INVERTIBILITY OF QUASICONFORMAL OPERATORS

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Abstract. The global homeomorphism theorem for quasiconformal maps describes the following specifically higher-dimensional phenomenon: Locally invertible quasiconformal mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ is globally invertible provided $n > 2$.

We prove the following operator version of the global homeomorphism theorem. If the operator $f : H \to H$ acting in the Hilbert space $H$ is locally invertible and is an operator of bounded distortion, then it is globally invertible.

1. Introduction.

The global homeomorphism theorem for quasiconformal maps describes the following specifically higher-dimensional phenomenon (formulated in [1] for $n = 3$; proved later in [2]):

Locally invertible quasiconformal mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ is globally invertible provided $n > 2$.

The theorem has several beautiful generalizations by a number of authors; they are mentioned in the surveys [3], [4], along with several open problems. Here we will discuss one of them: the possibility to extend the theorem to nonlinear operators of bounded distortion.

The necessary definitions and facts will be given below.

2. Several concepts and facts.

Let $f : H \to H$ be a nonlinear smooth operator acting in the Hilbert (or even Banach) space $H$. Let the tangent operator $f'(x)$ have a continuous inverse at each point $x \in H$. This ensures locally invertibility of the operator $f$.

The coefficient of quasiconformality of the operator $f$ at the point $x \in H$ is the value

$$k_{f'(x)} := \|f'(x)\| \cdot \|(f'(x))^{-1}\|.$$
Geometrically it means the ratio of the semi-major axis to the semi-
minor axis of the ellipsoid that is the image of the unit ball under the
linear mapping $f'(x)$.

(A similar value occurs in the operator theory, where it was intro-
duced and used by Banach to measure the distance between different
norms in a vector space\[1\].

The quasiconformality coefficient of the mapping $f$ at the point $x$ is
usually denoted by the symbol $k_f(x)$, and its upper bound $\sup_{x \in D} k_f(x)$
over the domain $D$ of the mapping $f$ is usually denoted by $k_f$. The
value $k_f$ is called the coefficient of quasiconformality of the mapping $f$ in the domain. In our case $D$ is the whole space $H$. If $k_f$ is finite,
then the mapping $f$ is said to have bounded distortion, or that it is
quasiconformal. If $k_f = 1$, the mapping $f$ is conformal.

In connection with the theorem on global homeomorphism for qua-
siconformal mappings of the Euclidean space, mentioned in the in-
troduction, the following question naturally arises: is such a theorem
also true in the infinite-dimensional setting of nonlinear operators of
bounded distortion? An operator analogue of the global homeomor-
phism theorem would give the following principle of invertibility:

\begin{quote}
If the operator $f : H \to H$ acting in the Hilbert space $H$ is locally
invertible and it is an operator of bounded distortion, then it is globally
invertible.
\end{quote}

Hence, in this case, the equation $f(x) = y$ has a solution for any
right-hand side $y \in H$, and that solution is unique.

It may happens (as it happens in the theory of operators) that
properties of the space (Hilbert, Banach) are essential. We recall
that Nevanlinna [5] proved the Liouville theorem on the Möbius group
of conformal mappings even in the case of a Hilbert space. How-
ever, I do not know any other results about just quasiconformal (not
quasi-isometric) mappings in the infinite-dimensional case. For quasi-
isometric operators (and for non-contractive ones) acting in Banach
spaces the following result by John [6] is known: if such an operator
is locally invertible, then it is also globally invertible. This statement
holds in any dimension (either finite or infinite). A quasi-isometric
operator is, of course, a bounded distortion operator; its quasiconfor-
mality coefficient is bounded, since the changes of length elements are
bounded at any point of the domain, and this estimate is uniform. But

\[1\] Banach considered the product of norms of the linear operator and its inverse
acting between two normed spaces, followed by taking the lower bound of this
product over all such linear operators.
Invertibility of quasiconformal operators

3. Invertibility of quasiconformal operators.

Generalizing the concept of conformal (and quasiconformal) mapping of domains of Riemannian manifolds of the same dimension M.Gromov suggested that a mapping of metric spaces is to be considered conformal (respectively, quasiconformal), for example the mapping $F : \mathbb{R}^m \to \mathbb{R}^n$ ($m \geq n$) if the image of any infinitely small ball at each point of the domain is transformed into an infinitesimal ball (respectively, into an ellipsoid of uniformly bounded eccentricity) [7]. For instance, any entire holomorphic function $f : \mathbb{C}^n \to \mathbb{C}$ defines a mapping conformal in the sense of Gromov.

In connection with this extension of the concepts of conformality and quasiconformality, Gromov naturally wondered what facts of the classical theory may apply to these mappings. In particular, is it true that if the mapping $F : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is conformal and bounded, then for $n \geq 2$ it is a constant?  

Investigating this problem we came to a construction which confirmed the validity of such a Liouville-type theorem [8]. Now it seems natural, although it did not immediately become clear that the same construction could work for investigating the problem of invertibility of operators of bounded distortion. The point is that mappings that are quasiconformal in the sense of Gromov, in contrast to classical quasiconformal mappings, can act between spaces of different dimensions. This circumstance forced us to expand the classical toolkit of conformal invariants adapting it to such a situation.

As a result it turned out to be possible to prove that if the operator $f : H \to H$ acting in the Hilbert space $H$ is locally invertible and an operator of bounded distortion, then the inverse image of any subspace $\mathbb{R}^3 \subset H$ of the target space is a non-singular connected three-dimensional surface $\tilde{S}^3$ properly embedded in the ambient space, while

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2For the information of the reader we note that on the very first page of the article [7], in the footnote, the author gives the reference where a more complete text of his work in English is presented.

In particular, there the reader can find an extended interpretation of conformity and quasiconformality, as well as a formulation of the question about the Liouville theorem for such mappings. This text is now available at: [https://www.ihes.fr/~gromov/wp-content/uploads/2018/08/problems-sept2014-copy.pdf](https://www.ihes.fr/~gromov/wp-content/uploads/2018/08/problems-sept2014-copy.pdf)
the restriction $f|_{\tilde{S}^3}$ of the original map onto the surface $\tilde{S}^3$ is quasiconformal, injective, and $f(\tilde{S}^3) = \mathbb{R}^3$.

This statement already implies that the mapping $f : H \to H$ is globally invertible.

The arguments proving this auxiliary key statement and its corollary indicated above will be given in the next section.

4. Outline of the proof.

We assume the following normalization by setting $f(0) = 0$. Take the germ of the inverse mapping at the point $0 \in H$. Let the germ be defined in some ball $B(r_0)$ of radius $r_0 > 0$. We will continue this germ along the rays emanating from the point $0$.

In some directions such an extension can go along the entire ray. Along other directions there may appear singular points that prevent continuation.

Let $\gamma$ denote the part of the ray from the boundary of the ball $B(r_0)$, where the original germ of the inverse mapping exits, going to the singular point (that part is finite if continuation along the entire ray is impossible).

We note right away that if the ray along which we continue the original germ of the inverse mapping abuts against a singular point that prevents further continuation, then preimage $\tilde{\gamma}$ of the specified piece $\gamma$ of this ray turns out to be a curve of infinite length, since such a curve cannot lead to any point of the mapped space $H = \tilde{H}$. (For further convenience the space-preimage will be denoted by $\hat{H}$).

We will take in the space of the image $H$ the subspace $\mathbb{R}^2$ — a plane, and we will construct the inverse image of $\mathbb{R}^2$ lifting the rays of this subspace emanating from the point $0$.

The union $\Gamma = \{\gamma\}$ of pieces $\gamma$ of such rays plus the circle $\mathbb{R}^2 \cap B(r_0)$ form a star subdomain $S^2$ of the plane $\mathbb{R}^2$, and the collection $\tilde{\Gamma} = \{\tilde{\gamma}\}$ of their preimages plus the preimage of this circle form a surface $\tilde{S}^2$ without boundary in the space $\tilde{H}$.

If we show (this will be done below) that the family $\tilde{\Gamma} = \{\tilde{\gamma}\}$ of curves $\tilde{\gamma}$ on the surface $\tilde{S}^2$ is a family of curves of conformal module equal to zero ($M_2(\tilde{\Gamma}) = 0$), then, due to the quasiconformality of the mapping $f$, the conformal module of the family $\Gamma = \{\gamma\}$ has to be equal to zero as well ($M_2(\Gamma) = 0$). Then there should be very few singular points preventing such an extension along the rays of the space $\mathbb{R}^2$, more precisely, this set should be a set of conformal capacity equal to zero.
Now, following the proof of the global homeomorphism theorem for locally invertible quasiconformal mappings of the space $\mathbb{R}^3$, one can show that the procedure for such an extension of the germ of the inverse mapping along the rays of any subspace $\mathbb{R}^3 \subset H$ cannot have singular points at all.

So, it happens that the inverse image of any subspace $\mathbb{R}^3 \subset H$ should be a three-dimensional surface $\tilde{S}^3 \subset \tilde{H}$, and the restriction of the original mapping $f$ onto this surface is not only quasiconformal but it is injective and $f(\tilde{S}^3) = \mathbb{R}^3$.

Now, as a corollary, we conclude that the initial mapping $f : H \to H$ is indeed the globally invertible one.

It remains to check the equality $M_2(\tilde{\Gamma}) = 0$.

Let us show first that if the mapping $f : H \to H$ is quasiconformal, then the area of the geodesic circle with center 0 of radius $r$ on the surface $\tilde{S}^2$ increases not faster than $O(r^2)$ as $r \to \infty$.

Indeed, since the increment of the area of the circle can be written in the form $L(r) dr$, where $L(r)$ is the length of the circle of radius $r$, then it suffices to check that the length of the circle increases no faster than $O(r)$ as $r \to \infty$.

By assuming the opposite we see that the following situation is inevitable. Steep waves should appear on this circle, steep in the sense that their height is much greater than their length. And this would contradict the quasiconformality of the mapping $f$.

This is useful and important observation, so we will reformulate it in a general way omitting details which are not essential.

Take the straight (real) line $\gamma$ in the Hilbert space $H$, and let $I = [-\varepsilon, \varepsilon]$ be a small segment on $\gamma$. Let $\tilde{H}$ be another copy of the same Hilbert space $H$. Take the curve $\tilde{\gamma}$ which differs from the straight line $\gamma$ only in the way that the segment $I$ is replaced by some curve (splash) $\tilde{I}$. Let $f : \tilde{H} \to H$ be a quasiconformal mapping such that the curve $\tilde{\gamma}$ goes over into the straight line $\gamma$ and the surge (wave) $\tilde{I}$ goes into the segment $I$. The statement is that if the diameter $\tilde{I}$ is much larger than the diameter $I$ (i.e. the height or the amplitude of the wave is much larger than its length), then the coefficient of quasiconformality of the mapping $f$ must be large. Moreover, the greater the specified ratio is, the greater will be the coefficient of quasiconformality.

$^3$Having indicated the key idea we do not stop here on the description of the entire technological chain which leads to the conclusion that in the space $\mathbb{R}^3$ there will be no singular points at all preventing the procedure of continuation of the germ of the inverse mapping. The details can be found in [2],
Let us clarify the assertion by an illustrative example.

Take two copies of the standard Euclidean plane \( \mathbb{R}^2 \) and \( \tilde{\mathbb{R}}^2 \) with Cartesian coordinates \((x, y)\) and \((\tilde{x}, \tilde{y})\) respectively. In the \( \mathbb{R}^2 \) plane mark the straight line \( \gamma \) — the \( x \) axis, and in the \( \tilde{\mathbb{R}}^2 \) plane instead of the \( \tilde{x} \) axis consider the curve \( \tilde{\gamma} \) that coincides with the \( \tilde{x} \) axis outside a small neighborhood of the origin and a small segment of this axis containing the origin is replaced by a deep loop (pit, wave) of a completely arbitrary shape. Suppose we have a quasiconformal mapping \( f : \tilde{\mathbb{R}}^2 \to \mathbb{R}^2 \) such that the curve \( \tilde{\gamma} \) transforms to the straight line \( \gamma \). We claim that the coefficient of quasiconformality of the mapping \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) must be very large, and it will be larger the larger will be depth of the well compared to its entrance (the greater will be the amplitude of the wave compared to its length).

Note that each of the two regions into which the curve \( \tilde{\gamma} \) divides the plane \( \tilde{\mathbb{R}}^2 \), admits even a conformal mapping onto its upper or lower half-plane, onto which the axis \( \gamma \) divides the plane \( \mathbb{R}^2 \). But we are talking on a mapping transforming the domain of the plane \( \tilde{\mathbb{R}}^2 \), containing the loop, into a neighborhood of the origin of the plane \( \mathbb{R}^2 \), and the map straightens a relatively deep well (or a wave of the amplitude large in comparison with its length).

This statement can be proved by comparing the conformal moduli of suitable families of curves. For example, in the \( \mathbb{R}^2 \) plane take the above-mentioned small neighborhood of the origin on the \( x \) axis (i.e., on the curve \( \gamma \)), take on the same axis \( x \) the exterior of the unit neighborhood of the origin, and consider the family of curves connecting these two sets in the \( \mathbb{R}^2 \) plane. The conformal modulus of such a family is the same as the conformal capacity of the corresponding condenser. The conformal modulus of this family of curves is small if the considered neighborhood of the origin is small (since one of the plates of the condenser is relatively small). However, the conformal modulus of the inverse image of this family of curves cannot be small if the wave amplitude on the \( \tilde{\gamma} \) curve is large compared to the wavelength.

The same considerations are applicable in the case when a similar type of curve \( \tilde{\gamma} \) containing a steep wave straightens under a quasiconformal mapping \( f : \mathbb{R}^n \to \mathbb{R}^n \) of a Euclidean space of any finite dimension. It is also clear that the above reasoning is essentially local. It will also remain valid if, instead of the mapping \( f : \mathbb{R}^n \to \mathbb{R}^n \), we consider the mapping \( f : \tilde{S}^n \to \mathbb{R}^n \) of the Riemannian manifold \( \tilde{S}^n \) obtained by the quasiconformal deformation of \( \tilde{\mathbb{R}}^n \).

Now let us return to the family of curves \( \tilde{\Gamma} \) on the surface \( \tilde{S}^2 \) and show that its conformal modulus is equal to zero \( (M_2(\tilde{\Gamma}) = 0) \).
Indeed, by construction curves of the family $\tilde{\Gamma}$ have infinite length, going to infinity along the surface $\tilde{S}^2$ (i.e. leaving any compact set of $\tilde{S}^2$). Thus, the function $\rho = \alpha (\ln r)^{-1}$ (where $r$ is the geodesic distance from a point of the surface $\tilde{S}^2$ to the starting point $\tilde{0} = 0$, and $\alpha > 0$) is an admissible function for this family of curves. Indeed, $\int_{\tilde{\gamma}} \rho = \infty$ for any curve $\tilde{\gamma} \in \tilde{\Gamma}$. On the other hand, the integral of the square of this function over that part of the surface that lies outside the neighborhood of the point $\tilde{0}$, is finite, and tends to zero as $\alpha \to 0$. Hence, $M_2(\tilde{\Gamma}) = 0.$

(Here we used the fact that the area of the geodesic circle of radius $r$ on the surface $\tilde{S}^2$ increases no faster than $O(r^2)$ as $r \to \infty$.)

Finally, let us make the last explanatory remark. If the procedure for constructing the inverse image of the subspace $\mathbb{R}^3 \subset H$ is done for each such subspace, then, on the one hand, we exhaust the entire space $H$ of the image. On the other hand, the corresponding preimages $\tilde{S}^3$ all together form a subdomain $\tilde{D} \subset \tilde{H}$ of the space-preimage $\tilde{H}$, and the restriction $f|_{\tilde{D}}$ of the mapping $f$ onto the domain $\tilde{D}$ is injective by construction and $f(\tilde{D}) = H$. This means that in fact $\tilde{D} = \tilde{H}$. Otherwise, one could take a point $p \in \tilde{H} \setminus \tilde{D}$ and connect it to the point $\tilde{0} = 0$ by a segment. The part of this segment adjacent to the point $\tilde{0} = 0$ certainly lies in the domain $\tilde{D}$. So, in this segment there is such a point, that in any its neighborhood there are points of the domain $\tilde{D}$ and points that do not belong to the region $\tilde{D}$. But this is not possible if the mapping $f : \tilde{H} \to H$ is locally invertible, $f(\tilde{D}) = H$ and $f|_{\tilde{D}}$ is injective.

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