Multidimensional Asymptotic Consensus in Dynamic Networks

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Abstract

We study the problem of asymptotic consensus as it occurs in a wide range of applications in both
man-made and natural systems. In particular, we study systems with directed communication graphs
that may change over time.

We recently proposed a new family of convex combination algorithms in dimension one whose weights
depend on the received values and not only on the communication topology. Here, we extend this
approach to arbitrarily high dimensions by introducing two new algorithms: the ExtremePoint and the
Centroid algorithm. Contrary to classical convex combination algorithms, both have component-wise
contraction rates that are constant in the number of agents. Paired with a speed-up technique for convex
combination algorithms, we get a convergence time linear in the number of agents, which is optimal.

Besides their respective contraction rates, the two algorithms differ in the fact that the Centroid
algorithm’s update rule is independent of any coordinate system while the ExtremePoint algorithm
implicitly assumes a common agreed-upon coordinate system among agents. The latter assumption may
be realistic in some man-made multi-agent systems but is highly questionable in systems designed for
the modelization of natural phenomena.

Finally we prove that our new algorithms also achieve asymptotic consensus under very weak con-
nnectivity assumptions, provided that agent interactions are bidirectional.

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1 Introduction

The problem of agents converging to a common final position, also known as asymptotic consensus, is of utmost importance in a wide range of networking problems. One can cite not only artificial, man-made, systems like sensor fusion [11], clock synchronization [12], formation control [10], rendezvous in space [13], or load balancing [7], but also the modelization of natural phenomena like flocking [22], firefly synchronization [16], or opinion dynamics [11].

Algorithms for asymptotic consensus repeatedly form convex combinations of their neighbors’ positions and move their current position there. Classically, these algorithms use weights in the convex combination that only depend on the communication topology, i.e., the set of the agent’s neighbors, but not on their current positions. This results in weights that are inversely proportional to the number of neighbors, e.g., the EqualNeighbor algorithm, the MaxDegree algorithm [17], or the Metropolis algorithm [23]. Their analysis consists in studying the stochastic matrices made up of the weights used by agents in their convex combinations. An important property of these associated stochastic matrices is that they inherit irreducibility properties from connectivity properties of the communication graph.

In the present article, we study the problem of asymptotic consensus in dynamic networks, in the challenging context of directed communication graphs that may change over time.

In a recent article [4], we proposed a new family of convex combination algorithms in dimension one whose weights depend on the received values. A particular example of such an algorithm is the MidPoint algorithm, whose contraction rate of $1/2$ is optimal. The analysis of these algorithms required the development of a new approach since the graphs associated to the stochastic matrices do not coincide anymore with the communication graphs and thus do not benefit from the same connectivity properties.

The goal of the present article is to extend this approach to multiple dimensions. For this, we present two generalizations of the MidPoint algorithm to the case of an arbitrary dimension $d$: the ExtremePoint and the Centroid algorithm. Contrary to classical convex combination algorithms like EqualNeighbor, both have component-wise contraction rates that are constant in the number of agents, namely $1 - \frac{1}{2d}$ for the ExtremePoint algorithm and $1 - \frac{1}{d+1}$ for the Centroid algorithm. Paired with a speed-up technique, we get a convergence time linear in the number of agents for both algorithms, which is optimal.

Besides their respective contraction rates, the two algorithms differ in the fact that the Centroid algorithm’s update rule is independent of any coordinate system while the ExtremePoint algorithm implicitly assumes a common agreed-upon coordinate system among agents. The latter assumption may be realistic in some man-made multi-agent systems but is highly questionable in systems designed for the modelization of natural phenomena.

The analysis of the two multi-dimensional algorithms that we propose is based on the notion of $\alpha$-safeness. This property guarantees that every agent stays in the convex hull of its neighbors and keeps a certain safety margin to its boundary which depends on the parameter $\alpha$. While the proof of safeness for the ExtremePoint algorithm is relatively straightforward, the proof for the Centroid algorithm uses a Steiner-type symmetrization and relies on the Brunn-Minkowski inequality. Apart from its application to asymptotic consensus, this last result may be of independent geometrical interest.

Finally we prove that our new algorithms share a remarkable property with the classical asymptotic consensus algorithms. Namely convergence is achieved under very weak connectivity assumptions, provided that agent interactions are bidirectional. This last point adds to a list of properties of the Centroid algorithm that makes it a well-suited candidate for the modelization of natural systems [6].

The paper is organized as follows. In Section 2 we introduce the model and problem statement. In Section 3 we recall results on one-dimensional asymptotic consensus. Section 4 generalizes the optimal one-dimensional algorithm to multiple dimensions in a component-wise fashion. The coordinate-free Centroid algorithm is presented in Section 5. Section 6 presents extensions to weaker connectivity assumptions with bidirectional communication graphs. We conclude the paper in Section 7.
2 The Model

We consider a set \([n] = \{1, \ldots, n\}\) of agents. We assume a distributed, round-based computational model in the spirit of the Heard-Of model \([3]\). Computation proceeds in rounds: in a round, each agent sends its state to its outgoing neighbors, receives messages from its incoming neighbors, and finally updates its state according to a deterministic local algorithm, i.e., a transition function that maps the collection of incoming messages to a new state. Rounds are communication closed in the sense that no agent receives messages in round \(t\) that are sent in a round different from \(t\).

Communications that occur in a round are modeled by a directed graph with a node for each agent. Since an agent can obviously communicate with itself instantaneously, every communication graph contains a self-loop at each node.

We fix a non-empty set of such directed graphs \(\mathcal{N}\) that determines the network model. To fully model dynamic networks in which topology may change continually and unpredictably, the communication graph at each round is chosen arbitrarily among \(\mathcal{N}\). Thus we form the infinite sequences of graphs in \(\mathcal{N}\) which we call communication patterns in \(\mathcal{N}\). In each communication pattern, the communication graph at round \(t\) is denoted by \(G_t = ([n], E_t)\), and \(\text{In}_p(t)\) and \(\text{Out}_p(t)\) are the sets of incoming and outgoing neighbors (in-neighbors and out-neighbors for short) of agent \(p\) in \(G_t\).

In the following, we use the product of two communication graphs \(G\) and \(H\), denoted \(G \circ H\), which is the directed graph with an edge from \(p\) to \(q\) if there exists \(r\) such that \((p, r) \in E(G)\) and \((r, q) \in E(H)\).

2.1 Asymptotic Consensus

The state or position of agent \(p\) is captured by a variable \(x_p\) in an Euclidean \(d\)-space, and we let \(x_p(t) \in \mathbb{R}^d\) denote the position of \(p\) at round \(t\). Thus the \(n\)-tuple \(x(t) = (x_1(t), \ldots, x_n(t))\) corresponds to the global configuration of the multi-agent system at round \(t\). We denote the \(k\)th component of \(x_p(t)\) by \(x_{p,k}(t)\).

We say an algorithm solves asymptotic consensus in a network model \(\mathcal{N}\) if the following holds for every initial configuration \(x(0)\) and every communication pattern in \(\mathcal{N}\):

Convergence. Each sequence \(x_p(t)\) converges.

Agreement. If \(x_p(t)\) and \(x_q(t)\) converge, then they have a common limit.

Validity. If \(x_p(t)\) converges, then its limit is in the convex hull of the initial states.

Our results can be easily translated to the approximate consensus problem, in which convergence is replaced by a decision in a finite number of rounds and where agreement should be achieved with an arbitrarily small error tolerance (see, e.g., \([14, 15]\)).

2.2 Convex Combination Algorithms

Because of the validity condition, the natural class of algorithms for solving asymptotic consensus is the class of the convex combination algorithms, also called averaging algorithms in the case of dimension one: at each round \(t\), every agent \(p\) updates \(x_p\) to some convex combination of the positions it has just received, i.e., the positions of its in-neighbors in the communication graph at round \(t\). That is

\[
x_p(t) = \sum_{q \in \text{In}_p(t)} w_{pq}(t) x_q(t-1),
\]

where weights \(w_{pq}(t)\) are non-negative real numbers with \(\sum_{q \in \text{In}_p(t)} w_{pq}(t) = 1\). In other words, at each round \(t\), every agent adopts a new position within the convex hull of its in-neighbors in the communication graph \(G_t\).

Since we strive for distributed implementations of convex combination algorithms, \(w_{pq}(t)\) is required to be locally computable by \(p\). For example, weights may depend only on the set of \(p\)’s in-neighbors, as is the case in the EqualNeighbor algorithm, with

\[
w_{pq}(t) = 1/|\text{In}_p(t)|,
\]
for every in-neighbor \( q \) of \( p \). Weight \( w_{pq}(t) \) may also depend on the positions of the in-neighbors of \( p \), as is the case, for instance, with the update rule

\[
w_{pq}(t) = \delta_{qq_0},
\]

where \( \delta \) is the Kronecker delta and \( q_0 \) is one in-neighbor of \( p \) in \( G_t \) with the largest first component, i.e.,

\[
x_{q_0,1}(t) = \max\{x_{q,1}(t) : q \in \text{In}_p(t)\}.
\]

When the structure of states allows each agent to record and to relay information it has received during any period of \( L \) rounds for some positive integer \( L \), we may be led to modify time-scale and to consider blocks of \( L \) consecutive rounds, called macro-rounds: macro-round \( s \) is the sequence of rounds \( (s-1)L+1, \ldots, sL \) and the corresponding information flow graph, called communication graph at macro-round \( s \), is the product of the communication graphs \( G_{(s-1)L+1} \circ \cdots \circ G_{sL} \).

### 2.3 Solvability of Asymptotic Consensus

In a previous paper [3], we proved the following characterization of network models in which asymptotic consensus is solvable.

**Theorem 1** ([3]). *In any dimension \( d \), the asymptotic consensus problem is solvable in a network model \( N \) if and only if each graph in \( N \) has a rooted spanning tree.*

The proof of the sufficient condition of rooted network model is based on a reduction to nonsplit network models: a directed graph is nonsplit if any two nodes have a common in-neighbor. Indeed we showed the following general proposition.

**Proposition 2.** *Every product of \( n-1 \) rooted graphs with \( n \) nodes and self-loops at all nodes is nonsplit.*

### 2.4 Convergence Rate and Convergence Time

Following [20], in the case convergence is achieved for some initial configuration \( x(0) \) and some communication pattern, we introduce

\[
\max_{p \in [n]} \lim_{t \to \infty} \sup \| x_p(t) - x_p^* \|^{1/t}
\]

(3)

where \( x_p^* = \lim_{t \to \infty} x_p(t) \) and \( \| \cdot \| \) is any norm on \( \mathbb{R}^d \). This quantity lies in \([0, 1]\). Moreover, it is independent of the norm \( \| \cdot \| \) because of the equivalence of norms in \( \mathbb{R}^d \).

For an algorithm that solves asymptotic consensus in a network model \( N \), we define its convergence rate \( \varrho \) as the supremum of (3) over all initial configurations and all communication patterns with graphs in \( N \).

Regarding approximate consensus and considering the infinity norm on \( \mathbb{R}^n \), we define the convergence time, \( T(\varepsilon) \), by

\[
\max_{k \in [d]} \inf \{ \tau : \forall t \geq \tau, \delta(x^k(t)) \leq \varepsilon \delta(x^k(0)) \}
\]

(4)

where \( \delta \) is the semi-norm on \( \mathbb{R}^n \) defined by \( \delta(u_1, \ldots, u_n) = \max_{p \in [n]} (u_p) - \min_{p \in [n]} (u_p) \).

### 3 The Case of Dimension One

We now briefly present our analysis techniques for the one-dimensional case, which we generalize to arbitrary dimensions in Sections 4 and 5. In [4], we proposed a new analysis of the convex combination algorithms in the specific case of dimension one: We considered the property of \( \alpha \)-safeness for averaging algorithms which is a generalization of the lower bound condition on positive weights. This property focuses on the interval of transmitted values and not on the linear functions (stochastic matrices) applied in the averaging steps, as done classically. It thus captures the essential properties needed for contracting the range of current values in the system. This approach led us to propose the first algorithm for asymptotic consensus in dynamic rooted networks, with a convergence time that is linear in the number of agents.
3.1 Nonsplit Network Models

Let \( \alpha \in [0, 1/2] \); an averaging algorithm is \( \alpha \)-safe if at any round \( t \), each agent adopts a new value within the interval formed by its neighbors in \( G_t \) not too close to the boundary:

\[
\alpha M_p(t) + (1 - \alpha) m_p(t) \leq x_p(t + 1) \leq (1 - \alpha)M_p(t) + \alpha m_p(t),
\]

where \( m_p(t) = \min_{q \in I_p(t)} (x_q(t)) \) and \( M_p(t) = \max_{q \in I_p(t)} (x_q(t)) \).

Besides, contracting the range of current values in the system is clearly a good mechanism to achieve agreement: an averaging algorithm is \( c \)-contracting in \( \mathcal{N} \) if at each round \( t \) of each of its executions with communication patterns in \( \mathcal{N} \), we have

\[
\delta(x(t)) \leq c \delta(x(t - 1)).
\]

A result from [2] states that the property of \( c \)-contraction is also sufficient to enforce the convergence of averaging algorithms. Then the main point lies in the fact that in a nonsplit network model, an \( \alpha \)-safe averaging algorithm is \((1 - \alpha)\)-contracting. We thus prove the following result.

**Theorem 3.** In a nonsplit network model, an \( \alpha \)-safe averaging algorithm solves asymptotic consensus with a convergence rate \( \varrho \leq 1 - \alpha \) and a convergence time \( T(\varepsilon) \leq \left\lceil \frac{\log_{1/n} \delta(0)}{\varepsilon} \right\rceil \).

As an immediate consequence of Theorem 3, we obtain that the EqualNeighbor algorithm, that is \((1/n)\)-safe, has a convergence rate bounded by \( 1 - 1/n \) and a convergence time in \( O(n \log (\delta(0)/\varepsilon)) \) in any nonsplit network model.

To improve these bounds, we introduced the **MidPoint algorithm** in which weights depend on the set of transmitted values and not on the sole communication graph: each agent adopts the mid-point of the range of values it has received, that is

\[
x_p(t + 1) = \frac{m_p(t) + M_p(t)}{2}.
\]

Clearly the MidPoint algorithm is \( 1/2 \)-safe, and so has a maximal contraction rate of \( 1/2 \) in any nonsplit network model, leading to a convergence rate of \( 1/2 \) and a convergence time \( T(\varepsilon) \leq \left\lceil \log_2 \frac{\delta(0)}{\varepsilon} \right\rceil \).

3.2 Rooted Network Models

One can easily show that if an averaging algorithm is \( \alpha \)-safe, then it is \( \alpha^{L} \)-safe with the coarser-grained granularity of macro-rounds composed of \( L \) consecutive rounds. Combined with Proposition 2 it follows that the EqualNeighbor algorithm solves asymptotic consensus in any rooted network model. Unfortunately, the convergence rate and convergence time satisfy

\[
1 - \varrho = \Omega(n^{-\varrho}) \quad \text{and} \quad T(\varepsilon) = O \left( n^{\varrho} \log \frac{\delta(0)}{\varepsilon} \right),
\]

and these exponential bounds have been proved to be tight [4].

To overcome this time-complexity lower bound of averaging algorithms, we introduced the amortization technique [4] which consists in inserting a value-gathering phase of \( n - 1 \) rounds before each averaging step. This additional phase transforms \( \alpha \)-safe algorithms into “turbo versions” of themselves in that convergence times pass from being exponential to being polynomial in the number of agents. Amortization assumes implicitly that all agents know the size \( n \) of the network. Moreover, it requires a priori to increase bandwidth channels and local storage capacities by a factor \( n \).

In anonymous networks, the amortization technique applies only to averaging algorithms with weights that depend only on the sets of received values without any multiplicity concern. In contrast to the EqualNeighbor algorithm, MidPoint thus admits an amortized version, called the **Amortized MidPoint** algorithm. For its correctness and time-analysis, we just need to observe that the Amortized MidPoint algorithm reduces to the MidPoint algorithm with the granularity of macro-rounds consisting in blocks of \( n - 1 \) consecutive rounds.
In round $t \geq 1$ do:
3: send $(m_p, M_p)$ to all agents in $\text{Out}_p(t)$ and receive $(m_q, M_q)$ from all agents $q$ in $\text{In}_p(t)$
4: $m_p \leftarrow \min \{m_q \mid q \in \text{In}_p(t)\}; \quad M_p \leftarrow \max \{M_q \mid q \in \text{In}_p(t)\}$
5: if $t \equiv 0 \mod n - 1$ then
6: $x_p \leftarrow (m_p + M_p)/2$
7: $m_p \leftarrow x_p; \quad M_p \leftarrow x_p$
8: end if

Algorithm 1 Amortized MidPoint algorithm for agent $p$

**Initialization:**
1: $x_p \leftarrow$ initial position of $p$
2: $m_p \leftarrow x_p; \quad M_p \leftarrow x_p$

**Theorem 4.** In a rooted network model, the Amortized MidPoint algorithm solves asymptotic consensus with convergence rate $\rho \leq 1 - \frac{1}{2n}$ and convergence time $T(\varepsilon) \leq (n - 1) \left\lceil \log_2 \frac{\delta(0)}{\varepsilon} \right\rceil$.

Under the assumption that all agents know $n$, the Amortized MidPoint algorithm thus solves asymptotic consensus in linear-time and with only two values per agent and per message. A similar result has been recently obtained by Olshevsky [19] with a linear-time algorithm, but this algorithm works only with a fixed communication graph that further ought to be bidirectional and connected.

### 4 Component-Wise Algorithms for the Multi-Dimensional Case

We now tackle the problem of multi-dimensional asymptotic consensus, and present several algorithms that are all generalizations of MidPoint to higher dimension. For the analysis of these algorithms, we proceed component by component: a $d$-dimensional execution is equivalent to $d$ one-dimensional executions. In particular, we extend the property of $\alpha$-safeness, $\alpha \in [0,1/2]$, to a higher dimension by enforcing \[5\] along each dimension. Formally, a convex combination algorithm in dimension $d$ is $\alpha$-safe if for any $t \in \mathbb{N}$,

$$\alpha M_{p,i}(t) + (1 - \alpha) m_{p,i}(t) \leq x_{p,i}(t+1) \leq (1 - \alpha) M_{p,i}(t) + \alpha m_{p,i}(t) \quad (6)$$

where $m_{p,i}(t)$ is the minimum and $M_{p,i}(t)$ the maximum of the values \{ $x_{p,i}(t) \mid q \in \text{In}_p(t+1)$ \} in the $i^{th}$ component of the positions of the in-neighbors of $p$ in round $t + 1$, respectively. Although this definition syntactically depends on the chosen coordinate system, it is in fact coordinate-free. This can be seen by applying, to the set of agent positions, the inverse of the transformation taking one coordinate system to another. Also note that, in contrast to the one-dimensional case, \[6\] does not guarantee that the algorithm is a convex combination algorithm.

With this definition, Theorem \[3\] holds in higher dimension. Its proof is exactly the same, applied in each component.

Like in one dimension, one may use the amortization technique in higher dimension to go from nonsplit to rooted network models by paying a multiplicative price of $n - 1$ in terms of convergence time. It requires all agents to know the size $n$ of the network and applies to convex combination algorithms in which multiplicity is not taken into account in the weights of the position update rules. Also, it requires a priori to increase channel bandwidth and local storage capacities by a factor $n$.

#### 4.1 Asymptotic Consensus in Dimension Two

A component-wise application of the MidPoint algorithm is obviously 1/2-safe. Unfortunately the following example shows that it is not a convex combination algorithm when $d \geq 3$, and thus may violate the validity clause: the convex hull of the points $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ in $\mathbb{R}^3$ does not contain the component-wise midpoint $M = (1/2, 1/2, 1/2)$.

Nonetheless, the following lemma shows that taking the component-wise midpoint does not exit the convex hull in dimension two.
Lemma 5. Let \( C \) be a nonempty compact convex set in \( \mathbb{R}^2 \). Then \( \left( \frac{x_i^+ - x_i^-}{2}, \frac{x_i^2 - x_i^+}{2} \right) \in C \) where
\[
x_i^+ = \max\{x_i \mid \exists y \in C: y_i = x_i\} \quad \text{and} \quad x_i^- = \min\{x_i \mid \exists y \in C: y_i = x_i\}.
\]

Proof. Without loss of generality, we assume \( x_i^- = 0 \) and \( x_i^+ = 1 \) for \( i = 1, 2 \) by scaling and translation, and we shall show that \( m = (1/2, 1/2) \in C \).

Let \( a = (0, a_2) \in C \) be a point with minimal first component and \( b = (b_1, 0) \in C \) one with minimal second component. Intersecting the segment \( \{(\lambda b_1, (1 - \lambda) a_2) \mid \lambda \in [0, 1]\} \) that joins \( a \) to \( b \) with the first median, we get the point \( c \in C \) with coordinates:
\[
c = \begin{cases} 
\left( \frac{b_1 a_2}{b_1 + a_2}, \frac{b_1 a_2}{b_1 + a_2} \right) & \text{if } b_1 \neq 0 \text{ or } a_2 \neq 0, \\
(0, 0) & \text{if } b_1 = a_2 = 0.
\end{cases}
\]

In both cases, since \( b_1 a_2 \leq \min\{b_1, a_2\} \leq b_1 + a_2 \), we have \( c = (\alpha, \alpha) \) with \( \alpha \leq 1/2 \).

A symmetric argument for two points with maximal coordinates yields a point \( c' \) in \( C \) such that \( c' = (\beta, \beta) \) with \( \beta \geq 1/2 \). Observing that \[
\frac{1}{2} = \frac{\beta - 1/2}{\beta - \alpha} \cdot \alpha + \frac{1/2 - \alpha}{\beta - \alpha} \cdot \beta,
\]
we then write \( m \) as a convex combination of the two points \( c \) and \( c' \), which shows that \( m \) is in \( C \).

Consequently, the component-wise MidPoint algorithm actually is a convex combination algorithm in dimension two. By analyzing each component separately, our results on the MidPoint algorithm carry over from the one-dimensional to the two-dimensional case. In particular, we can apply the amortization technique, which yields the following result.

Theorem 6. In the particular case of dimension two, the component-wise MidPoint algorithm solves asymptotic consensus in any rooted network model with convergence rate \( \rho \leq 1 - \frac{1}{2d} \) and convergence time \( T(\varepsilon) \leq (n - 1) \left\lfloor \log_2 \frac{2(0)}{\varepsilon} \right\rfloor \).

Observe that the component-wise mid-point depends on the chosen coordinate system.

4.2 The ExtremePoint Algorithm

We now introduce an algorithm, called the ExtremePoint algorithm, that generalizes the MidPoint algorithm in arbitrary dimension. In this algorithm, every agent collects its in-neighbors’ positions, identifies among them two extreme points in each component, and then averages over these \( 2d \) extreme positions.

For each component, the update rule is an average of exactly \( 2 \) \( d \)-real numbers. We thus easily check that the ExtremePoint algorithm is \( 1/(2d) \)-safe. From Theorem 3 we derive that in any non-split network model, the ExtremePoint algorithm achieves asymptotic consensus with \( \rho \leq 1 - \frac{1}{2d} \) and \( T(\varepsilon) \leq \left\lfloor \log_{2d/(2d-1)} \frac{\delta(0)}{\varepsilon} \right\rfloor \).

As with MidPoint, the weights in the ExtremePoint algorithm depend only on the sets of received positions without any multiplicity. The algorithm thus admits an amortized version given in Algorithm 2. During the position-gathering phase, \( p \) keeps track of the positions of two in-neighbors with the smallest and the largest \( i \)-th component, for every component \( i \). Hence, \( p \) records exactly \( 2d \) points in each round, a number independent of \( n \). Then \( p \) moves to the centroid of these \( 2d \) extreme points.

By Proposition 2 the communication graph in each macro-round of \( n - 1 \) rounds is non-split. Combined with Theorem 3 we obtain the following theorem.

Theorem 7. In a rooted network model, the Amortized ExtremePoint algorithm solves asymptotic consensus with convergence rate \( \rho \leq 1 - \frac{1}{2dn} \) and convergence time \( T(\varepsilon) \leq (n - 1) \left\lfloor \log_{2d/(2d-1)} \frac{\delta(0)}{\varepsilon} \right\rfloor \).

Observe that in the above algorithm, new positions at each round (line 9) depend both on non-deterministic choices for the points \( m_p^{(i)} \) and \( M_p^{(i)} \) (lines 5–6) and on the chosen coordinate system.
Theorem 9

it a priori requires capabilities to store and relay up to d positions. Since the Amortized Centroid algorithm relays all positions during its gathering phase, constructions exist if one fixes the dimension d. Besides, natural systems may be equipped with natural means to determine centroids. Since the Amortized Centroid algorithm relays all positions during its gathering phase, it a priori requires capabilities to store and relay up to n positions per round. This is in contrast to the

Algorithm 2 Amortized ExtremePoint algorithm for agent p

Initialization:
1: \( x_p \leftarrow \text{initial position of } p \)
2: \( m_p^{(1)}, m_p^{(2)}, \ldots, m_p^{(d)} \leftarrow x_p; M_p^{(1)}, M_p^{(2)}, \ldots, M_p^{(d)} \leftarrow x_p \)

In round \( t \geq 1 \) do:
3: send \((m_p^{(1)}, m_p^{(2)}, \ldots, m_p^{(d)})\) to all agents in Out\(_p(t)\) and receive \((m_q^{(1)}, m_q^{(2)}, \ldots, m_q^{(d)})\) from all agents q in In\(_p(t)\)
4: for \( i \leftarrow 1 \) to d do
5: \( m_p^{(i)} \leftarrow m_q^{(i)} \) with minimal \( i^{\text{th}} \) component \( m_q^{(i)} \) where \( q \in \text{In}_p(t) \)
6: \( M_p^{(i)} \leftarrow M_q^{(i)} \) with maximal \( i^{\text{th}} \) component \( M_q^{(i)} \) where \( q \in \text{In}_p(t) \)
7: end for
8: if \( t \equiv 0 \mod n - 1 \) then
9: \( x_p \leftarrow \frac{1}{d} \left( \sum_{i=1}^{d} m_p^{(i)} + \sum_{i=1}^{d} M_p^{(i)} \right) \)
10: for \( i \leftarrow 1 \) to d do
11: \( m_p^{(i)} \leftarrow x_p; M_p^{(i)} \leftarrow x_p \)
12: end for
13: end if

5 The Multi-Dimensional Case: A Coordinate-Free Algorithm

Both asymptotic consensus algorithms presented in Section 4 treat agent positions component-wise, thus intrinsically assuming a common, agreed-upon coordinate system. The same applies for the work on multi-dimensional approximate consensus [15] where convergence is obtained by cycling through the coordinate components, converging component by component. While the assumption of a common coordinate system, depending on the application, may be plausible in some man-made systems, the assumption is highly questionable in natural systems such as swarms of birds or bacteria and social models in opinion dynamics.

We now present the Centroid algorithm, a generalization of the MidPoint algorithm that is coordinate-free in the sense that it does not require an a priori agreed-upon coordinate system: Each agent moves to the centroid of the convex hull of the positions of its in-neighbors in the current communication graph, with uniform mass distribution over the convex hull. While the ExtremePoint algorithm computes the centroid of a finite set of points with equal mass, the Centroid algorithm computes the centroid of the whole convex hull of these points.

The main point of this section is to show that by spreading the mass to the convex hull, we obtain an algorithm that is \( 1/(d+1) \)-safe. We give the proof sketch in Section 5.1.

Theorem 8. The Centroid algorithm is a \( 1/(d+1) \)-safe convex combination algorithm.

From Theorem 3 we thus obtain a convergence rate of \( 1 - \frac{1}{d+1} \) in nonsplit network models instead of \( 1 - \frac{1}{2d} \) for the ExtremePoint algorithm. Since the algorithm’s update rule does not take into account any multiplicity, the Centroid algorithm admits an amortized version given in Algorithm 3. We use hull(A) to denote the convex hull of a set \( A \subseteq \mathbb{R}^d \).

From Proposition 2 and Theorem 8 we finally obtain the following result.

Theorem 9. In a rooted network model, the Amortized Centroid algorithm solves asymptotic consensus with convergence rate \( q \leq 1 - \frac{1}{n(d+1)} \) and convergence time \( T(\varepsilon) \leq (n - 1) \left[ \log_{d+1}(d+1) \right] \frac{\delta(0)}{\varepsilon} \).

While the centroid of a body \( A \) cannot be efficiently computed in general, we are in the case of \( A \) being a convex bounded \( d \)-polytope with at most \( n \) vertices. Although exact computation of the centroid has been shown to be \#P-hard even for these bodies [21], polynomial (in \( n \)) algorithms based on simplex decompositions exist if one fixes the dimension \( d \). Besides, natural systems may be equipped with natural means to determine centroids. Since the Amortized Centroid algorithm relays all positions during its gathering phase, it a priori requires capabilities to store and relay up to \( n \) positions per round. This is in contrast to the
Algorithm 3 Amortized Centroid algorithm for agent $p$

Initialization:
1. $x_p \leftarrow$ initial position of $p$
2. $C_p \leftarrow \{x_p\}$

In round $t \geq 1$ do:
3. send $C_p$ to all agents in $\text{Out}_p(t)$ and receive $C_q$ from all agents $q$ in $\text{In}_p(t)$
4. $C_p \leftarrow C_p \cup \bigcup_{q \in \text{In}_p(t)} C_q$
5. if $t \equiv 0 \mod n - 1$ then
6. $x_p \leftarrow$ centroid of $\text{hull}(C_p)$
7. $C_p \leftarrow \{x_p\}$
8. end if

MidPoint and the ExtremePoint Amortized algorithms. Optimizations, however, exist that may pay off in certain applications: in code line 4, the non-extreme points of $\text{hull}(C_p)$ can be removed from $C_p$. While the frame, i.e., the set of extreme points, can be computed in polynomial time by solving linear programs [9], one may not be willing to pay this additional overhead in each round. Alternatively, computationally less intensive heuristics can be applied to remove many of the non-extreme points, see, e.g., [8].

5.1 Safeness Proof

We now tackle the proof of Theorem 8. First let us introduce some notation. Let us denote the $d$-dimensional volume of set $A \subseteq \mathbb{R}^d$ by $\text{vol}_d(A)$. For $A \subseteq \mathbb{R}^d$ and $j \in [d]$, let $m_j(A) = \inf_{x \in A} x_j$ and $M_j(A) = \sup_{x \in A} x_j$. We next define sets, representing geometric bodies, that are symmetric around the first axis. For each $\xi \in \mathbb{R}$, let $H_\xi = \{x \in \mathbb{R}^d \mid x_1 = \xi\}$ be the hyperplane in $\mathbb{R}^d$ orthogonal to the first axis, intersecting it at $(\xi, 0, \ldots, 0)$. Let $C_{\xi}^d(\gamma)$ be the $(d - 1)$-cube of edge length $\gamma$ that lies within hyperplane $H_\xi$ and is centered at point $(\xi, 0, \ldots, 0)$, i.e., $C_{\xi}^d(\gamma) = \{x \in \mathbb{R}^d \mid x_1 = \xi \land \max_{2 \leq j \leq d} |x_j| \leq \gamma/2\}$. For a function $\ell : \mathbb{R} \rightarrow \mathbb{R}_0^+$ we define the symmetric body $S(\ell)$ as

$$S(\ell) = \bigcup_{\xi \in \mathbb{R}} C_{\xi}^d(\ell(\xi)).$$ (7)

Roughly speaking, we proceed as follows. Each $\text{hull}(C_p)$ in code line 6 is a bounded convex polytope in $\mathbb{R}^d$. Fix agent $p$ and component $i$ along which $\alpha$ in (6) is minimized. We then use a Steiner-type symmetrization along the $i$th axis: We transform polytope $A = \text{hull}(C_p)$ into polytope $A' = \text{hull}(C_p')$ which is highly symmetric around the $i$th axis and whose $i$th centroid component is invariant under the transformation. Figure 1 depicts the idea of the transformation in dimension two: the symmetric body $A'$ is constructed such that cuts orthogonal to the first axis of $A'$ have same volume as their corresponding cuts in $A$. This ensures invariance of the first component of the centroid $c$. We then reduce the problem to the class of those $A'$ that are formed by a hyperpyramid extended by a $d$-box at its base. Among these we show the hyperpyramids without $d$-boxes to minimize $\alpha$ in (6), finally reducing $A'$ to hyperpyramids. From a lower bound on the distance of centroid$_1(A')$ to its base, and the fact that the involved transformations and reductions did not shift centroid$_1(A')$ away from its base, we are finally able to prove safeness.

We start with some auxiliary lemmas on symmetrized bodies. First we show that if one removes parts from a body whose first component are left of the body’s centroid, then the first component of the centroid moves to the right.

Lemma 10. Let $\ell, \ell' : \mathbb{R} \rightarrow \mathbb{R}_0^+$. If $\ell'(\xi) \leq \ell(\xi)$ for $\xi \leq \text{centroid}_1(S(\ell))$ and $\ell'(\xi) = \ell(\xi)$ for $\xi > \text{centroid}_1(S(\ell))$ then $\text{centroid}_1(S(\ell')) \geq \text{centroid}_1(S(\ell))$.

Proof. Abbreviate $c = \text{centroid}_1(S(\ell))$ and $c' = \text{centroid}_1(S(\ell'))$. It is

$$c = \int_{-\infty}^{\xi} \ell(\xi) d\xi + \int_{\xi}^{\infty} \ell(\xi) d\xi = \int_{-\infty}^{c} \ell(\xi) d\xi + \int_{c}^{\infty} \ell(\xi) d\xi,$$

$$c' = \int_{-\infty}^{\xi} \ell'(\xi) d\xi + \int_{\xi}^{\infty} \ell'(\xi) d\xi = \int_{-\infty}^{c} \ell'(\xi) d\xi + \int_{c}^{\infty} \ell'(\xi) d\xi.$$


Figure 1: Symmetrization of the polytope $A = \text{hull}(C_p)$ around the first axis. The original polytope $A$ is transformed into the symmetric polytope $A' = \text{hull}(C'_p)$ such that cuts orthogonal to the first axis have same volume. The transformation ensures invariance of the first component of the centroid $C$.

Algebraic manipulation yields

$$\int_{-\infty}^{c} (c - \xi) \ell(\xi)^{d-1} d\xi = \int_{c}^{\infty} (\xi - c) \ell(\xi)^{d-1} d\xi.$$ \hfill (8)

Because $c - \xi \geq 0$ whenever $\xi \leq c$ and since $\ell'(\xi) \leq \ell(\xi)$ for those $\xi$, we get

$$\int_{-\infty}^{c} (c - \xi) \ell'(\xi)^{d-1} d\xi \leq \int_{-\infty}^{c} (c - \xi) \ell(\xi)^{d-1} d\xi.$$ 

Together with (8) this gives

$$\int_{-\infty}^{c} (c - \xi) \ell'(\xi)^{d-1} d\xi \leq \int_{c}^{\infty} (\xi - c) \ell(\xi)^{d-1} d\xi.$$ 

Again, algebraic manipulation yields

$$c \leq \frac{\int_{-\infty}^{c} \xi \ell(\xi)^{d-1} d\xi + \int_{c}^{\infty} \xi \ell(\xi)^{d-1} d\xi}{\int_{-\infty}^{c} \ell'(\xi)^{d-1} d\xi + \int_{c}^{\infty} \ell(\xi)^{d-1} d\xi}.$$ 

The latter term is, in fact, equal to $c'$ because $\ell(\xi) = \ell'(\xi)$ for $\xi > c$. This shows $c \leq c'$. \hfill $\square$

Lemma 10 will play a crucial role when proving that among all symmetric bodies, we can restrict our attention to those symmetric bodies composed of a hyperpyramid and a $d$-box.

The following lemma finds the one body in this class that moves the centroid the furthest to the right, i.e., away from the apex.

**Lemma 11.** Let $L \geq 0$. For $h \in [0, L]$, and $\vartheta > 0$ let $\ell_{h, \vartheta} : \mathbb{R} \to \mathbb{R}^+_0$ be the function with

$$\ell(\xi) = \begin{cases} 
0 & \text{if } \xi \leq 0 \text{ or } \xi > L \\
\frac{\xi \vartheta}{h} & \text{if } \xi \in (0, h] \\
\vartheta & \text{if } \xi \in (h, L].
\end{cases}$$

Among all symmetric bodies $S(\ell_{h, \vartheta})$ with $h \in [0, L]$, and $\vartheta > 0$, the symmetric bodies $S(\ell_{L, \vartheta})$, with arbitrary $\vartheta > 0$, maximize the first centroid component. It is $\text{centroid}_1(S(\ell_{L, \vartheta})) = \frac{L d}{d+1} = \frac{M_1(S(\ell_{L, \vartheta})))}{d+1}$. 

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Proof. We first observe that \( S(\ell_{h,\varnothing}) \) can be decomposed into a hyperpyramid, without a base, of height \( h \) and a \( d \)-box as follows: Define \( P(h) \) by

\[
P(h) = \bigcup_{\xi \in [0,h]} \text{Cb}_\xi(\ell_{h,\varnothing}(\xi)).
\]

We observe that \( P(h) \) is the hyperpyramid, without base, in \( \mathbb{R}^d \) whose height is \( h \), whose apex is at \((0, \ldots, 0)\) and whose base is the \((d-1)\)-cube with side length \( \varnothing \), centered at the first axis and lying within the hyperplane \( H_h \). Define \( B(h') \), with \( h' = L - h \), by

\[
B(h') = \bigcup_{\xi \in [h,L]} \text{Cb}_\xi(\ell_{h,\varnothing}(\xi)) = \bigcup_{\xi \in [h,L]} \text{Cb}_\xi(\varnothing).
\]

We observe that \( B(h') \) is a \( d \)-box in \( \mathbb{R}^d \), with

\[
\text{vol}_d(B(h')) = \varnothing^{d-1}(L - h).
\]

It is \( P(h) \cap B(h') = \emptyset \) and \( S(\ell_{h,\varnothing}) = P(h) \cup B(h') \).

We will next compute the first component of the centroids of \( P(h) \) and \( B(h') \), allowing us to compute the first component of the centroid of \( S(\ell_{h,\varnothing}) \). For any \( \alpha \in (0, h] \), the cut \( X_\alpha = P(h) \cap H_\alpha = \text{Cb}_\alpha(\ell_{h,\varnothing}(\alpha)) \) has volume \( \text{vol}_{d-1}(X_\alpha) = \left(\frac{\alpha h}{d}\right)^{d-1} \). From the volume of a pyramid in \( \mathbb{R}^d \), we obtain \( \text{vol}_d(P(h)) = \frac{h^{d+1}}{d} \).

The first centroid component of \( P(h) \) thus is at

\[
x'_1 = \frac{1}{\text{vol}_d(P(h))} \int_0^h \alpha \text{vol}_{d-1}(X_\alpha) d\alpha = \frac{hd}{d+1}.
\]

By symmetry arguments the first centroid component of \( B(h') \) is at

\[
x''_1 = h + \frac{h'}{2}.
\]

The first centroid component of the combined geometric body \( P(h) \cup B(h') \) thus is at

\[
x_1 = \frac{x'_1 \text{vol}_d(P(h)) + x''_1 \text{vol}_d(B(h'))}{\text{vol}_d(P(h) \cup B(h'))} = \frac{d(L^2d + d^2 - dh^2 + h^2)}{2(d+1)(Ld - dh + h)}. \tag{9}
\]

We next distinguish between two cases for dimension \( d \):

1. For \( d = 1 \), we obtain from \( \text{[1]} \) that \( x_1(h) = L/2 \); and the lemma follows.

2. Otherwise, \( d \geq 2 \). Algebraic manipulation yields,

\[
\frac{dx_1(h)}{dh} = \frac{d}{2(d+1)} \left( 1 - \frac{L^2}{(Ld - dh + h)^2} \right) > 0,
\]

for \( 0 \leq h < L \) and \( d \geq 2 \). Thus, \( \max_{h \in [0,L]} x_1(h) = x_1(L) = \frac{Ld}{d+1} \); and the lemma follows also in this case.

We are now in position to show our major result on the Centroid algorithm in Theorem \( \text{[8]} \) we prove that for any convex bounded polytope \( A \) in \( \mathbb{R}^d \) and for every \( j \in [d] \), we have

\[
\left( 1 - \frac{d}{d+1} \right) M_j(A) + \frac{d}{d+1} m_j(A) \leq \text{centroid}_j(A) \leq \left( 1 - \frac{d}{d+1} \right) m_j(A) + \frac{d}{d+1} M_j(A). \tag{10}
\]

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Choose an arbitrary component \( j \in [d] \). Without loss of generality assume that \( j = 1 \), \( m_j(A) = 0 \) and \( M_j(A) > 0 \). It suffices to prove the right inequality in (10) to show (10) by the following argument: assume by means of contradiction that that the right inequality is valid for all \( A \), but there is an \( A \) for which the left is invalid. Then negating all first components of points in \( A \) yields a polytope that violates the right inequality; a contradiction to the initial assumption. It thus suffices to show

\[
\text{centroid}_1(A) \leq \frac{d}{d+1} M_1(A). \tag{11}
\]

We now construct symmetrized body \( A_s \) from \( A \) that has the same volume and the same first centroid component as \( A \). For that purpose we do a Steiner-type symmetrization of \( A \).

By a simple reduction to a smaller dimension, we may assume \( \text{vol}_d(A) > 0 \). Let \( v_\xi = \text{vol}_{d-1}(H_\xi \cap A) \), and let \( \ell_A \) be the function \( \mathbb{R} \to \mathbb{R}_0^+ \) with

\[
\ell_A(\xi) = \begin{cases} 
  v_\xi^{1/(d-1)} & \text{if } v_\xi > 0, \\
  0 & \text{if } v_\xi = 0.
\end{cases}
\]

Then let \( A_s = S(\ell_A) \). From (7), we have \( \text{vol}_{d-1}(H_\xi \cap A_s) = \text{vol}_{d-1}(C_\xi(\ell_A(\xi))) = \ell_A(\xi)^{d-1} = v_\xi \). Thus

\[
\text{vol}_{d-1}(H_\xi \cap A_s) = \text{vol}_{d-1}(H_\xi \cap A) \quad \text{and further,} \quad \text{vol}_d(A) = \int_{-\infty}^{+\infty} \text{vol}_{d-1}(A \cap H_\xi) \, d\xi = \int_{-\infty}^{+\infty} \text{vol}_{d-1}(A_s \cap H_\xi) \, d\xi = \text{vol}_d(A_s). \tag{12}
\]

Combining both yields,

\[
\text{centroid}_1(A) = \frac{1}{\text{vol}_d(A)} \int_{-\infty}^{+\infty} \xi \text{vol}_{d-1}(A \cap H_\xi) \, d\xi = \text{centroid}_1(A_s),
\]

i.e., the first component of the centroid is invariant under symmetrization. For ease of notation, abbreviate the first component of the centroid by \( c = \text{centroid}_1(A) \geq 0 \).

Figure 2 depicts the process of symmetrization around the first axis at an example in dimension two: the body \( A' \) on the right is constructed from \( A \) by symmetric 1-cubes (line segments) centered at \( \xi \in [m_1(A), M_1(A)] \) such that their volume (length) \( v_\xi = \text{vol}_1(H_\xi \cap A) = \text{vol}_1(H_\xi \cap A') \).

![Figure 2: Symmetrization of the polytope \( A = \text{hull}(C_p) \) around the first axis. The original polytope \( A \) is transformed into the polytope \( A' = \text{hull}(C'_p) \) such that \( m_1, M_1 \) and the first component of the centroid \( C \) are invariant under the transformation.](image)

First observe, that \( H_\xi \cap A = \emptyset \) for all \( \xi < m_1(A) = 0 \) and all \( \xi > M_1(A) \). From the fact that \( A \) is a bounded polytope, \( M_1(A) < \infty \) and \( \ell_A \) is bounded. Thus \( \ell_A \) is zero outside of \([0, M_1(A)]\).

We next show that function \( \ell_A \) is concave in \([0, M_1(A)]\). In fact, concavity of \( \ell_A \) in \([m_1(A), M_1(A)]\) is equivalent to convexity of \( A_s \).

Arbitrarily choose \( \alpha, \beta \in [0, M_1(A)] \), and \( t \in [0, 1] \). From convexity of \( A \) we have, \( (H_{t\alpha+(1-t)\beta} \cap A) \supseteq t(H_\alpha \cap A) + (1-t)(H_\beta \cap A) \) where “+” on the right side denotes the Minkowski sum of sets. Hence

\[
\ell_A(t\alpha + (1-t)\beta)^{d-1} = \text{vol}_{d-1}(H_{t\alpha+(1-t)\beta} \cap A) \geq \text{vol}_{d-1} \left( t(H_\alpha \cap A) + (1-t)(H_\beta \cap A) \right). \tag{12}
\]
Applying the Brunn-Minkowski inequality we obtain,
\[
\text{vol}_{d-1} \left( t(H_\alpha \cap A) + (1 - t)(H_\beta \cap A) \right)^{1/(d-1)} \geq t \text{vol}_{d-1} \left( H_\alpha \cap A \right)^{1/(d-1)} + (1 - t) \text{vol}_{d-1} \left( H_\beta \cap A \right)^{1/(d-1)}.
\]
Together with (12) this yields, \(\ell_A(t\alpha + (1-t)\beta) \geq t\ell_A(\alpha) + (1-t)\ell_A(\beta)\), i.e., the concavity of function \(\ell_A\) in \([0, M_1(A)]\).

By the following reduction argument, we can further assume that \(\ell_A\) has constant positive slope within \([0, c]\), i.e., for all \(\xi \in [0, c]\), \(\ell_A(\xi) = \xi c / \ell_A(c)\). Assume by means of contradiction that this is not the case and consider the continuous function \(\ell_A' : \mathbb{R} \to \mathbb{R}_+^\ast\) with
\[
\ell_A'(\xi) = \begin{cases} 
\xi c / \ell_A(c) & \text{if } \xi \in [0, c], \\
\ell_A(\xi) & \text{else}.
\end{cases}
\] (13)

Note that \(\ell_A\) and \(\ell_A'\) differ only within \([0, c]\), and that \(\ell_A'\) has constant positive slope \(c / \ell_A(c)\) within \([0, c]\). By concavity of \(\ell_A\), we have \(\ell_A'(\xi) \leq \ell_A(\xi)\) for all \(\xi \in [0, c]\). From Lemma 10 centroid_1(S(\ell'_A)) \geq centroid_1(S(\ell_A)). We may thus consider \(\ell_A'\) instead of \(\ell_A\); the reduction follows.

Let the maximum value achieved by \(\ell_A\) be \(\vartheta = \max_{\xi \in [0, M_1(A)]} \{\ell_A(\xi)\}\) and let \(h \in [0, M_1(A)]\) be the smallest value where \(\ell_A(h) = \vartheta\), i.e., the maximum is reached.

By a reduction argument, we now show that \(\ell_A\) can be assumed to have constant positive slope within \([0, \max(c, h)]\) and has value \(\vartheta\) within \([\max(c, h), M_1(A)]\). Assume by means of contradiction that this is not the case and consider the continuous function \(\ell_A' : \mathbb{R} \to \mathbb{R}_+^\ast\) with
\[
\ell_A'(\xi) = \begin{cases} 
\xi c / \ell_A(c) & \text{if } \xi \in [0, \max(c, h)], \\
\vartheta & \text{else}.
\end{cases}
\] (14)

We distinguish between two cases for \(h\):

1. In case \(h < c\), function \(\ell_A\) and \(\ell_A'\) may differ only within \([c, M_1(A)]\). By definition of \(\vartheta\), it holds that \(\ell_A(\xi) \leq \vartheta = \ell_A'(\xi)\) for all \(\xi \in [c, M_1(A)]\). We may thus apply Lemma 10 and obtain that centroid_1(S(\ell'_A)) \geq centroid_1(S(\ell_A)); the reduction follows in this case.

2. Otherwise, \(h \geq c\) and function \(\ell_A\) and \(\ell_A'\) may differ only within \([h, M_1(A)]\). By concavity of \(\ell_A\), we have that \(\ell_A'(\xi) \leq \ell_A(\xi)\) for all \(\xi \in [c, h]\). Further, by definition of \(\vartheta\), we have \(\ell_A(\xi) \leq \vartheta = \ell_A'(\xi)\) for all \(\xi \in [h, M_1(A)]\). Again, we apply Lemma 10 and obtain that centroid_1(S(\ell'_A)) \geq centroid_1(S(\ell_A)); the reduction also follows in this case.

We thus obtain that \(\ell_A\) is of the form as required by Lemma 11, with \(L = M_1(A)\), \(h\) and \(\vartheta\). This yields (11), which concludes the proof.

6 ExtremePoint and Centroid with Disconnectivity

The aim of this section is to study the behavior of the ExtremePoint and the Centroid algorithms under very weak connectivity assumptions. Namely, we prove that the striking convergence properties of the convex combination algorithms with non-vanishing and bounded weights (e.g., EqualNeighbor) extend to both the ExtremePoint and the Centroid algorithms and, more generally, to every convex combination algorithm that is \(\alpha\)-safe. The result is based on the fundamental convergence theorem on infinite product of stochastic matrices proved by Moreau in (18) that we recall now.

Let \((A(t))_{t \in \mathbb{N}}\) be a sequence of stochastic matrices of size \(n\) and let \(G(t)\) denote the directed graph associated to \(A(t)\). The edges that appear infinitely often in the directed graphs \(G(t)\) define a directed graph denoted \(G^\infty\). The following assumptions are made about the matrices \(A(t)\):

**A1** Each matrix \(A(t)\) has a positive diagonal, i.e., \(A_{pp}(t) > 0\) for all \(p \in [n]\).
A2 There exists some \( a \in [0, 1] \) such that \( A_{pq}(t) \in \{0\} \cup [a, 1] \) for all \( p, q \in [n] \) and all \( t \in \mathbb{N}^* \).

A3 For each \( t \in \mathbb{N}^* \), the directed graph \( G(t) \) is bidirectional.

A4 The directed graph \( G^\infty \) is strongly connected.

**Theorem 12** \([15]\). Under assumptions A1–A4, the left-infinite product of stochastic matrices \( \prod_{t=1}^\infty A(t) \) converges to a stochastic matrix with identical rows.

This theorem is remarkable because it shows that in the case of bidirectional interactions, without any connectivity assumptions, every convex combination algorithm with non-vanishing and bounded weights converges and achieves asymptotic consensus among agents that are not disconnected from some time on.

Indeed, let \((G_t)_{t \geq 1}\) be a communication pattern composed of bidirectional directed graphs such that the directed graph of the edges that appear infinitely often is strongly connected. In other words, the agents are infinitely often connected. We consider a convex combination algorithm with non-vanishing weights that are lower bounded by some \( a > 0 \), i.e.,

\[
\forall (p, q) \in E_t, \ w_{pq}(t) \geq a.
\]  

(15)

Let \( W(t) \) denote the \( n \times n \) stochastic matrix with entries \( w_{pq}(t) \). The important point of \([15]\) lies in the fact that the associated graph of the matrix \( W(t) \) then coincides with the communication graph at round \( t \). Hence assumptions A1–A4 are fulfilled by all the matrices \( W(t) \). Theorem \([12]\) shows that with the communication pattern \((G_t)_{t \geq 1}\) and any initial configuration \( x(0) \in (\mathbb{R}^d)^n \), the convex combination algorithm achieves asymptotic consensus.

The key point now is that in the case of dimension one, every \( \alpha \)-safe averaging algorithm satisfies \([15]\) with \( a = \alpha/n \).

**Proposition 13.** Let \((v_1, \ldots, v_n)\) any \( n \)-tuple of real numbers such that \( v_1 \leq \ldots \leq v_n \), and let \( \alpha \) be a real number in \([0, 1/2]\). For every \( x \) in the interval \([[(1-\alpha) v_1 + \alpha v_n, \alpha v_1 + (1-\alpha) v_n] \), there exist \( n \) real numbers \( a_1, \ldots, a_n \) in the interval \([\alpha/n, 1] \) such that \( x = a_1 v_1 + \ldots + a_n v_n \) and \( a_1 + \ldots + a_n = 1 \).

**Proof.** Let us consider the simplex

\[
S_n^\alpha = \{(a_1, \ldots, a_n) \in [\alpha/n, 1] \mid a_1 + \ldots a_n = 1\},
\]

and let us denote

\[
S_n^\alpha \cdot v = \{a_1 v_1 + \cdots + a_n v_n \mid (a_1, \ldots, a_n) \in S_n^\alpha\}.
\]

We easily check that

\[
S_n^\alpha = \alpha \cdot (1/n, \ldots, 1/n) + (1-\alpha) \cdot S_n^0.
\]

Hence \( S_n^\alpha \cdot v \) is the compact interval

\[
S_n^\alpha \cdot v = [\alpha \overline{v} + (1-\alpha) v_1, \alpha \overline{v} + (1-\alpha) v_n]
\]

where \( \overline{v} = (v_1 + \cdots + v_n)/n \). Since \( v_1 \leq \overline{v} \leq v_n \), we have

\[
\alpha \overline{v} + (1-\alpha) v_1 \leq (1-\alpha) v_1 + \alpha v_n \leq \alpha v_1 + (1-\alpha) v_n \leq \alpha \overline{v} + (1-\alpha) v_n,
\]

and so \( S_n^\alpha \cdot v \) contains the interval \([[(1-\alpha) v_1 + \alpha v_n, \alpha v_1 + (1-\alpha) v_n]\), which completes the proof.

Then we derive the following theorem applying to the ExtremePoint and the Centroid algorithms.

**Theorem 14.** Asymptotic consensus is achieved in any execution of a convex combination algorithm that is \( \alpha \)-safe if communication graphs are all bidirectional and if the agents are infinitely often connected.
Proof. We apply Proposition 13 for each component: every round of an $\alpha$-safe algorithm thus corresponds to a $nd \times nd$ block diagonal matrix. The $k^{th}$ block for the $k^{th}$ component is an $n \times n$ stochastic matrix, denoted $A_k(t)$, and its associated graph is exactly $G_t^{-1}$.

Each matrix $A_k(t)$ satisfies assumptions A1–A4 with $a = \alpha/n$. By Theorem 12, all the agents converge to the same position $x^* \in \mathbb{R}^d$, and thus the convergence and agreement conditions are satisfied.

For the validity condition, we just observe that at round $t$, each agent moves within the convex hull of its neighbors. Hence the limit position $x^*$ is in the convex hull of the initial positions.

\[ \square \]

7 Conclusion

In this article we introduced three algorithms for multidimensional asymptotic consensus. All three of them work in dynamic networks with directed communication graphs that may change over time, with fast convergence rates. The algorithms are generalizations of the optimal MidPoint algorithm in dimension one. Two of them, the ExtremePoint and Centroid algorithms, work in an arbitrarily high dimension $d$ and are $\frac{1}{d}$ and $\frac{1}{d+1}$-safe, respectively. Our amortization technique thus makes their convergence time linear in the number of agents, which is optimal.

Moreover, we showed that all three algorithms solve asymptotic consensus under very weak connectivity assumptions in bidirectional communication graphs.

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\[4\text{Since Proposition 13 is applied component-by-component, there is no guarantee that for a given round } t, \text{ the matrices } A_1(t), \ldots, A_d(t) \text{ are equal.}\]
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