SINGULAR LEVI-FLAT HYPERSURFACES IN COMPLEX PROJECTIVE SPACE

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Abstract. We study singular real-analytic Levi-flat hypersurfaces in complex projective space. We give necessary and sufficient conditions for such a hypersurface to be a pullback of a real-analytic curve in $\mathbb{C}$ via a meromorphic function. We define the rank of a real hypersurface and study the connections between rank, degree, and the type and size of the singularity for Levi-flat hypersurfaces. Finally, we study degenerate singularities of algebraic Levi-flat hypersurfaces. Among other examples, we construct a nonalgebraic semianalytic Levi-flat hypersurface with compact leaves that is a perturbation of an algebraic Levi-flat variety.

1. Introduction

The purpose of this paper is to organize some basic results on Levi-flat hypersurfaces in complex projective space. First, we study when such hypersurfaces must be algebraic, and second we study some properties of algebraic Levi-flat hypersurfaces. Along the way we give several examples to illustrate the phenomena encountered.

A real smooth hypersurface in a complex manifold is said to be Levi-flat if it is pseudoconvex from both sides. If the hypersurface is real-analytic and nonsingular, then it is classical that in suitable local coordinates, it can be represented by $\text{Im } z_1 = 0$. Therefore, there are no local holomorphic invariants. The situation is different if we allow the hypersurface to have singularities. Local questions about singular Levi-flat hypersurfaces have been previously studied by Bedford [2], Burns and Gong [7], and the author [16,17]. See also the books [1,5,10] for the basic language and background.

Let $\mathbb{P}^n$ be the $n$-dimensional complex projective space. Lins Neto proved [18] that no nonsingular real-analytic Levi-flat hypersurfaces exist in $\mathbb{P}^n$, $n \geq 3$. There have since been much work on generalizing this result further. A different approach for the real-analytic case was taken by Ni and Wolfson [19], Siu [20], Cao and Shaw [9], and most recently Iordan and Matthey [13] improved the regularity requirement. The $n = 2$ case was studied by Siu [21].

Singular real-analytic Levi-flat hypersurfaces, however, are a different story and many such hypersurfaces exist. Instead of hypersurface, we will use the term hypervariety for a codimension one subvariety to emphasize the possibility of singularities, and to emphasize it is a closed subvariety. Also, unless specifically stated, subvarieties are analytic, not necessarily algebraic. Let $H \subset U \subset \mathbb{C}^k$ be a real-analytic hypervariety, i.e. a closed real subvariety of $U$ of real codimension one. Let $H^*$ be the set of points near which $H$ is a smooth hypersurface. A real hypervariety $H$ is said to be Levi-flat, if it is Levi-flat at all points of $H^*$. $H^*$ is foliated

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by complex hypersurfaces, and this foliation is called the Levi foliation. Any (real) algebraic Levi-flat hypervariety in \( \mathbb{C}^n \) can be extended to a Levi-flat hypervariety in \( \mathbb{P}^n \).

One method to obtain algebraic Levi-flat hypervarieties in \( \mathbb{P}^n \) is by taking a rational function \( R: \mathbb{P}^n \to \mathbb{C} \), and a real-algebraic one-dimensional subset \( S \subset \mathbb{C} \), and considering the set \( H = R^{-1}(S) \). Then \( H \) is an algebraic Levi-flat hypervariety. If \( H \) is algebraic, then all the leaves of the Levi foliation must be compact. A leaf is said to be compact if the closure is a subvariety in \( \mathbb{P}^n \) of the same dimension. If a hypervariety is to be given by \( H = R^{-1}(S) \), then at least locally it must be given by \( F^{-1}(T) \) for some local meromorphic function \( F \) and some real-analytic set \( T \subset \mathbb{C} \). We will relax this condition and suppose that \( F \) is constant along the leaves of \( H \). It turns out that these conditions are in fact sufficient.

**Theorem 1.1.** Let \( H \subset \mathbb{P}^n, n \geq 2 \), be an irreducible Levi-flat hypervariety with infinitely many compact leaves. Assume that for each \( p \in H^* \), there exists a neighbourhood \( U \) of \( p \) and a meromorphic function \( F \) defined on \( U \) such that \( F \) is constant along leaves of \( H^* \).

Then, there exists a global rational function \( R: \mathbb{P}^n \to \mathbb{C} \) and a real-algebraic one-dimensional subset \( S \subset \mathbb{C} \) such that \( H \subset R^{-1}(S) \). In particular, \( H \) is semialgebraic; it is contained in an algebraic Levi-flat hypervariety.

To prove the theorem we must find two objects. We must find an algebraic set \( S \subset \mathbb{C} \) and the function \( R \). We find a foliation of \( \mathbb{P}^n \) by using a result of Lins Neto. To find \( R \) we apply a result by Darboux and generalized by Jouanolou, which says that a foliation of \( \mathbb{P}^n \) with infinitely many compact leaves has a rational first integral. Next, we find the \( S \subset \mathbb{C} \) by proving Lemma 3.2, which says that the image of \( H \) under \( R \) must essentially be our algebraic curve \( S \).

We really need to only study semianalytic sets. A set is semianalytic if it is locally constructed from real-analytic sets by finite union, finite intersection, and complement. For a hypervariety \( H \), the set \( H^* \) is semianalytic. We will state a version of Theorem 1.1 for semianalytic hypersurfaces, see Theorem 4.1.

The hypothesis of compact leaves seems necessary. Example 8.1 is a perturbation of an algebraic Levi-flat hypervariety of \( \mathbb{P}^2 \), which is again Levi-flat, closed, semianalytic (thus contained near each point in a real-analytic subvariety), but not algebraic. The leaves of this hypersurface are complex hyperplanes, but do not extend to a foliation of \( \mathbb{P}^2 \). It also seems likely that a closure of a noncompact leaf of a foliation of \( \mathbb{P}^2 \) could be semianalytic, though no such example is known to the author.

Not all algebraic Levi-flat hypervarieties arise in the above way. One particular feature of hypervarieties defined using rational functions is the existence of a degenerate singularity (in the sense of Segre varieties) in dimension 2 or higher. There are, however, algebraic Levi-flat hypervarieties that do not have a degenerate singularity as we will show in Example 8.1.

To study algebraic Levi-flat hypervarieties we define their rank. It is the rank of the Hermitian form of the defining bihomogeneous polynomial. Equivalently, the rank is the minimum number of holomorphic polynomials needed to write the defining polynomial as a difference of squared norms. This definition of rank was used by the author together with D’Angelo to study a seemingly unrelated problem in [11]. See also the book by D’Angelo [10] for further applications of this circle of ideas. For example, it is useful to write a defining equation of a hypersurface as a
squared norm to characterize the complex varieties contained in the hypersurface. We will also study a local analytic version of the rank.

A simple argument shows that the dimension of the singular set of an algebraic Levi-flat hypervariety in $\mathbb{P}^n$ must be at least $2n - 4$. If the hypervariety has no nondegenerate singularities, the dimension of the singular set must be of maximal possible dimension, and the rank of the defining equation must also be large compared to the dimension. We will write $H_{\text{sing}}$ for the singular set of $H$. By the singular set we mean the set of points near which $H$ is not a smooth submanifold. $H_{\text{sing}}$ is not in general equal to the complement of $H^*$ as defined above, and is only a semialgebraic (or semianalytic if $H$ is not algebraic) set.

It is standard to also consider the algebraic singular set of an algebraic hypervariety. The algebraic singular set is the set of points where the defining polynomial has vanishing gradient. The algebraic singular set always contains the analytic singular set ($H_{\text{sing}}$ as defined above), and the containment can be proper. A classical example is $y^3 + 2x^2y - x^4 = 0$, which has no analytic singularities but has an algebraic singularity at the origin, see [4] for more on these issues. Therefore, we also have the concept of an algebraic degenerate singularity. Any degenerate singularity is an algebraic degenerate singularity, but not necessarily vice versa. We summarize what we can say about degenerate singularities in the following theorem.

**Theorem 1.2.** Let $H \subset \mathbb{P}^n$, $n \geq 2$, be an algebraic Levi-flat hypervariety of rank $r$.

(i) If $r \leq n$ then there exists a complex subvariety $S \subset H$ of dimension at least $n - r$ such that every point in $S$ is an algebraic degenerate singularity of $H$.

(ii) If $\dim H_{\text{sing}} < 2n - 2$ then $H$ has a degenerate singularity.

(iii) If $\dim H_{\text{sing}} = 2n - 4$ then there is a complex subvariety $S \subset H$ of dimension $n - 2$ such that every point in $S$ is a degenerate singularity of $H$.

The structure of this paper is as follows. In §2 we give some standard basic results about real subvarieties of complex projective space and Levi-flat hypervarieties in particular. In §3 we study Levi-flat hypervarieties defined by meromorphic functions. In §4 we study holomorphic foliations induced by Levi-flat hypervarieties and prove two alternate versions of Theorem 1.1. In §5 we prove that foliations extend from Levi-flat hypervarieties even without compact leaves. In §6 and §7 we introduce and discuss the rank of the hypersurface. In §8 we will prove Theorem 1.2 and study the set of degenerate singularities of an algebraic Levi-flat hypervariety. And finally in §9 we study nonalgebraic Levi-flat hypervarieties and semianalytic sets with compact leaves.

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## 2. Basic properties

Let $\sigma : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ be the natural projection. Suppose $X$ is a real-analytic subvariety of $\mathbb{P}^n$. Define the set $r(X)$ to be the set of points $z \in \mathbb{C}^{n+1}$ such that $\sigma(z) \in X$ or $z = 0$. A real-analytic subvariety $X \subset \mathbb{P}^n$ is said to be *algebraic* if $X = \sigma(V)$ for some real-algebraic complex cone $V$ in $\mathbb{C}^{n+1}$. A set $S$ is a complex cone when $p \in S$ implies $\lambda p \in S$ for all $\lambda \in \mathbb{C}$. We will say that an algebraic subvariety $X \subset \mathbb{P}^n$ is of degree $d$, if $d$ is the smallest integer such that you need real
polynomials of degree at most $d$ to define $\tau(X)$. We first establish some standard and easy to see properties of real-analytic subvarieties of $\mathbb{P}^n$. By a bihomogeneous polynomial we mean a polynomial that is separately homogeneous in $z$ and $\bar{z}$. That is, $P(tz, \overline{tz}) = t^{d/2}z^{d/2}P(z, \bar{z})$. Thus, by a degree $d$ bihomogeneous polynomial we mean of bi-degree $(d/2, d/2)$.

**Proposition 2.1.** Suppose $X \subset \mathbb{P}^n$ is a real-analytic subvariety.

(i) $\tau(X) \setminus \{0\}$ is a real-analytic subvariety of $\mathbb{C}^{n+1} \setminus \{0\}$.

(ii) $\tau(X)$ is subanalytic.

(iii) $X$ is algebraic if and only if $\tau(X)$ is a real-analytic subvariety.

(iv) If $X$ is an irreducible algebraic hypersurface of degree $d$, then $\tau(X)$ is defined by the vanishing of a single real valued bihomogeneous polynomial of degree $d$ (bi-degree $(d/2, d/2)$).

**Proof.** To see (i), take homogeneous coordinates $[z_1 : \cdots : z_{n+1}]$. Fix e.g. $z_1 = 1$ and find a set of defining functions $\rho_j$ for $X$ in some open set in the affine coordinates $z_2, \ldots, z_{n+1}$. Let $\rho_j(z_1, z_2, \cdots, z_{n+1}) = \rho_j(z_2/z_1, \cdots, z_{n+1}/z_1)$ to be our defining equation in some open subset of in $\mathbb{C}^{n+1} \setminus \{z_1 = 0\}$.

To see (ii) let again $[z_1 : \cdots : z_{n+1}]$ be the homogeneous coordinates, and let us work in the chart where $z_1 \neq 0$. Let $\tilde{X}$ be the subvariety in this chart. Take the semianalytic set $(\tilde{X} \cap B_n) \times \mathbb{D}$, where $\mathbb{D} \subset \mathbb{C}$ is the unit disc and $B_n \subset \mathbb{C}^n$ is the unit ball. Define the function $\varphi : \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}^{n+1}$ by $\varphi(w, \xi) = (\xi, \xi w)$. This map takes $(\tilde{X} \cap B_n) \times \mathbb{D}$ to a subanalytic set $Y \subset \tau(X)$. Furthermore, as $X$ is compact, then there are finitely many such charts and sets $Y_j$. The germ of $\cup_j Y_j$ at the origin agrees with the germ of $\tau(X)$ at the origin, which is what we needed to prove.

One direction of (iii) is clear, the other is the same as in the holomorphic case. Let $\rho$ be a real-analytic function that is zero on $\tau(X)$ near the origin. Let $\rho = \sum_j \rho_j$ be the decomposition into homogeneous parts. Take $t \in (-1, 1)$ and note $\rho(tz) = \sum_j \rho_j(tz) = \sum_j t^j \rho_j(z)$. For a fixed $z \in X$ we have a power series that is identically zero. Hence each $\rho_j$ must be zero on $X$ and $X$ is therefore algebraic.

Finally, let us prove (iv). Let $p$ be a real polynomial vanishing on $\tau(X)$. Write

$$p(z, \bar{z}) = \sum_{j,k} p_{jk}(z, \bar{z}) \quad (1)$$

where $p_{jk}$ is homogeneous of order $j$ in $z$ and of order $k$ in $\bar{z}$. Note that if $z \in \tau(X)$, then $\lambda z \in \tau(X)$ for all $\lambda \in \mathbb{C}$. Hence, if $z \in \tau(X)$ then for all $\lambda$

$$0 = \sum_{j,k} p_{jk}(\lambda z, \bar{\lambda} \bar{z}) = \sum_{j,k} \lambda^j \bar{\lambda}^k p_{jk}(z, \bar{z}). \quad (2)$$

If we complexify $\lambda$ and $\bar{\lambda}$, we get a polynomial in two variables that is identically zero. Therefore, $p_{jk}(z, \bar{z}) = 0$ for all $j$ and $k$.

Since $\tau(X)$ is a real cone, it must be defined by single irreducible real homogeneous polynomial of lowest degree. If the defining equation is not real homogeneous, then we can find a smaller degree homogeneous polynomial vanishing on $\tau(X)$. Call this polynomial $p$ and write $p_{jk}$ as above. Both the real and the imaginary parts of $p_{jk}$ must vanish on $X$, hence we can write $p_{jk}(z, \bar{z}) = A(z, \bar{z})p(z, \bar{z})$ for some (complex valued) polynomial $A$. The degree of $p$ is equal to the degree of $p_{jk}$ and both are real homogeneous. Plugging in $tz$ for $z$ and dividing by $t^k$ we notice that $p_{jk}(z, \bar{z}) = A(tz, t\bar{z})p(z, \bar{z})$ for all $t \in \mathbb{R}$, in particular when $t = 0$. Hence $p_{jk}$ is a
constant times $p$ and of course $p_{jk} = p$. As $p$ was real valued we are done. Notice also that $j = k$. □

It is equally easy to see that any real polynomial in $\mathbb{C}^n$ can be made into a bihomogeneous polynomial in $\mathbb{C}^{n+1}$ and defines a real subvariety of $\mathbb{P}^n$.

We will be using the Segre variety to study Levi-flat hypervarieties. Let $H \subset U \subset \mathbb{C}^k$ be a real hypervariety defined by $\rho(z, \bar{z}) = 0$ for some real-analytic function $\rho$ defined in $U$. Let $\text{conj}(U) = \{z \mid \bar{z} \in U\}$. Suppose that the power series for $\rho$ converges in $U \times \text{conj}(U)$ and hence we may complexify $\rho$. We define the Segre variety $\Sigma_p$ as the set

$$\Sigma_p := \{z \in U \mid \rho(z, \bar{p}) = 0\}. \tag{3}$$

The following is classical (see also for example [7]) but we prove it here for completeness.

**Proposition 2.2.** Suppose that $H \subset U \subset \mathbb{C}^k$ is a Levi-flat hypervariety ($U$ is small enough as above) and $p \in H^*$. Then one component $\Sigma_p'$ of $\Sigma_p$ agrees as a germ with the leaf of the Levi foliation of $H$ through $p$. The germ of $\Sigma_p'$ is the unique germ of a complex hypervariety through $p$.

**Proof.** Taking $U$ smaller can at most make $\Sigma_p$ smaller. Thus we can make $U$ small enough such that we can make a local change of coordinates such that $H$ is given near $p$ as $\{\text{Im} z_1 = 0\}$ and $p$ is the origin. Then $\rho(z, \bar{z}) = a(z, \bar{z})(1/2i)(z_1 - \bar{z}_1)$. $\Sigma_0$ then contains $\{z_1 = 0\}$. Let $f(z)$ be another holomorphic function such that $\{f = 0\} \subset H$ and $f(0) = 0$, then $f$ is real valued on $H$ and in particular on $\{z_1 = 0\}$, hence $f = 0$ when $z_1 = 0$, and uniqueness follows. □

In particular, if $H$ is a Levi-flat hypervariety and for some $p \in H$ there are two distinct germs of a complex hypervariety through $p$ contained in $H$, then $p$ must be a singular point of $H$. Similarly, if there is a germ of a complex analytic hypervariety contained in $H$ through $p$, singular at $p$, then $H$ itself must be singular at $p$. If $H$ is of a higher codimension at $p$, then the above two statements are obvious.

We now focus on Levi-flat hypervarieties $H \subset \mathbb{P}^n$. We get the following corollary. We say a leaf $L$ of the Levi foliation of $H^*$ is compact if the closure $\bar{L}$ has the same dimension. In this case, by Remmert and Stein $\bar{L}$ is a complex subvariety. We will generally abuse terminology and call $\bar{L}$ a leaf of $H$.

**Corollary 2.3.** Let $H \subset \mathbb{P}^n$ be an algebraic Levi-flat hypervariety. Then all leaves of $H$ are compact.

**Proof.** Take a defining polynomial for $\tau(H)$ and look at the Segre varieties. Note that any leaf is either contained in some Segre variety that is proper subset of $\mathbb{C}^{n+1}$, in which case it is a compact leaf, or it is contained in the set of points where the Segre variety is not a proper subset of $\mathbb{C}^{n+1}$ (what we will call the degenerate singularities). But that subset itself must be a proper complex subvariety. □

We have the following simple, and surely classical, observation.

**Proposition 2.4.** If $H \subset \mathbb{P}^n$ is an algebraic Levi-flat hypervariety. Then

$$2n - 4 \leq \dim H_{\text{sing}} \leq 2n - 2. \tag{4}$$

**Proof.** Any leaf of $\tau(H)$ must pass through the origin since $\tau(H)$ is a complex cone. Hence, any two leaves must meet on a complex subvariety of dimension $n - 1$ in $\mathbb{C}^n$ and this set must lie in the singular set. □
The singularity can also be of larger dimension than $2n - 4$. Pick any singular algebraic curve $S$ in $\mathbb{P}^1$ and look at $\tau(S) \subset \mathbb{C}^2$. The singular set is going to be a finite union of complex lines through the origin. Of course this argument also implies that if $n \geq 2$, then $H$ must be singular.

The canonical local example of a Levi-flat hypersurface in $\mathbb{C}^n$ is defined by $\text{Im } z_1 = 0$. This hypersurface can of course be extended to all of $\mathbb{P}^n$. If we bihomogenize this equation we will get a quadratic complex cone in $\mathbb{C}^{n+1}$ given by

$$z_1 \bar{z}_2 - \bar{z}_1 z_2 = 0. \quad (5)$$

Burns and Gong [7] have classified, up to local biholomorphism, all germs of quadratic Levi-flat hypervarieties. I.e. up to biholomorphism, there is only one quadratic complex cone that is a Levi-flat hypervariety, and that is given by (5). It is not hard to show this fact directly using Proposition 2.1 and it is equivalent to the following proposition.

**Proposition 2.5.** Suppose that $H$ is a quadratic Levi-flat hypervariety in $\mathbb{P}^n$. Then $H$ is biholomorphically equivalent to a hypervariety given by (5).

**Proof.** Let $\rho$ be the defining bihomogeneous polynomial of degree 2 for $\tau(H)$. Since it is of degree 2 and bihomogeneous it can be written as a Hermitian form, i.e. $\rho(z, \bar{z}) = \bar{z}^t A z$, where $z = (z_1, \ldots, z_{n+1})^t$. As $A$ is Hermitian, we can make a linear change of variables (a biholomorphic transformation of $\mathbb{P}^n$) such that $A$ is diagonal. Thus we can assume

$$\rho(z, \bar{z}) = \sum_j \epsilon_j z_j \bar{z}_j. \quad (6)$$

We can make further linear transformations to assume that $\epsilon_j = -1, 0, 1$. Being Levi-flat is equivalent to the Levi form vanishing at all smooth points. This condition is equivalent to the following differential equation

$$\text{rank } \begin{bmatrix} \rho & \rho_z \\ \rho_{\bar{z}} & \rho_{\bar{z}z} \end{bmatrix} \leq 2 \quad \text{on } \rho = 0. \quad (7)$$

All 3 by 3 subdeterminants of the matrix must be zero, hence all but two $\epsilon_j$ must be zero. It is not hard to see that at least two must be nonzero and of different sign, otherwise $H$ is not a hypersurface. Thus we can assume that $\rho(z, \bar{z}) = z_1 \bar{z}_1 - z_2 \bar{z}_2$, which is unitarily equivalent to (5). \qed

Therefore, there exist affine coordinates such that every quadratic Levi-flat hypervariety of $\mathbb{P}^n$ is given by $\text{Im } z_1 = 0$ in those affine coordinates. When $n \geq 2$, the hypervariety is singular and the singular set is a complex subvariety of dimension $n - 2$.

We end the section with an example which illustrates the subtlety of the geometry of the singular set. Levi-flat hypervarieties generally suffer from the same subtle issues as do real-analytic subvarieties in general.

**Example 2.6.** A classical example is the subvariety given by $y^2 + x^2 - x^3 = 0$ in $\mathbb{R}^2$. We get an irreducible (algebraically) curve for which the origin is an isolated point. We can think of $\mathbb{R}^2$ as $\mathbb{C}$ using $x + iy = z$ and let $X$ be the subvariety extended to $\mathbb{P}^1$. The equation then becomes

$$\bar{z}^3 + 3z \bar{z}^2 + 3z^2 \bar{z} - 8z \bar{z} + z^3 = 0. \quad (8)$$
We bihomogenize this equation to get a complex cone in \( \mathbb{C}^2 \). That is, we define the hypervariety \( H = \tau(X) \) by
\[
w^3z^3 + 3w^2\bar{w}z^2 + 3w\bar{w}^2z\bar{z} - 8w^2\bar{w}z\bar{z} + \bar{w}^3z^3 = 0. \quad (9)
\]
The left-hand side is irreducible as a polynomial, but also analytically at the origin. Suppose \( f \) is a real-analytic function defined on a neighbourhood of the origin that vanishes on a nontrivial part of \( H^* \). Write \( f = \sum f_j \) where \( f_j \) are real homogeneous of degree \( j \). Using the proof of Proposition 2.8 we see that each \( f_j \) vanishes on a nontrivial part of \( H^* \). Thus each \( f_j \) vanishes on all of \( H \), as \( H \) was defined by an irreducible polynomial. Hence, \( f \) vanishes on all of \( H \).

\( H \) is Levi-flat as it is a complex cone in \( \mathbb{C}^2 \). Near all points of the set \( \{z = 0, w \neq 0\} \), \( H \) is a complex line. Therefore \( H^* \) does not include the set \( \{z = 0, w \neq 0\} \). This set is colloquially called the “stick” of the “umbrella.”

Note also, that the one-dimensional part of \( X \) is a real-analytic subvariety, but it is only semialgebraic. So we have an example of a real-analytic Levi-flat hypervariety of \( \mathbb{P}^1 \), that is semialgebraic and not algebraic.

The “stick” of the umbrella need not be complex analytic. Brunella [6] gives the following example. Let \( z = x + iy \) and \( w = s + it \). Then the set given by \( t^2 = 4(y^2 + s)y^2 \) is Levi-flat and the “stick” is the set \( \{t = s = 0, s \leq 0\} \), which is totally real in \( \mathbb{C}^2 \).

Finally, we will need the following lemma of Burns and Gong (Lemma 2.2 in [7]) to see that for an irreducible hypervariety \( H \) we need only require it to be Levi-flat at one point of \( H^* \).

**Lemma 2.7** (Bruns-Gong). Let \( H \subset \mathbb{C}^n \) be a real-analytic hypervariety, locally irreducible at point \( p \in H \). Then there exists an open set \( U \subset \mathbb{C}^n \) containing \( p \) such that \( H \cap U \) is Levi-flat if and only if one of the components of \( H^* \cap U \) is Levi-flat.

The lemma is essentially proved by noticing that equation (7) must hold everywhere on \( H \) by complexification of the irreducible defining function of \( H \). By noticing again that the property of being Levi-flat is equivalent to the equation (7), we have the following trivial classical proposition which is useful in application of the Burns-Gong lemma.

**Proposition 2.8.** Let \( M \subset \mathbb{C}^n \) be a connected real-analytic submanifold of real dimension \( 2n - 1 \). \( M \) is Levi-flat if and only if there exists an open set \( N \subset M \) such that \( N \) is Levi-flat.

3. **Algebraic Levi-flat hypervarieties defined by meromorphic functions**

The following construction gives a large supply of Levi-flat hypervarieties of \( \mathbb{P}^n \), although it is not exhaustive.

**Proposition 3.1.** Let \( F \) be a meromorphic function on \( \mathbb{P}^n \). Let \( S \subset \mathbb{C} \) be a real-algebraic curve. Then the set \( F^{-1}(S) \) is an algebraic Levi-flat hypervariety. However, not all algebraic Levi-flat hypervarieties in \( \mathbb{P}^n \) are defined in this manner.

**Proof.** It is standard that any meromorphic function on \( \mathbb{P}^n \) is algebraic. Let \( P : \mathbb{C} \to \mathbb{R} \) be the defining polynomial of \( S \). Let \( H = \{z \mid P \circ F(z) = 0\} \). We only need to show locally that \( H \) is a subvariety and that it is Levi-flat at all points of \( H^* \). Write \( F \) in some set of affine coordinates as \( F = f/g \) for two relatively prime polynomials.
If $P$ is a polynomial of degree $d$ we notice that $\left|g^d\right|^2 (P \circ F)$ is a polynomial whose zero set is precisely $H$ in the affine chart we have chosen. Hence $H$ is a real-algebraic subvariety. To see that it is Levi-flat, note that locally it is always foliated by surfaces defined by the set $\{f = \lambda g\}$ for some constant $\lambda \in \mathbb{C}$.

To see the second part we refer to Example 8.1. In that example we construct an algebraic Levi-flat hypervariety such that there does not exist a point contained in infinitely many leaves of the Levi foliation. If $H$ is defined by a meromorphic function, there has to exist a point $p$ of indeterminacy since the dimension is at least 2. Define $\tilde{F}$ and notice that it is a finite map because if we replace $\bar{\zeta}$ and $\zeta$, the image is also a complex subvariety. We are really interested in picking a 2-dimensional subspace $V$.

**Lemma 3.2.** Let $H \subset U \subset \mathbb{C}^n$, $n \geq 2$, be an irreducible Levi-flat hypervariety of $U$, and let $p \in H$ be a point. Suppose there exists a meromorphic function $F$ defined in $U$ such that $F$ is constant along the leaves of $H^*$. Then there exists a one-dimensional algebraic subset $S \subset \mathbb{C}$ such that $H \subset F^{-1}(S)$.

**Proof.** First note that without loss of generality we can assume that $n = 2$. If we pick a 2-dimensional subspace $V$ and find an $S$ such that $F^{-1}(S) \cap V$ contains $H \cap V$, then since the inverse image of a single point under $F$ contains the whole relevant leaf of the Levi foliation, $F^{-1}(S)$ must then contain all of $H$ as $H$ is irreducible.

We can freely also pick a smaller neighbourhood $U$ of $p$. The conclusion of the lemma is true for a smaller neighbourhood, then it is true for the original $U$. So by perhaps picking a smaller $U$, we can assume that the neighbourhood $U$ is symmetric with respect to complex conjugation and assume that $H$ complexes to $U \times U$. That is, the Taylor series of defining equation $\rho$ of $H$ converges on $U \times U$ if we replace $\bar{z}$ with a new variable $w$.

Let $F = f/g$ in $U$ where $f$ and $g$ are relatively prime. If we look at the map

$$\psi(z, w) := (f(z), g(z), \tilde{f}(w), \tilde{g}(w))$$

and notice that it is a finite map because $f^{-1}(0) \cap g^{-1}(0)$ must be a set of codimension 2, hence a finite set. If $\tilde{H} \subset U \times U$ is the complexified $H$, then as $\psi$ is finite, the image is also a complex subvariety. We are really interested in $\varphi(H)$, where

$$\varphi(z) = (f(z), g(z)).$$

The image $\varphi(H)$ can be thought of as a (possibly proper) subset of $\psi(\tilde{H})$ intersected with the totally real submanifold $\{\zeta_1, \zeta_2, \zeta_3, \zeta_4 \mid \zeta_1 = \bar{\zeta}_3, \zeta_2 = \bar{\zeta}_4\}$. The point is that $\varphi(H)$ is semianalytic, that is, near the origin contained in a real-analytic subvariety $K$ of the same dimension.

Notice that $\varphi(H)$ is Levi-flat and $G(z) = z_1/z_2$ is constant along leaves of $\varphi(H)$. That means that $\varphi(H)$ contains complex lines through the origin. Take a defining
function \( r(z, \bar{z}) \) for \( K \). Write \( r \) as

\[
  r(z, \bar{z}) = \sum_{j,k} r_{jk}(z, \bar{z})
\]

(12)

where \( r_{jk} \) is homogeneous of order \( j \) in \( z \) and of order \( k \) in \( \bar{z} \). Suppose that \( z \in \varphi(H) \subset K \), then \( \lambda z \in \varphi(H) \subset K \) for some small open set of \( \lambda \) and so

\[
  0 = \sum_{jk} r_{jk}(\lambda z, \bar{\lambda} \bar{z}) = \sum_{jk} \lambda^j \bar{\lambda}^k r_{jk}(z, \bar{z}).
\]

(13)

By the same logic as in the proof of Proposition 2.1 the set \( K \) is defined by a bihomogeneous polynomial. In particular \( K \) is real-algebraic.

We look at the algebraic set

\[
  \{(z, \xi) \in \mathbb{C}^2 \times \mathbb{C} \mid z \in K, \text{ and } \xi z_2 = z_1\}.
\]

(14)

We project this set onto the \( \xi \) variable. By the theorem of Tarski-Seidenberg, the projection must be semialgebraic. It is not hard to see that the set must be of dimension one. A one-dimensional semialgebraic set is contained in a one-dimensional algebraic set \( S \subset \mathbb{C} \). Hence, \( K \subset G^{-1}(S) \), and as \( \varphi(H) \subset K \) and as \( F = G \circ \varphi \) then \( H \subset F^{-1}(S) \).

\[\square\]

4. Levi-flats and foliations

In this section we will prove Theorem 1.1. If a Levi-flat hypervariety of \( \mathbb{P}^n \) is locally defined by meromorphic functions and has infinitely many compact leaves, then it is algebraic and furthermore defined by a global meromorphic function on \( \mathbb{P}^n \). We will need the language of holomorphic foliations to prove this result.

A possibly singular holomorphic foliation \( F \) of codimension one of a complex manifold \( M \) is given by an open covering \( \{U_i\} \) with the following property. In each \( U_i \) a holomorphic one-form \( \omega_\i \). If \( U_i \cap U_\kappa \neq \emptyset \), then \( \omega_\i \) and \( \omega_\kappa \) must be proportional at every point of the intersection. A complex manifold is called a solution if it satisfies the differential equation \( \omega_\i = 0 \) in each \( U_\i \). The points where \( \omega_\i \) vanishes are called the singular set of \( F \) and denoted \( \text{sing}(F) \). The set \( M \setminus F \) is then a union of immersed complex hypersurfaces called leaves of the foliation. Note that the codimension of the singularity of the foliation can safely be taken to be at least 2, by dividing out the coefficients of the form by a common divisor. When talking about foliations of \( \mathbb{P}^n \), we will say a leaf is compact if its topological closure is of the same dimension. In this case we will also use the word leaf for the closure. As we assume the singularity is of codimension at least 2, a compact leaf is a complex analytic subvariety by the theorem of Remmert and Stein. See [8, 18] for more information on foliations in general. All foliations in the sequel will be holomorphic of codimension one.

The Levi foliation of a Levi-flat hypervariety does not necessarily extend (even locally) to a foliation of a neighbourhood of the hypervariety, at least not in the above sense, see Brunella [6]. If the Levi-flat hypervariety is such that locally there exists a meromorphic function \( F = f/g \) (in lowest terms) that is constant along leaves of \( H^* \), then the foliation extends locally. The leaves are defined by components of the sets \( \{f = \lambda g\} \) for a constant \( \lambda \) and the form is given by \( \omega = f(df) - g(dg) \).

Of course, the condition that the foliation extends is a necessary condition for a hypervariety to be defined in the same manner as in Theorem 1.1. If we further
know that the leaves of the Levi foliation are compact, these two conditions turn out to be sufficient.

As we said in the introduction, we will prove the theorem for semianalytic sets. We also need not require that the foliation extending that of $H$ be locally first integrable. The main feature of semianalytic sets we will use is that at each point, a germ of a semianalytic set is contained in a germ of a real-analytic set of the same dimension.

**Theorem 4.1.** Let $H \subset \mathbb{P}^n$, $n \geq 2$, be a connected semianalytic set of real dimension $2n - 1$. Suppose that $H = \bigcup L_i$, where $L_i$ are complex analytic hypervarieties of $\mathbb{P}^n$. Assume that for each $p \in H$, there exists a neighbourhood $U$ of $p$ and a holomorphic foliation on $U$ such that $L_i \cap U$ are invariant.

Then, there exists a global rational function $R: \mathbb{P}^n \to \mathbb{C}$ and a real-algebraic one-dimensional subset $S \subset \mathbb{C}$ such that $H \subset R^{-1}(S)$.

Essentially we are asking for a foliation of a neighbourhood of $H$, and $H$ to be an invariant set of the foliation. To extend the foliation into all of $\mathbb{P}^n$ we use the following result of Lins Neto [18].

**Theorem 4.2** (Lins Neto). Let $M$ be a Stein manifold, $\dim(M) \geq 2$. Let $K \subset M$ be compact, with $M \setminus K$ connected, and let $F$ be a singular holomorphic foliation of $M \setminus K$ where $\text{codim}(\text{sing}(F)) \geq 2$. Then $F$ extends to a singular holomorphic foliation on $M$.

To be able to apply Theorem 4.2 we need to find Stein manifolds inside $\mathbb{P}^n$.

**Theorem 4.3** (Takeuchi [22]). Let $U \subset \mathbb{P}^n$ be an open set such that $U \neq \mathbb{P}^n$. Suppose that $U$ is pseudoconvex (satisfies Kontinuitätssatz), then $U$ is Stein.

Take one complex variety $L = L_i$ that lies in $H$. The set $\mathbb{P}^n \setminus L$ is Stein. If we have a foliation of a neighbourhood of $H$, we have a foliation of a neighbourhood of $L$. We can then apply Theorem 4.2 to get a foliation of $\mathbb{P}^n$. The leaves of the foliation must coincide with the complex varieties near $L$ that are part of $H$. Note that $H$ is connected, so the $L_i$ are leaves of the extended foliation of $\mathbb{P}^n$.

Once we have the foliation extended to all of $\mathbb{P}^n$, we will need to find the rational function $R$. Therefore, we apply the following classical theorem of Darboux (see [12] page 29) generalized by Jouanolou, see [14] Theorem 3.3 page 102.

**Theorem 4.4** (Darboux-Jouanolou). If $F$ is a singular holomorphic foliation on $\mathbb{P}^n$ with infinitely many compact leaves, then $F$ has a rational first integral.

As $H$ is of dimension $2n - 1$, it must contain infinitely many complex varieties. As these coincide with the leaves of $F$ (the extended foliation), $F$ has infinitely many compact leaves and hence has a rational first integral. Therefore, the final piece of the proof of Theorem 4.1 is the following lemma, which may be of independent interest.

**Lemma 4.5.** Suppose that there exists a singular holomorphic foliation $F$ of $\mathbb{P}^n$, $n \geq 2$, with a rational first integral $R$. Let $H \subset \mathbb{P}^n$ be a connected semianalytic set of real dimension $2n - 1$ that is an invariant set of $F$. Then there exists a real-algebraic one-dimensional subset $S \subset \mathbb{C}$ such that $H \subset R^{-1}(S)$.

**Proof.** We can assume that $H$ is closed just by taking the closure, which is also semianalytic and invariant. We can write $H = \bigcup L_i$, where $L_i$ are irreducible
complex analytic hypervarieties (leaves of $\mathcal{F}$). Let $R$ be the first integral of $\mathcal{F}$. $R$ must be constant along the $L_i$. We find a point $p \in \mathbb{P}^n$ that is a point of indeterminacy for $R$, which exists because $n \geq 2$. Further a point $p$ of indeterminacy has to lie on $H$, since it must be in the closure of every leaf $L_i$. That is, write $R = f/g$, then without loss of generality there is some fixed $\lambda \neq 0$, such that $f/g = \lambda$ on $L_i$. The numerator $f$ must be zero on $L_i$ and then $g$ must also be zero at the same point and that must be a point of indeterminacy.

We can apply Lemma 4.2 near $p$ to find the required $S$. As $H \subset R^{-1}(S)$ locally near $p$, $H$ is a union of the $L_i$, and $p \in L_i$ for all $i$, then $H \subset R^{-1}(S)$.

Proof of Theorem 4.1. For every point $p \in H$ we have a neighbourhood $U$ and a foliation on $U$ extending the Levi foliation of $H^* \cap U$. We call $H^*$ the smooth part of real dimension $2n - 1$ just like for hypervarieties.

Suppose we have two connected neighbourhoods $U_1$ and $U_2$ such that $U_1 \cap U_2$ is nonempty and connected. Further, assume there exist holomorphic one-forms $\omega_j$ on $j = 1, 2$ that define a foliation extending the foliation of $H^*$. If we can show that $\omega_1$ is proportional to $\omega_2$ then we have a foliation of $U_1 \cap U_2$. Take a small neighbourhood of some point $p \in H^*$. We know that $\omega_1$ must be proportional to $\omega_2$ for all points of $H^*$ near $p$. $H^*$ is a real hypersurface, thus they are proportional in a whole neighbourhood as they are holomorphic. Since $U_1 \cap U_2$ is connected, we are done by analytic continuation.

We can choose a covering of $H$ that satisfies the above conditions for every pair of intersecting neighbourhoods. Hence, if the foliation of $H^*$ extends locally near every point of $H$, then there exists a neighbourhood $U$ of $H$ and a foliation $\mathcal{F}$ on $U$ that extends the foliation of $H^*$. Again, we can assume that the codimension of the singularity of the foliation is at least two.

We pick one complex hypervariety $L$ that lies in $H$ and we apply Theorem 4.2 to extend the foliation extends to a foliation on all of $\mathbb{P}^n$. As we said above, the foliation $\mathcal{F}$ has infinitely many compact leaves. We can apply Theorem 4.4 to get a rational first integral.

Finally we appeal to Lemma 4.5 which has the same conclusion as our theorem.

Once we have Theorem 4.1 it is not too hard to finish the proof of Theorem 4.1.

We can notice that $H^*$ is semianalytic, and we just need to show that it is a union of compact leaves. However, it is easier to modify the above proof. As above, we need not require the foliation to be locally first integrable.

Theorem 4.6. Let $H \subset \mathbb{P}^n$, $n \geq 2$, be an irreducible Levi-flat hypervariety with infinitely many compact leaves. Assume that for each $p \in \overline{H^*}$, there exists a neighbourhood $U$ of $p$ and a holomorphic foliation on $U$ extending the Levi foliation of $H^*$.

Then, there exists a global rational function $R: \mathbb{P}^n \rightarrow \mathbb{C}$ and a real-algebraic one-dimensional subset $S \subset \mathbb{C}$ such that $H \subset R^{-1}(S)$. In particular, $H$ is semialgebraic; it is contained in an algebraic Levi-flat hypervariety.

Proof. Follow the same argument as in the proof of Theorem 4.1. By exactly the same argument, we have a foliation of a neighbourhood of all of $\overline{H^*}$. If there are infinitely many compact leaves of $H$, one is contained in $\overline{H^*}$. We have a foliation of a neighbourhood of this compact leaf and we can extend the foliation to a foliation
of $\mathbb{P}^n$. So we have a foliation of $\mathbb{P}^n$ that extends the foliation of $H^\ast$. As it has infinitely many compact leaves, it has a first integral $R$. The set $\overline{\mathbb{P}^n}$ is invariant and satisfies the hypothesis of Lemma 4.5. □

5. Extending foliations

We have already proved Theorem 1.1, but it will be interesting to also prove the following stronger result about foliations, which does not use the compactness of leaves. This result is also of independent interest and is essentially an extension of a similar result by Lins-Neto to singular Levi-flat hypervarieties.

Theorem 5.1. Suppose $H \subset \mathbb{P}^n$, $n \geq 2$, is an irreducible Levi-flat hypervariety. Assume that for each $p \in H^\ast$, there exists a neighbourhood $U$ of $p$ and a meromorphic function $F$ defined on $U$ such that $F$ is constant along leaves of $H^\ast$.

Then, there exists a singular holomorphic foliation $\mathcal{F}$ of $\mathbb{P}^n$ that agrees with the foliation of $H^\ast$.

We already know we have a foliation of a neighbourhood of $H^\ast$. We notice the following corollary of the theorem of Takeuchi. Once the following lemma is proved, the proof of Theorem 5.1 follows at once.

Lemma 5.2. Let $H \subset \mathbb{P}^n$ be a Levi-flat hypervariety, such that for every $p \in H^\ast$ there exists a neighbourhood $U$ and a meromorphic function $F$ such that $F$ is constant along leaves of $H^\ast$. Then all the connected components of $\mathbb{P}^n \setminus H^\ast$ are Stein.

This corollary follows after we have shown that through every point of $H^\ast$ there exists a germ of a complex hypervariety contained in $H^\ast$, hence $H^\ast$ is pseudoconvex from all sides. A weaker theorem, that through every point of $H^\ast$ there exists a complex hypervariety contained in $H$ was essentially proven by Fornaess (see [15] Theorem 6.23). The statement by Fornaess assumes that $H$ is nonsingular, but that is not used in the proof. See also Burns and Gong [7] for more information regarding this point.

If $H$ was not a complex hypersurface near any point, we would be done by the theorem of Fornaess. However, Example 2.6 shows that it is possible to have an irreducible Levi-flat hypervariety with a component that is a complex hypervariety. We will prove the following lemma, which, together with Takeuchi’s theorem, implies Lemma 5.2 and hence Theorem 5.1.

Lemma 5.3. Let $H \subset U \subset \mathbb{C}^k$ be a Levi-flat hypervariety and $p \in \overline{H^\ast}$. Suppose there exists a meromorphic function $F$ defined in $U$ that is constant along leaves of $H^\ast$. Then there exists a germ of a complex hypervariety $(L, p)$ such that $(L, p) \subset (H^\ast, p)$.

Proof. First let us assume that $p$ is a point of indeterminacy of $F$. Let $F = f/g$ written in lowest terms. We can follow the proof of Lemma 3.2 to note that the image of $H$ under the map $z \mapsto (f(z), g(z))$ is a complex cone. Therefore, given a constant $\lambda$, set $\{f = \lambda g\}$ contains a leaf of $H$ going through the origin. I.e. there are infinitely many leaves of $H$ going through the origin. Only finitely many leaves can form a “stick” of an umbrella, and hence infinitely many are contained in $\overline{H^\ast}$.

If $p$ is not a point of indeterminacy of $F$, we can assume $F$ is holomorphic. By taking $U$ smaller, we could assume $F$ is holomorphic in all of $U$ and assume
Define the graph $\Gamma_F := \{(z, \xi) \mid \xi = F(z)\}$. After a possible linear change of coordinates in the $z$ variable we can apply the Weierstrass preparation theorem to get $\Gamma_F$ defined by
\[
z^d_k + \sum_{j=0}^{d-1} a_j(z', \xi)z^j_k = 0,
\]
where $z' = (z_1, \ldots, z_{k-1})$. The set $V_\lambda := \{z \mid F(z) = \lambda\}$ is a multigraph over the $z'$ of multiplicity at most $d$. That is, we have a holomorphic function $h: U' \subset \mathbb{C}^{k-1} \to \mathbb{C}^{\text{sym}}$ (a multifunction) and $V_\lambda$ is the set $\{z \mid z_k \in h(z')\}$. See [23] for more information on symmetric powers and multifunctions.

Pick a sequence of $\lambda_j \to 0$, such that $V_{\lambda_j}$ contains a branch $V'_{\lambda_j} \subset H$. As there can locally be at most finitely many branches of $H$ that are complex hypervarieties, we can assume that $V'_{\lambda_j} \subset H'$ for all $j$. As $V_\lambda$ is a multigraph of multiplicity $d$, there must exist a single integer $m$ such that each $V'_{\lambda_j}$ is a multigraph of multiplicity $m$. Assume $V'_{\lambda_j}$ is the multigraph of $h_j: U' \subset \mathbb{C}^{k-1} \to \mathbb{C}^{m \text{sym}}$. The functions $h_j$ are bounded and hence we can pass to a convergent subsequence. That is, there exists a complex hypervariety $V$ that is the limit of $V'_{\lambda_j}$. Since $V'_{\lambda_j} \subset H'$ then $V \subset H'$, furthermore, $p \in V$ as $\lambda_j \to 0$ and $F(p) = 0$. □

6. Rank

Let $H \subset \mathbb{P}^n$ be an algebraic Levi-flat hypervariety. Let $P$ be the defining bihomogeneous polynomial for $\tau(H)$. Using multiindex notation we write
\[
P(z, \bar{z}) = \sum_{\alpha \beta} c_{\alpha \beta} z^\alpha \bar{z}^\beta.
\]
Hence, if we order the multiindices in some way and write the column vector $Z = (z^{\alpha_1}, z^{\alpha_2}, \ldots, z^{\alpha_m})^t$, we can write the matrix $C = [c_{\alpha \beta}]_{\alpha \beta}$, and then
\[
P(z, \bar{z}) = Z^t C Z
\]
As $P$ is real valued, then $c_{\alpha \beta} = \overline{c_{\beta \alpha}}$, hence $C$ is Hermitian.

**Definition 6.1.** Let $H \subset \mathbb{P}^n$ be a real-algebraic hypervariety and $P$ the defining polynomial for $\tau(H)$. We form the matrix $C$ and define
\[
\text{rank } P := \text{rank } C, \quad (18)
\]
\[
\text{rank } H := \text{rank } C. \quad (19)
\]
It is standard that if $C$ is of rank $r$, then there exist $r$ column vectors $v_1, v_2, \ldots, v_r$ such that
\[
C = v_1 \overline{v_1} + \cdots + v_r \overline{v_r} - v_{r+1} \overline{v_{r+1}} - \cdots - v_r \overline{v_r}.
\]
Taking $p_j(z) := \overline{v_j} Z$ we can see that
\[
P(z, \bar{z}) = |p_1(z)|^2 + \cdots + |p_s(z)|^2 - |p_{s+1}(z)|^2 - \cdots - |p_r(z)|^2. \quad (21)
\]
The number $r$ is the minimum number of holomorphic polynomials $p_j$ we will need for such a decomposition. Hence, the rank $r$ can be also defined as the minimal number of holomorphic polynomials such that $P$ can be written as $\leq (d/2+n)$.

As $H$ is a hypersurface, there must be at least some positive and some negative eigenvalues of $C$. Therefore, $\text{rank } H \geq 2$. On the other hand, we have the trivial estimate $\text{rank } H \leq (d/2+n)$.
**Proposition 6.2.** If rank $H = 2$, then $H$ is Levi-flat.

**Proof.** $H$ is the set $|p_1(z)|^2 - |p_2(z)|^2 = 0$, and hence a Levi-flat hypervariety defined by the meromorphic function $p_1/p_2$. □

**Example 6.3.** Of course there exist Levi-flat hypervarieties with higher rank. For example, Let $z = x + iy$. The real curve $x^3 - y^2 = 0$ in $\mathbb{C}$ can be extended to $\mathbb{P}^1$ by bihomogenizing the equation (using the variable $w$) to get the polynomial

$$w^3 z^3 + 3zw^2 z^2 w + 2w^3 z^2 w + 3z^2 w^2 - 4zw^2 zw^2 + z^3 w^2 + 3z^2 w^3.$$  \hfill (22)

If we let $Z = (z^3, z^2 w, zw^2, w^3)^t$, we get the matrix

$$C = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 3 & 2 \\ 0 & 3 & -4 & 0 \\ 1 & 2 & 0 & 0 \end{bmatrix}.$$  \hfill (23)

The rank is 4, there are 2 positive and 2 negative eigenvalues. We can use the identity $ab + \bar{a}\bar{b} = |a + b|^2 - |a - b|^2$ to actually find a decomposition of the polynomial as follows

$$|z^3 + 2z^2 w + w^3|^2 - |z^3 + 2z^2 w - w^3|^2 + |3z^2 w + zw^2 + 2zw^2|^2 - |3z^2 w - zw^2 - w|^2.$$  \hfill (24)

Hence, we have found an example where the rank of $H$ is equal to the maximal rank possible and $H$ is still Levi-flat.

**Remark 6.4.** Note that Proposition 6.3 also says that any quadratic Levi-flat hypervariety of $\mathbb{P}^n$ must have rank 2. On the other hand, the generic quadratic hypervariety in $\mathbb{P}^n$ has rank $n + 1$. Hence, it is not always possible to construct examples that are Levi-flat and have the maximal rank $\binom{d/2+n}{n}$. However, Example 8.4 is a Levi-flat hypervariety of $\mathbb{P}^2$ of degree 4 and has rank 6, which is the maximum possible.

The above example however gives a way to construct examples of arbitrarily high rank. Given any real-algebraic curve in $\mathbb{C}$ we can bihomogenize the defining equation and get a Levi-flat cone in $\mathbb{C}^{n+1}$, for any $n \geq 1$, and thus get a Levi-flat hypervariety of $\mathbb{P}^n$. As we can choose an irreducible curve of degree $\delta$ such that the bihomogenized polynomial can have rank $\delta + 1$, the maximal possible rank for a curve in $\mathbb{C}$.

**Proposition 6.5.** Let $H \subset \mathbb{P}^n$ be a real-algebraic hypervariety. Then rank $H$ is invariant under automorphisms of $\mathbb{P}^n$.

**Proof.** Let $L$ be an invertible linear mapping and $P$ is given by (21), then $L^{-1}(\tau(H))$ is given by

$$P(Lz, \overline{Lz}) = |p_1(Lz)|^2 + \cdots + |p_s(Lz)|^2 - |p_{s+1}(Lz)|^2 - \cdots - |p_r(Lz)|^2.$$  \hfill (25)

Thus the rank cannot increase (and therefore cannot decrease) by composing with a linear transformation $\mathbb{C}^{n+1}$ and hence an automorphism of $\mathbb{P}^n$. □

It is also possible to work in some set of affine coordinates, rather than the homogeneous coordinates. The rank can be defined in generic affine coordinates and we will get the same number as we get in homogeneous coordinates. This procedure suggests that we might similarly define the rank (locally) for a nonalgebraic
We get a genuinely different notion of rank, which we study in the next section.

7. Analytic rank

Let $H \subset \mathbb{P}^n$ be a real-analytic hypervariety. Even when $H$ is not algebraic we can define the rank locally. Let $r(z, \bar{z})$ be a defining function for $H$ in a neighbourhood of a point $p \in \mathbb{P}^n$. Write

$$r(z, \bar{z}) = \sum \alpha \beta c_{\alpha \beta} z^\alpha \bar{z}^\beta$$

as before. $C_r = [c_{\alpha \beta}]_{\alpha \beta}$ is an infinite matrix, so rank $C_r$ can also be infinite. Let $I(H, p)$ be the ideal of real-analytic functions defined near $p$ vanishing on $H$. Define the (local) analytic-rank at $p$ as

$$\text{analytic-rank}(H, p) := \min \{\text{rank } C_r \mid r \in I(H, p)\}.$$  \hspace{1cm} (27)

Now we define the (global) analytic-rank of $H$ to be

$$\text{analytic-rank } H := \max \{\text{analytic-rank}(H, p) \mid p \in H\}.$$  \hspace{1cm} (28)

Note that the maximum is well defined; if the local analytic rank is unbounded, then we claim that there is a point $p$ where the local analytic rank is infinite. Let $p_j \in H$ be a sequence of points with increasing local analytic rank. We can go to a subsequence such that $p_j \to p \in H$. If rank at $p$ would be finite that it would also give a bound on the rank at all $p_j$ sufficiently close to $p$.

Remark 7.1. Following Remark 6.4 we note that in general for an algebraic $H$

$$\text{analytic-rank } H \neq \text{rank } H.$$  \hspace{1cm} (29)

Simply choose a real one-dimensional nonsingular curve in $\mathbb{C}$ or $\mathbb{P}^1$ of arbitrarily high rank. As the curve is nonsingular the local analytic rank is always 2.

However, the defining polynomial of an algebraic $H$ is in $I(H, p)$ for all $p \in H$.

Proposition 7.2. Let $H \subset \mathbb{P}^n$ be an algebraic Levi-flat hypervariety. Then

$$\text{analytic-rank } H \leq \text{rank } H.$$  \hspace{1cm} (30)

Obviously analytic-rank $H = \infty$ implies that $H$ is non-algebraic. We can state a weaker version of Theorem 1.1 in terms of analytic rank. By the same argument as in Proposition 6.2 if analytic-rank $H = 2$, then $H$ is automatically Levi-flat, and the foliation extends at every point of $H$.

Corollary 7.3. Let $H \subset \mathbb{P}^n$, $n \geq 2$, be a real-analytic hypervariety such that analytic-rank $H = 2$ and $H$ contains infinitely many complex hypervarieties of $\mathbb{P}^n$.

Then, there exists a global rational function $R: \mathbb{P}^n \to \mathbb{C}$ and a real-algebraic one-dimensional subset $S \subset \mathbb{C}$ such that $H \subset R^{-1}(S)$.

8. Degenerate singularities

Let $H \subset U \subset \mathbb{C}^k$ be a Levi-flat hypervariety. Let $\rho$ be a defining function for $H$ defined in a neighbourhood $U$ such that $\rho$ complexifies as in §2 and we may define the Segre variety for all $p \in U$.

We will say that $p \in H$ is a degenerate singularity if the Segre variety $\Sigma_p$ is open (of dimension $k$) for every local defining function. If we assume that $U$ is connected
\[ p \text{ is degenerate if and only if } \Sigma_p = U. \] Hence, \( p \) is a degenerate singularity whenever \( z \mapsto \rho(z, \bar{p}) \) is identically zero for \( z \) near \( p \) for all defining functions \( \rho \) of \( H \).

By the reality of \( \rho \) we note that if \( q \in \Sigma_p \) then \( p \in \Sigma_q \). Hence, if \( p \) is a degenerate singularity, then \( p \in \Sigma_q \) for all \( q \in U \). On the other hand if \( q \in \Sigma_p \) for all \( q \in H \cap U \), then \( q \in \Sigma_p \) for all \( q \in U \) and hence \( p \) is degenerate. Therefore by Proposition 2.2 if \( p \) is a singularity of \( H \) and there are infinitely many distinct germs of complex hypervarieties \((L, p) \subset (H, p)\), then \( p \) is a degenerate singularity.

The hypervariety defined by

\[ z_1 \bar{z}_2 - \bar{z}_1 z_2 = 0 \tag{31} \]

has a degenerate singularity at 0. By the reasoning above, whenever \( H \) is a complex cone, the origin is always a degenerate singularity.

For real-algebraic varieties, there is a distinction between an algebraic singularity and an analytic singularity. Similarly there is a distinction between degenerate singularities and algebraic degenerate singularities. Let \( P(z, \bar{z}) \) be a defining polynomial for a real-algebraic hypervariety \( H \). We then have the Segre variety induced by \( P \), where we take \( U = \mathbb{C}^k \). For a point \( p \) define the algebraic Segre variety as

\[ \Sigma_p = \{ z \mid P(z, \bar{p}) = 0 \}. \tag{32} \]

Suppose that \( p \) is an algebraic singularity of \( H \), i.e. gradient of \( P \) vanishes at \( p \). Then \( p \) is an algebraic degenerate singularity of \( H \) if \( P(z, \bar{p}) \equiv 0 \). If a point is a degenerate singularity, then it must also be an algebraic degenerate singularity.

It may happen that the Segre set for the local analytic defining function is different from the one given by the defining polynomial, as is illustrated by the following two examples. Let \( z_1 = x + iy, z_2 = s + it \). The classical example

\[ y^3 + 2x^2y - x^4 = 0 \tag{33} \]

is a Levi-flat hypervariety with algebraic (but not analytic) singularity along the set \( \{ z_1 = 0 \} \). The algebraic Segre variety is a triple plane at the origin union another disjoint complex hyperplane, while the analytic Segre variety is only a small piece of \( \{ z_1 = 0 \} \). To see a more dramatic example consider the equation

\[ s^3 + 2x^2s - x^4 = 0. \tag{34} \]

This equation defines a real-algebraic hypervariety (not Levi-flat) with an algebraic (but not analytic) singularity on the set \( \{ x = s = 0 \} \). The hypervariety is a smooth real-analytic submanifold and so the Segre variety induced by the local analytic defining equation is a nonsingular complex hypersurface. However, the algebraic Segre variety is locally an intersection of 3 smooth hypersurfaces at the origin.

We have found in Proposition 2.2 that the only quadratic algebraic Levi-flat hypervariety in \( \mathbb{P}^n \) is the quadratic cone given in homogeneous coordinates by \( \{ x_0 = 0 \} \). When \( n \geq 2 \), this hypervariety always has a degenerate singularity. On the other hand it is sufficient to consider degree 4 in \( \mathbb{P}^2 \) to have an example without a degenerate singularity.

**Example 8.1.** To construct a Levi-flat hypervariety of \( \mathbb{P}^2 \) without any (algebraic or analytic) degenerate singularities, we construct a Levi-flat hypervariety \( H \subset \mathbb{C}^3 \) that is a complex cone and such that the origin is the only degenerate singularity. We look at the equation

\[ z_1 + z_2 t + z_3 t^2 = 0. \tag{35} \]
We look the points $z = (z_1, z_2, z_3)$ where this polynomial has a real solution. We thus have the semialgebraic surface

$$\{z \in \mathbb{C}^3 \mid z_1 + z_2 t + z_3 t^2 = 0 \text{ for some } t \in \mathbb{R}\}$$

(36)

By applying the quadratic formula and finding where the solution is real, we obtain the following degree 4 real homogeneous polynomial defining a Levi-flat hypervariety, which contains all the planes defined in (35).

$$z_1^2 z_3^2 + z_1 z_2 z_3 + z_2^2 z_1 z_3 + z_1 z_3 z_2^2 - 2z_1 z_3 z_1 z_3 + z_2 z_3 z_1 z_2 + z_2^2 z_1^2 z_3 = 0.$$  

(37)

The hypervariety is Levi-flat by applying Lemma 2.7 and noticing that at least some subset of the hypervariety is foliation by the varieties defined by

$$z_1 + z_2 t + z_3 t^2 = 0$$

for some fixed real $t$. The only point that lies in all these varieties is the origin.

It is easy to see that even the entire hypervariety has no degenerate singularities (except the origin) by writing the defining equation as

$$z_1^2 z_3^2 + z_1 z_2 z_3 + z_2^2 z_1 z_3 + z_1 z_3 z_2^2 - 2z_1 z_3 z_1 z_3 + z_2 z_3 z_1 z_2 + z_2^2 z_1^2 z_3 = 0.$$  

(38)

We think of $z$ and $\bar{z}$ as independent. To find the (algebraic) Segre variety corresponding to this equation we would set $\bar{z}$ to a constant. So the expressions in parentheses are coefficients of a polynomial in $z$. The only place where they all vanish identically is when $z_1 = z_2 = z_3 = 0$. Thus $\sigma(H) \subset \mathbb{P}^2$ has no algebraic degenerate singularities, and hence no analytic degenerate singularities. The variety $\sigma(H)$ is of rank 6 as is easily seen from the defining equation (37).

The size of the algebraic degenerate singular set is related to the rank of $H$, which is itself related to the degree. The following lemma proves the first part of Theorem 1.2 from the introduction.

**Lemma 8.2.** Let $H \subset \mathbb{P}^n$, $n \geq 2$, be an algebraic Levi-flat hypervariety of rank $r$. If $r \leq n$ then there exists a complex subvariety $S \subset H$ of dimension at least $n - r$ such that every point in $S$ is an algebraic degenerate singularity of $H$.

In particular, if $H$ is nondegenerate then rank $H > n$. The proof of this lemma is essentially the following observation, which we state as a proposition. This result essentially gives us a method to find all algebraic degenerate singularities of $H$.

**Proposition 8.3.** Let $H$ be as above and $P$ the defining bihomogeneous polynomial, and let $r$ be the rank. Write

$$P(z, \bar{z}) = |p_1(z)|^2 + \cdots + |p_s(z)|^2 - |p_{s+1}(z)|^2 - \cdots - |p_r(z)|^2.$$  

(39)

Then $w$ is an algebraic degenerate singularity of $\tau(H)$ if and only if $p_j(w) = 0$ for all $j = 1, \ldots, r$.

Note that if $p_j(w) = 0$ for all $j$, then $w \in \tau(H)$, and $z \mapsto P(z, \bar{w})$ is identically zero. As the rank is $r$, the $p_j$ are linearly independent. The converse then follows. We finish the proof of Lemma 8.2 by applying Proposition 8.3 and taking $S$ to be the subvariety defined by $p_j = 0$ for all $j$.

It is not true that high rank guarantees lack of degeneracy. Since any real-algebraic curve in $\mathbb{C}$ extends to a Levi-flat hypervariety in $\mathbb{P}^n$ as in Remark 6.4, we can get Levi-flat hypersurfaces with arbitrarily high rank. However, all hypervarieties obtained in this way will have degenerate singularities.

By Proposition 2.7 we note that the singular set of an algebraic Levi-flat hypervariety of $\mathbb{P}^n$ has to be at least of real dimension $2n - 4$. This fact follows because
when two leaves of the foliation meet, they must meet in a set of complex dimension $n - 2$ and this set must be contained in the singular set of $H$. It therefore easy to see that if the singular set is only of dimension $2n - 4$ then all the leaves must meet on the same set of dimension $n - 2$. Hence we have the following lemma, which proves the second part of Theorem 1.2.

**Lemma 8.4.** Let $H \subset \mathbb{P}^n$, $n \geq 2$, be an algebraic Levi-flat hypervariety such that $\dim H_{\text{sing}} = 2n - 4$. Then there exists a complex subvariety $S$ of dimension $n - 2$ such that every point in $S$ is a degenerate singularity of $H$.

Obviously if $H$ is not to have any degenerate singularity, then the singular set must be large. The author essentially proved in [16] that if the singular set is a submanifold of dimension $2n - 2$, in the hypervariety, it is either complex or Levi-flat (i.e. locally equivalent to $\mathbb{C}^{n-2} \times \mathbb{R}^2$). The following lemma tells us that nondegeneracy must be compensated by such a singular set and proves the final part of Theorem 1.2.

**Lemma 8.5.** If $H \subset \mathbb{P}^n$, $n \geq 2$, is an algebraic Levi-flat hypervariety without degenerate singularities, then the singular set must be of real dimension $2n - 2$.

**Proof.** We look at $\tau(H) \subset \mathbb{C}^{n+1}$. We look at the leaves of the Levi foliation going through the origin. Any two such leaves must meet on a set of complex dimension $n - 1$, and this set must lie in the singular set of $\tau(H)$. As before, if all leaves met on the same set, then $\tau(H)$ would have a degenerate singularity away from the origin and hence $H$ would have a degenerate singularity. Thus suppose that the singular set of $\tau(H)$ is of dimension $2n - 1$. Let us look at a family of leaves $\{L_t\}$ parametrized by a real parameter $t$ in some small interval $(-\epsilon, \epsilon)$. That is, find

$$P(z, t) := \sum_{|\alpha| = d} a_\alpha(t)z^\alpha,$$  \hspace{1cm} (40)

where $a_\alpha(t)$ are real-analytic functions in $t$, and such that the sets $L_t = \{z | P(z, t) = 0\}$ are leaves of $\tau(H)$. We can find such a $P$ by considering the coefficients of a polynomial in $z$ as variables and then the set of polynomials whose zero sets are contained in $\tau(H)$ is a semialgebraic set.

Take two such parameters and look at the set $L_t \cap L_s$. The sets $L_t \cap L_s$ have real dimension $2n - 2$ (complex dimension $n - 1$). Fix $t$ and note $\bigcap_{s \neq t} L_s \cap L_t = \emptyset$. Hence, there must exist a submanifold $T_t \subset \tau(H)_{\text{sing}}$ of dimension $2n - 1$ that is foliated by $(n - 1)$-dimensional complex submanifolds (the $L_t \cap L_s$). We can pick a maximal such $T_t$ (not necessarily unique).

For each $t$ such a statement is true and as the singular set is of dimension $2n - 1$, there is some $t_0$ such that for infinitely $t$, the set $T_{t_0} \cap T_t$ is nonempty and of dimension $2n - 1$. But then infinitely many $L_t$ have the same nontrivial intersection with $L_{t_0}$, and hence the hypersurface would have a degenerate singularity. We obtain a contradiction. Consequently, the singular set of $\tau(H)$ must be of dimension $2n$, and the singular set of $H$ was real $(2n - 2)$-dimensional. \hfill \Box

9. Nonalgebraic hypervarieties with compact leaves

In this paper, we have mostly studied Levi-flat hypervarieties (or semianalytic sets) with compact leaves. Each compact leaf is algebraic. Therefore, the following
construction gives the most obvious type of Levi-flat hypersurface with compact leaves. For \( z \in \mathbb{C}^{n+1} \) let
\[
f(z, t) = \sum_{|\alpha| = d} c_\alpha(t)z^\alpha,
\]
where \( c_\alpha(t) \) are real-analytic functions of \( t \in (a, b) \subset \mathbb{R} \). If \( c_\alpha \) are analytic up to \( a \) and \( b \), then
\[
H = \{ z \in \mathbb{C}^{n+1} \mid f(z, t) = 0, \text{ for some } t \in (a, b) \} \tag{42}
\]
is a subanalytic Levi-flat hypersurface, which is a complex cone. Hence \( \sigma(H) \) is a subanalytic Levi-flat hypersurface in \( \mathbb{P}^n \).

Define the function of \((w, t) \in \mathbb{C}^{N+1} \times (a, b)\) by
\[
F(w, t) = \sum_{k=1}^{N+1} c_\alpha(t)w_k. \tag{43}
\]
The set
\[
H' = \{ w \in \mathbb{C}^{N+1} \mid F(z, t) = 0, \text{ for some } t \in (a, b) \} \tag{44}
\]
is a subanalytic Levi-flat hypersurface whose leaves are complex hyperplanes. As before \( \sigma(H') \subset \mathbb{P}^n \) is also subanalytic Levi-flat hypersurface. Let \( Z: \mathbb{C}^{n+1} \to \mathbb{C}^{N+1} \), where \( N+1 \) is the number of distinct degree \( d \) monomials, be the degree \( d \) Veronese mapping. That is, \( Z \) is the mapping \( z \mapsto \bigoplus_{|\alpha| = d} z^\alpha \). We then have
\[
H = Z^{-1}(H'). \tag{45}
\]
Therefore, to study hypersurfaces of the form (42) we need only study Levi-flat hypersurfaces of the form (44) with leaves being complex hyperplanes.

**Example 9.1.** Let us build a semianalytic Levi-flat hypersurface of \( \mathbb{P}^2 \) with compact leaves, which is a small perturbation of an algebraic Levi-flat hypervariety of \( \mathbb{P}^2 \), but is not algebraic itself. This example suggests that any analogue of Chow’s theorem for Levi-flat hypervarieties will likely have to require compact leaves.

First let us construct the algebraic Levi-flat hypervariety. Take
\[
H = \{ z \in \mathbb{C}^3 \mid z_1 + xz_2 + yz_3 = 0, x^2 + y^2 = 1, x \in \mathbb{R}, y \in \mathbb{R} \}. \tag{46}
\]
That is, \( H \) is the projection of a variety in \( \mathbb{C}^3 \times \mathbb{R}^2 \) onto \( \mathbb{C}^3 \). It is not just semialgebraic, it is in fact a real hypervariety in \( \mathbb{C}^3 \), and of course Levi-flat with leaves that are complex hyperplanes. To see that \( H \) is a variety, write \( z_j = s_j + it_j \). We have \( s_1 + xs_2 + ys_3 = 0 \) and \( t_1 + xt_2 + yt_3 = 0 \). Solve for \( x \) and \( y \) to get
\[
x = \frac{-s_3t_1 - s_1t_3}{s_3t_2 - s_2t_3}, \quad y = \frac{s_2t_1 - s_1t_2}{s_3t_2 - s_2t_3}. \tag{47}
\]
Therefore, \( H \) is defined by
\[
(s_3t_1 - s_1t_3)^2 + (s_2t_1 - s_1t_2)^2 = (s_3t_2 - s_2t_3)^2. \tag{48}
\]
This equation defines a Levi-flat complex cone in \( \mathbb{C}^3 \) and hence a Levi-flat hypervariety in \( \mathbb{P}^2 \).

To define a perturbation of \( H \), we want to perturb \( x^2 + y^2 = 1 \). Suppose we take a real-analytic \( f(x) \) that is a small perturbation of \( x \), and such that \( C = \{ \mathbb{R}^2 \mid f(x)^2 + y^2 = 1 \} \) is not contained in an algebraic curve. We can also ensure that near each point \( p \) on the curve \( C \) we can parametrize \( C \) by a one-to-one real-analytic \( \gamma: (-\epsilon, \epsilon) \to C \), and we only need to pick finitely many such \( \gamma \)'s to parameterize all of \( C \). That is, \( C \) is a compact topological manifold.
We now need only show that
\[ H' = \{ z \in \mathbb{C}^3 \mid z_1 + xz_2 + yz_3 = 0, f(x)^2 + y^2 = 1, x \in \mathbb{R}, y \in \mathbb{R} \} \quad (49) \]
is semianalytic for all \( p \in H' \) except \( p = 0 \). Then we need to show that \( H' \) is not contained in a real-algebraic variety. We then obtain a Levi-flat semianalytic set in \( \mathbb{P}^2 \) with compact leaves that is not contained in a real-algebraic Levi-flat.

Define \( H'' \subset \mathbb{C}^3 \times \mathbb{R}^2 \) by \( z_1 + xz_2 + yz_3 = 0 \) and \( f(x)^2 + y^2 = 1 \). Take a point \((\xi_1, \xi_2, \xi_3, x_0, y_0) \in H'' \) such that \((\xi_1, \xi_2, \xi_3) \neq (0, 0, 0)\). Find a \( \gamma = (\gamma_1, \gamma_2) \) as above for the \( C \) near the point \((x_0, y_0)\). The function
\[ t \mapsto \xi_1 + \gamma_1(t)\xi_2 + \gamma_2(t)\xi_3 \quad (50) \]
is not identically zero. Hence, we can apply Weierstrass preparation theorem to the function \( z_1 + \gamma_1(t)z_2 + \gamma_2(t)z_3 \) of \((z, t)\) with respect to the \( t \) variable. The projection of \( H'' \cap U \) to \( \mathbb{C}^3 \) for some neighbourhood \( U \) of \((\xi_1, \xi_2, \xi_3, x_0, y_0)\) is the same as the projection of \( \{ z_1 + \gamma_1(t)z_2 + \gamma_2(t)z_3 = 0 \} \) for some small interval of \( t \). If we know that the projection of this set to \( \mathbb{C}^3 \) is semianalytic, we are done.

The above claim is achieved by the following version of Tarski-Seidenberg theorem by Łojasiewicz, see Theorem 2.2 in [3]. Let us set up some terminology. Suppose \( A(U) \) is any ring of real valued functions on an open set \( U \subset \mathbb{R}^n \). Define \( S(A(U)) \) to be the smallest set of subsets of \( U \), which contain the sets \( \{ x \in U \mid f(x) > 0 \} \) for all \( f \in A(U) \), and is closed under finite union, finite intersection and complement. A set \( V \subset \mathbb{R}^n \) is semianalytic if and only if for each \( x \in \mathbb{R}^n \), there exists a neighbourhood \( U \) of \( x \), such that \( V \cap U \in S(\mathcal{O}(U)) \), where \( \mathcal{O}(U) \) denotes the real-analytic real valued functions. Let \( A(U)[t] \) denote the ring of polynomials in \( t \in \mathbb{R}^m \) with coefficients in \( A(U) \).

**Theorem 9.2** (Tarski-Seidenberg-Łojasiewicz), Suppose that \( V \subset U \times \mathbb{R}^m \subset \mathbb{R}^{n+m} \), is such that \( V \in S(A(U)[t]) \). Then the projection of \( V \) onto the first \( n \) variables is in \( S(A(U)) \).

Consequently, if we can locally Weierstrass the defining function with respect to the \( t \) variable, we can project onto the remaining variables and obtain a semianalytic set. Of course, the Weierstrass theorem will only apply in some neighbourhood, and hence for a small finite interval of the \( t \). We only need to do the projection for \( t \) in a compact interval for finitely many curves \( \gamma \). A finite union of semianalytic sets is semianalytic.

Finally we must show that \( H' \) is not contained in a real-algebraic hypervariety. Fix \( z_2 = -1 \) and \( z_3 = -i \). The defining equations become
\[ z_1 = x + iy, \quad f(x)^2 + y^2 = 1. \quad (51) \]
We picked \( f(x) \) precisely in such a way that this set projected onto \( z_1 \) is not contained in an algebraic curve.

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