Riesz transform on manifolds with ends of different volume growth for $1 < p < 2$

Ren-Jin Jiang, Hong-Quan Li, Hai-Bo Lin

Abstract. Let $M_1, \cdots, M_\ell$ be complete, connected and non-collapsed manifolds of the same dimension, where $2 \leq \ell \in \mathbb{N}$, and suppose that each $M_i$ satisfies a doubling condition and a Gaussian upper bound for the heat kernel. If each manifold $M_i$ has volume growth either bigger than two or equal to two, then we show that the Riesz transform $\nabla L^{-1/2}$ is bounded on $L^p(M)$ for each $1 < p < 2$ on the gluing manifold $M = M_1 \# M_2 \# \cdots \# M_\ell$.

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1 Introduction

In this paper, we consider a complete, connected and non-compact Riemannian manifold $M$, that is obtained by gluing together several complete manifolds of the same dimension. On the manifold $M$, we denote by $d$ the geodesic distance, by $\mu$ the Riemannian measure. We denote by $L$ the non-negative Laplace-Beltrami operator on $M$, and let $\{e^{-tL}\}_{t>0}$ be the heat semigroup. The corresponding Riesz transform is then given by

$$\nabla L^{-1/2} = \frac{1}{\sqrt{\pi}} \int_0^\infty \nabla e^{-sL} \frac{ds}{\sqrt{s}},$$

where $\nabla$ denotes the Riemannian gradient. For study and developments on the Riesz transform, we refer the readers to [1, 2, 4, 5, 7, 8, 9, 10, 11, 12, 13, 15, 17, 19, 20, 35, 36, 38, 39, 40, 41, 44, 45] and references therein. Notice that as a consequence of integration by parts, the Riesz transform $\nabla L^{-1/2}$ is always bounded on $L^2(M)$. We emphasize that the classical result on Euclidean spaces is not always valid in this general setting. More precisely, for any fixed $q_0 \geq 2$, there exists Riemannian manifolds of this type where the Riesz transform is bounded on $L^p$ for $1 < p < q_0$ but unbounded for $p > q_0$; if $q_0 > 2$ it is unbounded also on $L^q_0$ and not even of weak type $(q_0, q_0)$. Cf. e.g.\([17, 35]\) for early concrete examples.

To move further, let us recall some basic notation. We denote by $B(x, r)$, $B_i(x_i, r)$ the open ball with centre $x \in M$, $x_i \in M_i$ and radius $r > 0$ in $M$, $M_i$, and by $V(x, r)$, $V_i(x_i, r)$ their volume $\mu(B(x, r))$, $\mu_i(B_i(x_i, r))$, respectively. We say that $M_i$ satisfies the volume doubling property (in short is doubling) if there exists a constant $C_D > 1$ such that

$$(D) \quad V_i(x_i, 2r) \leq C_D V_i(x_i, r), \quad \forall r > 0, x_i \in M_i.$$

Let $L_i$ denote the non-negative Laplace-Beltrami operator on $M_i$. The heat semigroup on $M_i$ has a smooth positive and symmetric kernel $h_{i,t}(x, y)$, meaning that

$$e^{-tL_i}f(x) = \int_{M_i} h_{i,t}(x, y)f(y) d\mu_i(y)$$

for suitable functions $f$. One says that the heat kernel satisfies a Gaussian upper bound if there exists a constant $C > 0$ such that

$$(UE) \quad h_{i,t}(x, y) \leq \frac{C}{V_i(x, \sqrt{t})} \exp \left\{ -\frac{d^2(x, y)}{Ct} \right\}, \quad \forall t > 0, x, y \in M_i.$$
By Coulhon and Duong [17], the Riesz transform is bounded on $L^p(M)$ for all $p \in (1,2)$ if $(D)$ and $(UE)$ are satisfied. Chen et al. [14] and Li-Zhu [37] showed further that the Gaussian upper bound can be relaxed. We note that for a closely related operator, the Littlewood-Paley-Stein operator, Coulhon, Duong and Li in [18] proved that the operator is always $L^p$-bounded for $1 < p < 2$ on complete manifolds. Meanwhile, Coulhon-Duong [19] raised the following conjecture.

**Conjecture 1.1** (Coulhon-Duong). The Riesz transform is $L^p$-bounded for $1 < p < 2$ on complete manifolds.

The conjecture is far from being proved or disproved to the best of our knowledge, as there are no efficient tools to deal with the Riesz transform in general setting.

In this paper, we shall focus on $L^p$-boundedness of $\nabla L^{-1/2}$ for $1 < p < 2$. Let us consider a gluing manifold $M$, given as

$$M = M_1 \# M_2 \# \cdots \# M_\ell = E_0 \cup \bigcup_{i=1}^{\ell} E_i,$$

where each $M_i$ is a complete manifold and all $M_i$ are of the same dimension, $E_i = M_i \setminus K_i$, $K_i$ and $E_0$ are compact sub-manifolds; see Figure 1. We refer the reader to [28, § 2.2] for further details.

Figure 1: A gluing manifold.

If the gluing manifold $M$ also satisfies $(D)$, then $(UE)$ is preserved by the gluing result of Grigor’yan and Saloff-Coste [28], and the Riesz transform is bounded on $L^p(M)$ for $1 < p < 2$ by Coulhon and Duong [17]; see also Sikora [42].
We are interested in cases where the doubling condition fails on $M$. Let us review some closely related results. Hassell and Sikora [31, Theorem 7.1] proved the following. Suppose that for each $i$, $M_i$ has lower Ricci curvature bound and positive injectivity radius. Moreover, suppose $M_i$ satisfies $(D)$ and it holds that

$$V_i(x, r) \leq \begin{cases} Cr^N & r \leq 1, \\ Cr^{n_i} & r > 1, \end{cases} \quad \forall x \in M_i,$$  

and

$$\| \sqrt{t} \nabla e^{-tL_i} \|_{L^1 \to L^\infty} \leq \begin{cases} Ct^{-N/2} & t \leq 1, \\ Ct^{-n_i/2} & t > 1, \end{cases}$$  

where $n_i \geq 3$, $\| \cdot \|_{L^1 \to L^\infty}$ denotes the operator norm from $L^1$ to $L^\infty$. Then the Riesz transform is weakly $(1, 1)$ bounded, and $(p, p)$ bounded if $1 < p < \min_i \{n_i\}$. Very recently, Hassell, Nix and Sikora [30] further obtained weak $(1, 1)$ boundness on the same type manifold but allowing $n_i \geq 2$.

**Remark 1.2.** As pointed out in [31] (see (59) and the proof of Theorem 7.1 there), it follows from [22, Proposition 2.1 & Corollary 2.2] that, the conditions (1.1) and (1.2) together imply that, the heat kernel $h_{i,t}(x, y)$ of $e^{-tL_i}$ satisfies the Li-Yau estimate, i.e.,

$$\frac{1}{CV_i(x, \sqrt{t})} \exp \left\{ -C \frac{d^2(x, y)}{t} \right\} \leq h_{i,t}(x, y) \leq \frac{C}{V_i(x, \sqrt{t})} \exp \left\{ -\frac{d^2(x, y)}{Ct} \right\},$$

and the volume of balls has growth as

$$C^{-1}r^N \leq V_i(x, r) \leq Cr^N, \quad 0 < r \leq 1, \quad C^{-1}r^{n_i} \leq V_i(x, r) \leq Cr^{n_i}, \quad \forall r \geq 1,$$

Previous to [31, 30], Carron [10] obtained some stability result for $p \in (\nu^{-1}, \nu)$, where $\nu > 3$ is from the global Sobolev inequality. Devyver [24] further refined the Sobolev dimension $\nu$ to hyperbolic dimension.

It is then natural to wonder whether one can prove $L^p$-boundedness of the Riesz transform for all $1 < p < 2$ without requiring polynomial growth of the volume (1.3), or the gradient estimates of heat kernel (1.2)? Recall that a manifold $M$ is said to be non-collapsed, provided that the volume each ball with radius one in $M$ has a positive bottom, i.e., $\inf_{x \in M} V(x, 1) > 0$. In the paper, we shall prove the following:

**Theorem 1.3.** Let $2 \leq \ell \in \mathbb{N}$. Suppose that $\{M_i\}_{i=1}^\ell$ are complete, connected and non-collapsed manifolds of the same dimension, and each $M_i$ satisfies $(D)$ and $(UE)$. Moreover, assume that there exist constants $C, c > 0$ and some $x_i \in M_i$ for each $i$ such that for all $R \geq r \geq 1$, it holds either

$$c \left( \frac{R}{r} \right)^2 \leq \frac{V_i(x_i, R)}{V_i(x_i, r)} \leq C \left( \frac{R}{r} \right)^2,$$

or for some $n_i > 2$

$$c \left( \frac{R}{r} \right)^{n_i} \leq \frac{V_i(x_i, R)}{V_i(x_i, r)}.$$
Let \( M = M_1 \# M_2 \# \cdots \# M_\ell \). Then the Riesz transform \( \nabla L^{-1/2} \) is bounded on \( L^p(M) \) for each \( 1 < p < 2 \).

**Remark 1.4.** (i) We do not know how to prove weak \((1,1)\) boundedness. Our method uses the mapping property of the operators \( \nabla e^{-tL} \) (see Proposition 2.1 below) and \( tL e^{-tL} \) (see Proposition 3.6), which only have optimal bounds for \( p > 1 \).

(ii) The requirement of the case \( n_i = 2 \) is stronger than the case \( n_i > 2 \), since \( n_i = 2 \) corresponds to the critical case, the behavior of the heat kernel is much more complicate and will be not handful if we do not assume the upper bound of the volume growth; see [28]. The case \( n_i < 2 \) is missing from the main result due to the same reason. In fact, by assuming (stronger) two-side Gaussian bounds of the heat kernel on each \( M_i \), [28 Section 6] gives optimal heat kernel estimates including the cases that some \( n_i < 2 \), which we believe can be used to prove \( L^p\)-boundedness of the Riesz transform for \( 1 < p < 2 \). However, since the present proof is already quite long, we will deal these cases in future.

(iii) The case that all \( n_i \) equal two is obvious, since in this case, the manifold is doubling and the heat kernel satisfies \( (UE) \) by [28], the required conclusion follows from [17]. Therefore, we will only prove the case that some \( n_i > 2 \).

(iv) Since we only need a upper Gaussian bound of the heat kernel on \( M_i \), our result applies to any uniformly elliptic operators on these manifolds \((1.2)\) in general does not hold). Moreover, our result can be applied to the case where the manifolds have volume growth different from \( (1.3) \), which seems to be also new (see [31 Remark 7.3]). Let us take an example from [29]. For \( \alpha \in (0,2) \), consider \( R^\alpha := (\mathbb{R}^2, g_\alpha) \), where \( g_\alpha \) is a Riemannian metric such that, in the polar coordinates \((\rho, \theta)\), for \( p > 1 \) it equals

\[
g_\alpha = d\rho^2 + \rho^{2(\alpha - 1)} d\theta^2.
\]

The volume of balls \( B(x, r) \) on \( R^\alpha \), \( r > 1 \), has growth as

\[
V(x, r) \sim \begin{cases} r^\alpha, & |x| < r \\
\text{min}(r^2, r|x|^{\alpha - 1}), & |x| \geq r.
\end{cases}
\]

In particular, \( V(0, r) \sim r^\alpha \) for \( r > 1 \). Note that for \( \alpha \in (0,2) \), the exterior part \( \{ \rho > 1 \} \) of \( R^\alpha \) is isometric to a certain surface of revolution in \( \mathbb{R}^3 \) and the Li-Yau estimate holds on \( R^\alpha \); see [29] pp. 160-161. For \( n \geq 4 \), let \( M_1 \) and \( M_2 \) be closed manifolds of dimension \( n - 4 \) and \( n - 2 \) respectively. Our result then applies to the gluing manifolds \( \mathbb{R}^n \# (\mathbb{R}^2 \times R^\alpha \times M_1) \) and \( (\mathbb{R}^2 \times R^\alpha \times M_1) \# (\mathbb{R}^2 \times M_2) \).

We remark that the \( L^p \)-boundedness of \( \nabla L^{-1/2} \) for \( 1 < p \leq 2 \) is best possible in our setting, since we allow that some ends have volume growth as the plane at infinity. By Hassel et al. [30], on a manifold with at least two ends, if one end has volume growth as the plane at infinity, then the Riesz transform is not bounded for any \( p > 2 \); see also [17] for the case \( M = \mathbb{R}^2 \# \mathbb{R}^2 \) and [12].

In fact, even in the case that all ends are non-parabolic, i.e., \( \min\{n_i\} > 2 \), it is known that some additional condition are needed for the case \( p > 2 \) (cf. [5][33]).

The proof of the main result is rather long and delicate, we shall explain the method in the beginning of Section 4. Briefly speaking, we shall decompose the manifold into several subsets,
and deal with each subset with different methods. The heat kernel estimate from Grigor’yan and Saloff-Coste [28] and the original method of Coulhon and Duong [17] play important roles in our proof. We shall frequently use the $L^p$-Davies-Gaffney estimates for the operators $e^{-tL}$, $\nabla e^{-tL}$ and $Le^{-tL}$, and $L^p(E) \to L^2(F)$ type bounds for the operators $e^{-tL}$ and $Le^{-tL}$. Note that it is not possible to have a Gaussian upper bound for the time derivative of the heat kernel in our setting as in [17]; see [28]. We shall use a powerful tool established by Davies [23] to get an upper bound for the time derivative of the heat kernel $\partial_t h_t(x, y)$, which provides us the right norm bound of the operator $Le^{-tL}$ from $L^p(E)$ to $L^2(F)$ for some subsets $E$, $F$. We believe this is one of the main ingredients of the paper, see Section 3 and Subsection 4.4.

The paper is organized as follows, In Section 2, we provide some basic estimates for the heat kernel, and the local Riesz transform. In Section 3, we provide estimates and mapping property for the heat kernel and its time derivative. In Section 4, we provide the proof for the main result.

Throughout the work, we denote by $C$, $c$ positive constants which are independent of the main parameters, but which may vary from line to line. The symbol $A \lesssim B$ means that the implicit constant depends on $\alpha, \beta$, and $A \sim B$ means $cA \leq B \leq CA$, for some harmless constants $c, C > 0$. We use $\| \cdot \|_p$ to denote the $L^p(M)$ norm, and $\| \cdot \|_{p \to q}$ to denote the operator norm $\| \cdot \|_{L^p(M) \to L^q(M)}$.

2 Preliminaries

In this section, we collect some basic estimates of the heat semigroup and heat kernels.

2.1 Davies-Gaffney estimates

In this part, we provide some basic estimate for the heat semigroup. The first result is in fact a direct consequence of [19] Theorem 4.1 and [22] Proposition 3.6:

**Proposition 2.1.** The operator $\nabla e^{-tL}$ is bounded on $L^p(M)$ for $1 < p \leq 2$ with

$$\|\nabla e^{-tL}\|_{p \to p} \lesssim_p \frac{1}{\sqrt{t}}, \quad \forall t > 0.$$ 

**Proof.** By [19] Theorem 4.1], for any $f \in C^\infty_0(M)$ and $p \in (1, 2]$, we have

$$\|\nabla e^{-tL}f\|_p^2 \lesssim_p \|e^{-tL}f\|_p \|Le^{-tL}f\|_p \lesssim_p \frac{1}{t} \|f\|_p^2,$$

by the classical Littlewood-Paley-Stein theory (c.f. [43]).

For $1 \leq p < \infty$, we say that an operator $T$ satisfies the $L^p$-Davies-Gaffney estimate, if there exists $C > 0$ such that for any closed sets $E, F \subset M$, it holds

$$\|T(f\chi_E)\|_{L^p(F)} \leq C \exp\left( -\frac{d(E, F)^2}{Ct} \right) \|f\|_{L^p(E)}.$$ 

When $p = 2$, we shall say that $T$ satisfies the Davies-Gaffney estimate for short. The following result was proved in [5] p. 930, (3.1); see also [21].
Proposition 2.2. The operators $e^{-tL}$, $\sqrt{t}\nabla e^{-tL}$ and $tL e^{-tL}$ satisfy the Davies-Gaffney estimate.

Using the Riesz-Thorin interpolation theorem, we deduce the following $L^p$-Davies-Gaffney estimates.

Corollary 2.3. The operators $e^{-tL}$, $\sqrt{t}\nabla e^{-tL}$ and $tL e^{-tL}$ satisfy $L^p$-Davies-Gaffney estimate for $1 < p \leq 2$.

Proof. Note that $e^{-tL}$ and $tL e^{-tL}$ are $L^p$-bounded for all $1 < p < \infty$, and by Proposition 2.1, $\sqrt{t}\nabla e^{-tL}$ is bounded on $L^p$ for $1 < p \leq 2$. From this, together with the Riesz-Thorin theorem and Proposition 2.2, we deduce that for $1 < p < 2$,

$$\|e^{-tL}(f\chi_E)\|_{L^p(F)} + \|\sqrt{t}\nabla e^{-tL}(f\chi_E)\|_{L^p(F)} + \|tL e^{-tL}(f\chi_E)\|_{L^p(F)} \leq C(p) \exp \left( -c(p) \frac{d(E,F)^2}{t} \right) \|f\|_{L^p(E)},$$

as desired. □

From Corollary 2.3 and [3, Proposition 3.1], it follows that the composition of $\sqrt{t}\nabla e^{-tL}$ and $tL e^{-tL}$ satisfies:

Corollary 2.4. Let $1 < p \leq 2$. The operator $t^{3/2}\nabla L e^{-tL}$ satisfies the $L^p$-Davies-Gaffney estimate.

2.2 Notation and basic properties on volume growth

We will need the heat kernel estimates obtained in [28]. For this purpose, let us begin with some basic properties of the volume growth.

As in [28, § 4.3], let $B(x,r)$ (resp. $B_i(x,r)$ with $1 \leq i \leq \ell$) denote the geodesic ball in $M$ (resp. $M_i$). We set

\begin{align}
|x| &:= \sup_{y \in E_0} \{d(x,y)\}, \\
V(x,r) &:= V(B(x,r)), \\
V_i(x,r) &:= V_i(B_i(x,r)), \\
V_i(r) &:= V_i(o_i,r),
\end{align}

where $o_i \in \partial E_i$ is a fixed reference point. Note that $\partial E_i$ is the set that connecting $E_i = M_i \setminus K_i$ to $E_0$, and it always holds that

\begin{align}
d(x,y) &\leq |x| + |y|.
\end{align}

In what follows, for simplicity of notions, we shall assume that

\begin{align}
\text{diam}(E_0) = 1, \mu(E_0) = 1, 1 \leq V_i(1) = V_i(o_i,1) \leq 4, \forall 1 \leq i \leq \ell,
\end{align}

and

\begin{align}
1 \leq \mu(F_*) \leq 40\ell, \text{ with } F_* := \{x \in M; \text{ dist}(x, E_0) \leq 2\}.
\end{align}
It holds that
\[ V_i(x, R) \leq \left( \frac{R}{r} \right)^{N_i} \quad \forall x \in M_i, \ 0 < r \leq R. \tag{2.5} \]
Throughout the paper, we set
\[ N_\infty := \max_i \{ N_i : i = 1, \cdots, \ell \}. \tag{2.6} \]
We also set
\[ V_0(r) := \inf_{1 \leq i \leq \ell} V_i(r), \ r > 0; \quad F_i^{(r)} := \{ x \in E_i; \ \text{dist}(x, E_0) \leq 2r \}, \ r \geq 1, \ 1 \leq i \leq \ell. \tag{2.7} \]
And the following volume growth properties will be used:

**Lemma 2.5.** (i) It holds that
\[ \frac{V_i(R)}{V_i(r)} \geq \left( \frac{R}{r} \right)^{N_i} \geq \left( \frac{R}{r} \right)^2, \quad \forall 1 \leq r \leq R, \ 1 \leq i \leq \ell. \tag{2.8} \]
In particular, we have \( V_0(r) \geq r^2 \) for all \( r \geq 1 \).

(ii) We have
\[ \mu(F_i^{(r)}) \sim V_i(2r) \sim V_i(r), \quad \forall r \geq 1, \ 1 \leq i \leq \ell. \tag{2.9} \]

(iii) \( M \) satisfies the local doubling volume property
\[ \frac{V(x, 2r)}{V(x, r)} \leq 1, \quad \forall x \in M, \ 0 < r \leq 1, \]
and the volume is of at most polynomial growth in the sense that
\[ \frac{V(x, r)}{V(x, 1)} \leq r^{N_\infty}, \quad \forall x \in M, \ r \geq 1. \]

**Proof.** We start by proving (i). Indeed for each \( M_i \), recall that there exists \( x_i \in M_i \) such that
\[ \frac{V_i(x_i, R)}{V_i(x_i, r)} \geq \left( \frac{R}{r} \right)^{N_i}, \quad \forall 1 \leq r \leq R. \]
By the simple fact that
\[ B_i(o_i, R) \subset B_i(x_i, 2(R + d(x_i, o_i))) \subset B_i(o_i, 4(R + d(x_i, o_i))), \quad B_i(o_i, R) \subset B_i(x_i, 2(r + d(x_i, o_i))), \]
we can write
\[ \frac{V_i(R)}{V_i(r)} \geq \frac{V_i(R)}{V_i(x_i, 2(R + d(x_i, o_i)))} \frac{V_i(x_i, 2(R + d(x_i, o_i)))}{V_i(x_i, 2(r + d(x_i, o_i)))} \]
\[ \frac{V_i(R)}{V_i(4(R + d(x_i, o_i)))} \geq \frac{V_i(x_i, 2(R + d(x_i, o_i)))}{V_i(x_i, 2(r + d(x_i, o_i)))}. \]

Then (2.5) implies that
\[ V_i(R) V_i(r) \gtrsim \left( \frac{R}{r} \right)^{n_i} \left( \frac{1}{r + d(x_i, o_i)} \right)^{N_i-n_i} \left( \frac{r}{r + d(x_i, o_i)} \right)^{n_i} \gtrsim \left( \frac{R}{r} \right)^{n_i}, \]

which completes the proof.

Using (2.5) and the simple fact that \( B_i(o_i, 2r) \subset F_i(r) \cup K_i \subset B_i(o_i, 4r) \) for all \( r \geq 1 \), the claims (ii) and (iii) are clear. □

### 2.3 Heat kernel upper bounds

Let
\[ H(x, t) := \min \left\{ 1; \frac{|x|^2}{V_i(|x|)} + \left( \int_0^t \frac{ds}{V_i(\sqrt{s})} \right)_+ \right\}, \]

where \((\cdot)_+\) denotes the non-negative part and \( i_x \) denotes the index of the end that \( x \) belongs to.

Under our assumptions of the volume growth and the doubling condition,
\[ H(x, t) \sim \begin{cases} \frac{|x|^2}{V_i(|x|)}, & n_{i_x} > 2 \\ 1, & n_{i_x} = 2 \end{cases} \]

(see also [28 (4.21)] for the case where \( n_{i_x} > 2 \)). Therefore, we have the uniform bound
\[ (2.10) \quad H(x, t) \leq \frac{|x|^2}{V_i(|x|)} \leq 1. \]

We will use repeatedly the following heat kernel upper bounds on \( M \), which can be deduced from [28 § 4], especially [28 Theorem 4.9] and its remarks therein:

**Theorem 2.6** (Grigor’yan & Saloff-Coste). (i) *The small time heat kernel Gaussian upper bounds hold, namely*
\[ (2.11) \quad h_t(x, y) \leq \frac{1}{V(x, \sqrt{t})} \exp \left( -c \frac{d(x, y)^2}{t} \right), \quad \forall \ 0 < t \leq 1, \ x, y \in M. \]

(ii) *We have that*
\[ h_t(x, y) \leq \frac{1}{V_0(\sqrt{t})} \frac{|x|^2}{V_i(|x|)} \frac{|y|^2}{V_i(|y|)} e^{-\frac{|x+y|^2}{2t}}. \]
(2.12) \[ + \min \left( \frac{1}{V_i(x, \sqrt{t})}, \frac{1}{V_j(y, \sqrt{t})} \right) \exp \left\{ -c \frac{d(x, y)^2}{t} \right\}, \]

for any \( t \geq 1 \) and \( x, y \in E_i \) (\( 1 \leq i \leq \ell \)).

(iii) For \( r > 0 \) and \( g \in E_k \) (\( 1 \leq k \leq \ell \)), let

\[ V_*(g, r) := \max \{ V_k(g, r), V_k(r) \}. \]

It holds that

(2.13) \[ h_t(x, y) \leq \left( \frac{1}{V_0(\sqrt{t})} \frac{|x|^2}{V_i(|x|)} V_i(|y|) + \frac{1}{V_i(x, \sqrt{t})} \frac{|y|^2}{V_i(|y|)} + \frac{1}{V_*(y, \sqrt{t})} \frac{|x|^2}{V_i(|x|)} \right) e^{-c|\beta y|^2}, \]

for all \( t \geq 1 \), \( x \in E_j \), \( y \in E_i \) (\( 1 \leq i \neq j \leq \ell \)). Moreover,

(2.14) \[ h_t(x, y) \leq \left( \frac{1}{V_0(\sqrt{t})} \frac{|x|^2}{V_j(|x|)} + \frac{1}{V_*(x, \sqrt{t})} \right) e^{-c|\beta y|^2}, \]

provided \( t \geq 1 \), \( x \in E_j \), \( y \in E_0 \) (\( 1 \leq j \leq \ell \)).

**Proof.** The estimate (2.11) comes from [28, Corollary 4.16 0]. The claim (ii) follows from [28, (4.45)] and the symmetric property of the heat kernel. To get (2.13) and (2.14), it suffices to use [28, Theorem 4.9, Remark 4.10, (4.24), and Remark 4.12]. 

In particular, we have the following:

**Corollary 2.7.** Let \( 1 \leq i \leq \ell \), \( r \geq 1 \) and \( x, y \in E_i \). Let \( \beta \geq 8 \), then we have that

(2.15) \[ h_t(x, y) \leq_\beta \min \left( \frac{1}{V_i(x, \sqrt{t})}, \frac{1}{V_j(y, \sqrt{t})} \right) e^{-c \frac{d(x, y)^2}{t}}, \quad \forall \ 0 < t \leq \beta r^2, \ d(x, E_0) + d(y, E_0) \geq 2r. \]

**Proof.** Using the symmetric property of the heat kernel, it suffices to establish

\[ h_t(x, y) \leq_\beta \frac{1}{V_i(x, \sqrt{t})} e^{-c \frac{d(x, y)^2}{t}}, \quad \forall \ 0 < t \leq \beta r^2, \ d(x, E_0) + d(y, E_0) \geq 2r. \]

Notice that (cf. also [28, p. 1945]) \( V(x, r) = V_i(x, r) \) whenever \( B_i(x, r) \subset E_i \), otherwise \( V(x, r) \sim V_i(x, r) \) for all \( 0 < r \leq 1 \) and \( x \in E_i \). Hence according to (2.11), we may assume that \( t \geq 1 \). Without loss of generality we can suppose that \( d(y, E_0) \geq r \). By (2.12), (2.10), (2.2) and the fact that \( V_0(s) \geq s^2 \) for all \( s \geq 1 \), it remains to show that

\[ \frac{1}{t} \frac{|y|^2}{V_i(|y|)} e^{-c_1 \frac{|y|^2}{V_i(|y|)}} \leq c_1 \frac{1}{V_i(|y|)} e^{-c_1 \frac{|y|^2}{V_i(|y|)}} \leq c_1 \beta \frac{1}{V_i(x, \sqrt{t})}, \quad \forall \ 1 \leq t \leq \beta r^2, \ x \in E_i, \]

where we have used in the first inequality the fact that \( s e^{-s} \leq 1 \) for any \( s > 0 \).
Notice that $|y| \geq d(y, o_i) \geq d(y, E_0) \geq r \geq \sqrt t$. The doubling property implies that $V_i(|y|) \geq V_i(\sqrt t) = V_i(o_i, \sqrt t)$. By the fact that $|x| \geq d(o_i, x)$, it suffices to show

$$\frac{1}{V_i(o_i, \sqrt t)} e^{-c \frac{d(o_i, x)^2}{t}} \lesssim \frac{1}{V_i(x, \sqrt t)}.$$ 

Indeed, its proof is based on the standard trick of doubling property, which will be used repeatedly. More precisely,

$$\frac{1}{V_i(o_i, \sqrt t)} e^{-c \frac{d(o_i, x)^2}{t}} = \frac{1}{V_i(x, \sqrt t)} \frac{V_i(x, \sqrt t)}{V_i(o_i, \sqrt t)} e^{-c \frac{d(o_i, x)^2}{t}} \leq \frac{1}{V_i(x, \sqrt t)} \frac{V_i(o_i, \sqrt t + d(o_i, x))}{V_i(o_i, \sqrt t)} e^{-c \frac{d(o_i, x)^2}{t}} \lesssim \frac{1}{V_i(x, \sqrt t)} \left( \sqrt t + d(o_i, x) \right)^{N_\infty} e^{-c \frac{d(o_i, x)^2}{t}} \lesssim \frac{1}{V_i(x, \sqrt t)}.$$

This completes the proof of this lemma. \hfill \Box

In addition, using again $V_0(r) \geq r^2$ for all $r \geq 1$, we deduce from [28, (4.42) and (4.45)] that the following large-time on-diagonal upper bounds of the heat kernel:

**Lemma 2.8.** It holds that:

$$(2.17) \quad h_t(x, x) \leq \frac{1}{t} + \begin{cases} 0, & x \in E_0, \\ \frac{1}{V_i(x, \sqrt t)}, & x \in E_i (1 \leq i \leq \ell), \end{cases} \quad \forall t \geq \frac{1}{2}.$$ 

### 2.4 The ultracontractivity of the heat semigroup $e^{-tL}$ ($t \geq 1$) on $M$

Since each $M_i$ satisfies the non-collapsing condition, one has

$$\inf_{x \in M} V(x, 1) \geq c > 0.$$ 

Moreover, since each $M_i$ is a connected doubling manifold, there is a $\delta_i > 0$ such that:

$$\left( \frac{R_i^\delta}{r} \right) \leq \frac{V_i(x, R)}{V_i(x, r)}, \quad \forall x \in M_i, \ 0 < r \leq R,$$

see [25, p. 412] or [32, p. 213, Remark 8.1.15]. Note that $\delta_i$ might be different from $n_i$ but is not larger than $n_i$. From this, on-diagonal upper bounds (2.17) and the semigroup property imply that

$$\|e^{-tL}\|_{1 \rightarrow \infty} \lesssim t^{-\delta}, \quad \forall t \geq \frac{1}{2}, \text{ where } 2\delta = \min_{i \leq \ell} \delta_i.$$

Hence, from the Riesz-Thorin interpolation theorem and the contraction property of the heat semigroup, we get the following ultracontractivity:

$$(2.18) \quad \|e^{-tL}\|_{p \rightarrow q} \lesssim t^{-\delta \left( \frac{1}{p} - \frac{1}{q} \right)}, \quad \forall t \geq \frac{1}{2}, \ 1 \leq p \leq q \leq \infty.$$
2.5 Local estimates

Recall that
\[ \mathcal{L}^{-1/2} = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-s \mathcal{L}} \frac{ds}{\sqrt{s}}, \]
and
\[ (1 + \mathcal{L})^{-1/2} = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-s - s \mathcal{L}} \frac{ds}{\sqrt{s}}. \]

We prove in the following that, by splitting the integral in the Riesz operator into small time and large time, the part of the small time, i.e.,
\[ \int_0^1 \nabla e^{-s \mathcal{L}} \frac{ds}{\sqrt{s}}, \]
is bounded on \( L^p(M) \) for \( 1 < p < 2 \) under the assumptions of Theorem 1.3.

Lemma 2.9. (i) The operators \( \int_0^1 \nabla e^{-s \mathcal{L}} \frac{ds}{\sqrt{s}}, \int_1^\infty \nabla e^{-s \mathcal{L}} \frac{ds}{\sqrt{s}} \) are bounded on \( L^2(M) \).

(ii) Under the assumptions of Theorem 1.3, the operator
\[ \int_0^1 \nabla e^{-s \mathcal{L}} \frac{ds}{\sqrt{s}} \]
is bounded on \( L^p(M) \) for \( 1 < p < 2 \).

Proof. First, by Proposition 2.1, it follows from the semigroup property that for any \( 1 < p \leq 2 \),
\[ \left\| \int_1^\infty \nabla e^{-s \mathcal{L}} \frac{ds}{\sqrt{s}} \right\|_{L^p(M)} \leq \int_1^\infty \left\| \nabla e^{-s \mathcal{L}} \right\|_{L^p(M)} \frac{ds}{\sqrt{s}} \]
\[ \leq \int_1^\infty e^{-s} \left\| \nabla e^{-s \mathcal{L}} \right\|_{L^p(M)} \frac{ds}{\sqrt{s}} \]
\[ \leq \int_1^\infty e^{-s} \frac{ds}{\sqrt{s}} \]
\[ \lesssim 1. \]

Next, using the small time heat kernel Gaussian upper bound (cf. (2.11)) and Lemma 2.5 [17, Theorem 1.2] implies that the local Riesz transform \( \nabla (1 + \mathcal{L})^{-1/2} \) is bounded on \( L^p(M) \) for \( 1 < p \leq 2 \). This implies that
\[ \nabla (1 + \mathcal{L})^{-1/2} - \frac{1}{\sqrt{\pi}} \int_1^\infty \nabla e^{-s \mathcal{L}} \frac{ds}{\sqrt{s}} = \frac{1}{\sqrt{\pi}} \int_0^1 \nabla e^{-s \mathcal{L}} \frac{ds}{\sqrt{s}} \]
is bounded on \( L^p(M) \) for \( 1 < p \leq 2 \).

On the other hand, using Proposition 2.1 again, we get that
\[ \left\| \int_0^1 \nabla e^{-s \mathcal{L}} \frac{ds}{\sqrt{s}} - \int_0^1 \nabla e^{-s \mathcal{L}} \frac{ds}{\sqrt{s}} \right\|_{L^p(M)} \]

is bounded on \( L^p(M) \) for \( 1 < p \leq 2 \).
We conclude therefore that \( \int_0^1 \nabla e^{-sL} \frac{ds}{\sqrt{s}} \) is bounded on \( L^p(M) \) for all \( 1 < p \leq 2 \). Moreover, since \( \nabla L^{-1/2} \) is bounded on \( L^2(M) \), we also have \( \int_0^\infty \nabla e^{-sL} \frac{ds}{\sqrt{s}} \) is bounded on \( L^2(M) \). □

Remark 2.10. From the above proof, we actually see that, if the local Riesz transform \( \nabla (1 + L)^{-1/2} \) is \( L^p \)-bounded, \( 1 < p < 2 \), then the operator \( \int_0^1 \nabla e^{-sL} \frac{ds}{\sqrt{s}} \) is bounded on \( L^p(M) \).

3 Mapping property of the heat semigroup and its time derivative

Recall that for \( r \geq 1 \),

\[
F^{(r)}_i := \{ x \in E_i : \text{dist} (x, E_0) \leq 2r \}, \quad 1 \leq i \leq \ell.
\]

If \( \gamma \) is large enough, saying \( \gamma \geq 100 \ell \), we set in the sequels,

\[
R_i := R_i(\gamma) > 1 \text{ such that } \mu(F^{(R_i)}_i) = \gamma.
\]

It holds then from (2.9) that

\[
\mu(F^{(R_i)}_i) \sim V_i(R_i) \sim V_j(R_j) \sim \mu(F^{(R_j)}_j), \quad \forall \gamma \gg 1, \ 1 \leq i, j \leq \ell.
\]

See Figure 2.

3.1 Mapping properties of the heat semigroup with large time

The following proposition follows directly from Theorem 2.6.

Proposition 3.1. Under the assumptions of Theorem 1.3, for each \( p \in [1, 2) \), it holds that:

\[
\left\| e^{-tL} \right\|_{L^p(F_i^{(R_i)}) \to L^\infty(E_i \setminus F_i^{(R_i)})} \leq \frac{R_j}{t^{\frac{1}{2} - \frac{1}{p}}} \mu(F_i^{(R_i)})^{\frac{1}{2} - \frac{1}{p}}, \quad \forall t \geq R_j^2 = R_j(\gamma)^2, \ \gamma \gg 1,
\]

and

\[
\left\| e^{-tL} \right\|_{L^p(F_i^{(R_i)}) \to L^\infty(E_i \setminus F_i^{(R_i)})} \leq \frac{R_i^2}{t^{\frac{1}{2}}} \mu(F_i^{(R_i)})^{-\frac{1}{2}}, \quad \forall t \geq R_i^2 = R_i(\gamma)^2, \ \gamma \gg 1.
\]

Proof. Recall that (cf. Lemma 2.5 (i))

\[
V_k(r) \geq V_0(r) \geq r^2, \ \forall r \geq 1, \quad \frac{|g|^2}{V_k(|g|)} \leq 1, \ \forall g \in E_k (1 \leq k \leq \ell).
\]
We start with the case $i = j$. According to (2.12), we have that

$$h_t(x, y) \lesssim \frac{1}{t} |x|^2 + \frac{1}{V_i(y, \sqrt{t})}$$

$$= \frac{1}{t} \frac{s^2}{V_i(s)} \left( \left[ \frac{V_i(|x|)}{V_i(s)} \left| \frac{|x|}{s} \right| \right]^{-1} + \left[ \frac{V_i(\sqrt{t})}{V_i(s)} \left| \frac{\sqrt{t}}{s} \right| \right]^{-1} \frac{V_i(\sqrt{t})}{V_i(y, \sqrt{t})} \right)$$

$$\lesssim \frac{1}{t} \frac{s^2}{V_i(s)}, \quad \forall x \in E_i \setminus F_i^{(s)}, \quad y \in F_i^{(s)}, \quad t \geq s^2, \quad s \geq 1,$$

where we have used in the last inequality (2.8), the doubling property and the fact that

$$d(o_i, y) \leq d(y, E_0) + \text{diam } E_0 \leq 2s + 1,$$

$$B_t(o_i, \sqrt{t}) \subset B_t(y, \sqrt{t} + d(o_i, y)) \subset B_t(y, 4\sqrt{t}), \quad \forall t \geq s^2 \geq 1.$$
Similarly, for \(1 \leq i \neq j \leq \ell\), (2.13) implies that
\[
h_t(x, y) \leq \frac{1}{V_j(|x|)} \frac{|x|^2}{R_j} + \frac{1}{V_j(|y|)} \left\{ \frac{V_j(|x|) t}{V_j(s)} \right\}^{\frac{1}{2}} \left( \frac{V_j(|y|)}{V_j(s)} \right)^{\frac{1}{2}} + \frac{2}{V_j(s)}
\]
(3.4) \(\forall x \in E_j \setminus F_j, y \in F_j^{(s)}, t \geq s^2, s, r \geq 1\).

Hence, we obtain that
\[
\left\| e^{-t\mathbb{L}} \right\|_{L^1(F_j^{(R_j)}) \rightarrow L^\infty(E_j \setminus F_j^{(R_j)})} \leq \frac{1}{t} \frac{R_j^2}{V_j(R_j)} 
\]
Next, recall that \(\left\| e^{-t\mathbb{L}} \right\|_{L^1 \rightarrow L^1} \leq 1\). Hence, the Riesz-Thorin interpolation theorem implies that
\[
\left\| e^{-t\mathbb{L}} \right\|_{L^p(F_j^{(R_j)}) \rightarrow L^q(E_j \setminus F_j^{(R_j)})} \leq \frac{1}{\sqrt{t}} \frac{R_j^2}{V_j(R_j)} 
\]
By the fact that \(\mu(F_j^{(R_j)}) \sim V_j(R_j) \sim V_j(R_j)\) (cf. (3.2)), then it follows from the Hölder inequality that
\[
\left\| e^{-t\mathbb{L}} \right\|_{L^p(F_j^{(R_j)}) \rightarrow L^q(E_j \setminus F_j^{(R_j)})} \leq \frac{1}{t} \frac{R_j^2}{V_j(R_j)} \mu(F_j^{(R_j)})^{1-\frac{1}{p}} \sim \frac{R_j^2}{t} \mu(F_j^{(R_j)})^{1-\frac{1}{p}}.
\]
and
\[
\left\| e^{-t\mathbb{L}} \right\|_{L^p(F_j^{(R_j)}) \rightarrow L^q(E_j \setminus F_j^{(R_j)})} \leq \frac{1}{\sqrt{t}} \frac{R_j^2}{V_j(R_j)} \mu(F_j^{(R_j)})^{1-\frac{1}{p}} \sim \sqrt{\frac{R_j^2}{t}} \mu(F_j^{(R_j)})^{1-\frac{1}{p}}.
\]
This completes the proof. \(\square\)

**Remark 3.2.** From the proof above, via the doubling property, we see that the assumption that \(t \geq R_j^2\) can be relaxed as \(t \geq (cR_j)^2\) for some given positive constant \(c\). The same is valid for Proposition 3.6 below.

### 3.2 Mapping properties of \(t \mathbb{L} e^{-t\mathbb{L}} = -t \frac{d}{dt} e^{-t\mathbb{L}}\) with large time

For this purpose, let us begin with the following estimates for the time derivative of the heat kernel in general setting obtained by Davies [23, Theorem 4].

**Theorem 3.3** (Davies). Suppose that \(\delta \in (0, 1), \epsilon \in (0, \frac{1}{5}), x, y \in M\) and \(t > 0\). Let \(a, b, c\) be positive constants such that \(c \in (0, 1]\), and that
\[
h_{(1-\delta)p}(x, x) \leq a, \quad h_{(1-\delta)p}(y, y) \leq b, \quad |h_s(x, y)| \leq \sqrt{abc}
\]
for all \(s \in ((1-\delta)t, (1+\delta)t)\). Then for any \(m \in \mathbb{N}\), it holds
\[
\left| \frac{\partial^m}{\partial r^m} h_t(x, y) \right| \leq \frac{m!}{(\epsilon \delta t)^m} \sqrt{abc}^{1-3\epsilon}.
\]
The following estimates will play a key role in the study of mapping properties for \( t \mathcal{L} e^{-t\mathcal{L}} \) with \( t \geq 1 \):

**Lemma 3.4.** Let \( \epsilon \in (0, 1/8) \). Under the assumptions of Theorem 1.3 Then:

(i) We have that for \( 1 \leq i \leq \ell \)

\[
(3.5) \quad \left| t \partial_t h_i(x, y) \right| \leq \epsilon \frac{s^2}{t V_i(s)} \left( 1 + \left( \frac{V_i(s)}{s^2} \right) \frac{|y|^2}{V_i(|y|)} \right) \lesssim \frac{1}{V_i(s)} \epsilon^{\frac{3}{4}} - 1 \, \left( \frac{1}{V_i(|y|)} \right)^{\frac{3}{8}} , \quad \forall t \geq s^2 \geq 1, x \in E_i \setminus F_i^{(s)}, y \in F_i^{(s)}.
\]

(ii) It holds that for all \( 1 \leq i \neq j \leq \ell \)

\[
(3.6) \quad \left| t \partial_t h_i(x, y) \right| \leq \epsilon V_j(s)^{\frac{3}{4}} - 1 \left( \frac{1}{V_j(|y|)} \right)^{\frac{3}{8}} , \quad \forall t \geq s^2, x \in E_j \setminus F_j^{(s)}, y \in F_j^{(s)}, r, s \geq 1.
\]

**Proof.** Let us begin with the proof of (3.5). Recall (2.12) and (2.2). From the proof of (3.3), it is not hard to notice that

\[
(3.7) \quad h_i(x, y) \leq \frac{s^2}{t V_i(s)} e^{-c \frac{d(x, y)^2}{t}} := A(x, y, t), \quad \forall x \in E_i \setminus F_i^{(s)}, y \in F_i^{(s)}, t \geq s^2/2 \geq 1/2.
\]

Similarly, we have for all \( t \geq s^2/2 \geq 1/2, x \in E_i \setminus F_i^{(s)} \) and \( y \in F_i^{(s)} \)

\[
(3.8) \quad h_i(x, y) \leq \frac{1}{t} \frac{|x|^2}{V_i(|x|)} \leq \frac{1}{t} \frac{s^2}{V_i(s)} + \frac{1}{V_i(x, \sqrt{t})} := a(x, t),
\]

\[
(3.9) \quad h_i(y, y) \leq \frac{1}{t} \left( \frac{|y|^2}{V_i(|y|)} + \frac{s^2}{V_i(s)} \right) \leq \frac{1}{t} \left( \frac{V_i(s)}{s^2} \frac{|y|^2}{V_i(|y|)} \right) \quad := b(y, t).
\]

From (3.7)-(3.9), Theorem 3.3 implies that

\[
\left| t \frac{\partial}{\partial t} h_i(x, y) \right| \leq \epsilon A(x, y, t)^{1-3\epsilon} (a(x, t) b(y, t))^{\frac{3}{8}}
\]

Next we decompose \( e^{-c \frac{d(x, y)^2}{t}} \) as \( e^{-c \frac{d(x, y)^2}{t}} e^{-c \frac{d(x, y)^2}{t}} \), and write

\[
\left| t \frac{\partial}{\partial t} h_i(x, y) \right| \leq \epsilon \frac{s^2}{t V_i(s)} \left( 1 + \frac{V_i(s)}{s^2} \frac{|y|^2}{V_i(|y|)} \right) \lesssim \frac{1}{V_i(s)} \epsilon^{\frac{3}{4}} - 1 \, \left( \frac{1}{V_i(|y|)} \right)^{\frac{3}{8}} , \quad \forall x \in E_i \setminus F_i^{(s)}, y \in F_i^{(s)}.
\]

where we have used in the second inequality the standard trick of doubling property (cf. the proof of (2.16)), and (3.3) again in the last inequality. In conclusion, by the trivial inequality \((1 + a)^b \leq 1 + a^b\) for all \(a > 0\) and \(0 < b \leq 1\), we get that

\[
\left| t \frac{\partial}{\partial t} h_i(x, y) \right| \leq \epsilon \frac{s^2}{t V_i(s)} e^{-c \frac{d(x, y)^2}{t}} \left( 1 + \frac{V_i(s)}{s^2} \frac{|y|^2}{V_i(|y|)} \right)^{\frac{3}{8}}.
\]
Riesz Transform on Manifolds with Ends of Different Volume Growth

which is clearly majorized by constant times \( V_i(s)^{-1} (V_i(s)/V_i(|y|))^{3/2} \) under our assumptions, by the definition of \( F_i^{(r)} \) (cf. (2.7)) and the doubling property.

Now we turn to the proof of (3.6). In a very similar way, we have that for all \( t \geq s^2/2 \geq 1/2, x \in E_j \setminus F_j^{(s)} \) and \( y \in F_i^{(r)} \) with \( 1 \leq i \neq j \leq \ell, \)

\[
    h_t(x, y) \leq \frac{1}{t} \frac{s^2}{V_j(s)} e^{-c\frac{|y|^2}{V_j(s)}} := A_\ast(x, y, t), \quad \text{from the proof of (3.4),}
\]

\[
    h_t(x, x) \leq \frac{1}{t} \frac{s^2}{V_j(s)} + \frac{1}{V_j(x, V_j(t))} := a_\ast(x, t),
\]

\[
    h_t(y, y) \leq \frac{1}{t} \frac{|y|^2}{V_i(|y|)} + \frac{1}{V_i(y, V_i(t))} := b_\ast(y, t).
\]

Then, it deduces from Theorem 3.3 that

\[
(3.10) \quad \left| \frac{\partial}{\partial t} h_t(x, y) \right| \leq \epsilon \left( \frac{1}{t} \frac{s^2}{V_j(s)} \right)^{1-3\epsilon} e^{-c\frac{|y|^2}{V_j(s)}} \left( e^{-c\frac{|y|^2}{V_j(s)}} a_\ast(x, t) e^{-c\frac{|y|^2}{V_j(s)}} b_\ast(y, t) \right)^{\frac{3}{2}}
\]

Now by (2.8), the standard trick of doubling property implies that (e.g. in the proof of Corollary 2.7)

\[
    e^{-c\frac{|y|^2}{V_j(s)}} a_\ast(x, t) \leq \frac{1}{t} \frac{s^2}{V_j(s)}, \quad e^{-c\frac{|y|^2}{V_j(s)}} b_\ast(y, t) \leq \frac{1}{t} \frac{|y|^2}{V_i(|y|)}.
\]

Combining this with (3.10), we obtain that

\[
    \left| \frac{\partial}{\partial t} h_t(x, y) \right| \leq \epsilon \left( \frac{1}{t} \frac{s^2}{V_j(s)} \right)^{1-\frac{3}{2}} e^{-c\frac{|y|^2}{V_j(s)}} \left( \frac{1}{t} \frac{|y|^2}{V_i(|y|)} \right)^{\frac{1}{2}} e^{-c\frac{|y|^2}{V_j(s)}},
\]

then by the trivial inequality \( e^{-c\frac{|y|^2}{V_j(s)}} \leq \epsilon e^{-\frac{3}{2} - \epsilon} \), the fact that \( t \geq s^2 \) implies the desired result. \( \square \)

\textbf{Lemma 3.5.} Suppose that \( 0 < \frac{3}{2} \epsilon < \frac{1}{p'} \). Then we have for all \( 1 \leq i \leq \ell \) and \( s \geq 1 \)

\[
(3.11) \quad I(i, \epsilon, p', s) := \left\{ \int_{[y \in E_i; d(y, E_0) \leq 2s]} \left( \frac{1}{V_i(|y|)} \right)^{\frac{p}{2}} d\mu(y) \right\}^{\frac{1}{p'}} \leq \epsilon, p', V_i(s)^{\frac{1}{p'} - \frac{3}{2} \epsilon}.
\]

\textbf{Proof:} We split the domain of integration into the (not necessarily disjoint) sets

\[
\{ y \in E_i; d(y, E_0) \leq 2s \}, \quad \{ y \in E_i; 2^{-k}s < d(y, E_0) \leq 2^{-k+1}s \} \quad \text{with} \quad 0 \leq k \leq [\log_2 s],
\]

where \([a]\) denotes the integer part of \( a \in \mathbb{R} \). Hence, by (2.9), we can write

\[
I(i, \epsilon, p', s) \leq 1 + \sum_{k=0}^{[\log_2 s]} \left( \frac{1}{V_i(2^{-k}s)} \right)^{\frac{3}{2}} V_i(2^{-k+1}s)^{\frac{1}{p'}}.
\]
From the doubling property and the facts that $\frac{3}{2} \varepsilon < \frac{1}{p'}$, we get that

$$I(i, \varepsilon, p', s) \leq 1 + \sum_{k=0}^{[\log_2 s]} V_i(2^{-k} s)^{\frac{1}{p'} - \frac{3}{2} \varepsilon} \leq 2V_i(s)^{\frac{1}{p'} - \frac{3}{2} \varepsilon} \sum_{k=0}^{[\log_2 s]} \left( \frac{V_i(s)}{V_i(2^{-k} s)} \right)^{\frac{3}{2} \varepsilon - \frac{1}{p'}}$$

$$\leq V_i(s)^{\frac{1}{p'} - \frac{3}{2} \varepsilon} \sum_{k=0}^{\infty} 2^{-nk(\frac{1}{p'} - \frac{3}{2} \varepsilon)}$$

$$\lesssim \varepsilon, p' V_i(s)^{\frac{1}{p'} - \frac{3}{2} \varepsilon},$$

where the penultimate inequality follows from (2.8).

Finally one can state

**Proposition 3.6.** Under the assumptions of Theorem 1.3 for each $p \in (1, 2)$, it holds for all $1 \leq i, j \leq \ell$ that

$$\left\| t \mathcal{L} e^{-t\mathcal{L}} \right\|_{L^p(F_i^{(R_i)}) \to L^p(E_1 \cap F_i^{(R_i)})} \leq p \mu(F_i^{(R_i)})^{\frac{1}{2} - \frac{1}{p}}, \quad \forall t \geq R_j^2 = R_j(y)^2, \gamma \gg 1.$$

**Proof.** It follows from Lemma 3.4 that

$$\left\| t \mathcal{L} e^{-t\mathcal{L}} (fX_{F_i^{(R_i)}}) \right\|_{L^p(E_1 \cap F_i^{(R_i)})} \lesssim \varepsilon V_i(R_i)^{\frac{3}{2} \varepsilon - 1} \int_{F_i^{(R_i)}} |f(y)| \left( \frac{1}{V_i(y)} \right)^{\frac{3}{2} \varepsilon} d\mu(y).$$

Then by the Hölder inequality, we obtain from (3.11) that

$$\left\| t \mathcal{L} e^{-t\mathcal{L}} (fX_{F_i^{(R_i)}}) \right\|_{L^p(E_1 \cap F_i^{(R_i)})} \lesssim \varepsilon V_i(R_i)^{\frac{3}{2} \varepsilon - 1} V_i(R_i)^{\frac{1}{p'} - \frac{3}{2} \varepsilon} \left\| f \right\|_{L^p(F_i^{(R_i)})} \lesssim \varepsilon \frac{\left\| f \right\|_{L^p(F_i^{(R_i)})}}{\mu(F_i^{(R_i)})^{1/p}},$$

since $\mu(F_i^{(R_i)}) \sim V_i(R_i) \sim V_i(R_j)$ (cf. (3.2)).

In conclusion, for any $1 < p < 2$, by suitably choosing $\varepsilon$, we have that

$$\left\| t \mathcal{L} e^{-t\mathcal{L}} \right\|_{L^p(F_i^{(R_i)}) \to L^p(E_1 \cap F_i^{(R_i)})} \lesssim p \frac{1}{\mu(F_i^{(R_i)})^{1/p}}.$$

On the other hand, the classical Littlewood-Paley-Stein theory says that

$$\left\| t \mathcal{L} e^{-t\mathcal{L}} \right\|_{L^p(F_i^{(R_i)}) \to L^p(E_1 \cap F_i^{(R_i)})} \lesssim p 1, \quad \forall 1 < p < +\infty.$$

Hence the Hölder inequality implies that

$$\left\| t \mathcal{L} e^{-t\mathcal{L}} \right\|_{L^p(F_i^{(R_i)}) \to L^2(E_1 \cap F_i^{(R_i)})} \lesssim p \mu(F_i^{(R_i)})^{\frac{1}{2} - \frac{1}{p}},$$

which completes the proof. \qed
3.3 \( L^1 \) estimate of the space derivative of the heat kernel

The following result can be considered as a counterpart of the main result in [17, § 2.3], in other words, weighted estimates for the space derivative of the heat kernel (cf. [17, Lemma 2.4]). However, we will provide a direct proof by means of the main results in [26, 27]. And another proof similar to [17, Lemma 2.4] can be found in Subsection 4.5 below.

**Proposition 3.7.** Under the assumptions of Theorem 1.3 Let \( \beta \geq 16 \) and \( 1 \leq i \leq \ell \), it holds that:

\[
\int_{E_i \setminus F_i^{(s)}(y,r)} |\nabla_x h_t(x,y)| \, d\mu(x) \lesssim \frac{1}{\sqrt{t}} \, e^{-\frac{c}{r^2}}, \quad \forall t \leq \beta s^2, \; y \in E_i \setminus F_i^{(s)}, \; s \geq 1, \; r > 0.
\]

**Proof.** First by the H"older inequality, we can write that

\[
\int_{E_i \setminus F_i^{(s)}(y,r)} |\nabla_x h_t(x,y)| \, d\mu(x) \leq \left( \int_M |\nabla_x h_t(x,y)|^2 \, d\mu(x) \right)^{\frac{1}{2}} \left( \int_{E_i \setminus B(y,r)} e^{-\frac{d(x,y)^2}{2\beta}} \, d\mu(x) \right)^{\frac{1}{2}}
\]

\[
=: E_1(y,t)^{\frac{1}{2}} G(y,t,r)^{\frac{1}{2}}.
\]

Next, the doubling property and a standard method of decomposition in annuli imply that (e.g. [17, Lemma 2.1])

\[
G(y,t,r) \leq e^{-\frac{r^2}{2\beta}} V_i(y, \sqrt{t}), \quad \forall t, r > 0, \; y \in E_i.
\]

Moreover Corollary 2.7 says that

\[
\frac{1}{V_i(y, \sqrt{t})}, \quad \forall 0 < t \leq \beta s^2, \; y \in E_i \setminus F_i^{(s)}, \; s \geq 1.
\]

Using the doubling property, it is clear that the function \( h(t) := h(y,t) := V_i(y, \sqrt{t}) \) is regular on \((0, \beta R^2)\) in the sense of [27, p. 37] and \( h^{(1)}(t) := \int_0^t h(s) \, ds \sim t h(t) \) (e.g. [26, Lemma 3.1]). We then apply [27, Theorem 3.1] and [26, Theorem 1.1], and get that

\[
E_1(y,t) \lesssim \beta t^{-1} \frac{1}{V_i(y, \sqrt{t})}, \quad \forall 0 < t \leq \beta s^2, \; y \in E_i \setminus F_i^{(s)}, \; s \geq 1.
\]

Combining this with (3.13), we obtain the desired estimates. \( \square \)

4 Proof of Theorem 1.3

4.1 Outline of the proof

Let us outline the main approach for convenience of the reader. The heat kernel estimates (cf. Theorem 2.6) established in Grigor'yan and Saloff-Coste [28] is a key tool for the proof. Since we showed in Lemma 2.9 that the operator

\[
\frac{1}{\sqrt{\pi}} \int_0^1 \nabla e^{-sL} \frac{ds}{\sqrt{s}}
\]
is bounded on $L^p(M)$ for $1 < p < 2$, we only need to show that
\[
T := \frac{1}{\sqrt{\pi}} \int_1^{\infty} \nabla e^{-s^2} \frac{ds}{\sqrt{s}}
\]
is bounded on $L^p(M)$ for $1 < p < 2$. To this end, we will show that $T$ is weakly $(p, p)$ bounded for any $1 < p < p_0$, here $p_0$ is defined as
\[
(4.1) \quad p_0 := \min \left\{ 2, \frac{N_{\infty}}{N_{\infty} - 2} \right\},
\]
where $N_{\infty} = \max_i \{ N_i : i = 1, \ldots, \ell \}$ is defined in (2.6).

**Step 1: Reduction.** For $1 < p < p_0$ and any non-trivial $f \in L^p(M)$, from the non-collapsing assumption, it is not hard to see that for any $\lambda$ such that $\|f\|_{L^p(M)} \leq \lambda$,
\[
\mu \left( \{ x : |Tf| > \lambda \} \right) \leq \frac{\|f\|_{L^p(M)}^p}{\lambda^p};
\]
see Subsection 4.2. It is then enough to prove the same estimate holds for the case $0 < \lambda \ll \|f\|_{L^p(M)}$. To this end, in each end $E_i$, we choose a set $F_i^{(R)}$ around $E_0$ such that $\mu(F_i^{(R)}) \sim \|f\|_{L^p(M)}^p / \lambda^p$; see Figure 2. It is then enough to estimate that
\[
\mu \left( \left\{ x \in \bigcup_{j=1}^{\ell} \left( E_j \setminus F_j^{(R)} \right) : |Tf| > (2\ell + 1)\lambda \right\} \right)
\]
\[
\leq \mu \left( \left\{ x \in \bigcup_{j=1}^{\ell} \left( E_j \setminus F_j^{(R)} \right) : |T(f \chi_{E_0})| > \lambda \right\} \right) + \sum_{i=1}^{\ell} \mu \left( \left\{ x \in \bigcup_{j=1}^{\ell} \left( E_j \setminus F_j^{(R)} \right) : |T(f \chi_{F_i^{(R)}})| > \lambda \right\} \right)
\]
\[
+ \sum_{i=1}^{\ell} \mu \left( \left\{ x \in \bigcup_{j=1}^{\ell} \left( E_j \setminus F_j^{(R)} \right) : |T(f \chi_{E_0 \setminus F_i^{(R)}})| > \lambda \right\} \right).
\]

**Step 2: Estimate of the center part.** The part $f \chi_{E_0}$ is obvious, see Subsection 4.3.

**Step 3: Estimate of the part away from the center.** For the part $f \chi_{E \setminus F_i^{(R)}}$, we shall incorporate some ideas from [17] and use the Calderón-Zygmund decomposition to decompose $f \chi_{E \setminus F_i^{(R)}}$ as $g_i + \sum_k b_{ik}$, where $g_i$ and $b_{ik}$ are supported on $E_i$, and consider
\[
\mu \left( \left\{ x \in \bigcup_{j=1}^{\ell} \left( E_j \setminus F_j^{(R)} \right) : |T(f \chi_{E \setminus F_i^{(R)}})| > \lambda \right\} \right), \mu \left( \left\{ x \in E_i \setminus F_i^{(R)} : |T(f \chi_{E \setminus F_i^{(R)}})| > \lambda \right\} \right).
\]
Since we are dealing with small $\lambda$, $\lambda \ll \|f\|_{L^p(M)}$, the supporting balls of $b_{ik}$ are small enough, the $L^p$-Davies-Gaffney estimate of the operators $Ve^{-sL}$ and $Le^{-sL}$ established in Section 2 is sufficient to yield the estimate on the off-diagonal part, i.e., the first term. For the diagonal part (the second term), we shall further decompose the operator $T$ into the parts above or below the diagonal, and
deduce the required estimate by using the $L^p$-boundedness (Davies-Gaffney estimate) of $\nabla e^{-tL}$ and $Le^{-tL}$, and heat kernel estimate in Subsection 3.3.

**Step 4: Estimate of the part around the center.** For the part $f\chi_{F_i}(R_i)$, we shall use the boundedness of the operators $e^{-tL}$ and $Le^{-tL}$ from $L^p(F_i(R_i))$ to $L^2(E_j \setminus F_j(R_j))$ in Section 3, and the $L^p$ Davies-Gaffney estimate from Section 2.

**Step 5: Completion of the proof.** By combining previously obtained results, we show that $\nabla L^{-1/2}$ is weakly $(p, p)$ bounded for each $1 < p < p_0$. An application of the Marcinkowicz interpolation theorem gives the desired result.

### 4.2 Reduction

Let us begin with the following simple observation:

**Lemma 4.1.** Let $1 < p < 2$ and $0 < \beta_0 \leq 1$. It holds that

$$\mu (\{x : |Tf| > \lambda\}) \lesssim \beta_0 \frac{\|f\|_p^p}{\lambda^p}, \; \forall \lambda \geq \beta_0 \|f\|_p, \; f \in L^p. \tag{4.2}$$

**Proof.** Using Chebyshev’s inequality and Minkowski’s integral inequality, we have that

$$\mu (\{x : |Tf| > \lambda\}) \lesssim \frac{1}{\lambda^2} \left[ \int_1^{\infty} \|e^{-tL}f\|_2 \frac{dt}{\sqrt{t}} \right] \leq \frac{1}{\lambda^2} \left[ \int_1^{\infty} \|e^{-tL}f\|_2^2 \frac{dt}{\sqrt{t}} \right] \lesssim \frac{\|f\|^p_2}{\lambda^p}. \tag{4.3}$$

Next, the semigroup property and Proposition 2.1 imply that

$$\|e^{-tL}f\|_2 = \left\| e^{-\frac{t}{2}L} e^{-\frac{t}{2}L} f \right\|_2 \leq \frac{1}{\sqrt{t}} \left\| e^{-\frac{t}{2}L} f \right\|_2 \leq \frac{t^{\frac{1}{2}}}{\sqrt{t}} \frac{1}{\sqrt{t}} \|f\|_p$$

because of the ultracontractivity of the heat semigroup (cf. (2.18)). In conclusion,

$$\mu (\{x : |Tf| > \lambda\}) \lesssim \frac{\|f\|^2_2}{\lambda^2} \lesssim \beta_0 \frac{\|f\|^p_2}{\lambda^p}, \; \forall \lambda \geq \beta_0 \|f\|_p,$$

which finishes the proof. \qed

It is then enough to prove (4.2) holds for the case $0 < \lambda \ll \|f\|_p$ with $1 < p < p_0$ (see (4.1) for $p_0$). In such case, we decompose $M$ as

$$M = E_0 \cup (\bigcup_{i=1}^{\ell} F_i(R_i)) \cup \left( \bigcup_{j=1}^{\ell} (E_j \setminus F_j(R_j)) \right)$$

where

$$F_i(R_i) := \{x \in E_i : \text{dist}(x, E_0) \leq 2R_i\} \text{ with } R_i \gg 1 \text{ such that } \mu(F_i(R_i)) = 100 \ell \frac{\|f\|^p_2}{\lambda^p};$$
see Figure 2. By the convention $\mu(E_0) = 1$ (cf. (2.3)), it holds then

\begin{equation}
(4.4) \quad \mu(E_0 \cup (\bigcup_{i=1}^{\ell} F_i^{(R_i)})) = 1 + \sum_{i=1}^{\ell} \mu(F_i^{(R_i)}) \sim \frac{\|f\|_p^p}{\lambda^p}.
\end{equation}

Therefore, to prove that (4.2) holds for such $\lambda$, it is enough to prove that

\begin{equation}
(4.5) \quad \mu\left\{ x \in \bigcup_{j=1}^{\ell} \left( E_j \setminus F_j^{(R_j)} \right) : |T f| > \lambda \right\} \lesssim \frac{\|f\|_p^p}{\lambda^p}.
\end{equation}

For the simplicity of notation, we replace $\lambda$ by $(2\ell + 1)\lambda$ and write

\begin{align*}
\mu\left\{ x \in \bigcup_{j=1}^{\ell} \left( E_j \setminus F_j^{(R_j)} \right) : |T f| > (2\ell + 1)\lambda \right\} &
\leq \mu\left\{ x \in \bigcup_{j=1}^{\ell} \left( E_j \setminus F_j^{(R_j)} \right) : |T(f\chi_{E_0})| > \lambda \right\} + \sum_{i=1}^{\ell} \mu\left\{ x \in \bigcup_{j=1}^{\ell} \left( E_j \setminus F_j^{(R_j)} \right) : |T(f\chi_{F_i^{(R_i)}})| > \lambda \right\} \\
&\quad + \sum_{i=1}^{\ell} \mu\left\{ x \in \bigcup_{j=1}^{\ell} \left( E_j \setminus F_j^{(R_j)} \right) : |T(f\chi_{E_i \setminus F_i^{(R_i)}})| > \lambda \right\} =: C + \mathcal{A} + \mathcal{W}.
\end{align*}

The same trick will be used repeatedly. We shall call the first term as the center part, the second term as the part around the center, and the last term as the part away from the center. We shall treat them in the following subsections.

### 4.3 Estimate on the center

First we establish the following:

**Lemma 4.2.** Let $1 < p < 2$. It holds that

\begin{equation*}
C = \mu\left\{ x \in \bigcup_{j=1}^{\ell} \left( E_j \setminus F_j^{(R_j)} \right) : |T(f\chi_{E_0})| > \lambda \right\} \lesssim \frac{\|f\|_p^p}{\lambda^p}, \quad \forall \lambda \ll \|f\|_p, \ f \in L^p.
\end{equation*}

**Proof.** Similar to the proof of (4.2), we have that

\begin{align*}
C \lesssim \frac{1}{\lambda^p} \left\| T(f\chi_{E_0}) \right\|_p^p &\lesssim \frac{1}{\lambda^p} \left[ \int_1^\infty \left\| \nabla e^{-tL}(f\chi_{E_0}) \right\|_p \frac{dt}{\sqrt{t}} \right]^p \\
&\lesssim \frac{1}{\lambda^p} \left[ \int_1^\infty \left\| e^{-\frac{1}{2}tL}(f\chi_{E_0}) \right\|_p \frac{dt}{t} \right]^p \\
&\lesssim \frac{1}{\lambda^p} \left[ \int_1^\infty \lambda^{\frac{1}{2}-1} \left\| f\chi_{E_0} \right\|_1 \frac{dt}{t} \right]^p \\
&\lesssim \frac{\|f\chi_{E_0}\|_1^p}{\lambda^p} \lesssim \frac{\|f\|_p^p}{\lambda^p},
\end{align*}

as desired. \qed
4.4 The part away from the center

Recall that \( p_0 \) is defined by (4.1). The main aim of this subsection is to prove that

**Proposition 4.3.** Let \( 1 < p < p_0 \). It holds that

\[
\mathcal{W} = \sum_{i=1}^{\ell} \mu \left( \left\{ x \in \bigcup_{j=1}^{\ell} \left( E_i \setminus F_i^{(R_i)} \right) : |T(f X_{E_i \setminus F_i^{(R_i)}})| > \lambda \right\} \right) \leq \frac{\|f\|_p^p}{\lambda^p}, \quad \forall \lambda \ll \|f\|_p, \; f \in L^p.
\]

For each part \( E_i \setminus F_i^{(R_i)} \), we may run the Calderón-Zygmund decomposition (cf. [16]) of \( f X_{E_i \setminus F_i^{(R_i)}} \) on \( M_i \) at the height \( \delta \lambda \), where \( \delta > 1 \) is large enough to be fixed later; see also [17]. Since \( M_i \) is a doubling manifold, we obtain

\[
f X_{E_i \setminus F_i^{(R_i)}} = \sum_k b_{ik} + g_i,
\]

where \( \text{supp} b_{ik} \subset B_{ik}(x_{ik}, r_{ik}) \), \( \sum_k \chi_{B_{ik}} \leq 1 \), \( |g_i| \leq \delta \lambda \),

\[
\int_{M_i} |g_i|^p \, d\mu_i \leq \int_{M_i} |f X_{E_i \setminus F_i^{(R_i)}}|^p \, d\mu_i = \int_{M_i} |f X_{E_i \setminus F_i^{(R_i)}}|^p \, d\mu.
\]

(4.6)

\[
\int_{B_{ik}} |b_{ik}|^p \, d\mu_i \leq (\delta \lambda)^p,
\]

and

(4.7)

\[
\sum_k \mu(B_{ik}) \leq \frac{1}{(\delta \lambda)^p} \int_{M_i} |f X_{E_i \setminus F_i^{(R_i)}}|^p \, d\mu_i.
\]

We shall call \( g_i \) the good part, and \( b_{ik} \) the ‘bad’ part following tradition.

From (4.8) we further have

(4.9)

\[
\sum_k \mu(B_{ik}) \leq \frac{1}{(\delta \lambda)^p} \int_{M_i} |f X_{E_i \setminus F_i^{(R_i)}}|^p \, d\mu_i \leq \frac{\|f\|_p^p}{(\delta \lambda)^p} \leq \frac{\mu(F_i^{(R_i)})}{\delta^p},
\]

by (4.3). We claim that, for a large enough \( \delta \), it holds that

(4.10) \quad \text{dist} (B_{ik}, E_0) = \text{dist} (B_{ik}, \partial E_i) \geq \frac{1}{2} (R_i + r_{ik}) (\gg 1).

Indeed, if the radius \( r_{ik} < R_i/2 \), then the claim holds obviously, since \( B_{ik} \cap (E_i \setminus F_i^{(R_i)}) \neq \emptyset \) and

\[
F_i^{(R_i)} = \{ x \in E_i : \text{dist} (x, E_0) \leq 2R_i \}.
\]

If \( r_{ik} \geq R_i/2 \) and the claim fails, then

\[
\text{dist} (B_{ik}, E_0) < \frac{1}{2} (R_i + r_{ik}) < \frac{3}{2} r_{ik}.
\]
This together with the convention $\text{diam}(E_0) = 1$ (cf. (2.3)) and the doubling condition implies that
\[
\mu\left(F_j^{(R_j)}\right) \sim V_i(R_i) \leq V_i(o_i, 3r_{ik}) \leq V_i(x_{ik}, 3r_{ik}) \leq \mu(B_{ik}),
\]
which contradicts with (4.9) as soon as we choose $\delta$ large enough. Therefore, (4.10) holds, which further implies that
\[
\text{dist} (B_{ik}, E_0) \leq d(x_{ik}, o_i) \leq \text{dist} (B_{ik}, E_0) + \text{diam} (E_0) + r_{ik} \leq 3 \text{dist} (B_{ik}, E_0),
\]
and
\[
\frac{1}{2} (R_i + r_{ik}) + r_{ik} \leq \text{dist} (B_{ik}, E_0) + r_{ik} \leq d(x_{ik}, o_i).
\]

Note that (4.10) implies that $\text{supp} b_{ik} \subset E_i$, which together with the identity $f \chi_{E_i \setminus F_j^{(R_j)}}(x) = \sum_k b_{ik} + g_i$ implies that $\text{supp} g_i \subset E_i$. Since $\mu = \mu_i$ on $E_i$, the integrals of $g_i$ and $b_{ik}$ on $M_i$ are the same as on $M$, i.e.,
\[
\int_{M_i} |g_i|^p \, d\mu_i = \int_{M} |g_i|^p \, d\mu, \quad \int_{B_{ik}} |b_{ik}|^p \, d\mu_i = \int_{B_{ik}} |b_{ik}|^p \, d\mu.
\]

### 4.4.1 Estimate of the good part

The $L^2$-boundedness of $T$ (cf. Lemma 2.9(i)) together with the fact $|g_i| \leq \lambda$ implies
\[
\sum_{i=1}^{\ell} \mu\left( \left\{ x \in \bigcup_{j=1}^{\ell} \left( E_j \setminus F_j^{(R_j)} \right) : |T(g_i)| > \lambda \right\} \right) \leq \sum_{i=1}^{\ell} \frac{1}{\lambda^2} \int_{M} |g_i|^2 \, d\mu \leq \sum_{i=1}^{\ell} \frac{1}{\lambda^p} \int_{M_i} |g_i|^p \, d\mu
\]
\[
\leq \frac{\|f\|_p^p}{\lambda^p},
\]
because of (4.6).

### 4.4.2 Estimate of the ‘bad’ parts off the diagonal

For each $i, 1 \leq i \leq \ell$,
\[
\mu\left( \left\{ x \in \bigcup_{1 \leq j \neq i \leq \ell} \left( E_j \setminus F_j^{(R_j)} \right) : \left| T\left( \sum_k b_{ik} \right) \right| > \lambda \right\} \right)
\leq \mu\left( \left\{ x \in \bigcup_{1 \leq j \neq i \leq \ell} \left( E_j \setminus F_j^{(R_j)} \right) : \left| T\left( \sum_k (1 - e^{-r_{ik}^2b})b_{ik} \right) \right| > \lambda/2 \right\} \right)
\leq \mu\left( \left\{ x \in \bigcup_{1 \leq j \neq i \leq \ell} \left( E_j \setminus F_j^{(R_j)} \right) : \left| T\left( \sum_k e^{-r_{ik}^2b}b_{ik} \right) \right| > \lambda/2 \right\} \right)
\]
\[
\leq \mu\left( \left\{ x \in \bigcup_{1 \leq j \neq i \leq \ell} \left( E_j \setminus F_j^{(R_j)} \right) : \left| T\left( \sum_k e^{-r_{ik}^2b}b_{ik} \right) \right| > \lambda/2 \right\} \right)
\leq \mathcal{G}_1 + \mathcal{G}_2.
\]

For the term $\mathcal{G}_2$, we have
Proposition 4.4. Let $1 < p < 2$. It holds that

$$G_2 = \mu \left\{ x \in \bigcup_{1 \leq j \leq \ell} (E_j \setminus F_j^{(R_j)}) : \left| T \left( \sum_{k} e^{-r_k^2 \mathcal{L} b_{ik}} \right) > \lambda/2 \right) \right\} \leq \frac{\|f\|_p^p}{\lambda^p}, \quad \forall \lambda \ll \|f\|_p, \; f \in L^p.$$

Proof. The $L^2$-boundedness of $T$ implies that

$$G_2 \leq \frac{1}{\lambda^2} \int_M \left| \sum_{k} e^{-r_k^2 \mathcal{L} b_{ik}} \right|^2 \, d\mu$$

(4.15) $= \lambda^{-2} \int_{E_i} \left| \sum_{k} e^{-r_k^2 \mathcal{L} b_{ik}} \right|^2 \, d\mu + \lambda^{-2} \sum_{0 \leq j \neq \ell} \int_{E_j} \left| \sum_{k} e^{-r_k^2 \mathcal{L} b_{ik}} \right|^2 \, d\mu =: G_{21} + G_{22}.$

Let us begin with $G_{21}$. By (4.10), it follows from Corollary 2.7 that

$$h_{r_k^2}(x, y) \leq \frac{1}{V_i(x, r_k)} e^{-c(d(x, y))^2}, \quad \forall x \in E_i, \; y \in B_{ik} \subset E_i.$$

Let $M_i$ denote the centered Hardy-Littlewood Maximal function on $M_i$. Then the same arguments as Coulhon-Duong [17, pp. 1158-1160] gives for any $\|\psi\|_2 \leq 1$ that

$$\int_{E_i} \left( \sum_{k} e^{-r_k^2 \mathcal{L} b_{ik}} \right) \psi \, d\mu \leq \lambda \int_{E_i} \left( \sum_{k} \chi_{B_{ik}} \right) \mathcal{M}_i \psi \, d\mu \leq \lambda \left( \sum_{k} \chi_{B_{ik}} \right) L^2(M_i) \|\mathcal{M}_i \psi\|_{L^2(M_i)}$$

$$\leq \lambda \left( \sum_{k} \mu(B_{ik}) \right)^{1/2}.$$

Combining this with (4.8), we obtain that

$$G_{21} \leq \sum_{k} \mu(B_{ik}) \lesssim \frac{\|f\|_p^p}{\lambda^p}.$$

Consequently, what remains is to prove that

$$G_{22} = \lambda^{-2} \sum_{0 \leq j \neq \ell} \int_{E_j} \left| \sum_{k} e^{-r_k^2 \mathcal{L} b_{ik}} \right|^2 \, d\mu \leq \frac{\|f\|_p^p}{\lambda^p}, \quad \forall \lambda \ll \|f\|_p, \; f \in L^p, \; 1 \leq i \leq \ell.$$

Next, we claim that

(4.16) $\left\| e^{-r_k^2 \mathcal{L} b_{ik}} \right\|_{L^1(M \setminus E_i) \rightarrow L^\infty(B_{ik})} \lesssim \frac{1}{V_i(R_i)}.$
Let \( u \in B_{r_k} \subset E_i (1 \leq i \leq \ell) \) and \( v \notin E_i \). First consider the case where \( r_k \geq 1 \). Using (2.10) and the fact that \( V_j(s) \geq V_0(s) \geq s^2 \) for any \( s \geq 1 \) and \( 0 \leq j \leq \ell \) (cf. Lemma 2.5(i)), Theorem 2.6(iii) implies that

\[
h_{r_k}^2(u, v) \leq \left( \frac{d_{\text{diam}}^2}{r_k^2} \cdot \frac{1}{V_i(|u|)} + \frac{1}{V_i(r_k)} \right) \exp \left( -e^{\frac{|u|^2}{r_k^2}} \right) \lesssim \frac{1}{V_i(|u|)},
\]

by the trivial inequality \( r^\alpha e^{-r} \lesssim 1 \) for any \( \alpha > 0 \) and all \( r > 0 \), and the fact that \( V_i(|u|)/V_i(r_k) \) has at most polynomial growth w.r.t. \( |u|/r_k \) from the doubling property. Then using the doubling property again, (4.10) gets that \( h_{r_k}^2(u, v) \leq V_i(R_i)^{-1} \).

For the opposite case \( r_k < 1 \), similarly, using the symmetric property of the heat kernel, from (2.11) and (4.10), we can write

\[
h_{r_k}^2(u, v) = h_{r_k}^2(v, u) \leq \frac{1}{V(v, r_k)} \exp \left( -e^{\frac{d_{\text{diam}}^2}{r_k^2}} \right) \leq \frac{1}{V(v, r_k)} \exp \left( -e^{\frac{d_{\text{diam}}^2}{r_k^2}} \right) \lesssim \frac{1}{V_i(r_k)} \exp \left( -e^{\frac{d_{\text{diam}}^2}{r_k^2}} \right) \leq \frac{1}{V_i(R_i)},
\]

where the penultimate inequality follows from Lemma 2.5(iii) and a standard trick of doubling property (cf. e.g. the proof of (2.16)).

In conclusion, we yield (4.16), which together with \( \|e^{-r_k^2 \mathcal{L}} \|_{L^\infty(M \setminus E_i) \rightarrow L^\infty(B_k)} \leq 1 \) implies via the Riesz-Thorin interpolation theorem that

\[
\|e^{-r_k^2 \mathcal{L}} \|_{L^2(M \setminus E_i) \rightarrow L^\infty(B_k)} \lesssim \frac{1}{\sqrt{V_i(R_i)}}.
\]

Hence, for any \( 1 \leq i \leq \ell \) and \( \|\psi\|_2 \leq 1 \), we have that

\[
\left| \int_{\cup_{0 \leq j \leq \ell, i \neq j} E_j} \left( \sum_k e^{-r_k^2 \mathcal{L}} b_{ik} \right) \psi \, d\mu \right| \leq \int_M \sum_k |b_{ik}| e^{-r_k^2 \mathcal{L}}(\psi_X M \setminus E_i) \, d\mu \lesssim \sum_k \frac{|b_{ik}|}{V_i(R_i)} \, d\mu \lesssim \lambda \sum_k \mu(B_{ik}) \frac{|b_{ik}|}{V_i(R_i)},
\]

(4.17)

where we have used, in the last step, the Hölder inequality and (4.7). Combining this with the fact that \( \sum_k \mu(B_{ik}) \leq \mu(F_i(R_i)) \sim V_i(R_i) \) (see (4.9), (4.3) and (3.2)), we can conclude that

\[
\mathcal{G}_{22} \lesssim \frac{1}{\lambda^2} \left( \frac{\lambda}{\sqrt{V_i(R_i)}} \right)^2 \lesssim \sum_k \frac{\mu(B_{ik})}{\sqrt{V_i(R_i)}},
\]

which finishes the proof. \( \square \)

We now turn to \( \mathcal{G}_1 \). Recall that \( p_0 \in (1, 2] \) is given in (4.1).

**Proposition 4.5.** Let \( 1 < p < p_0 \). It holds uniformly for all \( f \in L^p(M) \) and \( 0 < \lambda \ll \|f\|_p \) that

\[
\mathcal{G}_1 = \mu \left( \left\{ x \in \bigcup_{1 \leq j \leq \ell} (E_j \setminus F_i^{(j)}) : \left| T \left( \sum_k (1 - e^{-r_k^2 \mathcal{L}} b_{ik}) \right)(x) \right| > \frac{\lambda}{2} \right\} \right) \lesssim \frac{\|f\|_p^p}{\lambda^p}.
\]
By the Chebyshev inequality, we can write
\[
\mathcal{G}_1 \leq \frac{1}{\lambda^p} \int_{\mathbb{R}^n} \left| T \left( \sum_k (1 - e^{-r_k^2}) b_{ik} \right) \right|^p d\mu
\]
\[
\leq \frac{1}{\lambda^p} \int_{\mathbb{R}^n} \left( \sum_k \left( \int_0^\infty \left| \nabla L e^{-(s + t) L} b_{ik} \right| ds dt \right) \right|^p d\mu,
\]
from the definition of \( T \). Then the Minkowski’s integral inequality yields that
\[
\mathcal{G}_1 \leq \frac{1}{\lambda^p} \left[ \sum_k \left( \int_0^\infty \left| \nabla L e^{-(s + t) L} b_{ik} \right| ds dt \right) \right]^p d\mu.
\]
Using (4.11), the double integral can be controlled by
\[
\int_1^{\infty} \int_0^r \frac{e^{-c \frac{d(x_k, o_2)^2}{t+s}}}{(t+s)^{3/2}} ds dt \leq \int_1^{\infty} \int_0^r \frac{e^{-c \frac{d(x_k, o_2)^2}{t+s}}}{(t+s)^{3/2}} ds dt \leq \int_1^{\infty} \int_0^r \frac{e^{-c \frac{d(x_k, o_2)^2}{t+s}}}{(t+s)^{3/2}} ds dt \leq \frac{r_k^2}{d(x_k, o_2)^2}.
\]
As a consequence,
\[
\mathcal{G}_1 \leq \frac{1}{\lambda^p} \left[ \sum_k \frac{r_k^2}{d(x_k, o_2)^2} \right]^p.
\]
Next notice that \( p' > N_{\infty}/2 \) provided \( 1 < p < p_0 \). From (4.12) and (2.5), we have that
\[
\left( \frac{r_k^2}{d(x_k, o_2)^2} \right)^{p'} \leq \left( \frac{r_k^2}{d(x_k, o_2)^2} \right)^{N_i} \leq \frac{V_i(x_k, r_k)}{V_i(x_k, d(x_k, o_i))} \leq \frac{\mu(B_{ik})}{V_i(R_i)} \leq \frac{\mu(B_{ik})}{V_i(R_i)}.
\]
by means of the doubling property. Therefore, using (4.9), (4.3) and (3.2), we obtain that
\[
\sum_k \left( \frac{r_k^2}{d(x_k, o_2)^2} \right)^{p'} \leq \sum_k \frac{\mu(B_{ik})}{V_i(R_i)} \leq 1.
\]
By the Hölder inequality, we conclude that
\[
\mathcal{G}_1 \leq \frac{1}{\lambda^p} \left[ \sum_k \frac{r_k^2}{d(x_k, o_2)^2} \right]^p \leq \frac{1}{\lambda^p} \left[ \sum_k \left( \frac{r_k^2}{d(x_k, o_2)^2} \right)^{p'/p'} \right]^p \left[ \sum_k \|b_{ik}\|_p^p \right].
\[ (4.18) \]
\[ \leq \frac{1}{\lambda^p} \left[ \lambda^p \sum_k \mu(B_{ik}) \right] \leq \frac{1}{\lambda^p} \| f \chi_{E_i \setminus F_i^R} \|_p^p \leq \frac{\| f \|_p^p}{\lambda^p}, \]

where we have used the properties of Calderón-Zygmund decomposition (4.7) and (4.8).

This completes the proof. \(\Box\)

4.4.3 Estimate of the ‘bad’ parts on the diagonal

It remains to estimate

\[ S := \mu \left( \left\{ x \in E_i \setminus F_i^R : \left| \sum_k b_{ik} \right| > \lambda \right\} \right). \]

Recall that

\[ T = \frac{1}{\sqrt{\pi}} \int_1^\infty \nabla e^{-t L} \frac{dt}{\sqrt{t}} \quad \nabla L^{-1/2} = \frac{1}{\sqrt{\pi}} \int_0^\infty \nabla e^{-s L} \frac{ds}{\sqrt{s}}. \]

We then have

\[ S \leq \mu \left( \left\{ x \in E_i \setminus F_i^R : \left| \nabla L^{-1/2} \left( \sum_k b_{ik} \right) \right| > \frac{1}{2} \lambda \right\} \right) \]

\[ + \mu \left( \left\{ x \in E_i \setminus F_i^R : \left| \int_0^1 \nabla e^{-t L} \left( \sum_k b_{ik} \right) \frac{dt}{\sqrt{t}} \right| > \frac{1}{2} \lambda \right\} \right). \]

Let us begin with the following:

**Lemma 4.6.** Let \( 1 < p < 2 \). We have uniformly for all \( f \in L^p(M) \) and all \( 0 < \lambda < \| f \|_p \) that

\[ S_1(\lambda) = \mu \left( \left\{ x \in E_i \setminus F_i^R : \left| \int_0^1 \nabla e^{-t L} \left( \sum_k b_{ik} \right) \frac{dt}{\sqrt{t}} \right| > \lambda \right\} \right) \leq_p \frac{\| f \|_p^p}{\lambda^p}. \]

**Proof.** Recall that the operator \( \int_0^1 \nabla e^{-t L} \frac{dt}{\sqrt{t}} \) is bounded on \( L^p(M) \), cf. Lemma 2.9. The Chebyshev inequality implies that

\[ S_1(\lambda) \leq_p \frac{1}{\lambda^p} \left\| \sum_k b_{ik} \right\|_p^p \leq \frac{1}{\lambda^p} \sum_k \left\| b_{ik} \right\|_p^p \leq \frac{\| f \|_p^p}{\lambda^p}, \]

where we have used in the second inequality the fact that the supports of \( \{ b_{ik} \} \) are of bounded overlap, (4.7) and (4.8) in the last one. \(\Box\)

For the remaining term, we write

\[ \mu \left( \left\{ x \in E_i \setminus F_i^R : \left| \nabla L^{-1/2} \left( \sum_k b_{ik} \right) \right| > \lambda \right\} \right) \]
Lemma 4.7. Let \( R \)

\[
\begin{align*}
&\leq \mu \left( \left\{ x \in E_i \setminus F_i^{(R)} : \left| \nabla L^{-1/2} \left( \sum_k e^{-r^2_{ik}L} b_{ik} \right) \right| > \frac{1}{2} \lambda \right\} \right) \\
&\quad + \mu \left( \left\{ x \in E_i \setminus F_i^{(R)} : \left| \nabla L^{-1/2} \left( \sum_k (1 - e^{-r^2_{ik}L}) b_{ik} \right) \right| > \frac{1}{2} \lambda \right\} \right) .
\end{align*}
\]

Proof. By the natural \( L^2 \)-boundedness of \( \nabla L^{-1/2} \), we obtain that

\[
S_{21}(\lambda) := \mu \left( \left\{ x \in E_i \setminus F_i^{(R)} : \left| \nabla L^{-1/2} \left( \sum_k e^{-r^2_{ik}L} b_{ik} \right) \right| > \lambda \right\} \right) \leq \frac{\|f\|_p^p}{\lambda^p}.
\]

by recalling that the last inequality is obtained in the proof of Proposition 4.4. \( \square \)

Proposition 4.8. Let \( 1 < p < p_0 \). We have that for all \( f \in L^p(M) \) and all \( 0 < \lambda < \|f\|_p \)

\[
S_{21}(\lambda) := \mu \left( \left\{ x \in E_i \setminus F_i^{(R)} : \left| \nabla L^{-1/2} \left( \sum_k (1 - e^{-r^2_{ik}L}) b_{ik} \right) \right| > \lambda \right\} \right) \leq \frac{\|f\|_p^p}{\lambda^p}.
\]

Proof. As in \([17] \) pp. 1160-1161], using (4.12), we write

\[
\sqrt{\pi} \nabla L^{-1/2}(1 - e^{-r^2_{ik}L}) = \int_0^\infty \left[ \frac{1}{\sqrt{t}} - \frac{X(t)}{\sqrt{t}} \right] \nabla e^{-tL} \, dt
\]

\[
= \int_0^{Rk_1} \left[ \frac{1}{\sqrt{t}} - \frac{X(t)}{\sqrt{t}} \right] \nabla e^{-tL} \, dt + \int_{Rk_1}^{4d(x_k, o)^2} \cdots + \int_{4d(x_k, o)^2}^\infty \cdots
\]

\[
=: T_{ik1} + T_{ik2} + T_{ik3}.
\]

It holds then

(4.19)

\[
S_{22}(\lambda) \leq \mu(\cup_k 2B_{ik}) + \sum_{j=1}^3 S_{22j},
\]

with

\[
S_{22j} := \mu \left( \left\{ x \in E_i \setminus \left( F_i^{(R)} \cup (\cup_k 2B_{ik}) \right) : \left| \sum_k T_{ikj} b_{ik} \right| > \frac{1}{3} \lambda \right\} \right) , \quad 1 \leq j \leq 3.
\]
The estimation of $\mu(\bigcup B_{ik})$ is standard, via the doubling property and (4.9). Next, we estimate $S_{221}$. Recall that (4.10) says $B_{ik} \subset E_i \setminus \{z \in E_i : \text{dist}(z, E_0) > R_t/2\} = E_i \setminus F_{i}^{(R_t/4)}$. Then for $t \leq R_t^2$ and $y \in B_{ik}$, by Proposition 3.7, it holds

$$
\int_{E_i \setminus (F_{i}^{(R_t/4)} \cup 2B_{ik})} |\nabla h_t(x, y)| \, d\mu(x) \leq \frac{1}{\sqrt{t}} e^{-\frac{c^2}{4}}.
$$

Therefore, following the argument in [17, p. 1161], we deduce that

$$
S_{221} \leq \frac{1}{A} \sum_k \int_0^{R_t^2} \int_{E_i \setminus (F_{i}^{(R_t/4)} \cup 2B_{ik})} \left| \frac{1}{\sqrt{t}} - \frac{X_{(x, y, o)}}{\sqrt{t - r_{ik}^2}} \right| |\nabla e^{-\mathcal{L}} b_{ik}| \, d\mu \, dt
$$

$$
\leq \frac{1}{A} \sum_k \int_0^{R_t^2} \frac{1}{\sqrt{t}} e^{-c^2\frac{t}{4}} \left| \frac{1}{\sqrt{t}} - \frac{X_{(x, y, o)}}{\sqrt{t - r_{ik}^2}} \right| \|b_{ik}\|_1 \, dt
$$

$$
\leq \frac{1}{A} \sum_k \|b_{ik}\|_1 \left( \int_0^{R_t^2} \frac{1}{\sqrt{t}} e^{-c^2\frac{t}{4}} \left| \frac{1}{\sqrt{t}} - \frac{X_{(x, y, o)}}{\sqrt{t - r_{ik}^2}} \right| \, dt + \int_{r_{ik}^2}^{\infty} \frac{1}{\sqrt{t}} - \frac{1}{\sqrt{t - r_{ik}^2}} \, dt \right)
$$

$$
\leq \frac{1}{A} \sum_k \|b_{ik}\|_1 \leq \frac{\|f\|_p^p}{A^p},
$$

where we have used (4.17) in the last inequality.

Aiming now at $S_{223}$. From the mapping property of $\nabla e^{-\mathcal{L}}$ (Proposition 2.1), we get that

$$
S_{223} \leq \frac{1}{A^p} \int_{E_i \setminus (F_{i}^{(R_t/4)})} \left| \sum_k T_{ik} b_{ik} \right|^p \, d\mu
$$

$$
\leq \frac{1}{A^p} \int_{E_i \setminus (F_{i}^{(R_t/4)})} \left( \int_0^{\infty} \sum_k \frac{X(4d(x_k, o), o)^2(t)}{\sqrt{t}} \left| \frac{1}{\sqrt{t}} - \frac{X_{(x_k, o)}}{\sqrt{t - r_{ik}^2}} \right| |\nabla e^{-\mathcal{L}} b_{ik}| \, dt \right)^p \, d\mu
$$

$$
\leq \frac{1}{A^p} \left[ \sum_k \int_0^{\infty} \left| \frac{1}{\sqrt{t}} - \frac{X_{(x_k, o)}}{\sqrt{t - r_{ik}^2}} \right| \left( \int_{E_i \setminus (F_{i}^{(R_t/4)})} |\nabla e^{-\mathcal{L}} b_{ik}|^p \, d\mu \right)^{1/p} \, dt \right]^p
$$

$$
\leq \frac{1}{A^p} \left[ \sum_k \|b_{ik}\|_p \int_0^{\infty} \left| \frac{1}{\sqrt{t}} - \frac{1}{\sqrt{t - r_{ik}^2}} \right| \, dt \right]^p,
$$

and in view of (4.12), the last integral here equals

$$
r_{ik}^2 \int_{4d(x_k, o)^2} \frac{1}{\sqrt{t + r_{ik}^2}} \frac{1}{\sqrt{t - r_{ik}^2}} \frac{dt}{\sqrt{t}} \leq \frac{r_{ik}^2}{d(x_k, o)^2}.
$$
Consequently, by (4.18), we obtain that

\[ S_{223} \lesssim \frac{1}{\Lambda^p} \left[ \sum_k \frac{r_{ik}^2}{d(x_{ik}, o_i)^2} \|b_{ik}\|_p \right]^p \lesssim \frac{\|f\|_p^p}{\Lambda^p}. \]

In conclusion, it remains to prove the lemma below. \(\square\)

**Lemma 4.9.** It holds that \( S_{222} \lesssim \Lambda^{-p} \|f\|_p \) provided \( 0 < \Lambda \ll \|f\|_p \).

**Proof.** By (4.11) and (4.12),

\[ \frac{1}{2}(R_i + r_{ik}) \leq \text{dist} (B_{ik}, E_0) < d(x_{ik}, o_i) \leq 1 + r_{ik} + \text{dist} (B_{ik}, E_0) \leq 3 \text{dist} (B_{ik}, E_0), \]

we split the set \( E_i \setminus (F_i^{(R)} \cup (\cup_k 2B_{ik})) \) into

\[ G_{1k} := \{ x \in E_i \setminus (F_i^{(R)} \cup (\cup_k 2B_{ik})) : d(x, E_0) \leq \frac{1}{2} \text{dist} (B_{ik}, E_0) \} \]

and

\[ G_{2k} := \{ x \in E_i \setminus (F_i^{(R)} \cup (\cup_k 2B_{ik})) : d(x, E_0) > \frac{1}{2} \text{dist} (B_{ik}, E_0) \}. \]

It holds then

\[ \text{dist} (G_{1k}, B_{ik}) \geq \frac{1}{2} \text{dist} (B_{ik}, E_0) \geq \frac{1}{6} d(x_{ik}, o_i), \]

\[ \text{dist} (G_{2k}, E_0) \geq \frac{1}{2} \text{dist} (B_{ik}, E_0) \geq \frac{1}{6} d(x_{ik}, o_i). \]

By Proposition 3.7, for \( t \leq 4d(x_{ik}, o_i)^2 \), \( y \in \{ x \in E_i : \text{dist} (x, E_0) \geq \frac{1}{6} d(x_{ik}, o_i) \} \cap B_{ik} \), it holds

\[ \int_{\{ x \in E_i : \text{dist} (x, E_0) \geq \frac{1}{6} d(x_{ik}, o_i) \} \cap (2B_{ik})} |\nabla h_t(x, y)| \, d\mu(x) \lesssim \frac{1}{\sqrt{t}} e^{-\frac{\sqrt{t}}{10}}. \]

Notice that

\[ S_{222} = \mu \left( \left\{ x \in E_i \setminus (F_i^{(R)} \cup (\cup_k 2B_{ik})) : \left| \sum_k (\chi G_{ik} + \chi G_{2k}) T_{ik2} b_{ik} \right| \right. \right. \left. \geq \frac{\Lambda}{3} \right) \]

\[ \lesssim \frac{1}{\Lambda^p} \int_{E_i \setminus F_i^{(R)}} \left| \sum_k \chi G_{ik} T_{ik2} b_{ik} \right|^p \, d\mu + \frac{1}{\Lambda} \int_{E_i \setminus F_i^{(R)}} \left| \sum_k \chi G_{2k} T_{2k} b_{ik} \right| \, d\mu =: \mathcal{F}_1 + \mathcal{F}_2. \]

We begin with \( \mathcal{F}_1 \). We argue as in the estimation of \( S_{223} \), by using the Davies-Gaffney estimate (cf. Corollary 2.3) instead of Proposition 2.1, and conclude that

\[ \mathcal{F}_1 \sim \frac{1}{\Lambda^p} \int_{E_i \setminus F_i^{(R)}} \left| \int_0^{\infty} \sum_k \chi (R_{ik}^2; Ad(x_{ik}, o_i^2))(t) \left( \frac{1}{\sqrt{t}} - \frac{\chi_{[r_{ik}]}(t)}{\sqrt{t}} \right) \right| \frac{d\mu}{\sqrt{t}} \leq \frac{\chi_{[r_{ik}]}(t)}{\sqrt{t}} \, \|\chi G_{ik} \nabla e^{-tE} b_{ik}\| \right| \frac{d\mu}{\sqrt{t}}. \]
Proof. For each $F$, inserting it in the last inequality and using (4.18), we get

\[
\left( \int_{G_{ik}} |\nabla e^{-t\mathcal{L} b_{ik}}|^p \, d\mu \right)^{1/p} \leq \frac{1}{\lambda^p} \left[ \sum_k |b_{ik}|_p \int_{R_i^2} e^{-c \frac{d(x_{ik}, o_i)^2}{r_{ik}^2}} \left| \frac{1}{\sqrt{t}} - \frac{X_{\{t > r_{ik}^2\}}}{\sqrt{t - r_{ik}^2}} \right| \, dt \right]^{1/p},
\]

and from (4.20), the last integral here can be controlled by

\[
\int_{R_i^2} e^{-c \frac{d(x_{ik}, o_i)^2}{r_{ik}^2}} \left| \frac{1}{\sqrt{t}} - \frac{X_{\{t > r_{ik}^2\}}}{\sqrt{t - r_{ik}^2}} \right| \, dt \lesssim \int_0^{r_{ik}^2} \frac{dt}{d(x_{ik}, o_i)^2} + \int_{R_i^2} e^{-c \frac{d(x_{ik}, o_i)^2}{r_{ik}^2}} \frac{r_{ik}^2}{d(x_{ik}, o_i)^3} \frac{dt}{\sqrt{t - r_{ik}^2}} \lesssim \frac{r_{ik}^2}{d(x_{ik}, o_i)^2}.
\]

Inserting it in the last inequality and using (4.18), we get $\mathcal{F}_1 \lesssim \lambda^{-p} \|f\|_p$.

For $\mathcal{F}_2$, by using (4.22), the argument used in the estimation of $S_{221}$ implies that

\[
\mathcal{F}_2 \lesssim \frac{1}{\lambda} \sum_k \int_{R_i^2} |\nabla e^{-t\mathcal{L} b_{ik}}| \, d\mu \, dt \lesssim \frac{1}{\lambda} \sum_k \|b_{ik}\|_1 \int_{R_i^2} e^{-c \frac{d(x_{ik}, o_i)^2}{r_{ik}^2}} \left| \frac{1}{\sqrt{t}} - \frac{X_{\{t > r_{ik}^2\}}}{\sqrt{t - r_{ik}^2}} \right| \, dt \lesssim \frac{\|f\|_p^p}{\lambda^p}.
\]

This completes the proof. \hfill \Box

4.5 The part around the center

For the part around the center, we have

**Proposition 4.10.** Let $1 < p < 2$. It holds that

\[
\mathcal{A} = \sum_{i=1}^{\ell} \mu \left( \left\{ x \in \bigcup_{j=1}^{\ell} E_j \setminus F_j^{(R_i)} : |T(f X_{\rho_i}(x))| > \lambda \right\} \right) \leq \frac{\|f\|_p^p}{\lambda^p}, \quad \forall \lambda \ll \|f\|_p, \ f \in L^p.
\]

**Proof.** For each $j$, we write

\[
\mu \left( \left\{ x \in E_j \setminus F_j^{(R_i)} : |T(f X_{\rho_i}(x))| > \lambda \right\} \right) \leq \mu \left( \left\{ x \in E_j \setminus F_j^{(R_i)} : \left| T \left( f X_{\rho_i} - e^{-R_i^2 \mathcal{L}} (f X_{\rho_i}) \right) (x) \right| > \frac{1}{2} \lambda \right\} \right).
\]

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We use a Caccioppoli's inequality as follows,

\[ (4.24) \]

\[
\int_M |\nabla e^{-sL}g|^2 \psi^2_j \, d\mu \\
= \int_M \nabla e^{-sL}g \cdot \nabla (\psi^2_j e^{-sL}g) \, d\mu - 2 \int_M (\nabla e^{-sL}g \cdot \nabla \psi_j) \psi_j e^{-sL}g \, d\mu \\
\leq \int_M (\mathcal{L} e^{-sL}g) (\psi^2_j e^{-sL}g) \, d\mu + \frac{1}{2} \int_M |\nabla e^{-sL}g| \psi_j^2 \, d\mu + 2 \int_{\{x \in E_j; R \leq |x| \leq 2R_j\}} |\nabla \psi_j|^2 (e^{-sL}g)^2 \, d\mu
\]

This implies

\[
\int_{E_j} |\nabla e^{-sL}g|^2 \psi^2_j \, d\mu \leq \int_M |\nabla e^{-sL}g|^2 \psi^2_j \, d\mu
\]

\[
\leq 2 \left( \int_M |\mathcal{L} e^{-sL}g|^2 \psi_j^2 \, d\mu \right)^{1/2} \left( \int_M |\psi_j e^{-sL}g|^2 \, d\mu \right)^{1/2} + \frac{C}{R_j^2} \int_{\{x \in E_j; R \leq |x| \leq 2R_j\}} (e^{-sL}g)^2 \, d\mu.
\]

Combining this with Propositions 3.1 and 3.6 we deduce that for \(1 < p < 2\),

\[
\left\| \nabla e^{-(t+R_j^2)}(f_{X_{F_i}^{(R_j)}}) \right\|_{L^p(E_j \setminus F_i^{(R_j)})}^2
\leq \frac{R_j}{t + R_j^2} \left( \frac{R_j}{\sqrt{t + R_j^2}} \right)^{1 - \frac{2}{p}} \left( \mu(F_i^{(R_j)}) \right)^{1 - \frac{2}{p}} \frac{R_j^2}{V_j(2R_j)}
\]

\[
\leq \frac{R_j}{t + R_j^2} \left( \frac{R_j}{\sqrt{t + R_j^2}} \right)^{1 - \frac{2}{p}} \left( \mu(F_i^{(R_j)}) \right)^{1 - \frac{2}{p}} \left( \frac{R_j^2}{t + R_j^2} \right)^{2} \frac{R_j^2}{V_j(2R_j)}
\]
Repeat the argument in the case $j$. It holds then
\[ \mu \left( \frac{R_j}{\sqrt{t + R_j^2}} \right)^{1 - \frac{2}{p}}. \]

Inserting this in (4.24), we obtain that
\[ K_{j,i,2} \lesssim \frac{\|f\|_p^p}{\lambda^p} \mu(F_{i}^{(R_j)})^{1 - \frac{2}{p}} \left( \int_1^\infty \left( \frac{R_j}{\sqrt{t + R_j^2}} \right)^{1/2} \frac{1}{\sqrt{t + R_j^2} \sqrt{t}} dt \right)^2. \]

(4.25)

since $\mu(F_{i}^{(R_j)}) \sim \|f\|_p^p$ (cf. (4.3)).

**Estimation of** $K_{j,i,1}$: For $1 \leq j \neq i \leq \ell$, it follows from Corollary 2.4 that
\[ K_{j,i,1} \lesssim \frac{1}{\lambda^p} \int_{E_i \setminus E_j^{(R_j)}} \|T \left( fX_{E_{i}^{(R_j)}} - e^{-R_j^2 \mathcal{L}}(fX_{E_{i}^{(R_j)}}) \right)(x) \|^p \mu \]
\[ \lesssim \frac{1}{\lambda^p} \left[ \int_1^\infty \int_0^{R_j} \left( \frac{\|fX_{E_{i}^{(R_j)}}\|_p}{(s + t)^{3/2}} \right) ds dt + \int_1^\infty \int_0^{R_j} \left( \frac{\|fX_{E_{i}^{(R_j)}}\|_p}{(s + t)^{3/2}} \right) e^{-\frac{r_j^2}{\lambda^2}} ds dt \right]^p \]
\[ \lesssim \frac{\|f\|_p^p}{\lambda^p}. \]

(4.26)

For the opposite case $j = i$, it suffices to slightly modify the above proof. More precisely, set
\[ \overline{F}_i := \{ x \in E_i : \text{dist}(x, E_0) \leq 4R_i \}. \]

It holds then
\[ \mu(\overline{F}_i) \sim V_i(R_i) \sim \lambda^{-p} \|f\|_p^p, \quad \text{dist}(E_i \setminus \overline{F}_i, F_{i}^{(R_j)}) \sim R_i. \]

Now we write
\[ K_{i,i,1} \leq \mu(\overline{F}_i) + \mu \left( \left\{ x \in E_i \setminus \overline{F}_i : \left| T \left( fX_{E_{i}^{(R_j)}} - e^{-R_j^2 \mathcal{L}}(fX_{E_{i}^{(R_j)}}) \right)(x) \right| > 2^{-1} \lambda \right\} \right). \]

Repeat the argument in the case $j \neq i$, one obtains immediately $K_{i,i,1} \lesssim \|f\|_p^p$. This together with (4.25) and (4.26) gives the desired estimate and completes the proof. \qed
4.6 Completion of the proof

Combining the estimates from the previous three subsection, i.e., the estimates on the center (Lemma 4.2), the estimates on the part away from the center (Proposition 4.3), and the estimates near the center (Proposition 4.10), we finally conclude that for \( \lambda < \| f \|_p \),

\[
\mu \left( \left\{ x \in \bigcup_{j=1}^{\ell} \left( E_j \setminus F_j^{(R)} \right) : |Tf| > (2\ell + 1)\lambda \right\} \right) \\
\leq \mu \left( \left\{ x \in \bigcup_{j=1}^{\ell} \left( E_j \setminus F_j^{(R)} \right) : |T(f\chi_{E_j})| > \lambda \right\} \right) + \sum_{i=1}^{\ell} \mu \left( \left\{ x \in \bigcup_{j=1}^{\ell} \left( E_j \setminus F_j^{(R)} \right) : |T(f\chi_{F_j^{(R)}})| > \lambda \right\} \right) \\
+ \sum_{i=1}^{\ell} \mu \left( \left\{ x \in \bigcup_{j=1}^{\ell} \left( E_j \setminus F_j^{(R)} \right) : |T(f\chi_{E_j \setminus F_j^{(R)}})| > \lambda \right\} \right) \\
\lesssim \frac{\| f \|_p^p}{\lambda^p},
\]

for each \( 1 < p < p_0 \), where \( p_0 \) is as in (4.1). This together with (4.4) yields

\[
\mu(\{ x \in M : |Tf| > (2\ell + 1)\lambda \}) \leq C \frac{\| f \|_p^p}{\lambda^p}.
\]

By this and (4.2), we see that the operator \( T \) is weakly \( L^p \) bounded for each \( 1 < p < p_0 \). Recall that

\[
T = \frac{1}{\sqrt{\pi}} \int_1^\infty \nabla e^{-tL} \frac{dt}{\sqrt{t}}.
\]

On the other hand, by Lemma 2.9

\[
\frac{1}{\sqrt{\pi}} \int_0^1 \nabla e^{-tL} \frac{dt}{\sqrt{t}}
\]

is bounded on \( L^p(M) \) for all \( 1 < p < 2 \). We finally conclude that \( \nabla L^{-1/2} \) is weakly \( L^p \) bounded for each \( 1 < p < p_0 \), and hence the Riesz transform \( \nabla L^{-1/2} \) is bounded on \( L^p(M) \) for all \( 1 < p < 2 \) by the Marcinkiewicz interpolation theorem. This completes the proof of Theorem 1.3.

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