Per-instance Differential Privacy and the Adaptivity of Posterior Sampling in Linear and Ridge regression

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Abstract

Differential privacy (DP), ever since its advent, has been a controversial object. On the one hand, it provides strong provable protection of individuals in a data set; on the other hand, it has been heavily criticized for being not practical, partially due to its complete independence to the actual data set it tries to protect. In this paper, we address this issue by a new and more fine-grained notion of differential privacy — per instance differential privacy (pDP), which captures the privacy of a specific individual with respect to a fixed data set. We show that this is a strict generalization of the standard DP and inherits all its desirable properties, e.g., composition, invariance to side information and closedness to postprocessing, except that they all hold for every instance separately. When the data is drawn from a distribution, we show that per-instance DP implies generalization. Moreover, we provide explicit calculations of the per-instance DP for the output perturbation on a class of smooth learning problems. The result reveals an interesting and intuitive fact that an individual has stronger privacy if he/she has small “leverage score” with respect to the data set and if he/she can be predicted more accurately using the leave-one-out data set. Using the developed techniques, we provide a novel analysis of the One-Posterior-Sample (OPS) estimator and show that when the data set is well-conditioned it provides $(\epsilon, \delta)$-pDP for any target individuals and matches the exact lower bound up to a $1 + O(n^{-1} \epsilon^{-2})$ multiplicative factor. We also propose AdaOPS which uses adaptive regularization to achieve the same results with $(\epsilon, \delta)$-DP. Simulation shows several orders-of-magnitude more favorable privacy and utility trade-off when we consider the privacy of only the users in the data set.

1 Introduction

While modern statistics and machine learning had seen amazing success, their applications to sensitive domains involving personal data remain challenging due to privacy issues. Differential privacy [11] is a mathematical notion that allows strong provable protection of individuals from being identified by an arbitrarily powerful adversary, and has been increasingly popular within the
machine learning community as a solution to the aforementioned problem [22, 6, 20, 1]. The strong privacy protection however comes with a steep price to pay. Differential privacy almost always lead to substantial and often unacceptable drop in utility, e.g., in contingency tables [16] and in genome-wide association studies [34]. This motivated a large body of research to focus on making differential privacy more practical [25, 10, 26, 31, 13, 4, 17] by exploiting local structures and/or revising the privacy definition.

Majority of these approaches adopt the “privacy-centric” model, which involves theoretically proving that an algorithm is differentially private for any data (within a data domain), then carefully analyzing the utility of the algorithm under additional assumptions on the data. For instance, in statistical estimation it is often assumed that the data is drawn i.i.d. from a family of distributions. In nonparametric statistics and statistical learning, the data are often assumed to having specific deterministic/structural conditions, e.g., smoothness, incoherence, eigenvalue conditions, low-rank, sparsity and so on. While these assumptions are strong and sometimes unrealistic, they are often necessary for a model to work correctly, even without privacy constraints. Take high-dimensional statistics for example, “sparsity” is almost never true, but if the true model is dense and unstructured, it is simply impossible to recover the true model anyways in the “small \( n \) large \( d \)” regime. That is why Friedman et al. [18] argued that one should “bet on sparsity” regardless and hope that it is a reasonable approximation of the reality. This is known as adaptivity in that an algorithm can perform provably better when some additional conditions are true.

The effect of these assumptions on privacy is unclear, mostly because there are no tools available to analyze such adaptivity in privacy. Since differential privacy is a worst-case quantity — a property of the randomized algorithm only (independent to the data) — it is unlikely that the obtained privacy loss \( \epsilon \) could accurately quantify the privacy protection on a given data set at hand. It is always an upper bound, but the bound could be too conservative to be of any use in practice (e.g., when \( \epsilon = 100 \)).

To make matter worse, the extent to which DP is conservative is highly problem-dependent. In cases like, releasing counting queries, the \( \epsilon \) clearly measures the correct information leakage, since the sensitivity of such queries do not change with respect to the two adjacent data sets; however, in the context of machine learning and statistical estimation (as we will show later), the \( \epsilon \) of DP can be orders of magnitude larger than the actual limit of information leakage that the randomized algorithm guarantees. That is why in practice, it is challenging even for experts of differential privacy to provide a consistent recommendation on standard questions such as:

“What is the value of \( \epsilon \) I should set in my application?”

In this paper, we take a new “algorithm-centric” approach of analyzing privacy. Instead of designing algorithms that take the privacy loss \( \epsilon \) as an input, we consider a fixed randomized algorithm and then analyze its privacy protection for every pair of adjacent data sets separately.

Our contribution is three-fold.

1. First, we develop per-instance differential privacy as a strict generalization of the standard pure and approximate DP. It provides a more fine-grained description of the privacy protection for each target individual and a fixed data set. We show that it inherits many desirable properties of differential privacy and can easily recover differential privacy for a given class of data and target users.
2. Secondly, we quantify the per-instance sensitivity in a class of smooth learning problems including linear and kernel machines. The result allows us to explicitly calculate per-instance DP of multivariate Gaussian mechanism. For an appropriately chosen noise covariance, the per-instance DP is proportional to the norm of the pseudo-residual in norm specified by the Hessian matrix. In particular, in linear regression, the per-instance sensitivity for a data point is proportional to its square root statistical leverage score and its leave-one-out prediction error.

3. Lastly, we analyze the procedure of releasing one sample from the posterior distribution (the OPS estimator) for ridge regression as an output perturbation procedure with a data dependent choice of covariance matrix. We show using the pDP technique that, when conditioning on a data set drawn from the linear regression model or having a well-conditioned design matrix, OPS achieves $(\epsilon, \delta)$-pDP for while matching the Cramer-Rao lower bound up to a $1 + \tilde{O}(n^{-1}\epsilon^{-2})$ multiplicative factor. OPS unfortunately cannot achieve DP with a constant $\epsilon$ while remaining asymptotically efficient. We fixed that by a new algorithm called AdaOPS , which provides $(\epsilon, \delta)$-DP and $1 + \tilde{O}(n^{-1}\epsilon^{-2})$-statistical efficiency at the same time.

### 1.1 Symbols and notations

Throughout the paper, we will use the standard notation in statistical learning. Data point $z \in \mathcal{Z}$. In supervised learning setting, $z = (x, y) \in \mathcal{X} \times \mathcal{Y} = \mathcal{Z}$. We use $\theta \in \Theta$ to denote either the predictive function $\mathcal{X} \rightarrow \mathcal{Y}$ or the parameter vector that specifies such a function. $\ell : \Theta \times \mathcal{Z} \rightarrow \mathbb{R}$ to denote the loss function or in a statistical model, $\ell$ represents the negative log-likelihood $-\log p_{\theta}(z)$. For example, in linear regression, $\mathcal{X} \subset \mathbb{R}^d$, $\mathcal{Y} \subset \mathbb{R}$, $\Theta \subset \mathbb{R}^d$ and $\ell(\theta, (x, y)) = (y - x^T \theta)^2$. We use $\mathcal{A} : \mathcal{Z}^n \rightarrow \mathcal{P}_{\Theta}$ to denote a randomized algorithm that outputs a draw from a distribution defined on a model space. Capital $Z$ denotes a data set, $\epsilon$ and $\epsilon(Z, z)$ will be used to denote privacy loss.

### 2 Per-instance differential privacy

In this section, we define notion of per-instance differential privacy, and derive its properties. We begin by parsing the standard definition of differential privacy.

**Definition 1** (Differential privacy [11]). We say a randomized algorithm $\mathcal{A}$ satisfies $(\epsilon, \delta)$-DP if for all data set $Z$ and data set $Z'$ that can be constructed by adding or removing one row $z$ from $Z$,

$$
\mathbb{P}_{\theta \sim \mathcal{A}(Z)}(\theta \in \mathcal{S}) \leq e^\epsilon \mathbb{P}_{\theta \sim \mathcal{A}(Z')}(\theta \in \mathcal{S}) + \delta, \ \forall \text{measurable set } \mathcal{S}.
$$

When $\delta = 0$, this is also known as pure differential privacy, and it is much stronger because for each data set $Z$, the protection holds uniformly over all privacy target $z$. When $\delta > 0$, then the protection becomes much weaker, in that the protection is stated for each privacy target separately.

It is helpful to understand what differential privacy is protecting against — a powerful adversary that knows everything in the entire universe, except one bit of information: whether a target $z$ is in the data set or not in the data set. The optimal strategy for such an adversary is to conduct a likelihood ratio test (or posterior inference) on this bit, and differential privacy uses randomization to limit the probability of success of such test [33].
In the above, we described the original “In-or-Out” version of DP definition (see, e.g., [12, Definition 2.3, 2.4]). There is also a “Replace-One” version of the DP definition, which assumes $Z'$ is constructed by replacing one row of $Z$ arbitrarily. This preserves the cardinality of the data set and makes it more convenient in certain settings. The “replace-one” differential privacy protects against a slightly stronger adversary who know data set $Z$ except one row and can limit the possibility of the unknown row to either $z$ or $z'$. Again, this is only 1-bit of information that the adversary tries to infer and the optimal strategy for the adversary is to conduct a likelihood ratio test. In this paper, we choose to work with the “In-or-Out” version of the differential privacy, although everything we derived can also be stated for the alternative version of differential privacy.

Note that the adversary always knows $Z$ and has a clearly defined target $z$, and it is natural to evaluate the winnings and losses of the “player”, the data curator by conditioning on the same data set and privacy target. This gives rise to the following generalization of DP.

**Definition 2** (Per-instance Differential Privacy). For a fixed data set $Z$ and a fixed data point $z$. We say a randomized algorithm $A$ satisfy $(\epsilon, \delta)$-per-instance-DP for $(Z, z)$ if for all measurable set $S \subset \Theta$, it holds that

$$P_{\theta \sim A(Z)}(\theta \in S) \leq e^\epsilon P_{\theta \sim A([Z,z])}(\theta \in S) + \delta,$$

$$P_{\theta \sim A([Z,z])}(\theta \in S) \leq e^\epsilon P_{\theta \sim A(Z)}(\theta \in S) + \delta.$$

This definition is different from DP primarily because DP is the property of the $A$ only and pDP is the property of both $A, Z$ and $z$. If we take supremum over all $Z \in \mathbb{Z}^n$ and $z \in Z$, then it recovers the standard differential privacy.

Similarly, we can define per-instance sensitivity for $(Z, z)$.

**Definition 3** (per-instance sensitivity). Let $\mathcal{H} = \mathbb{R}^d$, for a fixed $Z$ and $z$. The per-instance $\| \cdot \|_*$ sensitivity of a function $f : \text{Data} \to \mathbb{R}^d$ is defined as $\| f(Z) - f([Z,z]) \|_*$, where $\| \cdot \|_*$ could be $\ell_p$ norm or $\| \cdot \|_A = \sqrt{(\cdot)^T A (\cdot)}$ defined by a positive definite matrix $A$.

This definition also generalizes quantities in the classic DP literature. If we condition on $Z$ but maximize over all $z$, we get local-sensitivity [25]. If we maximize over all $Z$ and $z$ we get global sensitivity [12, Definition 3.1]. These two are often infinite in real-life problems, but for fixed data $Z$ and target $z$ to be protected, we could still get meaningful per-instance sensitivity.

Immediately, the per-instance sensitivity implies pDP for a noise adding procedure.

**Lemma 4** (Multivariate Gaussian mechanism). Let $\hat{\theta}$ be a deterministic map from a data set to a point in $\Theta$, e.g., a deterministic learning algorithm, and let the $A$-norm per-instance sensitivity $\Delta_A(Z, z)$ be $\| \hat{\theta}([Z,z]) - \hat{\theta}(Z) \|_A$. Then adding noise with covariance matrix $A^{-1}/\gamma$ obeys $(\epsilon(Z, z), \delta)$-pDP for any $\delta > 0$ with

$$\epsilon(Z, z) = \gamma \Delta_A(Z,z) \sqrt{\log(1.25/\delta)}.$$

The proof, which is standard and we omit, simply verifies the definition of $(\epsilon, \delta)$-pDP by calculating a tail bound of the privacy loss random variable and invokes Lemma 25.

### 2.1 Properties of pDP

We now describe properties of per-instance DP, which mostly mirror those of DP.
Fact 5 (Strong protection against identification). Let $\mathcal{A}$ obeys $(\epsilon, \delta)$-pDP for $(Z, z)$, then for any measurable set $S \subset \Theta$ where $\min\{\mathbb{P}_{\theta \sim \mathcal{A}(Z)}(\theta \in S), \mathbb{P}_{\theta \sim \mathcal{A}(Z)}(\theta \in S)\} \geq \delta/\epsilon$ then given any side information aux

$$-2\epsilon \leq \log \frac{\mathbb{P}_{\theta \sim \mathcal{A}(Z)}(\theta \in S|aux)}{\mathbb{P}_{\theta \sim \mathcal{A}([Z, z])}(\theta \in S|aux)} \leq 2\epsilon.$$  

Proof. Note that after fixing $Z$, $\theta$ is a fresh sample from $\mathcal{A}(Z)$, as a result, $\theta \perp aux|Z$. The claimed fact then directly follows from the definition.

Fact 6 (Convenient properties directly inherited from DP). For each $(Z, z)$ separately we have:

1. Simple composition: Let $\mathcal{A}$ and $\mathcal{B}$ be two randomized algorithms, satisfying $(\epsilon_1, \delta_1)$-pDP, $(\epsilon_2, \delta_2)$-pDP, then $(\mathcal{A}, \mathcal{B})$ jointly is $(\epsilon_1 + \epsilon_2, \delta_1 + \delta_2)$-pDP.

2. Advanced composition: Let $\mathcal{A}_1, ..., \mathcal{A}_k$ be a sequence of randomized algorithms, where $\mathcal{A}_i$ could depend on the realization of $\mathcal{A}_1(Z), ..., \mathcal{A}_i(Z)$, each with $(\epsilon, \delta)$-pDP, then jointly $\mathcal{A}_1: \mathcal{A}_k$ obeys $O(\sqrt{\log(1/\delta)\epsilon}), O(k\delta)$-pDP.

3. Closedness to post-processing: If $\mathcal{A}$ satisfies $(\epsilon_1, \delta_1)$-pDP, for any function $f$, $f(\mathcal{A}(\cdot))$ also obeys $(\epsilon_1, \delta_1)$-pDP.

4. Group privacy: If $\mathcal{A}$ obeys $(\epsilon, \delta)$-pDP with $\epsilon, \delta$ parameterized by (Data, Target), then

\[
P_{\theta \sim \mathcal{A}(Z)}(\theta \in S) \leq e^{(Z, z_1) + \epsilon([Z, z_1], z_2) + \cdots + \epsilon([Z, z_{k-1}], z_k)} P_{\theta \sim \mathcal{A}([Z, z_1:k])}(\theta \in S) + \tilde{\delta}.
\]

\[
P_{\theta \sim \mathcal{A}([Z, z_1:k])}(\theta \in S) \leq e^{(Z, z_1) + \epsilon([Z, z_1], z_2) + \cdots + \epsilon([Z, z_{k-1}], z_k)} P_{\theta \sim \mathcal{A}(Z)}(\theta \in S) + \tilde{\delta},
\]

for $\tilde{\delta} = \sum_{i=1:k} \delta([Z, z_1:k], z_i) \prod_{j=1:i-1} e^{(Z, z_1:j-1)}$.

Proof. These properties all directly follow from the proof of these properties for differential privacy (see e.g., [12]), as the uniformity over data sets is never used in the proof. The only property that gets slightly different for the new definition is group privacy, since the size of the data set changes as the size of the privacy target (now a fixed group of people) gets larger. The claim follows from a simple calculation that repeatedly apply the definition of pDP for a different $(Z, z)$.

2.2 Moments of pDP, generalization and domain adaptation

One useful notion to consider in practice is to understand exactly how much privacy is provided for those who participated in the data sets. This is practically relevant, because if a cautious individual decides to not submit his/her data, he/she would necessarily do it by rejecting a data-usage agreement and therefore the data collector is not legally obligated to protect this person and in fact does not have access to his/her data in the first place. After all, the only type of identification risk that could happen to this person is that the adversary can be quite certain that he/she is not in the data set. For instance, in a study of graduate student income, a group of 200 students are polled and their average income is revealed with some small noise added to it. While an adversary can be almost
certain based on the outcome that Bill Gates did not participate in the study, but that is hardly a any privacy risk to him. One advantage of pDP is that it offers a very natural way to analyze and also empirically estimate any statistics of the pDP losses over a data set or over a distribution of data points corresponding to a fixed randomized algorithm $A$.

**Definition 7** (Moment pDP for a distribution). Let $(Z, z)$ be drawn from some distribution (not necessarily a product distribution) $\mathcal{P}$, it induces a distribution of $\epsilon(Z, z)$. Then we say that the distribution obeys $k$th moment per-instance DP with parameter vector $(E\epsilon, E[\epsilon^2], \ldots, E[\epsilon^k], \delta)$.

For example, one can treat the problem of estimating privacy loss for a fixed data set $Z$ by choosing $\mathcal{P}$ to be a discrete uniform distribution supported on $\{(Z_{-i}, z_i)\}_{i=1}^n$ with probability $1/n$ for each $i$. Taking $k = 2$ allows us to calculate mean and variance of the privacy loss over the data set.

Similarly, if the data set is drawn iid from some unknown distribution $\mathcal{D}$ — a central assumption in statistical learning theory — then we can take $\mathcal{P} = \mathcal{D}^{n-1} \times \mathcal{D}$. This allows us to use the moment of pDP losses to capture on average, how well data points drawn from $\mathcal{D}$ are protected. It turns out that this also controls generalization error, and more generally cross-domain generalization.

**Definition 8** (On average generalization). Under the standard notations of statistical learning, the on average generalization error of an algorithm $A$ is defined as

$$\text{Gen}(A, \mathcal{D}, n) = \left| E_{Z \sim \mathcal{D}^n, z \sim \mathcal{D}} E_{\theta \sim A(Z)} \frac{1}{n} \sum_{i=1}^n \ell(\theta, z_i) - \ell(\theta, z) \right|$$

**Proposition 9** (Moment pDP implies generalization). Assume bounded loss function $0 \leq \ell(\theta, z) \leq 1$. Then the on-average generalization is smaller than

$$E_{Z \sim \mathcal{D}^n} (E_{z \sim \mathcal{D}} [\epsilon'(Z, z)]^2) - 1 + E_{Z \sim \mathcal{D}^n, z \sim \mathcal{D}} \delta(Z, z) + (E_{Z \sim \mathcal{D}^n} E_{z \sim \mathcal{D}} [\epsilon'(Z, z)] Z) E_{z \sim \mathcal{D}} [\delta(Z, z) | Z].$$

Note that this can also be used to capture the privacy and generalization of transfer learning (also known as domain adaptation) with a fixed data set or a fixed distribution. Let the training distribution be $\mathcal{D}$ and target distribution be $\mathcal{D}'$.

Take $\mathcal{P} = \mathcal{D}^n \otimes \mathcal{D}'$ or $\mathcal{P} = \delta_Z \otimes \mathcal{D}'$. In practice, this allows us to upper bound the generalization to the Asian demographics group, when the training data is drawn from a distribution that is dominated by white males (e.g., the current DNA sequencing data set). We formalize this idea as follows.

**Definition 10** (Cross-domain generalization). Assume $0 \leq \ell(\theta, z) \leq 1$. The on-average cross-domain generalization with base distribution $\mathcal{D}$ to target distribution $\mathcal{D}'$ is defined as:

$$\text{Gen}(A, \mathcal{D}, \mathcal{D}', n) \leq \left| E_{Z \sim \mathcal{D}^n, z \sim \mathcal{D}} E_{\theta \sim A(Z)} \left[ \frac{1}{n} \sum_{i=1}^n \rho_i \ell(\theta, z_i) - \ell(\theta, z) \right] \right|.$$

where $\rho_i = \mathcal{D}'(z_i) / \mathcal{D}(z_i)$ is the inverse propensity (or importance weight) to account for the differences in the two domains.

**Proposition 11**. The cross-domain on average generalization can be bounded as follows:

$$\text{Gen}(A, \mathcal{D}, \mathcal{D}', n) = E_{Z \sim \mathcal{D}^n-1, (z', z'') \sim \mathcal{D}, z' \sim \mathcal{D}'} [(\epsilon'(Z, z') + \epsilon(Z, z'') - 1) + \delta(Z, z') + \epsilon(Z, z') \delta(Z, z'')]$$

The expressions in Proposition 9 and 11 are a little complex, we will simplify it to make it more readable.
Corollary 12. Let \( \sup_{Z,z} \delta(Z,z) \leq \delta \), and \( \mathbb{E}_\mathcal{D}[e^{2\epsilon(Z,z)}] \leq 1 \) and for simplicity, we write \( \mathbb{E}_{Z \sim \mathcal{D}^n, z \sim \mathcal{D} \epsilon(Z,z)} = \mathbb{E}_\mathcal{D} f \) and \( \mathbb{E}_{Z \sim \mathcal{D}^n, z \sim \mathcal{D} \epsilon(Z,z)} = \mathbb{E}_\mathcal{D} f \). Then the cross domain on-average generalization is smaller than
\[
\frac{1}{2} [\mathbb{E}_\mathcal{D} e^{2\epsilon} + \mathbb{E}_\mathcal{D'} e^{2\epsilon}] - 1 + 2\delta = \frac{1}{2} \left[ \sum_{i=1}^{\infty} \frac{2^i}{i!} \mathbb{E}_\mathcal{D} e^i + \mathbb{E}_\mathcal{D'} e^i \right] + 2\delta.
\]

2.3 Related notions

We now compare the proposed privacy definition with existing ones in the literature. Most attempts to weaken differential privacy aims at more careful accounting of privacy loss by treating the \( \epsilon = \log[p(\theta)/p'(\theta)] \) as a random variable. This produces nice connection of \((\epsilon,\delta)\)-DP to concentration inequalities and in particular, it produces advanced composition of privacy loss via Martingale concentration. More recently, the idea is extended to defining weaker notions of privacy such as concentrated-DP [13, 4] and Rényi-DP [24] that allows for more fine-grained understanding of Gaussian mechanisms. Our work is complementary to this line of work, because we consider adaptivity of \( \epsilon \) to a fixed pair of data-set and privacy target, and in some cases, \( \epsilon \) being a random variable jointly parameterized by \( Z, z \) and \( \theta \). We summarize the differences of these definitions in the following table. It is clear from the table that if we ignore the differences in the probability metric used, per-instance DP is arguably the most general, and adaptive, since it depends on specific \( (Z, z) \) pairs.

| Data set | private target | probability metric | parametrized by |
|----------|----------------|--------------------|-----------------|
| Pure-DP[11]  | sup \( Z \) | sup \( z \) | \( D_\infty(P\|Q) \) | \( \mathcal{A} \) only |
| Approx-DP[9]  | sup \( Z \) | sup \( z \) | \( D^\delta(P\|Q) \) | \( \mathcal{A} \) only |
| \((z/m)\)-CDP[13, 4] | sup \( Z \) | sup \( z \) | \( D_{subG}(P\|Q) \) | \( \mathcal{A} \) only |
| Rényi-DP[24]  | sup \( Z \) | fixed \( z \) | \( D_\alpha(P\|Q) \) | \( \mathcal{A} \) only |
| Personal-DP[15, 20] | sup \( Z \) | sup \( z \) | \( ||P-Q||_{TV} \) | \( \mathcal{A} \) only |
| TV-privacy[2]   | sup \( Z \) | fixed \( z \) | \( D_{KL}(P\|Q) \) | \( \mathcal{A} \) only |
| KL-privacy[2]   | sup \( Z \) | sup \( z \) | \( D_{KL}(P\|Q) \) | \( \mathcal{A} \) and \( \mathcal{D} \) |
| On-Avg KL-privacy[32] | \( \mathbb{E}_{Z \sim \mathcal{D}^n} \) | \( \mathbb{E}_{z \sim \mathcal{D}} \) | \( D_{KL}(P\|Q) \) | \( \mathcal{A}, \mathcal{Z} \) and \( \mathcal{z} \) |
| Per-instance DP | fixed \( Z \) | fixed \( z \) | \( D_\infty(P\|Q) \) | \( \mathcal{A}, \mathcal{Z} \) and \( \mathcal{z} \) |

Table 1: Comparing variances of differential privacy.

The closest existing definition to ours is perhaps the personalized-DP, first seen in Ebadi et al. [15], Liu et al. [20]. It also tries to capture a personalized level of privacy. The difference is that personalized-DP requires the sensitivity of the private target to hold globally for all data sets.

On-Avg KL-privacy is also an adaptive quantity that measures the average privacy loss when the individuals in the data set and private target are drawn from the same distribution \( \mathcal{D} \). pDP on the other hand measures the approximate worst-case privacy for a fixed \( (Z, z) \) pair that is not necessarily random. Not surprisingly, on-Avg KL-privacy and expected per-instance DP are intricately related to each other, as the following remark suggests.

Remark 13. Let \( \mathcal{A}(Z) = f(Z) + \mathcal{N}(0, \sigma^2 I) \), namely, Gaussian noise adding. Then On-avg KL-privacy is
\[
\mathbb{E}_{Z \sim \mathcal{D}^n, z \sim \mathcal{D}} D_{KL}(p(Z)|p(Z')) = \mathbb{E}_{Z, z' \sim \mathcal{D}^n} ||f(Z) - f([Z_{-1}, z])||^2 / \sigma^2.
\]
Second moment of per-instance DP is
\[
\mathbb{E}_{Z \sim \mathcal{D}^n, z \sim \mathcal{D}} \left[ \left( \frac{\Delta f(Z) \sqrt{1.25/\delta}}{\sigma} \right)^2 \right] = \mathbb{E}_{Z \sim \mathcal{D}^n, z \sim \mathcal{D}} \| f(Z) - f([Z, z]) \|^2 \log(1.25/\delta)/\sigma^2.
\]

The two notions of privacy are almost equivalent. They differ only by a logarithmic term and by a minor change in the way perturbation is defined. In general, for Gaussian mechanism, \( \epsilon - \text{KL}-\text{privacy} \) implies \((\sqrt{\epsilon \log(1.25/\delta)}, \delta)\)-DP for any \( \delta \).

3 Per-instance sensitivity in smooth learning problems

In this section, we present our main results and give concrete examples in which per-instance sensitivity (hence per-instance privacy) can be analytically calculated. Specifically, we consider following regularized empirical risk minimization form:
\[
\hat{\theta} = \arg\min_{\theta} \sum_i \ell(\theta, z_i) + r(\theta),
\]

or in the non-convex case, finding a local minimum. \( \ell(\theta, z) \) and \( r(\theta) \) are the loss functions and regularization terms. We make the following assumptions:

A.1. \( \ell \) and \( r \) are differentiable in argument \( \theta \).

A.2. The partial derivatives are absolute continuous, i.e., they are twice differentiable almost everywhere and the second order partial derivatives are Lebesgue integrable.

Our results under these assumptions will cover learning problems such as linear and kernel machines as well as some neural network formulations (e.g., multilayer perceptron and convolutional net with sigmoid/tanh activation), but not non-smooth problems like lasso, \( \ell_1 \)-SVM or neural networks ReLU activation. We also note that these conditions are implied by standard assumptions of strong smoothness (gradient Lipschitz) and do not require the function to be twice differentiable everywhere. For instance, the results will cover the case when either \( \ell \) or \( r \) is a Huber function, which is not twice differentiable.

Technically, these assumptions allow us to take Taylor expansion and have an integral form of the remainder, which allows us to prove the following stability bound.

Lemma 14. Assume \( \ell \) and \( r \) satisfy Assumption A.1 and A.2. Let \( \hat{\theta} \) be a stationary point of \( \sum_i \ell(\theta, z_i) + r(\theta) \), \( \hat{\theta}' \) be a stationary point \( \sum_i \ell(\theta, z_i) + \ell(\theta, z) + r(\theta) \) and in addition, let \( \eta_t = t \hat{\theta} + (1 - t)\hat{\theta}' \) denotes the interpolation of \( \hat{\theta} \) and \( \hat{\theta}' \). Then the following identity holds:
\[
\hat{\theta} - \hat{\theta}' = \left[ \int_0^1 \left( \sum_i \nabla^2 \ell(\eta_t, z_i) + \nabla^2 \ell(\eta_t, z) + \nabla^2 r(\eta_t) \right) dt \right]^{-1} \nabla \ell(\hat{\theta}, z)
\]
\[
= - \left[ \int_0^1 \left( \sum_i \nabla^2 \ell(\eta_t, z_i) + \nabla^2 r(\eta_t) \right) dt \right]^{-1} \nabla \ell(\hat{\theta}', z).
\]
The proof uses first order stationarity condition of the optimal solution and apply Taylor’s theorem on the gradient. The lemma is very interpretable. It says that the perturbation of adding or removing a data point can be viewed as a one-step quasi-newton update to the parameter. Also note that \( \nabla \ell(\hat{\theta}', z) \) is the “score function” in parametric statistical models, and is called a “pseudo-residual” in gradient boosting [see e.g., 18, Chapter 10].

The result implies that the per-instance sensitivity in \( \| \cdot \|_A \) for some p.d. matrix \( A \) can be stated in terms of certain norm of the “score function” specified by a quadratic form \( H^{-1}AH^{-1} \), and therefore by Lemma 4, the output perturbation algorithm:

\[
\hat{\theta} \sim \mathcal{N}(\hat{\theta}(X), A^{-1}/\gamma),
\]

obeys \( (\epsilon, \delta) \)-pDP for any \( \delta > 0 \) and

\[
\epsilon(Z, z) = \sqrt{\nabla \ell(\hat{\theta}', z)^T H^{-1} A H^{-1} \nabla \ell(\hat{\theta}', z)} \log(1.25/\delta).
\]

This is interesting because for most loss functions the “score function” is often proportional to the prediction error of the fitted model \( \hat{\theta}' \) on data point \( z \) and this result suggests that the more accurately a model predicts a data point, the more private this data point is. This connection is made more explicit when we specialize to linear regression and the per-instance sensitivity

\[
\Delta_A(Z, z) = |y - x^T \hat{\theta}| \sqrt{x^T (X^T X)^{-1} A (X^T X)^{-1} x} = |y - x^T \hat{\theta}'| \sqrt{x^T (X^T X)^{-1} A (X^T X)^{-1} x}.
\]

is clearly proportional to prediction error. In addition, when we choose \( A \approx X^T X \), the second term becomes either \( \mu := x^T (X^T X)^{-1} x \) or \( \mu' := x^T (X^T X)^{-1} x \), which are “in-sample” and “out-of-sample” statistical leverage scores of \( x \). Leverage score measures the importance/uniqueness of a data point relative to the rest of the data set and it is used extensively in regression analysis [5] (for outlier detection and leverage score), compressed sensing (for adaptive sampling) [29] and numerical linear algebra (for fast matrix computation)[8]. For the best of our knowledge, this is the first time leverage scores are shown to be connected to differential privacy.

4 Case study: The adaptivity of OPS in ridge regression

So far we have described output perturbation algorithms with a fixed noise adding procedure. However in practice it is not known ahead of time how to choose \( A \). Assume all \( x \) are normalized to \( \|x\| = 1 \), denote \( \mu_2(x) := x^T (X^T X)^{-2} x \), \( \mu_1(x) := x^T (X^T X)^{-1} x \). We discuss the pros and cons of the three natural choices.

- \( A \approx \lambda_{\min} I \): This corresponds to the standard \( \ell_2 \)-sensitivity and it adds an isotropic noise and provides a uniform guarantee for all data-target pairs where \( X^T X \) has smallest eigenvalue \( \lambda_{\min} \), because \( \sup_x \sqrt{\mu_2(x)} \leq 1/\lambda_{\min} \), but it adds more noise than necessary for those with much smaller \( \mu_2(x) \).

- \( A \approx (X^T X)^2 \): We call this the “democratic” choice conditioned on the data set, as it homogenizes the “leverage” part of the per-instance sensitivity of points to \( \|x\| = 1 \) so any \( x \) gets about the same level of privacy. It however is not robust to if our data-independent choice of \( A \) is in fact far away from the actual \( (X^T X)^2 \).
- \( A \approx X^T X \): We call this the “Fisher”-choice, because the covariance matrix will be proportional to the inverse Fisher information, which is the natural estimation error of \( \hat{\theta} \) under the linear regression assumption. The advantage of this choice is that conducting statistical inference, e.g., t-test and ANOVA for linear regression coefficients would be trivial.

In fact, for linear and ridge regression, the third choice is closely related to the one-posterior-sampling (OPS) mechanism proposed in [7, 31] with an important difference being that in OPS, \( A \) is not fixed, but rather depends on the data. As a result, Lemma 4 does not work. In fact, if the data-target can be arbitrary and \( r = 0 \), the data-independent choice of \( A \) could imply an unbounded \( \epsilon \) (consider an arbitrarily near singular \( X \) and \( x \) in its null space).

Indeed, existing analysis of OPS requires additional assumption. [31] assumes that the loss function is bounded (by modifying it or constraining the domain \( \Theta \)) so that the exponential mechanism [23] would apply. It was later pointed out in [17] that OPS is asymptotically inefficient in that it has an asymptotic relative efficiency (ARE) inversely proportional to \( \epsilon \), while simple sufficient statistics perturbation can achieve asymptotic efficiency comparable to [28]. This is far from satisfactory.

Based on insight from pDP, we propose a direct analysis of OPS which reveals that if \( (\epsilon, \delta) \)-DP is all we need, then OPS is also asymptotically efficient under the same data assumption in [17]. In addition, it effectively converges to the “Fisher”-choice of noise adding in the same asymptotic regime and offers dimension and condition number independent expected pDP loss.

The first result calculates the pDP loss of OPS.

**Theorem 15** (The adaptivity of OPS in Linear/Ridge Regression). Consider the algorithm that samples from

\[
p(\theta | X, y) \propto e^{-\frac{1}{2} \| y - X \theta \|^2 + \lambda \| \theta \|^2}
\]

Let \( \hat{\theta} \) and \( \hat{\theta}' \) be the ridge regression estimate with data set \( X \times y \) and \([X, x] \times [y, y]\) and defined the out of sample leverage score \( \mu := x^T (X^T X + \lambda I)^{-1} x = x^T H^{-1} x \) and in-sample leverage score \( \mu' := x^T [(X')^T X' + \lambda I]^{-1} x = x^T (H')^{-1} x \). Then for every \( \delta > 0 \), privacy target \((x, y)\), the algorithm is \((\epsilon, \delta)\)-DP with

\[
\epsilon(Z, z) \leq \frac{1}{2} \left| - \log(1 + \mu) + \gamma \mu \left( y - x^T \hat{\theta} \right)^2 \right| + \frac{\mu}{2} \log(2/\delta) + \sqrt{\gamma \mu \log(2/\delta)} \| y - x^T \hat{\theta} \| \quad \tag{5}
\]

\[
= \frac{1}{2} \left| - \log(1 + \mu') - \gamma \mu' \left( y - x^T \hat{\theta}' \right)^2 \right| + \frac{\mu'}{2} \log(2/\delta) + \sqrt{\gamma \mu' \log(2/\delta)} \| y - x^T \hat{\theta}' \| \quad \tag{6}
\]

The two equivalent upper bounds are both useful. (5) is ideal for calculating pDP when \( x \) is not in the data set and (6) is perfect for the case when \( x \) is in the data set.

**Remark 16.** The bound (5) can be simplified to

\[
\frac{\mu}{2} (1 + \log(2/\delta)) + \frac{1}{2} \gamma \min(\mu, 1) \| y - x^T \hat{\theta} \|^2 + \sqrt{\gamma \mu \log(2/\delta)} \| y - x^T \hat{\theta} \|.
\]

If \( \mu = o(\log(2/\delta)) \) and we choose \( \gamma \) such that \( \sqrt{\gamma \mu \log(2/\delta)} \| y - x^T \hat{\theta} \| \leq 1 \), then the bound can be

\[
\frac{\mu}{2} (1 + \log(2/\delta)) + \frac{1}{2} \gamma \min(\mu, 1) \| y - x^T \hat{\theta} \|^2 + \sqrt{\gamma \mu \log(2/\delta)} \| y - x^T \hat{\theta} \|.
\]

This is not an unrealistic assumption because \( \mu \) and \( \mu' \) are \( o(1) \) as long as \( x \) is bounded and the minimum eigenvalue of \( X^T X + \lambda I \) is \( o(1) \). This is required for (agnostic) linear regression to be consistent and is implied by the condition that the the population covariance matrix is full rank.
This matches the order of Gaussian mechanism with a fixed (data-independent) covariance matrix.

The results in [17] are stated for general exponential family models under a set of assumptions that translate into the following for linear regression:

(a) data \( x_1, ..., x_n \) is drawn i.i.d. from \( \mathcal{D} \) supported on \( \mathcal{X} \) where \( \mathcal{X} \subset B_\| \cdot \|_2(1) \).

(b) population covariance matrix \( \frac{2}{n} I \succeq E_{\mathcal{D}x}x^T \succeq \frac{M}{d} I \) for constant \( m \) and \( M \),

(c) \( y_i \sim \mathcal{N}(x_i^T \theta_0, \sigma^2) \) for some \( \theta_0 \).

To simplify the presentation, we also assume \( n \) scales with respect to \( d \) such that

(d) with high probability, \( XX^T \succ \frac{m}{d} I \).

The last assumption measures how quickly the empirical covariance matrix \( \frac{1}{n} XX^T \) concentrates to \( E_{x \sim \mathcal{D}x} xx^T \). It has been shown that if \( X \) is an appropriately scaled subgaussian random matrix, this happens with probability \( 1 - n^{-10} \) whenever \( n \geq \max(10d, 10d^{-2/3} \log n) \).

**Proposition 17.** The sequence of OPS algorithm with parameter \( \gamma_n, \lambda_n \) obeys the following properties.

1. **pDP and DP in agnostic setting.** Assume \( \|x\| \leq 1 \) for every \( x \in \mathcal{X} \). The algorithm obeys \((\epsilon_n, \delta)-pDP\), for each data set \((X, y)\) and target \((x, y)\),

\[
\epsilon_n = \sqrt{\frac{\gamma_n \log(2/\delta)}{\lambda_n + \lambda_{\min}}} |y - x^T \hat{\theta}| + \frac{\gamma_n |y - x^T \hat{\theta}|^2}{2\max\{\lambda_n + \lambda_{\min}, 1\}} + \frac{\gamma_n(1 + \log(2/\delta))}{2(\lambda_n + \lambda_{\min})}.
\]  

If we further assume \( |y| < 1 \), then \( \sup_{(x,y),(x,y)} |y - x^T \hat{\theta}| = 1 + n^{1/2} \lambda_n^{-1/2} \) and the algorithm obeys \((\epsilon_n, \delta)-DP\) with

\[
\epsilon_n = \sqrt{\frac{2(n + \lambda_n) \gamma_n \log(2/\delta)}{\lambda_n^2}} + \frac{2(n + \lambda_n) \gamma_n}{\lambda_n \max\{1, \lambda_n\}} + \frac{\gamma_n(1 + \log(2/\delta))}{2\lambda_n}.
\]

2. **pDP under model assumption.** Assume conditions (a)(b)(c)(d) above are true, and also \( \gamma_n = o(1), \lambda_n = o(\sqrt{n}) \). Then with high probability over the joint distribution of \((X, y)\), the algorithm with \( \gamma_n \leq \frac{4n \log(2/\delta)}{\max\{d, (1+\log(2/\delta))^2\}} \) obeys \((\epsilon_n, \delta)-pDP\) with

\[
\epsilon_n = \begin{cases} 
O\left(\sqrt{\frac{(1 + \|\theta_0\|)^2 d \gamma_n}{m} \log \left(\frac{2}{\delta}\right)}\right) & \text{for all } (x, y) \text{ satisfying } \|x\| = O(1) \text{ and } y = O(1). \\
O\left(\sqrt{\frac{\sigma^2 d \gamma_n}{m} \log \left(\frac{2}{\delta}\right)}\right) & \text{for any } x \in \mathcal{X} \text{ with probability } 1 - \delta' \text{ over } y \sim \mathcal{N}(\theta_0 x, \sigma^2). 
\end{cases}
\]

Moreover, with probability \( 1 - n \delta' \) over the conditional distribution \( y | X \), the privacy loss of \((x_1, y_1), ..., (x_n, y_n)\) obeys

\[
\frac{1}{n} \sum_{i=1}^n \epsilon_n((X, y), (x_i, y_i))^2 = O\left(\frac{\sigma^2 d \gamma_n}{n} \log(2/\delta) \log(2/\delta')\right),
\]

which does not depend on \( m \) — the smallest eigenvalue of \( dX^T X/n \).
3. **Statistical efficiency.** For every realization of data set $X$ such that $n > d$ and let the smallest eigenvalue of $X^TX$ be $\lambda_{\text{min}}$, then

$$
\mathbb{E}_{y \sim \mathcal{N}(X\theta_0, \sigma^2 I_n)} \left[ \|\hat{\theta} - \theta_0\|_2^2 | X \right] = \sigma^2 \text{tr}((X^TX + \lambda_n I)^{-1})(1 + \gamma_n^{-1}) + \lambda_n^2 \|X^TX + \lambda_n I\|^{-1} \theta_0\|_2^2
$$

If $\lambda_{\text{min}} = \Omega(d/n)$ (this is true with high probability under assumption (b)(d)) Then we get

$$
\mathbb{E}_{y \sim \mathcal{N}(X\theta_0, \sigma^2 I_n)} \left[ \|\hat{\theta} - \theta_0\|_2^2 | X \right] = \sigma^2 \text{tr}((X^TX + \lambda_n I)^{-1})(1 + \gamma_n^{-1}) + O\left(\frac{\lambda_n^2 d^2 \|\theta_0\|_2^2}{n^2}\right)
$$

In other word, the estimator is asymptotically efficient, for all $\lambda_n = o(n^{1/2})$ and $\gamma_n = \omega(1)$. We now discuss a few aspects of the above results.

**pDP vs DP in agnostic setting.** Firstly, it highlights the key advantage of pDP over DP. DP is not able to take advantage of desirable structures in the data set, while pDP provides a principled framework to handle them.

In particular, let us compare the pDP and DP in the agnostic setting, for the OPS that uses the same randomization. DP measures something that is completely data independent, and corresponds specifically to a contrivedly constructed data set $(X,y)$ such that $y$ is an eigenvector of $XX^T$ corresponding to a specific eigenvalue of magnitude $\sqrt{\lambda_n}$, this makes $\|\hat{\theta}\|_2$ as large as $\sqrt{n}/\sqrt{\lambda_n}$. Moreover, a target data point is chosen so that $x$ match the direction of $\hat{\theta}$. While this is a legitimate construction in theory, but it does not directly correspond to the specific data set that a statistician just spent two years collecting, and it is unreasonable that he/she will have to calibrate the amount of noise to inject to provide more reasonable protection to a pathological case that has nothing to do with the reality.

pDP on the other hand, makes it possible for the statistician to condition on the data set. If the statistician finds out that $\|\hat{\theta}\|_2 = O(1)$, then the pDP loss is as small as $\sqrt{\gamma_n \log(2/\delta)}/\lambda_n$ for everyone in the population. With $\gamma_n = n^{\alpha/2}$ and $\lambda_n = n^{1/2 - \alpha/2}$ for any $\alpha > 0$, the algorithm remains to be statistically efficient with an ARE of $(1 + n^{-\alpha})$ yet can provide a strong privacy guarantee of $\epsilon_n = n^{-1/4 + \alpha/2}$. If in addition, the statistician realized that the data set is well-conditioned, that is, the maximum and minimum eigenvalue of $X^TX$ are on the same order of $n/d$, then we can further improve the bound by replacing $\lambda_n$ with $\lambda_{\text{min}} + \lambda_n$. The statistician can happily get away with the same privacy guarantee ($\epsilon_n = n^{-1/4}$) while not having to add too much noise or even regularize at all (setting $\gamma_n = n^{1/2}$ and $\lambda_n = 0$). Note that the condition number is a desirable property that governs how reliably one can hope to estimate the linear regression coefficients using the given data set.

We would like to emphasize that the pDP guarantee in the two cases we discussed above applies to everyone in the population $\{(x,y)||x|| \leq 1, |y| \leq 1\}$, therefore such $(\epsilon, \delta)$-pDP guarantee is as powerful as $(\epsilon, \delta)$-DP after the data set is collected.

**pDP of all vs average pDP on the data set.** Secondly, unlike DP which always provides a crude upper bound for everyone, pDP is able to reflect the differences in the protection of different target person. Under the model assumption, the average privacy loss of people in the data set, is
scale-invariant and interestingly, also independent to the condition number (smallest eigenvalue). It is a factor of \((1 + \|\theta_0\|)^2/m\) times smaller than the pDP guarantee for everyone in the population. This is significant for finite sample performance since \((1 + \|\theta_0\|)/m\) (although they do not change with \(n\)), can be quite large.

**pDP under covariate shift.** Lastly, if we consider a setting in between the above two, where the target \(x\) can be drawn from any distribution defined on \(X\) that could be arbitrarily different from the training data distribution, then the scale-invariant property remains (the factor of \((1 + \|\theta_0\|)\) is dropped). This is relevant in causal learning when the \(E(y|x)\) is specified by some physical principles that’s invariant to the distribution of \(x\). In this case, the moments of the pDP would imply a much stronger notion of cross-domain generalization than what we show in Proposition 11 since it does not depend on target covariate distribution of interest.

**Improved DP guarantee for OPS.** The proposition also improves the existing analysis for the OPS algorithm as a byproduct. The first statement shows that OPS preserves a meaningful (almost constant) differential privacy when \(\gamma_n = 1\) and \(\lambda_n = \sqrt{n/d}\) without requiring a constant boundedness in the domain \(\Theta\) or clipping the loss function like in Wang et al. [31]. As a matter of fact, the ridge regression solution \(\hat{\theta}\) could be in a ball of radius \(\Theta(n^{1/4})\), and even if we impose the smallest domain bound that covers \(\hat{\theta}\), by exponential mechanism, the algorithm only obeys a pure \(O(n^{1/2})\)-DP, in contrast to the \((O(\log(1/\delta)), \delta)\)-DP that we showed in the proposition above.

Despite the improvement, the DP guarantee is still a little unsatisfactory. If we require \((\epsilon, \delta)\)-DP with constant \(\epsilon\), then the OPS algorithm with \(\lambda_n = \sqrt{n}\) is not asymptotically efficient (although it does achieve the optimal \(O(1/n)\) rate).

Meanwhile, there are algorithms that attain asymptotic efficiency either by subsample-and-aggregate [28] or by simply adding noise to the sufficient statistics [14, 17].

So the question becomes: can we modify OPS such that it become asymptotically efficient with \((\epsilon_n, \delta)\)-differentially private with \(\epsilon_n = o(1)\)?

We address this issue in the next section.

## 5 Statistical efficient linear regression with differential privacy using AdaOPS

In this section, we resolve the dilemma described earlier using the idea of Dwork & Lei [10]. The new algorithm, which we call AdaOPS, adaptively and differentially privately chooses the tuning parameter \(\lambda_n\) and \(\gamma_n\) according to properties of the data set and privacy requirement. A pseudocode of AdaOPS is given in Algorithm 1. We acknowledge that the same idea of adaptively adding regularization term is not new and had been used by Kifer et al. [19], Blocki et al. [3], Sheffet [27] for analyzing other related differentially private algorithms. Our contribution here is only to assemble the ideas together into a working algorithm.
Algorithm 1 AdaOPS : One-Posterior Sample estimator with adaptive regularization

**input** Data $X, y$. Privacy target: $\epsilon, \delta$. And parameter $\kappa$ satisfying $\kappa \leq \frac{n \epsilon}{4d(1+\log(4/\delta))}$

1. Calculate the minimum eigenvalue $\lambda_{\text{min}}(X^T X)$.
2. Private release $\tilde{\lambda}_{\text{min}} = \lambda_{\text{min}} + \frac{\sqrt{\log(4/\delta)}}{\epsilon/2} Z$, where $Z \sim N(0, 1)$.
3. Get one sample $\tilde{\theta} \sim P(\theta | X, y) \propto e^{-\frac{\gamma_n}{2}(\|y - X\theta\|^2 + \lambda_n \|\theta\|^2)}$ with parameter

$$
\lambda_n = \min \left\{ 0, \frac{n}{d \kappa} - \frac{\log(4/\delta)}{\epsilon/2} \right\}
$$

$$
\gamma_n = \min \left\{ \frac{\epsilon^2}{16\kappa^2d^2 \log(4/\delta)}, \frac{\epsilon^2}{8\kappa^2d^2} \right\}
$$

**output** $\tilde{\theta}$

The $\kappa$ parameter is the largest acceptable condition number in the data set. Often it can be determined independent to the data. It is used in the algorithm to rule out the pathological case.

We now analyze the properties of AdaOPS.

**Proposition 18.**

1. Assume data domain is $\|x\|_2 \leq 1$ and $|y| \leq 1$. The AdaOPS estimator preserves $(\epsilon, \delta)$-DP.

2. If assumption (a)(b)(c) is true and in addition for the specific realization of $X$,

$$
\lambda_{\text{min}}(X^T X) > \frac{n}{kd} + \frac{\sqrt{\log(10n) \log(4/\delta)}}{\epsilon/2}
$$

(which is true with high probability if $m > 2/\kappa$ and $X$ is an appropriately scaled subgaussian random matrix), then, we have

$$
\mathbb{E}[\|\tilde{\theta} - \theta_0\|^2 | X] = [1 + \gamma_n] \sigma^2 \text{tr}[(X^T X)^{-1}] + O(n^{-10})\|\theta_0\|^2.
$$

In other word, since $\gamma_n \leq \min\{\frac{\epsilon^2 d^2 \log(4/\delta)}{n \epsilon^2}, \frac{\epsilon^2 d^2}{n \epsilon} \}$, the AdaOPS estimator achieves asymptotic efficiency whenever $\epsilon$ obeys that $\min\{n \epsilon^2, s \epsilon\} = o(\kappa^2 d^2 \log(4/\delta)/n)$.

This proposition reveals that AdaOPS improves over previous results in the literature [28, 17] in several ways. First of all, we only need $n \epsilon^2 = o(1)$ to achieve asymptotic efficiency. In contrast, [17] does not provide non-asymptotic results with explicit dependence and [28]’s bound for the subsample-and-aggregate method requires $n^{-1/5} \epsilon^{-6/5} = o(1)$ to achieve asymptotic efficiency.

Secondly, both [17] and [28] suffer from additional polynomial dependence on the dimension in the supposedly lower order term, and that affects finite sample performance. While we haven’t yet solved the problem, we managed to slightly improve the dimension dependence. In particular, our bound on the additive difference from exactly matching the Cramer-Rao lower bound of $(\sigma^2 \text{tr}((X^T X)^{-1}))$ translates into $\sigma^2 \text{tr}((X^T X)^{-1}) + d^3/(n^2 \epsilon^2)$. In contrast, there is a clear dependence of $d^5/n^2 \epsilon^2$
in the sufficient statistics perturbation approach\(^2\) and even higher polynomial dependence in the subsample-and-aggregate approach.

We conclude the section with two simulate experiments (shown in the two panes of Figure 1). In the first experiment, we consider the algorithm of adding isotropic Gaussian noise to linear regression coefficients, and then compare the worst-case DP and the distribution of per-instance DP for points in the data set (illustrated as box plots). In the second experiment, we compare different notions of privacy to utility (measured as excess risk) of the fixed algorithm that samples from a scaled posterior distribution. In both cases, the average per-instance differential privacy over the data sets is several orders of magnitude smaller than the worst-case differential privacy.

6 Conclusion

In this paper, we proposed to use per-instance differential privacy (pDP) for quantifying the fine-grained privacy loss of a fixed individual against randomized data analysis conducted on a fixed data set. We analyzed its properties and showed that pDP is proportional to well-studied quantities, e.g., leverage scores, residual and pseudo-residual in statistics and statistical learning theory. This formalizes the intuitive idea that the more one can “blend into the crowd” like a chameleon, the more privacy one gets; and that the better a model fits the data, the easier it is to learn the model differentially privately. Moreover, the new notion allows us to conduct statistical learning and inference and take advantage of desirable structures of the data sets to gain orders-of-magnitude more favorable privacy guarantee than the worst case. This makes it highly practical in applications.

\(^2\)This comes from the iid noise with standard deviation proportional to \(\sqrt{d^2 \log(2/\delta)}/\epsilon\) added to \(X^T X\) to obtain \((\epsilon, \delta)\)-DP, which results in a random noise matrix \(E_2\) with average eigenvalue not smaller than \(d^2/5\). By Lemma 20, we see that the difference of the OLS solution on the noisy sufficient statistics \(\tilde{\theta}\) and that on the true sufficient statistics \(\hat{\theta}\) is \((X^T X + E_2)^{-1}(E_2 - E_1)\), which if we assume \(\|\theta\|_2\) and a well-conditioned \(X\), gives rise to an MSE on the order of \(\sigma^2 \text{tr}[(X^T X)^{-1}] + O(d^3/n^2\epsilon^2)\).
Specifically, we conducted a detailed case-study on linear regression to illustrate how pDP can be used. The pDP analysis allows us to identify and account for key properties of the data set, like the well-conditionedness of the feature matrix and the magnitude of the fitted coefficient vector, thereby provides strong uniform differential privacy coverage to everyone in the population whenever such structures exist. As a byproduct, the analysis also leads to an improved differential privacy guarantee for the OPS algorithm [7, 31] and also a new algorithm called AdaOPS that adaptively chooses the regularization parameters and improves the guarantee further. In particular, AdaOPS achieves asymptotic statistical efficiency and differential privacy at the same time with stronger parameters than known before.

The introduction of pDP also raises many open questions for future research. First of all, how do we tell individuals what their $\epsilon$s and $\delta$s of pDP are? This is tricky because the pDP loss itself is a function of the data, thus needs to be privatized against possible malicious dummy users. Secondly, the problem gets substantially more interesting when we start to consider the economics of private data collection. For instance, what happens if what we tell the individuals would affect their decision on whether they will participate in the data set? In fact, it is unclear how to provide an estimation of pDP in the first place if we are not sure what would the data be at the end of the day. Thirdly, from the data collector’s point of view, the data is going to be “easier” and the model will have a better “goodness-of-fit” on the collected data, but that will be falsely so to some extent, due to the bias incurred during data collection according to pDP. How do we correct for such bias and estimate the real performance of a model on the population of interest? Addressing these problems thoroughly would require the joint effort of the community and we hope the exposition in this paper will encourage researchers to play with pDP in both theory and practical applications.

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References

[1] Abadi, M., Chu, A., Goodfellow, I., McMahan, H. B., Mironov, I., Talwar, K., & Zhang, L. (2016). Deep learning with differential privacy. In Proceedings of the 2016 ACM SIGSAC Conference on Computer and Communications Security, (pp. 308–318). ACM.

[2] Barber, R. F., & Duchi, J. C. (2014). Privacy and statistical risk: Formalisms and minimax bounds. arXiv preprint arXiv:1412.4451.

[3] Blocki, J., Blum, A., Datta, A., & Sheffet, O. (2012). The johnson-lindenstrauss transform itself preserves differential privacy. In Foundations of Computer Science (FOCS), 2012 IEEE 53rd Annual Symposium on, (pp. 410–419). IEEE.

[4] Bun, M., & Steinke, T. (2016). Concentrated differential privacy: Simplifications, extensions, and lower bounds. In Theory of Cryptography Conference, (pp. 635–658). Springer.

[5] Chatterjee, S., & Hadi, A. S. (1986). Influential observations, high leverage points, and outliers in linear regression. Statistical Science, (pp. 379–393).
[6] Chaudhuri, K., Monteleoni, C., & Sarwate, A. D. (2011). Differentially private empirical risk minimization. The Journal of Machine Learning Research, 12, 1069–1109.

[7] Dimitrakakis, C., Nelson, B., Mitrokotsa, A., & Rubinstein, B. I. (2014). Robust and private bayesian inference. In Algorithmic Learning Theory, (pp. 291–305). Springer.

[8] Drineas, P., Magdon-Ismail, M., Mahoney, M. W., & Woodruff, D. P. (2012). Fast approximation of matrix coherence and statistical leverage. Journal of Machine Learning Research, 13(Dec), 3475–3506.

[9] Dwork, C., Kenthapadi, K., McSherry, F., Mironov, I., & Naor, M. (2006). Our data, ourselves: Privacy via distributed noise generation. In Annual International Conference on the Theory and Applications of Cryptographic Techniques, (pp. 486–503). Springer.

[10] Dwork, C., & Lei, J. (2009). Differential privacy and robust statistics. In Proceedings of the forty-first annual ACM symposium on Theory of computing, (pp. 371–380). ACM.

[11] Dwork, C., McSherry, F., Nissim, K., & Smith, A. (2006). Calibrating noise to sensitivity in private data analysis. In Theory of cryptography, (pp. 265–284). Springer.

[12] Dwork, C., & Roth, A. (2013). The algorithmic foundations of differential privacy. Theoretical Computer Science, 9(3-4), 211–407.

[13] Dwork, C., & Rothblum, G. N. (2016). Concentrated differential privacy. arXiv preprint arXiv:1603.01887.

[14] Dwork, C., & Smith, A. (2010). Differential privacy for statistics: What we know and what we want to learn. Journal of Privacy and Confidentiality, 1(2), 2.

[15] Ebadi, H., Sands, D., & Schneider, G. (2015). Differential privacy: Now it’s getting personal. In ACM SIGPLAN Notices, vol. 50, (pp. 69–81). ACM.

[16] Fienberg, S. E., Rinaldo, A., & Yang, X. (2010). Differential privacy and the risk-utility tradeoff for multi-dimensional contingency tables. In International Conference on Privacy in Statistical Databases, (pp. 187–199). Springer.

[17] Foulds, J., Geumlek, J., Welling, M., & Chaudhuri, K. (2016). On the theory and practice of privacy-preserving bayesian data analysis. In Proceedings of the Thirty-Second Conference on Uncertainty in Artificial Intelligence, (pp. 192–201). AUAI Press.

[18] Friedman, J., Hastie, T., & Tibshirani, R. (2001). The elements of statistical learning, vol. 1. Springer series in statistics Springer, Berlin.

[19] Kifer, D., Smith, A., & Thakurta, A. (2012). Private convex empirical risk minimization and high-dimensional regression. Journal of Machine Learning Research, 1, 41.

[20] Liu, Z., Wang, Y.-X., & Smola, A. (2015). Fast differentially private matrix factorization. In Proceedings of the 9th ACM Conference on Recommender Systems, (pp. 171–178). ACM.

[21] Mackey, L., Jordan, M. I., Chen, R. Y., Farrell, B., Tropp, J. A., et al. (2014). Matrix concentration inequalities via the method of exchangeable pairs. The Annals of Probability, 42(3), 906–945.
A Proofs of technical results

Proof of Proposition 9. We first show that implies on-average stability and then on-average stability implies on-average generalization.
Let $Z' = [Z, z']$, $Z'' = [Z, z'']$ and fix $z$. We first prove stability. Let $S = \theta | p(\theta) \geq p'(\theta)$

\[ |\mathbb{E}_{\theta \sim A(Z')} \ell(\theta, z) - \mathbb{E}_{\theta \sim A(Z'')} \ell(\theta, z)| \]

\[ = \sup_{\theta, z} \ell(\theta, z)[P_{Z'}(\theta \in S) - P_{Z''}(\theta \in S)] \]

\[ \leq e^{\epsilon(Z, z')} P_{Z'}(\theta \in S) + \delta((Z, z')) - P_{Z''}(\theta \in S) \]

\[ \leq (e^{\epsilon(Z, z')} + \epsilon(Z, z'')) - 1) + \delta(Z, z') + \epsilon(Z, z') \delta(Z, z'') \]

\[ \leq (e^{\epsilon(Z, z')} + \epsilon(Z, z'')) - 1) + \delta(Z, z') + \epsilon(Z, z') \delta(Z, z'') \]

Note that the bound is independent to $z$.

Now we will show stability implies generalization using a "ghost sample" trick in which we resample $Z' \sim \mathcal{D}^n$ and construct $Z^{(i)}$ by replacing the $i$th data point from the $i$th data point of $Z'$.

\[ |\mathbb{E}_{Z \sim \mathcal{D}^n} \left( \mathbb{E}_{z \sim \mathcal{D}} \ell(\theta, z) - \mathbb{E}_{\theta \sim A(Z)} \frac{1}{n} \sum_{i=1}^{n} \ell(\theta, z_i) \right) | \]

\[ = |\mathbb{E}_{Z \sim \mathcal{D}^n, \{z'_1, \ldots, z'_n\} \sim \mathcal{D}^n} \left( \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}_{\theta \sim A(Z)} \ell(\theta, z'_i) - \mathbb{E}_{\theta \sim A(Z^{(i)})} \ell(\theta, z'_i) \right) | \]

\[ \leq |\mathbb{E}_{Z \sim \mathcal{D}^n, \{z'_1, \ldots, z'_n\} \sim \mathcal{D}^n} \left( \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}_{\theta \sim A(Z)} \ell(\theta, z'_i) - \mathbb{E}_{\theta \sim A(Z^{(i)})} \ell(\theta, z'_i) \right) | \]

\[ \leq \mathbb{E}_{Z \sim \mathcal{D}^n} \left( \mathbb{E}_{z \sim \mathcal{D}} [e^{\epsilon(Z, z')} | Z] + \mathbb{E}_{Z \sim \mathcal{D}^n} \mathbb{E}_{z \sim \mathcal{D}} [e^{\epsilon(Z, z')} | Z] \mathbb{E}_{Z \sim \mathcal{D}} [\delta(Z, z) | Z] \right) \]

\[ \textbf{Proof of Proposition 11.} \quad \text{The stability argument remains the same, because it is applied to a fixed pair of (Z, z). We will modify the ghost sample arguments with an additional change of measure.} \]

\[ |\mathbb{E}_{Z \sim \mathcal{D}^n} \left( \mathbb{E}_{\theta \sim A(Z)} \ell(\theta, z) - \mathbb{E}_{\theta \sim A(Z)} \frac{1}{n} \sum_{i=1}^{n} \rho(z_i) \ell(\theta, z_i) \right) | \]

\[ = |\mathbb{E}_{Z \sim \mathcal{D}^n} \left( \mathbb{E}_{\theta \sim A(Z)} \mathbb{E}_{z \sim \mathcal{D}} \rho(z) \ell(\theta, z) - \mathbb{E}_{\theta \sim A(Z)} \frac{1}{n} \sum_{i=1}^{n} \rho(z_i) \ell(\theta, z_i) \right) | \]

\[ = |\mathbb{E}_{Z \sim \mathcal{D}^n, \{z'_1, \ldots, z'_n\} \sim \mathcal{D}^n} \left( \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}_{\theta \sim A(Z)} \rho(z'_i) \ell(\theta, z'_i) - \mathbb{E}_{\theta \sim A(Z^{(i)})} \rho(z'_i) \ell(\theta, z'_i) \right) | \]

\[ \leq \mathbb{E}_{Z \sim \mathcal{D}^n, \{z'_1, \ldots, z'_n\} \sim \mathcal{D}^n} \frac{1}{n} \sum_{i=1}^{n} \rho(z'_i) \left| \mathbb{E}_{\theta \sim A(Z)} \ell(\theta, z'_i) - \mathbb{E}_{\theta \sim A(Z^{(i)})} \ell(\theta, z'_i) \right| \]

\[ \leq \mathbb{E}_{Z \sim \mathcal{D}^n, \{z'_1, \ldots, z'_n\} \sim \mathcal{D}^n} \frac{1}{n} \sum_{i=1}^{n} \rho(z'_i) \left[ e^{\epsilon(Z, z')} + \epsilon(Z, z'') - 1 \right] + \delta(Z, z') + \epsilon(Z, z') \delta(Z, z'') \]

\[ = \mathbb{E}_{Z \sim \mathcal{D}^n, \{z'_1, \ldots, z'_n\} \sim \mathcal{D}^n} \left[ e^{\epsilon(Z, z')} + \epsilon(Z, z'') - 1 \right] + \delta(Z, z') + \epsilon(Z, z') \delta(Z, z''). \]
Proof of Corollary 12.

\[ \mathbb{E} \left[ \mathbb{E}_D[e^{\epsilon(Z,z)}|Z] \mathbb{E}_{D'}[e^{\epsilon(Z,z')}|Z'] \right] - 1 + \delta(1 + \mathbb{E}[e^{\epsilon(Z,z)}]) \leq \sqrt{\mathbb{E}_D e^{2\epsilon} \mathbb{E}_{D'} e^{2\epsilon}} - 1 + 2\delta. \]

The inequality uses Jensen’s inequality \( \mathbb{E} \left[ \mathbb{E}[e^{\epsilon(Z,z)}|Z] \right]^2 \leq \mathbb{E} e^{2\epsilon(Z,z)} \) and the monotonicity of moment generating function on non-negative random variables. The statement is obtained by Taylor’s series on \( \mathbb{E} e^{2\epsilon(Z,z)} \). Lastly, we use the algebraic mean to upper bound the geometric mean in the first term and then use Taylor expansion.

Proof of Lemma 14. By the stationarity of \( \hat{\theta} \)

\[ \sum_i \nabla \ell(\hat{\theta}, z_i) + \nabla r(\hat{\theta}) = 0 \]

Add and subtract \( \ell(\hat{\theta}, z) \) and apply first order Taylor’s Theorem centered at \( \hat{\theta}' \) on \( \sum_i \nabla \ell(\hat{\theta}, z_i) + \nabla \ell(\hat{\theta}', z_i) + \nabla r(\hat{\theta}') \), we get

\[ \sum_i \nabla \ell(\hat{\theta}', z_i) + \nabla r(\hat{\theta}') + R - \nabla \ell(\hat{\theta}, z) = 0. \]

where if we define \( \eta_t = (1-t)\hat{\theta}' + t\hat{\theta} \), the remainder term \( R \in \mathbb{R}^d \) can be explicitly written as

\[ R = \left[ \int_0^1 \left( \sum_i \nabla^2 \ell(\eta_t, z_i) + \nabla^2 \ell(\eta_t, z) + \nabla^2 r(\eta_t) \right) dt \right] (\hat{\theta} - \hat{\theta}'). \]

By the mean value theorem for Fréchet differentiable functions, we know there is a \( t \) such that we can take \( \eta_t \) such that the integrand is equal to the integral.

Since \( \hat{\theta}' \) is a stationary point, we have

\[ \sum_i \nabla \ell(\hat{\theta}', z_i) + \nabla r(\hat{\theta}') = 0 \]

and thus under the assumption that \( \left[ \int_0^1 \left( \sum_i \nabla^2 \ell(\eta_t, z_i) + \nabla^2 \ell(\eta_t, z) + \nabla^2 r(\eta_t) \right) dt \right]^{-1} \) is invertible, we have

\[ \hat{\theta} - \hat{\theta}' = \left[ \int_0^1 \left( \sum_i \nabla^2 \ell(\eta_t, z_i) + \nabla^2 \ell(\eta_t, z) + \nabla^2 r(\eta_t) \right) dt \right]^{-1} \nabla \ell(\hat{\theta}, z). \]

The other equality follows by symmetry.

Proof of Theorem 15. Let \( X' = [X; x], y' = [y; y] \). Denote \( H := X'^T X + \lambda I, H' := (X')^T X' + \lambda I, g := X'^T y \) and \( g' := (X')^T y' \). Correspondingly, the posterior mean \( \hat{\theta} = H^{-1} g \) and \( \hat{\theta}' = [H']^{-1} g' \).
The covariance matrix of the two distributions are \( H/\gamma \) and \( H'/\gamma \). Using the fact that the normalization constant is known for Gaussian, the log-likelihood ratio at output \( \theta \) is

\[
\log \frac{|H^{-1}|^{-1/2} e^{-\frac{1}{2} ||\theta - \hat{\theta}||^2_H}}{|[H']^{-1}|^{-1/2} e^{-\frac{1}{2} ||\theta - \hat{\theta}'||^2_{H'}}} = \log \sqrt{\frac{|H|}{|H'|}} + \frac{\gamma}{2} \left[ ||\theta - \hat{\theta}'||^2_{H'} - ||\theta - \hat{\theta}||^2_H \right].
\]

Note that \( H' = H + xx^T \). By Lemma 22,

\[
\frac{|H|}{|H'|} = \frac{|H|}{|H|(1 + \mu)} = \frac{|H'|(1 - \mu')}{|H'|},
\]

so

\[
(\#) = \log (1 + \mu)^{-1} = \log 1 - \mu'.
\]

The second term in the above equation can be expanded into

\[
(\ast) = \theta^T[H' - H]\theta + (\hat{\theta}')^T H' \hat{\theta}' - \hat{\theta}^T H \hat{\theta} - 2(\hat{\theta}')^T H \theta + 2\hat{\theta}^T H \theta
\]

\[
= (x^T \theta)^2 + (y^T)^T X'[H']^{-1} X^T y' - y^T X(H)^{-1} X^T y - 2y(x^T \theta) \tag{9}
\]

\[
(\ast\ast) = [(y^T)^T X'[X'^T X' + \lambda I]^{-1} X^T y' - y^T X(X^T X + \lambda I)^{-1} X^T y] = [(y^T)^T \Pi y' - y^T \Pi y],
\]

where we denote the "hat" matrices \( \Pi := X(X^T X + \lambda I)^{-1} X^T \) and \( \Pi' = X'(X'^T X' + \lambda I)^{-1} (X')^T \). Also define \( v := X(X^T X + \lambda I)^{-1} x \). By Sherman-Morrison-Woodbury formula, we can write

\[
\Pi' = \begin{bmatrix} X \\ x^T \end{bmatrix} \left[ H^{-1} - H^{-1} x(1 + \mu)^{-1} x^T H^{-1} \right] \begin{bmatrix} X^T \\ x \end{bmatrix}
\]

\[
= \begin{bmatrix} \Pi - (1 + \mu)^{-1} v v^T, v - \mu(1 + \mu)^{-1} v \\ (1 + \mu)^{-1} v, \mu - \mu^2(1 + \mu)^{-1} \end{bmatrix}
\]

Note that \( v^T y = x^T \hat{\theta} \) and \( 1 - \mu(1 + \mu)^{-1} = (1 + \mu)^{-1} \), therefore

\[
(\ast\ast) = -(1 + \mu)^{-1} (x^T \hat{\theta})^2 + 2(1 + \mu)^{-1} x^T \hat{\theta} + \mu(1 + \mu)^{-1} y^2
\]

\[
= -(1 + \mu)^{-1} (y - x^T \hat{\theta})^2 + y^2.
\]

Substitute into (9), we get

\[
(\ast) = (y - x^T \theta)^2 - (1 + \mu)^{-1} (y - x^T \hat{\theta})^2.
\]

And the log-probability ratio is

\[
\log \frac{p(\theta | X, y)}{p(\theta | X', y')} = \log \sqrt{1 + \mu}^{-1} + \frac{\gamma}{2} \left[ (y - x^T \theta)^2 - (1 + \mu)^{-1} (y - x^T \hat{\theta})^2 \right]
\]

\[
= \log \sqrt{1 + \mu}^{-1} + \frac{\gamma}{2} \left[ (x^T \theta)^2 - 2(x^T \theta)(x^T \hat{\theta} - x^T \theta) + \frac{\mu}{1 + \mu} (y - x^T \hat{\theta})^2 \right]
\]

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Under the distribution of $\theta$ when the data is $(X, y)$, $x^T \theta - x^T \hat{\theta}$ follows a univariate normal distribution with mean 0 and variance $\mu/\gamma$. By the standard tail probability of normal random variable,

$$
P \left( \left| x^T \theta - x^T \hat{\theta} \right| > \sqrt{\frac{\mu}{\gamma} \log(2/\delta)} \right) \leq 2e^{-\log(2/\delta)} = \frac{\delta}{\log(2/\delta)} \leq \delta.
$$

we can calculate $(\epsilon, \delta)$-pDP for every $\delta > 0$. In particular, under $p(\theta|X, y)$

$$
P \left( \left| \log \frac{p(\theta|X, y)}{p(\theta|X', y')} \right| \geq \epsilon \right) < \delta
$$

for

$$
\epsilon = \frac{1}{2} \left| - \log(1 + \mu) + \frac{\mu \gamma}{(1 + \mu)} (y - x^T \hat{\theta})^2 \right| + \frac{\mu}{2} \log(2/\delta) + |y - x^T \hat{\theta}| \sqrt{\mu \gamma \log(2/\delta)}.
$$

By Lemma 25 this implies $(\epsilon, \delta)$-DP.

Now, we will work out an equivalent representation of the log-probability ratio that depends on $\hat{\theta}'$. Let $\mu'$ be the in-sample leverage score of $x$ with respect to $X'$, namely, $\mu' := x^T [H']^{-1} x$. By Sherman-Morrison-Woodbury formula

$$
H^{-1} = [H' - x x^T]^{-1} = [H']^{-1} + [H']^{-1} x (1 - \mu')^{-1} x^T [H']^{-1}.
$$

(10)

Standard matrix algebra gives us

$$
y^T \Pi y = (y')^T X' (X')^{-1} y' = y x^T [H']^{-1} y - 2 y x^T H^{-1} x^T y
$$

$$
= (y')^T X' (X')^{-1} y' - 2 y x^T [H']^{-1} X^T y + y x^T H^{-1} x y.
$$

Substitute (10) into the above, we get

$$
y^T \Pi y = (y')^T \Pi y + (1 - \mu')^{-1} (x^T \hat{\theta}')^2 - 2 y x^T \hat{\theta}' [1 + \mu' (1 - \mu')^{-1}] + y^2 \mu' + y^2 (\mu')^2 (1 - \mu')^{-1}
$$

$$
= (y')^T \Pi y + (1 - \mu')^{-1} (x^T \hat{\theta}')^2 - 2 y x^T \hat{\theta}' (1 - \mu')^{-1} + y^2 (1 - \mu')^{-1} - y^2.
$$

Therefore,

$$
(\star) = -(y - x^T \hat{\theta}')^2 (1 - \mu')^{-1} + y^2,
$$

and

$$
(\star\star) = (y - x^T \hat{\theta}')^2 - (1 - \mu')^{-1} (y - x^T \hat{\theta}')^2.
$$

The corresponding log-probability ratio

$$
\log \frac{p(\theta|X, y)}{p(\theta|X', y')} = - \log(\sqrt{1 - \mu'}) + \frac{\gamma}{2} \left[ (y - x^T \theta)^2 - (1 - \mu')^{-1} (y - x^T \hat{\theta}')^2 \right]
$$

$$
= - \log(\sqrt{1 - \mu'}) + \frac{\gamma}{2} \left[ (x^T \hat{\theta}' - x^T \theta)^2 + 2 (x^T \hat{\theta}' - x^T \theta) (y - x^T \hat{\theta}') - \frac{\mu'}{1 - \mu'} (y - x^T \hat{\theta}')^2 \right]
$$

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Under the posterior distribution of \((X', y')\), the mean of \(x^T \theta\) is centered at \(x^T \hat{\theta}'\) with variance \(\mu' / \gamma\). We can then derive a tail bound of the privacy loss random variable and it implies an \((\epsilon, \delta) - \text{pDP}\) guarantee by Lemma 25. Specifically, it implies that the method is \((\epsilon, \delta) - \text{pDP}\) with

\[
\epsilon = \frac{1}{2} \left\lvert \log(1 - \mu') - \frac{\gamma \mu'}{1 - \mu'} (y - x^T \hat{\theta}')^2 \right\rvert + \frac{\mu'}{2} \log(2/\delta) + \sqrt{\gamma \mu' \log(2/\delta)} |y - x^T \hat{\theta}'|.
\]

This complete the second statement of the proof.

Proof of Proposition 17. The proof mostly involves applying Theorem 15 and substituting bounds over either a bounded domain assumption (typical for DP analysis), or a model assumption of how data are generated (typical for statistical analysis).

Proof of Statement 1 in the agnostic setting. For any \(x \in X\), and any data set \(X\), using the choice of regularization term, we can bound \(\mu = 1 / \lambda_n\). Substitute that into Theorem 15, and use the inequality that \(\log(1 + x) \leq x\) we get the first expression.

Now, restricting ourselves to the bounded domain. Under the choice of \(\lambda_n\), we can choose an \(X, y\) with a singular value equal to \(\sqrt{\lambda_n}\) and the corresponding singular vector \(v \in \{-1, 1\}^n\) such that the following upper bounds are attained

\[
\| (X^T X + \lambda_n I)^{-1} X^T \| \leq \frac{1}{2\sqrt{\lambda_n}}.
\]

\[
\| \hat{\theta} \| \leq \| (X^T X + \lambda_n I)^{-1} X^T \| \| y \| \leq \frac{\sqrt{n}}{2\sqrt{\lambda_n}}.
\]

Now choose \((x, y)\) such that \(x^T \hat{\theta} = \|x\| \| \hat{\theta} \|\), we get that \(\sup_{(x, y), (x, y)} |y - \hat{\theta}^T x| = 1 + \frac{\sqrt{n}}{2\sqrt{\lambda_n}}\). The DP claim follows by substituting the upper bound into the pDP’s expression.

Proof of Statement 2 under the model assumption. To prove the second claim, note that by Assumption (b)(d), the smallest eigenvalue of \(X^T X\) is lower bounded by \(d/nm\). Also under the model assumption, the ridge regression estimator concentrates around \(\theta_0\).

In particular, under the model assumption, the ridge regression estimate

\[
\hat{\theta} = (X^T X + \lambda_n I)^{-1} X^T y = (X^T X + \lambda_n I)^{-1} X^T X \theta_0 + (X^T X + \lambda_n I)^{-1} X^T Z \quad (11)
\]

With high probability over the distribution of \(Z\)

\[
\| \hat{\theta} - \theta_0 \|^2 = O\left(\frac{d\sigma^2 \log(n)}{n} + \frac{\lambda_n^2 d^2 \| \theta_0 \|^2}{n^2}\right),
\]

thus for all \((x, y)\) satisfying \(\|x\| \leq 1 \ y \leq 1\), we get

\[
|y - x^T \hat{\theta}| \leq |y - x^T \theta_0| + |x^T (\hat{\theta} - \theta_0)| = O(1 + \| \theta_0 \|).
\]

Under the assumption that \(n > 10d \log n\), \(\| \theta_0 \| = O(1)\) and \(\sigma = O(1)\) this is effectively a constant.
For \( x \in \mathcal{X} \) and \( y \sim \mathcal{N}(\theta_0^T x, \sigma^2) \), using standard Gaussian tail bound, with high probability the perturbation is bounded, therefore \( |y - x^T \theta_0|^2 \leq \sigma^2 \log(2/\delta') \).

Lastly, for the case of the average pDP loss over the empirical data distribution. Besides taking into the above bound on \( |y - x^T \hat{\theta}| \), we further consider adding the different parts over the distributions. Since this is to deal with data points in the data set, we will instantiate the bound (6). Our assumption on \( \gamma_n, \lambda_n \) ensures that the dominant term is the third term, thus

\[
\frac{1}{n} \sum_{i=1}^{n} \epsilon_i((X, t), (x_i, y_i))^2 \leq C \gamma_n \frac{1}{n} \sum_{i=1}^{n} (y_i - x_i^T \hat{\theta})^2 x_i^T (X^T X + \lambda_n I)^{-1} x_i.
\]

Under the high probability event that the noise is bounded by \( \sigma \sqrt{2/\delta'} \) for all data points, we can extract them out then note that

\[
\frac{1}{n} \sum_{i=1}^{n} x_i^T (X^T X)^{-1} x_i = \frac{1}{n} \text{tr}(\sum_{i=1}^{n} x_i x_i^T (X^T X + \lambda_n I)) \leq \frac{1}{n} \text{tr}(I) = \frac{d}{n}.
\]

Substitute these bounds into Theorem 15, and we obtain the Statement 2.

**Proof of Statement 3 under the model assumption.** By (11) and the fact that OPS can be thought of as adding an independent multivariate Gaussian noise with covariance matrix \((X^T X + \lambda_n I)^{-1} X^T X (X^T X + \lambda_n I)^{-1} / \gamma_n \), we get

\[
\hat{\theta} = (X^T X + \lambda_n I)^{-1} X^T X \theta_0 + \sqrt{1 + \gamma_n} (X^T X + \lambda_n I)^{-1} X^T Z.
\]

By a bias-variance decomposition, we get

\[
\mathbb{E}(\|\hat{\theta} - \theta_0\|^2 | X) = \text{Var}(\hat{\theta} | X) + \mathbb{E}(\|\hat{\theta} - \theta_0\|^2)
\]

\[
= (1 + \gamma_n^{-1}) \sigma^2 \text{tr} \left[ (X^T X + \lambda_n I)^{-1} X^T X (X^T X + \lambda_n I)^{-1} \right] + \|I - (X^T X + \lambda_n I)^{-1} X^T X\| \theta_0 \|^2
\]

\[
= (1 + \gamma_n^{-1}) \sigma^2 \sum_{i=1}^{d} \frac{\sigma_i^2}{\sigma_i^2 + \lambda_n^2} + \lambda_n^2 \theta_0^T (X^T X + \lambda_n I)^{-2} \theta_0
\]

\[
\leq (1 + \gamma_n^{-1}) \sigma^2 \text{tr}(X^T X + \lambda_n I)^{-1} + \lambda_n^2 m^{-2} \|\theta_0\|^2
\]

The proof is complete by substitute the values of \( \gamma_n \) and \( \lambda_n \) into the inequality and noting that \( m = \Omega(1) \) and under the model assumption \( \|\theta_0\| \) does not grow with \( n \). Clearly, if \( \gamma_n = \omega(1) \) and \( \lambda_n = o(\sqrt{n}) \), then the algorithm is asymptotically efficient.

**Proof of Proposition 18.** We will first prove the claim on differential privacy and then analyze the statistical efficiency.

**Proof of differential privacy.** First of all, by Weyl’s theorem, and the assumption that \( \|xx^T\|_2 \leq 1 \), we get that the global sensitivity of \( \lambda_{\min}(X^T X) \) is 1. We will use \( \lambda_{\min} \) as the short hand of
\[ \lambda_{\min}(X^TX) \in \text{the rest of the proof}. \] So releasing \( \tilde{\lambda}_{\min} \) is \((\epsilon/2, \delta/2)\)-DP using the standard Gaussian mechanism. Secondly, under the same event with probability at least \(1 - \delta/2\), we have

\[ \lambda_{\min} - \frac{\log(4/\delta)}{\epsilon} \leq \tilde{\lambda}_{\min} \leq \lambda_{\min} + \frac{\log(4/\delta)}{\epsilon}. \]

Therefore, by our selection rule of the regularization parameter \( \lambda_n \),

\[ \frac{n}{d} \leq \lambda_{\min}(X^TX + \lambda_n I) \leq \max\{\lambda_{\min}, \frac{n}{d} + \frac{\log(4/\delta)}{\epsilon}\}. \]

The lower bound implies that for any \((x, y)\) satisfying the condition, the out of sample leverage score

\[ \mu = x^T(X^TX + \lambda_n I)^{-1}x \leq \frac{\kappa d}{n}. \tag{12} \]

It also implies an upper bound on the prediction error:

\[ |y - x^T \hat{\theta}| \leq 1 + \|\hat{\theta}\| \leq 1 + \|(X^TX + \lambda_n I)^{-1}X^T\|_2\|y\|_2 \leq \min \sqrt{2\kappa d}. \tag{13} \]

We will prove the final inequality above using the following lemma with \( h = \frac{n}{d}\kappa \) and then note that \(\|y\|_2 \leq \sqrt{n} \).

**Lemma 19.** For any matrix \( X \), and any \( \lambda \geq 0 \). If \( \lambda_{\min}(X^TX + \lambda I) \geq h \), then

\[ \| (X^TX + \lambda I)^{-1}X^T \| \leq \sqrt{2/h}. \]

The proof is technical so we defer it to later.

Now combine (12)(13) with Theorem 15, we get that the OPS step which obeys \((\tilde{\epsilon}, \delta/2)\)-pDP with

\[ \tilde{\epsilon}((X, y), (x, y)) \leq \frac{\mu}{2}(1 + \log(4/\delta)) + \frac{1}{2} \gamma_n \min(\mu, 1)\|y - x^T \hat{\theta}\|^2 + \sqrt{\gamma \mu \log(4/\delta)}\|y - x^T \hat{\theta}\| \]

\[ \leq \frac{\kappa d(1 + \log(4/\delta))}{2} + \frac{\gamma n \kappa d}{2n^2} + \sqrt{\frac{\gamma n \kappa d}{2}} \frac{\log(4/\delta)}{n} \]

\[ \leq \epsilon/8 + \epsilon/8 + \epsilon/4 \leq \epsilon/2 \]

Note that in the last step, we made use of the choice of \( \gamma_n \) and the condition that concerns \( \epsilon \) and \( \kappa \) as stated in the algorithm. Since this upper bound holds for all data set \((X, y)\) and all privacy target \((x, y)\). The OPS algorithm also satisfies \((\epsilon/2, \delta/2) - DP\).

The proof of the first claim is complete when we compose the two data access.

**Proof of the statistical efficiency.** Now we switch gear to analyze the estimation error bound.

Let event \( E \) be the event that \( \tilde{\lambda}_{\min} > \lambda_{\min} - \sqrt{10 \log(n) \log(4/\delta)} \), which happens with probability \(1 - n^{-10}\). Under \( E \), we have \( \lambda_0 = 0 \). By our assumption, this happens with

Applying the third claim in Proposition 17, we get that

\[ \mathbb{E}[\| \tilde{\theta} - \theta_0 \|^2 | X, E] \leq (1 + \gamma_n^{-1})\sigma^2 \text{tr}((X^TX)^{-1}). \]
Under the small probability event $E^c$, we use a crude upper bound that takes the sum of the maximum square bias and maximum variance.

$$\mathbb{E}[\|\hat{\theta} - \theta_0\|^2 | X, E^c] \leq (1 + \gamma_n^{-1})\sigma^2 \text{tr}((X^TX)^{-1}) + \|\theta_0\|^2$$

by law of total expectation, for an event $E$

$$\mathbb{E}[\|\hat{\theta} - \theta_0\|^2 | X] = \mathbb{E}[\|\hat{\theta} - \theta_0\|^2 | X, E] \mathbb{P}(E | X) + \mathbb{E}[\|\hat{\theta} - \theta_0\|^2 | X, E^c] \mathbb{P}(E^c | X)$$

$$\leq (1 + \gamma_n^{-1})\sigma^2 \text{tr}((X^TX)^{-1}) + \mathbb{P}(E^c)\|\theta_0\|^2 = (1 + \gamma_n^{-1})\sigma^2 \text{tr}((X^TX)^{-1}) + O(n^{-10}).$$

The proof is complete by substituting $\gamma_n$ into the bound.

**Proof of Lemma 19.** Take SVD of $X = U\Sigma V^T$, we can write

$$\|((X^TX + \lambda_n^2 I)^{-1}X^T\|_2 = \max_{i \in [d]} \frac{\Sigma_{ii}}{\Sigma_{ii}^2 + \lambda_n}$$

We now discuss two cases. First, for those $i \in [d]$ such that $\Sigma_{ii}^2 \leq \lambda_n$. In this case, adding $\lambda_n$ on both sides ensures that

$$h \leq \lambda_{\min}(X^TX + \lambda_n) = \lambda_{\min} + \lambda_n \leq \Sigma_{ii}^2 + \lambda_n \leq 2\lambda_n.$$

and therefore if $\Sigma_{ii} > 0$

$$\frac{\Sigma_{ii}}{\Sigma_{ii}^2 + \lambda_n} = \frac{1}{\Sigma_{ii} + \lambda_n/\Sigma_{ii}} \leq \frac{1}{\sqrt{\lambda_n}} \leq \sqrt{1/(2h)}.$$

The final inequality is also true for $\Sigma_{ii} = 0$. If on the other hand, for those $i \in [d]$ such that, $\Sigma_{ii}^2 > \lambda_n$. This time by adding $\Sigma_{ii}^2$ on both sides, we get

$$2\Sigma_{ii}^2 > \lambda_n + \Sigma_{ii}^2 \geq \lambda_n + \lambda_{\min} = \lambda_{\min}(X^TX + \lambda_n) \geq \frac{n}{\kappa}.$$ This implies that

$$\frac{\Sigma_{ii}}{\Sigma_{ii}^2 + \lambda_n} \leq \frac{1}{\Sigma_{ii}} \leq \sqrt{2/h}.$$ (15)

Combine (14) and (15) we get

$$\|(X^TX + \lambda_n I)^{-1}X^T\|_2 \leq \sqrt{2/h}$$

**B Technical lemmas**

**Lemma 20.** Let $\hat{\theta}' = (X^TX + E_1)^{-1}(Xy + E_2)$ for any matrix $E_1$, $E_2$. 

$$\hat{\theta}' - \hat{\theta} = (X^TX + E_1)^{-1}(E_2 - E_1\hat{\theta})$$
Proof.
\[
\hat{\theta}' = (X^T X + E_1)^{-1}(X^T y + E_2)
= (X^T X + E_1)^{-1}(X^T X)(X^T X)^{-1}X^T y + (X^T X + E_1)^{-1}E_2
= \hat{\theta} + [(X^T X + E_1)^{-1}(X^T X + E_1) - (X^T X + E_1)^{-1}E_1 - I_d] \hat{\theta} + (X^T X + E_1)^{-1}E_2
= \hat{\theta} - (X^T X + E_1)^{-1}E_1 \hat{\theta} + (X^T X + E_1)^{-1}E_2
= \hat{\theta} + (X^T X + E_1)^{-1}(E_2 - E_1 \hat{\theta})
\]

Lemma 21 (Sherman-Morrison-Woodbury Formula). Let \(A, U, C, V\) be matrices of compatible size, assume \(A, C\) and \(C^{-1} + VA^{-1}U\) are all invertible, then
\[(A + UCV)^{-1} = A^{-1} - A^{-1}U(C^{-1} + VA^{-1}U)^{-1}A^{-1}.
\]

Lemma 22 (Determinant of Rank-1 perturbation). For invertible matrix \(A\) and vector \(c, d\) of compatible dimension
\[
\det(A + cd^T) = \det(A)(1 + d^T A^{-1}c).
\]

Lemma 23 (Weyl’s eigenvalue bound [30, Theorem 1]). Let \(X, Y, E \in \mathbb{R}^{m \times n}\), w.l.o.g., \(m \geq n\). If \(X - Y = E\), then \(|\sigma_i(X) - \sigma_i(Y)| \leq \|E\|_2\) for all \(i = 1, ..., n\).

Lemma 24 (Gaussian tail bound). Let \(X \sim \mathcal{N}(0, 1)\). Then
\[
\mathbb{P}(|X| > \epsilon) \leq \frac{2e^{-\epsilon^2/2}}{\epsilon}.
\]

Lemma 25 (Tail bound to \((\epsilon, \delta)\)-DP conversion). Let \(\epsilon(\theta) = \log\left(\frac{p(\theta)}{p'(\theta)}\right)\) where \(p\) and \(p'\) are densities of \(\theta\). If
\[
\mathbb{P}(|\epsilon(\theta)| > t) \leq \delta
\]
then for any measurable set \(S\)
\[
\mathbb{P}_p(\theta \in S) \leq e^t \mathbb{P}_{p'}(\theta \in S) + \delta.
\]
and
\[
\mathbb{P}_{p'}(\theta \in S) \leq e^t \mathbb{P}_p(\theta \in S) + \delta
\]

Proof. Since \(\log\left(\frac{p(\theta)}{p'(\theta)}\right) = -\log\left(\frac{p'(\theta)}{p(\theta)}\right)\) and the tail bound is two-sided. It suffices for us to prove just one direction. Let \(E\) be the event that \(|\epsilon(\theta)| > t\).
\[
\mathbb{P}_p(\theta \in S) = \mathbb{P}_p(\theta \in S \cup E^c) + \mathbb{P}_p(\theta \in S \cup E) \leq \mathbb{P}_{p'}(\theta \in S \cup E)e^t + \mathbb{P}_p(\theta \in E) \leq e^t \mathbb{P}_{p'}(\theta \in S) + \delta.
\]

Lemma 26 (Matrix Hoeffding inequality [21]). Consider a finite sequence \(X_1, ..., X_n\) of independent random and self-adjoint matrices with dimension \(d\) and \(A_1, ..., A_n\) be a sequence of fixed self-adjoint matrices. In addition, let \(\mathbb{E}X_i = 0\) and \(X_i^2 \leq A_i^2\) almost surely for all \(i = 1, ..., n\). Then, for all \(t \geq 0\)
\[
\mathbb{P}\left\{\lambda_{\max}\left(\sum_{i=1}^n X_i\right) \geq t\right\} \leq de^{-t^2/2\sigma^2}
\]
where \(\sigma^2 = \|\sum_{i=1}^n A_i^2\|\).