TWO RESULTS ON CHARACTER CODEGREES

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Abstract. Let $G$ be a finite group and $\text{Irr}(G)$ be the set of irreducible characters of $G$. The codegree of an irreducible character $\chi$ of the group $G$ is defined as $\text{cod}(\chi) = |G : \ker(\chi)|/\chi(1)$. In this paper, we study two topics related to the character codegrees. Let $\sigma^c(G)$ be the maximal integer $m$ such that there is a member in $\text{cod}(G)$ having $m$ distinct prime divisors, where $\text{cod}(G) = \{\text{cod}(\chi) | \chi \in \text{Irr}(G)\}$. One is related to the codegree version of the Huppert’s $\rho$-$\sigma$ conjecture and we obtain the best possible bound for $|\pi(G)|$ under the condition $\sigma^c(G) = 2$, $3$, and $4$ respectively. The other is related to the prime graph of character codegrees and we show that the codegree prime graphs of several classes of groups can be characterized only by graph theoretical terms.

Keywords. finite group, character codegree, prime graph

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1. Introduction

Throughout this paper, $G$ is a finite group and $\text{Irr}(G)$ is the set of irreducible characters of $G$. The concept of character codegree (defined as $|G|/\chi(1)$ for any nonlinear irreducible character $\chi$ of $G$) was first introduced in [2] to characterize the structure of finite groups. However for nonlinear character $\chi \in \text{Irr}(G/N)$, where $N$ is a nontrivial normal subgroup of $G$, $\chi$ will have two different codegrees when it is considered as the character of $G$ and $G/N$ respectively. So the concept was improved as $\text{cod}(\chi) = |G : \ker(\chi)|/\chi(1)$ for any character $\chi \in \text{Irr}(G)$ by Qian, Wang, and Wei in [13]. Now many properties of codegree have been studied, such as the relationship between the codegrees and the element orders [8,14], codegrees of $p$-groups [3,4], and groups with few codegrees [1,11,12].

In this paper, we study two topics related to the character codegrees. Set $\text{cod}(G) = \{\text{cod}(\chi) | \chi \in \text{Irr}(G)\}$. Denote by $\pi(n)$ be the set of prime divisors of rational integer $n$ and we use $\pi(G)$ instead of $\pi(|G|)$. Let $\sigma^c(G)$ be the maximal integer $m$ such that there is a member in $\text{cod}(G)$ having $m$ distinct prime divisors. The first topic is about the following question which is raised in [13, Question A] and called the codegree version of the Huppert’s $\rho$-$\sigma$ conjecture: Is there any constant $k$ (independent of $G$) such that $|\pi(G)| \leq k\sigma^c(G)$? In particular, is it true that $|\pi(G)| \leq 4$ when $\sigma^c(G) = 2$?

We note that the first question was answered in [15] and further improved in [16]. In this paper, we show that the second question is not always true in general and we obtain the best possible bound for $|\pi(G)|$ under the condition $\sigma^c(G) = 2$, $3$, and $4$ respectively.

**Theorem 1.1.** Let $G$ be a finite group, then

(i) $|\pi(G)| \leq 5$ when $\sigma^c(G) = 2$.
(ii) $|\pi(G)| \leq 8$ when $\sigma^c(G) = 3$. 


(iii) $|\pi(G)| \leq 12$ when $\sigma^c(G) = 4$.

The element order prime graph (Gruenberg-Kegel graph) of a finite group is the (simple undirected) graph whose vertices are the prime numbers dividing the order of the group, with two vertices being linked by an edge if and only if their product divides the order of some element of the group. Gruber et al. characterized the element order prime graph of solvable groups only using graph theoretical terms in [6]. Recently Florez et al. gave complete characterizations for the element order prime graphs of several classes of groups, such as groups of square-free order, meta-nilpotent groups, groups of cube-free order, and, for any $n \in N$, solvable groups of $n^{th}$-power-free order in [5].

The codegree prime graph built on $\text{cod}(G)$, that we denote by $\Delta_{\text{cod}}(G)$, is the (simple undirected) graph whose vertices are the prime divisors of the numbers in $\text{cod}(G)$, and two distinct vertices $p, q$ are adjacent if and only if $pq$ divides some number in $\text{cod}(G)$. The second topic of this paper concerns the relation of solvable groups with the corresponding prime graph of character codegrees and we obtain several results parallel to the results in [5, 6].

2. THE CODEGREE ANALOGUE OF THE HUPPERT’S $\rho\sigma$ CONJECTURE

We first introduce some notation. Let $G$ be a group and $N \trianglelefteq G$. For $\chi \in \text{Irr}(G)$, $\text{Irr}(\chi|_N)$ is the set of irreducible constitutes of the restriction of $\chi$ to $N$.

**Lemma 2.1** (Lemma 1.8 of [9]). Let $|G| = q_1 \cdots q_n r_1 \cdots r_m$ with mutually distinct primes $q_i, r_j$ $(n, m \in \mathbb{N})$. Assume that $|F(G)| = r_1 r_2 \cdots r_m$ and $p$ is a prime with $|G| \mid p - 1$. Then there exists a faithful irreducible $GF(p)G$-module $V$ such that $F(G)$ acts fixed point freely on $V$.

**Example 2.1** (Example 1.9 of [9]).

Let $q_1, q_2, q_3, q_4, r_1, r_2, r_3, r_4$ be mutually distinct prime numbers where the $q_i$ may be arbitrarily chosen and the $r_i$ may have the following properties:

$q_1 \mid r_1 - 1, q_2 \mid r_2 - 1, q_1 q_2 q_3 \mid r_3 - 1, q_2 q_4 \mid r_4 - 1$.

The $r_i(i = 1, 2, 3, 4)$ surely exist because of the Dirichlet’s theorem on primes in arithmetic progression. We want $G/G''$ to have the following structure:

$G/G'' \cong Q_1 \times Q_2 \times Q_3 \times Q_4$ with $Q_i$ cyclic of order $q_i$,

$G'/G'' \cong R_1 \times R_2 \times R_3 \times R_4$ with $R_i$ cyclic of order $r_i$, and $Q_i \leq N_G(R_j)$ for all $i, j$. Then $Q_i$ can only act trivially or fixed point freely on $R_j$ $(i, j \in \{1, 2, 3, 4\})$, and we demand:

- $Q_1$ acts fixed point freely only on $R_1$ and $R_3$,
- $Q_2$ acts fixed point freely only on $R_2$, $R_3$, and $R_4$,
- $Q_3$ acts fixed point freely only on $R_3$, and finally
- $Q_4$ acts fixed point freely only on $R_4$.

This is possible because of our number theoretical restriction on the $r_i$. Now let $s_1, s_2, s_3, s_4$ be mutually distinct primes such that for $i = 1, 2, 3, 4$, $s_i - 1$ is divisible by $q_1 q_2 q_3 q_4 r_1 r_2 r_3 r_4$. (By Dirichlet’s theorem such $s_i$ exist.) Then let
G" = V_1 \times V_2 \times V_3 \times V_4 with irreducible GF(s_i)-modules V_i, where the V_i may be constructed as follows:

(1) Let V_1 be the faithful irreducible GF(s_1)Q_1R_1-module (existing by Lemma 2.1), on which R_1 acts fixed point freely. Let Q_2, Q_3, and Q_4 act fixed point freely on V_1 by multiplication with elements of GF(s_1) (i.e. Q_2Q_3Q_4 is represented by multiples of the identity matrix), and R_2R_3R_4 may act trivially on V_1. So we have
\[ C_{G/G''}(V_1) \cong R_2 \times R_3 \times R_4, \] and
\[ G/C_G(V_1) \cong Q_1R_1 \times Q_2 \times Q_3 \times Q_4. \]

(2) As in (1) let V_2 be the GF(s_2)Q_2R_2-module, on which R_2 acts fixed point freely, and we demand
\[ C_{G/G''}(V_2) \cong R_1 \times R_3 \times R_4, \] and
\[ G/C_G(V_2) \cong Q_2R_2 \times Q_1 \times Q_3 \times Q_4, \] where Q_1, Q_3 and Q_4 act fixed point freely on V_2 by multiplication with elements of GF(s_2).

(3) Let V_3 be the faithful, irreducible GF(s_3)Q_1Q_2Q_3R_1R_2R_3-module, on which R_1R_2R_3 = F(Q_1Q_2Q_3R_1R_2R_3) acts fixed point freely. Let Q_4 act fixed point freely on V_3 by multiplication with elements of GF(s_3), and let R_4 act trivially on V_3. Then
\[ C_{G/G''}(V_3) \cong R_4, \] and
\[ G/C_G(V_3) \cong Q_1Q_2Q_3R_1R_2R_3 \times Q_4. \]

(4) As in (3) let V_4 be the GF(s_4)Q_1Q_2Q_4R_1R_2R_4-module, so that R_1R_2R_4 acts fixed point freely on it, and we demand
\[ C_{G/G''}(V_4) \cong R_3, \] and
\[ G/C_G(V_4) \cong Q_1Q_2Q_4R_1R_2R_4 \times Q_3, \] where Q_3 acts fixed point freely on V_4 by multiplication with elements of GF(s_4).

**Proposition 2.1.** Let G be the group constructed in Example 2.1. Then |\sigma^{e}(G)| = 4.

**Proof.** It can be checked that F(G) = V_1V_2V_3V_4 and F_2(G)/F(G) = F(G/F(G)) = R_1R_2R_3R_4F(G)/F(G). Next we consider prime divisors of cod(\chi) case by case for any \chi \in Irr(G).

Case (a). G" \not\leq ker(\chi).
Choose \lambda \in Irr(\chi|_{F(G)}). We may assume \lambda = \lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdot \lambda_4, where \lambda_i \in Irr(V_i).

Subcase (a1). Suppose that \lambda_i \neq 1, i = 1, 2, 3, 4.
It can be checked that \eta_G(\lambda) = F(G) and \chi(1) = |G/F(G)|. Thus \pi(cod(\chi)) = \{s_1, s_2, s_3, s_4\}.

Subcase (a2). Suppose that one of the four characters \lambda_i is trivial and the other three characters are nontrivial.
If \lambda_4 = 1, it can be checked that \eta_G(\lambda) = F(G)R_4 and \chi(1) = |G/(F(G)R_4)|. Since V_4 \leq ker(\lambda), we have V_4 \leq ker(\chi) and \pi(cod(\chi)) \subseteq \{s_1, s_2, s_3, r_4\}. Similarly, we can check for subcases of \lambda_i = 1, i = 1, 2, 3 and also obtain that
\[ |\pi(\text{cod}(\chi))| \leq 4. \]

Subcase (a3). Suppose that two of the four characters \( \lambda_i \) are trivial and the other two characters are nontrivial.

If \( \lambda_3 \) and \( \lambda_4 \) are trivial characters, it can be checked that \( I_G(\lambda) \leq F(G)(R_3R_4) \) and \( |G|/|\chi(1)| = |F(G)R_3R_4| \). Since \( V_3V_4 \leq \ker(\lambda) \), we have \( V_3V_4 \leq \ker(\chi) \) and \( \pi(\text{cod}(\chi)) \subseteq \{s_1, s_2, r_3, r_4\} \). Similarly, we can check for other subcases and also obtain that \( |\pi(\text{cod}(\chi))| \leq 4 \).

Subcase (a4). Suppose that three of the four characters \( \lambda_i \) are trivial and the remaining character is nontrivial.

If \( \lambda_4 \neq 1 \), then \( T = I_G(\lambda) \leq F(G)R_3(Q_1Q_2Q_4) \) and \( \lambda \) extends to \( \theta \in \text{Irr}(T) \). By Clifford’s theorem, there exists \( \beta \in \text{Irr}(T/F(G)) \) such that \( \chi = (\theta \cdot \beta)^G \). Since \( V_1V_2V_3 \leq \ker(\lambda) \),

\[
\text{cod}(\chi) = \frac{|G|}{\ker(\chi)\chi(1)} = \frac{|G|}{\ker(\chi)|G: T|\beta(1)} = \frac{|T|}{\ker(\chi)|\beta(1)} |V_4R_3(Q_1Q_2Q_4)|.
\]

If \( T \not\leq F(G)R_3(Q_1Q_2Q_4) \) or \( \beta(1) > 1 \), then \( |\pi(\text{cod}(\chi))| \leq 4 \). Next we assume \( T = F(G)R_3(Q_1Q_2Q_4) \) and \( \beta(1) = 1 \). Since \( R_3 = [R_3, Q_1] \leq T' \leq \ker(\theta \cdot \beta) \) and \( V_1V_2V_3R_3 \) is a normal subgroup of \( G \), we have \( V_1V_2V_3R_3 \leq \ker(\theta \cdot \beta) \) and \( V_1V_2V_3R_3 \leq \ker(\chi) \). Therefore \( \pi(\text{cod}(\chi)) \subseteq \{s_4, q_1, q_2, q_4\} \).

Case (b). \( G'' \leq \ker(\chi) \).

Choose \( \lambda \in \text{Irr}(\chi|_{F_2(G)}) \). We may assume \( \lambda = \lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdot \lambda_4 \), where \( \lambda_i \in \text{Irr}(R_i) \). All cases can be handled with the same method as before and we take the following case as an example. Suppose that \( \lambda_1, \lambda_2 \) are nontrivial characters and \( \lambda_3, \lambda_4 \) are trivial, then \( I_G(\lambda) = F_2(G)Q_3Q_4 \) and \( \chi(1) = q_1 q_2 \). Since \( F(G)R_3R_4 \leq \ker(\chi) \), we have \( \pi(\text{cod}(\chi)) \subseteq \{r_1, r_2, q_3, q_4\} \).

By the same method, we can obtain the following result.

**Proposition 2.2.** Let \( H = Q_1Q_2R_1R_2V_1 \) and \( K = Q_2Q_3Q_4R_2R_4V_1V_3V_4 \), where \( Q_i, R_j, V_k \) are from Example 2.1, then \( |\sigma^e(H)| = 2 \) and \( |\sigma^e(K)| = 3 \).

By the previous examples and the main results of [9, 10], as well as the main result of [8], Theorem 1.1 can be easily checked. Note that all those upper bounds are the best possible.

3. **The codegree prime graphs of solvable groups**

The following three results are about the prime graphs (element order version) of solvable groups.

**Proposition 3.1** (Theorem 2 of [8]). An unlabeled graph \( F \) is isomorphic to the prime graph of some solvable group if and only if its complement \( \overline{F} \) is 3-colorable and triangle-free.
Proposition 3.2 (Theorem 3.1 of [5]). Let \( C \) be any class of groups such that \( \{ G : G \text{ has square-free order} \} \subseteq C \subseteq \{ G : G \text{ metanilpotent} \} \). Then \( F \) is isomorphic to the prime graph of some \( G \in C \) if and only if \( \overline{F} \) is bipartite.

Proposition 3.3 (Theorem 4.3 of [5]). \( F \) is isomorphic to the prime graph of a solvable group of \( n \)-th-power-free order if and only if \( \overline{F} \) satisfies the following conditions.

1. Triangle-free.
2. There exists a 3-coloring of \( \overline{F} \) by Red, Green, and Blue and a way to label each red vertex by a distinct prime number such that for any \( \pi \) a subset of those primes whose product is at least \( n \), we have that in the canonical orientation, no blue vertex is simultaneously the end of directed 2-paths starting from each of the red vertices in \( \pi \).

In [14], Qian proved that if \( G \) is a solvable group and \( g \in G \), then there is an irreducible character \( \chi \) of \( G \) such that \( p \) divides \( \operatorname{cod}(\chi) \) for any prime divisor \( p \) of the order of \( g \). A while later, Isaacs proved that the result is indeed true for any finite group in [8]. In view of this result, we see that the element order prime graph of a finite group \( G \) is a subgraph of its codegree prime graph. Next we show the element order prime graph and codegree prime graph coincide for various types of solvable groups.

Proposition 3.4. Let \( G \) be a solvable group such that every minimal normal subgroup is a Sylow subgroup and \( |G/F(G)| \) is square-free. For any prime pairs \( (p, q) \) if \( pq \mid \operatorname{cod}(\chi) \) for some \( \chi \in \operatorname{Irr}(G) \), then there exists an element \( g \in G \) such that \( pq \mid |g| \).

Proof. We proceed by the induction on the order of \( G \). If \( \ker(\chi) \) is nontrivial, then it can be checked that every minimal normal subgroup of \( G/\ker(\chi) \) is a Sylow subgroup and \( (G/\ker(\chi))/F(G/\ker(\chi)) \) is square-free. Then there exists an element \( g \ker(\chi) \in G/\ker(\chi) \) such that \( pq \mid |g\ker(\chi)| \) by induction. Therefore \( pq \mid |g| \). Next we assume that \( \ker(\chi) = 1 \).

Let \( F(G) = P_1 \times P_2 \times \cdots \times P_t \), where \( P_i \) is a Sylow \( p_i \)-subgroup. Choose \( \lambda \in \operatorname{Irr}(\chi|_{F(G)}) \) and set \( T = I_G(\lambda) \). Then \( \chi(1) = |G : T|\theta(1) \), where \( \theta \in \operatorname{Irr}(T|\lambda) \) by Clifford theory and thus \( \operatorname{cod}(\chi) = |T|/\theta(1) \). If \( pq \mid |F(G)| \), then there exists an element of order \( pq \). If \( p \mid |F(G)| \) and \( (q, |F(G)|) = 1 \), then the order of \( \lambda \) is divisible by \( p \) and \( q \mid |T| \). Since the actions of \( G/F(G) \) on \( \operatorname{Irr}(F(G)) \) and \( F(G) \) are permutation isomorphic (by Theorem 13.24 of [7]), there exists \( h \in F(G) \) such that \( p \mid |h| \) and \( q \mid |C_G(h)| \). Hence there exists an element \( g \in G \) such that \( pq \mid |g| \). If \( q \mid |F(G)| \) and \( (p, |F(G)|) = 1 \), the proposition is true similarly. Now we can assume \( pq \mid |G/F(G)| \) and \( (p, |F(G)|) = 1 \), the proposition is true similarly. Without loss of generality we assume that \( p \mid |F_2(G)/F(G)| \) and \( q \mid |G/F_2(G)| \). Choose a Sylow \( p \)-subgroup \( P \leq T \) and set \( N = F(G)P \). Let \( \alpha \in \operatorname{Irr}(\chi|_N) \) such that \( \lambda \in \operatorname{Irr}(\alpha|_{F(G)}) \), then \( \alpha \) is an
extension of \( \lambda \). Since \( N' \subseteq \ker(\alpha) \) and \( N \subseteq G \), \( N' \leq \ker(\chi) \) and \( N' = 1 \). Hence \( P \leq C_G(F(G)) = F(G) \), a contradiction with \( (p, |F(G)|) = 1 \).

Inspired by the previous results, we obtain similar results on the prime graph of codegrees.

**Proposition 3.5.** An unlabeled graph \( F \) is isomorphic to the codegree prime graph of some solvable group if and only if its complement \( \overline{F} \) is 3-colorable and triangle-free.

**Proof.** First, we assume \( F \) is isomorphic to the codegree prime graph of a solvable group \( G \). Let \( F_1 \) be the prime graph of \( G \). Then \( \overline{F_1} \) is 3-colorable and triangle-free by Proposition 3.1. Since \( F_1 \) is subgraph of \( F \), \( \overline{F} \) is a subgraph of \( \overline{F_1} \). Hence \( \overline{F} \) is 3-colorable and triangle-free.

Conversely, if \( \overline{F} \) is 3-colorable and triangle-free, then a solvable group \( G \) can be constructed such that \( F \) is isomorphic to the prime graph of \( G \) (The details on the structure of \( G \) can be found in Theorem 2.8 of [6].) It can be checked that every minimal normal subgroup of \( G \) is a Sylow subgroup and \( |G/F(G)| \) is square-free. Then \( \Delta_{\text{cod}}(G) \) is the same as its prime graph by Proposition 3.4. Therefore \( F \) is isomorphic to \( \Delta_{\text{cod}}(G) \).

**Proposition 3.6.** Let \( C \) be any class of groups such that
\[
\{G: G \text{ has square-free order}\} \subseteq C \subseteq \{G: G \text{ metanilpotent}\}.
\]
Then \( F \) is isomorphic to the codegree prime graph of some \( G \in C \) if and only if \( \overline{F} \) is bipartite.

**Proof.** First, we assume \( F \) is isomorphic to the codegree prime graph of a solvable group \( G \). Let \( F_1 \) be the prime graph of \( G \). Then \( \overline{F_1} \) is bipartite by Proposition 3.2. Since \( F_1 \) is subgraph of \( F \), \( \overline{F} \) is a subgraph of \( \overline{F_1} \). Hence \( \overline{F} \) is bipartite.

Conversely, if \( \overline{F} \) is 3-colorable and triangle-free, then a solvable group \( G \) of square-free order can be constructed such that \( F \) is isomorphic to the prime graph of \( G \) (The details on the structure of \( G \) can be found in Theorem 3.1 of [5]). Since \( G \) has square-free order, \( \Delta_{\text{cod}}(G) \) is the same as its prime graph. Therefore \( F \) is isomorphic to \( \Delta_{\text{cod}}(G) \).

**Proposition 3.7.** \( F \) is isomorphic to the codegree prime graph of a solvable group of \( n^{\text{th}} \)-power-free order if and only if \( \overline{F} \) satisfies the following conditions.

1. Triangle-free;
2. There exists a 3-coloring of \( \overline{F} \) by Red, Green, and Blue and a way to label each red vertex by a distinct prime number such that for any \( \pi \) a subset of those primes whose product is at least \( n \), we have that in the canonical orientation, no blue vertex is simultaneously the end of directed 2-paths starting from each of the red vertices in \( \pi \).

**Proof.** First, we assume \( F \) is isomorphic to the codegree prime graph of solvable group \( G \) with \( n^{\text{th}} \)-power-free order. Let \( F_1 \) be the prime graph of \( G \). Then \( \overline{F_1} \) satisfies the conditions in Proposition 3.3, so does \( \overline{F} \) for \( \overline{F} \) is a subgraph of \( \overline{F_1} \).
Conversely, if $\mathcal{F}$ satisfies the conditions, then by Theorem 4.3 of [5] a solvable group $G$ can be constructed such that $\mathcal{F}$ is isomorphic to the prime graph of $G$ which is still from Theorem 2.8 of [6]. Then $\Delta_{\text{cod}}(G)$ is the same as its prime graph by Proposition 3.3. Therefore $\mathcal{F}$ is isomorphic to $\Delta_{\text{cod}}(G)$. □

Similar with Corollaries 4.4-4.7 in [5], we can obtain the following results.

**Corollary 3.1.** $\mathcal{F}$ is isomorphic to the codegree prime graph of a solvable group of $n^{th}$-power-free order if and only if $\mathcal{F}$ satisfies the following conditions.

1. Triangle-free;
2. There exists a 3-coloring of $\mathcal{F}$ by Red, Green, and Blue such that we can label the red vertices by the first $m$ primes such that
   a. The primes are less than $n$;
   b. In the canonical orientation, no blue vertex is simultaneously the end of directed 2-paths starting from each of the red vertices in $\pi$, where $\pi$ is any subset of the first $m$ primes whose product is at least $n$.

**Corollary 3.2.** $\mathcal{F}$ is isomorphic to the codegree prime graph of a solvable group of square-free order if and only if $\mathcal{F}$ is 2-colorable.

**Corollary 3.3.** $\mathcal{F}$ is isomorphic to the codegree prime graph of a solvable group of cube free order if and only if $\mathcal{F}$ is triangle free and is 2-colorable after removing one vertex.

**Corollary 3.4.** The following are equivalent.

1. $\mathcal{F}$ is isomorphic to the codegree prime graph of a solvable group of fourth-power-free order.
2. $\mathcal{F}$ is isomorphic to the codegree prime graph of a solvable group of fifth-power-free order.
3. $\mathcal{F}$ is 3-colorable, triangle free and satisfies one of the following conditions.
   a. $\mathcal{F}$ is 2-colorable after removing one vertex;
   b. There exists a 3-coloring of $\mathcal{F}$ by Red, Green, and Blue such that there are exactly 2 red vertices. Furthermore, in the canonical orientation, no vertex is the end of a directed 2-path starting from both of the red vertices.

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