THE TWO-VALUED SUBSETS OF $L_p(\Omega, \mu)$ THAT HAVE STRICT $p$-NEGATIVE TYPE

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Abstract. Suppose $0 < p < 2$ and that $(\Omega, \mu)$ is a measure space for which $L_p(\Omega, \mu)$ is at least two-dimensional. Kelleher et al. [3] have shown that if a subset $B$ of $L_p(\Omega, \mu)$ does not have strict $p$-negative type, then $B$ is affinely dependent (when $L_p(\Omega, \mu)$ is considered as a real vector space). Examples show that the converse of this statement is not true in general. In this note we describe a class of subsets of $L_p(\Omega, \mu)$ for which the converse statement holds. We prove that if a two-valued set $B \subset L_p(\Omega, \mu)$ is affinely dependent (when $L_p(\Omega, \mu)$ is considered as a real vector space), then $B$ does not have strict $p$-negative type. This result is specific to two-valued subsets of $L_p(\Omega, \mu)$ and it generalizes an elegant theorem of Murugan [7]. As an application we deduce the non-existence of certain types of isometry with range in $L_p(\Omega, \mu)$.

1. Introduction

A set $B \subset L_p(\Omega, \mu)$ is two-valued if $|B| > 1$ and the essential range of each $f \in B$ is a subset of $\{0, 1\}$. Our interest in two-valued subsets of $L_p(\Omega, \mu)$ was piqued by the following elegant theorem.

Theorem 1.1 (Murugan [7]). Suppose $k, n \geq 1$. A subset $B = \{x_0, x_1, \ldots, x_k\}$ of the Hamming cube $\{0, 1\}^n \subset \ell_1^n$ does not have strict 1-negative type if and only if $B$ is affinely dependent.

Negative type was originally studied in relation to the problem of isometrically embedding metric and normed spaces into $L_p$-spaces, $0 < p \leq 2$. Cayley [2], Menger [6], Schoenberg [8] and Bretagnolle et al. [1] authored seminal papers on this embedding problem. Negative type is defined in the following manner.

Definition 1.1. Let $p \geq 0$ and let $(X, d)$ be a metric space with $|X| > 1$.

(1) $(X, d)$ has $p$-negative type if and only if for all integers $n \geq 2$, all finite subsets $\{z_1, \ldots, z_n\} \subseteq X$, and all scalar $n$-tuples $\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{R}^n$
that satisfy $\zeta_1 + \cdots + \zeta_n = 0$, we have:

$$\sum_{j,i=1}^n d(z_j, z_i)^p \zeta_j \zeta_i \leq 0. \quad (1.1)$$

(2) $(X, d)$ has strict $p$-negative type if and only if it has $p$-negative type and the inequalities (1.1) are strict except in the trivial case $\zeta = 0$.

Schoenberg [8] noted that if $0 < p \leq 2$ and if $(\Omega, \mu)$ is a measure space such that $L_p(\Omega, \mu)$ is at least two-dimensional, then $L_p(\Omega, \mu)$ has $p$-negative type but it does not have $q$-negative type for any $q > p$. Recently, Kelleher et al. [3] have shown that if a subset $B$ of $L_p(\Omega, \mu)$ does not have strict $p$-negative type, then $B$ is affinely dependent (when $L_p(\Omega, \mu)$ is considered as a real vector space). The converse of this statement is true when $p = 2$ but not when $p < 2$ (see Remark 1 for an example). However, in the case $p = 1$, we see from Theorem 1.1 that the converse statement can hold under additional assumptions. Murugan’s proof of Theorem 1.1 relies on novel properties of vertex transitive graphs and the special geometry of the Hamming cube. In this note we obtain a generalization of Murugan’s theorem. It turns out that the only thing one really needs to know about the Hamming cube is that it is a two-valued set in an $L_p$-space, $0 < p < 2$. Our argument is based on the notion of virtual degeneracy in $L_p$-spaces that was developed by Kelleher et al. [3]. We show that if $B$ is a two-valued subset of $L_p(\Omega, \mu)$, then the indicated converse statement holds for $p < 2$. As an application, we show how this result obstructs the existence of certain types of isometry with range in $L_p(\Omega, \mu)$. In Remark 1 we note that (without additional assumptions) one cannot expect a similar result to hold for subsets of $L_p(\Omega, \mu)$ that take more than two values.

In Section 2 we recall some key properties of virtually degenerate and balanced simplices in $L_p$-spaces. Our main results are stated and proven in Section 3.

Throughout the rest of this note we assume that $0 < p < 2$ and that $(\Omega, \mu)$ is a measure space for which $L_p(\Omega, \mu)$ is at least two-dimensional. In the case $0 < p < 1$, $L_p(\Omega, \mu)$ is endowed with the usual quasi-norm.

2. Virtually degenerate and balanced simplices in $L_p$-spaces

**Definition 2.1.** Let $X$ be a set and suppose that $s, t > 0$ are integers. A **signed** $(s,t)$-simplex in $X$ is a collection of (not necessarily distinct) points $x_1, \ldots, x_s, y_1, \ldots, y_t \in X$ together with a corresponding collection of real numbers $m_1, \ldots, m_s, n_1, \ldots, n_t$ that satisfy $m_1 + \cdots + m_s = n_1 + \cdots + n_t$. Such a configuration of points and real numbers will be denoted by $D = [x_j(m_j); y_i(n_i)]_{s,t}$ and will be called a **simplex** henceforth.
In the case of a metric space \((X, d)\), Definition 1.1 can be completely reformulated in terms of simplices. This approach to understanding negative type was first employed by Lennard et al. [4].

**Definition 2.2.** Given a simplex \(D = [x_j(m_j); y_i(n_i)]_{s,t}\) in a set \(X\) we denote by \(S(D)\) the set of distinct points in \(X\) that appear in \(D\). For each \(z \in S(D)\) we define the repeating numbers \(m(z)\) and \(n(z)\) as follows:

\[
m(z) = \sum_{j: z = x_j} m_j \quad \text{and} \quad n(z) = \sum_{i: z = y_i} n_i.
\]

We say that the simplex \(D\) is degenerate if \(m(z) = n(z)\) for all \(z \in S(D)\).

Informally, a simplex \(D\) is degenerate if each \(z \in S(D)\) is equally represented in both halves of the simplex. In the context of a metric space \((X, d)\), degenerate simplices may be equated in a precise way to the trivial case \(\zeta = 0\) in Definition 1.1. A related notion for \(L_p\)-spaces is that of virtually degenerate simplices.

**Definition 2.3.** A non-degenerate simplex \(D = [x_j(m_j); y_i(n_i)]_{s,t}\) in \(L_p(\Omega, \mu)\) is said to be virtually degenerate if the family of evaluation simplices \(D(\omega) = [x_j(\omega)(m_j); y_i(\omega)(n_i)]_{s,t}, \omega \in \Omega,\) are degenerate in the scalar field of \(L_p(\Omega, \mu)\) \(\mu\)-almost everywhere.

Given \(\omega \in \Omega\) and \(z \in S(D(\omega))\), we will let \(m_\omega(z)\) and \(n_\omega(z)\) denote the repeating numbers of \(z\) in the evaluation simplex \(D(\omega) = [x_j(\omega)(m_j); y_i(\omega)(n_i)]_{s,t}\). This notation will be used in the proof of Lemma 3.1.

Kelleher et al. [3] have shown that the subsets of \(L_p(\Omega, \mu)\) that do not have strict \(p\)-negative type may be characterized in terms of virtually degenerate simplices.

**Theorem 2.1 (Kelleher et al. [3]).** A non-empty subset of \(L_p(\Omega, \mu)\) has strict \(p\)-negative type if and only if it does not admit any virtually degenerate simplices.

A difficulty with Theorem 2.1 is that the description of the subsets of \(L_p(\Omega, \mu)\) that have strict \(p\)-negative type is not purely geometric. In practice, it can be a hard problem to determine if a set \(B \subset L_p(\Omega, \mu)\) admits a virtually degenerate simplex or not. However, virtually degenerate simplices satisfy the following definition.

**Definition 2.4.** Let \(D = [x_j(m_j); y_i(n_i)]_{s,t}\) be a simplex in a vector space \(X\). We say that \(D\) is balanced if \(\sum m_j x_j = \sum n_i y_i\).

Informally, a simplex \(D\) in a vector space \(X\) is balanced if the two halves of the simplex have the same center of gravity. In a real or complex vector space there is a direct link between non-degenerate balanced simplices and affinely dependent subsets. This theorem underpins the main considerations of this note.
Theorem 2.2 (Kelleher et al. [3]). Let $n \geq 1$ be an integer and let $X$ be a real or complex vector space. Then a subset $Z = \{z_0, z_1, \ldots, z_n\}$ of $X$ admits a non-degenerate balanced simplex if and only if the set $Z$ is affinely dependent (when $X$ is considered as a real vector space).

3. Two-valued subsets of $L_p(\Omega, \mu)$ that have strict $p$-negative type

Lemma 3.1. Let $B \subseteq L_p(\Omega, \mu)$ be a two-valued set. If $D = [x_j(m_j); y_i(n_i)]_{s,t}$ is a non-degenerate balanced simplex in $B$, then $D$ is virtually degenerate.

Proof. Suppose $D = [x_j(m_j); y_i(n_i)]_{s,t}$ is a non-degenerate balanced simplex in $B$. By definition, $\sum m_j x_j = \sum n_i y_i$. As a result, we have $\sum m_j x_j(\omega) = \sum n_i y_i(\omega)$ for almost all $\omega \in \Omega$. As $B$ is two-valued, we also have $x_j(\omega), y_i(\omega) \in \{0, 1\}$ for almost all $\omega \in \Omega$. Thus,

$$\sum_j m_j x_j(\omega) = \sum_{j : x_j(\omega) = 1} m_j = m_\omega(1) = n_\omega(1) = \sum_{i : y_i(\omega) = 1} n_i = \sum_i n_i y_i(\omega),$$

for almost all $\omega \in \Omega$. But $m_\omega(1) + m_\omega(0) = n_\omega(1) + n_\omega(0)$ for almost all $\omega \in \Omega$. So, in fact, we have $m_\omega(1) = n_\omega(1)$ and $m_\omega(0) = n_\omega(0)$ for almost all $\omega \in \Omega$. It is therefore the case that the evaluation simplices $D(\omega)$ are degenerate for almost all $\omega \in \Omega$. This proves that $D$ is virtually degenerate. □

Theorem 3.1. A two-valued set $B \subseteq L_p(\Omega, \mu)$ does not have strict $p$-negative type if and only if $B$ is affinely dependent (when $L_p(\Omega, \mu)$ is considered as a real vector space).

Proof. As noted in Section 1, the forward implication holds for any set $B \subseteq L_p(\Omega, \mu)$ with $|B| > 1$ by Corollary 4.10 in Kelleher et al. [3].

Suppose that a given set $B \subseteq L_p(\Omega, \mu)$ is two-valued and affinely dependent (when $L_p(\Omega, \mu)$ is considered as a real vector space). By Theorem 2.2, $B$ admits a non-degenerate balanced simplex $D$. By Lemma 3.1, $D$ is virtually degenerate. By Theorem 2.1, $B$ does not have strict $p$-negative type. □

Corollary 3.1.1. Suppose $q \in (p, 2]$. Let $B$ be any two-valued subset of $L_p(\Omega, \mu)$ that is affinely dependent (when $L_p(\Omega, \mu)$ is considered as a real vector space). Then no subset of any $L_q$-space is isometric to $B$. More generally, no metric space of strict $p$-negative type is isometric to $B$. 
Proof. If $A$ is a subset of an $L_q$-space such that $|A| > 1$ and $q \in (p, 2]$, then $A$ has $q$-negative type. Therefore $A$ has strict $p$-negative type by Li and Weston [5, Theorem 5.4]. However, $B$ does not have strict $p$-negative type by Theorem 3.1. Hence $A$ is not isometric to $B$. The general statement follows similarly. □

Remark 1. The reverse implication of Theorem 3.1 does not necessarily hold for subsets of $L_p(\Omega, \mu)$ that take more than two values. For example, consider the points $z_0 = (0, 0), z_1 = (1, 1), z_2 = (2, 1)$ and $z_3 = (2, 0) \in \ell_p(2)$. The set $Z$ takes the values $\{0, 1, 2\}$ and is affinely dependent. However, the set $Z$ clearly does not admit any virtually degenerate simplices and so it has strict $p$-negative type by Theorem 2.1.

In closing, it is worth noting that the above results hold for any $\{\alpha_1, \alpha_2\}$-valued subset $B$ of $L_p(\Omega, \mu)$, $\alpha_1 \neq \alpha_2$. Indeed, we may assume that $\alpha_1 = 0$ by a translation and then slightly modify the proof of Lemma 3.1. For example, by Theorem 3.1, it then follows that the classical system $B$ of Walsh functions in $L_p[0,1]$ forms a set of strict $p$-negative type.

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