Flux, Supersymmetry and M Theory on 7-Manifolds.

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Abstract

Various aspects of low energy M theory compactified to four dimensions are considered. If the supersymmetry parameter is parallel in the unwarped metric, then supersymmetry requires that the warp factor is trivial, the background four-form field strength is zero and that the internal 7-manifold has $G_2$ holonomy (we assume the absence of boundaries and other impurities). A proposal of Gukov - extended here to include M2-brane domain walls - for the superpotential of the compactified theory is shown to yield the same result. Finally, we make some speculative remarks concerning higher derivative corrections and supersymmetry breaking.

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1. Introduction.

At low energies, $M$ theory is described by eleven dimensional supergravity theory. The latter theory was presented in [1]. Over the past few years $M$ theory studies have led to a reconsideration of results in the supergravity theory. In the 1980’s much attention was given to Kaluza-Klein compactifications of eleven dimensional supergravity to four dimensions and we will reconsider some aspects of this work here. For a review article on Kaluza-Klein supergravity see [2].

The motivation for our work stemmed primarily from certain results concerning $M$ theory on Calabi-Yau fourfolds [3, 4, 5, 6], and in particular the observations due to Becker and Becker that one can include a non-trivial background expectation value for the four-form field strength in a supersymmetric fashion. Gukov, Vafa and Witten later proposed a simple formula for the superpotential of the compactified theory in such backgrounds, and showed that the resulting conditions for unbroken supersymmetry are precisely the conditions obtained by Becker and Becker who analysed the supersymmetry equations of eleven dimensional supergravity. Gukov later went on to propose similar formulae for the superpotential of $M$ theory compactified on various other special holonomy manifolds [7].

In this paper we will discuss the relationship between the four-form field strength and supersymmetry in compactifications of $M$ theory to four dimensions and the relationship (when it exists) to Gukov’s proposed superpotential. We will begin by analysing supersymmetry conditions in eleven dimensional supergravity backgrounds with a spacetime metric which is a warped product of a four-metric and a 7-metric. We will see that supersymmetry requires the warp factor to be trivial if the supersymmetry parameter is parallel in the unwarped metric. This implies that the 7-manifold has $G_2$-holonomy and that the four-form field vanishes identically$^4$.

We will go on to show how Gukov’s proposed superpotential for $G_2$-holonomy compactifications also gives rise to the same conclusion: that the only non-zero field in a supersymmetric background is the metric. This is in contrast to the situation in three dimensions where Becker and Becker showed that both the warp factor and the four-form field can be non-zero in a supersymmetric background.

The same analysis can be done for compactifications to three dimensions

$^4$For trivial warp factor, this is the calculation of Candelas and Raine [8]. For completeness, we extend their calculation to the case of non-zero cosmological constant. The only solution in this case appears to be the usual supersymmetric Freund-Rubin ansatz [9] in which both the four and seven manifolds admit Killing spinors. $AdS_4 \times S^7$ is the standard example
on $\text{Spin}(7)$ holonomy 8-manifolds and an agreement between Gukov’s proposal for that case and the supergravity analysis is also found. In this case, one can again turn on a non-trivial warp factor and four-form field strength in a manner consistent with supersymmetry. The details for $\text{Spin}(7)$ holonomy compactifications will be presented elsewhere [10].

In the next section we present the details of the supergravity calculation. In section three we show that Gukov’s proposed superpotential gives the same result. Finally we end with some discussion concerning higher derivative $M$ theoretic corrections to our results and speculate on possible scenarios with low energy supersymmetry breaking. Our conventions are detailed in an appendix.

There has been a previous attempt in the literature to consider the relationship between the supergravity equations and Gukov’s potential [11]. Unfortunately, we do not agree with the results of that paper.
2. Calculation.

We will consider backgrounds of low energy M theory in which the spacetime 11-manifold \( X \) decomposes into a warped product of a four-manifold \( M^4 \) with metric \( g_4(x) \) and a 7-manifold \( M^7 \) with metric \( g_7(y) \). The simplest case to consider would be when \( g_4 \) is the Minkowski metric on \( \mathbb{R}^4 \) and \( M^7 \) is compact in which case we can interpret the background as a compactification of \( M \) theory to four dimensional Minkowski space. We will take \( M^4 \) to have Lorentzian signature. The most general ansatz for the 11-metric is

\[
g_{11}(x, y) = \Delta^{-1}(y)(g_4(x) + g_7(y))
\]

(1)

which describes a warped product metric on \( M^4 \times M^7 \). We are interested in the constraints that supersymmetry imposes on such backgrounds, so we require that the expectation value of the gravitino field vanishes in the background. With this choice the equations for supersymmetry in the background are just that the supersymmetric variation of the gravitino must vanish. This means that there exists a spinor \( \eta \) for which

\[
\nabla_M \eta + Z_M \eta = 0
\]

(2)

where

\[
Z_M = \frac{1}{144}(\Gamma_M^{PQRS} - 8\delta_M^P \Gamma^{QRS})G_{PQRS}
\]

(3)

The above equations are valid in the extreme low energy limit of \( M \) theory and in principle receive higher derivative corrections. We will comment on these corrections later. We are also assuming that the background is free of boundaries, fivebranes or other ‘impurities’ which can give rise to additional terms (see, for instance [13]). The above metric has a group of symmetries which act on the frame bundle of \( X \) and this group is locally \( SO(3,1) \times SO(7) \) where the two factors act obviously on \( M^4 \) and \( M^7 \) respectively. The most general ansatz for the four-form field strength \( G \), consistent with the spacetime symmetries is:

\[
G_{\alpha\beta\gamma\delta} = 3m\epsilon_{\alpha\beta\gamma\delta} , \quad G_{mnpq} \neq 0
\]

(4)

with all other components vanishing. The factor of three is as usual for convenience and \( m \) is a constant. Our goal is to describe the constraints on \( m \), \( G_{mnpq} \) and \( \Delta(y) \) imposed by supersymmetry.

Consider first the \( \mu \) components of (2). After substituting (1) and (4), writing the connection as the spin connection for \( \Delta(y)g_{11}(x, y) \) plus terms involving derivatives of the warp factor (cf. [3]) we find:

\[
\nabla_\mu \eta + Z_\mu \eta = 0
\]

(5)
where
\[ Z_\mu = \gamma_\mu \otimes \alpha + \gamma_5 \gamma_\mu \otimes (\beta_1 + i \beta_2), \]  
and
\[ \alpha \equiv \frac{\Delta^{3/2}}{144} G_{mnpq} \gamma^{mnpq} \]  

\[ \beta_1 \equiv \frac{1}{4} \gamma^m \partial_m \log(\Delta), \quad \beta_2 \equiv -\Delta^{3/2} m \]  
Initially we will assume that \((M^4, g_4(x))\) are such that they admit parallel (i.e., covariantly constant) spinors (Minkowski space being the prime example) in which case (2) becomes
\[ Z_\mu \eta = 0 \]  
Since \(\gamma_\mu\) is invertible, we obtain
\[ \alpha \eta = \gamma_5 \otimes (\beta_1 + i \beta_2) \eta \]  
Now consider the \(p\)-components of (2). Substituting our ansatz we obtain
\[ \nabla_p \eta - \frac{1}{4} \gamma^p \partial_n \log(\Delta) \eta + \gamma_5 \otimes \gamma_p \alpha \eta - \Delta^{\frac{3}{2}} \left( \frac{1}{12} \gamma_5 \otimes G_{pqrs} \gamma^{pqrs} \eta + \frac{i}{2} m \otimes \gamma_m \eta \right) = 0 \]  
Contracting this equation with \(\gamma^p\) and substituting (10) we obtain,
\[ \gamma^p \nabla_p \eta - \frac{11}{4} \gamma^p \nabla_p \log(\Delta) \eta + \frac{3i}{2} m \Delta^{\frac{3}{2}} \eta = 0 \]  
Our assumption is that the supersymmetries are parallel in the unwarped metric, which implies that \(\eta\) is an eigenvector of the matrix \(\frac{11}{4} \gamma^p \nabla_p \log(\Delta)\) with eigenvalue \(\frac{3i}{2} m \Delta^{\frac{3}{2}}\). However, the eigenvalues of that matrix must be real, so we obtain that
\[ \nabla_p \Delta = m = 0 \]  
\[ \nabla_n \theta = m = G_{pqrs} = 0 \]  
\[ \text{This assumption is necessary in order to compare the conditions on } G \text{ with those which follow from [7]. This is because, [7] implicitly assumes that supersymmetric cycles in a } G_2 \text{-holonomy manifold are calibrated submanifolds. This assumption is valid for supersymmetries which are parallel.} \]
These equations can be proven simply from integrability of (2) with warp factor one. We do not give the details here since we will be performing a more general calculation below - of which the above is a special case. The fact that \((M^7, g_7)\) admits a parallel spinor is equivalent to the statement that the holonomy group of \(g_7\) is \(G_2\) or a subgroup thereof.

Note that a slightly stronger result can be achieved without assuming the supersymmetry is parallel but by assuming that the Freund-Rubin parameter \(m\) is zero. Then the integrability equations, derived from (11) can be used to show that when \(M^7\) is compact the \(G\)-field vanishes, the warp factor is trivial and that the spinor is in fact parallel. We will not give the details of this here, since it implies the same conditions on the background spacetime as above.

More generally we can require that \((M^4, g_4)\) admit Killing spinors. The most general such spinor obeys

\[
\nabla_\mu \epsilon = \Lambda_1 \gamma_\mu \epsilon + i \Lambda_2 \gamma_5 \gamma_\mu \epsilon
\]

where the constants \(\Lambda_1\) and \(\Lambda_2\) are real\(^7\). The case in which the right hand side of (13) is zero is essentially the case discussed above.

Here one again obtains an equation of the form of (6) but with \(\alpha\) and \(\beta\) given by

\[
\alpha \equiv \frac{1}{144} G_{mnpq} \gamma^{mnpq} + \Lambda_1
\]

\[
\beta_2 \equiv \Lambda_2 - m
\]

The warp factor is assumed constant in the remainder of this section. Contracting this new version of (6) with \(\gamma^\mu\) we find

\[
(\alpha - i \beta_2 \gamma_5) \eta = 0
\]

from which it follows, since \(\alpha\), \(\beta_2\) and \(\gamma_5\) are hermitian that

\[
\alpha \eta = \beta_2 \eta = 0
\]

Next we consider the \(n\)-components of (2). Taking our ansatz and substituting (19) we obtain

\[
\nabla_m \eta - \Lambda_1 \gamma_5 \otimes \gamma_m \eta - \frac{1}{12} \gamma_5 \otimes G_{mpqr} \gamma^{pqr} \eta - \frac{i}{2} \Lambda_2 \otimes \gamma_m \eta = 0
\]

\(^6\)Tensor products are to be understood throughout the remainder of the paper.

\(^7\)Reality follows from the Majorana condition on \(\eta \equiv \epsilon \otimes \theta\).
Contracting (20) with $\gamma^m$ and operating from the left with the Dirac operator, we find

$$\left(\nabla^2 + \frac{1}{4} R\right)\eta - 25\Lambda_1^2\eta + 35i\Lambda_1\Lambda_2\gamma_5\eta + \frac{49}{4}\Lambda_2^2\eta = 0$$

(21)

This equation implies that

$$\Lambda_1\Lambda_2 = 0$$

(22)

On the other hand, operating on (20) with $\nabla_n$, taking the skew-symmetric part and contracting the result with $\gamma^{mn}$, we obtain after some algebra and use of identities found in the appendix

$$\frac{1}{4} R = -12\Lambda_1^2 - \frac{21}{2}\Lambda_2^2 - \frac{7}{48}G^2_{pqrs}$$

(23)

In deriving the above a hermiticity argument also yields that

$$\nabla_n G^m_{pqrs} \gamma^{pqrs} \eta = 0$$

(24)

Combining equation (23) with (21) we obtain our main result:

$$\nabla^2\eta - \frac{7}{48}G^2_{pqrs}\eta - 37\Lambda_1^2\eta + \frac{7}{4}\Lambda_2^2\eta = 0$$

(25)

Recall (22) that either $\Lambda_1$ or $\Lambda_2$ must be zero. When $\Lambda_2$ is zero, the fact that $\nabla^2\eta$ is negative semi-definite implies that both $G$ and $\Lambda_1$ are also zero ie our first solution is summarised as

$$\nabla_m\eta = G_{pqrs} = m = \Lambda_1 = \Lambda_2 = 0$$

(26)

ie there is no flux and the pair $(M^7, g_7)$ are a $G_2$-holonomy 7-manifold. This is the solution of [8] stated earlier. Next we consider the case that $\Lambda_1$ is zero. In this case we have not been able to find a solution in which $G_{pqrs}$ is non-zero either. For instance, contraction of (18) with $\gamma^m$ yields in this case

$$\gamma^m\nabla_m\eta - \frac{7i}{2}\Lambda_2\eta = 0$$

(27)

This equation is implied by

$$\nabla_m\eta - \frac{i}{2}\Lambda_2\gamma_m\eta = 0$$

(28)

which obviously solves our main equation with $G_{pqrs} = 0$. Thus, we obtain our second solution which is summarised by the following three equations

$$G_{pqrs} = \Lambda_1 = 0$$

(29)
\[ m = \Lambda_2 \]  
and  
\[ \nabla_m \eta - \frac{i}{2} \Lambda_2 \gamma_m \eta = 0 \]  

This second solution is nothing but the original -well studied- supersymmetric Freund-Rubin ansatz.

In summary thus far, we have considered the most general metric on a spacetime of topology \( X = M^4 \times M^7 \). We assumed that the supersymmetry parameter is parallel in the unwarped metric. With this assumption, supersymmetry requires that the 11-metric on \( X \) is a Riemannian product i.e. that the warp factor is trivial.

With trivial warp factor, for completeness, we included a derivation of how supersymmetry then leads to two solution classes - a fact well known from the Kaluza-Klein supergravity studies, see [2] and references therein. Firstly when \((M^4, g_4)\) admits a parallel spinor, \((M^7, g_7)\) must do so as well and there is no background \(G\)-field. The second class of solutions are such that \((M^4, g_4)\) admits a non-trivial Killing spinor, in which case \((M^7, g_7)\) does too. In this case there is one component of \(G\) which is non-zero and proportional to the volume form of \(M^4\).
3. Relation to the Gukov Superpotential.

In [7] Gukov considered the relationship between calibrated submanifolds in $G_2$ holonomy manifolds, domain walls and the superpotential of the four dimensional theory. The superpotential he conjectured for $M$ theory compactified on a $G_2$ holonomy 7-manifold was obtained by extending an argument of Gukov, Vafa and Witten who studied $M$ theory on Calabi-Yau fourfolds. In [5] it was observed that the conditions for unbroken supersymmetry which follow from this conjectured form of the superpotential agree in the fourfold case with the constraints imposed by supersymmetry on the background four-form field strength which follow from solving (2) on fourfolds. This latter calculation was performed by Becker and Becker. We will consider a similar comparison here.

In [7] Gukov proposed a superpotential for compactifications of $M$ theory on a $G_2$ holonomy 7-manifold $M^7$. Such a manifold admits a parallel $G_2$-structure $\varphi$ (a locally $G_2$-invariant three form). The proposed superpotential is given by

$$W = \int_{M^7} G \wedge \varphi$$  \hspace{1cm} (32)

As it stands this cannot be precisely correct since the right hand side is manifestly real whereas in background Minkowski space the superpotential is a holomorphic function of the chiral superfields$^8$. This follows from the fact that the effective four dimensional supergravity has four dimensional $\mathcal{N} = 1$ supersymmetry. Recall [14, 15] that the massless complex scalar fields $\Phi_i$ in the low energy compactified theory are given by periods of the “complexified Bonan class” over a basis $\Sigma_j$ of $b_3(M^7)$ cycles spanning $H_3(M^7, \mathbb{Z})$:

$$\Phi_j = \int_{\Sigma_j} i\varphi + C$$  \hspace{1cm} (33)

where $C$ is the locally defined $M$ theory three-form potential $^9$

We should also consider the fact that $M$ theory compactified on a seven manifold of $G_2$-holonomy can possess $M2$-brane domain walls which reside at a point on the seven manifold. These are BPS saturated. Applying the arguments of [5, 7] to include the contribution from these walls, one must add a term in the superpotential proportional to $\int_{M^7} * G + G \wedge C$. The second

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$^8$Obviously, the value of the superpotential in a vacuum could be real.

$^9$The terminology “complexified Bonan class” stems from two facts: firstly that Bonan discovered that $G_2$-holonomy 7-manifolds admit a parallel, locally $G_2$ invariant three form [16] and secondly the analogy with “complexified Kahler class” in string theory on Calabi-Yau manifolds.
term is another reason why the superpotential is (33) and not (32). Lorentz invariance implies that the first term $\int_{M^7} G$ is proportional to $\text{Vol}(M^7)$.

Combining these observations we see that Gukov’s superpotential should properly be extended to

$$W(\Phi_j) = \text{Vol}(M^7) + \int_{M^7} G \wedge (i\varphi + C)$$

The real and imaginary parts correspond respectively to $M2$-brane and $M5$-brane domain walls in four dimensions. Note that the real part is the Page charge. In a four dimensional supergravity theory, the conditions for unbroken supersymmetry and zero cosmological constant are given by solutions to

$$W = dW = 0$$

Before we analyse these equations we must first make some remarks on the cohomology groups of $G_2$-holonomy 7-manifolds. Since the holonomy group is $G_2$, the spaces of tensor and spinor fields on $M^7$ decompose into irreducible $G_2$ representations (or more specifically modules). This decomposition, in the case of $\Lambda^*(TM^7)$ commutes with the Laplacian and hence descends to the cohomology groups of $M^7$. This gives the analogue [17] of the Hodge-Dolbeaut cohomology groups of a Kahler manifold. For instance, in the case of $H^4(M^7, \mathbb{Z})$ we have

$$H^4(M^7, \mathbb{Z}) = H^4_1(M^7, \mathbb{Z}) \oplus H^4_7(M^7, \mathbb{Z}) \oplus H^4_{27}(M^7, \mathbb{Z})$$

The subscripts denote the irreducible representations of $G_2$. Note that the above corresponds to the decomposition of the space of four-forms, which on a generic 7-manifold is a 35 of $SO(7)$, under $G_2$:

$$35 \longrightarrow 1 + 7 + 27$$

One should also note that Gukov’s potential should really be regarded as giving information about the cohomology class of $G$, since it was derived by considering the charges of domain walls and these are determined by cohomology classes. By abuse of notation below we will denote the cohomology class of $G$-field by $G$.

The first condition in (35) implies that

$$G \wedge \varphi = 0$$

This means that the singlet piece of $G$ under $G_2$ vanishes. We also have from the vanishing of the real part of $W$ that

$$\int G \wedge C + \text{Vol}(M^7) = 0$$
which states that the Page charge vanishes. We will return to this condition momentarily.

The variations of the real parts of the superfields correspond to variations of $\varphi$ in response to changes in the metric tensor. As is well known [18, 17], these generate $H^{3}_{27}(M^{7}, \mathbb{Z})$. Hence, this implies that the piece of $G$ transforming as a $27$ of $G_2$ is also zero. Finally, for manifolds whose holonomy is strictly $G_2$ and not a subgroup $H^{4}_{7}(M^{7}, \mathbb{Z})$ is trivial [17], since such manifolds have finite fundamental group. Hence, (the cohomology class of) $G$ is identically zero in supersymmetric backgrounds of low energy $M$ theory on $G_2$-holonomy 7-manifolds. With $G$ trivial in cohomology, the first term in the Page charge is zero and hence $m$ must also be zero.

Thus, we have confirmed that Gukov’s superpotential gives the same answer as the equations which follow from eleven dimensional supergravity.

For AdS space we have not been able to find a $G_2$ holonomy solution so we cannot compare to Gukov’s work.
4. Discussion.

Thus far we have only discussed the eleven dimensional supergravity approximation to $M$ theory. This receives quantum corrections which are higher order in derivatives. In particular, consider the equation of motion for the three-form potential. Schematically this has the form

$$d \ast G = G \wedge G$$

in the eleven dimensional supergravity theory. Both our solutions satisfy this equation. However, in $M$ theory there are corrections to this equation. To conclude, we will discuss these corrections in a very speculative light.

Our first interest will be in the topologically non-trivial corrections to this equation of which one term is known

$$d \ast G = G \wedge G + X_8$$

where $X_8$ represents, when restricted to any eight dimensional submanifold, $\frac{1}{24}$ times its Euler density. In fact following the work of [12] it is very tempting to believe that $X_8$ is the only cohomologically non-trivial correction to the equation of motion\textsuperscript{10}.

At the order in the derivative expansion at which the $X_8$ correction arises, apparently one cannot write down an action which is invariant under the supersymmetry transformation rules of the eleven dimensional supergravity theory [19]. Rather, the supersymmetry transformations themselves need to be corrected. We can ask whether or not the background field strength is zero in this corrected theory. In other words, is $G = 0$ implied by supersymmetry at higher orders in the derivative expansion? If the answer to that question is yes, then we arrive at an interesting conclusion: the equation of motion above gives a topological constraint on spacetime, since in that case,

$$X_8 = 0$$

in cohomology. For instance it implies that the Euler number of any 8-dimensional submanifold of spacetime is zero. If we now consider the path integral for $M$ theory regarded in a low energy approximation as a sum over spacetime topologies then we might come across an eleven manifold $M^4 \times M^7$ which admits parallel spinors but does not satisfy the above equation. Thus for such a manifold, we necessarily have to turn on $G$ to compensate for $X_8$ - but this breaks supersymmetry with our assumptions. Thus spacetime

\textsuperscript{10}The first author is indebted to G. Moore for discussions on this point.
topology might in this sense break supersymmetry. In the appendix we discuss examples of eleven manifolds with these properties.

However, a much more likely scenario is that at next non-trivial order in the derivative expansion, one can turn on $G$ whilst still maintaining supersymmetry. One would also require a modification of the $G_2$ holonomy metric. This scenario is much more in line with a similar discussion in the context of strongly coupled heterotic string theory on Calabi-Yau manifolds [13]. If a similar story exists at higher order there is also a possible scenario for low energy supersymmetry breaking.

Assume that at higher order in derivatives one has a supersymmetric solution of $M$ theory in which spacetime is topologically a product $M^4 \times M^7$ where the four manifold is Minkowski space and the 7-manifold admits $G_2$-holonomy metrics, even though the higher derivative solution does not have $G_2$-holonomy. This spacetime has trivial eighth de Rham cohomology group, so $X_8$ in this case is necessarily trivial. Then it is conceivable that the solution might in fact obey the lowest order equation of motion (38) and that the higher derivative corrections cancel amongst themselves. In such a scenario, a low energy (ie eleven dimensional supergravity) observer would observe that supersymmetry is in fact broken, whereas a high energy ($M$ theory) observer would not. These speculations are currently undergoing a more scientific investigation [20].

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5. Appendix.

We follow the conventions of [8]. These are related to the conventions of [2] by multiplying the gamma-matrices by $i$. The spacetime eleven manifold is denoted by $X$. This is always topologically the product of a (Lorentzian) four manifold $M^4$ and a (Euclidean) 7-manifold $M^7$.

* $M$, $N$, etc denote eleven dimensional world or tangent space indices.
* $\mu$, $\nu$ etc. denote four dimensional indices.
* $m$, $n$, $p$ etc denote seven dimensional indices.

The local coordinates of $M^4$ are denoted collectively by $x$ whilst those of $M^7$ are denoted by $y$.

The gamma-matrices $\Gamma_M$ are hermitian for $M = 1, \ldots, 10$ whereas $\Gamma_0$ is anti-hermitian. They obey

\[
\{\Gamma_M, \Gamma_N\} = 2g_{MN} \tag{43}
\]

where the metric has signature $(-, +, +, \ldots, +)$.

A decomposition of the matrices $\Gamma_M$ into gamma-matrices appropriate to $M^4$ and $M^7$ is

\[
\Gamma_{\mu} = i\gamma_{\mu} \otimes 1, \quad \Gamma_m = \gamma_5 \otimes \gamma_m \tag{44}
\]

where

\[
\gamma_5 = \frac{i}{4!} \epsilon_{\mu\nu\rho\sigma} \gamma^{\mu} \gamma^{\nu} \gamma^{\rho} \gamma^{\sigma} \tag{45}
\]

where $\gamma_5$ squares to one.

Defining

\[
G_n \equiv G_{npqr} \gamma^{pqrs}, \quad G \equiv G_{pqrs} \gamma^{pqrs} \tag{46}
\]

the identities we used in the calculation are:

\[
\gamma^m \gamma^n G_m G_n = -6G_n^2 - G^2 \tag{47}
\]
\[
\gamma^{mn} G_m G_n = -7G_n^2 - G^2 \tag{48}
\]
\[
\gamma^{mn} G_m \gamma_n = 0 \tag{49}
\]
\[
\gamma^{mn} \gamma_0 G_m = 6G \tag{50}
\]
\[
G_m^2 = -\frac{1}{8}G^2 - 3G_{pqrs}^2 \tag{51}
\]

As a simple example of an eleven manifold satisfying the criteria required in the discussion of section four, consider a sector of the Euclidean path
integral in which $M^4$ is a multi-center gravitational instanton and $M^7$ is a $G_2$-holonomy 7-manifold with finite but non-trivial fundamental group. Both types of manifolds exist. Then the product eleven-manifold admits 8-cycles of the form $S^2 \times N^6$ where $N^6$ is a 6-cycle in $M^7$ with non-zero Euler number. Another possible source of examples in the Lorentzian theory concerns four manifolds $(M^4, g_4)$ which have non-trivial $\mathbb{R}^2$ holonomy, as discussed in [21]. One again requires examples with non-trivial two-cycles of non-zero Euler number. We do not know of any examples of this type.

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