Spectral properties of the nonspherically decaying radiation generated by a rotating superluminal source

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The focusing of the radiation generated by a polarization current with a superluminally rotating distribution pattern is of a higher order in the plane of rotation than in other directions. Consequently, our previously published [J. Opt. Soc. Am. A 24, 2443 (2007)] asymptotic approximation to the value of this field outside the equatorial plane breaks down as the line of sight approaches a direction normal to the rotation axis, i.e., is non-uniform with respect to the polar angle. Here we employ an alternative asymptotic expansion to show that, though having a rate of decay with frequency (μ) that is by a factor of order μ0/3 slower, the equatorial radiation field has the same dependence on distance as the nonspherically decaying component of the generated field in other directions: It, too, diminishes as the inverse square root of the distance from its source. We also briefly discuss the relevance of these results to the giant pulses received from pulsars: The focused, nonspherically decaying pulses that arise from a superluminal polarization current in a highly magnetized plasma have a power-law spectrum (i.e., a flux density S ∝ μ−α) whose index (α) is given by one of the values −2/3, −2, −8/3, or −4. © 2008 Optical Society of America

1. INTRODUCTION

Radiation by polarization currents whose distribution patterns move faster than light in vacuo has been the subject of several theoretical and experimental studies in recent years [1–8]. When the motion of its source is accelerated, this radiation exhibits features that are not shared by any other known emission. In particular, the radiation from a rotating superluminal source consists, in certain directions, of a collection of subbeams whose azimuthal and polar widths narrow (as RP 2 and RP, respectively) with distance RP from the source [8]. Being composed of tightly focused wave packets that are constantly dispersed and reconstructed out of other waves, these subbeams neither diffract nor decay in the same way as conventional radiation beams. The field strength within each subbeam diminishes as RP 3/2, instead of RP, with increasing RP [6–8].

In earlier treatments [7,8], we evaluated the field of a superluminally rotating extended source by superposing the fields of its constituent volume elements, i.e., by convolving its density with the familiar Liénard–Wiechert field of a rotating point source. This Liénard–Wiechert field is described by an expression essentially identical to that encountered in the analysis of synchrotron radiation, except that its value at any given observation time receives contributions from more than one retarded time. The multivalued nature of the retarded time gives rise to the formation of caustics. The wavefronts emitted by each constituent volume element of a superluminally moving accelerated source possess a cusped envelope on which the field is infinitely strong (see Figs. 1 and 4 of [8]). Correspondingly, the Green’s function for the problem is non-integrally singular for those source elements that approach the observer along the radiation direction with the speed of light and zero acceleration at the retarded time (see Fig. 3 of [8]): the cusp of the envelope of wavefronts emanating from each such element is a spiraling curve extending into the far zone that passes through the position of the observer. When the source oscillates at the same time as rotating, the Hadamard finite part of the divergent integral that results from convolving the Green’s function with the source density has a rapidly oscillating kernel for a far-field observation point. The stationary points of the phase of this kernel turn out to have different orders depending on whether the observer is located in or out of the equatorial plane.

To reduce the complications posed by the higher-order stationary points of this phase, we restricted the asymptotic evaluation of the radiation integral thus obtained in [7,8] to observation points outside the plane of rotation, i.e., to spherical polar angles θp that do not equal π/2. The purpose of this paper is to evaluate the field of a superluminally rotating extended source also for the smaller class of observers at polar coordinate θp = π/2 and to obtain, thereby, a more global description of the nonspherically decaying radiation that is generated

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by such a source. The asymptotic expansion presented in our previous papers [7,8] breaks down in the case of a subbeam that is perpendicular to the rotation axis because there is a higher-order focusing associated with the waves emitted by those source elements whose actual speeds (rather than the line-of-sight components of their speeds) equal the speed of light as they approach the observer with zero acceleration.

Here, we present a brief account of the background material on the radiation field of a rotating superluminal source in Section 2, and the asymptotic evaluation of this field for an equatorial observer in Section 3. In Section 4, we give a description of the spectral properties of the non-spherically decaying component of this radiation in the light of the present results and those obtained in [7,8] and discuss the relevance of these properties to pulsar observations.

2. BACKGROUND: RADIATION FIELD OF A ROTATING SUPERLUMINAL SOURCE

We base our analysis on a generic superluminal source distribution [7,8], which has been created in the laboratory [2]. This source comprises a polarization current density \( \mathbf{j} = \mathbf{\partial P} / \mathbf{\partial t} \) for which

\[
P_{r, \varphi, z}(r, \varphi, z, t) = s_{r, \varphi, z}(r, z) \cos(m \phi) \cos(\Omega t),
\]

where \( P_{r, \varphi, z} \) are the components of the polarization \( \mathbf{P} \) in a cylindrical coordinate system based on the axis of rotation, \( s(r, z) \) is an arbitrary vector that vanishes outside a finite region of the \( (r, z) \) space, and \( m \) is a positive integer. For fixed \( t \), the azimuthal dependence of the density, Eq. (1), along each circle of radius \( r \) within the source is the same as that of a sinusoidal wave train of wavelength \( 2\pi/m \), whose \( m \) cycles fit around the circumference of the circle smoothly. As time elapses, this wave train both propagates around each circle with the velocity \( r \omega \) and oscillates in its amplitude with the frequency \( \Omega \). This is a generic source; one can construct any distribution with a uniformly rotating pattern \( P_{r, \varphi, z}(r, \varphi, z, t) \) by the superposition over \( m \) of terms of the form \( s_{r, \varphi, z}(r, \varphi, z, m) \cos(m \phi) \).

The electromagnetic fields

\[
\mathbf{E} = -\nabla \mathbf{\varphi}^0 - \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}
\]

arising from such a source are given, in the absence of boundaries, by the following classical expression for the retarded four-potential:

\[
A^\mu(x_p, t_p) = c^{-1} \int d^3x d^4j^\mu(x, t) \delta(t_p - t - R/c)R,
\]

where \( \mu = 0, \cdots, 3 \). Here, \((x_p, t_p) = (r_p, \varphi_p, z_p, t_p) \) and \((x, t) = (r, \varphi, z, t) \) are the space-time coordinates of the observation point and the source points, respectively; \( R \) stands for the magnitude of \( \mathbf{R} = x_p - x \); and \( \mu = 1, 2, 3 \) designate the spatial components \( \mathbf{A} \) and \( \mathbf{j} \) of \( A^\mu \) and \( j^\mu \) in a Cartesian coordinate system.

To find the retarded field that follows from Eq. (4) for the source described in Eq. (1), we first calculated in [7] the Liénard–Wiechert field arising from a circularly moving point source with a speed \( r \omega > c \), i.e., a generalization of the synchrotron radiation to the superluminal regime. We then evaluated the integral representing the retarded field of the extended source (1) by superposing the fields generated by the constituent volume elements of this source, i.e., by using the generalization of the synchrotron field as the Green’s function for the problem. In the superluminal regime, this Green’s function has extended singularities arising from the constructive interference of the emitted waves on the envelope of wavefronts and its cusp.

Labeling each element of the extended source (1) by its Lagrangian coordinate \( \hat{\mathbf{r}} \) and performing the integration with respect to \( t \) and \( \varphi \) (or equivalently \( \varphi \) and \( \phi \)) in the multiple integral implied by Eqs. (1)-(4), we showed in [7] that the resulting expression for the radiation field \( \mathbf{E} \) (or \( \mathbf{B} \)) consists of two parts: one that decays spherically (as \( R^1 \), as in a conventional radiation field) and another, \( \mathbf{B}^s \) (with \( \mathbf{E}^s = \hat{\mathbf{n}} \times \mathbf{B}^s \)), that decays nonspherically (as \( R^{1/2} \)) within the conical shell \( \arcsin(1/R_\alpha) < \theta < \arcsin(1/R_\gamma) \) in the far zone. Here, \((R_\alpha, \theta_\alpha, \varphi_\alpha) \) are the spherical polar coordinates of the observation point \( P \), \( \hat{\mathbf{r}} \) stands for \( r \omega \), \( \hat{\mathbf{n}} = \mathbf{R}/R \) is a unit vector in the radiation direction, and \( R_\alpha > 1 \) and \( R_\gamma > R_\alpha \) denote the radial boundaries of the support of the source density \( \mathbf{s} \).

The expression for the nonspherically decaying component of the field within this conical shell, in the far zone, is

\[
\mathbf{B}^s = -\frac{4}{3} i \exp[i(\Omega t)(\varphi_\alpha + 3\pi/2)] \sum_{\mu \neq \mu_\alpha} \mu \exp(-i\mu \varphi_\alpha) \\
\times \sum_{j=1}^3 \hat{\mathbf{q}}_j \int_{\Delta = 0}^{\Delta = \pi} \hat{\mathbf{r}} d\hat{\mathbf{r}} d\Delta A^{-1/2} u_j \exp(-i\mu \phi_\alpha),
\]

where \( \mu_\alpha = (\Omega t \omega \pm m, \varphi_\alpha = \varphi_\alpha - \omega t_\alpha, \)

\[
\hat{\mathbf{q}}_j = (1 - i\Omega t \omega)(1 \pm i\Omega t \omega),
\]

\[
u_j = s_\omega \cos \theta_\alpha \hat{\mathbf{e}}_\omega + s_\phi \hat{\mathbf{e}}_\phi,
\]

\[
u_3 = -s_\omega \sin \theta_\alpha \hat{\mathbf{e}}_\omega,
\]

\[
\Delta = (\tilde{r}_p^2 - 1)(\tilde{r}_p^2 - 1) - (\tilde{z} - \tilde{z}_p)^2,
\]

\[
\phi_\alpha = \frac{\tilde{r}_p}{\tilde{r}_p}, \varphi_\alpha - \varphi_p,
\]

\[
\varphi_\alpha = \varphi_\alpha + 2\pi - \arccos[(1 + \Delta^{1/2})(\tilde{r}_p)],
\]

\[
\tilde{R}_\alpha = [(\tilde{z} - \tilde{z}_p)^2 + \tilde{r}_p^2 + \tilde{r}_p^2 - 2(1 + \Delta^{1/2})^{1/2}],
\]

see Eq. (47) of [7]. In this expression, \((\hat{\mathbf{r}}, \tilde{\mathbf{r}}; \tilde{r}_p, \tilde{z}_p) \) stand for \((r \omega \), \(z \omega \), \(r \omega \), \(z \omega \)), and \( \hat{\mathbf{e}}_\alpha = \hat{\mathbf{n}} \times \hat{\mathbf{e}}_\phi / |\hat{\mathbf{e}}_\phi \times \hat{\mathbf{n}}| \) (which is parallel to the plane of rotation) and \( \hat{\mathbf{e}}_\zeta = \hat{\mathbf{n}} \times \hat{\mathbf{e}}_\alpha \) comprise a pair of unit vectors normal to the radiation direction \( \hat{\mathbf{n}} \) (\( \hat{\mathbf{e}}_z \) is the base vector associated with the coordinate \( z \)).
The domain of integration consists of the part of the source distribution \(s(r,z)\) that falls within \(\Delta \geq 0\) (see Fig. 4 of [8]).

Both derivatives, \(\partial \phi_r / \partial r\) and \(\partial \phi_z / \partial z\), of the function \(\phi_r(r,z)\) that appears in the phase of the integrand in Eq. (5) vanish at the point \(r=1, z=z_p\), where the cusp curve of the bifurcation surface is tangent to the light cylinder (see Figs. 3 and 4 of [8]). However, \(\partial^2 \phi_r / \partial \sigma^2\) diverges at this point, so that neither the phase nor the amplitude of the kernel of the integral in Eq. (5) are analytic at \(r=1, z=z_p\). Only for an observer who is located outside the plane of rotation, i.e., whose coordinate \(z_p\) does not match the coordinate \(z\) of any source element, is the function \(\phi_r(r,z)\) analytic throughout the domain of integration. To take advantage of the simplifications offered by the analyticity of \(\phi_r\) as a function of \(r\), we restricted the analyses in [7,8] to observation points for which \(\theta_p \neq \pi/2\).

In the calculation that follows, we find an asymptotic approximation to the integral

\[
I = \int_{\Delta > 0} \frac{r^2 r \, d\zeta}{\Delta^{1/2} \, u_i \exp(-i \mu \phi_{\sigma})} \tag{12}
\]

in Eq. (5) that is valid in the plane of rotation, i.e., for \(\theta_p = \pi/2\). We shall treat only the case of positive \(\mu\); \(I(\mu)\) for negative \(\mu\) can then be obtained via \(I(-\mu) = I(\mu)^*\).

3. ASYMPTOTIC VALUE OF THE FIELD FOR AN EQUATORIAL OBSERVER IN THE FAR ZONE

Since the main contribution toward the value of the field at \(\theta_p = \pi/2\) is made by the source elements that lie in the vicinity of the critical point \(r=1, z=z_p\), the first step in the asymptotic evaluation of \(B^\alpha\) is to replace \((\hat{r}, \hat{z})\) by a new pair of variables \((\rho, \sigma)\) for which the phase function \(\phi_{\sigma}(\rho, \sigma)\) is rendered analytic at this point:

\[
\hat{r} = (1 + \rho^2 \cosh^2 \sigma)^{1/2}, \tag{13}
\]

\[
\hat{z} = z_p + (\hat{r}_p^2 - 1)^{1/2} \rho \sinh \sigma. \tag{14}
\]

This transformation replaces \(\hat{r} \Delta^{1/2} d\hat{r} d\hat{z}\) by \(\rho \cosh \sigma d\rho d\sigma\) and yields

\[
\phi_{\sigma}(\rho, \sigma) = \left[\hat{r}_p^2 - 1 - 2(\hat{r}_p^2 - 1)^{1/2} \rho + (\hat{r}_p^2 \sinh^2 \sigma + 1) \rho^2\right]^{1/2} + 2 \pi - \arccos(\hat{r}_p^2(1 + \rho^2 \cosh^2 \sigma)^{-1/2}(1 + (\hat{r}_p^2 - 1)^{1/2} \rho)), \tag{15}
\]

which is analytic at \(\rho = 0\). In the plane of rotation, i.e., for \(z_p\) that equals the \(z\) coordinate of a cross section of the source distribution, \(\phi_{\sigma}(\rho, \sigma)\) doubly stationary as a function of both \(\rho\) and \(\sigma\). The two critical points designated as \(C\) and \(S\) in [7,8] coalesce in this plane, and as a result, all five of the derivatives \(\partial \phi_{\sigma} / \partial \rho\), \(\partial \phi_{\sigma} / \partial \sigma\), \(\partial^2 \phi_{\sigma} / \partial \rho^2\), \(\partial^2 \phi_{\sigma} / \partial \sigma^2\), and \(\partial^3 \phi_{\sigma} / \partial \rho \partial \sigma\) vanish at \(\rho = \sigma = 0\).

To see that applying the method of stationary phase to the integral in Eq. (5) results in a valid asymptotic approximation for large \(\hat{R}_p\), we begin by casting the \(\sigma\) dependence of the phase \(\phi_{\sigma}\) into a canonical form [9]. Since \(\sigma = 0\) is an isolated stationary point of \(\phi_{\sigma}\) (when regarded as a function of the single variable \(\sigma\)), we may employ the following transformation:

\[
\phi_{\sigma} = \phi_{\sigma\mid \sigma = 0} + \frac{1}{2} b \xi^2, \tag{16}
\]

in which

\[
b = \frac{\partial^2 \phi_{\sigma}}{\partial \sigma^2} \bigg|_{\sigma = 0} = \rho^2(\hat{r}_p^2 - 1 - (\hat{r}_p^2 - 1)^{1/2} \rho + \rho^2 \sinh^2 \sigma) \left(\hat{r}_p^2 - 1 - (\hat{r}_p^2 - 1)^{1/2} \rho\right). \tag{17}
\]

Equation (16) expresses \(\sigma\) as a function of \(\xi\) implicitly. Repeated differentiations of this equation with respect to \(\xi\) result in

\[
\frac{\partial \phi_{\sigma}}{\partial \sigma} = b \xi, \tag{18}
\]

\[
\frac{\partial^2 \phi_{\sigma}}{\partial \sigma^2} \frac{\partial \sigma}{\partial \xi} + \frac{\partial^2 \phi_{\sigma}}{\partial \sigma^2} \left(\frac{\partial \sigma}{\partial \xi}\right)^2 = b, \tag{19}
\]

and so on, which when evaluated at \(\xi = 0\) supply the coefficients \(\partial^3 \phi_{\sigma} \xi^{-3} |_{\xi = 0}, \partial^4 \phi_{\sigma} \xi^{-4} |_{\xi = 0}, \) etc., in the Taylor expansion of \(\sigma\) in powers of \(\xi\).

The integral \(I\) in Eq. (12) can therefore be written as

\[
I = \int \, d\rho d\xi Q(\rho, \xi) \exp(-i \beta \xi), \tag{20}
\]

where

\[
Q(\rho, \xi) = \rho \cosh \sigma u_i \exp(-i \mu \phi_{\sigma\mid \sigma = 0}) \partial \sigma / \partial \xi, \tag{21}
\]

\[
\frac{\partial \sigma}{\partial \xi} = \frac{b \xi \hat{R}_p(1 + \rho^2 \cosh^2 \sigma)}{\rho^2 \sinh \sigma \cosh \sigma} \left[\hat{r}_p^2 - 1 - (\hat{r}_p^2 - 1)^{1/2} \rho\right]^{-1}, \tag{22}
\]

and \(\beta = \frac{1}{2} \mu b\). The limits of integration are determined by the image of \(\Delta > 0\) under transformation (16).

The parameter \(b\) that multiplies the phase of the integrand in Eq. (20) has a large value in the far zone:

\[
b = \rho^2 \hat{R}_p, \quad \hat{R}_p \gg 1, \tag{23}
\]

[see Eq. (17)]. The asymptotic value of the integral \(I\) for large \(\hat{R}_p\) therefore receives its leading contribution from the immediate vicinity of \(\xi = 0\), where the phase of the integrand is stationary [9]. Replacing \(Q(\rho, \xi)\) in Eq. (20) by \(Q(\rho, 0)\) and extending the range of integration with respect to \(\xi\) to \((-\infty, \infty)\), we obtain

\[
I = (2\pi \mu)^{1/2} \int \, d\rho \rho u_{\rho \mid \rho = 0} \rho^{-1/2} \exp[-i \mu \phi_{\sigma\mid \sigma = 0}] \left(+ \pi/4\right) (\partial \sigma / \partial \xi)_{\xi = 0}
\]

\[
= (2\pi \mu)^{1/2} \hat{r}_p^{-1/2} \exp[-i \mu(\hat{r}_p + 3\pi/2 + \pi/4)]
\]

\[
\times \int \, d\rho \rho u_{\rho \mid \rho = 1 + \rho^2, \hat{z} = \hat{z}_p} \exp[-i \mu(\arctan \rho - \rho)],
\]

\[
= (2\pi \mu)^{1/2} \hat{r}_p^{-1/2} \exp[-i \mu(\hat{r}_p + 3\pi/2 + \pi/4)]
\]

\[
\times \int \, d\rho \rho u_{\rho \mid \rho = 1 + \rho^2, \hat{z} = \hat{z}_p} \exp[-i \mu(\arctan \rho - \rho)],
\]
where the integration extends over all values of \( \rho \) for which
the source density \( s_{|r=r_0, \rho|} \) is nonzero [see Eq. (7)].
In deriving this expression, we have inferred the value
\( \partial \rho / \partial \rho'_{|r_0=0} = 1 \) of the indeterminate Jacobian that appears
in \( Q(\rho, 0) \) from Eq. (19) (or, equivalently, from Eq. (22) and
H'ospital's rule), and expressed \( \rho_{|r=0} \) and \( b \) in terms of
their far-field values by means of Eqs. (15) and (23).

The contribution \( \mathbf{B}^{\text{ns}} \) toward the magnetic field \( \mathbf{B} \) of
the radiation is made by those volume elements of the source
that approach the observation point \( P \) along the radiation
direction with the speed of light and zero acceleration at
the retarded time, i.e., by the source elements for which
\( \Delta = 0 \). Hence, the amplitude of the integrand in Eq. (5) has
already been approximated by its leading term in powers
of \( \Delta^{1/2} = (r_0^2 - 1)^{1/2} \rho \) (see \( [7,8] \)). To be consistent, we must
also approximate the amplitude of the integrand in Eq. (24)
by its value for \( \rho \ll 1 \):

\[
\mathcal{I} = (2 \pi^2 \mu)^{1/2} r_P^{-1/2} \mathbf{u}_{|r=1, z=p} \exp[-i \mu (\hat{r}_P + 3 \pi/2) + \pi/4]]
\times \int_0^{(r_0^2 - 1)^{1/2}} \mathrm{d} \rho \exp[-i \mu (\arctan \rho - \rho)] \tag{25}
\]

where \( \hat{r}_P \) denotes the radial extent of the support of
the source density \( s \). This reduces to

\[
\mathcal{I} = 3^{-2/3} \frac{1}{3} (2 \pi)^{1/2} \mu^{-5/6} \mathbf{u}_{|r=1, z=p} \exp[-i \mu (\hat{r}_P + 3 \pi/2)]
+ \pi/12 |\mathbf{B}|^{1/2} \tag{26}
\]
in the regime \( \mu >> 1 \), where we can approximate \( \arctan \rho - \rho \)
in the argument of the exponential by \( -1/3 \rho^3 \) and replace
the upper limit of integration by \( \infty \) [9].

Thus, Eqs. (5), (12), and (25) jointly yield the following
expression for the \( \hat{R}_P \) of the magnetic field of the radiation
close to the plane \( \theta_P = \pi/2 \):

\[
\mathbf{B} = - \frac{4}{3} (2 \pi)^{1/2} r_P^{-1/2} \mathbf{u} \exp[i (\Omega / \omega) (\varphi_P
+ 3 \pi/2)] \sum_{\mu=\pm \mu_k} |\mu|^{1/2} \operatorname{sgn}(\mu) \exp[-i \mu (\hat{r}_P + 3 \pi/2)]
+ \pi/4 \operatorname{sgn}(\mu)] \sum_{j=1}^{3} \mathbf{q}_j \mathbf{u}_{|r=1, z=p} \mathcal{J}, \tag{27}
\]

where

\[
\mathcal{J} = \int_0^{(r_0^2 - 1)^{1/2}} \mathrm{d} \rho \exp[-i \mu (\arctan \rho - \rho)], \tag{28}
\]

for the contribution \( \mathbf{B}^{\text{ns}} \) toward the magnetic field \( \mathbf{B} \) of
the radiation is larger by a factor of the order of \( \hat{R}_P^{1/2} \) than
the spherically decaying contribution. This is the counterpart
of Eq. (55) of \( [7] \) and Eq. (61) of \( [8] \) (the electric field vector
of this radiation is given by \( \mathbf{n} \times \mathbf{B} \) as in any other radiation).

Note that the remaining integral in the above expression reduces to

\[
\hat{R}_P \gg 1, \tag{24}
\]

\[
\mathcal{J} = 3^{-2/3} \left( \frac{1}{3} \right) \exp(i \pi/6) \mu^{-1/3} \tag{29}
\]
in the limit \( |\mu| \ll 1 \) [see Eq. (26)].

4. SPECTRUM OF THE NONSPHERICALLY
DECAYING RADIATION: RELEVANCE
TO PULSAR OBSERVATIONS

Equation (27) shows that the radiation field of a rotating
superluminal source diminishes as \( \hat{R}_P^{1/2} \) with the distance
\( \hat{R}_P \) also in the equatorial plane \( \theta_P = \pi/2 \). This differs from
the corresponding result for \( \theta_P = \pi/2 \) (Eq. (55) of \( [7] \))
mainly in its dependence on frequency. The Fourier transform
\( \mathbf{s} \) in Eq. (57) of \( [7] \) has the asymptotic dependence
\( \mu^{-4} \) on \( \mu \) for a source density \( s_{|c(z)} \) that is approximately
constant within its finite support. Therefore, when \( s(r, z) \)
is of finite variation and support in \( z \) and the radiation frequency \( \mu \omega \)
appreciably exceeds the rotation frequency \( \omega \), the field in the plane of rotation decays more slowly
with frequency, by a factor of order \( \mu^{2/3} \), than does the field outside this plane.

Since the azimuthal width of the generated subbeams
(and hence the duration of the narrow signals that constitute
the overall pulse) is independent of frequency, the flux density \( \mathcal{S} \) of such signals (i.e., the power propagating
across a unit area per unit frequency) is proportional to
\( |\mathbf{B}|^2 / \mu \omega \), where \( \mu \omega \sim |\mu| \) is the bandwidth of the radiation.
The flux density of the emission described by Eq. (27)
thus depends on frequency as \( \mathcal{S} \propto \mu^{-2/3} \mathcal{Q}^2 |\mathbf{s}(\mu)|^2 \) for \( |\mu| \gg 1 \).
Here, \( |\mathbf{s}(\mu)| \) designates the frequency dependence of the factor \( s \), which enters the expression for the polarization
\( \mathbf{P} \) and the definitions of \( \mathbf{u} \) [see Eqs. (1) and (7)].
The flux density of the corresponding emission outside the equatorial
plane depends on frequency as \( \mathcal{S} \propto \mu^{-2/3} \mathcal{Q}^2 |\mathbf{s}(\mu)|^2 \) for
\( |\mu| \gg 1 \) since, apart from the dependence \( \mathbf{s}(\mu) \) of a multiplicative
factor (such as electric susceptibility) in \( s \), the Fourier transform \( \mathbf{s} \) would approximately decay as \( \mu^{-1} \) for
a source distribution that is of finite variation and support in \( z \).

To compare the predictions of Eq. (27) (and Eq. (55) of
\( [7] \)) with the observed spectra of the giant pulses from
pulsars, we therefore need to estimate the frequency dependence
of the electric susceptibility (contained in the factor \( s \)) for the magnetospheric plasma of a pulsar. The simple classical model of propagation of electromagnetic
disturbances in a cold magnetized plasma yields a dielectric
tensor, and hence an electric susceptibility, whose components decay with frequency as \( (\mu \omega)^{-1} \) when the frequency \( \mu \omega \) of the disturbance that polarizes the medium
is much lower than the gyration frequency of the electrons
in the magnetized plasma; see, e.g., Eq. (7.67) of
[10]. For a magnetic field as strong as that of a pulsar
\( (\sim 10^{12} \text{ G}) \), the Larmor frequency of an electron exceeds
the highest radio frequencies at which the pulses are observed
by a factor of order \( 10^6 \), so that \( s(\mu) \propto \mu^{-1} \) for pulsars.

Using this result, we obtain \( \mathcal{S} \propto \mathcal{Q}^2 \mathcal{S} \mu^{-6/3} \) for \( \theta_P = \pi/2 \) and
\( \mathcal{S} \propto \mathcal{Q}^2 \mu^{-4} \) for \( \theta_P \neq \pi/2 \). Depending on whether the modulation
frequency \( \Omega \) on which \( \mathbf{q}_j \) (of Eq. (6)) is comparable to or much smaller than the frequency \( m \omega \) of the
sinusoidal wave train characterizing the azimuthal distribution of the source, therefore, the spectral density of the nonspherically decaying radiation is given by

\[ S \propto \mu^{-2/3}, \quad \theta_p = \pi/2, \quad \Omega \omega = |\mu|; \]  

\[ S \propto \mu^{-2}, \quad \theta_p \neq \pi/2, \quad \Omega \omega = |\mu|; \]  

\[ S \propto \mu^{-8/3}, \quad \theta_p = \pi/2; \quad \Omega \omega \ll |\mu| \quad \text{or} \quad j = 1, \]  

or

\[ S \propto \mu^{-4}, \quad \theta_p \neq \pi/2; \quad \Omega \omega \ll |\mu| \quad \text{or} \quad j = 1. \]  

In other words, the spectral index of the pulses portraying the subbeams can have any of the values \(-2/3, -2, -8/3, \) or \(-4\).

The range of spectral indices \((-4 \leq \alpha \leq -2/3)\) implied by Eq. (27) and its counterpart, Eq. (57) of [7], is consistent with that which characterizes the observed power-law spectra of the giant pulses from pulsars [11–13]. For radio pulsars, the rotation frequency \(\omega\) of the distribution pattern of the radiating polarization current is of the order of 1 rad/s, and the oscillation frequency \(\mu \omega/2\pi\) of the source density of the order of 100 MHz, so that \(\mu\) has a large value of the order of 10^3. The coherent component of the radiation, i.e., the sharply focused subbeams that decay as \(R_P^{1/2}\), are emitted at the frequency \(\mu \omega\) [7]. The spherically decaying, incoherent component of the radiation arising from the polarization current described in Eq. (1), on the other hand, contains frequencies that are higher than \(\mu \omega\) by a factor of order \((\Omega \omega)^2\) [14]. In pulsars, \((\Omega \omega)^2 \sim 10^{18}\) when \(\Omega \omega\) is comparable to \(\mu\), i.e., when the frequency \(m \omega\) that characterizes the azimuthal fluctuations of the emitting plasma is of the order of, or smaller than, its modulation frequency \(\Omega\). Hence, not only the power-law indices of the coherent component, but the unusually broad spectral distribution of the incoherent component of this radiation, too, is consistent with the observational data from certain pulsars. The pulsed emission from the Crab pulsar, for example, extends over 53 octaves of the electromagnetic spectrum from radio waves to \(\gamma\) rays [15].

We note, finally, that neither the asymptotic expansion presented here nor that which was obtained in [7,8] is uniform with respect to the parameter \(\theta_p\). The present approximation receives contributions only from the volume elements in the vicinity of the single source point \(r = 1, \hat{z} = z_p\) at which the cusp surface \(\Delta = 0\) of the bifurcation surface touches the light cylinder (see Figs. 3 and 4 of [7]). This is in sharp contrast to the asymptotic expansion of the field outside the equatorial plane, for which the leading term receives contributions from a filamentary locus of source elements, i.e., from the intersection of the cusp curve of the bifurcation surface with the entire volume of the source. Comparison of Eq. (27) with its counterpart, Eq. (57) of [7], shows that the (smooth) transition from the nonequatorial to the equatorial regime occurs across \(\theta_p = \text{arccos}(\mu^{-2/3})\). However, the derivation of a uniform asymptotic approximation to integral \(I\) that would deter-

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