Quantum states of a neutral massive fermion with an anomalous magnetic moment in an external electric field

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The planar non-relativistic quantum dynamics of a neutral massive fermion with an anomalous magnetic moment (AMM) in the presence of the electric field of infinitely long and thin thread with a charge density distributed uniformly along it (an Aharonov–Casher field) is examined. The relevant Hamiltonian is singular and requires additional specification of a one-parameter self-adjoint extension, which can be given in terms of physically acceptable (self-adjoint) boundary conditions. We find all possible self-adjoint Hamiltonians with an Aharonov–Casher field (ACF) by constructing the corresponding Hilbert space of square-integrable functions, including the \( r = 0 \) region, for all their Hamiltonians. We determine the most relevant physical quantities, such as energy spectrum and wave functions, and discuss their correspondence with those obtained by the physical regularization procedure. We show that energy levels of bound states are simple poles of the scattering amplitude. Expressions for the scattering amplitude and cross section depending on the fermion spin are reexamined.

PACS numbers: 03.65-w, 03.65.Ge, 03.65.Nk
Keywords: Aharonov–Casher field; Singular Hamiltonian; Self-adjoint boundary conditions; Physical regularization; Anomalous magnetic moment; Bound states; Scattering
I. INTRODUCTION

The Aharonov–Bohm (AB) effect [1] is a quantum physical phenomenon demonstrating, in particular, the importance of potentials in quantum mechanics. Considering an electron travels in a region with a non-vanishing gauge vector potential (since it produces the magnetic flux in a thin solenoid) but where the magnetic field vanishes, the electron wave function may develop a quantum (geometric) phase. The AB vector potential can produce observable effects because the relative (gauge invariant) phase of the electron wave function, correlates with a non-vanishing gauge vector potential in the region with zero magnetic field [2].

An interesting and important corollary to the Aharonov–Bohm geometric phase is a phase acquired by the wave function of a neutral massive fermion with a magnetic moment when it propagates in an electric field of a uniformly charged long conducting thread aligned perpendicularly to the plane of fermion motion. The fermion transport is affected by the phase acquired with the fermion wave function and the resulting phase difference leads to a spin-field dependent effects in scattering (the Aharonov-Casher effect [3]).

The planar quantum dynamics of fermions in the AB potential and ACF are governed by singular Hamiltonians that require the supplementary definition in order for them to be treated as self-adjoint quantum-mechanical operators. Self-adjoint Hamiltonians are not unique but each of them can be specified a real ”self-adjoint extension” parameter by additional (self-adjoint) boundary conditions. To put it more exactly, a domain, including the singular $r = 0$ region, in the corresponding Hilbert space of square-integrable functions must be indicated for each self-adjoint Hamiltonian. Different choices of self-adjoint extension parameter values lead to inequivalent physical cases [4] so a physical interpretation of self-adjoint extensions is a purely physical problem and each extension can be understood through an appropriate physical regularization [5].

We note that self-adjoint Hamiltonians with the AB potential and ACF have been considered to show the presence of fermion bound states in [4–10]. Self-adjoint Schrödinger and Dirac Hamiltonians with singular potentials (such as the one-dimensional Calogero potential, the Coulomb potential, a superposition of the Aharonov-Bohm field and the so-called magnetic-solenoid field and the potentials localized at the origin, in particular, delta-like potentials) were considered in many works (see [3] and references there). Also, it should be noted that the quantum dynamics of a neutral fermion in the cosmic string space-time have been analyzed by self-adjoint extension method in Refs. [11, 12].

In given paper we find all the quantum-mechanical states (wave functions) for all possible self-adjoint Hamiltonians with the ACF and discuss their correspondence with those obtained by physical regularization, which are applied in the convenient quantum mechanics. These problems for the quantum system under investigation have not been discussed before in the literature. We address the non-relativistic case and find wave functions and energy levels for bound states. Energy levels for the bound states are derived as poles of the scattering amplitude. The scattering problem of spin-polarized neutral fermions in the ACF is briefly considered. Advanced treatment of the above problems is based on exact solutions of the self-adjoint Hamiltonians with the ACF, which enables one to examine these problems the most comprehensive and complete.

We shall adopt the units where $c = \hbar = 1$.

II. SELF-ADJOINT NON-RELATIVISTIC RADIAL DIRAC–PAULI HAMILTONIANS FOR A NEUTRAL FERMION WITH AMM IN THE PRESENCE OF ELECTRIC FIELD

The Dirac–Pauli equation for a neutral fermion of the mass $m$ with AMM $M$ in an electric field can be written in the form of the Schrödinger equation as follows

$$i\frac{\partial \Psi}{\partial t} = H_{DP}\Psi. \quad (1)$$

Here

$$H_{DP} = \alpha \cdot \mathbf{P} + iM \gamma \cdot \mathbf{E} + \beta m \quad (2)$$

is the Hamiltonian, $\mathbf{P} = -i \nabla$ is the canonical momentum operator, $\Psi$ is a bispinor, $\gamma^\mu = (\gamma^0, \gamma)$, $\alpha$ are the Dirac matrices $\mathbf{E}$ is the electric field strength.

Introducing the function

$$\Psi = \Psi_n e^{-imt} \quad (3)$$
and representing $\Psi_n$ in the form
\begin{equation}
\Psi_n = \left( \begin{array}{c} \psi \\ \chi \end{array} \right),
\end{equation}
where $\phi$ and $\chi$ are spinors, we obtain an equation, which governs the non-relativistic dynamics of a neutral fermion with the AMM $M$ in an electric field in the form
\begin{equation}
i \frac{\partial \psi}{\partial t} = \frac{(P - E \times M)^2 - M^2 E^2 + M \nabla \cdot E}{2m} \psi,
\end{equation}
where $M = M\sigma$, $\sigma$ are the Pauli matrices and the term $\nabla \cdot E$ is equal to $4\pi$ times the electric field charge density.

We shall suppose in what follows that the electric field configuration has the cylindrical symmetry and the system dynamics is planar. The planar dynamics is accessed by requiring that the momentum projection $p_z = 0$ together with the imposition of the electric field should not have a third direction. Then, a natural assumption is that the relevant quantum mechanical system is invariant along the symmetry $z$ axis and the quantum system moves in the $xy$ plane. The electric field generated by infinitely long straight thin (a zero radius) thread with a charge density $a/2$ distributed uniformly along the $z$-axis (the ACF) is known to be
\begin{equation}
E_x = \frac{ax}{r^2}, \quad E_y = \frac{ay}{r^2}, \quad E_z = 0, \quad E_r = \frac{a}{r}, \quad E_\varphi = 0,
\end{equation}
where $r = \sqrt{x^2 + y^2}$, $\varphi = \arctan(y/x)$ are polar coordinates. The Schrödinger equation for a neutral fermion with AMM in an electric field (10) takes the form
\begin{equation}
i \frac{\partial \psi}{\partial t} = \frac{P_x^2 + P_y^2 + 2M\sigma_3(E_x P_y - E_y P_x) + M^2(E_x^2 + E_y^2) + M \nabla \cdot E}{2m} \psi,
\end{equation}
The radial component of the (macroscopic) electric field is determined by the mean surface charge density as $\nabla \cdot E = 4\pi \rho$, and the expression $\rho = a\delta(r)/4\pi r$, therefore, well approximates $\rho$. We seek the solutions of (10) in the ACF in the polar coordinates in the form
\begin{equation}
\psi(t, r, \varphi) = \frac{\exp(-iEt + ik\varphi)}{\sqrt{2\pi r}} F_E(r),
\end{equation}
where $E$ is the particle energy, $k$ is an integer, and $F_E(r)$ is a doublet.

The Hamiltonian of a neutral fermion in the ACF contains the matrix $\sigma_3$. Therefore, the $\sigma_3$ matrix commutes with this Hamiltonian and doublet $F_E(r)$ satisfies equation $\sigma_3 F_{E,s}(r) = s F_{E,s}(r)$, where two eigenvalues $s = \pm 1$ correspond two spin projection on the $z$ axis. Let us denote the upper and lower components of doublet as follows $F_{E,s=1}(r) \equiv F_{E,1}(r)$ and $F_{E,s=-1}(r) \equiv F_{E,2}(r)$; then the scalar product of doublets $F_E(r)$ and $G_E(r)$ is determined by formula
\begin{equation}
(F_E(r), G_E(r)) = \int (\bar{F}_{E,1}(r)G_{E,1}(r) + \bar{F}_{E,2}(r)G_{E,2}(r)) dr.
\end{equation}
Here $\bar{F}$ is the complex conjugate function. Any doublet can be written as
\begin{equation}
\begin{pmatrix} A F_{E,1}(r) \\ B F_{E,2}(r) \end{pmatrix} = AF_{E,1}(r) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + BF_{E,2}(r) \begin{pmatrix} 0 \\ 1 \end{pmatrix},
\end{equation}
where $A, B$ are complex constants.

The radial Dirac-Pauli equation for a neutral fermion with AMM in the ACF in the non-relativistic approximation for two components of doublet $F_{E,s}(r)$ can be written in the form
\begin{equation}
h \begin{pmatrix} F_{E,1}(r) \\ F_{E,2}(r) \end{pmatrix} = 2mE \begin{pmatrix} F_{E,1}(r) \\ F_{E,2}(r) \end{pmatrix},
\end{equation}
where
\begin{equation}
h = h_0 + M a \frac{\delta(r)}{r}
\end{equation}
and
\begin{equation}
h_0 = -\frac{d^2}{dr^2} + \frac{k^2 + (Ma)^2 - 1/4}{r^2} + \frac{2kMa}{r^2}\sigma_3.
\end{equation}
It is seen from Eqs. (13) and (14) that the upper and lower components of doublet \( F_{E,s=1}(r) \) and \( F_{E,s=-1}(r) \), in fact, can be analyzed by means of scalar function \( F_E(r,s) \) depending explicitly on \( s \) (see, e.g., [13]). We shall mainly treat with this function in what follows.

The radial Hamiltonian \( h_0 \) is singular and so the supplementary condition is required in order for it to be treated as a self-adjoint quantum-mechanical operator. Therefore, we need construct all self-adjoint Hamiltonians associated with the differential expression in the right-hand side of Eq. (13), then specify correct self-adjoint extensions by means of physical conditions, i.e., in fact, indicate the Hamiltonian domain in the Hilbert space of square-integrable functions on the half-line, including the \( r = 0 \) region. The radial Hamiltonian \( h \) involves a singular potential term \((Ma\delta(r)/r)\), which influences the behavior of wave functions at \( r = 0 \) and takes into account by asymptotic self-adjoint boundary conditions at the origin.

Let us consider the differential expression in the right-hand side of Eq. (13) as an self-adjoint operator \( h^0 \) in the Hilbert space \( L^2(0, \infty) \) of quantum states for any \((k+Ma)^2-1/4\) without reservations about its domain. Then, let us just define the operator \( h^0 \) in the Hilbert space \( L^2(0, \infty) \) as

\[
\begin{align*}
\text{with } D(0, \infty) = C^\infty, \text{supp} F \subset [c,d], 0 < c < d < \infty.
\end{align*}
\]

This allows us to avoid the problems related to \( r \to \infty \). The operator \( h \) is symmetrical if for any \( F(r) \) and \( G(r) \)

\[
\int_0^\infty \bar{G}(r)hF(r)dr = \int_0^\infty [hG(r)]F(r)dr.
\]

It is evident that \( h^0 \) is the symmetrical operator.

Let \( h \) be the self-adjoint extension of \( h^0 \) in \( L^2(0, \infty) \) and let us consider the adjoint operator \( h^* \) given by (13) but defined as follows

\[
\begin{align*}
h^*:\begin{cases}
D(h^*) = \{F(r): F(r) \text{ are absolutely continuous in}(0, \infty), \\
h^*F(r) = h_0F(r),
\end{cases}
\end{align*}
\]

i.e. \( D(h^0) \subset D(h^*) \). A symmetric operator \( h \) is self-adjoint, if its domain \( D(h) \) coincides with that of its adjoint operator \( D(h^*) \equiv D^* \).

Integrating (13) by parts and checking that for any function \( F(r) \) of \( D(h^*) \) \( \lim_{r \to \infty} F(r) = 0 \), we write it in the form

\[
(\bar{G}'F - \bar{G}F')|_{r=0} = 0,
\]

where a prime denotes the derivative with respect to \( r \). If (16) is satisfied for any \( F(r) \) of \( D^* \) then the operator \( h^* \) is symmetric and, so, self-adjoint. This means that the operator \( h^0 \) is essentially self-adjoint, i.e., its unique self-adjoint extension is its closure \( h = h^0 \), which coincides with the adjoint operator \( h = h^* \). If (16) is not satisfied then the self-adjoint operator \( h = h^0 \) can be found as the narrowing of \( h^* \) on the so-called maximum domain \( D(h) \subset D^* \). Thus, any \( F(r) \) of \( D(h) \subset D^* \) must satisfy the self-adjoint boundary condition

\[
(\bar{F}'F - \bar{F}F')|_{r=0} = 0.
\]

Since signs of \( a \) and \( M \) are fixed it is enough to consider the only case \( Ma > 0 \). Let us rewrite \( Ma \) as follows

\[
Ma = [Ma] + \mu \equiv n + \mu,
\]

where \( n = 0,1,2,\ldots \) denotes the largest integer \leq Ma, and \( 1 > \mu \geq 0 \). It is convenient to change indexing as follows \( l \to k + sn \). Because any solution \( F_E(r) \) of Eq. (11) must satisfy asymptotic self-adjoint boundary condition (17), one can separate out three regions of the values of \((l + s\mu)^2\). In first region \((l + s\mu)^2 \geq 1\), or \( l + s\mu \geq 1 \) and \( l + s\mu \leq -1 \). Since \(-\infty < l < \infty \) and \( 1 > \mu \geq 0 \) it easily
to determine that all values of $l$ for $\mu > 0$ are allowed and $l = 0$, $s = \pm 1$ for $\mu = 0$ are not allowed. One can show that for such $l$, the initial symmetric operator $h^{0}$ is essentially self-adjoint, and its unique self-adjoint extension is the adjoint operator $h_{1} = h^{*}$; its domain $D(h^{*})$ is

$$D(h^{*}) = \{ F_{E}(r) : F_{E}(r), F'_{E}(r) \text{ are absolutely continuous in } (0, \infty), \quad F_{E}(r), h_{0}F_{E}(r) \in L^{2}(0, \infty), \quad h_{1}F_{E}(r) = h_{0}F_{E}(r) \}. \tag{19}$$

The generalized eigenfunctions $F_{E}(r)$ of the radial Hamiltonian are

$$F_{E}(r) = \sqrt{r}J_{\nu}(pr), \tag{20}$$

where $J_{\nu}(pr)$ is the Bessel function of the order $\nu = |l + s\mu|$ and $p = \sqrt{2mE}$. The energy spectrum is continuous $E > 0$.

In second region $(l + s\mu)^{2} < 1$; it is seen that no values $l$ are allowed for $\mu = 0$ and two values $l = 0, \mp 1$ for $s = \pm 1$ are allowed for $\mu > 0$. Then, for each $l = 0, \mp 1$ there exists one-parameter $U(1)$-family of self-adjoint Hamiltonians $h_{\xi}$ parameterized by the real parameter $-\infty \geq \xi \leq \infty$ (or $0 \geq \theta \leq 2\pi$, $\xi = \tan(\theta/2)$); the values $\xi = \pm \infty$, or $\theta = 0, 2\pi$ are equivalent. These Hamiltonians are specified by the asymptotic self-adjoint boundary conditions at the origin with the domain $D_{\xi}$

$$h_{\xi}: \quad D_{\xi} = \left\{ F_{E} : F_{E}, F'_{E} \text{ are absolutely continuous in } (0, \infty), \quad F_{E}(r) = A[(mr)^{1/2+\gamma_{l}} + \xi(mr)^{1/2-\gamma_{l}}] + O(r^{3/2}), r \to 0, -\infty < \xi < \infty, \quad F_{E}(r) = A[(mr)^{1/2-\gamma_{l}} + O(r^{1/2}), r \to 0, \xi = \infty \right\} \tag{21}$$

where $A$ is a complex constant and $\gamma_{l} = |l| - \mu$.

The energy spectrum of the radial self-adjoint Hamiltonian $h_{\xi}$ is continuous ($E \geq 0$) for any $\xi$ of $-\infty < \xi < \infty$ and the generalized eigenvalues are

$$F_{E}(r) = C\sqrt{r} \left[ J_{\gamma_{l}}(pr) + \xi \frac{\Gamma(1-\gamma_{l})}{\Gamma(1+\gamma_{l})} \left( \frac{-E}{2m} \right)^{\gamma_{l}} J_{-\gamma_{l}}(pr) \right], \tag{22}$$

where $C$ is a constant and $\Gamma(x)$ is the Euler gamma function of argument $x$.

For $-\infty < \xi < 0$, there exist bound fermion states (see, also [4]). In order for a quantum system to have a bound state, its energy must be negative, and, therefore, discrete levels with $E < 0$ have to exist in addition to continuous part of the energy spectrum. As was shown (see, for example, [13]) discrete energy levels are simple poles of the scattering amplitude in the complex plane of $E$ in the first (physical) sheet of the Riemann surface $Re\sqrt{-E} > 0$; these poles are the roots of equation $B_{l}(E) = 0$, where $B_{l}(E)$ is the coefficient before the ingoing wave in Eq. (22) at $r \to \infty$:

$$B_{l}(E) = 1 + \xi \frac{\Gamma(1-\gamma_{l})}{\Gamma(1+\gamma_{l})} \left( \frac{-E}{2m} \right)^{2\gamma_{l}}. \tag{23}$$

After simple calculations, we find the bound-state energy in the explicit form

$$E_{l}^{-} = -2m \left( -\xi \frac{\Gamma(1-\gamma_{l})}{\Gamma(1+\gamma_{l})} \right)^{-1/\gamma_{l}}. \tag{24}$$

The normalized eigenfunction of bound state $F_{l}^{-}$ is

$$F_{l}^{-}(r) = \sqrt{-2mE_{l}^{-}r \sin(\pi\gamma_{l})/\pi\gamma_{l}} K_{\gamma_{l}}(\sqrt{-2mE_{l}^{-}r}),$$

where $K_{\gamma_{l}}(\sqrt{-2mE_{l}^{-}r})$ is the so-called McDonald function. Separating Eq. (24) as follows

$$E_{0}^{-} = -2m \left( -\xi \frac{\Gamma(1-\mu)}{\Gamma(1+\mu)} \right)^{-1/\mu}, \quad l = 0, s = \pm 1, \gamma_{0} = \mu, 0 < \mu < 1, \tag{25}$$

$$E_{\pm 1}^{-} = -2m \left( -\xi \frac{\Gamma(\mu)}{\Gamma(2-\mu)} \right)^{-1/(1-\mu)}, \quad l = \pm 1, s = \mp 1, \gamma_{\pm 1} = 1 - \mu. \tag{26}$$
we see that $E_0^-(\mu) = E_0^- (\mu')$, where $\mu' = 1 - \mu$, $0 < \mu' < 1$.

Special case $l + s\mu = 0$ can be of some interest. In this region, no values $l$ are allowed for $\mu > 0$ and the only value $l = 0(k = -sn)$ is allowed for $\mu = 0$. For $l = 0$ there exists one-parameter $U(1)$-family of self-adjoint Hamiltonians $h_\xi$ parameterized by the real parameter $-\infty \geq \xi \geq \infty$. These Hamiltonians are specified by the asymptotic self-adjoint boundary conditions at the origin with the domain $D_\xi$

$$h_\xi \left\{ \begin{array}{l}
F_E(r), F'_E(r)\text{ are absolutely continuous in } (0, \infty); h_\xi F_E \in L^2(0, \infty),
F'_E(r) = C\sqrt{r} \ln(mr) + \xi + O(r^{3/2} \ln(mr)), r \to 0, \quad -\infty < \xi < +\infty,
F_E(r) = C(r)^{1/2}, r \to 0, \quad \xi = \infty
\end{array} \right.$$

(27)

where $C$ is a constant.

One can show that for $-\infty \geq \xi \geq \infty$ the energy spectrum is continuous and nonnegative as well as for $-\infty < \xi < 0$ there exists one negative level ($E_0$) in addition to the continuous part of the spectrum

$$E_0 = -4me^2(\xi - c),$$

(28)

where $C = 0.57721$ is the Euler constant \[14\]. The generalized eigenfunctions of the continuous spectrum are the linear combination of the Bessel ($J_0(pr)$) and Neumann ($N_0(pr)$) functions. The normalized wave function of bound state is $\sqrt{-2mE_0r}K_0(\sqrt{-2mE_0r})$.

III. PHYSICAL REGULARIZATION PROCEDURE

It should be noted that a choice of a self-adjoint Hamiltonian requires additional physical arguments \[5\]. We emphasize that the radial Hamiltonian $h$ contains a singular potential ($Ma\delta(r)/r$). Such a potential affects the behavior of wave functions at the origin but it not grasped by an initial (symmetric) radial Hamiltonian, whose domain includes functions vanishing at the origin. Mathematically, such a potential is grasped by constructing self-adjoint extensions of Hamiltonian that are parameterized by asymptotic self-adjoint boundary conditions at the origin so, physically, the nonzero extension parameter can be treated as a manifestation of additional singular ($\sim \delta(r)/r$) potentials \[5\]. For each $\xi$, we find a possible domain for a self-adjoint $h_\xi$ and different choices $\xi$ lead to inequivalent physical cases (see, also, \[4\] and \[15\]).

In order to see a correspondence for the model studied with the physical situation, we solve the problem using the physical regularization procedure, which is applied in the convenient quantum mechanics. We consider a model with the electric field configuration, which is preferable for the regularization of the $\delta$-function. For this, the singular electric field \[40\] is replaced by

$$E_z, E_r = 0, r \geq 0; E_r = 0, r < R, E_r = a/r, r > R, mR \ll 1$$

(29)

and the singular two-dimensional potential $Ma\delta(r)/r$ is replaced by a regularized one-dimensional potential $Ma\delta(r - R)/R$ (see, also \[19\]). Such a model implies that the electric field is generated by an infinitely long straight thin thread (for instance, a conductor) with a surface charge density distributed uniformly about it along the $z$-axis. We emphasize that the functional structures of these configurations are different and we shall use a well defined one-dimensional $Ma\delta(r - R)/R$ term, which can be taken account of by using continuity conditions at $r = R$. It is helpful to note that such a term has no $\delta(r)$-function contribution at the origin (see \[5, 21\]). In these field configuration, all the solutions of the radial Hamiltonian $h^0$ can be chosen satisfying the standard boundary condition $F_E(r) = 0$. Obviously, in the region $r < R$, they are the regular functions $F_E(r) = \sqrt{r}J_0(pr)$. Next, we must match the radial solutions and their derivatives in the region $r < R$ with those in the region $r > R$ at $r = R$ taking account of the $Ma\delta(r - R)/R$ potential as

$$F_E(R - \delta) = F_E(R + \delta), \quad RF'_E(r)|_{R-\delta}^{R+\delta} = MaF_E(R), \quad \delta \to 0.$$  

(30)

In the region $r > R$, the eigenfunctions

$$F_E(r) = N\sqrt{r}J_\nu(pr), \nu \geq 1; F_E(r) = N_{\pm \gamma_1}r^{\pm \gamma_1}(pr), 0 < \gamma_1 < 1$$

(31)

whose coefficients determined by the continuity relations \[30\] describe the scattering states, and the eigenfunctions $F'_E(r) = N_0\sqrt{r}K_\gamma(pr), 0 < \gamma < 1$ describe bound fermion states. We note that the limit $R \to 0$ is to be taken at the calculations of coefficients for wave functions.
Eliminating the normalization constants from Eq. (30) we obtain a transcendental equation implicitly determining the energy spectrum of the fermion in the form

$$\frac{[\sqrt{X} K_\nu(X)]'}{K_\nu(X)} - \frac{[\sqrt{X} J_\mu(X)]'}{J_\mu(X)} = \frac{Ma}{\sqrt{X}},$$

(32)

where $X = \sqrt{2m|E^-|}R$ and now a prime denotes the derivative with respect to $X$. Taking into account that the regularization parameter $R \ll 1/m$ we can use the following expansions for the Bessel and McDonald functions

$$J_0(z) = 1 - \frac{z^2}{2}, \quad J_\nu(z) = \frac{z^\nu}{2^{\nu} \Gamma(1 + \nu)}, \quad K_\nu(z) = -\frac{\pi}{2\sin(\pi\nu)} \left[ \frac{z^\nu}{2^{\nu} \Gamma(1 + \nu)} - \frac{z^{-\nu}}{2^{\nu} \Gamma(1 - \nu)} \right].$$

(33)

So, we substitute (33) into (32) to calculate the left-hand side of Eq. (32). After these manipulations, we find the relation:

$$\frac{X^\gamma(|l| + Ma - \gamma l)}{2^{\gamma} \Gamma(1 + \gamma l)} = \frac{X^{-\gamma}(|l| + Ma + \gamma l)}{2^{-\gamma} \Gamma(1 - \gamma l)}.$$  

(34)

Solving this equation for $E^-$, we find the following expression for the energy spectrum of bound states

$$E^- = -\frac{2}{mR^2} \left[ \frac{(|l| + Ma - \gamma l)\Gamma(1 - l)}{(|l| + Ma + \gamma l)\Gamma(1 + l)} \right]^{-1/\gamma}, \quad l = 0, 0 < \gamma_0 < 1/2; l = \pm 1, 1/2 < \gamma_0 < 1.$$ (35)

To be sure, the expression in the square brackets in Eq. (35) must be positive.

Besides, neutral massive fermions can be bound by the field under discussion, if the potential, which is the sum of the term $U(r)$ in Eq. (13) and a regularized one-dimensional $Ma\delta(r - R)/R$ potential, is attractive. Writing $U(r)$ as $U(r) = (\gamma^2 - 1/4)/r^2, r > R; U(r) = (l^2 - 1/4)/r^2, r < R$, we see that $\gamma_0$ has to be in intervals $0 < \gamma_0 < 1/2$ for $l = 0$ and $1/2 < \gamma_0 < 1$ for $|l| = 1$ as well as the coupling constant $Ma$ must be negative. Thus, there exists the only negative level $E^-_{age}$ with $l = 0, s = \pm 1$. Also, these physical arguments are valid regarding Eqs. (29) and (26): only the energy level Eq. (26) doubly degenerated in spin projection can exist in the AC background with regularized (at origin) effective potentials.

It should be emphasized that the expressions for energy levels of bound states (29) and (26) depend explicitly upon the self-adjoint extension parameter $\xi$ and the expression (35) contains the regularization parameter $R$. Moreover, it follows from parameter of Eq. (35) that the right-hand side of Eq. (35) diverges in the limit $R \rightarrow 0$ that is at removing the regularization. Such a behavior of the energy of bound states is caused by the fact that we treat with the Hamiltonian (12) in which the term $Ma\delta(r)/r$ is replaced by $Ma\delta(r - R)/R$. It is essential that the coupling constant $Ma = |\mu|$ is dimensionless and, therefore, the Hamiltonian does not contain an initial parameter of the dimension of mass but nevertheless, a bound state can emerge. So, the coupling constant must depend on $R$ so that the energy of bound state would remain finite as $R \rightarrow 0$ (the renormalization of coupling constant). Thus, the cutoff parameter $\Lambda = 1/R$, which (has the dimension of mass and) tends to infinity, transmutes in arbitrary energy of bound state $E^-$. We believe this is a non-relativistic analog of the phenomenon of dimensional transmutation that occurs in massless relativistic field theories (see, [21]).

IV. THE SCATTERING PROBLEM

Now we discuss the scattering of neutral fermions with AMM in the considered field. The particle wave functions for the scattering problem are constructed by the eigenfunctions given Eqs. (20) and (22) (in a self-adjoint approach and the functions (11) (in a physical regularization approach). Since, the fermion wave functions are composed by regular and irregular (but square integrable) functions we make some remarks. If the singular two-dimensional potential $Ma\delta(r)/r$ is attractive the singular solutions (more concentrated at the origin than the regular ones) are to exist, which is quite reasonable from the physical standpoint. Besides, insisting on regularity of all functions at the origin forces one to reject irregular solutions, which, in some cases, can entail a loss of completeness in the angular basis.

At first we consider the case when the spatial part of doublet (8) $\psi(r, \varphi)$ is the eigenfunction of operator $\sigma_3$ with the eigenvalues $s = \pm 1$. Then, corresponding doublets can be represented in the form

$$\begin{align*}
\psi_1(r, \varphi) = & \sum_{k=-\infty, k\neq sn}^{\infty} \frac{N_k}{2} J_\mu(pr)e^{ik\varphi} \begin{pmatrix} 1 + s \\ 1 - s \end{pmatrix} + \frac{1}{2} \left( \cos(\theta/2)J_\mu(pr)e^{-i\pi_\mu/2(1 + s)} + \sin(\theta/2)J_{-\mu}(pr)e^{i\pi_\mu/2(1 - s)} \right) e^{-in\varphi}.
\end{align*}$$

(36)
where \( N_k = e^{-i\nu \pi/2} e^{-ik\pi/2} \), \( \nu = |k + sn + \mu| \), \( 1 > \mu > 0 \), \( \xi = \tan(\theta/2) \). Expression (30) is the most general representation of the spatial wave function of a neutral fermion with AMM for the scattering problem in the AC background. We emphasize this representation is constructed of solutions (26) and (24), which were found by the self-adjoint extension method.

Now we shall consider the scattering of neutral fermions by electric field (29) and attractive potential \( Ma\delta(r-R)/R \). In which case we can essentially simplify the problem without violating a generality of its consideration. Applying the continuity relations (24), we present the upper (\( \psi_1 \)), lower (\( \psi_2 \)) component of spatial wave function respectively in the following form

\[
\left( \begin{array}{c} \psi_1(r, \varphi) \\ \psi_1(r, \varphi) \end{array} \right) = \sum_{k=-\infty}^{\infty} \frac{N_k}{2} J_{\nu}(pr) e^{ik\varphi} \begin{pmatrix} 1 + s \\ 1 - s \end{pmatrix} + \frac{1}{2} \left( J_{\mu}(pr) e^{-i\pi\mu/2}(1 + s) - J_{-\mu}(pr) e^{i\pi\mu/2}(1 - s) \right) e^{-in\varphi}. \tag{37} \]

It is important that the wave function (37) has been obtained in the limit \( R \to 0 \) that is as a result of removing the regularization. It is easily seen that Eq. (37) corresponds to the values of self-adjoint extension parameter \( \theta = 0 \) for \( s = 1 \) and \( \theta = \pi \) for \( s = -1 \).

The scattering amplitude is defined in the conventional manner. We consider the fermion scattering problem assuming that the fermion moves in the positive direction along the \( x \) axis before the scattering, i.e., \( e^{i\varphi x} \) is an incident wave. In this case, \( \varphi \) is the scattering angle. Then, the fermion wave function as \( r \to \infty \) can be written as a superposition of an incident plane wave and a (scattered) outgoing cylindrical wave

\[
\psi_p(r, \varphi) \to e^{i\varphi x} + \frac{f(\varphi)}{\sqrt{r}} e^{i(pr - \pi/4)}, \tag{38} \]

where \( f(\varphi) \) is the scattering amplitude.

The incident plane wave is represented as

\[
e^{i\varphi x} \to \frac{1}{\sqrt{2\pi pr}} \sum_{l=-\infty}^{\infty} e^{il\varphi} \left( e^{i(pr - \pi/4)} + (-1)^l e^{-i(pr - \pi/4)} \right). \]

Then, using the asymptotic expansion of Bessel functions at large values of argument in the form \( J_{\pm\mu}(z) = \sqrt{2/\pi z} \cos(z + \pi\mu/2 - \pi/4) \) the next formula can be easily derived

\[
\psi(r, \varphi) - e^{i\varphi x} = \frac{f_s(\varphi)}{\sqrt{r}} e^{i(pr - \pi/4)}, \tag{39} \]

in which case the scattering amplitude is given by

\[
f_s(\varphi) = -\frac{s}{\sqrt{2\pi p}} \frac{e^{i\pi|n|/2} e^{i\varphi + i|n|\pi} \sin(\pi\mu)}{\sin(\varphi/2)} u_s, \quad u_s = \frac{1}{2} \begin{pmatrix} 1 + s \\ 1 - s \end{pmatrix}. \tag{39} \]

It is well to note that the asymptotic expansion of scattering solutions (38), (39) does not include the logarithmic phase shift meanwhile the analogous expansion in the Coulomb field includes the above shift due to the long-range forces.

Consequently, if the initial-state spin of a fermion moving in the \( xy \) plane is oriented along the \( z \) axis, then the fermion scattering cross section in the AC configuration is determined by the formula

\[
\frac{d\sigma}{d\varphi} = |f(\varphi)|^2 = \frac{\sin^2(\pi\mu)}{2\pi p \sin^2(\varphi/2)}. \tag{40} \]

This formula also describes the scattering cross section of non-polarized fermions with an AMM in the ACF, which was first shown in (3).

Finally, we show that the scattering cross section depends on the initial-state spin of a fermion. For this, let us consider the case when the spin vector lies in the \( xy \) plane. If, for example, the spatial part of doublet (8) \( \psi(r, \varphi) \) is the eigenfunction of operator \( \sigma_1 \) with the eigenvalues \( s_1 = \pm 1 \) then the upper (\( \psi_1 \)) and lower (\( \psi_2 \)) components of spatial wave functions (doublets) can be determined as

\[
\psi_1(r, \varphi) = \sum_{k=-\infty}^{\infty} N_k J_{\nu}(pr) e^{ik\varphi} + J_{\mu}(pr) e^{i\pi\mu/2} + J_{-\mu}(pr) e^{-i\pi\mu/2} e^{-i|n|\varphi}, \quad \psi_2(r, \varphi) = s_1 \psi_1(r, \varphi).
\]
and two doublets are
\[
\begin{pmatrix}
\psi_1(r \varphi) \\
\psi_1(r \varphi)
\end{pmatrix}
, s_1 = 1;
\begin{pmatrix}
\psi_1(r \varphi) \\
-\psi_1(r \varphi)
\end{pmatrix}
, s_1 = -1.
\tag{41}
\]
Calculating the scattering amplitudes \( f_{\pm}(\varphi) \) for \( s_1 = \pm 1 \) and cross sections, we obtain (see, also [22])
\[
f_{\pm}(\varphi) = (-1)^{|n|} \frac{\sin(\pi \mu)}{\sqrt{2 \pi p \sin(\varphi/2)}} \sin(|n| - 1/2) \varphi |u_{\pm}, \quad u_{\pm} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}
\]
and
\[
\frac{d\sigma}{d\varphi} = \frac{\sin^2(\pi \mu)}{2 \pi p \sin^2(\varphi/2)} \sin^2(|n| - 1/2) \varphi.
\]

V. RESUME

We have analyzed the planar quantum dynamics of a neutral massive fermion with an anomalous magnetic moment in an Aharonov–Casher field. For this all the quantum states for all possible self-adjoint Hamiltonians with the ACF are constructed by applying physically acceptable (self-adjoint) boundary conditions at origin. Also, we have found quantum states of a neutral fermion with AMM by physical regularization procedure for a model with regularized (at the origin) potentials. The problems of bound states and scattering of a neutral massive fermion with the AMM in the ACF are considered in detail, with taking into consideration the fermion spin and discussed in terms of the physics of the problems. In particular, by physical arguments we show that only one energy level can exist in the AC background with regularized effective potentials.

It should be emphasized that the existence of bound state can affect the scattering process. Indeed, the bound state can modify the scattering states due the energy of bound states fixes the scale of energies of the quantum system as a whole. The total cross-section of non-relativistic particles scattered off a two-dimensional \(-\lambda \delta r\) potential, in fact, does not change due to the above modification (see, [2]). This result is also correct for the planar scattering of non-relativistic fermions with AMM in an Aharonov–Casher field.

There remains one more question. If the energy \( \epsilon \) of scattered fermion, being a small quantity \( (\epsilon R \ll 1) \), is close up the energy of bound state, then the scattering section can significantly increase; this is shown for scattering of slow particles in an attractive potential \( U(r) \) localized in the region of radius \( R \) when there exists a bound \( l = 0 \) - state with the negative energy small compared with potential \( U(r) \) [13]. An analysis of the effect of bound state on the planar scattering of fermions in potentials of an Aharonov–Bohm kind will be deferred to a future work.

Acknowledgment

The author thanks Prof. I.V. Tyutin for valuable comments.

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