A short survey on Lyapunov dimension for finite dimensional dynamical systems in Euclidean space.

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Abstract

Nowadays there are a number of surveys and theoretical works devoted to the Lyapunov exponents and Lyapunov dimension, however most of them are devoted to infinite dimensional systems or rely on special ergodic properties of the system. At the same time the provided illustrative examples are often finite dimensional systems and the rigorous proof of their ergodic properties can be a difficult task. Also the Lyapunov exponents and Lyapunov dimension have become so widespread and common that they are often used without references to the rigorous definitions or pioneering works.

The survey is devoted to the finite dimensional dynamical systems in Euclidean space and its aim is to explain, in a simple but rigorous way, the connection between the key works in the area: by Kaplan and Yorke (the concept of Lyapunov dimension, 1979), Douady and Oesterlé (estimation of Hausdorff dimension via the Lyapunov dimension of maps, 1980), Constantin, Eden, Foias, and Temam (estimation of Hausdorff dimension via the Lyapunov exponents and dimension of dynamical systems, 1985-90), Leonov (estimation of the Lyapunov dimension via the direct Lyapunov method, 1991), and numerical methods for the computation of Lyapunov exponents and Lyapunov dimension.

In this survey a concise overview of the classical results is presented, various definitions of Lyapunov exponents and Lyapunov dimension are discussed. An effective analytical method for the estimation of Lyapunov dimension is presented, its application to the self-excited and hidden attractors of well-known dynamical systems is demonstrated, and analytical formulas of exact Lyapunov dimension are obtained.

Keywords: Hausdorff dimension, Lyapunov dimension, Kaplan-Yorke dimension, Lyapunov exponents, finite-time Lyapunov exponents, Lyapunov characteristic exponents, dynamical system, self-excited attractor, hidden attractor, Henon map, Lorenz system, Glukhovsky-Dolzhansky system, Tigan system, Yang system, Shimizu-Morioka system

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1. Introduction: Hausdorff dimension

The theory of topological dimension [41, 49], developed in the first half of the 20th century, is of little use in giving the scale of dimensional characteristics of attractors. The point is that the topological dimension can take integer values only. Hence the scale of dimensional characteristics compiled in this manner turns out to be quite poor. For the analysis of attractors, the Hausdorff dimension of a set is much better. This dimensional characteristic can take any nonnegative value (not greater than the topological dimension of the space), and it coincides with the topological dimension for such typical objects in Euclidean space as a smooth curve, a smooth surface, or a countable set of points.

Let us give the definition of the Hausdorff dimension and its upper estimations based on the Lyapunov exponents following mainly ([6, 10, 16, 20, 27, 31, 40, 42, 45, 61, 73, 102, 106]).

Consider a set \( K \subseteq \mathbb{R}^n \) and numbers \( d \geq 0, \varepsilon > 0 \). We cover \( K \) by a countable set of balls \( B_{r_j} \) of radius \( r_j < \varepsilon \), and define

\[
\mu_H(K, d, \varepsilon) := \inf \left\{ \sum_{j \geq 1} r_j^d \mid r_j \leq \varepsilon, K \subseteq \bigcup_{j \geq 1} B_{r_j} \right\},
\]

where the infimum is taken over all such countable \( \varepsilon \)-coverings \( K \). It is obvious that \( \mu_H(K, d, \varepsilon) \) does not decrease with decreasing \( \varepsilon \). Therefore there exists a limit (perhaps infinite), namely

\[
\mu_H(K, d) = \lim_{\varepsilon \to 0^+} \mu_H(K, d, \varepsilon) = \sup_{\varepsilon > 0} \mu_H(K, d, \varepsilon).
\]

**Definition 1.** The function \( \mu_H(\cdot, d) \) is called the Hausdorff \( d \)-measure on \( \mathbb{R}^n \).

For a certain set \( K \), the function \( \mu_H(K, \cdot) \) has the following property. It is possible to find \( d_{cr} = d_{cr}(K) \in [0, n] \) such that

\[
\mu_H(K, d) = \infty, \forall d \in (0, d_{cr}); \quad \mu_H(K, d) = 0, \forall d > d_{cr}.
\]

We have \( d_{cr}(\mathbb{R}^n) = n \).

**Definition 2.** The Hausdorff dimension of the set \( K \) is defined as

\[
\dim_H K := d_{cr}(K) = \inf\{ d \geq 0 \mid \mu_H(K, d) = 0 \}.
\]
2. Singular value function and invariant sets of maps and dynamical systems

In the seminal paper [27] Douady and Oesterlé showed how to obtain an upper estimate of the Hausdorff dimension of set $K$. To demonstrate their approach, let us consider some definitions and auxiliary results.

Let $U$ be an open subset of $\mathbb{R}^n$ and $\varphi : U \to \mathbb{R}^n$ be a continuously differentiable map. With respect to the canonical basis in $\mathbb{R}^n$ the function $\varphi(u)$ has the $n \times n$ Jacobian matrix

$$D\varphi(u) = D_u \varphi(u) = \left( \frac{\partial \varphi_i(u)}{\partial u_j} \right)_{n \times n}, \quad u \in U.$$ 

Let $\sigma_i(u) = \sigma_i(D\varphi(u))$, $i = 1, 2, \ldots, n$, be the singular values of $D\varphi(u)$ (i.e. $\sigma_i(u) \geq 0$ and $\sigma_i(u)^2$ are the eigenvalues of the symmetric matrix $D\varphi(u)^T D\varphi(u)$ with respect to their algebraic multiplicity) ordered so that $\sigma_1(u) \geq \cdots \geq \sigma_n(u) \geq 0$ for any $u \in K$. If $\sigma_n(u) > 0$, then the unit ball $B$ is transformed by $D\varphi(u)$ into the ellipsoid $D\varphi(u)B$ and the lengths of its principal semiaxes coincide with the singular values.

**Definition 3.** The singular value function of $D\varphi(u)$ of order $d \in [0, n]$ at $u \in U$ is defined as

$$\omega_d(D\varphi(u)) := \begin{cases} 1, & d = 0, \\ \sigma_1(u) \cdots \sigma_d(u), & d \in \{1, \ldots, n\}, \\ \sigma_1(u) \cdots \sigma_{|d|}(u)\sigma_{|d|+1}(u)^{d-|d|}, & d \in (0, 1) \cup \ldots \cup (n-2, n-1), \end{cases}$$

where $|d|$ is the largest integer less or equal to $d$.

Remark that $|\det D\varphi(u)| = \omega_n(D\varphi(u))$.

Similarly, introducing the singular value function for arbitrary quadratic matrices, by the Horn inequality [39] for any two $n \times n$ matrices $A$ and $B$ and any $d \in [0, n]$ we have (see, e.g. [10], p.28)

**Lemma 1.**

$$\omega_d(AC) \leq \omega_d(A)\omega_d(C).$$

**Definition 4.** A set $K \subset U \subseteq \mathbb{R}^n$ with respect to the map $\varphi$ is said to be:

1) **positively invariant** if $\varphi(K) \subset K$,

2) **invariant** if $\varphi(K) = K$,

3) **and negatively invariant** if $\varphi(K) \supset K$, where $\varphi(K) = \{ \varphi(u) \mid u \in K \}$.

Consider an autonomous differential equation

$$\dot{u} = f(u), \quad (3)$$

where $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is a continuously differentiable vector-function. Suppose that any solution $u(t, u_0)$ of (3) such that $u(0, u_0) = u_0 \in U$ exists for $t \in [0, \infty)$, is unique, and stays in $U$. Then the evolutionary operator $\varphi^t(u_0) := u(t, u_0)$ is continuously differentiable and satisfies the semigroup property:

$$\varphi^{t+s}(u_0) = \varphi^t(\varphi^s(u_0)), \quad \varphi^0(u_0) = u_0 \quad \forall \ t, s \geq 0, \ \forall u_0 \in U.$$  

(4)

Thus $\{ \varphi^t \}_{t \geq 0}$ is a smooth dynamical system in the phase space $(U, || \cdot ||): \{ \{ \varphi^t \}_{t \geq 0} \mid (U \subseteq \mathbb{R}^n, || \cdot ||) \}$. Here $||u|| = \sqrt{u_1^2 + \cdots + u_n^2}$ is Euclidean norm of the vector $u = (u_1, \ldots, u_n) \in \mathbb{R}^n$. Similarly, we can consider a dynamical system generated by the difference equation

$$u(t + 1) = \varphi(u(t)), \quad t = 0, 1, \ldots,$$

(5)
where \( \varphi : U \subseteq \mathbb{R}^n \to U \) is a continuously differentiable vector-function. Here \( \varphi^t(u) = (\varphi \circ \varphi \circ \cdots \circ \varphi)(u) \), where \( \varphi^0(u) = u \), and the existence and uniqueness (in the forward-time direction) take place for all \( t \geq 0 \). Further \( \{\varphi^t\}_{t \geq 0} \) denotes a smooth dynamical system with continuous or discrete time.

### Definition 5
A set \( K \subseteq U \subseteq \mathbb{R}^n \) with respect to the dynamical system \( \{\varphi^t\}_{t \geq 0} \) is said to be positively invariant, invariant or negatively invariant if the corresponding property takes place with respect to the map \( \varphi^t \) for all \( t > 0 \).

Consider the linearizations of systems \([3]\) and \([5]\) along the solution \( \varphi^t(u) \):

\[
\dot{y} = J(\varphi^t(u))y, \quad J(u) = Df(u),
\]

\[
y(t + 1) = J(\varphi^t(u))y(t), \quad J(u) = D\varphi(u),
\]

where \( J(u) \) is the \( n \times n \) Jacobian matrix, all elements of which are continuous functions of \( u \). Consider the fundamental matrix

\[
D\varphi^t(u) = (y^1(t), \ldots, y^n(t)), \quad D\varphi^0(u) = I,
\]

which consists of linearly independent solutions \( \{y^i(t)\}_{i=1}^n \) of the linearized system. An important cocycle property of fundamental matrix \([8]\) is as follows

\[
D\varphi^{t+s}(u) = D\varphi^t(\varphi^s(u))D\varphi^s(u), \quad \forall t, s \geq 0, \forall u \in U \subseteq \mathbb{R}^n.
\]

Consider the singular values of the matrix \( D\varphi^t(u) \) sorted by descending for each \( t \in [0, +\infty) \) and \( u \in U \subseteq \mathbb{R}^n \):

\[
\sigma_i(t, u) := \sigma_i(D\varphi^t(u)), \quad \sigma_1(t, u) \geq \ldots \geq \sigma_n(t, u) \geq 0 \quad \forall t \geq 0, u \in U \subseteq \mathbb{R}^n.
\]

Similar to \([1]\), we introduce the singular value function of \( D\varphi^t(u) \) of order \( d \): \( \omega_d(D\varphi^t(u)) \).

For a fixed \( t \geq 0 \) one can consider the map defined by the evolutionary operator \( \varphi^t(u) : \varphi : U \subseteq \mathbb{R}^n \to U \).

Further we need the following auxiliary statements.

### Lemma 2
From formula \([1]\) it follows that for any \( u \in U \) and \( t \geq 0 \) the function \( d \mapsto \omega_d(D\varphi^t(u)) \) is a left-continuous function.

### Lemma 3
For any \( d \in [0, n] \) and \( t \geq 0 \) the function \( u \mapsto \omega_d(D\varphi^t(u)) \) is continuous on \( U \) (see, e.g. [33, p.554]). Therefore for a compact set \( K \subseteq U \) and \( t \geq 0 \) we have

\[
\sup_{u \in K} \omega_d(D\varphi^t(u)) = \max_{u \in K} \omega_d(D\varphi^t(u)).
\]

### Proof
It follows from the continuity of the functions \( u \mapsto \sigma_i(D\varphi(u)) \) \( i = 1, 2, \ldots, n \) on \( U \).

Next, unless otherwise stated, the invariance of the set \( K \subseteq U \subseteq \mathbb{R}^n \) is considered with respect to the dynamical system \( \{(\varphi^t)_{t \geq 0}, (U \subseteq \mathbb{R}^n, || \cdot ||)\} : \varphi^t(K) = K, \forall t \geq 0 \).

### Lemma 4
For a compact invariant set \( K \) and any \( d \in [0, n] \), the function \( t \mapsto \max_{u \in K} \omega_d(D\varphi^t(u)) \) is sub-exponential, i.e.

\[
\max_{u \in K} \omega_d(D\varphi^{t+s}(u)) \leq \max_{u \in K} \omega_d(D\varphi^t(u)) \max_{u \in K} \omega_d(D\varphi^s(u)) \quad \forall t, s \geq 0;
\]

If \( \max_{u \in K} \omega_d(D\varphi^t(u)) > 0 \) for \( t \geq 0 \), then \( \ln \max_{u \in K} \omega_d(D\varphi^{t+s}(u)) \) is subadditive, i.e.

\[
\ln \max_{u \in K} \omega_d(D\varphi^{t+s}(u)) \leq \ln \max_{u \in K} \omega_d(D\varphi^t(u)) + \ln \max_{u \in K} \omega_d(D\varphi^s(u)).
\]
Proof. By (9) and (2) we get
\[
\max_{u \in K} \omega_d(D\varphi^{t+s}(u)) = \max_{u \in K} \left( \omega_d(D\varphi(D\varphi^s(u))D\varphi^s(u)) \right) \leq \\
\leq \max_{u \in K} \omega_d(D\varphi^s(u)) \max_{u \in K} \omega_d(D\varphi^s(u)) \leq \max_{u \in K} \omega_d(D\varphi^s(u)) \max_{u \in K} \omega_d(D\varphi^s(u)).
\]
\[\blacksquare\]

Corollary 1. For an equilibrium point \( u_{eq} = \varphi^t(u_{eq}) \) we have
\[
\omega_d(D\varphi^t(u_{eq})) = (\omega_d(D\varphi(u_{eq})))^t, \quad t \geq 0.
\]

Corollary 2. Remark that for a compact invariant set \( K \)
\[
\inf_{t>0} \max_{u \in K} \omega_d(D\varphi^t(u)) < 1 \iff \lim_{t \to +\infty} \max_{u \in K} \omega_d(D\varphi^t(u)) < 1.
\]
In this case\[\blacksquare\]
\[
\inf_{t>0} \max_{u \in K} \omega_d(D\varphi^t(u)) = \lim_{t \to +\infty} \max_{u \in K} \omega_d(D\varphi^t(u)) = 0.
\]

Proof. Let \( \inf_{t>0} \max_{u \in K} \omega_d(D\varphi^t(u)) = M < 1 \). There are \( \delta > 0 \) and \( t_0 = t_0(\delta) \) such that
\[
\max_{u \in K} \omega_d(D\varphi^{t_0}(u)) \leq 1 - \delta.
\]
Thus by (12) we have for \( u \in K \) and \( n \geq 0 \)
\[
0 \leq \omega_d(D\varphi^{nt_0}(u)) \leq \max_{u \in K} \omega_d(D\varphi^{nt_0}(u)) \leq (\max_{u \in K} \omega_d(D\varphi^{t_0}(u)))^n \leq (1 - \delta)^n \to_{n \to +\infty} 0
\]
and therefore \( M = \lim_{n \to +\infty} \omega_d(D\varphi^{nt_0}(u)) = \lim_{t \to +\infty} \max_{u \in K} \omega_d(D\varphi^t(u)) = 0 \). The same is true if we consider \( \lim_{t \to +\infty} \max_{u \in K} \omega_d(D\varphi^t(u)) < 1 \) first. \(\blacksquare\)

Corollary 3. If for a fixed \( t > 0 \) and \( d \in [0, n] \) we have \( \max_{u \in K} \omega_d(D\varphi^t(u)) < 1 \), then
\[
\lim_{t \to +\infty} \max_{u \in K} \omega_d(D\varphi^t(u)) = \lim_{t \to +\infty} \omega_d(D\varphi^t(u)) = 0, \quad u \in K.
\]

Lemma 5. [27], p.33, [104], pp.359-360] From the sub-exponential behavior of singular value function (see (12)) on a compact invariant set \( K \) it follows that
\[
\inf_{t>0} \left( \max_{u \in K} \omega_d(D\varphi^t(u)) \right)^{1/t} = \lim_{t \to +\infty} \left( \max_{u \in K} \omega_d(D\varphi^t(u)) \right)^{1/t}, \quad t > 0.
\]

Proof. The proof of this result follows from Fekete’s lemma for the subadditive functions [47], pp.463-464] \(\blacksquare\)

Corollary 4. If \( \omega_d(D\varphi^t(u)) > 0 \), then
\[
\inf_{t>0} \max_{u \in K} \frac{1}{t} \ln(\omega_d(D\varphi^t(u))) = \lim_{t \to +\infty} \max_{u \in K} \frac{1}{t} \ln(\omega_d(D\varphi^t(u))).
\]

---

1 Considering additional properties of the dynamical system and the singular value function, one could get \( \lim_{t \to +\infty} \omega_d(D\varphi^t(u)) \) instead of \( \lim \inf_{t \to +\infty} \omega_d(D\varphi^t(u)) \), but we do not need it for our further consideration.

2 If \( f : \mathbb{R}^n \to \mathbb{R} \) is a measurable subadditive function, then for every \( u \in \mathbb{R}^n \) there exists the limit \( \lim_{t \to -\infty} \frac{f(tx)}{t} \).
For a compact set $K$, $t > 0$, $u \in K$, and $d \in [0, n]$ we consider two scalar functions $g_d(t, u)$ and $f_d(t, u)$. Suppose that $g_0(t, u) = f_0(t, u) = c$, therefore the following expressions

$$d_g^+(t, u) = \sup\{d \in [0, n] : g_d(t, u) \geq c\}, \quad d_f^+(t, u) = \sup\{d \in [0, n] : f_d(t, u) \geq c\}$$

are well defined. Also we consider

$$d_g^-(t, u) = \inf\{d \in [0, n] : g_d(t, u) < c\}, \quad d_f^-(t, u) = \inf\{d \in [0, n] : f_d(t, u) < c\}.$$

Here and further if the infimum on the empty set is considered, then we assume that the infimum is equal $n$. Define

$$d_f^+(t) = \sup\{d \in [0, n] : \sup_{u \in K} f_d(t, u) \geq c\}, \quad d_f^-(t) = \inf\{d \in [0, n] : \sup_{u \in K} f_d(t, u) < c\}.$$

**Lemma 6.** We have the following properties

**P1** If for fixed $t > 0$ and $u \in K$ the implication $(g_d(t, u) < c \Rightarrow f_d(t, u) < c)$ holds $\forall d \in [0, n]$, then

$$\inf\{d \in [0, n] : f_d(t, u) < c\} \leq \inf\{d \in [0, n] : g_d(t, u) < c\}; \quad (17)$$

**P2** If for fixed $t > 0$ and $u \in K$ the inequality $f_d(t, u) \leq g_d(t, u)$ holds $\forall d \in [0, n]$, then

$$\inf\{d \in [0, n] : f_d(t, u) < c\} \leq \inf\{d \in [0, n] : g_d(t, u) < c\}; \quad (19)$$

**P3** If

$$\sup\{d \in [0, n] : f_d(t, u) \geq c\} = \inf\{d \in [0, n] : f_d(t, u) < c\}; \quad (20)$$

then

$$\sup\{d \in [0, n] : \sup_{u \in K} f_d(t, u) \geq c\} = \sup\sup_{u \in K}\{d \in [0, n] : f_d(t, u) \geq c\}; \quad (21)$$

**P4** If for fixed $t > 0$ the equality

$$\sup_{u \in K} f_d(t, u) = \max_{u \in K} f_d(t, u) \quad \forall d \in [0, n] \quad (22)$$

is valid and [(20)] holds, then

$$\inf\{d \in [0, n] : \sup_{u \in K} f_d(t, u) < c\} = \sup\inf_{u \in K}\{d \in [0, n] : f_d(t, u) < c\}; \quad (23)$$

**P5**

$$\inf\{d \in [0, n] : \inf_{t>0} f_d(t, u) < c\} = \inf\inf_{t>0}\{d \in [0, n] : f_d(t, u) < c\}. \quad (24)$$

**Proof.**

**(P1), (P2):** Since in (P1) and (P2) the set of possible $d$, considered in the left-hand side of expression, involves the set of possible $d$, considered in the right-hand side of expression, we have the corresponding inequalities for the infimums of the sets. Similarly we get relation for suprema.

**(P3):** Since $f_d(t, u) \leq \sup_{u \in K} f_d(t, u)$, by [(19)] in (P2) we have $\sup_{u \in K} d_f^+(t, u) \leq d_f^+(t)$. Let $\sup_{u \in K} d_f^+(t, u) < d_f^+(t)$, then $\exists d_0 \in (\sup_{u \in K} d_f^+(t, u), d_f^+(t)) : f_d(t, u) < c \forall d' \in [d_0, n] \forall u \in K \Rightarrow d_f^+(t) \leq d_0$. Thus we get the contradiction.
For any \( t \) we have \( d_f(t,u) \leq d_f(t) \) for all \( u \in K \) by (P1) we have \( d_f(t,u) \leq d_f(t) \).

Let \( \sup_{u \in K} d_f(t,u) < d_f(t) \). Then \( \exists d_0 \in \{ \sup_{u \in K} d_f(t,u), d_f(t) \} \). Since \( d_0 < d_f(t) \), we have \( \sup_{u \in K} f_d(t,u) \geq c \). Therefore, from condition (22), \( \exists u_0 : f_d(t,u_0) \geq c \). Finally, according to condition (20), we have \( d_0 \leq \sup \{ d \in [0,n] : f_d(t,u_0) \geq c \} = d_f(t,u_0) \leq \sup_{u \in K} d_f(t,u) \). Thus we get the contradiction.

\( \Rightarrow \exists d > 0 \) such that \( d_f(t,u) \leq d_f(t) \).

Let \( d_f(t,u) < \inf_{t>0} d_f(t,u) \) \( \Rightarrow \exists d_0 \in [d_f(t), \inf_{t>0} d_f(t,u)] : \inf_{t>0} f_d(t,u) < c \) \( \Rightarrow \exists t_0 : f_d(t_0,u) < c \) \( \Rightarrow d_0 \geq d_f(t_0,u) \). Thus we get the contradiction. \( \blacksquare \)

Theorem 1. \cite{31, p.147, eq.3.21}, \cite{33, p.112, eq.4.19} Let \( \omega_d(D\varphi^t(u)) > 0 \). For a compact invariant set \( K \) and \( d \in [0,n] \) there is a point \( u^\varphi = u^\varphi(d) \in K \) (it may be not unique) such that

\[
\frac{1}{t} \ln \omega_d(D\varphi^t(u^\varphi(d))) \geq \lim \sup_{t \to +\infty} \frac{1}{t} \ln \omega_d(D\varphi^t(u)) \quad \forall t > 0. \tag{25}
\]

Relation (25) is presented in \cite{33, p.147, eq.3.21}, \cite{30, pp.114, eq.5.6} and its proof is based on the theory of positive operators \cite{14} (see also \cite{37}).

Corollary 5. \( \text{see, e.g.} \ [30, pp.113-114] \)

\[
\sup_{u \in K} \limsup_{t \to +\infty} \frac{1}{t} \ln \omega_d(D\varphi^t(u)) = \lim \sup_{t \to +\infty} \frac{1}{t} \ln \omega_d(D\varphi^t(u)) = \max \limsup_{u \in K} \frac{1}{t} \ln \omega_d(D\varphi^t(u)) = \lim \sup_{u \in K} \frac{1}{t} \ln \omega_d(D\varphi^t(u)). \tag{26}
\]

Proof. It is easy to check that (see, e.g. \cite{20, p.31})

\[
\sup_{u \in K} \limsup_{t \to +\infty} \frac{1}{t} \ln \omega_d(D\varphi^t(u)) \leq \lim \sup_{u \in K} \frac{1}{t} \ln \omega_d(D\varphi^t(u)). \tag{27}
\]

Thus, taking into account (25), we get (26). \( \blacksquare \)

3. Lyapunov dimension of maps

The concept of the Lyapunov dimension had been suggested in the seminal paper by Kaplan and Yorke \cite{15} and later it was rigorously developed in a number of papers (see, e.g. \cite{20, 34}).

The following two definitions are inspired by Douady–Oesterlé \cite{27}.

Definition 6. The (local) Lyapunov dimension\(^3\) of a continuously differentiable map \( \varphi : U \subseteq \mathbb{R}^n \to \mathbb{R}^n \) at the point \( u \in U \) is defined as

\[
dL(\varphi,u) := \sup \{ d \in [0,n] : \omega_d(D\varphi(u)) \geq 1 \}.
\]

For any \( u \in U \) this value is well-defined since \( \omega_0(D\varphi(u)) \equiv 1 \).

By Lemma 2 we get

\[
dL(\varphi,u) = \max \{ d \in [0,n] : \omega_d(D\varphi(u)) \geq 1 \}. \tag{28}
\]

\(^3\)This is not a dimension in a rigorous sense (see, e.g. \cite{8, 11, 39}).
Additionally, since the singular values in \( (1) \) are ordered by decreasing, we have
\[
d_L(\varphi, u) = \max \{ d \in [0, n] : \omega_d(D\varphi(u)) \geq 1 \} = \inf \{ d \in [0, n] : \omega_d(D\varphi(u)) < 1 \}
\] (29)
if the infimum exists (i.e. there exists \( d \in (0, n] \) such that \( \omega_d(D\varphi(u)) < 1 \)). Here and further in the similar constructions if the infimum does not exist, we assume that the infimum and considered dimension are taken equal to \( n \).

**Definition 7.** The Lyapunov dimension of a continuously differentiable map \( \varphi : U \subseteq \mathbb{R}^n \to \mathbb{R}^n \) of the compact set \( K \subset U \subseteq \mathbb{R}^n \) is defined as
\[
d_L(\varphi, K) := \sup_{u \in K} d_L(\varphi, u) = \sup \sup \{ d \in [0, n] : \omega_d(D\varphi(u)) \geq 1 \}.
\]
Remark that by Lemma 6 (property (21)) and Lemma 3 we have
\[
d_L(\varphi, K) = \sup_{u \in K} \sup \{ d \in [0, n] : \omega_d(D\varphi(u)) \geq 1 \} = \sup \{ d \in [0, n] : \max_{u \in K} \omega_d(D\varphi(u)) \geq 1 \}.
\] (30)
Additionally, by (29) and Lemma 6 (property (23)), we have
\[
d_L(\varphi, K) = \sup_{u \in K} \inf \{ d \in [0, n] : \omega_d(D\varphi(u)) < 1 \} = \inf \{ d \in [0, n] : \max_{u \in K} \omega_d(D\varphi(u)) < 1 \}
\] (31)
if the infimum exists (i.e. there exists \( d \in (0, n] \) such that \( \max_{u \in K} \omega_d(D\varphi(u)) < 1 \)).

**Theorem 2.** (Douady–Oesterlé, \([27, p.1135]\); see also \([106, p.369],[102, p.239],[10, p.332]\)) If the continuously differentiable map \( \varphi : U \subseteq \mathbb{R}^n \to \mathbb{R}^n \) has a negatively invariant or invariant compact set \( K \subset U \), i.e.
\[
\varphi(K) \supseteq K,
\]
then
\[
dim_H K \leq d_L(\varphi, K).
\]
Remark that under the assumptions of Theorem 2 if \( \omega_d(D\varphi(u)) < 1 \) for some \( d \leq 1 \), then \( \dim_H K = 0 \) (see, e.g. \([106, p.371]\)). Thus, taking into account Lemma 3 we have

**Lemma 7.** (see, e.g. \([35, p.554]\)) The functions \( u \mapsto d_L(\varphi, u) \) is continuous on \( U \) except at a point \( u \), which satisfies \( \sigma_1(D\varphi(u)) = 1 \), where it is still upper semi-continuous.

**Corollary 6.** By the Weierstrass extreme value theorem for the upper semi-continuous functions, there exists a critical point \( u_L \) (it may be not unique) such that
\[
\sup_{u \in K} d_L(\varphi, u) = \max_{u \in K} d_L(\varphi, u) = d_L(\varphi, u_L).
\] (32)
For an invariant compact set \( K \) of the dynamical system \( \{ \varphi^t \}_{t \geq 0}, (U \subseteq \mathbb{R}^n, \| \cdot \|) \) one may consider for a fixed \( t \) the evolutionary operator \( \varphi^t(u) \), then
\[
\varphi^t(K) = K
\]
and the corresponding Lyapunov dimension (finite time Lyapunov dimension)
\[
d_L(\varphi^t, K) = \sup_{u \in K} d_L(\varphi^t, u) = \inf \{ d \in [0, n] : \max_{u \in K} \omega_d(D\varphi^t(u)) < 1 \}.
\] (33)
Example. If for a nonempty compact set $K \subset U \subseteq \mathbb{R}^n$ it is considered the identical map $\varphi = \text{id}$, then $D\varphi(u) = I$ and by the definition of the Lyapunov dimension we have $d_L(\text{id}, K) = n$. Remark that for $t = 0$ we have $\varphi^0 = \text{id}$ and $d_L(\varphi^0, K) = n$, thus we further consider $t > 0$.

Remark 1. For the numerical estimations of dimension, the following remark is important: for any $t > 0$ the equality $([14]$ for a compact invariant set $K$ implies the existence of $s = s(t) > 0$ such that
\[ d_L(\varphi^{t+s}, K) \leq d_L(\varphi^t, K). \] (34)

While in the computations we can consider only finite time $t \leq T$ and evolutionary operator $\varphi^T(u)$, from a theoretical point of view, it is interesting to study the limit behavior of dynamical system $\{\varphi^t\}_{t \geq 0}$ as $t \to +\infty$. Next, unless otherwise stated, $K \subset U \subseteq \mathbb{R}^n$ denotes a compact invariant set with respect to the dynamical system $(\{\varphi^t\}_{t \geq 0}, (U \subseteq \mathbb{R}^n, \|\cdot\|))$: $\varphi^t(K) = K$, $t > 0$.

4. Lyapunov dimensions of dynamical system

According to the Douady-Oesterlé theorem it is natural to give the following generalization of Definition 7 for dynamical systems.

Definition 8. The Lyapunov dimension of the dynamical system $\{\varphi^t\}_{t \geq 0}$ with respect to a compact invariant set $K$ is defined as
\[ d_L(\{\varphi^t\}_{t \geq 0}, K) := \inf_{t > 0} d_L(\varphi^t, K) = \inf_{t > 0} \sup\{d \in [0, n] : \max_{u \in K} \omega_d(D\varphi^t(u)) \geq 1\}. \] (35)

By Theorem 2 we have
\[ \dim_H K \leq d_L(\{\varphi^t\}_{t \geq 0}, K) \leq d_L(\varphi^t, K). \] (36)

By [33] and Lemma 6 (property (24)) we have
\[ d_L(\{\varphi^t\}_{t \geq 0}, K) = \inf_{t > 0} d_L(\varphi^t, K) = \inf_{t > 0} \sup\{d \in [0, n] : \max_{u \in K} \omega_d(D\varphi^t(u)) \geq 1\}. \] (37)

and, finally, by (14) we have (see also [28, p.65])
\[ d_L(\{\varphi^t\}_{t \geq 0}, K) = \inf_{t > 0} d_L(\varphi^t, K) = \inf\{d \in [0, n] : \lim_{t \to +\infty} \inf_{u \in K} \omega_d(D\varphi^t(u)) = 0\}. \] (38)

It is interesting to consider a critical point $u_L(T) \in K$ such that the supremum of the local finite time Lyapunov dimension $d_L(\varphi^T, u)$ is achieved at this point.

---

4 While $\inf$ and $\sup$ give the same values for $\omega_d(D\varphi^t(u))$ in (30) and (31), for $\inf_{t > 0} \max_{u \in K} \omega_d(D\varphi^t(u))$ we need consider $\sup\{d \in [0, n] : \forall \tilde{u} \in [0, d] \inf_{t > 0} \max_{u \in K} \omega_d(D\varphi^t(u)) \geq 1\} = \inf\{d \in [0, n] : \inf_{t > 0} \max_{u \in K} \omega_d(D\varphi^t(u)) < 1\}$.  

5 Let there exists $\lim_{t \to +\infty} \max_{u \in K} \omega_d(D\varphi^t(u)) = 0$. It is interesting to study 1) the existence of critical point $u_0 \in K$ such that $\lim_{t \to +\infty} \max_{u \in K} \omega_d(D\varphi^t(u)) = \lim \sup_{t \to +\infty} \omega_d(D\varphi^t(u_0))$ and 2) the estimations $\dim_H K \geq \inf\{d \in [0, n] : \sup_{u \in K} \omega_d(D\varphi^t(u)) < 1\} \dim_H K \geq \sup \lim_{t \to +\infty} \sup_{u \in K} \omega_d(D\varphi^t(u)) \geq 1$. Remark, it is clear that $\lim_{t \to +\infty} \max_{u \in K} \omega_d(D\varphi^t(u)) \geq \lim \sup_{t \to +\infty} \omega_d(D\varphi^t(u))$.

From (32) it follows the existence of a critical point $u_L(t)$ such that $d_L(\varphi^t, u_L(t)) = \max_{u \in K} d_L(\varphi^t, u)$. Taking into account (34) we can consider a sequence $t_k \to +\infty$ such that $d_L(\varphi^t, u_L(t_k))$ is monotonically decreasing to $\inf_{t \geq 0} \max_{u \in K} d_L(\varphi^t, u)$. Since $K$ is a compact set, we can consider a subsequence $t_m = t_{k_m} \to +\infty$ such that there exists a limit critical point $u_L^\ast$: $u_L(t_m) \to u_L^\ast \in K$ as $t_m \to +\infty$. Thus we have $d_L(\varphi^{t^\ast}, u_L(t_m)) \leq d_L(\varphi^{t^\ast}, u_L^\ast)$ and $u_L(t_m) \to u_L^\ast \in K$ as $m \to +\infty$. One may guess that $u_L^\ast$ coincides with a critical point $u_L$ from (50).
Proposition 1. Suppose that for a certain \( t = T > 0 \) the supremum of the local finite time Lyapunov dimensions \( d_L(\varphi^T, u) \) is achieved at one of the equilibria points:
\[
d_L(\varphi^T, u^*_eq) = \sup_{u \in K} d_L(\varphi^T, u), \quad \varphi^t(u^*_eq) \equiv u^*_eq.
\] (39)
Then
\[
dim_H K \leq d_L(\varphi^T, u^*_eq) = d_L(\{\varphi^t\}_{t \geq 0}, K) = d_L(\varphi^T, K).
\] (40)

Proof. From (13) we have
\[
\omega_d(D\varphi^t(u^*_eq)) = (\omega_d(D\varphi(u^*_eq)))^t, \quad t \geq 0.
\]
Therefore
\[
(\omega_d(D\varphi^T(u^*_eq)) < 1) \Leftrightarrow (\omega_d(D\varphi^t(u^*_eq)) < 1) \Leftrightarrow (\omega_d(D\varphi^t(u^*_eq)) < 1, \forall t > 0)
\]
and
\[
(\lim \inf_{t \to +\infty} \omega_d(D\varphi^t(u)) < 1) \Rightarrow (\lim \inf_{t \to +\infty} \omega_d(D\varphi^t(u^*_eq)) < 1) \Leftrightarrow (\omega_d(D\varphi^T(u^*_eq)) < 1).
\]
By Lemma 6 (property (17)) we obtain
\[
\inf\{d \in [0, n] : \omega_d(D\varphi^T(u^*_eq)) < 1\} = d_L(\varphi^T, u^*_eq) \leq d_L(\{\varphi^t\}_{t \geq 0}, K).
\]
Finally, by from [36] we get the assertion of the proposition. ■
Further, to consider \( \ln \omega_d(D\varphi^t(u)) \), we suppose that \( \det J(u) \neq 0 \ \forall u \in U \) and thus
\[
\sigma_i(t, u) > 0, \quad i = 1, \ldots, n. \quad (41)
\]

The following definitions of Lyapunov dimension are inspired by Constantin, Foias, Temam [20] p.31,Remark 3.1., ii] and Eden [30] p.114[7]

Definition 9. The (global) Lyapunov dimension of the dynamical system \( \{\varphi^t\}_{t \geq 0} \) with respect to a compact invariant set \( K \) is defined as
\[
d_L^E(\{\varphi^t\}_{t \geq 0}, K) := \inf\{d \in [0, n] : \lim_{t \to +\infty} \max_{u \in K} \frac{1}{t} \ln \omega_d(D\varphi^t(u)) < 0\}. \quad (42)
\]
Correctness of the definition follows from (16).
By (41) we have \( (\lim \inf_{t \to +\infty} \max_{u \in K} (\omega_d(D\varphi^t(u))) < 1 \Leftrightarrow \lim \inf_{t \to +\infty} \max_{u \in K} \ln(\omega_d(D\varphi^t(u))) < 0) \) and
\[
(\lim \inf_{t \to +\infty} \max_{u \in K} \frac{1}{t} \ln (\omega_d(D\varphi^t(u))) < 0 \Rightarrow \lim \inf_{t \to +\infty} \max_{u \in K} \ln(\omega_d(D\varphi^t(u))) < 0).\]
Thus, taking into account (38) and (16), by Lemma 6 (property (17)) we have
\[
d_L(\{\varphi^t\}_{t \geq 0}, K) = \inf\{d \in [0, n] : \lim \inf_{t \to +\infty} \max_{u \in K} \frac{1}{t} \ln \omega_d(D\varphi^t(u)) < 0\} \leq \inf\{d \in [0, n] : \lim \max_{u \in K} \frac{1}{t} \ln \omega_d(D\varphi^t(u)) < 0\} = d_L^E(\{\varphi^t\}_{t \geq 0}, K).
\] (43)

6 In [20] p.31,Remark 3.1., ii] Constantin, Foias, Temam stated that if \( \sup_{u \in K} \lim_{t \to +\infty} (\omega_d(D\varphi^t(u)))^{1/t} < 1 \) or \( \lim_{t \to +\infty} \sup_{u \in K} (\omega_d(D\varphi^t(u)))^{1/t} < 1 \), then \( \dim_H K \leq d \). In [30] p.114 Eden considered the value \( d_K = \inf\{d > 0 : \sup_{u \in K} \omega_d(D\varphi^t(u)) \text{ converges to zero exponentially as } t \to \infty\} \) and called it the Douady-Oesterlé dimension of \( K \).

7 Comparing the expressions in the definitions [35] and [42], remark that we can change \( \frac{1}{t} \) in [42] to another scalar positive monotonically decreasing function \( q(t) \) such that \( \inf_{t \geq 0} q(t) \sup_{u \in K} \omega_d(D\varphi^t(u)) = \lim_{t \to +\infty} q(t) \max_{u \in K} \omega_d(D\varphi^t(u)). \) The last relation is important from a computational point of view.
Since for fixed $t > 0$ and $d \in [0, n]$

$$
\left(1 > \max_{u \in K} (\omega_d(D\varphi^t(u)))\right) \Rightarrow \left(0 > \max_{u \in K} \frac{1}{t} \ln \omega_d(D\varphi^t(u)) \geq \lim_{t \to +\infty} \max_{u \in K} \frac{1}{t} \ln \omega_d(D\varphi^t(u))\right),
$$

by Lemma 6 (property (17)) and (36) we have

**Proposition 2.**

$$\dim_H(K) \leq d_L(\{\varphi^t\}_{t \geq 0}, K) \leq d_L^E(\{\varphi^t\}_{t \geq 0}, K) \leq d_L(\varphi^t, K) \quad \forall t > 0. \quad (44)$$

**Corollary 7.** Taking $\inf_{t > 0}$ in (44), we obtain

$$d_L(\{\varphi^t\}_{t \geq 0}, K) = d_L^E(\{\varphi^t\}_{t \geq 0}, K). \quad (45)$$

**Definition 10.** The local Lyapunov dimension of the dynamical system $\{\varphi^t\}_{t \geq 0}$ at the point $u$ is defined as

$$d_L^E(\{\varphi^t\}_{t \geq 0}, u) := \inf\{d \in [0, n] : \limsup_{t \to +\infty} \frac{1}{t} \ln \omega_d(D\varphi^t(u)) < 0\}. \quad (46)$$

By (26) and Lemma 6 (property (23)) we have

$$\sup \inf_{u \in K} \{d \in [0, n] : \limsup_{t \to +\infty} \frac{1}{t} \ln \omega_d(D\varphi^t(u)) < 0\} = \inf \{d \in [0, n] : \sup \limsup_{u \in K} \frac{1}{t} \ln \omega_d(D\varphi^t(u)) < 0\}$$

Therefore, by (27) and Lemma 6 (property (18)) we get

$$\sup_{u \in K} d_L^E(\{\varphi^t\}_{t \geq 0}, u) = \sup \inf_{u \in K} \{d \in [0, n] : \limsup_{t \to +\infty} \frac{1}{t} \ln \omega_d(D\varphi^t(u)) < 0\} = \inf \{d \in [0, n] : \sup \limsup_{u \in K} \frac{1}{t} \ln \omega_d(D\varphi^t(u)) < 0\} \leq \inf \{d \in [0, n] : \sup \limsup_{u \in K} \frac{1}{t} \ln \omega_d(D\varphi^t(u)) < 0\} = d_L^E(\{\varphi^t\}_{t \geq 0}, K). \quad (47)$$

**Proposition 3.** If there is a critical equilibrium point $u_{eq}^{cr}$ such that (40) is valid, then

$$d_L(\varphi^T, u_{eq}^{cr}) = d_L^E(\{\varphi^t\}_{t \geq 0}, u_{eq}^{cr}) = \sup_{u \in K} d_L^E(\{\varphi^t\}_{t \geq 0}, K)$$

and from (44) it follows

$$\dim_H(K) \leq d_L(\{\varphi^t\}_{t \geq 0}, K) = d_L^E(\{\varphi^t\}_{t \geq 0}, K) = d_L^E(\{\varphi^t\}_{t \geq 0}, u_{eq}^{cr}) = d_L(\varphi^T, K). \quad (49)$$

In this case for the estimation of the Hausdorff dimension by (49) we need only the Douady-Oesterlé theorem (see Theorem 2). In the general case the existence of a critical point $u_{eq}^E$ (it may be not unique) such that

$$d_L^E(\{\varphi^t\}_{t \geq 0}, u_{eq}^E) = \sup_{u \in K} d_L^E(\{\varphi^t\}_{t \geq 0}, u) = d_L^E(\{\varphi^t\}_{t \geq 0}, K) \quad (50)$$

follows from (26) and the so-called Eden conjecture is that $u_{eq}^E$ corresponds to an equilibrium point or to a periodic orbit ([28, p.98, Question 1.]).

Finally, from (44) and (48) we have

**Theorem 3.**

$$\dim_H(K) \leq d_L(\{\varphi^t\}_{t \geq 0}, K) = \sup_{u \in K} d_L^E(\{\varphi^t\}_{t \geq 0}, u) \leq d_L(\varphi^t, K) = \sup_{u \in K} d_L(\varphi^t, u). \quad (51)$$
4.1. Lyapunov exponents: various definitions

**Definition 11.** The Lyapunov exponent functions of singular values (also called finite-time Lyapunov exponents [1]) of the dynamical system \( \{ \varphi^t \}_{t \geq 0}, (U \subseteq \mathbb{R}^n, \| \cdot \|) \) at the point \( u \in U \) are denoted by

\[
\nu_i(t, u) = \nu_i(D\varphi^t(u)), \quad i = 1, 2, \ldots, n,
\]

and defined as

\[
\nu_i(t, u) := \frac{1}{t} \ln \sigma_i(t, u).
\]

**Definition 12.** The Lyapunov exponents (LEs) of singular values of the dynamical system \( \{ \varphi^t \}_{t \geq 0} \) at the point \( u \) are defined (see, e.g. [87], [20, p.29, eq.3.26]) as

\[
\nu_i(u) := \limsup_{t \to +\infty} \nu_i(t, u) = \limsup_{t \to +\infty} \frac{1}{t} \ln(\sigma_i(t, u)), \quad i = 1, 2, \ldots, n.
\]

Often \( \nu_i(u) \) are called upper LEs and denoted as \( \nu_i(u) \), while \( \nu_i(u) := \liminf_{t \to +\infty} \nu_i(t, u) \) are called lower LEs. Remark that the Lyapunov exponents of singular values are the same for any fundamental matrices of the linearized systems (6) or (7).

**Proposition 4.** (see, e.g. [55]) For the matrix \( D\varphi^t(u)P \), where \( P \) is a nonsingular \( n \times n \) matrix (i.e. \( \det P \neq 0 \)), one has

\[
\lim_{t \to +\infty} \left( \nu_i(D\varphi^t(u)) - \nu_i(D\varphi^t(u)P) \right) = 0, \quad i = 1, 2, \ldots, n.
\]

**Definition 13.** The Lyapunov exponent functions of the fundamental matrix columns \((y^1(t, u), \ldots, y^n(t, u)) = D\varphi^t(u)\)

\[
\nu^L_i(t, u) = \nu^L_i(D\varphi^t(u)), \quad i = 1, 2, \ldots, n, \quad u \in U
\]

are defined as

\[
\nu^L_i(t, u) := \frac{1}{t} \ln ||y^i(t, u)||.
\]

The ordered Lyapunov exponent functions of the fundamental matrix columns at the point \( u \) (also called finite-time Lyapunov characteristic exponents) are given by the ordered set (for all \( t > 0 \)) of \( \nu^L_i(t, u) \):

\[
\nu^L_i(t, u) \geq \cdots \geq \nu^L_n(t, u), \quad \forall t \geq 0.
\]

**Definition 14.** The Lyapunov exponents of the fundamental matrix column\( \) are defined (see [83]) as

\[
\nu^L_i(u) := \limsup_{t \to +\infty} \nu^L_i(t, u), \quad i = 1, 2, \ldots, n.
\]

---

8 We add "of singular value" to distinguish this definition from other definitions of Lyapunov exponents; if the differences in the definitions are not significant for the presentation, we use the term "Lyapunov exponents" or "LEs".

9 Often they are called Lyapunov characteristic exponents (LCE) [76]. In [83] these values are defined with the opposite sign and called characteristic exponents at the point \( u \).
Remark 2. The Lyapunov exponents of fundamental matrix columns may be different for different fundamental matrices in contrast to the definition of Lyapunov exponents of singular values (see, e.g. Proposition 4). To get the set of all possible values of Lyapunov exponents of fundamental matrix columns (the set with the minimal sum of values), one has to consider the so-called normal fundamental matrices (see [83], [76]).

Definition 15. The relative Lyapunov exponents of singular value functions of the dynamical system \( \{\varphi^t\}_{t \geq 0} \) at the point \( u \) are defined (see, e.g. [87]) as

\[
\tilde{\nu}_1(u) := \limsup_{t \to +\infty} (\nu_1(t, u)),
\]

\[
\tilde{\nu}_{i+1}(u) := \limsup_{t \to +\infty} (\nu_1(t, u) + \cdots + \nu_{i+1}(t, u)) -
\]

\[
- \limsup_{t \to +\infty} (\nu_1(t, u) + \cdots + \nu_i(t, u)), \quad i = 1, ..., n - 1.
\]

For \( k = 1, 2, ..., n \) we have

\[
\tilde{\nu}_1(u) + \cdots + \tilde{\nu}_k(u) = \limsup_{t \to +\infty} (\nu_1(t, u) + \cdots + \nu_k(t, u)),
\]

\[
\nu_k(u) \leq \tilde{\nu}_k(u) = \limsup_{t \to +\infty} \sum_{i=1}^{k} \nu_i(t, u) - \limsup_{t \to +\infty} \sum_{i=1}^{k-1} \nu_i(t, u) \leq \limsup_{t \to +\infty} \nu_k(t, u) = \nu_k(u).
\]

From the Courant-Fischer theorem [39] it follows (see, e.g. [51]) that

\[
\nu_i(t, u) \leq \nu^v_i(t, u), \quad \forall t \geq 0, \quad \forall u \in U \quad i = 1, 2, ..., n.
\]  \hspace{1cm} (52)

Definition 16. [10, 20] The relative global (or uniform) Lyapunov exponents of singular value functions of the dynamical system \( \{\varphi^t\}_{t \geq 0} \) with respect to the compact invariant set \( K \subset U \) are defined as

\[
\tilde{\nu}_1(K) := \limsup_{t \to +\infty} \sup_{u \in K} \nu_1(t, u),
\]

\[
\tilde{\nu}_{i+1}(K) := \limsup_{t \to +\infty} \sup_{u \in K} \left( \nu_1(t, u) + \cdots + \nu_{i+1}(t, u) \right) -
\]

\[
- \limsup_{t \to +\infty} \sup_{u \in K} \left( \nu_1(t, u) + \cdots + \nu_i(t, u) \right), \quad i = 1, ..., n - 1.
\]

For \( i = 1, 2, ..., n \) we have

\[
\tilde{\nu}_1(K) + \cdots + \tilde{\nu}_i(K) = \limsup_{t \to +\infty} \sup_{u \in K} \left( \nu_1(t, u) + \cdots + \nu_i(t, u) \right).
\]

By (15) and (11) we get (see, e.g. [106] pp.360-361)

\[
\tilde{\nu}_1(K) = \lim_{t \to +\infty} \max_{u \in K} \nu_1(t, u),
\]

\[
\tilde{\nu}_{i+1}(K) = \lim_{t \to +\infty} \max_{u \in K} \left( \nu_1(t, u) + \cdots + \nu_{i+1}(t, u) \right) -
\]

\[
- \lim_{t \to +\infty} \max_{u \in K} \left( \nu_1(t, u) + \cdots + \nu_i(t, u) \right), \quad i = 1, ..., n - 1.
\]

10 For example [55], for the matrix \( u(t) = \begin{pmatrix} 1 & g(t) - g^{-1}(t) \\ 0 & 1 \end{pmatrix} \) we have the following ordered values: \( \nu^v_1 = \max \left( \limsup_{t \to +\infty} \frac{1}{2} \ln |g(t)|, \limsup_{t \to +\infty} \frac{1}{2} \ln |g^{-1}(t)| \right), \nu^v_2 = 0; \nu_{1.2} = \max \left( \limsup_{t \to +\infty} \frac{1}{2} \ln |g(t)|, \limsup_{t \to +\infty} \frac{1}{2} \ln |g^{-1}(t)| \right). \)
From (27) (see, e.g. [28, p.49], [31, p.146]) for \( u \in K \) we obtain the following inequality

\[
\tilde{\nu}_1(u) + \cdots + \tilde{\nu}_i(u) \leq \tilde{\nu}_1(K) + \cdots + \tilde{\nu}_i(K), \quad i = 1, 2, \ldots, n. \tag{53}
\]

At the same time, according to (26), there exists \( u^{cr}(m) \in K \) (it may be not unique) such that the above expressions in (53) coincide [28, 29, 31]:

\[
\tilde{\nu}_1(K) + \cdots + \tilde{\nu}_m(K) = \tilde{\nu}_1(u^{cr}(m)) + \cdots + \tilde{\nu}_m(u^{cr}(m)) = \max_{u \in K} \left( \tilde{\nu}_1(u) + \cdots + \tilde{\nu}_m(u) \right). \tag{54}
\]

Various characteristics of chaotic behavior are based on Lyapunov exponents (e.g., LEs are used in the Kaplan-Yorke formula of the Lyapunov dimension and the sum of positive LEs may be used [84, 90] as the characteristic of Kolmogorov-Sinai entropy rate [46, 101]). The properties of Lyapunov exponents and their various generalizations are studied, e.g., in [2, 6, 12, 22, 43, 48, 55, 56, 63, 76, 83, 87, 90].

The existence of different definitions of LEs, computational methods, and related assumptions led to the appeal: "Whatever you call your exponents, please state clearly how are they being computed" [21].

4.2. Kaplan-Yorke formula of the Lyapunov dimension

**Kaplan-Yorke formula with respect to the finite time Lyapunov exponents**

Consider the dynamical system \( \{\varphi^t\}_{t \geq 0}, (U \subseteq \mathbb{R}^n, || \cdot ||) \). For \( t > 0 \) we have

\[
\frac{1}{t} \ln(\omega_d(D\varphi^t(u))) = \begin{cases} 
0, & d = 0, \\
\sum_{i=1}^{[d]} \nu_i(t, u), & d = [d] \in \{1, \ldots, n\}, \\
\sum_{i=1}^{[d]} \nu_i(t, u) + (d - [d]) \nu_{[d]+1}(t, u), & d \in (0, n). 
\end{cases} \tag{55}
\]

If \( \ln(\omega_n(D\varphi^t(u))) \leq 0 \) for fixed \( t > 0 \) and point \( u \in K \), then by (29) for \( d(t, u) := d_L(\varphi^t, u) \) we have

\[
\frac{1}{t} \ln(\omega_{d(t,u)}(D\varphi^t(u))) = 0, \quad \frac{1}{t} \ln(\omega_{d(t,u)+\delta}(D\varphi^t(u))) < 0 \quad \forall \delta \in (0, n-d(t, u)]. \tag{56}
\]

Let for \( t > 0 \)

\[
j(t, u) = j(\{\nu_i(t, u)\}_{i=1}^n) := [d(t, u)] \in \{0, \ldots, n\}, \\
s(t, u) = s(\{\nu_i(t, u)\}_{i=1}^n) := d(t, u) - j(t, u) \in [0, 1). \]

Then for \( j(t, u) \leq n - 1 \) from (56) it follows that

\[
\sum_{i=1}^{j(t,u)} \nu_i(t, u) \geq 0, \quad \sum_{i=1}^{j(t,u)+1} \nu_i(t, u) < 0. \tag{57}
\]

We have

\[
\frac{1}{t} \ln(\omega_{d(t,u)}(D\varphi^t(u))) = \frac{1}{t} \ln \left( (\omega_{j(t,u)}(D\varphi^t(u)))^{1-s(t,u)}(\omega_{j(t,u)+1}(D\varphi^t(u)))^{s(t,u)} \right) = \]

\[
= (1-s(t,u)) \sum_{i=1}^{j(t,u)} \nu_i(t, u) + s(t,u) \sum_{i=1}^{j(t,u)+1} \nu_i(t, u) = 0. \]

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Therefore
\[
s(t, u) = \begin{cases} 
\frac{\nu_1(t, u) + \cdots + \nu_{j(t,u)}(t, u)}{|\nu_{j(t,u)+1}(t, u)|} < 1, & j(t, u) \in \{1, \ldots, n-1\}, \\
0, & j(t, u) = 0 \text{ or } j(t, u) = n.
\end{cases}
\]

The expression
\[
d^\text{KY}_L(\{\nu_i(t, u)\}_{1}^{n}) := j(t, u) + \frac{\nu_1(t, u) + \cdots + \nu_{j(t,u)}(t, u)}{|\nu_{j(t,u)+1}(t, u)|}
\]
(58)
corresponds to the Kaplan-Yorke formula [45] with respect to the finite time Lyapunov exponents, i.e. the ordered set \(\{\nu_i(t, u)\}_{1}^{n}\). The idea of \(d^\text{KY}_L\) construction may be used with other types of Lyapunov exponents (see below).

Further we assume that the relation \(s(t, u) = 0\) for \(j(t, u) = 0\) and \(j(t, u) = n\) follows from the first expression for \(s(t, u)\). Since \(\frac{1}{t} \ln(\omega_{d(t,u)}(D\varphi^t(u))) \leq 0 \Leftrightarrow \omega_{d(t,u)}(D\varphi^t(u)) \leq 1\) for \(t > 0\), from [33] we have

**Proposition 5.**

\[
d_L(\varphi^t, K) = \sup_{u \in K} d_L(\varphi^t, u) = \sup_{u \in K} d^\text{KY}_L(\{\nu_i(t, u)\}_{1}^{n}) = \sup_{u \in K} \left( j(t, u) + \frac{\nu_1(t, u) + \cdots + \nu_{j(t,u)}(t, u)}{|\nu_{j(t,u)+1}(t, u)|} \right). 
\]
(59)

While in computing we can consider only finite time \(t \leq T\), from a theoretical point of view, it may be interesting to study the limit behavior of \(\sup_{u \in K} d^\text{KY}_L(\{\nu_i(t, u)\}_{1}^{n})\) as \(t \to +\infty\).

**Kaplan-Yorke formula with respect to the relative global Lyapunov exponents of singular value functions**

Let
\[
j = j(\{\tilde{\nu}_i(K)\}_{1}^{n}) := \max\{m \in \{0, \ldots, n\} : \sum_{i=1}^{m} \tilde{\nu}_i(K) \geq 0\},
\]
\[
s = s(\{\tilde{\nu}_i(K)\}_{1}^{n}) := \begin{cases} 
0 \leq \tilde{\nu}_1(K) + \cdots + \tilde{\nu}_j(K) \left| \tilde{\nu}_{j+1}(K) \right| < 1, & j \in \{1, \ldots, n-1\}, \\
0, & j = 0 \text{ or } j = n.
\end{cases}
\]

The expression \(d^\text{KY}_L(\{\tilde{\nu}_i(K)\}_{1}^{n}) := j + \frac{\tilde{\nu}_1(K) + \cdots + \tilde{\nu}_j(K)}{|\tilde{\nu}_{j+1}(K)|}\) is the Kaplan-Yorke formula of Lyapunov dimension with respect to the relative global Lyapunov exponents of singular value functions. Then
\[
\lim_{t \to +\infty} \max_{u \in K} \frac{1}{t} \ln \left( \omega_{j+s}(D\varphi^t(u)) \right) = \lim_{t \to +\infty} \max_{u \in K} \left( \sum_{i=1}^{j} \nu_i(t, u) + s \nu_{j+1}(t, u) \right) \leq
\]
\[
= \lim_{t \to +\infty} \max_{u \in K} \left( (1 - s) \sum_{i=1}^{j} \nu_i(t, u) + s \sum_{i=1}^{j+1} \nu_i(t, u) \right) \leq
\]
(since, in general, the maximums may be achieved at different points \(u\))
\[
\leq \lim_{t \to +\infty} \max_{u \in K} (1 - s) \sum_{i=1}^{j} \nu_i(t, u) + \lim_{t \to +\infty} \max_{u \in K} s \sum_{i=1}^{j+1} \nu_i(t, u) =
\]
\[
= (1 - s) \lim_{t \to +\infty} \max_{u \in K} \sum_{i=1}^{j} \nu_i(t, u) + s \lim_{t \to +\infty} \max_{u \in K} \sum_{i=1}^{j+1} \nu_i(t, u) = 0.
\]

Thus, for any \(\tilde{s} : s < \tilde{s} < 1\), \(\lim_{t \to +\infty} \frac{1}{t} \ln(\omega_{j+s}(D\varphi^t(u))) < 0\) and from Definition [9] we have
Proposition 6. (see, e.g. [26], pp.30-31)

\[
d^E_t(\{\varphi^t\}_{t \geq 0}, K) \leq d^K_Y(\{\bar{\nu}_i(K)\})^n
\]

Under some conditions we can obtain the equality.

Corollary 8. If critical points \(u^{cr}(j)\) and \(u^{cr}(j+1)\) from \([26]\) coincide, i.e. \(u^{cr} = u^{cr}(j) = u^{cr}(j+1)\), then

\[
\lim_{t \to +\infty} \sum_{k=1}^j \nu_k(t, u^{cr}) = \lim_{t \to +\infty} \max_{u \in K} \sum_{k=1}^j \nu_k(t, u),
\]

\[
\lim_{t \to +\infty} \sum_{k=1}^{j+1} \nu_k(t, u^{cr}) = \lim_{t \to +\infty} \max_{u \in K} \sum_{k=1}^{j+1} \nu_k(t, u),
\]

and

\[
d_t(\{\varphi^t\}_{t \geq 0}, K) = d^K_Y(\{\bar{\nu}_i(K)\})^n.
\]

In [35, p.565] the systems, having property (60), are called “typical systems”.

Kaplan-Yorke formula with respect to relative Lyapunov exponents of singular value functions

Let

\[
j(u) = j(\{\bar{\nu}_i(u)\}) := \max\{m \in \{0, \ldots, n\} : \sum_{i=1}^m \bar{\nu}_i(u) \geq 0\}
\]

\[
s(u) = s(\{\bar{\nu}_i(u)\}) := \begin{cases} 0 \leq \frac{\bar{\nu}_1(u) + \cdots + \bar{\nu}_{j(u)}(u)}{|\bar{\nu}_{j(u)+1}(u)|} < 1, & j(u) \in \{1, \ldots, n-1\}, \\ 0, & j(u) = 0 \text{ or } j(u) = n. \end{cases}
\]

The expression \(d^K_Y(\{\bar{\nu}_i(u)\}) := j(u) + \frac{\bar{\nu}_1(u) + \cdots + \bar{\nu}_{j(u)}(u)}{|\bar{\nu}_{j(u)+1}(u)|}\) is the Kaplan-Yorke formula of Lyapunov dimension with respect to the relative Lyapunov exponents of singular value functions. We have

\[
\lim_{t \to +\infty} \sup \frac{1}{t} \ln \left(\omega_{j(u)+s(u)}(D\varphi^t(u))\right) = \lim_{t \to +\infty} \left(\sum_{i=1}^{j(u)} \nu_i(t, u) + s(u) \nu_{j(u)+1}(t, u)\right) =
\]

\[
= \lim_{t \to +\infty} \left(1 - s(u)\right) \sum_{i=1}^{j(u)} \nu_i(t, u) + \nu_{j(u)+1}(t, u) =
\]

\[
\leq \lim_{t \to +\infty} (1 - s(u)) \sum_{i=1}^{j(u)} \nu_i(t, u) + \lim_{t \to +\infty} s(u) \sum_{i=1}^{j(u)+1} \nu_i(t, u) =
\]

\[
= (1 - s(u)) \lim_{t \to +\infty} \sum_{i=1}^{j(u)} \nu_i(t, u) + s(u) \lim_{t \to +\infty} \sum_{i=1}^{j(u)+1} \nu_i(t, u) = 0.
\]

Thus, for any \(j(u) < n\) and \(\bar{s} : s(u) < \bar{s} < 1\), \(\lim_{t \to +\infty} \frac{1}{t} \ln \left(\omega_{j(u)+s(u)}(D\varphi^t(u))\right) < 0\) and from Definition [10] we have

Proposition 7.

\[
\sup_{u \in K} d_t(\{\varphi^t\}_{t \geq 0}, u) \leq \sup_{u \in K} d^K_Y(\{\bar{\nu}_i(u)\})^n = \sup_{u \in K} \left(j(u) + \frac{\bar{\nu}_1(u) + \cdots + \bar{\nu}_{j(u)}(u)}{|\bar{\nu}_{j(u)+1}(u)|}\right).
\]
Proposition 8. (see, e.g. [28, p.60])

\[
\sup_{u \in K} d_{L}^{\text{KY}}(\{\tilde{v}_i(u)\}_1^n) \leq d_{L}^{\text{KY}}(\{\tilde{v}_i(K)\}_1^n). \tag{62}
\]

\textbf{Proof.} The assertion follows from the relation (see [28, p.60])

\[
\sup_{u \in K} j(\{\tilde{v}_i(u)\}_1^n) = j(\{\tilde{v}_i(K)\}_1^n) \tag{63}
\]

and inequality (63). \(\blacksquare\)

Remark that there are examples in which inequality (62) is strict (see, e.g. [28, pp.49-51, 62-63]).

\textit{Kaplan-Yorke formula with respect to the Lyapunov exponents of singular values}

Let (see, e.g. [20, pp.32-34])

\[
j(u) = j(\{\nu_i(u)\}_1^n) := \max\{m \in \{0, \ldots, n\} : \sum_{i=1}^{m} \nu_i(u) \geq 0\},
\]

\[
s(u) = s(\{\nu_i(u)\}_1^n) := \begin{cases} 
0 \leq \frac{\nu_1(u) + \cdots + \nu_{j(u)}(u)}{|\nu_{j(u)+1}(u)|} < 1, & j(u) \in \{1, \ldots, n-1\}, \\
0, & j(u) = 0 \text{ or } j(u) = n.
\end{cases}
\]

The expression \(d_{L}^{\text{KY}}(\{\nu_i(u)\}_1^n) := j(u) + \frac{\nu_1(u) + \cdots + \nu_{j(u)}(u)}{|\nu_{j(u)+1}(u)|}\) is the Kaplan-Yorke formula of Lyapunov dimension with respect to the Lyapunov exponents of singular values.

Then

\[
\limsup_{t \to +\infty} \frac{1}{t} \ln \left(\omega_{j(u)+s(u)}(D\varphi^t(u))\right) \leq (1 - s(u)) \limsup_{t \to +\infty} \sum_{i=1}^{j(u)} \nu_i(t, u) + s(u) \limsup_{t \to +\infty} \sum_{i=1}^{j(u)+1} \nu_i(t, u) \leq
\]

\[
\leq (1 - s(u)) \sum_{i=1}^{j(u)} \nu_i(u) + s(u) \sum_{i=1}^{j(u)+1} \nu_i(u) = 0.
\]

For \(j(u) < n\) and any \(\overline{s} : s(u) < \overline{s} < 1\), \(\limsup_{t \to +\infty} \frac{1}{t} \ln \left(\omega_{j(u)+\overline{s}}(D\varphi^t(u))\right) < 0\) and from Definition 10 we have

\textbf{Proposition 9.}

\[
\sup_{u \in K} d_{L}(\{\varphi^t\}_{t \geq 0}, u) \leq \sup_{u \in K} d_{L}^{\text{KY}}(\{\nu_i(u)\}_1^n) = \sup_{u \in K} \left( j(u) + \frac{\nu_1(u) + \cdots + \nu_{j(u)}(u)}{|\nu_{j(u)+1}(u)|} \right). \tag{64}
\]

\footnote{Let \(\nu_1(t, u) = (e^u)^t\), \(\nu_2(t, u) = \left(\frac{1}{2}(1-u)\right)^t\) for all \(u \in K = [0, 1]\). Thus \(\nu_1(u) = \tilde{v}_1(u) = u\), \(\nu(u) = \tilde{v}_2 = \ln(1-u) - \ln 2\); \(\tilde{v}_1(K) = 1\), \(\tilde{v}_2 = -1 - \ln 2\); Here \(u^e(1) = 1\); \(\tilde{v}_1(1) = \tilde{v}_1(K) = 1\); \(u^e(2) = 0\); \(\tilde{v}_1(0) + \tilde{v}_2(0) = \tilde{v}_1(K) + \tilde{v}_2(K) = -\ln 2\). Then \(\sup_{u \in [0, 1]} d_{L}^{\text{KY}}(\{\tilde{v}_i(u)\}_1^n) = \frac{-\ln 2 - \ln(1-u)}{1+\ln 2} < 1 + \frac{1}{1+\ln 2} = d_{L}(\{\tilde{v}_i(K)\}_1^n).\)}
Kaplan-Yorke formula with respect to the Lyapunov exponents of fundamental matrix columns

Let

\[ j(u) = j(\{\nu_i^1(u)\}_1^n) := \max\{m \in \{0, \ldots, n\} : \sum_{k=1}^m \nu_i^k(u) \geq 0\}, \]

\[ s(u) = s(\{\nu_i^1(u)\}_1^n) := \begin{cases} 0 \leq \frac{\nu_i^1(u) + \cdots + \nu_i^{j(u)}(u)}{\nu_i^{j(u)+1}(u)} < 1, & j(u) \in \{1, \ldots, n-1\}, \\ 0, & j(u) = 0 \text{ or } j(u) = n. \end{cases} \]

The expression \( d_{\text{LY}}^\text{KY} (\{\nu_i^1(u)\}_1^n) := j(u) + \frac{\nu_i^1(u) + \cdots + \nu_i^{j(u)}(u)}{\nu_i^{j(u)+1}(u)} \) is the Kaplan-Yorke formula of Lyapunov dimension with respect to the Lyapunov exponents of fundamental matrix columns.

Then, similar to (61), by (52) we obtain

\[ \limsup_{t \to +\infty} \frac{1}{t} \ln (\omega_{j(u)+s(u)}(D\varphi^t(u))) \leq \]

\[ \leq (1 - s(u)) \sum_{i=1}^{j(u)} \nu_i^1(u) + s(u) \sum_{i=1}^{j(u)+1} \nu_i^1(u) = 0. \]

Thus, for \( j(u) < n \) and any \( \bar{s} : s(u) < \bar{s} < 1 \), \( \limsup_{t \to +\infty} \frac{1}{t} \ln (\omega_{j(u)+\bar{s}}(D\varphi^t(u))) < 0 \) and from Definition 10 we get

**Proposition 10.**

\[ \sup_{u \in K} d_{\text{LY}} (\{\varphi^t\}_{t \geq 0}, u) \leq \sup_{u \in K} d_{\text{LY}}^\text{KY} (\{\nu_i^1(u)\}_1^n) = \sup_{u \in K} \left( j(u) + \frac{\nu_i^1(u) + \cdots + \nu_i^{j(u)}(u)}{\nu_i^{j(u)+1}(u)} \right). \]

**Computation by the Kaplan-Yorke formulas**

For a given invariant set \( K \) and a given point \( u_0 \in K \) there are two essential questions related to the computation of Lyapunov exponents and the use of the Kaplan-Yorke formulas of local Lyapunov dimension \( \sup_{u \in K} d_{\text{LY}}^\text{KY} (\{\nu_i^1(u)\}_1^n) \) and \( \sup_{u \in K} d_{\text{LY}}^\text{KY} (\{\tilde{\nu}_i(u)\}_1^n) \):

(a) \( \limsup_{t \to +\infty} \nu_i(t, u_0) = \lim_{t \to +\infty} \nu_i(t, u_0) \) or \( \limsup_{t \to +\infty} (\sum_i \nu_i(t, u)) = \lim_{t \to +\infty} (\sum_i \nu_i(t, u)) \)

(b) if the above limits do not exist, then

\[ \sup_{u \in K} d_{\text{LY}}^\text{KY} (\{\nu_i(u)\}_1^n) = \sup_{u \in K \setminus \{\varphi^t(u_0), t \geq 0\}} d_{\text{LY}}^\text{KY} (\{\nu_i(u)\}_1^n) \]

or

\[ \sup_{u \in K} d_{\text{LY}}^\text{KY} (\{\tilde{\nu}_i(u)\}_1^n) = \sup_{u \in K \setminus \{\varphi^t(u_0), t \geq 0\}} d_{\text{LY}}^\text{KY} (\{\tilde{\nu}_i(u)\}_1^n). \]

In order to get rigorously the positive answer to these questions, from a theoretical point of view, one may use various ergodic properties of the dynamical system \( \{\varphi^t\}_{t \geq 0} \) (see, Oseledec [87], Ledrappier [61], and some auxiliary results in [8, 24]). However, from a practical point of view, the rigorous use of the above results is a challenging task (e.g. even for the well-studied Lorenz system) and hardly can be done effectively in the general case (see, e.g. the corresponding discussions in [7, 21] p.118, [89, 110] p.9) and the works on the Perron effects of the largest Lyapunov exponent sign reversals [56, 76]). For an example of the effective rigorous use of the ergodic theory for the estimation of the Hausdorff and Lyapunov dimensions see, e.g. [95].

Thus, in the general case, from a practical point of view, one cannot rely on the above relations (a) and (b) and shall use \( \limsup_{t \to +\infty} \) in the definitions of local Lyapunov exponents and the corresponding formulas for the Lyapunov dimension (see, e.g. Temam [106]).

However, if \( u_0 \) is an equilibrium point, then the expression “\( \limsup_{t \to +\infty} \)” in Definitions 12, 14 and 15 can be replaced by “\( \lim_{t \to +\infty} \)” and we have
Lemma 8. Let \( \varphi^t(u_0) \) be a stationary point, i.e. \( \varphi^t(u_0) \equiv u_0 \). Then for \( i = 1, 2, ..., n \) we have
\[
\lim_{t \to +\infty} \nu_i(t, u_0) = \nu_i(u_0) = \tilde{\nu}_i(u_0) = \nu^L_i(u_0).
\]

Thus, for \( j = j(\{\tilde{\nu}_i(K)\}^n_1) \), we get

Proposition 11. If critical points in (50) and (60) coincide with an equilibrium point \( u^c_{eq} \), i.e
\[
\varphi^t(u^c_{eq}) \equiv u^c_{eq} = u^c_L \equiv u^c(j) = u^c(j + 1),
\]

\[
d_L(\{\varphi^t\}^{t \geq 0}_{t \geq 0}, u^c_{eq}) = d_L(\{\varphi^t\}^{t \geq 0}_{t \geq 0}, K) = d_L^K(\{\tilde{\nu}_i(K)\}^n_1)
\]

and
\[
d_L(\{\varphi^t\}^{t \geq 0}_{t \geq 0}, u^c_{eq}) = \sup_{u \in K} d_L^K(\{\tilde{\nu}_i(u)\}^n_1) = \sup_{u \in K} d_L^K(\{\nu_i(u)\}^n_1) = \sup_{u \in K} d_L^K(\{\nu^L_i(u)\}^n_1) = \lim_{t \to +\infty} \max_{u \in K} d_L^K(\{\nu_i(t, u)\}^n_1)
\]

If \( u^c_T = u^c_T(j) = u^c_T(j + 1) \) belongs to a periodic orbit with period \( T \), then the same reasoning can be applied for \( (\varphi^T)^t \).

The last section of this survey is devoted to the examples in which the maximum of the local Lyapunov dimension achieves at an equilibrium point.

Taking into account the existence of different definitions of Lyapunov dimension and related formulas and following [21], we recommend that whatever you call your Lyapunov dimension, please state clearly how is it being computed.

5. Analytical estimates of the Lyapunov dimension and its invariance with respect to diffeomorphisms

Along with widely used numerical methods for estimating and computing the Lyapunov dimension (see, e.g. MATLAB realizations of the methods based on QR and SVD decompositions in [58] [67]) there is an effective analytical approach, proposed by G.A.Leonov in 1991 [71] (see also [10] [62] [67] [72] [74] [78]). The Leonov method is based on the direct Lyapunov method with special Lyapunov-like functions. The advantage of this method is that it allows one to estimate the Lyapunov dimension of invariant set without localization of the set in the phase space and in many cases get effectively exact Lyapunov dimension formula [62] [69] [70] [75] [78].

Following [51], next the invariance of Lyapunov dimension with respect to diffeomorphisms and its relation with the Leonov method are discussed. An analog of Leonov method for discrete time dynamical systems is suggested.

While topological dimensions are invariant with respect to Lipschitz homeomorphisms, the Hausdorff dimension is invariant with respect to Lipschitz diffeomorphisms and noninteger Hausdorff dimension is not invariant with respect to homeomorphisms [41]. Since the Lyapunov dimension is used as an upper estimate of Hausdorff dimension, the question arises whether the Lyapunov dimension is invariant under diffeomorphisms (see, e.g. [50] [88]).

Consider the dynamical system \( \{\{\varphi^t\}^{t \geq 0}_{t \geq 0}, (U \subseteq \mathbb{R}^n, ||\cdot||) \) under the change of coordinates \( w = h(u) \), where \( h : U \subseteq \mathbb{R}^n \to \mathbb{R}^n \) is a diffeomorphism. In this case the semi-orbit \( \gamma^+(u) = \{\varphi^t(u), t \geq 0\} \) is mapped to the semi-orbit defined by \( \varphi^t_h(w) = \varphi^t_h(h(u)) = h(\varphi^t(u)) \), the dynamical system \( \{\{\varphi^t\}^{t \geq 0}_{t \geq 0}, (U \subseteq \mathbb{R}^n, ||\cdot||) \) is transformed to the dynamical system \( \{\{\varphi^t_h\}^{t \geq 0}_{t \geq 0}, (h(U) \subseteq \mathbb{R}^n, ||\cdot||) \) \), and a compact set \( K \subseteq U \) invariant with respect to \( \{\varphi^t\}^{t \geq 0}_{t \geq 0} \) is mapped to the compact set \( h(K) \subseteq h(U) \) invariant with respect to \( \{\varphi^t_h\}^{t \geq 0}_{t \geq 0} \).

\[
D_w\varphi^t_h(w) = D_w(h(\varphi^t(h^{-1}(w)))) = D_u(h(\varphi^t(h^{-1}(w))))D_u\varphi^t(h^{-1}(w))D_u^1 h^{-1}(w),
\]
\[ D_u(\varphi_h^t(h(u))) = D_w \varphi_h^t(h(u)) D_u h(u) = D_u(h(\varphi^t(u))) = D_u h(\varphi^t(u)) D_u \varphi^t(u). \]

Therefore
\[ D_u h^{-1}(w) = (D_u h(u))^{-1} \]
and
\[ D \varphi_h^t(w) = Dh(\varphi^t(u)) D \varphi^t(u)(Dh(u))^{-1}. \quad (64) \]

If \( u \in K \), then \( \varphi^t(u) \) and \( \varphi_h^t(h(u)) \) define bounded semi-orbits. Remark that \( Dh \) and \( (Dh)^{-1} \) are continuous and, thus, \( Dh(\varphi^t(u)) \) and \( (Dh(\varphi^t(u)))^{-1} \) are bounded in \( t \). From (11) it follows that for any \( d \in [0, n] \) there is a constant \( c = c(d) \geq 1 \) such that for any \( t \geq 0 \)
\[ \max_{u \in K} \omega_d(Dh(u)) \leq c, \quad \max_{u \in K} \omega_d((Dh(u))^{-1}) \leq c, \quad t \geq 0. \quad (65) \]

**Lemma 9.** If for a fixed \( t > 0 \) there exist diffeomorphism \( h : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n \) and \( d \in [0, n] \) such that the estimation
\[ \max_{w \in h(K)} \omega_d(D \varphi_h^t(w)) = \max_{u \in K} \omega_d(Dh(\varphi^t(u))D \varphi^t(u)(Dh(u))^{-1}) < 1 \quad (66) \]
is valid\(^{12}\) then for \( u \in K \)
\[ \liminf_{t \to +\infty} \left( \omega_d(D \varphi^t(u)) - \omega_d(D \varphi_h^t(h(u))) \right) = 0 \]
and
\[ \liminf_{t \to +\infty} \omega_d(D \varphi_h^t(h(u))) = \liminf_{t \to +\infty} \omega_d(D \varphi^t(u)) = 0. \]

**Proof.** Applying (2) to (64), we get
\[ \omega_d(D \varphi_h^t(h(u))) \leq \omega_d(Dh(\varphi^t(u))) \omega_d(D \varphi^t(u)) \omega_d((Dh(u))^{-1}). \]

By (65) we obtain
\[ \omega_d(D \varphi_h^t(h(u))) \leq c^2 \omega_d(D \varphi^t(u)). \]

Similarly
\[ \omega_d(D \varphi^t(u)) \leq \omega_d((Dh(\varphi^t(u)))^{-1}) \omega_d(D \varphi_h^t(h(u))) \omega_d(Dh(u)) \]
and
\[ \omega_d(D \varphi^t(u)) \leq c^2 \omega_d(D \varphi_h^t(h(u))). \]

Therefore for any \( d \in [0, n] \), \( t \geq 0 \), and \( u \in K \)
\[ c^{-2} \omega_d(D \varphi_h^t(h(u))) \leq \omega_d(D \varphi^t(u)) \leq c^2 \omega_d(D \varphi_h^t(h(u))) \quad (67) \]
and
\[ (c^{-2} - 1) \omega_d(D \varphi_h^t(h(u))) \leq \omega_d(D \varphi^t(u)) - \omega_d(D \varphi_h^t(h(u))) \leq (c^2 - 1) \omega_d(D \varphi^t(h(u))). \]

If for a fixed \( t \geq 0 \) there is \( d \in [0, n] \) such that \( \sup_{u \in K} \omega_d(D \varphi_h^t(h(u))) < 1 \), then by (??) we have
\[ \liminf_{t \to +\infty} \omega_d(D \varphi_h^t(h(u))) = 0 \]
and
\[ 0 \leq \liminf_{t \to +\infty} \left( \omega_d(D \varphi^t(u)) - \omega_d(D \varphi_h^t(h(u))) \right) \leq 0. \]

\[^{12}\text{The expression in (66) corresponds to the expressions considered in [71, eq.(1)] for } p(u) = Dh(u), \quad [62, eq.(1)] \text{ and [72, p.99, eq.10.1] for } Q(u) = Dh(u).\]
Corollary 9. (see, e.g. [52]) For $u \in K$ we have
\[
\lim_{t \to +\infty} \left( \nu_i \left( D\varphi_h^t(h(u)) \right) - \nu_i \left( D\varphi^t(u) \right) \right) = 0, \quad i = 1, 2, \ldots, n
\]
and, therefore,
\[
\limsup_{t \to +\infty} \nu_i \left( D\varphi^t_h(h(u)) \right) = \limsup_{t \to +\infty} \nu_i \left( D\varphi^t(u) \right), \quad i = 1, 2, \ldots, n.
\]

**Proof.** For $t > 0$ from (67) we get
\[
\frac{1}{t} \ln c^{-2} + \frac{1}{t} \ln \omega_d \left( D\varphi^t_h(h(u)) \right) \leq \frac{1}{t} \ln \omega_d \left( D\varphi^t(u) \right) \leq \frac{1}{t} \ln c^2 + \frac{1}{t} \ln \omega_d \left( D\varphi^t_h(h(u)) \right).
\]
Thus for the integer $d = m$ we have
\[
\lim_{t \to +\infty} \left( \frac{1}{t} \ln \omega_m \left( D\varphi^t(u) \right) - \frac{1}{t} \ln \omega_m \left( D\varphi^t_h(h(u)) \right) \right) = \lim_{t \to +\infty} \left( \sum_{i=1}^{m} \nu_i \left( D\varphi^t(u) \right) - \sum_{i=1}^{m} \nu_i \left( D\varphi^t_h(h(u)) \right) \right) = 0.
\]

The above statements are rigorous reformulation from [52, 63] and implies the following

**Proposition 12.** The Lyapunov dimension of the dynamical system $\{\varphi^t\}_{t \geq 0}$ with respect to the compact invariant set $K$ is invariant with respect to any diffeomorphism $h : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$, i.e.
\[
d_L(\{\varphi^t\}_{t \geq 0}, K) = d_L(\{\varphi^t_h\}_{t \geq 0}, h(K)).
\]

**Proof.** Lemma 8 implies that if $\max_{w \in h(K)} \omega_d \left( D\varphi^t_h(w) \right) < 1$ for a fixed $t > 0$ and $d \in [0, n]$, then there exists $T > t$ such that
\[
\max_{w \in K} \omega_d \left( D\varphi^T(u) \right) < 1
\]
and vice versa. Thus the set of $d$, over which inf$_{t>0}$ is taken in (33), is the same for $D\varphi^t(u)$ and $D\varphi^t_h(w)$ and, therefore,
\[
\inf_{t>0} \inf \{d \in [0, n] : \max_{w \in K} \omega_d \left( D\varphi^t(u) \right) < 1 \} = \inf_{t>0} \inf \{d \in [0, n] : \max_{w \in h(K)} \omega_d \left( D\varphi^t_h(w) \right) < 1 \}.
\]

**Corollary 10.** Suppose $H(u)$ is a $n \times n$ matrix, the elements of which are scalar continuous functions of $u$ and $\det H(u) \neq 0$ for $u \in K$. If for a fixed $t > 0$ there is $d \in (0, n]$ such that
\[
\max_{w \in h(K)} \omega_d \left( D\varphi^t_h(w) \right) = \max_{u \in K} \omega_d \left( H(\varphi^t(u)) D\varphi^t(u) H(u)^{-1} \right) < 1,
\]
then by (66) with $H(u)$ instead of $Dh(u)$, (69) and (70) for sufficiently large $t = T > 0$ we have
\[
\dim_h K \leq d_L(\{\varphi^t\}_{t \geq 0}, K) \leq d_L(\varphi^T, K) \leq d.
\]
If it is considered $H(u) = p(u) S$, where $p(u) : U \subseteq \mathbb{R}^n \to \mathbb{R}^1$ is a continuous positive scalar function and $S$ is a nonsingular $n \times n$ matrix, then condition (71) takes the form
\[
\sup_{u \in K} \omega_d \left( H(\varphi^t(u)) D\varphi^t(u) H(u)^{-1} \right) = \sup_{u \in K} \left( (p(\varphi^t(u)) p(u)^{-1})^d \omega_d \left( SD\varphi^t(u) S^{-1} \right) \right) < 1.
\]
Consider now the Leonov method of analytical estimation of the Lyapunov dimension and its relation with the invariance of Lyapunov dimension with respect to diffeomorphisms. Following [62, 71], we consider the special class of diffeomorphisms such that $Dh(u) = p(u)S$, where $p(u) : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^1$ is a continuous scalar function and $S$ is a nonsingular $n \times n$ matrix. As it is shown below the multiplier of the type $p(\varphi'(u))(p(u))^{-1}$ in [72] plays the role of Lyapunov-like functions.\[13\]

Let us apply the linear change of variables $w = h(u) = Su$ with a nonsingular $n \times n$ matrix $S$. Then $\varphi'(u_0) = u(t, u_0)$ is transformed into $\varphi^t_S(u_0)$:

$$\varphi^t_S(u_0) = w(t, w_0) = S\varphi^t(u_0) = Su(t, S^{-1}w_0).$$

Consider the transformed systems (3) and (5)

$$\dot{w} = Sf(S^{-1}w) \text{ or } w(t + 1) = S\phi(S^{-1}w(t))$$

and their linearizations along the solution $\varphi^t_S(u_0) = w(t, w_0) = S\varphi^t(u_0)$:

$$\dot{v} = JS(w(t, w_0))v \text{ or } v(t + 1) = JS(w(t, w_0))v(t),$$

$$JS(w(t, w_0)) = S J(S^{-1}w(t, w_0)) S^{-1} = S J(u(t, u_0)) S^{-1}. \quad (73)$$

For the corresponding fundamental matrices we have $D\varphi^t_S(w) = SD\varphi^t(u)S^{-1}$.

First we consider continuous time dynamical system. Let $\lambda_i(u_0, S) = \lambda_i(S\varphi^t(u_0)), i = 1, 2, ..., n$, be eigenvalues of the symmetrized Jacobian matrix

$$\frac{1}{2} (SJ(u(t, u_0))S^{-1} + (SJ(u(t, u_0))S^{-1})^*) = \frac{1}{2} (JS(w(t, w_0)) + JS(w(t, w_0))^*) \quad (74)$$

ordered so that $\lambda_1(u_0, S) \geq \cdots \geq \lambda_n(u_0, S)$ for any $u_0 \in U$. The following theorem is rigorous reformulation of results from [62, 72, 73].

**Theorem 4.** Let $d = (j + s) \in [1, n]$, where integer $j = [d] \in \{1, \ldots, n\}$ and real $s = (d - [d]) \in [0, 1)$. If there are a differentiable scalar function $V(u) : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^1$ and a nonsingular $n \times n$ matrix $S$ such that

$$\sup_{u \in K} (\lambda_1(u, S) + \cdots + \lambda_j(u, S) + s\lambda_{j+1}(u, S) + \dot{V}(u)) < 0, \quad (75)$$

where $\dot{V}(u) = (\text{grad}(V))^*f(u)$, then

$$\dim H K \leq d_L(\{\varphi^t\}_{t \geq 0}, K) \leq d_L(\varphi^T, K) \leq j + s$$

for sufficiently large $T > 0$.

**Proof.** From the following relations (see Liouville’s formula and, e.g., [72, p.102])

$$\omega_{j+s}(SD\varphi^t(u)S^{-1}) = \exp\left(\int_0^t \lambda_1(S\varphi^\tau(u)) + \cdots + \lambda_j(S\varphi^\tau(u)) + s\lambda_{j+1}(S\varphi^\tau(u)) d\tau\right) \quad (76)$$

and

$$(p(\varphi'(u))(p(u))^{-1})^{j+s} = \exp\left(\dot{V}(\varphi^t(u)) - V(u)\right) = \exp\left(\int_0^t \dot{V}(\varphi^\tau(u)) d\tau\right)$$

\[13\] In [86] it is interpreted as changes of Riemannian metrics.
we get
\[
\left(p\left(\varphi^t(u)p(u)^{-1}\right)\right)^{j+s}\omega_{j+s}\left(SD\varphi^t(u)S^{-1}\right) \leq \\
\exp\left(\int_0^T \left(\lambda_1(S\varphi^\tau(u)) + \cdots + \lambda_j(S\varphi^\tau(u)) + s\lambda_{j+1}(S\varphi^\tau(u)) + \dot{V}(\varphi^\tau(u))\right) d\tau\right). \tag{77}
\]

Since \(\varphi^t(u) \in K\) for any \(u \in K\), for \(t > 0\) by [75] we have

\[
\max_{u \in K} \left(\left(p\left(\varphi^t(u)p(u)^{-1}\right)\right)^{j+s}\omega_{j+s}\left(SD\varphi^t(u)S^{-1}\right)\right) < 1, \quad t > 0.
\]

Therefore by Corollary 10 with \(H(u) = p(u)S\), where \(p(u) = \left(e^{V(u)}\right)^{\frac{1}{2}}\), we get the assertion of the theorem. ■

Now consider discrete time dynamical system. Let \(\lambda_i(u_0, S) = \lambda_i(S\varphi^t(u_0)), i = 1, 2, ..., n\), be positive square roots of the eigenvalues of the symmetrized Jacobian matrix

\[
\left((SJ(u(t, u_0))S^{-1})^*SJ(u(t, u_0))S^{-1}\right) = (J_S(w(t, w_0))^*J_S(w(t, w_0))),
\]

ordered so that \(\lambda_1(u_0, S) \geq \cdots \geq \lambda_n(u_0, S)\) for any \(u_0 \in U\).

**Theorem 5.** [50] Let \(d = (j + s) \in [1, n]\), where integer \(j = \lfloor d \rfloor \in \{1, \ldots, n\}\) and real \(s = (d - \lfloor d \rfloor) \in [0, 1)\). If there is a scalar continuous function \(V(u) : U \subseteq \mathbb{R}^n \to \mathbb{R}^1\) and a nonsingular \(n \times n\) matrix \(S\) such that

\[
\sup_{u \in K} \left(\ln \lambda_1(u, S) + \cdots + \ln \lambda_j(u, S) + s \ln \lambda_{j+1}(u, S) + (V(\varphi(u)) - V(u))\right) < 0, \tag{79}
\]

then

\[
\dim_H K \leq d_L(\{\varphi^t\}_{t \geq 0}, K) \leq d_L(\varphi^T, K) \leq j + s
\]

for sufficiently large \(T > 0\).

**Proof.** By [2] for \(D\varphi_S^t(u) = SD\varphi^t(u)S^{-1} = \prod_{\tau=0}^{t-1} \left(SJ(u(\tau, u_0))S^{-1}\right)\) we have

\[
\omega_{j+s}\left(SD\varphi^t(u)S^{-1}\right) \leq \prod_{\tau=0}^{t-1} \omega_{j+s}\left(SJ(u(\tau, u_0))S^{-1}\right). \tag{80}
\]

Therefore by the discrete analog of [76] we have

\[
\omega_{j+s}\left(SD\varphi^t(u)S^{-1}\right) \leq \prod_{\tau=0}^{t-1} \lambda_1(S\varphi^\tau(u)) \cdots \lambda_j(S\varphi^\tau(u))(\lambda_{j+1}(S\varphi^\tau(u)))^s. \tag{81}
\]

By the relation

\[
\left(p\left(\varphi^t(u)p(u)^{-1}\right)\right)^{j+s} = \exp (V(\varphi^t(u)) - V(u)) = \exp \left(\sum_{\tau=0}^{t-1} V(\varphi^{\tau+1}(u)) - V(\varphi^\tau(u))\right)
\]

and we get

\[
\ln \left(p\left(\varphi^t(u)p(u)^{-1}\right)\right)^{j+s} + \ln \omega_{j+s}\left(SD\varphi^t(u)S^{-1}\right) \leq \\
\leq \sum_{\tau=0}^{t-1} \left(\ln \lambda_1(S\varphi^\tau(u)) + \cdots + \ln \lambda_j(S\varphi^\tau(u)) + s \ln \lambda_{j+1}(S\varphi^\tau(u)) + V(\varphi(\varphi^\tau(u))) - V(\varphi^\tau(u))\right).
\]
Since $\varphi^t(u) \in K$ for any $u \in K$, by (79) and Corollary 10 with $H(u) = p(u)S$, where $p(u) = (e^{V(u)})^{1/2}$, we get the assertion of the theorem. ■

From (40) we have

**Corollary 11.** If at an equilibrium point $u_{eq}^{cr} \equiv \varphi^t(u_{eq}^{cr})$ for a certain $t > 0$ the relation

$$d_L(\varphi^t, u_{eq}^{cr}) = j + s$$

holds, then for any invariant set $K \ni u_{eq}^{cr}$ we get analytical formula of exact Lyapunov dimension

$$\dim_H K = d_L(\{\varphi^t\}_{t\geq 0}, K) = d_L(\{\varphi^t\}_{t\geq 0}, u_{eq}^{cr}) = j + s.$$  

Remark that in the above approach we need only the Douady-Oesterlé theorem (see Theorem 2) and do not use the results on the Lyapunov dimension developed by Eden, Constantin, Foias, Temam in [20, 31] (see (42),(46), Propositions 6 and 7).

In [9, 93] it is demonstrated, how a technique similar to the above can be effectively applied to derive constructive upper bounds of the topological entropy of dynamical systems.

For the study of continuous time dynamical system in $\mathbb{R}^3$ the following result is useful. Consider a certain open set $K_\varepsilon \subset U \subset \mathbb{R}^n$, which is diffeomorphic to a ball, whose boundary $\partial K_\varepsilon$ is transversal to the vectors $f(u)$, $u \in \partial K_\varepsilon$. Let the set $K_\varepsilon$ be a positively invariant for the solutions of system (3).

**Theorem 6.** (see, [72, 72]) Suppose a continuously differentiable function $V(u)$ and a non-degenerate matrix $S$ exist such that

$$\lambda_1(u, S) + \lambda_2(u, S) + \dot{V}(u) < 0, \quad \forall u \in K_\varepsilon. \quad (82)$$

Then any solution of system (3) with the initial data $u_0 \in K_\varepsilon$ tends to the stationary set as $t \to +\infty$.

### 6. Analytical formulas of exact Lyapunov dimension for well-known dynamical systems

Next we consider examples in which the critical point, corresponding to the maximum of the local Lyapunov dimension, is one of the equilibrium points (see (49)). Let us consider several examples of smooth dynamical systems generated by difference and differential equations (for an example of PDE see, e.g. [26]). In these examples we assume the existence of invariant set $K$ in which the corresponding dynamical system $\{\varphi^t\}_{t\geq 0}$ is defined, and use the compact notation $d_L(K)$ for the Lyapunov dimension instead of (35).

#### 6.1. Henon map

Consider the Henon map $F: \mathbb{R}^2 \to \mathbb{R}^2$

$$\left(\begin{array}{c} x \\ y \end{array}\right) \rightarrow \left(\begin{array}{c} a + by - x^2 \\ x \end{array}\right), \quad (83)$$

where $a > 0$, $b \in (0, 1)$ are the parameters of mapping. The stationary points $(x_+, x_-)$ of this map are the following

$$x_+ = \frac{1}{2} \left[ b - 1 + \sqrt{(b-1)^2 + 4a} \right],$$

$$x_- = \frac{1}{2} \left[ b - 1 - \sqrt{(b-1)^2 + 4a} \right].$$
Theorem 7. [63] For a bounded invariant set \( K \ni (x_-, x_-) \) with respect to (83) we have

\[
d_L(K) = 1 + \frac{1}{1 - \ln b / \ln \sigma_1(x_-)},
\]

where

\[
\sigma_1(x_-) = \sqrt{x_-^2 + b - x_-}.
\]

6.2. Lorenz system

Consider the classical Lorenz system suggested in [82]:

\[
\begin{align*}
\dot{x} &= \sigma (y - x), \\
\dot{y} &= rx - y - xz, \\
\dot{z} &= -bz + xy,
\end{align*}
\]

(84)

where \( \sigma > 0, r > 0, b > 0 \) because of their physical meaning (e.g., \( b = 4(1 + a^2)^{-1} \) is positive and bounded).

Since the system is dissipative and generates a dynamical system \( \{ \varphi_t \}_{t \geq 0} \) (to verify this, it is sufficiently to consider the Lyapunov function \( V(x,y,z) = \frac{1}{2}(x^2 + y^2 + (z - r - \sigma)^2) \); see, e.g., [10, 82]), it possesses a global attractor [10, 15].

Theorem 8. [63] Assume that the following inequalities

\[
r - 1 > 0, \quad (85)
\]

\[
r - 1 \geq \frac{b(b + \sigma - 1)^2 - 4\sigma(b + \sigma b - b^2)}{3\sigma^2} \quad (86)
\]

are satisfied. Suppose that one of the following two conditions holds:

a. \( \sigma^2(r - 1)(b - 4) \leq 4\sigma(\sigma b + b - b^2) - b(b + \sigma - 1)^2; \quad (87) \)

b. There are two distinct real roots of equations

\[
(2\sigma - b + \gamma)^2 (b(b + \sigma - 1)^2 - 4\sigma(b + \sigma b - b^2) + \sigma^2(r - 1)(b - 4)) + 4b\gamma(\sigma + 1) (b(b + \sigma - 1)^2 - 4\sigma(b + \sigma b - b^2) - 3\sigma^2(r - 1)) = 0 \quad (88)
\]

and

\[
\begin{align*}
\sigma^2(r - 1)(b - 4) &> 4\sigma(\sigma b + b - b^2) - b(b + \sigma - 1)^2, \\
\gamma^{(I)} &> 0
\end{align*}
\]

(89)

where \( \gamma^{(I)} \) is the greater root of equation (88).

In this case we have:

1. If

\[
(b - \sigma)(b - 1) < \sigma r < (b + 1)(b + \sigma), \quad (90)
\]

then any bounded on \([0; +\infty)\) solution of system (84) tends to a certain equilibrium as \( t \to +\infty \).
2. If
\[ \sigma r > (b + 1)(b + \sigma), \]  
then for a bounded invariant set \( K \ni (0, 0, 0) \)
\[ d_L(K) = 3 - \frac{2(\sigma + b + 1)}{\sigma + 1 + \sqrt{(\sigma - 1)^2 + 4\sigma r}}. \]  
(92)

If \( (0, 0, 0) \notin K \), then the right-hand side of (92) is an upper bound of \( d_L(\{\varphi^t\}_{t \geq 0}, K) \).

The existence of analytical formula for the exact Lyapunov dimension of the Lorenz system with classical parameters is known (see, e.g. [68]) as the Eden conjecture on the Lorenz system (see [29, p.411, Question 3.], [28, p.98, Question 2.], [30]).

**Remark 3.** It can be easily checked numerically that if all three equilibria are hyperbolic (see the theorem in [69]), then the conditions of Theorem 8 are satisfied. For example, for the standard parameters \( \sigma = 10 \) and \( b = \frac{8}{3} \) formula (92) is valid for \( r > \frac{209}{45} \).

### 6.3. Glukhovsky-Dolzhansky system

Consider a system, suggested by Glukhovsky and Dolghansky [36]
\[
\begin{align*}
\dot{x} &= -\sigma x + z + a_0 y z, \\
\dot{y} &= R - y - x z, \\
\dot{z} &= -z + x y,
\end{align*}
\]  
(93)
where \( \sigma, R, a_0 \) are positive numbers (here \( u = (x, y, x) \)). By the change of variables
\[
(x, y, z) \rightarrow (x, R - \frac{\sigma}{a_0 R + 1} z, \frac{\sigma}{a_0 R + 1} y)
\]  
(94)
system (93) becomes
\[
\begin{align*}
\dot{x} &= -\sigma x + \sigma y - \frac{a_0 \sigma^2}{(a_0 R + 1)^2} y z, \\
\dot{y} &= \frac{R}{\sigma} (a_0 R + 1) x - y - x z, \\
\dot{z} &= -z + x y.
\end{align*}
\]  
(95)
System (95) is a generalization of Lorenz system (84) and can be written as
\[
\begin{align*}
\dot{x} &= \sigma(y - x) - A y z \\
\dot{y} &= r x - y - x z \\
\dot{z} &= -b z + x y,
\end{align*}
\]  
(96)
where
\[
A = \frac{a_0 \sigma^2}{(a_0 R + 1)^2}, \quad r = \frac{R}{\sigma} (a_0 R + 1), \quad b = 1.
\]  
(97)

**Theorem 9.** [75] If
1. \( \sigma = Ar, 4\sigma r > (b + 1)(b + \sigma) \)
or

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2. $b = 1$, $r > 2$, and
\[
\begin{align*}
\sigma &> \frac{-3+2\sqrt{3}}{3} Ar, & \text{if } & 2 < r \leq 4, \\
\sigma &\in \left(\frac{-3+2\sqrt{3}}{3} Ar, \frac{3r+2\sqrt{r(2r+1)}}{r-4} Ar\right), & \text{if } & r > 4,
\end{align*}
\]
then for a bounded invariant set $K \ni (0, 0, 0)$ of system (96) with $b = 1$ or $\sigma = Ar$ we have
\[
d_L(K) = 3 - \frac{2(\sigma + 2)}{\sigma + 1 + \sqrt{(\sigma - 1)^2 + 4\sigma r}}, \tag{98}
\]
If $(0, 0, 0) \notin K$, then the right-hand side of the above relation is an upper bound of $d_L(\{\varphi^t\}_{t \geq 0}, K)$.

Note that this formula coincides with the formula for the classical Lorenz system \[73\]. Remark that system (93) is dissipative and possesses a global attractor (see, e.g. \[67\]).

6.4. Yang and Tigan systems

Consider the Yang system \[109\]:
\[
\begin{align*}
\dot{x} &= \sigma(y - x), \\
\dot{y} &= rx - xz, \\
\dot{z} &= -bz + xy,
\end{align*}
\tag{99}
\]
where $\sigma > 0$, $b > 0$, and $r$ is a real number. Consider also the T-system (Tigan system) \[107\]:
\[
\begin{align*}
\dot{x} &= a(y - x), \\
\dot{y} &= (c - a)x - axz, \\
\dot{z} &= -bz + xy.
\end{align*}
\tag{100}
\]
By the transformation $(x, y, z) \rightarrow (\frac{x}{\sqrt{a}}, \frac{y}{\sqrt{a}}, \frac{z}{a})$ the Tigan system takes the form of the Yang system with parameters $\sigma = a, r = c - a$.

**Theorem 10.** \[65\]

1. Assume $r = 0$ and the following inequalities $b(\sigma - b) > 0$, $\sigma - \frac{(\sigma + b)^2}{4(\sigma - b)} \geq 0$ are satisfied. Then any bounded on $[0; +\infty)$ solution of system (99) tends to a certain equilibrium as $t \to +\infty$.

2. Assume $r < 0$ and $r\sigma + b(\sigma - b) > 0$. Then any bounded on $[0; +\infty)$ solution of system (99) tends to a certain equilibrium as $t \to +\infty$.

3. Assume $r > 0$ and there are two distinct real roots $\gamma^{(II)} > \gamma^{(I)}$ of equation
\[
4br\sigma^2(\gamma + 2\sigma - b)^2 + 16b\gamma(r\sigma^2 + b(\sigma + b)^2 - 4\sigma(r + \sigma b - b^2)) = 0 \tag{101}
\]
such that $\gamma^{(II)} > 0$.

In this case
(a) if $b(b - \sigma) < r\sigma < b(\sigma + b)$,
then any bounded on $[0; +\infty)$ solution of system (99) tends to a certain equilibrium as $t \to +\infty$. 

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(b) if
\[ r\sigma > b(\sigma + b), \]
then
\[ d_L(K) = 3 - \frac{2(\sigma + b)}{\sigma + \sqrt{\sigma^2 + 4\sigma r}}, \]
where \( K \ni (0, 0, 0) \) is a bounded invariant set of system (99). If \( (0, 0, 0) \notin K \), then the right-hand side of the above relation is an upper bound of \( d_L(\{\varphi^t\}_{t \geq 0}, K) \).

6.5. Shimizu-Morioka system

Consider the Shimizu-Morioka system \([100]\) of the form
\[
\begin{align*}
\dot{x} &= y, \\
\dot{y} &= x - \lambda y - xz, \\
\dot{z} &= -\alpha z + x^2,
\end{align*}
\]
where \( \alpha, \lambda \) are positive parameters.

Using the diffeomorphism
\[
\begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrow \begin{pmatrix} x \\ y \\ z - \frac{x^2}{2} \end{pmatrix},
\]
system (104) can be reduced to the following system
\[
\begin{align*}
\dot{x} &= y, \\
\dot{y} &= x - \lambda y - xz + \frac{x^3}{2}, \\
\dot{z} &= -\alpha z + xy + \left(1 + \frac{\alpha}{2}\right)x^2,
\end{align*}
\]
where \( \alpha, \lambda \) are the positive parameters of system (104). We say that system (106) is a transformed Shimizu-Morioka system.

Theorem 11. \([63]\) Suppose, \( K \) is a bounded invariant set of system (106): \( (0, 0, 0) \in K \), and the following relations
\[
\lambda - 4 \leq \sqrt{10 + \frac{3}{\alpha} - 13\alpha}, \lambda < \frac{1}{\alpha} - \alpha, 4 - \lambda \leq \sqrt{\frac{8 + 15\alpha - 8\alpha^2 - 24\alpha^3}{2\alpha(\alpha + 1)}}
\]
are satisfied. Then
\[
d_L(K) = 3 - \frac{2(\lambda + \alpha)}{\lambda + \sqrt{\lambda^2 + 4}}.
\]
If \( (0, 0, 0) \notin K \), then the right-hand side of relation (108) is an upper bound of \( d_L(\{\varphi^t\}_{t \geq 0}, K) \).

In the proof there are used the Lyapunov function of the form
\[
V(x, y, z) = \frac{1 - s}{4\sqrt{4 + \lambda^2}}\vartheta,
\]
where
\[
\vartheta = \mu_1(2y^2 - 2xy - x^4 + 2x^2z) + \mu_2x^2 - \frac{4}{\alpha}z + \mu_3(z^2 - x^2z + \frac{x^4}{4} + xy) + \mu_4(z^2 + y^2 - \frac{x^4}{4} - x^2),
\]
and the nonsingular matrix
\[
S = \begin{pmatrix}
-\frac{1}{k} & 0 & 0 \\
\lambda - \alpha & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]
7. Attractors of dynamical systems

Compact invariant sets of dynamical systems are related with the notions of attractors (see, e.g. [4, 10, 15, 18, 59, 60, 72, 106]). Consider dynamical system \((\{\varphi^t\}_{t \geq 0}, (U \subseteq \mathbb{R}^n, ||\cdot||))\).

**Property 1.** An invariant set \(K \subset U \subseteq \mathbb{R}^n\) is said to be locally attractive if for a certain \(\varepsilon\)-neighborhood of the set \(K\): \(K_{\varepsilon} \subseteq U\),

\[
\lim_{t \to +\infty} \rho(K, \varphi^t(u)) = 0, \quad \forall u \in K_{\varepsilon}.
\]

Here \(\rho(K,u)\) is the distance from the point \(u\) to the set \(K\), defined as

\[
\rho(K,u) = \inf_{w \in K} ||w - u||,
\]

and \(K_{\varepsilon}\) is the set of points \(u\) for which \(\rho(K,u) < \varepsilon\).

**Property 2.** An invariant set \(K \subset U \subseteq \mathbb{R}^n\) is said to be globally attractive if

\[
\lim_{t \to +\infty} \rho(K, \varphi^t(u)) = 0, \quad \forall u \in U.
\]

**Property 3.** An invariant set \(K \subset U \subseteq \mathbb{R}^n\) is said to be uniformly locally attractive with respect to the dynamical system \(\{\varphi^t\}_{t \geq 0}\) if for a certain \(\varepsilon\)-neighborhood \(K_{\varepsilon} \subseteq U\), any number \(\delta > 0\), and any bounded set \(B \subseteq U \subseteq \mathbb{R}^n\), there exists a number \(t(\delta,B) > 0\) such that

\[
\varphi^t(B \cap K_{\varepsilon}) \subset K_{\delta}, \quad \forall t \geq t(\delta,B).
\]

Here

\[
\varphi^t(B \cap K_{\varepsilon}) = \{\varphi^t(u_0) \mid u_0 \in B \cap K_{\varepsilon}\}.
\]

**Property 4.** Invariant set \(K \subset U \subseteq \mathbb{R}^n\) is said to be uniformly globally attractive with respect to the dynamical system \(\{\varphi^t\}_{t \geq 0}\) if for any number \(\delta > 0\) and any bounded set \(B \subseteq U \subseteq \mathbb{R}^n\) there exists a number \(t(\delta,B) > 0\) such that

\[
\varphi^t(B) \subset K_{\delta}, \quad \forall t \geq t(\delta,B).
\]

**Definition 17.** For a dynamical system, a bounded closed invariant set \(K\) is

(1) an attractor if it is a locally attractive set (i.e., it satisfies Property 1);

(2) a global attractor if it is a globally attractive set (i.e., it satisfies Property 2);

(3) a B-attractor if it is a uniformly locally attractive set (i.e., it satisfies Property 3); or

(4) a global B-attractor if it is a uniformly globally attractive set (i.e., it satisfies Property 4).

**Remark 4.** In the above definition we assume the closedness for the sake of uniqueness. The reason is that the closure of a locally attractive invariant set \(K\) is also a locally attractive invariant set (for example, consider an attractor with excluded one of the embedded unstable periodic orbits). Note that if a dynamical system is defined for negative \(t\), then a locally attractive invariant set contains only whole trajectories, i.e. if \(u_0 \in K\), then \(\varphi^t(u_0) \in K\) for \(t \in \mathbb{R}\) (see [15]).
Remark 5. The definition under consideration implies that a global B-attractor is also a global attractor (and an attractor). Consequently, it is rational to introduce the notion of a minimal global attractor (and a minimal attractor) [15, 18]. This is the minimal bounded closed invariant set that possesses Property 3 (or Property 4, i.e. minimal local attractor is an attractor, which cannot be represented as a union of local attractors). Further, ”global attractor” means ”minimal global attractor”.

Definition 18. For an attractor $K$, the basin of attraction is the set $\beta(K) \subseteq U \subseteq \mathbb{R}^n$ of all $u_0 \in U$ such that
\[
\lim_{t \to +\infty} \rho(K, \varphi^t(u_0)) = 0.
\]

7.1. Computation of attractors and Lyapunov dimension

The study of a dynamical system typically begins with an analysis of the equilibria, which are easily found numerically or analytically. Therefore, from a computational perspective, it is natural to suggest the following classification of attractors, which is based on the simplicity of finding their basins of attraction in the phase space:

Definition 19. [67, 77, 79, 80] An attractor is called a self-excited attractor if its basin of attraction intersects with any open neighborhood of a stationary state (an equilibrium), otherwise it is called a hidden attractor.

Self-excited attractor in a system can be found using the standard computational procedure, i.e. by constructing a solution using initial data from a small neighborhood of the equilibrium, observing how it is attracted and, thus, visualizes the attractor. For example, in the Lorenz system (84) with classical parameters $\sigma = 10, \beta = 8/3, \rho = 28$ there is a chaotic attractor, which is self-excited with respect to all three equilibria and could have been found using the standard computational procedure with initial data in vicinity of any of the equilibria (see Fig. 1). Here it is possible to check numerically that for the considered parameters the local attractor is a global attractor (i.e. there are no other attractors in the phase space). In this case the global B-attractor involves the chaotic local attractor, three unstable equilibria and their unstable manifolds attracted to the chaotic local attractor.

However it is known that for other values of parameters, e.g. $\sigma = 10, \beta = 8/3, \rho = 24.5$ [103], the chaotic local attractor in the Lorenz system may be self-excited with respect to the zero unstable equilibrium only. In this case there are three coexisting minimal local attractors (see Fig. 2): chaotic local attractor and two trivial local attractors — stable equilibria $S_{1,2}$.
Figure 2: Numerical visualization of self-excited chaotic local attractor in the Lorenz system. Local B-attractor involves self-excited chaotic local attractor, unstable zero equilibrium and its unstable manifold attracted to the chaotic local attractor (left subfigure), $d_L(K) = \sup_{u \in K} d_L(u) = d_L(S_0) = 2.3727$ according to (92). Trajectories with the initial data $(\pm 1.3276, \mp 9.7014, 28.7491)$ tend to trivial local attractors — equilibria $S_{2,1}$ (middle subfigure), $d_L(S_{2,1}) = 1.9989$. Global attractor is the union of three coexisting local attractors: self-excited chaotic local attractor and two trivial local attractors (right subfigure), $d_L(K) \approx 2.0489$ by numerical computation. Parameters: $r = 24.5, \sigma = 10, b = 8/3$.

Self-excited attractors in a multistable system can be found using the standard computational procedure, whereas there is no standard way of predicting the existence of hidden attractors in a system.

While the multistability is a property of system, the self-excited and hidden properties are the properties of attractor and its basin. For example, hidden attractors are attractors in systems with no equilibria or with only one stable equilibrium (a special case of multistability and coexistence of attractors).

Figure 3: Numerical visualization of local B-attractor and hidden local attractor in the Glukhovsky-Dolghansky system. Local B-attractor involves outgoing separatrix (blue) of the saddle $S_0$ (red) attracted to the stable equilibria $S_{1,2}$ (green) (left subfigure). Hidden local attractor (magenta, $d_L(K) \approx 2.1322$ by numerical computation) coexists with local B-attractor ($d_L(K) = \sup_{u \in K} d_L(u) = d_L(S_0) = 2.8917$ by [98]). Global B-attractor involves the local B-attractor and the hidden local attractor.

In general, there is no straightforward way of predicting the existence or coexistence of hidden attractors in a system (see, e.g. [23, 51, 53, 54, 57, 66, 67, 77, 79, 80]). A numerical search of hidden attractors by evolutionary algorithms is discussed in [112, 113]. Recent examples of hidden attractors can be found in The European Physical Journal Special Topics: Multistability: Uncovering Hidden Attractors, 2015 (see [11, 32, 33, 44, 81, 91, 93, 97, 99, 104, 108, 114]).

For example, in the Glukhovsky-Dolghansky system and the corresponding generalized Lorenz system (96) with parameters $r = 700, a = 0.0052, \sigma = ra, b = 1$ a hidden chaotic local attractor can be found [66, 67] (see Fig. 3).
Remark that if a system is proved to be dissipative (i.e., it is possible to determine an absorbing bounded domain in the phase space such that all trajectories enter this domain within a finite time), then all self-excited or hidden local attractors of the system are inside this absorbing bounded domain and can be found numerically. However, in general, the determination of the number and mutual disposition of chaotic minimal local attractors in the phase space for a system may be a challenging problem [78] (see, e.g., the corresponding well-known problem for two-dimensional polynomial systems — the second part of 16th Hilbert problem on the number and mutual disposition of limit cycles [38] [43]). Thus the advantage of the analytical method for the Lyapunov dimension estimation, suggested in Theorem [4], is that it is useful not only for the dissipative systems (see, e.g., estimation of the Lyapunov dimension for one of the Rossler systems [74]) but also allows one to estimate the Lyapunov dimension of invariant set without localization of the set in the phase space.

Remark that, from a computational perspective, it is not feasible to numerically check Property [1] for all initial states of the phase space of a dynamical system. A natural generalization of the notion of an attractor is the consideration of the weaker attraction requirements: almost everywhere or on a set of positive measure (see, e.g., [85]). See also trajectory attractors [13] [17] [96]. In numerical computations, to distinguish an artificial computer generated chaos from a real behavior of the system, one can consider the shadowing property of the system (see, e.g., the survey in [92]).

We can typically see an attractor (or global attractor) in numerical experiments. The notion of a B-attractor is mostly used in the theory of dimensions, where we consider invariant sets covered by balls. The uniform attraction requirement in Property [3] implies that a global B-attractor involves a set of stationary points $S$ and the corresponding unstable manifolds $W^u(S) = \{u_0 \in \mathbb{R}^n | \lim_{t \to -\infty} \rho(S, \varphi^t(u_0)) = 0\}$ (see, e.g., [15] [18]). The same is true for B-attractor if the considered neighborhood $K_\varepsilon$ in Property [3] contains some of the stationary points from $S$. This allows one to get analytical estimations of the Lyapunov dimension for B-attractors and even formulas since the local Lyapunov dimension at a stationary point can be easily obtained analytically (but this does not help for chaotic minimal local attractors, hidden B-attractors since they do not involve any stationary points).

From a computational perspective, numerical check of Property [3] is also difficult. Therefore, if the basin of attraction involves unstable manifolds of equilibria, then computing the minimal attractor and the unstable manifolds that are attracted to it may be regarded as an approximation of minimal B-attractor. For example, consider the visualization of the classical Lorenz attractor from the neighborhood of the zero saddle equilibria. Note that a minimal global attractor involves the set $S$ and its basin of attraction involves the set $W^u(S)$.

For the computation of the Lyapunov dimension of an attractor $A$ we consider a sufficiently large time $T$ and a sufficiently dense grid of points $A_{\text{grid}}$ on the attractor, compute the local Lyapunov dimensions by the corresponding Kaplan-Yorke formula $d^\text{KY}_L(\{\nu_i(T, u)\}_{i=1}^n)$, and take maximum on the grid: $\max_{u \in A_{\text{grid}}} d^\text{KY}_L(\{\nu_i(T, u)\}_{i=1}^n)$.

Since numerically we can check only that all points of the grid belong to the basin of attraction, the following remark is useful. Let a point $u_0$ belongs to the basin of attraction of attractor $A$. Consider the union of the semi-orbit $\gamma^+(u_0) = \{\varphi^t(u_0), t \geq 0\}$ and attractor $A$: $\hat{K}(u_0) = A \cup \gamma^+(u_0)$. According to the definition of the basin of attraction, $\omega$-limit set of $\varphi^t(u_0)$ belong to $A$, thus the

\[ \text{Remark that if a system is proved to be dissipative (i.e., it is possible to determine an absorbing bounded domain in the phase space such that all trajectories enter this domain within a finite time), then all self-excited or hidden local attractors of the system are inside this absorbing bounded domain and can be found numerically. However, in general, the determination of the number and mutual disposition of chaotic minimal local attractors in the phase space for a system may be a challenging problem [78] (see, e.g., the corresponding well-known problem for two-dimensional polynomial systems — the second part of 16th Hilbert problem on the number and mutual disposition of limit cycles [38] [43]). Thus the advantage of the analytical method for the Lyapunov dimension estimation, suggested in Theorem [4], is that it is useful not only for the dissipative systems (see, e.g., estimation of the Lyapunov dimension for one of the Rossler systems [74]) but also allows one to estimate the Lyapunov dimension of invariant set without localization of the set in the phase space.

Remark that, from a computational perspective, it is not feasible to numerically check Property [1] for all initial states of the phase space of a dynamical system. A natural generalization of the notion of an attractor is the consideration of the weaker attraction requirements: almost everywhere or on a set of positive measure (see, e.g., [85]). See also trajectory attractors [13] [17] [96]. In numerical computations, to distinguish an artificial computer generated chaos from a real behavior of the system, one can consider the shadowing property of the system (see, e.g., the survey in [92]).

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set $K(u_0)$ is compact and invariant. Since $A \supseteq K(\phi^t(u_0)) \supseteq K(u_0)$, we have
\[
d_L(\phi^t, A) = \max_{u \in A} d_L(\phi^t, u) \leq \max_{u \in K(\phi^t(u_0))} d_L(\phi^t, u) \leq \max_{u \in K(u_0)} d_L(\phi^t, u).
\]
Since $\rho(K(u_0), K(\phi^t(u_0))) \to 0$ for $t \to +\infty$, from the properties of decreasing \cite{34} and continuity \cite{3}, it follows that
\[
d_L = \liminf_{t \to +\infty} \max_{u \in K(\phi^t(u_0))} d_L(\phi^t, u).
\]

8. Computation of the finite-time Lyapunov exponents and dimension in MATLAB

The singular value decomposition (SVD) of a fundamental matrix $D\phi^t(u_0)$ has the form
\[
D\phi^t(u_0) = U(t, u_0)\Sigma(t, u_0)V^T(t, u_0) : U(t, u_0)^TV(t, u_0) = I \equiv V(t, u_0)^TV(t, u_0),
\]
where $\Sigma(t) = \text{diag}\{\sigma_1(t, u_0), ..., \sigma_n(t, u_0)\}$ is a diagonal matrix with positive real diagonal entries — singular values. We now give a MATLAB implementation \cite{67} of the discrete SVD method for computing finite-time Lyapunov exponents $\{\nu_i(t, u_0)\}_{1}^{n}$ based on the product SVD algorithm (see, e.g., \cite{25, 105}). For the computation of the Lyapunov dimension of an attractor by the considered code one has to consider a sufficiently large time $T$ and a grid of points on the attractor $K_{grid}$, compute the local Lyapunov dimensions by the corresponding Kaplan-Yorke formula $d_L^H(\{\nu_i(T, u)\}_{1}^{n})$ (see, e.g. \cite{58}), and takes maximum on the grid: $\max_{u \in K_{grid}} d_L^H(\{\nu_i(T, u)\}_{1}^{n})$.

Listing 1: productSVD.m - product SVD algorithm

```matlab
function [U, R, V] = productSVD(initFactorization, nIterations)
% Parameters:
% initFactorization - the array contains factor matrices of the
% fundamental matrix X, such that: X = initFactorization(:,:,1) * ... * initFactorization(:,:,end);
% nIterations - the number of iterations in the product SVD algorithm.
% dimOde - dimension of the ODEs, nFactors - the number of factor matrices
[-, dimOde, nFactors] = size(initFactorization);
% A - 2d array of matrices storing the factor matrices at each iteration
A(:,:,:,:) = initFactorization;
% Q - array of matrices storing orthogonal matrices of the QR decomposition
Q = zeros(dimOde, dimOde, nFactors+1);
% U, V - orthogonal matrices in the SVD decomposition
U = eye(dimOde); V = eye(dimOde);
% R - array of upper triangular factor matrices, such that after
% the last iteration \Sigma = R(:,:,1) * ... * R(:,:,end)
R = zeros(dimOde, dimOde, nFactors);

% Main loop
for iIteration = 1 : nIterations
    Q(:,:,nFactors+1) = eye(dimOde, dimOde);
    for jFactor = nFactors : -1 : 1
        C = A(:,:,jFactor, iIteration) * Q(:,:,jFactor+1);
        [Q(:,:,jFactor), R(:,:,jFactor)] = qr(C);
        for kCoord = 1 : dimOde
            if R(kCoord, kCoord, jFactor) < 0
                R(kCoord, :, jFactor) = -1 * R(kCoord, :, jFactor);
                Q(:, kCoord, jFactor) = -1 * Q(:, kCoord, jFactor);
            end;
        end;
    end;
    if mod(iIteration, 2) == 1
        U = U * Q(:,:,1);
    else
        V = V * Q(:,:,1);
    end
    for jFactor = 1 : nFactors
        A(:,:,jFactor, iIteration + 1) = R(:,:,nFactors-jFactor+1);
    end
end
end
end
```
Listing 2: computeLEs.m – computation of the Lyapunov exponents

```matlab
function LEs = computeLEs( extOde, initPoint, tStep, ... 
    nFactors, nSvdIterations, odeSolverOptions )

% Parameters:
% extOde - extended ODE system (system of ODEs + var. eq.);
% initPoint - initial point;
% tStep - time-step in the factorization procedure;
% nFactors - number of factor matrices in the factorization procedure;
% nSvdIterations - number of iterations in the product SVD algorithm;
% odeSolverOptions - solver options (sover = ode45);

% Dimension of the ODE :
dimOde = length(initPoint);

% Dimension of the extended ODE (ODE + Var. Eq.):
dimExtOde = dimOde * (dimOde + 1);

tBegin = 0; tEnd = tStep;
tSpan = [tBegin, tEnd];
initFundMatrix = eye(dimOde);
initCond = [initPoint(:); initFundMatrix(:)];

X = zeros(dimOde, dimOde, nFactors);

% Main loop : factorization of the fundamental matrix
for iFactor = 1 : nFactors
    [~, extOdeSolution] = ode45( extOde, tSpan, initCond, odeSolverOptions);
    X(:, :, iFactor) = reshape( ... 
        extOdeSolution(end, (dimOde + 1) : dimExtOde), ... 
        dimOde, dimOde);
    currInitPoint = extOdeSolution(end, 1 : dimOde);
    currInitFundMatrix = eye(dimOde);
    tBegin = tBegin + tStep;
    tEnd = tEnd + tStep;
    tSpan = [tBegin, tEnd];
    initCond = [currInitPoint(:); currInitFundMatrix(:)];
end

% Product SVD of factorization X of the fundamental matrix
[~, R, ~] = productSVD(X, nSvdIterations);

% Computation of the Lyapunov exponents
LEs = zeros(1, dimOde);
for jFactor = 1 : nFactors
    LEs = LEs + log(diag(R(:, :, jFactor)));
end;
finalTime = tStep * nFactors;
LEs = LEs / finalTime;
end
```

Listing 3: lyapunovDim.m – computation of the Lyapunov dimension

```matlab
function LD = lyapunovDim( LEs )

% For the given array of finite-time Lyapunov exponents at a point the function 
% computes the local Lyapunov dimension by the Kaplan-Yorke formula.

% Parameters:
% LEs - array of the finite-time Lyapunov exponents.

% Initialization of the local Lyapunov dimension:
LD = 0;

% Number of LEs :
nLEs = length(LEs);

% Sorted LEs :
sortedLEs = sort(LEs, 'descend');

% Main loop :
leSum = sortedLEs(1);
if ( sortedLEs(1) > 0 )
    for i = 1 : nLEs-1
        if sortedLEs(i+1) ~= 0
            LD = i + leSum / abs( sortedLEs(i+1) );
            leSum = leSum + sortedLEs(i+1);
            if leSum < 0
                break;
            end
        end
    end
end
```

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Listing 4: genLorenzSyst.m – generalized Lorenz system along with the variational equation

```matlab
function OUT = genLorenzSyst(t, x, r, sigma, b, a)

% Generalized Lorenz system with
% parameters: r sigma b a

OUT(1) = sigma*(x(2) - x(1)) - a*x(2)*x(3);
OUT(2) = r*x(1) - x(2) - x(1)*x(3);
OUT(3) = -b*x(3) + x(1)*x(2);

% Jacobian at the point [x(1), x(2), x(3)]
J = [-sigma, sigma-a*x(3), -a*x(2);
    r-x(3), -1, -x(1);
    x(2), x(1), -b];

X = [x(4), x(7), x(10);
    x(5), x(8), x(11);
    x(6), x(9), x(12)];

% Variational equation
OUT(4:12) = J*X;
```

Listing 5: main.m – computation of the Lyapunov exponents and local Lyapunov dimension for the hidden attractor of generalized Lorenz system

```matlab
function main

% Parameters of generalized Lorenz system
% that correspond to the hidden attractor
r = 700; sigma = 4; b = 1; a = 0.0052;

% Initial point for the trajectory which visualizes the hidden attractor
x0 = [-14.551336132013954 -173.86811769236883 718.92035664071227);

tStep = 0.1;
nFactors = 10000;
nSvdIterations = 3;

% ODE solver parameters
acc = 1e-8; RelTol = acc; AbsTol = acc; InitialStep = acc/10;
odeSolverOptions = odeset('RelTol', RelTol, 'AbsTol', AbsTol, ...
    'InitialStep', InitialStep, 'NormControl', 'on');

LEs = computeLEs(@(t, x) genLorenzSyst(t, x, r, sigma, b, a), ...
    x0, tStep, nFactors, nSvdIterations, odeSolverOptions);

fprintf('Lyapunov exponents: %6.4f, %6.4f, %6.4f\n', LEs);

LD = lyapunovDim(LEs);
fprintf('Lyapunov dimension: %6.4f\n', LD);
end
```

Conclusions

In this survey for finite dimensional dynamical systems in Euclidean space we have tried to discuss rigorously the connection between the works by Kaplan and Yorke (the concept of Lyapunov dimension, 1979), Douady and Oesterlé (estimation of Hausdorff dimension via the Lyapunov dimension of maps, 1980), Constantin, Eden, Foias, and Temam (estimation of Hausdorff dimension via the Lyapunov exponents and dimension of dynamical systems, 1985-90), Leonov (estimation of the Lyapunov dimension via the direct Lyapunov method, 1991), and numerical methods for the computation of Lyapunov exponents and Lyapunov dimension. Remark that in the numerical estimations we can consider only finite time and get finite-time Lyapunov exponents, thus we have discussed the justification of Kaplan-Yorke formula with respect to finite-time Lyapunov exponent, by the Douady–Oesterlé theorem for maps. For various self-excited and hidden attractors of well-known dynamical systems, the numerical values, analytical estimations and formulas of the Lyapunov dimension are given.
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