THE NONMEASURABILITY OF BERNSTEIN SETS AND RELATED TOPICS

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Abstract. In this article, we shall explore the constructions of Bernstein sets, and prove that every Bernstein set is nonmeasurable and doesn’t have the property of Baire. We shall also prove that Bernstein sets don’t have the perfect set property.

1. Introduction

We know that any set obtained as the result of countably many applications of union, intersection, or complementation, starting from a countable family of closed, open, or nullsets, will be measurable. However, it’s not trivial to see that whether there exist nonmeasurable sets. Though it can be shown that any analytic set is measurable, (An analytic set is one that can be represented as the continuous image of a Borel set,) Gödel has shown that the hypothesis that there exists a nonmeasurable set that can be represented as the continuous image of the complement of some analytic set is consistent with the axioms of set theory, provided these axioms are consistent among themselves. No actual example of a non-measurable set that admits such a representation is known. Nevertheless, with the aid of the axiom of choice it is easy to show that non-measurable sets exist. As we shall proceed in this article, a type of non-measurable sets called Bernstein sets will be constructed and their lack of other properties, e.g. the property of Baire and the perfect set property, will be proved in a parallel manner.

The paper is divided into three sections. The second section will be devoted to introducing the necessary definitions, and certain useful lemmas will be proved within it. Readers who are familiar with the notions of measurability and the property of Baire can skip directly to the third section, in which the main theorems of this article together with their proofs will appear.

For a more detailed discussion of the duality of measurability and the property of Baire, see [1].

2. Definitions and Basic Properties

In this section, we shall introduce the notions of measure and category on R and give out some basic properties of them. Certain proofs that are not our main goals in this paper will be omitted.

2.1. Measure on R. Recall that the outer measure of a subset A of R is defined by

\[ m^*(A) = \inf \{ \sum |I_i| : A \subseteq \bigcup I_i \} \]

where the infimum is taken over all countable coverings of A by open intervals, and \(|I| = b - a\) if \(I = (a, b)\).
Here we list some properties of outer measures in \( \mathbb{R} \) without proving it. The proofs can be found in regular textbooks on real analysis, e.g. [2].

**Proposition 2.1.**

1. If \( A \subset B \) then \( m^*(A) < m^*(B) \).
2. If \( A = \bigcup A_i \) then \( m^*(A) \leq \sum m^*(A_i) \).
3. For any interval \( I \), \( m^*(I) = |I| \).
4. If \( F_1 \) and \( F_2 \) are disjoint bounded closed sets, then \( m^*(F_1 \cup F_2) = m^*(F_1) + m^*(F_2) \).

A subset \( E \) of \( \mathbb{R} \) is Lebesgue measurable, or simply measurable, if for each \( \epsilon > 0 \) there exists a closed set \( F \) and an open set \( G \) such that \( F \subset E \subset G \) and \( m^*(G - F) < \epsilon \). If \( E \) is measurable, we define its measure \( m(E) = m^*(E) \).

We also list some properties of measurable sets. The proofs can be found in [2].

**Proposition 2.2.**

1. If \( A \) is measurable, then \( A^c \) is measurable.
2. Any interval and any nullset is measurable. A set is called a nullset, if its outer measure is 0.
3. For any disjoint sequence of measurable sets \( A_i \), the set \( A = \bigcup A_i \) is measurable and \( m^*(A) = \sum m^*(A_i) \).

Next, we prove some useful lemmas which we shall encounter later in the paper.

**Theorem 2.3.** A set \( A \) is measurable if and only if it can be represented as an \( F_\sigma \) set plus a nullset, or as a \( G_\delta \) set minus a nullset. (A \( F_\sigma \) set is a countable union of closed sets and a \( G_\delta \) set is a countable intersection of open sets.)

**Proof.** If \( A \) is measurable, then for each \( n \) there exists a closed set \( F_n \) and an open set \( G_n \) such that \( F_n \subset A \subset G_n \) and \( m^*(G_n - F_n) < 1/n \). Put \( F = \bigcup F_n \), \( N = A - F \), \( G = \bigcap G_n \) and \( M = G - A \). Then \( F \) is a \( F_\sigma \) set and \( G \) is a \( G_\delta \) set. \( N \) and \( M \) are null sets, since \( N \cup M \subset G_n - F_n \) and \( m^*(N \cup M) < 1/n \) for every \( n \). Thus, \( A = F \cup N = G - M \) and we get the desired representation. \( \square \)

**Corollary 2.4.** If \( A \) is measurable, then \( m(A) = \inf \{ m(G) : A \subset G, G \text{ open} \} = \sup \{ m(F) : F \subset A, F \text{ closed} \} \). Especially, if every closed subset of \( A \) is a null set, then \( A \) is a null set.

The corollary follows directly from the proof of Theorem 2.3.

2.2. **Category on \( \mathbb{R} \).** A set \( A \) is dense in the interval \( I \) if \( A \) has a nonempty intersection with every subinterval of \( I \). The opposite notion of dense is the notion of nowhere dense.

**Definition 2.5.** A set \( A \) is nowhere dense if it is dense in no interval, that is, if every interval has a subinterval contained in the complement of \( A \).

There are two other useful ways to state the definition.

**Proposition 2.6.**

1. \( A \) is nowhere dense if and only if its complement \( A^c \) contains a dense open set
2. \( A \) is nowhere dense if and only if its closure \( \overline{A} \) has no interior points.
The equivalence between the three definitions are easy to prove, and we shall not proceed it here. By the second statement of the proposition above, the closure of a nowhere dense set is nowhere dense.

**Definition 2.7.** A set is said to be of first category if it can be represented as a countable union of nowhere dense sets. A set which is not of first category is said to be of second category.

Below are some basic properties of first category sets, the proofs are routine and can be found in [1].

**Proposition 2.8.**

1. Any subset of a set of first category is of first category.
2. The union of any countable family of first category sets is of first category.

We prove the theorem of Baire.

**Theorem 2.9 (Baire).**

1. The complement of any set of first category on the line is dense.
2. No interval in \( \mathbb{R} \) is of first category.
3. The intersection of any sequence of dense open sets is dense.

**Proof.** The three statements are essentially equivalent. To prove the first, let \( A = \bigcup A_n \) be a representation of \( A \) as a countable union of nowhere dense sets. For any interval \( I \), let \( I_1 \) be a closed subinterval of \( I - A_1 \). Let \( I_2 \) be a closed subinterval of \( I - A_2 \), and so on. Then \( \bigcap I_n \) is a non-empty subset of \( I - A \), hence \( A^c \) is dense.

The second statement is an immediate corollary of the first. The third statement follows from the first by complementation. \( \square \)

As a corollary, \( \mathbb{R} \) is not of first category, i.e. it’s of second category.

Now we introduce the property of Baire.

**Definition 2.10.** The symmetric difference \( A \triangle B \) of two sets \( A \) and \( B \) is defined by

\[
A \triangle B = (A \cup B) - (A \cap B) = (A - B) \cup (B - A).
\]

A subset of \( \mathbb{R} \) is said to have the property of Baire if it can be represented in the form \( A = G \triangle P \), where \( G \) is open and \( P \) is of first category.

**Lemma 2.11.** A set \( A \) has the property of Baire if and only if it can be represented in the form \( A = F \triangle Q \), where \( F \) is closed and \( Q \) is of first category.

**Proof.** If \( A = G \triangle P \), \( G \) open and \( P \) of first category, then \( N = \overline{G} - G \) is a nowhere dense closed set, and \( Q = N \triangle P \) is of first category. Let \( F = G \). Then \( A = G \triangle P = (G \triangle N) \triangle P = G \triangle (N \triangle P) = F \triangle Q \). Conversely, if \( A = F \triangle Q \), where \( F \) is closed and \( Q \) is of first category, let \( G \) be the interior of \( F \). Then \( N = F - G \) is nowhere dense, \( P = N \triangle Q \) is of first category, and \( A = F \triangle Q = (G \triangle N) \triangle Q = G \triangle (N \triangle Q) = G \triangle P \) \( \square \)

**Lemma 2.12.** If \( A \) has the property of Baire, then so does its complement.

**Proof.** For any two sets \( A \) and \( B \) we have \( (A \triangle B)^c = A^c \triangle B \). Hence if \( A = G \triangle P \), then \( A^c = G^c \triangle P \), and the conclusion follows from Lemma 2.11. \( \square \)

Similar to Theorem 2.3, we have a corresponding theorem on category.
Theorem 2.13. A set has the property of Baire if and only if it can be represented as a \(G_\delta\) set plus a set of first category, or as an \(F_\sigma\) set minus a set of first category.

Proof. Since the closure of any nowhere dense set is nowhere dense, any set of first category is contained in an \(F_\sigma\) set of first category. If \(G\) is open and \(P\) is of first category, let \(Q\) be an \(F_\sigma\) set of first category that contains \(P\). Then the set \(E = G - Q\) is a \(G_\delta\), and we have

\[G \triangle P = E \triangle [(G \triangle P) \cap Q].\]

The set \((G \triangle P) \cap Q\) is of first category and disjoint to \(E\). Hence any set having the property of Baire can be represented as the disjoint union of a \(G_\delta\) set and a set of first category. By taking the complement of the set it follows that any set having the property of Baire can also be represented as an \(F_\sigma\) set minus a set of first category. \(\square\)

3. Constructions of Bernstein Sets and Their Properties

In this section, we shall construct a certain type of sets called Bernstein sets, and prove that they neither are measurable nor have the property of Baire. Moreover, it will also be shown that Bernstein sets don’t have the perfect set property.

Definition 3.1. A subset \(B\) of \(\mathbb{R}\) is called a Bernstein set, if both \(B\) and \(B^c\) meet every uncountable closed subset of the line.

To construct Bernstein sets, we need the following lemma and the aid of the Axiom of Choice, which guarantees that every set can be well-ordered.

Lemma 3.2. The class of uncountable closed subsets of \(\mathbb{R}\) has cardinality \(\mathfrak{c}\).

Proof. The set of open intervals with rational endpoints is countable, and every open set is the union of some subset. Hence there are at most \(\mathfrak{c}\) open sets, and therefore at most \(\mathfrak{c}\) closed sets, since closed sets are complements of open sets. On the other hand, there are at least \(\mathfrak{c}\) uncountable closed sets, since there are that many closed intervals. Hence there are exactly \(\mathfrak{c}\) uncountable closed subsets of the line. \(\square\)

Now we can prove the existence of Bernstein sets.

Theorem 3.3. There exists a Bernstein set.

Proof. By the well ordering principle and Lemma 3.2, the set \(\mathcal{F}\) of uncountable closed subsets of the line can be indexed by the ordinal numbers less than \(\omega_c\), where \(\omega_c\) is the first ordinal having \(\mathfrak{c}\) predecessors, say \(\mathcal{F} = \{F_\alpha : \alpha < \omega_c\}\). We may assume that \(\mathbb{R}\), and therefore each member of \(\mathcal{F}\), has been well ordered. Note that each member of \(\mathcal{F}\) has cardinality \(\mathfrak{c}\), since any closed set is a \(G_\delta\). Let \(p_1\) and \(q_1\) be the first two members of \(F_1\). Let \(p_2\) and \(q_2\) be the first two members of \(F_2\) different from both \(p_1\) and \(q_1\). If \(1 < \alpha < \omega_c\) and if \(p_\beta\) and \(q_\beta\) have been defined for all \(\beta < \alpha\), let \(p_\alpha\) and \(q_\alpha\) be the first two elements of \(F_\alpha - \bigcup_{\beta < \alpha}(p_\beta, q_\beta)\). This set is non-empty (it has power \(\mathfrak{c}\)) for each \(\alpha\), and so \(p_\alpha\) and \(q_\alpha\) are defined for each \(\alpha < \omega_c\). Put \(B = \{p_\alpha : \alpha < \omega_c\}\). Since \(p_\alpha \in B \cap F_\alpha\) and \(q_\alpha \in B^c \cap F_\alpha\) for each \(\alpha < \omega_c\), the set \(B\) has the property that both it and its complement meet every uncountable closed set, i.e. \(B\) is a Bernstein set. \(\square\)

First, we shall show that Bernstein sets are non-measurable by proving the following theorem.
**Theorem 3.4.** Every measurable subset of either $B$ or $B^c$ is a null set, where $B$ is a Bernstein set.

*Proof.* Let $A$ be any measurable subset of $B$. Since every uncountable closed set meets $B^c$, any closed set $F$ contained in $A$ must be countable, hence $m(F) = 0$. By Corollary 2.4, we must have $m(A) = 0$. □

**Corollary 3.5.** Any Bernstein set $B$ is non-measurable.

*Proof.* Suppose $B$ is measurable. By Proposition 2.2, $B^c$ is also measurable. Then the theorem above asserts that both $B$ and $B^c$ are null sets, which is absurd since $\mathbb{R} = B \cup B^c$ but $\mathbb{R}$ cannot be represented as the union of two null sets. □

To prove that every Bernstein set doesn’t have the property of Baire, we need the following lemma.

**Lemma 3.6.** Any uncountable $G_\delta$ subset of $\mathbb{R}$ contains an uncountable closed subset.

The construction is similar to that of the Cantor set, here we omit the proof. With this lemma in hold, we can prove that:

**Theorem 3.7.** Any subset of $B$ or $B^c$ that has the property of Baire is of first category.

*Proof.* Suppose $B$ is a Bernstein set. If $A$ is a subset of $B$ having the property of Baire, then $A = E \cup P$, where $E$ is $G_\delta$ and $P$ is of first category. The set $E$ must be countable, since by the lemma above, every uncountable $G_\delta$ set contains an uncountable closed set, and therefore meets $B^c$. Hence $A$ is of first category. The same reasoning applies to $B^c$. □

As a corollary, every Bernstein set doesn’t have the property of Baire.

**Corollary 3.8.** Any Bernstein set $B$ lacks the property of Baire.

*Proof.* Suppose $B$ has the property of Baire. By Lemma 2.12, $B^c$ also has the property of Baire. Then the theorem above asserts that both $B$ and $B^c$ are of first category, which is absurd since $\mathbb{R} = B \cup B^c$ but as a corollary of Theorem 2.9, $\mathbb{R}$ is not of first category. □

Finally, we want to show that Bernstein sets don’t have the perfect set property.

**Definition 3.9.** A nonempty closed set is *perfect* if it has no isolated points. A set has the *perfect set property* if it either is countable or has a perfect subset.

By a theorem in set theory, we have

**Theorem 3.10.** Every perfect set has cardinality $\mathfrak{c}$.

The proof can be found in [3]. This leads to

**Theorem 3.11.** Any Bernstein set $B$ lacks the perfect set property.

*Proof.* If $B$ is countable, then $B$ is of first category, hence has the property of Baire, which contradicts Corollary 3.8. Therefore $B$ is uncountable. Every closed set contained in $B$ is countable, otherwise it’s an uncountable closed set that doesn’t meet $B^c$, which contradicts with $B$ being a Bernstein set. However, by the above theorem, every perfect set has cardinality $\mathfrak{c}$. Therefore $B$ doesn’t have a perfect subset. As a conclusion, $B$ lacks the perfect set property. □
References

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