Integrable Field Theories with Defects

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The structure of integrable field theories in the presence of defects is discussed in terms of boundary functions under the Lagrangian formalism. Explicit examples of bosonic and fermionic theories are considered. In particular, the boundary functions for the super sinh-Gordon model is constructed and shown to generate the Backlund transformations for its soliton solutions.

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1 Introduction

A quantum integrable theory of defects involving free bosonic and free fermionic fields was first studied in ref. [1] following the achievements obtained in studying the quantum field theory with boundaries [2], [3]. The Lagrangian formulation of a class of relativistic integrable field theories admitting certain discontinuities (defects) has been studied recently [4] - [6]. In particular, in ref. [4] the authors have considered a field theory in which different soliton solutions of the sine-Gordon model are linked in such a way that the integrability is preserved. The integrability of the total system imposes severe constraints specifying the possible types of defects. These are characterized by Backlund transformations which are known to connect two different soliton solutions.

The presence of the defect indicates the breakdown of space isotropy and henceforth of momentum conservation. The key ingredient to classify integrable defects is to impose certain first order differential relations between the different solutions (Backlund transformation). This introduces certain boundary functions (BF) which are specific of each integrable model and leads to the conservation of the total momentum.

Here, we analyze the structure of the possible boundary terms for various cases. We first consider the pure bosonic case studied by Corrigan et. al. [4] - [6] and derive the border functions by imposing conservation of the total momentum. Next, we consider a pure fermionic theory and propose Backlund transformation in terms of an auxiliary fermionic non local field. Such structure is then generalized to include both bosonic and fermionic fields. In particular we construct boundary functions for the supersymmetric sinh-Gordon model and show that it leads to the Backlund transformation proposed by Chaichian and Kulish [7].
2 General Formalism

In this section we introduce the Lagrangian approach proposed in [4]. Consider a system described by

\[ L = \theta(-x)L_{p=1} + \theta(x)L_{p=2} + \delta(x)L_D, \]

where \( L_p(\phi_p, \partial_\mu \phi_p) = \frac{1}{2}(\partial_x \phi_p)^2 - \frac{1}{2}(\partial_t \phi_p)^2 - V(\phi_p) \) describes a set of fields denoted by \( \phi_1 \) for \( x < 0 \) and \( \phi_2 \) for \( x > 0 \). A defect placed at \( x = 0 \), is described by

\[ L_D = \frac{1}{2} (\phi_2 \partial_t \phi_1 - \phi_1 \partial_t \phi_2) + B_0 \]

where \( B_0 \) is the border function. The equations of motion are therefore given by

\[ \begin{align*}
\partial_x^2 \phi_1 - \partial_t^2 \phi_1 & = \partial_{\phi_1} V(\phi_1), \quad x < 0 \\
\partial_x^2 \phi_2 - \partial_t^2 \phi_2 & = \partial_{\phi_2} V(\phi_2), \quad x > 0
\end{align*} \]

and \( x = 0 \),

\[ \begin{align*}
\partial_x \phi_1 - \partial_t \phi_2 & = -\partial_{\phi_1} B_0, \\
\partial_x \phi_2 - \partial_t \phi_1 & = \partial_{\phi_2} B_0.
\end{align*} \]

The momentum is

\[ P = \int_{-\infty}^{0} \partial_x \phi_1 \partial_t \phi_1 + \int_{0}^{\infty} \partial_x \phi_2 \partial_t \phi_2. \]

Acting with time derivative and inserting eqns. of motion (3) we find

\[ \frac{dP}{dt} = \int_{-\infty}^{0} \left( \frac{1}{2} \partial_x (\partial_t \phi_1)^2 + \frac{1}{2} \partial_x (\partial_x \phi_1)^2 - \partial_x \phi_1 \frac{\delta V_1}{\delta \phi_1} \right) dx \\
+ \int_{0}^{\infty} \left( \frac{1}{2} \partial_x (\partial_t \phi_2)^2 + \frac{1}{2} \partial_x (\partial_x \phi_2)^2 - \partial_x \phi_2 \frac{\delta V_2}{\delta \phi_2} \right) dx \]

Using eqns. (3) after integration, we find

\[ \frac{dP}{dt} = \left[ -\frac{\partial B_0}{\partial \phi_1} \phi_1 - \frac{\partial B_0}{\partial \phi_2} \phi_2 + \frac{1}{2} \left( \frac{\partial B_0}{\partial \phi_1} \right)^2 - \frac{1}{2} \left( \frac{\partial B_0}{\partial \phi_2} \right)^2 - V_1 + V_2 \right]_{x=0} \]

where \( \phi_\pm = \phi_1 \pm \phi_2 \). The modified momentum \( P = P + B_0 \) is then conserved if the border function \( B_0 \) satisfies its defining condition, i.e.,

\[ \left[ \frac{1}{2} \left( \frac{\partial B_0}{\partial \phi_1} \right)^2 - \frac{1}{2} \left( \frac{\partial B_0}{\partial \phi_2} \right)^2 - V_1 + V_2 \right]_{x=0} = 0 \]
Let us illustrate the above structure by first considering the free massive bosonic theory for which

\[ V_p = \frac{1}{2} m^2 \phi_p^2. \]

\[ \frac{1}{2} (\frac{\partial B_0}{\partial \phi_1})^2 - \frac{1}{2} (\frac{\partial B_0}{\partial \phi_2})^2 = \frac{m^2}{2} (\phi_1^2 - \phi_2^2) = \frac{m^2}{2} \phi_+ \phi_- \]  

(9)

The solution is easily found if we decompose \( B_0 = B_0^+ (\phi_+) + B_0^- (\phi_-) \) as

\[ B_0 = -m \beta^2 \phi_0^2 - m \phi_0^2 \phi_+ \]

(10)

and \( \beta^2 \) denotes a free (spectral) parameter.

As second example, consider the sinh-Gordon model for which \( V_p = 4m^2 \cosh(2\phi_p) \).

The defining eqn. (8) indicates the natural decomposition

\[ \frac{1}{2} (\frac{\partial B_0}{\partial \phi_1})^2 - \frac{1}{2} (\frac{\partial B_0}{\partial \phi_2})^2 = 4m^2 (\cosh(2\phi_1) - \cosh(2\phi_2)) \]

\[ = 8m^2 \sinh(\phi_+) \sinh(\phi_-) \]  

(11)

yielding

\[ B_0 = -m \beta^2 \cosh(\phi_-) - \frac{4m}{\beta^2} \cosh(\phi_+) \]

(12)

and hence we rederive the Backlund transformation for the sinh-Gordon model

\[ \partial_x \phi_1 - \partial_t \phi_2 = m \beta^2 \sinh(\phi_-) + \frac{4m}{\beta^2} \sinh(\phi_+), \]

\[ \partial_x \phi_2 - \partial_t \phi_1 = m \beta^2 \sinh(\phi_-) - \frac{4m}{\beta^2} \sinh(\phi_+), \]

(13)

3 Fermions and the Super sinh-Gordon Model

Before discussing the Super sinh-Gordon Model let us consider the pure fermionic prototype described by the Lagrangian density

\[ \mathcal{L}_p = \bar{\psi}_p \partial_t \psi - \bar{\psi}_p \partial_x \psi + \bar{\psi}_p \partial_t \psi + \bar{\psi}_p \partial_x \psi + W(\psi_p, \bar{\psi}_p) \]  

(14)

which, for the free fermionic theory, \( W(\psi_p, \bar{\psi}_p) = 2m \bar{\psi}_p \psi_p \). For the half line, \( x < 0 \) or \( x > 0 \) the equations of motion are given by

\[ \partial_x \psi_p + \partial_t \psi_p = \frac{1}{2} \partial_{\psi_p} W_p, \quad \partial_x \bar{\psi}_p - \partial_t \bar{\psi}_p = -\frac{1}{2} \partial_{\bar{\psi}_p} W_p \]  

(15)

according to \( p = 1 \) or \( 2 \) respectively.

Let us propose the following Backlund transformation

\[ \psi_1 + \psi_2 = -i \beta \sqrt{m} f_1 = \partial_{\psi_1} B_1, \quad \bar{\psi}_1 - \bar{\psi}_2 = -\frac{i \beta \sqrt{m}}{4} f_1 = \partial_{\bar{\psi}_1} B_1, \]

(16)
where \( f_1 \) satisfies
\[
\dot{f}_1 = -\frac{i\beta}{4}\sqrt{m}(\psi_1 - \psi_2) + \frac{i}{2\beta}\sqrt{m}(\bar{\psi}_1 + \bar{\psi}_2) = -\frac{1}{4}\partial_t B_1,
\]
\[
\partial_x f_1 = \frac{i\beta}{4}\sqrt{m}(\psi_1 - \psi_2) + \frac{i}{2\beta}\sqrt{m}(\bar{\psi}_1 + \bar{\psi}_2)
\]
(17)
written in terms of a border function \( B_1 = B_1(\bar{\psi}_1, \bar{\psi}_2, \psi_1, \psi_2, f_1) \), which now, due to the grassmanian character of the fermions, depends upon the non local fermionic field \( f_1 \). By considering the Lagrangian system (1) with \( L_p \) given by (14) and \( L_D = -\bar{\psi}_1 \psi_2 - \bar{\psi}_2 \psi_1 + 2f_1 \partial_t f_1 + B_1(\bar{\psi}_1, \bar{\psi}_2, \psi_1, \psi_2, f_1) \) (18) construct the momentum to be
\[
P = \int_{-\infty}^{0} (-\bar{\psi}_1 \partial_x \psi_1 - \psi_1 \partial_x \bar{\psi}_1)dx + \int_{0}^{\infty} (-\bar{\psi}_2 \partial_x \psi_2 - \psi_2 \partial_x \bar{\psi}_2)dx
\]
(19)
In analising its conservation, we find
\[
\frac{dP}{dt} = [W_1 - W_2 - (\bar{\psi}_1 \partial_t \bar{\psi}_1 + \bar{\psi}_2 \partial_t \bar{\psi}_2 - \psi_1 \partial_t \psi_1 + \psi_2 \partial_t \psi_2)]|_{x=0} = 0
\]
(20)
after using equations of motion (18) to eliminate time derivatives and integrating over \( x \). Using the Backlund transformation (19), eqn. (20) becomes
\[
\frac{dP}{dt} = [W_1 - W_2 - \partial_{\bar{\psi}_1} B_1 \bar{\psi}_1 + \partial_{\psi_1} B_1 \psi_1 - \partial_{\bar{\psi}_2} B_1 \bar{\psi}_2 + \partial_{\psi_2} B_1 \psi_2 - \partial_t (\bar{\psi}_2 \psi_1) - \partial_t (\psi_2 \bar{\psi}_1)]|_{x=0} = 0
\]
(21)
If we assume the border function to decompose as \( B_1 = B_1^+ (\bar{\psi}_+, f_1) + B_1^- (\psi_-, f_1) \), the modified momentum
\[
P = P - \bar{\psi}_2 \psi_1 + \psi_2 \psi_1 + B_1^+ - B_1^-
\]
(22)
is conserved provided
\[
[W_2 - W_1 - \partial_{\bar{\psi}_1} B_1^- \bar{\psi}_1 + \partial_{\psi_1} B_1^+ \psi_1]|_{x=0} = 0
\]
(23)
For the free fermi fields system (14)-(18) eqn. (23) becomes
\[
-\frac{1}{2}(\partial_{\bar{\psi}_1} B_1^+)(\partial_{\psi_1} B_1^-) = 2m(\bar{\psi}_1 \psi_1 - \bar{\psi}_2 \psi_2)
\]
(24)
The solution is
\[
B_1 = -\frac{2i}{\beta}\sqrt{mf_1 \bar{\psi}_+} + i\beta \sqrt{mf_1 \psi_-}
\]
(25)
Let us now consider the super sinh-Gordon model described by
\[
\mathcal{L}_p = \frac{1}{2} (\partial_x \phi_p)^2 - \frac{1}{2} (\partial_t \phi_p)^2 + \bar{\psi}_p \partial_t \psi - \bar{\psi}_p \partial_x \psi + \psi_p \partial_t \psi
\]
\[+ \psi_p \partial_x \psi + V(\phi_p) + W(\phi_p, \psi_p, \bar{\psi}_p) \tag{26}\]
and
\[
\mathcal{L}_D = \frac{1}{2} (\partial_x \phi_1 - \phi_1 \partial_x \phi_2) - \psi_1 \psi_2 - \bar{\psi}_1 \bar{\psi}_2 + 2 f_1 \partial_t f_1
\]
\[+ B_0(\phi_1, \phi_2) + B_1(\bar{\psi}_1, \bar{\psi}_2, \psi_1, \psi_2, f_1) \tag{27}\]
where \(V_p = 4m^2 \cosh(2\phi_p)\) and \(W_p = 8m \bar{\psi}_p \psi_p \cosh(\phi_p)\).
Propose the following backlund transformation \[8\],
\[
\partial_x \phi_1 - \partial_t \phi_2 = -\partial_{\psi_1}(B_0 + B_1), \quad \partial_x \phi_2 - \partial_t \phi_1 = \partial_{\psi_2}(B_0 + B_1)
\]
\[
\psi_1 + \psi_2 = \partial_{\psi_1}B_1, \quad \bar{\psi}_1 - \bar{\psi}_2 = -\partial_{\psi_2}B_1, \quad f_1 = -\frac{1}{4} \partial_{f_1}B_1 \tag{28}\]
Assuming the decomposition
\[
B_0 = B_0^+(\phi_+) + B_0^-(\phi_-), \quad B_1 = B_1^+(\bar{\psi}_+, f_1) + B_1^- (\psi_-, f_1) \tag{29}\]
we find that the modified momentum
\[
\mathcal{P} = P + B_0^+(\phi_+) - B_0^-(\phi_-) + B_1^+(\bar{\psi}_+, f_1) - B_1^- (\psi_-, f_1) \tag{30}\]
is conserved provided the border functions \(B_0\) and \(B_1\) satisfy
\[
W_1 - W_2 = \frac{1}{2} (\partial_{f_1}B_1^+)(\partial_{f_1}B_1^-) + 2 (\partial_{\phi_+}B_0^+)(\partial_{\phi_-}B_0^-) + 2 (\partial_{\phi_-}B_0^-)(\partial_{\phi_+}B_1^+) \tag{31}\]
The solution for \[31\] is given by
\[
B_0 = -m \beta^2 \cosh(\phi_-) - \frac{4m}{\beta^2} \cosh(\phi_+), \quad B_1 = -\frac{4i}{\beta} \sqrt{m} \cosh\left(\frac{\phi_+}{2}\right)f_1 \psi_+ + 2i \beta \sqrt{m} \cosh\left(\frac{\phi_-}{2}\right)f_1 \psi_- \tag{32}\]
The boundary functions \(B_0\) and \(B_1\) in \[32\] generate the Backlund transformation agrees with the one proposed in \[7\] for the super sinh-Gordon model. In ref. \[9\] the general one soliton solution was constructed and in \[8\] different solutions were analysed in the context of the Backlund transformation.

The integrability of the model is verified by construction of the Lax pair representation of the equations of motion. This is achieved by splitting the space into two overlapping regions, namely, \(x \leq b\) and \(x \geq a\) with \(a < b\) and defining a corresponding Lax pair within each of them. The integrability is ensured by the existence
of a gauge transformation relating the two sets of Lax pairs within the overlapping region. In ref. [8] we have explicitly constructed such gauge transformation for the super sinh-Gordon model in terms of the $SL(2,1)$ affine Lie algebra.

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