FRACTIONAL HARDY-LIEB-THIRRING AND RELATED INEQUALITIES FOR INTERACTING SYSTEMS

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Abstract. We prove analogues of the Lieb-Thirring and Hardy-Lieb-Thirring inequalities for many-body quantum systems with fractional kinetic operators and homogeneous interaction potentials, where no antisymmetry on the wave functions is assumed. These many-body inequalities imply interesting one-body interpolation inequalities, and we show that the corresponding one- and many-body inequalities are actually equivalent in certain cases.

1. Introduction

The uncertainty principle and the exclusion principle are two of the most important concepts of quantum mechanics. In 1975, Lieb and Thirring [29, 30] gave an elegant combination of these principles in a semi-classical lower bound on the kinetic energy of fermionic systems. They showed that there exists a constant $C_{LT} > 0$ depending only on the dimension $d \geq 1$ such that the inequality

$$
\left\langle \Psi, \sum_{i=1}^{N} -\Delta_i \Psi \right\rangle \geq C_{LT} \int_{\mathbb{R}^d} \rho_{\Psi}(x)^{1+2/d} \, dx
$$

holds true for every function $\Psi \in H^1((\mathbb{R}^d)^N)$ and for every $N \in \mathbb{N}$, provided that $\Psi$ is normalized and anti-symmetric, namely $\|\Psi\|_{L^2((\mathbb{R}^d)^N)} = 1$ and

$$
\Psi(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_N) = -\Psi(x_1, \ldots, x_j, \ldots, x_i, \ldots, x_N), \quad \forall i \neq j. \tag{2}
$$

The left hand side of (1) is the expectation value of the kinetic energy operator for $N$ particles, and for every $N$-body wave function $\Psi \in L^2((\mathbb{R}^d)^N)$, its one-body density is defined by

$$
\rho_{\Psi}(x) := \sum_{j=1}^{N} \int_{\mathbb{R}^{d(N-1)}} |\Psi(x_1, \ldots, x_{j-1}, x, x_{j+1}, \ldots, x_N)|^2 \prod_{i \neq j} dx_i.
$$

Note that $\int_Q \rho_{\Psi}$ can be interpreted as the expected number of particles to be found on a subset $Q \subset \mathbb{R}^d$ in the probability distribution given by $|\Psi|^2$. In particular, $\int_{\mathbb{R}^d} \rho_{\Psi} = N$.

The Lieb-Thirring inequality can be seen as a many-body generalization of the Gagliardo-Nirenberg inequality

$$
\left( \int_{\mathbb{R}^d} |\nabla u(x)|^2 \, dx \right)^{2/d} \left( \int_{\mathbb{R}^d} |u(x)|^2 \, dx \right)^{2/d} \geq C_{GN} \int_{\mathbb{R}^d} |u(x)|^{2(1+2/d)} \, dx, \tag{3}
$$

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for $u \in H^1(\mathbb{R}^d)$. Note that for $d \geq 3$, the Gagliardo-Nirenberg inequality \textup{(3)} is a consequence of Sobolev’s inequality
\[
\|\nabla u\|_{L^2(\mathbb{R}^d)} \geq C_S \|u\|_{L^{2d/(d-2)}(\mathbb{R}^d)}
\] (4)
and the Hölder interpolation inequality for $L^p$-spaces. Moreover, Sobolev’s inequality can actually be obtained from Hardy’s inequality
\[
\|\nabla u\|_{L^2(\mathbb{R}^d)}^2 \geq \frac{(d-2)^2}{4} \int_{\mathbb{R}^d} \frac{|u(x)|^2}{|x|^2} dx, \quad d > 2,
\] (5)
by a symmetric-decreasing rearrangement argument (see, e.g., [15, Sec. 4]).

All of the inequalities (3)-(4)-(5) are quantitative formulations of the uncer- tainty principle. On the other hand, the anti-symmetry (2), which is crucial for the Lieb-Thirring inequality \textup{(1)} to hold, corresponds to Pauli’s exclusion principle for fermions. In fact, inequality \textup{(1)} fails to apply to the product wave function
\[
\Psi(x_1, x_2, ..., x_N) = u(x_1)u(x_2)...u(x_N) =: u \otimes N(x_1, x_2, ..., x_N),
\]
which is a typical state of bosons. In this case $\rho_{u \otimes N}(x) = N|u(x)|^2$ and we only have the weaker inequality
\[
\left\langle u \otimes N, \left( \sum_{i=1}^N -\Delta_i \right) u \otimes N \right\rangle \geq C N^{-2/d} \int_{\mathbb{R}^d} \rho_{u \otimes N}(x)^{1+2/d} dx,
\] (6)
which is, however, equivalent to the Gagliardo-Nirenberg inequality \textup{(3)}.

The discovery of Lieb and Thirring goes back to the stability of matter problem (see [27] for a pedagogical introduction to this subject). It is often straightforward to derive the finiteness of the ground state energy of quantum systems from a formulation of the uncertainty principle such as \textup{(3)}, \textup{(4)} or \textup{(5)}. However, the fact that the energy does not diverge faster than proportionally to the number of particles — that is stability in a thermodynamic sense — is much more subtle and for this the exclusion principle is crucial. It was Dyson and Lenard [8, 23] who first proved thermodynamic stability for fermionic Coulomb systems, and their proof is based on a local formulation of the exclusion principle, which is a relatively weak consequence of \textup{(2)}. Later Lieb and Thirring [29] gave a much shorter proof of the stability of matter using their more powerful inequality \textup{(1)}.

Recently, Lundholm and Solovej [33] realized that the local exclusion principle in the original work of Dyson and Lenard [8, 23], when combined with local formulations of the uncertainty principle such as \textup{(3)}, \textup{(4)} or \textup{(5)}, actually implies the Lieb-Thirring inequality \textup{(1)}. From this point of view, they derived Lieb-Thirring inequalities for anyons, two-dimensional particles which do not satisfy the full anti-symmetry \textup{(2)} but still fulfill a fractional exclusion. The same approach was also employed to prove Lieb-Thirring inequalities for fractional statistics particles in one dimension by the same authors [34], as well as for fermions with certain point interactions by Frank and Seiringer [16].

Following the spirit in [33], Lundholm, Portmann and Solovej [32] found that Lieb-Thirring type inequalities still hold true for particles without any symmetry assumptions — and therefore in particular for bosons — provided

\[1\text{In general, bosonic wave functions satisfy \textup{(2)} with a plus instead of a minus sign.}\]
that the exclusion principle is replaced by a sufficiently strong repulsive interaction between particles. For example, they proved that there exists a constant $C > 0$ depending only on the dimension $d \geq 1$ such that for every normalized function $\Psi \in H^1((\mathbb{R}^d)^N)$ and all $N \in \mathbb{N},$

$$\left\langle \Psi, \left( \sum_{i=1}^{N} -\Delta_i + \sum_{1 \leq i < j \leq N} \frac{1}{|x_i - x_j|^2} \right) \Psi \right\rangle \geq C \int_{\mathbb{R}^3} \rho_{\Psi}(x)^{1+2/d} \, dx. \quad (7)$$

The appearance of the inverse-square interaction in (7) is natural as it makes all terms in the inequality scale in the same way.

The aims of our paper are threefold.

• We generalize the Lieb-Thirring inequality (7) to the fractional kinetic operator $(-\Delta)^s$ for an arbitrary power $s > 0$, with matching interaction $|x - y|^{-2s}$. The non-local property of $(-\Delta)^s$ for non-integer $s$ makes the inequality more involved. Nevertheless, the fermionic analogue of this inequality (without the interaction term) has been known for a long time in the context of relativistic stability [7]. For the interacting bosonic version we will follow the strategy of [32], using local uncertainty and exclusion, but we also develop several new tools. In particular, we will introduce a new covering lemma which provides an elegant way to combine the local uncertainty and exclusion into a single bound.

• We prove a stronger version of the Lieb-Thirring inequality (7) with the kinetic operator replaced by $(-\Delta)^s - C_{d,s}|x|^{-2s}$ and with the interaction $|x - y|^{-2s}$, for all $0 < s < d/2$. Here $C_{d,s}$ is the optimal constant in the Hardy inequality [18]

$$(-\Delta)^s - C_{d,s}|x|^{-2s} \geq 0.$$  

Our result can be seen as a bosonic analogue to the Hardy-Lieb-Thirring inequality for fermions found by Ekholm, Frank, Lieb and Seiringer [10, 13, 14].

• Just as the Lieb-Thirring inequality (11) implies the one-body interpolation inequality (3), the same will be shown to be true for these generalized many-body inequalities. For instance, our bosonic Hardy-Lieb-Thirring inequality implies the one-body interpolation inequality

$$\left\langle u, \left( (-\Delta)^s - C_{d,s}|x|^{-2s} \right) u \right\rangle^{1-2s/d} \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|u(x)|^2|u(y)|^2}{|x - y|^{2s}} \, dx dy \right)^{2s/d} \geq C \int_{\mathbb{R}^d} |u(x)|^{2(1+2s/d)} \, dx,$$

for $u \in H^s(\mathbb{R}^d)$ and $0 < s < d/2$. Moreover, we prove the equivalence between the (bosonic) Lieb-Thirring/Hardy-Lieb-Thirring inequalities and the corresponding one-body interpolation inequalities when $0 < s \leq 1$. Since one-body interpolation inequalities have been studied actively for a long time, we believe that this equivalence could inspire many new directions to the many-body theory.

In the next section our results will be presented in detail and an outline of the rest of the paper given.

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2. Main results

2.1. Fractional Lieb-Thirring inequality. Our first aim of the present paper is to generalize (7) to the fractional kinetic operator \((-\Delta)^s\) for an arbitrary power \(s > 0\), and with a matching interaction \(|x - y|^{-2s}\). The operator \((-\Delta)^s\) is defined as the multiplication operator \(|p|^{2s}\) in Fourier space, namely

\[
|(-\Delta)^s f|^2(p) = |p|^{2s} \hat{f}(p), \quad \hat{f}(p) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(x) e^{-ip\cdot x} \, dx.
\]

The associated space \(H^s(\mathbb{R}^d)\) is a Hilbert space with norm

\[
\|u\|_{H^s(\mathbb{R}^d)}^2 := \|u\|_{L^2(\mathbb{R}^d)}^2 + \|u\|_{H^s(\mathbb{R}^d)}^2, \quad \|u\|_{H^s(\mathbb{R}^d)}^2 := \langle u, (-\Delta)^s u \rangle.
\]

Our first result is the following

**Theorem 1** (Fractional Lieb-Thirring inequality). For all \(d \geq 1\) and \(s > 0\), there exists a constant \(C > 0\) depending only on \(d\) and \(s\) such that for all \(N \in \mathbb{N}\) and for every normalized function \(\Psi \in H^s((\mathbb{R}^d)^N)\),

\[
\left\langle \Psi, \sum_{i=1}^N (-\Delta_i)^s + \sum_{1 \leq i < j \leq N} \frac{1}{|x_i - x_j|^{2s}} \right\rangle \Psi \geq C \int_{\mathbb{R}^d} \rho_{\Psi}(x) x^{1+2s/d} \, dx. \quad (8)
\]

Since our result holds without restrictions on the symmetry of the wave function, and therefore in particular also for bosons, we consider it as a bosonic analogue to the fermionic inequality\(^2\)

\[
\left\langle \Psi, \sum_{i=1}^N (-\Delta_i)^s \Psi \right\rangle \geq C \int_{\mathbb{R}^d} \rho_{\Psi}(x) x^{1+2s/d} \, dx, \quad (9)
\]

which holds for wave functions \(\Psi\) satisfying the anti-symmetry. The original motivation for such a fermionic fractional Lieb-Thirring inequality has been its usefulness in the context of stability of relativistic matter (see [7] and the recent review [27]). Our inequality \((8)\) for \(s = 1/2\) and \(d = 3\) is relevant to the physical situation of relativistic particles (which could be identical bosons, or even distinguishable) with Coulomb interaction.

\(^2\)Throughout our paper, \(C\) denotes a generic positive constant. Two \(C\)’s in different places may refer to two different constants.
Remark 1. When $0 < s \leq 1$ we can also replace the one-body kinetic operator $(-\Delta)^s$ by $|i\nabla + A(x)|^{2s}$ with $A \in L^2_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^d)$ being a magnetic vector potential. By virtue of the diamagnetic inequality (see e.g. [13, Eq. (2.3)])

$$\langle u, |i\nabla + A|^{2s}u \rangle \geq \langle |u|, (-\Delta)^s |u| \rangle$$

the inequality (5) holds with the same constant (independent of $A$).

When $s \notin \mathbb{N}$, the Lieb-Thirring inequality (5) cannot be obtained from a straightforward modification of the proof of (7) in [32]. The non-local property of $(-\Delta)^s$ complicates the local uncertainty principle and a fractional interpolation inequality on cubes is required. We will follow the strategy in [32] but several technical adjustments are presented. The details are provided in Section 3. We believe that our presentation here provides a unified framework for proving Lieb-Thirring inequalities by means of local formulations of the uncertainty and exclusion principles, and can be used to simplify many parts of the previous works [33, 34, 16, 32]. For comparison, we also make a note about fermions and weaker exclusion principles in Section 3.5.

2.2. Hardy-Lieb-Thirring inequality. Recall that for every $0 < s < d/2$, we have the Hardy inequality

$$(-\Delta)^s - C_{d,s} |x|^{-2s} \geq 0 \text{ on } L^2(\mathbb{R}^d),$$

where the sharp constant is

$$C_{d,s} := 2^{2s} \left( \frac{\Gamma((d + 2s)/4)}{\Gamma((d - 2s)/4)} \right)^2.$$  

We will prove the following improvement of Theorem 1 when $0 < s < d/2$.

**Theorem 2** (Hardy-Lieb-Thirring inequality). For all $d \geq 1$ and $0 < s < d/2$, there exists a constant $C > 0$ depending only on $d$ and $s$ such that for every (normalized) function $\Psi \in H^s((\mathbb{R}^d)^N)$ and for all $N \in \mathbb{N}$, we have

$$\left\langle \Psi, \sum_{i=1}^N \left( (-\Delta)^s - \frac{C_{d,s}}{|x_i|^{2s}} \right) + \sum_{1 \leq i < j \leq N} \frac{1}{|x_i - x_j|^{2s}} \right\rangle \geq C \int_{\mathbb{R}^d} \rho_\Psi(x)^{1+2s/d} dx.$$  

(11)

For $s = 1/2$ and $d = 3$, the operator in (11) can be interpreted as the Hamiltonian of a system of $N$ equally charged relativistic particles (bosons, fermions or distinguishable) moving around a static ‘nucleus’ of opposite charge located at $x = 0$, where all particles interact via Coulomb forces.

Our result (11) can be considered as the interacting bosonic analogue to the following Hardy-Lieb-Thirring inequality for fermions:

$$\left\langle \Psi, \sum_{i=1}^N \left( (-\Delta_i)^s - \frac{C_{d,s}}{|x_i|^{2s}} \right) \Psi \right\rangle \geq C \int_{\mathbb{R}^d} \rho_\Psi(x)^{1+2s/d} dx,$$  

(12)

which holds for every wave function $\Psi$ satisfying the anti-symmetry (2). The bound (12) was proved for $s = 1$ by Ekholm and Frank [10], for $0 < s \leq 1$ by Frank, Lieb and Seiringer [13], and for $0 < s < d/2$ by Frank [14]. In fact,

3The case $s \geq d/2$ requires additional boundary conditions at $x = 0$ and will not be treated here. See [33] and [9] for corresponding fermionic Lieb-Thirring inequalities.
is dually equivalent to a lower bound on the sum of negative eigenvalues of the one-body operator \((-\Delta)^s - C_{d,s}|x|^{-2s} + V(x)\) and such a bound was proved in \([10, 13, 14]\). Unfortunately this duality argument (which has been the traditional route to proving Lieb-Thirring inequalities) does not apply in our interacting bosonic case.

Remark 2. The motivation for (12) was critical stability of relativistic matter in the presence of magnetic fields. In both (11) and (12) we can, for \(0 < s \leq 1\), replace \((-\Delta)^s\) with a magnetic operator \(|i\nabla + A(x)|^{2s}\); cf. Remark 1.

The proof of (12) in \([14]\) is based on the following powerful improvement of Hardy’s inequality: For every \(d \geq 1\) and \(0 < t < s < d/2\), there exists a constant \(C > 0\) depending only on \(d, s, t\) such that

\[
(-\Delta)^s - C_{d,s}/|x|^{2s} \geq C\ell^s - \ell^t (-\Delta)^t \quad \text{on } L^2(\mathbb{R}^d), \quad \forall \ell > 0. \tag{13}
\]

Note that by taking the expectation against a function \(u\) and optimizing over \(\ell > 0\), we can see that (13) is equivalent to the interpolation inequality

\[
\langle u, (-\Delta)^t u \rangle \geq C\|u\|_{L^q(\mathbb{R}^d)}^2, \quad q = \frac{2d}{d - 2t}, \quad 0 < t < d/2, \tag{14}
\]

By Sobolev’s embedding (see, e.g., \([25, 5]\) for the sharp constant)

\[
\langle u, (-\Delta)^t u \rangle \geq C\|u\|_{L^q(\mathbb{R}^d)}^2, \quad q = \frac{2d}{d - 2t}, \quad 0 < t < d/2, \tag{15}
\]

the bound (14) implies the Gagliardo-Nirenberg type inequality

\[
\langle u, (-\Delta)^s - C_{d,s}/|x|^{2s} \rangle^{t/s} \left( \int_{\mathbb{R}^d} |u|^2 \right)^{1-t/s} \geq C\|u\|_{L^q(\mathbb{R}^d)}^2, \quad q = \frac{2d}{d - 2t}. \tag{16}
\]

The bound (13) was first proved for \(s = 1/2, d = 3\) by Solovej, Sørensen and Spitzer \([39, \text{Lemma 11}]\) and was generalized to the full case \(0 < s < d/2\) by Frank \([14, \text{Theorem 1.2}]\).

In fact, (13) is also a key ingredient of our proof of (11). The overall strategy is similar to the proof of the fractional Lieb-Thirring inequality (8). However, since the system is not translation invariant anymore, the local uncertainty becomes much more involved. We need to introduce a partition of unity and use (14) and (16) to control the localization error caused by the non-local operator \((-\Delta)^s\). The details will be provided in Section 4.

2.3. Interpolation inequalities. Let us concentrate again on the case \(0 < s < d/2\). By applying the Lieb-Thirring inequality in Theorem 1 to the product wave function \(\Psi = u^{\otimes N}\) with \(\|u\|_{L^2(\mathbb{R}^d)} = 1\), we obtain

\[
N\langle u, (-\Delta)^s u \rangle + \frac{N(N - 1)}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|u(x)|^2 |u(y)|^2}{|x - y|^{2s}} \, dx \, dy \geq CN^{1+2s/d} \int_{\mathbb{R}^d} |u(x)|^{2(1+2s/d)} \, dx.
\]
Since the inequality holds for all $N \in \mathbb{N}$, it then follows that

$$
\lambda(u, (-\Delta)^s u) + \frac{\lambda^2}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|u(x)|^2 |u(y)|^2}{|x-y|^{2s}} \, dx \, dy \\
\geq C \lambda^{1+2s/d} \int_{\mathbb{R}^d} |u(x)|^{2(1+2s/d)} \, dx
$$

(17)

for all $\lambda \geq 1$ (possibly with a smaller constant). On the other hand, by using Sobolev’s embedding (15) and Hölder’s interpolation inequality for $L^p$-spaces, we get

$$
\langle u, (-\Delta)^s u \rangle \geq C \|u\|_{L^2} \|u\|_{L^{2s/d}}^{2s/d} = C \int_{\mathbb{R}^d} |u(x)|^{2(1+2s/d)} \, dx
$$

(18)

which implies (17) when $0 < \lambda < 1$. Thus (17) holds for all $\lambda > 0$, and optimizing over $\lambda$ gives the interpolation inequality

$$
\langle u, (-\Delta)^s u \rangle \geq C \int_{\mathbb{R}^d} |u(x)|^{2(1+2s/d)} \, dx
$$

(19)

for $u \in H^s(\mathbb{R}^d)$, $\|u\|_{L^2} = 1$. Note that in (19) the normalization $\|u\|_{L^2} = 1$ can be dropped by scaling.

The interpolation inequality (19) was first proved for the case $s = 1/2$, $d = 3$ by Bellazzini, Ozawa and Visciglia [3], and was then generalized to the general case $0 < s < d/2$ by Bellazzini, Frank and Visciglia [3]. The proofs in [3, 4] use fractional calculus on the whole space and are very different from our approach using the Lieb-Thirring inequality.

Remark 3. The inequality (19) is an end-point case of a series of interpolation inequalities in [3]. The existence of optimizer in this case is open. If a minimizer exists, by formally analyzing the Euler-Lagrange equation we expect that it belongs to $L^{2+\epsilon}(\mathbb{R}^d)$ for any $\epsilon > 0$ small, but not $L^2(\mathbb{R}^d)$. Thus (19) can be interpreted as an energy bound for systems of infinitely many particles.

Note that, when $s \geq d/2$, one has

$$
\iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|u(x)|^2 |u(y)|^2}{|x-y|^{2s}} \, dx \, dy = +\infty
$$

for all $u \neq 0$ since $|x|^{-2s}$ is not locally integrable. Therefore, the interpolation inequality (19) is trivial in this case. However, the Lieb-Thirring inequality (8) is non-trivial for all $s > 0$, since the wave function $\Psi$ may vanish on the diagonal set $\{(x_i)_{i=1}^N \in (\mathbb{R}^d)^N : x_i = x_j \text{ for some } i \neq j\}$.

In principle, the implication of a one-body inequality from a many-body inequality is not surprising. However, in the following result we show that the reverse implication also holds true under certain conditions.

Theorem 3. For $0 < s < d/2$ and $s \leq 1$, the Lieb-Thirring inequality (8) is equivalent to the one-body interpolation inequality (19).
As we explained above, the implication of (19) from (8) works for all \( 0 < s < d/2 \). The implication of (8) from (19) is more subtle and we obtain it from fractional versions of the Hoffmann-Ostenhof inequality [19], which requires \( 0 < s \leq 1 \), and a generalized version of the Lieb-Oxford inequality [24, 26] for homogeneous potentials. We will provide these details in Section 5.

By the same proof as that of Theorem 3, we also obtain the following equivalence for the Hardy-Lieb-Thirring inequality (11).

**Theorem 4.** For \( 0 < s < d/2 \) and \( s \leq 1 \), the Hardy-Lieb-Thirring inequality (11) is equivalent to the one-body interpolation inequality

\[
\left\langle u, \left( -\Delta - C_{d,s} |x|^{-2s} \right) u \right\rangle^{1-2s/d} \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|u(x)|^2 |u(y)|^2}{|x-y|^{2s}} \, dx \, dy \right)^{2s/d} \geq C \int_{\mathbb{R}^d} |u(x)|^{2(1+2s/d)} \, dx.
\]

The interpolation inequality (20) seems to be new. Note that the implication of (20) from (11) holds for all \( 0 < s < d/2 \) (by exactly the same argument as above), and hence (20) is also valid in this maximal range. There might be some way to prove (20) directly (as in the proof of (19) in [4, 3]), but we have not found such a proof yet.

Finally, we mention that our approach in this paper can be used to prove many other interpolation inequalities which do not really come from many-body quantum theory. For example, we have

**Theorem 5** (Isoperimetric inequality with non-local term). For any \( d \geq 2 \) and \( 1/2 \leq s < d/2 \) there exists a constant \( C > 0 \) depending only on \( d \) and \( s \), such that for all functions \( u \in W^{1,2s}(\mathbb{R}^d) \) we have

\[
\left( \int_{\mathbb{R}^d} |\nabla u|^{2s/d} \, dx \right)^{1-2s/d} \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|u(x)|^{2s} |u(y)|^{2s}}{|x-y|^{2s}} \, dx \, dy \right)^{2s/d} \geq C \int_{\mathbb{R}^d} |u|^{2s(1+2s/d)} \, dx.
\]

This inequality seems to be new and it could be useful in the context of isoperimetric inequalities with competing non-local term; see [22, Lemma 7.1], [21, Lemma 5.2] and [35, Lemma B.1] for relevant results. The proof of Theorem 5 will be given in Section 5.

3. Fractional Lieb-Thirring inequality

In this section we prove the fractional Lieb-Thirring inequality (8). We shall follow the overall strategy in [32], where we localize the interaction and kinetic energies into disjoint cubes, but we also introduce several new tools.

3.1. Local exclusion. The following result is a simplified version of the local exclusion principle in [32, Theorem 2 and Section 4.2].

**Lemma 6** (Local exclusion). For all \( d \geq 1 \), \( s > 0 \), for every normalized function \( \Psi \in L^2((\mathbb{R}^d)^N) \) and for an arbitrary collection of disjoint cubes \( Q \)'s in \( \mathbb{R}^d \), one has
\[
\left\langle \Psi, \sum_{1 \leq i < j \leq N} \frac{1}{|x_i - x_j|^{2s}} \Psi \right\rangle \geq \sum_{Q} \frac{1}{2d^s|Q|^{2s/d}} \left[ \left( \int_{Q} \rho_{\Psi} \right)^2 - \int_{Q} \rho_{\Psi} \right]_+.
\] (22)

Proof. The following argument goes back to Lieb’s work on the indirect energy [24]. Since the interactions between different cubes are positive and \(|x - y| \leq \sqrt{d}|Q|^{1/d}\) for all \(x, y \in Q\), we have

\[
\sum_{1 \leq i < j \leq N} \frac{1}{|x_i - x_j|^{2s}} \geq \sum_{Q} \sum_{1 \leq i < j \leq N} \frac{1}{|x_i - x_j|^{2s}} \mathbb{1}_Q(x_i) \mathbb{1}_Q(x_j)
\]

\[
\geq \sum_{Q} \frac{1}{2d^s|Q|^{2s/d}} \sum_{1 \leq i < j \leq N} \mathbb{1}_Q(x_i) \mathbb{1}_Q(x_j)
\]

\[
= \sum_{Q} \frac{1}{2d^s|Q|^{2s/d}} \left[ \left( \sum_{i=1}^{N} \mathbb{1}_Q(x_i) \right)^2 - \sum_{i=1}^{N} \mathbb{1}_Q(x_i) \right]_+.
\]

Taking the expectation against \(\Psi\) and using the Cauchy-Schwarz inequality

\[
\left\langle \Psi, \left( \sum_{i=1}^{N} \mathbb{1}_Q(x_i) \right)^2 \Psi \right\rangle \geq \left\langle \Psi, \sum_{i=1}^{N} \mathbb{1}_Q(x_i) \Psi \right\rangle^2 = \left( \int \rho_{\Psi} \right)^2,
\]

we obtain the desired estimate. \(\square\)

3.2. Local uncertainty. Now we localize the kinetic energy into disjoint cubes \(Q\)'s. For every \(s > 0\) we can write \(s = m + \sigma\) with \(m \in \{0, 1, 2, \ldots\}\) and \(0 \leq \sigma < 1\). Then for any one-body function \(u \in H^s(\mathbb{R}^d)\) we have

\[
\langle u, (-\Delta)^s u \rangle = \int_{\mathbb{R}^d} |p|^{2s} |\hat{u}(p)|^2 dp = \int_{\mathbb{R}^d} |p|^{2s} \left( \sum_{i=1}^{d} p_i^2 \right)^m |\hat{u}(p)|^2 dp
\]

\[
= \sum_{|\alpha| = m} \frac{m!}{\alpha!} \int_{\mathbb{R}^d} |p|^{2s} \prod_{i=1}^{d} p_i^{2\alpha_i} |\hat{u}(p)|^2 dp
\]

\[
= \sum_{|\alpha| = m} \frac{m!}{\alpha!} \left( D^\alpha u, (-\Delta)^\sigma D^\alpha u \right).
\] (23)

The last sum is taken over multi-indices \(\alpha = (\alpha_1, \ldots, \alpha_d) \in \{0, 1, 2, \ldots\}^d\) with

\[
|\alpha| = \sum_{i=1}^{d} \alpha_i, \quad \alpha! = \prod_{i=1}^{d} (\alpha_i !) \quad \text{and} \quad D^\alpha = \prod_{i=1}^{d} \frac{\partial^{\alpha_i}}{\partial^{\alpha_i} p_i}.
\]

Here we denoted by \(p = (p_1, p_2, \ldots, p_d) \in \mathbb{R}^d\) and \(r = (r_1, \ldots, r_d) \in \mathbb{R}^d\), the variables in the Fourier space and the configuration space, respectively.

If \(s = m\), we have

\[
\langle u, (-\Delta)^s u \rangle = \sum_{|\alpha| = m} \frac{m!}{\alpha!} \int_{\mathbb{R}^d} |D^\alpha u|^2 \geq \sum_{|\alpha| = m} \frac{m!}{\alpha!} \sum_{Q} \int_{Q} |D^\alpha u|
\] (24)
for disjoint cubes $Q$'s. On the other hand, if $m < s < m + 1$, then using the quadratic form representation \(^4\) (see e.g., \([13\) Lemma 3.1])

\[
(f, (-\Delta)^\sigma f) = c_{d, \sigma} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|f(x) - f(y)|^2}{|x - y|^{d+2\sigma}} \, dx \, dy,
\]

where

\[
c_{d, \sigma} := \frac{2^{2\sigma-1}}{\pi^{d/2}} \frac{\Gamma((d + 2\sigma)/2)}{\Gamma(-\sigma)},
\]

we have

\[
\langle u, (-\Delta)^\sigma u \rangle = c_{d, \sigma} \sum_{|\alpha|=m} \frac{m!}{\alpha!} \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x - y|^{d+2\sigma}} \, dx \, dy
\]

\[
\geq c_{d, \sigma} \sum_{|\alpha|=m} \frac{m!}{\alpha!} \int_{Q \times Q} \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x - y|^{d+2\sigma}} \, dx \, dy
\]

for disjoint cubes $Q$'s. It is convenient to combine \((24)\) and \((26)\) into a single formula

\[
\langle u, (-\Delta)^\sigma u \rangle \geq \sum_{Q} \|u\|^2_{H^\sigma(Q)},
\]

where the semi-norm $\|u\|^2_{H^\sigma(Q)}$ of $u \in L^2(Q)$ on a cube $Q$ is defined by

\[
\|u\|^2_{H^\sigma(Q)} := \left\{ \begin{array}{ll}
\sum_{|\alpha|=m} \frac{m!}{\alpha!} \int_{Q} |D^\alpha u|^2 & \text{if } s = m, \\
\frac{m!}{\alpha!} \int_{Q \times Q} \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x - y|^{d+2\sigma}} \, dx \, dy & \text{if } 0 < \sigma < 1.
\end{array} \right.
\]

The following estimate plays an essential role in our proof.

**Lemma 7** (Local uncertainty). For every $d \geq 1$, $s > 0$, cube $Q \subset \mathbb{R}^d$ and $u \in L^2(Q)$, one has

\[
\|u\|^2_{H^\sigma(Q)} \geq \frac{1}{C} \frac{\|u\|^2_{L^2(Q)}}{\|u\|^2_{L^2(Q)}} + \frac{C}{\|u\|^2_{L^2(Q)}} \int_{Q} |u|^2
\]

for a constant $C > 0$ independent of $Q$ and $u$.

Before proving Lemma 7, let us clarify a technical point concerning the Sobolev space $H^\sigma(Q) = W^{s,2}(Q)$, whose intrinsic norm can be defined by (see e.g., \([1\) Section 7.36 and Theorem 7.48])

\[
\|u\|^2_{H^\sigma(Q)} := \|u\|^2_{H^\sigma(Q)} + \sum_{|\alpha| \leq m} \int_{Q} |D^\alpha u|^2.
\]

Here recall that $s = m + \sigma$ with $m \in \{0, 1, 2, \ldots\}$ and $0 \leq \sigma < 1$. By Poincaré's inequality for $H^\sigma(Q)$ (see, e.g., \([20\) Lemma 2.2]) and the elementary inequality $|a - b|^2 \geq \frac{1}{2} |a|^2 - |b|^2$ for $a, b \in \mathbb{C}$, we have

\[
C \|u\|^2_{H^\sigma(Q)} \geq \sum_{|\alpha|=m} \left\| D^\alpha u - \frac{1}{|Q|} \int_{Q} D^\alpha u \right\|^2_{L^2(Q)} \geq \frac{1}{2} \|D^\alpha u\|^2_{L^2(Q)} - \frac{\|D^\alpha u\|^2_{L^2(Q)}}{|Q|}.
\]

\(^4\)Note that this formula only holds for $0 < \sigma < 1$. 
From the latter estimate and Sobolev’s embedding, it is straightforward to obtain the following equivalence of norms

$$\|u\|_{H^{s}(Q)}^{2} \geq \|u\|_{H^{s}(\Omega)}^{2} + \int_{Q}|u|^{2} \geq C_{Q}\|u\|_{H^{s}(Q)}^{2},$$

(29)

for a constant $C_{Q} > 0$ depending only on the the cube $Q$. Now we provide

**Proof of Lemma 4.** By translating and dilating, that is, replacing $u(x)$ by $u(\lambda(x - x_{0}))$ for $\lambda > 0$ and $x_{0} \in \mathbb{R}^{d}$, it suffices to consider the unit cube $Q = [0, 1]^{d}$. Then, thanks to (29), it remains to prove the fractional Gagliardo-Nirenberg inequality

$$\|u\|_{H^{s}(Q)}^{\theta}\|u\|_{L^{1}(Q)}^{1-\theta} \geq C\|u\|_{L^{q}(Q)}, \quad q = 2 + \frac{4s}{d}, \quad \theta = \frac{d}{d + 2s},$$

(30)

for a constant $C > 0$ independent of $u$. Since the (unit) cube $Q$ is regular, we may apply the extension theorem to $H^{s}(Q)$ (see [11, Theorem 7.41] or [12, Theorem 4.2.3]) and obtain for any function $u \in H^{s}(Q)$ a function $U \in H^{s}(\mathbb{R}^{d})$ satisfying

$$U|_{Q} = u, \quad \|U\|_{L^{2}(\mathbb{R}^{d})}^{2} \leq C\|u\|_{L^{2}(Q)}, \quad \|U\|_{H^{s}(\mathbb{R}^{d})} \leq C\|u\|_{H^{s}(Q)},$$

where $C > 0$ depends only on $d$ and $s$. We will show that

$$\|U\|_{H^{s}(\mathbb{R}^{d})}^{\theta}\|U\|_{L^{2}(\mathbb{R}^{d})}^{1-\theta} \geq C\|U\|_{L^{q}(\mathbb{R}^{d})}, \quad q = 2 + \frac{4s}{d}, \quad \theta = \frac{d}{d + 2s},$$

(31)

and (30) follows immediately. By Sobolev’s embedding (15)

$$\|U\|_{H^{s}(\mathbb{R}^{d})} \geq C\|U\|_{L^{q}(\mathbb{R}^{d})}, \quad q = 2 + \frac{4s}{d} = \frac{2d}{d - 2\theta s},$$

the estimate (31) follows from the following interpolation inequality

$$\|U\|_{H^{s}(\mathbb{R}^{d})}^{\theta}\|U\|_{L^{2}(\mathbb{R}^{d})}^{1-\theta} \geq \|U\|_{H^{s}(\mathbb{R}^{d})}, \quad \forall \theta \in (0, 1),$$

(32)

which is in turn a simple consequence of Hölder’s inequality

$$\left(\int_{\mathbb{R}^{d}}p^{2s}|\hat{U}(p)|^{2}dp\right)^{\theta}\left(\int_{\mathbb{R}^{d}}|\hat{U}(p)|^{2}dp\right)^{1-\theta} \geq \int_{\mathbb{R}^{d}}p^{2\theta s}|\hat{U}(p)|^{2}dp.$$

Remark 4. Note that to the semi-norm $\|\cdot\|_{H^{s}(\Omega)}$ there is a naturally associated operator, which for $s = 1$ coincides with $-\Delta_{\Omega}^{N}$, the Neumann Laplacian on $\Omega \subseteq \mathbb{R}^{d}$. It is a relevant question whether for $0 < s < 1$ and bounded domains $\Omega$ this operator coincides with $(-\Delta_{\Omega}^{N})^{s}$ (defined using the spectral theorem), something that was shown in [12] to be false in the case of the Dirichlet Laplacian $-\Delta_{\Omega}^{D}$ (see also [36, 35] for related results). In any case, the analogue of (28) for $(-\Delta_{\Omega}^{N/D})^{s}$ can be proved using the method in [37].

We will need the following many-body version of Lemma 7.
Lemma 8 (Many-body version of local uncertainty). For any normalized function \( \Psi \in L^2(\mathbb{R}^d)^N \) and for an arbitrary collection of disjoint cubes \( Q \)'s, the kinetic energy satisfies the estimate

\[
\left\langle \Psi, \sum_{i=1}^N (-\Delta_i)^s \Psi \right\rangle \geq \sum_Q \left[ \frac{1}{C} \left( \int_Q \rho^\perp \right)^{\frac{2s}{d}} - \frac{C}{|Q|^{\frac{2s}{d}}} \int_Q \rho \right],
\]

where \( C \) is the same constant as in Lemma 7.

Proof. Let \( \gamma^{(1)}(x, y) := \sum_{j=1}^N \int_{\mathbb{R}^{d(N-1)}} \Psi(x_1, \ldots, x_{j-1}, x, x_{j+1}, \ldots, x_N) \times \Psi(x_1, \ldots, x_{j-1}, y, x_{j+1}, \ldots, x_N) \prod_{i \neq j} dx_i. \)

Since \( \gamma^{(1)} \) is trace class, we can write

\[
\gamma^{(1)}(x, y) = \sum_{n \geq 1} u_n(x) u_n(y),
\]

where \( u_n \in L^2(\mathbb{R}^d) \) are not necessarily normalized. Then \( \rho = \sum_{n \geq 1} |u_n|^2 \) and

\[
\left\langle \Psi, \sum_{i=1}^N (-\Delta)^s \Psi \right\rangle = \text{Tr} \left[ (-\Delta)^s \gamma^{(1)} \right] = \sum_{n \geq 1} \langle u_n, (-\Delta)^s u_n \rangle \geq \sum_{n \geq 1} \sum_Q \| u_n \|_{H^s(Q)}^2,
\]

where we have used \( \text{Tr} \) in the last estimate. On the other hand, from the local uncertainty \( (27) \) we have

\[
\left( \int_Q |u_n|^2 \right)^{\frac{2s}{d+2s}} \left( \| u_n \|_{H^s(Q)}^2 + \frac{C}{|Q|^{\frac{2s}{d}}} \int_Q |u_n|^2 \right)^{\frac{d}{d+2s}} \geq C^{-d/(d+2s)} \| u_n \|_{L^{1+2s/d}(Q)}^2
\]

for all \( n \geq 1 \). Therefore, by Hölder’s inequality (for sums) and the triangle inequality we get

\[
\left( \int_Q \rho \right)^{\frac{2s}{d+2s}} \left( \sum_{n \geq 1} \| u_n \|_{H^s(Q)}^2 + \frac{C}{|Q|^{\frac{2s}{d}}} \int_Q \rho \right)^{\frac{d}{d+2s}} \geq \sum_{n \geq 1} \left( \int_Q |u_n|^2 \right)^{\frac{2s}{d+2s}} \left( \| u_n \|_{H^s(Q)}^2 + \frac{C}{|Q|^{\frac{2s}{d}}} \int_Q |u_n|^2 \right)^{\frac{d}{d+2s}}
\]
\[ \geq \sum_{n \geq 1} C^{-\frac{d}{d+2s}} \| u_n \|^2_{L^{1+2s/d}(Q)} \geq C^{-\frac{d}{d+2s}} \sum_{n \geq 1} |u_n|^2_{L^{1+2s/d}(Q)} = C^{-\frac{d}{d+2s}} \| \rho \|_{L^{1+2s/d}(Q)} \]

which is equivalent to
\[ \sum_{n \geq 1} \| u_n \|^2_{H^s(Q)} \geq \frac{1}{C} \left( \int_Q \rho_{\Psi}^{1+2s/d} \right)^{2s/d} - \frac{C}{|Q|^{2s/d}} \int_Q \rho_{\Psi}. \]

The latter estimate and (35) imply the desired inequality (33). \qed

Remark 5. By using the interpolation inequality (18) and the same argument of the proof of Lemma 8 (in this case one can work on the whole \( \mathbb{R}^d \) and no partition of cubes is needed), we obtain the following generalization of (6):
\[ \langle \Psi, \sum_{i=1}^N (-\Delta_i)^s \Psi \rangle \geq C N^{-2s/d} \int_{\mathbb{R}^d} \rho_{\Psi}^{1+2s/d} \tag{36} \]
for all normalized functions \( \Psi \in H^s((\mathbb{R}^d)^N) \) and for a constant \( C > 0 \) depending only on \( d \) and \( s \). When \( 0 < s \leq 1 \), (36) can also be proved using the Hoffmann-Ostenhof inequality in Lemma 14 and Sobolev’s embedding. We will use (36) to obtain the Lieb-Thirring inequality (8) when \( N \) is small.

3.3. A covering lemma. To combine the local uncertainty and exclusion principles, we need a nice choice of the partition of cubes \( Q \)'s. The following result is inspired by the work of Lundholm and Solovej [33, Theorem 11]. In fact, a similar result can be obtained by following their construction. However, our construction below is simpler to apply and results in improved constants.

Lemma 9 (Covering lemma). Let \( Q_0 \) be a cube in \( \mathbb{R}^d \) and let \( \Lambda > 0 \). Let \( 0 \leq f \in L^1(Q_0) \) satisfy \( \int_{Q_0} f \geq \Lambda > 0 \). Then \( Q_0 \) can be divided into disjoint sub-cubes \( Q \)'s such that:

- For all \( Q \), \( \int_Q f < \Lambda \).
- For all \( \alpha > 0 \) and integer \( k \geq 2 \)
  \[ \sum_{Q} \frac{1}{|Q|^\alpha} \left[ \left( \int_Q f \right)^2 - \frac{\Lambda}{a} \int_Q f \right] \geq 0, \tag{37} \]
  where
  \[ a := \frac{k^d}{2} \left( 1 + \sqrt{1 + \frac{1 - \frac{k-d}{d}}{k^d - 1}} \right). \]
- If \( k = 3 \), then the center of \( Q_0 \) coincides with the center of exactly one sub-cube \( Q \), and the distance from every other sub-cube \( Q \) to the center of \( Q_0 \) is not smaller than \( |Q|^{1/d}/2 \).

Note that the simplest choice is \( k = 2 \) and it is sufficient for the proof of the Lieb-Thirring inequality (8). However the case \( k = 3 \) will be more useful for the proof of the Hardy-Lieb-Thirring inequality (11) in Section 4.
Proof. First, we divide $Q_0$ into $k^d$ disjoint sub-cubes with $1/k$ of the original side length. For every sub-cube, if the integral of $f$ over it is less than $\Lambda$, then we will not divide it further; otherwise we divide this sub-cube into $k^d$ disjoint smaller cubes with $1/k$ of the side length, and then iterate the process. Since $f$ is integrable, the procedure must stop after finitely many steps and we obtain a division of $Q_0$ into finitely many sub-cubes $Q$’s.

It is obvious that for every sub-cube $Q$ one has $\int_Q f < \Lambda$ and $|Q| = k^{-\ell(Q)d}|Q_0|$ for some level $\ell(Q) \in \{0, 1, 2, \ldots\}$. By viewing the sub-cubes as the leaves of a full $k^d$-ary tree corresponding to the above division (cf. [33, Fig. 3]), we can distribute all sub-cubes into disjoint groups $\{F_i\}$ such that in each group $F_i$:

1. There are exactly $k^d$ smallest sub-cubes within $F_i$.
2. The integral of $f$ over the union of these $k^d$ smallest sub-cubes is greater than $\Lambda$.
3. There are at most $(k^d - 1)$ sub-cubes of every other volume.

Now we consider each group $F_i$. Let $m_i = \inf_{Q \in F_i} |Q|$ denote the minimal volume occurring in the group. By the Cauchy-Schwarz inequality we have

$$\sum_{Q \in F_i, |Q| = m_i} \frac{1}{|Q|^\alpha} \left[ \left( \int_Q f \right)^2 - \frac{\Lambda}{a} \int_Q f \right] \geq \frac{1}{m_i^\alpha} \left( \frac{\Lambda^2}{k^d} - \frac{\Lambda^2}{a} \right).$$  \hfill (38)

Here in the last inequality we have used the lower bound

$$\sum_{Q \in F_i, |Q| = m_i} \int_Q f \geq \Lambda > \frac{k^d \Lambda}{2a}$$

and that the function $t \mapsto t^2/k^d - (\Lambda/a)t$ is increasing when $t \geq k^d \Lambda/(2a)$. On the other hand, using the obvious lower bound

$$\left( \int_Q f \right)^2 - \frac{\Lambda}{a} \int_Q f \geq -\frac{\Lambda^2}{4a^2},$$

we find that

$$\sum_{Q \in F_i, |Q| > m_i} \frac{1}{|Q|^\alpha} \left[ \left( \int_Q f \right)^2 - \frac{\Lambda}{a} \int_Q f \right] \geq -\frac{\Lambda^2}{4a^2} \sum_{Q \in F_i, |Q| > m_i} \frac{1}{|Q|^\alpha} \sum_{j \geq 1} \frac{k^d - 1}{(k^d m_i)^\alpha} = -\frac{\Lambda^2}{4a^2} \frac{k^d - 1}{(k^d - 1)m_i^\alpha}. \hfill (39)$$

Here in the second inequality we have used the fact that in $F_i$, each sub-cube has volume $k^d m_i$ for some $j \in \{0, 1, 2, \ldots\}$ and there are at most $(k^d - 1)$ sub-cubes of every volume larger than $m_i$. Adding (38) and (39), we find
that
\[
\sum_{Q \in \mathcal{F}_i} \frac{1}{|Q|^{2s/d}} \left[ \left( \int_Q f \right)^2 - \frac{\Lambda}{a} \int_Q f \right] \geq \frac{\Lambda^2}{m_i^2} \left( \frac{1}{k^d} - \frac{1}{a} - \frac{k^d - 1}{4a^2(k^d - 1)} \right) = 0,
\]
where the last identity follows from the choice of \(a\). Since the latter inequality holds true for every group \(\mathcal{F}_i\), the conclusion follows immediately.

For \(k = 3\) (or any odd integer) there is at each level in the above division exactly one cube \(Q\) with its center at the center of \(Q_0\), and the statement follows by iteration. \(\square\)

3.4. **Proof of the Lieb-Thirring inequality.** Now we are able to give a proof of the Lieb-Thirring inequality (8).

**Proof of Theorem 1.** By a standard approximation argument we can assume that \(\rho \Psi\) is supported in a finite cube \(Q_0 \subset \mathbb{R}^d\). For every \(\Lambda \leq \int_{\mathbb{R}^d} \rho \Psi = N\), by applying the Covering Lemma 9 with \(f = \rho \Psi\), \(k = 2\) and \(\alpha = 2s/d\), we can divide \(Q_0\) into disjoint sub-cubes \(Q\)'s such that
\[
\int_Q \rho \Psi \leq \Lambda
\]
with
\[
a := \frac{2^d}{2} \left( 1 + \left[ 1 + \frac{1 - 2^{-d}}{2^{d\alpha - 1}} \right] \right).
\]

Next, from Lemma 6, Lemma 8 and (40), it follows that
\[
\left\langle \Psi, \left( \sum_{i=1}^N (-\Delta_i)^s + \sum_{1 \leq i < j \leq N} \frac{1}{|x_i - x_j|^{2s}} \right) \Psi \right\rangle \geq \sum_Q \frac{1}{|Q|^{2s/d}} \left[ \left( \int_Q \rho \Psi \right)^2 - \frac{\Lambda}{a} \int_Q \rho \Psi \right] \geq 0,
\]
with
\[
a := \frac{2^d}{2} \left( 1 + \left[ 1 + \frac{1 - 2^{-d}}{2^{d\alpha - 1}} \right] \right).
\]

for every \(0 < \Lambda \leq N\) and for some constant \(C > 0\) depending only on \(d \geq 1\) and \(s > 0\). Here in the last inequality in (41) we have used \(\int_Q \rho \Psi \leq \Lambda\) for all cubes \(Q\)’s.

Finally, using (41) for \(\Lambda = (2^d C + 1)a =: \Lambda_0\) if \(N > \Lambda_0\), and using (36) if \(N \leq \Lambda_0\), we find that
\[
\left\langle \Psi, \left( \sum_{i=1}^N (-\Delta_i)^s + \sum_{1 \leq i < j \leq N} \frac{1}{|x_i - x_j|^{2s}} \right) \Psi \right\rangle \geq C \int_{\mathbb{R}^d} \rho \Psi^{1+2s/d}
\]
for a constant \(C > 0\) depending only on \(d\) and \(s\). The proof is complete. \(\square\)
Remark 6 (Explicit constant). It is possible to derive an explicit constant $C$ in (8). Let us consider for example the case $s = 1$ and $d = 3$. By the Hoffmann-Ostenhof inequality (see Lemma 14) and Sobolev’s inequality,

$$
\left\langle \Psi, \sum_{i=1}^{N} -\Delta_{i} \Psi \right\rangle \geq \left( \sqrt{\rho_{\Psi}}, (-\Delta)^{1/2} \sqrt{\rho_{\Psi}} \right) \geq C_{S} \left( \int_{\mathbb{R}^{3}} \rho_{\Psi}^{3} \right)^{1/3} \geq C_{S} \frac{\int_{\mathbb{R}^{3}} \rho_{\Psi}^{5/3}}{\left( \int_{\mathbb{R}^{3}} \rho_{\Psi} \right)^{2/3}}.
$$

Moreover, combining the Hoffmann-Ostenhof inequality and the Poincaré-Sobolev inequality

$$
\| \nabla u \|_{L^2(Q)}^{2} \geq C_{P} \left\| u - \frac{1}{|Q|} \int_{Q} u \right\|_{L^{6}(Q)}^{2}
$$

as in [16], we get

$$
\left\langle \Psi, \sum_{i=1}^{N} -\Delta_{i} \Psi \right\rangle \geq \left( \sqrt{\rho_{\Psi}}, (-\Delta)^{1/2} \sqrt{\rho_{\Psi}} \right) \geq \sum_{Q} \| \nabla \sqrt{\rho_{\Psi}} \|_{L^2(Q)}^{2}
$$

$$
\geq C_{P} \sum_{Q} \left\| \sqrt{\rho_{\Psi}} - \frac{1}{|Q|} \int_{Q} \sqrt{\rho_{\Psi}} \right\|_{L^6(Q)}^{2}
$$

$$
\geq \sum_{Q} \left[ C_{P}(1-\varepsilon) \left( \int_{Q} \rho_{\Psi}^{5/3} \right)^{-2/3} - C_{P}(\varepsilon^{-1} - 1) \frac{1}{|Q|^{2/3}} \int_{Q} \rho_{\Psi} \right]
$$

for any $\varepsilon \in (0, 1)$. From these kinetic lower bounds, following the above proof of Theorem 1, we find that (8) holds true with

$$
C = \min\{ (1-\varepsilon) C_{P}, C_{S} \} \Lambda_{0}^{2/3}, \quad \Lambda_{0} = a(1+6C_{P}(\varepsilon^{-1} - 1)).
$$

Here we can take

$$
C_{S} = \frac{3}{4}(2\pi^{2})^{2/3}, \quad C_{P} = \frac{27}{16(1+3^{2/3})^{2}(2\pi)^{4/3}} \quad \text{and} \quad a = 4 + \sqrt{\frac{186}{3}}
$$

(the sharp value of $C_{S}$ can be inferred from [2, 41] and the value of $C_{P}$ is obtained by following [16, Lemma 1] but it may not be optimal). Then optimizing over $0 < \varepsilon < 1$ shows that (8) holds true with

$$
C = 0.002384.
$$

Although this explicit constant is far from optimal, it is already a significant improvement over [32].

3.5. A note about fermions and weaker exclusion. In this subsection we explain how to adapt our above proof to show the fermionic inequality (9)

$$
\left\langle \Psi, \sum_{i=1}^{N} (-\Delta_{i})^{s} \Psi \right\rangle \geq C \int_{\mathbb{R}^{d}} \rho_{\Psi}(x)^{1+2s/d} \, dx
$$

for all $d \geq 1$ and $s > 0$, where the wave function $\Psi$ satisfies the antisymmetry (2). In this case the kinetic energy not only contributes to a local uncertainty principle as in Lemma 8 but also to a local exclusion principle of the following weaker form:
Lemma 10 (Local exclusion for fermions). For any \( d \geq 1, s > 0 \) there is a constant \( C > 0 \) depending only on \( d \) and \( s \) such that for all \( N \in \mathbb{N} \), for every normalized function \( \Psi \in H^s((\mathbb{R}^d)^N) \) satisfying the anti-symmetry \( (2) \), and for an arbitrary collection of disjoint cubes \( Q \)'s in \( \mathbb{R}^d \),

\[
\left\langle \Psi, \sum_{j=1}^{N} (-\Delta)^s \Psi \right\rangle \geq \sum_{Q} \frac{C}{|Q|^{2s/d}} \left[ \int_{Q} \rho_{\Psi}(x) \, dx - q \right]_+, \tag{42}
\]

where \( q := \# \{ \text{multi-indices } \alpha : 0 \leq |\alpha| < s \} \).

Proof. First, consider one-body functions \( u \in H^s(Q) \) where \( s = m + \sigma \), \( m \in \mathbb{N} \), \( \sigma \in [0, 1) \). In the case that \( 0 < \sigma < 1 \), we have the fractional Poincaré inequality (see, e.g., [21] Lemma 2.2)

\[
\|u\|_{H^s(Q)}^2 \geq \frac{C}{|Q|^{2s/d}} \sum_{|\alpha|=m} \left\| D^\alpha u - \frac{1}{|Q|} \int_Q D^\alpha u \right\|_{L^2(Q)}^2,
\]

while for \( |\alpha| = m \) we have (by iteration of Poincaré’s inequality)

\[
\|D^\alpha u\|_{L^2(Q)}^2 \geq \frac{C}{|Q|^{2m/d}}\|u\|_{L^2(Q)}^2, \text{ if } \int_Q D^\beta u = 0 \text{ for all } 0 \leq |\beta| < m.
\]

Note that \( \int_Q D^\alpha u = \langle 1, T_\alpha u \rangle = \langle T_\alpha^* 1, u \rangle \), where the operator \( u \mapsto T_\alpha(u) := D^\alpha u, |\alpha| \leq m \), is relatively bounded w.r.t. the form domain \( H^s(Q) \). Hence we can treat these orthogonality conditions by considering the \( q \)-dimensional subspace \( V_s := \text{Span}\{T_\alpha^* 1 : 0 \leq |\alpha| < s \} \). On \( H^s(Q) \cap V_s^\perp \) we then have

\[
\|u\|_{H^s(Q)}^2 \geq \frac{C}{|Q|^{2s/d}}\|u\|_{L^2(Q)}^2,
\]

and in general, by taking out the projection onto \( V_s \),

\[
(-\Delta)^s|_{H^s(Q)} \geq \frac{C}{|Q|^{2s/d}}(\mathbb{I} - P_{V_s}).
\]

Now we proceed as in Lemma 8 although because of the anti-symmetry of \( \Psi \), the one-body functions \( u_n \) all have norm less than unity (again, see e.g. [27]). We then obtain

\[
\left\langle \Psi, \sum_{i=1}^{N} (-\Delta)^s \Psi \right\rangle \geq \sum_{n \geq 1} \sum_{Q} \|u_n\|_{H^s(Q)}^2 \geq \sum_{Q} \frac{C}{|Q|^{2s/d}} \left[ \sum_{n \geq 1} \|u_n\|_{L^2(Q)}^2 - q \right]_+,
\]

which proves the lemma. \( \square \)

We note that the Covering Lemma 4 can be also adapted to apply to the weaker form of the exclusion principle. This could be useful not only for fermions but also in situations when other types of interactions are present (cf. [33] [16] [34] [32]).

Lemma 11 (Covering lemma with weaker exclusion). Let \( Q_0 \) be a cube in \( \mathbb{R}^d \) and let \( 0 \leq f \in L^1(Q_0) \) satisfy \( \int_{Q_0} f \geq \Lambda > 0 \). Then \( Q_0 \) can be divided into disjoint sub-cubes \( Q \)'s such that

- For all \( Q \),

\[
\int_Q f < \Lambda.
\]
• For all $\alpha > 0$, $q \geq 0$ and integer $k \geq 2$,
\[
\sum_{Q} \frac{1}{|Q|^\alpha} \left( \left[ \int_{Q} f - q \right]_+ - b \int_{Q} f \right) \geq 0,
\]
where
\[
b := \left( 1 - \frac{q k^d}{\Lambda} \right) \frac{k^{d\alpha} - 1}{k^{d\alpha} + k^d - 2}.
\]

• If $k = 3$, then the center of $Q_0$ coincides with exactly one sub-cube $Q_0$, and the distance from every other sub-cube $Q$ to the center of $Q_0$ is not smaller than $|Q|^{1/d}/2$.

Proof. We proceed with the same division procedure as in the proof of Lemma 9. Instead of (38) we have
\[
\sum_{Q \in \mathcal{F}, |Q| = m_i} \frac{1}{|Q|^\alpha} \left( \left[ \int_{Q} f - q \right]_+ - b \int_{Q} f \right) \geq \frac{1}{m_i^\alpha} \left( (1 - b)\Lambda - q k^d \right),
\]
and instead of (39) we have
\[
\sum_{Q \in \mathcal{F}, |Q| > m_i} \frac{1}{|Q|^\alpha} \left( \left[ \int_{Q} f - q \right]_+ - b \int_{Q} f \right) \geq -b \Lambda \sum_{Q \in \mathcal{F}, |Q| > m_i} \frac{1}{|Q|^\alpha} \geq -b \Lambda \sum_{j \geq 1} \frac{k^d - 1}{(k^d m_i)^\alpha} = -b \Lambda \frac{k^d - 1}{m_i^\alpha k^{d\alpha} - 1}.
\]
Hence,
\[
\sum_{Q \in \mathcal{F}, |Q| > m_i} \frac{1}{|Q|^\alpha} \left( \left[ \int_{Q} f - q \right]_+ - b \int_{Q} f \right) \geq \frac{1}{m_i^\alpha} \left( \Lambda - q k^d - b \Lambda \left( 1 + \frac{k^d - 1}{k^{d\alpha} - 1} \right) \right),
\]
from which the lemma follows. \hfill \Box

From the local uncertainty in Lemma 8, the local exclusion in Lemma 10 and the Covering Lemma 11, one can prove the fermionic Lieb-Thirring inequality (9) by proceeding similarly as in the proof of Theorem 1. The details are left to the reader.

Remark 7. From Lemma 6 and the elementary inequality $(a^2 - a)_+ \geq (a - 1)_+, a \geq 0$, we obtain the following analogue of (42) for pair-interactions:
\[
\left\langle \Psi, \sum_{1 \leq i < j \leq N} \frac{1}{|x_i - x_j|^{2s}} \Psi \right\rangle \geq \sum_{Q} \frac{1}{2d^s|Q|^{2s/d}} \left[ \int_{Q} \rho \Psi(x) dx - 1 \right]_+
\]
for every normalized function $\Psi \in L^2((\mathbb{R}^d)^N)$. In our proofs of the Lieb-Thirring inequality (8) and the Hardy-Lieb-Thirring inequality (11) presented later, we can certainly use (46) instead of (22) (we then obtain similar inequalities but with worse constants).

4. HARDY-LIEB-THIRRING INEQUALITY

In this section we prove Theorem 2. We will need to strengthen the local uncertainty principle in Section 3 to account for the Hardy term, and to do this we also need a localization method for fractional kinetic energy.
4.1. Local uncertainty for centered cubes. The following local uncertainty principle is crucial for our proof.

**Lemma 12** (Local uncertainty for centered cubes). For every cube $Q \subset \mathbb{R}^d$ centered at 0, we have

$$
\|u\|^2_{H^s(\Omega)} = C_{d,s} \int_Q \frac{|u(x)|^2}{|x|^{2s}} \, dx \geq \frac{1}{C} \int_Q |u|^{2(1 + 2s/d)} - \frac{C}{|Q|^{2s/d}} \int_Q |u|^2 \quad (47)
$$

for a constant $C > 0$ depending only on $d \geq 1$ and $s > 0$.

Note that this local uncertainty principle is significantly stronger than the one in Lemma 7 because the left side of (47) can even be negative. Our strategy is to replace $u$ by $\chi u$ where $\chi$ is a smooth function supported in a neighborhood of the origin, and then apply the Hardy inequality with remainder term for $\chi u \in H^s(\mathbb{R}^d)$. To implement the localization procedure, we also need the following lemma which controls the error terms.

**Lemma 13** (A fractional IMS localization formula). Let $\Omega$ be a bounded open domain in $\mathbb{R}^d$ with $d \geq 1$. Let $\chi, \eta : \mathbb{R}^d \to [0, 1]$ be two smooth functions such that $\chi(x)^2 + \eta(x)^2 \equiv 1$ and $\chi$ is supported in a compact subset of $\Omega$. Then for every $s > 0$, there exists $t \in (0, s)$ and a constant $C > 0$ such that for every $u \in H^t(\Omega)$,

$$
\left| \|u\|^2_{H^t(\Omega)} - \|\chi u\|^2_{H^t(\Omega)} - \|\eta u\|^2_{H^t(\Omega)} \right| \leq C \left( \|\chi u\|^2_{H^t(\Omega)} + \|\eta u\|^2_{H^t(\Omega)} \right). \quad (48)
$$

**Remark 8.** It will be clear from the proof of Lemma 13 (provided below) that if $s \in \mathbb{N}$ then $t = s - 1$, and if $s = m + \sigma$ with $m \in \{0, 1, 2, \ldots\}$ and $0 < \sigma < 1$ then we can take $t = s - \varepsilon$ for any $0 < \varepsilon < \min\{\sigma, 1 - \sigma\}$.

Note that when $s = 1$, thanks to the IMS formula (cf. [6] Theorem 3.2)]

$$
|\nabla u|^2 = |\nabla (\chi u)|^2 + |\nabla (\eta u)|^2 - (|\nabla \chi|^2 + |\nabla \eta|^2)|u|^2,
$$

we obtain the estimate (48), with $t = 0$, immediately:

$$
\left| \|u\|^2_{H^1(\Omega)} - \|\chi u\|^2_{H^1(\Omega)} - \|\eta u\|^2_{H^1(\Omega)} \right| = \int_\Omega (|\nabla \chi|^2 + |\nabla \eta|^2)|u|^2 \leq C \int_\Omega |u|^2.
$$

When $0 < s < 1$, the estimate

$$
\left| \|u\|^2_{H^s(\Omega)} - \|\chi u\|^2_{H^s(\Omega)} - \|\eta u\|^2_{H^s(\Omega)} \right| \leq C \int_\Omega |u|^2
$$

follows from the representation (25)

$$
\|u\|^2_{H^s(\Omega)} = c_{d,s} \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{d+2s}} \, dx \, dy
$$

and the elementary identity (which goes back to a suggestion of Michael Loss and was used in [31])

$$
|\chi(x)u(x) - \chi(y)u(y)|^2 + |\eta(x)u(x) - \eta(y)u(y)|^2 - |u(x) - u(y)|^2
$$

$$
= \left[ (\chi(x) - \chi(y))^2 + (\eta(x) - \eta(y))^2 \right] |\Re (u(x)u(y))| \quad (49)
$$

However, the proof of (48) for $s > 1$ is rather involved and we defer it to the next subsection. In the following, let us provide a proof of Lemma 12 using Lemma 13.
Proof of Lemma 7. Since the inequality (47) that we wish to prove is dilation invariant, we can assume without loss of generality that $|Q| = 1$. Let $\chi, \eta : \mathbb{R}^d \to [0, 1]$ be two smooth functions such that $\chi^2(x) + \eta^2(x) \equiv 1$, $\chi(x) \equiv 1$ when $|x| \leq 1/4$ and $\chi(x) \equiv 0$ when $|x| \geq 1/3$. By using $\eta^2|u|^2/|x|^{2s} \leq 3^{2s}\eta^2|u|^2$ and Lemma 13 we obtain for some $t \in [0, s)$

$$\|\eta u\|_{H^s(Q)}^2 - C_{d,s} \int_Q \left| \frac{|u|^2}{|x|^{2s}} \right| \geq \|\chi u\|_{H^s(Q)}^2 - C_{d,s} \int_Q \left| \frac{|\chi|^2}{|x|^{2s}} \right|$$

$$+ \|\eta u\|_{H^s(Q)}^2 - C_1 \|\chi u\|_{H^s(Q)}^2 - C_1 \|\eta u\|_{H^s(Q)}^2$$

(50)

for some constant $C_1 > 0$ depending only on $d, s, t$ (and $\chi$).

Since $\chi$ has compact support, $\chi u$ can be considered as a function in $H^s(\mathbb{R}^d)$. Therefore, by the Gagliardo-Nirenberg type inequality (13) (there taking $t = s/(1 + 2s/d)$),

$$\frac{1}{2} \left( \|\chi u\|_{H^s(\mathbb{R}^d)}^2 - C_{d,s} \int_{\mathbb{R}^d} \left| \frac{|\chi|^2}{|x|^{2s}} \right| \right) \geq \frac{1}{C} \int_{\mathbb{R}^d} |\chi|^2|u|^{2(1+2s/d)}2^{s/d} \cdot \frac{1}{2s/d}$$. (51)

Moreover, by using the improved Hardy inequality (13) and the norm-equivalence (29), we find

$$\left( \|\chi u\|_{H^s(\mathbb{R}^d)}^2 - C_{d,s} \int_{\mathbb{R}^d} \left| \frac{|\chi|^2}{|x|^{2s}} \right|^t \|\chi u\|_{L^2(\mathbb{R}^d)}^{2(1-t/s)} \right)$$

$$\geq \frac{1}{C} \|\chi u\|_{H^s(\mathbb{R}^d)}^2 - C_1 \|\chi u\|_{H^s(\mathbb{R}^d)}^2 - C \|\chi u\|_{L^2(\mathbb{R}^d)}^2$$

(52)

which by Young’s inequality implies that

$$\frac{1}{2} \left( \|\chi u\|_{H^s(\mathbb{R}^d)}^2 - C_{d,s} \int_{\mathbb{R}^d} \left| \frac{|\chi|^2}{|x|^{2s}} \right| \right) \geq C_1 \|\chi u\|_{H^s(\mathbb{R}^d)}^2 - C \|\chi u\|_{L^2(\mathbb{R}^d)}^2,$$ (53)

for exactly $C_1$ in (50) and a (large) constant $C > 0$ depending only on $d, s, t$.

For the function $\eta u$, by the local uncertainty in Lemma 7,

$$\frac{1}{2} \|\eta u\|_{H^s(Q)}^2 \geq \frac{\int_Q |\eta u|^{2(1+2s/d)}}{(\int_Q |\eta u|^2)^{2s/d}} - C \|\eta u\|_{L^2(Q)}^2.$$ (54)

Moreover, by using the extension and interpolation arguments as in the proof of Lemma 7 we obtain

$$\|\eta u\|_{H^s(Q)}^{t/s} \|\eta u\|_{L^2(Q)}^{1-t/s} \geq C \|\eta u\|_{H^s(Q)}^2,$$

which, together with the norm-equivalence (29), leads again to

$$\frac{1}{2} \|\eta u\|_{H^s(Q)}^2 \geq C_1 \|\eta u\|_{H^s(Q)}^2 - C \|\eta u\|_{L^2(Q)}^2$$

(54)

for a (large) constant $C > 0$ depending only on $d, s, t$.

Summing all inequalities (50), (51), (52), (53), and (54), and using $\|\chi u\|_{L^2(Q)}^2 + \|\eta u\|_{L^2(Q)}^2 = \|u\|_{L^2(Q)}^2$ and estimating the denominators, we arrive at

$$\|u\|_{H^s(Q)}^2 - C_{d,s} \int_Q \left| \frac{|u|^2}{|x|^{2s}} \right| \geq \frac{1}{C} \int_Q \left( \|\chi|^2_{2(1+2s/d)} + |\eta u|_{2(1+2s/d)}^2 \right) \left( \int_Q |u|^2 \right)^{2s/d} - C \|u\|_{L^2(Q)}^2.$$
for a (large) constant \( C > 0 \) depending only on \( d, s \). The final conclusion then follows from the elementary inequality

\[
\chi^{2p} + \eta^{2p} \geq 2 \left( \frac{\chi^2 + \eta^2}{2} \right)^p = 2^{1-p}, \quad p = 1 + \frac{2s}{d} > 1.
\]

\( \square \)

**4.2. Proof of the fractional IMS localization formula.**

**Proof of Lemma 13**. **Step 1.** We start with the case \( s = m \in \mathbb{N} \). Recall that in our conventions

\[
\|u\|_{H^m(\Omega)}^2 = \sum_{|\alpha|=m} \frac{m!}{\alpha!} \int_{\Omega} |D^\alpha u|^2.
\]

Let us consider an arbitrary multi-index \( \alpha \) with \( |\alpha| = m \). Using

\[
D^\alpha (\chi u) = \chi D^\alpha u + \sum_{\beta < \alpha} \frac{\alpha!}{\beta!(\alpha - \beta)!} D^{\alpha - \beta} \chi D^\beta u
\]

and a similar formula for \( D^\alpha (\eta u) \), we find that

\[
\int |D^\alpha (\chi u)|^2 + |D^\alpha (\eta u)|^2 = (\chi^2 + \eta^2)|D^\alpha u|^2 +
\]

\[
+ \sum_{\beta < \alpha} \frac{\alpha!}{\beta!(\alpha - \beta)!} D^{\alpha - \beta} \chi D^\beta u \bigg|_{\Omega}^2 + \sum_{\beta < \alpha} \frac{\alpha!}{\beta!(\alpha - \beta)!} D^{\alpha - \beta} \eta D^\beta u \bigg|_{\Omega}^2 + 2 \sum_{\beta < \alpha} \frac{\alpha!}{\beta!(\alpha - \beta)!} (\chi D^{\alpha - \beta} \chi + \eta D^{\alpha - \beta} \eta) D^\alpha \pi D^\beta u
\]

(56)

Here for two multi-indices \( \alpha = (\alpha_1, ..., \alpha_d) \) and \( \beta = (\beta_1, ..., \beta_d) \), the notation \( \beta < \alpha \) means \( \beta \leq \alpha \), namely \( \beta_j \leq \alpha_j \) for all \( 1 \leq j \leq d \), and \( \beta \neq \alpha \). The first term of the right side of (56) is nothing but \( |D^\alpha u|^2 \) since \( \chi^2 + \eta^2 = 1 \). The next two terms can be bounded using the Cauchy-Schwarz inequality

\[
\left| \sum_{\beta < \alpha} \frac{\alpha!}{\beta!(\alpha - \beta)!} D^{\alpha - \beta} \chi D^\beta u \right|^2 + \left| \sum_{\beta < \alpha} \frac{\alpha!}{\beta!(\alpha - \beta)!} D^{\alpha - \beta} \eta D^\beta u \right|^2 \leq C \sum_{\beta < \alpha} |D^\beta u|^2.
\]

Therefore, by integrating (56) and using the triangle inequality we get

\[
\left| \|\chi u\|_{H^m(\Omega)}^2 + \|\eta u\|_{H^m(\Omega)}^2 - \|u\|_{H^m(\Omega)}^2 \right| \leq C \|u\|_{H^{m-1}(\Omega)}^2 +
\]

\[
+ 2 \sum_{|\alpha|=m} \sum_{\beta < \alpha} \frac{\alpha!}{\beta!(\alpha - \beta)!} \left| \int_{\Omega \times \Omega} (\chi D^{\alpha - \beta} \chi + \eta D^{\alpha - \beta} \eta) D^\alpha \pi D^\beta u \right|
\]

Now we estimate the last term of (57). For every \( \alpha \) with \( |\alpha|=m \), we can find \( 0 \leq \alpha' < \alpha \) and \( 1 \leq j \leq d \) such that \( D^{\alpha} = \partial_j D^{\alpha'} \). Since \( \chi D^{\alpha - \beta} \chi + \eta D^{\alpha - \beta} \eta \) has support in a compact subset of \( \Omega \), by using integration by parts with respect to the \( j \)-th coordinate we find that

\[
\int_{\Omega} (\chi D^{\alpha - \beta} \chi + \eta D^{\alpha - \beta} \eta) D^\alpha \pi D^\beta u = - \int_{\Omega} D^{\alpha'} \pi \partial_j \left( (\chi D^{\alpha - \beta} \chi + \eta D^{\alpha - \beta} \eta) D^\beta u \right)
\]

\[
= - \int_{\Omega} D^{\alpha'} \pi \left( \partial_j (\chi D^{\alpha - \beta} \chi + \eta D^{\alpha - \beta} \eta) D^\beta u + (\chi D^{\alpha - \beta} \chi + \eta D^{\alpha - \beta} \eta) \partial_j D^\beta u \right).
\]
Therefore, when $|\beta| \leq m - 2$, by the Cauchy-Schwarz inequality we can estimate
\[
\left| \int_{\Omega} (\chi D^{\alpha-\beta} x + \eta D^{\alpha-\beta} \eta) D^\alpha \nabla D^\beta u \right| \leq C\|u\|_{H^{m-1}(\Omega)}^2
\]
On the other hand, if $\beta < \alpha$ and $|\beta| = m - 1 = |\alpha| - 1$, then $D^{\alpha-\beta} = \partial_k$ for some $1 \leq k \leq d$ and hence
\[
\chi D^{\alpha-\beta} x + \eta D^{\alpha-\beta} \eta = \frac{1}{2} \partial_k (\chi^2 + \eta^2) = 0.
\]
In summary, (57) can be simplified to
\[
\sum_{\beta < \alpha} m! \frac{\alpha!}{\beta!(\alpha - \beta)!} \left( D^{\alpha-\beta} x D^{\beta} y(x) - D^{\alpha-\beta} y D^{\beta} x(y) \right) |x - y|^{-\sigma} dxdy.
\]
Step 2. Now we consider the case when $s = m + \sigma$ with $m \in \mathbb{N}$ and $0 < \sigma < 1$. Let us start by considering
\[
\|u\|_{H^s(\Omega)}^2 = C_{d, \sigma} \sum_{|\alpha| = m} \frac{m!}{\alpha!} \int_{\Omega \times \Omega} \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x - y|^{d+2\sigma}} dxdy.
\]
We will always denote by $\alpha$ an arbitrary multi-index with $|\alpha| = m$. Using (58) and the identity $|a + b|^2 = |a|^2 + 2\Re((a + b)b) - |b|^2$ (with complex numbers $a$ and $b$), we have
\[
|D^\alpha (\chi u)(x) - D^\alpha (\chi u)(y)|^2
= |\chi(x) D^\alpha u(x) - \chi(y) D^\alpha u(y) + \sum_{\beta < \alpha} \frac{\alpha!}{\beta!(\alpha - \beta)!} \left( D^{\alpha-\beta} \chi D^{\beta} u(x) - D^{\alpha-\beta} \chi D^{\beta} u(y) \right) |^2
= |\chi(x) D^\alpha u(x) - \chi(y) D^\alpha u(y)|^2
- \sum_{\beta < \alpha} \frac{\alpha!}{\beta!(\alpha - \beta)!} \left( D^{\alpha-\beta} \chi D^{\beta} u(x) - D^{\alpha-\beta} \chi D^{\beta} u(y) \right) |^2
+ 2\Re \sum_{\beta < \alpha} \frac{\alpha!}{\beta!(\alpha - \beta)!} \left( D^{\alpha} \chi D^{\beta} u(x) - D^{\alpha} \chi D^{\beta} u(y) \right) \times
\left( D^{\alpha-\beta} \chi D^{\beta} u(x) - D^{\alpha-\beta} \chi D^{\beta} u(y) \right).
\]
Now we estimate the right side of (59) using the Cauchy-Schwarz inequality. We have
\[
|D^{\alpha-\beta} \chi D^{\beta} u(x) - D^{\alpha-\beta} \chi D^{\beta} u(y)|^2
= |D^{\alpha-\beta} \chi D^{\beta} u(x) - D^{\alpha-\beta} \chi D^{\beta} u(y) + (D^{\alpha-\beta} \chi D^{\beta} u(x) - D^{\alpha-\beta} \chi D^{\beta} u(y))| \times
\]
Integrating this latter inequality against the weight \( |x - y|^{-d + 2\sigma} \) leads to

\[
\iint_{\Omega \times \Omega} \frac{\left| D^\alpha(\chi u)(x) - D^\alpha(\chi u)(y) \right|^2 - |\chi(x)D^\alpha u(x) - \chi(y)D^\alpha u(y)|^2}{|x - y|^{d + 2\sigma}} \, dx \, dy 
\leq C \left\| D^\alpha(\chi u) \right\|^2_{H^\sigma(\Omega)} + C \| u \|_{H^m(\Omega)}^2,
\]

where we also estimated difference quotients involving \( D^\beta u \) in terms of \( D^\alpha u \), \( |\alpha'| = m \). Combining with a similar inequality for \( D^\alpha(\eta u) \), we find that

\[
\iint_{\Omega \times \Omega} \frac{\left| D^\alpha(\chi u)(x) - D^\alpha(\chi u)(y) \right|^2 - |\chi(x)D^\alpha u(x) - \chi(y)D^\alpha u(y)|^2}{|x - y|^{d + 2\sigma}} \, dx \, dy 
+ \iint_{\Omega \times \Omega} \frac{\left| D^\alpha(\eta u)(x) - D^\alpha(\eta u)(y) \right|^2 - |\eta(x)D^\alpha u(x) - \eta(y)D^\alpha u(y)|^2}{|x - y|^{d + 2\sigma}} \, dx \, dy 
\leq C \left\| D^\alpha(\chi u) \right\|^2_{H^\sigma(\Omega)} + C \left\| D^\alpha(\eta u) \right\|^2_{H^\sigma(\Omega)} + C \| u \|_{H^m(\Omega)}^2.
\]
\[ \leq C|x - y|^2 \left( |D^\alpha u(x)|^2 + |D^\alpha u(y)|^2 \right). \]

Integrating the latter inequality against the weight \(|x - y|^{-(d+2\sigma)}\) we get
\[
\left| \int \int_{\Omega \times \Omega} \frac{\left| \chi(x)D^\alpha u(x) - \chi(y)D^\alpha u(y) \right|^2 + |\eta(x)D^\alpha u(x) - \eta(y)D^\alpha u(y)|^2}{|x - y|^{d+2\sigma}} \, dx \, dy \right| 
- \int \int_{\Omega \times \Omega} \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x - y|^{d+2\sigma}} \, dx \, dy \right| \leq C \int_{\Omega} |D^\alpha u|^2. \tag{61}
\]

From (60)-(61) and the triangle inequality, it follows that
\[
\int \int_{\Omega \times \Omega} \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x - y|^{d+2\sigma}} \, dx \, dy \leq C \int_{\Omega} |D^\alpha u|^2.
\]

for all \(|\alpha| = m\). By taking the sum over all \(\alpha\)’s with \(|\alpha| = m\), we get
\[
\left| \left| \chi + \eta \right|_H^2 \right|_{H^{s-\sigma}} + \left| \left| \eta \right|_H^2 \right|_{H^{s-\sigma}} - \left| \left| u \right|_H^2 \right|_{H^{s-\sigma}} \leq C \left( \left| \chi \right|_{H^{s-\sigma}}^2 + \left| \eta \right|_{H^{s-\sigma}}^2 + \left| u \right|_{H^{s-\sigma}}^2 \right).
\]

Combining with the estimate
\[
\left| \chi \right|_{H^s}^2 + \left| \eta \right|_{H^s}^2 - \left| u \right|_{H^s}^2 \leq C \left( \left| \chi \right|_{H^s}^2 + \left| \eta \right|_{H^s}^2 \right),
\]
which follows from the integer case in Step 1, we can conclude that
\[
\left| \left| \chi \right|_{H^s}^2 + \left| \eta \right|_{H^s}^2 - \left| u \right|_{H^s}^2 \right| \leq C \left( \left| \chi \right|_{H^s}^2 + \left| \eta \right|_{H^s}^2 \right).
\]

This is the desired inequality. \(\square\)

4.3. Proof of the Hardy-Lieb-Thirring inequality.

Proof of Theorem 2. By a standard approximation argument we can assume that \(\rho_\Psi\) is supported in a finite cube \(Q_0 \subset \mathbb{R}^d\) which centers at 0. Let an arbitrary \(0 < \Lambda \leq N\). By Lemma 9 with \(f = \rho_\Psi\), \(k = 3\) and \(\alpha = 2s/d\), there exists a division of \(Q_0\) into disjoint sub-cubes \(Q\)’s such that \(\int_Q \rho_\Psi \leq \Lambda\) and
\[
\sum Q \frac{1}{|Q|^\alpha} \left[ \left( \int_Q f \right)^2 - \frac{\Lambda}{b} \int_Q f \right] \geq 0, \tag{62}
\]
with
\[
b := \frac{3d}{2} \left( 1 + \sqrt{1 + \frac{1 - 3 - 3d}{3d^2} - 1} \right). \]

Moreover, for every sub-cube \(Q\) we have either that \(Q\) centers at 0 or that \(\inf_{x \in Q} |x| \geq |Q|^{1/d}/2\).
Now we claim that there exists a constant $C_1 > 0$ depending only on $d \geq 1$ and $s > 0$ such that for every sub-cube $Q$ and for every function $u \in H^s(Q)$ we have the uncertainty relation

$$
\|u\|^2_{H^s(Q)} - C_{d,s} \int_Q \frac{|u(x)|^2}{|x|^{2s}} \, dx \geq \frac{1}{C_1} \left( \int_Q |u|^{2(1+2s/d)} \right)^{2s/d} - \frac{C_1}{|Q|^{2s/d}} \int_Q |u|^2. \tag{63}
$$

In fact, if $Q$ centers at 0, then (63) is covered by Lemma 12. On the other hand, if $0 \not\in Q$, then using $|x| \geq |Q|^{1/d}/2$ we have

$$
\int_Q \frac{|u|^2}{|x|^{2s}} \, dx \leq \frac{2^{2s}}{|Q|^{2s/d}} \int_Q |u(x)|^2 \, dx
$$

and (63) is covered by Lemma 7. Using (63) and arguing in exactly the same way as in the proof of Lemma 8, we obtain the many-body estimate

$$
\left\langle \Psi, \sum_{i=1}^N \left( (-\Delta)^s + C_{d,s} |x|^{-2s} \right) \Psi \right\rangle \geq \sum_Q \left[ \frac{1}{C_1} \left( \int_Q \rho_\Psi^{1+2s/d} \right)^{2s/d} - \frac{C_1}{|Q|^{2s/d}} \int_Q \rho_\Psi \right] \\
\geq \frac{1}{C_1 A^{2s/d}} \int_{\mathbb{R}^d} \rho_\Psi^{1+2s/d} \, dx - \sum_Q \frac{C_1}{|Q|^{2s/d}} \int_Q \rho_\Psi \tag{64}
$$

Here in the last inequality of (64) we have used the bound $\int_Q \rho_\Psi \leq A$ for all $Q$. Combining (64), Lemma 6, and (62), we find that

$$
\left\langle \Psi, \left( \sum_{i=1}^N \left( (-\Delta)^s + C_{d,s} |x|^{-2s} \right) + \sum_{1 \leq i < j \leq N} \frac{1}{|x_i - x_j|^{2s}} \right) \Psi \right\rangle \\
\geq \frac{1}{C_1 A^{2s/d}} \int_{\mathbb{R}^d} \rho_\Psi^{1+2s/d} + \sum_Q \frac{1}{2^{2s}|Q|^{2s}} \left( \left( \int_Q \rho_\Psi \right)^2 - (2d^s C_1 + 1) \int_Q \rho_\Psi \right) \\
\geq \frac{1}{C_1 A^{2s/d}} \int_{\mathbb{R}^d} \rho_\Psi^{1+2s/d} + \left( \frac{\Lambda}{b} - 2d^s C_1 - 1 \right) \sum_Q \frac{1}{2^{2s}|Q|^{2s}} \int_Q \rho_\Psi \tag{65}
$$

for all $0 < \Lambda \leq N$.

On the other hand, using the interpolation inequality (16) with

$$
q = \frac{2d}{d - 2t} = 2 \left( 1 + \frac{2s}{d} \right), \quad \text{that is} \quad t = \frac{ds}{d + 2s},
$$

and the same argument of the proof of Lemma 8, we obtain the following strengthened version of (63):

$$
\left\langle \Psi, \sum_{i=1}^N \left( (-\Delta)^s + C_{d,s} |x|^{2s} \right) \Psi \right\rangle \geq C N^{-2s/d} \int_{\mathbb{R}^d} \rho_\Psi^{1+2s/d}, \tag{66}
$$

for a constant $C > 0$ depending only on $d$ and $s$.

Finally, using (65) with $\Lambda = (2d^s C_1 + 1)b =: \Lambda_0$ if $N > \Lambda_0$, and using (66) if $N \leq \Lambda_0$, we find the desired inequality. \hfill \Box
5. Interpolation Inequalities

5.1. Equivalence for the Lieb-Thirring inequality. In this subsection, we provide a proof of Theorem 3, i.e. the equivalence of the Lieb-Thirring inequality (8) and the one-body interpolation inequality (19). The implication of (19) from (8) was already explained in Section 2.3 and it holds for all $0 < s < d/2$. In the following, we show that the interpolation inequality (19) implies the Lieb-Thirring inequality (8) when $0 < s < d/2$ and $s \leq 1$.

We will use the Hoffmann-Ostenhof and Lieb-Oxford inequalities, which reduce the kinetic and interaction energies of a many-body state to those of its density.

**Lemma 14** (Hoffmann-Ostenhof inequality). For every $0 < s \leq 1$ and every normalized function $\Psi \in L^2((\mathbb{R}^d)^N)$, one has

$$\left\langle \Psi, \sum_{i=1}^N (-\Delta_i)^s \Psi \right\rangle \geq \left\langle \sqrt{\rho\Psi}, (-\Delta)^s \sqrt{\rho\Psi} \right\rangle. \quad (67)$$

The non-relativistic case $s = 1$ of (67) was first discovered by M. & T. Hoffmann-Ostenhof [19]. In fact, (67) is equivalent to the one-body inequality

$$\left\langle u, (-\Delta)^s u \right\rangle \geq \left\langle |u|, (-\Delta)^s |u| \right\rangle$$

(cf. the diamagnetic inequality (10)) and it is false when $s > 1$. See e.g. [27, Lemma 8.4] for a proof of (67) and further discussions.

**Lemma 15** (Lieb-Oxford inequality for homogeneous potentials). For every $0 < \lambda < d$ and for every normalized function $\Psi \in L^2((\mathbb{R}^d)^N)$, one has

$$\left\langle \Psi, \sum_{1 \leq i < j \leq N} \frac{1}{|x_i - x_j|^{\lambda}} \Psi \right\rangle \geq \frac{1}{2} \int \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{\rho\Psi(x)\rho\Psi(y)}{|x - y|^{\lambda}} dx dy - C_{LO} \int \rho_{\Psi} \frac{1+\lambda/d}{2s} \quad (68)$$

for a constant $C_{LO} > 0$ depending only on $d$ and $\lambda$.

The case $\lambda = 1$ and $d = 3$ of (68) was first studied in [24, 26]. The case $\lambda = 1$ and $d = 2$ was proved in [28, Lemma 5.3]. A proof of Lemma 15 following the strategy in [28] is provided in Appendix A.

We are now in a position to complete the proof of equivalence.

**Proof of Theorem 3**. We prove that (19) implies (8) when $0 < s < d/2$ and $s \leq 1$. By the Hoffmann-Ostenhof inequality (67) and the Lieb-Oxford inequality (68), one has

$$\left\langle \Psi, \left( \sum_{i=1}^N (-\Delta_i)^s + \sum_{1 \leq i < j \leq N} \frac{1}{|x_i - x_j|^{2s}} \right) \Psi \right\rangle \geq \left\langle \sqrt{\rho\Psi}, (-\Delta)^s \sqrt{\rho\Psi} \right\rangle + \varepsilon \int \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{\rho\Psi(x)\rho\Psi(y)}{|x - y|^{2s}} dx dy - \varepsilon C_{LO} \int \rho_{\Psi} \frac{1+2s/d}{2s} \quad (69)$$

for every $\varepsilon \in (0, 1]$. On the other hand, by using Young’s inequality and the interpolation inequality (19) with $u = \sqrt{\rho\Psi}$, we obtain

$$\left(1 - \frac{2s}{d}\right) \left\langle \sqrt{\rho\Psi}, (-\Delta)^s \sqrt{\rho\Psi} \right\rangle + \frac{2s}{d} \int \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{\rho\Psi(x)\rho\Psi(y)}{|x - y|^{2s}} dx dy \quad (70)$$
\[ \geq \varepsilon^{2s/d} \left( \sqrt{p_{\Psi}} (-\Delta)^s \sqrt{p_{\Psi}} \right)^{1-2s/d} \left( \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{\rho_{\Psi}(x) \rho_{\Psi}(y)}{|x-y|^{2s}} \, dx \, dy \right)^{2s/d} \]

\[ \geq C \varepsilon^{2s/d} \int \rho_{\Psi}^{1+2s/d} \]

for a constant \( C > 0 \) depending only on \( d \) and \( s \). Thus

\[ \left\langle \Psi, \left( \sum_{i=1}^{N} (-\Delta)^s + \sum_{1 \leq i < j \leq N} \frac{1}{|x_i - x_j|^{2s}} \right) \Psi \right\rangle \geq \left( C \varepsilon^{2s/d} - C_{\text{LO}} \varepsilon \right) \int \rho_{\Psi}^{1+2s/d} \]

for all \( \varepsilon \in (0, 1] \). As \( 2s/d < 1 \), we can choose \( \varepsilon > 0 \) small enough such that

\[ C \varepsilon^{2s/d} - C_{\text{LO}} \varepsilon > 0. \]

Then the Lieb-Thirring inequality (8) follows. \( \square \)

5.2. **Isoperimetric inequality with non-local term.** In the following we show how to use our local approach to Lieb-Thirring inequalities to prove the one-body interpolation inequality in Theorem 5.2.

**Proof of Theorem** By a standard approximation argument, we can assume that \( u \) is supported in a finite cube \( Q_0 \subset \mathbb{R}^d \). Let \( f(x) := |u(x)|^{2s} \). For an arbitrary \( 0 < \Lambda \leq \int_{\mathbb{R}^d} f \), we divide \( Q_0 \) into disjoint sub-cubes \( Q \)'s by applying Covering Lemma 9 with \( k = 2 \) and \( \alpha = 2s/d \). Thus we have \( \int_Q f \leq \Lambda \) for all cubes \( Q \)'s and

\[ \sum_Q \frac{1}{|Q|^\alpha} \left[ \left( \int_Q f \right)^2 - \frac{\Lambda}{a} \int_Q f \right] \geq 0, \quad a := \frac{2^d}{2} \left( 1 + \sqrt{1 + \frac{1}{2^{2d} - 1}} \right). \]  

(69)

Similarly to the proof of Lemma 9, by ignoring the interaction energy between different cubes and using \( |x-y| \leq \sqrt{d} |Q|^{1/d} \) for \( x, y \in Q \), we have

\[ \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{f(x)f(y)}{|x-y|^{2s}} \geq \sum_Q \int_{Q \times Q} \frac{f(x)f(y)}{|x-y|^{2s}} \geq \sum_Q \frac{1}{d^s |Q|^{2s/d}} \left( \int_Q f \right)^2. \]  

(70)

On the other hand, by the Sobolev inequality (recall that \( 1 \leq 2s < d \))

\[ \|u\|_{W^{1,2s}(Q)} \geq C \|u\|_{L^q(Q)}, \quad q = \frac{2sd}{d - 2s} > 2s, \]

we have

\[ \|u\|_{W^{2s}(Q)} \geq C \|f\|_{L^{\frac{d}{2s-2s}}(Q)} \geq C \left( \int_Q f^{1+2s/d} \right)^{2s/d}. \]

Hence,

\[ \int_{\mathbb{R}^d} |\nabla u|^{2s} + \sum_Q \frac{1}{|Q|^{2s/d}} \int_Q |u|^{2s} = \sum_Q \left( \int_Q |\nabla u|^{2s} + \frac{1}{|Q|^{2s/d}} \int_Q |u|^{2s} \right) \geq \sum_Q 2^{1-2s} \|u\|_{W^{1,2s}(Q)}^{2s} \geq C \sum_Q \left( \int_Q |u|^{2s(1+2s/d)} \right)^{2s/d}, \]

and, combining with (70) and (69),
Thus, if \( f \geq d^a \), then we can simply choose \( \Lambda = d^a \) and conclude that

\[
\int_{\mathbb{R}^d} |\nabla u|^{2s} \, dx + \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|u(x)|^{2s}|u(y)|^{2s}}{|x-y|^{2s}} \, dxdy \\
\geq \frac{C_1}{\Lambda^{2s/d}} \int_{\mathbb{R}^d} |u|^{2s(1+2s/d)} + \sum Q \left( \frac{1}{d^s} \left( \int_{\mathbb{R}^d} f \right)^2 - \int_Q f \right) \\
\geq \frac{C_1}{\Lambda^{2s/d}} \int_{\mathbb{R}^d} |u|^{2s(1+2s/d)} + \left( \frac{\Lambda}{(d^a)^{2s/d}} - 1 \right) \sum Q \frac{1}{|Q|^{2s/d}} \int_Q f.
\]

On the other hand, if \( f \leq d^a \), then using Sobolev’s inequality

\[
\|\nabla u\|_{L^{2s}(\mathbb{R}^d)} \geq C_2 \|u\|_{L^{2s/d/(d-2s)}(\mathbb{R}^d)}, \quad \forall u \in W^{1,2s}(\mathbb{R}^d)
\]

and Hölder’s inequality we have

\[
\int_{\mathbb{R}^d} |\nabla u|^{2s} \geq C_2 \|f\|_{L^{2s/d-(d-2s)}(\mathbb{R}^d)} \geq C_2 \int_{\mathbb{R}^d} f^{1+2s/d \left( \frac{2s}{d^a} \right)} \int_{\mathbb{R}^d} |u|^{2s(1+2s/d)}.
\]

In summary, it always holds that

\[
\int_{\mathbb{R}^d} |\nabla u|^{2s} \, dx + \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|u(x)|^{2s}|u(y)|^{2s}}{|x-y|^{2s}} \, dxdy \geq \min \{C_1, C_2\} \int_{\mathbb{R}^d} |u|^{2s(1+2s/d)} \, dx.
\]

By proceeding as for the Lieb-Thirring inequality in Section 2.3 that is rescaling \( u \mapsto \lambda u \) and optimizing over \( \lambda > 0 \), we obtain the interpolation inequality (21).

\[ \Box \]

**Appendix A. Lieb-Oxford inequality for homogeneous potentials**

In this appendix we prove Lemma 15. Note that the argument in the original papers [24, 26] uses Newton’s theorem and hence only works with the standard Coulomb interaction. The following proof is based on the strategy of Lieb, Solovej and Yngvason [28, Lemma 5.3].

**Proof of Lemma 15.** We start with the Fefferman-de la Llave representation

\[
\frac{1}{|x-y|^{\lambda}} = c_{d,\lambda} \int_0^\infty \int_{\mathbb{R}^d} \mathbb{1}_{B_R(x-u)} \mathbb{1}_{B_R(y-u)} \, du \, \frac{dR}{R^{d+\lambda+1}},
\]

where \( B_R = B(0, R) \) is the closed ball in \( \mathbb{R}^d \) and \( c_{d,\lambda} \) is a constant depending only on \( d \) and \( \lambda \) (see [11] for Coulomb potential, [25] Theorem 9.8 for homogeneous potentials and [17] Theorem 1] for more general cases). Consequently,

\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{\rho_\Psi(x) \rho_\Psi(y)}{|x-y|^{\lambda}} \, dxdy = \int_0^\infty \int_{\mathbb{R}^d} f_R(u)^2 \, du \, \frac{dR}{R^{d+\lambda+1}},
\]

where

\[
f_R := \rho_\Psi \ast \mathbb{1}_{B_R}
\]
Combining with the obvious inequality of the Littlewood maximal function of $g$, we find that
\[
\left\langle \Psi, \sum_{1 \leq i < j \leq n} \frac{1}{|x_i - x_j|^\lambda} \Psi \right\rangle = c_{d, \lambda} \int_0^\infty \int_{\mathbb{R}^d} g_R(u) du \frac{dR}{R^{d+\lambda+1}}
\] (75)

where
\[
g_R(u) := \left\langle \Psi, \sum_{1 \leq i < j \leq N} 1_{B_R(x_i - u)} 1_{B_R(x_j - u)} \Psi \right\rangle.
\]

Using the Cauchy-Schwarz inequality we find that
\[
g_R(u) = \frac{1}{2} \left\langle \Psi, \left( \sum_{i=1}^N 1_{B_R(x_i - u)} \right)^2 \Psi \right\rangle - \frac{1}{2} \left\langle \Psi, \sum_{i=1}^N 1_{B_R(x_i - u)} \Psi \right\rangle
\]
\[
\geq \frac{1}{2} \left\langle \Psi, \left( \sum_{i=1}^N 1_{B_R(x_i - u)} \right)^2 \Psi \right\rangle - \frac{1}{2} \left\langle \sum_{i=1}^N 1_{B_R(x_i - u)} \Psi \right\rangle
\]
\[
= \frac{1}{2} f_R^2(u) - \frac{1}{2} f_R(u).
\]

Combining with the obvious inequality $g_R(u) \geq 0$ we get
\[
g_R(u) \geq \frac{1}{2} f_R^2(u) - \frac{1}{2} \min\{f_R(u), f_R^2(u)\}.
\]

Inserting the latter inequality into (75) and using (74), we conclude that
\[
\left\langle \Psi, \sum_{1 \leq i < j \leq n} \frac{1}{|x_i - x_j|^\lambda} \Psi \right\rangle \geq \frac{1}{2} \int \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{\rho_{\psi}(x)\rho_{\psi}(y)}{|x - y|^2} dxdy
\]
\[
- \frac{c_{d, \lambda}}{2} \int_0^\infty \int_{\mathbb{R}^d} \min\{f_R(u), f_R^2(u)\} du \frac{dR}{R^{d+\lambda+1}}.
\] (76)

To estimate the second term of the right side, we introduce the Hardy-Littlewood maximal function of $\rho_{\psi}$:
\[
\rho^*(u) := \sup_{R > 0} \frac{1}{|B(0, R)|} \int_{|x-u| \leq R} \rho_{\psi}(x) dx
\]
\[
= |B_1|^{-1} \sup_{R > 0} \frac{f_R(u)}{R^d}.
\]

Using $f_R(u) \leq |B_1| R^d \rho^*(u)$, we find that
\[
\int_0^\infty \min\{f_R^2(u), f_R(u)\} \frac{dR}{R^{d+\lambda+1}} \leq \int_0^{R_*} f_R^2(u) \frac{dR}{R^{d+\lambda+1}} + \int_{R_*}^\infty f_R(u) \frac{dR}{R^{d+\lambda+1}}
\]
\[
\leq \int_0^{R_*} \left( |B_1| R^d \rho^*(u) \right)^2 \frac{dR}{R^{d+\lambda+1}} + \int_{R_*}^\infty |B_1| R^d \rho^*(u) \frac{dR}{R^{d+\lambda+1}}
\]
\[
= \frac{|B_1|^2}{d-\lambda} R_*^{d-\lambda} (\rho^*(u))^2 + \frac{|B_1|}{\lambda} R_*^{-\lambda} (\rho^*(u))^2
\]

for all $u \in \mathbb{R}^d$ and for all $R_* > 0$. Choosing $R_* = (|B_1| \rho^*(u))^{-1/d}$, we get
\[
\int_0^\infty \min\{f_R^2(u), f_R(u)\} \frac{dR}{R^{d+\lambda+1}} \leq \frac{d}{\lambda(d-\lambda)} |B_1|^{1+\lambda/d} (\rho^*(u))^{1+\lambda/d}
\]

for all $u \in \mathbb{R}^d$. Finally, by the maximal inequality (see, e.g. [40], p.58)
\[
\int_{\mathbb{R}^d} (\rho^*(u))^{1+\lambda/d} du \leq M_{d, \lambda} \int_{\mathbb{R}^d} \rho_{\psi}(u)^{1+\lambda/d} du,
\]
where $M_{d,\lambda}$ is a constant depending only on $d$ and $\lambda$, we conclude from (76) that

$$\left\langle \Psi, \sum_{1 \leq i < j \leq n} \frac{1}{|x_i - x_j|^{\lambda}} \Psi \right\rangle \geq \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{\rho_\Psi(x)\rho_\Psi(y)}{|x - y|^{\lambda}} \, dx \, dy - \frac{d_{c,\lambda} M_{d,\lambda}}{2\lambda(d - \lambda)} |B_1|^{1+\lambda/d} \int_{\mathbb{R}^d} \rho_\Psi^{1+\lambda/d}. $$

This is the desired inequality. □

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