Homological Equations for Tensor Fields and Periodic Averaging

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Abstract

Homological equations of tensor type associated to periodic flows on a manifold are studied. The Cushman intrinsic formula [4] is generalized to the case of multivector fields and differential forms. Some applications to normal forms and the averaging method for perturbed Hamiltonian systems on slow-fast phase spaces are given.

1 Introduction

The so-called homological equations usually appear in the context of normal forms and the method of averaging for perturbed dynamical systems (see, for example, [1, 16]). According to the Lie transform method [7, 9], the infinitesimal generators of normalization transformations for perturbed dynamics systems are defined as the solutions to homological equations for vector fields. In the Hamiltonian case, the normalization problem is reduced to the solvability of homological equations for functions. Let $X$ be a vector field on a manifold $M$ whose flow is periodic with period function $T: M \to \mathbb{R}$. Then, it is well-known [4] that for a given $G \in C^\infty(M)$ there exist smooth functions $F$ and $\bar{G}$ on $M$ satisfying the homological equation

$$\mathcal{L}_X F = G - \bar{G}$$

(1.1)

and the condition

$$\mathcal{L}_X \bar{G} = 0.$$  

(1.2)

The solvability of this problem follows from the decomposition

$$C^\infty(M) = \text{Ker} \mathcal{L}_X \oplus \text{Im} \mathcal{L}_X$$

(1.3)

and the corresponding global solutions are given by the formulas [4]

$$\bar{G} = \frac{1}{T} \int_0^T G \circ \Phi_t^{tx} \, dt,$$

(1.4)
\[
F = \frac{1}{T} \int_0^T (t - \frac{T}{2})G \circ \text{Fl}_X^t \, dt + K, \tag{1.5}
\]

where \(K\) is a first integral of \(X\). This result together with Deprit’s algorithm \[7\] implies that a perturbed Hamiltonian system admits a global normalization of arbitrary order relative to the unperturbed Hamiltonian vector field with periodic flow.

In this paper, we are interested in a generalized version of problem \((1.1), (1.2)\), when \(F, G\) and \(\bar{G}\) are tensor fields on \(M\) of arbitrary type. In the general case, when \(T \neq \text{const}\), the property like \((1.3)\) is no longer true and as a consequence there are obstructions to the solvability of \((1.1), (1.2)\). Using some algebraic properties of the \(S^1\)-averaging and calculus on exterior algebras, we generalize the free-coordinate formulas \((1.4), (1.5)\) to the case of multivector fields and differential forms on \(M\). These results are applied to the normalization problem for a special class of perturbed Hamiltonian dynamics on slow-fast phases spaces \([5, 6]\), which leads to the study of homological equations for non-Hamiltonian vector fields. Finally, we show how the homological equations for 1-forms appear in the context of Hamiltonization problem \([3, 18]\) and the construction of symplectic structures which are invariant with respect to skew-product \(S^1\)-actions \([17]\).

2 Algebraic properties of the \(S^1\)-averaging

Let \(M\) be a smooth manifold. Denote by \(T^k_s(M)\) the space of all tensor fields on \(M\) of type \((k, m)\). In particular, \(T^0_s(M) = \mathcal{C}^\infty(M)\) and \(T^1_s(M) = \mathfrak{X}(M)\) are the spaces of smooth functions and vector fields on \(M\), respectively. For every vector field \(X \in \mathfrak{X}(M)\), we denote by \(L_X : T^k_s(M) \to T^k_s(M)\) the Lie derivative along \(X\), that is, the unique differential operator on the tensor algebra of the manifold \(M\) which coincides with the standard Lie derivative \(L_X\) on \(\mathcal{C}^\infty(M)\) and \(\mathfrak{X}(M)\) (see, for example \([2]\)). The flow \(\text{Fl}_X^t\) of \(X\) and the Lie derivative \(L_X\) are related by the formula

\[
\frac{d}{dt}(\text{Fl}_X^t)^*\Xi = (\text{Fl}_X^t)^*(L_X \Xi),
\]

for any \(\Xi \in T^k_s(M)\).

Now, suppose that we are given an action of the circle \(S^1 = \mathbb{R}/2\pi\mathbb{Z}\) on \(M\) with infinitesimal generator \(\Upsilon\). Therefore, \(\Upsilon\) is a complete vector field on \(M\) whose flow \(\text{Fl}_\Upsilon^t\) is \(2\pi\)-periodic. We admit that the \(S^1\)-action is not necessarily free.

For every tensor field \(\Xi \in T^k_s(M)\), its average with respect to the \(S^1\)-action is a tensor field \(\langle \Xi \rangle \in T^k_s(M)\) of the same type which is defined as \[12\]

\[
\langle \Xi \rangle := \frac{1}{2\pi} \int_0^{2\pi} (\text{Fl}_\Upsilon^t)^*\Xi dt. \tag{2.1}
\]

A tensor field \(\Xi \in T^k_s(M)\) is said to be invariant with respect to the \(S^1\)-action if \((\text{Fl}_\Upsilon^t)^*\Xi = \Xi (\forall t \in \mathbb{R})\) or, equivalently, \(L_\Upsilon \Xi = 0\). In terms of the \(S^1\)-average of
Ξ, the $S^1$-invariance condition reads $\Xi = \langle \Xi \rangle$. Denote by $A : T^k_s(M) \to T^k_s(M)$ the averaging operator, $A(\Xi) = \langle \Xi \rangle$ which is a $\mathbb{R}$-linear operator with property $A^2 = A$. It is clear that the image of $A$ consists of all $S^1$-invariant tensor fields. A tensor field belongs to $\text{Ker } A$ if its $S^1$-average is zero. Therefore, we have the $S^1$-invariant splitting

$$T^k_s(M) = \text{Im } A \oplus \text{Ker } A. \quad (2.2)$$

Introduce also the $\mathbb{R}$-linear operator $S : T^k_s(M) \to T^k_s(M)$ given by

$$S(\Xi) := \frac{1}{2\pi} \int_0^{2\pi} (t - \pi)(\text{Fl}_t^\tau)^*\Xi dt. \quad (2.3)$$

It follows directly from definitions that the operators $\mathcal{L}_T, A$ and $S$ pairwise commute and satisfy the relations

$$A \circ \mathcal{L}_T = \mathcal{L}_T \circ A = 0, \quad (2.4)$$

$$A \circ S = S \circ A = 0. \quad (2.5)$$

Moreover, we have the following important property.

**Proposition 2.1** The following identity holds

$$\mathcal{L}_T \circ S = \text{id} - A, \quad (2.6)$$

*Proof.* For every tensor field $\Xi \in T^k_s(M)$, by definition (2.3), we have

$$(\text{Fl}_t^\tau)^*S(\Xi) = \frac{1}{2\pi} \int_0^{2\pi} (t - \pi)(\text{Fl}_t^\tau)^*\Xi dt$$

$$= \frac{1}{2\pi} \int_{2\pi + \tau}^{2\pi + \tau} (t - \tau - \pi)(\text{Fl}_t^\tau)^*\Xi dt$$

Differentiating the both sides of this equality in $\tau$ and using the $2\pi$-periodicity of the flow $\text{Fl}_t^\tau$, we get

$$\frac{d}{d\tau}(\text{Fl}_t^\tau)^*S(\Xi) = \frac{1}{2\pi}[(t - \tau - \pi)(\text{Fl}_t^\tau)^*\Xi]_{\tau}^{2\pi + \tau} - \frac{1}{2\pi} \int_{\tau}^{2\pi + \tau} (\text{Fl}_t^\tau)^*\Xi dt$$

$$= (\text{Fl}_t^\tau)^*(\Xi - \langle \Xi \rangle)$$

Comparing this equality with the identity

$$\frac{d}{d\tau}(\text{Fl}_t^\tau)^*S(\Xi) = (\text{Fl}_t^\tau)^*(\mathcal{L}_T S(\Xi)) \quad (2.7)$$

gives $\mathcal{L}_T(S(\Xi)) = \Xi - \langle \Xi \rangle$. \hfill \blacksquare

**Corollary 2.2** For every tensor field $\Xi \in T^k_s(M)$, the following assertions are equivalent.
• $S(Ξ) = 0$;
• $S(Ξ)$ is $S^1$-invariant;
• $Ξ$ is $S^1$-invariant.

Proof. The equivalence of the first two conditions follows from property (2.5) which says that $⟨S(Ξ)⟩ = 0$. Property (2.6) implies the equivalence of the last two assertions.

Proposition 2.3 The following relations hold
\[
\text{Ker} S = \text{Ker} L_Υ = \text{Im} A, \quad (2.8)
\]
\[
\text{Im} S = \text{Im} L_Υ = \text{Ker} A. \quad (2.9)
\]

Proof. Taking into account that the kernel of the Lie derivative $L_Υ : T^k(M) \to T^k(M)$ consists of all $S^1$-invariant tensor fields and by the Corollary 2.2, we derive (2.8). By (2.10), we have $\text{Im} L_Υ \subseteq \text{Ker} A$. On the other hand, it follows from (2.6) that
\[
L_Υ S(Ξ) = Ξ \quad \forall Ξ ∈ \text{Ker} A \quad (2.10)
\]
and hence $\text{Ker} A \subseteq \text{Im} L_Υ$. Therefore, $\text{Im} L_Υ = \text{Ker} A$. By (2.9) we have the identities $S = L_Υ \circ S^2$ and $L_Υ = S \circ L^2$ which say that $\text{Im} S = \text{Im} L_Υ$.

As a consequence of (2.2) and (2.8), we get also the decomposition
\[
T^k_s(M) = \text{Ker} L_Υ \oplus \text{Im} L_Υ \quad (2.11)
\]
which together with (2.10) implies that the restriction of $L_Υ$ to $\text{Im} L_Υ$ is an isomorphism whose inverse is just $S$.

3 Homological equations associated to periodic flows

Let $X$ be a complete vector field on a manifold $M$ with periodic flow. This means that there exists a a smooth positive function $T : M \to \mathbb{R}$, called a period function, such that $\text{Fl}_X^{t+T(m)}(m) = \text{Fl}_X^{t}(m)$ for all $m \in M$ and $t \in \mathbb{R}$. Introduce also the frequency function $ω : M \to \mathbb{R}$ given by $ω = \frac{2π}{T}$. It is clear that $ω$ is a first integral of $X$. The periodic flow $\text{Fl}_X$ induces the $S^1$-action on $M$ whose infinitesimal generator is the vector field $Υ = \frac{1}{ω}X$ with $2π$-periodic flow.

Homological equations for $k$-vector fields. Let $\chi^k(M) = \text{Sec}(\wedge^k TM)$ be the space of all $k$-multivector fields on $M$. In particular, $\chi^0(M) = C^∞(M)$ and $\chi^1(M) = \mathfrak{X}(M)$. It is clear that the operators $L_Υ, S$ and $A$ leave invariant
the subspaces $\chi^k(M) \subset T^k(M)$. For, every $k$-vector field $A \in \chi^k(M)$ and a 1-form $\alpha$ on $M$, denote by $i_\alpha A \in T^{k-1}(M)$ a $(k-1)$-vector field defined by

$$(i_\alpha A)(\alpha_1, ..., \alpha_{k-1}) = A(\alpha, \alpha_1, ..., \alpha_{k-1})$$

for all 1-forms $\alpha_1, ..., \alpha_{k-1}$ on $M$. By definition, $i_\alpha A = 0$, for every 0-vector field $A$.

Consider the $S^1$-action on $M$ associated to the periodic flow $Fl^X$. Denote by $\chi^k_{\text{inv}}(M) = \text{Ker} L^X$ the subspace of all $S^1$-invariant $k$-vector fields on $M$. Then, according to (2.11), we have the splitting

$$\chi^k(M) = \chi^k_{\text{inv}}(M) \oplus \chi^k_0(M),$$

where $\chi^k_0(M) = \text{Im} L^X$ denotes the subspace of all $k$-vector fields on $M$ with zero average.

**Theorem 3.1** Let $X$ be a vector field on $M$ with periodic flow and frequency function $\omega$. Then, for a given $B \in \chi^k(M)$, all $k$-vector fields $A$ and $\bar{B}$ on $M$ satisfying the homological equation

$$L_X A = B - \bar{B}$$

and the condition

$$\bar{B} \text{ is } S^1\text{-invariant}$$

are of the form

$$\bar{B} = \langle B \rangle + \frac{1}{\omega} X \wedge i_{d\omega} C,$$

$$A = \frac{1}{\omega} S(B) + \frac{1}{\omega^3} X \wedge S^2(i_{d\omega} B) + C,$$

where $C \in \chi^k_{\text{inv}}(M)$ is an arbitrary $S^1$-invariant $k$-vector field. Here, the average $\langle \rangle$ is taken with respect to the $S^1$-action on $M$ associated to the flow of $X$.

**Proof.** Using the identity, [2]

$$L_{\omega^X} A = \omega L^X A - \omega \wedge i_{d\omega} A,$$

we rewrite equation (3.2) in the form

$$L^X A - \frac{1}{\omega} Y \wedge i_{d\omega} A = \frac{1}{\omega} (B - \bar{B}).$$

Applying the averaging operator to the both sides of this equation and taking into account condition (3.3), we get

$$\bar{B} = \langle B \rangle = \langle B \rangle + Y \wedge i_{d\omega} \langle A \rangle.$$

According to decomposition (3.1), we have

$$A = \langle A \rangle + A_0, \quad \langle A_0 \rangle = 0,$$

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Putting these representations together with (3.8) into (3.7), we see that the original problem (3.2), (3.3) is reduced to the following equation for $A_0 \in \chi^0_k(M)$:

$$L_\mathcal{Y} A_0 - \frac{1}{\omega} \mathcal{Y} \wedge i_{dw} A_0 = \frac{1}{\omega^2} \mathcal{Y} \wedge i_{dw} S(B_0).$$

Looking for $A_0$ in the form $A_0 = \frac{1}{\omega} S(B_0) + \hat{A}_0$ and using property (2.6), we conclude that $\hat{A}_0 \in \chi^0_k(M)$ must satisfy the equation

$$L_\mathcal{Y} \hat{A}_0 - \frac{1}{\omega} \mathcal{Y} \wedge i_{dw} \hat{A}_0 = \frac{1}{\omega^2} \mathcal{Y} \wedge i_{dw} S(B_0)$$

(3.10)

Next, taking into account that $i_{dw} \mathcal{Y} = 0$ and putting $\hat{A}_0 = \mathcal{Y} \wedge S(D)$, we reduce (3.10) to the following equation for $D \in \chi^{k-1}_0(M)$:

$$\mathcal{Y} \wedge L_\mathcal{Y} S(D) = \frac{1}{\omega} \mathcal{Y} \wedge i_{dw} S(B_0).$$

By property (2.6) the tensor field $D = \frac{1}{\omega^2} S^2(i_{dw} B_0)$ satisfies the relation $L_\mathcal{Y} S(D) = \frac{1}{\omega^2} i_{dw} S(B_0)$. Therefore, the solutions to problem (3.2), (3.3) are given by (3.8) and (3.11), where

$$A_0 = \frac{1}{\omega} S(B_0) + \frac{1}{\omega^3} X \wedge S^2(i_{dw} B_0)$$

(3.11)

and $\langle A \rangle$ is an arbitrary $S^1$-invariant $k$-vector field on $M$. Finally, property (2.5) says that $S(B_0) = S(B)$ and hence formulas (3.4) and (3.5) follow from (3.8). Under this condition, every solution of (3.13) is given by (3.5), where $C = \langle A \rangle$.

As a straightforward consequence of Theorem 3.1, we get the following result.

**Corollary 3.2** The kernel of the Lie derivative $L_X : \chi^k(M) \to \chi^k(M)$ is

$$\text{Ker}(L_X) = \chi^{k}_{\text{inv}}(M) \cap \{ C \in \chi^k(M) \mid X \wedge i_{dw} C = 0 \}.$$  

(3.12)

For a given $B \in \chi^k(M)$ ($k \geq 1$), the homological equation

$$L_X A = B$$

(3.13)

is solvable relative to a k-vector field $A$ on $M$ if and only if

$$\langle B \rangle = X \wedge i_{dw} P$$

(3.14)

for a certain $S^1$-invariant k-vector field $P \in \chi^{k}_{\text{inv}}(M)$. Under this condition, every solution of (3.13) is given by (3.5), where $C = P + C'$, $C' \in \text{ker}(L_X)$.

It follows from (3.14) that the necessary conditions for the solvability of (3.13) are the following

$$X(m) = 0 \rightarrow \langle B \rangle(m) = 0,$$

(3.15)
\[ X \wedge \langle B \rangle = 0, \quad (3.16) \]
\[ \iota_{\omega} \langle B \rangle = 0. \quad (3.17) \]

Therefore, if one of these conditions does not hold, then equation (3.13) is unsolvable.

**Corollary 3.3** There exist \( k \)-vector fields \( A \) and \( \bar{B} \) on \( M \) satisfying the equations
\[ \mathcal{L}_X A = B - \bar{B}, \quad (3.18) \]
\[ \mathcal{L}_X \bar{B} = 0 \quad (3.19) \]
if and only if
\[ X \wedge \iota_{\omega} \langle B \rangle = 0. \quad (3.20) \]

Under this condition, all solutions \( (A, \bar{B}) \) to (3.18), (3.19) are given by formulas (3.4), (3.5). Moreover, the \( k \)-vector field \( A \) in (3.5) can be represented in the form
\[ A = \frac{1}{\omega} S(B) + (S^1 \text{-invariant } k \text{-vector field}) \] if and only if \( \Upsilon \wedge \iota_{\omega} \langle B \rangle = 0 \).

Let us consider some particular cases. In the case \( k = 0 \), formulas (3.4) and (3.5) coincide with (1.4), (1.5). The well-known solvability condition for the equation \( \mathcal{L}_X F = G \) reads \( \langle G \rangle = 0 \).

In the case \( k = 1 \), by Theorem 3.1, Corollary 3.2 and Corollary 3.3 (where \( A = Z \) and \( B = W \) are vector fields on \( M \)) we derive the following facts. For a given \( W \in \mathfrak{X}(M) \), all vector fields \( Z \in \mathfrak{X}(M) \) and \( \bar{W} \in \mathfrak{X}_{\text{inv}}(M) \) on \( M \) satisfying
\[ [X, Z] = W - \bar{W}, \quad (3.21) \]
are given by the formulas
\[ \bar{W} = \langle W \rangle + \frac{1}{\omega} \mathcal{L}_Y (\omega) X, \quad (3.22) \]
\[ Z = \frac{1}{\omega} S(W) + \frac{1}{\omega^2} S^2(\mathcal{L}_W \omega) X + Y, \quad (3.23) \]
where \( Y \in \mathfrak{X}_{\text{inv}}(M) \). The second term in (3.23) can be omitted only if \( \mathcal{L}_W \omega = 0 \). Moreover, the homological equation
\[ [X, Z] = W \]
for \( Z \) is solvable if and only if \( \langle W \rangle = \mathcal{L}_Y (\omega) X \) for a certain \( S^1 \text{-invariant } \) vector field \( \tilde{Y} \). In particular, the necessary condition (3.17) for the solvability of (3.24) takes the form
\[ \mathcal{L}_{\langle W \rangle} \omega = 0. \quad (3.25) \]

Let \( \text{Reg}(X) = \{ m \in M \mid X(m) \neq 0 \} \) be the set of points regular of \( X \). If \( \text{Reg}(X) \) is everywhere dense in \( M \), then the kernel of the Lie derivative \( \mathcal{L}_X : \mathfrak{X}(M) \to \mathfrak{X}(M) \) is
\[ \text{Ker}(\mathcal{L}_X) = \mathfrak{X}_{\text{inv}}(M) \cap \{ Y \in \mathfrak{X}(M) \mid \mathcal{L}_Y \omega = 0 \}. \quad (3.26) \]
Moreover, it follows from (3.20) that there exist vector fields $Z$ and $\bar{W}$ on $M$ satisfying homological equation (3.21) and the condition $[X, \bar{W}] = 0$ if and only if condition (3.25) holds.

**Homological equations for $k$-Forms.** Consider the space $\Omega^k(M) = \text{Sec}(\wedge^k T^* M)$ of $k$-forms on $M$. Then, subspace $\Omega^k(M) \subset T^0_k(M)$ is an invariant with respect to the action of the operators $L_\Upsilon, S$ and $A$. By $i_Y \alpha \in \Omega^{k-1}(M)$ we denote the interior product of a vector field $Y$ and a $k$-form $\alpha$ on $M$ which is defined by the usual formula: $(i_Y \alpha)(Y_1, ..., Y_{k-1}) = \alpha(Y, Y_1, ..., Y_{k-1})$. Let $\Omega^k_{\text{inv}}(M) = \text{Ker} L_\Upsilon$ and $\Omega^k_0(M) = \text{Im} L_\Upsilon$. Then, we have the $S^1$-invariant splitting

$$\Omega^k(M) = \Omega^k_{\text{inv}}(M) \oplus \Omega^k_0(M), \quad (3.27)$$

There is the following covariant analog of Theorem 3.1.

**Theorem 3.4** For a given $\eta \in \Omega^k(M)$, all $k$-forms $\theta$ and $\bar{\eta}$ on $M$ satisfying the homological equation

$$L_X \theta = \eta - \bar{\eta} \quad (3.28)$$

and the condition $\bar{\eta}$ is $S^1$-invariant,

are represented as

$$\bar{\eta} = \langle \eta \rangle - \frac{1}{\omega} d\omega \wedge i_X \mu, \quad (3.30)$$

$$\theta = \frac{1}{\omega} S(\eta) - \frac{1}{\omega^3} d\omega \wedge S^2(i_X \eta) + \mu, \quad (3.31)$$

where $\mu \in \Omega^k_{\text{inv}}(M)$ is an arbitrary $S^1$-invariant $k$-form.

The proof of this theorem goes in the same line as the proof Theorem 3.1, where instead of identity (3.6) we have to use its covariant analog: $L_\omega \Upsilon \theta = \omega L_\Upsilon \theta + d\omega \wedge \Upsilon \theta$.

Let $\eta = \eta_0 + \langle \eta \rangle$. Then, the general solution $\theta$ in (3.31) has the representation $\theta = \theta_0 + \mu$, where $\theta_0 \in \Omega^k_0(M)$ is uniquely determined by $\eta_0$,

$$\theta_0 = \frac{1}{\omega} S(\eta_0) - \frac{1}{\omega^3} d\omega \wedge S^2(i_X \eta_0) \quad (3.32)$$

From Theorem 3.4 we deduce the following consequences

**Corollary 3.5** The kernel of the Lie derivative $L_X : \Omega^k(M) \to \Omega^k(M)$ is

$$\text{Ker}(L_X) = \Omega^k_{\text{inv}}(M) \cap \{ \mu \in \Omega^k(M) \mid d\omega \wedge i_X \mu = 0 \}. \quad (3.33)$$

For a given $\eta \in \Omega^k(M)$ ($k \geq 1$), the homological equation

$$L_X \theta = \eta \quad (3.34)$$

is solvable relative to a $k$-form $\theta$ on $M$ if and only if

$$\langle \eta \rangle = d\omega \wedge i_X \alpha \quad (3.35)$$

for a certain $S^1$-invariant $k$-form $\alpha \in \Omega^k_{\text{inv}}(M)$.
It follows from (3.35) that the necessary conditions for the solvability of equation (3.34) are

\[ X(m) = 0 \implies \langle \eta \rangle(m) = 0, \]
\[ d\omega \wedge \langle \eta \rangle = 0, \]
\[ i_X \langle \eta \rangle = 0. \]

(3.36) \hspace{1cm} (3.37) \hspace{1cm} (3.38)

**Corollary 3.6** There exist k-forms \( \theta \) and \( \eta \) on \( M \) satisfying the equations

\[ \mathcal{L}_X \theta = \eta - \bar{\eta}, \]
\[ \mathcal{L}_X \bar{\eta} = 0, \]

if and only if the following condition holds

\[ d\omega \wedge i_X \langle \eta \rangle = 0 \] (3.41)

In the case \( k = 1 \), for a given 1-form \( \eta \in \Omega^1(M) \), formulas (3.30), (3.31) for solutions \((\theta, \bar{\eta})\) of problem (3.28), (3.29) can be written as follows

\[ \bar{\eta} = \langle \eta \rangle - \frac{1}{\omega} (i_X \mu) d\omega, \]
\[ \theta = \frac{1}{\omega} S(\eta) - \frac{1}{\omega^3} S^2(i_X \eta) d\omega + \mu, \]

with \( \mu \in \Omega^1_{\text{inv}}(M) \). The solvability condition for homological equation (3.34) reads

\[ \langle \eta \rangle = (i_X \alpha) d\omega \] (3.44)

for a certain \( S^1 \)-invariant 1-form \( \alpha \in \Omega^1_{\text{inv}}(M) \). Let \( \text{Reg}(\omega) = \{ m \in M \mid d_m \omega \neq 0 \} \) be the set of regular points of the frequency function \( \omega \). If \( \text{Reg}(\omega) \) is everywhere dense in \( M \), then

\[ \text{Ker}(\mathcal{L}_X) = \Omega^1_{\text{inv}}(M) \cap \{ \mu \in \Omega^k(M) \mid i_X \mu = 0 \}. \]

(3.45)

Moreover, condition (3.41) for the solvability of problem (3.39), (3.40) is equivalent to the following \( i_X \langle \eta \rangle = 0 \).

To end this section, let us consider the case of closed forms. Let \( d : \Omega^k(M) \to \Omega^{k+1}(M) \) be the exterior derivative. By standard properties of the exterior derivative, we conclude that \( d \) commutes with operators \( \mathcal{L}_X, S \) and \( A \), in particular, \( \langle d\eta \rangle = d(\langle \eta \rangle) \) for any \( \eta \in \Omega^k(M) \). Moreover, splitting (3.27) is invariant with respect to \( d \), in the sense that

\[ d\eta = \langle d\eta \rangle + (d\eta)_0 = d(\langle \eta \rangle) + d\eta_0, \]

where \( \eta = \langle \eta \rangle + \eta_0 \). It follows that if \( \eta \) is closed then, the components \( \langle \eta \rangle \) and \( \eta_0 \) are also closed k-forms. In this case, according to (3.32), a solution to the equation \( \mathcal{L}_X \theta_0 = \eta_0 \) is given by \( \theta_0 = S(\eta_0) \). Then, \( d\theta_0 = S(d\eta_0) = 0 \) and hence \( \eta_0 = \mathcal{L}_X \theta_0 = d \circ i_X \theta_0 \). This proves the following assertion.

**Proposition 3.7** For every closed k-form \( \eta \) on \( M \), we have the decomposition

\[ \eta = \langle \eta \rangle + d(i_X \theta_0), \]

(3.46)

where \( \theta_0 = \frac{1}{2\pi} \int_0^{2\pi} (t - \pi)(\text{Fl}_X\theta_0)^* \eta dt \).
4 Normal forms

Suppose we start again with a vector field $X$ on $M$ whose flow is periodic with frequency function $\omega : M \to \mathbb{R}$.

**Proposition 4.1** Let $P_\varepsilon = X + \varepsilon W$ be a perturbed vector field on $M$, where $\varepsilon$ is a small parameter. Let

$$\Phi_\varepsilon = \text{Fl}_Z |_{t=\varepsilon}$$

be the time-$\varepsilon$ flow of the vector field

$$Z = \frac{1}{\omega} S(W) + \frac{1}{\omega^3} S^2(L_W \omega)X + Y,$$

where $Y$ is an $S^1$-invariant vector field on $M$. Then, for a given open domain $N \subset M$ with compact closure, there exists a constant $\delta > 0$ such that formula

$$\Phi_\varepsilon$$

defines a near identity transformation $\Phi_\varepsilon : N \to M$ for $\varepsilon \in (-\delta, \delta)$ which brings $P_\varepsilon$ into the form

$$\Phi_\varepsilon^* P_\varepsilon = X + \varepsilon \hat{W} + O(\varepsilon^2)$$

(4.3)

$$\hat{W} := \langle W \rangle + L_Y \ln(\omega)X.$$

(4.4)

If the frequency function $\omega$ is a first integral of the $S^1$-average of $W$,

$$\mathcal{L}(W) \omega = 0$$

on $M$,

(4.5)

then the mapping $\Phi_\varepsilon$ is a normalization transformation of first order for $P_\varepsilon$ relative to $X$,

$$[X, \hat{W}] = 0,$$

(4.6)

for arbitrary choice of a vector field $Y \in \mathfrak{X}_{\text{inv}}(M)$ in (4.2).

The proof of this proposition follows from the standard Lie transform arguments and Corollary 3.6 for the case $k = 1$. Remark that if $\text{Reg}(X)$ is everywhere dense in $M$, then (4.5) becomes also a necessary condition for mapping $\Phi_\varepsilon$ (4.1) to be a normalization transformation.

Now, let us see how, in the context of the normalization procedure, one can use a freedom in the definition of $\Phi_\varepsilon$. Consider the perturbed vector field $P_\varepsilon = X + \varepsilon W$ and assume that the $S^1$-action with infinitesimal generator $\Upsilon = \frac{1}{\omega} X$ is free on $M$. Then, the orbit space $O = M/S^1$ is a smooth manifold and the projection $\rho : M \to O$ is a $S^1$-principle bundle. In this case, the frequency function is of the form $\omega = \omega_O \circ \rho$ for a certain $\omega_O \in C^\infty(O)$. Let $\text{Ver} = \text{Span}\{\Upsilon\}$ be the vertical subbundle and $D \subset TM$ an arbitrary subbundle which is complimentary to $\text{Ver}$. Then, for every vector field $u \in \mathfrak{X}(O)$ there exists a unique $e \in \text{Sec}(D)$ descending to $u$, $d\rho \circ e = u \circ \rho$. It follows that $[\Upsilon, e] = b \Upsilon$, where $b \in C^\infty(M)$ with $\langle b \rangle = 0$. Defining $\text{hor}(u) := e - \mathcal{S}(b)e$, by property (2.6), we get that $[\Upsilon, \text{hor}(u)] = 0$. Therefore, we have the splitting $TM = \text{Hor} \oplus \text{Ver}$ (a principle connection on $M$), where the horizontal subbundle $\text{Hor} = \text{Span}\{\text{hor}(u) \mid u \in \mathfrak{X}(O)\}$ is invariant with respect to the $S^1$-action (for
more details, see [12]. According to this splitting, the vector field \( \bar{W} \) in (4.4) has the decomposition \( \bar{W} = \bar{W}_{\text{hor}} + \bar{W}_{\text{ver}} \) into horizontal and vertical parts. The following statement shows that under an appropriate choice of \( Y \in \mathfrak{X}_{\text{inv}}(M) \), we can get \( \bar{W}_{\text{ver}} = 0 \).

**Proposition 4.2** If

\[
d\omega \neq 0 \text{ on } M,
\]
then one can choose an \( S^1 \)-invariant vector field \( Y \) (4.10) in a such way that the near identity transformation \( \Phi_\varepsilon \) (4.1) brings the perturbed vector field \( P_\varepsilon = X + \varepsilon W \) into the form \( \tilde{P}_\varepsilon = (\Phi_\varepsilon)^*P_\varepsilon = \tilde{P}_{\text{hor}} + \tilde{P}_{\text{ver}} \) with

\[
\tilde{P}_{\text{ver}} = X + O(\varepsilon^2),
\]
\[
\tilde{P}_{\text{hor}} = \varepsilon \text{hor}(w) + O(\varepsilon^2),
\]
where \( w \in \mathfrak{X}(O) \) is a unique vector field such that \( d\rho \circ \langle W \rangle = w \circ \rho \).

**Proof.** First, let us assume that \( O \) is parallelizable and pick a basis of global vector fields \( u_1, \ldots, u_n \) on \( O \). Then, we have the basis of global \( S^1 \)-invariant vector fields \( Y, \text{hor}(u_1), \ldots, \text{hor}(u_n) \) on \( M \). For the perturbation vector field \( W \), we have the decomposition \( W = W_{\text{hor}} + W_{\text{ver}} \), where \( W_{\text{hor}} = \sum_{i=1}^n c_i \text{hor}(u_i) \) and \( W_{\text{ver}} = c_0 Y \) for some \( c_i \in C^\infty(M) \). Then, its \( S^1 \)-average is given by

\[
\langle W \rangle = \sum_{i=1}^n \langle c_i \rangle \text{hor}(u_i) + \langle c_0 \rangle Y
\]

It follows that the condition \( \bar{W}_{\text{ver}} = 0 \) is equivalent to the algebraic equation

\[
i_Y d\omega = -\langle c_0 \rangle \text{ for } Y \in \mathfrak{X}_{\text{inv}}(M).
\]

Under assumption (4.7), a solution to this equation is

\[
Y = -\frac{\langle c_0 \rangle}{a^2} \sum_{i=1}^n a_i \text{hor}(u_i),
\]

where \( a_i = i_{\text{hor}(u_i)}d\omega \) are \( S^1 \)-invariant functions on \( M \) and \( a^2 = \sum_{i=1}^n a_i^2 \). In the general case, the statement follows from the partition of unity argument. \( \blacksquare \)

Remark that in terms of the averaged vector field \( w \) the normalization condition (4.5) reads \( L_w \omega_\Omega = 0 \) on \( O \). In this case, \([X, \text{hor}(w)] = 0\).

**The Hamiltonian case.** Let us show that, in the case when the perturbed vector field is Hamiltonian, the normalization condition (4.5) is satisfied. Let \( (M, \Omega) \) be a symplectic manifold. Suppose that a perturbed vector field \( P_\varepsilon = X + \varepsilon W \) is Hamiltonian relative to the symplectic structure \( \Omega \), \( i_X \Omega = -dH_0 \) and \( i_W \Omega = -dH_1 \) for some \( H_0, H_1 \in C^\infty(M) \). Assume that the flow of \( X \) is periodic with frequency function \( \omega \) and the corresponding \( S^1 \)-action is free. In particular \( dH_0 \neq 0 \) on \( M \). Then, according to the period-energy relation for Hamiltonian systems [3], [4], we have the identity

\[
dH_0 \wedge d\omega = 0
\]
saying that $\omega$ functionally depends only on $H_0$. It follows from (4.11) that the $S^1$-action preserves the symplectic form and hence the averaged vector field $\langle W \rangle$ is also Hamiltonian, $i_{\langle W \rangle} \Omega = -d\langle H_1 \rangle$. Then,

$$i_{\langle W \rangle} dH_0 = \Omega(\langle W \rangle, X) = -i_X d\langle H_1 \rangle = -\mathcal{L}_X \langle H_1 \rangle = 0.$$ 

Finally, from here and (4.11) we get the equality

$$0 = (i_{\langle W \rangle} dH_0) \omega - (i_{\langle W \rangle} d\omega) dH_0 = -(\mathcal{L}_{\langle W \rangle} \omega) dH_0$$

which implies (4.0). Moreover, one can show that formula (4.2) for $Y = 0$ gives a vector field $Z$ which is Hamiltonian relative to $\Omega$ and the function $S(H_1, \omega)$. Therefore, in the Hamiltonian case, Proposition 4.1 leads to the well-known result [4] on the global normalization of perturbed Hamiltonian dynamics relative to periodic flows.

## 5 The averaging on slow-fast phase spaces

Let $M = M_1 \times M_2$ be a product of two symplectic manifolds $(M_1, \sigma_1)$ and $(M_2, \sigma_2)$. Let $\pi_1 : M \to M_1$ and $\pi_2 : M \to M_2$ be the canonical projections and $d_1$ and $d_2$ the partial exterior derivatives on $M$ along $M_1$ and $M_2$, respectively. It is clear that $d = d_1 + d_2$ is the exterior derivative on $M$ and $d_1^2 = d_2^2 = d_1 \circ d_2 + d_2 \circ d_1 = 0$. Introduce the following $\epsilon$-dependent 2-form on $M$:

$$\sigma = \pi_1^* \sigma_1 + \varepsilon \pi_2^* \sigma_2$$

which is a symplectic structure for all $\varepsilon \neq 0$. For $H \in C^\infty(M)$, denote by $V_H$ the Hamiltonian vector field relative to $\sigma$. Then, $V_H = V_H^{(1)} + \frac{\varepsilon}{2} V_H^{(2)}$, where $V_H^{(1)}$ and $V_H^{(2)}$ are vector fields on $M$ uniquely defined by the relations

$$i_{V_H^{(1)}} (\pi_1^* \sigma_1) = -d_1 H,$$

$$d \pi_2 \circ V_H^{(1)} = 0$$

and

$$i_{V_H^{(2)}} (\pi_2^* \sigma_2) = -d_2 H,$$

$$d \pi_1 \circ V_H^{(2)} = 0.$$ 

It follows that, for all $m_1 \in M_1$ and $m_2 \in M_2$, the vector fields $V_H^{(1)}$ and $V_H^{(2)}$ are tangent to the symplectic slices $M_1 \times \{m_2\}$ and $\{m_1\} \times M_2$, respectively. For every $u \in \mathfrak{X}(M_1)$, denote by $\hat{u} = u \oplus 0 \in \mathfrak{X}(M)$ the lifting associated to the canonical decomposition $TM = TM_1 \oplus TM_2$.

On the slow-fast phase space $(M, \sigma)$, let us consider the following perturbed Hamiltonian model [5, 6, 17, 18]

$$H_\varepsilon = f \circ \pi_1 + \varepsilon F,$$
for some \( f \in C^\infty(M_1) \) and \( F \in C^\infty(M) \). The corresponding Hamiltonian vector field takes the form
\[
V_{H_\varepsilon} = V + \varepsilon W,
\]
where
\[
V = \hat{v}_f + V_F^{(2)}
\]
and
\[
W = V_F^{(1)}
\]
are unperturbed and perturbation vector fields, respectively. Here \( v_f \) denotes the Hamiltonian vector field on \((M_1, \sigma_1)\) of \( f \).

Remark that, in general, the unperturbed vector field \( V \) is not Hamiltonian relative to the symplectic structure \((5.1)\). Indeed, it is easy to show that this happens only if \( F = \pi_1^* f_1 + \pi_2^* f_2 \), for some \( f_1 \in C^\infty(M_1) \) and \( f_2 \in C^\infty(M_2) \).

The vector field \( V \) is \( \pi_1 \)-related with \( v_f \) and hence the trajectories of \( V \) are projected onto trajectories of the Hamiltonian vector field \( v_f \), \( \pi_1 \circ \text{Fl}_t^V = \varphi^t \circ \pi_1 \).

Here, \( \varphi^t \) denotes the flow of \( v_f \). Therefore, \( \text{Fl}_t^V \) is the skew-product flow,
\[
\text{Fl}_t^V(m_1, m_2) = (\varphi^t(m_1), G_{m_1}^t(m_2)),
\]
where \( G_{m_1}^t \) is a smooth family of symplectomorphisms on \((M_2, \sigma_2)\) determining as the solution of the time-dependent Hamiltonian system
\[
dG_{m_1}^t(m_2) = V_F^{(2)}(\varphi^t(m_1), G_{m_1}^t(m_2)),
\]
\[
G_{m_1}^0 = \text{id}_{M_2}.
\]
Assume that the flow \( \text{Fl}_t^V \) is periodic with frequency function \( \omega = \frac{2\pi}{T} \). Then,
\[
\varphi^{t+T(m_1, m_2)}(m_1) = \varphi^t(m_1)
\]
for all \( m_1 \in M_1, m_2 \in M_2 \) and \( t \in \mathbb{R} \). If \( v_f \neq 0 \) on \( M_1 \), then differentiating equality \((5.10)\) along \( M_2 \) says that the period function \( T \) is independent of \( m_2 \) and hence \( \omega = \varpi \circ \pi_1 \), for a certain smooth positive function \( \varpi \) on \( M_1 \). Therefore, the Hamiltonian flow \( \varphi^t \) of \( v_f \) is also periodic with frequency function \( \varpi \).

**Theorem 5.1** Let \( V_{H_\varepsilon} = V + \varepsilon W \) be the Hamiltonian vector field on \((M, \sigma)\), where the flow of the unperturbed vector field \( V \) \((5.3)\) is periodic with frequency function \( \omega \). Assume that the regular set \( \text{Reg}(v_f) \) is everywhere dense in \( M_1 \) and the \( S^1 \)-action associated to the Hamiltonian flow \( \varphi^t \) of \( v_f \) is free on \( \text{Reg}(v_f) \). Then, the perturbation vector field \( W \) \((5.9)\) satisfies the normalization condition
\[
\mathcal{L}_W \omega = 0 \text{ on } M
\]
and hence by Proposition \((4.1)\) \( V_{H_\varepsilon} \) admits the normalization of first order with respect to \( V \).
Proof. It is sufficient to show that (5.11) holds on the open domain \( \pi_{1}^{-1}(\text{Reg}(v_{f})) \) which is everywhere dense in \( M \). The period-energy relation for \( v_{f} \) says that
\[
d\omega \wedge d(f \circ \pi_{1}) = 0
\]
on \( \pi_{1}^{-1}(\text{Reg}(v_{f})) \). The hypotheses of the theorem imply that \( d(f \circ \pi_{1}) \neq 0 \). On the other hand, taking into account (5.2), (5.4), (5.9) and the identity \( \hat{v}_{f} = V_{f \circ \pi_{1}}^{(1)} \), we get
\[
\mathcal{L}_{\hat{v}_{f}} F = \pi_{1}^{*} \sigma_{1}(V_{f \circ \pi_{1}}^{(1)}, V_{F}^{(1)}) \\
= -\pi_{1}^{*} \sigma_{1}(V_{F}^{(1)}, V_{f \circ \pi_{1}}^{(1)}) = -\mathcal{L}_{\hat{v}_{f}} (f \circ \pi_{1})
\]
It follows from here that
\[
\mathcal{L}_{\langle W \rangle} (f \circ \pi_{1}) = \langle \mathcal{L}_{W} (f \circ \pi_{1}) \rangle = -\langle \mathcal{L}_{\hat{v}_{f}} F \rangle = -\langle \mathcal{L}_{\langle W \rangle} \omega \rangle d(f \circ \pi_{1}).
\]
which implies (5.11).

Remark 5.1 In the situation when \( v_{f} \equiv 0 \), we get a Hamiltonian model which appears in the theory of adiabatic approximation [11], [13]. In this case, the periodicity of the flow of \( V = V_{F}^{(2)} \) does not imply that the perturbation vector field \( W = V_{f \circ \pi_{1}}^{(1)} \) satisfies (6.11). The period-energy relation for the restriction of \( V_{F}^{(2)} \) to the symplectic slices \( \{m_{1}\} \times M_{2} \) implies only that \( d_{2} \omega \wedge d_{2} F = 0 \).

Periodicity conditions. The periodicity of the flow of vector field \( V \) (5.8) can be formulated as a resonance relation. Suppose that the flow \( \varphi^{t} \) of \( v_{f} \) satisfies all hypotheses of Theorem 5.1. Choose a frequency function \( \omega : M_{1} \to \mathbb{R} \) of \( \varphi^{t} \) in a such a way that the orbit through every point \( m_{1} \in \text{Reg}(v_{f}) \) is \( \tau(m_{1}) \)-minimally periodic. Here \( \tau(m_{1}) = \frac{2\pi}{\omega(m_{1})} \). It follows that \( V \) is complete and the group property of \( \text{Fl}_{V}^{t} \) implies the relations
\[
G_{m_{1}}^{t_{1}+t_{2}} = G_{v_{2}(m_{1})}^{t_{2}} \circ G_{m_{1}}^{t_{1}}
\]
for any \( m_{1} \in M_{1} \) and \( m_{2} \in M_{2} \). Define the monodromy of the flow \( \text{Fl}_{V}^{t} \) over a point \( m_{1} \in M_{1} \) as a symplectomorphism \( g_{m_{1}} : M_{2} \to M_{2} \) given by \( g_{m_{1}} := G_{m_{1}, m_{1}}^{\tau(m_{1})} \). Then, property (5.13) implies the identity \( G_{m_{1}}^{t_{1}+\tau(m_{1})} = G_{m_{1}}^{t_{1}} \circ g_{m_{1}} \). It follows that the flow \( \text{Fl}_{V}^{t} \) is periodic if and only if there exists an integer \( k \geq 1 \) such that
\[
g_{m_{1}}^{k} = \text{id} \ \forall m_{1} \in M_{1}.
\]
In this case, the corresponding frequency function can be defined as \( \omega = \frac{1}{c} \omega \circ \pi_1 \).

An important class of perturbed dynamics on slow-fast spaces comes from the linearization procedure for Hamiltonian systems around invariant symplectic submanifolds [11]. In this case, the unperturbed term \( V \) is a linear vector field on a symplectic vector bundle which represents the normal linearized dynamics. The verification of condition (5.14) is related to computing the monodromy of a time-periodic linear Hamiltonian system. For several Hamiltonian models with two degree of freedom, this problem was studied in [19], [14].

**Example 5.1** On the phase space \((M = \mathbb{R}^2 \times \mathbb{R}^2, \sigma = dp_1 \wedge dq_1 + \varepsilon dp_2 \wedge dq_2)\), consider the following perturbed Hamiltonian

\[
H_\varepsilon = \frac{1}{2} p_1^2 + \frac{1}{4} q_1^4 + \frac{\varepsilon}{2} (p_2^2 + \delta q_1^2 q_2^2) \quad (5.15)
\]

where \( \varepsilon \ll 1 \) is the perturbation parameter and \( \delta \in \mathbb{R} \). The corresponding Hamiltonian vector field has the representation (5.7), where the unperturbed and perturbed parts are of the form

\[
V = p_1 \frac{\partial}{\partial q_1} - q_1^3 \frac{\partial}{\partial p_1} + p_2 \frac{\partial}{\partial q_2} - \delta q_1^2 q_2 \frac{\partial}{\partial p_2}, \quad (5.16)
\]

\[
W = -\delta q_1 q_2 \frac{\partial}{\partial p_1}. \quad (5.17)
\]

Using the results in [19], one can show that the flow of the vector field \( V \) is periodic (condition (5.14)) if and only if the parameter \( \delta \) satisfies the relation

\[
\sqrt{2} \cos \left( \frac{\pi}{4} \sqrt{1 + 8\delta} \right) = \cos \left( \frac{\pi n}{k} \right)
\]

for arbitrary coprime integers \( n, k \in \mathbb{Z} \) such that \( 0 < n < k \). Solutions of this equation are represented in the form \( \delta = \frac{r(n+1)}{2} \), where \( r \) runs over the subset \( \Delta = \mathbb{Q} \cap \bigcup_{s=0}^{\infty} (4s, 4s+1) \). The frequency function is given by \( \omega = \frac{1}{c}(2 p_1^2 + q_1^4) \),

where \( c = \sqrt{2} \int_0^1 \frac{dz}{\sqrt{4z^2 + 1}} \). Therefore, for every \( r \in \Delta \), vector fields (5.10), (5.17) satisfy the hypotheses of Theorem 7.1 and hence Hamiltonian system (5.15) admits a normalization of first order. Notice that this system is non-integrable for all \( \varepsilon \neq 0 \) and \( r \in (4s, 4s+1), s \in \mathbb{Z}_+ \) [10, 14].

**Hamiltonian Structures.** As we have mentioned above the vector field \( V \) does not inherit any natural Hamiltonian structure from \( V_{H_\varepsilon} \). Here, we formulate a criterion for the existence of Hamiltonian structure for \( V \) which is based on Corollary 3.5 and the results in [5, 18].

Let \( \Omega^1_{\text{hor}}(M) \) be the subspace of horizontal 1-forms \( \eta \) on \( M \) with respect to the projection \( \pi_1 : M \to M_1 \). A 1-form \( \eta \) belongs to \( \Omega^1_{\text{hor}}(M) \) if \( \iota_Z \eta = 0 \) for every vector field \( Z \) on \( M \) such that \( d\pi_1 \circ Z = 0 \). In particular, \( d_1 F \in \Omega^1_{\text{hor}}(M) \). Since the infinitesimal generator \( \mathcal{T} = \frac{1}{2} V \) of the \( S^1 \)-action is \( \pi_1 \)-related with \( \frac{2p}{c} \omega_f \), the pull-back by the flow \( \text{Fl}_t \) of the \( S^1 \)-action is \( \pi_1 \)-related with \( \frac{2p}{c} \omega_f \), the pull-back by the flow \( \text{Fl}_t \) of the \( S^1 \)-action is \( \pi_1 \)-related with \( \frac{2p}{c} \omega_f \). It follows that \( \Omega^1_{\text{hor}}(M) \) is also an invariant subspace for operators \( \mathcal{A}, \mathcal{L}_\mathcal{T} \) and \( \mathcal{S} \).
Theorem 5.2 Suppose that the flow of the vector field \( \mathbb{V} = \dot{v} + V^{(2)}_F \) is periodic with frequency function \( \omega \) and there exists a \( S^1 \)-invariant horizontal 1-form \( \mu \in \Omega^1_{\text{hor}}(M) \) such that
\[
\langle d_1 F \rangle = (i_{\dot{v}_I} \mu)d_1 \omega.
\] (5.18)
Then, for a given open domain \( N \subset M \) with compact closure and small enough \( \varepsilon \neq 0 \), the vector field \( \mathbb{V} \) is Hamiltonian relative to the symplectic structure
\[
\hat{\sigma} = \sigma_1 + \varepsilon \sigma_2 - \varepsilon d\theta,
\] (5.19)
and the function
\[
\hat{H}_\varepsilon = f \circ \pi + \varepsilon (F - i_{\dot{v}_I} \theta).
\] (5.20)
Here \( \theta \in \Omega^1_{\text{hor}}(M) \) is a horizontal 1-form on \( M \) given by
\[
\theta = \frac{1}{\omega} S(d_1 F) - \frac{1}{\omega^3} S^2(\mathcal{L}_{\dot{v}_I} F)d_1 \omega + \mu.
\] (5.21)
Moreover, the functions \( f \circ \pi \) and \( F - i_{\dot{v}_I} \theta \) are Poisson commuting first integrals of \( \mathbb{V} \).

Proof. According to \[5, 18\], the vector field \( \mathbb{V} \) in \( \text{(5.18)} \) is Hamiltonian relative to 2-form \( \omega \) and function \( \text{(5.20)} \) if the horizontal 1-form \( \theta \in \Omega^1_{\text{hor}}(M) \) satisfies the homological equation
\[
\mathcal{L}_\mathbb{V} \theta = d_1 F.
\] (5.22)
Indeed, taking into account that \( i_{V^{(2)}} \theta = 0 \), we rewrite equation \( \text{(5.22)} \) in the form
\[
\mathcal{L}_{\dot{v}_I} \theta + i_{V^{(2)}} d\theta = d_1 F.
\] (5.23)
On other hand, we have
\[
i_{\dot{v}_I} \hat{\sigma} = -d(f \circ \pi_1) + \varepsilon (d_{\dot{v}_I} \theta - \mathcal{L}_{\dot{v}_I} \theta)
\] and \( i_{V^{(2)}} \hat{\sigma} = -\varepsilon (d_2 F + i_{V^{(2)}} d\theta) \). It follows form these relations and \( \text{(5.23)} \) that
\[
i_{\dot{v}_I} \hat{\sigma} = d(f \circ \pi_1) + \varepsilon (d_{\dot{v}_I} \theta - \mathcal{L}_{\dot{v}_I} \theta - d_2 F - i_{V^{(2)}} d\theta),
\]
\[
= -d(f \circ \pi_1) + \varepsilon (d_{\dot{v}_I} \theta - d_1 F - d_2 F).
\]
Finally, under condition \( \text{(5.18)} \), by Corollary 3.5 a solution \( \theta \) to equation \( \text{(5.22)} \) is given by formula \( \text{(5.21)} \).

It is clear that if \( \langle d_1 F \rangle = 0 \), then \( \text{(5.18)} \) is satisfied for \( \mu = 0 \). But, in general, one can not omit \( \mu \) in condition \( \text{(5.18)} \).

Example 5.2 Consider the phase space \( (M = M_1 \times M_2, \sigma) \), in the case when \( M_1 = \mathbb{R} \times S^1 \) is the 2-cylinder equipped with standard symplectic form \( \sigma_1 = ds \wedge d\varphi \). Let \( v_f = \varpi(s) \frac{\partial}{\partial s} \) where \( f = f(s) \), and \( \varpi = \frac{\partial f}{\partial s} \). Consider a vector field \( \mathbb{V} = \varpi(s) \frac{\partial}{\partial s} + V^{(2)}_F \), where \( F \in C^\infty(M) \) is a function which is independent of \( \varphi \) and \( \frac{\partial F}{\partial \varphi} \neq 0 \). Assuming that \( \frac{\partial \varpi}{\partial s} \neq 0 \), we get that \( \langle d_1 F \rangle = d_1 F \neq 0 \) and condition \( \text{(5.18)} \) holds for \( \mu = \frac{\partial \varpi}{\partial s} - \frac{\partial F}{\partial \varphi} d\varphi \).
Finally, we remark that if \( \langle d_1F \rangle \wedge d\omega \neq 0 \), then condition (5.18) does not hold.

**Invariant symplectic structures.** Consider again the phase space \((M = M_1 \times M_2, \sigma = \pi_1^*\sigma_1 + \varepsilon \pi_2^*\sigma_2)\) and suppose that we are given an \( S^1 \)-action on \( M \) (independent of \( \varepsilon \)) with infinitesimal generator of the form \( \Upsilon = \hat{v}_h + V^{(2)}_J \), for some \( h \in C^\infty(M_1) \) and \( J \in C^\infty(M) \). Therefore, \( \Upsilon \) is \( \pi_1 \)-related with a Hamiltonian vector field \( v_h \) on \((M_1, \sigma_1)\). The \( S^1 \)-action on \( M \) descends to a Hamiltonian \( S^1 \)-action on \( M_1 \) with infinitesimal generator \( v_h \).

Then, we have the relations

\[
i_\Upsilon \sigma = -d_1(h \circ \pi_1) - d_2J, \quad (5.24)
\]

\[
\mathcal{L}_\Upsilon \sigma = d i_\Upsilon \sigma = -\varepsilon d_1 d_2 J
\]

which say that the symplectic form \( \sigma \) is \( S^1 \)-invariant only in the case when \( J = \pi_1^* j_1 + \pi_2^* j_2 \) for some \( j_1 \in C^\infty(M_1) \) and \( j_2 \in C^\infty(M_2) \).

**Theorem 5.3** The \( S^1 \)-average of the symplectic form \( \sigma \) has the following representation

\[
\langle \sigma \rangle = \sigma - \varepsilon d\beta,
\]

where \( \beta \in \Omega^1_{\text{hor}}(M) \) is the horizontal 1-form given by

\[
\beta = S(d_1 J) \equiv \frac{1}{2\pi} \int_0^{2\pi} (t - \pi)(F^t \Upsilon)^* d_1 J dt.
\]

For every open domain \( N \subset M \) with compact closure and small enough \( \varepsilon \), \( \langle \sigma \rangle \) is symplectic form on \( N \). The \( S^1 \)-action is Hamiltonian relative to \( \langle \sigma_M \rangle \) if and only if

\[
\langle d_1 J \rangle = 0.
\]

Under this condition, the corresponding momentum map is

\[
\tilde{J}_\varepsilon = h \circ \pi_1 + \varepsilon (J - i_{v_h} \beta).
\]

**Proof.** By formulas (3.46) and (5.24), the \( S^1 \)-average of the symplectic form \( \sigma \) is given by \( \langle \sigma \rangle = \sigma - \varepsilon d\beta \), where

\[
\tilde{\beta} = \frac{1}{2\pi} \int_0^{2\pi} (t - \pi)(F^t \Upsilon)^* i_\Upsilon \sigma_M dt = -\frac{1}{2\pi} \int_0^{2\pi} (t - \pi)(F^t \Upsilon)^* d_2 J dt,
\]

\[
= -d_1\left( \frac{1}{2\pi} \int_0^{2\pi} (t - \pi)(F^t \Upsilon)^* J dt \right) + \frac{1}{2\pi} \int_0^{2\pi} (t - \pi)(F^t \Upsilon)^* d_1 J dt.
\]

This implies (5.25). The non-degeneracy of \( \langle \sigma_M \rangle \) for small enough \( \varepsilon \neq 0 \), follows from the evaluating of the 2-form \( \langle \sigma_M \rangle \) on the basis of vector fields \( \{\hat{v}_{\xi_i} + V^{(2)}_{x_\alpha}\} \), where \( \{\xi_i\} \) and \( \{x_\alpha\} \) are coordinates functions on \( M_1 \) and \( M_2 \), respectively (see, also [5]). Condition (5.27) and formula (5.28) follow directly from Theorem 5.2.

\[\square\]
Remark 5.2 If $v_h = 0$, then we can think of the $S^1$-action as a family of Hamiltonian actions on $(M_2, \sigma_2)$ with parameterized momentum map $J_{m_1}(m_2) = J(m_1, m_2)$. In this case, hypothesis (5.27), called the adiabatic condition, was introduced in [12, 13] in the context of the Hannay-Berry connections.

Example 5.3 Consider the phase space $(M = M_1 \times M_2, \sigma = \pi_1^*\sigma_1 + \varepsilon \pi_2^*\sigma_2)$, where $M_2 = S^2 \subset \mathbb{R}^3 = \{x = (x_1, x_2, x_3)\}$ is the unit sphere equipped with standard symplectic form $\sigma_2 = \frac{1}{2}\epsilon_{ijk}x^i dx^j \wedge dx^k$. Given a smooth mapping $\phi : M_1 \to S^2$, we define the function $J \in C^\infty(M)$ by $J(m_1, x) = \phi(m_1) \cdot x$ for $m_1 \in M_1$ and $x \in S^2$. Consider the $S^1$-action on $M$ with infinitesimal generator $V_2^{(2)} = -\left(\phi \times x\right) \cdot \frac{\partial}{\partial \phi}$, which is given by the rotations in $\mathbb{R}^3_x$ about the axis $\phi$. Then, one can show that the $S^1$-invariant symplectic form $\langle \sigma \rangle$ has the representation (5.27), where $\beta = \left(\phi \times x\right) \cdot d_1 \phi$. Moreover, it easy to see that (5.27) holds, $\langle d_1 J \rangle = \left(\phi \cdot x\right)(\phi \cdot d_1 \phi) = 0$, and hence the $S^1$-action is Hamiltonian relative to $\langle \sigma \rangle$ with momentum map $\varepsilon J$. Such a kind of invariant symplectic structures appears in the study of the particle dynamics with spin in the context of the averaging method [17].

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