Generating new classes of fixed-time stable systems with predefined upper bound for the settling time

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ABSTRACT
This paper aims to provide a methodology for generating autonomous and non-autonomous systems with a fixed-time stable equilibrium point where an Upper Bound of the Settling Time (UBST) is set a priori as a parameter of the system. Furthermore, some conditions for such an upper bound to be the least one are provided. This construction procedure is a relevant contribution compared with traditional methodologies for generating fixed-time algorithms satisfying time constraints since current estimates of an UBST may be too conservative. The proposed methodology is based on time-scale transformations and Lyapunov analysis. It allows the presentation of a broad class of fixed-time stable systems with predefined UBST, placing them under a common framework with existing methods using time-varying gains. To illustrate the effectiveness of our approach, we generate novel, autonomous and non-autonomous, fixed-time stable algorithms with predefined least UBST.

1. Introduction
In recent years, dynamical systems exhibiting convergence to their origin in some finite time, independent of the initial conditions of the system, have attracted a great deal of attention. For this class of dynamical systems, their origin is fixed-time stable, which is a stronger notion than finite-time stability (Bhat & Bernstein, 2000; Moulay & Perruquetti, 2006), because, in the latter, the settling time is, in general, an unbounded function of the initial conditions of the system. This research effort has derived several contributions on algorithms with predefined least settling time, such as synchronisation of complex networks (Khanzadeh & Pourgholi, 2017; Liu & Chen, 2018; Liu et al., 2019; Tian et al., 2018; Yang et al., 2017), stabilising controllers for deterministic systems (Basin et al., 2016; Gómez-Gutiérrez, 2020; Muñoz-Vázquez et al., 2019; Polyakov, 2012; Polyakov et al., 2015; Sánchez-Torres et al., 2020; Sánchez-Torres, Aldana-López et al., 2020; Sánchez-Torres et al., 2020; Zimenko et al., 2018; Zuo, 2019), stochastic systems (Yu et al., 2019), distributed resource allocation (Lin et al., 2020), fault detection schemes (Taoufik et al., 2021), optimisation (Ning et al., 2017; Torro et al., 2018), multi-agent coordination (Aldana-López et al., 2019; Defoort et al., 2016; Liu et al., 2019; Shi et al., 2018; Wang et al., 2018; Yu & Yu, 2020; Zuo & Tie, 2014), state observers (Ménard et al., 2017) and online differentiation algorithms (Angulo et al., 2013; Cruz-Zavala et al., 2011).

The fixed-time stability property is of great interest in developing algorithms for scenarios where real-time constraints need to be satisfied. In fault detection, isolation and recovery schemes (Tabatabaeipour & Blanke, 2014), failing to detect a fault on time may lead to an unrecoverable mode. In hybrid dynamical systems, it is frequently required that the observer (resp. controller) stabilises the observation error (resp. tracking error) before the next switching occurs (Defoort et al., 2011; Gómez-Gutiérrez et al., 2015). In the frequency control of an interconnected power network, it is of interest to control how long the frequency stays out of the bounds (Mishra et al., 2018).

A Lyapunov differential inequality for an autonomous system to exhibit fixed-time stability was presented in Polyakov (2012) and Zuo and Tie (2016), together with an Upper Bound of the Settling Time (UBST) of the system trajectory. However, in Aldana-López, Jiménez-Rodríguez et al. (2019) it was derived in detail that such an upper estimate can be made arbitrarily conservative with respect to the least upper bound of the settling time. Only recently, non-conservative UBST has been derived (Aldana-López, Jiménez-Rodríguez et al., 2019; Parsegov et al., 2012; Yu et al., 2020) for some scenarios. Based on homogeneity theory, an alternative characterisation was proposed in Andrieu et al. (2008), Polyakov et al. (2016) and Tian et al. (2018). Although it is a powerful tool for the design of high order fixed-time stable algorithms, it poses a challenging design problem for time-constrained scenarios since an UBST is often unknown. Thus the design of fixed-time stable systems where an UBST is set a priori explicitly as a parameter of the system and the reduction/elimination of the conservativeness of an UBST is of great interest. This problem has been partially addressed for autonomous systems, see, e.g. Aldana-López, Jiménez-Rodríguez et al. (2019), Sánchez-Torres...
et al. (2018) and Aldana-López et al. (2019), mainly focusing on the class of systems proposed in Polyakov (2012), Sánchez-Torres et al. (2018) and Jiménez-Rodríguez et al. (2018); and for non-autonomous systems, mainly focusing on time-varying gains that either become singular (Becerra et al., 2018; Kan et al., 2017; Morasso et al., 1997; Song et al., 2018; Wang et al., 2018; Yucelen et al., 2018) or induce Zeno behaviour (Liu et al., 2018; Ning & Han, 2018) as the predefined time is reached.

The main contributions of this paper can be summarised as follows:

- A methodology for generating new classes of autonomous and non-autonomous fixed-time stable systems, where an UBST is set a priori explicitly as a parameter of the system, is provided.
- The main result is a sufficient condition in the form of a Lyapunov differential inequality, for a nonlinear system to exhibit this property.
- It is shown that for any fixed-time stable system with continuous settling time function there exists a Lyapunov function satisfying such differential inequality.
- Based on this characterisation, it is shown how a fixed-time stable system with predefined UBST can be designed from a nonlinear asymptotically stable one, presenting sufficient conditions for such an upper bound to be the least one.
- To illustrate our approach, some examples are provided to show how one can derive autonomous and non-autonomous fixed-time stable systems, with predefined least UBST.

This is a significant contribution to the design of control systems satisfying time constraints since, even in the scalar case, the existing UBST estimates are often too conservative, as discussed in detail in Aldana-López, Jiménez-Rodríguez et al. (2019).

Notation: $\mathbb{R}$ is the set of real numbers, $\mathbb{R} = \mathbb{R} \cup \{-\infty, +\infty\}$, $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$ and $\mathbb{R}_+ = \mathbb{R}_+ \cup \{+\infty\}$. The Euclidean norm of $x \in \mathbb{R}^n$ is denoted as $|x|$. $h(z) = \frac{d\psi(t)}{dt}$ denotes the first derivative of the function $h : \mathbb{R} \rightarrow \mathbb{R}$, $C^k(I)$ is the class of functions $f : I \rightarrow \mathbb{R}$ with $k \geq 0$ and $I \subseteq \mathbb{R}$ which has continuous $k$th derivative in $I$, $\mathcal{C}^k$ is the class of strictly increasing $C^1((0, a))$ functions $h : [0, a) \rightarrow \mathbb{R}$ with $a, b \in \mathbb{R}$ satisfying $h(0) = 0$ and $\lim_{z \rightarrow a} h(z) = b$.

2. Preliminaries
We start by providing some standard definitions regarding fixed-time stability as found in the current literature. Consider the system

\[
x' = -\frac{1}{T_c} f(x, t), \quad \forall t \geq t_0, \quad f(0, t) = 0,
\]

where $x \in \mathbb{R}^n$ is the state of the system, $T_c > 0$ is a parameter, $t \in [t_0, +\infty)$ and $f : \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}^n$ is continuous on the first parameter. Moreover, in this work we allow $f(x, t)$ to be continuous almost everywhere on $t$. Due to this condition, solutions of (1) are better understood in the sense of Caratheodory (O’Regan, 1997). We assume that $f(x, t)$ is such that the origin of system (1) is asymptotically stable and system (1) has the properties of existence and uniqueness of solutions in forward-time on the interval $[t_0, +\infty)$ (Khalil & Grizzle, 2002). The solution of (1) for $t \geq t_0$ with initial condition $x_0$ is denoted by $x(t; x_0, t_0)$, and the initial state is given by $x(t_0; x_0, t_0) = x_0$.

**Remark 2.1:** For simplicity, throughout the paper, we assume that the origin is the unique equilibrium point of the systems under consideration. Thus, without ambiguity, we refer to the global stability (in the respective sense) of the system’s origin as the stability of the system. The extension to local stability is straightforward.

**Definition 2.1** (Polyakov & Fridman, 2014, Settling-time function): The settling-time function of system (1) is defined as

\[
UBST_{t_0} = \inf \{t \geq t_0 : \lim_{t \rightarrow \xi} x(t; x_0, t_0) = 0\} - t_0 \in \mathbb{R}.
\]

For autonomous systems ($f$ in (1) does not depend on $t$), the settling-time function is independent of $t_0$.

**Remark 2.2:** Notice that, if the system is at most asymptotically stable i.e. there does not exist any $\xi < +\infty$ such that $\lim_{t \rightarrow \xi} x(t; x_0, t_0) = 0$, $x_0 \neq 0$, then according to Definition 2.1, $\forall x_0 \neq 0, T(x_0, t_0) = \inf \mathbb{R} = +\infty$.

**Definition 2.2** (Polyakov & Fridman, 2014, Fixed-time stability): System (1) is said to be fixed-time stable if it is asymptotically stable (Khalil & Grizzle, 2002) and the settling-time function $T(x_0, t_0)$ is bounded on $\mathbb{R}^n \times \mathbb{R}_+$, i.e. there exists $T_{\infty} \in \mathbb{R}_+ \setminus \{0\}$ such that $T(x_0, t_0) \leq T_{\infty}$ if $t_0 \in \mathbb{R}_+$ and $x_0 \in \mathbb{R}^n$. Thus, $T_{\infty}$ is an UBST of $x(t; x_0, t_0)$.

In this work, we are interested on finding sufficient conditions on system (1) such that an UBST is given by the parameter $T_c$, i.e. $T_c = T_{\infty}$. Of particular interest is to find sufficient conditions such that $T_c$ is the least possible UBST, this is $T_c \equiv \inf \{T_{\infty} : T(x_0, t_0) \leq T_{\infty}, \forall (t_0, x_0) \in \mathbb{R}^n \times \mathbb{R}_+\} \equiv \sup_{(x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}_+} T(x_0, t_0)$.

2.1 Time-scale transformations
One of the tools we use in the approach presented in this work is time-scale transformations, introduced in the following. As in Picó et al. (2013), the trajectories corresponding to the system solutions in (1) are interpreted, in the sense of differential geometry (Kühnel, 2015), as regular parametrised curves. Since, in further developments, we only apply regular parameter transformations over the time variable, then without ambiguity, this reparametrisation is sometimes referred to as time-scale transformations.

**Definition 2.3** (Kühnel, 2015, Definition 2.1): A regular parametrised curve with parameter $t$, is a $C^1(I)$ immersion $c : I \rightarrow \mathbb{R}$, defined on a real interval $I \subseteq \mathbb{R}$. This means that $\frac{dc}{dt} \neq 0$ holds everywhere.

**Definition 2.4** (Kühnel, 2015, Pg. 8): (Regular parameter transformation) A regular curve is an equivalence class of regular parametrised curves, where the equivalence relation is given by regular (orientation preserving) parameter transformations.
where \( \mathcal{H} : \mathbb{R} \to \mathbb{R} \) is continuous in \( \mathbb{R} \), locally Lipschitz in \( \mathbb{R} \setminus \{0\} \), and satisfies \( \mathcal{H}(0) = 0 \) and \( \forall z \in \mathbb{R} \setminus \{0\}, z \mathcal{H}(z) > 0 \). Moreover, \( \tilde{x}(\tau; x_0, 0) \) is the unique solution of the asymptotically stable system

\[
\frac{d\tilde{x}}{d\tau} = -\mathcal{H}(\tilde{x}), \quad \tilde{x}(0; x_0, 0) = x_0, \quad \tilde{x} \in \mathbb{R},
\]

and \( T(x_0, 0) \) is its settling time. Such \( \mathcal{H}(\bullet) \) is called an admissible base function and (2) is called an admissible base system.

The following lemma presents the construction of the time-scale transformation that will be used hereinafter.

**Lemma 3.2:** Let \( \tilde{x}(\tau; x_0, 0) \) be a solution to the admissible base system (2) for some \( \mathcal{H}(\bullet) \) and let \( \mathcal{Y} : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}_+ \setminus \{0\} \) be any function satisfying that \( [\mathcal{Y}(x, \tilde{t})]^{-1} \) is continuous for all \( x \in \mathbb{R} \setminus \{0\} \) and \( \tilde{t} \in \mathbb{R}_+ \). Moreover, suppose that

\[
\psi(\tau) = T_c \int_0^\tau \frac{1}{\mathcal{Y}(\tilde{x}(\tilde{\xi}; x_0, 0), \psi(\tilde{\xi}))} \, d\tilde{\xi},
\]

has a unique solution \( \psi(\tau) \) on \( \mathcal{I}' = [0, T_c(x_0, 0)) \), where \( T_c > 0 \). Then, the following theorems are satisfied:

1. The map \( \psi : \mathcal{I}' \to J \), where \( J \) is the resulting range of \( \psi(\tau) \), is continuous and bijective.
2. The resulting range of \( \psi(\tau) \) is \( J = [0, \lim_{\tau \to T_c(x_0,0)} \psi(\tau)) \).
3. The bijective function \( \psi : \mathcal{I} = [t_0 + \inf J, t_0 + \sup J] \to \mathcal{I}' \) defined by \( \psi^{-1}(\tau) = \psi(\tau) + t_0 \) is a time-scale transformation.

**Proof:** Let \( \psi(\tau) \) be the solution of (3) in \( \mathcal{I}' \). Since \( \tilde{x}(\tau; x_0, 0) \), \( \tau \in \mathcal{I}' \), is continuous, then \( [\mathcal{Y}(\tilde{x}(\tau; x_0, 0), \psi(\tau))]^{-1} \) is continuous on \( \tau \in \mathcal{I}' \) and \( \psi(\tau) \) is \( C^1(\mathcal{I}') \). Moreover, for all \( x \in \mathbb{R} \setminus \{0\} \) and \( \tilde{t} \in \mathbb{R}_+ \), if \( \tilde{y}(\tau, \tilde{t}) > 0 \), then it satisfies \( \frac{d\psi}{d\tau} > 0 \), hence \( \psi \) is injective (Spivak, 1965, Pg. 34). On the other hand, \( \lim_{\tau \to \inf J} \psi(\tau) = \inf J \) and \( \lim_{\tau \to \sup J} \psi(\tau) = \sup J \), hence, by the continuity of \( \psi \), \( \psi \) is surjective and \( J = [0, \lim_{\tau \to T_c(x_0,0)} \psi(\tau)) \) which establishes item (2). Thus \( \psi : \mathcal{I}' \to J \) is bijective establishing item (1). Moreover, it follows that \( \psi \) is \( C^1(\mathcal{I}) \), satisfies \( \frac{d\psi}{d\tau} > 0 \) and is bijective. Thus \( \psi : \mathcal{I} \to \mathcal{I}' \) is a parameter transformation establishing item (3).
\[ \frac{d}{dt}(x \circ \varphi^{-1})(\tau; x_0, 0) = -\mathcal{H}(\tilde{x}(\tau; x_0, 0)), \] 

then for any solution of (4) on \( I \), there exists an equivalent curve on \( I' \) under the parameter transformation \( \varphi^{-1} \), that is solution of (2). Thus the uniqueness of the solution of (4) on \( I \) follows from the uniqueness of solutions of (2). Finally, since \( \tilde{x}(\tau; x_0, 0) \) reaches the origin at \( t = T(x_0, 0) \) then, \( x(t; x_0, 0) \) reaches the origin at \( t = I_0 + \lim_{q \to -\infty} T(x_0, 0) \). Moreover, since (4) has an equilibrium point at \( x = 0 \), then the solution of (4) remains at the origin for all \( t \in [0, I_0 + \lim_{q \to -\infty} T(x_0, 0)] \). Hence, we can conclude that (4) is asymptotically stable, (5) is the unique solution in the interval \([I_0, +\infty)\) and the settling time function is given by (6).

Up to now, we have established which properties the base system (2) and the time-scale transformation (3) must comply with in order to obtain a well-posed resulting system in (4) with a known settling time (6). However, the result depends on constructing a function \( \Upsilon(\bullet, \bullet) \) such that (3) has a unique solution. Moreover, to construct a fixed-time stable system, we must ensure the settling time in (6) to be bounded. In the following developments, we study how to construct \( \Upsilon(\bullet, \bullet) \) such that both of these properties are satisfied.

### 3.2 Uniqueness of solutions for time-scale generated systems

In the rest of the paper, we analyse the cases where \( \Upsilon(x, \hat{t}) \) is time-invariant or a function only of \( t \). In the following, we show that in these cases, (3) has a unique solution.

**Definition 3.4 (Autonomous System Generating Functions):**

Let \( \Phi : \mathbb{R}_+ \to \mathbb{R}_+ \setminus \{0\} \) be a continuous function on \( \mathbb{R}_+ \setminus \{0\} \) satisfying \( \Phi(0) = +\infty, \Phi(z) < +\infty \) \( \forall \in \mathbb{R}_+ \setminus \{0\} \) and

\[ \int_0^{+\infty} \Phi(z) \, dz = 1. \]  

(7)

Such function \( \Phi(\bullet) \) is called an Autonomous System Generating (ASG) function.

**Lemma 3.5:** Let \( \Upsilon(z, \hat{t}) \equiv (\Phi(|z|)\mathcal{H}(|z|))^{-1} \) where \( \mathcal{H}(\bullet) \) is an admissible base function and \( \Phi(\bullet) \) be an ASG function, then, (3) has a unique solution on \( I' = [0, T(x_0, 0)] \), given by

\[ \psi(\tau) = T_c \int_0^\tau \Phi(|\tilde{x}(\xi; x_0, 0)|) \mathcal{H}(|\tilde{x}(\xi; x_0, 0)|) \, d\xi. \]  

(8)

**Proof:** Let \( \Upsilon(z, \hat{t}) \equiv (\Phi(|z|)\mathcal{H}(|z|))^{-1} \) and notice that \( \Upsilon(x(t; x_0, 0), \psi^{-1}) \) is independent of \( \psi \). Therefore, it follows that

\[ \frac{d\psi}{d\tau} = T_c \Upsilon(x(t; x_0, 0), \psi), \quad (9) \]

has a unique solution given by \( \psi(\tau) = T_c \int_0^\tau \Phi(|\tilde{x}(\xi; x_0, 0)|) \mathcal{H}(|\tilde{x}(\xi; x_0, 0)|) \, d\xi \). Moreover, by Agarwal and Lakshmikantham (1993, Lemma 1.2.2), a solution of (9) is also a solution of (3) and vice versa.

**Definition 3.6 (Non-Autonomous System Generating Functions):**

Let \( \Phi : \mathbb{R}_+ \to \mathbb{R}_+ \setminus \{0\} \) be a continuous function on \( \mathbb{R}_+ \setminus \{0\} \) satisfying (7) and \( \forall \tau \in \mathbb{R}_+ \setminus \{0\}, \Phi(\tau) < +\infty \). Moreover, \( \Phi \) is either non-increasing or locally Lipschitz on \( \mathbb{R}_+ \setminus \{0\} \). Such function \( \Phi(\bullet) \) is called a Non-Autonomous System Generating (NASG) function.

**Lemma 3.7:** Let \( I' = [0, T(x_0, 0)] \), and consider the first-order ordinary differential equation

\[ \frac{d\psi}{d\tau} = T_c \Phi(\tau), \quad (10) \]

where \( \Phi(\bullet) \) is a NASG function, then (10) has a unique solution \( \psi : I' \to \mathbb{R} \) given by

\[ \psi(\tau) = T_c \int_0^\tau \Phi(\xi) \, d\xi, \quad \tau \in I'. \]

(11)

Moreover, let \( \Upsilon(z, \hat{t}) \equiv \Phi(\psi^{-1}(\hat{t})) \) then \( \psi(\tau) \) is also the unique solution of (3) on \( I' \).

**Proof:** It follows that (10) has a unique solution given by \( \psi(\tau) = T_c \int_0^\tau \Phi(\xi) \, d\xi \) by direct integration. Moreover, it follows that (11) is a solution of (3) since \( \Phi(\psi^{-1}(\psi(\tau))) = \Phi(\tau) \).

Now, we show that (11) is the only solution to (3). On the one hand, if \( \Phi \) is non-increasing, then \( \Phi \circ \psi^{-1} \) is non-increasing. To show this, let \( a > b \) which leads to \( \psi^{-1}(a) > \psi^{-1}(b) \) since \( \psi \) is strictly increasing. Hence, \( \Phi(\psi^{-1}(a)) < \Phi(\psi^{-1}(b)) \).

On the other hand, if \( \Phi \) is Lipschitz on \([\epsilon, +\infty), \forall \epsilon > 0, \) then \( \Phi \circ \psi^{-1} \) is Lipschitz on \([\epsilon, +\infty) \). To show this, note that there exists a constant \( M_\Phi \) such that \( |\Phi(\psi^{-1}(x_1)) - \Phi(\psi^{-1}(x_2))| \leq M_\Phi |\psi^{-1}(x_1) - \psi^{-1}(x_2)| \leq M|x_1 - x_2| \) where \( M = M_\Phi \max_{\epsilon \in [0,0]} \Phi(\psi^{-1}(x)) \).

Then, in the former case it follows from Peano’s uniqueness Theorem (Agarwal Lakshmikantham, 1993, Theorem 1.3.1) (resp. from Lipschitz uniqueness Theorem (Agarwal Lakshminikanth, 1993, Theorem 1.2.4)) that

\[ \frac{dz}{d\tau} = T_c \Phi(\psi^{-1}(\hat{t})), \quad \hat{t} = T_c \int_0^\epsilon \Phi(\xi) \, d\xi, \]

(12)

has a unique solution \( z = \psi(\tau), \forall \epsilon > 0, \tau \in [\epsilon, +\infty) \). Since \( \psi(0) = 0 \), then (12) with \( \epsilon = 0 \) has a unique solution \( z = \psi(\tau), \tau \in I' \). Moreover, by Agarwal Lakshminikanth (1993, Lemma 1.2.2), a solution of (12) is a solution of (3) and vice versa.

**3.3 Fixed-time stability of scalar systems with predefined least UBST**

In this section, it is shown that constructions in Lemmas 3.5 and 3.7 by which it is established that (4) has a unique solution, also lead directly to fixed-time stability. In particular, the following result shows that using the same conditions as in the autonomous case of Lemma 3.5 which use an ASG function \( \Phi(\bullet) \), the resulting characterisation for a map \( T : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}_+ \), leads to fixed-time stability of system (4) with \( T_c \) as the least UBST.
Lemma 3.8: (Characterisation of $\Upsilon(z, \hat{t})$ for fixed-time stability of autonomous systems with predefined least UBST) Let
\begin{equation}
\Upsilon(z, \hat{t}) = (\Phi(||z||)\mathcal{H}(||z||))^{-1}
\end{equation}
where $\hat{t} = t - t_0$, $\mathcal{H}(\bullet)$ is an admissible base function and $\Phi(\bullet)$ is an ASG function. Therefore, system (4) is fixed-time stable with $T_c$ as the predefined least UBST.

**Proof:** First, note that $\Phi(||z||) > 0, \forall z \geq 0$ since $\Phi(\bullet)$ is an ASG function and $\mathcal{H}(||z||)||z|| > 0, \forall z > 0$ since $\mathcal{H}(\bullet)$ is an admissible base function. Thus $\Upsilon(z, \hat{t}) > 0, \forall z > 0$. Now, by Lemma 3.5, $\psi(\tau)$ in (8) is the solution of (3). Thus using $\psi(\tau)$ in (8) and the change of variables $z = |\hat{x}(t; x_0, t_0)|$ with $dz = -\mathcal{H}(|\hat{x}(t; x_0, t_0)|) \, dt$, (6) leads to $T(x_0, t_0) = T_c \int_{x_0}^{x_0} \Phi(z) \, dz$. Since $\Phi(\bullet) > 0$ then $T(x_0, t_0)$ is increasing with respect to $|x_0|$. Moreover, since $\Phi(\bullet)$ satisfies (7), the settling-time function satisfies $\sup_{(x_0, t_0) \in \mathbb{R}^n \times [0, +\infty)} T(x_0, t_0) = \lim_{|x_0| \to +\infty} T_c \int_{x_0}^{x_0} \Phi(z) \, dz = T_c$. 

Similarly, the following result states the construction of fixed-time stable non-autonomous systems with predefined UBST using the conditions of Lemma 3.7 by making use of an NASG function $\Phi(\bullet)$.

**Lemma 3.9:** (Characterisation of $\Psi(z, \hat{t})$ for fixed-time stability of non-autonomous systems with predefined least UBST) Let $\psi(\tau), \tau \in [0, T(x_0, 0))$, be the solution of (10) and $\psi^{-1}(\hat{t})$ its inverse map. Then, with
\begin{equation}
\Upsilon(z, \hat{t}) = \frac{1}{\Phi(\psi^{-1}(\hat{t}))}
\end{equation}
where $\hat{t} = t - t_0$ and $\Phi(\bullet)$ a NASG function, system (4) is fixed-time stable with $T_c$ as the predefined UBST. Furthermore,

1. the settling time is exactly $T_c$ for all $x_0 \neq 0$ if $T(x_0, 0) = +\infty$, for all $x_0 \neq 0$;
2. if $T(x_0, t_0) < T_c$ if $T(x_0, 0) < +\infty$, but the least UBST is $T_c$ if, in addition, $T(x_0, 0)$ is radially unbounded, i.e. $T(x_0, 0) \to +\infty$ as $|x_0| \to +\infty$;
3. if (2) is fixed-time stable, then, there exists $\Psi_{\max} < +\infty$ such that for all $x_0$ and all $t \in [t_0, t_0 + T(x_0, t_0)]$, $\Psi(z, \hat{t}) \leq \Psi_{\max}$.

**Proof:** By Lemma 3.7, the solution of (3) is given by (11). Then, the settling time function of (4) is given by $T(x_0, t_0) = T_c \int_{x_0}^{x_0} \Phi(\xi) \, d\xi \leq T_c$ since $\Phi(\bullet)$ satisfies (7). Thus, fixed-time stability of (4) follows.

To show item (1), note that if $T(x_0, 0) = +\infty$, then $T(x_0, t_0) = T_c \int_{x_0}^{x_0} \Phi(\xi) \, d\xi = T_c, \forall x_0 \in \mathbb{R} \setminus \{0\}$. To show item (2) note that, since $T(x_0, 0) < +\infty$ then $T(x_0, t_0) = T_c \int_{x_0}^{x_0} \Phi(\xi) \, d\xi < T_c$. However, if $T(x_0, 0)$ is radially unbounded, then $\sup_{x_0 \in \mathbb{R}^n \times +\infty} T(x_0, t_0) = \lim_{|x_0| \to +\infty} T_c \int_{x_0}^{x_0} \Phi(\xi) \, d\xi = T_c$. Hence, $T_c$ is the least UBST. To show item (3), note that, since there exists $T_{\max} < +\infty$ such that for all $x_0 \in \mathbb{R}$, $T(x_0, 0) \leq T_{\max}$ then $T_c < T_{\max}$, then $\sup_{x_0 \in \mathbb{R}^n \times +\infty} T(x_0, t_0) \leq \lim_{|x_0| \to +\infty} T_c \int_{x_0}^{x_0} \Phi(\xi) \, d\xi = T_c$. Thus for all $t \in [t_0, t_0 + T_c]$, $\Psi(z, \hat{t}) \leq \Psi_{\max} := \Psi(z, \hat{T}_c) < +\infty$.

**Remark 3.1:** Fixed-time stability of non-autonomous systems has been applied for the design of stabilising controllers (Song et al., 2018), observers (Holloway & Krstic, 2019), consensus algorithms (Colunga et al., 2018; Ning & Han, 2018; Wang et al., 2017, 2018) and robot control (Delfin et al., 2016) with predefined settling-time at $T_c$, which uses time-varying gains that are either continuous in $[t_0, T_c + t_0]$ (Becerra et al., 2018; Morasso et al., 1997; Song et al., 2018; Wang et al., 2018) or piecewise continuous requiring Zeno behaviour (Liu et al., 2018; Ning & Han, 2018) as $t$ approaches $T_c + t_0$. Notice that, in this paper, we focus on the former case.

**Remark 3.2:** In the autonomous case, $T_c$ is the least UBST, whereas, in the non-autonomous case, if the item (1) is satisfied, every nonzero trajectory converges exactly at $T_c$. This feature has been referred in the literature as predefined-time (Becerra et al., 2018), appointed-time (Liu et al., 2018) or prescribed-time (Wang et al., 2018). However, note that $\lim_{t \to t_0 + T_c} \psi(z, \hat{t}) = +\infty$ and hence a singularity occurs. However, if item (2) or (3) in Lemma 3.9 is satisfied, then the origin is reached before the singularity in $\psi(z, \hat{t})$ occurs and system (4) switches to an autonomous right-hand side one as given in the definition of $\Psi(z, \hat{t})$ in Lemma 3.3. This is, contrary to previous approaches, when item (2) or (3) in Lemma 3.9 is satisfied, a bounded value of $\Psi(z, \hat{t})$ is obtained for all $\hat{t} > 0$ and any bounded initial conditions, thus removing the singularity.

### 3.4 Lyapunov analysis for fixed-time stability with predefined UBST

In this section, we take advantage of the previous results on fixed-time stability for scalar systems to develop a Lyapunov framework that can be used to analyse higher-order general systems as in (1). The following theorem provides a sufficient condition for a (general) nonlinear system to be fixed-time stable with predefined UBST. This result follows from the comparison lemma (Khalil Grizzle, 2002, Lemma 3.4) and applying the above results on the time derivative of the Lyapunov candidate function.

**Theorem 3.10:** (Lyapunov characterisation for fixed-time stability with predefined UBST) Let $\mathcal{H}(\bullet)$ be an admissible base function. Hence, if there exists a continuous, differentiable (except perhaps at the origin), positive definite and radially unbounded function $V : \mathbb{R}^n \to \mathbb{R}$, such that its time-derivative along the trajectories of (1) satisfies
\begin{equation}
\dot{V}(x) \leq -\frac{1}{T_c} \Psi(V(x), \hat{t})\mathcal{H}(V(x)), \quad x \in \mathbb{R}^n \setminus \{0\},
\end{equation}

where $\hat{t} = t - t_0$, and $\psi(z, \hat{t})$ is characterised by the conditions of Lemma 3.8 or Lemma 3.9, then, system (1) is fixed-time stable with $T_c$ as the predefined UBST.

**Proof:** Let $w(t)$ be a function satisfying $w(t) \geq 0$ and $\dot{w} = -\frac{1}{T_c} \psi(w, \hat{t})\mathcal{H}(w)$, and let $V(x_0) \leq w(0)$. Then, using Lemma 3.8 and Lemma 3.9, $T_c$ is the UBST of $w(t)$. Moreover, by the comparison lemma (Khalil Grizzle, 2002, Lemma 3.4), it follows that $V(x(t; x_0, t_0)) \leq w(t)$. Consequently, $V(x(t; x_0, t_0))$ will converge to the origin before $T_c$. 


In addition to the previous result, we show that for some fixed-time stable systems, the existence of a proper Lyapunov function satisfying (15) is ensured.

**Theorem 3.11:** If system (1) is autonomous fixed-time stable and has a continuous settling time function $T(x_0, t_0)$, then there exists a continuous positive definite function $V: \mathbb{R}^n \to \mathbb{R}$, such that its time-derivative along the trajectories of (1) satisfies (15) with $\dot{\Psi}(z, t)$ characterised by the conditions of Lemma 3.8. If in addition, $\lim_{\|x\| \to +\infty} T(x_0, t_0) = T_c$ then $V(x)$ is radially unbounded.

**Proof:** Let $G(z) = T_c \int_{\Psi_1(z)}^{\Psi_\infty(z)} \Phi(\xi) \, d\xi$, with $\Phi(\bullet)$ an ASG function. Note that $G' (z) > 0$ and hence $G: \mathbb{R}^+ \to [0, T_c]$ is a bijection (Spivak, 1965, Pg. 34). Moreover, note that $G(0) = 0$ and $\lim_{z \to -\infty} G(z) = T_c$. Hence, $V(x) = G^{-1}(T(x_0, t_0))$ is a continuous and positive definite function satisfying $V(0) = 0$. Furthermore, consider the trajectory $x(t; x_0, t_0)$, then, as noted in (Bhat Bernstein, 2000, Proposition 2.4), $T(x(t_0, t_0), t_0) = \max\{T(x_0, t_0), t_0\}$. Therefore, $\dot{V}(x) = -(G^{-1})'(T(x_0, t_0)) = -\frac{1}{T_c} \dot{\Psi}(V(x), \dot{t}) \dot{\mathcal{H}}(V(x))$, $\forall x \in \mathbb{R}^n \setminus \{0\}$. It follows that, if $\lim_{\|x\| \to +\infty} T(x_0, t_0) = T_c$ then $V(x) = G^{-1}(T(x_0, t_0))$ is radially unbounded.

As a consequence of the previous results, the following theorem allows generating fixed-time stable systems with predefined UBST from asymptotically stable ones with a Lyapunov function satisfying (17). By construction, such $V(x)$ will also be a Lyapunov function for system (18) satisfying (15).

**Theorem 3.12:** (Generating fixed-time stable systems with predefined UBST) Let the system

$$\dot{y} = -g(y),$$

be asymptotically stable, where $y \in \mathbb{R}^n$, $g: \mathbb{R}^n \to \mathbb{R}^n$ is continuous and locally Lipschitz everywhere except, perhaps, at $y = 0$ with $g(0) = 0$. If there exists a differentiable (except perhaps at the origin) Lyapunov function $V(\bullet)$ and an admissible base function $\mathcal{H}(\bullet)$ such that

$$\dot{V}(y) := -\frac{\partial V}{\partial y} g(y) \leq -\mathcal{H}(V(y)), \quad \forall y \in \mathbb{R}^n \setminus \{0\},$$

then, if $\dot{\Psi}(V(y), \dot{t}) g(y)$ is continuous on $y \in \mathbb{R}^n$, where $\dot{t} = t - t_0$ and $\dot{\Psi}(z, t)$ is a function satisfying the conditions of Lemma 3.8 or Lemma 3.9, the system

$$\dot{x} = -\frac{1}{T_c} \dot{\Psi}(V(x), \dot{t}) g(x), \quad x(t_0; x_0, t_0) = x_0$$

has a unique solution in the interval $[t_0, +\infty)$ and it is fixed-time stable with $T_c$ as the predefined UBST.

**Proof:** Since the conditions of Lemma 3.8 or Lemma 3.9 are satisfied, then (3) has a unique solution. Hence, the proof of the existence of a unique solution for (18) follows by the same arguments as those of the proof of Lemma 3.3. Let $V(y)$ be a Lyapunov function candidate for (16) such that (17) holds. Hence, the evolution of $V(x)$ is given by

$$\dot{V}(x) = -\frac{1}{T_c} \dot{\Psi}(V(x), \dot{t}) g(x) \leq -\frac{1}{T_c} \dot{\Psi}(V(x), \dot{t}) \mathcal{H}(V(x)), \forall x \in \mathbb{R}^n.$$  

Notice that the term $\dot{\Psi}(V(x), \dot{t}) \mathcal{H}(V(x))$ in (4) is continuous at $x = 0$ with any choice of $\dot{\Psi}(x, \dot{t})$ from either Lemma 3.8 or Lemma 3.9, since $\mathcal{H}(0) = 0$ and $\dot{\Psi}(0)^{-1} = 0$. However, an arbitrary selection of $\mathcal{H}(y)$ and $\dot{\Psi}(z, \dot{t})$ may lead to a right-hand side of (18) discontinuous at the origin. A construction from a linear system, guaranteeing continuity of the right-hand side of (18) is provided in the following result.

**Corollary 3.13:** Let $\dot{\Psi}(z, t)$ defined as in (13) with $\dot{\Phi}(\bullet)$ an ASG function and $\dot{H}(z) = (2\lambda_{\max}(P))^{-1}z$, $P \in \mathbb{R}^{n \times n}$ is the solution of $A^TP + PA = I$ with $-A \in \mathbb{R}^{n \times n}$ Hurwitz and $\lambda_{\max}(P)$ is the largest eigenvalue of $P$. Then, $\dot{\Psi}(V(x), \dot{t}) Ax$, where $\dot{V}(x) = \sqrt{x^T P x}$, is continuous. Furthermore, the system

$$\dot{x} = -\frac{1}{T_c} \dot{\Psi}(V(x), \dot{t}) Ax$$

where $\dot{t} = t - t_0$, is fixed-time stable with $T_c$ as the predefined UBST. Moreover, if $A = \alpha I + S$ with $\alpha$ a positive constant and $S$ a skew-symmetric matrix then $T_c$ is the least UBST.

**Proof:** Consider system (16) with $g(y) = Ay$ which has a Lyapunov function $V(y) = \sqrt{y^T P y}$ satisfying $\dot{V} \leq -(2\lambda_{\max}(P))^{-1} V(y) = -\mathcal{H}(V(y))$. Note that, $\dot{V}(\bullet)$ is continuous and $\dot{\Psi}(\bullet, \dot{t})$ is continuous except at the origin. Therefore, since $\dot{\Psi}(0, \dot{t}) A(0) = 0$, to check continuity, it is only sufficient to show that $\lim_{\|x\| \to +\infty} \|\dot{\Psi}(V(x), \dot{t}) Ax\| = 0$ which follows from $\lim_{\|x\| \to +\infty} \|\dot{\Psi}(V(x), \dot{t}) Ax\|^2 = 4\lambda_{\max}(P)^2 \lim_{\|x\| \to +\infty} (x^T A^T A x) (x^T P x)^{-1}$

$\dot{V}(x)\dot{V}(x)^{-2} \leq \frac{4\lambda_{\max}(P)^2 \lambda_{\max}(A^T A)}{\lambda_{\min}(P)} \lim_{\|x\| \to +\infty} \|\dot{\Psi}(V(x), \dot{t}) Ax\|^2 = 0$. Hence, $\dot{\Psi}(V(x), \dot{t}) Ax$ is continuous everywhere. It follows from Theorem 3.12 (19) that (12) is fixed-time stable with $T_c$ as the predefined UBST. Note that, if $A = \alpha I + S$ and $\dot{t} = t - t_0$, then $\dot{V}(y) = -\mathcal{H}(V(y)) = -\alpha V(y)$ and $\dot{V}(x) = -\frac{1}{T_c} \dot{\Psi}(V(x), \dot{t}) \mathcal{H}(V(x))$. Hence, by Theorem 3.10, (19) is fixed-time stable with $T_c$ as the least UBST.

**Remark 3.3:** Notice that Theorem 3.10 can be used for the design of first- and second-order controllers as in Aldana-López, Jiménez-Rodríguez et al. (2019), arbitrary order controllers as in Mishra et al. (2018), consensus protocols as in Aldana-López et al. (2019); Ning et al. (2017), non-autonomous arbitrary order controllers as in Gómez-Gutiérrez (2020); Pal et al. (2020) or non-autonomous state observers and online differentiation algorithms (Aldana-López et al., 2020).

4. **Examples of fixed-time stable systems with predefined least UBST**

In the following, we present some examples of ASG and NASG functions and construct fixed-time stable systems with predefined least UBST using such functions.
4.1 Examples of autonomous fixed-time stable systems with predefined least UBST

In this section, we present the design of some examples of ASG functions $\Phi(\bullet)$ for generating autonomous fixed-time stable systems with predefined least UBST. The result is mainly obtained by applying Corollary 3.13. For simplicity, we take $A = \frac{1}{2}I_4 \in \mathbb{R}^{4 \times 4}$.

**Proposition 4.1:** Let $h(z)$ be $K_{\infty}^\infty$. Then, the functions $\Phi(z)$ given in Table 3 are ASG functions. Moreover, the system $\dot{x} = -\frac{1}{T_c}(\Phi(||x||)||x||)^{-1}x$, where $x \in \mathbb{R}^n$ and $-\frac{1}{T_c}(\Phi(||x||)||x||)^{-1}x$ shown in Table 1 is fixed-time stable with $T_c$ as the least UBST.

**Proof:** First, consider the following functions $F_i(z) = (\alpha z + \beta z^p)^{-k}, F_2(z) = (\exp(2z) - 1)^{-1}, F_3(z) = \exp(-z), F_4(z) = (\sin(z) + a)/(1 + z)^2$. Using Proposition A.1, it follows that $\int_0^{\infty} F_i(z) \, dz = M_i$ with $M_1 = \gamma, M_2 = \pi/2, M_3 = 1, M_4 = \rho$ with $\gamma$ and $\rho$ given in Table 1. Now, note that $F_i(0) = +\infty$ for $i = 1, 2$ and $F_i(z) < +\infty, z \geq 0$ for $i = 3, 4$. Hence, Proposition A.2 is used to conclude that $1/(M_0)F_i(h(z))$, which coincide with functions $\Phi(z)$ in Table 1, are ASG functions if $h(z) = K_{\infty}^\infty$ and $\lim_{z \to +\infty} h(z) = +\infty$ when $i = 2, 3, 4$. Now, choose $\lambda = (1/2, 1, 0)$ and $P = I$ for which $A^2P + PA = I$ with $\lambda_{\max}(P) = 1$ and $\Phi(z)$ given in the first column of Table 1. Thus, with $g(x) = Ax = \frac{1}{2}x$, one obtains $V(x) = \sqrt{\frac{\gamma}{2}}P_{\lambda}x = ||x||, H(V(x)) = \frac{1}{2}||x||$ and $\Psi(V(x), i) = (\Phi(||x||)||x||)^{-1} = 2(\Phi(||x||)||x||)^{-1}$ for which one can conclude that systems of the form $\dot{x} = -\frac{1}{T_c}(\Phi(||x||)||x||)^{-1}x$ as shown in the second column of Table 1 are fixed-time stable with $T_c$ as the least UBST by applying Corollary 3.13.

**Remark 4.1:** The system in Table 1-(i) with $h(z) = z$ reduces to the system analysed in Lopez-Ramírez et al. (2019); Polyakov (2012). However, here $T_c$ is given as the predefined least UBST as a consequence of Lemma 3.8, ruling out any conservativeness in the UBST estimation. This feature is a significant advantage with respect to Lopez-Ramírez et al. (2019); Polyakov (2012), because, as illustrated in details in Aldana-López, Jiménez-Rodríguez et al. (2019), an UBST provided in Polyakov (2012) can be made arbitrarily conservative for suitable parameter selection. Similarly, the fixed-time stable system presented in Yu et al. (2020) can be obtained from Table 1-(i) by taking the limit $p \to 0$. Notice that, the fixed-time stable system with predefined UBST, analysed in Sánchez-Torres et al. (2018), is found from Table 1-(iii) with $h(z) = z^p$ with $0 < p < 1$. Thus the algorithms in Polyakov (2012), Yu et al. (2020) and Sánchez-Torres et al. (2018) are subsumed in our approach.

**Example 4.2:** From Table 1 new classes of fixed-time stable systems with predefined UBST, not present in the literature, can be derived. For instance, the systems

$$\dot{x} = -\frac{\gamma}{T_c} (1 + ||x||)(\alpha \log(1 + ||x||))^p + \beta \log(1 + ||x||)^p k \frac{x}{||x||}$$

and

$$\dot{x} = -\frac{\pi}{2T_c} (\exp(2||x||) - 1)^{1/2} \frac{x}{||x||}$$

are obtained from Table 1-(i) and Table 1-(ii) with $h(z) = \log(1 + z)$ and $h(z) = z$ respectively, which satisfy $h(\bullet) \in K_{\infty}^\infty$. Moreover, the system

$$\dot{x} = -\frac{\gamma}{T_c} (\sin(||x||)^2 + 2) \frac{x}{||x||}$$

is obtained from Table 1-(iv) with $h(z) = z^p$ with $0 < p < 1$, which satisfies $\lim_{z \to 0^+} h(z) = +\infty$ and $h(\bullet) \in K_{\infty}^\infty$. Simulations are shown in Figure 1.

4.2 Examples of non-autonomous fixed-time stable systems with predefined least UBST

In this section, we focus on the design of functions $\Phi(\psi^{-1}(\hat{t}))$ satisfying the conditions of Lemma 3.9. Based on these functions, we provide some examples of non-autonomous systems, with $T_c$ as the settling time for every nonzero trajectory as well as non-autonomous systems with $T_c$ as the least UBST.

**Proposition 4.3:** Let $\hat{t} = t - t_0$, then with $\Phi(\psi^{-1}(\hat{t}))$ given in Table 2, $\Psi(x, \hat{t})$ given in (14), satisfies the conditions of Lemma 3.9. Moreover, the system $\dot{x} = -\frac{1}{T_c} \Psi(||x||, \hat{t})x, x \in \mathbb{R}^n$, is fixed-time stable with $T_c$ as the settling time for every nonzero trajectory.

**Proof:** The fact that $\Psi(x, \hat{t})$ given in Table 2-(i) satisfies the condition of Lemma 3.9 follows by using Proposition A.3 directly. The same conclusion for Table 2-(ii) follows by choosing $h(z) = \frac{1}{\gamma}$ and $h(z) = h(z) = \eta(z)/(1 - \eta(z))^{-(\alpha+1)}$. Therefore, Proposition A.3 can be used with $\alpha > 0$.

To show that with $\Phi(\psi^{-1}(\hat{t}))$ given in Table 2-(iii)-(v) $\Psi(x, \hat{t})$ given in (14), satisfies the conditions of Lemma 3.9, let $F_3(z) = \frac{2}{(z^2 + 1)}$, $F_4(z) = \frac{1}{\sqrt{\pi}} \exp(-z^2)$ and $F_5(z) = \frac{1}{\gamma}(\alpha z + \beta z^p)^{-k}$ which verifies $\int_0^{\infty} F_i(z) \, dz = 1$. If $h(z) = K_{\infty}^\infty$ and $C(0, +\infty)$, by Proposition A.2, it follows that the functions $F_3(h(z))h'(z) = \frac{2\lambda_0}{\pi(\lambda_0^2 + 1)}, F_4(h(z))h'(z) = \frac{2\lambda_0}{\pi\lambda_0^2} \exp(-h^2(z))$ and $F_5(h(z))h'(z)$, satisfy (7). Now, we show that the remaining conditions in Definition 3.6 are satisfied for $F_i(h(z))h'(z), i = 3, 4, 5$. For $i = 3, 4$ let $h(z) = C(0, +\infty)$ so that $F_i(h(z))h'(z)$ is differentiable for $z \geq 0$ and thus the Lipschitz condition from Definition 3.6 is satisfied. For $i = 5$, let $h(\bullet)$ such that $h'(z) \leq 0, \forall z \geq 0$ and note that $F_5(z) < 0$ is decreasing. Thus $F_5(h(z))h'(z)$ is decreasing. Hence, $F_i(h(z))h'(z), i = 3, 4, 5$, which coincide with $\Phi(\bullet)$ as in Table 2-(iii)-(v), are NASG functions. Therefore, for $\Phi(\bullet) = F_i(h(z))h'(z), i = 3, 4, 5$, we have that $\psi_i(\tau) = T_c \int_0^\tau F_i(\xi) \, d\xi$ as in (11) leads to $\psi_i(\tau) = \frac{2\hat{t}_c}{\pi} \arctan(h(\tau)), \psi_i(\tau) = erf(h(\tau))$ and using Proposition A.1, $\psi_i(\tau) = \frac{T_c}{\sqrt{\pi}} \frac{(\alpha z + \beta z^p)}{\gamma(\alpha z + \beta z^p)} B((\alpha z + \beta z^p)^{1/2} - 1) - m_p, m_q$.

Moreover, $\psi_1^{-1}(\hat{t}) = \psi(\tau(T_{\hat{t}}))$, $\psi_2^{-1}(\hat{t}) = \psi(\tau(T_{\hat{t}}))$, $\psi_3^{-1}(\hat{t}) = \psi(\tau(T_{\hat{t}}))$, and $\psi_4^{-1}(\hat{t}) = \psi(\tau(T_{\hat{t}}))$, respectively, with $P(\bullet)$ defined in Table 2-(v) and $\eta(z) = h^{-1}(z)$, hence, using $\eta(z) = \frac{1}{h(h^{-1}(z))}$, we obtain $\Phi(\psi^{-1}(\hat{t}))^{-1}, i = 3, 4, 5$, given in Table 2-(ii)-(v), which satisfy the conditions of Lemma 3.9.
Remark 4.2: Let $\hat{t} = t - t_0$, then with $\alpha = 0$, $\Psi(z, \hat{t})$ in Table 2-(i) reduces to the class of time varying gains proposed in (Morasso et al., 1997) if $\hat{t}$ is chosen to be bell-shaped. Similar examples which can be derived from Table 2-(i), were used in Becerra et al. (2018), Colunga et al. (2018), Holloway and Krstic (2019), Kan et al. (2017), Song et al. (2018), Wang et al. (2017), Wang et al. (2018) and Yucelen et al. (2018). Thus, those mentioned examples show how Theorem 3.12, first presented in Aldana-López et al. (2019), generalises most of the representative cases of systems with predefined UBST. More recent studies attempt to provide new systems with that desired property of a predefined UBST. An example is the Theorem 1 in Pal et al. (2020). However, that interesting result, posterior to Aldana-López et al. (2019), is a particular case of Theorem 3.12, where $\Psi(x, \hat{t}) = \frac{1}{1-\frac{\hat{t}}{T_c}}$, $h(z) = (1 - e^{-|z|})\text{sign}(x)$, and $V(x) = |x|$, with $\hat{t} = t - t_0$, $t_0 = 0$, and $\eta \geq 1$. Note that, with such a function $H(z)$, the system (2) satisfies $T(x_0, 0) = +\infty$ for all $x_0 \in \mathbb{R}$.

Example 4.4: Let $\hat{t} = t - t_0$, then taking $\eta(z) = z/T_c$, which verifies that $\eta(z) = K_1^{1-\infty}$ and $C^2([0, T_c])$, and $\alpha = 0$ in Table 2-(i)
Proposition 4.5: Assume that, under a suitable selection of $k_1$, $k_2$, $g_1(\bullet)$ and $g_2(\bullet)$, the system
\[
\begin{align*}
\dot{y}_1 &= -k_1 g_1(y_1) + y_2 \\
\dot{y}_2 &= -k_2 g_2(y_1) - y_2
\end{align*}
\] is finite-time stable and there exists a Lyapunov function $V(y)$, satisfying (17). Then, the system
\[
\begin{align*}
\dot{z}_1 &= -\kappa(\hat{t}) k_1 g_1(z_1) + z_2 \\
\dot{z}_2 &= -\kappa(\hat{t})^2 k_2 g_2(z_1)
\end{align*}
\] where $\kappa(\hat{t}) = \frac{1}{Tc} \Psi(x, \hat{t})$ with $\Psi(x, \hat{t})$ given as in (14) with $\Phi(\psi^{-1}(\hat{t}))^{-1}$ given in (23) and $\hat{t} = t - t_0$, is fixed-time stable with $T_c$ as the predefined UBST.

Proof: Consider the coordinate change $x_1 = z_1$ and $x_2 = \kappa(\hat{t})^{-1} z_2$. Then, in the new coordinates $x = [x_1, x_2]^T$, the dynamics are represented by $\dot{x}_1 = \frac{1}{Tc} \Psi(x, \hat{t}) [-k_1 g_1(x_1) + x_2]$ and $\dot{x}_2 = \frac{1}{Tc} \Psi(x, \hat{t}) [-k_2 g_2(x_1) - x_2]$. Hence, the result follows from Theorem 3.12 by taking $g(y) = [k_1 g_1(y_1) - y_2, k_2 g_2(y_1) + y_2]^T$.

Proposition 4.6: Assume that, under a suitable selection of $k_1$, $k_2$, $g_1(\bullet)$ and $g_2(\bullet)$, the system
\[
\begin{align*}
\dot{y}_1 &= y_2 \\
\dot{y}_2 &= -k_1 g_1(y_1) - k_2 g_2(y_2) - y_2
\end{align*}
\] is finite-time stable and there exists a Lyapunov function $V(y)$, satisfying (17). Then, the system
\[
\begin{align*}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= -\kappa(\hat{t})^2 k_1 g_1(z_1) - k_2 \kappa(\hat{t})^2 g_2(\kappa(\hat{t})^{-1} z_2)
\end{align*}
\] where $\kappa(\hat{t}) = \frac{1}{Tc} \Psi(x, \hat{t})$ with $\Psi(x, \hat{t})$ given as in (14) with $\Phi(\psi^{-1}(\hat{t}))^{-1}$ given in (23) and $\hat{t} = t - t_0$, is fixed-time stable with $T_c$ as the predefined UBST.

Proof: The proof is similar to the one given for Proposition 4.5, considering the coordinate change $x_1 = z_1$ and $x_2 = \kappa(\hat{t})^{-1} z_2$.

Remark 4.3: Notice that the result in Proposition 4.5 can be applied straightforwardly to the design of predefined-time second-order observers, whereas the result in Proposition 4.6 can be applied straightforwardly to the design of second-order predefined-time controllers. These results can be extended to the high-order case. In fact, this approach was used in Gómez-Gutiérrez (2020) to design a controller which induced fixed-time convergence with predefined UBST for an arbitrary order integrator system.
design a time-independent fixed-time controller for the double integrator.

**Proposition 4.7:** Let us consider that $\Phi_1(\bullet)$ is an arbitrary ASG function and choose $\Phi_2(\bullet)$ as follows: $\Phi_2(\bullet)$ is a differentiable (except, perhaps at the origin) ASG function, and $\omega(z) = (\Phi_2(z))^{-2}$ verifies $0 < \omega(z) < +\infty, \forall z > 0$. Thus, for some $T_1, T_2 > 0$, the system

$$
\dot{x}_1 = x_2
$$

$$
\dot{x}_2 = -\left(\frac{1}{T_1 \Phi_1(|\sigma|)} + \frac{1}{T_2} \omega(|x_1|)\right) \text{sign}(\sigma) \tag{32}
$$

where

$$
\sigma = x_2 + \left[|x_2|^2 + \frac{2}{T_2} \omega(|x_1|) \text{sign}(x_1)\right]^{1/2}
$$

with $|\bullet|^\alpha = |\bullet|^\alpha \text{sign}(\bullet)$ for $\alpha > 0$, is fixed-time stable, with $T_c = T_1 + T_2$ as the predefined UBST.

**Proof:** We start by analysing the convergence of $\sigma$. Consider $V_1(\sigma) = |\sigma|$. Hence, for $\sigma 
eq 0$, we have

$$
\dot{V}_1(\sigma) = -\left(\frac{1}{T_1 \Phi_1(|\sigma|)} + \frac{1}{T_2} \omega(|x_1|)\right) \frac{|x_2|}{T_1 \Phi_1(|\sigma|)} + \frac{1}{T_2} \omega(|x_1|) \left(|x_2| - x_2 \text{sign}(\sigma)\right)
$$

$$
\leq -\frac{1}{T_1 \Phi_1(V_1(\sigma))}
$$

since $|x_2| - x_2 \text{sign}(\sigma) \geq 0$. Hence, using Theorem 3.10 with $\Psi(V_1(\sigma)), \tilde{t} \mathcal{H}(V_1(\sigma)) = \Phi_1(V_1(\sigma))^{-1}$ for any admissible base function $\mathcal{H}(\bullet)$ and ASG function $\Phi_1(\bullet)$, one can conclude that $\sigma$ converges to the origin before $t_0 + T_1$. Now, consider $\sigma = 0$ which is equivalent to

$$
x_2 = \dot{x}_1 = \frac{1}{T_c} \omega(|x_1|)^{-1/2} \text{sign}(x_1) = \frac{1}{T_c \Phi_2(|x_1|)} \text{sign}(x_1)
$$

Similarly as before, using $V_2(x_1) = |x_1|$, one can conclude using Theorem 3.10 that $x_1$ converges to the origin before $t_0 + T_1 + T_2$. Moreover, from $\sigma = 0$ and $x_1 = 0$ one obtains $x_2 = 0$. Hence, (32) is fixed-time stable towards the origin with predefined UBST $T_c = T_1 + T_2$.

**Remark 4.4:** Note that in the statement of Proposition 4.7, it is required that $\omega(z) < +\infty$ in order to ensure that the right hand side of (32) is bounded for any bounded state $(x_1, x_2)$. This condition can be fulfilled, for example, if we choose $\Phi_1(z) = \frac{1}{\gamma_1} (\alpha_1 |z|^3 + \beta_1 |z|^5)^{-1/2}$ with

$$
\gamma_1 = \frac{(1/2)^{2/(1/2)}(1/2)^{3/1/2}}{2^{1/2}2^{1/2}2^{1/2}}(\Phi_2)^{1/4}
$$

which is an ASG function according to Proposition 4.1. In this case, one has $\omega(z) = \gamma_1 (\alpha_1 |z|^3 + \beta_1 |z|^5)$ which satisfies $\omega(z) < +\infty, \forall z > 0$. This approach using the ASG function mentioned here, was already used in Aldana-López, Jiménez-Rodríguez et al. (2019).

**Remark 4.5:** An important consequence of Lemma 3.9 is that, based on the settling-time function of (28) (resp. (31)), $T_c$ can be the least UBST. Moreover, If (26) (resp. (30)) is fixed-time stable, then $\kappa(t)$ is bounded for all $t \in [t_0, t_0 + T(x_0, t_0)]$ and all $x_0 \in \mathbb{R}^2$.

5. Conclusion and future work

We presented a methodology for generating fixed-time stable algorithms such that an UBST is set a priori explicitly as a parameter of the system, providing conditions under which such upper bound is the least one. Our analysis is based on time-scaling and Lyapunov analysis. We have shown that this approach subsumes some existing methodologies for generating autonomous and non-autonomous fixed-time stable systems with predefined UBST and allows us to generate new systems with novel vector fields. Several examples are given showing the effectiveness of the proposed method. As future work, we consider the application/extension of these results to differentiators, control, and consensus algorithms.

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### Appendices

#### Appendix 1. Special functions and auxiliary identities

In the derived results, one uses the following special functions. For $z \in \mathbb{R}_+,$ $\Gamma(z) = \int_0^{+\infty} e^{-t} t^{z-1} \, dt$ is the Gamma function; for $x, a, b \in \mathbb{R}_+,$ $B(x, a, b) = \int_0^{+\infty} (1 - e^{-t})^{a-1} \, t^{b-1} \, dt$ and $B^{-1}(\bullet, \bullet, \bullet)$ are the incomplete Beta function and its inverse, respectively; for $x \in \mathbb{R}$ erf$(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \, dt$ is the Error function. For $x > 0,$ $\text{ci}(x) = -\frac{1}{\pi} \int_0^x \frac{\sin(t)}{t} \, dt$ and $\text{si}(x) = -\frac{1}{\pi} \int_0^x \frac{\sin(t)}{t} \, dt$ are the cosine and sine integrals respectively. See Abromowitz and Stegun (1965) for more details on these functions.

**Proposition A.1:** The following identities are satisfied:

1. \[
\int_0^\infty (x^p + \beta x^q)^{-k} \, dx = \frac{\Gamma(\frac{p}{k}) \Gamma(\frac{q}{k} + 1)}{\Gamma(1 + \frac{p}{k} + \frac{q}{k})} \cdot \frac{m_p}{m_q}, \quad \text{for } kp < 1, \quad k > 1, \quad a, \beta, p, q > 0, \quad m_p = \frac{1 - k^{-1}}{q P}, \quad m_q = \frac{k^{-1}}{q q} \quad \text{and } a > 1. \quad \text{Moreover,} \]

2. \[
\int_0^\infty \frac{1}{\sqrt{\text{exp}(2z) - 1}} \, dz = \frac{\pi}{2}.
\]

3. \[
\int_0^\infty \text{sin}(t \omega_{p+\epsilon}) d\omega = -a - \text{ci}(\frac{1}{2}(1 - s)) (1 - s) (\text{sin}(1) (1 - s) = \rho).
\]

**Proof:** For item (1) note that we can write \[
\int_0^\infty (x^p + \beta x^q)^{-k} \, dx = \beta^{-k} \int_0^\infty (x^p + \beta x^q)^{-k} \, x^{-k} \, dx \text{ and make the change of variables } \xi = (\frac{p}{\beta} \xi + q)^{-1} \text{ for which } z = (\frac{p}{\beta} \xi + q)^{-1} (\xi - 1)^{-1} \text{ and } dz = \frac{q}{\beta} (\frac{p}{\beta} \xi + q)^{-1} (\xi - 1)^{-1} \text{ and } \xi^{-1} \text{ and } \xi^{-2} \text{ dz. Using this and the fact that } m_p + m_q = k \text{ leads to}
\]

\[
\int_0^\infty (x^p + \beta x^q)^{-k} \, dx = \frac{\Gamma(\frac{p}{k}) \Gamma(\frac{q}{k} + 1)}{\Gamma(1 + \frac{p}{k} + \frac{q}{k})} \cdot \frac{m_p}{m_q},
\]

from which the definition of the incomplete beta function is used. For the last part of the item note that $B(1; m_p, m_q) = \Gamma(m_p) \Gamma(m_q) / \Gamma(m_p + m_q)$ similarly as in Aldana-López, Jiménez-Rodríguez et al. (2019). Item (2) follows directly by using the change of variables $\xi = \sqrt{\text{exp}(2z) - 1}$ with $d\omega = \frac{1}{\sqrt{\xi}} (\xi^2 + 1) d\xi.$ Item (3) follows by using $\xi = 1 + z$ to obtain \[
\int_0^\infty \text{sin}(t \omega_{p+\epsilon}) d\omega = \frac{(a/\beta)^{1/m_q}}{\text{exp}(1/t)} \int \frac{1}{\sqrt{\text{exp}(2z) - 1} + a} \, dz. \text{ Use } \int_0^1 \frac{1}{\sqrt{\xi}} d\xi = a \text{ and integration by parts to obtain } \int_0^\infty \text{sin}(t \omega_{p+\epsilon}) d\omega = \int_0^\infty \text{cos}(z)^{1/2} \, dz. \text{ Finally, one can use } \text{cos}(1) = -\text{cos}(1) \text{ and } \text{sin}(1) \text{ and the definition of } \text{ci} \text{ and } \text{si} \text{ in this case.}
\]

#### Appendix 2. Some results on the construction of $\Phi(z)$

**Proposition A.2:** Let $h(\bullet)$ be a $\mathcal{K}_{\infty}$ function and let $F : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{0\}$ a continuous function on $\mathbb{R}_+ \cup \{0\}$ satisfying $\int_0^\infty F(z) \, dz = M.$ Then, $\Phi(z) = \frac{1}{M} F(h(z))h'(z)$ satisfies (7). Furthermore, with such $\Phi(z),$ (11)
becomes \( \psi(\tau) = T_c \int_0^{h(\tau)} F(\xi) \, d\xi \). Moreover, if \( F(z) < +\infty, \forall z \in \mathbb{R}_+ \) and \( \lim_{z \to 0^+} h'(z) = +\infty \), or if \( F(0) = +\infty \) then \( \Phi(\bullet) \) is an ASG function.

**Proof:** Using \( \xi = h(z) \), it follows \( \int_0^{+\infty} \Phi(z) \, dz = \frac{1}{M} \int_0^{+\infty} F(h(z)) h'(z) \, dz = \frac{1}{M} \int_0^{+\infty} F(\xi) \, d\xi = 1 \). Moreover, if \( F(z) < +\infty, \forall z \in \mathbb{R}_+ \) and \( \lim_{z \to 0^+} h'(z) = +\infty \), then \( \lim_{z \to 0^+} \Phi(z) = F(0) \lim_{z \to 0^+} h'(z) = +\infty \). Hence, \( \Phi(\bullet) \) is an ASG function. The rest of the proof follows from (11) and the change of variables \( \xi = h(z) \).

**Proposition A.3:** Let \( h(z) \) be a \( K^\infty_+ \) function. Then, the function \( \Phi(z) = \frac{1}{T_c} [h'(h^{-1}(z))]^{-1} \) is a NASG function. Moreover, let \( T(\hat{z}, \hat{t}) = \Phi(h^{-1}(\hat{t})) \) then, the solution of (3) is given by \( \psi(\tau) = h^{-1}(\tau) \) and \( \Phi(\psi^{-1}(\hat{t})) = T_c h'(\hat{t}) \).

**Proof:** Using the change of variables \( \xi = h^{-1}(z) \) with \( d\xi = [h'(h^{-1}(z))]^{-1} \)
\( d\xi = \int_0^{+\infty} \Phi(z) \, dz = T_c^{-1} \int_0^{+\infty} [h'(h^{-1}(z))]^{-1} \, dz = T_c^{-1} \int_0^{+\infty} d\xi = 1 \). Then, from Lemma 3.7, (11) is the solution of (3). Moreover, from (11) we have \( \psi(\tau) = \int_0^{+\infty} h'[h^{-1}(z)]^{-1} \, dz = \int_0^{h^{-1}(\tau)} d\xi = h^{-1}(\tau) \). Hence, \( \psi^{-1}(\hat{t}) = h(\hat{t}) \) and \( \Phi(\psi^{-1}(\hat{t})) = \Phi(h(\hat{t})) = [T_c h'(\hat{t})]^{-1} \).