Logic Functions and Quantum Error Correcting Codes

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Abstract. In this paper, based on the relationship between logic functions and quantum error correcting codes (QECCs), we unify the construction of QECCs via graphs, projectors and logic functions. A construction of QECCs over a prime field \( F_p \) is given, and one of the results given by Ref[8] can be viewed as a corollary of one theorem in this paper. With the help of Boolean functions, we give a clear proof of the existence of a graphical QECC in mathematical view, and find that the existence of an \([n, k, d]\) QECC over \( F_p \) requires similar conditions with that depicted in Ref[9]. The result that under the correspondence defined in Ref[17], every \([n, 0, d]\) QECC over \( F_2 \) corresponding to a simple undirected graph has a Boolean basis state, which is closely related to the adjacency matrix of the graph, is given. After a modification of the definition of operators, we find that some QECCs constructed via projectors depicted in Ref[11] can have Boolean basis states. A necessary condition for a Boolean function being used in the construction via projectors is given. We also give some examples to illustrate our results.

1 Introduction

Quantum error correcting codes (QECCs) have received more and more attention for near few decades since the theory of quantum error correction was put forward[12,14]. One of the central tasks in the theory of QECCs is how to construct them, and the first systematic mathematical construction is given in[4] in the binary case and then generalized in [5,6]. Many good non-binary QECCs have been constructed by using classical error-correcting codes over \( F_q \) or \( F_{q^2} \) (\( q \) is a power of an odd prime) with special orthogonal properties. Besides these, many quantum codes are constructed via tools including graphs and Boolean functions[7,8,10]. In this paper, we unify the two tools.

In Ref[7], the author constructed \([n, 0, d]\) QECCs using Boolean functions with \( n \) variables and aperiodic propagation criterion (APC) distance \( d \), and gave an algorithm to compute the APC distance of a Boolean function, orbits of Boolean functions of the same APC distance are also studied. In this paper, a construction of quantum codes of dimension more than 0 is given basing on the
relationship between logic functions and quantum codes. In Ref\[8\], the authors constructed quantum codes by giving $K$ basis states based on a graph state associated with a graph, and each basis state corresponds to a subset of the vertexes, which can be viewed as a corollary of this paper with weaker requirements.

In Ref\[9\], the authors initiated the construction of QECCs via the construction of matrixes with some properties and proved the necessary and sufficient conditions for the graph that the resulting code corrects a certain number of errors in physics view. In Ref\[10\], the author gave another proof in mathematical view. Based on the mathematical proof and the close relationship between logic functions and matrixes, properties that a logic function should have in order to construct a QECC is systematically studied. and find that the existence of an $[n,k,d]$ QECC over $F_p$ requires similar conditions with that depicted in Ref\[9\]. Under the correspondence defined in Ref\[17\], we find that every $[n,0,d]$ over $F_2$ corresponding to a simple undirected graph with the adjacency matrix $\Gamma_{n \times n}$ has the basis state $|\psi_f\rangle = 2^{-n/2} \sum_{x \in F_2^n} (-1)^{f(x)} |x\rangle$, where $f(x) = \frac{1}{2} x \Gamma x^T$. We also give an example to illustrate our result.

In Ref\[11\], the author describes a common mathematical framework for the design of QECCs basing on the correspondence between Boolean functions and projection operators. We point out that in some conditions, the basis states of the QECCs under the framework can have the probability vector of the form $2^{-n/2} (-1)^{f(x)}$, where $f(x)$ is a Boolean function.

2 Preliminaries

We often express an error (also can be viewed as an operator) operating on $C^q^n$ as $E_{(a,b)} = i^a X_a Z_b$, where $a, b$ are vectors of length $n$ over $F_q$, where $q$ is a power of a prime $p$. And

$$E_{(a,b)} |x_1 x_2 \cdots x_n\rangle = i^a \zeta^{tr_{q/p}(\sum_{j=1}^{n} b_j x_j)} |x'_1 x'_2 \cdots x'_n\rangle$$

where $x'_j = x_j + a_j$, $\zeta$ is a $p$-th primitive root of 1.

Especially, an error acting on a n-qubit state in $C^2^n$ has simpler forms.

**Definition 1.** Operators $E_{(a,b)}$ associated with binary vectors $(a,b) \in F_2^{2n}$ are defined by

$$E_{(a,b)} = e_1 \otimes \cdots \otimes e_m = i^{a \cdot b} X_a Z_b$$

where $e_i = \left\{ \begin{array}{ll} I_2, & a_i = 0, b_i = 0. \\ \sigma_x, & a_i = 1, b_i = 0. \\ \sigma_z, & a_i = 0, b_i = 1. \\ \sigma_y, & a_i = 1, b_i = 1. \end{array} \right.$$

**Definition 2.** The weight of an error $E_{(a,b)} = i^a X_a Z_b$ is defined by the symplectic weight of two vectors $a, b$ of length $n$, i.e.,

$$W_k(a,b) = \sharp \{ i | 1 \leq i \leq n, (a_i, b_i) \neq (0,0) \}$$
We generalize the definition of APC (aperiodic propagation criterion) distance of a Boolean function in Ref[7] to $F_p$.

**Definition 3.** The APC (aperiodic propagation criterion) distance of a logic function $f$ over $F_p$ is defined by the smallest nonzero $w_s(a,b)$, where $a,b \in F_p^n$ such that

$$\sum_{x \in F_p^n} \zeta^{f(x)+f(x-a)+b \cdot x} \neq 0$$

(4)

where $\zeta$ is a $p$-th primitive root of 1.

Let $G = (V,E)$ be a graph with vertex set $V = X \cup Y$ and edge set $E = V \times V$. Each edge $(u,v) \in E(u,v \in V)$ is assigned a weight $a_{uv} (= a_{vu}) \in F_p$. Therefore, such a graph $G$ corresponds to a symmetric matrix over $F_p$ $A_{n \times n} = (a_{uv})_{u,v \in V}$.

For two subsets $S$ and $T$ of $V$, we denote $A_{S,T}$ as the submatrix of $A$ with size $|S| \times |T|

$$A_{S,T} = (a_{uv})_{u \in S,v \in T}$$

Similarly, a vector in the vector space $F_p^{|V|} = F_p^{n+k}$ is denoted by a column vector

$$d^v = \begin{pmatrix} d^{v_1} \\ d^{v_2} \\ \vdots \\ d^{v_{n+k}} \end{pmatrix} = \{d^v|v \in V\}$$

where $d^{v_i} \in F_p$. For a subset $S$ of $V$, we denote $d^S = \{d^v|s \in S\} \in F_p^{|S|}$, and $O^S$ a vector of length $|S|$ with every coordinate equal to 0.

Let $E$ be a subset of $Y$ with $d - 1$ elements, $I = Y \setminus E$, then

$$A = \begin{pmatrix} A_{XX} & A_{XE} & A_{XI} \\ A_{EX} & A_{EE} & A_{EI} \\ A_{IX} & A_{IE} & A_{II} \end{pmatrix}$$

**Lemma 1.** Suppose $X,Y$ are two disjoint sets. $|X| = k$, $|Y| = n \geq k + 2d - 2$, $d \geq 2$, $A = (a_{ij})_{i,j \in X \cup Y}$ is a symmetric matrix with vanishing diagonal entries. For arbitrary $E \subseteq Y$, $|E| = d - 1$, if

$$A_{IX}d^X + A_{IE}d^E = O^I$$

(5)

with $I = Y \setminus E$ implies that

$$d^X = O^X \text{ and } A_{XE}d^E = O^X$$

(6)

Then there exists an $[n,k,d]$ quantum code.
Lemma 3. We define the $Z$ set $f$ of a Boolean function $f$ by

$$Zset_f = \{a\mid \sum_{x \in F_p^2} f(x)f(x+a) = 0\}$$

Lemma 2. If the weight of the Boolean function $f$ with $n$ variables is $M$, and $M \leq 2^{n-1}$, then $Zset_f = \{a\mid r_f(a) = 2^n - 4M\}$, where $r_f(a)$ is the autocorrelation function of $f(v)$ at $a$, i.e., $r_f(a) = \sum_{x \in F_p^2} (-1)^{f(x)+f(x+a)}$.

Lemma 4. We define the $Z$ set $f$ of a Boolean function $f$ with the following properties

1) $f$ is a function of $n$ variables and has weight $M$.
2) The $Z$ set $f$ contains the set $\{\alpha_1, \alpha_2, \ldots, \alpha_{2^n}, \alpha_1 + \alpha_{n+1}, \ldots, \alpha_n + \alpha_{2^n}\}$ and the matrix $A_f = (\alpha_1, \alpha_2, \ldots, \alpha_{2^n})_{n \times 2^n}$ has the property that any two rows have symplectic inner product zero and all the rows are independent.

In the above lemma, the QECC is constructed by giving the projector $P = f(P_1, P_2, \ldots, P_n)$ onto the code, and $P$ is constructed in the sense of a logic of projection operators given in Ref. 11, where $\{P_{n+i} = \frac{1}{2}(I + E_{\gamma_i})|1 \leq i \leq n\}, \gamma_i$ is the $i$-th row of the matrix $A_f$. From the symplectic orthogonality of the rows of $A_f$, we have $\{P_i|1 \leq i \leq n\}$ are pairwise commutative, and the error-correcting ability of the QECC is ensured by the properties of $Zset_f$. An arbitrary error $e$ acting nontrivially on one qubit only takes the projector $P_{f(x)}$ to $P_{f(x+\delta)}$, i.e., $eP_{f(x)}e = P_{f(x+\delta)}$, where $\delta$ is an element in $Zset_f$, and $P_{f(x+\delta)}$ is orthogonal to $P_{f(x)}$.

3 logic functions and quantum states

For a logic function $f(x)$ with $n$ variables over $F_p$, it corresponds to a vector $s = p^{-\frac{n}{2}}\xi^{f(x)}$, which can be interpreted as the probability distribution vector of the quantum state

$$|\psi_f\rangle = p^{-\frac{n}{2}} \sum_{x \in F_p^n} \xi^{f(x)}|x\rangle$$

Specially, if a state has the form of $2^{-\frac{n}{2}} \sum_{x \in F_p^n} (-1)^{f(x)}|x\rangle$, where $f(x)$ is a Boolean function, we call it a Boolean state corresponding with $f(x)$.

Then if an error $E_{(a,b)}$ acts on the state $|\psi_f\rangle$, it takes it to another state which is proportional to $p^{-\frac{n}{2}} \sum_{x \in F_p^n} \xi^{f(x-a)+b \cdot x}|x\rangle$, which can also be expressed in terms of a logic function, $f(x) \rightarrow f(x - a) + b \cdot x$.

Let $K$ Boolean functions be $g_i(x) = f(x) + \beta_i \cdot x$, and for $1 \leq i < j \leq K$, $\beta_i \neq \beta_j$, define $K$ quantum states as $|\psi_i\rangle = p^{-\frac{n}{2}} \sum_{x \in F_p^n} \xi^{g_i(x)}|x\rangle$, then we have the following theorem.
4.1 quadratic Boolean functions of the form \( \frac{1}{2} x \Gamma x^T \)

Now we consider a class of quadratic logic function corresponding with a simple undirected graph.

If \( f(x) \) is a quadratic Boolean function and can be represented as \( f(x) = \frac{1}{2} x \Gamma x^T \), where \( \Gamma_{n \times n} \) is a symmetric matrix with elements in \( F_2 \) and vanishing diagonal entries, then the state \( \langle \psi \rangle \) can be viewed as a graph state because \( \Gamma \) can be viewed as the adjacency matrix of a graph \( G = (V,E) \), where \( V \) and \( E \) denote the set of vertices and edges respectively and \( |V| = n \). If we label every vertex of the graph \( G \) of \( n \) vertexes from 1 to \( n \), then every vertex corresponds with one qubit, and the error \( \varepsilon_d = X_a Z_b \) can be written as \( X_\omega Z_\delta \), where \( \omega, \delta \) are subsets of \( V \).

Consider the operator \( \mathcal{G}_a = X_a \prod_{b \in N_a} Z_b \), where \( N_a \) represents the neighborhood of the vertex \( a \) and is denoted by \( N_a = \{ v \in V | \Gamma_{av} = 1 \} \), and it was shown in Ref\([12]\) that \( \mathcal{G}_a |\psi_f\rangle = |\psi_f\rangle \), so \( X_\omega Z_\delta |\psi_f\rangle = |\psi_f\rangle \).

Choose \( K \) subsets of \( V \) \( \{ C_i | 1 \leq i \leq K \} \), we define \( K \) pair-wise orthogonal quantum states \(([8])\) as

\[
|\psi_i\rangle = 2^{-\frac{n}{2}} \sum_{x \in F_2^n} (-1)^{g_i(x)} |x\rangle = Z_{C_i}|\psi\rangle.
\]

where and corresponds to a vector \( \beta_i \) of length \( n \), therefore \( g_i(x) \) can be expressed as \( g_i(x) = f(x) + \beta_i \cdot x \). It was shown that \([8]\) \( \mathcal{G}_a |\psi_i\rangle = -|\psi_i\rangle \) if \( a \in C_i \) and \( \mathcal{G}_a |\psi_i\rangle = |\psi_i\rangle \) if \( a \in C_i \) otherwise.

Theorem 1. The subspace spanned by \( \{|\psi_i\rangle | 1 \leq i \leq K \} \) is an \(( (n,K,d') )_p \) quantum code, where \( d' = \min \{ W_s(u,v) | \text{there exist } 1 \leq i \leq K \text{ such that } W_s(u,v + \beta_i + \beta_j) \geq d \} \), where \( d \) is the APC distance of \( f(x) \).

**Proof.** We only need to prove that for any error \( \varepsilon_d \) acting nontrivially on less than \( d \) qubits, \( \langle \psi_i | \varepsilon_d | \psi_j \rangle = f(\varepsilon_d) \delta_{ij} \) for all \( 1 \leq i, j \leq K \). Without lose of generosity we assume that \( \varepsilon_d = X_a Z_b \) for some pair of vectors in \( F_2^n \) with \( W_s(u,v) < d' \).

We have

\[
\langle \psi_i | \varepsilon_d | \psi_j \rangle \propto \sum_{x \in F_2^n} \zeta^f(x-u)+(\beta_i+\beta_j+v)\cdot x+f(x).
\]

Since \( W_s(u,v + \beta_i + \beta_j) < d \), so \( \langle \psi_i | \varepsilon_d | \psi_j \rangle = 0 \).

Then we verify that \( \langle \psi_i | \varepsilon_d | \psi_i \rangle \) only depends on \( \varepsilon_d \).

\[
\langle \psi_i | \varepsilon_d | \psi_i \rangle \propto \sum_{x \in F_2^n} \zeta^f(x-u)+v\cdot x+f(x) = 0
\]

So, \( \{|\psi_i\rangle | 1 \leq i \leq K \} \) span an \(( (n,K,d') )_p \) quantum code.\[
\]
Definition 5. The $d$-uncoverable set $\mathbb{D}_d$ that contains all the subsets of $V$ which can’t be covered by less than $d$ vertices is denoted by
\[
\mathbb{D}_d = 2^V - \{ \delta \Delta N_{\omega} | |\omega \cup \delta| < d \}
\]
where $\omega \Delta \delta$ denotes the symmetric difference of two sets $\omega, \delta$, i.e., $\omega \Delta \delta = \omega \cup \delta - \omega \cap \delta$, and $N_{\omega}$ denotes the neighborhood of the $\omega$, i.e., for every element $v$ in $N_{\omega}$, there exist an element $v'$ in $\omega$ such that $\Gamma_{v,v'} = 1$.

Corollary 1. If $C = \{C_1, C_2, \ldots, C_K\}$ satisfies the following two conditions,
\[
\begin{align*}
(1) \varnothing & \in C; (2) C_i \Delta C_j \in \mathbb{D}_d.
\end{align*}
\]
then the subspace spanned by the basis $\{ |\psi_i \rangle = Z_{C_i} |\psi \rangle |1 \leq i \leq K\}$ is an $(n, K, d)$ code,

Proof. We choose a Boolean function $f(x) = \frac{1}{2} x T x^T$ with APC distance $d$, and from Theorem 1, $\{ |\psi_i \rangle |1 \leq i \leq K\}$ span a $(n, K, d)$, where $d = \min \{ W_s(u,v) \}$ such that $W_u(u, v + \beta_i + \beta_j) \geq d$. For an correctable error $\varepsilon_d = X_u Z_v = X_u Z_{\delta}$, $W_s(u,v) \leq d - 1$, we have
\[
\begin{align*}
\langle \psi_j | \varepsilon_d | \psi_i \rangle & \propto \langle \psi | Z_{\delta \Delta N_{\omega}} Z_{C_i \Delta C_j} | \psi \rangle \\
& \propto \sum_{x \in F_2^n} (-1)^{f(x) + f(x + u) + (v + \beta_i + \beta_j) \cdot x}
\end{align*}
\]
(11)

And $W_s(u, v + \beta_i + \beta_j) < d$, so
\[
\sum_{x \in F_2^n} (-1)^{f(x) + f(x + u) + (v + \beta_i + \beta_j) \cdot x} = 0.
\]
Then we have $\delta \Delta N_{\omega} \neq C_i \Delta C_j$, in other words, $C_i \Delta C_j \in \mathbb{D}_d$.

It should be noted here that in Ref[8], the authors gave three conditions for the existence of an $(n, K, d)$ over $F_2$ quantum code, and we consider them unnecessary.

4.2 quadratic logic functions of the form $\frac{1}{2} (c, x) A(c, x)^T$

Consider an $(n + k) \times (n + k)$ symmetric matrix $A$ with elements in $F_p$ and vanishing diagonal entries, then for every vector $c$ of length $k$ with elements in $F_p$, $f(c, x) = \frac{1}{2} (c, x) A(c, x)^T$ is a logic function of $n$ variables, where $x = (x_1, x_2, \cdots, x_n)$ is a vector of $n$ variables. Notice that the degree of $f(x)$ is at most two.

Now, we consider the sufficient conditions for the set of states $\{ |\psi_i \rangle = p^{-\frac{1}{2}} \sum_{x \in F_p^n} \zeta f(c_i, x) |x \rangle \} |c_i \in F_k^1\}$ that can span an $[n, k, d]_p$ quantum code, i.e., the required properties of the Boolean function. Basing on Lemma 1, we have the following theorem.
Theorem 2. Suppose $C, X$ are two disjoint sets. $|C| = k, |X| = n, d \geq 2$, $A = (a_{ij})_{i,j \in C \cup X}$ is a symmetric matrix with elements in $F_p$ and vanishing diagonal entries. For arbitrary $E \subseteq X, |E| = d - 1, I = X \setminus E$, if the rows of $A_{EI}$ are linear independent, and

$$A_{IC}d^C + A_{IE}d^E = O^I$$

implies that

$$d^C = O^C$$

Then the subspace spanned by $\{\ket{\psi_i}\}$ is an $[[n, k, d]]_p$ code over $F_p$, where $\zeta$ is a $p$-th primitive root of 1.

Proof. We first prove that if $A_{IC}d^C + A_{IE}d^E = O^I$ where $I = X \setminus E$ implies that $d^X = O^X$, then for different $i, j$, $\langle \psi_j | \psi_i \rangle = 0$.

$$\langle \psi_j | \psi_i \rangle \propto \sum_{x \in F_p^n} \zeta^{f(c_i, x) - f(c_j, x)}$$

Since $A$ can be expressed as $A = \begin{pmatrix} A_{CC} & A_{CX} \\ A_{XC} & A_{XX} \end{pmatrix}$, then

$$\sum_{x \in F_p^n} \zeta^{f(c_i, x) - f(c_j, x)} \propto \sum_{x \in F_p^n} \zeta^{x A_{XC}(c_i - c_j)^T}$$

so $\langle \psi_j | \psi_i \rangle \neq 0$ if $A_{XC}(c_i - c_j)^T = O^X$.

Seeking a contradiction, we suppose $A_{XC}(c_i - c_j)^T = O^X$, then there exist $I \subseteq X$ such that $A_{IC}(c_i - c_j)^T = O^I$. Let $d^E = O^E$, then $A_{IC}(c_i - c_j)^T + A_{IE}d^E = O^I$, which satisfies Eq.(13), so $(c_i - c_j)^T = O^C$ which is impossible because of $i \neq j$. We come to the result that $\{\ket{\psi_i}\}$ span a subspace of dimension $p^k$.

Then we prove the subspace spanned by $\{\ket{\psi_i}\}$ is an $[[n, k, d]]_p$ quantum code, i.e., for an error $\varepsilon_d = X_a Z_b$ with $W_s(a, b) \leq d - 1$, $\langle \psi_j | \varepsilon_d | \psi_i \rangle = f(\varepsilon_d) \delta_{ij}$.

$$\langle \psi_j | \varepsilon_d | \psi_i \rangle \propto \sum_{x \in F_p^n} \zeta^{bx + x A_{XC}(c_i - c_j)^T + a A_{XX}x^T}$$

Let $\varepsilon_d$ acts on qubits corresponding with a subset $E \subseteq X$, and $a_E, b_E$ are vectors of length $d - 1$. For simplicity, we denote variables in $I$ as $y$, variables in $E$ as $z$. Then

$$\langle \psi_j | \varepsilon_d | \psi_i \rangle \propto \sum_{x \in F_p^n} \zeta^{b_Ez + z A_{EC}(c_i - c_j)^T + z A_{EE}y^T + y A_{IC}(c_i - c_j)^T + y A_{IE}a^T}$$

For different $i, j$, consider linear terms of $y$, if $A_{IC}(c_i - c_j)^T + A_{IE}a^T \neq O^I$, then $\langle \psi_j | \varepsilon_d | \psi_i \rangle = 0$. If $A_{IC}(c_i - c_j)^T + A_{IE}a^T = O^I$, which satisfies Eq.(13), so $(c_i - c_j)^T = O^C$ which contradicts the fact that $c_i, c_j$ are different.

Then we verify $\langle \psi_i | \varepsilon_d | \psi_i \rangle$ only depends on $\varepsilon_d$. 
\[ \langle \psi_i | \varepsilon_d | \psi_i \rangle \propto \sum_{x \in \mathbb{F}_p} \zeta^{b_E x + z A_{E E} x} + y A_{EE} T \]

Since the rows of \( A_{EI} \) are independent, then if \( A_{IE} a_E T = 0 \), we can know \( a_E = 0 \), thus \( \langle \psi_i | \varepsilon_d | \psi_i \rangle \propto \sum_{x \in \mathbb{F}_p} \zeta^{b_E x} \). Because \( b_E \neq 0 \), \( \langle \psi_i | \varepsilon_d | \psi_i \rangle = 0 \).

So, \( \{|\psi_i\rangle|1 \leq i \leq 2^k\} \) span an \([n, k, d]_p\) quantum code.

### 4.3 Graphical \([n, 0, d]\) QECC

Consider the adjacency matrix \((n \times n) \Gamma\) of a graph, then the rows of \( A = (\omega I | \Gamma)\) can span an self-dual additive code \( C \) over \( GF(4) = \{0, 1, \omega, \omega^2\} \), where \( \omega^2 + \omega + 1 = 0 \). And \( C \) is equivalent to a graph code \( D \).

Let \( \alpha_i, \beta_i \) are the \( i \)-th column of \( I \) and \( \Gamma \) respectively, in fact, \( X_{\alpha_i} Z_{\beta_i} \) are stabilizers of \( D \), if we can find a Boolean function \( f(x) \) satisfying the following equations:

\[ f(x + \alpha_i) = \beta_i x, \quad \text{for } 1 \leq i \leq n \quad (19) \]

then we can state that \( |\psi\rangle = 2^{-\frac{d}{2}} \sum_{x \in \mathbb{F}_2} (-1)^{f(x)} |x\rangle \) is the basis state. So we find that the graph code is equivalent to \([n, 0, d]\), where \( d \) is the APC distance of \( f(x) \).

**Example 1.** Consider a complete graph of 4 vertices, then matrix

\[
A = \begin{pmatrix}
\omega & 0 & 0 & 0 \\
0 & \omega & 0 & 0 \\
0 & 0 & \omega & 0 \\
0 & 0 & 0 & \omega
\end{pmatrix}
\]

After computation, we find that \( f(x) = x_1x_2 + (x_1 + x_2)(x_3 + x_4) + x_3x_4 \), the APC distance of \( f(x) \) is 2, so \( \{|\psi\rangle = 2^{-2} \sum_{x \in \mathbb{F}_2} (-1)^{f(x)} |x\rangle \} \) span a \([4, 0, 2]\) QECC.

In fact, under the correspondence defined in Ref[17], every simple undirected graph with adjacency matrix \( \Gamma_{n \times n} \) corresponds to an \([n, 0, d]\) QECC over \( F_2 \) with the basis state \( |\psi\rangle = 2^{-\frac{d}{2}} \sum_{x \in \mathbb{F}_2} (-1)^{f(x)} |x\rangle \), where \( f(x) = \frac{1}{2} x \Gamma x^T \).

### 5 Boolean functions and projectors

In Ref[11], the authors constructed quantum error correcting codes via the tools of projectors and Boolean functions. They first redefine a logic of projectors, then on the assumption that they can construct a certain matrix, which satisfies some properties corresponding to the \( Z_{set f} \) of a Boolean function \( f \), finally they construct a projector onto a quantum code.

In this section, we refine the projectors another logic of projectors, and come to the result that in some conditions, the quantum codes under the construction which is similar to that given Ref[11] has Boolean basis states. Now we give our definition of operator, which is denoted by \( E'_{(a,b)} \).
Definition 6. Operators $E'_{(a,b)}$ associated with binary vectors $(a,b) \in F_2^n$ are redefine by

$$E'_{(a,b)} = e'_1 \otimes \ldots \otimes e'_n$$

where $e'_j = i^{a_j} b_j$ for $1 \leq j \leq n$. In other words, $E'_{(a,b)} = X_a Z_b$.

Basing on the definition of the logic of projection operators in Ref.\[11], we define another logic as the following definition.

Definition 7. Let $P = X_a Z_b, P' = X_{a'} Z_{b'}, P'' = X_{a''} Z_{b''}$ are three projection operators, where $a, a', a'', b, b', b''$ are vectors of length $n$. Then we define $P \lor P' = X_a Z_b + X_{a'} Z_{b'}, P \land P' = (-1)^{a'b} X_{a+a'} Z_{b+b'}, \hat{P} = I - P$ and $(P \lor P') \land P'' = (-1)^{a'b} X_{a+a''} Z_{b+b''} + (-1)^{a''} X_{a+a''} Z_{b+b''}$.

Definition 8.\[11] Given an arbitrary Boolean function $f(x_1, x_2, \ldots, x_n)$, we define the Projection function $\hat{f}(P_1, P_2, \ldots, P_n)$ in which $x_i$ is replaced by $P_i$, multiplication, summation and not operation in Boolean logic are replaced by the meet, join and tilde operation in the projection logic described in Definition 7 respectively.

We denote $P_i^{c_i}$ as $P_i$ if $c_i = 0$, and $\hat{P}_i$ if $c_i = 1$.

If we can construct matrix $A_f = (A|B)$ as in Lemma 3, where $A$ and $B$ are blocks of $A_f$ of size $n \times n$ with the $i$-th row vectors $\alpha_i, \beta_i$, corresponding with a Boolean function $f(x)$ with $n$ variables, then we redefine the operation operator $P_{n+1-t} = \frac{1}{2}(I + E'_{n-t})$. The projector $\hat{f}(P_1, P_2, \ldots, P_n)$ is still a projector onto an $(<n, M, 2>)$ QECC, where $M$ is the Hamming weight of $f(x)$.

The projector $P$ onto a QECC $Q$ has the form $P = \sum |\psi\rangle \langle \psi|$, where $|\psi\rangle$ run over all the basis states of $Q$. Without lose of generosity, we assume the vector $(t_1, t_2, \ldots, t_n)$ is an element of the support of $f$, in fact, every element in the support of $f$ corresponds to a basis state. Then the term corresponds to $(t_1, t_2, \ldots, t_n)$ in $P = f(P_1, P_2, \ldots, P_n)$ is $P_1^{t_1}, P_2^{t_2}, \ldots, P_n^{t_n}$, which can be written as

$$2^{-n} \sum_{d \in F_2^n} \sum_{x \in F_2^n} (-1)^{\lambda(d,t)} |x + \sum_{i=1}^{n} d_i \alpha_i\rangle \langle x|$$

$$+ \sum_{i=1}^{n} t_i |d_i\rangle x + \sum_{1 \leq j \leq k \leq n} d_j d_k \alpha_j \beta_k.$$

Now we consider properties that the Boolean function $f(t_1, t_2, \ldots, t_n)$ should have if $P_1^{t_1}, P_2^{t_2}, \ldots, P_n^{t_n}$ can be written as $|\psi(t_1, t_2, \ldots, t_n)\rangle \langle \psi(t_1, t_2, \ldots, t_n)|$, where $|\psi(t_1, t_2, \ldots, t_n)\rangle$ is a Boolean state corresponding with $f(t_1, t_2, \ldots, t_n)$.

For simplicity, we write $\hat{f}$ in place of $f(t_1, t_2, \ldots, t_n)$, $|\tilde{\psi}\rangle$ in place of $|\psi(t_1, t_2, \ldots, t_n)\rangle$, and

$$|\tilde{\psi}\rangle \langle \tilde{\psi}| \propto \sum_{x \in F_2^n} \sum_{s \in F_2^n} (-1)^{f(x)+f(x+s)} |x + s\rangle \langle x|$$

Then we have the following theorem.
**Theorem 3.** A is invertible, \( \tilde{f}(x) \) is is quadratic and \( \tilde{f}(x) + \tilde{f}(x + \alpha_i) = \beta_i x + t_i. \)

**Proof.** Since \( s \) (in Eq. (22)) and \( d \) (in Eq. (21)) run over \( F_2^n \), we require that \( \alpha_1, \alpha_2, \cdots, \alpha_n \) are linear independent, and

\[
\tilde{f}(x) + \tilde{f}(x + \alpha_i) = \beta_i x + t_i.
\]  

Then for arbitrary \( d \in F_2^n \), \( \tilde{f}(x) + \tilde{f}(x + \sum_{i=1}^{n} d_i \alpha_i) = \left( \sum_{i=1}^{n} d_i \beta_i \right) x + \sum_{i=1}^{n} d_i t_i + \sum_{1 \leq j < k \leq n} d_j d_k \alpha_j \beta_k \), which coincides with \( \lambda(d, t) \).

Since the right part of Eq. (23) is an affine Boolean function, we know that \( \tilde{f}(x) \) is quadratic.

If a QECC has Boolean states, then the study of the QECC can again be converted to the study of Boolean functions corresponding with them, we say it is possible.

**Example 2.** Define a Boolean function \( g(y_1, y_2, y_3, y_4) = (y_1 + y_2 + y_3)(y_1 + y_2 + y_4) \), then \( g(y) \) is partially bent, and \( |\text{Supp}(g)| = 4 \), then for every \( a \in F_2^4 \) with \( r_g(a) = 0 \) is in \( Zsetg \), and \( \text{Supp}(g) = \{s_1 = (1000), s_2 = (0100), s_3 = (0011), s_4 = (1111)\} \). Let \( a_i \) be a unitary vector of length 4 with 1 in the \( i \)-th coordinate and 0 elsewhere. Since \( g(y + a_i) = g(y) + b_i x + c_i (1 \leq i \leq 4, c = (1100)) \), we construct the matrix \( A_g \) as

\[
A_g = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0
\end{pmatrix}
\]

where the \( i \)-th row of \( A_g \) is \( w_i = (a_i, b_i) \). We can easily verify that all the rows of \( A_g \) are independent and any two rows have symplectic product zero because the right four columns of \( A_g \) form a symmetric matrix. Express \( A_g = [x_1, x_2, \cdots, x_8] \), then for every \( \omega \) with \( W_s(\omega) \leq 1 \), \( A_g \omega^T \) is in \( Zsetg \). After computation, we have \( \tilde{f}_1 = g + y_2, \tilde{f}_2 = g + y_1, \tilde{f}_3 = g + y_1 + y_2 + y_3 + y_4, \tilde{f}_4 = g + y_3 + y_4 \). We can see that for different \( i, j \), \( \tilde{f}_i - \tilde{f}_j \) are linear terms. And \( \{2^{-2} \sum_{x \in F_2^2} (-1)^{\tilde{f}_i(x)} |x\} \) spans a \([4, 2, 2]\) code, which meets the quantum singleton bound, and therefore is an MDS code.

In fact, for every function \( f(y) \) with \( 2m \) variables of the form \( f(y) = (y_1 + y_2 + \cdots + y_{2m-2} + y_{2m-1})(y_1 + y_2 + \cdots + y_{2m-2} + y_{2m}) \) (which is a partially bent function [18]), we can find \( |\text{Supp}(f)| \) Boolean functions \( \tilde{f}_i \) satisfying that \( \{2^{-m} \sum_{x \in F_2^{2m}} (-1)^{f_i(x)} |x\} \) spans a \([2m, 2m-2, 2]\) MDS code.

Because a quantum code with Boolean basis state is interesting, it is natural to question what kind of properties of the Boolean functions used in Lemma 3 should satisfy.
Lemma 4. \[ r_f(s) = \begin{cases} 2^n & \text{if } s = 0 \\ 0 & \text{else} \end{cases} \text{, and if } f(x) \text{ is a bent function, then } |\text{Supp}(f)| = 2^{n-1} \pm 2^{n/2-1}. \]

Theorem 4. For arbitrary \((c_1, c_2, \cdots, c_n) \in \text{Supp}(f)\), the Boolean function \(f\) with more than 2 variables used in Lemma 3 can't be a bent function.

Proof. Seeking a contradiction, we assume that \(f\) is bent, then for every \(s \in F^n_2\), \(s \neq 0\), \(r_f(s) = 0\). If \(Z\text{set}f \neq \emptyset\), then for every \(a \in Z\text{set}f\), \(r_f(a) = 0\). From Lemma 2, the weight of \(f\) is equal to \(2^{n-2}\), which contradicts the property of bent functions described in Lemma 4, so \(f\) is not bent.

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Logic Functions and Quantum Error Correcting Codes

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Abstract. In this paper, based on the relationship between logic functions and quantum error correcting codes (QECCs), we unify the construction of QECCs via graphs, projectors and logic functions. A binary construction which can be generalized to any prime field of QECCs is given, and one of the results given by Ref\cite{8} can be viewed as a corollary of one theorem in this paper. With the help of Boolean functions, we give a clear proof of the existence of a graphical QECC in mathematical view, and find that the existence of a binary \([n, k, d]\) QECC requires similar conditions with that of an \([n, k, d]\) graphical QECC over \(F_p\). The existence of an \(((n, \frac{1}{2}(p^m + 1), d))\) QECC over \(F_p\) requires weaker conditions. The result that under the correspondence defined in Ref\cite{17}, every \([n, 0, d]\) QECC over \(F_2\) corresponding to a simple undirected graph has the Boolean basis state, which is closely related to the adjacency matrix of the graph, is given.

After a modification of the definition of operators, we find that some QECCs constructed via projectors can have Boolean basis states. A necessary condition for a Boolean function being used in the construction via projectors is given. We also give some examples to illustrate our results.

1 Introduction

Quantum error correcting codes (QECCs) have received more and more attention for nearest few decades since the theory of quantum error correction was put forward\cite{12, 34}. One of the central tasks in the theory of QECCs is how to construct them, and the first systematic mathematical construction is given in\cite{4} in the binary case and then generalized in\cite{5, 10}. Many good non-binary QECCs have been constructed by using classical error-correcting codes over \(F_q\) or \(F_{q^2}\) (\(q\) is a power of odd prime number) with special orthogonal properties. Besides these, many quantum codes are constructed via tools including graphs and Boolean functions\cite{7, 8, 9, 10}.

In Ref\cite{7}, the author constructed \([n, 0, d]\) QECCs using Boolean functions with \(n\) variables and aperiodic propagation criterion (APC) distance \(d\), and gave an algorithm to compute the APC distance of a Boolean function, orbits of Boolean functions of the same APC distance are also studied. In this paper,
a construction of quantum codes of dimension more than 0 is given basing on the relationship between Boolean functions and quantum codes. In Ref[8], the authors constructed quantum codes by giving $K$ basis states based on a graph state associated with a graph, and each basis state corresponds to a subset of the vertexes, which can be viewed as a corollary of this paper.

In Ref[9], the authors initiated the construction of QECCs via the construction of matrixes with some properties and proved the necessary and sufficient conditions for the graph that the resulting code corrects a certain number of errors in physics view. In Ref[10], the author gave another proof in mathematical view. Based on the mathematical proof and the close relationship between Logic functions and matrixes, properties that a logic function should have in order to construct a QECC is systematically studied. We first study the relationship between Boolean functions and $[n,k,d]$ QECC over $F_2$, and find that the existence of an $[n,k,d]$ QECC over $F_2$ requires similar conditions with that of an $[n,k,d]$ graphical QECC over $F_p$. When we try to generalize the result to logic functions over $F_p$, we find the existence of an $(n,1,2(p^n + 1),d)$ QECC over $F_p$ requires similar conditions. Under the correspondence defined in Ref[17], we find that every $[n,0,d]$ over $F_2$ corresponding to a simple undirected graph with $n \times n$ adjacency matrix $\Gamma$ has the basis state $|\psi_f\rangle = 2^{-n/2} \sum_{x \in F_n^2} (-1)^{f(x)} |x\rangle$, where $f(x) = \frac{1}{2} x \Gamma x^T$. We also give an example to illustrate our result.

In Ref[11], the author describes a common mathematical framework for the design of QECCs basing on the correspondence between Boolean functions and projection operators. We point out that in some conditions, the basis states of the QECCs under the framework can have the probability vector of the form $2^{-n/2} (-1)^{f(x)}$, where $f(x)$ is a Boolean function.

2 Preliminaries

Definition 1. The symplectic weight of two vectors $a,b$ of length $n$ is defined by

$$W_s(a,b) = \sharp \{i | 1 \leq i \leq n, (a_i,b_i) \neq (0,0) \}$$

Definition 2. The weight of an error $e = i^\lambda x_a Z b$ is defined by

$$W(e) = W_s(a,b)$$

Definition 3. The autocorrection function of a Boolean function $f(v)$ with $n$ variables at $a$ is $r_f(a) = \sum_{v \in F_2^n} (-1)^{f(v)+f(v+a)}$.

Definition 4. [7] The APC(aperiodic propagation criterion) distance of a Boolean function $f$ is defined by the smallest nonzero $w_s(a,b)$ such that

$$\sum_{x \in F_2^n} (-1)^{f(x)+f(x+a)+b x} \neq 0$$

(1)
The graph (or matrix) description of quantum codes given by Ref\[9\] can be stated in following elementary way. Let $X$ and $Y$ be disjoint sets with $k$ and $n$ elements, respectively. Let $G = (V(G), E(G))$ be a graph with vertex set $V = V(G) = X \cup Y$ and edge set $E(G) = V \times V$. Each edge $uv \in E(u, v \in V)$ is assigned a weight $a_{uv} (= a_{vu}) \in F_p$. Therefore, such a graph $G$ corresponds to a symmetric matrix over $F_p$ with size $(n + k) \times (n + k)$

$$A = A(G) = (a_{uv})_{u,v \in V}$$

For two subsets $S$ and $T$ of $V$, we denote $A_{S,T}$ as the submatrix of $A$ with size $|S| \times |T|

$$A_{S,T} = A_{S,T}(G) = (a_{uv})_{u \in S, v \in T}$$

Similarly, a vector in the vector space $F_p^{|V|} = F_p^{n+k}$ is denoted by a column vector

$$d^V = \begin{pmatrix} d^V_1 \\ d^V_2 \\ \vdots \\ d^V_{n+k} \end{pmatrix} = \{ d^v | v \in V \}$$

where $V = \{ v_1, \cdots, v_{n+k} \}$ and $d_{v_i} \in F_p$. For a subset $S$ of $V$, we denote

$$d^S = \{ d^s | s \in S \} \in F_p^S$$

For example, let $E$ be a subset of $Y$ with $d - 1$ elements, $I = Y \setminus E$. Then

$$d^V = \begin{pmatrix} d^X \\ d^E \\ d^I \end{pmatrix}$$

and the multiplication of $A = A(G)$ and $d^V$ can be expressed as

$$Ad^V = \begin{pmatrix} A_{XX} & A_{XE} & A_{XI} \\ A_{EX} & A_{EE} & A_{EI} \\ A_{IX} & A_{IE} & A_{II} \end{pmatrix} \begin{pmatrix} d^X \\ d^E \\ d^I \end{pmatrix}$$

Lemma 1. \[9\] Suppose $X, Y$ are two disjoint sets, $|x| = k, |Y| = n \geq k + 2d - 2, d \geq 2$, $A = (a_{ij})_{i,j \in X \cup Y}$ is a symmetric matrix with vanishing diagonal entries. For arbitrary $E \subseteq Y, |E| = d - 1$, if

$$A_{IX}d^X + A_{IE}d^E = O^I$$

with $I = Y \setminus E$ implies that

$$d^X = O^X \text{ and } A_{XE}d^E = O^X$$

Then there exist an $[n, k, d]$ code.
Definition 5. [11] We define the Zset\(_f\) of a Boolean function \(f\) by
\[
\text{Zset}_f = \{ a | \sum_{v \in F_2^n} f(v)f(v + a) = 0 \}
\]

Lemma 2. [11] If the weight of the Boolean function \(f\) with \(n\) variables is \(M\), and \(M \leq 2^{n-1}\), then \(\text{Zset}_f = \{ a | r_f(a) = 2^n - 4M \}\).

Definition 6. Operators \(E_{(a,b)}\) associated with binary vectors \((a,b) \in F_2^n\) are defined by
\[
E_{(a,b)} = e_1 \otimes \ldots \otimes e_m
\]
where \(e_i = \begin{cases} I_2, & a_i = 0, b_i = 0. \\ \sigma_x, & a_i = 1, b_i = 0. \\ \sigma_z, & a_i = 0, b_i = 1. \\ \sigma_y, & a_i = 1, b_i = 1. \end{cases}\)
In other words, \(e_i = i^{a \cdot b} X_a Z_b\).

Then \(E_{(a,b)}E_{(a',b')} = (-1)^{a' \cdot b + a \cdot b'} E_{(a + a', b + b')}\), and \(E_{(a,b)}E_{(a',b')} = (-1)^{a' \cdot b + a \cdot b'} E_{(a',b')}E_{(a,b)}\).

Lemma 3. [11] A \(((n, M, 2))\)-QECC is determined by a Boolean function \(f\) with the following properties

1) \(f\) is a function of \(n\) variables and has weight \(M\).
2) The Zset associated with \(f\) contains the set \([[x_1, x_2, \ldots, x_{2n}]^T | \omega^T \omega = 2^n \text{ bit vector of symplectic weight } 1\}]\) (or in other words the set \(\{ x_1, x_2, \ldots, x_{2n}, x_1 + x_{n+1}, \ldots, x_n + x_{2n} \}\)) and the matrix \(A_f = [x_1, x_2, \ldots, x_{2n}]_{n \times 2n}\) has the property that any two rows have symplectic product zero and all the rows are independent.

In the above lemma, the QECC is constructed by giving the projector \(P\) onto the code, and the projector is a Boolean function of \(n\) projection operators \(\{P_{n+1-i} = \frac{1}{2}(I + E_{v_i})|1 \leq i \leq n\}\), i.e., \(P = f(P_1, P_2, \ldots, P_n)\) in the sense of a logic of projection operators given in Ref[11], where \(v_i\) is the \(i-th\) row of the matrix \(A_f\). From the symplectic orthogonality of the rows of \(A_f\), we have \(\{P_i|1 \leq i \leq n\}\) are pairwise commutative, and the error-correcting ability of the QECC is ensured by the properties of \(\text{Zset}_f\). An arbitrary error \(e\) satisfying \(W(e) = 1\) acting on one qubit only takes the projector \(P_{f_0}e = P_{f(v + t)}e\), i.e., \(eP_{f_0}e = P_{f(v + t)}e\), where \(t\) is an element in \(\text{Zset}_f\), and \(P_{f(v + t)}\) is orthogonal to \(P_{f(v)}\).

3 logic functions and quantum states

For simplicity, we take Boolean functions as an example to illustrate our work, and we state that all the results about quantum codes over \(F_2\) given in this section can be generalized to quantum codes over a prime field \(F_p\).
For a Boolean function \( f(x) \) with \( n \) variables, it corresponds to a vector \( s = 2^{-n}(-1)^{f(x)} \), which can be interpreted as the probability distribution vector of the quantum state

\[
|\psi_f\rangle = 2^{-\frac{n}{2}} \sum_{x \in F_2^n} (-1)^{f(x)} |x\rangle
\]

(4)

If a state has the form of Eq.(4), then we call it a Boolean state. Consider a \( n \)-variable Boolean function \( f(x) \), which can also be expressed in terms of a Boolean function, \( f |x\rangle \). When acting on the quantum state \( |\psi\rangle \), it corresponds to a quantum state

\[
|\psi\rangle \rightarrow 2^{-\frac{n}{2}} \sum_{x \in F_2^n} (-1)^{(x+a)+bx} |x\rangle.
\]

(6)

In fact, a logic function \( g(x) \) with \( n \) variables over a prime field \( F_p \) also corresponds to a quantum state \( |\psi_g\rangle = 2^{-\frac{n}{2}} \sum_{x \in F_2^n} \zeta^{g(x)} |x\rangle \), where \( \zeta \) is a \( p \)-th primitive root of 1. Then an arbitrary error \( \epsilon = \lambda_{u}Z_{v} \), where \( u,v \in F_2^n \), transforms \( |\psi_g\rangle \) to

\[
|\psi'\rangle \rightarrow 2^{-\frac{n}{2}} \sum_{x \in F_2^n} \zeta^{g(x)+u+vx} |x\rangle.
\]

(7)

Let \( K \) Boolean functions be \( g_i(x) = f(x) + b_i x \), and for \( 1 \leq i < j \leq K \), \( b_i \neq b_j \), define \( K \) quantum states as \( |\psi_i\rangle = 2^{-\frac{n}{2}} \sum_{x \in F_2^n} (-1)^{g_i(x)} |x\rangle \). Since

\[
\sum_{x \in F_2^n} (-1)^{f(x)+f(x)+(b_i+b_j)x} = 0,
\]

it is trivially to verify that \( \langle \psi_i | \psi_j \rangle = 0 \).

**Theorem 1.** Let \( |\psi_f\rangle \) be a quantum state with probability distribution vector \( 2^{-\frac{n}{2}}(-1)^{f(x)} \), where \( f(x) \) is a Boolean function with \( n \) variables and APC distance \( d \), i.e.,\( |\psi_f\rangle = 2^{-\frac{n}{2}} \sum_{x \in F_2^n} (-1)^{f(x)} |x\rangle \). Then the subspace spanned by \( \{|\psi_i\rangle | 1 \leq i \leq K \} \) is an \((n, K, d')\) quantum code, where \( d' = \min\{W_{a}(u,v)|W_{a}(u,v+b_i+b_j) < d \text{ for all } 1 \leq i < j \leq K\} \).

**Proof.** We need only to prove that for any error \( \epsilon_{k} \) acting nontrivially on less than \( d \) qubits, we have \( \langle \psi_i | \epsilon_{k} | \psi_j \rangle = f(\epsilon_{k}) \delta_{ij} \) for all \( 1 \leq i, j \leq K \). Without lose of generosity we assume that \( \epsilon_{k} = \lambda_{u}Z_{v} \) for some pair of vectors in \( F_2^n \), which represents that there are \( \lambda \), \( Z \), and \( \lambda Z \) errors on the qubits with index \( \{k|u_k = 1, v_k = 0\}, \{k|u_k = 0, v_k = 1\} \), and \( \{k|u_k = 1, v_k = 1\} \) respectively.

When acting on the quantum state \( |\psi_j\rangle \), the error takes it to

\[
|\psi'_j\rangle = 2^{-\frac{n}{2}} \sum_{x \in F_2^n} (-1)^{(x+a)+(u+x)b_j} |x\rangle
\]

(8)

\[
\propto 2^{-\frac{n}{2}} \sum_{x \in F_2^n} (-1)^{f(x+a)+b_j(x+a)+vx} |x\rangle
\]

(9)
We have
\[
\langle \psi_i | \varepsilon_d | \psi_j \rangle \propto \sum_{x \in \mathbb{F}_2^n} (-1)^{f(x+u) + b_j(x+u) + v_x + f(x) + b_x}.
\]

Since \( W_s(u, v + b_i + b_j) < d \), and the APC distance of \( f \) is \( d \), so
\[
\sum_{x \in \mathbb{F}_2^n} (-1)^{f(x+u) + (b_j + b_i + v)x + ub_j + f(x)} = 0. \tag{10}
\]

Then we consider whether \( \langle \psi_i | \varepsilon_d | \psi_i \rangle \) only depends on \( \varepsilon_d \) or not.
\[
\langle \psi_i | \varepsilon_d | \psi_i \rangle \propto \sum_{x \in \mathbb{F}_2^n} (-1)^{f(x+u) + v_x + f(x)} = 0 \tag{11}
\]

So, \( \{|\psi_i\rangle| 1 \leq i \leq K\} \) span an \((n,K,d')\) quantum code.

**Corollary 1.** Let \( A = (a_{ij})_{n \times n} \) be a nondegenerate matrix on \( \mathbb{F}_2 \), \( f(x) \) be a Boolean function with \( n \) variables and APC distance \( d \), and the vector \( s \) run over \( \mathbb{F}_k^2 \), we define \( |\psi_s\rangle = 2^{-\frac{n}{2}} \sum_{x \in \mathbb{F}_2^n} (-1)^{f(x) + \sum_{i=1}^n s_i \sum_{j=1}^n a_{ij} x_j} |x\rangle \), then \( |\{\psi_s\}\rangle| = 2^k \). we have the following fact that the subspace spanned by \( \{|\psi_s\rangle\} \) is an \([n,k,d']\) quantum code, where \( d' = \min \{W_s(u,v) | W_s(u,v + \sum_{i=1}^k t_i a_i) < d, \forall t \in \mathbb{F}_2^k\} \), where \( a_i \) denotes the \( i \)-th row of \( A \).

We should note here that if we generalize theorem 1 to QECCs over \( \mathbb{F}_p \), where \( p \) is a prime, we need to generalize the definition of APC distance to \( \mathbb{F}_p \).

**Definition 7.** The APC distance of a logic function \( f \) is defined by the smallest nonzero \( w_s(a,b) \), where \( a, b \in \mathbb{F}_p^n \) such that
\[
\sum_{x \in \mathbb{F}_p^n} \zeta^{f(x) + f(x+a) + b \cdot x} \neq 0 \tag{12}
\]
where \( \zeta \) is a \( p \)-th primitive root of 1.

### 4 Logic functions and graphical QECCs

Just as before, we first take Boolean functions as an example to illustrate our work, and then generalize it to logic functions over \( \mathbb{F}_p \), where \( p \) is a prime.

If \( f(x) \) is a quadratic Boolean function and can be represented as \( f(x) = \frac{1}{4} x \Gamma x^T \), where \( \Gamma \) is an \( n \times n \) symmetric matrix with elements in \( \mathbb{F}_2 \) and vanishing diagonal entries, then the state \( |\psi\rangle \) can be viewed as a graph state because \( \Gamma \) can be viewed as the adjacency matrix of a graph \( G = (V, E) \), where \( V \) and \( E \)
denote the set of vertices and edges respectively and $|V| = n$. The graphs talked about in this paper are limited to simple undirected ones.

Consider the operator $G_a = X_a \prod_{b \in N_a} Z_b$, where $N_a$ represents the neighborhood of the vertex $a$ and is denoted by $N_a = \{ v \in \Gamma | \Gamma_a = 1 \}$, it operates on the qubit which corresponds to the vertex $a$ as the Pauli operator $\sigma_x$ and operates on the qubits which corresponds to the vertex in $N_a$ as the Pauli operator $\sigma_z$. And if we write $G_a$, it means $X_a Z_{N_a}$. And it was shown [12] that $G_a \psi = \psi$.

If we label every vertex of the graph $G$ of $n$ vertexes from 1 to $n$, then we can write $G_a \psi = 2^{-\frac{n}{2}} \sum_{x \in F_2^n} (-1)^{f(x+e_a)+f_a(x)} |x\rangle$, where $e_a$ denotes the vector with 1 in the coordinate corresponds to the vertex $a$ and 0 otherwise, $\Gamma_a$ denotes the row corresponds to the vertex $a$. And the error $\epsilon_d = X_a Z_v$ can be written as $X_a Z_{\Delta}$, where $\omega, \delta$ are subsets of $V$.

We define $K$ pair-wise orthogonal quantum states [8] as

$$|\psi_i\rangle = 2^{-\frac{n}{2}} \sum_{x \in F_2^n} (-1)^{g_i(x)} |x\rangle = Z_{C_i} |\psi\rangle,$$

where $\{C_i|1 \leq i \leq K\}$ are subsets of $V$ and corresponds to a vector $c_i$ of length $n$, therefore $g_i(x)$ can be expressed as $g_i(x) = f(x) + c_i x$. It is shown that $[8] G_a |\psi_i\rangle = -|\psi_i\rangle$ if $a \in C_i$ and $G_a |\psi_i\rangle = |\psi_i\rangle$ if $a \not\in C_i$ otherwise.

**Definition 8.** [8] The $d$–uncoverable set $D_d$ that contains all the subsets of $V$ which can’t be covered by less than $d$ vertices is denoted by

$$D_d = 2^V - \{ \delta \Delta N_{\omega} | \omega \cup \delta < d \}$$

**Corollary 2.** [8] If $C = \{C_1, C_2, \ldots, C_K\}$ satisfies the following two conditions,

1. $\emptyset \in C$; 2. $C_i \Delta C_j \in D_d$.

then the subspace spanned by the basis $\{|\psi_i\rangle = Z_{C_i} |\psi\rangle | 1 \leq i \leq K\}$ is an $((n, K, d))$ code, where $C_i \Delta C_j \in D_d$ denotes the symmetric difference of two sets $C_i, C_j$, i.e., $C_i \Delta C_j = C_i \bigcup C_j - C_i \bigcap C_j$.

It should be noted here that in Ref[8], the authors gave three conditions of which we consider unnecessary.

Consider a matrix $(n + k) \times (n + k)$ symmetric matrix $A$ with elements in $F_2$ and vanishing diagonal entries, then for every vector $c$ of length $k$ with elements in $F_2$, $f(c, x) = \frac{1}{2} (c, x) A (c, x)^T$ is a Boolean function of $n$ variables, where $x = (x_1, x_2, \ldots, x_n)$ is a vector of $n$ variables. Notice that the degree of $f(x)$ is at most two.

Now, we consider the sufficient conditions for the set of states $\{|\psi_i\rangle = 2^{-\frac{n}{2}} \sum_{x \in F_2^n} (-1)^{f(c_i, x)} |x\rangle | c_i \in F_2^k\}$ that can span a $[n, k, d]$ code, i.e., the required properties of the Boolean function. Basing on Lemma 1, we have the following theorem.
Theorem 2. Suppose $C, X$ are two disjoint sets, $|C| = k, |X| = n \geq k + 2d - 2, d \geq 2$, $A = (a_{ij})_{i,j \in C \cup X}$ is a symmetric matrix with elements in $F_2$ and vanishing diagonal entries. For arbitrary $E \subseteq X, |E| = d - 1$, if the rows of $A_{E^I}$ are linear independent, and

$$A_{IC}d^C + A_{IE}d^E = O^I$$

where $I = Y \setminus E$ implies that

$$d^C = O^C$$

where $A_{IC} = (a_{ij})_{i,j \in I \cup X}$ and $A_{IE}, d^E, d^C, O^C, O^I$ are defined as in lemma 1. Then subspace spanned by $\{|\psi_i\rangle = 2^{-\frac{d}{2}} \sum_{x \in F_2^n} (-1)^{\frac{1}{2}(c_i, x)A(c_i, x)^T} |x\rangle |c_i \in F_2^n\}$ is an $\lceil n, k, d \rceil$ code.

Proof. We first prove that if $A_{IC}d^C + A_{IE}d^E = O^I$ where $I = Y \setminus E$ implies that $d^X = O^X$, then for different $i, j$(which implies $c_i \neq c_j$), $\langle \psi_j | \psi_i \rangle = 0$.

$$\langle \psi_j | \psi_i \rangle \propto \sum_{x \in F_2^n} (-1)^{f(c_i, x) + f(c_j, x)}$$

Express $A$ as $A = \begin{pmatrix} A_{CC} & A_{CX} \\ A_{XC} & A_{XX} \end{pmatrix}$, where $A_{CC}, A_{XC}, A_{CX}, A_{XX}$ are $k \times k, n \times k, k \times n, n \times n$ blocks of $A$, then

$$f(c_i, x) + f(c_j, x) = \frac{1}{2}(c_i, x)A(c_i, x)^T + \frac{1}{2}(c_j, x)A(c_j, x)^T$$

and

$$\sum_{x \in F_2^n} (-1)^{f(c_i, x) + f(c_j, x)} \propto \sum_{x \in F_2^n} (-1)^{xA_{CC}(c_i + c_j)^T}$$

so $\langle \psi_j | \psi_i \rangle \neq 0$ iff $A_{XX}(c_i + c_j)^T = O^X$.

Seeking a contradiction, we suppose $A_{XX}(c_i + c_j)^T = O^X$, then there exist $I \subseteq X$ such that $A_{IC}(c_i + c_j)^T = O^I$. Let $d^E = O^E$, then $A_{IC}(c_i + c_j)^T + A_{IE}d^E = O^I$, which satisfies Eq.(13), so $(c_i + c_j)^T = O^C$ which contradicts the fact that $c_i, c_j$ are different. We come to the result that $\{|\psi_i\rangle\}$ span a dimension $2^k$.

Then we prove the subspace spanned by $\{|\psi_i\rangle\}$ is an $\lceil n, k, d \rceil$ code, i.e., for an error $e = X_aZ_b$ with $W_\delta(e) \leq d - 1$, $\langle \psi_j | e | \psi_i \rangle = f(e)\delta_{ij}$.

$$\langle \psi_j | e | \psi_i \rangle \propto \sum_{x \in F_2^n} (-1)^{bx + xA_{CC}(c_i + c_j)^T + aA_{XX}x^T}$$

Let $e$ acts on qubits corresponding with a subset $E \subseteq X$, and $a_E, b_E$ are vectors of length $d - 1$. For simplicity, we denote variables in $I$ as $y$, variables in $E$ as $z$. Then
\[ \langle \psi_j | e | \psi_i \rangle \propto \sum_{x \in F_2^n} (-1)^{b_E x + z A_{EC}(c_i + c_j)^T + z A_{EE} a_E^T + y A_{IC}(c_i + c_j)^T + y A_{IE} a_E^T} \] (19)

i.e., only includes linear terms.

For different \( i, j \), consider linear terms of \( y \), if \( A_{IC}(c_i + c_j)^T + A_{IE} a_E^T \neq O^T \), then \( \langle \psi_j | e | \psi_i \rangle = 0 \). If \( A_{IC}(c_i + c_j)^T + A_{IE} a_E^T = O^T \), which satisfies Eq.(13), so \((c_i + c_j)^T = O^C \) which contradicts the fact that \( c_i, c_j \) are different.

Then we consider whether \( \langle \psi_i | e_d | \psi_i \rangle \) only depends on \( e \) or not.

\[ \langle \psi_i | e | \psi_i \rangle \propto \sum_{x \in F_2^n} (-1)^{b_E x + z A_{EE} a_E^T + y A_{IE} a_E^T} \]

Since the rows of \( A_{E1} \) are independent, then if \( A_{IE} a_E^T = 0 \), we can know \( a_E = 0 \), thus \( \langle \psi_i | e | \psi_i \rangle \propto \sum_{x \in F_2^n} (-1)^{b_E x} \). Because \( b_E \neq 0 \), \( \langle \psi_i | e | \psi_i \rangle = 0 \).

So, \( \{|\psi_i| | 1 \leq i \leq 2^k \} \) span an \((|n, K, d|)\) quantum code.

Now we consider an undirected weighted graph \( G' = (V, E) \) with the adjacency matrix \((n \times n) \) \( \Gamma \), which is symmetric and with elements in \( F_p \) and vanishing diagonal entries, then we have the following theorem.

**Theorem 3.** Suppose \( C, X \) are two disjoint sets. \(|C| = k, |X| = n, d \geq 2, A = (a_{ij})_{i,j \in C \cup X} \) is a symmetric matrix with elements in \( F_p \) and vanishing diagonal entries. For arbitrary \( E \subseteq X, |E| = d - 1 \), if

\[ A_{IC} d^C + A_{IE} d^E = O^I \] (20)

where \( I = Y \setminus E \) implies that

\[ d^C = O^C \] (21)

where \( A_{IC} = (a_{ij})_{i,j \in C \cup X} \) and \( A_{IE}, d^E, d^C, O^C, O^I \) are defined as in lemma 1. Then subspace spanned by \( \{|\psi_i| = 2^{-\frac{k}{2}} \sum_{x \in F_2^n} \zeta^4 A(c_i, x)^T |x| |c_i \in S\} \) is an \((|n, \frac{1}{2}(1 + p^k), d|)\) code over \( F_p \), where \( S \subseteq F_2^n, |S| = \frac{1}{2}(1 + p^k) \), and if \((c_{i1}, c_{i2}, \ldots, c_{ik}) \in S \), then \((p - c_{i1}, p - c_{i2}, \ldots, p - c_{ik}) \notin S \), \( \zeta \) is a \( p \)-th primitive root of 1.

### 4.1 Graphical \([n, 0, d]\) QECC

Consider the adjacency matrix \((n \times n) \) \( \Gamma \) of a graph, then the rows of \( A = (\omega^I \Gamma) \) can span an self-dual additive code \( C \) over \( GF(4) = \{0, 1, \omega, \omega^2\} \), where \( \omega^2 + \omega + 1 = 0 \). And \( C \) is equivalent to a graph code \( D[17] \).

Let \( a_i, b_i \) are the \( i \)-th column of \( I \) and \( \Gamma \) respectively, in fact, \( X_{a_i}, Z_{b_i} \) are stabilizers of \( D \), if we can find a Boolean function \( f(x) \) satisfying the following equations:

\[ f(x + a_i) = b_i x, \text{ for } 1 \leq i \leq n \] (22)
then we can state that $|\psi\rangle = 2^{-\frac{n}{2}} \sum_{x \in F_2^n} (-1)^{f(x)} |x\rangle$ is the basis state. So we find that the graph code is equivalent to $[[n, 0, d]]$, where $d$ is the APC distance of $f(x)$.

**Example 1.** Consider a complete graph of 4 vertices, then matrix

$$
A = \begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 0
\end{pmatrix}
$$

After computation, we find that $f(x) = x_1x_2 + (x_1 + x_2)(x_3 + x_4) + x_3x_4$, the APC distance of $f(x)$ is 2, so $\{|\psi\rangle = 2^{-\frac{n}{2}} \sum_{x \in F_2^n} (-1)^{f(x)} |x\rangle\}$ span a $[[4, 0, 2]]$ QECC.

Notice that $f(x) = \frac{1}{2} x^T \Gamma x^T$, in fact, under the correspondence defined in Ref[17], every simple undirected graph with adjacency matrix $\Gamma_{n \times n}$ corresponds to an $[[n, 0, d]]$ QECC over $F_2$ with the basis state $|\psi\rangle = 2^{-\frac{n}{2}} \sum_{x \in F_2^n} (-1)^{f(x)} |x\rangle$, where $f(x) = \frac{1}{2} x^T \Gamma x^T$.

5 Boolean functions and projectors

In order to unify the constructions of QECCs via projectors and Boolean functions, we redefine $E_{(a, b)}$ as $E'_{(a, b)}$.

**Definition 9.** Operators $E'_{(a, b)}$ associated with binary vectors $(a, b) \in F_2^{2n}$ are redefine by

$$
E'_{(a, b)} = e'_1 \otimes \ldots \otimes e'_m
$$

where $e'_i = \begin{cases}
I_2, & a_i = 0, b_i = 0. \\
\sigma_x, & a_i = 1, b_i = 0. \\
\sigma_z, & a_i = 0, b_i = 1. \\
\sigma_x \sigma_z, & a_i = 1, b_i = 1.
\end{cases}$

In other words, $e'_i = \chi_a \zeta_b$.

Basing on the definition of the logic of projection operators in Ref[11], we define another logic as the following definition.

**Definition 10.** Let $P = \chi_a \zeta_b$, $P' = \chi_{a'} \zeta_{b'}$, $P'' = \chi_{a''} \zeta_{b''}$ be three projection operators, where $a, a', d', b, b', b''$ are vectors of length $n$. Then we define

- **join:** $P \lor P' = \chi_a \zeta_b + \chi_{a'} \zeta_{b'}$

- **meet:** $P \land P' = (-1)^{a' \cdot b} \chi_{a+a'} \zeta_{b+b'}$

- **tilde:** $\tilde{P} = I - P$

- **distributive law:** $(P + P') \land P'' = P \land P'' + P' \land P''$
Definition 11. Given an arbitrary Boolean function $f(v_1, v_2, \ldots, v_n)$, we define the Projection function $\tilde{f}(P_1, P_2, \ldots, P_n)$ in which $v_i$ is replaced by $P_i$, multiplication, summation and not operation in Boolean logic are replaced by the meet, join and tilde operation in the projection logic described in Definition 11 respectively.

We denote $P_i^{c_i}$ as $P_i$ if $c_i = 0$, and $P_i^{c_i}$ as $\tilde{P}_i$ if $c_i = 1$.

Then given a Boolean function $f$ with $n$ variables, construct matrix $A_f$ as in Lemma 3, and redefine the operation operator $P_{n+1-i} = \frac{1}{2}(I + E_{(a_i, b_i)})$, where $(a_i, b_i) = v_i$ is the $i$-th row of $A_f$, whose rows are independent and are pair-wise symplectic orthogonal. The projector $f(P_1, P_2, \ldots, P_n)$ is still a projector onto a QECC ($(n, M, 2)$), where $M$ is the Hamming weight of $f(v)$.

As the projector $P$ onto a QECC $Q$ has the form $P = \sum |\psi\rangle\langle\psi|$, where $|\psi\rangle$ is a basis state. Without lose of generosity, we assume $(c_1, c_2, \ldots, c_n)$ is an element of the support of $f$, in fact, every element in the support of $f$ corresponds to a basis state. Then the term in $P = f(P_1, P_2, \ldots, P_n)$ corresponds to $(c_1, c_2, \ldots, c_n)$ is $P_1^{c_1}, P_2^{c_2}, \ldots, P_n^{c_n}$, which can be written as $2^{-n} \sum_{d \in F_2^n} \sum_{x \in F_2^n} (-1)^{\sum_{i=1}^n d_i b_i} x + \sum_{i=1}^n c_i d_i \lambda(d_i)$

$\langle x + \sum_{i=1}^n d_i a_i | x \rangle$, where $\lambda(d_i)$ is 1 or -1 depending on $d_i$, and

$$\lambda(d_i) = (-1)^{\sum_{j \leq k \leq n, j \neq k} d_j d_k a_j b_k} \quad (24)$$

It is reasonable to suppose every basis state $|\psi(c_1, c_2, \ldots, c_n)\rangle$ is proportional to $\sum_{x \in F_2^n} (-1)^{f(c_1, c_2, \ldots, c_n)(x)} |x\rangle$, where $f(c_1, c_2, \ldots, c_n)(x)$ is a Boolean function (for simplicity, we write $\tilde{f}$ in place of $f(c_1, c_2, \ldots, c_n)$, $\tilde{\psi}$ in place of $\psi(c_1, c_2, \ldots, c_n)$). Then

$$|\tilde{\psi}\rangle\langle\tilde{\psi}| \propto \sum_{s \in F_2^n} \sum_{x \in F_2^n} (-1)^{\tilde{f}(x) + f(x + s)} |x + s\rangle\langle x| \quad (25)$$

Since $s, d$ run over $F_2^n$, we require that $a_1, a_2, \ldots, a_n$ are linear independent, and

$$\tilde{f}(x) + \tilde{f}(x + a_i) = b_i x + c_i \quad (26)$$

Since the right part of Eq.(26) is an affine Boolean function, we suppose $\tilde{f}(x)$ is quadratic.

Notice that $a_1, b_1 \in F_2^n$, it is reasonable to assume $\tilde{f}$ has the form $\tilde{f}(x) = \frac{1}{2}(X \Gamma X^T + \sum_{i=1}^n \Gamma_{ii} x_i)$, where $X = (x_1, x_2, \ldots, x_n)$ is a vector of $n$ variables, $\Gamma = (\gamma_{ij})_{n \times n}$ is a symmetric matrix, and the coefficient of $x_i x_j$ in $\tilde{f}$ is $\gamma_{ij}$, the coefficient of $x_i$ in $\tilde{f}$ is $\gamma_{ii}$. Our following task is try to find the exact expression of $\tilde{f}$.

Let the matrices $A, B$ are $n \times n$ matrix blocks of $A_f$, i.e., $A_f = \langle A | B \rangle$ and let $F_i, B_i$ denote the $i-th$ column of $\Gamma$ and $B$ respectively, then $\tilde{f}$ satisfying the following equations:
- $A_{i} = B_{i}$, $1 \leq i \leq n$

- $\sum_{1 \leq i,j \leq n} a_{si}a_{sj} = c_{s}$, for $1 \leq s \leq n$

If we can find a Boolean function having the required properties, then we can write all of the basis states in the form of $2^{-\frac{m}{2}} \sum_{x \in \mathbb{F}_{2}^{m}} (-1)^{f(x)} |x\rangle$ (i.e., Boolean state). Thus the study of the QECC can again be converted to the study of Boolean functions corresponding with them. We say that it is possible, and we will illustrate an example to show this.

**Example 2.** Define a Boolean function $g(y_{1}, y_{2}, y_{3}, y_{4}) = (y_{1} + y_{2} + y_{3})(y_{1} + y_{2} + y_{3})$, then $g(y)$ is partially bent, and $|\text{Supp}(g)| = 4$, then for every $a \in \mathbb{F}_{2}^{4}$ with $r_{\omega}(a) = 0$ is in $\text{Zsetg}$, and $\text{Supp}(g) = \{s_{1} = (1000), s_{2} = (0100), s_{3} = (0011), s_{4} = (1111)\}$. Let $a_{i}$ be a unitary vector of length 4 with 1 in the $i$-th coordinate and 0 elsewhere. Since $g(y + a_{i}) = g(y) + b_{i}x + c_{i}(1 \leq i \leq 4, \; c = (1100))$, we construct the matrix $A_{g}$ as

$$A_{g} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 0
\end{pmatrix}$$

where the $i$-th row of $A_{g}$ is $v_{i} = (a_{i}, b_{i})$. We can easily verify that all the rows of $A_{g}$ are independent and any two rows have symplectic product zero because the right four columns of $A_{g}$ form a symmetric matrix. Express $A_{g}$ as $A_{g} = [x_{1}, x_{2}, \cdots, x_{8}]$, then for every $\omega$ with $W_{s}(\omega) \leq 1$, $A_{g} \ast \omega^{T}$ is in $\text{Zsetg}$. After computation, we have $\tilde{f}_{1} = g + y_{2}$, $\tilde{f}_{2} = g + y_{1}$, $\tilde{f}_{3} = g + y_{1} + y_{2} + y_{4}$, $\tilde{f}_{4} = g + y_{3} + y_{4}$. We can see that for different $i, j$, $\tilde{f}_{i} - \tilde{f}_{j}$ are linear terms. And $\{2^{-2} \sum_{x \in \mathbb{F}_{2}^{m}} (-1)^{f_{i}(x)} |x\rangle\}$ spans a $[4, 2, 2]$ code, which meets the quantum singleton bound, and therefore is an MDS code.

In fact, for every function $f(y)$ with $2m$ variables of the form $f(y) = (y_{1} + y_{2} + \cdots + y_{2m-2} + y_{2m-1})(y_{1} + y_{2} + \cdots + y_{2m-2} + y_{2m})$ (which is a partially bent function[16]), we can find $|\text{Supp}(f)|$ Boolean functions $\tilde{f}_{i}$ as the foregoing example satisfying that $\{2^{-m} \sum_{x \in \mathbb{F}_{2}^{2m}} (-1)^{h_{i}(x)} |x\rangle\}$ spans a $[2m, 2m - 2, 2]$ code, which is also an MDS code.

Because a quantum code with Boolean basis state is interesting, it is natural to question what kind of properties of the Boolean functions used in Lemma 3 should satisfy. We have known that a partially bent function of the form $f(y) = (y_{1} + y_{2} + \cdots + y_{2m-2} + y_{2m-1})(y_{1} + y_{2} + \cdots + y_{2m-2} + y_{2m})$ can be used in Lemma 3, which is a sufficient condition. Next, we will give a necessary condition, before our following theorem, we give one lemma first.
Lemma 4. \[ f(x) = \begin{cases} 2^n, & s = 0 \\ 0, & \text{else} \end{cases}, \text{if and only if } f \text{ is a bent function, then } |\text{supp}(f)| = 2^{n-1} \pm 2^{n/2-1}.

Theorem 4. For arbitrary \((c_1, c_2, \cdots, c_n) \in \text{supp}(f)\), the Boolean function \(f\) with more than 2 variables used in Lemma 3 can’t be a bent function.

Proof. Seeking a contradiction, we assume that \(f\) is bent, then for every \(s \in \mathbb{F}_2^n, s \neq 0, r_f(s) = 0\). If \(Zsetf \neq \emptyset\), then for every \(a \in Zsetf, r_f(a) = 0\). From Lemma 2, the weight of \(f\) is equal to \(2^{n-2}\), which contradicts the property of bent functions described in Lemma 5, so \(f\) is not bent.

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