Second-order Stable Finite Difference Schemes for the Time-fractional Diffusion-wave Equation

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Abstract We propose two stable and one conditionally stable finite difference schemes of second-order in both time and space for the time-fractional diffusion-wave equation. In the first scheme, we apply the fractional trapezoidal rule in time and the central difference in space. We use the generalized Newton-Gregory formula in time for the second scheme and its modification for the third scheme. While the second scheme is conditionally stable, the first and the third schemes are stable. We apply the methodology to the considered equation with also linear advection-reaction terms and also obtain second-order schemes both in time and space. Numerical examples with comparisons among the proposed schemes and the existing ones verify the theoretical analysis and show that the present schemes exhibit better performances than the known ones.

Keywords Fractional diffusion-wave equation · fractional linear multi-step method · Fourier analysis · stability · second-order in time.

1 Introduction

In this work, we consider second-order finite difference schemes in both time and space for the following time-fractional diffusion-wave equation, see e.g. \[20,21,24,30,33\],

\[
\begin{aligned}
\cfrac{cD^\beta_0 U(x,t)}{t} &= \mu \partial^2_x U(x,t) + f(x,t), \quad (x,t) \in I \times (0,T], I = (a,b), T > 0, \\
U(x,0) &= \phi_0(x), \quad \partial_t U(x,0) = \psi_0(x), \quad x \in I, \\
U(a,t) &= U_a(t), \quad U(b,t) = U_b(t), \quad t \in (0,T],
\end{aligned}
\]

where \(1 < \beta < 2\), \(\mu > 0\), and \(cD^\beta_0\) is the \(\beta\)th-order Caputo derivative operator defined by

\[
cD^\beta_0 U(x,t) = D_{0,t}^{(2-\beta)} \left[ \partial^2_x U(x,t) \right] = \frac{1}{T(2-\beta)} \int_0^t (t-s)^{1-\beta} \partial^2_x U(x,s) \, ds,
\]

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in which \( D^\gamma_{0,t} \) is the fractional integral operator defined by, see for example \[28\].

\[
D^\gamma_{0,t} U(x,t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} U(x,s) \, ds, \quad \gamma > 0.
\] (3)

Temporal finite difference schemes for the time-fractional diffusion-wave equation \[1\] and its equivalent form are mostly of first-order, \((3 - \beta)\)-order and second-order convergence in time. First-order schemes can be based on either \(L2\) method and its generalization, see e.g. \[25\] or the first- and second-order fractional backward difference methods, see e.g. \[10\] \[32\]. The \((3 - \beta)\)-th order time discretization techniques are based on the \(L1\) method \[31\], see also \[8\] \[15\] \[29\] \[37\]. Second-order schemes are either generalized Crank–Nicolson schemes \[23\] or based on fractional backward difference methods, see \[4\] \[7\] \[13\] \[34\]. In \[14\], the \(\beta\)-th order method was derived based on the Crank–Nicolson scheme and the second-order fractional backward difference method. There have existed other related works on the time-fractional diffusion equations, see e.g. \[1\] \[4\] \[7\] \[12\] \[17\] \[22\] \[27\].

In this paper, we adopt different time discretization approaches to the time-fractional diffusion-wave equation of the form \[1\], which yields three schemes with second-order accuracy both in time and space. The key of our discretization is that we use three different second-order generating functions for the time discretization of \[1\] which are different from those in all the aforementioned works. Our first scheme for time discretization is based on the second-order fractional trapezoidal rule as that used in \[35\]. The second and third schemes are based on the second-order generalized Newton-Gregory formula in time and its modification. With the second-order central difference method in space discretization, we can prove that the first and the third schemes are stable and that the second one is conditionally stable through the Fourier analysis, and all the schemes are convergent of order two both in time and space.

One important feature of our schemes is that these schemes respectively reduce to classical difference schemes when \(\beta \to 2\) while the second-order schemes in \[4\] \[7\] \[13\] \[23\] do not. Specifically, when \(\beta \to 2\), our discretization in time for \[1\] (with \(f(x,t) = 0\)) is reduced to

\[
\frac{1}{\tau^2} \left( \frac{9}{4} u^{n+1}_j - 6u^n_j + \frac{11}{2} u^{n-1}_j - 2u^{n-2}_j + \frac{1}{4} u^{n-3}_j \right) = \mu \delta_x^2 u^{n+1}_j, \quad n \geq 3.
\]

The second-order schemes in \[4\] \[23\] do not reduce to the central difference scheme in time when \(\beta = 2\). Though the discretization of time derivative in \[32\] can lead to the central difference scheme in time for \(1 < \beta \leq 2\), the method in \[32\] has only first-order accuracy in time for \(1 < \beta < 2\).

Spatial discretization for \[1\] can be finite difference methods, see e.g. \[8\] \[15\] \[29\] \[31\] \[37\] and finite element methods, see e.g. \[13\] \[24\]. Here, we consider finite difference methods while the finite element methods can be also applied.

The remainder of this paper is outlined as follows. In Section 2, we present a fully discrete finite difference scheme for \[1\] and establish the analysis of the stability, consistency, and convergence. In Section 3, we propose two more fully schemes for \[1\], one is conditionally stable and the other is stable. We present numerical schemes for the time-fractional diffusion-wave equation with linear advection-reaction term in Section 4. Numerical experiments are provided in Section 5 before the conclusion in the last section.
2 The finite difference scheme based on the fractional trapezoidal rule

In this section, we first present the time discretization for (1) based on the fractional trapezoidal rule. With space discretization by the central finite difference, we prove the stability, consistency, and convergence of the fully discrete scheme.

2.1 The finite difference scheme in time

Let \( \tau \) be the time step size and \( n_T \) be a positive integer with \( \tau = T/n_T \) and \( t_n = n\tau \) for \( n = 0, 1, ..., n_T \). For the function \( y(t) \in C([0, T]) \), denote by \( y^n = y(t_n) \). Denote by \( h \) as the space step size with \( h = (b-a)/N \), where \( N \) is a positive integer. The space grid point \( x_j \) is defined as \( x_j = a + jh \), \( j = 0, 1, ..., N \). For the function \( u(x,t) \in C(\bar{I}; [0, T]) \), we also denote by \( u^0 = u^0(\cdot) = u(\cdot, t_0) \) and \( u^n = u(x_j, t_n) \). For simplicity, we also introduce the following notations

\[
\delta_t^\beta u^n_j = \frac{u^n_{j+1} - 2u^n_j + u^n_{j-1}}{h^2}, \quad \delta_t u^n_j = \frac{u^n_{j+1} - u^n_{j-1}}{2h}.
\]

We discretize the time of (1) through the fractional linear multistep methods (FLMMs) developed by Lubich [18]. The \( p \)-th order FLMMs for \( D^\beta u(t) \) are given by

\[
D^\beta u(t)|_{t=n\tau} = \tau^p \sum_{k=0}^{\infty} w_{n,k}^{(p)} u(t_k) + O(\tau^p), \quad (4)
\]

where \( \{w_{n,k}^{(p)}\} \) can be the coefficients of the Taylor expansions of the following generating functions

\[
w_{p}(z) = \left[ \sum_{j=1}^{p} \frac{1}{j} (1 - z)^j \right]^\beta, \quad p = 1, 2, ..., 6, \quad (5)
\]

\[
w_{p}(z) = (1 - z)^\beta \left\{ \gamma_0 + \gamma_1 (1 - z) + \gamma_2 (1 - z)^2 + ... + \gamma_p (1 - z)^{p-1} \right\}, \quad (6)
\]

\[
w_{p}(z) = \left( \frac{1 + z}{1 - z} \right)^\beta, \quad (7)
\]

in which \( \{\gamma_k\} \) in (6) satisfy the following relation

\[
\left( \frac{\ln z}{z - 1} \right)^\beta = \sum_{k=0}^{\infty} \gamma_k (1 - z)^k, \quad \gamma_0 = 1, \gamma_1 = -\frac{\beta}{2}.
\]

The starting weights \( \{w_{n,k}^{(p)}\} \) are chosen such that the asymptotic behavior of the function \( u(t) \) near the origin \( (t = 0) \) are taken into account [6]. One way to determine \( \{w_{n,k}^{(p)}\} \) for the sufficiently smooth function \( u(t) \) is given as follows [18]

\[
\sum_{k=1}^{\infty} w_{n,k}^{(p)} k^q = \frac{\Gamma(q+1)}{\Gamma(q+\beta+1)} t^{\beta q - 1} - \sum_{k=1}^{n} w_{n,k}^{(p)} k^q, \quad q = 0, 1, ..., p - 1. \quad (8)
\]

The FLMM (4) (also called the fractional trapezoidal rule) has second-order accuracy if the generating function (7) is used. In this section, we will discretize the time of the fractional wave equation (1) with the generating function (4).
We first consider the following fractional ordinary differential equation (FODE)
\[ cD_{0}^\beta y(t) = \mu y(t) + g(t), \quad y(0) = y_0, y'(0) = y'_0, \quad 1 < \beta < 2. \tag{9} \]
We also assume that \( y(t) \) is sufficiently smooth. Let \( \hat{\varphi}(t) = y(0) + y'(0)t \). Then the above FODE is equivalent to the following Volterra integral equation \[ y(t) - \hat{\varphi}(t) = \mu D_{0}^\beta y(t) + D_{0}^\beta g(t) = \mu D_{0}^\beta (y(t) - \hat{\varphi}(t)) + \mu D_{0}^\beta \hat{\varphi}(t) + D_{0}^\beta g(t). \tag{10} \]

The kernel \( \frac{1}{\Gamma(\beta)} (t - \tau)^{\beta - 1} \) (see also Eq. (3) in above Eq. (10) has no singularity for \( \beta \geq 1 \). There exist several difference methods to solve (10) and the error estimates can be proved by the generalized Gronwall inequality for any \( \beta > 0 \), see e.g. [14,16]. For \( \beta \geq 1 \), the error estimate can be also proved by the classical Gronwall inequality due to the nonsingularity of the kernel, see e.g. [16]. Here, we use another way to discretize (10) that will be used to discretize the time of (1).

Before discretizing (10), we introduce three lemmas.

**Lemma 1** ([18,35]) If \( y(t) = t^\nu, \nu \geq 0, \beta > 0, \) then
\[ D_{0}^\beta y(t) \big|_{t = t_k} = \tau^\beta \sum_{k=0}^{n} \omega^{(\beta)}_k y(t_k) + O(t^{\beta - 1} \tau^\nu) + O(t^{\beta - 1} \tau^{\nu + 1}), \tag{11} \]
where \( \{ \omega^{(\beta)}_k \} \) can be the coefficients of the Taylor series of the generating functions defined as (5)–(7), and \( p = 2 \) if (7) is used.

**Lemma 2** ([35]) Denote by
\[ y_n = \sum_{k=0}^{n} \omega^{(\beta)}_k G_k, \tag{12} \]
where \( G_k (k = 0, 1, \ldots) \) is any number and \( \{ \omega^{(\beta)}_k \} \) are the coefficients of Taylor expansions of the generating functions \( w^{(\beta)}(z) \) defined by Eq. (5), Eq. (6), or Eq. (7). Then, Eq. (12) is equivalent to the following form
\[ \sum_{k=0}^{n} \alpha_k y_{n-k} = \sum_{k=0}^{n} \theta_k G_k \tag{13} \]
where \( \alpha_k \) and \( \theta_k \) are the coefficients of Taylor expansions of \( \alpha(z) \) and \( \theta(z) \), respectively, with \( w^{(\beta)}(z) = \theta(z)/\alpha(z) \).

**Lemma 3** ([36]) Suppose that \( \beta > 0 \). Let \( \{ \alpha_k \} \) be the coefficients of Taylor expansions of the generating function \( \alpha(z) = (1 - z)^\beta \), i.e., \( \alpha_k = (-1)^k \frac{\beta!}{k!} \). Then
\[ \sum_{k=1}^{n} \alpha_{n-k} k^{\gamma-1} = O(n^{\gamma-1-\beta}) + O(n^{\gamma-1}), \quad \gamma \in \mathbb{R}, \quad \gamma \neq 0, -1, -2, \ldots. \]

Now, we are in a position to discretize (10). If \( y(t) \) is smooth enough, then we have \( y(t) - \hat{\varphi}(t) = \frac{1}{\nu} y''(0) t^2 + D_{0}^\beta y''(t) \). Therefore, by Lemma 1 we can have the following discretization for \( D_{0}^\beta (y(t) - \hat{\varphi}(t)) \big|_{t = t_k} \) as
\[ D_{0}^\beta (y(t) - \hat{\varphi}(t)) \big|_{t = t_k} = \tau^\beta \sum_{k=0}^{n} \omega^{(\beta)}_k (y(t_k) - \hat{\varphi}(t_k)) - R^\beta, \tag{14} \]
where \(\{a_j^{(0)}\}\) are the coefficients of Taylor expansions of the generating function (7), and the truncation error \(\tilde{R}^n\) satisfies \(\tilde{R}^n = O(l^{(0)}_t \tau^2) + O(l^{(0)}_\sigma \tau^3) + O(\tau^{2+\beta}) + O(\tau^{2+\beta})\), where Lemma 1 with \(\nu = p (p \geq 2)\) is used to obtain \(\tilde{R}^n\).

Hence, Eq. (10) has the following discretization

\[
y^n = \tilde{y}^n = \tilde{\mu}^n t^n \sum_{k=0}^{n} a_{n-k} (y^k - \tilde{y}^k) + \tilde{\mu} \left[ D^{\alpha}_{y} \tilde{\varphi}(t) \right]_{t=k} + \left[ D^{\alpha}_{x} g(t) \right]_{t=k} + \tilde{R}^n,
\]

(15)

Applying Lemma 2 yields the equivalent form of (15) as

\[
\sum_{k=0}^{n} \alpha_{n-k} (y^k - \tilde{y}^k) = \mu t^n \sum_{k=0}^{n} \theta_{n-k} (y^k - \tilde{y}^k) + \sum_{k=0}^{n} \alpha_{n-k} \left\{ \mu \left[ D^{\alpha}_{y} \tilde{\varphi}(t) \right]_{t=k} + \left[ D^{\alpha}_{x} g(t) \right]_{t=k} + \tilde{R}^n \right\},
\]

(16)

where \(\alpha(z)\) and \(\theta(z)\) in Lemma 2 can be chosen as \(\alpha(z) = (1-z)^\beta = \sum_{k=0}^{\infty} \alpha_k z^k = \sum_{k=0}^{\infty} (-1)^k \beta C_{\alpha}^k\), and \(\theta(z) = \frac{(1-z)^\beta}{1-z} = \sum_{k=0}^{\infty} \theta_k z^k = \beta \sum_{k=0}^{\infty} \left( \frac{\beta}{2} \right)^k \). One can also find that \(\theta_k = 2^{-\beta} (-1)^\beta \alpha_k\).

Rewriting (16) into the following form

\[
\sum_{k=0}^{n} \alpha_{n-k} (y^k - \tilde{y}^k) = \mu t^n \sum_{k=0}^{n} \theta_{n-k} (y^k - \tilde{y}^k) + \sum_{k=0}^{n} \alpha_k \tilde{y}^k + \sum_{k=0}^{n} \alpha_{n-k} G^k + R^n,
\]

(17)

where \(G^k = \left[ D^{\alpha}_{y} g(t) \right]_{t=k}, \tilde{y}^k = \left[ D^{\alpha}_{y} \tilde{\varphi}(t) \right]_{t=k} = \frac{\mu t^{(0)} t^{(2)} t_{\beta-1}^k}{(\beta+1)(\beta+2)}, \) and \(R^n = \sum_{k=1}^{n} \alpha_{n-k} \tilde{R}^k\).

Next, we analyse the truncation error \(R^n = \sum_{k=0}^{\infty} \alpha_{n-k} \tilde{R}^k\) defined in (17) when the generating function (7) is used. We can obtain a bound of \(R^n\) in (17) as follows

\[
R^n = \sum_{k=0}^{n} \alpha_{n-k} \tilde{R}^k = \sum_{k=0}^{n} \alpha_{n-k} (O(t^{\alpha(2,1)} \tau^{2+\beta}) + O(t^{(2,1)} \tau^{2+\beta}) + O(\tau^{2+\beta})) = O(\tau^{2+\beta}),
\]

(18)

where we have used Lemma 3.

Assume that \(U(x, t)\) is sufficiently smooth in time. From (17), we can obtain the time discretization of the wave equation (1) as follows.

- **Time discretization 1**: Applying the time discretization (17) with the generating function (7) to Eq. (1) yields

\[
\sum_{k=0}^{n} \alpha_{n-k} (U^k - \tilde{y}^k) = \mu t^n \sum_{k=0}^{n} \theta_{n-k} (\partial_t^2 U^k - \tilde{y}^k) + \sum_{k=0}^{n} \alpha_{n-k} \tilde{y}^k + \sum_{k=0}^{n} \alpha_{n-k} F^k + R^n,
\]

(19)

where \(\alpha_k = (-1)^k \frac{\beta}{2} \), \(\theta_k = 2^{-\beta} (-1)^k \alpha_k\), \(\varphi(x, t) = U(x, 0) + \partial_t U(x, 0) = \phi_0(x) + \psi_0(x)t\), \(\tilde{y}^k = \left[ D^{\alpha}_{y} \tilde{\varphi}(x, t) \right]_{t=k}, \tilde{y}^k = \left[ D^{\alpha}_{y} \varphi(x, t) \right]_{t=k} = \frac{\phi_0(x) + \psi_0(x)t_{\beta-1}^k}{(\beta+1)(\beta+2)}, \) \(F^k = \left[ D^{\alpha}_{y} f(x, t) \right]_{t=k}\), and \(R^n\) is the discretization error in time satisfying \(|R^n| \leq C \tau^{2+\beta}\).

Next, we present the fully discrete approximation for equation (1). From the time discretization (15) with the second-order central difference discretization of the space derivative operator, we present the corresponding fully discrete approximations for (1) as follows.
2.2 Stability, consistency, and convergence

This subsection mainly focuses on the stability, consistency, and convergence of the scheme (20). We first rewrite the scheme (20) into the following form

\[
\begin{align*}
\sum_{k=0}^{n} \alpha_{n-k}(u_k^j - \varphi_j^k) &= \mu \tau^p \sum_{k=0}^{n} \theta_{n-k} \left( \delta^2 u_k^j - \delta^2 \varphi_j^k \right) + \mu \sum_{k=0}^{n} \alpha_{n-k} \delta_\tau \Phi_j^k + \sum_{k=0}^{n} \alpha_{n-k} F_j^k, \\
u_0^j &= U_a(t_k), \quad u_N^k = U_b(t_k), \quad k = 0, 1, ..., n_T. \\
u_0^j &= \phi_0(x_j),
\end{align*}
\]

where \( \alpha_k = (-1)^k \frac{\beta^k}{k!}, \ \theta_k = 2^{-\beta}(-1)^k \alpha_k, \ \Phi_j^k = \left[ D_{\Omega j}^{\beta} \varphi(x_j, t) \right]_{t=t_k} = \frac{\phi_0(x_j)}{\tau^{\beta+1}} + \frac{\phi_0(x_j) \varphi_j^k}{\tau^{\beta+2}} \). \( \varphi_j^k = \phi_0(x_j) + \phi_0(x_j) \mu_k, \) and \( F_j^k = \left[ D_{\Omega j}^{\beta} f(x_j, t) \right]_{t=t_k} \).

**Remark 1** If \( \beta \to 2 \), then the scheme (20) reduces to the unconditionally stable central difference scheme of second-order accuracy both in time and space, i.e.,

\[
\frac{\nu_{j+1}^n - 2 \nu_j^n + \nu_{j-1}^n}{\tau^2} = \frac{\mu}{4} \left( \delta^2 \nu_j^n - 2 \delta^2 \nu_j^{n-1} + \delta^2 \nu_j^{n-2} \right) + \left( F_{j+1}^n - 2 F_j^n + F_{j-1}^{n-1} \right), \quad n \geq 1.
\]

**Calculation of \( F^n \):** In (20), we do not illustrate how to calculate \( \left[ D_{\Omega j}^{\beta} f(x, t) \right]_{t=t_k} \) in \( F^n \). If \( U(x, t) \) is sufficiently smooth in time, then \( f(x, t) - f(x, 0) \) has the form \( f(x, t) - f(x, 0) = t^{2-\beta} f_1(x, t) + t f_2(x, t) \), where \( f_1(x, t) \) and \( f_2(x, t) \) are sufficiently smooth in time. Hence, we can use the following second-order formula to approximate \( \left[ D_{\Omega j}^{\beta} f(x, t) \right]_{t=t_k} \)

\[
\left[ D_{\Omega j}^{\beta} f(x, t) \right]_{t=t_k} = \left[ D_{\Omega j}^{\beta} (f(x, t) - f(x, 0)) \right]_{t=t_k} + \frac{\tau^\beta}{f(1 + \beta)} f(x, 0)
\]

\[
= \tau^\beta \sum_{k=0}^{n} \left[ w_{n-k}^{(\beta)} (f(x, t_k) - f(x, 0)) \right]_{t=t_k} + \frac{\tau^\beta}{f(1 + \beta)} f(x, 0) + O(\tau^2),
\]

where \( \{w_{n-k}^{(\beta)}\} \) are the coefficients of the Taylor expansions of the generating function (6). The coefficients \( \{w_{n-1}^{(\beta)}\} \) and \( \{w_{n-2}^{(\beta)}\} \) are chosen such that (21) is exact for \( f(x, t) - f(x, 0) = t^{2-\beta}, t \).

Hence, one has

\[
\nu_{n-1}^{(\beta)} + 2 \nu_{n-2}^{(\beta)} = \frac{\Gamma(q + 1)}{\Gamma(q + \beta + 1)} n^{q+\beta} - \sum_{k=1}^{n} \omega_{n-k}^{(\beta)} k^q, \quad q \in [2 - \beta, 1].
\]  

(22)

2.2.1 Stability, consistency, and convergence
in which
\[
A_n = \frac{1}{\Gamma(\beta + 1)} \sum_{k=0}^{n} \alpha_{n-k} k^{\beta} - \sum_{k=0}^{n} \theta_{n-k}
\]
\[
B_n = \frac{1}{\Gamma(\beta + 2)} \sum_{k=0}^{n} \alpha_{n-k} k^{\beta+1} - \sum_{k=0}^{n} \theta_{n-k}.
\] (25)

Consider (11) with \(y(t) = t^\alpha, \alpha \geq 0\), one has
\[
\frac{\Gamma(v + 1) t^{\alpha v}}{\Gamma(\beta + v + 1)} = \left[ D_{0+}^{\beta} \right]_{t=0}^t = t^\beta \sum_{k=0}^{n} \omega_{n-k}^{(\beta)} (t)^{\alpha v + O(t^{v-2} t^2) + O(t^{v-1})}.
\] (26)

The above equation implies
\[
\frac{\Gamma(v + 1) t^{\alpha v}}{\Gamma(\beta + v + 1)} = \sum_{k=0}^{n} \omega_{n-k}^{(\beta)} t^k + O(t^{v-2} t^2) + O(t^{v-1}).
\] (27)

Applying Lemma 2 yields
\[
\frac{\Gamma(v + 1) t^{\alpha v}}{\Gamma(\beta + v + 1)} \sum_{k=0}^{n} \alpha_{n-k} k^{\alpha v} = \sum_{k=0}^{n} \theta_{n-k} = \sum_{k=0}^{n} \alpha_{n-k} \left( O(k^{v-2} t^2) + O(k^{v-1}) \right).
\] (28)

By Lemma 3 and (25) with \(v = 0, 1\), one has
\[
A_n = O(n^{-1}), \quad B_n = O(n^{-1}).
\] (29)

Next, we prove the stability of the scheme (20) through Fourier analysis.

**Theorem 1** The finite difference scheme (20) is stable.

**Proof** Suppose that \(u_t^j = \rho^j \exp(i j \sigma h), \varphi^2 = -1, \psi_0(x_0) = d_0 \exp(i j \sigma h), \) and \(F^j = 0\). Then we have
\[
\exp(i j \sigma h) \sum_{k=0}^{n} \alpha_{n-k} (\rho^j - \rho^0 - d_0 h)
\]
\[
= \frac{\mu h^2}{2} \sum_{k=0}^{n} \theta_{n-k} (\rho^j) + A_d \rho^0 + B_d d_0 \tau \left( \exp(i(j + 1) \sigma h) - 2 \exp(i j \sigma h) + \exp(i(j - 1) \sigma h) \right)
\]
\[
= \exp(i j \sigma h) \left( \frac{\mu h^2}{2} \sin^2 \left( \frac{\sigma h}{2} \right) \right) \sum_{k=0}^{n} \theta_{n-k} \rho^j + A_d \rho^0 + B_d d_0 \tau.
\]

where \(\alpha_k = (-1)^k \theta_k\) and \(\theta_k = 2^{-\beta} (-1)^k \alpha_k\). Eliminating \(\exp(i j \sigma h)\) from the above equation leads to
\[
\sum_{k=0}^{n} \alpha_{n-k} (\rho^j - \rho^0 - d_0 h) = S \sum_{k=0}^{n} \theta_{n-k} \rho^j + A_d \rho^0 + B_d d_0 \tau,
\] (30)

where
\[
S^* = \frac{4\mu h^2}{\beta^2} \sin^2 \left( \frac{\sigma h}{2} \right).
\]

Now, we need to investigate the stability of the difference equation (30). Let
\[
A(z) = \sum_{k=0}^{\infty} A_k z^k, \quad B(z) = \sum_{k=0}^{\infty} B_k z^k, \quad \rho(z) = \sum_{k=0}^{\infty} \rho^k z^k, \quad |z| \leq 1.
\]
From (30), one can obtain
\[
\sum_{n=0}^{\infty} \sum_{k=0}^{n} a_{n-k}(\rho^k - \rho^0 - d_0 \tau k) \big| z^n = S^* \sum_{n=0}^{\infty} A_n \rho^0 + B_n \rho d_0 \tau + \sum_{k=0}^{n} \theta_{n-k} \rho^k \big| z^n,
\]
which implies
\[
\alpha(z) \left( \frac{\rho^0}{1 - z} - d_0 \tau K(z) \right) = S^* \left( \theta(z) \rho(z) + A(z) \rho^0 + B(z) d_0 \tau \right),
\]
where
\[
K(z) = \sum_{k=0}^{\infty} k z^k = \frac{z}{(1 - z)^2}.
\]
Hence,
\[
\rho(z) = \frac{\alpha(z) \left( (1 - z)^{-1} \rho^0 + K(z) d_0 \tau \right) - S^* \left( A(z) \rho^0 + B(z) d_0 \tau \right)}{\alpha(z) - S^* \theta(z)}.
\]
Denote by \(\alpha(z) \left( (1 - z)^{-1} \rho^0 + d_0 \tau K(z) \right) - S^* \left( A(z) \rho^0 + B(z) d_0 \tau \right) = \sum_{k=0}^{\infty} \delta_k z^k\). From (29), (31), (32), and \(\alpha(z) = (1 - z)^{\rho}\), we can derive that \(g_n \to 0\) as \(n \to \infty\) for any given \(S^*\) (see the first part in the proof of Lemma 3.5 in [15] and Eq. (2.8) in [19]), which implies \(\rho^0 \to 0\) as \(n \to \infty\) for any given \(S^*\) (see also Eq. (2.8) in [19]). If there exists a number \(\sigma\) such that \(\sin^2 \left( \frac{\pi \rho}{\alpha(z)} \right) = 0\), i.e., \(S^* = 0\), then we can also obtain \(\rho^0 = \rho^0 + dt_k\) from (30). Like Theorem 2.1 in [19], we can obtain the following region
\[
\mathbb{S} = \mathbb{C} \setminus \left\{ \frac{\alpha(z)}{\theta(z)} : |z| \leq 1 \right\}
\]
such that \(\rho^0 (0 \leq n \leq nt)\) is bounded for any \(\tau\) and \(h\) and given \(T\). That is to say, for any \(S^* \in \mathbb{S}\), the difference equation (30) is stable. Since \(\alpha(z) = (1 - z)^{\rho} \geq 0\) and \(\theta(z) = 2^{\rho}(1 + z)^{\rho} \geq 0\) for all \(z \in [-1, 1]\), the value of \(\frac{\alpha(z)}{\theta(z)}\) is always nonnegative. Therefore, the stability region \(\mathbb{S}\) contains the whole of the left-half plane (Of course, it contains the negative semi axis), which implies that the difference relation (30) is stable for every \(S^* < 0\), i.e., \(\rho^0\) is bounded as \(n \to \infty\). Hence, the scheme (20) is stable for any \(\tau^2/h^2\), which completes the proof.

Next, we investigate the consistency of the scheme (20). Letting \(x = x_j\) in (19) and applying the central difference method to the space derivative, we can derive
\[
\sum_{k=0}^{n} a_{n-k} (U_j^k - \varphi^k) = \mu \tau^2 \sum_{k=0}^{n} \theta_{n-k} (\delta_x^2 U_j^k - \delta_x^2 \varphi^k) + \mu \sum_{k=0}^{n} \alpha_{n-k} \delta_x^2 \varphi_j^k + \sum_{k=0}^{n} \alpha_{n-k} F_j^k + R_j^0,
\]
where \(R_j^0 = O(\tau^3 \tau^2 + h^2)\). In order to prove that the scheme (20) is consistent of order \(O(\tau^2 + h^2)\), we just need to prove the following result
\[
\lim_{\tau \to 0, h \to 0} \mathcal{P}(\tau, h) = \psi D_{tt} U(x_j, t) - \mu \frac{\partial^3 U(x_j, t)}{\partial t^3} f(x_j, t) = 0, \quad n \tau = t,
\]
where
\[
\mathcal{P}(\tau, h) = \frac{1}{\tau^2} \left\{ \sum_{k=0}^{n} a_{n-k} (U_j^k - \varphi^k) - \mu \tau^2 \sum_{k=0}^{n} \theta_{n-k} (\delta_x^2 U_j^k - \delta_x^2 \varphi^k) \right. \\
+ \left. \mu \sum_{k=0}^{n} \alpha_{n-k} \delta_x^2 \varphi_j^k + \sum_{k=0}^{n} \alpha_{n-k} F_j^k + R_j^0 \right\}, \quad n \tau = t.
\]
According to the Grünwald–Letnikov formula \[28\], we have
\[
\lim_{{\tau \to 0, h \to 0}} \frac{1}{\tau^\beta} \sum_{k=0}^{n} a_n \alpha_k \tau^\beta (U_j - \varphi_j) = RL D_0^\beta_{\tau, h} (U(x_j, t) - \varphi(x_j, t)) = C D_0^\beta_{\tau, h} U(x_j, t),
\]
where
\[
\lim_{{\tau \to 0, h \to 0}} \frac{1}{\tau^\beta} \sum_{k=0}^{n} a_n \alpha_k F_j = RL D_0^\beta_{\tau, h} f(x_j, t) = f(x_j, t),
\]
(36)

where \( RL D_0^\beta_{\tau, h} \) is the Riemann–Liouville fractional derivative operator, see e.g. \[28\].

From \[24\], \[45\], and \[29\], we derive
\[
\lim_{{\tau \to 0, h \to 0}} \frac{1}{\tau^\beta} \left( \tau^\beta \sum_{k=0}^{n} \alpha_n \alpha_k \tau^\beta (U_j - \varphi_j) - \sum_{k=0}^{n} a_n \alpha_k \tau^\beta \varphi_j \right) = 0.
\]
(37)

For \( \sum_{k=0}^{n} \alpha_n \alpha_k \tau^\beta U_j \), we have
\[
\lim_{{\tau \to 0, h \to 0}} \frac{1}{\tau^\beta} \sum_{k=0}^{n} \alpha_n \alpha_k \tau^\beta U_j = \lim_{{\tau \to 0}} \sum_{k=0}^{n} \alpha_n \alpha_k \tau^\beta U(x_j, t_k)
\]
\[
= \lim_{{\tau \to 0}} \sum_{k=0}^{n} \alpha_n \alpha_k \tau^\beta \left( U(x_j, t) + (k - n) \tau \partial_1 U(x_j, \xi(k, n)) \right) \quad (0 < \xi(k, n) < t)
\]
\[
= \alpha_n \alpha_k \tau^\beta U(x_j, t) + \lim_{{\tau \to 0}} \sum_{k=0}^{n} \alpha_n \alpha_k \tau^\beta \partial_1 U(x_j, \xi(k, n)) = \alpha_n \alpha_k \tau^\beta U(x_j, t),
\]
(38)

where we have used \( \sum_{k=0}^{n} \alpha_n \to 1 \) as \( n \to \infty \) and
\[
\lim_{{\tau \to 0}} \left| \sum_{k=0}^{n} \alpha_n \alpha_k \tau^\beta \partial_1 U(x_j, \xi(k, n)) \right| \quad (t = n\tau)
\]
\[
\leq C_1 \| \lim_{{\tau \to 0}} \sum_{k=0}^{n} \alpha_n \alpha_k \tau^\beta \partial_1 U(x_j, \xi(k, n)) \| \leq C_2 \| \lim_{{\tau \to 0}} \sum_{k=0}^{n} (n - k)^\beta n^{-1} = 0,
\]
(39)
in which \( \theta_j = 2^\beta \left( \frac{1}{\tau^\beta} \right) = O(k^\beta \xi) \) and \( \sum_{k=0}^{n} (n - k)^\beta \leq C_3 \) have been used, \( C_1, C_2, C_3 \) are positive constants independent of \( n, \tau \) and \( h \). Combining \( \Box \) yields \( \Box \).

We now give the following consistency theorem.

**Theorem 2** Suppose that \( U(x, t) \) is the solution to \( \Box \). \( U \in C^2(C^4(I); [0, T]) \). The finite difference scheme \( \Box \) is consistent of order \( O(\tau^2 + h^\beta) \).

Next, we discuss the convergence for the scheme \( \Box \). Denote by \( e_n^0 = U(x_j, t_n) - u_n^0 \).

From \( \Box \) and \( \Box \), we obtain the error equation of \( \Box \) as follows
\[
\sum_{k=0}^{n} \alpha_n \alpha_k e_j^0 = \mu \tau^\beta \sum_{k=0}^{n} \alpha_n \alpha_k \tau^\beta e_j^0 + R_j^0,
\]
(40)

where \( R_j^0 = O(\tau^2 + h^\beta) \) and \( R_j^0 \) is bounded.

Using the identity \( R_j^0 = \sum_{k=0}^{n} \alpha_n + \sum_{k=0}^{n} \alpha_k \cdot R_j^0 \) and Lemma \[2\] or applying the similar reasoning as Lemma 3.4 in \[25\], we can derive the equivalent form of \( \Box \) as
\[
e_j^0 = \mu \tau^\beta \sum_{k=0}^{n} \omega_{n-k}^\beta \delta_{x}^\beta e_j^0 + G_j^0,
\]
(41)
where \( G_j^\alpha = \sum_{n=0}^{\infty} \tilde{a}_{n-k} R_j^\alpha \) is the coefficient of the Taylor expansion of the generating function \( \tilde{a}(z) = (z(\sigma(z)))^{-1} = (1 - z)^{-\alpha}, \) \( \omega_{j}^{(0)} \) is the coefficient of the Taylor expansion of the generating function \( \theta(z)/\alpha(z). \) Let \( e_j^n = e^n \exp(i j \tau h) \) and \( G_j^\alpha = e_j^n \exp(i j \tau h), \) \( e_j^n \) is bounded. Similar to (30), we can obtain from (41)

\[
e^n = S^* \sum_{k=0}^{n} \omega_{n-k}^{(0)} e^n + \eta^n (\tau^2 + h^2), \quad S^* = -\frac{4\mu \tau h}{h^2} \sin^2 \left( \frac{\sigma h}{2} \right), \tag{42}
\]

which yields

\[
e(z) = S^* e(z) \theta(z)/\alpha(z) + \eta(z) (\tau^2 + h^2), \tag{43}
\]

where

\[
e(z) = \sum_{k=0}^{\infty} e^k \alpha_k, \quad \eta(z) = \sum_{k=0}^{\infty} \eta^k \beta_k.
\]

From (43), we have

\[
e(z) = \frac{\eta(z)(\tau^2 + h^2)}{1 - S^* \theta(z)/\alpha(z)} = \frac{(1 - z)^{\beta} \eta(z)(\tau^2 + h^2)}{(1 - z)^{\beta} - S^* 2^{\beta} (1 + z)^{\beta}} \tag{44}
\]

Denote by \( D(z) = (1 - z)^{\beta} - S^* 2^{\beta} (1 + z)^{\beta} = \sum_{k=0}^{\infty} d_k z^k. \) Then we have \( d_k = O(k^{-\beta - 1}), 1 < \beta < 2. \) So the sequence \( \{d_k\} \) is in \( \ell^1. \) From Theorem 1, we know that \( D(z) \neq 0 \) for all \( S^* \in \mathbb{S} \) and \( |\alpha| \leq 1. \) Denote by \( 1/D(z) = \sum_{k=0}^{\infty} \tilde{d}_k z^k. \) Then the sequence \( \{\tilde{d}_k\} \) is also in \( \ell^1, \) see Eq. (2.5) in 12. Let \( (1 - z)^{\beta} \eta(z) = \sum_{k=0}^{\infty} c_k z^k. \) Then it is easy to obtain that \( |c_k| = |\sum_{k=0}^{\infty} \alpha_k \eta_{n-k}| \leq \max_{0 \leq k \leq n} \eta_k \sum_{k=0}^{\infty} |\alpha_k| \leq 2^\beta \max_{0 \leq k \leq n} |\eta_k|. \) From (44), one has

\[
|e^n| = (\tau^2 + h^2) \left\| \sum_{k=0}^{n} \tilde{d}_k c_{n-k} \right\| \leq 2^\beta \max_{0 \leq k \leq n} |\eta_k| (\tau^2 + h^2) \sum_{k=0}^{n} |d_k| \leq C(\tau^2 + h^2). \tag{45}
\]

If \( S^* = 0, \) then we directly have \( |e^n| \leq C(\tau^2 + h^2) \) from (42). Now, we give the following convergence theorem.

**Theorem 3.** Let \( U(x,t) \) be the solution to (1) and \( u_j^n (j = 0, 1, ..., N, n = 0, 1, ..., n_T) \) be the solutions to (20). Then there exists a positive constant \( C \) independent of \( n, \tau, \) and \( h, \) such that

\[
\|e^n\| \leq C(\tau^2 + h^2),
\]

where \( e^n = (e_0^n, e_1^n, ..., e_N^n)^T, e_j^n = U(x_j, t_n) - u_j^n, \) and \( ||e^n|| = \sqrt{h \sum_{j=0}^{N-1} (e_j^n)^2}. \)

**Proof.** From \( e_j^n = e^n \exp(i j \tau h) \) and (45), one has \( |e_j^n| \leq C(\tau^2 + h^2). \) So

\[
||e^n||^2 = h \sum_{j=0}^{N-1} (e_j^n)^2 \leq C(\tau^2 + h^2)^2,
\]

which completes the proof.
3 The finite difference schemes based on the generalized Newton-Gregory formula and its modification

In this section, we construct another second-order difference scheme for (1) with the help of the generating function \( w^{(p)}(z) = \frac{z^{p-1}}{(1-z)^{p}} \), see (5) with \( p = 2 \). Similar to (19), we can derive the following time discretization with the help of the generating function \( w^{(p)}(z) = \frac{z^{p-1}}{(1-z)^{p}} \).

\[
\sum_{k=0}^{n} \alpha_{n-k}(U^k - \phi^k) = \mu \rho \sum_{k=0}^{n} \theta_{n-k} \left( \frac{\partial^2}{\partial x^2} U^k - \frac{\partial^2}{\partial x^2} \phi^k \right) + \mu \sum_{k=0}^{n} \alpha_{n-k} \frac{\partial^2}{\partial x^2} \phi^k + \sum_{k=0}^{n} \alpha_{n-k} F^k + R^n,
\]

(46)

where \( \alpha_k = (-1)^k \left( \frac{\beta}{2} \right) \), \( \theta_0 = 1 - \frac{\beta}{2}, \theta_1 = \frac{\beta}{2}, \theta_k = 0 (k \geq 2), \phi(x, t) = \phi_0(x) + \psi_0(x)t, \phi^k = \left[ D^\beta_{0,t} \phi(x, t) \right]_{t=a_k}, F^k = \left[ D^\beta_{0,t} f(x, t) \right]_{t=a_k}, \) and \( R^n \) is the discretization error in time satisfying \( |R^n| \leq C \tau^{p+1} \).

From (46), we can derive the fully discrete finite difference scheme for (1) as: Find \( u_j^r \) for \( j = 1, 2, ..., N-1, n = 1, 2, ..., n_T \), such that

\[
\left\{ \begin{align*}
\sum_{k=0}^{n} \alpha_{n-k}(u_j^k - \phi^k) &= \mu \rho \sum_{k=0}^{n} \theta_{n-k} \left( \frac{\partial^2}{\partial x^2} u_j^k - \frac{\partial^2}{\partial x^2} \phi^k \right) + \mu \sum_{k=0}^{n} \alpha_{n-k} \frac{\partial^2}{\partial x^2} \phi^k + \sum_{k=0}^{n} \alpha_{n-k} F_j^k, \\
\theta_0 u_0^0 &= U_0(t_0), \quad \theta_n u_n^0 = U_N(t_0), \quad k = 0, 1, ..., n_T, \n\end{align*} \right.
\]

(47)

where \( \alpha_k = (-1)^k \left( \frac{\beta}{2} \right) \), \( \theta_0 = 1 - \frac{\beta}{2}, \theta_1 = \frac{\beta}{2}, \theta_k = 0 (k \geq 2), \phi^k = \phi_0(x) + \psi_0(x)t, \phi^k = \left[ D^\beta_{0,t} \phi(x, t) \right]_{t=a_k}, F^k = \left[ D^\beta_{0,t} f(x, t) \right]_{t=a_k}, \) and \( R^n \) is the discretization error in time satisfying \( |R^n| \leq C \tau^{p+1} \).

If \( \beta \to 2 \), then the method (47) a conditionally stable scheme \( \frac{\mu \rho}{1-\beta} = \mu \delta^2 \frac{u_j^0}{\tau^2} + \left( F_j^{r-1} - 2F_j^r + F_j^{r+1} \right) \) requiring \( \frac{\tau^2}{\beta} \leq \frac{1}{\beta} \). We may think that the method (47) is also conditionally stable. Similar to Theorem 1 we can indeed obtain the stability region for the method (47) below

\[
S = \mathbb{C} \setminus \left\{ \frac{\alpha(z)}{\beta(z)} : |z| \leq 1 \right\} = \mathbb{C} \setminus \left\{ \left( \frac{1-\beta}{1-\beta} \right)^{\frac{1}{\beta}} : |z| \leq 1, 1 < \beta < 2 \right\}.
\]

The above stability region contains the interval \( \left( \frac{2}{1-\beta}, 0 \right) \), which implies

\[
\frac{2^\beta}{1-\beta} \leq \frac{4 \mu \rho}{h^2} \sin^{-1} \left( \frac{\sigma h}{2} \right) < 0.
\]

From the above inequality, we can derive a CLF condition for the method (47) as follows

\[
\frac{2^\beta}{1-\beta} \leq \frac{4 \mu \rho}{h^2} < 0, \quad \iff \quad 0 < \frac{\mu (\beta - 1) \tau^\beta}{2(\beta-2)h^2} \leq 1.
\]

Next, we make a slight modification of the scheme (47) such that the derived scheme is stable for any given real value of \( \tau^2/h^2 \). We make a slight modification of the first term \( \sum_{k=0}^{n} \theta_{n-k} \left( \frac{\partial^2}{\partial x^2} U^k - \frac{\partial^2}{\partial x^2} \phi^k \right) \) in the right hand side of (46) as follows

\[
\sum_{k=0}^{n} \theta_{n-k} \left( \frac{\partial^2}{\partial x^2} U^k - \frac{\partial^2}{\partial x^2} \phi^k \right) = (1 - \frac{\beta}{2}) \left( \frac{\partial^2}{\partial x^2} U^k - \frac{\partial^2}{\partial x^2} \phi^k \right) + \frac{\beta}{2} \left( \frac{\partial^2}{\partial x^2} U^{k-1} - \frac{\partial^2}{\partial x^2} \phi^{k-1} \right)
\]

(49)
Combining (44) and (47), we obtain the following new time discretization approach.

- **Time discretization II:****
  \[
  \sum_{k=0}^{n} \alpha_{n-k} (U^k - \varphi^k) = \mu \sum_{k=0}^{n} \theta_{n-k} \left( \partial_{x}^{2} U^k - \partial_{x}^{2} \varphi^k \right) + \mu \sum_{k=0}^{n} \alpha_{n-k} \partial_{x}^{2} \Phi^k + \sum_{k=0}^{n} \alpha_{n-k} F^k + R^n, \tag{50}
  \]
  where \(\alpha_k = (-1)^k \rho_k, \theta_0 = 1 - \frac{\beta}{2}, \theta_1 = 0, \theta_2 = \frac{\beta}{2}, \theta_k = 0(k \geq 3), \varphi(x, t) = U(x, 0) + \partial_{t} U(x, 0) t = \phi_0(x) + \psi_0(x) t, \Phi^k = \left[ D_{0}^{\beta} \varphi(x, t) \right]_{t=\tau_k} = \frac{\phi_0(x) \rho_k^{\beta}}{t^{\beta+1}} + \frac{\phi_0(x) \rho_k^{\beta-1}}{t^{\beta+2}}, F^k = \left[ D_{0}^{\beta} f(x, t) \right]_{t=\tau_k} \]
  and \(R^n\) is the discretization error in time satisfying \(|R^n| \leq C \tau^{1+\beta} \).

From (50), we can derive the following fully discrete finite difference scheme.

- **Scheme II:** Find \(u^k_j\) for \(j = 1, 2, ..., N-1, n = 1, 2, ..., n_T\), such that
  \[
  \left\{ \begin{array}{l}
  \sum_{k=0}^{n} \alpha_{n-k} (u^k_j - \varphi^k_j) = \mu \sum_{k=0}^{n} \theta_{n-k} \left( \partial_{x}^{2} u^k_j - \partial_{x}^{2} \varphi^k_j \right) + \mu \sum_{k=0}^{n} \alpha_{n-k} \partial_{x}^{2} \Phi^k + \sum_{k=0}^{n} \alpha_{n-k} F^k_j, \\
  u^0_0 = U_0(t_0), \quad u^k_N = U_N(t_k), \quad k = 0, 1, ..., n_T.
  \end{array} \right. \tag{51}
  \]
  where \(\alpha_k = (-1)^k \rho_k, \theta_0 = 1 - \frac{\beta}{2}, \theta_1 = 0, \theta_2 = \frac{\beta}{2}, \theta_k = 0(k \geq 3), \Phi^k_j = \left[ D_{0}^{\beta} \varphi(x_j, t) \right]_{t=\tau_k} = \frac{\phi_0(x_j) \rho_k^{\beta}}{t^{\beta+1}} + \frac{\phi_0(x_j) \rho_k^{\beta-1}}{t^{\beta+2}}, \Phi^k_j = \phi_0(x_j) + \psi_0(x_j) t_k, \) and \(F^k_j = \left[ D_{0}^{\beta} f(x_j, t) \right]_{t=\tau_k} \).

**Remark 2** If \(\beta \to 2\), then the scheme (51) is reduced to the following unconditionally stable scheme

\[
\frac{u^{n+1} - 2u^n + u^{n-1}}{\tau^2} = \frac{\mu}{2} \left( \partial_{x}^{2} u^{n+1} + \partial_{x}^{2} u^{n-1} \right) + (F^n_{n+1} - 2F^n_{n} + F^n_{n-1}), \quad n \geq 1.
\]

Similar to Theorem I, we can prove that the scheme (51) is stable, we just need to replace \(\theta(z) = 2^{-\beta}(1 + z)^{\beta}\) in (20) with \(\theta(z) = 1 - \frac{\beta}{2} + \frac{\beta^2}{2!} z^2\) to get the desired result. Like Theorems II and III, we can easily prove that the scheme (51) is consistent and convergent of order \(O(\tau^{1+\beta})\).

**Remark 3** In fact, we use a new generating function \(w^{(\beta)}(z) = \frac{1 - \frac{\beta^2}{2} z^2}{(1 - z)^{\beta}}\) in the construction of the scheme (51).

### 4 Fractional diffusion-wave with linear advection-reaction term

In this section, we extend the time discretization techniques used in (20) and (51) to the following equation

\[
\begin{cases}
\partial_{t}^{\beta} U(x, t) + \nu \partial_{x}^{2} U(x, t) + K_1 U(x, t) + K_2 \partial_{x} U(x, t) = \mu \partial_{x}^{2} U(x, t) + f(x, t), & (x, t) \in \mathbb{R} \times (0, T], I = (a, b), T > 0, \\
U(x, 0) = \phi_0(x), \quad \partial_{t} U(x, 0) = \psi_0(x), & x \in I, \\
U(a, t) = U_{a}(t), \quad U(b, t) = U_{b}(t), & t \in (0, T],
\end{cases} \tag{52}
\]

where \(1 < \beta < 2, \mu > 0, K_1, K_2 \geq 0\). See e.g. [3] for the case of \(K_2 = 0\).

The time in (52) is discretized similarly to the technique used in (20) or (51), the first-order and second-order space derivative operators are both discretized by the central difference method, we directly give the full scheme for (52) as follows.
– **Scheme III** \(m\): Find \(u_j^k\) for \(j = 1, 2, ..., N - 1, n = 0, 1, 2, ..., n_f - 1\), such that

\[
\begin{align*}
\sum_{k=0}^{n} \alpha_{n-k}(u_j^k - \varphi_j^k) &= \mu t^2 \sum_{k=0}^{n} \theta_{n-k} (\delta^2_x u_j^k - \delta^2_x \varphi_j^k) + \mu \sum_{k=0}^{n} \alpha_{n-k} \delta_x \Phi_j^k + \sum_{k=0}^{n} \alpha_{n-k} F_j^k, \\
- K_2 \sum_{k=0}^{n} \theta_{n-k} (\delta_t u_j^k - \delta_t \varphi_j^k) &= - K_2 \sum_{k=0}^{n} \alpha_{n-k} \delta_t \Phi_j^k + \sum_{k=0}^{n} \alpha_{n-k} \Phi_j^k, \\
- K_1 \sum_{k=0}^{n} \theta_{n-k} (u_j^k - \varphi_j^k) &= - K_1 \sum_{k=0}^{n} \alpha_{n-k} \Phi_j^k, \quad \Phi_j^k \bigg\vert_{n_i+1} = \Phi_j^{k-1} + \frac{\phi_{n+1}^{k+1} - \phi_{n+1}^{k}}{h^2}, \quad \Phi_j^k = \phi_0(x_j) + \phi_0(x_j) \delta_k, \\
F_j^k &= \left[D_{x,x} g(x_j, t) \right]_{n_i+1}, \quad \text{and} \\
m &= \begin{cases} 
1, & \theta_k = \frac{(-1)^k}{2^m} \alpha_k, k = 0, 1, \ldots; \\
2, & \theta_0 = 1 - \frac{\beta}{4}, \theta_1 = 0, \theta_2 = \frac{\beta}{4}, \theta_k = 0 (k \geq 3). 
\end{cases}
\end{align*}
\]

where \(\alpha_k = \binom{h^4}{k}\), \(\Phi_j^k = \left[D_{x,x} \varphi(x_j, t) \right]_{n_i+1} = \frac{\varphi(x_{j+1})}{h^{2(m+1)}} + \frac{\varphi(x_{j-1})}{h^{2(m+1)}}, \quad \varphi_j^k = \phi_0(x_j) + \phi_0(x_j) \delta_k, \quad F_j^k = \left[D_{x,x} f(x_j, t) \right]_{n_i+1}\). 

Similar to Theorem 1 the finite difference method \(53\) can be proven to be stable, we just need to replace \(S^* \) in the proof of Theorem 1 with

\[
S^* = - \frac{4m\mu t^2}{h^2} \sin^2 \left(\frac{\sigma h}{2}\right) - K_1 t^2 - i \frac{K_2 \mu t^2}{h} \sin(\sigma h)
\]

to reach the conclusion. The consistency of order \(O(x^2 + h^2)\) of \(53\) can be also similarly proved as that of Theorem 2. The stability and convergence rate are also shown numerically in the following section.

5 Numerical examples

In this section, we present numerical examples to verify the theoretical analysis in the previous sections. We first numerically verify the error estimates and the convergence orders of Scheme I (see Eq. 20), the scheme 47, and Scheme II (see Eq. 51).

**Example 1** Consider the following diffusion-wave equation \(51\)

\[
\begin{align*}
\frac{d}{dx} \frac{\partial^4}{\partial x^4} U(x, t) &= \partial_t^2 U(x, t) + f(x, t), \quad (x, t) \in (0, 1) \times (0, 1), \\
U(x, 0) &= 2 \exp(x), \quad \partial_t U(x, 0) = \exp(x), \quad x \in (0, 1), \\
U(0, t) &= t^2 + t^2 + 2, \quad U(1, t) = (t^2 + t^2 + t + 2) \exp(1) \quad t \in (0, 1),
\end{align*}
\]

Choose a suitable right hand side function \(f(x, t)\) such that the exact solution to \(55\) is

\[
U(x, t) = (t^2 + t^2 + t + 2) \exp(x).
\]


where \( \tau, \tau \) and steps sizes are chosen as the method (47) is stable if \( 0 < \tau \leq \frac{1}{2(\beta + 1)} \) inline with the theoretical analysis. In Table 2, we show the convergence rates in space for \( \max_0 \) second-order accuracy in time for di

In Table 1, we show the convergence rates in space for the time and space steps sizes are chosen as \( h = 1/1000 \) and \( \tau = 1/16, 1/32, 1/64, 1/128, 1/256 \), the \( L^2 \) error \( \max_{0 \leq x < h} ||e^n|| \) is shown in Table 1. It is found that Scheme I (20) and II (51) both show second-order accuracy in time, especially when \( \tau > 1/2 \). The numerical result is inline with the theoretical result (48). The numerical results also show second-order accuracy both in time and space by the simple calculation using (56).

We first check the accuracy of the schemes (20) and (51) in time, and the space and time steps sizes are chosen as \( h = 1/1000 \) and \( \tau = 1/16, 1/32, 1/64, 1/128, 1/256 \), the \( L^2 \) error \( \max_{0 \leq x < h} ||e^n|| \) is shown in Table 1. It is found that Scheme I (20) and II (51) both show second-order accuracy in time for different fractional order \( \beta (\beta = 1.1, 1.5, 1.9) \), which is inline with the theoretical analysis. In Table 2, we show the convergence rates in space for the two schemes (20) and (51), from which the second-order accuracy is observed.

We also test the accuracy and stability of the method (47). From (48), one knows that the method (47) is stable if \( 0 < r = \frac{\beta + 1}{2\beta} \leq 1 \). We use the method (47) to solve (55), the numerical results are shown in Table 3 which shows that the method (47) is stable for \( r \leq 1 \), and unstable for \( r > 1 \) (see stars * in Table 3) which means the numerical solutions blow up when \( r > 1 \). The numerical result is inline with the theoretical result (48). The numerical results in Table 3 also show second-order accuracy both in time and space by the simple calculation using (56).

Here, we also compare Scheme I and Scheme II with the finite difference scheme developed in (31) with convergence of order \( O(\tau^{3\beta} + h^2) \), the results are shown in Table 3. Obviously, the present methods show better performances because of their high-order convergence in time, especially when \( \beta \) tends to 2.

### Example 2

Consider the following equation

\[
\begin{aligned}
&cD^{\beta}_{\tau} U(x, t) + U(x, t) + \partial_t U(x, t) = \partial^2_x U(x, t) + f(x, t), \quad (x, t) \in (0, 1) \times (0, 1), \\
&U(x, 0) = 1, \quad \partial_t U(x, 0) = -x, \quad x \in (0, 1), \\
&U(0, t) = 1, \quad U(1, t) = \exp(-t) \quad t \in (0, 1),
\end{aligned}
\]  

\[(57)\]
Table 2 The $L^2$ errors max $\|e^n\|$ for Example [1] $\tau = 5 \times 10^{-4}$.

| Methods | $N$ | $\beta = 1.1$ order | $\beta = 1.5$ order | $\beta = 1.9$ order |
|---------|-----|-------------------|-------------------|-------------------|
| Scheme I | 16  | 2.2415e-4 | 2.2608e-4 | 2.7949e-4 |
|          | 32  | 5.6079e-5 | 5.6560e-5 | 6.9901e-5 |
|          | 64  | 1.4040e-4 | 1.4137e-4 | 1.7486e-4 |
|          | 128 | 3.5287e-6 | 3.5826e-6 | 3.9846e-6 |
|          | 256 | 9.0088e-7 | 9.3364e-7 | 1.1050e-6 |

Table 3 The $L^2$ errors max $\|e^n\|$ for Example [1] with method [9], $r = \frac{\beta(\beta-1)}{\tau^2}$.

Table 4 Comparison of the $L^2$ errors max $\|e^n\|$ of different methods, $N = 1000$.

| Methods | $1/\tau$ | $\beta = 1.1$ | $\beta = 1.3$ | $\beta = 1.5$ | $\beta = 1.65$ | $\beta = 1.8$ | $\beta = 1.95$ |
|---------|---------|---------------|---------------|---------------|---------------|---------------|---------------|
| Scheme I | 16      | 6.056e-4     | 8.1295e-4     | 1.0579e-3     | 1.2092e-3     | 1.0682e-3     | 1.0472e-3     |
|          | 32      | 5.128e-4     | 6.5810e-5     | 8.9083e-4     | 2.5239e-4     | 1.9743e-4     |               |
|          | 64      | 3.8249e-5    | 4.6231e-5     | 7.1239e-5     | 6.1129e-5     | 4.1599e-5     |               |
|          | 128     | 1.9771e-5    | 2.9876e-5     | 4.7582e-5     | 3.5031e-5     | 9.4623e-5     |               |
|          | 256     | 4.3960e-6    | 6.4510e-6     | 9.8782e-6     | 6.7264e-6     | 2.2504e-6     |               |

| Scheme II | 16    | 6.1808e-3 | 2.3332e-3 | 2.6654e-3 | 2.9169e-3 | 2.8080e-3 | 1.8335e-3 |
|          | 32    | 5.7531e-4 | 6.6574e-4 | 7.2324e-4 | 6.9427e-4 | 4.6041e-4 |               |
|          | 64    | 4.5731e-4 | 5.6724e-4 | 6.5515e-4 | 7.2324e-4 | 6.9427e-4 |               |
|          | 128   | 1.1497e-4 | 1.4126e-4 | 1.6621e-4 | 1.7216e-4 | 1.1465e-4 |               |
|          | 256   | 7.2146e-5 | 8.8547e-5 | 1.0385e-5 | 1.1835e-5 | 7.3180e-6 |               |

| Method 31 | 16    | 8.2832e-4 | 2.4337e-3 | 8.0335e-4 | 1.0385e-2 | 4.3317e-2 | 9.0354e-2 |
|          | 32    | 2.1707e-4 | 7.4321e-4 | 2.8287e-3 | 7.6281e-3 | 1.9237e-2 | 4.4283e-2 |
|          | 64    | 5.7161e-5 | 2.2527e-4 | 9.4990e-4 | 2.9957e-3 | 8.4268e-3 | 2.1535e-2 |
|          | 128   | 1.5065e-4 | 8.8356e-5 | 3.5040e-4 | 1.1750e-3 | 3.6782e-3 | 1.0435e-2 |
|          | 256   | 3.973e-6  | 2.0894e-5 | 1.2354e-4 | 4.6078e-4 | 1.6029e-3 | 5.0475e-3 |
where $1 < \beta < 2$. Choose the suitable $f(x,t)$ satisfies

$$f(x,t) = x^2 t^{2-\beta} \sum_{k=0}^{\infty} \frac{(-xt)^k}{\Gamma(k+3-\beta)} + \exp(-xt) - t \exp(-xt) - t^2 \exp(-xt)$$

such that (57) has the following analytical solution

$$U(x,t) = \exp(-xt).$$

In this example, we test the convergence rates of Scheme III (1) and Scheme III (2), and we also compare the present methods with the existing time discretization used in [7], see also [13], where the second-order fractional backward difference formula was used to discretize the Caputo derivative operator. We choose the time step size and space step size as $\tau = h$, the $L^2$ errors for different fractional order $\beta$ are shown in Table 5. Clearly, the two methods of the present paper show second-order accuracy both in time and space, while the second-order method in [7,13] do not show second-order accuracy, especially when $\beta$ tends to 2, the method in [7] is reduced to

$$\frac{1}{\tau^2} \left( \frac{9}{4} u_{j+1} - 6u_j + \frac{11}{2} u_{j-1} - 2u_{j-2} + \frac{1}{4} u_{j-3} \right) = \mu \delta^2 u_j + f_j^{n+1},$$

where $u_j^k (k = 1, 2, 3)$ should be derived with any known high-order methods, or the second-order accuracy will possibly be lost.

6 Conclusion

In this paper, we propose three finite difference schemes for the fractional diffusion-wave equation (1). The first one is based on the fractional trapezoidal formula in time and the central difference in the space. This scheme is proven to be stable by Fourier analysis with convergence order two in both time and space. The second scheme is based on a second-order generalized Newton-Gregory formula in time. The second scheme is only conditionally stable, while a slight modification of the second scheme leads to the third scheme that is stable. The last two schemes are also of order two in both time and space. We extend the two of these time discretization techniques to a class of fractional differential equations, and derived stable schemes with second-order convergence both in time and space.

When $\beta \to 2$, the present methods (20), (47), and (51) becomes the corresponding classical methods for the classical diffusion-wave equation, which is an important feature different from the time discretization techniques used in previous papers, see for example [4,7,10,13,23,34].

We present numerical experiments to verify the theoretical analysis, and comparisons with other methods exhibit better accuracy than many of the existing numerical methods. The present methods can be readily extended to two- and three-dimensional problems and the stability and convergence analysis are similar to those given here.

References

1. Bhrawy, A.H., Doha, E.H., Baleanu, D., Ezz-Eldien, S.S.: A spectral tau algorithm based on Jacobi operational matrix for numerical solution of time fractional diffusion-wave equations. J. Comput. Phys., in press, 2014
Table 5 Comparison of the $L^2$ errors at $t = 1$ for Example 2, $r = h^1/N$.

| $\beta$ | $N$ | Scheme III (1) order | Scheme III (2) order | Method [7] order |
|---------|-----|----------------------|----------------------|------------------|
| 1.2     | 32  | 8.4630e-6            | 6.7927e-6            | 5.7930e-6        |
|         | 64  | 2.1115e-6            | 1.6853e-6            | 1.4888e-6        |
|         | 128 | 5.2733e-7            | 4.1958e-7            | 3.8205e-7        |
|         | 256 | 1.3178e-7            | 1.0467e-7            | 9.8097e-8        |
|         | 512 | 3.2939e-8            | 2.0006               | 2.0015           |
| 1.5     | 32  | 6.2643e-6            | 6.8147e-6            | 3.2195e-5        |
|         | 64  | 1.6018e-6            | 1.9675               | 1.6981e-6        |
|         | 128 | 4.0577e-7            | 4.2395e-7            | 4.0675e-6        |
|         | 256 | 1.0225e-7            | 1.0592e-7            | 1.4322e-6        |
|         | 512 | 2.5688e-8            | 1.9930               | 1.5046e-7        |
| 1.7     | 32  | 4.8058e-6            | 8.0483e-6            | 1.1786e-4        |
|         | 64  | 1.2490e-6            | 2.0105e-6            | 5.5140e-5        |
|         | 128 | 3.2035e-7            | 5.0129e-7            | 2.3829e-5        |
|         | 256 | 8.1431e-8            | 1.2508e-7            | 9.9913e-6        |
|         | 512 | 2.0586e-8            | 3.1233e-8            | 4.1302e-6        |
| 1.8     | 32  | 4.8725e-6            | 7.1377e-6            | 1.5921e-4        |
|         | 64  | 1.2457e-6            | 1.8058e-6            | 8.5657e-5        |
|         | 128 | 3.1645e-7            | 4.5167e-7            | 4.0687e-6        |
|         | 256 | 7.9933e-8            | 1.1286e-7            | 1.8473e-5        |
|         | 512 | 2.0129e-8            | 2.8213e-8            | 5.0467e-7        |
| 1.9     | 32  | 5.5981e-6            | 6.8122e-6            | 1.1319e-4        |
|         | 64  | 1.4231e-6            | 1.9769               | 8.657e-5         |
|         | 128 | 3.5907e-7            | 4.2691e-7            | 5.2516e-5        |
|         | 256 | 9.006e-8             | 1.0665e-7            | 1.0959           |
|         | 512 | 2.2531e-8            | 2.6620e-8            | 1.2548e-5        |
| 2       | 32  | 7.9469e-6            | 1.1951e-5            | 1.1319e-4        |
|         | 64  | 2.0114e-6            | 2.9856e-6            | 6.0261e-5        |
|         | 128 | 5.0676e-7            | 7.4851e-7            | 3.6628e-5        |
|         | 256 | 1.2720e-7            | 1.8752e-7            | 2.0343e-5        |
|         | 512 | 2.5231e-8            | 2.6620e-8            | 1.2548e-5        |
| 2.99    | 32  | 9.2889e-6            | 1.3606e-5            | 1.3064e-4        |
|         | 64  | 2.2342e-6            | 3.3105e-6            | 5.9485e-5        |
|         | 128 | 5.4612e-7            | 8.1441e-7            | 3.4066e-5        |
|         | 256 | 1.3491e-7            | 2.0190e-7            | 1.8740e-5        |
|         | 512 | 3.3520e-8            | 5.0258e-8            | 9.8582e-6        |

2. Cao, J.Y., Xu, C.J.: A high order schema for the numerical solution of the fractional ordinary differential equations. J. Comput. Phys., 238, 154–168 (2013)
3. Chen, J., Liu, F., Anh, V., Shen, S., Liu, Q., Liao, C.: The analytical solution and numerical solution of the fractional diffusion-wave equation with damping. Appl. Math. Comput., 219, 1737–1748 (2012)
4. Cuesta, E., Lubich, C., Palencia, C.: Convolution quadrature time discretization of fractional diffusion-wave equations. Math. Comp., 75, 673–696 (2006)
5. Diethelm, K., Ford, N.J., Freed, A.D.: Detailed error analysis for a fractional Adams method. Numer. Algorithms, 36, 31–52 (2004)
6. Diethelm, K., Ford, N.J., Freed, A.D., Weilbeer, M.: Pitfalls in fast numerical solvers for fractional differential equations. J. Comput. Appl. Math., 186, 482–503 (2006)
7. Deng, H.F., Li, C.P.: Numerical algorithms for the fractional diffusion-wave equation with reaction term. Abstr. Appl. Anal., 2013, 493406 (2013)
8. Du, R., Cao, W.R., Sun, Z.Z.: A compact difference scheme for the fractional diffusion-wave equation. Appl. Math. Model., 34, 2998–3007 (2010)
9. Hanyyad, A.: Multidimensional solutions of time-fractional diffusion-wave equations. Proc. R. Soc. Lond. A, 458, 933–957 (2002)
10. Huang, J., Tang, Y., Vázquez, L., Yang, J.: Two finite difference schemes for time fractional diffusion-wave equation. Numer. Algorithms, 64, 707–720 (2013)
11. Jafari, M.A., Aminataei, A.: An algorithm for solving multi-term diffusion-wave equations of fractional order. Comput. Math. Appl., 62, 1091–1097 (2011)
12. Jafari, H., Momani, S.: Solving fractional diffusion and wave equations by modified homotopy perturbation method. Physics Letters A 370, 388–396 (2007)
13. Jin, B., Lazarov, R., Zhou, Z.: On two schemes for fractional diffusion and diffusion-wave equations. arXiv:1404.3800 (2014)
14. Li, C.P., Zeng, F.H.: The finite difference methods for fractional ordinary differential equations. Numer. Funct. Anal. Opt. 34, 149–179 (2013)
15. Li, L.M., Xu, D., Luo, M.: Alternating direction implicit Galerkin finite element method for the two-dimensional fractional diffusion-wave equation. J. Comput. Phys. 255, 471–485 (2013)
16. Lin, R., Liu F.: Fractional high order methods for the nonlinear fractional ordinary differential equations. Nonlinear Analysis 66, 856–869 (2007)
17. Liu, F., Meerschaert, M.M., McGough, R.J., Zhuang, P., Liu, Q.: Numerical methods for solving the multi-term timefractional wave-diffusion equation. Fract. Calc. Appl. Anal. 17, (2014) 69–719
18. Lubich, C.: A stability analysis of convolution quadratures for Abel–Volterra integral equations. IMA J Numer. Anal. 6, (1986) 87–101
19. Luchko, Y. Mainardi, F., Povstenko, Y.: Propagation speed of the maximum of the fundamental solution to the fractional diffusion-wave equation. Comput. Math. Appl. 66, (2013) 774–784
20. McLean, W., Mustapha, K.: A second-order accurate numerical method for fractional diffusion and wave equations. SIAM J. Numer. Anal. 51, 491–515 (2013)
21. Metzler, R., Nonnenmacher, T.F.: Space- and time-fractional diffusion and wave equations, fractional Fokker-Planck equations, and physical motivation. Chemical Physics 284, 67–90 (2002)
22. Mustapha, K., McLean, W.: Superconvergence of a discontinuous Galerkin method for fractional diffusion and wave equations. SIAM J. Numer. Anal. 51, 491–515 (2013)
23. Podlubny, I.: Fractional Differential Equations. Academic Press, San Diego (1999)
24. Ren, J.C., Sun, Z.Z.: Numerical algorithm with high spatial accuracy for the fractional diffusion-wave equation with Neumann boundary conditions. J. Sci. Comput. 56, 381–408 (2013)
25. Schneider, W.R., Wyss, W.: Fractional diffusion and wave equations. J. Math. Phys. 30, 134–144(1989)
26. Sun, Z.Z., Wu X.N.: A fully discrete difference scheme for a diffusion-wave system. Appl. Numer. Math. 56, 193–209 (2006)
27. Sweilam, N.H., Khader, M.M., Adel, M.: On the stability analysis of weighted average finite difference methods for fractional wave equation. Fractional Differential Calculus 2, 17–29 (2012)
28. Vázquez, L.: From Newton’s equation to fractional diffusion and wave equations. Adv. Differ. Equ. 2011, 169421 (2011)
29. Yang, J.Y., Huang, J.F., Liang, D.M., Tang, Y.F.: Numerical solution of fractional diffusion-wave equation based on fractional multistep method. Appl. Math. Model. 38, 3652–3661 (2014)
30. Zeng, F.H., Li, C.P., Liu, F., Turner, I.: The use of finite difference/element approaches for solving the time-fractional subdiffusion equation. SIAM J. Sci. Comput. 35, A2976–A3000 (2013)
31. Zeng, F.H., Li, C.P., Liu, F., Turner, I.: Numerical algorithms for time-fractional subdiffusion equation with second-order accuracy. SIAM J. Sci. Comput. (2014), in press.
