Solution of the Boussinesq equation using evolutionary vessels

Andrey Melnikov
Drexel University
December 11, 2013

Abstract

In this work we present a solution of the Boussinesq equation. The derived formulas include solitons, Schwartz class solutions and solutions, possessing singularities on a closed set $Z$ of $\mathbb{R}^2 (x,t)$ domain, obtained from the zeros of the tau function. The idea for solving the Boussinesq equation is identical to the (unified) idea of solving the KdV and the evolutionary NLS equations: we use a theory of evolutionary vessels. But a more powerful theory of non-symmetric evolutionary vessels is presented, inserting flexibility into the construction and allowing to deal with complex-valued solutions. A powerful scattering theory of Deift-Tomei-Trubowitz for a three dimensional operator, which is used to solve the Boussinesq equation, fits into our setting only in a particular case. On the other hand, we create a much wider class of solutions of the Boussinesq equation with singularities on a closed set $Z$.

Contents

1 Introduction 2

2 Scattering theory of the operator $L$ 4
   2.1 Non-symmetric regular vessel, realizing a scattering theory of $L$ 4
   2.2 Structure of the moment $H_0(x)$ 8

3 Vessels with unbounded operators. Standard construction of a vessel 10

4 “Uniqueness” of the scattering data 12

5 Choice of the parameters realizing the Boussinesq equation 13
   5.1 Realizing the Boussinesq equation by a Boussinesq vessel 13
   5.2 Standard construction of a Boussinesq vessel 13

6 Examples of solutions of the Boussinesq equation 15
   6.1 Solitons 15
   6.2 Solutions, belonging to the Schwartz class 16
   6.3 General solutions 17
1 Introduction

The Boussinesq equation \[ B_{ou72} \]

\[
q_{tt} = \frac{\partial^2}{\partial x^2} [3q_{xx} - 12q^2]
\] (1)

is a foundation of a shallow water theory. A fundamental solution of this equation was presented by P. Deift, C. Tomei and E. Trubowitz in \[ DTT82 \]. The basic idea in this work, presented by V. E. Zakharov \[ Zac74 \], is a development of a scattering theory of a three dimensional operator

\[
\tilde{L} = i \frac{d^3}{dx^3} + \frac{1}{i} (\frac{d}{dx} q + \frac{d}{dx} \bar{q}) + p
\]

and using the fact that for \( Q = i(3 \frac{d^2}{dx^2} - 4q) \), the operators \( \tilde{L}, Q \) constitute a Lax pair:

\[
\frac{d}{dt} \tilde{L} = [Q, \tilde{L}] = Q\tilde{L} - \tilde{L}Q.
\]

It is assumed in \[ DTT82 \] that \( p(x), q(x) \) are in a Schwartz space, but the theory goes through for \( q(x), \bar{q}(x) \) with only a finite number of derivatives and a finite order of decay. The scattering data in this case consists of a list of 6 functions, whose evolving with \( t \) under an analogue of \( Q \) is studied producing a solution of the Boussinesq equation (1) with a given initial value \( q(x, 0) = q(x) \). Since we will use instead of \( \tilde{L} \) its multiplication on \(-i\), we define

\[
L = -i\tilde{L} = \frac{d^3}{dx^3} - 2q \frac{d}{dx} - (q' + ip)
\] (2)

and a very particular (inverse) scattering of this operator will be researched. More precisely, we will discuss solutions of

\[
Lu = \frac{d^3}{dx^3} - 2q \frac{d}{dx} - (q' + ip) = k^3 u, \quad p'(x) = P(\int q, q, q', \ldots, q^{(n)})
\] (3)

In this setting \( q(x) \) is an “arbitrary” function, but \( p(x) \) is derived from \( q(x) \) (see formula (38)) using the formula \( p'(x) = P(\int q, q, q', \ldots, q^{(n)}) \) for a polynomial \( P \), which is not derived explicitly, since we do not use it.

In order to solve (1), we present a similar to \[ DTT82 \] scheme, where we use a special case of the inverse scattering theory: we create inverse scattering of (2), where \( p(x) \) is not arbitrary and then evolve \( q(x) \) with \( t \). Let us explain first the (inverse) scattering theory. The scattering data is encoded in our setting in a matrix-valued function. In fact, there is an almost “one-to-one” correspondence (Theorem \[ BGR90 \]) between the coefficients \( q(x) \), defining \( L \) (2) and \( 3 \times 3 \) matrix-functions \( S(\lambda) \) possessing a realization

\[
S(\lambda) = I - C_0 X_0^{-1}(\lambda I - A)^{-1} B_0 \sigma_1,
\]

\[
\sigma_1 = \begin{bmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{bmatrix}, \quad AX_0 + X_0 A_z + B_0 \sigma_1 C_0 = 0.
\]

1 The full Boussinesq equation \( q_{tt} = \frac{\partial^2}{\partial x^2} [q + q_{xx} - 4q^2] \) was shown by McKean \[ McK78 \] to be equivalent to (1), significantly simplifying algebra computations.

2 We will usually denote by \( q' \) the partial derivative \( \frac{\partial}{\partial x} q \), and by \( \dot{q} \) the partial derivative \( \frac{\partial}{\partial t} q \). Similarly, for the higher derivatives: \( q'' \) stands for \( \frac{\partial^2}{\partial x^2} q \), etc.
Here for an auxiliary Hilbert space $\mathcal{H}$ the linear operators act as follows: $C_0 : \mathcal{H} \to \mathbb{C}^3$, $A_\zeta, X_0, A : \mathcal{H} \to \mathcal{H}$, $B_0 : \mathbb{C}^3 \to \mathcal{H}$. Let us assume for the simplicity of the presentation, that all the operators are bounded. In order to construct $q(x)$, uniquely defined from $S(\lambda)$, use the following 4 steps, fixing $\sigma_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$,

$$\gamma = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} :$$

1. solve for $B(x)$ from $B' \sigma_1 = -AB \sigma_2 - B \gamma$ (5) with initial $B(x_0) = B_0$,
2. solve for $C(x)$ from $\sigma_1 C' = -\sigma_2 CA_\zeta + C \gamma$ (6) with initial $C(x_0) = C_0$,
3. solve for $X(x)$ from $X'(x) = B(x) \sigma_2 C(x)$ (7), $X(x_0) = X_0$,
4. define $\gamma(x) = \gamma + \sigma_2 C(x) X^{-1}(x) B(x) \sigma_1 - \sigma_1 C(x) X^{-1}(x) B(x) \sigma_2$ (5), for all points where $X(x)$ is invertible.

Then we prove in Theorem 4 that the function

$$S(\lambda, x) = I - C(x) X^{-1}(x) (\lambda I - A)^{-1} B(x) \sigma_1$$

is a Bäcklund transformation for the operator $L$ from the trivial $L_0 = \frac{d^3}{dx^3}$ to a more complicated one $L$ (2), in which $q(x) = -\frac{3}{2} \frac{d^2}{dx^2} \ln \det (X^{-1}_0 X(x))$ and $p(x)$ is defined from $q(x)$ up to a constant. In the regular case, explained here the coefficients $q(x), p(x)$ are analytic functions at all points, where $X(x)$ is invertible.

Letting the operators $C, B, X$ further evolve with respect to $t$, we will obtain a solution of the Boussinesq equation (1). In order to show it, we take three additional matrices (15)

$$\bar{\sigma}_1 = \sigma_1, \quad \bar{\sigma}_2 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \bar{\gamma} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & i \end{bmatrix}$$

and follow these steps (last step is the same as the previous fourth step):

4. solve for $B(x, t)$ from $\dot{B} \bar{\sigma}_1 = -AB \bar{\sigma}_2 - B \bar{\gamma}$ (50) (see footnote 2) with initial $B(x, t_0) = B(x)$,
5. solve for $C(x, t)$ from $\dot{\bar{\sigma}}_1 C = -\bar{\sigma}_2 CA_\zeta + C \bar{\gamma}$ (51) with initial $C(x, t_0) = C(x)$,
6. solve for $X(x, t)$ from $\dot{X}(x) = B(x, t) \bar{\sigma}_2 C(x, t)$ (7), $X(x, t_0) = X(x)$,
7. define $\gamma(x, t) = \gamma + \sigma_2 C(x, t) X^{-1}(x, t) B(x, t) \sigma_1 - \sigma_1 C(x, t) X^{-1}(x, t) B(x, t) \sigma_2$ (5), for all points where $X(x, t)$ is invertible.

It turns out that $q(x, t) = -\frac{3}{2} \frac{\partial^2}{\partial x^2} \ln \det (X^{-1}_0 X(x, t))$ (37) and satisfies the Boussinesq equation (1) (see Theorem 17 for details). Construction of the coefficient $q(x)$ from a realized function $S(\lambda)$ is called the standard construction of a vessel and is presented in Section 3.

The simplicity and richness of this construction is best revealed in soliton formulas (Section 6.1). By choosing the inner space $\mathcal{H} = \mathbb{C}$ - the one dimensional Hilbert space, we create a classical soliton $q(x, t) = -\frac{9 \mu^2}{2 \cosh^2(\sqrt{\frac{3}{2}} \mu(x + t \mu))}$ (58) and another one $q(x, t) = -\frac{18 e^{\sqrt{\frac{3}{2}} \mu(x + 2t \mu)} \mu^2}{(e^{\sqrt{\frac{3}{2}} \mu} + e^{4\sqrt{\frac{3}{2}} \mu})^2}$ (59). Here $\mu$ is an arbitrary complex parameter.

This construction seems to be more concrete and suitable for the solution of the Boussinesq equation (4), because it uses just “enough” of the very powerful and complicated inverse scattering theory,
developed in [DTT82]. Notice that in this later work the coefficients \(q(x), p(x)\) are quite arbitrary. In our work, on the other hand, the coefficient \(p(x)\) is uniquely determined (up to a constant) from \(q(x)\). Still, the formulas enable to produce solutions, applying the standard construction of a vessel, explained above, to different \(S(\lambda)\). Moreover, at the same section [3] we show that much more general classes of solutions arise, as we impose as few as possible restrictions on \(S(\lambda)\). The solutions, presented in [DTT82] are either in the Schwartz class or exponentially decaying and correspond in our setting to an analytic \(S(\lambda)\), possessing jumps along the real negative axis.

Finally, this work presents a very general setting for a construction of solutions of (1). We find necessary regularity assumptions (11) on the operator \(B(x, t)\) such that we can create a vessel, and hence a solution of (1). This is proved in Theorem [19]. Following the remark after Theorem [19] one can easily construct a solution, which fails to be five times \(x\)-differentiable (four times are necessary for the existence of (1)).

2 Scattering theory of the operator \(L\)

We start from the definition of the vessel parameters, which create an inverse scattering theory of \(L\) [2].

**Definition 1.** The vessel parameters are defined as follows

\[
\sigma_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \gamma = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}.
\]

2.1 Non-symmetric regular vessel, realizing a scattering theory of \(L\)

**Definition 2.** A (regular, non-symmetric) vessel, associated to vessel parameters (see Definition [7]) is a collection of operators, spaces and an interval \(I\)

\[
\mathfrak{W}_{reg} = (C(x), A_\zeta, X(x), A, B(x); \sigma_1, \sigma_2, \gamma, \gamma_*(x); \mathcal{H}, \mathbb{C}^3; I),
\]

where the bounded operators \(C(x) : \mathcal{H} \to \mathbb{C}^3, A_\zeta, X(x), A : \mathcal{H} \to \mathcal{H}, B(x) : \mathbb{C}^3 \to \mathcal{H}\) and a \(3 \times 3\) matrix function \(\gamma_*(x)\) satisfy the following vessel conditions:

\[
\frac{\partial}{\partial x} B = -(A B \sigma_2 + B \gamma) \sigma_1^{-1}, \quad (5)
\]

\[
\frac{\partial}{\partial x} C = \sigma_1^{-1} (\gamma C - \sigma_2 CA_\zeta), \quad (6)
\]

\[
\frac{\partial}{\partial x} X = B \sigma_2 C, \quad (7)
\]

\[
\gamma_* = \gamma + \sigma_2 CX^{-1} B \sigma_1 - \sigma_3 CX^{-1} \sigma_1 C, \quad (8)
\]

\[
A X + X A_\zeta = -B \sigma_1 C. \quad (9)
\]

The operator \(X(x)\) is assumed to be invertible on the interval \(I\). If \(A^*_\zeta = A^*\) and \(C = B^*\) we call such a vessel symmetric.

**Remarks:** 1. Notice that the operators \(C(x), X(x), B(x)\) are globally defined for all \(x \in \mathbb{R}\): \(C(x), B(x)\) are solutions of operator-valued differential equations with constant coefficients and \(X(x)\) is obtained from them by a simple integration. 2. For the definition of the matrix-function \(\gamma_*(x)\) we need the invertability of the operator \(X(x)\), so in general we may suppose that \(\gamma_*(x)\) is defined for all \(x \in \mathbb{R}\), except for those points where \(X(x)\) is not invertible. For simplicity, we take an interval \(I\) and in a more general setting we consider \(\gamma_*(x)\) on \(\mathbb{R}\), except fro the points where \(X(x)\) is not invertible.
Remark: Notice that it is enough to prove that the transfer function $s$ satisfies the following ODE:

$$AX(x) + X(x)A_x + B(x)s_1 C(x) = 0 \text{ holds for a fixed } x_0 \in I, \text{ then it holds for all } x \in I.$$  

Proof: By differentiating the left hand side of (9), we will obtain that it is zero.

By the definition, the transfer function of the vessel $\mathcal{V}_{x_0}$ is defined as follows:

$$S(\lambda, x) = I - C(x)X^{-1}(x)(A - A)^{-1}B(x)s_1. \quad (10)$$

Notice that poles and singularities of $S$ with respect to $\lambda$ are determined by $A$ only. We would like to show that the function $S(\lambda, x)$ realizes a Bäcklund transformation of the corresponding LDEs: multiplication by the function $S(\lambda, x)$ maps $[Ls01, AMV12, Mel11]$ a solution of the input Linear Differential Equation (LDE) with the spectral parameter $\lambda$

$$\lambda s_2 u(\lambda, x) - \sigma_1 \frac{\partial}{\partial x} u(\lambda, x) + \gamma u(\lambda, x) = 0. \quad (11)$$

to a solution of the output LDE with the same spectral parameter

$$\lambda s_2 y(\lambda, x) - \sigma_1 \frac{\partial}{\partial x} y(\lambda, x) + \gamma^* y(\lambda, x) = 0. \quad (12)$$

The function $\gamma^*$ is defined by the Linkage condition (3). The fundamental solutions of (11) and (12), which are equal to $I$ (the identity matrix matrix at $x = 0$) are denoted usually by $\Phi(\lambda, x)$ and $\Phi^*(\lambda, x)$.

In other words, these are matrix functions satisfying:

$$\lambda s_2 \Phi(\lambda, x) - \sigma_1 \frac{\partial}{\partial x} \Phi(\lambda, x) + \gamma \Phi(\lambda, x) = 0, \quad \Phi(\lambda, 0) = I, \quad (13)$$

$$\lambda s_2 \Phi^*(\lambda, x) - \sigma_1 \frac{\partial}{\partial x} \Phi^*(\lambda, x) + \gamma^* \Phi^*(\lambda, x) = 0, \quad \Phi^*(\lambda, 0) = I. \quad (14)$$

Theorem 4 (Vessel=Bäcklund transformation $[Ls01, AMV12, Mel11]$). Suppose that $u(\lambda, x)$ satisfies (11), then $y(\lambda, x) = S(\lambda, x)u(\lambda, x)$ satisfies (12).

Remark: Notice that it is enough to prove that the transfer function satisfies the following ODE:

$$\frac{\partial}{\partial x} S(\lambda, x) = s_2^{-1}(\sigma_2 \lambda + \gamma^*) S(\lambda, x) - S(\lambda, x) s_2^{-1}(\sigma_2 \lambda + \gamma). \quad (15)$$

Proof: First we calculate

$$\frac{d}{dx} (C(x)X^{-1}(x)) = s_2^{-1}(\gamma C - \sigma_2 C A_x)X^{-1}(x) - CX^{-1}B_s C X^{-1} = \text{ by (9)} \quad (16)$$

$$= s_2^{-1}(\gamma C - \sigma_2 C A_x)X^{-1} - CX^{-1}B_s C X^{-1} \quad (10)$$

$$= s_2^{-1}(\gamma C - \sigma_2 C A_x)X^{-1} - CX^{-1}B_s C X^{-1} \quad \text{ by (8)}$$

$$= s_2^{-1}(\gamma C - \sigma_2 C A_x)X^{-1} - CX^{-1}B_s C X^{-1} \quad \text{ by (8)}$$

So we obtain that

$$\frac{d}{dx} (C X^{-1}) = s_2^{-1}(\sigma_2 C X^{-1} A + \gamma^* C X^{-1}). \quad (16)$$
Let us differentiate next the transfer function using (10):

\[
\frac{d}{dx} S(\lambda, x) = -\frac{d}{dx} (C^{-1})(\lambda - A)^{-1}B\sigma_1 - C^{-1}(\lambda - A)^{-1} \frac{d}{dx} B\sigma_1 = \text{by (15), (16)}
\]

\[
= (\sigma_1^{-1} \sigma_2 C^{-1} A + \sigma_1^{-1} \gamma \sigma_2 C^{-1})(\lambda - A)^{-1} B\sigma_1 - C^{-1}(\lambda - A)^{-1}(A B\sigma_2 + B\gamma)
\]

\[
= \sigma_1^{-1} \gamma_s (S - I) - (S - I) \sigma_1^{-1} \gamma + \sigma_1^{-1} \sigma_2 C^{-1} A(\lambda - A)^{-1} B\sigma_1 - C^{-1}(\lambda - A)^{-1} A B\sigma_2
\]

\[
= \text{insert } A = A + \lambda \text{ and expand}
\]

\[
= \sigma_1^{-1} \gamma_s (S - I) - (S - I) \sigma_1^{-1} \gamma + \sigma_1^{-1} \sigma_2 C^{-1} B\sigma_1 - \lambda \sigma_1^{-1} \sigma_2 C^{-1}(\lambda - A)^{-1} B\sigma_1 - \lambda \sigma_1^{-1} \sigma_2 C^{-1}(\lambda - A)^{-1} B\sigma_2
\]

\[
= \text{using (5), (10)}
\]

\[
= \sigma_1^{-1} \gamma_s (S - I) - (S - I) \sigma_1^{-1} \gamma + \sigma_1^{-1} (\gamma_s - \gamma) + \lambda \sigma_1^{-1} \sigma_2 S - S \sigma_1^{-1} \lambda \sigma_2
\]

\[
= \sigma_1^{-1} (\lambda \sigma_2 + \gamma) S - S \sigma_1^{-1} (\lambda \sigma_2 + \gamma).
\]

Expanding the transfer function \(S(\lambda, x)\) into a Taylor series around \(\lambda = \infty\), we obtain a notion of the moment:

\[
S(\lambda, x) = I - \sum_{n=0}^{\infty} \frac{H_n(x) \sigma_1}{\lambda^{n+1}},
\]

where by the definition the \(n\)-th moment \(H_n(x)\) of the function \(S(\lambda, x)\) is

\[
H_n(x) = C(x) X^{-1}(x) A^n B(x).
\]

(17)

Using the zero moment, for example, we obtain that the so called “linkage condition” \(\Box\) is equivalent to

\[
\gamma_s(x) = \gamma + \sigma_2 H_0(x) \sigma_1 - \sigma_1 H_0(x) \sigma_2.
\]

There is also a recurrent relation between the moments \(H_n(x)\), arising from \(\Box\).

**Theorem 5.** The following recurrent relation between the moments of the vessel \(\mathcal{V}_{reg}\) holds

\[
\sigma_1^{-1} \sigma_2 H_{n+1} - H_{n+1} \sigma_1^{-1} = (H_n)' - \sigma_1^{-1} \gamma_s H_n + \gamma \sigma_1^{-1} H_n.
\]

(18)

**Proof:** Follows from the differential equation \(\Box\) by plugging \(S(\lambda, x) = I - \sum_{n=0}^{\infty} \frac{H_n(x) \sigma_1}{\lambda^{n+1}}\).

Let us investigate more carefully the LDEs (11) and (12). Denote \(u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\), then (11) becomes

\[
\begin{bmatrix} \lambda u_1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} u_1' \\ u_2' \\ u_3' \end{bmatrix} + \begin{bmatrix} 0 \\ u_3 \\ -u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.
\]

Solving this we obtain that

\[
\begin{aligned}
\begin{cases}
u_2 = -u_1', \\
u_3 = u_2' = -u_1'', \\
u_1''' = -\lambda u_1.
\end{cases}
\end{aligned}
\]

(19)

We can see that actually this equation is equivalent to a third-order differential equation with the spectral parameter \(\lambda\):

\[
u_1''' = -\lambda u_1.
\]

(20)
In order to analyze (12), we denote first moment $H_0(x) = [\pi_{ij}] = \begin{bmatrix} \pi_{11} & \pi_{12} & \pi_{13} \\ \pi_{21} & \pi_{22} & \pi_{23} \\ \pi_{31} & \pi_{32} & \pi_{33} \end{bmatrix}$, and as a result, the linkage condition becomes

$$\gamma_e = \gamma + \begin{bmatrix} \pi_{13} - \pi_{31} & \pi_{12} & \pi_{11} \\ -\pi_{21} & 0 & 0 \\ -\pi_{11} & 0 & 0 \end{bmatrix}.$$ 

Denote next $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ and plugging the expression for $\gamma_e$ just derived into (12), we will obtain that

$$\begin{align*}
\begin{bmatrix} \lambda y_1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix} + \begin{bmatrix} (\pi_{13} - \pi_{31})y_1 + \pi_{12}y_2 + \pi_{11}y_3 \\ -\pi_{21}y_1 + y_3 \\ -\pi_{11}y_1 - y_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{align*}$$

or solving this:

$$\begin{cases}
q_2 = -\pi_{11}y_1 - y_1', \\
y_3 = \pi_{21}y_1 + y_2' = \pi_{21}y_1 - y_1y_1' - \pi_{11}y_1'' - y_1'', \\
y_1'' - 2qy_1 - (q' + p)y_1 = -\lambda y_1,
\end{cases}$$

where

$$q(x) = \frac{\pi_{11} + \pi_{12} + \pi_{21} - 2\pi_{11}}{2}, \quad p(x) = -i(-\pi_{13} + \pi_{31} + \pi_{11}(\pi_{12} - \pi_{21}) - \frac{\pi_{12} - \pi_{21}}{2}).$$

In other words the equation (12) is equivalent to

$$y_1'' - 2qy_1 - (q' + ip)y_1 = -\lambda y_1.$$  \hspace{1cm} (23)

Now we are ready to justify the term “scattering matrix” attached to $S(\lambda, 0)$. An independent set

$$y(\lambda, x) = \begin{bmatrix} y_1(\lambda, x) \\ y_2(\lambda, x) \\ y_3(\lambda, x) \end{bmatrix}$$

of solutions of (2), can be derived from (12) in the following form

$$y(\lambda, x) = S(\lambda, x)\Phi(\lambda, x) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} \Phi_e(\lambda, x)S(\lambda, 0) \end{bmatrix}.$$

The fundamental matrix $\Phi(\lambda, x)$, solving (11) with the initial condition $\Phi(\lambda, 0) = I$ (the identity matrix) is obtained from (10) and can be explicitly written as

$$\Phi(\lambda, x) = \frac{1}{3\alpha^2} \begin{bmatrix} R_1 & \frac{R_2}{k} & -\frac{R_3}{k} \\ \frac{kR_3}{k^2} & R_1 & -\frac{R_2}{k} \\ -k^2R_2 & -kR_3 & R_1 \end{bmatrix},$$

where $E_1 = e^{-kx}, E_2 = e^{-\alpha kx}, E_3 = e^{-\alpha^2 kx}$ for $\alpha = e^{2\pi i/3}$ ($\alpha^3 = 1$). This matrix is analytic in $\lambda$, because examining Taylor series of $E_i$, we will come to the conclusion that all the entries of $\Phi$ depend on $k^3 = \lambda$. The structure of $S(\lambda, x)$ is also known from (10), so we can study solutions of (24) or equivalently of (12), creating in this manner the (inverse) scattering of $L$. \hspace{1cm} (24)
2.2 Structure of the moment $H_0(x)$

Let us examine the recurrence relation (18). We will research for the simplicity of the presentation the structure of the first moment $H_0(x)$, but almost the same structure will actually apply for all moments.

Let us denote

$$H_1(x) = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix}. $$

Then the left hand side of (18) becomes

$$\sigma_1^{-1} \sigma_2 H_1(x) - H_1(x) \sigma_2 \sigma_1^{-1} = \begin{bmatrix} 0 & 0 & -g_{11} \\ 0 & 0 & -g_{21} \\ g_{11} & g_{12} & g_{13} - g_{31} \end{bmatrix},$$

which must be equal to

$$(H_0)_g - \sigma_1^{-1} \gamma_e H_0 + H_0 \gamma \sigma_1^{-1} = \begin{bmatrix} \pi_1^2 + \pi_{12} + \pi_{21} + \pi_{11} & \pi_{11} \pi_{12} - \pi_{13} + \pi_{22} + \pi_{12} & \pi_{11} \pi_{13} + \pi_{23} + \pi_{11}' \\ \pi_{11} \pi_{21} - \pi_{31} + \pi_{22} + \pi_{21} & \pi_{12} \pi_{21} - \pi_{23} - \pi_{32} + \pi_{22}' & \text{Res}(23) \\ -\pi_{11} \pi_{13} - \pi_{12} \pi_{21} + \pi_{32} + \pi_{31}' & \text{Res}(32) & \text{Res}(33) \end{bmatrix}$$,

where

$$\text{Res}(23) = \pi_{13} \pi_{21} - \pi_{33} + \pi_{23}',$$

$$\text{Res}(32) = -\pi_{12} (\pi_{13} - \pi_{31} + \pi_{22}) - \pi_{11} \pi_{32} - \pi_{33} + \pi_{32}',$$

$$\text{Res}(33) = -\pi_{12} \pi_{23} + \pi_{13} (-\pi_{13} + \pi_{31}) - \pi_{11} \pi_{33} + \pi_{33}'. $$

Equating we obtain that

$$g_{11} = -\pi_{11} \pi_{13} - \pi_{23} - \pi_{11}', \quad g_{21} = -\text{Res}(23), \quad g_{12} = \text{Res}(32), \quad g_{13} - g_{31} = \text{Res}(33)$$

and the following system of equations:

$$\begin{cases} \pi_{11}^2 + \pi_{12} + \pi_{21} + \pi_{11}' = 0, \\
\pi_{11} \pi_{12} - \pi_{13} + \pi_{22} + \pi_{12}' = 0, \\
\pi_{11} \pi_{21} - \pi_{31} + \pi_{22} + \pi_{21}' = 0, \\
\pi_{12} \pi_{21} - (\pi_{23} + \pi_{32}) + \pi_{22}' = 0, \\
-\pi_{11} \pi_{13} - \pi_{23} - \pi_{13}' = -\pi_{11} \pi_{13} - \pi_{12} \pi_{21} + \pi_{32} + \pi_{31}' (= g_{11}). \end{cases}$$

or rearranging

$$\begin{cases} \pi_{21} = -\left(\pi_{12} + \pi_{11}' + \pi_{11}\right), \\
\pi_{22} = -\pi_{11} \pi_{12} + \pi_{13} - \pi_{12}', \\
\pi_{22}' = -\pi_{11} \pi_{21} + \pi_{31} - \pi_{21}', \\
\pi_{32} = \pi_{12} \pi_{21} - \pi_{23} + \pi_{22}', \\
\pi_{32}' = \pi_{12} \pi_{21} - \pi_{23} - \pi_{31}' - \pi_{13}'. \end{cases}$$

Plugging the fourth equation of (20) into the last one, we will obtain that the later becomes $\pi_{31} + \pi_{13}' = -\pi_{22}'$ or requiring a normalization

$$\pi_{31}(x_0) + \pi_{13}(x_0) = -\pi_{22}(x_0)$$

we obtain

$$\pi_{31} + \pi_{13} = -\pi_{22}. $$

(28)
In a similar manner, one can solve for some other entries and we summarize these intermediate results in the next lemma. Additional relations between the entries of \(H_0(x)\) are obtained if we consider (28) for \(n = 1\). For example, formulas similar to (29) are as follows:

\[
\begin{cases}
\pi_{11}g_{11} + g_{12} + g_{21} + g_{11}' = 0, \\
\pi_{11}g_{12} - g_{13} + g_{22} + g_{12}' = 0, \\
g_{11}π_{21} - g_{31} + g_{22} + g_{21}' = 0, \\
g_{12}π_{21} - (g_{23} + g_{32}) + g_{22}' = 0, \\
-π_{11}g_{13} - g_{23} - g_{13}' = -π_{11}g_{31} + g_{11}(π_{31} - π_{13}) - π_{12}g_{21} + g_{32} + g_{31}'.
\end{cases}
\]

From these formulas follow the following relations

\[
\begin{cases}
π_{11}g_{11} + g_{12} + g_{21} + g_{11}' = 0, \\
π_{11}g_{12} - π_{11}g_{21} - (g_{13} - g_{31}) + g_{12}' - g_{21}' = 0,
\end{cases}
\]

which can be rewritten as relations on the entries of \(H_0(x)\) in view of (25). Other three relations, which are obtained are

\[
\begin{cases}
π_{11}g_{12} - g_{13} + g_{22} + g_{12}' + g_{11}π_{21} - g_{31} + g_{22} + g_{21}' = 0, \\
g_{12}π_{21} - (g_{23} + g_{32}) + g_{22}' = 0, \\
-π_{11}g_{13} - g_{23} - g_{13}' = -π_{11}g_{31} + g_{11}(π_{31} - π_{13}) - π_{12}g_{21} + g_{32} + g_{31}'.
\end{cases}
\]

which serve to find \(g_{22}', g_{23} + g_{32}, g_{13} + g_{31}'.\) It turns out that actually the relations (25) are not independent and a formula for \(π_{12}π_{11}'\) is derived from them. All these results are summarized in the next Lemma and we notice that the exact formulas for \(π_{23}', π_{33}'\) are omitted, because we are not interested in their form:

**Definition 7.** Tau-function of the vessel \(V\) is defined as follows:

\[
τ(x) = \det(X^{-1}(x_0)X(x)),
\]

where \(x_0 \in I\) is an arbitrary point.

The fact that this object is well-defined follows from an equivalent to (7) equation

\[
X(x) = X_0 + \int_0^x B(y)σ_2C(y)dy,
\]

so that \(X^{-1}_0X(x) = I + T(x)\) for a trace-class operator \(T\). Using this notion we obtain the following
Theorem 8. The coefficient \( q(x) \) possesses the following formula:

\[
q(x) = -\frac{3}{2} \frac{d^2}{dx^2} \ln \tau(x) = -\frac{3}{2} \frac{d}{dx} \pi_{11}. \tag{37}
\]

The derivative of the coefficient \( p(x) \) is a differential polynomial in \( \pi_{11} \):

\[
p'(x) = P(\pi_{11}, \pi_{11}', \ldots), \quad P \text{ - a polynomial.} \tag{38}
\]

Proof: Inserting (30) into the formula (22) for \( q(x) \) we find that

\[
q(x) = \frac{\pi_{11}^2 + \pi_{12} + 2\pi_{11} - 2\pi_{11}'}{2} = -\frac{3}{2} \pi_{11}.
\]

Using a formula for the determinant of an operator, we obtain that (see [GK69, AMV12] for details)

\[
\frac{\tau'(x)}{\tau(x)} = \text{tr}(X(x)X^{-1}(x)) = \text{tr}(B(x)\sigma_2B^*(x)X^{-1}(x)) = \text{tr}(\sigma_2H_0(x)) = \pi_{11}
\]

and the result follows for \( q(x) \). As for \( p(x) \), we differentiate the formula for \( p(x) \), appearing in (22):

\[
p'(x) = -i(-\pi_{13} + \pi_{31}) - i \frac{d}{dx}(\pi_{11}(\pi_{12} - \pi_{21}) - \frac{\pi_{12}' - \pi_{21}'}{2}).
\]

Then using (32), (44), (30) and (35) we will obtain a differential polynomial in \( \pi_{11} \).

One can also derive the following formulas, corresponding to the symmetric case \( C = B^*, A_{C} = A^* \), plugging the definition (36) and using the fact that \( H_0(x) \) is self-adjoint:

Lemma 9. For the symmetric case, \( C = B^*, A_{C} = A^* \), the following relations between the entries of the first moment \( H_0(x) = H_0^*(x) \) hold

\[
\Re \pi_{12} = -\frac{1}{2} \frac{\tau''}{\tau}, \quad \pi_{22} = \frac{1}{3} \frac{\tau''}{\tau}, \quad \Re \pi_{13} = -\frac{1}{6} \frac{\tau''}{\tau},
\]

\[
\Im \pi_{13} = \pi_{11} \Im \pi_{12} + \Im \pi_{12}',
\]

\[
\Re \pi_{23} = \frac{3\pi_{12}^2}{2} + \frac{2}{9} (q(x)^2 - \frac{1}{4} q''(x)) + \frac{1}{8} \frac{\tau^{(4)}}{\tau}
\]

3 Vessels with unbounded operators. Standard construction of a vessel

The ideas presented in this Section can be found in [Mela] for the symmetric case. The class of functions serving as “initial conditions” for the transfer functions of vessels is defined as follows

Definition 10. Class \( \mathcal{R}(\sigma_1) \) consist of \( p \times p \) matrix-valued functions \( S(\lambda) \) of the complex variable \( \lambda \), possessing the following representation:

\[
S(\lambda) = I - C_0 X_0^{-1}(\lambda I - A)^{-1}B_0 \sigma_1 \tag{39}
\]

where for an auxiliary Hilbert space \( \mathcal{H} \) there are defined operators \( C_0 : \mathcal{H} \to \mathbb{C}^3, A_\lambda X_\lambda, A : \mathcal{H} \to \mathcal{H}, B_0 : \mathbb{C}^3 \to \mathcal{H}. \) A general matrix-function \( S(\lambda) \), representable in such a form is called realized. Moreover, the operators are subject to the following assumptions:

1. the operators \( A, A_\lambda \) have dense domains \( D(A), D(A_\lambda) \). \( A, A_\lambda \) are generators of \( C_0 \) semi-groups on \( \mathcal{H} \). Denote the resolvents as follows \( R(\lambda) = (\lambda I - A)^{-1}, R_\lambda(\lambda) = (\lambda I + A_\lambda)^{-1}, \)

2. the operator \( B_0 \) satisfies \( R(\lambda)B_0 c \in \mathcal{H} \) for all \( \lambda \notin \text{spec}(A), c \in \mathbb{C}^3, \)
3. the operator $X_0$ is bounded and invertible,

4. the Lyapunov equation holds for all $\lambda \notin \text{spec}(A) \cup \text{spec}(-A_\zeta)$:

$$R(\lambda)X_0 - X_0R_\zeta(\lambda) + R(\lambda)B_0\sigma_1C_0R_\zeta(\lambda) = 0.$$  \hspace{1cm} (40)

We call an element of $\mathcal{R}(\sigma_1)$ as scattering matrix-function. The subclass $\mathcal{U}(\sigma_1) \subseteq \mathcal{R}(\sigma_1)$ consists of symmetric functions, i.e. satisfying $S(\lambda)\sigma_1^{-1}S^*(-\lambda^*) = \sigma_1^{-1}$. The Schur class $\mathcal{SU}(\sigma_1) \subseteq \mathcal{U}$ consists of symmetric functions, for which $X_0$ is a positive operator. The sub-classes of rational functions in $\mathcal{SU}, \mathcal{U}, \mathcal{R}$ are denoted by $r\mathcal{SU}, r\mathcal{U}, r\mathcal{R}$ respectively.

When $S(\lambda)$ is just analytic at infinity (hence $A$ must be bounded), there is a very well known theory of realizations developed in [BGR90]. For analytic at infinity and symmetric, i.e. satisfying $S^*(-\lambda)\sigma_1 S(\lambda) = \sigma_1$, functions there exists a good realization theory using Krein spaces ($\mathcal{H}$ is a Krein space), developed in [DLdS]3. Such a realization is then translated into a function in $\mathcal{U}(\sigma_1)$. The sub-classes $\mathcal{U}, \mathcal{SU}$ appear a lot in the literature and correspond to the symmetric case. We will not consider these two classes in this work and refer to [AMV12].

Now we present the standard construction of a vessel $\mathfrak{V}$ from the given realized matrix-function $S(\lambda)$:

1. Let $B(x)$ be the unique solution of the following equation ($R(\lambda) = (\lambda I - A)^{-1}$)

$$\frac{\partial}{\partial x}R(\lambda)B = -(A R(\lambda)B + R(\lambda)B\gamma)\sigma_1^{-1}$$  \hspace{1cm} (41)

satisfying $B(x_0) = B_0$. This equation is solvable because the coefficients $\sigma_1, \sigma_2, \gamma$ are constant and $A$ is a generator of a $C_0$ semigroup. For this equation to hold we must require the following regularity assumptions:

$$\forall \lambda \notin \text{spec}(A): R(\lambda)B(x)\sigma_2 \mathbb{C}^3 \subseteq D(A),$$

$$\forall \lambda \notin \text{spec}(A): R(\lambda)B(x)\gamma \mathbb{C}^3 \subseteq \mathcal{H},$$  \hspace{1cm} (42), (43)

2. Let $C(x)$ be the unique solution of (6) with the initial condition $C(x_0) = C_0$, when we consider the equation (4), applied to vectors in the dense set $D(A_\zeta)$ only.

3. Solve (7) for $X(x)$, satisfying $X(x_0) = X_0$ and let $I$ be an interval, including $x_0$ on which $X(x)$ is invertible. Notice that $B(x)\sigma_2 \mathbb{C}^3 \subseteq \mathcal{H}$ follows from (42).

4. Define $\gamma_\ast(x)$ on I by (8).

The main reason, why we call $S(\lambda)$ as the “scattering data” is the fact that $\gamma_\ast(x)$ (generalized potential) is uniquely determined from $S(\lambda)$ by this construction. The question of uniqueness of $S(\lambda)$ for a given potential $\gamma_\ast(x)$ will be studied in Section 4. A vessel in a more general form is defined as follows

**Definition 11.** The collection of operators and spaces

$$\mathfrak{V} = (C(x), A_\zeta, X(x), A, B(x); \sigma_1, \sigma_2, \gamma, \gamma_\ast(x); \mathcal{H}, \mathbb{C}^p; 1),$$  \hspace{1cm} (44)

is called a vessel, if $C(x): \mathbb{C}^3 \to \mathcal{H}$, $A_\zeta, X(x), A: \mathcal{H} \to \mathcal{H}$, $B(x): \mathbb{C}^p \to \mathcal{H}$ are differentiable linear operators, subject to the following conditions:

1. the operators $A, A_\zeta$ have dense domains $D(A), D(A_\zeta)$. $A, A_\zeta$ are generators of $C_0$ semi-groups on $\mathcal{H}$. Denote the resolvents as follows $R(\lambda) = (\lambda I - A)^{-1}, R_\zeta(\lambda) = (\lambda I + A_\zeta)^{-1}$,

2. $B(x)$ satisfies regularity assumptions (42), (43), and the equation (41).

---

3At the paper [DLdS] a similar result is proved for functions symmetric with respect to the unit circle, but it can be translated using Calley transform into $S^*(-\lambda)\sigma_1 S(\lambda) = \sigma_1$ and was done in [Melb] [AMV12].
3. $C(x)$ satisfies (4) on $D(A_{\zeta}),$
4. $\mathcal{X}(x)$ is bounded, and invertible on $I$ and satisfies (7),
5. the Lyapunov equation holds for all $x \in I, \lambda \not\in \text{spec}(A) \cup \text{spec}(-A_{\zeta}):$
   \[ R(\lambda)\mathcal{X}(x) - \mathcal{X}(x)R_{\zeta}(\lambda) + R(\lambda)B(x)\sigma_{1}C(x)R_{\zeta}(\lambda) = 0. \] (45)
6. $\gamma_{\ast}(x)$ satisfies (5).

The class of the transfer functions is defined as follows

**Definition 12.** Class $\mathcal{I} = \mathcal{I}(\sigma_{1}, \sigma_{2}, \gamma; I)$ consist of $3 \times 3$ matrix-valued (transfer) functions $S(\lambda, x)$ of the complex variable $\lambda$ and $x \in I$ for an interval $I = [a, b]$ possessing the following representation:

\[ S(\lambda, x) = I - C(x)X^{-1}(x)(\lambda I - A)^{-1}B(x)\sigma_{1} \] (46)

where for an auxiliary Hilbert space $\mathcal{H}$, the operators $C(x) : \mathcal{H} \to \mathcal{C}^{3}, A_{\zeta}, \mathcal{X}(x), A : \mathcal{H} \to \mathcal{H}$ and $B(x) : \mathcal{C}^{p} \to \mathcal{H}$ constitute a vessel $\mathfrak{Y}$ (14) for some $\sigma_{2}, \gamma$.

An analogues of the Bäcklund transformation Theorem 14 for this new setting can be found in [Mela] for the symmetric case. Actually, the assumptions on the operators were found in such a manner that this theorem still holds. We omit its proof, since we are interested in solutions of (1).

## 4 “Uniqueness” of the scattering data

Let us consider now the uniqueness of the scattering matrix $S(\lambda, 0)$. First we prove the following

**Lemma 13.** Suppose that we are given a regular vessel (4)

\[ \mathfrak{Y}_{\text{reg}} = (C(x), A, \mathcal{X}(x), A_{\zeta}, B(x); \sigma_{1}, \sigma_{2}, \gamma, \gamma_{\ast}(x); \mathcal{H}, \mathcal{C}^{3}; I), \]

realizing coefficients $q(x), p(x)$. Let $S(\lambda, x)$ be its transfer function, defined in (14). Let $Y(\lambda)$ be an arbitrary $3 \times 3$ matrix function, commuting with the fundamental solution $\Phi(\lambda, x)$ of (11). Then $\tilde{S}(\lambda, x) = S(\lambda, x)Y(\lambda)$ is the transfer function of a vessel realizing the same coefficients $q(x), p(x)$.

**Proof:** By the definition it follows that

\[ S(\lambda, x) = \Phi_{\ast}(\lambda, x)S(\lambda, 0)\Phi^{-1}(\lambda, x). \]

So,

\[ \tilde{S}(\lambda, x) = S(\lambda, x)Y(\lambda) = \Phi_{\ast}(\lambda, x)S(\lambda, 0)\Phi^{-1}(\lambda, x)Y(\lambda) = \Phi_{\ast}(\lambda, x)S(\lambda, 0)Y(\lambda)\Phi^{-1}(\lambda, x) \]

and realizes the same coefficients $q(x), p(x)$. By the standard construction, there is a vessel $\tilde{\mathfrak{Y}}$, whose transfer functions is $\tilde{S}(\lambda, x)$.

Let us investigate the structure of a matrix $Y(\lambda)$, commuting with $\Phi(\lambda, x)$. Using the form (24), it is easy to conclude that a matrix, which commutes with $\Phi(\lambda, x)$ must be of the form

\[ Y(\lambda) = a(\lambda)I + b(\lambda) \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -\lambda & 0 & 0 \end{bmatrix} + c(\lambda) \begin{bmatrix} 0 & 0 & 1 \\ 0 & -\lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}, \] (47)

by considering the coefficients of pure exponents in $\Phi(\lambda, x)Y(\lambda) = Y(\lambda)\Phi(\lambda, x)$.

**Theorem 14** (Uniqueness of the scattering matrix). *Suppose that $S(\lambda, x)$, $\tilde{S}(\lambda, x)$ are the transfer functions of two regular vessels $\mathfrak{Y}_{\text{reg}}, \tilde{\mathfrak{Y}}_{\text{reg}}$, defined in (14), realizing the same potential $\gamma_{\ast}(x)$. Then there exists a matrix $Y(\lambda) \in \mathcal{R}$ such that$\tilde{S}(\lambda, x) = S(\lambda, x)Y(\lambda)$. 

\[ \tilde{S}(\lambda, x) = S(\lambda, x)Y(\lambda). \]
Proof: Let us consider the function \( S^{-1}(\lambda, x) \tilde{S}(\lambda, x) \). By the definition this function maps solutions of the input LDE (11) to itself:

\[
S^{-1}(\lambda, x) \tilde{S}(\lambda, x) = (\Phi_*(\lambda, x)S(\lambda, 0)\Phi^{-1}(\lambda, x))^{-1} \Phi_*(\lambda, x) \tilde{S}(\lambda, 0)\Phi^{-1}(\lambda, x) = \Phi(\lambda, x)S^{-1}(\lambda, 0) \tilde{S}(\lambda, 0)\Phi^{-1}(\lambda, x)
\]

Plug here, the formula (24) and find conditions so that the coefficients of the exponents \( e^{-kx}, e^{-\alpha kx}, e^{-\alpha^2 kx} \) vanish. This is necessary for making this function bounded at infinity out of the spectrum of \( A \). Then calculations show that actually \( S^{-1}(\lambda, 0) \tilde{S}(\lambda, 0) \) must commute with \( \Phi(\lambda, x) \) so that this function cancels all the singularities at infinity. As a result, by the preceding arguments it must be a function \( Y(\lambda) \) of the form (47). And we obtain that

\[
S^{-1}(\lambda, 0) \tilde{S}(\lambda, 0) = Y(\lambda),
\]

from where the result follows.

Another, weaker form of the uniqueness is used later in the text and is presented in the next Lemma. We emphasize that a similar theorem lemma was proved in the Sturm-Liouville case in [Melb] and in [Fad74] for purely continuous spectrum.

Lemma 15. Suppose that two functions \( S(\lambda, x), \tilde{S}(\lambda, x) \) are in class \( \mathcal{I}(\sigma_1, \sigma_2, \gamma) \), possessing the same initial value

\[
S(\lambda, 0) = \tilde{S}(\lambda, 0)
\]

and are bounded at a neighborhood of infinity, with a limit value \( I \) there. Then the corresponding outer potentials are equal:

\[
\gamma_*(x) = \tilde{\gamma}_*(x).
\]

Proof: Suppose that

\[
S(\lambda, x) = \Phi_*(\lambda, x)S(\lambda, 0)\Phi^{-1}(\lambda, x), \quad \tilde{S}(\lambda, x) = \tilde{\Phi}_*(\lambda, x)S(\lambda, 0)\Phi^{-1}(\lambda, x).
\]

Then

\[
\tilde{S}^{-1}(\lambda, x)S(\lambda, x) = \tilde{\Phi}_*(\lambda, x)\Phi^{-1}(\lambda, x)
\]

is entire (the singularities appear in \( S(\lambda, 0) = \tilde{S}(\lambda, 0) \) only and are cancelled) and equal to \( I \) (the identity matrix) at infinity. By a Liouville theorem, it is a constant function, namely \( I \). So \( \tilde{\Phi}_*(\lambda, x)\Phi^{-1}(\lambda, x) = I \) or

\[
\tilde{\Phi}_*(\lambda, x) = \Phi_*(\lambda, x).
\]

If we differentiate this, we obtain that \( \tilde{\gamma}_*(x) = \gamma_*(x) \).

5 Choice of the parameters realizing the Boussinesq equation (1)

5.1 Realizing the Boussinesq equation by a Boussinesq vessel

Let us choose the following parameters

\[
\tilde{\sigma}_1 = \sigma_1, \quad \tilde{\sigma}_2 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \tilde{\gamma} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & i \end{bmatrix}.
\] (48)

Suppose also that the operators \( C, X, B \) evolve with respect to \( t \) using the same formulas as for the vessel with vessel parameters \( \sigma_1, \sigma_2, \gamma \) substituted with \( \tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\gamma} \). More precisely, we define the following vessel:

13
Definition 16. Suppose that the parameters $\sigma_1, \sigma_2, \gamma$ are defined in Definition 15 and $\bar{\sigma}_1, \bar{\sigma}_2, \bar{\gamma}$ are defined in (15). Then a (regular non-symmetric) Boussinesq vessel $\mathfrak{B}_{\text{Bouss,reg}}$ is a collection of operators and spaces and a rectangle $R$

$$\mathfrak{B}_{\text{Bouss,reg}} = (C(x,t), A, \mathfrak{X}(x,t), A, B(x,t); \sigma_1, \sigma_2, \gamma, \gamma_*(x,t), \bar{\sigma}_1, \bar{\sigma}_2, \bar{\gamma}; \mathcal{H}, \mathbb{C}^3; R),$$

(49)

where the bounded operators $C(x,t) : \mathcal{H} \to \mathbb{C}^3$, $A, \mathfrak{X}(x,t), A : \mathcal{H} \to \mathcal{H}$, $B(x,t) : \mathbb{C}^3 \to \mathcal{H}$ and a $3 \times 3$ matrix function $\gamma_*(x,t)$ satisfy the vessel conditions (1), (2), (4), (5) and the following evolutionary equations

$$\frac{\partial}{\partial t} B = -(AB\bar{\sigma}_2 + B\bar{\gamma})\bar{\sigma}_1^{-1},$$

(50)

$$\frac{\partial}{\partial t} C = \bar{\sigma}_1^{-1} \left( \bar{\gamma}C - \bar{\sigma}_2 CA_\mathfrak{X} \right),$$

(51)

$$\frac{\partial}{\partial t} \mathfrak{X} = B\bar{\sigma}_2 C.$$  

(52)

the operator $\mathfrak{X}(x,t)$ is assumed to be invertible on the rectangle $R$. In the case $A_\mathfrak{X} = A^*$ and $C = B^*$, we call such a vessel symmetric.

Theorem 17. Suppose that $\mathfrak{B}_{\text{Bouss,reg}}$ is a Boussinesq regular vessel, defined in (15) and $\tau(x) = \det(\mathfrak{X}(x_0,t_0)^{-1} \mathfrak{X}(x,t))$ is its tau function, defined for arbitrary point $(x_0,t_0) \in R$. Then the coefficient $g(x) = -\frac{3}{2} \frac{\partial^2}{\partial x^2} \ln(\tau(x,t))$ satisfies the Boussinesq equation (1) on $R$.

Proof: Consider the Boussinesq equation (1):

$$q_{tt} = \frac{\partial^2}{\partial x^2} [q_{xx} - 4q^2].$$

From the formula (37) we find that it can be rewritten as

$$-\frac{3}{2}(\pi_{11})_{xxt} = \frac{\partial^2}{\partial x^2} \left[ -\frac{3}{2} \pi_{xxx} - 4(-\frac{3}{2} \pi_{11})_{x}^2 \right]$$

or integrating with respect to $x$ once and multiplying by $-\frac{2}{3}$ it is enough to show that:

$$(\pi_{11})_{tt} = (\pi_{11})_{xxxx} + 12(\pi_{11})_{x}(\pi_{11})_{xx}. 

(53)

The formula (15), where we substitute the vessel parameters with tilde notation, and substitute $x$-derivative with $t$-derivative, implies that

$$\bar{\sigma}_1^{-1} \bar{\sigma}_2 H_{n+1} - H_{n+1} \bar{\sigma}_2 \bar{\sigma}_1^{-1} = (H_n)' - \bar{\sigma}_1^{-1} (\bar{\gamma} + \bar{\sigma}_2 H_0 \bar{\sigma}_1 - \bar{\sigma}_1 H_0 \bar{\sigma}_2) H_n + H_n \bar{\gamma} \bar{\sigma}_1^{-1}. 

(54)

Since $\pi_{11} = \text{tr}(\sigma_2 H_0)$, we can use (54) to obtain:

$$\frac{\partial}{\partial t} \pi_{11} = \frac{\partial}{\partial t} \text{tr}(\sigma_2 H_0) = \text{tr}(\sigma_2 \frac{\partial}{\partial t} H_0) = i(\pi_{12} - \pi_{21}) = \frac{\partial}{\partial x} \text{tr} \left[ \begin{array}{ccc} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{array} \right] H_0,$$

after cancellations and using (30). So, differentiating again and using (54) we obtain:

$$\frac{\partial^2}{\partial t^2} \pi_{11} = \frac{\partial}{\partial t} \frac{\partial}{\partial x} \text{tr} \left[ \begin{array}{ccc} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{array} \right] H_0 = \frac{\partial}{\partial x} \text{tr} \left[ \begin{array}{ccc} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{array} \right] \frac{\partial}{\partial t} H_0 =$$

$$= -i[\pi_{12}' \pi_{11}'' - \pi_{11}' \pi_{12}'' - \frac{1}{2} \pi_{11}'' \pi_{11}'' - \frac{1}{2} \pi_{11}'' \pi_{11}'' + \frac{1}{2} \pi_{11}'' \pi_{11}'' + \pi_{11}'' \pi_{11}'' + \frac{1}{2} \pi_{11}'' \pi_{11}''].$$

Plugging here (54) we obtain after cancellations (55).
5.2 Standard construction of a Boussinesq vessel

We may revise now the standard construction of a vessel, presented in Section 3 in order to show how to construct a Boussinesq vessel and hence how to produce a solution of the Boussinesq equation (1). For this we have to continue the process of construction, using the parameters \( \tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\gamma} \) defined in (48):

1-3 use old steps to construct \( C(x), X(x), B(x) \) with the same assumptions, as in Section 3,

4. solve the equation

\[
\frac{\partial}{\partial x} R(\lambda)B = -(A R(\lambda)B \tilde{\sigma}_2 + R(\lambda)B \tilde{\gamma}) \tilde{\sigma}_1^{-1}
\]

with the initial condition \( B(x) \) resulting in a function \( B(x, t) \). We have to require, similarly to the construction of \( B(x) \) that

\[
\forall \lambda \not\in \text{spec}(A) : R(\lambda)B(x, t) \tilde{\sigma}_2 C^3 \subseteq D(A),
\]

\[
\forall \lambda \not\in \text{spec}(A) : R(\lambda)B(x, t) \tilde{\gamma} C^3 \subseteq H,
\]

1. solve for \( C(x, t) \) the equation (51) with initial condition \( C(x) \) on \( D(A \zeta) \),

2. solve for \( X(x, t) \) the equation (52) with initial \( X(x) \),

3. define \( \gamma_\ast(x, t) \) by (8) at all points, where \( X(x, t) \) is invertible.

Meanwhile, we present a weaker, regular case form of a general Theorem 19 on producing solutions of (1):

**Theorem 18.** Suppose that a collection \( (C(x, t), \zeta, X(x, t), A, B(x, t); \sigma_1, \sigma_2, \gamma, \gamma_\ast(x, t), \tilde{\sigma}_2, \tilde{\gamma}; H, C^3; R) \) is obtained from a regular (i.e. all the initial operators are bounded) \( S(\lambda) \in R \) by the standard construction around a point \((x_0, t_0)\). Then this collection is a Boussinesq vessel on a rectangle \( R \), including \((x_0, t_0)\). The coefficient \( q(x) = -\frac{3}{2} \frac{\partial^2}{\partial x^2}(X^{-1}(x_0, t_0)X(x, t)) \) satisfies the Boussinesq equation (1) on \( R \).

**Proof:** Notice that since \( X(x_0, t_2) \) is invertible, by a continuity, since all the operators are bounded, it will be invertible on a rectangle \( R \), including \((x_0, t_0)\). Then the Theorem follows from the definitions and Theorem 17.

6 Examples of solutions of the Boussinesq equation (1)

6.1 Solitons

If we take \( H = C \), we will obtain solitons. Use the following choice of solutions of the vessel equations

\[
A = (2\mu)^3, \quad A \zeta = (i\mu)^3, \\
B(x, t) = \begin{bmatrix} E_B & -2i\alpha \mu E_B & (2\alpha \mu)^2 E_B \end{bmatrix}, \quad E_B = \exp(2i\alpha \mu(x + 2\alpha \mu t)) \\
C(x, t) = \begin{bmatrix} -i\alpha \mu E_C \\ (\alpha \mu)^2 E_C \end{bmatrix}, \quad E_C = \exp(i\alpha \mu(x - \alpha \mu t)),
\]

\[
X(x, t) = \frac{2\alpha \cosh(\sqrt{\frac{3}{2}} \mu(x + \mu t)) E_X}{\sqrt{3} \mu}, \quad E_X = \exp \left( \frac{\sqrt{3}}{2} \mu x (2\alpha^2 - 1) + \frac{\sqrt{3}}{2} \mu^2 t (2\alpha^2 + 5) \right).
\]
where \( \alpha = e^{\frac{2\pi i}{3}} \) is the basic third root of 1. Then simple calculations show that

\[
q(x) = -\frac{9\mu^2}{2\cosh^2\left(\frac{\sqrt{3}}{2} \mu(x + t\mu)\right)}
\]

(58)
is the Boussinesq soliton solution of (1) (see [SCM73]).

Another non-trivial soliton, obtained in this form is as follows.

\[
A = (2\mu)^3, \quad \alpha = (2i\mu)^3, \quad k_1 = \alpha 2i\mu, \quad f_1 = \alpha^2 2i\mu,
\]

\[
B(x, t) = \begin{bmatrix} b \frac{E_B}{k_1} + c \frac{E_f}{k_1} & -b \frac{E_B}{k_1} - c \frac{E_f}{k_1} \end{bmatrix} = (2\mu)^3, \quad \Psi = \exp(k_1 x - ik_1^2 t),
\]

\[
c = -i, \quad b = \frac{1}{i + \sqrt{3}}
\]

\[
C(x, t) = \begin{bmatrix} \frac{E_B}{k_1} \frac{E_f}{k_1} \end{bmatrix},
\]

\[
\Omega(x, t) = \alpha e^{-2i\pi t} (e^{2\sqrt{\pi\mu}x} + e^{4\sqrt{\pi\mu^2}})
\]

and the coefficient \( q(x, t) \) is

\[
q(x, t) = -\frac{18e^{2\sqrt{\pi\mu(x + t\mu)}\mu^2}}{(e^{2\sqrt{\pi\mu}x} + e^{4\sqrt{\pi\mu^2})^2}}
\]

(59)

### 6.2 Solutions, belonging to the Schwartz class.

Suppose that we are given real coefficients \( q(x), p(x) \) in the Schwartz class and define \( \pi_{11} \) by formula (47). Suppose that the functions \( F_1, F_2, F_3 \) are solutions of (3) satisfying

\[
F_1(x, k) = e^{-kx}(1 + O(1)), \quad \text{as} \quad x \to +\infty,
\]

\[
F_2(x, k) = e^{-\alpha kx}(1 + O(1)), \quad \text{as} \quad x \to +\infty,
\]

\[
F_3(x, k) = e^{-\alpha^2 kx}(1 + O(1)), \quad \text{as} \quad x \to +\infty
\]

where \( k \in \Omega_1 = \{k | -\frac{2\pi}{3} < \arg(k) < \frac{2\pi}{3}\} \). For the coefficients \( q(x), p(x) \in S \), where \( S \) - the Schwartz class of rapidly decreasing functions, the existence of \( F_1, F_2, F_3 \) is shown in [DTT82], for example.

Then a function-matrix \( \Psi_s(\lambda, x) \) which is a solution of (12) can be constructed from solutions \( \Psi_{s,1}, \Psi_{s,2}, \Psi_{s,3} \) of (3) and is given in view of (21) by \( (k = \sqrt{\lambda}) \)

\[
\Psi_s(\lambda, x) = \frac{1}{3a^2} \left[ \begin{array}{ccc} \Psi_{s,1} & \Psi_{s,2} & \Psi_{s,3} \\
-\pi_{11}\Psi_{s,1} - \Psi_{s,1}' & -\pi_{11}\Psi_{s,2} - \Psi_{s,2}' & -\pi_{11}\Psi_{s,3} - \Psi_{s,3}' \\
\pi_{21}\Psi_{s,1} - (\pi_{11}\Psi_{s,1} + \Psi_{s,1})' & \pi_{21}\Psi_{s,2} - (\pi_{11}\Psi_{s,2} + \Psi_{s,2})' & \pi_{21}\Psi_{s,3} - (\pi_{11}\Psi_{s,3} + \Psi_{s,3})'
\end{array} \right]
\]

(60)

where

\[
\Psi_{s,1} = \alpha^2 (F_1 + F_2 + F_3), \quad \Psi_{s,2} = \frac{\alpha^2}{k} F_1 + \frac{\alpha}{k^2} F_2 + \frac{1}{k^2} F_3, \quad \Psi_{s,3} = \frac{\alpha^2}{k^2} F_1 + \frac{1}{k^3} F_2 + \frac{\alpha}{k^3} F_3.
\]
As a function of $\lambda$ the matrix-function $\Psi_*(\lambda, x)$ is analytic in the whole plane except for the cut along the negative axis $(-\infty, 0]$. Notice that this function is globally bounded as there are asymptotic formulas for such solutions appearing in [DTT82]. Actually, the three functions $F_1, F_2, F_3$ have the asymptotic behavior $\exp(\alpha'kx)(1+O(k)), i=0,1,2$ as $k \to \infty$ for all $x \in \mathbb{R}$ and these asymptotics can be differentiated infinitely many times.

So we define

$$S(\lambda, x) = \Psi_*(\lambda, x)\Phi^{-1}(\lambda, x),$$

which becomes

$$S(\lambda, x) = \Phi_*(x, \lambda)S(\lambda, 0)\Phi^{-1}(\lambda, x)$$

with $\Phi_*(x, \lambda)$ - the fundamental solution of $[12]$ equal to identity at $x = 0$. Since $\Psi_*$ and $\Phi$ are analytic functions of $k$ in $\Omega$, they will be analytic functions of $\lambda$, except for the cut along the negative real axis, where it will have jumps, but is remained bounded at the infinity. These jumps are expressible by parameters $R_1, R_2$, appearing at [DTT82] in a quite general case [DTT82] Theorem 14 of real coefficients $q(x), p(x)$.

Applying the standard construction to $S(\lambda, 0)$ we will obtain a vessel realizing $q_\nu(x)$ and derived from it $p_\nu(x)$, which is equal by Lemma [15] to the given $q(x)$: $q_\nu(x) = q(x)$. Evolving further the operators with respect to $t$, we will create a solution of (1), coinciding with $q(x)$ at time $t = 0$.

6.3 General solutions

We can obtain more general solutions of (1) if we choose to work in a more general setting, presented in Section 3. Applying the standard vessel construction to a function $S(\lambda) \in \mathcal{R}$, we obtain

**Theorem 19.** Suppose that a Boussinesq vessel $\mathcal{V}_{\text{Bouss}}$ is obtained from

$$S(\lambda) = I - C_0\lambda^{-1}(\lambda I - A)^{-1}B_0\sigma_1$$

by the standard construction, satisfies the following assumptions ($\forall(x, t) \in \mathbb{R}^2$):

$$B(x, t) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \in D(A), \quad B(x, t) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \in D(A), \quad B(x, t) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \in \mathcal{H}. \quad (61)$$

Then the coefficient $q(x, t)$ is twice differentiable with respect to $t$ and four times differentiable with respect to $x$ on an open subset $\Omega \subseteq (\mathbb{R} \times \mathbb{R}) \setminus Z$, obtained after removing a closed subset $Z \subseteq \mathbb{R} \times \mathbb{R}$ of the points $(x, t) \in Z$, in which $\mathcal{X}(x, t)$ is not invertible. Moreover, the coefficient $q(x, t)$ satisfies the Boussinesq equation (1) on $\Omega$.

**Remarks:** 1. Notice that in the regular case the operator $\mathcal{X}(x, t)$ and hence the tau function $\tau(x, t) = \det(\mathcal{X}^{-1}(x_0, t_0)\mathcal{X}(x, t))$ are analytic on $\mathbb{R} \times \mathbb{R}$. So, the zeros of $\tau(x, t)$ are analytic sets of $\mathbb{R} \times \mathbb{R}$, but in general their structure is complicated and out of the scope of this work. 2. The assumptions (61) imply the regularity assumptions $[12], [13], [50], [57]$.

**Proof:** Notice that the regularity assumptions $[12], [13]$ and $[50], [57]$ imply that the operators $B(x, t) : \mathbb{C}^3 \to \mathcal{H}, C(x, t) : \mathbb{C}^3 \to D(A_c)$ are defined for all $(x, t) \in \mathbb{R}^2$ (actually, for $C(x) = \left[ c_1(x, t) \quad c_2(x, t) \quad c_3(x, t) \right]$ one has to verify that the three entries $c_1(x, t), c_2(x, t), c_3(x, t)$ are in $D(A_c)$ for all $x$). The operator $\mathcal{X}(x, t)$ is the unique solution of (1) (52) with the initial condition $\mathcal{X}_0$. Notice that this is an operator of the form $\mathcal{X}_0 + T(x, t)$ for a trace-class operator $T(x, t)$ and we can define $\tau(x, t) = \det(\mathcal{X}_0^{-1}\mathcal{X}(x, t))$ (59). Moreover, since $\mathcal{X}(x)$ is $x$-differentiable

$$\frac{\tau'(x, t)}{\tau(x, t)} = \text{tr}(\mathcal{X}'(x, t)\mathcal{X}^{-1}(x, t)) = \text{tr}(B(x, t)\sigma_2C(x)\mathcal{X}^{-1}(x, t))$$
for all points \((x, t) \in Z = \{(x, t) \mid \tau(x, t) \neq 0\}\) = \{(x, t) \mid X(x, t) is not invertible\}. On the other hand, the differentiability of the coefficient \(q(x) = \frac{3}{2} \frac{\partial^2}{\partial x^2} \ln \tau(x, t)\) follows from the differentiability of \(\frac{\tau'(x, t)}{\tau(x, t)}\).

So, we investigate the existence of the derivatives for \(\frac{\tau'(x, t)}{\tau(x, t)}\).

Applying the formulas (5), (16) we obtain that

\[
\frac{\tau''(x, t)}{\tau(x, t)} = \frac{\partial}{\partial x} \frac{\tau'(x, t)}{\tau(x, t)} + \left(\frac{\tau'(x, t)}{\tau(x, t)}\right)^2 = \text{tr}(B(x, t)(\sigma_2\sigma_1^{-1}\gamma - \gamma\sigma_1^{-1}\sigma_2)C(x, t)X^{-1}(x, t)) = \\
= - \text{tr} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} C(x, t)X^{-1}(x, t)B(x),
\]

which is well-defined on \(\Omega\). Then, similarly,

\[
\frac{\tau'''(x, t)}{\tau(x, t)} = \frac{\partial}{\partial x} \frac{\tau''(x, t)}{\tau(x, t)} + \frac{\tau''(x, t)}{\tau(x, t)} \frac{\tau'(x, t)}{\tau(x, t)} = - \text{tr} \begin{bmatrix} 0 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 0 \end{bmatrix} C(x, t)X^{-1}(x, t)B(x)).
\]

Then, differentiating and plugging (5), (16) again we can obtain that

\[
\frac{\partial}{\partial x} \frac{\tau''(x, t)}{\tau(x, t)} = 3(\pi_{12}\pi_{21} - \pi_{23} - \pi_{32}).
\]

Since \(\pi_{12} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} C(x, t)X^{-1}(x, t)B(x, t) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\) in view of (43) it is a well-defined function on \(\Omega\). Similarly, \(\pi_{21}, \pi_{23}, \pi_{32}\) are well defined from (43). Finally, differentiating the last expression, we will obtain that

\[
\frac{\partial^2}{\partial x^2} \frac{\tau''(x, t)}{\tau(x, t)} = 3(\pi_{12}'\pi_{21} + \pi_{12}\pi_{21}' - \pi_{23}' - \pi_{32}')
\]

Here, the derivative of \(\pi_{12}\) is as follows

\[
\pi_{12}' = \frac{\partial}{\partial x} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} C(x, t)X^{-1}(x, t)B(x, t) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \\
= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \sigma_1\gamma_1(x, t)C(x, t)X^{-1}(x, t)B(x, t) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} C(x, t)X^{-1}(x, t)B(x, t)\gamma_1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},
\]

where we used (5), (16). Thus it is a well-defined function on \(\Omega\). Similarly for \(\pi_{21}'\). The derivatives of \(\pi_{21}, \pi_{32}\) are as follows

\[
\pi_{21}' = - \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} C(x, t)X^{-1}(x, t)AB(x, t) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \pi_{12}\pi_{13} + \pi_{33},
\]

\[
\pi_{32}' = - \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} C(x, t)X^{-1}(x, t)AB(x, t) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \pi_{12}(\pi_{13} + \pi_{23} - \pi_{31}) + \pi_{11}\pi_{32} + \pi_{33},
\]

which are well defined by the assumption (41) on \(\Omega\).
The fact that we can $t$-differentiate $q(x,t)$ twice follows exactly the same lines, using the parameters $\tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\gamma}$ and the regularity assumption (61). Using (54), the first derivative is

$$\dot{\pi}_{11} = i(\pi_{13} - \pi_{31} + \pi_{11}(\pi_{21} - \pi_{12})),$$

which is well-defined on $\Omega$. For the second derivatives, it is enough to check that $\dot{\pi}_{13} - \dot{\pi}_{31}, \dot{\pi}_{21} - \dot{\pi}_{12}$ are well defined. Using (54) again

$$i\dot{\pi}_{13} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} C(x,t)X^{-1}(x,t)AB(x,t) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \pi_{21}\pi_{22} + \pi_{11}(\pi_{23} + \pi_{32}) - \pi_{33}$$

$$-i\dot{\pi}_{12} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} C(x,t)X^{-1}(x,t)AB(x,t) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \pi_{12}\pi_{21} + \pi_{23} + \pi_{11}(\pi_{13} - \pi_{22} + \pi_{31}),$$

which are well-defined on $\Omega$ by (61). Similarly for $\dot{\pi}_{21}, \dot{\pi}_{31}$ and the proof is finished.

**Remark:** Notice that one more derivative of $\pi_{23}$ produces a term of the form

$$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} C(x,t)X^{-1}(x,t)AB(x,t) \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

which may fail to be a well-defined function, because $B(x,t) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ may fail to be at the domain of $A$.

**References**

[AMV12] D. Alpay, A. Melnikov, and V. Vinnikov. Schur algorithm in the class I of J-contractive functions intertwining solutions of linear differential equations. *IEEE*, 74(3):313–344, 2012.

[BGR90] J. Ball, I. Gohberg, and L. Rodman. *Interpolation of rational matrix functions*. Operator Theory: Advances and Applications. Birkh"auser Verlag, Basel, 1990.

[Bou72] J. Boussinesq. Théorie des ondes et des remous qui se propagent le long d’un canal rectangulaire horizontal, en communiqant au liquide contenu dans ce canal des vitesses sensiblement pareilles de la surface au fond. *Journal de Mathématique Pures et Appliquées*, 17(2):55–108, 1872.

[DLdS] A. Diksma, H. Langer, and H.S.V. de Snoo. Representations of holomorphic operator functions by means of resolvents of unitary or self-adjoint operators in Krein spaces. *Operator Theory: Adv. and App.*, 24:123–143. Birkhauser Verlag, Berlin.

[DTT82] P. Deift, C. Tomei, and E. Trubowitz. Inverse scattering and the Boussinesq equation. *Comm. on Pure and Appl. math.*, 35(5):567–628, 1982.

[Fad74] L.D. Fadeev. The inverse problem in the quantum theory of scattering, II. *Itogi Nauk. i Techn.*, 4:93–180, 1974.

[GK69] I. Gohberg and M. Krein. *Introduction to the theory of linear non-selfadjoint operators*. translations of AMS, 1969.

[LS01] M.S. Livšic. Vortices of 2D systems. *Operator Theory: Advances and Applications*, 123:7–41, 2001.

[McK78] H. P. McKean. Boussinesq’s equation as a Hamiltonian system. *Topics in Func. Anal.*, *Adv. Math. Suppl. Studies*, 3:217–226, 1978.
[Mela] A. Melnikov. Construction of a Sturm-Liouville vessel using Gelfand-Levitan theory. On solution of the Korteweg-de Vries equation in the first quadrant. http://arxiv.org/abs/1212.1730.

[Melb] A. Melnikov. On a theory of vessels and the inverse scattering. http://arxiv.org/abs/1103.2392.

[Mel11] A. Melnikov. Finite dimensional Sturm Liouville vessels and their tau functions. *IEOT*, 71(4):455–490, 2011.

[SCM73] A. C. Scott, F. Y. F. Chu, and D. W. McLaughlin. The soliton: A new concept in applied science. *Proc. IEEE*, 61(10):1443–1483, 1873.

[Zac74] V. E. Zacharov. On stochastization of one-dimensional chains of nonlinear oscillators. *Sov. Phys. JETP*, 38(1):108–110, 1974.