Three Random Intercepts of a Segment

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Abstract. We construct random triangles via uniform sampling of certain families of lines in the plane. Two examples are given. The word “uniform” turns out to be vague; two competing models are examined. Everything we write is well-known to experts. Which model is more appropriate? Our hope is to engage a larger audience in answering this question.

Let $\ell$ denote a planar random line with slope $\tan(\omega)$ and x-intercept $\xi$, where $\omega \sim \text{Uniform}[0, \pi]$ and $\xi \sim \text{Uniform}[-1, 1]$ are independent. Let $\ell_1, \ell_2, \ell_3$ be independent copies of $\ell$. The three lines determine a compact triangle $\Delta$ almost surely. The probability density function for the maximum angle in $\Delta$ is

\[
  f(\alpha) = \begin{cases} 
  6(3\alpha - \pi)/\pi^2 & \text{if } \pi/3 \leq \alpha < \pi/2, \\
  6(\pi - \alpha)/\pi^2 & \text{if } \pi/2 \leq \alpha \leq \pi, \\
  0 & \text{otherwise}
  \end{cases}
\]

and hence the probability that $\Delta$ is obtuse is $3/4$.

A variation on the preceding is to require $\omega \sim \text{Uniform}[\pi/4, 3\pi/4]$, that is, the lines $\ell_1, \ell_2, \ell_3$ each have $|\text{slope}|$ exceeding 1. The maximum angle density here is

\[
  f(\alpha) = \begin{cases} 
  24(\pi - \alpha)(2\alpha - \pi)/\pi^3 & \text{if } \pi/2 \leq \alpha \leq \pi, \\
  0 & \text{otherwise}.
  \end{cases}
\]

The random triangle $\Delta$ is almost surely obtuse.

Gates [4] examined the same two problems, for “triangles generated by uniform random lines”, but adopted a different probability model than the preceding. He did not elaborate on the quoted phrase, but referred to an earlier paper [5], where it is apparent that the density for $\omega$ should be proportional to $\sin(\omega)$. On the one hand, his model is standard in the sense that the measure is invariant under rigid motions [6, 7]. On the other hand, it possesses a feature that vertical lines are weighted more than horizontal lines. This curious tradeoff raises an interesting question: which model is more appropriate when constructing random triangles?

For the unrestricted case ($0 \leq \omega < \pi$), the inclination angle density is

\[
  g(\omega) = \frac{1}{2} \sin(\omega)
\]

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and the maximum angle density is consequently [4]

\[
f(\alpha) = \begin{cases} 
3 & [(3\alpha - \pi) \cos(\alpha) + 2 \sin(\alpha) - 2 \sin(2\alpha) + \sin(3\alpha)] \\
\frac{3}{4} & [3(\pi - \alpha) \cos(\alpha) + 3 \sin(\alpha) - 2 \sin(2\alpha)] \\
0 & \text{if } \pi/3 \leq \alpha < \pi/2, \\
\frac{1}{4} & \text{if } \pi/2 \leq \alpha \leq \pi, \\
0 & \text{otherwise.}
\end{cases}
\]

It follows that

\[
\mathbb{P}\{\Delta \text{ is obtuse}\} = 2 - \frac{3\pi}{8} = 0.8219...
\]

which is larger than 3/4. For the restricted case \((\pi/4 \leq \omega \leq 3\pi/4)\), the inclination angle density is

\[
g(\omega) = \frac{1}{\sqrt{2}} \sin(\omega)
\]

and the maximum angle density is [4]

\[
f(\alpha) = \begin{cases} 
\frac{1}{2} & [\cos(\alpha) + \sin(\alpha) + \cos(2\alpha) - 2 \sin(2\alpha)] \\
0 & \text{if } \pi/2 \leq \alpha \leq \pi, \\
\frac{1}{2} & \text{otherwise.}
\end{cases}
\]

The expressions for \(f\) when \(\omega\) enjoys constant weighting are simpler than those for \(f\) when \(\omega\) enjoys sinusoidal weighting. This statement alone does not imply that the first model is preferable to the second model; there are other issues to consider too. To generate random triangles according to [4] is only slightly more complicated than according to [1, 2]: if \(U \sim \text{Uniform}[0, 1]\), then by the inverse CDF method,

\[
\omega = \arccos (1 - 2U)
\]
gives inclination angles for the unrestricted case and

\[
\omega = \arccos \left(\frac{(1 - 2U)/\sqrt{2}}{2}\right)
\]
gives inclination angles for the restricted case.

The most compelling argument for sinusoidal weighting is its theoretical consistency with the Poisson line process [8, 9]. Let us focus on the unrestricted case. By Example 20 of [10], the inclination angles \(\omega_j\) of the lines relative to the \(x\)-axis are independent and identically distributed with density \(\sin(\omega)/2\) on \([0, \pi]\). In words, acute angles \(\approx 0^\circ\) and obtuse angles \(\approx 180^\circ\) are less likely than near-right angles \(\approx 90^\circ\). Vertical rain wets more than slanted rain [11]. We quote [10]:
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... although the lines of the line process have “uniformly distributed orientations” in some sense, the angles of incidence with any fixed axis are not uniformly distributed... the probability of ‘catching’ a random line in a given sampling interval of the $x$-axis depends on the orientation of the line...

and, further, [12]:

This is a classic paradox. If you consider the random lines which intersect a given, fixed line, then these random lines have angles which are non-uniformly distributed with probability density proportional to the sine of the incidence angle. If you consider the random lines which intersect a given circle then these random lines have uniformly-distributed orientation angles. In each case the bold text describes a selection or sampling operation, and sampling operations introduce bias.

We sketch a proof of this theorem in Appendix 1. Proofs of the four density formulas for $f$ are not provided here; in the following section, we choose instead to examine only a special scenario for illustration’s sake.

1. Diagonal Line

Let us examine the restricted case ($\pi/4 \leq \omega \leq 3\pi/4$), initially with constant weighting and subsequently with sinusoidal weighting. We follow [3] closely. Let $\omega_1 = \pi/4$, $\omega_2$, $\omega_3$ be the inclination angles of the three lines, hence the first line is fixed as the diagonal $y = x$. Clearly $\omega_1 < \omega_2$ and $\omega_1 < \omega_3$ almost surely. The angles $\omega_2$, $\omega_3$ are independent and identically distributed, thus $\mathbb{P}\{\omega_2 < \omega_3\} = 1/2$. The triangle formed by the three lines has angles $\omega_2 - \omega_1$, $\omega_3 - \omega_2$, $\pi - \omega_3 + \omega_1$. Since $\pi/2 = \pi - 3\pi/4 + \pi/4 \leq \pi - \omega_3 + \omega_1$, the maximum angle is obviously $\alpha = 5\pi/4 - \omega_3$.

We have

$$\mathbb{P}(\alpha < a) = \mathbb{P}\left\{ 5\pi/4 - \omega_3 < a \mid \omega_2 < \omega_3 \right\}$$

$$= \frac{\mathbb{P}\{\omega_3 > 5\pi/4 - a, \ \omega_2 < \omega_3\}}{\mathbb{P}\{\omega_2 < \omega_3\}}$$

$$= 2 \mathbb{P}\{\omega_3 > \max(\omega_2, 5\pi/4 - a)\}$$

$$= 2 \left[ \int_{\pi/4}^{5\pi/4-a} \int_{5\pi/4-a}^{3\pi/4} g(\omega_3)g(\omega_2)d\omega_3 d\omega_2 + \int_{5\pi/4-a}^{\pi} \int_{\pi/4}^{3\pi/4} g(\omega_3)g(\omega_2)d\omega_3 d\omega_2 \right].$$

For $g(\omega) = 2/\pi$, evaluating the double integrals yields

$$\mathbb{P}(\alpha < a) = \frac{(2a - \pi)(3\pi - 2a)}{\pi^2}.$$
and, upon differentiation,

\[ f(\alpha) = \frac{8(\pi - \alpha)}{\pi^2}, \quad \frac{\pi}{2} \leq \alpha \leq \pi. \]

For \( g(\omega) = \sin(\omega)/\sqrt{2} \), evaluating the double integrals yields

\[ P(\alpha < a) = \frac{1}{4} \left[ 2 - 2 \cos(a) - 2 \sin(a) - \sin(2a) \right] \]

and, upon differentiation,

\[ f(\alpha) = \frac{1}{2} \left[ -\cos(\alpha) + \sin(\alpha) - \cos(2\alpha) \right], \quad \frac{\pi}{2} \leq \alpha \leq \pi. \]

Moments are easily calculated; the mode is \( \pi/2 \) for the former and

\[ 2 \arctan \left[ \frac{1}{2} \left( -3 + \sqrt{17} + \sqrt{2 \left( 5 - \sqrt{17} \right)} \right) \right] = 1.7713... \]

for the latter. Identical results apply when instead the third line is fixed as the anti-diagonal \( y = -x \).

On a personal note, I had intended this article to be a quick follow-up to my 2011 article on random tangents to a circle \[13\]. Who would have suspected that random intercepts of a segment might be so much more hazardous than the preceding? Uncovering Gates’ model \[4, 5\] constituted a turning point in my writing. This humble contribution is the uncertain outcome of several years of hesitation and delay.

The R package \texttt{spatstat} \[14\] has planar random process simulation capabilities. I can generate Poisson lines in a sampling window via \texttt{rpoisline} and determine their inclination angles \( \omega_j \) via \texttt{angles.psp} (with option \texttt{directed=FALSE}). An elliptical window of eccentricity \( \varepsilon \approx 1 \) is less likely to be hit by lines almost parallel to the major axis than by almost perpendicular lines. In contrast, for a circular window \( (\varepsilon = 0) \), all directions are equally likely. Clarifying these observations more rigorously would be worthwhile and I welcome thoughts on how this should be done.

2. Appendix 1

The ordered pair \((\xi, \omega)\) offers one representation of a line \( L \), involving the \( x \)-intercept \( \xi \) and inclination angle \( \omega \). Another representation \((p, \theta)\) where \(-\infty < p < \infty \) and \(0 \leq \theta < \pi\), called the Hesse normal form, involves the length \( |p| \) of the perpendicular segment from \((0, 0)\) to \( L \) and the orientation angle \( \theta \) of this segment. In the definition of a Poisson line process, it is usually assumed that \( \theta \sim \text{Uniform}[0, \pi] \). From

\[ x \cos(\theta) + y \sin(\theta) = p \]
we see that
\[ p = \begin{cases} 
-\xi \sin(\omega) & \text{if } \omega < \pi/2, \\
\xi \sin(\omega) & \text{if } \omega \geq \pi/2
\end{cases} \quad \text{and} \quad \theta = \begin{cases} 
\omega + \pi/2 & \text{if } \omega < \pi/2, \\
\omega - \pi/2 & \text{if } \omega \geq \pi/2
\end{cases} \]
since \( \cos(\omega \pm \pi/2) = \mp \sin(\omega) \). At first glance, it would seem that \( \omega \sim \text{Uniform}[0, \pi] \) immediately because \( \theta \sim \text{Uniform}[0, \pi] \). In fact, the \( 2 \times 2 \) Jacobian determinant of the transformation \((\xi, \omega) \mapsto (p, \theta)\) is \( \mp \sin(\omega) \), which implies that the density of \( \omega \) is \( \sin(\omega)/2 \). Reason for the factor of 2: both \((\xi, \omega)\) and \((-\xi, \pi - \omega)\) are mapped to the same \((p, \theta)\). Details of the proof in a more general setting appear in [9, 15, 16].

3. Appendix 2
We present R simulation output results (ten histograms in blue) graphed against density expressions found herein (six curves in red). The first four plots correspond to the first four expressions for \( f \), given without proof. The next two plots correspond to those associated with the diagonal line \( y = x \) scenario. Analysis of other scenarios involving the vertical line \( x = 0 \) or the horizontal line \( y = 0 \) are left to the reader.

4. Appendix 3
Given a convex region \( C \) in the plane, a width is the distance between a pair of parallel \( C \)-supporting lines. Fix an inclination angle \( 0 \leq \omega < \pi \) relative to the \( x \)-axis. A measure of all lines of angle \( \omega \) hitting a \( C \)-window is proportional to the corresponding width. For example, if \( C \) is the square \([-1, 1] \times [-1, 1] \), we obtain a bimodal inclination angle density [17]

\[
\frac{1}{4} \max \left\{ \sqrt{1 + \sin(2\omega)}, \sqrt{1 - \sin(2\omega)} \right\}
\]

with modes at \( \pi/4 \) and \( 3\pi/4 \). It is easier to obtain the \( \sin(\omega)/2 \) density for the interval \([-1, 1]\), but harder to examine \( 2 \times 2\sqrt{1 - \varepsilon^2} \) rectangles of eccentricity \( 0 < \varepsilon < 1 \).

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Figure 1: Top row: $f$ for unrestricted. Bottom row: $f$ for restricted.
Figure 2: All restricted. Top row: diagonal $y = x$. Middle row: vertical $x = 0$. Bottom row: horizontal $y = 0$. 
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