On global existence for semilinear wave equations with space-dependent critical damping

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Abstract. The global existence for semilinear wave equations with space-dependent critical damping $\partial_t^2 u - \Delta u + \frac{V_0}{|x|} \partial_t u = f(u)$ in an exterior domain is dealt with, where $f(u) = |u|^{p-1}u$ and $f(u) = |u|^p$ are in mind. Existence and non-existence of global-in-time solutions are discussed. To obtain global existence, a weighted energy estimate for the linear problem is crucial. The proof of such a weighted energy estimate contains an alternative proof of energy estimates established by Ikehata–Todorova–Yordanov [J. Math. Soc. Japan (2013), 183–236] but the argument in this paper clarifies the precise dependence of the location of the support of initial data. The blowup phenomena are verified by using a test function method with positive harmonic functions satisfying the Dirichlet boundary condition.

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1 Introduction

In this paper we consider the following initial-boundary value problem

$$\begin{cases}
\partial_t^2 u(x, t) - \Delta u(x, t) + \frac{V_0}{|x|} \partial_t u(x, t) = f(u(x, t)) & \text{in } \Omega \times (0, T), \\
u(x, t) = 0 & \text{on } \partial \Omega \times (0, T), \\
(u, \partial_t u)(0) = (u_0, u_1) & \text{in } \Omega,
\end{cases}
$$

(1.1)

where $\Omega$ is an exterior domain in $\mathbb{R}^N$ ($N \geq 3$) with smooth boundary $\partial \Omega$ and $0 \notin \Omega$, and $f(u) = |u|^{p-1}u$ (or $f(u) = |u|^p$). The constant $V_0 > 0$ describes the effect of the damping term. If we only focus on the scaling structure of the equation in (1.1), then the solution $u$ and the scale parameter $\lambda > 0$ give another solution $\lambda^{-\frac{2}{p-1}} u(\lambda x, \lambda t)$. In this sense, the damping term of the form $|x|^{-1}$ is scale-critical and therefore the constant in front of $|x|^{-1}$ plays a crucial role. Despite of this, the damping coefficient $V_0 |x|^{-1}$ is bounded because of the setting for the domain $\Omega$. Our interest is if the problem (1.1) having the scale-critical damping term possesses nontrivial global-in-time solutions or not.

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If \( \Omega = \mathbb{R}^N \) and \( V_0 = 0 \), then the problem (1.1) is the semilinear wave equation with power type nonlinearities

\[
\begin{aligned}
\begin{cases}
\partial_t^2 u(x, t) - \Delta u(x, t) = |u(x, t)|^p & \text{in } \mathbb{R}^N \times (0, T), \\
(u, \partial_t u)(0) = (u_0, u_1) & \text{in } \mathbb{R}^N,
\end{cases}
\end{aligned}
\]

(1.2)

It is proved in John [19] for the three dimensional case that

- if \( 1 < p < 1 + \sqrt{2} \), then (1.2) does not have nontrivial global-in-time solutions;
- if \( p > 1 + \sqrt{2} \), then (1.2) possesses nontrivial global-in-time solutions.

There are many subsequent papers dealing with the semilinear wave equations in \( N \)-dimensional space (see e.g., Kato [20], Yordanov–Zhang [31], Zhou [33] and their references therein), and then the critical exponent, that is, the threshold for dividing existence and non-existence of global-in-time solutions is clarified as the positive root of the quadratic equation

\[
\gamma(N, p) = 2 + (N + 1)p - (N - 1)p^2 = 0
\]

which is so-called Strauss exponent given by

\[
p_S(N) = \frac{N + 1 + \sqrt{N^2 + 10N - 7}}{2(N - 1)} \quad (N \geq 2).
\]

In the case of the semilinear wave equation with constant damping

\[
\begin{cases}
\partial_t^2 u(x, t) - \Delta u(x, t) + \partial_t u(x, t) = |u(x, t)|^p & \text{in } \Omega \times (0, T), \\
u(x, t) = 0 & \text{in } \partial \Omega \times (0, T), \\
(u, \partial_t u)(0) = (u_0, u_1) & \text{in } \Omega,
\end{cases}
\]

(1.3)

in Todorova–Yordanov [30], Zhang [32] and Ikehata–Tanizawa [16], it is proved for (1.3) with \( \Omega = \mathbb{R}^N \) that existence of global-in-time solutions if \( 1 + \frac{2}{N} < p < \frac{N+2}{N-2} \) and non-existence of those if \( 1 < p \leq 1 + \frac{2}{N} \). This threshold \( 1 + \frac{2}{N} \) is exactly the same as the Fujita exponent for the semilinear heat equation found in Fujita [9]. A certain low regularity solution of (1.3) in the class \( (H^{\alpha,0} \cap H^{0,\delta}) \times (H^{\alpha-1,0} \cap H^{0,\delta}) \) is considered in Hayashi–Kaikina–Naumkin [10], where \( \mathcal{F} \) is the Fourier transform and

\[
H^{\ell,m} = \{ \phi \in L^2(\mathbb{R}^N) ; \langle x \rangle^{m} |\nabla|^{\ell} \phi \in L^2(\mathbb{R}^N) \}, \quad \langle |\nabla| \rangle = \mathcal{F}^{-1} \langle \xi \rangle \mathcal{F}, \quad \langle \xi \rangle = \sqrt{1 + |\xi|^2}.
\]

The exterior domain case has been considered in Ono [26] by using the result of Dan–Shibata [7]. The existence result for the exterior problem (1.3) with \( N = 2, 2 < p < \infty \) is given in Ikehata [13]. The non-existence result for this problem with \( N \geq 2 \) and \( 1 < p < 1 + \frac{2}{N} \) is proved in Takeda–Ogawa [25]. Non-existence for the critical case was discussed in Fino–Ibrahim–Wehbe [8]. Therefore the critical exponent for (1.3) is given by the Fujita exponent \( p = 1 + \frac{2}{N} \).
We move to the problem with space-dependent damping. The problem is the following:

\[
\begin{aligned}
\partial_t^2 u(x, t) - \Delta u(x, t) + \frac{a}{|x|^{\alpha}} \partial_t u(x, t) &= |u(x, t)|^p & \text{in } \mathbb{R}^N \times (0, T), \\
(u, \partial_t u)(0) &= (u_0, u_1) & \text{in } \mathbb{R}^N,
\end{aligned}
\] (1.4)

where \(a > 0\) is a positive constant. In this case, Ikehata–Todorova–Yordanov [17] proved that \(p = 1 + \frac{2}{N-\alpha}\) is the critical exponent for this problem for \(\alpha \in (0, 1)\). This result can be regarded as a generalization of constant damping case \(\alpha = 0\). The critical exponent for the case \(\alpha < 0\) is still given by \(p = 1 + \frac{2}{N-\alpha}\) in Nishihara–Sobajima–Wakasugi [24].

On the other hand, if \(\alpha > 1\), then Lai–Tu [21] proved that the critical exponent for (1.4) is given by \(p = p_S(N)\). This means that the case \(\alpha > 1\) is close to the problem (1.2). Therefore \(\alpha = 1\) is the threshold for the structure of the critical exponent. If \(\alpha = 1\), then Ikehata–Todorova–Yordanov [18], proved that the linear solution with compactly supported initial data satisfies the following energy estimate

\[
\int_{\mathbb{R}^N} \left( |\nabla u(x, t)|^2 + (\partial_t u(x, t))^2 \right) dx \leq \begin{cases} C(1 + t)^{-a}, & \text{if } 1 < a < N, \\
C_\varepsilon(1 + t)^{-N+\varepsilon}, & \text{if } a \geq N,
\end{cases}
\]

where the constants \(C\) and \(C_\varepsilon\) (\(\varepsilon > 0\) is arbitrary) depend on the location of the support of initial data. One can observe that the behavior of solutions strongly depends on the size of the constant \(V_0\). For the semilinear problem, non-existence of global-in-time solutions for the case \(1 < p \leq 1 + \frac{2}{N-1}\) can be found in Li [22].

Then we come back to our problem (1.1). In the whole space case \(\Omega = \mathbb{R}^N (N \geq 3)\), in Ikeda–Sobajima [12], non-existence of global-in-time solutions to (1.1) when \(0 < V_0 < \frac{(N-1)^2}{N+1}\) and \(\frac{N}{N-1} < p \leq p_S(N + V_0)\) is proved. It should be notice that although the critical exponent is not determined yet, one can find that the behavior of solutions to (1.1) strongly depends on the parameter \(V_0\). A further development about non-existence results with singular dampings and potentials can be found in Dai–Kubo–Sobajima [6].

As a summary, existence of global-in-time solutions to (1.1) is still open so far. To discuss such a situation, we need to clarify the dependence of the behavior of linear solutions with respect to the parameter \(V_0\). The purpose of the present paper is to address existence of global-in-time solutions to (1.1). The two-dimensional case is excluded from our discussion because of the difficulty which comes from the absence of the usual Hardy inequality. Although we can treat the singular damping case \(\Omega = \mathbb{R}^N (N \geq 3)\), to avoid a complicated discussion, we only deal with the case of exterior domain \(\Omega\) with smooth boundary \(\partial \Omega\) and \(0 \notin \Omega\).

Before stating our result, we give the definition of solutions to (1.1).

**Definition 1.1.** The function \(u\) is called a weak solution of the problem (1.1) in \((0, T)\) if

\[
u \in C([0, T); H^1_0(\Omega)) \cap C^1([0, T); L^2(\Omega)) \cap C([0, T); L^{2p}(\Omega))
\]

and the pair \(U(t) = (u(t), \partial_t u(t))\) satisfies

\[
U(t) = e^{-tA} \left( u_0 \atop u_1 \right) + \int_0^t e^{-(t-s)A}[N(U(s))] \, ds,
\]
where \( \{e^{-tA}\}_{t \geq 0} \) is the \( C_0 \)-semigroup on the Hilbert space \( \mathcal{H} = H^1_0(\Omega) \times L^2(\Omega) \) generated by \( -A = \begin{pmatrix} 0 & 1 \\ \Delta & -V_0/|x| \end{pmatrix} \) endowed with the domain \( D(A) = [H^2(\Omega) \cap H^1_0(\Omega)] \times H^1_0(\Omega) \) and \( \mathcal{N}(t(u,v)) = t(0, |u|^{p-1}u) \). Additionally, if \( T < \infty \), then \( u \) is called a local-in-time weak solution, and if \( T = \infty \), then \( u \) is called a global-in-time weak solution.

By the standard argument based on the semigroup theory (see e.g., Cazenave–Haraux [5], or Ikeda–Sobajima [12]), it is not difficult to find local-in-time (weak) solutions of (1.1) for every initial data. Moreover, it is well-known that the problem (1.1) has the finite propagation property.

**Proposition 1.1.** Assume that \( V_0 \geq 0 \) and \( 1 < p < \frac{N}{N-2} \). Then for every \( t(u_0,u_1) \in H^1_0(\Omega) \times L^2(\Omega) \), there exists a positive constant \( T = T(u_0,u_1) \in (0,\infty) \) such that (1.1) has a unique weak solution \( u \) in \( (0,T) \). Moreover, if \( \text{supp } u_0 \cup \text{supp } u_1 \subset B(0,R_0) \), then for every \( t \in (0,T) \), \( \text{supp } u(t) \subset B(0,R_0 + t) \).

Now we are in a position to state our result about existence of global-in-time weak solutions to (1.1).

**Theorem 1.2.** Assume that \( V_0 > N-2 \) and \( f(u) = |u|^{p-1}u \) (or \( f(u) = |u|^p \)) with \( 1 + \frac{4}{N-2+\min\{N,V_0\}} < p < \frac{N}{N-2} \). Then there exists a positive constant \( \delta_0 \) such that if the pair \( t(u_0,u_1) \in H^1_0(\Omega) \times L^2(\Omega) \) satisfies

\[
\int_{\Omega} \left( |\nabla u_0|^2 + u_1^2 \right) (1 + |x|)^{\frac{4}{N-1} - 1} dx \leq \delta
\]  

(1.5)

for some \( \delta \in (0,\delta_0] \), then the problem (1.1) possesses a unique global-in-time (weak) solution \( u \). Moreover, \( u \) satisfies

\[
\int_{\Omega} \left( |\nabla u(t)|^2 + (\partial_t u(t))^2 \right) (1 + t + |x|)^{\frac{4}{N-1} - 1} dx \leq C \delta,
\]  

(1.6)

where \( C \) is a positive constant depending only on \( N, p \) and \( V_0 \).

**Remark 1.1.** The same assertion as Theorem 1.2 for (1.4) with \( \alpha = 1 \) can be verified by the almost same procedure.

**Remark 1.2.** The rate for the energy decay of global-in-time solutions to (1.1) is (almost) the same as the linear estimates proved by Ikehata–Todorova–Yordanov [18]. This suggests that the solution of the semilinear problem with small initial data behaves like the one of the linear problem.

**Remark 1.3.** The weight \( |x|^{\frac{4}{N-1} - N+2} \) may be reasonable in view of scaling structure for the problem (1.1).

If we focus our attention to the case \( p > 1 + \frac{2}{N-1} \), we can deduce the following corollary from Theorem 1.2.
Corollary 1.3. Assume that \( V_0 \geq N, \frac{N+1}{N-1} < p < \frac{N}{N-2} \). Then there exist positive constants \( \delta_0 \) and \( C \) such that if the pair \( \{u_0, u_1\} \in H^1_0(\Omega) \times L^2(\Omega) \) satisfies (1.5) for some \( \delta \in (0, \delta_0] \), then the problem (1.1) possesses a unique global-in-time (weak) solution \( u \) satisfying (1.6).

The following is a part of non-existence result for (1.1) with the positive definite nonlinearity \( f(u) = |u|^p \) with \( 1 < p \leq 1 + \frac{2}{N-1} \). This clarifies that the critical exponent for the problem (1.1) with \( V_0 \geq N \) is \( p = 1 + \frac{2}{N-1} \).

Proposition 1.4. Assume that \( N \geq 3, V_0 > 0 \) and \( f(u) = |u|^p \) with \( 1 < p \leq 1 + \frac{2}{N-1} \). If \( \Omega = \mathbb{R}^N \setminus B(0,1) \), then the following assertion holds: for every \( \{u_0, u_1\} \in [C^\infty_0(\Omega)]^2 \) satisfying
\[
\int_\Omega \left( u_1 + \frac{V_0}{|x|} u_0 \right) \left( 1 - |x|^{2-N} \right) \, dx > 0,
\]
the corresponding solution \( u \) of (1.1) blows up in finite time.

Here we briefly describe the (very rough) idea of the proof. At first, we consider the estimates of weighted energy functional
\[
\int_\Omega \left( |\nabla w|^2 + (\partial_t w)^2 \right) (1 + t + |x|)^m \, dx
\]
(\( 0 < m < N \)) for the following inhomogeneous problem
\[
\begin{cases}
\partial_t^2 w - \Delta w + \frac{V_0}{|x|} \partial_t w = F & \text{in } \Omega \times (0, T), \\
w = 0 & \text{on } \partial \Omega \times (0, T), \\
(w, \partial_t w)(0) = (w_0, w_1) & \text{in } \Omega,
\end{cases}
\tag{1.7}
\]
where \( w_0, w_1, F \) are compactly supported. If \( V_0 < N - 1 \), then we can prove the desired estimate (Proposition 3.1) by using a similar idea in Ikehata–Todorova–Yordanov [18]. To treat the case \( V_0 > N - 1 \), we introduce the decomposition of the solution to (1.7) as follows:
\[
\begin{cases}
\frac{\lambda}{|x|^2} \psi_1 - \Delta \psi_1 = w_1 & \text{in } \Omega, \\
\psi_1 = 0 & \text{on } \partial \Omega,
\end{cases}
\quad
\begin{cases}
\frac{V_0}{|x|} \partial_t v - \Delta v = F & \text{in } \Omega \times (0, T), \\
v = 0 & \text{on } \partial \Omega, \\
v(0) = w_0 + \frac{\lambda}{V_0|x|} \psi_1 & \text{in } \Omega,
\end{cases}
\]
and
\[
\begin{cases}
\partial_t^2 U - \Delta U + \frac{V_0}{|x|} \partial_t U = -\partial_t v & \text{in } \Omega \times (0, T), \\
U = 0 & \text{on } \partial \Omega \times (0, T), \\
(U, \partial_t U)(0) = (-\psi_1, -\frac{\lambda}{V_0|x|} \psi_1) & \text{in } \Omega.
\end{cases}
\]
Then we can find the relation $w = v + \partial_t U$. The idea of such a decomposition is based on Sobajima [28] and Ikehata–Sobajima [15] (motivated by so-called modified Morawetz method in Ikehata–Matsuyama [14]). The weighted estimates for $v$ (and $\partial_t v$) are valid via a weighted energy estimate due to Sobajima–Wakasugi [29]. Then combining these estimates and energy estimates for $0 < V_0 < N - 1$, we can deduce the desired energy estimates for the case $V_0 > N - 1$. To justify the above procedure, we need to use the restriction on the bounded region $D = \{x \in \Omega ; |x| < R_0 + T\}$ for $t \in (0, T)$ via the finite propagation property.

To apply the above estimates to the semilinear problem (1.1), we use the Caffarelli–Kohn–Nirenberg inequality of the form

$$C \int_\Omega |u|^{2p} |x|^{\mu'} \, dx \leq \int_\Omega |\nabla u|^2 |x|^{\mu} \, dx.$$  

A priori estimate for the weighted energy with the blowup alternative provides the global existence. The proof of small data blowup result is an application to the test function method with the positive harmonic function satisfying Dirichlet boundary condition, which is used in Ikeda–Sobajima [11]. To justify the above procedure, we introduce the problem in a bounded domain in view of finite propagation property.

The paper is organized as follows. In Section 2, we collect some functional inequalities (weighted Hardy inequalities and Caffarelli–Kohn–Nirenberg inequalities) and a family of special solutions to the parabolic equation $\frac{V_0}{|x|} \partial_t \Phi = \Delta \Phi$. Section 3 is devoted to the proof of weighted energy estimates for the inhomogeneous problem (1.7). In Section 4, we prove existence of global–in–time solutions to (1.1) via a priori estimate with the blowup alternative. The small data blowup (non-existence) of solutions to (1.1) is discussed in Section 5.

2 Preliminaries in bounded domains

In this section, we collect some important lemmas to analyse the problem (1.1).

2.1 Functional inequalities

Here we give some functional inequalities on bounded domain $D \subset \mathbb{R}^N$ such that $\partial D$ is smooth enough and $0 \notin \partial D$.

The first inequality is so-called weighted Hardy inequality. For the proof, we refer Arendt–Goldstein–Goldstein [1, Theorem 3.1] and also Metafune–Sobajima–Spina [23, Proposition 8.1].

**Lemma 2.1.** If $N - 2 + \beta > 0$, then for every $w \in H^1_0(D)$,

$$\left(\frac{N - 2 + \beta}{2}\right)^2 \int_D w^2 |x|^{\beta - 2} \, dx \leq \int_D |\nabla w|^2 |x|^{\beta} \, dx.$$
The second is a special case of the classical Sobolev inequality (see e.g., Brezis [3, Theorem 9.9]).

**Lemma 2.2.** If $N \geq 3$, then there exists a positive constant $C_{GN}$ depending only on $N$ such that for every $H^1_0(D)$,

$$
\left( \int_D |w|^{\frac{2N}{N-2}} \, dx \right)^{\frac{N-2}{2N}} \leq C_{GN} \left( \int_D |\nabla w|^2 \, dx \right)^{\frac{1}{2}}.
$$

The third is a part of the Caffarelli–Kohn–Nirenberg inequality (see Caffarelli–Kohn–Nirenberg [4]), which is crucial to treat the nonlinear effect from $|u|^{p-1}u$ in (1.1). For the reader’s convenience, we will give a proof of that via the Hardy and Gagliardo–Nirenberg inequalities.

**Lemma 2.3.** If $\mu > 2 - N$ and $q \in (2, \frac{2N}{N-2})$, then there exists a positive constant $C_{GN,\mu}$ (depending only on $N, \mu$) such that for every $w \in H^1_0(D)$,

$$
\int_D |w|^{q} |x|^{\mu} \, dx \leq C_{GN,\mu} \left( \int_D |\nabla w|^{2} |x|^{\mu} \, dx \right)^{\frac{q}{2}},
$$

where $\mu' = \mu - 2 + \frac{N-2+\mu}{2}(q-2)$.

**Proof.** Let $\gamma \in \mathbb{R}$ be given later. We see from the Hölder inequality that

$$
\int_D |w|^{q} |x|^{\gamma} \, dx \leq \left( \int_D |w|^{2} |x|^{\gamma-2} \, dx \right)^{1-\frac{N-2(q-2)}{2}} \left( \int_D |x|^{\frac{\mu'}{2}} w^{\frac{q}{2}-2} \, dx \right)^{\frac{N-2(q-2)}{2}},
$$

where $\gamma' = \gamma + 2 - \frac{N-2+\gamma}{2}(q-2)$. Now we choose $\gamma' = \mu$, that is, $\gamma = \mu'$. Then using the Hardy inequality (Lemma 2.1) and the Gagliardo–Nirenberg inequality (Lemma 2.2), we obtain (2.1). The proof is complete.

### 2.2 Special solutions to the corresponding parabolic equation

Next we introduce a family of special solutions of the homogeneous parabolic equation

$$
\frac{V_0}{|x|} \partial_t \Phi - \Delta \Phi = 0 \quad \text{in } \mathbb{R}^N \times [0, \infty),
$$

which is used in Sobajima–Wakasugi [29].

**Definition 2.1.** Define a family of functions $\{\Phi_\beta\}_{\beta \in \mathbb{R}}$ as follows:

$$
\Phi_\beta(x,t) = (1+t)^{-\beta} \varphi_\beta \left( \frac{|x|}{V_0(1+t)} \right), \quad \varphi_\beta(z) = e^{-z} M(N-1-\beta, N-1; z),
$$

where $M(a,c;z)$ denotes the Kummer confluent hypergeometric function

$$
M(a,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{(c)_n n!}, \quad \text{for } a \in \mathbb{R}, \quad -c \notin \mathbb{N} \cup \{0\}
$$

and $(d)_n$ is the Pochhammer symbol given by $(d)_0 = 1$ and $(d)_n = \prod_{k=1}^{n} (d + k - 1)$ for $n \in \mathbb{N}$ (for the detail, see e.g., Beals–Wong [2]).

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The properties of the functions $\Phi_\beta$ are listed as follows.

**Lemma 2.4 ([29, Lemma 2.4])**. The following assertions hold:

(i) For every $\beta \in \mathbb{R}$,
\[
\frac{V_0}{|x|} \partial_t \Phi_\beta - \Delta \Phi_\beta = 0, \quad x \in \mathbb{R}^N, \quad t \geq 0.
\]

(ii) For every $\beta \in \mathbb{R}$,
\[
\partial_t \Phi_\beta = -\beta \Phi_{\beta+1}, \quad x \in \mathbb{R}^N, \quad t \geq 0.
\]

(iii) For every $\beta \in \mathbb{R}$,
\[
|\Phi_\beta| \leq C_{\Phi_\beta} \left(1 + t + V_0|x|\right)^{-\beta}, \quad x \in \mathbb{R}^N, \quad t \geq 0.
\]

(iv) For every $\beta < N - 1$,
\[
\Phi_\beta \geq c_{\Phi_\beta} \left(1 + t + V_0|x|\right)^{-\beta}, \quad x \in \mathbb{R}^N, \quad t \geq 0.
\]

Moreover, we need the following two lemmas which come from integration by parts.

**Lemma 2.5 ([27])**. Assume that $\Phi \in C^2(D)$ is positive and $\delta \in (0, \frac{1}{2})$. Then for every $z \in H^2(D) \cap H^1_0(D)$,
\[
\int_D \frac{z \Delta z}{\Phi^{1-2\delta}} \, dx \leq \frac{\delta}{1-\delta} \int_D \frac{|\nabla z|^2}{\Phi^{1-2\delta}} \, dx + \frac{1-2\delta}{2} \int_D \frac{z^2 \Delta \Phi}{\Phi^{2-2\delta}} \, dx.
\]

**Lemma 2.6 ([29, Lemma 3.5])**. If $m > 2 - N$, then for every $z \in H^1_0(D)$ and $t \geq 0$,
\[
\int_D \frac{z^2 \Psi(t)^{m-1}}{|x|} \, dx \leq \min\left\{\frac{N-1}{2}, \frac{N-2+m}{2}\right\}^{-2} \int_D |\nabla z|^2 \Psi(t)^m \, dx,
\]
where $\Psi = 1 + t + |x|$.

## 3 The inhomogeneous problems in bounded domains

In this section, we consider the following inhomogeneous problem in the bounded domain $D$ ($\partial D$ is smooth enough and $0 \notin \overline{D}$):

\[
\begin{cases}
\partial_t^2 w(x,t) - \Delta w(x,t) + \frac{V_0}{|x|} \partial_t w(x,t) = F(x,t) & \text{in } D \times (0,T), \\
w(x,t) = 0 & \text{on } \partial D \times (0,T), \\
(w, \partial_t w)(0) = (w_0, w_1) & \text{in } D,
\end{cases}
\]

(3.1)
where \((w_0, w_1) \in H^1_0(D) \times L^2(D)\) and \(F \in C([0, T]; L^2(D))\). If \(F \equiv 0\) and \(D = \mathbb{R}^N\), then the energy estimate of \(w\) with compactly supported initial data is proved in Ikehata–Todorova–Yordanov [18]. In contrast, we will show the estimate of the functional with space-time dependent weight

\[
E^\Psi_m(w; t) = \int_D \left( |\nabla w(x,t)|^2 + (\partial_t w(x,t)) \right) \Psi(x,t)^m \, dx, \quad \Psi(x,t) = 1 + t + |x|.
\]

To apply such estimates in the bounded domain \(D\) to the case of exterior domain, it is crucial to derive them having constants which are independent of the shape of \(D\).

**Proposition 3.1.** Assume that \(t^i(w_0, w_1) \in H^1_0(D) \times L^2(D)\) and \(F \in C([0, T]; L^2(D))\). If \(0 < m < \min\{N - 1, V_0 - 1\}\), then there exist positive constants \(\delta_m, K_m\) (depending only on \(N\) and \(m\)) such that

\[
E^\Psi_{m+1}(w; t) + \delta_m \int_0^t E^\Psi_m(w; s) \, ds 
\leq K_m \left( E^\Psi_{m+1}(w; 0) + \int_0^t \int_D F(x, s)^2 \Psi(x, s)^{m+1} |x| \, dx \, ds \right).
\]

By using the resolvent \(J_n = (1 - \frac{1}{n} \Delta)^{-1}\) of the operator \(-\Delta\) with the domain \(H^2(D) \cap H^1_0(D)\), we can verify that \(t^i(J_n w_0, J_n w_1) \in (H^2(D) \cap H^1_0(D)) \times H^1_0(D)\) and \(J_n F \in C([0, T]; H^1_0(D))\) and

\[
\begin{cases}
J_n w_0 \to w_0 & \text{in } H^1_0(D), \\
J_n w_1 \to w_1 & \text{in } L^2(D), \\
J_n F \to F & \text{in } C([0, T]; L^2(D)).
\end{cases}
\]

Therefore in the energy method, (by the above approximation) we can use the integration by parts.

### 3.1 Energy estimates for \(0 < m < N - 2\)

We will give estimates of the weighted energy functionals

\[
E_{\mu+1}(w; t) = \int_D \left( |\nabla w(x,t)|^2 + (\partial_t w(x,t)) \right) |x|^{\mu+1} \, dx
\]

for \(\mu \in \mathbb{R}\). The following (auxiliary) functionals play an essential role:

\[
\widetilde{E}_{\mu+1}(w; t) = E_{\mu+1}(w; t) - \frac{(\mu + 1)(N - 2 + \mu)}{2} \int_D w(x,t)^2 |x|^{\mu-1} \, dx,
\]

\[
E^*_\mu(w; t) = \int_D \left( 2w(x,t)\partial_t w(x,t) + \frac{V_0}{|x|} w(x,t)^2 \right) |x|^{\mu} \, dx
\]

and their linear combination

\[
E^t_{\mu+1}(w; t) = \widetilde{E}_{\mu+1}(w; t) + \frac{V_0}{2} E^*_\mu(w; t).
\]
The following lemma asserts that $E_{\mu+1}$ and $E_{\mu+1}^*$ are equivalent under a suitable restriction on $\mu$.

**Lemma 3.2.** If $1 - N < \mu < \sqrt{(N - 2)^2 + 1} + V_0^2$, then there exist positive constants $c_{\mu+1}^*$ and $C_{\mu+1}$ depending only on $N$, $\mu$, and $V_0$ such that

$$c_{\mu+1}^* E_{\mu+1}(w; t) \leq E_{\mu+1}^*(w; t) \leq C_{\mu+1} E_{\mu+1}(w; t).$$

**Proof.** Since for every $\varepsilon > 0$, the Young inequality gives

$$V_0 \left| \int_D w \partial_t w |w|^\mu \, dx \right| \leq \frac{1}{1 + \varepsilon} \int_D (\partial_t w)^2 |w|^\mu + \frac{V_0^2}{4} (1 + \varepsilon) \int_D w^2 |w|^\mu - 1 \, dx,$$

we have

$$E_{\mu+1}^*(w; t) = \int_D |\nabla w|^2 |w|^\mu \, dx - \frac{(\mu + 1)(N - 2 + \mu)}{2} \int_D w^2 |w|^\mu - 1 \, dx$$

$$+ \int_D (\partial_t w)^2 |w|^\mu \, dx + V_0 \int_D w \partial_t w |w|^\mu \, dx + \frac{V_0^2}{2} \int_D w^2 |w|^\mu - 1 \, dx.$$

Moreover, using Lemma 2.1 with $\beta = \mu + 1 > 2 - N$, we see

$$E_{\mu+1}^*(w; t) \geq \varepsilon \int_D |\nabla w|^2 |w|^\mu \, dx + \frac{\varepsilon}{1 + \varepsilon} \int_D (\partial_t w)^2 |w|^\mu \, dx$$

$$+ \frac{1}{4} \left( (N - 2)^2 + 1 + V_0^2 - \mu^2 - ((N - 1 + \mu)^2 + V_0^2) \varepsilon \right) \int_D w^2 |w|^\mu - 1 \, dx.$$

Choosing $\varepsilon > 0$ small enough, we deduce the desired lower bound. The calculation for the upper bound is almost the same as above. $\square$

To provide estimates for the derivatives of $\tilde{E}_{\mu+1}$ and $E_{\mu}^*$, we need to introduce a weighted gradient

$$\nabla_{\mu} w = \nabla w + \frac{N - 2 + \mu}{2} \frac{x}{|x|^2} w$$

(if we consider the problem (1.4) with $\alpha = 1$, then we choose $\nabla_{\mu} w = \nabla w + \frac{N - 2 + \mu}{2} \frac{x}{|x|^2} w$).

The following lemma describes the relation between $\nabla$ and $\nabla_{\mu}$ in the sense of weighted $L^2$-norms.

**Lemma 3.3.** If $\mu > 2 - N$, then for every $w \in H_0^1(D)$,

$$\frac{1}{2} \int_D |\nabla w|^2 |w|^\mu \, dx \leq \int_D |\nabla_{\mu} w|^2 |w|^\mu \, dx + \left( \frac{N - 2 + \mu}{2} \right)^2 \int_D w^2 |w|^\mu - 1 \, dx,$$

$$5 \int_D |\nabla w|^2 |w|^\mu \, dx \geq \int_D |\nabla_{\mu} w|^2 |w|^\mu \, dx + \left( \frac{N - 2 + \mu}{2} \right)^2 \int_D w^2 |w|^\mu - 1 \, dx.$$
Proof. Using the Young inequality, we have

\[
\int_D |\nabla w|^2 |x|\mu \, dx = \int_D \left| \nabla_\mu w - \frac{N - 2 + \mu}{2} \frac{x \cdot w}{|x|^2} \right|^2 |x|\mu \, dx
\]

\[
= \int_D \left( |\nabla_\mu w|^2 - (N - 2 + \mu) \nabla_\mu w \cdot \frac{x}{|x|^2} w + \left( \frac{N - 2 + \mu}{2} \right)^2 \frac{w^2}{|x|^2} \right) |x|\mu \, dx
\]

\[
\leq 2 \int_D \left( |\nabla_\mu w|^2 + \left( \frac{N - 2 + \mu}{2} \right)^2 \frac{w^2}{|x|^2} \right) |x|\mu \, dx.
\]

In the same way, we see that

\[
\int_D |\nabla w|^2 |x|\mu \, dx \leq 2 \int_D \left( |\nabla w|^2 + \left( \frac{N - 2 + \mu}{2} \right)^2 \frac{w^2}{|x|^2} \right) |x|\mu \, dx.
\]

Using Lemma 2.1 with \( \beta = \mu \), we obtain

\[
\int_D |\nabla_\mu w|^2 |x|\mu \, dx + \left( \frac{N - 2 + \mu}{2} \right)^2 \int_D w^2 |x|^{\mu-2} \, dx \leq 5 \int_D |\nabla w|^2 |x|\mu \, dx.
\]

The proof is complete. \( \square \)

Here we consider the estimate for \( E_{\mu+1}^2 \). The following two lemmas are the estimates for the derivatives of \( E_{\mu+1}^\mu \) and \( E_{\mu+1}^\mu \), respectively.

**Lemma 3.4.** Assume that \((w_0, w_1) \in H_0^1(D) \times L^2(D), F \in C([0, T]; L^2(D)) \) and \( \mu \in \mathbb{R} \). Let \( w \) be the solution of (3.1). Then for every \( t \in (0, T) \),

\[
\frac{d}{dt} E_{\mu+1}^\mu(w; t) \leq -2V_0 \int_D (\partial_t w|^2 |x|\mu \, dx + 2 \int_D (\partial_t w)F |x|^\mu+1 \, dx
\]

\[
+ 2(\mu + 1) \left( \int_D |\nabla_\mu w|^2 |x|\mu \, dx \right)^{\frac{1}{2}} \left( \int_D |\nabla_\mu w|^2 |x|\mu \, dx \right)^{\frac{1}{2}}.
\]

Proof. By the definitions of \( E_{\mu+1}^\mu \) and \( \nabla_\mu \), we see from integration by parts that

\[
\frac{d}{dt} E_{\mu+1}^\mu(w; t) = 2 \int_D \left( \partial_t w_\mu \partial_\mu^2 w + \nabla \partial_\mu w \cdot \nabla w \right) |x|^{\mu+1} \, dx
\]

\[
- (\mu + 1)(N - 2 + \mu) \int_D w_\mu \partial_\mu w |x|^{\mu-1} \, dx
\]

\[
= 2 \int_D \partial_t w \left( \partial_\mu^2 w - \Delta w \right) |x|^{\mu+1} \, dx - 2(\mu + 1) \int_D \partial_\mu w \nabla w \cdot x |x|^{\mu-1} \, dx
\]

\[
- (\mu + 1)(N - 2 + \mu) \int_D w_\mu \partial_\mu w |x|^{\mu-1} \, dx
\]

\[
= 2 \int_D \partial_t w \left( \partial_\mu^2 w - \Delta w \right) |x|^{\mu+1} \, dx - 2(\mu + 1) \int_D \nabla_\mu w \cdot \frac{x}{|x|} \partial_\mu w |x|^{\mu} \, dx.
\]

Using the equation in (3.1) and the Young inequality, we deduce the desired inequality. \( \square \)
Lemma 3.5. Assume that \( t(w_0, w_1) \in H^1_0(D) \times L^2(D) \), \( F \in C([0,T];L^2(D)) \) and \( \mu \in \mathbb{R} \). Let \( w \) be the solution of (3.1). Then for every \( t \in (0,T) \),

\[
\frac{d}{dt} E^\mu_t(w; t) = 2 \int_D (\partial_t w)^2 |x|^\mu \, dx - 2 \int_D |\nabla w|^2 |x|^\mu \, dx - \frac{(N-2)^2 - \mu^2}{2} \int_D w^2 |x|^\mu-2 \, dx + 2 \int_D w F |x|^\mu \, dx.
\]  

(3.2)

Proof. By using the equation in (3.1), we have

\[
\frac{d}{dt} E^\mu_t(w; t) = 2 \int_D (\partial_t w)^2 |x|^\mu \, dx + 2 \int_D w \left( \partial_t^2 w + \frac{V_0}{|x|} \partial_t w \right) |x|^\mu \, dx
\]

\[
= 2 \int_D (\partial_t w)^2 |x|^\mu \, dx + 2 \int_D w \left( \Delta w + F \right) |x|^\mu \, dx
\]

\[
= 2 \int_D (\partial_t w)^2 |x|^\mu \, dx + 2 \int_D w \Delta w |x|^\mu \, dx + 2 \int_D w F |x|^\mu \, dx.
\]

Observe that integration by parts provides

\[
- \int_D w \Delta w |x|^\mu \, dx = \int_D |\nabla w|^2 |x|^\mu \, dx + \mu \int_D w \nabla w \cdot x |x|^\mu-2 \, dx
\]

\[
= \int_D \left| \nabla w + \frac{N-2 + \mu}{2} \frac{x}{|x|^2} w \right|^2 |x|^\mu \, dx
\]

\[
- (N-2) \int_D w \nabla w \cdot x |x|^\mu-2 \, dx - \left( \frac{N-2 + \mu}{2} \right)^2 \int_D w^2 |x|^\mu-2 \, dx
\]

\[
= \int_D |\nabla w|^2 |x|^\mu \, dx + \frac{(N-2)^2 - \mu^2}{4} \int_D w^2 |x|^\mu-2 \, dx,
\]

we obtain (3.2). \qed

The following lemma is the estimate for the derivative of \( E^\mu_{t+1} \), which is a summary of Lemmas 3.4 and 3.5.

Lemma 3.6. Assume that \( t(w_0, w_1) \in H^1_0(D) \times L^2(D) \), \( F \in C([0,T];L^2(D)) \) and \( \mu \in \mathbb{R} \). Let \( w \) be the solution of (3.1). Then for every \( t \in (0,T) \),

\[
\frac{d}{dt} E^\mu_{t+1}(w; t) \leq -(V_0 - |\mu + 1|) \int_D \left( (\partial_t w)^2 + |\nabla w|^2 \right) |x|^\mu \, dx
\]

\[
- \frac{(N-2)^2 - \mu^2}{4} V_0 \int_D w^2 |x|^\mu-2 \, dx
\]

\[
+ 2 \int_D (\partial_t w) F |x|^\mu+1 \, dx + V_0 \int_D w F |x|^\mu \, dx.
\]
In particular, if $|\mu + 1| < V_0$, then there exist positive constants $\delta'_\mu$ and $C'_{\mu+1}$ such that

$$\frac{d}{dt} E^\delta_{\mu+1}(w; t) + \delta'_\mu E^\delta_\mu(w; t) \leq \begin{cases} C'_{\mu+1} \left( \int_D F^2 |x|^\mu + 2 \, dx \right) & \text{if } |\mu| < N - 2, \\ C'_{\mu+1} \left( \int_D w^2 |x|^\mu - 2 \, dx + \int_D F^2 |x|^\mu + 2 \, dx \right) & \text{if } |\mu| \geq N - 2. \end{cases}$$

**Remark 3.1.** In the homogeneous case $F \equiv 0$, we have from the first inequality in Lemma 3.6 that if $V_0 \leq N - 1$, then choosing $\mu = V_0 - 1$, we have

$$c^\delta_0 E_{V_0}(w, t) \leq c^\delta_0 E_{V_0}(w, 0).$$

Proceeding as in the proof of Ikehata–Todorova–Yordanov [18, Proposition 2.2], we can deduce the energy decay estimate

$$E_0(w; t) \leq C t^{-V_0} E_{V_0}(w, 0)$$

which is the same as [18, Theorem 1.1] (for $1 < V_0 \leq N - 1$).

Here we prove Proposition 3.1 under the restriction $m < N - 2$.

**Proof of Proposition 3.1 when $0 < m < N - 2$.** We see from Lemma 3.6 with $\mu = m$ that

$$E^\delta_{m+1}(w; t) + \delta'_m \int_0^t E^\delta_m(w; s) \, ds \leq E^\delta_{m+1}(w; 0) + C'_{m+1} \int_0^t \int_D F(s)^2 |x|^{m+2} \, dx \, ds$$

Moreover, Lemma 3.6 with $\mu = m - 1 > -1$ gives

$$\frac{d}{dt} \left[ (1 + t) E^\delta_m(w; t) \right] + \delta'_{m-1} (1 + t) E^\delta_{m-1}(w; t) \leq (1 + t) \left[ \int_0^t E^\delta_m(w; t) + \delta'_{m-1} E^\delta_{m-1}(w; t) \right] + E^\delta_m(w; t) \leq C'_m (1 + t) \int_D F^2 |x|^{m+1} \, dx + E^\delta_m(w; t)$$

and therefore we have

$$(1 + t) E^\delta_m(w; t) + \delta'_{m-1} \int_0^t (1 + s) E^\delta_{m-1}(w; s) \, ds \leq E^\delta_m(w; 0) + C'_m \int_0^t (1 + s) \int_D F(s)^2 |x|^{m+1} \, dx \, ds + \frac{C'_{m+1}}{\delta'_m} \int_0^t \int_D F(s)^2 |x|^{m+2} \, dx \, ds.$$
we have
\[(1 + t)^k E_{m+1-k}(w; t) + \bar{\delta} \int_0^t (1 + s)^k E_{m-k}(w; s) ds \]
\[\leq \bar{C} \left( E_{m}(w; 0) + \int_0^t \int_D F(s)^2 |x| \Psi^{m+1} dx ds \right)\]
for some positive constants \(\bar{\delta}\) and \(\bar{C}\). Since
\[(1 + s)^m E_{0}(w; s) \leq \left( (1 + s)^k E_{m-k}(w; s) \right)^\frac{m}{k} \left( E_{m}(w, s) \right)^{1 - \frac{m}{k}},\]
we have
\[\int_0^t E_{m}(w; s) ds \leq \bar{C}' \left( E_{m}(w; 0) + \int_0^t \int_D F(s)^2 \Psi^{m+1} |x| dx ds \right)\]
for some positive constant \(\bar{C}'\). Furthermore, noting that
\[\frac{d}{dt} \left[ (1 + t)^{m+1} E_{0}(w; t) \right] \]
\[= (m + 1)(1 + t)^m E_{0}(w; t) + 2(1 + t)^{m+1} \int_D \partial_t w \left( - \frac{V_0}{|x|} \partial_t w + F \right) dx \]
\[\leq (m + 1)(1 + t)^m E_{0}(w; t) + \frac{1}{2V_0} (1 + t)^{m+1} \int_D F^2 |x| dx \]
\[\leq (m + 1)(1 + t)^m E_{0}(w; t) + \frac{1}{2V_0} \int_D F^2 \Psi^{m+1} |x| dx,\]
we obtain the desired estimate for \(E_{m+1}(w; t)\). The proof is complete.

\[\Box\]

### 3.2 Energy estimates for \(N - 2 \leq m < N - 1\)

In this case, we introduce auxiliary functions \(\psi_1, v\) and \(U\) which are given as the solutions of the following elliptic, parabolic and hyperbolic problems, respectively:

\[
\begin{aligned}
&\left\{ \begin{array}{l}
\frac{\lambda}{|x|^2} \psi_1 - \Delta \psi_1 = w_1 \quad \text{in} \ D, \\
\psi_1 = 0 \\
\end{array} \right. \\
&\begin{array}{l}
\frac{V_0}{|x|} \partial_t v - \Delta v = F \\
v = 0 \\
v(0) = v_0 := w_0 + \frac{\lambda}{V_0 |x|} \psi_1 \\
\end{array} \\
&\begin{array}{l}
\partial_t^2 U - \Delta U + \frac{V_0}{|x|} \partial_t U = -\partial_t v \\
U = 0 \\
(U, \partial_t U)(0) = \left( -\psi_1, -\frac{\lambda}{V_0 |x|} \psi_1 \right) \\
\end{array} \quad \text{in} \ D, \\
\end{aligned}
\tag{3.3, 3.4, 3.5}
\]
where \( t(w_0, w_1) \in H^1_0(D) \times L^2(D) \), \( F \in C([0, T); L^2(D)) \) and \( \lambda = \max_{1 \leq \mu \leq m} \lambda_{\mu} \) with \( \lambda_{\mu} = \frac{(\mu - 1)(N - 3 + \mu)}{2} + 1 \).

The following lemma provides the existence of a unique solution \( \psi_1 \) of (3.3) and its estimate.

**Lemma 3.7.** Assume that \( w_1 \in L^2(D) \). Then there exists a unique solution \( \psi_1 \in H^2(D) \cap H^1_0(D) \) of (3.3) for every \( \lambda \geq 0 \). Moreover, if \( \mu \in \mathbb{R} \) and \( \lambda \geq \lambda_{\mu} = \frac{(\mu - 1)(N - 3 + \mu)}{2} + 1 \), then

\[
\int_D \psi_1^2 |x|^{\mu - 3} \, dx + \int_D |\nabla \psi_1|^2 |x|^{\mu - 1} \, dx \leq \int_D w_1^2 |x|^{\mu + 1} \, dx.
\]

In particular, if \( w_0 \in H^1_0(D) \), then one has

\[
\int_D |\nabla w_0|^2 |x|^{\mu + 1} \, dx \leq 3 \int_D |\nabla w_0|^2 |x|^{\mu + 1} \, dx + \frac{3\lambda^2}{\lambda_{\mu}} \int_D w_1^2 |x|^{\mu + 1} \, dx.
\]

**Proof.** Noting that 0 is in the resolvent of \(-\Delta\) in bounded domain and \( \frac{\lambda}{\lambda_{\mu}} \) is nonnegative and bounded, we see that there exists a unique solution \( \psi_1 \in H^2(D) \cap H^1_0(D) \). Then by integration by parts twice, we see

\[
\int_D (-\Delta \psi_1) \psi_1 |x|^{\mu - 1} \, dx = \int_D |\nabla \psi_1|^2 |x|^{\mu - 1} \, dx + (\mu - 1) \int_D \psi_1 \nabla \psi_1 \cdot x |x|^{\mu - 3} \, dx
\]

\[
= \int_D |\nabla \psi_1|^2 |x|^{\mu - 1} \, dx - \frac{(\mu - 1)(N - 3 + \mu)}{2} \int_D \psi_1^2 |x|^{\mu - 3} \, dx.
\]

Therefore if \( \lambda \geq \frac{(\mu - 1)(N - 3 + \mu)}{2} + 1 \), we have

\[
\int_D \psi_1^2 |x|^{\mu - 3} \, dx + \int_D |\nabla \psi_1|^2 |x|^{\mu - 1} \, dx \leq \int_D \left( \frac{\lambda}{|x|^2} \psi_1 - \Delta \psi_1 \right) \psi_1 |x|^{\mu - 1} \, dx
\]

\[
\leq \left( \int_D w_1^2 |x|^{\mu + 1} \, dx \right)^{\frac{1}{2}} \left( \int_D \psi_1^2 |x|^{\mu - 3} \, dx \right)^{\frac{1}{2}}.
\]

Neglecting the second term of the left-hand side of the above inequality implies

\[
\int_D \psi_1^2 |x|^{\mu - 3} \, dx \leq \int_D w_1^2 |x|^{\mu + 1} \, dx
\]

and then

\[
\int_D \psi_1^2 |x|^{\mu - 3} \, dx + \int_D |\nabla \psi_1|^2 |x|^{\mu - 1} \, dx \leq \left( \int_D w_1^2 |x|^{\mu + 1} \, dx \right)^{\frac{1}{2}} \left( \int_D \psi_1^2 |x|^{\mu - 3} \, dx \right)^{\frac{1}{2}}
\]

\[
\leq \int_D w_1^2 |x|^{\mu + 1} \, dx.
\]

This is nothing but the first desired inequality. The second inequality can be proved directly via the definition of \( v_0 \) and the first inequality. \( \square \)
Next we consider the existence of a unique solution $v$ of (3.4) and space-time weighted estimates of $v$ and $\nabla v$.

**Lemma 3.8.** Assume that $^t(w_0, w_1) \in H_0^1(D) \times L^2(D)$ and $F \in C([0, T); L^2(D))$. Then there exists a unique solution $v$ of (3.4). Moreover, $v$ satisfies

$$
\int_D \frac{|v(t)|^2}{|x|} \Psi(t)^m dx + \int_0^t \int_D |\nabla v(s)|^2 \Psi(s)^m \, dx \, ds \\
\leq C_{m,1} \left( \int_D \left(|\nabla w_0|^2 + w_1^2 \right)(1 + |x|)^{m+1} \, dx + \int_0^t \int_D F(s)^2 \Psi(s)^{m+1} |x| \, dx \, ds \right)
$$

for some positive constant $C_{m,1}$ depending only on $N$, $m$ and $V_0$.

**Proof.** Choose $\beta$ satisfying $m < \beta < N - 1$ and $\beta(1 - 2\delta) = m$.

Then using the equation in (3.4) and Lemmas 2.4 (i) and 2.5, we have

$$
dt \int_D \frac{v^2}{|x|} \frac{1}{\Phi_\beta^{-2\delta}} dx = 2 \int_D \frac{\partial v}{|x|} \frac{1}{\Phi_\beta^{-2\delta}} dx - (1 - 2\delta) \int_D \frac{v^2}{\Phi_\beta^{-2\delta}} \partial_\beta \Phi_\beta dx \\
= 2 \int_D \frac{v(\Delta v + F)}{\Phi_\beta^{-2\delta}} dx - (1 - 2\delta) \int_D \frac{v^2}{\Phi_\beta^{-2\delta}} \Delta \Phi_\beta dx \\
\leq - \frac{2\delta}{1 - \delta} \int_D \frac{|\nabla v|^2}{\Phi_\beta^{-2\delta}} dx + 2 \left( \int_D \frac{v^2}{|x|} \Phi_\beta^2 \, dx \right)^{\frac{1}{2}} \left( \int_D \frac{F^2|x|\Psi}{\Phi_\beta^{-2\delta}} \, dx \right)^{\frac{1}{2}}
$$

Using the above estimate and integrating it over $[0, t]$, we deduce

$$
\int_D \frac{|v(t)|^2}{|x|} \Phi_\beta(t)^{1-2\delta} dx + \frac{2\delta}{1 - \delta} \int_0^t \int_D |\nabla v(s)|^2 \Phi_\beta(s)^{1-2\delta} \, dx \, ds \\
\leq \int_D \frac{v_0^2}{|x|} \Phi_\beta(0)^{1-2\delta} dx + \frac{2\delta}{1 - \delta} \int_0^t \int_D \frac{v(s)^2}{|x|} \Phi_\beta(s)^{1-2\delta} \, dx \, ds + \frac{1}{\varepsilon} \int_0^t \int_D \frac{F^2(s)^2 \Psi(s)^2}{\Phi_\beta(s)^{1-2\delta}} \, dx \, ds
$$

and therefore, it follows from Lemma 2.4 (iii), (iv) that

$$
V_0 \int_D \frac{|v(t)|^2}{|x|} \frac{\Psi(t)^m}{|x|} dx + \frac{2\delta}{1 - \delta} \int_0^t \int_D |\nabla v(s)|^2 \Psi(s)^m \, dx \, ds \\
\leq C \left( \int_D v_0^2 \frac{\Psi(0)^m}{|x|} dx + \frac{2\delta}{1 - \delta} \int_0^t \int_D v(s)^2 \frac{\Psi(s)^{m-1}}{|x|} \, dx \, ds + \frac{1}{\varepsilon} \int_0^t \int_D \frac{F^2(s)^2 \Psi(s)^{m+1}}{\Phi_\beta(s)^{1-2\delta}} \, dx \, ds \right),
$$

where $C = (C_{\Phi_\beta} c_{\Phi_\beta})^{1-2\delta}$ with $C_{\Phi_\beta} = C_{\Phi_\beta} \max \{1, V_0 \}$ and $c_{\Phi_\beta} = c_{\Phi_\beta} \min \{1, V_0 \}$.

Noting that by Lemma 2.6 one has

$$
\int_D v_0^2 \frac{\Psi(0)^m}{|x|} dx \leq \min \left\{ \frac{N - 1}{2}, \frac{N - 1 + m}{2} \right\} \int_D |\nabla v_0|^2 \Psi(0)^{m+1} \, dx,
$$

$$
\int_0^t \int_D \frac{v(s)^2}{|x|} \frac{\Psi(s)^m}{|x|} \, dx \, ds \leq \min \left\{ \frac{N - 1}{2}, \frac{N - 2 + m}{2} \right\} \int_0^t \int_D |\nabla v(s)|^2 \Psi(s)^m \, dx \, ds,
$$

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applying Lemma 3.7 with \( \mu = -1, m \), and choosing \( \varepsilon \) small enough, we obtain the desired estimate.

Next lemma gives a weighted estimate for \( \partial_t v \) and \( \nabla v \).

**Lemma 3.9.** Assume that \( (w_0, w_1) \in H_0^1(D) \times L^2(D) \) and \( F \in C([0,T);L^2(D)) \). Let \( v \) be the solution of (3.4). Then

\[
\int_D |\nabla v(t)|^2 \Psi(t)^{m+1} dx + \int_0^t \int_D |\partial_t v(s)|^2 \frac{\Psi(s)^{m+1}}{|x|} dx ds \\
\leq C_{m,2}' \left( \int_D (|\nabla w_0|^2 + w_1^2) (1 + |x|)^{m+1} dx + \int_0^t \int_D F(s)^2 \Psi(s)^{m+1} |x| dx ds \right)
\]

for some positive constant \( C_{m,2}' \) depending only on \( N, m \) and \( V_0 \).

**Proof.** Using the equation in (3.4), we have

\[
\frac{d}{dt} \int_D |\nabla v|^2 \Psi^{m+1} dx = 2 \int_D \nabla \partial_t v \cdot \nabla v \Psi^{m+1} dx + (m+1) \int_D |\nabla v|^2 \Psi^m dx \\
= -2 \int_D \partial_t v \Delta \Psi^{m+1} dx - 2(m+1) \int_D \partial_t v \frac{x}{|x|} \Psi^m dx \\
+ (m+1) \int_D |\nabla v|^2 \Psi^m dx \\
\leq -2V_0 \int_D \frac{(\partial_t v)^2}{|x|} \Psi^{m+1} dx + (m+1) \int_D |\nabla v|^2 \Psi^m dx \\
+ 2(m+1) \int_D |\partial_t v| |\nabla v| \Psi^m dx + 2 \int_D \partial_t v F \Psi^{m+1} dx.
\]

By the Young inequality, we have

\[
(m+1) \int_D |\partial_t v| |\nabla v| \Psi^m dx \leq \frac{V_0}{2} \int_D (\partial_t v)^2 \Psi^m dx + \frac{2(m+1)^2}{V_0} \int_D |\nabla v|^2 \Psi^m dx \\
\leq \frac{V_0}{2} \int_D (\partial_t v)^2 \frac{\Psi^{m+1}}{|x|} dx + \frac{2(m+1)^2}{V_0} \int_D |\nabla v|^2 \Psi^m dx
\]

and

\[
2 \int_D \partial_t v F \Psi^{m+1} dx \leq \frac{V_0}{2} \int_D (\partial_t v)^2 \frac{\Psi^{m+1}}{|x|} dx + \frac{2}{V_0} \int_D F^2 \Psi^{m+1} |x| dx.
\]

Combining the above three estimates, we obtain the desired estimate. \( \Box \)

The following is the reason why we introduce the problems (3.3)–(3.5). The solution \( w \) of (3.1) can be represented by the sum of two parts which are the solutions of (3.4) and (3.5). The idea of this kind of decomposition for the abstract evolution equations is introduced in Sobajima [28] and Ikehata–Sobajima [15].
Lemma 3.10. Assume that $t(w_0, w_1) \in H^1_0(D) \times L^2(D)$ and $F \in C([0, T); L^2(D))$. Let $w, v$ and $U$ be the solutions of (3.1), (3.4) and (3.5), respectively. Then $w = v + \partial_t U$.

Proof. Observe that $t(-\psi, -\frac{\lambda}{\|v\|_x}\psi) \in (H^2(D) \cap H^1_0(D)) \times H^1_0(\Omega)$. Let $U_n$ be a solution of

$$
\begin{cases}
\partial_t^2 U_n - \Delta U_n + \frac{V_0}{|x|} \partial_t U_n = -\partial_t v & \text{in } D \times (0, T), \\
U_n(x, t) = 0 & \text{on } \partial D \times (0, T), \\
(U_n, \partial_t U_n)(0) = (-J_n \psi_1, -J_n(\frac{\lambda}{\|v\|_x}\psi_1)) & \text{in } D,
\end{cases}
$$

(3.6)

where $J_n = (1 - \frac{1}{n})^{-1}$. Then we have $\Delta U_n \in C^1([0, T); L^2(D))$ and $t(U_n(t), \partial_t U_n(t)) \rightarrow t(U(t), \partial_t U(t))$ in $(H^2(D) \cap H^1_0(D)) \times H^1_0(D)$ as $n \rightarrow \infty$.

Put $w_n = v + \partial_t U_n$, and therefore, we have $w_n(t) \rightarrow v(t) + \partial_t U(t)$ in $H^1_0(D)$ as $n \rightarrow \infty$. On the other hand, we see from (3.6) that

$$
\partial_t w_n = \partial_t v + \partial_t^2 U_n = \Delta U_n - \frac{V_0}{|x|} \partial_t U_n = \Delta U_n - \frac{V_0}{|x|}(w_n - v).
$$

This gives $w_n(0) = w_0 + U_1 - J_n U_1 \rightarrow w_0$ in $H^1_0(D)$ and $\partial_t w(0) = -J_n \Delta \psi_1 + \frac{\lambda}{|x|} J_n(\psi_1) \rightarrow -\Delta \psi_1 + \frac{\lambda}{|x|} \psi_1 = \psi_1$ in $L^2(D)$ as $n \rightarrow \infty$. Moreover, we have

$$
\partial_t^2 w_n + \frac{V_0}{|x|} \partial_t w_n = \Delta \partial_t U_n + \frac{V_0}{|x|} \partial_t v = \Delta \partial_t U_n + \Delta v + F = \Delta w_n + F.
$$

The uniqueness of solutions for the homogeneous problem of (3.6) implies $w_n(t) \rightarrow w(t)$ in $H^1_0(D)$ as $n \rightarrow \infty$. The proof is complete. \hfill \Box

Using the property of $U$, now we prove the weighted $L^2$-estimates for the solution $w$ of (3.1).

Proposition 3.11. Assume that $t(w_0, w_1) \in H^1_0(D) \times L^2(D)$ and $F \in C([0, T); L^2(D))$. Let $w$ be the solution of (3.1). Then

$$
\int_D w(t)^2 |x|^{m-1} \, dx + \int_0^t \int_D w(s)^2 |x|^{m-2} \, dx \, ds 
\leq C_{m,3}' \left( \int_D (|\nabla w_0|^2 + w_1^2)(1 + |x|)^m \, dx + \int_0^t \int_D F(s)^2 |x|^{m+1} \, dx \, ds \right)
$$

for some positive constant $C_{m,3}'$ depending only on $N, m$ and $V_0$. 

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Applying Lemmas 3.7, 3.9 and 3.2 provides

Noting \( \tilde{E}_0(U; t) \) we already have

\[
\begin{align*}
\int_D v(t)^2 |x|^{m-1} dx + \int_0^t \int_D v(s)^2 |x|^{m-2} dx ds \\
\leq \left( \frac{2}{N-1+m} \right)^2 \int_D |\nabla v(t)|^2 |x|^{m+1} dx + \left( \frac{2}{N-2+m} \right)^2 \int_0^t \int_D |\nabla v(s)|^2 |x|^m dx ds \\
\leq \left( \frac{2}{N-1+m} \right)^2 \int_D |\nabla v(t)|^2 \Psi(t)|x|^{m+1} dx + \left( \frac{2}{N-2+m} \right)^2 \int_0^t \int_D |\nabla v(s)|^2 \Psi(s)^m |x| dx ds \\
\leq C \left( \int_D \left( |\nabla w_0|^2 + w_1^2 \right) (1 + |x|)^{m+1} dx + \int_0^t \int_D F(s)^2 \Psi(s)|x| dx ds \right)
\end{align*}
\]

(3.7)

for some positive constant \( C \).

Here here we divide the proof into two cases where \( m > 1 \) and \( m = 1 \). If \( m > 1 \), then applying Lemma 3.6 with \( \mu = m - 2 \in (-1, N-2) \) to the solution \( U \) of (3.5), we have

\[
E^2_{m-1}(U; t) + \delta_{m-2} \int_0^t E^2_{m-2}(U; s) ds \\
\leq C'_{m-1} \left( \int_D \left( |\nabla U(0)|^2 + (\partial_t U(0))^2 \right) |x|^{m-1} dx + \int_0^t \int_D |\partial_t v(s)|^2 |x|^m dx ds \right) \\
\leq C'_{m-1} \left( \int_D |\nabla \psi_1|^2 |x|^{m-1} dx + \frac{\lambda^2}{V_0^2} \int_D \psi_1^2 |x|^{m-2} dx + \int_0^t \int_D |\partial_t v(s)|^2 \frac{\psi(s)^{m+1}}{|x|} dx ds \right) .
\]

Applying Lemmas 3.7, 3.9 and 3.2 provides

\[
E_{m-1}(U; t) + \delta_{m-2} \int_0^t E_{m-2}(U; s) ds \\
\leq C' \left( \int_D \left( |\nabla w_0|^2 + w_1^2 \right) |x|(1 + |x|)^m dx + \int_0^t \int_D F(s)^2 \Psi(s)|x| dx ds \right)
\]

for some positive constant \( C' \). Noting that \( w = v + \partial_t U \) and

\[
E_{m-1}(U; t) \geq \int (\partial_t U)^2 |x|^{m-1} dx,
\]

we obtain the desired inequality.

If \( m = 1 \) (and therefore \( N = 3 \)), then we see from Lemma 3.4 with \( \mu = -1 \) that

\[
\frac{d}{dt} \tilde{E}_0(U; t) \leq -2V_0 \int_D (\partial_t U)^2 |x|^{-1} dx - 2 \int_D \partial_t U \partial_v dx \\
\leq -V_0 \int_D (\partial_t U)^2 |x|^{-1} dx + \frac{1}{V_0} \int_D (\partial_t v)^2 \frac{\Psi(s)^2}{|x|} dx .
\]

Noting \( \tilde{E}_0(U; t) = E_0(U; t) \) and integrating it over \([0, t]\) and using Lemma 3.9, we have

\[
E_0(U; t) + V_0 \int_0^t \int_D (\partial_t U)^2 |x|^{-1} dx ds \\
\leq E_0(U; 0) + \frac{C'^{m,2}}{V_0} \left( \int_D \left( |\nabla w_0|^2 + w_1^2 \right) (1 + |x|)^2 dx + \int_0^t \int_D F(s)^2 \Psi(s)^2 |x| dx ds \right) .
\]
The proof is complete.

At the end of this subsection, we prove Proposition 3.1 in the rest case.

**Proof of Proposition 3.1 when** \( m \in [N-2, N-1) \). In view of Lemma 3.6 for \( \mu = m \geq N-2 \), we already have

\[
\frac{d}{dt}E_{m+1}^\mu(w; t) + \delta_m E_{m}^\mu(w; t) \leq C_{m+1}' \left( \int_D w^2|x|^{m-2} \, dx + \int_D F^2|x|^{m+2} \, dx \right)
\]

\[
\leq C_{m+1}' \left( \int_D w^2|x|^{m-2} \, dx + \int_D F^2\Psi^{m+1}|x| \, dx \right).
\]

By virtue of Proposition 3.11, integrating the above inequality on \([0,t]\) and using Lemma 3.2, we deduce

\[
E_{m+1}^\mu(w; t) + \delta_m \int_0^t E_{m}^\mu(w; s) \, ds \leq C \left( E_{m+1}^\Psi(w; 0) + \int_0^t \int_D F(s)^2\Psi(s)^{m+1}|x| \, dx \, ds \right)
\]

for some positive constant \( C \). Then noting that \( m+1 < V_0 \) yields the validity of assumption of Lemma 3.2. Proceeding the same argument as in the case of \( 0 < m < N-2 \), we can obtain the desired estimate for \( N-2 \leq m < N-1 \). The proof is complete.

\[\square\]

4 The semilinear problem in exterior domains

To prove global existence for the semilinear problem (1.1), we use the following lemma which is so-called blowup alternative.

**Lemma 4.1.** Assume that \((u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)\). Let \( T_{\text{max}} \in (0, \infty] \) be the corresponding lifespan of the weak solution \( u \) of (1.1), that is,

\[
T_{\text{max}} = \sup \{ T \in (0, \infty) \mid \text{there exists a weak solution of (1.1) in } (0, T) \}.
\]

If \( T_{\text{max}} < \infty \), then \( \lim_{t \uparrow T_{\text{max}}} (\|u(t)\|_{H_0^1(\Omega)} + \|\partial_t u(t)\|_{L^2(\Omega)}) = \infty \).

**Proof of Theorem 1.2.** The proof is divided into two steps which are for the case of compactly supported initial data and the one of initial data with non-compact supports.

**Step 1 (compactly supported initial data).** Let \( u \) be a weak solution of (1.1) in \((0, T)\) and set \( \text{supp } u_0 \cup \text{supp } u_1 \subset \overline{B(0, R_0)} \) for some \( R_0 > 0 \) satisfying \( \mathbb{R}^N \setminus \Omega \subset B(0, R_0) \). Choose \( D = \Omega \cap B(0, R_0 + T) \). Since the problem (1.1) has the finite propagation property, \( u_D = u|_D \) (the restriction on \( D \)) can be regarded as the solution of

\[
\begin{cases}
\partial_t^2 u_D(x, t) - \Delta u_D(x, t) + \frac{V_0}{|x|} \partial_t u_D(x, t) = f(u_D(x, t)) & \text{in } D \times (0, T) \\
u_D(x, t) = 0 & \text{on } \partial D \times (0, T), \\
(u_D, \partial_t u_D)(0) = (u_0|_D, u_1|_D) & \text{in } D.
\end{cases}
\]

\[\square\]
Here we take \( m = \frac{4}{p-1} - N + 1 \in (N-3, N-1) \), that is, \( m \) satisfies \( p = 1 + \frac{4}{N-1+m} \). Then the assumption for the initial value can be written as follows:

\[
E_{m+1}^\Psi(u_D; 0) = \int_\Omega \left( |\nabla u_0|^2 + u_0^2 \right) (1 + |x|)^{m+1} \, dx \leq \delta,
\]

where \( \delta \) will be chosen later. Noting that \(|f(u_D)| = |u_D|^p\), by Proposition 3.1 we have

\[
E_{m+1}^\Psi(u_D; t) + \delta m \int_0^t E_{m}^\Psi(u_D; s) \, ds
\leq K_m \left( E_{m+1}^\Psi(u_D; 0) + \int_0^t \int_D |u_D(s)|^2 |x|^m \, dx \, ds \right)
\]

for \( t \in [0, T) \). Observe that Lemma 2.3 with \( \mu = m + \frac{p-1}{p} \) and \( \mu = 0 \) respectively implies

\[
\int_D |u_D|^{2p}|x|^{m+2} \, dx \leq C_{GN,m+\frac{p-1}{p}} \left( \int_D |\nabla u_D|^2 |x|^{m+\frac{p-1}{p}} \, dx \right)^p
\]

\[
\leq C_{GN,m+\frac{p-1}{p}} \left( \int_D |\nabla u_D|^2 |x|^{m+1} \, dx \right)^{p-1} \int_D |\nabla u_D|^2 |x|^{m} \, dx
\]

\[
\leq C_{GN,m+\frac{p-1}{p}} \left( E_{m+1}^\Psi(u_D; t) \right)^{p-1} \int_D |\nabla u_D|^2 \Psi^m \, dx
\]

and

\[
(1 + t)^{m+1} \int_D |u_D|^{2p}|x| \, dx
\]

\[
\leq C_{GN,0}^\phi (1 + t)^{m+1} \left( \int_D |u_D|^{2p}|x|^{m+2} \, dx \right)^{1-\theta} \left( \int_D |u_D|^{2p}|x|^{-2+(N-2)(p-1)} \, dx \right)^{\theta}
\]

\[
\leq C_{GN,0}^\phi (1 + t)^{(m+1)\theta-\theta} \left( \int_D |u_D|^{2p}|x|^{m+2} \, dx \right)^{1-\theta} \left( \int_D |\nabla u_D|^2 \, dx \right)^{\theta}
\]

\[
\leq C_{GN,0}^\phi \left( \int_D |u_D|^{2p}|x|^{m+2} \, dx \right)^{1-\theta} \left( \int_D |\nabla u_D|^2 \Psi^{m+1} \, dx \right)^{(p-1)\theta} \left( \int_D |\nabla u_D|^2 \Psi^m \, dx \right)^{\theta},
\]

where \( \theta = \frac{m+1}{m+4-(N-2)(p-1)} \). In view of the above inequalities, by \( \frac{1}{2m} \Psi^{m+1} \leq (1 + t)^{m+1} + |x|^{m+1} \) we deduce

\[
\int_0^t \int_D |u_D(s)|^{p-1} u_D(s)|^2 \Psi(s)^{m+1} |x| \, dx \, ds \leq C \left( E_{m+1}^\Psi(u_D; t) \right)^{p-1} \int_0^t E_{m}^\Psi(u_D; s) \, ds
\]

for some positive constant \( C \). Therefore putting

\[
M_{m+1}(t) = E_{m+1}^\Psi(u_D; t) + \delta_m \int_0^t E_{m}^\Psi(u_D; s) \, ds,
\]

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by (4.2) we obtain
\[ M_{m+1}(t) \leq C\left(\delta + \left(M_{m+1}(t)\right)^p\right) \]
for some positive constant \(C\). If we choose \(\delta\) sufficiently small, then by the continuity of \(M_m\) we obtain that there exists a positive constant \(C''\) independent of \(T\) such that
\[ \int_{\Omega} \left(\left|\nabla u(t)\right|^2 + (\partial_t u(t))^2\right) (1 + t + |x|)^{m+1} \, dx \leq M_{m+1}(t) \leq C'' \delta, \quad t \in [0, T). \]

The blowup alternative (Lemma 4.1) with the above estimate implies \(T_{\text{max}} = \infty\), that is, there exists a global-in-time (weak) solution \(u\) which satisfies
\[ \int_{\Omega} \left(\left|\nabla u\right|^2 + (\partial_t u)^2\right) (1 + t + |x|)^{m+1} \, dx \leq C'' \delta, \quad t \in [0, \infty). \]

**Step 2 (initial data with non-compact supports).** In this case we use a family of cut-off functions \(\{\zeta_n\}_{n \in \mathbb{N}}\) defined by
\[ \zeta_n(x) = \eta\left(\frac{\log |x|}{n}\right), \quad x \in \Omega \]
where \(\eta \in C^1(\mathbb{R})\) satisfies \(\eta(s) = 1\) on \((-\infty, 1]\), \(\eta(s) = 0\) on \([2, \infty)\), \(\eta'(s) \leq 0\) on \(\mathbb{R}\).
Assume that
\[ \int_{\Omega} \left(\left|\nabla u_0\right|^2 + u_1^2\right) (1 + |x|)^{m+1} \, dx \leq \frac{\delta}{3}. \]

By Lemma 2.1, we see by \(1 + |x| \leq 2 \frac{m+2}{m+1} (1 + |x|^{m+1})\) that
\[
\begin{align*}
\int_{\Omega} \left(\left|\nabla (\zeta_n u_0)\right|^2 + (\zeta_n u_1)^2\right) (1 + |x|)^{m+1} \, dx \\
&\leq \int_{\Omega} \zeta_n^2 \left(\left|\nabla u_0\right|^2 + u_1^2\right) (1 + |x|)^{m+1} \, dx \\
&\quad + 2 \frac{\|\eta\|_{L^\infty}}{n} \int_{\Omega} \zeta_n |\nabla u_0| \frac{(1 + |x|)^{m+1}}{|x|} \, dx + \frac{\|\eta\|_{L^\infty}^2}{n^2} \int_{\Omega} u_0^2 \frac{(1 + |x|)^{m+1}}{|x|^2} \, dx \\
&\leq 2 \int_{\Omega} \left(\left|\nabla u_0\right|^2 + u_1^2\right) (1 + |x|)^{m+1} \, dx + 2 \frac{m+1}{m+2} \frac{\|\eta\|^2_{L^\infty}}{n^2} \int_{\Omega} u_0^2 \left(\frac{1}{|x|^2} + |x|^{m-1}\right) \, dx \\
&\leq 2 \int_{\Omega} \left(\left|\nabla u_0\right|^2 + u_1^2\right) (1 + |x|)^{m+1} \, dx + \frac{C'''}{n^2} \int_{\Omega} \left|\nabla u_0\right|^2 (1 + |x|^{m+1}) \, dx
\end{align*}
\]
for some positive constant \(C'''\). Therefore if \(n \geq \sqrt{C'''}\), then
\[ \int_{\Omega} \left(\left|\nabla (\zeta_n u_0)\right|^2 + (\zeta_n u_1)^2\right) (1 + |x|)^{m+1} \, dx \leq \delta. \]

Then by the conclusion of Step 1, for every \(n \in \mathbb{N}\), the problem
\[
\begin{cases}
\partial_t^2 u_n(x, t) - \Delta u_n(x, t) + \frac{V_0}{|x|} \partial_t u_n(x, t) = f(u_n(x, t)) & \text{in } \Omega \times (0, \infty), \\
u_n(x, t) = 0 & \text{on } \partial \Omega \times (0, \infty), \\
(u_n, \partial_t u_n)(0) = (\zeta_n u_0, \zeta_n u_1) & \text{in } \Omega
\end{cases}
\]
has a unique (global-in-time) solution \( u_n \) satisfying

\[
\int_\Omega \left( |\nabla u_n|^2 + (\partial_t u_n)^2 \right)(1 + t + |x|)^{m+1} \, dx \leq C'' \delta, \quad t \in [0, \infty).
\]

It is worth noticing that the constant \( C'' \) is independent of \( n \) (the size of support of initial data). Since the solution of (1.1) has a continuous dependence on the initial data \( u \in \mathcal{P} \), we can obtain that the sequence \( \{u_n\}_{n \in \mathbb{N}} \) converges to \( u \) in \( C([0, \infty); H^1_0(\Omega)) \cap C^1([0, \infty); L^2(\Omega)) \) (uniformly in any compact interval), which is the global-in-time weak solution of (1.1). The proof is complete. \( \Box \)

5 Remark on small data blowup when \( 1 < p \leq 1 + \frac{2}{N-1} \)

To end the present paper, we prove the blowup phenomena for arbitrary small initial data when \( f(u) = |u|^p \) (Proposition 1.4).

**Proof of Proposition 1.4.** We may assume \( T_{\text{max}} > T_0 = \max\{1, 2R_0, R_0^2\} \), where \( R_0 > 1 \) satisfies \( \text{supp } u_0 \cup \text{supp } u_1 \subset \overline{B(0, R_0)} \). Observe that \( \psi(x) = 1 - |x|^{2-N} \) is the harmonic function satisfying the Dirichlet boundary condition on \( \partial \Omega \). Fix \( \eta \in C^\infty(\mathbb{R}) \) satisfying \( \eta'(s) \leq 0, \eta(s) = 1 \) for \( s \in (-\infty, \frac{1}{2}] \) and \( \eta(s) = 0 \) for \( s \in [1, \infty) \). Setting \( \eta_T(t) = \eta(t/T) \), we have

\[
\int_\Omega |u|^p \psi \eta^2_T \, dx = \int_\Omega \left( \partial_t^2 u - \Delta u + \frac{V_0}{|x|} \partial_t u \right) \psi \eta^2_T \, dx
\]

\[
= \frac{d}{dt} \int_\Omega \left( \partial_t u \eta^2_T - u \partial_t (\eta^2_T) + \frac{V_0}{|x|} u \eta^2_T \right) \psi \, dx
\]

\[
+ \int_\Omega u \left( \partial_t^2 (\eta^2_T) - \frac{V_0}{|x|} \partial_t (\eta^2_T) \right) \psi \, dx,
\]

where \( p' = p/(p-1) \) is the Hölder conjugate of \( p \). Noting the finite propagation property \( \text{supp } u(t) \subset B(0, R_0 + t) \) and integrating the above inequality over \([0, T]\), we see by the Young inequality that

\[
\int_\Omega \left( u_1 + \frac{V_0}{|x|} u_0 \right) \psi \, dx + \int_0^T \int_\Omega |u|^p \psi \eta^2_T \, dx \, dt
\]

\[
= 2p' \int_{T/2}^T \int_\Omega \left( \eta (\partial_t^2 \eta_T) + (p' - 1)|\partial_t \eta_T|^2 - \frac{V_0}{|x|} \eta_T \partial_t \eta_T \right) \eta^2_T \eta^{2p'-2} \psi \, dx \, dt
\]

\[
\leq C\|\psi\|_{W^{2,\infty}}^2 \left( \int_{T/2}^T \int_\Omega |u|^p \psi \eta^2_T \, dx \, dt \right)^{\frac{1}{p'}} \left( \int_{T/2}^T \int_{B(0,R_0+t)\setminus B(0,1)} \left( \frac{1}{T^2} + \frac{1}{|x|} \right)^{p'} \, dx \, dt \right)^{1-\frac{1}{p'}}.
\]

Noting that

\[
\int_{T/2}^T \int_{B(0,R_0+t)\setminus B(0,1)} \left( \frac{1}{T^2} + \frac{1}{T|x|} \right)^{p'} \, dx \, dt \leq \begin{cases} \frac{C T^{-\frac{1}{p-1}}}{p-1} & \text{if } 1 < p < \frac{N}{N-1}, \\ \frac{C T^{N-1-rac{2}{p-1}} \log T}{p-1} & \text{if } p = \frac{N}{N-1}, \\ \frac{C T^{N-1-\frac{2}{p-1}}}{p-1} & \text{if } p > \frac{N}{N-1}, \end{cases}
\]

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we can see that if $1 < p < 1 + \frac{2}{N-1}$, then

$$0 < \int_{\Omega} \left( u_1 + \frac{V_0}{|x|} u_0 \right) \psi \, dx \leq \begin{cases} 
CT^{-\frac{1}{p-1}} & \text{if } 1 < p < \frac{N}{N-1}, \\
CT^{N-1-\frac{2}{p-1}} \log T & \text{if } p = \frac{N}{N-1}, \\
CT^{N-1-\frac{2}{p-1}} & \text{if } p > \frac{N}{N-1}.
\end{cases}$$

Since the right-hand side of the above inequality converges to 0 as $T \to \infty$, and the choice of $T$ is arbitrary in $(T_0, T_{\max})$, the lifespan $T_{\max}$ must be finite. In the critical case $p = 1 + \frac{2}{N-1}$, taking an auxiliary function

$$Y(T) = \int_{\Omega} \left( u_1 + \frac{V_0}{|x|} u_0 \right) \psi \, dx + \int_0^T \left( \int_{\tau/2}^\tau \int_{\Omega} |u|^p \psi \eta^2 \frac{d}{\sqrt{r}} \eta \, dx \, dt \right) \frac{d\tau}{\tau}$$

as in [11, Lemma 3.10], we can deduce $Y(T)^p \leq CTY'(T)$ for $T \in (T_0, T_{\max})$. This implies that $Y(T)^{1-p}$ becomes negative if $T$ can be arbitrary large, which contradicts $Y(T) > 0$. Therefore, we obtain an upper bound for $T_{\max}$. The proof is complete.

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References

[1] W. Arendt, G.R. Goldstein, J.A. Goldstein, Outgrowths of Hardy’s inequality, Recent advances in differential equations and mathematical physics, 51–68, Contemp. Math. 412, Amer. Math. Soc., Providence, RI, 2006. MR2259099

[2] R. Beals, R. Wong, “Special functions,” A graduate text. Cambridge Studies in Advanced Mathematics 126, Cambridge University Press, Cambridge, 2010. MR3524801

[3] H. Brezis, “Functional analysis, Sobolev spaces and partial differential equations,” Universitext. Springer, New York, 2011. xiv+599 pp. MR2759829

[4] L. Caffarelli, R. Kohn, L. Nirenberg, First order interpolation inequalities with weights, Compositio Math. 53 (1984), 259–275. MR0768824

[5] T. Cazenave, A. Haraux, “An introduction to semilinear evolution equations”. Oxford Lecture Series in Mathematics and its Applications 13. The Clarendon Press, Oxford University Press, New York, 1998. MR1691574

[6] W. Dai, H. Kubo, M. Sobajima, Blow-up for Strauss type wave equation with damping and potential, Nonlinear Anal. Real World Appl. 57 (2021), 103195. MR4130094

[7] W. Dan, Y. Shibata, On a local energy decay of solutions of a dissipative wave equation, Funkcial. Ekvac. 38 (1995), 545–568. MR1374437

24
[8] A.Z. Fino, H. Ibrahim, A. Wehbe, *A blow-up result for a nonlinear damped wave equation in exterior domain: the critical case*, Comput. Math. Appl. 73 (2017), 2415–2420. MR3648022

[9] H. Fujita, *On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$*. J. Fac. Sci. Univ. Tokyo Sect. I 13 (1966), 109–124. MR0214914

[10] N. Hayashi, E.I. Kaikina, P.I. Naumkin, *Damped wave equation with super critical nonlinearities*, Differential Integral Equations 17 (2004), 637–652. MR02054939

[11] M. Ikeda, M. Sobajima, *Sharp upper bound for lifespan of solutions to some critical semilinear parabolic, dispersive and hyperbolic equations via a test function method*, Nonlinear Anal. 182 (2019), 57–74. MR3894246

[12] M. Ikeda, M. Sobajima, *Life-span of blowup solutions to semilinear wave equation with space-dependent critical damping*, Funkcial. Ekvac. 64 (2021), 137–162. MR4316930

[13] R. Ikehata, *Global existence of solutions for semilinear damped wave equation in 2-D exterior domain*, J. Differential Equations 200 (2004), 53–68. MR2046317

[14] R. Ikehata, T. Matsuyama, *$L^2$-behaviour of solutions to the linear heat and wave equations in exterior domains*, Sci. Math. Jpn. 55 (2002), 33–42. MR1885774

[15] R. Ikehata, M. Sobajima, *Singular limit problem of abstract second order differential equations*, preprint. (arXiv:1912.10181v1 [math.AP])

[16] R. Ikehata and K. Tanizawa, *Global existence of solutions for semilinear damped wave equations in $\mathbb{R}^N$ with noncompactly supported initial data*, Nonlinear Anal. 61 (2005), 1189–1208. MR2131649

[17] R. Ikehata, G. Todorova, B. Yordanov, *Critical exponent for semilinear wave equations with space-dependent potential*, Funkcial. Ekvac. 52 (2009), 411–435. MR2589664

[18] R. Ikehata, G. Todorova, B. Yordanov, *Optimal decay rate of the energy for wave equations with critical potential*. J. Math. Soc. Japan 65 (2013), 183–236. MR3034403

[19] F. John, *Blow-up of solutions for quasi-linear wave equations in three space dimensions*, Commun. Pure Appl. Math. 34 (1981), 29–51. MR0526180

[20] T. Kato, *Blow up of solutions of some nonlinear hyperbolic equations*, Commun. Pure Appl. Math. 33 (1980), 501–505. MR0575735

[21] N.-A. Lai, Z. Tu, *Strauss exponent for semilinear wave equations with scattering space dependent damping*, J. Math. Anal. Appl. 489 (2020), 124189. MR4095813

[22] X. Li, *Critical exponent for semilinear wave equation with critical potential*, NoDEA Nonlinear Differential Equations Appl. 20 (2013), 1379–1391. MR3057181

[23] G. Metafune, M. Sobajima, C. Spina, *Weighted Calderón–Zygmund and Rellich inequalities in $L^p$, Math. Ann. 361 (2015), 313–366. MR3302622
K. Nishihara, M. Sobajima, Y. Wakasugi, *Critical exponent for the semilinear wave equations with a damping increasing in the far field*, NoDEA Nonlinear Differential Equations Appl. 25 (2018), no. 6, Paper No. 55, 32 pp. MR3878674

T. Ogawa, H. Takeda, *Non-existence of weak solutions to nonlinear damped wave equations in exterior domains*, Nonlinear Anal. 70 (2009), 3696–3701. MR2504456

K. Ono, *Decay estimates for dissipative wave equations in exterior domains*, J. Math. Anal. Appl. 286 (2003), 540–562. MR2008848

M. Sobajima, *Global existence of solutions to semilinear damped wave equation with slowly decaying initial data in exterior domain*, Differential Integral Equations 32 (2019), 615–638. MR4021256

M. Sobajima, *Higher order asymptotic expansion of solutions to abstract linear hyperbolic equations*, Math. Ann. 380 (2021), 1–19. MR4263676

M. Sobajima, Y. Wakasugi, *Weighted energy estimates for wave equation with space-dependent damping term for slowly decaying initial data*, Commun. Contemp. Math. 21 (2019), 1850035. MR3980691

G. Todorova, B. Yordanov, *Critical exponent for a nonlinear wave equation with damping*, J. Differential Equations 174 (2001), 464–489. MR1846744

B. Yordanov, Q.S. Zhang, *Finite time blow up for critical wave equations in high dimensions*, J. Funct. Anal. 231 (2006), 361–374. MR2195336

Q.S. Zhang, *A blow-up result for a nonlinear wave equation with damping: the critical case*, C. R. Acad. Sci. Paris Sér. I Math. 333 (2001), 109–114. MR1847355

Y. Zhou, *Blow up of solutions to semilinear wave equations with critical exponent in high dimensions*, Chin. Ann. Math., Ser. B 28 (2007), 205–212. MR3169791