INTRODUCTION OF FEEDBACK LINEARIZATION TO ROBUST LQR AND LQI CONTROL – ANALYSIS OF RESULTS FROM AN UNMANNED BICYCLE ROBOT WITH REACTION WHEEL

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ABSTRACT

In this paper, the Jacobian-linearization- and feedback-linearization-based techniques of obtaining linearized model approaches are combined with a family of robust LQR control laws to identify the pairing which results in superior control performance of the bicycle robot, despite uncertainty and constraints, what is the main contribution of the paper. The control performance is analyzed using various indices, related, e.g. to energy consumption of the considered laws, with the experiments conducted on a real bicycle robot. As a result, the easily-implementable controller is obtained, which requires only to perform a set of off-line computations with a single additional parameter $\delta$ in comparison with a standard linear-quadratic controller, to obtain a state-feedback vector, which, when implemented to the control system, ensures proper regulation of the output signal of the plant, despite uncertainty or possible actuator failures, obtaining energy-efficient control law.

Key Words: LQR/LQI control, feedback linearization, robustness, actuator failure, unmanned bicycle robot.

I. INTRODUCTION

The bicycle is a well-known mechanical structure that has been around for over one hundred years. Officially, it was invented on June 12, 1817 by Karl von Drais [9,12,14]. The nominal operating point of bicycles is the upright unstable equilibrium point, as in human body movement. This analogy is drawn from many animal species in natural evolution. The inverted pendulum-like objects are characterized by a various dynamics. They are usually underactuated, that might be the reason why people enjoy riding bicycles rather than three- or four-wheeled vehicles. Human bicycle riding makes a new object, which is easily controllable by a human, nevertheless, the performance of the new system can be extended by adding stabilization units.

By adding them, the comfort of riding a bicycle can be increased. It is possible to stop the bicycle and keep balance without any external help. The futuristic vision is to let robots travel by bicycles. Any supporting system in stabilization increases robustness. Today, the best known application of the reaction wheel is in geostationary satellites (for example neural control of the reaction wheel [37] or fault-tolerant control of the reaction wheel [2]). Usually, there are three reaction wheels in three different axes used to rotate the whole system to any orientation. This is the solution to the problem of actuating satellites using electric power from solar panels in no gravity regions. Alternatively, jet propulsion can be used, though it uses extremely precious fuel gases which cannot be refueled in space.

The reaction phenomenon can be found everywhere where Newtons third law applies (the basic and important references concerning physics principles are [1,13,28]). Almost in every part of nature when bodies act on one another using force then the same force is created but in the opposite direction and is called the reaction force (the same applies to torques). The effect is almost ubiquitous and without it the world would not work properly at all. In a correct configuration, it can be used as driving force.

There are several possible ways to design the control law which is able to stabilize the reaction wheel pendulum using feedback signals from sensors. A basic proportional–integral–derivative (PID) controller is certainly insufficient to deal with this task, because it has one input and one output. In this case, it is necessary to stabilize all state-space variables simultaneously. The considered plant is nonlinear; therefore, the control system needs to be correctly designed. There are advanced techniques which are specially designed for nonlinear
systems such as: sliding mode control [19], Lyapunov redesign [35], feedback linearization (FBL) [10], fuzzy logic control [5,11], neural control [27] and some kinds of predictive control [26]. There is also a control algorithm based on Lyapunov theory, namely the Backstepping method, invented in 1990 by Petar V. Kokotović and well described in [20]. It was tested in the real application of the reaction wheel pendulum in [29], however, the results were not satisfactory. It is necessary to describe the plant in the strict feedback form which is impossible for under-actuated systems – like the reaction wheel pendulum. For this reason the backstepping algorithm is not able to control the whole state vector.

In this paper, the considered bicycle robot has the same equilibrium properties as a stand-alone bicycle [40], and has been built and used for experimental tests. It has been decided to use the linear-quadratic regulator (LQR) method [36] to control the reaction wheel pendulum, i.e. the considered robot, as it is capable of stabilizing it. Nevertheless, this approach requires one to obtain a linear model of the considered plant.

Control of the systems that are nonlinear, which may exhibit phenomena like limit cycles, input or output multiplicities, bifurcations or chaotic behaviour, are still challenging for control algorithms designing. Often, control algorithms are tailored for a given nonlinear system. Linearizing systems in the operating point and using well developed linear control methods is a frequent approach, but it does not work properly if the distance from the operating point is too large. However, feedback linearization method allows to overcome those problems, at least for some large class of nonlinear systems, as it provides linear model that exactly represents nonlinear one.

Feedback linearization is presented, e.g., in [18,19]; comprehensive description can be found also in [35]. Some of the newest applications of the method show that it successfully helps to design control systems in the wide range of nonlinear problems. Several examples of those applications are the following: [38] where the authors use feedback linearization with linear-quadratic regulator to control Stewart robot; [22,24] where feedback linearization in combination with sliding mode control is used to control induction motor drives; or [34] where, for spring loaded inverted pendulum as a model for legged locomotion, the authors use partial feedback linearization. Additionally, feedback linearization is applied also when the problem of uncertainties occurs, as in [43] where the authors consider control of nonlinear hydraulic generator with external disturbances and system uncertainty; in [25] where feedback linearization and extended state observer based control is proposed that deals with uncertainties of rotor-active magnetic bearings system; or in [46] where a neural network-based supple-

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mentary control system for a nonlinear plant is build on the basis of feedback linearization. The feedback linearization method is also still developed, as in [23], where the authors propose the variant of the method for systems with time varying delays.

This paper is a result of prior research made on this topic, binding all the prior results with the new ones. The authors have initially focused on 4DoF simulation model, see [16], by introducing robustness into LQR control scheme, or by comparing different control strategies, i.e. LQR, LQI and LMI-based LQR control in [30]. The results turned the attention of the authors to a problem with lesser DoF to be concerned [32]. First, by introducing robustness into LQR/LQI control laws with results based on experiments [31], then introducing feedback linearization to a simulation model [44], with in-depth analysis of impact of initial conditions and robustness parameter [45], and different linearization schemes [17]. This paper is mainly focused on extending the results referring to the best combination of control approaches at the research stage of [31], when feedback linearization is used, and in the case of possible actuator failure, which mimics uncertainty introduced by the linearization of the robot model, or by possible constraints in the control signal. This control scheme, at the same time, guarantees the cost of the selected control strategy, as in [8,33,39].

By considering robust approach, with actuator failure-like behaviour of the control system, the capabilities of the system are extended in comparison with a plain LQR, as has been shown in the previous research of the authors, giving rise to the need to introduce FBL linearization to the system, being the last stage in the search for the best control algorithm.

The paper is structured as follows: Section II introduces the mathematical model of the robot, Section III presents the approach to considering actuator failures in the system. In Section IV, two control strategies for Jacobian-linearized model are described, and a pair of strategies of FBL-linearized model is presented in Section V. Section VI describes the way, in which LQR approach can be adopted to a new linearized model of the system. Finally, Section VII describes the experimental setup in brief, and Section VIII presents the experimental comparison of the considered control strategies, accompanied by Summary in Section IX.

II. MATHEMATICAL MODEL OF THE 2DOF BICYCLE ROBOT

2.1 Preliminaries

Various mathematical models describe bicycle dynamics [4], and differ in extent to which the actual
behaviour of the object is reflected, or in the number of degrees of freedom. In this paper, the simplified model of the bicycle with 2DoF (the angle from the vertical and the angle of rotation of the reaction wheel) in the form of an inverted pendulum is used. An important assumption is that the bicycle considered here is not able to turn the handlebar, therefore, centrifugal forces do not affect it. This approach has allowed the authors to focus fully on the actuator which is the reaction wheel acting on the whole bicycle robot. The bicycle robot also has an inverted pendulum-like structure, and presented considerations form an introduction to its further development such as increasing the number of degrees of freedom by, e.g., including the handlebar angle actions into account.

The design of a control law for a bicycle with classical control (moment of forces exerted on a handlebar) for relatively low velocities (below 1 m/sec) is problematic, and often impossible, taking such physical constraints as maximum moment of force of an actuator. Stabilisation by reaction wheel is independent of the velocity of the bicycle (robot), as it does not use centrifugal force effect. This supports the reason to use this stabilisation system at low velocities.

Currently, there are two solutions to be found for automatic stabilisation of one-track vehicles in a vertical position. Firstly, the solution of Lit Motors Inc. [6], which developed an electric one-track vehicle to transport people using stabilisation based on moments of forces resulting from gyroscopic effect generated by two rotating masses with variable rotation axes. Secondly, the Honda Riding Assist [7] enabling self-stabilisation of a motor by taking complete control over the handlebar, to change dynamically the deflection from vertical axis. The drawback of the first are high angular speeds of the rotating masses required (loss of energy efficiency), and in the second solution, stabilisation system works with velocities up to 5 kmph.

The reaction wheel used for stabilisation, forcing the pendulum-like model to be used, solves both latter problems, and this is the basic reason why the model presented in the next subsection is used.

2.2 Continuous-time model of the robot. Solution of equations describing its dynamics

The complete mathematical description of the 2DoF robot model is given by the following non-linear differential Eqs. (1)–(4):

\[
\dot{x}_1(t) = x_2(t), \tag{1}
\]

\[
\dot{x}_2(t) = \frac{g h_r m_r \sin(x_1(t))}{I_{rg}} - \frac{b_1 x_2(t)}{I_{rg}} + \frac{b_1 x_4(t)}{I_{rg}} + \frac{k_m u(t)}{I_{rg}}, \tag{2}
\]

\[
\dot{x}_3(t) = x_4(t), \tag{3}
\]

\[
\dot{x}_4(t) = \frac{k_m u(t)}{I_I + I_{mr}} - \frac{b_1 x_4(t)}{I_I + I_{mr}}, \tag{4}
\]

where (see Fig. 1 for the kinematic scheme of the robot): \( \dot{x} \in \mathbb{R}^n \), state vector; \( x_1 \), vertical deflection angle of the robot; \( x_2 \), angular velocity of the robot; \( x_3 \), angle of rotation of the reaction wheel; \( x_4 \), angular velocity of the reaction wheel; \( u \in \mathbb{R} \), control signal (current of the motor); \( m_r \), weight of the robot; \( I_I \), moment of inertia of the reaction wheel; \( I_{mr} \), moment of inertia of the rotor of

Fig. 1. Kinematic scheme of the considered bicycle robot model. [Color figure can be viewed at wileyonlinelibrary.com]
the motor; \( I_m \), moment of inertia of the robot (rel. to the ground); \( h_r \), distance between the ground and the center of mass of the robot; \( g \), gravity force; \( k_m \), constant of the motor; \( b_f \), friction coefficient in rotational movement; \( b_r \), friction coefficient in the rotation of the reaction wheel; \( P_1, P_2 \), contact points of the wheels with the ground; \( C_1 \), center of the rear wheel; \( C_2 \), center of the front wheel; \( COM \), center of mass.

In the presented model forces such as centrifugal, gravitation forces, as well as reaction momentum \( \vec{m} \) and \( \vec{t} \) have been taken into consideration. The model, to the best knowledge of the authors, has been derived with reference to the rules described in [3], and presents a good approximation of the real physical structure.

The parameters of the physical model of the 2DoF robot take on the following values: \( m_r = 3.96 \text{ kg}, h_r = 0.13 \text{ m}, I_r = 0.0094 \text{ kg} \cdot \text{m}^2, I_m = 0.0008 \text{ kg} \cdot \text{m}^2, I_{rg} = 0.0931 \text{ kg} \cdot \text{m}^2, g = 9.80665 \text{ m} / \text{sec}^2, k_{m1} = 0.421 \text{ N m} / \text{A}, b_r = 0.0013 \text{ N} \cdot \text{m} \cdot \text{sec}, b_f = 0.0003 \text{ N} \cdot \text{m} \cdot \text{sec} \), which are given on the basis of prior experiments or measurements. The current in the motor is constrained to the level of \( \pm 2.1 \text{ A} \).

In order to apply the considered control approach, firstly, a linearized model must be obtained and, secondly, it needs to be discretized in due course. In this paper, two approaches to linearization are considered, namely Jacobian matrix-based linearization, and feedback linearization (FBL). The both considered methods result in continuous-time linearized models which require the discretization operation to be performed, in order to apply the proposed control algorithms.

### 2.3 Discrete-time model of the plant

The below multivariable model of the linearized plant must be obtained in order to apply the controllers that are described in the further parts of the text:

\[
\bar{x}_{t+1} = A \bar{x}_t + bu_t, \quad (5)
\]

\[
y_t = c^T \bar{x}_t, \quad (6)
\]

where the above matrix and two vectors have known values, sizes and are the result of step-invariant discretization of the continuous-time linearized model of the plant, with the sample period \( T_S \). The output signal is denoted by \( y \in \mathbb{R} \), the constrained control signal by \( u \in \mathbb{R} \), and the state vector \( \bar{x} \in \mathbb{R}^n \) are in discrete-time domain (denoted by subscript \( t \), where \( t \) is a sample number, and refers to time instant \( tT_S \)).

### III. ACTUATOR FAILURE MODELS

The paper considers the approach to the LQR control ensuring robustness against selected actuator failure models, with the performance index [21]

\[
J = \sum_{t=0}^{\infty} (\bar{x}_t^T Q \bar{x}_t + Ru_t^2), \quad (7)
\]

with \( Q \geq 0 \), and \( R \geq 0 \) as design criteria. At this stage, the third element of the sum, namely \( x_t^T N u_t \) is not introduced here for compatibility with the rest of the paper. The general actuator failure case [47], i.e.,

\[
u_t^k = (1 - \rho_t^k) \text{sat} (v_t; \alpha) \quad (k = 1, 2, \ldots, g_F),
\]

where \( \rho_t^k \) is the unknown constant from the range defined in the further part of the paper, the index \( k \) identifies the \( k \)th failure model, and \( g_F \) is the total number of failure models considered. The constrained control signal is denoted as \( u_t^k \), whenever actuator failure occurs. In other cases, \( u_t^k = v_t \), i.e. the constrained control signal is equal to the calculated control signal. For any failure model considered, \( \rho_t^k \) is constant and lies within the range \( \rho_{t,\min}^k \leq \rho_t^k \leq \rho_{t,\max}^k \), and function sat defines the method of applying constraints (e.g., cut-off constraint).

Having taken a single model of failure into account, (8) can be transformed [15,31,41,42] to

\[
u_t^k = \phi v_t, \quad (8)
\]

where the parameters defining failure model satisfy

\[
0 \leq \phi_- \leq \phi \leq \phi_+ \quad (9)
\]

with \( \phi_- \leq 1 \) and \( \phi_+ \geq 1 \). By introducing this notation one can take possible failures into account, and introduce, e.g., presence of the uncertainty of the linearized model as a sort of actuator failure, when the applied control signal has other impact on the real plant as can be told from the linearized model equations. In the case of the robot used in experiments, current constraints mimic failure-like behavior, when demanded torque cannot be generated by the motor.

Referring to (8), there are no active constraints whenever \( \phi_- = \phi_+ \) and \( u_t^k = v_t \), partial failure takes place when \( \phi_- > 0 \), and \( \phi_- = 0 \) corresponds to the outage case. The following notation is henceforth adopted from [31,41]: \( u_t^k = u_t^k \).

The control action must take possible actuator failure into account, to generate the control signal, in order to ensure the smallest performance deterioration when failure occurs. Generation of the control signal is the...
topic of the following sections. As the reference to the algorithm ensuring robustness see [31].

IV. CONTROL STRATEGIES FOR JACOBIAN-BASED LINEARIZATION

4.1 Introduction

In the first method of linearization, the Jacobian matrix is used, and the resulting model is given by \( \dot{x}(t) = A_jx(t) + b_ju(t) \), where:

\[
A_j = \begin{bmatrix}
0 & 1 & 0 & 0 \\
\frac{gh_m}{I_x} & 0 & -\frac{b_i}{I_x} \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -\frac{b_i}{I_x} + l_{mr}
\end{bmatrix},
\]

(10)

\[
b_j = \begin{bmatrix}
0 \\
\frac{k_m}{I_x} \\
0 \\
\frac{k_m}{I_x + l_{mr}}
\end{bmatrix},
\]

(11)

with linearization point (units omitted and coherent with prior considerations from Section 2.2) \( \bar{x}_i = 0 \).

4.2 Considered control approaches

As the reference point to all other considerations, the standard LQR control law of the form

\[
v_t = k^T \bar{x}_t,
\]

(12)

is taken into account, where \( \bar{x}_t \) is the sampled state vector from the robot, measured using suitable fusion from filters and estimates of signals from accelerometer, gyroscope and encoder. This sampled state vector originates from the nonlinear model from (1)–(4).

It is said that the control law (12) is called reliable when it assures a specified value of the performance index (7) which is not exceeded for the considered model of the plant, and when it is connected to a matrix \( P \), the system (10), (11), and if \( P \) satisfies [31,41]

\[
(A + \rho bk^T)^TP(A + \rho bk^T) - P + R\rho^2k^TP + Q \leq 0.
\]

(13)

When the control law is established to be reliable, the closed-loop model

\[
\bar{x}_{t+1} = (A + \rho bk^T)\bar{x}_t
\]

(14)

is stable, and in an infinite horizon it holds that

\[
J = \sum_{t=0}^{\infty} x^T_t (Q + R\rho^2k^TP) x_t \leq x_0^TPx_0.
\]

(15)

If robustness aspects are not considered, the optimal vector \( k \) from (12) is the solution for

\[
k^T = -(b^TPb + R)^{-1}b^TPA,
\]

(16)

\[
A = Q + A\rho kPA - A^TPb(b^TPb + R)^{-1}b^TPA,
\]

(17)

and the optimal value of \( J_k \) (7) is attained for \( k \) from (16) and (17). The upper bound of (15):

\[
J_k = \bar{x}_0^TP\bar{x}_0.
\]

(18)

Secondly, the LQI (linear-quadratic-integral) control law is considered, where:

\[
v_t = k_\bar{x} \bar{x}_t + k^T \bar{x}_{1,t},
\]

(19)

where \( \bar{x}_t \) is the sampled state vector of the robot (measurements), and \( \bar{x}_{1,t} \) is an appropriate integral of the state vector, whereas \( k_\bar{x} \) for the described problem is the zero vector with its third element of value \( a \neq 0 \) (compensation of gyroscopic drift in \( \bar{x}_3 \)), as in [31].

V. CONTROL STRATEGIES FOR FEEDBACK LINEARIZATION

5.1 Feedback linearization of the model

On the contrary to Jacobian linearization, a new description of the linearized plant can be obtained, resulting in the need to recalculate standard LQR control parameters defined in its performance index with respect to obtained linearization. Feedback linearization (FBL) enables getting an exact linear model, mapping nonlinear dynamics of the system [18,35].

In this paper, the FBL transformation is applied to the following description:

\[
\dot{x}(t) = f(x(t)) + gu(t),
\]

(20)

with \( x \in \mathbb{R}^n, u \in \mathbb{R}^m \), and smooth functions \( f(x) : \mathbb{R}^n \to \mathbb{R}^n, g \in \mathbb{R}^m \), referring to (1)–(4).

Two types of FBL linearization might be performed, namely input-state linearization or output-output linearization of the system. As in [45], and for the considered robot, it is possible linearize three selected state variables, what is the result of choosing the artificial output at first

\[
y(t) = -\psi_1x_1(t) - \psi_2x_3(t) + x_2(t),
\]

(21)
as the first variable of the new state $z$. Using the procedure of calculating the new FBL model one has:

$$z_1(t) = \psi_1 x_4(t) - \psi_2 x_1(t) + x_2(t),$$  

(22)

$$z_2(t) = \psi_3 \sin x_1(t),$$  

(23)

$$z_3(t) = \psi_4 x_2(t) \cos x_1(t),$$  

(24)

what results in the alternative linear model:

$$\dot{z}_1(t) = z_2(t),$$  

(25)

$$\dot{z}_2(t) = z_3(t),$$  

(26)

$$\dot{z}_3(t) = v(t),$$  

(27)

where

$$v(t) = \psi_2 \cos x_1(t) \psi_3 x_4(t) +$$

$$+ \psi_2 x_2(t) + \psi_3 \sin x_1(t) +$$

$$+ \psi_4 u(t) - \frac{x_2^2(t) \sin x_1(t)}{\psi_1},$$  

(28)

and [45]: $\psi_1 = \frac{I_{\alpha} + I_w}{I_k}, \psi_2 = \frac{-b}{I_k}, \psi_3 = \frac{g h m}{I_k}, \psi_4 = \frac{k_x}{I_f + I_w}$.

$\psi_3 = \frac{-b h}{I_{\alpha} + I_w}$.

To complement the new state, the forth state variable can be used:

$$z_4 = x_3,$$  

(29)

what leads to obtain internal dynamics,

$$\dot{z}_4 = \frac{z_3}{\psi_1 \sqrt{\psi_3^2 - z_2^2}} - \frac{z_1}{\psi_1} \text{ as in } \frac{z_2}{\psi_3},$$  

(30)

which is stable when $x_1, x_2, x_4$ converge to zero.

The feedback linearization scheme is presented in Fig. 2. The linear model is constructed by diffeomorphism, $\tilde{z} = \varphi(x)$, presented by Eqs. (22), (23), (24) and (29), and the nonlinear feedback, $u = \phi(\tilde{x}, \nu)$, that can be calculated from equation (28).

5.2 FBL-linearized discrete-time model of the plant

In order to apply the LQR controller in discrete-time, the following multivariable model of the linearized plant must be introduced:

$$\zeta_{t+1} = A_{\text{FBL}} \zeta_t + b_{\text{FBL}} \mu_t,$$  

(31)

$$\eta_t = c_{\text{FBL}}^T \zeta_t,$$  

(32)

where the appropriate matrix $A_{\text{FBL}}$ and vectors $b_{\text{FBL}}, c_{\text{FBL}}$ are of known values, sizes and are the result of the ZOH discretization of the continuous-time feedback-linearized model of the plant (see[45]), with selected sample period $T_S$. In this representation, $\zeta = z$, $\mu = v$, and the artificial output refers to new variables resulting from FBL transformation. Definition of $\eta$ is not necessary, since output signals are not considered in the presented approaches.

5.3 Considered control approaches to the FBL-linearized system

Assuming that uncertainty of the nonlinear model introduced by linearization (here: by means of FBL), can be treated by the control algorithm as an actuator failure-like behavior of the model, one can apply the approach from the Section IV, to FBL-linearized case. By substituting matrices $A, Q, b, \text{ state and state-feedback vectors and R}$ from (13)–(18) one can use the same formulas to FBL-linearized system.

The third considered case of a control law is the LQR control algorithm of the form (12) applied to the FBL system,

$$v_t = k_{\text{FBL}}^T \tilde{z}_t,$$  

(33)

and the fourth control strategy refers to the LQI case,

$$v_t = k_{\text{FBL}}^T \tilde{z}_t + k_{\text{LQI}}^T \tilde{x}_t,$$  

(34)

where $\tilde{z}_t$ is a state vector of FBL representation, $\tilde{x}_t$ is the sample of an appropriate integral of the state vector from the model, and $k_{\text{LQI}}$ for the described problem is again a zero vector with its third element of value $a \neq 0$ (compensation of gyroscopic drift in $x_3$).

VI. OPTIMAL STATE-FEEDBACK FOR FBL CONTROL

In general, FBL of a nonlinear plant results in reformulation of the classical LQR-type control law. The form
of the control law remains unchanged, as in (12) or (33), but matrices defining performance indices need to be reformulated. The initial control performance index for control law with Jacobian-linearized discretized model (10), (11) and matrices \( Q, R \), and, in addition, \( N \) are used now to formulate new performance index for diagonal \( Q \) only

\[
J_{\text{FBL}} = \sum_{t=0}^{\infty} (z_t^T Q z_t + R v_t^2 + 2z_t^T N_v v_t),
\]

where from (7) and [45] \( q_k = Q(k, k), k = 1, 2, 4 \):

\[
Q = \frac{1}{(\psi_1 \psi_3 \psi_4)^2} \begin{bmatrix}
\psi_3^2 (q_4 \psi_3^2 + R \psi_3^2) \\
\psi_3 (q_4 \psi_3 \psi_4^2 + R \psi_5 (\psi_2 \psi_5 - \psi_3)) \\
-\psi_3 (q_4 \psi_4^2 + R \psi_5 (\psi_2 + \psi_3)) \\
\psi_5^2 (q_4 \psi_3^2 + R \psi_5 (\psi_2 \psi_5 - \psi_3)) \\
\psi_5 (q_4 \psi_2^2 + q_4 \psi_4^2) + R (\psi_2 \psi_5 - \psi_3)^2 \\
-\psi_5 (q_4 \psi_2 \psi_4^2 + R (\psi_2 + \psi_3)) \\
-\psi_5 (q_5 \psi_2^2 + R \psi_3 (\psi_2 + \psi_3)) \\
\psi_2^2 (q_3 \psi_2^2 + q_4) + R (\psi_2 + \psi_3)^2
\end{bmatrix},
\]

\[
R_z = \frac{R}{(\psi_1 \psi_3 \psi_4)^2},
\]

\[
N_z = \frac{1}{(\psi_1 \psi_3 \psi_4)^2} \begin{bmatrix}
R \psi_5 \\
R (\psi_2 \psi_5 - \psi_3) \\
-R (\psi_2 + \psi_3)
\end{bmatrix}.
\]

It can be noted that the last term of the expression under sum operator is not present in (7), and similarly is not taken into account into iterative algorithm [31]. However, as it has resulted from initial simulations of the system, its current value is neglectful, as it usually is at least three orders less than other terms in \( J_{\text{FBL}} \).

The iterative algorithm used to calculate the state-feedback vector, for both linearization schemes enabling to obtain the optimal \( k \) to increase robustness of the system against actuator failure, can be found in [31], and is not presented here for the sake of brevity.

**VII. EXPERIMENTAL SETUP – THE 2DOF UNMANNED BICYCLE ROBOT**

In Fig. 3, the real robot is presented, which has 2DoF, namely the deflection angle from the unstable equilibrium point and angular displacement of the reaction wheel. The electric motor presented in the figure, is the only actuator.

The control system is responsible for stabilization of the robot in the upright position, with energy effort maintained at the lowest level possible. The DC brushed electric motor is mounted in the construction to accelerate or decelerate the rotating mass of the reaction wheel. Its shaft torque creates the reaction force that is used to stabilization. The current of the DC motor is the input signal to the plant, and is proportional to the torque. This way, the direct control of the torque is of prime importance here.

The one-axis accelerometer, one-axis gyroscope and encoder are in the measurement system of the robot. By using fusion of measurements (filtering and estimation process) in the form of the fitted combination of Kalman and FIR filters (see [31] and references therein), the current state of the machine can be estimated. The core of the embedded system is the high performance 32-bit STM32 microcontroller.

In the real system, the maximum feasible deflection angle from the vertical position, the maximum angular velocity of the reaction wheel and the maximum control effort must be taken into consideration. In addition, the DC motor also has limits, such as the maximum rotational velocity and its maximum torque, which when exceeded result in performance deterioration, and the only actuator gets difficult to control, what can be treated as the failure of the actuator.

In the paper, the actuator failures are tackled out using the modified LQR/LQI control for the two considered linearization schemes.

**VIII. EXPERIMENTAL COMPARISON OF CONTROL STRATEGIES**

**8.1 Experiment conditions**

In Fig. 4, the experimental setup is presented. As can be seen, there is a small cylinder on the right to
assumed). However, when \( \delta \) is 0, the design procedure yields a plain LQR law (no failure

When the control system is turned on, and data acquisition starts, the robot restores its unstable equilibrium position. All measurements are stored in the SD card with a frequency of 200 Hz.

The considered controllers work in discrete-time with their output fed to ZOH and are based on the discretized model of the linearized plant (1)–(4) (either on Jacobian, or FBL linearization).

During the experiment the following parameters have been fixed: \( T_s = 0.02 \text{sec} \), \( \Omega = \text{diag } \{1, 1, 0.01, 0.4\} \), \( R = 10 \), \( 0 \leq \delta \leq 0.8 \), \( \rho_+ = 1 + \delta \), \( \rho_- = 1 - \delta \), \( a = 2.1 \text{A}, \)

\( R_0 = 0.1 \), \( k_I = [0, 0, a, 0]^T \) (\( a = 0.2 \)), \( x_0 = [\pm 0.06 \text{ rad}, 0 \text{ rad} \cdot \text{sec}^{-1}, 0 \text{ rad}, 0 \text{ rad} \cdot \text{sec}^{-1}]^T \). For each of 18 values of \( \delta \) a series of 5 experiments has been carried out, for LQR and LQI controllers and for both linearization schemes, giving 4 sets of 90 experiments. The results presented in subsequent sections are the mean values from every experiment series for every value of \( \delta \) (360 experiments in total).

The introduction of the parameter \( \delta \) gives rise to changing properties of the control law. Starting with \( \delta = 0 \), the design procedure yields a plain LQR law (no failure assumed). However, when \( \delta > 0 \), the control law gives a reliable solution supplied by the cited procedure, taking possible failures into account, or, preparing the controller for mismodeling errors. The experimental results depict performance indices versus \( \delta \), to show in what range of this parameter, and for what values, the closed-loop system has appealing properties.

It is to be stressed that \( \Omega \) and \( R \) have been chosen to ensure that the optimal state-feedback vector \( k \) computed with standard LQR procedure results in acceptable dynamics of the closed-loop system, control signal does not saturate frequently and the robot itself is far from resonant-line behavior.

8.2 Performance indices

In order to considerably extend the results presented in the previous work of the authors [31], the following performance indices are considered, referring to the measured signals:

- sum of absolute values of the \( i \)th entry of the state-space vector (\( J_{\|e_i\|} \)),
- sum of squared values of the \( i \)th entry of the state-space vector (\( J_{\|x_i\|} \)), or control signal (\( J_u \)),
- time when control signal is saturated (\( J_{\text{sat}} \)),
- steady-state values of the \( i \)th entry of the state-space vector (\( J_{\|x_i\|}^{\infty} \)), at the end of experiments,
- maximum values of the \( i \)th entry of the state-space vector (\( J_{\|x_i\|} \)),
- time of stabilization of the \( i \)th entry of the state-space vector in the around-zero horizontal envelope of pre-specified width (\( J_{\text{stab}} \)).

All the signals in the above list are collected every \( T_s \) from the sampled-data control system, and the experiment time has been equal in all cases and set to 7.5 sec, \( i.e. \) sufficient to stabilize the system.

In all the cases, solid lines refer to LQR, and dashed refer to LQI, whereas thick to Jacobian- and thin FBL-linearized control strategies, and the performance indices are presented as functions of \( \delta \), referring to degree of robustness against actuator failure.

8.3 Analysis of experimental results

Fig. 5 presents performance indices related to sums of absolute values of signal samples, whereas Fig. 6 depicts indices related to sums of squared values (large amplitudes cause larger increase of the performance index). It can be said that LQR with FBL has a similar performance for \( x_i \) and \( x_4 \) for both indices, since from the time plots that are presented later, the intermediate values of samples are larger in comparison with the 2 first proposed approaches, what does not have to be a drawback of the last 2 approaches. The improvement is visible for velocity signals (see \( x_2 \) and \( x_3 \), especially for increasing \( \delta \)). From Fig. 6e and f it is obvious that increasing \( \delta \) results in smaller control effort (energy efficient control law) in the both FBL approaches, and control signal

![Fig. 4. The picture of the real robot during the experiment.](image-url)
gets faster desaturated, what causes the FBL-linearized system to exhibit linear behaviour (no active constraints).

Fig. 7 presents steady-state values of the state vector calculated as the mean value from the last 100 samples (units omitted). From Fig. 7c it can be seen that LQI action actually eliminates gyroscopic drift in the considered control horizon, and the proposed FBL+LQR and FBL+LQI combinations result in improvement with respect to Jacobian-linearized control laws in a wide range of $\delta$. The performance of two pairs of control approaches is absolutely comparable for $x_1$, $x_2$ and $x_4$, but outperforms other approaches in the case of $x_3$. It is disadvantageous that small values of $\delta$ for (c) cause performance deterioration. FBL-systems have similar static accuracy, as Jacobian-linearized, and since two of the approaches have no integral action, the zero steady-state errors cannot be attained, taking nonlinear phenomena present in the system. The LQI approach is a remedy to this problem, however, a proper choice of $a$ is required.
Fig. 7. (a) $J_1$, (b) $J_2$, (c) $J_3$, (d) $J_4$.

Fig. 8. (a) $J_{\text{max}}^1$, (b) $J_{\text{max}}^2$, (c) $J_{\text{max}}^3$, (d) $J_{\text{max}}^4$.

Fig. 9. (a) $J_{\text{stab}}^1$, (b) $J_{\text{stab}}^2$, (c) $J_{\text{stab}}^3$, (d) $J_{\text{stab}}^4$.
Fig. 8 presents maximum values for the four elements of the state vector. As can be seen, FBL-linearized systems have wider bandwidth and are faster (see Fig. 10), as the maximum values are bigger in comparison with Jacobian-linearized systems. The control system, however, causes the robot to be stabilized in the upright equilibrium point. The maximum values of FBL control laws are not dramatically large, but when results presented in Figs. 6 and 9 are taken into consideration, the wider bandwidth of the system becomes obvious. Jacobian-linearized systems are connected to smaller maximum values of the signals, and have slower dynamics, thus indices presented in Figs 5 and 6 do not mirror the actual behaviour of the system.

Fig. 9 presents time instants (in [sec]) from which the signals fall inside the horizontal envelope of the following widths: (a) ±0.008, (b) ±0.02, (c) ±1.5, (d) ±1. In the case of $x_1$, the stabilization for Jacobian-linearized systems is much slower, similarly in the case of $x_2$. For FBL control laws these signals change initially more and stabilize much faster when FBL is used. The decay ratio of transients is larger here, thus the dominating time constant of the closed-loop system is shorter, though the control signal desaturates faster. For a fictitious situation where disturbances act on the system not at a single instant at the beginning of the experiments, but, say, periodically, the FBL-linearized control law should result in better performance. Plots (c) and (d) present only signals where stabilization in the horizontal envelope has been possible. As can be seen, FBL+LQI approach performance here strongly depends on the value of $\alpha$.

The increase in $\delta$ gives visible improvement of the performance indices of FBL-linearized systems for:

- $J_{|x_1|}$ and $J_{|x_4|}$ in the case of LQI control laws,
- $J_{x_1^2}$, $J_{x_2^2}$, $J_{x_4^2}$, $J_{u^2}$ and $J_{\text{sat}_u}$.

In Fig. 10, time responses for two $\delta$s are shown, where dynamic properties of the closed-loop FBL system are shown. By selecting $\delta > 0$ improves performance for the both considered control schemes, maintaining superior properties of the FBL system.

IX. SUMMARY

The paper presents the analysis of control performance versus $\delta$, at the final stage of research on the search of the control law resulting in superior performance, and yet computationally simple enough to be implemented on the STM32 microcontroller, for the considered robot. Earlier papers compared plain LQR to robust LQR approaches (with LQI considered), see [44], analyzed the impact of $\delta$ on robust versions of LQR and LQI algorithms, see [31], or the impact of initial condition of $x_1$ [17] versus $\delta$ on similar performance indices as in this paper.

It has been shown that FBL-based robust LQR control scheme enables the designer of the control system to improve its performance obtaining computationally simple control law and the implementable controller. In this approach the simplicity of tuning the design parameters of the plain LQR control law is maintained, and, at the same time, it results in obtaining an energy-efficient control signal, enabling to regulate the output of the system despite actuator failures, or uncertainty introduced by the chosen linearized model.

In the paper, the analysis of the impact of introduction of FBL method into the control system has been presented, but there is no simple answer which performance indices should be chosen to evaluate the performance. Since large deviations from the equilibrium point might be more or less tolerated by the user, it is better to focus on energy efficiency of the control law, or evaluating stabilization time in some envelope. In all the cases, selecting $\delta > 0$ allows one to compensate for possible mismodelling errors, decreasing performance indices, extending the applicability area to systems with imperfect models, where the case of $\delta = 0$ states the algorithm has no degree of robustness. However, this approach originates limits of FBL transformation, applicable whenever state-space model description is affine in control input (see (20)), $f$ and $g$ vector fields are smooth, and the output is selectable to obtain appropriate relative order conditions.
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