GENERALIZED SOLUTIONS TO THE DIRICHLET PROBLEM OF TRANSLATING MEAN CURVATURE EQUATIONS

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Abstract. In this paper we study the Dirichlet problem of translating mean curvature equations over domains in Riemannian manifolds with dimension $n$. Imitating the generalized solution theory of Miranda-Giusti, we define a new conformal area functional and a generalized solution to this Dirichlet problem. The existence of generalized solutions to this problem on bounded Lipschitz domains is established. If the domain is mean convex and bounded with $C^2$ boundary, its closure does not contain any closed minimal hypersurface except a singular set with its Hausdorff dimension at most $n - 7$ and the boundary data is continuous, the generalized solution is the desirable classical smooth solution. The non-minimal condition could not be removed in general.

1. Introduction

In this paper we consider the Dirichlet problem of the following quasilinear equation

\[
\begin{aligned}
\text{div} \left( \frac{Du}{\omega} \right) &= \frac{\alpha}{\omega}, \quad x \in \Omega, \quad \omega = \sqrt{1 + |Du|^2}, \\
u(x) &= \psi(x) \quad x \in \partial \Omega
\end{aligned}
\]

Here $\Omega$ is a bounded open set with Lipschitz boundary in a Riemannian manifold $N$, $Du$ denotes the gradient of $u$, $\alpha$ is a fixed constant, $\text{div}$ is the divergence of $N$ and $\psi(x)$ is a continuous or $L^1$ function on $\partial \Omega$. The equation in (1.1) is called as a translating mean curvature equation.

1.1. Background. The motivation of our paper are two-fold. First it naturally arises in the study of type II mean convex singularities of mean curvature flows or translating mean curvature flow in Euclidean spaces (i.e see White [24], Wang [23], Spruck-Xiao [21], Hoffman-Ilmanen-Martin-White [12] etc). Thus we name (1.1) in terms of “translating mean curvature equation”. Second we are interested in the Dirichlet problem of mean curvature equations in Riemannian manifolds. The generalization of various boundary problems of mean curvature equations is not trivial (see Appendix C).
The Dirichlet problem of (1.1) on mean convex domains in $\mathbb{R}^n$ with continuous boundary data is solved by [24]. Ma [16] and [23] via geometric measure theory, mean curvature type flows and the Monge-Ampere equation theory respectively. A key fact in $\mathbb{R}^n$ is that there is a global solution to the translating mean curvature equation $\text{div}(Du\omega) = \frac{\alpha}{\omega}$ in $\mathbb{R}^n$. This provides a $C^0$ bound for any $C^2$ solution to (1.1).

However in Riemannian manifolds no similar fact exists in general except manifolds with a similar warped metric structure as $\mathbb{R}^n$. As a result no similar $C^0$ bound estimate is known for the solution to (1.1) in general Riemannian manifolds. In Appendix C we give an example of a domain such that no $C^2$ solution exists for (1.1) when $\alpha = n$. We need an approach to attack the Dirichlet problem (1.1) without requiring $C^0$ a priori estimate different with the previous methods in [7, 24, 23, 16].

One candidate for our purpose is the generalized solution theory of the Dirichlet problem to the following mean curvature equation

\begin{equation}
\text{div}(Du\omega) = f(x), \quad x \in \Omega \quad u(x) = \psi(x) \quad x \in \partial \Omega
\end{equation}

developed by Miranda and Giusti in [17, 18, 8, 9]. Its key idea is described as follows. One defines an area functional on bounded variation (BV) functions with the property that if $u$ is smooth and minimizes this functional then $u$ satisfies (1.2). One defines a generalized solution $u(x)$ to (1.2) taking possible infinity values which locally minimizes the corresponding functional. Without requiring $C^0$ estimate they showed the existence of the generalized solution to the corresponding Dirichlet problem. Suppose the domain is mean convex in $\mathbb{R}^n$ and the boundary data is continuous they showed that such generalized solutions are classical. More details are given in [10]. For other applications of the Miranda-Giusti theory see Scholz [19] and Liang [15].

1.2. Main Results. Now we follow the idea of Miranda-Giusti generalized solution theory. Before stating the definition, we need some notation. Suppose $N$ is a Riemannian manifold with metric $g$ and $\text{dim} \, N = n$. Let $\Omega$ be an open set in $N$. Fix $\alpha > 0$. Suppose $BV(\Omega)$ is the set of bounded variation (BV) functions in $\Omega$ (Definition 2.1). Let $C^0(\Omega)$ and $T_0\Omega$ be the set of all smooth functions and smooth vector fields with compact support in $\Omega$ respectively.

For $u \in BV(\Omega)$, define a conformal area functional

$$\mathcal{F}_\alpha(u, \Omega) = \sup_{h \in C^0(\Omega), X \in T_0\Omega} \left\{ \int_\Omega e^{\alpha u}(h + \frac{1}{\alpha}\text{div}(X)) \text{dvol} : h^2 + \langle X, X \rangle \leq 1 \right\}$$

Observe that when $u \in C^2(\Omega)$ and minimizes $\mathcal{F}_\alpha(u, \Omega)$ locally, it solves $\text{div}(Du\omega) = \frac{\alpha}{\omega}$. Moreover for $u \in C^1(\Omega)$, $\mathcal{F}_\alpha(u, \Omega)$ is the area of the graph of $u(x)$ in $Q_\alpha$ which is the manifold $N \times \mathbb{R}$ with the metric $e^{2\alpha}(g + dr^2)$.

Let $U$ be the subgraph of $u(x)$ defined by $\{ (x, t) : x \in \Omega, t < u(x) \}$ in $Q_\alpha$. Inspired by Miranda-Giusti [10] we proposed the following definition.
Definition 1.1. Let \( \Omega \subset \subset B \) be two open bounded set in \( N \). Let \( u(x), \psi(x) \) be two measurable functions taking values in \([ -\infty, \infty ]\) such that \( \Psi \), the subgraph of \( \psi(x) \), is a Caccioppoli set (Definition 5.1) in \( Q_\alpha \). We say that \( u(x) \) is a generalized solution to the Dirichlet problem (1.1) on \( \Omega \) with boundary data \( \psi(x) \) on \( \partial \Omega \) if

1. the subgraph of \( u(x) \), \( U \), coincides with \( \Psi \) outside \( \bar{\Omega} \times \mathbb{R} \);
2. for any Caccioppoli set \( F \) satisfying \( F \Delta U \subset K \) where \( K \) is a compact set in \( \bar{\Omega} \times \mathbb{R} \), it holds that

\[
\int_K |D\lambda_U|_{Q_\alpha} \leq \int_K |D\lambda_F|_{Q_\alpha}
\]

where \( |D\lambda_U|_{Q_\alpha} \) and \( |D\lambda_F|_{Q_\alpha} \) are Radon measures of generated by \( \lambda_U \) and \( \lambda_F \) in \( Q_\alpha \) respectively (See Theorem 2.6).

A key point of this definition is Condition (2) above. It says that \( u \) locally minimizes \( \mathfrak{Z}_\alpha(\cdot, \Omega) \) (see Theorem 5.5). In our setting a generalized solution always exists.

Theorem 1.2 (Theorem 6.6). Let \( \Omega \subset \subset B \) be two open bounded set in a Riemannian manifold \( N \). Suppose \( \partial \Omega \) is Lipschitz and \( \psi(x) \) is a measurable function such that its subgraph \( \Psi \) is a Caccioppoli set in \( Q_\alpha \). Then there is a generalized solution to the Dirichlet problem (1.1) with boundary data \( \psi(x) \).

Remark 1.3. The assumption on \( \psi(x) \) is not restrictive. For any \( \psi(x) \in L^1(\partial \Omega) \) we can extend \( \psi(x) \) as the trace of a BV function on \( B \) such that its subgraph is a Caccioppoli set in \( Q_\alpha \). For its proof see Remark 6.7. Remark C.3 gives an example of a nontrivial generalized solution to the Dirichlet problem (1.1) with finite boundary data.

Whether generalized solutions are classical depends heavily on the geometry of the closure of \( \Omega \), the condition (2) below.

Theorem 1.4 (Theorem 8.1). Let \( N \) be a Riemannian manifold with dimension \( n \). Suppose \( \Omega \) is a bounded open domain with \( C^2 \) boundary satisfying

1. \( H_{\partial \Omega} = \text{div}(\vec{v}) \geq 0 \) on \( \partial \Omega \) where \( \vec{v} \) is the outward normal vector of \( \partial \Omega \);
2. (a) if \( n \leq 7 \), no closed embedded minimal hypersurface exists in \( \bar{\Omega} \);
   (b) if \( n > 7 \), no closed embedded minimal hypersurface with a closed singular set \( S \) with \( H^k(S) = 0 \) for \( k > n - 7 \) exists in \( \bar{\Omega} \) where \( H^k \) denotes the k-dimensional Hausdorff measure on \( N \);

Then the Dirichlet problem (1.1) admits an unique solution \( u \in C^2(\Omega) \cap C(\bar{\Omega}) \) for any continuous function \( \psi(x) \) on \( \partial \Omega \). Here \( C^k \) denotes the property of k-th differential.

Remark 1.5. In the Miranda-Giusti generalized solution theory [10], the condition (1) is sufficient to transfer a generalized solution into a classical solution. However in our setting the condition (2) can not be removed in
general. In Appendix C we show that in the upper half sphere $S^+_n$ in the sphere $S^n$ no classical solution to the Dirichlet problem of (1.1) exists for continuous boundary data for $\alpha = n$. Its boundary $\partial S^+_n$ is minimal.

Remark 1.6. In Euclidean spaces and Hyperbolic spaces all bounded domains satisfy the condition (2) by the maximum principle of singular stationary hypersurface in Ilmanen [13]. Thus Theorem 1.4 generalizes the corresponding results in Euclidean spaces by [23, 24, 16].

Now we present the ideas to show Theorem 1.2 (Theorem 6.6) and Theorem 1.4 (Theorem 8.1).

One observes that $\tilde{c}_\alpha(u, \Omega)$ is equal to the perimeter of the subgraph of $u$ in $Q_\alpha$ (Theorem 5.5). Thus the locally minimizer of $\tilde{c}_\alpha(u, \Omega)$ corresponds to a locally minimizer of the perimeter in $Q_\alpha$ (Theorem 5.8). We called those results as the Miranda’s observation. Then the strategy in Theorem 16.11 of [10] shall work in our setting and yields Theorem 6.6.

With the assumption of Theorem 8.1 it admits a generalization $u(x)$ with continuous boundary data $\psi(x)$ by Theorem 6.6. Consider the set $P_+$ given as $\{x \in \Omega : u(x) = \infty\}$. The finiteness of $\psi(x)$ and the smooth boundary of $\Omega$ implies that $\partial P_+ \times \mathbb{R}$ is a locally almost minimal set in $Q_\alpha$. This is very different with the case of [10] (see Remark 7.12). By the mean convex condition and the property of generalized solutions, the regularity of the almost minimal set implies that $\partial P_+$ satisfies the opposite of Condition (2) in Theorem 5.1 (see Theorem 7.11). This means $P_+ (P_-)$ is empty and $u(x)$ is locally bounded. By Theorem 8.3 $u(x) \in \mathcal{C}^2(\Omega)$. The boundary continuity of $u(x)$ is established in Lemma 8.9. This finally yields Theorem 8.1.

1.3. Outline. Our paper is organized as follows. In Section 2 we discuss the properties of BV functions. In Section 3 we discuss various properties of product area functional $\tilde{c}(u, \Omega)$ and conformal area funcional $\tilde{c}_\alpha(u, \Omega)$ (see Definition 3.1). It is technically involved because the method of Theorem 1.17 in [10] can be directly applied.

In Section 4 we record results of the trace of BV functions on Lipschitz domains in Riemannian manifolds. In Section 5 we show the Miranda’s observation upon the perimeter in $Q_\alpha$ and $\tilde{c}_\alpha(u, \Omega)$. In Section 6 we show Theorem 5.6. In Section 7 we study various properties of generalized solutions only taking infinity values from a given generalized solution (Lemma 7.2). All results are summarized in Theorem 7.11. In Section 8 we prove Theorem 1.4 (Theorem 8.1).

In Appendix A we record a decomposition result for Radon measures in Riemannian manifolds. It is useful to prove the $C^\infty$ approximation of $\tilde{c}_\alpha(u, \Omega)$ in Section 3. In Appendix B we show that the Dirichlet problem (1.1) is solvable in sufficiently small normal balls in Riemannian manifolds. It is a key result to show the regularity result in Theorem 8.3. In Appendix C we give an example of a domain such that no classical solution to
the Dirichlet problem exists. In Appendix D we collect some facts on mean curvatures of hypersurfaces.

At last we should remind the reader that our paper depends deeply on the book of Giusti [10].

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2. BV functions in Riemannian manifolds

In this section we discuss the definition of BV function on Riemannian manifold. Our viewpoint is that a BV function corresponds a Radon measure. We also define the convolution of functions and vector fields in a normal ball for later use. We mainly follow from Chapter 1 of Simon [20], the books of [10] and [5].

2.1. BV functions. Let $N$ be a Riemannian manifold with a metric $g$. Let $\langle , \rangle$ be the corresponding inner product. Write $\text{div}$ and $d\text{vol}$ for the divergence and volume of $N$ respectively. Suppose $\Omega \subset N$ is an open set. Let $T_0\Omega$ be the set of smooth vector fields with compact support in $\Omega$ and $d\text{vol}$ be the volume form of $\Omega$.

**Definition 2.1.** Let $u \in L^1(\Omega)$, we define

$$\|Du\|_N(\Omega) = \sup \{ \int_{\Omega} u\text{div}(X)d\text{vol}, X \in T_0\Omega \quad \langle X, X \rangle \leq 1 \}$$

If $\|Du\|_N(\Omega) < \infty$, we say that $u$ has bounded variation in $\Omega \subset N$ and $u \in BV(\Omega)$ or $u \in BV_N(\Omega)$ to emphasize the manifold $N$.

If $u \in BV_N(\Omega')$ for any bounded open set $\Omega' \subset \Omega$, we say $u \in BV_{\text{loc},N}(\Omega)$.

**Remark 2.2.** If $u \in C^1(\Omega)$, the divergence theorem implies that

$$\int_{\Omega} u\text{div}(X)d\text{vol} = -\int_{\Omega} \langle X, Du \rangle d\text{vol}$$

for any $X \in T_0\Omega$ where $Du$ is the gradient of $u$ on $\Omega$.

A straightforward verification shows that BV functions have a lower semi-continuity as follows.

**Theorem 2.3.** Suppose $\{u_j\}_{j=1}^{\infty} \in BV(\Omega)$ and converges to $u$ in $L^1(\Omega)$ as $j \to +\infty$. Then

$$\|Du\|_N(\Omega) \leq \lim_{j \to +\infty} \|Du_j\|_N(\Omega)$$
2.2. **Radon measures.** Now we shall see that BV functions naturally induce Radon measures. For more details we refer to Chapter 1 in [20].

**Definition 2.4.** Let $X$ be a locally compact Hausdorff measure. A Radon measure on $X$ is an outer measure $\nu$ on $X$ having three properties:

1. $\nu$ is Borel regular and $\nu(K) < \infty$ for any compact set $K \subset X$;
2. $\nu(A) = \inf\{\nu(U) : A \subset U, \ U \text{ open}\}$ for each subset $A \subset X$;
3. $\nu(U) = \sup\{\nu(K) : K \text{ compact } \subset U\}$ for each open $U$ in $X$.

For any set $X$ let $K_+(X)$ denote the set of non-negative continuous functions $f : X \rightarrow [0, \infty)$ with compact support.

**Theorem 2.5 (Remark 4.3 in [20]).** Suppose $X$ is a locally compact Hausdorff space and $\lambda : K_+(X) \rightarrow [0, \infty)$ satisfies $\lambda(cf) = c\lambda(f)$, $\lambda(f + h) = \lambda(f) + \lambda(h)$ for any constant $c \geq 0$ and $f, g \in K_+(\Omega)$. Then there is a Radon measure $\nu$ on $X$ given by

$$\nu(U) = \sup\{\lambda(f), f \in K_+(X), \text{supp}(f) \subset U, f \leq 1\} \forall \text{ open } U \in X$$

such that

$$\lambda(f) = \int_X f d\nu \ \forall f \in K_+(\Omega)$$

Suppose $u(x) \in BV(\Omega)$. Set a nonnegative functional $\lambda_u : K_+(\Omega) \rightarrow [0, +\infty)$ as

$$\lambda_u(h) = \sup\{\int u\text{div}(X)dvol, X \in T_0\Omega, \langle X, X \rangle \leq h^2\}$$

for every $h \in K_+(\Omega)$. It is clear that

$$\lambda_u(ch) = c\lambda_u(h), \ \lambda_u(h + h_1) = \lambda_u(h) + \lambda_u(h_1)$$

where $c$ is any positive constant, $h$ and $h_1 \in K_+(\Omega)$. Thus Theorem 2.5 gives a Radon measure from a BV function as follows.

**Theorem 2.6.** Let $\Omega$ be an open set in a Riemannian manifold $N$. Suppose $u \in BV_{\text{loc}, N}(\Omega)$.

1. There is a Radon measure $|Du|_N$ on $\Omega$ such that

$$\int_{\Omega'} fd|Du|_N = \sup\{\int_{\Omega'} u\text{div}(X)dvol, X \in T_0\Omega', \langle X, X \rangle \leq f^2\}$$

for any bounded open set $\Omega' \subset \Omega$ and any nonnegative function $f \in L^1(|Du|_N, \Omega)$.

2. Moreover there is a vector field $\nu$ on $\Omega$ satisfies

$$\int_{\Omega} u\text{div}(X)dvol = -\int_{\Omega} \langle X, \nu \rangle d|Du|_N$$

where $\langle \nu, \nu \rangle = 1$ a.e. $|Du|_N$. 
Proof. The existence and definition of $|Du|_N$ are from Theorem 2.5. Then $|Du|_N$ is a Radon measure. Similar as the proof of Theorem 5.10 in Chapter 1 of [20], there is a monotone nonegative increasing sequence $\{f_j\}_{j=1}^\infty$ such that each $f_j \in K^+(\Omega)$, $f_j \leq f$ and $f_j$ converges to $f$ in $L^1(|Du|_N, \Omega)$. Let $\Omega'$ be any open set in $\Omega$. By (2.5),

$$\int_{\Omega'} f_j d|Du|_N = \sup \{ \int_{\Omega'} u \text{div}(X) d\text{vol}, X \in T_0\Omega', \langle X, X \rangle \leq f_j^2 \}$$

Letting $j \to +\infty$ on both sides we obtain the conclusion. The conclusion (2) is from a version of the Riesz representation theorem (see Theorem 4.1 in [20]). □

A relationship between BV functions on the manifold and its conformal manifold is given as follows.

**Definition 2.7.** Let $M$ be a Riemannian manifold with a metric $g$. Let $\phi(x) > 0$ be a smooth positive function on $M$. A conformal manifold $M_\phi$ is the smooth manifold $M$ with the metric $\phi^2 g$.

**Theorem 2.8.** Take the notation in Definition 2.7. Suppose $\Omega$ is an open set in $M$ and $u \in BV_{loc,M}(\Omega)$. Then

$$||Du||_{M_\phi}(\Omega) = \int_{\Omega} \phi^{m-1}(x) d|Du|_M$$

where $m$ is the dimension of $M$, $|Du|_M$ is the Radon measure given in Theorem 2.6.

**Remark 2.9.** Notice that the metric $g$ of $M$ can be written as $\phi^{-2} \phi^2 g$. A consequence of Theorem 2.8 is that $u \in BV_{loc,M}(\Omega)$ if and only if $u \in BV_{loc,M_\phi}(\Omega)$.

**Proof.** Let $\text{div}_\phi$ and $d\text{vol}_\phi$ be the divergence and the volume of $M_\phi$ respectively. Then $d\text{vol}_\phi = \phi^m d\text{vol}$ where $d\text{vol}$ is the volume form of $M$. By the definition of the divergence (Page 423 in [14]), we have

$$\text{div}_\phi(X) d\text{vol}_\phi = d(X, d\text{vol}_\phi) = (\phi^m \text{div}(X) + m\phi^{m-1}(X, \nabla \phi)) d\text{vol} = \text{div}(\phi^m X) d\text{vol}$$

(2.11)

where $\nabla \phi$ is the gradient of $\phi$ in $M$. By Theorem 2.6 one has that

$$||Du||_{M_\phi}(\Omega) = \sup \{ \int_{\Omega} f \text{div}_\phi(X) d\text{vol}_\phi : \phi^2 \langle X, X \rangle \leq 1, X \in T_0\Omega \}$$

$$= \sup \{ \int_{\Omega} f \text{div}(X') d\text{vol} : \langle X', X' \rangle \leq \phi^{2m-2}, X \in T_0\Omega \}$$

$$= \int_{\Omega} \phi^{m-1}(x) d|Du|_M$$

(2.12)

The proof is complete. □
2.3. The convolution of functions and vector fields. Now we consider how to approximate a function and a smooth vector field in a sufficiently small normal embedded ball in a Riemannian manifold.

**Definition 2.10.** Fix any point $p$ in a Riemannian manifold $M$. Let $\exp_p$ be the exponential map near $p$. There is a Euclidean ball $B_r(0)$ centered at 0 in $\mathbb{R}^n$ such that $\exp_p : B_r(0) \to B_r(p) \subset M$ is a diffeomorphism. Via the exponential map we can identify $B_r(p)$ with $B_r(0)$. Moreover the metric of $M$ is represented with

$$g = g_{ij}dx^idx^j$$

with the coordinate in $\mathbb{R}^n$. Such ball $B_r(p)$ is called as a normal (open) ball.

Let $\varphi(x)$ be a symmetric smooth mollifier in $\mathbb{R}^n$. i.e. $\varphi(x) = \varphi(-x)$ and $\varphi(x)$ has a compact support in the Euclidean unit ball $B_n(1)$ and

$$\int_{\mathbb{R}^n} \varphi(x)dx = 1$$

where $dx$ is the standard Euclidean volume in $\mathbb{R}^n$.

Suppose $W$ is an open set in $\mathbb{R}^n$. Let $h(x)$ denote a measurable function on $W$ and $X$ denote a tangent vector field on $W$ written as

$$X = X^i \frac{\partial}{\partial x^i}$$

where $\{ \frac{\partial}{\partial x^i} \}_{i=1}^n$ is the standard orthonormal coordinate vector fields in $\mathbb{R}^n$.

**Definition 2.11.** Let $\sigma > 0$ be a sufficiently small positive constant. Then $\varphi_\sigma * h(x)$, the convolution of $h(x)$, is given as follows.

$$\varphi_\sigma * h(x) = \int_{\mathbb{R}^n} \frac{1}{\sigma^n} \varphi\left(\frac{x-y}{\sigma}\right) h(y) dy, \quad x \in W$$

where we extend $h(x)$ outside $W$ as $h(x) = 0$ for $x \notin W$. For $X$ in (2.14), $\varphi_\sigma * X(x)$ is defined as

$$\varphi_\sigma * X(x) = \varphi_\sigma * X^i \frac{\partial}{\partial x^i}, \quad x \in W$$

A useful property about the convolution is

$$\int_{\mathbb{R}^n} u(x) \varphi_\sigma * h(x) dx = \int_{\mathbb{R}^n} h(x) \varphi_\sigma * u(x) dx$$

**Theorem 2.12.** Let $B$ be a normal open ball in a Riemannian manifold with a metric $g = g_{ij}dx^idx^j$. Let $f$ be a nonnegative continuous function on $B$. Let $h \in C(B)$ and let $X$ be a smooth vector field satisfying

$$h^2(x) + \langle X, X \rangle(x) \leq f^2(x) \quad \forall \quad x \in B$$

where $\langle \cdot, \cdot \rangle$ is the inner product determined by $g$. Then for any $\varepsilon > 0$ and any compact set $K \subset B$ there exists a $\sigma_0 = \sigma_0(f, K, g, \varepsilon)$ s.t. for all $\sigma < \sigma_0$,

$$h^2(x) + \langle Y, Y \rangle(x) \leq (f(x) + \varepsilon)^2 \quad x \in K$$
where \( \det(g) := \det(g_{ij}) \), \( h' \) and \( Y \) are given by

\[
(2.20) \quad h' = \frac{1}{\sqrt{\det(g)}} \varphi_{\sigma} \ast (\sqrt{\det(g)} h)
\]

\[
(2.21) \quad Y = \frac{1}{\sqrt{\det(g)}} \varphi_{\sigma} \ast (\sqrt{\det(g)} X)
\]

**Proof.** Let \( \sigma_1 \) be a positive constant less than the Euclidean distance between \( \partial \Omega \) and \( K \). Since \( K \) is compact, for all \( \sigma < \frac{a}{2} \) the function \( h' \) in (2.20) and the tangent vector \( Y \) in (2.21) are well-defined for \( x \in K \subset B \).

Let \( \varepsilon' \) be a small constant determined later. For any \( x_0 \in K \), there is a positive constant \( \sigma_2 = \sigma_2(f, g, K, \varepsilon') < \sigma_1 \), such that for all \( \sigma < \frac{a}{2} \) and \( y, y' \in B_{x_0}(2\sigma) \)

\[
(2.22) \quad \frac{1}{1 + \varepsilon} g_{ij}(y') \leq g_{ij}(y) \leq (1 + \varepsilon') g_{ij}(y')
\]

\[
(2.23) \quad \max_{y, y' \in B_{x_0}(x_0)} \frac{\sqrt{\det(g)(y')}}{\sqrt{\det(g)(y)}} \leq 1 + \varepsilon'
\]

\[
(2.24) \quad f(y) \leq f(y') + \varepsilon' \quad \text{for } y, y' \in B_{2\sigma}(x_0)
\]

Here \( B_{2\sigma}(x_0) \) is the Euclidean ball of \( x_0 \) with radius \( 2\sigma \) in \( B \). By Definition 2.11 and (2.14), we have

\[
(2.25) \quad Y^i = \frac{1}{\sqrt{\det(g)}} \varphi_{\sigma} \ast (\sqrt{\det(g)} X^i) \quad \text{and} \quad Y = Y^i \frac{\partial}{\partial x_i}
\]

Fix any point \( y \in B_{x_0}(\sigma) \). With a rotation we can assume that \( g_{ij}(y) = \sigma_{ik} \sigma_{kj} \) where \( (\sigma_{ik}) \) is a positive definite matrix. By (2.21), (2.22), (2.23) and (2.25) for any \( \sigma < \frac{a}{2} \) we have

\[
g_{ij}(y) Y^i Y^j(y) = \frac{1}{\det(g)(y)} (\varphi_{\sigma} \ast (\sqrt{\det(g)} \sigma_{ik} X^i))^2 (y)
\]

\[
\leq (1 + \varepsilon')^2 (\varphi_{\sigma} \ast (\sigma_{ik} X^i))^2
\]

\[
= (1 + \varepsilon')^2 (\varphi_{\sigma} \ast (g_{ij}(y) X^i X^j))
\]

\[
\leq (1 + \varepsilon')^3 \varphi_{\sigma} \ast (g_{ij} X^i X^j)
\]

(2.26)

By (2.20) a similar derivation implies that

\[
(2.27) \quad (h')^2(y) \leq (1 + \varepsilon')^3 \varphi_{\sigma} \ast h^2
\]

Combining (2.26) with (2.27) we obtain

\[
(h')^2(y) + g_{ij} Y^i Y^j(y) \leq (1 + \varepsilon')^2 \varphi_{\sigma} \ast (h^2 + g_{ij} X^i X^j)
\]

\[
\leq (1 + \varepsilon')^3 \varphi_{\sigma} \ast f^2
\]

\[
\leq (1 + \varepsilon')^3 (f(y) + \varepsilon')^2 \quad \text{by (2.24)}
\]

Because \( K \) is compact, we can choose \( \varepsilon' \) small enough such that \( (1 + \varepsilon')^3 (f(y) + \varepsilon')^2 \leq (f(y) + \varepsilon)^2 \) for all \( y \in B_{\sigma}(x_0) \) and \( x_0 \in K \). For such
fixed $\varepsilon'$, define $\sigma_0 = \sigma_2(f, g, K, \varepsilon')$. Thus for any $x_0 \in K$, $y \in B_\sigma(x_0)$ and $\sigma < \frac{1}{2} \sigma_0$ we have
\begin{equation}
(2.28) \quad (h')^2(y) + g_{ij} Y_i^j Y_j(y) \leq f(y) + \varepsilon
\end{equation}
where $\sigma_1 = \sigma_1(f, g, K, \varepsilon)$. We complete the proof.

The following technique result will be very useful in the next section.

**Lemma 2.13.** Let $B$ be a normal open ball with a metric $g = g_{ij} dx^i dx^j$. Suppose $u \in BV(B)$ and $q(x)$ is a smooth function with compact support in $B$. Let $X$ be a smooth vector field on $B$ satisfying $(X, X) \leq 1$. Then for any $\varepsilon > 0$, there is a $\sigma_0 = \sigma_0(u, g, q) > 0$ such that for all $\sigma \in (0, \sigma_0)$
\begin{equation}
(2.29) \quad \int_B \varphi_\sigma \ast (qu) \text{div}(X) d\text{vol} \leq \int_B u \text{div}(q Y_\sigma) d\text{vol} - \int_B u \langle X, \nabla q \rangle d\text{vol} + \varepsilon
\end{equation}
where $Y_\sigma$ is $\frac{1}{\sqrt{\text{det}(g)}} \varphi_\sigma \ast (\sqrt{\text{det}(g)} X)$ and we assume $X = 0$ outside $B$.

**Proof.** Notice that $d\text{vol} = \sqrt{\text{det}(g)} dx$ where $d\text{vol}$ and $dx$ are the volume form of $B$ with respect to $g$ and the Euclidean metric respectively. Moreover $\text{div}(X) d\text{vol} = \text{div}_{\mathbb{R}^n} (\sqrt{\text{det}(g)} X) dx$ where $\text{div}$ and $\text{div}_{\mathbb{R}^n}$ are the divergence of $B$ and $\mathbb{R}^n$ respectively. We also view $B$ as an open set in Euclidean space $\mathbb{R}^n$. Thus $\varphi_\sigma \ast (qu)$ is well-defined if we choose sufficiently small $\sigma$. Then
\begin{equation}
(2.30) \quad \int_B \varphi_\sigma \ast (qu) \text{div}(X) d\text{vol} = \int_{\mathbb{R}^n} \varphi_\sigma \ast (qu) \text{div}_{\mathbb{R}^n} (\sqrt{\text{det}(g)} X) dx
\end{equation}
\begin{equation}
= \int_{\mathbb{R}^n} q(x) u(x) \text{div}_{\mathbb{R}^n} (\varphi_\sigma \ast (\sqrt{\text{det}(g)} X))(x) dx
\end{equation}
\begin{equation}
= \int_B u(x) \text{div}(q Y_\sigma) d\text{vol} - \int_B u \langle Y_\sigma, \nabla q \rangle d\text{vol}
\end{equation}
where $Y_\sigma = \frac{1}{\sqrt{\text{det}(g)}} \varphi_\sigma \ast (\sqrt{\text{det}(g)} X)$. On the other hand, we have
\begin{equation}
(2.31) \quad \int_B u \langle Y_\sigma, \nabla q \rangle d\text{vol} = \int_{\mathbb{R}^n} u g_{ij} \varphi_\sigma \ast (\sqrt{\text{det}(g)} X^i) \nabla^j q dx
\end{equation}
\begin{equation}
= \int_{\mathbb{R}^n} X^i \varphi_\sigma \ast (u g_{ij} \nabla^j q) \sqrt{\text{det}(g)} dx
\end{equation}
Since $(X, X) \leq 1$, there is a $\sigma_0 = \sigma_0(u, g, q) > 0$ independent of $X$ such that
\begin{equation}
- \int_{\mathbb{R}^n} X^i \varphi_\sigma \ast (u g_{ij} \nabla^j q) \sqrt{\text{det}(g)} dx \leq - \int_{\mathbb{R}^n} X^i u g_{ij} \nabla^j q \sqrt{\text{det}(g)} dx + \varepsilon
\end{equation}
\begin{equation}
= - \int_B u \langle X, \nabla q \rangle d\text{vol} + \varepsilon
\end{equation}
Combining the above two inequalities together we obtain $(2.29)$. The proof is complete.
3. Conformal area functionals and their $C^\infty$ approximation

In this section we define product area functionals and conformal area functionals. Then we obtain their $C^\infty$ approximation properties in Theorem 3.6. Our proof deeply depends on a decomposition result of Radon measures from the Besicovitch Covering Theorem in Riemannian manifolds (see Theorem A.4).

3.1. The conformal area functional. Throughout this section we adopt the following notation. Let $N$ be a complete Riemannian manifold with metric $g$. Write $\text{div}$ and $dvol$ for the divergence and volume form of $N$ respectively. Fix $\Omega$ as an open bounded set in $N$. Let $C_0(\Omega)$ and $T_0(\Omega)$ denote the sets of all smooth functions and smooth vector fields with compact supports in $\Omega$ respectively.

**Definition 3.1.** Let $u(x)$ be a measurable function on $\Omega$. The product area functional of $u$, $F(u,\Omega)$, is defined by

$$F(u,\Omega) : \equiv \sup \{ \int_\Omega u(h + \text{div}(X))dvol : h \in C_0(\Omega), X \in T_0(\Omega), $$
$$h^2 + \langle X, X \rangle \leq 1 \}$$

(3.1)

Let $\alpha > 0$ be a fixed constant. The conformal area product functional $\mathfrak{F}_\alpha(u,\Omega)$, is defined by

$$\mathfrak{F}_\alpha(u,\Omega) : \equiv \sup \{ \int_\Omega e^{\alpha u}(h + \frac{1}{\alpha}\text{div}(X))dvol : h \in C_0(\Omega), X \in T_0(\Omega), $$
$$h^2 + \langle X, X \rangle \leq 1 \}$$

(3.2)

The terms “product area functional” and “conformal area functional” come from the following fact.

**Lemma 3.2.** Suppose $u \in C^1(\Omega)$. Let $Du$ be the gradient of $u$ in $\Omega$ and $|Du|^2 = \langle Du, Du \rangle$.

1. Then $\mathfrak{F}(u,\Omega)$ is the area of the graph of $u(x)$ in the product manifold $N \times \mathbb{R}$ with the metric $g + dr^2$ which is $\int_\Omega \sqrt{1 + |Du|^2}dvol$.
2. And $\mathfrak{F}_\alpha(u,\Omega)$ is the area of the graph of $u(x)$ in the product manifold $N \times \mathbb{R}$ with the metric $e^{2\alpha u}(g + dr^2)$ which is $\int_\Omega e^{\alpha u}\sqrt{1 + |Du|^2}dvol$.

**Remark 3.3.** The proof is straightforward so we skip it here. If $u \in C^2(\Omega)$ locally minimizes of $\mathfrak{F}_\alpha(u,\Omega)$, then $u$ satisfies the translating mean curvature equation

$$\text{div}(\frac{Du}{\omega}) = \frac{\alpha}{\omega}$$

on $\Omega$ where $\omega = \sqrt{1 + |Du|^2}$. This is the reason we study the conformal area functional $\mathfrak{F}_\alpha(u,\Omega)$ in this paper.
For any $f \in K^+(\Omega)$, we define two nonnegative functionals:

$$
\lambda_{u,0}(f) \equiv \sup \{ \int_{\Omega} u(h + \text{div}(X)) \text{dvol} : h \in C_0(\Omega), X \in T_0 \Omega, \\
h^2 + \langle X, X \rangle \leq f^2(x) \}
$$

(3.3)

$$
\lambda_{u,\alpha}(f) \equiv \sup \{ \int_{\Omega} e^{\alpha u}(h + \frac{1}{\alpha} \text{div}(X)) \text{dvol} : h \in C_0(\Omega), X \in T_0 \Omega, \\
h^2 + \langle X, X \rangle \leq f^2(x) \}
$$

(3.4)

It is clear that both $\lambda_{u,0}(\cdot)$ and $\lambda_{u,\alpha}(\cdot)$ are linear on $K^+(\Omega)$. Rewriting the definitions of $\mathcal{F}(u, \Omega)$ and $\mathcal{F}_\alpha(u, \Omega)$ we obtain the following two representations:

$$
\mathcal{F}(u, \Omega) : \equiv \sup \{ \lambda_{u,0}(f) : f \in K^+(\Omega), f \leq 1 \}
$$

(3.5)

$$
\mathcal{F}_\alpha(u, \Omega) : \equiv \sup \{ \lambda_{u,\alpha}(f) : f \in K^+(\Omega), f \leq 1 \}
$$

(3.6)

In fact the above two formulas are true for any open set in $\Omega$. By Theorem 2.5, as in the case of BV functions, we obtain two Radon measures from $\mathcal{F}(u, \Omega)$ and $\mathcal{F}_\alpha(u, \Omega)$ as follows.

**Theorem 3.4.** Let $\Omega$ be an open set in $N$.

1. Suppose $u \in BV_{loc,N}(\Omega)$ such that $\mathcal{F}(u, \Omega')$ is finite for any bounded open set $\Omega' \subset \Omega$. Then there is a unique Radon measure $\mu_0$ on $\Omega$ satisfying

$$
\mu_0(\Omega') = \mathcal{F}(u, \Omega') \quad \forall \Omega' \subset \Omega
$$

2. Suppose $u \in BV(\Omega)$ such that $\mathcal{F}_\alpha(u, \Omega')$ is finite for any bounded open set $\Omega' \subset \Omega$. Then there is a unique Radon measure $\mu_\alpha$ on $\Omega$ satisfying

$$
\mu_\alpha(\Omega') = \mathcal{F}_\alpha(u, \Omega') \quad \forall \Omega' \subset \Omega
$$

Proof. By Theorem 2.5 on $\lambda_{u,0}(\cdot)$ in (3.3) and $\lambda_{u,\alpha}(\cdot)$ in (3.4) we obtain the existence of $\mu_0$ and $\mu_\alpha$. The two conclusions follow from the definitions of $\mathcal{F}(u, \cdot)$ in (3.5) and $\mathcal{F}_\alpha(u, \cdot)$ in (3.6) respectively. \(\square\)

The semicontinuous properties are also valid for $\mathcal{F}(u, \Omega)$ and $\mathcal{F}_\alpha(u, \Omega)$.

**Theorem 3.5.** Let $\Omega$ be a bounded open set in $N$.

1. Suppose $u_k$ converges to $u$ in $L^1(\Omega)$. Then

$$
\mathcal{F}(u, \Omega) \leq \liminf_{k \to \infty} \mathcal{F}(u_k, \Omega)
$$

2. Suppose $e^{\alpha u_k}$ converges to $e^{\alpha u}$ in $L^1(\Omega)$. Then

$$
\mathcal{F}_\alpha(u, \Omega) \leq \liminf_{k \to \infty} \mathcal{F}_\alpha(u_k, \Omega)
$$

3. If $u \in BV_N(\Omega)$, then

$$
\max \{ ||Du||_N(\Omega), \text{vol}(\Omega) \} \leq \mathcal{F}(u, \Omega) \leq \text{vol}(\Omega) + ||Du||_N(\Omega)
$$

where $\text{vol}(\Omega)$ denotes the volume of $\Omega$ in $N$. 

Proof. The conclusion (1) and (2) follow directly from the definitions of \( \mathfrak{F}(u, \Omega) \) and \( \mathfrak{F}_\alpha(u, \Omega) \) respectively. As for the left inequality in (3), we just let \( h \equiv 0 \) or \( X \equiv 0 \) and take the supremum in (3.1). The right inequality in (3) just follows from the definition of BV functions. \( \square \)

3.2. The \( C^\infty \) approximation. In this subsection we show the \( C^\infty \) approximation properties of \( ||Du||_N(\Omega), \mathfrak{F}(u, \Omega) \) and \( \mathfrak{F}_\alpha(u, \Omega) \) together. Notice that the method in (Theorem 1.17, \[10\]) can not be applied directly into the domains in general Riemannian manifolds because they may be not simply connected. Neither does the method of the weighted BV function in Baldi \[2\] because its definition of weighted BV functions does not consider the Riemannian metric.

**Theorem 3.6.** Let \( \Omega \subset N \) be a bounded open set and \( u(x) \in BV_N(\Omega) \).

1. Then there is a sequence \( \{u_k\}_{k=1}^\infty \) in \( C^\infty(\Omega) \) s.t \( u_k \) converges to \( u \) in the \( L^1(\Omega) \) sense and

\[
\lim_{k \to \infty} ||Du_k||_N(\Omega) = ||Du||_N(\Omega)
\]

2. There is a sequence \( \{u_k\}_{k=1}^\infty \) in \( C^\infty(\Omega) \) s.t \( u_k \) converges to \( u \) in the \( L^1(\Omega) \) sense and

\[
\lim_{k \to \infty} \mathfrak{F}(u_k, \Omega) = \mathfrak{F}(u, \Omega)
\]

3. In addition suppose \( \alpha > 0 \) and \( \mathfrak{F}_\alpha(u, \Omega) \) is finite. Then there is a sequence \( \{u_k\}_{k=1}^\infty \) in \( C^\infty(\Omega) \) s.t \( e^{\alpha u_k} \) converges to \( e^{\alpha u} \) in the \( L^1(\Omega) \) sense and

\[
\lim_{k \to \infty} \mathfrak{F}_\alpha(u_k, \Omega) = \mathfrak{F}_\alpha(u, \Omega)
\]

Proof. The case of \( \mathfrak{F}(u, \Omega) \):

Since \( u \in BV_N(\Omega) \) and \( \Omega \) is bounded, then \( \mathfrak{F}(u, \Omega) \) is finite by (3) in Theorem 3.3. From Theorem 3.4 there is a Radon measure \( \mu_0 \) satisfying \( \mu_0(\Omega^\prime) = \mathfrak{F}(u, \Omega^\prime) \) for any open set \( \Omega^\prime \subset \Omega \).

Fix \( \varepsilon > 0 \). By Theorem \[A.3\] there is a collection of normal open balls \( \{B_i\}_{i=1}^\infty \) such that \( (1) \Omega \subset \bigcup_{i=1}^\infty B_i \); \( 2 \) there is an integer \( \kappa(\varepsilon) > 0 \) such that \( \{B_1, \cdots, B_n\}_{i=1}^{\kappa(\varepsilon)} \) is a pairwise disjoint collection with the estimate

\[
(3.7) \quad \mu_0(\Omega) - \varepsilon \leq \sum_{i=1}^{\kappa(\varepsilon)} \mu_0(B_i) \leq \mu_0(\Omega) ; \quad \sum_{i=\kappa(\varepsilon)+1}^\infty \mu_0(B_i) \leq \varepsilon
\]

Let \( \{q_i(x)\}_{i=1}^\infty \) be a unit partition subordinate to the cover \( \{B_i\}_{i=1}^\infty \), that is, \( q_i \in C_0^\infty(B_i) \), \( 0 \leq q_i \leq 1 \) and \( \sum_{i=1}^\infty q_i = 1 \). Assume \( \tilde{h} \in C_0(\Omega) \) and \( X \in T_0\Omega \) satisfying

\[
(3.8) \quad \tilde{h}^2 + \langle X, X \rangle \leq 1
\]

Since each \( B_i \) is an open normal ball in \( N \), we can view \( B_i \) as an open set in \( \mathbb{R}^{n+1} \) with the metric \( g = g_{kl}dx^kdx^l \). In what follows we use the same notation for the functions and vector fields on \( B_i \) as those in the
corresponding open set in Euclidean spaces. Suppose $X$ is written as $X^k \frac{\partial}{\partial x_k}$.

Then on $B_i$ we have

$$\tilde{h}^2 + g_{kl}X^kX^l \leq 1$$

(3.9)

For each $i > 0$ we can choose $\sigma_i > 0$ such that

$$\int_{B_i} |\varphi_{\sigma_i}*(uq_i) - uq_i|dvol \leq \frac{\varepsilon}{2}$$

(3.10)

$$(q_i\tilde{h})^2 + (q_i)^2(Y_{\sigma_i}, Y_{\sigma_i}) \leq 1 + \varepsilon$$

by Theorem 2.12

(3.11)

$$\int_{B_i} \varphi_{\sigma_i}*(uq_i)div(X))dvol \leq \int_{B_i} (udiv(q_iY_{\sigma_i})dvol$$

(3.12)

by Lemma 2.13

where $Y_{\sigma_i}^k = \frac{\varphi_{\sigma_i}*(\sqrt{det(g)}X^k)}{sqrt{det(g)}}$ and $Y_{\sigma_i} = Y_{\sigma_i}^k \partial_k$.

Now we define $u_\varepsilon$ as

$$u_\varepsilon = \sum_{i=1}^{\infty} \varphi_{\sigma_i}*(uq_i)$$

(3.13)

By (3.10) one sees that

$$\int_{\Omega} |u_\varepsilon - u|dvol \leq \sum_{i=1}^{\infty} \int_{B_i} |\varphi_{\sigma_i}*(uq_i) - uq_i|dvol \leq \varepsilon$$

(3.14)

Consider

$$\int_{\Omega} (\tilde{h} + u_\varepsilon \text{div}(X))dvol \leq \sum_{i=1}^{\infty} \int_{B_i} (\tilde{h}q_i + \varphi_{\sigma_i}*(uq_i) \text{div}(X))dvol$$

(3.15)

From (3.12) we have

$$\int_{B_i} (\tilde{h}q_i + \varphi_{\sigma_i}*(uq_i) \text{div}(X))dvol$$

$$\leq \int_{B_i} (\tilde{h}q_i + u\text{div}(q_iY_{\sigma_i}))dvol - \int_{B_i} u(X, \nabla q_i)dvol + \frac{\varepsilon}{2}$$

$$\leq (1 + \varepsilon)\mu_0(B_i) - \int_{B_i} u(X, \nabla q_i)dvol + \frac{\varepsilon}{2}$$

Combining the condition (3.7) with (3.15) we obtain

$$\int_{\Omega} (\tilde{h} + u_\varepsilon \text{div}(X))dvol \leq (1 + \varepsilon)\sum_{i=1}^{\infty} \mu_0(B_i) + \varepsilon$$

(3.16)

$$\leq (1 + \varepsilon)(\mu_0(\Omega) + \varepsilon) + \varepsilon$$

In the first inequality we apply the fact that $\sum_{i=1}^{\infty} \int_{B_i} u(X, \nabla q_i)dvol = 0$ because of $\sum_{i=1}^{\infty} q_i(x) = 1$. Taking supremum for all $\tilde{h}, X$ satisfying (3.8)
yields that
\[ \mathcal{F}(u, \Omega) \leq (1 + \varepsilon)(\mathcal{F}(u, \Omega) + \varepsilon) + \varepsilon \]
Now take a sequence \( \varepsilon_k \to 0 \) as \( k \to \infty \). By (3.14) and (3.17) we obtain a smooth sequence \( \{u_{\varepsilon_k}\}_{k=1}^{\infty} \) such that \( u_{\varepsilon_k} \) converges to \( u \) in the \( L^1 \) sense and
\[ \lim_{k \to \infty} \sup \mathcal{F}(u_{\varepsilon_k}, \Omega) \leq \mathcal{F}(u, \Omega) \]
From Theorem 3.5 we have \( \lim_{k \to \infty} \inf \mathcal{F}(u_{\varepsilon_k}, \Omega) \geq \mathcal{F}(u, \Omega) \). Thus \( \{u_{\varepsilon_k}\}_{k=1}^{\infty} \) is the desirable sequence. We obtain the conclusion (1).

**The case of \( ||Du||_N(\Omega) \):**

The proof of the conclusion (1) is similar to that of the conclusion (2). We just let \( \tilde{h} \equiv 0 \) in the derivation from (3.8) to (3.18). Recall that
\[ ||Du||_N(\Omega) = \sup\{ \int_{\Omega} u \text{div}(X) d\text{vol}, X \in T_0(\Omega), \langle X, X \rangle \leq 1 \} \]
Now we define \( u_\varepsilon \) as
\[ u_\varepsilon = \sum_{i=1}^{\infty} \varphi_{\sigma_i} \ast (u q_i) \]
where \( \sigma_i \) satisfies (3.10)–(3.12). With the exact same derivation in the proof of the conclusion (2) we can find a sequence \( \{\varepsilon_i\}_{i=1}^{\infty} \) converging to 0 such that
\[ \lim_{i \to +\infty} ||Du_{\varepsilon_i}||_N(\Omega) = ||Du||_N(\Omega) \]
This is the conclusion (1).

**The case of \( \mathcal{F}_\alpha(u, \Omega) \):**
The idea to derive the conclusion (3) is also similar to the proof of conclusion (2). But the approximating sequence shall have a little modification.

By Theorem 3.4 there is a unique Radon measure \( \mu_\alpha \) on \( \Omega \) such that \( \mu_\alpha(\Omega') = \mathcal{F}_\alpha(u, \Omega') \) for all open \( \Omega' \subset \Omega \). Moreover \( \mu_\alpha(\Omega) \) is finite. Fix \( \varepsilon > 0 \). By Theorem A.4 there is a collection of open sets \( \{B_i\}_{i=1}^{\infty} \) such that (1) each \( B_i \) is an open normal ball in \( \Omega \), \( \Omega \subset \bigcup_{i=1}^{\infty} B_i \); (2) there is an integer \( N(\varepsilon) \) such that \( \{B_1, \ldots, B_{N(\varepsilon)}\} \) is a pairwise disjoint collection with the estimate
\[ \mu_\alpha(\Omega) - \varepsilon \leq \sum_{i=1}^{N(\varepsilon)} \mu_\alpha(B_i) \leq \mu_\alpha(\Omega); \quad \sum_{i=N(\varepsilon)+1}^{\infty} \mu_\alpha(B_i) \leq \varepsilon \]
Now assume \( h \in C_0(\Omega) \) and \( X \in T_0\Omega \) satisfying
\[ h^2 + \langle X, X \rangle \leq 1 \]
Each \( B_i \) can be viewed as an open set in \( \mathbb{R}^n \) with the metric \( g = g_{kl} dx^k dx^l \).

On each \( B_i \) assume \( X = X^k \partial_k \). Then
\[ h^2 + g_{kl} X^k X^l \leq 1 \]
Let \( \{q_i(x)\}_{i=1}^\infty \) be a unit partition subordinate to the cover \( \{B_i\}_{i=1}^\infty \). For each \( i > 0 \) we can choose \( \sigma_i > 0 \) such that

\[
(3.24) \quad \int_{B_i} |\varphi_{\sigma_i} * (e^{\alpha u} q_i) - e^{\alpha u} q_i| \, dvol \leq \frac{\varepsilon}{2^n}
\]

\[
(3.25) \quad (q_i h')^2 + (q_i)^2 g_{kl} Y^k_{\sigma_i} Y^l_{\sigma_i} \leq 1 + \varepsilon
\]

\[
(3.26) \quad \int_{B_i} \varphi_{\sigma_i} * (e^{\alpha u} q_i) \text{div}(X) \, dvol \leq \int_{B_i} (e^{\alpha u} \text{div}(q_i Y_{\sigma_i})) \, dvol - \int_{B_i} e^{\alpha u} \langle X, \nabla q_i \rangle \, dvol + \frac{\varepsilon}{2^n}
\]

where \( Y^k_{\sigma_i} = \varphi_{\sigma_i} * (\sqrt{\det(g)} X^k) \sqrt{\det(g)} \) and \( Y_{\sigma_i} = Y^k_{\sigma_i} \partial_k \). The proof of the above arguments is similar as that of the case of \( \mathcal{F}(u, \Omega) \) just replacing \( u \) with \( e^{\alpha u} \). In particular (3.26) is from Lemma 2.13.

Now we define \( u_\varepsilon \) as

\[
(3.27) \quad e^{\alpha u_\varepsilon} = \sum_{i=1}^\infty \varphi_{\sigma_i} * (e^{\alpha u} q_i)
\]

This definition is well-defined because the right hand is a finite positive summation at any point \( x \in \Omega \). According to (3.24) we have

\[
(3.28) \quad \int_{\Omega} |e^{\alpha u_\varepsilon} - e^{\alpha u}| \, dvol \leq \sum_{i=1}^\infty \int_{B_i} |\varphi_{\sigma_i} * (e^{\alpha u} q_i) - e^{\alpha u} q_i| \, dvol \leq \varepsilon
\]

Replacing \( u \) with \( e^{\alpha u} \) in (3.12) and taking the same derivation we obtain

\[
(3.29) \quad \int_{B_i} \varphi_{\sigma_i} * (e^{\alpha u} q_i) \text{div}(X) \, dvol = \int_{\mathbb{R}^n} (e^{\alpha u} \text{div}(q_i Y_{\sigma_i})) - e^{\alpha u} \langle Y_{\sigma_i}, \nabla q_i \rangle \, dvol
\]

On the other hand

\[
(3.30) \quad \int_{B_i} \varphi_{\sigma_i} * (e^{\alpha u} q_i) h \, dvol = \int_{B_i} \varphi_{\sigma_i} * (e^{\alpha u} q_i) h \sqrt{\det(g)} \, dx
\]

\[
= \int_{B_i} \varphi_i e^{\alpha u} h' \, dvol
\]
where \( h' = \frac{\psi_{\sigma_i}(\sqrt{\det(g)h})}{\sqrt{\det(g)}} \). By (3.29) and (3.30) we compute

\[
\int_{\Omega} e^{\alpha u}(h + \frac{1}{\alpha} \text{div}(X))dvol = \sum_{i=1}^{\infty} \int_{B_i} \varphi_{\sigma_i} * (e^{\alpha u} q_i)(h + \frac{1}{\alpha} \text{div}(X))dvol
\]

\[
\leq \sum_{i=1}^{\infty} \left\{ \int_{B_i} e^{\alpha u}(q_i h' + \frac{1}{\alpha} \text{div}(q_i Y_{\sigma_i}))dvol - \frac{1}{\alpha} \int_{B_i} e^{\alpha u} \langle X, \nabla q_i \rangle dvol + \frac{1}{\alpha} \frac{\varepsilon}{2^i} \right\}
\]

\[
\leq (1 + \varepsilon) \sum_{i=1}^{\infty} \mu_\alpha(B_i) + \frac{\varepsilon}{\alpha}
\]

For the first term in the last inequality we combine (3.25) and the definition of \( \mathfrak{F}_\alpha(u, B_i) \) together. For the second term in the last inequality we apply (3.26) and the fact \( \sum_{i=1}^{\infty} q_i \equiv 1 \) on \( \Omega \). Continuing applying the assumption (3.21) into the above estimate, we obtain

\[
(3.31) \quad \int_{\Omega} e^{\alpha u}(h + \frac{1}{\alpha} \text{div}(X))dvol \leq (1 + \varepsilon)(\mu_\alpha(\Omega) + \varepsilon) + \frac{\varepsilon}{\alpha}
\]

Now we arrive a similar position as in (3.16) when we show the conclusion (1). With a similar derivation we can achieve the conclusion (3).

The proof of Theorem 3.6 is complete. \( \square \)

A corollary of the conclusion (2) of Theorem 3.6 is the compactness of BV function on Lipschitz domains in \( N \). Its proof is similar to that in the Euclidean spaces. (See Theorem 2, Section 5.2.2 in [5]).

**Theorem 3.7.** Suppose \( \Omega \) is an open bounded set in \( N \) with Lipschitz boundary. Suppose a sequence \( \{u_i\}_{i=1}^{\infty} \in BV(\Omega) \) satisfies that

\[
(3.32) \quad \int_{\Omega} |u_i|dvol + ||Du_i||_N(\Omega) \leq c \quad \text{for all } i
\]

where \( c \) is a fixed constant. Then there is a \( u \in BV(\Omega) \) such that there is a subsequence of \( \{u_i\}_{i=1}^{\infty} \) converging to \( u \) in \( L^1(\Omega) \).

**Proof.** For any \( \varepsilon > 0 \), (2) in Theorem 3.6 implies that there is a set given by

\[
A_\varepsilon = \{u'_j \in C^\infty(\Omega) : \int_{\Omega} |u_j - u'_j|dvol \leq \varepsilon, ||Du'_j||_N(\Omega) = \int_{\Omega} |Du'_j|dvol \leq ||Du_j||_N(\Omega) + \varepsilon \}
\]

Thus \( A_\varepsilon \in W^{1,1}(\Omega') \). Here \( W \) denotes the Sobolev space. For more details see Appendix B. Arguing as Theorem 7.25 of Chapter 7 in [7], the fact that \( \Omega \) is Lipschitz implies \( W^{1,1}(\Omega) \) can be isometrically embedded into \( W^{1,1}_0(\Omega') \). Here \( \Omega' \) is a larger open set containing \( \Omega \). By the Sobolev embedding theorem, \( W^{1,p}_0(\Omega') \) is isometrically embedded into \( L^1(\Omega') \). This implies that the
set \( \bigcup_{1 > \varepsilon > 0} A_{\varepsilon} \) is a precompact set in \( L^1(\Omega) \). Letting \( \varepsilon \to 0 \) we can subtract the desirable sequence. The proof is complete.

\[ \square \]

4. The trace of BV functions

The traces of BV functions in Riemannian manifolds play an important role in our computation on generalized solutions to the Dirichlet problem (1.1) (see Definition 6.1). We will collect related results on this topic. Most of their proofs will be skipped because of no essential modification comparison with those in corresponding Euclidean versions. For exact proofs, we refer to Chapter 2 in [10] and Chapter 5 in [5]. An application is given in Lemma 4.9.

4.1. The trace of BV functions.

The definition of the trace is from the following divergence theorem.

**Theorem 4.1** (Theorem 5.9 [5] or Lemma 2.4 [10]). Let \( \Omega \) be a bounded open set in a Riemannian manifold \( N \) with Lipschitz boundary \( \partial \Omega \). Then there is a linear functional \( T : BV_{loc,N}(\Omega) \to L^1(\partial \Omega) \) such that for any \( u \in BV_{loc,N}(\Omega) \) it holds that

\[
\int_{\Omega} u \text{div}(X) d\text{vol} = -\int_{\Omega} \langle X, \nu \rangle d|Du|_N + \int_{\partial \Omega} T u \langle X, \vec{v} \rangle d\text{vol}_{\partial \Omega}
\]

for any smooth vector field \( X \) with compact support in \( N \). Here \( |Du|_N \) is the Radon measure generated by \( u \) (see Theorem 2.6), \( \vec{v} \) is the inward normal vector of \( \partial \Omega \) and \( d\text{vol}_{\partial \Omega} \) is the volume form of \( \partial \Omega \), \( \nu \) is a vector field on the tangent bundle of \( \Omega \) such that \( \langle \nu, \nu \rangle = 1 \) a.e.-\( |Du|_N \).

Now we define the trace as follows.

**Definition 4.2.** With the above result, for \( u \in BV_{loc,N}(\Omega) \) the function \( Tu \) up to a measure zero set in \( \partial \Omega \) is called the trace of \( u \) on \( \partial \Omega \). \( T \) is referred as the trace operator.

The following two properties are not hard to check.

**Lemma 4.3.** Let \( \Omega \) be given in Theorem 4.1. Suppose \( u \in BV_N(\Omega) \) and \( Tu \) is its trace on \( \partial \Omega \). It holds that

1. \( T(u - j) = Tu - j \) for a fixed constant \( j \);
2. (Proposition 2.6 in [10]) for a sequence \( \{u_k\}_{k=1}^\infty \in BV_N(\Omega) \) such that \( u_k \) converges to \( u \) in \( L^1(\Omega) \) and \( ||Du_k||_N(\Omega) \) converges to \( ||Du||_N(\Omega) \), \( Tu_k \) converges to \( Tu \) as \( k \to +\infty \) in \( L^1(\partial \Omega) \).

**Proof.** By Theorem 2.6 two Radon measures \( |Du|_N \) and \( |D(u - j)|_N \) are equal to each other. Fix \( j \in \mathbb{R} \). By (4.1), we have

\[
\int_{\Omega} (u - j) \text{div}(X) d\text{vol} = -\int_{\Omega} \langle X, \nu \rangle d|Du|_N + \int_{\partial \Omega} (Tu - j) \langle X, \vec{v} \rangle d\text{vol}_{\partial \Omega}
\]

\[
= -\int_{\Omega} \langle X, \nu \rangle d|Du - j|_N + \int_{\partial \Omega} (Tu - j) \langle X, \vec{v} \rangle d\text{vol}_{\partial \Omega}
\]
for any smooth vector field $X$. Due to the arbitrariness of $X$, (4.11) implies that the trace of $u - j$ is $Tu - j$. We obtain the conclusion (1). As for the proof of the conclusion (2), we refer to Proposition 2.6 in [10] or the proof of step (5) of Theorem 5.9 in [5].

Now we illustrate the trace is indeed the “boundary value” of BV functions in the approximation sense.

Since $\partial \Omega$ is Lipschitz and bounded, there is a $r_0 > 0$ such that for every $r \in (0, r_0)$ if $y \in \Omega$ satisfies $d(y, \partial \Omega) \leq r$ there is only one $x \in \partial \Omega$ with the property $d(x, y) = d(x, \partial \Omega)$. Thus we define a set

$$\Gamma_r := \partial \Omega \times (0, \varepsilon) = \{(x, s) : x \in \partial \Omega, s \in (0, r)\}$$

for $r < r_0$ with the corresponding metric $g = \sigma(x, r) + dr^2$. Here $\sigma(x, 0)$ is just the induced metric of $\partial \Omega$ and $\sigma(x, r)$ is smooth with respect to $x$ and Lipschitz with respect to $r$. With this local coordinate on $\Gamma_{r_0}$ near $\partial \Omega$, for every $f \in BV(\Omega)$ we define a function on $\partial \Omega$ as

$$f_r(x) := f(p) \quad \text{where } p = (x, r) \in \Gamma_{r_0}$$

**Lemma 4.4.** Let $\Omega$ be an open bounded domain in a Riemannian manifold $N$ with Lipschitz boundary. Suppose $f \in BV_{loc,N}(\Omega)$. Let $f_r$ be the function given in (4.2) defined on $\partial \Omega$. Then there is a Lebesgue measure zero set $E$ in a small interval $(0, r_0)$

$$\lim_{r \to 0, r \notin E} f_r(x) = Tf(x) \quad \text{locally in } L^1(\partial \Omega)$$

If $\partial \Omega$ is unbounded, then $f_r$ is defined locally on $\partial \Omega$ and (4.3) is valid locally. That is, $r_0$ and $E$ are locally determined near a fixed point on $\partial \Omega$.

**Proof.** We can assume that $\Omega$ is bounded. The general case could be easily derived by the unity partition technique.

Our proof just follows from the the step 4 and step 5 in that of Theorem 5.9 [5]. Thus we only describe its sketch here.

First assume $f \in C^\infty(\Omega) \cap BV(\Omega)$. Then $\{f(x, r)\}_{r_0 > r > 0}$ is a Cauchy sequence as $r \to 0^+$ because $f \in BV(\Omega)$ and

$$\int_{\partial \Omega} |f_r - f_s|dvol_{\partial \Omega} \leq C||Df||_N(\Gamma_r)$$

for any $r > s \in (0, r_0)$. Here $C$ is a constant only depending on the metric on $\Omega$ and $dvol_{\partial \Omega}$ is the induced volume form from $N$. Thus $\{f_r\}$ is the desirable Cauchy sequence in $L^1(\partial \Omega)$ as $r \to 0^+$ converging to $Tf$. In (4.4) fixing $r$ and letting $s$ go to zero, one sees that

$$\int_{\partial \Omega} |f_r - Tf|dvol_{\partial \Omega} \leq C||Df||_N(\Gamma_r)$$

for all $r \in (0, r_0)$. Let $E$ be the empty set. This gives (4.3).

Next consider $f \in BV(\Omega)$. By the conclusion (1) in Theorem 3.6 there is a sequence $\{f_k\}_{k=1}^\infty \in C^\infty \cap BV(\Omega)$ such that $f_k$ converges to $f$ in $L^1(\Omega)$ and $||Df_k||_N(\Omega)$ converges to $||Df||_N(\Omega)$ as $k \to \infty$. 

By the Fubini theorem we conclude that except a Lebesgue measure zero set $E$ in $(0, r_0)$ such that for every $r \in (0, r_0) \setminus E$ it holds that

1. $\|Df\|_N(\partial \Omega_r) = 0$ where $\partial \Omega_r = \{y \in \Omega : d(y, \partial \Omega) = r\}$.
2. $f_k(x, r)$ converges to $f(x, r)$ in $L^1(\partial \Omega)$ because $f_k(p)$ converges to $f(p)$ in $L^1(\Gamma_{r_0})$ and $p = (x, r) \in \Gamma_{r_0}$.

Because of (1), Proposition 1.13 in [10] implies that

$$\lim_{k \to +\infty} \|Df_k\|_N(\Gamma_r) = \|Df\|_N(\Gamma_r)$$

for every $r \in (0, r_0) \setminus E$. Now we apply (4.5) for $\{f_k\} \in C^\infty(\Omega) \cap BV(\Omega)$.

Letting $r \to 0^+$ in $(0, r_0) \setminus E$ we obtain the conclusion. $\square$

First the trace only depends the conformal equivalent class of metrics.

**Lemma 4.5.** The trace operator $\mathcal{T}$ in Definition 4.2 only depends on the conformal equivalent class of the metric $g$.

**Proof.** Suppose $\phi > 0$ be a smooth function on $N$. Let $N$ and $N_\phi$ be two conformal Riemannian manifolds with the metric $g$ and $\phi^2 g$ respectively. Let $\Omega$ be an open set in $N$ with Lipschitz boundary.

In the followings, the qualities with (without) the low index $\phi$ denote the ones related to the manifold $(N, \phi^2 g)$ ($(N, g)$).

Let $\mathcal{T}u$ be the trace of $u$ on $\partial \Omega$ where $u \in BV_{loc,N}(\Omega)$. To show the Lemma, it is suffice to show that $\mathcal{T}u = \mathcal{T}_\phi u$.

By Remark 2.9, we have $u \in BV_{loc,N_\phi}(\Omega)$. Let $X$ be a smooth vector field. By (4.1)

$$\int_\Omega u \text{div}(\phi^2 X) d\nu = -\int_\Omega \phi^2 \langle X, \nu \rangle d|Du|_N$$

(4.7)

where $n = \text{dim } N$. By (2.11), the divergence theorem and Theorem 2.8 we have

$$\text{div}(\phi^2 X) d\nu = \text{div}_{\phi}(X) d\nu_{\phi}$$

(4.8)

$$\langle X, \nu \rangle_{\phi} d|Du|_{N_\phi} = \phi^n \langle X, \nu \rangle d|Du|_N$$

(4.9)

$$\langle X, \nu \rangle_{\phi} d\nu_{\phi,\Omega,\phi} = \phi^n \langle X, \nu \rangle d\nu_{\phi,\Omega}$$

(4.10)

Based on the above identities we compare (4.7) with (4.1) in the case of $\mathcal{T}_\phi u$. This gives that $\mathcal{T}_\phi u = \mathcal{T}u$ in the sense of $L^1(\partial \Omega)$. The proof is complete. $\square$

The trace is helpful to compute the bounded variation on BV functions on Lipschitz boundaries.
Theorem 4.6 (Proposition 2.8 in [10]). Suppose \( \Omega^+ \) and \( \Omega^- \) are two open sets sharing a Lipschitz boundary \( \partial \Omega \) in a Riemannian manifold \( N \). Let \( f_1 \in BV(\Omega^+) \) and \( f_2 \in BV(\Omega^-) \). Let \( \Omega = \Omega^+ \cup \Omega^- \). Define a function \( f : \Omega \to \mathbb{R} \) by

\[
(4.11) \quad f = \begin{cases} 
  f_1 & \text{in } \Omega^+ \\
  f_2 & \text{in } \Omega^-
\end{cases}
\]

Then \( f \in BV(\Omega) \) and

\[
(4.12) \quad \int_{\partial \Omega} |\mathcal{T} f_1 - \mathcal{T} f_2| d\text{vol}_{\partial \Omega} = \int_{\partial \Omega} d|Df|_N
\]

Here \( \mathcal{T} \) denotes the trace and \( d\text{vol}_{\partial \Omega} \) is the volume form of \( \partial \Omega \) with the induced metric.

The following extension lemma shows that the trace could be any \( L^1 \) function on Lipschitz functions.

Lemma 4.7 (Proposition 2.15 [10]). Let \( \Omega \) be a bounded open set with Lipschitz boundary. Let \( \psi(x) \in L^1(\partial \Omega) \). Then there is a function \( f \in W^{1,1}(\Omega) \) such that \( \mathcal{T} f = \psi(x) \) on \( \partial \Omega \). In particular if \( \psi(x) \in C(\partial \Omega) \) then such \( f \) is continuous and bounded. Here \( W \) denotes the sobolev space (see Definition B.2) and it is clear that \( W^{1,1}(\Omega) \subset BV(\Omega) \).

Proof. The proof follows exactly from that of Proposition 2.15 in [10]. In particular from the construction of \( f \) it is clear if \( \psi(x) \in C(\partial \Omega) \) then \( f \) is bounded. Thus we skip its proof here. \( \square \)

4.2. An application. Now we see a direct application of the trace of \( BV \) function. The following definition will be repeatedly used in the remainder of this paper.

Definition 4.8. Suppose \( \Omega \) is a bounded open set in \( N \). For \( T > 0 \) define

\[
(4.13) \quad u_T(x) := \max\{\min\{u(x),T\},-T\}
\]

Let \( U_T \) and \( U \) be the subgraph of \( u_T(x) \) and \( u(x) \) respectively. Then

\[
\begin{align*}
(1) \quad \lim_{T \to +\infty} ||D\lambda_{U_T}||_{Q(\Omega \times \mathbb{R})} &= ||D\lambda_U||_{Q(\Omega \times \mathbb{R})} \\
(2) \quad \lim_{T \to +\infty} ||D\lambda_{U_T}||_{Q_\alpha(\Omega \times \mathbb{R})} &= ||D\lambda_U||_{Q_\alpha(\Omega \times \mathbb{R})}.
\end{align*}
\]

Remark 4.10. Suppose \( u \in BV_N(\Omega) \). Arguing as in Theorem 2.3 of [10] we conclude

\[
\lim_{T \to +\infty} ||Du_T||_N(\Omega) = ||Du||_N(\Omega)
\]

Proof. We can assume ||\( D\lambda_U ||_Q(\Omega \times \mathbb{R}) \) and ||\( D\lambda ||_{Q_\alpha(\Omega \times \mathbb{R})} \) are finite. Otherwise the conclusion (1) and (2) are trivial by Theorem 2.3. This means that \( u \in L^1(\Omega) \) when we show the conclusion (1) and \( e^{au} \in L^2(\Omega) \) when we
show the conclusion (2).

Now we begin to show the conclusion (1). Fix $T > 0$. Since $\lambda_{U_T}$ converges to $\lambda_U$ locally in $L^1(\Omega \times \mathbb{R})$. From Theorem 2.3 it is sufficient to show that

$$||D\lambda_U||_{Q(\Omega \times \mathbb{R})} \geq \lim_{T \to +\infty} \sup \ ||D\lambda_{U_T}||_{Q(\Omega \times \mathbb{R})}$$

Moreover beacause $\lambda_{U_T}$ is equal to $\lambda_U$ on the set $\Omega \times (-T, T)$, we have

$$||D\lambda_{U_T}||_{Q(\Omega \times (\infty, -T))} = ||D\lambda_{U_T}||_{Q(\Omega \times (T, +\infty))} = 0$$

We set

$$E_{T+} = \{ x \in \Omega : u(x) > T \} \quad E_{T-} = \{ x \in \Omega : u(x) < -T \}$$

By Theorem 4.6

$$||D\lambda_{U_T}||_{Q(\Omega \times \{\pm T\})} = \int_{E_{T\pm}} dvol$$

where $dvol$ is the volume of $\Omega$. Since $u \in L^1(\Omega)$ and $vol(\Omega)$ is bounded. Then the limit of (4.17) is 0 as $T \to +\infty$. Continuing the computation in (4.15) we obtain

$$||D\lambda_{U_T}||_{Q(\Omega \times (\infty, -T))} = \lim_{T \to +\infty} \sup \ ||D\lambda_{U_T}||_{Q(\Omega \times (\infty, -T))}$$

This gives (6.9). Thus we arrive the conclusion (1).

The proof of the conclusion (2) is similar to that of the conclusion (1). The differences come from the metric in $Q_\alpha$. By Theorem 4.6

$$||D\lambda_{U_T}||_{Q_\alpha(\Omega \times \{\pm T\})} = \int_{E_{T\pm}} e^{\pm \alpha T} dvol$$

Notice that

$$\int_{E_{T+}} e^{\alpha T} dvol \leq \int_{E_{T+}} e^{\alpha u(x)} dvol \ \int_{E_{T-}} e^{-\alpha T} dvol \leq e^{-\alpha T} vol(\Omega)$$

Since $e^{\alpha u(x)} \in L^1(\Omega)$ and $\Omega$ is bounded, the limit of (4.18) is 0 as $T \to +\infty$. Now arguing as in the proof of the conclusion (1) we will show the conclusion (2). The proof is complete. □

5. Miranda’s observation

In this section we study the relationship between two area functionals and the perimeter in corresponding manifolds. In Theorem 5.9 we show that the Miranda’s observation mentioned in the introduction is true for the conformal product functional $\mathcal{F}_\alpha(u, \Omega)$ in $Q_\alpha$. 
5.1. The perimeter. We start with the definition of the perimeter.

**Definition 5.1.** Let $\Omega$ be an open set in a Riemannian manifold $N$. Suppose $E$ is a Borel set in $N$ and $\lambda_E$ is its characteristic function. The perimeter of $E$ in $\Omega$ is given by

$$
||D\lambda_E||_N(\Omega) = \sup\left\{ \int_{\Omega} \lambda_E \text{div}(X) \text{dvol} : X \in T_0 \Omega, \langle X, X \rangle \leq 1 \right\}
$$

(5.1)

If $E$ has locally finite perimeter, $||D\lambda_E||_N(\Omega') < \infty$ for each bounded open set $\Omega'$ in $\Omega$ (i.e. $\lambda_E \in BV_{loc,N}(\Omega)$), then $E$ is called a Caccioppoli set.

**Remark 5.2.** By Remark 2.9 the fact that $E$ is a Caccioppoli set only depends the conformal class of the metric in $N$. See also Lemma 4.5.

Next we define the Hausdorff measure on Riemannian manifolds.

**Definition 5.3.** Let $N$ be a complete Riemannian manifold with dimension $n$. By the Nash Embedding Theorem, there is an isometric embedding $i : N \rightarrow \mathbb{R}^{n+s}$ for some positive integer $s$. Let $k > 0$. For any set $E \subset N$, the $k$-dimensional Hausdorff measure of $E$, $H^k(E)$ is given by

$$
H^k(E) := H^k(i(E))
$$

where in the right side $H^k$ denote the $k$-dimensional Hausdorff measure in $\mathbb{R}^{n+s}$.

A particular case is that for any open set $A$ in $N$, $H^n(A) = \text{vol}(A)$ according to the area formula where $\text{vol}(A)$ is the volume of $A$ with respect to the metric in $N$. For more details see Subsection 8.6 Example (2) in [20].

For a Caccioppoli set its perimeter and other properties are unchanged if we make alternations of measure zero. In other words we are really concerned with equivalence classes of Caccioppoli sets.

The following result is a generalization of Proposition 3.1 in [10] with exactly the same proof by the Nash Embedding Theorem.

**Proposition 5.4.** For any $x \in N$, $B_x(r)$ be the ball in $N$ centered at $x$ with radius $r$. Let $\text{inj}(x)$ denote the injective radius of $x$, i.e. the supremum of $r$ such that $B_r(x)$ is an embedded normal ball in $N$.

If $E$ is a Borel set in $N$, there exists a Borel set $\tilde{E}$ equivalent to $E$ (that is, differs only by a set of $H^n$ measure zero) and such that

$$
0 < H^n(\tilde{E} \cap B_\rho(x)) < \text{vol}(B_\rho(x))
$$

for all $x \in \partial \tilde{E}$ and $\rho \in (0, \text{inj}(x))$.

In the remainder of this paper we always assume (5.2) holds for every Caccioppoli set mentioned.
5.2. The perimeter of subgraphs. First we show that the area functional of a BV function is the perimeter of its subgraph in corresponding product manifolds.

The following result generalizes Theorem 14.6 in [10] in Euclidean space into general product manifold $Q$ and conformal product manifold $Q_{\alpha}$ (Definition 4.8).

**Theorem 5.5.** Use the notation in Definition 3.1. Suppose $\Omega$ is an open bounded set in a Riemannian manifold $N$ with Lipschitz boundary. Let $u$ be a measurable function on $\Omega$ and $U$ be its subgraph.

1. If $u \in BV(\Omega)$, then $F(u, \Omega) = ||D\lambda_U||_{Q(\Omega \times \mathbb{R})}$;
2. If $\alpha > 0$ and $e^{\alpha u} \in BV(\Omega)$, then $F_{\alpha}(u, \Omega) = ||D\lambda_U||_{Q_{\alpha}(\Omega \times \mathbb{R})}$.

First we obtain one side of the identities in Theorem 5.5 as follows.

**Lemma 5.6.** Take the assumptions and notation in Theorem 5.5.

1. Suppose $u \in BV(\Omega)$. Then $||D\lambda_U||_{Q(\Omega \times \mathbb{R})} \leq F(u, \Omega)$.
2. Suppose $\alpha > 0$ and $e^{\alpha u} \in BV(\Omega)$. Then $||D\lambda_U||_{Q_{\alpha}(\Omega \times \mathbb{R})} \leq F_{\alpha}(u, \Omega)$

**Proof.** First assume $u \in C^1(\Omega)$. Then its subgraph $U$ has a $C^1$ boundary. According to the definition of the perimeter we have

\[
||D\lambda_U||_{Q_{\alpha}(\Omega \times \mathbb{R})} = vol_{Q_{\alpha}}(\partial U \cap (\Omega \times \mathbb{R}))
\]

\[
||D\lambda_U||_{Q(\Omega \times \mathbb{R})} = vol_Q(\partial U \cap (\Omega \times \mathbb{R}))
\]

where $vol$ denotes the volume and the lower index indicates the corresponding ambient manifold. From the definition of $F(u, \Omega)$ and $F_{\alpha}(u, \Omega)$ we have

\[
vol_{Q_{\alpha}}(\partial U \cap (\Omega \times \mathbb{R})) = \int_{\Omega} e^{\alpha u} \sqrt{1 + |Du|^2} dvol = F_{\alpha}(u, \Omega)
\]

\[
vol_Q(\partial U \cap (\Omega \times \mathbb{R})) = \int_{\Omega} \sqrt{1 + |Du|^2} dvol = F(u, \Omega)
\]

Thus it holds that for $u \in C^1(\Omega)$

\[
||D\lambda_U||_{Q_{\alpha}(\Omega \times \mathbb{R})} = \mathcal{F}_{\alpha}(u, \Omega)
\]

\[
||D\lambda_U||_{Q(\Omega \times \mathbb{R})} = \mathcal{F}(u, \Omega)
\]

By (2) in Theorem 3.6 for $u \in BV(\Omega)$ there exists a smooth sequence $\{u_i\}_{i=1}^{\infty}$ in $C^\infty(\Omega)$ such that $u_i$ converges to $u$ in $L^1(\Omega)$ and

\[
\lim_{i \to \infty} \mathcal{F}(u_i, \Omega) = \mathcal{F}(u, \Omega)
\]

Let $U_i$ be the subgraph of $u_i$. It is easy to see that $\lambda_{U_i}$ converges to $\lambda_U$ in the $L^1_{loc}(Q)$. Thus with (5.5) we obtain

\[
||D\lambda_U||_{Q(\Omega \times \mathbb{R})} \leq \liminf_{i \to \infty} ||D\lambda_{U_i}||_{Q(\Omega \times \mathbb{R})}
\]

\[
= \lim_{i \to +\infty} \inf \mathcal{F}(u_i, \Omega) = \mathcal{F}(u, \Omega)
\]

This gives the conclusion (1).

As for the conclusion (2) notice that if $e^{\alpha u} \in BV(\Omega)$ then the definition
of \( \mathfrak{F}_\alpha(u, \Omega) \) implies that it is finite. By (3) in Theorem 3.6 there is a smooth sequence \( \{u_i\}_{i=1}^{\infty} \) in \( C^\infty(\Omega) \) such that \( e^{\alpha u_i} \) converges to \( e^{\alpha u} \) in \( L^1(\Omega) \) and
\[
\lim_{i \to \infty} \mathfrak{F}_\alpha(u_i, \Omega) = \mathfrak{F}_\alpha(u, \Omega)
\]
Thus with a similar argument as the proof of the conclusion (1) we conclude the conclusion (2).

Now we are ready to show Theorem 5.5. Our proof is similar as that of Theorem 14.6 in [10].

Proof. Let \( T \) be a fixed positive constant. First we assume \( u \in [-T, T] \) and \( u \in BV(\Omega) \). Suppose \( h \in C_0(\Omega) \) and \( X \in T_0(\Omega) \) satisfying
\[
e^{-2\alpha(T+1)}(h^2 + \langle X, X \rangle) \leq 1
\]
where \( \langle \cdot, \cdot \rangle \) is the inner product on \( N \).

Let \( \eta(r) \) be a smooth function on \( \mathbb{R} \) with its support in \( [-T, T] \) such that \( \eta \equiv 1 \) in \( [-T, \sup \Omega u + 1] \) and \( |\eta(r)| \leq 1 \). Let \( \eta_1(r) \) be a smooth function with a compact support on \( \mathbb{R} \) satisfying \( \eta_1(r) = e^{-\alpha r} \) on \( [-T, T] \). Now we define a smooth vector field
\[
X' = \eta_1(r)\eta(r)(h \partial_r + X)
\]
on \( Q_{\alpha} \). The inner product of \( X' \) on \( Q_{\alpha} \) is given by
\[
\langle X', X' \rangle_{g_{\alpha}} = e^{-2\alpha(T+1)}\eta_1^2(r)(h^2 + \langle X, X \rangle) \leq 1
\]
where \( g_{\alpha} \) denotes the metric of \( Q_{\alpha} \).

Let \( dvol_{\alpha} \) and \( dvoldr \) be the volume form of \( Q_{\alpha} \) and \( \Omega \). We have
\[
dvol_{\alpha} = e^{\alpha \frac{(n+1)r}{n}} dvoldr
\]
Let \( div_{\alpha} \) and \( div \) be the divergence of \( Q_{\alpha} \) and \( \Omega \) respectively. Notice that \( X' \) has compact support in \( \Omega \times \mathbb{R} \). By the definition of the perimeter we have
\[
||D\lambda_U||_{Q_{\alpha}(\Omega \times \mathbb{R})} \geq \int_{\Omega \times \mathbb{R}} \lambda_U div_{\alpha}(X') dvol_{\alpha}
\]
From the definition of \( \eta_1(r) \) and \( \eta(r) \) we have
\[
X'_\perp dvol_{\alpha} = e^{-\alpha(T+1)} \eta(r) e^{\alpha r} ((-1)^n h(x) dvoldr + X_\perp dvol_{\alpha})
\]
This yields that
\[
div_{\alpha}(X') dvol_{\alpha} = d(X'_\perp dvol_{\alpha})
\]
\[
e^{-\alpha(T+1)} (\eta(r) e^{\alpha r})' h(x) dvoldr
\]
\[
+ e^{-\alpha(T+1)} e^{\alpha r} \eta(r) div(X) dvoldr
\]
Notice that
\[
\int_{-\infty}^{u(x)} (\eta(r) e^{\alpha r})' dr = e^{\alpha u(x)}
\]
\begin{equation}
\int_{-\infty}^{u(x)} \eta(r)e^{\alpha r} dr = \begin{cases}
e^{\alpha u(x)}/\alpha + C, & \alpha \neq 0 \\
u(x) + C, & \alpha = 0
\end{cases}
\end{equation}

where \(C\) is a fixed constant. Now \(F_{in} := \int_{\Omega \times \mathbb{R}} \lambda U \text{div}_x (X') dvol_x\). Therefore

\begin{equation}
F_{in} = \begin{cases}
\int_{\Omega} e^{-\alpha(T+1)} (e^{\alpha u(x)} h(x) + \frac{e^{\alpha u(x)}}{\alpha} \text{div}(X)) dvol, & \alpha \neq 0 \\
\int_{\Omega} (h(x) + u(x) \text{div}(X)) dvol, & \alpha = 0
\end{cases}
\end{equation}

From (5.12) and the condition (5.8), we obtain

\begin{equation}
||D\lambda U||_{Q_\alpha(\Omega \times \mathbb{R})} \geq \sup \limits_{h, X \in (5.8)} F_{in} = \begin{cases}
\mathcal{F}_\alpha(u, \Omega), & \alpha > 0 \\
\mathcal{F}(u, \Omega), & \alpha = 0
\end{cases}
\end{equation}

for \(|u| \leq T\). Combining Lemma 5.6 and (5.18) together we obtain Theorem 5.5 in the case that \(u\) is uniformly bounded.

For \(T > 0\) we define \(u_T(x) := \max\{\min(u(x), T), -T\}\). Now suppose \(u \in BV(\Omega)\), then \(u \in L^1(\Omega)\). Thus \(u_T\) converges to \(u(x)\) in \(L^1(\Omega)\) as \(T \to +\infty\). By Theorem 3.5 and (1) in Lemma 4.9 we have

\begin{equation}
||D\lambda U||_{Q_\alpha(\Omega \times \mathbb{R})} = \lim \limits_{T \to +\infty} ||D\lambda U_T||_{Q(\Omega \times \mathbb{R})}
\end{equation}

for \(u \in BV(\Omega)\). This is the conclusion (1).

Now assume \(e^{\alpha u(x)} \in BV(\Omega)\). Thus \(\mathcal{F}_\alpha(u, \Omega)\) is finite. Moreover \(\Omega\) is bounded. By Theorem 3.5 and (2) in Lemma 4.9 we conclude that

\begin{equation}
||D\lambda U||_{Q_\alpha(\Omega \times \mathbb{R})} = \lim \limits_{T \to +\infty} ||D\lambda U_T||_{Q_\alpha(\Omega \times \mathbb{R})}
\end{equation}

with the conclusion (2) of Lemma 5.6 we conclude

\begin{equation}
||D\lambda U||_{Q_\alpha(\Omega \times \mathbb{R})} = \mathcal{F}_\alpha(u, \Omega)
\end{equation}

for \(u \in BV(\Omega)\). This is the conclusion (2). The proof is complete.

\begin{corollary}
Suppose \(u\) is a measurable function on \(\Omega\) with \(|u| \leq T\) such that \(\mathcal{F}_\alpha(u, \Omega)\) is finite. Then

\begin{equation}
\mathcal{F}_\alpha(u, \Omega) \geq e^{-\alpha T} \max\{\text{vol}(\Omega), ||Du||_{N}(\Omega)\}
\end{equation}

\end{corollary}
Proof. By Theorem 2.8 and Theorem 5.5, we have
\[ \mathfrak{F}_\alpha(u, \Omega) = \|D\lambda_U\|_{Q_\alpha(\Omega \times \mathbb{R})} \]
\[ = \int_{\Omega \times \mathbb{R}} e^{\alpha t} d|D\lambda_U|_Q \]
\[ \geq e^{-\alpha T} \|D\lambda_U\|_{Q(\Omega \times \mathbb{R})} \]
\[ = e^{-\alpha T} \mathfrak{F}(u, \Omega) \]
In the third line above, we use the fact that the support of the Radon measure \(|D\lambda_U|_Q\) is on the set \((x, u(x))\). The conclusion follows from (3) in Theorem 3.5.

5.3. Miranda’s observation. In this subsection we show that a local minimizer of the conformal area functional \(\mathfrak{F}_\alpha(u, \Omega)\) is a local minimizer of the perimeter of the perimeter in \(Q_\alpha\). According to Giusti [9] it is Miranda [17] firstly to observe this phenomenon for product area functional in the product manifold.

The following result is similar to Theorem 14.8 in [10].

**Lemma 5.8.** Let \(u(x)\) be a measurable function with \(e^{\alpha u(x)} \in BV(\Omega)\). Let \(U\) be the subgraph of \(u(x)\) in \(Q_\alpha\). Let \(\Omega\) be an open set in \(N\). Let \(F \subset Q_\alpha\) be a Caccioppoli set with finite perimeter satisfying for a.e. \(x \in \Omega\), \(\lambda_F(x, t) = 0\) for all \(t > T_x\) where \(T_x\) is a constant depending on \(x\).

Then the function \(\omega(x)\)
\[ e^{\alpha \omega(x)} = \alpha \lim_{k \to +\infty} \left( \int_{-k}^{k} e^{\alpha t} \lambda_F(x, t) dt \right) \]
is a.e. well-defined and
\[ \mathfrak{F}_\alpha(\omega, \Omega) \leq \|D\lambda_F\|_{Q_\alpha(\Omega \times \mathbb{R})} \]
Proof. Due to the assumption in Lemma 5.8 it is obvious that \(\omega(x)\) is well-defined. Suppose \(h \in C_0(\Omega)\) and \(X \in T_0(\Omega)\) satisfying
\[ h^2 + \langle X, X \rangle \leq 1 \]
where \(\langle \cdot, \cdot \rangle\) is the inner product with respect to the metric \(g\).

Let \(\eta(r)\) be a smooth function such that \(0 \leq \eta(r) \leq 1\) with compact support in \(\mathbb{R}\). Set \(X' = e^{-\alpha r} \eta(r)(X + h(r) \partial_r)\). Then \(\langle X', X' \rangle_\alpha \leq 1\) where \(\langle \cdot, \cdot \rangle_\alpha\) denotes the inner product of \(Q_\alpha\). By Definition 2.1 we have
\[ \|D\lambda_F\|_{Q_\alpha(\Omega \times \mathbb{R})} \geq \int_{\Omega \times \mathbb{R}} \lambda_F(x, r) \text{div}_\alpha(X') d\text{vol}_\alpha \]
where \(\text{div}_\alpha\) and \(d\text{vol}_\alpha\) are the divergence and the volume form of \(Q_\alpha\). Arguing as in (5.11), (5.12), (5.13) and (5.14) we obtain
\[ \text{div}_\alpha(X') d\text{vol}_\alpha = \{(e^{\alpha \eta(r)})' h(x) + e^{\alpha \eta(r)} \text{div}(X)\} d\text{vol} dr \]
where $\text{div}$ is the divergence of $\Omega$. Thus expanding (5.25) gives that

$$
||D\lambda F||_{Q_\alpha(\Omega \times \mathbb{R})} \geq \int_\Omega h(x) \left\{ \int_{-\infty}^{\infty} (e^{ar}\eta(r))' \lambda F(x,r) dr \right\} dvol 
+ \int_\Omega \text{div}(X) \left\{ \int_{-\infty}^{\infty} e^{ar}\eta(r) \lambda F(x,r) dr \right\} dvol
$$

Replacing $\eta(t)$ with a sequence of $\{\eta_k(t) : 0 \leq \eta_k \leq 1$ with compact support $\}_{k=1}^{\infty}$ which converges to the constant function 1 as $k \to +\infty$, we obtain

$$
||D\lambda F||_{Q_\alpha(\Omega \times \mathbb{R})} \geq \int_\Omega h(x) \left\{ \int_{-\infty}^{\infty} \alpha e^{ar} \lambda F(x,r) dr \right\} dvol 
+ \int_\Omega \text{div}(X) \left\{ \int_{-\infty}^{\infty} e^{ar} \lambda F(x,r) dr \right\} dvol
$$

Here we apply the fact for a.e. $x \in \Omega$, $\lambda F(x,t) = 0$ for all $t > T_x$ where $T_x$ is a constant depending on $x$. Now taking the supremum of all $h$ and $X$ satisfying (5.24) and applying the definition of $\mathcal{F}_\alpha(\cdot,\Omega)$, one sees that

$$
(5.28) \quad ||D\lambda F||_{Q_\alpha(\Omega \times \mathbb{R})} \geq \mathcal{F}_\alpha(\omega,\Omega)
$$

The proof is complete. \hfill \Box

Now we can conclude the Miranda’s observation in the conformal product manifold $Q_\alpha$ given in Definition 4.8 as follows.

**Theorem 5.9 (Miranda’s observation).** Let $\Omega \subset N$ be an open bounded set and $\alpha > 0$ be a fixed positive constant. Let $u$ be a measurable function. If $e^{au} \in BV(\Omega)$ and $u$ is a local minimum of the conformal area functional $\mathcal{F}_\alpha(u,\Omega)$. Let $U$ be the subgraph of $u(x)$. Then $U$ locally minimizes the perimeter in the manifold $Q_\alpha$.

**Proof.** Suppose $u(x)$ is a measurable function on $\Omega$ with $e^{au(x)} \in BV(\Omega)$ and $u(x)$ is a local minimum of $\mathcal{F}_\alpha(u,\Omega)$. Let $U$ denote the subgraph of $u(x)$ in $Q_\alpha$.

Let $F$ be a Caccioppoli set with finite perimeter satisfying $F \Delta U \subset K$ where $K$ is a compact set in $\Omega \times \mathbb{R} \subset Q_\alpha$. Thus $F$ has the property that for a.e. $x \in \Omega$, $\lambda F(x,t) = 0$ for all $t > T_x$ where $T_x$ is a constant depending on $x$. Now from Lemma 5.8 we obtain

$$
(5.29) \quad \mathcal{F}_\alpha(\omega,\Omega) \leq ||D\lambda F||_{Q_\alpha(\Omega \times \mathbb{R})}
$$

where $\omega(x)$ is given by

$$
e^{a\omega(x)} = \alpha \lim_{k \to +\infty} \left( \int_{-k}^{k} e^{at} \lambda F(x,t) dt \right)
$$

Since $F \Delta U \subset K$, then $\omega(x) = u(x)$ outside a compact set in $\Omega$. By Theorem 5.5 one sees that

$$
(5.30) \quad ||D\lambda U||_{Q_\alpha(\Omega \times \mathbb{R})} = \mathcal{F}_\alpha(u,\Omega) \leq \mathcal{F}_\alpha(\omega,\Omega)
$$
Here we apply the fact that \( u(x) \) is a local minimum of \( F(.,\Omega) \). Combining (5.29) and (5.30) together we obtain
\[
(5.31) \quad ||D\lambda_U||_{Q_\alpha(\Omega \times \mathbb{R})} \leq ||D\lambda_F||_{Q_\alpha(\Omega \times \mathbb{R})}
\]
The proof of Theorem 5.9 is complete. \( \square \)

6. Existence of generalized solutions

In this section we define a generalized solution of the Dirichlet problem (1.1) following from the Miranda-Giusti’s generalized solution theory in [10].

6.1. A generalized solution. We continue to use the notation in Definition 4.8. Now fix two bounded open sets \( B, \Omega \) in \( N \) satisfying \( \overline{\Omega} \subset \subset B \).

Moreover \( \Omega \) has Lipschitz boundary \( \partial \Omega \).

Definition 6.1. Let \( \Omega \subset \subset B \) be two open bounded set in \( N \). Let \( u(x), \psi(x) \) be two measurable functions taking values in \( [-\infty, \infty] \) such that the subgraph \( \Psi \) of \( \psi(x) \) is a Caccioppoli set in \( Q_\alpha \). We say that \( u(x) \) is a generalized solution to the Dirichlet problem (1.1) on \( \Omega \) with boundary data \( \psi(x) \) on \( \partial \Omega \) if

1. the subgraph of \( u(x), U \), coincides with \( \Psi \) outside \( \overline{\Omega} \times \mathbb{R} \);
2. for any Caccioppoli set \( F \) satisfying \( F \Delta U \subset K \) where \( K \) is a compact set in \( \overline{\Omega} \times \mathbb{R} \), it holds that
\[
(6.1) \quad \int_K d|D\lambda_U|_{Q_\alpha} \leq \int_K d|D\lambda_F|_{Q_\alpha}
\]
where \( |D\lambda_U|_{Q_\alpha} \) and \( |D\lambda_F|_{Q_\alpha} \) are Radon measures of generated by \( \lambda_U \) and \( \lambda_F \) in \( Q_\alpha \) respectively (See Theorem 2.6).

Remark 6.2. In the followings such \( u(x) \) is also referred as a generalized solution if without any confusion.

In order to understand (6.1), we need to define the generalized trace of \( u \) when its subgraph \( U \) is a Caccioppoli set in \( Q_\alpha \). It is useful in the proof of Lemma 7.2.

Lemma 6.3. Suppose \( \Omega \) is an open bounded set in \( N \) with Lipschitz boundary. Let \( u(x) \) be a measurable function on \( N \) such that its subgraph \( U \) is a Caccioppoli set in \( Q_\alpha \). Then

1. Let \( u_k(x) \) be the function \( \max\{\min(u(x),k),-k\} \). Then \( \{u_k(x)\} \in BV(\Omega) \) and their trace \( \{u_k(x)\} \) converges to a measurable function \( GT_u(x) \) taking value in \( [-\infty, \infty] \) a.e. on \( \partial \Omega \) as \( k \) goes to \( +\infty \).
2. Let \( U_{GT} \) be the subgraph of \( GT_u(x) \) in \( \partial \Omega \times \mathbb{R} \). The trace of \( \lambda_U \) on \( \partial \Omega \times \mathbb{R} \) satisfies that
\[
\mathcal{T}\lambda_U = \lambda_{U_{GT}}
\]

Definition 6.4. When the subgraph of \( u(x) \) is a Caccioppoli set in \( \Omega \times \mathbb{R} \subset Q_\alpha \), \( GT_u(x) \) defined in the above lemma is called the generalized trace of \( u(x) \) from \( \Omega \).
Remark 6.5. If \( u \in BV(\Omega) \) and \( \Omega \) is Lipschitz, the generalized trace of \( u(x) \) is just the trace of \( u \) by Remark 4.10 and Lemma 4.3. In general the generalized trace of \( u(x) \) can take the value in \([−∞, ∞]\) and may not belong to \( L^1(∂\Omega) \). For example see Remark 7.3.

Proof. Let \( U_k \) be the subgraph of \( u_k(x) \) in \( B \times \mathbb{R} \). Thus \( \lambda_{U_k} = \lambda_U \) on the open set \( \Omega_k \subset Q_α \). Since \( U \) is a Caccioppoli set in \( Q_α \), then

\[
(6.2) \quad \infty > ||D\lambda_U||_{Q_α}(\Omega \times (-k, k)) = ||D\lambda_{U_k}||_{Q_α}(\Omega \times (-k, k)) = \mathfrak{F}_α(u_k, \Omega) \quad \text{by Theorem 5.5}
\]

Thus \( u_k \in BV(\Omega) \). By Theorem 5.1 its trace \( T u_k \) is well-defined. Let \( U_k \) be the subgraph of \( u_k \). Let \( T \lambda_{U_k} \) be the trace of \( U_k \) on \( ∂\Omega \times \mathbb{R} \subset Q_α \). By Lemma 6.3 there is a measure zero set \( E \) in a open interval \((0, r_0)\) such that

\[
(6.3) \quad \lim_{r \to 0^+, r \notin E} u_k(x, r) = T u_k(x) \quad x \in ∂\Omega \quad \text{in} \quad L^1(∂\Omega)
\]

\[
(6.4) \quad \lim_{r \to 0^+, r \notin E} \lambda_{U_k}(p, r) = T \lambda_{U_k}(p) \quad p \in ∂\Omega \times \mathbb{R} \quad \text{in} \quad L^1(∂\Omega \times (-k, k))
\]

Thus \( T \lambda_{U_k} \) is the subgraph of \( T u_k \) on \( ∂\Omega \) in \( ∂\Omega \times (-k, k) \).

By (6.2) and the definition of \( T u_k \) it is clear that \( T \lambda_{U_k} \) converges to \( T \lambda_U \) almost everywhere on \( ∂\Omega \times \mathbb{R} \) as \( k \to +∞ \). Thus arguing as Lemma 16.3 in [10], \( T \lambda_U \) on \( ∂\Omega \times \mathbb{R} \) is a subgraph of some measurable function \( \mathcal{G} T u(x) \) on \( ∂\Omega \). Thus \( T u_k(x) \) converges to \( \mathcal{G} T u(x) \) a.e. on \( ∂\Omega \). \( \square \)

6.2. The existence of generalized solutions. The main result of this section is stated as follows.

Theorem 6.6. Let \( \Omega \subset \subset B \) be two open bounded set in a Riemannian manifold \( N \). Suppose \( ∂\Omega \) is Lipschitz. Let \( \psi(x) \) be any measurable function such that its subgraph \( \Psi \) is a Caccioppoli set in \( Q_α \). Then there is a generalized solution to the Dirichlet problem (1.1) with boundary data \( \psi(x) \).

Remark 6.7. The restriction on \( \psi(x) \) is sufficient general. Suppose \( \psi(x) \in L^1(∂\Omega) \) and \( ∂\Omega \) is Lipschitz. By Lemma 4.7 there is a function, still denoted by \( \psi(x) \), in \( BV(B) \) such that its trace on \( ∂\Omega \) from \( \Omega \) and \( B \setminus \Omega \) are \( \psi(x) \). Since \( B \) is bounded, the conclusion (3) in Theorem 5.5 implies that \( \mathfrak{F}(\psi, B) \) is finite. From the conclusion (1) in Theorem 5.5 \( \Psi \) is a Caccioppoli set in \( Q \). Since \( Q \) and \( Q_α \) are conformal to each other (see Definition 4.8), then \( \Psi \) is also a Caccioppoli set in \( Q_α \) by Remark 2.9 and Remark 6.2. In particular if \( \psi(x) \) is continuous on \( ∂\Omega \) and bounded, then the extension function is still bounded by Lemma 4.7.

Proof. For any \( k > 0 \) we set

\[
\psi_k(x) = \min\{k, \max\{\psi(x), -k\}\}
\]
Let $\Psi_k$ be the subgraph of $\psi_k$. Because $\lambda_{\Psi_k} = \lambda_{\Psi}$ on $\Omega \times (-k, k)$. On the other hand since $\Psi$ is a Caccioppoli set in $Q_{\alpha}$, then

$$||D\lambda_{\Psi}||_{Q_{\alpha}((\bar{\Omega} \times (-k, k)) = ||D\lambda_{\Psi_k}||_{Q_{\alpha}((\bar{\Omega} \times (-k, k)) < \infty$$

By (2) in Theorem 5.5 this implies $\bar{\cal{F}}_{\alpha}(\psi_k, \Omega)$ is finite. By Corollary 5.7 $\psi_k \in BV(\mathcal{B})$ because $|\psi_k(x)| \leq k$. Thus we can consider the following minimizing problem

$$\alpha_k = \min\{\bar{\cal{F}}_{\alpha}(u, \mathcal{B}) : u \in BV(\Omega), |u| \leq k, u = \psi_k \text{ on } \mathcal{B}\setminus\Omega\}$$

Let $\{u_j\}_{j=1}^{\infty} \in BV(\Omega)$ be the sequence satisfying $|u_j| \leq k, u = \psi_k$ on $\mathcal{B}\setminus\Omega$ such that

$$\lim_{j \to +\infty} \bar{\cal{F}}_{\alpha}(u_j, \mathcal{B}) = \alpha_k$$

Again by Corollary 5.7 we have

$$\max\{|D\alpha_j|_{N}(\mathcal{B}) : j = 1, \cdots, \infty\} \leq C(k, \alpha_k)$$

Since the boundary of $\mathcal{B}$ is Lipschitz, the compactness of $BV$ functions implies that there is a subsequence of $\{u_j\}_{j=1}^{\infty}$ with $|u_j| \leq k$ and $u_j = \psi_k$ on $\mathcal{B}\setminus\Omega$, still denoted as $\{u_j\}_{j=1}^{\infty}$, such that $u_j \to u_k$ in $L^1(\mathcal{B})$ as $j \to +\infty$. By the semicontinuity of $\bar{\cal{F}}_{\alpha}(u, \mathcal{B})$, we have

$$\alpha_k \leq \bar{\cal{F}}_{\alpha}(u_k, \mathcal{B}) \leq \liminf_{j \to \infty} \bar{\cal{F}}_{\alpha}(u_j, \mathcal{B}) = \alpha_k$$

with the property that

$$|u_k| \leq k \quad u_k = \psi_k \quad \text{on } \mathcal{B}\setminus\Omega$$

Thus $\bar{\cal{F}}_{\alpha}(u_k, \mathcal{B}) = \alpha_k$.

Now let $U_k$ be the subgraph of $u_k$ in $\mathcal{B} \times \mathbb{R} \subset Q_{\alpha}$. Let $W$ be the set $U_k \cup \Omega \times (-T, T)$. Suppose $k > 2T$. By Theorem 5.9 we obtain the estimate

$$\int_{B \times (-2T, 2T)} |D\alpha_{U_k}|_{Q_{\alpha}} \leq \int_{B \times (-2T, 2T)} |D\alpha_W|_{Q_{\alpha}} \leq \int_{B \times (-2T, 2T)} |D\alpha_{\Psi}|_{Q_{\alpha} + 2\text{vol}(\partial(\Omega \times (-2T, 2T)))} = c(T)$$

Arguing as Lemma 16.3 in [10] we can extract a subsquence, still denote again by $u_k$, converging almost everywhere to a measurable function $u_\infty$.

Now let $U_\infty$ be the subgraph of $u_\infty$. It is clear that $U_\infty$ coincides with $\Psi$ outside $\bar{\Omega} \times \mathbb{R}$. Suppose $V$ is a Caccioppoli set in $B \times \mathbb{R}$ coinciding with $U_\infty$ except some compact set $K \subset \Omega \times (-T, T)$. Let $A$ be an open set satisfying $\Omega \subset A \subset B$. Now set

$$V_k = \begin{cases} V \text{ in } A \times (-T, T) \\ U_k \text{ outside } A \times (-T, T) \end{cases}$$
By Theorem 5.9 $U_k$ locally minimizes the perimeter in $B \times (-T, T)$.

\begin{equation}
\int_{A \times [-T, T]} |D\lambda U_k|_{Q_\alpha} \leq \int_{A \times [-T, T]} |D\lambda V_k|_{Q_\alpha}
\end{equation}

If $k > T$, it is clear that $V_k$ coincides with $U_k$ outside $\Omega \times (-T, T)$. By Theorem 4.6 and the definition of traces of BV functions we have

\begin{equation}
\int_{(\partial A) \times (-T, T)} |D\lambda U_k|_{Q_\alpha} = \int_{(\partial A) \times (-T, T)} |D\lambda V_k|_{Q_\alpha}
\end{equation}

Notice that $\partial (A \times [T, T]) = \{ (\partial A) \times (-T, T) \} \cup (A \times \{ \pm T \})$. By Theorem 4.6 the inequality (6.9) is equivalent to

\begin{equation}
\int_{A \times (-T, T)} |D\lambda U_k|_{Q_\alpha} \leq \int_{A \times (-T, T)} |D\lambda V|_{Q_\alpha}
\end{equation}

\begin{equation}
+ \int_{A \times \{ \pm T \}} |\mathcal{T} \lambda V - \mathcal{T} \lambda U_k|_{d\text{vol}Q_\alpha}
\end{equation}

where $\mathcal{T}$ denotes the corresponding traces of $\lambda V$ and $\lambda U_k$ on $\mathcal{T} \lambda V$ and $\mathcal{T} \lambda U_k$.

Because $U_k$ and $U_\infty$ are Caccioppoli set in $Q_\alpha$ for all $k = 1, \cdots, \infty$. By the Fubiniz theorem for a.e. $T \in \mathbb{R}$ we have

\begin{equation}
\int_{A \times \{ T \}} |Df|_{Q_\alpha} = 0 \quad \text{for} \quad f = \lambda U_k \quad \text{or} \quad \lambda U_\infty
\end{equation}

This implies that for a.e. $T$

\begin{equation}
\mathcal{T} f(x, \pm T) = f(x, \pm T) \quad \text{for} \quad f = \lambda U_k \quad \text{or} \quad \lambda U_\infty \quad \text{a.e.} x \in A
\end{equation}

and $\mathcal{T} \lambda V(x, \pm T) = \lambda U_\infty(x, \pm T)$ for a.e. $x \in A$. Since $\lambda U_k$ converges to $\lambda U_\infty$ locally with the $L^1(Q)$ norm,

\begin{equation}
\lim_{k \to +\infty} \lambda U_k(x,T) = \lambda U_\infty(x,T)
\end{equation}

for a.e. $T \in \mathbb{R}$ and $x \in A$ as $k \to +\infty$. Now we choose a $T \in \mathbb{R}$ satisfying in (6.11), (6.13) for all $k = 1, \cdots, \infty$ and the compact set $K \subset \Omega \times [-T, T]$. Let $k \to \infty$ in (6.10). By Theorem 3.5 it becomes

\begin{equation}
\int_{A \times (-T, T)} |D\lambda U_\infty|_{Q_\alpha} \leq \int_{A \times (-T, T)} |D\lambda V|_{Q_\alpha}
\end{equation}

which gives

\begin{equation}
\int_{K} |D\lambda U_\infty|_{Q_\alpha} \leq \int_{K} |D\lambda V|_{Q_\alpha}
\end{equation}

From the assumption of $V$ we obtain that $u$ is a generalized solution to the Dirichlet problem (1.1) on $\Omega$ with boundary data $\psi(x)$. This completes the proof. \qed
7. THE INFINITY VALUE OF GENERALIZED SOLUTIONS

In this section we study generalized solutions that take the infinity values. First we construct generalized solutions only taking infinity values. Then we recall some facts on the almost minimal set. With its regularity theory we describe the boundary property of the set in which a generalized solution takes the infinity value in Theorem 7.11.

7.1. Generalized solutions with infinity boundary data. We continue to use the notation in Definition 4.8. First in $Q_{\alpha}$ the translating motion of generalized solutions also gives generalized solutions.

Lemma 7.1. Take the notation in Definition 6.1. Suppose $u(x)$ is a generalized solution with boundary data $\psi$ and $a \in \mathbb{R}$. Then $u(x) + a$ is a generalized solution on $\Omega$ with boundary data $\psi(x) + a$ in $Q_{\alpha}$.

Proof. Fix $a \in \mathbb{R}$. We define a translating motion $T_a : N \times \mathbb{R} \to N \times \mathbb{R}$ as $T_a(x, t) = (x, t + a)$ where $x \in N$ and $t \in \mathbb{R}$. It is clear $T_a$ is an isometry of $Q$ with respect to the metric $g + dr^2$.

Suppose $F$ is a Caccioppoli set in $Q$. For any smooth vector field $X$ with compact support and any Borel set $C$ in $Q$, we have

$$\int_C \lambda_F \text{div}_Q(X) dvol_Q = \int_{T_aC} \lambda_{T_aF} \text{div}_Q(T_a^*X) dvol_Q$$

where $\text{div}_Q$ and $dvol_Q$ are the divergence and the volume form of $Q$ respectively. By Theorem 2.6 for any Borel set $C$ in $Q$, (7.1) implies that

$$\int_C d|D\lambda_F|_Q = \int_{T_aC} d|D\lambda_{T_aF}|_Q$$

Recall that the metric of $Q_{\alpha}$ is $e^{2\alpha n}(g + dr^2)$. Combining Theorem 2.8 with (7.2) together we obtain

$$e^{\alpha a} \int_C d|D\lambda_F|_{Q_{\alpha}} = \int_{T_aC} d|D\lambda_{T_aF}|_{Q_{\alpha}}$$

Here the function $\psi^{m-1}$ in Theorem 2.8 is just $e^{\alpha r}$.

Let $U$ be the subgraph of $u(x)$. Let $F$ be a Caccioppoli set in $Q_{\alpha}$ satisfying $F \Delta T_a U \subset K$ where $K$ is a compact set in $\bar{\Omega} \times \mathbb{R}$. Thus $T_{-a}F \Delta U \subset T_{-a}K$. Because $u(x)$ is a generalized solution on $\Omega$, by Definition 6.1 we have

$$\int_K d|D\lambda_{T_aF}|_{Q_{\alpha}} = e^{\alpha a} \int_{T_{-a}K} d|D\lambda_U|_{Q_{\alpha}}$$

$$\leq e^{\alpha a} \int_{T_{-a}K} d|D\lambda_{T_{-a}F}|_{Q_{\alpha}}$$

$$= \int_K d|D\lambda_F|_{Q_{\alpha}} \quad \text{by (7.3)}$$
Obviously outside $\Omega \times \mathbb{R}$ the subgraph of $u + a$ is just the subgraph of $\psi(x) + a$. By Definition 6.1 $u(x) + a$ is the generalized solution on $\Omega$ with boundary data $\psi(x) + a$. The proof is complete. \[ \square \]

Suppose $u(x)$ is a generalized solution on $\Omega$ with boundary data $\psi(x)$ as in Definition 6.1. We set

$$P_+ = \{ x \in \Omega : u(x) = +\infty \}$$

$$P_- = \{ x \in \Omega : u(x) = -\infty \}$$

Now we give more assumptions on the boundary data $\psi(x)$. Assume that there are three open sets $A_\pm, A_0$ in $\mathcal{B}$ with the following properties:

1. $\psi(x) = \pm \infty$ in $A_\pm$;
2. $\psi(x)$ is continuous on $A_0$;
3. $\partial \Omega = \Gamma_0 \cup \Gamma_+ \cup \Gamma_- \cup \mathcal{R}$ where $\Gamma_0 = \partial \Omega \cap \partial A_0, \Gamma_\pm = \partial \Omega \cap \partial A_\pm$ and $H^{n-1}(\partial \Omega) = 0$.
4. $\Gamma_+$ and $\Gamma_-$ are $C^2$-hypersurfaces.

**Lemma 7.2.** Let $u(x)$ be a generalized solution to the Dirichlet problem (1.1) on $\Omega$ with boundary data $\psi(x)$ on $\partial \Omega$. Then it holds that

1. the function $u^+_\infty$ given by

   $$u^+_\infty = +\infty \text{ on } P_+ \quad u^+_\infty = -\infty \text{ on } B \setminus P_+$$

   is a generalized solution to the Dirichlet problem (1.1) on $\Omega$ with boundary data

   $$\psi^+_\infty = +\infty \text{ on } A_+ \quad \psi^+_\infty = -\infty \text{ on } B \setminus A_+$$

2. the function $u^-\infty$ given by

   $$u^-\infty = -\infty \text{ on } P_- \quad u^-\infty = +\infty \text{ on } B \setminus P_-$$

   is a generalized solution to the Dirichlet problem (1.1) on $\Omega$ with boundary data

   $$\psi^-\infty = -\infty \text{ on } A_- \quad \psi^-\infty = +\infty \text{ on } B \setminus A_-$$

**Remark 7.3.** Let $\Psi^\pm \subset$ be the subgraph of $\psi^\pm \subset$ in $Q$. Following the proof of Lemma 6.3 the generalized trace of $\psi^\pm \subset$ on $\partial \Omega$ is

$$\mathcal{G}T \psi^\pm_\infty(x) = +\infty \text{ on } \Gamma_+ ; \quad \mathcal{G}T \psi^\pm_\infty(x) = -\infty \text{ on } \partial \Omega \setminus \Gamma_+$$

Thus the trace of $\lambda^\pm \subset$ on $\partial \Omega \times \mathbb{R}$ in $Q$ is the characteristic function of the subgraph of $\mathcal{G}T \psi^\pm_\infty$ over $\partial \Omega$.

By Lemma 6.5 the manifold $Q$ can be replaced with $Q_\alpha$. A similar conclusion also holds for $\psi^\infty$.

**Proof.** We only show the conclusion (1). The proof of the conclusion (2) is similar to that of the conclusion (1). So we skip it here.

Suppose $u$ is a generalized solution on $\Omega$ with boundary data $\psi(x)$ on $\partial \Omega$. By Lemma 7.1 $u(x) - j$ is a generalized solution of the Dirichlet problem with boundary $\psi(x) - j$. 

Let $U_j, \ U_j^+, \ \Psi_j$ and $\Psi_j^+$ be the subgraph of $u(x) - j, \ u_j^+(x), \ \psi(x) - j$ and $\psi_j^+$ respectively.

Let $\mathcal{V}$ be a Caccioppoli set satisfying $\mathcal{V} \Delta U_{\infty}^+ \subset K$ where $K$ is a compact set in $\Omega \times [-T, T]$. Here $T$ is a positive constant chosen such that

$$\lim_{j \to +\infty} \lambda_{U_j}(x, T) = \lambda_{U_{\infty}^+}(x, T) \quad a.e. \quad x \in \Omega$$

The proof of the above equalities are similar to those of (6.11) and (6.13). Now set

$$V_j = \begin{cases} 
V \text{ in } \Omega \times (-T, T) \\
U_j \text{ outside } \Omega \times (-T, T)
\end{cases}$$

Since $u(x) - j$ is a generalized solution, by Definition (6.1) we have

$$\int_{\Omega \times [-T, T]} d|D\lambda_{U_j}|_{Q_\alpha} \leq \int_{\Omega \times [-T, T]} d|D\lambda_{V_j}|_{Q_\alpha}$$

Now with (7.5) and Theorem 4.6 (7.8) gives that

$$\int_{\Omega \times (-T, T)} d|D\lambda_{U_j}|_{Q_\alpha} + \int_{\partial \Omega \times (-T, T)} |T\lambda_{\Psi_j} - T\lambda_{U_j}|dvol_{Q_\alpha}$$

$$\leq \int_{\Omega \times (-T, T)} d|D\lambda_{V}|_{Q_\alpha} + \int_{\partial \Omega \times (-T, T)} |T\lambda_{\Psi_j} - T\lambda_{V}|dvol_{Q_\alpha}$$

$$+ \int_{\Omega \times \{\pm T\}} |T\lambda_{V} - \lambda_{U_j}|(x, \pm T)dvol_{Q_\alpha}$$

In the last line we use the fact $T\lambda_{U_j} = \lambda_{U_j}$ on $\Omega \times \{\pm T\}$ by (7.3).

Let $\mathcal{G}\mathcal{T}\psi(x)$ be the generalized trace of $\psi(x)$ on $\partial \Omega$ from $\mathcal{B}\setminus \bar{\Omega}$. By its definition Lemma (6.3) implies that

$$\mathcal{G}\mathcal{T}\psi(x) = +\infty, \quad x \in \Gamma_+, \quad \mathcal{G}\mathcal{T}\psi(x) < +\infty \quad x \in \partial \Omega \setminus \Gamma_+$$

Let $T\lambda_{\Psi}$ and $T\lambda_{\Psi_j}$ be the trace of $\lambda_{\Psi}$ and $\lambda_{\Psi_j}$ on $\partial \Omega \times \mathbb{R}$. By the definition of $\Psi$ and $\Psi_j$, we have $\lambda_{\Psi}(y, t) = \lambda_{\Psi_j}(y, t - j)$ for every $y \in \mathcal{B}\setminus \bar{\Omega}$. By Lemma (6.4) it is clear that $T\lambda_{\Psi_j}(x, t - j) = T\lambda_{\Psi}(x, t)$ for a.e. $x \in \partial \Omega$. By Lemma (6.3) $T\lambda_{\Psi}$ is the characteristic function of the subgraph of $\mathcal{G}\mathcal{T}\psi$ in $\partial \Omega \times \mathbb{R}$. Thus $T\lambda_{\Psi_j}$ is the characteristic function of the subgraph of $\mathcal{G}\mathcal{T}\psi - j$ in $\partial \Omega \times \mathbb{R}$. By (6.10) and Remark (6.3) letting $j$ go to $\infty$ we obtain that

$$\lim_{j \to +\infty}(\mathcal{G}\mathcal{T}\psi - j) = \mathcal{G}\mathcal{T}\psi_\infty^+$$

This implies that

$$\lim_{j \to +\infty}T\lambda_{\Psi_j}(p) = T\lambda_{\Psi_\infty^+}(p) \quad a.e. p \in \partial \Omega \times \mathbb{R}$$
where $\mathcal{T}\lambda_{\Psi^+}$ is the trace of $\lambda_{\Psi^+}$ on $\partial\Omega\times\mathbb{R}$. With a similar derivation we shall obtain on $\partial\Omega\times\mathbb{R}$

$$\lim_{j \to +\infty} \mathcal{T}\lambda U_j = \mathcal{T}\lambda U^+_{\infty} \ a.e.$$  

Applying (7.11), (7.12), (7.5) and (7.6) into (7.9) and taking $j \to +\infty$, one sees that

$$\int_{\Omega \times (-T,T)} d|D\lambda_{U^+_{\infty}}| Q_\alpha + \int_{\partial\Omega \times (-T,T)} |\mathcal{T}\lambda_{\Psi^+} - \mathcal{T}\lambda U^+_{\infty}| d\nu Q_\alpha$$

$$\leq \int_{\Omega \times (-T,T)} d|D\lambda V| Q_\alpha + \int_{\partial\Omega \times (-T,T)} |\mathcal{T}\lambda_{\Psi^+} - \mathcal{T}\lambda V| d\nu Q_\alpha$$

$$+ \int_{\Omega \times \{\pm T\}} |\mathcal{T}\lambda V - \mathcal{U}^+_{\infty}|(x,\pm T) d\nu Q_\alpha$$

By (7.5) and (7.6) this implies that

$$\int_{\Omega \times [-T,T]} d|D\lambda_{U^+_{\infty}}| Q_\alpha \leq \int_{\Omega \times [-T,T]} d|D\lambda V| Q_\alpha$$

Since $V \Delta U^+_{\infty} \subset K$ where $K$ is an arbitrary compact set in $\Omega \times [-T,T]$, $u^+_{\infty}$ is a generalized solution to the Dirichlet problem (1.1) with boundary data $\psi^+_{\infty}$. The proof is complete. □

7.2. Almost minimal set. We shall recall some basic facts on almost minimal sets for later use. The papers of Duzzar-Steffen [3] and Tamanini [22] are our main references. Although their results are discussed in Euclidean space, their versions in Riemannian manifolds can be easily obtained following the technique in Section 36 in [20].

**Definition 7.4.** Let $W$ be an open set in a Riemannian manifold $G$ with $\text{dim} G = n + 1$. Let $\text{inj}_W$ denote the injective radius of $W$ in $G$. Let $E$ be a Caccioppoli set in $W$. We say $E$ is an almost minimal set in $W$ if it holds that

$$\int_{B_\rho(x)} d|D\lambda E| G \leq \int_{B_\rho(x)} d|D\lambda F| G + C \rho^{n+\beta}$$

for every point $x$ in any compact set $A \subset W$ and any Caccioppoli set $F \Delta E \subset B_\rho(x)$. Here $\beta \in [0, 1)$ is a given constant, $\rho < \min\{\text{inj}_W, \text{dist}(x, G \setminus W)\}$, $C$ is a positive constant depending on $W$.

The boundary $\partial E$ (See Proposition 5.4) is called as an almost minimal boundary. If $C = 0$, $\partial E$ is called the minimal boundary and $E$ is a minimal set.

**Remark 7.5.** By Proposition 5.4 we always require $E$ satisfying $0 < H^n(E \cap B_\rho(x)) < \text{vol}(B_\rho(x))$ for all $x \in \partial E$ and sufficiently small $\rho$.

An example of almost minimal boundaries is the boundary of smooth domains. Our proof imitates that of Example A.1 in Appendix A of [4] by
Eichmair and applies the fact that a $C^2$ boundary has locally bounded mean curvature.

Lemma 7.6. Let $\Omega$ be an open set in a Riemannian manifold $G$. Suppose $\Gamma \subset \partial \Omega$ is a $C^2$ connected hypersurface in $G$. For each point $x \in \Gamma$, there exists an open set $W$ near $x$ such that $\Omega \cap W$ is an almost minimal set in $W$.

Proof. We just give the sketch of the proof here. And the notation in Appendix A in [4] will be used without explanation.

Notice that the above result is a local result. Without loss of generality, we can assume $\Omega$ is bounded and $\Gamma = \partial \Omega$. Set

$$s(x) = \begin{cases} -\text{dist}(x, \partial \Omega) & x \in \bar{\Omega} \\ +\text{dist}(x, \partial \Omega) & x \in N \setminus \bar{\Omega} \end{cases}$$

Arguing as Lemma 14.16 in [7], $d(x)$ is smooth on $\Gamma_\mu$ for sufficiently small $\mu$. Taking $\mu$ small enough such that for each $y \in \Gamma_\mu$ there is only one $x \in \partial \Omega$ with $d(y) = \text{dist}(x, y)$. Thus $Dd$ is a smooth unit vector field on $\Gamma_\mu$. Now we view any boundary of Caccioppoli as a varifold. For any Caccioppoli set $F$

$$M_{B_r(x)}(\partial F) = \int_{B_r(x)} d|D\lambda_F|_G \tag{7.16}$$

Now let $x \in \partial \Omega$ and take $r > 0$ such that $B_r(x) \subset \Gamma_\mu$. Here $M$ denotes the mass of the varifold (see Simon [20]).

Suppose $F$ is a Caccioppoli set such that $F \Delta \Omega$ containing a compact set in $B_r(x)$. Notice that $\partial F = \partial \Omega + \partial (F \Delta \Omega)$. Let $\omega^* = Ds \, d\text{vol}$ where $d\text{vol}$ is the volume form of $G$ and $Ds$ is the gradient of $s(x)$. Suppose $\phi$ is any smooth function with compact support on $B_r(x)$ with $\phi \leq 1$.

$$M_{B_r(x)}(\partial F) = \sup_{\omega \in D^\nu(\Gamma_\mu), |\omega'| \leq 1} (\partial \Omega + \partial (F \Delta \Omega))(\phi \omega')$$

$$\geq \sup_{\phi(x)} (\partial \Omega)(\phi \omega') - \sup \partial (F \Delta \Omega)(\phi \omega')$$

$$\geq M_{B_r(x)}(\partial \Omega) - (F \Delta \Omega)(d\omega^*)$$

$$\geq M_{B_r(x)}(\partial \Omega) - C\text{vol}(B_r(x))$$

because $d\omega^* = \text{div}(Ds) d\text{vol}$ and $|\text{div}(Ds)| \leq C$ on $\Gamma_\mu$. With (7.16), the above estimate is just (7.15). Let $W$ be $B_r(x)$. The proof is complete. □

Next we define the regular set of a Caccioppoli set.

Definition 7.7. Suppose $F$ is a Caccioppoli set in a Riemannian manifold $G$. Define the regular set $\text{reg}(\partial F) := \{ x \in \partial F : \exists \rho > 0 \text{ with } \partial F \setminus B_\rho(x) \text{ is a } C^{1,\beta} \text{ graph} \}$ where $\beta \in (0,1)$. The singular set in $\partial F$ is its complement, written as $\text{sing}(\partial F)$.

The following two facts about almost minimal boundaries are standard.
Theorem 7.8. Let $\partial F$ be an almost minimal boundary in a Riemannian manifold $G$ with $\dim G = n + 1$.

(1) (Theorem 1 in [22] and Theorem 5.6 in [3]) Then $\text{sing}(\partial F) = \emptyset$ for $n \leq 6$. If $n = 7$ $\text{sing}(\partial F)$ consists of isolated points. If $n > 7$, $H^{n-7+\beta}(\text{sing}T) = 0$ for $\forall \beta > 0$.

(2) For any compact set $K \subset F$, there exists a $r_0 := r_0(K) > 0$ s.t. for all $r \in (0, r_0)$

$$\text{vol}(\partial F \cap B_r(x)) \geq Cr^n \quad \forall x \in K \cap \partial F$$

where $C$ is a positive constant only depending on $r_0$ and the metric $g$ on $K$.

The proof of the conclusion (2) above is exactly to that of Proposition 5.14 in [10] if we take $r_0$ as sufficiently small as possible. Thus we skip its proof here.

7.3. The property of $P_\pm$. First we define the mean curvature of a smooth boundary as follows.

Definition 7.9. Let $W$ be an open domain in a Riemannian manifold $G$ with a $C^2$ boundary $\partial W$. Let $\vec{v}$ be the outward normal vector of $\Gamma$. The mean curvature of $\partial W$ is $\text{div}(\vec{v})$ written as $H_{\partial W}$. Here $\text{div}$ is the divergence of $G$.

Remark 7.10. In Appendix D we collect some facts of mean curvature.

The main result of this section is given as follows.

Theorem 7.11. Let $\Omega$ and $\mathcal{B}$ be two bounded open domains in a Riemannian manifold $N$ satisfying $\Omega \subset\subset \mathcal{B}$ and $\partial \Omega$ is $C^2$. Let $n$ be the dimension of $N$. Suppose $u(x)$ is a generalized solution to the Dirichlet problem in (1.1) on $\Omega$ with bounded boundary data $\psi(x)$. Define

$$P_+ = \{x \in \Omega : u(x) = +\infty\}$$

$$P_- = \{x \in \Omega : u(x) = -\infty\}$$

The following statements hold:

(1) $P_+ \times \mathbb{R}$ is an almost minimal set in $\mathcal{B} \times \mathbb{R} \subset Q_\alpha$. Moreover the part of $P_+ \times \mathbb{R}$ in $\Omega \times \mathbb{R} \subset Q_\alpha$ is a minimal set. The same conclusion holds for $P_- \times \mathbb{R}$.

(2) if in addition $\partial \Omega$ is mean convex, that is, $H_{\partial \Omega} \geq 0$,

(a) then $\partial P_+$ and $\partial P_-$ are closed smooth minimal surfaces in $\Omega$ when $n \leq 7$;

(b) then $\partial P_+$ and $\partial P_-$ are closed smooth minimal surfaces in $\bar{\Omega}$ except a singular set $S$ with Hausdorff measure $H^k(S) = 0$ for $k > n - 7$ when $n > 7$.

Remark 7.12. In Theorem 16.6 [10] $P_+$ is a locally minimal set in $\mathcal{B}$. But in our case we can only obtain that $P_+ \times \mathbb{R}$ is a locally almost minimal set in $Q_\alpha$. This difference is the essential reason that we need Condition (2) to
guarantee generalized solutions are classical. The example in Appendix C says such condition is necessary.

**Proof.** We just prove the case of $P_+ \times \mathbb{R}$ and $\partial P_+$. The case of $P_- \times \mathbb{R}$ and $\partial P_-$ is similar so we skip their proofs here.

**The proof of the conclusion (1):**

By Lemma 7.2 $u^+_{\infty}$ is a generalized solution to the Dirichlet problem of (1.1) with boundary data $\psi^+_{\infty}$ on $\partial \Omega$. Because $\psi(x)$ is finite, we have $\psi^+_{\infty}(x) = -\infty$ outside $\Omega$. Thus $P_+ \times \mathbb{R}$ is the subgraph of $u^+_{\infty}$ with boundary data $\psi^+_{\infty}$ in $Q_\alpha$. Thus we also write $U_\infty$ for $P_+ \times \mathbb{R}$ in the following proof.

Now the discussion is divided into two cases. The first case is $p \in \Omega \times \mathbb{R}$. There is a $r > 0$ with $B_r(p) \subset \Omega \times \mathbb{R}$ where $B_r(p)$ is the ball centered at $p$ with radius $r$ in $Q_\alpha$. By Definition 6.1 we have

$$\int_{B_r(p)} d|D\lambda_{\infty}|_{Q_\alpha} \leq \int_{B_r(p)} d|D\lambda_F|_{Q_\alpha}$$

for any Caccioppoli set $F$ satisfying $F \Delta U_\infty \subset B_r(p)$. Thus $P_+ \times \mathbb{R}$, namely $U_\infty$, is a minimal set in $\Omega \times \mathbb{R}$.

The second case is $p \in \partial \Omega \times \mathbb{R}$. Let $E$ be any Caccioppoli set coinciding with $U_\infty$ outside some compact set in $B_r(p)$. Since $U_\infty$ locally minimizes the perimeter in $\Omega \times \mathbb{R}$, then

$$\int_{B_r(p)} d|D\lambda_{\infty}|_{Q_\alpha} \leq \int_{B_r(p)} d|D\lambda_{(\Omega \times \mathbb{R}) \cap E}|_{Q_\alpha}$$

Because $\partial \Omega$ is smooth, by Lemma 7.3 there is a $r_0 > 0$ such that $\Omega \times \mathbb{R}$ is an almost minimal set in the ball $B_r(p)$ for all $r < r_0$. Thus there are two constants $C = C(r_0) > 0$ and $\beta \geq 0$ such that

$$\int_{B_r(p)} d|D\lambda_{(\Omega \times \mathbb{R}) \cup E}|_{Q_\alpha} \leq \int_{B_r(p)} d|D\lambda_{(\Omega \times \mathbb{R}) \cap E}|_{Q_\alpha} + C(r_0)r^{n+\beta}$$

By Lemma 15.1 in [10] one sees that

$$\int_{B_r(p)} d|D\lambda_{(\Omega \times \mathbb{R}) \cup E}|_{Q_\alpha} + \int_{B_r(p)} d|D\lambda_{(\Omega \times \mathbb{R}) \cap E}|_{Q_\alpha}$$

$$\leq \int_{B_r(p)} d|D\lambda_{(\Omega \times \mathbb{R})}|_{Q_\alpha} + \int_{B_r(p)} d|D\lambda_E|_{Q_\alpha}$$

Combining (7.18), (7.19) and (7.20) together we obtain

$$\int_{B_r(p)} d|D\lambda_{U_\infty}|_{Q_\alpha} \leq \int_{B_r(p)} d|D\lambda_E|_{Q_\alpha} + C(r_0)r^{n+\beta}$$

Thus $P_+ \times \mathbb{R}$, i.e. $U_\infty$, is an almost minimal set in $B \times \mathbb{R} \subset Q_\alpha$. Together with (7.17) we obtain the conclusion (1).

**The proof of the conclusion (2):**

Since $P_+ \times \mathbb{R}$ is an almost minimal set in $Q_\alpha$, Theorem 7.8 implies that

(a) If $n \leq 6$, $\partial P_+ \times \mathbb{R}$ is regular. It is locally a $C^{1,\beta}$ embedded hypersurface in $Q_\alpha$ for some $\beta > 0$. 

(b) If $n = 7$, the singular set of $\partial P_+ \times \mathbb{R}$ is a collection of isolated points.

(c) If $n > 7$, let $S$ be the singular set of $\partial P_+ \times \mathbb{R}$. Then $H^k(S) = 0$ for $k > n - 7$.

The case of $n = 7$ should be examined more carefully. Let $(x_0, r_0)$ be a isolated singular point in $\partial P_+ \times \mathbb{R}$. Then there is an open set $V$ in $Q_\alpha$ containing $(x_0, r_0)$ such that $V \cap (\partial P_+ \times \mathbb{R})$ is $C^{1,\beta}$ except the point $(x_0, r_0)$. Thus there is a neighborhood $V_0$ of $x_0$ in $N$ such that $\partial P_+ \cap V_0$ is $C^{1,\beta}$ embedded. But this implies that near $(x_0, r_0)$ $\partial P_+ \times \mathbb{R}$ is $C^{1,\beta}$ embedded. Thus all points in $\partial P_+ \times \mathbb{R}$ are regular.

It is clear that $p = (x, r)$ is a regular point in $\partial P_+ \times \mathbb{R}$ if and only if $x$ is a regular point in $\partial P_+$. The above derivation yields that

(e) If $n \leq 7$, $\partial P_+$ is a $C^{1,\beta}$ embedded hypersurface in $N$ for some $\beta > 0$.

(f) If $n > 7$, let $S$ be the closed singular set of $\partial P_+$. Then $H^k(S) = 0$ for $k > n - 7$.

The last step is to show that the regular part of $\partial P_+$ is smooth and minimal. This is equivalent to show $H_{\partial P_+} = 0$ a.e. on the regular part of $\partial P_+$.

First we collect two facts on the regular part of $\partial P_+ \times \mathbb{R}$. Just noticing that the normal vector of $\partial P_+ \times \mathbb{R}$ is perpendicular to $\partial_r$ in $Q$ and $Q_\alpha$ we obtain

\begin{align}
\tag{7.22} H^3_{\partial P_+ \times \mathbb{R}}(p) = e^{-\varphi} H_{\partial P_+ \times \mathbb{R}}(p) & \quad \text{by Lemma D.3} \\
\tag{7.23} H_{\partial P_+ \times \mathbb{R}}(p) = H_{\partial P_+}(x)
\end{align}

where $p = (x, r)$ belongs to the regular part of $\partial P_+ \times \mathbb{R}$ and one side of these identities exists. Here $H_{\partial P_+ \times \mathbb{R}}$ is the mean curvature of $\partial P_+ \times \mathbb{R}$ in $Q (Q_\alpha)$. Our discussion is divided into two cases.

The first case is when $x \in \partial P_+ \cap \Omega$. In the conclusion (1) $P_+ \times \mathbb{R}$ is a minimal set in $\Omega \times \mathbb{R}$. Thus in the regular part of $\partial P_+ \times \mathbb{R}$ we have $H^3_{\partial P_+ \times \mathbb{R}}(p) = 0$ a.e. Thus by (7.23) $H_{\partial P_+}(x) = 0$ a.e. $x \in \partial P_+ \cap \Omega$.

The second case is when $x \in \partial P_+ \cap \partial \Omega$. Let $x$ be a regular point in $\partial P_+ \cap \partial \Omega$. By our assumption, $H_{\partial \Omega}(x) \geq 0$ with respect to the outward normal vector. On the other hand, $P_+ \times \mathbb{R}$ locally minimizes the perimeter in $\Omega \times \mathbb{R}$. Thus $H^3_{\partial P_+ \times \mathbb{R}} \leq 0$ a.e. on the regular part of $(\partial P_+ \cap \partial \Omega) \times \mathbb{R}$ with respect to the outward normal vector. By (7.22) and (7.23), we have $H_{\partial P_+}(x) \leq 0$ a.e. $x \in \partial P_+ \cap \partial \Omega$ with respect to the outward normal vector. Notice that $\partial P_+$ is tangent to $\partial \Omega$ at every point in $\partial P_+ \cap \partial \Omega$. Fix one such $x \in \partial P_+ \cap \partial \Omega$. In a neighborhood of $x$, $\partial P_+$ and $\partial \Omega$ are two $C^{1,\alpha}$ graphs satisfying $H_{\partial \Omega} \geq 0$ and $H_{\partial P_+} \leq 0$. Applying the weak version of the strong maximum principle (Theorem 8.19 in [7]), we obtain that $\partial \Omega$ should coincide with $\partial P_+$ near $x$ and $H_{\partial P_+} = 0$ a.e.

In summary in the regular part of $\partial P_+$ we have $H_{\partial P_+} = 0$ a.e. Together with the statements in (e) and (f) we obtain the conclusion (2). The proof is complete. \(/\)
In this section we study the regularity property of a generalized solution to the Dirichlet problem of (1.1). Then we study the condition in which generalized solutions of the Dirichlet problem of (1.1) is classical.

All of them are summarized as follows.

Theorem 8.1. Let $N$ be a Riemannian manifold with $\dim N = n$. Suppose $\Omega$ is a bounded open domain in $N$ with $C^2$ boundary satisfying

(1) $H_{\partial\Omega} = \text{div}(\vec{v}) \geq 0$ on $\partial \Omega$ where $\vec{v}$ is the outward normal vector of $\partial \Omega$;

(2) (a) if $n \leq 7$, no closed embedded minimal hypersurface exists in $\bar{\Omega}$;
(b) if $n > 7$, no closed embedded minimal hypersurface with a closed singular set $S$ with $H^k(S) = 0$ for $k > n - 7$ exists in $\Omega$ where $H^k$ is the $k$-dimensional Hausdorff measure on $N$;

Then the Dirichlet problem of (1.1) admits a unique solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$ for any continuous function $\psi(x)$ on $\partial \Omega$.

Remark 8.2. The condition (2) can be viewed as an obstacle to the solvability of the Dirichlet problem of (1.1). For example see Appendix C.

8.1. The interior regularity. Now we show the interior regularity of locally bounded generalized solutions to the Dirichlet problem in (1.1). Let $C^k(A)$ denote the set of $k$-th differential functions on the open set $A$. Let $W^{1,1}(A)$ be the Sobolev space on $A$. For more details see Appendix B.

Theorem 8.3. Fix $\alpha > 0$. Suppose $v(x)$ is a locally bounded generalized solution to the Dirichlet problem in (1.1) on $\Omega$ with boundary data $\psi(x)$ in $\partial \Omega$. Then $v(x) \in C^2(\Omega)$.

Remark 8.4. Our proof follows the spirit of Theorem 14.13 in [10].

Proof. Let $\Sigma$ be the graph of $u(x)$ in $Q_\alpha$. Because $u(x)$ is locally bounded, $\Sigma$ locally minimizes least perimeter in $\Omega \times \mathbb{R} \subset Q_\alpha$. Thus (1) in Theorem 7.8 implies that $\Sigma$ is smooth except a singular closed set $\Sigma'$ satisfying $H^{n-7}(\Sigma') = 0$.

Let $S$ denote the projection of $\Sigma'$ from $Q_\alpha$ into $\Omega$. Then $S \subset \Omega$ is a closed set with $H^{n-7}(S) = 0$.

Set $S = \Omega \setminus S$. Let $\Sigma_S$ be the part of $\Sigma$ restricted on $L \times \mathbb{R}$. Thus $\Sigma_S$ is smooth. We claim that

Lemma 8.5. $\Sigma_S$ is a smooth graph in $Q_\alpha$ with $v \in C^2(S)$.

Proof. Notice that $Q_\alpha$ is conformal to $Q$. For the convenience of computations, we work $Q$ instead of $Q_\alpha$. Let $\vec{v}$ be the downward normal vector of $\Sigma_S$ in $Q$. Since $v(x)$ locally minimizes the functional $\mathcal{F}_\alpha(v, \Omega)$. Aruging as in Lemma 2.2 of [25], $\Theta = \langle \vec{v}, \partial_r \rangle$ satisfies

$$\Delta \Theta + \langle |A|^2 + \bar{\text{Ric}}(\vec{v}, \vec{v}) \rangle \Theta + \langle \nabla \Theta, \partial_r \rangle = 0$$

where $\bar{\text{Ric}}$ is the Ricci curvature of $Q$, $\Delta$ is the Laplacian operator on $\Sigma$, $\nabla$ is the covariant derivative of $\Sigma_S$. Thus on each connected component of
Again we use the fact that $v(8.2)$

But this gives a contradiction. Thus $\Theta > (8.5)\max$

estimate functions $\{\}$

is uniquely solvable in $C_{\{\}$

Fix any $x_0 \in S$. By Theorem 3.9 there is a constant $r^* > 0$ such that for every $r \in (0, r^*)$ the Dirichlet problem

\begin{equation}
\begin{aligned}
\begin{cases}
\text{div} (\frac{Du}{\omega}) = \frac{\alpha}{\omega}, & x \in B_r(x_0) \subset \Omega \\
u(x) = \psi(x) & x \in \partial B_r(x_0)
\end{cases}
\end{aligned}
\end{equation}

is uniquely solvable in $C^2(B_r(x_0)) \cap C(\bar{B}_r(x_0))$ for any continuous function $\psi(x)$ on $\partial B_r(x_0)$.

Next we show $v(x) \in \mathcal{W}^{1,1}(B_{r^*}(x_0))$. Let $V$ be the subgraph of $v(x)$ in $Q_{\alpha}$. Lemma 8.3 says that

$$
\int_{\Omega \times \mathbb{R}} |D\lambda_\nu|_{Q_{\alpha}} \geq \int_{\Omega \setminus S \times \mathbb{R}} |D\lambda_\nu|_{Q_{\alpha}} = \int_{\Omega \setminus S} e^{\alpha v} \sqrt{1 + |Dv|^2} dv
$$

where $dv$ is the volume on $\Omega$. Since $v(x)$ is locally finite, thus $\int_{\Omega \setminus S} |Dv| dv$ is finite. Because $H^{n-1}(S) = 0$, we have $v(x) \in \mathcal{W}^{1,1}(B_{r^*}(x_0))$.

Choose one $r_0 \in (0, r^*)$ such that

\begin{equation}
Tv(x) = v(x) \quad x \in \partial B_0
\end{equation}

where $B_0$ is the ball $B_{r_0}(x_0) \subset \Omega$ and $Tv$ is the traces of $v(x)$ from $B_0$.

Since the singular closed set $S$ satisfies that $H^{n-1}(S) = 0$, from the definition of the Radon measure we can find a sequence of open sets $\{S_i\}$ in $\Omega$ such that

$$
S_i \subset S_{i+1}, \quad i = 1, 2, \ldots, \cap_{i=1}^{\infty} S_i = S
$$

with $H^{n-1}(S_i \cap \partial B_0) \rightarrow 0$ as $i \rightarrow +\infty$. Suppose a sequence of smooth functions $\{\psi_i\}_{i=1}^{\infty}$ on $\partial B_0$ satisfies

\begin{equation}
\psi_i = v(x) \quad \text{in} \quad \partial B_0 \setminus S_i, \quad \sup_{\partial B_0} |\psi_i| \leq 2 \sup_{\partial B_0} |v(x)|
\end{equation}

Again we use the fact that $v(x)$ is locally bounded in $\Omega$.

Let $\{u_i\}_{i=1}^{\infty}$ be the classical solution of the Dirichlet problem 8.2 with smooth boundary data $\{\psi_i\}_{i=1}^{\infty}$ on $\partial B_0$. By 8.3 and 8.4, we have the estimate

\begin{equation}
\max_{B_0} |u_i(x)| \leq C(r^*, n, \max_{\partial B_0} |\psi_i|) = C(r^*, n, 2 \sup_{\partial B_0} |v(x)|)
\end{equation}
Following the derivation in (B.27), we conclude
\[(8.6) \qquad \max_{B_0}\{|u_i|, |Du_i|, |D^2u_i|\} \leq C(\sup_{\partial B_0} |\psi(x)|, r^\sigma)\]

By the Ascoli-Arzela theorem there is a subsequence, still denoted by \(u_j\), will converge uniformly on compact subsets of \(B_0\) to a locally \(C^2\) function \(u(x)\) satisfying
\[(8.7) \qquad \text{div}(\frac{Du}{\omega}) = \frac{\alpha}{\omega} \quad \text{on} \quad B_0\]

By (8.6), \(u(x) \in W^{1,1}(B_0)\). Let \(T : W^{1,1}(B_0) \to L^1(\partial B_0)\) denote the trace operator. Notice that \(u_i \in W^{1,1}(B_0)\) and \(u_i\) also converges to \(u(x)\) in \(W^{1,1}(B_0)\) as \(i \to +\infty\) by the Dominated Convergence Theorem. Thus
\[Tu_i = \psi_i \to Tu \quad \text{as} \quad i \to +\infty \quad \text{in} \quad L^1(\partial B_0)\]

On the other hand, from the definition in (8.4) and \(H^{n-1}(S_1 \cap \partial B_0) \to 0\), \(Tu_i\) converges to \(v(x)\) in \(L^1(\partial B_0)\). Hence \(Tu(x) = Tv(x)\) on \(\partial B_0\) in the \(L^1(\partial B_0)\) sense.

Now we claim \(u(x) = v(x)\) on \(B_0\) under the condition \(Tu = Tv\). Since \(u \in C^2(B_0)\) satisfying \(\text{div}(\frac{Du}{\omega}) = \frac{\alpha}{\omega}\) where \(\omega = \sqrt{1 + |Du|^2}\). Now we define a vector field in \(\Omega \times \mathbb{R}\) in the product manifold \(Q\) (not the conformal product manifold \(Q_\alpha\)) as
\[X = e^{\omega r} \frac{\partial r - Du}{\omega}\]

Let \(\text{div}_p\) be the divergence of \(Q\). Suppose \(\{e_1, \cdots, e_n\}\) is a local orthormal frame on \(\Omega\). Let \(\nabla\) deote the covariant derivative of \(Q\). Then
\[\text{div}_p(X) = -e^{\omega r} \langle \nabla e_i \frac{Du}{\omega}, e_i \rangle + \alpha \langle \partial_r, X \rangle\]
\[= e^{\omega r} (-\text{div}(\frac{Du}{\omega}) + \frac{\alpha}{\omega}) = 0\]

Let \(f \in C(\bar{B}_0) \cap C^2(B_0)\). Let \(\Sigma_f\) be the graph of \(f\) over \(B_0\). Let \(U\) be the domain in \(\Omega \times \mathbb{R}\) enclosed by \(\Sigma_f, \Sigma_u\) (the graph of \(u(x)\)) and \(\partial B_0 \times \mathbb{R}\). Applying the divergence theorem for \(X\) in \(U\), we have
\[0 = \int_U \text{div}_p(X) dvoldr\]
\[= \int_{B_0} e^{\omega u(x)} \omega dvoldr - \int_{\partial B_0} e^{\alpha f} \langle \frac{\partial_r - Du}{\omega}, (\partial_r - Df)\rangle dvold\partial B_0\]
\[(8.8) \quad + \int_{\partial B_0} (f - Tu)(X, \gamma) dvold\partial B_0\]

where \(\gamma\) is the inward normal vector of \(\partial B_0\) and \(Df\) is the gradient of \(f\) in \(\Omega\).

Since \(C(\bar{B}_0) \cap C^2(B_0)\) is dense in \(W^{1,1}(B_0)\), there is a sequence \(\{v_j\}_{j=1}^\infty \in C(\bar{B}_0) \cap C^2(B_0)\) such that \(v_j\) converges to \(v\) in \(W^{1,1}(B)\) such that \(Tv_j \to Tv\).
as \( j \to \infty \). Recall that \( T u = T v \). Replacing \( f \) with \( v_j \) in (8.8) and letting \( j \to \infty \) give that

\[
0 = \int_{B_0} e^{\alpha u(x)} \sqrt{1 + |Du|^2} d\mu - \int_{B_0} e^{\alpha v(x)} \left( \frac{\partial_r - Du}{\omega} , (\partial_r - Dv) \right) d\mu
\]

Define the function

\[
(8.10) \tilde{v} = \begin{cases} u(x) & x \in B_0 \\ v(x) & \text{otherwise} \end{cases}
\]

By (8.3), we have

\[
(8.11) T^+ \tilde{v} = T^- \tilde{v}
\]

where \( T^+ (T^-) \) is the trace of \( \tilde{v} \) from \( B_0 \) (outside \( B_0 \)). Thus combining Lemma 6.3 and Theorem 4.6 together, we obtain

\[
(8.12) \int_{\partial B_0 \times \mathbb{R}} d|D\lambda_{\tilde{V}}|Q_\alpha = 0
\]

where \( \tilde{V} \) is the subgraph of \( \tilde{v} \). On the other hand because \( v(x) \) is a generalized solution in \( \Omega \), we obtain

\[
(8.13) \int_{B_0 \times \mathbb{R}} d|D\lambda_V|Q_\alpha \leq \int_{B_0 \times \mathbb{R}} d|D\lambda_{\tilde{V}}|Q_\alpha
\]

By (8.12), this implies that

\[
\int_{B_0 \times \mathbb{R}} d|D\lambda_V|Q_\alpha \leq \int_{B_0 \times \mathbb{R}} d|D\lambda_{\tilde{V}}|Q_\alpha
\]

Since \( v \in W^{1,1}(B_0) \) and \( u \in C^2(\mathcal{B}) \), the above inequality is equivalent that

\[
(8.14) \int_{B_0} e^{\alpha v(x)} \sqrt{1 + |Dv|^2} dvol \leq \int_{B_0} e^{\alpha u(x)} \omega dvol
\]

Due to (8.9), the above equality should hold and implies that

\[
(8.15) \frac{\partial_r - Du}{\omega} = \frac{\partial_r - Dv}{\sqrt{1 + |Dv|^2}} \ a.e. \ x \in B_0
\]

Recall that \( \omega = \sqrt{1 + |Du|^2} \). Thus \( Du = Dv \) a.e. \( x \in B_0 \). Thus \( u = v + C \) for a constant \( C \) on \( B_0 \). The fact \( Tu = Tv \) on \( \partial B_0 \) implies that \( C = 0 \).

We obtain that \( u = v \) on \( B_0 \). Thus \( v(x) \in C^2(B_0) \). Recall that \( B_0 = B_{r_0}(x_0) \) where \( x_0 \in S \). Due to the arbitrariness of \( x_0 \) we conclude that \( v(x) \in C^2(\Omega) \). \( \square \)

8.2. The proof of Theorem 8.1. The first step is to show that the corresponding generalized solution is locally bounded.

**Lemma 8.6.** Under the assumption of Theorem 8.1, there is a locally bounded generalized solution \( u(x) \) on \( \Omega \) with bounded boundary data \( \psi(x) \).

**Remark 8.7.** In this lemma we do not use the fact that \( \partial \Omega \) is mean convex.
Proof. Notice that $\partial \Omega$ is $C^2$. Then by Lemma 4.7 and Remark 6.7 we can extend $\psi(x)$ is a bounded BV function (still written as $\psi(x)$) on a larger bounded open set containing $\Omega$ such that its subgraph is a Caccioppoli set in $Q_\alpha$ and its trace on $\partial \Omega$ is $\psi(x)$. By Theorem 6.6 there is a generalized solution $u(x)$ with the continuous boundary data $\psi(x)$.

Recall that $P_\pm$ are the sets $\{x \in \bar{\Omega} : u(x) = \pm \infty\}$. By Theorem 7.11, $\partial P_\pm$ are closed embedded minimal surfaces with a closed singular set $S$ satisfying $H^k(S) = 0$ for $k > n - 7$ in $\bar{\Omega}$ if $n > 7$ and $\partial P_\pm$ are embedded minimal surfaces for $n \leq 7$. By the assumption (2) in Theorem 8.1 $P_\pm$ are empty in the sense of perimeter. Namely $P_\pm$ is a $H^{n-1}$ measure zero set where $n$ is the dimension of $N$.

Now assume $P_+$ is not empty. Then $u(x)$ is not locally bounded near some point in $\bar{\Omega}$. Without loss of generality we can assume that there is a sequence $\{x_j\} \subset \Omega$ converging to $x_0$ in $\bar{\Omega}$ such that $u(x_j) > j$ as $j \to +\infty$. Let $z_j = (x_j, u(x_j))$.

Because $\psi(x)$ is bounded there is a $R > 0$ such that the normal ball $B_R(z_j)$ in $Q_\alpha$ does not intersect the graph of $\psi(x)$. The following proposition is useful.

**Proposition 8.8.** It holds that $U \cap B_R(z_j)$ is an almost minimal set in $B_R(z_j)$.

**Proof.** The definition in Definition 6.1 implies that $U$ has the least perimeter in $\bar{\Omega} \times \mathbb{R} \cap B_R(z_j)$. Recall that $\partial \Omega \times \mathbb{R}$ is $C^2$ in $Q_\alpha$. Arguing as in (7.17), (7.18), (7.19) and (7.20), we will obtain (7.21) for $U$. Thus we obtain the conclusion. 

Let $U_j$ be the subgraph of $u_j(x) = u(x) - j$. By Proposition 8.8 each $U_j$ is an almost minimal set in $B_{R_1}(z_0)$.

By Conclusion (2) in Theorem 7.8 we have

\begin{equation}
\text{vol}(\partial U_j \cap B_r(z_0)) > cr^n
\end{equation}

for some $c > 0$ and all $r < C(R)$ where $C(R)$ is a positive constant depending on $R$. Thus

\begin{equation}
\text{vol}(\partial U_j \cap B_{2R}(z_0)) \geq cr^n
\end{equation}

where $z_0 = (x_0, 0)$ and any $r \in (0, C(R))$. As $j \to \infty$ $\lambda_{U_j}$ converges weakly to $\lambda_{P_+ \times \mathbb{R}}$ in the BV function sense of $Q_\alpha$. Thus $\text{vol}((\partial P_+ \times \mathbb{R}) \cap B_{2R}(z_0)) \geq cr^n$. This gives a contradiction since $P_+$ is a $H^{n-1}$ measure zero set. Thus $P_+$ is empty in $\Omega$.

A similar derivation yields that $P_-$ is also empty in $\Omega$. Thus $u(x)$ is locally bounded. 

Now we conclude the boundary continuity when $\psi(x)$ is continuous on $\partial \Omega$. 

Lemma 8.9. Let \( u(x) \) be the generalized solution on \( \Omega \) with boundary data \( \psi(x) \) under the assumption in Theorem 8.7. Then \( u(x) \) is continuous on \( \Omega \) and is equal to \( \psi(x) \) on \( \partial \Omega \).

Proof. By the assumption of Theorem 8.3 \( \psi(x) \) is continuous on \( \partial \Omega \). By Lemma 8.6 \( u(x) \) is locally bounded on \( \Omega \).

Suppose \( x_0 \in \partial \Omega \) and \( \lambda = \limsup_{x \in \Omega, x \to x_0} u(x) > \psi(x_0) \). Then there is a sequence \( \{x_j\} \) and \( \lambda > 0 \) in \( \Omega \) converging to \( x_0 \) and

\[
\lim_{j \to + \infty} u(x_j) = \lambda > \psi(x_0)
\]

Let \( z_0 \) be the point \( (x_0, \lambda) \) in \( Q_\alpha \). By Lemma 4.7 we can view \( \psi(x) \) is a bounded continuous BV function on a larger open set containing \( \Omega \). There is a \( R > 0 \) such that there is a \( R > 0 \) such that the normal ball \( B_R(z_0) \) in \( Q_\alpha \) does not intersect the graph of \( \psi(x) \).

By the Nashing isometric embedding theorem, we can view \( Q_\alpha \) as a smooth submanifold in certain \( \mathbb{R}^{n+k} \) with the induced metric for some positive integer \( k \). Now we blow up \( U \cap B_R(z_0) \) in \( \mathbb{R}^{n+k} \) as follows:

\[
U_j = \{ z \in \mathbb{R}^{n+k} : j^{-1}z + z_0 \in U \cap B_R(z_0) \}
\]

\[
S_j = \{ z \in \mathbb{R}^{n+k} : j^{-1}z + z_0 \in \bar{\Omega} \times \mathbb{R} \}
\]

Similar as the derivation in Theorem 37.4 in [20] \( U_j \) will converge weakly to a minimal cone \( C \) in \( T_{z_0}Q_\alpha \) in \( \mathbb{R}^{n+k} \) and \( S_j \) converges to a closed plane \( S \) in \( T_{z_0}Q_\alpha \). Then \( C \) is contained in the half-space determined by \( S \). By Theorem 15.5 in [10] \( C \) is just a closed half-space in \( T_{z_0}Q_\alpha \). Consequently \( \partial C \) is \( C^{1,\beta} \) near \( z_0 \) and can be written as a graph of a \( C^{1,\beta} \) function \( w(x) \) over \( \partial \Omega \times \mathbb{R} \) with \( \omega(z_0) = 0 \).

Since \( H_{\partial \Omega} \geq 0 \), (7.22) implies that

\[
H^\alpha_{\partial \Omega \times \mathbb{R}} = e^{-\pi} H_{\partial \Omega} \geq 0
\]

with respect to the outward normal vector of \( \partial \Omega \times \mathbb{R} \) in \( Q_\alpha \). Because \( \partial U \) is smooth, we can assume that \( \vec{v} \) be the normal vector of \( \partial U \) near \( z_0 \) which points outward to \( (\bar{\mathcal{B}}(\Omega) \times \mathbb{R} \) at \( z_0 \). The fact that \( U \) locally minimizes the perimeter in \( \bar{\Omega} \times \mathbb{R} \subset Q_\alpha \) yields that

\[
H^\alpha_{\partial U} = \text{div}_{\Gamma_\alpha}(\vec{v}) \leq 0
\]

near \( z_0 \). Notice that \( \partial U \) is tangent to \( \partial \Omega \times \mathbb{R} \) at \( z_0 \). By the weak version of the strong maximum principle (see Theorem 8.19 in [7]) \( \partial U \) coincides \( \partial \Omega \times \mathbb{R} \) near \( z_0 \). This contradicts to the fact \( \lambda = \limsup_{x \in \Omega, x \to x_0} u(x) \). Thus we conclude

\[
\lim_{j \to \infty} u(x_j) \leq \psi(x_0)
\]

With a similar argument \( \lim_{j \to \infty} u(x_j) \geq \psi(x_0) \). Set \( u(x) = \lim_{x_j \to x_0} u(x_j) \) for \( x \in \partial \Omega \). Then \( u(x) \) is continuous until the boundary and \( u(x) = \psi(x) \) for each \( x \in \partial \Omega \).
By Theorem 6.6 there is a generalized solution \( u(x) \) with continuous boundary data \( \psi(x) \). Theorem 8.3 implies \( u \in C^2(\Omega) \). By Lemma 8.9 \( u(x) \) is continuous to the boundary. Namely \( u(x) \in C(\bar{\Omega}) \). Since \( u(x) \) locally minimizes the conformal area functional \( F_\alpha(u, \Omega) \), \( u(x) \) satisfies \( \text{div}(\frac{\partial u}{\partial \nu}) = \frac{\alpha}{\psi} \) on \( \Omega \). The uniqueness of the solution to the Dirichlet problem [1.1] is obvious. We complete the proof of Theorem 8.1.

**Appendix A. A decomposition result of Radon measures**

Now we consider a decomposition of Radon measures on Riemannian manifolds. The reason we derive it here is that a domain in Riemannian manifold may not be simply connected any more. Many techniques in Euclidean spaces cannot be applied directly. The main references of this section are Chapter 1 in [20] and Section 2.8 in [6].

Throughout this section let \( N \) be a complete Riemannian manifold with \( \text{dim} N = n \). For every point \( x \in N \) we denote the open (closed) embedded normal ball (see Definition 2.10) centered at \( x \) with radius \( r \) by \( B_r(x) \/ \bar{B}_r(x) \).

**Definition A.1.** Let \( F \) be a collection of closed normal balls such that the radius of these balls is a bounded set. Let \( A \) denote the set of all centers of these balls. If \( F \) covers \( A \) finely, then there are \( \kappa \) subcollections \( \{F_i\}_{i=1}^\kappa \) of \( F \) such that the balls in each \( F_i \) are pairwise disjoint and \( A \subset \bigcup_{i=1}^\kappa \bar{B}_{\in F_i} \).

A straightforward verification shows that

**Corollary A.3.** Suppose \( \Omega \) is a bounded open set in \( N \). Let \( \mu \) be a Radon measure on \( \Omega \) with \( \mu(\Omega) < \infty \). Let \( F \) be a collection of closed normal balls covering \( \Omega \) finely. Then there is a countable pairwise disjoint collection of closed normal balls \( \{\bar{B}_{r_j}(x_j) \in F : j = 1, \cdots, \infty\} \) with \( \mu(\Omega \setminus \bigcup_{j=1}^\infty \bar{B}_{r_j}(x_j)) = 0 \).

Now we obtain a useful decomposition of Radon measures in Riemannian manifolds as follows.

**Theorem A.4.** Let \( \Omega \) be an open bound set in a Riemannian manifold. Fix any \( \varepsilon > 0 \) and \( r_0 > 0 \). Suppose \( \mu \) is a Radon measure satisfying \( \mu(\Omega) < \infty \). Then there is a collection of countable open normal balls in \( \Omega \) defined by

\[
\mathcal{B} = \{B_k = B_{r_k}(x_k) : k = 1, \cdots, \infty, x_k \in \Omega, r_k \leq r_0, \mu(\partial B_k) = 0\}
\]

and an positive integer \( \kappa_0 = \kappa_0(\varepsilon, n, \Omega) \) such that
(1) \( \{B_1, \ldots, B_{\kappa_0}\} \) is a pairwise disjoint subcollection of \( \mathcal{B} \) with
\[
\mu(\Omega) - \varepsilon \leq \sum_{k=1}^{\kappa_0} \mu(B_k) = \mu(\bigcup_{k=1}^{\kappa_0} B_k) \leq \mu(\Omega)
\]

(2) the subcollection \( \{B_k : k = \kappa_0 + 1, \ldots, \infty\} \) of \( \mathcal{B} \) satisfies that
\[
\sum_{k=\kappa_0+1}^{\infty} \mu(B_k) \leq \kappa \varepsilon
\]

where \( \kappa = \kappa(n, \Omega) \) is the positive integer given in Theorem A.2.

Proof. Let \( d \) be the distance given by the metric on \( \Omega \). We define a collection of closed normal balls as follows.

(A.2) \( \mathcal{F} = \{\bar{B}_r(x) : x \in \Omega, r < \min\{r_0, d(x, \partial \Omega)\}, \mu(\partial \bar{B}_r(x)) = 0\} \)

Since \( \mu(\Omega) < \infty \), the Fubini's theorem implies that \( \mu(\partial \bar{B}_r(x)) = 0 \) for any \( x \in \Omega \) and a.e. \( r \in (0, \min\{r_0, d(x, \partial \Omega)\}) \). Thus \( \mathcal{F} \) covers \( \Omega \) finely.

Fix \( \varepsilon > 0 \). By Corollary A.3 there is \( \kappa_0 = \kappa_0(\varepsilon, n, \Omega) \) and a pairwise disjoint subcollection of closed balls \( \{\bar{B}_{r_1}(x_i)\} \) in \( \mathcal{F} \) such that

(A.3) \( \mu(\Omega) - \varepsilon \leq \sum_{k=1}^{\kappa_0} \mu(B_{r_k}(x_k)) = \mu(\bigcup_{k=1}^{\kappa_0} B_{r_k}(x_k)) \leq \mu(\Omega) \)

because \( \mu(\partial \bar{B}_r(x)) = 0 \) for each \( \bar{B}_r(x) \in \mathcal{F} \).

Namely there is a pairwise disjoint collection of finite open balls

(A.4) \( \{B_{r_1}(x_1), \ldots, B_{r_{\kappa_0}}(x_{\kappa_0})\} \)

satisfying

(A.5) \( \mu(\Omega \setminus \bigcup_{k=1}^{\kappa_0} \bar{B}_{r_k}(x_k)) \leq \frac{\varepsilon}{4} \)

Now define an open set \( \Omega_\eta \) as

(A.6) \( \Omega_\eta \equiv \{x \in \Omega : d(x, \Omega \setminus \bigcup_{k=1}^{\kappa_0} \bar{B}_{r_k}(x_k)) < \eta\} \)

where \( \eta \) is a sufficiently small positive constant such that \( \mu(\Omega_\eta) \leq \frac{\varepsilon}{2} \). Similar as in (A.2) we define a collection of closed normal balls in \( \Omega_\eta \) as

(A.7) \( \mathcal{F}_\eta = \{\bar{B}_r(x) : x \in \Omega_\eta, r < \min\{r_0, d(x, \partial \Omega_\eta)\}, \mu(\partial \bar{B}_r(x)) = 0\} \)

By Theorem A.2 there are \( \kappa = \kappa(\Omega, n) \) subcollections \( \{\mathcal{F}_{\eta,k}\}_{k=1}^{\kappa} \) such that the closed balls in each \( \mathcal{F}_{\eta,k} \) are pairwise disjoint and

(A.8) \( \Omega_\eta \subset \bigcup_{k=1}^{\kappa} \bigcup_{\bar{B}_r(x) \in \mathcal{F}_{\eta,k}} \bar{B}_r(x) \)

Moreover for each \( k = 1, \ldots, \kappa \)

(A.9) \( \sum_{\bar{B}_r(x) \in \mathcal{F}_{\eta,k}} \mu(\bar{B}_r(x)) \leq \mu(\Omega_\eta) \leq \frac{\varepsilon}{2} \)
Notice that there are only countable closed normal balls in each subcollection \( \mathcal{F}_{\eta,k} \). For each ball \( \overline{B}_r(x) \) in each collection \( \mathcal{F}_{\eta,k} \) we can replace it with a large open ball \( B_{r_x}(x) \) such that

\[
\sum_{\overline{B}_r(x) \in \mathcal{F}_{\eta,k}} \mu(\overline{B}_r(x)) \leq \sum_{\overline{B}_r(x) \in \mathcal{F}_{\eta,k}} \mu(B_{r_x}(x)) \leq \varepsilon
\]

This gives \( \kappa \) collections of open normal balls as follows

\[
\mathcal{F}'_{\eta,k} := \{ B_{r_x}(x) : \overline{B}_r(x) \in \mathcal{F}_{\eta,k}, \overline{B}_r(x) \subset B_{r_x}(x) \subset \Omega_\eta \}
\]

with the condition (A.10) for \( k = 1, \ldots, \kappa \). Now we relabel all open balls in \( \{\mathcal{F}_{\eta,k}\}_{k=1}^\kappa \) and list them as follows

\[
\{B_{r_k}(x_k) : k = \kappa_0 + 1, \ldots, \infty\}
\]

Obviously \( \Omega_\eta \subset \bigcup_{k=\kappa_0+1}^\infty B_{r_k}(x_k) \) according to our definition. The condition (A.10) yields that

\[
\sum_{k=\kappa_0+1}^\infty \mu(B_{r_k}(x_k)) \leq \kappa \varepsilon
\]

Combining the open balls in (A.4) with the property (A.5) and the open balls in (A.12) with the property (A.13) together we obtain the desirable collection of open normal balls. The proof is complete.

**Appendix B. Local existence of the Dirichlet problem of mean curvature equations**

In this section we show the existence of the Dirichlet problem (1.1) on a sufficiently small normal ball in Riemannian manifolds.

Throughout this section \( N \) is a Riemannian manifold with dimension \( n \). For a set \( A \) in \( N \), \( C(A), C^k(A), C^\infty(A) \) denote the sets of continuous functions, \( k \)-differential functions and smooth functions on \( A \) respectively. For a point \( x \in N \), \( B_r(x) \) denotes the normal ball centered at \( x \) with radius \( r \).

The main result of this section is described as follows.

**Theorem B.1.** Fix a point \( x_0 \) in \( N \) and a function \( f(x) \in C(N) \). Then there is a positive constant \( r^* = r^*(f, \text{Ric}, n) > 0 \) depending on the Ricci curvature near \( x_0 \) such that for every \( r \in (0, r^*) \) the Dirichlet problem

\[
\begin{cases}
\text{div} \left( \frac{Du}{\omega} \right) = f(x), & x \in B_r(x_0) \\
u(x) = \psi(x), & x \in \partial B_r(x_0)
\end{cases}
\]

is uniquely solvable in \( C^2(B_r(x_0)) \cap C(\overline{B}_r(x_0)) \) for any \( \psi(x) \) in \( C(\partial B_r(x_0)) \).

In the following we give some basic definitions, which is from Section 2.1 of Chapter 2 in Hebey [11].
Let $g$ be the Riemannian metric of $N$. Let $k$ be an integer and $u \in C^\infty(N)$ and $\nabla^k$ denote the $k$th covariant derivative of $u$. By definition one has that
\[
|\nabla^k u|^2 = g^{i_1j_1} \cdots g^{i_kj_k} (\nabla^k u)_{i_1 \cdots i_k} (\nabla^k u)_{j_1 \cdots j_k}
\]
where $g = g_{ij} dx^i dx^j$ and $(g^{ij}) = (g_{ij})^{-1}$. The $L^p$ norm of $u(x)$ is
\[
||u||_p = \int_N |u|^p dvol
\]
In local coordinates $dvol = \sqrt{\det(g_{ij})} dx$ is the volume form of $N$ where $dx$ stands for the Lebesgue's volume element of Euclidean space $\mathbb{R}^n$. The $L^p$ space is the completion of $C^\infty(N)$ with respect to the norm $||.||_p$.

For $k$ an integer and $p \geq 1$ real, we denote by $C^p_k(N)$ the sets given by
\[
C^p_k(N) = \{ u \in C^k(N) : s.t. \forall j = 0, \cdots, k, \int_N |\nabla^j u|^p dvol < \infty \}
\]
**Definition B.2.** The Sobolev space $W^{k,p}(N)$ is the completion of $C^p_k(N)$ with respect to the norm
\[
||u||_{W^{k,p}} = \sum_{j=0}^k \left( \int_N |\nabla^j u|^p dvol \right)^{\frac{1}{p}}
\]
The Sobolev space $W^{0,p}(N)$ is the completion of the functions in $C^p_k(N)$ having compact supports with respect to the norm $||.||_{W^{k,p}}$.

First we see the sobolev inequalities from Euclidean spaces are valid on small normal balls. We denote the constant $C$ only depending on $i,j,k$ by $C(i,j,k)$.

**Theorem B.3.** Fix $x_0 \in N$. The dimension of $N$ is $n$. There is a $r_1 > 0$ such that the following Sobolev inequalities hold for any $u \in W^{1,p}_0(B_r(x_0))$ with every $r < r_1$,
\[
\begin{align*}
||u||_{W^{1,p}} & \leq C(n,p,r_1)||Du||_p \quad \text{for } 1 \leq p < n, \\
||u||_p & \leq C(r_1)||Du||_p \quad \text{for } 1 \leq p < \infty,
\end{align*}
\]
Here $r_1$ depends on the metric near $x_0$.

**Remark B.4.** Let $\Omega$ be an open set in $\mathbb{R}^n$. Suppose $u \in W^{1,p}_0(\Omega), 1 \leq p < \infty$. According to (7.44) and Theorem 7.10 of Chapter 7 in [7], we have
\[
\begin{align*}
||u||_{W^{1,p}} & \leq C(n,p)||Du||_p \quad \text{for } 1 \leq p < n, \\
||u||_p & \leq \frac{1}{|\Omega|^\frac{1}{n}}||Du||_p \quad \text{for } 1 \leq p < \infty,
\end{align*}
\]
where $|\Omega|$ is the volume of $\Omega$ and $\omega_n$ is the area of the unit sphere in $\mathbb{R}^n$.

**Proof.** Fix $x_0 \in N$. First we take $r$ sufficiently. The metric on $B_r(x_0)$ can be written as
\[
g = g_{ij}(y)dy^idy^j
\]
Here \( y \in B_r(0) \), in the Euclidean ball in \( \mathbb{R}^n \) with radius \( r \). By Lemma 1.4 of Chapter 1 in [11] by Hebey there is a constant \( r_1 \) such that for all \( r \in (0, r_1) \) it holds that

\[
(B.7) \quad \frac{1}{4} \delta_{ij} \leq g_{ij}(y) \leq 4 \delta_{ij}
\]

Since \( C^0_k(B_{r_1}(x_0)) \) is complete in \( W^{1,p}_0(B_{r_1}(x_0)) \), we can \( u \in C^0_k(B_{r_1}(x_0)) \).

Without confusion we also write \( u \) for the corresponding function on the Euclidean ball \( B_r(0) \). Let \( D_E u \) denote the Euclidean gradient of \( u \), that is, \( |D_E u|^2 = \sum_{i=1}^n (\partial_i u)^2 \). According to (B.7) we have the estimate

\[
(B.8) \quad \frac{1}{4} |D_E u|^2(y) \leq |u|^2 = g^{ij} \partial_i u \partial_j u \leq 4 |D_E u|^2(y)
\]

\[
(B.9) \quad \frac{1}{2^n} \leq \sqrt{\det(g_{ij})}(y) \leq 2^n,
\]

for \( y \in B_{r_0}(0) \).

Let \( dx \) be the Lebesgue volume form in \( \mathbb{R}^n \). Suppose \( 1 \leq p < n \). With (B.4), (B.8) and (B.9) we derive

\[
|u|^\frac{np}{n-p} = \left( \int_{B_{r_0}(0)} |u|^{\frac{np}{n-p}} \sqrt{\det(g_{ij})} |dx| \right)^{\frac{n-p}{np}} \leq C(n,p)||Du||_p^{\frac{n-p}{np}}
\]

We obtain (B.2). Now assume \( 1 \leq p < \infty \). Applying (B.5) a similar derivation as above should show (B.3) because \( \text{vol}(B_r(x_0)) = Cr^n \). \( \square \)

Next we obtain a \( C^0 \) estimate for the solution to the Dirichlet problem (B.1) in sufficiently small normal balls.

**Proposition B.5.** Fix \( x_0 \in N \) and \( f(x) \in C(N) \). Let \( r_1 \) be the constant given in Theorem B.3. Let \( \mu_1 \) denote the constant \( \max_{B_{r_1}(x_0)} |f| \). Then there is a \( r_2 = r_2(\mu_1, r_1) < r_1 \) such that the following properties holds. For every \( r \in (0, r_2) \) if \( u(x) \in C(B_r(x_0)) \cap C^2(B_r(x_0)) \) is the solution to the Dirichlet problem to (B.1) on \( B_r(x_0) \), then

\[
(B.10) \quad \sup_{B_r(x_0)} |u| \leq C(n,r_1,\mu_1)( \sup_{\partial B_r(x_0)} |\psi(x)| + 1)
\]

**Proof.** Our proof follows from Theorem 10.10 of Chapter 10 in [7].

For any function \( h(x) \), let \( h^+(x) = \max\{h(x), 0\} \) and \( h^-(x) = \max\{-h, 0\} \). First we claim that there is a constant \( r^*(\mu_1, r_1) < r_1 \) such that for every \( r \in (0, r^*) \)

\[
(B.11) \quad ||u^+||_1 \leq C(r_1,\mu_1)( \sup_{\partial B_r(x_0)} \psi^+(x) + 1)
\]

where \( u \) is the solution of (B.10). Here and in the followings all \( ||.||_p \) are computed for measurable functions on \( B_r(x_0) \).

It is sufficient to show (B.11). Without loss of generality, we assume that
$r < r_1$ and $\psi(x) \leq 0$ on $\partial B_r(x_0)$. Since $u^+ \in W^1_0(B_r(x_0))$. Recall that $\omega = \sqrt{1 + |Du|^2}$. Let $dvol$ be the volume form of $(N, g)$. Then by (B.11)

$$\int_{B_r(x_0)} \frac{Du}{\omega}, Du^+ dvol \leq \int_{B_r(x_0)} |f(x)u^+| dvol \leq \mu_1 ||u^+||_1$$

With the Sobolev inequality (B.3) this implies that

$$||u^+||_1 \leq C(r_1)r||Du^+||_1 \leq C(r_1)r\mu_1 ||u^+||_1 + C(r_1)r_1 vol(B_{r_1}(x_0))$$

Now we take $r_2 = r_2(r_1, \mu_1) < r_1$ such that $C(r_1)r\mu_1 \leq \frac{1}{2}$. Then for all $r < r_2$ we conclude (B.11) for $\psi(x) \leq 0$.

Now for every $r \in (0, r_2)$ we claim that

$$(B.13) \sup_{B_r(x_0)} u^+ \leq C(n, r_1, \mu_1)(\sup_{\partial B_r(x_0)} \psi^+(x) + 1)$$

First we assume $\psi^+(x) = 0$. Then $u^+(x) \in W^1_{0,p}(B_r(x_0))$ for any $p$.

We set $v(x) = u^+(x) + 1$. Let $\beta > 1$ be any fixed constant. Let $v'(x) = v^\beta(x) - 1$. Replacing $u^+$ with $v'(x)$ in (B.12) and applying (B.11) we obtain

$$\beta \int_{B_r(x_0)} v^{\beta - 1} |Du^+| dvol \leq \mu_1 \int_{B_r(x_0)} (v^\beta - 1) dvol + \beta \int_{B_r(x_0)} v^\beta dvol$$

On the other hand applying (B.2) on $v^\beta(x) - 1$ we obtain

$$(B.14) ||v^\beta(x) - 1||_{\frac{n}{\beta}} \leq \beta C(n, 1) \int_{B_r(x_0)} v^{\beta - 1} |Du^+| d\mu$$

Combining them together we obtain

$$||v||_{\beta, \frac{n}{\beta}} \leq \beta^\frac{1}{\beta} C(n, \mu_1, r_1)^\frac{1}{\beta} ||v||_\beta$$

Repeating the above process with $v^{\beta_k}$ where $k \in \mathbb{Z}^+$ and letting $k \to \infty$ gives the estimate in (B.13) with $\sup_{\partial B_r(x_0)} u^+ = 0$. For exact details we refer to Theorem 8.15 in [7]. For the general case of $\psi^+$, we replace $u$ with $u - L$ with $L = \sup_{\partial B_r(x_0)} \psi^+$.

With a similar derivation as above we can obtain that

$$(B.15) \sup_{B_r(x_0)} u^- \leq C(n, r_1, \mu_1)(\sup_{\partial B_r(x_0)} \psi^-(x) + 1)$$

Notice that $r_2$ is the same constant as above. Combining (B.13) with (B.15) we conclude (B.10). The proof is complete. \hfill $\square$

The following proposition is a corollary of Theorem 15.53 in [1] by Aubin.

**Proposition B.6.** Fix $x_0 \in N$ and $f \in C(N)$. There is a constant $r_3 = r_3(n, \text{Ric})$ such that for $r \in (0, r_3)$ the ball $B_r(x_0) \subset N$ is embedded and the mean curvature of $\partial B_r(x_0)$ with respect to the outward normal vector satisfies

$$H_{\partial B_r(x_0)} > \max_{B_{r_3}(x_0)} |f(x)|$$

$$(B.16)$$
Proof. Theorem 15.53 in the Aubin’s book [1] implies that $H_{\partial B_r(x_0)}$ satisfies
\begin{equation}
H_{\partial B_r(x_0)} = \frac{n}{r} + O(r), \quad \text{as} \quad r \to 0
\end{equation}
when $r$ is sufficiently small. Notice $f(x)$ is a continuous function on $N$. Taking $r_3$ be sufficiently small shall imply the conclusion. \hfill \square

Recall that $\omega = \sqrt{1 + |Du|^2}$. Let $\{\partial_i\}_{i=1}^n$ be a local coordinate vector field on $B_r(x_0)$. Denote $\partial_i u$ by $u_i$ and $\tilde{g}^{ij} u_i$ by $u^i$. We define
\begin{equation}
\tilde{g}^{ij} = \sigma^{ij} - \frac{u^i u^j}{\omega^2}
\end{equation}
For a $C^2$ function $u$ in $N$, $u_{ij}$ is the covariant derivative $\nabla_i \nabla_j u$. In fact
\begin{equation}
\text{div}(\frac{Du}{\omega}) = \frac{1}{\omega} \tilde{g}^{ij} u_{ij}
\end{equation}
The following lemma is very useful for the gradient estimate of mean curvature equation.

Lemma B.7 (Lemma 2.8 in [26]). It holds that
\begin{equation}
\tilde{g}^{ij} \omega_{ij} = (|A|^2 + \text{Ric}(\frac{Du}{\omega}, \frac{Du}{\omega})) \omega + \langle \frac{Du}{\omega}, \nabla(H \omega) \rangle + \frac{2}{\omega} \tilde{g}^{ij} \omega_{ij} + \nabla_i \nabla_j u
\end{equation}
where $|A|^2 = \frac{1}{2} \tilde{g}^{ik} \tilde{g}^{jl} u_{kj} u_{lj}$. $\text{Ric}$ is the Ricci curvature of $N$ and $H = \text{div}(\frac{Du}{\omega})$.

Let us see a result on the interior estimate of the gradient in terms of boundary estimates of the gradient and $C^0$ estimate on mean curvature equations.

Theorem B.8. Fix $x_0 \in N$. Suppose $u(x) \in C^2(B_r(x_0)) \cap C^1(\overline{B}_r(x_0))$ satisfying
\begin{equation}
\tilde{g}^{ij} \omega_{ij} = \psi(x,u,Du)
\end{equation}
where $\psi(x,u,Du) = 1$ or $f(x)\omega$. Then there is a positive constant $K > 0$ such that
\begin{equation}
\sup_{B_r(x_0)} |Du| \leq e^{2K \max_{\partial B_r(x_0)} |u|} \sup_{\partial B_r(x_0)} (1 + |Du|)
\end{equation}
Here $K$ is a constant depending on the Ricci curvature on $B_r(x_0)$ (and $\max_{\partial B_r(x_0)} |f(x)|$ in the case of $\psi(x,u,Du) = f(x)\omega$).

Proof. Let $K$ be a fixed positive constant determined later. Let $\eta$ be the function $e^{Ku}$. Suppose the function $\eta \omega$ achieves the maximum in $\Omega$ at some point $y_0 \in B_r(x_0)$. Let $L$ denote the operator $\tilde{g}^{ij} \nabla_i \nabla_j$. First observe that $\omega_i \eta + \omega \eta_i = 0$ at $y_0$. Moreover
\begin{align}
0 &\geq L(\omega \eta)(y_0) = (L \omega - \frac{2}{\omega} \tilde{g}^{ij} \omega_{ij} \eta) + \omega L \eta \\
&\geq \eta(\frac{L \eta}{\eta} + \text{Ric}(\frac{Du}{\omega}, \frac{Du}{\omega}) + |A|^2 + \langle \frac{Du}{\omega^2}, \nabla(\psi(x,u,Du)) \rangle)
\end{align}
We can assume $\omega(y_0) \geq \sqrt{2}$. Otherwise nothing needs to be proven. Thus
\[
\tilde{g}^{ij}u_iu_j(y_0) = (\sigma^{ij} - \frac{u^iu^j}{\omega^2})u_iu_j = \frac{|Du|^2}{\omega^2} \geq \frac{1}{2}
\]
Notice that at $y_0$ we have
\[
\frac{L\eta}{\eta} = K^2\tilde{g}^{ij}u^iu^j + K\psi(x, u, Du)
\]
\[\tag{B.22}
\geq \frac{1}{2}K^2 + K\psi(x, u, Du)
\]
Since $D\omega = -D\eta = -KDu$ at $y_0$. No matter in the case of $\psi(x, u, Du) = 1$ or $f(x)\omega$ it is clear that at $y_0$.
\[\tag{B.23}
0 \geq L(\omega\eta)(y_0) \geq \eta(\frac{1}{2}K^2 - \max_{B_r(x_0),|X| \leq 1} |\text{Ric}(X, X)| - C(\mu_1)K)
\]
Now we choose choose $K$ sufficiently large such that \eqref{B.21} becomes
\[\tag{B.24}
0 \geq L(\omega\eta)(y_0) \geq 0
\]
Here $K$ depends on the Ricci curvature (and $\max_{B_r(x)} |f(x)|$ in the case of $\psi(x, u, Du) = f(x)\omega$).

Since $L$ is an elliptic operator, this leads to a contradiction to the strong maximum principle of elliptic operators. For such fixed $K$, then
\[\tag{B.25}
e^{Ku}\omega(x) \leq \sup_{\partial B_r(x_0)} e^{Ku}\omega \quad \text{for} \quad x \in \bar{B}_r(x_0)
\]
It naturally gives the conclusion \eqref{B.20}. The proof is complete. \hfill \Box

Now we are ready to show Theorem B.1.

\begin{proof}
Our proof is exactly the same in the Dirichlet problem of minimal surface equation in Theorem 16.9 in [7]. First let $r^*$ be $\min\{r_3, r_2\}$ where $r_3$ and $r_2$ be the positive constant in Proposition B.6 and Proposition B.5 respectively. It is obvious that $r^* < r_1$ and $r^* = r^*(n, \text{Ric}, \mu_1)$ where $r_1$ is the constant in Theorem B.3.

Now assume $r \in (0, r^*)$ and $\psi \in C^2(\partial \Omega)$. If for $u_\sigma \in C(\bar{B}_r(x)) \cap C^2(B_r(x))$ is the solution
\[\tag{B.26}
\begin{cases}
\div(\frac{Du}{\omega}) = \sigma f(x), & x \in B_r(x) \\
u(x) = \sigma \psi(x) & x \in \partial B_r(x)
\end{cases}
\]
for every $\sigma \in [0, 1]$ we claim that
\[\tag{B.27}
\max_{\Omega} \{|u_\sigma|, |Du_\sigma|, |D^2u_\sigma|\} \leq \mu_0 = \mu_0(\sup_{\partial \Omega} |\psi(x)|, \sup_{B_r(x)} |f(x)|)
\]
The proof is complete. \hfill \Box
\end{proof}
Since \( r < r^* < r_2 \), Proposition \([B.5]\) implies that
\[
\text{(B.28)} \quad \max_{\Omega} |u_\sigma| \leq C(n, r_1, \mu_1)(\sup_{\partial B_r(x)} |\psi(x)| + 1) \quad \text{for all } \sigma \in [0, 1]
\]

Notice that \( u_\sigma \) satisfies that
\[
\tilde{g}^{ij} u_{ij} = f(x) \omega
\]

Therefore by Theorem \([B.8]\) we have
\[
\text{(B.29)} \quad \sup_{\bar{B}_r(x)} |Du_\sigma| \leq e^{2K \max_{\partial B_r(x)} |u_\sigma|} \sup_{\partial B_r(x)} (1 + |Du_\sigma|)
\]

On the other hand arguing as Theorem 14.6 in \([7]\) we conclude
\[
\text{(B.30)} \quad \sup_{\partial B_r(x)} |Du_\sigma| \leq C(n, \mu_1, \sup_{\partial B_r(x)} |\psi(x)|)
\]

because of \( r < r^* < r_3 \) and \([B.16]\). This gives a uniform estimate of the gradient \( |Du_\sigma| \) for \( \sigma \in [0, 1] \). The classical Schauder estimate in Riemannian manifolds gives the estimate of \( |D^2 u_\sigma| \) in \([B.27]\).

Applying Theorem 11.3 in \([7]\) we obtain the existence of the solution to the Dirichlet problem \([B.1]\) assuming \( \psi(x) \in C^2(\partial B_r(x)) \) for all \( r \in (0, r^*) \).

For general \( \psi(x) \in C(\partial B_r(x)) \) since \( \partial B_r(x) \) is smooth, there is a sequence \( \{\psi_k(x)\}_{k=1}^\infty \in C^2(\partial B_r(x)) \) such that \( \psi_k(x) \) converges to \( \psi(x) \) uniformly as \( k \to \infty \) on \( \partial B_r(x) \). With the above argument, there is a sequence \( \{u_k(x)\} \) as the existence of the solutions to the Dirichlet problem \([B.1]\) with \( u_k(x) = \psi_k(x) \) on \( \partial B_r(x) \). Since \( \{\psi_k\}_{k=1}^\infty \) and \( \psi \) are uniformly bounded, \( \{u_k\} \) satisfies \([B.27]\) uniformly. By the Arzela-Ascoli theorem \( \{u_k\} \) converges to \( u(x) \) in the \( C^2 \) norm. And \( u(x) \) is a solution of the Dirichlet problem \([B.1]\) with \( u(x) = \psi(x) \) for \( \psi(x) \in C(\partial B_r(x)) \).

The proof is complete. \( \square \)

A direct consequence of Theorem \([B.1]\) is given as follows. It says that the Dirichlet problem \([1.1]\) is solvable if the domain is a sufficiently small normal ball in Riemannian manifolds. This plays a key role in the proof of Theorem \([8.3]\).

**Theorem B.9.** Fix \( x_0 \in N \) and \( \alpha > 0 \). Then there is a positive constant \( r^* > 0 \) depending on the sectional curvature near \( x_0 \), \( \alpha \) and \( n \) such that for every \( r \in (0, r^*) \) the Dirichlet problem
\[
\text{(B.31)} \quad \begin{cases}
\text{div}(\frac{Du}{\omega}) = \frac{\alpha}{\omega} & x \in B_r(x_0) \\
u(x) = \psi(x) & x \in \partial B_r(x_0)
\end{cases}
\]
is uniquely solvable in \( C^2(B_r(x_0)) \cap C(\partial B_r(x_0)) \) for any \( \psi(x) \) in \( C(\partial B_r(x_0)) \). Here \( B_r(x_0) \) is the ball centered at \( x_0 \) with radius \( r \) in \( N \).
Proof. Fix $x_0 \in N$. By Theorem [B.1] there is a constant $r^* = r^*(\text{Ric}, n, \alpha) > 0$ such that for every $r$ in $(0, r^*)$ the Dirichlet problem

\begin{equation}
\text{div}\left( \frac{Du}{\omega} \right) = \pm \alpha \quad \text{on} \quad B_r(x_0), \quad u(x) = \psi(x) \quad \text{on} \quad \partial B_r(x_0)
\end{equation}

is solvable in $C^2(B_r(x_0)) \cap C(\partial B_r(x_0))$ for any $\psi(x)$ in $C(\partial B_r(x_0))$. Now we write $u^+(x)$ ($u^-(x)$) for its solution as in the case that the right hand side is $+\alpha$ ($-\alpha$) respectively.

For any $\sigma \in [0, 1]$, suppose $u_\sigma \in C^2(B_r(x_0)) \cap C(\bar{B}_r(x_0))$ is the solution of the following Dirichlet problem:

\begin{equation}
\begin{cases}
\text{div}\left( \frac{Du}{\omega} \right) = \sigma \frac{\alpha}{\omega} & x \in B_r(x_0) \\
u(x) = \psi(x) & x \in \partial B_r(x_0)
\end{cases}
\end{equation}

By the comparison principle, we have $u^+(x) \leq u_\sigma \leq u^-(x)$ on $B_r(x)$. Namely we have the estimate

\begin{equation}
\max_{B_r(x_0)} |u_\sigma(x)| \leq C(r^*, n, \max_{\partial B_r(x_0)} |\psi|, \alpha)
\end{equation}

according to Proposition [B.3]. Proceeding with the same derivation in the Theorem [B.1] we obtain

\begin{equation}
\max_{B_r(x_0)} \left\{ |u_\sigma|, |Du_\sigma|, |D^2u_\sigma| \right\} \leq C(\sup_{\partial \Omega} |\psi(x)|, r^*, \alpha)
\end{equation}

Thus the existence of the solution to the Dirichlet problem (B.31) follows from the continuation methods in Theorem 11.3, [7] if assuming $\psi(x) \in C^2(\Omega)$. The general case can be derived as similar as that in Theorem [B.1]. The proof is complete. \hfill \Box

**Appendix C. An example**

We give an example to illustrate the condition (2) in Theorem 8.1 can not be removed in general.

**Theorem C.1.** The sphere $S^n$ in $\mathbb{R}^{n+1}$ is given by

\[ S_n = \{(x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_{n+1}^2 = 1\} \]

with the induced metric $\sigma_n$. Let $S^+_n$ be the open upper half sphere given by

\[ S^+_n = \{(x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_{n+1}^2 = 1, x_{n+1} > 0\} \]

Then the Dirichlet problem

\begin{equation}
\text{div}\left( \frac{Du}{\omega} \right) = \frac{n}{\omega} \quad \text{on} \quad S^+_n; \quad u(x) = \psi(x) \quad \text{on} \quad \partial S^+_n
\end{equation}

for any continuous function $\psi(x)$ on $\partial S^+_n$ admits no solution in $C(\bar{S}^+_n) \cap C^2(S^+_n)$. Here $\omega = \sqrt{1 + |Du|^2}$.
Remark C.2. Notice that the set
\[ \partial S_n^+ := \{(x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_{n+1}^2 = 1, x_{n+1} = 0\} \]
is a minimal smooth hypersurface in \((S_n, \sigma_n)\). The above theorem satisfies the assumptions of Theorem 8.1 except its condition (2).

**Proof.** Suppose there is a continuous function \(\psi(x)\) on \(\partial S_n^+\) such that the Dirichlet problem (C.1) has a solution \(u(x) \in C(\bar{S}_n^+) \cap C^2(S_n^+)\). It is clear that \(\Sigma = (x, u(x))\) is a minimal surface in the conformal product manifold
\[ M := (S_n \times \mathbb{R}, e^{2r} (\sigma_n + dr^2)) \]
Let \(s = e^r\). Then \(M\) is isometric to the Euclidean space \(\mathbb{R}^{n+1}\setminus\{0\}\) with its polar coordinate representation
\[ (S_n \times (0, +\infty), s^2 \sigma_n + ds^2) \]
via the map \(f(x, r) = (x, e^r)\) from \(M\) to \(\mathbb{R}^n\setminus\{0\}\). Thus \(f(\Sigma)\) is an embedded minimal surface in \(\mathbb{R}^{n+1}\) not touching the origin with its boundary on the plane \(x_{n+1} = 0\) in \(\mathbb{R}^{n+1}\). Then \(x_{n+1} = 0\) on the boundary of \(f(\Sigma)\) and not the 0 constant function. Therefore there is a point \(p = (p_1, \cdots, p_{n+1})\) in the interior of \(f(\Sigma)\) such that \(p_{n+1} \neq 0\) attains the minimum or maximum of the \(x_{n+1}\)-coordinate on \(f(\Sigma)\). From the maximum principle then \(f(\Sigma)\) should be contained in the totally geodesic plane \(x_{n+1} = p_{n+1}\). This gives a contradiction since the boundary of \(f(\Sigma)\) belongs to the plane \(x_{n+1} = 0\).

Thus no classical solution to the Dirichlet problem (C.1) exists. \(\Box\)

**Remark C.3.** By Theorem 6.6 the generalized solution to (C.1) exists. Define \(u_0(x) = +\infty\) on \(S_n^+\) and \(u_0(x)\) is a continuous function on \(S_n \setminus S_n^+\) such that \(u_0(x) = \psi(x)\) on \(\partial S_n^+\). From the above proof \(u_0(x)\) is a generalized solution to (C.1) with boundary data \(\psi(x)\).

Thus \(P_+\) is equal to \(S_n^+\). Moreover the above proof shows that \(\partial S_n^+ \times \mathbb{R}\) has locally least perimeter in \(M = \mathbb{R}^{n+1}\setminus\{0\}\). But \(S_n^+\) is not an almost minimal set in \(S_n\) because its boundary \(\partial S_n^+\) is unstable in \(S_n\).

**Appendix D. Some facts on mean curvatures**

In this section we collect some facts on mean curvature. Our reference is [25].

Let \(M\) be a Riemannian manifold with a metric \(\sigma\). Let \(\Sigma\) be a smooth hypersurface in \(M\) with a normal vector \(\vec{v}\). Suppose a local orthonormal frame \(\{e_1, e_2, \cdots, e_n\}\). Then the mean curvature vector of \(\Sigma\), \(\vec{H}\), is given by
\[ \vec{H} = (\nabla_{e_i} e_i, \vec{v})\vec{v} = -div_M(\vec{v})\vec{v} \]
where \(\nabla\) and \(div_M\) are the covariant derivative and the divergence of \(M\) respectively.

**Definition D.1.** \(H = div_M(\vec{v})\) is called the mean curvature of \(\Sigma\) with respect to the normal vector \(\vec{v}\).
For example, the mean curvature of the standard sphere $S_n$ in $\mathbb{R}^{n+1}$ with the normal vector pointed to the infinity is $n$.

Now we collect two classical facts on mean curvatures.

**Lemma D.2** (Lemma 3.1 in [25]). Let $f$ be a smooth function on $M$. Let $\tilde{M}$ denote the same smooth manifold as $M$ equipped with the metric $e^{2f}g$. Let $\Sigma$ be a smooth hypersurface in $M$ (so is in $\tilde{M}$). Then the mean curvatures of $\Sigma$ in $\tilde{M}$ and in $M$ have the relation that

\[
\tilde{H} = e^{-f}(H + (m-1)df(\vec{v}))
\]

Here $\vec{v}$ is the normal vector of $\Sigma$ in $M$, $m-1 = \dim\Sigma$, $H$ ($\tilde{H}$) denotes the mean curvature of $\Sigma$ in $M$ ($\tilde{M}$) with respect to the normal vector $\vec{v}$ ($e^{-f}\vec{v}$).

Let $N$ be a complete manifold with a metric $g$. Let $Q$ be a product manifold $N \times \mathbb{R}$ with the metric $g + dr^2$ and $Q_\alpha$ be the conformal product manifold $N \times \mathbb{R}$ equipped with the metric $e^{2\alpha}(g + dr^2)$. From Lemma D.2, we conclude that

**Lemma D.3.** Let $\Omega$ be a smooth hypersurface in $Q$ with the normal vector $\vec{v}$. Let $H$ be the mean curvature of $\Sigma$ in $Q$ with respect to $\vec{v}$. Then the mean curvature of $\Sigma$ in $Q_\alpha$, $H^\alpha$, is

\[
H^\alpha = e^{-\alpha}\langle H + \alpha(\vec{v}, \partial_r) \rangle
\]

where $\langle , \rangle$ is the inner product in $Q$.

Suppose $u \in C^2(\Omega)$ and $\Sigma$ is the graph of $u(x)$. Let $\vec{v}$ be the downward normal vector of $\Sigma$. Then the mean curvature of $\Sigma$ with respect to $\vec{v}$ in $Q$ is $H = \text{div}(\frac{Du}{\omega})$ where $\text{div}$ is the divergence of $N$ and $\omega = \sqrt{1 + |Du|^2}$. Lemma 2.2 in [25] indicates that

**Lemma D.4.** If $u \in C^2(\Omega)$ satisfies $\text{div}(\frac{Du}{\omega}) = \frac{\omega}{\omega}$, then its subgraph is minimal in $Q_\alpha$.

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