$E_7$ as $D = 10$ space-time symmetry
— Origin of the twistor transform

MARTIN CEDERWALL

Institute of Theoretical Physics
Chalmers University of Technology and University of Gothenburg
S-412 96 Gothenburg, Sweden

ABSTRACT

Massless particle dynamics in $D = 10$ Minkowski space is given an $E_7$-covariant formulation, including both space-time and twistor variables. $E_7$ contains the conformal algebra as a subalgebra. Analogous constructions apply to $D = 3, 4$ and 6.

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It is well known that massless particle actions are conformal invariant (with
the possible exception of $D = 10$ superparticles, in which case the matter is not
clearly understood). However, there is a still larger symmetry present, as will
be demonstrated in this paper. Only bosonic particles will be treated. Explicit
calculations apply to $D = 10$ – completely analogous constructions are valid in
$D = 3, 4$ and $6$. Some technical details and conventions are found in the appendix.

One has traditionally two options for manifestly conformal formulations of
particle dynamics, the space-time picture and, in $D = 3, 4, 6$ or $10$, the twistor
picture. The space-time formulation, on one hand, with $P_m P^m \approx 0$ as only con-
straint, is easily made conformally covariant by enlarging the vectors $X^m, P^m$ of
$SO(1, 9)$ to become vectors $X^\mu, P^\mu$ of $SO(2, 10)$. The constraints are taken to be
$X_\mu X^\mu \approx 0, X_\mu P^\mu \approx 0$ and $P_\mu P^\mu \approx 0$. With gauge choices $X^{\oplus} = c$ (=constant)
and $P^{\oplus} = 0$, the ordinary space-time picture is recovered (this of course applies to
any dimensionality).

The twistor picture[1-4], on the other hand, is reached via the twist or transform
$P_m = \frac{1}{2} \gamma_\alpha^m \gamma_\beta^n \psi \chi$ which implies that the conformal spinor $Z^A = \begin{pmatrix} \psi_\alpha \\ \omega_\alpha \end{pmatrix}$ satisfies seven constraints,
generating $S^7$ (a covariantly conformal form of these constraints is given in ref.4).

Consider now a set of phase-space variables $\Xi^M \in 56$ of $E_7$. When $E_7 \to
Sp(2) \times SO(2, 10)$ the the branching rule[5] is $56 \to (2, 12) + (1, 32)$ and for the
adjoint $133 \to (3, 1) + (2, 32') + (1, 66)$, so that $\Xi^M = (S^a \mu, Z^A)$ and $133 \ni T^A =
(T^{ab}, T^A, T^{[\mu \nu]}).$ If $\{\Xi^M, \Xi^N\} = g^{MN}, E_7$ is generated by $T^A = \frac{1}{2} \Xi^M \Omega^A_{MN} \Xi^N,$
$\Omega^A_{MN}$ being Clebsh-Gordan coefficients for $56 \times 56 \times 133 \to 1$. $S^a \mu = (X^\mu, P^\mu)^t$
is to be interpreted as the conformal space-time vectors and $Z^A$ as the twistor
variables. The set of constraints is chosen to be

$$T^A \approx 0$$

which is obviously first class. The content of eq.(2) will now be analyzed.
The pure $Sp(2)$ part $T^{ab} \approx 0$ contains exactly the constraints stated above for the space-time picture. I make the same gauge choices, with $c = 1/\sqrt{2}$ (this reduces manifest covariance to $SO(1, 9)$), solve for $X^\ominus$, $P^\ominus$ and insert in the remaining constraints. The resulting set of equations is highly reducible. With some help from formulas in the appendix, it is easy to verify that $T^{\ominus m} \approx 0$ and the second component of $T^1_A \approx 0$ are exactly the twistor transform equations (1). Once these are fulfilled, all other identities in eq.(2) hold. The system described by eq.(2) is thus equivalent to a massless particle in $D = 10$, and may be gauge-fixed to either the space-time or the twistor picture.

In the dimensionalities $D = 3, 4$ and 6, the constructions are analogous to the one above (in $D = 4$ and 6 the twistor variables obey bilinear constraints generating $S^1$ and $S^3$ that also are part of the symmetry). The resulting algebras are $Sp(6)$, $SU(6)$ and $SO(12)$, respectively. The algebras can be collectively described as $Sp(6; K_\nu)$ in $D = \nu + 2$, where $K_\nu$, $\nu = 1, 2, 4, 8$ is the division algebra of dimension $\nu$. They are conformal algebras of the Jordan algebras of $3 \times 3$ hermitean matrices with entries in $K_\nu[6, 7]$. The conformal algebra in $D = \nu + 2$ is $SO(2, \nu + 2) \approx Sp(4; K_\nu)$, so one sees how it is contained in the present larger structure.

The real forms of the algebras are defined by their maximal compact subalgebras, which are the symmetric subalgebras $SU(3) \times U(1) \subset Sp(6)$, $SU(3)^2 \times U(1) \subset SU(6)$, $SU(6) \times U(1) \subset SO(12)$ and $E_6 \times U(1) \subset E_7$ respectively. It is worth mentioning that the real form of $SO(12)$ does not belong to the class $SO(n, 12 - n)$, and that it, in contrast to the $D = 10$ conformal algebra $SO(2, 10)$, does possess a superextension[7].

It is probably motivated to regard the extended conformal symmetries considered in this paper not only as convenient algebraic constructions, but as fundamental geometric properties of combined space-time/twistor spaces underlying $D = \nu + 2$ Minkowski space.
APPENDIX

$Sp(2)$ spinor indices are raised and lowered with $\epsilon_{ab}$ as $y_a = \epsilon_{ab} y^b$. The representation 3 consists of symmetric matrices $M^{ab}$.

$SO(1,9)$ gamma-matrices are denoted $\gamma^m_{\dot{a}\dot{a}}$, $\tilde{\gamma}^m a\dot{a}$ and obey

$$\gamma^m \gamma^n = \eta^{mn} 1, \quad \tilde{\gamma}^m \gamma^n = \eta^{mn} 1$$

with $\eta^{mn} = \text{diag}(-1,1,\ldots,1)$. From these, $SO(2,10)$ gamma-matrices $\Gamma_{A'B}$ are constructed:

$$\Gamma^{\oplus} = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix}, \quad \Gamma^{\ominus} = \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{2} \end{bmatrix}, \quad \Gamma^m = \begin{bmatrix} 0 & \tilde{\gamma}^m \\ \gamma^m & 0 \end{bmatrix}.$$

Then,

$$\Gamma^{[\mu A} \Gamma^{\nu]A'} = \eta^{\mu\nu} \delta_A^A', \quad \Gamma^{[\mu A} \Gamma^{\nu]B} = \eta^{\mu\nu} \delta_A^B'$$

with $\eta^{\ominus} = -1$, $\eta^{mn}$ as above. $\Gamma^\mu_{AB}$ is defined as $\Gamma^{[\mu A} \Gamma^{\nu]B}$. The invariant spinor metrics

$$g_{AB} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = g_{A'B'}$$

are used to raise and lower spinor indices as $Y_A = g_{AB} Y^B$ and analogously for primed indices.

The non-zero Clebsh-Gordan coefficients for $56 \times 56 \rightarrow 133$ of $E_7$ are, with indices according to the maximal subalgebra $Sp(2) \times SO(2,10)$ (no attention paid to normalization of the generators):

$$\Omega^{ab}_{\epsilon\mu\nu,\delta \lambda} = \frac{1}{2} \eta_{\mu\nu} \delta^c_{\delta a} \delta^d_{d} \delta^\epsilon_b, \quad \Omega^A_{\mu\nu,A'} = \Omega^{A'}_{\mu\nu,A}, \quad \delta^a_{b} \Gamma^A_{A}, \quad \Omega^{\mu\nu}_{A\dot{a}\dot{a}} = \epsilon_{ab} \delta^{\mu\nu}_{\kappa\lambda}, \quad \Omega^{\mu\nu}_{A,B} = -\frac{1}{4} \Gamma^{\mu\nu}_{A'B}.$$

The invariant metric for $56 \times 56 \rightarrow 1$ is $g_{MN}$, with $g_{\mu\nu} = \eta_{\mu\nu} \epsilon_{ab}$ and $g_{AB}$ as above.
REFERENCES

1. R. Penrose and M.A.H. McCallum, *Phys.Rep.* 6(1972), 241 and references therein.

2. I. Bengtsson and M. Cederwall, *Nucl.Phys.* B302(1988), 81.

3. N. Berkovits, *Phys.Lett.* 247B(1990), 45.

4. M. Cederwall, *J.Math.Phys.* 33(1992), 388.

5. R. Slansky, *Phys.Rep.* 79(1981), 1.

6. A. Sudbery, *J.Phys.* A17(1984), 939.

7. M. Cederwall, *Phys.Lett* 210B(1988), 169.