A GATHERING PROCESS IN ARTIN BRAID GROUPS

EVGEINJ S. ESYP AND ILYA V. KAZACHKOV

Abstract. In this paper we construct a gathering process by the means of which we obtain new normal forms in braid groups. The new normal forms generalise Artin-Markoff normal forms and possess an extremely natural geometric description. In the two last sections of the paper we discuss the implementation of the introduced gathering process and the questions that arose in our work. This discussion leads us to some interesting observations, in particular, we offer a method of generating a random braid.

KEYWORDS: Braid groups; normal forms; rewriting systems.

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1. Preliminaries

Recall, that Artin braid group on \( n + 1 \) strands is the group given by the following generators and relations:

\[
B_{n+1} = \langle x_1, \ldots, x_n \mid [x_i, x_j] = 1, |i - j| \geq 2; x_ix_{i+1}x_i = x_{i+1}x_i x_{i+1} \rangle.
\]

The braid group admits a geometric presentation which we shall use throughout this paper. We refer to [5, 6, 12] for details. The idea is to associate to every word in the \( x_i \)'s a plane diagram, which is obtained by successively concatenating the diagrams of letters, they are given by Figure 1.

We shall make use of another definition, closely linked to the geometric presentation of braids. If two strands cross, then this we call a crossing (on Figure 1 the crossings of the \( i \)-th and the \( i + 1 \)-th strands are shown). To every letter \( y \) of a braid word \( \xi \in B_{n+1} \) we associate a ‘crossing’ of two strands. Consider the diagram of \( \xi \). The letters of \( \xi \) are in one-to-one correspondence with the crossings of the diagram of \( \xi \). So, depending on the position of the letter \( y \) in the word \( \xi \) the crossing corresponding to \( y \) may be different. For instance, to the letter \( x_1 \) in the
word $\xi_1 = x_1 x_2$ corresponds the crossing of the first and the second strand, while in the word $\xi_2 = x_2 x_1$ the crossing linked to $x_1$ is the crossing of the first and the third strands (see Figure 2). Depending on the sign of the generator the sign of crossing can be positive or negative.

**Figure 1.** The diagrams of generators

**Figure 2.** Correspondence of crossings and letters

**Definition 1** (notation of crossing). Let $|r, s|$ denote the crossing of the $r$-th and the $s$-th strands. Thus the crossing is completely defined when we fix the numbers $r$ and $s$ and the sign. We say that a crossing is negative, whenever the corresponding letter of the word is the inverse of one of the generators $x_1, \ldots, x_n$ and positive otherwise. Thus, the natural notion of the ‘sign’ $\epsilon = \pm 1$ of a crossing arises.

From the foregoing discussion we conclude that braid words can be treated as sequences of crossings.

Note that any crossing, consider $|1, 2|^{-1}$, for instance, can correspond to different generators $x_1, \ldots, x_n$, depending on the word and the position of a generator in the word (see Figure 3).

Furthermore, we offer the reader to check that the following remark holds

**Remark 1.** Let $|i, j|$ be an arbitrary crossing. Then there exists a braid word $w$ and a letter $y$ in $w$ so that the crossing corresponding to $y$ in $w$ is $|i, j|$, here $y \in X \cup X^{-1}$.

We next formulate the main result of the current paper.

**Theorem 1** (Normal form of elements of $B_{n+1}$). Let $B_{n+1}$ be the braid group on $n + 1$ strands. Every element $w \in B_{n+1}$ can be uniquely written in the form

$$w = x_{m}^{n} \cdot w_{3}(x_{1}, x_{2}) \cdot w_{4}(x_{1}, x_{2}, x_{3}) \cdots w_{n+1}(x_{1}, \ldots, x_{n}); \ m \in \mathbb{Z}$$

Where for every $3 \geq k \leq n + 1$ the words $w_k$ are freely reduced and the crossings involved into $w_k$ are the crossings of the form $|i, k|$ $(k = 3, \ldots, n + 1; i < k)$ only.
This is roughly saying that for given braid there exists a unique braid, which is equivalent to the initial one and is ‘constructed’ in the following way: first one entangles the first and the second strand $m$ times and the first and the second strands do not cross further. Then one entangles the third strand with the second and the first (this is $w_3$) so that furtheron there are no crossings of the first and the second strands with the third strand, then one entangles the fourth strand with the third, the second and the first (this is $w_4$) and so on.

**Example 1.** Let $\xi \in B_4$, $\xi = x_3x_2^{-2}x_1$. Then normal form (2) of the word $\xi$ is $\xi^* = x_1x_3x_2x_1^{-2}x_2^{-1}$, see Figure 4.

In the case of $B_3$ Theorem 1 can be reformulated as follows:

**Corollary 2.** Every element $w \in B_3$ can be uniquely written in the form

$$w = x_1^m \cdot (x_2^{k_1}x_1^{k_2}x_2^{k_3} \cdots x_l^k)$$

here $l = 1, 2, k, m \in \mathbb{Z}$, $k_r$ is even if and only if $r$ is even and is odd if and only if $r$ is odd.
Remark 2. We draw reader’s attention to the fact that in the normal form given in the above Corollary the power of the first letter \( x_1 \) has an arbitrary degree \( m \) and that the terminal letter of \( w \) may be any letter \( (x_1 \text{ or } x_2) \) in any power \( k \).

Consider the word \( w = x_2^{k_1} x_1^{k} \in B_3 \), where \( k_1 \) is an odd integer and \( k \in \mathbb{Z} \). This braid is in normal form, since only the third strand is entangled. If we now look at \( w \) as a word written as in Corollary, we have \( m = 0 \) and, since \( x_1 \) is the terminal letter of \( w \), it may be in an arbitrary power (not necessarily even).

Moreover, the word \( x_1^m x_2^k \), where \( m, k \in \mathbb{Z} \), is in normal form.

In paper [1] E. Artin (and A. Markov in [10]) proves that every pure braid can be transformed into the form (2). He suggests to introduce a normal form in the braid group \( B_{n+1} \) as follows. Every element of \( B_{n+1} \) can be written as a product of a pure braid and a coset representative of \( B_{n+1} \) modulo the group of pure braids \( I_{n+1} \). Normal form (2), therefore, generalises Artin-Markoff normal form (see [1] and [10]) to all (not necessarily pure) braids and admits a natural geometric description. Moreover, in Section 2, we construct a gathering process which transforms a given word into the form (2).

As mentioned above in the case when \( w \) is a pure braid, normal form (2) coincides with Artin-Markoff normal form, given by Theorem 17, [1] (see as well [10]). For the sake of convenience and completeness we expose this theorem below.

By the definition set
\[
A_{i,i+1} = x_i^2, \quad i = 1, \ldots, n.
\]

For \( i < j, i, j = 1, \ldots, n + 1 \) set
\[
A_{j,i} = A_{i,j} = x_i^{-1} \cdots x_{j-1}^{-1} \cdot x_j^2 \cdot x_{j-1} \cdots x_i = x_{j-1} \cdots x_{i+1} \cdot x_i^2 \cdot x_{i+1}^{-1} \cdots x_{j-1}^{-1}
\]

Theorem 3 ([1], Theorem 17). The \( A_{i,k} \)’s are generators of the group of pure braids. Every pure braid can be uniquely written in the form:
\[
A = U_1 \cdots U_{n-1}
\]
where each \( U_j \) is a uniquely determined power product of the \( A_{i,j} \) using only those with \( i > j \).

Remark 3. In [1] the author reads the words from right to left and draws diagrams moving upwards. The diagrams (in our interpretation) of elements \( A_{j,i} = A_{i,j} \) take the form shown on Figure 5.

In [1] the author introduces the elements \( A_{i,j} \)’s to make normal form (2) a unique word in the \( A_{i,j} \)’s (however the elements \( A_{i,j} \)’s can be presented by different braid words). Choose the word representative of the element \( A_{i,j} \) to be \( x_{j-1} \cdots x_{i+1} \cdot x_i^{-1} \cdot x_{i+1}^{-1} \cdots x_{j-1}^{-1} \), where \( j > i \). Notice next that in this case the braid word \( A_{i,j} \) entangles the \( j \)-th strand only, leaving the other \( n \) strands unentangled (see Figure 4).

Finally, minding the insignificant difference in the geometric interpretation of braid words and noticing that the elements \( U_{n-1}, \ldots, U_1 \) in (3) are uniquely determined elements of \( B_{n+1} \) (in [1] the \( U_j \)'s are words in the symbols \( A_{i,j} \)'s), we conclude that for pure braids the elements \( x_1^m, w_3, \ldots, w_{n+1} \) in normal form (2) coincide with, correspondingly, \( U_{n-1}, \ldots, U_1 \) in Equation (4).
2. Proof of Theorem 1

In this section we prove Theorem 1. We also provide an algorithm (a gathering process) for constructing normal form (2).

We first prove that every braid word \( w \in B_{n+1} \) can be taken to the word in the form (2). And then prove that there exists only a unique word of the form (2) equal to \( w \) in \( B_{n+1} \).

We say that a crossing that involves the \( n+1 \)-th strand is \textit{big}. Otherwise we term a crossing \textit{small}.

In the following by \( \xi_1 \equiv \xi_2 \) we shall mean the equality in the free group and by \( \xi_1 \cdot \xi_2 \) we denote cancellation-free multiplication of two words.

2.1. \textbf{Existence.} Below we introduce a gathering process that takes the word \( w \) into the form (2). First we transform the word \( w \) into the form in which every big crossing is gathered in the end of the word, \( w = \xi_n w_{n+1} \). Then we apply a similar process to \( \xi_n \) and take it in the form in which every crossing that involves the \( n \)-th strand is collected in the end of the word, \( \xi_n w_{n+1} = \xi_{n-1} w_n w_{n+1} \) and so on.

Before we begin to give a formal description of our algorithm we give a less formal description of the idea it uses. Given an arbitrary word \( w \in B_n \) we consider it as a sequence of crossings. Take the first small crossing \( \rho \) in \( w \) so that there are big crossings preceding \( \rho \) (i.e. the corresponding letters are closer to the beginning of \( w \)). The word \( w \) then has the following form \( T \cdot U \cdot V \cdot W \). Where in \( T \) all the crossings are small, in \( U \) all the crossings are big, \( V \) is a letter and the corresponding crossing is \( \rho \) and \( W \) is simply the rest of the word. We then, using transformations given by Figures 10 and 11 move this crossing upwards (preserving the braid) till there are no big crossings in \( w \) preceding \( \rho \). Applying the same procedure to every small crossing in \( w \) we get its decomposition in the form \( w = \xi_n w_{n+1} \)

We begin to construct a gathering process that transforms an arbitrary braid word \( w \) into the form \( w = \xi_n w_{n+1} \), where every crossing in \( w_{n+1} \) is big and every crossing in \( \xi_n \) is small.

By the diagram shown on Figure 6 we denote an arbitrary braid on \( n+1 \) strands. Let us agree that the \( n+1 \)-th strand is designated bold. By the diagram shown on Figure 7 we denote a braid, in which no small crossings occur. And by the diagram shown on Figure 8 we denote a braid, in which no big crossings occur.
Now, an arbitrary braid word $w$ can be represented as follows:

$$T \cdot U \cdot V \cdot W$$

where the corresponding braid diagram takes the form shown on Figure 9. Note that between $T$ and $U$, $U$ and $V$, $V$ and $W$ no free cancellation occurs. I.e. $T$ is a word in which only small crossings occur. In $U$ every crossing involves the $n+1$-th strand and therefore is big. The word $V$ is a letter, the corresponding crossing $\rho$ is small and $W$ is the rest of the braid word $w$. We shall further move the only crossing in $V$, that is $\rho$, upwards (towards the beginning of the word). Thereby obtaining the following word $TV \cdot W'$, where $TV$ is a word in which every crossing is small and $W'$ counts as many small crossings as $W$.

We prove the statement using double induction. First is on the length $j$ of the word $U$ (the number of big crossings preceding small crossing $\rho$) and the second one on the number $k$ of letters in $W$ that correspond to small crossings (the number of small crossings that stand further from the beginning of the word than $\rho$).

Consider two cases.

1. The letter $V$ commutes with the terminal letter in $U$. In which case we permute the letter $V$ and the terminal letter in $U$ then freely reduce the word and
obtain the new word $w = T \cdot U' \cdot V \cdot U'' \cdot W$ (or $w = T'UW$, provided that $UV = VU$) and, therefore, the number $j$ is decreased and the statement follows by induction.

2. The terminal letter in $U$ does not commute with $V$. Since every crossing in $T$ and $V$ is small and every crossing in $U$ is big the only possibilities for $V$ and the two crossings preceding $V$ are shown on Figure 10. Replace the configurations shown on Figure 10 by the corresponding braid on Figure 11 and then freely reduce the obtained word.
Figure 10. Possible variants
Figure 11. Replacements
Denote by $U'''$ the following word $U''' \cdot z_1 \cdot z_2 = U$, where $z_1, z_2 \in \{x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}\}$. In terms of words the transformations shown on Figures 10 and 11 can be rewritten as follows.

\[
TU''' \cdot x_i^\delta x_{i+1}^\epsilon x_i x_{i+1}^\delta W \rightarrow TU''' \cdot x_i x_{i+1} x_i x_{i+1}^\delta W
\]  

(4)

\[
TU''' \cdot x_i^\epsilon x_{i+1} x_i x_{i+1}^\delta W \rightarrow TU''' \cdot x_i^\epsilon x_{i+1} x_i x_{i+1}^\delta W
\]  

(5)

\[
TU''' \cdot x_i^\epsilon x_{i+1} x_i x_{i+1}^\delta x_i x_{i+1}^\delta W \rightarrow TU''' \cdot x_i^\delta x_i x_{i+1} x_i x_{i+1}^\delta x_i x_{i+1}^\delta W
\]  

(6)

\[
TU''' \cdot x_i^\epsilon x_{i+1} x_i x_{i+1}^\delta W \rightarrow TU''' \cdot x_i^\epsilon x_{i+1} x_i x_{i+1}^\delta x_i x_{i+1}^\epsilon W
\]  

(7)

\[
TU''' \cdot x_i^\epsilon x_{i+1} x_i^\epsilon x_{i+1}^\epsilon x_i x_{i+1}^\delta x_i^\epsilon x_{i+1}^\epsilon x_i x_{i+1}^\epsilon W
\]  

(8)

\[
TU''' \cdot x_i^\epsilon x_{i+1} x_i x_{i+1}^\delta W \rightarrow TU''' \cdot x_i^\epsilon x_{i+1} x_i x_{i+1}^\delta x_i x_{i+1}^\epsilon W
\]  

(9)

\[
TU''' \cdot x_i^\epsilon x_{i+1} x_i x_{i+1}^\delta W \rightarrow TU''' \cdot x_i^\epsilon x_{i+1} x_i x_{i+1}^\delta x_i x_{i+1}^\epsilon W
\]  

(10)

\[
TU''' \cdot x_i^\epsilon x_{i+1} x_i^\epsilon x_{i+1}^\epsilon x_i x_{i+1}^\epsilon x_i^\epsilon x_{i+1}^\epsilon x_i x_{i+1}^\epsilon W
\]  

(11)

Furthermore, in the constructed procedure whenever the letter $V$ commutes with the terminal letter in $U$ we permute the letter $V$ and the terminal letter in $U$. In this case the word $U$ may consist of a single letter. This transformation rewrites in terms of words as follows:

\[
T\hat{U}z_1 V W \rightarrow TUVz_1 W,
\]

where $\hat{U} z_1 = U$ and $\hat{U}$ is possibly empty.

Finally on the frontier of either $U'''$ (or $\hat{U}$) or $W$ or $z_1$ and the replaced segment free cancellation might occur. For the sake of completeness, we write the expression of free reduction

\[
x_i^\epsilon x_i^{-\epsilon} \rightarrow 1, \; \epsilon = \pm 1
\]  

(13)

We leave the reader to check that Equations (4)-(11), (12) and (13) are in fact equalities in the braid group.

Suppose that after a step of the procedure from the initial word $w$ we obtain a new word $w'$, and consider its $T \cdot U \cdot V \cdot W$-decomposition. In the obtained $T \cdot U \cdot V \cdot W$-decomposition the length $j$ of $U$ is lower, provided that the crossing $\rho$ is not in $T$ (in which case the statement follows by induction on $k$). The word $V$ is a letter and it corresponds to the same crossing $\rho$, provided that the letter $V$ has not cancelled with a letter from $T$. Finally, since all of the above replacements do not create any new small crossings in $w$ (though replacements given by Expressions (6), (7), (10), (11) create new big crossings), the above process stops and the statement follows by induction on $k$.

As an output we obtain the following decomposition of an arbitrary braid $w = \xi_n w_{n+1}$, where in the word $\xi_n$ every crossing is small and in $w_{n+1}$ all big crossings
of \( w \) are collected. We next apply a similar process to \( \xi_n \) and take it in the form in which all crossings involving the \( n \)-th strand are gathered in its end, \( w = \xi_n w_{n+1} = \xi_{n-1} w_n w_{n+1} \) and so on. Finally, we obtain a word in the form (2).

2.2. Uniqueness. Below we show that if \( \xi_1 \) and \( \xi_2 \) are two braid words written in the form (2) and \( \xi_1 = \xi_2 \) in the braid group \( B_{n+1} \) then \( \xi_1 \equiv \xi_2 \). Firstly, note that, on account of Theorem 3 (Theorem 17, [1]) or Theorem 6 in Section 11 [10] if \( \xi_1 \) and \( \xi_2 \) are two pure braid words written in the form (2) and \( \xi_1 = \xi_2 \) then \( \xi_1 \equiv \xi_2 \).

Consider two braid words \( \xi_1 = p_2 \cdots p_n w \) and \( \xi_2 = q_2 q_3 \cdots q_n w \) written in the form (2). Denote \( p_2 \cdots p_n \) and \( q_2 q_3 \cdots q_n \) by correspondingly, \( t \) and \( v \). Note that \( t \) and \( v \) consist of small crossings only, while \( u \) and \( w \) of big crossings only.

In the above notation we have

\[
t \circ u \circ w^{-1} \circ v^{-1} = 1
\]

In what follows that \( v^{-1} \circ t \circ u \circ w^{-1} = 1 \), where by ‘\( \circ \)’ we denote the concatenation (without free reduction) of words.

We next want to show that the subword \( v^{-1} \circ t \circ u \circ w^{-1} \) is a pure braid every crossing in which is small and that the subword \( u \circ w^{-1} \) of the word \( v^{-1} \circ t \circ u \circ w^{-1} \) is a pure braid every crossing in which is big. Obviously \( v^{-1} t \) is a braid in which every crossing is small (by the definition both \( v \) and \( t \) involve letters \( x_1, \ldots, x_{n-1} \) and their inverse only).

**Remark 4.** Let \( \xi \) be a braid every crossing in which is big (in which case the initial letter of \( \xi \) is \( x_n \)) and let \( \zeta \) be an arbitrary braid every crossing in which is small (in which case \( \zeta \) does not contain the letter \( x_{n+1} \)). Then in the word \( \zeta \cdot \xi \) every crossing in the subword \( \zeta \) is small, while every crossing in the subword \( \xi \) is big. By

![Figure 12. The braid \( \xi \) is big regardless of the initial segment of the word \( \zeta \xi \), provided that every crossing in \( \zeta \) is small](image)

the definition the braid \( \xi \) entangles the \( n + 1 \)-th strand, leaving the first \( n \) strands parallel (regardless of the permutation defined by the subword \( \zeta \) of the word \( \zeta \xi \)), see Figure 12.

We next show that every crossing in the subword \( u \circ w^{-1} \) of the word \( v^{-1} \circ t \circ u \circ w^{-1} \) is big. Since every crossing in \( u \) is big, by Remark 4 every crossing in the subword \( u \) of the word \( v^{-1} \circ t \circ u \circ w^{-1} \) is big. We are left to show that every crossing in the \( w^{-1} \) passage of the word \( v^{-1} \circ t \circ u \circ w^{-1} \) is big. To show the latter we need to study the structure of crossings of the inverse of a braid that entangles the \( n + 1 \)-th strand only. Consider a braid \( \xi \) so that every its crossing is big (it therefore entangles the \( n + 1 \)-th strand, leaving the first \( n \) strands unentangled)
and so that the permutation defined by $\xi$ takes the $n+1$-th strand into the $k$-th position then the inverse $\xi^{-1}$ entangles the $k$-th strand only, leaving the other $n$ strands unentangled. Furthermore, by the definition, the permutation defined by $\xi^{-1}$ is the inverse of the permutation defined by $\xi$ and consequently the permutation defined by $\xi^{-1}$ takes the $k$-th strand of $\xi^{-1}$ into the $n$-th position.

The above argument can also be illustrated by the following diagrams. The diagram of an arbitrary braid word $\xi$ has the form given on Figure 13 (two strands are designated and a corner is marked to demonstrate the connections between a braid and its inverse) and, consequently, the diagram of the word $w^{-1}$ has the form given on Figure 14 i.e. the diagram of the word $\xi^{-1}$ is a two times reflected diagram of $\xi$. Let us consider an example.

**Example 2.** Let $\xi = x_3x_2^{-2}x_1$, the respective diagram is given by Figure 15. The inverse of $\xi$ is $\xi^{-1} = x_1^{-1}x_2^2x_3^{-1}$ and the corresponding diagram is given by Figure 16.

![Figure 13](image13.png)

**Figure 13.** An arbitrary braid

![Figure 14](image14.png)

**Figure 14.** The inverse of an arbitrary braid

We now return to the consideration of the word $v^{-1} \circ t \circ u \circ w^{-1}$. Recall that, by our assumption, the word $u$ takes the $n+1$-th strand into the $k$-th position.
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Figure 16. The diagram of $\xi^{-1} = x_1^{-1}x_2^2x_3^{-1}$

Since $tu = vw$ thus the permutations defined by $tu$ and $vw$ coincide, and both $u$ and $w$ permute the $n + 1$-th strand into the $k$-th position. Now, from the foregoing discussion we know that:

- Every crossing in $v^{-1} \circ t$ is small,
- Every crossing in $u$ is big,
- The word $w^{-1}$ entangles the $k$-th strand only.

Consequently, in the word $v^{-1} \circ t \circ u \circ w^{-1}$ the subword $w^{-1}$ entangles the strand, which is on the $k$-th position. By Remark 4 since every crossing in the braid $v^{-1}t$ is small and since the permutation defined by the word $u$ takes the $n + 1$-th strand into the $k$-th position, so does the permutation defined by the word $v^{-1} \circ t \circ u$. We thereby obtain that all crossings in the $v^{-1} \circ t$ subword of the word $v^{-1} \circ t \circ u \circ w^{-1}$ are small and all the crossings in the subword $u \circ w^{-1}$ are big.

The latter argument can be illustrated by diagrams. The diagrams corresponding to Equation (14) take the form shown on Figure 17. From Figure 17 we see that,

![Diagrams](image)

Figure 17. Diagrams of $t \circ u \circ w^{-1} \circ v^{-1}$ and $t \circ u \circ w^{-1} \circ v^{-1}$

since the permutation defined by the word $v^{-1} \circ t \circ u \circ w^{-1}$ is the identity and since $v^{-1} \circ t$ defines a permutation of the first $n$ strands, while $u \circ w^{-1}$ entangles the $n + 1$-th strand only both $v^{-1} \circ t$ and $u \circ w^{-1}$ are pure braids.
From the above discussion, since \( v^{-1} \circ t \circ u \circ w^{-1} \) is a pure braid, by Theorem 3 (Theorem 17, [1]) or Theorem 6 in Section 11, [10], the equality \( v^{-1} \circ t \circ u \circ w^{-1} = 1 \) therefore implies that
\[
v^{-1} \circ t = 1 \text{ and } u \circ w^{-1} = 1.
\]
Using induction on the number of strands \( n + 1 \) we may assume that the equality \( v = t \) in \( B_n \) implies that \( v \equiv t \).

Finally, we are left to show that if the braid \( uw^{-1} \) is trivial, every its crossing is big and the word \( uw^{-1} \) is freely reduced then \( uw^{-1} \) is the empty word. Since \( uw^{-1} \) is a pure braid every crossing of which is big, it can be written as a product of the following words
\[
A_{i,n+1} = x_{n+1}^{-1} \cdots x_{i+1}^{-1} \cdot x_i^2 \cdot x_{i+1} \cdots x_n; \quad i = 1, \ldots, n.
\]
By the definition \( A_{n,n+1} = x_n^2 \). Consequently normal form \((2)\) of a pure braid \( uw^{-1} \) is \( U_1 \) (in the notation of Theorem 3). Therefore, since \( uw^{-1} = 1 \), on behalf of Theorem 6 in Section 11, [10], \( uw^{-1} \) is the empty word and \( u \equiv w \). Consequently, normal form \((2)\) is unique.

3. Actual Implementation and Random Braid

In this section we explain another approach to the computation of normal form \((2)\). This approach turns out to be efficient in the actual implementation of the algorithm on a computer. The current and the following sections are less formal and hold some of our ideas and comments on the issue.

In the actual implementation of the algorithm for computing the normal form \((2)\) of a braid word \( \xi \) it is more convenient to treat elements of \( B_{n+1} \) as sequences of crossings. Let us consider an example.

Example 3. Let \( \xi = x_3 x_2^{-2} x_1 \). This element rewrites in terms of crossings as follows:
\[
\xi_1 = |3, 4 | 2, 4 |^{-1} | 2, 4 |^{-1} | 1, 2 |.
\]
On the other hand, given a proper sequence of crossings, we can form the corresponding braid word. Let us consider the following sequence of crossings:
\[
\xi_1^* = |1, 2 | 3, 4 | 1, 4 | 2, 4 |^{-1} | 2, 4 |^{-1} | 1, 4 |^{-1}.
\]
It is fairly obvious that the corresponding braid word is \( \xi^* = x_1 x_3 x_2 x_1^{-2} x_2^{-1} \).

Remark 5. The correspondence between sequences of crossings and braid words is not one-to-one. Let \( \zeta_1 = |1, 3 |^{-1} \) be a sequence of crossings. There is no braid word \( \zeta \) corresponding to the sequence \( \zeta_1 \).

Remark 6. The set of all sequences of crossings that represent elements of a fixed braid group \( B_n \) is regular, i.e. recognised by a finite automaton.

Proof. The automaton has \( n! \) states which are numbered by the elements of the group of permutations \( S_n \) on \( n \) symbols. The initial state is the one that corresponds to the trivial permutation, and all the states are fail states. There are two edges, which are labelled by crossings \( |i, j |^{-1} \) from a state labelled \( \theta \in S_n \) to a state labelled \( \theta \in S_n \) whenever \( \theta = \theta \cdot (i, j) = \theta \cdot (l, l+1)^{\theta} = (l, l+1)\theta \), for some \( l = 1, \ldots, n \). We leave the reader to check that this automaton recognises the set of all sequences of crossings that correspond to braid words (not necessarily freely reduced).
We next rewrite the replacements given by Figures 10 and 11 (Equations (4) - (11)) in terms of sequences of crossings.

Let $k > j, l$, $1 \leq k, j, l \leq n + 1$ and let $\epsilon, \delta \in \{-1, 1\}$. Replacements given by Figures 10 and 11 (Equations (4) - (11)) take the form (we assume that $k > j, l$):

\begin{align*}
(15) & \quad | j, k |^\delta | l, k |^\epsilon | l, j |^\epsilon \rightarrow | l, j |^\epsilon | l, k |^\epsilon | j, k |^\delta \\
(16) & \quad | j, k |^\epsilon | l, k |^\epsilon | l, j |^\delta \rightarrow | l, j |^\delta | l, k |^\epsilon | j, k |^\epsilon \\
(17) & \quad | j, k |^\epsilon | j, k |^\delta | l, j |^\epsilon | l, k |^\delta \rightarrow | l, j |^\delta | l, k |^\epsilon | j, k |^\epsilon | l, k |^\delta
\end{align*}

Recall that in Subsection 2.1 we used yet another transformation of a braid. In the $T \cdot U \cdot V \cdot W$ decomposition of a braid if the letter $V$ commutes with the terminal letter in $U$ we permute the letter $V$ and the terminal letter in $U$ and obtain the new word $w = T U V U V W$, see Equation (12). To perform such a transformation we introduce the following transformation of sequences of crossings.

Let $i < \max \{l, k\}$ and $j < \max \{l, k\}$; $1 \leq i, j, k, l \leq n + 1$ and let $\epsilon, \delta \in \{-1, 1\}$. The transformation mentioned in Subsection 2.1 takes the form:

\begin{align*}
(19) & \quad | l, k |^\epsilon | i, j |^\delta \rightarrow | l, k |^\delta | i, j |^\epsilon
\end{align*}

We, therefore replace the sequence of crossings $\rho \varrho$ by $\varrho \rho$, whenever $\rho$ and $\varrho$ commute and $\varrho$ is ‘smaller’ than $\rho$.

Finally, since in Subsection 2.1 we used reduction in the free group (see Equation (13)) we need to introduce the following transformation of sequences of crossings:

\begin{align*}
(20) & \quad | i, j |^\epsilon \rightarrow 1
\end{align*}

We next reformulate Theorem 4 for elements of braid groups viewed as sequences of crossings.

**Theorem 4.** Every sequence of crossings which corresponds to an element $w \in B_{n+1}$ can be taken (by the means of transformations (15) - (20)) to the form

\begin{align*}
(21) & \quad w = | 1, 2 |^m \cdot w_3 | 1, 3 | \cdots w_{n+1} | 1, n |, \ldots, | n, n + 1 |; \quad m \in \mathbb{Z}
\end{align*}

Where for every $3 \leq k \leq n + 1$ the words $w_k$ do not contain subwords of the form $| i, j |^\epsilon | i, j |^\epsilon$, $\epsilon = \pm 1$. Under the above assumptions presentation in the form (21) is unique.

**Remark 7.** Let $w$ be an arbitrary sequence of crossings, which represents a braid word $\xi$. Suppose next that in $\xi$ there is a subword, which coincides with one of the left-hand sides of rules (15) - (20). Then the word $\xi'$, which is obtained from $\xi$ by replacing the left-hand side of a rule (15) - (20) by the respective right-hand side, is a sequence of crossings which rewrites into a braid word. I. e. rules (15) - (20) preserve the property of being a representative of a braid word.

**Remark 8.** Not only did the above approach turned out to be very useful in actual implementation of the algorithm of computation of normal form (2), but also hints at us the idea to construct a term rewriting system (a Knuth-Bendix like algorithm) for elements of braid groups viewed as sequences of crossings. We suppose that
normal form \((2)\) can be generalised and a similar normal form can be constructed for an arbitrary Artin group \(A\) of finite type. First one needs to fix an ordering of the natural generators of the corresponding (to \(A\)) Coxeter group \(A\). Next one needs to rewrite an arbitrary word \(x \in A\) as a sequence of elements of \(A\). Then transform the element \(x\) (agreeing with the introduced order) to a form whose presentation as a sequence of generators of \(A\) is in some sense small (an analogue of the form \((2)\)). And consequently there exist a nice geometric normal form for an arbitrary Artin group of finite type. Below, in Section 5 we show how one can construct such a form for the group \(A = \langle a, b \mid abab = baba\rangle\).

One of the most important problems in a struggle to construct cryptography on braid groups is to give a method for generating a random braid (see [6]). We, therefore, can not but notice that Theorem 1 gives a method of generating a random braid.

A naive approach to generating a random braid is to generate a random freely reduced word and claim this a random braid. However there is a dramatic difference between a random braid and a random freely reduced word. For example, the results of [11] and [12] show that a random braid has got a non-trivial centraliser (its centraliser differs from the center of \(B_{n+1}\)) with a non-zero probability, while computational results show, that the centraliser of a random reduced word in the free group, treated as an element of \(B_{n+1}\) is trivial.

To generate a random braid on \(n + 1\) strands we can use Theorem 1 for instance, as follows. Here we use the ideas of [2], whereto we refer the reader for details and justification of the method. We do not in any way insist that the suggested method is better than any other method known, though we believe that it may turn out to be useful for computer scientists.

We suggest the following random process with weak interferences. One has:

\[
w = x_1^{n_1} \cdot w_3(x_1, x_2) \cdot w_4(x_1, x_2, x_3) \cdots w_{n+1}(x_1, \ldots, x_n).
\]

1. We generate a random power of \(x_1\) as follows, thus entangling the first two strands. We start at the identity element of \(B_{n+1}\) and either do nothing with probability \(s_2 \in (0, 1]\) and then go to step 2 or move to one of the two elements \(x_1\) or \(x_1^{-1}\) with equal probabilities \(\frac{1-\epsilon_2}{2}\). If we are at an element \(x_1^{\epsilon_2} = v \neq 1\) we either stop at \(v\) with probability \(s_2\) (and proceed to step 2), or move, with probability \(1 - s_2\) to the vertex \(x_1^{m+1}\), if \(m > 0\) and to the vertex \(x_1^{-m-1}\), if \(m < 0\).

2. On this step we generate \(w_3(x_1, x_2)\) (in the notation of Equation \((2)\)), thus entangling the third strand into the first two. This process can be treated as a discrete random walk on two points, which are linked to the first and the second strands. We start at the identity element of \(B_{n+1}\) and either do nothing with probability \(s_3 \in (0, 1]\) and then go to step 3 or move to one of the two elements \(x_2\) or \(x_2^{-1}\) with equal probabilities \(\frac{1-\epsilon_3}{2}\). If we are at an element \(x_2^{\epsilon_3} \cdot x_1^{n_2} = v \neq 1\), \(l = 1, 2\), we either stop at \(v\) with probability \(s_3\) (and proceed to step 3), or:

A. If \(l = 2\) and \(m_l > 0\) is odd, move, with probability \(\frac{1-\epsilon_3}{3}\), to one of the vertices \(x_2^{m_1} \cdot x_1^{m_2} \cdots x_2^{m_{l+1}}\) or \(x_2^{m_1} \cdot x_1^{m_2} \cdots x_1^{m_l} x_1\) or \(x_2^{m_1} \cdot x_1^{m_2} \cdots x_l^{m_l} x_1^{-1}\).

B. If \(l = 2\) and \(m_l < 0\) is odd, move, with probability \(\frac{1-\epsilon_3}{3}\), to one of the vertices \(x_2^{m_1} \cdot x_1^{m_2} \cdots x_2^{m_l} x_1\) or \(x_2^{m_1} \cdot x_1^{m_2} \cdots x_l^{m_l} x_1^{-1}\).

C. If \(l = 2\) and \(m_l > 0\) is even, move, with probability \(1 - s_3\), to the vertex \(x_2^{m_1} \cdot x_1^{m_2} \cdots x_2^{m_{l+1}}\).

D. If \(l = 2\) and \(m_l < 0\) is even, move, with probability \(1 - s_3\), to the vertex \(x_2^{m_1} \cdot x_1^{m_2} \cdots x_2^{m_l} x_1^{-1}\).
E. If \( l = 1 \) and \( m_1 > 0 \) is odd, move, with probability \( 1 - s_3 \), to the vertex \( x_1^{m_1} \cdot x_1^{m_2} \cdot \ldots \cdot x_1^{m_1-1} \).

F. If \( l = 1 \) and \( m_1 < 0 \) is odd, move, with probability \( 1 - s_3 \), to the vertex \( x_1^{m_1} \cdot x_1^{m_2} \cdot \ldots \cdot x_1^{m_1-1} \).

G. If \( l = 1 \) and \( m_1 < 0 \) is even, move, with probability \( \frac{1-s_3}{3} \), to one of the vertices \( x_1^{m_1} \cdot x_1^{m_2} \cdot \ldots \cdot x_1^{m_1-1} \) or \( x_1^{m_1} \cdot x_1^{m_2} \cdot x_1^{m_1} \cdot x_2 \) or \( x_1^{m_1} \cdot x_1^{m_2} \cdot \ldots \cdot x_1^{m_1} \cdot x_2^{-1} \).

H. If \( l = 1 \) and \( m_1 > 0 \) is even, move, with probability \( \frac{1-s_3}{3} \), to one of the vertices \( x_1^{m_1} \cdot x_1^{m_2} \cdot \ldots \cdot x_1^{m_1+1} \) or \( x_1^{m_1} \cdot x_1^{m_2} \cdot x_1^{m_1} \cdot x_2 \) or \( x_1^{m_1} \cdot x_1^{m_2} \cdot \ldots \cdot x_1^{m_1} \cdot x_2^{-1} \).

3. On this step we generate \( w_4(x_1, x_2, x_3) \) (in the notation of Equation (2)), thus entangling the fourth strand into the first three. This process can be treated as a discrete random walk on three points, which are linked to the first three strands. We start at the identity element of \( B_{n+1} \) and either do nothing with probability \( s_4 \in (0, 1] \) and then go to step 4 or move to one of the two elements \( x_3 \) or \( x_3^{-1} \) with equal probabilities \( \frac{1-s_4}{2} \). If we are at an element \( w_1 \cdot x_3^m = v \neq 1 \) we either stop at \( v \) with probability \( s_4 \) (and proceed to step 4), or:

A. If \( l = 3 \) and \( m > 0 \) is odd, move, with probability \( \frac{1-s_4}{3} \), to one of the vertices \( x_1 \cdot x_3^{m+1} \) or \( x_1 \cdot x_3^m \cdot x_2 \) or \( x_1 \cdot x_3^{m+1} \).

B. If \( l = 3 \) and \( m < 0 \) is odd, move, with probability \( \frac{1-s_4}{3} \), to one of the vertices \( x_1 \cdot x_3^{m-1} \) or \( x_1 \cdot x_3^m \cdot x_2 \) or \( x_1 \cdot x_3^{m+1} \).

C. If \( l = 3 \) and \( m > 0 \) is even, move, with probability \( 1 - s_3 \), to the vertices \( x_1 \cdot x_3^{m+1} \).

D. If \( l = 3 \) and \( m < 0 \) is even, move, with probability \( 1 - s_3 \), to the vertices \( x_1 \cdot x_3^{m+1} \).

E. If \( l = 2 \) and \( m > 0 \) is odd, move, with probability \( \frac{1-s_4}{3} \), to one of the vertices \( x_1 \cdot x_3^{m+1} \) or \( x_1 \cdot x_3^m \cdot x_1 \) or \( x_1 \cdot x_3^{m+1} \).

F. If \( l = 2 \) and \( m < 0 \) is odd, move, with probability \( \frac{1-s_4}{3} \), to one of the vertices \( x_1 \cdot x_3^{m-1} \) or \( x_1 \cdot x_3^m \cdot x_1 \) or \( x_1 \cdot x_3^{m+1} \).

G. If \( l = 2 \) and \( m > 0 \) is even, move, with probability \( \frac{1-s_4}{3} \), to one of the vertices \( x_1 \cdot x_3^{m+1} \) or \( x_1 \cdot x_3^m \cdot x_3 \) or \( x_1 \cdot x_3^{m+1} \).

H. If \( l = 2 \) and \( m < 0 \) is even, move, with probability \( \frac{1-s_4}{3} \), to one of the vertices \( x_1 \cdot x_3^{m-1} \) or \( x_1 \cdot x_3^m \cdot x_3 \) or \( x_1 \cdot x_3^{m+1} \).

I. If \( l = 1 \) and \( m > 0 \) is odd, move, with probability \( 1 - s_3 \), to the vertex \( x_1 \cdot x_1^{m+1} \).

J. If \( l = 1 \) and \( m < 0 \) is odd, move, with probability \( 1 - s_3 \), to the vertex \( x_1 \cdot x_1^{m+1} \).

K. If \( l = 1 \) and \( m > 0 \) is even, move, with probability \( \frac{1-s_4}{3} \), to one of the vertices \( x_1 \cdot x_1^{m+1} \) or \( x_1 \cdot x_1^m \cdot x_2 \) or \( x_1 \cdot x_1^{m+1} \).

L. If \( l = 1 \) and \( m < 0 \) is even, move, with probability \( \frac{1-s_4}{3} \), to one of the vertices \( x_1 \cdot x_1^{m+1} \) or \( x_1 \cdot x_1^m \cdot x_2 \) or \( x_1 \cdot x_1^{m+1} \).

4. And so on.

In other words on the \( k \)-th step we generated the word \( w_{k+1}(x_1, \ldots, x_k) \), thus entangling the \( k + 1 \)-th strand into the first \( k \). Since the words \( w_j \)’s from Equation (2) are determined uniquely and since, by the construction, the obtained words \( w_j \)’s are freely reduced, we obtain a ‘random’ braid.
4. A Term Rewriting System for Elements of Braid Groups

In this section we view elements of $B_{n+1}$ as sequences of crossings, which for the most part will refer as strings. We construct a confluent string rewriting system, whose output will be a string in the form $w^{\pm 1}$.

To construct a rewriting system one is to define the strings and the rules. In our case, strings are the sequences of crossings that correspond to braids and the rules of the rewriting system are the rules given by (15)–(20). We also term these rules by transformations or rewrites.

Let us number all the crossings for the group $B_{n+1}$ as follows: $\rho_1 = \begin{array}{|c|} 1, 2 \end{array}, \rho_2 = \begin{array}{|c|} 1, 3 \end{array}, \rho_3 = \begin{array}{|c|} 2, 3 \end{array}, \rho_4 = \begin{array}{|c|} 1, 4 \end{array}, \ldots, \rho_{\frac{n(n+1)}{2}} = \begin{array}{|c|} n, n+1 \end{array}$: naturally $\rho_1^{-1} = \begin{array}{|c|} 1, 2 \end{array}, \rho_2^{-1} = \begin{array}{|c|} 1, n+1 \end{array}, \ldots, \rho_{\frac{n(n+1)}{2}}^{-1} = \begin{array}{|c|} n, n+1 \end{array}$.

We now prove that the rewriting system defined on the set of all sequences of crossings that correspond to braids by transformations (15)–(20) is confluent, i.e. that any chain of rewrites terminates and that there is the only residue and that it is the one corresponds to normal form (21).

To prove the confluence of our rewriting system we first prove that for any string one can sequentially apply only a finite number of transformations. First of all note that all rewrites (15)–(20) either have the form

\begin{equation}
\rho_1(p_{k_1}, \ldots, p_{k_m}) \cdot \rho_{i_1} \rightarrow \rho_{i_1} \cdot \rho_1(p_{k_1}, \ldots, p_{k_m}),
\end{equation}

where $g_1(p_{k_1}, \ldots, p_{k_m}), g_2(p_{k_1}, \ldots, p_{k_m})$ are strings of crossings that involve crossings $p_{k_1}, \ldots, p_{k_m}$ and their inverses only, $k_i < k_{i+1}, \ldots, k_j$, or the form

\begin{equation}
\rho_i \cdot \rho_i^{-\epsilon} \rightarrow 1,
\end{equation}

where $\epsilon = \pm 1$.

Denote by $n(w)$ the maximal length of a chain of transformations applied to a string $w$, so that $n(w)$ is either a positive integer or the symbol $\infty$, in the case that the length of chains of transformations is not bounded above.

**Remark 9.** Let $w = v \cdot u$ be a presentation of a sequence of crossings $w$ so that no free cancellation between $v$ and $u$ occurs and $n(w) \neq \infty$. Then $n(w) \geq n(v), n(u)$.

We now intend to prove the following

**Lemma 5.** Let $R$ be the set of rules of the form (22) and (23). Let $w$ be an arbitrary string then any sequence of transformations of the string $w$ terminates.

**Proof.** Suppose first that a string involves $\rho_{\frac{n(n+1)}{2}}$ and its inverse only. Then any chain of transformations has transformations given by Equation (23) only, and is clearly finite. Thus, without loss of generality, we may assume that any chain of transformations of any sequence of crossings that involves $\rho_2, \ldots, \rho_{\frac{n(n+1)}{2}}$ (and their inverses) only counts a finite number of steps.

We now use induction on the number of occurrences of $\rho_1^\pm 1$ in a string. If $\rho_1^\pm 1$ does not occur in a string, the statement is straightforward. Consider next a sequence of crossings $t = w(\rho_2, \ldots, \rho_m)\rho_1^i v(\rho_2, \ldots, \rho_m), m = \frac{n(n+1)}{2}$. By the induction assumption, $n(w)$ and $n(v)$ are finite. Consider an arbitrary chain of transformations of $t$. Suppose that the $t$-th rewrite involves the $\rho_1^i$ and the string rewrites as follows $w(\rho_2, \ldots, \rho_m)\rho_1^i v(\rho_2, \ldots, \rho_m) \rightarrow w''(\rho_2, \ldots, \rho_m)\rho_1^i v''(\rho_2, \ldots, \rho_m)$. Where $w''(\rho_2, \ldots, \rho_m)$ is a substring of
Theorem 6. The set of rules \[ \rho_1, \ldots, \rho_m \] gives rise to a confluent term rewriting system on the set of sequences of crossings, which represent braid words.

Proof. By the definition, a rewriting system is confluent whenever the order of application of rules does not matter, i.e., we have the following diagrams:

\[
\begin{align*}
  w &= x \cdot l_1 \cdot y \cdot l_2 \cdot z \\
  \text{a rule} \\
  w &= x \cdot l_1 \cdot y \cdot l_2 \cdot z \\
  \text{a rule} \\
  w &= x \cdot l_1' \cdot y \cdot l_2' \cdot z \\
  \text{a rule} \\
  w &= x \cdot l_1' \cdot r_2 \cdot z \\
  \text{a rule} \\
  w &= x \cdot l_1' \cdot r_2 \cdot z \\
  \text{a rule} \\
  w &= x \cdot l_1' \cdot r_2 \cdot l_1'' \cdot z \\
  \text{a rule}
\end{align*}
\]

Here \( l_1, l_2, l_1', y \) and \( y \cdot l_2', l_1' \cdot l_2 \cdot l_1'' \) and \( l_2 \) are left parts of the rules, \( r_1 \) and \( r_2 \) are right-hand sides of the rules and \( \text{res}(w) \) is the residue of \( w \), i.e., such a word
to which no transformation can be applied. This is roughly saying that if there is a choice of which rule to apply then there exists a common residue of the resulting words.

To show that our term rewriting system is confluent suppose that there exist two distinct residues \( \text{res}_1(w) \) and \( \text{res}_2(w) \) of a string \( w \). By Theorem 4 there exist a residue \( w^\ast \) of \( w \) in the form (21) (since none of the rules (15)–(20) can be applied to \( w^\ast \)). Since \( \text{res}_1(w) \neq \text{res}_2(w) \) at least one of them does not coincide with \( w^\ast \).

Suppose that \( \text{res}_1(w) \neq w^\ast \). Then, by Remark 7, \( \text{res}_1(w) \) corresponds to a braid, and therefore, on account of Theorem 1 (see also Section 2.1), we can take \( \text{res}_1(w) \) to \( w^\ast \) by the means of the rules (15)–(20) — a contradiction.

\[ \blacksquare \]

**Corollary 7.** The set of all sequences of crossings that correspond to normal form (21) of elements of the braid group \( B_{n+1} \) is regular.

**Proof.** By Remark 6 the set of all sequences of crossings that correspond to elements of \( B_{n+1} \) is regular. Since the collection of rules (15)–(20) is finite, the set of all sequences of crossings that do not contain the left-hand sides of rules is regular. The statement now follows, for the intersection of two regular sets is regular. \[ \blacksquare \]

5. **AN ANALOGUE OF THE NORMAL FORM FOR ANOTHER ARTIN GROUP**

In this Section we show how one can obtain an analogue of the form (2) for an Artin group of finite type \( A = \langle a, b \mid abab = baba \rangle \).

Any Artin group of type \( B \) embeds into a braid group on sufficiently many strands and the embedding is fairly natural, see [12] for details. For example, the group \( A \) embeds into \( B_3 \) as follows: \( \phi(a) = x_1, \phi(b) = x_2^2 \). Using this embedding and the normal form constructed in this paper one can obtain similar geometric normal forms for \( A \) (and any other Artin group of type \( B \)). However, we want to explicitly demonstrate the generalisation of the idea used to construct normal forms in braid groups and show how one can construct normal forms in \( A \) ‘barehanded’.

Consider the corresponding Coxeter group \( A \). As it is well-known, see for instance [3], \( A \) can be treated as a group, generated by the following reflections of a square: \( \bar{a} = \begin{array}{c} 1 \end{array} \begin{array}{c} 2 \end{array} \begin{array}{c} 3 \end{array} \begin{array}{c} 4 \end{array} \) and \( \bar{b} = \begin{array}{c} 1 \end{array} \begin{array}{c} 2 \end{array} \begin{array}{c} 3 \end{array} \begin{array}{c} 4 \end{array} \), where the corresponding square has the form:

\[
\begin{array}{c}
1 \\
\| \\
4 \\
\| \\
3 \\
\|
\end{array}
\]

We use this notation to express, that \( \bar{a} \) is the reflection of the square that permutes vertices 1 and 3 and \( \bar{b} \) is the reflection of the square that permutes 1 with 2, and 3 with 4:

\[
\begin{array}{c}
1 \\
\| \\
4 \\
\|
\end{array} \begin{array}{c}
2 \\
\| \\
3 \\
\|
\end{array} \begin{array}{c}
2 \\
\| \\
1 \\
\|
\end{array} \begin{array}{c}
2 \\
\| \\
1 \\
\|
\end{array} \begin{array}{c}
1 \\
\| \\
3 \\
\|
\end{array} \begin{array}{c}
2 \\
\| \\
4 \\
\|
\end{array} \begin{array}{c}
1 \\
\| \\
4 \\
\|
\end{array} \begin{array}{c}
2 \\
\| \\
3 \\
\|
\end{array}
\]

We use this notation to express, that \( \bar{a} \) is the reflection of the square that permutes vertices 1 and 3 and \( \bar{b} \) is the reflection of the square that permutes 1 with 2, and 3 with 4:

\[
\begin{array}{c}
1 \\
\| \\
4 \\
\|
\end{array} \begin{array}{c}
2 \\
\| \\
3 \\
\|
\end{array} \begin{array}{c}
2 \\
\| \\
1 \\
\|
\end{array} \begin{array}{c}
2 \\
\| \\
1 \\
\|
\end{array} \begin{array}{c}
1 \\
\| \\
3 \\
\|
\end{array} \begin{array}{c}
2 \\
\| \\
4 \\
\|
\end{array} \begin{array}{c}
1 \\
\| \\
4 \\
\|
\end{array} \begin{array}{c}
2 \\
\| \\
3 \\
\|
\end{array}
\]

We next consider every element of \( A \) as a sequence of elements of \( A \). The group \( A \) consists of 8 elements: \( 1, 1, 2, 3, 4, 1, 2, 3, 4, 1, 2, 3, 4, 1, 2, 3, 4 \). All the above elements, but \( 1, 3 \) involve the 4-th vertex of the square.
Remark 10. In the case of braid groups, the crossing $|1,3|$ of two strands can correspond to any generator (since all the permutations $(i, j)$ are conjugate in the Coxeter group, which corresponds to the braid group, i.e., the group of permutations on $n + 1$ symbols). However, only the elements $|1,3|$ and $|2,4|$ of the group $\mathcal{A}$ can correspond to the generator $a$ (treated as a subword of a word of $\mathcal{A}$), since $|1,3|$ is the image of $a$ in $\mathcal{A}$ and $|2,4|$ is the only element of $\mathcal{A}$ conjugate to $|1,3|$.

We next prove the following theorem.

**Theorem 8.** In the above notation, every element $w \in \mathcal{A}$ can be taken to the form

$$w = a^m \cdot w_1(a, b); m \in \mathbb{Z}$$

Where $w_1$ is freely reduced and the reflections, which correspond to the letters in $w_1$ are the following: $|2,4|, |1,2||3,4|, |1,3||1,2||3,4|, |1,2||3,4||1,3|, |1,4||2,3|, |1,3||2,4|$ only.

**Proof.** The proof of the theorem repeats the argument given in Subsection 2.3. We only need to introduce analogues of transformations (11)-(11) and Figures 10 and 11. Algebraic analogues of Figures 10 and 11 take the form:

$$
\begin{align*}
\left(\left|1,2\right||3,4\right)^{\epsilon} \cdot \left(\left|1,2\right||3,4\right)^{\epsilon} & \cdot \left|1,3\right|^\delta \cdot \left(\left|1,4\right||2,3\right)^\delta, \\
\cdot \left(2,4\right)^{\delta} \cdot \left(\left|1,2\right||3,4\right)^{\epsilon} & \cdot \left(\left|1,2\right||3,4\right)^{\epsilon} \cdot \left(2,4\right)^{-\delta} \cdot \left(\left|1,2\right||3,4\right)^{-\delta}, \\
\left(\left|1,4\right||2,3\right)^{\epsilon} & \cdot \left(\left|1,4\right||2,3\right)^{\epsilon} \cdot \left|1,3\right|^\delta \cdot \left(\left|1,2\right||3,4\right)^{\delta}, \\
\cdot \left(2,4\right)^{\delta} & \cdot \left(\left|1,4\right||2,3\right)^{\epsilon} \cdot \left(\left|1,4\right||2,3\right)^{\epsilon} \cdot \left(2,4\right)^{-\delta} \cdot \left(\left|1,4\right||2,3\right)^{-\delta}, \\
\left(2,4\right)^{\epsilon} \cdot \left(\left|1,4\right||2,3\right)^{-\epsilon} & \cdot \left|1,3\right|^\epsilon \rightarrow \\
\rightarrow & \left(\left|1,2\right||3,4\right)^{\epsilon} \cdot \left|1,3\right|^\epsilon \cdot \left(\left|1,4\right||2,3\right)^{\epsilon} \cdot \left(2,4\right)^{\epsilon} \cdot \left(\left|1,2\right||3,4\right)^{-\epsilon}, \\
\left(2,4\right)^{\epsilon} & \cdot \left(\left|1,4\right||2,3\right)^{-\epsilon} \cdot \left|1,3\right|^\epsilon \rightarrow \\
\rightarrow & \left(\left|1,4\right||2,3\right)^{\epsilon} \cdot \left|1,3\right|^\epsilon \cdot \left(\left|1,4\right||2,3\right)^{\epsilon} \cdot \left(2,4\right)^{\epsilon} \cdot \left(\left|1,4\right||2,3\right)^{-\epsilon}, \\
\left(2,4\right)^{\epsilon} & \cdot \left(\left|1,4\right||2,3\right)^{-\epsilon} \cdot \left|1,3\right|^\epsilon \rightarrow \\
\rightarrow & \left(\left|1,4\right||2,3\right)^{\epsilon} \cdot \left|1,3\right|^\epsilon \cdot \left(\left|1,4\right||2,3\right)^{\epsilon} \cdot \left(2,4\right)^{\epsilon} \cdot \left(\left|1,4\right||2,3\right)^{-\epsilon}, \\
\left(2,4\right)^{-\epsilon} & \cdot \left(\left|1,4\right||2,3\right)^{-\epsilon} \cdot \left|1,3\right|^\epsilon \rightarrow \\
\rightarrow & \left(\left|1,4\right||2,3\right)^{\epsilon} \cdot \left|1,3\right|^\epsilon \cdot \left(\left|1,4\right||2,3\right)^{\epsilon} \cdot \left(2,4\right)^{\epsilon} \cdot \left(\left|1,4\right||2,3\right)^{-\epsilon}, \\
\left(2,4\right)^{-\epsilon} & \cdot \left(\left|1,4\right||2,3\right)^{-\epsilon} \cdot \left|1,3\right|^\epsilon \rightarrow \\
\rightarrow & \left(\left|1,4\right||2,3\right)^{\epsilon} \cdot \left|1,3\right|^\epsilon \cdot \left(\left|1,4\right||2,3\right)^{\epsilon} \cdot \left(2,4\right)^{\epsilon} \cdot \left(\left|1,4\right||2,3\right)^{-\epsilon}, \\
\left(2,4\right)^{-\epsilon} & \cdot \left(\left|1,4\right||2,3\right)^{-\epsilon} \cdot \left|1,3\right|^\epsilon \rightarrow \\
\rightarrow & \left(\left|1,4\right||2,3\right)^{\epsilon} \cdot \left|1,3\right|^\epsilon \cdot \left(\left|1,4\right||2,3\right)^{\epsilon} \cdot \left(2,4\right)^{\epsilon} \cdot \left(\left|1,4\right||2,3\right)^{-\epsilon}.
\end{align*}
$$
In terms of words analogues of transformations (4)–(11) take the form:

\[
TU''' \cdot b' b' a^\delta a W \rightarrow TU''' \cdot a^\delta b' a a^\delta a W
\]

(25)

\[
TU''' \cdot a' b' a' W \rightarrow TU''' \cdot b' a' b' a' b' a' W
\]

(26)

\[
TU''' \cdot a' b' a' W \rightarrow TU''' \cdot b' a' b' a' a' W
\]

(27)

\[
TU''' \cdot a' b' a' W \rightarrow TU''' \cdot a' b' a' b' a' W
\]

(28)

\[
TU''' \cdot a' b' a' W \rightarrow TU''' \cdot b' a' b' a' W
\]

(29)

One can see that each rule in the list above moves the reflection \(|1, 3| \pm 1\) to the beginning of the word and is so that in the right-hand side of the rules there is only one occurrence of \(a\), which corresponds to the reflection \(|1, 3| \pm 1\). Furthermore, the rules above enumerate all the possibilities, when \(|1, 3| \pm 1\) corresponds to the terminal letter of a subword of a word from \(A\) and so that only the following elements of \(A\) correspond to the 2 letters preceding the letter that corresponds to \(|1, 3| \pm 1\):

\[
|2, 4|, |1, 2|, |3, 4|, |1, 3|, |2, 3|, |3, 4|, |1, 4|, |2, 3|, |1, 3|, |2, 4|.
\]

\[\blacksquare\]

**Remark 11.** We did not show that the form \((24)\) is unique. To prove the uniqueness of normal form \((24)\) for braids we used results of E. Artin. We do not know similar results for arbitrary Artin groups of finite type. However, it may be derived from the fact that \(A\) embeds into some braid group.

6. Questions and Final Remarks

In the current section we list some of the questions that arose in our work. We also briefly discuss some of advantages and disadvantages of normal form \((2)\).

**Question 1.** In Section 3 we have given a description of a procedure of generating a random braid. That procedure involves \(n\) random walks, each of which is defined by a parameter \(s_j\). Choose the \(s_j\)'s so that the obtained word generator would be useful for practical needs of computer scientists. We suppose that a solution of this problem can be piloted by papers 4 and 11.

Consider a list of transformations (4) of the words of a free monoid with the alphabet \(X \cup X^{-1}\).

- \(x_i^\epsilon x_i^{-\epsilon} \rightarrow 1\), \(\epsilon = \pm 1\);
- \(x_i^\epsilon x_j^{\eta} \rightarrow x_j^{\eta} x_i^\epsilon\), \(|i - j| \geq 2; \epsilon, \eta = \pm 1\);
- \(x_i^\epsilon x_{i+1}^\eta x_i^{-\epsilon} \leftrightarrow x_{i+1}^\epsilon x_i^\eta x_{i+1}^{-\epsilon}\), \(\epsilon = \pm 1\);
- \(x_i^\epsilon x_i^k x_i^{-\epsilon} \leftrightarrow x_i^{-\epsilon} x_i^k x_i^\epsilon\), \(\epsilon, \eta = \pm 1, \epsilon \cdot \eta = -1, k \in \mathbb{N}\);
- \(x_i^\epsilon x_i^k x_i^{-\epsilon} \leftrightarrow x_i^\eta x_i^{k+1} x_i^{-\epsilon}\), \(\epsilon, \eta = \pm 1, \epsilon \cdot \eta = -1, k \in \mathbb{N}\);
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Question 2. I. Consider two geodesic words $\xi_1, \xi_2$ from a braid group $B_n$ such that $\xi_1 B_n = \xi_2$. Is the list of transformations (♦) sufficient to obtain the word $\xi_1$ from the word $\xi_2$?

II. Let $\xi_1, \xi_2$ be arbitrary words from a braid group $B_n$. Assume that $\xi_1$ is written in geodesic form and $\xi_1 B_n = \xi_2$. Is the list of transformations (♦) sufficient to obtain the word $\xi_1$ from the word $\xi_2$?

III. Analogues of questions I and II for an arbitrary Artin group.

A positive answer to parts I and II of Question 2 was announced in [9]. However, as far as the authors are concerned, proof never appeared.

Question 3. We presume that the introduced normal forms do not form an automatic structure (this was conjectured in our conversation with G. A. Noskov). Let $\xi$ be a braid word. Our experiments show that normal form (2) $\xi^*$ of $\xi$ may have an exponential length on the length of the input word (consider the word $\xi = x_2^2 x_1^2 x_2^2$ and its powers, for instance). In what follows that this normal form can not be automatic (see [7]). However, resulting from our experiments, we conjecture that in the class of words conjugate to $\xi$ (and even among all cyclic permutation of $\xi$) there exists an element $\zeta$ so that its normal form $\zeta^*$ is polynomial (linear) on the length of its geodesic form $\zeta_1$.

Question 4. As mentioned in Question 3 normal form (2) $\xi^*$ of $\xi$ may have an exponential length on the length of the input word. Is it true that normal form (2) of $\xi$ is generically polynomial (linear) on the length of a geodesic word $\xi_1 = \xi$? For a detailed explanation of the term ‘generically polynomial’ the reader may consult [8].

In [11] E. Artin wrote the following regarding his normal forms. We can not but agree.

Though it has been proved that every braid can be transformed to a similar normal form the writer is convinced that any attempt to carry this out on a living person would only lead to violent protests and discrimination against mathematicians. He would therefore discourage such an experiment.

E. Artin

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REFERENCES

[1] E. Artin “Theory of Braids” Ann. Math. Vol. 48 No. 1 (1947), p. 101-126
[2] A. V. Borovik, A. G. Myasnikov and V. N. Remeslennikov, Multiplicative Measures on Free Groups, Int. J. of Alg. and Comp. Vol. 13 No. 6 (2003) p. 705-731.
[3] N. Bourbaki, “Groupes et Algèbres de Lie” 2me Partie, Hermann, 1968
[4] J. Debois and S. Nechaev, Statistics of reduced words in locally free and braid groups, J. Statist. Phys. Vol. 88 (1997) p.2767-2789.
[5] P. Dehornoy, “Braids and Self Distributivity Progress in Mathematics.” Vol. 192; Birkhauser (2000)
[6] P. Dehornoy, Cryptography on Braids. Contemporary Mathematics, to appear
[7] D. B. A. Epstein, with J. W. Cannon, D. F. Holt, S. V. F. Levy, M. S. Paterson and W. P. Thurston, “Word Processing in Groups”; Jones and Bartlett, Boston-London, 1992.
[8] I. Kapovich, A. Myasnikov, P. Schupp and V. Shpilrain Generic-case complexity, decision problems in group theory and random walks, J. Algebra Vol. 264 (2003), No. 2, p. 665-684
[9] G. S. Makanin, *Conjugacy Problem in Braid Groups*. Proceedings of Russian Academy of Science, Vol. 182 (1968), No. 3.

[10] A. A. Markoff, “Foundation of the Algebraic Theory of Tresses.”, Tr. Mat. Inst. Steklova 16, 53 S. (1945).

[11] A. Vershik, S. Nechaev, R. Bikbov, *Statistical properties of braid groups in locally free approximation*, Comm. Math. Phys., Vol. 212 (2000) p. 469-501.

[12] V. V. Vershinin, *Braid Groups and Spaces of Loops*, Advances in Mathematics, Vol. 54 (1999), No. 2(326), p. 3-84

OSHK BRANCH OF INSTITUTE OF MATHEMATICS, (SIBERIAN BRANCH OF RUSSIAN ACADEMY OF SCIENCE), PEVTSOVA ST. 13, OMSK, 644099, RUSSIA; EMAIL: ESYP@IITAM.OMSK.NET.RU

DEPARTMENT OF MATHEMATICS AND STATISTICS, MCGILL UNIVERSITY, 805 SHERBROOKE ST. WEST, MONTREAL, QC, H3A 2K6, CANADA; EMAIL: ILYA.KAZACHKOV@GMAIL.COM