Amplification of Primordial Gravitational Waves by a Geometrically Driven non-canonical Reheating Era

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In order to describe inflation in general relativity, scalar fields must inevitably be used, with all the setbacks of that description. On the other hand, $f(R)$ gravity and other modified gravity theories seem to provide a unified description of early and late-time dynamics without resorting to scalar or phantom theories. The question is, can modified gravity affect directly the mysterious radiation domination era? Addressing this question is the focus in this work, and we shall consider the case for which in the early stages of the radiation domination era, namely during the reheating era, the background equation of state parameter is different from $w = 1/3$. As we show, in the context of $f(R)$ gravity, an abnormal reheating era can affect the primordial gravitational wave energy spectrum today. Since future interferometers will exactly probe this era, which consists of subhorizon modes that reentered the horizon during the early stages of the radiation domination era, the focus in this work is how a short abnormal reheating era that deviates from the standard perfect fluid pattern with $w \neq 1/3$, and generated by higher order curvature terms, can affect the primordial gravitational wave energy spectrum. Using a WKB approach, we calculate the effect of an $f(R)$ gravity generated abnormal reheating era, and as we show the primordial gravitational wave spectrum is significantly amplified, a result which is in contrast to the general relativistic case, where the effect is minor.

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I. INTRODUCTION

To date general relativity (GR) seems to be a successful description of the Universe at astrophysical levels, and only a few events seem to deviate from the standard GR description at an astrophysical levels \cite{1}, still this deviation is debatable. It however seems to be difficult for GR to consistently describe large scales of the Universe at a cosmological level, since dark energy cannot be described in a rigid and non-problematical way by GR. Specifically, a dark energy era with phantom divide crossing in GR would require the presence of phantom scalar fields, a feature not so desirable in theoretical physics models. On the other hand, the inflationary era \cite{2,3,4,5} can be described in the context of GR, but again the presence of a scalar field which drives the evolution is required. Although scalar fields are in general expected to be present in the low-energy regime of some fundamental underlying string theory, these are the string moduli, the description of inflation with some fundamental scalar other than the Higgs field has some shortcomings. Indeed, the inflaton particle properties, like its mass, its coupling to fundamental particles, its potential energy, make difficult the identification of the inflaton in terms of fundamental particle physics models. In some sense the inflaton is introduced in an ad hoc way, and its properties seem to be the result of a suitable fine tuning of its parameters relevant to inflation.

A natural extension of GR which overcomes the scalar field problems we mentioned, is offered by modified gravity in its various forms \cite{6,7,8,9,10,11}. The extension of GR is a natural choice, since the fundamental curvature term is present in the Einstein-Hilbert Lagrangian, thus it is possible that higher order curvature terms might be present, originating from the underlying fundamental quantum gravity theory. With modified gravity it is possible to describe both the inflationary era and the dark energy era, without the introduction of scalar fields, see for example \cite{12,13,22} for works in the context of the most fundamental modified gravity theory, namely $f(R)$ gravity.

While in most cases the focus in modified gravity theories is on inflation and dark energy and how these two eras can be realized by modified gravity, the question is how does modified gravity affect the intermediate eras, namely the radiation and the matter domination era. With regard to the latter, post-recombination we know that the Universe
should mimic the Λ-Cold-Dark-Matter (ΛCDM) model, however before that and specifically during the radiation domination era, the plot thickens. In principle, the radiation domination era, which starts with the reheating era, is quite mysterious. We have no clue on what actually happened in the Universe during this evolutionary patch of it. We have theoretical hints and theoretical requirements, but this era is quite mysterious and the temperature is quite large of the order \( T > 150 \text{GeV} \). Hence the theoretical question is, can higher curvature correction terms affect in some way this era? Can modified gravity have a direct imprint on some stages of this evolutionary era? Usually it is assumed that \( w = 1/3 \) during radiation era, which is a not however a necessary assumption, just a theoretical expectation. In fact, for this era, the background EoS parameter \( w \) can be found in the range \( 0 < w < 1/3 \) \cite{26}. Hence, an abnormal reheating era with \( w \neq 1/3 \) can have implications on the present day primordial gravitational waves energy spectrum, if the relevant modes reentered the horizon during this abnormal reheating era. In the context of GR however the modification of the gravitational wave energy spectrum is minor, so the question is whether this pattern is repeated in modified gravity. Specifically, if a modified gravity generates a short abnormal reheating era, is the gravitational wave energy spectrum significantly affected? The answer lies in the affirmative for \( f(R) \) gravity as we will show in this paper. The motivation for modifying the reheating era is two fold: Firstly it is theoretically more likely for the higher order curvature terms to operate during this mysterious era, and secondly, it is the era where the modes that will be probed by future high frequency experiments reenter the horizon after inflation. Thus if an abnormal reheating era occurs, this will be detectable and perhaps verified.

To be specific, in order to discriminate which model drives a possible future observation, several experiments must be combined. Since inflation plays a prominent role in future experiments, it will be tested by both Cosmic Microwave Background (CMB) experiments \cite{27, 28} and by high frequency interferometers \cite{29–36}. These modes will verify either the existence of a \( B \)-mode pattern in the CMB temperature fluctuations \cite{37}, or they will verify the existence of a stochastic primordial gravitational wave background. The CMB experiments will hopefully measure the tensor-to-scalar ratio and possibly will determine the tensor spectral index. If the tensor spectral index is red-tilted and at the same time a stochastic cosmological signal is observed by the gravitational waves experiments, the theories that can describe such a physical situation are quite narrowed down. In fact, single scalar field theories and their Jordan frame counterpart theories, namely \( f(R) \) gravities, will be too difficult to describe the physics, if not impossible, assuming a standard reheating era. However, with the present paper we want to stress the fact that if an abnormal reheating era is generated by \( f(R) \) gravity, the amplification of the primordial gravitational wave energy spectrum is significant enough to be detected by most of the future experiments, even if the abnormal reheating era lasts for a short period of time after the end of inflation. Hence, the plot seems to thicken in modified gravity cosmology, if such alternative reheating-radiation domination era scenarios are taken into account. In this work we shall describe the different evolutionary patches of the Universe using an underlying \( f(R) \) gravity. The inflationary patch, if it is a quasi-de Sitter evolution, then it will be generated by an \( R^2 \) gravity, and matter and radiation fluids are ignored. We shall assume that during early stages of the radiation domination era, specifically during the reheating era, a short period of abnormal reheating takes place, with the total EoS parameter being \( w \neq 1/3 \), in the presence of radiation and dark matter perfect fluids. We find which \( f(R) \) gravity can generate such an evolution and accordingly we examine the matter and late-time eras of the Universe, which are controlled by an appropriate \( f(R) \) gravity term, so as for the \( f(R) \) model to mimic the ΛCDM model. This late-time \( f(R) \) gravity term does not affect at all the inflationary nor the abnormal reheating era. Accordingly, we find numerically the amplification of the primordial gravitational wave energy spectrum caused by this abnormal reheating era, and as we show it is quite large to make the signal detectable by all future experiments. In fact, the larger the duration of the abnormal reheating is, the larger the amplification of the signal becomes.

This paper is organized as follows: In section II we discuss the various evolutionary patches of the Universe, from inflation to dark energy and discuss how these can be realized by \( f(R) \) gravity in the presence of dark matter and radiation fluids. During inflation, the matter fluids are neglected, but post-inflationary these must be taken into account. We also add a short period of abnormal reheating with EoS parameter \( w = 0.1 \) and we examine which \( f(R) \) gravity in the presence of dark matter and radiation perfect fluids can realize such an exotic epoch. The late-time properties of the proposed \( f(R) \) gravity is examined too. In section III we calculate the effects of the abnormal reheating era on the primordial gravitational wave energy spectrum. A discussion along with the conclusions follow in the conclusions section.

II. EVOLUTION OF THE UNIVERSE WITH \( f(R) \) GRAVITY: FROM INFLATION TO THE DARK ENERGY ERA AND NON-CANONICAL REHEATING REALIZATION

To date, the basic assumptions for the evolution of the Universe mainly concern four evolutionary regimes, the inflationary era, the reheating-radiation domination era, the matter domination era and the late-time acceleration era, the so-called dark energy era. The question how the Universe smoothly passes through these era is not firmly
answered, although several appealing models can provide a unified description of most of the Universe’s evolution eras. Modified gravity in its various forms seems to be an inevitable choice for describing the inflationary and dark energy era, mainly the latter to be honest, since GR fails to describe consistently the late-time era. Also in the context of GR, the inflationary era description relies on scalar fields, and this has several shortcomings as we mentioned in the introduction. Thus, modified gravity seems to provide a consistent description for the inflationary era and the dark energy era, the two acceleration eras of the Universe. But the question then is, which other eras may be described by modified gravity, or equivalently, why should modified gravity affect only inflation and the dark energy eras? In principle it should be present and control all the evolutionary eras. For the matter domination era, it should provide a $\Lambda$CDM like evolution for small redshifts near the end of the matter domination eras, and the major question then is what it happens during reheating-radiation domination era. The reheating and the evolution of it to the radiation domination era, is quite mysterious, and we know almost nothing for this era. The only way to have our grasp on it will be offered by future interferometer experiments like the LISA mission, or DECIGO, which will probe frequencies corresponding to inflationary modes that reentered the horizon during the reheating-radiation domination eras. If an underlying $f(R)$ gravity controls the evolution of the Universe, synergistically with cold dark matter and radiation fluids post-inflationary, then the $f(R)$ gravity should have its imprints during both the radiation and the matter domination eras. Among the matter and radiation domination eras, it is highly likely that the total EoS of the Universe might be non-standard during the early stages of the radiation domination era, hence during reheating and beyond. During the matter domination era, the effects of the underlying $f(R)$ gravity should provide a $\Lambda$CDM model like evolution, at least for small redshifts post-recombination.

Hence, if exotic scenarios should occur, it is highly likely that these occurred post-inflationary and during the early stages of the radiation domination era. This is the scenario that we will describe in this work, in the context of $f(R)$ gravity. Although during inflation, the effects of cold dark matter and radiation fluids are neglected, post-inflationary these fluids cannot be neglected. Hence our approach for the post-inflationary early stages of the reheating era, should include radiation and cold dark matter fluids. Finding the exact form of the underlying $f(R)$ gravity which describes all the evolution eras of our Universe, is a rather formidable task, however we can find the leading order term of $f(R)$ gravity which can describe the different patches of the Universe’s evolution. During the inflationary era, the Universe is described by a quasi-de Sitter evolution for example, which by neglecting the matter fields, as we show can be described by an $R^2$ gravity. Accordingly, for the radiation domination era we shall assume that the Universe has a constant EoS parameter $w$ different from that of radiation.

Let us start with the gravitational action of $f(R)$ gravity in the presence of perfect matter fluids,

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} f(R) + S_m,$$

with $\kappa^2$ denoting as usual $\kappa^2 = 8\pi G = \frac{1}{M_p^2}$, where $G$ is Newton’s gravitational constant and $M_p$ stands for the reduced Planck mass. In the metric formalism, the field equations can be found by varying the action with respect to the metric, and these are,

$$f_R(R) R_{\mu\nu}(g) - \frac{1}{2} f(R) g_{\mu\nu} - \nabla_{\mu} \nabla_{\nu} f(R) + g_{\mu\nu} \Box f(R) = 0 \kappa^2 T^m_{\mu\nu},$$

where $T^m_{\mu\nu}$ is the energy momentum tensor of the matter perfect fluids, and we introduced $f_R = \frac{df}{dR}$. For a flat Friedmann-Robertson-Walker (FRW) metric, the Friedmann equation becomes,

$$-18 \left( 4H(t)^2 \dot{H}(t) + H(t) \ddot{H}(t) \right) f_{RR}(R) + 3 \left( H^2(t) + \dot{H}(t) \right) f(R) - \frac{f(R)}{2} + \kappa^2 (\rho_m + \rho_r) = 0,$$

where $\rho_m, \rho_r$ denote the energy density of the cold dark matter and radiation respectively.

Let us now consider the primordial patch of the Universe’s evolution, which comprises from the inflationary patch and the radiation domination era. We shall assume that the early stages of the radiation domination era is described by a constant EoS $w_i$, so the evolution during the early radiation era is basically a power-law. Also during the inflationary patch, the Universe is described by a quasi-de Sitter evolution, so the scale factor during the inflationary and early post-inflationary era is the following,

$$a(t) = a_0 e^{H_0 t - \frac{H_i^2 t^2}{2}} + a_r t^{\frac{2}{3(1+w_i)}},$$

where $a_0$ is the scale factor at the beginning of the inflationary era, and $a_r$ is the scale factor at the beginning of the radiation era, and at the end of inflation, and $H_i$ are free parameters with mass dimensions $[H_0] = [H_i] = [m]$. With regard to the value of the total EoS parameter $w$, we shall assume that it is not $w = 1/3$, but it is similar to
a deformed matter domination era EoS parameter, so \( w = 0.1 \). The whole analysis works with other values different than \( w = 1/3 \), but let us fix \( w = 0.1 \), so that the Universe does not commence the radiation domination era with a pure \( w = 1/3 \) but more closely to the matter domination total EoS parameter \( w = 0 \). In the literature, the values of the total EoS parameter during the reheating process, which is the early radiation domination era period, are expected to be found in the range \( 0 < w < 1/3 \). In fact, the value of the total EoS parameter we chose, which is very close to the \( w = 0 \) case, is considered motivated in the literature\(^{26} \) and in the context of GR the effect of such EoS parameter on the primordial gravitational wave energy spectrum would be similar to having massive relics.

Coming back to the evolution\(^{3} \), it describes a quasi de-Sitter evolution at early times, which is the exponential part, followed by the power-law evolution which describes the Universe during the early post-inflationary era. The quasi de-Sitter evolution dominates at early times, during inflation, and after that, the power-law evolution dominates. This can be seen clearly in Fig. 1, where with blue dotted curve we plot the scale factor corresponding to the pure quasi-de Sitter evolution dominates at early times, during inflation, and after that, the power-law evolution dominates. Finding the combined form of the scale factor is defined as some initial value of the scale factor. By using the \( e^{-N} = a_i / a \),

\begin{equation}
-18 \left[ 4H^3(N)H'(N) + H^2(N)(H')^2 + H^3(N)H''(N) \right] f_{RR}(R) \\
+ 3 \left[ H^2(N) + H(N)H'(N) \right] f_R(R) - \frac{f(R)}{2} + \kappa^2 \rho = 0,
\end{equation}

where \( \rho = \rho_m + \rho_r \). Introducing the function \( G(N) = H^2(N) \), the Ricci scalar is written as,

\begin{equation}
R = 3G'(N) + 12G(N).
\end{equation}

Now by specifying the scale factor and correspondingly the Hubble rate, using Eq. (7) one can invert it and find the function \( N(R) \), thus upon substituting the resulting expression in Eq. (6), one ends up with a second order differential equation with the dynamical variable being the Ricci scalar \( R \), which in the absence of perfect matter fluids, in terms of \( G(N) \) takes the form,

\begin{equation}
-9G(N(R)) \left[ 4G'(N(R)) + G''(N(R)) \right] f_{RR}(R) + \left[ 3G(N) + \frac{3}{2}G'(N(R)) \right] f_R(R) - \frac{f(R)}{2} = 0,
\end{equation}

where \( G'(N) = dG(N)/dN \) and \( G''(N) = d^2G(N)/dN^2 \) and since we consider the quasi. Thus upon solving it, one may obtain the \( f(R) \) gravity which realizes the given scale factor.
Let us first consider the quasi-de Sitter part of the scale factor (3) $a(t) \sim e^{H_0 - H_i^2 t^2}$, so in this case we have,

$$G(N) = H_0^2 - 4H_i^2 N.$$  

Using Eqs. (7) and (14), the e-foldings number $N$ in terms of the Ricci scalar is found to be,

$$N = \frac{12H_0^2 - 12H_i^2 - R}{48H_i^2}.$$  

Using the above, the Friedmann equation becomes,

$$(12H_i^2 (12H_i^2 + R)) \frac{d^2 f(R)}{dR^2} + \left( \frac{R}{4} - 3H_i^2 \right) R \frac{df(R)}{dR} - \frac{f(R)}{2} = 0,$$

the solution of which is,

$$f(R) = R + \frac{R^2}{72H_i^2 + 2H_i^2 - \frac{C_2 (144H_i^4 + 72H_i^2 R + R^2)}{3981312H_i^4}},$$

where $C_2$ has mass dimensions $[C_2] = [m]^7$ and it is simply an integration constant. Apparently the model (12) is a deformation of the $R^2$ model and it can be shown (the calculation will be presented elsewhere), the model is quantitatively a deformation of the $R^2$ model. Indeed, for $N \sim 60$, and irrespective of the values of the free parameters, the spectral index of the primordial scalar curvature perturbations for this model is $n_s \sim 0.967078$, while the tensor-to-scalar ratio and the tensor spectral index can be $r = 0.00327846$ and $n_T \simeq -0.000135483$. All the values are very close to the standard $R^2$ model. A term which can be added in the inflationary and has an insignificant effect during inflation, is the following,

$$f_{DE}(R) = \delta R e^{-\frac{\Lambda_1}{R}} e^{-15\Lambda_2 R},$$

where $\delta$ is a dimensionless parameter, and $\Lambda_1$, $\Lambda_2$ are parameters with cosmological constant dimensions and will be chosen to be of the same order of magnitude. The term (15) has an insignificant effect during inflation, thus the quasi-de Sitter evolution is still generated by the $f(R)$ (12) effectively. However its effect at late times is important, since as we show later on in this work, it can generate a deformed $\Lambda$CDM evolution, while its contribution during the radiation domination era is also insignificant.

Now we shall derive which $f(R)$ gravity can realize the power-law evolutionary patch of the evolution (4). In this case we shall take into account the presence of the cold dark matter and radiation, thus the Friedmann equation has the form (3). In the case at hand, with the scale factor being of the form $a(t) \sim t^{2/(3(w_i+1))}$, the function $G(N)$ reads,

$$G(N) = \frac{4e^{-3N(w_i+1)}}{9(w_i+1)^2},$$  

where we have set $a_r = 1$ for convenience. Thus upon combining Eqs. (7) and (14), we can obtain the e-foldings number $N$ as a function of $R$,

$$N = \log \left( \frac{4(1-3w_i)}{3(1+w_i)} \right).$$  

Inserting the above in the Friedmann equation (5), we can solve the differential equation, however before that there is an important step related with the perfect matter fluids. The total matter energy density $\rho_{tot}$ must be expressed in terms of $N(R)$, thus it will be a function of the Ricci scalar eventually. Since the matter and radiation fluids are perfect fluids, then the energy densities $\rho_i$, $i = r, m$, satisfy independently the continuity equation $\dot{\rho}_i = 3H(1+w_i)\rho_i = 0$, with $w_i$ their corresponding EoS parameters. Thus,

$$\rho_{tot} = \sum_i \rho_0 a^0 e^{-3(1+w_i)} e^{-3N(R)(1+w_i)},$$

and therefore, by inserting all the above in the Friedmann equation (5), we obtain the following differential equation for $f(R)$,

$$a_1 R^2 \frac{d^2 f(R)}{dR^2} + a_2 R \frac{df(R)}{dR} - \frac{f(R)}{2} + \sum_i S_i R^{2(1+w_i)} = 0,$$
where $a_1$ and $a_2$ are defined as follows,

$$a_1 = \frac{3(1 + w)}{4 - 3(1 + w)}$$
$$a_2 = \frac{2 - 3(1 + w)}{2(4 - 3(1 + w))},$$

and recall that the index “$i$” takes the values $i = (r, m)$ with $i = r$ denoting radiation and $i = m$ the cold dark matter perfect fluid. Also the parameters $S_i$ stand for,

$$S_i = \frac{\kappa^2 \rho_0 a_0^{-3(1+w_i)}}{\left[3A(4 - 3(1 + w))\right]^{\frac{3(1 + w)}{w - w_i}}},$$

and the parameter $A$ is defined as follows,

$$A = \frac{4}{3(w + 1)}$$

The solution of the differential equation (17) provides the exact $f(R)$ gravity that produces the power-law patch of the evolution (14). The general solution of the differential equation (17) is easily found to be,

$$f_p(R) = \left[\frac{c_2 \rho_1}{\rho_2} - \frac{c_1 \rho_1}{\rho_2(\rho_2 - \rho_1 + 1)}\right] R^{\rho_2 + 1} + \sum_i \left[\frac{c_1 S_i}{\rho_2(\delta_i + 2 + \rho_2 - \rho_1)}\right] R^{\delta_i + 2 + \rho_2} - \sum_i B_i c_2 R^{\delta_i + \rho_2} + c_1 R^{\rho_1} + c_2 R^{\rho_2},$$

where $c_1, c_2$ arbitrary integration constants, and we defined $\delta_i$ and $B_i$, $i = (r, m)$ as follows,

$$\delta_i = \frac{3(1 + w_i) - 23(1 + w)}{3(1 + w)} - \rho_2 + 2, \quad B_i = \frac{S_i}{\rho_2 \delta_i}.$$  

Thus the $f_p(R)$ gravity of Eq. (21) can generate an early radiation era which is different from the GR pattern in which $w = 1/3$. This geometrically generated exotic early radiation domination era with $w = 0.1$ as we said, can last sufficiently long, but we shall discuss this important issue in the next section. As we will show in the next section, the geometrically generated exotic radiation domination era, can affect significantly the energy spectrum of the primordial gravitational waves, leading to an amplification of the predicted signal at present day. An observation we made is that the inclusion of the early dark energy term (13) does not affect at all the resulting energy spectrum. Thus we can safely assume that this early dark energy term does not affect the evolution aspects of the model until late times, where it is responsible for a viable $\Lambda$CDM-like dark energy era. Before going to the late-time era analysis, let us summarize our findings, and make some assumptions on the issue of the duration of the exotic patch of the radiation domination era. Primordially in our scenario, the Universe is described by a quasi-de Sitter patch which is realized by $R^2$ gravity, which is followed by a short period of a non-canonical reheating, with the background EoS parameter being $w$ instead of the standard $w = 1/3$. In this case, the dominant form of the $f(R)$ gravity which realizes this short non-canonical reheating era is given by Eq. (21). It is conceivable that this short abnormal reheating era is geometrically driven by $f(R)$ gravity and the physics of it is different compared to GR. This feature will be strongly justified in the next section, where a comparison to the GR pattern of effects shall be given. Also it is notable that the addition of the dark energy term (13) does not affect at all both the inflationary era and the abnormal short radiation domination era period. For the short abnormal reheating era, this feature will be justified in the next section. After the geometrically driven non-canonical reheating era, the Universe enters the standard radiation domination era, followed by the dark matter domination era, for which the dark energy $f(R)$ gravity term also affects the evolution, especially at late times. We can give a schematic description of the different evolution patches of our Universe, and the corresponding $f(R)$ gravity description, which we quote below,

$$T \sim 10^{16}\text{GeV} - 10^{12}\text{GeV}, \quad f(R) \sim R + R^2 + f_{DE}(R),$$
$$T \sim 10^{12}\text{GeV} - 10^{10}\text{GeV}, \quad f(R) \sim f_p(R) + f_{DE}(R) + \text{radiation and dark matter fluids},$$
$$T < 10^{10}\text{GeV}, \quad f(R) \sim R + f_{DE}(R) + \text{radiation and dark matter fluids},$$

In the schematic description (23) we assumed that the short abnormal reheating era lasts for a short period of time, in the temperature range $10^{12}\text{GeV}$ to $10^{10}\text{GeV}$. So this geometrically driven era occurs at the beginning of the radiation domination era, during the reheating era and for a short period of time.
Now let us consider the late-time evolution of the model, which is affected by the dark energy $f(R)$ gravity term and by the radiation and matter perfect fluids. Considering the dark energy $f(R)$ gravity term, the parameters $\Lambda_1$ and $\Lambda_2$ will be considered to be of the order of the present day cosmological constant. For the numerical analysis that will follow, we specifically assume that $\Lambda_1 \sim 19.5\Lambda$ and $\Lambda_2 \sim 15\Lambda$, where $\Lambda \simeq 11.895 \times 10^{-67}eV^2$. Also the parameter $\delta$ in Eq. (13) will be chosen to be $\delta = 2.01$. With this fine tuning, the late-time era will prove to be viable and compatible with several observational constraints on the dark energy era. Now let us quantify the late-time era description of our model, firstly considering the field equations in the presence of matter fluids, which can be written in the Einstein-Hilbert form as follows,

$$3H^2 = \kappa^2 \rho_{\text{tot}},$$

$$-2\dot{H} = \kappa^2 (P_{\text{tot}} + \rho_{\text{tot}}),$$

where in the case at hand, the total energy density is equal to $\rho_{\text{tot}} = \rho_m + \rho_r$. The energy density of dark energy $\rho_{DE}$ is a geometric energy density contribution, cause by $f(R)$ gravity and controls the late-time dynamics. The dark energy density is defined to be,

$$\kappa^2 \rho_{DE} = \frac{f_R R - f}{2} + 3H^2(1 - f_R) - 3H\dot{f}_R,$$

and the corresponding dark energy pressure is defined as,

$$\kappa^2 P_{DE} = \dot{f}_R - H\dot{f}_R + 2\dot{H}(f_R - 1) - \kappa^2 \rho_{DE},$$

while the total pressure is $P_{\text{tot}} = P_r + P_{DE}$. We shall rewrite the field equations in terms of the redshift parameter $1 + z = \frac{1}{a}$ and the statefinder function $y_H(z)$ [6,33,11],

$$y_H(z) = \frac{\rho_{DE}}{\rho_m^{(0)}},$$

where $\rho_m^{(0)}$ stands for the energy density of cold dark matter today. The function $y_H(z)$ is written as follows,

$$y_H(z) = \frac{H^2}{m_s^2} - (1 + z)^3 - \chi(1 + z)^4,$$

where $\rho_m = \rho_m^{(0)}(1 + z)^3$ and also $\chi$ is defined as $\chi = \frac{\rho_r^{(0)}}{\rho_m^{(0)}} \simeq 3.1 \times 10^{-4}$, with $\rho_r^{(0)}$ being the radiation energy density today. We also introduced the parameter $m_s$ defined as $m_s^2 = \kappa^2 \rho_m^{(0)} = H_0\Omega_c = 1.37201 \times 10^{-67}eV^2$, and the Hubble rate is constrained by the CMB to have approximately the value $H_0 \simeq 1.37187 \times 10^{-35}eV$ [42]. The Friedmann equation in terms of the redshift is written as,

$$\frac{d^2y_H(z)}{dz^2} + J_1 \frac{dy_H(z)}{dz} + J_2 y_H(z) + J_3 = 0,$$

where the functions $J_1$, $J_2$ and $J_3$ are defined as follows,

$$J_1 = \frac{1}{z + 1} \left(-3 - \frac{1 - F_R}{(y_H(z) + (z + 1)^3 + \chi(1 + z)^4) 6m_s F_{RR}}\right),$$

$$J_2 = \frac{1}{(z + 1)^2} \left(\frac{2 - F_R}{(y_H(z) + (z + 1)^3 + \chi(1 + z)^4) 3m_s F_{RR}}\right),$$

$$J_3 = -3(z + 1) - \frac{(1 - F_R) (z + 1)^3 + 2\chi(1 + z)^4 + \frac{R - F}{4m_s^2}}{(1 + z)^2 (y_H(z) + (1 + z)^3 + \chi(1 + z)^4) 6m_s^2 F_{RR}},$$

and also we defined $f(R) = R + F(R)$ and $F_{RR} = \frac{\partial^2 F}{\partial R^2}$, while $F_R = \frac{\partial F}{\partial R}$. In our case, $F(R) = f_{DE}(R)$, with $f_{DE}(R)$ being defined in Eq. (13). For our numerical analysis of the late-time era, the following initial conditions shall be assumed in the redshift interval $z = [0, 10],$

$$y_H(z_f) = \frac{\Lambda}{3m_s^2} \left(1 + \frac{(1 + z_f)}{1000}\right), \quad \frac{dy_H(z)}{dz} \bigg|_{z = z_f} = \frac{1}{1000} \frac{\Lambda}{3m_s^2},$$

(31)
which are physically motivated by matter domination era \cite{6,33,41}. In terms of the statefinder function $y_H(z)$, the Ricci scalar is written as,

$$R(z) = 3m_\chi^2 \left( 4y_H(z) - (z + 1) \frac{dy_H(z)}{dz} + (z + 1)^3 \right).$$

(32)

and accordingly the dark energy density parameter $\Omega_{DE}$ is,

$$\Omega_{DE}(z) = \frac{y_H(z)}{y_H(z) + (z + 1)^3 + \chi(z + 1)^4}.$$  

(33)

Also the dark energy EoS parameter $\omega_{DE}$, is written in terms of $y_H(z)$ as follows,

$$\omega_{DE}(z) = -1 + \frac{1}{3} (z + 1) \frac{1}{y_H(z)} \frac{dy_H(z)}{dz},$$

(34)

and the total EoS parameter takes the form,

$$\omega_{tot}(z) = \frac{2(z + 1)H'(z)}{3H(z)} - 1.$$  

(35)

The dark energy EoS parameter and the dark energy density parameter are constrained by the Planck 2018 constraints \cite{42}, so these are important quantities. Finally, we shall consider the deceleration parameter $q$ in order to compare our model with the $\Lambda$CDM model, which is defined as follows,

$$q = -1 - \frac{\dot{H}}{H^2} = -1 + (z + 1) \frac{H'(z)}{H(z)}.$$  

(36)

Finally, the results of our numerical analysis shall be compared with the base $\Lambda$CDM model for which the Hubble rate is,

$$H_\Lambda(z) = H_0 \sqrt{\Omega_\Lambda + \Omega_M(z + 1)^3 + \Omega_r(1 + z)^4},$$

(37)

where $\Omega_\Lambda \simeq 0.681369$ and $\Omega_M \sim 0.3153$ \cite{42}, while $\Omega_r/\Omega_M \simeq \chi$. By numerically solving the differential equation (29), we obtain the function $y_H(z)$ and from it the Hubble rate $H(z)$. The results of our analysis are presented in Figs. 2 and 3. In Fig. 2 we plot the total EoS parameter $\omega_{tot}$ (left plot) and the dark energy EoS parameter (right plot) as functions of the redshift. In Fig. 3 we plot the deceleration parameter versus the redshift and in all the plots, the red curves correspond to the $f(R)$ model, while the blue curves to the base $\Lambda$CDM model. Overall, the $f(R)$ gravity model qualitatively behaves as the $\Lambda$CDM model, but it is surely distinct from it. At present day however, the $f(R)$ model is quite similar with the $\Lambda$CDM model. Also quite good compatibility properties with the Planck constraints \cite{42} are obtained for the $f(R)$ gravity model, since for the $f(R)$ model yields $\omega_{DE}(0) \simeq -1.04573$ and the Planck constraints are $\omega_{DE} = -1.018 \pm 0.031$. Also with regard to the dark energy density parameter the $f(R)$ model yields $\Omega_{DE}(0) \simeq 0.690065$ and the Planck constraints are $\Omega_{DE} = 0.6847 \pm 0.0073$. The deceleration parameter for the $f(R)$
model at present day is $q(0) = -0.582385$, while for the ΛCDM model is $q(0) = -0.52701$. Hence the $f(R)$ gravity model at hand behaves quite similarly to the ΛCDM and is deemed a viable dark energy model. Having described the qualitative behavior of the Universe in the various patches of its evolution, in the next section we shall consider the produced energy spectrum of the primordial gravitational waves. Specifically we shall show that the geometrically originating short in duration abnormal reheating era can significantly enhance the energy spectrum of the primordial gravitational waves. This is to be contrasted with the GR case, where a similar abnormal reheating era does not amplify the energy spectrum of the gravitational waves significantly.

III. PRIMORDIAL GRAVITATIONAL WAVE ENERGY SPECTRUM AMPLIFICATION DUE TO NON-CANONICAL $f(R)$ REHEATING

As we mentioned in the introduction, in about a decade from now, several experiments will probe directly the inflationary era, seeking for a stochastic background of primordial gravitational waves. Thus, the theoretical predictions on the energy spectrum of primordial gravitational waves which have been developing for decades [26, 43–74], will be put to test. The predicted energy spectrum of single scalar field inflationary models and of $f(R)$ gravity is too small to be detected from future experiments, thus in the case that some future stochastic primordial signal is detected, these two theories will not provide a good fit to the data. This consideration however is based on the assumption that a standard reheating and the subsequent radiation domination era take place. In this scenario, the EoS parameter during the whole radiation domination era is $w = 1/3$. If a non-standard reheating era takes place, the spectrum will be affected, see for example [26]. Specifically, the EoS parameter during reheating can take values $0 < w < 1/3$ [26] and an EoS $w = 0$ during reheating would mimic the effects of massive relics on primordial gravitational waves [26]. In the context of GR, the impact of a different from $w = 1/3$ EoS parameter during radiation domination, is small in magnitude though, as we evince shortly. This is not true in the case that the abnormal reheating era has a geometric origin, as we show in this section. Indeed, we will show that if the abnormal reheating is generated the $f(R)$ gravity [21] of the previous section, the amplification of the primordial gravitational wave energy spectrum is significant and can be measurable by most of the future interferometers. Before we start our analysis, we shall need to specify the duration of the abnormal reheating era. As we assumed in Eq. (21) the abnormal reheating era commences just after the end of the inflationary era with temperature $T_e \sim 10^{12}\text{GeV}$ and lasts for a short period until the temperature drops to $T_a \sim 10^{10}\text{GeV}$. Now we should relate the temperatures to redshifts, so using the approximate relation $T = T_0 (1 + z)^{[73]}$, where $T_0$ is the present day temperature $T_0 = 2.58651 \times 10^{-4}\text{eV}$, we may obtain the redshifts corresponding to the beginning and the end of the abnormal reheating era. The temperature at the end of inflation corresponds to $z_e = 3.86621 \times 10^{24}$ and the redshift at the end of the abnormal reheating is $z_a = 3.86621 \times 10^{22}$. After that, we assume that the standard radiation domination takes place, where the radiation perfect fluid drives the evolution, and subsequently the matter fluid and the $f_D\text{e}^2(R)$ drive the late-time era. Let us now recall the formalism of $f(R)$ gravity cosmological gravitational waves, see [69] and references therein for details. For a perturbed flat FRW background, the Fourier transformed tensor perturbation satisfies the following,

$$\frac{1}{a^3 f_R} \frac{d}{dt} \left( a^3 f_R h(k) \right) + \frac{k^2}{a^2} h(k) = 0,$$

(38)
or equivalently,
\[ \ddot{h}(k) + (3 + \alpha M) H \dot{h}(k) + \frac{k^2}{a^2} h(k) = 0, \]  
(39)

where parameter \( \alpha M \) for a general \( f(R) \) gravity is defined as follows,
\[ a_M = \frac{f_{RR} R}{f_R H}. \]  
(40)

In order to quantify the \( f(R) \) gravity effects on the gravitational waves, we adopt a WKB approach \[55, 60\], in which the WKB solution of the tensor perturbation differential equation reads,
\[ h = e^{-D} h_{GR}, \]  
(41)

where \( h_{GR} \) is the GR waveform which corresponds to \( a_M = 0 \). The physical quantity \( D \) contains the \( f(R) \) gravity effects on the gravitational waves, and it is defined as,
\[ D = \frac{1}{2} \int \! a_M \mathcal{H} dz = \frac{1}{2} \int_0^z \frac{a_M}{1 + z'} dz'. \]  
(42)

In order to find the overall effect of \( f(R) \) gravity on the gravitational waves at present day, the quantity \( D \) has to be evaluated from present day at \( z = 0 \) until the end of inflation at \( z_e \). However, as we now show, the most important integration periods are: from present day until recombination and from the end of the abnormal reheating era, until its start. We discuss this important issue shortly. The GR energy spectrum of primordial gravitational waves is,
\[ \Omega_{gw}(f) = \frac{k^2}{12H_0^2} \Delta_h^2(k), \]  
(43)

where \( \Delta_h^2(k) \) is \[26, 53, 54, 59, 61, 69\],
\[ \Delta_h^2(k) = \Delta_h^{(p)}(k)^2 \left( \frac{\Omega_m}{\Omega_x} \right)^2 \left( \frac{g_s(T_{in})}{g_{s0}} \right)^2 \left( \frac{g_{s0}}{g_{ss}(T_{in})} \right)^{4/3} \left( \frac{3j_1(k\tau_0)}{k\tau_0} \right)^2 T^2_1(x_{eq}) T^2_2(x_R), \]

where \( \Delta_h^{(p)}(k)^2 \) stands for the inflationary tensor power spectrum,
\[ \Delta_h^{(p)}(k)^2 = A_T(k_{ref}) \left( \frac{k}{k_{ref}} \right)^{n_T}, \]  
(44)

evaluated at the CMB pivot scale \( k_{ref} = 0.002 \) Mpc\(^{-1}\). Also, \( n_T \) denotes the inflationary tensor spectral index and \( A_T(k_{ref}) \) stands for amplitude of the tensor perturbations, which is,
\[ A_T(k_{ref}) = rP_\zeta(k_{ref}), \]  
(45)

where \( r \) denotes the tensor-to-scalar ratio and \( P_\zeta(k_{ref}) \) denotes the amplitude the scalar perturbations. Thus finally, we have,
\[ \Delta_h^{(p)}(k)^2 = rP_\zeta(k_{ref}) \left( \frac{k}{k_{ref}} \right)^{n_T}, \]  
(46)

hence the energy spectrum of the primordial gravitational waves for the GR and the \( f(R) \) gravity waveforms takes the form,
\[ \Omega_{gw}(f) = e^{-2D} \times \frac{k^2}{12H_0^2} rP_\zeta(k_{ref}) \left( \frac{k}{k_{ref}} \right)^{n_T} \left( \frac{\Omega_m}{\Omega_x} \right)^2 \left( \frac{g_s(T_{in})}{g_{s0}} \right)^2 \left( \frac{g_{s0}}{g_{ss}(T_{in})} \right)^{4/3} \left( \frac{3j_1(k\tau_0)}{k\tau_0} \right)^2 T^2_1(x_{eq}) T^2_2(x_R), \]

where \( D \) can be found in Eq. (42) and let us now discuss the redshift intervals for which it will be evaluated. Apparently, the redshift intervals will be from present day up to a redshift where the Universe entered the radiation domination era, and from \( z_e \) when the abnormal reheating era ends, up to \( z_e \) when inflation ends and the abnormal reheating era starts. For the two intervals, the dominant \( f(R) \) gravity which drives the evolution is different, for
example for the first interval $R + f_{DE}(R)$ drives the evolution, while for the second interval, $f_p(R)$ appearing in Eq. (21) drives the evolution. In both cases, the $f(R)$ gravity synergistically with matter and radiation fluids drive the evolution of the Universe. Thus the parameter $D$ has to be numerically evaluated in the following way,

$$D = \frac{1}{2} \left( \int_0^{z_a} \frac{a_{M_1}}{1 + z'} dz' + \int_{z_a}^{z_e} \frac{a_{M_2}}{1 + z'} dz' \right),$$

(47)

with the parameter $a_{M_1}$ and $a_{M_2}$ being defined in Eq. (40), but with $a_{M_1}$ evaluated for $f(R) = R + f_{DE}(R)$ and $a_{M_2}$ being evaluated for $f_p(R)$ appearing in Eq. (21). For the first integration, the integral $\int_0^{z_a} \frac{a_{M_1}}{1 + z'} dz'$ receives contribution from present day up to the recombination era but it is truly minor and of the order $\sim 10^{-8}$. Beyond the recombination redshift the numerical integration yields zero, and by also considering that when the Universe enters the radiation domination era, $\dot{R} = 0$, this means that the integral $\int_0^{z_a} \frac{a_{M_1}}{1 + z'} dz'$ vanishes up to redshift $z_a$. This feature is model dependent, however this is also true in most of the dark energy models of $f(R)$ gravity, see the discussion and models of [69]. Thus the only significant contribution to the parameter $D$ comes from the second integral, namely $\int_{z_a}^{z_e} \frac{a_{M_2}}{1 + z'} dz'$ which is evaluated to be $\int_{z_a}^{z_e} \frac{a_{M_2}}{1 + z'} dz' \sim -33$, thus an amplification of the gravitational wave energy spectrum occurs of the order $\mathcal{O}(10^{14})$ occurs. Hence, overall the signal is significantly amplified and thus detectable by all the future experiments seeking high frequency gravitational waves. This can be seen in Fig. 4 where we present the predicted $f(R)$ gravity $h^2$-scaled energy spectrum with purple, blue and red curves and also the sensitivity curves of most of the future high frequency experiments, and also the low frequency Litebird experiment. The $f(R)$ gravity curves correspond to three different reheating temperatures, and specifically, the purple curve to $T_R = 10^{12}$GeV, the red curve to $T_R = 10^7$GeV and the blue curve to $T_R = 10^6$GeV. The three curves are indistinguishable up to frequencies of the order $\mathcal{O}(10^{-4})$Hz and thereafter the low reheating temperature blue curve breaks off, while the red curve breaks off around $\mathcal{O}(1)$Hz. Thus the effect of a geometrically originating short lasting abnormal reheating era, with $w = 0.1$

\[ h = 10^{12} \text{GeV}, \]

\[ h = 10^7 \text{GeV}, \]

\[ h = 10^6 \text{GeV}. \]

FIG. 4: The $h^2$-scaled gravitational wave energy spectrum for pure $f(R)$ gravity. The $f(R)$ gravity curves correspond to three different reheating temperatures, the purple curve to $T_R = 10^{12}$GeV, the red curve to $T_R = 10^7$GeV and the blue curve to $T_R = 10^6$GeV. The $f(R)$ gravity gravitational wave energy spectrum is significantly amplified due to the geometrically originating short lasting abnormal reheating era.

results to a significant amplification of the gravitational wave energy spectrum. We need to note two things: firstly the amplification occurs for other values of $w$ too, but the final amplification is different of course. Secondly the duration of this abnormal reheating era greatly affects the amplification of the gravitational wave energy spectrum, and the longer the abnormal reheating era lasts, the larger the amplification is. Also at this point, let us note that the $f(R)$ gravity effect in the whole process is significant, and this is in contrast with the GR situation. Indeed, in the latter case, the change in the gravitational wave energy spectrum in the case that the background EoS parameter is $w \neq 1/3$ is a multiplicative factor $\sim \frac{1}{1 + 3w}$. Thus for $w = 0.1$ one gets an overall damping of
the order $O(1/2)$ in the gravitational wave energy spectrum. Hence the effect of a geometric term like $f(R)$ gravity, which causes a deviation from the standard $w = 1/3$ radiation domination era pattern, can be quite significant and measurable. Thus if a signal is detected in some future high frequency experiment, the scenario of an abnormal reheating era caused by some higher curvature gravity might play a prominent role for the identification of the theory that caused signal. Even in this case though, the road toward understanding the exact theory behind a future signal, is long and thorny.

Before closing this section, we need to address another important issue related to the superhorizon modes of the modified gravity we considered in this paper. So far we considered the subhorizon modes and the effects caused on these modes by an abnormal reheating era with EoS parameter $w$. We need to clarify an important issue before discussing any possible effects of the abnormal reheating era on the superhorizon modes. The superhorizon modes are the ones with small wavelength which were subhorizon modes during inflation, basically they became subhorizon just after inflation started. Eventually these were the first modes that reentered the horizon after inflation ended. Exactly for these modes, the WKB method we used applies, so we were able to measure the overall amplification caused by the abnormal reheating era on them. On the antipode of these modes, lie the superhorizon modes, and specifically the ones with wavelength $\lambda \geq 10 \, \text{Mpc}$, correspond to modes probed by the CMB experiments and the LiteBird experiment. These modes were superhorizon during reheating and during the inflationary era, and became subhorizon during the era probed by the CMB experiments and the LiteBird experiment. Now interestingly enough, one may ask the reasonable question, is the enhancement of the superhorizon modes comparable to the one obtained for the subhorizon modes? In such a case, one may claim that scenarios which will be probed by the high-frequency experiments are already ruled out by the CMB experiments, since the LiteBird curve in the sensitivity curves lies below the direct detection curves of the high-frequency gravitational wave experiments. Also notably, similar considerations apply for previous CMB experiments as well, see for example Fig. 2 in Ref. [76].

This question is very interesting so let us address in brief the superhorizon modes evolution, for the whole time that these are actually superhorizon modes. As we already mentioned, the superhorizon modes relevant to the CMB experiments are those with $\lambda \geq 10 \, \text{Mpc}$, and these are superhorizon during the whole inflationary era and remain superhorizon until $z \sim 1100$ which the redshift corresponding to the CMB. Thus the superhorizon modes during the abnormal reheating era, are still superhorizon modes. For these modes, since for superhorizon modes we have $k \ll H\alpha$, the evolution differential equation for the tensor modes, namely Eq. (49), becomes,

$$\ddot{h}_\ell(k) + \left(3 + \alpha_M\right) H\dot{h}_\ell(k) = 0 \, .$$

The general solution of the above differential equation is,

$$h_\ell(k) = C_\ell(k) + D_\ell(k) \int_1^\tau \exp\left(\int_1^\tau \left(-a_M(\tau) - 3H(\tau)\right) d\tau\right) \, d\eta \, ,$$

thus it is apparent that the solution describes a time-independent frozen term $C_\ell(k)$ and the second term in Eq. (49), which is an exponentially decaying mode. Hence during the whole abnormal reheating era, the superhorizon modes remain frozen and thus there should be no effect by the abnormal reheating era on them. However, at this point it is crucial to note two things: first, if the contribution from $a_M(\tau)$ during the reheating era is negative in Eq. (49), then it is possible that superhorizon do not freeze and evolve. This would break the linear approximation though, and such effects should be carefully addressed. In this paper though we did not consider these modes, but we aimed for the study of the subhorizon modes, so this study is deferred to a future work. Secondly, it should be noted that for these low-frequency-large wavelength modes, several effects caused by free-streaming relativistic particles like neutrinos should be taken into account, which however we did not take into account. Certainly such effects should be taken into account, so the part of the curves we presented in the plots we presented in this section, should be corrected at low-frequencies.

IV. CONCLUSIONS

In this paper we studied the effects of an short abnormal reheating era generated by higher order curvature terms, on the primordial gravitational wave energy spectrum. Specifically we focused on $f(R)$ gravity theory and we discussed how $f(R)$ gravity may affect the various evolutionary patches of the Universe. Specifically, $f(R)$ gravity may drive inflation, ignoring perfect matter fluids, but can also affect the late dark matter and the dark energy era, providing a $\Lambda$CDM-like evolution in the presence of matter and radiation fluids. The era for which $f(R)$ gravity may not play a significant role at all is the radiation domination era for a flat FRW Universe. This however only holds true in the case that the background EoS parameter is that of radiation $w = 1/3$, since for a FRW Universe, $R = 0$ in this case. If an alternative EoS parameter governs the early stages of the radiation domination era, then $f(R)$ gravity might be
the actual generator of this era. We examined how and which $f(R)$ gravity can generate each evolutionary patch of the Universe, from inflation, the abnormal reheating era, and finally the late-time era. Actually the late-time era $f(R)$ gravity term can be present from primordial times and has no significant effect on the evolution, until late times where it drives the late-time era. Using a WKB approach, valid for subhorizon modes, which are the modes that will actually be probed by most high frequency future experiments, we quantified the effect of the $f(R)$ gravity on the primordial gravitational wave energy spectrum. We calculated numerically the amplification factor, and the only significant contribution came only from the short abnormal reheating era. In fact, the longer this short abnormal reheating era lasts, the larger the amplification of the signal is. Now let us briefly discuss some possible future scenarios. The main problem of non-tachyon single scalar field theories, and of their Jordan frame counterpart theories, namely $f(R)$ gravity, is that they predict an undetectable by the future experiments signal. Thus if inflation is verified in the CMB, and also a signal of a stochastic gravitational wave background is detected by future experiments, this would rule out both single scalar and $f(R)$ gravities. However, with the present paper we showed that in the context of $f(R)$ gravity, alternative scenarios related to the mysterious radiation domination era, might enhance significantly the predicted signal. Similar scenarios were discussed briefly in [24], in the context of GR, however in GR the amplification-damping is insignificant compared to the geometrically originating on generated by $f(R)$ gravity. Hence one may not rule out easily $f(R)$ gravity, and the quest for future theorists and observational cosmologists, is to distinguish the theory that produces a detected signal. Is it a blue-tilted theory or it is a modified gravity theory with abnormal radiation domination era? This is a difficult question to answer, and for the moment model dependent, but the clue point will be from how many interferometer experiments will the signal be captured. These questions will possibly occupy the minds of theorists and experimentalists for the next two decades.

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