Abstract

The problem of classical particle in linear potential is studied by using the formalism of Hilbert space and tomographic probability distribution. The Liouville equation for this problem is solved by finding the density matrix satisfying von Newmann-like equation in the form of product of wave functions. The relation to quantum mechanics is discussed.

Keywords: Wigner function, classical mechanics, quantum tomography, probability distribution.

1 Introduction

Recently the tomographic probability representation of quantum states was introduced. In the probability representation the states are identified with the fair probability distributions connected with wave functions by integral transforms. The density matrix of mixed quantum states can be also related to the tomographic probability representation.

The Wigner functions is determined by the tomographic probability distribution in view of Radon transform. In the tomographic probability distribution was introduced to describe the states of classical particles. This approach was based on using the Radon transform of the probability distribution function $f(q,p,t)$ on the phase space. The tomographic probability was called the state tomogram and its properties were discussed in the context of studying the classical and quantum chaos. Thus, there exists the duality of considering classical and quantum systems. One can formulate quantum mechanics by using the mathematical tools of classical statistics like probability distribution functions considered as alternative of wave functions. On the other hand, one can introduce the quantum-like formalism of Hilbert space, wave functions and density operators considering classical systems. The idea of the introducing the Hilbert space formalism for classical system states was suggested in. But in the tomographic approach the quantum-like formalism used in classical domain is different from suggested by Koopman, though the general idea is related to the approach presented in and other works. Some relations of quantum-like and classical descriptions were studied in.
Tomographic representation of quantum and classical mechanics was reviewed in [14]. It was shown that quantum states with discrete variables like spin also can be described by using probability distribution functions called tomograms [15] [16]. For continuous variables we got an idea of describing classical states by using formalism of quantum mechanics, but for discrete variables such approach is not developed yet. Thus, in our work we concentrate on introducing description of the states of classical systems with continuous variables by using formalism of quantum mechanics and following [17].

In this work we consider one example of classical systems. It is the problem of a particle moving in the linear potential. On this example we consider some connections between the standard description and Hilbert space representations of classical mechanics.

The paper is organized as follows. In next section Sec. 2 we describe our classical system using standard approach. Then, in Sec. 3 we introduce density matrix by using the Wigner function and its similarity to classical probability distribution function $f(q, p, t)$. In next two sections, Sec. 4 and Sec. 5, classical wave function for gaussian state is found and also relation between evolution operator of Liouville equation and Green’s function of wave equation is discussed. The conclusions are presented in Sec. 6.

## 2 Review of the classical problem

Let us formulate our problem. We consider classical particle motion in the field with linear potential $V = q$. Here $q$ is a position of the classical particle. We use positive sign for potential. For instance, our problem can be considered as the problem of electron’s moving between plates of flat condenser. Also we take the mass of our particle and its charge $m = Q = 1$. Thus, we take the Hamiltonian of particle moving in linear potential in the form

$$H = \frac{p^2}{2} + q.$$  \hspace{1cm} (1)

So, we get the equations of motions for the position $q$ and the momentum $p$

$$\frac{\partial H}{\partial p} = \dot{q}, \quad - \frac{\partial H}{\partial q} = \dot{p}.$$  \hspace{1cm} That yields

$$\ddot{q} + 1 = 0.$$  \hspace{1cm} (2)

It is easy to solve this equation, so let us write the solution

$$p = p_0 - t,$$

$$q = q_0 + p_0 t - \frac{t^2}{2}.$$  \hspace{1cm} (3)

Here $p_0$ is initial momentum, $q_0$ is initial position.

In case position and momentum fluctuate, the state of our system is described by a probability distribution function $f(q, p, t)$ on the phase space. This function is nonnegative and it satisfies the normalization condition, i. e.

$$f(q, p, t) \geq 0,$$

$$\int f(q, p, t) dq dp = 1.$$  \hspace{1cm} (4)
The function is a solution of the Liouville kinetic equation

$$\frac{\partial f}{\partial t} + \frac{\partial H}{\partial p} \frac{\partial f}{\partial q} - \frac{\partial H}{\partial q} \frac{\partial f}{\partial p} = 0.$$ 

In case of linear potential the Liouville equation reads

$$\frac{\partial f}{\partial t} + p \frac{\partial f}{\partial q} - \frac{\partial f}{\partial p} = 0.$$ (5)

In next sections we will consider solutions of the above equation by applying the quantum-like formalism of the Hilbert space vectors and the corresponding wave functions.

3 Classical wave function

So, we want to consider solution of equation (5) by using quantum-like formalism. But before we start we should show how we introduce this formalism in classical mechanics. In quantum mechanics, the states are described by a wave function or a density matrix. But also we can use other representations. For instance there exists the phase-space quasidistribution $W(q,p,t)$ called the Wigner function. It is given by the Fourier transform of a density matrix $\rho(x,x',t)$ introduced in [18]

$$W(q,p,t) = \int \rho \left(q + \frac{u}{2}, q - \frac{u}{2}, t\right) e^{-ipu} du. \tag{6}$$

The inverse Fourier transform yields

$$\rho(x,x',t) = \frac{1}{2\pi} \int W \left(\frac{x + x'}{2}, p, t\right) e^{ip(x-x')} dp. \tag{7}$$

So we get the relation between this two representations of quantum mechanics. They provide two ways to describe quantum states. Doing the transform (6) we associate the quantum state with the function on phase space.

In classical mechanics we are usually (we can say always) work in phase space. It is the standard tool. But it is possible to make transform to use the Hilbert space representation by introducing density matrix (density operator) given formally by (7). So, let us consider probability distribution $f(q,p,t)$ which is an analog of the Wigner function. It means that we replace $W(q,p,t)$ by $2\pi f(q,p,t)$ in formula (7). Then we can write expression for density matrix of classical motion

$$\rho(x,x',t) = \int f \left(\frac{x + x'}{2}, p, t\right) e^{ip(x-x')} dp. \tag{8}$$

Then it is important to introduce the evolution equation. In our case we get the equation which resembles the quantum equation. In fact, doing some transformations with Liouville equation (5), finally we get von Neumann-like equation for the density matrix

$$i \frac{\partial \rho(x,x',t)}{\partial t} = -\frac{1}{2} \left( \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial x'^2}\right) \rho(x,x',t) + (x-x')\rho(x,x',t), \tag{9}$$
where we used the substitution rules

\[
\begin{align*}
\frac{\partial f}{\partial t} & \to \frac{\partial \rho (x, x', t)}{\partial t}, \\
\frac{\partial f}{\partial p} & \to i(x - x') \rho (x, x't), \\
p \frac{\partial f}{\partial q} & \to \frac{1}{2i} \left( \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial x'^2} \right) \rho (x, x', t).
\end{align*}
\]

(10)

This equation is equation with separable variables \(x, x'\). In view of this one can find the solution of the equation factorizing the density matrix in the form

\[\rho (x, x', t) = \Psi (x, t) \Psi^* (x', t).\]  (11)

Then, one can show that the function \(\Psi(x, t)\), which can be called the wave function of classical particle satisfies the Shrödinger-like equation.

But let us pay our attention on the following circumstance. The formula for the density matrix provides the possibility to make the gauge transformation of the wave function

\[\Psi(x, t) \to \Psi(x, t) e^{i\phi(t)}.\]  (12)

Here \(\phi(t)\) is a real phase factor depending on time. From this point of view, the classical probability density \(f(q, p, t)\) can be mapped onto density matrix of the factorized form \(\rho(x, x', t) = \Psi(x, t) \Psi^*(x', t)\). The corresponding wave function is determined up of this gauge factor. But we want the function \(\Psi(x, t)\) to satisfy Shrödinger-like equation. So, when we use this condition we get equation for the wave function \(\Psi(x, t)\)

\[i \frac{\partial \Psi (x, t)}{\partial t} = -\frac{1}{2} \frac{\partial^2 \Psi (x, t)}{\partial x^2} + x \Psi (x, t).\]  (13)

It is worthy to point on that we lose time-dependent phase-factor when we get Shrödinger-like equation from the von Neumann-like equation for density matrix.

Also we must take into account that not each solution of wave equation provides the real probability distribution function. The function \(f(q, p, t)\) expressed in terms of solution \(\Psi(x, t)\) of Shrödinger-like always satisfies Liouville equation, but this solution can take negative values which is not admissible for classical probability distribution.

### 4 Gaussian solution

In this section we will consider gaussian solution of our problem and show some connections between these two representations. Firstly we will find solution of the Liouville equation, in another words we will get probability density function and using (7) we will find the density matrix. The Liouville equation can be reduced, by using the Fourier transform, to the von Neumann-like equation as it was demonstrated.

Let us suppose that the probability distribution at \(t = 0\) has the Gaussian form

\[f_0 (q_0, p_0) = f(q, p, 0) = \frac{1}{\pi} e^{-q_0^2 - p_0^2}.\]  (14)
We want to find evolution of the probability distribution function. This problem can be solved easily. In fact, the equations of motion read
\[ p = p_0 - t, \]
\[ q = q_0 + p_0 t - \frac{t^2}{2}. \]

For the initial momentum and the initial position we find the expressions
\[ p_0 = p + t, \]
\[ q_0 = q - pt - \frac{t^2}{2}. \] (15)

Since the probability density is the integral of motion from the Liouville theorem we get the probability distribution function for arbitrary time moment
\[ f(q, p, t) = \frac{1}{\pi} e^{-\left(p + t\right)^2 - \left(q - pt - \frac{t^2}{2}\right)^2}. \] (16)

Also we can find the evolution of statistical parameters of our Gaussian probability distribution like variances and covariance of random position and momentum.

We have the initial dispersion matrix
\[ C_0 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \] (17)

The transformation matrix, corresponding to the evolution (16) reads
\[ A = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}. \] (18)

Then we get the dispersion matrix for arbitrary time moment \( t \)
\[ C = AC_0 A^T = \frac{1}{2} \begin{pmatrix} 1 + t^2 & t \\ t & 1 \end{pmatrix}. \] (19)

Our aim is to find the wave function that is associated with this probability distribution function. We put expression for \( f(q, p, t) \) (16) in (8), and get the density matrix
\[ \rho(x, x', t) = \frac{1}{\pi} \int e^{-(p+it)^2 - \left(q - pt - \frac{t^2}{2}\right)^2} e^{ip(x-x')} dp. \] (20)

Factorizing the expression for the density matrix \( \rho(x, x', t) \) we get this wave function \( \Psi(x, t) \)
\[ \Psi(x, t) = \exp \left\{ -\frac{1}{4} \frac{t^2/2 + 2ix^2 + 2t^2 x + 4ix t - 2ix^2 t + 2it^3 x}{(1+t^2)} \right\} \frac{1}{\sqrt{\pi} (1+t^2)}. \] (21)
There is the problem of the time-dependent phase-factor as pointed out in the end of Sec. 3. That’s why we have one problem, this wave function (21) doesn’t satisfy Shrödinger-like equation. So, we can see it
\[
\left( -\frac{1}{2} \frac{\partial^2}{\partial x^2} - i \frac{\partial}{\partial t} + x \right) \Psi(x, t) = \frac{1}{8} \left( \frac{4 + 8t^2 + 3t^4 + t^6}{(1 + t^2)^2} \right) \Psi(x, t) \neq 0. \tag{22}
\]

But we want the wave function of classical particle to satisfy the Shrödinger-like equation. Because of the gauge invariance, in general, the wave function has the form
\[
\Psi(x, t) = \exp\left\{ -\frac{1}{4} \frac{t^2}{2 + 2t^2 + 4t^4 + 4ixt - 2ix^2t + 2it^3x}{1 + t^2} + i\phi(t) \right\} \sqrt{\frac{\pi}{1 + t^2}}. \tag{23}
\]

As we require, this function has to satisfy Shrödinger-like equation (13). Using this condition and putting (23) in (13), we find the equation for time dependent phase \(\phi(t)\)
\[
\left( \frac{4 + 8t^2 + 3t^4 + t^6}{8(1 + t^2)^2} + \dot{\phi}(t) \right) \Psi(x, t) = 0. \tag{24}
\]

Solving the differential equation for the phase \(\phi(t)\), we finally get
\[
\phi(t) = -\frac{t}{8} - \frac{t^3}{24} - \frac{1}{2} \arctan(t) + \frac{1}{8} \left( \frac{t}{1 + t^2} \right) + \text{const}. \tag{25}
\]

In our case, for this choice of time-dependent phase, the wave function \(\Psi(x, t)\) satisfies the Shrödinger-like equation (13).

5 Relation between propagator of the Liouville equation and Green function of the wave equation

In previous section we found gaussian solution of our classical problem. We presented two ways to describe our classical system. We found probability distribution (16) and corresponding wave function (23). Also we can find this wave function using Green function of the Shrödinger-like equation. In this section we want get it.

We are going to find the relation between the propagator of the Liouville equation (5) and the Green function of the Shrödinger-like equation. In this section we want get it.

We are going to find the relation between the propagator of the Liouville equation (5) and the Green function of the Shrödinger-like equation (13).

Let us introduce the following maps:
\[
\rho \mapsto \hat{F}, \quad \rho_0 \mapsto \hat{F}_0, \quad f \mapsto \hat{\Pi}, \quad \rho \mapsto \hat{B}, \quad \rho_0 \mapsto \rho. \tag{26}
\]

The operators \(\hat{F}, \hat{F}^{-1}, \hat{\Pi}, \hat{B}\) can be given in matrix form expressed as kernels of the integral transforms. These maps are determined by the following expressions
\[
\rho(x, y, t) = \int F^{-1}(x, y, q, p, t) f(q, p, t)dqdp
\]
and
\[ f(q, p, t) = \int F(x, y, q, p, t) \rho(x, y, t) dx dy, \]

here from (6), (7), (8) we find
\[ F^{-1}(x, y, q, p, t) = e^{ip(x-y)} \delta \left( \frac{x + y}{2} - q \right), \]
\[ F(x, y, q, p, t) = \frac{1}{2\pi} e^{-ip(x-y)} \delta \left( \frac{x + y}{2} - q \right), \]
\[ f(q, p, t) = \int \Pi(q, p, q', p', t) f_0(q', p') dq dp', \]

the matrix \( \hat{B} \) is expressed in terms of the Shrödinger-like equation
\[ \hat{B}(x, y, x', y', t) = 1 \]

Then we get identity
\[ \hat{B} \rho_0 = \hat{F}^{-1} \hat{\Pi} \hat{F} \rho_0, \]

that provide this expression
\[ \int B(x, y, x', y', t) \rho_0(x', y') dx' dy' = \frac{1}{2\pi} \int F^{-1}(x, y, q, p, t) dq dp \int dq' dp' \Pi(q, p, q', p', t) \int F(x', y', q', p', t) \rho_0(x', y') dx' dy'. \]

We compare both part of this identity and get expression for function \( B(x, y, x', y', t) \). So, after some transformations we find expression for function \( B(x, y, x', y', t) \)
\[ B(x, y, x', y', t) = \frac{1}{2\pi} \int e^{ip(x-y)} e^{-ip'(x'-y')} \Pi(q, p, q', p', t) \delta \left( q - \frac{x + y}{2} \right) \delta \left( q' - \frac{x' + y'}{2} \right) dq dq' dp dp'. \]

Propagator for the Liouville equation (5) reads
\[ \Pi(q, p, q', p', t) = \delta \left( q' - q_0(q, p, t) \right) \delta \left( p' - p_0(q, p, t) \right). \]

Then, we get
\[ B(x, y, x', y', t) = \int \frac{1}{2\pi t} \exp \left\{ i \frac{(q - q' - \frac{t^2}{2})}{2t} (x - y) \right\} \delta \left( q - \frac{x + y}{2} \right) \delta \left( q' - \frac{x' + y'}{2} \right) dq dq'. \]

That provides Green function
\[ G(x, x', t) = \frac{1}{\sqrt{2\pi t}} e^{i \frac{(x-x')^2 - t^2(x+x')}{2t}}. \]
In fact, because of gauge invariance Green function has the form
\[
G(x, x', t) = \frac{1}{\sqrt{2\pi t}} e^{\frac{(x-x')^2 - t^2(x+x')}{2t}} e^{i\phi(t)}. \tag{33}
\]

Indeed, we remember that we lose the time-dependent phase. We remember our demand that Green function like wave function must satisfy the Shrödinger-like equation. So, using last condition we can find the time-dependent phase up to a constant. Expression for the time-dependent phase reads
\[
\phi(t) = -\frac{t^3}{24} + \text{const.} \tag{34}
\]

Thus we find Green function of the Shrödinger-like equation. It gives us opportunity to get all possible wave functions and these functions can describe or determine the classical states. But we should take into account the nonnegativity property of probability distribution on phase space. Not each wave function provides us with a real probability distribution. Using transform (6) we can find solution of the Liouville equation. But possibly this solution will not satisfy conditions (4) (for instance this function can take negative values) and that is why it will not be probability distribution function, in another words it will not describe real classical state. In general, solution of the Shrödinger equation always provides us with solution of the Liouville equation but it does not have to correspond to real classical state.

6 Conclusions

To resume we formulate the main results of our work. We have constructed the solution of Liouville equation of a classical particle moving in linear potential in the form of Fourier transform of the product of two ”wave functions”. The wave functions satisfy the Shrödinger-like equations identical to Shrödinger equations for quantum particle moving in the linear potential. We demonstrated on this example that in classical mechanics one can use formalism of Hilbert spaces and formalism of quantum density operator. It is another example in addition to examples considered previously in [9] and [17] which provides the possibility to use the same formalism both in classical and quantum mechanics.

Acknowledgments

The author acknowledges the financial support provided within the Project RFBR 11-02-00456.

References

[1] S. Mancini, V. I. Manko and P. Tombesi, Phys. Lett. A, 213, 1 (1996).
[2] V. I. Manko and R. V. Mendes, Phys. Lett. A, 263, 53 (1999).
[3] Vladimir Manko, Marcos Moshinsky, and Anju Sharma, Phys. Rev. A, 59, 1809, (1999).
[4] S. Mancini, V. I. Manko, and P. Tombesi, Found. Phys., 27, 801 (1997).
[5] E. Wigner, Phys. Rev., 40, 749 (1932).
[6] J. Radon, *Ber. Sachs. Akad. Wiss., Leipzig*, **69**, 262 (1917).

[7] O. V. Manko and V. I. Manko, *J. Russ. Laser Res.*, **118**, 407 (1997).

[8] V. I. Manko and R. V. Mendes, *Physica D*, **145**, 330 (2000).

[9] B. O. Koopman, *Proc. Natl. Acad. Sci. U.S.A.*, **17**, 315 (1931).

[10] D. Mauro, *Int. J. Mod. Phys. A*, **17**, 1301, (2002).

[11] E. Gozzi and D. Mauro, *Int. J. of Mod. Phys. A*, **19**, 1475, (2004).

[12] Denys I. Bondar, Renan Cabrera and Herschel A. Rabitz, arXiv:1202.3628v1 [quant-ph] (16 Feb 2012).

[13] Elliott Francesco Tammaro, *Found. Phys.*, **42**, Issue 2, pp 284-290, (2012).

[14] A. Ibort, V.I. Manko, G. Marmo, A. Simoni, F. Ventriglia, *Phys. Scr.*, **79** 065013 (2009).

[15] V. V. Dodonov, and V. I. Man’ko, *Phys. Lett. A*, **229**, 335 (1997).

[16] V. I. Man’ko and O. V. Man’ko, *J. Exp. Theor. Phys.*, **85**, 430, (1997).

[17] V. N. Chernega and V. I. Manko, *J. Russ. Laser Res.*, **28**, 6 (2007).

[18] L. Landau, *Z. Phys*. **45**, 430 (1927).