ON NONLINEAR AND QUASILINEAR ELLIPTIC FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. We consider nonlinear elliptic functional differential equations. The corresponding operator has the form of a product of nonlinear elliptic differential mapping and linear difference mapping. It were obtained sufficient conditions for solvability of the Dirichlet problem. A concrete example shows that a nonlinear differential–difference operator may not be strongly elliptic even if the nonlinear differential operator is strongly elliptic and the linear difference operator is positive definite. The analysis is based on the theory of pseudomonotone–type operators and linear theory of elliptic functional differential operators.

1. Introduction. Nonlinear elliptic differential equations have been considered since the middle of the 20th century by many mathematicians (see [16, 1, 2, 9]). Abstract nonlinear elliptic functional differential equations were studied in [5]. At that moment it was no explicit example of nonlinear elliptic functional differential equations. In the 80s-90s, the basis of the theory of linear elliptic functional differential equations was created, see [12, 13]. This theory was applied for study of elliptic and parabolic functional differential equations with nonlinearity in lower order terms, see [10, 14]. In this paper we study elliptic functional differential equations with nonlinearity in terms with higher order derivatives. We note that the essentially nonlinear elliptic functional differential operators equal to the product of the p-Laplacian and positive definite difference operators were considered in [15].

Let $Q \subset \mathbb{R}^n$ be a bounded domain with a boundary $\partial Q \in C^\infty$, or $Q = (0, d) \times G$, $G \subset \mathbb{R}^{n-1}$ be a bounded domain (with a boundary $\partial G \in C^\infty$ if $n \geq 3$). If $n = 1$ we denote $Q = (0, d)$. We consider the problem

$$A_R u(x) = f(x) \quad (x \in Q),$$

with the boundary condition

$$u(x) = 0 \quad (x \notin Q),$$

where $f \in W_q^{-1}(Q)$, $1/q + 1/p = 1$, and $1 < p < \infty$. Here functional differential operator $A_R$ is a composition of nonlinear differential operator $A$ given by the

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formula
\[ A_u(x) = - \sum_{1 \leq i \leq n} \partial_i A_i (x, u, \nabla u) + A_i (x, u, \nabla u), \] (1.3)

and of linear difference operator \( R \) given by the formula
\[ Ru(x) = \sum_{h \in M} a_h u(x + h), \] (1.4)

where \( a_h \in \mathbb{R}, M \subset \mathbb{Z}^n \) is a finite set of vectors with integer coordinates (similarly we can consider the case of commensurable shifts).

Section 2 deals with the properties of difference operator \( R_Q \), which is uniquely defined by operator \( R \) and domain \( Q \).

In Section 3, we prove solvability of problem (1.1)–(1.4). For this we study such properties of operator \( A_R \) as pseudo–monotonocity, coercivity, and property \((S_+)\).

In Section 4, we formulate the solvability conditions for \( 2m \)-order nonlinear elliptic functional differential problems (1.1)–(1.2) with
\[ \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha A_{\alpha} (D^\beta R_Q u) \quad |\beta| \leq m. \] (1.5)

Section 5 is devoted to second order quasilinear elliptic functional differential equations. We assume that coefficients of differential operator are differentiable and satisfy algebraic condition of strong ellipticity.

Section 6 consists of examples. We show that a product of positive definite difference operator and strongly elliptic differential operator can form functional differential operator that does not satisfy the strong ellipticity condition. For linear theory such situation is not the case.

2. Difference operators. We consider properties of difference operators in the spaces \( L_p(Q) \) and \( W_p^k(Q) \), \( 1 < p < \infty \). Here \( W_p^k(Q) \) is the Sobolev space of functions \( u \in L_p(Q) \) having all generalized derivatives \( D^\alpha u \in L_p(Q) \) \((|\alpha| \leq k)\) with the form
\[ \|u\|_{W_p^k(Q)} = \left\{ \sum_{|\alpha| \leq k} \left( \int_Q |D^\alpha u(x)|^p dx \right)^{1/p} \right\}^{1/p}. \]

If \( p = 2 \), these properties were studied in \([12, 13]\). For \( 1 < p < \infty \), corresponding generalizations were obtained in \([15]\).

Denote by \( M \) the additive group generated by the set \( M \). Let \( Q_r \) be the open connected components of the set
\[ Q \setminus \left( \bigcup_{h \in M} (\partial Q + h) \right). \]

**Definition 2.1.** The set \( Q_r \) is called a subdomain. The family \( \mathcal{R} \) of all subdomains \( Q_r, r = 1, 2, \ldots \), is called the decomposition of the domain \( Q \).

It is easy to see that the set \( \mathcal{R} \) is at most countable and, in addition,
\[ \bigcup_r \partial Q_r = \left( \bigcup_{h \in M} (\partial Q + h) \right) \cap \overline{Q} \quad \text{and} \quad \bigcup_r Q_r = \overline{Q}. \]

For any subdomain \( Q_{r_1} \) and arbitrary vector \( h \in M \), either there exists \( Q_{r_2} \) such that \( Q_{r_2} = Q_{r_1} + h \), or \( Q_{r_1} + h \subset \mathbb{R}^n \setminus \overline{Q} \), see Lemma 7.2 in \([13, \text{ Ch.2, Sec.7}]\). Thus, the family \( \mathcal{R} \) can be decomposed into disjoint classes as follows: subdomains \( Q_{r_1}, Q_{r_2} \in \mathcal{R} \) belong to the same class if \( Q_{r_2} = Q_{r_1} + h \) for some \( h \in M \). We denote
the subdomains $Q_r$ by $Q_{sl}$, where $s$ is the number of class and $l$ is the number of subdomain in the $s$th class. Obviously, each class consists a finite number $N = N(s)$ of subdomains $Q_{sl}$ and $N(s) \leq (\text{diam}(Q) + 1)^n$. The set of classes may be either finite or countable, see Secton 7 in [13]. Let us illustrate these decompositions by two examples.

**Example 2.2.** Let $R u(x_1, x_2) = \gamma_0 u(x_1, x_2) + \gamma_1 u(x_1 + 1, x_2) + \gamma_{-1} u(x_1 - 1, x_2)$, and $Q = (0, 3) \times (0, 1)$. For domain $Q$, we define three subdomains from one class that correspond to the operator $R_Q$:

- $Q_{11} = (0, 1) \times (0, 1)$
- $Q_{12} = (1, 2) \times (0, 1)$
- $Q_{13} = (2, 3) \times (0, 1)$

![Fig. 2.1](image)

**Example 2.3.** Let the difference operator is the same as in Example 2.1 and $Q = (0, 2.2) \times (0, 1)$. For domain $Q$, we define two classes of subdomains that correspond to the operator $R_Q$, see Fig. 2.2.

![Fig. 2.2](image)

The first class of subdomains contains

- $Q_{11} = (0, 0.2) \times (0, 1)$
- $Q_{12} = (1, 1.2) \times (0, 1)$
- $Q_{13} = (2, 2.2) \times (0, 1)$

The second class of subdomains contains

- $Q_{21} = (0.2, 1) \times (0, 1)$
- $Q_{22} = (1.2, 2) \times (0, 1)$

Consider the properties of the difference operator $R : L_p(\mathbb{R}^n) \rightarrow L_p(\mathbb{R}^n)$, $1 < p < \infty$. Introduce $R_Q = P_Q R I_Q : L_p(Q) \rightarrow L_p(Q)$, where $I_Q : L_p(Q) \rightarrow L_p(\mathbb{R}^n)$ is the operator of extension of functions from $L_p(Q)$ by zero to $\mathbb{R}^n \setminus Q$ and $P_Q : L_p(\mathbb{R}^n) \rightarrow L_p(Q)$ is the operator of restriction of functions from $L_p(\mathbb{R}^n)$ to $Q$.

Recall that the operator $R$ is nonlocal. Translation by a vector $h$ can map a point $x \in Q$ into $\mathbb{R} \setminus Q$. Therefore, boundary conditions (1.2) for differential difference equation (1.1) are specified not only on the boundary $\partial Q$ but on the whole set $\mathbb{R}^n \setminus Q$. To take into account boundary conditions (1.2) in the difference operator $R$, we introduce the operator $I_Q$. Thus, a function $u(x)$ defined on $Q$ is replaced
by the function \((I_Qu)(x)\) defined on the whole space \(\mathbb{R}^n\). After the action of the operator \(R\), we again obtain a function defined on the whole space \(\mathbb{R}^n\). The operator \(P_Q\) is introduced to obtain the restriction of the function \((RI_Qu)(x)\) onto \(Q\).

**Lemma 2.4.** (cf. Lemma 8.1 [13, Ch.2, Sec.8]). The operators \(I_Q : L_p(\mathbb{R}^n) \to L_p(\mathbb{R}^n)\) and \(P_Q : L_p(\mathbb{R}^n) \to L_p(\mathbb{R}^n)\) are bounded for \(1 < p < \infty\).

**Lemma 2.5.** (cf. Lemma 8.2 [13, Ch.2, Sec.8]). The operators \(R : L_p(\mathbb{R}^n) \to L_p(\mathbb{R}^n)\) and \(R_Q : L_p(\mathbb{R}^n) \to L_p(\mathbb{R}^n)\) are bounded for \(1 < p < \infty\).

Denote by \(L_p(\bigcup_l Q_{sl})\) the subspace of functions in \(L_p(\mathbb{R}^n)\) that vanish outside of \(\bigcup_l Q_{sl}\) \((l = 1, \ldots, N(s))\). We introduce a bounded operator \(P_s : L_p(\mathbb{R}^n) \to L_p(\bigcup_l Q_{sl})\) by the formula \(P_su(x) = u(x)\) for \(x \in \bigcup_l Q_{sl}\) and \(P_su(x) = 0\) for \(x \in \mathbb{R}^n \setminus \bigcup_l Q_{sl}\). It is obvious that \(P_s\) is a projection operator onto \(L_p(\bigcup_l Q_{sl})\). Since \(\text{mes}_n(\partial Q_{sl}) = 0\), we have

\[
L_p(\mathbb{R}^n) = \sum_s L_p \left( \bigcup_l Q_{sl} \right). \tag{2.1}
\]

**Lemma 2.6.** (cf. Lemma 8.5 [13, Ch.2, Sec.8]). \(L_p(\bigcup_l Q_{sl})\) is an invariant subspace of the operator \(R_Q\).

Let us introduce the isomorphism of the reflexive Banach spaces

\[
U_s : L_p \left( \bigcup_l Q_{sl} \right) \to L_p^N(Q_{s1})
\]

by the formula

\[
(U_su)(x) = u(x + h_{s1}) \quad (x \in Q_{s1}), \tag{2.2}
\]

where \(l = 1, \ldots, N = N(s)\) and the vector \(h_{s1}\) is such that \(Q_{s1} + h_{s1} = Q_{sl}\) \((h_{s1} = 0)\), \(L_p^N(Q_{s1}) = \prod_l L_p(Q_{sl})\).

**Lemma 2.7.** (cf. Lemma 8.6 [13, Ch.2, Sec.8]). The operator \(R_Q : L_p^N(Q_{s1}) \to L_p^N(Q_{s1})\) defined by the relation

\[
R_Q = U_s R Q U_s^{-1}, \tag{2.3}
\]

is the operator of multiplication by the matrix \(R_s = R_s(x) \quad (x \in \overline{Q}_{s1})\) of order \(N(s) \times N(s)\) with entries

\[
r_{ij}^s(x) = \begin{cases} a_h(x + h_{s1}) & (h = h_{sj} - h_{si} \in \mathcal{M}), \\ 0 & (h_{sj} - h_{si} \notin \mathcal{M}). \end{cases} \tag{2.4}
\]

In order to illustrate this lemma, we construct the matrices \(R_s\) for Examples 2.2 and 2.3.

**Example 2.8.** In Example 2.2, we have one class of subdomains with three elements. Then the matrix \(R_1\) corresponding to operator \(R_Q\) has the form

\[
R_1 = \begin{pmatrix} \gamma_0 & \gamma_1 & 0 \\ \gamma_1 & \gamma_0 & \gamma_1 \\ 0 & \gamma_1 & \gamma_0 \end{pmatrix}.
\]
Example 2.9. In Example 2.3, we have two classes of subdomains with three and two elements. The matrices $R_1$ and $R_2$ corresponding to operator $R_Q$ have the form

$$R_1 = \begin{pmatrix} \gamma_0 & \gamma_1 & 0 \\ \gamma_1 & 0 & \gamma_1 \\ 0 & \gamma_1 & \gamma_0 \end{pmatrix} \quad \text{and} \quad R_2 = \begin{pmatrix} \gamma_0 & \gamma_1 \\ \gamma_1 & 0 \\ 0 & \gamma_0 \end{pmatrix}.$$  

Since $Q$ is a bounded domain, by virtue of (2.4), a number of different matrices $R_s$ is finite if the coefficients $a_h$ are constants. Let $n_1$ denote this number, and let $R_{s'}$ denote all different matrices $R_s$ ($\nu = 1, \ldots, n_1$).

Lemma 2.10. (cf. Lemma 8.7 [13, Ch.2, Sec.8]). Let the coefficients $a_h$ be constants. Then $\sigma(R_Q) = \sum_{1 \leq \nu \leq n_1} \sigma(R_{s'})$.

Now we consider properties of difference operators in Sobolev spaces. Let $\bar{W}^m_p(Q)$ be the closure of the set of compactly supported infinitely differentiable in $Q$ functions $C_0^\infty(Q)$ in the space $W^m_p(Q)$.

Lemma 2.11. (cf. Lemma 8.14 [13, Ch.2, Sec.8]). Let the coefficients $a_h$ be constants, and $k \geq 0$ be integer. Then, $D^\alpha R_Q u = R_Q D^\alpha u$ for all $u \in \bar{W}^m_p(Q)$, where $|\alpha| \leq m$.

Lemma 2.12. (cf. Lemma 8.15 [13, Ch.2, Sec.8]). Let the coefficients $a_h$ be constants. Then, for all $u \in L_p(Q)$ such that $u \in \bar{W}^m_p(Q_{st})$ ($s = 1, 2, \ldots; l = 1, \ldots, N(s)$), we have $R_Q u \in \bar{W}^m_p(Q_{st})$ and

$$\|R_Q u\|_{\bar{W}^m_p(Q_{st})} \leq c_1 \sum_{1 \leq j \leq N(s)} \|u\|_{\bar{W}^m_p(Q_{sj})}. \quad (2.5)$$

Moreover, if $\det R_{s'} \neq 0$ ($\nu = 1, \ldots, n_1$), then $R_Q^{-1} u \in \bar{W}^m_p(Q_{st})$ and

$$\|R_Q^{-1} u\|_{\bar{W}^m_p(Q_{st})} \leq c_2 \sum_{1 \leq j \leq N(s)} \|u\|_{\bar{W}^m_p(Q_{sj})}. \quad (2.6)$$

Here the constants $c_1, c_2 > 0$ are independent of $s$ and $u$.

Corollary 2.13. It is easy to see that

$$R_Q^{-1} = \sum_s U_s^{-1} R_s^{-1} U_s P_s, \quad U_s P_s R_Q^{-1} = R_s^{-1} U_s P_s. \quad (2.7)$$

3. General criteria of solvability for elliptic functional differential equations. Since Lebesgue spaces and Sobolev spaces are reflexive Banach ones for $1 < p < \infty$, then in order to formulate general properties of operators we introduce some abstract reflexive Banach space $X$. We denote its topological dual space by $X^*$ and the corresponding duality by $\langle \cdot, \cdot \rangle : X \times X^* \to \mathbb{R}$.

Definition 3.1. A mapping $A : X \to X^*$ is called demicontinuous, if it is continuous from the strong topology of $X$ into the weak topology of $X^*$. A mapping $A : X \to X^*$ is called semicontinuous, if for any $u, v, w \in X$ the function $\lambda \mapsto \langle A(u + \lambda v), w \rangle$ is continuous from $\mathbb{R}$ to $\mathbb{R}$.

Obviously, any demicontinuous operator is semicontinuous.
Definition 3.2. A mapping \( A : X \to X^* \) has property \((S_+)\), if for arbitrary sequence \( u_n \to u \) weakly in \( X \) and sequence \( Au_n \to w \) weakly in \( X^* \) such that
\[
\lim_{n \to \infty} \langle Au_n, u_n - u \rangle \leq 0,
\]
we have \( w = A(u) \) and \( u_{n_k} \to u \) strongly in \( X \) for some subsequence \( \{u_{n_k}\} \subset \{u_n\} \).

Definition 3.3. A mapping \( A : X \to X^* \) is called pseudomonotone, if for arbitrary \( u_n \to u \) weakly in \( X \) such that condition (3.1) holds, we have
\[
\lim_{n \to \infty} \langle Au_n, u_n - \xi \rangle \geq \langle Au, u - \xi \rangle \quad \forall \xi \in X.
\]

Remark 3.4. Demicontinuous operators with property \((S_+)\) are pseudomonotone.

Definition 3.5. A mapping \( A : X \to X^* \) is called coercive if there exists \( u_0 \in X \) such that
\[
\|u\|_X^{-1} \langle A(u), u - u_0 \rangle \to \infty \text{ if } \|u\|_X \to \infty.
\]

We define the operator \( A_R \) by the formula
\[
A_Ru(x) = - \sum_{1 \leq i \leq n} \partial_i A_i(x, RQ_u, RQ \partial_1 u, \cdots, RQ \partial_n u) + A_0(x, RQ_u, RQ \partial_1 u, \cdots, RQ \partial_n u),
\]
i.e. for any \( u, y \in \dot{W}^1_p(Q) \)
\[
\langle A_Ru, y \rangle = \sum_{1 \leq i \leq n} \int_Q A_i(x, RQ_u, RQ \partial_1 u, \cdots, RQ \partial_n u) \partial_i y \, dx + \int_Q A_0(x, RQ_u, RQ \partial_1 u, \cdots, RQ \partial_n u) y \, dx.
\]

In order to formulate the necessary conditions for solvability of functional-differential equations we remind the classical conditions from theory of elliptic differential equations with operator of pseudomonotone-type. Let \( A_i \) be such that the following conditions hold:

**I)** Integrability condition: the functions \( A_i \) satisfy the Caratheodory conditions, i.e. \( A_i(\cdot, \xi) \) are measurable for a.a. \( \xi \in \mathbb{R}^{n+1} \), and \( A_i(\cdot, \cdot) \) are continuous for a.a. \( x \in Q \); moreover, for a.a. \( x \in Q \) and for all \( \xi \in \mathbb{R}^{n+1} \) we have
\[
|A_i(x, \xi)| \leq c_1 \sum_{0 \leq i \leq n} |\xi_i|^{p-1} + h(x), \quad i = 0, \ldots, n,
\]
where \( c_1 > 0, h \in L_q(Q) \).

**II)** Ellipticity condition: for \( x \in Q \) and \( \xi, \eta \in \mathbb{R}^{n+1} \) such that \( \xi_0 = \eta_0 \) and \( \xi \neq \eta \), we have
\[
\sum_{1 \leq i \leq n} (A_i(x, \xi) - A_i(x, \eta))(\xi_i - \eta_i) > 0.
\]

**III)** Coercivity condition: there exist \( c_2, c_3, c_4 \in \mathbb{R} \), and \( c_4 \in \mathbb{R} \) such that
\[
\sum_{1 \leq i \leq n} A_i(x, \xi) \xi_i \geq c_2 \sum_{1 \leq i \leq n} |\xi_i|^{p} - c_3 |\xi_0|^p - c_4.
\]

The differential operator \( A : \dot{W}^1_p(Q) \to \dot{W}_q^{-1}(Q) \) given by the formula
\[
Au = - \sum_{1 \leq i \leq n} \partial_i A_i(x, u, \nabla u) + A_0(x, u, \nabla u)
\]
and satisfying conditions I– III) is bounded, demicontinuous, pseudomonotone, and coercive, moreover, it has the property \((S_+),\) see \([8, 4, 11]\) for example.

In order to describe conditions for the functions \(A_i(x, u, \partial_1 u, \ldots, \partial_n u)\) and for the matrices \(R_s\) \((\dim R_s = N(s) \times N(s))\) we use

\[
\zeta = \begin{pmatrix}
\zeta_{10} & \zeta_{11} & \zeta_{12} & \cdots & \zeta_{1n} \\
\zeta_{20} & \zeta_{21} & \zeta_{22} & \cdots & \zeta_{2n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\zeta_{N(s)0} & \zeta_{N(s)1} & \zeta_{N(s)2} & \cdots & \zeta_{N(s)n}
\end{pmatrix}.
\]

(3.3)

We denote the elements of \(\zeta\) by \(\zeta_{ii}\), where \(\zeta_{ii}\) is the \(i\)th column of \(\zeta\), and \(\zeta_{ii}\) is the \(l\)th line of \(\zeta\), here \(\dim \zeta = N(s) \times (n + 1)\), \(\dim \zeta = N(s)\), \(\dim \zeta = n + 1\).

We assume that for any \(s\) the coefficients \(A_i, i = 0, 1, \ldots, n\) and the matrices \(R_s\) satisfy the following conditions:

(A0) Nondegeneracy condition: the coefficients \(a_{ii} \in \mathbb{R}\) are constants and \(\det R_s \neq 0\) \((\nu = 1, \ldots, n_1)\).

(A1) Integrability condition: the functions \(A_i\) satisfy the Caratheodory conditions, i.e. \(A_i(x, \xi)\) are measurable in \(x\) for any \(\xi \in \mathbb{R}^{n+1}\) and continuous in \(\xi \in \mathbb{R}^{n+1}\) for a.a. \(x \in Q\); moreover, for a.a. \(x \in Q\) and for all \(\xi \in \mathbb{R}^{n+1}\), we have

\[
|A_i(x, \xi)| \leq c_1 \sum_{0 \leq i \leq n} |\xi_i|^p - 1 + h(x), \quad i = 0, \ldots, n,
\]

where \(c_1 > 0, h \in L_q(Q)\).

(A2) Ellipticity condition: for any \(s\), for a.a. \(x \in Q_{s1}\), and for all \(\zeta, \eta \in \mathbb{R}^{N(s) \times (n+1)}\) such that \(\eta_0 = \zeta_0\) and \(\eta \neq \zeta\) the following estimate holds:

\[
\sum_{1 \leq i \leq N(s)} \sum_{1 \leq i \leq n} (A_i(x + h_{sl}, \zeta_i) - A_i(x + h_{sl}, \eta_i)) (R_s^{-1}(\zeta_i - \eta_i))_t > 0.
\]

(A3) Coercivity condition: for any \(s\), for a.a. \(x \in Q_{s1}\), and for all \(\zeta \in \mathbb{R}^{N(s) \times (n+1)}\), there exist \(c_2 > 0\) and \(c_3, c_4 \in \mathbb{R}\) such that

\[
\sum_{1 \leq i \leq N(s)} \sum_{1 \leq i \leq n} A_i(x + h_{sl}, \zeta_i) (R_s^{-1}(\zeta_i))_t \geq c_2 \sum_{1 \leq i \leq N(s)} \sum_{1 \leq i \leq n} |\zeta_{ii}|^p - c_3 \sum_{1 \leq i \leq N(s)} |\zeta_{0i}|^p - c_4.
\]

Ellipticity condition (A2) is the modification of ellipticity condition II) for differential operators. In contrast to differential operators, see condition II), in ellipticity condition for functional–differential operators we must consider the connections of coordinates corresponding to the values of functions in different subdomains of the \(s\)th class. For this purpose in ellipticity condition (3.4) we use the weights in the form of matrix \(R_s^{-1}\).

**Theorem 3.6.** Let conditions (A0), (A1), (A3) hold. Then the operator \(A_R: \dot{W}_p^1(Q) \rightarrow \dot{W}_q^{-1}(Q)\) given by formula (3.2) is bounded, demicontinuous, and coercive.

**Proof.** By Lemmas 2.11 and 2.12 operator \(R_Q: \dot{W}_p^1(Q) \rightarrow \dot{W}_q^1(Q)\) is bounded. By property (A1) the operator \(A: \dot{W}_p^1(Q) \rightarrow \dot{W}_q^{-1}(Q)\) is bounded, see \([8, 4, 11]\). Thus, the operator \(A_R = AR_Q: \dot{W}_p^1(Q) \rightarrow \dot{W}_q^{-1}(Q)\) is bounded as a composition of bounded operators.
Operator $\mathcal{A}_R = \mathcal{A}R_Q : \dot{W}^1_p(Q) \to W^{-1}_q(Q)$ is demicontinuous as composition of linear bounded operator $R_Q$ and demicontinuous bounded operator $\mathcal{A}$.

Let $w = R_Q u$, where $u \in W^1_p(Q)$. By Lemma 2.5, there exists a bounded inverse operator $R^{-1}_Q : L_p(Q) \to L_p(Q)$. Then we can use equality (2.7):

$$\sum_i \int_Q A_i(x, R_Q u, R_Q \partial_1 u, \ldots, R_Q \partial_n u) \partial_i u \, dx$$

$$= \sum_i \int_Q A_i(x, w, \partial_1 w, \ldots, \partial_n w) \partial_i R^{-1}_Q w \, dx$$

$$= \sum_i \sum_j \int_{\cup_{Q_{sl}}} A_i(x, P_s w, P_s \nabla w) (U_s^{-1} R^{-1}_w U_s P_s \partial_j w) \, dx$$

$$= \sum_i \sum_j \int_{Q_{sl}} (U_s A_i(x, P_s w, P_s \nabla w), R^{-1}_w U_s P_s \partial_j w) \, dx$$

$$= \sum_{s,l} \sum_{Q_{sl}} \int A_i(x + h_{sl}, (U_s P_s w)_t, \nabla(U_s P_s w)_t) (R^{-1}_w U_s P_s \partial_j w)_t \, dx.$$  

(3.6)

Here and further we sum over $i = 1, \ldots, n$, if the opposite is not specified.

If we denote $\zeta_0 = \zeta_0(x) = U_s P_s w(x)$, $\zeta_1 = \zeta_1(x) = \partial_s U_s P_s w(x)$, from equalities (3.6), Lemmas 2.10, 2.11 and theorem on the equivalent norms in $W^1_p(Q)$ we obtain

$$\sum_i \int_Q A_i(x, R_Q u, R_Q \partial_1 u, \ldots, R_Q \partial_n u) \partial_i u \, dx$$

$$= \sum_{s,l} \sum_i \int_{Q_{sl}} A_i(x + h_{sl}, \zeta_s)(R^{-1}_s \zeta_s)_t \, dx$$

$$\geq c_2 \sum_{s,l} \sum_i \int_{Q_{sl}} |\zeta_s|^p \, dx - c_3 \sum_{s,l} \int_{Q_{sl}} |\zeta_0|^p \, dx - c_4 \text{mes}(Q)$$

$$= c_2 \sum_i \int_Q |\partial_i w|^p \, dx - c_3 \int_Q |w|^p \, dx - c_4 \text{mes}(Q)$$

$$\geq k_1 \sum_i \int_Q |\partial_i u|^p \, dx - k_2 \int_Q |u|^p \, dx - k_3$$

$$= k_1 \|u\|_{W^1_p(Q)}^p - k_2 \|u\|_{L_p(Q)}^p - k_3,$$

where constants $k_1, k_2, k_3$ do not depend on $u$; moreover, $k_1 > 0$. By Lemma 2.12 and condition (A1),

$$\left\| \int_Q A_0(x, R_Q u, \nabla R_Q u) u \, dx \right\| \leq k_4 \|u\|_{L_p(Q)} \left( k_5 + \|u\|_{L_p(Q)}^{p-1} + \sum_i \|\partial_i u\|_{L_p(Q)}^{p-1} \right),$$

where constants $k_4, k_5$ do not depend on $u$. Using well–known formula $ab \leq \frac{(\mu a)^p + b^q}{\mu^p a^q}$, we get
where constants \( k_0, k_7 \) do not depend on \( u \). Thus,

\[
\langle A_R u, u \rangle = \sum_{0 \leq i \leq n} \int_Q A_i(x, R_Q u, \nabla R_Q u) \partial_i u \, dx \geq \frac{k_1}{2} \sum_i \|\partial_i u\|_{L^p(Q)}^p + k_6 \|u\|_{L^p(Q)}^p + k_7,
\]

From (3.7) it follows that \( \|u\|_{W_0^2(Q)}^{-1} \langle A_R u, u \rangle \to \infty \) as \( \|u\|_{W_0^2(Q)} \to \infty \). Coercivity is proved.

We need some auxiliary function \( H_s : Q_{s1} \times \mathbb{R}^{N(s) \times (n+1)} \times \mathbb{R}^{N(s) \times (n+1)} \to \mathbb{R}^1 \),

\[
H_s(x, \xi, \eta) = \sum_{1 \leq i \leq N(s)} \sum_{1 \leq l \leq n} (A_i(x+h_{sl}, \xi_i) - A_i(x+h_{sl}, \eta_i)) (R_s^{-1}(\xi_i - \eta_i)_l),
\]

which is defined for any \( \xi, \eta \) such that \( \xi_0 = \eta_0 \). Analogous function was proposed in [7]. For the convenience of the reader we prove its property completely.

**Lemma 3.7.** For any \( \kappa, C, C_1 > 0 \), there exists a positive function \( c(x) \) such that on the bounded set

\[
U = \left\{ \eta \in \mathbb{R}^{N(s) \times (n+1)} : |\eta_0| \equiv \sum_{1 \leq i \leq N(s)} |\eta_i| \leq C, \sum_{1 \leq i \leq n} |\eta_i| \leq C_1 \right\}
\]

the following estimate holds:

\[
H_s(x, \xi, \eta) \geq c(x)|\xi - \eta| \quad \forall \xi \in U, \eta \in \mathbb{R}^{N(s) \times (n+1)} : |\xi - \eta| \geq \kappa > 0.
\]

Here \( c(x) > 0 \) is defined for a.a. \( x \in Q_{s1} \), it depends on \( \kappa, C, C_1 \) only, and it does not depend on \( \xi \) and \( \eta \).

**Proof.** Let \( \xi = \eta + \chi \xi^0 \), where \( |\xi^0| = \kappa, c_0^0 = 0 \). We denote \( h(\chi) = H_s(x, \eta + \chi \xi^0, \eta) \), here arguments \( x, \eta, \xi_0 \) are fixed. Thus,

\[
h(\chi) = \sum_{1 \leq i \leq N(s)} \sum_{1 \leq l \leq n} (A_i(x+h_{sl}, (\eta + \chi \xi^0)_l) - A_i(x+h_{sl}, \eta_l)) \chi(R_s^{-1}\xi^0_i)_l
\]

\[
= \sum_{l,i} \left( A_i(x+h_{sl}, (\eta + \chi \xi^0)_l) - A_i(x+h_{sl}, (\eta + \xi^0)_l) \right) \chi(R_s^{-1}\xi^0_i)_l
\]

\[
+ \chi \sum_{l,i} \left( A_i(x+h_{sl}, (\eta + \xi^0)_l) - A_i(x+h_{sl}, \eta_l) \right) (R_s^{-1}\xi^0_i)_l
\]

\[
= \sigma(\chi) + \chi h(1).
\]

By ellipticity condition (A2) \( h(1) > 0 \). We introduce \( \xi^0 = \eta + \xi^0 \). In this notation \( \chi \xi^0 = \frac{\chi}{\chi - 1} (\xi - \xi^0) \). From condition (A2) we obtain

\[
\sigma(\chi) = \sum_{l,i} \left( A_i(x+h_{sl}, \xi_l) - A_i(x+h_{sl}, \xi^0_l) \right) (R_s^{-1}(\xi_i - \xi^0_i))_l \frac{\chi}{\chi - 1} > 0 \quad \forall \chi > 1.
\]
Thus, \( h(\chi) = \sigma(\chi) + \chi h(1) \geq \chi h(1) \) for any \( \chi \geq 1 \). This result may be written in the following form:

\[
H_s(x, \xi, \eta) = h(\chi) \geq \chi h(1) = \kappa^{-1} |\xi - \eta| H_s(x, \eta + \zeta^0, \eta),
\]

where \( \xi = \eta + \chi \zeta^0 \), \( |\zeta^0| = \kappa \), \( \chi \geq 1 \). We introduce

\[
c(x) = \kappa^{-1} \min_{U_\kappa} H_s(x, \eta + \zeta^0, \eta),
\]

where \( U_\kappa = \{ \eta \in U, \zeta^0 \in \mathbb{R}^{N(s) \times (n+1)} : |\zeta^0| = \kappa, |\zeta^0| = 0 \} \). There exists the minimum of function \( c(x) \) on the closed bounded set \( U_\kappa \). Moreover, the value of this minimum is strongly positive because two last arguments of function \( H_s(x, \eta + \zeta^0, \eta) \) are different and estimate (3.4) holds. \( \square \)

**Theorem 3.8.** Let conditions (A0) – (A2) hold. Then the operator \( A_R : \tilde{W}_p^1(Q) \rightarrow W_q^{-1}(Q) \) given by formula (3.2) is pseudomonotone and has property \((S_+)\).

**Proof.** We assume that \( u_j \rightarrow u \) weakly in \( X = \tilde{W}_p^1(Q) \) and

\[
\lim_{j \rightarrow \infty} \langle A_R u_j, u_j - u \rangle \leq 0. \tag{3.8}
\]

Since the imbedding \( W_p^1(Q) \subset L_p(Q) \) is compact, there exists a subsequence \( \{ u_{j_k} \} \) that converges in \( L_p(Q) \) to a function \( u \). Without loss of generality we assume that \( u_{j_k} = u_j \). Using Lemma 2.5, condition (A1), and the Lebesgue theorem on a limit under integral, we have

\[
A_i(x, R_Q u_j, \nabla R_Q u) \rightarrow A_i(x, R_Q u, \nabla R_Q u) \quad \text{in} \quad L_q(Q),
\]

\[
A_0(x, R_Q u_j, \nabla R_Q u_j) \rightarrow A_0(x, R_Q u, \nabla R_Q u) \quad \text{weakly in} \quad L_q(Q).
\]

In the second relation we again consider the sequence \( \{ u_j \} \) instead of some subsequence. Let us rewrite

\[
\langle A_R u_j, u_j - u \rangle = \sum_{1 \leq i \leq n} \int_Q A_i(x, R_Q u_j, \nabla R_Q u) \partial_i (u_j - u) \, dx + \]

\[
+ \int_Q A_0(x, R_Q u_j, \nabla R_Q u_j) (u_j - u) \, dx + \sum_{1 \leq i \leq n} \int_Q \left( A_i(x, R_Q u_j, \nabla R_Q u_j) - A_i(x, R_Q u_j, \nabla R_Q u) \right) \partial_i (u_j - u) \, dx \tag{3.9}
\]

\[
= I_{1j} + I_{2j} + I_{3j}.
\]

The first term of (3.9) \( I_{1j} \) tends to zero as \( j \rightarrow \infty \), since it is a product of convergent sequence in \( L_q(Q) \) and weakly convergent to zero sequence in \( L_p(Q) \). The second term of (3.9) \( I_{2j} \) tends to zero as \( j \rightarrow \infty \), since it is a product of weakly convergent sequence in \( L_q(Q) \) and convergent to zero sequence in \( L_p(Q) \). Hence, by virtue of (3.8), we have

\[
0 \geq \lim_{j \rightarrow \infty} (I_{1j} + I_{2j} + I_{3j}) = \lim_{j \rightarrow \infty} I_{3j}. \tag{3.10}
\]
Now we consider the third term of (3.9). Let \( w = R_Q u \), \( w_j = R_Q u_j \). By Lemma 2.10, there exists a bounded inverse operator \( R_Q^{-1} \). Then we can use equality (2.7):

\[
I_{3j} = \sum_i \int_Q \left( A_i(x, w_j, \nabla w_j) - A_i(x, w_j, \nabla w) \right) R_Q^{-1} \partial_i (w_j - w) \, dx
\]

\[
= \sum_i \int_Q \left( A_i(x, P_s w_j, P_s \nabla w_j) - A_i(x, P_s w_j, P_s \nabla w) \right) U_s^{-1} R_s^{-1} U_s \partial_i (w_j - w) \, dx
\]

\[
= \sum_i \int_Q \left( A_i(x, P_s w_j, P_s \nabla w_j) - A_i(x, P_s w_j, P_s \nabla w) \right) R_s^{-1} U_s \partial_i (w_j - w) \, dx
\]

\[
= \sum_{s,l} \int_{Q_{s,l}} \left( A_i(x + h_{sl}, (U_s P_s w_j)_l, (U_s P_s \nabla w_j)_l) - A_i(x + h_{sl}, (U_s P_s w_j)_l, (U_s P_s \nabla w)_l) \right) R_s^{-1} U_s \partial_i (w_j - w) \, dx. 
\]

Denote

\[ \zeta^l = (U_s P_s w_j, U_s P_s \partial_1 w_j, \ldots, U_s P_s \partial_n w_j), \quad \zeta = (U_s P_s w, U_s P_s \partial_1 w, \ldots, U_s P_s \partial_n w). \]

Then we obtain

\[
I_{3j} = \sum_{s,l} \int_{Q_{s,l}} \left( A_i \left( x + h_{sl}, \zeta^l \right) - A_i \left( x + h_{sl}, \zeta \right) \right) R_s^{-1} \partial_i (\zeta^l - \zeta) \, dx
\]

\[
= \sum_s \int_{Q_s} H_s(x, \zeta^l, \zeta) \, dx, \tag{3.11}
\]

Then we continue estimate (3.10) using equality (3.11):

\[
0 \geq \lim_{j \to \infty} \langle A_R u_j, u_j - u \rangle = \lim_{j \to \infty} \sum_s \int_{Q_s} H_s(x, \zeta^l, \zeta) \, dx.
\]

By Lemma 3.7 the values of \( H_s(x, \zeta^l, \zeta) \) are nonnegative. Using countable additivity and positivity of the Lebesgue integral, we get that

\[
\lim_{j \to \infty} \langle A_R u_j, u_j - u \rangle \geq 0.
\]

From (3.8) and last estimate, it follows that

\[
\lim_{j \to \infty} \langle A_R u_j, u_j - u \rangle = 0.
\]

At the same time by Lemma 3.7 we obtain that for any \( s \)

\[
0 = \lim_{j \to \infty} \int_{Q_s} H_s(x, \zeta^l, \zeta) \, dx \geq \lim_{j \to \infty} \int_{Q_s} c(x) |\zeta^l - \zeta| \, dx \\
= \lim_{j \to \infty} \int_{Q_s} c(x) |\partial_i u_j - \partial_i u| \, dx.
\]
This is possible if and only if
\[ \partial_i u_j(x) \rightarrow \partial_i u(x) \text{ as } j \rightarrow \infty \text{ at a.a. } x \in Q. \]
Thus \( u_j \rightarrow u \) in \( \dot{W}^1_p(Q) \). We proved that \( A_R \) has property \((S_+)\).

Therefore from Remark 3.4 it follows that the operator \( A_R \) is pseudomonotone.

Unfortunately, if nonlinear differential operator satisfies conditions I)–III) we cannot propose general condition on difference operator \( R_Q \) such that \( A_R \) satisfies conditions \((A0) - (A3)\). We will illustrate this by examples, see Section 6.

Now we consider the solvability of the Dirichlet problem
\[
A_R u(x) = f(x) \quad (x \in Q), \tag{3.12}
\]
\[
u = 0 \quad (x \in \partial Q), \tag{3.13}
\]
where \( f \in W^{-1}_q(Q) \).

**Definition 3.9.** A function \( u \in \dot{W}^1_p(Q) \) is called a generalized solution of the problem \((3.12),(3.13)\), if the following integral identity holds for any \( \xi \in \dot{W}^1_p(Q) \):
\[
\sum_{1 \leq i \leq n} \int_Q A_i(R_Q u, R_Q \partial_1 u, \cdots, R_Q \partial_n u) \partial_i \xi \, dx
+ \int_Q A_0(R_Q u, R_Q \partial_1 u, \cdots, R_Q \partial_n u) \xi \, dx = \int_Q f \xi \, dx. \tag{3.14}
\]

**Theorem 3.10.** If conditions \((A0) - (A3)\) hold, then problem \((3.12),(3.13)\) has at least one generalized solution. Moreover, the set of generalized solutions is weakly compact.

**Proof.** Conditions \((A0) - (A3)\) guarantee that operator \( A_R \) is bounded, demicontinuous and pseudomonotone, see Theorem 3.6 and Theorem 3.8, then problem \((3.12),(3.13)\) has at least one generalized solution, see Theorem II.2.7 [8].

Let \( U \) be a set of generalized solutions of problem \((3.12),(3.13)\). By Theorem 3.6, the operator \( A_R \) is coercive. Hence the set \( U \) is bounded in \( \dot{W}^1_p(Q) \). Therefore there is a sequence \( \{u_j\} \subset U \) such that \( u_j \rightarrow u \) weakly in \( \dot{W}^1_p(Q) \). We prove that \( u \in \dot{W}^1_p(Q) \) is a generalized solution of problem \((3.12),(3.13)\).

Since \( \{u_j\} \) are solution of \((3.12),(3.13)\), and \( u_j \rightarrow u \) weakly in \( \dot{W}^1_p(Q) \), we have
\[
\lim_{j \rightarrow \infty} \langle A_R u_j, u - u \rangle = \lim_{j \rightarrow \infty} \langle f, u_j - u \rangle = 0.
\]
By virtue of Theorem 3.8 the operator \( A_R \) has property \((S_+)\). Hence \( A_R u = f \). Therefore the function \( u \in \dot{W}^1_p(Q) \) is a generalized solution of problem \((3.12),(3.13)\). We have proved that the set of solutions of problem \((3.12),(3.13)\) is weakly compact.

Note that conditions \((A0) - (A3)\) are sufficient but not necessary for solvability of problem \((3.12),(3.13)\). In [15] the problem with p-Laplacian was considered and conditions on \( R_s \) was obtained. These conditions guarantee that the problem with p-Laplacian has at least one generalized solution.

4. **Solvability of 2m–th order nonlinear functional differential equations.**

For the convenience of readers, in this section we formulate the results only. Proofs are similar to the ones in the previous section.
Theorem 4.2. We assume that conditions order operator, see Section 3. A is bounded, demicontinuous, pseudomonotone, coercive, and has the property (\(1\)).

Denote by \(N_0\) the number of all \(\alpha\) such that \(|\alpha| \leq m\). The differential operator \(A : X = \dot{W}_{p}^{m}(Q) \to W_{q}^{-m}(Q)\) is given by the formula

\[
(Au, \xi) = \sum_{|\alpha| \leq m} \int_{Q} A_{\alpha}(x, D^{\beta} u) D^{\alpha} \xi \, dx \quad |\beta| \leq m
\]

(4.1)

for arbitrary \(\xi \in \dot{W}_{p}^{m}(Q)\).

We denote by \(\zeta\) the matrix of order \(N_0 \times N(s)\) with column \(\zeta_{\alpha}\), \(|\alpha| \leq m\). Let \(\zeta_{l, \alpha}\) be the elements of this matrix and \(\zeta_{l}\) be the \(l\)th line of \(\zeta\).

Example 4.1. Let \(Q = (0,3) \times (0,1)\) (see Example 2.1) and \(m = 2\). Then we have one class of subdomains with \(N(1) = 3\) and functions \(A_{\alpha}\) depend on \(N_0 = 6\) arguments: \(u, \partial_{1} u, \partial_{2} u, D_{1}^{2} u, D_{2}^{2} u, D_{12} u\). Thus, we must consider

\[
\zeta = \begin{pmatrix}
\zeta_{1,0} & \zeta_{1,1} & \zeta_{1,2} & \zeta_{1,11} & \zeta_{1,12} & \zeta_{1,22} \\
\zeta_{2,0} & \zeta_{2,1} & \zeta_{2,2} & \zeta_{2,11} & \zeta_{2,12} & \zeta_{2,22} \\
\zeta_{3,0} & \zeta_{3,1} & \zeta_{3,2} & \zeta_{3,11} & \zeta_{3,12} & \zeta_{3,22}
\end{pmatrix}.
\]

Assumptions on \(A_{R}\): 

(A0\(_m\)) Nondegeneracy condition: The coefficients \(a_{h} \in \mathbb{R}\) are constants and \(\det R_{\alpha} \neq 0\) (\(\nu = 1, \ldots, n_1\)).

(A1\(_m\)) Integrability condition: The functions \(A_{\alpha}\) satisfy the Caratheodory conditions, i.e. \(A_{\alpha}(x, \xi)\) is measurable in \(x \in Q\) for any \(\xi \in \mathbb{R}^{N_0}\) and continuous in \(\xi\) for a.a. \(x \in Q\); moreover, there exists a constant \(c_1 > 0\) such that

\[
|A_{\alpha}(x, \xi)| \leq c_1 \left(1 + \sum_{|\beta| \leq m} |\xi_{\beta}|^{p-1}\right) \quad (|\alpha| \leq m).
\]

(A2\(_m\)) Ellipticity condition: For arbitrary \(s\), for a.a. \(x \in Q_{s1}\), and for any \(\zeta, \eta \in \mathbb{R}^{N(s) \times N_0}\) such that \(\eta \neq \zeta\) and \(\zeta_{\beta} = \eta_{\beta}\) for \(|\beta| \leq m - 1\) the following estimate holds

\[
\sum_{1 \leq l \leq N(s)} \sum_{|\alpha| = m} (A_{\alpha}(x + h_{sl}, \zeta_{l}) - A_{\alpha}(x + h_{sl}, \eta_{l}))(R_{s}^{-1}(\zeta_{\alpha} - \eta_{\alpha}))_{l, \nu} > 0.
\]

(A3\(_m\)) Coercivity condition: For arbitrary \(s\), for a.a. \(x \in Q_{s1}\), and for any \(\zeta \in \mathbb{R}^{N(s) \times N_0}\), there exist \(c_2 > 0\), and \(c_3, c_4 \in \mathbb{R}\) such that

\[
\sum_{1 \leq l \leq N(s)} \sum_{|\alpha| = m} A_{\alpha}(x, \zeta_{l})(R_{s}^{-1}\zeta_{\alpha})_{l, \nu} \geq c_2 \sum_{1 \leq l \leq N(s)} \sum_{|\alpha| = m} |\zeta_{\alpha}|^{p} - c_3 \sum_{1 \leq l \leq N(s)} \sum_{|\alpha| \leq m - 1} |\zeta_{\alpha}|^{p} - c_4
\]

Conditions (A0\(_m\))–(A3\(_m\)) are similar to corresponding conditions for a second order operator, see Section 3.

Theorem 4.2. We assume that conditions (A0\(_m\))–(A3\(_m\)) hold. Then operator \(A_{R} : W_{p}^{m}(\Omega) \to W_{q}^{-m}(\Omega)\), given by the formula

\[
A_{R} u(x) = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^{\alpha} A_{\alpha}(R_{Q} D^{\beta} u) \quad |\beta| \leq m,
\]

is bounded, demicontinuous, pseudomonotone, coercive, and has the property \((S_{+})\).
Theorem 4.3. We assume that conditions (A0m) – (A3m) hold. Then problem
\[ \mathcal{A}_R u = f \]
has a nonempty and weakly compact set of generalized solutions from \( \dot{W}^m_p(Q) \).

5. Criteria of solvability for quasilinear equations.

Definition 5.1. Operator \( A : \dot{W}^m_p(Q) \to \dot{W}^{-m}_q(Q) \) is called strongly elliptic, if there exist \( c_A > 0 \) and \( \alpha > 0 \) such that
\[ \langle A(u) - A(y), u - y \rangle \geq c_A \| u - y \|_{\dot{W}^m_p(Q)}^{1+\alpha} \]
for any \( u, y \in \dot{W}^m_p(Q) \).

Let us remind that in linear case a product of strongly elliptic operator \( A \) and positive definite operator \( R_Q \) is strongly elliptic too. Thus, we show that in quasilinear case a composition of strongly elliptic operator \( A \) that satisfies algebraic condition of strong ellipticity, see [4], and symmetric positive definite operator \( R_Q \) has useful properties such that the corresponding equation has a generalized solution.

We consider symmetric positive definite matrices \( R_s \). Then there exists symmetric positive definite matrices
\[ T_s = \sqrt{R_s} = \{ t^s_{ml} \}_{m,l=1}^N(s) \]
and
\[ \hat{T}_s = \sqrt{R_s^{-1}} = \{ \hat{t}^s_{ml} \}_{m,l=1}^N(s). \]

We denote \( A_{ij}(x, \xi) = \frac{\partial A_i(x, \xi)}{\partial \xi_j} \).

Theorem 5.2. Let \( p \in [2, \infty) \), and let \( \{ R_s \} \) be the set of matrices that correspond to operator \( R_Q \). We suppose that, for any \( s \), the matrices \( R_s \) are symmetric and positive definite. Moreover, we assume that the operator
\[ \mathcal{A} u(x) = - \sum_{1 \leq i \leq n} \partial_i A_i (x, \partial_1 u, \ldots, \partial_n u) \]
has differentiable coefficients \( A_i(x, \xi) \) with respect to \( \xi \) and satisfies algebraic ellipticity condition: for any \( s \), for a.a. \( x \in Q_s \), \( \xi \in \mathbb{R}^{N(s) \times n} \), and \( \eta \in \mathbb{R}^{N(s) \times n} \),
\[ \sum_{1 \leq m,l \leq N(s)} \sum_{1 \leq i,j \leq n} \hat{t}^s_{ml} A_{ij} (x + h_{sm}, \xi_m) t^s_{ml} \eta_i \eta_j \]
\[ \geq c_5 \sum_{1 \leq i \leq N(s)} \sum_{1 \leq i \leq n} \left| \sum_{1 \leq m \leq N(s)} \hat{t}^s_{mi} \xi_m \right|^{p-2} \eta_i^2, \]
where \( c_5 > 0 \) does not depend on \( x, \xi, \) and \( \eta \), and
\[ |A_{ij}(x, \xi)| \leq c_6 \left( 1 + \sum_{1 \leq i \leq n} |\xi_i|^{p-2} \right) \quad (i, j = 1, \ldots, n) \]
for a.a. \( x \in Q \) and any \( \xi \in \mathbb{R}^n \), \( c_6 > 0 \) does not depend on \( x \) and \( \xi \).

Then operator \( \mathcal{A}_R : \dot{W}^1_p(Q) \to \dot{W}^{-1}_q(Q) \) given by
\[ \mathcal{A}_R u(x) = - \sum_{1 \leq i \leq n} \partial_i A_i (x, \partial_1 R_Q u, \ldots, \partial_n R_Q u) \]
is strongly elliptic.
Recall that the differential operator $A : \dot{W}_p^1 (Q) \to W_q^{-1} (Q)$ given by

$$Au(x) = - \sum_{1 \leq i \leq n} \partial_i A_i (x, \partial_1 u, \ldots, \partial_n u)$$

is strongly elliptic if for almost all $x \in Q$ this operator satisfies the algebraic condition of strong ellipticity:

$$\sum_{1 \leq i,j \leq n} A_{ij} (x, \xi) \eta_i \eta_j \geq \hat{c} \sum_{1 \leq i \leq n} |\xi_i|^{p-2} |\eta_i|^2 \quad \forall \xi \in \mathbb{R}^{n+1}, \eta \in \mathbb{R}^n,$$

see [4, Ch.2, §2]. Thus estimate (5.2) is a generalization of the algebraic condition of strong ellipticity for nonlinear differential operator. The coefficients $t_{lm}$ and $\hat{t}_{lm}$ of the matrices $T_s = \sqrt{R_s}$ and $\hat{T}_s = \sqrt{R_s^{-1}}$ corresponding to the difference operator $R_Q$ allow to consider a contribution of different subdomains $Q_{sm}$.

**Proof of Theorem 5.2.** Let $w = R_Q (u - y)$, $v = R_Q y$, where $u, y \in \dot{W}_p^1 (Q)$. By Lemma 2.10, there exists a bounded inverse operator $R_Q^{-1} : L_p (Q) \to L_p (Q)$. From Lemma 2.11 it follows that $\partial_i (u - y) = R_Q^{-1} \partial_i w$. Integrating by parts and substituting $\nabla R_Q u = \nabla v + \nabla w$, $\nabla R_Q y = \nabla v$, we obtain

$$\langle A_R u - A_R y, u - y \rangle = \sum_i \int_Q \bigl( A_i (x, \nabla R_Q u) - A_i (x, \nabla R_Q y) \bigr) (\partial_i u - \partial_i y) \, dx$$

$$= \sum_i \int_Q \bigl( A_i (x, \nabla v + \nabla w) - A_i (x, \nabla v) \bigr) R_Q^{-1} \partial_i w \, dx = I_1.$$

Then, using (2.7), we have

$$I_1 = \sum_i \sum_s \int_{\bigcup_j Q_{ij}} P_s \bigl( A_i (x, \nabla (v + w)) - A_i (x, \nabla v) \bigr) R_s^{-1} U_s P_s \partial_i w \, dx.$$

By virtue of (2.2), we can rewrite $I_1$ in the following form

$$I_1 = \sum_i \sum_s \int_{Q_{ij}} \left( U_s P_s \left( A_i (x, \nabla (v + w)) - A_i (x, \nabla v) \right), R_s^{-1} U_s P_s \partial_i w \right) \, dx.$$  

From differentiability of $A_i$ we obtain

$$I_1 = \sum_{i,j} \sum_s \int_{Q_{ij}} \left( U_s P_s \left( \int_0^1 A_{ij} (x, \nabla v + \tau \nabla w) d\tau \partial_j w \right), R_s^{-1} U_s P_s \partial_i w \right) \, dx.$$

Let us consider the integrand of $I_1$. Since $R_s$ are symmetric and positive definite, $R_s^{-1}$ are symmetric and positive definite too. Then there exist symmetric and positive definite matrices $\sqrt{R_s^{-1}}$ and

$$\left( U_s P_s \left( \int_0^1 A_{ij} (x, \nabla v + \tau \nabla w) d\tau \partial_j w \right), R_s^{-1} U_s P_s \partial_i w \right)$$

$$= \left( \sqrt{R_s^{-1}} U_s P_s \int_0^1 A_{ij} (x, \nabla v + \tau \nabla w) d\tau \partial_j w, \sqrt{R_s^{-1}} U_s P_s \partial_i w \right).$$
Using the rules of the matrix multiplication, the left factor in this scalar product can be written as following:

\[
\hat{T}_s U_s P_s \int_0^1 A_{ij}(x, \nabla v + \tau \nabla w) d\tau \partial_j w
\]

\[
= \hat{T}_s \text{diag} \left\{ U_s P_s \int_0^1 A_{ij}(x, \nabla v + \tau \nabla w) d\tau \right\} \hat{T}_s \left( \hat{T}_s U_s P_s \partial_j w \right),
\]

where \( \text{diag} \left\{ U_s P_s \int_0^1 A_{ij}(x, \nabla v + \tau \nabla w) d\tau \right\} \) is the diagonal matrix of order \( N(s) \times N(s) \) with the diagonal elements \( \int_0^1 A_{ij}(x + h_{sl}, (\nabla v + \tau \nabla w)(x + h_{sl})) d\tau \). We denote

\[
\hat{\omega}^s_{ij}(x) = \sum_{1 \leq m \leq N(s)} \hat{t}_{lm}^s \partial_j w(x + h_{sm})
\]

\[
(x \in Q_{s4}; l = 1, \ldots, N(s); j = 1, \ldots, n).
\]

Now we can use property (5.2):

\[
I_2 = \sum_{1 \leq l, m \leq N(s)} \sum_{1 \leq i, j \leq n} \hat{t}_{lm} \int_0^1 A_{ij}(x + h_{sm}, (\nabla v + \tau \nabla w)(x + h_{sm})) d\tau t_{ml} \hat{\omega}^s_{ij} \hat{\omega}^s_{li}
\]

\[
\geq c_5 \sum_{1 \leq l, m \leq N(s)} \sum_{1 \leq l, m \leq N(s)} \int_0^1 \left| \sum_{1 \leq m \leq N(s)} \hat{t}_{lm} (\partial_i v + \tau \partial_i w)(x + h_{sm}) \right|^{p-2} d\tau |\hat{\omega}^s_{li}|^2.
\]

Using the well known estimate

\[
\int_0^1 |a + \tau b|^{p-2} d\tau \geq c_7 |b|^{p-2},
\]

we obtain that

\[
\int_0^1 \left| \sum_{1 \leq m \leq N(s)} \hat{t}_{lm} (\partial_i v + \tau \partial_i w)(x + h_{sm}) \right|^{p-2} d\tau
\]

\[
\geq c_7 \left| \sum_{1 \leq m \leq N(s)} \hat{t}_{lm} \partial_i w(x + h_{sm}) \right|^{p-2} = c_7 |\hat{\omega}^s_{li}|^{p-2}.
\]

Hence,

\[
I_1 \geq c_8 \sum_{s, l, i} \int_{Q_{s4}} |\hat{\omega}^s_{li}|^p dx = c_8 \sum_{s, l, i} ||\hat{\omega}^s_{li}||_{L^p(Q_{s4})}^p.
\]

Since the matrices \( \hat{T}_s \) and \( R_s \) are nondegenerate, by virtue of (2.6), we have

\[
c_8 \sum_{s, l, i} ||\hat{\omega}^s_{li}||_{L^p(Q_{s4})}^p \geq c_9 \sum_{s, l, i} \int_{Q_{s4}} |\partial_i w(x + h_{sl})|^p dx
\]

\[
= c_9 ||w||_{W^{1,p}_s(Q)}^p \geq c_{10} ||u - y||_{W^{1,p}_s(Q)}^p.
\]

Strong ellipticity is proved. \( \square \)

Obviously, strongly elliptic operator is coercive. Moreover, strongly elliptic operator has the property \( (S_+) \), see Remark 1.5 in [3, Ch.III, §1].
Theorem 5.3. Let conditions of Theorem 5.2 hold. Then there exists a unique generalized solution of problem (3.12), (3.13).

Proof. Operator $\mathcal{A}$ is demicontinuous, see [6, 3.2], for example. Since $R_Q: W^1_p(Q) \to W^1_p(Q)$ is continuous, see Lemma 2.12, then the operator $\mathcal{A}_R$ is demicontinuous too. Demicontinuous, strongly elliptic operator $\mathcal{A}_R: W^m_p(Q) \to W^{-m}_q(Q)$ is a homeomorphism, see Corollary 1.1.1 in [4].

Theorem 5.4. Let $p \in [2, \infty)$, for any $s$, the symmetric and positive definite matrices $R_s$ correspond to $R_Q$. Moreover, let operator $\mathcal{A}$ be given by the formula

$$\mathcal{A}u(x) = -\sum_{1 \leq i \leq n} \partial_i A_i(x, u, \partial_1 u, \ldots, \partial_n u) + A_0(x, u, \partial_1 u, \ldots, \partial_n u).$$

We assume that functions $A_i(x, \xi)$ are differentiable with respect to $\xi$ and satisfy algebraic ellipticity condition: for any $s$, for a.a. $x \in Q_{s1}$, $\xi \in \mathbb{R}^{N(s) \times n}$, and $\eta \in \mathbb{R}^{N(s) \times n}$,

$$\sum_{1 \leq m, l \leq N(s)} \sum_{1 \leq i, j \leq n} \tilde{t}^s_{lm} A_{ij}(x + h_{sm}, \xi_m) t^s_{ml} \eta_i \eta_j \geq c_3 \sum_{1 \leq i \leq N(s)} \sum_{1 \leq l \leq n} \sum_{1 \leq m \leq N(s)} \tilde{t}^s_{lm} \xi_{mi} |\eta_i|^2,$$  (5.6)

where $c_3 > 0$ does not depend on $x, \xi,$ and $\eta; A_{ij}(x, \xi) = \frac{\partial A_i(x, \xi)}{\partial \xi_j}$ are continuous and

$$|A_{ij}(x, \xi)| \leq c_6 \left(1 + \sum_{0 \leq i \leq n} |\xi_i|^{p-2}\right) \quad (i, j = 0, 1, \ldots, n)$$  (5.7)

for a.a. $x \in Q$ and any $\xi \in \mathbb{R}^{n+1}, c_6 > 0$ does not depend on $x$ and $\xi$.

Then the operator $\mathcal{A}_R = \mathcal{A}_R : W^1_p(Q) \to W^{-1}_q(Q)$ is demicontinuous, pseudomonotone, and coercive. Moreover, it has property $(S_+)$.\[\square\]

Proof. 1. Note that conditions (5.7) are stronger than (A1). Hence, by Theorem 3.6, operator $\mathcal{A}_R$ is demicontinuous and bounded.

2. We show that the operator $\mathcal{A}_R$ has property $(S_+)$. Clearly,

$$\left(\mathcal{A}_R u - \mathcal{A}_R y, u - y\right) = \sum_{1 \leq i \leq N} \int_{Q} (A_i(x, R_Q u, \nabla R_Q u) - A_i(x, R_Q y, \nabla R_Q y)) (\partial_i u - \partial_i y) \, dx$$

$$+ \int_{Q} (A_0(x, R_Q u, \nabla R_Q u) - A_0(x, R_Q y, \nabla R_Q y)) (u - y) \, dx$$

$$= \left(\mathcal{A}_R^u(u, u) - \mathcal{A}_R^u(u, y), u - y\right) + \left(\mathcal{A}_R^y(u, y) - \mathcal{A}_R^y(y, y), u - y\right),$$

where

$$\left(\mathcal{A}_R^u(u, u) - \mathcal{A}_R^u(u, y), u - y\right) = \sum_{1 \leq i \leq N} \int_{Q} (A_i(x, R_Q u, \nabla R_Q u) - A_i(x, R_Q u, \nabla R_Q y)) (\partial_i u - \partial_i y) \, dx.$$
Using condition (5.6), similarly to Theorem 5.2 we can show that the operator $\partial$ that is a bounded inverse operator written as following:

$$\sum_{1 \leq i \leq n} \int_Q \left( A_i(x, RQ u, \nabla RQ y) - A_i(x, RQ u, \nabla RQ y) \right) (\partial_i u - \partial_i y) \, dx$$

$$+ \int_Q \left( A_0(x, RQ u, \nabla RQ y) - A_0(x, RQ y, \nabla RQ y) \right) (u - y) \, dx.$$

Using condition (5.6), similarly to Theorem 5.2 we can show that the operator $A^v_R : \dot{W}^1_p(Q) \to W^{-1}_p(Q)$ is strongly elliptic. For the convenience of readers, we shall present the proof.

Let $w = RQ(u - y), v = RQy$, where $u, y \in \dot{W}^1_p(Q)$. By Lemma 2.10, there exists a bounded inverse operator $R^{-1}_Q : L_p(Q) \to L_p(Q)$. From Lemma 2.11 it follows that $\partial_i(u - y) = R^{-1}_Q \partial_i w$. Integrating by parts and substituting $\nabla RQ u = \nabla v + \nabla w$, $\nabla RQ y = \nabla v, \nabla$, we obtain

$$\sum_{1 \leq i \leq n} \int_Q \left( A_i(x, RQ u, \nabla RQ u) - A_i(x, RQ u, \nabla RQ y) \right) (\partial_i u - \partial_i y) \, dx$$

$$= \sum_{1 \leq i \leq n} \int_Q \left( A_i(x, v + w, \nabla v + \nabla w) - A_i(x, v + w, \nabla v) \right) R^{-1}_Q \partial_i w \, dx = I_3.$$

Then, using (2.7), we have

$$I_3 = \sum_{1 \leq i \leq n} \int_{Q_{st}} \left( U_i P_s (A_i(x, v + w, \nabla(v + w)) - A_i(x, v + w, \nabla v)) U^{-1}_s R^{-1}_s U_s P_s \partial_i w \right) \, dx.$$

By virtue of (2.2), we can rewrite $I_3$ in the following form

$$I_3 = \sum_{1 \leq i \leq n} \int_{Q_{st}} \left( U_i P_s (A_i(x, v + w, \nabla(v + w)) - A_i(x, v + w, \nabla v)) U^{-1}_s R^{-1}_s U_s P_s \partial_i w \right) \, dx.$$

From differentiability of $A_i$ we obtain

$$I_3 = \sum_{1 \leq i \leq n} \int_{Q_{st}} \left( U_i P_s \left( \int_0^1 A_{ij}(x, v + w, \nabla v + \tau \nabla w) \, d\tau \partial_j w \right) \right. \left. U^{-1}_s R^{-1}_s U_s P_s \partial_i w \right) \, dx.$$

Let us consider the integrand of $I_3$. Since $R_s$ are symmetric and positive definite, $R^{-1}_s$ are symmetric and positive definite too. Then there exist symmetric and positive definite matrices $\sqrt{R^{-1}_s}$ and

$$U_i P_s \left( \int_0^1 A_{ij}(x, v + w, \nabla v + \tau \nabla w) \, d\tau \partial_j w \right) \right. \left. U^{-1}_s R^{-1}_s U_s P_s \partial_i w \right) = \left( \sqrt{R^{-1}_s} U_i P_s \int_0^1 A_{ij}(x, v + w, \nabla v + \tau \nabla w) \, d\tau \partial_j w, \sqrt{R^{-1}_s} U_s P_s \partial_i w \right).$$

Using the rules of the matrix multiplication, left vector in this product can be written as following:

$$\hat{T}_i U_i P_s \int_0^1 A_{ij}(x, v + w, \nabla v + \tau \nabla w) \, d\tau \partial_j w.$$
where诊 \{ U_s P_s \int_0^1 A_{ij}(x, v + w, \nabla v + \tau \nabla \nabla w)\} is the diagonal matrix with the diagonal elements \( \int_0^1 A_{ij}(x + h_s t, (v + w)(x + h_s t), (\nabla v + \tau \nabla w)(x + h_s t)) d\tau \). Using notation (5.4), by property (5.6) we obtain
\[
I_3 = \sum_{1 \leq l, m \leq N(s)} \sum_{1 \leq i, j \leq n} \int_0^1 A_{ij}(x, v + w, \nabla v + \tau \nabla \nabla w) \left( \int_{m} A_{sl}(x, v + w, \nabla v + \tau \nabla \nabla w) \right) \left| x, v + w, \nabla v + \tau \nabla \nabla w \right| d\tau \mid \hat{W}_{t_m l}^s \hat{W}_{t_l}^s \geq c_5 \sum_{1 \leq l, m \leq N(s)} \sum_{1 \leq i, j \leq n} \int_0^1 \int_{1 \leq m \leq N(s)} \hat{W}_{t_m l}^s (\partial_i v + \tau \partial_i w) (x + h_s m) d\tau \mid \hat{W}_{t_i}^s | p-2.
\]
Using the well known estimate \( \int_0^1 |a + \tau b|^{p-2} d\tau \geq c_7 |b|^{p-2} \), we obtain that
\[
\int_0^1 \sum_{1 \leq m \leq N(s)} \hat{W}_{t_m l}^s (\partial_i v + \tau \partial_i w) (x + h_s m) d\tau \mid p-2 \\
\geq c_7 \sum_{1 \leq m \leq N(s)} \hat{W}_{t_m l}^s | x, v + w, \nabla v + \tau \nabla \nabla w | p-2 = c_7 | \hat{W}_{t_i}^s |^{p-2}.
\]
Hence,
\[
I_3 \geq c_8 \sum_{s,l} \sum_{i} \int \hat{W}_{t_i}^s | x, v + w, \nabla v + \tau \nabla \nabla w | dx = c_8 \sum_{s,l} \| \hat{W}_{t_i}^s \|_{L^p(Q)}
\]
Since the matrices \( T_t \) and \( R_t \) are nondegenerate, by virtue of (2.6), we have
\[
c_9 \sum_{s,l,i} \| \hat{W}_{t_i}^s \|_{L^p(Q)} \geq c_9 \sum_{s,l,i} \int \| \partial_i w (x + h_s) \|_{L^p(Q)} dx \\
\geq c_9 \| w \|_{L^p(Q)} \geq c_{10} \| u - y \|_{W^p(Q)},
\]
i.e.
\[
(A_R^w(u, u) - A_R^w(u, y), u - y) \geq c_{11} \| u - y \|_{W^p(Q)}, \quad (5.8)
\]
Hence the operator \( A_R^w \) is strongly elliptic. Therefore it is pseudomonotone and has property \( (S_+) \).

Let us show that \( A_R \) has property \( (S_+) \). We assume that \( y_m \to y \) weakly in \( W^1_p(Q) \). Then, passing to a subsequence and using the imbedding theorem, we have \( y_m \to y \) in \( L^p(Q) \). For continuous operator \( R_Q \), we have that \( R_Q y_m \to R_Q y \) weakly in \( W^1_p(Q) \) and \( R_Q y_m \to R_Q y \) in \( L^p(Q) \). At the same time, \( A_i(\cdot, R_Q y_m, \nabla R_Q y_m) \to A_i(\cdot, R_Q y, \nabla R_Q y) \) in \( L^q(Q) \) by continuity of \( A_i \), see the proof of Theorem 3.8. Then we have
\[
\lim_{m \to \infty} \langle A_R^w(y_m, y) - A_R^w(y, y), y_m - y \rangle = \lim_{m \to \infty} \int_Q (A_0(x, R_Q y_m, \nabla R_Q y_m) - A_0(x, R_Q y, \nabla R_Q y)) (y_m - y) dx
\]
+ \sum_{i}^{\infty} \int_{Q} \left( A_{i}(x, RQy_{m}, \nabla RQy) - A_{i}(x, RQy, \nabla RQy) \right) (\partial_{i} y_{m} - \partial_{i} y) \, dx \right) = 0.

Hence,

\lim_{m \to \infty} \langle A_{R} y_{m}, y_{m} - y \rangle = \lim_{m \to \infty} \langle A_{R} y_{m} - A_{R} y, y_{m} - y \rangle

= \lim_{m \to \infty} \left\{ \langle A_{R}^{u}(y_{m}, y_{m}) - A_{R}^{u}(y_{m}, y), y_{m} - y \rangle + \langle A_{R}^{l}(y, y), y_{m} - y \rangle \right\}

= \lim_{m \to \infty} \langle A_{R}^{u}(y_{m}, y_{m}) - A_{R}^{u}(y_{m}, y), y_{m} - y \rangle \geq \lim_{m \to \infty} c_{11} \| y_{m} - y \|_{\tilde{W}^{1}_{p}(Q)}^{p} \geq 0.

We have proved that if $y_{m} \to y$ weakly in $\tilde{W}^{1}_{p}(Q)$ and $\lim_{m \to \infty} \langle A_{R} y_{m}, y_{m} - y \rangle \leq 0$, then

\lim_{m \to \infty} \langle A_{R} y_{m}, y_{m} - y \rangle = 0.

Therefore, repeating the above arguments, we derive

$c_{11} \lim_{m \to \infty} \| y_{m} - y \|_{\tilde{W}^{1}_{p}(Q)}^{p} \leq \lim_{m \to \infty} \langle A_{R} y_{m}, y_{m} - y \rangle = 0.$

Thus the operator $A_{R}$ has property $(S_{+}).$

3. Let us show that $A_{R}$ is coercive. Clearly,

$\langle A_{R} u - A_{R} 0, u \rangle = \langle A_{R}^{u}(u, u) - A_{R}^{u}(u, 0, 0) \rangle + \langle A_{R}^{l}(u, 0, 0) - A_{R}^{l}(0, 0, 0) \rangle = I_{3} + I_{4}.$

For the first term, we have estimate (5.8):

$I_{3} = \sum_{1 \leq i \leq n} \int_{Q} (A_{i}(x, RQy, \nabla RQy) - A_{i}(x, RQy, 0)) \partial_{i} y \, dx \geq c_{12} \| u \|_{\tilde{W}^{1}_{p}(Q)}^{p}.$

For the second term, from the continuity of $A_{i}$ and inequality (5.7) we obtain

$|I_{4}| = \left| \sum_{1 \leq i \leq n} \int_{Q} (A_{i}(x, RQy, 0) - A(x, 0, 0)) \partial_{i} y \, dx \right|

+ \int_{Q} (A_{0}(x, RQy, \nabla RQy) - A_{0}(x, 0, 0)) \, u \, dx |

\leq \sum_{1 \leq i \leq n} \int_{Q} \tilde{c}_{i} |RQy|^{p-1} |\partial_{i} y| \, dx + \sum_{1 \leq i \leq n} \int_{Q} \tilde{c}_{i} |RQy|^{p-1} |u| \, dx + \int_{Q} \tilde{c}_{0} |u|^{p} \, dx.

Using boundedness of $RQ$ (see estimate (2.5)) and the well–known formula $ab \leq \frac{(\mu a)^{p}}{p} + \frac{b^{q}}{\mu^{q}}$, we can find constants $c_{13}, c_{14}$ such that $c_{13} < c_{12}$ and

$|I_{4}| \leq c_{13} \| u \|_{\tilde{W}^{1}_{p}(Q)}^{p} + c_{14} \| u \|_{L^{q}(Q)}^{q}.$

Then

$\langle A_{R} u - A_{R} 0, u \rangle \geq (c_{12} - c_{13}) \| u \|_{\tilde{W}^{1}_{p}(Q)}^{p} - c_{14} \| u \|_{L^{q}(Q)}^{q}.$

We have proved that the operator $A_{R}$ is coercive.

**Theorem 5.5.** Let conditions of Theorem 5.4 hold. Then problem (3.12), (3.13) has a nonempty, weakly compact set of generalized solutions.

**Proof.** By Theorem 5.4, the operator $A_{R}$ is bounded, demicontinuous, pseudomonotone, and coercive. Thus, there exists at least one generalized solution of (3.12), (3.13), see Theorem 2.7 [8, Chap.II]. The proof of the weak compactness for the set of solutions is similar as to Theorem 3.10.  \[\Box\]
6. Examples. In this section, we consider examples with strongly elliptic differential operators. We formulate conditions for a difference operator, under which Theorem 3.10 holds. We also demonstrate that even in the case of symmetric positive definite matrices the equation can have several solutions. It will be considered an example of nonsymmetric difference operator. We shall demonstrate that in this example the condition of ellipticity is not fulfilled.

Example 6.1. Let \( p = 4, Q = (0, 2) \times (0, 1) \), and let

\[
Ru(x_1, x_2) = u(x_1, x_2) + \gamma u(x_1 + 1, x_2) + \gamma u(x_1 - 1, x_2).
\]

We consider the problem

\[
- \sum_{i=1,2} \partial_i (\partial_i Ru)^3 + 3 \sum_{i=1,2} \partial_i^2 w (\partial_i Ru)^2 = f \quad (x \in Q),
\]

\[
R^2 = f \quad (x \notin Q).
\]

where \( w \in C^2(Q) \) is a given function.

Operator \( Au = - \sum_{i=1,2} \partial_i (\partial_i u)^3 + 3 \sum_{i=1,2} \partial_i^2 u (\partial_i u)^2 \) was considered in [4, Chap.1, §3]. It was shown that for \( \gamma = 0 \) problem (6.1)–(6.2) has at least one generalized solution. Let us calculate under such \( |\gamma| < 1 \) problem (6.1)–(6.2) has at least one generalized solution.

For domain \( Q \), we define two subdomain from one class that correspond to the operator \( R_Q \):

\[
Q_1 = (0, 1) \times (0, 1) \quad \text{and} \quad Q_2 = (1, 2) \times (0, 1),
\]

see Fig. 1.

The matrix \( R_1 \) corresponding to \( R_Q \) has the form

\[
R_1 = \begin{pmatrix}
1 & \gamma \\
\gamma & 1
\end{pmatrix}.
\]

This matrix is nondegenerate, if \( |\gamma| \neq 1 \).

Obviously, condition (A1) holds. It is sufficient to show that conditions (A2) and (A3) are fulfilled. Note that

\[
R_1^{-1} = \frac{1}{1 - \gamma^2} \begin{pmatrix}
1 & -\gamma \\
-\gamma & 1
\end{pmatrix}.
\]
The left part of ellipticity condition (A2) for $A_R$ is given by the formula:

$$\sum_{1 \leq i \leq 2} \sum_{1 \leq l \leq 2} (A_i(\xi_{li}) - A_i(\eta_{li})) \left( R_1^{-1} \begin{bmatrix} \xi_{li} - \eta_{li} \\ \xi_{2l} - \eta_{2l} \end{bmatrix} \right)_l$$

$$= \frac{1}{1 - \gamma^2} \sum_{1 \leq i \leq 2} \left\{ (\xi_{1i}^3 - \eta_{1i}^3)(\xi_{1i} - \gamma\xi_{2i} - \eta_{1i} + \gamma\eta_{2i}) + (\xi_{2i}^3 - \eta_{2i}^3)(\xi_{2i} - \gamma\xi_{1i} - \eta_{2i} + \gamma\eta_{1i}) \right\}$$

$$= \frac{1}{1 - \gamma^2} \sum_{1 \leq i \leq 2} \left\{ (\xi_{1i} - \eta_{1i})^2(\xi_{1i}^2 + \eta_{1i}^2) - \gamma(\xi_{1i} - \eta_{1i})(\xi_{2i} - \eta_{2i})(\xi_{1i}^2 + \eta_{1i}^2 + \xi_{2i}^2 + \gamma\eta_{2i}^2) + (\xi_{2i} - \eta_{2i})^2(\xi_{2i}^2 + \eta_{2i}^2 + \gamma\eta_{2i}^2) \right\} = I.$$  

We introduce the new variables:

$$a := \xi_{1i} - \eta_{1i}; \quad b := \xi_{2i} - \eta_{2i},$$

$$c := \xi_{1i}^2 + \eta_{1i}^2 + \eta_{2i}^2,$$

$$d := \xi_{2i}^2 + \eta_{2i}^2.$$  

Note that $c > 0$ and $d > 0$. We have

$$I = a^2 c - \gamma ab(c + d) + b^2 d$$

$$= \frac{|\gamma|(c + d)}{2} \left( \frac{2c}{|\gamma|(c + d)} a^2 - \text{sign}(\gamma)2ab + \frac{2d}{|\gamma|(c + d)} b^2 \right)$$

$$= \frac{|\gamma|(c + d)}{2} \left( \left( \sqrt{\frac{2c}{|\gamma|(c + d)}} a - \text{sign}(\gamma) \sqrt{\frac{|\gamma|(c + d)}{2c}} b \right)^2 + \frac{|\gamma|(c + d)b^2}{2} \left( \frac{2d}{|\gamma|(c + d)} - \frac{|\gamma|(c + d)}{2c} \right) \right).$$

This expression is positive for any $a, b, c > 0, d > 0$, if

$$\frac{2d}{|\gamma|(c + d)} > \frac{|\gamma|(c + d)}{2c} \iff |\gamma|^2 < \frac{4cd}{(c + d)^2}.$$  

But $c + d \geq 2\sqrt{cd}$. We have proved, that if $|\gamma| < 1$, then for any $a, b \in \mathbb{R} \setminus \{0\}, c > 0$, and $d > 0$

$$a^2 c - \gamma ab(c + d) + b^2 d > 0.$$  

Hence, in (6.3) we have the sum of positive summands, if $|\gamma| < 1$. Ellipticity condition (3.4) is true if $|\gamma| < 1$.

Let us show that coercivity condition (A3) holds. We have

$$\sum_{1 \leq i \leq 2} \sum_{1 \leq l \leq 2} A_i(\xi_{li}) \left( R_1^{-1} \begin{bmatrix} \xi_{li} \\ \xi_{2l} \end{bmatrix} \right)_l$$

$$= \frac{1}{1 - \gamma^2} \sum_{1 \leq i \leq 2} \left\{ \xi_{1i}^3(\xi_{1i} - \gamma\xi_{2i}) + \xi_{2i}^3(\xi_{2i} - \gamma\xi_{1i}) \right\}$$

...
\[
\begin{align*}
\sum_{1 \leq i,l \leq 2} \xi_i^4 - \gamma \left[ \xi_{11} \xi_{21} (\xi_{11}^2 + \xi_{21}^2) + \xi_{21} \xi_{22} (\xi_{21}^2 + \xi_{22}^2) \right] \\
\geq \frac{1}{1 - \gamma^2} \sum_{1 \leq i,l \leq 2} \xi_i^4 - \frac{1}{2} |\gamma| \left[ (\xi_{11}^2 + \xi_{21}^2)^2 + (\xi_{21}^2 + \xi_{22}^2)^2 \right]
\end{align*}
\]

Conditions (A0)–(A3) hold, if |\gamma| < 1. Then problem (6.1)–(6.2) has at least one solution for any \( f \in W^{-1}_{4/3}(Q) \).

**Example 6.2.** Let us construct some solutions of problem (6.1), (6.2) for \( \gamma = \frac{1}{2} \).

Let \( p = 4, Q = (0,2) \times (0,1) \) and let

\[ Ru(x_1, x_2) = u(x_1, x_2) + \frac{1}{2} u(x_1 + 1, x_2) + \frac{1}{2} u(x_1 - 1, x_2). \]

We consider the problem

\[
\begin{align*}
- \sum_{i=1,2} \partial_i (\partial_i Ru)^3 + 3 \sum_{i=1,2} \partial_i^2 w (\partial_i Ru)^2 & = 0 \quad (x \in Q), \\
\Gamma w & = 0 \quad (x \notin Q),
\end{align*}
\]

where \( w(x) = \sin(\pi x_1)\sin(\pi x_2) \).

The symmetric matrix

\[ R_1 = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix} \]

corresponds to the operator \( R_Q \). This matrix is positive definite. Thus, problem (6.4), (6.5) has nonempty, weakly compact set of solutions. In particular, it is easy to verify that

\[ u = 0 \]

and

\[ u = 2w(x) = 2\sin(\pi x_1)\sin(\pi x_2) \]

are solutions of (6.4), (6.5).

In the following examples, the nonlinear differential operator is strongly elliptic, while the symmetric part of difference operator corresponds to positive definite matrix. In linear case these conditions are sufficient for differential–difference operator to be strongly elliptic. Hence a corresponding linear problem has a unique generalized solution, see [13, Chap.II, §9]. For nonlinear problem it is not true. In particular, nonlinear differential–difference operator can be not strongly elliptic, moreover, ellipticity condition (A2) can be false for them.

**Example 6.3.** Let us consider the problem with nonsymmetric operator \( R_Q \). Let \( p = 4, Q = (0,2) \times (0,1) \), and let

\[ Ru(x_1, x_2) = u(x_1, x_2) + \gamma u(x_1 + 1, x_2). \]

We consider properties of the operator \( A_R = AR_Q \), if \( A_R u(x) = - \sum_{1 \leq i \leq 2} \partial_i (\partial_i u(x))^3 \).

Note that the matrices

\[ R_1 = \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix}, \quad R_1^{-1} = \begin{pmatrix} 1 & -\gamma \\ 0 & 1 \end{pmatrix} \]
correspond to the operator $R_Q$. The left part of ellipticity condition (A2) for $A_R$ is given by the formula:

$$
\sum_{1 \leq i, l \leq 2} \sum_{1 \leq i, l \leq 2} (A_i(\xi_{1i}) - A_i(\eta_{1l})) \left( R^{-1}_i \left[ \begin{array}{c} \xi_{1i} - \eta_{1l} \\ \xi_{2i} - \eta_{2l} \end{array} \right] \right)_l
$$

$$
= \sum_{1 \leq i, l \leq 2} \left\{ (\xi_{1i} - \eta_{1l}) (\xi_{1i} - \gamma \xi_{2i} - \eta_{1l} + \gamma \eta_{2l}) + (\xi_{2i} - \eta_{2l}) (\xi_{2i} - \eta_{2l}) \right\}
$$

$$
= \sum_{1 \leq i, l \leq 2} \left\{ (\xi_{1i} - \eta_{1l})^2 (\xi_{1i} + \eta_{1i}) (\xi_{2i} - \eta_{2l}) - \gamma (\xi_{1i} - \eta_{1l}) (\xi_{2i} - \eta_{2l}) 
\times (\xi_{1i} + \eta_{1i} \xi_{1i} + \eta_{1i}) + (\xi_{2i} - \eta_{2l}) (\xi_{2i} + \eta_{2i}) (\xi_{2i} + \eta_{2i}) \right\} = I.
$$

(6.6)

We introduce the new variables:

$$
a := \xi_{1i} - \eta_{1l}; \\
b := \xi_{2i} - \eta_{2l}; \\
c := \xi_{1i}^2 + \eta_{1i} \xi_{1i} + \eta_{1i}^2; \\
d := \xi_{2i}^2 + \eta_{2i} \xi_{2i} + \eta_{2i}^2.
$$

We get

$$
I = a^2 c - \gamma abc + b^2 d = \frac{\gamma|c|}{2} \left( \frac{2}{|\gamma|} a^2 - \text{sgn}(\gamma) 2ab + \frac{2d}{|\gamma|} b^2 \right)
$$

$$
= \frac{\gamma|c|}{2} \left( \left( \frac{2}{|\gamma|} a - \text{sgn}(\gamma) \sqrt{|\gamma| \frac{|\gamma|}{2} b} \right)^2 + b^2 \left( \frac{2d}{|\gamma|} - \frac{|\gamma|}{2} \right) \right).
$$

This expression is positive for any $a, b, c > 0, d > 0$, if

$$
\frac{2d}{|\gamma|c} > \frac{|\gamma|}{2} \quad \Leftrightarrow \quad \gamma^2 < \frac{4d}{c}.
$$

For any $\gamma \neq 0$, there exist $\{\xi_i, \eta_i\}_{1 \leq i, l \leq 2}$ such that $\gamma^2 \geq \frac{4d}{c} = 4 \frac{\xi_{1i}^2 + \eta_{1i} \xi_{1i} + \eta_{1i}^2}{\xi_{2i}^2 + \eta_{2i} \xi_{2i} + \eta_{2i}^2}$.

Hence, for any $\gamma$ there exists the pair of vectors $(\xi, \eta)$ such that ellipticity condition is not fulfilled for this pair.

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