STRONG EQUIVALENCES OF APPROXIMATION NUMBERS AND TRACTABILITY OF WEIGHTED ANISOTROPIC SOBOLEV EMBEDDINGS*

Jidong HAO (郝季东)  Heping WANG (汪和平)†
School of Mathematical Sciences, Capital Normal University, Beijing 100048, China
E-mail: 1047695025@qq.com; wanghp@cnu.edu.cn

Abstract  In this article, we study multivariate approximation defined over weighted anisotropic Sobolev spaces which depend on two sequences \(a = \{a_j\}_{j \geq 1}\) and \(b = \{b_j\}_{j \geq 1}\) of positive numbers. We obtain strong equivalences of the approximation numbers, and necessary and sufficient conditions on \(a, b\) to achieve various notions of tractability of the weighted anisotropic Sobolev embeddings.

Key words  strong equivalences; tractability; approximation numbers; weighted anisotropic spaces; analytic Korobov spaces

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1 Introduction

This article is devoted to investigating sharp constants of approximation numbers and the tractability of embeddings of weighted anisotropic Sobolev spaces on \([0, 1]^d\) into \(L_2([0, 1]^d)\). The approximation numbers of a bounded linear operator \(T : X \to Y\) between two Banach spaces are defined as

\[
a_n(T : X \to Y) := \inf_{\text{rank } A < n} \sup_{\|x\|_X \leq 1} \|Tx - Ax\|_Y
\]

\[
= \inf_{\text{rank } A < n} \|T - A\|_{X \to Y}, \quad n \in \mathbb{N}.
\]

These describe the best approximation of \(T\) by finite rank operators. If \(X\) and \(Y\) are Hilbert spaces and \(T\) is compact, then \(a_n(T)\) is the \(n\)-th singular value of \(T\). Also, \(a_n(T)\) is just the \((n - 1)\)-st minimal worst-case error with respect to arbitrary algorithms and general linear information in the Hilbert setting.

Recently, Kühn and many other authors investigated and obtained strong equivalences, preasymptotics and asymptotics of the approximation numbers and the tractability of the classical isotropic Sobolev embeddings, Sobolev embeddings of dominating mixed smoothness, Gevrey space embeddings, anisotropic Sobolev embeddings on the torus \(T^d = [0, 2\pi]^d\) (see

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†Corresponding author: Heping WANG.
We note that unlike on the torus $T^d = [0, 2\pi]^d$, the anisotropic spaces $W^d_2([0,1]^d)$ on the torus $[0,1]^d$ naturally induce weighted norms given in terms of Fourier coefficients (see (2.4) in Section 2.1), where $b = \{b_j\}_{j \geq 1}$ is a smoothness parameter sequence. In this article we consider general weighted anisotropic Sobolev spaces $W^d_2(a,b)([0,1]^d)$ on the torus $[0,1]^d$ whose definitions are given in Section 2.1, where the positive sequence $a = \{a_j\}_{j \geq 1}$ is a scaling parameter sequence, and $b = \{b_j\}_{j \geq 1}$ is a smoothness parameter sequence. We discuss the approximation numbers and the tractability of the weighted anisotropic Sobolev embedding

$$I_d : W^d_2(a,b)([0,1]^d) \rightarrow L_2([0,1]^d),$$

where $I_d$ is the identity (embedding) operator.

We obtain strong equivalences of the approximation numbers $a_n(I_d) \equiv a_n(I_d : W^d_2(a,b)([0,1]^d) \rightarrow L_2([0,1]^d))$ as $n \rightarrow \infty$. We remark that the sharp orders of $a_n(I_d)$ depend only on the smoothness parameter sequence $b$, and the sharp constants are closely related to the volume of the generalized ellipsoid defined by $a$, $b$, and $d$. Our result generalizes Theorem 2.4 in [1]. However, we do not obtain results about the preasymptotics of $a_n(I_d)$ as in [1].

We also consider the tractability of the approximation problem $I = \{I_d\}$ of the weighted anisotropic Sobolev embeddings. We consider algorithms that use finitely many continuous linear functionals. The information complexity $n(\varepsilon, I_d)$ is defined as the minimal number of linear functionals which are needed to find an approximation to within an error threshold $\varepsilon$. There are two kinds of tractability: that based on polynomial convergence, and that based on exponential convergence. The classical tractability describes how the information complexity behaves as a function of $d$ and $\varepsilon^{-1}$, while the exponential convergence-tractability (EC-tractability) does as one of $d$ and $(1 + \ln \varepsilon^{-1})$. Nowadays, the study of tractability and EC-tractability has attracted much interest, and a great number of interesting results have been obtained (see [5, 6, 15–18, 22] and the references therein).

Denote by $H(K_{d,a,2b})$ the analytic Korobov space which is a reproducing kernel Hilbert space with the reproducing kernel $K_{d,a,2b}$, and whose definition will be given in Section 2.2. Such spaces $H(K_{d,a,2b})$ have been widely investigated in the study of tractability and EC-tractability (see [4–8, 13, 14, 23]). In particular, the articles [4, 23] considered different notions of EC-tractability of the approximation problems $APP = \{APP_d\}_{d \in \mathbb{N}}$, and obtained the corresponding necessary and sufficient conditions, where

$$APP_d : H(K_{d,a,2b}) \rightarrow L_2([0,1]^d), \quad \text{with } APP_d(f) = f. \quad (1.2)$$

In this article, we establish the relationship of the information complexities $n(\varepsilon, I_d)$ and $n(\varepsilon, APP_d)$. On the basis of this relationship, we obtain the necessary and sufficient conditions for various notions of tractability of the approximation problem $I = \{I_d\}_{d \in \mathbb{N}}$. The article is organized as follows: in Section 2 we introduce the weighted anisotropic Sobolev spaces, the analytic Korobov spaces, the properties of the approximation numbers, the tractability, and then state our main results. Section 3 is devoted to proving the strong
equivalence of the approximation numbers of the weighted anisotropic embeddings. In Section 4 we prove the tractability of the weighted anisotropic embeddings.

2 Preliminaries and Main Results

2.1 Weighted anisotropic Sobolev spaces on $[0, 1]^d$

Denote by $L_2([0, 1]^d)$ the collection of measurable functions $f$ on $[0, 1]^d$ with finite norm

$$
\|f\|_2 = \left( \int_{[0, 1]^d} |f(x)|^2 \, dx \right)^{\frac{1}{2}} < +\infty.
$$

It is well known that any $f \in L_2([0, 1]^d)$ can be expressed by its Fourier series

$$
f(x) = \sum_{\mathbf{k} \in \mathbb{Z}^d} \hat{f}(\mathbf{k}) e^{2\pi i \mathbf{k} \cdot \mathbf{x}}, \quad \mathbf{x} \in [0, 1]^d,
$$

where $i = \sqrt{-1}$, $\mathbf{kx} = \sum_{j=1}^d k_j x_j$, and

$$
\hat{f}(\mathbf{k}) = \int_{[0, 1]^d} f(x) e^{-2\pi i \mathbf{k} \cdot \mathbf{x}} \, dx, \quad \mathbf{k} \in \mathbb{Z}^d
$$

are the Fourier coefficients of the function $f$. We have the following Parseval equality:

$$
\|f\|_2 = \left( \sum_{\mathbf{k} \in \mathbb{Z}^d} |\hat{f}(\mathbf{k})|^2 \right)^{1/2}.
$$

For $r > 0$, denote by $D^r_j f = \frac{\partial^r}{\partial x_j^r} f$ the $r$-th order partial derivative of $f$ with respect to $x_j$ in the sense of Weyl, i.e.,

$$
D^r_j f(x) = \sum_{\mathbf{k} \in \mathbb{Z}^d} (2\pi i k_j)^r \hat{f}(\mathbf{k}) e^{2\pi i \mathbf{k} \cdot \mathbf{x}}, \quad (2\pi i k_j)^r = |2\pi k_j|^r \exp \left( \frac{r\pi i}{2} \text{sign} k_j \right).
$$

It follows from the Parseval equality that for $D^r_j f \in L_2([0, 1]^d)$,

$$
\|D^r_j f\|_2 = \left( \sum_{\mathbf{k} \in \mathbb{Z}^d} |2\pi k_j|^r |\hat{f}(\mathbf{k})|^2 \right)^{1/2}. \tag{2.1}
$$

Now we define weighted anisotropic Sobolev spaces. Let $\mathbf{a} = \{a_k\}_{k \geq 1}$ and $\mathbf{b} = \{b_k\}_{k \geq 1}$ be two sequences of positive numbers. Usually, we assume that the sequences $\mathbf{a} = \{a_k\}_{k \geq 1}$ and $\mathbf{b} = \{b_k\}_{k \geq 1}$ satisfy

$$
0 < a_1 \leq a_2 \leq \cdots \leq a_k \leq \cdots, \quad \text{and} \quad \inf k \geq 1 b_k > 0. \tag{2.2}
$$

The weighted anisotropic Sobolev space $W^{\mathbf{a}, \mathbf{b}}_2([0, 1]^d)$ is defined by

$$
W^{\mathbf{a}, \mathbf{b}}_2([0, 1]^d) = \left\{ f \in L_2([0, 1]^d) : D_j^{b_j} f \in L_2([0, 1]^d), \quad j = 1, 2, \cdots, d \right\},
$$

with the norm

$$
\|f\|_{W^{\mathbf{a}, \mathbf{b}}_2} = \left( \|f\|_2^2 + \sum_{j=1}^d \frac{a_j}{(2\pi)^{2b_j}} \|D_j^{b_j} f\|_2^2 \right)^{1/2}.
$$

Clearly, $W^{\mathbf{a}, \mathbf{b}}_2([0, 1]^d)$ is a Hilbert space. We remark that $\mathbf{b}$ is a smoothness parameter sequence, and $\mathbf{a}$ is a (regulated) scaling parameter sequence with respect to the sequence $\mathbf{b}$. 

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It follows from (2.1) that
\[
\|f\|_{W_2^{a,b}} = \left( \sum_{k \in \mathbb{Z}^d} \left( 1 + \sum_{j=1}^d a_j |k_j|^{2b_j} \right) |\hat{f}(k)|^2 \right)^{1/2}.
\] (2.3)

If \(a_j = (2\pi)^{2b_j}, \ j \in \mathbb{N}\), then \(W_2^{a,b}([0,1]^d)\) reduces to the usual anisotropic Sobolev spaces \(W_2^b([0,1]^d)\) on the torus \([0,1]^d\). We emphasize that the anisotropic Sobolev spaces given in [1] are defined on the torus \(T^d = [0, 2\pi]^d\), not on \([0,1]^d\). It is easily seen that
\[
\|f\|_{W_2^b} = \left( \|f\|_2^2 + \sum_{j=1}^d \|D_j f\|_2^2 \right)^{1/2} = \left( \sum_{k \in \mathbb{Z}^d} \left( 1 + \sum_{j=1}^d |2\pi k_j|^{2b_j} \right) |\hat{f}(k)|^2 \right)^{1/2},
\] (2.4)
so if we write \(b = \{b_j\}\), \(\tilde{b}_j = (2\pi)^{2b_j}, \ j \in \mathbb{N}\), then \(W_2^b([0,1]^d) = W_2^{\tilde{b},b}([0,1]^d)\).

### 2.2 Analytic Korobov spaces

Let \(a = \{a_j\}_{j \geq 1}\) and \(b = \{b_j\}_{j \geq 1}\) be sequences satisfying (2.2). Fix \(\omega \in (0,1)\). We define the analytic Korobov kernel \(K_{d,a,2b}\) by
\[
K_{d,a,2b}(x,y) = \sum_{k \in \mathbb{Z}^d} \omega_k e^{2\pi i k(x-y)}, \quad \text{for all} \ x, y \in [0,1]^d,
\]
where
\[
\omega_k = \omega_j^{a_j |k_j|^{2b_j}}, \quad \text{for all} \ k \in \mathbb{Z}^d.
\]
Denote by \(H(K_{d,a,2b})\) the analytic Korobov space which is a reproducing kernel Hilbert space with the reproducing kernel \(K_{d,a,2b}\). The inner product of the space \(H(K_{d,a,2b})\) is given by
\[
(f,g)_{H(K_{d,a,2b})} = \sum_{k \in \mathbb{Z}^d} \hat{f}(k) \overline{\hat{g}(k)} \omega_k^{-1}, \quad f, g \in H(K_{d,a,2b}),
\]
where \(\hat{f}(k), \ \hat{g}(k), \ k \in \mathbb{Z}^d\) are the Fourier coefficients of the functions \(f\) and \(g\). The norm of a function \(f\) in \(H(K_{d,a,2b})\) is given by
\[
\|f\|_{H(K_{d,a,2b})} = \left( \sum_{k \in \mathbb{Z}^d} |\hat{f}(k)|^2 \omega_k^{-1} \right)^{1/2} < \infty.
\]
Obviously, \(\{e_k\}_{k \in \mathbb{Z}^d}\) is an orthonormal basis for \(H(K_{d,a,2b})\) with \(e_k(x) = e^{2\pi i kx} \omega_k^{-1/2}\).

### 2.3 Approximation numbers

Let \(H\) and \(G\) be two Hilbert spaces and let \(T\) be a compact linear operator from \(H\) to \(G\). The basic properties of the approximation numbers \(a_n(T : H \to G)\) are well known; see e.g., Pietsch [19, Chapter 11] and [20, Chapter 2], König [9, Section 1.1b], Pinkus [21, Theorem IV.2.2], and Novak and Woźniakowski [15, Corollary 4.12].

Let \(H\) be a separable Hilbert space, \(\{e_k\}_{k=1}^\infty\) an orthonormal basis in \(H\), and \(\tau = \{\tau_k\}_{k=1}^\infty\) a sequence of positive numbers with \(\tau_1 \geq \tau_2 \geq \cdots \geq \tau_k \geq \cdots > 0\). Let \(H^\tau\) be a Hilbert space defined by
\[
H^\tau = \left\{ x \in H : \|x\|_{H^\tau} = \left( \sum_{k=1}^\infty \frac{|(x,e_k)|^2}{\tau_k^2} \right)^{1/2} < \infty \right\}.
\]
According to [21, Corollary 2.6], we have the following lemma:
Lemma 2.1  Let $H$, $\tau$ and $H^\tau$ be defined as above. Then
\[ a_n(I_d : H^\tau \to H) = \tau_n, \quad n \in \mathbb{N}. \]

Let \( \{W^*_{a,b,d}(l)\}_{l=1}^\infty \) be the non-increasing rearrangement of
\[ \left\{ \left( 1 + \sum_{j=1}^d a_j|k_j|^{2b_j} \right)^{-\frac{1}{b_j}} \right\}_{k=(k_1,\ldots,k_d) \in \mathbb{Z}^d}, \]
and let \( \{\lambda_{d,k}\}_{k=1}^\infty \) be the non-increasing rearrangement of
\[ \{\omega_k\}_{k \in \mathbb{Z}^d} = \left\{ \sum_{j=1}^d a_j|k_j|^{2b_j} \right\}_{k=(k_1,\ldots,k_d) \in \mathbb{Z}^d}, \]
with fixed $\omega \in (0,1)$. According to Lemma 2.1, we obtain
\[ a_n(I_d : W^a_{2,b}([0,1]^d) \to L_2([0,1]^d)) = W^*_{a,b,d}(n), \] \hfill (2.5)
and
\[ a_n(\text{APP}_d : H(K_{d,a,2b}) \to L_2([0,1]^d)) = \lambda_{d,n}^{1/2}. \] \hfill (2.6)

2.4 General notations of tractability

Let $H_d$ and $G_d$ be two sequences of Hilbert spaces and for each $d \in \mathbb{N}$, and $F_d$ be the unit ball of $H_d$. Assume a sequence of bounded linear operators (solution operators)
\[ S_d : H_d \to G_d \]
for all $d \in \mathbb{N}$. For $n \in \mathbb{N}$ and $f \in F_d$, $S_df$ can be approximated by algorithms
\[ A_{n,d}(f) = \phi_{n,d}(L_1(f), L_2(f), \ldots, L_n(f)), \]
where $L_j$, $j = 1,2,\cdots,n$ are continuous linear functionals on $H_d$ which are referred to as general information, and $\phi_{n,d} : \mathbb{R}^n \to G_d$ is an arbitrary mapping. The worst case error $e(A_{n,d})$ of the algorithm $A_{n,d}$ is defined as
\[ e(A_{n,d}) = \sup_{f \in F_d} \|S_d(f) - A_{n,d}(f)\|_{G_d}. \]
Furthermore, the n-th minimal worst-case error is defined as
\[ e(n,S_d) = \inf_{A_{n,d}} e(A_{n,d}), \]
where the infimum is taken over all algorithms using $n$ information operators $L_1, L_2, \cdots, L_n$.
For $n = 0$, we use $A_{0,d} = 0$. The error of $A_{0,d}$ is called the initial error and is given by
\[ e(0,S_d) = e(A_{0,d}) = \sup_{f \in F_d} \|S_d(f)\|_{G_d}. \]

From [15, p.118], we know that the n-th minimal worst-case error $e(n,S_d)$ with respect to arbitrary algorithms and general information in the Hilbert setting is just the $(n+1)$-st approximation number $a_{n+1}(S_d : H_d \to G_d)$, i.e.,
\[ e(n,S_d) = a_{n+1}(S_d : H_d \to G_d). \]

In this article, we consider the embedding operators $S_d = I_d$ and $S_d = \text{APP}_d$ which are defined by (1.1) and (1.2). We note that
\[ e(0, I_d) = \|I_d\| = 1 \quad \text{and} \quad e(0, \text{APP}_d) = \|\text{APP}_d\| = 1. \]
In both cases, \( e(0, S_d) = 1 \). In other words, the normalized error criterion and the absolute error criterion coincide for the approximation problems \( I = \{I_d\} \) and \( \text{APP} = \{\text{APP}_d\} \).

For \( \varepsilon \in (0, 1) \) and \( d \in \mathbb{N} \), let \( n(\varepsilon, S_d) \) be the information complexity defined by

\[
n(\varepsilon, S_d) = \min\{ n : e(n, S_d) \leq \varepsilon \},
\]

(2.7)

where

\[
e(n, I_d) = a_{n+1}(I_d : W_2^{a,b}([0, 1]^d) \rightarrow L_2([0, 1]^d)),
\]

\[
e(n, \text{APP}_d) = a_{n+1}(\text{APP}_d : H(K_{d,a,2b}) \rightarrow L_2([0, 1]^d)).
\]

Now, we list different notions of tractability. We say that the approximation problem \( S = \{S_d\}_{d \in \mathbb{N}} \) is

- strongly polynomially tractable (SPT) if there exist non-negative numbers \( C \) and \( p \) such that, for all \( d \in \mathbb{N}, \varepsilon \in (0, 1) \),

\[
n(\varepsilon, S_d) \leq C(\varepsilon^{-1})^p;
\]

- polynomially tractable (PT) if there exist non-negative numbers \( C, p \) and \( q \) such that, for all \( d \in \mathbb{N}, \varepsilon \in (0, 1) \),

\[
n(\varepsilon, S_d) \leq Cd^q(\varepsilon^{-1})^p;
\]

- quasi-polynomially tractable (QPT) if there exist two constants \( C, t \geq 0 \) such that, for all \( d \in \mathbb{N}, \varepsilon \in (0, 1) \),

\[
n(\varepsilon, S_d) \leq C \exp[t(1 + \ln \varepsilon^{-1})(1 + \ln d)];
\]

- uniformly weakly tractable (UWT) if, for all \( \alpha, \beta > 0 \),

\[
\lim_{\varepsilon^{-1} + d \rightarrow \infty} \frac{\ln n(\varepsilon, S_d)}{(\varepsilon^{-1})^\alpha + d^\beta} = 0;
\]

- weakly tractable (WT) if

\[
\lim_{\varepsilon^{-1} + d \rightarrow \infty} \frac{\ln n(\varepsilon, S_d)}{\varepsilon^{-1} + d} = 0;
\]

- \((s, t)\)-weakly tractable ((s, t)-WT) for fixed positive numbers \( s \) and \( t \) if

\[
\lim_{\varepsilon^{-1} + d \rightarrow \infty} \frac{\ln n(\varepsilon, S_d)}{(\varepsilon^{-1})^s + d^t} = 0.
\]

In the above definitions of SPT, PT, QPT, UWT, WT, and \((s, t)\)-WT, if we replace \( \frac{1}{\varepsilon} \) by \( (1 + \ln \frac{1}{\varepsilon}) \), we get the definitions of exponential convergence-strong polynomial tractability (EC-SPT), exponential convergence-polynomial tractability (EC-PT), exponential convergence-quasi-polynomial tractability (EC-QPT), exponential convergence-uniform weak tractability (EC-UWT), exponential convergence-weak tractability (EC-WT), and exponential convergence-\((s, t)\)-weak tractability (EC-(s, t)-WT), respectively. We now give the above notions of EC-tractability in detail.

We say that \( S = \{S_d\}_{d \in \mathbb{N}} \) is

- Exponential convergence-strong polynomially tractable (EC-SPT) if and only if there exist non-negative numbers \( C \) and \( p \) such that, for all \( d \in \mathbb{N}, \varepsilon \in (0, 1) \),

\[
n(\varepsilon, S_d) \leq C(1 + \ln \varepsilon^{-1})^p;
\]

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We shall show that the volume of the generalized ellipsoid $B_a$ is known (see [24]). The authors of [1] investigated, among other things, the strong equivalence of the approximation numbers $a_n$ in these spaces. We use the volume argument to obtain the asymptotic behavior of $a_n(I_d : W^{n,b}_2([0,1]^d) \to L_2([0,1]^d))$. Our result can be formulated as follows:

2.5 Main results

Let $a = \{a_k\}_{k \geq 1}$ and $b = \{b_k\}_{k \geq 1}$ be two sequences of positive numbers. For $t > 0$ and $d \in \mathbb{N}$, denote by

$$B_{a,b}^d(t) = \{x \in \mathbb{R}^d : \sum_{j=1}^d a_j x_j^b \leq t\}$$

the generalized ellipsoid in $\mathbb{R}^d$. We write $B_{a,b}^d$ instead of $B_{a,b}^d(1)$ for brevity. Clearly, when $a_1 = a_2 = \cdots = a_d = 1$, $B_{a,b}^d$ recedes to the generalized unit ball

$$B_b^d = \{x \in \mathbb{R}^d : \sum_{j=1}^d x_j^b \leq 1\}.$$ 

We shall show that the volume of the generalized ellipsoid $B_{a,b}^d$ is

$$\text{vol}(B_{a,b}^d) = 2^d a_1^{-\frac{1}{b_1}} a_2^{-\frac{1}{b_2}} \cdots a_d^{-\frac{1}{b_d}} \frac{\Gamma(1 + \frac{1}{b_1}) \Gamma(1 + \frac{1}{b_2}) \cdots \Gamma(1 + \frac{1}{b_d})}{\Gamma(1 + \frac{1}{b_1} + \frac{1}{b_2} + \cdots + \frac{1}{b_d})} \text{vol}(B_b^d),$$

where $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, dt$ is the Gamma function. The volume of $B_b^d$ is known (see [24]).

The authors of [1] investigated, among other things, the strong equivalence of the approximation numbers $a_n$ in these spaces. We use the volume argument to obtain the asymptotic behavior of $a_n(I_d : W^{n,b}_2([0,1]^d) \to L_2([0,1]^d))$. Our result can be formulated as follows:

$$\lim_{n \to \infty} n^{g(R)} a_n(I_d : W^{n,b}_2([0,1]^d) \to L_2([0,1]^d)) = (\text{vol}(B_{2,\infty}^d))^{g(R)},$$

where $g(R) = \frac{1}{\frac{1}{b_1} + \frac{1}{b_2} + \cdots + \frac{1}{b_d}}$. In this article, we generalize the above result to the weighted anisotropic spaces $W^{n,b}_2([0,1]^d)$. We use the volume argument to obtain the asymptotic behavior of $a_n(I_d : W^{n,b}_2([0,1]^d) \to L_2([0,1]^d))$. Our result can be formulated as follows.
Theorem 2.2 Let \( a = \{a_j\}_{j \geq 1} \) and \( b = \{b_j\}_{j \geq 1} \) be two sequences of positive numbers. Then we have

\[
\lim_{n \to \infty} n^{g_d(b)} a_n(I_d : W_2^n b([0, 1]^d) \to L_2([0, 1]^d)) = (\text{vol}(B_{\| \mathbf{a}, 2\|b})^{g_d(b)},
\]  

(2.9)

where \( g_d(b) = \frac{1}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_d}} \).

In particular, we have

Corollary 2.3 Let \( b = \{b_j\}_{j \geq 1} \) be a sequence of positive numbers. Then

\[
\lim_{n \to \infty} n^{g_d(b)} a_n(I_d : W_2^n b([0, 1]^d) \to L_2([0, 1]^d)) = ((2\pi)^{-\frac{2d}{2}} \text{vol}(B_{2b}^d))^{g_d(b)}.
\]  

(2.10)

Remark 2.4 Let \( a = \{a_j\}_{j \geq 1} \) and \( b = \{b_j\}_{j \geq 1} \) be two sequences of positive numbers, and \( g_d(b) = \frac{1}{\frac{1}{b_1} + \cdots + \frac{1}{b_n}} \). Theorem 2.2 indicates that the exact decay rate in \( n \) of the approximation numbers \( a_n(I_d : W_2^n b([0, 1]^d) \to L_2([0, 1]^d)) \) is \( n^{-g_d(b)} \), which is independent of \( a \), and the sharp constant is \( (\text{vol}(B_{\| \mathbf{a}, 2\|b})^{g_d(b)}) \). We can rephrase (2.9) as strong equivalences

\[
a_n(I_d : W_2^n b([0, 1]^d) \to L_2([0, 1]^d)) \sim n^{-g_d(b)} (\text{vol}(B_{\| \mathbf{a}, 2\|b})^{g_d(b)}).
\]

The novelty of Theorem 2.2 is that it gives strong equivalences and provides asymptotically optimal (sharp) constants for arbitrary fixed \( d, a, \) and \( b \).

Remark 2.5 Comparing (2.8) with (2.10), we get that the sharp constants of the approximation numbers of anisotropic Sobolev embeddings on the torus \([a, b]^d\) depend on the volume of the torus. We can show that

\[
\lim_{n \to \infty} n^{g_d(b)} a_n(I_d : W_2^n b([a, b]^d) \to L_2([a, b]^d)) = \left( \frac{b - a}{2\pi} \text{vol}(B_{b}^d) \right)^{g_d(b)}.
\]

Remark 2.6 According to [3, Theorem 3.1], we know that the condition \( g_d(b) > \frac{1}{2} \) is a sufficient and necessary condition for the embedding from \( W_2^n b([0, 1]^d) \) into \( L_\infty([0, 1]^d) \) or \( C([0, 1]^d) \). Indeed, it follows from [3, Theorem 3.1] that a sufficient and necessary condition for the embedding from \( W_2^n b([0, 1]^d) \) into \( L_\infty([0, 1]^d) \) or \( C([0, 1]^d) \) is that

\[
\sum_{k \in \mathbb{Z}^d} \left( 1 + \sum_{j=1}^d a_j |k_j|^{2b_j} \right)^{-1} = \sum_{n=1}^\infty \left( a_n(I_d : W_2^n b([a, b]^d) \to L_2([a, b]^d)) \right)^2 < \infty,
\]

which is equivalent to

\[
g_d(b) > \frac{1}{2}.
\]

Furthermore, for \( g_d(b) > \frac{1}{2} \), using the proof technique of [3, Theorem 4.3], we can show that

\[
\lim_{n \to \infty} n^{g_d(b)-\frac{1}{2}} a_n(I_d : W_2^n b([0, 1]^d) \to L_\infty([0, 1]^d)) = (2g_d(b) - 1)^{-\frac{1}{2}} (\text{vol}(B_{\| \mathbf{a}, 2\|b})^{g_d(b)}).
\]

Note that the above equality also holds if we replace \( L_\infty([0, 1]^d) \) by \( C([0, 1]^d) \).

Next, we establish a relationship between the information complexities \( n(\varepsilon, I_d) \) and \( n(\varepsilon, \text{APP}_d) \). Such a relationship is crucial for obtaining sufficient and necessary conditions for various notions of tractability of the approximation problem \( I = \{I_d\} \).

Theorem 2.7 For \( \varepsilon \in (0, 1), \ n \in \mathbb{N} \), we have that

\[
n(\varepsilon, \text{APP}_d) = n\left( \frac{\ln \varepsilon^{-2}}{\ln \omega^{-1}} + 1 \right)^{-\frac{1}{2}}, I_d \right)
\]

(2.11)
and

\[ n(\varepsilon, I_d) = n(\omega^{-2} - 1, \text{APP}_d), \]

(2.12)

where \( n(\varepsilon, S_d) \) is given in (2.7), \( S_d = I_d \) or \( \text{APP}_d \).

We know that the classical tractability notions of the multivariate problem APP were solved completely in [6, 8, 13]. For the EC-tractability of APP, the sufficient and necessary conditions for EC-SPT, EC-PT, EC-QPT, EC-UWT, EC-WT and EC-(s,t)-WT with max(s,t) \( > 1 \) were given in [6], and for EC-(s,t)-WT with max(s,t) \( \leq 1 \) and min(s,t) \( < 1 \) in [23]. On the basis of Theorem 2.3 we obtain the following tractability results of the approximation problem \( I = \{ I_d \} \):

**Theorem 2.8** Consider the approximation problem \( I = \{ I_d \} \) in the worst case setting with the sequences \( a \) and \( b \) satisfying (2.2). Then

(i) SPT holds if and only if PT holds if and only if \( \sum_{j=1}^{\infty} b_j^{-1} < \infty \) and \( \lim_{j \to \infty} \frac{\ln a_j}{j} > 0 \);

(ii) QPT holds if and only if \( \sup_{d \in \mathbb{N}} \frac{\sum_{j=1}^{d} b_j^{-1}}{1+\ln d} < \infty \) and \( \lim_{j \to \infty} \frac{(1+\ln j) \ln a_j}{j} > 0 \);

(iii) UWT holds if and only if \( \lim_{j \to \infty} \frac{\ln a_j}{\ln j} = \infty \);

(iv) WT holds if and only if \( \lim_{j \to \infty} a_j/j = \infty \);

(v) \( (s,t) \)-WT with \( \max(s/2, t) > 1 \) always holds;

(vi) \( (2,1) \)-WT holds if and only if \( \lim_{j \to \infty} a_j/j = \infty \);

(vii) \( (2,t) \)-WT with \( t < 1 \) holds if and only if \( \lim_{j \to \infty} a_j/j = \infty \);

(viii) \( (s,t) \)-WT with \( s < 2 \) and \( t \leq 1 \) holds if and only if \( \lim_{j \to \infty} a_j/j^{|s-2|/t} = \infty \).

In particular, let \( b = \{ b_j \} \) satisfy

\[ 0 < b_1 \leq b_2 \leq \cdots \leq b_j \leq \cdots, \]

(2.13)

and let \( \tilde{b} = \{ \tilde{b}_j \}, \tilde{b}_j = (2\pi)^{2b_j}, j \in \mathbb{N} \). Then \( W_2^b([0,1]^d) = W_2^{\tilde{b}, b}([0,1]^d) \). By Theorem 2.4, we have the following result:

**Theorem 2.9** Consider the approximation problem \( \tilde{I} = \{ \tilde{I}_d \}_{d \in \mathbb{N}} \) in the worst case setting with the sequence \( b \) satisfying (2.13), where

\( \tilde{I}_d : W_2^b([0,1]^d) \to L_2([0,1]^d) \) with \( \tilde{I}_d(f) = f \).

Then we have that

(i) \( \tilde{I} \) is SPT if and only if \( \tilde{I} \) is PT if and only if \( \sum_{j=1}^{\infty} b_j^{-1} < \infty \);

(ii) \( \tilde{I} \) is QPT if and only if \( \sup_{d \in \mathbb{N}} \frac{\sum_{j=1}^{d} b_j^{-1}}{1+\ln d} < \infty \);

(iii) UWT holds if and only if \( \lim_{j \to \infty} b_j/j = \infty \);

(iv) WT holds if and only if \( \lim_{j \to \infty} \frac{(2\pi)^{2b_j}}{j} = \infty \);

(v) \( (s,t) \)-WT with \( \max(s/2, t) > 1 \) always holds;

(vi) \( (2,1) \)-WT holds if and only if \( \lim_{j \to \infty} b_j = \infty \);

(vii) \( (2,t) \)-WT with \( t < 1 \) holds if and only if \( \lim_{j \to \infty} \frac{(2\pi)^{2b_j}}{\ln j} = \infty \);

(viii) \( (s,t) \)-WT with \( s < 2 \) and \( t \leq 1 \) holds if and only if \( \lim_{j \to \infty} \frac{(2\pi)^{2b_j}}{j^{s-2}|t|} = \infty \).
3 Strong Equivalences of Approximation Numbers

This section is devoted to studying the strong equivalence of the approximation numbers $a_n(I_d: W^{2,b}_2([0,1]^d) \to L_2([0,1]^d))$, where $a = \{a_j\}_{j \geq 1}$ and $b = \{b_j\}_{j \geq 1}$ are two sequences of positive numbers. In this section we do not need to assume that $a, b$ satisfy (2.2).

We know from [24] that the volume of the generalized unit ball $B^d_b$ is

$$\text{vol}(B^d_b) = \text{vol}\left\{ x \in \mathbb{R}^d : \sum_{j=1}^d |x_j|^{b_j} \leq 1 \right\} = 2^d \frac{\Gamma(1 + \frac{1}{b_1}) \Gamma(1 + \frac{1}{b_2}) \cdots \Gamma(1 + \frac{1}{b_d})}{\Gamma(1 + \frac{1}{b_1} + \frac{1}{b_2} + \cdots + \frac{1}{b_d})}.$$  

**Lemma 3.1** Let $a = \{a_j\}_{j \geq 1}$ and $b = \{b_j\}_{j \geq 1}$ be two sequences of positive numbers. Then

$$\text{vol}(B^d_{a,b}) = a_1^{-\frac{1}{a_1}} a_2^{-\frac{1}{a_2}} \cdots a_d^{-\frac{1}{a_d}} \text{vol}(B^d_b). \quad (3.1)$$

**Proof** We make a change of variables

$$y_1 = x_1 a_1^{-\frac{1}{a_1}}, \quad y_2 = x_2 a_2^{-\frac{1}{a_2}}, \quad \ldots, \quad y_d = x_d a_d^{-\frac{1}{a_d}}$$

that deforms $B^d_{a,b}$ into $B^d_b$. The Jacobian determinant is $J(y) = a_1^{-\frac{1}{a_1}} a_2^{-\frac{1}{a_2}} \cdots a_d^{-\frac{1}{a_d}}$. By the change of variables formula, we obtain that

$$\text{vol}(B^d_{a,b}) = \int_{B^d_{a,b}} 1 \, dx = \int_{B^d_b} J(y) \, dy = a_1^{-\frac{1}{a_1}} a_2^{-\frac{1}{a_2}} \cdots a_d^{-\frac{1}{a_d}} \text{vol}(B^d_b).$$

The proof of Lemma 3.1 is finished. \qed

**Lemma 3.2** Let $a = \{a_j\}_{j \geq 1}$ and $b = \{b_j\}_{j \geq 1}$ be two sequences of positive numbers. Then

$$\text{vol}(B^d_{a,b}(t)) = t^\frac{1}{a_1} t^\frac{1}{a_2} \cdots t^\frac{1}{a_d} \text{vol}(B^d_{a,b}) = t^\frac{1}{g_d(b)} \text{vol}(B^d_{a,b}). \quad (3.2)$$

where $g_d(b) = \frac{1}{a_1 + \frac{1}{a_2} + \cdots + \frac{1}{a_d}}$.

**Proof** We make a change of variables

$$y_1 = x_1 t^{-\frac{1}{a_1}}, \quad y_2 = x_2 t^{-\frac{1}{a_2}}, \quad \ldots, \quad y_d = x_d t^{-\frac{1}{a_d}}$$

that deforms $B^d_{a,b}(t)$ into $B^d_{a,b}$. The Jacobian determinant is $J(y) = t^\frac{1}{a_1} t^\frac{1}{a_2} \cdots t^\frac{1}{a_d}$. By the change of variables formula, we obtain that

$$\text{vol}(B^d_{a,b}(t)) = \int_{B^d_{a,b}(t)} 1 \, dx = \int_{B^d_{a,b}} J(y) \, dy = t^\frac{1}{a_1} t^\frac{1}{a_2} \cdots t^\frac{1}{a_d} \text{vol}(B^d_{a,b}).$$

The proof of Lemma 3.2 is finished. \qed

**Lemma 3.3** Let $a = \{a_j\}_{j \geq 1}$ and $b = \{b_j\}_{j \geq 1}$ be two sequences of positive numbers, and let $p_d = \max\{1, 2b_1, 2b_2, \ldots, 2b_d\}$. Then for any $x, y \in \mathbb{R}^d$, we have that

$$\left( \sum_{j=1}^d a_j |x_j + y_j|^{2b_j} \right)^{\frac{1}{p_d}} \leq \left( \sum_{j=1}^d a_j |x_j|^{2b_j} \right)^{\frac{1}{p_d}} + \left( \sum_{j=1}^d a_j |y_j|^{2b_j} \right)^{\frac{1}{p_d}}. \quad (3.3)$$

**Proof** It follows from [1, Lemma 3.2] that for $x = (x_1, \ldots, x_d)$, $y = (y_1, \ldots, y_d)$, $x, y \in \mathbb{R}^d$, we have that

$$\left( \sum_{j=1}^d |x_j + y_j|^{2b_j} \right)^{\frac{1}{p_d}} \leq \left( \sum_{j=1}^d |x_j|^{2b_j} \right)^{\frac{1}{p_d}} + \left( \sum_{j=1}^d |y_j|^{2b_j} \right)^{\frac{1}{p_d}}.$$
If we replace \( x_j, y_j, j = 1, 2, \cdots, d \) by \( \frac{1}{p_j} x_j, \frac{1}{p_j} y_j, j = 1, 2, \cdots, d \) in the above inequality, then we obtain that

\[
\left( \frac{1}{p_d} \sum_{j=1}^{d} \left| a_j \frac{1}{p_j} x_j + a_j \frac{1}{p_j} y_j \right|^{2b_j} \right)^{1/p_d} \leq \left( \frac{1}{p_d} \sum_{j=1}^{d} \left| a_j \frac{1}{p_j} x_j \right|^{2b_j} \right)^{1/p_d} + \left( \frac{1}{p_d} \sum_{j=1}^{d} \left| a_j \frac{1}{p_j} y_j \right|^{2b_j} \right)^{1/p_d},
\]

which gives (3.3). Lemma 3.3 is proved.

**Proof of Theorem 2.2** It follows from (2.5) that

\[
a_n(I_d : W^a_{2, b}([0, 1]^d) \rightarrow L_2([0, 1]^d)) = W^a_{2, b}(n),
\]

where \( \{W^a_{2, b}(l)\}_{l=1}^{\infty} \) is the non-increasing rearrangement of

\[
\left\{ \left( 1 + \sum_{j=1}^{d} |a_j|^{2b_j} \right)^{-\frac{1}{2}} \right\}_{k=(k_1, \cdots, k_d) \in \mathbb{Z}^d}.
\]

For \( m \in \mathbb{N} \), let \( C(m, a, b, d) \) denote the cardinality of the set

\[
\left\{ \mathbf{k} : \sum_{j=1}^{d} a_j |k_j|^{2b_j} \leq m^{p_d}, \quad \mathbf{k} \in \mathbb{Z}^d \right\},
\]

where \( p_d = \max\{1, 2b_1, 2b_2, \cdots, 2b_d\} \). It follows that for \( n > C(m, a, b, d) \),

\[
a_n(I_d : W^a_{2, b}([0, 1]^d) \rightarrow L_2([0, 1]^d)) \leq (1 + m^{p_d})^{-\frac{1}{2}},
\]

and for \( n \leq C(m, a, b, d) \), that

\[
a_n(I_d : W^a_{2, b}([0, 1]^d) \rightarrow L_2([0, 1]^d)) \geq (1 + m^{p_d})^{-\frac{1}{2}}.
\]

For any \( m \in \mathbb{N} \), let \( Q_k \) be a cube with center \( \mathbf{k} \), sides parallel to the axes and side-length 1. For

\[
\mathbf{x} \in \bigcup_{\mathbf{k} \in \mathbb{Z}^d} Q_k \quad \text{subject to} \quad \sum_{j=1}^{d} a_j |k_j|^{2b_j} \leq m^{p_d}
\]

there exists a \( \mathbf{k} \in \mathbb{Z}^d \) such that \( \sum_{j=1}^{d} a_j |k_j|^{2b_j} \leq m^{p_d} \) and \( \mathbf{x} \in Q_k \). It follows from the definition of \( Q_k \) that

\[
|x_j - k_j| \leq \frac{1}{2}, \quad j = 1, 2, \cdots, d.
\]

By (3.3) we have

\[
\left( \sum_{j=1}^{d} a_j |x_j|^{2b_j} \right)^{1/p_d} \leq \left( \sum_{j=1}^{d} a_j |x_j - k_j|^{2b_j} \right)^{1/p_d} + \left( \sum_{j=1}^{d} a_j |k_j|^{2b_j} \right)^{1/p_d} \leq \left( \sum_{j=1}^{d} a_j 2^{-2b_j} \right)^{1/p_d} + m.
\]

It follows that

\[
\bigcup_{\mathbf{k} \in \mathbb{Z}^d} Q_k \subset B_{a, 2b} \left( m + \left( \sum_{j=1}^{d} a_j 2^{-2b_j} \right)^{1/p_d} \right), \quad (3.4)
\]

\( \Box \) Springer
Using the same technique, we obtain that

\[ B^d_{a,2b}\left((m - \left(\sum_{j=1}^{d} a_j 2^{-2b_j}\right) \frac{\rho_d}{m}\right)^{\frac{\rho_d}{m}} \right) \subseteq \bigcup_{k \in \mathbb{Z}^d} Q_k, \tag{3.5} \]

where \( a_* \) is equal to \( a \) if \( a \geq 0 \) and \( 0 \) if \( a < 0 \). We note that the volume of the set

\[ \text{vol}\left(\bigcup_{k \in \mathbb{Z}^d} Q_k\right) = \#\{k : \sum_{j=1}^{d} a_j |k_j|^{2b_j} \leq m^{p_d}, \ k \in \mathbb{Z}^d\} \]

is just \( C(m, a, b, d) \), where \( \#A \) denotes the number of elements in a set \( A \). We set

\[ C_{a,b,d} = \left(\sum_{j=1}^{d} a_j 2^{-2b_j}\right)^{\frac{\rho_d}{m}}. \]

By (3.5), (3.4) and (3.2) we obtain that

\[ (m - C_{a,b,d})^{\frac{\rho_d}{m}} \text{vol}(B^d_{a,2b}) \leq C(m, a, b, d) \leq (m + C_{a,b,d})^{\frac{\rho_d}{m}}, \tag{3.6} \]

where \( g_d(2b) = 2g_d(b) = \frac{2}{\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_d} \). We write

\[ a_n(I_d) = a_n(I_d : W^a_{2,1}([0,1]^d) \to L_2([0,1]^d)). \]

We also write

\[ A(m, a, b, d) = (m + C_{a,b,d})^{\frac{\rho_d}{m}} \text{vol}(B^d_{a,2b}), \]

and

\[ B(m, a, b, d) = (m - C_{a,b,d})^{\frac{\rho_d}{m}} \text{vol}(B^d_{a,2b}). \]

On the one hand, we suppose that \( A(m, a, b, d) < n \leq A(m+1, a, b, d) \) for sufficiently large \( n \). By (3.6) we obtain that

\[ n > C(m, a, b, d), \]

which implies that

\[ a_n(I_d) \leq (1 + m^{p_d})^{-\frac{1}{2}}. \]

It follows that

\[ n^{g_d(b)} a_n(I_d) \leq \frac{(A(m + 1, a, b, d))^{g_d(b)}}{(1 + m^{p_d})^{2}} \leq \frac{(m + 1 + C_{a,b,d})^{\frac{\rho_d}{m}} \text{vol}(B^d_{a,2b})^{g_d(b)}}{(1 + m^{p_d})^{2}}. \]

Obviously, we have

\[ \lim_{m \to \infty} \frac{(m + 1 + C_{a,b,d})^{\frac{\rho_d}{m}}}{(1 + m^{p_d})^{2}} = 1, \]

which implies that

\[ \lim_{n \to \infty} n^{g_d(b)} a_n(I_d) \leq \left(\text{vol}(B^d_{a,2b})\right)^{g_d(b)}. \tag{3.7} \]

On the other hand, we suppose that \( B(m, a, b, d) \leq n < B(m+1, a, b, d) \) for sufficiently large \( n \). By (3.6) we obtain that

\[ n < C(m + 1, a, b, d), \]

\[ \Box \]
which implies that
\[ a_n(I_d) \geq (1 + (m + 1)^{p_d})^{-\frac{1}{2}}. \]

It follows that
\[ n^{g_d(b)}a_n(I_d) \geq \frac{(B(m, a, b, d))^{g_d(b)}}{(1 + (m + 1)^{p_d})^{\frac{1}{2}}} \geq \frac{(m - C_{a, b, d})^{\frac{1}{2}}}{(1 + (m + 1)^{p_d})^{\frac{1}{2}}} (\text{vol}(B_{a, 2b}^d))^{g_d(b)}. \]

We have
\[ \lim_{m \to \infty} \frac{(m - C_{a, b, d})^{\frac{1}{2}}}{(1 + (m + 1)^{p_d})^{\frac{1}{2}}} = 1, \]
which implies that
\[ \lim_{n \to \infty} n^{g_d(b)}a_n(I_d) \geq (\text{vol}(B_{a, 2b}^d))^{g_d(b)}. \]

Combining (3.7) with (3.8), we obtain (2.9).

Theorem 2.2 is proved.

\[ \square \]

### 4 Tractability Results of Weighted Anisotropic Embeddings

In this section, we first give the proof of Theorem 2.3. Next we establish the relationship between the tractability of \( I = \{I_d\}_{d \in \mathbb{N}} \) and the EC-tractability of \( \text{APP} = \{\text{APP}_d\}_{d \in \mathbb{N}} \), based on Theorem 2.3, where \( I_d \) and \( \text{APP}_d \) are given by (1.1) and (1.2), respectively. Finally, we show Theorems 2.4 and 2.5.

**Proof of Theorem 2.3** For any \( \varepsilon \in (0, 1), \ d \in \mathbb{N} \), by (2.7), (2.5) and (2.6) we have that
\[ n(\varepsilon, I_d) = \min \{ n : a_{n+1}(I_d : W^a_{2} \to L_2([0,1]^d)) \leq \varepsilon \} = \min \{ n : W^a_{a,b,d}(n + 1) \leq \varepsilon \}, \]
and
\[ n(\varepsilon, \text{APP}_d) = \min \{ n : a_{n+1}(\text{APP}_d : H(K_{a,\text{APP}_d}^d) \to L_2([0,1]^d)) \leq \varepsilon \} = \min \{ n : \lambda_{d,n+1} \leq \varepsilon^2 \}, \]
where \( \{W^a_{a,b,d}(l)\}_{l=1}^\infty \) is the non-increasing rearrangement of
\[ \left\{ \left( 1 + \sum_{j=1}^{d} a_j |k_j|^{2b_j} \right)^{-\frac{1}{2}} \right\}_{k=(k_1, \ldots, k_d) \in \mathbb{Z}^d}, \]
and \( \{\lambda_{d,k}\}_{k=1}^\infty \) is the non-increasing rearrangement of
\[ \{\omega_k\}_{k \in \mathbb{Z}^d} = \left\{ \omega_{\sum_{j=1}^{d} a_j |k_j|^{2b_j}} \right\}_{k=(k_1, \ldots, k_d) \in \mathbb{Z}^d} \]
with fixed \( \omega \in (0, 1). \)

For any \( \varepsilon_1 \in (0, 1) \), by (4.2) we have that
\[ n(\varepsilon_1, \text{APP}_d) = \min \{ n : \lambda_{d,n+1} \leq \varepsilon_1^2 \} = \# \{ j \in \mathbb{N} : \lambda_{d,j} \geq \varepsilon_1^2 \} \]
\[ = \# \left\{ k \in \mathbb{Z}^d : \omega_{\sum_{j=1}^{d} a_j |k_j|^{2b_j}} > \varepsilon_1^2 \right\}, \]
\[
\begin{align*}
&= \# \{ k \in \mathbb{Z}^d : \omega \sum_{j=1}^d a_j |k_j|^{2b_j} < \varepsilon_1^{-2} \} \\
&= \# \{ k \in \mathbb{Z}^d : \sum_{j=1}^d a_j |k_j|^{2b_j} < \varepsilon_1^{-2} / \ln \omega^{-1} \}.
\end{align*}
\]

We set \( \varepsilon_2 = \left( \frac{\ln \varepsilon_1^{-2}}{\ln \omega^{-1}} + 1 \right)^{\frac{1}{2}} \). Then \( \varepsilon_2 \in (0, 1) \), and
\[
\varepsilon_2^{-2} - 1 = \frac{\ln \varepsilon_1^{-2}}{\ln \omega^{-1}}.
\]

By (4.1) we obtain that
\[
n(\varepsilon_2, I_d) = \min \{ n : W_{a,b,d}^*(n+1) \leq \varepsilon_2 \} = \# \{ l \in \mathbb{N} : W_{a,b,d}^*(l) > \varepsilon_2 \}
\]
\[
= \# \{ k \in \mathbb{Z}^d : \left( 1 + \sum_{j=1}^d a_j |k_j|^{2b_j} \right)^{-\frac{1}{2}} > \varepsilon_2 \}
\]
\[
= \# \{ k \in \mathbb{Z}^d : \sum_{j=1}^d a_j |k_j|^{2b_j} < \varepsilon_2^{-2} - 1 \}
\]
\[
= \# \{ k \in \mathbb{Z}^d : \sum_{j=1}^d a_j |k_j|^{2b_j} < \frac{\ln \varepsilon_1^{-2}}{\ln \omega^{-1}} \}.
\]

It follows that
\[
n(\varepsilon_1, \text{APP}_d) = n(\varepsilon_2, I_d) = n\left( \left( \frac{\ln \varepsilon_1^{-2}}{\ln \omega^{-1}} + 1 \right)^{\frac{1}{2}}, I_d \right),
\]
which gives (2.11). Using the same method, we can prove (2.12).

Theorem 2.3 is proved. \( \square \)

Theorem 4.1 Consider the approximation problems \( I = \{ I_d \} \) and \( \text{APP} = \{ \text{APP}_d \} \) in the worst case setting, where \( I_d \) and \( \text{APP}_d \) are given by (1.1) and (1.2). We have that

(i) for fixed \( s, t > 0 \), \( I \) is \((2s, t)\)-WT if and only if \( \text{APP} \) is EC-(\((s, t)\))-WT;

(ii) \( I \) is UWT if and only if \( \text{APP} \) is EC-UWT;

(iii) \( I \) is QPT if and only if \( \text{APP} \) is EC-QPT;

(iv) \( I \) is PT if and only if \( \text{APP} \) is EC-PT;

(v) \( I \) is SPT if and only if \( \text{APP} \) is EC-SPT.

Proof (i) First we prove the sufficiency. Assume that \( I \) is \((2s, t)\)-WT. For \( \varepsilon \in (0, 1) \), we set
\[
\varepsilon_1 = \left( \frac{\ln \varepsilon_1^{-2}}{\ln \omega^{-1}} + 1 \right)^{\frac{1}{2}} \in (0, 1).
\]

Then we have that
\[
d + \varepsilon_1^{-1} \to \infty \quad \text{if and only if} \quad d + \varepsilon_1^{-1} \to \infty.
\]

By the definition of \((s, t)\)-WT, we have that
\[
\lim_{d + \varepsilon_1^{-1} \to \infty} \frac{\ln n(\varepsilon_1, I_d)}{d^s + (\varepsilon_1^{-1})^2s} = 0.
\]

It follows from (2.11) that
\[
\frac{\ln n(\varepsilon, \text{APP}_d)}{d^s + (1 + \varepsilon)^{-1}s} = \frac{\ln n(\varepsilon_1, I_d)}{d^s + \varepsilon_1^{-2s}} \frac{d^s + (\ln \varepsilon_1^{-2})^s}{d^s + (1 + \ln \varepsilon_1^{-1})^s}.
\]

\( \square \) Springer
Noting that
\[ \varepsilon_1^{-2} = \frac{\ln \varepsilon^{-2}}{\ln \omega^{-1}} + 1 \leq C_1 (1 + \ln \varepsilon^{-1}), \] (4.4)
where \( C_1 = \max \{ 1, \frac{2}{\ln \omega^{-1}} \} \), we obtain that
\[ \frac{d^t + (\ln \varepsilon^{-2} + 1)^s}{d^t + (1 + \ln \varepsilon^{-1})^s} \leq \frac{d^t + C_1 (1 + \ln \varepsilon^{-1})^s}{d^t + (1 + \ln \varepsilon^{-1})^s} \leq C_1. \]
It follows that
\[ \lim_{d+\varepsilon^{-1} \to \infty} \frac{\ln n(\varepsilon, \text{APP}_d)}{d^t + (1 + \ln \varepsilon^{-1})^s} = 0, \]
which yields that APP is EC-(s, t)-WT.

Next we show the necessity. Suppose that APP is EC-(s, t)-WT. For \( \varepsilon \in (0, 1) \), we set
\[ \varepsilon_2 = \omega^{-\frac{2}{\ln \omega^{-1}}} \in (0, 1). \] (4.5)
Then we have that \( d + \varepsilon^{-1} \to \infty \) if and only if \( d + \varepsilon_2^{-1} \to \infty \). By the definition of EC-(s, t)-WT we have that
\[ \lim_{d+\varepsilon_2^{-1} \to \infty} \frac{\ln n(\varepsilon_2, \text{APP}_d)}{d^t + (1 + \ln \varepsilon_2^{-1})^s} = 0. \]
By (2.12), we have that
\[ \frac{\ln n(\varepsilon, I_d)}{d^t + (\varepsilon^{-1})^{2s}} = \frac{\ln n(\varepsilon_2, \text{APP}_d)}{d^t + (1 + \ln \varepsilon_2^{-1})^s} \frac{d^t + (1 + \varepsilon_2^{-1}) \ln \omega^{-1})^s}{d^t + (\varepsilon^{-1})^{2s}}. \]
Noting that
\[ 1 + \ln \varepsilon_2^{-1} = 1 + \varepsilon^{-2} - \frac{1}{2} \ln \omega^{-1} \leq 1 + \varepsilon^{-2} \ln \omega^{-1} \leq C_2 \varepsilon^{-2}, \] (4.6)
where \( C_2 = 1 + \ln \omega^{-1} \), we obtain that
\[ \frac{d^t + (1 + \varepsilon_2^{-1}) \ln \omega^{-1})^s}{d^t + (\varepsilon^{-1})^{2s}} \leq \frac{d^t + C_2 \varepsilon^{-2s}}{d^t + \varepsilon^{-2s}} \leq C_2. \]
It follows that
\[ \lim_{d+\varepsilon^{-1} \to \infty} \frac{\ln n(\varepsilon, I_d)}{d^t + (\varepsilon^{-1})^{2s}} = 0, \]
which implies that \( I \) is (2s, t)-WT. Hence (i) is proved.

(ii) We note that \( I \) is UWT if and only if, for all \( \alpha, \beta > 0 \), \( I \) is \((\alpha, \beta)\)-WT. By (i) we obtain that \( I \) is UWT if and only if for all \( \alpha, \beta > 0 \), APP is EC-(2s, \beta)-WT. This is equivalent to APP being EC-UWT. Hence (ii) is proved.

(iii) We first assume that \( I \) is QPT. Then there exist two constants \( C, t \geq 0 \) such that, for all \( d \in \mathbb{N}, \varepsilon \in (0, 1), \)
\[ n(\varepsilon, I_d) \leq C \exp[t(1 + \ln \varepsilon^{-1})(1 + \ln d)]. \]
By Theorem 2.3 and (4.4) we have that
\[ n(\varepsilon, \text{APP}_d) = n(\varepsilon_1, I_d) \leq C \exp[t(1 + \ln \varepsilon_1^{-1})(1 + \ln d)] \leq C \exp \left\{ t \left[ 1 + \frac{1}{2} \ln(C_1(1 + \ln \varepsilon^{-1})) \right] (1 + \ln d) \right\} \leq C \exp \left\{ t_1 \left[ 1 + \ln(1 + \ln \varepsilon^{-1}) \right] (1 + \ln d) \right\}, \]
where $\varepsilon_1$ is given by (4.3), $t_1 = t(1 + \frac{\ln C_1}{2})$, $C_1 = \max\{1, \frac{2}{\ln \omega - 1}\}$. This means that APP is EC-QPT.

Next, we assume that APP is EC-QPT. We want to show that $I$ is QPT. There exist two constants $C, t \geq 0$ such that, for all $d \in \mathbb{N}$, $\varepsilon \in (0, 1)$,

$$n(\varepsilon, \text{APP}_d) \leq C \exp\{t[1 + \ln(1 + \ln \varepsilon^{-1})(1 + \ln d)]\}.$$

According to Theorem 2.3 and (4.6), we obtain that

$$n(\varepsilon, I_d) = n(\varepsilon, \text{APP}_d) \leq C \exp\{t[1 + \ln(1 + \ln C_2)^{-2}](1 + \ln d)\} \leq C \exp[t_2(1 + \ln \varepsilon^{-1})(1 + \ln d)],$$

where $\varepsilon_2$ is given by (4.5), $t_2 = t \max\{1 + \ln C_2, 2\}$, $C_2 = 1 + \ln \omega^{-1}$. This means that $I$ is QPT. Hence (iii) is proved.

(iv) First we assume that $I$ is PT. There exist non-negative numbers $C, p$ and $q$ such that, for all $d \in \mathbb{N}$, $\varepsilon \in (0, 1)$,

$$n(\varepsilon, I_d) \leq Cd^q \varepsilon^{-p}.$$

By Theorem 2.3, (4.3) and (4.4) we have that

$$n(\varepsilon, \text{APP}_d) = n(\varepsilon, I_d) \leq C \exp\{t[1 + \ln(1 + \ln \varepsilon^{-1})](1 + \ln d)\} \leq C \exp[t_1(1 + \ln \varepsilon^{-1})(1 + \ln d)],$$

which means that APP is EC-PT.

Next, suppose that APP is EC-PT. There exist non-negative numbers $C, p$ and $q$ such that, for all $d \in \mathbb{N}$, $\varepsilon \in (0, 1)$,

$$n(\varepsilon, \text{APP}_d) \leq C \exp[t_2(1 + \ln \varepsilon^{-1})(1 + \ln d)].$$

By Theorem 2.3, (4.5) and (4.6) we have that

$$n(\varepsilon, I_d) = n(\varepsilon, \text{APP}_d) \leq C \exp[t_2(1 + \ln \varepsilon^{-1})(1 + \ln d)] \leq C \exp[t_2(1 + \ln \varepsilon^{-1})\varepsilon^{-p}],$$

which implies that $I$ is PT. Hence (iv) is proved.

(v) The proof is analogous to the one of (iv).

The proof of Theorem 4.1 is finished.

\[\square\]
Proof of Theorem 2.5  We note that if \( \tilde{b} = \{ \tilde{b}_j \}, \tilde{b}_j = (2\pi)^{2b_j}, \ j \in \mathbb{N} \), then
\[
W_2^b([0,1]^d) = W_2^{2,\tilde{b}}([0,1]^d).
\]
It follows from Theorem 2.4 that

1. \( \tilde{I} \) is SPT if and only if \( \tilde{I} \) is PT if and only if
\[
\sum_{j=1}^{\infty} b_j^{-1} < \infty \quad \text{and} \quad \lim_{j \to \infty} \frac{b_j}{j} > 0;
\]

2. \( \tilde{I} \) is QPT if and only if
\[
\sup_{d \in \mathbb{N}} \frac{\sum_{j=1}^{d} b_j^{-1}}{1 + \ln d} < \infty \quad \text{and} \quad \lim_{j \to \infty} \frac{(1 + \ln j) b_j}{j} > 0;
\]

3. (iii), (iv), (v), (vi), (vii) and (viii) hold.

Hence, in order to show Theorem 2.5, it suffices to prove that

(a) if \( \sum_{j=1}^{\infty} b_j^{-1} < \infty \), then we have \( \lim_{j \to \infty} \frac{b_j}{j} > 0 \);

(b) if \( \sup_{d \in \mathbb{N}} \frac{\sum_{j=1}^{d} b_j^{-1}}{1 + \ln d} < \infty \), then \( \lim_{j \to \infty} \frac{(1 + \ln j) b_j}{j} > 0 \).

First we prove (a). Assume that \( B := \sum_{j=1}^{\infty} b_j^{-1} < \infty \). We have
\[
B \geq \sum_{j=1}^{k} b_j^{-1} \geq k b_k^{-1}.
\]
It follows that
\[
\frac{b_k}{k} \geq \frac{1}{B},
\]
which yields that
\[
\lim_{k \to \infty} \frac{b_k}{k} \geq \frac{1}{B} > 0.
\]
Hence (a) holds.

Next, we show (b). Assume that \( D := \sup_{d \in \mathbb{N}} \frac{\sum_{j=1}^{d} b_j^{-1}}{1 + \ln d} < \infty \). We have
\[
D \geq \frac{\sum_{j=1}^{k} b_j^{-1}}{1 + \ln k} \geq \frac{k b_k^{-1}}{1 + \ln k}.
\]
It follows that
\[
\frac{(1 + \ln k) b_k}{k} \geq \frac{1}{D},
\]
which yields that
\[
\lim_{k \to \infty} \frac{(1 + \ln k) b_k}{k} \geq \frac{1}{D} > 0.
\]
Hence (b) holds.

This completes the proof of Theorem 2.5. \( \square \)
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