ON THE THRESHOLD DYNAMICS OF THE STOCHASTIC SIRS EPIDEMIC MODEL USING ADEQUATE STOPPING TIMES

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(Co-Communicated by Peter Hinow)

Abstract. As it is well known, the dynamics of the stochastic SIRS epidemic model with mass action is governed by a threshold $R_S$. If $R_S < 1$ the disease dies out from the population, while if $R_S > 1$ the disease persists. However, when $R_S = 1$, classical techniques used to study the asymptotic behaviour do not work any more. In this paper, we give answer to this open problem by using a new approach involving some adequate stopping times. Our results show that if $R_S = 1$ then, small noises promote extinction while the large one promote persistence. So, it is exactly the opposite role of the noises in case when $R_S \neq 1$.

1. Introduction. Investigation of asymptotic analysis of epidemic models has become focus in mathematical epidemiology [1, 5]. In this framework, diseases can be modeled by deterministic or stochastic SIR, SIRS or SIRI models. The effect of environmental noise on epidemic models has been studied intensively. For instance, in [6], authors studied the dynamical properties of a stochastic SIRI epidemic model with relapse and media coverage around both equilibria points and proved the existence of a stationary distribution (see also [2, 10, 11, 12]). In [18], authors introduced a stochastic SIRS epidemic model with general incidence rate in a population of varying size and developed sufficient conditions for the extinction and the existence of a unique stationary distribution. In [22], Lu presented a SIRS model with or without distributed time delay influenced by random perturbations and studied stability conditions of the disease-free equilibrium of the associated stochastic SIRS system. In [16], authors discussed a multigroup SIR model with stochastic perturbation, where extinction and persistence in mean are derived according to values of the basic reproduction number. In [7], a random and a stochastic version of a SIR [15] non-autonomous model is considered. In particular, the existence

2010 Mathematics Subject Classification. 92B05, 60G51, 60H30, 60G57.
Key words and phrases. Stochastic threshold, SIRS epidemic model, Wiener process.
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of a random attractor is proved for the random model and the persistence of the disease is analyzed as well. In [26], a stochastic SIS epidemic model [13] with vaccination is presented where authors derived a threshold of the stochastic model which determines the outcome of the disease in case the white noises are small. In [15], the authors showed that, when the noise is small, the threshold determines the extinction and persistence of the epidemic. On the other hand, they obtained that the large noise intensity will also suppress the epidemic to prevail, which never happens in the deterministic system. In this paper, we reconsider the stochastic model investigated by Lahrouz and Settati [19]. This model can be described by the following stochastic SIRS epidemic model studied by many scholars (see, e.g., [3, 4, 18, 19, 20, 22, 23] and the references cited therein)

$$\begin{align*}
&\begin{cases}
  dS(t) = (\mu - \mu S(t) - \beta S(t) I(t) + \gamma R(t)) \, dt - \sigma S(t) I(t) dB(t), \\
  dI(t) = (-\mu + \lambda) I(t) + \beta S(t) I(t)) \, dt + \sigma S(t) I(t) dB(t), \\
  dR(t) = (-(\mu + \gamma) R(t) + \lambda I(t)) \, dt,
\end{cases}
\end{align*}$$

(1.1)

where $S(t)$, $I(t)$ and $R(t)$ are the population fractions of susceptible, infective and removed at time $t$, respectively. The constants $\mu$, $\beta$, $\lambda$, and $\gamma$ are positive constants that stand for birth and death rates, infection coefficient, recovery rate of the infective individuals, and losing immunity rate, respectively. $B(t)$ is a Brownian motion that models the random fluctuations of the environment where the intensity is $\sigma > 0$. It is assumed in System (1.1) that environmental disturbances are mainly manifested in the infection coefficient $\beta$. That is, model (1.1) is obtained by replacing $\beta dt$ by $\beta dt + \sigma dB(t)$ in the classical deterministic SIRS model with mass action incidence given by the following system of ordinary differential equations:

$$\begin{align*}
&\begin{cases}
  dS(t) = (\mu - \mu S(t) - \beta S(t) I(t) + \gamma R(t)) \, dt, \\
  dI(t) = (-\mu + \lambda) I(t) + \beta S(t) I(t)) \, dt, \\
  dR(t) = (-(\mu + \gamma) R(t) + \lambda I(t)) \, dt.
\end{cases}
\end{align*}$$

(1.2)

The stability and the asymptotic behavior of the deterministic version (1.2) is completely characterized using the basic reproduction number $R_0 = \frac{\beta}{\mu + \gamma}$. Precisely, if $R_0 \leq 1$, the free-disease equilibrium state $E_0(1, 0, 0)$ is globally asymptotically stable. While $R_0 > 1$, $E_0$ becomes unstable and the model admits a unique endemic equilibrium state $(\frac{\lambda(R_0-1)}{(\mu+\lambda+\gamma)R_0-1}, \frac{\lambda(R_0-1)}{(\mu+\lambda+\gamma)R_0}, \frac{\lambda(R_0-1)}{(\mu+\lambda+\gamma)R_0})$ which is globally asymptotically stable. For detailed works concerning deterministic SIRS models, the reader may refer to [8, 9, 14, 25] and the book by Capasso [5]. The stochastic model (1.1) was proposed by Tornatore et al. [24] in the case when $\gamma = 0$. They gave a sufficient condition for the asymptotic stability in probability of the free-disease steady state $E_0$. Furthermore, the authors conjectured from computer simulations that the perturbation of the infection coefficient by white noises modify the deterministic threshold $R_0$ giving rise to the new threshold

$$R_S = \frac{\beta}{\mu + \lambda + \frac{1}{2} \sigma^2}.$$

Several attempt were made to extend the model and improve the sufficient condition given by Tornatore et al. [24]. However, a partial answer to the conjecture by Tornatore and their co-workers is given by Lahrouz and Settati [19] when the intensity of noises is small relative to the infection coefficient $\beta$. The authors proved that under the assumption $\sigma^2 \leq \beta$, if $R_S < 1$ the free-disease $E_0$ is globally asymptotically stable in probability. While if $R_S > 1$ the disease persists with probability

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one. In addition, they have established the persistence in mean of model solutions. The existence, uniqueness and asymptotic stability of the stationary distribution under condition $R_S > 1$ is given by Lin and Jiang [21] using the Markov semigroup theory. However, to our knowledge, there is no global threshold results, compared to deterministic models, for stochastic epidemic models in literature and especially for the limiting case when $R_S = 1$. In this case, classical techniques used to study the asymptotic behaviour do not work anymore. In order to unblock this situation, we will use a new approach involving some adequate stopping times. Precisely, in this paper, we investigate this critical case under large and small intensities of noises. We shall prove that for large intensities of noises, that is, $\beta < \sigma$ and $R_S = 1$, the disease continues to exist in the sense that each component of the so-

solutions ($S(t)$, $I(t)$, $R(t)$) rises to or above certain positive level infinitely often with probability one, while for small intensities of noises that is $\sigma^2 < 1$ and $R_S = 1$, the disease-free equilibrium state $E_0 (1, 0, 0)$ is asymptotically stable in probability, that is to say, system (1.1) is extinctive and stable around the disease-free equilibrium state $E_0$:

$$\lim_{t \to \infty} S(t) = 1, \quad \lim_{t \to \infty} I(t) = 0, \quad \lim_{t \to \infty} R(t) = 0 \quad a.s.,$$

and for all $\varepsilon > 0$

$$\lim_{(S_0, I_0, R_0) \to (1, 0, 0)} \mathbb{P} \left( \sup_{t \geq 0} (1 - S(t) + I(t) + R(t)) > \varepsilon \right) = 0.$$

where $(S_0, I_0, R_0)$ stands for any initial value of the solution $(S(t), I(t), R(t))$. To begin with, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions and denote

$$\Delta = \{ x \in \mathbb{R}^3 \mid x_i > 0, i = 1, 2, 3, x_1 + x_2 + x_3 = 1 \}.$$

The set $\Delta$ is almost surely positively invariant by the system (1.1). That is, if $(S_0, I_0, R_0) \in \Delta$, then

$$\mathbb{P} ( (S(t), I(t), R(t)) \in \Delta) = 1 \quad \forall t \geq 0.$$

2. Small noise. In this section, we investigate the global asymptotic stability of the free-disease equilibrium state $E_0 (1, 0, 0)$. Firstly, we show that system (1.1) is extinctive.

**Theorem 2.1.** For any initial values $(S_0, I_0, R_0) \in \Delta$, if $\beta \geq \sigma^2$ and $R_S = 1$ then the solution of SDE (1.1) obeys

$$\lim_{t \to \infty} S(t) = 1, \quad \lim_{t \to \infty} I(t) = 0, \quad \lim_{t \to \infty} R(t) = 0 \quad a.s..$$

**Proof.** Let $\varepsilon < I_0$. Define the stopping times $\tau_\varepsilon = \inf \{ t > 0, I(t) \leq \varepsilon \}, \quad \tau_\varepsilon' = \inf \{ t > \tau_\varepsilon, I(t) \geq \varepsilon \}.$

First, we claim that $\mathbb{E}(\tau_\varepsilon) < \infty$. Indeed, for all $T > 0$ and $t \leq T \land \tau_\varepsilon$, we have

$$I(t) \geq \varepsilon \quad a.s..$$

Applying Itô’s formula to the function $\log I$ on $t \in (0, T \land \tau_\varepsilon)$, gives that if $\sigma^2 \leq \beta$ and $R_S = 1$,

$$d \log(I) = \left( -\mu - \lambda \right) + \beta (1 - I - R) - \frac{1}{2} \sigma^2 (1 - I - R)^2 \right) dt + \sigma (1 - I - R) dB$$
\[
\begin{align*}
&\leq \left(-\mu + \lambda + \beta(1 - I) - \frac{1}{2}\sigma^2(1 - I)^2\right) dt + \sigma(1 - I - R) dB \\
&= -I \left(\beta - \sigma^2 + \frac{1}{2}\sigma^2\right) dt + \sigma(1 - I - R) dB \quad (2.1) \\
&\leq -\varepsilon \left(\beta - \sigma^2 + \frac{1}{2}\sigma^2\varepsilon\right) dt + \sigma(1 - I - R) dB, \quad (2.2)
\end{align*}
\]
where these inequalities are obtained by studying the function
\[
\Phi(x) = -\frac{1}{2}\sigma^2 x^2 + \beta x - (\mu + \lambda).
\]
So, \(\Phi(x)\) is increasing on \((0, 1)\) and then \(\Phi(1 - I - R) < \Phi(1 - I)\). Integrating the last inequality between 0 and \(T \wedge \tau_e\), and taking expectation in both sides, the result is
\[
\varepsilon \left(\beta - \sigma^2 + \frac{1}{2}\sigma^2\varepsilon\right) E(T \wedge \tau_e) \leq -E(\log I(T \wedge \tau_e)) + E(\log I_0),
\]
which gives by letting \(T \to \infty\) and using Fatou’s lemma to
\[
E(\tau_e) \leq -\frac{\log \varepsilon}{\varepsilon (\beta - \sigma^2 + \frac{1}{2}\sigma^2\varepsilon)}.
\]
Thus
\[
P(\tau_e < \infty) = 1. \quad (2.3)
\]
Second, we claim that
\[
P(\tau'_e = \infty) = 1. \quad (2.4)
\]
Assume that \((2.4)\) is not true. That is, \(P(\tau'_e < \infty) > 0\). Then, define the stopping time
\[
\tau''_e = \inf \{t > \tau'_e, I(t) < \varepsilon\}.
\]
Let \(t > 0\), integrating \((2.1)\) between \(t \wedge \tau'_e\) and \(t \wedge \tau''_e\) and taking expectation, gives
\[
E\left[\log I(t \wedge \tau''_e) - \log I(t \wedge \tau'_e)\right] \leq -E\left[\int_{t \wedge \tau'_e}^{t \wedge \tau''_e} \left(\beta - \sigma^2 + \frac{1}{2}\sigma^2 I(u)\right) I(u) du\right].
\]
Since
\[
E\left[\log I(t \wedge \tau''_e) \mathcal{X}_{\tau'_e = \infty}\right] = E\left[\log I(t \wedge \tau'_e) \mathcal{X}_{\tau'_e = \infty}\right],
\]
and
\[
E\left[\mathcal{X}_{\tau'_e = \infty} \int_{t \wedge \tau'_e}^{t \wedge \tau''_e} \left(\beta - \sigma^2 + \frac{1}{2}\sigma^2 I(u)\right) I(u) du\right] = 0,
\]
we deduce that
\[
E\left[\log I(t \wedge \tau''_e) \mathcal{X}_{\tau'_e < \infty}\right] - E\left[\log I(t \wedge \tau'_e) \mathcal{X}_{\tau'_e < \infty}\right]
\leq -E\left[\mathcal{X}_{\tau'_e < \infty} \int_{t \wedge \tau'_e}^{t \wedge \tau''_e} \left(\beta - \sigma^2 + \frac{1}{2}\sigma^2 I(u)\right) I(u) du\right].
\]
Letting \(t \to \infty\), using the fact that \(I(\tau'_e) = I(\tau''_e) = \varepsilon\), by Lebesgue’s dominated convergence theorem, we get
\[
0 \leq -E\left[\mathcal{X}_{\tau'_e < \infty} \int_{t \wedge \tau'_e}^{t \wedge \tau''_e} \left(\beta - \sigma^2 + \frac{1}{2}\sigma^2 I(u)\right) I(u) du\right]. \quad (2.5)
\]
On the other hand, for \( \tau_e' < \infty \),
\[ I(u) \geq \epsilon, \quad \tau_e' \leq u \leq \tau_e'', \]
which implies with (2.5) that
\[ 0 \leq -\frac{1}{2}(\epsilon\sigma)^2 \mathbb{E} \left[ (\tau_e'' - \tau_e') \mathcal{X}_{(\tau_e' < \infty)} \right]. \]
Thereby
\[ \mathbb{E} \left[ (\tau_e'' - \tau_e') \mathcal{X}_{(\tau_e' < \infty)} \right] = 0. \]
Consequently
\[ \tau_e'' - \tau_e' = 0 \quad \text{for almost } w \in (\tau_e' < \infty). \]
This contradicts the definition of \( \tau_e' \) and \( \tau_e'' \). Thus, our claim is true. Now, combining (2.3) and (2.4), one can write that for every \( \epsilon > 0 \), and for almost all \( w \in \Omega \), there exists \( \tau_e(w) > 0 \) such that
\[ I(t, w) < \epsilon \quad \forall t \geq \tau_e(w). \]
This means that \( \lim_{t \to \infty} I(t) = 0 \) as. Now, for the convergence of \( R(t) \), rewriting the third equation of system (1.1) in the integral form as follows.
\[ R(t) = R_0 e^{-(\mu + \gamma)t} + \lambda \int_0^t I(s) e^{-(\mu + \gamma)(t-s)} ds, \]
which implies easily that \( \lim_{t \to \infty} R(t) = 0 \) and then \( \lim_{t \to \infty} S(t) = 1 \).

Now, we will prove that the solution \((S(t), I(t), R(t))\) is stable around the disease-free equilibrium state \( E_0(1, 0, 0) \).

**Theorem 2.2.** Let \((S_0, I_0, R_0) \in \Delta\). If \( \beta \geq \sigma^2 \) and \( R_S = 1 \) then for all \( \epsilon > 0 \), we have
\[ \lim_{(S_0, I_0, R_0) \to (1, 0, 0)} \mathbb{P} \left( \sup_{t \geq 0} (1 - S(t) + I(t) + R(t)) > \epsilon \right) = 0, \quad (2.6) \]
that is, the disease-free equilibrium state \( E_0(1, 0, 0) \) is stable in probability.

**Proof.** For \( \epsilon > 0 \), the family \( \left( \sup_{0 \leq t \leq n} (1 - S(t) + I(t) + R(t)) > \epsilon \right) \) is increasing and
\[ \bigcup_n \left( \sup_{0 \leq t \leq n} (1 - S(t) + I(t) + R(t)) > \epsilon \right) = \left( \sup_{t \geq 0} (1 - S(t) + I(t) + R(t)) > \epsilon \right). \]
Therefore,
\[ \mathbb{P} \left( \sup_{t \geq 0} (1 - S(t) + I(t) + R(t)) > \epsilon \right) \]
\[ = \lim_{n \to \infty} \mathbb{P} \left( \sup_{0 \leq t \leq n} (1 - S(t) + I(t) + R(t)) > \epsilon \right), \]
which implies that for all \( \epsilon > 0 \) and \( \epsilon > 0 \), there exists \( n_0 > 0 \)
\[ \mathbb{P} \left( \sup_{t \geq 0} (1 - S(t) + I(t) + R(t)) > \epsilon \right) \]
\[ \leq \mathbb{P} \left( \sup_{0 \leq t \leq n_0} (1 - S(t) + I(t) + R(t)) > \epsilon \right) + \frac{\epsilon}{2}. \quad (2.7) \]
On other hand, from the third equation of (1.1) we have

\[ R(n_0) \leq R_0 + \lambda \int_0^{n_0} I(s)ds, \]

which implies with \((S_0, I_0, R_0) \in \Delta,\)

\[ \sup_{0 \leq t \leq n_0} (1 - S(t) + I(t) + R(t)) \leq 2R_0 + 2(1 + \lambda n_0) \sup_{0 \leq t \leq n_0} I(t). \quad (2.8) \]

From (2.7) and (2.8), for all \(\varepsilon > 0\) and \(\epsilon > 0,\) there exists \(n_0 > 0\)

\[ P \left( \sup_{t \geq 0} (1 - S(t) + I(t) + R(t)) > \varepsilon \right) \leq 1_R_0 > \frac{\varepsilon}{4} + P \left( \sup_{0 \leq t \leq n_0} I(t) > \frac{\varepsilon}{4(1 + \lambda n_0)} \right) + \frac{\epsilon}{2}, \quad (2.9) \]

where \(1_A\) denote the characteristic function of \(A.\) Let \(0 < \kappa < 1,\) By Itô’s formula one can easily show that if \(\beta \geq \sigma^2\) and \(R_S = 1,\) that

\[ dI\kappa = \kappa I\kappa \left[ -(\mu + \lambda) + \beta(1 - I - R) + \left( \frac{\kappa}{2} - \frac{1}{2} \right) \sigma^2(1 - I - R)^2 \right] dt + \kappa \sigma I\kappa (1 - I - R) dB \quad (2.10) \]

\[ \leq \kappa I\kappa \left[ \sup_{0 < s \leq 1} \left( -(\mu + \lambda) + \beta - \frac{1}{2} \sigma^2 \delta^2 \right) + \frac{\kappa}{2} \sigma^2 \right] dt + \kappa \sigma I\kappa (1 - I - R) dB \]

\[ \leq \frac{\kappa^2}{2} \sigma^2 I\kappa dt + \kappa \sigma I\kappa (1 - I - R) dB. \]

Hence, by using \(I^\kappa < 1,\) we get

\[ \sup_{0 \leq t \leq n_0} I^\kappa(t) \leq I_0^\kappa + \frac{\kappa^2}{2} \sigma^2 n_0 + \kappa \sigma \sup_{0 \leq t \leq n_0} \int_0^t I^\kappa(s)(1 - I(s) - R(s))dB(s). \]

Using \(I < 1,\) we get for all \(\varepsilon > 0\) and \(\epsilon > 0,\) there exists \(n_0 > 0\) such that for all \(\kappa < 1\) we have

\[ P \left( \sup_{0 \leq t \leq n_0} I(t) > \frac{\varepsilon}{4(1 + \lambda n_0)} \right) \leq P \left( \kappa \sigma \sup_{0 \leq t \leq n_0} M_t > \frac{\varepsilon}{12(1 + \lambda n_0)} \right) + 1_{I_0^\kappa > \frac{\varepsilon}{12(1 + \lambda n_0)}} + 1_{\kappa^2 \sigma^2 n_0 > \frac{\varepsilon}{12(1 + \lambda n_0)}}. \quad (2.11) \]

where

\[ M_t = \int_0^t I^\kappa(s)(1 - I(s) - R(s))dB(s). \]

Moreover \(M_t\) is a real-valued continuous martingale, so by Doob’s inequality we have

\[ P \left( \kappa \sigma \sup_{0 \leq t \leq n_0} M_t > \frac{\varepsilon}{12(1 + \lambda n_0)} \right) \leq \frac{144(1 + \lambda n_0)^2 \kappa^2 \sigma^2}{\varepsilon^2} \mathbb{E} \left( \int_0^{n_0} I^\kappa(s)(1 - I(s) - R(s))dB(s) \right)^2 \]

\[ = \frac{144(1 + \lambda n_0)^2 \kappa^2 \sigma^2}{\varepsilon^2} \mathbb{E} \left( \int_0^{n_0} (I^\kappa(s)(1 - I(s) - R(s)))^2 ds \right) \]

\[ \leq \frac{144(1 + \lambda n_0)^2 \kappa^2 \sigma^2}{\varepsilon^2} n_0. \quad (2.12) \]
Taking
\[ \kappa = \kappa(\epsilon, \epsilon) = \left( \frac{\epsilon}{6\sigma^2n_0(1 + \lambda n_0)} \right) \wedge \left( \frac{\epsilon^2}{288(1 + \lambda n_0)^2\sigma^2n_0} \right)^{\frac{1}{2}} \wedge \frac{1}{2}, \]
we get from (2.9), (2.11) and (2.12) that for all \( \epsilon > 0 \) and \( \epsilon > 0 \), that if
\[ \| (1 - S_0, I_0, R_0) \| < \frac{\epsilon}{4} \wedge \left( \frac{\epsilon}{12(1 + \lambda n_0)} \right)^{\frac{1}{2}}, \]
where \( \| . \| \) is the norm defined on for all \( x = (x_1, x_2, x_3) \in \mathbb{R}^3 \) by \( \| x \| = |x_1| + |x_2| + |x_3| \). Then
\[ P \left( \sup_{t \geq 0} (1 - S(t) + I(t) + R(t)) > \epsilon \right) \leq \epsilon. \]
This makes end to the proof of the theorem.

From Theorem 2.1 and 2.2 we have the following corollary ensuring the asymptotic stability of the disease-free equilibrium state \( E_0(1, 0, 0) \).

Corollary 1. Let \( (S_0, I_0, R_0) \in \Delta \). If \( \beta \geq \sigma^2 \) and \( R_S = 1 \) then the disease-free equilibrium state \( E_0(1, 0, 0) \) is asymptotically stable in probability.

3. Large noise. In this section, we show that for large intensity the disease continues to exist for any initial conditions. Firstly, let us prove the following theorem which yields easily to our main theorem cited thereafter.

Theorem 3.1. Let \( (S_0, I_0, R_0) \in \Delta \). If \( \beta < \sigma^2 \) and \( R_S = 1 \), then
\[ \frac{2(\sigma^2 - \beta)}{\sigma^2} \leq \limsup_{t \to \infty} (I(t) + R(t)) \quad a.s. \]

Proof. It suffices to show that for all sufficiently small \( \epsilon > 0 \) we have
\[ \frac{2(\sigma^2 - \beta)}{\sigma^2} - \epsilon \leq \limsup_{t \to \infty} (I(t) + R(t)) \quad a.s. \]
To do it, suppose that
\[ I_0 + R_0 < \frac{2(\sigma^2 - \beta)}{\sigma^2} - \epsilon, \]
and show that
\[ P(\rho_\epsilon < \infty) = 1 \quad \text{where} \quad \rho_\epsilon = \inf \left\{ t \geq 0, \ I(t) + R(t) \geq \frac{2(\sigma^2 - \beta)}{\sigma^2} - \epsilon \right\}. \quad (3.1) \]
From (2.10) and \( R_S = 1 \), one has
\[ dI^\kappa = I^\kappa \Phi_\kappa(I + R)dt + \kappa \sigma I^\kappa(1 - I - R)dB, \quad (3.2) \]
where
\[ \Phi_\kappa(x) = \left( -\frac{1}{2}(1 - \kappa)\sigma^2x^2 + ((1 - \kappa)\sigma^2 - \beta)x + \frac{\kappa}{2}\sigma^2 \right). \]
Let \( \kappa_0 \) be a sufficiently small positive number in \( (0, 1) \) such that \( \beta < (1 - \kappa_0)\sigma^2 \).
For \( \kappa \leq \kappa_0 \), one can easily verify that there exists a unique positive root \( \xi_\kappa \) on \( (0, 1) \) of \( \Phi_\kappa(x) \) given by
\[ \xi_\kappa = \frac{(1 - \kappa)\sigma^2 - \beta \pm \sqrt{((1 - \kappa)\sigma^2 - \beta)^2 + (1 - \kappa)\kappa \sigma^2}}{(1 - \kappa)\sigma^2}, \quad (3.3) \]
and for all $x$ and all sufficiently small $\varepsilon > 0$ such that $0 < x \leq \xi_\kappa - \varepsilon$, we have
\[
\Phi_\kappa(x) \geq \alpha_{\kappa,\varepsilon}, \quad \text{where} \quad \alpha_{\kappa,\varepsilon} = \min \left\{ \Phi_\kappa(\xi_\kappa - \varepsilon), \frac{\kappa}{2} \sigma^2 \right\} > 0. \tag{3.4}
\]
For $\kappa \leq \kappa_0$ and $\varepsilon > 0$ sufficiently small, we define the stopping times
\[
\rho_{\kappa,\varepsilon} = \inf \{ t \geq 0, \ I(t) + R(t) \geq \xi_\kappa - \varepsilon \}.
\]
By (3.2) and Dynkin’s formula, we have
\[
EI^{\kappa}(t \wedge \rho_{\kappa,\varepsilon}) - I_0^\kappa = E \int_0^{t \wedge \rho_{\kappa,\varepsilon}} \Phi_\kappa(I(s) + R(s)) I^{\kappa}(s) ds, \quad t > 0, \tag{3.5}
\]
which implies by (3.4) that for all sufficiently small $\varepsilon > 0$ and $\kappa \leq \kappa_0$
\[
EI^{\kappa}(t \wedge \rho_{\kappa,\varepsilon}) - I_0^\kappa \geq \alpha_{\kappa,\varepsilon} E \int_0^{t \wedge \rho_{\kappa,\varepsilon}} I^{\kappa}(s) ds. \tag{3.6}
\]
On the other hand we can easily verify
\[
EI^{\kappa}(t \wedge \rho_{\kappa,\varepsilon}) = E \left( I^{\kappa}(\rho_{\kappa,\varepsilon})1_{\rho_{\kappa,\varepsilon} < t} \right) + E \left( I^{\kappa}(t)1_{t \leq \rho_{\kappa,\varepsilon}} \right) \tag{3.7}
\]
and
\[
E \int_0^{t \wedge \rho_{\kappa,\varepsilon}} I^{\kappa}(s) ds = \int_0^t E \left( I^{\kappa}(s)1_{s \leq \rho_{\kappa,\varepsilon}} \right) ds. \tag{3.8}
\]
So, we get from (3.6), (3.7) and (3.8),
\[
E \left( I^{\kappa}(\rho_{\kappa,\varepsilon})1_{\rho_{\kappa,\varepsilon} < t} \right) + \left( E \left( I^{\kappa}(t)1_{t \leq \rho_{\kappa,\varepsilon}} \right) - \alpha_{\kappa,\varepsilon} \int_0^t E \left( I^{\kappa}(s)1_{s \leq \rho_{\kappa,\varepsilon}} \right) ds \right) \geq I_0^\kappa, \tag{3.9}
\]
which means
\[
E \left( I^{\kappa}(\rho_{\kappa,\varepsilon})1_{\rho_{\kappa,\varepsilon} < t} \right) e^{-\alpha_{\kappa,\varepsilon}t} + \left( e^{-\alpha_{\kappa,\varepsilon}t} \int_0^t E \left( I^{\kappa}(s)1_{s \leq \rho_{\kappa,\varepsilon}} \right) ds \right) \geq I_0^\kappa e^{-\alpha_{\kappa,\varepsilon}t}. \tag{3.10}
\]
Integrating (3.10) from 0 to $t$ and using $I(\rho_{\kappa,\varepsilon}) \leq I(\rho_{\kappa,\varepsilon}) + R(\rho_{\kappa,\varepsilon}) = \xi_\kappa - \varepsilon$ yields
\[
(\xi_\kappa - \varepsilon)^\kappa \int_0^t P(\rho_{\kappa,\varepsilon} < s) e^{-\alpha_{\kappa,\varepsilon}s} ds + e^{-\alpha_{\kappa,\varepsilon}t} \int_0^t E \left( I^{\kappa}(s)1_{s \leq \rho_{\kappa,\varepsilon}} \right) ds \geq I_0^\kappa \frac{1 - e^{-\alpha_{\kappa,\varepsilon}t}}{\alpha_{\kappa,\varepsilon}}. \tag{3.11}
\]
Using
\[
\int_0^t P(\rho_{\kappa,\varepsilon} < s) e^{-\alpha_{\kappa,\varepsilon}s} ds = E \int_0^t e^{-\alpha_{\kappa,\varepsilon}s} ds = \frac{1}{\alpha_{\kappa,\varepsilon}} \left( E \left( e^{-\alpha_{\kappa,\varepsilon}\rho_{\kappa,\varepsilon}} \right) - e^{-\alpha_{\kappa,\varepsilon}t} \right) \tag{3.12}
\]
and letting $t \to \infty$ into (3.11), gives
\[
E \left( e^{-\alpha_{\kappa,\varepsilon}\rho_{\kappa,\varepsilon}} \right) \geq \left( \frac{I_0}{\xi_\kappa - \varepsilon} \right)^\kappa > I_0^\kappa. \tag{3.13}
\]
Moreover
\[
\lim_{\kappa \to 0} \xi_\kappa = \frac{2(\sigma^2 - \beta)}{\sigma^2}, \quad \lim_{\kappa \to 0} \rho_{\kappa,\varepsilon} = \rho_\varepsilon,
\]
and
\[
\lim_{\kappa \to 0} \alpha_{\kappa,\varepsilon} = \Phi \left( \frac{2(\sigma^2 - \beta)}{\sigma^2} - \varepsilon \right) = \frac{\sigma^2}{2} \left( \frac{2(\sigma^2 - \beta)}{\sigma^2} - \varepsilon \right),
\]
Finally, which is equivalent to solution (3.13) and using the Reverse Fatou’s lemma

$$E \left( e^{-\frac{\sigma^2}{2t} (2(\sigma^2 - \beta) - \varepsilon) \rho_\varepsilon} \right) \geq 1,$$

which is equivalent to

$$E \left( e^{-\frac{\sigma^2}{2t} (2(\sigma^2 - \beta) - \varepsilon) \rho_\varepsilon} 1_{\rho_\varepsilon < \infty} \right) \geq 1,$$

and then

$$\mathbb{P}(\rho_\varepsilon < \infty) \geq E \left( e^{-\frac{\sigma^2}{2t} (2(\sigma^2 - \beta) - \varepsilon) \rho_\varepsilon} 1_{\rho_\varepsilon < \infty} \right) \geq 1.$$

Finally $\mathbb{P}(\rho_\varepsilon < \infty) = 1$. This completes the proof. □

Before citing the main theorem, we put forward the following Lemma which is a special case of Proposition 5.1 in Settati et al. [23].

**Lemma 3.2.** For any initial values $(S_0, I_0, R_0) \in \Delta$, it holds that

$$\frac{\lambda}{\mu + \gamma} \lim \inf I(t) \leq \lim \inf R(t) \leq \lim \sup R(t) \leq \frac{\lambda}{\mu + \gamma} \lim \sup I(t).$$

Now, let us cite the following main theorem showing that each component of the solution $(S(t), I(t), R(t))$ rises to or above certain positive level infinitely often with probability one.

**Theorem 3.3.** For any initial values $(S_0, I_0, R_0) \in \Delta$, if $\beta < \sigma^2$ and $R_S = 1$ then the solution of the stochastic differential equation (1.1) obeys

(i): $\lim \inf S(t) \leq \frac{2\beta - \sigma^2}{\sigma^2} \leq \lim \sup S(t)$ a.s.,

(ii): $\lim \inf I(t) \leq \frac{2(\sigma^2 - \beta)(\mu + \gamma)}{\sigma^2(\mu + \gamma + \lambda)} \leq \lim \sup I(t)$ a.s.,

(iii): $\lim \inf R(t) \leq \frac{2(\sigma^2 - \beta)\lambda}{\sigma^2(\mu + \gamma + \lambda)} \leq \lim \sup R(t)$ a.s.

**Proof.** (i) It is easy to see that the left hand of (i) is an immediate consequence of $S = 1 - I - R$ and Theorem 3.3, so

$$\lim \inf S(t) = 1 - \lim \inf (I(t) + R(t)) \leq \frac{2\beta - \sigma^2}{\sigma^2}.$$

For the right hand of (i), we will proceed by contradiction. Suppose that the right hand of (i) is not true, then there is a sufficiently small $\varepsilon > 0$ such that $\mathbb{P}(\Omega_1) > 0$ where

$$\Omega_1 = \left\{ \lim \sup S(t) \leq \frac{2\beta - \sigma^2}{\sigma^2} - 2\varepsilon \right\}.$$

Hence, for every $w \in \Omega_1$, there is a $T(w) > 0$ such that

$$0 \leq S(s) \leq \frac{2\beta - \sigma^2}{\sigma^2} - \varepsilon < 1, \quad s \geq T(w). \quad (3.14)$$

By Itô’s formula, we have if $R_S = 1$

$$\log(I(t)) = \log(I_0) + \int_0^t \Phi(S(s)) ds + \int_0^t \sigma S(s) dB(s), \quad (3.15)$$
where $\Phi(x) = (1-x) \left( \frac{1}{2} \sigma^2 x - \beta \right)$. From $R_S = 1$, one can easily verify that for all sufficiently small $\varepsilon > 0$ and all $x$ such that $0 < x \leq \frac{2\beta - \sigma^2}{\sigma^2} - \varepsilon$, we have

$$\Phi(x) \leq \Phi \left( \frac{2\beta - \sigma^2}{\sigma^2} - \varepsilon \right) < 0. \quad (3.16)$$

From (3.14) and (3.16) we get, for any $s \geq T$,

$$\Phi(S(s)) \leq \Phi \left( \frac{2\beta - \sigma^2}{\sigma^2} - \varepsilon \right) < 0. \quad (3.17)$$

Hence, by (3.15) and (3.17) and the large number theorem for martingales, there is a $\Omega^2 \subset \Omega$ with $P(\Omega^2) = 1$ such that for every $w \in \Omega^1 \cap \Omega^2$,

$$\limsup_{t \to \infty} \frac{1}{t} \log(I(t)) \leq \Phi(\xi_s - \varepsilon) < 0.$$

Therefore, $\lim_{t \to \infty} I(t) = 0$, which implies together with Lemma 3.2 that $\lim_{t \to \infty} R(t) = 0$ and thereby $\lim_{t \to \infty} S(t) = 1$. But this contradicts (3.14). The right hand of (i) must therefore hold.

(ii) It is easy to see that (ii) and (iii) are immediate consequences of $S + I + R = 1$ and (i) and Lemma 3.2. 

4. **Conclusion and simulations.** From the analytical results established by Lahrouz and Settati [19], the introduction of noise in the deterministic SIRS model modifies the basic reproductive number $R_0 = \frac{\beta}{\mu + \lambda + \gamma}$ giving rise to a new threshold quantity $R_S = \frac{\beta}{\mu + \lambda + \gamma} + \frac{1}{2} \sigma^2$. So, if $R_S < 1$ the disease dies out from the population, while if $R_S > 1$ the disease persists. However the case when $R_S = 1$ remains an open question, so the classical techniques used to study the asymptotic behavior do not work any more. In this paper, we gave answer to this open problem by using a new approach involving some adequate stopping times. Ours results extend the deterministic result in Hethcote [14] for the critical case $R_S = 1$, where the disease dies out from the host population. So, in the presence of environmental noises, the long time behaviour of the disease depends on the intensity of noise. Precisely, we have shown that in the stochastic framework, when $R_S = 1$, there are two cases. If the intensity of noise is small enough such that $\sigma^2 \leq \beta$, the environmental perturbation of the transmission rate does not change the asymptotic behavior of the deterministic model, that is to say the free-disease equilibrium state $E_0(1,0,0)$ is asymptotically stable in probability which extends the deterministic result in Hethcote [14] by letting the noise $\sigma$ to 0, so, we have $\lim_{\sigma \to 0} R_S(\sigma) = R_0$ (see Fig.1 ).

On the other hand, for the large intensity case such that $\sigma^2 > \beta$, the disease in the perturbed SIRS epidemic model persists around some positive constant in the sense that each component of the solutions $(S(t), I(t), R(t))$ rises to or above certain positive level infinitely often with probability one and this occurs regardless of the initial values $(S_0, I_0, R_0)$ (see Fig.2 ). In other words, our Theorems 2.1 and 3.3 show that if $R_S = 1$ then, small noises promote extinction while the large one promote persistence. So, it is exactly the opposite role of the noises in case when $R_S \neq 1$.

Finally we point out that some issues deserve further theoretical and numerical investigation. For instance, what is the long-time behavior of the epidemic model (1.1) in the case when $R_S < 1$ and $\sigma^2 > \beta$? Another interesting continuation
of this work might be to introduce Lévy noise into some parameters of system (1.1). We leave these for future investigation.

\textbf{Figure 1.} simulation results of $S(t), I(t), R(t)$ respectively for the SDE model (1.1) with different initial conditions $(S_0, I_0, R_0)$ and the parameters: $\mu = 0.1, \beta = 0.62, \lambda = 0.5, \gamma = 0.5, \sigma = 0.2$. Here $R_S = 1$ and $\sigma^2 < \beta$. For the different values of initial conditions the infection-free equilibrium $E_0(1, 0, 0)$ is asymptotically stable in probability.
Figure 2. Simulation results of $S(t), I(t), R(t)$ respectively for the SDE model (1.1) with different initial conditions $(S_0, I_0, R_0)$ and the parameters: $\mu = 0.1, \beta = 0.7802, \lambda = 0.2, \gamma = 0.2, \sigma = 0.98$. Here $R_S = 1$ and $\beta < \sigma^2$. For the different values of initial conditions, respectively, $S(t), I(t)$ and $R(t)$ rises to or above the level $\xi_s = 0.6247, \xi_i = 0.2252$ and $\xi_r = 0.1501$ infinitely often with probability one.

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Received April 2019; revised July 2019.

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