Sinc-Galerkin method for solving system of singular perturbed reaction-diffusion problems

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INTRODUCTION

Mathematicians can model many real-life problems mathematically. They generally use differential equations for modeling. Briefly, electrical systems and circuits, mechanics, population growth, the spread of disease are a few of them. On the other hand, applied mathematics interest physical systems that has no explicit solution. If domain of the problem is too large or the problem has a small parameter then its numerical solution gets harder. In these problems, model restricts the domain to small or sets the parameter to zero. As a result, the singular problem arises.

In this paper, we investigate one of those models, singularly perturbation problems (SPP). SPP's are very common problem types in science and engineering. SPP can occur in the various science field as solid mechanics, fluid dynamics, quantum mechanics, optimal control, chemical reactor, reaction-diffusion process [1]. As you can imagine SPP's can have systems like the system of differential
equations. Systems of SPP are used commonly in electrochemical analysis and population dynamic models. Some actual studies about SPP are as follows; theoretical studies on singularly perturbed Kirchhoff problem [2], investigation of asymptotical convergence on singularly perturbed integro-differential equations [3], producing symmetric collocation scheme for linear SPP with two boundary conditions [4] and developing the numerical scheme for singularly perturbed quasilinear equations [5].

In literature, there are various solving methods for SPP. Parameter-uniform numerical method [6], finite element method [7,8], dual finite element method [9], finite difference method [10,11], 6-point interpolatory subdivision method [12], second-order adaptive grid method [13] and also Sinc-Galerkin method [14]. However, the last reference is about singularly perturbation problems not system of SPP.

Sinc function and Sinc-Galerkin method (SGM) is very common in last decades. Frank Stenger was published an article about solving boundary value problems in 1979 [15], after this main article sinc functions became popular for solving boundary value problems (BVP) numerically. SGM is very effective way for solving linear, non-linear, ODE and PDE. Researchers can find articles which uses Sinc-Galerkin method for linear singular BVP [16], nonlinear BVP [17], hyperbolic PDE [18], parabolic PDE [19], fractional [20] equations. Also can be found the solution of Troesch [21], Euler-Bernoulli [22], Bratu [23] and Schrödinger [24] equations as phenomena. Besides the citations given, there are also up-to-date studies on SGM; solution of fourth-order partial integro-differential equation [25], solution of fractional convection-diffusion equation [26], mediated bioelectrocatalysis process [27] and SGM approach for thermal analysis of moving porous fin subject to nanoliquid flow [28]. Considering the citations given, there are quite a lot paper about Sinc-Galerkin Method. Despite all this research, the lack of an article about solution of singularly perturbed reaction-diffusion problems with SGM is the main motivation for this article.

In this paper, solution of the system of singular perturbed reaction-diffusion problems were investigated. Hereby also, the solution of linear and nonlinear system of differential equations with Sinc-Galerkin Method were obtained. The systems have type which are investigated are

\[
P_0(x)u_1(x) + P_1(x)u_1'(x) + P_2(x)u_2'(x) + R_0(x)u_1(x) + R_1(x)u_1'(x) = f_1(x)
\]

\[
Q_0(x)u_2(x) + Q_1(x)u_2'(x) + Q_2(x)u_2''(x) + T_0(x)u_2(x) + T_1(x)u_2'(x) + T_2(x)u_2''(x) = f_2(x)
\]

with boundary conditions as

\[
u_1(a) = u_1(b) = u_2(a) = u_2(b) = 0\text{ where } a < b < 0.
\]

Thus, problem can be described as System of Singular Perturbed Problems with Dirichlet Type Boundary Conditions.

**SINC – BASIS FUNCTION**

The function sinc(z) is defined [15] on \( \mathbb{C} \) complex plane which \( z \in \mathbb{C} \) by

\[
sinc(z) = \begin{cases} 
sin(\pi z) / (\pi z) & z \neq 0 \\
0 & z = 0 \end{cases}
\]

for \( h > 0 \) and \( k = 0, 1, 2 \ldots \) translated sinc functions with evenly spaced nodes are given by [15]

![Sinc functions with different values of k](image1.png)

Figure 1. The basis function \( S(k,h)(x) \) with \( h = \pi / 4 \) [16].
The Whittaker Cardinal Function is defined for any $h > 0$ by (3):

$$S(k,h)(x) = \frac{\sin(\pi x - kh)}{\pi(x - kh)} \begin{cases} x \neq kh \\ 1 \quad x = kh \end{cases}$$

The function converges also. Because of the examination onto the interval $(0,1)$ in this article, conformal mapping function was selected as

$$\phi = \ln\left(\frac{x}{1-x}\right).$$

Interval of problem can be different from $(0,1)$. In this situation, conformal mapping can be obtained as shown in Table 1 [29]. Also evenly, space nodes are denoted by

$$x_k = \phi^{-1}(x) = \frac{e^{kh}}{1 + e^{kh}}.$$  

This conformal mapping $\phi(x)$ carries domain $D_e$ onto domain $D_s$. Here $D_e$ and $D_s$ defined as (7) and show in Figure 2.

$$D_s = \left\{ w = u + vi, \quad \left| v \right| < d \leq \frac{\pi}{2} \right\}$$

and

$$D_e = \left\{ x + yi, \quad \arg\left(\frac{z}{1-z}\right) < d \leq \frac{\pi}{2} \right\}.$$  

This conformal mapping $\phi(x)$ carries domain $D_e$ onto strip domain $D_s$ [16].

**Definition 2.1** [14] Let $D_e$ be a simply connected domain in the complex plane and let $\partial D_e$ be boundaries. Let $a,b$ be these boundary points and $\phi(x)$ be a conformal mapping from $D_e$ onto $D_s$ such that $\phi(a) = -\infty$ and $\phi(b) = \infty$. If the inverse map of $\phi(x)$ is denoted by $\psi(x)$, define

$$\Gamma = \{ \psi(u) \in D_e : -\infty < u < \infty \} = \{z_k = \psi(kh), k = 0, \pm 1, \pm 2, \ldots\}.$$  

**Definition 2.2** [14] Let $B(D_e)$ be the class of function $F$ that is analytic in $D_e$ and satisfy

$$\int_{\psi(1+0)} F(z) \, dz \to 0, \quad as \ u = \mp \infty,$$  

where

$$L = \left\{ \arg\left(\frac{z}{1-z}\right) < d \leq \frac{\pi}{2} \right\}$$

and satisfy

$$T(F) = \int_{D_s} |F(z)| \, dz < \infty.$$  

**Table 1.** Conformal mapping and nodes for subintervals of $\mathbb{R}$ [16]

| Interval | $\phi(x)$ | $z_k$ |
|----------|-----------|-------|
| $a$ | $\ln\left(\frac{z-a}{b-z}\right)$ | $\frac{a + be^{kh}}{1 + e^{kh}}$ |
| $0$ | $\ln\left(\frac{z}{1-z}\right)$ | $\frac{e^{kh}}{1 + e^{kh}}$ |
| $0$ | $\ln(\sinh(z))$ | $(e^{kh} + \sqrt{e^{kh} + 1})$ |
| $-\infty$ | $\infty$ | $z$ |
| $-\infty$ | $\sinh^{-1}(z)$ | $kh$ |

**Figure 2.** Conformal mapping $\phi(x)$ from eye-shaped domain $D_e$ onto strip domain $D_s$ [16].
Theorem 2.1 [14] Let $\Gamma$ be $(0,1)$, $F \in B(D_\Gamma)$, then for $h > 0$ sufficiently small

$$\int_{\Gamma} F(z) dz - h \sum_{j \neq 0} \frac{F(z_j)}{\phi(z_j)} = \frac{i}{2} \int_{\partial D_\Gamma} (\tilde{F}(z)k(\phi, h)(z) \sin(\pi \phi(z)) / h) dz = I_\phi, \quad (11)$$

where

$$[k(\phi, h)]_{j \neq 0} = \left[ \frac{\sin(\pi \phi(z_j))}{\pi \phi(z_j)} \right]_{j \neq 0} = \frac{-e^{i\phi}}{\pi \phi} \quad (12).$$

Proof of Theorem 2.1 can be found in [15].

Theorem 2.2 [14] If there exists positive constants $\alpha$, $\beta$ and $C$ such that

$$\left| \frac{F(x)}{\phi(x)} \right| \leq C \left( e^{-\alpha|x||} \quad x \in \gamma\left((-\infty, \infty)\right) e^{-\beta|x||} \quad x \in \gamma\left((0, \infty)\right) \right) \quad (13)$$

then the error bound for quadrature rule (11) is given by

$$\left| \int_{\Gamma} F(z) dz - h \sum_{j \neq 0} \frac{F(z_j)}{\phi(z_j)} \right| \leq C \left( \frac{e^{-\alpha Nh}}{\alpha} + \frac{e^{-\beta Nh}}{\beta} \right) + |I_\phi|. \quad (14)$$

The infinite sum (11) is truncated with (13) to obtain (14).

Making selections $h = \frac{\pi d}{\sqrt{\alpha N}}$ and $N = \lceil \frac{\alpha}{\beta} N + 1 \rceil$ where $[.]$ is an integer part of the statement and $N$ is the integer value which specifies the grid size and lastly

$$\int_{\Gamma} F(x) dx = h \sum_{j \neq 0}^{N} \frac{F(x_j)}{\phi(x_j)} + O(e^{-\pi \alpha N d^2}) \quad (15).$$

Theorem 2.1 and 2.2 were used to approximating the integrals that arise in the formulation of discrete systems corresponding to the second order system of singular perturbed boundary value problem.

Theorem 2.3 [14] Let $\phi$ be a conformal one-to-one map of simply connected domain $D_\Gamma$ onto $D_\gamma$. Then

$$\delta^0_{\phi} = \left[ S(j, h) \phi(x) \right]_{k \neq j} = \left\{ \begin{array}{ll} 1 & k = j \\ 0 & k \neq j \end{array} \right. \quad (16)$$

$$\delta^1_{\phi} = h \frac{d}{d\phi} \left[ S(j, h) \phi(x) \right]_{k \neq j} = \left\{ \begin{array}{ll} 1 & k = j \\ (-1)^{k-j} & k \neq j \end{array} \right. \quad (17)$$

$$\delta^2_{\phi} = h \frac{d^2}{d\phi^2} \left[ S(j, h) \phi(x) \right]_{k \neq j} = \left\{ \begin{array}{ll} \frac{-\pi^2}{3} & k = j \\ -2(-1)^{k-j} & k \neq j \end{array} \right. \quad (18)$$

Proof of Theorem (2.2) and Theorem (2.3) can be found in [29].

CONVERGENCE ANALYSIS

Consider the problem in type (1) with Dirichlet Boundary Conditions $u_1(a) = u_1(b) = u_2(a) = u_2(b) = 0$ where $a < x < b$ also where $P_i$, $Q_j$ and $T_j$ for $i = 0, 1, 2$ are analytic on $D_\Gamma$. Sinc approximation was considered by the formula of Whittaker

$$u_1(x) = \sum_{k=-N}^{N} c_k S(k, h) \phi(x) \quad \text{and} \quad u_2(x) = \sum_{k=-N}^{N} d_k S(k, h) \phi(x) \quad (19)$$

with

$$S(k, h)(x) = \frac{\sin \left( \pi \frac{x-kh}{h} \right)}{\pi \frac{x-kh}{h}}. \quad (20)$$

The unknown coefficients in equation (19) are determined by orthogonalizing the residual with respect to the sinc basis functions. The Galerkin method enables to determine the $c_k$ and $d_k$ coefficients by solving the system of equations

$$\langle Lu_{i_n} - f_i, S(k, h) \phi(x) \rangle = 0 \quad \text{and} \quad \langle Lu_{j_n} - f_j, S(k, h) \phi(x) \rangle = 0 \quad (k = -N \ldots N). \quad (21)$$

Let $f(x)$ and $g(x)$ be arbitrary functions on $D_\Gamma$. The inner product of these two function is defined as follows:

$$\langle f(x), g(x) \rangle = \int_{\Gamma} w(x) f(x) g(x) dx, \quad (21)$$

where $w(x)$ is weight function. It's convenient [15] to take $w(x) = \frac{1}{\phi(x)}$.

For equation (1) using the notations (16)–(18) and definition of inner product and using quadrature rule, the followings are obtained

$$\langle P_0(x) u(x), S(k, h) \phi(x) \rangle = \int_{\Gamma} w(x) P_0(x) u(x) S(k, h)(x) dx \quad \equiv h \frac{w(x_j) P_0(x_j)}{\phi(x_j)} c_k, \quad (22)$$

$$\langle P_1(x) u'(x), S(k, h) \phi(x) \rangle = -\frac{h}{N} \sum_{j=-N}^{N} c_j \times \left[ \frac{(w(x_j) P_0(x_j))'}{\phi'(x_j)} \delta^0_{\phi} + \delta^1_{\phi} \left( \frac{w(x_j) P_0(x_j)}{\phi(x_j)} \right) \right], \quad (23)$$

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\[
\langle P(x)u(x), S(k, h) \circ \phi(x) \rangle \\
\approx h \sum_{j=-N}^{N} c_j \left[ (w(x_j)P(x_j))^{\star} \phi(x_j) - \frac{\delta^2 w}{h^2} \right] \\
+ \frac{\delta^2 w}{h} \left[ \frac{2(w(x_j)P(x_j))^{\star} + w(x_j)P(x_j)}{\phi(x_j)} \right] \\
+ w(x_j)P(x_j) \phi'(x_j) \frac{\delta^2 w}{h^2}.
\]

Here only approximations of \( P \) were wrote but same formulation can be obtained with changing the function \( P \) by \( R, Q \) and \( T \).

\[
\langle f(x), S(k, h) \circ \phi(x) \rangle \equiv h \frac{w(x_f(x_j))}{\phi(x_j)} c_j.
\]

Approximation of \( f(x) \) is equation (25) and same formulation can be obtained for \( f(x) \). The choices of \( h \) and \( N \) is same with Theorem (2.2).

These error approximation can be found by using Theorem (2.2) and (29):

- Let \( v, w \in B(D) \) for \( v = f(x) \) or \( p(x)w(x) \) then

\[
\int_{-M}^{M} \left[ \frac{1}{\phi(x)} \left( \frac{\partial^2 v}{\partial x^2} \right)(x)dx - h \left( \frac{\partial v}{\partial x} \right)_x \right] \\
\leq L \epsilon e^{-\left(\frac{1}{m}\right)^{\frac{1}{3}}},
\]

where \( L \) is positive constant that depends on \( v, w \) and \( d \).

- Let \( u(p[S(j, h) \circ \phi])w) \in B(D) \) for

\[
\int_{-M}^{M} \left[ \left( \frac{\partial^2 u}{\partial x^2} \right)(x)dx - h \sum_{j=-M}^{M} (upw)(x_j) \frac{\delta^2 w}{h} \\
+ h \left( \frac{u(pw)}{\phi} \right)(x_j) \right] \\
\leq L \epsilon e^{-\left(\frac{1}{m}\right)^{\frac{1}{3}}},
\]

where \( L \) is positive constant that depends on \( u, p, w, \phi \) and \( d \).

Sinc-Galerkin System with (20) becomes

\[
0 = (Lu_x + Lu_{xx} - f_x, S(k, h) \circ \phi),
0 = (Qu_x, S(j, h) \circ \phi),
0 = (P_u, S(j, h) \circ \phi),
0 = (P_{uu}, S(j, h) \circ \phi)
\]

\[
= (P_{uu}, S(j, h) \circ \phi) + (R_{uu}, S(j, h) \circ \phi)
\]

\[
= (R_{uu}, S(j, h) \circ \phi) - (R_{uu}, S(j, h) \circ \phi)
\]

\[
= (f_x, S(j, h) \circ \phi).
\]

Approximation of Sinc-Galerkin equation is below and bounded by error term is \( CM \epsilon e^{-\left(\frac{1}{m}\right)^{\frac{1}{3}}} \).
sys_{3} = \begin{bmatrix}<Lu_{1,n} + Lu_{2,n} - f_{1}, S(k, h) \cdot \phi(x) = 0 > \\
<Lu_{1,n} + Lu_{2,n} - f_{1}, S(k, h) \cdot \phi(x) = 0 > \\
<Lu_{1,n} + Lu_{2,n} - f_{1}, S(k, h) \cdot \phi(x) = 0 > \\
\end{bmatrix}_{2 \times 1} \tag{34}

with \( vars = [c_{n}, c_{n-1}, \ldots, c_{0}, d_{n}, d_{n-1}, \ldots, d_{0}]^{T}. \)

So if we use \( Ax = b \) equation, here formulation turns

\[ [sys_{3}]_{2 \times 1} \cdot X = [vars]_{2 \times 1}. \tag{35} \]

We can obtain coefficients \( c \) and \( d \). The coefficients will be used in Whittaker’s Cardinal Function for obtaining approximate solution of \( u_{1}(x) \) and \( u_{2}(x) \) which defined in equation (19).

**NUMERICAL RESULTS**

In this chapter of paper, system of SPPs are examined. The equation (35) was solved via Maple.

**Example 1**

Consider the singularly perturbed system of coupled reaction-diffusion two-point BVPs [30]:

\[
\begin{align*}
-\varepsilon u_{1}'(x) + 4u_{1}(x) - 2u_{2}(x) &= 1 \\
-\varepsilon u_{2}'(x) - u_{1}(x) + 3u_{2}(x) &= 2 \\
u_{1}(0) &= u_{1}(1) = u_{2}(0) = u_{2}(1) = 0
\end{align*}
\tag{36}
\]

Table 2 shows absolute error for \( N = 32 \) and \( \varepsilon = 10^{-4} \). Also Figure 3 shows exact and approximate graphs of functions with same values. Figure 4 shows error between exact and SGM solution for Example 1 (36). Table 3 shows absolute error of SGM and absolute error of asymptotic initial value method in [30] for Example 1 (36).

**Example 2**

Consider the constant coefficient problem [11]

\[
\begin{align*}
-\varepsilon u_{1}'(x) + 4u_{1}(x) - 2u_{2}(x) &= 1 \\
-\varepsilon (u_{1}'(x)) - u_{1}(x) + 3u_{2}(x) &= 2 \\
u_{1}(0) &= u_{1}(1) = u_{2}(0) = u_{2}(1) = 0 \\
\end{align*}
\tag{37}
\]

In paper [11], there is a table that shows the maximum error of their approximation. They noted the maximum
Figure 4. Error between exact solution and approximate solution for Example 1 (36) \( N = 32 \) and \( \varepsilon = 10^{-4} \).

Table 3. Comparison of absolute error of \( u_i(x) \) between SGM and [30]

| \( x \)   | Absolute Error of SGM | Absolute Error of [30] |
|-----------|------------------------|------------------------|
| \( \sqrt{\varepsilon} \) | 3.17346E-05            | 1.82E-02               |
| 2\( \sqrt{\varepsilon} \) | 5.8823E-06            | 6.36E-03               |
| 3\( \sqrt{\varepsilon} \) | 2.79308E-05           | 1.75E-03               |
| 4\( \sqrt{\varepsilon} \) | 1.76096E-05           | 4.48E-04               |
| 5\( \sqrt{\varepsilon} \) | 1.93274E-05           | 1.11E-04               |
| 6\( \sqrt{\varepsilon} \) | 1.2414E-07            | 2.73E-05               |
| 7\( \sqrt{\varepsilon} \) | 1.12516E-05           | 6.68E-06               |
| 8\( \sqrt{\varepsilon} \) | 1.06398E-05           | 1.64E-06               |
| 9\( \sqrt{\varepsilon} \) | 4.88453E-06           | 4.07E-07               |

error as 0.175908. Figure 5 shows that the graph of exact solution and SGM approximation. Figure 6 shows that the maximum error of the Sinc-Galerkin Method for \( u_1(x) \) is around 0.0026 also as seen in Table 4. This means that Sinc-Galerkin Approximation gives a better result for \( \varepsilon = 2^{-22} \) according to [11].

Example 3
Consider the constant coefficient problem [30]:

\[
\begin{align*}
-\varepsilon u_1'(x) + 4u_1(x) - 1u_1(x) &= 2 \\
-\varepsilon u_2'(x) - u_2(x) + 3u_2(x) &= 3 \\
u_1(0) = u_2(1) &= u_1(0) = u_2(1) = 0
\end{align*}
\]

(38)

Figure 5. Exact solutions and SGM Approximations for Example 2 (37) \( \varepsilon = 2^{-22} \) \( N = 64 \).

Figure 7 shows that the approximate plots of \( u_i(x) \) and \( u_0(x) \) for \( N = 32 \) and \( \varepsilon = 10^{-4} \).

Figure 8 shows that maximum error of \( u_i(x) \) and \( u_0(x) \) is almost \( 10^{-4} \). For getting significant results we will compare with paper [30]. Table 5 shows this comparison.
CONCLUSION

In this paper linear systems of singularly perturbed problems are examined. The main purpose is to obtain an approximate solution with Sinc-Galerkin Method. The absolute error of method in Example 1 (36) was calculated approximately as $10^{-7}$. In Example 2 (37) was calculated approximately as $10^{-3}$ and in Example 3 (38) was calculated approximately as $10^{-6}$. Considering the low error rate and other studies in the literature, it can be said that the method works properly. Since there is no article about Sinc-Galerkin Method related to SPP's, this paper completes a
previously untouched area in literature. When compared the effectiveness of Sinc-Galerkin Method with parameter robust method [11] or asymptotical initial value method [30], it's clearly seen that SGM gives better results. In addition to this paper, systems of singularly perturbed nonlinear problems can be investigated as different research.

AUTHORSHIP CONTRIBUTIONS

Concept; A.S.; Design; I.O.; Supervision; A.S.; Materials; A.S.; Data; I.O.; Analysis; I.O.; Literature Search; I.O.; Writing; I.O.; Critical Revision; M.O.

DATA AVAILABILITY STATEMENT

The published publication includes all graphics and data collected or developed during the study.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

Figure 8. Error between exact solution and SGM Approximation for $u_1(x)$ (above) and $u_2(x)$ (below) of Example 3 (38).

ETHICS

There are no ethical issues with the publication of this manuscript.

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