Schild Action and Space-Time Uncertainty Principle in String Theory

TAMIAKI YONEYA

Institute of Physics, University of Tokyo, Komaba, Tokyo

Abstract

We show that the path-integral quantization of relativistic strings with the Schild action is essentially equivalent to the usual Polyakov quantization at critical space-time dimensions. We then present an interpretation of the Schild action which points towards a derivation of superstring theory as a theory of quantized space-time where the squared string scale, $\ell_s^2 \sim \alpha'$, plays the role of the minimum quantum for space-time areas. A tentative approach towards such a goal is proposed, based on a microcanonical formulation of large $N$ supersymmetric matrix model.

*Internet address: tam@hep1.c.u-tokyo.ac.jp
Introduction

Modern quantum theory of relativistic (super) strings usually starts from an action which is quadratic with respect to the space-time coordinates \( X^\mu \)

\[
S_P = -\frac{1}{4\pi\alpha'} \int d^2\xi \sqrt{-g} g^{ab} \frac{1}{2} \partial_a X \cdot \partial_b X + \cdots \tag{1}
\]

In contrast with older and more geometrically motivated Nambu-Goto action,

\[
S_{NG} = -\frac{1}{4\pi\alpha'} \int d^2\xi \sqrt{-\det(\partial_a X \cdot \partial_b X)} + \cdots \tag{2}
\]

the former is appropriate for the path-integral quantization of strings [1]. In fact, the former seems to be the unique action which is tractable in a direct path-integral approach. Here and in what follows, by the ellipses we imply the additional terms which must enter to make the theory supersymmetric.

There are an infinite number of different actions which are classically equivalent to the Nambu-Goto action. For example,

\[
S_n = -\int d^2\xi e \left\{ e^\lambda \left[ -\frac{1}{2\alpha'} (e^{ab} \partial_a X^\mu \partial_b X^\nu)^2 \right]^{n/2} + n - 1 \right\} + \cdots \tag{3}
\]

with arbitrary nonzero \( n \) gives the same classical equations of motion, where the auxiliary field \( e(\xi) (> 0) \) have the same transformation property as \( \sqrt{-g} \) under the world-sheet reparametrizations, and \( \lambda = 4\pi\alpha' \). The action \( S_{NG} \) is a special limit with \( n \to 1 \), where the auxiliary field decouples. Another special case which has been studied in the past is \( n = 2 \), which is essentially the case first proposed by Schild [2].

Some interesting remarks on the \( n = 2 \) case have been made by Nambu [3]. In particular, he suggested an interesting interpretation for the quantity \( \sigma^{\mu\nu} \equiv e^{ab} \partial_a X^\mu \partial_b X^\nu \) as a formal analog of the Poisson bracket, and further speculated a quantization of the world-sheet by regarding the space-time coordinates as a sort of gauge fields. His argument is a precursor of some of recent proposals concerning possible matrix representations [4, 5] of the space-time coordinates, as has been motivated by string-theory dualities and D-branes [6]. In particular, the Schild action plays an important role in a more recent proposal made in [7]. To the best of our knowledge, however, quantum string theory based on the Schild action has never been concretely developed, except for an early attempt made in [8].

\[\text{‡} \quad \text{The metric in the present note is Minkowskian (+, +, \ldots, +, -) for both space-time and world sheet unless specified otherwise.} \]

\[\text{‡} \quad \text{See a note added at the end.}\]
The purpose of the present note is to show that the action $S_2$ is quantum-mechanically equivalent to the action $S_P$ at critical space-time dimensions, and then to discuss its relevance for a formulation of a space-time uncertainty relation which has been proposed [9] as a qualitative characterization of the space-time structure in fundamental string theory. We have recently argued [10] that the space-time uncertainty relation of [9] and [11] nicely fits the short distance structure of the D-particle dynamics [12]. We will emphasize a viewpoint that the above Poisson structure is a manifestation of conformal structure of the Polyakov quantization and is related to a possible realization of space-time uncertainty principle. Based on these observations, we will propose a tentative approach towards nonperturbative definition of string theory in terms of a microcanonical matrix model.

**Equivalence of the Schild and Polyakov quantizations**

Classical equivalence between the actions $S_n$ and $S_{NG}$ is trivially established by using the variational equation for the auxiliary field $e$

$$\frac{1}{e} \sqrt{-\sigma^2} = \lambda, \quad \sigma^2 \equiv \frac{1}{2}(\sigma^{\mu\nu})^2$$

which leads to $S_n = nS_{NG}$. We will call this equation as the conformal constraint for a reason that will become clear below. In the Hamiltonian formalism, the conjugate momenta to the world sheet coordinates $X^\mu$ are

$$\mathcal{P}^\mu = -\frac{e}{\sigma^2} (\dot{X}^\mu (\dot{X})^2 - \dot{X}^\mu (\dot{X} \cdot \dot{X})).$$

Here of course the dot $(\dot{X})$ and the prime $(\dot{X})$ are derivatives with respect to the world-sheet time and space coordinates, respectively. Combined with the conformal constraint (4), it is easy to see that the momenta satisfy the usual classical Virasoro conditions

$$\mathcal{P}^2 + \frac{1}{\lambda^2} \dot{X}^2 = 0,$$
$$\mathcal{P} \cdot \dot{X} = 0.$$ 

The Virasoro conditions in general express the conformal invariance of the world sheet theory. When we start from the Schild-type action, they are thus a consequence of the equation (4) (hence, the name “conformal constraint”). More precisely, the conformal constraint is responsible only for the first Virasoro condition (6) and the second one follows from the definition (5) alone. This reflects the fact that the second condition is kinematical in its nature: It only represents invariance of the theory under an arbitrary
reparametrization of the string at fixed world-sheet time. Conversely, under the kinematical condition (7), the first constraint (6) necessarily leads to the conformal constraint (4) since the general solution to (7) takes the form $-a(\xi)(\dot{X}^\mu(\dot{X})^2 - \dot{X}^\mu(\dot{X} \cdot \dot{X}))$ for some scalar function $a(\xi)$, leading to the identification $a = 1/e$ on imposing the first condition (6). Given these primary constraints (6) and (7), following Dirac’s procedure after introducing two lagrange-multiplier fields corresponding to the non-trace part of the world sheet metric in (1), we arrive at the same Hamiltonian as for the quadratic action $S_P$. This establishes equivalence of three different actions at the formal classical level.

It is not obvious that classical equivalence in the above sense automatically leads to equivalence quantum mechanically. A typical argument for a quantum mechanical derivation of the quadratic action starting from the Nambu-Goto action is given in [13] which generically shows that the former is obtained after making a rescaling for the string coordinates $X^\mu$ appropriately by a divergent constant and fine-tuning the cosmological constant. In general, quantum string theories based on the Nambu-Goto action requires to make renormalization of the string-theory constants such as the tension and the cosmological constant ($\sim$ tachyon condensation). As is well known by now, a precise control over this problem is only available for systems with one or lower target space-time dimensions, which are most conveniently formulated in terms of old matrix models with $c \leq 1$.

In the case of the Schild action with $n = 2$, we can derive the Polyakov action directly without making connection with the Nambu-Goto action. To see this, let us introduce auxiliary fields $t^{12}(= t^{21}), t^{11}, t^{22}$, which form a tensor density of weight two and rewrite the action

$$S(t, e, X) = \int d^2 \xi \frac{1}{e} [\det t - \frac{1}{\lambda} t^{ab} \partial_a X \cdot \partial_b X] - \int d^2 \xi e.$$  

(8)

This action is equivalent to the original Schild action $S_2$, even quantum mechanically, since the difference is a quadratic form of the auxiliary fields:

$$S_2 = S(t, e, X) - \int d^2 \xi \frac{1}{e} \det \tilde{t}$$  

(9)

where

$$\tilde{t}^{ab} = t^{ab} - \frac{1}{\lambda} e^{ac} e^{bd} \partial_c X \cdot \partial_d X.$$  

(10)

Let us then make a change of variables $t^{ab} \rightarrow g^{ab}, e \rightarrow \tilde{e}$ where $g^{ab}$ transforms as the standard world-sheet metric and $\tilde{e}$ as a scalar, $t^{ab} = g^{ab} e^2, e = \tilde{e} \sqrt{-g}$. Then the action reduces to

$$S(t, e, X) = \int d^2 \xi (\tilde{e}^3 - \tilde{e}) \sqrt{-g} - \frac{1}{\lambda} \int d^2 \xi \tilde{e} \sqrt{-g} g^{ab} \partial_a X \cdot \partial_b X$$  

(11)
At this point, we can assume that the measure in the partition function has been defined such that the total integration measure \( [d\tilde{e}[dX][dg] \) obtained in these transformations is reparametrization invariant. We then decompose the measure \( [dg] \) as usual
\[
\frac{[dg][dX]}{[d(\text{diff}_2)]} = [d\text{Weyl}] \frac{[dg][dX]}{[d\text{Weyl}][d(\text{diff}_2)]}
\]
to separate the Weyl mode. When the conformal anomaly which generically appears in this process cancels, we see that the conformal mode is contained only in the first term of (11). Thus at critical space-time dimensions, we have the equation of motion \( \ddot{\tilde{e}}^3 - \ddot{\tilde{e}} = 0 \) from the variation of the conformal factor. Only allowed solution of this equation is \( \ddot{\tilde{e}} = 1 \) since we assume that \( e > 0 \). Thus the action (11) reduces to the quadratic action, and the quantization of the Schild action \( S_2 \) is essentially equivalent to the Polyakov quantization.

Rigorously speaking, however, we have to be more careful with respect to the definitions of the continuum path-integrals, including more precise (regularized) definition of the measure and, in particular, the ranges of various auxiliary fields. Despite these subtleties, we propose to take the above observation as a hint for a new interpretation of string theory towards its nonperturbative formulation.

Conformal constraint and space-time uncertainty principle

We now want to clarify the meaning of the conformal constraint (4). It was already emphasized that, from the view point of the Schild action, conformal invariance of world-sheet string theory is a consequence of the conformal constraint. Now, what is the most crucial physical property of string theory resulting from the world-sheet conformal invariance? In some of earlier works [9][11], we have proposed a possible answer for this question that the conformal invariance is related with a space-time uncertainty principle of the form
\[
\Delta X \Delta T \sim \lambda
\]
for the minimum uncertainties with respect to the measurements of the lengths of the space (\( \Delta X \)) and time (\( \Delta T \)) intervals [9]. If we remember the reparametrization invariant Poisson bracket structure
\[
\{ X^\mu, X^\nu \} = \frac{1}{e} \epsilon^{ab} \partial_a X^\mu \partial_b X^\nu,
\]
the conformal constraint takes the following very suggestive form
\[
- \frac{1}{2} \left( \{ X^\mu, X^\nu \} \right)^2 = \lambda^2.
\]

\[\text{§ There has been other proposals for extended uncertainty relations [14] motivated from string theory.}\]
To exhibit the relation between (12) and (14), let us have recourse to a simple qualitative example considered in [11]. We consider an amplitude for the mapping from a rectangular region of the world sheet with the side lengths $a, b$ in the conformal gauge to a corresponding rectangular region of the target space-time whose lengths are $A$ and $B$ in the time ($\mu = 0$) and the space ($\mu = 1$) directions, respectively. The boundary conditions for the space-time coordinates are

$$X^\mu(\xi^0, 0) = X^\mu(\xi^0, b) = \delta^{\mu 0} A \xi^0 / a,$$  \hspace{1cm} (15)

$$X^\mu(0, \xi^1) = X^\mu(a, \xi^1) = \delta^{\mu 1} B \xi^1 / b.$$  \hspace{1cm} (16)

Then the quadratic action in the Euclidean metric gives the following factor for the amplitude, apart from a power behaved measure factor which is irrelevant for the present qualitative discussion,

$$\exp\left[-\frac{1}{\lambda} (A^2 b a + B^2 a b)\right].$$  \hspace{1cm} (17)

The ratio $a/b$ is nothing but the unique conformal invariant (namely, modular parameter) for the rectangle on the world sheet which is known as the “extremal length” [13]. We note that two forms, $\Gamma = a/b$ or $\Gamma^\ast = b/a$, of the extremal lengths are nothing but the duality relation between the extremal length $\Gamma$ and its conjugate extremal length $\Gamma^\ast$ satisfying $\Gamma \Gamma^\ast = 1$. The expression (17) clearly shows that there is an uncertainty relation of the form (12) with

$$\Delta T \sim \langle A \rangle \sim \sqrt{\lambda \Gamma}, \quad \Delta X \sim \langle B \rangle \sim \sqrt{\lambda \Gamma^\ast}.$$

Now, making the correspondence $(\lambda, A, B, \sqrt{a/b}, \sqrt{b/a}) \leftrightarrow (\hbar, x, y, \Delta x/\sqrt{\hbar}, \Delta p/\sqrt{\hbar})$, the form (17) is very analogous to Wigner’s phase space representation of a density operator $O_{\Delta x} = |g(\Delta x) > < g(\Delta x)|$

$$O(x, p) \equiv \int dy e^{\frac{ipy}{\hbar}} < x - \frac{1}{2} y | O_{\Delta x} | x + \frac{1}{2} y > \propto \exp - \frac{1}{2} \left( \frac{x}{\Delta x} \right)^2 - \frac{1}{2} \left( \frac{p}{\Delta p} \right)^2,$$

representing a Gaussian wave packet state $|g(\Delta x) >$ in ordinary particle quantum mechanics, where $< x | g(\Delta x) > \sim \exp - \frac{1}{2} \left( \frac{x}{\Delta x} \right)^2$ and $\Delta x \Delta p = \hbar$. If we further suppose to integrate out the density operator with respect to the parameter $\Delta x$ in analogy with the string modular integral, and take the classical limit $\hbar \to 0$, the integral is dominated by the classical solution $|\frac{x}{p}| = |\frac{\Delta x}{\Delta p}|$. This relation can be regarded as the classical counterpart of the Heisenberg uncertainty relation, corresponding to the Poisson bracket relation $\{x, p\} = 1$. The integration over the parameter $\Delta x$ of course means that we are now considering a statistical density operator.
On the other hand, if we calculate the same amplitude using the Schild action in the
gauge \( e = 1 \), we have the area law factor

\[
\exp\left(-\frac{1}{\lambda^2} \frac{(AB)^2}{a'b'} + a'b'\right) \Rightarrow \exp\left(-\frac{2}{\lambda} AB\right)
\]

(18)

under the conformal constraint

\[
\frac{AB}{a'b'} = \lambda,
\]

(19)
or more precisely, after integrating over the parameter \( s \equiv a'b' \), making the change of
variable \( s \to \sqrt{s} - \frac{AB}{\lambda \sqrt{s}} \). Here we put prime on the world-sheet length parameters to dis-
criminate them from those of the quadratic action. The argument of the previous section
requires that we must have the same result as for the quadratic action after integrating
over the modular parameter \( \frac{a}{b} \). This becomes obvious by making a correspondence of the
integration parameters \( a \leftrightarrow a' \), \( b \leftrightarrow B^2 / (\lambda b') \), by which the conformal constraint (19) is
reduced to \( \frac{A}{B} = \frac{s}{\lambda} \), the “classical solution” for (17). Thus, the conformal constraint can
be interpreted as the classical counter part for the space-time uncertainty relation (12), in
the same sense as in the ordinary classical limit \( \hbar \to 0 \) for the statistical density operator
given by \( \rho \equiv \int d(\Delta x) \mathcal{O}_{\Delta x} \).

These elementary considerations strongly suggest the existence of a theory in which
the Poisson structure (13) is replaced by a commutator between coordinate operators
\( \{X^\mu, X^\nu\} \to [X^\mu, X^\nu] \), and also that such a theory should take into account some form of
quantum condition which reduces to the conformal constraint (14) in a certain classical
limit. Our discussion above shows that, in this kind of theories of quantized space-time,
the ordinary space-time continuum should be interpreted as something analogous to the
classical phase space in quantum particle mechanics. A strong form of possible quantum
conditions \( \bullet \) would be to demand that the operator \([X^\mu, X^\nu]\) satisfies

\[
- \frac{1}{2}([X^\mu, X^\nu])^2 = \lambda^2 I \tag{20}
\]

where \( I \) is the identity operator in some operator algebra and we assume the Minkowski
metric,

\[
\frac{1}{2}([X^\mu, X^\nu])^2 \equiv - \sum_{i=1}^{9} [X_0, X_i]^2 + \frac{1}{2} \sum_{i,j=1}^{9} [X_i, X_j]^2.
\]

\( \bullet \) A similar quantum condition has been considered in ref. [16] in an even more stronger form in 4
dimensions and rigorous inequalities similar to our uncertainty relations have been proved. I would like
to thank S. Doplicher and A. Jevicki for bringing this work into my attention.
A weaker form of the quantum condition would be to demand (20) only for appropriately defined expectation values \( \langle \cdots \rangle \). This leads to the following inequality,

\[
\langle ([X^0, X^i])^2 \rangle \geq \lambda^2,
\]

and hence is consistent with the space-time uncertainty relation (12) in a way that is invariant under Lorentz transformations. Note that, without further conditions, no definite lower bound exits for the space-space components. This conforms to the analysis in [10] in string theory based on the results [12] on the dynamics of D-particles.

**Microcanonical matrix model**

We have seen that a key feature of the conformal invariance is encoded in the space-time uncertainty principle. It is natural to postulate that non-perturbative string theory should be formulated on the basis of the space-time uncertainty principle. We now want to briefly discuss a tentative approach towards such a goal. To motivate our proposal, we first remark a striking analogy of the above structure to the matrix algebra [4] appearing as the zero dimensional reduction of 10-dimensional super Yang-Mills theory as describing the effective weak-coupling dynamics of D-instantons (namely, Dirichlet-(-1) branes) in the type IIB superstring theory. Assuming the validity of the conjectured self-duality of 10-dimensional type IIB theory, it is natural to suppose that the microscopic space-time structure seen through the D-branes is basically the same as that seen in the elementary-string excitation picture.

Our postulate then is to identify the algebra for the D-instanton coordinate matrices \( X_\mu (\mu = 1, 2, \ldots, 10) \) which are \( N \times N \) hermitian (anti-hermitian for \( \mu = 10 \)) matrices describing \( N \) D-instantons with the operator algebra discussed in the previous section. As is well known, these matrices are nothing but the Yang-Mills fields coupled to open strings at their end points where the D-instantons are located.

The quantum condition is assumed to be

\[
\frac{1}{2} \langle ([X_\mu, X_\nu])^2 \rangle \equiv \lim_{N \to \infty} \frac{1}{N} \text{Tr} \frac{1}{2} ([X_\mu, X_\nu])^2 = -\lambda^2.
\]

Here, \( \langle \cdot \rangle \) denotes the expectation value with respect to the \( U(N) \) trace as indicated. We adopt a weaker form of the possible quantum conditions and assume the large \( N \) limit to include the case of arbitrary number of D-instantons. From the work [17], we know that the parameter \( 1/N \) plays the role of Planck constant in relating the Poisson structure to
the commutator. Then it is natural to define the fundamental partition function as

$$Z = \int \left( \prod_{\mu=1}^{10} d^{\infty} x_{\mu} \right) \mathcal{J}[X] \delta \left( \frac{1}{2} \left( [X_{\mu}, X_{\nu}]^2 \right) + \lambda^2 \right)$$  \hspace{1cm} (23)

where $d^{\infty} x_{\mu}$ is the large N limit of the standard $U(N)$ invariant Haar measure and $\mathcal{J}[X]$ is an additional measure factor to be determined below.

Let us consider how to determine the measure factor $\mathcal{J}[X]$. The quantum condition has the gauge symmetry under

$$X_{\mu} \to UX_{\mu}U^{-1} + a_{\mu}I$$  \hspace{1cm} (24)

where $U \in U(N)$ and $a_{\mu}$ are arbitrary real constants. The measure factor should respect this gauge symmetry. A further requirement to make the theory sensible is that the partition function should satisfy the cluster property. That is, distant D-branes must behave independently except for the possible power-behaved long-range forces between them. A more stronger condition is that the partition function be consistent with the known behavior of the classical BPS saturated solutions. From the viewpoint of Yang-Mills theory reduced to a point, the classical solutions are characterized as the solutions of the bosonic equations

$$[X_{\mu}, [X_{\mu}, X_{\nu}]] = 0$$  \hspace{1cm} (25)

which are nothing but the variational equation for the quantum condition \((22)\). These solutions can be regarded as D-branes as composites of the D-instantons. Explicit solution for the case of D-strings (i.e., D-1 brane) are constructed in ref. \cite{7}. The higher-dimensional cases have also been discussed in several recent works. \cite{4}

The cluster property for distant D-branes requires that the fluctuations around the solutions cancel to the leading order. Otherwise the distant D-branes would in general interact through logarithmic long-range forces which cannot be accepted. Together with the gauge symmetry, only conceivable way to satisfy the above criteria is to choose the measure factor $\mathcal{J}$ such that the partition function be supersymmetric. The largest space-time dimensions in which this is possible is 10. This follows from a well known fact about supersymmetric Yang-Mills theory, after making the reduction to a point (i.e., reduction to $-1$ dimensions). Thus we are naturally led to the measure factor,

$$\mathcal{J}[X] = \int d^{16} \psi' \exp \left( \frac{1}{2} \langle \psi' \Gamma_{\mu} [X_{\mu}, \psi] \rangle \right)$$  \hspace{1cm} (26)

\footnote{See, e.g., \cite{8}. In particular, \cite{9} has discussed the solutions emphasizing close affinity between the properties of the solutions and the space-time uncertainty relation.}
where $\psi$ is the $U(\infty)$-matrix whose elements are Majorana-Weyl spinors in 10 dimensions and the prime in the integration volume denotes that possible fermion zero-modes should be removed for the partition function. The supersymmetry is easily established after rewriting the partition function by introducing an auxiliary constant multiplier $c$,

$$Z = \int dc \left( \prod_\mu d^{16}\psi' \exp \left[ c \left( \frac{1}{2} \langle [X_\mu, X_\nu]^2 \rangle + \lambda^2 \right) + \frac{1}{2} \langle \bar{\psi} \Gamma_\mu [X_\mu, \psi] \rangle \right] \right).$$  \hspace{1cm} (27)

The action in this expression is invariant under two supersymmetry transformations,

$$\delta_\epsilon \psi = ic [X_\mu, X_\nu] \Gamma_{\mu\nu} \epsilon, \hspace{1cm} \text{(28)}$$

$$\delta_\epsilon X_\mu = i\pi \Gamma_\mu \psi, \hspace{1cm} \text{(29)}$$

$$\delta_\epsilon c = 0, \hspace{1cm} \text{(30)}$$

$$\delta_\eta \psi = \eta, \hspace{1cm} \text{(31)}$$

$$\delta_\eta X_\mu = 0, \hspace{1cm} \text{(32)}$$

$$\delta_\eta c = 0. \hspace{1cm} \text{(33)}$$

This looks similar to the model considered in ref. [1] which we call the IKKT model except for an additional auxiliary variable $c$ which enters to realize the space-time uncertainty principle in a weak form. We now explain some features of our model. First we shall argue that the IKKT model can be interpreted from our model as an effective theory for D-branes.

Let us first consider the effective action for many distant clusters of D-brane systems. This corresponds to introducing the background $X^b_\mu$ in the block-diagonal form

$$X^b_\mu \equiv \begin{pmatrix} Y_\mu^{(1)} & 0 & 0 & \cdots \\ 0 & Y_\mu^{(2)} & 0 & \cdots \\ 0 & 0 & Y_\mu^{(3)} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$ \hspace{1cm} (34)

where each block is assumed to be $N_a \times N_a$ hermitian matrix. The separation between the clusters $Y^{(a)}_\mu$ and $Y^{(b)}_\mu$ is measured by $\ell_{a,b} \equiv |\frac{1}{N_a} \text{Tr}_a Y^{(a)}_\mu - \frac{1}{N_b} \text{Tr}_b Y^{(b)}_\mu|$ which are here assumed to be sufficiently large compared to $\sqrt{\lambda}$. The notation $\text{Tr}_a$ means the trace operation within each block. The backgrounds must satisfy the conditions

$$- \sum_{a=1}^n \text{Tr}_a \frac{1}{2} |Y^{(a)}_\mu, Y^{(a)}_\nu|^2 = N\lambda^2,$$ \hspace{1cm} (35)
\[
\sum_{a=1}^{n} N_a = N \rightarrow \infty.
\]  

(36)

Now let us suppose to evaluate the fluctuations around the backgrounds by decomposing as \(X_\mu = Y_\mu + \tilde{X}_\mu\). For fixed \(c\), the calculation is entirely the same as the IKKT model \([7]\) where it is shown that the leading order one-loop effective action for the backgrounds decreases as \(O(\frac{1}{\ell_{ab}})\) in the limit of large separation. To one-loop order, the effective action is independent of the parameter \(c\). Thus the distant D-brane systems can be treated as independent objects in this approximation, and hence we can take into account the conditions \((35), (36)\) in a statistical way by introducing two Lagrange multipliers \(\alpha\) and \(\beta\). Namely, we can derive the effective action for the D-brane sub systems by the same argument as we use in going from the microcanonical ensemble to the canonical distribution for subsystems. Then the effective partition function \(Z_{\text{eff}}\) for the D-brane sub system described \(Y_\mu\) within the semi-classical approximation is given by a grand canonical form

\[
Z_{\text{eff}} = \sum_{N} \left( \prod_\mu \int d^{N^2} Y_\mu \right) d^{16} \psi \exp S[Y, \psi, \alpha, \beta],
\]  

(37)

\[
S[Y, \psi, \alpha, \beta] \equiv -\alpha N - \beta \text{Tr}_N \frac{1}{2} [Y_\mu, Y_\nu]^2 - \frac{1}{2} \text{Tr} \bar{\psi} [\Gamma_\mu, Y_\mu] \psi,
\]  

(38)

where by \(N\) we denote the order of the background submatrix \(Y\) and \(\text{Tr}_N\) is the corresponding trace. This form is identical with the IKKT model, except that in our case the following condition must be satisfied,

\[
- \langle \frac{1}{2} [Y_\mu, Y_\nu]^2 \rangle \equiv - \frac{\sum_{N} \left( \prod_\mu \int d^{N^2} Y_\mu \right) d^{16} \psi \exp S[Y, \psi, \alpha, \beta]}{\sum_{N} \left( \prod_\mu \int d^{N^2} Y_\mu \right) d^{16} \psi \exp S[Y, \psi, \alpha, \beta]} = \lambda^2
\]  

(39)

in order to respect the quantum condition. Note that the constant \(\beta\) can be identified with the mean value of the original Lagrange multiplier \(c\) in \((27)\). The Lagrange multipliers can in principle be determined by requiring the equivalence with the microcanonical form in the present approximation. The two constants of the effective theory should be related to the vacuum expectation values of scalar background fields of type IIB superstring theory, such as dilaton and/or its dual partner, scalar axion.

Our model is thus approximately equivalent with the IKKT model as an effective theory describing sufficiently distant D-brane systems in the nearly classical limit. However, we should not expect complete equivalence in full quantum theory. First we note that the completely diagonal background is not a good background in the present case, except in the limit \(\lambda \to 0\), since it cannot satisfy the quantum condition \((35)\). The quantum condition gives a rationale for choosing the solution satisfying the condition \([Y_\mu, Y_\nu] \propto \mathcal{I}\)
for some pairs of the components $\mu, \nu$ where at least one pair must contain the time component and $I$ is a constant matrix whose square has a nonzero trace in the large $N$ limit. The authors of ref. [7] interpreted their model as an effective theory for the reduced Yang-Mills theory in 10 dimensions. In perturbation theory, this interpretation suffers from nonrenormalizable infrared divergencies, reflecting nonrenormalizable ultraviolet difficulty of 10 dimensional super Yang-Mills theory. Here we would like to recall a remark [21] that the equivalence between the microcanonical and canonical formulations of non-renormlizable field theories is not at all obvious. In a similar vein, our microcanonical matrix model is most probably not exactly equivalent with the reduced Yang-Mills theory. An obvious flaw of the grand canonical form (37) as the definition of the theory is that the action appeared in it is not bounded in the Minkowski metric. To make sense out of this form, it is necessary to make a Wick rotation $Y_0 \rightarrow iY_{10}$ and assume a positive value for $\beta$. It is not clear how this is justified within the logic of the present formalism. Note that the quantum condition in the form (22) is meaningful only in the Minkowski metric.

*Further discussions*

It seems that, through exploring the meaning of the Schild action in string theory, we have opened up innumerable new problems.

First of all, the microcanonical form is not the unique possibility. We may try to impose a stronger form of the quantum condition such as (20) by generalizing the classical action $S_2$ with the auxiliary field $e(\xi)$. In that case, however, a naive introduction of an auxialiry matrix corresponding to the variable $e$ in general violates the supersymmetry, except for the naive continuum theory in the large $N$ limit. Although such a model may still satisfy the cluster property in the one-loop approximation, lack of exact supersymmetry would make the theory ambiguous. For example, if we cannot assume exact supersymmetry, it is not clear how to fix the measure $\mathcal{J}(X)$. To implement a stronger quantum condition, a higher symmetry than $[U(N)\text{-gauge} + \text{SUSY}]$ seems necessary, unless we rely upon an argument assuming universality even if we start from the non supersymmetric models in the continuum limit. Considering various other possibilities, our model should yet be regarded as a tentative working hypothesis as a first step for further exploration of space-time uncertainty relation as a fundamental principle underlying the string theory. We postpone more detailed investigations of the present model and other possibilities to forthcoming works.

12
Secondly, one of the crucial questions is whether and in what sense the fundamental perturbative string theories are contained in the present formalism. If our interpretation that the matrix $X_\mu$ represents the D-branes is correct, it is natural to introduce the quantity

$$A(s)ds + \cdots \equiv X_\mu \frac{\partial x^{\mu}(s)}{\partial s} ds + \cdots$$

as the “string-bit” operator, whose matrix element $A_{ab}(s)ds$ is an infinitesimal open segment of a bosonic string connecting two D-instantons labeled by $a$ and $b$, respectively, of which the infinitesimal space-time element is given by $\dot{x}^{\mu}(s) ds$. In the perturbative string theory, this is nothing but the vertex operator for an open string connecting two D-instantons, and the dynamics of string coordinate $x^\mu(s)$ would encode the dynamics of the matrix variable $X_\mu$ as collective coordinates. The matrix interpretation of the D-brane coordinates has first been obtained in this picture \cite{4} as an effective low-energy theory. It is important to develop systematic formalism for deriving a full-fledged dynamical theory for the matrices along this line. Conversely, if the full dynamics of D-brane matrix $X_\mu$ were known, the dynamics of the fundamental strings should be derived from the former by treating in turn the string variables $x^\mu(s)$ as the collective coordinate. Then the operator representing a closed string configuration $x^\mu(s)$ ($x^\mu(0) = x^\mu(2\pi)$) would be the Wilson loop,

$$\frac{1}{N} \text{Tr} P \exp \left( i \frac{1}{\lambda} \oint ds X_\mu \frac{\partial x^{\mu}(s)}{\partial s} \right)$$

apart from some appropriate fermionic contribution, with $P$ being the ordering operator along the string. The derivation of string dynamics is now reduced to the long-standing question of deriving the string dynamics as effective theory from matrix models. This is essentially a dual transformation. Note that the commutator $[X_\mu, X_\nu]$ and the area derivative $\frac{\delta}{\delta \sigma^{\mu\nu}}$ corresponding to an infinitesimal area deformation $\delta \sigma^{\mu\nu} = \delta x^\mu \wedge \delta x^\nu$ of the Wilson loop are dual to each other in this dual transformation. At present, it is not clear whether the microcanonical formulation provides new handles on this old and difficult question. One thing evident from the structure of the Schwinger-Dyson equation for the matrix model, however, is that the mean value of the Lagrange multiplier $c \sim \beta$ in the fundamental partition function is inversely proportional to the string coupling constant, $c \propto \frac{1}{g_s \lambda^2}$, of string perturbation theory, being consistent with well known effective low-energy theories of D-branes \cite{4}. It should also be mentioned that under the dual transformation, the stronger quantum condition \cite{29} reduces to the Virasoro-type condition $(\delta / \delta \sigma^{\mu\nu})^2 \propto \lambda^2$, whose classical counterpart is \cite{3} as it should be, at least for a smooth string configuration $x^\mu(\sigma)$. This can be a further motivation in favor of the stronger quantum condition.

Another important question is what is, if any, the relation of the present approach to

\[**\] A slightly different suggestion has been made in \cite{7} where $\frac{\partial x^{\mu}(s)}{\partial s}$ is replaced by the momentum density of the string.
the matrix model (so-called, M(atrix) theory) proposed in [3] as a nonperturbative light-cone formulation of the M-theory (∼ type IIA superstring) which requires 11 (=10+1) dimensions. Our present approach requires 9+1 dimensions as a largest possible dimensions and conforms to the type IIB superstring. To connect to type IIA and/or M theory, we may compactify one dimension and invoke the T-duality, or may add one additional dimension. For the latter possibility, we might go over to the stochastic quantization of the model.

Finally, a remaining crucial question is the problem of background independence. Ever since the first discovery that the string theory is a quantum theory of gravity [22][23], the geometrical formulation of string theory as a natural quantum extension of Einstein’s general relativity has been a big mystery. In the present paper, we always assumed flat space-time background. It is not at all clear how to interpret the quantum conditions in a manner consistent with the principles of general relativity, which should ultimately be reconciled with the principles of string theory at least in the long distance regime. In this respect, it may be interesting to extend our interpretation of the conformal constraint to the case with non-trivial space-time background fields. The real issue, however, is how to understand the dynamics of space-time geometry itself (not just a different fixed curved space-time) within the present framework.

Obviously much work remains to be done in order to realize the present proposals.

I would like to thank A. Jevicki for sharing together an idea that D-instantons may be the fundamental building block for string theory and for collaborative discussions on this and related topics during my stay at Brown University in the summer in 1996, which was made possible under the US-Japan Collaborative Program for Scientific Research supported by the Japan Society for the Promotion of Science. Also I wish to thank my colleagues at Komaba, especially, M. Kato and Y. Kazama, for stimulating discussions on D-brane dynamics at our seminars.

Note added: After the paper first appeared on hep-th, Dr. Spallucci called my attention to the work [24] which used the Schild action in a loop space approach to string quantization. Also, Dr. Tseytlin pointed out his work [25] which established the equivalence of the Schild and Polyakov quantizations in the semi-classical approximation.

References

[1] A. M. Polyakov, Phys. Lett. 103B (1981) 207.
[2] A. Schild, Phys. Rev. D16 (1977) 1722.

[3] Y. Nambu, p. 1 in “Quark Confinement and Field Theory” (John Wiley & Sons, New York, 1977).

[4] E. Witten, Nucl. Phys. B460 (1995) 335.

[5] T. Banks, W. Fischler, S. H. Shenker and L. Susskind, hep-th/9610043.

[6] J. Polchinski, Phys. Rev. Lett. 75 (1995) 4724.

[7] N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, hep-th/9612115.

[8] T. Eguchi, Phys. Rev. Lett. 44 (1980) 126.

[9] T. Yoneya, p. 419 in “Wandering in the Fields”, eds. K. Kawarabayashi and A. Ukawa (World Scientific, 1987) ; p. 23 in “Quantum String Theory”, eds. N. Kawamoto and T. Kugo (Springer, 1988).

[10] M. Li and T. Yoneya, Phys. Rev. Lett. 78 (1997) 1219; hep-th/9611072.

[11] T. Yoneya, Mod. Phys. Lett. A4 (1989) 1587.

[12] For a recent extensive discussion on D-particle dynamics, we would like to refer the reader to M. R. Douglas, D. Kabat, P. Pouliot and S. H. Shenker, hep-th/9608042.

[13] A. M. Polyakov, Chapter 9 in “Gauge Fields and Strings” (Harwood Scientific Pub., Switzerland, 1987).

[14] D. Amati, M. Ciafaloni and G. Veneziano, Phys. Lett. 197B, 81(1987).  
D. Gross and P. Mende, Nucl. Phys. B303, 407(1988).  
For recent discussions, see, e.g., G. Amelio-Camelin, J. Ellis, N. E. Mavromatos and D. V. Nanopoulos, hep-th/9701144 and references therein.

[15] L. V. Alfors, Ch. 5 in “Conformal Invariants : Topics in Geometric Function Theory” (McGraw-Hill, New York, 1973).

[16] S. Doplicher, K. Fredenhagen and J. E. Roberts, Comm. Math. Phys. 172 (1995) 187; Phys. Lett. B331 (1994) 39.

[17] B. de Wit, J. Hoppe and H. Nicolai, Nucl. Phys. B305 (1988) 545.

[18] A. Fayyazuddin and D. J. Smith, hep-th/9701168.  
I. Chepelev, Y. Makeenko and K. Zarembo, hep-th/9701151.

[19] M. Li, hep-th/9612222.

[20] M. Green and J. H. Schwarz, Phys. Lett. 136B (1984) 367.

[21] A. Strominger, Ann. Phys. (N.Y.) 146 (1983) 419.

[22] J. Scherk and J. H. Schwarz, Nucl. Phys. B81 (1974) 118; Phys. Lett. 57B (1975) 463.

[23] T. Yoneya, Prog. Theor. Phys. 51 (1974) 1907; Nuovo Cim. Lett. 8 (1973) 951.

[24] S. Ansoldi, A. Aurilia and E. Spallucci, Phys. Rev. D53 (1996) 870.

[25] E. S. Fradkin and A. A. Tseytlin, Ann. Phys. (NY) 143 (1982) 413.