ON THE SPECTRUM OF A MATRIX MODEL FOR THE $D=11$ SUPERMEMBRANE COMPACTIFIED ON A TORUS WITH NON-TRIVIAL WINDING.

L. Boulton$^1$, M. P. García del Moral$^2$, I. Martín$^2$, A. Restuccia$^2$

$^1$Departamento de Matemáticas, Universidad Simón Bolívar
Caracas, Venezuela
$^2$Departmento de Física, Universidad Simón Bolívar
Caracas, Venezuela

email: lboulton@ma.usb.ve, mgarcia@fis.usb.ve, isbeliam@usb.ve, arestu@usb.ve

Abstract. The spectrum of the Hamiltonian of the double compactified $D=11$ supermembrane with non-trivial central charge or equivalently the non-commutative symplectic super Maxwell theory is analyzed. In distinction to what occurs for the $D=11$ supermembrane in Minkowski target space where the bosonic potential presents string-like spikes which render the spectrum of the supersymmetric model continuous, we prove that the potential of the bosonic compactified membrane with non-trivial central charge is strictly positive definite and becomes infinity in all directions when the norm of the configuration space goes to infinity. This ensures that the resolvent of the bosonic Hamiltonian is compact. We find an upper bound for the asymptotic distribution of the eigenvalues.

1. Introduction

In [1], the spectrum of the Hamiltonian of the double compactified $D=11$ supermembrane on $D=11$ Minkowski target space was shown to be the whole interval $[0, \infty)$. There are two properties of the model which render the spectrum continuous: the existence of local string-like spikes and supersymmetry. The string-like spikes (cf. [4]) are local configurations with zero Hamiltonian density of the bosonic sector. Its existence imply that the minimal configurations of the Hamiltonian are degenerate along some directions. The potential has the shape of valleys extending to infinite in those directions. Consequently the bosonic theory is classically unstable, however its associated quantum mechanical system has discrete spectrum. There are several ways to explain this behaviour (cf. [11], [4] and [5]). In the directions transverse to the valleys the system may be described in terms of harmonic oscillators. Its zero
point energy generates a quantum effective potential which goes to infinity in all directions when the norm in the configuration space goes to infinite, as a consequence the quantum theory becomes stable. This argument may be extended in a rigorous way to show that under very mild assumptions on the potential the spectrum of the corresponding Schrödinger operator becomes discrete. The situation changes drastically when the supersymmetry is introduced into the model. This is so, because the fermionic sector cancels the bosonic contribution to the zero point energy. In [1] it was proven that the spectrum of the $D = 11$ supermembrane in a Minkowski target space is continuous. We remark that the construction in [1] is valid in the interior of an open cone in the configuration space. A well posed spectral problem may be defined by imposing boundary conditions to the wave functions on the cone surface, in their case Dirichlet boundary conditions. Its extension to the whole configuration space requires at least the proof of existence of a well posed spectral problem in the whole space. This problem has not been addressed in the literature.

In this paper we will consider as in [1] wavefunctions with support in the interior of an open cone in configuration space. The analysis of the spectrum for the compactified supermembrane is not yet conclusive. The main problem has been that the SU($N$) regularization of the compactified supermembrane seems not to be possible. The close but not exact modes present in the compactified case apparently do not fit into an SU($N$) formulation of the theory [2]. It was, however, pointed out there that the spectrum of the compactified supermembrane should also be continuous, because of the presence of string-like spikes in the configuration space of the compactified membrane. In [3] by considering particular configurations of the compactified supermembrane it was argued that a non-trivial central charge should render a discrete spectrum for the compactified supermembrane.

In [6] a formulation of the double compactified $D = 11$ supermembrane in terms of a non-commutative geometry was obtained. In this formulation the Hamiltonian may be expressed in terms of a non-commutative super Maxwell theory plus the integral of the curvature of the non-commutative connection over the world volume. In [3] it was explicitly shown that the presence of this curvature term introduces string-like spikes into the configuration space of the compactified supermembrane in agreement with [2]. If however we restrict the theory to have a fixed central charge, which describes a sector of the full compactified theory then the integral of the curvature becomes zero and the theory reduces to a non-commutative super Maxwell theory coupled to
seven scalar fields which represent the transverse directions to the supermembrane. It was also shown in the later paper that in this sector there are no string-like spikes. The explicit computation was possible because in that sector of the compactified supermembrane the closed non-exact model can be properly handled and an SU($N$) regularization of the Hamiltonian may be obtained.

In section 3 we go one step forward and characterize completely the quantum problem. We show that the potential of the noncommutative Maxwell theory coupled to the scalar field is strictly positive definite and becomes infinite in any direction when the norm in the configuration space goes to infinity. Consequently (cf. [8]) the corresponding quantum Hamiltonian has compact resolvent so that its spectrum consists of a discrete set of eigenvalues of finite multiplicity. The speed at which the potential escapes to infinity is controlled by the quadratic and quartic powers of the norm in the configuration space, we determine upper bounds for the asymptotic distribution of these eigenvalues by comparing with those of the harmonic oscillator. Finally we conjecture (based on the form of the bosonic potential) that the noncommutative supermaxwell theory should have a discrete spectrum. This problem is currently under investigation by some of the authors.

2. THE HAMILTONIAN OF THE DOUBLE COMPACTIFIED SUPERMEMBRANE

The Hamiltonian of the $D = 11$ supermembrane, with target space $M_q \times S^1 \times S^1$, describing non-trivial wrapping over the compactified directions was obtained in [8], [7]. It was formulated in terms of a symplectic noncommutative geometry. The resulting Hamiltonian is exactly equivalent to the one of the $D = 11$ supermembrane dual over $M_q \times S^1 \times S^1$. A symplectic structure is introduced in the formulation by considering the connection 1-form which minimizes the Hamiltonian. The winding number $n$ characterizes a $U(1)$ bundle over the world-volume where the minimal connection is defined. It corresponds to a monopole type connection. The explicit expression of the bosonic Hamiltonian is

$$H = \int_{\Sigma} \frac{1}{2\sqrt{W}}[(P_m)^2 + (\Pi_r)^2 + (1/2)W\{X^m, X^n\}^2 + W(D_r X^m)^2 + (1/2)W(F_{rs})^2] + \int_{\Sigma} [(1/8)\sqrt{W}n^2 - \Lambda(D_r \Pi_r + \{X^m, P_m\})] + \int_{\Sigma} \sqrt{W} n^* F, \quad n \neq 0$$

(1)
together with its supersymmetric extension

\[ \int_{\Sigma} \sqrt{W} \left[ -\partial \Gamma \cdot \partial r \theta + \partial \Gamma \cdot \partial m \{ X^m, \theta \} + \Lambda \{ \partial \Gamma \cdot \theta \} \right] \]  

(2)

where \( m = 1, \ldots, 7 \) are indices denoting the scalar fields once the supermembrane is formulated in the light cone gauge. They describe the transverse directions to the world volume. The indices \( r, s = 1, 2 \) are the ones related to the two compactified directions of the tangent space.

We denote by \( \Sigma \) the spatial part of the world volume, which is assumed to be a closed Riemann surface of genus \( g \). By \( P_m \) and \( \Pi_r \) we denote the conjugate momenta to \( X^m \) and the connection 1-form \( A_r \) respectively.

By \( \vartheta \) we denote the Majorana spinors of the \( D = 11 \) formulation which may be decomposed in terms of a complex 8-component spinor of \( SO(7) \times U(1) \).

The covariant derivative is defined by

\[ D_r = D_r + \{ A_r, \} \]

and the field strength

\[ F_{rs} = D_r A_s - D_s A_r + \{ A_r, A_s \} \]

The bracket \( \{ , \} \) is defined by

\[ \{ \ast, \phi \} = \frac{2e^{sr}}{n} (D_r \ast) (D_s \phi) \]

where \( n \), the winding number, denotes the integer which characterizes the non trivial \( U(1) \) principle bundle over \( \Sigma \).

In the above, \( D_r \) is a tangent space derivative defined by

\[ D_r = \frac{\hat{\Pi}_a}{\sqrt{W}} \partial_a \]

where \( \partial_a \) denotes derivatives with respect to the local coordinates on \( \Sigma \). By \( \hat{\Pi}_a \) we denote a zwei-vein defined from the minimum of the Hamiltonian \( (1) \). It satisfies

\[ \hat{\Pi}_a = \epsilon^{ab} \partial_b \hat{\Pi}_r \]

\[ \{ \hat{\Pi}_r, \hat{\Pi}_s \} = (1/2) n \epsilon_{rs} \]

In [3] an \( SU(N) \) regularization of \( (1), (2) \), for the case in which the symplectic connection \( A_r \) has no transitions over \( \Sigma \), was obtained. In that case the latter term of \( (1) \), which is the integral of a total derivative of a single-valued object over \( \Sigma \), vanishes.
The resulting SU($N$) model is

$$H = \text{Tr} \left( \frac{1}{2N^3} (P_0^m T_0 P_0^m T_0 + \Pi_r^0 T_0 \Pi_r^0 T_0 + (P_m)^2 + (\Pi_r)^2) + 
+ \frac{n^2}{16\pi^2 N^3} [X^m, X^n]^2 + \frac{n^2}{8\pi^2 N^3} \left( \frac{i}{N} [T_{V_r}, X^m] T_{-V_r} - [A_r, X^m] \right)^2 + 
+ \frac{n^2}{16\pi^2 N^3} \left( [A_r, A_s] + \frac{i}{N} ([T_{V_r}, A_r] T_{-V_s} - [T_{V_r}, A_s] T_{-V_r}) \right)^2 + \frac{1}{8} n^2 + 
+ \frac{n}{4\pi N^3} \Lambda \left( [X^m, P_m] - \frac{i}{N} [T_{V_r}, \Pi_r] T_{-V_r} + [A_r, \Pi_r] \right) + 
+ \frac{in}{4\pi N^3} \left( \overline{\psi} \gamma_\gamma [X^m, \psi] - \overline{\psi} \gamma_\gamma [A_r, \psi] + \Lambda [\overline{\psi} \gamma_\gamma, \psi] + 
- \frac{i}{N} \overline{\psi} \gamma_\gamma [T_{V_r}, \psi] T_{-V_r} \right) \right) \right)$$

(3)

subject to

$$A_1 = A_1^{(a_1,0)} T_{(a_1,0)},$$
$$A_2 = A_2^{(a_1,a_2)} T_{(a_1,a_2)} \text { with } a_2 \neq 0. \hspace{1cm} (4)$$

We use the following notation

$$X^m = X^m A^T_A \hspace{1cm} P_m = P A_m A^T_A$$
$$A_r = A^{A_r} T_A \hspace{1cm} \Pi_r = \Pi_r^A T_A$$

where $T_A$ are the generators of the SU($N$) algebra:

$$[T_A, T_B] = f^{C}_{AB} T_C.$$  

It was shown in [3] that this regularized Hamiltonian has an associated mass operator with no string-like spikes. That is, the local conditions
on the bosonic sector

\[ \text{Tr} \left[ \frac{n^2}{16\pi^2N^3} [X^m, X^n]^2 + \frac{n^2}{8\pi^2N^3} \left( \frac{i}{N} [T_{V_r}, X^m] T_{-V_r} - [A_r, X^m] \right)^2 + \right. \]
\[ \left. + \frac{n^2}{16\pi^2N^3} \left( [A_r, A_s] + \frac{i}{N} ([T_{V_s}, A_r] T_{-V_s} - [T_{V_r}, A_s] T_{-V_r}) \right)^2 \right] = 0, \]

(5)

\[ [X^m, P_m] - \frac{i}{N} [T_{V_r}, \Pi_r] T_{-V_r} + [A_r, \Pi_r] = 0, \]

(6)

\[ A_1 = A_1^{(a_1,0)} T_{(a_1,0)} \]

(7)

\[ A_2 = A_2^{(a_1,a_2)} T_{(a_1,a_2)}, \quad \text{with} \quad a_2 \neq 0, \]

(8)

imply

\[ X^{Bm} = 0, \quad A_r^B = 0, \quad P_m^B = 0, \quad \Pi_r^B = 0. \]

(9)

The constraint (6) determines \( \Pi_1^{(a,b)}, b \neq 0 \) and \( \Pi_2^{(a,0)} \) which together with (7) and (8) allow a canonical reduction \( H_R \) for the Hamiltonian (3). The same canonical reduction may be performed in (2) when the geometrical objects are expressed in a complete orthonormal basis of the space \( L^2(\Sigma) \). After this reduction, the term \( |\Pi_1^{(a,b)}|^2, b \neq 0 \) and \( |\Pi_2^{(a,0)}|^2 \) become non-trivial, however since they are positive, we can bound the mass operator

\[ \mu_R = H_R - \text{Tr} \left( \frac{1}{2N^3} P_m^0 T_0 P_m^0 T_0 + \Pi_r^0 T_0 \Pi_r^0 T_0 \right) \]

by an operator \( \mu \) without such terms. If the resulting \( \mu \) is bounded from below and has a compact resolvent, the same properties are valid for \( \mu_R \). We will show the former in the forthcoming section. Notice that \( \mu \) is of Schrödinger type with potential the left hand side of (5). In what follows it will be understood without further mention that the centre of mass terms have been removed. This restrictions (4) associated to gauge fixing conditions of (1), (2) are equivalent to the ones used in [1]. In their analysis they use the gauge freedom to diagonalize one of the \( X \) maps, the residual gauge freedom when \( X \) is regular consists of an arbitrary element of the Cartan subgroup.

The imposition of (3) together with the elimination of the associated conjugate momenta is valid only in the interior of an open cone \( K \). Our model thus considers as in [1], wavefunctions with support on the interior of \( K \). For technical reasons, in order to show that the spectrum of \( \mu \) is discrete, in the following section we also consider \( \mu \) as an operator acting on the whole configuration space. We shall see that
discreteness of the spectrum for the latter implies the same property for the restriction to any hyper-cone.

3. Discretness of the spectrum of $\mu$

We introduce the following notation. The Hamiltonian

$$\mu = -\Delta_X - \Delta_A + V(X, A)$$

acts on $L^2((X, A) \in \mathbb{R}^M)$ for suitably large $M$. The potential $V$ is given by

$$V(X, A) = V_1(X) + V_2(A) + V_3(X, A)$$

where

$$V_1(X) = 4 \sum_{D,m,n} \left| N \sin \left( \frac{B \times C}{N} \pi \right) X^{Bm} X^{Cn} \delta_{B+C}^D \right|^2,$$

$$V_2(A) = 4 \sum_{r,s,D} \left| \omega^{(D \times V_r)/2} N \sin \left( \frac{V_r \times D}{N} \pi \right) A_s^D + \omega^{(D \times V_s)/2} N \sin \left( \frac{V_s \times D}{N} \pi \right) A_r^D \right|^2,$$

$$V_3(X, A) = 2 \sum_{D,r,m} \left| \omega^{(D \times V_r)/2} N \sin \left( \frac{V_r \times D}{N} \pi \right) X^{Dm} + i \sum_{B,C} N \sin \left( \frac{B \times C}{N} \pi \right) A_r^B A_c^C \delta_{B+C}^D \right|^2,$$

$$\omega = \exp \left( \frac{2\pi i}{N} \right).$$

We define rigorously $\mu$ as the self-adjoint non-negative Friedrichs extension of $(\mu, C_c^\infty(\mathbb{R}^M))$. Our aim is to show that $\mu$ has compact resolvent. The proof depends upon the fact that $V$ is a basin shape potential.

As we mentioned in section 2, the operator $\mu$ that bounds our model should not be defined on the whole space $\mathbb{R}^M$ but only on the open hyper-cone $K \subset \mathbb{R}^M$. Every sequence of approximate eigenfunctions squared integrable on $K$ is also a sequence of approximate eigenfunctions squared integrable on the whole space. Therefore, if the spectrum of $\mu$ as an operator acting on $\mathbb{R}^M$ is discrete, it also has spectrum discrete as an operator acting on $K$. We use this without further mention.
Notice that the same argument applies to any open hyper-cone on the configuration space.

**Lemma 1.** The potential $V(X, \mathcal{A}) = 0$, if and only if $X^{Bm} = 0$ and $\mathcal{A}_r^B = 0$ for all indices $B, m, r$.

**Proof.** The condition $V_2(\mathcal{A}) = 0$, yields

$$k^{-<(V_1 \times \mathcal{A})>/2}\lambda_r A_r^4 - k^{-<(V_1 \times \mathcal{A})>/2}\lambda_s A_s^4 + f_{BC}^A A_r^B A_s^C = 0.$$  

Using (7) and (8) we obtain

$$(1/2)k^{-<(V_1 \times \mathcal{A})>/2}N \sin\left(\frac{N \pi}{V_1 \times \mathcal{A}}\right) A_2^B - k^{-<(V_2 \times \mathcal{A})>/2}N \sin\left(\frac{N \pi}{V_2 \times \mathcal{A}}\right) A_1^B +$$

$$+ iN \sin\left(\frac{b_1 V_1 \times \mathcal{A}}{N \pi}\right) A_1^{b_1, 0} A_2^{A-b_1 V_1} = 0$$

where the $b_i$ are integers. In particular for $\mathcal{A} = l V_1$, $l$ integer, we get $A^{m V_1} \equiv A_1^{l, 0} = 0$ hence $A_1^A = 0$. We then obtain from (3) and (8), $A_2^A = 0$. The condition

$$V_3(X, \mathcal{A}) = 0$$

then reduces to

$$k^{<(A \times V_1)/2} \sin\left(\frac{N \pi}{V_1 \times \mathcal{A}}\right) X^{mA} = 0,$$

$$k^{<(A \times V_2)/2} \sin\left(\frac{N \pi}{V_2 \times \mathcal{A}}\right) X^{mA} = 0,$$

which yields $X^{mA} = 0$. □

The following proposition is the main result of this section.

**Lemma 2.** The potential $V(X, \mathcal{A}) \to \infty$ as $(X, \mathcal{A}) \to \infty$.

**Proof.** Let $\mathbb{T}$ be the unit ball of $\mathbb{R}^M$. We write $X^{Bm}$ and $\mathcal{A}_r^B$ in polar coordinates as

$$X^{Bm} = R \phi^{Bm}, \quad \mathcal{A}_r^B = R \psi_r^B$$

or

$$X = R \phi, \quad \mathcal{A} = R \psi$$

where $R \geq 0$, $\phi = (\phi^{Bm})$, $\psi = (\psi_r^B)$ and $(\phi, \psi) \in \mathbb{T}$. In order to show the desired limit, we shall show

$$\inf_{(\phi, \psi) \in \mathbb{T}} V(R \phi, R \psi) \to \infty \quad \text{as} \quad R \to \infty. \quad (10)$$

Elementary computations yield

$$V(R \phi, R \psi) = R^4 k_1(\phi, \psi) + R^3 k_2(\phi, \psi) + R^2 k_3(\phi, \psi),$$
where \( k_1(\phi, \psi) \geq 0, k_3(\phi, \psi) \geq 0, k_2(\phi, \psi) \in \mathbb{R} \) can be negative and if \( k_1(\phi, \psi) = 0 \), then \( k_2(\phi, \psi) = 0 \). The \( k_j \) are continuous in \((\phi, \psi) \in \mathcal{T}\) and by virtue of lemma 1,

\[
K := \inf_{(\phi, \psi) \in \mathcal{T}} [k_1(\phi, \psi) + k_2(\phi, \psi) + k_3(\phi, \psi)] = \inf_{(\phi, \psi) \in \mathcal{T}} V(\phi, \psi) > 0.
\]

When clear from the context, below we will write \( k_j \equiv k_j(\phi, \psi) \).

Let

\[
\mathcal{T}_1 = \{ (\phi, \psi) \in \mathcal{T} : k_1(\phi, \psi) = 0 \}
\]

\[
\mathcal{T}_2 = \{ (\phi, \psi) \in \mathcal{T} : k_1(\phi, \psi) > 0 \}.
\]

Then \( \mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2 \) and since \( k_1 \) is continuous \( \mathcal{T}_2 = \mathcal{T} \). Put

\[
V(R_\phi, R_\psi) = R^2(R^2k_1(\phi, \psi) + RK_2(\phi, \psi) + K_3(\phi, \psi)) = R^2P_{\phi,\psi}(R).
\]

Then \( P_{\phi,\psi}(R) \) is a family of paraboles parameterized by \((\phi, \psi) \in \mathcal{T}\) such that

\[
P_{\phi,\psi}(0) = k_3(\phi, \psi),
\]

\[
P_{\phi,\psi}''(R) = k_1(\phi, \psi) \geq 0 \quad \text{and} \quad P_{\phi,\psi}(1) = k_1(\phi, \psi) + k_2(\phi, \psi) + k_3(\phi, \psi) \geq K > 0.
\]

Now

\[
\inf_{(\phi, \psi) \in \mathcal{T}} V(R_\phi, R_\psi) = R^2 \inf_{(\phi, \psi) \in \mathcal{T}} P_{\phi,\psi}(R) \geq R^2 \inf_{(\phi, \psi) \in \mathcal{T}} M(\phi, \psi),
\]

where

\[
M(\phi, \psi) := \min_{R \geq 1} P_{\phi,\psi}(R).
\]

If \( M : \mathcal{T} \rightarrow \mathbb{R} \) was a strictly positive continuous function, since \( \mathcal{T} \) is compact necessarily

\[
\inf_{(\phi, \psi) \in \mathcal{T}} M(\phi, \psi) \geq \bar{M} > 0,
\]

thus the right hand side of \((11)\) escapes to \( \infty \) as \( R \rightarrow \infty \) and so \((10)\) is proven.

In order to complete the proof, we show first that \( M \) is a strictly positive function. If \((\phi, \psi) \in \mathcal{T}_1\), then \( k_1 \) and hence \( k_2 \) vanish so \( M(\phi, \psi) = k_3(\phi, \psi) > 0 \). On the other hand, if \( M(\phi, \psi) = 0 \) for some \((\phi, \psi) \in \mathcal{T}_2\), we would have \( P_{\phi,\psi}(R_0) = 0 \) for some \( R_0 > 0 \) so that

\[
V(R_0\phi, R_0\psi) = R_0^2P_{\phi,\psi}(R_0) = 0
\]

hence we contradict lemma \([1]\). Therefore necessarily \( M(\phi, \psi) > 0 \) for all \((\phi, \psi) \in \mathcal{T}_2\).
Finally we show that $M$ is continuous. For this we consider separately the regions $T_1$ and $T_2$. If $(\phi, \psi) \in T_2$, the minimum of the polynomial $P_{\phi, \psi}$ is attained at

$$R_0 \equiv R_0(\phi, \psi) = -\frac{k_2}{2k_1}.$$ 

At this point

$$P_{\phi, \psi}(R_0) = k_3 - \frac{k_2^2}{4k_1^2}.$$ 

If $k_2 \geq -2k_1$, then $R_0 < 1$ so

$$M(\phi, \psi) = P_{\phi, \psi}(1) = k_1 + k_2 + k_3.$$ 

If $k_2 < -2k_1$, then

$$M(\phi, \psi) = k_3 - \frac{k_2^2}{4k_1^2}.$$ 

In both cases the continuity of $M$ at $(\phi, \psi) \in T_2$ follows from the continuity of the $k_1, k_2, k_3$ and the fact that $k_1(\phi, \psi) \neq 0$.

Since $T_1 \subset T_2$, in order to show the continuity of $M$ in $T_1$, it is enough to prove that for any sequence $(\phi_n, \psi_n) \in T_2$, such that

$$(\phi_n, \psi_n) \to (\phi, \psi) \in T_1 \quad \text{as} \quad n \to \infty,$$ 

we also have

$$M(\phi_n, \psi_n) \to M(\phi, \psi). \quad (13)$$

For this let the family of lines

$$L_n(R) = [k_1(\phi_n, \psi_n) + k_2(\phi_n, \psi_n)]R + k_3(\phi_n, \psi_n) \quad R > 0.$$ 

Since

$$P''_{(\phi_n, \psi_n)}(R) > 0, \quad L_n(0) = P_{(\phi_n, \psi_n)}(0)$$ 

and

$$L_n(1) = P_{(\phi_n, \psi_n)}(1),$$ 

one has

$$L_n(R) \leq P_{(\phi_n, \psi_n)}(R) \quad \text{for all} \quad R \geq 1.$$ 

Then for all $n$ large

$$L_n \left( \frac{1}{k_1^2(\phi_n, \psi_n)} \right) \leq P_{(\phi_n, \psi_n)} \left( \frac{1}{k_1^2(\phi_n, \psi_n)} \right). \quad (14)$$

Since $k_1(\phi_n, \psi_n) \to 0$ as $n \to \infty$, 

$$k_1(\phi_n, \psi_n) + k_2(\phi_n, \psi_n) + k_3(\phi_n, \psi_n) \geq K$$
and the $k_2(\phi_n, \psi_n)$ are bounded, both the left hand side and the right hand side of (14) tend to $k_3(\phi, \psi)$ as $n \to \infty$. Hence the family of straight lines $L_n(R)$ approaches to the horizontal line $y \equiv k_3(\phi, \psi)$ as $n \to \infty$. An elementary geometric argument (notice $-k_2/(2k_1) < 1/k_1^2$ for all large $n$) confirms that this is enough to guarantee (13) and so the proof is complete.

As a consequence of lemma 2 and from a standard result in the spectral theory of Schrödinger operators (cf. [8]), the resolvent of $\mu$ is compact so the spectrum consists purely of isolated eigenvalues of finite multiplicity. By the positivity of $V$, we also know that all such eigenvalues are non-negative.

Let us conclude this section by describing how the estimates in the proof of lemma 2 provide information about the distribution of the eigenvalues of the original model $H$. For simplicity we introduce the following standard notation. If the linear operator $T$ is bounded below and has discrete spectrum, we define $N_T(\lambda)$ (the counting function of $T$) as the number of eigenvalues of $T$ counting multiplicity which are less or equal to $\lambda > 0$.

We find upper bounds for $N_H(\lambda)$ as follows. Each eigenvalue of $\mu$ as an operator acting on an hyper-cone will also be an eigenvalue of $\mu$ as an operator acting on the whole space $\mathbb{R}^M$, thus the counting function of the former will always be below the counting function of the latter. Since $H$ only differs from $\mu$ (acting on $K$) by a positive term, the above yields

$$N_H(\lambda) \leq N_\mu(\lambda)$$

where now the $\mu$ at the right hand side acts on $\mathbb{R}^M$.

Explicit estimates on the eigenvalues of $\mu$ come from bounds on the potential. By virtue of (11) and (12), there exist constants $\tilde{M} > 0$ and $b > 0$ such that

$$V(R\phi, R\psi) \geq \tilde{M}R^2 - b \quad \phi, \psi \in \mathbb{T}, R > 0.$$

According to the min-max principle (cf. [8]), the eigenvalues of $\mu$ are bounded from below by those of the harmonic oscillator

$$\tilde{\mu} = -\Delta_X - \Delta_A + \tilde{M}(|X|^2 + |A|^2) - b.$$

This yields $N_\mu(\lambda) \leq N_{\tilde{\mu}}(\lambda)$. Now, the spectrum of $\tilde{\mu}$ can be computed explicitly, indeed $N_{\tilde{\mu}}(\lambda)$ is of the order of $\lambda$ as $\lambda \to \infty$. By collecting all these estimates, we gather that

$$N_H(\lambda) = O(\lambda)$$

as $\lambda \to \infty$. 
We should mention that the leading term in $V(X, A)$ is not quadratic but quartic. This suggests that the estimate found above is not very sharp. At the present moment an investigation on better bounds for the eigenvalues of $H$ is being carried out by some of the authors.

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