Abstract. For $k \geq 1$, denote by $p_k(n)$ the number of partitions of an integer $n$ into $k$-th powers. In this note, we apply the saddle-point method to provide a new proof for the well-known asymptotic expansion of $p_k(n)$. This approach turns out to significantly simplify those of Wright (1934), Vaughan (2015) and Gafni (2016).

1. Introduction

Let $p(n)$ denote, as usual, the number of unrestricted partitions of an integer $n$, i.e. the number of solutions to the equation

$$n = a_1 + a_2 + \cdots + a_d,$$

where $d \geq 1$ and the $a_j$ are positive integers such that $a_1 \geq a_2 \geq \cdots \geq a_d \geq 1$. In 1917, Hardy and Ramanujan [3] proved the asymptotic formula

$$(1.1) \quad p(n) \sim \frac{\exp(\pi \sqrt{2n/3})}{4\sqrt{3}n} \quad (n \to \infty).$$

by using modular properties of Jacobi’s $\Delta$-function.

More generally, given an integer $k \geq 1$, let $p_k(n)$ denote the number of partitions of the integer $n$ into $k$-th powers, i.e. the number of solutions to the equation

$$n = a_1^k + \cdots + a_d^k,$$

where, as before, $d \geq 1$ and $a_1 \geq a_2 \geq \cdots \geq a_d \geq 1$. Thus, $p_1(n) = p(n)$.

In 1918, Hardy and Ramanujan [4] stated without proof the asymptotic formula

$$(1.2) \quad p_k(n) \sim \frac{b_k \exp\{c_k n^{1/(k+1)}\}}{n^{(3k+1)/(2k+2)}} \quad (n \to \infty),$$

where the constants $b_k$ and $c_k$ are defined by

$$(1.3) \quad a_k := \{k^{-1}\zeta(1+k^{-1})\Gamma(1+k^{-1})\}^{k/(k+1)},$$

$$(1.4) \quad b_k := \frac{a_k}{(2\pi)^{(k+1)/2} \sqrt{(1+1/k)}},$$

$$(1.5) \quad c_k := (k+1)a_k,$$

and $\zeta$ is the Riemann zeta-function. In 1934, introducing a number of complicated objects including generalised Bessel functions, Wright [8] obtained an asymptotic expansion of $p_k(n)$: for any integer $k \geq 1$, there is a real sequence $\{\alpha_{kj}\}_{j \geq 1}$ such that, for any $J \geq 1$, we have

$$(1.6) \quad p_k(n) = \frac{b_k \exp(c_k(n+h_k)^{1/(k+1)})}{(n+h_k)^{(3k+1)/(2k+2)}} \left\{1 + \sum_{1 \leq j < J} \frac{(-1)^j \alpha_{kj}}{(n+h_k)^j/(k+1)} + O\left(\frac{1}{n^{J/(k+1)}}\right)\right\},$$

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where

\[ h_k := \begin{cases} 
0 & \text{if } k \text{ is even}, \\
\frac{(-1)^{(k+1)/2}(2\pi)^{-(k+1)}k!\zeta(k+1)}{(2\pi)^{(k+1)/2}k^{(k+1)/2}} & \text{if } k \text{ is odd},
\end{cases} \]

and the implied constant depends at most on \( J \) and \( k \). Apart from an explicit formula for \( a_{k1} \), no further information was given about the \( \alpha_{kj} \) beyond the statement that they depend only on \( k \) and \( j \) and that they “may be calculated with sufficient labour for any given values of \( k, j \).”

Of course, taking \( J = 1 \) in (1.6) yields an effective form of (1.2).

More recently, appealing to a relatively simple implementation of the Hardy-Littlewood circle method, Vaughan [6] obtained an explicit version of (1.6) in the case \( k = 2 \) and Gafni [2] generalised the argument to arbitrary, fixed \( k \).

Gafni states his result in the following way. Let \( X_k(n) \) denote the real solution to the equation

\[ n = (a_k X)^{1+1/k} - \frac{1}{2}X - \frac{1}{2}\zeta(-k), \]

and write

\[ Y_k(n) := (1 + 1/k)a_k^{1+1/k}X^{1/k} - \frac{1}{4}. \]

Then, given any \( k \geq 1 \), there is a real sequence \( \{\beta_{kj}\}_{j \geq 1} \) such that for any fixed \( J \geq 1 \), we have

\[ p_k(n) = \exp\left\{ \frac{(k+1)}{(2\pi)^{(k+1)/2}X^{3/2}Y^{1/2}} \left( 1 + \sum_{1 \leq j < J} \frac{\beta_{kj}}{Y^j} + O\left( \frac{1}{Y^J} \right) \right) \right\}. \]

It may be checked that the asymptotic formulae (1.6) and (1.9) match each other.

In this note, our aim is to provide a new proof of (1.2), and indeed also of (1.6) and (1.9), by applying the saddle-point method along lines very similar to those employed in [5] in the case of \( p(n) \). Our approach appears to be significantly simpler than those of the quoted previous works.

The constants \( b_k \) and \( c_k \) being defined as in (1.4) and (1.5), we can state the following.

**Theorem 1.** Let \( k \geq 1 \) be a fixed integer. There is a real sequence \( \{\gamma_{kj}\}_{j \geq 1} \) such that, for any given integer \( J \geq 1 \), we have

\[ p_k(n) = \frac{b_k \exp(c_k n^{1/(k+1)})}{n^{(3k+1)/(2k+2)}} \left( 1 + \sum_{1 \leq j < J} \frac{\gamma_{kj}}{n^{j/(k+1)}} + O\left( \frac{1}{n^{J/(k+1)}} \right) \right) \]

uniformly for \( n \geq 1 \). The implied constant depends at most on \( J \) and \( k \).

The coefficients \( \gamma_{kj} \) can be made explicit directly from the computations in our proof. For instance, we find that \( \gamma_{k1} = -(11k^2 + 11k + 2)/(24k\epsilon_k) \) when \( k \geq 2 \), in accordance with the expression given by Wright. (It can be checked, after some computations, that it matches Gafni’s formula too.) We also have

\[ \gamma_{11} = -\frac{1}{12}\epsilon_1 - 1/\epsilon_1 = -\sqrt{\frac{2}{3}} \left( \frac{\pi}{48} + \frac{3}{2\pi} \right). \]

It may be seen that \( |\gamma_{kj}| \) grows like \( \Gamma(j)e^{O(j)} \) and thus that the series \( \sum_{j \geq 1} \gamma_{kj}z^j \) has radius of convergence 0.
2. Technical preparation

Define

\[ F_k(s) := \sum_{n \geq 0} p_k(n) e^{-ns} \quad (\Re s > 0), \]

so that

\[ p_k(n) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} F_k(s) e^{ns} ds = \frac{1}{2\pi} \int_{-\pi}^{\pi} F_n(\sigma + i\tau) e^{n\sigma + i\tau} d\tau. \]

According to the principles of the saddle-point method, we aim at selecting the integration abscissa \( \sigma \) as a solution \( \sigma_n \) of \( -F'_k(\sigma)/F_k(\sigma) = n \). We plainly have

\[ F_k(s) = \prod_{m \geq 1} (1 - e^{-ms})^{-1} \quad (\Re s > 0). \]

Thus, in the same half-plane, we may define a determination of \( \log F_k(s) \) by the formula

\[ \Phi_k(s) := \sum_{m \geq 1} \log \left( \frac{1}{1 - e^{-ms}} \right) \]

where the complex logarithms are taken in principal branch. Expanding throughout and inverting summations, we get

\[ \Phi_k(s) = \sum_{m \geq 1} \sum_{n \geq 1} \frac{e^{-msn}}{n} = \sum_{r \geq 1} \frac{w_k(r)}{r} e^{-rs}, \quad -\Phi_k'(s) = \sum_{r \geq 1} w_k(r)e^{-rs} \quad (\Re s > 0), \]

where

\[ w_k(r) := \sum_{m^r \geq 1} m^k \quad (r \geq 1). \]

Hence \( -\Phi_k'(\sigma) \) decreases from \( +\infty \) to \( 0^+ \) on \((0, \infty)\), and so the equation \( -\Phi_k'(\sigma) = n \) has for each integer \( n \geq 1 \) a unique real solution \( \sigma_n = \sigma_n(k) \). Moreover, the sequence \( \{\sigma_n\}_{n \geq 1} \) is decreasing and the trivial estimates \( 1 \leq w_k(r) \leq r^2 \) yield \( 1/n \ll \sigma_n \ll 1/\sqrt{n} \).

We start with an asymptotic expansion for the derivatives \( \Phi_k^{(m)}(\sigma_n) \) in terms of powers of \( \sigma_n \). It turns out that all coefficients but a finite number vanish.

**Lemma 2.1.** Let \( J \geq 1 \), \( k \geq 1 \). As \( n \to \infty \), we have

\[ \Phi_k(\sigma_n) = \frac{k \alpha_1^{1/k}}{\sigma_n^{1/k}} + \frac{1}{2} \log \left( \frac{\sigma_n}{2\pi k^k} \right) + \frac{1}{2} \zeta(-k)\sigma_n + O(\sigma_n^J), \]

Moreover, for fixed \( m \geq 1 \),

\[ (-1)^m \Phi_k^{(m)}(\sigma_n) = \prod_{1 \leq \ell \leq m} \left( \ell + \frac{1}{k} \right) \frac{\alpha_1^{\ell+1/k}}{\sigma_n^{\ell+1/k}} - \frac{(m-1)!}{2\sigma_n^m} - \frac{1}{2} \delta_1 \zeta(-k) + O(\sigma_n^J), \]

where \( \delta_1m \) is Kronecker’s symbol.

**Proof.** Considering Mellin’s inversion formula

\[ e^{-s} = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \Gamma(z) s^{-z} \, dz \quad (\Re s > 0) \]

and the convolution identity

\[ \sum_{r \geq 1} \frac{w_k(r)}{r^{1+z}} = \zeta(z+1)\zeta(kz) \quad (\Re z > 1/k), \]
we derive from the series representation (2.4) the integral formula
\begin{equation}
\Phi_k(s) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \zeta(z+1)\zeta(kz)\Gamma(z) \frac{dz}{sz},
\end{equation}
and in turn
\begin{equation}
(-1)^m \Phi_k^{(m)}(\sigma_n) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \zeta(z+1)\zeta(kz)\Gamma(z+m) \frac{dz}{\sigma_n z+m},
\end{equation}
\(m \geq 0\).

Using the classical fact that \(\zeta(z)\) has finite order in any vertical strip \(a \leq \Re z \leq b\) \((a, b \in \mathbb{R}\) with \(a < b\)), or, in other words, satisfies
\[
\zeta(x+iy) \ll_{a,b} 1 + |y|^A \quad (a \leq x \leq b, |y| \geq 1),
\]
for suitable \(A = A(a, b)\), and invoking Stirling’s formula in the form
\[
|\Gamma(x+iy)| = \sqrt{2\pi |y|^{1-1/2}e^{-\pi|y|/2}} \{1 + O_{a,b}(1/y)\}
\]
we may move the line of integration to \(\Re z = -J - m - \frac{1}{2}\).

The shifted integral is clearly \(\ll \sigma_n^J\).

Let us first consider the case \(m = 0\). Then the crossed singularities are a pole of order 2 at \(z = 0\), and two simple poles at \(z = 1/k\) and \(z = -1\). Indeed, \(\zeta(z+1)\zeta(kz)\) vanishes at all negative integers \(\leq -2\), so the corresponding zeros compensate the poles of \(\Gamma(z)\) at negative integers \(\leq -2\).

The residue at \(z = 1/k\) is equal to
\[
k^{-1}\zeta(1+k^{-1})\Gamma(k^{-1})\sigma_n^{-1/k} = ka_k^{1+1/k}\sigma_n^{-1/k}.
\]
The residue at \(z = 0\) is the coefficient of \(z\) in the Taylor expansion of
\[
z^2 \zeta(z+1)\zeta(kz)\Gamma(z)\sigma_n^{-z} = z\zeta(z+1)\zeta(kz)\Gamma(z+1)\sigma_n^{-z}
\]
\[
= (1-\gamma z)\{\zeta(0)+k\zeta'(0)z\}(1+\gamma z)(1-z \log \sigma_n) + O(z^2)
\]
\[
= \zeta(0) + \{-\zeta(0) \log \sigma_n + k\zeta'(0)\}z + O(z^2).
\]
Since \(\zeta(0) = -\frac{1}{2}\) and \(\zeta'(0) = -\frac{1}{2} \log (2\pi)\), this residue equals \(\frac{1}{2} \log \{\sigma_n/(2\pi)^k\}\).

The residue at \(z = -1\) equals \(\frac{1}{2} \zeta(-k)\sigma_n\).

This completes the proof of (2.5).

When \(m = 1\), the three crossed singularities are simple poles. The residues at \(z = 1/k\), \(z = 0\) and \(z = -1\) are respectively \((1/k)\Gamma(1+1/k)\zeta(1+1/k)\sigma_n^{-1/k},-1/2\sigma_n\) and \(-\frac{1}{2} \zeta(-k)\).

When \(m \geq 2\), the only crossed singularities are two simple poles, at \(z = 1/k\) and \(z = 0\), with respective residues \((1/k)\Gamma(m+1/k)\zeta(1+1/k)\sigma_n^{-m-1/k}\) and \(-\frac{1}{2} \log \{\sigma_n/(2\pi)^k\}\). This proves (2.6).

**Lemma 2.2.** Let \(J \geq 1, k \geq 1, m \geq 1\).

(i) There is a real sequence \(\{a_k\}_{j \geq 1}\) with \(a_{k1} = -k/(2\pi), a_{k2} = k/(8\pi^2),\) such that
\begin{equation}
\sigma_n = \frac{a_k}{n^{k/(k+1)}} \left\{ 1 + \sum_{1 \leq j < k} \frac{a_{kj}}{n^{j/(k+1)}} + O\left(\frac{1}{n^{j/(k+1)}}\right) \right\} \quad (n \to \infty).
\end{equation}

(ii) There is a real sequence \(\{b_k\}_{j \geq 1}\) with \(b_{k1} = -a_{k1}/k\) such that, as \(n \to \infty\), we have
\begin{equation}
\Phi_k(\sigma_n) = ka_k n^{1/(k+1)} \left\{ 1 + \sum_{1 \leq j < k} \frac{b_{kj}}{n^{j/(k+1)}} + O\left(\frac{1}{n^{j/(k+1)}}\right) \right\} + \frac{1}{2} \log \left(\frac{\sigma_n}{(2\pi)^k}\right).
\end{equation}
We have Lemma 2.4. assumed, unless otherwise stated, to depend at most upon (2.2), we need to show that it is dominated by a small neighborhood of the saddle-point. The next result meets this requirement. Here and in the sequel, all constants

\[ \| \|_{(2.14)} \text{ and } \| \|_{(2.11)} \]

Noticing that

Proof. We infer from (2.6) that

\[ (2.12) \]

\[ n = \frac{a_k}{\sigma_n^{1+1/k}} - \frac{1}{2\sigma_n} - \frac{1}{2}\zeta(-k) + O(\sigma_n^J). \]

This immediately implies (2.9) by Lagrange’s inversion formula — see, e.g. [7, §7.32]. We may obtain an explicit expression for the \( a_{kj} \) from the formula

\[ (2.13) \]

\[ \sigma_n = \frac{a_k}{2\pi n^{k/(k+1)}} \int_{|z-1|=\varrho} \frac{zG(z)}{G(z)} \, dz + O\left( \frac{1}{n^{(k+1)/(k+1)}} \right) \]

where \( \varrho \) is a fixed, small positive constant and

\[ G(z) := z^{-1-1/k} - 1 - \frac{1}{2a_k \varrho^{1/(k+1)}} - \frac{\zeta(-k)}{2n} \]

This is classically derived from Rouché’s theorem and we omit the details. The values of \( a_{k1} \) and \( a_{k2} \) may be retrieved from the above or by formally inserting (2.9) into (2.12).

Inserting (2.13) back into (2.5) and (2.6) immediately yields (2.10) and (2.11).

The following statement appears in [1, Lemma 6.3]. Here and in the sequel, we employ the standard notation \( \| t \| := \min_{n \in \mathbb{Z}} |n - t| \) \( (t \in \mathbb{R}) \).

Lemma 2.3. Let \( \vartheta \in \mathbb{R}, a \in \mathbb{Z}, q \in \mathbb{N} \) with \( (a, q) = 1 \), \( \vartheta = a/q + \beta, \| \beta \| \leq 1/q^2, t \in \mathbb{N}, 0 \leq v < q \). Then there exist at most six integers \( r \) with \( 0 \leq r < q \) such that

\[ (2.14) \]

\[ \| \vartheta(tq + r) \| \in (v/q, (v + 1)/q). \]

With the aim of applying Laplace’s method to evaluate the integral on the right-hand side of (2.2), we need to show that it is dominated by a small neighbourhood of the saddle-point \( \sigma_n \). The next result meets this requirement. Here and in the sequel, all constants \( c_j \) \( (j \geq 0) \) are assumed, unless otherwise stated, to depend at most upon \( k \).

Lemma 2.4. We have

\[ (2.15) \]

\[ \frac{|F_k(\sigma_n + i\tau)|}{|F_k(\sigma_n)|} \leq \begin{cases} e^{-c_1 \tau^2 \sigma_n^{-2+1/k}} & \text{if } |\tau| \leq 2\pi \sigma_n, \\ e^{-c_2 \sigma_n^{-1/k}} & \text{if } 2\pi \sigma_n < |\tau| \leq \pi. \end{cases} \]

Proof. Noticing that

\[ 1 - e^{-m^k(\sigma_n+i\tau)} \]

\[ = 1 - e^{-m^k \sigma_n} \]

\[ + 4e^{-m^k \sigma_n} \sin^2 \left( \frac{1}{4} m^k \tau \right) \]

\[ \geq 1 - e^{-m^k \sigma_n} \]

\[ + 16e^{-m^k \sigma_n} \| m^k \tau/(2\pi) \| \]

\[ \geq 1 - e^{-m^k \sigma_n} \]

\[ + 16e^{-m^k \sigma_n} \| m^k \tau/(2\pi) \| \]

we can write

\[ \frac{|F_k(\sigma_n + i\tau)|^2}{|F_k(\sigma_n)|^2} \leq \prod_{m \geq 1} \left( 1 + \frac{16\| m^k \tau/(2\pi) \|^2}{e^{m^k \sigma_n} (1 - e^{-m^k \sigma_n})} \right)^{-1} \]

\[ \leq \prod_{(4\sigma_n)^{-1/k} < m \leq (2\sigma_n)^{-1/k}} \left( 1 + \frac{16\| m^k \tau/(2\pi) \|^2}{e^{m^k \sigma_n} (1 - e^{-m^k \sigma_n})} \right)^{-1}. \]
Thus, there is an absolute positive constant $c_3$ such that
\begin{equation}
\frac{|F_k(\sigma_n + i\tau)|}{|F_k(\sigma_n)|} \leq e^{-c_3 S(\tau; \sigma_n)}
\end{equation}
with
\[ S(\tau; \sigma_n) := \sum_{(4\sigma_n)^{-1/k} < m \leq (2\sigma_n)^{-1/k}} \| m^{k\tau}/(2\pi) \|^2. \]

If $|\tau| \leq 2\pi\sigma_n$ and $m \leq (2\sigma_n)^{-1/k}$, we have $|m^{k\tau}/(2\pi)| \leq \frac{1}{2}$. Thus
\begin{equation}
S(\tau; \sigma_n) = \sum_{(4\sigma_n)^{-1/k} < m \leq (2\sigma_n)^{-1/k}} m^{2k\tau^2/4\pi^2} \asymp \tau^2 \sigma_n^{-2-1/k}.
\end{equation}

When $2\pi\sigma_n < |\tau| \leq \pi$, Dirichlet’s approximation lemma guarantees that there exist integers $a \in \mathbb{Z}^*$ and $q \in [1, \sigma_n^{-1/k}]$ such that
\[ |\tau/(2\pi) - a/q| \leq \sigma_n^{1/k}/q \leq 1/q^2. \]

According to Lemma 2.3, there are $\gg q$ integers $v \in [1, q)$ such that the inequality
\[ \| m^{k\tau}/(2\pi) \| \gg v/q \]
holds for $\gg \sigma_n^{-1/k}$ integers $m$ from the interval $[(4\sigma_n)^{-1/k}, (2\sigma_n)^{-1/k}]$. Hence
\begin{equation}
S(\tau; \sigma_n) \gg \sigma_n^{-1/k}. \tag{3.18}
\end{equation}

3. Completion of the proof

Proposition 3.1. Let $k \geq 1$, $J \geq 1$. Then there is a real sequence $\{e_{kj}\}_{j \geq 1}$ such that for any integer $J \geq 1$ we have
\begin{equation}
p_k(n) = \frac{\exp(n\sigma_n + \Phi_k(\sigma_n))}{\sqrt{2\pi}\Phi'_k(\sigma_n)} \left( 1 + \sum_{2 \leq j < J} \frac{e_{kj}}{n^{j/(k+1)}} + O\left( \frac{1}{n^{J/(k+1)}} \right) \right) \quad (n \to \infty).
\end{equation}

Proof. By (2.2), we have
\begin{equation}
p_k(n) = \frac{e^{n\sigma_n}}{2\pi} \int_{-\pi}^{\pi} e^{\Phi_k(\sigma_n + i\tau) + i\tau^2} \, d\tau.
\end{equation}

From (2.15), we deduce that
\begin{equation}
\int_{\pi \sigma_n < |\tau| \leq \pi} e^{\Phi_k(\sigma_n + i\tau) + i\tau^2} \, d\tau \ll e^{\Phi_k(\sigma_n) - c_4 \sigma_n^{-1/k}}
\end{equation}
\begin{equation}
\int_{\sigma_n^{1-1/3k} < |\tau| \leq 2\pi \sigma_n} e^{\Phi_k(\sigma_n + i\tau) + i\tau^2} \, d\tau \ll e^{\Phi_k(\sigma_n) - c_4 \sigma_n^{-1/3k}}.
\end{equation}

Since these bounds are exponentially small with respect to the expected main term, it only remains to estimate the contribution of the interval $J := [-\sigma_n^{1+1/(3k)} \sigma_n^{1+1/(3k)}]$, corresponding to a small neighbourhood of the saddle-point.

In this range, we have
\[ \Phi_k(\sigma_n + i\tau) = \sum_{0 \leq m \leq 2J+1} \Phi_k^{(m)}(\sigma_n) (i\tau)^m + O\left( \frac{\tau^{2J+2}}{\sigma_n^{1/k+2J+2}} \right), \]
where the estimate for the error term follows from (2.6). The same formula ensures that $|\Phi_k^{(m)}(\sigma_n)r^m| \ll 1$ for $m \geq 3$. Thus for $\tau \in I$, we can write

$$e^{\Phi_k(\sigma_n+ir)+in\tau} = e^{\Phi_k(\sigma_n)-\frac{i}{2}\Phi'_k(\sigma_n)r^2}\left\{1 + \sum_{1 \leq \ell \leq 2J} \frac{1}{\ell!} \left( \sum_{3 \leq m \leq 2J+1} \frac{\Phi_k^{(m)}(\sigma_n)}{m!} (i\tau)^m \right) + O\left(\frac{\tau^{2J+2}}{\sigma_n^{1/2J+2}}\right)\right\}$$

$$= e^{\Phi_k(\sigma_n)-\frac{i}{2}\Phi'_k(\sigma_n)r^2}\left\{1 + \sum_{1 \leq \ell \leq 2J} \frac{1}{\ell!} \sum_{3 \leq m \leq 2J+1} \lambda_{k,\ell,m}(n)r^m + O\left(\frac{\tau^{2J+2}}{\sigma_n^{1/2J+2}}\right)\right\},$$

where

$$\lambda_{k,\ell,m}(n) := i^m \sum_{3 \leq m_1, \ldots, m_\ell \leq 2J+1} \prod_{r \leq \ell} \frac{\Phi_k^{(m_r)}(\sigma_n)}{m_r!}. \tag{3.4}$$

Since the contributions from odd powers of $\tau$ vanish, we get

$$\int_I e^{\Phi_k(\sigma_n+ir)+in\tau} d\tau = e^{\Phi_k(\sigma_n)} \left\{ I_0 + \sum_{1 \leq \ell \leq 2J} \frac{1}{\ell!} \sum_{3 \leq m \leq 2J+1} \lambda_{k,\ell,2m}(n) I_m + O(R) \right\}, \tag{3.5}$$

with

$$I_m := \int_I e^{\frac{i}{2}\Phi'_k(\sigma_n)r^2} r^{2m} d\tau, \quad R := \sigma_n^{-1/2J-2} \int_I e^{\frac{i}{2}\Phi'_k(\sigma_n)r^2} \tau^{2J+2} d\tau.$$ 

Extending the range of integration in $I_m$ involves an exponentially small error, so we get from the classical formula for Laplace integrals

$$I_m = \frac{\sqrt{2\pi}(2m)!}{m!2^m \Phi_k''(\sigma_n)^m m^{1/2}} + O(e^{-c_mn^{1/3(k+3)}}), \quad R \asymp \sigma_n^{1+(J+1/2)/k} \frac{1}{\sqrt{\Phi_k''(\sigma_n)n^{J/(k+1)}}}.$$ 

Inserting these estimates back into (3.5) and expanding all arising factors $\Phi_k^{(m)}(\sigma_n)$ by (2.11), we obtain (3.1). \hfill \Box

Remark. From (3.3) and (3.5) we see that, when $k \geq 2$,

$$p_k(n) = \frac{e^{\sigma_{n+\Phi_k(\sigma_n)}}}{\sqrt{2\pi \Phi_k''(\sigma_n)}} \left\{ 1 - \frac{2k^2 + 5k + 2}{24k} \left( \frac{\sigma_n}{\alpha_k} \right)^{1/k} + O\left(\sigma_n^{2/k}\right) \right\} \tag{3.6}$$

where, in view of (2.6), the quantity inside curly brackets may be replaced by an asymptotic series in powers of $\sigma_n^{1/k}$. Inserting (2.5) and (2.9) in the main term, we thus get a formula which is very close to, but simpler than (1.9), since it follows from (2.12) that $X$ and $1/\sigma_n$ agree to any power of $\sigma_n$.

We are now in a position to complete the proof of Theorem 1.

We infer from (2.9) and (2.10) that

$$n\sigma_n + \Phi_k(\sigma_n) = \zeta_k n^{1/(k+1)} + \sum_{1 \leq j < J} \frac{a_{kj}}{n^{j/(k+1)}} + O\left(\frac{1}{n^{j/(k+1)}}\right) + \frac{1}{2} \log \left(\frac{\sigma_n}{(2\pi)^k}\right)$$
with $a^*_k j := a_k (ka_{k,j+1} + b_{k,j+1})$. Exponentiating and expanding, we get
\[
\exp(n\sigma_n + \Phi(n)) \\
= \frac{\sqrt{\sigma_n}}{(2\pi)^{k/2}} \exp \left( \zeta_k n^{1/(k+1)} \right) \left\{ 1 + \sum_{1 \leq \ell < J} \frac{1}{\ell !} \left( \sum_{1 \leq j < J} \frac{a^*_k j}{n^j/(k+1)} \right) + O\left( \frac{1}{n^{J/(k+1)}} \right) \right\}
\]
(3.7)
\[
= \frac{\sqrt{\sigma_n}}{(2\pi)^{k/2}} \exp \left( \zeta_k n^{1/(k+1)} \right) \left\{ 1 + \sum_{1 \leq j < J} \frac{f_k j}{n^j/(k+1)} + O\left( \frac{1}{n^{J/(k+1)}} \right) \right\}
\]
with
\[
f_k j := \sum_{1 \leq \ell < J} \frac{1}{\ell !} \sum_{1 \leq j_1, \ldots, j_\ell < J} a^*_k j_1 \cdots a^*_k j_\ell.
\]

It remains to insert back into (3.1) and expand $\sqrt{\sigma_n / \Phi_k'(\sigma_n)}$ according to (2.9) and (2.11) with $m = 2$ to obtain the required asymptotic formula.

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