Transformation and summation formulas for Kampé de Fériet series $F_{1:1}^{0:3}(1, 1)$

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Abstract

The double hypergeometric Kampé de Fériet series $F_{1:1}^{0:3}(1, 1)$ depends upon 9 complex parameters. We present three cases with 2 relations between those 9 parameters, and show that under these circumstances $F_{1:1}^{0:3}(1, 1)$ can be written as a $4 \, F_3(1)$ series. Some limiting cases of these transformation formulas give rise to new summation results for special $F_{1:1}^{0:3}(1, 1)$’s. The actual transformation results arose out of the study of 9-$j$ coefficients.

1 Introduction

The results given in this paper are basically mathematical but the subject area where they have arisen is physics. Therefore we shall devote this introduction to giving some relevant references and to describing the field in which these results have appeared naturally.

We shall present a number of hypergeometric series transformation and summation formulas which were discovered when systematically studying the 9-$j$ coefficient. These coefficients, depending upon 9 integer or half-integer parameters and usually denoted by

$$\begin{bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{34} \\ j_{13} & j_{24} & J \end{bmatrix}$$

appear in the quantum mechanical treatment of angular momentum in physics. The 9-$j$ coefficient can be considered as a transformation coefficient connecting two different ways in which four angular momenta $j_1, j_2, j_3, j_4$ can be coupled: either $j_1$ and $j_2$ to $j_{12}$, $j_3$ and $j_4$ to $j_{34}$, and $j_{12}$ with $j_{34}$ to $J$; or else $j_1$ and $j_3$ to $j_{13}$, $j_2$ and $j_4$ to $j_{24}$, and $j_{13}$ with $j_{24}$ to $J$. By identifying the representation theory of angular momentum with the representation theory of the Lie group SU(2), one can also interpret the 9-$j$ coefficient as the transformation coefficient relating irreducible constituents of the SU(2) tensor products $(V_1 \otimes V_2) \otimes (V_3 \otimes V_4)$ and $(V_1 \otimes V_3) \otimes (V_2 \otimes V_4)$.

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From equations expressing the $9$-$j$ coefficient in terms of $3$-$(n,m)$ coefficients (cfr. [3]), it follows that the $9$-$j$ coefficient has 72 symmetries [13, 4]: up to a sign, the array [5] is invariant for row and column permutations and for transposition. On the other hand, there exists an alternative expression of the $9$-j coefficient: this is the triple sum series expression obtained by Ališauskas and Jucys [1] and independently proved in [9]. This triple sum series is further discussed in Ref. [16]. In Ref. [17], a certain “doubly stretched” $9$-j coefficient for which a single term expression exists was considered and it was shown how the symmetries which do not yield a single term can give rise to single, double and triple hypergeometric summation formulas.

Presently, we have investigated those symmetries of doubly stretched $9$-j coefficients which give rise to double sums (the reason for considering doubly stretched coefficients is that singly stretched or unstretched coefficients would be too general to lead to summation or transformation results). The basic formulas for doubly stretched $9$-j coefficients can be found in Refs. [13] and [18]. The clue for finding new transformation or summation formulas lies in comparing the expressions to which the triple sum series reduces for two different symmetries. In the following sections we shall present and prove the three types of special Kampé de Fériet series transformation formulas without further reference to the background of $9$-j coefficients.

2 Transformation formulas

The Kampé de Fériet function $F_{1;1}^{0;3}$, a generalization of Appell series [2], is defined as follows [15]:

$$F(x,y) = F_{1;1}^{0;3} \left[ \begin{array}{c} a, b, c, a', b', c' \\ \frac{d}{e} \end{array} ; x, y \right] = \sum_{m,n=0}^{\infty} \frac{(a,b,c)_m (a',b',c')_n x^m y^n}{(d)_m (d+e)_n (e)_n} \frac{1}{m!n!}, \quad (2)$$

where the notation is as in Ref. [14]: $(a)_m$ is a Pochammer symbol, and $(a,b,c)_m = (a)_m (b)_m (c)_m$. The region of convergence is given by $|x| < 1$ and $|y| < 1$, and the series is absolutely convergent for $|x| = |y| = 1$ provided [3]

$$\Re(d + e - a - b - c) > 0 \quad \text{and} \quad \Re(d + e' - a' - b' - c') > 0. \quad (3)$$

As usual, it is understood that no denominator parameters are zero or negative integers.

The series $F(1,1)$ depends upon 9 parameters $a, b, \ldots, e'$. We present three transformation formulas with 7 free parameters, i.e. when 2 relations hold among the 9 parameters. The first of our formulas is when $a' = d - a$ and $e' = d + e - a - b - c$; then we find:

$$F(1,1) = \Gamma \left[ \begin{array}{c} d, d + e' - a' - b' - c' \\ a', d + e' - a' - b', d + e' - a' - c' \end{array} \right] \times _4F_3 \left[ \begin{array}{c} a, e - b, e - c, d + e' - a' - b' - c' \\ e, d + e' - a' - b', d + e' - a' - c' \end{array} \right]. \quad (4)$$

Herein, $\Gamma$ is the classical Gamma function with the convention

$$\Gamma \left[ \begin{array}{c} a_1, a_2, \ldots \\ b_1, b_2, \ldots \end{array} \right] = \frac{\Gamma(a_1)\Gamma(a_2)\ldots}{\Gamma(b_1)\Gamma(b_2)\ldots}.$$
and \(\text{$_4F_3$}\) is a generalized hypergeometric function [3 4]. Eq. (4) holds when (3) and \(\Re(a') > 0\) are satisfied.

The second formula is for \(a' = d - a\) and \(d + e - a - b - c = 1\). Now we find:

\[
F(1,1) = \Gamma \left[ \begin{array}{c} e, e - a - b, e - a - c, e - b - c, e', e' - b' - c' \\ e - a, e - b, e - c, e - a - b - c, e' - b', e' - c' \\ \end{array} \right] \\
\times \text{$_4F_3$} \left[ \begin{array}{c} a, b, c - d - b - c \\ d - b, d - c, 1 + b' + e' - e' \\ \end{array} \right].
\]

This formula is shown to be valid under one of the following conditions:

(i) \(a\) is a negative integer and the second inequality of (3) is satisfied;

(ii) \(c\) and \(c'\) are negative integers.

The third and last of our transformation formulas is for \(a' = d - a\) and \(b' = d - b\); moreover we assume that \(a'\) or \(b'\) is a negative integer. Then:

\[
F(1,1) = \Gamma \left[ \begin{array}{c} d, e, a + b + d, d + a - b - c \\ a, b, e - c, d + e - a - b \\ \end{array} \right] \\
\times \text{$_4F_3$} \left[ \begin{array}{c} a', b', c' - d - c \dfrac{e' + d - a - b - c}{e' + 1 + d - a - b} \\ \end{array} \right].
\]

In this case, the termination of the \(\text{$_4F_3$}\) series implies that conditions (3) are sufficient.

From the three transformation formulas (1)-(3) we believe that the first two are new. The third is not new; in fact it is the terminating form of a formula given by Karlsson [11] in his proof of one of our earlier double hypergeometric summation formulas [7] (see eq. (23) of this paper). Karlsson’s formula holds for \(a' = d - a\), \(b' = d - b\), without the assumption that \(a'\) or \(b'\) should be negative integral. It reads:

\[
F(1,1) = \Gamma \left[ \begin{array}{c} d, e, a + b + d, d + e - a - b - c \\ a', b', c' - b - c \dfrac{e' + d - a - b - c}{e' + 1 + d - a - b} \\ \end{array} \right] \\
\times \text{$_4F_3$} \left[ \begin{array}{c} a, b, e - c, d + e - a - b \\ e, 1 + d - a' - b', d + e' - a' - b' \\ \end{array} \right] \\
+ \Gamma \left[ \begin{array}{c} d, e, a + b + d, d + a - b - c \\ a, b, e - c, d + e - a - b \\ \end{array} \right] \\
\times \text{$_4F_3$} \left[ \begin{array}{c} a', b', c' - d - c \dfrac{e' + d - a - b - c}{e' + 1 + d - a - b} \\ \end{array} \right] + \cdots.
\]

and holds under the extra condition \(\Re(1 - d + c + c') > 0\).

So there remain (4) and (5) to prove. Our proof of (4) is inspired by Gasper’s proof [8] of the \(q\)-analogue of one of our earlier results (eq. (40) of Ref. [17]), and uses the Beta function:

\[
B(x, y) = \Gamma \left[ \begin{array}{c} x+y \\ x+y \end{array} \right] = \int_0^1 t^{x-1} (1 - t)^{y-1} dt, \quad \Re(x) > 0, \Re(y) > 0.
\]

Considering \(B(x + m, y + n)\) and using \(\Gamma(x + m) = (x)_m \Gamma(x)\), one finds

\[
\frac{(x)_m (y)_n}{(x + y)_{m+n}} = \Gamma \left[ \begin{array}{c} x+y \\ x+y \end{array} \right] \int_0^1 t^{x+m-1} (1 - t)^{y+n-1} dt.
\]
We start now from the series $F(1, 1)$ with one constraint $a' = d - a$ and apply (8) to $(a)_m(d - a)_n/(d)_{m+n}$:

$$F(1, 1) = \sum_{m,n=0}^{\infty} \frac{(a, b, c)_m (d - a, b', c')_n}{(d)_{m+n}(e)_m(e')_n m! n!}.$$

$$= \Gamma \left[ \frac{d}{a, d - a} \right] \int_0^1 \sum_{m,n} \frac{(b, c)_m (b', c')_n}{(e)_m (e')_n m! n!} a^{m+1}(1-t)^{d-a+n-1} dt. \quad (9)$$

In this last expression, we use Euler’s identity (see Ref. [14], eq. (1.3.15)):

$$\sum_m \frac{(b, c)_m}{(e)_m} = _2F_1 \left[ \frac{b, c}{e}; \frac{1}{t} \right] = (1-t)^{-b-c} \left( \frac{1-t}{e} \right)^c \left( \frac{1-t}{e} \right)^t. \quad (10)$$

We obtain

$$\Gamma \left[ \frac{d}{a, d - a} \right] \int_0^1 \sum_{m,n} \frac{(b - e, e, c)_m (b', c')_n}{(e)_m (e')_n m! n!} a^{m+1}(1-t)^{d-e-a+b-c+n-1} dt. \quad (11)$$

For the last integral, we apply again (8). This leads to

$$\Gamma \left[ \frac{d + e - a - b - c}{a', d + e - b - c} \right] \sum_{m,n=0}^{\infty} \frac{(a, e - b, e - c)_m (b', c')_n (d + e - a - b - c)_n}{(e)_m (e')_n (d + e - b - c)_{m+n} m! n!}. \quad (12)$$

So far, we have used only the condition $a' = d - a$. Suppose now a second constraint is satisfied: $e' = d + e - a - b - c$. Then (12) simplifies and using $(d + e - b - c)_{m+n} = (a + e')_{m+n} = (a + e')_m (a + e' + m)_n$, we find that (12) can be written as

$$\Gamma \left[ \frac{d, e'}{a', a + e'} \right] \sum_{m=0}^{\infty} \frac{(a, e - b, e - c)_m}{(e)_m (a + e')_m m!} \frac{(b', c')_n}{(e')_n [1 + a + m + n]} \_2F_1 \left[ \frac{b', c'}{a + e' + m}; 1 \right]. \quad (13)$$

Applying Gauss’s theorem for the $_2F_1$ series yields, after some elementary manipulations, the transformation formula (14). The extra condition $\Re(a') > 0$ comes from the convergence requirements of the $_4F_3$ series. The other conditions used to apply the Beta function integral (8) disappear by analytic continuation.

Next, consider again (12) but now with the extra constraint $d + e - a - b - c = 1$. Then (12) can be rewritten as

$$\Gamma \left[ \frac{d}{d - a} \right] \sum_{m=0}^{\infty} \frac{(a, e - b, e - c)_m}{(e)_m m!} \sum_{n=0}^{\infty} \frac{(b', c')_n}{(e')_n [1 + a + m + n]} \_2F_1 \left[ \frac{b', c'}{a + e' + m}; 1 \right]. \quad (14)$$

Consider now condition (i) where $a$ is a negative integer, say $a = -N$. Then the sum over $m$ is finite going from 0 upto $N$, and due to the Gamma function the summation over $n$ goes from $N - m$ upto $\infty$. Replacing $n$ by $k + N - m$ and using $(x)_k = (x + N - m)_k \Gamma(x + N - m)/\Gamma(x)$ for $x = b', c', e'$, (14) reduces to

$$\Gamma \left[ \frac{d, e'}{d + N, b', c'} \right] \sum_{m=0}^{N} \frac{(-N, e - b, e - c)_m}{(e)_m m!} \sum_{k=0}^{\infty} \frac{(b' + N - m, c' + N - m)_k}{(e' + N - m)_k k!} \_2F_1 \left[ \frac{b', c'}{e' + N - m}; 1 \right]. \quad (15)$$
The $k$-summation is a $\,_{2}F_{1}$ series and can be summed using Gauss’s theorem. Performing this explicitly, and using some elementary manipulations with Gamma functions and Pochammer symbols, one arrives at

$$\Gamma \left[ \begin{array} { c c c c c } { d, e', b' + N, c' + N, -N - b' - c' + e'} \\ { d + N, b', c', e' - b', e' - c'} \end{array} \right] \,_{4}F_{3} \left[ \begin{array} { c c c c c } { -N, e - b, e - c, -N - b' - c' + e' } \\ { e, 1 - N - b', 1 - N - e' } \end{array} ; 1 \right]. \quad (16)$$

The above $\,_{4}F_{3}$ series is finite because $N$ is a positive integer, and thus one can apply reversal of series to it:

$$\,_{4}F_{3} \left[ \begin{array} { c c c c c } { A, B, C, -N } \\ { D, E, F } \end{array} ; 1 \right] = (-1)^{N} \frac{(A, B, C)_{N}}{(D, E, F)_{N}} \times \,_{4}F_{3} \left[ \begin{array} { c c c c c } { 1 - D - N, 1 - E - N, 1 - F - N, -N } \\ { 1 - A - N, 1 - B - N, 1 - C - N } \end{array} ; 1 \right]. \quad (17)$$

This gives rise to (3), thus proving it under the condition (i).

In order to prove (5) – with $a' = a - d$ and $d + e - a - b - c = 1$ – under the conditions (ii), i.e. $c = -N$ and $c' = -N'$ with $N$ and $N'$ positive integers, we shall make use of the following identity ($N$ positive integer),

$$\,_{3}F_{2} \left[ \begin{array} { c c c c } { A, B, -N } \\ { C, D ; 1 } \end{array} \right] = \frac{(C - A)_{N}}{(C)_{N}} \,_{3}F_{2} \left[ \begin{array} { c c c c } { A, D - B, -N } \\ { 1 + A - C - N, D ; 1 } \end{array} \right], \quad (18)$$

which follows, for example, from eq. (4.3.4.2) of Slater [14]. Starting from (2) and using $(d)_{m+n} = (d)_{n}(d + n)_{m}$, $\,_{3}F_{2}(1, 1)$ can be rewritten as

$$\sum_{n=0}^{N'} \,_{3}F_{2} \left[ \begin{array} { c c c c } { a, b, -N } \\ { d + n, e ; 1 } \end{array} \right] \frac{(d')_{n} n!}{(d, e')_{n} n!}. \quad (19)$$

Applying (18) to this $\,_{3}F_{2}$ leads to

$$\,_{3}F_{2} \left[ \begin{array} { c c c c } { a, b, -N } \\ { d + n, e ; 1 } \end{array} \right] = \frac{(d + n - a)_{N}}{(d + n)_{N}} \,_{3}F_{2} \left[ \begin{array} { c c c c } { a, e - b, -N } \\ { e - b - n, e ; 1 } \end{array} \right]$$

$$= \frac{(d - a)_{N}}{(d)_{N}} \frac{(d, d - a + N)_{n}}{(d - a + N)_{n}} \,_{3}F_{2} \left[ \begin{array} { c c c c } { a, e - b, -N } \\ { e - b - n, e ; 1 } \end{array} \right]. \quad (20)$$

Plugging this in (13), it becomes

$$\frac{(d - a)_{N}}{(d)_{N}} \sum_{m,n} \frac{(a, -N, e - b)_{m}}{(e, e - b - n)_{m} m!} \frac{(d - a + N, b', -N')_{n}}{(d + N, e')_{n} n!}. \quad (21)$$

Since $d + e - a - b - c = d + e - a - b + N = 1$, we have

$$\frac{(e - b)_{m}(d - a + N)_{n}}{(e - b - n)_{m}} = \frac{(e - b)_{m}(1 + b - e)_{n}}{(e - b - n)_{m}} = (1 - e + b - m)_{n}, \quad (22)$$

and thus (21) reduces to

$$\frac{(d - a)_{N}}{(d)_{N}} \sum_{m} \frac{(a, -N)_{m}}{(e)_{m} m!} \,_{3}F_{2} \left[ \begin{array} { c c c c } { b', 1 - e + b - m, -N' } \\ { e', d + N } \end{array} ; 1 \right]. \quad (23)$$
Since also \( N' \) is a positive integer, we can apply (18) to the last expression, and obtain

\[
\frac{(d-a)_N (e' - b')_{N'}}{(d)_N (e')_{N'}} \sum_{m,n} \frac{(a,-N)_m (b',a+m,-N')_n}{(e)_m m! (1 + b' - N' - e', d + N)_n n!}.
\]  

(24)

Making the replacement \((a)_m (a + m)_n = (a)_n (a + n)_m\) implies that the last sum can be rewritten as

\[
\frac{(d-a)_N (e' - b')_{N'}}{(d)_N (e')_{N'}} \sum_{n} \frac{(b',a,-N')_n (e')_n a, d + N)_n n!}{(1 + b' - N' - e', d + N)_n n!}  \quad \text{}_2\text{F}_1 \left[ \frac{a + n, -N'}{e} \right],
\]

(25)

to which Vandermonde’s theorem can be applied. Replacing \((e - a - n)_N\) by \((e - a)_N(1 - e + a)_n/(1 - e + a - N)_n\) leads to

\[
\frac{(d-a)_N (e' - b')_{N'}}{(d)_N (e')_{N'}} \sum_{n} \frac{(b',a,-N', 1 - e + a)_n N}{(1 + b' - N' - e', d + N, 1 - e + a - N)_n n!}.
\]

(26)

Using \(d + e - a - b + N = 1\) this can finally be rewritten in the form (3), providing a proof under the condition (ii).

3 Special cases and summation formulas

Some limiting cases of (3)–(6) are worth considering. If we assume that there are three relations among the 9 parameters in (3), i.e.,

\[
e' = d + e - a - b - c, \quad a' = d - a, \quad b' = d - b,
\]

(27)

then, using Dixon’s theorem (eq. (2.3.3.7) of Ref. [14]) one can deduce from (3) that

\[
F(1,1) = \Gamma \left[ \frac{e, e'}{e - c, e' + c} \right] 3\text{F}_2 \left[ \frac{d-a, d-b, c + e'}{d, e' + c}; 1, 1 \right],
\]

\[
\Re(e') > 0, \quad \Re(e - c - e') > 0.
\]

(28)

In particular, for \(e' = -c\), the rhs simply reduces to a product of Gamma functions, and we obtain the summation formula

\[
F_{0:3}^{1:1} \left[ \frac{-a, b, c, d-a, d-b, d-c}{d, e; d + e - a - b - c}; 1, 1 \right] = \Gamma \left[ \frac{e, e + d - a - b - c}{e - c, e + d - a - b} \right],
\]

\[
\Re(e) > 0, \quad \Re(d + e - a - b - c) > 0.
\]

(29)

This special formula was already obtained earlier [14] in the context of 9-j coefficients and proved by Karlsson [11]; its \(q\)-analogue was proved by Gasper [3].

If on the other hand, we take in (28) the extra condition \(e' = d - c\), the \(3\text{F}_2\) series reduces to a \(2\text{F}_1\), which can be summed with Gauss’s theorem. There results the following summation formula :

\[
F_{0:3}^{1:1} \left[ \frac{-a, b, c, d-a, d-b, d-c}{d, e; d + e - a - b - c}; 1, 1 \right] = \Gamma \left[ \frac{e, e + d - a - b - c, e - d'}{e - a, e - b, e - c} \right],
\]

\[
\Re(e - d) > 0, \quad \Re(d + e - a - b - c) > 0.
\]

(30)
Another interesting set of three relations among the 9 parameters is
\[ e' = d + e - a - b - c, \quad a' = d - a, \quad d = b + c + b' + c'. \] (31)

The transformation formula (3) now becomes
\[ F(1,1) = \Gamma \left[ \begin{array}{cc} d, e, e' \\ a', e + b', e + c' \end{array} \right] \frac{\binom{a, e - b, e - c}{d + e - a - b - c}}{3F2 \left[ \begin{array}{cc} a, e - b, e - c \\ e + b', e + c' : 1 \end{array} \right]}, \]
\[ \Re(e') > 0, \quad \Re(e) > 0. \] (32)

The summation formulas that can be deduced from here are again (29) and (30). In principle one can deduce further summation formulas by requiring the 3F2 in (25) or (26) to be terminating and Saalschützian or by requiring the above 3F2’s or 4F3’s to be of Karlsson-Minton type \[10, 12\] (see also Eq. (1.9.1) of \[7\]). We give one example: consider (28) with \( c' = e - c - 1 \) and \( d - a \) or \( d - b \) a negative integer. Then the Pfaff-Saalschütz formula can be used and one obtains:
\[ \binom{a, b, c}{d} = \binom{d, e'}{d + e - a - b - c} \Gamma \left[ \begin{array}{cc} 1 - a, 1 - b, e, e - d, d + e - a - b - c \\ 1 - d, e - a, e - b, e - c, 1 + d - a - b \end{array} \right], \]
\[ \Re(d + e - a - b - c) > 0 \quad \text{and} \quad d - a \text{ or } d - b \text{ a negative integer.} \] (33)

Some special summation formulas follow by specializing (3) under one of the two conditions. In the case (i), one can specialize \( b' = d - b \) and \( c' = d - c \); or \( b' = d - b \) and \( c' = 1 + c + c' \). In the case (ii), one can choose \( b' = d - b \) and \( c' = 1 - a - b + c' + d; \) or \( b' = d - b \) and \( c' = 1 + c + c' \). In terms of 5 independent parameters, such specializations give rise to the following four summation formulas:
\[ \binom{a, b, c}{d} = \binom{-N, b, c}{1 - N + b + c - d} \Gamma \left[ \begin{array}{cc} e', e' + b + c - 2d \\ e' + b - d, e' + c - d \end{array} \right], \]
\[ \Re(e' - N + b + c - 2d) > 0 \quad \text{and} \quad N \text{ a positive integer;} \] (34)
\[ \binom{-N, b, c}{d} = \binom{-N, b, c}{1 - N + b + c - d} \Gamma \left[ \begin{array}{cc} 1 + c + c', 1 + b + c - d \\ 1 + c, 1 + b + c + c' - d \end{array} \right], \]
\[ \Re(1 - N + b + c - d) > 0 \quad \text{and} \quad N \text{ a positive integer;} \] (35)
\[ \binom{a, b, c}{d} = \binom{a, b, c}{d + a, d - b, c'} \Gamma \left[ \begin{array}{cc} d - a, d - b, -N' \\ 1 - a - b - N' + d \end{array} \right], \]
\[ \Re(d + a, d - b)_{N(a + b - d)_{N'}}, \]
\[ N \text{ and } N' \text{ positive integers;} \] (36)
\[
F^{a,b,-N}_{d-a,d-b,-N'} \left[ \begin{array}{c} d; 1 + a + b - N - d; 1 - N - N' \end{array} ; 1, 1 \right] = \\
\frac{(d - a, d - b)_{N + N'}}{(d)_{N + N'}(d - a - b)_{N(N')}^N} \ N(\neq 0) \ \text{and} \ N' \ \text{positive integers.} \quad (37)
\]

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