Critical branching processes in random environment with immigration: the size of the only surviving family

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Abstract

We consider a critical branching process $Y_n$ in an i.i.d. random environment, in which one immigrant arrives at each generation. Let $A_i(n)$ be the event that all individuals alive at time $n$ are offspring of the immigrant which joined the population at time $i$. We study the conditional distribution of $Y_n$ given $A_i(n)$ when $n$ is large and $i$ follows different asymptotics which may be related to $n$ ($i$ fixed, close to $n$, or going to infinity but far from $n$).

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1 Introduction and main result

We consider a branching process with immigration evolving in a random environment. Individuals in this process reproduce independently of each other according to random offspring distributions which vary from one generation to the other. In addition, an immigrant enters the population at each generation. A formal definition of such a process looks as follows. Let $\Delta$ be the space of all probability measures on $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$. Equipped with a metric, $\Delta$ is a Polish space. Let $F$ be a random variable taking values in $\Delta$, and let $F_n, n \in \mathbb{N} := \mathbb{N}_0 \setminus \{0\}$ be a sequence of independent copies of $F$. The infinite sequence $\mathcal{E} = \{F_n, n \in \mathbb{N}\}$ is called a random environment.

A sequence of $\mathbb{N}_0$-valued random variables $Y = \{Y_n, n \in \mathbb{N}_0\}$ specified on the respective probability space $(\Omega, \mathcal{F}, P)$ is called a branching process with one
immigrant joining each generation and evolving in random environment (BPIRE for short), if $Y_0 = 1$ and, given $E$ the process $Y$ is a Markov chain with
\[
\mathcal{L}(Y_n|Y_{n-1} = y_{n-1}, F_i = f_i, i = 1, 2, \ldots) = \mathcal{L}(\xi_{n1} + \ldots + \xi_{ny_{n-1}} + 1)
\]
for every $n \in \mathbb{N}$, $y_{n-1} \in \mathbb{N}_0$ and $f_1, f_2, \ldots \in \Delta$, where $\xi_{n1}, \xi_{n2}, \ldots$ are i.i.d. random variables with distribution $f_n$. We assume in the sequel that if $Y_{n-1} = y_{n-1} > 0$ is the population size of the $(n-1)$th generation of $Y$ then first $\xi_{n1} + \ldots + \xi_{ny_{n-1}}$ individuals of the $n$th generation are born and afterwards one immigrant enters the population.

The tail distribution of the life-periods of BPIRE’s were considered in [4] and [9] under weaker assumptions than the ones we impose here. An $(i, n)$-clan of a BPIRE is the set of individuals alive at generation $n$ and being descendants of the immigrant which entered the population at generation $i$. We say that only the $(i, n)$-clan survives in $Y$ at moment $n$ if $Y_n^- := \xi_{n1} + \ldots + \xi_{ny_{n-1}} > 0$ and all the $Y_n^-$ particles belong to the $(i, n)$-clan.

Let $\mathcal{A}_i(n)$ be the event that only the $(i, n)$-clan survives in $Y$ at moment $n$. The asymptotic behavior of the probability $P(\mathcal{A}_i(n))$ as $n \to \infty$ and $i$ varies with $n$ in an appropriate way was investigated in [11] for the critical BPIRE’s (see Assumption A2) and in [13] for the subcritical BPIRE’s. The present paper complements the results of [11] by describing the distribution of the population size of the process at moment $n$ given the event $\mathcal{A}_i(n)$. The estimation of the population size in this case is important from an evolutionary point of view because it provides information on the probability of survival of the population in a variable environment. Successive migrants bring genetic diversity to the population, and when there is only one surviving lineage, this implies that the population is poor in genetic diversity. If the population size of the process at moment $n$, given the event $\mathcal{A}_i(n)$, is small, the population will therefore be very vulnerable to an environmental change that could make the present genetic type less adapted. The question of quantifying the size and the genetic diversity of a population is also related to the concept of founder effect in population genetics. It is the loss of genetic variation that occurs when a new population is established by a very small number of individuals from a larger population. Because of the loss of genetic variation, the new population may be distinctly different, both genotypically and phenotypically, from the parent population from which it was derived. In extreme cases, the founder effect is thought to lead to speciation and subsequent evolution of new species (see [10] for more details on founder effect).

We need to consider, along with the process $Y$, a standard branching process $Z = \{Z_n, n \in \mathbb{N}_0\}$ in the random environment $E$ which, given $E$ is a Markov chain with $Z_0 = 1$ and
\[
\mathcal{L}(Z_n|Z_{n-1} = z_{n-1}, F_i = f_i, i = 1, 2, \ldots) = \mathcal{L}(\xi_{n1} + \ldots + \xi_{nz_{n-1}})
\]
for $n \in \mathbb{N}$, $z_{n-1} \in \mathbb{N}_0$ and $f_1, f_2, \ldots \in \Delta$. 

2.
To formulate our results we introduce the so-called associated random walk $S = \{S_n, n \in \mathbb{N}_0\}$ (see [1] for instance). This random walk has increments $X_n = S_n - S_{n-1}$, $n \geq 1$, defined as

$$X_n = \log m(F_n)$$

which are i.i.d. copies of the logarithmic mean offspring number $X := \log m(F)$ with

$$m(F) := \sum_{j=1}^{\infty} jF(\{j\}).$$

With each measure $F$ we associate the respective probability generating function

$$F(s) := \sum_{j=0}^{\infty} F(\{j\}) s^j.$$

Introduce the following assumptions:

**Hypothesis A1.** The probability generating function $F(s)$ is geometric with probability 1:

$$F(s) = \frac{q}{1 - ps} = \frac{1}{1 + m(F)(1 - s)}$$

with random $p, q \in (0, 1)$ satisfying $p + q = 1$, and the random variable

$$X := \log m(F) = \log \frac{p}{q}$$

has a nonlattice distribution.

**Hypothesis A2.** The branching process $Z$ is critical: $E[X] = 0$, and

$$E[e^X + e^{-X}] < \infty.$$

**Hypothesis A3.** The distribution of $X$ is continuous.

Denote $Z_{i,n}$ the size of the $(i, n)$-clan at moment $n$ and, introducing the associated random walk $S$ with $S_0 = 0$, set

$$Y_{i,n} := e^{S_i - S_n} Z_{i,n}.$$

The main result of the present paper is the following theorem.

**Theorem 1** If Hypotheses A1–A2 are valid then

1) for any fixed $N$ and $s \in [0, 1]$

$$\lim_{n \to \infty} E[s^{Z_{n-N,n}} | A_{n-N}(n)] =: \Theta_N(s) = E[s^{\vartheta_N}],$$

where $\vartheta_N$ is a proper nondegenerate random variable;
2) for any fixed \( i \) and \( \beta \geq 0 \)

\[
\lim_{n \to \infty} \mathbb{E} \left[ e^{-\beta Y_{i,n}} \mid A_i(n) \right] =: \Lambda_i(\beta) = \mathbb{E} \left[ e^{-\beta \hat{Y}_{i,n}} \right],
\]

where \( \hat{Y}_{i,n} \) a proper strictly positive random variable;

3) if \( \min (i, n - i) \to \infty \) and, in addition, Hypotheses A3 is valid then for any \( \beta \geq 0 \)

\[
\lim_{n \to \infty} \mathbb{E} \left[ e^{-\beta Y_{i,n}} \mid A_i(n) \right] =: \Lambda(\beta) = \mathbb{E} \left[ e^{-\beta \hat{Y}} \right],
\]

where \( \hat{Y} \) a proper strictly positive random variable.

Roughly speaking, Theorem 1 establishes that on the event \( A_i(n) \), the population size \( Z_{i,n} \) behaves as \( e^{S_n - S_i} \). Previous works have shown that, conditioned on the event of survival at time \( n \) of a Galton-Watson process evolving in an environment with independent identically distribute components, the population size at this moment behaves as \( e^{S_n - S_{\tau(n)}} \), where \( S_{\tau(n)} \) is the minimum of the random walk \( S \) on \( \{0, ..., n\} \) (see for instance Theorem 1.3 in [1], or Theorem 1.4 in [2] or [3]). As previously observed under different assumptions on the random environment (see, for instance, [12] for a comprehensive review on the critical and subcritical cases (before 2013) or the recent monograph [407]?) the survival of a branching process in random environment until a distant time \( n \) is essentially determined by its survival until the moment when the associated random walk \( S \) attains its infimum on the interval \([0, n]\). The idea is that if we divide the trajectory of the process on the interval \([0, n]\) into two parts, before the running infimum \( \tau(n) \) of the associated random walk \( S \) on \([0, n]\), and after this moment, the process will live in a favorable environment after the moment \( \tau(n) \), and will thus survive with a non-negligible probability until time \( n \), provided it survived until \( \tau(n) \). For the process with immigration we consider here, moment \( \tau(n) \) should be, as a rule, close to close time \( i \). Indeed, on one hand, on the event \( A_i(n) \), every family generated by an immigrant joined the population before time \( i \) is extinct at the time \( n \), and thus has undergone bad environments before observation time \( n \). On the other hand, the fact that the \((i, n)\)-clan is nonempty at time \( n \) implies that \( \inf\{S_k - S_i, i \leq k \leq n\} \) is likely to be non-negative. Finally, the difference \( S_n - S_i \) is also likely to be not too big, otherwise some of the immigrants arrived after time \( i \) would have a positive line of descents at time \( n \). These nonrigorous considerations indicate that on the event \( A_i(n) \):

- \( S_n - S_i \) is likely to be of the same order as \( S_n - S_{\tau(n)} \)
- \( S_n - S_i \) is likely to be relatively small

The results of Theorem 1 confirm to a certain extend these hypothesis and similar in spirit to Theorem 1.3 in [1], Theorem 1.4 in [2] and paper [3].

The rest of the paper is organised as follows. In Section 2 we collect some auxiliary results dealing with the probability of the event \( A_i(n) \). Section 3 is
dedicated to the proof of point 1) of Theorem 1. The proof of point 3) of Theorem 1 is provided in Section 4. Finally, the proof of Theorem 1 is completed in Section 5 by considering the case when \( i \) is fixed.

In the sequel we will denote by \( C, C_1, C_2, \ldots \) constants which may vary from line to line and by \( K_1, K_2, \ldots \) some fixed constants.

## 2 Auxiliary results

Given the environment \( \mathcal{E} = \{ F_n, n \in \mathbb{N} \} \), we introduce the i.i.d. sequence of generating functions

\[
F_n(s) := \sum_{j=0}^{\infty} F_n(\{j\}) s^j, \quad s \in [0, 1],
\]

and use below the compositions of \( F_1, \ldots, F_n \) specified for \( 0 \leq i \leq n-1 \) by the equalities

\[
\begin{align*}
F_i,n(s) &:= F_{i+1}(F_{i+2}(\ldots F_n(s)\ldots)), \\
F_{n,i}(s) &:= F_n(F_{n-1}(\ldots F_{i+1}(s)\ldots)),
\end{align*}
\]

and \( F_{n,n}(s) := s \) for \( i = n \).

Set

\[
\mathcal{H}_{i,n}(s) := (1 - F_{i,n}(s)) \prod_{j \neq i} F_j(n), \quad \mathcal{H}_{i,n} := \mathcal{H}_{i,n}(0).
\]

For \( 1 \leq i \leq n \) introduce the notation

\[
\begin{align*}
a_{i,n} &:= e^{S_i - S_n}, \quad a_n := a_{0,n} = e^{-S_n}, \quad b_0 := 0, \\
b_{i,n} &:= \sum_{k=i}^{n-1} e^{S_i - S_k}, \quad b_n := b_{0,n} = \sum_{k=0}^{n-1} e^{-S_k} =: 1 + B_{1,n}.
\end{align*}
\]

We can check by induction that if condition \( \mathbb{H} \) is valid then for \( i \leq n \)

\[
1 - F_{i,n}(s) = \frac{a_i}{a_n (1 - s)^{-1} + b_n - b_i}.
\]

In particular,

\[
F_{i,n}(0) = \frac{a_n + b_n - b_{i+1}}{a_n + b_n - b_i}.
\]

Now we provide an expression in terms of \( a_k \) and \( b_k \), \( k \geq 0 \), for the random variable \( \mathcal{H}_{i,n}(s) \).

**Lemma 2** Under Hypothesis A1 for any \( i = 0, 1, \ldots, n-1 \)

\[
\mathcal{H}_{i,n}(s) = \frac{1}{a_{i,n}(1-s)^{-1} + b_{i,n}} \frac{a_n}{a_{i,n} + b_{i,n} - 1} \frac{a_n}{a_n} = \frac{a_i}{a_n(1-s)^{-1} + b_n - b_i} \frac{a_n}{a_n + b_n - b_{i+1}} \frac{a_n}{b_{i+1}}.
\]
In particular,
\[ P(A_i(n)|S) = E \left[ \mathcal{H}_{i,n}(0) \right] = \frac{a_i}{a_n + b_n - b_{i+1} b_{n+1}}. \]

**Proof.** The desired statements are direct consequences of (4) and (5):
\[ H_{i,n}(s) = \frac{(1 - F_{i,n}(s))}{F_{i,n}(0)} \prod_{j=0}^{n-1} F_{j,n}(0) \]
\[ = \frac{a_i}{a_n(1-s)^{-1} + b_n - b_i a_n + b_n - b_{i+1}} \prod_{j=0}^{n-1} a_n + b_n - b_j \]
\[ = \frac{a_i}{a_n(1-s)^{-1} + b_n - b_i a_n + b_n - b_{i+1} b_{n+1}}. \]

We recall that the asymptotic behavior of the probability \( P(A_i(n)) \) as \( n \to \infty \) and \( i \) varies with \( n \) in an appropriate way is described by the following theorem (see [11]).

**Theorem 3** If Hypotheses A1–A2 are valid then
1) for any fixed \( N \)
\[ \lim_{n \to \infty} n^{1/2} P(A_{n-N}(n)) = r_N \in (0, \infty); \]
2) for any fixed \( i \)
\[ \lim_{n \to \infty} n^{3/2} P(A_i(n)) = w_i \in (0, \infty); \] (6)
3) if, in addition, Hypothesis A3 is valid then
\[ \lim_{\min(n, n-i) \to \infty} i^{1/2} (n-i)^{3/2} P(A_i(n)) = K \in (0, \infty), \] (7)
where an expression for \( K = K(h) \) is given by formula [13] below.

The next statement is a particular case of a theorem established in [13] (see [7] for a previous result, more general than what is needed in our case).

**Lemma 4** Let \( h : [0, \infty) \times [0, \infty) \to [0, \infty) \) be a nonnegative continuous function not identically equal zero and such that there exist constants \( 0 < \lambda \) and \( C > 0 \) such that
\[ h(x, y) \leq \frac{C}{(1 + x + y)^{\lambda}} \]
for all \( x \geq 0, y \geq 0. \)

If the distribution of \( X \) is nonlattice, \( EX = 0 \) and \( \text{var} X \in (0, \infty) \) then there exists a positive constant \( K(h) \) such that
\[ \lim_{n \to \infty} n^{1/2} E[h(a_n, B_{1,n})] = K(h). \]

With these results in hands, we will now be able to prove our main result, Theorem [11].
3 The case $i = n - N$

Let $N$ be a fixed integer and consider the case $i = n - N$. Taking the expectation with respect to the $\sigma$-algebra generated by the sequence $F_i, F_{i+1}, \ldots, F_n$ and making the changes $F_j \to \tilde{F}_{j-i}$ for $j = i, \ldots, n$ we write

$$E[H_{i,n}(s)] = E\left[\left(1 - F_{i,n}(s)\right) \prod_{j=i+1}^{n-1} F_{j,n}(0) \prod_{j=0}^{i-1} F_{j,n-N}(F_{i,n}(0))\right]$$

$$= E\left[\left(1 - F_{0,N}(s)\right) \prod_{j=1}^{N} \tilde{F}_{0,j}(0) \prod_{j=0}^{i-1} F_{j,i}(\tilde{F}_{0,N}(0))\right]$$

$$= E\left[\left(1 - \tilde{F}_{0,N}(s)\right) \prod_{j=1}^{N} \tilde{F}_{0,j}(0) \Psi_i(\tilde{F}_{0,N}(0))\right],$$

where

$$\Psi_i(z) := (1 - z)E\left[\prod_{j=0}^{i-1} F_{j,i}(z)\right].$$

Observe that as the environments are i.i.d.,

$$\prod_{k=1}^{i-1} F_{k,i}(z) \overset{d}{=} \prod_{k=1}^{i-1} F_{k,0}(z).$$

Now from Lemma 2 in [4]

$$\prod_{k=1}^{i-1} F_{k,0}(z) = \frac{(1 - z)^{-1}}{(1 - z)^{-1} + \sum_{k=1}^{i-1} e^{-S_k}} = \frac{(1 - z)^{-1}}{(1 - z)^{-1} + a_{i-1} + \tilde{B}_{i,i-1}}.$$ 

By a direct application of Lemma 4 with $\lambda = 1$ we obtain that for any $z \in [0, 1),

$$\lim_{i \to \infty} \sqrt{i} \Psi_i(z) := \psi(z) \in (0, \infty).$$

Note also that in view of Theorem 1.1 in [6]

$$\sqrt{i} \Psi_i(z) \leq \sqrt{i} E\left[\frac{1}{1 + \sum_{k=1}^{i-1} e^{-S_k}}\right] = \sqrt{i} P(Z_i > 0) \leq C. \quad (8)$$

Proof of point 1) of Theorem [1]. By conditioning with respect to the piece of the environment $F_1, ..., F_n$, we may check that, for all $s \in [0, 1]$

$$E[s^{Z_i,n} | A_i(n)] = 1 - \frac{E[1 - s^{Z_i,n}; A_i(n)]}{P(A_i(n))} = 1 - \frac{E[H_{i,n}(s)]}{P(A_i(n))}. \quad (9)$$

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From (8), we get

$$\sqrt{i} \frac{(1 - \tilde{F}_{0,N}(s))}{(1 - \tilde{F}_{0,N}(0))} \sum_{k=1}^{N-1} \tilde{F}_{k,N}(0) \Psi_i(\tilde{F}_{0,N}(0)) \leq \sqrt{i} \Psi_i(\tilde{F}_{0,N}(0)) \leq C.$$ 

Applying the dominated convergence theorem we conclude that, for \( i = n - N \)

$$\lim_{n \to \infty} \frac{E[H_{i,n}(s)]}{P(A_i(n))} = \lim_{n \to \infty} n^{1/2} \frac{E[H_{i,n}(s)]}{E[H_{i,n}(0)]} = \lim_{n \to \infty} i^{1/2} \frac{E\left[ \prod_{k=1}^{N} \tilde{F}_{k,N}(0) \psi(\tilde{F}_{0,N}(0)) \right]}{E\left[ \prod_{k=1}^{N} \tilde{F}_{k,N}(0) \psi(\tilde{F}_{0,N}(0)) \right]} =: \phi_N(s).$$

Let

$$\Theta_N(s) = E[s^{\tilde{\nu}_N}] := \lim_{n \to \infty} E[s^{\tilde{\nu}_{1,n}} | A_i(n)] = 1 - \phi_N(s).$$

Since \( N \) is fixed, it follows from the previous relation that \( \Theta_N(0) = 0, \Theta_N(1) = 1 \) and \( \Theta_N(s) \in (0,1) \) for all \( s \in (0,1) \). Hence, \( \tilde{\nu}_N \) is a proper nondegenerate random variable.

This proves (2).

4 The case \( \min(i, n - i) \to \infty \)

We now consider the case \( \min(i, n - i) \to \infty \). Introduce the running maximum and minimum of the associated random walk \( S \) :

$$M_n := \max(S_1, \ldots, S_n), \quad L_n := \min(S_0, S_1, \ldots, S_n)$$

and denote by

$$\tau(n) := \min\{0 \leq k \leq n : S_k = L_n\}$$

the moment of the first random walk minimum up to time \( n \).

It is known that if Hypothesis A2 is valid then (see, for instance, [5], Ch.XII, Sec 7, Theorem 1a) there exist positive constants \( K_1 \) and \( K_2 \) such that, as \( n \to \infty \)

$$P(\tau(n) = n) = P(M_n < 0) \sim K_1 n^{-1/2}, \quad P(L_n \geq 0) \sim K_2 n^{-1/2}.$$ 

(10)

and (see, for instance, Proposition 2.1 in [2]) there exist positive constants \( K_3 \) and \( K_4 \) such that, as \( n \to \infty \)

$$E[e^{S_n}; \tau(n) = n] \sim K_3 n^{-3/2}, \quad E[e^{-S_n}; L_n \geq 0] \sim K_4 n^{-3/2}.$$ 

(11)
The proof of the third statement of Theorem 1 is based on two changes of measure performed by means of the right-continuous functions \( U : \mathbb{R} \to [0, \infty) \) and \( V : \mathbb{R} \to [0, \infty) \) specified by

\[
U(x) := I\{x \geq 0\} + \sum_{n=1}^{\infty} P(S_n \geq -x, M_n < 0),
\]

\[
V(x) := I\{x < 0\} + \sum_{n=1}^{\infty} P(S_n < -x, L_n \geq 0),
\]

where \( I\{A\} \) is the indicator of the set \( A \).

It is known (see, for instance, [1] and [2]) that for any oscillating random walk

\[
E[U(x + X); X + x \geq 0] = U(x), \quad x \geq 0,
\]

and

\[
E[V(x + X); X + x < 0] = V(x), \quad x \leq 0.
\]

Let \( \mathcal{E} = \{F_n, n \in \mathbb{N}\} \) be a random environment and let \( \mathcal{F}_n \) be the \( \sigma \)-algebra of events generated by the random variables \( F_1, F_2, ..., F_n \) and the sequence \( Y_0, Y_1, ..., Y_n \). The sequence of these \( \sigma \)-algebras forms a filtration \( \mathcal{F} \). Using the martingale property (12)-(13) of \( U, V \) one can introduce (see, for instance, [8], Chapter 7) a sequence of probability measures \( \{P_{(n)}^+, n \geq 1\} \) on the \( \bigvee_{n \geq 1} \) by means of the equalities

\[
dP_{(n)}^+(A) := U(S_n)I\{L_n \geq 0\} dP(A), \quad A \in \mathcal{F}_n.
\]

Using this definition and Kolmogorov’s extension theorem one can specify on a suitable probability space a probability measure \( P^+ \) on \( \bigvee_{n \geq 1} \) such that

\[
P^+|\mathcal{F}_n = P_{(n)}^+, \quad n \in \mathbb{N}.
\]

We write \( P_x \) and \( E_x \) for the corresponding probability measures and expectations if \( S_0 = x \). Thus, \( P = P_0 \). With this notation, (14) may be rewritten as follows: for every \( \mathcal{F}_n \)-measurable random variable \( O_n \) such that \( E_x [O_n U(S_n); L_n \geq 0] < \infty, \quad x \geq 0, \)

\[
E_x^+ [O_n] := \frac{1}{U(x)} E_x [O_n U(S_n); L_n \geq 0], \quad x \geq 0.
\]

Similarly, \( V \) gives rise to probability measures \( P_x^-, x \leq 0 \), which can be defined via:

\[
E_x^- [O_n] := \frac{1}{V(x)} E_x [O_n V(S_n); M_n < 0], \quad x \leq 0.
\]

By means of the measures \( P_x^+ \) and \( P_x^- \), we investigate the limit behavior of certain conditional distributions.
For \( \lambda > 0 \), let \( \mu_\lambda \) and \( \nu_\lambda \) be the probability measures on \([0, +\infty)\) and \((-\infty, 0)\):

\[
\mu_\lambda(dz) := c_1 e^{-\lambda z} U(z) 1_{\{z \geq 0\}} dz, \quad \nu_\lambda(dz) := c_2 e^{\lambda z} V(z) 1_{\{z < 0\}} dz
\]

with

\[
c_1^{-1} = c_1^{-1} := \int_0^{\infty} e^{-\lambda z} U(z) dz, \quad c_2^{-1} = c_2^{-1} := \int_{-\infty}^{0} e^{\lambda z} V(z) dz.
\]

The next three lemmas are proven in [11] and are natural variations of Lemmas 7.3 and 7.5 in [8], Chapter 7. We recall them for the sake of readability. We use the agreement \( \delta n := |\delta n| \) for \( 0 < \delta < 1 \) in their formulations and write below \( f^{(m)} \) for \( m \) times integration \( \int \cdots \int \).

Let \( \mathcal{G}, \mathcal{H}, \mathcal{T} \) be three Euclidian (or Polish) spaces. For each \( r \in \mathbb{N} \) consider three functions \( g_r : \Delta_r \to \mathcal{G} \), \( h_r : \Delta_r \to \mathcal{H} \), and \( t_r : \Delta_r \to \mathcal{T} \), measurable with respect to the corresponding \( \sigma \)-algebras of Borel sets. **We assume in the three lemmas to follow that Hypotheses A2 and A3 hold.**

**Lemma 5 (Lemma 9 in [11])** Let \( G_r := g_r(F_1, \ldots, F_\delta_r) \), \( r \in \mathbb{N} \) be random variables with values in \( \mathcal{G} \) such that, as \( r \to \infty \)

\[ G_r \to G_\infty \quad \mathbb{P}^+ \text{ a.s.} \]

for some \( \mathcal{G} \)-valued random variable \( G_\infty \). Also let \( H_r := h_r(F_1, \ldots, F_\delta_r) \), \( r \in \mathbb{N} \), be random variables with values in \( \mathcal{H} \) such that, as \( r \to \infty \)

\[ H_r \to H_\infty \quad \mathbb{P}_x^+ \text{ a.s.} \]

for all \( x \leq 0 \) and some \( \mathcal{H} \)-valued random variable \( H_\infty \). Denote

\[ \hat{H}_r := h_r(F_1, \ldots, F_{\delta_r + 1}). \]

Let \( T_r := t_r(F_1, \ldots, F_r) \), \( r \in \mathbb{N} \) be random variables with values in \( \mathcal{T} \) such that, as \( r \to \infty \)

\[ T_r \to T_\infty \quad \mathbb{P}_x^+ \text{ a.s.} \]

for all \( x \geq 0 \) and some \( \mathcal{T} \)-valued random variable \( T_\infty \). Denote \( T_{n-r} := t_{n-r}(F_{r+1}, \ldots, F_n) \) for \( r \leq n \). Then for \( \lambda > 0 \) and any bounded continuous function \( \varphi : \mathcal{G} \times \mathcal{H} \times \mathbb{R} \times \mathcal{T} \to \mathbb{R} \), as \( \min(r, n-r) \to \infty \),

\[
\frac{\mathbb{E}[\varphi(G_r, \hat{H}_r, S_r; \hat{T}_{n-r}) e^{-\lambda S_r} : L_{n} \geq 0]}{\mathbb{E}[e^{-\lambda S_r} : L_{r} \geq 0]} \mathbb{P} \left( L_{n-r} \geq 0 \right) \\
\to \int_{\{4\}} U(-z) \varphi(u, v, -z, t) \mathbb{P}_x^+ (G_\infty \in du) \mathbb{P}_z^- (H_\infty \in dv) \mathbb{P}_{-z}^+ (T_\infty \in dt) \nu_\lambda(dz).
\]

**Lemma 6 (Lemma 10 in [11])** Let \( G_r, H_r, \hat{H}_r, T_r, \hat{T}_r, r \in \mathbb{N} \), be as in Lemma 5 now fulfilling, as \( r \to \infty \)

\[ G_r \to G_\infty \quad \mathbb{P}_x^+ \text{ a.s.} \forall x \geq 0, \quad H_r \to H_\infty \quad \mathbb{P}^+ \text{ a.s.}, \quad T_r \to T_\infty \quad \mathbb{P}_x^+ \text{ a.s.} \]
Then, for $\lambda > 0$ and any bounded continuous function $\varphi : G \times H \times \mathbb{R} \times T \to \mathbb{R}$, as $\min(r, n-r) \to \infty$

$$\frac{\mathbb{E}[\varphi(G_r, \tilde{H}_r, S_r; \tilde{T}_{n-r})e^{\lambda S_r}; \tau(n) = r]}{\mathbb{E}[e^{\lambda S_r}; \tau(r) = r] \mathbb{P}(L_{n-r} \geq 0)} \to \int \varphi(u, v, -z, t) \mathbf{P}_z^+ (G_\infty \in du) \mathbf{P}_z^- (\bar{H}_\infty \in dv) \mathbf{P}_z^+ (T_\infty \in dt) \mu_\lambda (dz).$$

**Lemma 7 (Lemma 11 in [11])** Let $r, n, r \in \mathbb{N}$, be as in Lemma 4 and

$$H_{N,r} = h_{N,r}(F_N, F_{N+1}, \ldots, F_{r-1}), \quad \tilde{H}_{r,N} = h_{N,r}(F_{r-N}, F_{N-1}, \ldots, F_{-r+1})$$

now fulfilling as $r \to \infty$

$$G_r \to G_\infty \mathbf{P}_x^+ - \text{a.s.}, \quad \forall x \geq 0, \quad (H_r, H_{N,r}) \to (H_\infty, H_{N,\infty}) \mathbf{P}^+ - \text{a.s.}$$

and $T_r \to T_\infty \mathbf{P}_x^+ - \text{a.s.}$ Then, for $\lambda > 0$ and for any bounded continuous function $\varphi : G \times H \times \mathbb{R} \times T \to \mathbb{R}$, as $\min(r, n-r) \to \infty$

$$\frac{\mathbb{E}[\varphi(G_r, \tilde{H}_r, \tilde{H}_{N,r}, \tilde{T}_{n-r})e^{\lambda S_r}; \tau(n) = r]}{\mathbb{E}[e^{\lambda S_r}; \tau(r) = r] \mathbb{P}(L_{n-r} \geq 0)} \to \int \varphi(u, v_1, v_2, -z, t) \mathbf{P}_z^+ (G_\infty \in du)$$

$$\times \mathbf{P}_z^- ((H_\infty, H_{\infty,N}) \in (dv_1, dv_2)) \mathbf{P}_z^+ (T_\infty \in dt) \mu_\lambda (dz).$$

We will need one more statement related to the driftless random walks $\{S_n, n \in \mathbb{N}\}$.

For a fixed positive integer $N \leq \min(j/2, n-j)$ set

$$K_1 = K_1(j, N) := [N, j-N] \cap \mathbb{N}, \quad K_2 = K_2(j, n, N) := [j + N, n] \cap \mathbb{N}. \quad (15)$$

**Lemma 8 [see Lemmas 13 and 14, and Corollary 15 in [11]]** If

$$\mathbb{E}X_i = 0, \quad \sigma^2 = \text{Var}X_i \in (0, \infty),$$

then there exists a constant $C > 0$ such that for all $n \geq j \geq 1$

$$\mathbb{E} e^{-S_i}e^{S_{r(j-1)}e^{S_{r(n)}}} \leq \frac{C}{j^{3/2} (n - j + 1)^{1/2}}. \quad (16)$$

and for every $\varepsilon > 0$ there exists $N = N(\varepsilon)$ such that

$$\mathbb{E} e^{-S_i}e^{S_{r(j-1)}e^{S_{r(n)}}}; \tau(n) \in K_1 \cup K_2 \leq \frac{\varepsilon}{j^{3/2} \sqrt{n - j + 1}}$$

for all $j \geq j_0 = j_0(\varepsilon), n \geq n_0 = n_0(\varepsilon)$. 

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We now prove point 3) of Theorem 1. Recall the definition of $a_{i,n}$ in $3$, and consider the rescaled process $Y_{i,n} = a_{i,n} Z_{i,n}$. Let $\beta > 0$. If we take $s = \exp \{-\beta a_{i,n}\}$ in $3$, we get

$$
E \left[ e^{-\beta Y_{i,n}} | A_i(n) \right] = 1 - \frac{E [H_{i,n}(\exp \{-\beta a_{i,n}\})]}{P(A_i(n))}.
$$

(17)

Note that in view of Lemma 2

$$
H_{i,n}(\exp \{-\beta a_{i,n}\}) = \frac{1}{a_{i,n}/(1 - \exp \{-\beta a_{i,n}\}) + b_{i,n}} \frac{a_n}{a_{i,n} + b_{i,n} - 1} a_n + b_n.
$$

Replacing $a_{i,n}, b_{i,n}, a_n$ and $b_n$ by their definition we obtain,

$$
\frac{1}{a_{i,n}/(1 - \exp \{-\beta a_{i,n}\}) + b_{i,n}} \frac{a_n}{a_{i,n} + b_{i,n} - 1} a_n + b_n = \frac{1}{e^{S_{i,n}\!-\!S_n}/(1 - \exp \{-\beta e^{S_{i,n}\!-\!S_n}\}) + \sum_{j=1}^{n-1} e^{S_{i,n}\!-\!S_j} e^{-S_{n-j}}} \times \frac{\sum_{j=1}^{n-1} e^{S_{i,n}\!-\!S_j} e^{-S_j}}{\sum_{j=0}^{n-1} e^{-S_j}} \times \frac{1}{\sum_{j=i+1}^{n} e^{S_{n-j}} \sum_{j=0}^{n} e^{-S_{n-j}}}.
$$

Using the duality property of random walks (5, Ch.XII, Section 2) we obtain the representation

$$
E [H_{i,n}(\exp \{-\beta a_{i,n}\})]
= E \left[ \frac{e^{S_{n-i}}}{1/(1 - \exp \{-\beta e^{S_{n-i}}\}) + \sum_{k=1}^{n-i} e^{S_k} \sum_{k=0}^{n-i-1} e^{S_k} \sum_{k=0}^{n} e^{S_k}} \right].
$$

(18)

For the sake of readability, we now introduce the reflection of the random walk $S$,

$$
\bar{S} := \{S_n, n \in \mathbb{N}_0\} = \{-S_n, n \in \mathbb{N}_0\}.
$$

Functions and measures related to $\bar{S}$ will be indicated with bars 7. For instance, we write

$$
\bar{a}_k = e^{-\bar{S}_k}, \ L_n := \min_{0 \leq r \leq n} \bar{S}_k, \ \bar{M}_n := \max_{1 \leq k \leq n} \bar{S}_k, \ \bar{\tau}(n) := \min \{ k \geq 0 : \bar{S}_k = \bar{L}_n \}
$$

and

$$
\bar{U}(x) := I \{ x \geq 0 \} + \sum_{n=1}^{\infty} P( \bar{S}_n \geq -x, \bar{M}_n < 0 )
$$
and specify the measure $\bar{P}_x^+$ by the relation

$$\bar{E}_x^+ [O_n] := \frac{1}{U(x)} E_x [O_n \bar{U}(\bar{S}_n); \bar{L}_n \geq 0], \ x \geq 0.$$  

We also write $j$ instead of $n - i$ in the remaining part of the proofs. The agreements above allow us to rewrite (18) as

$$E \left[ H_{i,n}(\exp\{-\beta a_{i,n}\}) \right] = E \left[ V_{j,n}(\beta) \right], \tag{19}$$

where

$$V_{j,n}(\beta) := \frac{\tilde{a}_j}{(1 - \exp\{-\beta/\bar{a}_j\})^{-1} + B_{1,j+1}} \frac{1}{\bar{b}_{j+1}} \bar{b}_j \bar{b}_{n+1}, \tag{20}$$

It will be also convenient to consider

$$V_{j,n}(\infty) := \frac{\tilde{a}_j}{\bar{b}_j \bar{b}_{n+1}}.$$  

According to this agreement

$$E \left[ V_{j,n}(\infty) \right] = E \left[ H_{i,n}(0) \right] = P(\mathcal{A}_i(n)).$$

It will be clear from the arguments to follow that all our estimates and limiting expressions are valid for all $\beta \in (0, \infty]$, i.e., $\beta = \infty$ is included.

We fix some positive integer $N$, and recall the definition of $K_1$ and $K_2$ in (15). We have the decomposition

$$E \left[ V_{j,n}(\beta) \right] = E \left[ V_{j,n}(\beta); \bar{\tau}(n) \in K_1 \cup K_2 \right]$$

$$+ E \left[ V_{j,n}(\beta); \bar{\tau}(n) < N \right] + E \left[ V_{j,n}(\beta); \bar{\tau}(n) \in (j - N, j] \right]$$

$$+ E \left[ V_{j,n}(\beta); \bar{\tau}(n) \in (j, j + N] \right]. \tag{21}$$

Since

$$E \left[ V_{j,n}(\beta) \mid \bar{S} \right] \leq E \left[ V_{j,n}(\infty) \mid \bar{S} \right] \overset{d}{=} P(\mathcal{A}_i(n) \mid \bar{S})$$

and

$$E \left[ V_{j,n}(\beta) \right] \leq P(\mathcal{A}_i(n)), \quad \beta \in (0, \infty],$$

it follows from Lemma 14 in [11] that for all $n \geq j + 1 \geq 2$ and for any $\varepsilon > 0$ there exists $N(\varepsilon)$ such that, for all $N \geq N(\varepsilon)$ and all sufficiently large $j$ and $n - j$

$$E \left[ V_{j,n}(\beta) \mid \bar{\tau}(n) \in K_1 \cup K_2 \right] \leq \frac{\varepsilon}{j^{3/2} (n - j)^{1/2}}, \quad \beta \in (0, \infty].$$

We now focus on the last three summands in the right-hand side of (21).

We write

$$\bar{b}_n = \bar{b}_k + \bar{a}_k \bar{b}_{k,n}$$

and use below the equality $\bar{b}_{k,n} \overset{d}{=} \bar{b}_{n-k}$ and the independence of $(\bar{b}_k, \bar{a}_k)$ and $\bar{b}_{k,n}$ many times.
1) We first evaluate \( E[\varphi_{j,n}(\beta); \bar{\tau}(n) < N] \). Let \( k < N < j \). By conditioning on the trajectory of the random walk \( \bar{S} \) until time \( k \) we obtain
\[
E[\varphi_{j,n}(\beta); \bar{\tau}(n) = k] = E\left[ e^{-\bar{S}_k} \varphi_{j-k,n-k}(\beta; e^{-\bar{S}_k}, \tilde{b}_k); \bar{\tau}(k) = k \right]
\]
where
\[
\varphi_{j,n}(\beta; u, q) := \frac{1}{(1 - \exp \{-\beta/\tilde{u}a_j\})^{-1} + q - 1 + ub_{j+1} \quad q + ub_j \quad q + ub_{n+1}} \quad ; \bar{L}_n \geq 0
\]
for \( \beta \in (0, \infty) \) and
\[
\tilde{b}_j := \sum_{r=0}^{n} e^{-\bar{S}_r} =: \bar{H}_{\tilde{b},j}, \quad \quad \tilde{\xi}_j,j,n := \sum_{r=j}^{n} e^{-\bar{S}_r} =: \tilde{\xi}_{n-j}. \tag{22}
\]
Set \( t = [j/2] \) and denote for \( w \geq t + 2 \)
\[
G_t := \sum_{r=0}^{w-1} e^{-\bar{S}_r}, \quad H_{t+1, w-1} := \sum_{r=t+1}^{w-1} e^{\bar{S}_r} =: H_{t, w-t}, \\
T_{j,n} := \sum_{r=j}^{n} e^{-\bar{S}_r} =: T_{n-j}.
\]
Observe that
\[
\tilde{b}_{j+1} = G_t + e^{-\bar{S}_j} \bar{H}_{t+1, j-1} + e^{-\bar{S}_j} = \tilde{b}_j + e^{-\bar{S}_j}
\]
and
\[
\tilde{b}_{n+1} = G_t + e^{-\bar{S}_j} \bar{H}_{t+1, j-1} + e^{-\bar{S}_j} \tilde{\xi}_{j,n}.
\]
Using these equalities we may check that
\[
\varphi_{j,n}(\beta; u, q) = E\left[ e^{-\bar{S}_j} \varphi_{\beta, u, q}(G_t, \tilde{H}_{t+1, j-1}, \tilde{\xi}_j; \tilde{\xi}_{j,n}); \bar{L}_n \geq 0 \right],
\]
where
\[
\varphi_{\beta, u, q}(g, h, s; t) := \frac{(1 - \exp \{-\beta/\tilde{u}e^{-s}\})^{-1} + q - 1 + u (g + e^{-s}(h + 1))}{q + u (g + e^{-s}h)} \quad \times \quad \frac{1}{q + u (g + e^{-s}(h + t))}.
\]
Clearly, \( \varphi_{\beta, u, q}(g, h, s; t) \) is a continuous function for \( \beta \geq 0, q \geq 1 \) and \( u, g, h, s, t \geq 0 \) and, by monotonicity in \( \beta \)
\[
\varphi_{\beta, u, q}(g, h, s; t) \leq \varphi_{\infty, u, q}(g, h, s; t) := \frac{1}{q + u (g + e^{-s}h)} \quad \times \quad \frac{1}{q + u (g + e^{-s}(h + t))} \leq 1. \tag{23}
\]
Since the random walk \( \bar{S} \) is driftless and \( \text{var} \bar{X}_i = \text{var} X_i \), it follows from Lemma 5.5 in [?] that, as \( \min(j, n - j) \to \infty \),

\[
G_t \to G_\infty := \sum_{r=0}^{\infty} e^{-\bar{S}_r} \bar{P}^+ - a.s.,
\]

\[
H_{1,j-t} \to H_{1,\infty} := \sum_{r=1}^{\infty} e^{\bar{S}_r} \bar{P}_x^- - a.s., \quad \forall x \leq 0,
\]

\[
T_{n-j} \to T_\infty := \sum_{r=0}^{\infty} e^{-\bar{S}_r} \bar{P}_x^+ - a.s., \quad \forall x \geq 0.
\]

Estimate (23) allows us to apply Lemma 5 and to obtain that

\[
\lim_{\min(j, n - j) \to \infty} \Psi_{j,n}(\beta; u, q) = 0
\]

exists for each fixed tuple \((\beta; u, q)\), where, for \(\beta \in (0, \infty]\)

\[
\Psi_\infty(\beta; u, q) := \int_{H_1,\infty \in dh} U(-s) \varphi_{\beta,u,q}(g, h, -s, t) \bar{P}^+(G_\infty \in dg) \times \bar{P}^-_{s}(H_{1,\infty} \in dh) \bar{P}^+_{s}(T_\infty \in dt) \bar{v}(ds).
\]

Since

\[
\lim_{\beta \to 0} \varphi_{\beta,u,q}(g, h, s; t) = 0
\]

and the inequality (23) is valid, we conclude by the dominated convergence theorem that

\[
\lim_{\beta \to 0} \Psi_\infty(\beta; u, q) = 0.
\]

Furthermore, setting

\[
C_k(\beta) := \mathbb{E} \left[ e^{-\bar{S}_k} \Psi_\infty \left( \beta; e^{-\bar{S}_k}, \bar{B}_{1,k} \right) ; \bar{\tau}(k) = k \right]
\]

we have, again by monotonicity of \(\Psi_\infty(\beta; u, q)\) in \(\beta\) and the dominated convergence theorem that

\[
\lim_{\beta \to 0} C_k(\beta) = 0
\]

and

\[
\lim_{\beta \to \infty} C_k(\beta) = \mathbb{E} \left[ e^{-\bar{S}_k} \Psi_\infty \left( \infty; e^{-\bar{S}_k}, \bar{B}_{1,k} \right) ; \bar{\tau}(k) = k \right] = C_k(\infty) < \infty.
\]

Invoking the dominated convergence theorem once more we conclude by (24) and (25) that for any \(k < N\), as \(\min(j, n - j) \to \infty\)

\[
\mathbb{E} \left[ e^{-\bar{S}_k} \Psi_{j,k-n+k} \left( \beta; e^{-\bar{S}_k}, \bar{B}_{1,k} \right) ; \bar{\tau}(k) = k \right] \sim \mathbb{E} \left[ e^{-\bar{S}_k} \Psi_\infty \left( \beta; e^{-\bar{S}_k}, \bar{B}_{1,k} \right) ; \bar{\tau}(k) = k \right] \mathbb{E} \left[ e^{-\bar{S}_j} ; \bar{L}_j \geq 0 \right] \mathbb{P} \left( \bar{L}_{n-j} \geq 0 \right)
\]

\[
\sim \frac{K_2 K_4 C_k(\beta)}{j^{3/2} (n - j)^{1/2}},
\]

(27)
Recalling (19) and summing (27) over the $k$'s in $[0, N - 1]$, gives, as $\min(j, n - j) \to \infty$

$$j^{3/2} (n - j)^{1/2} E \{ V_{j,n}(\beta); \tilde{\tau}(n) < N \} \sim K_2 K_4 C(\beta, N)$$

(28)

where

$$C(\beta, N) := \sum_{k=0}^{N} E \left[ e^{-\tilde{S}_k \Psi_{\infty}} \left( \beta; e^{-\tilde{S}_k, \tilde{B}_{1,k}} \right); \tilde{\tau}(k) = k \right].$$

(29)

2) We now evaluate $E \{ V_{j,n}(\beta); \tilde{\tau}(n) \in (j - N, j) \}$. To this aim we fix $1 \leq k < N$, recall that $t = [j/2]$, as well as the definitions of $\tilde{H}$ and $\tilde{T}$ in (22). We also introduce

$$\tilde{D}_{j,k} := \tilde{S}_{j-k} - \tilde{S}_j \overset{d}{=} -\bar{\tilde{S}}_k.$$

The same as before, as $\min(j, n - j) \to \infty$

$$G_t \to G_\infty, \quad \tilde{P}_x^+ - a.s. \forall x \geq 0, \quad H_{t,j-t-k-1} \to H_{1,\infty}, \quad \tilde{P}_- - a.s., \quad T_{0,n-j+k} \to T_\infty \quad \tilde{P}_- - a.s.$$

Besides, using (20) and multiplying both numerator and denominator by $e^{\bar{s}_{j-k}}$, we obtain

$$E \{ V_{j,n}(\beta); \tilde{\tau}(n) = j - k \}
= E \left[ e^{\bar{s}_{j-k}} \frac{e^{\bar{s}_{j-k} - \bar{s}_j}}{e^{\bar{s}_{j-k}} (1 - \exp \{ -\beta e^{\bar{s}_j} \})^{-1} + e^{\bar{s}_{j-k}} \bar{b}_{1,j+1}} \frac{\bar{b}_{j+1}}{e^{\bar{s}_{j-k}} \bar{b}_{n+1}}; \tilde{\tau}(n) = j - k \right]$$

Observing that

$$e^{\bar{s}_{j-k}} \bar{B}_{1,j+1} = e^{\bar{s}_{j-k}} \sum_{r=1}^{t} e^{-\bar{s}_r} + \sum_{r=t+1}^{j-k-1} e^{\bar{s}_{j-k} - \bar{s}_r} + \sum_{r=j-k}^{j} e^{\bar{s}_{j-k} - \bar{s}_r}
= e^{\bar{s}_{j-k}} (G_t - 1) + \tilde{H}_{t+1,j-k} + \tilde{T}_{j-k,j}$$

we write

$$e^{\bar{s}_{j-k} - \bar{s}_j}
= e^{\bar{s}_{j-k} \left( 1 - \exp \{ -\beta e^{\bar{s}_j} \} \right)^{-1} + e^{\bar{s}_{j-k}} \bar{B}_{1,j+1}}
\frac{\bar{b}_{j+1}}{e^{\bar{s}_{j-k}} \bar{b}_{n+1}}$$

and, similarly,

$$e^{\bar{s}_{j-k}} \bar{b}_{j+1} = e^{\bar{s}_{j-k}} G_t + \tilde{H}_{t+1,j-k} + \tilde{T}_{j-k,j},$$
$$e^{\bar{s}_{j-k}} \bar{b}_{j} = e^{\bar{s}_{j-k}} G_t + \tilde{H}_{t+1,j-k} + \tilde{T}_{j-k,j-1},$$
$$e^{\bar{s}_{j-k}} \bar{b}_{n+1} = e^{\bar{s}_{j-k}} G_t + \tilde{H}_{t+1,j-k} + \tilde{T}_{j-k,n}.$$
Thus,

\[ E[V_{j,n}(\beta); \tilde{\tau}(n) = j - k] = E[e^{S_{j-k}} \varphi_{\beta}(G_t, H_{t+1,j-k}, S_{j-k}, D_{j,k}, T_{j-k,j}, T_{j-k,j-1}, T_{j-k,n}); \tilde{\tau}(n) = j - k], \]

where, for \( \beta \in (0, \infty) \)

\[ \varphi_{\beta}(g, h, s_1, s_2, t_1, t_2, t_3) = \frac{e^{s_2}}{e^{s_1}(1 - \exp\{-\beta e^{-s_2} e^{s_1}\})^{-1} + e^{s_1}(g - 1) + h + t_1} \times \frac{1}{e^{s_1}g + h + t_2 e^{s_1}g + h + t_3} \]

and, by monotonicity of the function with respect to \( \beta \)

\[ \varphi_{\beta}(g, h, s_1, s_2, t_1, t_2, t_3) \leq \varphi_{\beta}(g, h, s_1, s_2, t_1, t_2, t_3) = \frac{e^{s_2}}{e^{s_1}g + h + t_2 e^{s_1}g + h + t_3} \leq 1 \]

in the domain

\[ \{g \geq 1, h \geq 0, s_1 \leq 0, s_2 \leq 0, t_1 \geq 0, t_2 \geq 0, t_3 \geq 0\} \cap \{e^{s_2} \leq e^{s_1}g + h + t_2, e^{s_1}g + h + t_3 \geq 1\}. \]

The same as before, as \( \min(j, n - j) \to \infty \)

\[ G_t \to G_\infty \quad \tilde{\mathbf{P}}^+_x \text{ - a.s.} \quad \forall x \geq 0, \quad H_{1,j} \to H_{1,\infty} \quad \tilde{\mathbf{P}}^- \text{ - a.s.}, \]

\[ (T_{k-1}, T_k, T_{n-j+k}) \to (T_{k-1}, T_k, T_\infty) \quad \tilde{\mathbf{P}}^+ \text{ - a.s.}. \]

Recalling Lemma 7 we see that, for each \( k \) there exists a constant \( J_{-k}(\beta) \geq 0 \) such that, as \( \min(j, n - j) \to \infty \)

\[ E[V_{j,n}(\beta); \tilde{\tau}(n) = j - k] \sim J_{-k}(\beta) E[e^{S_{j-k}}; \tilde{\tau}(j - k) = j - k](\tilde{E}_{n-j+k} \geq 0) \]

\[ \sim \frac{K_2 K_3 J_{-k}(\beta)}{j^{3/2} (n - j)^{1/2}}, \quad (30) \]

where

\[ J_{-k}(\beta) := \int (7) \varphi_{\beta}(g, h, -s_1, s_2, t_1, t_2, t_3) \tilde{\mathbf{P}}^+ (G_\infty \in dg) \tilde{\mathbf{P}}^-_{S_1} ((H_{1,\infty}, -S_k) \in d(h, s_2)) \times \tilde{\mathbf{P}}^+ ((T_{k-1}, T_k, T_\infty) \in d(t_1, t_2, t_3)) \tilde{\mu}_1 (ds_1). \]

Monotonicity of \( \varphi_{\beta}(g, h, -s_1, -s_2, t_1, t_2, t_3) \) in \( \beta \) and the dominated convergence theorem show that, for each \( k \)

\[ \lim_{\beta \to 0} J_{-k}(\beta) = 0 \quad (31) \]

and

\[ \lim_{\beta \to \infty} J_{-k}(\beta) = J_{-k}(\infty) < \infty. \quad (32) \]
Summing (30) over the $k$'s in $[1, N - 1]$, gives, as $\min(j, n - j) \to \infty$

$$j^{3/2} (n - j)^{1/2} \mathbb{E} [\mathcal{V}_{j,n}(\beta); \tilde{\tau}(n) \in (j - N, j)]$$

$$\sim K_2 K_3 j^{-\frac{1}{2}} N \sum_{k=1}^{N-1} J_{-k}(\beta), \tag{33}$$

3) We finally evaluate $\mathbb{E} [\mathcal{V}_{j,n}(\beta); \tilde{\tau}(n) \in (j, j + N)]$. As before, we fix $1 \leq k \leq N$, set $t = \lfloor j/2 \rfloor$ and introduce a more complicated notation

$$\tilde{H}_{t,w,v} := \sum_{r=t+1}^{w} e^{S_{w+v-S_r}} \frac{d}{d + v - t} \sum_{r=v}^{w} e^{\tilde{S}_r} =: H_{v,w,v-t},$$

$$\tilde{C}_{j,k} := \tilde{S}_{j+k} - \tilde{S}_j \equiv \tilde{S}_k.$$

With this notation in view we write

$$e^{\tilde{S}_{j+k} - \tilde{S}_j} = e^{\tilde{S}_{j+k} \left(1 - \exp \left\{-\beta e^{\tilde{S}_j} \right\} \right)^{-1} + e^{\tilde{S}_{j+k} \tilde{B}_{1,j+1}}}$$

$$e^{\tilde{S}_{j+k} \tilde{B}_{j+1} e^{\tilde{S}_{j+k}}}$$

$$e^{\tilde{S}_{j+k} \tilde{B}_{j+1} G_t + \tilde{H}_{t,j,k} = e^{\tilde{S}_{j+k} \tilde{B}_j + e^{\tilde{S}_{j+k} - \tilde{S}_j}}$$

and

$$e^{\tilde{S}_{j+k} \tilde{B}_{n+1}} = e^{\tilde{S}_{j+k} G_t + \tilde{H}_{t,j+k-1,1} + \tilde{T}_{j+k,n}}.$$

As a result we have, using again (20) and multiplying both numerator and denominator by $e^{2\tilde{S}_{j+k}}$

$$\mathbb{E} [\mathcal{V}_{j,n}(\beta); \tilde{\tau}(n) = j + k]$$

$$= \mathbb{E} \left[ \frac{e^{\tilde{S}_{j+k} - \tilde{S}_j} \tilde{b}_{j+1} + 1}{e^{\tilde{S}_{j+k} \tilde{B}_{n+1}} \tilde{b}_j} \right] \tilde{\tau}(n) = j + k$$

$$= \mathbb{E} \left[ e^{\tilde{S}_{j+k} \varphi_j(G_t, \tilde{H}_{t,j,k}, \tilde{H}_{t,j-1,k+1}, \tilde{H}_{t,j+k-1,1}, \tilde{S}_{j+k}, \tilde{C}_{j,k}, \tilde{T}_{j+k,n}); \tilde{\tau}(n) = j + k \right],$$

where

$$\varphi_j^+(g, h_1, h_2, h_3, s_1, s_2, t) := \frac{e^{s_2}}{e^{s_1} \left(1 - \exp \{-\beta e^{s_2} e^{s_1} \} \right)^{-1} + e^{s_1} (g - 1) + h_1} \times \frac{e^{s_1} g + h_1}{e^{s_1} g + h_2 e^{s_1} g + h_3 + t}$$

and

$$\varphi_j^+(g, h_1, h_2, h_3, s_1, s_2, t) \leq \varphi_{j,\infty}^+(g, h_1, h_2, h_3, s_1, s_2, t)$$

$$: = \frac{e^{s_2}}{e^{s_1} g + h_2 e^{s_1} g + h_3 + t} \leq 1.$$
Similarly to the previous case, for each \( k \) we have \( \sum_{j,n} \) for some constant \( J \) where

\[
\{ g \geq 1, h_2 \geq h_1 \geq 0, h_3 \geq 0, s_1 \leq 0, s_2 \leq 0, t \geq 0 \} \cap \{ e^{s_2} \leq e^{s_1}g + h_2, e^{s_1}g + h_3 + t \geq 1 \}.
\]

We know that for any fixed \( k \), as \( \min(j, n - j) \to \infty \)

\[
G_t \to G_\infty \quad \hat{P}^+ - a.s., \quad \forall x \geq 0,
\]

\[
(H_{k,j,t-1}, H_{k,1,j,t-1}, H_{j,k+1,t-1}, S_k) \to (H_{k,\infty}, H_{k+1,\infty}, H_{1,\infty}, S_k) \quad \hat{P}^+ - a.s.,
\]

\[
T_{n-j-k} \to T_{\infty} \quad \hat{P}^+ - a.s.
\]

Hence, using Lemma 7 and relations (10) and (11) we get that, for each fixed \( k \),

\[
\mathbb{E} \left[ \mathcal{V}_{j,n}(\beta); \bar{\tau}(n) = j + k - \beta \right] \sim \frac{J_{++k}(\beta)}{j^{3/2}(n-j)^{1/2}}.
\]

where

\[
J_{++k}(\beta) := \int (7) \varphi_{\beta}^+(g, h_1, h_2, h_3, -s_1, -s_2, t_1) \hat{P}^+ (G_\infty \in dg) \times \hat{P}^+ (T_\infty \in dt_1) \hat{P}^+ \left( (H_{k,\infty}, H_{k+1,\infty}, H_{1,\infty}, S_k) \in d(h_1, h_2, h_3, s_2) \right) \bar{\mu}_1(ds_1).
\]

Similarly to the previous case, for each \( k \)

\[
\lim_{\beta \to 0} J_{++k}(\beta) = 0.
\]

and

\[
\lim_{\beta \to \infty} J_{++k}(\beta) = J_{++k}(\infty) < \infty.
\]

Summing (34) over the \( k \)'s in \([0, N - 1] \), give, as \( \min(j, n - j) \to \infty \)

\[
\mathbb{E} \left[ \mathcal{V}_{j,n}(\beta); \bar{\tau}(n) \in [j, j + N] \right] \sim \frac{J_{++}(\beta, N)}{j^{3/2}(n-j)^{1/2}}.
\]

for some constant \( J_{++}(\beta, N) \).

Combining (28), (29) and (37) with (35) shows that, for any \( \beta \in (0, \infty] \)

\[
\lim_{\min(j, n-j) \to \infty} j^{3/2}(n-j)^{1/2} \mathbb{E} \left[ \mathcal{V}_{j,n}(\beta) \right] = K(\beta) \in (0, \infty).
\]

In particular,

\[
K(\infty) = \lim_{\min(j, n-j) \to \infty} j^{3/2}(n-j)^{1/2} \mathbb{E} \left[ \mathcal{V}_{j,n}(\infty) \right]
\]

\[
= \lim_{\min(i, n-i) \to \infty} (n-i)^{3/2} t^{1/2} \mathbb{E} \left[ \mathcal{H}_{i,n}(0) \right]
\]

\[
= \lim_{\min(i, n-i) \to \infty} (n-i)^{3/2} t^{1/2} \mathbb{P} \left( \mathcal{A}_i(n) \right) \in (0, \infty),
\]

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where we used the substitution $i \rightarrow n - j$ and \textbf{7}. Recalling \textbf{17} and \textbf{19}, we conclude that

$$
\lim_{\min(i, n-i) \rightarrow \infty} \mathbb{E} \left[ e^{-\beta Y_{i,n}} \mid A_i(n) \right] = 1 - \lim_{\min(i, n-i) \rightarrow \infty} \frac{\mathbb{E} [H_{i,n}(\exp \{-\beta a_{i,n}\})]}{\mathbb{P} (A_i(n))}
$$

$$
= 1 - \lim_{\min(i, n-i) \rightarrow \infty} \frac{i^{1/2} (n-i)^{3/2} \mathbb{E} [H_{i,n}(\exp \{-\beta a_{i,n}\})]}{i^{1/2} (n-i)^{3/2} \mathbb{P} (A_i(n))}
$$

$$
= 1 - \frac{K(\beta)}{K(\infty)} := \Lambda (\beta) = \mathbb{E} \left[ e^{-\beta \hat{Y}} \right].
$$

To complete the proof of point 3) of Theorem \textbf{1} it remains to show that

$$
\lim_{\beta \uparrow \infty} \Lambda (\beta) = \mathbb{P} \left( \hat{Y} = 0 \right) = 0 \quad \text{and} \quad \lim_{\beta \downarrow 0} \Lambda (\beta) = \mathbb{P} \left( \hat{Y} < \infty \right) = 1.
$$

To check the validity of these statements we first observe that, for all $\beta \in (0, \infty)$

$$
K(\beta) = K(\infty) - K(\infty) \Lambda (\beta) = \sum_{k=0}^{\infty} C_k(\beta) + \sum_{k=1}^{\infty} J_{-k}(\beta) + \sum_{k=0}^{\infty} J_{+k}(\beta) \leq K(\infty).
$$

Combining this estimate with \textbf{25}, \textbf{31} and \textbf{35}, and applying the monotone convergence theorem we see that

$$
\lim_{\beta \downarrow 0} (K(\infty) - K(\infty) \Lambda (\beta)) = K(\infty) - K(\infty) \lim_{\beta \downarrow 0} \Lambda (\beta) = 0.
$$

Thus, $\lim_{\beta \downarrow 0} \Lambda (\beta) = 1$ and, therefore, $\hat{Y}$ is a proper random variable.

Moreover, in view of \textbf{26}, \textbf{32} and \textbf{36}

$$
\lim_{\beta \uparrow \infty} (K(\infty) - K(\infty) \Lambda (\beta)) = K(\infty)(1 - \lim_{\beta \uparrow \infty} \Lambda (\beta))
$$

$$
= \sum_{k=0}^{\infty} C_k(\infty) + \sum_{k=1}^{\infty} J_{-k}(\infty) + \sum_{k=0}^{\infty} J_{+k}(\infty) = K(\infty) = K. \quad (38)
$$

These relations justify the equalities

$$
\mathbb{P} \left( \hat{Y} = 0 \right) = \lim_{\beta \uparrow \infty} \Lambda (\beta) = 0.
$$

Thus, $\hat{Y}$ is positive with probability 1.

This completes the proof of point 3) of Theorem \textbf{1}.

\section{The case of fixed $i$}

The proof of point 2) of Theorem \textbf{1} is similar to the proof of point 3) of the theorem and is shorter.
Set, for $x \geq 1$
$$\mathcal{M}_j(\beta; x) := \frac{\bar{a}_j}{1 - \exp \{-\beta/\bar{a}_j\}^{-1} + B_{1,j+1}} \frac{\bar{b}_{j+1}}{b_j} \frac{1}{b_j + \bar{a}_j x}. $$

We write as earlier $j$ instead of $n - i$ and taking the expectation with respect to $(\bar{X}_{j+1}, \bar{X}_{j+2}, \ldots, \bar{X}_n) \sim (\bar{X}_1, \bar{X}_2, \ldots, \bar{X}_i)$

and recalling (20) obtain that
$$E[H_{i,n}(\exp \{-\beta a_i,n\})] = E[H_{j,n}(\beta)] = E[Y_j(\beta; b_i)],$$
where
$$Y_j(\beta; x) := E[M_j(\beta; x)].$$

Note that in view of (11) for all $\beta \geq 0$ and $x \geq 1$
$$E[M_j(\beta; x)] \leq E\left[\frac{\bar{a}_j}{b_j b_j + \bar{a}_j}\right] \leq E\left[e^{-S_j}e^{S_{i+1}}e^{S_{j+1}}\right] \leq \frac{C}{\bar{N}^{3/2}}. \quad (39)$$

Recall the definition of $K_1$ and $K_2$ in (15). Now we fix some positive integer $N$ and use the decomposition

$$Y_j(\beta; x) = E[M_j(\beta; x); \bar{\tau}(j) \in K_1] + E[M_j(\beta; x); \bar{\tau}(j) < N] + E[M_j(\beta; x); \bar{\tau}(j) \in (j - N, j)].$$

First we observe that in view of (11) for any $\varepsilon > 0$ one can find $N_0 = N_0(\varepsilon)$ such that
$$E[M_j(\beta; x); \bar{\tau}(j) \in K_1] \leq E[e^{-S_j}; \bar{\tau}(j) \in K_1] = \sum_{r=N}^{j-N} E\left[e^{S_r}; \bar{\tau}(j) = r\right] \leq \frac{C}{\bar{N}^{3/2}} \sum_{r=N}^{j-N} \frac{1}{r^{3/2}} \leq \frac{\varepsilon}{\bar{N}^{3/2}} \quad (40)$$

for all sufficiently large $j$ and $N \geq N_0$.

Now we evaluate $E[M_j(\beta; x); \bar{\tau}(j) < N]$. By conditioning on the trajectory of the random walk $\bar{S}$ until time $k$ we obtain
$$E[M_j(\beta; x); \bar{\tau}(j) = k] = E\left[e^{-S_k}\Psi_{j-k}(\beta; x, e^{-S_k}, \bar{b}_k); \bar{\tau}(k) = k\right]$$

where
$$\Psi_{j} (\beta; x, u, q) = e^{-S_j} \frac{1}{(1 - \exp \{-\beta/u\bar{a}_j\})^{-1} + q - 1 + ub_{j+1}} \frac{1}{q + ub_j} \frac{1}{q + u(b_j + \bar{a}_j x)}; \bar{L}_j \geq 0. $$
Recall that $t = \lfloor j/2 \rfloor$, as well as the definition of $\tilde{H}$ in \cite{22}. A direct computation gives that

$$
\Psi_j (\beta; x, u, q) = E \left[ e^{-S_j} \varphi_{\beta, x, u, q}(G_t, \tilde{H}_{t+1,j-1}, S_j); \tilde{L}_j \geq 0 \right],
$$

where, for $\beta \in (0, \infty)$

$$
\varphi_{\beta, x, u, q}(g, h, s) \equiv \frac{1}{(1 - \exp \{-\beta/ue^{-s}\})^{-1} + q - 1 + u (g + e^{-s}(h + 1))} \times \frac{1}{q + u (g + e^{-s}(h + 1))}\frac{1}{q + u (g + e^{-s}(h + x))}
$$

and, for $\beta = \infty$

$$
\varphi_{\infty, x, u, q}(g, h, s) := \frac{1}{q + u (g + e^{-s}(h + x))} \leq 1 \quad (41)
$$

if $q \geq 1$.

Repeating now the arguments similar to those used to prove \eqref{24} and applying Theorem 2.7 from \cite{2} we obtain

$$
\lim_{j \to \infty} \frac{\Psi_j (\beta; x, u, q)}{E[e^{-S_j}; \tilde{L}_j \geq 0]} = \Psi_{\infty} (\beta; x, u, q),
$$

where for $\beta \in (0, \infty]$

$$
\Psi_{\infty} (\beta; x, u, q) := \int_{(3)} \varphi_{\beta, x, u, q}(g, h, -s) \tilde{P}^+ (G_\infty \in dg) \tilde{P}_-^- (H_{1,\infty} \in dh) \tilde{v}_1 (ds).
$$

Note that in view of \eqref{11}

$$
\Psi_{\infty} (\beta; x, u, q) \leq \int_{(3)} \tilde{P}^+ (G_\infty \in dg) \tilde{P}_-^- (H_{1,\infty} \in dh) \tilde{v}_1 (ds) \leq \int \tilde{v}_1 (ds) = 1.
$$

Since

$$
\lim_{\beta \downarrow 0} \varphi_{\beta, x, u, q}(g, h, -s) = 0
$$

and $\varphi_{\beta, x, u, q}$ is monotone decreasing as $\beta \downarrow 0$, we conclude by the dominated convergence theorem that

$$
\lim_{\beta \downarrow 0} \Psi_{\infty} (\beta; x, u, q) = 0.
$$

Furthermore, setting

$$
C'_k (\beta, x) := E \left[ e^{-S_k} \Psi_{\infty} (\beta; x, e^{-S_k}, B_{1,k}); \tilde{\tau}(k) = k \right], \quad \beta \in (0, \infty],
$$

we have, again by monotonicity of $\Psi_{\infty} (\beta; \cdot, \cdot, \cdot)$ in $\beta$ and the dominated convergence theorem that

$$
\lim_{\beta \downarrow 0} C'_k (\beta, x) = 0 \quad (42)
$$

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We now evaluate $E$ where

$$\{ \text{in the domain} \}$$

Summing (43) over the $k$ and $\beta$, we conclude by (41) that for any fixed $k$, as $j \to \infty$

$$E \left[ e^{-\tilde{S}_k \Psi_k} \left( \beta; x, e^{-\tilde{S}_k}, \tilde{B}_{1,k} \right) ; \tilde{\tau}(k) = k \right] \sim E \left[ e^{-\tilde{S}_k \Psi_{\infty}} \left( \beta; x, e^{-\tilde{S}_k}, \tilde{B}_{1,k} \right) ; \tilde{\tau}(k) = k \right] E [e^{-\tilde{S}_{j_1}} ; \tilde{L}_j \geq 0] \sim \frac{K_4 C'_k(\beta, x)}{j^{3/2}}. \quad (43)$$

Summing (43) over the $k$’s in $[0, N-1]$, gives, as $j \to \infty$

$$j^{3/2} E [\mathcal{M}_j(\beta; x); \tilde{\tau}(n) < N] \sim K_4 C'(\beta, x, N) \quad (44)$$

where

$$C'(\beta, x, N) := \sum_{k=0}^{N-1} E \left[ e^{-\tilde{S}_k \Psi_{\infty}} \left( \beta; x, e^{-\tilde{S}_k}, \tilde{B}_{1,k} \right) ; \tilde{\tau}(k) = k \right].$$

We now evaluate $E [\mathcal{M}_j(\beta; x); \tilde{\tau}(n) = (j - N, j)]$ and write

$$E [\mathcal{M}_j(\beta; x); \tilde{\tau}(n) = j - k] = E \left[ e^{\tilde{S}_{j-k} - k} \varphi_{\beta,x}^{-}(G_1, \tilde{H}_{t,j-k-1,1}, \tilde{S}_{j-k}, \tilde{S}_{j-k} - \tilde{S}_j, \tilde{T}_{j-k,j}, \tilde{T}_{j-k,j-1}); \tilde{\tau}(n) = j - k \right],$$

where, for $\beta \in (0, \infty)$

$$\varphi_{\beta,x}^{-}(g, h, s_1, s_2, t_1, t_2) := e^{s_2} \times \frac{e^{s_1}(1 - \exp \{ -\beta e^{-s_2} e^{s_1} \})^{-1} + e^{s_1}(g - 1) + h + t_1}{e^{s_1}g + h + t_1} \times \frac{1}{e^{s_1}g + h + t_2 e^{s_1}g + h + t_2 + x}$$

and, by monotonicity

$$\varphi_{\beta,x}^{-}(g, h, s_1, s_2, t_1, t_2) \leq \varphi_{\infty,x}^{-}(g, h, s_1, s_2, t_1, t_2) \leq e^{s_2} \frac{1}{e^{s_1}g + h + t_2 e^{s_1}g + h + t_2 + x} \leq 1 \quad (45)$$

in the domain

$$\{ g \geq 1, h \geq 0, s_1 \leq 0, s_2 \leq 0, t_1 \geq 0, t_2 \geq 0 \} \cap \{ e^{s_2} \leq e^{s_1}g + h + t_2, e^{s_1}g + h + t_2 + x \geq 1 \}.$$

Since

$$G_t \to G_{\infty} \stackrel{\tilde{P}}{-} a.s. \forall x \geq 0, (H_{1,j}, S_{k}) \to (H_{1,\infty}, S_{k}) \stackrel{\tilde{P}}{-} a.s.,$$

$$\left( T_{k-1}, T_k \right) \to \left( T_{k-1}, T_k \right), \quad \tilde{P}^+ - a.s.$$
as \( j \to \infty \) we may apply Lemma 7 in [2] and conclude that, for each \( k \), as \( j \to \infty \)

\[
E \[ M_j(\beta, x); \tau(j) = j - k \] \sim J_{-k}(\beta, x) E[e^{\tilde{\beta} j - k}; \tilde{\tau}(j - k) = j - k] \\
\sim \frac{K_3 J_{-k}(\beta, x)}{j^{3/2}},
\]

(46)

where

\[
J_{-k}(\beta, x) = P \left( L_k \geq 0 \right) \int_{(6)} \varphi_{\tilde{\beta}, x}(g, h, -s_1, -s_2, t_1, t_2) \\
\times \hat{P}_{s_1, t_1} (G_\infty \in dg) \hat{P}^+ ((T_{k-1}, T_k) \in dt(t_1, t_2)) \\
\times \hat{P}^- ((H_{1, \infty}, S_k) \in d(h, s_2)) \hat{\mu}(ds_1).
\]

Monotonicity of \( \varphi_{\tilde{\beta}, x}(g, h, -s_1, -s_2, t_1, t_2) \) in \( \beta \) and the dominated convergence theorem show that, for each fixed \( k \)

\[
\lim_{\beta \downarrow 0} J_{-k}(\beta, x) = 0 \quad (47)
\]

and

\[
\lim_{\beta \uparrow \infty} J_{-k}(\beta, x) = J_{-k}(\infty, x) < \infty.
\]

Summing (46) over the \( k \)'s in \([0, N - 1]\), gives, as \( j \to \infty \)

\[
j^{3/2} E \left[ M_j(\beta, x); \tilde{\tau}(j) \in (j - N, j] \right] \sim K_3 J_{-}(\beta, x, N) = K_3 \sum_{k=0}^{N-1} J_{-k}(\beta, x). \quad (48)
\]

Combining (44) and (48) with (39) and (40) shows that

\[
\lim_{j \to \infty} j^{3/2} Y_j(\beta; x) = \lim_{j \to \infty} j^{3/2} E \left[ M_j(\beta, x) \right] = K(\beta, x)
\]

where, for all \( \beta \in (0, \infty) \)

\[
K(\beta, x) := K_3 \sum_{k=0}^{\infty} C'_k(\beta, x) + K_3 \sum_{k=0}^{\infty} J_{-k}(\beta, x) \leq K(\infty, x) \in (0, \infty) .
\]

Hence, using the dominated convergence theorem we conclude that

\[
\lim_{j \to \infty} j^{3/2} E \left[ H_{i,n}(\exp \{-\beta a_{i,j}\}) \right] = \lim_{j \to \infty} j^{3/2} E \left[ Y_j(\beta; \tilde{b}_i) \right] \\
= E \left[ \lim_{j \to \infty} j^{3/2} Y_j(\beta; \tilde{b}_i) \right] = E \left[ K(\beta; \tilde{b}_i) \right] .
\]

Observe that in view of (45) the arguments above are valid for the case \( \beta = \infty \), i.e. for the case when the expression \( 1 - \exp \{-\beta u \tilde{a}_j\} \) is everywhere replaced by \( 1 \). This corresponds to studying the asymptotic behavior of \( P (A_i(n)) \) as
\[ n \to \infty. \text{ Therefore, } w_i = \mathbb{E}[K(\infty; \bar{b}_i)] \text{ in [6]. Thus, for each fixed } i \text{ and } j = n - i \]

\[
\lim_{n \to \infty} \mathbb{E}[e^{-\beta \hat{Y}_{i,n}}|A_i(n)] = 1 - \lim_{n \to \infty} \frac{\mathbb{E}[H_{i,n}(\exp\{-\beta a_{i,n}\})]}{\mathbb{P}(A_i(n))} = 1 - \lim_{n \to \infty} (n - i)^{3/2} \mathbb{E}[\Upsilon_j(\beta; \bar{b}_i)] = 1 - \lim_{n \to \infty} (n - i)^{3/2} \mathbb{P}(A_i(n)) = 1 - \lim_{n \to \infty} (n - i)^{3/2} \mathbb{E}[\Upsilon_j(\beta; \bar{b}_i)] = 1 - \lim_{n \to \infty} (n - i)^{3/2} \mathbb{P}(A_i(n)).
\]

Again by monotonicity of \( K(\beta, x) \) and (42) and (47) it follows that

\[
\lim_{\beta \downarrow 0} K(\beta, x) = 0
\]

and

\[
\lim_{\beta \downarrow 0} \mathbb{E}[K(\beta; \bar{b}_i)] = \mathbb{E}\left[\lim_{\beta \downarrow 0} K(\beta; \bar{b}_i)\right] = 0.
\]

Thus,

\[
\mathbb{P}\left(\hat{Y}_{i,\infty} < \infty\right) = 1.
\]

On the other hand, the inequality \( K(\beta; x) \leq K(\infty; x) \leq K(\infty; 1) < \infty \) and the dominated convergence theorem give

\[
\lim_{\beta \uparrow \infty} \mathbb{E}[K(\beta; \bar{b}_i)] = \mathbb{E}\left[\lim_{\beta \uparrow \infty} K(\beta; \bar{b}_i)\right] = \mathbb{E}[K(\infty; \bar{b}_i)]
\]

implying

\[
\lim_{\beta \uparrow \infty} \Lambda(\beta) = \mathbb{P}\left(\hat{Y}_{i,\infty} = 0\right) = 0.
\]

This completes the proof of point 2) of Theorem [1].

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