) \mapsto (e_1, e_2, e_3, e_4, e_5, e_6)$ and Casimir functions are $c_1(x) = x^1, c_2(x) = x^3, c_3(y) = y^1, c_4(x, y) = x^2 y^1 + x^3 y^3 - x^1 y^2$.

**Acknowledgments**

This article has received financial support from the Polish Ministry of Science and Higher Education under subsidy for maintaining the research potential of the Faculty of Mathematics and Informatics, University of Bialystok (BST-148).

**References**

[1] M. Ammar, G. Kass, N. Poncin, *The structure of Poisson cohomology*, Universitatis Iagellonicae Acta Mathematica, Fasciculus XLVII, 2009.

[2] A.V. Bolsinov, A.V. Borisov, *Compatible Poisson brackets on Lie algebras*, Mat. Zametki, 72(1), 11-34, 2002.

[3] A.V. Borisov, I.S. Mamaev, *Poisson Structures and Lie Algebras in Hamiltonian Mechanics*, Izhevsk: Izd. UdSU, 1999.

[4] A. Cannas da Silva, A. Weinstein, *Geometric models for noncommutative algebras*, Berkeley Math. Lecture Notes, Amer. Math. Soc., 1999.
[5] T. Courant, *Tangent Lie algebroids*, J. Phys. A: Math. Gen., 27, 4527-4536, 1994.

[6] M. Crainic; I. Moerdijk, *Deformations of Lie brackets: cohomological aspects*, Journal of the European Mathematical Society 010.4 (2008): 1037-1059.

[7] M. Crainic, Rui Fernandes, *Lectures on integrability of Lie brackets*, in Lectures on Poisson Geometry: Proceedings of the Summer School on Poisson Geometry, ICTP, Trieste, 2005, Geometry and Topology Monographs Series, editors: T.S. Ratiu, A. Weinstein, N.T. Zung, Vol. 17, 2011.

[8] A. Dobrogowska, G. Jakimowicz, *Tangent lifts of bi-Hamiltonian structures*, J. Math. Phys., 58, 083505, 2017.

[9] A. Dobrogowska, G. Jakimowicz, K. Wojciechowicz, *Deformation of algebroid bracket of differential forms and Poisson manifold*, arXiv:1806.08142, 2018.

[10] J-P. Dufour, N.T. Zung, *Poisson Structures and Their Normal Forms*, Birkhäuser Verlag, 2005.

[11] R.L. Fernandes, *Completely Integrable Bi-Hamiltonian Systems*, Journal of Dynamics and Differential Equation, 6, No.1, 53-69, 1994.

[12] J. Grabowski, P. Urbanski, *Tangent lifts of Poisson and related structures*, J. Phys. A: Math. Gen., 28, 6743-6777, 1995.

[13] W. Hong, *Poisson Cohomology of holomorphic toric Poisson manifolds. I.*, arXiv:1611.08485v3.

[14] M. Karasev, *Analogaes of the objects of Lie group theory for nonlinear Poisson brackets*, Math. USSR Izvest., 28, 497-527, 1987.

[15] A. Lichnerowicz, *Les variétés de Poisson et leurs algèbres de Lie associées*, J. Diff. Geom., 12, No.2, 253-300, 1977.

[16] K.C.H. Mackenzie, *General Theory of Lie Groupoids and Lie Algebroids*, Cambridge U. Press, 2005.
[17] F. Magri, *A simple model of integrable Hamiltonian equation*, J. Math. Phys., 19, 1156-1162, 1978.

[18] F. Magri, C. Morosi, *A geometrical characterization of integrable Hamiltonian systems through the theory of Poisson-Nijenhuis manifolds*, Quaderno S., Universitá di Milano, 19, 1984.

[19] Ch.-M. Marle, *Differential calculus on a Lie algebroid and Poisson manifolds*, arXiv:0804.2451, 2008.

[20] A. Mba, P. M. K. Wamba, R. P. Nimpa, *Vertical and horizontal lifts of multivector fields and applications*, Lobachevskii Journal of Mathematics, Vol.38, No. 1, 1-15, 2017.

[21] G. Mitric, I. Vaisman, *Poisson structures on tangent bundles*, Differential Geometry and its Applications, 18, 207-228, 2003.

[22] J.A. Nijenhuis, *Jacobi-type identities for bilinear differential concomitants of certain tensor fields*, Indag. Math., 17, 390-403, 1955.

[23] A. Odzijewicz, A. Dobrogowska, *Integrable Hamiltonian systems related to the Hilbert-Schmidt ideal*, J. Geom. Phys., 61, 1426-1445, 2011.

[24] J. Pradines, *Théorie de Lie pour les groupoïdes différentiables. Calcul différentiel dans la catégorie des groupoïdes infinitésimaux*, C.R. Acad. Sci. Paris 264 A, 245-248, 1967.

[25] J. Patera, R.T. Sharp, P. Winternitz, H. Zassenhaus, *Invariants of real low dimension Lie algebras*, J. Math. Phys., 17, 986, doi: 10.1063/1.522992, 1976.

[26] J.A. Schouten, *On the differential operators of first order in tensor calculus*, Convegno Int. Geom. Diff. Italia, 1953, Ed. Cremonese, Roma, 1-7, 1954.

[27] V.V. Trofimov, A.T. Fomenko, *Algebra and Geometry of Integrable Hamiltonian Differential Equations*, Factorial, Moscow (in Russian), 1995.

[28] A.V. Tsiganov, *On bi-integrable natural Hamiltonian systems on Riemannian manifolds*, J. of Nonlinear Mathematical Physics, 18:2, 245-268, 2013.
[29] A. Weinstein, *The local structure of Poisson manifolds*, J. Differential Geometry, 18, 523-557, 1983.

[30] A. Weinstein, *Symplectic groupoids and Poisson manifolds*, Bull. Amer. Math. Soc., 16, 101-103, 1987.

[31] K. Yano and S. Ishihara, *Tangent and Cotangent Bundles*, M. Dekker Inc., New York, 1973.