ON THE $b$-FUNCTIONS OF HYPERGEOMETRIC SYSTEMS

THOMAS REICHELT, CHRISTIAN SEVENHECK, ULI WALTHER

ABSTRACT. For any integer $d \times (n+1)$ matrix $A$ and parameter $\beta \in \mathbb{C}^d$ let $M_A(\beta)$ be the associated $A$-hypergeometric (or GKZ) system in the variables $x_0, \ldots, x_n$. We describe bounds for the (roots of the) $b$-functions of both $M_A(\beta)$ and its Fourier transform along the hyperplanes ($x_j = 0$). We also give an estimate for the $b$-function for restricting $M_A(\beta)$ to a generic point.

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Let $D$ be the ring of algebraic $\mathbb{C}$-linear differential operators on $\mathbb{C}^{n+1}$ with coordinates $x_0, \ldots, x_n$.

Definition 0.1 (Compare [Kas77, MM04]). Let $M$ be a left $D$-module and pick an element $m \in M$ with annihilator $I \subseteq D$. If $(V^i D)$ is the vector space spanned by the monomials $x^{\alpha} \partial^{\beta}$ with $\alpha_0 - \beta_0 \geq i$ then the $b$-function of $m \in M$ along the coordinate hyperplane $x_0 = 0$ is the minimal monic polynomial $b(s)$ that satisfies:

$$b(x_0 \partial_0) \cdot m \in (V^1 D) \cdot m \text{ in } M,$$

which is to say $b(x_0 \partial_0) \in I + (V^1 D)$ in $D$.

If $M$ is cyclic, i.e., $M = D/I$, then we call $b$-function of $M$ the $b$-function in the above sense of the element $1 + I \in M$.

The $b$-function exists in greater generality along any hypersurface ($f = 0$), as long as the module $M$ is holonomic, cf. [Kas77]. The $V$-filtration of Kashiwara and Malgrange then takes the form $(V^i D) = \{ P \in D \mid f^{i+k} \text{ divides } P \cdot f^k \text{ for } k \gg 0 \}$. Both the $V$-filtration and the $b$-function are intimately connected to the restriction of the given $D$-module to the hypersurface. The purpose of this note is to give, for any $A$-hypergeometric system as well as its Fourier transform, an explicit arithmetic description of a bound for the root set of the $b$-function along any coordinate hyperplane that involves the parameter $\beta$ in a very elementary way.

We have several applications in mind: first, it is a longstanding question to understand the monodromy of $A$-hypergeometric systems, and for this purpose the roots of the $b$-function as considered above can be of some use. On the other
hand, the Fourier transform of an $A$-hypergeometric system often (see [SW09b]) appears as a direct image module under a natural torus embedding given by the columns of the matrix $A$. This point of view turns out to be extremely useful for Hodge theoretic considerations of $A$-hypergeometric systems (see [Rei14]). It is one of the fundamental insights of Morihiko Saito (see [Sai88, Section 3.2]) that the boundary behavior of variations of Hodge structures (or, more generally, of mixed Hodge modules) is controlled by the Kashiwara–Malgrange filtration along such a boundary divisor. In the case of a cyclic $D$-module, such as $A$-hypergeometric systems or their Fourier transforms, one can often deduce a large part of this filtration from the values of the $b$-function. We refer to [RS15] for an immediate application of our results. In a third direction, one can also see our calculation of the $b$-function of the Fourier transform as a refinement of [SW09b, FFW11] geared towards restriction of $A$-hypergeometric systems.

In the last part we compute an upper bound for the $b$-function of restriction of the $A$-hypergeometric system to a generic point, again in elementary terms of $A$ and $\beta$. Since the restriction of a $D$-module to a point is a dual object to the 0-th level solution functor, our estimate can be viewed as a step towards a sheafification in $\beta$ of the solution space, a problem that remains unsolved.

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1. Basic notions and results

Notation. Throughout, the base field is $\mathbb{C}$ and we consider a $\mathbb{C}$-vector space $V$ of dimension $n+1$.

In this introductory section we review basic facts on $A$-hypergeometric systems as well as the Euler–Koszul functor. Readers are advised to refer to [MMW05] for more detailed explanations.

Notation 1.1. For any integer matrix $A$, let $R_A$ (resp. $O_A$) be the polynomial ring over $\mathbb{C}$ generated by the variables $\partial_j$ (resp. $x_j$) corresponding to the columns $a_j$ of $A$. We identify $O_A$ with the symmetric algebra on $\text{Hom}_\mathbb{C}(V, \mathbb{C}) \cong \bigoplus \mathbb{C} \cdot x_j$. Further, let $D_A$ be the ring of $\mathbb{C}$-linear differential operators on $O_A$, where we identify $\frac{\partial}{\partial x_j}$ with $\partial_j$ and multiplication by $x_j$ with $x_j$ so that both $R_A$ and $O_A$ become subrings of $D_A$.

1.1. $A$-hypergeometric systems. Let $A = (a_0, \ldots, a_n)$ be an integer $d \times (n+1)$ matrix, $d \leq n+1$. For convenience we assume that $ZA = \mathbb{Z}^d$. For $(v_1, \ldots, v_r) = \mathbf{v} \in \mathbb{Z}^r$ we denote by $\mathbf{v}_+, \mathbf{v}_-$ the vectors given by

$$(\mathbf{v}_+)_j = \max(0, v_j) \quad \text{and} \quad (\mathbf{v}_-)_j = \max(0, -v_j).$$

For the complex parameter vector $\beta \in \mathbb{C}^d$ consider the system of $d$ homogeneity equations

$$(1.1) \quad E_i \cdot \phi = \beta_i \cdot \phi,$$
where $E_i = \sum_{j=0}^{n} a_{i,j} x_j \partial_j$ is the $i$-th Euler operator, together with the toric (partial differential) equations

\[(1.2) \quad \{(\partial^x - \partial^y) \cdot \phi = 0 \mid A \cdot v = 0\}.
\]

In $R_A$, the toric operators $\{\Delta_v | A \cdot v = 0\}$ generate the toric ideal $I_A$. The quotient $S_A := R_A/I_A$

is naturally isomorphic to the semigroup ring $\mathbb{C}[\mathbb{N}A]$. In $D_A$, the left ideal generated by all equations (1.1) and (1.2) is the hypergeometric ideal $H_A(\beta)$. We put $M_A(\beta) := D_A/H_A(\beta)$;

this is the $A$-hypergeometric system introduced and first investigated by Gelfand, Graev, Kapranov, and Zelevinsky, in [Gel86] and a string of other papers.

1.2. $A$-degrees. If the rowspan of $A$ contains $1_A$ we call $A$ homogeneous. Homogeneity is equivalent to $I_A$ defining a projective variety, and also to the system $H_A(\beta)$ having only regular singularities [Hot98, SW08]. A more general $A$-degree function on $R_A$ and $D_A$ is induced by:

$$\deg_A(x_j) := a_j =: \deg_A(\partial_j).$$

We denote $\deg_A, i(\cdot)$ the $A$-degree function associated to the weight given by the $i$-th row of $A$, so $\deg_A = (\deg_A, 1, \ldots, \deg_A, d)$.

An $R_{A^*}$ (resp. $D_{A^*}$)-module $M$ is $A$-graded if it has a decomposition $M = \bigoplus_{\alpha \in \mathbb{N}A} M_{\alpha}$ such that the module structure respects the grading $\deg_A(\cdot)$ on $R_A$ (resp. $D_A$) and $M$. If $N$ is an $A$-graded $R_{A^*}$-module, then we denote $\deg_A(N) \subseteq \mathbb{Z}^d$ the set of all degrees of all non-zero homogeneous elements of $N$. The quasi-degrees $q \deg_A(N)$ of $N$ are the points in the Zariski closure in $\mathbb{C}^d$ of $\deg_A(N)$.

As is common, if $M$ is $A$-graded then $M(b)$ denotes for each $b \in \mathbb{Z}A$ its shift with graded structure $(M(b))_{b'} = M_{b+b'}$.

1.3. Euler–Koszul complex. Since

$$x^u E_i - E_i x^u = -(A \cdot u)_i x^u,$$

$$\partial^u E_i - E_i \partial^u = (A \cdot u)_i \partial^u,$$

we have

\[(1.3) \quad E_i P = P(E_i - \deg_A, i(P))\]

for any $A$-homogeneous $P \in D_A$ and all $i$.

On the $A$-graded $D_{A^*}$-module $M$ one can thus define commuting $D_{A^*}$-linear endomorphisms $E_i$ via

$$E_i \circ m := (E_i + \deg_A, i(m)) \cdot m$$

for $A$-homogeneous elements $m \in M$. In particular, if $N$ is an $A$-graded $R_{A^*}$-module one obtains commuting sets of $D_{A^*}$-endomorphisms on the left $D_{A^*}$-module $D_A \otimes_{R_A} N$ by

$$E_i \circ (P \otimes Q) := (E_i + \deg_A, i(P) + \deg_A, i(Q))P \otimes Q.$$

The Euler–Koszul complex $\mathcal{K}_{(*)}(N; \beta)$ of the $A$-graded $R_{A^*}$-module $N$ is the homological Koszul complex induced by $E - \beta := \{(E_i - \beta_i) \circ \}^d_{i=1}$ on $D_A \otimes_{R_A} N$. In particular, the terminal module $D_A \otimes_{R_A} N$ sits in homological degree zero. We denote the homology groups of $\mathcal{K}_{(*)}(N; \beta)$ by $H_{(*)}(N; \beta)$. Implicit in the notation
is “A”: different presentations of semigroup rings that act on $N$ yield different Euler–Koszul complexes.

If $N(b)$ denotes the usual shift-of-degree functor on the category of graded $R_A$-modules, then $\mathcal{K}_*(N; \beta)(b)$ and $\mathcal{K}_*(N(b); \beta - b)$ are identical.

1.4. The toric category. There is a bijection between faces $\tau$ of the cone $\mathbb{R}_{\geq 0}A$ and $A$-graded prime ideals $I_\tau^A = I_A + RA_\beta$ of $RA$ containing $I_A$. If the origin is a face of $\mathbb{R}_{\geq 0}A$, it corresponds to the ideal $I_A^0 = (\partial_1, \ldots, \partial_n)$. In general, $R_A/I_A^\tau \cong \mathbb{C}[\mathbb{N}\tau]$.

An $R_A$-module $N$ is toric if it is $A$-graded and has a (finite) $A$-graded composition chain

$$0 = N_0 \subset N_1 \subset N_2 \cdots \subset N_k = N$$

such that each composition factor $N_i/N_{i-1}$ is isomorphic as $A$-graded $R_A$-module to an $A$-graded shift $(R_A/I_A^\tau)(b)$ for some $b \in \mathbb{Z}A$ and some face $\tau$. The category of toric modules is closed under the formation of subquotients and extensions.

For toric input $N$, the modules $\mathcal{K}_*(N; \beta)$ are holonomic. As $D_A$ is $R_A$-free, any short exact sequence $0 \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow 0$ of $A$-graded $R_A$-modules produces a long exact sequence of Euler–Koszul homology. If $\beta$ is not a quasi-degree of $N$ then the complex $\mathcal{K}_*(N; \beta)$ is exact, and if $N$ is a maximal Cohen–Macaulay module then $\mathcal{K}_*(N; \beta)$ is a a resolution of $\mathcal{K}_0(N; \beta)$.

1.5. The Euler space.

Notation 1.2. The $\mathbb{C}$-linear span of the Euler operators $\{E_i\}_{1}^d$ is called the Euler space. Let $E$ be in the Euler space. Then $E$ is in a unique fashion (as $\text{rk}(A) = d$) a linear combination $E = \sum c_i E_i$. With $\beta_E := \sum c_i \beta_i$ we have $E - \beta_E \in H_A(\beta)$. We further write $\deg_E(-)$ for the degree function $\sum c_i \deg_{A,i}(-)$.

Denote $\theta_j = x_j \partial_j$ and $\theta = (\theta_0, \ldots, \theta_n)$. A linear combination $\sum_j v_j \theta_j$ is in the Euler space if and only if the coefficient vector $v = (v_0, \ldots, v_n)$, interpreted as a linear functional on $\mathbb{C}^{n+1}$ via $v((q_0, \ldots, q_n)) := \sum v_j q_j$, is the pull-back via $A$ of a linear functional on $\mathbb{C}^d$. In other words,

$$[v \cdot \theta^T = \sum_j v_j \theta_j \text{ is in the Euler space}] \iff [v = c \cdot A \text{ for some } c \in \mathbb{C}^d].$$

If $L: \mathbb{C}^d \rightarrow \mathbb{C}$ is a linear functional then the Euler operator in $H_A(\beta)$ corresponding to its image under $\text{Hom}_\mathbb{C}(\mathbb{C}^d, \mathbb{C}) \rightarrow \text{Hom}_\mathbb{C}(\mathbb{C}^{n+1}, \mathbb{C})$ is denoted $E_L - \beta_L$.

Lemma 1.3. For any set $F$ of columns of $A$ contained in a hyperplane that passes through the origin of $\mathbb{C}^d$ but does not contain $a_k$, there is an Euler operator $E_F - \beta_F$ in $H_A(\beta)$ such that the coefficient of $\theta_j$ in $E_F$ is zero for all $j \in F$, and equal to 1 for $j = k$. If $R_{\geq 0}F$ is a facet of $R_{\geq 0}A$ then $E_F - \beta_F$ is unique.

Proof. Choose for any such set $F$ a linear functional $L: \mathbb{Q}^d \rightarrow \mathbb{Q}$ that vanishes on $F$ while $L(a_k) = 1$. The corresponding Euler operator $E_L - \beta_L$ has the desired properties, and if we define numbers $a_{L,j}$ by

$$E_L =: \sum_j a_{L,j} x_j \partial_j$$

then $a_{L,j} = L(a_j)$. The uniqueness in the facet case is obvious. \qed
2. Restricting the Fourier transform

The Fourier transform $\mathcal{F}(-)$ is a functor from the category of $D$-modules on $V$ to the category of $D$-modules on the dual space $V^* = \text{Hom}_C(V, \mathbb{C})$. In this section we bound the $b$-function along a coordinate hyperplane of the Fourier transform $\mathcal{F}(M_A(\beta))$ of the hypergeometric system. Note that this module is called $\tilde{M}_A^\beta$ in [RS15].

The square of the Fourier transform is the involution induced by $x \mapsto -x$, which has no effect on the analytic properties of the modules we study. In particular, $b$-functions along coordinate hyperplanes are unaffected by this involution and we therefore consider $\mathcal{F}^{-1}(M_A(\beta))$ without harm.

We start with introducing some notation.

**Notation 2.1.** Let $\{y_j\}$ be the coordinates on $V^*$ such that $\mathcal{F}^{-1}(\partial_j) = y_j$ on the level of differential operators. We let $\check{D}_A$ be the ring of $\mathbb{C}$-linear differential operators on $\check{O}_A := \mathbb{C}[y_0, \ldots, y_n]$, generated by $\{y_j, \delta_j\}_{\delta_j}$ where $\delta_j$ denotes $\frac{\partial}{\partial y_j}$. Then $\mathcal{F}^{-1}(x_j) = -\delta_j$. The subring $\mathbb{C}[\delta_1, \ldots, \delta_n]$ of $\check{D}_A$ is denoted $\check{R}_A$. The isomorphism $(-): D_A \rightarrow \check{D}_A$ induced by $\check{\partial}_j := y_j$ and $\check{x}_j = \delta_j$ sends $O_A$ to $\check{R}_A$ and $R_A$ to $\check{O}_A$.

Thus, $\check{I}_A := \mathcal{F}^{-1}(I_A)$ is an ideal of $\check{O}_A$; the advantage of considering $\mathcal{F}^{-1}$ rather than $\mathcal{F}$ is that $I_A$ retains the shape of the generators of $I_A$ as differences of monomials. For each $j$ set $\check{\theta}_j := \mathcal{F}^{-1}(\partial_j) = -\delta_j y_j$. The $i$-th level $V$-filtration on $\check{D}_A$ along $y_i$ is spanned by $\delta^\alpha y^\beta$ with $\beta_i - \alpha_i \geq i$.

Before we get into the technical part, let us show by example an outline of what is to happen.

**Example 2.2.** Let $A = \begin{pmatrix} -1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$, a matrix whose associated semigroup ring is a normal complete intersection. We will estimate the $b$-function for restriction to the hyperplane $y_1 = 0$ (corresponding to the middle column) of $\mathcal{F}^{-1}(M_A(\beta))$.

![Figure 1: Restriction of the Fourier transform to $y_1 = 0$.](image)

The ideal $\check{H}_A(\beta) := \mathcal{F}^{-1}(H_A(\beta))$ is generated by

$$-\check{\theta}_0 + \check{\theta}_2 - \beta_1, \quad \check{\theta}_0 + \check{\theta}_1 + \check{\theta}_2 - \beta_2, \quad y_0 y_2 - y_1^2. \quad (2.1)$$

Since $y_1 \in (V^1 \check{D}_A)$, $y_0 y_2$ and hence also $\check{\theta}_0 \check{\theta}_2$ are in $(V^1 \check{D}_A) + \check{H}_A(\beta)$. The strategy of the example, and of the theorem in this section, is to multiply the element $1 \in \check{D}_A$ by suitable Euler operators so that the result is a sum of a polynomial $p(\check{\theta}_1)$ with an element of $\mathbb{C}[\check{\theta}_0, \check{\theta}_1, \check{\theta}_2] \cdot \check{\theta}_0 \check{\theta}_2$; this certifies $p(\check{\theta}_1)$ to be in $\check{H}_A(\beta) + (V^1 \check{D}_A)$.
In the case at hand, the relevant Euler operators are $2\tilde{\theta}_0 + \tilde{\theta}_1 + \beta_1 - \beta_2$ and $\tilde{\theta}_1 + 2\tilde{\theta}_2 - \beta_1 - \beta_2$. Modulo $H_A(\beta)$ we can rewrite $(V^1\hat{D}_A) \ni 4\delta_0\delta_2y_1^2 \equiv 4\delta_0\delta_2 \equiv (-\tilde{\theta}_1 - \beta_1 + \beta_2)(\tilde{\theta}_1 + \beta_1 + \beta_2)$. It follows that $(\tilde{s} + \beta_1 - \beta_2)(\tilde{s} - \beta_1 - \beta_2)$ is a multiple of the $b$-function, where $\tilde{s} = \tilde{\theta}_1 = -y_1\delta_1 - 1$. This Fourier twist in the argument of the $b$-function occurs naturally throughout and we will make our computations in this section in terms of $b(\tilde{s})$.

The expressions $\tilde{\theta}_1 + 2\tilde{\theta}_2$ and $2\tilde{\theta}_0 + \tilde{\theta}_1$ that appear in the Euler operators we used can be found systematically as follows. Let $d_1, d_2$ denote the coordinates on the degree group $\mathbb{Z}^2$ corresponding to $E_1$ and $E_2$; compare the discussion following Notation 1.2. An element of $S_\alpha$ has degree on the facet $\mathbb{R}_{\geq 0}a_0$ if and only if the functional $L_1(d_1, d_2) = d_1 + d_2$ vanishes, and the Euler field that corresponds to this functional in the spirit of Lemma 1.3 is exactly $\theta_1 + 2\theta_2 - \beta_1 - \beta_2$. The elements of $S_\alpha$ with degree on the facet $\mathbb{R}_{\geq 0}a_2$ are determined by the vanishing of $L_2(d_1, d_2) = d_2 - d_1$ and the Euler field corresponding to this functional is exactly $2\theta_0 + \theta_1 - \alpha - \beta_2$. It is no coincidence that the union of the kernels of these two functionals is exactly the set of quasi-degrees of $S_\alpha/\partial_1 \cdot S_\alpha$. The point is that modulo $H_A(\beta)$ all monomials in $S_\alpha$ with degree in $\mathbb{R}_{\geq 0}A$ are already in $(V^1\hat{D}_A)$. The task is then to deal with those with degree on the boundary through multiplication with suitable expressions.

The picture shows in blue the elements of $A$, in black the other elements of $NA$, and in red the quasi-degrees of $S_\alpha/\partial_1 \cdot S_\alpha$. Note finally that $(\beta_2 - \beta_1)a_1$ and $(\beta_1 + \beta_2)a_1$ are the intersections of $\mathbb{R} \cdot a_1$ with $qdeg_A(S_\alpha) + \beta$.

We now generalize the computation of the example to the general case.

**Convention 2.3.** For the remainder of this section we consider restriction to the hyperplane $y_0$ in order to save overhead (in terms of a further index variable).

Consider the toric module $N = S_\alpha/\partial_0 S_\alpha$, and take a toric filtration

$$(N) \quad 0 = N_0 \subset N_1 \subset \ldots \subset N_k = N$$

with composition factors

$$N_\alpha := N_\alpha/N_{\alpha-1},$$

each isomorphic to some shifted face ring $S_{F'_\alpha}(b_\alpha)$, $F'_\alpha = \tau_\alpha \cap A$, attached to a face $\tau_\alpha$ of $\mathbb{R}_{\geq 0}A$. (We will call such $F'_\alpha$ also a face.) Lifting the $N_\alpha$ to $S_\alpha$ yields an increasing sequence of $A$-graded ideals $J_\alpha \supset \partial_0$ of $S_\alpha$ with $N_\alpha = J_\alpha/\partial_0 \cdot S_\alpha$.

Choose for each composition factor a facet $F_\alpha$ containing $F'_\alpha$. Note that none of the faces $F'_\alpha$ will contain $a_0$ (as $\partial_0$ is zero on $N$ but not nilpotent on any face ring of a face containing $a_0$) and hence we can arrange that the corresponding facets do not contain $a_0$ either.

Lemma 1.3 produces for each $N_\alpha$ a facet $F_\alpha$ and corresponding functional $L_{F_\alpha}$ (which we abbreviate to $L_\alpha$) that vanishes on the facet and evaluates to 1 on $a_0$. The associated Euler operator in $H_F(\beta)$ is $E_{F_\alpha} - \beta_{F_\alpha}$. Since $L_\alpha$ is zero on all $A$-columns in $F_\alpha$ and since $N_\alpha$ is a shifted quotient of $S_{F_\alpha}$, there is a unique value for $L_\alpha$ on the $A$-degrees of all nonzero $A$-homogeneous elements of $N_\alpha$. We denote this value by $L_\alpha(N_\alpha)$. Note, however, that $L_\alpha(N_\alpha)$ does very much depend on the choice of the facet $F_\alpha$ even though the notation does not remember this.

Now let $T_0$ be the image in $\mathcal{F}^{-1}(M_A(\beta))$ of $\mathcal{F}^{-1}(J_\alpha)$ under the map induced by $\hat{O}_A \longrightarrow \hat{D}_A \longrightarrow \mathcal{F}^{-1}(M_A(\beta))$. Note that the image of $T_0 = y_0\hat{O}_A$ in $\mathcal{F}^{-1}(M_A(\beta))$ is in $(V^1\hat{D}_A) \cdot \bar{T}$, the bar denoting cosets in $\mathcal{F}^{-1}(M_A(\beta))$. 
Lemma 2.4. In the context above, let $\kappa_\alpha$ be the constant $L_\alpha(\mathbf{N}_\alpha)$. Then in $\mathcal{F}^{-1}(M_\alpha(\beta))$, modulo the image of $(V^1 \bar{D}_A)$,

$$(\bar{\theta}_0 + \kappa_\alpha - \beta_\alpha) \cdot (V^0 \bar{D}_A) \cdot T_\alpha = (V^0 \bar{D}_A) \cdot (\bar{\theta}_0 + \kappa_\alpha - \beta_\alpha) \cdot T_\alpha \subseteq (V^0 \bar{D}_A) \cdot T_{\alpha-1}.$$

Proof. Since the commutators $[\bar{\theta}_0, (V^0 \bar{D}_A)]$ are in $(V^1 \bar{D}_A)$, it suffices to show that $\bar{\theta}_0 + \kappa_\alpha - \beta_\alpha \cdot T_\alpha \subseteq (V^0 \bar{D}_A) \cdot T_{\alpha-1}$ modulo $\mathcal{F}^{-1}(H_A(\beta))$.

By definition, $E_\alpha \cdot \beta_\alpha := \mathcal{F}^{-1}(E_\alpha - \beta_\alpha)$ is zero in $\mathcal{F}^{-1}(M_\alpha(\beta))$. Take a monomial $\mathbf{m} \in \hat{O}_A$ whose cost lies in $T_\alpha \setminus T_{\alpha-1}$. By Equation (1.3), $E_\alpha \cdot \mathbf{m} = \mathbf{m}(E_\alpha - \kappa_\alpha)$ since $\mathcal{F}^{-1}(\cdot)$ is a homomorphism. Now write $E_\alpha = \sum a_{\alpha, j} \theta_j$; as before we have $a_{\alpha, j} = L_\alpha(a_j)$.

Since the coefficient of $\theta_0$ in $E_\alpha$ is 1, it follows that in $\mathcal{F}^{-1}(M_\alpha(\beta))$:

$$\begin{align*}
\bar{\theta}_0 \bar{\mathbf{m}} &= (\bar{E}_\alpha + \theta_0) \bar{\mathbf{m}} + \bar{E}_\alpha \bar{\mathbf{m}} \\
&= \sum_{\substack{j \neq 0 \\
\alpha_j \neq 0}} a_{\alpha, j} \delta_j \gamma_j \bar{\mathbf{m}} + \bar{\mathbf{m}}(\bar{E}_\alpha - \kappa_\alpha) \\
&= \sum_{\substack{j \neq 0 \\
\alpha_j \neq 0}} a_{\alpha, j} \delta_j \gamma_j \bar{\mathbf{m}} + \bar{\mathbf{m}}(\beta_\alpha - \kappa_\alpha).
\end{align*}$$

Recall that $F_\alpha$ contains $F'_\alpha$ and that $\mathbf{N}_\alpha$ is a $\mathbb{Z}A$-shift of $S_{F'_\alpha} = R_A/I_A$, whence each $y_j$ with $a_j \notin F'$ annihilates $\mathcal{F}^{-1}(\mathbf{N}_\alpha)$. Therefore, each term $a_{\alpha, j} \delta_j (y_j m)$ in the last sum of the display is in $(V^0 \bar{D}_A)T_{\alpha-1}$. It follows that in $\mathcal{F}^{-1}(M_\alpha(\beta))$ we have $(\bar{\theta}_0 + \kappa_\alpha - \beta_\alpha)T_\alpha \subseteq (V^0 \bar{D}_A)T_{\alpha-1}$ as claimed. $\square$

Theorem 2.5. For $t = 0, \ldots, n$, the number $\varepsilon \in \mathbb{C}$ is a root of the b-function $b(\hat{s})$ (with $\hat{s} = \theta_1 = -\delta \gamma_1$) of $\mathcal{F}^{-1}(M_\alpha(\beta))$ along $y_0 = 0$, only if $\varepsilon \cdot a_0$ is a point of intersection of the line $\mathbb{C} \cdot a_0$ with the set $\beta - \text{qdeg}_A(N)$, the quasi-degrees of the toric module $N = S_A/\partial_0 S_A$ multiplied by $-1$ and shifted by $\beta$.

Proof. Without loss of generality we shall suppose that $t = 0$ by way of re-indexing.

We will show that a divisor of $\textstyle\prod_{\alpha} (\bar{\theta}_0 + \kappa_\alpha - \beta_\alpha)$ is inside $H_A(\beta) + (V^1 \bar{D}_A)$, in notation from the previous lemma.

Indeed, it follows from Lemma 2.4 that $\textstyle\prod_{\alpha} (\bar{\theta}_0 + \kappa_\alpha - \beta_\alpha)$ multiplies $T \in \mathcal{F}^{-1}(M_\alpha(\beta))$ into $(V^0 \bar{D}_A) \cdot y_0 \cdot T \subseteq (V^1 \bar{D}_A) \cdot T$. Hence the root set of the b-function $b(\bar{\theta}_0)$ in question is a subset of $\{ \beta_\alpha - \kappa_\alpha \}$, $\alpha$ running through the indices of the chosen composition series of $N$. This set is determined by the composition series $(N)$ and the choices of the facets $F_{\alpha}$ for each $N_{\alpha}$. Varying over all choices of facets $\{ F_{\alpha} \}$ for a given chain $(N)$, the root set of $b(\bar{\theta}_0)$ is in the intersection $\rho_N$ of all possible sets $\{ \beta_\alpha - \kappa_\alpha \}_{\alpha \in (N)}$.

Since $L_\alpha(a_0) = 1$, the point $(\beta_\alpha - \kappa_\alpha) \cdot a_0$ is the intersection of the hyperplane $L_\alpha = \beta_\alpha - \kappa_\alpha$ with the line $\mathbb{C} \cdot a_0$. Thus, $\rho_N$ is inside the intersection of $\mathbb{C} \cdot a_0$ with all arrangements $\text{Var} \textstyle\prod_{\alpha} (L_\alpha - \beta_\alpha + \kappa_\alpha)$. The intersection of the arrangements $\text{Var} \textstyle\prod_{\alpha} (L_\alpha - \beta_\alpha + \kappa_\alpha)$ is the union of the quasi-degrees of all $\mathbf{N}_\alpha$ of the composition chain $(N)$, multiplied by $-1$ and shifted by $-\beta_\alpha$. As $N$ is finitely generated, $\text{qdeg}_A(N) = \bigcup_{\alpha} \text{qdeg}_A(\mathbf{N}_\alpha)$. Hence the root set of $b(\bar{\theta}_0)$ is contained in the intersection $-\text{qdeg}_A(S_A/\bar{\theta}_0 S_A) + \beta$ with $\mathbb{C} \cdot a_0$. $\square$
Remark 2.6. The quantity $\hat{t}_i$ is the more natural argument for the $b$-function here. Note that the roots of $b(y_t \delta_i)$ are those of $b(\hat{t}_i)$ shifted up by 1 and then multiplied by $-1$.

Example 2.7. Let $A = (a_0, a_1, a_2) = \left( \begin{array}{ccc} -1 & 0 & 3 \\ 1 & 1 & 1 \end{array} \right)$ and $\beta = (\beta_1, \beta_2)$. The ring $S_A$ is a complete intersection but not normal.

Consider restriction to $y_1 = 0$ (the middle column). Then $N = S_A/\partial_1 \cdot S_A$ has a toric filtration involving 4 steps, given by the ideals $0 \subseteq \delta_1^3 \cdot N \subseteq \delta_0^2 \cdot N \subseteq \delta_0 \cdot N \subseteq N$. The corresponding $A$-graded composition factors are $S_A(-3 \cdot a_0)/\langle \partial_1, \partial_2 \rangle S_A$ and $\{ S_A(-\alpha \cdot a_0)/\langle \partial_0, \partial_1 \rangle S_A \}_a = 0$. The $b$-function $b(\hat{t}_1)$ for the inverse Fourier transform is $(\hat{t}_1 - \beta_1 - \beta_2) \prod_{a=0}^{2} (\hat{t}_1 - 3\beta_2 - 3\beta_1 - 4a)$.

Explicitly, $y_1^2 - y_0 y_2 \in H_A(\beta)$ gives $(V^1 D_A) \ni \delta_0^3 \delta_2 y_0^2 y_2 = \hat{t}_3 \hat{t}_0 (\hat{t}_0 - 1) (\hat{t}_0 - 2)$ which modulo $H_A(\beta)$ equals $(-1)^2 (\hat{t}_1 - \beta_1 - \beta_2) \prod_{a=0}^{2} (\hat{t}_1 - 3\beta_2 - 3\beta_1 - 4a)$. The relevant Euler operators are $\hat{t}_1 + 4\hat{t}_2 - \beta_1 - \beta_2$ and $3\hat{t}_1 + 4\hat{t}_0 - 3\beta_2 + \beta_1$.

Figure 2: Restriction of the Fourier transform to $y_1 = 0$.

The picture shows in blue the columns of $A$, in black the other elements of $NA$, in red the quasi-degrees of $N = S_A/\partial_1 \cdot S_A$. The roots of $b(\delta_1 y_1)$ (which are opposite to the roots of $b(\hat{t}_1)$) are the intersections of the line $C \cdot \left( \begin{array}{c} 0 \\ \beta \end{array} \right)$ with the shift of the red lines by $-\beta$.

In this example, each composition factor corresponds to a facet and to a component of the quasi-degrees of $N$. One checks that each composition chain must have these four lines as quasi-degrees. Note, however, that composition chains are far from unique and in general such correspondence will not exist.

Remark 2.8. The $b$-function for $\mathcal{F}^{-1}(M_A(\beta))$ along a coordinate hyperplane is generally not reduced, and its degree may be lower than the length of the shortest toric filtration for $N = S_A/\partial_t \cdot S_A$ would suggest. (Not every component of $\beta - \text{qdeg}_A(N)$ needs to meet the line $C \cdot a_t$).

Corollary 2.9. The roots of the $b$-function $b(\delta_i y_i)$ of $\mathcal{F}^{-1}(M_A(\beta))$ along $y_i = 0$ are in the field $\mathbb{Q}(\beta)$.

Consider $\mathcal{F}^{-1}(M_A(0))$; then:

1. the roots of the $b$-function $b(\hat{t}_i)$ are non-negative rationals;
2. if $S_A$ is normal, all roots are in the interval $[0, 1]$;
3. if the interior ideal of $S_A$ is contained in $\partial_t \cdot S_A$ then zero is the only root.

Proof. The first claim is a consequence of the intersection property in Theorem 2.5: the defining equations for the quasi-degrees are rational.

Let $N = S_A/\partial_t S_A$. For items 1.-3., we need to study the intersection of $\text{qdeg}_A(N)$ with $C \cdot a_t$, since $\beta = 0$ and $\delta_i y_i = -\hat{t}_i$. The quasi-degrees of $N$ are covered by
hyperplanes of the sort $L_\alpha = \varepsilon$ where $L_\alpha$ is a rational supporting functional of the facet $F_\alpha$. In particular, we can arrange $L_\alpha$ to be zero on $F_\alpha$, positive on the rest of $A$, and $L_\alpha(a) = 1$. As $\deg_A(N) \subseteq \deg_A(S_A)$, $\varepsilon \geq 0$. Hence $\text{Var}(L_\alpha - \varepsilon)$ meets $C \cdot a_\ell$ in the non-negative rational multiple $\varepsilon a_\ell$ of $a_\ell$. If $S_A$ is normal, $\deg_A(S_A/\partial_A S_A)$ is covered by hyperplanes $\text{Var}(L_\alpha - \varepsilon)$ that do not meet the cone $a_\ell + \mathbb{R}_{\geq 0}a$. These are precisely the ones for which $\varepsilon < 1$.

If $\partial_1 S_A$ contains the interior ideal then $\deg_A(N)$, and hence $\text{qdeg}_A(N)$, is inside the supporting hyperplanes of the cone, which meet $C \cdot a_\ell$ at the origin. □

Remark 2.10. One special case in which case 3 of Corollary 2.9 applies is when $S_A$ is Gorenstein and where further $\partial_1$ generates the canonical module. The matrix

$$A = (a_0, \ldots, a_3) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

with the interior ideal being generated by $\partial_1 \partial_3$, provides an example that case (3) can occur in a Gorenstein situation without the boundary of $\mathbb{N}A$ being saturated. See [SW09a] for a discussion on Cohen–Maculayness of face rings of Cohen–Macaulay semigroup rings.

3. $b$-FUNCTIONS FOR THE HYPERGEOMETRIC SYSTEM

3.1. Restriction along a hyperplane. We are here interested in the $b$-function for the hypergeometric module $M_A(\beta)$ along the hyperplane $x_t = 0$. As in the previous section, apart from examples, we actually carry out all computations for $t = 0$, in order to have as few variables around as possible. On the other hand, the natural argument for expressing the $b$-function will be $s = x_0 \partial_0$.

Notation 3.1. With $A = (a_0, \ldots, a_n)$ and distinguished index 0, we denote $A' := (a_1, \ldots, a_n)$. Via $N A' \subseteq N A$ we consider $S_{A'}$ as a subring of $S_A$.

For $k \in \mathbb{N}$ let $J_{A,0,k} \subseteq S_{A'}$ be the vector space spanned by the monomials $\partial^u$ with $u_0 = 0$ (so that $\partial^u \in S_{A'}$) that satisfy $\partial_0^k \cdot \partial^u \in S_{A'}$. We denote $J_{A,0,k} \subseteq R_{A'}$ the preimage of $J_{A,0,k}$ under the natural surjection $R_{A'} \twoheadrightarrow S_{A'}$. Put $J_{A,0} = \sum_{k \geq 1} J_{A,0,k}$ and $J_{A,0} = J_{A,0}/I_{A'} \subseteq S_{A'}$.

Each $J_{A,0,k}$ is a monomial ideal of $S_{A'}$ since $\partial_0^k(\partial^u \partial^m) = \partial^u(\partial_0^k \partial^m)$. Note, however, that $J_{A,0,k}$ need not be contained in $J_{A,0,k+1}$. If $a_0 \in \mathbb{R}_{\geq 0}A'$ then some power of $\partial_0$ is in $S_{A'}$ and so $J_{A,0} = S_{A'}$.

Definition 3.2. For $a_0 \in \mathbb{R}^d$ outside $\mathbb{R}_{\geq 0}A'$, a point $a \in \mathbb{R}_{\geq 0}A'$ is $a_0$-visible if $a + \lambda \cdot a_0, 0 < \lambda \ll 1$ is outside $\mathbb{R}_{\geq 0}A'$. (The idea behind the choice of language is that the observer stands at the point of projective space given by the line $\mathbb{R}a_0$.)

By abuse of notation, we say that $\partial^a$ is $a_0$-visible if $a$ is.

Lemma 3.3. Assume that $a_0$ is not in the cone $\mathbb{R}_{\geq 0}A'$. Then the radical of $J_{A,0}$ is generated by the $a_0$-invisible elements of $S_{A'}$, and in consequence the quasi-degrees of $S_{A'}/J_{A,0}$ are a union of shifted face spans where each face is in its entirety visible from $a_0$.

Proof. If $ZA'/ZA'$ has positive rank then all points of $\mathbb{N}A$ are $a_0$-visible while $J_{A,0}$ is clearly zero, so that in this case there is nothing to prove. We therefore assume that $ZA'/ZA'$ is finite.

It is immediate that $a$ is $a_0$-visible if and only if any positive integer multiple of it is. This implies that no power of an $a_0$-visible element $\partial^a$ of $S_{A'}$ can be in the radical of $J_{A,0}$ since $\partial^{m \cdot a + ka_0}$ can’t have its degree in the cone of $A'$. 
For the converse, suppose \( a \) is not \( a_0 \)-visible, so that there are positive integers \( p < q \) with \( a+(p/q) \cdot a_0 \in \mathbb{R}_{\geq 0}A' \). Then a high power of \( \partial^\alpha \cdot a_0 \) is in \( \mathbb{C}[Z\!A'] \mathbb{R}_{\geq 0}A' \) and a suitable power \( \partial^b \) of that will be in \( \mathbb{C}[Z\!A'] \mathbb{R}_{\geq 0}A' \) because of the finiteness of \( Z\!A'/A' \). Now let \( \tau \) be the smallest face of \( \mathbb{R}_{\geq 0}A' \) that contains \( b \); this makes \( b \) an interior point of \( \tau \). Since \( \mathbb{C}[\tau \cap Z\!A'] \) is a finitely generated \( \mathbb{C}[\tau \cap NA'] \)-module, some power of \( \partial^b \) is in \( \mathbb{C}[\tau \cap NA'] \subseteq A' \). This shows that some power of \( \partial^\alpha \cdot a_0 \) and some power of \( \partial^\alpha \cdot a_0 \) is in \( A' \), establishing the first claim of the lemma.

In every composition chain for \( S_{A'}/J_{A,0} \), each composition factor is an \( S_{A'}/\sqrt{J_{A,0}} \)-module. Thus the quasi-degrees of \( S_{A'}/J_{A,0} \) are inside a union of shifted quasi-degrees of \( S_{A'}/\sqrt{J_{A,0}} \) and hence all \( a_0 \)-visible, which implies the second claim. \( \square \)

Our main theorem in this section is:

**Theorem 3.4.** The root locus of the \( b \)-function \( b(x_0\partial_0) \) for restriction of \( M_{A}(\beta) \) along \( x_0 = 0 \) is, up to inclusion of non-negative integers, contained in the locus of intersection \( (-\qdeg(A)S_{A'/J_{A,0}} + \beta) \cap C \cdot a_0 \). The set of integers needed can be taken to be the integers \( 0, \ldots, k-1 \) such that \( J_{A,0} = \sum_{1 \leq i \leq k} J_{A,0,i} \).

In two extreme cases one can be explicit:

1. If \( \dim S_{A} - 1 = \dim S_{A'} \) then the \( b \)-function is linear with root given by the intersection of \( (-\qdeg(A)S_{A'}) + \beta) \cap C \cdot a_0 \); that is, \( v_i = (v_1, \ldots, v_d) \) such that the Euler operator

\[
E - \beta_E = \sum_{i=1}^{d} v_i (E_i - \beta_i)
\]

is in \( H_{\alpha}(\beta) \) and equals \( t_0 - \beta_E \). In particular, the \( b \)-function is \( s - \beta_E \). On the other hand: \( \mathfrak{I}_{A,0} \) is zero in this case, \( v = (v_1, \ldots, v_d) \) is in the kernel of \( A'^T \), and \( a_0^T v = 1 \). Therefore, the quasi-degrees of \( S_{A'}/\sqrt{J_{A,0}} \) form the hyperplane given as the kernel of \( v \) and \( (v^T) \beta a_0 = \beta_E a_0 \) is the intersection of \( -\qdeg(A)S_{A'}) + \beta \) with \( C a_0 \).

2. If \( \mathfrak{I}_{A,0} = \mathbb{R}_{\geq 0}A' \) then \( \mathfrak{N}a_0 \) meets \( NA' \) and so \( \partial_0^k = \partial^u \) with \( u = (0, u_1, \ldots, u_n) \in \mathbb{N}A' \). In particular, \( J_{A,0} = S_{A'} \) in this case. Moreover, \( (x_0\partial_0)(x_0\partial_0-1) \cdots (x_0\partial_0-k+1) = x_0^k \partial_0^k - x_0^k (\partial_0^k - \partial^u) + x_0^k \partial^u \in H_{\alpha}(\beta) + V^1(D_A) \) shows the claim made in this case.

Now suppose that \( A \) and \( A' \) have equal rank but \( a_0 \notin \mathbb{R}_{\geq 0}A' \). In that case, \( \mathfrak{I}_{A,0} \) is a non-trivial ideal of \( S_{A'} \). We shall use a toric filtration

\[
(N) : 0 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_i = S_{A'}/\mathfrak{I}_{A,0}
\]

and let \( J_{\alpha} \supseteq J_{A,0} \) be the \( R_{A'} \)-ideal such that \( \alpha = J_{\alpha}/J_{A,0} \). We will view \( J_{\alpha} \) as subset of \( D_{A'} \) or even \( D_A \). In analogy to the previous case, for any \( \partial^u \) in \( J_{A,0;k} \) the \( b \)-function along \( x_0 \) of the coset of \( \partial^u \) in \( M_{\alpha}(\beta) \) divides \( s(s-1) \cdots (s-k+1) \). Indeed, \( \partial^u \in J_{A,0;k} \) implies that \( \partial_0^k \partial^u - \partial^u \in \mathfrak{I}_{A} \) for some \( v \) with \( v_0 = 0 \), and so \( x_0^k \partial_0^k \partial^u \in H_{\alpha}(\beta) + V^1(D_A) \). In particular, the root set of the \( b \)-function of the coset of \( \partial^u \) in \( M_{\alpha}(\beta) \) is inside the set of integers described in the statement of the theorem.

For each composition factor \( \mathfrak{N}_\alpha = \alpha/\alpha_{\alpha-1} \) choose now a facet \( \tau_\alpha \) of \( A' \) and an element \( \partial^u \) of \( S_{A'} \) \( u_\alpha \in \{0\} \times \mathbb{N}^n \) such that \( \alpha \) is a quotient of \( S_{A'}/\partial^u \) and such
that the annihilator of \(\partial^{\mu_0}\) in \(\mathbf{T}_0\) contains the toric ideal \(\mathcal{I}_A^{\tau_0}\). Then \(\text{qdeg}_A(\mathbf{T}_0)\) is contained in \(A' \cdot \mathbf{u} + \text{qdeg}_{A'}(S_{\tau_0})\).

Since \(\mathbf{a}_0\) is not in \(\mathbb{R}_{>0} A'\), Lemma 3.3 shows that the facet \(\tau_0\) can be chosen such that \(\mathbf{a}_0 \not\in \mathbb{Q} \cdot \tau_0\). Indeed, if an entire face of \(\mathbb{R}_{>0} A'\) is visible from \(\mathbf{a}_0\) then it sits in at least one facet whose span does not contain \(\mathbf{a}_0\). By Lemma 1.3 there is an element \(E_0\) of the Euler space of \(A\) that does not involve any element of \(\tau_0\) but which has coefficient 1 for \(\theta_0\). Notation 1.2 then associates a degree function \(\text{deg}_{E_0}(-)\) to \(\alpha\).

As \(\partial_j \cdot \partial^{\mu_0} \in \mathcal{N}_{\alpha-1}\) for \(j \notin \tau_0\) it follows that the difference of \(((E_\alpha - \beta_\alpha) \cdot \partial^{\mu_0})\) and \((\theta_0 - \beta_\alpha) \cdot \partial^{\mu_0}\) is inside \((V^0 D_A)\mathcal{N}_{\alpha-1}\). Since \(E_\alpha - \beta_\alpha\) is in \(H_A(\beta)\), so is \(\partial^{\mu_0}(E_\alpha - \beta_\alpha) = (E_\alpha - \beta_\alpha + \text{deg}_{E_0}(\partial^{\mu_0}))\partial^{\mu_0}\). Therefore, \((\theta_0 - \beta_\alpha + \text{deg}_{E_0}(\partial^{\mu_0}))\partial^{\mu_0}\) is in \(H_A(\beta) + (V^0 D_A)\mathcal{N}_{\alpha-1}\). Then, in parallel to how Lemma 2.4 was used in the proof of Theorem 2.5, the product

\[
\prod_{\alpha}((\theta_0 - \beta_\alpha + \text{deg}_{E_0}(\partial^{\mu_0})))
\]

multiplies \(1 \in D_A\) into \(H_A(\beta) + (V^0 D_A)J_{A,0} + (V^1 D_A)\). Multiplying by \(x^k_0 \partial^k_0\) for suitable \(k\) one obtains the desired bound for the \(b\)-function as in the second paragraph of the proof.

It follows as in Theorem 2.5 (with the modification that we have here \(\theta_0\) rather than \(\mathcal{F}^{-1}(\theta_0)\), which affects signs) that the intersection of the roots of all such bounds is the intersection of \((-\text{qdeg}_A(S_A/\mathcal{T}_{A,0}) + \beta)\) with the line \(\mathbb{C} \cdot \mathbf{a}_0\).

**Example 3.5.** With \(A = (\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2) = \left(\begin{array}{ccc} -1 & 0 & 3 \\ 1 & 1 & 1 \end{array}\right)\), consider the \(b\)-function along \(x_1\) of the \(A\)-hypergeometric system. The ideal \(J_{A,1}\) is generated by \(1 \in S_A = \mathbb{C}[\mathbf{N}(\mathbf{a}_0, \mathbf{a}_2)]\) since \(\partial^{\mu_1}_{0}\) is in \(S_A\). The set of necessary integer roots is then \(\{0, 1, 2, 3\}\). No other roots are needed since \(S_A/J_{A,1}\) is zero, irrespective of \(\beta\).

Figure 3: The elements of \(S_A \setminus S_A'\) (black) and \(S_A'\) (green) for restriction to \(x_1\)

Restriction to \((x_2 = 0)\) behaves differently. As \(S_A' = \mathbb{C}[\mathbf{N}(\mathbf{a}_0, \mathbf{a}_1)]\) now, \(J_{A,2} = J_{A,2,1}\) is generated by \(\partial^{\mu_1}_{0}\), and the quasi-degrees of \(S_A'/J_{A,2}\) are the lines \(\mathbb{C} \cdot (0, 1) + (i, 0)\) with \(i = 0, -1, -2\). The intersection of the negative of three these lines, shifted by \(\beta\), with the line \(\mathbb{C} \cdot \mathbf{a}_2\) is \(\mathbf{a}_2 \cdot \{(i + \beta_1)/3\}_{i=0,1,2}\). So the \(b\)-function has (at worst) roots \(\{0, \beta_1, \beta_1 + 1, \beta_1 + 2\}/3\).

**Remark 3.6.** We believe that both bounds in Theorems 2.5 (as is) and 3.4 (up to integers) are sharp.

3.2. **Restriction to a generic point.** We suppose here that \(A\) is homogeneous; in other words, the Euler space contains a homothety. Let \(p = (p_0, \ldots, p_n)\) be a point of \(\mathbb{C}^{n+1}\). We wish to estimate here the \(b\)-function for restriction of \(M_A(\beta)\) to the point \(-p\) if \(p\) is generic. As a holonomic module is a connection near any generic...
point, this restriction yields a vector space isomorphic to the space of solutions to $H_A(\beta)$ near $-p$, see [SST00, Sec. 5.2].

**Definition 3.7.** Let $\theta_p = (x_0 + p_0)\partial_0 + \ldots + (x_n + p_n)\partial_n$ and write $\theta$ for $\theta_p$ if $p = 0$. The $b$-function for restriction of a principal $D$-module $M = D/I$ to the point $x + p = 0$ is the minimal polynomial $b_p(s)$ such that $b_p(\theta_p) \in I + (V^1_p D)$ where $V^k_p D$ is the Kashiwara–Malgrange $V$-filtration along $\Var(x + p)$:

$$V^k_p D = \mathbb{C} \cdot \{(x + p)^u \partial^v \mid |u| - |v| \geq k\}.$$

**Remark 3.8.** (1) For any pair of manifolds $Y \subseteq X$ and given a $D$-module $M$ on $X$ one can define a $b$-function of restriction for the section $m \in M$ along $Y$ by a formula generalizing both Definition 0.1 and Definition 3.7. Kashiwara proved their existence for holonomic $M$.

(2) The roots of this $b$-function here relate to restriction of solution sheaves as follows. Near a generic point $x + p = 0$, a $D$-module $M$ is a connection whose solution space has a basis consisting of a certain number of holomorphic functions. The germs of these functions form a vector space that can be identified with the dual of the 0-th homology group of $(D/(x + p)D) \otimes_{D} M$. Filtering this complex by $V^*_p D$, $b_p(k)$ annihilates the $k$-th graded part of its homology, compare [Oak97, OT01, Wal00]. In particular, $b_p(s)$ carries information on the starting terms of the solution sheaf of $M$ near $x + p = 0$.

The purpose of this section is to bound $b_p(s)$ for $I = H_A(\beta)$ and generic $p$ with the following strategy. We first show that a polynomial $b(s)$ is a multiple of $b_p(s)$ if $b(\theta)$ is in $D_A(I_A, A \cdot \mathcal{E} \cdot \partial)$ where

$$\mathcal{E} = \begin{pmatrix} p_0 & 0 & \cdots & 0 \\ 0 & p_1 & \vdots & \\ \vdots & \ddots & 0 & \\ 0 & \cdots & 0 & p_n \end{pmatrix},$$

provided that $p$ is component-wise nonzero. The generators of $D_A(I_A, A \cdot \mathcal{E} \cdot \partial)$ are independent of $x$ and we next observe that the radical of $R_A(I_A, A \cdot \mathcal{E} \cdot \partial)$ is $R_A \cdot \partial$, provided that $p$ is generic. Thus, $b_p(s)$ will be a factor of any polynomial that annihilates the finite length module $R_A/(I_A, A \cdot \mathcal{E} \cdot \partial)$ as long as $p$ is generic. We exhibit a particular such polynomial with all roots integral. In the case of a normal semigroup ring, we show that the (necessarily integral) roots of $b_p(s)$ are in the interval $[0, d - 1]$.

We begin with pointing out that $b(\theta_p) \in I + (V^1_p D)$ is equivalent to $b(\theta) \in I_p + (V^1_0 D)$ where $I_p$ is the image of $I$ under the morphism induced by $x \mapsto x - p$.
\( \partial \to \partial \) and \((V_0^k D)\) is the Kashiwara–Malgrange filtration along the origin. Among the generators of \( I = H_A(\beta) \), only the Euler operators depend on \( x \) while \((I_A)_p = I_A\) for any \( p \); one has \((E_i - \beta_i)_p = \sum a_{i,j}(x_j - p_j)\partial_j - \beta_i = E_i - \beta_i - \sum a_{i,j}p_j\partial_j\). We hence seek a relation \( b(\theta) \in D_A \cdot (I_A, E - \beta - A \cdot \mathcal{E} \cdot \partial) + (V_0^1 D_A) \) with \( \mathcal{E} \) as above.

Generally, a statement \( b(\theta) \in I + (V_0^1 D_A) \) is equivalent to \( b(\theta) \) being in the degree zero part \( \text{gr}^0_{V_0}(I) \) of the associated graded object. Note that \( \text{gr}_{V_0}(D_A) \) is a Weyl algebra again (although of course the symbol map \( D_A \to \text{gr}_{V_0}(D_A) \) is not an isomorphism). Abusing notation, we denote \( x \) and \( \partial \) also the symbols in \( \text{gr}_{V_0}(D_A) \) of the respective elements of \( D_A \). By the previous paragraph then, the graded ideal \( \text{gr}_{V_0}(H_A(\beta)_p) \) contains the elements that generate \( I_A \) (since \( I_A \) is homogeneous!), as well as the elements \( A \cdot \mathcal{E} \cdot \partial \) which arise as the \( V_0 \)-symbols of \( E_p - \beta \).

We need the following folklore result (for which we know no explicit reference).

**Claim.** The \( R_A \)-ideal generated by \( I_A \) and \( A \cdot \mathcal{E} \cdot \partial \) has, for generic \( \mathcal{E} \), radical \( R_A \cdot \partial \).

**Proof.** As \( I_A \) and \( A \cdot \mathcal{E} \cdot \partial \) are standard graded, \( \text{Var}(I_A, A \cdot \mathcal{E} \cdot \partial) \) is a conical variety. It thus suffices to show that the ideal \( \text{Var}(I_A, A \cdot \mathcal{E} \cdot \partial) \) is of height \( n + 1 \).

The ideal \( R_A[x](I_A, A \cdot \theta) \) in the polynomial ring \( R_A[x] \) defines in the cotangent bundle \( \text{Spec}(R_A[x]) \) of \( \mathbb{C}^{n+1} \) the union of the conormals to each torus orbit since the Euler fields are tangent to the torus and span a space of the correct dimension in each orbit point. Suppose the claim is false, so that there is a nonzero point \( y \in \text{Var}(I_A) \) such that (the generically chosen vector) \( p \) is a conormal vector to the orbit of \( y \). If \( y \) is in a torus orbit \( O_\tau \) associated to a proper face \( \tau \) of \( A \) then its coordinates corresponding to \( A \setminus \tau \) are zero and we can reduce the question to the case where \( A = \tau \). It is hence enough to show that there is \( p \in \mathbb{C}^{n+1} \) such that \( p \) is not a conormal vector to any smooth point of \( \text{Var}(I_A) \).

Let \( X \subseteq \mathbb{C}^{n+1} \) be any reduced affine variety and denote \( X_0 \) its smooth locus. We define a set \( C(X) \) inside \( \mathbb{C}^{n+1} \) by setting

\[
[\eta \in C(X)] \iff (\exists y \in X_0, \eta \in (T^*_{X_0}(\mathbb{C}^{n+1}))_y)\]

where \((T^*_{X_0}(\mathbb{C}^{n+1}))_y\) is the fiber of the conormal bundle at \( y \) of the pair \( X_0 \subseteq \mathbb{C}^{n+1} \). This is a constructible, analytically parameterized union of a \( \dim(X) \)-dimensional family of vector spaces of dimension \( n + 1 - \dim(X) \), which hence might fill \( \mathbb{C}^{n+1} \).

Now suppose that \( X \) is a conical variety; then the conormals of \( y \) and \( \lambda y \) agree for all \( \lambda \in \mathbb{C}^* \). In particular,

\[
C(X) = \bigcup_{\mathcal{P} \in \text{Proj}(X)} (T^*_{X_0}(\mathbb{C}^{n+1}))_y
\]

where \( \text{Proj}(X) \) is the associated projective variety. But this is now an analytically parameterized union of a \( (\dim(X) - 1) \)-dimensional family of vector spaces of dimension \( n + 1 - \dim(X) \). It follows that most elements of \( \mathbb{C}^{n+1} \) are outside \( C(X) \) in this case, and the claim follows. \( \square \)

It follows from the Claim that \( \text{gr}_{V_0}(H_A(\beta)_p) \) contains all monomials in \( \partial \) of a certain degree \( k \) that depends on \( A \). Let \( E = \theta_0 + \ldots + \theta_n \); by hypothesis \( E - \beta E \in H_A(\beta) \).
Theorem 3.11. Denote $\partial_A^k$ the set of all monomials of degree $k$ in $\partial_0, \ldots, \partial_n$, and $D_A \cdot \partial_A^k$ the left $D_A$-ideal generated by $\partial_A^k$. Then in $D_A/D_A \cdot \partial_A^k$, the identity $E(E - 1) \cdots (E - k + 1) \equiv 0$ holds.

Proof. This is clear if $k = 1$. In general, by induction,

$$E(E - 1) \cdots (E - k + 1) \in D_A \cdot \partial_A^{k-1} \cdot (E - k + 1) = D_A \cdot E \cdot \partial_A^{k-1} \subseteq D_A \cdot \partial_A^k.$$

$\square$

Remark 3.10. The homogeneity of $X$ is necessary in the Claim, since otherwise $C(X)$ does not need to be contained in a hypersurface. Consider, for example, $A = (2, 1)$ in which case the union of all tangent lines (nearly) fills the plane, and where the zero locus of $I_A$ and $A \cdot \mathcal{E} \cdot \partial$ contains always at least two points.

The lemma implies that $\text{gr}_{(1, p)}^0(H_A(\beta)_p)$ contains $E(E - 1) \cdots (E - k + 1)$ if $p$ is generic. In other words, the $b$-function for restriction of $M_A(\beta)$ to a generic point divides $s(s - 1) \cdots (s - k + 1)$.

In some cases one can be more explicit about $k - 1$, the top degree in which $R_A/R_A(I_A, A \cdot \mathcal{E} \cdot \partial)$ is nonzero. Suppose $S_A$ is a Cohen–Macaulay ring, then systems of parameters are regular sequences. In particular, the Hilbert series of $Q_A := R_A/R_A(I_A, A \cdot \mathcal{E} \cdot \partial)$ is that of $S_A$ multiplied by $(1 - t)^d$. Suppose in addition, that $S_A$ is normal. Since we already assume that $S_A$ is standard graded, let $P$ be the polytope that forms the convex hull of the columns of $A$. The Hilbert series of $S_A$ is then of the form $\sum_{m=0}^\infty p_m \cdot t^m$ where $p_m$ is the number of lattice points in the dilated polytope $m \cdot P$. This number of lattice points is counted by the Erhart polynomial $E_P(m)$ of $P$, a polynomial of degree $d - 1 = \dim(P)$. If one writes the Hilbert series of $S_A$ in standard form $Q(t)/(1 - t)^d$ then the Hilbert series of $Q_A$ is just the polynomial $Q(t)$. In particular, the highest degree of a non-vanishing element of $Q_A$ is the degree of $Q(t)$.

In order to determine $\deg(Q(t))$ let $E_P(m) = e_{d-1}m^{d-1} + \ldots + e_0$. Now in

$$\sum_{m=0}^\infty E_P(m)t^m = \sum_{i=0}^{d-1} \left( e_i \cdot \sum_{m=0}^\infty m^i \cdot t^m \right),$$

each term $\sum_{m=0}^\infty m^i \cdot t^m$, for $m > 0$, is a polylogarithm $\text{Li}_{-i}(t)$ given by $(1/t)^i \log(1/t)$.

A simple calculation shows that $\text{Li}_{-i}(t)$ is the quotient of a polynomial of degree $i - 1$ by $(1 - t)^i$. Hence the sum in the display is the quotient of a polynomial of degree at most $d - 1$ by $(1 - t)^d$. The degree is truly $d - 1$ as one can check from the differential expression for $\text{Li}_{-i}(t)$ above.

Therefore, the Hilbert series $Q(t)$ of $Q_A$ is a polynomial of degree $d - 1$. We have proved

Theorem 3.11. Let $S_A$ be standard graded. The $b$-function for restriction of $M_A(\beta)$ to a generic point $x + p = 0$ divides $s(s - 1) \cdots (s - k + 1)$ where $k$ denotes the highest degree in which the quotient $S_A/S_A \cdot (A \cdot \mathcal{E} \cdot \partial)$ is nonzero. If, in addition, $S_A$ is normal then one may take $k = d$. $\square$

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T. Reichelt, Universität Heidelberg, Mathematisches Institut, Im Neuenheimer Feld 205, 69120 Heidelberg, Germany

E-mail address: reichelt@mathi.uni-heidelberg.de

C. Sevenheck, Technische Universität Chemnitz, Fakultät für Mathematik, 09107 Chemnitz, Germany

E-mail address: christian.sevenheck@mathematik.tu-chemnitz.de

U. Walther, Purdue University, Dept. of Mathematics, 150 N. University St., West Lafayette, IN 47907, USA.

E-mail address: walther@math.purdue.edu