Symbolic powers of monomial ideals and Cohen-Macaulay vertex-weighted digraphs

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Dedicated to Professor Antonio Campillo on the occasion of his 65th birthday

Abstract In this paper we study irreducible representations and symbolic Rees algebras of monomial ideals. Then we examine edge ideals associated to vertex-weighted oriented graphs. These are digraphs having no oriented cycles of length two with weights on the vertices. For a monomial ideal with no embedded primes we classify the normality of its symbolic Rees algebra in terms of its primary components. If the primary components of a monomial ideal are normal, we present a simple procedure to compute its symbolic Rees algebra using Hilbert bases, and give necessary and sufficient conditions for the equality between its ordinary and symbolic powers. We give an effective characterization of the Cohen–Macaulay vertex-weighted oriented forests. For edge ideals of transitive weighted oriented graphs we show that Alexander duality holds. It is shown that edge ideals of weighted acyclic tournaments are Cohen–Macaulay and satisfy Alexander duality.
1 Introduction

Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over a field $K$ and let $I \subset R$ be a monomial ideal. The Rees algebra of $I$ is

$$R[I] := R \oplus I t \oplus \cdots \oplus I^k t^k \oplus \cdots \subset R[t],$$

where $t$ is a new variable, and the symbolic Rees algebra of $I$ is

$$R_s(I) := R \oplus I^{(1)} t \oplus \cdots \oplus I^{(k)} t^k \oplus \cdots \subset R[t],$$

where $I^{(k)}$ is the $k$-th symbolic power of $I$ (see Definition 2).

One of the early works on symbolic powers of monomial ideal is [35]. Symbolic powers of ideals and edge ideals of graphs where studied in [1]. A method to compute symbolic powers of radical ideals in characteristic zero is given in [36].

In Section 2 we recall the notion of irreducible decomposition of a monomial ideal and prove that the exponents of the variables that occur in the minimal generating set of a monomial ideal $I$ are exactly the exponents of the variables that occur in the minimal generators of the irreducible components of $I$ (Lemma 1). This result indicates that the well known Alexander duality for squarefree monomial ideals could also hold for other families of monomial ideals.

We give algorithms to compute the symbolic powers of monomial ideals using Macaulay2 [16] (Lemma 2, Remarks 1 and 5). For a monomial ideal with no embedded primes we classify the normality of its symbolic Rees algebra in terms of the normality of its primary components (Proposition 3).

The normality of a monomial ideal is well understood from the computational point of view. If $I$ is minimally generated by $x^{v_1}, \ldots, x^{v_r}$ and $A$ is the matrix with column vectors $v^t_1, \ldots, v^t_r$, then $I$ is normal if and only if the system $xA \geq 1, x \geq 0$ has the integer rounding property [9, Corollary 2.5]. The normality of $I$ can be determined using the program Normaliz [4]. For the normality of monomial ideals of dimension 2 see [6, 12] and the references therein.

To compute the generators of the symbolic Rees algebra of a monomial ideal one can use the algorithm in the proof of [21, Theorem 1.1]. If the primary components of a monomial ideal are normal, we present a procedure that computes the generators of its symbolic Rees algebra using Hilbert bases and Normaliz [3] (Proposition 4, Example 4), and give necessary and sufficient conditions for the equality between its ordinary and symbolic powers (Corollary 3).

In Section 3 we study edge ideals of weighted oriented graphs. A directed graph or digraph $\mathcal{D}$ consists of a finite set $V(\mathcal{D})$ of vertices, together with a prescribed collection $E(\mathcal{D})$ of ordered pairs of distinct points called edges or arrows. An oriented graph is a digraph having no oriented cycles of length two. In other words an oriented graph $\mathcal{D}$ is a simple graph $G$ together with an orientation of its edges. We call $G$ the underlying graph of $\mathcal{D}$. If a digraph $\mathcal{D}$ is endowed with a function $d : V(\mathcal{D}) \to \mathbb{N}_+$, where $\mathbb{N}_+ := \{1, 2, \ldots\}$, we call $\mathcal{D}$ a vertex-weighted digraph.
Monomial ideals and Cohen-Macaulay digraphs

Edge ideals of edge-weighted graphs were introduced and studied by Paulsen and Sather-Wagstaff [33]. In this work we consider edge ideals of graphs which are oriented and have weights on the vertices. In what follows by a weighted oriented graph we shall always mean a vertex-weighted oriented graph.

Let $D$ be a vertex-weighted digraph with vertex set $V(D) = \{x_1, \ldots, x_n\}$. The weight $d(x_i)$ of $x_i$ is denoted simply by $d_i$. The edge ideal of $D$, denoted $I(D)$, is the ideal of $R$ given by

$$I(D) := (x_i x_j^{d_j} \mid (x_i, x_j) \in E(D)).$$

If a vertex $x_i$ of $D$ is a source (i.e., has only arrows leaving $x_i$) we shall always assume $d_i = 1$ because in this case the definition of $I(D)$ does not depend on the weight of $x_i$. In the special case when $d_i = 1$ for all $i$, we recover the edge ideal of the graph $G$ which has been extensively studied in the literature [8, 11, 14, 18, 20, 30, 38, 39, 40, 42]. A vertex-weighted digraph $D$ is called Cohen–Macaulay (over the field $K$) if $R/I(D)$ is a Cohen–Macaulay ring.

Using a result of [24], we answer a question of Aron Simis and a related question of Antonio Campillo by showing that an oriented graph $D$ is Cohen–Macaulay if and only if the oriented graph $U$, obtained from $D$ by replacing each weight $d_i > 3$ with $d_i = 2$, is Cohen–Macaulay (Corollary 6). Seemingly, this ought to somewhat facilitate the verification of this property.

It turns out that edge ideals of weighted acyclic tournaments are Cohen–Macaulay and satisfy Alexander duality (Corollaries 7 and 8). For transitive weighted oriented graphs it is shown that Alexander duality holds (Theorem 4). Edge ideals of weighted digraphs arose in the theory of Reed-Muller codes as initial ideals of vanishing ideals of projective spaces over finite fields [4, 17, 25].

A major result of Pitones, Reyes and Toledo [34] shows an explicit combinatorial expression for the irredundant decomposition of $I(D)$ as a finite intersection of irreducible monomial ideals (Theorem 2). We will use their result to prove the following explicit combinatorial classification of all Cohen–Macaulay weighted oriented forests.

**Theorem 5.** Let $D$ be a weighted oriented forest without isolated vertices and let $G$ be its underlying forest. The following conditions are equivalent:

(a) $D$ is Cohen–Macaulay.

(b) $I(D)$ is unmixed, that is, all its associated primes have the same height.

(c) $G$ has a perfect matching $\{x_1, y_1\}, \ldots, \{x_r, y_r\}$ so that $\deg_G(y_i) = 1$ for $i = 1, \ldots, r$ and $d(x_i) = d_i = 1$ if $(x_i, y_i) \in E(D)$.

All rings considered here are Noetherian. For all unexplained terminology and additional information, we refer to [2] for the theory of digraphs, and [14, 20, 30, 42] for the theory of edge ideals of graphs and monomial ideals.
2 Irreducible decompositions and symbolic powers

In this section we study irreducible representations of monomial ideals and various aspects of symbolic Rees algebras of monomial ideals. Here we continue to employ the notation and definitions used in Section 1.

Recall that an ideal $L$ of a Noetherian ring $R$ is called irreducible if $L$ cannot be written as an intersection of two ideals of $R$ that properly contain $L$. Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over a field $K$. Up to permutation of variables the irreducible monomial ideals of $R$ are of the form

$$(x_1^{a_1}, \ldots, x_r^{a_r}),$$

where $a_1, \ldots, a_r$ are positive integers. According to [42, Theorem 6.1.17] any monomial ideal $I$ of $R$ has a unique irreducible decomposition:

$$I = I_1 \cap \cdots \cap I_m,$$

where $I_1, \ldots, I_m$ are irreducible monomial ideals and $I \neq \cap_{i \neq j} I_i$ for $j = 1, \ldots, m$, that is, this decomposition is irredundant. The ideals $I_1, \ldots, I_m$ are called the irreducible components of $I$.

By [42, Proposition 6.1.7] a monomial ideal $\mathcal{I}$ is a primary ideal if and only if, after permutation of the variables, it has the form:

$$\mathcal{I} = (x_1^{a_1}, \ldots, x_r^{a_r}, x_{r+1}^{b_1}, \ldots, x_{r+s}^{b_s}),$$

where $a_i \geq 1$ and $\cup_{r+1}^{r+s} \supp(x^{b_j}) \subset \{x_1, \ldots, x_r\}$. Thus if $\mathcal{I}$ is a monomial primary ideal, then $\mathcal{I}^k$ is a primary ideal for $k \geq 1$. Since irreducible ideals are primary, the irreducible decomposition of $I$ is a primary decomposition of $I$. Notice that the irreducible decomposition of $I$ is not necessarily a minimal primary decomposition, that is, $I_i$ and $I_j$ could have the same radical for $i \neq j$. If $I$ is a squarefree monomial ideal, its irreducible decomposition is minimal. For edge ideals of weighted oriented graphs one also has that their irreducible decompositions are minimal [34].

**Definition 1.** An irreducible monomial ideal $L \subset R$ is called a minimal irreducible ideal of $I$ if $I \subset L$ and for any irreducible monomial ideal $L'$ such that $I \subset L' \subset L$ one has that $L = L'$.

**Proposition 1.** If $I = I_1 \cap \cdots \cap I_m$ is the irreducible decomposition of a monomial ideal $I$, then $I_1, \ldots, I_m$ are the minimal irreducible monomial ideals of $I$.

**Proof.** Let $L$ be an irreducible ideal that contains $I$. Then $L_i \subset L$ for some $i$. Indeed if $L_i \not\subset L$ for all $i$, for each $i$ pick $x_i^{a_i}$ in $I \setminus L$. Since $L \subset L_i$, setting $x^a = \operatorname{lcm}(x_i^{a_i})_{i=1}^m$ and writing $L = (x_1^{b_1}, \ldots, x_r^{b_r})$, it follows that $x^a$ is in $L$ and $x_i^{a_i}$ is a multiple of $x_i^{b_i}$ for some $1 \leq i \leq m$ and $1 \leq t \leq r$. Thus $x_i^{a_i}$ is in $L$, a contradiction. Therefore if $L$ is minimal one has $L = L_i$ for some $i$. To complete the proof notice that $I_i$ is a minimal irreducible monomial ideal of $I$ for all $i$. This follows from the first part of the proof using that $I = I_1 \cap \cdots \cap I_m$ is an irredundant decomposition. \( \square \)
The unique minimal set of generators of a monomial ideal \( I \), consisting of monomials, is denoted by \( G(I) \). The next result tells us that in certain cases we may have a sort of Alexander duality obtained by switching the roles of minimal generators and irreducible components [42, Theorem 6.3.39] (see Example 7 and Theorem 4).

**Lemma 1.** Let \( I \) be a monomial ideal of \( R \), with \( G(I) = \{x^{v_1}, \ldots, x^{v_r}\} \) and \( v_i = (v_{i1}, \ldots, v_{in}) \) for \( i = 1, \ldots, r \), and let \( I = I_1 \cap \cdots \cap I_m \) be its irreducible decomposition. Then

\[
V := \{x^{vi_j}|v_{ij} \geq 1\} = G(I_1) \cup \cdots \cup G(I_m).
\]

**Proof.** "\( \subset \)" Take \( x^{vi_j} \) in \( V \), without loss of generality we may assume \( i = j = 1 \). We proceed by contradiction assuming that \( x^{vi_1} \) is not in \( \bigcup_{l=1}^m G(I_l) \). Setting \( M = x^{v_1i_1-1}x^{v_1i_2} \cdots x^{v_1i_m} \), notice that \( M \) is in \( I \). Indeed for any \( I_j \) not containing \( x^{v_1i_2} \cdots x^{v_1i_m} \), one has that \( x^{v_1i_1} \) is in \( I \) because \( x^{v_1i_1} \) is in \( I \). Thus there is \( x^{v_1i_1} \) in \( G(I_j) \) such that \( v_1i_1 > c_j \geq 1 \) because \( x^{v_1i_1} \) is not in \( G(I_j) \). Thus \( M \) is in \( I \). This proves that \( M \) is in \( I \), a contradiction to the minimality of \( G(I) \) because this monomial that strictly divides one of the elements of \( G(I) \) cannot be in \( I \). Thus \( x^{vi_1} \) is in \( \bigcup_{l=1}^m G(I_l) \), as required.

"\( \supset \)" Take \( x^{vi_j} \) in \( G(I_l) \) for some \( i, j \). Without loss of generality we may assume that \( i = j = 1 \) and \( G(I_1) = \{x^{a_1}, \ldots, x^{a_l}\} \). We proceed by contradiction assuming that \( x^{a_1} \notin V \). Setting \( L = (x^{a_1-1}, x^{a_2}, \ldots, x^{a_l}) \), notice that \( I \subset L \). Indeed take any monomial \( x^{v_k} \) in \( G(I) \) which is not in \( (x^{a_2}, \ldots, x^{a_l}) \). Then \( x^{v_k} \) is a multiple of \( x^{a_1} \) because \( I \subset I_1 \). Hence \( v_{11} > a_1 \) because \( x^{a_1} \notin V \). Thus \( x^{v_k} \) is in \( L \). This proves that \( I \subset L \subseteq I_1 \), a contradiction to the fact that \( I_1 \) is a minimal irreducible monomial ideal of \( I \) (see Proposition 1).

Let \( I \subset R \) be a monomial ideal. The *Alexander dual* of \( I \), denoted \( I^* \), is the ideal of \( R \) generated by all monomials \( x^a \), with \( a = (a_1, \ldots, a_n) \), such that \( \{x^{ai}|ai \geq 1\} \) is equal to \( G(L) \) for some minimal irreducible ideal \( L \) of \( I \). The *dual* of \( I \), denoted \( I^\dual \), is the intersection of all ideals \( \{x^{ai}|ai \geq 1\} \) such that \( x^a \in G(I) \). Thus one has

\[
I^\dual = \left( \prod_{f \in G(I_1)} f, \ldots, \prod_{f \in G(I_m)} f \right) \quad \text{and} \quad I^* = \bigcap_{x^a \in G(I)} \{x^{ai}|ai \geq 1\},
\]

where \( I_1, \ldots, I_m \) are the irreducible components of \( I \). If \( I^* = I^\dual \), we say that *Alexander duality* holds for \( I \). There are other related ways introduced by Ezra Miller [23, 27, 28, 29] to define the Alexander dual of a monomial ideal. It is well known that \( I^* = I^\dual \) for squarefree monomial ideals [42, Theorem 6.3.39].

**Definition 2.** Let \( I \) be an ideal of a ring \( R \) and let \( p_1, \ldots, p_r \) be the minimal primes of \( I \). Given an integer \( k \geq 1 \), we define the \( k \)-th *symbolic power* of \( I \) to be the ideal

\[
I^k := \bigcap_{i=1}^r q_i = \bigcap_{i=1}^r (I^k R_{p_i} \cap R),
\]

where \( q_i \) is the \( p_i \)-primary component of \( I^k \).
In other words, one has $I^{(k)} = S^{-1}I^k \cap R$, where $S = R \setminus \cup_{i=1}^{r} p_i$. An alternative notion of symbolic power can be introduced using the whole set of associated primes of $I$ instead (see, e.g., [5, 7]):

$$I^{(k)} = \bigcap_{p \in \text{Ass}(R/I)} (I^k R_p \cap R) = \bigcap_{p \in \text{maxAss}(R/I)} (I^k R_p \cap R),$$

where $\text{maxAss}(R/I)$ is the set of associated primes which are maximal with respect to inclusion [5, Lemmas 3.1 and 3.2]. Clearly $I^k \subset I^{(k)} \subset I^{(1)}$. If $I$ has no embedded primes, e.g. for radical ideals such as squarefree monomial ideals, the two last definitions of symbolic powers coincide. An interesting problem is to give necessary and sufficient conditions for the equality “$I^k = I^{(k)}$ for $k \geq 1$”.

For prime ideals the $k$-th symbolic powers and the $k$-th usual powers are not always equal. Thus the next lemma does not hold in general but the proof below shows that it will hold for an ideal $I$ in Noetherian ring $R$ under the assumption that $\mathcal{J}^k_i = \mathcal{J}^{(k)}_i$ for $i = 1, \ldots, r$. The next lemma is well known for radical monomial ideals [41, Propositions 3.3.24 and 7.3.14].

**Lemma 2.** Let $I \subset R$ be a monomial ideal and let $I = \mathcal{J}_1 \cap \cdots \cap \mathcal{J}_r \cap \cdots \cap \mathcal{J}_m$ be an irredundant minimal primary decomposition of $I$, where $\mathcal{J}_1, \ldots, \mathcal{J}_r$ are the primary components associated to the minimal primes of $I$. Then

$$I^{(k)} = \mathcal{J}^k_1 \cap \cdots \cap \mathcal{J}^k_r \text{ for } k \geq 1.$$  

**Proof.** Let $p_1, \ldots, p_r$ be the minimal primes of $I$. By [42, Proposition 6.1.7] any power of $\mathcal{J}_j$ is again a $p_j$-primary ideal (see Eq. (1) at the beginning of this section). Thus $\mathcal{J}^k_j = \mathcal{J}^{(k)}_j$ for any $i, k$. Fixing integers $k \geq 1$ and $1 \leq i \leq r$, let

$$I^k = q_1 \cap \cdots \cap q_r \cap \cdots \cap q_s$$

be a primary decomposition of $I^k$, where $q_j$ is $p_j$-primary for $j \leq r$. Localizing at $p_i$ yields $I^k R_{p_i} = q_i R_{p_i}$, and from $I = \mathcal{J}_1 \cap \cdots \cap \mathcal{J}_r \cap \cdots \cap \mathcal{J}_m$ one obtains:

$$I^k R_{p_i} = (IR_{p_i})^k = (\mathcal{J}_i R_{p_i})^k = \mathcal{J}^k_i R_{p_i}.$$

Thus $\mathcal{J}^k_i R_{p_i} = q_i R_{p_i}$, and contracting to $R$ one has $\mathcal{J}^k_i = q_i$. Therefore

$$I^{(k)} = \mathcal{J}^{(k)}_1 \cap \cdots \cap \mathcal{J}^{(k)}_r = \mathcal{J}^k_1 \cap \cdots \cap \mathcal{J}^k_r. \quad \square$$

It was pointed out to us by Ngô Viêt Trung that Lemma 2 is a consequence of [21, Lemma 3.1]. This lemma also follows from [5, Proposition 3.6].

**Remark 1.** To compute the $k$-th symbolic power $I^{(k)}$ of a monomial ideal $I$ one can use the following procedure for Macaulay2 [16].

```plaintext
SPG = (I,k) -> intersect(for n from 0 to #minimalPrimes(I)-1 list localize(Iˆk, (minimalPrimes(I))#n))
```
Example 1. Let $I$ be the ideal $(x_2x_3, x_4x_5, x_3x_4, x_2x_5, x_1^2x_3, x_1x_2^2)$. Using the procedure of Remark 1, we obtain $I^{(2)} = I^2 + (x_1x_2^2x_3, x_1x_2x_3)$. 

Remark 2. If one uses Ass$(R/I)$ to define the symbolic powers of a monomial ideal $I$, the following function for Macaulay2 $^{16}$ can be used to compute $I^{(k)}$.

```
SPA=I,k->intersect(for n from 0 to #associatedPrimes(I)-1 list localize(I^n,(associatedPrimes(I))#n))
```

Example 2. Let $I$ be the ideal $(x_1x_2^2, x_3x_1^2, x_2x_3^2)$. Using the procedures of Remarks 1 and 2, we obtain

$I^{(1)} = I + (x_1x_2x_3)$ and $I^{(1)} = I$.

Remark 3. The following formula is useful to study the symbolic powers $I^{(k)}$ of a monomial ideal $I$ $^{[5]}$ Proposition 3.6]:

$$I^k R_p \cap R = (IR_p \cap R)^k$$

for $p \in$ Ass$(R/I)$ and $k \geq 1$.

Definition 3. An ideal $I$ of a ring $R$ is called normally torsion-free if $\text{Ass}(R/I^k)$ is contained in $\text{Ass}(R/I)$ for all $k \geq 1$.

Remark 4. Let $I$ be an ideal of a ring $R$. If $I$ has no embedded primes, then $I$ is normally torsion-free if and only if $I^k = I^{(k)}$ for all $k \geq 1$.

Lemma 3. $^{[43]}$ Lemma 5, Appendix 6] Let $I \subset R$ be an ideal generated by a regular sequence. Then $I^k$ is unmixed for $k \geq 1$. In particular $I^k = I^{(k)}$ for $k \geq 1$.

One can also compute the symbolic powers of vanishing ideals of finite sets of reduced projective points using Lemma 2 because these ideals are intersections of finitely many prime ideals that are complete intersections. It is well known that complete intersections are normally torsion-free (Lemma 3).

Remark 5. (Jonathan O’Rourke) If $I$ is a radical ideal of $R$ and all associated primes of $I$ are normally torsion-free, then the $k$-th symbolic power of $I$ can be computed using the following procedure for Macaulay2 $^{16}$.

```
SP1 = (I,k) -> ((temp = primaryDecomposition I; temp2 = (temp_0)ˆk); for i from 1 to #temp-1 do(temp2 = intersect(temp2, (temp_i)ˆk)); return temp2)
```

Example 3. Let $X$ be the set $\{[e_1], [e_2], [e_3], [e_4], [[1, 1, 1, 1]]\}$ of 5 points in general linear position in $\mathbb{P}^3$, over the field $\mathbb{Q}$, where $e_i$ is the $i$-th unit vector, and let $I = I(X)$ be its vanishing ideal. Using Macaulay2 $^{16}$ and Remark 5, we obtain

$I = (x_2x_4 - x_3x_4, x_1x_4 - x_3x_4, x_2x_3 - x_3x_4, x_1x_3 - x_3x_4, x_1x_2 - x_3x_4),$  

$I^2 = I^{(2)}, I^3 \neq I^{(3)}$ and $I$ is a Gorenstein ideal. This example (in greater generality) has been used in $^{[31]}$ proof of Proposition 4.1 and Remark 4.2(2).

Proposition 2. $^{[21]}$ If $I \subset R$ is a monomial ideal, then the symbolic Rees algebra $R_s(I)$ of $I$ is a finitely generated $K$-algebra.
Proof. It follows at once from Lemma 2 and [21, Corollary 1.3]. □

To compute the generators of the symbolic Rees algebra of a monomial ideal one can use the procedure given in the proof of [21, Theorem 1.1]. Another method will be presented in this section that works when the primary components are normal.

Remark 6. The symbolic Rees algebra of a monomial ideal $I$ is finitely generated if one uses the associated primes of $I$ to define symbolic powers. This follows from [21, Corollary 1.3] and the following formula [5, Theorem 3.7]:

$$I^{(k)} = \bigcap_{p \in \text{maxAss}(R/I)} (IR_p \cap R)^k$$

for $k \geq 1$.

Corollary 1. If $I$ is a monomial ideal, then $R_s(I)$ is Noetherian and there is an integer $k \geq 1$ such that $[I^{(k)}] = I^{(k)}$ for $i \geq 1$.

Proof. It follows at once from [15, p. 80, Lemma 2.1] or by a direct argument using Proposition 2. □

For convenience of notation in what follows we will often assume that monomial ideals have no embedded primes but some of the results can be stated and proved for general monomial ideals.

Proposition 3. Let $I \subset R$ be a monomial ideal without embedded primes and let $I = \cap_{i=1}^r I_i$ be its minimal irredundant primary decomposition. Then $R_s(I)$ is normal if and only if $R[\mathfrak{J}_i]$ is normal for all $i$.

Proof. $\Rightarrow$: Since $R_s(I)$ is Noetherian and normal it is a Krull domain by a theorem of Mori and Nagata [26, p. 296]. Therefore, by [37, Lemma 2.5], we get that $R_p[I_p,t] = R_p[I_p] \cap R[t]$ is normal. Let $p_i$ be the radical of $\mathfrak{J}_i$. Any power of $\mathfrak{J}_i$ is a $p_i$-primary ideal. This follows from [42, Proposition 6.1.7] (see Eq. (1) at the beginning of this section). Hence it is seen that $R_p[(\mathfrak{J}_i)p,t] \cap R[t] = R[\mathfrak{J}_i,t]$. As $R[t]$ is normal it follows that $R[\mathfrak{J}_i,t]$ is normal.

$\Leftarrow$: By Lemma 2 one has $\cap_{i=1}^r R[\mathfrak{J}_i,t] = R_s(I)$. As $R[\mathfrak{J}_i,t]$ and $R_s(I)$ have the same field of quotients it follows that $R_s(I)$ is normal. □

In general, even for monomial ideals without embedded primes, normally torsion-free ideals may not be normal. For instance $I = (x_1^2, x_2^3)$ is normally torsion-free and is not normal. As a consequence of Proposition 3 one recovers the following well known result.

Corollary 2. Let $I$ be a squarefree monomial ideal. Then $R_s(I)$ is normal and $R[\mathfrak{J}]$ is normal if $I$ is normally torsion-free.

Let $I$ be a monomial ideal and let $G(I) = \{x^{v_1}, \ldots, x^{v_m}\}$ be its minimal set of generators. We set

$$\mathcal{A}_I = \{e_1, \ldots, e_n, (v_1, 1), \ldots, (v_m, 1)\},$$
where $e_1, \ldots, e_n$ belong to $\mathbb{Z}^{n+1}$, and denote by $\mathbb{R}_+(I)$ or $\mathbb{R}_+(\mathfrak{A}_t)$ (resp. $\mathbb{N}_+(\mathfrak{A}_t)$) the cone (resp. semigroup) generated by $\mathfrak{A}_t$. The integral closure of $R[\mathfrak{A}_t]$ is given by $R[\mathfrak{A}_t] = K[\mathbb{R}_+(I) \cap \mathbb{Z}^{n+1}]$. Recall that a finite set $\mathcal{H}$ is called a Hilbert basis if $\mathbb{N}_+(\mathcal{H}) = \mathbb{R}_+(\mathcal{H}) \cap \mathbb{Z}^{n+1}$, and that $R[\mathfrak{A}_t]$ is normal if and only if $\mathfrak{A}_t$ is a Hilbert basis [42 Proposition 14.2.3].

Let $C \subset \mathbb{R}^{n+1}$ be a rational polyhedral cone. A finite set $\mathcal{H}$ is called a Hilbert basis of $C$ if $C = \mathbb{R}_+(\mathcal{H})$ and $\mathcal{H}$ is a Hilbert basis. A Hilbert basis of $C$ is minimal if it does not strictly contain any other Hilbert basis of $C$. For pointed cones there is unique minimal Hilbert basis [42 Theorem 1.3.9].

If the primary components of a monomial ideal are normal, the next result gives a simple procedure to compute its symbolic Rees algebra using Hilbert bases.

**Proposition 4.** Let $I$ be a monomial ideal without embedded primes and let $I = \cap_{i=1}^r \mathfrak{I}_i$ be its minimal irredundant primary decomposition. If $R[\mathfrak{I}_i]$ is normal for all $i$ and $\mathcal{H}$ is the Hilbert basis of the polyhedral cone $\cap_{i=1}^r \mathbb{R}_+(\mathfrak{I}_i)$, then $R_s(I)$ is $K[\mathbb{N}_+(\mathcal{H})]$, the semigroup ring of $\mathbb{N}_+(\mathcal{H})$.

**Proof.** As $R[\mathfrak{I}_i] = K[\mathbb{N}_+(\mathfrak{A}_t)]$ is normal for $i = 1, \ldots, r$, the semigroup $\mathbb{N}_+(\mathfrak{A}_t)$ is equal to $\mathbb{R}_+(\mathfrak{I}_i) \cap \mathbb{Z}^{n+1}$ for $i = 1, \ldots, r$. Hence, by Lemma 2 we get

$$R_s(I) = \cap_{i=1}^r R[\mathfrak{I}_i] = \cap_{i=1}^r K[\mathbb{N}_+(\mathfrak{A}_t)] = K[\cap_{i=1}^r \mathbb{N}_+(\mathfrak{A}_t)]$$

$$= K[\mathbb{R}_+(\mathfrak{I}_1) \cap \cdots \cap \mathbb{R}_+(\mathfrak{I}_r) \cap \mathbb{Z}^{n+1}] = K[\mathbb{N}_+(\mathcal{H})].$$

**Definition 4.** The rational polyhedral cone $\cap_{i=1}^r \mathbb{R}_+(\mathfrak{I}_i)$ is called the Simis cone of $I$ and is denoted by $C_n(I)$.

For squarefree monomial ideals the Simis cone was introduced in [10]. In particular from Proposition 4 we recover [10 Theorem 3.5].

**Example 4.** The ideal $I = (x_2x_3, x_4x_5, x_3x_4, x_2x_5, x_1^2x_3, x_1x_2^2)$ satisfies the hypothesis of Proposition 4. Using Normaliz we obtain that the minimal Hilbert basis of the Simis cone is:

18 Hilbert basis elements:

0 0 0 0 1 0 1 2 0 0 0 1
0 0 0 1 0 0 2 0 1 0 0 1
0 0 1 0 0 0 1 2 0 0 1 2
0 1 0 0 0 0 1 2 1 0 0 2
1 0 0 0 0 0 2 2 1 0 1 3
0 0 0 1 1 1 2 2 2 0 0 3
0 0 1 1 0 1 2 4 1 0 2 5
0 1 0 0 1 1 2 4 2 0 1 5
0 1 1 0 0 1 2 4 3 0 0 5

Hence $R_s(I)$ is generated by the monomials corresponding to these vectors.

Let $I$ be an ideal of $R$. The equality “$I^k = I^{(k)}$ for $k \geq 1$” holds if and only if $I$ has no embedded primes and is normally torsion-free (see Remark 3). We refer the reader to [7] for a recent survey on symbolic powers of ideals.
In [13, Corollary 3.14] it is shown that a squarefree monomial ideal \( I \) is normally torsion-free if and only if the corresponding hypergraph satisfies the max-flow min-cut property. As an application we present a classification of the equality between ordinary and symbolic powers for a family of monomial ideals.

**Corollary 3.** Let \( I \) be a monomial ideal without embedded primes and let \( J_1, \ldots, J_r \) be its primary components. If \( R[J_i] \) is normal for all \( i \), then \( I^k = I^{(k)} \) for \( k \geq 1 \) and only if \( \text{Cn}(I) = \mathbb{R}_+(I) \) and \( R[It] \) is normal.

**Proof.** \( \Rightarrow \): As \( R_s(I) = R[It] \), by Proposition[4] \( R[It] \) is normal. Therefore one has

\[
K[\text{Cn}(I) \cap \mathbb{Z}^{n+1}] = R_s(I) = R[It] = \overline{R[It]} = K[\mathbb{R}_+(I) \cap \mathbb{Z}^{n+1}].
\]

Thus \( \text{Cn}(I) = \mathbb{R}_+(I) \).

\( \Leftarrow \): By the proof of Proposition[4] one has \( R_s(I) = K[\text{Cn}(I) \cap \mathbb{Z}^{n+1}] \). Hence

\[
R_s(I) = K[\text{Cn}(I) \cap \mathbb{Z}^{n+1}] = K[\mathbb{R}_+(I) \cap \mathbb{Z}^{n+1}] = \overline{R[It]}.
\]

As \( R[It] \) is normal, we get \( R_s(I) = R[It] \), that is, \( I^k = I^{(k)} \) for \( k \geq 1 \). \( \square \)

## 3 Cohen–Macaulay weighted oriented trees

In this section we show that edge ideals of transitive weighted oriented graphs satisfy Alexander duality. It turns out that edge ideals of weighted acyclic tournaments are Cohen–Macaulay and satisfy Alexander duality. Then we classify all Cohen–Macaulay weighted oriented forests. Here we continue to employ the notation and definitions used in Sections[1] and [2].

Let \( G \) be a graph with vertex set \( V(G) \). A subset \( C \subseteq V(G) \) is a minimal vertex cover of \( G \) if: (i) every edge of \( G \) is incident with at least one vertex in \( C \), and (ii) there is no proper subset of \( C \) with the first property. If \( C \) satisfies condition (i) only, then \( C \) is called a vertex cover of \( G \).

Let \( \mathcal{D} \) be a weighted oriented graph with underlying graph \( G \). Next we recall a combinatorial description of the irreducible decomposition of \( I(\mathcal{D}) \).

**Definition 5.**[4] Let \( C \) be a vertex cover of \( G \). Consider the set \( L_1(C) \) of all \( x \in C \) such that there is \( (x, y) \in E(\mathcal{D}) \) with \( y \notin C \), the set \( L_3(C) \) of all \( x \in C \) such that \( N_G(x) \subseteq C \), and the set \( L_2(C) = C \setminus (L_1(C) \cup L_3(C)) \), where \( N_G(x) \) is the neighbor set of \( x \) consisting of all \( y \in V(G) \) such that \( \{x, y\} \) is an edge of \( G \). A vertex cover \( C \) of \( G \) is called a strong vertex cover of \( \mathcal{D} \) if \( C \) is a minimal vertex cover of \( G \) or else for all \( x \in L_3(C) \) there is \( (y, x) \in E(\mathcal{D}) \) such that \( y \in L_2(C) \cup L_3(C) \) with \( d(y) \geq 2 \).

**Theorem 1.**[4] Let \( \mathcal{D} \) be a weighted oriented graph. Then \( L \) is a minimal irreducible monomial ideal of \( I(\mathcal{D}) \) if and only if there is a strong vertex cover of \( \mathcal{D} \) such that

\[
L = (L_1(C) \cup \{x_i^d \mid x_i \in L_2(C) \cup L_3(C)\}).
\]
**Theorem 2.** If $\mathcal{D}$ is a weighted oriented graph and $\Upsilon(\mathcal{D})$ is the set of all strong vertex covers of $\mathcal{D}$, then the irreducible decomposition of $I(\mathcal{D})$ is

$$I(\mathcal{D}) = \bigcap_{C \in \Upsilon(\mathcal{D})} I_C,$$

where $I_C = (L_1(C) \cup \{x_i^{d_i} | x_i \in L_2(C) \cup L_3(C)\})$.

**Proof.** This follows at once from Proposition 1 and Theorem 1. \qed

**Corollary 4.** Let $\mathcal{D}$ be a weighted oriented graph. Then $p$ is an associated prime of $I(\mathcal{D})$ if and only if $p = (C)$ for some strong vertex cover $C$ of $\mathcal{D}$.

**Example 5.** Let $K$ be the field of rational numbers and let $\mathcal{D}$ be the weighted digraph of Fig. 1 whose edge ideal is $I = I(\mathcal{D}) = (x_1^2x_3, x_1x_2^2, x_3x_2^2, x_3x_4^2, x_2x_5, x_2^2x_5)$. By

Fig. 1: A Cohen–Macaulay digraph

Theorem 2, the irreducible decomposition of $I$ is

$$I = (x_1^2, x_2^2, x_3^2) \cap (x_1, x_3, x_5) \cap (x_2^2, x_3, x_4^2) \cap (x_2^2, x_3, x_5).$$

Using Macaulay 2 [16], we get that $I$ is a Cohen–Macaulay ideal whose Rees algebra is Cohen–Macaulay and whose integral closure is

$$7 = I + (x_1x_2x_3, x_1x_3x_4, x_2x_3x_4, x_2x_4x_5).$$

We note that the Cohen–Macaulayness of both $I$ and its Rees algebra is destroyed (or recovered) by a single stroke of reversing the edge orientation of $(x_5,x_2)$. This also destroys the unmixedness property of $I$.

In the summer of 2017 Antonio Campillo asked in a seminar at the University of Valladolid if there was anything special if we take an oriented graph $\mathcal{D}$ with underlying graph $G$ and set $d_i$ equal to $\deg_G(x_i)$ for $i = 1, \ldots, n$. It will turn out that in determining the Cohen–Macaulay property of $\mathcal{D}$ one can always make this canonical choice of weights.
Lemma 4. Let $I \subset R$ be a monomial ideal, let $x_i$ be a variable and let $h_1, \ldots, h_r$ be the monomials of $G(I)$ where $x_i$ occurs. If $x_i$ occurs in $h_j$ with exponent 1 for all $j$ and $m$ is a positive integer, then $I$ is Cohen–Macaulay of height $g$ if and only if $((G(I) \setminus \{h_j\}_{j=1}^r) \cup \{x_i^mh_j\}_{j=1}^r)$ is Cohen–Macaulay of height $g$.

Proof. It follows at once from [32] Lemmas 3.3 and 3.5. □

It was pointed out to us by Ngô Viêt Trung that the next proposition follows from the fact that the map $x_i \to y_i^{d_i}$ (replacing $x_i$ by $y_i^{d_i}$) defines a faithfully flat homomorphism from $K[X]$ to $K[Y]$.

Proposition 5. Let $I$ be a squarefree monomial ideal and let $d_i = d(x_i)$ be a weighting of the variables. If $G'$ is a set of monomials obtained from $G(I)$ by replacing each $x_i$ with $x_i^{d_i}$, then $I$ is Cohen–Macaulay if and only if $I' = (G')$ is Cohen–Macaulay.

Proof. It follows applying Lemma 4 to each $x_i$. □

If a vertex $x_i$ is a sink (i.e., has only arrows entering $x_i$), the next result shows that the Cohen–Macaulay property of $I(\mathcal{D})$ is independent of the weight of $x_i$.

Corollary 5. If $x_i$ is a sink of a weighted oriented graph $\mathcal{D}$ and $\mathcal{D}'$ is the digraph obtained from $\mathcal{D}$ by replacing $d_i$ with $d_i = 1$. Then $I(\mathcal{D})$ is Cohen–Macaulay if and only if $I(\mathcal{D}')$ is Cohen–Macaulay.

That is, to determine whether or not an oriented graph $\mathcal{D}$ is Cohen–Macaulay one may assume that all sources and sinks have weight 1. In particular if all vertices of $\mathcal{D}$ are either sources of sinks and $G$ is its underlying graph, then $I(\mathcal{D})$ is Cohen–Macaulay if and only if $I(G)$ is Cohen–Macaulay.

Let $I$ be a monomial ideal and let $x_i$ be a fixed variable that occurs in $G(I)$. Let $q$ be the maximum of the degrees in $x_i$ of the monomials of $G(I)$ and let $B_i$ be the set of all monomial of $G(I)$ of degree in $x_i$ equal to $q$. For use below we set

$$\mathcal{A}_i := \{x^i | \deg_{x_i}(x^i) < q\} \cap G(I) \setminus B_i;$$

$$p := \max\{\deg_{x_i}(x^i) | x^i \in \mathcal{A}_i\}$$

and $L := (\{x^i/x_i | x^i \in B_i \cup \mathcal{A}_i\})$.

Theorem 3. Let $I$ be a monomial ideal. If $p \geq 1$, and $q - p \geq 2$, then

$$\text{depth}(R/I) = \text{depth}(R/L).$$

Proof. To simplify notation we set $i = 1$. We may assume that $G(I) = \{f_1, \ldots, f_r\}$, where $f_1, \ldots, f_m$ are all the elements of $G(I)$ that contain $x_1^i$ and $f_{m+1}, \ldots, f_k$ are all the elements of $G(I)$ that contain some positive power $x_1^i$ of $x_1$ for some $1 \leq \ell < q$. Let $X' = \{x_{1,2}, \ldots, x_{1,q-1}\}$ be a set of new variables. If $f = x_1^if'$ is a monomial with $\gcd(x_1, f') = 1$, we write $f^{\text{pol}} = x_{1,2} \cdots x_{1,t} + x_{1}^{s-t}f'$ where $t = \min(q - s, s)$. Making a partial polarization of $x_1^i$ with respect to the new variables $x_{1,2}, \ldots, x_{1,q-1}$ [42, p. 203], gives that $f_1$ polarizes to $f_1^{\text{pol}} = x_{1,2} \cdots x_{1,q-1}x_1^if_1'$ for $i = 1, \ldots, m$, where
Monomial ideals and Cohen-Macaulay digraphs

\[ f_1, \ldots, f_m \] are monomials that do not contain \( x_1 \) and \( f_i = x_i^q f_i' \) for \( i = 1, \ldots, m \). Hence, using that \( q - p \geq 2 \), one has the partial polarization

\[ \mathcal{P}_{\text{pol}} = (x_{1,2} \cdots x_{1,q-1} f_1' \cdots, \ldots, x_{1,q-1} f_m', f_{s+1}, \ldots, f_r), \]

where \( f_m', \ldots, f_r \) do not contain \( x_1 \) and \( \mathcal{P}_{\text{pol}} \) is an ideal of \( R_{\text{pol}} = R[x_{1,2}, \ldots, x_{1,q-1}] \).

On the other hand, one has the partial polarization

\[ \mathcal{L}_{\text{pol}} = (x_{1,2} \cdots x_{1,q-1} f_1' \cdots, \ldots, x_{1,q-1} f_m', f_{s+1}, \ldots, f_r). \]

By making the substitution \( x_1^q \rightarrow x_1 \) in each element of \( G(\mathcal{P}_{\text{pol}}) \) this will not affect the depth of \( R_{\text{pol}} / \mathcal{P}_{\text{pol}} \) (see [32, Lemmas 3.3 and 3.5]). Thus

\[ q - 2 + \text{depth}(R/I) = \text{depth}(R_{\text{pol}} / \mathcal{P}_{\text{pol}}) = \text{depth}(R_{\text{pol}} / \mathcal{L}_{\text{pol}}) = q - 2 + \text{depth}(R/L), \]

and consequently \( \text{depth}(R/I) = \text{depth}(R/L) \). □

**Corollary 6.** Let \( I = I(\mathcal{D}) \) be the edge ideal of a vertex-weighted oriented graph with vertices \( x_1, \ldots, x_n \) and let \( d_i \) be the weight of \( x_i \). If \( \mathcal{W} \) is the digraph obtained from \( \mathcal{D} \) by assigning weight 2 to every vertex \( x_i \) with \( d_i \geq 2 \), then \( I \) is Cohen-Macaulay if and only if \( I(\mathcal{W}) \) is Cohen-Macaulay.

**Proof.** By applying Theorem[3] to each vertex \( x_i \) of \( \mathcal{D} \) of weight at least 3, we obtain that \( \text{depth}(R/I(\mathcal{D})) = \text{depth}(R/I(\mathcal{W})) \). Since \( I(\mathcal{D}) \) and \( I(\mathcal{W}) \) have the same height, then \( I(\mathcal{D}) \) is Cohen-Macaulay if and only if \( I(\mathcal{W}) \) is Cohen-Macaulay. □

**Lemma 5.** [19, Theorem 16.3(4), p. 200] Let \( \mathcal{D} \) be an oriented graph. Then \( \mathcal{D} \) is acyclic, i.e., \( \mathcal{D} \) has no oriented cycles, if and only if there is a linear ordering of the vertex set \( \mathcal{V}(\mathcal{D}) \) such that all the edges of \( \mathcal{D} \) are of the form \((x_i, x_j)\) with \( i < j \).

A complete oriented graph is called a tournament. The next result shows that weighted acyclic tournaments are Cohen–Macaulay.

**Corollary 7.** Let \( \mathcal{D} \) be a weighted oriented graph. If the underlying graph \( \mathcal{G} \) of \( \mathcal{D} \) is a complete graph and \( \mathcal{D} \) has no oriented cycles, then \( I(\mathcal{D}) \) is Cohen–Macaulay.

**Proof.** By Lemma[5] \( \mathcal{D} \) has a source \( x_i \) for some \( i \). Hence \( \{x_1, \ldots, x_n\} \) is not a strong vertex cover of \( \mathcal{D} \) because there is no arrow entering \( x_i \). Thus, by Corollary[4] the maximal ideal \( m = (x_1, \ldots, x_n) \) cannot be an associated prime of \( I(\mathcal{D}) \). Therefore \( R/I(\mathcal{D}) \) has depth at least 1. As \( \text{dim}(R/I(\mathcal{D})) = 1 \), we get that \( R/I(\mathcal{D}) \) is Cohen–Macaulay. □

The next result gives an interesting family of digraphs whose edge ideals satisfy Alexander duality. Recall that a digraph \( \mathcal{D} \) is called transitive if for any two edges \((x_i, x_j), (x_j, x_k) \) in \( E(\mathcal{D}) \) with \( i, j, k \) distinct, we have that \((x_i, x_k) \in E(\mathcal{D}) \). Acyclic tournaments are transitive and transitive oriented graphs are acyclic.

**Theorem 4.** If \( \mathcal{D} \) is a transitive oriented graph and \( I = I(\mathcal{D}) \) is its edge ideal, then Alexander duality holds, that is, \( I^* = I' \).
Proof. \( \Rightarrow \): Take \( x^\alpha \in G(I') \). According to Theorem 2 there is a strong vertex cover \( C \) of \( \mathcal{D} \) such that

\[
x^\alpha = \left( \prod_{x_i \in L_1} x_i \right) \left( \prod_{x_k \in L_2 \cup L_3} x_k^{d_k} \right),
\]

where \( L_i = L_i(C) \) for \( i = 1, 2, 3 \). Fix a monomial \( x_i x_j^{d_j} \) in \( G(I(\mathcal{D})) \), that is, \( (x_i, x_j) \in E(\mathcal{D}) \). It suffices to show that \( x^\alpha \) is in the ideal \( I_{i,j} := \langle \{x_i, x_j^{d_j}\} \rangle \). If \( x_i \in C \), then by Eq. (2) the variable \( x_i \) occurs in \( x^\alpha \) because \( C \) is equal to \( L_1 \cup L_2 \cup L_3 \). Hence \( x^\alpha \) is a multiple of \( x_i \) and \( x^\alpha \) is in \( I_{i,j} \), as required. Thus we may assume that \( x_i \notin C \). By Theorem 2 the ideal

\[
I_C = (L_1 \cup \{ x_k^{d_k} | x_k \in L_2 \cup L_3 \})
\]
is an irreducible component of \( I(\mathcal{D}) \) and \( x_i x_j^{d_j} \in I_C \).

Case (I): \( x_i x_j^{d_j} \in (L_1) \). Then \( x_i x_j^{d_j} = x_k x^\alpha \) for some \( x_k \in L_1 \). Hence, as \( x_k \notin C \), we get \( j = k \). Therefore, as \( x_j \in L_1 \), there is \( x_k \notin C \) such that \( (x_j, x_k) \) is in \( E(\mathcal{D}) \). Using that \( \mathcal{D} \) is transitive gives \( (x_i, x_k) \in \mathcal{D} \) and \( x_i x_k^{d_k} \in I(\mathcal{D}) \). In particular \( x_i x_k^{d_k} \in I_C \), a contradiction because \( x_i \) and \( x_k \) are not in \( C \). Hence this case cannot occur.

Case (II): \( x_i x_j^{d_j} \in \langle \{ x_k^{d_k} | x_k \in L_2 \cup L_3 \} \rangle \). Then \( x_i x_j^{d_j} = x_k^{d_k} x^\alpha \) for some \( x_k \in L_2 \cup L_3 \). As \( x_k \notin C \), we get \( j = k \) and by Eq. (2) we obtain \( x^\alpha \in I_{i,j} \), as required.

\( \Rightarrow \): Take a minimal generator \( x^\alpha \) of \( I' \). By Lemma (1) for each \( i \) either \( \alpha_i = 1 \) or \( \alpha_i = d_i \). Consider the set \( A = \{ x_k | \alpha_k \geq 1 \} \). We can write \( A = A_1 \cup A_2 \), where \( A_1 \) (resp. \( A_2 \)) is the set of all \( x_k \) such that \( \alpha_k = 1 \) (resp. \( \alpha_k = d_k \geq 2 \)). As \( A \) contains \( I \), from the proof of Proposition (1) and using Theorem 2 there exists a strong vertex cover \( C \) of \( \mathcal{D} \) contained in \( A \) such that the ideal

\[
I_C = (L_1(C) \cup \{ x_i^{d_i} | x_i \in L_2(C) \cup L_3(C) \})
\]
is an irreducible component of \( I(\mathcal{D}) \). Thus it suffices to show that any monomial of \( G(I_C) \) divides \( x^\alpha \) because this would give \( x^\alpha \in I' \).

Claim (I): If \( x_k \in A_1 \), then \( d_k = 1 \) or \( x_k \in L_1(A) \). Assume that \( d_k \geq 2 \). Since \( x^\alpha \) is a minimal generator of \( I' \), the monomial \( x^\alpha / x_k \) is not in \( I' \). Then there is and edge \( (x_j, x_k) \) such that \( x^\alpha / x_k \) is not in the ideal \( I_{i,j} := \langle \{x_i, x_j^{d_j}\} \rangle \). As \( x^\alpha \) is in \( I' \) and \( d_k \geq 2 \), one has that \( x^\alpha \) is in \( I_{i,j} \) and \( i = k \). Notice that \( x_k \) is not in \( A_2 \) because \( x^\alpha / x_k \) is not in \( I_{k,j} \). If \( x_j \) is in \( A_1 \) the proof is complete because \( x_k \in L_1(A) \). Assume that \( x_k \) is in \( A_1 \). Then \( d_j \geq 2 \) because \( x^\alpha / x_k \) is not in \( I_{k,j} \). Setting \( k_1 = k \) and \( k_2 = j \) and applying the previous argument to \( x^\alpha / x_k \), there is \( x_{k_1} \notin A_2 \) such that \( (x_{k_1}, x_k) \) is in \( E(\mathcal{D}) \). Since \( \mathcal{D} \) is transitive, \( (x_{k_1}, x_{k_2}) \) is in \( E(\mathcal{D}) \). If \( x_{k_2} \) is in \( A_1 \) the proof is complete. If \( x_{k_1} \) is in \( A_1 \) and \( s < r \) such that \( x_{k_s} \notin A_2 \), and \( (x_{k_1}, x_{k_s-1}) \) and \( (x_{k_s}, x_{k_r}) \) are in \( E(\mathcal{D}) \). Since \( \mathcal{D} \) is transitive, \( (x_{k_1}, x_{k_r}) \) is in \( E(\mathcal{D}) \). If \( x_{k_r} \) is not in \( A_1 \) the proof is complete. If \( x_{k_1} \) is in \( A_1 \) and \( s = r \), that is, \( A_1 = \{ x_{k_1}, \ldots, x_{k_r} \} \), then applying the
previous argument to $x^a/x_k$ there is $x_{r+1}$ not in $A$ such that $(x_r, x_{r+1})$ is in $E(D)$. Thus by transitivity $(x_k, x_{r+1})$ is in $E(D)$, that is, $x_k$ is in $L_1(A)$.

Claim (II): If $x_k \notin A_2$, then $x_k \in L_2(A)$. Since $x^a \in G(I^*)$ and $\alpha_k = d_k \geq 2$, there is $(x_i, x_k)$ in $E(D)$ such that $x^a/x_k$ is not in $I_{x_k} = (\{x_i, x_k\})$. In particular, $x_i$ is not in $A$. To prove that $x_k$ is in $L_2(A)$ it suffices to show that $x_k$ is not in $L_1(A)$. If $x_k$ is in $L_1(A)$, there is $x_j$ not in $A$ such that $(x_k, x_j)$ is in $E(D)$. As $D$ is transitive, we get that $(x_i, x_j)$ is in $E(D)$ and $A \cap \{x_i, x_j\} = \emptyset$, a contradiction because $(A)$ contains $I$.

Take a monomial $x_k^{\alpha_k}$ of $G(I_C)$.

Case (A): $x_k \in L_1(C)$. Then $\alpha_k = 1$. There is $(x_k, x_j) \in E(D)$ with $x_j \notin C$. Notice $x_k \in A_1$. Indeed if $x_k \in A_2$, then $x_k$ is in $L_2(A)$ because of Claim (II). Then there is $(x_i, x_k)$ in $E(D)$ with $x_i \notin A$. By transitivity $(x_i, x_j) \in E(D)$ and $(x_i, x_j) \cap C = \emptyset$, a contradiction because $(C)$ contains $I$. Thus $x_k \in A_1$, that is, $\alpha_k = 1$. This proves that $x_k^{\alpha_k}$ divides $x^a$.

Case (B): $x_k \in L_2(C)$. Then $x_k^{\alpha_k} = x_k^{d_k}$. First assume $x_k \in A_1$. Then, by Claim (I), $d_k = 1$ or $x_k \notin L_1(A)$. Clearly $x_k \notin L_1(A)$ because $L_1(A) \subseteq L_1(C)$ and $x_k$—being in $L_2(A)$—cannot be in $L_1(C)$. Thus $d_k = 1$ and $x_k^{d_k}$ divides $x^a$. Next assume $x_k \in A_2$. Then, by construction of $A_2$, $x_k^{d_k}$ divides $x^a$.

Case (C): $x_k \in L_3(C)$. Then $x_k^{\alpha_k} = x_k^{d_k}$. First assume $x_k \in A_1$. Then, by Claim (I), $d_k = 1$ or $x_k \notin L_1(A)$. Clearly $x_k \notin L_1(A)$ because $L_1(A) \subseteq L_1(C)$ and $x_k$—being in $L_3(A)$—cannot be in $L_1(C)$. Thus $d_k = 1$ and $x_k^{d_k}$ divides $x^a$. Next assume $x_k \in A_2$. Then, by construction of $A_2$, $x_k^{d_k}$ divides $x^a$. □

**Corollary 8.** If $D$ is a weighted acyclic tournament, then $I(D)^* = I(D)^\vee$, that is, Alexander duality holds.

**Proof.** The result follows readily from Theorem 4 because acyclic tournaments are transitive. □

**Example 6.** Let $D$ be the weighted oriented graph whose edges and weights are

$$(x_2, x_1), (x_3, x_2), (x_3, x_4), (x_3, x_1),$$

and $d_1 = 1, d_2 = 2, d_3 = 1, d_4 = 1$, respectively. This digraph is transitive. Thus $I(D)^* = I(D)^\vee$.

**Example 7.** The irreducible decomposition of the ideal $I = (x_1x_2^2, x_3x_2^2, x_4x_3^2)$ is

$$I = (x_1, x_2) \cap (x_1, x_3^2) \cap (x_2, x_3^2),$$

in this case $I^\vee = (x_1x_2, x_1x_3^2, x_2x_3^2) = (x_1, x_2^2) \cap (x_1, x_3^2) \cap (x_2, x_3^2) = I^*.$

**Example 8.** The irreducible decomposition of the ideal $I = (x_1x_2^2, x_3x_1^2, x_2x_3^2)$ is

$$I = (x_1^2, x_2) \cap (x_1, x_3^2) \cap (x_2^2, x_3) \cap (x_1^2, x_2^2, x_3^2),$$

in this case $I^\vee = (x_1^2x_2, x_1^2x_3^2, x_2x_3^2) \subseteq (x_1, x_2^2) \cap (x_3, x_1^2) \cap (x_2, x_3^2) = I^*.$
Example 9. The irreducible decomposition of the ideal \( I = (x_1x_2^3, x_1x_3^2) \) is

\[
I = (x_1) \cap (x_1^2, x_2^2) \cap (x_3, x_3^2),
\]
in this case \( I^v = (x_1, x_2^2x_3) \supset I^e = (x_1, x_2^3) \cap (x_1^2, x_3) = (x_1^3, x_1x_3, x_2^3x_3). \)

We come to the main result of this section.

Theorem 5. Let \( \mathcal{D} \) be a weighted oriented forest without isolated vertices and let \( G \) be its underlying forest. The following conditions are equivalent:

(a) \( \mathcal{D} \) is Cohen–Macaulay.
(b) \( I(\mathcal{D}) \) is unmixed, that is, all its associated primes have the same height.
(c) \( G \) has a perfect matching \( \{x_1, y_1\}, \ldots, \{x_r, y_r\} \) so that \( \deg_G(y_i) = 1 \) for \( i = 1, \ldots, r \) and \( d(x_i) = d_i = 1 \) if \( (x_i, y_i) \in E(\mathcal{D}). \)

Proof. It suffices to show the result when \( G \) is connected, that is, when \( \mathcal{D} \) is an oriented tree. Indeed \( \mathcal{D} \) is Cohen–Macaulay (resp. unmixed) if and only if all connected components of \( \mathcal{D} \) are Cohen–Macaulay (resp. unmixed) \([34, 40]\).

(a) \( \Rightarrow \) (b): This implication follows from the general fact that Cohen–Macaulay graded ideals are unmixed \([42, \text{Corollary 3.1.17}]\).

(b) \( \Rightarrow \) (c): According to the results of \([40]\) one has that \( |V(G)| = 2r \) and \( G \) has a perfect matching \( \{x_1, y_1\}, \ldots, \{x_r, y_r\} \) so that \( \deg_G(y_i) = 1 \) for \( i = 1, \ldots, r \).

Consider the oriented graph \( \mathcal{H} \) with vertex set \( V(\mathcal{H}) = \{x_1, \ldots, x_r\} \) whose edges are all \( (x_i, x_j) \) such that \( (x_i, x_j) \in E(\mathcal{D}) \). As \( \mathcal{H} \) is acyclic, by Lemma 5 we may assume that the vertices of \( \mathcal{H} \) have a “topological” order, that is, if \( (x_i, x_j) \in E(\mathcal{H}) \), then \( i < j \). If \( (y_i, x_i) \in E(\mathcal{D}) \) for \( i = 1, \ldots, r \), there is nothing to prove. Assume that \( (x_k, y_k) \in E(\mathcal{D}) \) for some \( k \). To complete the proof we need only show that \( d(x_k) = d_k = 1 \). We proceed by contradiction assuming that \( d_k \geq 2 \). In particular \( x_k \) cannot be a source of \( \mathcal{H} \).

Case (I): Assume that \( (y_r, x_r) \in E(\mathcal{D}) \). Then \( x_r \) is a sink of \( \mathcal{D} \) (i.e., has only arrows entering \( x_r \)). Using the equalities

\[
C = (X \setminus N_{\mathcal{H}'}(x_k)) \cup \{y_i \mid x_i \in N_{\mathcal{H}'}(x_k)\} \cup \{y_k\},
\]

where \( N_{\mathcal{H}'}(x_k) \) is the in-neighbor set of \( x_k \) consisting of all \( y \in V(\mathcal{H}) \) such that \( (y, x_k) \in E(\mathcal{H}) \). Clearly \( C \) is a vertex cover of \( G \) with \( r + 1 \) elements because the set \( N_{\mathcal{H}'}(x_k) \) is an independent set of \( G \). Let us show that \( C \) is a strong cover of \( \mathcal{D} \). The set \( N_{\mathcal{H}'}(x_k) \) is not empty because \( x_k \) is not a source of \( \mathcal{D} \). Thus \( x_k \) is not in \( L_3(C) \). Since \( L_3(C) \subset \{x_k, y_k\} \subset C \), we get \( L_3(C) = \{y_k\} \). There is no arrow of \( \mathcal{D} \) with source at \( x_k \) and head outside of \( C \), that is, \( x_k \) is in \( L_2(C) \). Hence \( (x_k, y_k) \) is in \( E(\mathcal{D}) \) with \( x_k \in L_2(C) \) and \( d(x_k) \geq 2 \). This means that \( C \) is a strong cover of \( \mathcal{D} \).

Applying Theorem 4 gives that \( p = (C) \) is an associated prime of \( I(\mathcal{D}) \) with \( r + 1 \) elements, a contradiction because \( I(\mathcal{D}) \) is an unmixed ideal of height \( r \).

(c) \( \Rightarrow \) (a): We proceed by induction on \( r \). The case \( r = 1 \) is clear because \( I(\mathcal{D}) \) is a principal ideal, hence Cohen–Macaulay. Let \( \mathcal{H} \) be the graph defined in the proof of the previous implication. As before we may assume that the vertices of \( \mathcal{H} \) are in topological order and we set \( R = K[x_1, \ldots, x_r, y_1, \ldots, y_r]. \)

Case (I): Assume that \( (y_r, x_r) \in E(\mathcal{D}) \). Then \( x_r \) is a sink of \( \mathcal{D} \) (i.e., has only arrows entering \( x_r \)). Using the equalities
Theorem 6. Let $\mathcal{D}$ be a weighted oriented graph and let $G$ be its underlying graph. Suppose that $G$ has a perfect matching $\{x_1,y_1\}, \ldots, \{x_r,y_r\}$ where $\deg_G(y_i) = 1$ for each $i$. The following conditions are equivalent:

(a) $\mathcal{D}$ is Cohen–Macaulay.
(b) $I(\mathcal{D})$ is unmixed, that is, all its associated primes have the same height.
(c) $d(x_i) = 1$ for any edge of $\mathcal{D}$ of the form $(x_i, y_i)$.

The equivalence between (b) and (c) was also proved in [34, Theorem 4.16].

Remark 7. If $\mathcal{D}$ is a Cohen–Macaulay weighted oriented graph, then $I(\mathcal{D})$ is unmixed and $\text{rad}(I(\mathcal{D}))$ is Cohen–Macaulay. This follows from the fact that Cohen–Macaulay ideals are unmixed and using a result of Herzog, Takayama and Terai [22, Theorem 2.6] which is valid for any monomial ideal. It is an open question whether the converse is true [34, Conjecture 5.5].

Example 10. The radical of the ideal $I = (x_2x_1, x_3x_2^2, x_3x_4)$ is Cohen–Macaulay and $I$ is not unmixed. The irreducible components of $I$ are $(x_1, x_3)$, $(x_2, x_3)$, $(x_1, x_2^3, x_4)$, $(x_2, x_4)$.

Example 11. (Terai) The ideal $I = (x_1, x_2)^2 \cap (x_2, x_3)^2 \cap (x_3, x_4)^2$ is unmixed, $\text{rad}(I)$ is Cohen–Macaulay, and $I$ is not Cohen–Macaulay.
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