Using Covariant Polarisation Sums in QCD

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Abstract Covariant gauges lead to spurious, non-physical polarisation states of gauge bosons. In QED, the use of the Feynman gauge, $\sum_{\lambda} \varepsilon_{\mu}^{(\lambda)} \varepsilon_{\nu}^{(\lambda)*} = -\eta_{\mu\nu}$, is justified by the Ward identity which ensures that the contributions of non-physical polarisation states cancel in physical observables. In contrast, the same replacement can be applied only to a single external gauge boson in squared amplitudes of non-abelian gauge theories like QCD. In general, the use of this replacement requires to include external Faddeev-Popov ghosts. We present a pedagogical derivation of these ghost contributions applying the optical theorem and the Cutkosky cutting rules. We find that the resulting cross terms $A(c_1, \bar{c}_1; \ldots) A(\bar{c}_1, c_1; \ldots)^*$ between ghost amplitudes cannot be transformed into $(-1)^{n/2} |A(c_1, \bar{c}_1; \ldots)|^2$ in the case of more than two ghosts. Thus the Feynman rule stated in the literature holds only for two external ghosts, while it is in general incorrect.

1 Introduction

The traditional way to derive physical observables like decay widths and cross sections from Feynman amplitudes $A$ is to calculate the squared amplitude $|A|^2$ and then to sum over the final and to average over the initial spin and polarisation states of the external particles. In the case of fermions, one obtains for each sum a projection operator on the positive or negative energy solutions of the Dirac equations, and a fermion line is thereby converted into a trace of spinors which is relatively easy to calculate. For particles with spin $s \geq 1$, the summation over polarisation states is complicated by the fact there is a mismatch between the physical number of polarisation states and field variables. In particular, a massless spin-one field $A_\mu$ has four components but only two physical polarisation states. Excluding the non-physical states requires to choose a non-covariant form of the polarisation sum, what in turn results in lengthy intermediate expressions for the squared amplitude $|A|^2$. For instance, the squared amplitude of the four-gluon amplitude in QCD contains 228 420 terms before simplifications reduce them to four.
In the case of photons, one can avoid this complication by choosing for the sum over polarisation states a covariant gauge as, e.g., the Feynman gauge, \( \sum_{\lambda} \varepsilon^{(\lambda)} \varepsilon^{(\lambda)^*} = -\eta_{\mu\nu} \). Although both time-like and longitudinal photons are included in this polarisation sum, the Feynman gauge leads in QED to the same results as a physical gauge where only transversely polarised photons propagate: The photon couples to a conserved current, and thus the resulting Ward identity, \( k_{\mu} A^{\mu} = 0 \) for an external photon with four-momentum \( k_{\mu} \), ensures that the contributions of the two redundant polarisation states cancel in physical observables.

In the non-abelian case, the gauge boson couples to a current which is the sum of the conserved fermion current and a piece induced by the self-coupling of the gauge field. This current is neither gauge-invariant nor conserved. Therefore the simple QED Ward identity does not hold. In particular, the contribution of the two redundant polarisation states do not cancel any more, if covariant polarisation sums are used for more than one external gauge boson. If one insists to use the simpler covariant gauges, one has therefore to add Faddeev-Popov ghosts. In the case of internal gluons, this approach is the standard procedure applied in almost all loop calculations. The use of covariant gauges and Faddeev-Popov ghosts for external gauge bosons is, on the contrary, rarely employed. An exception is the work of Cutler and Sivers [1] who performed the first calculation of four-gluon scattering in QCD. Their result differs however from the now accepted one, first found in the same year by Combridge, Kripfganz and Ranft using a different method [2]. Employing \(-\eta^{\mu\nu}\) as polarisation sum method in QCD is also described briefly in Nachtmann’s textbook [3]. There it is stated that the covariant polarisation sum can be used, given that we also add squared ghost amplitudes, modified by factors of \((-1)^n\) for 2\(n\) external Faddeev-Popov ghosts. All possible amplitudes replacing some even number of external gluons by ghosts should be included.

In this work, we examine the use of covariant gauges for external gauge bosons and in particular the Feynman rules for external ghosts. The possibility to use covariant gauges also for the polarisation sums of external gluons adding also amplitudes with external ghosts is guaranteed by the optical theorem or, equivalently, by the invariance of the amplitude under gauge transformations of internal lines. Applying the cutting rules of Cutkosky and Landau, we can connect in turn the properties of internal lines to those of external states. We first introduce these cutting rules in a QED example in Section 2. This example contains most of the concepts needed to understand also the QCD case. The additional subtleties arising in QCD are treated in Section 3. Finally, we apply the method to four-gluon scattering in Section 4. This example demonstrates that, when more than two polarisation sums are replaced by \(-\eta^{\mu\nu}\), the ghost contributions can no longer be expressed as squared amplitudes times a sign factor. Instead the cross terms between ghost amplitudes and their \(c \leftrightarrow \bar{c}\) counterpart have to be used. This implies that the generalisation of the Feynman rule for external ghosts derived from the \(qq \rightarrow gg\) amplitude using squared ghost amplitudes is incorrect. Finally, we summarize in Section 5.
2 Cutting Rules in a QED Example

A well-known consequence of the unitarity of the S-matrix is the optical theorem \[4, 5\],

\[
2 \text{Im} \{ A(i \to i) \} = \sum_{N=2}^{\infty} \sum_{\xi_1 \ldots \xi_N} \int |A(i \to \xi_1 \ldots \xi_N)|^2 \, d\Phi^{(N)}. \tag{1}
\]

It relates the imaginary part of the forward scattering amplitude \(A(i \to i)\) to the total cross section of the initial state scattering into any final state. The sum over \(N\) has to be truncated at a finite \(N\) corresponding to the considered order in perturbation theory, while the sum over \(\xi_i\) includes all quantum numbers describing the final state.

We consider the optical theorem in the specific example of \(e^- e^+ \to e^- e^+\) scattering at order \(\alpha^2\). The imaginary part of the amplitude for this process is the sum of the imaginary parts of the contributing Feynman diagrams. We will consider only the one-loop diagram shown in Figure 1 in detail. For later convenience, we choose to represent the diagram in terms of two off-shell sub-amplitudes as follows,

\[
i A^{(\gamma \gamma)} = \int \frac{d^4k}{(2\pi)^4} \, i A^{(u)}_{\mu \nu} (e^- (p_1) e^+ (p_2) \to \gamma (p_1 + p_2 - k) \gamma (k)) \times \]

\[
\times i A^{(u)}_{\nu' \mu'} (\gamma (k) \gamma (p_1 + p_2 - k) \to e^+ (p_3) e^- (p_4)) \times N^{\mu \nu} (k) \times N^{\nu' \mu'} \frac{(p_1 + p_2 - k)^2 + i\epsilon}{(p_1 + p_2 - k)^2 + i\epsilon}.
\]

Here, \(i A^{(\gamma \gamma)}\) corresponds to the diagram in Figure 1, while \(i A^{(u)}_{\mu \nu}\) represents the \(u\)-channel diagram for either \(e^- e^+ \to \gamma \gamma\) or \(\gamma \gamma \to e^- e^+\) scattering with the external polarisation vectors removed. The photon momenta \(k\) and \(p_1 + p_2 - k\) in the latter two amplitudes are off-shell. Finally, \(N^{\mu \nu}\) denotes the numerator of a photon propagator. After seeing how the imaginary part can be computed for this diagram, it should be clear how the same method can be applied to the remaining one-loop diagrams for \(e^- e^+ \to e^- e^+\).

The imaginary part of a Feynman diagram is proportional to the discontinuity across its \(s\)-channel branch cut \([4, 5]\). Such discontinuities can be computed elegantly by using the cutting rules of Cutkosky and Landau \([6, 7]\). We illustrate how the rules work in our present example; the diagram of Figure 1. First we should find all sets of internal lines such that the corresponding momenta \(q_i\) satisfy \(s = (\sum q_i)^2\). Additionally, every pair of lines \(\{q_i, q_j\}\) must share at least one
common closed loop. In our example we see that the set of internal lines with momenta \( k \) and \( p_1 + p_2 - k \) is the only set that works. Next, the Cutkosky rules tell us that to compute the discontinuity we should perform the replacement

\[
\frac{1}{q_1^2 - M_1^2} \rightarrow 2\pi i \delta^{(+)}(q_1^2 - M_1^2),
\]

(3)
in each of the propagators corresponding to our chosen internal lines. Here \( \delta^{(+)} \) means we take only the principal root of \( q_1^2 = M_1^2 \). From our expression (2), now specialized to forward scattering, we obtain then

\[
2i \text{Im} \{ A^{(\gamma\gamma)} \} = \text{Disc}_{s-channel} \{ A^{(\gamma\gamma)} \} = i \int \frac{d^4 k}{(2\pi)^2} \delta^{(+)}(k^2) \delta^{(+)}((p_1 + p_2 - k)^2) \times A^{(u)}_{\nu\mu}(e^-(p_1)e^+(p_2) \rightarrow \gamma(p_1 + p_2 - k)\gamma(k)) \times A^{(u)}_{\nu'\mu'}(\gamma(k)\gamma(p_1 + p_2 - k) \rightarrow e^+(p_2)e^-(p_1)) \times N^{\nu\mu'}(k) N^{\nu'\mu}(p_1 + p_2 - k).
\]

The way we have chosen to express \( A^{(\gamma\gamma)} \) makes it clear that its form is already quite close to the one of the optical theorem. The \( \delta^{(+)}(k^2) \) sets the external photon momentum \( k \) on-shell in the two sub-amplitudes. The second \( \delta^{(+)}((p_1 + p_2 - k)^2) \) sets the other external photon on-shell. The loop integral is transformed by the delta functions into the two-particle phase-space integral. To be precise,

\[
\int \frac{d^4 k}{(2\pi)^2} \delta^{(+)}(k^2) \delta^{(+)}((p_1 + p_2 - k)^2) = \int \frac{d^3 k}{(2\pi)^3 2\omega_k} \frac{d^3 k'}{(2\pi)^3 2\omega_{k'}} (2\pi)^4 \delta^{(4)}(p_1 + p_2 - k - k'),
\]

where the right-hand side is twice the phase-space integral of two photons, since they are identical particles. This factor of two means that the single loop diagram in Figure 1 gives rise to the squares of both the \( u \)- and \( t \)-channel tree diagrams for \( e^+e^- \rightarrow \gamma\gamma \). This holds since we can turn the \( u \)-channel sub-amplitudes above into \( t \)-channel sub-amplitudes by interchanging \( k \) and \( k' \), which does not change the value of the integral.

Although close, we have not quite arrived at the squared amplitude expected from the optical theorem yet. Firstly, we need to realize that

\[
A^{(u)}_{\nu\mu}(\gamma(k)\gamma(k') \rightarrow e^+(p_2)e^-(p_1)) = A^{(u)}_{\nu'\mu'}(e^-(p_1)e^+(p_2) \rightarrow \gamma(k')\gamma(k))^*.
\]

(5)

This follows from the property that complex conjugation of the amplitude turns an incoming physical particle into an outgoing one and vice versa. Technically, this is facilitated for a Dirac chain by relations like \((\bar{v}\gamma^\mu_1...\gamma^\mu_n)\bar{v}^* = \bar{v}\gamma^\mu_1...\gamma^\mu_n\bar{v}\). Using this relation, we obtain

\[
2 \text{Im} \{ A^{(\gamma\gamma)} \} = 2 \int d\phi^{(2)} A^{(u)}_{\mu\nu}(e^-(p_1)e^+(p_2) \rightarrow \gamma(k')\gamma(k)) \times N^{\nu\mu'}(k) N^{\nu'\mu}(k') A^{(u)}_{\nu'\mu'}(e^-(p_1)e^+(p_2) \rightarrow \gamma(k')\gamma(k))^*.
\]

(6)
The right hand side of this equation is the sum of the squared \( u \)- and \( t \)-channel diagrams for \( e^+e^- \rightarrow \gamma\gamma \), integrated over the two particle phase space and with the sum over polarisations replaced by \( N^{\nu\mu'}(k) \). Had we repeated the analysis for
the one-loop diagram similar to Figure 1 but with the photon lines crossed, we would have found the cross term between the $u$- and $t$-channel diagrams, again with the sum over polarisations replaced by $N_{\mu\nu}(k)$. Thus the optical theorem (1) connects the numerator of the photon propagator with the sum over polarisations of external photons. In particular, the optical theorem implies that the Feynman propagator $N_{\mu\nu}(k) = -\eta_{\mu\nu}$ can be used to sum over polarisations.

The sum over transverse, physical polarisations neither equals $-\eta_{\mu\nu}$ nor it is gauge dependent like the numerator of the propagator $N_{\mu\nu}(k)$. The optical theorem then requires the amplitudes to have such a form that any non-transverse part of $N_{\mu\nu}(k)$ is cancelled. This is equivalent to the freedom that we may use the propagator in any gauge in internal lines. The connection between the covariant and physical polarisation sum is

$$-\eta_{\mu\nu} = \sum_{\lambda} \varepsilon_{\lambda}^{\mu} \varepsilon_{\lambda}^{\nu} \frac{k^\mu \tilde{k}^\nu + \tilde{k}^\mu k^\nu}{2\omega k},$$

where $\tilde{k}^\mu = (\omega_k, -k)$ and $\lambda$ are the two transverse polarisation states [5]. Thus the Ward identity $k^\mu A_\mu = 0$ is the required property, ensuring that only the transverse part of $N_{\mu\nu}(k)$ contributes.

3 Extension to QCD

We now turn to the equivalent QCD process $\bar{q}q \rightarrow \bar{q}q$ in order to see how the discussion of the previous section has to be changed in the case of a non-abelian gauge theory. The QED-like diagrams of the previous section contribute now as before, being only altered by the added SU(3) colour structure. Since the kinematics is the same, twice the imaginary part of the QCD equivalent of Figure 1 again gives the squares of the $t$- and $u$-channel diagrams, now for $\bar{q}q \rightarrow gg$. However, the $\bar{q}q \rightarrow gg$ process has an additional diagram compared to its QED counterpart: the $s$-channel diagram. In correspondence with the optical theorem we have the diagram in Figure 2, whose imaginary part gives the square of that $s$-channel diagram. Additionally, we have two one-loop diagrams with a single three-gluon vertex that give the cross terms between the $s$-channel and the $t$- and $u$-channel diagrams.

Fig. 2 The diagram $A^{(g-loop)}$ contributing to $\bar{q}q \rightarrow \bar{q}q$ at the one-loop level.

Repeating the argument of the previous section to find the imaginary part of the diagram in Figure 2 we are left with a difficulty. Again we should be able to use any gauge for the internal gluon propagator numerators $i\delta^{ab}N_{\mu\nu}(k)$, but now the QED Ward identity does not hold. The non-transverse part of $N_{\mu\nu}(k)$ gives a non-zero contribution when contracted into the two sub-amplitudes of the
This non-transverse contribution is not present on the right-hand side of the optical theorem (1), so we have an apparent contradiction.

We consider this non-transverse contribution in detail in the Feynman gauge, following the same steps as in the QED example. Inserting the expression (7) for $N_{\mu\nu} = -\eta_{\mu\nu}$, we obtain for the imaginary part of the diagram in Figure 2,

$$2\text{Im}\left\{A^{(g\text{-loop})}\right\} = \sum_{a,a'} \sum_{\lambda,\lambda'} \int d\Phi^{(2)} \left| A^{(s\text{-channel})}(\bar{q}q \to gg)\right|^2$$

$$+ 2 \int d\Phi^{(2)} \left[ g_s^2 \frac{i f_{baa' T_{ij}^b}}{(p_1 + p_2)^2} \bar{v}(2)\bar{u}(1) \right] \left[ g_s^2 \frac{i f_{baa' T_{ji}^b}}{(p_1 + p_2)^2} \bar{v}(2)(-k')\bar{u}(1) \right].$$

Here, $g_s$ is the strong coupling, $\lambda, \lambda'$ are the physical polarisation states of the two final state gluons, $a, a'$ are their colour states and $k, k'$ are their momenta. Now consider the following amplitudes involving external Faddeev-Popov ghosts, which we denote by $c$ and $\bar{c}$,

$$A(q(p_1)\bar{q}(p_2) \to c(k)\bar{c}(k')) = g_s^2 \frac{i f_{baa' T_{ij}^b}}{(p_1 + p_2)^2} \bar{v}(2)\bar{u}(1),$$

(9)

$$A(c(k)\bar{c}(k') \to q(p_1)\bar{q}(p_2)) = g_s^2 \frac{i f_{baa' T_{ji}^b}}{(p_1 + p_2)^2} \bar{v}(2)(-k')\bar{u}(1).$$

(10)

Remembering also that ghosts are fermions and thus the ghost loop contains an additional factor of $-1$, we see that the non-transverse contribution on the second line of (8) is exactly minus the cut of the diagram in Figure 3. The factor of two accounts for the fact that, unlike the final state gluons, $c$ and $\bar{c}$ are not identical.

![Fig. 3 The diagram with Faddeev-Popov ghosts contributing to $\bar{q}q \to \bar{q}q$ at the one-loop level.](image)

As expected, the optical theorem is saved by the inclusion of Faddeev-Popov ghosts. Thus covariant propagator numerators can, like in the previous section, be used to sum over polarisation, if we also include ghosts. For the present case of $q(p_1)\bar{q}(p_2) \to g(k)g(k')$ the relevant ghost term is

$$-2A(q(p_1)\bar{q}(p_2) \to c(k)\bar{c}(k'))A(c(k)\bar{c}(k') \to q(p_1)\bar{q}(p_2))$$

$$+ 2A(q(p_1)\bar{q}(p_2) \to c(k)\bar{c}(k'))A(q(p_1)\bar{q}(p_2) \to \bar{c}(k)c(k'))^*,$$

(11)

which follows from the combination of (8) with (9) and (10). Note that unlike before we no longer get a square, but rather the cross term between a ghost amplitude and its $c \leftrightarrow \bar{c}$ counterpart. For physical particles complex conjugation exchanges incoming and outgoing particles, like we saw for Dirac fermions in the previous section. While complex conjugating equation (10) gives the correct colour structure for exchanging incoming and outgoing particles due to $T_{ij}^{a*} = T_{ji}^a$, the
−k′ from the ghost vertex is unchanged. Reinterpreting the ghosts as outgoing we then get an additional sign and the final state is ¯c(k)c(k′), not the c(k)c(k′) that would give a square.

This example with only two cut gluons allows for a peculiar, further transformation of the result in Eq. (11), since the ghost momenta sit next to an on-shell spinor: Using the Dirac equation to transform ¯v(2)(−k′)u(1) → ¯v(2)ku(1) in (10), we can write the ghost term (11) as a square with a sign −1 relative to our term. Such a transformation is clearly not possible in general, as we will also see in the next section in a concrete example.

4 Cutting More Than Two Gluons

Now that we understand how a covariant propagator numerator can be used to sum over polarisations, we will apply the method to a final example. We will illustrate that there is no need to first consider the loop amplitudes in order to write down the relevant ghost terms. By choosing the four-gluon amplitude we will also be able to see how the discussion of the previous section generalizes to more than two cut gluons. To gradually increase the complexity we will start by using −ημν as the polarisation sum for three of the gluons, before moving on to all four. We adopt a notation where g stands for a gluon which we sum over purely transverse polarisations, while ˜g is a gluon where we replace the sum over polarisations by −ημν. Keeping one gluon transverse we then find

\[ \sum_{\text{pol.}} \text{tr} |A(g ˜g \rightarrow ˜g ˜g)| = \sum_{\text{pol.}} \text{tr} |A(gg \rightarrow gg)|^2 \]

\[ + 324g_4^4 - 288g_4^4 \frac{st}{u^2} - 288g_4^4 \frac{su}{t^2} - 432g_4^4 \frac{tu}{s^2} , \]

where s, t and u denote the usual Mandelstam variables. There are now three possible positions for a ghost ¯c, and having fixed that there are two remaining positions for the second external ghost. In other words there are 3 · 2 = 6 possible ghost amplitudes. Of the six corresponding ghost cross terms only three are unique since the remaining three are related by a complex conjugation. These three ghost cross terms are

\[ \sum_{\text{pol.}} \text{tr} A(g ˜g \rightarrow ˜c)A(g ˜g \rightarrow ˜c)^* = -90g_4^4 + 216g_4^4 \frac{tu}{s^2} , \] (13)

\[ \sum_{\text{pol.}} \text{tr} A(gc \rightarrow ˜g)A(g ˜c \rightarrow ˜g)^* = -18g_4^4 + 36g_4^4 \frac{t}{s} + 144g_4^4 \frac{su}{t^2} , \] (14)

\[ \sum_{\text{pol.}} \text{tr} A(gc \rightarrow ˜c)A(gc \rightarrow ˜c)^* = -54g_4^4 - 36g_4^4 \frac{t}{s} + 144g_4^4 \frac{st}{u^2} . \] (15)

A FORM file for computing the three terms required for using the Feynman gauge on all external lines in the four gluon amplitude can be found at https://gitlab.com/magnunm/yang-mills-scattering-amplitudes/-/blob/master/gluon-gluon-scattering/4-feynman-gauge.frm. We have also written a Mathematica program that illustrates in a pedagogical manner the ideas presented in the text with concrete calculations in the four gluon case, it can be found at https://gitlab.com/magnunm/yang-mills-scattering-amplitudes/-/blob/master/gluon-gluon-scattering/gluon-gluon-scattering-ghost-cancellation.wl
Summing over all the six ghost cross terms gives minus the second line of (12), as expected from the optical theorem.

Computing next the square

$$\sum_{\text{pol.}} \text{tr} |A(g\bar{g} \rightarrow c\bar{c})|^2 = 108g_4^4 + 36g_4^4 \frac{1}{s^2} - 216g_4^4 \frac{1}{s^2},$$

(16)

and comparing it to (13), we see that the two expressions no longer have a simple relationship: Writing the ghost terms arising in the optical theorem as squares is not possible when one goes beyond two cut gluons.

Finally, we substitute $-\eta^{\mu\nu}$ for the sum over polarisations for all four external gluons. We must then also include all amplitudes with four external ghosts. There are six such amplitudes and six corresponding cross terms. As before only three need to be computed, the remaining three being complex conjugates of the first ones. Now there are twelve amplitudes with two external ghosts. The six amplitudes in (13)–(15) plus an additional six where there is a ghost at position 1. The last six are however just given by relabellings of momenta in the first. The full equality implied by unitarity is then

$$\sum_{\text{pol.}} \text{tr} |A(gg \rightarrow gg)|^2 = \sum_{\text{pol.}} \text{tr} |A(\tilde{g}\tilde{g} \rightarrow \tilde{g}\tilde{g})|^2$$

(17)

$$+ 4 \sum_{\text{pol.}} \text{tr} A(\tilde{g}\tilde{g} \rightarrow c\bar{c})A(\tilde{g}\tilde{g} \rightarrow c\bar{c})^* + 4 \sum_{\text{pol.}} \text{tr} A(\tilde{g}c \rightarrow \tilde{g}c)A(\tilde{g}c \rightarrow \tilde{g}c)^*$$

$$+ 4 \sum_{\text{pol.}} \text{tr} A(\tilde{g}c \rightarrow c\bar{g})A(\tilde{g}c \rightarrow c\bar{g})^* + 2 \sum_{\text{pol.}} \text{tr} A(c\bar{c} \rightarrow c\bar{c})A(c\bar{c} \rightarrow c\bar{c})^*$$

$$+ 2 \sum_{\text{pol.}} \text{tr} A(c\bar{c} \rightarrow c\bar{c})A(c\bar{c} \rightarrow c\bar{c})^* + 2 \sum_{\text{pol.}} \text{tr} A(c\bar{c} \rightarrow c\bar{c})A(c\bar{c} \rightarrow c\bar{c})^*.$$

Considering that crossing symmetry holds for these amplitudes, we can get all the ghost terms above by crossing the expressions for $A(\tilde{g}\tilde{g} \rightarrow c\bar{c})A(\tilde{g}\tilde{g} \rightarrow c\bar{c})^*$ and $A(c\bar{c} \rightarrow c\bar{c})A(c\bar{c} \rightarrow c\bar{c})^*$ into the appropriate channels. The amplitudes with four external ghosts arise from loop amplitudes with two ghost loops. Since they also involve four ghosts being reinterpreted as outgoing/incoming, the signs cancel out as with two external ghosts. This clearly continues to any number of external ghosts.

5 Summary

We have reviewed the use of covariant gauges for the polarisation states of external gauge bosons. While in QED the application of such gauges is justified by the Ward identity, $k_\mu A^\mu = 0$, the use of these gauges in non-abelian gauge theories like QCD requires to include external Faddeev-Popov ghosts. Since covariant gauges and Faddeev-Popov ghosts for external gauge boson have been rarely employed in QCD, it seems that the Feynman rules for external ghosts have not been previously checked beyond the simplest case of two external ghosts.

We have used the optical theorem and the cutting rules of Cutkosky and Landau to show how the non-physical contributions due to non-transverse polarisation states present in covariant gauges are cancelled by the Faddeev-Popov ghosts. This
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cancellation requires to take into account all possible amplitudes replacing an even number of external gluons by ghosts. These amplitudes should be multiplied by the complex conjugate of their $c \leftrightarrow \bar{c}$ counterparts; they have to be summed with no additional sign. Only in the simplest case of two external ghost is the transformation $A(c, \bar{c})\bar{A}(\bar{c}, c_1)^* = -|A(c, \bar{c})|^2$ possible. In the past, this peculiar case has been probably the basis to argue that the general Feynman rule for external ghosts is $(−1)^{n/2}|A(c_1, \bar{c}_1; \ldots)|^2$, as it is given, e.g., in the textbook by Nachtmann [3]. Using this wrong rule one reproduces also the incorrect result for $gg \rightarrow gg$ scattering obtained in Ref. [1].

In this work, we have used the unitarity of the S-matrix as our starting point. This allows one to derive the form of the ghost terms at any order of perturbation theory. The existence of Slavnov-Taylor identities that underlie the cancellation of unphysical degrees of freedom is in this approach a consequence of the assumption of unitarity. Alternatively, we could have started by deriving these identities. Then the steps used here to find the ghost terms may be generalised into ’t Hooft’s method of proving the unitarity of the S-matrix [8]. Mathematically more sophisticated are proofs based on the BRST-formalism as, e.g., the one presented by Kugo and Ojima [9]. The more elementary diagrammatic approach followed here has the advantage with respect to Ref. [9] to expose explicitly the Feynman rules required for a cancellation of external ghost particles.

Finally, we should warn the reader that there is a good reason why the use of covariant gauges combined with ghosts for external gluons has been unpopular: Firstly, the increase in the number of diagrams is not really compensated by the simplicity of the polarisation sum in the Feynman gauge. More importantly, the advent of new powerful methods to calculate helicity amplitudes involving gauge bosons in the last two decades has made this approach obsolete for most practical purposes.

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