Eliminating ultraviolet divergence in quantum field theory through use of the Boltzmann factor

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Abstract

In this paper, we suggest that quantum fluctuations are suppressed by a relativistic Boltzmann factor $e^{-\beta |\bar{\omega}| p}$, and demonstrate that the factor results in finite values for the zero-point energy density and one-particle-irreducible self-energy of the scalar field.
§1. Introduction

First, the zero-point energy of quantum field theory is reconsidered. A simple free scalar field is used to demonstrate our concepts. The Lagrangian density is given by

\[ \mathcal{L} = \frac{1}{2} \partial_\mu \phi(x) \partial^\mu \phi(x) - \frac{1}{2} m^2 (\phi(x))^2, \]  

(1)

and the Fourier expansion of the field \( \phi(x) \) is written as

\[ \phi(t, x) = \int \frac{d^3 p}{\sqrt{(2\pi)^3 2 \omega(p)}} \left\{ a(p)e^{-ipx} + a^\dagger(p)e^{ipx} \right\} \]

(2)

with \( \omega(p) = \sqrt{p^2 + m^2} \). The coefficients of the expansion, \( a(p) \) and \( a^\dagger(p) \) satisfy the canonical commutation relations

\[ [a(p), a(q)] = \delta^3(p - q), \quad [a(p), a(q)] = [a^\dagger(p), a^\dagger(q)] = 0. \]

(3)

The Hamiltonian is given by

\[ H = \int d^3 x \{ (\dot{\phi}(x))^2 - \mathcal{L} \} \]

\[ = \int d^3 p \frac{1}{2} \omega(p) \left\{ a^\dagger(p) a(p) + a(p) a^\dagger(p) \right\} \]

\[ = \int d^3 p \frac{1}{2} \omega(p) \{ 2a^\dagger(p)a(p) + \delta(p = 0) \} \]

\[ = \int d^3 p \omega(p) \left\{ n(p) + \frac{1}{2} \int \frac{d^3 x}{(2\pi)^3} e^{ipx} \big|_{p \to 0} \right\} \]

\[ = \int d^3 p \omega(p) \left\{ n(p) + \frac{1}{2} \frac{V_\infty}{(2\pi)^3} \right\} \]

\[ = \int d^3 p \omega(p) \left\{ n(p) + \frac{1}{2} \frac{V_\infty}{(2\pi)^3} \right\}, \]

(4)

where \( V_\infty \equiv \int d^3 x \) and \( n(p) = a^\dagger(p)a(p) \) is a number operator with momentum \( p \). The zero-point energy density is then calculated as

\[ \frac{<0|H|0>}{V_\infty} = \int_{-\infty}^{\infty} \frac{d^3 p}{(2\pi)^3} \frac{1}{2} \omega(p) = \infty. \]

(5)

The energy \( \omega(p) \) can be equal to any value between zero and infinity, and all possibilities need to be included in the calculation. Hence, the zero-point energy density diverges. However, the observed vacuum energy is finite, meaning that the above integration requires correction.

There have been several attempts to control this divergence. One method using continuous wavelet transforms\(^{\text{[1-3]}}\) results in the effective regulation of Feynman graphs, which
renders each internal line decay an effective factor $\propto e^{-p^2A^2}$, where $A$ is the minimal scale of all the internal lines.

If the zero-point energy density takes a finite value and the number of states is maximized, the energy distribution of the zero-point motion should follow a canonical distribution. In thermal quantum field theory, energy eigenstates $|n>$ are excited with Boltzmann weight factors $e^{-\beta E_n}$. The weight factor suppresses energy divergence. However, the Boltzmann weight factor is not applied to the zero-point motion, so the zero-point energy density is still infinite.

In this paper, we assumed that quantum fluctuations are caused by energy exchange with a thermal bath, and request (5) to be

$$<0|H|0>=\int_{-\infty}^{\infty}\frac{d^3p}{(2\pi)^3}\frac{1}{\omega(p)}\frac{e^{-\beta\omega(p)}}{Z}.$$  
(6)

The zero-point energy density is now finite. This study aims to investigate how the Boltzmann factor affects quantum field theory.

§2. The Model

In the Fourier transforms of position and momentum space, we request that a relativistic Boltzmann factor be multiplied as follows:

$$f(x) = \int_{-\infty}^{\infty}\frac{d^3p}{\sqrt{(2\pi)^3}}F(p)e^{ip\cdot x} \times e^{-|u_\mu p^\mu(p)|/(am_c)}$$  
(7)

for position space and

$$F(p) = \int_{-\infty}^{\infty}\frac{d^3x}{\sqrt{(2\pi)^3}}f(x)e^{-ip\cdot x} \times e^{i|u_\mu p^\mu(p)|/(am_c)}$$  
(8)

for momentum space, where

$$a = \begin{cases} 2 & \text{if } f(x) \text{ and } F(p) \text{ are field operators,} \\ 1 & \text{otherwise,} \end{cases}$$  
(9)

$m_c$ is a constant with mass as a dimension and $p^\mu(p) \equiv (\omega(p),p)$. $\bar{u}_\mu$ is an average of four-velocities of real particles (on-shell particles) created by the scalar field $\phi$. Notice that $\bar{u}_\mu \neq p^\mu(p)$ in general, because $p$ in (7) and (8) contains the momentum of virtual particles.

The spatial distribution of matter in the universe is homogeneous and isotropic, which means that the average real particle four-velocities $\bar{u}_\mu$ can be written in any coordinate system as

$$\bar{u}_\mu = (1,0,0,0),$$  
(10)
because the spatial component is canceled with plus and minus appearing equally. So in any coordinate system, we can always rewrite the relativistic Boltzmann factor to

\[ e^{-|u_\mu p^\mu(p)|/(amc)} = e^{-\omega(p)/(amc)}. \]  

(11)

From (7) with \( a = 2 \), the scalar field is expanded as follows:

\[ \phi(t, x) = \int \frac{d^3p}{\sqrt{(2\pi)^32\omega(p)}} \{ a(p)e^{-ipx} + a^\dagger(p)e^{ipx} \} e^{-|u_\mu p^\mu(p)|/(2mc)}. \]  

(12)

Substituting (12) into the Hamiltonian (4), we have

\[ H = \int d^3p \omega(p) \left\{ n(p) + \frac{1}{2} \frac{V_\infty}{(2\pi)^3} \right\} e^{-|u_\mu p^\mu(p)|/mc}. \]  

(13)

If we define vacuum as

\[ |0'\rangle \equiv \frac{1}{\sqrt{Z}} |0 \rangle, \quad <0'|0'\rangle = 1, \]  

(14)

we have the zero-point energy density

\[ \frac{<0'|H|0'}{V_\infty} = \int_{-\infty}^{\infty} \frac{d^3p}{(2\pi)^3/2} \omega(p) e^{-|u_\mu p^\mu(p)|/mc} \frac{Z}{Z} <0|0 \rangle, \]  

(15)

where \( Z \) is the state sum

\[ Z \equiv \int d^3p d^3x e^{-|u_\mu p^\mu(p)|/mc}, \]  

(16)

which is Lorentz invariant because \( d^3p d^3x \) is Lorentz invariant. (15) represents that the energy \( \omega(p) \) is canonically distributed. (15) is calculated as

\[ \frac{<0'|H|0'}{V_\infty} = \int_{-\infty}^{\infty} \frac{d^3p}{(2\pi)^3/2} \omega(p) e^{-|u_\mu p^\mu(p)|/mc} \]  

\[ = \int_{-\infty}^{\infty} \frac{d^3p}{(2\pi)^3/2} \omega(p) e^{-\omega(p)/mc} \]  

\[ = \frac{m^4}{(2\pi)^2} \int_A y^2 \sqrt{y^2 - A^2} e^{-y} dy \quad (A = m/m_c) \]  

\[ = \frac{m^4}{2(2\pi)^3} \left\{ \left(\frac{m}{m_c}\right)^3 K_3 \left(\frac{m}{m_c}\right) - \left(\frac{m}{m_c}\right)^2 K_2 \left(\frac{m}{m_c}\right) \right\} \]  

\[ \leq \frac{m^4}{(2\pi)^2} \int_A y^3 e^{-y} dy \]  

\[ = \frac{m^4 e^{-m/m_c}}{(2\pi)^2} \left\{ \left(\frac{m}{m_c}\right)^3 + 3 \left(\frac{m}{m_c}\right)^2 + 6 \frac{m}{m_c} + 6 \right\}, \]  

(17)
where we used (10). The equal sign can be used only when \( m = 0 \). \( K_2(x) \) and \( K_3(x) \) are the modified Bessel function of the second kind. We obtained a finite zero-point energy density.

Using (12), the Feynman propagator function is calculated,

\[
\Delta_F(x, y) = -i \langle 0\vert T(\phi(x)\phi(y))\vert 0'\rangle > = -i \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} e^{-\vert a_\mu p^\mu(p)\vert/(2m_c)} e^{-\vert a_\mu q^\mu(q)\vert/(2m_c)} e^{-ipx} a_\mu(q) e^{iqy} + \theta(y^0 - x^0) a_\mu(q) e^{-iqy} a_\mu^\dagger(p) e^{ipx} \vert 0'\rangle >
\]

which can be rewritten as

\[
\Delta_F(x, y) = \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} \frac{1}{p^2 - m^2 + i\epsilon} e^{-\vert a_\mu p^\mu(p)\vert/m_c}
\]

using \( \omega(-p) = \omega(p) \).

Consider the Feynman propagator function in momentum-space:

\[
i\Delta_F(p) = \int d^4x e^{ipx} i\Delta_F(x, 0) e^{\vert a_\mu p^\mu(p)\vert/m_c}
\]

\[
= \int \frac{d^4q}{(2\pi)^4} \frac{i}{q^2 - m^2 + i\epsilon} e^{-\vert a_\mu q^\mu(q)\vert/m_c} \frac{(2\pi)^4}{4} \delta^4(q - p) e^{\vert a_\mu p^\mu(p)\vert/m_c}
\]

\[
= \frac{i}{p^2 - m^2 + i\epsilon}.
\]

Since (20) is the Fourier transform of \( \Delta_F(x, 0) \), we used the modified Fourier transform (8) with \( a = 1 \).

The equation (20) picking out the specific energy means that the origin of canonical distribution is shifted by \( u_\mu p^\mu(p) \). Similarly, we define the one-particle state with specific momentum \( p \) as

\[
\vert p > \equiv e^{\vert a_\mu p^\mu(p)\vert/(2m_c)} a_\mu^\dagger(p) \vert 0'\rangle >
\]

Then, we find that the amplitude associated with an external leg with \( p \) is

\[
< 0\vert \phi(x)\vert p > = \frac{e^{-ipx}}{\sqrt{(2\pi)^3 2\omega(p)}}.
\]

Eventually, from (20) and (22), we find that the Boltzmann factor \( e^{-\vert a_\mu p^\mu(p)\vert/m_c} \) has no effect on the Feynman rules at the tree level (zero-loop).
Next, consider the case where there is an interaction term $\mathcal{L}_{\text{int}} = -\frac{1}{4} \phi^4$. For a first-order two-point function, we have

\[
G^{(2)}_1(x_1, x_2) = -\frac{i\lambda}{2!} \int d^4 y \left\{ i \Delta_F(x_1, y) \right\} \left\{ i \Delta_F(y, x_2) \right\} [i \Delta_F(x_2, y)]
\]

\[
eq \int d^4 y \int \frac{d^4 p_1}{(2\pi)^4} \frac{e^{-i\mu \lambda (p_1)/m_c}}{p_1^2 - m^2 + i\epsilon} \times \frac{-i\lambda}{2!} \int \frac{d^4 l}{(2\pi)^4} \frac{e^{-i\mu \lambda (l)/m_c}}{l^2 - m^2 + i\epsilon}
\]

\[
\times \int \frac{d^4 p_2}{(2\pi)^4} \frac{i e^{-ip_2(x_2-y)}}{p_2^2 - m^2 + i\epsilon} e^{-i\mu \lambda (p_2)/m_c} \times \frac{-i\lambda}{2!} \int \frac{d^4 l}{(2\pi)^4} \frac{e^{-i\mu \lambda (l)/m_c}}{l^2 - m^2 + i\epsilon}
\]

\[
\times \frac{i}{p_2^2 - m^2 + i\epsilon} e^{-i\mu \lambda (p_2)/m_c}.
\]  \hspace{1cm} (23)

Using the modified Fourier transform \([8]\) with $a = 1$, we obtain the two-point Green's function in momentum-space

\[
i \Delta_F(p) = \int d^4 x e^{ipx} \left\{ i \Delta_F(x, 0) + G^{(2)}_1(x, 0) + \cdots \right\} e^{i\mu \lambda (p)/m_c}
\]

\[
= \frac{i}{p^2 - m^2 + i\epsilon} + \frac{i}{p'^2 - m^2 + i\epsilon} \left\{ -i\Sigma(p^2) \right\} \frac{i e^{-i\mu \lambda (p)/m_c}}{p^2 - m^2 + i\epsilon} + \cdots
\]

\[
= \frac{i}{p^2 - m^2 - \Sigma(p^2)} e^{-i\mu \lambda (p)/m_c} + i\epsilon.
\]  \hspace{1cm} (24)

where $-i\Sigma(p^2)$ is the one-particle-irreducible self-energy function

\[
-i\Sigma(p^2) = \frac{-i\lambda}{2} \int \frac{d^4 l}{(2\pi)^4} \frac{i}{l^2 - m^2 + i\epsilon} e^{-i\mu \lambda (l)/m_c}.
\]  \hspace{1cm} (25)

Let us calculate $\Sigma(p^2)$ in \([25]\). We have

\[
\Sigma(p^2) = \frac{\lambda}{2} \int \frac{d^4 l}{(2\pi)^4} \frac{i}{l^2 - m^2 + i\epsilon} e^{-i\mu \lambda (l)/m_c}
\]

\[
= \frac{i\lambda}{2} \int \frac{d^3 l}{(2\pi)^3} e^{-i\mu \lambda (l)/m_c} \int d\omega \frac{1}{l^0 - \omega + i\epsilon} \int d^3 l' \frac{2\pi i}{l'^0 - \omega + i\epsilon} \left\{ 1 \right\}_{l'^0 = -\omega + i\epsilon}
\]

\[
= \frac{\lambda}{2} \int \frac{d^3 l}{(2\pi)^3} e^{-i\mu \lambda (l)/m_c}.
\]  \hspace{1cm} (26)
Again, from the homogeneous and isotropic of the real particles,
\[
\bar{u}^{\mu} = (1, 0, 0, 0).
\] (27)

Then, (26) is calculated as
\[
\Sigma(p^2) = \frac{\lambda}{2} \int_{-\infty}^{\infty} \frac{d^3 l}{(2\pi)^3} \frac{d l}{2} \omega(l) e^{-\omega(l)/m_c}
= \frac{\lambda m_c^2}{4(2\pi)^3} \int_{A}^{\infty} \sqrt{y^2 - A^2} e^{-x} dy \quad (A = m/m_c)
= \frac{\lambda m_c^2 m}{8\pi^2 m_c K_1(m/m_c)}
\leq \frac{\lambda m_c^2 e^{-m/m_c}}{8\pi^2} \left( \frac{m}{m_c} + 1 \right),
\] (28)

where the equal sign can be used only when \( m = 0 \). \( K_1(x) \) is the modified Bessel function of the second kind. We obtained a finite self-energy.

Finally, let us give one of the four-point vertex correction to see an example of the Boltzmann factor:
\[
i\tilde{\Gamma}^{(4)}((p_1 + p_2)^2)
= \frac{(-i\lambda)^2}{2} \int \frac{d^4 l}{(2\pi)^4} \frac{i e^{-|\bar{u}^{\mu}(l)|/m_c}}{l^2 - m^2 + i\epsilon} \times \frac{i e^{-|\bar{u}^{\mu}(p_1^\mu + p_2^\mu - l^\mu)|/m_c}}{(l + p_1 + p_2)^2 - m^2 + i\epsilon}.
\] (29)

### §3. Conclusion

In this study, we introduced the relativistic Boltzmann factor \( e^{-|\bar{u}^{\mu}(p)|/m_c} \) into quantum field theory. We assumed that quantum fluctuations are caused by energy exchange with a thermal bath. \( \bar{u} \) is the average of four-velocities of real particles created by the scalar field \( \phi \). We demonstrated that the factor results in finite values for the zero-point energy density and one-particle-irreducible self-energy of the scalar field, and they do not depend on \( \bar{u}^{\mu} \) thanks to the homogeneous and isotropic of the real particles (matter). We found that the Boltzmann factor has no effect on the Feynman rules at the tree level thanks to the modified Fourier transform (8) and (21), and appears only in loop integrals.

### References

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