A Subspace of Maximal Dimension with Bounded Schmidt Rank

Priyabrata Bag & Santanu Dey

Abstract

We study Schmidt rank for a vector (i.e., a pure state) and Schmidt number for a mixed state which are entanglement measures. We show that if a subspace of a certain bipartite system contains no vector of Schmidt rank \( k \), then any state supported on that space has Schmidt number at least \( k + 1 \). A construction of subspace of \( \mathbb{C}^m \otimes \mathbb{C}^n \) of maximal dimension, which does not contain any vector of Schmidt rank less than 3, is given here.

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1 Introduction

A state (also known as the density matrices) is a positive matrix whose trace is equal to 1. A pure state \( |v\rangle \in \mathbb{C}^m \otimes \mathbb{C}^n \) (or \( |v\rangle \langle v| \)) is said to be a product state if it can be written as \( |v\rangle = |v_1\rangle \otimes |v_2\rangle = |v_1\rangle |v_2\rangle \) for some \( |v_1\rangle \in \mathbb{C}^m \) and \( |v_2\rangle \in \mathbb{C}^n \); otherwise it is called entangled. The states, which are not pure, are also referred to as mixed states. If \( \{|e_0\rangle, |e_1\rangle\} \) is the standard basis of \( \mathbb{C}^2 \), then the states

\[
\frac{|e_0\rangle |e_0\rangle \pm |e_1\rangle |e_1\rangle \sqrt{2}}{\sqrt{2}} \quad \text{and} \quad \frac{|e_0\rangle |e_1\rangle \pm |e_1\rangle |e_0\rangle \sqrt{2}}{\sqrt{2}}
\]

are famous entangled states (cf. Sections 1.3.7 and 2.3 of [5]), which are known as Bell states or EPR pairs.

A mixed state \( \rho \in M_m \otimes M_n \) is called separable if it is convex combination of pure product states; otherwise it is called entangled. Entanglement is the key property of quantum systems which is responsible for the higher efficiency of quantum computation and tasks like teleportation, super-dense coding, etc (cf. [4]). The Schmidt rank of vectors and Schmidt number of states in a bipartite finite dimensional Hilbert space are measures of entanglement. In Section 2, we establish a sufficient condition for states on a bipartite finite dimensional Hilbert space to be of Schmidt number bounded below by some positive integer. A construction of subspace of \( \mathbb{C}^m \otimes \mathbb{C}^n \) of maximal dimension, which does not contain any vector of Schmidt rank less than 3, is given in Section 3.

In [2], it was proved using algebraic geometric techniques that for a bipartite system \( \mathbb{C}^m \otimes \mathbb{C}^n \), the dimension of any subspace of Schmidt rank greater than or equal to \( k \) is bounded above by \( (m-k+1)(n-k+1) \). In [2, Proposition 10] it was also shown that this bound is attained. In this article, motivated by the analysis done in [1], we present an alternate approach to prove that this bound is attained for \( k = 3 \). We construct a subspace of \( \mathbb{C}^m \otimes \mathbb{C}^n \) of Schmidt rank greater than or equal to 3 and, unlike [2], we also have a basis of this subspace consisting of elements of Schmidt rank 3. For the case when a subspace of \( \mathbb{C}^m \otimes \mathbb{C}^n \) is of Schmidt rank greater than or equal to 2 (i.e., the subspace does not contain any product vector), the maximum dimension of that subspace is \( (m-1)(n-1) \), and this was first proved in [5] and [8] (cf. [1]).
2 Schmidt Rank and Schmidt Number

Let $\mathcal{H}$ denote the bipartite Hilbert space $\mathbb{C}^m \otimes \mathbb{C}^n$. By Schmidt decomposition theorem (cf. [5, Section 2.5]), any pure state $|\psi\rangle \in \mathcal{H}$ can be written as

$$|\psi\rangle = \sum_{j=1}^{k} \alpha_j |u_j\rangle \otimes |v_j\rangle$$

(2.1)

for some $k \leq \min\{m, n\}$, where $\{u_j\} : 1 \leq j \leq k$ and $\{v_j\} : 1 \leq j \leq k$ are orthonormal sets in $\mathbb{C}^m$ and $\mathbb{C}^n$ respectively, and $\alpha_j$’s are nonnegative real numbers satisfying $\sum_j \alpha_j^2 = 1$.

**Definition 2.1.** In the Schmidt decomposition (2.1) of a pure bipartite state $|\psi\rangle$ the minimum number of terms required in the summation is known as the Schmidt rank of $|\psi\rangle$, and it is denoted by $SR(|\psi\rangle)$.

In the bipartite Hilbert space $\mathbb{C}^m \otimes \mathbb{C}^n$, for any $1 \leq r \leq \min\{m, n\}$, there is at least some state $|\psi\rangle$ with $SR(|\psi\rangle) = r$. Any state $\rho$ on a finite dimensional Hilbert space $\mathcal{H}$ can be written as

$$\rho = \sum_j p_j |\psi_j\rangle \langle \psi_j|,$$

(2.2)

where $|\psi_j\rangle$’s are pure states in $\mathcal{H}$ and $\{p_j\}$ forms a probability distribution. The following notion was introduced in [7].

**Definition 2.2.** The Schmidt number of a state $\rho$ on a bipartite finite dimensional Hilbert space $\mathcal{H}$ is defined to be the least natural number $k$ such that $\rho$ has a decomposition of the form given in (2.2) with $SR(|\psi_j\rangle) \leq k$ for all $j$. The Schmidt number of $\rho$ is denoted by $SN(\rho)$.

The next theorem gives a sufficient condition for a state to have Schmidt number greater than $k$.

**Theorem 2.1.** Let $S$ be a subspace of $\mathcal{H} = \mathbb{C}^m \otimes \mathbb{C}^n$ which does not contain any vector of Schmidt rank lesser or equal to $k$. Then any state $\rho$ supported on $S$ has Schmidt number at least $k + 1$.

**Proof.** Let $\rho$ be a state with Schmidt number, $SN(\rho) = r \leq k$. So, $\rho$ can be written as

$$\rho = \sum_j p_j |\psi_j\rangle \langle \psi_j|,$$

where $SR(|\psi_j\rangle) \leq k$ for all $j$ and $\{p_j\}$ forms a probability distribution. First we show that each of $|\psi_j\rangle$ is in the range of $\rho$. With out loss of generality, we show this for $j = 1$. Write

$$\rho = p_1 |\psi_1\rangle \langle \psi_1| + T,$$

(2.3)

where $p_1 > 0$ and $T$ is a nonnegative operator.

Let $|\psi\rangle \neq 0$ in $\mathcal{H}$ be such that $T|\psi\rangle = 0$ and $\langle \psi_1|\psi\rangle \neq 0$. Clearly, $\rho|\psi\rangle$ is a nonzero multiple of $|\psi_1\rangle$ and hence $|\psi_1\rangle$ belongs to the range of $\rho$. Now suppose the null space of $T$ is contained in $\{|\psi_1\rangle\}^\perp$. Then $|\psi_1\rangle$ is in the range of $T$. So, there exists $|\psi\rangle \neq 0$ such that $\langle \psi|\psi\rangle = |\psi_1\rangle$. Thus, (2.3) yields

$$\rho|\psi\rangle = (p_1 \langle \psi_1|\psi\rangle + 1) |\psi_1\rangle.$$

If $\rho|\psi\rangle = 0$, then by the positivity of $\rho$, $p_1 |\psi_1\rangle \langle \psi_1|$ and $T$, and (2.3), we obtain $\langle T|\psi\rangle = 0$, and hence $T|\psi\rangle = 0$. This is a contradiction. Thus, we deduce that $\rho|\psi\rangle \neq 0$. Therefore, it follows that $|\psi_1\rangle$ is in the range of $\rho$.

If such a $\rho$ is supported on $S$, then the above statement gives a contradiction to the fact that $S$ does not contain any vector of Schmidt rank lesser or equal to $k$. This completes the proof. □
3 A Subspace of Maximal Dimension of Schmidt Rank $\geq 3$

Let $\mathcal{H} = \mathbb{C}^m \otimes \mathbb{C}^n$ as before. Let us fix an infinite dimensional Hilbert space $\mathcal{K}$ with orthonormal basis $\{|e_0\rangle, |e_1\rangle, \ldots\}$, and identify $\mathbb{C}^m$ and $\mathbb{C}^n$ with $\text{span}\{|e_0\rangle, |e_1\rangle, \ldots, |e_{m-1}\rangle\}$ and $\text{span}\{|e_0\rangle, |e_1\rangle, \ldots, |e_{n-1}\rangle\}$, respectively. Define $N = n + m - 2$, and for $2 \leq d \leq N - 2$ define

$$S^{(d)} = \text{span}\{|e_{i-1}\rangle \otimes |e_{j+1}\rangle - 2 |e_i\rangle \otimes |e_j\rangle + |e_{i+1}\rangle \otimes |e_{j-1}\rangle : 1 \leq i \leq m-2, 1 \leq j \leq n-2, i+j = d\},$$

(3.1)

$$S^{(0)} = S^{(1)} = S^{(N-1)} = S^{(N)} = \{0\}, \text{ and } S := \bigoplus_{d=0}^{N} S^{(d)}.$$  

(3.2)

We claim that $S$ does not contain any vector of Schmidt rank less than 3.

**Lemma 3.1.** The columns of the $(t+2) \times t$ matrix

$$A_t = \begin{bmatrix}
1 & 0 & \ldots & 0 & 0 \\
-2 & 1 & \ldots & 0 & 0 \\
1 & -2 & \ldots & 0 & 0 \\
0 & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & 0 \\
0 & 0 & \ldots & -2 & 1 \\
0 & 0 & \ldots & 1 & -2 \\
0 & 0 & \ldots & 0 & 1
\end{bmatrix}$$

are linearly independent such that any linear combination of these columns has at least 3 nonzero entries.

**Proof.** At first, we show that all the order-$t$ minors of $A_t$ are nonzero. We would prove this statement by induction. The statement clearly holds for $l = 1$, i.e., for the matrix $A_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$. Assume that the statement holds for $l = t - 1$. We would show that the statement is true for $l = t$ by observing the following cases:

1. If the two rows deleted from $A_t$ to obtain the order-$t$ minor does not include the first row, then let us denote the $t \times t$ matrix by $F$ whose determinant gives this order-$t$ minor. By two elementary row operations (involving the first row) on $F$ which does not change the determinant, we can obtain $(1, 0, 0, \ldots, 0)^T$ as the first column of $F$, where superscript $T$ denotes the transpose. The order-$(t-1)$ minor obtained by deleting the first row and the first column of $F$ is nonzero by induction hypothesis because it is a order-$(t-1)$ minor of $A_{t-1}$. It follows that the det $F$ is nonzero.

2. If the two rows deleted from $A_t$ to obtain the order-$t$ minor does not include the last row, then let us denote the $t \times t$ matrix by $L$ whose determinant gives this order-$t$ minor. By two elementary row operations (involving the last row) on $L$ which does not change the determinant, we can obtain $(0, \ldots, 0, 0, 1)^T$ as the last column of $L$. The order-$(t-1)$ minor obtained by deleting the last row and the last column of $F$ is nonzero by induction hypothesis because it is a order-$(t-1)$ minor of $A_{t-1}$. It follows that the det $L$ is nonzero.
3. If the first and the last row is deleted from \( A_t \) to obtain the order-\( t \) minor, then we need to show that the determinant of

\[
C_s = \begin{bmatrix}
-2 & 1 & \ldots & 0 & 0 \\
1 & -2 & \ldots & 0 & 0 \\
0 & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & 0 \\
0 & 0 & \ldots & -2 & 1 \\
0 & 0 & \ldots & -2 & -2 \\
\end{bmatrix}
\]

is nonzero. We claim that \( \det C_s = (-1)^s(s+1) \). Once again induct on \( s \). Verifying for \( s = 2,3 \) directly. Assume that the formula holds for all \( k < s \) and \( s > 3 \). Then expand along the first row to see that

\[
\det C_s = -2(\det C_{s-1}) - \det C_{s-2} \\
= -2(-1)^{s-1}s - (-1)^{s-2}(s-1) \\
= (-1)^s(s+1).
\]

Thus, all the order-\( t \) minors of \( A_t \) are nonzero. Let us assume that a linear combination of columns of \( A_t \) has less than 3 non zero entries, i.e., a linear combination of columns of \( A_t \) has \( t \) or more zero entries. Let \( I \) be the set of indices of any \( t \) of those zero entries. So, the \( t \times t \) submatrix formed from the rows of \( A_t \) which are indexed by \( I \), is such that the columns of this submatrix is linearly dependent. We deduce that the corresponding order-\( t \) minor is zero and this is a contradiction. This proves that any linear combination of the columns of \( A_t \) has at least 3 nonzero entries. It also follows that the columns of the matrix \( A_t \) are linearly independent. \( \square \)

**Theorem 3.2.** Let \( m \) and \( n \) be natural numbers such that \( 3 \leq \min\{m, n\} \). The space \( \mathcal{S} \) defined by equations (3.1) and (3.2) does not contain any vector of Schmidt rank \( \leq 2 \) and \( \dim \mathcal{S} = (m-2)(n-2) \).

**Proof.** Let us fix an infinite dimensional Hilbert space \( \mathcal{K} \) with orthonormal basis \( \{|e_0\}, \{|e_1\}, \ldots\} \), and identify \( \mathbb{C}^m \) and \( \mathbb{C}^n \) with \( \text{span}\{|e_0\}, \{|e_1\}, \ldots, |e_{m-1}\} \) and \( \text{span}\{|e_0\}, \{|e_1\}, \ldots, |e_{n-1}\} \), respectively, as done before. If for any element \( |v\rangle \in \mathbb{C}^m \otimes \mathbb{C}^n \), we have \( |v\rangle = \sum_{ij} c_{ij} |e_i\rangle \otimes |e_j\rangle \), then we identify the bipartite system \( \mathbb{C}^m \otimes \mathbb{C}^n \) with the space \( M_{m\times n} \) of \( m \times n \) matrices by the isometric isomorphism \( \phi : \mathbb{C}^m \otimes \mathbb{C}^n \rightarrow M_{m\times n} \) defined by \( \phi(|v\rangle) = [c_{ij}]_{m\times n} \). It follows from the standard proof of Schmidt decomposition based on the singular value decomposition that, an element of \( \mathbb{C}^m \otimes \mathbb{C}^n \) has Schmidt rank at least \( r \), if and only if the corresponding \( m \times n \) matrix is of rank at least \( r \). Also, it is known that a matrix has rank at least \( r \) if and only if it has a nonzero minor of order \( r \) (cf. [33 page 18]). Thus, it is enough to construct a set of \( (m-2)(n-2) \) linearly independent matrices, which are image under \( \phi \) of a basis of \( \mathcal{S} \), such that any linear combination of these matrices has a nonzero minor of order \( 3 \).

Label the anti-diagonals of any \( m \times n \) matrix by non-negative integers \( k \), such that the first anti-diagonal from upper left (of length one) is labelled \( k = 0 \) and value of \( k \) increases from upper left to lower right. Let the length of the \( k^{th} \) anti-diagonal be denoted by \( |k| \). If we assume, without loss of generality, that \( m \leq n \), then the formula for \( |k| \) is

\[
|k| = \begin{cases}
    k + 1 & \text{for } 0 \leq k \leq m - 1 \\
    m & \text{for } m - 1 \leq k \leq n - 1 \\
    m + n - (k + 1) & \text{for } n - 1 \leq k \leq N.
\end{cases}
\]
Recall that for $2 \leq k \leq N - 2$, $S^{(k)}$ is generated by the set

$$B_k = \{|e_{i-1}\otimes|e_{j+1}\rangle - 2|e_i\rangle \otimes |e_j\rangle + |e_{i+1}\rangle \otimes |e_{j-1}\rangle : 1 \leq i \leq m - 2, 1 \leq j \leq n - 2, i + j = k\}.$$  

For $2 \leq k \leq N - 2$, let $\tilde{B}_k$ denote the image of $B_k$ under the map $\phi$. Note that any element of $\tilde{B}_k$ is the matrix obtained from the $m \times n$ zero matrix by replacing the $k^{th}$ anti-diagonal by $(0, \ldots, 0, 1, -2, 1, 0, \ldots, 0)$. For $2 \leq k \leq N - 2$ and $t = |k| - 2$, from Lemma 3.1 $\tilde{B}_k$ is a set of $t$ linearly independent matrices. Also, by Lemma 3.1 it follows that if $M$ is a matrix obtained by taking any linear combination of matrices from $\tilde{B}_k$, then $M$ has at least 3 nonzero entries in the $k^{th}$ anti-diagonal and the entries of $M$, other than those on $k^{th}$ anti-diagonal, are zeros. Since the determinant of the $3 \times 3$ submatrix with these 3 nonzero elements in its principal anti-diagonal is clearly nonzero, any linear combination of the matrices in $\tilde{B}_k$ has at least one nonzero order-3 minor, thus has rank at least 3.

Let $\tilde{B} = \bigcup_{k=2}^{N-2} \tilde{B}_k$. Since elements from different $\tilde{B}_k$ have different nonzero anti-diagonal, $\tilde{B}$ is linearly independent. Thus $B = \bigcup_{k=2}^{N-2} B_k$ is a basis for $S$. Let $C$ be a matrix obtained by an arbitrary linear combination from the elements of $\tilde{B}$. Let $\kappa$ be the largest $k$ for which the linear combination involves an element from $\tilde{B}_k$. The $\kappa^{th}$ anti-diagonal of $C$ has at least 3 nonzero elements. Because $\kappa$ labels the bottom-rightmost anti-diagonal of $C$ that contains nonzero elements, the $3 \times 3$ submatrix of $C$, with these 3 nonzero elements in the principal anti-diagonal, has only zero entries in all its anti-diagonals which are below the principal anti-diagonal. Hence the $3 \times 3$ submatrix has nonzero determinant. Thus, the rank of $C$ is at least 3. We conclude that $S$ does not contain any vector of Schmidt rank $\leq 2$.

The dimension of $S$ is equal to the cardinality of $B$. Without loss of generality, assume $m \leq n$. Then, the cardinality of $B$ is given by

$$|B| = \sum_{k=2}^{N-2} |B_k| = \sum_{k=2}^{N-2} (|k| - 2)$$

$$= \sum_{k=2}^{m-2} (k - 1) + \sum_{k=m-1}^{n-1} (m - 2) + \sum_{k=n}^{m+n-4} (m + n - 2 - (k + 1))$$

$$= (n - m + 1)(m - 2) + 2 \sum_{k=2}^{m-2} (k - 1)$$

$$= (m - 2)(n - 2).$$

From the above theorem it follows that the basis

$$B = \bigcup_{k=2}^{N-2} \{|e_{i-1}\otimes|e_{j+1}\rangle - 2|e_i\rangle \otimes |e_j\rangle + |e_{i+1}\rangle \otimes |e_{j-1}\rangle : 1 \leq i \leq m - 2, 1 \leq j \leq n - 2, i + j = k\},$$

of $S$ is of Schmidt rank 3 (i.e., all the elements have Schmidt rank 3).
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Priyabrata Bag  
Department of Mathematics  
Indian Institute of Technology Bombay  
Mumbai, Maharashtra 400076, India  
E-mail: priyabrata@iitb.ac.in

Santanu Dey  
Department of Mathematics  
Indian Institute of Technology Bombay  
Mumbai, Maharashtra 400076, India  
E-mail: santanudey@iitb.ac.in