A Quantum Teleportation Game

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We investigate a game where a sender (Alice) teleports coherent states to two receivers (Bob and Charlie) through a tripartite Gaussian state. The aim of the receivers is to optimize their teleportation fidelities by means of local operations and classical communications. We show that a non-cooperative strategy corresponding to the standard telecloning protocol can be outperformed by a cooperative strategy which gives rise to a novel (cooperative) telecloning protocol.

I. INTRODUCTION

The theory of games has recently entered the domain of quantum mechanics [1], giving rise to the so-called “quantum games”, a new way of looking at concepts of quantum information and computation. Beside the advantages connected to novel quantum strategies, it is possible to consider a set of classical strategies to be optimized over a shared quantum signal. This is the case of the present paper, where the interpretation of a quantum teleportation network as a game played by more parties leads to the definition of a new (cooperative) telecloning protocol which can outperform the standard (non-cooperative) one.

II. THE GAME

In our game, Alice has a large set of coherent states \{\ket{\phi}_i\} which she wants to teleport to both Bob and Charlie. For every input \ket{\phi}_i, she can exploit a shared quantum channel given by a 3-mode Gaussian state \rho_{abc} with modes \(a, b, c\) belonging to Alice, Bob and Charlie respectively (see Fig. 1). In order to make the game fair, such state must be symmetric for Bob and Charlie, i.e. \(\rho_{abc} = \rho_{acb}\). Following the standard continuous variable teleportation protocol [2], Alice mixes her part of the channel with the input state through a balanced beam-splitter and performs an homodyne detection of the output modes, i.e. she measures the quadratures \(\hat{X}_a = 2^{-1/2}(\hat{X}_a - \hat{X}_\text{in})\) and \(\hat{P}_a = 2^{-1/2}(\hat{P}_a + \hat{P}_\text{in})\). After this Bell measurement, Alice announces the result \(\eta = -\hat{X}_a + i\hat{P}_a\) over a public classical channel so that both Bob and Charlie can accomplish teleportation by a set of local operations and classical communications (Fig. 1). The aim of Bob and Charlie is to maximize the fidelity \(F_{AB}, F_{AC}\) of the copy teleported to their respective modes \(b\) and \(c\) over a large number of instances of the game.

The study of this problem is hard in general, but, if we restrict to a particular set of shared channels, we are already able to show remarkable effects of cooperation. For simplicity consider a zero-drift channel characterized by

![Fig. 1: Set-up of the game. Circles are local operations and dashed lines are classical communications.](image-url)
a correlation matrix (CM) of the form

\[ V = \begin{pmatrix} \alpha I & \delta Z & \delta Z \\ \delta Z & \beta I & \gamma I \\ \delta Z & \gamma I & \beta I \end{pmatrix} \]  

(1)

where \( I \) is the \( 2 \times 2 \) identity matrix, \( Z \equiv \text{diag}(1, -1) \) the \( \sigma_z \) Pauli matrix and \( \alpha, \beta, \gamma, \delta \) real parameters. Such form is symmetric under the exchange Bob→Charlie and, tracing out one mode of the three, it implies bipartite entanglement only between the sender (Alice) and the receiver (Bob or Charlie). The CM (1) is genuine (i.e. it corresponds to a physical state [3]) if and only if the uncertainty principle \( V - J/2 \geq 0 \) holds, where \( J \equiv \sum_{k=a,b,c} Y_k \) with \( Y_k \) the \( \sigma_y \) Pauli matrix. We can achieve this condition if we choose

\[ \alpha \geq \frac{1}{2}, \quad \beta = \frac{1}{2}(\alpha + 1), \quad \gamma = \frac{\alpha}{2}, \quad \delta = \frac{1}{2}\sqrt{(2\alpha - 1)(\alpha + 1)} \]  

(2)

so that we simply deal with a one-variable CM \( V = V(\alpha) \).

**III. NON-COOPERATIVE STRATEGY**

A non-cooperative strategy for this game is simply given by the standard teleporting protocol [4]. In this strategy, Bob and Charlie ignore each other and apply suitable conditional displacements on their modes which optimize their teleportation fidelities \( F_{tr}^{AB} \) and \( F_{tr}^{AC} \). Such conditional displacements are \( \hat{D}_b(\eta) = \exp(\eta \hat{b}^\dagger - \eta^* \hat{b}) \) and \( \hat{D}_c(\eta) \), where the shift \( \eta \) exactly balances the drift created by Alice’s measurement on modes \( b \) and \( c \) (see Fig. 2). The

![FIG. 2: Non-cooperative strategy.](image)

symmetry of the channel implies \( F_{tr}^{AB} = F_{tr}^{AC} \equiv F_{tr} \) and \( F_{tr} \) is simply related to the CM of the reduced channel \( \rho_{tr} = T_{tr}(\rho) = T_{tr}(\rho_{abc}) \).

Quantitatively we have \( F_{tr} = (\det \Gamma_{tr})^{-1/2} \) where \( \Gamma_{tr} = (1 + \alpha + \beta - 2\delta)I \). Setting \( \kappa(\alpha) \equiv (1 + \alpha + \beta - 2\delta) \), we simply have \( \Gamma_{tr} = \kappa(\alpha)I \) and therefore \( F_{tr} = \kappa(\alpha)^{-1} \).

**IV. COOPERATIVE STRATEGY**

A simple cooperative strategy for the two receivers can be derived if we include measurements in the set of local operations. Suppose that, after Alice’s declaration of measurement result \( \eta \), Charlie performs the usual conditional displacement \( \hat{D}_c(\eta) \), but then, he heterodynes his mode and reconstructs the state from the result \( \mu \). This subsequent operation surely worsens his fidelity \( F_{tr}^{AC} < F_{tr}^{AB} \) but, if he now communicates the result to Bob, then Bob can actually improve his teleportation \( F_{AB} = F_{tr}^{AB} \). It is sufficient that Bob performs a conditional displacement \( \hat{D}_b(\eta') \) where \( \eta' = \eta + (\beta + 1/2)^{-1}(\delta - \gamma)(\mu - \eta) \) is a modified shift which balances both the drift caused by Alice’s measurement and the drift acquired from the Charlie’s local operations [5] (see Fig. 3).

Quantitatively we have for Charlie \( F_{AC} = (\det \Gamma_{AC})^{-1/2} \) where \( \Gamma_{AC} = I + \Gamma_{tr} \). It is easy to prove that \( F_{AC} < F_{tr}^{AC} \) and \( F_{AC} \leq 1/2 \) (threshold for quantum teleportation [7]), and introducing \( \kappa(\alpha) \), as before, we have \( F_{AC} = [\kappa(\alpha)+1]^{-1} \). Meanwhile, for Bob, it is possible to prove that

\[
F_{AB} = \frac{\alpha + 2}{(\alpha + 2)\kappa(\alpha) - 2(\delta - \gamma)^2}
\]  

(3)
Fig. 3: Single instance of the cooperative strategy. Charlie displaces his mode, heterodynes it and communicates the result to Bob, who performs a modified displacement.

which clearly satisfies $F_{AB} \geq F_{tr}^{AB}$ for every $\alpha \geq 1/2$.

Due to the symmetry of the channel, the performances are exactly inverted if the roles of Bob and Charlie are inverted (i.e. Bob displaces, measures and informs Charlie, who performs a modified displacement). Obviously in the case where always the same receiver performs the measurement, the strategy is lossy for that receiver while it is highly convenient for the other one. However a fair cooperation between the receivers is possible alternating the roles during the various instances of the game (or choosing the roles by a truly random generator). In such a case both receivers achieve the same fidelity $F = (F_{AB} + F_{AC})/2$ which can be greater than $F_{tr}$.

The two fidelities $F$ and $F_{tr}$ as functions of parameter $\alpha$ are plotted in Fig. 4, where we can see a threshold value $\alpha_{th} \sim 5.76$. For $\alpha > \alpha_{th}$, i.e. for a noisy channel, the cooperative strategy ($F$) is better, and it remains above the classical teleportation value ($1/2$) up to high values of $\alpha$. The non-cooperative strategy ($F_{tr}$) is instead better for $\alpha < \alpha_{th}$ where the channel is more pure and where it is possible to reach the maximum fidelities permitted by the no-cloning theorem ($F_{tr} = 2/3$ for $\alpha = 2$). The depicted situation clearly shows that a cooperative telecloning protocol, as naturally defined by the cooperative strategy, can be more efficient of the standard one in presence of noisy channels.

A final remark has to be made about the measurement. The choice of the heterodyne detection, as local measurement for the receivers (see Fig. 3), is strictly connected with the particular form of the CM, since it just represents an optimal local measurement in order to maximize teleportation fidelity in that case. If we consider a Gaussian channel with arbitrary CM (in general asymmetric for the receivers), then an optimal local measurement (in general different for the two receivers) is equivalent to a suitable squeezing transformation followed by an heterodyne detection. For this reason the cooperative protocol can be extended to more general channels if we suitably modify the local measurement performed by the receivers.
V. CONCLUSION

A simple teleportation game between a sender and two receivers has been investigated. A first non-cooperative strategy is given by the standard telecloning protocol, but this one can be outperformed by a cooperative strategy which represents a new (cooperative) telecloning protocol. This is surely true for channels of the type $\text{II}$ with a certain amount of noise ($\alpha > \alpha_{\text{th}}$), but more general channel could be considered.

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