Network Shrinkage Estimation

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Abstract

Networks are a natural representation of complex systems across the sciences, and higher-order dependencies are central to the understanding and modeling of these systems. However, in many practical applications such as online social networks, networks are massive, dynamic, and naturally streaming, where pairwise interactions become available one at a time in some arbitrary order. The massive size and streaming nature of these networks allow only partial observation, since it is infeasible to analyze the entire network. Under such scenarios, it is challenging to study the higher-order structural and connectivity patterns of streaming networks. In this work, we consider the fundamental problem of estimating the higher-order dependencies using adaptive sampling. We propose a novel adaptive, single-pass sampling framework and unbiased estimators for higher-order network analysis of large streaming networks. Our algorithms exploit adaptive techniques to identify edges that are highly informative for efficiently estimating the higher-order structure of streaming networks from small sample data. We also introduce a novel James-Stein-type shrinkage estimator to minimize the estimation error. Our approach is fully analytic with theoretical guarantees, computationally efficient, and can be incrementally updated in a streaming setting. Numerical experiments on large networks show that our approach is superior to baseline methods.

1 Introduction

Network analysis has been central to the understanding and modeling of large complex systems in various domains, e.g., social, biological, neural, and technological systems [8, 38]. These complex systems are usually represented as a network (graph) where the vertices represent the components of the system, and the edges represent their direct (observed) interactions over time. The success of network analysis throughout the sciences rests on the ability to describe the complex structure and dynamics of arbitrary systems using only observed pairwise interaction data among the components of the system. Many network systems indeed exhibit rich structural and connectivity patterns that can be captured at the level of pairwise links (edges) or individual vertices. However, higher-order dependencies that capture complex forms of interactions remain largely unknown, since they are beyond the reach of methods that focus primarily on pairwise links. Recently, there has been a surge of studies on higher-order network analysis [11, 55, 6, 57, 45, 20]. In general, these methods focus on generalizing the analysis of network data to more complex forms of relations such as multi-node relationships (e.g., motif patterns) and higher-order network paths that depend on more history. Higher-order structural and connectivity patterns were shown to change node rankings [10, 43, 47, 59, 7], reshape the community structure [55, 11, 58], unveil the hub structure [6, 44], learn more accurate embeddings [11, 42], and better network models [17].

Many networks are massive, dynamic, and naturally streaming [34, 5, 46], where pairwise interactions (i.e., edges) become available one at a time in some arbitrary order (e.g., online social networks, Emails, Twitter data, recommendation engines). The massive size and streaming nature of these
networks allow only partial observation, since it is infeasible to analyze the entire network. Under such scenarios, it is challenging to study the higher-order structural and connectivity patterns of streaming network data, and the majority of existing methods focus on static networks that can fit entirely in memory. Our work is motivated by large-scale streaming network data that are generated by measurement processes, and we study higher-order network analysis in these settings.

Randomization and sampling techniques are fundamental in the context of graph and matrix approximations in both static and streaming settings [34, 30, 27, 3], where the general problem is setup as follows: given a graph $G$ and a budget $m$, find a sparser graph $G_s$ such that the (expected) number of edges (non-zero entries) is at most $m$ and $G_s$ is a good proxy for $G$. In the data streaming model, the input graph $G$ is partially observed as the edges become available to the algorithm one at a time in some arbitrary order. The streaming model is fundamental to applications of online social networks, social media, and recommendation systems where network data become available one at a time (e.g., friendship links, emails, Twitter feeds, user-item preferences, purchase transactions, etc). Moreover, the streaming model is also crucial where network data is streaming from storage and random accesses of edges are too expensive.

We propose a novel topologically adaptive, single-pass priority sampling framework and unbiased estimators for higher-order network analysis of large streaming networks, where edges become available one at a time in any arbitrary order. We establish adaptive techniques for preferential selection of edges within target classes of subgraphs and motifs (Alg 1). We derive unbiased estimators for local subgraph counts (Theorem 1), and show how to efficiently compute both these (Theorem 2) and their unbiased variance estimators during stream processing (Alg 2). We also propose a novel James-Stein-type shrinkage estimator [24, 21] to reduce the variance, and provide an algorithm for their streaming computation (Alg 2). Our approach is fully analytic, computationally efficient, and can be incrementally updated as the edges become available one at a time. The proposed methods are generally applicable to a wide variety of networks, including directed, undirected, weighted, and heterogeneous networks with multiple link types.

Prior Work. Higher-order Network Analysis: there has been an increasing interest in higher-order network analysis [35, 11, 55, 58, 6, 57, 45, 20, 42, 17], in particular to generalize pairwise links to many-body relationships with arbitrary node sets and motifs. The majority of these methods focus on small static networks that fit in memory. Graph Approximations: randomization in the context of graph approximations is well-studied topic [49, 14, 22, 31, 52]. See [34] for a survey. Much work was devoted for triangle count approximation and other motifs for static graphs [12, 54, 48, 29, 26, 53, 18, 44, 39], and for streaming graphs [9, 50, 3, 25, 33, 2, 4]. In the streaming setting, most work focused on estimating point statistics using fixed probabilities, e.g., the global triangle or motif count using reservoir-based approaches [50]. Here, we focus instead on estimating motif-weighted graph from a stream of unweighted edges, and We propose a general methodology for adaptive priority sampling. We also adapt reservoir sampling to our setting in Section 4.

Stream Sampling: our work is inspired by multiple classes of prior work, e.g., single-pass reservoir sampling [28, 37, 50], priority, order and threshold sampling [15, 40, 13, 16], and probability proportional to size sampling (IPPS) [51]. Most of these methods were designed for sampling IID data streams (e.g., application in streams of IP network packets, database transactions, etc). Here, we focus instead on the rich streaming network (graph) data that exhibit structure, higher-order dependencies, and attributes (e.g., online social networks, web graph, Email communications, etc). We propose adaptive priority sampling that generalizes many of the above methods to the streaming network settings.

2 Theoretical Methodology for Adaptive Priority Sampling

Notation and Problem Definition. Let $G = (V, K)$ be an undirected graph and $H$ be a class of subgraphs of $G$ that are isomorphic to a motif pattern $M$, e.g., triangles, cliques of a given size, feedforward loop. The $H$-weighted graph of $G$ is the weighted graph $(V, K, N)$ where the edge weights $N = \{n_k : k \in K\}$ comprise the local subgraph counts $n_k = \{|h \in H : h \ni k, h \cong M\}$ i.e., the number of subgraphs in $H$ that contain $k$ as an edge. (For brevity we will identify $h \in H$ with its edge set). We refer to it as motif-weighted graph, and we denote $A$ as its motif adjacency matrix [11]. Suppose the edges of $G$ are labelled in an arbitrary order and presented in a stream. Let $G_t = (V_t, K_t)$ denote the subgraph of $G$ formed by the first $t$ edges in this order, $H_t = \{|h \in H : h \subset K_t\}$ be the subgraphs in $H$ all of whose edges have arrived by $t$, and $(V_t, K_t, N_t)$ the corresponding $H_t$-weighted graph of $G_t$ with weights $N_t = \{n_{k,t} : k \in K_t\}$. The problem we address in this paper...
is to maintain a reservoir sample $\hat{K}$ of $m$ edges from the unweighted edge stream from which we can compute an unbiased estimate of the motif-weighted graph $(V_t, \hat{K}_t, N_t)$ at any time $t \in \|K\|$. and its adjacency matrix $A$. We propose a variable weight adaptive priority sampling for streaming non-IID network data, that enables preferential retention of edges with higher-order dependencies, i.e., those edges that play a more important role in formation of motifs and subgraphs of interest $H$.

**Topologically Adaptive Priority Sampling (APS).** We describe weight adaptive priority sampling of the edge stream. Consider a generic reservoir sample $\hat{K}$ taken progressively from the edge stream labelled $K = \|K\| = \{1, 2, \ldots, \|K\|\}$. (We will also use $k_t$ to denote the edge arriving at $t$). The first $m$ edges are admitted: $\hat{K}_t = \{t\}$ for $t \leq m$. Each subsequent edge $t$ is provisionally included with the current sample to form $\hat{K}_t = \hat{K}_{t-1} \cup \{t\}$, from which an edge is discarded to produce the sample $\hat{K}_t$.

Our method preferentially retains edges in the reservoir according to their topological relevance for non-IID network data, that enables preferential retention of edges with higher-order dependencies, $A$. We can compute an unbiased estimate of the motif-weighted graph whose edges are in $\hat{K}_t$.

**Unbiased Subgraph Estimation.** Observe that the distribution of $\{u_i : i \in J\}$, conditional on $J_t \subset K_t$ and $Z_{J,t}$, are independent, with each $u_i$ uniformly distributed on $(0,1]$. The priority of an edge $i \in \hat{K}_t$ is defined as $r_{i,t} = w_{i,t}/u_i$, and the edge of minimum priority $z_t = \min_{j \in \hat{K}_t} r_{j,t}$ is discarded from $\hat{K}_t$.

**Local Count Estimation from the Last Arriving Edge.** Let $\tau_j = \max J$ denote the time of the last arriving edge of $J \subset K$, set $J^{(0)} = J \setminus \{\tau_j\}$, and define $\hat{S}_{J,t} = \hat{S}_{J^{(0)},\tau_j-1}$. (We also denote $p_{i,t}$ for $p_{i,t,0}$). Set $t_J = \min J$ and define $S_{J,t} = I(J_t \in \hat{K}_t)/P_{J,t}$ (note the difference in denominator from $\hat{S}_{J,t}$) and the set of variables $Z_{J,t} = \{z_{s,t} : t \leq s \leq t\}$.

**Theorem 1.** (i) The distributions of $\{u_i : i \in J\}$, conditional on $J_t \subset \hat{K}_t$ and $Z_{J,t}$, are independent, with each $u_i$ uniformly distributed on $(0, p_{i,t}|J|]$. (ii) $E[I(J_t \subset K_t)|Z_{J,t}, J_{t-1} \subset \hat{K}_{t-1}] = P_{J,t}/P_{J,t-1}$ (iii) $E[\hat{S}_{J,t}|Z_{J,t}, J_{t-1} \subset \hat{K}_{t-1}] = \hat{S}_{J,t-1}$ and hence $E[\hat{S}_{J,t}] = 1$ for $t > t_J$.

**Corollary 1.** $E[\hat{S}_{J}] = 1$ and hence $\hat{n}_{k,t} = \sum_{J \in H_{k,t}} \hat{S}_{J}$ is an unbiased estimator of the local subgraph count $n_{k,t}$ for all $k \in \hat{K}_t$.

In the streaming context, then for each arriving edge $t$, for each $J \in H$ that is completed by $t$, for each $k \in J$, we increment the estimate of $P_{k,t}$ by the inverse probability $1/P_{J^{(0)},\tau_j-1}$. This allows us to "bank" the estimate for $n_k$ at time $t$, rather than risk loss of some edge $J$ during subsequent sampling. Estimation variance will be discussed in Section 5.

**Nondecreasing Topological Weights.** For fixed edge weights $\hat{K}_t$ established that the variance of the increment the number of subgraphs in class $H$ created by an arriving edge is minimized by edge weights proportional to that increment. Therefore we give each edge $k$ a fixed initial weight, to which we add at each time step $t \geq k$ the number of new subgraphs in $H$ whose edges are in $\hat{K}_t$.
and that contain (and are hence completed by) edge \( t \). The use of nondecreasing weights brings an unrelated computational benefit. Computing the probabilities \( p_{i,t} \) according to \([1]\) requires an update for each edge at each time step. We now show this cost can be reduced when \( w_{i,t} \) is nondecreasing in \( t \). Let \( d_t \leq t \) denote the edge that is deleted at step \( t > m \), i.e., \( \{d_t\} = \hat{K}_t \setminus \hat{K}_i \). Define \( z^*_t \) iteratively by \( z^*_{m-1} = 0 \) and \( z^*_t = \max\{z^*_{t-1}, z_t\} \) for \( t > m \). Define \( p^*_t = \min\{1, w_{i,t}/z^*_t\} \) and \( p^*_{t+1} = \min\{p^*_{t,t}, w_{i,t+1}/z^*_{t+1}\} \), for \( t \geq i \), i.e., similar to \([1]\) but with \( z_t \) replaced by \( z^*_t \).

**Theorem 2.** When \( w_{i,t} \) is non-decreasing in \( t \) for all \( i \) then (i) \( d_t \neq t \) implies \( z^*_t = z_t \); and (ii) \( p^*_{t,i} = p_{i,t} \) for all \( t \geq i \).

**Algorithm Specification.** We take advantage of Theorem \([2]\) to reduce the number of updates of the \( p^*_{t,i} \). Since \( w_{i,t} \) is nondecreasing and \( z^*_t \) nondecreasing, \( w_{i,t}/z^*_t \) can only increase when \( w_{i,t} \) increases. In intervals of constant \( w_{i,t} \), \( w_{i,t}/z^*_t \) is nonincreasing, and so provided we update \( p^*_{i,t} \) at times when \( w_{i,t} \) increases, all other updates of \( p^*_{i,t} \) can be deferred until needed for estimation; see line \([9]\) of Algorithm \([1]\) that specifies Topologically Adaptive Priority Sampling. \( \Delta_k \) is the set of subgraphs from \( H \) containing the arrival \( k \) whose remaining edges lie within the sample \( \hat{K} \).

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**Algorithm 1** Topologically Adaptive Priority Sampling (APS)

**Require:** Edge stream, sample size \( m \), Motif pattern \( M \)

**Ensure:** \( \hat{K} \leftarrow 0, z^* \leftarrow 0 \)

1: for a new edge \( k \) do
2: \( u(k) \sim \text{Uni}(0, 1) \)
3: \( w(k) \leftarrow W(k, \hat{K}) \)
4: \( r(k) \leftarrow w(k)/u(k) \) \hspace{1cm}\triangleright \text{Priority variable for edge } k
5: \( p(k) \leftarrow 1 \)
6: \( \hat{K} \leftarrow \hat{K} \cup \{k\} \) \hspace{1cm}\triangleright \text{Add } k \text{ to the sample}
7: \( \Delta \leftarrow \{h \subset \hat{K} : h \ni k, h \cong M\} \) \hspace{1cm}\triangleright \text{Set of subgraphs containing } k \text{ and isomorphic to } M
8: for \( h \in \Delta \) do
9: \( p(j) \leftarrow \min\{p(j), w(j)/z^*\} \) if \( z^* > 0 \), \( \forall j \in h : j \neq k \)
10: \( w(j) \leftarrow w(j) + 1, \forall j \in h : j \neq k \)
11: \( p(h) \leftarrow \prod_{j \in h} p(j) \)
12: \( n(j) \leftarrow n(j) + 1/p(h), \forall j \in h \) \hspace{1cm}\triangleright Update estimated count of \( M \) incident to edge \( j \)
13: if \(|K| > m\) then
14: \( k^* \leftarrow \arg \min_{j \in K} r(j) \)
15: Update sample threshold \( z^* \leftarrow \max\{z^*, r(k^*)\} \)
16: Remove \( k^* \) from \( \hat{K} \)

3 Estimation Error Reduction using James-Stein Shrinkage

Unbiased estimators of local subgraph counts are subject to high relative variance when the motif counts are small, because in this case the individual count estimates, scaled by the inverse probabilities, are less smoother by aggregation. More generally, James and Stein originated the observation that unbiased estimators do not necessarily minimize mean square error \([24]\). In their study, unbiased estimates of high dimensional Gaussian random variable are adjusted through scaling-based regularization and linear combination with dimensional averages. In this paper we examine shrinkage for the \( \hat{n}_k \) by convex combination with a simple estimator that represent the unnormalized count obtained from the edge sampling weight \( w_k \).

Define \( \eta_w = \lambda \n + \lambda w \) where \( w \) represents the weight \( w_k \) of any edge \( k \), i.e., its unnormalized subgraph count as maintained in Algorithm \([1]\). The loss \( L_w(\lambda) \) is the mean square error:

\[
\text{Var}(\eta_w) + (E[\eta_w] - n)^2 = \lambda^2 \text{Var}(\n) + \lambda^2 \text{Var}(w) + 2\lambda \lambda \text{Cov}(\n, w) + \lambda^2 \text{Var}(w) + \lambda^2 \text{Var}(n - w)^2
\]

A straightforward computation shows that \( L_w(\lambda) \) is minimized when \( 1 - \lambda = \text{Cov}(\n - w, \n)/E[(\n - w)^2] \). A plug-in estimator \( \lambda_w \) for \( \lambda \) is obtained by substituting \( (\n - w)^2 \) in the denominator, and an estimate for \( \text{Cov}(\n - w, \n) \) whose computation we now describe.
3.1 Covariance Relations Amongst Subgraph Estimators

Define the unnormalized subgraph estimator \( \hat{I}_{J,t} = I(J \subset \hat{K}_t) \), and analogous to to \( S'_J \), let \( I'_J = I_{J(0),t_j-1} \). This is 1 if all the edges of \( J(0) \) are present in \( \hat{K}_{t_j-1} \), i.e., immediately prior to the arrival of the last edge. When this occurs, each edge in \( J(0) \) has its weight increments; see line 10 of Algorithm 1. Thus, the weight associated with edge \( k \) at time \( t \) is \( w_{k,t} = \sum_{J \in \hat{K}_{t}} I'_J \). Estimates of covariances amongst the \( \hat{n}_{k,t} \) and \( w_{k,t} \) need to compute the shrinkage coefficient \( \lambda_w \) will then follow by linearity from estimates of covariances of the \( \hat{S}_J \) and \( I'_J \).

**Theorem 3.** \( \text{Cov}(\hat{S}_{J_1,t_1}, \hat{S}_{J_2,t_2}) \) has unbiased estimator
\[
C_{J_1,t_1;J_2,t_2} = \hat{S}_{J_1 \setminus J_2,t_1} \hat{S}_{J_1 \cap J_2,t_1} \hat{S}_{J_2 \setminus J_1,t_2} (\hat{S}_{J_1 \cap J_2,t_1} \hat{S}_{J_2 \setminus J_1,t_2} - 1) \hat{S}_{J_2 \setminus J_1,t_2}
\]

**Theorem 4.**
(i) When \( t_1 \geq t_2 \), \( \text{Cov}(\hat{S}_{J_1,t_1}, I_{J_2,t_2}) \) has unbiased estimator
\[
D_{J_1,t_1;J_2,t_2} = \hat{S}_{J_1 \setminus J_2,t_1} I_{J_2,t_2} - \hat{S}_{J_1 \setminus J_2,t_1} \hat{S}_{J_2 \setminus J_1,t_2} \hat{S}_{J_2 \setminus J_1,t_2} I_{J_2,t_2}
\]
(ii) \( D_{J_1,t_1;J_2,t_2} > 0 \) iff \( \hat{S}_{J_1,t_1} > 0 \) and \( I_{J_2,t_2} > 0 \). Hence \( D_{J_1,t_1;J_2,t_2} \) can be computed from samples that have been taken.
(iii) For the special case \( J_1 = J_2 = J \) and \( t_1 = t_2 = t \) then \( D_{J_1,t_1;J_2,t_2} = \hat{S}_{J_1 \setminus J_2,t_1} I_{J_2,t_2} = I_t(p_{j_1}^1 - 1) \).

Central to both the proof Theorem 4 and the computation of covariance estimates is the following:

**Lemma 2.** For \( J_1 \cap J_2 = \emptyset \) and \( t_1 \geq t_2 \), then \( \mathbb{E}[\hat{S}_{J_1,t_1} I_{J_2,t_2}] = \mathbb{E}[I_{J_2,t_2}] \) and hence \( \text{Cov}(\hat{S}_{J_1,t_1}, I_{J_2,t_2}) = 0 \).

3.2 Computation of Shrinkage Coefficients for Local Triangle Counts

We now outline how to compute the estimator \( \hat{\lambda}_w \) for each sampled edge progressively during the stream in the case that \( M \) is the triangle motif. Specifically we produced an unbiased estimator \( v_t(k) \) for \( \text{Var}(\hat{n}_{k,t}) \) and an estimator \( c_t(k) \) for \( \text{Cov}(\hat{n}_{k,t}, w_{k,t}) \) from which the shrinkage estimator \( \hat{n}_{w,t} \) can be computed as \( \hat{\lambda}_k \hat{n}_{k,t} + (1 - \hat{\lambda}_k)w_{k,t} \) where \( \hat{\lambda}_k = 1 - c_t(k)/(\hat{n}_{k,t} - w_{k,t})^2 \). This is presented in Algorithm 2 as a supplement to Algorithm 1 immediately following its line 12.

**Algorithm 2** Supplemental Iterative Variance Computation Following Line 12 of Algorithm 1

**Require:** New edge \( k \), current sample set \( \hat{K} \supset k \), triangle \( h = (j_1, j_2, k) \subset \hat{K} \).

1: for \( j \in h \) do
2: \( v(j) \leftarrow v(j) + ((p(j_1)p(j_2))^{-1} - 1)/(p(j_1)p(j_2)) \)
3: \( c(j) \leftarrow c(j) + (p(j_1)p(j_2))^{-1} - 1 \)
4: for \( j \in h : j \neq k \) do
5: \( v(j) \leftarrow v(j) + 2U(j)/(p(j_1)p(j_2)) \)
6: \( c(j) \leftarrow c(j) + U(j) + D(j)/(p(j_1)p(j_2)) \)
7: \( U(j) \leftarrow U(j) + (p(j)^{-1} - 1)/p(j) \)
8: \( D(j) \leftarrow D(j) + 1 - p(j) \) \( j' \in h, j' \neq j, k \)

**Estimating \( \text{Var}(\hat{n}_{k,t}) \).** Denote by \( \Delta_{j,t} = H_{j,t} \setminus H_{j,t-1} \) the set of triangles in \( K_t \) containing \( j \) that are created by the arrival of \( k_t \). Then \( \hat{n}_{j,t} = \hat{n}_{j,t-1} + \sum_{j \in \Delta_{j,t}} \hat{S}_{J_j} \). Let \( \Delta_t \) denote the (possibly empty) set of triangles in \( \hat{K}_t \) created by the arrival at \( t \). For convenience we denote \( p_{j_1,t_1} \) by \( q_{j_1,t} \). Any distinct \( J_1, J_2 \in \Delta_{k,t} \) have only the edge \( k_t \) in common, and hence \( J_{1}^{(0)} \cap J_{2}^{(0)} = \emptyset \) and so \( \text{Cov}(\hat{S}_{J_1,t_1}, \hat{S}_{J_2,t_2}) = 0 \). Thus \( \text{Var}(\hat{n}_{j,t}) = \text{Var}(\hat{n}_{j,t-1}) + \sum_{J \in \Delta_{j,t}} (\text{Var}(\hat{S}_{J_j,t}) + 2 \text{Cov}(\hat{n}_{j,t-1}, \hat{S}_{J_j,t})) \) and the increment to \( \text{Var}(\hat{n}_{j,t}) \) for each \( J \in \Delta_{j,t} \) is
\[
\text{Var}(\hat{S}_{J_j,t}) + 2 \text{Cov}(\hat{n}_{j,t-1}, \hat{S}_{J_j,t})
\]

For \( j = k_t \) only the first term in (5) can be non-zero since \( \hat{n}_{k_t,t-1} = 0 \) because \( k_t \) arrives at \( t \). Thus each \( J = \{j_1, j_2, k\} \in \Delta_t \) results in an increment \((1/(q_{j_1,t}q_{j_2,t}) - 1)/(q_{j_1,t}q_{j_2,t}) \) to the unbiased
where the sum \( \sum \) The first two terms on the RHS of (7) are 0
\[
\text{Cov}(\hat{n}_{j,t-1}) = \sum_{s < t} \sum_{L \in \Delta_{j,s}} \text{Cov}(\hat{S}_{L,s}, \hat{S}_{j,t}).
\] This has unbiased estimator

\[
U_{j,t} = (q_{j,t} q_{j',t}), \quad \text{where } U_{j,t} = \sum_{s < t} (q_{j,s}^{-1} - 1) / q_{t,s}
\]  
where the sum \( \sum' \) is over times \( t_j < s < t \) for which \( I'_{L,s} > 0 \) if there exists \( L = (j, \ell, k_s) \in \Delta_{j,s}; \) see lines 5 and 7 of Alg. 2.

**Estimating** \( \text{Cov}(\hat{n}_k, w_k) \). Any distinct triangles \( J_1, J_2 \in \hat{\Delta} \) have only edge \( k_t \) in common and so \( \text{Cov}(\hat{S}_{J_1,t}, \hat{S}_{J_2,t}) = 0 \) by Lemma 2. The increment of \( \text{Cov}(\hat{n}_j, w_j) \) due to the arrival of \( k \) is

\[
\sum_{J \in \Delta_{j,t}} \text{Cov}(\hat{n}_j, I_{J,t}) + \text{Cov}(w_j, \hat{S}_{j,t}) + \text{Cov}(\hat{S}_{J,t}, I_{J,t})
\]  
In last term on the RHS of (7), for each \( J = \{j_1, j_2, k_t\} \in \Delta_t \) with \( I'_{J,t} > 0 \), each \( j \in J \) receives an increment gives an increment \( (q_{j,t} q_{j',t})^{-1} - 1 \) to the estimator of \( \text{Cov}(\hat{n}_j, w_j); \) see line 3 of Alg. 2.

The first two terms on the RHS of (7) are 0 for \( j = k_t \) since \( \hat{n}_{j,t-1} = w_{k,t-1} = 0 \). For \( j \neq k_t \) there is at most one \( J = \{j_1, j_2, k_t\} \in \Delta_t \) and hence by Theorem 4.1) the second term on the RHS of (7) is \( \text{Cov}(\hat{S}_{J_1,t}, \hat{n}_{J_2,t-1}) = \sum_{s < t} \sum_{L_j \in \Delta_{j,s}} \text{Cov}(\hat{S}_{L_j,s}, I_{J,t}) \). This has unbiased estimator

\[
D_{j,t} = (q_{j,t} q_{j',t}), \quad \text{where } D_{j,t} = \sum_{s < t} (1 - q_{j,s})
\]  
by Theorem 4, where the sum \( \sum' \) is over times \( t_j < s < t \) for which \( I'_{L,s} > 0 \) if there exists \( L = (j, \ell, k_s) \in \Delta_{j,s}. \) These correspond to lines 6 and 8 of Alg. 2. The first term on the RHS of (7) is \( \text{Cov}(\hat{n}_{j,t-1}, I_{J,t}) = \sum_{s < t} \sum_{L_j \in \Delta_{j,s}} \text{Cov}(\hat{S}_{L_j,s}, I_{J,t}) \) Although Theorem 4 (i) does not provide an expression for \( \text{Cov}(\hat{S}_{L_j,s}, I_{J,t}) \) when \( s < t \), we adopt (4) as an estimate, resulting in an increment of \( U_{j,t} \) to \( \text{Cov}(\hat{n}_j, w_j) \). This increment is also reflected in line 6 of Alg. 2.

**4 Experiments & Discussion**

| Table 1: Summary of graph Statistics |
|--------------------------------------|
| **data** | | | | |
| SOC-Flickr | 514K | 3.2M | 58.8M | 2236 |
| SOC-LiveJournal | 4.03M | 27.9M | 83.6M | 586 |
| SOC-Youtube | 1.13M | 2.98M | 3.05M | 4034 |
| Wiki-Talk | 2.4M | 4.7M | 9.2M | 1631 |
| Web-BerkStan-Dir | 685K | 6.7M | 64.7M | 45057 |
| Cit-Patents | 3.8M | 16.5M | 7.5M | 591 |
| Soc-Orkut-Dir | 3.07M | 117.2M | 627.6M | 9145 |

**Setup.** We experimented with graphs from different domains and with different characteristics (see [41] for downloads). See Table [I] for a summary of the main characteristics, and [41] for downloads.

- **soc-flickr**: Crawl of the Flickr photo-sharing social network from May 2006. Nodes are users and edges represent that a user added another user to their list of contacts [19].
- **soc-livejournal**: LiveJournal is an online social community publishing platform. Nodes are users and edges are user-to-user links [36].
- **soc-youtube**: Youtube social network. Nodes are users and edges are user-to-user friendship links [36].
- **wiki-Talk**: Wikipedia network of user discussions from the inception of Wikipedia till January 2008. Nodes are Wikipedia users and edges are user-to-user edits of talk pages [31].
- **web-BerkStan-dir**: Web network where nodes represent webpages from Berkeley and Stanford and edges represent hyperlinks among them [32].
- **cit-Patents**: The citation graph of US Patents includes all citations made by patents granted between 1975 and 1999 [30].
- **soc-orkut-dir**: Orkut online social network, where nodes represent users and edges represent user-to-user friendship links [36].

For all graph datasets, we consider an undirected, unweighted, simplified graph without self-loops. Edges streams are obtained by randomly permuting the set of edges in each graph, and the same edge order is used for all compared methods. We repeat the experiment ten different times with sample fractions \{0.10, 0.20, 0.40, 0.50\}. All experiments were performed using a server with two Intel Xeon E5-2687W 3.1GHz CPUs, 256GB of memory. The experiments are executed independently for each sample fraction. Summary of steps:

1. For each sample fraction, we use Alg 1 to collect a sample \( \hat{K} \subset K \), from edge stream \( K \).
2. We use triangles in the experiments as an example of the motif pattern \( M \). The approach itself is general and applicable to any motif patterns.
3. Using the collected sample and variance estimates, we compute the unbiased estimators and James-Stein shrinkage estimators of triangle counts for sampled edges only.
4. Given a sample \( \hat{K} \subset K \), we compute the mean square error (MSE), and the relative spectral norm \( \|A - \hat{A}\|_2/\|A\|_2 \), where \( A \) is the exact triangle-weighted adjacency matrix of the input graph, \( \hat{A} \) is the average estimated triangle-weighted adjacency matrix, and \( \|A\|_2 \) is the spectral norm of \( A \) [1].
5. We use uniform sampling (reservoir sampling [56]) as the baseline with the same setup above and the same estimators (i.e., Horvitz-Thompson, and James-Stein shrinkage estimator).

**Baseline Comparisons.** We collect a sample of edges \( \hat{K} \subset K \) from the edge stream \( K \) in a single pass, which we use to construct the triangle-weighted adjacency matrix \( \hat{A} \) that serves as an estimate for the true triangle-weighted adjacency matrix \( A \). We compute the shrinkage estimator as in Sec. 3. We report the MSE at sample fraction \( f = 0.20 \) in Table 2 which demonstrates two main insights. First, the shrinkage estimator applied to adaptive priority (APS) sampling significantly improves the performance of the vanilla APS which uses Horvitz-Thompson estimator for all graphs. This is particularly clear for soc-flickr and soc-orkut for which the APS shrinkage is far superior than other methods. Second, when applied to uniform sampling, the shrinkage estimator did not result in any improvement. Note that the major difference between APS and uniform sampling in their ability to estimate the underlying distribution of the data stream. While APS approach adaptively samples edges proportional to their size (expected triangle count), uniform sampling is a reservoir sampling which uses uniform probabilities. These leads to very sparse samples with high variance where the shrinkage estimator does not help.

Although we derive the shrinkage estimator to minimize the mean square error, we consider another measure of approximation quality in addition to MSE. The spectral norm \( \|A - \hat{A}\|_2 \) was previously used for matrix approximation [1]. \( \|A - \hat{A}\|_2 \) measures the strongest linear trend of \( A \) not captured by the estimate \( \hat{A} \). This is different from the mean square error which focused on the magnitude of the estimates. We report the relative spectral norm (i.e., \( \|A - \hat{A}\|_2/\|A\|_2 \)) at sample fraction \( f = 0.20 \) for various graphs in Table 2. The experiments demonstrate that in all of the example graphs, both APS and APS with shrinkage significantly outperform uniform sampling. One observed exception is the soc–flickr graph, where the estimates using APS blows up due the high variance of Horvitz-Thompson estimation for edges with small counts. Under such scenarios, the APS with shrinkage significantly helps and improves the original APS estimates. We also notice the difference between how MSE ranks the best methods versus the relative spectral norm. A good example of this is the soc–orkut graph, for which APS performs worse than uniform sampling. However, APS is superior to uniform sampling for the relative spectral norm. Thus, despite of the large mean square error, APS (even without shrinkage) captures the linear trend and structure of the data better than uniform sampling. Finally, Figures 1 and 2 show the convergence performance of mean square error and relative spectral norm respectively as a function of the sampling fraction. Both APS and APS with shrinkage converge faster than uniform sampling, and APS with shrinkage proved to significantly improve the vanilla APS.
Table 2: Left: Mean Square Error. Right: Relative Spectral Norm. (sampling frac. 0.2). JS: James-Stein Shrinkage, APS: Adaptive Priority Sampling, Unif: Uniform Reservoir Sampling.

| data            | APS   | APS JS | Unif | Unif JS | APS   | APS JS | Unif | Unif JS |
|-----------------|-------|--------|------|---------|-------|--------|------|---------|
| SOC-Flickr      | 22.30K| 295.13 | 6.3K | 11.70K  | 0.5793| 0.0478 | 0.4321| 0.7848  |
| SOC-LiveJournal | 214.80| 16.11  | 257.60| 457.03  | 0.0269| 0.0089 | 0.429 | 0.7803  |
| SOC-Youtube-Snap | 11.35 | 6.68   | 119.79| 218.92  | 0.0455| 0.079  | 0.4159| 0.7548  |
| Wiki-Talk       | 7.70  | 5.32   | 589.92| 1.062K  | 0.0105| 0.0359 | 0.4315| 0.7802  |
| Web-BerkStan-Dir| 7.32K | 561.20 | 10.70K| 19.53K  | 0.1169| 0.0557 | 0.4381| 0.764   |
| Cit-Patents     | 6.02  | 3.03   | 10.59 | 14.30   | 0.0187| 0.0428 | 0.4325| 0.7639  |
| Soc-Orkut-Dir   | 2.08K | 70.79  | 467.90| 830.33  | 0.1086| 0.0726 | 0.4385| 0.7826  |

Figure 1: Each Plot corresponds to one graph for all methods, and shows MSE vs the sampling fraction. x-axis: sampling fraction $f$, y-axis: MSE in log scale.

Figure 2: Each Plot corresponds to one graph for all methods, and shows relative spectral norm $\|A - \hat{A}\|_2/\|A\|_2$ vs the sampling fraction. x-axis: sampling fraction $f$, y-axis: relative spectral norm.

Analysis of Estimated Distributions. We take the top-k non-zero weights of the true triangle-weighted adjacency matrix $A$, and we compare them against their corresponding estimates. Figures 4 and 5 show the top-1K weights for APS with shrinkage estimation and uniform sampling respectively. The results demonstrate the superior performance of APS with shrinkage estimation over uniform sampling, where APS with shrinkage estimation preserves the distribution and ranks of the top-k weights compared to uniform sampling. We report the analysis for two sampling fractions $f = \{0.20, 0.40\}$. In Figure 6 we report the normalized weights (probabilities) of the the top-10K edges of $A$. This experiment demonstrates how the sample distribution at sampling fraction $f = 0.20$ converges to the true distribution at sampling fraction $f = 0.40$ using APS with shrinkage. We observe that $\alpha$s collected at sampling fraction $f = 0.40$ using APS with shrinkage are almost indistinguishable from the true distribution in all of the example graphs. Figure 3 compares APS against APS with shrinkage, and shows how the shrinkage estimator reduces the variance of APS, in particular for the small local counts with high variance.
Figure 3: soc-livejournal graph, sample size $f = 0.4$. Left: APS estimate vs exact. Right: APS with Shrinkage estimator (James-Stein JS) vs exact. UB: upper bound, LB: lower bound.

Figure 4: Each Plot corresponds to one graph at sampling fractions $f = \{0.20, 0.40\}$, and shows the raw count of the top-1K edges for APS with Shrinkage Estimation vs the actual count. The top-1K edges are ranked based on their true counts. $x$-axis: the rank of top edges $1$–$1K$ in $\log_{10}$ scale, $y$-axis: weights (triangle count per edge).

Figure 5: Each Plot corresponds to one graph at sampling fractions $f = \{0.20, 0.40\}$, and shows the raw count of the top-1K edges for Uniform Sampling with Horvitz-Thompson Estimation vs the actual count. The top-1K edges are ranked based on their true counts. $x$-axis: the rank of top edges $1$–$1K$ in $\log_{10}$ scale, $y$-axis: weights (triangle count per edge).

Figure 6: Each Plot corresponds to one graph at sampling fractions $f = \{0.20, 0.40\}$, and shows the normalized count of the top-10K edges for APS with Shrinkage Estimation vs the actual normalized count. The top-10K edges are ranked based on their true normalized counts. The $x$-axis: the rank of top edges $1$–$10K$ in $\log_{10}$ scale, the $y$-axis: normalized weights (normalized triangle count per edge).

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A Proofs

Suppose $J_i \subset \tilde{K}_t'$. Then $J_i \subset K_t$ if some edge $j \in \tilde{K}_t'$ \ $J_t$ has minimum priority, i.e., if $r_{i,t} > z_{j,t}$, or equivalently, $u_i \leq w_{i,t}/z_{j,t}$ for all $i \in J_i$. Denote $A_{i,t,s} = \{u_i < w_{i,t}/z_{j,t}\}$ when $i \in J_i \cap \tilde{K}_t'$. Then for $t \geq \max J \geq t_J$ the event \{\$J \subset \tilde{K}_t\$\} decomposes as $\bigcap_{t \leq s \leq t} \bigcup_{i \in J} A_{i,t,s}$ where $B_{i,t} = \bigcup_{i \in J} A_{i,t,s}$.

**Proof of Theorem 2** (i) The proof is by induction on $t$. For $t < t_J$ the conditioning is trivial and $u_i$ are IID on $(0,1) = (0,p_{i,t,J})$. The same property holds at general $t$ for all $i \in J$ which have not yet arrived, i.e., for $i \in J \setminus J_i$. Consider now $t \geq t_J$ and assume that the result holds for $t - 1$. Then by (A1), the weights $w_{i,t}$ for $i \in J_i \cap \tilde{K}_t'$ are fixed by the conditioning on the event $\{J_{t-1} \subset \tilde{K}_t'\}$. Further conditioning on $z_{j,t}$ and $J_i \subset \tilde{K}_t$ requires $x_i < w_{i,t}/z_{j,t}$ for all $i \in J_i \subset \tilde{K}_t$. Imposing this condition on the assumed independent uniform distributions of $u_i$ on $(0,p_{i,t,J-1})$ results in independent uniform distributions of $u_i$ on $(0,\min\{p_{i,t,J-1},w_{i,t}/z_{j,t}\}) = (0,p_{i,t,J})$.

(ii) \[ \mathbb{E}[I(J_t \subset \tilde{K}_t) I_{J,t-1} \subset \tilde{K}_{t-1}] = \mathbb{P}[B_{J,t} I_{J,t-1} \subset \tilde{K}_{t-1}] \]

where in the last step we have used the statement of part (i) for the distribution of $u_i$ conditioning on $Z_{J,t-1}$ and $\{J_{t-1} \subset \tilde{K}_{t-1}\}$, since $w_{i,t}$ is determined given $\tilde{K}_{t-1}$.

(iii) \[ \mathbb{E}[S_{J,t} I_{J,t-1}] = \tilde{S}_{J,t-1} \]

that is independent of the conditioning on $z_{J,t}$ and hence \[ \mathbb{E}[S_{J,t} I_{J,t-1}] = \tilde{S}_{J,t-1}. \]

The initial value is \[ \tilde{S}_{J,t} = I(t_J \in \tilde{K}_t)/p_{t_J,J} = I(u_{t_J} \leq w_{t_J,t_J}/z_{t_J})/p_{t_J,J}, \]

and so by (i) \[ \mathbb{E}[S_{J,t} I_{J,t-1}] = 1 \]

Finally \[ \mathbb{E}[S_{J,t}] = 1 \]

for all $t \geq t_J$ by chaining the conditional expectations.

(iv) Trivially \[ \tilde{S}_{J,t} = S_{J,t} = 0 \]

for $t \leq \max J$. Since $z_{J,t} = z_t$ when $J \subset \tilde{K}_t$, \[ \tilde{S}_{J,t} = \tilde{S}_{J,t} \]

for $t \geq \max J$ and \[ \mathbb{E}[S_{J,t}] = 1 \]

by (iii).

**Proof of Lemma 2** Let $J = J_1 \cup J_2$. Chaining conditional expectations from Theorem 1 (i) \[ \mathbb{E}[\tilde{S}_{J_{t1},t} I_{J_{t2},t} | Z_{J,t2}, J_{t2-1} \subset \tilde{K}_{J,t2-1}] \]

\[ \mathbb{E}[\tilde{S}_{J_{t2},t} I_{J_{t1},t} | Z_{J,t1}, J_{t1-1} \subset \tilde{K}_{J,t1-1}] \]

\[ \mathbb{E}[\tilde{S}_{J_{t1},t} I_{J_{t2},t} | Z_{J,t2}, J_{t2-1} \subset \tilde{K}_{J,t2-1}] \]

\[ \mathbb{E}[\tilde{S}_{J_{t2},t} I_{J_{t1},t} | Z_{J,t1}, J_{t1-1} \subset \tilde{K}_{J,t1-1}] \]

for all $s \in [d_i,t]$. The first inequality follows from the nondecreasing property or $w_{i,t}$, the second inequality since edge $d_t$ survives selection from $d_i$ until $t$ and hence its priority cannot be lower than the threshold $z_s$ for any $s$ in that interval. But since $d_i$ was admitted, we have $d_i \neq d_t$ and hence we apply the argument back recursively to the first sampling time.

(ii) By assumption $i$ is admitted to $\tilde{K}_t$, and hence $i \neq d_i$ and so by (i) $p_{i,i} = \min\{1,w_{i,i}/z_i\} = \min\{1,w_{i,i}/z\} = p_{i,s}$. The general case is by induction. Assume $p_{i,s} = p_{i,s}$ for all $s \in [i,t]$, and $z_{t+1} > z_i$. Then $z_{i+1} = z_{i+1}$ hence $p_{i,s} = p_{i,s}$ for all $s \in [i,t]$, and $z_{i+1} = z_{i+1}$ hence $p_{i,s} = p_{i,s}$ for all $s \in [i,t]$, and hence $z_{i+1} = z_{i+1}$. Thus we can replace $z_{i+1}$ by $z_{i+1}$ in (i) but use of either leaves the iterated value unchanged, since by the induction hypothesis, both are greater than $p_{i,t} \leq w_{i,t}/z_{i,t}$.

**Proof of Lemma 2** Let $J = J_1 \cup J_2$. Chaining conditional expectations from Theorem 1 (i) \[ \mathbb{E}[\tilde{S}_{J_{t1},t} I_{J_{t2},t} | Z_{J,t2}, J_{t2-1} \subset \tilde{K}_{J,t2-1}] \]

\[ \mathbb{E}[\tilde{S}_{J_{t2},t} I_{J_{t1},t} | Z_{J,t1}, J_{t1-1} \subset \tilde{K}_{J,t1-1}] \]

\[ \mathbb{E}[\tilde{S}_{J_{t1},t} I_{J_{t2},t} | Z_{J,t2}, J_{t2-1} \subset \tilde{K}_{J,t2-1}] \]

\[ \mathbb{E}[\tilde{S}_{J_{t2},t} I_{J_{t1},t} | Z_{J,t1}, J_{t1-1} \subset \tilde{K}_{J,t1-1}] \]
using Theorem 1(i). Hence $E[\hat{S}_{J_1,t_1}I_{J_2,t_2}] = E[I_{J_2,t_2}]$ and since $E[\hat{S}_{J_1,t_1}] = 1$ the result follows.

\[\square\]

Proof of Theorem 1

(i) Since $E[\hat{S}_{J_1,t_1}] = 1$ it suffices to show that (the negative of) the second term on the RHS of (4) has expectation $E[I_{J_2,t_2}]$. When $t_1 \geq t_2$ then repeating the conditioning argument of Lemma 2 this term has conditional expectation

$$E[\hat{S}_{J_1,t_1}p_{J_1 \cap J_2,t_2}I_{J_2,t_2} \mid Z_{J,t_2}, J_{t_2-1} \in \hat{K}_{t_2-1}] \quad (18)$$

$$= E[\hat{S}_{J_1,t_2}p_{J_1 \cap J_2,t_2}I_{J_2,t_2} \mid Z_{J,t_2}, J_{t_2-1} \in \hat{K}_{t_2-1}] \quad (19)$$

$$= E[I_{J_2,t_2} \mid Z_{J,t_2}, J_{t_2-1} \in \hat{K}_{t_2-1}] \quad (20)$$

and hence the stated property holds. (ii) Holds since $\hat{S}_{J,t} > 0$ implies $I_{J,t} > 0$ for $t \geq t' \geq t_J$ and (iii) is a special case of (i).

\[\square\]