Exact solutions and residual regulator dependence
in functional renormalisation group flows

Benjamin Knorr

Perimeter Institute for Theoretical Physics, 31 Caroline St N, Waterloo, ON N2L 2Y5, Canada

We construct exact solutions to the functional renormalisation group equation of the $O(N)$ model and the Gross-Neveu model at large $N$ for $2 < d < 4$, without specifying the form of the regulator. This allows to investigate which quantities are independent of the choice of regulator without being plagued by truncation artefacts. We find that only universal quantities, like critical exponents, and qualitative features, like the existence of a finite vacuum expectation value, are regulator-independent, whereas values of coupling constants are generically arbitrary. We also provide a general algorithm to construct a concrete operator basis for truncations in the derivative expansion and the Blaizot-Méndez-Wschebor scheme.

I. INTRODUCTION

The large $N$ limit is a useful and well-known expansion scheme in quantum field theory [1]. By introducing an infinite number of a specific kind of field, one can control the leading order quantum effects, allowing for analytic expressions for, e.g., critical exponents characterising a second order phase transition as a series in $1/N$. In some cases, the (typically asymptotic) series can then be used to make predictions for a finite, physical number of fields. As a downside, it is generally hard to prove statements about the accuracy of such an expansion, and independent, non-perturbative computations are desirable to arrive at trustworthy estimates of physical quantities.

One such tool to perform non-perturbative calculations is the functional renormalisation group (FRG) [2–8]. In this approach, one considers the so-called effective average action $\Gamma_k$, which depends on a fiducial momentum scale $k$. It interpolates between the classical action at a microscopic scale $\Lambda$, and the standard effective action at $k = 0$. The $k$-dependence is dictated by the Wetterich equation [2, 3],

$$\dot{\Gamma}_k = \frac{1}{2} \text{Tr} \left[ (\Gamma_k^{(2)} + R_k)^{-1} \dot{R}_k \right]. \tag{1}$$

In this, an overdot denotes the scale derivative $k \partial_k$, the trace includes a functional and an index trace as well as a minus sign for fermions, $\Gamma_k^{(2)}$ is the second variation of the effective average action, and $R_k$ is a regulator, which essentially acts as a $k$-dependent mass term, making the right-hand side well-defined. In particular, in the limit $k \to 0$, the regulator vanishes. This equation defines a vector field on theory space, and the points where this vector field vanishes are called fixed points. Fixed points correspond physically to second order phase transitions. The flow around such a fixed point is dictated by the universal critical exponents, which are related to the eigenvalues of the stability matrix at the fixed point.

For a general theory, the flow equation (1) typically has to be solved in an approximation which is not closed, i.e., in general one has to deal with systematic errors in the form of truncation errors. As a consequence, estimates of physical quantities obtained in truncated renormalisation group flows will rather generically show a residual regulator dependence [5, 10–13]. It is thus of interest to understand what quantities are truly independent of the choice of regulator in exact solutions to give a good hint for what kinds of regulator dependencies one should study in truncated renormalisation group flows. We will exemplify this by considering the $O(N)$ limit of some well-known theories: the $O(N)$ model and the (partially bosonised) Gross-Neveu model. These models are relevant to condensed matter and high energy physics, since they describe well-known second order phase transitions [1, 14–17]. As we will see, the large $N$ limit allows for a non-perturbative discussion of the existence and both universal and non-universal properties of the regular (Wilson-Fisher type) fixed point in these models. As a by-product, we will discuss some structural aspects of the flow equations in these models that carry over to finite $N$.

This work is structured as follows. In section II we discuss the $O(N)$ model, focussing on the large $N$ limit. We will first derive the simplified exact flow equation obtained in this limit in subsection II A. In subsection II B, we will construct an operator basis that parameterises the effective average action in the form of a derivative expansion, which reduces the functional flow equation to an infinite tower of first order ordinary differential equations. With that in place, in subsection II C we show that the derivative expansion is a perfect truncation in the sense that the flow of any $n$-th order operator only depends on the solution of operators of lower or the same order. Since the effective potential plays a central role in

1 See however [9] for recent developments in deriving general exact solutions to the flow equation.
2 Recently, non-regular solutions in this limit have been discussed in [18–21]. These solutions are based on a more subtle large $N$ limit where naively sub-leading terms nevertheless contribute due to their divergence structure.
the discussion of fixed points, we will investigate it in detail in subsection II.D. Subsection II.E is devoted to a partial resummation of the derivative expansion, which however still represents a perfect truncation. The resummation allows us to directly study momentum- and field-dependent correlation functions. With the complete fixed point solution, we are in the situation to discuss the regulator dependence in subsection II.F. Finally, we discuss some aspects of the model at finite N in subsection II.G. As a second example, we discuss the Gross-Neveu model at large N in section III. We first present the exact flow equation in the large N limit in III.A, then we discuss the lowest order critical correlation functions and critical exponents in III.B and III.C, respectively, finishing the section with a short discussion of the regulator dependence in III.D. We close with a summary of the main results in section IV.

II. O(N) MODEL

A. Exact flow equation

We will start our investigation by considering an O(N)-symmetric scalar field \( \phi^a \) in \( d \) Euclidean dimensions with \( 2 < d < 4 \). Due to its simplicity and interesting applications, it is potentially the best-studied model in the context of the FRG [22–38]. Let us derive the exact flow equation in the large N limit for the effective action\(^3\)

\[
\Gamma_{N \to \infty}[\phi] = \frac{N}{2} \int d^d x \left( \partial_\mu \phi^a \right) \left( \partial^\mu \phi^a \right) + N\bar{\Gamma}[\rho], \tag{2}
\]

where all non-trivial behaviour is carried by the second term \( \bar{\Gamma} \). Here we introduced \( \rho = \phi^a \phi^a / 2 \). The regulator is chosen to respect the \( O(N) \) symmetry,

\[
\Delta S_k = \frac{N}{2} \int d^d x \phi^a R_k (-\partial^2) \phi^a. \tag{3}
\]

First we show that in the regular large N limit the effective action is indeed of this form (2). The crucial observation for the simplification in the large N limit is that for the right-hand side of the flow equation to yield a factor of \( N \), the trace over the \( O(N) \) bundle indices must be a trace over the corresponding field space identity. For that reason, we only need the part of the two-point function which is proportional to the identity:

\[
\frac{\delta^2 \Gamma_{N \to \infty}}{\delta \phi^a_\mu \delta \phi^{\rho}_{\nu}} = N \left[ -\partial^2 + \frac{\delta \bar{\Gamma}}{\delta \rho} \right] \delta^{ab} + \ldots. \tag{4}
\]

\( ^3 \) To improve the readability, we will suppress the index \( k \) on all quantities from hereon (except the regulator), and in a slight abuse of language refer to \( \Gamma_k \) as the effective action. Since we will not discuss the limit \( k \to 0 \) in this work, this should not lead to any confusion.

Here, the dots indicate terms that, when traced over, do not give rise to a factor of \( N \), i.e., their bundle indices are carried by \( \phi \) and derivatives of it. Note that the identity component of the inverse of this operator only involves the inverse of the identity component of the operator itself. With this observation, we can immediately write down the exact flow equation at large N:

\[
\dot{\bar{\Gamma}} = \frac{1}{2} \int d^d x \int d^d \rho \frac{\bar{R}_k(q^2)}{q^2 + \frac{\delta \bar{\Gamma}}{\delta \rho}}. \tag{5}
\]

In this we introduced the momentum integral measure

\[
\int d^d \rho = \int \frac{q^d}{(2\pi)^d}. \tag{6}
\]

Two points deserve being mentioned with regards to (5): first, the full trace is already performed, and only a standard momentum integral is left. Second, in contrast to standard flows, only the first functional derivative of the action appears in the propagator on the right-hand side. This is indeed at the heart of the simplifications in the large N limit, and allows to write down perfect truncations where increasing the truncation order doesn’t alter the flow equations of previously calculated orders.

We still have to argue that the standard kinetic term used above is the only possibility for a term which doesn’t depend on \( \rho \) only. In particular, one could imagine a field- or momentum-dependent wave function renormalisation factor, also giving rise to a potentially non-vanishing anomalous dimension. However, since the flow equation doesn’t give any contribution to the flow of the kinetic term, requiring locality and regularity at vanishing fields immediately fixes a trivial wave function renormalisation and a vanishing anomalous dimension.

B. Spanning an operator basis

Even though the flow equation (5) is much simpler than it is in the generic case, a direction solution is still involved. To make things tractable, we will now discuss spanning the non-trivial part of the effective action, \( \bar{\Gamma} \), in a derivative expansion. For this, we need to specify a unique basis at every fixed number of derivatives where we took partial integrations into account. Since derivatives always have to come in pairs to form a Lorentz scalar, we will refer to the \( n \)-th order as all terms having \( 2n \) derivatives. The convergence properties of this expansion at finite N have been thoroughly discussed in [36, 37].

At a given order \( n \), a general operator in the derivative expansion of \( \bar{\Gamma} \) has the form

\[
X(\rho) \left( \partial_\mu \cdots \rho \right) \left( \partial_\nu \cdots \rho \right) \cdots, \tag{7}
\]

that is, it is a product of a \( \rho \)-dependent function and a series of derivatives acting on different \( \rho \)s so that for
FIG. 1. The graph indicating the partial integration rules and the basis for the first order of the derivative expansion for the $O(N)$ model at large $N$. The pictograph in the first line correspond to an operator $X(\rho)\partial^2\rho$, which is related to the operator represented by the pictograph in the second line, $Y(\rho)(\partial_{\mu}\rho)(\partial^\nu\rho)$, by a partial integration. The blue dashed line indicates that $X$ and $Y$ are related by $Y(\rho) = -X'(\rho)$.

constant $\rho$, all interaction terms except for the potential vanish. We will now describe a graphical procedure to obtain a unique basis at order $n$, and all partial integration rules to write an arbitrary operator in this basis. For this, we draw a box for every $n$ on which derivatives act, and a link connecting two boxes (or a box with itself) which corresponds to a pair of derivatives which is contracted, the end points marking the fields on which they act. With this, the algorithm at order $n$ is as follows:

1. In the first row, draw all possible graphs with any number of boxes and exactly $n$ links, where all links start and end on the same box, and no box can have zero links attached to it. This corresponds to all operators of the form (7) where we only have $\partial^2$ as derivative operators, each acting on a single $\rho$.

2. To arrive at the next row of the graph, employ the following partial integration rules. For any graph of the first row, from the box with the most links, disconnect one of the links which starts and ends on the box, and draw all distinct graphs where the loose link is connected to either one of the existing boxes, or a new box, with the following weights:

   (a) if the link is connected to an already existing box, multiply with a factor of $-1$,

   (b) if the link is connected to a new box, multiply with a factor of $-x$.

   If connecting the loose link to different boxes gives the same new diagram, multiply the weight by this multiplicity.

3. Draw lines from every “parent” box of the given level of the graph to its “children” as obtained by the previous integration rules, indicating the weights.

4. Perform the two previous steps for all other links, but don’t attach weights to the lines from the parents to these “illegitimate children”. This step ensures that all distinct graphs on a given row are generated, since not all graphs of a given row need to have parents in the above sense. The first time that new operators are generated in this way for the $O(N)$ model is at fourth order.

5. Repeat the above steps until all links connect different boxes.

The bottom line of the graph represents the independent basis at order $n$ that we will use, and corresponds to the basis where all $\partial^2$ operators are partially integrated. To write any element in this basis, follow the graph starting from this element down to the bottom row, multiplying the weights from row to row. Once the bottom row is reached, any factor of $x$ corresponds to a $\rho$-derivative of the operators’ prefactor, that is the function $X$ in (7). To illustrate this procedure, we show the graphs for orders $n = 1, 2, 3$ in Figure 1, Figure 2 and Figure 3, the graph for $n = 4$ can be found in the supplemental material as it is too bulky to present it here. In these graphs, we only indicate multiplicities explicitly. Factors of $-1$ are indicated by olive, full lines while factors of $-x$ are indicated by blue, dashed lines. Illegitimate relations are indicated by red, dotted lines.

We can now span the effective action in terms of our basis in a derivative expansion:

$$\bar{\Gamma} = \int d^dx \left[ V(\rho) + Y(\rho) (\partial_{\mu}\rho) (\partial^\mu\rho) + W_1(\rho) (\partial_{\mu}\partial_{\nu}\rho) (\partial^\mu\partial^\nu\rho) + W_2(\rho) (\partial_{\mu}\rho) (\partial^\mu\partial^\nu\rho) (\partial_{\nu}\rho) + W_3(\rho) (\partial^\mu\rho) (\partial_{\mu}\rho) (\partial_{\nu}\rho) + \mathcal{O}(\partial^6) \right].$$  \hfill (8)

The potential $V$ has no graphical representation. The wave function renormalisation for the field $\rho$ is represented in the bottom line of Figure 1, and the functions $W_{1,2,3}$ correspond to the bottom line of Figure 2, from left to right. The number of terms increases rapidly with the order $-$ for $n = 3$, there are eight coupling functions, whereas for $n = 4$, there are already 23 coupling functions.
C. Derivative expansion as a perfect truncation

In this section we will show that the derivative expansion in the large $N$ limit is a perfect truncation, that is, the flow of the $n$-th order coupling functions doesn’t depend on the coupling functions of any higher order. In fact, we can show that for every coupling function $X$ except the potential, the flow equation reads

$$
\dot{X}(\rho) = -\frac{1}{2} X'(\rho) \int d\mu_q \frac{\hat{R}_k(q^2)}{(q^2 + R_k(q^2) + V'(\rho))^2} 
+ n_X V''(\rho) X(\rho) \int d\mu_q \frac{\hat{R}_k(q^2)}{(q^2 + R_k(q^2) + V'(\rho))^3}
+ f_X(\rho),
$$

where $n_X$ is the number of $\rho$s in the operator on which derivatives act. Graphically, this is the number of boxes in the graph associated to the coupling function $X$. Furthermore, $f_X(\rho)$ is a function which doesn’t depend on $X$, but depends on some operators of lower order, and can depend linearly on coupling functions of the same order, but not their derivatives. Let us point out that the flow equations are thus first order linear ordinary differential equations.

To show this form of the flow equation, we will use the basis introduced in the previous section. For a general operator that appears in the action as

$$
X(\rho) \left( \mathcal{D}_1 \rho \right) \cdots \left( \mathcal{D}_{n_X} \rho \right),
$$

where the $\mathcal{D}_i$ are collections of contracted derivatives in the form of the basis of the previous section, the variation w.r.t. $\rho$ reads

$$
\frac{\delta}{\delta \rho} \int d^d x \ X(\rho) \left( \mathcal{D}_1 \rho \right) \cdots \left( \mathcal{D}_{n_X} \rho \right)
= X'(\rho) \left( \mathcal{D}_1 \rho \right) \cdots \left( \mathcal{D}_{n_X} \rho \right) + \sum_{i=1}^{n_X} \left[ (-1)^{|\mathcal{D}_i|} \mathcal{D}_i \left( X(\rho) \left( \mathcal{D}_1 \rho \right) \cdots \left( \mathcal{D}_{i-1} \rho \right) \left( \mathcal{D}_{i+1} \rho \right) \cdots \left( \mathcal{D}_{n_X} \rho \right) \right) \right].
$$

In this, $|\mathcal{D}_i|$ is the number of derivatives contained in the operator $\mathcal{D}_i$. Since this expression is still of the same derivative order, it cannot contribute to the flow of operators of lower orders, and we can expand the flow equation to linear order in this expression to obtain its contribution to the same order. The first term is still in the correct basis, so that it only contributes to the flow of itself at that order, and is indeed the first term of (9). The additional minus sign comes from the expansion of the propagator. The second term gives a contribution

$$
- \frac{1}{2} \int d^d x \int d\mu_q \frac{\hat{R}_k(q^2)}{(q^2 + R_k(q^2) + V'(\rho))^2} \sum_{i=1}^{n_X} \left[ (-1)^{|\mathcal{D}_i|} \mathcal{D}_i \left( X(\rho) \left( \mathcal{D}_1 \rho \right) \cdots \left( \mathcal{D}_{i-1} \rho \right) \left( \mathcal{D}_{i+1} \rho \right) \cdots \left( \mathcal{D}_{n_X} \rho \right) \right) \right]
$$

We have to perform a partial integration on $\mathcal{D}_i$ to arrive again in our chosen basis. Since the $\mathcal{D}_i$ don’t include $\partial^2$ by construction, performing the partial integration directly gives an expression in the correct basis. In particular, we can easily isolate the term which contributes to the flow of $X$ itself: it is that term where all the deriv-
tives of $D_l$ act on a single $\rho$. From this, we get a factor of $-2V''(\rho)$ and another propagator from taking the derivative of the squared propagator, and a factor of $n_X$ from the sum, so that we arrive at the second line of (9). All other terms that $X$ contributes to are different operators of the same order, where it contributes linearly and without derivatives, or higher orders. This completes the proof of (9).

D. Fixed point potential

We will now start to discuss the fixed point, that is the point where the flow of all dimensionless coupling functions vanishes. For this, we rescale the field and coupling functions by appropriate powers of $k$, e.g.,

$$\rho = k^{d-2} r, \quad V(\rho) = k^d v(k^{2-d} \rho). \quad (13)$$

Here, $r$ is the dimensionless version of the field $\rho$ and $v$ the dimensionless potential.\footnote{Since the wave function renormalisation factor $Z$ doesn’t renormalise in the large $N$ limit of the $O(N)$ model, we don’t have to include it in the rescaling. This is different in the Gross-Neveu model discussed below, where we assume that proper powers of this factor have been included in all dimensionless couplings.} With this, at the fixed point we have to solve

$$d v(r) - (d - 2) r v'(r) = \frac{1}{2} \int d\mu_q \frac{\hat{R}_k(q^2)}{q^2 + R_k(q^2)} \cdot (14)$$

to arrive at the critical potential. In fact, it is easier to solve the derivative of this equation for $u(r) \equiv v'(r)$:

$$2u(r) - (d - 2) r u'(r) = \frac{1}{2} u'(r) \int d\mu_q \frac{\hat{R}_k(q^2)}{(q^2 + R_k(q^2))^2}(15)$$

The Wilson-Fisher type solution of this equation can be given implicitly,

$$r = \frac{1}{2(d-2)} \int d\mu_q \frac{\hat{R}_k(q^2)}{(q^2 + R_k(q^2))^2} \times$$

$$2F_1 \left( 2, 1 - \frac{d}{2}; \frac{d}{2} - \frac{d}{2} \left| \frac{u(r)}{q^2 + R_k(q^2)} \right. \right) \cdot (16)$$

The complete set of solutions of (15) for a specific choice of regulator is discussed, e.g., in [35].

In fact, (16) establishes a bijective map between $r \in (-\infty, \infty)$ and $u \in (u_{-\infty}, \infty)$, even though a priori physical values of $r$ are non-negative. To see this, we calculate $u'(r)$ and show that it is positive. Taking the derivative of (16) with respect to $r$ and solving for $u'$, we get

$$u'(r) = \left[ \frac{1}{4} \int d\mu_q \frac{\hat{R}_k(q^2)}{(q^2 + R_k(q^2))^3} \frac{q^2 + R_k(q^2)}{u(r)} \right] \left( 2F_1 \left( 2, 1 - \frac{d}{2}; \frac{d}{2} - \frac{d}{2} \left| \frac{u(r)}{q^2 + R_k(q^2)} \right. \right) - \left( 1 + \frac{u(r)}{q^2 + R_k(q^2)} \right)^2 \right)^{-1}. \quad (17)$$

which is positive for any regulator, so that the minimum is always in the physical regime.

By evaluating (16) at $r = 0$, we get an implicit equation which fixes $u(0)$. This indeed also gives an upper bound on $u(0)$ since for the integral to vanish, the argument of the hypergeometric function has to have a minimum value, as it is positive for all arguments $x$ larger than some small negative value. The bound for $u(0)$ is shown in Figure 4 for general dimensions $d$. In $d = 3$, the hypergeometric function vanishes at approximately 0.388. This extremal value is exactly reached for the Litim cutoff [11], for which the hypergeometric function is independent of the integration variable and thus its argument must be its zero.

The bijective relation between $r$ and $u$ will play a central role in the following discussion, since as it turns out, $u$ is a much more natural variable than $r$.

At this point it is also worthwhile to note the special form of the fixed point equations for all the higher order operators. Translating (9) into dimensionless form, we
have

\[
d_x x(r) - (d - 2) r x'(r) = \frac{1}{2} x'(r) \int \mu_q \frac{\hat{R}_k(q^2)}{(q^2 + R_k(q^2) + u(r))^2} + n_X u'(r)x(r) \int \mu_q \frac{\hat{R}_k(q^2)}{(q^2 + R_k(q^2) + u(r))^3} + f_X(r),
\]

(20)

where \(d_x\) and \(x\) are the canonical dimension and the dimensionless version of the coupling function \(X\), respectively. Using (15) to replace the first integral on the right-hand side gives

\[
d_x x(r) - 2 \frac{u(r)}{u'(r)} x'(r) = n_X u'(r)x(r) \int \mu_q \frac{\hat{R}_k(q^2)}{(q^2 + R_k(q^2) + u(r))^3} + f_X(r).
\]

(21)

This is a linear first order differential equation for \(x\), so a priori it has one free integration constant. We notice however that at \(r = r_0\) the prefactor of \(x'\) vanishes, which fixes the integration constant uniquely by requiring that the solution is regular.\(^5\)

Before we go on with the discussion of the other operators, let us recalculate the critical exponents, showing that they are indeed regulator-independent. Perturbing the flow equation for the potential linearly about the fixed point gives

\[
(d - \theta)\delta v(r) - (d - 2) r \delta v'(r) = -\frac{1}{2} \delta v'(r) \int \mu_q \frac{\hat{R}_k(q^2)}{(q^2 + R_k(q^2) + v'(r))^2} = \left(2 \frac{v'(r)}{v''(r)} - (d - 2) r \right) \delta v'(r),
\]

(22)

where \(\delta v\) is the perturbation of the fixed point potential \(v\) and \(\theta\) is the critical exponent. In the last line, we used the fixed point equation (15) to replace the integral. We can now use the aforementioned variable transformation to map \(r\) to \(u\) and get

\[
(d - \theta)\delta v(u) - 2u \delta v'(u) = 0,
\]

(23)

which has the solution

\[
\delta v(u) = c \frac{1}{u^\theta},
\]

(24)

with \(c\) being a normalisation constant. Since \(u(r_0) = 0\), and \(u(r) < 0\) for \(0 \leq r < r_0\), we have to demand that the exponent is a non-negative integer \(n\) for the perturbations to be globally well-defined and real, which gives the well-known quantised spectrum

\[
\theta = d - 2n, \quad n \in \mathbb{N}.
\]

(25)

To complete the argument, we still have to show that there are perturbations for all higher order operators such that they fulfill their perturbed flow equations for the same critical exponents, this however follows from similar arguments as the existence of the fixed point itself.

E. Partial resummation of the derivative expansion

With the results provided so far, in principle the full fixed point effective action can be reconstructed, order by order in the derivative expansion. As it turns out, this procedure can be optimised by resumming certain subclasses of operators. This comes from the observation that an operator corresponding to a diagram element with \(n\) boxes only contributes to diagram elements with \(n\) or more boxes, that is, there is a further type of hierarchy which gives again rise to a perfect truncation. To prove this statement, we use momentum space techniques, and consider the \(n\)-th \(\rho\)-derivative of the flow equation, evaluated at constant \(\rho\). Because in the large \(N\) limit, the propagator only involves the first derivative of the effective action, the flow of the \(n\)-point function involves correlators up to order \(n + 1\). In particular, the only dependence on the higher order correlator is by a tadpole diagram. Explicitly, this dependence is

\[
\Gamma^{(n)}(p_1, \ldots, p_n|\rho) \supset -\frac{1}{2} \int \mu_q \frac{\hat{R}_k(q^2)}{(q^2 + R_k(q^2) + v'(|\rho|))^2} \times \Gamma^{(n+1)}(p_1, \ldots, p_n, -(p_1 + \cdots + p_n)|\rho).
\]

(26)

\(^5\) Strictly speaking, since \(f_X\) contains coupling functions of the same order, one has to deal with coupled linear equations. Showing existence of a solution is easier for the resummed correlation functions discussed below, so we will not dwell on it here.
For better readability, we introduced a vertical bar to separate the momentum arguments from the field argument. The dependence of the $n$-point functions $\Gamma^{(n)}$ on the momenta $p_i$ is fully symmetric. Using momentum conservation for the last momentum argument of the $(n+1)$-point function, we find

$$\hat{\Gamma}^{(n)}(p_1, \ldots, p_n)|\rho\rangle \supset -\frac{1}{2} \Gamma^{(n+1)}(p_1, \ldots, p_n, 0)|\rho\rangle \int \frac{d\mu_q}{(q^2 + R_k(q^2) + V'(\rho))^2} R_k(q^2).$$

(27)

The key observation is that the last momentum in the $(n+1)$-point function is zero, so that there is no contribution from operators with $(n+1)$ boxes.

Readers familiar with FRG techniques will realise that this partial resummation is a concrete realisation of the so-called Blaizot-Méndez-Wschebor (BMW) approximation [22, 39–42], which resolves both momentum and field dependence. In fact, the diagrammatic representation can be used as a way to give a concrete action as a starting point to the BMW scheme, and motivates the origin of the truncation rules employed there. Concretely, the resummed ansatz reads in our case

$$\hat{\Gamma} = \int d^d x \left[ V(\rho) + \sum_{l \geq 1} Y_l(\rho) (\partial_{\mu_1} \cdots \partial_{\mu_l} \rho) (\partial^{\mu_1} \cdots \partial^{\mu_l} \rho) \right]$$

$$+ \sum_{n \geq 2} \sum_{l=1}^{n-1} W_{n,l}(\rho) (\partial_{\mu_1} \cdots \partial_{\mu_l} \rho) (\partial^{\mu_{l+1}} \cdots \partial^{\mu_{n}} \rho) (\partial^{\mu_{l+1}} \cdots \partial^{\mu_{n}} \rho)$$

$$+ \sum_{n \geq 3} \sum_{l=1}^{n-2} \sum_{j=l}^{n-1} W_{n,l,j}(\rho) (\partial_{\mu_1} \cdots \partial_{\mu_l} \partial_{\mu_{l+1}} \cdots \partial_{\mu_j} \rho) (\partial^{\mu_{l+1}} \cdots \partial^{\mu_{j+1}} \partial^{\mu_{n}} \rho) (\partial^{\mu_{l+1}} \cdots \partial^{\mu_{j+1}} \partial^{\mu_{n}} \rho)$$

$$+ O((\partial\rho)^4) \right].$$

(28)

The first line corresponds graphically to 0 and 2 boxes, the second line comprises the diagrams with three boxes arranged in a line, whereas the third line lists all operators with three boxes arranged in a triangle. To our knowledge, such an explicit expression for the effective action giving rise to the BMW scheme hasn’t been put forward before.

We will now iteratively study the flow equations for the $n$-point functions to gain some structural insights. The zero-point function is identical to the potential and has been dealt with in the last subsection. The first non-trivial momentum dependence arises for the two-point function.\(^6\)

1. Two-point function

Taking the second $\rho$-derivative of the flow equation and evaluating it for constant field gives

$$\hat{\Gamma}^{(0,2)}(p^2|\rho) = -\frac{1}{2} \frac{\delta^2 \hat{\Gamma}}{\delta \rho(p) \delta \rho(-p) \delta \rho(0)} \int d\mu_q \frac{\hat{R}_k(q^2)}{(q^2 + R_k(q^2) + V'(\rho))^2} R_k(q^2)$$

$$+ \left[ \hat{\Gamma}^{(0,2)}(p^2|\rho) \right]^2 \int d\mu_q \frac{\hat{R}_k(q^2)}{(q^2 + R_k(q^2) + V'(\rho))^2} \frac{1}{(p + q)^2 + R_k((p + q)^2) + V'(\rho)}.$$  

(29)

Note that no loop momentum appears in the vertex functions. This is a general feature in the large $N$ limit and once again arises due to the fact that the exact flow equation (5) depends on the first, not the second derivative of the effective action. For that reason, the two internal $\phi$-legs of any vertex actually count as a single $\rho$-leg, with a momentum equal to minus all external legs due to momentum conservation, and hence no loop momentum.

\(^6\) We emphasise that the two-point function of the composite $\rho$ corresponds to a combination of parts of the two-, three- and four-point functions of the elementary field $\phi$. Similarly, higher order composite correlators correspond to combinations of higher order elementary correlators.
As a consequence, we still only have to deal with (now partial) differential equations and not integro-differential equations, as would be expected when resolving momentum dependencies.

We will call the dimensionless version of the two-point vertex $\gamma_2$. Furthermore we will introduce the integrals

\begin{equation}
\mathcal{I}_n(u(r)) = \int \frac{d\mu_q}{(q^2 + R_k(q^2) + u(r))^n},
\end{equation}

Taking care of the correct regular boundary conditions, we find that the fixed point equation for $\gamma_2$ reads

\begin{equation}
(4 - d)\gamma_2(p^2|r) - (d - 2) r \gamma_2^{(1,0)}(p^2|r) - 2p^2 \gamma_2^{(1,0)}(p^2|r) = -\frac{1}{2} \gamma_2^{(0,1)}(p^2|r) \mathcal{I}_2(u(r)) + \gamma_2(p^2|r)^2 \mathcal{I}_{2,1}(p^2|u(r)).
\end{equation}

Note that we made it explicit in the argument of $\mathcal{I}_{2,1}$ that because of Lorentz invariance, the integral must be a function of the squared dimensionless momentum, which we denote by the same letter $p^2$. We can simplify the equation in two steps. First, we again use the fixed point equation for the derivative of the potential to replace the first integral on the right-hand side of the equation via (15). Second, we use the bijective map from $r$ to $u$ and make a coordinate transformation to replace all occurrences of $r$ by $u$. This results in the fixed point equation

\begin{equation}
(4 - d)\gamma_2(p^2|u) - 2p^2 \gamma_2^{(1,0)}(p^2|u) - 2u \gamma_2^{(0,1)}(p^2|u) = \gamma_2(p^2|u)^2 \mathcal{I}_{2,1}(p^2|u).
\end{equation}

Taking care of the correct regular boundary conditions, this gives rise to the solution

\begin{equation}
\gamma_2(p^2|u) = \frac{1}{2} \int_0^1 d\omega \omega^{1-\frac{d}{2}} \mathcal{I}_{2,1}(\omega p^2|\omega u).
\end{equation}

The integral exists for all $d \in (2,4)$. It can be checked straightforwardly by comparison with (17) and using identities for $\mathcal{I}_1$ that for $p^2 = 0$ indeed $\gamma_2(0, u) = u'(r)$, as it should be. Expanding the solution in a Taylor series in $x$ around 0, one can directly read off the (dimensionless) fixed point coefficient functions $y_l(r)$ of the ansatz (28). Explicitly,

\begin{equation}
\gamma_2(p^2|u) = u'(r) + 2 \sum_{l \geq 1} y_l(r) p^{2l},
\end{equation}

where both $r$ and $u'(r)$ have to be replaced according to (16) and (17).

We can also derive the large momentum limit at finite $u$. In this limit,

\begin{equation}
\gamma_2(p^2|u) \propto (p^2)^{2-\frac{d}{2}}, \quad u \to \infty.
\end{equation}

Comparing to the standard behaviour of a two-point function at large momentum,

\begin{equation}
\gamma_2(p^2|u) \propto (p^2)^{1-\frac{d}{2}},
\end{equation}

this suggests to define an anomalous dimension with respect to the field $\rho$ with value

\begin{equation}
\eta_\rho = d - 2.
\end{equation}

This is to be contrasted with the anomalous dimension of the fundamental field $\phi$, which vanishes. Note also that while the power law exponent is regulator-independent, the overall prefactor of the limit does depend on the choice of the regulator.

### 2. Higher order correlation function

For the higher order correlation functions, it is convenient to choose the inner product of distinct momenta as a basis for the momentum dependence,

\begin{equation}
y_{ij} = p_{i\mu} p_{j}^{\mu}, \quad 1 \leq i < j \leq n.
\end{equation}

Any squared momentum can be replaced by a combination of these by imposing momentum conservation,

\begin{equation}
\sum_{l=1}^{n} p_l = 0,
\end{equation}

where we have the convention that all momenta are incoming. This implies the replacement rules

\begin{equation}
p_i^2 = -p_{i\mu} \sum_{l=1, l \neq i}^{n} p_l^{\mu} = -\sum_{l=1}^{n} y_{li} - \sum_{l=i+1}^{n} y_{il}.
\end{equation}

Whenever squared momenta appear in the following, they are to be understood as a shorthand for their expression in terms of the $y_{ij}$.

Following the same steps as for the two-point function, we get the fixed point equation for the three-point function,

\begin{equation}

\begin{align*}
&[(6 - 2d) - 2y_{ij} \partial_{y_{ij}} - 2u \partial_u] \gamma_3(y_{12}, y_{13}, y_{23}|u) = \gamma_3(y_{12}, y_{13}, y_{23}|u) \sum_{i=1}^{3} \gamma_2(p_i^2|u) \mathcal{I}_{2,1}(p_i^2|u) \\
&= \frac{1}{2} \gamma_2(p_1^2|u) \gamma_2(p_2^2|u) \gamma_2(p_3^2|u) \sum_{i,j=1}^{3} \mathcal{I}_{2,2}(p_i, p_j|u).
\end{align*}
\end{equation}

By requiring momentum conservation and again using Lorentz symmetry, it is clear that all terms are symmetric
functions of the three squared momenta $x_i = p_i^2$, and correspondingly of the $y_{ij}$. Structurally, we thus have
\[
[(6 - 2d) - 2y_{ij} \partial y_{ij} - 2u \partial u] \gamma_3(y_{12}, y_{13}, y_{23}|u) \\
= \gamma_3(y_{12}, y_{13}, y_{23}|u) \mathcal{L}_3(y_{12}, y_{13}, y_{23}|u) \\
+ C_3(y_{12}, y_{13}, y_{23}|u).
\]

The solution with the correct boundary condition at vanishing momenta is
\[
\gamma_3(y_{12}, y_{13}, y_{23}|u) \\
= - \frac{1}{2} \int_0^1 d\omega \omega^{d-4} C_3(\omega y_{12}, \omega y_{13}, \omega y_{23}|\omega u) \times \\
\exp \left[ - \frac{1}{2} \int_0^1 \frac{dt}{\tau} \mathcal{L}_3(\tau y_{12}, \tau y_{13}, \tau y_{23}|\tau u) \right].
\]

The solution can be mapped to the coupling functions $W$ appearing in (28), but the expression is rather lengthy and provides no additional insight, so we refrain from spelling it out.

Indeed, this structure carries over in the same way for the higher order correlation functions. The generalisation of the above for the $n$-point function, with $n \geq 3$, reads
\[
[(2n - (n - 1) - 2y_{ij} \partial y_{ij} - 2u \partial u] \gamma_n(\{y_{ij}\}|u) \\
= \gamma_n(\{y_{ij}\}|u) \mathcal{L}_n(\{y_{ij}\}|u) + C_n(\{y_{ij}\}|u),
\]

where
\[
\mathcal{L}_n = \sum_{i=1}^n \gamma_2(p_i^2|u) \mathcal{I}_{2,1}(p_i^2|u),
\]

comes from the self-energy type diagrams. All squared momenta should be replaced by combinations of the $y_{ij}$, and $C_n$ arises from all flow diagrams which involve only correlators of order less than $n$, and thus are fixed already. The corresponding solution to the equation is
\[
\gamma_n(\{y_{ij}\}|u) = - \frac{1}{2} \int_0^1 d\omega \omega^{d-2(n-1)-2} C_n(\{\omega y_{ij}\}|\omega u) \times \\
\exp \left[ - \frac{1}{2} \int_0^1 \frac{d\tau}{\tau} \mathcal{L}_n(\{\tau y_{ij}\}|\tau u) \right].
\]

Since the flow diagrams entering $C_n$ can be calculated easily in an iterative way, this completes the calculation of the exact fixed point action for the $O(N)$ model at large $N$. We emphasise that while the general idea of an exact solution in the large $N$ limit has appeared in the literature before [39], the systematic discussion of the fixed point correlation functions, filling in all intermediate details, has been missing.

\section{Regulator dependence}

We will now discuss the regulator dependence of the exact solution and its properties. Clearly, as it must be, the existence of the fixed point and all critical exponents are regulator-independent, see (25). The latter are physically observable, at least in principle, and characterise the second order phase transition. Further regulator-independent features are the existence of a positive vacuum expectation value at $r = r_0 > 0$, the monotonicity of the derivative of the potential, and the bound on the value of $u(0)$. These are however qualitative, not quantitative features.

The more interesting question is thus about quantitatively regulator independence, e.g., whether any combination of coupling constants is regulator-independent as well. If such a combination would be found, it would be possible to assess the quality of a given truncation in a new way. On the other hand, if this is not the case, so that coupling constants are essentially free and completely independent of each other, then any general comparison of values for coupling constants obtained with different regulators is meaningless. In particular, any strong regulator dependence of strictly non-universal quantities would not be a sign of a bad truncation whatsoever. We will find that indeed the latter option is the case: fixed point values of coupling constants are not universal.

To make this more precise, let us study the coefficients of the Taylor expansion of the critical potential around the minimum. Taking $n$ derivatives of the fixed point equation and using Faà di Bruno's formula gives
\[
(2n - (n - 1)d)v^{(n)}(r) - (d - 2)v^{(n+1)}(r) \\
= \frac{1}{2} \sum_{k=1}^n (-1)^k k! \mathcal{I}_{k+1}(v'(r)) B_{n,k}(v''(r), \ldots, v^{(n-k+2)}(r)).
\]

Here, the $B_{n,k}$ denote the Bell polynomials. We now notice that the only term on the right-hand side which involves $v^{(n+1)}(r)$ is the term with $k = 1$. Evaluating this equation at the minimum $r = r_0$, we find that any dependence on $v^{(n+1)}(r_0)$ drops out of the equation, and we get a linear equation for $v^{(n)}(r_0)$ in terms of the lower derivatives at the minimum. The first few derivatives read
\[
v(r_0) = \frac{1}{2} \mathcal{I}_1(0),
\]
\[
v'(r_0) = 0,
\]
\[
v''(r_0) = \frac{4 - d}{\mathcal{I}_3(0)}.
\]

In general, one finds that $v^{(n)}(r_0)$ depends on the threshold integral $\mathcal{I}_{n+1}(0)$. For $n \geq 3$, the explicit dependence on this integral reads
\[
v^{(n)}(r_0) = \frac{1}{2} \frac{n!}{d - 2n} \left( \frac{d - 4}{\mathcal{I}_3(0)} \right)^n \mathcal{I}_{n+1}(0) + \ldots,
\]

where the dots indicate terms which only include threshold integrals with indices $\leq n$. Since these threshold integrals are in general independent of each other, we conclude that all couplings are independent of each other.
be extracted, but we haven’t found a way to do so. We conclude that to derive exact expressions for higher order terms of the large $N$ expansion of critical quantities, standard large $N$ methods [1, 43] are more efficient.

2. Finite $N$

Let us now discuss some implications and benefits of the presented results for studies at finite $N$. From the analysis of the $O(N)$ model at varying $N$, it was found that the fixed point has a smooth dependence on $N$ [37]. We thus propose the following new way to close the flow equation at finite $N$. Instead of setting correlation functions that are not resolved to zero, the large $N$ solution might be substituted instead. For large enough $N$, this should still be an excellent approximation, and improve the convergence of any other truncation scheme. This gives a practically feasible implementation of the idea presented in [44] to use information on higher order operators to stabilise lower order truncations.

On a conceptual level, at finite $N$ it is indeed more useful to employ a somewhat different basis than that of the large $N$ limit, since we now have to deal with a bundle index. It is most convenient to then let all derivatives act on fields $\phi^a$ only instead of both $\phi^a$ and $\rho$, since that way we can avoid having to introduce two kinds of boxes and deal with more complicated integration rules. This modifies our algorithm to obtain the basis and partial integration rules in the following way:

- boxes now stand for fields $\phi^a$, and we have a second kind of link, which we represent by a wavy link, indicating contraction of the bundle index,
- only an even number of boxes can exist, since all fields must be contracted to form a scalar,
- when adding a new element by partial integration corresponding to a weight of $-x$, we have to add two boxes which are linked by a wavy link, and the derivative link connects to one of them; no factors of two arise because of the normalisation $\rho = \phi^a\phi^a/2$ so that $(\partial_\mu \rho) = \phi^a(\partial_\mu \phi^a)$.

We illustrate these changes in the graphs corresponding to the first two non-trivial orders in the derivative expansion in Figure 5 and Figure 6. This choice of basis coincides with the choice made in the recent investigation of the $n = 2$ truncation of the $O(N)$ model at finite $N$ [37], and we indicate the map to the prefactors of that work to our basis in the caption of the graphs. As has been noted in [37], for the case $N = 1$ once again different rules apply since the wavy links don’t represent a contraction anymore and can be dropped. Then, any pair of completely unconnected boxes can be reabsorbed into the prefactor coupling function, so that the number of independent operators at any given order of the derivative expansion is lower for $N = 1$ than for $N > 1.$
A unit strength symmetrisation over the external momenta is implied on the right-hand side of the equation.

We will close this section by commenting again on the relation of this basis to the BMW scheme [22, 39–42]. At finite $N$ and order $n$, this corresponds to the resummation of diagram elements where exactly $n$ boxes are connected by at least one derivative link. In contrast to the large $N$ limit, this includes graphs with up to $2n$ boxes.

III. GROSS-NEVEU MODEL

A. Exact flow equation

As a second example for an exact fixed point solution, we will discuss the partially bosonised Gross-Neveu model. In this, we discuss $N_f$ Dirac fermions in a $d_z$-dimensional representation of the Clifford algebra, coupled to a $\mathbb{Z}_2$-symmetric scalar field. For investigations at finite $N_f$ in different dimensions and in both a condensed matter and a high energy physics context, see [25, 29, 45–67]. The large $N$ limit consists in taking the flavour number to infinity, $N_f \to \infty$, with the consequence that only diagrams with a complete fermion loop contribute to the flow in this limit. By assumption, the four-fermion coupling is set to zero in this model at the microscopic level since it has been transformed to a Yukawa coupling by the Hubbard-Stratonovich transformation. In that way, the fermionic sector of the action doesn’t flow in the large $N$ limit, but remains classical. Employing again locality and regularity arguments, this fixes the form of the action to

$$\Gamma_{N \to \infty} [\phi, \Psi] = d_z N_f \int d^d x \left[ i \bar{\Psi} \gamma^\mu \partial_\mu \Psi + i \bar{\Psi} \gamma^\mu \gamma^5 \partial_\mu \Psi \right] + \bar{\Gamma}[\phi].$$

(51)

Both the fermionic wave function renormalisation and the Yukawa interaction are restricted to be constant by locality and regularity. With the regulator choice

$$\Delta S_k = d_z N_f \int d^d x i \bar{\Psi} \gamma^\mu \partial_\mu \partial_\nu \Psi,$$

(52)

where $r_k$ is a dimensionless shape function, we can write down the exact flow as

$$\hat{\Gamma} = - \int d^d x \int d\mu_q \text{tr} \left[ (i \bar{\partial} - q) (1 + r_k ((q - i \partial)^2)) + i \bar{\Psi} \gamma^\mu \partial_\mu \Psi \right]^{-1} \left( -q r_k (q^2) \right) \left|_{d_z N_f} \right.$$

(53)

where the remaining trace over the Dirac indices has to be projected onto the term proportional to $d_z N_f$. To bring this into a more useful form, we first introduce the fermionic propagator at constant scalar field $\phi$,

$$G_0(q|\phi) = \left[ -q (1 + r_k (q^2)) + i \bar{\phi} \gamma^\mu \partial_\mu \phi \right]^{-1}$$

$$= \frac{1}{q^2 (1 + r_k (q^2)) + i \bar{\phi} \gamma^\mu \partial_\mu \phi}.$$

(54)

A unit strength symmetrisation over the external momenta is implied on the right-hand side of the equation.
Let us point out two key differences to the $O(N)$ model: first, since the flow only contributes to bosonic operators but is driven purely by fermionic fluctuations, the resulting fixed point equations for the correlation functions are linear first order differential equations that can be integrated directly. Second, as is well-known, since the flow contributes to all bosonic operators, we will have a nontrivial bosonic anomalous dimension $\eta_\phi \neq 0$.

**B. Low order critical correlators**

We will now discuss the lowest order correlation functions and the spectrum. In the following we will occasionally use the identification

$$q^2(1 + r_k(q^2))^2 = q^2 + R_k(q^2),$$

(56)

to bring the flow equation into a form which resembles a bosonic flow.

Going somewhat out of order, we will first discuss the fixed point equation for the Yukawa coupling $\bar{g}$. The reason for this is that it is dimensionful, so that requiring a fixed point in fact determines the bosonic anomalous dimension. For this we assume that $\bar{g} \neq 0$, which is reasonable since otherwise the flow is a constant, and we are only left with the Gaussian fixed point. In general dimensions, the flow equation for the dimensionless Yukawa coupling $g$ reads

$$\dot{g} = \frac{d - 4 + \eta_\phi}{2} g = 0.$$  

(57)

We conclude that for $g \neq 0$, the bosonic anomalous dimension is

$$\eta_\phi = 4 - d.$$  

(58)

Next, we will discuss the bosonic potential. Its flow reads

$$\dot{v}(r) = -d,v(r) + (d - 2 + \eta_\phi)r v'(r) - \frac{1}{2} \int dp^2 q^2 + R_k(q^2) + 2g^2 r.$$  

(59)

Here we used the rewriting (56). Using (58), the regular fixed point solution reads

$$v(r) = -\frac{1}{2d} \int dp^2 \frac{\dot{R}_k(q^2)}{q^2 + R_k(q^2)} 2F_1 \left(1, -\frac{d}{2}; \frac{1 - d}{2}; -\frac{2g^2 r}{q^2 + R_k(q^2)} \right).$$  

(60)

For non-negative $r$ the potential is monotonically increasing, so that the fixed point potential is in the symmetric regime. Let us now discuss the bosonic two-point function. At constant $r$, the flow equation reads

$$\dot{g_2}(p^2|r) = (-2 + \eta_\phi) g_2(p^2|r) + (d - 2 + \eta_\phi)r g_2^{(0,1)}(p^2|r) + 2p^2 g_2^{(1,0)}(p^2|r) + 2g^2 \int dp^2 q^2 + R_k(q^2) [q^2(1 + r_k(q^2))^2 + 2g^2 r]^2 [(p + q)^2(1 + r_k((p + q)^2))^2 + 2g^2 r] \times \left\{ \left( q^2 + p \cdot q \right)(1 + r_k((p + q)^2))(q^2(1 + r_k(q^2))^2 - 2g^2 r) - 4q^2(1 + r_k(q^2))^2g^2 r \right\}.$$  

(61)

Denoting the integral (without the prefactor) as $I_2^\psi(p^2|r)$, the solution to the equation with the correct boundary conditions reads

$$g_2(p^2|r) = \frac{2g^2}{d - 2} I_2^\psi(0|0) - g^2 \int_0^1 d\omega \omega^{-\frac{d}{2}} \left[ I_2^\psi(\omega p^2|\omega r) - I_2^\psi(0|0) \right].$$  

(62)

One can easily check that this reduces to $v'(r) + 2rv''(r)$

---

8 We stress that in this section, $\gamma_2$ denotes the second variation of the effective action with respect to $\phi$, in contrast to the discussion on the $O(N)$ model, where the variation was with respect to $\rho$.

9 Here we again used that by Lorentz invariance, the integral must be a function of $p^2$.

It is worthwhile to discuss the equation for the anomalous dimension, which is related to the canonical normalisation of the scalar field. For this, we take the $p^2$-derivative of (61) and set $p = r = 0$. Using the normalisation condition $\gamma^{(1,0)}(0,0) = 1$ to fix the wave function

$$\gamma_2(p^2|r) \propto (p^2)^{\frac{d}{2} - 1} = (p^2)^{1 - \frac{d}{2}}, \quad \text{as } p \to \infty,$$  

(63)

which is consistent with the scaling expected from the anomalous dimension (58).
renormalisation, we find
\[
\frac{\eta_0}{g^2} = \frac{1}{2} \int d\mu q^2 (q^2 + R_k(q^2))^{3/2} \left( \frac{\delta R_k(q^2)}{q^2 (q^2 + R_k(q^2))^{1/2}} \right) + \frac{3}{2} \int d\mu q^2 (q^2 + R_k(q^2))^{1/2} \left( \frac{\delta R_k(q^2)}{q^2 (q^2 + R_k(q^2))^{1/2}} \right) - \frac{15}{2d} \int d\mu q^2 (q^2 + R_k(q^2))^{1/2} \left( \frac{\delta R_k(q^2)}{q^2 (q^2 + R_k(q^2))^{1/2}} \right).
\]
(64)

While usually this equation fixes the anomalous dimension, in our case it actually fixes the value of the Yukawa coupling \(g\), since \(\eta_0\) is already fixed by (58). This equation will be important in the next subsection to discuss the spectrum of the theory.

The fixed point equations for the higher order correlation functions can be found in a similar way, and solutions have a similar form as (62). We note that for all correlators, including the two-point function, general solutions are only defined up to a multiple of the homogeneous solution, but this is however uniquely fixed by the consistency condition at vanishing momenta and regularity. We thus will not present the higher order correlators explicitly.

### C. Critical exponents

We are now prepared to discuss the critical exponents of the fixed point. First, we will discuss the spectrum that is related to variations of the potential, with vanishing variation of the Yukawa coupling and anomalous dimension. With these conditions, we find
\[
-\theta \delta v(r) = -d\delta v(r) + (d - 2 + \eta_0) r \delta v'(r),
\]
(65)
with the solution
\[
\delta v(r) = cr^{\frac{d-a}{2}}.
\]
(66)
Here, \(c\) is a constant. In fact, these are the same perturbations as for the \(O(N)\) model, with the replacement \(r \to u\), once again reinforcing that for the \(O(N)\) model \(u\) is the more natural variable. In conclusion, this part of the spectrum agrees with the spectrum of the \(O(N)\) model,
\[
\theta = d - 2n, \quad n \in \mathbb{N}.
\]
(67)
There is one additional critical exponent which corresponds to a finite perturbation of the Yukawa coupling. Taking the variation of (57), we get
\[
-\theta \delta g = \frac{1}{2} g \delta \eta_0, \quad \delta \left( \frac{\eta_0}{g^2} \right) = 0,
\]
(69)
so that
\[
\frac{\delta \eta_0}{g} = \frac{2\eta_0 \delta g}{g}. \tag{70}
\]
Combining (68) with (70), we find the final critical exponent
\[
\theta_\eta = -\phi_0 = d - 4. \tag{71}
\]
Notice that due to the linear nature of the fixed point equations, it is clear that one can find solutions to the perturbation equations of all \(n\)-point correlation functions for all critical exponents. For example, for \(\delta g = 0\), we find
\[
(d - \theta - 2) \delta \gamma_2(p^2|r) = 2r \delta \gamma_{2(0)}(p^2|r) + 2p^2 \delta \gamma_{1(0)}(p^2|r).
\]
(72)
The general solution to this equation reads
\[
\delta \gamma_2(p^2|r) = \left( p^2 \right)^{\frac{d-2-a}{2}} c_n \left( \frac{r}{p^2} \right),
\]
(73)
where \(c_n\) is a free function. The set of critical exponents (67) then makes the first factor regular, and requiring that the perturbation is well-defined for all momenta constrains \(c_n\) to be a polynomial of order at most \((n - 1)\). Since \(\delta \gamma_2(0|r)\) is related to \(\delta \nu(r)\), there will be conditions on certain coefficients of this polynomial in relation to the normalisation constant \(c\) of the perturbations of the potential.

It is interesting to point out explicitly that almost all critical exponents of the \(O(N)\) model and the Gross-Neveu model at large \(N\) agree. The only exception is the extra critical exponent \(\theta_\eta\) which is exclusive to the latter model. This has the interesting consequence that to uniquely characterise any universality class, the complete set of critical exponents has to be determined. Any finite or infinite subset is, in general, insufficient to specify the universality class.

### D. Regulator dependence

Once again we discuss the regulator dependence of the solution. The existence of the fixed point and its spectrum are independent of the regulator, as required. By contrast, the situation for the coupling constants is the exact same as for the \(O(N)\) model. Again taking the example of the couplings making up the potential, the Taylor coefficients can be directly read off from the functional solution (60), since the hypergeometric function can be defined in terms of a power series around vanishing argument. In complete similarity to the \(O(N)\) model, the \(n\)-th coupling constant is related to an independent threshold integral, \(\mathcal{I}_n(0)\). We conclude that all remarks made earlier also apply to the Gross-Neveu model, and indicate that indeed generically, fixed point couplings carry no regulator-independent information at all.
IV. SUMMARY

In this work we studied the $O(N)$ model and the Gross-Neveu model within functional renormalisation in dimensions between two and four. In both models, we derived the complete non-perturbative fixed point effective action in the large $N$ limit without specifying the regulator function. That was possible because of the simplification of the flow equation in the large $N$ limit which allows for perfect truncations, where improving the truncation doesn’t alter the renormalisation group equations of previous orders.

For the $O(N)$ model, with regards to a large $N$ expansion, we have found indications that already at next-to-leading order, the perfect truncation property of the large $N$ limit is lost. It thus seems difficult to derive the $1/N$ corrections to critical quantities from the functional renormalisation group analytically. Nevertheless, the form of the effective action at infinite $N$ might serve as an approximate closure of the flow equation at finite $N$, potentially improving the convergence of common approximation schemes.

The Gross-Neveu model is even simpler than the $O(N)$ model in the large $N$ limit, since the flow is exclusively driven by fermionic fluctuations contributing to the running of bosonic operators. This results in linear partial differential equations for the correlation functions. Notably, the spectrum of the Gross-Neveu model and the $O(N)$ model is almost the same.

A key result of the study of these models is that only observables and qualitative aspects are regulator-independent. Concretely, this concerns the existence of the fixed point and its spectrum, and qualitative results like the existence or absence of a non-trivial vacuum expectation value. By contrast, generically fixed point couplings depend on the choice of the regulator even for non-truncated solutions to the flow equation, and thus are essentially arbitrary. This is not entirely surprising, but having concrete calculations without systematic errors supporting this view is reassuring.

To our knowledge, this is the first time that a complete non-perturbative fixed point action was derived. This serves as a showcase to test general properties of the effective action independent of truncation errors. The explicit fixed point correlators represent a useful testing bed for numerical methods that aim to resolve field and momentum dependencies.

As a new technical result, we have put forward an algorithm to derive an operator basis at $n$-th order of the derivative expansion, which has an intuitive graphical representation, and that at the same time provides the partial integration rules to map any operator of a given order into the basis. Its application to higher orders, while cumbersome, is entirely straightforward, and the algorithm can be implemented in a computer code to automatise these computations.

It would be interesting to derive the conformal data [68–70], such as the coefficients of the operator product expansion, for these analytically solvable cases to gain some knowledge about what techniques work best to extract the data from a non-perturbative renormalisation group flow, but we leave this for future work.

ACKNOWLEDGEMENTS

This work was supported by Perimeter Institute for Theoretical Physics. Research at Perimeter Institute is supported in part by the Government of Canada through the Department of Innovation, Science and Industry Canada and by the Province of Ontario through the Ministry of Colleges and Universities.
[47] C. Luperini and P. Rossi, Three loop Beta function(s) and effective potential in the Gross-Neveu model, Annals Phys. 212, 371 (1991).
[48] B. Rosenstein, H.-L. Yu, and A. Kovner, Critical exponents of new universality classes, Phys. Lett. B314, 381 (1993).
[49] L. Karkkainen, R. Lacaze, P. Lacock, and B. Petersson, Critical behavior of the 3-d Gross-Neveu and Higgs-Yukawa models, Nucl.Phys. B415, 781 (1994), arXiv:hep-lat/9310020 [hep-lat].
[50] J. A. Gracey, Computation of Beta-prime (g(c)) at O(1/N**2) in the O(N) Gross-Neveu model in arbitrary dimensions, Int. J. Mod. Phys. A9, 567 (1994), arXiv:hep-th/9306106 [hep-th].
[51] J. A. Gracey, Computation of critical exponent eta at O(1/N**3) in the four Fermi model in arbitrary dimensions, Int.J.Mod.Phys. A9, 727 (1994), arXiv:hep-th/9306107 [hep-th].
[52] J. Braun, H. Gies, and D. D. Scherer, Asymptotic safety: a simple example, Phys.Rev. D83, 085012 (2011), arXiv:1011.1456 [hep-th].
[53] L. Janssen and I. F. Herbut, Antiferromagnetic critical point on graphene’s honeycomb lattice: A functional renormalization group approach, Phys. Rev. B89, 205403 (2014), arXiv:1402.6277 [cond-mat.str-el].
[54] J. M. Pawlowski and F. Rennecke, Higher order quarkmesonic scattering processes and the phase structure of QCD, Phys. Rev. D90, 076002 (2014), arXiv:1403.1179 [hep-ph].
[55] J. Braun, L. Fister, J. M. Pawlowski, and F. Rennecke, From Quarks and Gluons to Hadrons: Chiral Symmetry Breaking in Dynamical QCD, Phys. Rev. D94, 034016 (2016), arXiv:1412.1045 [hep-ph].
[56] L. Wang, P. Corboz, and M. Troyer, Fermionic Quantum Critical Point of Spinless Fermions on a Honeycomb Lattice, New J. Phys. 16, 103008 (2014), arXiv:1407.0029 [cond-mat.str-el].
[57] G. P. Vacca and L. Zambelli, Multimeson Yukawa interactions at criticality, Phys. Rev. D91, 125003 (2015), arXiv:1503.09136 [hep-th].
[58] J. A. Gracey, T. Luthe, and Y. Schroder, Four loop renormalization of the Gross-Neveu model, Phys. Rev. D94, 125028 (2016), arXiv:1609.05071 [hep-th].
[59] J. Borchardt, H. Gies, and R. Sondenheimer, Global flow of the Higgs potential in a Yukawa model, Eur. Phys. J. C76, 472 (2016), arXiv:1603.05861 [hep-ph].
[60] L. N. Mihaila, N. Zerf, B. Ihrig, I. F. Herbut, and M. M. Scherer, Gross-Neveu-Yukawa model at three loops and Ising critical behavior of Dirac systems, Phys. Rev. D96, 165133 (2017), arXiv:1703.08801 [cond-mat.str-el].
[61] L. Iliesiu, F. Kos, D. Poland, S. S. Pufu, and D. Simmons-Duffin, Bootstrapping 3D Fermions with Global Symmetries, JHEP 01, 036, arXiv:1705.03484 [hep-th].
[62] N. Zerf, L. N. Mihaila, P. Marquard, I. F. Herbut, and M. M. Scherer, Four-loop critical exponents for the Gross-Neveu-Yukawa models, Phys. Rev. D96, 096010 (2017), arXiv:1709.05057 [hep-th].
[63] E. Huffman and S. Chandrasekharan, Fermion bag approach to Hamiltonian lattice field theories in continuous time, Phys. Rev. D96, 114502 (2017), arXiv:1709.03578 [hep-lat].
[64] B. Ihrig, L. N. Mihaila, and M. M. Scherer, Critical behavior of Dirac fermions from perturbative renormalization, Phys. Rev. B98, 125109 (2018), arXiv:1806.04977 [cond-mat.str-el].
[65] L. Dabelow, H. Gies, and B. Knorr, Momentum dependence of quantum critical Dirac systems, Phys. Rev. D 99, 125019 (2019), arXiv:1903.07388 [hep-th].
[66] J. Lenz, L. Pannullo, M. Wagner, B. Wellegehausen, and A. Wipf, Inhomogeneous phases in the Gross-Neveu model in 1+1 dimensions at finite number of flavors, Phys. Rev. D 101, 094512 (2020), arXiv:2004.00295 [hep-lat].
[67] J. J. Lenz, L. Pannullo, M. Wagner, B. Wellegehausen, and A. Wipf, Baryons in the Gross-Neveu model in 1+1 dimensions at finite number of flavors, arXiv:2007.08382 [hep-lat] (2020).
[68] K. G. Wilson, Nonlagrangian models of current algebra, Phys. Rev. 179, 1499 (1969).
[69] A. Codello, M. Safari, G. P. Vacca, and O. Zanusso, Functional perturbative RG and CFT data in the e-expansion, Eur. Phys. J. C78, 30 (2018), arXiv:1705.05558 [hep-th].
[70] G. Pagani and H. Sonoda, Operator product expansion coefficients in the exact renormalization group formalism, Phys. Rev. D 101, 105007 (2020), arXiv:2001.07015 [hep-th].