On spacelike surfaces in four-dimensional Lorentz–Minkowski spacetime through a light cone

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On any spacelike surface in a light cone of four-dimensional Lorentz–Minkowski space, a distinguished smooth function is considered. We show how both extrinsic and intrinsic geometry of such a surface are codified by this function. The existence of a local maximum is assumed to decide when the spacelike surface must be totally umbilical, deriving a Liebmann-type result. Two remarkable families of examples of spacelike surfaces in a light cone are explicitly constructed. Finally, several results that involve the first eigenvalue of the Laplace operator of a compact spacelike surface in a light cone are obtained.

1. Introduction

The classical Liebmann theorem states that the only compact surfaces with constant Gauss curvature of the Euclidean three-dimensional space are the totally umbilical round spheres. This property is also satisfied if the Euclidean space is replaced by the three-dimensional hyperbolic space $\mathbb{H}^3$ or for an open hemisphere of the three-dimensional sphere (see, for example, [2]).

In the Lorentzian setting, the result remains true for compact spacelike surfaces in the three-dimensional De Sitter space $S^3_1$ [1]. Note that $\mathbb{H}^3$ and $S^3_1$ can be seen as hyperquadrics of the four-dimensional Lorentz–Minkowski space $L^4$. From this point of view, the Liebmann theorems for $\mathbb{H}^3$ and $S^3_1$ could be enunciated as follows. Every compact spacelike surface in $L^4$ through the hyperquadrics $\mathbb{H}^3$ and $S^3_1$ with constant Gauss curvature is a totally umbilical round sphere. Besides $\mathbb{H}^3$ and $S^3_1$, there is another hyperquadric in $L^4$ with relevant geometry: the light cone. Recall that the induced metric from $L^4$ on a light cone is degenerate. However, among the spacelike surfaces in $L^4$, those that lie in a light cone constitute an outstanding class.

In fact, any simply connected two-dimensional Riemannian manifold can be isometrically immersed in a light cone of $L^4$ [15]. Therefore, any point of an arbitrary two-dimensional Riemann manifold $(M^2, g)$ has a neighbourhood that can be isometrically immersed in a light cone. Alternatively, this fact also follows from the local existence of isothermal parameters and the argument at the beginning of § 4.
Hence, we have no local intrinsic information for \((M^2, g)\) if it admits an isometric immersion in a light cone. On the contrary, the higher-dimensional case goes in a different direction. In fact, an \(n(\geq 3)\)-dimensional Riemannian manifold is conformally flat (i.e. each point lies in a neighbourhood which is conformally equivalent to an open subset of the \(n\)-dimensional Euclidean space, with its canonical metric) if and only if it can be locally isometrically immersed in a light cone of the \((n + 2)\)-dimensional Lorentz–Minkowski space \(\mathbb{L}^{n+2}\); see [4], [6, corollary 7.6]. This result is an important motivation for the study of spacelike hypersurfaces in the light cone in [10,11,14].

It is not difficult to see that a compact spacelike surface in a light cone is diffeomorphic to a two-dimensional sphere \(S^2\) (proposition 5.1). This is also the case for a compact spacelike surface of \(S^3\) [1]. However, whereas any Riemannian metric on \(S^2\) can be obtained from a spacelike immersion in a light cone of \(\mathbb{L}^4\), it is known that if a Riemannian metric \(g\) on \(S^2\) has Gauss curvature \(K > 1\), then there is no isometric immersion from \((S^2, g)\) in the (unit) De Sitter space \(S^3\) [1, corollary 10].

In this paper, we will mainly focus on the global geometry of spacelike surfaces in \(\mathbb{L}^4\) through a light cone. Thus, the following problem arises in a natural way.

Is a complete spacelike immersion of constant Gauss curvature in a light cone totally umbilical in \(\mathbb{L}^4\)?

An answer to this question is given in theorem 5.4, where it is shown that, under the assumptions of completeness and the existence of a local maximum of a certain smooth function, the spacelike surface must be a totally umbilical round sphere.

This paper has the following structure. First, \(\S 2\) gives the basic background formulae. For any spacelike orientable surface \(M^2\) in \(\mathbb{L}^4\), two independent normal lightlike vector fields \(\xi\) and \(\eta\) are introduced. We then give expressions, in terms of \(\xi\) and \(\eta\), for the Gauss curvature \(K\) of \(M^2\), (2.7), and for the mean curvature vector field \(H\), (2.5).

The local geometry of spacelike surfaces through a light cone is studied in \(\S 3\). We begin by showing a characterization of spacelike surfaces \(M^2\) of \(\mathbb{L}^4\) in a light cone; see proposition 3.1. Then, we explicitly construct, in this case, the lightlike normal vector fields \(\xi\) and \(\eta\) (lemma 3.2) and compute the corresponding Weingarten operators. Next, the mean curvature vector field and the Gauss curvature are shown. It is a remarkable fact that for any spacelike surface in a light cone, its Gauss curvature and mean curvature vector field are related by (3.4),

\[
K = \langle H, H \rangle.
\]

This formula, essentially obtained in [10,13], shows an interesting relation between intrinsic and extrinsic geometry of a spacelike surface in a light cone and has a nice consequence in the compact case. In fact, from proposition 5.1 and making use of the Gauss–Bonnet theorem, we get that the Willmore integral on any compact spacelike surface in a light cone is constant, independently of the spacelike immersion (remark 5.3).

In \(\S 4\), we deal with the construction of two remarkable families of examples of spacelike surfaces in a light cone. In order to do so, we recall that the light cone at the origin is invariant under conformal transformations. Therefore, for every
spacelike immersion \( \psi: M^2 \to \mathbb{L}^4 \) through the light cone at the origin and every smooth function \( \sigma \) on \( M^2 \), a new spacelike immersion is constructed by \( \psi_\sigma = e^\sigma \psi \). The first family comes from an isometric immersion \( \psi \) of the Euclidean plane \( \mathbb{E}^2 \) into the future light cone \( \Lambda^+ \), and the second one from an isometric immersion \( \psi \) of the unit two-dimensional sphere into \( \Lambda^+ \). In the non-compact case, the total umbilicity of \( \psi_\sigma \) is described by means of a partial differential system (4.2). Several particular solutions are then listed. In the compact case, all the totally umbilical spacelike immersions of \( S^2 \) in a light cone are explicitly constructed.

Finally, in § 5, we deal with the first eigenvalue \( \lambda_1 \) of the Laplace operator of a compact spacelike surface \( M^2 \) in a light cone. We point out that in this case we have (5.3),

\[
\lambda_1 \leq 2 \frac{\int_{M^2} (H^2) \, dA}{\text{area}(M^2)},
\]

that is, the Reilly extrinsic bound of \( \lambda_1 \) [18] for a compact surface in \( m \)-Euclidean space holds true in this setting. However, this is not true for a general compact spacelike surface in \( \mathbb{L}^4 \) (see remark 5.9). Using this inequality for \( \lambda_1 \), we are able to characterize the totally umbilical round spheres in a light cone (theorem 5.10).

Moreover, a comparison area result is obtained and another distinguishing property of totally umbilical round spheres in a lightlike cone is given (proposition 5.8).

### 2. Preliminaries

Let \( \mathbb{L}^4 \) be the Lorentz-Minkowski spacetime, that is, \( \mathbb{R}^4 \) endowed with the Lorentzian metric

\[
\langle \cdot, \cdot \rangle = -(dx_0)^2 + (dx_1)^2 + (dx_2)^2 + (dx_3)^2,
\]

where \((x_0, x_1, x_2, x_3)\) are the canonical coordinates of \( \mathbb{R}^4 \). A smooth immersion \( \psi: M^2 \to \mathbb{L}^4 \) of a two-dimensional (connected) manifold \( M^2 \) is said to be spacelike if the induced metric tensor via \( \psi \) (denoted also by \( \langle \cdot, \cdot \rangle \)) is a Riemannian metric on \( M^2 \). In this case, we call \( M^2 \) a spacelike surface.

Let \( \nabla \) and \( \bar{\nabla} \) be the Levi-Civita connections of \( M^2 \) and \( \mathbb{L}^4 \), respectively, and let \( \nabla^\perp \) be the connection on the normal bundle. The Gauss and Weingarten formulae are

\[
\bar{\nabla}_X Y = \psi_\ast (\nabla_X Y) + II(X, Y) \quad \text{and} \quad \bar{\nabla}_X \xi = -\psi_\ast (A_\xi X) + \nabla^\perp_X \xi
\]

for any \( X, Y \in \mathfrak{X}(M^2) \) and \( \xi \in \mathfrak{X}^\perp(M^2) \), where \( II \) denotes the second fundamental form of \( \psi \). The shape (or Weingarten) operator corresponding to \( \xi \), \( A_\xi \) is related to \( II \) by

\[
\langle A_\xi X, Y \rangle = \langle II(X, Y), \xi \rangle
\]

for all \( X, Y \in \mathfrak{X}(M^2) \). The mean curvature vector field of \( \psi \) is given by \( H = \frac{1}{2} \text{tr} \langle \cdot, \cdot \rangle \, II \), and the Gauss and Codazzi equations are, respectively,

\[
R(X, Y)Z = A_{II(Y, Z)}X - A_{II(X, Z)}Y, \tag{2.1}
\]

\[
(\bar{\nabla}_X II)(Y, Z) = (\nabla_Y II)(X, Z), \tag{2.2}
\]

where \( R \) stands for the curvature tensor (our convention on the sign is \( R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z \)) of the induced metric and

\[
(\bar{\nabla}_X II)(Y, Z) = \nabla^\perp_X II(Y, Z) - II(\nabla_X Y, Z) - II(Y, \nabla_X Z)
\]
for any \( X, Y, Z \in \mathfrak{X}(M^2) \). For each \( \xi \in \mathfrak{X}^\perp(M^2) \), the Codazzi equation gives
\[
(\nabla_X A_{\xi}) Y - (\nabla_Y A_{\xi}) X = A_{\nabla_X \xi} Y - A_{\nabla_Y \xi} X. \tag{2.3}
\]
We denote by \( \Pi_{\xi} \) the symmetric tensor field on \( M^2 \) defined by
\[
\Pi_{\xi}(X, Y) = -\langle A_{\xi} X, Y \rangle = -\langle \Pi(X, Y), \xi \rangle.
\]

**Remark 2.1.** In contrast with the case of a spacelike hypersurface in \( \mathbb{L}^4 \), a spacelike surface \( M^2 \) in \( \mathbb{L}^4 \) may be non-orientable.

If we assume \( M^2 \) is orientable, then we can globally take two independent lightlike normal vector fields \( \xi, \eta \in \mathfrak{X}^\perp(M^2) \) with \( \langle \xi, \eta \rangle = 1 \), and the following (global) formula holds:
\[
\Pi(X, Y) = -\Pi_{\eta}(X, Y) \xi - \Pi_{\xi}(X, Y) \eta \tag{2.4}
\]
for every \( X, Y \in \mathfrak{X}(M^2) \). Therefore,
\[
H = \frac{1}{2} (\text{tr} A_{\eta}) \xi + \frac{1}{2} (\text{tr} A_{\xi}) \eta. \tag{2.5}
\]
Contracting (2.1) we obtain
\[
\text{Ric}(Y, Z) = 2 \langle A_{H} Y, Z \rangle - \langle (A_{\xi} A_{\eta} + A_{\eta} A_{\xi}) Y, Z \rangle, \tag{2.6}
\]
that is,
\[
K I = 2 A_{H} - A_{\xi} A_{\eta} - A_{\eta} A_{\xi}, \tag{2.7}
\]
where \( K \) is the Gauss curvature of \( M^2 \) and \( I \) is the identity transformation. Therefore, we obtain
\[
2K = 4 \langle H, H \rangle - 2 \text{tr}(A_{\xi} A_{\eta}) = 4 \langle H, H \rangle - \langle \Pi, \Pi \rangle, \tag{2.8}
\]
where
\[
\langle \Pi, \Pi \rangle_q = \sum_{i,j=1}^2 \langle \Pi(e_i, e_j), \Pi(e_i, e_j) \rangle,
\]
for an orthonormal basis \( \{e_1, e_2\} \) of \( T_q M^2 \), is the squared length of the second fundamental form at \( q \in M^2 \).

### 3. Local geometry of a spacelike surface in a light cone

We write that
\[
A^+ = \{ v \in \mathbb{L}^4 : \langle v, v \rangle = 0, \ v_0 > 0 \}
\]
for the future light cone of \( \mathbb{L}^4 \). For every \( p \in \mathbb{L}^4 \), the future (respectively, past) light cone at \( p \) is given by \( A^+(p) = p + A^+ \) (respectively, \( A^-(p) = p - A^+ \)). A spacelike surface \( \psi : M^2 \rightarrow \mathbb{L}^4 \) factors through a light cone at \( p \in \mathbb{L}^4 \) if \( \psi(M^2) \subset A^+(p) \) or \( \psi(M^2) \subset A^-(p) \).

We begin by recalling the following result [16, (0.9)] which characterizes when a spacelike surface in \( \mathbb{L}^4 \) lies in some light cone (cf. [12, theorem 4.3]).
Proposition 3.1. Let $\psi: M^2 \to \mathbb{L}^4$ be a spacelike surface. Then, the following conditions are equivalent.

(i) The immersion $\psi$ factors through a light cone.

(ii) There exist a lightlike normal vector field $\xi \in \mathfrak{X}^\perp(M^2)$ and $\lambda \in C^\infty(M^2)$, $\lambda > 0$, such that
\[
A_\xi = -\lambda I \quad \text{and} \quad \nabla^\perp \xi = d(\log \lambda)\xi.
\]

(iii) There exists a lightlike normal vector field $\xi$, parallel to the normal connection and such that $A_\xi = -I$.

After a suitable translation, we may always assume a spacelike surface in a light cone is contained in a light cone at the origin. Moreover, there is no loss of generality to assume this light cone is $\Lambda^+$.

In what follows, we set $\partial_0 = \partial/\partial x_0$, for $\partial_0 \circ \psi$ the vector field $\partial_0$ along the immersion $\psi$, $\psi_0$ the first component of $\psi$ and $\nabla$ the gradient operator of $M^2$.

Lemma 3.2. Let $\psi: M^2 \to \mathbb{L}^4$ be a spacelike surface that factors through the light cone $\Lambda^+$. Then, $\xi = \psi$ and $\eta = 1 + \frac{\|\nabla \psi_0\|^2}{2\psi_0^2} + \frac{1}{\psi_0}(\partial_0 \circ \psi + \psi_0(\nabla \psi_0))$ are two lightlike normal vector fields with $\langle \xi, \eta \rangle = 1$.

Proof. It is clear that $\xi$ is a lightlike normal vector field. Let $T \in \mathfrak{X}^\perp(M^2)$ be the normal component of $\partial_0$. A direct computation shows that
\[
T = \partial_0 \circ \psi + \psi_0(\nabla \psi_0).
\]
Now, taking into account that $\xi$ and $T$ span the normal bundle of $M^2$ and $\langle \xi, T \rangle = -\psi_0$, $\langle T, T \rangle = -1 - \|\nabla \psi_0\|^2$, we deduce the formula for $\eta$. \qed

Remark 3.3.

(a) Consequently, a spacelike surface in $\mathbb{L}^4$ that factors through $A^+$ must be orientable.

(b) On the other hand, taking into account $\nabla^\perp \xi = 0$, the normal vector field $\eta$ in lemma 3.2 also satisfies $\nabla^\perp \eta = 0$. Therefore, the normal connection of $\psi$ must be flat.

(c) Compare our approach with [10], where $\xi$ and $\eta$ were chosen with $\langle \xi, \eta \rangle = -2$.

Proposition 3.4. Suppose $\psi: M^2 \to \mathbb{L}^4$ is a spacelike surface that factors through $A^+$ and let $\xi, \eta$ be as above. Then, we have that
\[
A_\xi = -I \quad \text{and} \quad A_\eta = -\frac{1 + \|\nabla \psi_0\|^2}{2\psi_0^4} I + \frac{1}{\psi_0} \nabla^2 \psi_0,
\]
where $\nabla^2 \psi_0(v) = \nabla_v(\nabla \psi_0)$ for every $v \in T_q M^2$, $q \in M^2$. 
Proof. Clearly, we have that $\nabla_v \psi = \psi_*(v)$. Therefore, $\nabla_v \frac{1}{\psi} \psi = 0$ and the Weingarten formula directly gives $A_{\xi} = -I$. On the other hand, since $\nabla_v T = \nabla_v (\psi_*(\nabla \psi_0)) = \psi_*(\nabla_v \nabla \psi_0) + \Pi(v, (\nabla \psi_0)_q)$, we obtain $A_T = -\nabla^2 \psi_0$. Now, the formula for $A_\eta$ follows from lemma 3.2.

Remark 3.5. The previous result implies that a spacelike surface $M^2$ in $\mathbb{L}^4$ that factors through a light cone has no point where $H$ vanishes (see (2.5)), and $M^2$ is totally umbilical if and only if $\eta$ is umbilical.

A direct computation from proposition 3.4 shows the following.

Corollary 3.6. A spacelike surface $\psi: M^2 \to \mathbb{L}^4$ that factors through a light cone is totally umbilical in $\mathbb{L}^4$ if and only if

$$\nabla^2 \psi_0 = \frac{1}{2} \Delta \psi_0 I.$$

Now, from (2.5) and (2.8) the following formulae for the mean curvature vector field $H$ and the Gauss curvature are obtained.

Corollary 3.7. Suppose $\psi: M^2 \to \mathbb{L}^4$ is a spacelike surface that factors through $\Lambda^+$. Then,

$$H = \left( \frac{\Delta \psi_0}{2 \psi_0} - \frac{1 + \|\nabla \psi_0\|^2}{\psi_0^2} \right) \xi + \frac{1}{\psi_0} T,$$

$$K = \frac{1 + \|\nabla \psi_0\|^2}{\psi_0^2} - \Delta \psi_0 \psi_0^{-1},$$

and therefore

$$K = \langle H, H \rangle.$$

Remark 3.8. In the terminology of [13, definition 1.1], the function $\frac{1}{2} \langle \Delta \psi, \eta \rangle$ is called the mean curvature of the spacelike surface in a light cone. Using the well-known Beltrami formula $\Delta \psi = 2H$ and

$$K = -\text{tr} A_\eta,$$

which follows from (3.3) and (2.5), we obtain $\frac{1}{2} \langle \Delta \psi, \eta \rangle = -\langle H, H \rangle$. Note now that proposition 3.4 implies that, for a spacelike surface in a light cone, (2.5) reduces to

$$H = -\frac{1}{2} K \xi - \eta.$$

As a direct consequence of the previous formula we have that a spacelike surface in a light cone is pseudo-umbilical if and only if it is totally umbilical.

Remark 3.9. A direct computation from corollary 3.7 shows that

$$K = -\Delta \log \psi_0 + \frac{1}{\psi_0^2}.$$

This formula means that the new metric on $M^2$ defined by $\psi_0^{-2}\langle \cdot, \cdot \rangle$ has constant Gauss curvature 1. This fact has an obvious topological consequence: if a two-dimensional manifold $S$ admits a spacelike immersion in a light cone of $\mathbb{L}^4$ and $S$ is
compact, then from the Gauss–Bonnet theorem and remark 3.3(a), we have that $S$ must be homeomorphic to $S^2$. This can be also deduced from a direct topological argument in proposition 5.1. On the other hand, the well-known uniformization theorem implies that every simply connected two-dimensional Riemannian manifold is conformally embedded in the unit sphere $S^2$. This property has been used to show that every simply connected two-dimensional Riemannian manifold admits an isometric embedding in the light cone $\Lambda^+$ of $\mathbb{L}^4$ [15].

We get [5, theorem 4.3] directly from (3.6) and (3.4).

**Corollary 3.10.** Let $\psi: M^2 \to \mathbb{L}^4$ be a spacelike surface that factors through a light cone. Then, the following conditions are equivalent.

(i) $K$ is a constant.

(ii) The mean curvature vector field satisfies $\nabla^\perp H = 0$.

The following result shows how the sign of the Gauss curvature influences the existence of relative extreme points of the function $\psi_0$.

**Proposition 3.11.** Let $\psi: M^2 \to \mathbb{L}^4$ be a spacelike surface that factors through a future (respectively, past) light cone. Assume $K \leq 0$. Then, the function $\psi_0$ attains no local maximum (respectively, minimum) value.

**Proof.** On the contrary, if there exists a local maximum point $q \in M^2$ of $\psi_0$, then from (3.3) the Gauss curvature satisfies $K > 0$ on a neighbourhood of $q$. The proof for past light cones works in a similar way.

**4. Several examples**

This section is devoted to describing two families of spacelike surfaces in $\mathbb{L}^4$ through the light cone $\Lambda^+$. Let $\psi: M^2 \to \mathbb{L}^4$ be a spacelike immersion with $\psi(M^2) \subset \Lambda^+$. Now, for each $\sigma \in C^\infty(\mathbb{R}^2)$, we construct a new spacelike surface in $\Lambda^+$ taking $\psi_\sigma = e^{\sigma} \psi$. If we define $g = \psi^\ast (\cdot, \cdot)$, then we set $g_\sigma = \psi_\sigma^\ast (\cdot, \cdot) = e^{2\sigma} g$ (cf. [6, proposition 7.5]). Therefore, we have that

$$K_\sigma = \frac{K - \Delta_\sigma}{e^{2\sigma}},$$

(4.1)

where $K$ and $K_\sigma$ are the Gauss curvature of $g$ and $g_\sigma$, respectively.

**Example 4.1.** Consider the following spacelike immersion:

$$\psi(x, y) = (\cosh x, \sinh x, \cos y, \sin y), \quad (x, y) \in \mathbb{R}^2.$$ 

It is not difficult to show that $\psi$ is an isometric immersion from the Euclidean plane $\mathbb{E}^2$ in $\mathbb{L}^4$ through the light cone $\Lambda^+$. From (3.6), it follows that $H$ is lightlike everywhere. That is, $\psi$ is a marginally trapped surface, in the terminology of [17].

For each $\sigma \in C^\infty(\mathbb{R}^2)$, we will identify with a superscript $\sigma$ the differential operators associated with the metric $g_\sigma$. Direct computations show that

$$\nabla^\sigma \psi_0 = \frac{1}{e^{\sigma}}[(\sigma_x \cosh x + \sinh x) \partial_x + (\sigma_y \cosh x) \partial_y],$$

$$\Delta^\sigma \psi_0 = \frac{1}{e^{\sigma}}[(1 + \Delta^0 \sigma + \|\nabla^0 \sigma\|^2_0) \cosh x + (2\sigma_x) \sinh x],$$

where $\nabla$ and $\Delta$ denote the gradient and Laplacian operators of $\mathbb{L}^4$, respectively.
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Table 1. Several solutions of (4.2) and the Gauss curvature of $\psi_\sigma$ on the corresponding open subset of $\mathbb{R}^2$.

| Solution $e^\sigma$ | Gauss curvature $K_\sigma$ |
|---------------------|-----------------------------|
| $e^x$               | 0                           |
| $a \text{sech}(x)$  | $1/a^2$                     |
| $a \text{cosech}(x)$| $-1/a^2$                    |
| $e^x/(a^{2x} - 1)$  | $-4$                        |
| $a \text{sec}(y)$   | $-1/a^2$                    |
| $a \text{cosec}(y)$ | $-1/a^2$                    |

where $\|\cdot\|_0$, $\Delta^0$ and $\nabla^0$ denote the corresponding Euclidean operators. Note that the above formulae allow us to deduce that $K_\sigma = -\Delta^0_\sigma/e^{2\sigma}$ from (3.3). Therefore, we obtain

\[
(\nabla^\sigma)^2_{\partial_x} \psi_0 = \frac{1}{e^{\sigma}}[((1 + \sigma_y^2 + \sigma_{xx}) \cosh x + \sigma_x \sinh x) \partial_x + ((\sigma_{xy} - \sigma_x \sigma_y) \cosh x) \partial_y],
\]

\[
(\nabla^\sigma)^2_{\partial_y} \psi_0 = \frac{1}{e^{\sigma}}[((\sigma_{xy} - \sigma_x \sigma_y) \cosh x) \partial_x + ((\sigma_y^2 + \sigma_{yy}) \cosh x + \sigma_x \sinh x) \partial_y].
\]

Proposition 3.4 can be used to deduce that, with respect to the basis $\{\partial_x, \partial_y\}$, the Weingarten endomorphism associated with the corresponding normal lightlike section $\eta_\sigma$ is characterized by

\[
\text{II}_{\eta_\sigma} = \frac{1}{2} \begin{pmatrix}
\sigma_x^2 - \sigma_y^2 - 2\sigma_{xx} - 1 & 2(\sigma_x \sigma_y - \sigma_{xy}) \\
2(\sigma_x \sigma_y - \sigma_{xy}) & \sigma_y^2 - \sigma_x^2 - 2\sigma_{yy} + 1
\end{pmatrix}.
\]

Corollary 3.6 implies that the immersion $\psi_\sigma$ is totally umbilical if and only if $\sigma$ satisfies the following system of partial differential equations:

\[
\sigma_x^2 - \sigma_y^2 - \sigma_{xx} + \sigma_{yy} = 1, \quad \sigma_{xy} = \sigma_x \sigma_y. \quad (4.2)
\]

Table 1 shows several solutions of (4.2) and the Gauss curvature of $\psi_\sigma$ on the corresponding open subset of $\mathbb{R}^2$.

Note that, if we replace $x$ with $y$ in the solutions $\sigma$ of table 1, we obtain immersions $\psi_\sigma$ with Gauss constant curvature, which are not totally umbilical.

Example 4.2. Now, consider the following embedding:

$\psi: \mathbb{S}^2 \to \Lambda^+, \quad \psi(x, y, z) = (1, x, y, z)$.

Clearly, $\psi$ is a totally umbilical spacelike surface with mean curvature vector field $H = -\psi + \partial_0 \circ \psi$. It is not difficult to see that the induced metric is the usual one, $\langle \cdot, \cdot \rangle_0$, of constant Gauss curvature 1.

For each $\sigma \in C^\infty(\mathbb{S}^2)$, we denote with the superscript $\sigma$ the geometric operators associated with $g_\sigma = e^{2\sigma} \langle \cdot, \cdot \rangle_0$. The Levi-Civita connection of $g_\sigma$ satisfies

\[
\nabla^\sigma_X Y = \nabla^0_X Y - \langle X, Y \rangle_0 \nabla^0 \sigma + (X\sigma)Y + (Y\sigma)X + (1 + P\sigma)(X, Y)_0 P
\]

for every $X, Y \in \mathfrak{X}(\mathbb{S}^2)$. Here, we denote with the superscript 0 the differential operators of $\mathbb{E}^3$ and set $P$ as the position vector field. Now, from proposition 3.4
we obtain
\[
\Pi_n = \frac{1}{2}[(1 + P\sigma)^2 - \|\nabla^0\sigma\|^2]_0 - \text{Hess}^0(\sigma) + d\sigma \otimes d\sigma. \tag{4.3}
\]

We end this section by showing an application of the previous formula. Let \(u \in \mathbb{L}^4\) be a vector that satisfies \(\langle u, u \rangle = -1\) and \(u_0 < 0\), and let \(r\) be a positive real number. Set
\[
\mathbb{S}^2(u, r) = \{x \in \mathbb{L}^4 : \langle x, x \rangle = 0, \langle u, x \rangle = r\}.
\]
The surface \(\mathbb{S}^2(u, r)\) may be parametrized by \(\psi = e^u\), where \(\sigma = \log r - \log \langle u, \psi \rangle\). In this case, \(\text{Hess}^0(\sigma) = d\sigma \otimes d\sigma\) and (4.3) reduces to \(\Pi_n = (1/2r^2)g_\sigma\). Therefore, \(A_{n\sigma} = -(1/2r^2)I\) and \(K_\sigma = 1/r^2\). In particular, the surfaces \(\mathbb{S}^2(u, r)\) are totally umbilical. Conversely, if \(\psi : \mathbb{S}^2 \to \mathbb{L}^4\) is a totally umbilical spacelike immersion which factors through the light cone \(A^+\), then \(A_{n\sigma} = -(1/2r^2)I\), where \(K = 1/r^2\). Moreover, \(w = -(1/2r^2)\psi + \eta\) is timelike and constant in \(\mathbb{L}^4\). Now, it is not difficult to show that
\[
\psi(M^2) = \mathbb{S}^2(u, r),
\]
where \(u = rw\). We will refer to \(\mathbb{S}^2(u, r)\) as the totally umbilical round spheres of the light cone \(A^+\).

5. Compact spacelike surfaces in a light cone

**Proposition 5.1.** Every compact spacelike surface in \(\mathbb{L}^4\) that factors through a light cone is a topological 2-sphere.

**Proof.** Let \(\psi : M^2 \to \mathbb{L}^4\) be a compact spacelike immersion with \(\psi(M^2) \subset A^+\). Consider the map \(F : M^2 \to \mathbb{S}^2\) given by \(F = \pi \circ \alpha \circ \psi\), where \(\pi : (0, +\infty) \times \mathbb{S}^2 \to \mathbb{S}^2\) is the projection onto the second factor and \(\alpha : A^+ \to (0, +\infty) \times \mathbb{S}^2\) is the diffeomorphism defined by
\[
\alpha(v) = \left(v_0, \frac{1}{v_0}(v_1, v_2, v_3)\right).
\]
The map \(F\) is a local diffeomorphism. The compactness of \(M^2\) and the connectedness of \(\mathbb{S}^2\) imply that \(F\) is a covering map. Finally, due to the simple connectedness of \(\mathbb{S}^2\), \(F\) is a diffeomorphism (see [7, proposition 5.6.1] for details). The proof for past light cones works in a similar way. \(\square\)

**Remark 5.2.** It should be noted that the same argument as in the previous result shows that every compact \(n(\geq 2)\)-dimensional submanifold in \(\mathbb{L}^n+2\) which factors through a light cone is a topological \(n\)-sphere. However, that is not the case if the codimension of the spacelike submanifold is assumed to be \(\geq 3\).

**Remark 5.3.** Proposition 5.1 and (3.4) allow us, making use of the Gauss–Bonnet theorem, to obtain
\[
\int_{M^2} \langle H, H \rangle \, dA = 4\pi
\]
for any compact spacelike surface \(M^2\) in a light cone of \(\mathbb{L}^4\). Note that the integrand may be negative somewhere, as well-known examples show. For a general compact spacelike surface with \(H\) non-zero everywhere, the existence of some point \(p\) where \(\langle H, H \rangle(p) > 0\) is already known [3, remark 4.2]. On the other hand, non-compact marginally trapped surfaces in a light cone of \(\mathbb{L}^4\) are shown to exist in §4.
Theorem 5.4. Suppose \( \psi : M^2 \to \mathbb{L}^4 \) is a complete spacelike surface that factors through the light cone \( \Lambda^+ \). Assume \( K \) is constant. If \( \psi_0 \) attains a local maximum value, then \( M^2 \) is a totally umbilical round sphere.

Proof. From proposition 3.11 we have that \( K > 0 \). The classical Myers theorem is now invoked to get that \( M^2 \) is compact. Therefore, from proposition 5.1, \( M^2 \) is isometric to a sphere of Gauss curvature \( K \). Now, consider the quadratic differential \( \Omega \) on \( M^2 \) locally given by

\[
\Omega = \langle \Pi(\partial_z, \partial_z), \eta \rangle dz^2,
\]

where \( z = x + iy \) and \((x, y)\) are local isothermal parameters on \( M^2 \) with \( \langle \partial_x, \partial_x \rangle = \langle \partial_y, \partial_y \rangle = F > 0 \). Then, \( \Omega \) is well defined and \( \Omega = 0 \) if and only if \( M^2 \) is totally umbilical (see, for example, [9, \S 2]). Now, from the Codazzi equation it follows that

\[
\nabla^+_{\partial_z} \Pi(\partial_z, \partial_z) = \nabla^+_{\partial_z} \Pi(\partial_z, \partial_z) - \frac{1}{F} \frac{\partial F}{\partial z} \Pi(\partial_z, \partial_z) = \frac{1}{F} \nabla^+_{\partial_z} \nabla^+_{\partial_z} \Pi(\partial_z, \partial_z).
\]

Using \( \nabla^+_{\partial_z} \eta = 0 \) we obtain

\[
\partial_z \langle \Pi(\partial_z, \partial_z), \eta \rangle = \frac{1}{F} \partial_z \langle \nabla^+_{\partial_z} \Pi(\partial_z, \partial_z), \eta \rangle = \frac{1}{F} \partial_z \langle \Pi(\partial_z, \partial_z), \eta \rangle.
\]

Therefore, \( \Omega \) is holomorphic if and only if the function \( \langle \Pi(\partial_z, \partial_z), \eta \rangle \) is constant. But this is the case, because \( \langle \Pi(\partial_z, \partial_z), \eta \rangle = -K/2 \). Consequently, since \( M^2 \) is a topological sphere, it follows that \( \Omega = 0 \). \( \square \)

Remark 5.5. Under the assumption of theorem 5.4, we know from corollary 3.10 that the mean curvature vector field is parallel. Alternatively, as soon as we know that \( M^2 \) is a topological sphere, the proof follows from [3, corollary 4.5].

Remark 5.6. Of course, there exist non-totally umbilical isometric immersions of the unit round sphere \( S^2 \) in \( \mathbb{L}^4 \). For instance, \( \psi : S^2 \to \mathbb{L}^4 \) given by \( \psi(x, y, z) = (\cosh x, \sinh x, y, z) \). In fact, it is not difficult to see that \( N_1 \) and \( N_2 \), given by

\[
N_1(x, y, z) = (\cosh x, \sinh x, 0, 0), \quad N_2(x, y, z) = (x \sinh x, x \cosh x, y, z),
\]

\((x, y, z) \in S^2\), are normal vector fields such that \( \langle N_1, N_1 \rangle = -\langle N_2, N_2 \rangle = -1 \), \( \langle N_1, N_2 \rangle = 0 \) and the corresponding shape operators satisfy

\[
A_{N_1}(v_1, v_2, v_3) = v_1(x^2 - 1, xy, xz), \quad A_{N_2}(v_1, v_2, v_3) = -(v_1, v_2, v_3)
\]

for all \((v_1, v_2, v_3) \in T_{(x,y,z)} S^2 \) and \((x, y, z) \in S^2\).

Remark 5.7. Formula (3.7) may be generalized as follows:

\[
K = -\Delta \log(\psi, u) + \frac{1}{(\psi, u)^2},
\]

where \( u \in \mathbb{L}^4 \) satisfies \( \langle u, u \rangle = -1 \), \( u_0 < 0 \). In particular, in the compact case this gives

\[
\int_{S^2} \frac{1}{(\psi, u)^2} dA = 4\pi. \tag{5.1}
\]

A direct consequence of (5.1) and the Schwarz inequality gives the following.
Proposition 5.8. Let $\psi: S^2 \to L^4$ be a spacelike immersion that factors through $A^+$. Then, for every $u \in L^4$ which satisfies $\langle u, u \rangle = -1$ with $u_0 < 0$, we have the following upper bound for the area of the induced metric:

$$\text{area}(S^2, \langle \cdot, \cdot \rangle) \leq 2\sqrt{\pi} \|\psi, u\|,$$

where $\|\cdot\|$ denotes the $L^2$ norm. The equality holds for some $u$ if and only if the surface is the totally umbilical round sphere $S^2(u, r)$, $r = \langle \psi, u \rangle \in \mathbb{R}^+$. 

Consider $S^2$ endowed with an arbitrary Riemannian metric $g$, and denote by $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$ the spectrum of the Laplace operator of $g$. The Hersch inequality [8] states that

$$\lambda_1 \leq \frac{8\pi}{\text{area}(S^2, g)} \tag{5.2}$$

and the equality holds if and only if $(S^2, g)$ has constant Gauss curvature.

Remark 5.9. The Hersch inequality, taking into account (3.4) and the Gauss–Bonnet formula, may be rewritten for a compact spacelike surface $S^2$ as

$$\lambda_1 \leq 2 \min \frac{\text{area}(S^2, \langle \cdot, \cdot \rangle)}{\text{area}(S^2, \langle \cdot, \cdot \rangle)} \tag{5.3}$$

which is formally equal to the well-known Reilly extrinsic bound for $\lambda_1$ in Euclidean space [18]. However, the Reilly inequality does not hold in general for a compact spacelike surface in $\mathbb{L}^4$. As a counter-example, consider the isometric immersion $\psi: S^2 \to L^4$ given in remark 5.6. The mean curvature vector field is $H(x, y, z) = -\frac{1}{2}(x^2 - 1)N_1 - N_2, (x, y, z) \in S^2$. Hence, $\langle H, H \rangle = -\frac{1}{4}(x^2 - 1)^2 + 1$ and therefore

$$\int_{S^2} \langle H, H \rangle \, dA < 4\pi.$$

Theorem 5.10. Let $\psi: S^2 \to L^4$ be a spacelike immersion that factors through $A^+$. Then, for every $u \in L^4$ that satisfies $\langle u, u \rangle = -1, u_0 < 0$, we have that

$$\lambda_1 \leq \frac{2}{\min \psi_0^2}$$

and the equality holds for some $u$ if and only if the surface is the totally umbilical round sphere $S^2(u, r)$, $r = \langle \psi, u \rangle \in \mathbb{R}^+$.

Proof. The aforementioned inequality is directly deduced from (5.1) and (5.2). Assume now that the equality holds for the timelike vector $u$. The Hersch inequality gives that $(S^2, \langle \cdot, \cdot \rangle)$ has constant Gauss curvature and therefore the result follows from theorem 5.4. Conversely, since $S^2(u, r)$ has constant Gauss curvature $1/r^2$ we have that $\lambda_1 = 2/r^2$. \square

Remark 5.11. As a particular case of theorem 5.10, for every compact spacelike immersion $\psi$ which factors through $A^+$ we have that

$$\lambda_1 \leq \frac{2}{\min \psi_0^2}$$

and the equality holds if and only if $\psi_0$ is a constant.
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