On the breakdown of space-time via constraint quantization of $d \geq 2$ General Relativity

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Abstract.
Based on the canonical quantization of $d \geq 2$ dimensional General Relativity (GR) via the Dirac constraint formalism (also termed as ‘constraint quantization’), we propose the loss of covariance as a fundamental property of the theory. This breakdown occurs for the first-order Einstein Hilbert action, whereby besides first class constraints, second class constraints also exist leading to non-standard ghost fields which render the path integral non-covariant. We also attempt, for the first time, the canonical quantization via calculation of the path integral for the equivalent Hamiltonian formulation of GR for which only first class constraints exist. However, loss of covariance still happens in this action due to structures arising from non-covariant constraints in the path integral. In contrast, we find that covariance as a symmetry is restored and quantization with perturbative calculations are possible in the weak limit of the gravitational field of these actions. Hence, we firstly infer that the breakdown in space-time is a property of GR itself (for $d \geq 2$ dimensions). We further propose that the breakdown of space-time occurs as a non-perturbative feature of GR in the strong field limit of the theory. Besides GR, we also note that covariance is preserved when constraint quantization is conducted for non-Abelian gauge theories, such as the Yang-Mills theory. These findings are novel from a canonical gravity formalism standpoint, and are consistent with GR singularity theorems which indicate breakdown at a strong field limit of the theory. They also support emergent theories of spacetime and gravity, though do not require thermodynamics such as entropic gravity. From an effective field theory view, these indicate that new degrees of freedom in the non-perturbative sector of the full theory are a requirement, whereby covariance as a symmetry is broken in the high energy (strong field) sector. Our findings are also consistent with the recent resolution of the information loss paradox in black holes.

Keywords: General Relativity, Quantum Gravity, Space-time

1. Introduction

The quantization of General Relativity (GR) leading to a full and coherent theory of gravitation at quantum mechanical scales is an ongoing and active area of theoretical physics research. Quantum Gravity (QG), a theory which extends GR to such scales is
being developed by a diversity of ways including quantum field theoretical approaches. In this regard, Dirac laid the foundation for quantizing gravity via his constraint formalism [1,2]. Pirani et al. [3] formulated a Hamiltonian formulation of GR action based on this methodology, and Dirac later developed another Hamiltonian based formulation (HGR) aimed to quantize gravity via quantum field theory [4]. Einstein [5] also formulated GR into what is now known as the first order (1EH) Einstein-Hilbert action. In this letter, based on recent attempts to quantize GR via the Dirac constraint formalism and path integral approach (which we will term as ‘constraint quantization’), for actions, we present the finding loss of covariance in quantization, though different canonical constraint structures arise from each type of representation. We arrive at the conclusion that this breakdown in space-time is non-perturbative, as via the same constraint quantization successful quantization is possible in the weak limit of the gravitational field, which indicates that covariance is only upheld in this limit. Hence, the breakdown in space-time is a non-perturbative property of GR (for \( d \geq 2 \)) itself. In the following section, we present a summary of results derived in [9-13] and a first attempt of canonical quantization of the HGR action in [3,4]. Based on these findings we propose that the loss of manifest covariance using the constraint quantization approach is due to its application on the full action. To support our claims, we demonstrate that this breakdown is actually a non-perturbative property of \( d \)-dimensional GR theory (\( d \geq 2 \)) itself from the recovery of covariance and renormalizability from linearized versions (which have different constraint structures for 1EH and HGR actions [14,15]). This is recent work focusing on 1EH and second order (2EH) actions using background and Lagrange multiplier fields and the path integral approach [18-25]. These findings are consistent with existing work on space-time breakdown in GR in the case of black holes by Penrose [26,27], whereby this breakdown in GR happens in the strong limit of the gravitational field of a black hole, and also with Effective Field Theory (EFT) results which hold at one loop order in the low-energy limit of the theory [28, 29]. Moreover, we find it to also be consistent with the recent resolution of the Black Hole Information paradox by Page [30] and Marloff and Maxfield [31], which also indicates breakdown in the space-time fabric as an essential ingredient of their explanation. We also note that space-time is an emergent feature, however there is loss of covariance in the high-energy (strong field limit) of the theory, and that this property does not require thermodynamics as is required by entropic gravity [32]. We further present a counterexample regarding the successful quantization of non-Abelian gauge Yang Mills theories, where covariance is recovered in the path integral [13]. We finally provide three novel findings based on the results in earlier sections.

2. Canonical quantization of \( d \geq 2 \) dimensional GR via canonical constraint formalism

The \( d \)-dimensional 1EH action is defined as:
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\[ \mathcal{L}^1_{dEH} = h^{\mu\nu} \left( G^\lambda_{\mu\nu,\lambda} + \frac{1}{d-1} G^\lambda_{\mu\nu} G^\sigma_{\sigma\nu} - G^\lambda_{\sigma\nu} G^\sigma_{\lambda\mu} \right) \]  

where \( h^{\mu\nu} = \sqrt{-g} g^{\mu\nu} \), \( g_{\mu\nu} \) is the metric, \( G^\lambda_{\mu\nu,\lambda} = \Gamma^\lambda_{\mu\nu} - \frac{1}{2}(\delta^\lambda_{\mu} \Gamma^\sigma_{\sigma\nu} + \delta^\lambda_{\nu} \Gamma^\sigma_{\sigma\mu}) \) and \( \Gamma^\lambda_{\mu\nu} \) is the affine connection. This choice of this action for QG is based on the property that it is trinomial in power of the metric, thereby rendering perturbation theory possible via the Feynman diagrammatic approach. In contrast, in the second order (2EH) action, increasing powers of the metric tensor in perturbation thereby rendering calculation via Feynman diagrams and renormalization impossible \([6, 7]\). In \([8]\) and \([9]\), for \( d > 2 \), using the Dirac constraint formalism to quantize this action, it was found that besides first class constraints, second class constraints also existed. For constrained systems with second class constraints, the generating functional via the path integral formalism derived by Senjanovic \([16]\) is:

\[ Z[J] = \int D\Phi D\Pi D\lambda D\kappa \det \{ \phi_a, \chi_b \} \det^{1/2} \{ \theta_a, \theta_b \} \delta(\chi_b) \times \exp i \int dx \left( \Pi \frac{\partial}{\partial t} \Phi - \mathcal{H}_c(\Phi, \Pi) - \lambda_a \phi_a(\Phi, \Pi) - \kappa_a \theta_a(\Phi, \Pi) + J\Phi \right) \]  

Here \( \Phi \) are the canonical fields, \( \Pi \) are the conjugate momenta, \( \mathcal{H}_c \) is the canonical Hamiltonian, \( \chi_b \) is the gauge condition associated with the first-class constraints \( \phi_a, \lambda_a \) and \( \kappa_a \) are lagrange multipliers, while \( \theta_a \) and \( \theta_b \) are second-class constraints. Chishtie and Mckeon \([9]\) found that these second class constraints were non-covariant, while the quantity \( p^i q_i - \mathcal{H}_c \) also displayed this breakdown in spacetime, rendering the path integral impossible to proceed further. This loss of manifest covariance in the path integral rendered any feasible calculation impossible. These non-standard ghost fields were also found in the case of scalar tensor theory by Chishtie and McKeon \([10]\).

We next attempt a first time canonical quantization for the action of the Hamiltonian based GR (HGR) \([3,4]\) which is as follows:

\[ \mathcal{L}^H_d = \sqrt{-g} g^{\alpha\beta} \left( \Gamma^\mu_{\alpha\beta} \Gamma^\nu_{\beta\mu} - \Gamma^\nu_{\alpha\beta} \Gamma^\mu_{\beta\nu} \right) = \sqrt{-g} B^{\alpha\beta\gamma\mu\nu} g_{\alpha\beta,\gamma} g_{\mu\nu,\rho} \]  

where \( B^{\alpha\beta\gamma\mu\nu} = g^{\alpha\beta} g^{\gamma\rho} g^{\mu\nu} - g^{\alpha\mu} g^{\beta\nu} g^{\gamma\rho} + 2 g^{\alpha\nu} g^{\beta\rho} g^{\gamma\mu} - 2 g^{\alpha\beta} g^{\gamma\mu} g^{\rho\nu} \).

For this action, it was found that the constraints are first class \([11]\) for \( d > 2 \) dimensions, and similarly for \( d = 2 \) dimensions \([12]\) for which the following path integral by Fadeev \([17]\) applies:

\[ Z[J] = \int D\Phi D\Pi D\lambda \det \{ \phi_a, \chi_b \} \delta(\chi_b) \times \exp i \int dx \left( \Phi \frac{\partial\Pi}{\partial t} - \mathcal{H}_c(\Phi, \Pi) - \lambda_a \phi_a(\Phi, \Pi) + J\Phi \right) \]

Here, the gauge condition as well as the constraints are obtained to be non-covariant, and this in turn renders the path integral non-covariant as well. We therefore note that this same breakdown in covariance occurs in the \( d \)-dimensional space as the 1EH action as was noted in earlier work cited above. Interestingly, the canonical structure for the \( d = 2 \) 1EH action was found to contain only first class constraints,
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unlike for the $d > 2$ case [13], however, these these are non-covariant as well, leading to the same behavior as for higher dimensions. Though with a different canonical structure in the $d = 2$ case, there is a loss of manifest covariance as for 1EH action, hence this property applies for $d \geq 2$ dimensions in GR.

While there is loss of covariance in the full GR actions with the application of the Dirac constraint formalism, however this symmetry is restored in the weak limit of these actions. Linearized GR actions are derived by expanding the field metric tensor around the flat metric as follows,

$$ h^{\mu\nu} = \eta^{\mu\nu} + \kappa \phi^{\mu\nu}, $$

(5)

where $h^{\mu\nu} = h^{\nu\mu}$, $\eta^{\mu\nu}$ is the flat metric ($\eta^{\mu\nu} = diag(+ + +...-)$ and $\phi^{\mu\nu}$ is the perturbation around this metric.

In [14] and [15], the constraints derived for linear 1EH action and HGR action are shown to be different from those derived from the full theory action, and the resulting path integral is shown to be covariant as a result. Recovery of covariance in the weak limit of the metric tensor is an important finding, whereby this symmetry is restored and field path integral quantization is indeed then made possible. Recently, McKeon and collaborators et al. have used background field theory approach in the weak limit of the gravitational field to render the 1EH and 2EH actions renormalizable, finite and restricted it to one-loop order using background and lagrange multiplier fields [18-25]. These are also consistent with EFT results whereby quantum corrections are possible at 1-loop order in the low-energy limit of 4-dimensional GR, however there is no indication of breakdown in covariance in this approach, except that renormalizability beyond one-loop in this formalism is not possible [28,29]. In this regard, [25] aims to derive a one-loop Beta function based on the quantized and renormalized 1EH action, which would indicate the extent to which GR holds in the weak limit and strong dynamics begin. We intend to follow this work with this investigation of the dynamics responsible for this onset of breakdown in covariance [33] and also coupling the 1EH action with Yang Mills theory using the canonical formalism [34].

While both of the full EH actions indicate breakdown in covariance, in [13] it was shown that application of the Dirac constraint and the path integral formalism in fact does recover covariance for gauge theories such as Yang Mills theory, though the space and time components are separated at the start of the analysis. So, for Yang-Mills theory for a vector field $A_\mu^a$

$$ \mathcal{L}^{YM} = -\frac{1}{4} F_{\mu\nu}^{a} F^{a\mu\nu} $$

(6)

with the covariant derivative, $D_\mu^{ab} = \partial_\mu \delta^{ab} + f^{apb} A_\mu^p$ and $[D_\mu, D_\nu]^{ab} = f^{apb} F_{\mu\nu}^p$, it is found that the constraints are first class. Upon application of the path integral for first class systems, the following covariant path integral is obtained:

$$ Z[J] = \int DA_\mu^a \det(-\partial^\mu D_\mu^{ab}) \exp i \int dx \left[ -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} - \frac{1}{2\alpha} (\partial \cdot A^a)^2 + J^a A_\mu^a \right] $$

(7)
With this counter example, we want to illustrate that the constraint quantization does work out for covariant gauge theories such as Yang-Mills theory. Hence, the breakdown of covariance in the case of $d \geq 2$ dimensional GR is actually a property of the theory itself, rather than an artifact of the methodology. This does not mean that GR is totally intractable at quantum mechanical scales; rather, in the weak field limit of the metric tensor, it is shown to yield different canonical constraint structure than the full theory, covariance is restored rendering constraint quantization possible in this limit, which is provided as a main argument for proof of the claims made in this work.

3. Findings and Implications

Our finding of the breakdown in space-time as a non-perturbative feature of the GR theory are consistent with the singularity theorems by Penrose which indicates the breakdown of the theory in the strong limit of the field in black holes [26, 27]. Our present findings supports QG scenarios whereby space-time and gravity is emergent without requiring thermodynamics as is required by entropic gravity [30]. The recent work on black holes whereby the information paradox loss is explained via quantum mechanical models, hinting that entanglement is perhaps a key feature in the non-perturbative sector of QG [31,32]. Based on the canonical gravity findings from contraint quantization, we therefore arrive at three novel findings postulated as follows:

(i) The breakdown of spacetime in General Relativity is a non-perturbative property, whereby covariance is lost in the high energy limit (strong field limit) of the theory.

(ii) Covariance as a symmetry is restored in the low energy (weak limit) of the GR theory, indicating it as non-fundamental and spacetime as an emergent property.

(iii) Physical principles more fundamental than covariance and/or new degrees of freedom and dynamics are required for a coherent and comprehensive theory of Quantum Gravity.

4. Conclusions

In this letter, we propose three novel findings based on the canonical structures and the loss of manifest covariance when quantization is conducted for various formulations of $d$-dimensional GR. We firstly propose that this is a non-perturbative property of the theory itself (valid for $d \geq 2$). These are deduced from different canonical constraint structures for the GR 1EH action and the Hamiltonian formulation of GR, whereby there is loss of covariance in both actions of GR when constraint quantization is utilized. Secondly, we propose that covariance is restored in the weak limit of the actions (though the canonical structure is different from the full GR actions) and quantization is acheived in this limit of the theory, which indicates space-time as a non-fundamental
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and an emergent property, without requiring thermodynamical conditions as is required in entropic gravity. Finally, we propose that in the onset of non-perturbative dynamics in the high-energy (strong field limit), new degrees of freedom and/or physical principles are required for a full theory of QG as space-time (covariance) via GR are no longer valid across all energy scales of this theory.

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6. References

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