Framework
The physics of $\text{sp}(2, \mathbb{R})$

by

Ian Hawthorn
William Crump & Matthew Ussher
Abstract

A mathematical framework for relativistic quantum mechanics is constructed with natural symmetry $\mathfrak{so}(2, 3) = \mathfrak{sp}(2, \mathbb{R})$. In this framework gravity and electromagnetism unify as aspects of the geometry. The source equation for gravity differs from Einstein's equation and permits behavior that could explain dark matter.

Note on the Title Page Illustration: The diagram on the title page uses the Petersen graph to represent the Lie algebra $\mathfrak{so}(2, 3) = \mathfrak{sp}(2, \mathbb{R})$. The labelled vertices represent basis elements. The edges connect those which commute. Those which are not connected are linked by a unique path of length 2. The other vertex connected to the middle vertex of the path gives the Lie bracket with sign specified by whether a physical turn to the left (positive) or to the right (negative) is made in traversing the path. For example to compute $[T, X]$ from the diagram, note that the path from $T$ to $X$ turns left at $I$, and the other vertex connected to $I$ is $A$. Hence $[T, X] = A$. 

SVN version 434
This is a book of original research written about physics by mathematicians. My task in this preface is to explain how such an odd thing came to be, and the reasons for publishing in this unusual form.

I am Ian Hawthorn and the main author of this book. I am the sole author in a legal copyright sense. The words here are mine and any mistakes and omissions are my sole responsibility. However some of this work was carried out in collaboration with two excellent students, William Crump and Matthew Ussher, who both made important and significant contributions at different stages and hence have been named as coauthors.

I have also taken the liberty of naming aspects of the theory after them to ensure that they receive the recognition they deserve. Crump scalars are named after William Crump as the necessity for these was the main topic of his Masters thesis; and Ussher’s equation is named after Matthew Ussher who came up with it in his Masters thesis.

Why the choice of a book format? Most original research these days is published in journals. In this case however the article format seemed very confining. The scope of the work is large, so we would have needed to divide it into a series of journal articles. Matthew and I made an attempt to subdivide the work in this way at one point. However the early parts seemed too elementary and unmotivated, while the later parts didn’t make sense without the earlier work.

The book format allows us the space to build gradually from a leisurely consideration and discussion of ideas; through the careful construction of an appropriate axiomatic structure to express those ideas; to the consideration of the mathematical theory of that structure; and finishing with a discussion of the physical implications. The book format also enabled us to include unoriginal background material in the introductory chapter, and allowed us
to write in a more relaxed and conversational style rather than in the terse style expected in journals. I believe this makes it a much easier read than if we had tried to force the material into the journal format.

The reader should note that the first chapter contains a lot of background material presented in a rather elementary way as I hope that the book will be accessible to graduate students in both mathematics and physics. If you find this material too elementary I suggest you skip ahead to Chapter 2. The final chapter is a summary of the rest of the book. If you just want a quick idea of what the book is about, you could look at that. Otherwise the best way to read this book is to start at the beginning and read it through to the end. I have tried very hard to make everything self-contained, readable and simple. It is my belief that correct explanations are simple explanations and that the best academic writing makes the subject seem easy.

The rest of this preface discusses the history of the work. Those who are not interested in such things should skip it.

I first started thinking about the things in this book in the 1990s. However it was not until 2007 that I found a way into the problem. And it was not until my academic leave in 2009 that significant progress was first made. By the end of 2009 I had a rather beautiful mathematical structure which did most of what I wanted. It combined curvature with $so(2,3)$ symmetry in an elegant way allowing a natural expression of the Dirac equation, Maxwell’s equations and Einstein’s equation.

However there was a small problem. Maxwell’s equations appeared in this structure with an additional constraint which I had hoped initially was a gauge condition. I wish to thank Dr Yuri Litvinenko, my colleague at Waikato, for convincing me that it was instead an unphysical constraint on the electromagnetic field. This meant that after all my work, all I had was a broken model. It was an elegant and beautiful broken model based on a very small set of natural assumptions. But it was making an unphysical prediction so it was, as Feynman would say, wrong!

A model of this type is very specific and rigid with almost no adjustable parameters. It is quite the opposite in that respect to string theory which is a very general and flexible model with many adjustable parameters. The advantage of using a more rigid approach is that it tends to make strong predictions. The disadvantage is that if those predictions fail the test of reality the problem can be very hard to fix. Rigid models don’t bend well. I therefore put the work aside and devoted my attention to other things, mainly administration since I had just taken over as chairperson of the mathematics department at that time.
In 2010 I took on a master’s student, William Crump, with interests in mathematical physics. The task assigned to him for his Master’s thesis was to read and understand the work as it existed at that time, and to explore and discuss the nature of the failure with respect to Maxwell’s equations. In the course of working on this problem together we came to realise that the electromagnetic field was much more closely linked to the geometry of the model than I had previously thought. We discovered that the problem arose from a very subtle assumption that I had made which constrained the geometry effectively creating a zero field condition for the electromagnetic field.

Removing this assumption freed up the model and eliminated the unphysical constraint on Maxwell’s equations. Furthermore our new insights about the link between electromagnetism and the geometry then allowed us to also link Maxwell’s equations to the Bianchi identities for the gravitational field. The dynamical equations for both electromagnetism and gravity were now unified as components of a single equation for the spinor curvature. These changes required a complete rewrite which I commenced in 2011. The added flexibility made the mathematics much more complicated however, which slowed progress.

In 2012 I was fortunate to have another excellent Master’s student, Matthew Ussher. Matthew explored in his thesis the nature of the source equations for electromagnetism and gravity. Whereas the dynamical equations for these forces were unified, the source equations still looked very different. In discussing this issue Matthew came up with a unified source equation which gave the Ampere-Gauss equation for electromagnetism and another equation for gravity which I have called Ussher’s equation. Ussher’s equation is quite different from Einstein’s equation. We were not able to link the two equations and were not sure quite what to make of this result.

In 2014 I was again on academic leave, and during that time I looked at the Lagrangian analysis in more detail. Matthew had attempted a Lagrangian analysis in his thesis with limited success. I was able to push the Lagrangian analysis through to a full conclusion and discovered that Ussher’s equation was only a special case; and the resulting full equations offered a possible explanation for dark matter.

At that time I realised I had something pretty special. However the work was far from complete. It was clear by the nature of the work that I would be unable to tie off every loose end, but there were certain things I wanted to finish up before publication. In particular I wanted to find the source terms for the force equations so that I would have a complete and unified relativistic field theoretic description of electromagnetism and
gravity. Source terms for fermions at least should be obtainable by applying an appropriate variation to the correct Dirac Lagrangian density.

I did not expect this to be difficult, however it proved considerably harder than I had anticipated. My Dirac Lagrangian densities were persistently misbehaving despite the fact that I had proved equivalence to the standard Dirac Lagrangian density. It took me much longer than it should have done to realise that the problems originated with the standard Dirac Lagrangian density which is complex when it needs to be real, and which is not well defined since it has an unstated conformal dependence.

Having discovered these issues it took time to explore them properly which delayed final publication to the end of 2015. And I still didn’t get the source terms I was seeking.

I have chosen to release the work on ArXiv while I search for a suitable publisher. I also intend to submit shorter summary papers to peer reviewed journals.

- Ian Hawthorn
## Contents

1 **Spacetime Symmetry**  
   1.1 The Lie algebra $\mathfrak{so}(2, 3)$  
   1.2 The Lie algebra $\mathfrak{sp}(2, \mathbb{R})$  
   1.3 The Enveloping Algebra  
   1.4 Roots and Weights  
   1.5 The Charge Multiplier

2 **Symmetry and Manifolds**  
   2.1 Tensor Derivations and Manifolds  
   2.2 Torsion and Curvature  
   2.3 Lie Groups  
   2.4 Lie Manifolds

3 **Spinor Manifolds and Frameworks**  
   3.1 Matrix Lie Groups  
   3.2 $V$-tensors and $V$-tensor Derivations  
   3.3 Matrix Lie Manifolds  
   3.4 Complex matters  
   3.5 Reality Check

4 **Generalised Tensors**  
   4.1 Definition  
   4.2 Basics  
   4.3 Locally Trivial Spaces  
   4.4 Parallel Transport  
   4.5 Indecomposable Generalised Tensors

5 **Specific Tensors**  
   5.1 Notation and General Discussion  
   5.2 Spinor Transformations  
   5.3 Action of $\mathfrak{so}(3, 3)$  
   5.4 Tensors with Two Versor Indices  
   5.5 Tensors with one Versor and one Spinor Index  
   5.6 Symmetric 3 and 4 Component Spinors  
   5.7 Tensors with Two Vector Indices
6 **Crump Scalars and Beyond**
- 6.1 Casimir Identities ........................................ 58
- 6.2 The Symplectic Form .................................... 59
- 6.3 Tensors with Two Spinor Indices ....................... 62
- 6.4 The Bullet Index ......................................... 63
- 6.5 Scalars and the Zero Curvature Test ................... 65
- 6.6 Superalgebra Identities .................................. 68

7 **Curvature and Forces**
- 7.1 Components of Curvature ................................. 70
- 7.2 Bianchi Identities Revisited ........................... 73
- 7.3 Connection Components ................................. 75
- 7.4 The Field and Potential .................................. 76
- 7.5 Connections, Curvature and Gauge ..................... 76
- 7.6 Vector, Versor and Crump Connections ................. 78
- 7.7 Curvature and Potentials ................................. 80

8 **Einstein’s Equation**
- 8.1 Invariant Operators ...................................... 83
- 8.2 Operator Identities ...................................... 84
- 8.3 Contractions of the Curvature Tensor ................. 86
- 8.4 Einstein’s Equation ....................................... 88

9 **The Ampere-Gauss equation and Ussher’s equation**
- 9.1 Ussher’s Identity .......................................... 92
- 9.2 Extended Ampere-Gauss Equation ....................... 94
- 9.3 Ussher’s equation ......................................... 95
- 9.4 Further Identities ......................................... 98

10 **Lagrangian Methods and Forces**
- 10.1 Integration and Stokes’ Theorem ..................... 100
- 10.2 Lagrange Methods: Electromagnetism ................. 103
- 10.3 Lagrange Methods: Gravity ........................... 105
- 10.4 The Exact Equation ..................................... 115
- 10.5 The Dark side of the Force ............................... 118

11 **Matter**
- 11.1 The Extended Dirac Equation ......................... 124
- 11.2 The Dirac Lagrangian ................................... 129
- 11.3 The Electromagnetic Source Term ..................... 137
- 11.4 The Gravitational Source Term ....................... 141
- 11.5 Discussion ................................................ 146
# 12 Summary

| Section                          | Page |
|---------------------------------|------|
| 12.1 Objectives                 | 149  |
| 12.2 The Mathematics            | 150  |
| 12.3 The Physics of Forces      | 152  |
| 12.4 The Physics of Matter      | 157  |
| 12.5 Conclusion                 | 159  |

Bibliography

160
Chapter 1

Spacetime Symmetry

Symmetry is absolutely central to modern physics. Not only does symmetry play a vital and direct role in many physical theories, it also serves as a bridge between the various mathematical formalisms and the everyday reality in which our physical intuition lives. All mathematical structures have a natural symmetry and consequently anything defined in one context purely in terms of symmetry can be translated into another context having the same symmetry group quite easily.

Of the various symmetries of use to modern physics the symmetries of space-time itself are the most fundamental. Whereas other symmetries may apply only to some objects or in some circumstances, everything that lives in our universe must conform to the symmetries of space-time.

The symmetry group of space-time is ten dimensional which is to say that ten continuous parameters are needed to situate an event. The time of the event gives us one parameter and three more are required to locate its position in space. We need three parameters to describe how the event is oriented, and finally a further three parameters (Lorentz boost coordinates) are needed to specify an instantaneous reference velocity for the event.

We may not need all ten parameters if the event we wish to locate is itself possessed of symmetry. For example if the event is spherically symmetric (appears the same in all orientations), we do not need the orientation parameters to unambiguously locate it. If the event can be described without reference to velocity, then we won’t need the three Lorentz boost coordinates.

It is usual in physics to regard the four dimensions required to specify location in space and time as being more physically real (in some sense) than the other six coordinates. That is because our everyday experience of space-
time symmetry involves its 4-D representation. However the distinction is not a natural one to make from the point of view of the group itself and in other representations these four coordinates look exactly the same as the other six. We will accord all ten coordinates equal status.

The space-time symmetry group is continuous and connected. Sometimes various discrete symmetries; reflections and inversions; are considered as part of the space-time symmetry group but we choose to reject this point of view. There are known physical events, most notably involving the weak force, which are not conserved under these inversions. The true symmetries of space-time should conserve the form of all possible physical events.

Continuous symmetries can be performed gradually and hence it is enough to understand the symmetry processes which perform these gradual transformations. These can be described using a basis of ten processes; translation through time and space, rotation about each of the three principal axes, and Lorentz boost along the three principal directions.

Symmetries have an algebraic structure. The symmetries themselves form a group under the binary operation of composition while the symmetry processes constitute a Lie algebra with three operations; an addition and scalar multiplication giving a vector space, and a commutator or Lie bracket which is bilinear, antisymmetric and obeys the Jacobi bracket condition.

Our understanding of the algebraic structure of the symmetries of space-time has evolved through history. The Galilean group of classical mechanics has been supplanted by the Poincaré group of special relativity. Whereas in the Galilean group boost operators commute, in the Poincaré group their commutators are multiples of rotation operators where the multiplying factor is inversely proportional to the speed of light. As the speed of light is large, it follows that these commutators are close to zero and hence the Galilean Lie algebra approximates the Poincaré Lie algebra in the regime where velocities are much less than the speed of light. Indeed we can regard the Galilean Lie algebra as being in some sense the limit as \( c \to \infty \) of the Poincaré Lie algebra. This kind of limiting process is called a contraction.

The Poincaré Lie algebra itself can be obtained via contraction from other Lie algebras, in particular from the DeSitter Lie algebra \( \mathfrak{so}(1, 4) \) and the Anti-DeSitter algebra \( \mathfrak{so}(2, 3) \). For reasons we shall discuss later it is the second of these two possibilities that is of greatest interest. The contracting parameter in this case is an absolute distance scale \( r \) which we will colloquially refer to as the radius of the universe, and in this group the commutator of two translation operators is a multiple of a rotation operator where the multiplying factor is inversely proportional to \( r \). If \( r \) (as measured in everyday units) is large then the Poincaré Lie algebra is
very closely approximated by $\mathfrak{so}(2, 3)$, especially in the regime of distance and time scales much less than $r$.

**Obvious Question:** Which group is correct?

It would be very difficult for us to answer this question directly by making measurements on space-time if $r$ is significantly bigger than the regime of distance and time accessible to us for direct measurement. Furthermore on large distance scales we cannot ignore the effects of curvature and cosmological model.

Indeed the very notion of a global symmetry group for space-time becomes rather dubious in the presence of curvature. And while local symmetry retains its meaning it is very unclear how we might go about measuring it accurately enough to distinguish between the two groups.

The main problem is that the measurements that might help distinguish between $\mathfrak{so}(2, 3)$ and the Poincaré Lie algebra are also going to be effected by our choice of cosmological constant or cosmological model.

Our obvious question does not therefore have an obvious answer. It appears unlikely that we can determine which group is correct by direct measurements of spacetime, at least in the realm of classical physics. Can quantum mechanics rescue our question?

Considering how these two groups act on spacetime we find that the problem of measuring the difference is just as difficult, but it manifests in a different way. Energy for example would be quantised in a universe based on $\mathfrak{so}(2, 3)$. However as the quanta would be associated with periods of cosmological duration, we are not going to be able to detect this.

However this does not completely answer the question since in quantum mechanics the symmetry group of space-time has a second action.

Rotations for example act on particles in two ways. They act in an ordinary fashion on space-time which is the domain space of the wave functions, and they also act intrinsically on the range space of wave functions. The eigenvalues of the first action give us angular momentum while the eigenvalues of the second action give an intrinsic angular momentum or spin.

The reasons for this double action are not clear. Why should rotations act on this other space? We know that they do and indeed are so used to it that we may have come to view it as simply the natural order of things. But it is worth remembering that mathematically speaking the existence of this second action on an unrelated space is an unanticipated surprise.

One very strange feature of intrinsic actions is that we only allow part of the symmetry group to act in this way. Translation operators are not
permitted to act intrinsically (equivalently we require that their intrinsic action is trivial). That is because in the Poincaré group translation operators would have a continuous spectrum of eigenvalues. If the translation operators in the Poincaré group were free to act intrinsically as rotations do, we would expect to see continuously varying intrinsic versions of momentum and energy, which we do not.

It is strange enough that symmetries of space-time act intrinsically at all. It is doubly strange that only some of the symmetries of space-time should act in this way, and the constraint that excludes only translations from acting intrinsically seems arbitrary and ugly, if not actually broken. Are things any better in this regard if we use $\mathfrak{so}(2,3)$ instead?

The Lie algebra $\mathfrak{so}(2,3)$ has a very different spectrum of eigenvalues. In particular the fundamental representation of $\mathfrak{so}(2,3)$ is four dimensional with two quantum numbers, a spin (eigenvalue for rotation) taking values $\pm \frac{1}{2}$, and an intrinsic energy (eigenvalue for translation through time) taking values $\pm \frac{1}{2}$ in natural units. How well does this spectrum of eigenvalues accord with observation?

Fermions, and in particular solutions to the Dirac equation, are characterised by two quantum numbers, spin and charge. If we identify charge with intrinsic energy this fits perfectly with the intrinsic spectrum of $\mathfrak{so}(2,3)$. Furthermore this identification makes a great deal of sense. It has long been clear that time and charge are linked in some fashion. Positrons for example are often described as electrons travelling backwards in time. As we will see later the Dirac equation is very natural when viewed from the point of view of local symmetry group $\mathfrak{so}(2,3)$. Maxwell’s equations are also natural from this perspective.

We conclude therefore that there is good reason to expect that $\mathfrak{so}(2,3)$ may fit better with observation than the Poincaré group particularly with regard to intrinsic action on wave-functions in quantum mechanics. It would therefore be worthwhile to explore the consequences of using mathematical structures which naturally express this symmetry group in our physics.

1.1 The Lie algebra $\mathfrak{so}(2,3)$

Choose coordinates $\lambda, t, x, y, z$ in $\mathbb{R}^5$ and define $O(2,3)$ as the $5 \times 5$ real matrices conserving the bilinear form $\lambda_1 \lambda_2 + t_1 t_2 - x_1 x_2 - y_1 y_2 - z_1 z_2$. For $u, v \in \mathbb{R}^5$ this form can be written as $\langle u, v \rangle = u^t \Omega v$ where

---

1. Rest mass is a function of (and constrains) the ordinary momentum and energy of a particle. In particular it should not be confused with intrinsic energy.
\[
\Omega = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & -1 \\
\end{pmatrix}
\]

Hence for \( A \in O(2, 3) \) we have \( A^t \Omega A = \Omega \) and thus \(|A| = \pm 1\). The matrices with determinant one give the subgroup \( SO(2, 3) \) which is the connected component of the identity.

The 4-D manifold in \( \mathbb{R}^5 \) where \( \lambda^2 + t^2 - x^2 - y^2 - z^2 = 1 \) forms an invariant 4-D manifold in \( \mathbb{R}^5 \) for this group. This manifold is called Anti-DeSitter spacetime, and \( SO(2, 3) \) is called the Anti-DeSitter group when acting on this manifold.

Consider a small neighbourhood of the point on the manifold where \( \lambda = 1 \). Near this point \( \lambda \) is a function of the other coordinates allowing us to identify this neighbourhood with a local region in space-time. Note that \( \lambda = 1 \) up to second order in the coordinates, and for small regions the resulting transformation group is almost indistinguishable from the Poincaré group.

The operators \( \{ \frac{1}{r}T, \frac{1}{rc}X, \frac{1}{rc}Y, \frac{1}{rc}Z, \frac{1}{c}A, \frac{1}{c}B, \frac{1}{c}C, I, J, K \} \) give an alternative basis for the Lie algebra using operators defined using ordinary units. To obtain a description of the Lie algebra in ordinary units we should rename \( T, X, Y, Z, A, B, C, I, J, K \) to these scaled versions. Table 1.3 gives the Lie algebra in ordinary units. Table 1.2 can be recovered by choosing units for distance and time so that \( r = c = 1 \).

Looking at table 1.3 and letting \( r \to \infty \) we obtain the Lie algebra for the Poincaré group. We say that the ADS group contracts to the Poincaré group as the radius of the universe tends to infinity. Letting \( c \to \infty \) as well gives the non-relativistic Galilean group of classical mechanics. We say
\[
T = \begin{pmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
X = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
Y = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}
\]

\[
Z = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
A = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
B = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
C = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
I = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
J = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
K = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

Table 1.1: The canonical representation of \(\mathfrak{so}(2, 3)\)

|   | T | X | Y | Z | A | B | C | I | J | K |
|---|---|---|---|---|---|---|---|---|---|---|
| T | 0 | A | B | C | -X | -Y | -Z | 0 | 0 | 0 |
| X | -A | 0 | -K | J | -T | 0 | 0 | 0 | Z | -Y |
| Y | -B | K | 0 | -I | 0 | -T | 0 | -Z | 0 | X |
| Z | -C | -J | I | 0 | 0 | 0 | -T | Y | -X | 0 |
| A | X | T | 0 | 0 | 0 | -K | J | 0 | C | -B |
| B | Y | 0 | T | 0 | K | 0 | -I | -C | 0 | A |
| C | Z | 0 | 0 | T | -J | I | 0 | B | -A | 0 |
| I | 0 | 0 | Z | -Y | 0 | C | -B | 0 | K | -J |
| J | 0 | -Z | 0 | X | -C | 0 | A | -K | 0 | I |
| K | 0 | Y | -X | 0 | B | -A | 0 | J | -I | 0 |

Table 1.2: The Lie algebra \(\mathfrak{so}(2, 3)\) in natural units
that the Poincaré group contracts to the Galilean group as the speed of light tends to infinity.

It is also possible to allow $c \to \infty$ while leaving $r$ alone. The result describes a non-relativistic universe in which objects with velocity acquire distance over time (as one would expect); but also objects which are distant acquire velocity over time. In other words we’d have a classical universe with a cosmological constant.

Historically the ADS group arose as a cosmological model for a space of constant curvature. As a cosmological model it suffers from causality violating time-like loops. We will use the group only to describe local symmetry and this does not require acceptance of either ADS cosmology or time-like loops. The ADS group and the Poincaré group are indistinguishable as local symmetry groups for large $r$ so this is consistent with observation.

### 1.2 The Lie algebra $\text{sp}(2, \mathbb{R})$

$\text{Sp}(2, \mathbb{R})$ is the group of $4 \times 4$ symplectic matrices. Symplectic matrices are those which conserve a symplectic form – in this case a fixed antisymmetric non-degenerate bilinear form on $\mathbb{R}^4$. Such a form is given with respect to a Darboux basis by $\langle u, v \rangle = u^t \Omega v$ where $\Omega$ is the $4 \times 4$ matrix.

$p$-Notation for symplectic groups varies. We use the notation from [2].
$$\Omega = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

The group $\text{Sp}(2, \mathbb{R})$ thus consists of all $4 \times 4$ matrices $M$ with $M^t \Omega M = \Omega$. The associated Lie algebra $\text{sp}(2, \mathbb{R})$ consists of the $4 \times 4$ matrices $D$ with $D^t \Omega + \Omega D = 0$. This Lie algebra is ten dimensional with basis specified as in table 1.4.

Commutators of these matrices give precisely the relationships in table 1.2, hence $\text{so}(2, 3)$ and $\text{sp}(2, \mathbb{R})$ are isomorphic.

Note that each matrix $M$ in this representation is invertible with $M^{-1} = \pm 4M$. The sign here is positive for $\{X, Y, Z, A, B, C\}$ and negative for $\{T, I, J, K\}$. We can therefore express this in terms of the invariant bilinear form on the Lie algebra.

$$T = \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad I = \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad J = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad K = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$X = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad Y = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad Z = \frac{1}{2} \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$A = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad B = \frac{1}{2} \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad C = \frac{1}{2} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$
basis \( \{P_{\lambda}, P_t, P_x, P_y, P_z\} \) where the action of \( \{T, X, Y, Z, A, B, C, I, J, K\} \) on this basis generates precisely the matrices in table 1.1.

The matrices \( \{P_{\lambda}, P_t, P_x, P_y, P_z\} \) can be expressed in terms of the matrices \( \{T, X, Y, Z, A, B, C, I, J, K\} \) as follows.

\[
P_{\lambda} = -2AI = -2IA = -2BJ = -2JB = -2CK = -2KC \quad (1.1a)
\]

\[
P_t = 2XI = 2IX = 2YJ = 2JY = 2ZK = 2KZ \quad (1.1b)
\]

\[
P_x = 2TI = 2IT = -2ZB = -2BZ = 2YC = 2CY \quad (1.1c)
\]

\[
P_y = 2TJ = 2JT = -2XC = -2CX = 2ZA = 2AZ \quad (1.1d)
\]

\[
P_z = 2TK = 2KT = -2YA = -2AY = 2XB = 2BX \quad (1.1e)
\]

As all the products listed commute we can also express these matrices in terms of Jordan brackets; for example \( P_t = XI + IX = \{X, I\} \).

The matrices \( \{P_{\lambda}, P_t, P_x, P_y, P_z, T, X, Y, Z, A, B, C, I, J, K\} \) form a basis of the conformal Lie algebra \( \mathfrak{so}(3, 3) \). This Lie algebra is given in table 1.6.

The adjoint action of \( \{T, X, Y, Z, A, B, C, I, J, K\} \) on \( \{P_{\lambda}, P_t, P_x, P_y, P_z\} \) agrees with the usual physical interpretation. This suggests that the adjoint action of the operators \( \{P_{\lambda}, P_t, P_x, P_y, P_z\} \) might allow us to physically interpret these operators.

The operator \( P_{\lambda} \) for example rotates \( P_t \) through \( T \), \( P_x \) through \( X \), \( P_y \) through \( Y \) and \( P_z \) through \( Z \). Hence we might view \( P_{\lambda} \) as the operator for reflection or inversion of the \( t,x,y \), and \( z \) coordinates via rotation through additional dimensions. Similarly \( P_t \) can be viewed as inverting all coordinates except the \( t \) coordinate.

\[
1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]

\[
P_{\lambda} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}
\]

\[
P_t = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}
\]

\[
P_x = \frac{1}{2} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]

\[
P_y = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}
\]

\[
P_z = \frac{1}{2} \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}
\]

Table 1.5: Basis for a 6-D representation of \( \mathfrak{sp}(2, \mathbb{R}) \) under adjoint action
The operators $P_x$, $P_y$ and $P_z$ are not compact, so would give hyperbolic rotations through additional dimensions. It is not clear how these should be interpreted.

The operator $P_x$ commutes with $T$ and $I$ and $\{T, I, P_x\}$ is the basis of a maximally compact Cartan subalgebra of $\mathfrak{so}(3, 3)$.

### 1.3 The Enveloping Algebra.

The enveloping algebra is the non-commutative polynomial Lie algebra generated abstractly by symbols $\{T, X, Y, Z, A, B, C, I, J, K\}$ which commute according to the relations in table 1.2. The commutator is linear and satisfies Leibniz conditions

$$[P, QR] = [P, Q]R + Q[P, R]$$

$$[PQ, R] = [P, R]Q + P[Q, R]$$

|    | $P_\lambda$ | $P_t$ | $P_x$ | $P_y$ | $P_z$ | $T$ | $X$ | $Y$ | $Z$ | $A$ | $B$ | $C$ | $I$ | $J$ | $K$ |
|----|-------------|-------|-------|-------|-------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $P_\lambda$ | 0 | $T$ | $X$ | $Y$ | $Z$ | $-P_t$ | $-P_x$ | $-P_y$ | $-P_z$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $P_t$ | $-T$ | 0 | $A$ | $B$ | $C$ | $P_\lambda$ | 0 | 0 | 0 | $-P_x$ | $-P_y$ | $-P_z$ | 0 | 0 | 0 |
| $P_x$ | $-X$ | $-A$ | 0 | $-K$ | $J$ | 0 | $-P_\lambda$ | 0 | 0 | $-P_t$ | 0 | 0 | $P_z$ | $-P_y$ |
| $P_y$ | $-Y$ | $-B$ | $K$ | 0 | $-I$ | 0 | 0 | $-P_\lambda$ | 0 | 0 | $-P_t$ | 0 | $-P_z$ | 0 | $P_x$ |
| $P_z$ | $-Z$ | $-C$ | $-J$ | $I$ | 0 | 0 | 0 | 0 | $-P_\lambda$ | 0 | 0 | $-P_t$ | 0 | $P_y$ | $-P_x$ | 0 |
| $T$ | $P_t$ | $-P_\lambda$ | 0 | 0 | 0 | 0 | $A$ | $B$ | $C$ | $-X$ | $-Y$ | $-Z$ | 0 | 0 | 0 |
| $X$ | $P_x$ | 0 | $P_\lambda$ | 0 | 0 | $-A$ | 0 | $-K$ | $J$ | $-T$ | 0 | 0 | 0 | $Z$ | $-Y$ |
| $Y$ | $P_y$ | 0 | 0 | $P_\lambda$ | 0 | $-B$ | $K$ | 0 | $-I$ | 0 | $-T$ | 0 | $-Z$ | 0 | $X$ |
| $Z$ | $P_z$ | 0 | 0 | 0 | $P_\lambda$ | $-C$ | $-J$ | $I$ | 0 | 0 | 0 | $-T$ | $Y$ | $-X$ | 0 |
| $A$ | 0 | $P_x$ | $P_t$ | 0 | 0 | $X$ | $T$ | 0 | 0 | 0 | $-K$ | $J$ | 0 | $C$ | $-B$ |
| $B$ | 0 | $P_y$ | 0 | $P_t$ | 0 | $Y$ | 0 | $T$ | 0 | $K$ | 0 | $-I$ | $-C$ | 0 | $A$ |
| $C$ | 0 | $P_z$ | 0 | 0 | $P_\lambda$ | $Z$ | 0 | 0 | $T$ | $-J$ | $I$ | 0 | $B$ | $-A$ | 0 |
| $I$ | 0 | 0 | 0 | $P_z$ | $-P_y$ | 0 | 0 | $Z$ | $-Y$ | 0 | $C$ | $-B$ | 0 | $K$ | $-J$ |
| $J$ | 0 | 0 | $-P_z$ | 0 | $P_x$ | 0 | $-Z$ | 0 | $X$ | $-C$ | 0 | $A$ | $-K$ | 0 | $I$ |
| $K$ | 0 | 0 | $P_y$ | $-P_x$ | 0 | 0 | $Y$ | $-X$ | 0 | $B$ | $-A$ | 0 | $J$ | $-I$ | 0 |

Table 1.6: The Lie algebra $\mathfrak{so}(3, 3)$
where $P$, $Q$ and $R$ are arbitrary polynomials. The commutator (Lie bracket) of any pair of polynomials in the enveloping algebra can be determined from these conditions.

The enveloping algebra is graded by degree. The original Lie algebra (in our case $\mathfrak{so}(2,3)$) is the subalgebra of polynomials of degree one. Any representation of the enveloping algebra restricts to a representation of $\mathfrak{so}(2,3)$ and conversely any representation of $\mathfrak{so}(2,3)$ can be extended uniquely to a representation of the enveloping algebra. In particular the adjoint representation of the enveloping algebra on itself gives us a representation of $\mathfrak{so}(2,3)$ on the enveloping algebra.

It is not hard to show using table 1.2 and the Leibniz conditions for commutators that this representation conserves the space $V_d$ of polynomials of degree at most $d$. Hence $\mathfrak{so}(2,3)$ is also naturally represented on the quotient space $V_{d+1}/V_d$.

The polynomial

$$Q = -T^2 + X^2 + Y^2 + Z^2 + A^2 + B^2 + C^2 - I^2 - J^2 - K^2$$

commutes with all of \{T, X, Y, Z, A, B, C, I, J, K\} and hence lies in the centre of the enveloping algebra. Any representation of $\mathfrak{so}(2,3)$ can be extended to a representation of the enveloping algebra and in particular $Q$ is so represented. The eigenspaces of this operator are conserved by $\mathfrak{so}(2,3)$, so an irreducible representation of $\mathfrak{so}(2,3)$ consists of a single eigenspace of $Q$ with eigenvalue denoted $\mu$.

The polynomials

$$P_\lambda = \frac{2}{3}(-AI - BJ - CK)$$

$$P_t = \frac{2}{3}(XI + YJ + ZK)$$

$$P_x = \frac{2}{3}(TI - BZ + CY)$$

$$P_y = \frac{2}{3}(TJ - CX + AZ)$$

$$P_z = \frac{2}{3}(TK - AY + BX)$$

generate a subspace of the enveloping algebra which is stable under the action of $\mathfrak{so}(2,3)$ and which is isomorphic to the canonical 5-D representation. The constant of $\frac{2}{3}$ has been added to ensure that when evaluated in the case of the 4-D fundamental representation, these definitions agree with those in equations 1.1 in the last section. However the commutators of these operators in the enveloping algebra are not as described in table 1.6, which is valid only for the 4-D representation. In particular the set $\{P_\lambda, P_t, P_x, P_y, P_z, T, X, Y, Z, A, B, C, I, J, K\}$ does not form a basis for a copy of $\mathfrak{so}(3,3)$ in the enveloping algebra.

3 However the commutators of these operators in the enveloping algebra are not as described in table 1.6, which is valid only for the 4-D representation. In particular the set $\{P_\lambda, P_t, P_x, P_y, T, X, Y, Z, A, B, C, I, J, K\}$ does not form a basis for a copy of $\mathfrak{so}(3,3)$ in the enveloping algebra.
The fourth degree polynomial
\[ R = P_\lambda^2 + P_t^2 - P_x^2 - P_y^2 - P_z^2 \] (1.4)
is invariant and lies in the centre of the enveloping algebra. The eigenspaces of this operator will also be conserved, hence an irreducible representation consists of a single eigenspace with eigenvalue denoted \( \rho \).

The two operators \( Q \) and \( R \) are algebraically independent and together generate the centre of the enveloping algebra. They are known as the second and fourth degree Casimir operators respectively.

## 1.4 Roots and Weights

The Lie algebra \( \mathfrak{so}(2,3) \simeq \mathfrak{sp}(2,\mathbb{R}) \) is simple. We can therefore compute its roots and use those to obtain information about irreducible representations. The general theory of roots and weights can be found in any introductory Lie algebra text (for example [2] or [3]). In this section we summarise the consequences of that general theory for the specific Lie algebra \( \mathfrak{so}(2,3) \).

Consider a finite dimensional complex irreducible representation. Such a representation associates matrices to elements of the enveloping algebra. The matrix for the Casimir element \(-T^2 + X^2 + Y^2 + Z^2 + A^2 + B^2 + C^2 - I^2 - J^2 - K^2\) must commute with the matrices for all elements of the Lie algebra, hence eigenspaces of this matrix are invariant. Since by assumption our representation is irreducible we conclude that the matrix of the Casimir element is scalar.

The Casimir element and Killing form to which it is closely related, can be used to separate the Lie algebra into two subspaces. The compact subspace \( \text{span}\{T, I, J, K\} \) is spanned by basis elements with a negative coefficient in the Casimir element, while the non-compact subspace \( \text{span}\{X, Y, Z, A, B, C\} \) is spanned by elements with a positive coefficient.

Our next task is to choose a Cartan subalgebra (CSA)- a maximal commutative subalgebra of \( \mathfrak{so}(2,3) \). There are many possibilities but we single out two of particular utility, a compact CSA \( \text{span}\{T, I\} \) contained in the compact subspace, and a non-compact CSA \( \text{span}\{Y, C\} \) contained in the non-compact subspace.

The non-compact CSA is useful in proving certain mathematical properties of the Lie algebra. However the compact CSA has the greatest physical utility since compact elements have eigenvalues that are more easily interpreted. For example the eigenvalues of a rotation operator which is
compact gives angular momentum or spin, whereas it is much less clear what the eigenvalues of a non-compact operator such as a Lorentz boost might be describing.

$$\lambda_T, \lambda_I$$

| Root | $$(\lambda_T, \lambda_I)$$ | Root element |
|------|--------------------------|--------------|
| $\cdot$ | (0, 0) | $T$ |
| $\cdot$ | (0, 0) | $I$ |
| $\rightarrow$ | (i, 0) | $\frac{1}{\sqrt{2}}(A + iX)$ |
| $\leftarrow$ | (−i, 0) | $\frac{1}{\sqrt{2}}(-A + iX)$ |
| $\uparrow$ | (0, i) | $\frac{1}{\sqrt{2}}(J - iK)$ |
| $\downarrow$ | (0, −i) | $\frac{1}{\sqrt{2}}(J + iK)$ |
| $\nearrow$ | (i, i) | $\frac{1}{2}(C - Y + iB + iZ)$ |
| $\nwarrow$ | (−i, i) | $\frac{1}{2}(C + Y + iB - iZ)$ |
| $\searrow$ | (i, −i) | $\frac{1}{2}(-C - Y + iB - iZ)$ |
| $\swarrow$ | (−i, −i) | $\frac{1}{2}(-C + Y + iB + iZ)$ |

Table 1.7: Roots for compact CSA

Since the elements of a CSA commute we may seek simultaneous eigenvectors for the basis of the CSA. We call these weight vectors and the associated pair of simultaneous eigenvalues is called a weight. The weights and weight vectors for the adjoint representation of the Lie algebra have special names. We call these roots and root elements respectively. Roots and root elements for the compact CSA are given in table 1.7. The components of a root $$(\lambda_T, \lambda_I)$$ for the compact CSA are imaginary. Ignoring the imaginary unit we can depict roots graphically in a root diagram as shown in figure 1.1. We have chosen to name our root elements by using

Figure 1.1: The Root System for $\mathfrak{so}(2,3)$
the corresponding vectors in the root diagram. The root elements extend
the basis of the CSA to a full basis of the Lie algebra.

Suppose \((r_1, r_2)\) is a root for root element \(R\) with respect to some CSA
span\((Q_1, Q_2)\). Let the Lie algebra be represented on \(V\) and let \(v \in V\) be a
weight vector for weight \((\mu_1, \mu_2)\). Then

\[
Q_i (R(v)) = R(Q_i(v)) + [Q_i, R](v) = \mu_i R(v) + r_i R(v) = (\mu_i + r_i) R(v) \tag{1.5}
\]

so \(R(v)\) is either zero or is a weight vector for weight \((\mu_1 + r_1, \mu_2 + r_2)\).
We conclude that root elements map weight vectors to weight vectors, and
map corresponding weights by adding the root.

As a corollary of this the set of all weight vectors spans an invariant
subspace, and so any irreducible representation has a basis of weight vectors.
In such a basis the matrices for elements of the CSA are diagonal. As every
element of \(\mathfrak{so}(2, 3)\) belongs to at least one CSA we conclude that all matrices
in an irreducible representation are diagonalisable.

If we start with one given weight vector \(v\), then the set of all weight
vectors obtained by repeated application of root elements to \(v\) spans a non-
trivial invariant subspace and hence there is a basis of such vectors in the
irreducible case. Hence the set of weights for an irreducible representation
is contained in a lattice in \(\mathbb{C}^2\) invariant under translation by root vectors.
For each weight we may look at the dimension of the space of weight vectors
belonging to that weight. We call this dimension the degree of the weight.
A weight diagram for an irreducible representation is a plot of the set of all
weights labelled with their degrees.

For a finite dimensional representation only finitely many weights ap-
ppear. The components of each weight are pure imaginary and can be or-
dered. We can then order the weights themselves via the library ordering
of their components and identify a maximal weight and a minimal weight.

The same ordering can be applied to roots by classifying the non-zero
roots as either positive and negative depending on whether translation by
this root constitutes an increase or a decreases under the weight ordering.
Figure 1.1 shows a depiction of the roots with positive roots marked.

The root diagram and all the weight diagrams are symmetrical with
respect to the Weyl group. The Weyl group is given by the natural action
of the Lie group on the Lie algebra. In particular it is the quotient of the
stabiliser by the centraliser of the set of roots under this natural action.
The Weyl group for \(\mathfrak{so}(2,3)\) is the symmetry group \(D_8\), the dihedral group
of order 8, and is generated by reflections of the root diagram which map a
root to its negative.
In particular the minimal weight must be the negative of the maximal weight since inversion is an element of the Weyl group. Also the set of weights must be invariant under exchange of coordinates since this also is an element of the Weyl group. For an irreducible representation all weights, and in particular the maximal weight, can be obtained by a sequence of translations by positive roots from the minimal weight.

We conclude that the maximal weights for an irreducible representation take the form \((m, n)\) or \(((m + \frac{1}{2})i, (n + \frac{1}{2})i\) where \(m, n \in \mathbb{Z}\).

It is possible to show (we not do so here) that all finite dimensional irreducible complex representations of \(\mathfrak{so}(2, 3)\) have a basis in which the matrices for elements of \(\mathfrak{so}(2, 3)\) are all real. As a consequence of this result the real irreducible representations can be identified with the complex ones.

Any irreducible representation has a basis of weight vectors. The degree of each weight in the representation is defined to be the dimension of the corresponding subspace of weight vectors. In a finite dimensional representation the degrees can be calculated using Kostant’s formula (Hall [2] Theorem 7.42). The sum of the degrees of all weights is the dimension of the representation. The weights together with their degrees can be depicted in a weight diagram for the representation. Weight diagrams for some small dimensional irreducible representations are given in figure 1.2. The representations are labelled by their maximal weights.

Consider now a finite dimensional irreducible representation with maximal weight \((q_0, s_0)\). Then \(Q\) and \(R\) act as scalar operators on this representation with eigenvalues \(\mu\) and \(\rho\) respectively. Clearly \(\mu\) and \(\rho\) are functions of \(q_0\) and \(s_0\). We seek now to make this explicit.

The following operators are compositions of roots which begin with a positive root and hence act as zero operators on the maximal weight-space.

\[
(Y - iZ + iB + C)(Y + iZ - iB + C) = Y^2 + Z^2 + B^2 + C^2 + \{Y, C\} - \{Z, B\} - 2iI + 2iT
\]

\[
(Y + iZ + iB - C)(Y - iZ - iB - C) = Y^2 + Z^2 + B^2 + C^2 - \{Y, C\} + \{Z, B\} + 2iI + 2iT
\]

\[
(X + iA)(X - iA) = X^2 + A^2 + iT
\]

\[
(J + iK)(J - iK) = J^2 + K^2 - iI
\]

These combine to give useful identities on the maximal weight-space

\[
Q = -T^2 - I^2 - iI - 3iT
\]  

(1.6)
Figure 1.2: Weight diagrams for small dimensional irreducible representations of \( \mathfrak{so}(2,3) \). Each representation is labelled by its dimension and highest weight \((q_0, s_0)\)
\{Y, C\} - \{Z, B\} = 2iI \quad (1.7)

Since \(T\) and \(I\) have eigenvalues \(iq_0\) and \(is_0\) respectively while \(Q\) has eigenvalue \(\mu\), equation 1.6 gives

\[
\mu = q_0(q_0 + 3) + s_0(s_0 + 1) \quad (1.8)
\]

Evaluating this at \((q_0, s_0) = (\frac{1}{2}, \frac{1}{2})\) gives \(\mu = \frac{5}{2}\) in agreement with the value directly computed using the matrices in table 1.4 while evaluating at \((q_0, s_0) = (1, 0)\) gives \(\mu = 4\) which can be verified using the matrices in table 1.1. The 10 dimensional regular representation with \((q_0, s_0) = (1, 1)\) has \(\mu = 6\).

To compute a similar formula for the 4th degree Casimir invariant we begin by examining the adjoint action of \(\mathfrak{so}(2, 3)\) on the 5-D space with basis \(\{P_{\lambda}, P_t, P_x, P_y, P_z\}\). As the operators \(T\) and \(I\) commute, this space has a basis of simultaneous eigenvectors which gives a second set of root-like vectors on the weights. We call these \(P\)-roots.

| \((\lambda_T, \lambda_I)\) | Basis element |
|--------------------------|---------------|
| (0 , 0 )                | \(P_x\)       |
| (i , 0 )                | \(P_\lambda - iP_t\) |
| (-i , 0 )               | \(P_\lambda + iP_t\) |
| (0 , i )                | \(P_y + iP_z\)  |
| (0 , -i )               | \(P_y - iP_z\)  |

Table 1.8: \(P\)-roots

Products of \(P\)-roots which begin with a positive \(P\)-root must act as zero operators on the maximal weight-space. Hence the following operators are zero on the maximal weight-space.

\[
(P_\lambda + iP_t)(P_\lambda - iP_t) = P_\lambda^2 + P_t^2 - iT
\]

\[
(P_y - iP_z)(P_y + iP_z) = P_y^2 + P_z^2 - iI
\]

giving two useful identities on the maximal weight-space

\[
P_\lambda^2 + P_t^2 = iT \quad (1.9)
\]

\[
P_y^2 + P_z^2 = iI \quad (1.10)
\]

The definition of \(P_x\) together with equation 1.7 gives another identity

\[
P_x = \frac{2}{3}(TI + iI) \quad (1.11)
\]
Combining these we obtain
\[-P_\lambda^2 - P_t^2 + P_{x}^2 + P_{y}^2 + P_{z}^2 = -iT + iI + \frac{4}{3} I^2 (T + i)^2\]  \hspace{1cm} (1.12)

which gives a formula for the eigenvalue \(\rho\) of the 4th order Casimir operator \(\hat{R}\).
\[\rho = q_0 - s_0 + \frac{4}{3} s_0^2 (q_0 + 1)^2\]  \hspace{1cm} (1.13)

Evaluating this at \((q_0, s_0) = (\frac{1}{2}, \frac{1}{2})\) gives \(\rho = \frac{1}{4}\); while evaluating at \((q_0, s_0) = (1, 0)\) gives \(\rho = 1\). The 10 dimensional regular representation with \((q_0, s_0) = (1, 1)\) has \(\rho = \frac{16}{9}\).

### 1.5 The Charge Multiplier

We have suggested that we should interpret intrinsic energy as charge with units chosen so that \(e = q = \frac{1}{2}\). While this works nicely for electrons and positrons, it works considerably less well for the other particles in the modern particle physics menagerie.

In particular this interpretation of charge would require quarks to have representations including weights \((q, s) = (\pm \frac{1}{6}, \pm \frac{1}{2})\) or \((q, s) = (\pm \frac{1}{3}, \pm \frac{1}{2})\) none of which appear in any finite dimensional representation of \(so(2,3)\). Even neutrinos are problematic as they would need the weight \((q, s) = (0, \pm \frac{1}{2})\) which doesn’t appear either.

This problem must be addressed as we simply can’t have particles with these kinds of problematic weights if our model is to make sense. Imagine for example finding a free particle with spin \(\frac{1}{4}\). How would rotations even act on this thing? Our current difficulty is of a similar order of magnitude.

To escape this trap which we have constructed for ourselves we must revisit the identification of intrinsic energy \(q\) with charge \(Q\) which is what has caused this problem. We do want intrinsic energy and charge to be closely related, but perhaps we went too far when we asserted that they should actually be the same.

Instead of asserting that \(Q = q\) let us instead suppose that \(Q = q m\) where \(m\) is some constant multiplying factor which depends on the nature of the particle.

All elementary fermions will fit into this revised picture using the fundamental \((q_0, s_0) = (\frac{1}{2}, \frac{1}{2})\) representation with \(m\) in the set \(\{0, \frac{1}{3}, \frac{2}{3}, 1\}\).

The same fix works for bosons as well. Photons and gluons can be described using the \((q_0, s_0) = (1, 1)\) and \((q_0, s_0) = (1, 0)\) representations with \(m = 0\). Weak bosons would belong to a \((q_0, s_0) = (1, 1)\) representation
with $m = \frac{1}{2}$, while Gravitons would presumably be described using one of the three $q_0 = 2$ representations with $m = 0$.

There are some issues remaining with the multiplicities of various states in this picture, particularly in the case of bosons. We also must explain the origin of this mysterious multiplying factor $m$. Without trivialising these issues however, they are not grounds for abandoning our model at such an early stage. We can hope to find explanations for these things later if we continue. They are not fatal flaws which would cause us to stop.
Chapter 2

Symmetry and Manifolds

In the first chapter we discussed why it would make a great deal of sense to use the group \( \text{so}(2,3) \) instead of the Poincaré group to describe local symmetry. In this chapter we try to attach a precise mathematical meaning to this vague physical statement.

It is not difficult to simply pin a Lie algebra to an arbitrary manifold. However this would not serve our purpose. We want the Lie algebra to be attached to the manifold in a natural fashion so that it arises from the structure of the manifold itself. In some sense we want the Lie algebra at every point to describe local symmetry.

To make this work we will need to make some fairly subtle and careful distinctions. On a manifold with curvature, translations do not commute. Indeed curvature itself is essentially a description of the failure of parallel transport to commute. On the other hand Lie algebras also describe the nature of a failure to commute. The Lie algebra we wish to attach to the manifold involves translations which don’t commute. It seems like we have two mathematical structures competing to describe the same thing.

Resolving this apparent clash requires a deeper understanding of the relationship between curvature and local symmetry. We will discover that we are able to do this in very nice way provided the manifold has the same dimension as the Lie algebra. Hence the theory for \( \text{so}(2,3) \) arising from this approach will work only when the manifold is ten dimensional.

This is not really a problem. The ‘extra’ dimensions in this case are simply those of rotation and Lorentz boost and the ten dimensional manifold is just the manifold of inertial frames. Hence, unlike some other theoretical approaches which introduce extra dimensions, in our case the additional six dimensions are quite physical and we need make no attempt to explain why
they are not observed.

Once we have built a ten dimensional curved manifold with local symmetry so(2,3) the next task will be to look at how other mathematical structures, specifically spinor valued wave functions on the manifold can be added in a consistent way. Finally we show that all of this mathematical structure flows from a small number of very simple and quite physical assumptions.

2.1 Tensor Derivations and Manifolds

We will begin by setting up some basic notation describing curvature on a manifold. The approach we follow is inspired by the approach taken in Wald’s book [6].

Definition 2.1.1 A tensor derivation $D$ on a manifold $M$ is a map $D: \text{tensors} \rightarrow \text{tensors}$ satisfying

• Linearity.

• Leibniz condition on tensor products.

• Commutes with contraction.

We do not require tensor derivations to conserve degree. The composition of two tensor derivations usually fails to obey the Leibniz condition and is therefore not a tensor derivation. However we can easily verify the following

Proposition 2.1.2 If $D$ and $E$ are tensor derivations, then the commutator $[D, E]$ is also a tensor derivation where $[D, E](X) = D(E(X)) - E(D(X))$.

The commutator of two linear operators obeys the Jacobi identity, so we might expect tensor derivations to form a Lie algebra. However addition of two arbitrary tensor definitions is problematic. The sum $D + E$ of two tensor derivations is given by $(D + E)(X) = D(X) + E(X)$. But this only makes sense if $D(X)$ and $E(X)$ have the same rank so that they can be added. If this is not always the case then $D + E$ is undefined.

Since it lacks a well defined addition the set of all tensor derivations is not a Lie algebra. However any subset of the set of tensor derivations
on which addition is defined will be a Lie algebra if it is closed under
commutator, addition, and under the obvious scalar multiplication.

Let $D$ be a tensor derivation. A function $f$ on $M$ is a tensor of rank zero,
hence $D(f)$ is a tensor. Suppose $D(f)$ has rank $i^j$. Let $T$ be a tensor of
rank $k$. Now $D(fT) = D(f)T + fD(T)$ using the Leibniz condition. The
first term has rank $i^j + k$, hence the second term must have the same rank
since the Leibniz condition requires that addition be well defined. It follows
that the rank of $D(T)$ is $i^j + k$. We call $i^j$ the rank of $D$ and express
$D$ using index notation in the form $D^\beta_{\alpha_1 \alpha_2 ... \alpha_j}$. We have proved

**Proposition 2.1.3** Every tensor derivation has a rank $i^j$ and maps ten-
sors of rank $k^l$ to tensors of rank $i^j + k^l$.

**Proposition 2.1.4** If $D$ is a tensor derivation and $S$ is any tensor, then
$S \otimes D$ is a tensor derivation where $(S \otimes D)(T) = S \otimes D(T)$.

**Proof** Just check the definition.

Every ordinary derivation can be extended to a tensor derivation of rank
$0^j$ by allowing it to act on components. We use $T \mapsto \partial \partial_i T$ to denote
this tensor derivation where $\partial_i = \frac{\partial}{\partial x_i}$.

Conversely every tensor derivation $D$ of rank $0^j$ acts on functions as an
ordinary derivation. Hence we may associate to it a tangent vector field so
that with respect to some coordinate system we have $D(f) = a^\partial_i \partial(f)$.

Note that the two operators need only agree in their action on functions.
It follows that the difference $D - a^\partial_i \partial_i$ is a tensor derivation of rank $0^j$ which
maps all functions to the zero function.

**Proposition 2.1.5** Let $E$ be a tensor derivation of rank $0^j$ with $E(f) = 0$
for all functions $f$ on $M$. Then there exists a tensor $\Gamma_j$ of rank $i^j$ so that

$$E(x_{\beta_1 \beta_2 ... \beta_n}^{\alpha_1 \alpha_2 ... \alpha_m}) = \sum_s \Gamma_s^\alpha_{\alpha_1 ... \alpha_s ... \alpha_m} x_{\beta_1 \beta_2 ... \beta_n}^{\alpha_1 \alpha_2 ... \alpha_m} - \sum_t \Gamma_t^\beta_{\beta_1 ... \beta_t ... \beta_n} x_{\beta_1 \beta_2 ... \beta_n}^{\alpha_1 \alpha_2 ... \alpha_m}$$

**Proof** If $v$ is a vector field and $f$ is a scalar field then $E(fv) = E(f)v + fE(v) = fE(v)$. If follows that $E$ acts linearly on the tangent vector fields
on $M$. Hence its action on vector fields is contraction with a tensor of rank
$0^j$ (locally matrix multiplication).

Indeed if the vector fields $\{e_i\}$ form a basis of the tangent spaces at
each point so that we can write $v = v^i e_i$ in terms of this basis, then we can
explicitly find this tensor, since $E(v) = E(v^i e_i) = v^i E(e_i) = v^i \Gamma^j e_j$ where
\( \Gamma^j_i \) is the \( j \)-th component of \( E(e_i) \). Using coordinates to describe tensors, we can thus write

\[
E(v^j) = \Gamma^j_i v^i
\]

Now \( 0 = E(u_i v^j) = E(u_i) v^j + u_i \Gamma^i_j v^j \), and since this is true for all \( v \) it follows that

\[
E(u_i) = -\Gamma^i_j u_t
\]

We can now argue inductively on the total rank of tensor \( X \) by contracting with either \( u_i \) or \( v^j \) to diminish the total rank and applying \( E \) to the resulting tensor.

Conversely, let \( \Gamma \) be a tensor of rank \((\frac{1}{1})\). We can define a tensor derivation denoted \( \Gamma(\ast) \) by

\[
\Gamma(\ast) (X^{\alpha_1 \alpha_2 \ldots \alpha_m}_{\beta_1 \beta_2 \ldots \beta_n}) = \sum_s \Gamma^s_{\alpha_s} X^{\alpha_1 \ldots \hat{\alpha}_s \ldots \alpha_m}_{\beta_1 \beta_2 \ldots \beta_n} - \sum_t \Gamma^t_{\beta_t} X^{\alpha_1 \alpha_2 \ldots \alpha_m}_{\beta_1 \ldots \hat{\beta}_t \ldots \beta_n}
\]  

(2.1)

We have proved that all rank \((\frac{0}{0})\) tensor derivations are of the form

\[
a^i \partial_i + \Gamma(\ast)
\]

for some choice of vector field \( a^i \) and tensor \( \Gamma_i \).

Now let \( D_{\mu_1 \ldots \mu_n}^{\lambda_1 \ldots \lambda_m} \) be a tensor derivation of rank \((\frac{m}{n})\). By contracting this with component tensors we see that each of the operators obtained by fixing the values of all indices \( \lambda_i \) and \( \mu_j \) is a tensor derivation of rank \((\frac{0}{0})\). This leads us to the following.

**Proposition 2.1.6** Every tensor derivation of rank \((\frac{m}{n})\) takes the form

\[
D_{\mu_1 \ldots \mu_n}^{\lambda_1 \ldots \lambda_m} = a^i_{\mu_1 \ldots \mu_n} \partial_i + \Gamma_{\mu_1 \ldots \mu_n}^{\lambda_1 \ldots \lambda_m} (\ast)
\]

where

\[
\Gamma_{\mu_1 \ldots \mu_n}^{\lambda_1 \ldots \lambda_m} (\ast) (T^{\alpha_1 \ldots \alpha_p}_{\beta_1 \ldots \beta_q}) = \sum_s \Gamma_{\mu_1 \ldots \mu_n \alpha_s}^{\lambda_1 \ldots \lambda_m} T^{\alpha_1 \ldots \hat{\alpha}_s \ldots \alpha_p}_{\beta_1 \ldots \beta_q} - \sum_t \Gamma_{\mu_1 \ldots \mu_n \beta_t}^{\lambda_1 \ldots \lambda_m} T^{\alpha_1 \ldots \alpha_p}_{\beta_1 \ldots \hat{\beta}_t \ldots \beta_q}
\]

The tensor derivations of rank \((\frac{0}{1})\) are of particular importance. These are of the form

\[
D_i = a^k_i \partial_k + \Gamma_i (\ast)
\]  

(2.2)

In the special case that \( a^k_i = 1^k_i \) we call this a **covariant derivative**. Much of the subsequent work will take place in the context of a manifold with a distinguished covariant derivative \( \nabla_i \).
2.2 Torsion and Curvature

Let $M$ be a manifold with distinguished covariant derivative $\nabla_i$. Since the commutator of tensor derivations is a tensor derivation $[\nabla_i, \nabla_j]$ is a tensor derivation of rank $(0,2)$. Hence we can write

$$[\nabla_i, \nabla_j] = T_{ij}^k \partial_k + K_{ij}(\_\_\_\_)$$  
(2.3)

Applying both sides to a function $f$ we see that

$$T_{ij}^k \frac{\partial f}{\partial x_k} = -\Gamma_{ij}^k \frac{\partial f}{\partial x_k} + \Gamma_{ji}^k \frac{\partial f}{\partial x_k}$$  
(2.4)

Hence

$$T_{ij}^k = - (\Gamma_{ij}^k - \Gamma_{ji}^k)$$  
(2.5)

which is the negative of the torsion tensor as usually defined. There are reasons to believe that the torsion tensor would be better defined with opposite sign - see for example [6] page 31.

Similarly by applying $[\nabla_i, \nabla_j]$ to a vector field $v^k$ and comparing the terms not involving the partial derivative on both sides, we see that

$$K_{ijy}^x = [\partial_i \Gamma_{jy}^x - \partial_j \Gamma_{iy}^x] + [\Gamma_{it}^x \Gamma_{jy}^t - \Gamma_{jt}^x \Gamma_{iy}^t] + T_{ij}^t \Gamma_{ty}^x$$  
(2.6)

In the torsion-free case the last term vanishes and $K_{ijy}^x$ becomes the usual Riemann tensor. However we will predominantly not be working in the torsion-free case.

The commutator of operators obeys the Jacobi identity.

$$[[\nabla_i, \nabla_j], \nabla_k] + [[\nabla_j, \nabla_k], \nabla_i] + [[\nabla_k, \nabla_i], \nabla_j] = 0$$  
(2.7)

Resolving the tensor and partial derivative terms of this equation will give identities involving the torsion and the curvature tensor which turn out to be the well known and very important Bianchi identities. However in evaluating these identities we note that expressing $[\nabla_i, \nabla_j]$ in terms of $\partial_k$ is not very helpful when we wish to subsequently take another commutator. It would be much more useful to express $[\nabla_i, \nabla_j]$ in terms of $\nabla_k$. We define

$$[\nabla_i, \nabla_j] = T_{ij}^k \nabla_k + R_{ij}(\_\_\_)$$  
(2.8)

Since $\nabla_k$ and $\partial_k$ act identically on functions, the coefficients $T_{ij}^k$ are as given in equation 2.5 (the negative of the usual torsion tensor). Hence

$$T_{ij}^k \partial_k + K_{ij}(\_\_\_) = T_{ij}^k \nabla_k + R_{ij}(\_\_\_)$$

$$= T_{ij}^k \partial_k + T_{ij}^k \Gamma_k(\_\_\_) + R_{ij}(\_\_\_)$$

24
and so

$$R^x_{ijy} = K^x_{ijy} - T^k_{ijy} \Gamma^x_{kjy} = \left[ \partial_i \Gamma^x_{jy} - \partial_j \Gamma^x_{iy} \right] + \left[ \Gamma^x_{iky} \Gamma^x_{kjy} - \Gamma^x_{iky} \Gamma^x_{kjy} \right]$$  \hspace{1cm} (2.9)

Hence $R^x_{ijy}$ is the usual Riemann tensor in all cases, even when the manifold is not torsion free.

We now derive the Bianchi identities by looking at the Jacobi identity for the derivations $\nabla_i, \nabla_j$ and $\nabla_k$.

$$[[\nabla_i, \nabla_j], \nabla_k](v^x) = \left[ T^t_{ij} \nabla_t + R^s_{ij} \left( T^t_{sk} \nabla_k \right) \right] (v^x)$$

$$= \left( T^t_{ij} \nabla_s \nabla_k (v^x) + R^t_{ij} \nabla_k (v^x) - R^t_{ijk} \nabla_t (v^x) \right)$$

$$- \left( \nabla_k T^t_{ij} \nabla_t (v^x) + T^s_{ij} \nabla_k (v^x) + \nabla_k (R^t_{ij}) v^t + R^s_{ij} \nabla_k (v^t) \right)$$

$$= T^s_{ij} \nabla_s \nabla_k (v^x) - R^t_{ijk} \nabla_t (v^x) - \nabla_k (T^t_{ij} \nabla_t (v^x)) - \nabla_k (R^t_{ij}) v^t$$

$$= \left( T^s_{ij} T^t_{sk} - \nabla_k (T^t_{ij}) - R^t_{ijk} \right) \nabla_t (v^x) + \left( T^s_{ij} R^t_{kst} - \nabla_k (R^t_{ij}) \right) v^t$$

If we cyclically permute the indices $\{i, j, k\}$ in this expression and add, then by the Jacobi identity the result must be zero. Separating out the coefficients of $v^t$ and $\nabla_t (v^x)$ we obtain the first and second Bianchi identities.

The first Bianchi identity is

$$R^t_{ijk} + T^t_{kx} T^x_{ij} + \nabla_k (T^t_{ij}) \overset{(ijk)}{=} 0$$  \hspace{1cm} (2.10)

where we introduce the notation $\overset{(ijk)}{=} \equiv$ to mean that applying the permutation $(ijk)$ cyclically to the left hand side and summing gives the expression on the right.

The second Bianchi identity is

$$\nabla_k (R^t_{ij}) + T^x_{ij} R^t_{kxs} \overset{(ijk)}{=} 0$$  \hspace{1cm} (2.11)

### 2.3 Lie Groups

Now that we have developed sufficient mathematical tools to allow us to intelligently talk about the problem, let us consider the question of local symmetry, and in particular how we might specify that a manifold has local symmetry $\text{so}(2, 3)$. 

25
We turn for inspiration to the one manifold which we can unambiguously say must have local symmetry \( \mathfrak{so}(2,3) \), namely the manifold which is the ten dimensional Lie group \( \text{SO}(2,3) \) itself.

There are many ways to describe the relationship between a Lie group and its Lie algebra. For example the Lie algebra is commonly described as the algebra of left invariant vector fields. This uniquely identifies the Lie algebra with the tangent space at each point. Note that it also provides us with an identification of tangent spaces at different points. Consequently every Lie group comes equipped with a natural connection arising from viewing left multiplication in the group as parallel transport.

The description of the Lie algebra as left invariant vector fields does not address its relationship with this natural connection which will be our main focus in what follows. The existence of this connection means that there is a natural covariant derivative \( \nabla \) defined on every Lie group. We observe the following.

**Theorem 2.3.1** The natural connection on a Lie group satisfies 
\[
[\nabla_i, \nabla_j] = T^k_{ij} \nabla_k
\]
where \( T^k_{ij} \) are the structure coefficients of the associated Lie algebra on each tangent space. Furthermore \( \nabla_i(T^k_{ij}) = 0 \).

Hence the Lie structure is the negative of the torsion as usually defined. As noted in the comments following equation \( 2.5 \), there are many reasons to believe that torsion would have been better defined with opposite sign. Rather than live with this annoying negative sign we choose to correct the definition. Hence we will call \( T^k_{ij} \) as defined in equation \( 2.5 \) the torsion and not its negative.

With this definition the torsion with respect to the natural connection on a Lie group is precisely the Lie structure while the curvature tensor is zero.

This suggests how we might be able to have a curved space which accommodates a local symmetry group in which translations do not commute. We can use the torsion to specify the local symmetry. Curvature and Torsion have a clearly defined relationship and can coexist on the same manifold. Hence we can specify local symmetry using the Torsion and we can do it on a curved manifold without any confusion between the local symmetry and the curvature.
2.4 Lie Manifolds

Definition 2.4.1 A **Lie manifold** is a connected real manifold with invariant torsion. That is with $\nabla_i (T^a_{bj}) = 0$.

The covariant derivatives $\{\nabla_i\}$ for fixed indices $i$ are tensor derivations of rank $(0,0)$, and this set of tensor derivations is closed under addition and commutator and so forms a Lie algebra. The Jacobi identity now gives

$$[\nabla_k, [\nabla_i, \nabla_j]] (f)^{(ijk)} = 0 \quad (2.12)$$

Expanding this out and using $\nabla_k (T^t_{ij}) = 0$ we obtain

$$T^t_{kx} T^x_{ij}^{(ijk)} = 0 \quad (2.13)$$

and hence the torsion obeys the Jacobi condition defining a Lie algebra on each tangent space. By invariance (and connectivity) these Lie algebras are all isomorphic, so there is essentially a unique Lie algebra associated with each connected Lie manifold.

Equation (2.13) implies that on a Lie manifold the first Bianchi identity takes the more familiar form

$$R^t_{ijk} (ijk) = 0 \quad (2.14)$$

A Lie algebra has a natural bilinear form, the Killing form, which is invariant under the adjoint action. This means every Lie manifold has defined on it a natural pseudometric given by the Killing form.

$$k_{ij} = T^a_{ib} T^b_{ja} \quad (2.15)$$

The Jacobi identity gives $T_k (k_{ij}) = 0$ confirming that $k_{ij}$ is invariant under the adjoint action of the Lie algebra on each tangent space. If the Lie algebra is semisimple (or indeed simple as is the case for $\mathfrak{so}(2,3)$) then $k_{ij}$ will be non-degenerate. Furthermore

$$\nabla_t (k_{ij}) = \nabla_t (T^a_{ib} T^b_{ja}) = \nabla_t (T^a_{ib}) T^b_{ja} + T^a_{ib} \nabla_t (T^b_{ja}) = 0 \quad (2.16)$$

so this pseudometric is also globally invariant under the action of the covariant derivative. As is common in the area of general relativity we will stop calling it a pseudometric beyond this point and will simply call it the metric.

Let us quickly examine the nature of this metric in the case that the Lie algebra is $\mathfrak{so}(2,3)$. We will use the basis $\{T, X, Y, Z, A, B, C, I, J, K\}$
for the Lie algebra and coordinates \( \{ t, x, y, z, a, b, c, i, j, k \} \) so that a typical element of the Lie algebra takes the form 
\[ tT + xX + yY + zZ + aA + bB + cC + iI + jJ + kK. \]
We use ordinary units (because we want to see the effect of \( r \)) and compute the Killing form directly from table 1.3. We obtain
\[
-\frac{6}{r^2} \left( t_1 t_2 - \frac{1}{2} (x_1 x_2 + y_1 y_2 + z_1 z_2) \right) \\
- 6 \left( i_1 i_2 + j_1 j_2 + k_1 k_2 - \frac{1}{2} (a_1 a_2 + b_1 b_2 + c_1 c_2) \right)
\] (2.17)

This gives us a Lorentz metric across the four space-time dimensions \( \{ T, X, Y, Z \} \) together with a spin or helicity term on the six Lorentz dimensions. This is precisely the kind of metric we'd expect to have on the manifold of frames.

**Axiom 1** *Our universe defines a Lie manifold for \( \mathfrak{so}(2, 3) \). We interpret this as the 10-D manifold of local inertial frames.*

We call this an axiom because such a statement cannot be proved mathematically. Instead it must be evaluated in terms of the usefulness and physicality of the resulting mathematical model.

In a Lie manifold for \( \mathfrak{so}(2, 3) \) the Lie algebra \( \mathfrak{so}(2, 3) \) can act on vectors in two ways. The global action is given by \( v^k \mapsto \nabla_i (v^k) \). There is also a local action given by \( v^k \mapsto T^k_{ij} v^j \) which describes the adjoint action of \( \mathfrak{so}(2, 3) \). It is a map from the manifold into the 10-D representation.

At this point we have almost everything we need. However tensors alone are not enough for us to do physics. We need spinors to act as the wave functions of fermions. This requires us to extend our ideas to allow other local actions.

---

1 or possibly a cover of the manifold of inertial frames

28
Chapter 3

Spinor Manifolds and Frameworks

In the last chapter we looked at how local symmetry \( \mathfrak{so}(2, 3) \) could be specified on a curved manifold. We were able to do this by using torsion to describe the local symmetry. A vector on such a manifold can be viewed as a function on which the symmetry group has two actions; an extrinsic action specified by the covariant derivative, and an intrinsic action specified by the torsion.

This is exactly the kind of thing we want in order to do physics. However tensors have representations with integral values for spin. We need to be able to use functions into other representations, and in particular into spinors. We need to extend our model in a natural way so that this becomes possible.

To see how this might be done we return to the canonical example where the manifold is the Lie group itself. This time however we will begin with the assumption that we are working with a matrix Lie group – a closed subgroup of the group \( \text{End}(V) \) of endomorphisms some vector space \( V \). This means our Lie algebra comes equipped with a local action on \( V \). In the case that \( V \) is the 4-D representation of \( \mathfrak{sp}(2, \mathbb{R}) \) we hope to learn from this example how to naturally include spinor valued functions on a Lie manifold for \( \mathfrak{so}(2, 3) \).

3.1 Matrix Lie Groups

Consider a matrix Lie group \( M \) on a (real or complex) vector space \( V \). Choose a basis for \( V \) and use Greek indices to denote components with respect to this basis. A \( V \)-field is a \( V \) valued function on \( M \) denoted in
terms of our basis as $v^\alpha$ where it is understood that these coordinates are functions on $M$. Choose a coordinate system for $M$ treating it as a real manifold. We will use roman indices for this coordinate system, which gives a basis at each point of the Lie algebra. An ordinary vector field on $M$ is denoted $v^i$ where it is understood that these coordinates are functions on $M$.

The elements of the Lie algebra are matrices over $V$. Note that the Lie algebra is always a real Lie algebra, but the matrices that represent it may be complex in the case that $V$ is a complex vector space. Hence the Lie algebra elements can be written in terms of our bases as $T^\alpha_i$ where the coefficients are functions of $M$. Since these matrices represent the Lie algebra we have

$$T^\alpha_i T^\lambda_j - T^\lambda_i T^\alpha_j = T^k T^\alpha_k$$

where $T^k_{ij}$ is the structure constants for the Lie algebra. We call this representation the local action on $V$-fields. The notation is chosen to be consistent with the local action on vector fields which is the adjoint action given by $T^k_{ij}$. This allows us to describe the natural local action as $T^k (\cdot \cdot \cdot)$ regardless of whether vector fields or $V$-fields are being acted on. This local action can be extended naturally to tensor products and duals of $V$-fields and vector fields.

For example a $V^*$-field is a map from $M$ into the dual space $V^*$ of $V$ and can be denoted $u_\alpha$ with respect to our chosen basis of $V$. It maps $v^\alpha$ to $u_\alpha v^\alpha$ calculated under the summation convention. The natural local action on $V^*$-fields is given by

$$T^k (u_\alpha) = -T^\lambda_k u_\beta$$

which ensures that $T(u)(v) + u(T(v)) = T(u(v)) = 0$ consistent with a trivial local action on scalar fields.

Similarly a $V \otimes V$-field is a map from $M$ to $V \otimes V$ denoted $X^{\alpha \beta}$ with respect to our chosen basis of $V$. The local action is defined by requiring a Leibniz condition and is given by

$$T^k (X^{\alpha \beta}) = T^\alpha_{k \lambda} X^{\lambda \beta} + T_{k \lambda} X^{\alpha \lambda}$$

which ensures that $T(v \otimes w) = T(v) \otimes w + v \otimes T(w)$.

The most general objects that can be formed using dual and tensor product from $V$-fields and vector fields are called $V$-tensors. A typical $V$-tensor can be described in terms of coordinates for $V$ and the tangent spaces $T$ using a combination of greek and roman, upper and lower indices.
We define the rank of such a $V$-tensor to be $(m,p_{n,q})$. Thus $T_{i\beta}^\alpha$ is a $V$-tensor of rank $(0,1)$.  

The local action on $V$-tensors is denoted $T_k(\ast)$ and is calculated by contracting $T^i_{kj}$ or $T^\alpha_{k\beta}$ as appropriate with each upper and lower index in turn; adding the results in the case of the upper indices, and subtracting in the case of the lower ones. This will ensure that the local action obeys a Leibniz condition on tensor products and respects contraction. Furthermore we have

$$[T_i(\ast), T_j(\ast)] = T^k_{ij} T_k(\ast)$$

(3.4)

which means the local action on $V$-tensors of arbitrary rank represents the Lie algebra.

We finish our consideration of the local action by looking at the local action on the $V$-tensor $T_{i\beta}^\alpha$. We see

$$T_k(\ast) (T_{i\beta}^\alpha) = T^\lambda_{k\beta} T^\alpha_{i\lambda} - T^\lambda_{k\beta} T^\alpha_{i\lambda} - T^\alpha_{ki} T^\lambda_{x\beta} = 0$$

(3.5)

This follows directly from equation 3.1. We say that $T_{i\beta}^\alpha$ is locally invariant.

We now turn to the global action. As we noted earlier left multiplication by elements of $G$ can be regarded as defining a natural parallel transport of vector fields on a Lie group. In the present case where the elements of $G$ are matrices over $V$, left multiplication also defines a natural parallel transport of $V$-valued functions. This can be expressed via a connection $\Gamma_{i\beta}^\alpha$. This notation has been chosen so that we can describe the natural connection as $\Gamma_i(\ast)$ regardless of what is being acted on. This definition also extends naturally to $V$-tensors and can be used to define a global action $\nabla_k = \partial_k + \Gamma_k(\ast)$ on $V$-tensors. This global action also represents the Lie algebra and we have

$$[\nabla_i, \nabla_j] = T^k_{ij} \nabla_k$$

(3.6)

where $T^k_{ij}$ is the Lie structure and also obviously the torsion.

Finally we observe that the local and global actions commute. Hence we have

$$\nabla_k (T_{i\beta}^\alpha) = 0$$

(3.7)

and thus $T_{i\beta}^\alpha$ is globally as well as locally invariant.

So far we have simply been exploring the structure of an ordinary matrix Lie group from a somewhat unorthodox perspective. It is now time to consider how we might we might add curvature to this picture. But that
should not be difficult. We just need to weaken equation 3.6 and replace it with the more general equation

\[ [\nabla_i, \nabla_j] = T^k_{ij} \nabla_k + R_{ij}(\ast) \]

where \( R_{ij}(\ast) \) is a linear operator on arbitrary \( V \)-tensors which we will call the curvature.

### 3.2 \( V \)-tensors and \( V \)-tensor Derivations

In this section we implement the idea at the end of the last section, and clarify our notation.

Let \( M \) be a manifold and choose a local coordinate system on \( M \). Use the roman alphabet for the indices so that a vector field is written \( v^i \) with respect to the coordinate system.

Let \( V \) be any vector space (real or complex) with basis \( \{ e_\alpha \} \) where we use greek indices to distinguish it from the coordinate system on \( M \). A \( V \)-field on \( M \) is a map from \( M \) to \( V \) and is denoted \( v^\alpha \) in terms of its components with respect to this basis.

Other types of functions on the manifold can be constructed from \( V \)-fields and ordinary vector fields using the operations of dual and tensor product. Such functions are called \( V \)-tensors. A typical \( V \)-tensor is described with respect to our coordinate system on \( M \) and basis of \( V \) via a combination of greek and roman, upper and lower indices. We define the rank of the \( V \)-tensor \( a_{j_1\ldots j_n\lambda_1\ldots \lambda_p} \) to be \( (m,p) \times (n,q) \).

**Definition 3.2.1** A \( V \)-tensor derivation is a map from \( V \)-tensors to \( V \)-tensors which satisfies the following conditions

- Linearity.
- Leibniz condition on tensor products.
- Commutes with contraction (trace) of Greek and Roman indices.

The results on tensor derivations obtained in section 2.1 extend to \( V \)-tensor derivations without difficulty. We summarise omitting proofs which are essentially identical to those given earlier.

**Proposition 3.2.2** If \( D \) and \( E \) are \( V \)-tensor derivations, then \([D,E]\) is also a \( V \)-tensor derivation where \([D,E](X) = D(E(X)) - E(D(X))\).
Proposition 3.2.3 If $D$ is a $V$-tensor derivation and $A$ is a $V$-tensor, then $A \otimes D$ is an $V$-tensor derivation where $(A \otimes D)(X) = A \otimes D(X)$.

Proposition 3.2.4 Every $V$-tensor derivation has a rank $\left( \begin{array}{c} m \\ n \end{array} \right)$ and maps $V$-tensors of rank $\left( \begin{array}{c} p \\ q \end{array} \right)$ to $V$-tensors of rank $\left( \begin{array}{c} m+p \\ n+q \end{array} \right)$.

Every ordinary derivation can be extended to a $V$-tensor derivation of rank $\left( \begin{array}{c} 0 \\ 0 \end{array} \right)$ by allowing it to act on components. We use $T \mapsto a^i \partial_i(T)$ to denote this $V$-tensor derivation.

Conversely every $V$-tensor derivation $D$ of rank $\left( \begin{array}{c} 0 \\ 0 \end{array} \right)$ acts on functions as an ordinary derivation. Hence we may associate to it a tangent vector field so that with respect to some coordinate system on $M$ we have $D(f) = a^i \partial_i(f)$.

The difference $D - a^i \partial_i$ is a $V$-tensor derivation of rank $\left( \begin{array}{c} 0 \\ 0 \end{array} \right)$ which maps all functions to the zero function.

Proposition 3.2.5 Let $E$ be an $V$-tensor derivation of rank $\left( \begin{array}{c} 0 \\ 0 \end{array} \right)$ which maps all functions to the zero function. Then there exists a $V$-tensor $\Gamma^\beta_\alpha$ of rank $\left( \begin{array}{c} 0 \\ 1 \end{array} \right)$ and an $V$-tensor $\Gamma^i_j$ of rank $\left( \begin{array}{c} 1 \\ 0 \end{array} \right)$ so that $E = \Gamma^(*)$ defined in the obvious way.

Proposition 3.2.6 All $V$-tensors of rank $\left( \begin{array}{c} m \\ n \end{array} \right)$ can be expressed in the form

$$a^i_{j_1...j_m \lambda_1...\lambda_p} \partial_k + A^i_{j_1...j_n \mu_1...\mu_q} \partial_k(\ast)$$

In particular a $V$-tensor derivation of rank $\left( \begin{array}{c} 0 \\ 1 \end{array} \right)$ takes the form

$$a^i_k \partial_k + \Gamma^i_k(\ast)$$

In the case where $a^i_k = 1^i_k$ we call this a covariant derivative. The expressions $\Gamma^\alpha_{k\beta}$ and $\Gamma^i_{kj}$ are called connections and these define a notion of parallel transport of $V$-fields and vector fields respectively.

3.3 Matrix Lie Manifolds

We are now in a position to define the structure we need.

Definition 3.3.1 A Matrix Lie Manifold is

(a) a real manifold $M$ and a real or complex vector space $V$ together with
(b) a **local action** \( T_k(\ast) \) of tangent vectors on \( V \)-tensors with

\[
[T_i(\ast), T_j(\ast)] = T^k_{ij}(\ast)
\]

(c) a **global action** \( \nabla_k = \partial_k + \Gamma_k(\ast) \) on \( V \)-tensors with

\[
[\nabla_i, \nabla_j] = T^k_{ij} \nabla_k + R_{ij}(\ast)
\]

for some linear operator on \( V \)-tensors \( R_{ij}(\ast) \)

(d) where local and global actions commute

\[
\nabla_k(T^i_{j\ell}) = 0
\]

Part (c) tells us that the Lie structure \( T^k_{ij} \) is also the torsion\(^1\) and parts (d) and (b) give \( \nabla_i(T^i_{j\ell}) = 0 \). It follows that every matrix Lie manifold is a Lie manifold with a natural local action on \( V \)-valued functions.

Every matrix Lie manifold defines a unique matrix Lie algebra. A tangent space gives the Lie algebra, the torsion defines a Lie structure, and the local action gives a representation on \( V \). Furthermore the same matrix Lie algebra is obtained at each point on the manifold.

Matrix Lie manifolds provide a mathematical framework for building curved models with functions mapping into a chosen local representation. We now need to decide which local representation of \( \mathfrak{so}(2, 3) \) to use.

### 3.4 Complex matters

Spinors are the 4-D representation of our group \( \mathfrak{so}(2, 3) \). We want to be able to have spinor valued functions on our manifold so that we can use them as wave functions for elementary particles. We can do this by looking at a matrix Lie manifold for the appropriate representation. But do we want to use real spinors or complex ones? What difference does it make?

We make the following temporary definitions to aid our discussion.

**Definition 3.4.1** A **real spinor manifold** is defined to be a matrix Lie manifold for the 4-D real irreducible representation of the real Lie algebra \( \mathfrak{sp}(2, \mathbb{R}) \).

A **complex spinor manifold** is defined to be a matrix Lie manifold for the 4-D complex irreducible representation of the real Lie algebra \( \mathfrak{sp}(2, \mathbb{R}) \).

\(^1\)remember that our torsion is the negative of the usual definition
Note that every real spinor manifold is also a complex spinor manifold since we can construct the required actions on complex spinors by simply taking tensor product with complex scalars. However the converse appears not to be true. There is no obvious way to build a local and global action on real spinors from the actions on complex spinors.

Let us think about how we might try to do this to understand the nature of the problem. The obvious approach would be to seek a separation of complex spinors into real and imaginary parts in a manner respected by both the local and global action. The actions on the real part would then meet the requirements for a real spinor manifold.

This is equivalent to finding a basis in which all $T^\alpha_{k\beta}$ and $\Gamma^\beta_{k\alpha}$ are real. While we can find a basis that makes all $T^\beta_{k\alpha}$ real, there is no obvious way to ensure that all $\Gamma^\beta_{k\alpha}$ are real as well.

We can also restate the problem in terms of the search for a conjugation map; a conjugate linear involution $v \mapsto \overline{v}$ on complex spinors, which commutes with both local and global actions. The complex spinors in a real spinor manifold constructed by tensor product with $\mathbb{C}$ have such an involution. But there is no obvious reason why complex spinors on a complex spinor manifold should have such a map.

Conjugating a complex vector is not a trivial matter. Simply conjugating coordinates does not give a well defined map since this depends on the choice of basis. In order to have a well defined conjugation (or conjugate transpose) additional structure must be added to the space. Often this is done via some sort of inner product. However in our case the best approach seems to be to simply to require that our complex spinor manifold come equipped with a natural invariant conjugation map. As discussed above we can then choose bases so that $T^\alpha_{k\beta}$ and $\Gamma^\beta_{k\alpha}$ are real.

A complex spinor manifold equipped with such a conjugation is a real spinor manifold since we can use the conjugation to identify real components preserved by the local and global actions. Since conjugating complex spinors seems to be something we will need to be able to do, we will therefore choose a real spinor manifold to build our model. Complex spinor manifolds remain an obvious generalisation which is definitely worthy of investigation at a later date.

**Definition 3.4.2** We define a framework to be a real spinor manifold. Real spinor manifolds are the structure we will be looking at throughout the rest of the book, and it is useful to have a shortened name to describe them.

**Axiom 2** Our Universe is a framework.
An axiom is simply an assumption. We will be exploring the consequences of this assumption throughout the rest of this book. We next discuss whether this assumption is physically reasonable.

3.5 Reality Check

In ordinary general relativity the fundamental structure is the metric $g_{ij}$. This is assumed to be invariant.

A metric determines the nature of the manifold by distinguishing between two types of coordinate; positive and negative definite; space-like and time-like. Global invariance of the metric ensures that this distinction is conserved under parallel transport.

On a Lie manifold for $\mathfrak{so}(2, 3)$ the fundamental structure is the Lie structure or torsion $T^k_{ij}$. This is assumed to be invariant.

A natural metric, the Killing form, is then given by $g_{ij} = \frac{1}{6} T^x_{iy} T^y_{jx}$ (the constant of $\frac{1}{6}$ is included for consistency with later results). Invariance of the Lie structure implies invariance of this metric. Hence the assumption that the Lie structure is invariant is a generalisation of the idea of metric invariance.

The Lie structure identifies the physical nature of the ten coordinates on the manifold enabling us to distinguish between a time coordinate and a rotation coordinate for example, which a metric alone cannot do. Invariance of the Lie structure implies that these identifications will be conserved under parallel transport, which seems physically reasonable.

In a framework (real spinor manifold) the fundamental structure is the local action $T_{i\alpha}^\beta$ which is assumed to be invariant.

From the equation $T_{i\alpha}^\beta T_{j\beta}^{\lambda} = T_{ij}^k T_{k\alpha}^{\beta}$ it is clear that global invariance of $T_{i\alpha}^\beta$ implies global invariance of $T_{ij}^k$ and hence of $g_{ij}$ so this is a further extension of the idea of metric invariance. This follows since

$$g_{ij} = \frac{1}{6} T^x_{iy} T^y_{jx} = T_{ij}^\alpha T_{\alpha j}^\beta$$

(3.9)

The requirement that $T_{i\alpha}^\beta$ is globally invariant means that the action of the Lie algebra on spinors is conserved under parallel transport. If $\phi = X(\psi)$ for a tangent vector $X$ and spinors $\psi$ and $\phi$ defined at a point; and if these are parallel transported to $X'$, $\psi'$ and $\phi'$ respectively at an adjacent point, then we must have $\phi' = X'(\psi')$. 36
Invariance of $T^\beta_{\alpha\alpha}$ also is equivalent to the requirement that the local action $T_k(\alpha)$ and the global action $\nabla_k$ of our symmetry group commute. It is not unreasonable for us to ask that the global action respect the local action.

**Question 1** Are axiom 1 and axiom 2 equivalent? Can every Lie manifold for $\mathfrak{so}(2, 3)$ be extended to a framework?

To answer this question we need to know whether the existence of the spinor action places any constraints on the nature of the manifold and in particular its curvature.

We will later prove many interesting and significant results about the nature of curvature on a framework which depend critically on the existence of the spinor action. Since we don’t see how we might otherwise prove these results our intuition is that the answer to question II is most probably “No”.

For example the spinor action will later be used to prove equation 7.18

$$R^l_{ijk} = R^l_{ij} T^i_k$$

which tells us that on a framework, the curvature operator $R_{ij}(\alpha)$ on vectors can be always be expressed as a linear combination of torsion operators $T_k(\alpha)$. A Lie manifold for $\mathfrak{so}(2, 3)$ where this was not true could not be a framework. However the question remains open awaiting the construction of an explicit example.
Chapter 4

Generalised Tensors

A Lie manifold for $\mathfrak{so}(2, 3)$ is a manifold with the invariant Lie structure of $\mathfrak{so}(2, 3)$. It is a generalisation of the manifold which is the Lie group $\text{SO}(2, 3)$. A framework additionally has a natural local action on spinors. It generalises the matrix Lie group $\text{Sp}(2, \mathbb{R})$ and as such comes equipped with a local and global action on spinors. Using dual and tensor product we were able to extend these actions to $V$-tensors.

In all of this local and global actions play a central role. All $V$-tensors have both a local and global action, and the two actions commute. These are very physical properties. We need a global action to discuss extrinsic properties like Energy. We need a local action to discuss intrinsic properties like spin. And we need the two actions to commute if these are to be simultaneously determined.

This leads us to define a generalised tensor, a quantity on the manifold with a well defined local and global action where these commute. All $V$-tensors are generalised tensors, however as we will see the converse is not true. In particular if we analyse a $V$-tensor in terms of its indecomposable or irreducible components, we will discover that these need not be $V$-tensors. They will however be generalised tensors.

Generalised tensors represent the end of our search for a suitable mathematical structure to do physics with. Tensors and $V$-tensors were steps along the journey to generalised tensors.

There appears to be a very nice mathematical theory of generalised tensors on a Lie manifold, and in this chapter we will look at some of the basics of this theory. However much more remains to be done.
4.1 Definition

We make the following definition.

**Definition 4.1.1** A *Generalised Tensor Space* on a Lie manifold $M$ is any vector space $V$ with

1. a smooth local action defined on $V$-fields.
2. a global action defined on $V$-fields.
3. where the local and global actions commute

A **local action** is a representation of each tangent space (as a Lie algebra) on $V$. Hence it can be described using matrices $T_k(\ast)$ with

$$T_i(\ast)T_j(\ast) - T_j(\ast)T_i(\ast) = T^kT_k(\ast)$$

A **global action** is specified by a covariant derivative on $V$-valued functions which weakly represents the Lie algebra. The covariant derivative

$$\nabla_i = \partial_i + \Gamma_i(\ast)$$

is described in terms of a connection $\Gamma_i(\ast)$ which describes parallel transport of $V$-valued functions on $M$. It weakly presents the Lie algebra in the sense that

$$\nabla_i\nabla_j - \nabla_j\nabla_i = T^k\nabla_k + R_{ij}(\ast)$$

where $R_{ij}(\ast)$ are linear maps on $V$ defined on $M$.

Clearly all tensors belong to generalised tensor spaces, as do all $V$-tensors on a framework. However this does not preclude other generalised tensors from existing on the same manifold. Indeed we will see that generalised tensors which are not $V$-tensors always exist.

The question of which generalised tensors exist on a Lie manifold is an interesting one. The general answer to this question appears complicated and leads us away from our objectives, so ultimately we will provide only a partial answer in the case of a framework. In working towards that result however we will continue to argue generally for as long as possible.

4.2 Basics

**Proposition 4.2.1** The space of ordinary vector fields on a Lie manifold is a generalised tensor space which we denote $\mathcal{T}$.
Proposition 4.2.2 The direct sum $U \oplus V$ of two generalised tensor spaces is a generalised tensor space with

- local action $T_k(u, v) = (T_k(u), T_k(v))$.
- global action $\nabla_k(u, v) = (\nabla_k(u), \nabla_k(v))$.

This can be checked directly. We also note that $R_{ij}(u, v) = (R_{ij}(u), R_{ij}(v))$.

Proposition 4.2.3 The tensor product $U \otimes V$ of two generalised tensor spaces is a generalised tensor space with

- local action $T_k(u \otimes v) = T_k(u) \otimes v + u \otimes T_k(v)$.
- global action $\nabla_k(u \otimes v) = \nabla_k(u) \otimes v + u \otimes \nabla_k(v)$.

This can also be checked directly. We observe that $R_{ij}(u \otimes v) = R_{ij}(u) \otimes v + u \otimes R_{ij}(v)$.

Proposition 4.2.4 The dual $V^*$ of a generalised tensor space is a generalised tensor space with

- local action $T_k(f)(v) = f(T_k(v)) - T_k(f(v))$.
- global action $\nabla_k(f)(v) = f(\nabla_k(v)) - \nabla_k(f(v))$.

This can be checked directly, and in this case we have $R_{ij}(f)(v) = R_{ij}(f(v)) - f(R_{ij}(v))$.

Under these definitions the matrices $T_{k\beta}^\alpha$ defining the local action for a generalised tensor space $V$ belong to the generalised tensor space $T^* \otimes V^* \otimes V$. Hence there is a well defined global action on these. The condition that local and global actions commute can be written

$$\nabla_m (T_{k\beta}^\alpha) = 0$$

We next consider the question of isomorphisms and homomorphisms between generalised tensor spaces. In the case of representations of a Lie algebra, the appropriate notion is that of an intertwining map. In the case of a generalised tensor space we have two actions to consider, and both must be preserved in an isomorphism of generalised tensor spaces.

Definition 4.2.5 If $U$ and $V$ are generalised tensor spaces on the same Lie manifold $M$ then an intertwining map between $U$ and $V$ is a linear function $\theta : U \rightarrow V$ defined smoothly on $M$ which maps $U$-fields to $V$-fields and which satisfies
1. \( T_k(\theta(u)) = \theta(T_k(u)) \)

2. \( \nabla_k(\theta(u)) = \theta(\nabla_k(u)) \)

for all \( U \)-fields \( u \). If only the first of these properties holds we call it a \textit{local intertwining map}.

Two generalised tensor spaces are said to be \textit{equivalent} if there is an invertible intertwining map between them. They are said to be \textit{locally equivalent} if there is an invertible local intertwining map between them.

Let \( V \) be a generalised tensor space on \( M \), and let \( U \leq V \) be a vector subspace of \( V \). Then the local and global action on \( V \)-fields both act on \( U \)-fields by restriction. If the set of \( U \)-fields is conserved by the local action we call \( U \) a \textit{locally invariant} subspace. If this set is conserved by both actions we say that \( U \) is \textit{fully invariant} or just \textit{invariant}. If \( U \) is an invariant subspace of \( V \) then it is a generalised tensor space under the restriction of the local and global actions on \( V \).

**Proposition 4.2.6** If \( \theta : U \to V \) is an intertwining map then the kernel and image of \( \theta \), as obviously defined, are invariant in \( U \) and \( V \) respectively. If \( \theta \) is a local intertwining map then the kernel and image are locally invariant.

If \( V \) has a non-trivial invariant subspace we call it \textit{reducible}, otherwise it is called \textit{irreducible}. If a non-trivial locally invariant subspace exists we call it \textit{locally reducible} otherwise it is called \textit{locally irreducible}.

If \( V \) is equivalent to a non-trivial direct sum of invariant subspaces we call it \textit{decomposable}, otherwise it is called \textit{indecomposable}. If \( V \) is locally equivalent to a direct sum of locally invariant subspaces we call it \textit{locally decomposable} otherwise it is called \textit{locally indecomposable}. If the Lie algebra is semisimple then all locally indecomposable generalised tensor spaces are locally irreducible. We will assume that the Lie algebra is semisimple in the remainder of this discussion.

We are interested in understanding the nature of irreducibles and indecomposables. The obvious approach is to decompose in the first instance into a direct sum of local irreducibles. Since local and global actions commute we might hope that this decomposition is also respected by the global action.

This is \textit{almost} but not quite true. The problem case involves generalised tensor spaces where the local action is trivial. Such spaces are easy to construct as we will see in the next section. If the local action is trivial (the
zero map) then the commuting of the two actions does not constrain the global action which is then free to misbehave.

4.3 Locally Trivial Spaces

A generalised tensor space is said to be **locally trivial** if $T_k(\ast) = 0$ at every point so that the local action is trivial. These are easy to construct. Indeed choose any generalised tensor space with local and global action given by $T_k(\ast)$ and $\nabla_m$, and simply replace the local action with the trivial one. It is easy to verify that the result is a generalised tensor space.

The global action still exists and represents (with curvature) the Lie algebra. The new (trivial) local action also represents (trivially) the Lie algebra. And the global and local actions commute since the zero map commutes with everything. Using this trick we can construct a large number of different locally trivial generalised tensor spaces. However we should note that not all locally trivial generalised tensors are constructed in this way. In particular the Crump scalars introduced in chapter 6 are not of this type.

In traditional terms a locally trivial generalised tensor space would be interpreted as a scalar field or collection of scalar fields with non-trivial parallel transport. The obvious question is whether these play any role in the physics. Crump scalars are an example of locally trivial generalised tensors which are very physical. Indeed we will find that these are essential for a proper description of the electromagnetic field.

We finish this section with a useful example. Let $U$ be a locally trivial irreducible generalised tensor space of dimension $d$ and let $V$ be a generalised tensor space with irreducible local action. Then the tensor product $W = U \otimes V$ is a generalised tensor space. Under the local action we see that it consists of a direct sum of $d$ irreducible components each locally equivalent to $V$.

Consider a local intertwining map from $V$ to $W$. Then by Schur’s lemma this map gives a scalar map into each of the $d$ components. So the space $E$ of all local intertwining maps from $V$ to $W$ is a $d$-dimensional vector space. There is a natural local and global action on functions between generalised tensors and because these maps are local intertwining maps, the natural local action on $E$ is trivial. In fact it isn’t hard to show that $E$ and $U$ are isomorphic, and that $W$ is isomorphic to $E \otimes V$.
4.4 Parallel Transport

The connection defines a notion of parallel transport of generalised tensors. In this section we consider parallel transport in more detail. We will begin with the notion of parallel transport along a path.

Consider a smooth path $\pi : [0,1] \rightarrow M$ from point $p$ to point $q$ so that $\pi(0) = p$ and $\pi(1) = q$. Denote the tangent vectors to this path by $\pi^k$. Note that $\pi^k$ is only defined at points on the path, however we could extend it to a vector field if we wished. Alternatively we could start with a vector field $\pi^k$ and choose $\pi$ to be a trajectory. Let $\hat{X}$ be a generalised tensor.

Then we will say that $\hat{X}$ parallel transports along $\pi$ if

$$\pi^k \nabla_k (\hat{X}) = 0 \quad (4.1)$$

at each point on the path. This definition is the standard one from general relativity.

**Proposition 4.4.1** We list some properties of parallel transport along a path.

1. Any globally invariant generalised tensor parallel transports along all paths. In particular the generalised tensors describing the local action $T_k (\ast \ast) \parallel$ parallel transport along all paths.

2. If $\hat{X}$ and $\hat{Y}$ are generalised tensors of the same type which parallel transport along $\pi$, then any linear combination $a\hat{X} + b\hat{Y}$ also parallel transports along $\pi$.

3. If $\hat{X}$ and $\hat{Y}$ parallel transport along $\pi$ then so does $\hat{X} \otimes \hat{Y}$.

4. If $\hat{X}$ parallel transports along $\pi$ then any contraction or trace of $\hat{X}$ also parallel transports along $\pi$.

Proofs are elementary.

Considering $\hat{X}$ evaluated at $\pi(s)$ as a function of $s$ we have

$$\frac{d}{ds} \hat{X} = -\pi^k \Gamma_k (\ast) (\hat{X}) \quad (4.2)$$

Specifying the value of $\hat{X}$ at $p$ gives us a DE initial value problem which has unique solution. If the initial value at $p$ is obtained from another generalised tensor $X$, then the value of $\hat{X}$ at $q$ is the value of $X$ at $p$ parallel transported along the path $\pi$ to $q$. 43
Parallel transport along a path forces us to work at specific points. In some cases it is easier to work with the related notion of parallel transport of a generalised tensor along a vector field which can be defined in a region or even on the entire manifold. If $X$ is a generalised tensor and $\pi^k$ is a suitable vector field, then $\text{Port}(X, \pi^k, t)$ denotes the generalised tensor obtained by transporting $X$ along the vector field $\pi^k$ with transport parameter $h$. If the path $\pi$ is a trajectory of $\pi^k$ then the value of $\text{Port}(X, \pi^k, t)$ at $\pi(t)$ is defined to be the value of $X$ at $\pi(0)$ parallel transported along the path $\pi$ from $\pi(0)$ to $\pi(t)$.

We state without proof a number of important properties of this parallel transport operator

**Proposition 4.4.2** The parallel transport operator on generalised tensors has the following properties

1. $\text{Port}(X, \pi^k, 0) = X$
2. $\text{Port}(\text{Port}(X, \pi^k, t), \pi^k, s) = \text{Port}(X, \pi^k, t + s)$
3. $\text{Port}(X, \pi^k, -h) = \text{Port}(X, -\pi^k, h)$
4. $\frac{d}{dt} (\text{Port}(X, \pi^k, t)) = -\pi^k \nabla_k (\text{Port}(X, \pi^k, t))$
5. $\left. \frac{d}{dt} (\text{Port}(X, \pi^k, t)) \right|_{t=0} = -\pi^k \nabla_k (X)$.
6. If $X$ is globally invariant then $\text{Port}(X, \pi^k, t) = X$ for all $\pi^k$ and $t$. In particular the generalised tensors describing the local action $T_k(\ast)$ have this property.
7. $\text{Port}(aX + bY, \pi^k, t) = a \text{Port}(X, \pi^k, t) + b \text{Port}(Y, \pi^k, t)$
8. $\text{Port}(X \otimes Y, \pi^k, t) = \text{Port}(X, \pi^k, t) \otimes \text{Port}(Y, \pi^k, t)$
9. Transport preserves contraction or trace. If $Y$ is a contraction of $X$ then $\text{Port}(Y, \pi^k, t)$ is a contraction of $\text{Port}(X, \pi^k, t)$.
10. $T_k(\ast) \text{Port}(X, \pi^k, t) = \text{Port}(T_k(\ast) X, \pi^k, t)$ and hence transport is a local intertwining map.

---

\[^1\text{the carpet under which we will sweep any technical issues}\]
4.5 Indecomposable Generalised Tensors

From property 10 of proposition 4.4.2 parallel transport preserves the decomposition of a generalised tensor into locally homogeneous components as this decomposition is preserved by intertwining maps. From property 5 we can then conclude that the global action also preserves the decomposition into locally homogeneous components so that this decomposition is also global. We have proved the following.

**Proposition 4.5.1** The decomposition of a generalised tensor into locally homogeneous components is also respected by the global action. Hence all indecomposable generalised tensors are locally homogeneous.

To go beyond this we must consider the question of which irreducible representations of the Lie algebra can be realised as the local action for a generalised tensor.

**Definition 4.5.2** Let $M$ be a Lie manifold for the Lie algebra $\mathfrak{g}$. Let $V$ be an irreducible representation of $\mathfrak{g}$. Then we say that $V$ is local on $M$ if there is a generalised tensor on $M$ where the local action is equivalent to the action of $\mathfrak{g}$ on $V$.

Note that a framework or framework or spinor manifold $M$ is precisely a Lie manifold for $\text{so}(2,3)$ where the four dimensional irreducible representation is local on $M$.

The question of which irreducible representations are local on a given Lie manifold is an interesting one. As we have already noted requiring that the spinor representation is local appears to introduce non-trivial constraints on the nature of the curvature on $M$. The fully general situation therefore appears complicated and beyond the scope of this work. We therefore restrict our attention to the case that $M$ is a framework.

Proposition 4.5.1 is very useful for determining whether an irreducible $V$ is local on $M$. If we can find a generalised tensor on $M$ which has a single copy of $V$ in its local decomposition into irreducibles, then the decomposition into homogeneous components will give us a generalised tensor with local action equivalent to $V$ and we can conclude that $V$ is local on $M$.

All irreducible representations of $\text{so}(2,3)$ can be ultimately formed from the spinor representation by taking tensor products and picking out irreducible components. All the small dimensional representations appearing in figure 1.2 can be formed in such a way that they arise with multiplicity one in a homogeneous component and hence are local on $M$. We will look
at the details of how to do this shortly. So far no example of an irreducible which is not local on a framework $M$ has been found, which leads us to make the following conjecture.

**Conjecture 4.5.3** *All irreducibles are local on a framework $M$.*

Suppose that $W$ is an indecomposable generalised tensor. Then $W$ is homogeneous. Suppose that the local irreducible appearing in the local decomposition of $W$ is local on $M$ and let $V$ be a generalised tensor with this local action.

Consider the set $F$ of all functions from $V$ to $W$. There is a well defined local and global action on $F$ under which $F$ is a generalised tensor. If $f \in F$ then the local action is given by

$$
(T_k(\ast)(f))(v) = T_k(\ast)(f(v)) - f(T_k(\ast)(v)) \quad (4.3)
$$

while the global action is

$$
(\nabla_k(f))(v) = \nabla_k(f(v)) - f(\nabla_k(v)) \quad (4.4)
$$

Let $E$ be the set of all functions in $F$ which are local intertwining maps. Then $E$ is precisely the homogeneous component belonging to the locally trivial action on $F$ and hence $E$ is invariant also under the global action giving a generalised tensor.

**Proposition 4.5.4** *The generalised tensor $E$ is indecomposable and $W$ is isomorphic to the generalised tensor space $E \otimes V$.***

As a consequence of this proposition, if conjecture 4.5.3 holds, then every indecomposable generalised tensor on a framework arises as the tensor product of a locally irreducible generalised tensor with an indecomposable locally trivial generalised tensor. This reduces the problem of finding indecomposable generalised tensors on a framework to two cases; the locally trivial case and the locally irreducible case.
In the interests of conservation of ink and provided there is no potential for confusion, the word tensor by itself will be used to mean generalised tensor in what follows. Note that a tensor must have a specified local and global action.

Our main objection in this chapter is to show that the small dimensional representations of \( \mathfrak{so}(2,3) \) in figure 1.2 are local on a framework \( M \) by constructing tensors with these local actions. We will also develop the algebra of tensors.

5.1 Notation and General Discussion

Consider a generalised tensor space \( V \). Given a basis of \( V \) we can describe elements of this space using index notation. We will distinguish between different types of generalised tensor by our choice of alphabet for the indices. We can however use repeated indices to denote tensor product and the same alphabet with lowered indices to denote the dual.

Some choices of alphabet are fixed at the outset. We will use upper indices from the lowercase roman alphabet for vector fields, and upper indices from the lowercase greek alphabet for spinor fields.

For the purposes of this discussion let us use upper indices with the boldface lowercase roman alphabet to describe some arbitrary generalised space. Hence a generalised tensor from this space would be written as \( v^a \) in terms of some chosen basis of \( V \). Should we have need to consider a different basis of \( V \) we would use primes to denote this. For example we
would write a change of basis equation as
\[ v^{a'} = \delta^a_{a'} v^a \] (5.1)

where \( \delta^a_{a'} \) denotes the change of basis matrix. We will allow the change of basis matrix be a function of location in \( M \). There is no canonical way to identify \( V \) values at one point with those at another; the best we can do is to use parallel transport which is path dependent. Hence a fully general treatment should allow basis changes of this type.

The local action is given by
\[ T^b_{ia}(x^a) = T^a_{ka} v^b \] (5.2)

using the summation convention. The connection for the global action is \( \Gamma^b_{ma} \) and the global action is
\[ \nabla_k (v^a) = \partial_k (v^a) + \Gamma^a_{kb} v^b \] (5.3)

Hence we have
\[ T^b_{ic} T^c_{ja} - T^b_{jc} T^c_{ia} = T^k_{ij} T^b_{ka} \] (5.4)

and
\[ [\nabla_i, \nabla_j] (x^a) = T^b_{ij} \nabla_k (x^a) + R^a_{jb} x^b \] (5.5)

where \( R^b_{ij} \) expresses the curvature in terms of our bases.

A change of coordinates is an intertwining map for both actions. Hence
\[ T^b_{ka}(1^a) = 0 \]
\[ \nabla_{m}(1^a) = 0 \]

These give equations describing the effect of a change of bases on \( T^a_{kb} \) and \( \Gamma^a_{kb} \).

\[ T^a_{kb'} = 1^a_b 1^b_{b'} 1^k_t T^a_{kt} \] (5.6)

\[ \Gamma^a_{kb'} = 1^a_b 1^b_{b'} 1^k_t \Gamma^a_{kt} - 1^k_t \partial_k (1^a_t) 1^b_{b'} \] (5.7)

We see that \( T^a_{kb} \) changes basis as expected for a generalised tensor in the space \( V \otimes T^* \otimes V^* \). Indeed it is such a generalised tensor. However the connection \( \Gamma^a_{kb} \) has different behavior under a change of basis. Although we write it using index notation which would suggest that it belongs to the space \( V \otimes T^* \otimes V^* \), it is not a generalised tensor, and we therefore need to be careful with it particularly when changing basis.
This completes our general discussion. We end this section by listing the tensors we have met so far.

 Scalars are tensors with trivial local action and where the global action is simply partial differentiation (the scalar connection is zero). Scalars are usually denoted without index. However we will find it sometimes expedient to use a dummy index for scalars, and in such situations $a$ may be used; for example $f^a = f = f_0$. Hence $T_{ka}^0 = 0$ and $\Gamma_{m0}^0 = 0$. The existence of scalars shows that the $(q_0, s_0) = (0, 0)$ representation is local on $M$. 

 Spinors are defined on a framework; they are mentioned in the definition of a spinor manifold. We denote them $x^\alpha$ with a greek lowercase index. The local action is given by $T^\beta_k$ and global action $\nabla_k$ (with connection $\Gamma_{ka}^\beta$). Their existence shows that the $(q_0, s_0) = (1, \frac{1}{2})$ representation is local on $M$. 

 Vectors are vectors on the manifold in the usual sense of the word. This means they are ten dimensional. We may from time to time need to use the word vector to refer to other things, but unless some other meaning is clear from the context, the word vector by itself will refer to a ten dimensional vector. We denote vectors $v^i$ with a roman lower case index. Vectors are tensors with local action $T_{ij}^k$ and global action $\nabla_k$ (with connection $\Gamma_{ij}^k$). Their existence is specified in the definition of a spinor manifold and shows that the $(q_0, s_0) = (1, 1)$ representation is local on $M$. 

5.2 Spinor Transformations

A real spinor transformation is a $V$-tensor of rank $(0, 1)$. A typical tensor of this type is denoted $X_\beta^\alpha$. We begin by considering a smooth decomposition into irreducibles under the local action. This will mirror the decomposition of $4 \times 4$ matrices into irreducible representations carried out in chapter 1. As there is no degeneracy in this decomposition (no two irreducibles are equivalent) it follows that each irreducible component is conserved by both the local and global actions and hence constitutes a type of generalised tensor.

The ten dimensional irreducible component is spanned by the set of $V$-tensors $T_{k\beta}^\alpha$ for various $k$. If we choose $\{T_{k\beta}^\alpha\}$ as our basis, then the maps

$$X_\beta^\alpha \mapsto g^{ki}T_{i\beta}^\alpha = x^k$$

$$x^k \mapsto T_{k\beta}^\alpha x^k = X_\beta^\alpha$$
represent the projection and injection maps to and from this component. We should define local and global actions on the vector component so that these maps are intertwining. However since both $T^\alpha_{\beta\gamma}$ and $g^{ij}$ are totally invariant, these maps are intertwining with respect to the usual local and global action on vectors. We conclude that this component gives precisely the space of vectors and not some other type of tensor. The composition of projection and injection gives the totally invariant idempotent projection onto the vector component

$$X_\beta^\alpha \mapsto g^{ij}T^\alpha_{ij}T^\lambda_{\mu\nu}X^\mu_\lambda$$

The transformation $\frac{1}{2}1^\alpha_\beta$ is locally invariant and spans a 1 dimensional irreducible component of the space of spinor transformations. Elements of this representation need no basis and will be denoted as scalars. In fact as we will see they are precisely scalars. To show this we need to check the nature of the local and global actions.

Projection maps to and from this component are given by

$$X_\beta^\alpha \mapsto \frac{1}{2}1^\beta_\alpha \cdot X_\beta^\alpha = \frac{1}{2}X_\alpha^\alpha = x$$

$$x \mapsto \frac{1}{2}1^\alpha_\beta \cdot x = X_\beta^\alpha$$

The factor of $\frac{1}{2}$ was included to ensure idempotency of the composition

$$X_\beta^\alpha \mapsto \frac{1}{4}1^\alpha_\beta \cdot X^\lambda_\lambda$$

Local and global actions are defined with respect to our chosen basis so that these the change of basis maps are totally invariant. This gives $T_k(x) = 0$ and $\nabla_m(x) = \partial_m(x)$ which we recognise as the usual local and global actions on scalars. We conclude that the one dimensional component is actually scalar and not some other kind of one dimensional generalised tensor.

We can find an idempotent projection onto the remaining 5-D component by subtracting the idempotent projections onto the other two components from the identity.

$$X_\beta^\alpha \mapsto X_\beta^\alpha - \frac{1}{4}1^\alpha_\beta \cdot X^\lambda_\lambda - g^{ij}T^\alpha_{ij}T^\lambda_{\mu\nu}X^\mu_\lambda$$

Choose a basis for this component $\{T^\alpha_{\lambda\beta}\}$ which we will denote with a capital roman index. Assume such a choice of basis has been made smoothly on the manifold so that we can discuss the derivative using this basis.
These basis elements project trivially onto the other components giving

\[ T^\alpha_{A\alpha} = 0 \quad \text{(5.15)} \]
\[ T^\alpha_{A\beta} T^\beta_{k\alpha} = 0 \quad \text{(5.16)} \]

At this point it is also useful to define

\[ g_{AB} = T^\alpha_{A\beta} T^\beta_{B\alpha} \quad \text{(5.17)} \]

Projection maps to and from this component are given by

\[ X^A \mapsto T^\alpha_{A\beta} X^\beta = X^\alpha \quad \text{(5.18)} \]
\[ X^\alpha \mapsto S^\alpha_{A\beta} X^\beta = X^A \quad \text{(5.19)} \]

A combination of these two maps gives the idempotent projection, hence

\[ T^\alpha_{A\beta} S^\beta_{A\mu} = 1^\alpha_{\mu} 1^\beta_{\mu} - g^{ij} T^\alpha_{i\beta} T^\nu_{j\mu} - \frac{1}{4} 1^\alpha_{\beta} 1^\mu_{\mu} \quad \text{(5.20)} \]

Contracting both sides of equation (5.20) with \( T_B^\beta \) we obtain

\[ S^\alpha_{A\mu} g_{AB} = T^\nu_{B\mu} \quad \text{(5.21)} \]

While contracting equation (5.20) with \( S^C_{\nu\mu} \) gives

\[ (S^C_{\nu\mu} S^\alpha_{A\nu}) g_{AB} = 1^C_B \quad \text{(5.22)} \]

hence \( g_{AB} \) is non-singular with inverse \( g^{AB} = S^A_{\nu\mu} S^B_{\mu\nu} \) and

\[ S^\alpha_{A\beta} = T^\beta_{B\alpha} g^{AB} \quad \text{(5.23)} \]

The local and global actions in our new basis are such that these maps are all invariant. The local action \( T^B_{iA} \) is thus given by

\[ T_i(^*) (T^\alpha_{A\beta}) = 0 \quad \text{(5.24)} \]

and the connection \( \Gamma^B_{mA} \) giving parallel transport is defined by the equation

\[ \nabla_m (T^\alpha_{A\beta}) = 0 \quad \text{(5.25)} \]

Since \( T^\alpha_{A\beta} \) is totally invariant it follows that \( g_{AB} \) and \( g^{AB} \) are also totally invariant and We may use the metric \( g_{AB} \) and its dual \( g^{AB} \) to raise and lower these indices. This process will commute with both the local and global actions.
The 5-dimensional local representation on \( \{ x^A \} \) is the canonical representation of \( \mathfrak{so}(2, 3) \) while the symmetric bilinear form \( g_{AB} \) is the canonical metric. Hence the irreducible representation of \( \mathfrak{so}(2, 3) \) with maximal weight \( (q_0, s_0) = (1, 0) \) is local on \( M \).

We call this type of tensor a versaor. Versors are denoted \( k^A \) with an upper-case roman index. The word versaor was originally coined by Hamilton to refer to a specific type of quaternion associated with an axis of reversal, but has fallen into disuse. The discussion at the end of section 1.2 where the 5-D canonical representation was associated with axes of reversal makes this choice of notation seem appropriate. It is also a great word which deserves to be recycled.

We finish this section with some identities obtained by applying our decomposition to the product \( T^\alpha_{j^\lambda} T^\lambda_{j^\beta} \). We note that we can initially decompose this product into components which are symmetric and antisymmetric in terms of the indices \( i \) and \( j \), and this decomposition will be totally invariant.

The antisymmetric component is obtained from the commutator

\[
T^\alpha_{i^\lambda} T^\lambda_{j^\beta} - T^\alpha_{j^\lambda} T^\lambda_{i^\beta} = T^k_{ij} T^\alpha_{k^\beta}
\]  

and is the vector component. The symmetric component is obtained from the Jordan bracket. As the Jordan bracket projects trivially onto the vector component it must decompose into the other components. The scalar component of the Jordan bracket is \( g_{ij} \). We denote the 5-D component \( g_{ij}^A \) so that

\[
T^\alpha_{i^\lambda} T^\lambda_{j^\beta} + T^\alpha_{j^\lambda} T^\lambda_{i^\beta} = \frac{1}{2} g_{ij}^A + g_{ij}^A T^\alpha_{A^\beta}
\]  

Since all other quantities in this equation are globally invariant it follows that \( g_{ij}^A \) is also globally invariant. It is symmetric in \( i \) and \( j \) and behaves similarly to the metric in many respects. We can obtain an explicit expression for it by projecting the Jordan bracket.

\[
g_{ij}^A = g^{AB} T^\beta_{B^\alpha} \left( T^\alpha_{i^\lambda} T^\lambda_{j^\beta} + T^\alpha_{j^\lambda} T^\lambda_{i^\beta} \right)
\]  

Putting these results together the product can be decomposed as follows

\[
T^\alpha_{i^\lambda} T^\lambda_{j^\beta} = \frac{1}{2} T^k_{ij} T^\alpha_{k^\beta} + \frac{1}{4} g_{ij}^A T^\alpha_{A^\beta} + \frac{1}{2} g_{ij}^A T^\alpha_{A^\beta}
\]  

In the case that the spinors are complex then as discussed in section 3.4 there is a conjugation map conserved by both the local and global actions on spinors. This extends naturally to give a conjugation on spinor transformations which is also conserved by the local and global action.
spinor transformations are therefore simply the tensor product of real spinor transformations with \( \mathbb{C} \).

The conjugation map will preserve the decomposition into irreducibles and hence the complex irreducible components are simply the tensor product with \( \mathbb{C} \) of the real irreducibles identified in this section.

### 5.3 Action of \( \mathfrak{so}(3,3) \)

The operators \( T^{\beta}_{A\alpha} \) together with the operators \( T^{\beta}_{i\alpha} \) define a natural local representation of the Lie algebra \( \mathfrak{so}(3,3) \) on the manifold. Local invariance of \( T^{\beta}_{A\alpha} \) gives

\[
T^{\beta}_{i\mu} T^{\mu}_{A\alpha} - T^{\beta}_{A\mu} T^{\mu}_{i\alpha} = T^{B}_{iA} T^{\beta}_{B\alpha}
\]

(5.30)

We can thus regard \( T^{B}_{iA} \) in two ways, firstly as defining the action of \( \mathfrak{so}(2,3) \) on versors, and secondly as specifying some of the mixed index structure coefficients for the Lie algebra \( \mathfrak{so}(3,3) \). Examining table 1.6 we note that most other mixed index structure coefficients will be identically zero. Using compatible \( T \) notation in particular we observe

\[
T^{k}_{iA} = 0 \quad T^{A}_{ij} = 0 \quad T^{C}_{AB} = 0
\]

(5.31)

However mixed index structure coefficients of the form \( T^{k}_{AB} \) need not be zero and we can put them on our manifold by defining them via

\[
T^{\beta}_{A\lambda} T^{\lambda}_{B\alpha} - T^{\beta}_{B\lambda} T^{\lambda}_{A\alpha} = T^{k}_{AB} T^{\beta}_{k\alpha}
\]

(5.32)

The invariant metric on \( \mathfrak{so}(3,3) \) defined in the usual way to be

\[
g_{\Lambda\Pi} = T^{\beta}_{\Lambda\beta} T^{\beta}_{\Pi\alpha}
\]

(5.33)

where the indices \( \Lambda \) and \( \Pi \) are lower-case or upper-case. This simply evaluates to \( g_{ij} \) on lower-case indices and \( g_{AB} \) on upper-case indices. The mixed terms are all zero.

Note that the totally invariant tensors \( g^{ij}, g_{ij}, g^{AB}, g_{AB} \) can be used to raise and lower vector and versor indices in the obvious way. Contraction with these defines a bijective map which respects both local and global action.
5.4 Tensors with Two Versor Indices

Consider the space \{X^{AB}\} of tensors with two versor indices. This space is 25 dimensional and decomposes into irreducible representations under the local action of dimensions 1, 10 and 14. Since there is no degeneracy in this decomposition each of these irreducibles is preserved by the local as well as the global action and defines a type of tensor. The antisymmetric subspace is 10 dimensional and is irreducible. We can construct a totally invariant operator mapping this component to vectors as follows.

\[ X^{AB} \mapsto g^{ki} g_{AC} T_{iB}^C X^{AB} = x^k \]  

(5.34)

As this map is invariant under the local and global action this identifies the ten dimensional component as being ordinary vectors.

The symmetric tensors with two versor indices form a 15 dimensional space which obviously has the trivial representation as a component. The intertwining operator here is

\[ X^{AB} \mapsto g_{AB} X^{AB} = x \]  

(5.35)

which is invariant. Hence this 1-D component is scalar.

The trace free symmetric tensors with two versor components give the irreducible 14 dimensional representation, as can be confirmed by checking eigenvalues for \( T \) and \( I \). This demonstrates that the representation of \( \mathfrak{so}(2,3) \) with \((q_0, s_0) = (2, 0)\) is local on \( M \).

We will rarely have need to talk about these tensors so we do not reserve an exclusive alphabet for them. Instead we will assign a temporary alphabet at the time of use.

5.5 Tensors with one Versor and one Spinor Index

Consider the space \{X^{A\alpha}\} of tensors with one versor and one spinor index. This is 20 dimensional. There are natural invariant maps

\[ X^{A\alpha} \mapsto T_{\alpha\beta}^{B} X^{A\alpha} = x^{\beta} \]  

(5.36)

\[ x^{\beta} \mapsto g^{AB} T_{B\beta}^{\alpha} x^{\beta} = X^{A\alpha} \]  

(5.37)

which identify a 4-D spinor component. The kernel of the projection onto spinors is 16 dimensional. If \( v^A \) is a versor with weight \((q, s) = (1, 0)\)
and $x^\alpha$ is a spinor with weight $(q, s) = (1/2, 1/2)$ then the tensor product $v^A x^\alpha = X^{A\alpha}$ has weight $(3/2, 1/2)$. Hence this weight must appear as a weight in one of the irreducible components. The only way this can happen is if the entire 16 dimensional kernel of the projection into spinors is the irreducible component with $(q_0, s_0) = (3/2, 3/2)$. So this representation is also local on $M$.

We will rarely have need to talk about these tensors so we do not reserve an exclusive alphabet for them. Instead we will assign a temporary alphabet at the time of use.

5.6 Symmetric 3 and 4 Component Spinors

Consider the space of three component spinors $\{X^{\alpha\beta\gamma}\}$. This is a 64 dimensional space of tensors which splits into a symmetric component of dimension 20, an antisymmetric component of dimension 4, and a mixed symmetry component of dimension 40. We focus our attention here on the totally symmetric component.

If $v^\alpha$ has weight $(q, s) = (1/2, 1/2)$ then the tensor product of three of these $v^\alpha v^\beta v^\gamma$ is totally symmetric and hence lies in our space. We can compute its weight to be $(q, s) = (3/2, 3/2)$ so this must be a weight of one of the irreducible components of the 20 dimensional space of totally symmetric 3 component spinors. But the only way that can happen is if the space of totally symmetric 3 component spinors is irreducible with maximal weight $(q_0, s_0) = (3/2, 3/2)$. Hence this irreducible is also local on $M$.

Similarly the space of symmetric 4-component spinors is 35 dimensional. It includes the tensor $v^\alpha v^\beta v^\gamma v^\delta$ where $v^\alpha$ has weight $(q, s) = (1/2, 1/2)$. This tensor has weight $(q, s) = (2, 2)$ it follows that the space of symmetric 4-component spinors is locally irreducible with maximal weight $(q_0, s_0) = (2, 2)$. Hence this irreducible is local on $M$.

5.7 Tensors with Two Vector Indices

Consider the space $\{X^{ij}\}$ of tensors with two vector indices. This space is 100 dimensional and decomposes into six different irreducibles under the local action. Due to the lack of degeneracy each is globally invariant and we may discover a local and global action for it.

The antisymmetric subspace is 45 dimensional and is the sum of two irreducible components of dimensions 10 and 35 corresponding to $(q_0, s_0) =$
(1, 1) and \((q_0, s_0) = (2, 1)\) respectively. The tensor \(T^k_{ij}\) is totally invariant and defines an intertwining map \(X^{ij} \mapsto T^k_{ij}X^{ij}\) which projects onto the 10-dimensional irreducible component and identifies it as having the correct global and local action to be the space of vectors. The antisymmetric tensors which lie in the kernel of this map give the 35 dimensional irreducible component.

The symmetric subspace is 55 dimensional and is the sum of irreducibles of dimension 1, 5, 14 and 35 corresponding to \((q_0, s_0) = (0, 0), (1, 0), (2, 0)\) and \((2, 2)\) respectively. We seek totally invariant maps which will enable us to perform this decomposition.

The trace map

\[ X^{ij} \mapsto g_{ij}X^{ij} \] (5.38)

is a totally invariant map into the scalar component.

A totally invariant map into the versor component can be constructed using Jordan Brackets.

In equation 5.27 we defined the totally invariant tensor \(g^A_{ij}\) via a Jordan bracket of \(T\) operators.

The map

\[ X^{ij} \mapsto X^{ij}g^A_{ij} \] (5.39)

is a fully invariant map from two component vectors to versors and identifies the versor component.

An intertwining map which takes two dimensional symmetric tensors to two dimensional symmetric versors can be constructed as follows.

\[ s^{AB}_{ij} = T^{\alpha}_{ij}T^{\beta}_{jk}T^{\gamma}_{k\delta}T^{\delta}_{\gamma\alpha}g^{AX}_i g^{BY}_j \] (5.40)

This allows us to use our previous identification of the 14 dimensional irreducible as the space of two dimensional symmetric versors of trace zero to pick out the 14 dimensional component here.

The 35 dimensional component of the symmetric part can be identified as the kernel of all the maps mentioned so far. This representation, which is locally irreducible with maximal weight \((q_0, s_0) = (2, 2)\) is therefore locally equivalent to the space of fully symmetric spinors with four indices. The global actions differ however. We will look at this in more detail once we understand Crump scalars which are introduced in the next chapter.

In section 4.5 we claimed that all the small dimensional representations appearing in figure 1.2 are local on \(M\). We have now proved this by finding
generalised tensors for each of these local actions. This means proposition 4.5.4 applies to indecomposable generalised tensors whose local action consists of these irreducibles. Such generalised tensors can thus be written as a tensor product of the generalised tensors we have found with locally trivial indecomposable generalised tensors.
Chapter 6

Crump Scalars and Beyond

In this chapter we continue our exploration of tensors applicable to physics. We will also continue to develop algebraic techniques and will discuss a number of useful identities.

Crump scalars are an important topic in this chapter. These locally trivial tensors arise naturally from a consideration of the invariant symplectic form. Crump scalars turn out to be particularly important in understanding electromagnetism.

We begin by looking at the Casimir identities which are useful in identifying locally homogeneous components of a tensor.

6.1 Casimir Identities

The quadratic and quartic Casimir invariants for representations of so(2, 3) were introduced in equations 1.2 and 1.4. We can extend these to operators on the manifold which commute not only with the local action but also with the global action.

The operator $g^{ij}T_i(\epsilon)T_j(\epsilon)$ expresses the quadratic Casimir operator $Q$ discussed in the first chapter at every point on our manifold and is both globally and locally locally invariant. By Schur’s lemma it is scalar on every local irreducible. Furthermore its eigenspaces are globally invariant and give a decomposition of any generalised tensor. Hence this operator is scalar on any indecomposable generalised tensor space.
Applying this to specific spaces gives the useful Casimir identities.

\[ g^{ij} T^\beta_{i\lambda} T^\alpha_{j\beta} = \frac{5}{2} T^\alpha_{\alpha} \]
\[ g^{ij} T^B_{i\lambda} T^X_{j\lambda} = 4 T^B_{A} \]
\[ g^{ij} T^b_{i\lambda} T^x_{j\lambda} = 6 T^b_{a} \]

(6.1)

There is a Casimir identity for every finite dimensional irreducible representation. In the general case, using the formula in equation 1.8 we have

\[ g^{ij} T^A_{i\alpha} T^\alpha_{jA} = (q(q + 3) + s(s + 1)) T^A_{A} \]

(6.2)

where the constant is a function of the maximum weight vector \((q, s)\) for the irreducible local representation.

The quartic Casimir operators are also globally invariant and give rise to a similar set of identities. The 4th degree Casimir invariant takes the form

\[ g^{AB} g^{ij} T^A_{i\alpha} T^B_{j\alpha} = (q(q + 3) + s(s + 1)) T^A_{A} \]

(6.3)

and the general identity arising from it is

\[ g^{AB} g^{ij} g^{kl} T^A_{i\alpha} T^B_{j\alpha} T^A_{k\alpha} T^B_{l\alpha} = (q - s + \frac{4}{9} s^2 (q + 1)^2) T^A_{A} \]

(6.4)

where the irreducible representation has maximal weight vector \((q, s)\).

6.2 The Symplectic Form.

In section 5.2 we defined totally invariant bilinear forms

\[ g_{ij} = T^\beta_{i\alpha} T^\alpha_{j\beta} \]
\[ g_{AB} = T^\beta_{A\alpha} T^\alpha_{B\beta} \]

on vectors and versors. We now look at whether we can similarly define an invariant bilinear form on spinors. In the local case we have such a form, the canonical symplectic form defining \(sp(2, \mathbb{R})\) which is unique up to a choice of scalar. Consequently a locally invariant form \(s_{\alpha\beta}\) can be chosen smoothly on the manifold and is unique up to a choice of scalar field. Explicitly, if \(t_{\alpha\beta}\) is another such invariant form then

\[ t_{\alpha\beta} = f(s_{\alpha\beta}) \]

(6.5)
for some scalar field \( f \). Local invariance of \( s_{\alpha\beta} \) gives the equation

\[
T^\lambda_{k\alpha} s_{\lambda\beta} + T^\lambda_{k\beta} s_{\alpha\lambda} = 0 \quad (6.6)
\]

Taking a covariant derivative we obtain

\[
T^\lambda_{k\alpha} \nabla_m (s_{\lambda\beta}) + T^\lambda_{k\beta} \nabla_m (s_{\alpha\lambda}) = 0 \quad (6.7)
\]

Thus \( \nabla_m (s_{\alpha\beta}) \) is also a locally invariant bilinear form for each \( m \). We conclude that there is a vector \( A_m \) with

\[
\nabla_m (s_{\alpha\beta}) = -A_m \cdot s_{\alpha\beta} \quad (6.8)
\]

The negative sign here anticipates results that will follow later. For the dual form defined by \( 1^\alpha_{\beta} = s^{\alpha\lambda} s_{\lambda\beta} = s_{\beta\lambda} s^{\lambda\alpha} \) we will then have

\[
\nabla_m (s^{\alpha\beta}) = A_m \cdot s^{\alpha\beta} \quad (6.9)
\]

Of course the vector field \( A_m \) here depends on the choice of \( s_{\alpha\beta} \). Choose another locally invariant non-degenerate form \( t_{\alpha\beta} \) and denote the associated vector field \( B_m \). Then \( t_{\alpha\beta} \) is a multiple of \( s_{\alpha\beta} \) at every point by a non-zero (due to non-degeneracy) scalar. Assume (with very little loss of generality) that the scalar is positive. Then we may write

\[
t_{\alpha\beta} = e^f s_{\alpha\beta} \quad (6.10)
\]

for some scalar function \( f \). And

\[
\nabla_m (t_{\alpha\beta}) = (\nabla_m (f) - A_m) \cdot t_{\alpha\beta} \quad (6.11)
\]

We conclude that

\[
B_m = A_m - \nabla_m (f) \quad (6.12)
\]

We are interested in the question of whether or not it is possible to choose \( s_{\alpha\beta} \) in such a way that it is globally invariant. We will be able to do this if we can find a scalar function \( f \) solving the differential equation

\[
\nabla_m (f) = A_m \quad (6.13)
\]

If a solution \( f \) to equation \( 6.13 \) exists, then applying \([\nabla_i, \nabla_j] \) to \( f \) gives the necessary condition

\[
\nabla_i (A_j) - \nabla_j (A_i) = T^k_{ij} A_k \quad (6.14)
\]
We define
\[ F_{ij} = \nabla_i(A_j) - \nabla_j(A_i) - T^k_{ij}A_k \]  
so that our condition becomes
\[ F_{ij} = 0 \]

A differential equation like equation (6.13) can be solved by taking line integrals from a fixed point provided that this gives a well defined answer independent of path. For a simply connected region this will be true when line integrals around a small loop are zero, which is true if and only if condition (6.16) holds.

Hence condition (6.16) is both a necessary and sufficient condition that a globally invariant bilinear form \( s_{\alpha\beta} \) exists in a simply connected region.

Having gone as far as we can go via a purely mathematical argument it is tempting at this point to appeal to a physical argument and simply assert that a globally invariant symplectic form should exist. Such a form would be very useful. It would for example allow us to raise and lower spinor indices in an invariant manner. Indeed in the initial development of this theory this was the path we followed.

However the notation above was not chosen by chance. We later will find reason to interpret \( A_i \) as the electromagnetic potential. This makes \( F_{ij} \) the electromagnetic field and condition (6.16) is thus effectively a zero field condition and constrains the physics in an undesirable way.

Hence it isn’t simply a case of not being able to find a globally invariant symplectic form. The physics is telling us that the non-existence of such a form is essential for a proper physical description of electromagnetism.

This link between the electromagnetic field and the lack of a globally invariant symplectic form was discovered by the lead author and William Crump during work on his thesis. An early account of this work can thus be found in [1]. Prior to this revelation we had been falsely assuming that a unique global action should exist for each local action. Since the local action on the symplectic form is trivial we had therefore identified it as scalar and thus globally invariant.

This error meant that the version of Maxwell’s equations we had obtained included a non-physical constraint; effectively a zero field condition. We also wish to thank Dr. Yuri Litvinenko for pointing out that this extra constraint was indeed unphysical and was not simply a gauge condition as we had at first hoped.
6.3 Tensors with Two Spinor Indices

Consider the space \( \{ X^{\alpha\beta} \} \) of tensors with two spinor indices. This space is 16 dimensional and, like the space of spinor transformations, decomposes into irreducible representations under the local action of dimensions 1, 5 and 10. Indeed the space of tensors with two spinor indices and the space of spinor transformations are locally equivalent. Contraction with \( s_{\alpha\beta} \) gives an isomorphism with respect to the local action. This map however does not conserve the global action and so these two types of tensors are not equivalent.

Choose an element (a basis) of the trivial component smoothly at each point and denote it \( e^{\alpha\beta} \). We will use a bullet as our alphabet for this trivial representation (as the space is one dimensional we only need an alphabet with one letter). A projection map onto this component is denoted \( s_{\alpha\beta} \). Hence \( s_{\alpha\beta} e^{\alpha\beta} = 1 \).

We will call tensors of the same type as the trivial component Crump scalars or sometimes bullet scalars after the index used to describe them. I have named them after my student William Crump who demonstrated in his thesis that these are essential for a proper description of electromagnetism. Local and global actions on Crump scalars are then defined to ensure that \( e^{\alpha\beta} \) and \( s_{\alpha\beta} \) are totally invariant.

In the case where our spinors are complex, our Crump scalars will also be complex. In this case we will want to extend the definition of the natural conjugation map on spinors discussed in section 3.4 to Crump scalars. This is done by requiring that the basis elements \( s_{\alpha\beta} \) are invariant under spinor conjugation (in other words are real). Choosing a real Crump scalar as the bullet basis will ensure that conjugation of Crump scalars is simply complex conjugation of the single component.

As the local action on Crump scalars is trivial, we have \( T_{\alpha} = 0 \). Local invariance of \( s_{\alpha\beta} \) then gives

\[
T_{\alpha}^{\lambda} s_{\lambda\beta} + T_{\beta}^{\lambda} s_{\alpha\lambda} = 0
\]

We conclude that \( s_{\alpha\beta} = s_{\alpha\beta} \) is a locally invariant symplectic form. From section 6.2 we can write

\[
\nabla_m (s_{\alpha\beta}) = -A_m \cdot s_{\alpha\beta}
\]

for some vector field \( A_m \). Note however that this equation ignores any effect of the connection on the bullet index which is suppressed in this equation.
By definition we must have $\nabla_m (s_{a\beta}) = 0$ and indeed this condition defines the Crump connection $\Gamma^\bullet_m$ which cannot then be zero. We conclude that the bullet index is subject to non-trivial parallel transport with

$$\Gamma^\bullet_m = A_m$$  \hspace{1cm} (6.19)

Crump scalars $x^\bullet$ are locally trivial and behave like scalars under the local action. However they transform differently under the global action. Note that this is an example of a locally trivial generalised tensor not formed using the trick from section 4.3.

We may take tensor products with Crump scalars to form bullet versions of other tensors. These will have the same local action, but the global action will be modified by the Crump connection.

We can now identify the remaining irreducible components of the two component spinors as bullet vectors $X^{k\bullet}$ and bullet versors $X^{A\bullet}$ respectively via the invariant maps

$$X^{\alpha\beta} \mapsto X^{\alpha\lambda} s_{\lambda\beta} T^\bullet_{\nu\alpha} g^{ik} \hspace{1cm} (6.20)$$

$$X^{\alpha\beta} \mapsto X^{\alpha\lambda} s_{\lambda\beta} T^\bullet_{\nu\alpha} g^{BA} \hspace{1cm} (6.21)$$

projecting onto each component.

### 6.4 The Bullet Index

Additional types of scalar can be obtained from Crump scalars using the dual and tensor product operations. In our notation these would be denoted $x_\bullet$ or $x^{\bullet\bullet}$. Note however that upper and lower bullets appearing together will cancel via what is essentially a process of automatic contraction. We see this for example in equation 6.19.

We define the bullet index to be the number of bullets with the number being negative if the bullets are contravariant. So $x_\bullet$ will have bullet index $-1$ while $x^{\bullet\bullet}$ has bullet index $2$. We may also combine bullets with other alphabets. So for example a quantity like $v^{k\bullet}$ would be described as a vector with bullet index $-2$.

We can extend this definition of bullet index to tensors with a combination of vector, versor, spinor and bullet indices by ignoring the vector and versor indices, and counting each spinor index as half a bullet. Hence the bullet index of $X^{\alpha\beta\gamma}_{k\bullet}$ is $\frac{1}{2}$. This will ensure that contraction with $s_{a\beta}^\bullet$ does not change the bullet index, which is reasonable since it is an intertwining map and does not change the nature of the generalised tensor.
The one element set \( \{1^\bullet\} \) is a basis of the 1-D trivial representation we have chosen in order to quantify Crump scalars. We shouldn’t get too attached to it however as, like all bases, it is somewhat arbitrary. Consider a different basis \( 1^\bullet' \) with change of basis matrices \( \delta^\bullet \) and \( \delta^\bullet' \). Then the change of basis matrix is non-singular everywhere so we can write (after adjusting the sign if needed) \( \delta^\bullet' = e^f \) for some scalar \( f \) and so
\[
s'^{\alpha\beta} = \delta^\bullet s^{\alpha\beta} = e^f s^{\alpha\beta}
\]
which is equation 6.10. We conclude that changing the basis for Crump scalars is equivalent to choosing a different symplectic form, and hence Crump scalars can be regarded as specifying a choice of symplectic form.

To raise and lower spinor indices we would need invariant bilinear forms \( s_{\alpha\beta} \) and \( s^{\alpha\beta} \) where
\[
s^{\alpha\lambda} s_{\beta\lambda} = 1^\alpha_{\beta}
\]
However as we have seen no such invariant forms exist. The symplectic form is in general only locally invariant. To ensure global invariance we must instead use the bilinear form \( s^\bullet_{\alpha\beta} \) which maps a pair of spinors onto a Crump scalar and not to an ordinary scalar. The dual of this will then be \( s^\bullet^\alpha^\beta \) where
\[
s^\bullet^\alpha^\lambda s_{\beta\lambda} = 1^\alpha_{\beta}
\]
(6.23)

It follows that
\[
s^\bullet^\alpha^\beta s_{\alpha\beta} = 4
\]
(6.24)
and that the map \( \Pi^\alpha^\beta = \frac{1}{3} s^\bullet^\alpha^\beta s^\bullet_{\gamma\delta} \) on the set of generalised tensors with two spinor indices satisfies \( \Pi^2 = \Pi \) and hence is a projection. Hence we can identify
\[
s^\bullet^\alpha^\lambda = 4e^\alpha^\beta
\]
(6.25)
where \( e^\alpha^\beta \) is the basis of the bullet component for generalised tensors with two spinor indices specified earlier. Essentially we have adjusted our notation for \( s^\bullet_{\alpha\beta} \) and \( s^\bullet^\alpha^\beta \) to avoid having the constant 4 appear every time we raise and then subsequently lower a spinor index.

Note that raising or lowering a spinor index in this way will leave a bullet behind. Also as these forms are antisymmetric care is needed when raising and lowering to avoid possibly uncertainty over sign. If, when we raise and lower, we place the contracted indices adjacent, then a consistent application of the rule

\textbf{Raise on the Right ; Lower on the Left}
will ensure that raising and lowering are opposite operations.

\[ v^\gamma = (s^a_{\alpha} v^a) s^{\beta \gamma} \]

\[ v_\gamma = s^a_{\gamma} (v_\alpha s^{a \beta}) \]

In section 5.7 we noted that the 35 dimensional irreducible \((q_0, s_0) = (2, 2)\) component of the symmetric two component vectors had a different global action to the 35 dimensional irreducible \((q_0, s_0) = (2, 2)\) space of symmetric four component spinors. Hence they are not the same kind of tensor. We can now describe the nature of this difference which turns out to be simply a difference in bullet index.

We can construct intertwining maps between the space \(\{X_{ij}\}\) of two component vectors and the space \(\{Y_{\alpha \beta \gamma \delta}^{\cdots}\}\) of tensors with four spinor and two bullet indices by

\[
X_{ij} \mapsto X_{ij} T_{i\lambda}^{\alpha} T_{j\mu}^{\beta} s^{\lambda \gamma} s^{\mu \delta} = Y_{\alpha \beta \gamma \delta}^{\cdots}
\]

\[
Y_{\alpha \beta \gamma \delta}^{\cdots} \mapsto s_{\beta \lambda} s_{\delta \mu} Y_{\alpha \beta \gamma \delta}^{\cdots} T_{\alpha \gamma}^{\lambda} T_{\mu \beta}^{\lambda} g^{xi} g^{yj} = X_{ij}
\]

These intertwining maps identify the two component vectors with the four component spinors of bullet index two which are symmetric under the exchange of the first two and last two spinor indices. Schur’s lemma then tells us that these maps give an isomorphism between the 35 dimensional irreducible components.

### 6.5 Scalars and the Zero Curvature Test

Suppose that we come across a type of one dimensional generalised tensor and want to know if it is scalar. Clearly any one dimensional tensor is locally trivial. But as Crump scalars demonstrate, one dimensional tensors need not be globally trivial and therefore scalar.

One way to show that our tensor is indeed scalar would be to find a recognisable intertwining map to scalars. That is what we did in sections 5.4 and 5.7 when we identified the 1-D components of the symmetric tensors with two versor, or with two vector indices as scalar. However not being able to find such an intertwining map does not answer the question. What we need is a definitive test.

Scalars of course are the ones with trivial global action. So it would seem that all we need to do is look at the global action and see if it is
trivial. However, recognizing when the global action is trivial is not easy since the global action is specified by the connection which changes basis in a complicated way. What is clearly trivial in one basis may not appear so in another.

The usual basis for scalars is denoted with the symbol $\circ$, and in that basis we have $\Gamma^\circ_k = 0$. Suppose we have a different basis $\circ'$ with change of basis matrix $\delta^\circ_{\circ'}$ and connection $\Gamma^\circ'_{k\circ'} = B_k$.

As change of basis is an intertwining map we have $\nabla_k (\delta^\circ_{\circ'}) = 0$ which gives

$$\partial_k (\delta^\circ_{\circ'}) = \Gamma^\circ'_{k\circ'} \delta^\circ_{\circ'}$$  \hspace{1cm} (6.28)

Since our change of basis matrix is $1 \times 1$ it is a non-zero scalar function on the manifold and we can write it as either $e^f$ or $-e^f$ for some function $f$. In both cases we have $\partial_k (\delta^\circ_{\circ'}) = \partial_k (f). \delta^\circ_{\circ'}$ and can conclude

$$\Gamma^\circ'_{k\circ'} = B_k = \partial_k (f)$$  \hspace{1cm} (6.29)

So a one-dimensional tensor will be scalar if and only if its connection is the gradient of a function.

A simpler test can be found by looking at the curvature which is a tensor and therefore changes basis in a simple fashion. The curvature of scalars is $R^\circ_{ij} = 0$. It follows that it is zero in any basis and so $R^\circ'_{ij\circ'} = 0$. This gives a necessary condition for a one-dimensional tensor to be scalar which we call the **zero curvature test**. We need

$$\partial_i (B_j) - \partial_j (B_i) = 0$$  \hspace{1cm} (6.30)

if equation (6.29) is to hold. Furthermore, if the manifold is simply connected the zero curvature test is also sufficient. It ensures that equation (6.29) can be solved by path integration with the result independent of path.

The constraint that the manifold is simply connected is a technical one that can be satisfied by restriction to a simply connected region. Alternatively, we could ensure that our entire manifold is simply connected by switching to a simply connected cover.

However, this does raise the interesting possibility of tensors that are scalar in any simply connected region but where there is a global topological obstruction which prevents them from being scalar on the entire manifold.

Now that we have found a test to determine if a representation is scalar it is time to use it by applying it to some one-dimensional tensors. Consider the completely antisymmetric tensors with

- four spinor indices \{\(e^{\alpha\beta\gamma\delta}\)\}.
• five versor indices \( \{k^{ABCDE}\} \).
• ten vector indices \( \{m^{abcdefghij}\} \).

To apply the zero curvature test to these tensors we need to understand the respective connections. The argument in all three cases is similar.

Consider \( \Gamma_k(a^\beta) (e^{\alpha\beta\gamma\delta}) \). Each spinor index can take four values which we choose to represent by the numbers 1 through 4. By antisymmetry it is enough to look at one component. Looking at the 1234 component we have

\[
\Gamma_k(a^\beta) (e^{1234}) = \Gamma_1^k e^{1234} + \Gamma_2^k e^{134} + \Gamma_3^k e^{124} + \Gamma_4^k e^{134} = \Gamma_k^\lambda e^{1234} \quad (6.31)
\]

By antisymmetry the coefficients of \( e^{\alpha\beta\gamma\delta} \) with repeated indices are zero. Hence the only non-zero components in this equation are

\[
\Gamma_k(a^\beta) (e^{1234}) = \Gamma_k^\lambda e^{1234} \quad (6.32)
\]

We conclude that

\[
\Gamma_k(a^\beta) (e^{\alpha\beta\gamma\delta}) = \Gamma_k^\lambda e^{\alpha\beta\gamma\delta} \quad (6.33)
\]

The same argument in the other two cases gives

\[
\Gamma_k(a^\beta) (k^{ABCDE}) = \Gamma_k^M k^{ABCDE} \quad (6.34)
\]
\[
\Gamma_k(a^\beta) (m^{abcdefghij}) = \Gamma_k^t m^{abcdefghij} \quad (6.35)
\]

Knowing the connections allows us to compute the curvature and apply the zero curvature test to check whether these tensors are scalar. The three curvatures are

\[
\partial_i (\Gamma_j^\lambda) - \partial_j (\Gamma_i^\lambda) = R_{ij\lambda} \quad (6.36)
\]
\[
\partial_i (\Gamma_j^M) - \partial_j (\Gamma_i^M) = R_{ijM} \quad (6.37)
\]
\[
\partial_i (\Gamma_j^t) - \partial_j (\Gamma_i^t) = R_{ijt} \quad (6.38)
\]

which we recognise as the traces of the spinor, versor and vector curvature tensors respectively. We will be looking at the nature of these curvature tensors in more detail in the next chapter. Equations 7.8, 7.18 and 7.24 from that chapter show that the first of these expressions is \( F_{ij} \) (not usually zero), while the other two are indeed zero.

We conclude that the completely antisymmetric tensors with four spinor indices are not scalar, but that the completely antisymmetric tensors with five versor or with ten vector indices are scalar.
The fact that completely antisymmetric tensors with ten vector components are scalar has important consequences. We can use these tensors to build differential forms and define a measure and hence an integration on the manifold. Because tensors of this type are scalar we can define this measure and integration to be invariant and to commute with both the local and global action. Such an invariant integration will be useful later when we look at Lagrangian methods.

This leaves us however with a question. If the completely antisymmetric tensors with four spinor indices are not scalar then what are they? Have we found a new type of locally trivial tensor? In the next section we will answer this question by finding intertwining maps that allow us to identify these tensors as two-bullet scalars of the form \( \{x^{**}\} \).

### 6.6 Superalgebra Identities

The basic idea of a Lie superalgebra is that \( T_{ijk}^{i}, T_{i\alpha}^{j}\) and \( T_{i\alpha\beta}^{\lambda} = s_{\alpha\lambda} T_{i\beta}^{k} g^{ij} \) can be viewed as structure coefficients for a \( \mathbb{Z}_2 \)-graded algebra on vectors and spinors which satisfies a ‘super-Jacobi’ identity.

Note however that in doing this we suppressed the bullet index. Putting it back to restore the proper global action breaks the interpretation in terms of a \( \mathbb{Z}_2 \)-graded algebra. Consequently the definition of Lie superalgebra is broken in a fairly fundamental way by the global action and is not natural on a framework.

We can however salvage an interesting identity from the wreckage. The super-Jacobi identity still holds and the various components of the super-Jacobi identity give many identities we are already familiar with, plus one new identity which is the main subject of this section.

**Theorem 6.6.1 (The Superalgebra Identity)**

\[
 s_{\alpha\lambda} T_{i\beta}^{i} T_{i\gamma}^{j} g^{ij} (\alpha\beta\gamma) \equiv 0 \quad (6.39)
\]

Where the notation \( (\alpha\beta\gamma) \) means the left hand side of the expression should be cyclically summed over the permutation \( (\alpha\beta\gamma) \) before being equated to the right.

**Proof** This can be proved by direct computation using the explicit matrices given in chapter 1. Each tensor in the sum on the left is formed as the sum of ten 256 component tensors. Fortunately these tensors are sparse and can
be described by simply listing non-zero components. The indices $\alpha$, $\beta$ and $\gamma$ in this list can then be cyclically permuted and added. The calculation is simple enough to be done by hand, although it would be rather messy to present here.

We can also generalise this result to obtain two further identities. Consider the expressions

$$s_{\alpha\beta}^\ast s_{\gamma\delta}^\ast \epsilon_{\alpha\beta\gamma\delta} = E_{\alpha\beta\gamma\delta}^\ast\ast$$
$$s_{\alpha\lambda}^\ast T^\Lambda_{\alpha\beta} A^B T^\mu_{\mu\gamma B} \epsilon_{\alpha\beta\gamma\delta} = B_{\alpha\beta\gamma\delta}^\ast\ast$$

By examining the effect of transposing two of the spinor indices it is not hard to show that both of these expressions are completely antisymmetric. Hence they are completely determined by one of their coefficients and in particular they differ by a scalar field. Furthermore both of these tensors are invariant. Hence they actually differ by a constant scalar.

Note that the invariance of these tensors also means that we could use them to define an intertwining map between antisymmetric tensors with four spinor indices, and two-bullet scalars.

To understand more we must choose bases and compute. If, with respect to our chosen bases, we define $\epsilon_{\alpha\beta\gamma\delta}$ to be the unique completely antisymmetric tensor with $\epsilon^{1234} = 1$ then we can write

$$E_{\alpha\beta\gamma\delta}^\ast\ast = k_{\alpha\beta\gamma\delta}$$
$$B_{\alpha\beta\gamma\delta}^\ast\ast = c k_{\alpha\beta\gamma\delta}$$

for a suitable two-bullet scalar $k_{\alpha\beta\gamma\delta}$ and constant scalar $c$. Suppose we choose bases so that at that at a specified point $s_{\alpha\beta}^\ast$ takes the form of the matrix $\Omega$ from section 1.2 while the other matrices take the form of the corresponding matrices in chapter 1. In these bases and at that point we can compute $k_{\alpha\beta\gamma\delta} = -1$ and $c = -\frac{5}{4}$.

Since constant scalars are unchanged under a change of basis we will have $c = -\frac{5}{4}$ at any point in any basis.
Chapter 7

Curvature and Forces

In this chapter we look more closely at the connection and the curvature which describe forces in this mathematical context. In particular the methods of the last chapter enable us to decompose these into components with respect to the local action in a natural way. From this decomposition we will obtain dynamical equations for the components of curvature which are very physical. It is worth emphasising that these equations are not assumed. They are mathematical identities that arise naturally and must hold on every framework. There is enough structure here to potentially account for all observed forces however we will focus our attention in this chapter on recognising electromagnetism and gravity.

7.1 Components of Curvature

The curvature $R_{ij}^{\beta}$ can be considered as a set of spinor transformations indexed by $i$ and $j$. We can decompose these into components and write

$$R_{ij}^{\beta} = F_{ij}^{\beta} + R_{ij}^{\alpha} T_{\alpha}^{\beta} + R_{ij}^{A} T_{A}^{\beta}$$  \hspace{1cm} (7.1)$$

In fact we can do much better than this. As we will now show the third component in this decomposition is always zero, while the first can be directly computed from the curvature action on Crump scalars. Our first objective is to prove this result which will follow from the following useful Lemma.
Lemma 7.1.1 Let $M(\ast)$ be an operator defined on both spinors and Crump scalars, and suppose that $M(\ast) \left( s^{\ast}_{\alpha \beta} \right) = 0$. Then

$$M^\alpha_\beta = M^k T^\alpha_{k\beta} + \frac{1}{2} M.1^\alpha_\beta$$

Where $M^k$ is some vector and where $M = M^{\ast}$. Furthermore $M = \frac{1}{2} M^\alpha_{\ast \alpha}$.

Proof Let $h_\ast$ be any non-zero Crump scalar and define $s_{\alpha \beta} = h_\ast s^{\ast}_{\alpha \beta}$. Applying $M(\ast)$ we obtain

$$M(\ast) s_{\alpha \beta} = M(\ast) (h_\ast) s^{\ast}_{\alpha \beta} = -M s_{\alpha \beta}$$

Note that $1(\ast) (s_{\alpha \beta}) = -2s_{\alpha \beta}$. Hence we can rewrite this equation in the form

$$\left( M(\ast) - \frac{1}{2} M.1(\ast) \right) s_{\alpha \beta} = 0$$

This means the symplectic form $s_{\alpha \beta}$ is invariant under the action of $M(\ast) - \frac{1}{2} M.1(\ast)$ and so this operator acts as an element of the Lie algebra $\mathfrak{sp}(2, \mathbb{R})$ under local action. Hence there are coefficients $M^k$ so that

$$M^\alpha_\beta - \frac{1}{2} M.1^\alpha_\beta = M^k T^\alpha_{k\beta}$$

which gives the desired equation.

We already have $M = M^{\ast}$. Contracting $\alpha$ and $\beta$ in equation (7.5) gives us $M = \frac{1}{2} M^\alpha_{\ast \alpha}$ which completes the proof.

Now we consider the defining equation for torsion and curvature

$$[\nabla_i, \nabla_j] = T^k_{ij} \nabla_k + R_{ij}(\ast)$$

and in particular consider what happens when we apply this to the globally invariant tensor $s^{\ast}_{\alpha \beta}$. Since the covariant derivative acts trivially on $s^{\ast}_{\alpha \beta}$, the terms containing a covariant derivative all vanish leaving the equation

$$R_{ij}(\ast) \left( s^{\ast}_{\alpha \beta} \right) = 0$$

It follows from Lemma 7.1.1 that we can write

$$R^\theta_{ij\sigma} = R^\theta_{ij} T^\sigma_{\theta \sigma} + F_{ij} 1^\theta$$

Where $F_{ij} = \frac{1}{2} R^{\ast}_{ij} = \frac{1}{2} R^\alpha_{ij\alpha}$ and hence

$$R^{\ast}_{ij\ast} = 2 F_{ij}$$
We will call the tensor $R_{ij}$ the **reduced curvature tensor**. The tensor $F_{ij}$ giving the scalar component will be called the **field tensor**. Our notation anticipates later results where we will identify this as the electromagnetic field tensor, but we are not quite in a position to reach that conclusion at this stage.

We also conclude that there is no versor component of curvature.

The decomposition above was for the curvature on spinors. But we are also interested in similar decompositions of the curvature $R_{ij}^\alpha$ on other types of objects. We begin by seeking a decomposition of the curvature on vectors, that is of the ordinary Riemannian tensor.

We begin by applying the defining equation, equation 7.6 to the globally invariant tensor $T^{\alpha}_{\alpha l\beta}$. Once again only the curvature term remains giving us

$$R_{ij}^\alpha \left( T^{\alpha}_{k\beta} \right) = 0 \quad (7.10)$$

This gives

$$R_{ijk}^\alpha T^{\alpha}_{k\beta} = R^{\alpha}_{ij\lambda} T^{\lambda}_{k\beta} - R^{\lambda}_{ij\beta} T^{\alpha}_{k\lambda} \quad (7.11)$$

Contracting this equation with $T^{\beta}_{\beta m\alpha}$ gives

$$R_{ijk}^\alpha g_{lm} = R^{\alpha}_{ij\lambda} T^{\lambda}_{k\beta} T^{\beta}_{\beta m\alpha} - R^{\lambda}_{ij\beta} T^{\alpha}_{k\lambda} T^{\beta}_{\beta m\alpha} \quad (7.12)$$

which simplifies to the equation

$$R_{ijk}^\alpha T^{\alpha}_{k\beta} = R^{\alpha}_{ij\lambda} T^{\lambda}_{k\beta} - R^{\lambda}_{ij\beta} T^{\alpha}_{k\lambda} \quad (7.13)$$

It follows that

$$R_{ijk}^\alpha = R^{\alpha}_{ij\lambda} T^{\lambda}_{k\beta} \quad (7.14)$$

We see that the reduced curvature tensor is totally responsible for the curvature of the manifold. The field tensor plays no role here.

We next look at the decomposition of the versor curvature.

The tensors $T^{\alpha}_{A\beta}$ are totally invariant. Applying equation 7.6 to this leaves only the curvature term once again

$$R_{ij}^\alpha \left( T^{\alpha}_{A\beta} \right) = 0 \quad (7.15)$$

which simplifies to the equation

$$R_{ij}^B T^{\alpha}_{B\beta} = R^{\alpha}_{ij\lambda} T^{\lambda}_{A\beta} - R^{\lambda}_{ij\beta} T^{\alpha}_{A\lambda} \quad (7.16)$$
Applying equation \(7.8\) we obtain

\[
R^B_{ijA} T^\alpha_{B\beta} = (R^l_{ij} T^\alpha_{l\lambda} + F^l_{ij} T^\lambda_{1\beta}) T^\alpha_{A\lambda} - (R^l_{ij} T^\lambda_{l\beta} + F^l_{ij} T^\lambda_{1\beta}) T^\alpha_{A\lambda} \quad (7.21)
\]

\[
= R^l_{ij} (T^\alpha_{l\lambda} T^\lambda_{A\beta} - T^\alpha_{l\beta} T^\lambda_{A\lambda}) + F^l_{ij} (1^\alpha T^\lambda_{A\beta} - 1^\lambda T^\alpha_{A\lambda}) \quad (7.22)
\]

\[
= R^l_{ij} T^B T^\alpha_{B\beta} \quad (7.23)
\]

and hence, as the operators \(T^\alpha_{B\beta}\) are linearly independent we must have

\[
R^B_{ijA} = R^l_{ij} T^B \quad (7.24)
\]

Decompositions for the curvature action on other types of generalised tensors can be obtained by similar arguments.

### 7.2 Bianchi Identities Revisited

Bianchi identities arise directly from the Jacobi identity for the action of the covariant derivative. We now understand that curvature arises from the commutator of the covariant derivative and can act on things other than tensors. In particular we have a description of the action of curvature on Crump scalars, spinors and versors. We also can decompose curvature and express it in terms of curvature components. In this section we revisit the Bianchi identities using these new ideas.

Applying the Jacobi identity to the covariant derivative action on vectors gives the two ordinary Bianchi identities we obtained earlier. However we can now express these as identities on the reduced curvature tensor.

\[
R^m_{ij} T^l_{mk} \quad (ijk) = 0 \quad (7.25)
\]

\[
R^l_{im} T^m_{jk} + \nabla_i (R^l_{jk}) \quad (ijk) = 0 \quad (7.26)
\]

Applying the Jacobi identity to the action on Crump scalars gives equation \((7.25)\) again as the first Bianchi identity, and a new second Bianchi identity for the field tensor

\[
F^m_{im} T^m_{jk} + \nabla_i (F^m_{jk}) \quad (ijk) = 0 \quad (7.27)
\]

Applying the Jacobi identity to the covariant derivative action on spinors and separating into components we obtain equation \((7.25)\) again for the first
Bianchi identity. And both equations 7.26 and 7.27 as components for the second Bianchi identity.

Applying the Jacobi identity to the covariant derivative action on versors and separating into components also gives us nothing new. We obtain equation 7.25 for the first Bianchi identity and equation 7.26 as the second Bianchi identity.

We can also further expand the second Bianchi identities by expressing the covariant derivative in terms of the partial derivative and connection. The second Bianchi identity for the Field tensor is particularly interesting in this form, giving

\[ F_{im} T_{jk} + \nabla_i F_{jk} \stackrel{(ijk)}{=} 0 \]

\[ \Rightarrow \]

\[ F_{im} T_{jk} + \partial_i F_{jk} - \Gamma^m_{ij} F_{mk} - \Gamma^m_{ik} F_{jm} \stackrel{(ijk)}{=} 0 \]

\[ \Rightarrow \]

\[ F_{im} T_{jk} + \partial_i F_{jk} + \Gamma^m_{jk} F_{im} - \Gamma^m_{kj} F_{im} \stackrel{(ijk)}{=} 0 \]

\[ \Rightarrow \]

\[ F_{im} T_{jk} + \partial_i (F_{jk}) - (-\Gamma^m_{jk} + \Gamma^m_{kj}) F_{im} \stackrel{(ijk)}{=} 0 \]

The term in brackets is the torsion allowing us to cancel and finally obtain the equation

\[ \partial_i (F_{jk}) \stackrel{(ijk)}{=} 0 \] (7.28)

In this form we easily recognise the Faraday-Gauss equations, (actually a 10-D extension) the source free part of Maxwell equations. In our model these are purely geometric identities.

The second Bianchi identity for the reduced curvature tensor can also be simplified and expressed in terms of partial derivatives.

\[ R^l_{im} T^{m}_{jk} + \nabla_i (R^l_{jk}) \stackrel{(ijk)}{=} 0 \]

\[ \Rightarrow \]

\[ R^l_{im} T^{m}_{jk} + \partial_i (R^l_{jk}) + \Gamma^m_{im} R^l_{jk} - \Gamma^m_{ij} R^l_{mk} - \Gamma^m_{ik} R^l_{jm} \stackrel{(ijk)}{=} 0 \]

\[ \Rightarrow \]

\[ R^l_{im} T^{m}_{jk} + \partial_i (R^l_{jk}) + \Gamma^m_{im} R^l_{jk} + \Gamma^m_{jk} R^l_{im} - \Gamma^m_{kj} R^l_{im} \stackrel{(ijk)}{=} 0 \]

\[ \Rightarrow \]

\[ \partial_i (R^l_{jk}) + \Gamma^m_{im} R^l_{jk} \stackrel{(ijk)}{=} 0 \]

The first Bianchi identity and the definition of curvature can be used to rewrite it in the form.

\[ \partial_i (R^l_{jk}) + R^m_{ij} \Gamma^l_{mk} \stackrel{(ijk)}{=} 0 \] (7.29)
7.3 Connection Components

The connection $\Gamma^\beta_{\kappa\alpha}$ is not a tensor. However the reason why it fails to be a tensor lies in the behaviour under transformation of the index $k$. For fixed $k$ each $\Gamma^\beta_{\kappa\alpha}$ defines a linear transformation of spinors at each point. This can be decomposed into vector, versor and scalar components allowing us to write

$$
\Gamma^\beta_{\kappa\alpha} = A_k 1^\beta_{\alpha} + G^j_k T^\beta_{\alpha j} + N^A_k T^\beta_{A\alpha} 
$$

(7.30)

The three quantities $A_k$, $G^j_k$ and $N^A_k$ are called the scalar, vector and versor connection components respectively. These do not transform as tensors. They should be viewed as facets of the connection.

Contracting with $1^\gamma_{\beta}$, $T^\alpha_{\beta\gamma}$ and $T^A_{B\beta}$ we obtain explicit expressions for the connection components as follows.

$$
A_k = \frac{1}{4} \Gamma^\alpha_{k\alpha} 
$$

(7.31)

$$
N^A_k = \Gamma^\beta_{k\alpha} T^A_{B\beta} g^{BA} 
$$

(7.32)

$$
G^j_k = \Gamma^\alpha_{k\alpha} T^\alpha_{mj} g^{mj} 
$$

(7.33)

To determine the transformation properties for connection components choose different local bases for all quantities involved. We will denote that a new basis has been used by attaching a prime to its index. At every point on the manifold we can define change of basis matrices $\delta^i_j$, $\delta^A_B$ and $\delta^\alpha_\alpha'$; and their inverses $\delta^i_j$, $\delta_A^B$ and $\delta^\alpha'_\alpha$.

Then the spinor connection transforms according to the equation

$$
\Gamma'^{\beta'}_{k'\alpha'} = \delta^\beta_{\beta'} \delta^\alpha_{\alpha'} \delta^k_k \Gamma^\beta_{k\alpha} - \delta^k_k \delta^\lambda_{\alpha'} \partial_k (\delta^\beta_{\lambda}) 
$$

(7.34)

and this allows us to determine the transformation properties of the connection components $A_k$, $N^A_k$ and $G^j_k$. We obtain

$$
A'_k = \delta^k_k A_k - \frac{1}{4} \delta^k_k \delta^\alpha_{\alpha'} \partial_k (\delta^\alpha_{\alpha'}) 
$$

(7.35)

$$
G^{j'}_{k'} = \delta^k_{k'} \delta^j_j G^j_k - \delta^k_k \delta^\alpha_{\alpha'} \partial_k (\delta^\alpha_{\alpha'}) T^\lambda_{\alpha j} g^{lj} \delta^\lambda_j 
$$

(7.36)

$$
N'^A_{k'} = \delta^k_{k'} \delta^A_A N^A_k - \delta^k_k \delta^\alpha_{\alpha'} \partial_k (\delta^\alpha_{\alpha'}) T^\alpha_{BA} g^{AB} \delta^A_A 
$$

(7.37)

Note that if the only change is to the coordinate system of the manifold and in particular if there is no change in the basis of spinor space then the last terms disappear and all three transform as if they were tensors. We can therefore regard them as tensors with a hidden dependence on the choice of spinor basis. We could think of this as a choice of gauge.
7.4 The Field and Potential

We defined the Field tensor \( F_{ij} \) to be the scalar component of the spinor curvature.

\[
R^\alpha_{ij\beta} = \left[ \partial_i (\Gamma^\alpha_{j\beta}) - \partial_j (\Gamma^\alpha_{i\beta}) \right] + \left[ \Gamma^\alpha_{i\lambda} \Gamma^\lambda_{j\beta} - \Gamma^\alpha_{j\lambda} \Gamma^\lambda_{i\beta} \right]
\]

We can obtain an expression for \( F_{ij} \) in terms of connection components by contracting this equation with \( 1^\alpha_{\alpha} \). We obtain

\[
F_{ij} = \partial_i (\Gamma^\alpha_{j\beta}) 1^\beta_{\alpha} - \partial_j (\Gamma^\alpha_{i\beta}) 1^\beta_{\alpha} \tag{7.38}
\]

\[
= \partial_i (\Gamma^\alpha_{j\alpha}) - \partial_j (\Gamma^\alpha_{i\alpha}) \tag{7.39}
\]

\[
= \partial_i A_j - \partial_j A_i \tag{7.40}
\]

Note that we were able to move the identities \( 1^\beta_{\alpha} \) inside the partial derivatives because \( \partial_i (1^\beta_{\alpha}) = 0 \).

Equation (7.40) gives the expected relationship between the electromagnetic field and the electromagnetic potential. This adds weight to the notion that we should interpret the components of \( A_i \) and of \( F_{ij} \) indexed by the four translation dimensions as being the electromagnetic potential and electromagnetic field tensor respectively.

The components of \( A_i \) and \( F_{ij} \) indexed by the Lorentz dimensions must also describe forces and we will need to identify the nature of these at some point. There are several possible explanations. The most likely explanation is that they represent adjustments to the electromagnetic forces to account for things like the flux of spin. However a more intriguing possibility is that they could describe additional forces such as the weak force. If so then demonstrating this will not be an easy matter.

We would also like to find a similar equation linking \( R^k_{ij} \) to curvature components. This proves to be a little more complicated. Before we do this we need to take a deeper look at connection components, specifically the components of connections other than the spinor connection.

7.5 Connections, Curvature and Gauge

At the end of section 7.3 we commented that the hidden dependence of connection components on the choice of spinor basis could be regarded as a choice of gauge. Now that we have discovered the equation relating the field \( F_{ij} \) and the potential \( A_k \), we are able to clarify that remark.
In classical electromagnetism the gauge describes the extent to which the potential $A_k$ can vary without changing the field $F_{ij}$. Consider for example any scalar function $f$ on the manifold and define a new potential $A'_k = A_k + \partial_k(f)$. Applying equation 7.40

$$F'_{ij} = \partial_i(A'_j) - \partial_j(A'_i)$$

$$= F_{ij} + \partial_i\partial_j(f) - \partial_j\partial_i(f)$$

$$= F_{ij}$$

we observe that this change in the potential has no effect on the field. This describes the classical notion of gauge. However we can also view gauge in terms of the effect of a change of basis.

In section 7.3 we noted that under a change of basis $A_k$ transforms as

$$A'_k = \delta'^k_l A_l - \frac{1}{4} \delta'^k_\alpha \partial_k (\delta'^\alpha)$$

and hence it has a hidden dependence on the choice of spinor basis. To see the effects of this hidden dependence let us assume that only the spinor basis is changed. This gives the equation

$$A'_k = A_k - \frac{1}{4} \delta^\alpha \partial_k (\delta'^\alpha)$$

$$= F_{ij}$$

we observe that this change in the potential has no effect on the field. This describes the classical notion of gauge. However we can also view gauge in terms of the effect of a change of basis.

In section 7.3 we noted that under a change of basis $A_k$ transforms as

$$A'_k = \delta'^k_l A_l - \frac{1}{4} \delta'^k_\alpha \partial_k (\delta'^\alpha)$$

and hence it has a hidden dependence on the choice of spinor basis. To see the effects of this hidden dependence let us assume that only the spinor basis is changed. This gives the equation

$$A'_k = A_k - \frac{1}{4} \delta^\alpha \partial_k (\delta'^\alpha)$$

Now $F_{ij}$ is a tensor with no dependence on the spinor basis, hence it will be left unchanged by this change in $A_k$.

The classical abelian gauge is a special case which arises by choosing a basis change of the form $\delta'^\alpha = -4f \delta^\alpha$, where $f$ is a non-zero scalar function on the manifold. This simplifies to give $A'_k = A_k + \partial_k(f)$ exactly as before.

The other connection components $G^i_k$ and $N^A_k$ also have a hidden dependence on the choice of spinor basis and we can also regard these as gauge symmetries as well. However these gauges for $A_k$, for $G^i_k$ and for $N^A_k$ are all related as they simply express the consequence of changing the spinor basis. What we don’t see here is three independent gauge groups each giving rise to a different force.

In the standard model gauge groups are the key to classifying and understanding the different forces. In our approach the gauge groups do not seem to serve this function. We distinguish forces by extracting components of the curvature and the connection instead.
7.6 Vector, Versor and Crump Connections

The indices in the Crump connection $\Gamma^\bullet_{\cdot\cdot}$ auto-contract to give a vector-like quantity which we denote $H_i$. The Crump curvature obtained from the Crump connection also simplifies by auto-contraction to give

$$R^\bullet_{ij\cdot} = \partial_i(H_j) - \partial_j(H_i) \quad (7.45)$$

In equation 7.39 we showed that $R^\bullet_{ij\cdot} = 2F_{ij}$. Hence

$$F_{ij} = \partial_i \left( \frac{1}{2} H_j \right) - \partial_j \left( \frac{1}{2} H_i \right) \quad (7.46)$$

However equation 7.40 states

$$F_{ij} = \partial_i(A_j) - \partial_j(A_i)$$

this suggests a close relationship between $A_i$ and $H_i$. In particular we are tempted to conjecture that perhaps $H_i = 2A_i$. Before we are tempted to proceed down this path however we need to consider their transformation properties since neither is a tensor.

In new bases for vectors, spinors and Crump scalars we have

$$H_i' = \delta^i_i H_i - \partial_i' \left( \delta^\alpha_\alpha' \right) s_{\alpha\beta} \quad (7.47)$$

So $H_i$ transforms as a vector, but has a hidden (gauge) dependence on the choice of basis for Crump scalars. The basis changing behavior of $A_k$ was described earlier in equation 7.35. We have

$$A_{i'} = \delta_{i'}^i A_i - \frac{1}{4} \partial_{i'} (\delta^\alpha_{\alpha'}) \delta^\alpha_{\alpha'}$$

so that $2A_i$ is a vector with a hidden (gauge) dependence on the choice of basis for spinors.

Because $H_i$ and $2A_i$ transform differently they cannot be equal. They are related however, and describing this relationship is our next task.

The equation $\nabla_k(s_{\alpha\beta}) = 0$ gives

$$\partial_k(s_{\alpha\beta}) + H_k s_{\alpha\beta} - \Gamma^\lambda_{k\alpha} s_{\lambda\beta} - \Gamma^\lambda_{k\beta} s_{\alpha\lambda} \quad (7.48)$$

Expanding this out in terms of connection components we obtain

$$\partial_k(s_{\alpha\beta}) + H_k s_{\alpha\beta} = (A_k 1^\lambda_{\alpha} s_{\lambda\beta} + G^m_k T^\lambda_{ma} s_{\lambda\beta} + N_k^A T^\lambda_{Aa} s_{\lambda\beta})$$

$$+ \left( A_k 1^\lambda_{\beta} s_{\alpha\lambda} + G^m_k T^\lambda_{m\beta} s_{\alpha\lambda} + N_k^A T^\lambda_{A\beta} s_{\alpha\lambda} \right) \quad (7.49)$$

$$= 2A_k s_{\alpha\beta} + 2G^m_k (T^\lambda_{ma} s_{\lambda\beta} + T^\lambda_{m\beta} s_{\alpha\lambda})$$

$$+ N_k^A (T^\lambda_{Aa} s_{\lambda\beta} + T^\lambda_{A\beta} s_{\alpha\lambda}) \quad (7.50)$$

$$= 2A_k s_{\alpha\beta} + 2N_k^A T^\lambda_{Aa} s_{\lambda\beta} \quad (7.51)$$
Contracting both sides with $s^{\alpha \beta}$ to extract the scalar component we obtain
\[ H_k = 2A_k - \frac{1}{4} \partial_k (s^{\alpha \beta}) s^{\alpha \beta} \quad (7.52) \]

The last term can be regarded as a basis adjustment term which compensates for the different hidden basis dependencies.

The vector and versor components of equation 7.51 give the identities
\[ \partial_k (s^{\alpha \lambda}) s^{\lambda \beta} T_{m \beta} = 0 \quad (7.53) \]
\[ \partial_k (s^{\alpha \lambda}) s^{\lambda \beta} T_{B \beta}^a = 2N^A_k g_{AB} \quad (7.54) \]

We now consider the versor connection $\Gamma^B_{iA}$. Since versors can be viewed as maps on spinors, we expect to be able to relate this to the spinor connection and its components.

The equation $\nabla_i (T_{A \alpha}^\beta) = 0$ gives
\[ \Gamma^B_{iA} T_{B \alpha}^\beta - \partial_i (T_{A \alpha}^\beta) = \Gamma^\lambda_{iA} T_{A \alpha}^\lambda - \Gamma^\lambda_{iA} T_{A \lambda}^\beta \quad (7.55) \]
\[ = (A_i T_{A \alpha}^\beta + G^i_{\alpha \lambda} T_{A \alpha}^\lambda + N^B_i T_{B \lambda}^\beta T_{A \lambda}) \]
\[ - (A_i T_{A \alpha}^\beta + G^i_{\alpha \lambda} T_{A \alpha}^\lambda + N^B_i T_{B \lambda}^\beta T_{A \lambda}) \quad (7.56) \]
\[ = G^i_{\alpha \lambda} T_{A \alpha}^\lambda - N^B_i T_{A \lambda}^\beta T_{A \lambda} \quad (7.57) \]

Contracting both sides with $T_{C \beta}^\alpha g^{CK}$ we
\[ \Gamma^K_{iA} = G^i_{\alpha \lambda} T_{A \alpha}^\lambda + \partial_i (T_{A \alpha}^\beta) T_{C \beta}^\alpha g^{CK} \quad (7.58) \]

where the last term can be viewed as an adjustment term for the different behavior under a change of basis. To within such an adjustment we see that the vector connection is also essentially determined by the vector component $G^i_{\alpha}$ of the spinor connection.

If we contract instead with the term $T_{m \beta}^\alpha g^{mk}$ we obtain
\[ \partial_i (T_{A \alpha}^\beta) T_{m \beta}^\alpha g^{mk} = N^B_i T_{A \beta}^k \quad (7.59) \]
Yet another interesting equation to add to our collection of interesting identities involving $N^A_i$.

Finally we turn our attention to the vector connection $\Gamma^k_{ij}$ This describes curvature on the manifold and we expect it to describe gravity. However as vectors can be viewed as maps on spinors, we expect the vector connection will also be closely related to the spinor connection and its components.
The equation $\nabla_i(T^\beta_{ja}) = 0$ gives

$$
\Gamma^\beta_{ij} T^\beta_{sa} - \partial_i(T^\beta_{ja}) = \Gamma^\beta_{ia} T^\lambda_{ja} - \Gamma^\lambda_{ja} T^\beta_{ia} \tag{7.60}
$$

$$
= (A_i T^\beta_{ja} + G^i T^\lambda_{ja} T^\lambda_{ja} + N_i A T^\beta_{Aa} T^\lambda_{ja})
- (A_i T^\beta_{ja} + G^i T^\lambda_{ja} T^\lambda_{ja} + N_i A T^\beta_{Aa} T^\lambda_{ja}) \tag{7.61}
$$

$$
= G^i T^s_{lj} T^\beta_{sa} - N_i A T^B_{jA} T^\beta_{B\alpha} \tag{7.62}
$$

Contracting both sides with $T^\alpha_{m\beta} g^{mk}$ we obtain

$$
\Gamma^k_{ij} = G^i T^k_{lj} + \partial_i(T^\beta_{ja}) T^\alpha_{m\beta} g^{mk} \tag{7.63}
$$

where the last term can be viewed as a basis adjustment term compensating for the different hidden basis dependencies. To within such an adjustment we see that the vector connection is essentially determined by the vector component $G^i_1$ of the spinor connection.

If we contracted instead with the term $T^\alpha_{Cj} g^{CK}$ we would obtain the equation

$$
\partial_i(T^\beta_{ja}) T^\alpha_{Cj} g^{CK} = N_i A T^K_{jA} \tag{7.64}
$$

If we set $j = k$ in equation 7.63 we obtain the interesting and useful equation

$$
\Gamma^k_{ik} = \partial_i(T^\beta_{ko}) T^\alpha_{m\beta} g^{mk} \tag{7.65}
$$

$$
= \frac{1}{2} \partial_i(g_{mk}) g^{mk} \tag{7.66}
$$

we could also have obtained this equation directly.

### 7.7 Curvature and Potentials

The curvature is determined by the connection and so the curvature components should be determined by connection components. We have already found one equation of this type, namely equation 7.40 which states $F_{ij} = \partial_i(A_j) - \partial_j(A_i)$. We seek a similar equation for the reduced curvature tensor.

Equation 7.40 was obtained as the scalar component of equation 7.31

$$
R^\alpha_{ij\beta} = \left[ \partial_i(\Gamma^\alpha_{j\beta}) - \partial_j(\Gamma^\alpha_{i\beta}) \right] + \left[ \Gamma^\alpha_{ik} \Gamma^k_{j\beta} - \Gamma^\alpha_{ik} \Gamma^k_{i\beta} \Gamma^\alpha_{j\beta} \right] + \left[ \Gamma^\alpha_{ij} \Gamma^\alpha_{j\beta} \right] \tag{7.31}
$$
One obvious approach might be to expand out every instance of the connection in this equation in terms of connection components via the equation

\[ \Gamma_{k\alpha}^\beta = A_k 1_\alpha^\beta + G_k^i T_{i\alpha}^\beta + N_k^A T_{A\alpha}^\beta \]

Unfortunately this approach is complicated by the fact that partial derivatives \( \partial_i \) and \( \partial_j \) act non-trivially on the tensors \( T_{k\alpha}^\beta \) and \( T_{A\alpha}^\beta \) used to identify components. The resulting extra terms introduce further instances of the connection which must in turn be expanded in terms of connection components.

To avoid this problem it is helpful to restate equation 7.4 using covariant derivatives and work with covariant derivatives as much as possible. The connection \( \Gamma_{j\beta}^\alpha \) is not a generalised tensor, however we can define a formal covariant differentiation on it to be

\[
\nabla_i \left( \Gamma_{j\alpha}^\beta \right) = \partial_i \left( \Gamma_{j\alpha}^\beta \right) + \Gamma_{i\lambda}^\beta \Gamma_j^\lambda - \Gamma_{j\lambda}^\beta \Gamma_{i\alpha}^\lambda - \Gamma_{ij}^\beta \Gamma_{k\alpha}^\beta
\]

We similarly define formal covariant differentiation on connection components. This notation allows us to write equation 7.4 in the form

\[
R_{ij\beta}^\alpha = \left[ \nabla_i (\Gamma_{j\beta}^\alpha) - \nabla_j (\Gamma_{i\beta}^\alpha) \right] - \left[ \Gamma_{i\lambda}^\alpha \Gamma_{j\beta}^\lambda - \Gamma_{j\lambda}^\alpha \Gamma_{i\beta}^\lambda \right] - T_{ij}^k \Gamma_{k\beta}^\alpha
\]

If we use this form of the equation to extract components the complications discussed earlier will not arise.

We already know what the scalar component should look like, but let’s check our method by obtaining it again from equation 7.68. Contracting with \( 1_\alpha^\beta \) we obtain

\[
F_{ij} = \nabla_i (A_j) - \nabla_j (A_i) - T_{ij}^k A_k
\]

Rewriting this using partial derivatives gives equation 7.40 as expected.

Extracting the vector component by contracting equation 7.68 with \( T_{ma} g^{mk} \) gives

\[
R_{ij}^k = \left[ \nabla_i (G_j^k) - \nabla_j (G_i^k) \right] - G_j^x G_i^y T_{xy}^k - N_i^A N_j^B T_{AB}^k - G_m^k T_{ij}^m
\]

This equation is in a form which is easy to state and use since everything in it is well behaved. However while equation 7.70 has many good features in some respects it is less than ideal. The covariant derivative in this equation involves a connection which hides additional connection components. We
might prefer an equation written in terms of partial derivatives where all instances of the connection in the equation appear explicitly.

\[ R^k_{ij} = \left[ \partial_i (G^k_j) - \partial_j (G^k_i) \right] - G^z_i G^y_j T^k_{xy} - N^A_i N^B_j T^k_{AB} \\
+ (\Gamma^k_{ij} G^m_j - \Gamma^k_{jm} G^m_i) \quad (7.71) \]

however this equation involves the vector connection which also involves connection components. We can relate this explicitly to connection components using equation 7.63 however we are then forced to introduce extra terms which were earlier described as basis adjustment terms.

\[ R^k_{ij} = \left[ \partial_i (G^k_j) - \partial_j (G^k_i) \right] + G^z_i G^y_j T^k_{xy} - N^A_i N^B_j T^k_{AB} \\
+ (G^m_j Q^k_{im} - G^m_i Q^k_{jm}) \quad (7.72) \]

Where \( Q^c_{ab} = \partial_a (T^\beta_{b\alpha}) T^s_{\alpha\beta} g^{sc} \) is a basis adjustment term.

In this section we have found three equations that express, in different ways, the gravitational field in terms of potentials; equations 7.70, 7.71 and 7.72. None are ideal. The relationship between field and potential is considerably more complicated for gravitation than it is for electromagnetism.
Chapter 8

Einstein’s Equation

In this chapter we want to see whether Einstein’s equation for gravity is meaningful on a framework. Einstein’s equation is usually stated in terms of the Ricci tensor, a contraction of the curvature tensor. So first we will need to look at contractions of the curvature tensor.

We begin however by looking at some interesting and very useful operator identities.

8.1 Invariant Operators

An inner product space is a vector space equipped with a generalisation of the familiar dot product in $\mathbb{R}^n$. In analogous fashion a Lie algebra can be viewed as a vector space equipped with a generalisation of the familiar cross product in $\mathbb{R}^3$. If viewed in this manner an invariant metric is a generalised dot product compatible with the generalised cross product.

A local Lie manifold has an invariant metric $g_{ij}$ and a Lie algebra structure $T^k_{ij}$. These can be interpreted as equipping the manifold with a generalised cross product $u \bullet v = T^k_{ij} u^i v^j$ and a compatible dot product $u \cdot v = g_{ij} u^i v^j$ defined on tangent vectors. By combining these with the contravariant derivative $\nabla^k$, we can define generalisations of the operators Div Grad and Curl.

The generalised gradient operator

$$\nabla(X) = \nabla^k(X) \quad (8.1)$$

can be applied to any tensor $X$. It is a tensor derivation of rank $(1,0)$ and in particular it maps scalar fields to vector fields on our manifold.
The **Divergence** operator

\[ \nabla \cdot (v) = g_{ij} \nabla^i (v^j) = \nabla_i (v^i) \]  

maps vector fields to scalar fields.

A versor variant of the divergence can be defined by replacing \( g_{ij} \) with the related tensor \( g^A_{ij} \).

The **Curl** operator

\[ \nabla \times (X) = \nabla^i \left( T_i^j(X) \right) \]  

combines the gradient operator with the Lie structure (generalised cross product). It maps tensors to tensors or the same type, and in particular gives a map from vectors to vectors.

In \( \mathbb{R}^3 \) the divergence of the gradient gives the Laplacian operator. This leads us to define a generalised Laplacian

\[ \Box (X) = g_{ij} \nabla^i \nabla^j (X) \]  

A versor variant of the Laplacian can be defined by replacing \( g_{ij} \) with the related tensor \( g^A_{ij} \) giving

\[ \Box^A (X) = g^A_{ij} \nabla^i \nabla^j (X) \]  

The operator

\[ \blacksquare (X) = g_{AB} \Box^A \Box^B (X) = g_{AB} g^A_{ij} g^B_{kl} \nabla^i \nabla^j \nabla^k \nabla^l (X) \]  

is an invariant 4th order operator. The operators \( \Box \) and \( \blacksquare \) relate to the second and fourth degree Casimir operators respectively.

### 8.2 Operator Identities

In this section we examine identities for the divergence, gradient and curl defined in the last section. Many of these identities are suggested by the notation and are complicated to express without it.

The identities we discover are not as simple as those for the ordinary divergence gradient and curl, as we are working on a space which has both torsion and curvature. Furthermore our Curl operator is more general than the others as it acts on tensor fields of all types, and this too will complicate our identities.
Proposition 8.2.1 If $f$ is a scalar on the manifold then
\[ \nabla \times (\nabla f) = -3 \nabla f \]  
(8.7)

Proof
\[ \nabla \times (\nabla f) = g^{js} T_{st}^k \nabla_i (g^{ij} \nabla_j (f)) \]
\[ = g^{js} g^{jt} T_{st}^k \nabla_i \nabla_j (f) \]
\[ = -g^{js} g^{jt} T_{st}^k \nabla_j \nabla_i (f) \]
where the last follows by renaming $i \leftrightarrow j$ and $s \leftrightarrow t$. Averaging these two expressions gives
\[ \nabla \times (\nabla f) = \frac{1}{2} g^{js} g^{jt} T_{st}^k [\nabla_i, \nabla_j] (f) \]
\[ = \frac{1}{2} g^{js} g^{jt} T_{st}^k \nabla_x (f) \]
\[ = \frac{1}{2} g^{js} g^{jt} T_{st}^k \nabla_x (f) \]

however $T_{st}^i T_{ij}^t = -6g_{sj}$ from equation [3.9] hence
\[ \nabla \times (\nabla f) = -3 \nabla f \] as claimed

Proposition 8.2.2 If $v = v^k$ is any vector field then
\[ \nabla \bullet (\nabla \times v) = -3 \nabla \bullet v + 6 R_k v^k \] 
(8.8)

Where $R_k = R_{kt}^t$.

Proof
\[ \nabla \bullet (\nabla \times v) = g^{ij} T_{is}^k \nabla_k \nabla_j (v^s) \]
\[ = -g^{ik} T_{is}^j \nabla_k \nabla_j (v^s) \]
\[ = -g^{ij} T_{is}^k \nabla_k \nabla_j (v^s) \]
where the last follows by renaming $j \leftrightarrow k$. Averaging gives
\[ \nabla \bullet (\nabla \times v) = \frac{1}{2} g^{ij} T_{is}^k [\nabla_i, \nabla_j] (v^s) \]
\[ = \frac{1}{2} g^{ij} T_{is}^k T_{kj}^t \nabla_t (v^s) + \frac{1}{2} g^{ij} T_{is}^k R_{jt}^l T_{tm}^s v^m \]
\[ = -\frac{1}{2} (6.1^s) \nabla_t (v^s) - \frac{1}{2} g^{ij} T_{is}^k (R_{jm}^l T_{tk}^s + R_{mk}^l T_{sj}^t) v^m \]
\[ = -3 \nabla_k (v^k) - \frac{1}{2} g^{ij} (T_{is}^k T_{tk}^s) R_{jm}^l v^m + \frac{1}{2} (g^{ij} T_{is}^k T_{jt}^s) R_{mk}^l v^m \]
\[ = -3 \nabla_k (v^k) - \frac{1}{2} g^{ij} 6g_{kl} R_{jm}^l v^m + \frac{1}{2} 6.1^l R_{mk}^l v^m \]
\[ = -3 \nabla \bullet v + 6 R_k v^k \]
using the first Bianchi and Casimir identities.

\[ ^1 \text{We will later show that } R_k = 0. \]
The combination of the last two propositions is extremely powerful as, for any scalar field \( f \) on our manifold, we can write

\[
\bigtriangledown \bullet (\bigtriangledown \times (\bigtriangledown f)) = -3 \bigtriangledown (f)
\]  

(8.9)

using proposition 8.2.1. But by proposition 8.2.2 we have

\[
\bigtriangledown \bullet (\bigtriangledown \times (\bigtriangledown f)) = -3 \bigtriangleup (f) + 6 R_k \bigtriangledown^k (f)
\]  

(8.10)

We conclude that \( R_k \bigtriangledown^k (f) = 0 \) for all scalar functions \( f \) on our manifold. In particular we can choose \( f \) to be a coordinate function in the neighbourhood of any point, and it follows that all coordinates of \( R_k \) are zero at every point. We have proved the useful identity

\[
R^m_{km} = 0
\]  

(8.11)

We can use this to simplify proposition 8.2.2.

**Proposition 8.2.3** If \( v^k \) is any vector field then

\[
\bigtriangledown \bullet (\bigtriangledown \times v) = -3 \bigtriangledown \bullet v
\]  

(8.12)

### 8.3 Contractions of the Curvature Tensor

In this section we look at various contractions of the reduced curvature tensor \( R^k_{ij} \) with a view to obtaining something like Einstein’s equation. We have already seen that the obvious contraction \( R_k = R^l_{kl} = -R^l_{lk} \) is identically zero. However other contractions can be obtained by using the fully invariant tensors \( g_{ij}, g^{ij} \) and \( T^k_{ij} \).

We define

- **The Ricci Tensor**

\[
R_{ij} = R^x_{iy} T^y_{xj}
\]  

(8.13)

- **The Curvature Scalar**

\[
R = R_{ij} g^{ij}
\]  

(8.14)

Other contractions are also possible although most reduce to one of these. Note that the Ricci tensor is the usual Ricci tensor since

\[
R^k_{ikj} = R^x_{ik} T^x_{kj} = R_{ij}
\]  

(8.15)

The curvature scalar can also be written as

\[
R = R^k_{ij} T^a_{ab} g^{ia} g^{jb} g_{kc} = R^k_{ij} T^k_{ij}
\]  

(8.16)

where \( T^k_{ij} = T^a_{ab} g^{ia} g^{jb} g_{kc} \) is the Lie structure with all indices raised/lowered.

The Bianchi identities give identities on these contractions.
Proposition 8.3.1  Contractions of the Bianchi identities give

\[ R_{ij} = R_{ji} \quad \text{(8.17)} \]

\[ \nabla_k (R^i_j) = 0 \quad \text{(8.18)} \]

\[ 6 R^m_{ij} g_{mk} + R_{im} T^m_{jk} \overset{(ijk)}{=} 0 \quad \text{(8.19)} \]

\[ \nabla_k (R) = 2 \nabla^l (R_{lk}) \quad \text{(8.20)} \]

**Proof** The first Bianchi identity gives

\[ R_{ij} = R^a_{ib} T^b_{aj} = - R^a_{bj} T^b_{ai} = R^a_{ji} T^a_{ib} = R_{ji} \]

hence the Ricci tensor is symmetric as claimed in equation 8.17.

Contracting the second Bianchi identity we have

\[ \nabla_k (R^i_j) + \nabla_i (R^i_j) + \nabla_j (R^i_j) + R^k_{im} T^m_{ij} + R^k_{im} T^m_{jk} + R^k_{jm} T^m_{ki} = 0 \]

This simplifies since \( R^k_{ik} = 0 \) to give

\[ \nabla_k (R^i_j) - R_{ij} + R_{ji} = 0 \]

and the last two terms cancel since the Ricci tensor is symmetric proving equation 8.18.

To prove equation 8.19 we start with the expression on the left and simplify by expanding \( 6 g_{mk} \) and \( R_{ij} \). This allows us to usefully apply the first Bianchi and Jacobi identities.

\[ 6 R^m_{ij} g_{mk} + R_{im} T^m_{jk} = R^m_{ij} T^a_{mk} T^b_{ka} + R^m_{ib} T^b_{am} T^m_{jk} \]

\[ = (R^m_{jb} T^a_{mi} T^b_{ka} + R^m_{bi} T^m_{mj} T^b_{ak}) + (R^a_{ia} T^b_{mk} T^a_{ja} + R^b_{ia} T^b_{mj} T^a_{ka}) \]

\[ = R^a_{ik} T^a_{j} T^b_{i} - R^a_{ix} T^a_{jy} T^b_{z} \]

Cycling the indices \( i, j \) and \( k \) and adding gives equation 8.19.

Finally we have

\[ \nabla_k (R) = g^{ij} T^a_{b} \nabla_i (R^b_{ka}) \]

\[ = - g^{ij} T^a_{b} \nabla_i (R^b_{ka}) - g^{ij} T^b_{a} \nabla_a (R^b_{ki}) \]

\[ - g^{ij} T^a_{b} R^b_{ks} T^s_{ia} - g^{ij} T^a_{b} R^b_{is} T^s_{ak} - g^{ij} T^a_{b} R^b_{as} T^s_{ki} \]

\[ = g^{ij} \nabla_i (R^b_{ka}) + g^{ij} \nabla_a (R^b_{ki}) \]

\[ + g^{ij} T^a_{b} T^a_{i} R^b_{ks} - g^{ij} T^a_{b} R^b_{i} T^s_{ak} - g^{ij} T^a_{b} R^b_{s} T^a_{ik} \]

\[ = 2 g^{ij} \nabla_i (R_{kj}) + 6 g^{is} g_{bi} R^b_{ks} - 2 g^{ij} T^a_{b} R^b_{i} T^s_{ak} \]

\[ = 2 \nabla^j (R_{kj}) + 6 R^s_{ks} + 2 g^{aj} T^a_{b} R^b_{i} T^s_{ak} \]

\[ = 2 \nabla^j (R_{kj}) - 2 g^{aj} R^a_{jk} T^s_{ak} \]
However $-2g^{a_j}R_{s_j}T^s_{a_k} = 2g^{a_a}R_{s_j}T^j_{a_k} = 2g^{a_j}R_{s_j}T^s_{a_k}$ by relabelling $s$ and $j$, hence the last term is identically zero and we conclude

$$\nabla_k(R) = 2\nabla^j(R_{kj})$$

proving equation 8.20.

An immediate consequence of equation 8.20 is the following Corollary which states that the Einstein tensor, if defined on our manifold in the obvious way, is divergence free.

**Corollary 8.3.2**

$$\nabla^l(R_{lk} - \frac{1}{2}g_{lk}R) = 0 \quad (8.21)$$

### 8.4 Einstein’s Equation

Equation 8.21 can be rewritten as

$$R_{ij} - \frac{1}{2}g_{ij}R = \Theta_{ij} \quad (8.22)$$

Where $\Theta_{ij}$ is a divergence free symmetric tensor.

If we choose to identify $\Theta_{ij}$ so that the non-Lorentz components are the stress energy tensor we would then have the Einstein equation, or rather a ten dimensional extension of it.

We must decide whether this identification is physically reasonable. This is not something that can be determined mathematically. One must verify that the equation is consistent with observation.

Einstein’s equation represents the current paradigm for gravitation so is generally regarded as consistent with observation. However there are some issues of concern with regard to anomalous galactic rotation and the need for dark matter and energy to fit observations to the equation.

Also of concern to us is the absence of the anticipated unified equation with electromagnetism.

Before proceeding we note that our equation 8.22 differs from Einstein’s equation in several respects.

The most obvious difference is that our version of the equation is ten dimensional. The manner in which we decide to deal with the additional dimensions then becomes important. Note that our model is explicitly
quantum mechanical - we built it in order to accommodate wave functions on a curved manifold. Those wave functions can be viewed as quantum mechanical trajectories.

The Lorentz components are unimportant for waves which are constant across Lorentz dimensions. Classical point-like particles with well defined energy and momentum are approximated by such wave functions. Hence the Lorentz components can be ignored when passing to the classical (non-quantum) approximation.

Another difference is that the curvature in Einstein’s equation uses as its connection the Christoffel symbols, the unique symmetric connection conserving the metric. Our connection also conserves the metric but has torsion.

The covariant derivative $\nabla_k + \frac{1}{2} T_k(\ast)$ is torsion free and conserves the metric. Hence the associated connection

$$C_{ij}^k = \Gamma_{ij}^k + \frac{1}{2} T_{ij}^k$$

(8.23)

is the Christoffel symbols for the metric $g_{ij}$ as it conserves the metric and is torsion free. If we use these Christoffel symbols to construct a curvature tensor we obtain a torsion free version of the Riemannian.

$$\hat{R}_{ij}(\ast) = \left[ \nabla_i + \frac{1}{2} T_i(\ast), \nabla_j + \frac{1}{2} T_j(\ast) \right]$$

(8.24)

$$\hat{R}_{ij}(\ast) = \left[ \nabla_i, \nabla_j \right] + \left[ \frac{3}{2} T_i(\ast), \nabla_j \right] + \left[ \nabla_i, \frac{1}{2} T_j(\ast) \right] + \left[ \frac{3}{2} T_i(\ast), \frac{1}{2} T_j(\ast) \right]$$

(8.25)

$$\hat{R}_{ij}(\ast) = \left[ \nabla_i, \nabla_j \right] - \frac{1}{2} T_{ij}^k \nabla_k - \frac{1}{2} T_{ij}^k \nabla_k + \frac{1}{4} [T_i(\ast), T_j(\ast)]$$

(8.26)

$$\hat{R}_{ij}(\ast) = R_{ij}(\ast) + \frac{1}{2} T_{ij}^k T_k(\ast)$$

(8.27)

As expected there is no torsion term and we have

$$\hat{R}'_{ijk} = R'_{ijk} + \frac{1}{2} T_{ij}^m T_{mk}$$

(8.28)

where $\hat{R}'_{ijk}$ is the Riemannian obtained from the Christoffel symbols. Contracting to obtain torsion free versions of the Ricci tensor and scalar curvature we obtain

$$\hat{R}_{ij} = R_{ij} - \frac{3}{2} g_{ij}$$

(8.29)

$$\hat{R} = R - 15$$

(8.30)

(8.31)

If we now rewrite equation 8.22 in terms of the torsion free Ricci tensor we obtain

$$\hat{R}_{ij} - \frac{1}{2} g_{ij} \hat{R} - 6 g_{ij} = \Theta_{ij}$$

(8.32)
Which is Einstein’s equation with a cosmological constant of $-6$. Of course this is 6 in natural units. The usual units for the cosmological constant are $m^{-2}$. Switching to those units we obtain $\Lambda = -\frac{6}{r^2}$.

If equation 8.32 is assumed then measurements of the cosmological constant will give information about $r$. As of 2015 the cosmological constant is estimated to be on the order of $|\Lambda| \approx 10^{-52} m^{-2}$. This would give

$$r \approx 2.6 \times 10^{17} \text{ seconds} \simeq 8.2 \text{ billion years} 
(8.33)$$

as our theory has been based on the assumption that $r$ is large it is reassuring that our first estimate of $r$ is indeed large.

Note that $r$ is a structure constant for the geometry. It measures the scale at which torsion becomes significant. In particular although we assigned the catchy name ‘radius of the universe’ to this quantity, we did not intend to relate it to the universe currently estimated at 13.772 ± 0.059 billion years.

This estimate of $r$ depends on assuming that equation 8.32 is the correct source equation for gravity. And that would seem on the face of it to be a very reasonable assumption to make. Einstein’s equation after all is the current paradigm for gravitation, so we would expect to find that the gravitational equation in the context of a framework should resemble it. On these grounds the extended Einstein equation seems to be exactly what we are looking for. However in the next chapter we will discover another quite different looking equation which also seems to be a source equation for gravitation. If we believe that this alternative equation is correct then the calculation of $r$ we have just performed is invalidated.

\footnote{thanks to Prof. Malte Henkel for inviting me to update the estimate used in an earlier draft}
Chapter 9

The Ampere-Gauss equation and Ussher’s equation

We begin this chapter by looking for an appropriate source equation for electromagnetism. Much of this work can be found in [5].

Maxwell’s equations in a relativistic context are given by two equations; a dynamical equation called the Faraday-Gauss equation, and a source equation called the Ampere-Gauss equation.

In our new context a generalised 10-D version of the Faraday-Gauss equation 7.28 arises directly from the Bianchi identities, and hence is a geometric identity on our manifold. So half the work of finding a suitable form of Maxwell’s equations has already been done.

To find the Ampere-Gauss equation however we will need a different argument. The standard Ampere-Gauss equation equates the divergence of the field tensor $F_{ij}$ to a charge-current tensor $J_k$. So we might guess that the form of the equation should be something like

$$\nabla^i F_{ik} = J_k$$  \hspace{1cm} (9.1)

However this is really just a wild guess which we are not very confident about. Should we be using the partial derivative here instead of the covariant one for instance? We are not sure.

In the usual context the source term $J_k$ is divergence free and this hints at a method for finding the correct form of this equation. We need merely look for a divergence free expression on the left hand side. In fact this is precisely the way that we obtained the extended Einstein equation.

We seek an identity of the form $\nabla^k (\text{expression}_k) = 0$, where the expression is some function of the field tensor. We can then write this identity in
the form expression $k = J_k$ where $J_k$ is divergence free. If we interpret $J_k$ as a source term this will give us a source equation.

This kind of argument will still require us to make a physical assumption, namely we must decide on the physical interpretation of $J_k$. But having an argument of this type is at least a little bit more convincing than just guessing at the form of the equation.

An alternative approach would be to use a Lagrangian method. We will look at the Lagrangian approach in chapter 10.

### 9.1 Ussher’s Identity

In this section we seek an identity which we can use to construct a source equation for the field tensor $F_{ij}$. We this identity to construct the same kind of argument as was used in chapter 8 to obtain the generalised Einstein equation, equation 8.22. We begin with some lemmas.

**Lemma 9.1.1**

$$g^{is} g^{jt} R_{st} (\ast) (F_{ij}) = 0 \tag{9.2}$$

**Proof**

$$g^{is} g^{jt} R_{st} (\ast) (F_{ij}) = g^{is} g^{jt} R_{st}^x T_x (\ast) (F_{ij}) = -g^{is} g^{jt} R_{st}^x T_x F_{aj} - g^{is} g^{jt} R_{st}^x T_x F_{ia} = g^{ia} g^{jt} R_{st}^x T_x F_{aj} + g^{ia} g^{jt} R_{st}^x T_x F_{ia} = R_{aj} F_{ia} + R_{ia} F_{aj} = 0$$

as each term consists of a contraction of an antisymmetric tensor with a symmetric one.

An analogous result also holds for the other component of curvature $R^k_{ij}$ and this is the content of our next lemma.

**Lemma 9.1.2**

$$g^{is} g^{jt} R_{st} (\ast) (R^k_{ij}) = 0 \tag{9.3}$$

**Proof**

$$g^{is} g^{jt} R_{st} (\ast) (R^k_{ij}) = g^{is} g^{jt} R_{st}^x T_x (\ast) (R^k_{ij}) = g^{is} g^{jt} R_{st}^x T_x R^k_{aj} - g^{is} g^{jt} R_{st}^x T_x R^k_{ia} - g^{is} g^{jt} R_{st}^x T_x R^k_{aj} = g^{is} g^{jt} R_{st}^x T_x R^k_{ia}$$

We now show that each of these three terms is zero and the result with then follow.
Case 1. Consider the term $g^{is}g^{jt}R^x_{st}T^k_{xa}R^a_{ij}$.

By renaming fully contracted indices $s \leftrightarrow i$, $t \leftrightarrow j$, and $x \leftrightarrow a$ we can make use of the antisymmetry of $T$ to equate this term to its own negative. Hence it is identically zero.

Case 2. Consider the term $-g^{is}g^{jt}R^x_{st}T^a_{xi}R^k_{aj}$.

Applying the first Bianchi identity we obtain

$$-g^{is}g^{jt}R^x_{st}T^a_{xi}R^k_{aj} = g^{is}g^{jt}R^x_{ts}T^a_{ri}R^k_{aj} + g^{is}g^{jt}R^x_{xt}R^a_{ri}R^k_{aj}$$

$$= g^{is}g^{jt}R^x_{ts}T^a_{ri}R^k_{aj} - g^{is}g^{jt}R^x_{xt}R^a_{ri}R^k_{aj}$$

$$= g^{is}g^{jt}R^x_{ts}T^a_{ri}R^k_{aj} + g^{as}g^{jt}R_{ta}R^k_{aj}$$

We now observe that the first term is zero as it involves a contraction of the symmetric tensor $g^{is}$ with the antisymmetric tensor $R^x_{ts}$, while the second term is zero as it includes a contraction of the symmetric tensor $R^a_{ri}$ with the antisymmetric tensor $R^k_{aj}$.

Case 3. Consider the term $-g^{is}g^{jt}R^x_{st}T^a_{xt}R^k_{aj}$.

By renaming fully contracted indices $i \leftrightarrow j$ and $s \leftrightarrow t$ this reduces to the previous case.

Hence all three terms are zero completing the proof.

Lemma 9.1.3

$$g^{si}g^{aj}R_{st}(\alpha) (R^\alpha_{ij}) = 0 \quad (9.4)$$

Proof: Simply write $R^\beta_{ij} = F_{ij}1^\beta_\alpha + R^k_{ij}T^\beta_{ka}$ and apply the tensor derivation $R_{st}(\alpha) = R^\beta_{st}T^\alpha_{\beta}$). The result follows from the previous two lemmas and from the local invariance of $T^\beta_{ka}$ and $1^\beta_\alpha$.

Theorem 9.1.4 (Ussher’s Identity)

$$\nabla^i(\nabla^j R^\beta_{jia} - \frac{1}{2}g^{is}T^\alpha_{ri}R^\beta_{ras}) = 0 \quad (9.5)$$

hence in particular

$$\nabla^i(\nabla^j R^k_{jii} - \frac{1}{2}g^{is}T^\alpha_{ri}R^k_{ras}) = 0 \quad (9.6)$$

$$\nabla^i(\nabla^j F_{jii} - \frac{1}{2}g^{is}T^\alpha_{ri}F_{ras}) = 0 \quad (9.7)$$
**Proof** Consider the double divergence of the curvature tensor $\nabla^i \nabla^j (R^\beta_{ij\alpha})$. By renaming $i \leftrightarrow j$ and using the antisymmetry of $R$ we obtain

\begin{align}
2\nabla^i \nabla^j (R^\beta_{ij\alpha}) &= [\nabla^i, \nabla^j] (R^\beta_{ij\alpha}) \\
&= g^{is}g^{jt} T^s_{it} \nabla_x (R^\beta_{ij\alpha}) + g^{is}g^{jt} R_{st}(\star) (R^\beta_{ij\alpha})
\end{align}

where the second term is zero by lemma 9.1.3. Relabelling and rearranging we can write this in the form

\begin{align}
\nabla^i \left( \nabla^j R^\beta_{jia} - \frac{1}{2} g^{ts} T^r_{it} R^\beta_{rsa} \right) &= 0
\end{align}

thereby proving equation 9.5. Equations 9.6 and 9.7 now follow by taking components.

As an immediate consequence of Ussher's identity we can write

**Corollary 9.1.5**

\begin{align}
\nabla^j R^\beta_{jia} - \frac{1}{2} g^{ts} T^r_{it} R^\beta_{rsa} &= \Theta^\beta_{ia}
\end{align}

Where the divergence $\nabla^i \Theta^\beta_{ia} = 0$.

Ussher's identity is a purely geometric identity. It must hold on any Lie manifold obeying the very weak assumptions of the model. We cannot choose to disregard it but we can choose how to interpret it. In particular we must decide whether it is reasonable to interpret $\Theta^\beta_{ia}$ as a source term determined from the distribution of matter.

### 9.2 Extended Ampere-Gauss Equation

Ussher's identity is the right kind of identity to give source equations. In particular we hope to find the Ampere-Gauss equation by looking at the scalar component. So we need to start by separating this equation into component equations. We begin by writing $\Theta^\beta_{ia}$ in terms of its components.

\begin{align}
\Theta^\beta_{ia} = J^\beta_i + K^k_i T^\beta_{ka}
\end{align}

Note that equation 9.10 implies there is no versor component. The equation $\nabla^i \Theta^\beta_{ia} = 0$ now gives two component equations $\nabla^i J^\beta_i = 0$ and $\nabla^i K^k_i = 0$.

The scalar component of equation 9.10 is

\begin{align}
\nabla^j F_{ji} - \frac{1}{2} g^{ts} T^r_{it} F_{rs} &= J^\beta_i
\end{align}
which is in the right form to be a suitable generalisation of the Ampere-
Gauss equation and to act as a source equation for the field tensor $F_{ij}$. We
will call this the **extended Ampere-Gauss equation**.

Note that this equation seems to have an extra term when compared to
our initial guess, equation 9.1. This extra term is not a problem for three
reasons.

1. $T^i_{kb}$ is zero when all three indices $i$, $k$ and $b$ are non-Lorentz. Hence
   this extra term will have no effect on the behavior of the translation
   components which give the classical fields.

2. All spacetime components of the extra term vanish in the Poincaré
   limit and are insignificant over distance and time scales less than $r$.

3. If we rewrite the equation in terms of the torsion free covariant deriva-
tive $\tilde{\nabla}_k = \nabla_k + \frac{1}{2} T_k(\cdot)$ it becomes simply

   $$g^{ij} \tilde{\nabla}_i F_{jk} = J_k$$

   (9.13)

So the extra term is simply a consequence of writing the equation
using a covariant derivative with torsion.

This achieves our objective of obtaining a plausible source equation for
electromagnetism. However we obtained it as a component of a larger
equation, and we now have a leftover component which we must decide
what to do with. We need to take a very hard look at that component
equation because if we accept the Ampere-Gauss equation it will be difficult
to argue that we should not also accept this other component.

### 9.3 Ussher’s equation.

The vector component of equation 9.10 is

$$\nabla^j R^k_{ji} - \frac{1}{2} T^j_{is} g^{st} R^k_{jt} = K^k_i$$

(9.14)

which looks like a source equation for the gravitational field described
given by the reduced curvature tensor $R^k_{ij}$. That is unexpected and is very
interesting. This equation is claiming to do the same job as Einstein’s equa-
tion; it can be viewed as a source equation for gravity. Is that reasonable?
Which source equation should we be using?
In our picture, the gravitational and electromagnetic fields are components of the overall field specified by the curvature tensor $R_{ij\beta}$. It is entirely reasonable therefore to expect that the source equations for electromagnetism and gravity should be the components of a combined source equation for the curvature tensor.

Equations 9.14 and 9.12 are indeed components of equation 9.10 exactly as we would expect. But Einstein’s equation 8.22 does not seem to unify with a source equation for electromagnetism in this way.

The form of equation 9.14

$$\nabla^j R_{ji}^k - \frac{1}{2} T^j_{is} g^{st} R_{jt}^k = K^k_i$$

where the divergence of a field is related to the presence of a source, is actually typical of what we would expect the source equation for a field to look like. Einstein’s equation is the odd one out. It doesn’t fit this pattern.

If we were to ignore history and imagine that we were meeting these equations for the first time, we would probably regard equation 9.14 as a stronger candidate for a source equation for gravity than Einstein’s equation for these reasons.

Of course we still do have Einstein’s equation, 8.22

$$R_{ij} - \frac{1}{2} g_{ij} R = \Theta_{ij}$$

as a geometric identity on our manifold. Indeed equations 8.22 and 9.14 are both geometric identities on our manifold. It therefore isn’t a matter of choosing one of these equations over the other equation. Rather the issue is how we decide how to interpret the two divergence free quantities $\Theta_{ij}$ and $K^k_i$. The question is which of these quantities, if any, is to be interpreted as a source term determined by the distribution of matter.

The ultimate test is of course whether equation 9.14 is consistent with the experimental tests of general relativity which have been carried out to date. This may not be as unlikely as one would think given that Ussher’s equation and Einsteins equation look very different. Einstein’s equation is notoriously difficult to test. Most experimental verifications have involved looking at small fields in the absence of sources, so we need simply show that the two equations are compatible in those limited circumstances.

One way to validate equation 9.14 would be to relate it in some way to Einstein’s equation, the existing paradigm for gravity. But if the two equations make slightly different predictions we won’t succeed in linking them exactly via a mathematical identity. We would need to use some kind of approximation argument and these are often difficult to construct.
Furthermore there are instances such as anomalous galactic rotation where Einstein’s equation does not seem to do very well. Assorted dark quantities have been conjectured to explain the difference between prediction and observation in these instances. Our inability to directly observe these dark things or indeed to understand their nature is however embarrassing, and gives us reason to suspect that Einstein’s equations might not be the final answer to the question of gravitation, even in the weak field case. For this reason perhaps trying to link Ussher’s equation directly to Einstein’s equation is not the best approach.

It would instead be simpler and indeed more certain to seek direct validation from observation. That would require us to construct the equivalent of Schwarzschild and Kerr solutions for equation 9.14 which would then allow direct validation from experimental data.

We will not attempt to construct specific solutions, however we will comment briefly on the issues that arise in attempting to do so. We begin by noting the similarity between the equations for gravity in the absence of sources

\[ R_{ia} - \nabla_i R_{jk} + \nabla^i R_{ik} - \frac{1}{2} R_{ij} g^{jk} T_{kt} = 0 \]

and the equations for electromagnetism in the absence of sources

\[ F_{ia} - \nabla_i F_{jk} + \nabla^i F_{ik} - \frac{1}{2} F_{ij} g^{jk} T_{kt} = 0 \]

For each fixed value of \( m \) in the first set of equations the resulting tensor satisfies the second set of equations. Hence solutions to the gravitational equations consist of ten solutions to Maxwells equations, one for each value of \( m \). These are not independent but must be compatible with the first Bianchi identity.

\[ R_{ij} T_{mk} = 0 \]

This suggests that we might be able to construct a Schwartzchild type solution for the gravitational equations from central charge solutions to Maxwell’s equations.

However things are not as simple as they seem. The problem of course is that the covariant derivative in the above equations involves the connection which also defines the curvature. Hence the equations above are considerably more complicated than they appear and finding specific solutions is likely to be difficult. This looks like a job for an applied mathematician.
Finally of course there is the issue of how we deal with the extra six dimensions. How these are perceived will depend on the nature of the matter on which they act. We do not envisage a universe filled with pointlike particles whose trajectories are described by geodesics across all ten dimensions of the manifold. Instead matter will be described by a spinor wave function satisfying a wave equation.

As we will see in chapter 11, electrons seem to be described by wave functions that are constant (at least locally) across the Lorentz dimensions. For this reason we expect that electrons will be insensitive to what is happening there. We also expect protons to be similar to electrons in this respect. Hence it seems likely that components of curvature involving the Lorentz dimensions will exert only a second order effect on ordinary matter. To first order we should be able to ignore the extra dimensions and just look at what is happening in the first four.

9.4 Further Identities

The tensors in equation 9.14 satisfy a number of interesting algebraic constraints. These may also prove useful in our later work. To facilitate exploring these identities we define

$$U_{km}^j = \nabla^j R_{jm}^k - \frac{1}{2} T_{ms}^j g_{st} R_{kt}^s$$  \hspace{1cm} (9.15)

Hence Ussher’s equation 9.14 takes the form

$$U_{km}^j = K_{km}^j$$

where the left hand side is a tensor related to curvature and constraining gravity as described in equation 9.15 while the right hand side is a tensor related to the distribution of energy and matter, and acting as a source.

The tensor $U_{km}^j$ satisfies a number of algebraic identities which must therefore also constrain $K_{km}^j$.

**Lemma 9.4.1** The tensor $U_{km}^j$ satisfies

$$\nabla^m U_{mk}^j = 0$$  \hspace{1cm} (9.16)

$$U_{km}^j = \frac{1}{2} R$$  \hspace{1cm} (9.17)

$$T_{ik}^m U_{mk}^k = -\nabla^j R_{ji}$$  \hspace{1cm} (9.18)

$$T_{ik}^m U_{mk}^k = -\frac{1}{2} \nabla^i R$$  \hspace{1cm} (9.19)

$$T_{ik}^m U_{mk}^k = -\nabla^i U_{km}^j$$  \hspace{1cm} (9.20)

$$\nabla_k U_{mk}^j + T_{mb}^m U_{bn}^j = g^{ij} R_{ik}^l T_{lm}^s R_{js}^k$$  \hspace{1cm} (9.21)
Proof Equation 9.16 is simply Ussher’s identity 9.6.

Contracting the two indices in equation 9.15 gives the equation

\[ U^k_k = \nabla^j R_j - \frac{1}{2} g^{ts} T^r_{st} R^k_{rs} \]

The curvature vector \( R_j \) is identically zero while the other term on the right hand side simplifies to give equation 9.17

Next contract equation 9.15 with \( T^a_{ab} \) to give

\[ \nabla^j R^k_{ji} T^a_{ab} - \frac{1}{2} g^{ts} T^r_{st} R^k_{rs} T^a_{rb} = U^k_a T^a_{kb} \]

Applying the 1st Bianchi identity to the term in brackets we obtain

\[ \nabla^j R^k_{ji} T^a_{ab} + \frac{1}{2} g^{ts} T^r_{st} R^k_{sr} T^a_{rb} = U^k_a T^a_{kb} \]

By swapping the contracted indices \( s \) and \( r \) in the last term on the left, we see that this term is the same as the one before it giving the equation

\[ \nabla^j R^k_{ji} T^a_{ab} + \frac{1}{2} g^{ts} T^r_{st} R^k_{sr} T^a_{rb} = U^k_a T^a_{kb} \]

contracting the indices \( a \) and \( i \) in this equation we thus obtain

\[ \nabla^j R^k_{jb} = U^k_a T^a_{kb} \]

which rearranges to give 9.18. Equation 8.21 can now be applied to obtain equation 9.19.

Equation 9.20 follows directly from equations 9.17 and 9.19.

Starting again from the definition we have

\[ \nabla_k U^k_m = g^{ij} \nabla_k \nabla_i R^k_{jm} - \frac{1}{2} T^j_{ms} g^{st} \nabla_k R^k_{jt} \]

The last term vanishes as \( \nabla_k R^k_{jt} = 0 \) by equation 8.18. We would also like to exploit this fact in the first term which requires us to commute the two covariant derivatives. We obtain

\[ \nabla_k U^k_m = g^{ij} [\nabla_k, \nabla_i] R^k_{jm} + g^{ij} \nabla_i \nabla_k R^k_{jm} \]

where the last term will vanish once again by equation 8.18. Hence

\[ \nabla_k U^k_m = g^{ij} T^k_{ki} \nabla_i R^k_{jm} + g^{ij} R^k_{ki} (T^k_{ls} R^s_{jm} - T^s_{jm} R^k_{ls} - T^s_{im} R^k_{js}) \]

\[ = g^{ij} \nabla_i (T^k_{ij} R^k_{jm}) + g^{ij} (R^k_{ki} T^k_{ij}) R^k_{jm} + g^{ij} (R^k_{ki} T^k_{ij}) R^k_{im} - g^{ij} R^k_{ki} T^s_{ms} R^k_{js} \]

\[ = g^{ij} \nabla_i R^k_{im} - g^{ij} R^k_{is} R^s_{jm} + g^{ij} R^k_{kj} R^k_{sm} - g^{ij} R^k_{ki} T^s_{ms} R^k_{js} \]

\[ = \nabla^i R^k_{im} + g^{ij} R^k_{ik} T^s_{im} R^k_{js} \]

Applying 9.18 and rearranging we obtain 9.21.
Chapter 10

Lagrangian Methods and Forces

In this chapter we use Lagrangian methods to find source equations for the forces. Since electromagnetism and gravity are simply components of curvature we might expect to find a single source equation for curvature that gives equations for electromagnetism and gravity as its components. We begin by looking at the mathematical machinery required to apply a Lagrangian method.

10.1 Integration and Stokes’ Theorem

Lagrangian methods depend on Stokes theorem which relates the integral in an oriented simply connected\(^1\) compact region \(\Omega\) to an integral on its boundary \(\partial \Omega\).

\[
\int_{\Omega} \nabla_k v^k \, dx^o = \int_{\partial \Omega} v^k \, dx_k^o \tag{10.1}
\]

Stokes’ theorem is a more general result, but this special case written in terms of the natural measure and integration on our framework is sufficient to our purpose.

Integration is the measure weighted summation of values at different points. To add values at different points we must be able to compare them. To compare values at different points we need a notion of parallel transport, which we do have on a framework.

The integral on a framework is naturally defined by partitioning the region of integration into small pieces; computing a value for the integrand

\(^1\)assuming simply connected will simplify the discussion. We will only have need to integrate on regions of this type
at each piece by combining the quantity to be integrated with the measure of the piece; and then parallel transporting these values to a single location to be added. The integral is a limit obtained by refining the partition. We omit the technical details of the limiting process.

For this procedure to be well defined, parallel transport of values from one point to another must give the same answer regardless of path. Equivalently parallel transport of these values around a loop must be trivial. If the region of integration is simply connected it is sufficient that this be the case for small loops. Hence the associated curvature is zero and via the same arguments that were used in section [6.5] there must exist a basis in which the global action on these values is trivial and the associated connection is zero.

So this kind of natural integration is only well defined where the integrand parallel transports in the same way as a scalar. Note that both integrands in equation 10.1 are scalar.

A measure on the manifold is defined by specifying its value on a small parallelepiped. Consider the 10-D parallelepiped at a point \( p \) spanned by the ordered set of small vectors \( \{dx_0^i, \ldots, dx_9^i\} \) (indexing from 0 to 9 to avoid double digits). We define its measure to be

\[
dx_0 \wedge \cdots \wedge dx_9 = \eta_{i_0 \cdots i_9} dx_0^{i_0} \cdots dx_9^{i_9}\tag{10.2}
\]

where \( \eta_{i_0 \cdots i_9} \) is a completely antisymmetric tensor with ten indices. Different measures will arise by choosing different antisymmetric tensors. One obvious choice is to pick \( \eta_{i_0 \cdots i_9} \) so that all non-zero entries are \( \pm 1 \). This has the advantage that it is easy to specify given a basis and that \( \partial_k \eta_{i_0 \cdots i_9} = 0 \). However this is basis dependent.

The completely antisymmetric tensors with ten indices are scalar and can be written as scalar functions via a change of basis. Conversely any scalar function gives a completely antisymmetric tensor with ten indices and will define a measure. If we start with a constant function then the measure we obtain will be invariant under parallel transport. Such a measure is called a Haar measure and will be denoted \( dx^\circ \).

Every framework is equipped with a Haar measure which is unique up to a choice of scale. The scale is unimportant for Lagrangian calculations. To find the Haar measure we use the fact that all completely antisymmetric ten dimensional tensors differ by a scalar function. Hence we can write

\[
dx^\circ = \epsilon \eta_{i_0 \cdots i_9} dx_0^{i_0} \cdots dx_9^{i_9}\tag{10.3}
\]

where \( \eta_{i_0 \cdots i_9} \) has entries \( \pm 1 \) and where \( \epsilon \) is a scalar function chosen so that \( \eta^\circ = \epsilon \eta_{i_0 \cdots i_9} \) is constant. Since \( \eta^\circ \) is constant we must have \( \nabla_k \eta^\circ = 0 \).
Since $\partial_k \eta_{i_0 \cdots i_9} = 0$ this simplifies to give

$$\partial_k \epsilon = \epsilon \Gamma^a_{ka}$$ \hspace{1cm} (10.4)

which determines $\epsilon$ uniquely to within a choice of scale.

The Haar measure can also be defined from the metric. Choose oriented bases so that $T^a_{ij}$ look like the matrices in table 1.4 on page 8. Then the Haar measure of a parallelepiped spanned by small vectors $dx_k$ along the axes in this basis can be defined as the product of their lengths computed from the metric. The invariance of the metric means this construction will be invariant. The $\sqrt{g}$ which appears in some notations for the integral refers to this procedure.

We now turn to the right hand side of equation 10.1 which involves integration on the nine dimensional boundary of our region. A measure on a nine dimensional surface is defined as

$$dx_1 \wedge \cdots \wedge dx_9 = \eta_{i_1 \cdots i_9} \ dx_1^{i_1} \cdots dx_9^{i_9}$$ \hspace{1cm} (10.5)

where $\eta_{i_1 \cdots i_9}$ is a completely antisymmetric tensor with nine components. The completely antisymmetric tensors with nine components are vectors. Indeed we can use a completely antisymmetric tensor with ten components as an intertwining map

$$m^k \mapsto m^k \eta_{k \ i_1 \cdots i_9} = \eta_{(i_1 \cdots i_9)}$$ \hspace{1cm} (10.6)

If we use the completely antisymmetric tensor chosen to construct the Haar measure to do this then the measure given by $\eta_{i_1 \cdots i_9}$ will parallel transport as a vector. We denote this measure $dx_k^\circ$ where the superscript reminds us that we used the scalar Haar measure to define it, and the subscript reminds us that it transforms as a vector. We can also write

$$dx_k^\circ = \epsilon \eta_{k \ i_1 \cdots i_9} \ dx_1^{i_1} \cdots dx_9^{i_9}$$ \hspace{1cm} (10.7)

where $\epsilon$ is determined by equation 10.4, and $\eta_{i_0 \cdots i_9}$ is the completely antisymmetric tensor with values $\pm 1$.

Integration is only well defined where the integrand parallel transports as a scalar. Hence the vector measure on the boundary must combine with other quantities in the integral to form a scalar integrand. We observe this in equation 10.1. The parallel transport in the integral can be around the boundary. However we will also get the same result by parallel transporting across the interior of the region.

Apart from the unusual notation which was chosen to relate to the natural structures on a framework, equation 10.1 is simply the standard Stokes’ theorem.
10.2 Lagrange Methods: Electromagnetism

A Lagrangian approach uses a scalar function $\mathcal{L}$ known as the Lagrangian density. Obtaining this function is an art. The Lagrangian density is a function of quantities on the manifold for which we are seeking some kind of dynamical equation. The Lagrangian hypothesis is that these quantities will arrange themselves so that the integral of the Lagrangian density is extremal under any small variation on a compact region.

Consider for example the Lagrangian density

$$\mathcal{L}_e = F_{ij} F^{ij} = g^{ia} g^{jb} F_{ij} F_{ab} \quad (10.8)$$

which is the usual Lagrangian density for electromagnetism, albeit with a few extra dimensions. This is a function of the field tensor which in turn is a function of the potential $A_i$. Equation 7.69 on page 81 gives

$$F_{ij} = \nabla_i A_j - \nabla_j A_i - T^k_{ij} A_k$$

We prefer this rather than equation 7.40 as we prefer to work with the covariant derivative.

Assume now that $A_i$ undergoes a variation $A_i \mapsto A_i + \delta A_i$, where $\delta A_i$ is small and is zero outside the interior of a compact region $\Omega$. In particular $\delta A_i = 0$ on the boundary $\partial \Omega$. Then other quantities that depend on $A_i$ will also vary and in particular $F_{ij}$ and the Lagrangian density $\mathcal{L}_e$ will also vary. Note that other quantities like $g_{ij}$ which do not depend on $A_i$ will not vary. To first order the variation in $F_{ij}$ is

$$\delta F_{ij} = \nabla_i \delta A_j - \nabla_j \delta A_i - T^k_{ij} \delta A_k \quad (10.9)$$

while the variation in $\mathcal{L}_e$ is

$$\delta \mathcal{L}_e = g^{ia} g^{jb} \delta F_{ij} F_{ab} + g^{ia} g^{jb} F_{ij} \delta F_{ab} = 2 g^{ia} g^{jb} \delta F_{ij} F_{ab} \quad (10.10)$$

the Lagrangian principle now states that

$$\mathcal{L}_e = \int_\Omega \mathcal{L}_e \, dx^0 \quad (10.11)$$

is extremal under this variation. That is we have $\delta \mathcal{L}_e = 0$ to first order.
in $\delta A_i$. But

$$\delta L_e = \int_\Omega 2F^{ij}\delta F_{ij} \, dx^o$$

$$= \int_\Omega 2F^{ij}\nabla_i\delta A_j - 2F^{ij}\nabla_j\delta A_i - 2F^{ij}T^k_{ij}\delta A_k \, dx^o$$

$$= \int_\Omega 4F^{ij}\nabla_i\delta A_j - 2F^{ij}T^k_{ij}\delta A_k \, dx^o$$

$$= \int_\Omega -4\nabla_iF^{ij}\delta A_j - 2F^{ij}T^k_{ij}\delta A_k + 4\nabla_i(F^{ij}\delta A_j) \, dx^o$$

$$= \int_\Omega -4\nabla_iF^{ij}\delta A_j - 2F^{ij}T^k_{ij}\delta A_k \, dx^o + 4\int_{\partial\Omega} F^{ij}\delta A_j \, dx^i$$

The integral on the boundary $\partial\Omega$ is zero because the variation is zero there. Thus

$$\delta L_e = \int_\Omega (-4\nabla_iF^{ik} - 2F^{ij}T^k_{ij}) \delta A_k \, dx^o = 0 \quad (10.12)$$

and since this must be true for any variation, we must have

$$2\nabla_iF^{ik} + F^{ij}T^k_{ij} = 0 \quad (10.13)$$

This is interpreted as the dynamic equation for electromagnetic fields in the absence of sources. When sources are present the right hand side will be replaced by a source term.

We would prefer an equation in terms of $F_{ij}$. Rearranging we obtain

$$\nabla^iF_{ik} - \frac{1}{2}F_{ia}g^{ab}T^i_{kb} = J_k \quad (10.14)$$

where $J_k$ is a source term. This is equation 9.12, the scalar component of Ussher’s equation, which reinforces our belief that this is the correct extension of the Ampere-Gauss equation in our geometry.

The advantage of obtaining this equation via a Lagrangian approach rather than as a consequence of Ussher’s identity is that it gives us a method of finding the source term. In particular the source term $J_k$ can be determined for a particle by varying its Lagrangian with respect to $A_i$.

The disadvantage of a Lagrangian approach is that we must guess at the form of the Lagrangian. The Lagrangian that we used in this case was the obvious extension of the usual electromagnetic Lagrangian into our model, formed by contracting $F_{ij}$ with its dual $F^{ij}$. Contracting a tensor with its dual is one of the most obvious and natural ways to obtain a scalar from it.

We call this scalar the norm of $F_{ij}$, and denote it $||F_{ij}||^2$. Such a norm is defined whenever we can define a unique dual.
10.3 Lagrange Methods: Gravity

Equation (10.14) is the dynamic equation for $F_{ij}$ obtained via a Lagrangian principle by variation of $A_k$. But $F_{ij}$ is merely a component of the overall spinor curvature tensor $R_{ij}^{\alpha \beta}$ while $A_k$ is merely a component of the spinor connection $\Gamma_{k\beta}^{\alpha}$.

This suggests that we should seek a dynamical equation for the entire spinor curvature tensor by varying the spinor connection. To do this we need a Lagrangian constructed from the curvature $R_{ij}^{\alpha \beta}$.

One natural way to do this is to contract it with its dual

$$R_{ij}^{\alpha \beta} = s_{\alpha \lambda} R_{ab\mu}^{\lambda} s_{\mu \beta} g^{ia} g^{jb}$$

(10.15)

Note the use of the convention for spinor indices that they are lowered on the left and raised on the right. This ensures that raising and lowering are opposite operations. This gives

$$\mathcal{L} = ||R_{ij}^{\alpha \beta}||^2 = R_{ij}^{\alpha \beta} R_{ab\mu}^{\lambda} g^{ia} g^{jb} s_{\alpha \lambda} s_{\mu \beta}$$

(10.16)

Resolving the spinor curvature into its components we obtain

$$\mathcal{L} = ||R_{ij}^{\alpha \beta}||^2 = R_{ij}^{k} R_{abc}^{\mu} g^{ia} g^{jb} g_{kc} - 4 F_{ij} F_{ab}^{ia} g^{jb} = ||R_{ij}^{k}||^2 - 4||F_{ij}||^2 = \mathcal{L}_g - 4 \mathcal{L}_e$$

(10.17)

Where $\mathcal{L}_e = ||F_{ij}||^2$ is the electromagnetic Lagrangian we considered previously, and $\mathcal{L}_g = ||R_{ij}^{k}||^2$ is a Lagrangian constructed from the reduced curvature tensor responsible for the gravitational field.

This is very promising. It is a natural Lagrangian constructed from the curvature tensor, and it extends the Lagrangian for electromagnetism in exactly the way that we wanted. It contains two terms, the expected electromagnetic term and a new and very natural gravitational term which we might hope to use to obtain a source equation for gravity.

Unfortunately at this point the weakness in the Lagrangian method is about to bite us, because there is another natural way in which we can form a scalar from the curvature tensor $R_{ij}^{\alpha \beta}$. Since this tensor has one upper and one lower spinor index, these can be contracted without raising or lowering to obtain the Lagrangian

$$\mathcal{L} = R_{ij}^{\alpha \beta} g^{ia} g^{jb} R_{abc}^{\alpha} = \mathcal{L}_g + 4 \mathcal{L}_e$$

(10.18)

The gravitational and electromagnetic components $\mathcal{L}_g$ and $\mathcal{L}_e$ are the same as before, but there is a difference in sign in the way they are combined.
We expect this to lead to a measurable difference in physical phenomenology. In particular we expect the resulting gravitational field equations will have a different source term for the electromagnetic contribution to the gravitational field. In principle we should eventually be able to rule out one of these Lagrangians on that basis.

Meanwhile we continue the analysis by working with the components $\mathcal{L}_g$ and $\mathcal{L}_e$ which will enable us to defer the question of how these should combine until later. It will also have the benefit of minimising algebraic complexity. We define Lagrangians to go along with these Lagrangian densities

\[ L_g = \int \mathcal{L}_g \, dx^o \]
\[ L_e = \int \mathcal{L}_e \, dx^o \]  

The overall Lagrangian will be $L = L_g \pm 4L_e$, with the sign determined by our choice of Lagrangian density $\mathcal{L}$.

Having chosen a Lagrangian we must decide what to vary and how to vary it. The obvious approach is to simply vary the connection. However the extent to which the connection by itself can vary without breaking the structure of our framework is very limited. For example if the local action $T^\alpha_{i\beta}$ does not vary then this fixes both the torsion $T^k_{ij}$ and the metric $g_{ij}$. The metric determines the Christoffel symbols so both the antisymmetric and symmetric parts of $\Gamma^k_{ij}$ are fixed which implies $R^l_{ijk} = R^l_{ij}T^i_{lk}$ is also fixed. Hence no variation in the gravitational field is possible.

This implies that any non-trivial variation which does not break the structure of the framework must vary both the local and global actions. In considering the general situation the following Lemma will be useful.

**Lemma 10.3.1** Consider a tensor derivation $M(\ast)$ acting on spinors and vectors, given by $M^\alpha_{\beta}$ and $M^i_j$ respectively.

Then $M(\ast)(T^\alpha_{i\beta}) = 0$ if and only if there is a scalar $a$ and vector $b^k$ with

\[ M^\alpha_{\beta} = a1^\alpha_\beta + b^kT^\alpha_{k\beta} \]  
\[ M^i_j = b^kT^i_{kj} \]  

**Proof** First, since $M(\ast)(T^\alpha_{i\beta}) = 0$ we have

\[ M^\lambda_{\alpha}T^\lambda_{i\beta} - T^\alpha_{\lambda i}M^\lambda_{\beta} = M^k_iT^\alpha_{k\beta} \]
the left hand side here can be viewed as the commutator \([M, T_i]\) of two matrices. Next we write \(M^\beta_\beta\) in terms of the decomposition of spinor transformations into components to obtain

\[
M^\alpha_\beta = a 1^\alpha_\beta + b^k T^\beta_k + c^A T^\beta_A
\]  

(10.23)

putting these things together we must have

\[
b^k T^m_{ki} T^\alpha_m - c^A T^B_i T^B_{\alpha} = M^m_{m_\beta} T^\alpha_m
\]  

(10.24)

Comparing versor components on each side of this equation we must have \(c^A T^B_i = 0\) for all \(i\) and \(B\) which gives \(c^A = 0\). Comparing vector components we have \(b^k T^m_k = M^m_i\) completing the proof.

Now suppose that both the local and global actions vary in a manner consistent with the structure of a framework.

\[
T^\beta_{i\alpha} \rightarrow T^\beta_{i\alpha} + \delta T^\beta_{i\alpha} \\
\Gamma^\beta_{i\alpha} \rightarrow \Gamma^\beta_{i\alpha} + \delta \Gamma^\beta_{i\alpha}
\]  

(10.25)

(10.26)

Where the expressions on the right specify valid local and global actions for the Lie algebra \(so(2,3)\) and where these commute.

On any framework bases can be chosen so that the matrices \(T^\beta_{i\alpha}\) at a chosen point take the form of the matrices on page 84. Since after variation we have a valid spinor manifold we must therefore be able to describe the variation in \(T^\alpha_{i\beta}\) as if it were a small change of basis.

Note that we are not saying here that the variation is simply a change of basis. A variation is a real change whereas a change of basis is only a change in description. However these two things will have the same algebraic form. We have met this kind of thing before in basic linear algebra where a rotation is a transformation having the same algebraic description as a change of orthonormal basis.

Hence there are small tensors \(\delta^\alpha_a\) and \(\delta^\beta_\beta\) so that to first order

\[
T^\beta_{i\alpha} + \delta T^\beta_{i\alpha} = (1^k_i - \delta^k_i) T^\mu_{k\nu} (1^\beta_\mu + \delta^\beta_\mu) (1^\nu_\alpha - \delta^\nu_\alpha)
\]  

(10.27)

where the negative signs arise from taking the inverse assuming \(\delta^\alpha_a\) and \(\delta^\beta_\beta\) are small. Expanding out we find that to first order we have

\[
\delta T^\beta_{i\alpha} = \delta^\alpha_\mu T^\mu_{i\alpha} - \delta^k_i T^\beta_k - \delta^\nu_\alpha T^\beta_{i\nu} = \delta (\ast) (T^\beta_{i\alpha})
\]  

(10.28)

Hence the variation in the local action can be described via small tensor derivation \(\delta(\ast)\).
The variation in the global action given by $\delta \Gamma_k(\ast)$ must be consistent with the variation in the local action. Requiring that $\nabla_k T_{ij}^\alpha = 0$ after variation gives (to first order)

$$[\delta \Gamma_k(\ast) + \nabla_k(\delta(\ast))] T_{ij}^\alpha = 0$$

(10.29)

Hence by lemma [10.3.1] we can write

$$\delta \Gamma_k^\alpha + \nabla_k(\delta(\ast)) = a_k T^\alpha_{i\beta} + b_k T_{i\beta}^\alpha$$

(10.30)

$$\delta \Gamma^i_k + \nabla_k(\delta(\ast)) = b_k T_{ij}^i$$

(10.31)

for scalar fields $a_k$ and vector fields $b_k$. Using this description a general variation in the local and global actions will be written in the form

$$\delta T^\alpha_{k\beta} = \delta(\ast) T^\alpha_{k\beta}$$

$$\delta T^i_{k\beta} = \delta(\ast) T^i_{k\beta}$$

$$\delta \Gamma^\alpha_{k\beta} = \delta^\alpha_{\lambda} \Gamma^\lambda_{k\beta} - \delta^\lambda_{\beta} \Gamma^\alpha_{k\lambda} - \partial_k(\delta^\alpha_{\beta}) + a_k 1^\alpha_{\beta} + b_k T^\alpha_{t\beta}$$

$$\delta \Gamma^i_{k\beta} = \delta^i_{m} \Gamma^m_{k\beta} - \delta^m_{\beta} \Gamma^i_{km} - \partial_k(\delta^i_{\beta}) + b_k T^i_{t\beta}$$

(10.32)

Such variations preserve the requirements that the local action represent $so(2, 3)$ and that the global action commute with it. We now need to require that the global action represent (albeit with curvature) the Lie algebra even after variation. This is true if the variation in the torsion agrees with the variation in the Lie structure. Hence we must have

$$\delta \Gamma^i_{kj} - \delta \Gamma^i_{jk} = \delta(\ast) T^i_{jk}$$

(10.33)

But from equation [10.31] we also have

$$\delta \Gamma^i_{k} - \delta \Gamma^i_{jk} = (b_k T^i_{ij} - b_j T^i_{ik}) + (\nabla_j \delta^i_k - \nabla_k \delta^i_j)$$

(10.34)

Putting these together gives a constraint relating $b_k$ to $\delta(\ast)$.

$$b_k T^i_{ij} - b_j T^i_{ik} = (\nabla_k \delta^i_j - \nabla_j \delta^i_k) + \delta(\ast) T^i_{jk}$$

$$= (\partial_k \delta^i_j - \partial_j \delta^i_k) - (\delta^i_k \Gamma^i_{kj} - \delta^i_j \Gamma^i_{jk})$$

(10.35)

Variations which satisfy equation [10.35] are called physical variations as the variation conserves all axioms of a framework. The physical variations form an abelian group under addition. Our task is to require that the Lagrangian is invariant under an arbitrary physical variation.

First however we discuss the effect of these variations on the measure which we will need to know about since the variation $\delta L$ in the Lagrangian
depends on both the variation $\delta L$ in the Lagrangian density and on the variation $\delta dx^0$ in the measure.

The measure is specified by equation 10.3 to be

$$dx^0 = \epsilon \eta_{i_0 \cdots i_9} dx_0^{i_0} \cdots dx_9^{i_9}$$

Its variation $\delta dx^0$ can be described in terms of the variation $\delta \epsilon$ in the multiplying factor.

$$\delta dx^0 = \delta \epsilon \eta_{i_0 \cdots i_9} dx_0^{i_0} \cdots dx_9^{i_9}$$

This can be simplified by writing $\delta \epsilon$ as a multiple of epsilon

$$\delta \epsilon = \delta. \epsilon$$

where $\delta$ is a small scalar function specifying the variation. We obtain

$$\delta dx^0 = \delta. \epsilon \eta_{i_0 \cdots i_9} dx_0^{i_0} \cdots dx_9^{i_9} = \delta. dx^0$$

(10.37)

Applying the variation to equation 10.4 we must also have

$$\partial_k (\delta \epsilon) = (\delta \epsilon) \Gamma^a_{ka} + \epsilon (\delta \Gamma^a_{ka})$$

$$\Rightarrow (\partial_k \delta) \epsilon + \delta. (\partial_k \epsilon) = (\delta \epsilon) \Gamma^a_{ka} + \epsilon (\delta \Gamma^a_{ka})$$

$$\Rightarrow \partial_k \delta = \delta \Gamma^a_{ka}$$

(10.38)

and 10.32 then can be applied to obtain

$$\partial_k \delta = - \partial_k (\delta^a_a)$$

(10.39)

It follows that $\delta = -\delta^a_a + c$ for some constant $c$, and since both are zero on the boundary $\partial \Omega$ the constant of $c$ must be zero. We have proved that

$$\delta dx^0 = -\delta^a_a dx^0$$

(10.40)

Now that we understand the effect of the variation on the measure we are at last ready to look at the Lagrangian problem.

We break the problem up into three cases, each involving a different type of physical variation. We may consider these separately since an arbitrary physical variation can be written as a sum of variations from these three cases.
Case 1. Variations given by $a_k$ only with $b_k' = 0$, $\delta^\alpha_\beta = 0$ and $\delta^i_j = 0$.

\begin{align*}
\delta T_{k\beta}^\alpha &= 0 \\
\delta T_{kj}^i &= 0 \\
\delta \Gamma_{k\beta}^\alpha &= a_k T_{\beta k\lambda}^\alpha \\
\delta \Gamma_{kj}^i &= 0
\end{align*}
(10.41)

These are physical. There is no variation on the measure and since they only affect the $F_{ij}$ component of the spinor curvature only the $L_e$ component of the Lagrangian density can change under a variation of this type. The Lagrangian method in this cases reduces to the situation in section 10.2 giving equation 9.12, the extended Ampere-Gauss equation.

Case 2. Variations given by $\delta^\beta_\alpha$ only with $a_k = 0$, $b_k' = 0$ and $\delta^i_j = 0$.

\begin{align*}
\delta T_{k\beta}^\alpha &= \delta^\alpha_\lambda T_{k\beta}^\lambda - \delta^\beta_\lambda T_{k\lambda}^\alpha \\
\delta T_{kj}^i &= 0 \\
\delta \Gamma_{k\beta}^\alpha &= \delta^\alpha_\lambda \Gamma_{k\beta}^\lambda - \delta^\beta_\lambda \Gamma_{k\lambda}^\alpha - \partial_k \delta^\alpha_\beta \\
\delta \Gamma_{kj}^i &= 0
\end{align*}
(10.42)

All variations of this type are physical. They fix the measure, and as we will now show they also fix the Lagrangian density. Hence the Lagrangian does not change under a variation of this type and they play no role in determining dynamic equations via a Lagrangian method.

To determine the variation of the Lagrangian we need to know the variations in $R_{ij\beta}^\alpha$, $g^{ij}$, $s_{\alpha\beta}$ and $s_{\alpha\beta}^\ast$. But

$$\delta g^{ij} = 0 = \delta^{ij}(\ast) g^{ij}$$
(10.43)

as it is determined by $T_{ij}^k$ and $\delta T_{kj}^i = 0$. The last part follows since $\delta^i_j = 0$.

The symplectic form $s_{\alpha\beta}$ is only determined up to a scalar by $T_{ij}^\alpha$, so we don’t expect to find a unique variation for it. However the variations

\begin{align*}
\delta s_{\alpha\beta} &= \delta^{ij}(\ast) s_{\alpha\beta} \\
\delta s_{\alpha\beta}^\ast &= \delta^{ij}(\ast) s_{\alpha\beta}^\ast
\end{align*}
(10.44)

where $\delta^{ij}(\ast)$ is defined in terms of $\delta^i_j$ and an arbitrary scalar $\delta_{\alpha\beta}^\ast$, ensure the proper relationships are conserved under variation and so must be correct. Note that the scalar $\delta_{\alpha\beta}^\ast$ can take any value and we can if we wish choose it
to be zero. It represents a local variation only in our choice of symplectic form. To find $\delta R^\alpha_{ij\beta}$ we write

$$\delta R^\alpha_{ij\beta} = (\partial_i \delta \Gamma^\alpha_{j\beta} - \partial_j \delta \Gamma^\alpha_{i\beta}) + (\delta \Gamma^\alpha_{i\lambda} \Gamma^\lambda_{j\beta} - \Gamma^\alpha_{j\lambda} \delta \Gamma^\lambda_{i\beta})$$

$$+ (\delta \Gamma^\alpha_{i\lambda} \Gamma^\lambda_{j\beta} - \delta \Gamma^\alpha_{j\lambda} \Gamma^\lambda_{i\beta})$$  (10.46)

We next substitute for the variations using (10.42). This gives an equation with 24 terms. Fortunately 16 of these cancel and the remaining 8 terms can be collected up to show

$$\delta R^\alpha_{ij\beta} = \delta \alpha^\lambda R^\lambda_{ij\beta} - \delta \lambda^\beta R^\alpha_{ij\lambda}$$  (10.47)

and since $\delta^i_j = 0$ we can write this in the form

$$\delta R^\alpha_{ij\beta} = \delta^*(\alpha) R^\alpha_{ij\beta}$$  (10.48)

At this point we can conclude that the variation acts as the tensor derivation $\delta^*(\alpha)$ on $R^\alpha_{ij\beta}$, $g^{ij}$, $T^\alpha_{k\beta}$, $s^\alpha_{\alpha\beta}$, $s^\alpha_{\beta\beta}$ and $1^\alpha_{\alpha\beta}$. Hence it will act as the tensor derivation $\delta^*(\alpha)$ on all tensors constructed from these via tensor product and contraction.

As our Lagrangian densities $L_e$ and $L_g$ are constructed in precisely this way, the variation will act as $\delta^*(\alpha)$ on these too. But the tensor derivation $\delta^*(\alpha)$ acts trivially on scalars. Hence variations of this type have no effect on $L_e$ and $L_g$ as claimed.

**Case 3.** Variations given by $b^\alpha_k$ and $\delta^i_j$ only with $a_k = 0$ and $\delta^\alpha_j = 0$.

$$\delta T^\alpha_{k\beta} = \delta^*(\alpha) T^\alpha_{k\beta}$$

$$\delta T^i_{kj} = \delta^*(\alpha) T^i_{kj}$$

$$\delta \Gamma^\alpha_{k\beta} = b^k_i T^i_{kj}$$

$$\delta \Gamma^i_{kj} = b^k_i T^i_{kj} - \nabla^k \delta^i_j$$  (10.49)

Not all such variations are physical so we must also require equation (10.35). This is the interesting and difficult case.

Since the variation in the connection $\Gamma^\alpha_{k\beta}$ is expressed only in terms of $b^k_i$, we can obtain an expression for the variation in the curvature purely in terms of $b^k_i$. After a little algebraic heroism we obtain

$$\delta R^\alpha_{ij\beta} = \left[ (\partial_i b^m_j - \partial_j b^m_i) + (b^i_j \Gamma^m_{it} - b^i_j \Gamma^m_{jt}) \right] T^\alpha_{m\beta}$$  (10.50)
which, when expressed in terms of the covariant derivative, gives

$$\delta R^{\alpha}_{ij\beta} = \left[ \nabla_i b_j^{m} - \nabla_j b_i^{m} - T^{\alpha}_{ij} b^{m}_{i} \right] T^m_{\beta}$$  \hspace{1cm} (10.51)

separating out components we obtain

$$\delta F_{ij} = 0$$

$$\delta R^{k}_{ij} = \nabla_i b_{j}^{m} - \nabla_j b_{i}^{m} - T^{k}_{ij} b^{m}_{i} + R^{k}_{ij} \delta^{m}_{m}$$  \hspace{1cm} (10.52)

The variations in the Lagrangian densities $\delta L_e$ and $\delta L_g$ depend on the variations $\delta g_{ij}$ and $\delta g^{ij}$ in the metric, while the variation in the Lagrangian depends also on the variation $\delta dx^o$ in the measure. The metric variation is

$$\delta g_{ij} = \delta (\delta)_{ij}$$

$$\delta g^{ij} = \delta (\delta)_{ij}$$  \hspace{1cm} (10.53)

We can now determine $\delta L_e$ and $\delta L_g$. We get

$$\delta L_e = F_{ij} F_{ab} \left( \delta g^{ia} g^{jb} + g^{ia} \delta g^{jb} \right)$$

$$= \left[ 4 F_{ij} F_{ab} g^{ia} g^{jm} \right] \delta^{m}_{m}$$  \hspace{1cm} (10.54)

$$= \left[ 4 F_{ij} F_{ab} g^{ia} g^{jm} \right] \delta^{m}_{m}$$  \hspace{1cm} (10.55)

for the variation of the electromagnetic Lagrangian density, where the symmetry properties of the tensors have been exploited to show all four terms are equal. The gravitational Lagrangian density gives

$$\delta L_g = 2 \delta R^{k}_{ij} R^{c}_{ab} g_{kc} g^{ia} g^{jb} + 2 \delta R^{k}_{ij} R^{c}_{ab} \delta g_{kc} g^{ia} g^{jb} + R^{k}_{ij} \delta R^{c}_{ab} g_{kc} g^{ia} g^{jb}$$

$$= 2 \delta R^{k}_{ij} R^{c}_{ab} g_{kc} g^{ia} g^{jb} + 4 \delta R^{k}_{ij} R^{c}_{ab} g_{kc} g^{ia} g^{jm} \delta^{b}_{m} - 2 \delta R^{k}_{ij} R^{c}_{ab} g_{kc} g^{ia} g^{jb} \delta^{b}_{k}$$

$$= 2 \left( \nabla_i b_{j}^{k} - \nabla_j b_{i}^{k} - T^{k}_{ij} b^{m}_{i} \right) R^{c}_{ab} g_{kc} g^{ia} g^{jb}$$

$$+ 4 \delta R^{k}_{ij} R^{c}_{ab} g_{kc} g^{ia} g^{jm} \delta^{b}_{m} - 2 R^{k}_{ij} R^{c}_{ab} g_{kc} g^{ia} g^{jb} \delta^{b}_{k}$$

$$= 2 \left( \nabla_i b_{j}^{k} - \nabla_j b_{i}^{k} - T^{k}_{ij} b^{m}_{i} \right) R^{c}_{ab} g_{kc} g^{ia} g^{jb}$$

$$+ 4 \delta R^{k}_{ij} R^{c}_{ab} g_{kc} g^{ia} g^{jm} \delta^{b}_{m}$$

$$= 2 \left( \nabla_i b_{j}^{k} - T^{k}_{ij} b^{m}_{i} \right) R^{c}_{ab} g_{kc} g^{ia} g^{jb}$$

$$+ 4 \delta R^{k}_{ij} R^{c}_{ab} g_{kc} g^{ia} g^{jm} \delta^{b}_{m}$$  \hspace{1cm} (10.56)

We can now integrate these to obtain the variations in the Lagrangians $L_e$ and $L_g$. In doing so we must also use $\delta dx^o = -\delta^a dx^a$ from equation[10.40]

$$\delta L_e = \int \delta L_e dx^o + \int \delta L_e \delta dx^o$$

$$= \int 4 F_{ij} F_{an} g^{ia} g^{jm} \delta^{m}_{m} - F_{ij} F_{ab} g^{ia} g^{jm} dx^o$$

$$= \int 4 F_{ij} F_{an} g^{ia} (4g^{jm} 1_n - g^{jm} 1_n) \delta^{m}_{m} dx^o$$  \hspace{1cm} (10.57)

$$= \int F_{ij} F_{ab} g^{ia} (4g^{jm} 1_n - g^{jm} 1_n) \delta^{m}_{m} dx^o$$  \hspace{1cm} (10.58)
The integrand here involves the obvious extension of the stress energy tensor for the electromagnetic field to our ten dimensional context. This is the expected source term for the contribution to gravity of the energy in an electromagnetic field.

The gravitational Lagrangian \( L_g \) is as follows. Note the use of Stoke’s theorem in the third step.

\[
\delta L_g = \int_\Omega \delta L_g \, dx^\circ + \int_\Omega L_g \delta x^\circ \\
= \int_\Omega \left( 4 \nabla_i b^k_j R_{ab}^c g_{kc} g^{ia} g^{jb} \right) \frac{1}{2} T_{ij}^m R_{ab}^c g^{jb} \delta_m^o \\
+ 4 \nabla_i R_{ab}^c g_{kc} g^{ia} g^{jb} \delta_m^o - R_{ij} R_{ab}^c g_{kc} g^{ia} g^{jb} \delta_m^o \, dx^o \\
= \int_\Omega \left( -4 \nabla_i R_{ab}^c g^{mb} - 2 T_{ij}^m R_{ab}^c g^{jb} \right) g_{nc} g^{ia} b^n_m \\
+ \left( 4 R_{ij} R_{an}^c g^{jm} - R_{ij} R_{ab}^c g^{jb} 1^m_n \right) g_{kc} g^{ia} \delta_m^o \, dx^o \\
= \int_\Omega \left( -4 \left( \nabla^j R_{ij}^k - \frac{1}{2} T_{ij}^m R_{j}^k \right) g^{mi} g_{nk} b^n_m \\
+ R_{ij} R_{ab}^c g^{ia} g_{kc} \left( 4 g^{jm} 1^b_n - g^{jb} 1^m_n \right) \delta_m^o \right) \\
= \int_\Omega \left( -4 U^k_i g^{mi} g_{nk} b^n_m \\
+ R_{ij} R_{ab}^c g^{ia} g_{kc} \left( 4 g^{jm} 1^b_n - g^{jb} 1^m_n \right) \delta_m^o \right) \\
= \int_\Omega \left( -4 U^k_i g^{mi} g_{nk} b^n_m \\
+ R_{ij} R_{ab}^c g^{ia} g_{kc} \left( 4 g^{jm} 1^b_n - g^{jb} 1^m_n \right) \delta_m^o \right) dx^o \tag{10.59}
\]

Where \( U^k_i \) is Uscher’s tensor from equation 9.15. Note the similarity of the last term to equation 10.58. This term describes the energy held in the gravitational field and suggests the presence of non-linear phenomena where gravity acts as a source of gravity.

Combining these two into the Lagrangian \( L = L_g + 4L_e \) we obtain

\[
\delta L = \int_\Omega \left( -4 U^k_i g^{mi} g_{nk} b^n_m \\
+ R_{ij} R_{ab}^c g^{ia} g_{kc} \left( 4 g^{jm} 1^b_n - g^{jb} 1^m_n \right) \delta_m^o \right) dx^o \tag{10.60}
\]

To go beyond this we will need to use an equation relating \( b^n_m \) and \( \delta_m^o \). The algebra involved is long and complicated; we will work our way through it in section 10.4.

However we are now in a position to obtain a good approximate solution in the small field case where \( F_{ij} \) and \( R_{ij}^k \) are small. Under these conditions

\[\text{calling it the ‘weak field case’ might be misunderstood} \]
we can drop all terms which are second order in \( F_{ij} \) or \( R_{ij} \). Equation 10.60 then simplifies to

\[
\delta L_{\text{small field}} = \int_{\Omega} -4U_i^k g^{mi} g_{nk} b_m^n \, dx^o \tag{10.61}
\]

which we can now solve quite easily since all the terms involving \( \delta^n_m \) have been eliminated. If this is to be zero for all \( b_m^n \) we must have

\[
-4U_i^k g^{mi} g_{nk} = 0 \tag{10.62}
\]

Rewriting this in terms of \( R_{ij}^k \), adding in a source term and rearranging we obtain

\[
\nabla_j R_{ji}^k - \frac{1}{2} R^{ij} g^{st} R_{st}^k = K_i^k
\]

Which is precisely Ussher’s equation 9.14 for gravitation which we obtained earlier from Ussher’s identity. We have now obtained this equation again via a Lagrangian approach. However we obtained it as an approximate solution in the small field case. Fortunately this is likely to be sufficient for many applications.

We end this section with some words of caution about the small field solution we have just obtained.

1. Eliminating \( \delta^n_m \) from the Lagrangian problem by making the small field assumption does not relieve us of our obligation to use only physical variations. To the extent that this constrains the possible values of \( b_m^n \), additional solutions may be possible in the small field case. We will revisit this very subtle technicality once we have solved the exact equation.

2. We have not yet specified the nature of the source term in Ussher’s equation. We were hoping to find this by varying the Lagrangian for matter. However the small field approximation we have just made will make this difficult as we have no reason to expect that such a variation will depend only on \( b_m^n \). We would need to express any \( \delta^n_m \) term as a function of \( b_m^n \) and to the extent that \( \delta^n_m \) is not fully determined by \( b_m^n \) this may not be possible. We may have to seek a best approximation.
10.4 The Exact Equation

We resume our efforts to obtain an exact solution for the full Lagrangian problem continuing on from equation 10.60.

\[ \delta L = \int_\Omega -4U^k_i g^{mi} g_{nk} b_m^i + (R^k_{ij} R^c_{ab} g^{ia} g_{kc} \pm 4F_{ij} F_{ab} g^{ia}) (4g^{im} 1^n_b - g^{jb} 1^m_n) \delta_m^a d^x \]

The variation is described in terms of two quantities \( \delta_b^m \) and \( b_m^a \) which are small in the interior of a region \( \Omega \) and zero on its boundary; if the variation is to be physical these are constrained by equation 10.35 which states.

\[ (b_t^i T_{ij}^k - b_t^j T_{ik}^i) = (\nabla_k \delta_j^i - \nabla_j \delta_k^i) + \delta(\ast) T_{jk}^i = (\partial_k \delta_j^i - \partial_j \delta_k^i) - (\delta_k^i \Gamma_{ij}^k - \delta_j^i \Gamma_{ik}^k) \]

However in this form the constraint is difficult to use. We would prefer a constraint with \( b_m^a \) or \( \delta_n^m \) as the subject which might allow us to substitute. We begin this section by seeking an equivalent constraint in this form.

The vector connection \( \Gamma_{ij}^k \) can be written in terms of the Christoffel connection and the torsion as given in equation 8.23. Furthermore the Christoffel connection can be written purely in terms of the metric and its partial derivatives. Hence the connection can be expressed in terms of the torsion, the metric and its partial derivatives.

\[ \Gamma_{ij}^k = \frac{1}{2} \left[ (\partial_i g_{jm} + \partial_j g_{im} - \partial_m g_{ij}) g^{mk} - T_{ij}^k \right] \quad (10.63) \]

Since the variations of the quantities on the right hand side only depend on \( \delta_n^m \) we can use this to write \( \delta \Gamma_{ij}^k \) in terms of \( \delta_n^m \) and its partial derivatives. We can then use this to write \( b_m^a \) as a function of \( \delta_n^m \).

We begin by writing equation 10.63 in a form which minimises contractions.

\[ 2\Gamma_{ij}^m g_{mk} = \partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij} - T_{ij}^m g_{mk} \quad (10.64) \]

We do this in order to minimise the number of terms in the equation after variation. We can also simplify the algebra by working using lowered index versions, \( \delta_{ab} = \delta_i^a g_{ib} \) and \( b_{ab} = b_i^a g_{ib} \), of our variation. In particular we have

\[ \delta g_{ij} = -\delta_{ij} - \delta_{ji} \quad (10.65) \]
\[ b_t^i T_{ij}^m g_{mk} = b_t^m T_{jk}^m \quad (10.66) \]
\[-\delta (T_{ij}^m g_{mk}) = T_{ij}^m \delta_{mk} + T_{jk}^m \delta_{mi} + T_{ki}^m \delta_{mj} \quad (10.67) \]
Applying the variation we obtain in the first instance

\[
2b_{im} T_{jk}^m = 2\nabla_i (\delta_j^m) g_{mk} - 2\Gamma_{ij}^m \delta_{mk} - 2\Gamma_{ij}^m \delta_{km}
\]

\[
\begin{align*}
= & \ -\partial_i \delta_{jk} - \partial_i \delta_{kj} - \partial_j \delta_{ik} - \partial_j \delta_{ki} + \partial_k \delta_{ij} + \partial_k \delta_{ji} \\
& + T_{ij}^m \delta_{mk} + T_{jk}^m \delta_{mi} + T_{km}^m \delta_{mj}
\end{align*}
\]

(10.68)

The next step is to replace all of the partial derivatives with covariant derivatives, and to cancel and collect up all the resulting terms involving the connection. These pair up nicely to give torsion terms yielding the equation

\[
2b_{im} T_{jk}^m = \left(\nabla_i \delta_{jk} - \nabla_i \delta_{kj} \right) + \left(\nabla_k \delta_{ij} - \nabla_j \delta_{ik} \right) + \left(\nabla_j \delta_{ki} - \nabla_k \delta_{ij} \right) + 2\delta_{mi} T_{jk}^m + \delta_{im} T_{jk}^m + \left(\delta_{jm} T_{ik}^m - \delta_{km} T_{ij}^m \right)
\]

(10.69)

Note that the equation is antisymmetric with respect to the indices \(j\) and \(k\) and has been written to illustrate this. We now contract with the tensor

\[
\frac{1}{2} T_{ji}^k = \frac{1}{2} T_{ts}^j g^{sk} \text{ which is also antisymmetric with respect to the indices } j \text{ and } k
\]

to obtain the equation

\[
6b_{it} = \nabla_i \delta_{jk} T_{ts}^j g^{sk} + \nabla_k \delta_{ij} T_{ts}^j g^{sk} + \nabla_j \delta_{ki} T_{ts}^j g^{sk} + 6\delta_{it} + 3\delta_{it} + \delta_{jm} T_{ik}^m g^{sk} T_{ts}^j
\]

(10.70)

We finish up by renaming contracted indices in a more systematic fashion.

\[
6b_{mn} = \nabla_m \delta_{ij} T_{ns}^i g^{sj} + \nabla_j \delta_{mi} T_{ns}^i g^{sj} + \nabla_j \delta_{im} T_{ns}^i g^{sj} + 3\delta_{mn} + \delta_{mn} T_{ms} g^{st} T_{nt}^i
\]

(10.71)

We now prepare equation \[10.60\] for substitution by rewriting it in the form

\[
\delta L = \int_{\Omega} -4U^n_p g^{km} b_{mn} + W_p^m g^{pm} \delta_{mn} \, dx^c
\]

(10.72)

where \(U^n_p\) is Ussher’s tensor from equation \[9.15\] and

\[
W_p^m = \left( R_{ai}^s R_{bij}^d g_{st} g^{ab} + 4F_{aj} F_{bt} g^{ab} \right) \left( 4g^{jm} g^{i1}_p - g^{ij1}_p \right)
\]

(10.73)

We now at last are ready to substitute equation \[10.71\] into equation \[10.72\] to obtain a Lagrangian we can solve. We obtain the following equation. Note the use of Stokes’ theorem to simplify the first three terms.

\[
3\delta L = \int_{\Omega} 2\nabla^k U^n_k T_{ns}^i g^{sj} \delta_{ij} + 2\nabla^s U^n_k T_{ns}^i g^{km} \delta_{mi} + 2\nabla^s U^n_k T_{ns}^i g^{km} \delta_{im} - 6U^n_k g^{km} \delta_{mn} - 12U^n_k g^{km} \delta_{nm} - 2U^n_k g^{km} T_{ms} g^{st} T_{nt}^i \delta_{ij} + 3W_p^m g^{pm} \delta_{mn} \, dx^c
\]

(10.74)
We next rename contracted indices and rewrite it in terms of $\delta^n_m$

$$
\delta L = \int_\Omega \frac{2}{3} \nabla^k U^i_k T^m_{in} \delta^n_m + \frac{2}{3} \nabla^s U^i_k T^p_{is} g^{km} g_{pn} \delta^n_m + \frac{2}{3} \nabla^s U^i_k T^m_{is} \delta^n_m \\
- 2U^p_k g^{km} g_{pn} \delta^n_m - 4U^m_n \delta^n_m + \frac{2}{3} U^j_k g^{ji} T^l_{in} T^m_{jl} \delta^n_m \\
+ W^m_n \delta^n_m \, dx^\circ 
$$

(10.75)

This puts the equation in the form

$$
\delta L = \int_\Omega [\text{expression}]^m_n \delta^n_m \, dx^\circ 
$$

(10.76)

which is zero for all variations if and only if $[\text{expression}]^m_n = 0$. We can now solve the Lagrangian problem to obtain

$$
\nabla^k U^i_k T^m_{in} + \nabla^s U^i_k T^p_{is} g^{km} g_{pn} + \nabla^s U^i_n T^m_{is} \\
- 3U^p_k g^{km} g_{pn} - 6U^m_n + U^j_k g^{ji} T^l_{in} T^m_{jl} + \frac{3}{2} W^m_n = 0 
$$

(10.77)

Ussher’s identity [10.16] gives $\nabla^k U^i_k = 0$ allowing us to drop the first term. We collect up the remaining terms and write it in the form of a differential equation in Ussher’s tensor.

$$
(-T^t_{in} g^{km} + 1^k T^m_{is} g^{ts}) \nabla^t U^i_k \\
- (3g^{km} g_{in} + 6,1^m i^k - g^{kj} T^l_{jn T^m_{it}}) U^i_k + \frac{3}{2} W^m_n = 0 
$$

(10.78)

This is the long sought exact solution to the Lagrangian problem. We can simplify the form of this equation by defining the coefficient tensors

$$
A^{km}_{in} = T^t_{in} g^{km} - 1^k T^m_{is} g^{ts} 
$$

(10.79)

$$
B^{km}_{in} = 3g^{km} g_{in} + 6,1^m i^k - g^{kj} T^l_{jn T^m_{it}} 
$$

(10.80)

In the presence of sources it will take the form

$$
A^{km}_{in} \nabla^i U^i_k + B^{km}_{in} U^i_k = \frac{3}{2} (S^m_n + W^m_n) 
$$

(10.81)

where the source term $S^m_n$ will be the coefficient of $\delta^m_n$ under a type 3 variation of the Lagrangian for matter.

Consider the small field approximation in the absence of sources obtained by dropping the second order term $W^m_n$ and the source term $S^m_n$ from equation [10.81].

$$
A^{km}_{in} \nabla^i U^i_k + B^{km}_{in} U^i_k = 0 
$$

(10.82)
At the end of section 10.3 we obtained Ussher’s equation 9.14 as a small field approximate equation for gravity in the absence of sources. Now we have a different small field approximation. How can we reconcile the two?

Ussher’s equation (in the absence of sources) takes the form \( U_{ik} = 0 \), so clearly any solution to Ussher’s equation will also be a solution to equation 10.82. But the converse will not be true. Equation 10.82 will have additional solutions.

In the discussion on page 114 we noted that additional small field solutions might be possible if the variation \( b^m_n \) is partially constrained by the requirement that the variation be physical. That does indeed seem to be the origin of the additional solutions in equation 10.82.

In the presence of sources the small field equation takes the form of differential equation in Ussher’s tensor.

\[ A_{lm}^{km} \nabla_i U_{ik}^l + B_{lm}^{km} U_{ik}^l = \frac{3}{2} S_{lm}^{mn} \]  

(10.83)

This equation is not quite as simple as it looks since there are connections hiding in the covariant derivative. Nevertheless it invites us to view the situation in terms of a two stage process, where equation 10.83 determines the value of Ussher’s tensor, and Ussher’s equation then determines the curvature.

10.5  The Dark side of the Force

The title of this section is perhaps a temptation that should have been resisted. However one of the things we are particularly interested in looking at is whether there is room in these equations for dark phenomena — behavior which could give the appearance of the presence of dark matter and/or energy. The answer seems to be yes. These equations do indeed seem to predict dark phenomena as we shall now explain.

We will use an analogy with the suspension system of a car to help describe the situation.

Ussher’s equation in the presence of sources is \( U_{ik}^l = S_{ik}^l \) where \( S_{ik}^l \) is a source term determined by the distribution of matter. It is an equation of type \( u = s \), where the value of \( s \) directly determines the value of \( u \). A step change in \( s \) will thus result in a step change in \( u \). We may think of it as a car with no suspension. Every bump in the road has immediate effect.

The small field equation 10.83 introduces a derivative term to this relationship. It gives an equation of type \( \ddot{u} = -k(u - s) \). The value of \( s \) still
determines the value of $u$, but in a less direct fashion. Our allegorical car is now equipped with a suspension system smoothing out the bumps in the road.

This suspension analogy is a useful one. In a car with suspension the car no longer directly ‘sees’ the road. Instead its behavior is determined by the state of the suspension. It is the suspension that ‘sees’ the road.

Hence gravity, in the weak field case, can be described by two equations. Firstly we have the dark equation, which is essentially just the small field equation 10.83

$$A^{k}_{m n} \nabla_{k} X^{i}_{k} + B^{k}_{m n} X^{i}_{k} = \frac{3}{2} S^{m}_{n}$$

(10.84)

This in principle determines $X^{i}_{k}$, which is then used as the effective source in Ussher’s equation $U^{i}_{k} = X^{i}_{k}$.

$$\nabla^{j} R^{k}_{j i} - \frac{1}{2} T^{j}_{i k} g^{s t} R^{k}_{j t} = X^{k}_{i}$$

(10.85)

Ussher’s equation describes what we might think of as the ordinary behavior of gravity. We expect to obtain Newton’s law from it for example as a non-relativistic approximation. But since Ussher’s equation ‘sees’ the effective source $X^{i}_{k}$ and not the usual source $K^{i}_{k}$ it will seem to us at times as if gravity is responding to a different distribution of energy and matter than we can actually observe.

Sometimes the gravitational force predicted by these equations may respond as though matter were present in places where there isn’t any, because while $K^{i}_{k} = 0$ there, $X^{i}_{k}$ is not. We might interpret this as dark matter.

It may also be possible for the effective source to take values which cannot be achieved by the apparent source. For example $X^{i}_{k}$ might take values which could only be achieved by $K^{i}_{k}$ if negative mass were present. We might interpret this as dark energy.

We need a better name for $X^{i}_{k}$. Because it controls the dark side of the force we propose to call it the sith. While this is a little cheesy at first sight, it is a good short word and you rapidly get used to it. The sith is effectively the suspension system of the universe. It mediates between the source terms and Ussher’s equation which describes the more usual behaviors of gravity.

If we want to talk only about Dark matter, we could define it to be the difference between the sith $X^{i}_{k}$ and the usual source term $K^{i}_{k}$ for Ussher’s equation determined directly from the distribution of matter.

$$D^{i}_{k} = X^{i}_{k} - K^{i}_{k}$$

(10.86)
Of course in order to do this we would need to know how to compute $K^i_k$ which is something we have not yet determined. We noted in the discussion on page 114 that we can’t expect to find this via a Lagrangian argument. However we might be able to find it by simplifying the exact equation. To do this we need to look more closely at the nature of the relationship between Ussher’s equation and the exact one.

Ussher’s equation is algebraic in $U^i_k$. It is local in the sense that Ussher’s tensor at each point is a function of the source at that point. The dark equation relates $U^i_k$ to the source via a differential equation. It is not local since Ussher’s tensor depends also on what is happening at other points. The derivative terms in the equation induce this non-local behavior.

We expect Ussher’s equation to approximate the exact one. This suggests we should look at the best (algebraic) local approximation to the exact equation. The easiest way to obtain an algebraic equation from the exact equation is to simply drop all the terms involving derivatives, since it is these that induce the non-local behavior.

Is this reasonable? Returning to our simple suspension analogy, we are effectively trying to remove the suspension from the car. We want to start with the equation for the behavior with suspension, and deduce the equation without suspension. Will simply dropping derivative terms do the trick? If we drop the derivative term from the equation $\ddot{u} = -k(u - s)$ we get $0 = -k(u - s)$ which simplifies to $u = s$. This is indeed the original equation for a car without suspension. So yes the procedure does seem to be a reasonable one.

This suggests the apparent source term for Ussher’s equation should be the solution to the linear equation

$$B^{km}_n K^i_k = \frac{3}{2} S^m_n$$  \hspace{1cm} (10.87)

where $S^m_n$ is the source term obtained from the Lagrangian for matter which we hope to be able to compute in chapter 11. Note that in the source free case where $S^m_n = 0$ that $K^i_k = 0$ is a solution.

We have now ‘found’ dark matter and energy in the sense of having discovered the requisite phenomena within the equations for gravity. So does dark matter exist and if so what is it? Whether dark matter can be said to exist or not is really a question of philosophy; what does it mean for anything to exist? Such questions generate more headaches than answers. It is much more productive to ask what we would expect to find if we were to second quantise these equations. What quanta would we get and could we detect them as particles?
As dark matter and energy are aspects of the gravitational field, the associated particles would have to be gravitons of some sort, since gravitons are by definition the quanta of the gravitational field. Similarly the magnetic field is an aspect of the electromagnetic field and the particles of magnetism are simply photons.

Having considered the small field case we now briefly discuss the large field situation. When the fields are large we cannot ignore the second order term \( W^{m}_{mn} \). In fact for very large fields this term will dominate.

We can write down a large field approximation by dropping linear terms. Ussher’s tensor thus becomes \( U^{i}_{ik} = \nabla^{p} R^{i}_{pk} \). We will also drop the electromagnetic field terms from \( W^{m}_{mn} \) under the assumption that gravitational fields completely dominate in this situation. This gives the second order differential equation

\[
\left( T^{m}_{is} g^{km} g^{ts} + T^{m}_{is} g^{kn} g^{ts} \right) \nabla^{l} \nabla^{p} R^{i}_{pk} + \left( 3g^{km} g_{in} + 6.1 T^{i}_{jn} T^{m}_{it} \right) \nabla^{p} R^{i}_{pk} + \frac{3}{2} R_{ai} R_{bj} g_{st} g^{ab} \left( 4g^{jm} g^{jv} - g^{ij} g^{mp} \right) g_{pn} = 0 \quad (10.88)
\]

This equation will control behavior around black holes and any place where the curvature is very large. It should determine whether singularities are possible in such situations and will describe how they form if they arise. We have become used to the notion that singularities exist in black holes. However in most physical theories singularities are a symptom that our model has broken down. We should be skeptical about the existence of singularities in any physical theory. If singularities do not form, something else must happen when a very large concentration of matter forms. Equation \([10.88]\) is where we should look for answers.

We started this section by noting that the exact equations for gravity are unlikely to give exact solutions. We can however simplify them by looking at component equations.

Our equations constrain tensors with two vector components which are 100 dimensional. Hence these equations are really systems of 100 equations in 100 variables. That is a lot of equations and a lot of variables. However all the operations in our equations are natural in the context of a framework and hence in particular they respect decomposition into irreducibles. By looking at irreducible components of our 100 dimensional tensors we can obtain much simpler component equations.

Tensors with two vector components decompose into irreducibles of dimension 1, 5, 10, 14 and 35 (two different types). For example the scalar
component of the dark equation is

$$\nabla^i X^i_k t^k_{it} - 3 X^k_k = -\frac{3}{2} S^m_m$$ \hspace{1cm} (10.89)

The dark equation separates into component equations very naturally since the constant tensors in the equation are linear combinations of component projection maps. Hence they will disappear or simplify when looking at a single component.

We end with a word of caution. The equations in these last two chapters are not geometric identities arising directly from the structure of the framework as most of the equations in the earlier chapters were. They depend on an additional assumption of some sort, whether it be a choice of Lagrangian or a decision to attach a particular physical interpretation to a certain divergence free quantity. These equations are only as good as the assumptions used to generate them.
Chapter 11

Matter

We expect matter to be described by tensors giving the wave function, which we will regard as a matter field.

The particles which mediate forces have, as their wave functions, tensors associated with the geometry; specifically the connection, which acts as the potential, and the curvature which gives the associated field. Dynamical equations for these tensors arise from the structure of the geometry, and source equations can be obtained via a Lagrangian argument by varying the geometry. Second quantisation would be necessary for a complete theory of these particles, but that is beyond the scope of this work which focuses on the field theory before second quantisation.

Other elementary particles do not arise from the geometry. Elementary fermions will have spinor wave functions. We can therefore think of a spinor on our framework as describing a fermion field. We seek dynamical and source equations for the fermion field. Once again second quantisation would be needed to obtain a complete multiparticle theory, but this is beyond the scope of this work. Spinors will thus be our main focus in this Chapter.

The fundamental physical equation governing the dynamics of fermions is the Dirac equation. We therefore begin by finding what the Dirac equation looks like in our ten dimensional context. Initially we will simply state an appropriate equation and explore some of its properties. Later we will look at deriving it from a natural Lagrangian.
11.1 The Extended Dirac Equation

Let $\psi = \psi^\alpha$ be a 4-D vector complex spinor defined on our 10-D spinor manifold which we have identified as the manifold of frames. Consider the equation

$$\nabla \varepsilon \psi = \lambda \psi \quad (11.1)$$

We can also write in the form

$$T^\mu_{\kappa\nu} \nabla^k \psi^\nu = \lambda \psi^\mu \quad (11.2)$$

We will call this the extended Dirac equation. The constant $\lambda$ is left without interpretation at this point, although we expect it to relate to rest mass and possibly charge. Note that this equation uses natural operations in our context and is quite obviously covariant. We now consider how this equation relates to the standard Dirac equation.

We firstly claim that the extended Dirac equation becomes the standard Dirac equation in the Poincaré limit as $r \to \infty$. Ignoring curvature and using a standard basis with natural units, the Curl operator on spinors is

$$\nabla \varepsilon ( ) = -T\partial_t + X\partial_x + Y\partial_y + Z\partial_z$$

$$+ A\partial_a + B\partial_b + C\partial_c - I\partial_i - J\partial_j - K\partial_k \quad (11.3)$$

where $T, X, Y, Z, A, B, C, I, J, K$ are the $4 \times 4$ matrices $T_{\alpha\beta}$. In ordinary units this becomes

$$\nabla \varepsilon ( ) = -rT\partial_t + rX\partial_x + rY\partial_y + rZ\partial_z$$

$$+ cA\partial_a + cB\partial_b + cC\partial_c - I\partial_i - J\partial_j - K\partial_k \quad (11.4)$$

Assuming that $r$ is very large, dropping insignificant terms gives

$$\frac{1}{rc} \nabla \varepsilon ( ) = -\frac{1}{c} T\partial_t + X\partial_x + Y\partial_y + Z\partial_z \quad (11.5)$$

and equation \[11.1\] becomes

$$\left(-\frac{1}{c} T\partial_t + X\partial_x + Y\partial_y + Z\partial_z\right) \psi^\alpha = rc\lambda \psi^\alpha \quad (11.6)$$

which for an appropriate choice of $\lambda$ is in the right form to be the standard Dirac equation. Let us do a more detailed comparison.

The usual Dirac equation (in ordinary units) can be written as

$$\left(\frac{1}{c} \gamma^0 \partial_0 + \gamma^1 \partial_1 + \gamma^2 \partial_2 + \gamma^3 \partial_3\right) \psi = \frac{mc}{i\hbar} \psi$$

where
\[ \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu} \]

- \[\eta^{\mu\nu} = 0\] for all \(\mu \neq \nu\).
- \(\eta^{00} = 1\) and \(\eta^{11} = \eta^{22} = \eta^{33} = -1\).
- \(\gamma^0\) is Hermitian and Unitary.
- \(\gamma^i\) is anti-Hermitian and Unitary for \(i = 1, 2, 3\).

If we make the identifications

\[ \gamma^0 = -2iT \quad \gamma^1 = 2iX \quad \gamma^2 = 2iY \quad \gamma^3 = 2iZ \]

then the matrices so defined satisfy all the properties above and multiplying equation \(\text{(11.6)}\) on both sides by \(2i\) gives precisely the Dirac equation provided that

\[ -2\lambda \gamma = \frac{m}{\hbar} \]

So the standard Dirac equation can be regarded as the Poincaré limit of equation \(\text{(11.4)}\). However while mention of the Lorentz dimensions disappears from the extended Dirac equation in the Poincaré limit, those additional dimensions have not gone away. The equations just ignore how the wave functions behave along those directions as in the Poincaré limit that behavior becomes insignificant.

A better way to view the relationship between the extended and standard Dirac equations is to consider spinors \(\psi^\alpha\) which are functions of space and time only, and have no dependence on the Lorentz coordinates. Space-time and Lorentz coordinates are mixed by large translations so this distinction between types of coordinates is not possible globally. However it is quite meaningful on laboratory scales. Under this restriction the partial derivatives with respect to the Lorentz coordinates in the extended Dirac equation will be zero giving the standard Dirac equation.

Consequently we can identify solutions to the standard Dirac equation with solutions of the extended Dirac equation which are locally constant on the Lorentz coordinates. We expect that the extended Dirac equation will also have solutions that vary along the Lorentz coordinates. The identification of these additional solutions is something that we will leave for later.

Equation \(\text{(11.1)}\) has many advantages over the standard Dirac equation. We pause to list a few of them here.
For one thing equation 11.1 is obviously invariant since we built it from the invariant Curl operator. Furthermore it is already expressed in coordinate free fashion on a space with curvature. Contrast this with usual approaches to the Dirac equation where a demonstration of relativistic invariance and the correct transformation properties of Dirac matrices often requires a laborious search for S-matrix transformations and is a fairly non-trivial exercise.

The most important difference however is that every quantity in equation 11.1 including the Dirac matrices themselves comes equipped with a direct physical interpretation. Contrast this with the usual approach to the Dirac equation where the Dirac matrices are initially chosen purely for their algebraic properties and physical interpretations are obtained later and only with considerable effort. Indeed the Dirac matrices themselves still lack a generally accepted physical interpretation.

Our gamma matrices $T, X, Y, Z$ are matrices representing the intrinsic action of translation by one natural unit along the $t, x, y, z$ directions. We don’t have to interpret them as such. That is what they are. We thus know what their eigenvalues should represent. They should give us intrinsic energy and momentum in natural units. We expect these to relate to ordinary energy and momentum in the same way that spin relates to angular momentum. We may think of them as properties internal to the particle. Note that intrinsic energy is unrelated to rest mass which is a function of ordinary energy and momentum.

Since $T, X, Y, Z$ anti-commute, intrinsic energy and momentum are not simultaneously observable. The eigenvalues of $T$ are $\pm \frac{1}{2}i$ while those of $X, Y$ and $Z$ are $\pm \frac{1}{2}$. The lack of a factor of $i$ in the eigenvalues of $X, Y$ and $Z$ is of note. This occurs because $X$ for example is Hermitian while $T$ is anti-Hermitian. In other words the difference arises because the associated representation of $SO(2, 3)$ is not unitary. Disregarding any factors of $i$ that arise, the intrinsic energy or momentum as measured along any chosen axis is always $\pm \frac{1}{2}$ in natural units. In ordinary units the intrinsic energy is $\pm \frac{\hbar}{2r}$ and the intrinsic momentum measured along an axis is $\pm \frac{\hbar}{2rc}$. Note that only one of these can be observed at a time.

Intrinsic velocity along a given direction, were we interested in defining such a thing, would be most naturally defined as the quotient of intrinsic energy with intrinsic momentum and we could therefore obtain it from appropriate quotients of the Dirac matrices. The operators so obtained will not commute so intrinsic velocity could only be measured along one direction at a time. The intrinsic velocity measured along any direction would be either $c$ or $-c$. 

126
The operators \((\gamma^0)^{-1}\gamma^i\) have in fact long been recognised as velocity operators. Their eigenvalues of \(\pm c\) have thus been regarded as problematic since the electron, having finite mass, should not be moving at the speed of light. The usual explanation for this paradox envisages the instantaneous electron velocity as oscillating very rapidly between \(+c\) and \(-c\) in such a way that the observed overall velocity is finite, an effect commonly known as the Zitterbewegung or “shaking motion” of the electron. Getting this explanation to work has however proved difficult.

The realisation that the velocity measured by these operators should be interpreted as intrinsic solves the problem completely. An intrinsic velocity of \(c\) can coexist with an ordinary finite velocity quite easily since we no longer have to reconcile the two. The intrinsic velocity is free to do whatever it likes independent of the ordinary velocity just as spin is independent of angular momentum. The intrinsic velocity need not even oscillate. The problem of the Zitterbewegung is thus resolved.

As can be seen in table 1.2 commutators of the Dirac matrices \(\{X, Y, Z\}\) of translations are precisely the operators \(\{I, J, K\}\) of rotation. We do not need to interpret them this way. That is what they are. Hence it is immediate and trivial that their eigenvalues should describe spin. Justification that these operators should be interpreted as spin operators and that the equation therefore describes particles with spin \(\frac{1}{2}\) is much more difficult in the standard approach to the Dirac equation.

As another example consider Dirac’s \(\gamma_5\) matrix. We can compute this now directly from the matrices in table 1.4. Making appropriate adjustments for the differing conventions with regard to the factor \(\frac{1}{2}\) and the complex unit we obtain the matrix

\[
8T_{XYZ} = \frac{1}{2} \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix}
\]

This is precisely the matrix \(P_\lambda\) in table 1.5. As we noted on page 9 this matrix can be interpreted as a matrix for an inversion of the \(t, x, y\) and \(z\) coordinates, which is consistent with the usual understanding of the \(\gamma_5\) operator.

Generally using equation 11.1 is a much more transparent process than using the standard Dirac approach. The quantities are well defined with clear physical meanings, their coordinate dependence is clear, their transformation properties are known, and we have a rigorous algebraic framework for working with them.
We finish this section by considering the probability current. We define

\[ J_k^* = s_{\mu\alpha} \bar{\psi} T^\mu_{k\beta} \psi^\beta \]  

(11.9)

This is real since

\[ \bar{\bar{J}}_k^* = s_{\mu\alpha} \bar{\psi}^\alpha T^\mu_{k\beta} \bar{\psi}\beta = J_k^* \]

(11.10)

Furthermore if \( \psi^\alpha \) is a solution to equation 11.2 then

\[ \nabla^k J_k^* = s_{\mu\alpha} \nabla^k \bar{\psi}^\alpha T^\mu_{k\beta} \psi^\beta + s_{\mu\alpha} \bar{\psi}^\alpha T^\mu_{k\beta} \nabla^k \psi^\beta = \lambda s_{\mu\beta} \bar{\psi}^\mu \psi^\beta + \lambda s_{\mu\alpha} \bar{\psi}^\mu \psi^\alpha \]

(11.11)

Hence \( J_k^* \) is conserved and it would therefore seem reasonable to identify it as the probability current. It is however a bullet vector and not a true vector. We therefore call \( J_k^* \) the bullet probability vector.

The fact that \( J_k^* \) is a bullet vector and not a vector makes it difficult to see how we might use it as the source term for our field equations. In particular this cannot be the \( J_i \) of equation 9.12. In our theory bullet vectors and true vectors are quite distinct and behave differently under parallel transport and we cannot simply substitute one for the other.

The presence of a bullet index indicates an object with a functional dependence on the choice of symplectic form. We can obtain an object without bullet indices by specifying a symplectic form. A symplectic form is specified by choosing a non-zero Crump scalar \( h\bullet \) which we will call the Crump factor allowing us to define \( s_{\alpha\beta} = h\bullet s_{\alpha\beta} \). The corresponding probability current vector with respect to the symplectic form specified by the Crump factor will then be \( J_k = h\bullet J_k^* \).

Of course things are not quite that simple. A probability current vector \( J_k \) defined in this way from a solution \( \psi^\alpha \) to equation 11.2 will usually not be conserved since

\[ \nabla^k J_k = \nabla^k (h\bullet J_k^*) = H^k J_k \]

(11.12)

where \( H^k \) is given by \( \nabla^k (h\bullet) = H^k h\bullet \). This is not surprising since \( J_k \) is a function of the Crump factor \( h\bullet \), and \( h\bullet \) does not appear in equation 11.2. If we want \( J_k \) to be conserved we would need to modify equation 11.2 to explicitly include terms involving \( h\bullet \).
11.2 The Dirac Lagrangian

We next seek to obtain the Dirac equation via a Lagrangian argument. This will provide a further check on the form of the equation. A Lagrangian density for fermions should also give fermion source terms for our force equations when we subject it to variations in the geometry.

Most texts on relativistic field theory present a derivation of the Dirac equation from a Lagrangian density and, as we shall see, the standard Dirac Lagrangian density translates quite naturally into our framework. Indeed we will find that some things work considerably better; for one thing our Dirac Lagrangian is an obviously well defined scalar whereas showing that the this is the case for the standard Dirac Lagrangian density requires considerable effort.

Another issue with the standard Dirac Lagrangian density is that it is complex. The translation of this Lagrangian into our context will also be complex. This is awkward since a complex Lagrangian density makes physical interpretation of the Lagrangian density unclear and threatens us with the prospect of imaginary source terms for our force equations.

The problem is not specific to our version and afflicts standard Dirac Lagrangian theory as well. The obvious cure would be to simply take the real or imaginary part. Unfortunately one component generates the Dirac equation in the Lagrangian problem while the other is responsible for generating the electromagnetic source terms under the appropriate variation of the geometry. Hence both are needed for a full treatment.

Because this issue is critical for a proper understanding of source terms we will look at the two components separately to assist us in fully understanding these issues.

We begin by seeking a Lagrangian density function $L_D$ for the fermion field given by a complex spinor $\psi^\alpha$. We want our Lagrangian density to be a real scalar.

To obtain a real scalar from a complex spinor we will need to use a bilinear form and a conjugation map. The natural invariant bilinear form for spinors is the symplectic form $s^*_{\alpha\beta}$. A framework also has a natural conjugation map as discussed in section 3.4. This enables us to separate a complex spinor into real and imaginary parts in a way which is respected by the local and global actions.

If we choose a spinor basis consisting of spinors which have no imaginary part then natural conjugation corresponds to conjugation of coordinates...
with respect to this basis; in particular in such a basis the components of \( T_{\alpha \beta}^{\gamma} \) and \( \Gamma_{k \beta}^{\alpha} \) will all be real.

The natural conjugation map is defined on Crump scalars by requiring that the symplectic form be invariant. If we use a real Crump scalar as our bullet basis then the natural conjugation map on Crump scalars will correspond to ordinary conjugation; and in particular the components of \( s^*_\alpha \beta \) will be real. Using these tools we can construct

\[
\frac{s^*_\alpha \beta}{\psi^\beta} \overline{\psi^\alpha} \quad (11.13)
\]

Conjugating this we obtain

\[
s^*_\alpha \beta \overline{\psi^\beta} \psi^\alpha = -s_{\beta \alpha} \overline{\psi^\alpha} \psi^\beta = -s_{\alpha \beta} \overline{\psi^\beta} \psi^\alpha
\]

hence this is imaginary. Multiplying by \(-i\) to extract the imaginary part gives a real Crump scalar.

For a dynamical term, the form of the standard Dirac Lagrangian suggests that we should look at

\[
s^*_\alpha \beta T_{k \lambda}^{\alpha \beta} \nabla^k \psi^\lambda \quad (11.14)
\]

As we will see later this is indeed equivalent to the dynamical term in the standard Dirac Lagrangian density. Unfortunately it is complex, just like the dynamical term in the standard Dirac Lagrangian density.

Physics texts that comment on this issue may mention that the problem can be addressed by taking the real part. They will also typically explain that this is unnecessary since when the Lagrangian problem is solved the integral of the unwanted component reduces via Stokes theorem to an integral on the boundary – a surface term – which will vanish.

This reasoning seems odd. Why persist then in using a complex Lagrangian density if an adequate real alternative is available? In fact as we will see physicists have a very good reason for wanting to hang on to the apparently superfluous component since dropping it causes a problem later when seeking source terms.

We will will use a real Lagrangian density which we will obtain by taking an appropriate component – real or imaginary – of \(11.14\). Just as it does in the standard approach we will find that this causes an issue later on when we seek source terms. We could try to sidestep that issue at this point by arguing (insincerely) that complex Lagrangian densities are not so bad after

\[\text{or imaginary, depending on how things have been defined}\]
all, but it is more illuminating to proceed in a completely straightforward manner, allow ourselves to run head on into the problem, and deal with the issue then.

To see which component of \(11.14\) we want we first look at the conjugate.

\[
s_{\alpha \beta} T_{\kappa \lambda} \psi^{\beta} \nabla^{k} \psi^{\lambda} = s_{\alpha \beta} T_{\kappa \lambda} \psi^{\beta} \nabla^{k} \psi^{\lambda} \tag{11.15}
\]

Hence the real part is

\[
\text{Re}\left(s_{\alpha \beta} T_{\kappa \lambda} \psi^{\beta} \nabla^{k} \psi^{\lambda}\right) = \frac{1}{2} s_{\alpha \beta} T_{\kappa \lambda} \left(\psi^{\beta} \nabla^{k} \psi^{\lambda} + \nabla^{k} \psi^{\beta} \psi^{\lambda}\right) = \frac{1}{2} \nabla^{k} \left(s_{\alpha \beta} T_{\kappa \lambda} \psi^{\beta} \psi^{\lambda}\right) \tag{11.16}
\]

This is a divergence and when integrating, Stokes’ theorem will allow us to convert it to an integral on the boundary\(^2\). This therefore isn’t the component we are interested in; we want the other one.

\[
\text{Im}\left(s_{\alpha \beta} T_{\kappa \lambda} \psi^{\beta} \nabla^{k} \psi^{\lambda}\right) = -\frac{1}{2} i s_{\alpha \beta} T_{\kappa \lambda} \left(\psi^{\beta} \nabla^{k} \psi^{\lambda} - \nabla^{k} \psi^{\beta} \psi^{\lambda}\right) = -\frac{1}{2} i s_{\alpha \beta} T_{\kappa \lambda} \left(\psi^{\beta} \nabla^{k} \psi^{\lambda} - \nabla^{k} \psi^{\beta} \psi^{\lambda}\right) \tag{11.17}
\]

We thus consider the expression

\[
\mathcal{L}_D = \text{Im}\left(s_{\alpha \beta} T_{\kappa \lambda} \psi^{\beta} \nabla^{k} \psi^{\lambda} + \lambda s_{\alpha \beta} \psi^{\beta} \psi^{\lambda}\right) \tag{11.21}
\]

where \(\lambda\) is a real constant relating the two terms. This is certainly real, however it is a Crump scalar and a Lagrangian density must be a true scalar.

To obtain a true scalar from a Crump scalar we must contract with a distinguished non-zero Crump scalar \(h\), which we call the Crump factor. We can think of \(h\) as fixing the choice of invariant symplectic form \(h s_{\alpha \beta}\) at every point on the manifold. We will assume for now that the Crump factor \(h\) is real\(^3\). Hence we will use for our Lagrangian density the scalar function

\[
\mathcal{L}_D = \text{Im}\left(h s_{\alpha \beta} T_{\kappa \lambda} \psi^{\beta} \nabla^{k} \psi^{\lambda} + \lambda s_{\alpha \beta} \psi^{\beta} \psi^{\lambda}\right) \tag{11.22}
\]

If we define

\[
\psi_{\nu} = h s_{\nu \beta} \psi^{\beta} \tag{11.23}
\]

\(^2\)As it is a Crump scalar however, there is a residual term which does not vanish.

\(^3\)the consequences of allowing it to be complex should be explored
this will take the form
\[ \mathcal{L}_D = \text{Im}(\bar{\psi}_\mu T^\mu_{\nu\rho} \nabla^k \psi^\nu + \lambda \bar{\psi}_\nu \psi^\nu) \] (11.24)

We now can compare this to the usual Dirac Lagrangian. We take the real part since we wish to compare it to \( \mathcal{L}_D \) which is real.
\[ \mathcal{L}_U = \text{Re}(i \psi_\nu \gamma^k \partial_k \psi^\nu - m \psi_\nu \psi^\nu) \] (11.25)

Care is needed when working with this equation since the notation differs from the notation in the rest of the book. With allowances for the notation the two expressions appear quite similar. We note the following differences.

1. The standard Lagrangian density is defined on spacetime and not the ten dimensional manifold of frames. However the additional dimensions in equation (11.24) disappear if we restrict our attention to wave functions with no Lorentz coordinate dependence.

2. Equation (11.25) uses partial derivatives while equation (11.24) uses covariant ones. This simply means that equation (11.24) is equipped with interaction terms.

Otherwise the two expressions would appear to differ only in the placement of imaginary units and the choice of constant \( \lambda \). To see if indeed this is the case we must unwrap the definition of \( \psi_\nu \) in equation (11.25). This is typically defined by
\[ \psi_\nu = (\psi^\nu)^\dagger \gamma^0 \] (11.26)

where the dagger operation is the conjugate transpose and where \( \gamma^0 \) is the Dirac matrix associated with time.

This definition is problematic for a number of reasons. The conjugate transpose of a complex vector depends on the choice of basis and so is not well defined. We really should be using the adjoint with respect to an invariant inner product of some kind, but there isn’t one. Also the \( \gamma^0 \) matrix is explicitly linked to the time coordinate. Hence \( \mathcal{L}_U \) would seem very unlikely to be scalar.

Demonstrating that \( \mathcal{L}_U \) is indeed a well defined scalar is not easy. Lengthy arguments in support of this can be found in some of the better physics texts (for example [4]). Note that by contrast \( \mathcal{L}_D \) is manifestly a well defined scalar. Not only is \( \mathcal{L}_U \) difficult to define, it is also awkward to work with. Fortunately we need only relate it to \( \mathcal{L}_D \).

In equation (11.24) we noted that \( \gamma^0 = -2iT \) where \( T \) is the local action on spinors for translation through time. In section (1.2) an explicit real matrix
is given for $T$ and also for $\Omega$, the matrix of the symplectic form. Comparing these matrices we note that $2T = \Omega$. This equality is coincidental and basis dependent since $\Omega$ is the matrix of a bilinear form while $T$ is the matrix of a linear transformation. Nevertheless in this basis we can write $\gamma^0 = -i\Omega$ and have it be true in the sense that the matrices are the same. Updating our notation for the symplectic form, $\Omega$ becomes $h \cdot \sigma_{\alpha\beta}$. Putting all this together gives $\gamma^0 = -i h \cdot \sigma_{\alpha\beta}$. Hence

$$\psi_{\mu} = (\psi^{\nu})^\dagger \gamma^0 = -ih \cdot \sigma_{\nu\alpha} \bar{\psi}_{\alpha} = -i\bar{\psi}_{\nu}$$  \hspace{1cm} (11.27)

We will also need to substitute for $\gamma^k$. Rewriting (11.7) in terms of the local action we obtain (for the four translation dimensions)

$$\gamma^k = 2i g^{km} T_m (\ast) \hspace{1cm} (11.28)$$

We can now rewrite $\mathcal{L}_U$ in terms of our notation and directly compare it to $\mathcal{L}_D$. We have

$$\mathcal{L}_U = \text{Re} \left( 2i \bar{\psi}_{\nu} g^{km} T_m \partial_k \psi^\lambda + im \bar{\psi}_{\nu} \psi^\nu \right) = \text{Im} \left( -2\bar{\psi}_{\nu} g^{km} T_m \partial_k \psi^\lambda - m \bar{\psi}_{\nu} \psi^\nu \right) \hspace{1cm} (11.29)$$

Provided therefore that we choose $\lambda = \frac{m}{2}$, ignore interactions by identifying $\partial^k$ with $\nabla^k$, and restrict our attention to spinors with no Lorentz coordinate dependence, we have shown

$$\mathcal{L}_U = -2\mathcal{L}_D \hspace{1cm} (11.30)$$

While our proof of this depended on a coincidence in a specific basis, if both sides are scalar they should be equal in any basis.

There are some subtleties here however. Our Lagrangian has an explicit dependence on the Crump factor $h \cdot$ which defines the choice of symplectic form. Changing $h \cdot$ gives a different Lagrangian and since there is no preferred symplectic form there is really no way to choose among them. This raises the awkward question of which one of these is the one that is supposed to be equivalent to the standard Dirac Lagrangian.

The answer would seem to be that they all are. It looks like the rather awkward definition of the standard Dirac Lagrangian has a hidden conformal degree of freedom arising from the choice of spinor basis and Dirac matrices. In mathematical technical terms we would say that the standard
Dirac Lagrangian is not well defined since the definition has a hidden dependence on a conformal factor. This could be corrected by making the dependence explicit.

This is not a problem created by our approach. It is a problem with the way that the definition of the standard Dirac Lagrangian is stated which our notation has simply made more apparent.

We will not consider this issue further here. We simply note at this point that our Lagrangian is consistent with the standard one and we can therefore expect to obtain comparable solutions for the Lagrangian problem, which we now seek.

Let $\Omega$ be a compact set and define $L_D = \int_\Omega L_D dx^\alpha$. Consider a variation $\psi^\alpha \mapsto \psi^\alpha + \delta \psi^\alpha$ defined on the interior of $\Omega$ and zero on the boundary, and let $\delta L_D = \int_\Omega \delta L_D dx^\alpha$ be the resulting change. Then

$$\delta L_D = \text{Im} \left( \delta \overline{\psi}_\mu T^\mu_{k\nu} \nabla^k \psi^\nu + \overline{\psi}_\mu T^\mu_{k\nu} \nabla^k (\delta \psi^\nu) + \lambda \delta \overline{\psi}_\nu \psi^\nu + \lambda \overline{\psi}_\nu \delta \psi^\nu \right)$$

Using Stokes’ theorem we obtain

$$\delta L_D = \int_\Omega \text{Im} \left( (T^\mu_{k\nu} \nabla^k \psi^\nu + \lambda \psi^\nu) \delta \overline{\psi}_\mu - (T^\mu_{k\nu} \nabla^k (\overline{\psi}_\mu) - \lambda \overline{\psi}_\nu) \delta \psi^\nu \right) dx^\alpha$$

Now $\delta \overline{\psi}_\mu$ is a function of $\delta \psi^\nu$. Expressing it as such we have

$$\psi^\mu \delta \overline{\psi}_\mu = \psi^\mu h_\bullet \delta \overline{\psi}^\nu = -\psi_\nu \delta \overline{\psi}^\nu$$

A similar treatment of the covariant derivative term is complicated by the fact that, since $\nabla^k h_\bullet \neq \mathbb{1}$, spinor index lowering does not commute with the covariant derivative. This will generate an extra term. We define a vector $H^k$ by writing $\nabla^k h_\bullet$ as a multiple of $h_\bullet$.

$$\nabla^k h_\bullet = H^k h_\bullet$$

Note that if we choose the Crump factor $h_\bullet$ to have a constant numerical value in our bullet basis, then $H^k$ is simply the Crump connection. We can then express the extra term cleanly in terms of $H^k$.

$$T^\mu_{k\nu} \nabla^k \psi^\nu \delta \overline{\psi}_\mu = T^\mu_{k\lambda} \nabla^k \psi^\nu \delta \overline{\psi}_\mu = H^k T^\mu_{k\lambda} \psi^\nu \delta \overline{\psi}^\lambda$$

\(^4\text{In no way are these independent}\)

\(^5\text{else } h_\bullet \text{ would be a true scalar and not a Crump scalar}\)
Substituting equations \([11.33\) and \([11.35\) into equation \([11.32\) gives

\[
\delta L_D = \int_\Omega \text{Im} \left( \left( T_{\kappa\mu}^\nu \nabla^k \psi_\nu - H^k T_{\kappa\mu}^\nu \psi_\nu - \lambda \psi_\mu \right) \overline{\delta \psi}^\mu - \left( T_{\kappa\mu}^\nu \nabla^k \psi_\nu - \lambda \overline{\psi}^\mu \right) \delta \psi^\mu \right) \, dx^0 \tag{11.36}
\]

The imaginary part complicates this expression making it difficult to see what we must equate to zero. We use \(\text{Im}(z) = \frac{1}{2i}(z - \overline{z})\) to rewrite it in terms of the conjugate.

\[
\delta L_D = \frac{1}{2i} \int_\Omega \left( 2T_{\kappa\mu}^\nu \nabla^k \psi_\nu - H^k T_{\kappa\mu}^\nu \psi_\nu - 2\lambda \psi_\mu \right) \overline{\delta \psi}^\mu - \left( 2T_{\kappa\mu}^\nu \nabla^k \psi_\nu - H^k T_{\kappa\mu}^\nu \overline{\psi}^\mu - 2\lambda \overline{\psi}^\mu \right) \delta \psi^\mu \, dx^0 \tag{11.37}
\]

and this must be zero for all variations \(\delta \psi^\alpha\). By considering the cases where the variations are real and imaginary respectively we see that this can only occur if both of the expressions in brackets are zero. As these are conjugate we obtain from this the single equation

\[
T_{\kappa\mu}^\nu \nabla^k \psi_\nu - \frac{1}{2} H^k T_{\kappa\mu}^\nu \psi_\nu - \lambda \psi_\mu = 0 \tag{11.38}
\]

The equation is expressed in terms of \(\psi_\nu\). Writing it in terms of \(\psi^\nu\) gives

\[
T_{\kappa\mu}^\nu \nabla^k \psi^\mu + \frac{1}{2} H^k T_{\kappa\mu}^\nu \psi^\mu + \lambda \psi^\mu = 0 \tag{11.39}
\]

Which is the extended Dirac equation, albeit with an extra term. To understand the nature of this extra term it is helpful to consider the more general equation

\[
T_{\kappa\mu}^\nu \nabla^k \psi^\mu = \alpha H^k T_{\kappa\mu}^\nu \psi^\mu - \lambda \psi^\nu \tag{11.40}
\]

which becomes equation \([11.39\) when \(\alpha = -\frac{1}{2}\). By varying \(\alpha\) we can see what effect this term has. We are particularly interested seeing if there is any physical reason why we would want to have \(\alpha = -\frac{1}{2}\).

The key turns out to be the probability current. We need this to be divergence free to ensure that matter is conserved. So how should the probability current be defined? The usual definition is something like

\[
j^k = \psi_\nu \gamma^k \psi^\nu \tag{11.41}
\]

which when translated into our notation suggests

\[
J_k = h \star s^{\lambda \alpha} T_{\kappa\beta}^\lambda \overline{\psi}^\kappa \psi^\beta \tag{11.42}
\]
This is the lowered index version. The upper index version is defined by \( J^k = g^{ki}J_i \). We will also find the following related quantities useful

\[
J = h_{\bullet} \star_{\alpha\beta} \bar{\psi}^\alpha \psi^\beta \\
J_A = h_{\bullet} \star_{\lambda\alpha} T^\lambda_{\alpha\beta} \bar{\psi}^\beta
\]  

(11.43)  

(11.44)

The fact that the Crump factor \( h_{\bullet} \) appears in the definition of \( J_k \) is interesting. It seems that the probability current can only be defined relative to a choice of symplectic form, which \( h_{\bullet} \) encodes. The implications of this are unclear and will need to be considered later. For now let’s just see if it is conserved. Assume that \( \psi^\alpha \) is a solution to equation (11.40) and consider the divergence.

\[
\nabla^k J_k = \nabla^k h_{\bullet} \star_{\alpha\beta} T^\lambda_{\beta\alpha} \bar{\psi}^\alpha \psi^\beta + h_{\bullet} \star_{\lambda\alpha} T^\lambda_{\alpha\beta} \nabla^k \bar{\psi}^\beta \psi^\beta + h_{\bullet} \star_{\lambda\alpha} \psi^\alpha \nabla^k \bar{\psi}^\beta \\
= H^k J_k + h_{\bullet} \star_{\lambda\beta} \left( \alpha H^k T^\lambda_{\beta\alpha} \bar{\psi}^\alpha - \lambda \bar{\psi}^\lambda \right) \psi^\beta + h_{\bullet} \star_{\lambda\alpha} \psi^\alpha \left( \alpha H^k T^\lambda_{\beta\alpha} \psi^\beta - \lambda \psi^\lambda \right) \\
= H^k J_k + \left( \alpha H^k J_k - \lambda J \right) + \left( \alpha H^k J_k + \lambda J \right) \\
= (1 + 2\alpha) H^k J_k
\]  

(11.45)

So the probability current \( J_k \) is conserved if and only if \( \alpha = -\frac{1}{2} \). The extra term in equation (11.39) therefore ensures that the probability current is conserved, which is a very good physical reason for including it.

There are clearly mysteries in this approach. What physical interpretation should we place on the Crump factor \( h_{\bullet} \) for example? We included this term purely in order to account for the explicit choice of symplectic form needed to obtain a true scalar Lagrangian. However our solutions depend on it as does the probability current. This suggests it is something more than simply an arbitrary choice of coordinates. Things with physical consequences are by definition physical things.

If \( h_{\bullet} \) is a physical field of some sort we should look at the consequences of varying it on the interior of a compact region \( \Omega \). Any such variation \( \delta h_{\bullet} \) can be expressed in terms of \( h_{\bullet} \) in the form \( \delta h_{\bullet} = \delta h_{\bullet} \) where \( \delta \) is a small real scalar field. This will then give \( \delta \mathcal{L}_D = \delta \mathcal{L}_D \) and we will have \( \delta \mathcal{L}_D = 0 \) for all such \( \delta \) if and only if \( \mathcal{L}_D = 0 \). But if \( \psi^\mu \) is a solution to our Dirac equation then

\[
\mathcal{L}_D = \text{Im} \left( \bar{\psi}^\mu \left( T^\mu_{\kappa\nu} \nabla^k \bar{\psi}^\nu + \lambda \psi^\kappa \right) \right) \\
= \text{Im} \left( \bar{\psi}^\mu \left( -\frac{1}{2} H^k T^\mu_{\kappa\nu} \psi^\nu \right) \right) = 0
\]  

(11.46)
so solutions to the Dirac equation minimise the Lagrangian under variation of of the Crump factor $h_\bullet$ as well as under variation of $\psi^\mu$. This is somewhat reassuring, but it doesn’t give us a dynamical equation for $h_\bullet$. However perhaps we should not expect this since it is doubtful that $\mathcal{L}_D$ is a full and complete Lagrangian for $h_\bullet$ anyway. We note in particular the absence of a dynamical term involving $\nabla_k(h_\bullet)$ which we would expect to see in such a Lagrangian.

If the Crump factor $h_\bullet$ is physical, what physics is being described? More precisely, after second quantisation what particle would we expect to get from $h_\bullet$? Since $h_\bullet$ is a (Crump) scalar it is tempting to think that we might get a Higgs\(^6\), but as there is no symmetry breaking or Higgs mechanism in sight at the moment this would be extremely premature. In fact it would be a wild guess.

Setting aside these interesting questions, we note that we have now fully solved the Lagrangian problem for the extended Dirac Lagrangian. Solutions are functions which obey equation [11.39] which is the extended Dirac equation from the previous section with an extra term. This extra term is necessary to ensure that the probability current is divergence free. The solution did however require us to introduce a Crump scalar field $h_\bullet$ of uncertain physical significance.

So far everything has been going reasonably well. In the next section the wheels will fall off.

### 11.3 The Electromagnetic Source Term.

In this section we look at how the Dirac Lagrangian behaves under the variations to the geometry discussed starting on page[109]. We expect this to yield source contributions from the fermion field towards the field equations for electromagnetism and gravity.

We begin by considering a Case 1 variation of the type given in equation [10.41]. Variations of this type gave us the Ampere-Gauss equation so we expect to be able to obtain a source term for that equation from the fermion field. The variation takes the form

\[ \delta T_{k\beta}^\alpha = 0 \quad \delta T_{kj}^i = 0 \quad \delta \Gamma_{k\beta}^\alpha = a_{k+1}^\beta \quad \delta \Gamma_{kj}^i = 0 \]

\(^6\)hence the rather hopeful choice of the letter ‘h’
This variation in the geometry does not cause any variation in the symplectic form. We are also not varying the matter fields. Hence we can add the equations
\[ \delta \psi^a = 0 \quad \delta h_\bullet = 0 \quad \delta s^\bullet_{\alpha \beta} = 0 \]

We require that \( \nabla_k s^\bullet_{\alpha \beta} = 0 \) is still true after variation. This determines the variation in the Crump connection and hence the variation in \( H^k \).

\[ \delta \Gamma^\bullet_{k \cdot} = 2a_k \quad \delta H_k = \Gamma^\bullet_{k \cdot} = 2a_k \quad \delta H^k = 2a^k \]

Under such a variation the only part of the Dirac Lagrangian to vary is the connection in the covariant derivative term which gives
\[ \delta \nabla^k \psi^\lambda = a^k \psi^\lambda \quad (11.47) \]

Hence the variation in the Dirac Lagrangian is
\[ \delta \mathcal{L}_D = \text{Im}(a^k J_k) = 0 \quad (11.48) \]

What?\footnote{The sound of wheels falling off.}

The problem, which also afflicts the standard Lagrangian, is that while the imaginary part generates the Dirac equation in the Lagrangian problem, it is the real part that gives the expected source terms. This is the reason why physicists are so keen to hang on to both components despite the difficulties of being forced to work with a complex Lagrangian density. Let’s look then at the real part which previously we discarded.

\[ \mathcal{L}_R = \text{Re}(\bar{\psi}_\mu T^\mu_{k\nu} \nabla^k \psi^{\nu} + \lambda \bar{\psi}_\mu \psi^{\nu}) \quad (11.49) \]

If \( \lambda \) is real then \( \lambda \bar{\psi}_\nu \psi^{\nu} \) is imaginary. Hence

\[ \mathcal{L}_R = \text{Re}(\bar{\psi}_\mu T^\mu_{k\nu} \nabla^k \psi^{\nu}) \]
\[ = \frac{1}{2} (\bar{\psi}_\mu T^\mu_{k\nu} \nabla^k \psi^{\nu} + \psi_\mu T^\mu_{k\nu} \nabla^k \psi^{\nu}) \quad (11.50) \]
\[ = \frac{1}{2} (\bar{\psi}_\mu T^\mu_{k\nu} \nabla^k \psi^{\nu} + \nabla^k \bar{\psi}_\mu T^\mu_{k\nu} \psi^{\nu}) \]
\[ = \frac{1}{2} (\nabla^k J_k - H^k J_k) \]

Applying a type 1 variation to this we obtain
\[ \delta \mathcal{L}_R = -a^k J_k \quad (11.51) \]

which would give us the desired source term of \( J_k \) to within a constant.

138
So should we add the real part back in and follow the crowd by using the complex Lagrangian

\[ \mathcal{L} = \bar{\psi}_\mu T^\mu_{k\nu} \nabla^k \psi^\nu + \lambda \bar{\psi}_\nu \psi^\nu \] (11.52)

That would certainly give us the right source term thus solving our problems in this section. But it would break our work in the last section since the real part in our construction is not simply a divergence. It is close to a divergence as can be seen in equation (11.50), but the extra \(- H^k J_k\) term is going to change the answer to our Lagrangian problem. To see how we must consider what happens to this term under a variation in \(\psi^\alpha\).

\[ \delta J_k = \delta \bar{\psi}_\mu T^\mu_{k\nu} \psi^\nu + \bar{\psi}_\mu T^\mu_{k\nu} \delta \psi^\nu = \bar{\psi}_\mu T^\mu_{k\nu} \delta \bar{\psi}^\nu + \bar{\psi}_\mu T^\mu_{k\nu} \delta \psi^\mu \] (11.53)

and hence

\[ \delta \mathcal{L}_R = \frac{1}{2} \left( \nabla^k (\delta J_k) - H^k \delta J_k \right) \]
\[ = \frac{1}{2} \nabla^k (\delta J_k) - \frac{1}{2} H^k \bar{\psi}_\mu T^\mu_{k\nu} \delta \bar{\psi}^\nu - \frac{1}{2} H^k \bar{\psi}_\mu T^\mu_{k\nu} \delta \psi^\mu \] (11.54)

When we solve the Lagrangian problem for a complex Lagrangian the variations in both the real and imaginary parts must be zero. The variation in the imaginary part gives us the Dirac equation as in the last section. Setting the variation in the real part to zero however gives

\[ \delta L_R = \int_\Omega \delta \mathcal{L}_R \, dx^\circ \]
\[ = \int_\Omega -\frac{1}{2} H^k \bar{\psi}_\mu T^\mu_{k\nu} \delta \bar{\psi}^\nu - \frac{1}{2} H^k \bar{\psi}_\mu T^\mu_{k\nu} \delta \psi^\mu \, dx^\circ \] (11.55)

where the divergence term vanishes using Stoke’s theorem. This must be zero for all variations. Hence the coefficients of both \(\delta \bar{\psi}^\nu\) and \(\delta \psi^\mu\) must be zero. As these are conjugate we obtain the single equation

\[ H^k \bar{\psi}_\mu T^\mu_{k\nu} = 0 \] (11.56)

which when rewritten in terms of \(\psi^{\nu'}\) gives

\[ H^k h^{\nu}_{\mu} s^{\bullet}_{\mu\nu} T^\mu_{k\lambda} \psi^\lambda = 0 \] (11.57)

This is a problematic and severely constraining equation. Since the symplectic form \(h_{\bullet} s^{\bullet}_{\alpha\beta}\) is non-singular this simplifies to the equation

\[ H^k T^\mu_{k\lambda} \psi^\lambda = 0 \] (11.58)
which asks that at each point $\psi^\lambda$ lies in the kernel of $H^k T_k(\ast)$. Hence in particular at any point where the wave function can take non-zero values we must have the vector $H^k$ pointing in a direction where the corresponding local action on spinors is singular. While this doesn’t rule out the possibility of solutions altogether, it comes close. It certainly seems to be a severe and apparently unphysical constraint on the possible values of both $H^k$ and $\psi^\alpha$.

The solution to the Lagrangian problem for the complex Lagrangian density specified in equation 11.52 is thus any solution to the two equations

$$T_{\nu k}^\nu \nabla^k \psi^\mu + \lambda \psi^\mu = 0$$

$$H^k T_{\nu k}^\nu \psi^\mu = 0 \tag{11.59}$$

Note that for such a solution we have $\mathcal{L} = 0$. While the first of these equations seems quite reasonable, the second does not.

The second equation involves the vector $H_k$ arising from the Crump factor $h^\ast$. Since we doubt that our Lagrangian density is a full and complete Lagrangian density for $h^\ast$ (it lacks a dynamical term) perhaps the cure might be to simply correct this by adding on an additional term. This would add extra terms to our solution which could help. How would this help? Note that equation 11.58 would not be nearly so problematic were it to include an extra term involving $\nabla^k H^k$.

So what might this missing dynamical term look like? Note that since $\nabla^k(h^\ast) = H^k h^\ast$ dynamical terms can be written in terms of $H^k$. The following seem plausible and worthy of investigation.

$$H^k \psi^\nu T_{\nu k}^\nu \psi^\mu \tag{11.60}$$

$$H^k \psi^\nu \nabla^k \psi^\mu \tag{11.61}$$

As both are complex we would also need to also worry about them generating imaginary source terms for our force equations. And as we are treating the Crump factor as a physical quantity we should also consider the effect of varying $h^\ast$.

Would such terms break the link to the standard Dirac Lagrangian? The standard Dirac Lagrangian does not mention the Crump factor $h^\ast$ (effectively setting it equal to 1), and ignores the conformal degree of freedom that it represents. Hence terms which involve $H^k$ in our formulation are invisible to the standard approach. That means we are free to include such terms without having to worry about breaking the link to the standard Dirac Lagrangian.

Our efforts to derive both a Dirac equation and source terms for our forces from a plausible Dirac Lagrangian have thus run into difficulty. We
have not run out of options, but there are no clear answers. What can we conclude from this?

Firstly we should note that the problem we have run into here is not a consequence of our method. The same issues exists also with the standard Dirac Lagrangian although it is considerably harder to see them in that context. Our more rigorous definitions and clearer notation have simply exposed an issue with the Dirac Lagrangian that was previously obscure.

Secondly we should note that being unable to find a Lagrangian that behaves as we would like is not a fatal flaw. The great weakness of the Lagrangian method has always been the lack of a rigorous procedure for determining the Lagrangian. We don’t know what the Lagrangian should be and have no method other than guessing to find it. We don’t even know that a correct Lagrangian exists; the physics we are seeking to describe need not arise from a Lagrangian at all.

11.4 The Gravitational Source Term.

In this section we will apply type 2 and type 3 variations to the complex Lagrangian density $\mathcal{L}$ from equation 11.52 to see what sort of source terms it generates. This Lagrangian is equivalent to the usual one and delivers the correct source term for electromagnetism. However solutions to the Lagrangian problem for this Lagrangian density must satisfy two equations 11.59. And while the first of these is the expected generalisation of the Dirac equation, the second appears unphysical.

Consequently we are not confident that $\mathcal{L}$ is the correct Lagrangian density. That means we cannot place much weight on the answers obtained in this section. They should be viewed as qualitative indicators of the possible nature of the source term for gravity only.

We first look at a Case 2 variation to the geometry. The Lagrangians $L_g$ and $L_e$ were both found to be invariant under this this type of variation and no force equations were obtained as a result. Since we don’t have any equations in need of source terms we rather hope we don’t obtain any by applying this type of variation to the complex Dirac Lagrangian $L$ since we wouldn’t know what to do with them.

Case 2 variations are specified by $\delta^\alpha_\beta$ with $\delta^a_b = 0$ and take the form

$$
\delta T^\alpha_{k\beta} = \delta^\alpha_\beta T^\alpha_{k\beta} \quad \delta T^i_{kj} = 0 = \delta^i_\ast T^i_{kj} \quad \delta \Gamma^\alpha_{k\beta} = -\nabla_k (\delta^\alpha_\beta) \quad \delta \Gamma^i_{kj} = 0
$$
The discussion starting on page 110 adds the equations

\[ \delta g^{ij} = 0 = \delta (\cdot) g^{ij} \quad \delta s_{\alpha\beta}^\bullet = \delta (\cdot) s_{\alpha\beta} \]

where an arbitrary \( \delta \cdot \cdot \) is used to define the variation in \( s_{\alpha\beta} \). As noted on page 110, \( \delta \cdot \cdot \) can take any value here including zero.

Assume for a moment that in addition to the variation in the geometry we apply compatible variations to the wave function \( \psi^\alpha \) and the Crump factor \( h_\cdot \) given by

\[ \delta \psi^\alpha = \delta (\cdot) \psi^\alpha = \delta^\alpha_\lambda \psi^\lambda \quad \delta h_\cdot = \delta (\cdot) h_\cdot \quad (11.62) \]

Then this suite of variations gives

\[ \delta (\nabla_k \psi^\alpha) = \delta (\cdot) (\nabla_k \psi^\alpha) \quad (11.63) \]

All components in the complex Dirac Lagrangian \( \mathcal{L} \) vary according to the tensor derivation \( \delta (\cdot) \), and as \( \mathcal{L} \) is scalar it follows that \( \delta \mathcal{L} = \delta (\cdot) \mathcal{L} = 0. \)

So variations of this type are non-trivial only to the extent that the variations in \( \psi^\alpha \) and \( h_\cdot \) differ from the compatible variations specified in equation (11.62). And to the extent that they do differ from those compatible variations they will simply give solutions to the Dirac Lagrangian problem. In particular, as expected, no source terms arise from applying this type of variation to the Dirac Lagrangian.

Finally we consider a Case 3 variation to the geometry. Case 3 variations are the most difficult and are responsible for the gravitational equations. They are specified by \( b_j^k \) and \( \delta_j^i \) and give

\[ \delta T^{\alpha}_{k\beta} = -\delta^m_k T^\alpha_{m\beta} \quad \delta \Gamma^\alpha_{k\beta} = b_j^k T^\alpha_{j\beta} \quad \delta g^{ij} = \delta^i_m g^{mj} + \delta^j_m g^{mi} \quad \delta s_{\alpha\beta}^\bullet = 0 \]

Assuming also that \( \delta \psi^\alpha = 0 \) and \( \delta h_\cdot = 0 \) we obtain

\[ \delta \mathcal{L} = \overline{\psi}_\nu T^\nu_{\lambda \mu} \psi^\mu b_j^k \nabla \psi^\mu \delta_a^b + \overline{\psi}_\nu T^\nu_{\mu 0} \nabla \psi^\mu \delta_a^b \quad (11.64) \]

We can now use equation (10.71) to replace \( b_j^k \) with a function of \( \delta_a^b \). Equation (10.71) is written in terms of quantities with lowered indices because that form was easiest to obtain. Putting indices back in standard position we have

\[ b_j^k = \frac{1}{a} \left( T^{\alpha}_{m\beta} T^{\alpha}_{n\gamma} g^{kn} - T^{\mu}_{j\nu} T^{\lambda}_{m\gamma} g^{kn} + T^{\alpha}_{m\gamma} T^{\alpha}_{n\gamma} g^{kn} \right) \nabla_m \delta_a^b + \left( \frac{1}{a} T^{\alpha}_{j\nu} T^{\alpha}_{m\gamma} g^{kn} + g^{ak} g_{bj} - \frac{1}{a} T^{\alpha}_{j\nu} T^{\alpha}_{m\gamma} g^{kn} \right) \delta_a^b \quad (11.65) \]
We are interested in $\delta L$ which is related to $\delta \mathcal{L}$ via
\[ \delta L = \int_{\Omega} \delta \mathcal{L} \, dx^\circ + \int_{\Omega} \mathcal{L} \delta dx^\circ \] (11.66)

When $\psi^a$ is a solution to the Dirac Lagrangian problem then $\mathcal{L} = 0$ as noted on page 140. Hence the last integral is zero and can be ignored. Hence, substituting for $b^k_j$, we obtain

\[ \delta L = \frac{1}{b} \int_{\Omega} \overline{\psi}_\nu T^\nu_{ik} g^{ij} T_{k\mu}^\lambda \psi^\mu \left( 1^{m}_{j} T^n_{nb} g^{kn} - 1^{a}_{j} T^n_{nb} g^{kn} + T^n_{mt} g^{tm} g^{kn} g_{bj} \right) \nabla_m \delta^b_a \, dx^\circ \]
\[ + \int_{\Omega} \overline{\psi}_\nu T^\nu_{ik} g^{ij} T_{k\mu}^\lambda \psi^\mu \left( \frac{1}{6} b^{ka} \right) \nabla^a \delta^b_a \, dx^\circ \]
\[ + \int_{\Omega} \overline{\psi}_\nu T^\nu_{ik} g^{ia} \nabla_b \psi^a \delta^b_a \, dx^\circ \]

The next step is to apply Stoke’s theorem to the first integral. We obtain

\[ \delta L = \frac{1}{b} \int_{\Omega} T^\nu_{ik} T_{k\mu} \left( g^{im} T^n_{nb} g^{kn} - g^{ia} T^n_{nb} g^{kn} + 1^{a}_{b} T^n_{mt} g^{tm} g^{kn} \right) \nabla_m \overline{\psi}_\nu \psi^\mu \delta^b_a \, dx^\circ \]
\[ + \int_{\Omega} T^\nu_{ik} T_{k\mu} \left( \frac{1}{6} b^{ka} \right) \nabla^a \delta^b_a \, dx^\circ \]
\[ + \int_{\Omega} \overline{\psi}_\nu T^\nu_{ik} g^{ia} \nabla_b \psi^a \delta^b_a \, dx^\circ \]

The coefficient of $\delta^b_a$ will be our source term.

\[ S^a_b = -\frac{1}{6} T^\nu_{ik} T_{k\mu} \left( g^{im} T^n_{nb} g^{kn} - g^{ia} T^n_{nb} g^{kn} + 1^{a}_{b} T^n_{mt} g^{tm} g^{kn} \right) \nabla_m \overline{\psi}_\nu \psi^\mu \]
\[ + T^\nu_{ik} T_{k\mu} \left( \frac{1}{6} b^{ka} \right) \nabla^a \delta^b_a \]
\[ + \left( \overline{\psi}_\nu T^\nu_{ik} \nabla_b \psi^a \right) g^{ia} \]

We next apply equation 5.2.9, which states
\[ T^\nu_{ik} T_{k\mu} = \frac{1}{2} T^C_{ik} T^\nu_{cp} + \frac{1}{2} g_{ik} \delta^\nu_{cp} + \frac{1}{2} g_{ik} T^\nu_{cp} \]
to obtain the equation

\[ S^a_b = -\frac{1}{b} \nabla_m \left( \overline{\psi}_\nu \psi^\mu \right) \left( \frac{1}{2} T^C_{ik} T^\nu_{cp} + \frac{1}{2} g_{ik} \delta^\nu_{cp} + \frac{1}{2} g_{ik} T^\nu_{cp} \right)
\[ + \overline{\psi}_\nu \psi^\mu \left( g^{im} T^n_{nb} g^{kn} - g^{ia} T^n_{nb} g^{kn} + 1^{a}_{b} T^n_{mt} g^{tm} g^{kn} \right)
\[ + \left( \overline{\psi}_\nu T^\nu_{ik} \nabla_b \psi^a \right) g^{ia} \]
\[ = \frac{1}{2} T^C_{ik} T^\nu_{cp} + \frac{1}{2} g_{ik} \delta^\nu_{cp} + \frac{1}{2} g_{ik} T^\nu_{cp} \] (11.67)
We now expand this out (there will be nineteen terms) and simplify. The Dirac equations (11.59) can be used to simplify some of the terms. We hope to express our answer in terms of the probability currents

\[ J = \bar{\psi}_\nu \psi^\nu \]  
\[ J^k = g^{k\nu} \bar{\psi}_\nu T^\nu_{\mu} \psi^\mu \]  
\[ J_A = g^{AB} \bar{\psi}_\nu T^\nu_{B\mu} \psi^\mu \]

and their covariant derivatives. Note that \( J \) and \( J^A \) are imaginary while \( J^k \) is real. Since a source term for gravity should be real we would hope that our imaginary terms cancel.

The first term in equation (11.67) is

\[ -\frac{1}{12} \nabla_m (\bar{\psi}_\nu \psi^\mu) (T^\nu_{\mu i} T^i_{\nu k} + \frac{1}{2} g_{ik} \psi^\nu T^\nu_{\mu} + g_{ik} T^\nu_{A\mu}) (g^{im} T^m_{nb} - g^{ia} T^m_{nb} + 1_i T^m_{nb} g^{kn}) \]

Expanding will give nine terms, which we will simplify and write in terms of \( J, J^A \) and \( J^k \). We will also use \( g^{ab} = g_{AK} g_K^{ia} \) to simplify some of the answers. The nine terms are

\[ -\frac{1}{12} \nabla_m (\bar{\psi}_\nu \psi^\mu) T^\nu_{\mu i} T^i_{\nu k} g^{im} T^m_{nb} g^{kn} = \frac{1}{12} T_{bn} T_{ik} \nabla^i J^k \]  
\[ \frac{1}{12} \nabla_m (\bar{\psi}_\nu \psi^\mu) T^\nu_{ik} T^i_{\mu} g^{ia} T^m_{nb} g^{kn} = \frac{1}{12} T_{kn} T_{bi} \nabla^i J^k \]  
\[ -\frac{1}{12} \nabla_m (\bar{\psi}_\nu \psi^\mu) T^\nu_{ik} T^i_{\nu} T^a_{\mu} T^m_{nb} g^{kn} = \frac{1}{12} T_{ma} T_{kn} \nabla^i J^k \]  
\[ -\frac{1}{24} \nabla_m (\bar{\psi}_\nu \psi^\mu) g_{ik} \psi^\nu T^m_{nb} g^{kn} = \frac{1}{24} T_{nb} \nabla^i J \]  
\[ \frac{1}{24} \nabla_m (\bar{\psi}_\nu \psi^\mu) g_{ik} \psi^\nu T^m_{nb} g^{ia} g^{kn} = \frac{1}{24} T_{bi} \nabla^i J \]  
\[ -\frac{1}{24} \nabla_m (\bar{\psi}_\nu \psi^\mu) g_{ik} \psi^\nu T^m_{nb} T^a_{\mu} T^m_{nb} g^{kn} = \frac{1}{24} T_{ma} T_{bi} \nabla^i J \]  
\[ -\frac{1}{12} \nabla_m (\bar{\psi}_\nu \psi^\mu) g_{ik} A^\nu_{A\mu} \psi^\nu T^m_{nb} T^a_{\mu} T^m_{nb} g^{kn} = \frac{1}{12} T_{ma} T_{bi} \nabla^i J^A \]  
\[ \frac{1}{12} \nabla_m (\bar{\psi}_\nu \psi^\mu) g_{ik} A^\nu_{A\mu} \psi^\nu T^m_{nb} T^a_{\mu} T^m_{nb} g^{kn} = \frac{1}{12} T_{ma} T_{bi} \nabla^i J^A \]

The first three terms cancel via the Jacobi identity. Collecting up the remaining terms we get

\[ \frac{1}{24} T_{bi} \nabla^i J + \frac{1}{12} (T_{bn} g^a_{Ai} + T_{bn} g^a_{Ai} + T_{mn} g^a_{Aj}) \nabla^i J^A \]

These are all imaginary.

The second term in equation (11.67) is

\[ \bar{\psi}_\nu \psi^\mu \left( \frac{1}{2} T^\nu_{\mu i} T_{cp} + \frac{1}{2} g_{ik} \psi^\nu T^\nu_{\mu} + g_{ik} T^\nu_{A\mu} \right) \left( \frac{1}{2} g^{ai} 1_b + g^{ak} 1_b - \frac{1}{6} g^{ij} T^j_b T^a_{mn} g^{kn} \right) \]
Expanding will give nine terms which simplify as follows.

\[
\frac{1}{4} T^k_{ik} T^\nu_{\mu} g^{ai} 1_b (\overline{\psi}_i \psi_\mu) = -\frac{1}{4} T^a_{kb} J^k
\]  \hspace{1cm} (11.81)

\[
\frac{1}{2} T^k_{ik} T^\nu_{\mu} g^{ak} 1_b (\overline{\psi}_i \psi_\mu) = \frac{1}{2} T^a_{kb} J^k
\]  \hspace{1cm} (11.82)

\[
-\frac{1}{12} T^k_{ik} T^\nu_{\mu} g^{ij} T^a_{ia} T^m_{nt} g^{kn} (\overline{\psi}_i \psi_\mu) = -\frac{1}{12} T^a_{ik} T^i_{bj} T^k_{nt} g^{mn} J^i
\]  \hspace{1cm} (11.83)

\[
\frac{1}{8} g_{ik} T^\nu_{\mu} g^{ai} 1_b (\overline{\psi}_i \psi_\mu) = \frac{1}{8} T^a_{ib} J^i
\]  \hspace{1cm} (11.84)

\[
\frac{1}{4} g_{ik} T^\nu_{\mu} g^{ak} 1_b (\overline{\psi}_i \psi_\mu) = \frac{1}{4} T^a_{ib} J^i
\]  \hspace{1cm} (11.85)

\[
-\frac{1}{12} g_{ik} T^\nu_{\mu} g^{ij} T^a_{ia} T^m_{nt} g^{kn} (\overline{\psi}_i \psi_\mu) = -\frac{1}{12} T^a_{ik} T^i_{bj} T^k_{nt} g^{mn} J^i
\]  \hspace{1cm} (11.86)

\[
\frac{1}{4} T^a_{ib} A_{\mu} g^{ai} 1_b (\overline{\psi}_i \psi_\mu) = \frac{1}{4} T^a_{ib} J^i
\]  \hspace{1cm} (11.87)

\[
\frac{1}{2} T^a_{ik} T^\nu_{\mu} g^{ak} 1_b (\overline{\psi}_i \psi_\mu) = \frac{1}{2} T^a_{ib} J^i
\]  \hspace{1cm} (11.88)

\[
-\frac{1}{12} T^a_{ik} T^\nu_{\mu} g^{ij} T^a_{ia} T^m_{nt} g^{kn} (\overline{\psi}_i \psi_\mu) = -\frac{1}{12} T^a_{ik} T^i_{bj} T^k_{nt} g^{mn} J^i
\]  \hspace{1cm} (11.89)

Collecting up terms we obtain two real terms

\[
\frac{1}{4} T^k_{kb} J^k + \frac{1}{12} T^i_{ik} T^a_{ia} T^k_{nt} g^{mn} J^i
\]  \hspace{1cm} (11.90)

and three imaginary terms

\[
\frac{1}{8} T^a_{ib} J^i + \frac{1}{8} T^a_{ib} J^i + \frac{1}{12} T^a_{ik} T^i_{bj} T^k_{nt} g^{mn} J^i
\]  \hspace{1cm} (11.91)

Finally the last term of equation \ref{11.67} gives

\[
(\overline{\psi}_i T^\nu_{\mu} \nabla_b t^j_{\mu}) g^{ia}
\]  \hspace{1cm} (11.92)

The real part of this term can be written in terms of current vectors in the form \(\frac{1}{2} \nabla_b J^a - H_b J^a\). However the imaginary part cannot be so simply expressed.

Assembling all the components, the real part of the source term is

\[
\frac{1}{4} T^k_{kb} J^k + \frac{1}{12} T^i_{ik} T^a_{ia} T^k_{nt} g^{mn} J^i + \frac{1}{2} \nabla_b J^a - H_b J^a
\]  \hspace{1cm} (11.93)

and the imaginary part is

\[
\frac{1}{12} T^a_{ik} T^\nu_{\mu} \nabla^i J^a + \frac{1}{12} T^a_{ik} T^\nu_{\mu} g^{ai} 1_b (\overline{\psi}_i \psi_\mu) + \frac{1}{8} T^a_{ib} J^i + \frac{1}{4} T^a_{ib} J^i + \frac{1}{12} T^a_{ik} T^i_{bj} T^k_{nt} g^{mn} J^i + \frac{1}{2} \nabla_b J^a - H_b J^a
\]  \hspace{1cm} (11.94)

The gravitational equations we are using expect a real source term so this imaginary part is problematic. It seems very unlikely to be zero. The terms are sufficiently diverse in nature that it would be extraordinary if via some prodigious feat of algebra-fu we could get them all to cancel. The existence of these imaginary gravitational source terms is one more reason to doubt that our Lagrangian density is correct.
11.5 Discussion

This has been an interesting chapter. Some things have worked very well. Some things have proved more difficult. It is therefore worthwhile to sum up the situation with regard to the Dirac equation in our model.

We began the chapter by observing that the Dirac equation could be very simply and nicely incorporated into our framework. The Dirac matrices themselves turned out to be simply intrinsic translation operators and the Dirac operator could therefore be viewed as a curl operator.

Adopting this viewpoint simplifies and clarifies many things about the Dirac matrices that are otherwise obscure. For example the $S$-matrix transformations of the Dirac matrices become simply the expected change of coordinate behavior for the intrinsic translation operators. That products of Dirac matrices give rotation operators is now a simple observation that can be directly calculated. The Zitterbewegung also has a simple explanation since the velocity being described is intrinsic.

Our extended Dirac equation involved wave functions defined on all ten dimensions of the manifold of frames. For those functions which are constant across the six Lorentz dimensions the equation reduces to the ordinary four dimensional Dirac equation, particularly in the Poincaré limit. Presumably these functions then represent electrons.

Functions which are not constant across the six Lorentz dimensions must therefore represent other types of fermion. Since the three rotation dimensions are compact we expect solutions across these dimensions to be discretely quantised. This opens up the possibility of a fermion field describing all fermions with a single spinor function.

The Dirac Lagrangian also was much more easily understood in our context. The version in our framework was also easier to work with. In particular it was obviously a well defined scalar, something which requires considerable work in the standard approach. We were able to show that, under the assumption that the standard Dirac Lagrangian density is scalar, it was equal to our version for functions $\psi^\alpha$ constant across the Lorentz dimensions.

However problems started to appear at this point. Our clearer notation revealed two difficulties with the Dirac Lagrangian which exist also within the standard theory but which are usually obscured by the notation.

Firstly, our Lagrangian depended on a Crump scalar $h_\cdot$ encoding the choice of symplectic form $s_{\alpha\beta} = h_\cdot s^\alpha_\beta$. This prompted us to ask, for which choice of $h_\cdot$ was our Lagrangian equal to the standard one? Our proof, which assumed only that the standard Lagrangian was a well defined scalar,
seemed to work for all of them. Looking more closely at the effect of chang-
ing the Crump factor $h_\bullet$ on the correspondence we found that a conformal
degree of freedom equivalent to a choice of $h_\bullet$ also exists within the standard
Dirac Lagrangian arising from a combination of choice of Dirac matrices and
a conformal change of spinor basis. Mathematically therefore the standard
Dirac Lagrangian density is not well defined. To correct this problem the
conformal degree of freedom in its definition should be made explicit.

In a sense we can view uncovering this issue as a positive for our frame-
work. While we might not be happy about discovering a problem, we should
credit the added clarity of our notation and approach that enabled us to
discover it.

The second obvious problem with the Dirac Lagrangian density is the
fact that it is complex. A Lagrangian density strictly speaking should be
real. Allowing it to be complex makes it difficult to attach to it a rea-
sonable physical interpretation, and threatens to generate complex source
terms for our force equations. This is a thoroughly explored issue with the
standard Dirac Lagrangian where the use of a complex Lagrangian density
is usually justified on the basis that it gives the expected answers. Complex
Lagrangian densities are therefore now commonly accepted in modern
physics.

The way the theory is normally presented the imaginary component
gives the Dirac equation when solving the Lagrangian problem but makes a
zero contribution towards source terms for forces, which is fortunate since
we wouldn’t know what to do with an imaginary source term. The
real component on the other hand is a divergence and has no effect on the
Lagrangian problem, but gives the correct real source terms. The complex
Lagrangian combines both components and thus gives both a Dirac equation
and also the expected source terms.

Unfortunately when we translate all this to our framework things don’t
work out quite so well. The Crump factor $h_\bullet$, needed to explicitly deal with
the conformal degree of freedom gives us a real part that is no longer a
divergence. Hence including the real part changes the solution to the La-
grangian problem by introducing an additional constraint, equation [11.58]
which appears unphysical. It is possible that this extra constraint may be
more palatable if a dynamical term for $h_\bullet$ is included in our Lagrangian.
However we did not pursue this idea further.

This same issue must exist even in standard Dirac Lagrangian theory
and should become apparent there if the conformal degree of freedom de-
scribed in our notation by $h_\bullet$ is made explicit and treated rigorously. Hence

---

\footnote{physics generally has an ‘ends justifies the means’ attitude toward mathematics}
it is not simply the case that Dirac Lagrangian theory failed to work in our framework. It looks like it has never actually worked properly even in the standard case. It only seems to work because the parts that are broken are hidden by the use of inadequate notation.

Finally we found that the much more complicated variations which generate gravity in our framework do not act trivially on the imaginary component of the Dirac Lagrangian. This means they generate imaginary source terms for gravity which would break our gravitational equations. This is one more reason to doubt that we have the correct Lagrangian density.

Not being able to find the correct Lagrangian density does not invalidate or disprove the framework approach. Indeed what our approach has done is clarified the situation revealing problems that were previously hidden and pointing the way towards their solution. So where should we look to solve these issues?

One bright light in the darkness here is that using the mathematical tools in our framework makes it very easy to construct scalar functions to use as candidate Lagrangians. We already have noted possible dynamical terms for $h_s$ and the effect of adding these should be explored.

Furthermore the curvature acts on the spinors which opens up the possibility of Lagrangians that combine both curvature and the wave functions in more interesting ways. Consider for example the following constructions.

\[ g^{ia}g^{jb}\overline{\psi}_\nu R^\nu_{ij\lambda}R^\lambda_{\alpha\beta} \psi_\mu \]  
\[ h_s\gamma_{a\beta}g^{ia}g^{jb}R_{ij\mu}\overline{\psi}_\nu R^\nu_{ab\nu} \psi_\mu \]  
\[ g^{ia}g^{jb}[\nabla_i, \nabla_j]\overline{\psi}_\nu [\nabla_a, \nabla_b]\psi_\mu \]  
\[ h_s\gamma_{a\beta}g^{ia}g^{jb}[\nabla_i, \nabla_j]\overline{\psi}_\nu [\nabla_a, \nabla_b]\psi_\mu \]

All are trivially scalar and look like reasonable candidates for applying the Lagrangian method. They are complex, but one could correct that by taking their real or imaginary components. Indeed our problem is not that it is difficult to construct interesting looking scalar functions, rather it is all too easy. What we really need is more thought as to the physical meaning of the Lagrangian density which might restrict our choices and help guide us to the appropriate expression.
Chapter 12

Summary

It is time for this book to end, although obviously there is a huge amount remaining to be done. In this chapter we will reflect on what we set out to do; on what we actually achieved and its implications; and on the loose ends and unanswered questions that should be the subject of future research.

12.1 Objectives

We began by noting, as a great many others have noted, that the anti-deSitter group $SO(2, 3)$ seems to work better in some physical theories than the Poincaré group. Since the one contracts into the other, so long as the contraction parameter $r$ is large enough we cannot distinguish between the two groups from the point of view of classical physics and the extrinsic action. However the intrinsic action of $so(2, 3)$ gives realistic quantum numbers which the Poincaré Lie algebra does not.

If indeed $SO(2, 3)$ is the correct symmetry group for physics then we have a problem because all the mathematical tools that we use to do physics were built originally in Euclidean space and naturally express Euclidean symmetry. A manifold for example is defined via an atlas of homeomorphisms into Euclidean space; our calculus and linear algebra were all initially defined on Euclidean space; and our notions of curvature measure the departure from the properties of flat Euclidean space.

Of course the tools of modern mathematics are very flexible and it is certainly possible to add $so(2, 3)$ symmetry in a variety of ways to any given mathematical construct. However it isn’t clear how to do this consistently for all the different types of mathematical structures we need in modern physics.
What we really need are mathematical tools that have the symmetry group $\text{so}(2,3)$ baked into them from the start, so that every natural structure expressed using these tools immediately manifests this symmetry group, both extrinsically and intrinsically, in a natural and consistent way. The construction of such a mathematical toolkit was our primary objective.

Wigner [7] and others talk about what they call “the unreasonable effectiveness of mathematics in the natural sciences”, however I prefer to think that mathematics is just unreasonably effective. What mathematics particularly excels at is in revealing the unexpected logical consequences of a small set of assumptions or axioms. Our hope was that natural axioms incorporating $\text{so}(2,3)$ symmetry would not only allow the physics to be expressed more clearly, but might also allow mathematics to demonstrate a bit of that unreasonable effectiveness for us by means of some unexpected consequences.

At the outset we needed to decide what type of mathematical structures we were looking to define. We chose to direct our efforts to developing the mathematical tools required in order to do relativistic quantum field theory in curved space. We therefore sought to define curved manifolds with a natural extrinsic $\text{so}(2,3)$ symmetry; along with wave functions from those manifolds into a complex spinor space with intrinsic $\text{so}(2,3)$ symmetry. That seemed to us to be sufficient to enable a description of most physical phenomena. We stopped short of second quantisation however. Hence forces would be described in terms of fields, potentials and curvature; and fermions in terms of a spinor wave function which we can think of as specifying the fermion field.

We decided not to consider second quantisation for several reasons. Firstly we simply didn’t have time. Secondly we lacked the background knowledge needed to tackle this difficult topic. And finally second quantisation is notorious for its mathematical abuses and as mathematicians we therefore thought it prudent to steer clear of it.

12.2 The Mathematics

If you have read through the rest of the book I hope you will agree that our efforts to create a consistent axiomatic mathematical structure with the required symmetry were successful. The axioms of a framework are natural, as all good axiom systems should be. Indeed if you start from the assumption that $\text{so}(2,3)$ is the natural symmetry group for physics (which
I do admit is a big assumption) then it is hard not to believe that it can be described using a framework.

The approach we ended up taking was to use as our guide the matrix Lie group $\text{Sp}(2, \mathbb{R})$ itself, which already has most of the features we need. The matrix Lie algebra $\mathfrak{sp}(2, \mathbb{R})$ acts on this manifold as the collection of left invariant vector fields. These vector fields define derivations and the identification of the Lie algebra with each tangent space defines a consistent notion of parallel transport, and hence a connection which defines a covariant derivative. The torsion for this covariant derivative is (up to an annoying sign which we adjusted the definition of torsion to discard) the Lie algebra structure constants. The curvature is zero.

The adjoint action of the Lie algebra on itself gives a local action on vector fields which is conserved by the covariant derivative. Furthermore the $4 \times 4$ matrices of the Lie algebra defines a local action of each tangent space on $\mathbb{R}^4$. Extension of this local spinor action to an action on $\mathbb{C}^4$ is trivial. Left multiplication by matrices in $\text{Sp}(2, \mathbb{R})$ also defines a natural parallel transport and hence a covariant derivative on spinor valued functions. The global action once again respects the local action. One can of course extend this basic structure using tensor product and dual to other types of tensor, each of which has a local and global action which commute.

This example therefore has all of the features we are looking for except curvature. Our approach was to generalise from the properties of this specific example by simply adding curvature to the extrinsic action in a consistent fashion. This does however restrict us to manifolds with the same dimension as the Lie algebra. Hence we must talk always about the 10-D manifold of inertial frames and not 4-D spacetime.

Having constructed our axiomatic system we then spent some time discussing its features and in particular its mathematical properties. We have only barely scratched the surface. It is in fact a bit embarrassing to leave so many questions unanswered, but if we had stopped to explore all of these mathematical issues properly we would never have had a chance to look at the physics. And we needed to look at the physics to justify the utility of our axiom system.

The biggest mathematical omission is the lack a complete classification theorem for generalised tensors. We have only a partial classification in the small dimensional case. In particular we would really like to know what the locally trivial generalised tensors all look like. We would also like to know whether a compatible global action exist for every local action. This seems likely to be true and indeed we have proved that it is true for all the small dimensional representations we have looked at in this book. We have most
of a proof in the general case; however the proof is incomplete so this is not yet a theorem.

There are also some interesting open questions with regard to the extent to which the algebraic properties of our structure constrain the geometry and vice versa. And there are also some obvious generalisations which should be explored, the most obvious one being the generalisation to other Lie algebras.

The generalisation of greatest potential physical relevance would be to use complex spinor manifolds instead of real ones. This is equivalent to dropping the requirement than an invariant conjugation map on spinors exists, which is the most tenuous of our axioms. If we dropped this requirement we would need to consider the possibility that invariant conjugation maps did not exist, and the set of all local conjugation maps would then constitute a new kind of locally trivial tensor, possibly involving a non-trivial curvature. This would have consequences particularly for the description of fermions.

A more radical mathematical generalisation would be to drop the requirement that spinors exist from our set of axioms altogether.

In applied mathematical terms however the most glaring omission is probably the lack of specific solutions in this book. In our defence these do seem rather difficult to construct.

12.3 The Physics of Forces

The first test of our new framework was in how it coped with the physics of forces. And in that regard I would say that it has been extremely successful.

Forces should arise from the connection and from curvature. The connection is the most fundamental description, but is not a tensor which can make working with it difficult. The curvature is a less immediate description of the geometry but has the advantage of being a tensor. The action on spinors is fundamental, and hence the spinor connection and curvature are the most important objects for describing curvature.

The mathematical process of decomposition into irreducibles allowed us to separate both the spinor connection and the curvature into components in a very natural way:

\[ \Gamma_{k\alpha}^\beta = A_k^1 \Gamma_{1\alpha}^\beta + G^i_k T_{i\alpha}^\beta + N^A_k T_{A\alpha}^\beta \]
\[ R_{ij\alpha}^\beta = F_{ij}^1 \Gamma_{1\alpha}^\beta + R_{ij}^k T_{k\alpha}^\beta \]
Hence the unified force described by the spinor connection and curvature could be separated into components forces.

With regard to the scalar components we identified the four translation components of $A_k$ as the electromagnetic potential, and the translation components of $F_{ij}$ as the electromagnetic field tensor. The connection determines the curvature, which in terms of scalar components gave

$$F_{ij} = \partial_i A_j - \partial_j A_i$$

which is the expected relationship between the potential and the field and the scalar component of the Jacobi identity for the global action was

$$\partial_i (F_{jk})^{(ijk)} = 0$$

which we recognise as the Faraday-Gauss equation. This is almost miraculous. These equations of electromagnetism are just there naturally in the geometry and we don’t have to do anything to obtain them. Furthermore when we later come to look at matter we find that the covariant derivative in the Dirac equation naturally inserts $A_k$ into the expected place. Hence the action of the field on matter as specified by the connection also seems correct. We lack only a source equation to have a complete set of equations for electromagnetism.

The potential $A_k$ is a connection and not a vector, and has a hidden dependence on the choice of spinor basis, which seems to be how gauge symmetries enter into this picture. However we caution that gauge groups are not the main event here and it would a mistake to overemphasise them and attempt to force the mathematics to go along that path.

Of course we were only looking at the four translation components. What of the other six? Presumably these describe other forces. We might hope that they describe the weak and perhaps also the strong force. The other possibility is that these additional dimensions simply provide minor adjustments to the electromagnetic force to account for things like the effect on spin.

We chose not to investigate these interesting possibilities further for several reasons. Firstly we were not sure that we could recognise the field equations for the weak or strong force if we tripped over them, which in a sense we may just have done. Typically these short range forces are not described in terms of fields and we don’t know what their field equations ought to look like. These forces are more typically described in terms of their interactions which requires an understanding of matter and probably also requires second quantisation. And of course we just didn’t have time to go down this road.
We have left this therefore as unexplored territory. However we are not aware of anything here that would have the effect of invalidating our approach.

The vector components of the curvature and potential seem to describe gravity, although once again there are extra components which could suggest either additional forces or adjustment factors to take account of such things as spin. The vector component of the Jacobi identity gives the first and second Bianchi identities

\[
\begin{align*}
R_{ij}^m T_{mk}^{(ijk)} &= 0 \\
R_{im}^l T_{jk}^m + \nabla_i (R_{jk}^l)^{(ijk)} &= 0
\end{align*}
\]

and while we have written these in terms of the reduced curvature tensor, if you rewrite them in terms of the Riemann tensor they are indeed precisely the usual first and second Bianchi identities, albeit with extra dimensions.

The potential \( G^i_k \) determines the vector connection \( \Gamma^k_{ij} \) via the equation

\[
\Gamma^k_{ij} = G^l_i T^k_{lj} + \partial_i (T^\beta_{jm}) T^\alpha_{m\beta} g^{mk}
\]

where the last term can be viewed as a basis adjustment term compensating for the different hidden basis dependencies. Hence \( G^i_k \) determines the geometry of the manifold and should also therefore determine the gravitational field \( R^k_{ij} \).

However when we sought such an equation explicitly we obtained

\[
R^k_{ij} = \left[ \partial_i (G^k_j) - \partial_j (G^k_i) \right] + G^s_i G^y_j T^k_{xy} - N^A_i N^B_j T^k_{AB}
+ \left( G^m_k Q^k_{im} - G^m_i Q^k_{jm} \right)
\]

Where \( Q^\alpha_{ab} = \partial_a (T^\beta_{\alpha b}) T^\alpha_{\beta d} g^{dc} \).

This would suggest that the gravitational field depends on both the vector potential \( G^i_k \) and the versor potential \( N^A_k \).

There is an inconsistency here that we have yet to fully resolve. We suspect that the versor potential \( N^A_k \), which as a connection component has a hidden basis dependence on the spinor basis, can be made to vanish by a suitable choice of spinor basis.

The lack of an associated versor component of curvature tells us that the effect on spinors of parallel translation around a loop has no versor component. Hence we suspect that it is possible, perhaps by using path integration, to redefine our spinor bases consistently across the manifold in such a way that it eliminates the versor component of parallel translation.
If this can indeed be done then the gravitational field $R^k_{ij}$ will indeed be determined only by the gravitational potential $G^i_k$, albeit in a rather more complicated fashion than we might have hoped for.

With regards to source equations for our forces it is not difficult using our toolkit to write down versions of the Ampere-Gauss and Einstein equations. However these equations are very different and in particular they clearly are not simply components of an overall equation for $R^\alpha_{ij\beta}$. This suggests that perhaps one of these equations is not correct.

We then considered the equation

$$\nabla^j R^\beta_{j\alpha} - \frac{1}{2} g^{ts} T^r_{it} R^\beta_{rs\alpha} = \Theta^\beta_{\alpha}$$

where $\nabla^i \Theta^\beta_{\alpha} = 0$; which has components

$$\nabla^j F_{ji} - \frac{1}{2} g^{ts} T^r_{it} F_{rs} = J_i$$
$$\nabla^j R^k_{ji} - \frac{1}{2} T^s_{is} g^{st} R^k_{jt} = K^k_i$$

The scalar component is the Ampere-Gauss equation for electromagnetism while the vector component, which we call Ussher’s equation, is an alternative source equation for gravity.

It is rather bold to suggest such a different looking equation as an alternative to Einstein’s equation. However the close parallel to the equations for electromagnetism makes us confident that Ussher’s equation will approximate Newtonian gravitation in the weak field non-relativistic limit.

The only loose end at this point seemed to be determining the nature of the source terms. In preparation for this we looked at an alternative Lagrangian approach to the source equations which would enable us to later calculate source terms from the Lagrangians for matter.

The Lagrangian approach however turned out to be very interesting indeed. Finding a suitable Lagrangian density for the curvature $R^\alpha_{ij\beta}$ was not difficult. The function

$$\mathcal{L} = ||R^\alpha_{ij\beta}||^2 = \mathcal{L}_g - 4\mathcal{L}_e$$

for example is an obvious choice and separates naturally into components $\mathcal{L}_e = ||F_{ij}||^2$ and $\mathcal{L}_g = ||R^k_{ij}||^2$ for the electromagnetic and gravitational fields respectively.

The interesting part was describing all possible variation to the geometry which leave the axiomatic properties of our framework intact. This complicated our analysis.
We were able to categorise the variations into three cases. Case 1 variations only involved the scalar component of the field and minimising the Lagrangian for such variations gave the Ampere-Gauss equation. Case 2 variations were variations with the same form as a change of basis and left the Lagrangian invariant. Case 3 variations were the most complicated.

We found that these variations could be expressed in terms of two small but not independent parameters $b^n_m$ and $\delta^n_m$, and that the variation in the Lagrangian in terms of these was

$$\delta L = \int_{\Omega} -4U^k_i g'^{mi} g_{nk} b^n_m$$

$$+ \left( R^k_i R^c_{ab} g^{ia} g_{kc} \pm 4F_{ij} F^a_{ab} g^{ia}\right) \left( 4g^{jm} 1^n_b - g^{jb} 1^m_n \right) \delta^n_m \, dx^o$$

where $U^k_i$ is Ussher’s tensor

$$U^k_i = \nabla^j R^k_{ji} - \frac{1}{2} T^j_{is} g^{st} R^k_{jt}$$

In the weak field case the second term of the integral is insignificant compared to the first. Hence $U^k_i = 0$ which is Ussher’s equation in the absence of sources, solves the Lagrangian problem in the weak field case.

This however is deceptive because the dependence between $b^n_m$ and $\delta^n_m$ prevents the former from varying arbitrarily and hence additional weak field solutions may be possible.

Solving the Lagrangian problem fully we obtained the equation

$$A^{km}_m \nabla^i U^k_i + B^{km}_m U^i_k = \frac{3}{2} (S^m_n + W^m_n)$$

where $U^i_k$ is Ussher’s tensor and $W^m_n$ is a second order term. In the weak field case we can conceptualise this in terms of a two step process; firstly solving the dark equation

$$A^{km}_m \nabla^i X^k_i + B^{km}_m X^i_k = \frac{3}{2} S^m_n$$

to find the sith $X^i_k$; and then solving Ussher’s equation $U^k_i = X^k_i$, or equivalently

$$\nabla^j R^k_{ji} - \frac{1}{2} T^j_{is} g^{st} R^k_{jt} = X^k_i$$

to find the field.

What these equations are telling us is that the influence of matter on gravity is less direct than we had supposed. In the weak field case we can conceptualise this as a two stage process whereby the distribution of matter determines an intermediate quantity we call the sith; and the sith then acts
as the effective source of gravity. It is quite possible for the sith to differ from the source, and when this occurs we are likely to interpret it as dark matter. However there really is no matter present.

We have stumbled upon a possible explanation for one of the greatest mysteries of modern physics, the nature of dark matter. It seems that gravity can indeed behave in some circumstances as though matter were present when it is not, and we even have a dynamical equations to describe this behaviour.

Of course this is only a possible explanation. It must be tested against reality. That means we are going to need to actually simulate those rather horrible looking differential equations and compare the results to observation. Only once this is done if the results agree, can we truly say that we have found and explained dark matter. We might also hope to explain dark energy in the same way.

In the spirit of optimism we end this section with a conjecture. The sith satisfies the same identities as Ussher’s tensor, hence $\nabla^k X_k = 0$ and sith is conserved. We conjecture that the amount of extra sith in galaxies (dark matter) exactly balances the deficit of sith (dark energy) in intergalactic space. Hence dark energy is the result of sith that ‘should’ be present in intergalactic space going off and hanging around a galaxy pretending to be dark matter; leaving behind a deficit.

12.4 The Physics of Matter

We next turned to the equations of matter and in particular the Dirac equation which should describe fermions. It was largely the Dirac equation and its natural compatibility with $so(2,3)$ that prompted this work in the first place. We therefore expected to have no difficulty adapting the Dirac equation to our framework.

Indeed the Dirac equation does adapt very well. The Dirac operator itself becomes simply a Curl operator; the Dirac matrices are intrinsic translation operators; and everything works beautifully. Indeed because our framework affords an interpretation to the Dirac matrices it is considerably easier to work with the equation in this context than it is normally.

For wave functions which are constant across the six Lorentz dimensions the parallel to the standard Dirac equation is exact, and we can therefore describe electrons in this way. Wave functions that vary across the Lorentz dimensions need explanation. We did not investigate these further, but it is possible that they describe additional fermions.
An analysis of these more general solutions should start by using some kind of separation of variables on the equation to isolate the behaviour across the Lorentz dimensions from the behaviour across the translation dimensions. Since the rotation coordinates are compact we might hope that the behaviour across Lorentz dimensions can be described using special functions with discrete parameters, which could be regarded as determining particle type. However we did not attempt to do this.

We also desired a Lagrangian derivation of the Dirac equation in order to extract from the Lagrangian density source terms for gravity. This turned out to be considerably more complicated than we had anticipated.

The problem was not with our framework which actually made the work considerably clearer. It was much easier for example to check that our Lagrangian density was scalar. However this added clarity revealed difficulties with the standard Lagrangian approach to the Dirac equation that are typically obscured in the notation.

In particular as a result of our analysis we now doubt that the standard Dirac Lagrangian density is a well defined scalar. It seems to have an undeclared conformal degree of freedom. In our notation this degree of freedom is described by the Crump factor $h_\bullet$, which specifies the choice of invariant symplectic form and is needed to define the probability vector. This Crump factor seems to be a physical field of some sort.

Crump scalars differ from true scalars in their parallel transport properties and are linked to the electromagnetic field. Hence we can’t simply drop this factor or set it to a constant and ignore it. The non-trivial parallel transport will introduce an extra term involving the Crump connection $H^k$ every time we take a covariant derivative.

Solving the Lagrangian problem properly with the Crump factor present introduces an additional equation involving $H^k$ which seems severely constraining and non-physical. The cure seems to be to introduce a dynamical term for $h_\bullet$ which the standard Dirac Lagrangian lacks. This will convert this additional equation into a more physical looking wave equation for $h_\bullet$. The precise nature of this extra term and the interpretation of $h_\bullet$ (is it a Higgs?) were not investigated further.

Also causing difficulties with the Dirac Lagrangian density was the fact that it is complex. This is a well known annoyance with the standard theory justified on the basis that the complex Lagrangian density can be made to give the expected real answers. However the more complicated gravitational variations in our framework make the issue more critical since it appears that the complex Lagrangian gives imaginary source terms for gravity under these variations. Since we can’t have that, this is yet another
reason for thinking that we need a better Lagrangian density.

12.5 Conclusion

I hope you will agree that the mathematical tools developed in this book have proved their worth by clarifying much that is obscure and by making unexpected predictions. Possibly the most striking prediction is the new description of gravitation and in particular the explanation of dark matter. What is most striking is the minimal nature of the assumptions from which these were generated. Our description of dark matter assumed only the axioms of a framework which naturally encode the $\mathfrak{so}(2,3)$ symmetry, and the fairly natural Lagrangian $||R_{\alpha \beta}||^2$. That was all. The power of the predictions arising from such minimal assumptions demonstrates the value of this approach.

We now hope many others will come and play in this playground since there is much work remaining to be done. In particular second quantisation or similar is obviously needed. However in doing so we ask that an effort be made to respect the mathematics as much as possible. Mathematics is a powerful tool for predicting the unexpected consequences of a set of assumptions, but it can only do its job if the mathematics is not abused. The abuse of mathematics in modern mathematical physics destroys the utility of what should be one of its most powerful tools.
[1] William Crump, *Maxwells equations on a 10-dimensional manifold with local symmetry so(2,3)*, Master’s thesis, University of Waikato, 2012.

[2] Brian C. Hall, *Lie groups, lie algebras and representations: An elementary introduction*, Springer, 2003.

[3] Anthony W. Knapp, *Lie groups beyond an introduction, 2nd ed.*, Birkhauser, 2002.

[4] M. Robinson, *Symmetry and the standard model*, Springer, 2011.

[5] Matthew Ussher, *Investigating electromagnetism and gravity on a 10-dimensional manifold with local symmetry so(2,3)*, Master’s thesis, University of Waikato, 2013.

[6] Robert M. Wald, *General relativity*, Chicago Press, 1984.

[7] Eugene P. Wigner, *The unreasonable effectiveness of mathematics in the natural sciences*, Communications on Pure and Applied Mathematics XIII (1960), 001–14.