Connections between deep-inelastic and annihilation processes at next-to-next-to-leading order and beyond\textsuperscript{1}

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Abstract We have discovered 7 intimate connections between the published results for the radiative corrections, $C_K$, to the Gross–Llewellyn Smith (GLS) sum rule, in deep-inelastic lepton scattering, and the radiative corrections, $C_R$, to the Adler function of the flavour-singlet vector current, in $e^+e^-$ annihilation. These include a surprising relation between the scheme-independent single-electron-loop contributions to the 4-loop QED $\beta$-function and the zero-fermion-loop abelian terms in the 3-loop GLS sum rule. The combined effect of all 7 relations is to give the factorization of the 2-loop $\beta$-function in

$$\Delta_S \equiv C_K C_R - 1 = \frac{\beta(\overline{\alpha_s})}{\overline{\alpha_s}} \left( S_1 C_F \overline{\alpha_s} + [S_2 T_F N_F + S_A C_A + S_F C_F] C_F \overline{\alpha_s}^2 \right) + O(\overline{\alpha_s}^4),$$

where $\overline{\alpha_s} = \alpha_s (\mu^2 = Q^2) / 4\pi$ is the $\overline{MS}$ coupling of an arbitrary colour gauge theory, and

$S_1 = -\frac{21}{2} + 12\zeta_3; \quad S_2 = \frac{326}{3} - \frac{304}{3} \zeta_3; \quad S_A = -\frac{320}{2} + \frac{884}{3} \zeta_3; \quad S_F = \frac{397}{6} + 136\zeta_3 - 240\zeta_5$

specify the sole content of $C_K$ that is not already encoded in $C_R$ and $\beta(\overline{\alpha_s}) = Q^2 d\overline{\alpha_s} / dQ^2$ at $O(\overline{\alpha_s}^4)$. The same result is obtained by combining the radiative corrections to Bjorken’s polarized sum rule with those for the Adler function of the non-singlet axial current. We suggest possible origins of $\beta$ in the ‘Crewther discrepancy’, $\Delta_S$, and determine $\Delta_S / (\beta(\overline{\alpha_s}) / \overline{\alpha_s})$, to all orders in $N_F \overline{\alpha_s}$, in the large-$N_F$ limit, obtaining the entire series of coefficients of which $S_1$ and $S_2$ are merely the first two members.

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1 Introduction

In 1972, Crewther [1] related three fundamental constants of an arbitrary parton model: the anomalous constant $S$, associated with the amplitude for $\pi^0 \rightarrow \gamma\gamma$ decay [2]; the coefficient $K$ in Bjorken’s sum rule for polarized deep-inelastic electron scattering [3]; and the constant $R'$ in the annihilation channel, giving the asymptotic value of the Adler function [4] for the correlator of the iso-vector axial current. His non-perturbative derivation relied on conformal and chiral invariance of the leading short-distance singularity, with coefficient $S$, in the operator product expansion (OPE) of the 3-point function $AVV$, for $\pi^0$ decay, where $A = J_5^\mu$ is the iso-vector axial current and $V = J_{EM}^\mu$ is the electromagnetic current. To obtain $3S = KR'$ [1], one first takes the OPE of the 2-point function $VV$, in which one encounters the axial current, $A$, with the coefficient $K$ of Bjorken’s polarized sum rule. Then one obtains $R'$ in the leading term of the resultant OPE of the $AA$-correlator, corresponding to the Adler function of the iso-vector axial current.

The relation $3S = KR'$ is, necessarily, satisfied by the standard quark-parton model, which gives $S = \frac{1}{2}$, $K = 1$, $R' = \frac{3}{2}$, for an iso-doublet of $u$ and $d$ quarks, each having $N_C = 3$ colours. The chiral symmetry of the quark-parton model means that $K$ also occurs in the Gross–Llewellyn Smith (GLS) sum rule [5] of deep-inelastic neutrino scattering, corresponding to the coefficient of the vector current in the vector-axial correlator. Invoking both chiral symmetry and SU(3)-flavour symmetry [1], one obtains $R' = \frac{3}{4}R$, where $R = 2$ corresponds to the Adler function of the $VV$-correlator, giving the quark-parton model prediction for $e^+e^-$ annihilation (below the charm threshold).

It is not at all clear what the theoretical status and phenomenological consequences of the Crewther connection might be in QCD, where radiative corrections to the naive quark-parton model give a dependence on the running coupling $\alpha_s(\mu^2) = Q^2/4\pi$ that is appreciable at presently accessible values of $-q^2 \equiv Q^2$. In this paper we study radiative corrections to deep-inelastic [6] and annihilation [7] processes, at next-to-next-to-leading order (NNLO) in the \(\overline{\text{MS}}\) scheme, discovering that they are intimately connected, in a manner that is profoundly related to the Crewther connection [1].

2 Deep-inelastic and annihilation results at NNLO

In deep-inelastic lepton scattering, radiative corrections to the GLS sum rule [6],

$$\frac{1}{2} \int_0^1 dx \ F_3^{ep+\nu p}(x, Q^2) = 3 C_{\text{GLS}}(\alpha_s),$$

and to Bjorken’s polarized sum rule [8],

$$\int_0^1 dx \ g_1^{ep-en}(x, Q^2) = \frac{1}{3} \left| \frac{g_s}{g_V} \right| C_{\text{Bj}}(\alpha_s),$$

have been obtained [8] to NNLO in the \(\overline{\text{MS}}\) scheme, where the dependence on $Q^2 = -q^2$ is absorbed, at large $Q^2$, into the coupling $\alpha_s(\mu^2) = Q^2/4\pi$. A difference between $C_{\text{GLS}}$ and $C_{\text{Bj}}$ is first encountered at $O(\alpha_s^3)$, where so called ‘light-by-light-type’ diagrams contribute to $C_{\text{GLS}}$, but not to $C_{\text{Bj}}$. 

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In annihilation, at large $Q^2$, the Adler functions $D^V_{\text{EM}}$ and $D^A_{\text{NS}}$ have been calculated \[^4\] at NNLO, again in the $\overline{\text{MS}}$ scheme, for the electromagnetic current $J^\mu_{\text{EM}} = \sum_f Q_f \overline{\psi}_f \gamma^\mu \psi_f$, where the sum is over $N_F$ flavours of quark, with charges $Q_f$, and for the non-singlet axial current $J^\mu_a = \overline{\psi}_i \gamma^\mu \gamma_5 \psi_j$, where $i$ and $j$ are different quark flavours. In \[^6\] it was assumed that $D^A_{\text{NS}}$, relevant to $\tau$ decay, is identical to the Adler function $D^V_{\text{NS}}$, of the non-singlet vector current, whose radiative corrections are obtained by omitting the NNLO light-by-light-type terms that contribute to $D^V_{\text{EM}}$. Analytic continuation of $D^V_{\text{EM}}$ and $D^A_{\text{NS}}$ to the time-like region yields contributions to the processes $e^+e^- \rightarrow$ hadrons and $\tau \rightarrow \nu_\tau +$ hadrons. Note that the correlator of the singlet axial current, related to $Z^0 \rightarrow$ hadrons, receives anomalous contributions from diagrams with gluons in the intermediate state, considered in \[^5\].

To investigate the possibility of a perturbative Crewther connection, in colour gauge theories, we study the radiative corrections

\begin{align}
C_K &\equiv C_{\text{GLS}}(\overline{\alpha}_s) = 1 - 3C_F \overline{\alpha}_s + O(\overline{\alpha}_s^2), \\
C_R &\equiv \frac{D^V(\overline{\alpha}_s)}{N_F N_C} = 1 + 3C_F \overline{\alpha}_s + O(\overline{\alpha}_s^2), \quad D^V \equiv -12\pi^2 Q^2 \frac{d\Pi^V}{dQ^2},
\end{align}

to the GLS sum rule \[^1\] and the Adler function, $D^V$, of the correlator $(g_{\mu\nu} - q^2 g_{\mu\nu})\Pi^V$ of the flavour-singlet vector current $J_\mu = \sum_f \overline{\psi}_f \gamma^\mu \psi_f$, with $N_F$ active quark flavours.

The NNLO results for $C_K$ \[^3\] and $C_R$ \[^2\], in the $\overline{\text{MS}}$ scheme, are given in Table 1, for an arbitrary colour gauge group. (The colour factors take the values $T_F = \frac{1}{2}$, $C_A = N_C = 3$, $C_F = \frac{4}{3}$, $d^2_{abc} = \frac{40}{3}$, in the particular case of QCD.) To obtain the radiative corrections \[^4\] to the flavour-singlet Adler function $D^V$, one has merely to give the quarks equal charges, $Q_f = \text{constant}$, in $D^V_{\text{EM}}$, corresponding to setting $(\sum_f Q_f)^2 = N_F^2 \sum_f Q_f^2$ in the results of \[^6\]. (Note that we consider $D^V$ at large space-like $Q^2 = -q^2$ and hence omit the $\pi^2$ terms of Eq. (12) of \[^7\], which result from analytic continuation to the time-like region.)

The NNLO radiative corrections to $C_{\text{Bj}}$, are obtained by dropping light-by-light-type terms, proportional to $d^2_{abc}$, from $C_K \equiv C_{\text{GLS}}$; the corrections to $D^A_{\text{NS}}/N_C = D^V_{\text{NS}}/N_C$ are obtained by dropping them from $C_R \equiv D^V/N_F N_C$.

At first glance, one is tempted to conclude from \[^1\] that the radiative corrections \[^3\][^4] might give a product, $C_K C_R$, that is free of radiative corrections, in the spirit of the non-renormalization theorem \[^10\] for the axial anomaly that determines $\pi_0$ decay in the zero-mass limit. However, the NLO corrections to $C_K$ \[^11\] and $C_R$ \[^12\] give $C_K C_R \neq 1$, when one absorbs the $\alpha_s^2 \ln(Q^2/\mu^2)$ term of each process in the $\overline{\text{MS}}$ coupling $\overline{\alpha}_s = \alpha_s(\mu^2 = Q^2)/4\pi$. The recent availability of highly non-trivial NNLO results, for both $C_K$ \[^2\] and $C_R$ \[^6\], prompted us to study the ‘Crewther discrepancy’ $\Delta_S \equiv (C_K C_R - 1)$, at $O(\overline{\alpha}_s^4)$.

## 3 Anatomy of a discovery

After much investigation of Table 1, we discovered the following remarkable relation between the $\overline{\text{MS}}$ results of \[^3\][^7], for any colour gauge theory, renormalized at $\mu^2 = Q^2$:

\[
\Delta_S \equiv C_K C_R - 1 = \frac{\beta(\overline{\alpha}_s)}{\overline{\alpha}_s} \left\{ S_1 C_F \overline{\alpha}_s + \left[ S_2 T_F N_F + S_A C_A + S_F C_F \right] C_F \overline{\alpha}_s^2 \right\} + O(\overline{\alpha}_s^4),
\]

(5)
where $\beta(\bar{\alpha}_s) \equiv Q^2 d\bar{\alpha}_s/dQ^2 = \bar{\alpha}_s \sum_{n \geq 1} \beta_n \bar{\alpha}_s^n$, and

$$S_1 = -\frac{21}{2} + 12\zeta_3; \quad S_2 = \frac{326}{3} - \frac{304}{3}\zeta_3; \quad S_A = -\frac{629}{6} + \frac{884}{3}\zeta_3; \quad S_F = \frac{397}{6} + 136\zeta_3 - 240\zeta_5$$

specify the sole NNLO content of $C_K$ that is not derivable from $C_R$ and from the coefficients

$$\beta_1 = -\frac{11}{3}C_A + \frac{4}{3}T_F N_F; \quad \beta_2 = -\frac{21}{3}C_A^2 + \frac{29}{3}C_A T_F N_F + 4C_T T_F N_F$$

of the two-loop $\beta$-function. The same result is obtained by combining the NNLO corrections to Bjorken’s polarized sum rule (2) with those for the Adler function $D_{\beta S\beta}$ of the non-singlet axial current, which differ from $C_K$ and $C_R$, respectively, merely by omitting $d_{abc}^2 \bar{\alpha}_s^3$ terms that cancel in (3).

Since $C_K$ and $C_R$, taken up to $O(\bar{\alpha}_s^3)$, each involve the 11 distinct colour factors of the terms $\{T_n | n = 1, 11\}$, defined in Table 1, the existence of a relation of the form of (4) entails the following ‘seven wonders’ of the Crewther discrepancy $\Delta_S \equiv (C_K C_R - 1)$:

1. The leading-order terms cancel in $\Delta_S$.
2. The NLO corrections give no $C_A^2 \bar{\alpha}_s^2$ term in $\Delta_S$.
3. The NLO corrections give $C_F C_A \bar{\alpha}_s^2$ and $C_F T_F N_F \bar{\alpha}_s^2$ terms in $\Delta_S$ that are in the same ratio as the $C_A$ and $T_F N_F$ terms in $\beta_1$.
4. The NNLO corrections give no $C_F^3 \bar{\alpha}_s^3$ term in $\Delta_S$. This leads to the astonishing observation that the scheme-independent single-electron-loop contributions, $\beta_{\text{QED}}^{[1]}$, to the QED $\beta$-function, are obtained, up to 4-loops, by taking the reciprocal of the zero-fermion-loop abelian terms in the 3-loop GLS result of (5), giving

$$\beta_{\text{QED}}^{[1]}(a) = \frac{\frac{4}{3}a^2}{1 - 3a + \frac{21}{2}a^2 - \frac{3}{2}a^3 + O(a^4)} = \frac{4}{3}a^2 + 4a^3 - 2a^4 - 46a^5 + O(a^6),$$

in precise agreement with (13). Such is the power of relation (6).
5. The NNLO light-by-light-type terms of (3) and (5), involving $T_{11} \equiv \frac{N_c}{N_c} d_{abc} \bar{\alpha}_s^3$, cancel in $\Delta_S$ (taking equal quark charges in (7), to obtain the singlet Adler function $D^V$).
6. The NNLO corrections in $\Delta_S$ are expressible as the sum of multiples of the one-loop and two-loop contributions, $\beta_1 \bar{\alpha}_s^2$ and $\beta_2 \bar{\alpha}_s^3$, to the $\beta$-function.
7. At NNLO, $\beta_2 \bar{\alpha}_s^3$ occurs in $\Delta_S$ with the same coefficient that multiplies $\beta_1 \bar{\alpha}_s^2$ at NLO, allowing one to factor out $\beta(\bar{\alpha}_s)$ in (3). Moreover, this factorization is independent of the momenta in the two processes: if one takes the GLS sum-rule results at a momentum transfer $-q^2 = Q_R^2$, and the Adler function $D^V$ at $-q^2 = Q_R^2$, the factorization of (3) still occurs, with the replacements $\bar{\alpha}_s \rightarrow \alpha_s(\mu^2 = Q_R^2)/4\pi$ and

$$S_1 \rightarrow S_1 - 3\lambda; \quad S_2 \rightarrow S_2 + 16\lambda - 4\lambda^2; \quad S_A \rightarrow S_A - 46\lambda + 11\lambda^2; \quad S_F \rightarrow S_F + 12\lambda;$$

where $\lambda = \ln(Q_K^2/Q_R^2)$. [9]
The corresponding 7 relations between the coefficients of $C_K = \sum k_n T_n + O(\bar{\pi}_s^4)$ and $C_R = \sum r_n T_n + O(\bar{\pi}_s^4)$, given in Table 1, can be divided into two groups. Observations 1,2,4,5, above, correspond to the 4 conditions

$$0 = k_1 + r_1 = k_2 + r_2 + k_1 r_1 = k_5 + r_5 + k_1 r_2 + r_1 k_2 = k_{11} + r_{11},$$

which are required to ensure the absence of terms in $\Delta_S$ that cannot occur in the factorization (10). Observations 3,6,7, above, correspond to the 3 remaining conditions

$$-\frac{3}{11}(k_3 + r_3) = \frac{3}{4}(k_4 + r_4)
\quad = \frac{1}{7}(k_7 + r_7) + \frac{1}{86}(k_9 + r_9) + \frac{12}{112}(k_{10} + r_{10})
\quad = \frac{1}{11}(k_6 + r_6 + k_1 r_3 + r_1 k_3) + \frac{1}{4}(k_8 + r_8 + k_1 r_4 + r_1 k_4),$$

which relate equivalent ways of evaluating $S_1 = -\frac{21}{2} + 12\zeta_3$, consistent with (10). We invite any reader who may still doubt the significance of the factorization of the two-loop $\beta$-function in (3) to use the coefficients of Table 1 to verify that the 7 necessary conditions in (10) are satisfied in a highly non-trivial manner.

We relate our discovery (3) to (1) by observing that it seems rather natural to obtain $C_K C_R = 1$ at any fixed point, where $\beta = 0$, since Crewther’s assumptions of conformal and chiral invariance should hold in that scale-free limit. We note that the anomalous dimension of the pseudo-scalar operator $G^{\mu\nu}\bar{G}_{\mu\nu}$, in the anomalous divergence of the singlet axial current, is $-\beta(a_s)/a_s$ (14). This makes it reasonable that $C_K C_R = 1$, when $\beta = 0$, corresponding to no renormalization of the anomaly at any fixed point, and hence suggesting that one may expect to find a Crewther discrepancy $\Delta_S \equiv (C_K C_R - 1) \propto \beta$. (See (14, 13) for recent studies of corrections to the one-loop axial anomaly equation.)

Though we cannot yet be sure that $\beta$ can be factored out of $(C_K C_R - 1)$, beyond NNLO, it seems most likely to us that at any given order, $\bar{\pi}_s^n$, one will encounter in $\Delta_S$ only the coefficients $\{\beta_n | n < N\}$, multiplied by linear combinations of colour factors. We believe that it is the scale-dependent procedure of renormalization that modifies the naive result $\Delta_S = (C_K C_R - 1) = 0$, suggested by the conformal arguments of (1) and the essentially one-loop nature of the anomaly (10). Since the coefficients $\beta_n$ multiply all scale-dependent perturbative artefacts of the renormalization procedure, one expects them to occur in $\Delta_S$. This does not, of itself, require that $\Delta_S \propto \beta$. However, the existence of the relations (11), between the highly non-trivial coefficients of Table 1, may be taken as powerful circumstantial evidence in favour of this stronger hypothesis.

We leave these considerations for later work and now obtain all the $O(1/N_F)$ terms of

$$\frac{C_K C_R - 1}{\beta(\bar{\pi}_s)/\bar{\pi}_s} = \frac{C_F}{T_F N_F} \sum_{n=1}^{\infty} S_n (T_F N_F \bar{\pi}_s)^n + O(1/N_F^2),$$

in the $\overline{\text{MS}}$ scheme, taking the limit $N_F \rightarrow \infty$, with $N_F \bar{\pi}_s$ fixed. In obtaining all the coefficients $S_n$, we provide an all-orders consistency test of the procedures of (3).

4 All-orders results at large $N_F$

We follow (3, 4, 14, 16, 17, 18) in defining an axial current, within the framework of dimensional regularization, by calculating Green functions of the renormalized non-singlet
antisymmetric-tensor current

\[ A_{\kappa \lambda \mu} \equiv \frac{1}{2} Z_A \bar{\psi}_i (\gamma_\kappa \gamma_\lambda \gamma_\mu - \gamma_\mu \gamma_\lambda \gamma_\kappa) \psi_j, \quad Z_A = 1 + \sum_{n=1}^{\infty} a_s^n \sum_{p=0}^{n-1} \frac{Z_{n,p}}{\varepsilon^p} \]  \hspace{1cm} \text{(13)}

where \( i \) and \( j \) are different quark flavours, \( d \equiv 4 - 2\varepsilon \) is the spacetime dimension, \( a_s = \alpha_s / 4\pi \) is the \( \overline{\text{MS}} \) coupling, renormalized at scale \( \mu \), and \( Z_A \) is a non-minimal renormalization constant, constructed so as to preserve chiral symmetry and to have vanishing anomalous dimension. The condition

\[ \frac{d \ln Z_A}{d \ln \mu^2} = \left( -\varepsilon + \frac{\beta(a_s)}{a_s} \right) \frac{d \ln Z_A}{d \ln a_s} = O(\varepsilon) \]  \hspace{1cm} \text{(14)}

thus relates the singular terms in (13) to the finite terms, giving NNLO singular terms

\[ Z_{2,1} = \frac{1}{7} \beta_1 Z_{1,0}, \quad Z_{3,2} = \frac{1}{3} \beta_1^2 Z_{1,0}, \quad Z_{3,1} = \frac{1}{3} \beta_2 Z_{1,0} + \frac{1}{6} \beta_1 (Z_{1,0}^2 + 4Z_{2,0}), \]  \hspace{1cm} \text{(15)}

in terms of \( \beta_1 \) and \( \beta_2 \), in (11), and the NLO finite terms (10).

\[ Z_{1,0} = -4C_F, \quad Z_{2,0} = 22C_F^2 - \frac{197}{9} C_F C_A + \frac{1}{3} C_F T_F N_F. \]  \hspace{1cm} \text{(16)}

The procedure for obtaining Green functions involving the non-singlet axial current is to combine the non-minimal renormalization of (13) with the standard \( \overline{\text{MS}} \) renormalization of the bare coupling constant, \( g_0 \). One then subtracts any polynomial in the momenta that is singular at \( \varepsilon = 0 \), and takes the limit \( \varepsilon \to 0 \). Thereafter, one multiplies by the appropriate number of 4-dimensional Levi-Civita tensors (one for each axial vertex), to obtain renormalized Green functions of the conventional 4-dimensional axial current, which may be written schematically as

\[ A_\nu = \bar{\psi}_i \gamma_\nu \gamma_5 \psi_j \cong \frac{1}{3!} \varepsilon_{\kappa \lambda \mu \nu} \lim_{d \to 4} \left[ \frac{1}{2} Z_A \bar{\psi}_i (\gamma_\kappa \gamma_\lambda \gamma_\mu - \gamma_\mu \gamma_\lambda \gamma_\kappa) \psi_j \right], \]  \hspace{1cm} \text{(17)}

with the limit \( d \to 4 \) taken after all renormalization. Before renormalization, all reference to the Levi-Civita tensor, and hence to \( \gamma_5 \), is resolutely avoided. This enables covariant \( d \)-dimensional calculation, albeit at the expense of large traces over \( \gamma \)-matrices. With traces involving only an even number of axial vertices, it is presumed (19) that all physical results are identical to those that would have been obtained by naively using the 4-dimensional identities \( \gamma_\mu \gamma_5 = -\gamma_5 \gamma_\mu \) and \( \gamma_5^2 = 1 \). We now test this in an all-orders calculation.

### 4.1 Annihilation processes

The renormalized correlator of the vector current \( V_\mu \equiv \bar{\psi}_i \gamma_\mu \psi_j \) may be written as

\[ i \int dx \, e^{ix \cdot \xi} \langle T \{ V_\mu(x)V_\nu^\dagger(0) \} \rangle = -q^2 g_\nu^\mu \Pi_F^\nu + q^\nu q_\nu (\Pi_L^\nu + \Pi_T^\nu), \]  \hspace{1cm} \text{(18)}

with \( \Pi_L^\nu = 0 \), for massless quarks. The decomposition of the correlator of (13) may be written as

\[ i \int dx \, e^{ix \cdot \xi} \langle T \{ A^{\alpha \beta \gamma}(x)A_\alpha^\dagger(x) \} \rangle = -q^2 G_{\alpha \beta \gamma}^\nu \Pi_F^\nu + G_{\alpha \beta \gamma}^\nu q^\nu (\Pi_L^\nu + \Pi_T^\nu), \]  \hspace{1cm} \text{(19)}
with a tensor structure given by the determinants

\[ G^{\alpha \beta \gamma}_{\kappa \lambda \mu} \equiv \left| \begin{array}{ccc} g_\kappa & g_\lambda & g_\mu \\ g_\kappa & g_\lambda & g_\mu \\ g_\kappa & g_\lambda & g_\mu \end{array} \right|, \quad C^{\alpha \beta \gamma \delta}_{\kappa \lambda \mu \nu} \equiv \left| \begin{array}{cccc} g_\kappa & g_\lambda & g_\mu & g_\nu \\ g_\kappa & g_\lambda & g_\mu & g_\nu \\ g_\kappa & g_\lambda & g_\mu & g_\nu \\ g_\kappa & g_\lambda & g_\mu & g_\nu \end{array} \right|. \]  

(20)

For massless quarks, chiral symmetry requires that \( \Pi^A = (\Pi^A_T - \Pi^V) \) be constants. Note that they may be non-zero, since the non-minimal renormalization of \( (13) \), combined with minimal subtraction of infinities, may still leave chiral-symmetry-breaking finite terms in the renormalized expressions for divergent quantities. Only the subtraction-free non-singlet Adler functions are required to be equal:

\[ D^A_{NS} = -12\pi^2 Q^2 \frac{d\Pi^A}{dQ^2} = -12\pi^2 Q^2 \frac{d\Pi^V}{dQ^2} = D^V_{NS} = N_C (1 + 3C_F \alpha_s + O(\alpha_s^2)). \]  

(21)

In \[ 17 \] it was shown that the equality of \( D^A_{NS} \) and \( D^V_{NS} \), at the two-loop level, requires the leading-order renormalization \( Z_A = 1 - 4C_F \alpha_s + O(\alpha_s^2) \), which then gives an infinite NLO renormalization in \( (13) \), with \( Z_{2,1} = -2\beta_0 C_F \), according to \( (15) \). We now investigate the situation, to all orders in the coupling \( \alpha_s \), in the large-\( N_F \) limit.

As \( N_F \to \infty \), with \( N_F \alpha_s \) fixed, all radiative corrections to the parton model are suppressed by at least one factor of \( 1/N_F \). From the \( O(1/N_F) \) corrections to \( D^V_{NS} \), we obtain those in

\[ C_R \equiv \frac{D^V}{N_F N_C} = \frac{D^V_{NS}}{N_C} + O(1/N_F^2) = 1 + \frac{C_F}{T_F N_F} \sum_{n=1}^{\infty} R_n(T_F N_F \alpha_s)^n + O(1/N_F^2), \]  

(22)

in the \( \overline{\text{MS}} \) scheme, with \( O(1/N_F) \) coefficients given in closed form by

\[ R_n = \frac{3}{2} 4^n (n-1)! \sum_{p=1}^{n} \frac{(-5)^{n-p}}{(n-p)! (p-1)!} \Psi_{p+1}^{[p]}, \]  

(23)

in terms of the recently obtained momentum-scheme coefficients \[ 24 \]

\[ \Psi_n^{[n-1]} = \frac{(n-1)!}{(-3)^{n-1}} \left[ -2n + 4 - \frac{n+4}{2^n} + \frac{16}{n-1} \sum_{n/2 > s > 0} s \left( 1 - 2^{-2s} \right) \left( 1 - 2^{2s-n} \right) \zeta_{2s+1} \right], \]  

(24)

that specify the \( O(1/N_F) \) terms of \( \Psi \equiv \beta_{\text{MOM}} \), the QED \( \beta \)-function in the momentum (MOM) subtraction scheme. The all-orders result \( (24) \) reproduces the \( O(1/N_F) \) 4-loop results of \( (13) \), for \( n \leq 4 \). To transform to the \( \overline{\text{MS}} \) Adler-function coefficients \( (24) \), one has merely to observe that, at \( O(1/N_F) \), a MOM-scheme subtraction at \( -q^2 = Q^2 \) is equivalent to a \( \overline{\text{MS}} \) renormalization at \( \mu^2 = e^{-5/3} Q^2 \). From \( (23,24) \), we readily obtain the results for \( R_n \) in Table 2. The first 3 coefficients agree with \[ 4 \]; the remainder are new.

### 4.2 Axial renormalization constant

We determine the renormalization constant of \( (13) \), at \( O(1/N_F) \), by imposing the chiral-symmetry relations \( D^A_{NS} = D^V_{NS} \) and \( d\Pi^A_T/dQ^2 = 0 \). First we calculate the contributions
to the axial correlator of the generic $n$-loop bare diagrams with $n - 1$ quark loops, keeping $n$ as an algebraic variable. This result involves an $F_{3,2}$ hypergeometric function, whose expansion about $\varepsilon = 0$ cannot be effected in terms of $\zeta$-functions. Fortunately, coupling-constant renormalization ensures that one needs the function only in the limit $\varepsilon \to 0$, with $n\varepsilon$ fixed, where it is a tri-gamma function whose expansion yields $\zeta$-functions [21].

Analyzing the residual, analytically simpler, bare contributions, we encounter the 3 anticipated problems that must be solved by the non-minimal renormalization (13): there are non-subtractable singular bare terms, involving $\ln(Q^2/\mu^2)/\varepsilon$; the bare transverse axial and vector contributions differ by logarithmic terms; the bare longitudinal axial contributions also have a logarithmic $Q^2$-dependence.

For these problems, a single cure is available: the renormalization (13), which acts only on the one-loop term, at $O(1/N_F)$. At $n + 1$ loops, only one new constant is at our disposal: the leading term in the large-$N_F$ expansion of the coefficient $Z_{n,0}$. Following the methods of [21], we have explicitly verified that this suffices to solve all 3 problems. The required all-orders solution to (13) is

$$Z_A = 1 + \frac{C_F \varepsilon}{6 T_F N_F} \hat{L} \left\{ \frac{\ln(1 - \frac{4}{3} T_F N_F a_s/\varepsilon)}{B(2 - \varepsilon, 2 - \varepsilon) B(3 - \varepsilon, 1 + \varepsilon)} \right\} + O(1/N_F^2),$$

(25)

where $\hat{L}$ is the Laurent operator, which removes non-singular terms from the perturbative expansion of the term in braces, in accordance with the Ansatz of (13). Note that the Euler $B$-functions, in (25), result from the residue at $n = 1$ of the analytical expression for the bare $n$-loop contribution, in much the same way that the $O(1/N_F)$ QED $\beta$-function of any MS-like scheme results from a residue at $n = 0$ [24, 22]. Expanding (25) to order $a_s^3$, we verify the large-$N_F$ terms in the axial-current renormalization used in [4]. Using it to all orders, we verify the chiral-symmetry relations $D_{NS}^A = D_{NS}^V$ and $d\Pi_L^A/dQ^2 = 0$, at $O(1/N_F)$ in the $\overline{\text{MS}}$ scheme.

### 4.3 Deep-inelastic processes

We now calculate all the $O(1/N_F)$ radiative corrections to the sum rules (12), which differ only by terms of $O(1/N_F^2)$. For the polarized deep-inelastic electron scattering sum rule (23), we calculate the generic $n$-loop $O(1/N_F)$ bare diagrams for forward Compton scattering of a vector current, with momentum $q$, off a zero-momentum quark [6, 11, 23], by inserting $n - 1$ quark loops into the one-loop diagrams, obtaining simple $\Gamma$-functions. The Compton diagrams must then be divided by (25), to obtain the coefficient $C_{\text{Bip}}$ of the axial current (13) in the OPE of $VV$. After the coupling-constant renormalization

$$\left( \frac{g_0}{4\pi} \right)^2 = \left( \frac{\mu^2 e^\gamma}{4\pi} \right)^\varepsilon \frac{a_s}{1 - \frac{4}{3} T_F N_F a_s/\varepsilon} + O(1/N_F^2),$$

(26)

we obtain the $O(1/N_F)$ contributions to

$$C_K \equiv C_{\text{GLS}} = C_{\text{Bip}} + O(1/N_F^2) = 1 + \frac{C_F}{T_F N_F} \sum_{n=1}^{\infty} K_n(T_F N_F a_s)^n + O(1/N_F^2),$$

(27)

in the $\overline{\text{MS}}$ scheme, with $O(1/N_F)$ coefficients given in closed form by

$$K_n = \lim_{\varepsilon \to 0} \left( -\frac{4}{3} \frac{d}{dz} \right)^{n-1} \overline{\mathcal{K}}(z), \quad \overline{\mathcal{K}}(z) = -\frac{(3 + z) \exp(5z/3)}{(1 - z^2)(1 - z^2/4)},$$

(28)
where $\mathcal{K}(z)$ is obtained from the $\varepsilon \to 0$ limit of the bare $n$-loop contribution, with $z = n\varepsilon$ fixed. The renormalization constant (25) precisely cancels the infinities of the bare terms, obtained from the residue of the pole at $n = 0$. The same results are obtained for $C_{\text{GLS}}$, in the large-$N_F$ limit, since the diagrams that distinguish the sum rules have three (or more) gluons in the $t$-channel [6] and hence are (at least) of order $1/N_F^2$.

From (28), we readily obtain the results for $K_n$ in Table 2. The first 3 coefficients agree with [6]; the remainder are new. Combining (23,28), we obtain the $O(1/N_F)$ MS coefficients $S_n = \frac{3}{4}(K_{n+1} + R_{n+1})$ in (13), also given in Table 2, which may be extended, *ad libitum*. The first 2 coefficients, $S_1$ and $S_2$, agree with [6]; the remainder are new.

5 Conclusions

We have discovered the 7 intricate connections of (10,11), between the highly non-trivial NNLO radiative corrections, $C_K$ and $C_R$, to the GLS sum rule [3] and the Adler function [4] of the flavour-singlet vector current, given in Table 1. Forming $\Delta_S \equiv (C_K C_R - 1)$, we find the remarkable result of (5), namely a linear function of the coupling, multiplying the two-loop $\beta$-function, and hence reducing from 11 to 4 the number of independent colour structures in $\Delta_S$, up to $O(\pi^0)$. Two of the 4 coefficients in (5), namely $S_1$ and $S_2$, have been obtained *ab initio*, as the first two members of the series of large-$N_F$ coefficients in (12), whose higher-order members can be obtained from our new all-orders results (23,28). Values are given in Table 2 for $n < 10$.

The consistency of the prescription (17) with chiral symmetry has been demonstrated, in the large-$N_F$ limit, using the all-orders axial renormalization constant (25). A validation of $S_A$ and $S_F$, the $O(1/N_F^2)$ coefficients in (5), has not been attempted here. We note, however, that recent progress with $O(1/N_F^2)$ corrections to QED [24], and to the Gross-Neveu model [25], suggests that one may eventually be able to obtain the entire series of coefficients that have $S_A$ and $S_F$ as their leading members. In any case, the relations (10,11) give one a high degree of confidence in the accuracy of (5,6). Section 3 offers some general observations, suggesting that the relation $\Delta_S \propto \beta$ may persist beyond NNLO. In any case, we confidently expect the expansion of $\Delta_S$ to involve only the coefficients of the $\beta$-function, multiplied by linear combinations of colour factors.

We believe that the hypothesis $\Delta_S \propto \beta$ merits close attention. Its proof (or disproof) could contribute significantly to an area of field theory that combines deep principles [1,10] with calculational achievement [3,4,13,24] and phenomenological relevance [27]. We recommend careful re-examination of the Crewther connection [1] in the light of our findings. The new connections (5,8) suggest that gauge theories know much more about it than has been supposed.

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Table 1 NNLO results of $[4, 7]$ for $C_K = \sum_n k_n T_n + O(\alpha_s^4)$, $C_R = \sum_n r_n T_n + O(\alpha_s^4)$.

| $n$ | $T_n$ | $k_n$ | $r_n$ |
|-----|-------|-------|-------|
| 1   | $C_F \alpha_s^3$ | -3    | 3     |
| 2   | $C_F^2 \alpha_s^3$ | $\frac{1}{2}$ | $-\frac{7}{12}$ |
| 3   | $C_F C_A \alpha_s^3$ | -23   | $\frac{123}{2}$ | -44$\zeta_3$ |
| 4   | $C_F T_F N_F \alpha_s^3$ | 8     | -22   | $+16 \zeta_3$ |
| 5   | $C_F^2 \alpha_s^5$ | $-\frac{9}{7}$ | $-\frac{37}{7}$ |
| 6   | $C_F^2 C_A \alpha_s^3$ | $\frac{1241}{27}$ | $\frac{176}{3} \zeta_3$ | $-127 - 572 \zeta_3 + 880 \zeta_5$ |
| 7   | $C_F C_A \alpha_s^3$ | $\frac{10874}{27}$ | $\frac{440}{3} \zeta_5$ | $\frac{90445}{54}$ | $-10948 \zeta_3 - 440 \zeta_5$ |
| 8   | $C_F^2 T_F N_F \alpha_s^3$ | $-\frac{27}{9}$ | $-\frac{30}{3} \zeta_3$ | $-29 + 304 \zeta_3 - 320 \zeta_5$ |
| 9   | $C_F C_A T_F N_F \alpha_s^3$ | $\frac{7070}{27}$ | $+ \frac{48 \zeta_3 - 160}{3} \zeta_5$ | $\frac{31040}{27}$ | $+ \frac{7168 \zeta_3 + 160}{3} \zeta_5$ |
| 10  | $C_F T_F^2 N_F \alpha_s^3$ | $-\frac{920}{27}$ | $\frac{1216}{9} \zeta_3$ | $\frac{4872}{27}$ | $-\frac{1216}{9} \zeta_3$ |
| 11  | $\frac{N_F}{N_c} d^2_{abc} \alpha_s^3$ | $-\frac{11}{3}$ | $+ 8 \zeta_3$ | $\frac{11}{3}$ | $- 8 \zeta_3$ |

Table 2 Large-$N_F$ expansions $[23, 27, 23]$, obtained from $[23, 25]$, with $x \equiv T_F N_F \alpha_s$.

\[
\begin{align*}
\sum_{n<10} R_n \alpha^n &= 3x + \left[ -22 + 16 \zeta_3 \right] x^2 + \left[ \frac{4832}{27} - \frac{1216}{9} \zeta_3 \right] x^3 + \left[ -\frac{392384}{243} + \frac{25984}{27} \zeta_3 + \frac{1280}{3} \zeta_5 \right] x^4 \\
&+ \left[ \frac{1178720}{729} - \frac{5073920}{729} \zeta_3 - \frac{194560}{27} \zeta_5 \right] x^5 + \left[ \frac{24969720}{19683} + \frac{357201920}{6561} \zeta_3 + \frac{20787200}{243} \zeta_5 + \frac{71680}{3} \zeta_7 \right] x^6 \\
&+ \left[ \frac{38155997760}{177147} - \frac{9308446720}{19683} \zeta_3 - \frac{2029568000}{2187} \zeta_5 - \frac{5447680}{9} \zeta_7 \right] x^7 \\
&+ \left[ \frac{5056220794880}{177147} + \frac{245582254080}{331441} \zeta_3 + \frac{200020075200}{19683} \zeta_5 + \frac{814858240}{81} \zeta_7 + \frac{19496600}{81} \zeta_9 \right] x^8 \\
&+ \left[ \frac{508327309475840}{14348907} - \frac{23972713062400}{4782969} \zeta_3 - \frac{20850926052800}{177147} \zeta_5 - \frac{3182360262400}{2187} \zeta_7 - \frac{59270758400}{279} \zeta_9 \right] x^9 \\
\sum_{n<10} K_n \alpha^n &= -3x + 8x^2 - \frac{920}{27} x^3 + 38720 \frac{243}{243} x^4 - \frac{239876}{243} x^5 + \frac{130862080}{19683} x^6 - \frac{10038092800}{177147} x^7 \\
&+ \frac{247493587200}{531441} x^8 - \frac{825190094739200}{14348907} x^9 \\
\sum_{n<10} S_n \alpha^n &= -\left[ \frac{21}{2} + 12 \zeta_3 \right] x + \left[ \frac{326}{3} - \frac{304}{3} \zeta_3 \right] x^2 + \left[ \frac{9824}{9} + \frac{646}{9} \zeta_3 + 320 \zeta_5 \right] x^3 \\
&+ \left[ \frac{2760448}{243} - \frac{126840}{243} \zeta_3 - \frac{4840}{9} \zeta_5 \right] x^4 + \left[ -\frac{280736320}{2187} + \frac{80900480}{2187} \zeta_3 + \frac{5196800}{81} \zeta_5 + 17920 \zeta_7 \right] x^5 \\
&+ \left[ \frac{10320047360}{6561} - \frac{232711680}{6561} \zeta_3 - \frac{50739200}{729} \zeta_5 - \frac{1361920}{3} \zeta_7 \right] x^6 \\
&+ \left[ -\frac{3723517199360}{177147} + \frac{613955563520}{177147} \zeta_3 + \frac{5008268800}{6561} \zeta_5 + \frac{203174560}{27} \zeta_7 + \frac{48742400}{27} \zeta_9 \right] x^7 \\
&+ \left[ \frac{4854840175000160}{1594323} - \frac{59933178265000}{1594323} \zeta_3 - \frac{5127230163200}{1594323} \zeta_5 - \frac{79559065600}{279} \zeta_7 - \frac{14817689600}{243} \zeta_9 \right] x^8 \\
&+ \left[ \frac{761610928349960}{1594323} + \frac{72657564193280}{1594323} \zeta_3 + \frac{195646580326400}{177147} \zeta_5 + \frac{112018521120}{279} \zeta_7 \right] x^9 \\
&+ \frac{31660360400}{243} \zeta_9 + \frac{7281721600}{27} \zeta_{11} \right] x^9
\end{align*}
\]
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