Command-filtered adaptive neural network backstepping quantized control for fractional-order nonlinear systems with asymmetric actuator dead-zone via disturbance observer

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Abstract An adaptive neural network backstepping quantized control of fractional-order nonlinear systems with asymmetric actuator dead-zone and unknown external disturbance is investigated in this paper. An adaptive NN mechanism is designed to estimate uncertain functions. A command filter is introduced to estimate the virtual control variable as well as its derivative, so that the “explosion of complexity” problem existed in the classical backstepping method can be avoided. To handle the unknown external disturbance, a fractional-order disturbance observer is developed. Moreover, a hysteresis-type quantizer is used to quantify the final input signal to overcome the system performance damage caused by the actuator dead-zone. The quantized input signal can ensure that all the involved signals stay bounded and the tracking error converges to an arbitrarily small region of the origin. Finally, two examples are presented to verify the effectiveness of the proposed method.

Keywords Adaptive neural network control · Asymmetric actuator dead-zone · Quantized control · Disturbance observer

1 Introduction

In recent decades, due to magnificent heredity and memorability, fractional calculus has been increasingly used to characterize nonlinear dynamic behaviors in control, signal processing, material and thermal systems, system identification [1–3]. In recent years, many achievements on the theory of fractional derivatives have been reported, some of which are used to discuss the problem of stability and control for fractional-order nonlinear systems (FONSs) with known models. For example, Ref. [4] proposed a Lyapunov direct method, which is useful in the stability analysis of FONSs. Ref. [5] developed a fractional-order non-fragile state observer, and used continuous frequency distribution method to derive the Lyapunov stability. On the other hand, due to the existence of measurement errors, unknown external disturbances, estimation errors and other factors in the process of system modeling, uncertainties are almost universal in actual systems. Therefore, it is meaningful to consider the stability analysis and control for uncertain FONSs, and some effective methods have been proposed, such as adaptive fuzzy control and neural network (NN) control. A novel adaptive fuzzy inversion dynamic surface control method was presented for SISO uncertain FONSs in [6].
improved adaptive backstepping control method was proposed for uncertain FONSs with unlimited actuator failures in [7], where the actuator fault is compensated by introducing a compensation signal in the control law. An adaptive NN fractional-order sliding mode controller was established for FONSs with disturbances in [8], where the NN is presented to estimate the boundary of the disturbance. An adaptive terminal sliding mode control with NN estimator was designed for fractional-order hyperchaotic economic systems to achieve finite-time stabilization and synchronization of FONSs in [9].

It is worth emphasizing that many systems can be classified as strict-feedback systems in practical applications, such as inverted pendulum control systems and permanent magnet synchronous electromechanical systems. Among many solutions, backstepping control is a powerful tool for controller design of strict-feedback systems, and recently, it has been extended to strict-feedback FONSs. However, in the process of backstepping control, in order to design the final controller, the virtual controller needs to be repeatedly differentiated step by step, which will lead to the increasing number of the controller items and then cause the “explosion of complexity” problem. In particular, for FONSs, the fractional derivative of a compound function is a complicated infinite series, so the derivative of the virtual controller will be more complex. For the mismatched-channel situation, a fractional-order backstepping method was proposed to design a sliding mode controller for FONSs in [10]. In [11], in every step, the virtual controller and its fractional-order derivative were estimated by the fuzzy logic system to avoid the problem of “repeated derivation of the virtual controller” in the backstepping; however, as the order of the system increases, the accumulated fuzzy estimation errors will damage the control performance. That is, the aforementioned literature still does not well-resolve the “explosion of complexity” problem. Therefore, Ref. [12] established a first-order command filter to solve this problem, where the output of the command filter is used to estimate the virtual controller in every step of the virtual control input design. A second-order command filter was also developed to solve this problem in [13]. Although the estimation accuracy of the second-order command filter is higher than the first-order one, it usually produces chattering and makes the design of the controller very complicated.

As is well-known, unknown disturbances exist in most of actual control systems, which reduce the control performance of the system and even result in instability of the control system. It is usually possible to reduce the impact of unknown disturbances by designing a larger control input, but this will greatly increase the control cost and resource loss. In order to solve this problem, disturbance observers provide a feasibility method, as they are commonly used to estimate unknown external disturbances, and they also can realize control, parameter identification or monitoring in turn. In the past few decades, the exploitation of disturbance observers for integer-order systems has yielded fruitful results, for example, see [14–17]. Since the operation properties of FONSs are different from integer-order systems, integer-order disturbance observers are usually invalid for FONSs. Recently, some researchers have made efforts to solve this problem. For instance, a sliding mode fractional-order disturbance observer was constructed to estimate the unknown disturbance in [18]. An adaptive sliding mode disturbance observer was proposed for the fractional-order chaotic systems with time-varying disturbance in [19]. A fractional-order disturbance observer based on sliding mode control was developed to deal with the control problem of FONSs with matched and mismatched disturbances in [20]. It should be noted that the uncertainty considered in the literature mentioned above needs to satisfy some strict conditions, and their control methods may take up a large bandwidth, which prevent the real applications.

In order to effectively utilize limited bandwidth, it is often necessary to quantify the transmitted signal in digital or network control systems. Therefore, using a quantized control signal to realize the tracking control of the closed-loop system has become an urgent problem. In the past few years, many issues concerning quantized control have been discussed for integer-order nonlinear systems, for example, in [21–26]. A logarithmic-type quantizer (LTQ) was first proposed in [23], and it was used to reduce the communication ratio in [24–26]. Subsequently, a hysteresis-type quantizer (HTQ) was used to overcome the chattering phenomenon caused by the rapid switching of the quantized signals in [27–30]. In addition, when the quantized control signal passes through an actuator, the dead-zone nonlinearity may significantly reduce the system performance. Thus, how to effectively couple the quantizer and the dead-zone nonlinearity becomes
a meaningful topic. There are some research results about the quantized dead-zone of integer-order systems. Combining sector-bounded characteristics of the HTQ and the simplified dead-zone model, a quantized control method was established for an integer-order stochastic nonlinear quantization system with actuator dead-zone in [31]. A quantized controller was designed for uncertain integer-order systems with unknown time-varying disturbances and dead-zone nonlinearity, and a tuning function was constructed to estimate the maximum upper bounds of unknown system parameters and disturbances in [32]. An adaptive quantized control was proposed for MIMO switching nonlinear system with unknown asymmetric dead-zone in [33], where different subsystem inputs use different quantizers. However, only few works have been reported on the quantized control for FONSs up to now. For example, an NN quantized control method was proposed for FONSs with time-delay in [34], where nonlinear functions with time-delay are unknown, and the NN is adopted to approximate them. An LTO was introduced to quantify the input signal for FONSs with differentiable time-varying time-delay linearity and actuator failure in [35]. According to the indirect fractional-order Lyapunov stability theory and fractional-order dynamic surface control technology, an adaptive backstepping control mechanism was established for FONSs with input quantizer and unknown control direction in [36]. Yet, to the authors’ knowledge, there is almost no literature on the quantized dead-zone case for FONSs. As is well-known, it is difficult to deal with the coupling between the quantizer and the dead-zone nonlinearity in FONSs, so designing a suitable quantized control method is a meaningful but challenging work, which is worth being investigated.

Motivated by the above discussion, this paper proposed an adaptive NN backstepping mechanism via command filter and disturbance observer applied on FONSs with quantized asymmetric actuator dead-zone input. A disturbance observer is designed to estimate the unknown disturbance. An HTQ is introduced to quantize the final control signal to effectively use the limited bandwidth. The proposed control method can ensure that the tracking error converges to an arbitrarily small neighborhood of the origin. The main advantages of this approach over current results can be summarized as follows. (1) A quantized controller is designed for FONSs which can deal with both system uncertainty and unknown dead-zone nonlinearity. (2) A fractional-order disturbance observer is established to estimate the unknown disturbance effectively.

The rest of this paper is organized as follows. Section 2 gives some preliminaries and the problem description. In Sect. 3, a fractional-order disturbance observer is used to estimate the unknown external disturbance, and an adaptive backstepping quantized control mechanism based on command filter is proposed. Furthermore, an HTQ is introduced to quantify the final control signal, and the stability analysis of the system is given. In Sect. 4, simulation results are given to verify the effectiveness of this proposed method. Finally, Sect. 5 summarizes this paper.

2 Preliminaries and problem description

2.1 Preliminaries

There are two commonly used definitions of fractional calculus, i.e., Riemann–Liouville and Caputo fractional calculus. Because of its satisfactory physical meanings, for example, the initial conditions for fractional-order differential equations with Caputo derivatives take on the same form as for integer-order differential equations, the Caputo definition is adopted in our work. The $\beta$-th Caputo fractional integral can be given by [1]

$$
^C \int_0^t y(\tau) d\tau = \frac{1}{\Gamma(\beta)} \int_0^t \frac{y(\xi)}{(t-\xi)^{1-\beta}} d\xi,
$$

(1)

where $\Gamma(\cdot)$ is the Gamma function, $\Gamma(\beta) = \int_0^{+\infty} \xi^{\beta-1} e^{-\xi} d\xi$.

The $\beta$-th Caputo fractional derivative can be described as [1]

$$
^C D_t^\beta y(t) = \frac{1}{\Gamma(n-\beta)} \int_0^t \frac{y^{(n)}(\xi)}{(t-\xi)^{\beta+1-n}} d\xi,
$$

(2)

where $n-1 \leq \beta < n$.

For convenience, in the following of this paper, $^C D_t^\beta y(t)$ is replaced by $D_t^\beta y(t)$, and only the case $\beta \in (0, 1)$ is considered.

**Definition 1** ([4]) A continuous function $\omega : [0, c) \rightarrow [0, +\infty)$ is called to be class-K, if it is strictly increasing and satisfies $\omega(0) = 0$. 
Lemma 1 ([4]) Let an equilibrium point of $D_t^\beta s(t) = g(t, s(t))$ be the origin, in which $g : [0, +\infty) \times \Omega \to R$ ($\Omega \subseteq R^n$) is Lipschitz continuous on $s(t)$, $s(t) \in R^n$. If there exist a Lyapunov function $V(t, s(t))$ and three class-K functions $\omega_i(t) (i = 1, 2, 3)$ satisfying

$$\omega_1(||s(t)||) \leq V(t, s(t)) \leq \omega_2(||s(t)||),$$

$$D_t^\beta V(t, s(t)) \leq -\omega_3(||s(t)||),$$

then \( \lim_{t \to +\infty} s(t) = 0. \)

Lemma 2 ([5]) Assume that $s(t) \in C^1([0, +\infty)) \subseteq R^n$, then

$$\frac{1}{2} D_t^\beta (s^T(t)s(t)) \leq s^T(t)D_t^\beta s(t).$$

Lemma 3 ([6]) For arbitrary positive real constant $M$ and real number $W$, the following inequality holds

$$0 \leq |W| - \frac{W^2}{\sqrt{M^2 + W^2}} < M.$$

Lemma 4 ([13]) For $\Sigma_0 > 0$, the inequality holds

$$|\Phi| - \Phi \tanh \left( \frac{\Phi}{\Sigma_0} \right) \leq 0.2785 \Sigma_0 = \tilde{\Sigma}_0.$$

Lemma 5 ([37]) Suppose that $h(t)$ and $\varphi(t)$ are $C^1 \cup L^\infty$ and satisfy

$$D_t^\beta h(t) = -\kappa(h(t) - \varphi(t))$$

then, for any constant $\sigma \in R^+$, it holds

$$|h(t) - \varphi(t)| \leq \sigma$$

for all $t \geq 0$ if the parameter $\kappa$ is chosen sufficiently large ($\kappa \in R^+$).

Lemma 6 ([37]) Suppose that $V(t) \in C^1$ satisfies

$$D_t^\beta V(t) \leq -a_1 V(t) + a_2$$

where $a_1, a_2 > 0$ are constants. Then, for every constant $\varepsilon > 1$, there exist $t^* > 0$, such that

$$|V(t)| \leq \frac{\varepsilon a_2}{a_1},$$

holds for all $t \geq t^*$.

2.2 The radial-basis-function NN

Let $z(\chi(t))$ be a continuous unknown nonlinear function defined over a compact set $\Omega$, where $\chi(t) = [\chi_1(t), \chi_2(t), \cdots, \chi_n(t)]^T \in R^n$, then it can be approximated by a radial-basis-function NN as

$$\hat{z}(\chi(t)) = \theta^T(t) \psi(\chi(t)), \quad (3)$$

in which $\hat{z} : R^n \to R (n \in N), t \geq 0, \theta(t) = [\theta_1(t), \theta_2(t), \cdots, \theta_N(t)]^T \in R^N$ ($N > 1$) is an adjustable weight vector, $\psi(\chi(t)) = [\psi_1(\chi(t)), \psi_2(\chi(t)), \cdots, \psi_N(\chi(t))]^T \in R^N$ is a regressor, and $N$ is the number of NN nodes. The regression variable $\psi_i(\chi_j(t))$ is defined as

$$\psi_i(\chi_j(t)) = \exp \left( -\frac{\|\chi_j(t) - \delta_{i,j}\|^2}{2\iota_i} \right),$$

where $i = 1, 2, \cdots, N, j = 1, 2, \cdots, n, \delta_{i,j}$ represents the $i$-th center of the $j$-th Gaussian function, and $\iota_i \in R^+$ represents the width of the Gaussian function.

From the universal approximation theorem of NN [38], for any nonlinear function $z(\chi(t))$, there is always an optimal estimation of NN, i.e.,

$$z(\chi(t)) \to^{\psi} \theta^*(\chi(t)) - \epsilon(\chi(t)). \quad (4)$$

in which $\epsilon(\chi(t))$ is the optimal approximation error, and $\theta^*$ is the optimal constant weight vector satisfying

$$\theta^* = \arg \min_{\theta \in R^N} \left[ \sup_{\chi(t) \in R^n} |z(\chi(t)) - \hat{z}(\chi(t))| \right].$$

Let

$$\bar{\theta}(t) = \theta(t) - \theta^* \quad (5)$$

be the parameter estimation error. From the universal approximation theorem of NN, it is reasonable to assume that the optimal approximation error $\epsilon(\chi(t))$ keeps bounded. That is, there exists a positive parameter $\bar{\epsilon}$, such that

$$|\epsilon(\chi(t))| \leq \bar{\epsilon}. \quad (6)$$

Thus, one has

$$\bar{z}(\chi(t), \theta(t)) - z(\chi(t))$$

$$= \bar{z}(\chi(t), \theta(t)) - \hat{z}(\chi(t), \theta^*) + \hat{z}(\chi(t), \theta^*) - z(\chi(t))$$

$$= \bar{\theta}(t)^T \psi(\chi(t)) + \epsilon(\chi(t)). \quad (7)$$

2.3 Problem description

Consider FONSs with unknown external disturbance and asymmetric actuator dead-zone described by
\[
\begin{aligned}
\mathcal{D}_t^\beta x_j &= g_j(\bar{x}_j)x_{j+1} + f_j(\bar{x}_j), \quad (j = 1, 2, \ldots, n - 1), \\
\mathcal{D}_t^\beta x_n &= g_n(x)u + f_n(x) + d(t), \\
u &= H(q(v)), \\
y &= x_1,
\end{aligned}
\]

where \( \bar{x}_j = [x_1, x_2, \ldots, x_j]^T \in \mathcal{R}^j, x = \bar{x}_n = [x_1, x_2, \ldots, x_n]^T \in \mathcal{R}^n \) is the state vector of the system, \( g_j(\bar{x}_j) > \tilde{g}_j \) and \( g_n(x) > \tilde{g}_n \) are known smooth nonlinear functions \( (\tilde{g}_j, \tilde{g}_n > 0) \), \( f_j(\bar{x}_j) \in \mathcal{R} \) and \( f_j(x) \in \mathcal{R} \) are unknown smooth nonlinear functions, \( d(t) \in \mathcal{R} \) is an unknown external disturbance, \( y \in \mathcal{R} \) is the output, \( u = H(q(v)) \in \mathcal{R} \) is the dead-zone control input with quantized control input function \( q(v) \), \( v(t) \in \mathcal{R} \) is the control signal to be designed, and \( q(v) \in \mathcal{R} \) is the final control signal after being quantized. In addition, the order \( \beta \in (0, 1) \) is an uncertain constant.

**Assumption 1** Assume that \( d(t) \) and \( \mathcal{D}_t^\beta d(t) \) are bounded and unknown, that is, there exist two unknown constants \( \tilde{d} \) and \( \tilde{\theta} \) such that \( |d(t)| \leq \tilde{d} \) and \( |\mathcal{D}_t^\beta d(t)| \leq \tilde{\theta} \) for all \( t \geq 0 \).

The dead-zone control input of FONSs is designed as

\[
H(q(v)) = \tilde{w}[q(v) + m(q(v))],
\]

in which \( \tilde{w} > 0 \) is an unknown bounded constant, \( m(q(v)) \) is the nonlinear function of the dead-zone, and \( q(v) \) is an HTQ which can be expressed as

\[
q(v) = \begin{cases} 
\psi \text{sign}(v), & \frac{v}{v_1} + b < |v| \leq v_1, \tilde{\theta} < 0, \\
\psi \text{sign}(v), & v_1 < |v| \leq \frac{v_1}{1 - b}, \tilde{\theta} > 0, \\
\psi (1 + b) \text{sign}(v), & \frac{v}{v_1} - b < |v| \leq \frac{v_1}{1 - b}, \tilde{\theta} < 0, \\
0, & 0 \leq |v| \leq v_{\text{min}} + \frac{v_{\text{min}}}{1 - b}, \tilde{\theta} > 0, \\
q(v), & \frac{v_{\text{min}}}{1 + b} \leq |v| \leq v_{\text{min}}, \tilde{\theta} > 0, \\
0, & \tilde{\theta} = 0,
\end{cases}
\]

where \( v_i = r^{1-i}v_{\text{min}} \) \((i = 1, 2, \ldots, \) and \( v_{\text{min}} > 0) \) represents the size of dead-zone for the HTQ \( q(v) \), \( r \in (0, 1) \), \( b = (1 - r)/(1 + r) \), \( b \in (0, 1) \), and \( q(v) \) is in the set \( V = \{0, \pm v_1, \pm (1 + b)v_i\} \), which is described in Fig. 1.

**Remark 1** In the case of limited bandwidth, considering that the actual channel needs to quantify the transmitted data, an HTQ (see Fig. 1) is used to quantize the input signal in this paper. Unlike the LTQ, the HTQ can be regarded as a combination of two LTQs, which will greatly reduce the severe chattering presented in the LTQ.

**Remark 2** In many practical systems, it should be noted that the dead-zone often exists, which will increase the steady-state error and the control performance of the nonlinear system will be impaired. On the other hand, due to the transmission bandwidth in the network control system is always limited [39], it is important to effectively use the bandwidth of the transmission channel in the actual control system. Therefore, in order to improve bandwidth utilization rate, this paper proposes an adaptive backstepping quantized control method for FONSs with unknown control direction and asymmetric actuator dead-zone input. In order to complete the control target, the HTQ \( q(v) \) is subject to some processing.

Noting that the definition of \( q(v) \) in (10) is very complicated. In order to facilitate the design of the actual controller \( v \), here \( q(v) \) is transformed into

\[
q(v) = v + \zeta,
\]

where \( \zeta \) is the quantization nonlinearity satisfying

\[
\begin{cases} 
\zeta^2 \leq b^2v^2, & |v| \geq v_{\min}, \\
\zeta^2 \leq v_{\text{min}}^2, & |v| < v_{\text{min}}.
\end{cases}
\]

The function \( m(q(v)) \) in (9) can be presented as

\[
m(q(v)) = \begin{cases} 
-p_r, & q(v) \geq p_r, \\
-q(v), & p_l < q(v) < p_r, \\
-p_l, & q(v) \leq p_l,
\end{cases}
\]

in which \( p_l \leq 0 \) and \( p_r \geq 0 \) are all bounded and unknown constants. From (13), noting that \( m(q(v)) \) is bounded, then there exists \( N = \max\{-p_{\text{min}}, p_{\text{max}}\} \geq 0 \), such that \( |m(q(v))| \leq N \).

**Remark 3** Due to the dead-zone nonlinearity, the asymmetric actuator dead-zone input cannot be directly quantized. Therefore, the HTQ is divided into two parts in (11) to couple the dead-zone nonlinearity with the HTQ. As shown in Fig. 2, the goal is to design an actual controller \( v \) that can be quantified as \( q(v) \), then put it into (9) to get the quantized input dead-zone. Finally, the final designed control signal \( v \) can ensure that all
signals of the closed-loop system are bounded, the reference signal $x_d$ is tracked by $y$, and the tracking error $e_1$ eventually converges to an arbitrarily small region of the origin.

3 Main results

3.1 The fractional-order disturbance observer design

This part will construct a fractional-order disturbance observer to effectively estimate the unknown external disturbance in the system (8). Note that the unknown external disturbance cannot be directly estimated, one tries to use the system information to design a fractional-order disturbance observer. In the light of [19,20], an auxiliary function $\varphi(t)$ is constructed, which can be expressed as

$$\varphi(t) = d(t) - \xi_1 x_n,$$

where $\xi_1$ is a design parameter. From (14), one obtains

$$D_t^\beta \varphi(t) = D_t^\beta d(t) - \xi_1 D_t^\beta x_n$$
$$= D_t^\beta d(t) - \xi_1 \left[ g_n(x)u + f_n(x) + d(t) \right]$$
$$= D_t^\beta d(t) - \xi_1 \left[ g_n(x)u + f_n(x) + \xi_1 x_n + \varphi(t) \right]$$

where $\hat{\varphi}(t), \hat{d}(t)$ and $\hat{f}_n(x)$ are the estimate of $\varphi(t), d(t)$ and $f_n(x)$, respectively. Define the disturbance error as $\tilde{d}(t) = d(t) - \hat{d}(t)$. It follows from (7) that

$$D_t^\beta \tilde{d}(t) = D_t^\beta \varphi(t) - D_t^\beta \hat{\varphi}(t)$$
$$= -\xi_1 \left[ \tilde{d}(t) + f_n(x) - \hat{f}_n(x, \theta_n) \right]$$
$$+ D_t^\beta d(t)$$

Remark 4 The disturbance observer can estimate the external disturbance through the known information of the controlled object, such as state variables, control output or input, and the output of the disturbance

$$\hat{d}(t) = d(t) - \hat{\varphi}(t)$$
$$= -\xi_1 \left[ \hat{\varphi}(t) - \xi_1 x_n \right]$$

$$+ D_t^\beta d(t).$$

The designed disturbance observer is presented as

$$\begin{cases}
\hat{\varphi}(t) = \hat{d}(t) - \xi_1 x_n, \\
D_t^\beta \hat{\varphi}(t) = -\xi_1 \hat{\varphi}(t) - \xi_1 \left[ g_n(x)u + \hat{f}_n(x, \theta_n) + \xi_1 x_n \right] + D_t^\beta d(t).
\end{cases}$$

(15)
observer can be used to design the control law. According to the ingenious design of (15) and (16), the disturbance observation error \( \hat{d}(t) \) can be obtained. Based on (5) and (6), \( \tilde{\theta}_j \) and \( e_{n_l}(x) \) are bounded. From Assumption 1, one knows that \( D^\beta_t d(t) \) is bounded. Therefore, if the parameter \( \xi_2 \) is properly selected, the disturbance estimation error \( \hat{d}(t) \) will remain bounded.

Remark 5 The purpose of this paper is to design an appropriate controller so that all the involved signals keep bounded and the tracking error converges to an arbitrarily small region of the origin. However, since \( d(t) \) is unknown, it should be estimated. In the traditional method, it is assumed that \( |d(t)| \) has an upper bound, which can be estimated by an adjustable variable. But this method often results in a relatively large estimation error, which affects the control performance. In this paper, a fractional-order disturbance observer (16) is designed, which can vary with the variation of \( d(t) \), and then \( d(t) \) can be directly estimated. Thus, better estimation performance can be expected.

3.2 The adaptive NN backstepping control

In order to avoid the “repeated derivation of the virtual controller” problem, a command filter is used to estimate the virtual controller and its fractional-order derivative in this paper, which can be described by

\[
D^\beta_t x_{j,c} = -\kappa_j (x_{j+1,c} - \alpha_j),
\]

where \( \alpha_j \) is the input signal (the virtual control signal), \( x_{j+1,c} (j = 1, 2, \ldots, n - 1) \) is the output signal, and \( k_j \) is a positive constant.

Let the smooth reference signal be \( x_d = x_{1,c} \). Then, the tracking error signals are defined as

\[
e_j = x_j - x_{j,c},
\]

\[
e_n = x_n - x_{n,c}.
\]

The virtual control controller is constructed as

\[
\alpha_j = \frac{1}{g_j(x)} \left[ \left( k_{j+1} + \frac{g_j(x)}{2} \right) e_j + \theta_j^T \psi_j(x_j) ight. + k_{2j} \tanh \left( \frac{e_j}{\Xi_j} \right) + \left. g_{j-1}(x) e_{j-1} - D^\beta_t x_{j,c} \right],
\]

where \( e_0 = 0, k_{1j} > 0, \Xi_j > 0 \) and \( k_{2j} > \bar{e}_j \), and \( \bar{e}_j \) will be given later.

The actual controller \( v \) is established as

\[
v = -\frac{1}{g_n(x)(1 - b)} \left( e_n p^2(t) \hat{\lambda}^2(t) \right. + \left. \frac{\delta_1 e_n}{\epsilon_0^2 + \delta_2^2} + g_n(x) \hat{\phi}(t) \tanh \left( \frac{e_n}{\Xi_n} \right) \right),
\]

in which \( \hat{\lambda}(t) \) and \( \hat{\phi}(t) \) are the estimation of \( \lambda \) and \( \phi \), respectively, \( \lambda = \frac{1}{\sigma}, \phi = \Xi + \nu_{\text{min}} \), \( \Xi_n > 0 \), \( \delta_1 \) and \( \delta_2 \) are positive design parameters, and \( p(t) \) will be designed later. Thus, the adaptive NN backstepping control mechanism which contains \( n \) steps are shown as follows.

Step 1:

Let \( v_1 = \frac{1}{2} e_1^2 \). According to Lemma 2, and combining (7) and (8), one obtains

\[
D^\beta_t v_1 \leq e_1 D^\beta_t e_1 = e_1 \left[ g_n(x) \hat{\lambda}^2(t) + D^\beta_t x_d \right]
\]

\[
= e_1 \left[ g_1(\tilde{x}_1) x_2 + f_1(\tilde{x}_1) - D^\beta_t x_d \right]
\]

\[
= e_1 \left[ g_1(\tilde{x}_1) (x_2 - x_{2,c} + x_{2,c} - \alpha_1 + \alpha_1) + f_1(\tilde{x}_1) - f_1(\tilde{x}_1, \theta_1) + f_1(\tilde{x}_1, \theta_1) - D^\beta_t x_d \right]
\]

\[
= e_1 \left[ g_1(\tilde{x}_1) (e_2 + \alpha_1) + g_1(\tilde{x}_1) \tilde{\alpha}_1 - \theta_1^T \psi_1(\tilde{x}_1) \right. \]

\[
- \left. e_1(\tilde{x}_1) + \theta_1^T \psi_1(\tilde{x}_1) - D^\beta_t x_d \right].
\]

with \( \tilde{\alpha}_1 = x_{2,c} - \alpha_1, |e_1(\tilde{x}_1)| \leq \bar{e}_1 \), and \( \tilde{\theta}_1 = \theta_1 - \theta_1^* \).

If \( k_{11} \) and \( k_{21} \) are chosen such that \( k_{11} > 0 \) and \( k_{21} > \bar{e}_1 \), then substituting (21) into (23) yields

\[
D^\beta_t v_1 \leq e_1 \left[ g_1(\tilde{x}_1) e_2 + g_1(\tilde{x}_1) \tilde{\alpha}_1 - \left( k_{11} + \frac{g_1(\tilde{x}_1)}{2} \right) e_1 \right.
\]

\[
- \left. \tilde{\theta}_1^T \psi_1(\tilde{x}_1) - e_1(\tilde{x}_1) - k_{21} \tanh \left( \frac{e_1}{\Xi_1} \right) \right]
\]

\[
\leq g_1(\tilde{x}_1) e_2 + g_1(\tilde{x}_1) \tilde{\alpha}_1 e_1 - \left( k_{11} + \frac{g_1(\tilde{x}_1)}{2} \right) e_1^2
\]

\[
- e_1 \tilde{\theta}_1^T \psi_1(\tilde{x}_1) + \bar{e}_1 |e_1| - k_{21} e_1 \tanh \left( \frac{e_1}{\Xi_1} \right).
\]

According to Lemma 4, the following inequality

\[
\bar{e}_1 |e_1| - k_{21} e_1 \tanh \left( \frac{e_1}{\Xi_1} \right) = \bar{e}_1 |e_1| - k_{21} |e_1|
\]

\( \Xi \) Springer
\[ D \] from \( (5) \) that \[ \text{Lemma} 2 \text{, one has} \]
\[ \leq 0.2785k_{21} \Xi = k_{21} \Xi, \] \quad (25)
holds.

From the Young’s inequality, one obtains
\[ \tilde{\alpha} e \leq \frac{1}{2} \tilde{\alpha}^2 + \frac{1}{2} e^2. \] \quad (26)

According to Lemma 5, substituting (25) and (26) into (24) yields
\[ \mathcal{D}_t^\beta V_1 \leq g_{1}(\tilde{x}_1)e_1 e_2 - k_{11} e_1^2 + \frac{g_{1}(\tilde{x}_1)}{2} \tilde{\alpha}_1^2 \]
\[ - e_1 \tilde{\theta}_1^T \psi_1 (\tilde{x}_1) + k_{21} \Xi, \]
\[ \leq g_{1}(\tilde{x}_1)e_1 e_2 - k_{11} e_1^2 + \frac{g_{1}(\tilde{x}_1)}{2} \sigma_1^2 \]
\[ - e_1 \tilde{\theta}_1^T \psi_1 (\tilde{x}_1) + k_{21} \Xi, \] \quad (27)
where \( |\tilde{\alpha}_1| \leq \sigma_1 \), and \( \sigma_1 \in \mathcal{R}^+ \).

Choose the Lyapunov function \( V_1 \) as
\[ V_1 = v_1 + \frac{1}{2\rho_1} \tilde{\theta}_1^T \tilde{\theta}_1, \] \quad (28)
where \( \rho_1 \) is a positive design parameter. Thus, according to Lemma 2, one has
\[ \mathcal{D}_t^\beta v_1 \leq \frac{1}{\rho_1} \tilde{\theta}_1^T \mathcal{D}_t^\beta \tilde{\theta}_1 \]
\[ \leq g_{1}(\tilde{x}_1)e_1 e_2 - k_{11} e_1^2 + \frac{g_{1}(\tilde{x}_1)}{2} \sigma_1^2 \]
\[ - e_1 \tilde{\theta}_1^T \psi_1 (\tilde{x}_1) + k_{21} \Xi + \frac{1}{\rho_1} \tilde{\theta}_1^T \mathcal{D}_t^\beta \tilde{\theta}_1. \] \quad (29)

The adaptive law is designed as
\[ \mathcal{D}_t^\beta \theta = \rho_1 e_1 \psi_1 (\tilde{x}_1) - \gamma_1 \theta, \] \quad (30)
where \( \gamma_1 \) is a positive design parameter. Then, it follows from (5) that
\[ \mathcal{D}_t^\beta \tilde{\theta}_1 = \mathcal{D}_t^\beta \theta_1. \] \quad (31)

Substituting (30) and (31) into (29), one obtains
\[ \mathcal{D}_t^\beta V_1 \leq -k_{11} e_1^2 + \frac{g_{1}(\tilde{x}_1)}{2} \sigma_1^2 + g_{1}(\tilde{x}_1)e_1 e_2 + k_{21} \Xi \]
\[ - \frac{\gamma_1}{\rho_1} \tilde{\theta}_1^T \theta, \]
\[ \leq -k_{11} e_1^2 + \frac{g_{1}(\tilde{x}_1)}{2} \sigma_1^2 + g_{1}(\tilde{x}_1)e_1 e_2 + k_{21} \Xi \]
\[ - \frac{\gamma_1}{\rho_1} \tilde{\theta}_1^T \theta_1 - \frac{\gamma_1}{\rho_1} \tilde{\theta}_1^T \theta_1. \] \quad (32)

From the Young’s inequality, one knows
\[ - \frac{1}{\rho_1} \tilde{\theta}_1^T \theta_1 \leq \frac{1}{2\rho_1} \theta_1^T \theta_1 + \frac{1}{2\rho_1} \tilde{\theta}_1^T \tilde{\theta}_1. \] \quad (33)

Substituting (32) into (33) yields
\[ \mathcal{D}_t^\beta V_1 \leq -k_{11} e_1^2 + \frac{g_{1}(\tilde{x}_1)}{2} \sigma_1^2 + g_{1}(\tilde{x}_1)e_1 e_2 \]
\[ + k_{21} \Xi - \frac{\gamma_1}{2\rho_1} \tilde{\theta}_1^T \tilde{\theta}_1 + \frac{\gamma_1}{2\rho_1} \theta_1^T \theta_1 \]
\[ \leq -k_{11} V_1 + g_{1}(\tilde{x}_1)e_1 e_2 + \Theta_1, \] \quad (34)
where \( k_1 = \min\{2k_{11}, \gamma_1\} > 0 \), and \( \Theta_1 = (g_{1}(\tilde{x}_1)/2) \)
\[ \sigma_1^2 + (\gamma_1/2\rho_1) \theta_1^T \theta_1 + k_{21} \Xi > 0 \] are two constants.

**Step j** \( (j = 2, 3, \ldots, n - 1) \):

Let \( v_j = \frac{1}{2} e_j^2 \), one obtains
\[ \mathcal{D}_t^\beta v_j \leq e_j \mathcal{D}_t^\beta e_j \]
\[ = e_j \left[ g_{j}(\tilde{x}_j)x_{j+1} + f_{j}(\tilde{x}_j) - D_{t,x}^\beta e_j \right] \]
\[ = e_j \left[ g_{j}(\tilde{x}_j) (x_{j+1} - x_{j+1,c} + x_{j+1,c} - \alpha_j + \alpha_j) + f_{j}(\tilde{x}_j) - f_{j}(\tilde{x}_j, \theta_j) + f_{j}(\tilde{x}_j, \theta_j) - D_{t,x}^\beta e_j \right] \]
\[ = e_j \left[ g_{j}(\tilde{x}_j) (e_j + \alpha_j) + g_{j}(\tilde{x}_j) \tilde{\alpha}_j - \tilde{\theta}_j^T \psi_j (\tilde{x}_j) + \tilde{\theta}_j^T \psi_j (\tilde{x}_j) - D_{t,x}^\beta e_j \right] \]
\[ = e_j \left[ g_{j}(\tilde{x}_j) (e_j + \alpha_j) + g_{j}(\tilde{x}_j) \tilde{\alpha}_j - \tilde{\theta}_j^T \psi_j (\tilde{x}_j) - D_{t,x}^\beta e_j \right] \]
\[ \leq e_j \left[ g_{j}(\tilde{x}_j) (e_j + \alpha_j) + g_{j}(\tilde{x}_j) \tilde{\alpha}_j - \tilde{\theta}_j^T \psi_j (\tilde{x}_j) - D_{t,x}^\beta e_j \right]. \] \quad (35)

where \( \tilde{\alpha}_j = x_{j+1,c} - \alpha_j, |e_j(\tilde{x}_j)| \leq \tilde{e}_j, \) and \( \tilde{\theta}_j = \theta_j - \theta_0^* \).

Let \( k_{1j} > 0 \) and \( k_{2j} > \tilde{e}_j \), then substituting (21) into (35) yields
\[ \mathcal{D}_t^\beta v_j \leq e_j \left[ g_{j}(\tilde{x}_j)e_{j+1} - \left( k_{1j} + \frac{g_{j}(\tilde{x}_j)}{2} \right) e_j \right. \]
\[ + g_{j}(\tilde{x}_j) \tilde{\alpha}_j - \tilde{\theta}_j^T \psi_j (\tilde{x}_j) \]
\[ - e_j (\tilde{x}_j) - k_{2j} \tanh\left( \frac{e_j}{\varepsilon_j} \right) - g_{j-1}(\tilde{x}_{j-1})e_{j-1} \right] \]
\[ \leq g_{j}(\tilde{x}_j)e_{j+1} + g_{j}(\tilde{x}_j) \tilde{\alpha}_j e_j \]
\[ \left. - \left( k_{1j} + \frac{g_{j}(\tilde{x}_j)}{2} \right) e_j - \tilde{\theta}_j^T \psi_j (\tilde{x}_j) \right\} \]
\[ + \tilde{e}_j |e_j| - k_{2j} e_j \tanh\left( \frac{e_j}{\varepsilon_j} \right) - g_{j-1}(\tilde{x}_{j-1})e_{j-1} \]
\[ \leq g_{j}(\tilde{x}_j) |e_{j+1}| - \left( k_{1j} + \frac{g_{j}(\tilde{x}_j)}{2} \right) e_j \]
\[ + g_{j}(\tilde{x}_j) \tilde{\alpha}_j e_j + \tilde{e}_j |e_j| + k_{2j} e_j \]
\[ -k_{j_2}|e_j| - k_{j_2}e_j \tanh \left( \frac{e_j}{\xi_j} \right) \\
\leq g_j(\bar{x}_j)e_j e_{j-1} + \left( k_{j_1} + \frac{g_j(\bar{x}_j)}{2} \right) e_j^2 \\
+ g_j(\bar{x}_j)\bar{a}_j e_j - e_j \bar{\theta}^T \psi_j(\bar{x}_j) \\
- g_j(\bar{x}_j-1)e_{j-1}e_j + k_{j_2}\hat{z}_j. \tag{36} \]

where \( \hat{z}_j = 0.2785\xi_j \).

Similar to the step 1, one has

\[ \bar{a}_j e_j \leq \frac{1}{2} \bar{a}_j^2 + \frac{1}{2} e_j^2. \tag{37} \]

From (37), (36) can be transformed into

\[ D^\beta_i v_j \leq g_j(\bar{x}_j)e_j e_{j+1} - k_{j_1}e_j^2 \\
+ \frac{g_j(\bar{x}_j)}{2} \bar{a}_j^2 - e_j \bar{\theta}^T \psi_j(\bar{x}_j) \\
- g_j(\bar{x}_j-1)e_{j-1}e_j + k_{j_2}\hat{z}_j \\
\leq g_j(\bar{x}_j)e_j e_{j+1} - k_{j_1}e_j^2 \\
+ \frac{g_j(\bar{x}_j)}{2} \bar{a}_j^2 - e_j \bar{\theta}^T \psi_j(\bar{x}_j) \\
- g_j(\bar{x}_j-1)e_{j-1}e_j + k_{j_2}\hat{z}_j, \tag{38} \]

where \( |\hat{a}_j| \leq \sigma_j \), and \( \sigma_j \in \mathbb{R}^+ \).

The Lyapunov function \( V_j \) can be designed as

\[ V_j = V_{j-1} + v_j + \frac{1}{2\rho_j} \bar{\theta}^T \bar{\theta}_j, \tag{39} \]

in which \( \rho_j > 0 \) is a designed parameter. Thus, from (38), one has

\[ D^\beta_i V_j \leq D^\beta_i V_{j-1} + D^\beta_i v_j + \frac{1}{\rho_j} \bar{\theta}^T \bar{\theta}_j \\
\leq -k_{j_1}V_{j-1} + \Theta_{j-1} + g_j(\bar{x}_j)e_j e_{j+1} \\
- k_{j_1}e_j^2 + \frac{g_j(\bar{x}_j)}{2} \sigma_j^2 \\
- e_j \bar{\theta}^T \psi_j(\bar{x}_j) + k_{j_2}\hat{z}_j \\
+ \frac{1}{\rho_j} \bar{\theta}^T D^\beta_i \bar{\theta}_j. \tag{40} \]

The adaptive law is designed as

\[ D^\beta_i \bar{\theta}_j = \rho_j e_j \psi_j(\bar{x}_j) - \gamma_j \bar{\theta}_j, \tag{41} \]

where \( \gamma_j \) is a positive design parameter. Substituting (41) into (40), one has

\[ D^\beta_i V_j \leq -k_{j_1}V_{j-1} + \Theta_{j-1} + g_j(\bar{x}_j)e_j e_{j+1} \\
- k_{j_1}e_j^2 + \frac{g_j(\bar{x}_j)}{2} \sigma_j^2 + k_{j_2}\hat{z}_j - \frac{\gamma_j}{\rho_j} \bar{\theta}^T \bar{\theta}_j \\
\leq -k_{j_1}V_{j-1} + \Theta_{j-1} - k_{j_1}e_j^2 + \frac{g_j(\bar{x}_j)}{2} \sigma_j^2 \\
+ g_j(\bar{x}_j)e_j e_{j+1} + k_{j_2}\hat{z}_j - \frac{\gamma_j}{\rho_j} \bar{\theta}^T \bar{\theta}_j - \frac{\gamma_j}{\rho_j} \bar{\theta}^* \bar{\theta}_j, \tag{42} \]

Noting that

\[ -\frac{1}{\rho_j} \bar{\theta}^T \bar{\theta}_j \leq \frac{1}{2\rho_j} \theta_j^T \theta_j^* + \frac{1}{2\rho_j} \bar{\theta}^T \bar{\theta}_j, \tag{43} \]

it follows from (42) and (43) that

\[ D^\beta_i V_j \leq -k_{j_1}V_{j-1} + \Theta_{j-1} - k_{j_1}e_j^2 + \frac{g_j(\bar{x}_j)}{2} \sigma_j^2 \\
+ g_j(\bar{x}_j)e_j e_{j+1} + k_{j_2}\hat{z}_j - \frac{\gamma_j}{\rho_j} \bar{\theta}^T \bar{\theta}_j \\
+ \frac{\gamma_j}{2\rho_j} \theta_j^* \theta_j^* \\
\leq -k_{j_1}V_j + g_j(\bar{x}_j)e_j e_{j+1} + \Theta_{j}, \tag{44} \]

in which \( k_j = \min(k_{j_1}, 2k_{j_1}, \gamma_j) \) and \( \Theta_{j} = \Theta_{j-1} + (g_j(\bar{x}_j)/2) \sigma_j^2 + (\gamma_j/2\rho_j) \theta_j^* \theta_j^* + k_{j_2}\hat{z}_j \) are two positive constants.

**Step n:**

Let \( v_n = \frac{1}{2}e_n^2 \). Similar to the previous steps, one has

\[ D^\beta_i v_n \leq e_n D^\beta_i e_n = e_n \left[ f_n(x) + d(t) + g_n(x)u - D^\beta_i x_{n,c} \right] \\
= e_n \left[ f_n(x) - \hat{f}_n(x, \theta_n) + \hat{f}_n(x, \theta_n) + d(t) \right] \\
+ g_n(x)u - D^\beta_i x_{n,c} \right]. \tag{45} \]

where \( |e_n(x)| \leq \bar{e}_n \), and \( \bar{\theta}_n = \theta_n - \theta_n^* \). Substituting (9) into (45) yields

\[ D^\beta_i v_n \leq e_n \left[ -\bar{\theta}_n^T \psi_n(x) - e_n(x) + \theta_n^T \psi_n(x) \right] \\
+ d(t) + g_n(x)u - D^\beta_i x_{n,c} \right]. \tag{46} \]

Combining with (11) and (46), one obtains

\[ D^\beta_i v_n \leq e_n \left[ -\theta_n^T \psi_n(x) - e_n(x) + \theta_n^T \psi_n(x) + d(t) \right] \\
+ g_n(x)\bar{w}q(v) + g_n(x)\bar{w}q(v) + m(q(v)) - D^\beta_i x_{n,c} \tag{46} \]
\begin{equation}
\leq -e_n \tilde{\theta}_n^T \psi_n(x) - e_n(x) e_n + e_n \theta_n^T \psi_n(x) + d(t) e_n + g_n(x) \bar{w} e_n v + g_n(x) \bar{w} |e_n| |m(q(v))| + g_n(x) \bar{w} v e_n v + v_{\min} |\bar{w} e_n|,
\end{equation}

According to (22), for all positive initial conditions \( \hat{\phi}(0) \), one has \( e_n(t) < 0 \). From (12), the following inequality holds

\begin{equation}
\bar{w} e_n \leq b |\bar{w} e_n v| + v_{\min} |\bar{w} e_n|.
\end{equation}

Consequently, from (48), (47) can be transformed into

\begin{equation}
D_1^\beta v_n \leq -e_n \tilde{\theta}_n^T \psi_n(x) + e_n(x) e_n + e_n \theta_n^T \psi_n(x) + d(t) e_n + g_n(x) \bar{w} e_n v + g_n(x) \bar{w} |e_n| |m(q(v))| + g_n(x) \bar{w} v e_n v + v_{\min} |\bar{w} e_n| - e_n D_1^\beta x_n.\nonumber
\end{equation}

According to Lemma 3, one has

\begin{equation}
\bar{w} e_n \leq -e_n \frac{e_n p^2(t) \tilde{\lambda}^2(t)}{\sqrt{e_n^2 p^2(t) \tilde{\lambda}^2(t) + \delta_1^2}} - \frac{\bar{w} e_n}{e_n} = -\frac{\bar{w} e_n p^2(t) \tilde{\lambda}^2(t)}{\sqrt{e_n^2 p^2(t) \tilde{\lambda}^2(t) + \delta_1^2}} < \bar{w} \delta_1 - \bar{w} e_n p(t) \tilde{\lambda}(t),
\end{equation}

and

\begin{equation}
\bar{w} e_n \left( -\frac{\delta_1 e_n}{e_n^2 + \delta_2} \right) < \bar{w} \delta_1.
\end{equation}

Then, in the light of (52) and (53), (51) can be simplified to

\begin{equation}
D_1^\beta v_n \leq -e_n \tilde{\theta}_n^T \psi_n(x) + e_n(x) e_n + e_n \theta_n^T \psi_n(x) + d(t) e_n + g_n(x) \bar{w} e_n v + \left( k_1 n + \frac{\delta_2}{2} \right) e_n^2 - g_n(x) \bar{w} e_n v (1 - b) + g_n(x) \bar{w} |e_n| |m(q(v))| + v_{\min} |\bar{w} e_n| + g_n(x) \bar{w} v e_n v + v_{\min} |\bar{w} e_n|.
\end{equation}

From Lemma 4, one has

\begin{equation}
g_n(x) \bar{w} |e_n| - g_n(x) \bar{w} \tilde{\phi}(t) e_n \tan(e_n) = g_n(x) \bar{w} \tilde{\phi}(t) e_n |\bar{w} e_n| + g_n(x) \bar{w} \tilde{\phi}(t) |e_n| - g_n(x) \bar{w} \tilde{\phi}(t) e_n \tan(e_n) \leq -g_n(x) \bar{w} \tilde{\phi}(t) e_n |\bar{w} e_n| + g_n(x) \bar{w} \tilde{\phi}(t) \tilde{e}_n \leq -g_n(x) \bar{w} \tilde{\phi}(t) e_n |\bar{w} e_n| + g_n(x) \bar{w} \left( \phi(t) \tilde{e}_n \right) \tilde{e}_n.
\end{equation}

where \( \tilde{e}_n = 0.2785 \tilde{e}_n \).

The Lyapunov function \( V_n \) is chosen as

\begin{equation}
V_n = V_{n-1} + v_n + \frac{1}{2 \rho_n} \tilde{\theta}_n^T \tilde{\theta}_n + \frac{1}{2 \rho_2} \tilde{\lambda}^2(t) + \frac{\bar{w}}{2 \sigma_1} \tilde{\lambda}^2(t) + \frac{\bar{w}}{2 \tilde{\lambda}_1} \phi^2(t),
\end{equation}

where \( \tilde{\theta}_n = \theta_n - \theta_n \), \( \tilde{\lambda}(t) = \dot{\lambda}(t) - \lambda \), \( \dot{\phi}(t) = \phi(t) - \dot{\phi} \), \( \rho_n, \sigma_2, \sigma_1 \) and \( l_1 \) are designed parameters. Then,
combining with (54) and (55), one gets
\[
\mathcal{D}_t^\beta V_n \leq -k_{n-1}V_{n-1} + \Theta_{n-1} - e_n\theta_n^T\psi_n(x) \\
+ \frac{1}{2\xi_2}d^2(t) - g_n(x)\bar{w}\phi(t)|e_n| \\
+ g_n(x)\bar{w}(\phi + \bar{\phi}(t))\Xi_n + p(t)e_n + 2\bar{w}\delta_1 \\
- \bar{w}e_n p(t)\hat{\lambda}(t) - k_{n-1}e_n^2 \\
+ \frac{1}{\rho_n}\theta_n^T\mathcal{D}_t^\beta\theta_n + \frac{1}{\xi_2}\tilde{d}(t)\mathcal{D}_t^\beta\tilde{d}(t) \\
+ \frac{\bar{w}}{\sigma_1}\tilde{\lambda}(t)\mathcal{D}_t^\beta\tilde{\lambda}(t) + \frac{\tilde{w}}{l_1}\bar{\phi}(t)\mathcal{D}_t^\beta\bar{\phi}(t).
\]  
(57)

The adaptive laws can be designed as
\[
\begin{aligned}
\frac{d}{dt}\theta_n &= \rho_n e_n\psi_n(x) - \gamma_n\theta_n, \\
\frac{d}{dt}\lambda(t) &= \sigma_1 p(t)e_n - \sigma_2\hat{\lambda}(t), \\
\frac{d}{dt}\phi(t) &= l_1 g_n(x)(|e_n| - \bar{k}) - l_2\hat{\phi}(t),
\end{aligned}
\]  
(58)
where \(\gamma_n, \sigma_2, l_2 > 0\) and \(\bar{k} > \Xi_n\). Substituting (17) and (58) into (57), one obtains
\[
\mathcal{D}_t^\beta V_n \leq -k_{n-1}V_{n-1} + \Theta_{n-1} - e_n\theta_n^T\psi_n(x) \\
+ \frac{1}{2\xi_2}d^2(t) - g_n(x)\bar{w}\phi(t)|e_n| \\
+ g_n(x)\bar{w}(\phi + \bar{\phi}(t))\Xi_n - g_n(x)\bar{k}\bar{w}\phi(t) \\
+ p(t)e_n + 2\bar{w}\delta_1 - \bar{w}e_n p(t)\hat{\lambda}(t) \\
- k_{n-1}e_n^2 + \frac{1}{\rho_n}\theta_n^T\mathcal{D}_t^\beta\theta_n + \frac{1}{\xi_2}\tilde{d}(t)\mathcal{D}_t^\beta\tilde{d}(t) \\
+ \frac{\bar{w}}{\sigma_1}\tilde{\lambda}(t)\mathcal{D}_t^\beta\tilde{\lambda}(t) \\
+ \frac{\tilde{w}}{l_1}\bar{\phi}(t)\mathcal{D}_t^\beta\bar{\phi}(t) \\
\leq -k_{n-1}V_{n-1} + \Theta_{n-1} + \frac{1}{2\xi_2}d^2(t) \\
+ p(t)e_n + 2\bar{w}\delta_1 + g_n(x)\bar{w}\phi\Xi_n \\
- \bar{w}e_n p(t)\hat{\lambda}(t) - k_{n-1}e_n^2 \\
- \frac{\gamma_n}{\rho_n}\theta_n^T\theta_n - \frac{\xi_1}{\xi_2}\bar{d}(t) \\
+ \frac{\xi_1}{\xi_2}\hat{d}(t)\theta_n^T\psi_n(x) + \frac{\xi_1}{\xi_2}\tilde{d}(t)e_n(x) \\
+ \bar{w}\tilde{\lambda}(t)p(t)e_n + \frac{1}{\xi_2}\tilde{d}(t)\mathcal{D}_t^\beta\tilde{d}(t) \\
- \frac{\bar{w}\sigma_2}{\sigma_1}\tilde{\lambda}(t)\tilde{\lambda}(t) + \frac{\tilde{w}l_2}{l_1}\bar{\phi}(t)\bar{\phi}(t).
\]
(59)

From the Young’s inequality and Assumption 1, one can get
\[
\begin{aligned}
- \frac{1}{\rho_n}\theta_n^T\theta_n^* &\leq \frac{1}{2\rho_n}\theta_n^T\theta_n^* + \frac{1}{2\rho_n}\bar{\theta}_n^T\bar{\theta}_n, \\
\bar{d}(t)\theta_n^T\psi_n(x) &\leq \frac{1}{2}q_1w_0\bar{d}^2(t) + \frac{1}{2q_1}\bar{d}^2(t), \\
\tilde{d}(t)e_n(x) &\leq \frac{1}{2}q_2\bar{d}^2(t) + \frac{1}{2q_2}\bar{d}^2(t), \\
\tilde{d}(t)\mathcal{D}_t^\beta\tilde{d}(t) &\leq \frac{1}{2}q_3\bar{d}^2(t) + \frac{1}{2q_3}\bar{d}^2(t), \\
- \frac{1}{\xi_2}\bar{d}(t) &\leq \frac{1}{2\xi_2}\bar{d}(t) + \frac{1}{2\xi_2}\bar{d}(t), \\
- \frac{1}{l_1}\bar{\phi}(t)\phi &\leq \frac{1}{2l_1}\bar{\phi}^2(t) + \frac{1}{2l_1}\phi^2(t),
\end{aligned}
\]  
(60)
where \(w_0 = \|\psi_n(x)\|^2, q_1 > 0, q_2 > 0\) and \(q_3 > 0\).

Furthermore, if follows from (59) and (60) that
\[
\mathcal{D}_t^\beta V_n \leq -k_{n-1}V_{n-1} + \Theta_{n-1} + 2\bar{w}\delta_1 + g_n(x)\bar{w}\phi\Xi_n \\
- k_{n-1}\bar{e}_n^2 - \frac{1}{2}\left(\frac{\gamma_n}{\rho_n} - \frac{\xi_1}{\xi_2}\right)\bar{\theta}_n^T\bar{\theta}_n \\
+ \frac{\gamma_n}{2\rho_n}\theta_n^*\theta_n^* + \frac{\xi_1}{2\xi_2}\bar{d}(t)e_n(x) \\
- \frac{1}{\xi_2q_3}\bar{d}(t) - \frac{\bar{w}\sigma_2}{\sigma_1}\tilde{\lambda}(t)\tilde{\lambda}(t) + \frac{\tilde{w}l_2}{l_1}\bar{\phi}(t)\bar{\phi}(t).
\]
where \( w_1 = \xi_1 (2 - q_1 w_0 + q_2 + q_3) - 1 > 0 \),
\( k_n = \min\{k_{n-1}, 2k_{1n}, \gamma_n - \rho_n \xi_1 / q_1 \xi_2, w_1, \sigma_2, l_2\} \), and
\( \Theta_n = \Theta_{n-1} + (\gamma_n / 2 \rho_n) \theta_n^T \theta_n + \left( \xi_1 / 2 q_3 \xi_2 \right) \xi_n^2 (x) + (1 / q_3 \xi_2) \eta_n^2 + (\omega \sigma_2 / 2 \sigma_1) \lambda_n^2 + (\omega \delta / 2 l_1) \phi_n^2 + 2 \hat{\omega} \delta_1 + g_n (x) \hat{\omega} \phi_n \xi_n \) are two positive constants.

According to the above analysis, the stability analysis of FONNs is shown in the following theorem.

**Theorem 1** For FONNs (8) under Assumption 1, if the virtual control signal is designed as (21), the actual controller is established as (22), and the adaptive laws are designed as (30), (41), and (58), then the tracking error \( e_1 \) converges to an arbitrarily small neighborhood of the origin if appropriate control parameters are chosen.

**Proof** Similar to [37], for every constant \( \varepsilon > 1 \), there exists \( t^* > 0 \), such that

\[
|V_n| < \frac{\varepsilon \Theta_n}{k_n}
\]

(62)

for all \( t > t^* \).

From (62) and the definition of \( V_n \), one knows that all signals of the closed-loop system keep bounded, and \( |e_1| \leq \sqrt{2 \varepsilon \Theta_n / k_n} \) holds for all \( t > t^* \). \( \square \)

**Remark 6** From (62), it is possible to reduce the range of \( |e_1| \) and improve the system control performance by increasing \( k_n \) or reducing \( \Theta_n \). As can be seen from the definition of \( k_n \) and \( \Theta_n \) in (61), the values of \( k_n \) and \( \Theta_n \) are determined by \( k_{1j}, k_{1n}, \sigma_1, l_1, \xi_2, \rho_j \) and \( \delta_1 \), i.e., one can increase the values of \( k_{1j}, k_{1n}, \sigma_1, l_1, \xi_2 \) and reduce the values of \( \rho_j \) and \( \delta_1 \) to increase \( k_n \) and reduce \( \Theta_n \). Therefore, in the simulation, the values of these parameters can be adjusted to reduce the tracking error \( |e_1| \) and improve the control performance of FONNs. However, if too large \( k_{1j} \) and \( k_{1n} \) are selected, it may result in too large control signal which may exceed the tolerance of the system; and if the values of design parameters \( \sigma_1, l_1, \xi_2 \) are too large, the parameter drift problem may occur. In a word, based on the adjustable design parameters variation feature, the appropriate parameters are often gradually adjusted in the actual simulation.

**Remark 7** Most existing literature ignores disturbances in nonlinear control systems or requires some strict conditions to ensure that unknown disturbances are bounded. However, these methods are hard to implement in the actual uncertain system. Therefore, a disturbance observer is designed to estimate the system unknown disturbance. In addition, the controller (22) is designed, which can simultaneously handle the uncertainty of the system as well as the dead-zone nonlinearity and disturbance observer. Therefore, on the design thread of the tracking control scheme, the framework diagram of the overall closed-loop system is given in Fig. 3.

### 4 Simulation results

Two examples will be given to demonstrate the effectiveness of the adaptive NN backstepping quantized control based on command filter for FONNs with asymmetric actuator dead-zone and disturbance observer in this section.

#### 4.1 Example 1

Consider the following fractional-order Duffing’s Oscillator chaotic system

\[
\begin{align*}
D^\beta_{\xi} x_1 &= x_2, \\
D^\beta_{\xi} x_2 &= x_1 - x_1^3 - 0.5 x_1 x_2 + 1.3 \cos t \\
&+ d(t) + H(q(v)),
\end{align*}
\]

(63)

in which \( f_1(x_1) = 0, f_2(x_1, x_2) = x_1 - x_1^3 - 0.5 x_1 x_2 + 1.3 \cos t \) are unknown functions, and \( g_1(x_1) = g_2(x_1, x_2) = 1 \).

Let the initial condition be \( x(0) = [0.21, 0.13]^T \). In addition, Fig. 4 shows that the system (63) has chaotic phenomenon when \( \beta = 0.95 \), \( H(q(v)) = 0 \) and \( d(t) = 0 \).

It should be mentioned that the LTQ is used in many literature, such as [24–26]. In this paper, the HTQ is introduced due to its fascinating properties. One performs a comparative experiment to display the difference in control effect and quantitative performance of the LTQ and the HTQ. In the simulation, in order to ensure the fairness of the comparison, all selected control parameters are the same in addition to the quantizer. The control method using the HTQ will be compared to the case using the LTQ to demonstrate the effectiveness of the method proposed in this paper.
On the other hand, the order $\beta$ is randomly chosen in the interval $(0.8, 1)$. The smooth reference signal $x_d$ and the unknown external disturbance signal $d(t)$ are selected as $\sin t$ and $\sin t + \cos t$, respectively. First, a set of design parameters are randomly selected in the process of simulation, which is prepared for regulating the control performance of the system gradually. Next, according to Remark 6, one can manually adjust these design parameters. In the process of parameters adjustment, it is not difficult to find that $k_{11}$, $k_{12}$, $k_{21}$ and $k_{22}$ play main control roles. If $k_{11}$, $k_{12}$, $k_{21}$ and $k_{22}$ take too small, the system will not achieve satisfactory control effects; on the contrary, too large parameters will cause the control input exceed the tolerance of the system. During the process of simulation, one founds that the system can achieve satisfactory control effects when $k_{11} \in [3, 6]$, $k_{12} \in [3, 6]$, $k_{21} \in [3, 6]$ and $k_{22} \in (0, 0.5)$. Then, the main design parameters can be manually chosen as $k_{11} = k_{12} = k_{21} = 5$ and $k_{22} = 0.1$. Finally, similar to the choice of the main parameters, the minor parameters are selected as $\rho_1 = \rho_2 = 5$, $\gamma_1 = \gamma_2 = 0.2$.
\( \xi_1 = 20, \xi_2 = 1, \kappa_1 = 30, r = 0.3, i = 50, v_{\text{min}} = 0.2, \sigma_1 = \sigma_2 = l_1 = l_2 = 0.1, \) and \( \delta_1 = \delta_2 = 1. \) The initial values of the adaptive law are selected as \( \theta(0) = [1, 1, 1, 1, 1]^T \in \mathbb{R}^5, \theta_2(0) = [1, 1, \ldots, 1]^T \in \mathbb{R}^{25}, \) \( \hat{\lambda}(0) = 0.1, \) and \( \hat{\phi}(0) = 0.2. \)

Case 1: The HTQ is used to quantify the dead-zone input in the system (63).

Case 1.1: When the dead-zone sizes are \( p_r = 0.2 \) and \( p_l = -0.3 \), simulation results are shown in Figs. 5a, 6a and 7a. From Fig. 5a, the reference signal \( x_d \) and the filtered signal \( x_{2,c} \) are tracked by the state variables \( x_1 \) and \( x_2 \) of the system (63), respectively, and
both tracking effects are relatively well. As shown in Fig. 5a, the tracking error $e_1$ rapidly converges to a relatively small neighborhood of the origin. All adjustable parameters are smooth and bounded in Fig. 6a. According to Fig. 6b, the filtered signal $x_{2,c}$ is tracked by the virtual controller $\alpha_1$, then one can see that tracking effect is consistent with the result of theoretical analysis. Finally, one considers the observation effect of disturbance observer $\hat{d}(t)$ in Fig. 6c, d, in which Fig. 6c is the trace of $d(t)$ and $\hat{d}(t)$, and Fig. 6d is the disturbance observation error $\tilde{d}(t)$. From Fig. 7a, the actual control input $v$, the quantized actual control input $q(v)$ and the quantized asymmetric actuator dead-zone $H(q(v))$ are large at first, then decrease rapidly, and when the system reaches stability, they keep in a relatively small region. Consequently, the effectiveness of the proposed method is verified for FONSS with asymmetric actuator dead-zone and unknown disturbance.
Case 1.2: Consider the case of $p_r = 2$, $p_l = -2$, simulation results of the system (63) are shown in the Figs. 5b and 7b. From Fig. 5a, b, when the changes in the dead-zone sizes are relatively obvious, the impact on the control performance and the tracking errors of the system has little effect. According to Fig. 7a, b, the smaller the dead-zone size, the greater the quantitative intensity. Although the difference between the selected dead-zone sizes in Case 1.1 and Case 1.2 is large, the changes in the control performance of the system (63) are not obvious. Therefore, the proposed method has strong robustness for the variation of dead-zone size in this paper.

Case 2: When the LTQ is selected and the dead-zone sizes are chosen as $p_r = 0.2$ and $p_l = -0.3$, simulation results of the system (63) are shown in Figs. 5c and 7c. It can be seen that the LTQ also has satisfactory control effects and quantitative effects.

However, compare simulation results of Case 1.1 and Case 2 from Fig. 5a, c, it is obvious that the control effect of using the HTQ is better than the case of the LTQ. It follows from Fig. 7a, c that the chattering is more obvious when using the LTQ. That is, the HTQ can effectively reduce the chattering phenomenon. What’s more, when the quantization level $i$ is larger, the quantization density of the HTQ is better than the LTQ. Therefore, the proposed method using the HTQ is more effective than the case of the LTQ in this paper.

4.2 Example 2

Consider the following fractional-order Chua-Hartley’s chaotic system

$$
\begin{align*}
D_\beta^\alpha x_1 &= x_2 + \frac{10}{7}(x_1 - x_1^3), \\
D_\beta^\alpha x_2 &= x_3 + 10x_1 - x_2, \\
D_\beta^\alpha x_3 &= -\frac{100}{7}x_2 + d(t) + H(q(v)), 
\end{align*}
$$

where $x(0) = [0.8, -2, 1]^T$, $f_1(x_1) = \frac{10}{7}(x_1 - x_1^3)$, $f_2(x_1, x_2) = 10x_1 - x_2$, and $f_3(x_1, x_2, x_3) = -\frac{100}{7}x_2$. 

Fig. 9 Simulation results of the system (64)

Fig. 10 Adjustable parameters of the system (64) with the HTQ
If \( \beta = 0.98 \), \( H(q(v)) = 0 \) and \( d(t) = 0 \), the chaotic phenomenon of the system (64) is shown in Fig. 8.

One chooses \( x_d = \cos t \) and \( d(t) = \sin t \cos t + \cos t \). The order \( \beta \) is randomly selected in the interval (0.9, 1). Similar to the parameters adjustment process of Example 1, the design parameters are manually chosen as \( k_{11} = k_{12} = k_{13} = k_{21} = k_{22} = 6, k_{23} = 0.1, \rho_1 = \rho_2 = \rho_3 = 10, \gamma_1 = \gamma_2 = \gamma_3 = 0.1, \xi_1 = 30, \xi_2 = 1, \kappa_1 = 2, \kappa_2 = 40, r = 0.2, i = 50, v_{\text{min}} = 0.1, \sigma_1 = 0.01, \sigma_2 = l_1 = l_2 = 0.1, \) and \( \delta_1 = \delta_2 = 0.1 \). The initial values of the adaptive law are selected as \( \theta_1(0) = [1, 1, 1, 1]^T \in \mathcal{R}^5, \theta_2(0) = [1, 1, \ldots, 1]^T \in \mathcal{R}^{25}, \theta_3(0) = [1, 1, \ldots, 1]^T \in \mathcal{R}^{125}, \hat{\lambda}(0) = 0.1, \) and \( \hat{\phi}(0) = 0.1 \).

When the above conditions are satisfied, simulation results similar to Example 1.

Case 1: The HTQ is used to quantify the actual control input \( v \) in the system (64).

![Fig. 11 Simulation results of the system (64) in Case 1.1](image1)

![Fig. 12 Control input of the system (64)](image2)
Case 1.1: The dead-zone sizes are selected as $p_r = 0.1$, $p_l = -0.2$, simulation results of the system (64) are shown in Figs. 9a, 10, 11 and 12a.

Case 1.2: When $p_r = 2$, $p_l = -2$, simulation results of system (64) are shown in Figs. 9b and 12b. From Figs. 9a, b and 12a, b, it is not difficult that the result is similar to Example 1.

Case 2: The LTQ is adopted and dead-zone sizes are chosen as $p_r = 0.1$, $p_l = -0.2$, Figs. 9c and 12c are simulation results of system (64). Compared to the Case 1.1, good tracking effects and quantization effects are obtained when using the HTQ.

5 Conclusion

In this paper, an adaptive NN backstepping quantized control observer method based on command filter and disturbance observer are proposed for FONSs with asymmetric dead-zone. The results show that the virtual control input and its fractional-order derivative can be estimated with a fractional-order command filter. This paper proposes a control method to coupling the dead-zone nonlinearity and the HTQ, which can solve the problem in the process of generating the dead-zone during quantized control for FONSs. The theoretical analysis shows that the proposed control method guarantees that the tracking error converges in an arbitrarily small region of the origin, and all signals of the closed-loop system remain bounded. However, it should be noted that although the HTQ has been used in this paper, the chattering phenomenon still exists. In addition, the NN is used to estimate the uncertain function, but the estimation accuracy is still not very satisfactory. Therefore, the further work will focus on improving the estimation accuracy for the unknown function and eliminating the chattering phenomenon.

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Declaration

Conflict of interest The authors declare that they have no conflict of interest.
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