Sharp Interface Limit for a Navier-Stokes/Allen-Cahn System with Different Viscosities
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March 9, 2023

Abstract
We discuss the sharp interface limit of a coupled Navier-Stokes/Allen-Cahn system in a two dimensional, bounded and smooth domain, when a parameter $\varepsilon > 0$ that is proportional to the thickness of the diffuse interface tends to zero, rigorously. We prove convergence of the solutions of the Navier-Stokes/Allen-Cahn system to solutions of a sharp interface model, where the interface evolution is given by the mean curvature flow with an additional convection term coupled to a two-phase Navier-Stokes system with surface tension. This is done by constructing an approximate solution from the limiting system via matched asymptotic expansions together with a novel Ansatz for the highest order term, and then estimating its difference with the real solution with the aid of a refined spectral estimate of the linearized Allen-Cahn operator near the approximate solution.

Mathematics Subject Classification (2000): Primary: 76T99; Secondary: 35Q30, 35Q35, 35R35, 76D05, 76D45
Key words: Two-phase flow, diffuse interface model, sharp interface limit, Allen-Cahn equation, Navier-Stokes equation

1 Introduction and Main Result
The study of two-phase flows of macroscopically immiscible fluids is a challenging and important problem with many applications in the sciences and engineering applications. There are two classes of mathematical models to deal with two-phase flows: so-called sharp interface models and diffuse interface or phase field models. In sharp interface models the interface separating two fluids is treated as a sufficiently smooth surface with zero width and the equations of motion that hold in each fluid are supplemented by boundary conditions at the sharp interface. In diffuse interface models a mixing of the macroscopically immiscible fluids on a small length scale is taken into account. Hence the interface is treated as a transition layer of finite (but small) width $\varepsilon > 0$, which is described by a suitable scalar function (the so-called order parameter or phase field) $c_\varepsilon$. Typically it is related to the concentrations or volume fractions of the fluids. Diffuse interface models have many advantages in numerical simulations of the interfacial motion since it can describe topological singularities of interfaces such as pinch-off and reconnection. One of the important and natural problems is to investigate whether the diffuse interface model can be related to the corresponding sharp interface model in the limit in which the interfacial width $\varepsilon$ tends to zero (i.e., by the so called the sharp-interface limit). This sharp interface limit is in fact a question about the consistency of sharp and diffuse interface models.

In this paper we consider the singular limit $\varepsilon \to 0$ of the following Navier-Stokes/Allen-Cahn system:

$$
\partial_t v_\varepsilon + v_\varepsilon \cdot \nabla v_\varepsilon - \text{div}(2\nu(c_\varepsilon)Dv_\varepsilon) + \nabla p_\varepsilon = -\varepsilon \text{div}(\nabla c_\varepsilon \otimes \nabla c_\varepsilon) \quad \text{in } \Omega \times (0, T_0),
$$

(1.1)

$$
\text{div } v_\varepsilon = 0 \quad \text{in } \Omega \times (0, T_0),
$$

(1.2)
\[ \partial_t c_\varepsilon + \mathbf{v}_\varepsilon \cdot \nabla c_\varepsilon = \Delta c_\varepsilon - \frac{1}{\varepsilon^2} f'(c_\varepsilon) \quad \text{in } \Omega \times (0, T_0), \]  
(1.3)

\[ (\mathbf{v}_\varepsilon, c_\varepsilon)_{|\partial \Omega} = (0, -1) \quad \text{on } \partial \Omega \times (0, T_0), \]  
(1.4)

\[ (\mathbf{v}_\varepsilon, c_\varepsilon)_{|t=0} = (\mathbf{v}_{0,\varepsilon}, c_{0,\varepsilon}) \quad \text{in } \Omega. \]  
(1.5)

Here \( \mathbf{v}_\varepsilon, p_\varepsilon \) are the velocity and the pressure of the fluid mixture, \( c_\varepsilon \) is the order parameter, which is related to the concentration difference of the fluids, \( \nabla c_\varepsilon \otimes \nabla c_\varepsilon \) models the extra reactive stress, \( \nu(c_\varepsilon) \) describes the viscosity in dependence on \( c_\varepsilon \), and \( f \) is a suitable smooth double well potential, e.g., \( f(c) = \frac{1}{4}(c^2 - 1)^2 \). Moreover, \( D\mathbf{v}_\varepsilon = \frac{1}{2} (\nabla \mathbf{v}_\varepsilon + (\nabla \mathbf{v}_\varepsilon)^T) \), \( \nabla = \nabla_x \), \( \text{div} = \text{div}_x \), \( \Delta = \Delta_x \) are always taken with respect to \( x \in \Omega \). Furthermore, \( \Omega \subseteq \mathbb{R}^2 \) is a bounded domain with smooth boundary. We note that this model was suggested by Liu and Shen in \cite{35} as an alternative approximation of a classical sharp interface model for a two-phase flow of viscous, incompressible, Newtonian fluids. Later the model was derived by Jiang et al. \cite{22} in a more general version for fluids with different densities and phase transitions. Moreover, existence of weak solutions and long time behaviour of the model was studied. The longtime behavior of solutions of this model was studied by Gal and Grasselli \cite{21}. Recent analytic results on a mass-conserving Navier-Stokes/Allen-Cahn system and further reference can be found in Giorgini et al. \cite{22}.

We will prove the convergence of (1.3)-(1.5) to the following sharp interface limit system:

\[ \partial_t \mathbf{v}_0^\pm + \mathbf{v}_0^\pm \cdot \nabla \mathbf{v}_0^\pm - \nu^\pm \Delta \mathbf{v}_0^\pm + \nabla p_0^\pm = 0 \quad \text{in } \Omega^\pm(t), t \in (0, T_0), \]  
(1.6)

\[ \text{div} \mathbf{v}_0^\pm = 0 \quad \text{in } \Omega^\pm(t), t \in (0, T_0), \]  
(1.7)

\[ [2\nu^\pm D\mathbf{v}_0^\pm - p_0^\pm \mathbf{n}_\Gamma] = -\sigma H_\Gamma \mathbf{n}_\Gamma \]  
(1.8)

\[ [\mathbf{v}_0^\pm] = 0 \quad \text{on } \Gamma_t, t \in (0, T_0), \]  
(1.9)

\[ \mathbf{V}_\Gamma - \mathbf{n}_\Gamma \cdot \mathbf{v}_0^\pm = H_\Gamma \]  
(1.10)

\[ \mathbf{v}_0^\pm |_{\partial \Omega} = 0 \quad \text{on } \partial \Omega \times (0, T_0) \]  
(1.11)

\[ (\mathbf{v}_0^\pm, \Gamma_t)_{|t=0} = (\mathbf{v}_{0,0}^\pm, \Gamma_0). \]  
(1.12)

where \( \nu^\pm = \nu(\pm 1) \), \( \Omega \) is the disjoint union of \( \Omega^+(t), \Omega^-(t) \), and \( \Gamma_t \) for every \( t \in [0, T_0] \). \( \Omega^\pm(t) \) are smooth domains, \( \Gamma^\pm = \partial \Omega^\pm(t) \), and \( \mathbf{n}_\Gamma \) is the normal velocity of \( \Gamma_t \), with respect to \( \Omega^\pm(t) \). Moreover,

\[ [u](p, t) = \lim_{h \to 0^+} [u(p + \mathbf{n}_\Gamma(p)h) - u(p - \mathbf{n}_\Gamma(p)h)] \]

is the jump of a function \( u : \Omega \times [0, T_0] \to \mathbb{R}^2 \) at \( \Gamma_t \) in direction of \( \mathbf{n}_\Gamma \). \( \mathbf{V}_\Gamma \) and \( \mathbf{n}_\Gamma \) are the curvature and the normal velocity of \( \Gamma_t \), both with respect to \( \mathbf{n}_\Gamma \). Furthermore, \( D\mathbf{v} = \frac{1}{2} (\nabla \mathbf{v} + (\nabla \mathbf{v})^T) \) and \( \sigma = \int_\Omega \theta_0(\rho)^2 d\rho \), where \( \theta_0 \) is the so-called optimal profile that is the unique solution of

\[ -\theta_0''(\rho) + f'(\theta_0(\rho)) = 0 \quad \text{for all } \rho \in \mathbb{R}, \]  
(1.13)

\[ \lim_{\rho \to \pm \infty} \theta_0(\rho) = \pm 1, \quad \theta_0(0) = 0. \]  
(1.14)

Using the method of formally matched asymptotic expansion, this sharp interface limit was formally discussed in \cite{2} together with arguments in \cite{3}. In the present contribution we will verify this limit rigorously for well-prepared initial data as long as a sufficiently smooth solution of the limit system (1.6)-(1.12). Here it is assumed that the interfaces \( \Gamma_t \) and \( \partial \Omega \) do not intersect for all \( t \in [0, T_0] \). We note that existence of strong solutions of (1.6)-(1.12) for sufficiently smooth initial data was proved by Moser and the first author in \cite{5}. By standard parabolic theory one can show that the solution is indeed smooth for smooth initial data satisfying the necessary compatibility conditions. Existence of weak solutions for a non-Newtonian variant of (1.6)-(1.12) was proved by Liu et al. \cite{34}. The result of the present contribution generalizes a corresponding result by the first author and Liu in \cite{6}, where \( \nu^+ = \nu^- \equiv \nu \) was assumed and the terms \( \partial_t \mathbf{v}_\varepsilon + \mathbf{v}_\varepsilon \cdot \nabla \mathbf{v}_\varepsilon, \partial_t \mathbf{v}_\varepsilon^\pm + \mathbf{v}_\varepsilon^\pm \cdot \nabla \mathbf{v}_\varepsilon^\pm \), respectively, on the left-hand sides of the Navier-Stokes equation were neglected and a quasi-stationary Stokes flow was considered. Moreover, convergence was only verified for sufficiently small times \( T_0 > 0 \). A more detailed comparison will be given below.
In order to put our result into the relevant scientific context, we briefly summarize some important related results and models. A fundamental phase field model was introduced by Cahn and Hilliard to model the process of phase separation and coarsening of a binary alloy at a fixed temperature \[15\]. In the Cahn-Hilliard equation, there is no coupling between diffusion and fluid mechanics. In the case of fluids with same densities this coupling was first treated in the so-called model H, which was derived in \[21\], and leads to a Navier-Stokes/Cahn-Hilliard (is the so-called model H) system. This model describes the flow of two macroscopically immiscible, viscous, incompressible Newtonian fluids. A Boussinesq approximation of the model H in a Hele-Shaw cell is the so-called Hele-Shaw-Cahn-Hilliard system \[33\]. Existence of solutions for the model H was e.g. shown in \[10\]. One can see \[19\] for the existence of solutions for the Hele-Shaw-Cahn-Hilliard system. Based on the method of formally matched asymptotic expansion, the sharp interface limit for the model H was studied in \[8\]. The existence of solutions for the sharp interface limit system was given in \[11\] \[12\]. Despite the formal analysis for the sharp interface limit, there are only few rigorous results on proving the sharp interface limit for the model H. Using the notion of varifold solutions as discussed in \[3\] such results for large times were shown in \[14\] Appendix of for the model H, in \[19\] for the Hele-Shaw-Cahn-Hilliard system, and in \[8\] also for a generalization of the model H for fluids with different densities, which was derived in \[8\]. But in sense of varifold solutions the convergence is rather weak and holds only for a suitable subsequence and no rates of convergence were obtained.

For the Allen-Cahn equation, i.e., \(\epsilon^2 \frac{\partial c}{\partial t} = \nabla \cdot (\nabla c - \frac{c - c_0}{\epsilon}) \) with \(c_0 \equiv 0\), De Mottoni and Schatzman \[17\] proved convergence to the mean curvature flow \(V_t = N \cdot H_t\), for well-prepared initial data as long as a smooth solution of the mean curvature flow exists. They used the matched asymptotic expansion method to construct an approximate solution, and then estimated the difference between the approximate solution and the real solution with the aid of a spectral estimate of the linearized Allen-Cahn operator. The result also provides convergence rates in strong norms. This result was extended to the case of constant contact angle at the boundary of \(\pi/2\) and close to it in \[30\] \[31\]. Alternative approaches to the sharp interface limit of the Allen-Cahn equation can be found in \[13\] \[31\] (based on viscosity solutions), \[27\] \[37\] \[30\] (based on varifold solutions), \[32\] (conditional result in a \(BV\)-setting) and in \[20\] (relative entropy method, locally in time). Recently, the latter result was extended by Hensel and Liu \[25\] to the Navier-Stokes/Allen-Cahn system with constant viscosity in space dimension \(d = 2, 3\).

Moreover, Alikakos, Bates, and Chen in \[13\] proved that classical solutions of the Cahn-Hilliard equation tend to solutions of the Mullins-Sekerka problem (also called the Hele-Shaw problem) with a modification of the method used by De Mottoni and Schatzman. However, only few results with convergence rates in strong norms are known for two-phase flow models in fluid mechanics. In \[5\] and \[6\] \[7\] the authors considered a coupled Stokes/Allen-Cahn system and Stokes/Cahn-Hilliard system in two dimensions, respectively. It is shown that smooth solutions of the diffuse interface systems converge for short times to solutions of the corresponding sharp interface model, the so-called two-phase Stokes system and two-phase Stokes-Hele-Shaw system respectively, where the evolution of the free surface is governed by a convective mean curvature and the jump of the stress tensor, accounting for capillary forces, which complies the Young-Laplace law. Finally, in \[29\] Jiang, Su, Xie extended recently the result of \[5\] to the case of the instationary Stokes/Allen-Cahn system with some improvements in the error estimates.

Our main result can be stated as follows:

**THEOREM 1.1** Let \(N \geq 3\), \(N \in \mathbb{N}\), \((v_0^\epsilon, c_0^\epsilon, \Gamma)\) be a smooth solution of \(1\) \(1\) \(12\) for some \(T_0 \in (0, \infty)\). Then there are smooth \(c_{A,0} : \Omega \to \mathbb{R}\) and \(v_{A,0} : \Omega \to \mathbb{R}^2\), depending on \(\epsilon \in (0, 1)\), such that the following is true: Let \((v_\epsilon, c_\epsilon)\) be strong solutions of \(1\) \(1\) \(1\) \(5\) with initial values \(c_{0, \epsilon} : \Omega \to [-1, 1], v_{0, \epsilon} : \Omega \to \mathbb{R}^2, 0 < \epsilon \leq 1\), satisfying

\[
\|c_{0, \epsilon} - c_{A,0}\|_{L^2(\Omega)} + \epsilon^2 \|
abla (c_{0, \epsilon} - c_{A,0})\|_{L^2(\Omega)} + \|v_{0, \epsilon} - v_{A,0}\|_{L^2(\Omega)} \leq C_{\epsilon} N_{3/\delta + 1},
\]

(1.15)

for all \(\epsilon \in (0, 1)\) and some \(C > 0\). Then there are some \(\epsilon_0 \in (0, 1), R > 0\), and \(c_A : \Omega \times [0, T_0] \to \mathbb{R}, v_A : \Omega \times [0, T_0] \to \mathbb{R}^2\) (depending on \(\epsilon\)) such that

\[
\sup_{0 \leq t \leq T_0} \|c\epsilon(t) - c_A(t)\|_{L^2(\Omega)} + \|
abla (c_\epsilon - c_A)\|_{L^2(\Omega \times (0, T_0) \setminus \{0\})} \leq R_{\epsilon} N_{3/\delta + 1},
\]

(1.16a)

Our main result can be stated as follows:

**THEOREM 1.2** Let \(N \geq 3\), \(N \in \mathbb{N}\), \((v_0^\epsilon, c_0^\epsilon, \Gamma)\) be a smooth solution of \(1\) \(1\) \(12\) for some \(T_0 \in (0, \infty)\). Then there are smooth \(c_{A,0} : \Omega \to \mathbb{R}\) and \(v_{A,0} : \Omega \to \mathbb{R}^2\), depending on \(\epsilon \in (0, 1)\), such that the following is true: Let \((v_\epsilon, c_\epsilon)\) be strong solutions of \(1\) \(1\) \(1\) \(5\) with initial values \(c_{0, \epsilon} : \Omega \to [-1, 1], v_{0, \epsilon} : \Omega \to \mathbb{R}^2, 0 < \epsilon \leq 1\), satisfying

\[
\|c_{0, \epsilon} - c_{A,0}\|_{L^2(\Omega)} + \epsilon^2 \|
abla (c_{0, \epsilon} - c_{A,0})\|_{L^2(\Omega)} + \|v_{0, \epsilon} - v_{A,0}\|_{L^2(\Omega)} \leq C_{\epsilon} N_{3/\delta + 1},
\]

(1.15)

for all \(\epsilon \in (0, 1)\) and some \(C > 0\). Then there are some \(\epsilon_0 \in (0, 1), R > 0\), and \(c_A : \Omega \times [0, T_0] \to \mathbb{R}, v_A : \Omega \times [0, T_0] \to \mathbb{R}^2\) (depending on \(\epsilon\)) such that

\[
\sup_{0 \leq t \leq T_0} \|c\epsilon(t) - c_A(t)\|_{L^2(\Omega)} + \|
abla (c_\epsilon - c_A)\|_{L^2(\Omega \times (0, T_0) \setminus \{0\})} \leq R_{\epsilon} N_{3/\delta + 1},
\]

(1.16a)
\[
\begin{align*}
\|\nabla(v_t - c_A)\|_{L^2(\Omega \times (0, T_0) \cap \Gamma(2\delta))} + \varepsilon \|\partial_t(u_t - c_A)\|_{L^2(\Omega \times (0, T_0) \cap \Gamma(2\delta))} & \leq R \varepsilon^{N + \frac{1}{2}}, & (1.16b) \\
\|\nabla(c_t - c_A)\|_{L^2(\Omega \times (0, T_0) \cap \Gamma(2\delta))} & \leq R \varepsilon^{N - \frac{1}{2}}, & (1.16c)
\end{align*}
\]

and
\[
\|v_t - v_A\|_{L^2(\Omega \times (0, T_0); L^2(\Omega))} + \|v_t - v_A\|_{L^2(\Omega \times (0, T_0); H^1(\Omega))} \leq C(R)\varepsilon^{N + \frac{1}{2}}
\]

hold true for all \(\varepsilon \in (0, \varepsilon_0]\) and some \(C(R) > 0\). Here \(\Gamma(\delta), \Gamma(2\delta)\) are as in Section 2.1 below for some \(\delta > 0\) depending only on \(\Gamma\). Moreover,

\[
\lim_{\varepsilon \to 0} c_A = \pm 1 \quad \text{uniformly on compact subsets of} \ \Omega^\pm = \bigcup_{t \in [0, T_0]} \Omega^\pm(t) \times \{t\}
\]

and
\[
v_A = v_0^\pm + O(\varepsilon) \quad \text{in} \ L^\infty(\Omega \times (0, T_0)) \text{as} \ \varepsilon \to 0.
\]

Remark 1.2 In principle one can get convergence in \(L^\infty(\Omega \times (0, T_0))\) and any \(C^k\)-norm, \(k \geq 1\), if one chooses \(N\) sufficiently large and uses an interpolation argument as in the proof of Theorem 2.3 in \[13\].

Remark 1.3 Here \(c_{A,0} = c_A|_{t=0}\) and \(v_{A,0} = v_A|_{t=0}\), where the construction of \((c_A, v_A)\) is discussed in Section 3 below. In particular we will have \(c_{A,0} = \delta_{A,0} + O(\varepsilon^2)\) with
\[
c_{A,0}(x) = \zeta(d_{\Gamma_0}(x)\theta) \left(\frac{d_{\Gamma_0}(x)}{\varepsilon} \right)^{k-1} \left(1 - \zeta(d_{\Gamma_0}(x)) \left(\chi_{\Omega^+}(x) - \chi_{\Omega^-}(x)\right)\right) \quad \text{for all} \ x \in \Omega,
\]

where \(d_{\Gamma_0} = d_{\Gamma}|_{t=0}\) is the signed distance function to \(\Gamma_0\) and \(\zeta \in C^\infty(\mathbb{R})\) such that
\[
\zeta(z) = 1 \quad \text{if} \ |z| \leq \delta; \ \zeta(z) = 0 \quad \text{if} \ |z| \geq 2\delta; \ 0 \leq -z\zeta'(z) \leq 4 \quad \text{if} \ \delta \leq |z| \leq 2\delta.
\]

Finally, we remark that every sufficiently smooth solution of \[1.11-1.15\] satisfies the energy identity
\[
\frac{d}{dt} \left(\int_{\Omega} \frac{1}{2}|v_t|^2 \, dx + \int_{\Omega} \frac{1}{2}(|\nabla c_t|^2 + \frac{1}{2}f(c_t)) \, dx\right) = -\int_{\Omega} (|Dv_t|^2 + \frac{1}{2}|\mu_t|^2) \, dx
\]

for all \(t \in (0, T_0)\), where \(\mu_t = -\varepsilon \Delta c_t + \frac{1}{2}f'(c_t)\). In particular,
\[
\sup_{t \in [0, T_0]} \int_{\Omega} \frac{1}{2}|v_t(t)|^2 \, dx + \int_{\Omega} \left(\frac{1}{2}(|\nabla c_t(t)|^2 + \frac{1}{2}f(c_t(t)))\right) \, dx \\
+ \int_0^{T_0} \int_{\Omega} (|Dv_t|^2 + \frac{1}{2}|\mu_t|^2) \, dx \, dt \leq E_{0,\varepsilon},
\]

where
\[
E_{0,\varepsilon} := \int_{\Omega} \left(\frac{1}{2}|v_{0,\varepsilon(t)}|^2 + \int_{\Omega} \left(\frac{1}{2}(|\nabla c_{0,\varepsilon}(t)|^2 + \frac{1}{2}f(c_{0,\varepsilon}(t)))\right) \, dx\right). \quad (1.19)
\]

Hence the left-hand side in \[1.19\] is uniformly bounded in \(\varepsilon \in (0, 1)\) if \(\sup_{t \in (0, T_0)} E_{0,\varepsilon} < \infty\). From the form of \(c_A, v_A\) given in Section 3 and a Taylor expansion of \(f(c_t)\) it is easy to see that this is the case under the assumption \[1.15\].

Compared to \[5\] we consider the coupling of the Allen-Cahn and the Navier-Stokes system with general, possibly non-equal viscosities \(\nu^+, \nu^-\), \(\nu(c) > 0\). This brings new difficulties coming from the the nonlinear terms \(v_x \cdot \nabla v_x\) and \(-\text{div}(2\nu(c)Dv_x)\). In particular, estimates of errors in the velocity \(v_x\) are more involved since the non-constant viscosity limits them to estimates in \(L^\infty(0, T_0; L^2(\Omega)^d) \cap L^2(0, T_0; H^1(\Omega)^d)\) obtained by an energy method, cf. Theorem 1.1 below. E.g. estimates in \(L^r(0, T_0; L^q(\Omega))\) as in \[5\] Proposition 3.6 are not available. Moreover, in \[5\] only the convergence rate \(N = 2\) was considered and smallness of the time interval had to be assumed. Using an approximation of higher order \(N > 3\) we are able to prove the result as long as a smooth solution of the limit system exists. In order to deal with these difficulties, we will construct suitable approximate solution to \[1.11-1.13\] with the method of matched asymptotic expansions used in \[5\] and \[6, 7\], see Theorem 3.1 below. But there is a crucial difference in the
We use ideas from [6, 7] to improve the estimate for the error in the velocity by a factor linearly. It might be helpful for future results on sharp interface limits as well. Furthermore, function construction compared to the latter contribution concerning the highest order term of the height compared to [5] as well as [6, 7] since the most critical terms related to novel ansatz, which does not follow standard ways for formally matched asymptotics. The idea is to use an ansatz based on linearization for the term instead of including it in the stretched variable, cf. Section 3 below for details. This new ansatz simplifies a lot of technical difficulties compared to [5] and parametrize \((\Gamma_t)_{t \in [0,T_0]}\) with the aid of a family of smooth diffeomorphisms \(X_0: \mathbb{T}^1 \times [0,T_0] \to \Omega\) such that \(\partial_s X_0(s,t) \neq 0\) for all \(s \in \mathbb{T}^1, t \in [0,T_0]\). Moreover, let
\[
\tau(s,t) = \frac{\partial_s X_0(s,t)}{|\partial_s X_0(s,t)|}, \quad \text{and} \quad n(s,t) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tau(s,t)
\]
be the normalized tangent and normal vectors on \(\Gamma_t\) at \(X_0(s,t)\), where we choose the orientation of \(\Gamma_t\) (induced by \(X_0(\cdot,t)\)) such that \(n(s,t)\) is the exterior normal with respect to \(\Omega^-(t)\). Furthermore, we define
\[
n_{\Gamma_t}(x) := n(s,t) \quad \text{for all} \quad x = X_0(s,t) \in \Gamma_t, \quad (2.1)
\]
\[
V(s,t) := V_{\Gamma_t}(X_0(s,t)), \quad H(s,t) := H_{\Gamma_t}(X_0(s,t)) \quad \text{for all} \quad s \in \mathbb{T}^1, t \in [0,T_0], \quad (2.2)
\]
where \(V_{\Gamma_t}\) and \(H_{\Gamma_t}\) are the normal velocity and (mean) curvature of \(\Gamma_t\) (with respect to \(n_{\Gamma_t}\)). In particular,
\[
V_{\Gamma_t}(X_0(s,t)) = V(s,t) = \partial_s X_0(s,t) \cdot n(s,t) \quad \text{for all} \quad (s,t) \in \mathbb{T}^1 \times [0,T_0].
\]
We use tubular neighborhoods of \(\Gamma_t\): For \(\delta > 0\) sufficiently small, the orthogonal projection \(P_{\Gamma_t}(x)\) of all
\[
x \in \Gamma_t(3\delta) = \{ y \in \Omega : \text{dist}(y, \Gamma_t) < 3\delta \}
\]
is well-defined and smooth for all \(t \in [0,T_0]\). We choose \(\delta\) so small that \(\text{dist}(\partial\Omega, \Gamma_t) > 3\delta\) for every \(t \in [0,T_0]\). Every \(x \in \Gamma_t(3\delta)\) has a unique decomposition
\[
x = P_{\Gamma_t}(x) + r n_{\Gamma_t}(P_{\Gamma_t}(x)),
\]
where \(r = \text{sdist}(\Gamma_t, x)\). Here
\[
dr(x,t) := \text{sdist}(\Gamma_t, x) = \begin{cases} \text{dist}(\Omega^-(t), x) & \text{if} \ x \not\in \Omega^-(t), \\ -\text{dist}(\Omega^+(t), x) & \text{if} \ x \in \Omega^-(t). \end{cases}
\]
For the following we define for \(\delta' \in (0,3\delta]\)
\[
\Gamma(\delta') = \bigcup_{t \in [0,T_0]} \Gamma_t(\delta') \times \{t\}, \quad \Omega^\pm = \bigcup_{t \in [0,T_0]} \Omega^\pm(t) \times \{t\}.
\]
We will frequently use
\[
\int_{\Gamma_1(x_t')} f(x) \, dx = \int_{-\delta'}^{\delta'} \int_{\Gamma_1} f(p + r n_{\Gamma_1}(p)) J(r, p, t) d\sigma(p) dr
\]
for any \(\delta' \in (0, 3\delta]\), where \(J: (-3\delta, 3\delta) \times \Gamma \to (0, \infty)\) is a smooth function depending on \(\Gamma\).

In particular, we use new coordinates in \(\Gamma(3\delta)\) with the aid of the mapping
\[
X: (-3\delta, 3\delta) \times T^1 \times [0, T_0] \to \Gamma(3\delta) \quad \text{by} \quad X(r, s, t) := X_0(s, t) + r n(s, t),
\]
where
\[
r = s \, \text{dist}(\Gamma_1, x), \quad s = X_0^{-1}(P_{\Gamma_1}(x), t) =: S(x, t).
\] (2.3)

Moreover, we use
\[
\nabla d r(x, t) = n_{\Gamma_1}(P_{\Gamma_1}(x)), \quad \partial_{\delta} d r(x, t) = -V_{\Gamma_1}(P_{\Gamma_1}(x)), \quad \Delta d r(q, t) = -H_{\Gamma_1}(q)
\] (2.4)
for all \((x, t) \in \Gamma(3\delta)\), \((q, t) \in \Gamma\), resp., cf. Chen et al. \[16\] Section 4.1, and define
\[
\partial_\tau u(x, t) := \tau(S(x, t), t) \nabla_\tau u(x, t), \quad \nabla_\tau u(x, t) := \partial_\tau u(x, t) \tau(S(x, t), t)
\] (2.5)
for all \((x, t) \in \Gamma(3\delta)\).

In the following we associate a function \(\phi(x, t)\) to \(\tilde\phi(r, s, t)\) such that
\[
\phi(x, t) = \tilde\phi(d r(x, t), S(x, t), t) \quad \text{or} \quad \phi(X_0(s, t) + r n(s, t), t) = \tilde\phi(r, s, t).
\] (2.6)

As in \[5\] we have
\[
\partial_\delta \tilde\phi(x, t) = -V_{\Gamma_1}(P_{\Gamma_1}(x)) \partial_\delta \tilde\phi(x, s, t) + \partial^X_\delta \tilde\phi(x, s, t),
\]
\[
\nabla \tilde\phi(x, t) = n_{\Gamma_1}(P_{\Gamma_1}(x)) \partial_\delta \tilde\phi(x, s, t) + \nabla^X_\delta \tilde\phi(x, s, t),
\] (2.7)
\[
\Delta \tilde\phi(x, t) = \partial^2_\delta \tilde\phi(x, s, t) + \Delta d r(x, t) \partial_\delta \tilde\phi(x, s, t) + \Delta^X_\delta \tilde\phi(x, s, t),
\]
where \(r, s\) are as in \[\[2.3\]\] and we use the notation
\[
\partial^X_\delta \tilde\phi(x, s, t) = \partial_\delta \tilde\phi(x, s, t) + \partial_\delta S(x, t) \cdot \partial_\delta \tilde\phi(x, s, t),
\]
\[
\nabla^X_\delta \tilde\phi(x, s, t) = \nabla S(x, t) \partial_\delta \tilde\phi(x, s, t),
\] (2.8)
\[
\Delta^X_\delta \tilde\phi(x, s, t) = (\Delta S)(x, t) \partial_\delta \tilde\phi(x, s, t) + |\nabla S(x, t)|^2 \partial^2_\delta \tilde\phi(x, s, t),
\]
cf. \[16\] Section 4.1 for more details. In \[\[2.8\]\] \(x\) is understood via \(x = n(s, t) r + X_0(s, t)\). Note that \(\nabla^X g\) is a function of \((r, s, t)\):
\[
\nabla^X g(r, s, t) = (\nabla S)(x, t) \partial_\delta g(s, t), \quad \text{where} \quad x = X(r, s, t).
\] (2.9)

Therefore we define for every \(h: T^1 \times [0, T_0] \to \mathbb{R}\)
\[
(\nabla^X h)(s, t) := (\nabla^X h)(0, s, t),
\]
\[
(\Delta^X h)(s, t) := (\Delta^X h)(0, s, t), \quad (2.10)
\]
and
\[
(L^h)(r, s, t) := (\nabla h)(0, r, s) - (\nabla h)(s, t),
\]
\[
(L^h)(r, s, t) := (\Delta h)(0, r, s) - (\Delta h)(s, t), \quad (2.11)
\]
where the coefficients of the latter operators vanish for \(r = 0\), which corresponds to \(x \in \Gamma_1\).

Finally, we denote
\[
(X_0^u)(s, t) := u(X_0(s, t), t) \quad \text{for all} \quad s \in T^1, \quad t \in [0, T_0],
\]
\[
(X_0^{-1} v)(p, t) := v(X_0^{-1}(p, t), t) \quad \text{for all} \quad (p, t) \in \Gamma
\]
if \(u: \Gamma \to \mathbb{R}^N\) and \(v: \Gamma_0 \times [0, T_0] \to \mathbb{R}^N\) for some \(N \in \mathbb{N}\).
2.2 Function Spaces

Lemma 2.1 For any $u \in H^1_0(\Gamma_1(2\delta))$, $v \in H^1_0(\Gamma_1(2\delta) ; \mathbb{R}^2)$ we have

$$[\partial_u, \nabla v]u = \tau(\partial_u n \cdot \nabla u)$$

and

$$\int_{\Gamma_1(2\delta)} u \text{div}_v v \text{d}x = - \int_{\Gamma_1(2\delta)} \nabla u \cdot v \text{d}x - \int_{\Gamma_1(2\delta)} \kappa n \cdot v \text{d}x,$$

where $\kappa = - \text{div} n$ and

$$\nabla = (I - n(S(\cdot, \cdot)) \otimes n(S(\cdot, \cdot))) \nabla.$$

Proof: This lemma is a consequence of [5, Corollary 2.7].

2.2 Function Spaces

We denote by $L^p(U)$ the usual Lebesgue space with respect to the Lebesgue measure and the $L^p$-Sobolev space of order $m \in \mathbb{N}$ on an open set $U \subseteq \mathbb{R}^d$ is denoted by $W^m_p(U)$. Furthermore, $H^s(U)$ is the $L^2$-Sobolev space of order $s \in \mathbb{R}$ and $H^s_0(U)$ is the closure of $C_c^\infty(U)$ in $H^s(U)$. The $X$-valued variants are denoted by $W^m_p(U; X)$, $L^p(U; X)$, and $H^s(U; X)$, respectively. As in [16] we define

$$L^{p,\infty}(\Gamma_1(2\delta)) := \left\{ f : \Gamma_1(2\delta) \to \mathbb{R} \text{ measurable } : \| f \|_{L^{p,\infty}(\Gamma_1(2\delta))} < \infty \right\},$$

where

$$\| f \|_{L^{p,\infty}(\Gamma_1(2\delta))} := \left( \int_{\Gamma_1(2\delta)} \text{ess sup}_{|r| \leq 2\delta} |f(X_0(s, t) + r n(s, t))|^p \text{d}s \right)^{\frac{1}{p}}.$$

We will often use the embedding

$$H^1(\Gamma_1(2\delta)) \hookrightarrow L^{4,\infty}(\Gamma_1(2\delta)),
$$

which is a consequence of the interpolation inequality

$$\| f \|_{L^{\infty}(-\delta, \delta)} \leq C \| f \|_{L^{2}(-\delta, \delta)}^{\frac{1}{2}} \| f \|_{H^{1}(\Gamma_1(2\delta))}^{\frac{1}{2}} \text{ for } f \in H^{1}(\Gamma_1(2\delta)).$$

2.3 The Stretched Variable and Remainder Terms

In the following we will use a “strecheted variable”, which is defined by

$$\rho = \frac{dr(x, t)}{\varepsilon} - h_\varepsilon(s, t) \text{ for } (x, t) \in \Gamma(3\delta), \varepsilon \in (0, \varepsilon_0),$$

where $s = S(x, t)$ as in (2.3). Here we assume that $h_\varepsilon : \mathbb{T}^1 \times [0, T] \to \mathbb{R}$ is given and (sufficiently) smooth with bounded $C^k$-Norms for sufficiently large $k \in \mathbb{N}$ uniformly in $\varepsilon \in (0, \varepsilon_0)$ for some $\varepsilon_0 > 0$.

As in [16] Section 4.2] a Taylor expansion of $\Delta dr$ in the normal direction yields

$$\Delta dr(x, t) = -H_1 \kappa_1(s, t) + \sum_{k=2}^{K-1} \kappa_k(s, t) dr(x, t)^k + dr(x, t) \kappa_K(x, t)$$

$$= -H_1 \kappa_1(s, t) - \varepsilon (\rho + h_\varepsilon(s, t)) \kappa_1(s, t) + \sum_{k=2}^{K-1} \varepsilon^k \kappa_k(s, t) (\rho + h_\varepsilon(s, t))^k + \varepsilon^K \kappa_{K, \varepsilon}(\rho, s, t),$$

where $K$ is some large integer, $s$ is understood via (2.3) and

$$\kappa_k(s, t) = -\nabla dr(X_0(s, t)) \cdot \nabla \Delta dr(X_0(s, t)),$$

$\{\kappa_j(s, t)\}_{2 \leq j \leq K-1}$ are smooth and $\kappa_{K, \varepsilon}$ is a smooth function satisfying

$$|\kappa_{K, \varepsilon}(\rho, s, t)| \leq C|\rho + h_\varepsilon(s, t)|^K \text{ for all } \rho \in \mathbb{R}, s \in \mathbb{T}^1, t \in [0, T_0], \varepsilon \in (0, 1).$$

The following lemma follows from the chain rule and (2.7), cf. [16] Section 4.2]:
Lemma 2.2 Let \( \hat{w} : \mathbb{R} \times \Omega \times [0, T_0] \rightarrow \mathbb{R} \) be sufficiently smooth and let
\[
w(x, t) = \hat{w}(\rho(x, t), x, t) \quad \text{for all } (x, t) \in \Gamma(2\delta).
\]
Then for each \( \varepsilon > 0 \)
\[
\partial_t w(x, t) = -\left( \frac{R_{\varepsilon}(P_{\varepsilon}(\varepsilon))}{\varepsilon} + \partial_t^\Gamma h_\varepsilon(r, s, t) \right) \partial_t \hat{w}(\rho(x, t), x, t) + \partial_t \hat{w}(\rho(x, t), x, t),
\]
\[
\nabla w(x, t) = \left( \frac{\partial_{\varepsilon}(P_{\varepsilon}(\varepsilon))}{\varepsilon} - \nabla^\Gamma h_\varepsilon(r, s, t) \right) \partial_t \hat{w}(\rho(x, t), x, t) + \nabla \hat{w}(\rho(x, t), x, t),
\]
\[
\Delta w(x, t) = (\varepsilon^{-2} + |\nabla^\Gamma h_\varepsilon(r, s, t)|^2) \frac{\partial_{\varepsilon}^2 \hat{w}(\rho(x, t), x, t)}{\varepsilon} + 2 \left( \frac{\partial_{\varepsilon}}{\varepsilon} - \nabla^\Gamma h_\varepsilon(r, s, t) \right) \cdot \nabla \hat{w}(\rho(x, t), x, t) + \Delta \hat{w}(\rho(x, t), x, t),
\]
where \( \rho \) is as in \( \mathbb{E} \) and \( (r, s) \) is understood via \( \mathbb{E} \).

For a systematic treatment of the remainder terms, we introduce:

Definition 2.3 For any \( k \in \mathbb{R} \) and \( \alpha > 0 \), \( \mathcal{R}_{k, \alpha} \) denotes the vector space of families of continuous functions \( \tilde{r}_\varepsilon : \mathbb{R} \times \Gamma(2\delta) \rightarrow \mathbb{R} \), indexed by \( \varepsilon \in (0, 1) \), which are continuously differentiable with respect to \( n_{r, \varepsilon} \) for all \( \varepsilon \in [0, T_0] \) such that
\[
|\partial_{n_{r, \varepsilon}} \tilde{r}_\varepsilon(\rho, x, t)| \leq Ce^{-\alpha|\rho|} \varepsilon^k \quad \text{for all } \rho \in \mathbb{R}, (x, t) \in \Gamma(2\delta), j = 0, 1, \varepsilon \in (0, 1)
\]
for some \( C > 0 \) independent of \( \rho \in \mathbb{R}, (x, t) \in \Gamma(2\delta), \varepsilon \in (0, 1) \). \( \mathcal{R}_{k, \alpha}^0 \) is the subclass of all \( (\tilde{r}_\varepsilon)_{\varepsilon \in (0, 1)} \) in \( \mathcal{R}_{k, \alpha} \) such that \( \tilde{r}_\varepsilon(\rho, x, t) = 0 \) for all \( \rho \in \mathbb{R}, x \in \Gamma(t), \varepsilon \in (0, 1) \).

We remark that \( \mathcal{R}_{k, \alpha} \) and \( \mathcal{R}_{k, \alpha}^0 \) are closed under multiplication and \( \mathcal{R}_{k, \alpha} \subset \mathcal{R}_{k-1, \alpha} \).

Lemma 2.4 Let \( 0 < \varepsilon \leq \varepsilon_0 \), \( h_\varepsilon \) be defined by \( \mathbb{E} \) and satisfy
\[
M := \sup_{0 < \varepsilon \leq \varepsilon_0, (x, t) \in \Gamma^1} |h_\varepsilon(s, t)| < \infty
\]
for some \( T_\varepsilon \in (0, T_0], \varepsilon_0 \in (0, 1), \) and \( \tilde{r}_\varepsilon \in R_{k, \alpha} \) for some \( \alpha > 0 \), \( k \in \mathbb{R} \) and let \( j = 1 \) if even \( (\tilde{r}_\varepsilon)_{\varepsilon \in (0, 1)} \in \mathcal{R}_{k, \alpha} \) and \( j = 0 \) else. Then there is some \( C > 0 \), independent of \( T_\varepsilon \), \( 0 < \varepsilon \leq \varepsilon_0, \varepsilon_0 \in (0, 1) \) such that
\[
r_\varepsilon(x, t) := \tilde{r}_\varepsilon(\rho, x, t) \quad \text{for all } (x, t) \in \Gamma(2\delta)
\]
with \( \rho \) as in \( \mathbb{E} \) satisfies
\[
\|a(P_{\varepsilon}(\cdot))r_\varepsilon \varphi\|_{L^1(\Gamma(2\delta))} \leq C(1 + M)^j \varepsilon^{1+k+j} \|\varphi\|_{H^1(\Omega)} \|a\|_{L^2(\Gamma_1)},
\]
\[
\|a(P_{\varepsilon}(\cdot))r_\varepsilon\|_{L^2(\Gamma(2\delta))} \leq C(1 + M)^j \varepsilon^{1+k+j} \|a\|_{L^2(\Gamma_1)}
\]
uniformly for all \( \varphi \in H^1(\Omega), a \in L^2(\Gamma_1), t \in [0, T_\varepsilon], \varepsilon \in (0, \varepsilon_0] \).

We refer to \([5]\) Corollary 2.7 for the proof.

Lemma 2.5 Let \( f \in S(\mathbb{R}) \) such that \( f' \in \mathcal{R}_{0, \alpha} \) for some \( \alpha > 0 \). Then there is a constant \( C > 0 \) such that for all \( t \in [0, T_0], a \in H^1(\Gamma_1) \) and \( \varphi \in C_0^{\infty}(\Omega) \) we have
\[
\left| \int_{\Gamma(2\delta)} f'(\rho(x, t))a(S(x, t))n_{r_\varepsilon} \cdot n_{r_\varepsilon} \cdot \nabla \varphi \, dx \right| \leq C\varepsilon^{2\theta} \|a\|_{H^1(\Gamma_1)} \|\varphi\|_{H^1(\Omega)}.
\]
2.4 Parabolic Equations on Evolving Hypersurfaces

Proof: First of all, since \( h_x \) is uniformly bounded, there is some \( \varepsilon_0 \in (0, 1) \) such that

\[
\left| \frac{\pm \delta}{\varepsilon} - h_x(s, t) \right| \geq \frac{\delta}{2} \quad \text{for all } s \in T^1, t \in [0, T_0], \varepsilon \in (0, \varepsilon_0]. \tag{2.21}
\]

We use that \( f'(\rho) = \varepsilon \partial_a f(\rho) \) and \( n_r \circ n_{r_1} : \nabla \varphi = -\text{div} \varphi \) since \( \text{div} \varphi = 0 \). To treat the remaining integral, we may use Lemma 2.11 to get

\[
\left| \int_{r_1(2\delta)} f'(\rho(x, t))a(S(x, t))n_r \circ n_{r_1} : \nabla \varphi \, dx \right|
\leq \int_{r_1(2\delta)} \epsilon \partial_a (\rho(x, t))a(S(x, t)) : \nabla \varphi \, dx + \int_{r_1(2\delta)} \varepsilon \partial_a f(\rho(a)(S(x, t)) : \nabla \varphi \, dx
\]

\[
+ C \sum \int \left| \frac{\pm \delta}{\varepsilon} - h_x(s, t) \right| a(s) \varphi (X_0(\pm \delta, s, t)) \right| \, ds
\]

\[
:= J_1 + J_2 + J_3^+ + J_3^-.
\]

Now

\[
J_3^\pm \leq C_{\epsilon} e^{-\frac{\delta}{\alpha}} \int_{\gamma^1} |a(s)| \sup_{\varepsilon \in [-\delta, \delta]} |\varphi(X_0(r, s, t))| \, ds \leq C_{\epsilon} \frac{\delta}{\alpha} \|a\|_{L^2(T^1)} \|\varphi\|_{H^1(r_1(2\delta))},
\]

where we used (2.21) and \( H^1(\Gamma_1(2\delta)) \hookrightarrow L^{2, \infty}(\Gamma_1(2\delta)) \) (cf. (2.12)). For the second term we use integration by parts and get

\[
J_2 \leq \left| \varepsilon \int_{r_1(2\delta)} f(\rho(a)(S(x, t)) \partial_a (\varphi k(x, t)) : n(S(x, t), t) \, dx \right| + C e^{-\frac{\delta}{\alpha}} \|a\|_{L^2(T^1)} \|\varphi\|_{H^1(r_1(2\delta))}
\]

\[
\leq C \varepsilon \|a\|_{L^2(T^1)} \|\varphi\|_{H^1(r_1(2\delta))} e^{\frac{\delta}{\alpha}} \|f\|_{L^2(\mathbb{R})} + Ce^{-C_{\epsilon} \frac{\delta}{\alpha}} \|a\|_{L^2(T^1)} \|\varphi\|_{H^1(r_1(2\delta))}
\]

\[
\leq C \varepsilon \frac{\delta}{\alpha} \|a\|_{L^2(T^1)} \|\varphi\|_{H^1(r_1(2\delta))},
\]

where the exponentially decaying term comes from the appearing boundary integral, which is estimated as before. Moreover, we used a change of variables \( r \mapsto \frac{r}{\varepsilon} - h_x \).

Finally, we have

\[
J_1 \leq \left| \varepsilon \int_{r_1(2\delta)} \partial_a \nabla (f(\rho(x, t))a(S(x, t)) : \varphi(x) \, dx \right|
\]

\[
+ \left| \varepsilon \int_{r_1(2\delta)} [\partial_a, \nabla] f(\rho(x, t))a(S(x, t)) : \varphi(x) \, dx \right|
\]

\[
\leq C \varepsilon \left| \int_{r_1(2\delta)} \nabla (f(\rho(x, t))a(S(x, t)) : \partial_n \varphi(x) \, dx \right|
\]

\[
+ C \epsilon \|f\|_{L^2(\mathbb{R})} \|a\|_{H^1(T^1)} \|\varphi\|_{L^2(r_1(2\delta))} + Ce^{-C_{\epsilon} \frac{\delta}{\alpha}} \|a\|_{L^2(T^1)} \|\varphi\|_{H^1(r_1(2\delta))}
\]

\[
\leq C \epsilon \frac{\delta}{\alpha} \|a\|_{H^1(T^1)} \|\varphi\|_{H^1(r_1(2\delta))},
\]

Here we used that \([\partial_a, \nabla] \) is a differential operator in tangential direction and integration by parts. This finishes the proof.

2.4 Parabolic Equations on Evolving Hypersurfaces

Let \( 0 < T < \infty \) be arbitrary. We define

\[
X_T := L^2(0, T; H^{3/2}(\mathbb{T}^1)) \cap H^1(0, T; H^{1/2}(\mathbb{T}^1)) \tag{2.22}
\]
equipped with the norm
\[ \|u\|_{X_T} = \|u\|_{L^2(0,T;H^{3/2}(\Omega))} + \|u\|_{H^1(0,T;H^{1/2}(\Omega))} + \|u\|_{t=0} \|H^{3/2}(\Omega)). \]
We note that
\[ X_T \hookrightarrow BUC([0,T];H^{3/2}(\Omega)) \cap L^1(0,T;H^2(\Omega)) \] (2.23)
and the operator norm of the embedding is uniformly bounded in \( T \).

THEOREM 2.6 Let \( w: \mathbb{T}^1 \times [0,T] \rightarrow \mathbb{R}^2 \) and \( a: \mathbb{T}^1 \times [0,T] \rightarrow \mathbb{R} \) be smooth. For every \( g \in L^2(0,T;H^\frac{3}{2}(\mathbb{T}^1)) \) and \( h_0 \in H^\frac{3}{2}(\mathbb{T}^1) \) there is a unique solution \( h \in X_T \) of

\[
\begin{align*}
D_t h + w \cdot \nabla h - \Delta_T h + ah &= g \quad \text{on } \mathbb{T}^1 \times [0,T], \\
|t|=0 &= h_0 \quad \text{on } \mathbb{T}^1.
\end{align*}
\] (2.24) (2.25)

Proof: See [5].

2.5 Spectral Estimate

In this subsection we assume that \( \Omega \subseteq \mathbb{R}^2 \) is a bounded domain and \( \Gamma_t \subseteq \Omega, t \in [0,T_0] \), \( T_0 > 0 \), are given smooth evolving closed and compact \( C^\infty \)-hypersurfaces, dividing \( \Omega \) in disjoint domains \( \Omega^+ (t) \) and \( \Omega^- (t) \) as before, and

\[
\begin{align*}
c_{A}(x) &= c_{A,0}(x) + \varepsilon^2 c_{A,2+}(x), \\
c_{A,0}(x) &= \zeta \circ d_t \theta \rho (\rho) + (1 - \zeta \circ d_T) (\chi_{\Omega^+ (t)} - \chi_{\Omega^- (t)}) \quad \text{for all } x \in \Omega,
\end{align*}
\]
where \( \zeta \in C^\infty (\mathbb{R}) \) is as in (2.18). Moreover, we assume that \( \text{dist} (\Gamma_t, \partial \Omega) > 2\delta \) for all \( t \in [0,T_0] \). For given continuous functions \( (\tilde{h}_s)_{0 < s < 1}: \Gamma \rightarrow \mathbb{R} \) with \( \Gamma := \bigcup_{t \in [0,T_0]} \Gamma_t \times \{t\} \), we define the stretched variable \( \rho \) as in (2.14) with \( \tilde{h}_s(s,t) = \tilde{h}_s(X_0(s,t),t) \). Furthermore, we assume that

\[
\sup_{\varepsilon \in (0,1)} \left( \sup_{(p,t) \in \Gamma} |\tilde{h}_s(p,t)| + \sup_{x \in \Omega, t \in [0,T_0]} |c_{A,2+}(x,t)| \right) \leq M
\] (2.26)
for some \( M > 0 \). We will apply the results of this subsection to \( \tilde{h}_s(p,t) = h(s,X_0^{-1}(p,t),t) \) for some \( h_0: \mathbb{T}^1 \times [0,T_0] \rightarrow \mathbb{R} \).

The following spectral estimate due to [5, Theorem 2.13] is a key ingredient for the proof of convergence.

THEOREM 2.7 Let \( c_A \) be as above and (2.26) be satisfied for some \( M > 0 \). Then there are some \( C_L, \varepsilon_0 > 0 \), independent of \( \tilde{h}_s, c_A \), such that for every \( \psi \in H^1 (\Omega), t \in [0,T_0] \), and \( \varepsilon \in (0,\varepsilon_0) \) we have

\[
\int_{\Omega} \left( |\nabla \psi (x)|^2 + \frac{f''(c_{A}(x,t))}{\varepsilon^2} \psi^2 (x) \right) dx \geq -C_L \int_{\Omega} \psi^2 dx + \int_{\Omega, \Gamma_\varepsilon (t)} |\nabla \psi|^2 dx + \int_{\Gamma_\varepsilon (t)} |\nabla \tau \psi|^2 dx.
\]

The following refinement will be essential for our proof as well.

**Corollary 2.8** Let the previous assumptions hold true and let \( t \in [0,T] \), let \( \psi \in H^1 (\Gamma_t (\delta)) \) and \( \Lambda_\varepsilon \in \mathbb{R} \) be such that

\[
\int_{\Gamma_\varepsilon (\delta)} \varepsilon |\nabla \psi (x)|^2 + \varepsilon^{-1} f'' (c_{A}(x,t)) \psi (x)^2 dx \leq \Lambda_\varepsilon
\] (2.27)
and denote \( I^{\delta}_{\varepsilon} := (-\frac{\varepsilon}{2} - h(s,t), \frac{\varepsilon}{2} - h(s,t)) \). Then, for \( \varepsilon > 0 \) small enough, there exist functions \( Z \in H^1 (\mathbb{T}^1) \), \( \psi^R \in H^1 (\Gamma_\varepsilon (\delta)) \) and smooth \( \Psi: I^{\delta}_{\varepsilon} \times \mathbb{T}^1 \rightarrow \mathbb{R} \) such that

\[
\psi (X(r,s,t)) = \varepsilon^{-\frac{1}{2}} Z(s) \left( \beta(s) \theta \rho (\rho (r,s)) + \Psi (\rho (r,s),s) \right) + \psi^R (r,s)
\] (2.28)
for almost all \((r, s) \in (-\delta, \delta) \times \mathbb{T}^1\), where \(\rho(r, s) = \frac{\rho}{s} - h_s(s, t)\) and \(\beta(s) = \left(\int_{\Gamma_s} \left(\theta_0^t(\rho)\right)^2 \, d\rho\right)^{-\frac{1}{2}}\).

Moreover,
\[
\|u^R\|_{L^2(\Gamma_s(\delta))}^2 \leq C \left(\varepsilon \Lambda_s + \varepsilon^2 \|u\|_{L^2(\Gamma_s(\delta))}^2\right),
\]
\[
\|Z\|_{H^1(\mathbb{T}^1)} + \|
abla T\|_{L^2(\Gamma_s(\delta))} + \|\psi^R\|_{L^2(\Gamma_s(\delta))} \leq C \left(\|\psi\|_{L^2(\Gamma_s(\delta))} + \frac{\Lambda_s}{\varepsilon}\right),
\]
and
\[
\sup_{s \in \mathbb{T}^1} \left(\int_{\Gamma_s(\delta)} (\Psi(\rho, s)^2 + \Psi(\rho, s)^2) J(\varepsilon(\rho - h_s(s, t), s) \, d\rho) \right) \leq C\varepsilon^2.
\]

We refer to \([6]\) Corollary 2.12 for the proof and note that it is easy to verify that Assumption 2.11 in \([6]\) is satisfied in our situation.

**Remark 2.9** In the following we will apply Corollary 2.8 to \(\psi = u = \frac{\varepsilon x - \varepsilon A(\rho)}{\varepsilon x - \varepsilon A(\rho)}\) satisfying (1.10) and
\[
\int_0^T \int_{\Gamma_s(\delta)} \left(\varepsilon |\nabla u|^2 + \varepsilon^{-1} f''(\varepsilon(\rho, s))u^2\right) \, dx \, dt \leq R \varepsilon^{2N+2/2}.
\]

Then we obtain that
\[
\bar{u}(r, s) = \varepsilon^{-\frac{1}{2}} Z(s) \Psi_r \left(\frac{\rho}{s} - h_s(s, s)\right) + \psi^R(r, s),
\]
where \(\Psi_1(\rho, s) = \beta(s)\theta_0(\rho) + \Psi(\rho, s)\) and
\[
\|Z\|_{L^2(0, T, H^1(\Gamma_s(\delta)))} + \|
abla T\|_{L^2(\Gamma_s(\delta))} + \|\psi^R\|_{L^2(0, T, H^1(\Gamma_s(\delta)))} \leq CR\varepsilon^{N+1/2},
\]
and
\[
\|\psi^R\|_{L^2(\Gamma_s(\delta))} \leq C\varepsilon^{N+3/2}.
\]

**Remark 2.10** For \(u \in H^1(\Gamma_s(\delta))\) let us introduce the \(\varepsilon\)-dependent norms
\[
\|u\|_{x_\varepsilon} = \inf \left\{\|Z\|_{H^1(\Gamma_s(\delta))} + \|v\|_{H^1(\Gamma_s(\delta))} + \varepsilon^{-1}\|v\|_{L^2(\Gamma_s(\delta))} : v \in H^1(\Gamma_s(\delta), v \in H^1(\Gamma_s(\delta))\right\}.
\]

Then, choosing \(\Lambda_s\) such that equality holds in (2.27), Corollary 2.8 yields
\[
\|u\|_{x_\varepsilon} + \|
abla T\|_{L^2(\Gamma_s(\delta))}^2 \leq C \left(\int_{\Gamma_s(\delta)} \left(|\nabla u|^2 + \frac{1}{\varepsilon^2} f''(\varepsilon(\rho, s))u^2\right) \, dx + \|u\|_{L^2(\Gamma_s(\delta))}^2\right).
\]
As a consequence we obtain

**Lemma 2.11** Let the assumptions above hold true and \(f: \Gamma_s(3\delta) \to \mathbb{R}\) be such that
\[
f(x, t) = a(\rho, s, t)w(s)\quad \text{in} \quad \Gamma_s(3\delta), \quad \text{where} \quad x = X(r, s, t), \rho = \frac{\rho}{s} - h_s(s, t),
\]
with \(w \in L^2(\mathbb{T}^1)\) and \(a \in \mathcal{R}_{\alpha, \alpha}\)
\[
\int_{\mathbb{R}} a(\rho, s, t)\theta_0^t(\rho) \, d\rho = 0 \quad \text{for all} \quad s \in \mathbb{T}^1, t \in [0, T_0].
\]

Then there are constants \(C(T_0), c_0 > 0, \varepsilon_0 > 0\) independent of \(t \in [0, T_0]\) and \(w\) such that
\[
\|f\|_{x_\varepsilon} \leq C\varepsilon^{2/3}\|w\|_{L^2(\mathbb{T}^1)}
\]
for every \(\varepsilon \in (0, \varepsilon_0)\).
Proof: Let $u \in X_\varepsilon$ with $\|u\|_{X_\varepsilon} \leq 1$ and $Z \in H^1(\mathbb{T}^1)$, $v \in H^1(\Gamma_\varepsilon(35))$ with
\[ u(x) = \bar{u}(r,s) = Z(s) e^{-\frac{s}{\varepsilon}} \delta_0'(\rho) + \bar{v}(r,s), \quad \text{where } x = X(r,s,t), \rho = \frac{r}{\varepsilon} - h_\varepsilon(s,t), \]
v(x) = \bar{v}(r,s), and
\[ \|Z\|_{H^1(\mathbb{T}^1)} + \|v\|_{H^1(\Gamma_\varepsilon(35))} + \varepsilon^{-1} \|v\|_{L^2(\Gamma_\varepsilon(35))} \leq 2. \]

Then, using $J(r,s,t) = J(0,s,t) + r \bar{J}(r,s,t)$,
\[
\int_{\Gamma_\varepsilon(34)} f(x,t) u(x) \, dx = \int_{\mathbb{T}^1} \int_{-\frac{3\delta}{\varepsilon}}^{\frac{3\delta}{\varepsilon}} a \left( \frac{r}{\varepsilon} - h_\varepsilon(s,t), s, t \right) w(s) \bar{u}(r,s) J(r,s,t) \, dr \, ds \\
= \varepsilon \int_{\mathbb{T}^1} \int_{-\frac{3\delta}{\varepsilon}}^{\frac{3\delta}{\varepsilon}} a \left( \frac{r}{\varepsilon} - h_\varepsilon(s,t), s, t \right) w(s) Z(s) e^{-\frac{s}{\varepsilon}} \delta_0'(\rho) J(0,s,t) \, d\rho \, ds \\
+ \varepsilon^2 \int_{\mathbb{T}^1} \int_{-\frac{3\delta}{\varepsilon}}^{\frac{3\delta}{\varepsilon}} a \left( \frac{r}{\varepsilon} - h_\varepsilon(s,t), s, t \right) w(s) Z(s) e^{-\frac{s}{\varepsilon}} \delta_0'(\rho) w(s) (\rho + \varepsilon h_\varepsilon(s,t)) \bar{J}(\rho + \varepsilon h_\varepsilon(s,t), s, t) \, d\rho \, ds \\
+ \int_{\mathbb{T}^1} \int_{-\frac{3\delta}{\varepsilon}}^{\frac{3\delta}{\varepsilon}} a \left( \frac{r}{\varepsilon} - h_\varepsilon(s,t), s, t \right) w(s) \bar{v}(r,s) J(r,s,t) \, dr \, ds \\
= I_1 + I_2 + I_3.
\]

Moreover, using (3.33), we can estimate
\[
|I_1| = \varepsilon \int_{\mathbb{T}^1} \int_{|\rho| \geq \frac{3\delta}{\varepsilon}} a \left( \frac{r}{\varepsilon} - h_\varepsilon(s,t), s, t \right) w(s) Z(s) e^{-\frac{s}{\varepsilon}} \delta_0'(\rho) J(0,s,t) \, d\rho \, ds \\
\leq C \varepsilon \frac{2}{\varepsilon} \|Z\|_{L^2(\mathbb{T}^1)} \|w\|_{L^2(\mathbb{T}^1)} \leq C \varepsilon \frac{2}{\varepsilon} \|w\|_{L^2(\mathbb{T}^1)}.
\]

Furthermore,
\[
|I_2| \leq \varepsilon \int_{\mathbb{R}^3} \sup_{r \in \mathbb{T}^1, t \in [0,T_\varepsilon]} a \left( \frac{r}{\varepsilon} - h_\varepsilon(s,t), s, t \right) w(s) \delta_0'(\rho) (\rho + \varepsilon h_\varepsilon(s,t)) \, d\rho \|w\|_{L^2(\Gamma_{\varepsilon}(35))} \|Z\|_{L^2(\mathbb{T}^1)} \\
\leq C \varepsilon \frac{2}{\varepsilon} \|w\|_{L^2(\mathbb{T}^1)}
\]
and
\[
|I_3| \leq \left( \int_{\mathbb{T}^1} \int_{-\frac{3\delta}{\varepsilon}}^{\frac{3\delta}{\varepsilon}} a \left( \frac{r}{\varepsilon} - h_\varepsilon(s,t), s, t \right) w(s)^2 J(r,s,t) \, dr \, ds \right) \frac{1}{\varepsilon} \|v\|_{L^2(\Gamma_{\varepsilon}(35))} \\
\leq C \left( \int_{-\frac{3\delta}{\varepsilon}}^{\frac{3\delta}{\varepsilon}} \exp\left(-\frac{x_1}{\alpha_\varepsilon}\right) \, dx \right) \frac{1}{\varepsilon} \|v\|_{L^2(\mathbb{T}^1)} \|v\|_{L^2(\Gamma_{\varepsilon}(35))} \\
\leq C \varepsilon \frac{2}{\varepsilon} \|w\|_{L^2(\mathbb{T}^1)} \|v\|_{L^2(\Gamma_{\varepsilon}(35))} \leq C' \varepsilon \frac{2}{\varepsilon} \|w\|_{L^2(\mathbb{T}^1)},
\]
for all $\varepsilon \in (0,\varepsilon_0)$ for some $\varepsilon_0 > 0$ sufficiently small.

3 Construction of the Approximate Solutions

The main goal of this section is to prove:

THEOREM 3.1 Let $\mathbf{u} = \mathbf{u}(\varepsilon) \in L^2(0,T_\varepsilon;H^1(\Omega)^d)$, $\varepsilon \in (0,1)$, be given such that for some $T_\varepsilon \in (0,T_0)$, $M > 0$ and $\varepsilon_0 > 0$ we have
\[
\|\mathbf{u}\|_{L^2(0,T_\varepsilon;H^1(\Omega))} \leq M \quad \text{for all } \varepsilon \in (0,\varepsilon_0).
\]
Then there are smooth $c_A, p_A: \Omega \times [0,T_\varepsilon] \to \mathbb{R}$, $v_A: \Omega \times [0,T_\varepsilon] \to \mathbb{R}^2$ such that
\[
\partial_t v_A + v_A \cdot \nabla v_A - \text{div}(2\varepsilon(c_A)Dv_A) + \nabla p_A = -\varepsilon \text{div}(\nabla c_A \otimes \nabla c_A) + R_1^1 + R_2^1, \quad (3.1)
\]
\[ \text{div } v_A = G, \]  
\[ \partial_t c_A + v_A \cdot \nabla c_A + \varepsilon^{N+\frac{1}{2}} u \cdot \nabla c_A = \Delta c_A - \frac{1}{\varepsilon^2} f'(c_A) + S, \]  
\[ (v_A, c_A)|_{\partial\Omega} = (0, -1), \]  

where

\[ \|R^A_k\|_{L^2(0, T; L^2(\Omega))} \leq C\varepsilon \|u\|_{L^2(0, T; L^2(\Gamma_1))} \varepsilon^{N+\frac{1}{2}}, \]
\[ \|G^A\|_{L^2(0, T; L^2(\Omega))} \leq C(M) \varepsilon^{N+\frac{1}{2}}, \]
\[ \|\nabla c_A\|_{L^2(0, T; L^2(\Omega))} \leq C(M) \varepsilon^N. \]

for some \( C(M), C' > 0 \) independent of \( \varepsilon, T \). Here \( c_A \) is of the form \( \hat{\varepsilon} \) below for some \( h_{N+\frac{1}{2}} \in X_T \), which is determined by \( \delta \) below in dependence on \( u \). In particular, \( c_A(\cdot, t) \equiv \pm 1 \) in \( \Omega^\varepsilon(t) \setminus \Gamma(t, 25) \) and \( \nabla c_A \) is supported in \( \Gamma(t, 25) \) for \( t \in [0, T] \). Moreover, \( M \mapsto C(M) \) is increasing.

As before we introduce in \( \Gamma(3\delta) \) the stretched variable
\[ \rho(x, t) = \frac{d \rho(x, t)}{\varepsilon} - h_0(S(x, t), t) \]
with
\[ h_0(s, t) = \sum_{k=0}^N \tilde{c}_k h_{k+1}(s, t), \]

where \( \{h_k\}_{0 \leq k \leq N-1} \subseteq C^\infty(T^1 \times [0, T]) \) are smooth functions (independent of \( \varepsilon \)). Moreover, we will use a function \( h_{N+\frac{1}{2}} \in T^1 \times [0, T] \to \mathbb{R} \), which may depend on \( \varepsilon \), such that \( h_{N+\frac{1}{2}} \in X_T \) is bounded (with respect to \( \varepsilon \)), where
\[ X_T = L^2(0, T; H^{5/2}(T^1)) \cap H^1(0, T; H^{1/2}(T^1)), \]
normed by
\[ \|h\|_{X_T} := \|h\|_{L^2(0, T; H^{5/2}(T^1))} + \|h\|_{H^1(T; H^{1/2}(T^1))} + \|h\|_{H^1(T^1)}. \]

We construct approximate solutions of the Navier-Stokes/Allen-Cahn system in the following form
\[ c_A(x, t) = \zeta \circ d \psi c^\varepsilon_A(x, t) + (1 - \zeta \circ d \psi) (c^\varepsilon_A^+ + c^\varepsilon_A^-), \]
\[ v_A(x, t) = \zeta \circ d \psi v^\varepsilon_A(x, t) + (1 - \zeta \circ d \psi) (v^\varepsilon_A^+ + v^\varepsilon_A^-) - \nabla_s(t), \]
\[ p_A(x, t) = \zeta \circ d \psi p^\varepsilon_A(x, t) + (1 - \zeta \circ d \psi) (p^\varepsilon_A^+ + p^\varepsilon_A^-) \]

where \( \zeta \) as in \( \text{(12)} \), \( c^\varepsilon_A = \pm 1, \chi^\pm = \chi^\pm(\varepsilon) \), \( \nabla : \Omega \to \mathbb{R}^2 \) is a smooth vector field such that \( \nabla|_{\partial\Omega} = n_{\partial\Omega} \) and \( \tilde{a} : (0, T) \to \mathbb{R} \) is some suitable function related to the compatibility condition
\[ \int_{\Omega} \text{div } v_A \, dx = \int_{\partial\Omega} n_{\partial\Omega} \cdot v_A \, d\sigma. \]

It will be essential that we use the following ansatz
\[ c_A^{\text{in}}(x, t) = c^\varepsilon_A(\rho, s, t) + (\varepsilon^{N+\frac{1}{2}} \delta_{0\varepsilon}(\rho) + \varepsilon^{N+\frac{1}{2}} \partial_\rho \hat{c}_2(\rho, S(x, t), t)) h_{N+\frac{1}{2}}(S(x, t), t), \]
\[ v_A^{\text{in}}(x, t) = v^\varepsilon_A(\rho, s, t) + \varepsilon^{N+\frac{1}{2}} \hat{w}(\rho, x, t), \]
\[ p_A^{\text{in}}(x, t) = p^\varepsilon_A(\rho, s, t) + \varepsilon^{N+\frac{1}{2}} \hat{q}(\rho, x, t) \]

for the inner expansions of \( c_A, v_A, \) and \( p_A, \) where we use the following standard ansatz for the first terms \( \hat{c}_A, \hat{v}_A \) and \( \hat{p}_A \)
\[ \hat{c}_A^{\varepsilon}(\rho, s, t) = \delta_0(\rho) + \sum_{k=2}^{N+2} \varepsilon^k \hat{c}_k(\rho, x, t), \]
\[ \hat{v}_A^{\varepsilon}(\rho, s, t) = \delta_0(\rho) + \sum_{k=2}^{N+2} \varepsilon^k \hat{v}_k(\rho, x, t), \]
\[ \hat{p}_A^{\varepsilon}(\rho, s, t) = \delta_0(\rho) + \sum_{k=2}^{N+2} \varepsilon^k \hat{p}_k(\rho, x, t), \]
3 Construction of the approximate solutions

\[ \psi_A^{in}(p, x, t) = \sum_{k=0}^{N+2} \varepsilon^k \hat{\psi}_k(p, x, t), \]  
\[ \tilde{p}_A^{in}(p, x, t) = \sum_{k=-1}^{N+1} \varepsilon^k \tilde{p}_k(p, x, t). \]  

Here \( \hat{c}_k, \hat{v}_k, \) and \( \tilde{p}_k \) will be smooth functions that are independent of \( \varepsilon \). Furthermore, \( \hat{w} \) and \( \hat{q} \) will be chosen later with the property that \( \mathbf{n} \cdot \hat{w} = 0 \) in \( \Gamma(3) \). For the following \( \tilde{c}_A^{in}, \tilde{v}_A^{in}, \) \( \tilde{p}_A^{in} \) are defined as in (3.11), but with \( (h_{N+\frac{1}{2}}, \hat{w}, \hat{q}) \equiv 0 \). Moreover, \( (\tilde{c}_A, \tilde{v}_A, \tilde{p}_A) \) denote the corresponding approximate solution in \( \Omega \) (with \( (h_{N+\frac{1}{2}}, \hat{w}, \hat{q}) \equiv 0 \)).

The construction is done by the following scheme:

1. First we construct approximate solutions \( (\tilde{c}_A, \tilde{v}_A, \tilde{p}_A) \) such that in \( \Gamma(3) \) we have

\[
\partial_t \psi_A^{in} + \psi_A^{in} \cdot \nabla \psi_A^{in} - \text{div}(2\nu(c_A^{in})D\psi_A^{in}) + \nabla \tilde{p}_A^{in} = -\varepsilon \text{div} (\nabla \tilde{c}_A^{in} \otimes \nabla \tilde{c}_A^{in}) + R, \\
\text{div} \tilde{\psi}_A^{in} = G, \\
\partial_t \tilde{c}_A^{in} + \tilde{c}_A^{in} \cdot \nabla \tilde{c}_A^{in} - \Delta \tilde{c}_A^{in} + \frac{1}{\varepsilon^2} f'(\tilde{c}_A^{in}) = \rho_s(p, s, t) + S, 
\]

with suitable estimates for the remainder terms. The construction is done similarly as in [30].

2. Now \( h_{N+\frac{1}{2}} \) and therefore the additional terms

\[
(\varepsilon^{N+\frac{1}{2}} \partial_0(p) + \varepsilon^{N+\frac{1}{2}} \partial_0 \tilde{c}_2(p, S(x, t), t) ) h_{N+\frac{1}{2}}(S(x, t), t) 
\]

are chosen such that they give \( \varepsilon^{N+\frac{1}{2}} \mathbf{u} \cdot \nabla c_A^{in} \) up to lower order terms, cf. Theorem 3.3 below.

3. Finally, \( \hat{w} \) and \( \hat{q} \) are chosen such that a leading error term on the right-hand side of the Navier-Stokes equation cancels and all additional terms give only lower order terms of order \( \mathcal{O}(\varepsilon^{N+\frac{1}{2}}) \) in \( L^2(0, T; (H^1(\Omega))^3) \).

Here the first step can be done in a standard manner. The outcome is summarized in the following theorem:

**THEOREM 3.2** Let \( N \in \mathbb{N} \). Then there are smooth \( (\tilde{c}_A^{in}, \tilde{v}_A^{in}, \tilde{p}_A^{in}) \) defined in \( \Gamma(3) \) and \( (c_A^{\pm}, v_A^{\pm}, p_A^{\pm}) \) defined on \( \Omega \times [0, T_0] \), which are smooth, such that:

1. Inner expansion: In \( \Gamma(3) \) we have

\[
\partial_t \psi_A^{in} + \psi_A^{in} \cdot \nabla \psi_A^{in} - \text{div}(2\nu(c_A^{in})D\psi_A^{in}) + \nabla \tilde{p}_A^{in} = -\varepsilon \text{div} (\nabla \tilde{c}_A^{in} \otimes \nabla \tilde{c}_A^{in}) + R, \\
\text{div} \tilde{\psi}_A^{in} = G, \\
\partial_t \tilde{c}_A^{in} + \tilde{c}_A^{in} \cdot \nabla \tilde{c}_A^{in} - \Delta \tilde{c}_A^{in} + \frac{1}{\varepsilon^2} f'(\tilde{c}_A^{in}) = \rho_s(p, s, t) + S, 
\]

where

\[
\|(R, \partial_0 G, S)\|_{L^\infty((0, T_0) \times \Omega)} \leq C\varepsilon^{N+1}, \tag{17} \\
\|G_s\|_{L^\infty((0, T_0) \times \Omega)} \leq C\varepsilon^{N+2}. \tag{18} 
\]

2. Outer expansion: In \( \Omega^\pm \) we have \( c_A^{\pm} \equiv \pm 1 \) and

\[
\partial_t v_A^{\pm} + v_A^{\pm} \cdot \nabla v_A^{\pm} - \nu^{\pm} \Delta v_A^{\pm} + \nabla p_A^{\pm} = R_A^{\pm}, \\
\text{div} v_A^{\pm} = 0, \\
v_A^{\pm}|_{\partial\Omega} = \nu \mathbf{n}|_{\partial\Omega}, \tag{20} 
\]

where \( \nu : (0, T) \rightarrow \mathbb{R} \) is continuous and

\[
\|R_A^{\pm}\|_{L^\infty((0, T_0) \times \Omega)} \leq C\varepsilon^{N+2} \quad \text{for all } \varepsilon \in (0, 1). 
\]
3. Matching condition: We have
\[
\|\partial_t (\tilde{v}^\varepsilon_A - \tilde{v}_A x^+ - \tilde{v}_A x^-)\|_{L^\infty(\Gamma(2\beta) \setminus \Gamma(\delta))} \leq Ce^{-\varepsilon \beta},
\]
\[
\|\partial_x (p\tilde{v}^\varepsilon_A - p\tilde{x} (x^+ - p\tilde{x} (x^-))\|_{L^\infty(\Gamma(2\beta) \setminus \Gamma(\delta))} \leq Ce^{-\varepsilon \beta},
\]
\[
\|\partial_x (\tilde{c}^\varepsilon_A - c_A x^+ - c_A x^-)\|_{L^\infty(\Gamma(2\beta) \setminus \Gamma(\delta))} \leq Ce^{-\varepsilon \beta},
\]
for all \(\varepsilon \in (0, 1)\) and \(\beta \in \mathbb{N}_0\).

**Proof:** The proof is done in the appendix.

Let us denote \(u^{in}_A := c^{in}_A - \tilde{c}^{in}_A\) and \(w^{in}_A := v^{in}_A - \tilde{v}^{in}_A\). Then we have
\[
\partial_t u^{in}_A + v^{in}_A \cdot \nabla c^{in}_A - \Delta c^{in}_A + \frac{1}{\varepsilon^2} f'(c^{in}_A)
\]
\[
= \partial_t c^{in}_A + v^{in}_A \cdot \nabla c^{in}_A - \Delta c^{in}_A + \frac{1}{\varepsilon^2} f'(c^{in}_A) + \partial_t u^{in}_A + v^{in}_A \cdot \nabla u^{in}_A + w^{in}_A \cdot \nabla c^{in}_A - \Delta u^{in}_A
\]
\[
+ \frac{1}{\varepsilon^2} f''(c^{in}_A) u^{in}_A + \tilde{s}_A
\]
\[
= \partial_t u^{in}_A + v^{in}_A \cdot \nabla u^{in}_A - \Delta u^{in}_A + \frac{1}{\varepsilon^2} f''(c^{in}_A) u^{in}_A + S_{\varepsilon} + \tilde{s}_A + w^{in}_A \cdot \nabla \tilde{c}^{in}_A
\]
in \(\Gamma_1(2\beta), t \in [0, T]\), where \(\tilde{s}_A\) contains terms that are quadratic in \(u^{in}_A\) times \(\varepsilon^{-2}\). Hence \(\tilde{s}_A\) is \(O(\varepsilon^{2N+\frac{3}{2}}) = O(\varepsilon^{N+1})\) in \(L^\infty(0, T; L^2(\Gamma_1(2\beta)))\) and \(O(\varepsilon^{N+1})\) in \(L^\infty(0, T; L^2(\Gamma_1(2\beta)))\) if \(N \geq 3\) due to (2.20), resp., where the constants are uniform if \(\|h_{N+\frac{1}{2}}\|_{L^\infty(\Gamma)} \leq \chi M\) for some \(M > 0\). Moreover, one can show \(w^{in}_A \cdot \nabla c^{in}_A = O(\varepsilon^{N+1})\) in \(L^\infty(0, T; L^2(\Gamma_1(2\beta)))\) with the aid of Lemma 2.4 in a straightforward manner since \(n \cdot w(\rho, x, t) = 0\) by the construction below.

Here \(w(\rho, x, t)\) is as in (3.11).

For the first terms we have:

**THEOREM 3.3** Let \(T_\varepsilon \in (0, T_0)\) and \(h_{N+\frac{1}{2}} \in X_{T_\varepsilon}\) be the solution of
\[
D_t h_{N+\frac{1}{2}} - X_0(\mathbf{v}) \cdot \nabla r h_{N+\frac{1}{2}} - \Delta h_{N+\frac{1}{2}} - \int_{\Gamma_1(2\beta)}(\rho \cdot \nabla u^{in}_A + \varepsilon h_{N+\frac{1}{2}}) = -X_0(n \cdot u)\text{ on } T^1 \times [0, T],
\]
(3.21)
\[
h_{N+\frac{1}{2}}|_{t=0} = 0,
\]
(3.22)
where \(g_0 : \Gamma \rightarrow \mathbb{R}\) is a smooth function, which is given by (A.66) in the appendix. Then for any \(M > 0\) there is some \(C(M) > 0\) such that, if \(\|h_{N+\frac{1}{2}}\|_{L^\infty(\Gamma)} \leq \chi M\), it holds
\[
\|\partial_t u^{in}_A + v^{in}_A \cdot \nabla u^{in}_A - \Delta u^{in}_A + \frac{1}{\varepsilon^2} f''(c^{in}_A) u^{in}_A + \varepsilon^{N+\frac{1}{2}} u^{in}_A \cdot \nabla c^{in}_A \|_{L^2(0, T; X_{T_{\varepsilon}})} \leq C(M)\varepsilon^{N+1},
\]
\[
\|\partial_t u^{in}_A + v^{in}_A \cdot \nabla u^{in}_A - \Delta u^{in}_A + \frac{1}{\varepsilon^2} f''(c^{in}_A) u^{in}_A + \varepsilon^{N+\frac{1}{2}} u^{in}_A \cdot \nabla c^{in}_A \|_{L^2(0, T; L^2(\Gamma_1(2\beta)))} \leq C(M)\varepsilon^{N},
\]

**Proof:** First of all, we have
\[
u^{in}_A(n, x, t) = \left(\varepsilon^{N+\frac{1}{2}} \delta_0(\rho) + \varepsilon^{N+\frac{1}{2}} \partial_{\rho}\delta_2(\rho, S(x, t), t)\right) h_{N+\frac{1}{2}}(S(x, t), t), \text{ where } \rho = \rho(x, t),
\]
in \(\Gamma_1, t \in [0, T]\). Because of Lemma 2.2, Lemma 2.4 and (3.6), we have
\[
\partial_t u^{in}_A = -\left(\varepsilon^{-\frac{1}{2}} \delta_0(\rho) + \partial_t \delta_2(r, s, t)\right) h_{N+\frac{1}{2}}(s, t)
\]
\[
+ \varepsilon^{N+\frac{1}{2}} \partial_{\rho}\delta_2(\rho, S(x, t), t) h_{N+\frac{1}{2}}(s, t) + \varepsilon^{N+\frac{1}{2}} \partial_{\rho}\delta_2(\rho, S(x, t), t) h_{N+\frac{1}{2}}(s, t)
\]
\[
= -\varepsilon^{N+\frac{1}{2}} \partial_r \delta_2(\rho, S(x, t), t) h_{N+\frac{1}{2}}(s, t)
\]
\[
- \varepsilon^{N+\frac{1}{2}} \partial_{\rho}\delta_2(\rho, S(x, t), t) h_{N+\frac{1}{2}}(s, t) + \varepsilon^{N+\frac{1}{2}} \delta_0(\rho) \partial_{\rho} h_{N+\frac{1}{2}}(s, t) + O(\varepsilon^{N+1})
\]
in \(L^\infty(0, T; L^2(\Gamma_1(2\beta)))\) and similarly
\[
v^{in}_A \cdot \nabla u^{in}_A.
\]
\[
\begin{align*}
\Delta u^n_A &= \varepsilon^{N-2} \theta''_0(\rho) h_{N+\frac{1}{2}}(s,t) + \varepsilon^{N-2} \theta''_0(\rho) \nabla^T h_{N+\frac{1}{2}} + \nabla \theta''_0(\rho) \nabla^T h_{N+\frac{1}{2}} + O(\varepsilon^{N+1}) \\
&= \varepsilon^{N-2} \v_{\text{net}}(\rho) h_{N+\frac{1}{2}}(s,t) + \varepsilon^{N-2} \v_{\text{net}}(\rho) \nabla^T h_{N+\frac{1}{2}} + \varepsilon^{N-2} \v_{\text{net}}(\rho) \nabla^T h_{N+\frac{1}{2}}(r,s,t) + \varepsilon^{N-2} \v_{\text{net}}(\rho) \nabla^T h_{N+\frac{1}{2}}(r,s,t) + O(\varepsilon^{N+1})
\end{align*}
\]
in \( L^\infty(0, T; L^2(\Gamma(2\delta))) \), where \( s = S(x,t), r = d_r(x,t), \rho = \rho(x,t) \), and we have used
\[ n r_{\text{net}}(\rho, x,t) = v_{\text{net}}(\rho, x,t) - \varepsilon \nabla^T \v_{\text{net}}(\rho, x,t) + O(\varepsilon^2), \]
cf. (4.17)ff. in [5] proof of Lemma 4.4. Moreover,
\[
\Delta u^n_A = \sum_{s,t} \v_{\text{net}}(\rho) h_{N+\frac{1}{2}}(s,t) + \v_{\text{net}}(\rho) h_{N+\frac{1}{2}}(s,t)
\]
and
\[
\sum_{s,t} \v_{\text{net}}(\rho) h_{N+\frac{1}{2}}(s,t) + \v_{\text{net}}(\rho) h_{N+\frac{1}{2}}(s,t)
\]
in \( L^\infty(0, T; L^2(\Gamma(2\delta))) \) because of (2.16), and lastly
\[
\frac{1}{\varepsilon^2} f''(c^n_A) u^n_A = \varepsilon^{N-2} f''(\theta(\rho)) \theta'(\rho) h_{N+\frac{1}{2}}(s,t) + \varepsilon^{N-2} \theta''_0(\rho) h_{N+\frac{1}{2}}(s,t) + \varepsilon^{N-2} \theta''_0(\rho) h_{N+\frac{1}{2}}(s,t) + O(\varepsilon^{N+1})
\]
in \( L^\infty(0, T; L^2(\Gamma(2\delta))) \).
In the following we use that
\[
\partial^2_{\rho} c_2(\rho, s,t) + f''(\theta(\rho)) \theta'(\rho) c_2 = |\nabla h_{1}(s,t)|^2 \theta''_0(\rho) - \theta'_0(\rho) \rho_0(\rho) - \theta''_0(\rho) \rho_0(\rho)
\]
for all \( s \in \mathbb{T}_1, t \in [0, T] \) and \( \rho \in \mathbb{R} \) (cf. Remark A.3 in the appendix below), which yields after differentiation with respect to \( \rho \)
\[
\partial^2_{\rho} c_2 + f''(\theta(\rho)) \partial^2_{\rho} c_2 + f''(\theta(\rho)) \theta''_0(\rho) c_2 = |\nabla h_{1}(s,t)|^2 \theta''_0(\rho) - \theta''_0(\rho) \rho_0(\rho) - \theta''_0(\rho) \rho_0(\rho)
\]
for all \( s \in \mathbb{T}_1, t \in [0, T] \) and \( \rho \in \mathbb{R} \). Using additionally (1.10),
\[
\theta''_0(\rho) + f''(\theta(\rho)) \theta'(\rho) = 0 \quad \text{for all } \rho \in \mathbb{R},
\]
and
\[
\partial^2_{\rho} h_{N+\frac{1}{2}} - |\nabla h_{1}(s,t)|^2 \theta''_0(\rho) - \theta''_0(\rho) \rho_0(\rho) + \varepsilon^{N-2} \theta''_0(\rho) a(\rho, x,t) + O(\varepsilon^{N+1})
\]
for all \( s \in \Gamma, t \in [0, T] \), we obtain
\[
\partial^2_{\rho} u^n_A + \nabla u^n_A - \Delta u^n_A + \frac{1}{\varepsilon^2} f''(c^n_A) u^n_A = -\varepsilon^{N-2} \n \cdot \nabla^T h_{N+\frac{1}{2}} + \varepsilon^{N-2} \theta''_0(\rho) + \varepsilon^{N-2} \theta''_0(\rho) a(\rho, x,t) + O(\varepsilon^{N+1})
\]
in \( L^\infty(0, T; L^2(\Gamma(2\delta))) \) since the \( O(\varepsilon^{N-\frac{1}{2}}) \)- and \( O(\varepsilon^{N-\frac{3}{2}}) \)-terms cancel, where
\[
\int_{\mathbb{T}_1} a(\rho, s,t) \psi d\rho = 0 \quad \text{for all } s \in \mathbb{T}_1, t \in [0, T]
\]
because of \( \int_{\mathbb{T}_1} \partial^2_{\rho} \theta''_0(\rho) \theta'(\rho) d\rho = 0 \). Next we use \( a(\rho, x,t) = f_1(x,t) + f_2(x,t) \) with
\[
f_1(x,t) = \varepsilon^{N-2} a(\rho, S(x,t), t), \quad f_2(x,t) = \varepsilon^{N-2} (a(\rho, x,t) - a(\rho, S(x,t), t)) + O(\varepsilon^{N+1})
\]
in $L^\infty(0,T_r;L^2(\Gamma(2\delta)))$. Then, because of Lemma 2.11 and Lemma 2.14,
\[
\|f_1\|_{L^2(0,T_r;X')} \leq C\varepsilon^{N+\frac{1}{2}+\frac{1}{2}} = C\varepsilon^{N+1},
\]
\[
\|f_2\|_{L^2(0,T_r;L^2(\Omega_{(2\delta)}))} \leq C\varepsilon^N,
\]
and
\[
\|f_2\|_{L^2(0,T_r;X')} \leq C\|f_2\|_{L^2(0,T_r;L^2(\Gamma(2\delta)))} \leq C\varepsilon^{N+\frac{1}{2}+\frac{1}{2}} = C\varepsilon^{N+1}
\]
because of [5, Corollary 2.7]. Altogether, the previous estimates imply the statements of the theorem.

**Proof of Theorem 3.1** Let $(c_A, v_A, p_A)$ be as in (3.18) - 3.10. Moreover, we define
\[
a(\rho, x, t) := \partial_t(\theta_0(\rho)\theta_0(\rho)) h_{N+\frac{1}{2}}(s, t)(\nabla^T h_1)(r, s, t) + \theta''_0(\rho)\theta_0(\rho)(\nabla^T h_{N+\frac{1}{2}})(r, s, t)
\]
\[-2\theta''_0(\rho)\theta_0(\rho) h_{N+\frac{1}{2}}(s, t) \text{ div } (n \otimes n) \]
(3.24)
with $s = S(x, t)$, $r = d_T(x)$, and choose $\tilde{w}$ and $\tilde{q}$ as solution of
\[
-\partial_s(2\nu(\theta_0(\rho))\partial_s w(x, t)) + \partial_s q(x, t) n = n(\rho, x, t),
\]
\[-\partial_s\tilde{w}(x, t) \cdot n = 0
\]
for all $\rho \in \mathbb{R}$, $x \in \Gamma(3\delta)$, $t \in [0, T_r]$, where we use that $\int_\mathbb{R} n(\rho, x, t) d\rho = 0$ and Lemma A.3 in the appendix. Actually, $\tilde{q}$ is determined such that
\[
\partial_s\tilde{q}(x, t) = n \cdot a(\rho, x, t)
\]
and
\[
-\partial_s(2\nu(\theta_0(\rho))\partial_s w_\nu(x, t)) = a_\nu(\rho, x, t),
\]
\[n \cdot \tilde{w}(x, t) = 0
\]
for all $\rho \in \mathbb{R}$, $x \in \Gamma(3\delta)$, $t \in [0, T_r]$. Since $a$ depends linearly on $(h_{N+\frac{1}{2}}, \partial_s h_{N+\frac{1}{2}}) \in L^2(0, T_r; H^1(\Gamma_1)) \cap H^{1/2}(0, T_r; L^2(\Gamma_1))$, we have that
\[
\tilde{w}, \tilde{q} \in L^2(0, T_r; H^1(\Gamma_1)) \cap H^{1/2}(0, T_r; L^2(\Gamma_1))
\]
for some smooth $\tilde{b}_j : \mathbb{R} \times \Gamma(3\delta) \to \mathbb{R}^2$, $j = 1, 2$, with uniformly bounded $C^k$-norms such that $\partial_s^k \tilde{b}_j \in (\mathcal{R}_{\theta_0})^2$ for every $k \in \mathbb{N}$ and $j = 1, 2$. Since $h_{N+\frac{1}{2}}, \partial_s h_{N+\frac{1}{2}} \in L^2(0, T_r; H^1(\Gamma_1)) \cap H^{1/2}(0, T_r; L^2(\Gamma_1))$ are bounded by some $C(M)$, we also obtain
\[
\|\tilde{w}\|_{L^2(0, T_r; H^1(\mathbb{R} \times \Gamma_1))} + \|\tilde{q}\|_{H^1(0, T_r; L^2(\mathbb{R} \times \Gamma_1))} \leq C(M)
\]
for all $\varepsilon \in (0, \varepsilon_0)$. Then
\[
\text{div}(v_A(x, t) + \tilde{N} \tilde{a}_k(t)) = \zeta(d_T(x, t))(G_\varepsilon + \varepsilon^{N+1}) \big((\text{div}, \tilde{w})(\rho, x, t) - \nabla h_4(S(x, t), t)\partial_s \tilde{w}(\rho, x, t)\big)
\]
\[+ \zeta'(d_T(x, t))(v^\nu_A - v^\nu_A(x, t)\chi_{\Omega^+}(x) - v^-A(x, t)\chi_{\Omega^-}(x)) = O(\varepsilon^{N+1})
\]
(3.25)
in $L^\infty(0, T_r; L^2(\Omega))$ because of (3.18), the matching condition in Theorem 3.2 and $n \cdot \tilde{w}(\rho, x, t) = 0$. Hence
\[
\sup_{0 \leq t \leq T_r} |\tilde{a}_k(t)| = \sup_{0 \leq t \leq T_r} \frac{1}{H^1(\partial \Omega)} \left| \int_{\partial \Omega} (v_A + \tilde{N} \tilde{a}_k(t)) \, d\sigma \right|
\]
\[= \sup_{0 \leq t \leq T_r} \frac{1}{H^1(\partial \Omega)} \left| \int_{\Omega} \text{div}(v_A + \tilde{N} \tilde{a}_k) \, dx \right| \leq C\varepsilon^{N+1}
\]
and similarly
\[ \|u_k\|_{H^{1/2}(0,T;L^2)} = \frac{1}{H^{1/2}(\Omega)} \left\| \int_\Omega (v_A + N\tilde{u}_k) \, dx \right\|_{H^{1/2}(0,T;L^2)} \leq C \varepsilon^{N+1}. \]

This shows (3.22) for some (different) \( G_\varepsilon : \Omega \times (0,T) \to \mathbb{R} \), which is given by the sum of the right-hand side of (3.23) and \(-\text{div}(N\tilde{u}_k(t))\), such that
\[ \|G_\varepsilon\|_{H^{1/2}(0,T;L^2(\Omega))} \leq C(M)\varepsilon^{N+1}. \]

In the same manner as for the divergence equation one shows that
\[ \partial_t v_A - \text{div}(2\varepsilon(c_A)Dv_A) + \nabla p_A = -\varepsilon (d_0) \text{div}(\nabla \varepsilon_A^n \otimes \nabla c_A^n) \]
\[ + \varepsilon^{N-\frac{2}{3}} (\partial_\rho (2\varepsilon_0(\rho)) \partial_\rho \tilde{w}(\rho,x,t) + \partial_\rho \tilde{q}(\rho,x,t)n) + \varepsilon^{N-\frac{2}{3}} r(\rho,x,t) + O(\varepsilon^{N+1}) \]
\[ \text{in } L^2(0,T;L^2(\Gamma_\varepsilon(2\delta))) \]
by using (3.17) and the matching condition in Theorem 3.2. Moreover, one can use (2.19) in Lemma 2.3 to show that \( \varepsilon^{N-\frac{2}{3}} r(\rho,x,t) \) is of order \( O(\varepsilon^{N+1}) \) in \( L^2(0,T;L^2(\Gamma_\varepsilon(T\delta))) \).

Hence
\[ \partial_t v_A - \text{div}(2\varepsilon(c_A)Dv_A) + \nabla p_A = -\varepsilon (d_0) \text{div}(\nabla \varepsilon_A^n \otimes \nabla c_A^n) \]
\[ + \varepsilon^{N-\frac{2}{3}} (\partial_\rho (2\varepsilon_0(\rho)) \partial_\rho \tilde{w}(\rho,x,t) + \partial_\rho \tilde{q}(\rho,x,t)n) + O(\varepsilon^{N+\frac{2}{3}}) \quad (3.26) \]
\[ \text{in } L^2(0,T;L^2(\Gamma_\varepsilon(T\delta))) \]
Now we use that
\[ \varepsilon \nabla c_A^n \otimes \nabla c_A^n - \varepsilon \nabla c_A^n \otimes \nabla c_A^n \]
\[ = \varepsilon^{N-\frac{2}{3}} 2\theta_0'(\rho) \theta_0(\rho) n \otimes n h_{N+\frac{1}{3}}(S(x,t),t) \]
\[ + \varepsilon^{N-\frac{2}{3}} \theta_0'(\rho) \theta_0(\rho) \left( h^\Gamma h_1(S(x,t),t) \otimes n + n \otimes h^\Gamma h_1(S(x,t),t) \right) h_{N+\frac{1}{3}}(S(x,t),t) \]
\[ + \varepsilon^{N-\frac{2}{3}} \theta_0'(\rho) \left( \nabla h^\Gamma h_{N+\frac{1}{3}}(S(x,t),t) \otimes n + n \otimes \nabla h^\Gamma h_{N+\frac{1}{3}}(S(x,t),t) \right) \]
\[ + \varepsilon^{N-\frac{2}{3}} r_s(\rho,x,t) \cdot a_s(s,t) \]
for some \( r_s \in (R_0,a)^N, a_s \in L^\infty(0,T;L^2(T^\Gamma)) \) and \( N \in \mathbb{N} \) with uniformly bounded norms in \( \varepsilon \in (0,\varepsilon_0) \). Here one uses that \( h_{N+\frac{1}{3}} \in X_T \mapsto L^\infty(0,T;W^{1,2}_2(T^\Gamma)) \) is bounded and that \( \partial_s h_{N+\frac{1}{3}} \) enters at most quadratically. Hence, using Lemma 2.3 we obtain
\[ -\varepsilon \text{div}(\nabla c_A^n \otimes \nabla c_A^n) + \varepsilon \text{div}(\nabla \varepsilon_A^n \otimes \nabla c_A^n) \]
\[ = \varepsilon^{N-\frac{2}{3}} \text{div} \left( 2\theta_0'(\rho) \theta_0(\rho)(I - n \otimes n) h_{N+\frac{1}{3}} \right) + \nabla p_s \]
\[ - \varepsilon^{N-\frac{2}{3}} \text{div} \left( \theta_0'(\rho) \theta_0(\rho)(\nabla h^\Gamma h_1 \otimes n + n \otimes \nabla h^\Gamma h_1) h_{N+\frac{1}{3}} \right) \]
\[ - \varepsilon^{N-\frac{2}{3}} \text{div} \left( \theta_0'(\rho) \left( \nabla h^\Gamma h_{N+\frac{1}{3}}(S(x,t),t) \otimes n + n \otimes \nabla h^\Gamma h_{N+\frac{1}{3}}(S(x,t),t) \right) \right) + O(\varepsilon^{N+1}) \]
\[ \text{in } L^2(0,T;L^2(\Gamma_\varepsilon(2\delta))) \], where \( p_s(x,t) = \varepsilon^{N-\frac{2}{3}} 2\theta_0'(\rho) \theta_0(\rho) h_{N+\frac{1}{3}}(S(x,t),t), a_s \) as in (3.21) and
\[ \|R_s\|_{L^2[0,T;L^2(\Gamma_\varepsilon(2\delta))]} \leq C \varepsilon^{N+\frac{1}{3}} h_{N+\frac{1}{3}} \|\varepsilon^{N+\frac{2}{3}} a_s \|_{L^2[0,T;H^2(\Omega)]} \leq C' \varepsilon^{N+\frac{1}{3}} \|u\|_{L^2(\Gamma_\varepsilon(2\delta))} \]
for some \( C > 0 \) independent of \( \varepsilon \in (0,\varepsilon_0), M > 0 \). Here one estimates the additional remainder terms in the same way as before with the aid of (3.20) and uses (3.21) - (3.23) together with standard estimates for parabolic equations. Altogether the terms in (3.21) with the factor \( \varepsilon^{N-\frac{2}{3}} \) cancel and all remaining terms are of order \( O(\varepsilon^{N+\frac{1}{3}}) \) in \( L^2(0,T;H^1(\Gamma_\varepsilon(T\delta))) \). Therefore combining (3.22) and (3.24) and the previous identity and replacing \( p_A \) by \( p_A + p_s \) we obtain (3.1).
4 Proof of Main Result

4.1 Preparations and the Error in the Velocity

Throughout this subsection we assume that \((c_A, v_A, p_A)\) and \((\tilde{c}_A, \tilde{v}_A, \tilde{p}_A)\) are given as in Section 3, where \((c_A, v_A, p_A)\) still depends on the choice of \(u\), which will be specified next, but \((\tilde{c}_A, \tilde{v}_A, \tilde{p}_A)\) do not. In the following we will often write \(n\) instead of \(n_f\).

Let \(\tilde{w} : \Omega \times [0, T_0] \to \mathbb{R}^2\) be such that
\[
v_s = v_A + \tilde{w}.
\]
Then \(\tilde{w}\) solves
\[
\begin{aligned}
\partial_t \tilde{w} + v_s \cdot \nabla \tilde{w} - \text{div}(2\nu(c_s)D\tilde{w}) + \nabla q &= -\varepsilon \text{div}(\nabla u \otimes \nabla c_A) - \varepsilon \text{div}(\nabla u \otimes \nabla u) \\
&\quad + \text{div}(2(\nu(c_s) - \nu(c_s))Dv_A) - \tilde{w} : \nabla v_A - R_1^1 - R_2^2, \\
\text{div} \tilde{w} &= -G_s, \\
\tilde{w}|_{t=0} &= v_{0,s} - v_{A,0}, \\
\tilde{w}|_{\partial \Omega} &= 0,
\end{aligned}
\]
for some \(q : \Omega \times [0, T_0] \to \mathbb{R}\), where \(u = c_\varepsilon - c_A, a \otimes b = a \otimes b + b \otimes a\) and \(R_s, G_s\) are as in Theorem 4.3. Now we choose
\[
u = \frac{\tilde{w}}{n + 1/2}.
\]
More precisely, since the right-hand side depends on \(c_A\) and therefore on \(u\), this \(\nu\) is determined by a non-linear (and non-local) evolution equation with a globally Lipschitz nonlinearity, which can be solved by the same arguments as in [3] Proof of Lemma 4.2. Then \(c_A\) solves
\[
\begin{aligned}
\partial_t c_A + v_A \cdot \nabla c_A + \varepsilon^{N+h_\sigma} \tilde{w} \cdot \nabla c_A &= \Delta c_A - \frac{f'(c_A)}{\varepsilon^2} + S_s, \\
\end{aligned}
\]
in \(\Omega \times (0, T_0)\). For the following we consider the estimates
\[
\begin{aligned}
\sup_{0 \leq t \leq \tau} \|c_\varepsilon(t) - c_A(t)\|_{L^2(\Omega)} + \|\nabla (c_\varepsilon - c_A)\|_{L^2(\Omega \times (0, \tau))} &\leq R\varepsilon^{N+\frac{1}{2}}, \\
\|\nabla (c_\varepsilon - c_A)\|_{L^2(\Omega \times (0, \tau))} + \|\nabla^2 (c_\varepsilon - c_A)\|_{L^2(\Omega \times (0, \tau))} &\leq R\varepsilon^{N+\frac{1}{2}}, \\
\int_0^\tau \int_{\Gamma(t)} |\nabla u|^2 + \varepsilon^{-2} f''(c_A(x, t))u^2 dx dt &\leq R^2\varepsilon^{2N+1}.
\end{aligned}
\]
hold true for some \(\tau = \tau(\varepsilon) \in (0, T_0], \varepsilon_0 \in (0, 1]\), and all \(\varepsilon \in (0, \varepsilon_0]\), where \(R = R(\theta) > 0\) is chosen such that
\[
\|v_{0,s} - v_{A,0}\|_{L^2(\Omega)} + \varepsilon^4 \|\nabla (v_{0,s} - c_A)\|^2_{L^2(\Omega)} + \|v_{0,s} - v_{A,0}\|_{L^2(\Omega)} \leq \frac{\theta^2 R^2}{4} \varepsilon^{2N+1} e^{-C_L T_0}
\]
for all \(\varepsilon \in (0, 1]\), where \(C_L > 0\) is the constant from the spectral estimate in Theorem 2.7 and \(\theta \in (0, 1]\) is a suitable constant to be chosen later. Moreover, we define
\[
T_\varepsilon := \sup\{\tau \in [0, T_0] : \text{(4.5) holds true.}\}
\]
Then \(T_\varepsilon > 0\).

The following theorem provides the essential estimate for the error in the velocity.

\textbf{Theorem 4.1} Assume that \(c_A, \tilde{w}\) are as above and \(u\) satisfies \((4.3)\) for some \(R > 0, \tau = T(\varepsilon)\), and \(N \geq 3\). Then there are some \(C(R) > 0, \varepsilon_0 > 0\), and \(M > 0\) independent of \(\varepsilon \in (0, \varepsilon_0]\) and \(T \in (0, T_\varepsilon]\) such that \(h_{N+\frac{1}{2}} \|v_A\|_{X_T} \leq M\) and
\[
\|\tilde{w}\|_{L^\infty(0, T; L^2(\Omega))} + \|\tilde{w}\|_{L^2(0, T; H^1(\Omega))} \leq C(R)\varepsilon^{N+\frac{1}{2}},
\]
where \(h_{N+\frac{1}{2}}\) is as in Theorem 4.3 with \(u\) as in \((4.3)\).
Proof: First assume that $h_{N+\frac{1}{2}} \in X_T$ is some function with $\|h_{N+\frac{1}{2}}\|_{X_T} \leq M$ and $c_A, v_A, p_A$ is constructed with this choice of $h_{N+\frac{1}{2}}$. We note that $\tilde{c}_A$ and $\tilde{v}_A$ are independent of $h_{N+\frac{1}{2}}$.

To proceed we introduce $(w_0, q_0)$ satisfying the following system

$$\begin{align*}
\partial_t w_0 - \Delta w_0 + \nabla q_0 &= 0 \quad \text{in } \Omega \times [0, T], \\
\operatorname{div} w_0 &= G_\varepsilon \quad \text{in } \Omega \times [0, T], \\
w_0|_{\partial \Omega} &= 0 \quad \text{on } \partial \Omega \times [0, T], \\
w_0|_{t=0} &= 0 \quad \text{in } \Omega.
\end{align*}$$

(4.9)

We note that

$$\int_\Omega G_\varepsilon \, dx = \int_\Omega \operatorname{div} v_A \, dx = 0 \quad \text{for all } t \in [0, T]$$

since $G_\varepsilon$ is as in Theorem 3.1. Therefore the latter system possesses a unique solution for every $t \in [0, T]$, for which we have

$$\|w_0\|_{L^2(0, T; H^1(\Omega))} + \|\partial_t w_0\|_{L^2(0, T; V')} \leq C \left( \|G_\varepsilon\|_{L^2(0, T; L^2(\Omega))} + \|G_\varepsilon\|_{H^1(0, T; L^2(\Omega))} \right) \leq C(M)\varepsilon^{N+1}$$

(4.10)

because of [39] Theorem 3.3, where $V = H_0^1(\Omega)^2 \cap L^6(\Omega)$ and $H_0^1(\Omega) = (H_0^1(\Omega))'$. Using

$$L^2(0, T; V) \cap H^1(0, T; V') \hookrightarrow L^\infty(0, T; L^2(\Omega)),$$

we obtain additionally

$$\|w_0\|_{L^\infty(0, T; L^2(\Omega))} \leq C(M)\varepsilon^{N+1}.$$  (4.11)

Now we define $\tilde{w} = \tilde{w} + w_0$ and then the system for $\tilde{w}$ reads

$$\begin{align*}
\partial_t \tilde{w} + v_\varepsilon \cdot \nabla \tilde{w} - \operatorname{div}(2\nu(c_\varepsilon)D\tilde{w}) + \nabla q = -\varepsilon \operatorname{div}(\nabla u \otimes \nabla c_A) - \tilde{w} \cdot \nabla v_A + w_0 \cdot \nabla v_A \\
&+ \operatorname{div}(2(\nu(c_\varepsilon) - \nu(c_A))Dv_A) \\
&- \varepsilon \operatorname{div}(\nabla u \otimes \nabla u) - R_\varepsilon^1 - R_\varepsilon^2 \\
&+ \partial_t w_0 + v_\varepsilon \cdot \nabla w_0 - \operatorname{div}(2\nu(c_\varepsilon)Dw_0),
\end{align*}$$

(4.12)

where the system has to be understood in a weak sense. Since $\nabla v_A$ is uniformly bounded, we have

$$\left| \int_0^T \int_\Omega (\tilde{w} \cdot \nabla v_A) \cdot \tilde{w} \, dx dt \right| \leq C\|\tilde{w}\|_{L^2(0, T; L^2(\Omega))}^2.$$

Moreover, Korn’s inequality leads to

$$\int_\Omega \nu(c_\varepsilon)|D\tilde{w}|^2 \, dx \geq C \int_\Omega |\nabla \tilde{w}|^2 \, dx.$$  (4.13)

Testing (4.12) with $\tilde{w}$ and integrating by parts yield that

$$\begin{align*}
\frac{1}{2} \frac{d}{dt} \int_\Omega |\tilde{w}|^2 \, dx - C \int_\Omega |\tilde{w}|^2 \, dx + 2 \int_\Omega \nu(c_\varepsilon)|D\tilde{w}|^2 \, dx \\
&\leq \varepsilon \int_\Omega \langle \nabla u \otimes \nabla c_A : \nabla \tilde{w} \rangle \, dx - 2 \int_\Omega \left( \nu(c_\varepsilon) - \nu(c_A) \right)Dv_A : \nabla \tilde{w} \, dx \\
&+ \varepsilon \int_\Omega \langle \nabla u \otimes \nabla u : \nabla \tilde{w} \rangle \, dx - \int_\Omega R_\varepsilon \cdot \tilde{w} \, dx \\
&- \int_\Omega (\tilde{w} \cdot \nabla (v_A - \tilde{v}_A)) \cdot \tilde{w} \, dx + \langle \partial_t w_0(t), \tilde{w} \rangle_{V', V} +
\end{align*}$$
4.1 Preparations and the Error in the Velocity

\[ + \int_{\Omega} \left( v_s \cdot \nabla w_0 + w_0 \cdot \nabla v_A \right) \cdot \bar{w} + \left( 2\nu(c_e) Dw_0 : \nabla \bar{w} \right) \, dx. \quad (4.14) \]

Thanks to Gronwall’s inequality we get

\[
\begin{align*}
& \sup_{0 \leq t \leq T} \frac{1}{2} \int_{\Omega} |\bar{w}|^2 \, dx + 2 \int_0^T \int_{\Omega} \nu(c_e) |D\bar{w}|^2 \, dx \, dt \\
& \leq e^{CT} \left( \frac{1}{2} \int_{\Omega} |\bar{w}|^2 |_{t=0} \, dx + \varepsilon \int_0^T \int_{\Omega} \left( \nabla u \otimes^{\delta} \nabla c_A : \nabla \bar{w} \right) \, dx \, dt \\
& + 2 \int_0^T \int_{\Omega} \left( (\nu(c_e) - \nu(c_0)) \nabla v_A : \nabla \bar{w} \right) \, dx \, dt \\
& + \varepsilon \int_0^T \int_{\Omega} \left( \nabla u \otimes \nabla v_A : \nabla \bar{w} \right) \, dx \, dt + \int_0^T \int_{\Omega} R_s \cdot \bar{w} \, dx \, dt \\
& + \int_0^T \int_{\Omega} \left( \bar{w} \cdot \nabla (v_A - \bar{v}_A) \cdot \bar{w} \right) \, dx \, dt + \int_0^T \int_{\Omega} \left( \partial_t w_0(t) \cdot \bar{w} \middle| \Gamma \right) \, dt \\
& + \int_0^T \int_{\Omega} \left( v_s \cdot \nabla w_0 + w_0 \cdot \nabla v_A \right) \cdot \bar{w} + \left( 2\nu(c_e) Dw_0 : \nabla \bar{w} \right) \, dx \, dt \right).
\end{align*}
\]

To proceed the proof will be divided into three steps.

**Step 1:** In this step we prove

\[
\varepsilon \int_0^T \int_{\Omega} \nabla u \otimes \nabla c_A : \nabla \bar{w} \, dx \, dt \leq \left( C_0(R) \varepsilon^{N+\frac{1}{2}} + C(R, M) \varepsilon^{N+1} \right) \|\nabla \bar{w}\|_{L^2(0,T;L^2(\Omega))} \quad (4.16)
\]

for some \( C_0(R) \) independent of \( M \). We decompose \( \nabla c_A = \nabla \bar{c}_A |_{\Omega \setminus \Gamma_1(\varepsilon\delta)} + \nabla \bar{c}_A |_{\Gamma_1(\varepsilon\delta)} + \nabla (c_A - \bar{c}_A) \) and then the proof of (4.16) consists of three parts. Firstly, obviously there holds

\[
\varepsilon \int_0^T \int_{\Omega \setminus \Gamma_1(\varepsilon\delta)} \nabla u \otimes \nabla \bar{c}_A : \nabla \bar{w} \, dx \, dt \leq C\varepsilon \|\nabla u\|_{L^2(0,T;L^2(\Omega \setminus \Gamma_1(\varepsilon\delta)))} \|\nabla \bar{w}\|_{L^2(0,T;L^2(\Omega))}. \quad (4.17)
\]

Secondly, noting that

\[
|\partial_t (x, t) - \varepsilon h_e (S(x, t), t)| \geq \frac{\delta}{2} \quad \text{for all } x \in \Gamma_1(\varepsilon\delta) \setminus \Gamma_1(\delta), \, t \in [0,T], \varepsilon \in (0,\varepsilon_0) \quad (4.18)
\]

if \( \varepsilon_0 \in (0,1) \) is sufficiently small, we have for \( x \in \Gamma_1(\varepsilon\delta), t \in [0,T] \)

\[
\nabla \bar{c}_A(x, t) = \nabla (\zeta \circ \partial_t) \left( \bar{c}_A^\varepsilon (\rho, s, t) - c_A^\varepsilon \chi_+ - c_A^\varepsilon \chi_- \right) + \varepsilon^{-1} \zeta \circ \partial_t \bar{\theta}_0(\rho) \, \mathbf{n} = O(\varepsilon^{\frac{1}{2}})
\]

\[
- \zeta \circ \partial_t \bar{\theta}_0(\rho) \sum_{k=0}^{N} \varepsilon^k \nabla \tau h_{k+1}(s, t) + \zeta \circ \partial_t \sum_{k=2}^{N+2} \varepsilon^k \nabla \tau \hat{c}_k(\rho, s, t) = O(1)
\]

\[
+ \zeta \circ \partial_t \sum_{k=2}^{N+2} \varepsilon^{k-1} \partial_\rho \hat{c}_k(\rho, s, t) \left( \mathbf{n} - \sum_{k=0}^{N} \varepsilon^{k+1} \nabla \tau h_{k+1}(s, t) \right). \quad (4.19)
\]

Using (4.19) we obtain

\[
\varepsilon \int_0^T \int_{\Gamma_1(\varepsilon\delta)} \nabla u \otimes \nabla \bar{c}_A : \nabla \bar{w} \, dx \, dt \leq \int_0^T \int_{\Gamma_1(\varepsilon\delta)} \zeta \circ \partial_t \bar{\theta}_0(\rho) \partial_n u \left( \mathbf{n} \otimes \nabla \bar{w} \right) \, dx \, dt + C\|\nabla u\|_{L^2(0,T;L^2(\Gamma_1(\varepsilon\delta)))} \|\nabla \bar{w}\|_{L^2(0,T;L^2(\Omega))}
\]
Finally, we note that

\[ \text{Consequently, if } L \leq \text{div } \mathbf{w} = (\mathbf{n} \otimes \mathbf{n} : \nabla \mathbf{w}) + \text{div} \tilde{\mathbf{w}}. \]

With the aid of (2.30) and using similar arguments as in the proof of Lemma 3.4 in [6] one shows

\[ I \leq C \left( \| Z \|_{L^2(0,T,H^1(\Omega))} + \| \psi \|_{L^2(0,T,H^1(\Gamma_t(2\delta)))} \right) \| \nabla \mathbf{w} \|_{L^2(0,T,L^2(\Omega))}, \]

where

\[ \Lambda_\varepsilon = \int_{\Gamma_t(\delta)} \left( \varepsilon |\nabla u|^2 + \frac{1}{\varepsilon} \int_0^\varepsilon (c_\delta(x,t)) u(x,t)^2 \right) dx. \]

Combining (4.22) with (1.20) one has

\[ \varepsilon \int_0^T \int_{\Gamma_t(2\delta)} \nabla u \otimes \nabla \tilde{u} : \nabla \mathbf{w} \mathbf{d}x \mathbf{d}t \]

\[ \leq C \left( \| \Lambda_\varepsilon \|_{L^1(0,T)} + \| u \|_{L^2(0,T,L^2(\Omega))} \right) \| \nabla \mathbf{w} \|_{L^2(\Omega \times (0,T))}, \]

Combining (4.22) with (1.20) one has

\[ \varepsilon \int_0^T \int_{\Gamma_t(2\delta)} \nabla u \otimes \nabla \tilde{u} : \nabla \mathbf{w} \mathbf{d}x \mathbf{d}t \]

\[ \leq C \left( \| \Lambda_\varepsilon \|_{L^1(0,T)} + \| u \|_{L^2(0,T,L^2(\Omega))} \right) \| \nabla \mathbf{w} \|_{L^2(\Omega \times (0,T))}, \]

Thirdly, we note that

\[ \| \nabla u \|_{L^4(\Omega)} \leq C \left( \| \nabla u \|_{L^2(\Omega)}^{\frac{1}{2}} \| \Delta u \|_{L^2(\Omega)}^{\frac{1}{2}} + \| \nabla u \|_{L^2(\Omega)} \right). \]

Therefore

\[ \| \nabla u \|_{L^4(\Omega)} \leq C \left( \| \nabla u \|_{L^2(\Omega)}^{\frac{1}{2}} \| \Delta u \|_{L^2(\Omega)}^{\frac{1}{2}} + \int_0^T \| \nabla u \|_{L^2(0,T,L^2(\Omega))} \right) \]

\[ \leq C(R) \varepsilon^{N - \frac{5}{2}}. \]

Moreover, since

\[ \nabla (c_A - \tilde{c}_A) = \zeta \left( \varepsilon^{N - \frac{5}{2}} \theta_0(\rho) + \varepsilon^{N + \frac{1}{2}} \partial_s \tilde{c}_2(\rho, s, t) \right) \nabla (h_N + \frac{1}{\varepsilon} S(x, t, t)) + O(\varepsilon^{N - \frac{3}{2}}) \]

in \( L^\infty ((0, T) \times \Omega) \), we obtain

\[ \| \nabla (c_A - \tilde{c}_A) \|_{L^4(\Omega \times (0,T))} \leq C(M) \varepsilon^{N - \frac{5}{2}}. \]

Consequently, if \( N \geq 3 \), we conclude

\[ \varepsilon \int_0^T \int_{\Omega} \nabla u \otimes \nabla (c_A - \tilde{c}_A) : \nabla \mathbf{w} \mathbf{d}x \mathbf{d}t \]
4.1 Preparations and the Error in the Velocity

\[ \leq C \varepsilon \| \nabla u \|_{L^4(\Omega \times (0,T))} \| \nabla (c_A - \tilde{c}_A) \|_{L^4(\Omega \times (0,T))} \| \nabla \tilde{w} \|_{L^2(\Omega \times (0,T))} \]
\[ \leq (R,M) \varepsilon^{N-2} \| \nabla \tilde{w} \|_{L^2(\Omega \times (0,T))} \leq (R,M) \varepsilon^{N+1} \| \nabla \tilde{w} \|_{L^2(\Omega \times (0,T))}. \] (4.25)

Here we have used \( X_T \to C([0,T]; H^2(T^1)) \to C([0,T]; W^{1,p}(T^1)) \) for all \( 1 \leq p < +\infty \).

Since \( N \geq 3 \), \( (4.10) \) is a consequence of \( (4.17), (4.20), (4.28) \) and \( (4.1) \).

Step 2: In this step we show

\[ \int_0^T \left| \int_\Omega \left( (\nu(c_+ - \nu(c_A)) D\tilde{v}_A : \nabla \tilde{w} \right) dx \right| dt \leq C(R,M) \varepsilon^{N+1} \| \nabla \tilde{w} \|_{L^2(\Omega \times (0,T))}. \] (4.26)

We use the decomposition \( D\tilde{v}_A = D\tilde{v}_A + D(v_A - \tilde{v}_A) \) and estimate each term separately.

Firstly, we note that the leading order of \( \tilde{v}_A \) is

\[ v_0(x,t) = \zeta \circ dt \tilde{v}_0(\rho, x, t) + (1 - \zeta \circ dt \tilde{v}_0(\rho, x, t) \chi_+ + v_0 (x, t) \chi_- \] (4.27)

with \( \tilde{v}_0 \) defined in \( (A.54) \). It follows from \( (4.27), (A.54) \) and that \( v_0, h_k, k = 0, \ldots, N \), are smooth that one has

\[ |\nabla v_0| \leq \frac{1}{\varepsilon} \zeta \circ dt \left| \frac{v_0^+ - v_0^-}{dt} \cdot dt \nu_0'(\rho) \right| + \zeta \circ dt \left| (v_0^+ - v_0^-) \nu_0'(\rho) \nu_1 \right| + C \]
\[ \leq C \zeta \circ dt |\nu_0'(\rho + h_\varepsilon)| + C \leq C. \] (4.28)

From this it easily follows that \( \nabla \tilde{v}_A \) is uniformly bounded. Accordingly we get

\[ \int_0^T \left| \int_\Omega \left( (\nu(c_+ - \nu(c_A)) D\tilde{v}_A : \nabla \tilde{w} \right) dx \right| dt \leq C(T^\frac{1}{2} \| u \|_{L^\infty(0,T;L^2(\Omega)))} \| \nabla \tilde{w} \|_{L^2(\Omega \times (0,T))}) \] (4.29)

here we have used \( |\nu(c_+ - \nu(c_A))| \leq \| u' \|_{L^\infty(0,T)} |u| \).

Secondly, since

\[ \| \nabla (v_A - \tilde{v}_A) \|_{L^4(0,T;L^2(\Omega)))} \leq C(M) \varepsilon^N \]
due to \( X_T \to L^4(0,T; H^2(T^1))) \), we conclude

\[ \varepsilon \int_0^T \left| \int_\Omega \left( (\nu(c_+ - \nu(c_A)) D(v_A - \tilde{v}_A) : \nabla \tilde{w} \right) dx \right| dt \]
\[ \leq C(R,M) \varepsilon T^\frac{1}{2} \| \nabla (v_A - \tilde{v}_A) \|_{L^4(0,T;L^2(\Omega)))} \| \nabla \tilde{w} \|_{L^2(\Omega \times (0,T))} \]
\[ \leq C(R,M) \varepsilon^{N+1} \| \nabla \tilde{w} \|_{L^2(0,T;H^1(\Omega))}. \] (4.30)

Thanks to \( (4.29), (4.30) \) and \( (4.28) \) we derive \( (4.26) \).

Step 3: Similarly as in the proof of \( (4.25) \) we conclude

\[ \varepsilon \int_0^T \left| \int_\Omega \left( \nabla u \cdot \nabla \tilde{u} : \nabla \tilde{w} \right) dx \right| dt \leq C \varepsilon \| \nabla u \|_{L^4(0,T;L^4(\Omega)))} \| \nabla \tilde{w} \|_{L^2(\Omega \times (0,T))} \]
\[ \leq C(R) \varepsilon^{N+2} \| \nabla \tilde{w} \|_{L^2(\Omega \times (0,T))}. \] (4.31)

Since \( \| R^2 \|_{L^2(0,T;H^1(\Omega)^2)} \leq C(M) \varepsilon^{N+1} \) and

\[ \| R^1 \|_{L^2(0,T;H^1(\Omega)^2)} \leq C \varepsilon^{N+1} \| w \|_{L^2(0,T;L^2(\Omega))} \leq C \varepsilon^{N+1} \| \nabla \tilde{w} \|_{L^2(0,T;H^1(\Omega))}, \]

cf. e.g. \( [23] \) Lemma 2.4], we obtain by noting that \( w = \tilde{w} - w_0 = \tilde{w} - w_0 = \tilde{w} - \tilde{w} \)

\[ \int_0^T \left| \int_\Omega R^1 \cdot \nabla dx \right| dt \leq C \varepsilon^{N+1} \| \tilde{w} \|_{L^2(0,T;L^2(\Omega))} \| \tilde{w} \|_{L^2(0,T;H^1(\Omega))} \]
\[ \leq C \| \tilde{w} \|_{L^2(0,T;L^2(\Omega))} \| \tilde{w} \|_{L^2(0,T;H^1(\Omega))} \]
\[ \leq C \| \tilde{w} \|_{L^2(0,T;L^2(\Omega))} \| \tilde{w} \|_{L^2(0,T;H^1(\Omega))} \]
\[ + C \| w_0 \|_{L^2(0,T;H^1(\Omega))} \| \tilde{w} \|_{L^2(0,T;H^1(\Omega))} \]
Due to (4.10) one has

\[ \| w_0 \|_{L^2(\Omega)} \leq C(M) \epsilon^{N+1} \| w \|_{L^2(\partial \Omega)} \]

Moreover,

\[ \int_0^T \int \left( \frac{\partial w_0}{\partial t} \cdot \nabla (v_A - \bar{v}_A) \right) \cdot \bar{w} \, dt \leq C(M) \epsilon^{N+1} \| \nabla w \|_{L^2(\partial \Omega)} + C(M) \epsilon^{N+1} \| \nabla w \|_{L^2(\Omega)} \]

as well as

\[ \int_0^T \int 2\nu(c_A) D w_0 : \nabla \bar{w} \, dx \, dt \leq C \| w_0 \|_{L^2(\partial \Omega)} \| \nabla w \|_{L^2(\partial \Omega)} \]

Plugging (4.15), (4.20), (4.33), (4.35) and utilizing (4.15), (4.34) and Young’s inequality we can derive

\[ \| \bar{w} \|_{L^2(\partial \Omega)} + \| \bar{w} \|_{L^2(\partial \Omega)} \leq C_1(R) \epsilon^{2N+1} + C_2(R, M) \epsilon^{2N+2} + C_3(M) \epsilon^{4N} \int_0^T \| \bar{w}(t) \|_{L^2(\Omega)}^2 \, dt. \]

Then the Gronwall inequality and (4.10), (4.11) yield

\[ \| w \|_{L^\infty(\partial \Omega)} + \| w \|_{L^2(\partial \Omega)} \leq \exp(C_3(M) \epsilon^{4N} T_0) C_1(R) \epsilon^{N+\frac{1}{2}} + C_2(R, M) \epsilon^{N+1}. \]
4.1 Preparations and the Error in the Velocity

for all $T \in (0,T_0)$. Now we choose $\varepsilon_0 \in (0,1)$ (in dependence of $M$) so small that $C_3(M)\varepsilon_0^N \leq 1$ and $C_2(R,M)\varepsilon_0^{\frac{1}{2}} \leq C_1(R)$. Then

$$\|w\|_{L^\infty(0,T;L^2(\Omega))} + \|w\|_{L^2(0,T;H^1(\Omega))} \leq 2C_1(R)\varepsilon^{N+\frac{1}{2}}.$$  

(4.36)

for all $T \in (0,T_0)$ and $\varepsilon \in (0,\varepsilon_0)$. Moreover, since $h_{N+\frac{1}{2}} \in X_T$ is determined by (4.21) with $u$ as in (3.3) we have

$$\|h_{N+\frac{1}{2}}\|_{X_T} \leq C_4(T_0)\varepsilon^{-N-\frac{1}{2}}\|w\|_{L^2(0,T;H^1(\Omega))} \leq 2C_4(T_0)C_1(R).$$

Hence we can finally choose $M = 2C_4(T_0)C_1(R)$. This also determines $\varepsilon_0 \in (0,1)$. Altogether this implies (4.5).

In the following let $M$ and $\varepsilon_0 > 0$ be as in Theorem 4.1 (in dependence on $R$). Noting $w = \frac{w}{\varepsilon^{N+\frac{1}{2}}}$, we get for $T \in (0,T_2)$

$$\|w\|_{L^\infty(0,T;L^2(\Omega))} + \|w\|_{L^2(0,T;H^1(\Omega))} \leq C(R).$$  

(4.37)

Theorem 4.2 For $T \in (0,T_0)$ there holds

$$\int_0^T \left| \int_{\Omega} (w - w|_T) \cdot \nabla_{\mathcal{A}}u \, dx \right| dt \leq C(R)\varepsilon^{N+1}.$$  

(4.38)

Proof: We set

\[
\begin{align*}
\zeta^{(0)}(x,t) &= \zeta \circ d_T(x,t) \left( \theta_0(\rho) + \varepsilon^{N-\frac{1}{2}}\theta_0''(\rho)h_{N+\frac{1}{2}}(S(x,t),t) \right) \\
&\quad + (1 - \zeta \circ d_T(x,t)) \left( c_A'(x,t)\chi_+(x,t) + c_A'(x,t)\chi_-(x,t) \right).
\end{align*}
\]

(4.39)

Using that

$$\left( \theta_0(\rho) - \left( \pm 1 \right)_{|_{\Omega_T^{\pm}}} \right) \nabla \zeta(\rho) = O(e^{-\frac{\varepsilon N}{2}}),$$  

(4.40)

due to (4.13), we get

$$\int_{\Gamma_T^{(2)}} (w - w|_T) \cdot \nabla c^{(0)} u \, dx = J_1 - J_2 + J_2 + O(e^{-\frac{\varepsilon N}{2}}\|w\|_{H^1(\Omega)}\|u\|_{L^2(\Gamma_T^{(2)})}).$$  

(4.41)

where

\[
\begin{align*}
J_1 &= \varepsilon^{-1}\int_{\Gamma_T^{(2)}} \zeta \circ d_T(w - w|_T) \cdot \nabla (\theta_0'(\rho) + \theta_0''(\rho)\varepsilon^{N-\frac{1}{2}}h_{N+\frac{1}{2}}) \, u \, dx, \\
J_2 &= \int_{\Gamma_T^{(2)}} \zeta \circ d_T(w - w|_T) \cdot \nabla h_{N+\frac{1}{2}}(\theta_0(\rho) + \theta_0''(\rho)\varepsilon^{N-\frac{1}{2}}h_{N+\frac{1}{2}}) \, u \, dx, \\
J_3 &= \int_{\Gamma_T^{(2)}} \zeta \circ d_T(w - w|_T) \cdot \varepsilon^{N-\frac{1}{2}}\nabla h_{N+\frac{1}{2}}(\theta_0''(\rho)) \, u \, dx.
\end{align*}
\]

It follows from $\|h_{N+\frac{1}{2}}\|_{X_T} \leq M$ and $X_T \hookrightarrow BUC([0,T];C^0(T^1))$ that

$$\|\varepsilon^{N-\frac{1}{2}}h_{N+\frac{1}{2}}(\theta_0''(\rho))\|_{C^0([0,T];C^0(T^1))} \leq C(M).$$  

(4.42)

Based on (4.42) and the procedure of proving [3] Lemma 5.1 one has

$$\int_0^T |J_1| \, dt \leq C\varepsilon^{\frac{1}{2}}\int_0^T \|w(-,t)\|_{L^2(\Omega)}^\frac{1}{2}\|w(\cdot,t)\|_{H^1(\Omega)}^\frac{1}{2}\|w(\cdot,t)\|_{L^2(\Gamma_T^{(2)})} + \|u\|_{L^2(\Gamma_T^{(2)})}) \, dt$$

$$\leq C(R,M)T^{\frac{1}{2}}\left( \|\nabla u\|_{L^2(0,T;L^2(\Omega))} + T^\frac{1}{2}\|u\|_{L^\infty(0,T;L^2(\Omega))} \right)$$

$$\cdot \|w\|_{L^\infty(0,T;L^2(\Omega))} \|w\|_{L^2(0,T;H^1(\Omega))} \leq C(R,M)T^{\frac{1}{2}}(1 + T^\frac{1}{2})\varepsilon^{N+1}$$  

(4.43)
Proof of Theorem 1.1

In order to estimate the error due to linearization of $f'(c)$ we use:

**Proposition 4.3**

$$\int_0^T \int_{\Omega} |u|^3 dx dt \leq C(R)T^2 \varepsilon^{3N+1}. \quad (4.46)$$

**Proof:** Thanks to [6] Lemma 3.9, Hölder’s inequality and assumption (4.5) one has

$$\int_0^T \int_{\Gamma(\delta)} |u|^3 dx dt \leq C(\|u\|_{L^2(0, T; L^2(\Omega, \Gamma(\delta)))} + \|\nabla u\|_{L^2(0, T; L^2(\Omega, \Gamma(\delta)))}) \frac{1}{2}$$

$$\cdot (\|u\|_{L^2(0, T; L^2(\Omega, \Gamma(\delta)))} + \|\partial_u u\|_{L^2(0, T; L^2(\Omega, \Gamma(\delta)))}) \frac{1}{2} \|u\|_{L^4(0, T; L^2(\Omega, \Gamma(\delta)))}^2$$

$$\leq C(R)T^2 \varepsilon^{3N+1}. \quad (4.47)$$

Moreover, due to the Gagliardo-Nirenberg inequality and assumption (4.5) one gets

$$\int_0^T \int_{\Omega \setminus \Gamma(\delta)} |u|^3 dx dt \leq C(\|u\|_{L^2(0, T; L^2(\Omega, \Gamma(\delta)))} \|u\|_{L^4(0, T; L^2(\Omega, \Gamma(\delta)))}^2)$$

$$\leq C(R)(T^{\frac{1}{2}} + T^2) \varepsilon^{3N+\frac{3}{2}}. \quad (4.48)$$

Consequently we arrive at (4.46).

Noting that

$$v_\varepsilon = v_A + \varepsilon^{N+\frac{3}{2}} w \quad (4.49)$$

with the help of (1.3) and (1.4) we then find that

$$\partial_t u + v_\varepsilon \cdot \nabla u + \varepsilon^{N+\frac{1}{2}} (w - w|\Gamma) \cdot \nabla c_A = \Delta u - \frac{f''(c_A)}{\varepsilon^2} u - \varepsilon^{-2} N(c_A, u) - S_\varepsilon, \quad (4.50)$$

where

$$N(c_A, u) = f'(c) - f'(c_A) - f''(c_A)u.$$

Multiplying (4.50) by $u$ and integrating by parts yield that

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} u^2 dx + \int_{\Omega} (\|\nabla u\|^2 + \frac{f''(c_A)}{\varepsilon^2} u^2) dx$$

$$\leq \varepsilon^{N+\frac{1}{2}} \int_{\Omega} (w - w|\Gamma) \cdot \nabla c_A u dx + \varepsilon^{-2} C \int_{\Omega} |u|^3 dx + \left| \int_{\Omega} S_\varepsilon u dx \right|. \quad (4.51)$$
Here we have used
\[ \int \mathcal{N}(c_A, u) dx \geq -C \int |u|^3 dx. \]

By using the spectral estimate of Lemma 2.7 in [111], one gets
\[
\frac{1}{2} \frac{d}{dt} \int \Omega u^2 dx - C_L \int \Omega u^2 dx + \int_{\Omega \setminus \Gamma(t)} |\nabla u|^2 dx + \int_{\Gamma(t)} |\nabla u|^2 dx \\
\leq \varepsilon^{N+\frac{1}{2}} \left( \int \omega - \omega_1 |r| \cdot \nabla c_A u dx + \varepsilon^{-2} \int \Omega |u|^3 dx + \int \int_{S_u} u dx \right),
\]
(4.52)

Applying Gronwall’s inequality and using (4.6) we obtain for all \( T \in [0, T_0] \)
\[
\sup_{0 \leq t \leq T} \frac{1}{2} \int_\Omega u^2 dx + \int_0^T \int_{\Omega \setminus \Gamma(t)} |\nabla u|^2 dx dt + \int_0^T \int_{\Gamma(t)} |\nabla u|^2 dx dt \\
\leq e^{C_1 T_0 \left( \frac{1}{2} \int_\Omega u^2 dx + \varepsilon^{N+\frac{1}{2}} \left( \int \omega - \omega_1 |r| \cdot \nabla c_A u dx + \varepsilon^{-2} \int \Omega |u|^3 dx + \int \int_{S_u} u dx \right) \right) + \varepsilon^{-2} \int_\Omega u^2 dx dt + \int_0^T \int \int_{S_u} u dx dt}
\]
\[
\leq \theta^2 \frac{R^2}{8} \varepsilon^{2N+1} + (R, T_0)(\varepsilon^{2N+\frac{1}{2}} + \varepsilon^{3N-1} + \varepsilon^{2N+\frac{1}{2}}).
\]
(4.53)

Hence, if \( \varepsilon \in (0, \varepsilon_0) \) and \( \varepsilon_0 > 0 \) is sufficiently small,
\[
\theta^2 \frac{R^2}{8} \varepsilon^{2N+1} + (R, T_0)(\varepsilon^{2N+\frac{1}{2}} + \varepsilon^{3N-1} + \varepsilon^{2N+\frac{1}{2}}) \leq \theta^2 \frac{R^2}{4} \varepsilon^{2N+1}.
\]
(4.54)

Thus there holds
\[
\sup_{0 \leq t \leq T} \frac{1}{2} \int_\Omega u^2(x, t) dx + \int_0^T \int_{\Omega \setminus \Gamma(t)} |\nabla u|^2 dx dt + \int_0^T \int_{\Gamma(t)} |\nabla u|^2 dx dt \leq \theta^2 \frac{R^2}{4} \varepsilon^{2N+1}.
\]
(4.55)

We note that (4.61) implies for \( 0 \leq t \leq T_0 \)
\[
\int_0^T \int \left( |\nabla u|^2 + \frac{f''(c_A)}{\varepsilon^2} u^2 \right) dx dt \\
\leq \frac{1}{2} \int_\Omega u^2|_{t=0} dx + \varepsilon^{N+\frac{1}{2}} \left( \int_0^T \int \omega - \omega_1 |r| \cdot \nabla c_A u dx \right) dt \\
+ \varepsilon^{-2} \int_0^T \int_\Omega u^2 dx dt + \int_0^T \int S_u u dx dt \\
\leq \theta^2 \frac{R^2}{8} \varepsilon^{2N+1} + (R, T_0)(\varepsilon^{2N+\frac{1}{2}} + \varepsilon^{3N-1}).
\]
(4.56)

Then for small \( \varepsilon_0 > 0 \), if \( T \leq T_0 \) and \( \varepsilon \in (0, \varepsilon_0) \), we have
\[
\int_0^T \int \left( |\nabla u|^2 + \frac{f''(c_A)}{\varepsilon^2} u^2 \right) dx dt \leq \theta^2 \frac{R^2}{4} \varepsilon^{2N+1}
\]
(4.57)

and
\[
\varepsilon^2 \int_0^T \int \partial_\tau u^2 dx dt \leq \varepsilon^2 \int_0^T \int |\nabla u|^2 dx dt \\
\leq \int_0^T \int \left( \varepsilon^2 |\nabla u|^2 + f''(c_A) u^2 \right) dx dt + C_0 \int_0^T \int u^2 dx dt \\
\leq \frac{R^2}{4} \varepsilon^{2N+1} + C_0 T_0 \theta^2 R^2 \frac{R^2}{2} \varepsilon^{2N+1} \leq \frac{3R^2}{4} \varepsilon^{2N+1}
\]
(4.58)
where $C_0 = \min_{s\in]\mathbb{R}} f''(s)$ and we choose $\theta \in (0, 1]$ so small that $C_0 T_0 \theta^2 \leq 1$. Multiplying (4.60) by $-\varepsilon^4 \Delta u$ and integrating by parts yield that

$$
\sup_{0 \leq t \leq T} \int_{\Omega} |\nabla u|^2(x, t) dx + \varepsilon^4 \int_{0}^{T} \int_{\Omega} |\Delta u|^2 dx dt 
\leq \varepsilon^4 \int_{0}^{T} \int_{\Omega} |\nabla u|^2 dx dt + \varepsilon^2 \int_{0}^{T} \int_{\Omega} |f''(c_A)u|dx dt 
+ \varepsilon \int_{0}^{T} \int_{\Omega} |v \cdot \nabla u| dx dt 
+ \varepsilon \int_{0}^{T} \int_{\Omega} |(w - w|_{T}) \cdot \nabla c_A u| dx dt 
+ \varepsilon^2 \int_{0}^{T} \int_{\Omega} S_{\varepsilon} u dx dt.
$$

(4.59)

Thanks to Hölder’s inequality and (4.55) one gets

$$
\varepsilon^2 \int_{0}^{T} \int_{\Omega} |f''(c_A)u|dx dt \leq C \varepsilon^2 T_0^3 \|u\|_{L^\infty(0, T; L^2(\Omega))} \|\Delta u\|_{L^2(0, T; L^2(\Omega))} 
\leq C \varepsilon^2 T_0^3 \|u\|_{L^\infty(0, T; L^2(\Omega))} \|\Delta u\|_{L^2(0, T; L^2(\Omega))}.
$$

(4.60)

Noting (4.49) we then have

$$
\varepsilon^4 \int_{0}^{T} \int_{\Omega} |v \cdot \nabla u| dx dt 
\leq \varepsilon^4 \int_{0}^{T} \int_{\Omega} |v \cdot \nabla u| dx dt + \varepsilon^2 \int_{0}^{T} \int_{\Omega} |w \cdot \nabla u| dx dt.
$$

(4.61)

It follows from Hölder’s inequality, the Gagliardo-Nirenberg interpolation inequality and (4.60) and (4.55) that

$$
\varepsilon^4 \int_{0}^{T} \int_{\Omega} |v \cdot \nabla u| dx dt \leq C \varepsilon^4 \|\nabla u\|_{L^2(0, T; L^2(\Omega))} \|\Delta u\|_{L^2(0, T; L^2(\Omega))} 
\leq C \varepsilon^4 \|\nabla u\|_{L^2(0, T; L^2(\Omega))} \|\Delta u\|_{L^2(0, T; L^2(\Omega))}.
$$

(4.62)

and

$$
\varepsilon^2 \int_{0}^{T} \int_{\Omega} |w \cdot \nabla u| dx dt 
\leq \varepsilon^2 \int_{0}^{T} \int_{\Omega} |w \cdot \nabla u| dx dt 
+ \varepsilon^2 \int_{0}^{T} \int_{\Omega} |w \cdot \nabla u| dx dt.
$$

(4.63)

Using (4.62), (4.63) in (4.53) implies that

$$
\varepsilon^4 \int_{0}^{T} \int_{\Omega} |v \cdot \nabla u| dx dt \leq C \varepsilon^4 \|\nabla u\|_{L^2(0, T; L^2(\Omega))} \|\Delta u\|_{L^2(0, T; L^2(\Omega))} 
+ C(\varepsilon) \varepsilon^2 \|\nabla u\|_{L^2(0, T; L^2(\Omega))} \|\Delta u\|_{L^2(0, T; L^2(\Omega))}.
$$

(4.64)

Moreover, we have

$$
\varepsilon^2 \int_{0}^{T} \int_{\Omega} |(w - w|_{T}) \cdot \nabla c_A u| dx dt \leq C(\varepsilon) \varepsilon^2 \|\nabla u\|_{L^2(0, T; L^2(\Omega))} \|\Delta u\|_{L^2(0, T; L^2(\Omega))}.
$$

(4.65)

and

$$
\varepsilon^2 \int_{0}^{T} \int_{\Omega} |N(c_A, u) u| dx dt \leq C \varepsilon^2 \int_{0}^{T} \int_{\Omega} (|u^2 u| + |u^3 \Delta u|) dx dt.
$$
and (1.16) holds. Thus there holds

\[ T \leq \frac{C\varepsilon^{2}}{||\Delta u||_{L^{2}(0,T;L^{2}(\Omega))}}. \] (4.66)

Combining (1.60), (1.65) and (1.64)-(1.67) with (1.69) leads to

\[ \sup_{0 \leq t \leq T} \varepsilon^{4} \int_{\Omega} |\nabla u(x,t)|^{2} dx + \varepsilon^{4} \int_{0}^{T} \int_{\Omega} |\Delta u(t)|^{2} dxdt \leq \frac{R^{2}}{8} \varepsilon^{2N+1} + C(R)\varepsilon^{2N+1} + C(||u||_{L^{\infty}(0,T;L^{6}(\Omega))})||u||_{L^{2}(0,T;L^{2}(\Omega))}^{2} \] (4.68)

Applying Young’s inequality in (1.68) we get for \( T \leq T_{\varepsilon} \) and \( \varepsilon \in (0,\varepsilon_{0}) \)

\[ \sup_{0 \leq t \leq T} \varepsilon^{4} \int_{\Omega} |\nabla u(x,t)|^{2} dx + \varepsilon^{4} \int_{0}^{T} \int_{\Omega} |\Delta u(t)|^{2} dxdt \leq \frac{R^{2}}{8} \varepsilon^{2N+1} + C(R)\varepsilon^{2N+1} + C(T_{\varepsilon})\varepsilon^{2N+3} \leq \frac{R^{2}}{4} \varepsilon^{2N} \] (4.69)

if we first choose \( \theta > 0 \) small enough (which finally determines \( R = R(\theta) \)) and afterwards \( \varepsilon_{0} > 0 \) sufficiently small. Thus there holds

\[ \sup_{0 \leq t \leq T} \frac{1}{2} \int_{\Omega} |\nabla u(x,t)|^{2} dx + \int_{0}^{T} \int_{\Omega} |\Delta u(t)|^{2} dxdt \leq \frac{2R^{2}}{3} \varepsilon^{2N-3}. \] (4.70)

According to the definition of \( T_{\varepsilon} \) in (1.4), then (1.5d), (1.55), (1.56) and (1.70) implies \( T_{\varepsilon} = T_{0} \) and then (1.4) holds.

Finally, (1.71) is a direct consequence of (1.4) and (1.8). The remaining two statements in Theorem 1.1 follow from the constructions of \( c_{A} \) and \( v_{A} \). We then complete the proof of Theorem 1.1.

### A Formally Matched Asymptotics

In this appendix we discuss the construction of approximate solutions with the aid of the method of formally matched asymptotics. The scheme is similar to the scheme in [13] with adaptations similar to the schemes in [10] and [5]. Moreover, it is an adaption of the scheme presented in [7] for the “integer order part” to the case of a Navier-Stokes/Allen-Cahn system instead of a Stokes/Cahn-Hilliard system, which can also be found in [36] in more detail.

First of all, we note that, since

\[ \text{div}(\nabla c^{\varepsilon} \otimes \nabla c^{\varepsilon}) = \frac{1}{2} \text{div}(|\nabla c^{\varepsilon}|^{2}) + \Delta c^{\varepsilon} \nabla c^{\varepsilon}, \]

we can rewrite (1.1) as follows

\[ \partial_{t} c^{\varepsilon} + \nu^{\varepsilon} \cdot \nabla c^{\varepsilon} - \text{div}(2\nu^{(c^{\varepsilon})}Dc^{\varepsilon}) + \nabla p^{\varepsilon} = -\varepsilon \Delta c^{\varepsilon} \nabla c^{\varepsilon}, \] (A.1)

\[ \text{div} c^{\varepsilon} = 0, \] (A.2)

\[ \partial_{t} c^{\varepsilon} + \nu^{\varepsilon} \cdot \nabla c^{\varepsilon} = \Delta c^{\varepsilon} - \varepsilon^{-2} f^{\varepsilon}(c^{\varepsilon}), \] (A.3)

in \( \Omega \times (0,T_{0}) \) by changing \( p^{\varepsilon} \) by a scalar function.
A.1 The Outer Expansion

We assume that in $\Omega^\pm$ the solutions of (A.1)-(A.3) have the expansions

$$c^\pm(x,t) \approx \sum_{k \geq 0} \varepsilon^k c_k^\pm(x,t), \quad v^\pm(x,t) \approx \sum_{k \geq 0} \varepsilon^k v_k^\pm(x,t), \quad p^\pm(x,t) \approx \sum_{k \geq -1} \varepsilon^k p_k^\pm(x,t),$$

i.e., we have up to higher-order terms

$$c^\pm(x,t) = \sum_{k = 0}^{N+2} \varepsilon^k c_k^\pm(x,t), \quad v^\pm(x,t) = \sum_{k = 0}^{N+2} \varepsilon^k v_k^\pm(x,t), \quad p^\pm(x,t) = \sum_{k = 0}^{N+2} \varepsilon^k p_k^\pm(x,t),$$

for some $N \in \mathbb{N}$, $c_k^\pm$, $v_k^\pm$ and $p_k^\pm$ are smooth functions defined in $\Omega^\pm$. Plugging this ansatz into (A.1)-(A.3) yields in $\Omega^\pm$

$$\sum_{k \geq 0} \varepsilon^k \partial_k v_k^\pm + \sum_{k \geq 0} \varepsilon^k \sum_{0 \leq j \leq k} v_j^\pm \cdot \nabla v_{k-j}^\pm - \text{div} \left( 2 \sum_{k \geq 0} \varepsilon^k \sum_{0 \leq j \leq k} \nu_j^\pm \nabla v_{k-j}^\pm \right) + \sum_{k \geq -1} \varepsilon^k \nabla p_k^\pm = - \sum_{k \geq 0} \varepsilon^{k+1} \sum_{0 \leq j \leq k} \Delta \epsilon_j^\pm \nabla c_k^\pm, \quad (A.4)$$

$$\sum_{k \geq 0} \varepsilon^k \text{div} v_k^\pm = 0, \quad (A.5)$$

$$\sum_{k \geq 0} \varepsilon^k \partial_k c_k^\pm + \sum_{k \geq 0} \varepsilon^k \sum_{0 \leq j \leq k} v_j^\pm \cdot \nabla c_{k-j}^\pm = - \frac{1}{\varepsilon} f'(c_0^\pm) - \frac{1}{\varepsilon^2} f''(c_0^\pm) c_1^\pm + \sum_{k \geq 0} \varepsilon^k \left( \Delta c_k^\pm - f''(c_0^\pm) c_k^\pm - f_k(c_0^\pm, \ldots, c_{k+1}^\pm) \right). \quad (A.6)$$

Here we have used for $g = \nu, f$

$$g(c^\pm) = g(c_0^\pm) + \varepsilon g'(c_0^\pm) c_1^\pm + \sum_{k = 2}^{N+2} \varepsilon^k \left( g'(c_0^\pm) c_k^\pm + g_k(c_0^\pm, \ldots, c_{k-1}^\pm) \right) + \varepsilon^{N+3} \sum_{k = N+3} g_k(c_0^\pm, \ldots, c_{N+2}^\pm) = \sum_{k \geq 0} \varepsilon^k g_k, \quad (A.7)$$

where for fixed $c_0^\pm$ the functions $\nu_k, f_k$ are polynomials in $(c_1^\pm, \ldots, c_k^\pm)$ and are the result of a Taylor expansion. Moreover, they do not depend on $\varepsilon$, except for the remainder term $g_k$.

Matching the $\mathcal{O}(\varepsilon^{-2}), \mathcal{O}(\varepsilon^{-1})$ terms in (A.6) yields $f(c_0^\pm) = f''(c_0^\pm) c_1^\pm = 0$ and in view of the Dirichlet boundary data for $c^\pm$ we set

$$c_0^\pm = \pm 1. \quad (A.8)$$

Then

$$c_1^\pm = 0. \quad (A.9)$$

Applying an induction argument to (A.6) implies

$$c_k^\pm = 0, \quad k \geq 2. \quad (A.10)$$

Substituting (A.8)-(A.10) into (A.7) leads to

$$\nu_0^\pm = \nu(c_0^\pm), \quad \nu_k^\pm = 0 \quad \text{for } k \geq 1. \quad (A.11)$$

Comparing the order terms $\mathcal{O}(\varepsilon^k)$ (with $k \geq -1$) in (A.4)-(A.5) yields in $\Omega^\pm$

$$\nabla p_{-1}^\pm = 0, \quad (A.12)$$

$$\partial_k v_k^\pm + v_j^\pm \cdot \nabla v_k^\pm + v_k^\pm \cdot \nabla v_j^\pm - \nu(c_0^\pm) \Delta v_k^\pm + \nabla p_k^\pm = - \sum_{1 \leq j \leq k-1} v_j^\pm \cdot \nabla v_{k-j}^\pm, \quad k \geq 0, \quad (A.13)$$

$$\text{div} v_k^\pm = 0, \quad k \geq 0. \quad (A.14)$$

For simplicity we take

$$p_{-1}^\pm = 0. \quad (A.15)$$
A.2 The Inner Expansion

We will need \((c_k^\pm, v_k^\pm, p_k^\pm)\), for \(k \geq 0\), to not only be defined in \(\Omega^\pm\), but we have to extend them onto \(\Omega^\pm \cup \Gamma(2\delta)\). For \(p_k^\pm\) we may use any smooth extension. One possibility is to use the extension operator defined in [30, Part VI, Theorem 5]. It is trivial to extend \(c_k^\pm\).

For \(v_k^\pm\) we employ the same extension operator and then use the Bogovskii operator to ensure that the extension is divergence free in \(\Gamma(2\delta)\). In particular we may construct a divergence free extension \(\mathcal{E}^\pm(v_k^\pm)\) such that \(\mathcal{E}^\pm(v_k^\pm) = v_k^\pm\) in \(\Omega^\pm\) (t) and
\[
\|\mathcal{E}^\pm(v_k^\pm)\|_{H^2(\Omega^\pm; \Gamma(2\delta))} \leq C\|v_k^\pm\|_{H^2(\Omega^\pm)}.
\]

For later use we define
\[
W_1^\pm(x, t) = \partial_t v_k^\pm(x, t) + \sum_{j=0}^{k} \nabla v_j^\pm \cdot \nabla v_k^\pm - \nu(c_0^\pm)\Delta v_k^\pm(x, t) + \nabla p_k^\pm(x, t)
\]
\[
W_p^\pm = \sum_{k \geq 0} \varepsilon^k W_k^\pm
\]
for \((x, t) \in \Omega^\pm \cup \Gamma(2\delta)\). Note that by \((A.2)\) and \((A.13)\) we have \(W_k^\pm(x, t) = 0\) for all \((x, t) \in \Omega^\pm\).

A.2 The Inner Expansion

Close to the interface \(\Gamma\) we introduce a stretched variable
\[
\rho^\varepsilon(x, t) := \frac{d\varepsilon^\varepsilon(x, t) - \varepsilon^{\varepsilon}(S(x, t), t)}{\varepsilon}
\]
for all \((x, t) \in \Gamma(2\delta)\) \((A.19)\)
for \(\varepsilon \in (0, 1)\). Here \(h^\varepsilon: \mathbb{T}^1 \times [0, T_0] \rightarrow \mathbb{R}\) is a given smooth function and can heuristically be interpreted as the distance of the zero level set of \(c^\varepsilon\) to \(\Gamma\), see also [15, Chapter 4.2]. In the following, we will often drop the \(\varepsilon\)-dependence and write \(\rho(x, t) = \rho^\varepsilon(x, t)\).

Now assume that, in \(\Gamma(2\delta)\), the identities
\[
u^\prime(x, t) = \delta \left( \frac{d\nu^\prime(x, t)}{\varepsilon} - \varepsilon^{\nu^\prime}(S(x, t), t), x, t \right) \quad \text{for all } (x, t) \in \Gamma(2\delta)
\]
hold for the solutions of \((A.1)\) \((A.2)\) and some smooth functions \(\delta^\varepsilon, \nu^\varepsilon: \mathbb{R} \times \Gamma(2\delta) \rightarrow \mathbb{R}\), \(\delta^\varepsilon: \mathbb{R} \times \Gamma(2\delta) \rightarrow \mathbb{R}^2\). Furthermore, we assume that we have the expansions
\[
c_k^\varepsilon(\rho, x, t) \approx \sum_{k \geq 0} \varepsilon^k c_k(\rho, x, t), \quad \delta^\varepsilon(\rho, x, t) \approx \sum_{k \geq 0} \varepsilon^k \delta_k(\rho, x, t), \quad \nu^\varepsilon(\rho, x, t) \approx \sum_{k \geq 0} \varepsilon^k \nu_k(\rho, x, t),
\]
for all \((\rho, x, t) \in \mathbb{R} \times \Gamma(2\delta)\) and also
\[
h^\varepsilon(s, t) \approx \sum_{k \geq 0} \varepsilon^k h_{k+1}(s, t) \quad \text{for all } s \in \mathbb{T}^1, t \in [0, T_0],
\]
where \(c_k, p_k: \mathbb{R} \times \Gamma(2\delta) \rightarrow \mathbb{R}\), \(v_k: \mathbb{R} \times \Gamma(2\delta) \rightarrow \mathbb{R}^2\) and \(h_{k+1}: \mathbb{T}^1 \times [0, T_0] \rightarrow \mathbb{R}\) are smooth functions for all \(k \geq 0\). When referring to \(\delta, \nu, \delta\) and the expansion terms we write \(\delta = \delta^\varepsilon\) and \(\Delta = \Delta^\varepsilon\) for the gradient, Laplacian, resp., with respect to \(x\). The expressions \(\partial_t^2 \varepsilon^\varepsilon(x, t), \partial_t \varepsilon^\varepsilon(x, t), \partial_t^2 \varepsilon^\varepsilon(x, t)\) and \(D_2^k \varepsilon^\varepsilon(x, t)\) are for \((x, t) \in \Gamma(2\delta)\) to be understood in the sense of \((2.11)\).

Moreover, for \(g = \nu, f^\prime\)
\[
g(c^\varepsilon) = g(c_0) + \varepsilon g^\prime(c_0) c_1 + \sum_{k=2}^{N+2} \varepsilon^k \left( g^\prime(c_0) c_k + g_{k-1}(c_0, \ldots, c_{k-1}) \right) + \varepsilon^{N+3} g_{N+3}(c_0, \ldots, c_{N+2}) =: \sum_{k \geq 0} \varepsilon^k g_k,
\]
\[
(A.21)
\]
In order to match the inner and outer expansions, we require that for all \(k\) the so-called inner-outer matching conditions
\[
\sup_{(x, t) \in \Gamma(2\delta)} \left| \partial_{\nu}^n \partial_{\rho}^m \partial_{\rho}^n \left( \varphi(\pm \rho, x, t) - \varphi^\pm(x, t) \right) \right| \leq Ce^{-\alpha \rho},
\]
\[
(A.22)
\]
where \( \varphi = c_h, v_k, p_k \) and \( k \geq 0 \) hold for constants \( \alpha, C > 0 \) and all \( \rho > 0, m, n, t \geq 0 \).

If \( \nu'(x, t) = \nabla \nu'(x, t) \), then one has by direct computations

\[
\begin{align*}
\text{div}(2\nu(c\varepsilon)\nabla \nu) &= \varepsilon^{-2} \partial_{\rho} (\nu(c\varepsilon) \partial_{\rho} \nabla \nu) + \varepsilon^{-1} \partial_{\rho} (2\nu(c\varepsilon) \nabla D\nu) \cdot \nabla d_r + \varepsilon^{-1} \text{div} (2\nu(c\varepsilon) D\nu \nabla h^\varepsilon) \\
&- \varepsilon^{-1} \partial_{\rho} (\nu(c\varepsilon) \text{div} \nabla \nu) d_r + \partial_{\nu} (\nu(c\varepsilon) \partial_{\nu} \nabla h^\varepsilon) |\nabla \nu|^2 - \partial_{\rho} (2\nu(c\varepsilon) D\nu \nabla h^\varepsilon) \cdot \nabla^\nu h^\varepsilon \\
&+ \text{div} (2\nu(c\varepsilon) (D\nu - D_h \nu \nabla h^\varepsilon)) + \partial_{\nu} (\nu(c\varepsilon) \text{div} \nabla \nu) \nabla^\nu h^\varepsilon,
\end{align*}
\]

where

\[
\begin{align*}
D\nu^\varepsilon &= \frac{1}{2} \left( \nabla \nu^\varepsilon + (\nabla^\nu \nu^\varepsilon)^T \right), \\
D_h \nu^\varepsilon &= \frac{1}{2} \left( \partial_{\rho} \nu^\varepsilon \nabla^\nu h^\varepsilon + (\partial_{\nu} \nu^\varepsilon \nabla^\nu h^\varepsilon)^T \right), \\
D_h \nu^\varepsilon &= \frac{1}{2} \left( \partial_{\rho} \nu^\varepsilon \nabla d_r + (\partial_{\nu} \nu^\varepsilon \nabla d_r)^T \right).
\end{align*}
\]

For the following we use

\[
\begin{align*}
\varepsilon \Delta c^\varepsilon \nabla c^\varepsilon &= \varepsilon^{-2} \partial_{\rho} c^\varepsilon \partial_{\rho} c^\varepsilon \nabla d_r - \varepsilon^{-1} \partial_{\rho} c^\varepsilon \partial_{\nu} c^\varepsilon \nabla^\nu h^\varepsilon - \partial_{\nu} c^\varepsilon)^2 \left( \Delta d_r \nabla^\nu h^\varepsilon + \Delta^\nu h^\varepsilon \nabla d_r \right) \\
&- 2 \partial_{\rho} c^\varepsilon \cdot \nabla^\nu h^\varepsilon \partial_{\nu} c^\varepsilon \nabla d_r + \varepsilon^{-1} h^\varepsilon + B^\varepsilon + \varepsilon C^\varepsilon,
\end{align*}
\]

where \( h^\varepsilon, B^\varepsilon \) are polynomials in \( \partial_{\rho} c^\varepsilon, \nabla c^\varepsilon, \nabla d_r \) and their derivatives and \( C^\varepsilon \) is a polynomial in \( \partial_{\rho} c^\varepsilon, \nabla c^\varepsilon, \nabla d_r, \nabla^\nu h^\varepsilon \). Inserting the expansions for \( c^\varepsilon \) and \( h^\varepsilon \) one obtains

\[
\begin{align*}
\mathcal{A}^\varepsilon &= \sum_{k \geq 0} \varepsilon^k \mathcal{A}_k, \\
\mathcal{B}^\varepsilon &= \sum_{k \geq 0} \varepsilon^k \mathcal{B}_k, \\
\mathcal{C}^\varepsilon &= \sum_{k \geq 0} \varepsilon^k \mathcal{C}_k.
\end{align*}
\]

In the new coordinate \( (\rho, x, t) \) the system (A.4)-(A.3) reduces to

\[
\begin{align*}
\partial_{\rho} (\nu(c^\varepsilon) \partial_{\rho} \nu) &= \partial_{\rho} c^\varepsilon \partial_{\rho} c^\varepsilon \nabla d_r + \varepsilon \left( \partial_{\rho} \nu^\varepsilon \partial_{\rho} d_r + \nabla^\nu \nabla d_r \partial_{\rho} \nu^\varepsilon - \partial_{\rho} c^\varepsilon \partial_{\rho} c^\varepsilon \nabla^\nu h^\varepsilon + A^\varepsilon \right) \\
&- \text{div} (2\nu(c^\varepsilon) D\nu^\varepsilon) + \partial_{\rho} (\nu(c^\varepsilon) \text{div} \nabla \nu) d_r + \partial_{\nu} \nu^\varepsilon \nabla d_r - \partial_{\rho} c^\varepsilon \partial_{\rho} c^\varepsilon \nabla^\nu h^\varepsilon + h^\varepsilon + B^\varepsilon + \varepsilon C^\varepsilon,
\end{align*}
\]

\[
\begin{align*}
\partial_{\rho} \nu^\varepsilon \cdot n &= \varepsilon \partial_{\rho} \nu^\varepsilon \cdot \nabla \nu^\varepsilon - \varepsilon \text{div} \nu^\varepsilon, \\
\partial_{\rho} \nu^\varepsilon - f'(c^\varepsilon) &= \frac{\partial_{\rho} c^\varepsilon \partial_{\rho} d_r + \partial_{\rho} c^\varepsilon \nabla d_r - 2 \nabla \partial_{\rho} c^\varepsilon \cdot \nabla d_r - \partial_{\rho} c^\varepsilon \Delta d_r}{2} \\
&+ \varepsilon \left( \frac{2 \nabla \partial_{\rho} c^\varepsilon \cdot \nabla h^\varepsilon + \partial_{\rho} c^\varepsilon \Delta ^\nu h^\varepsilon - \partial_{\rho} c^\varepsilon |\nabla \nu|^2 - \Delta c^\varepsilon}{2} \\
&+ \partial_{\rho} c^\varepsilon - \partial_{\rho} c^\varepsilon \partial_{\rho} \nu \nabla \nu^\varepsilon + \nabla \nu^\varepsilon \cdot (\nabla \nu^\varepsilon - \partial_{\rho} c^\varepsilon \nabla^\nu h^\varepsilon) \right).
\end{align*}
\]

We interpret \( \{x(t), t \in \Gamma(2\delta) \} \) as an approximation of the 0-level set of \( c^\varepsilon \). Thus, we normalize \( c_h \) such that

\[
\begin{align*}
c_h(0, x, t) = 0 & \quad \text{for all } (x, t) \in \Gamma(\delta), k \geq 0.
\end{align*}
\]

In a similar manner as in [13], we introduce auxiliary functions \( g'(x, t) \), \( u'(x, t) \), and \( \Gamma'(x, t) \) for \( (x, t) \in \Gamma(2\delta) \). Here \( g' \) is added to \( \Delta 20 \) and \( \Gamma' \) is added to \( \Delta 24 \) in order to enable to fulfill the compatibility conditions in \( \Gamma(2\delta) \setminus \Gamma \). Furthermore, adding \( u' \) to \( \Delta 24 \) is important.
to ensure the matching conditions in $\Gamma(2\delta) \setminus \Gamma$. Moreover, we choose $\eta : \mathbb{R} \rightarrow [0,1]$ such that $\eta = 0$ in $(-\infty,-1]$, $\eta = 1$ in $[1,\infty)$ and $\eta' \geq 0$ in $\mathbb{R}$ and define

$$\eta^{C,\pm}(\rho) = \eta(-C \pm \rho)$$

for a suitable constant $C > 0$ to be chosen later and $\rho \in \mathbb{R}$.

Now we may rewrite (A.23)-(A.26) as

$$\partial_{\rho}(\nu(\tilde{c}^\varepsilon) \partial_{\rho} \rho^\varepsilon) = \partial_{\rho} \rho^\varepsilon \partial_{\rho,\rho} \rho^\varepsilon \nabla d\tau + \varepsilon \left( \partial_{\rho} \rho^\varepsilon \partial_{\rho} \rho^\varepsilon + \nabla \rho^\varepsilon \cdot \nabla d\tau - \partial_{\rho} \rho^\varepsilon \partial_{\rho} \rho^\varepsilon \nabla \Gamma h^\varepsilon + \kappa^\varepsilon \right)$$

$$+ \varepsilon^2 \left( \partial_{\rho} \rho^\varepsilon + \nabla \rho^\varepsilon \cdot \nabla \rho^\varepsilon - \varepsilon \rho^\varepsilon \partial_{\rho} \rho^\varepsilon - \tilde{c} \rho^\varepsilon \Delta \rho^\varepsilon + \varepsilon \rho^\varepsilon \partial_{\rho} \rho^\varepsilon \nabla \Gamma h^\varepsilon 
- \partial_{\rho} \rho^\varepsilon \partial_{\rho} \rho^\varepsilon |\nabla \Gamma h^\varepsilon|^2 + \partial_{\rho} \rho^\varepsilon \partial_{\rho} \rho^\varepsilon \nabla \Gamma h^\varepsilon 
- \partial_{\rho} \rho^\varepsilon \partial_{\rho} \rho^\varepsilon \nabla \Gamma h^\varepsilon - \Delta \rho^\varepsilon \right)$$

$$+ \varepsilon^2 \left( \rho^\varepsilon \partial_{\rho} \rho^\varepsilon - \varepsilon \rho^\varepsilon \partial_{\rho} \rho^\varepsilon + \varepsilon \rho^\varepsilon \partial_{\rho} \rho^\varepsilon \nabla \Gamma h^\varepsilon 
+ \varepsilon \rho^\varepsilon \partial_{\rho} \rho^\varepsilon \nabla \Gamma h^\varepsilon 
+ \varepsilon \rho^\varepsilon \partial_{\rho} \rho^\varepsilon \nabla \Gamma h^\varepsilon \right)$$

$$+ \varepsilon^2 \left( W^\varepsilon \eta_{C^*,+} + W^- \eta_{C^*,-} \right), \quad (A.28)$$

and

$$\partial_{\rho} \rho^\varepsilon \cdot \nabla d\tau = \varepsilon \partial_{\rho} \rho^\varepsilon \cdot \nabla \rho^\varepsilon - \varepsilon \rho^\varepsilon \partial_{\rho} \rho^\varepsilon + \left( u^\varepsilon \cdot (\nabla d\tau - \varepsilon \nabla \rho^\varepsilon) \right) \eta'(\rho) (d\tau - \varepsilon (\rho + h^\varepsilon)), \quad (A.29)$$

$$\partial_{\rho} \rho^\varepsilon - f'(\tilde{c}^\varepsilon) = \varepsilon \left( \partial_{\rho} \rho^\varepsilon \partial_{\rho} \rho^\varepsilon + \partial_{\rho} \rho^\varepsilon \nabla \rho^\varepsilon - 2 \partial_{\rho} \rho^\varepsilon \nabla \rho^\varepsilon - \partial_{\rho} \rho^\varepsilon \Delta \rho^\varepsilon \right)$$

$$+ \varepsilon^2 \left( 2 \partial_{\rho} \rho^\varepsilon \cdot \nabla \rho^\varepsilon + \partial_{\rho} \rho^\varepsilon \Delta \rho^\varepsilon - \partial_{\rho} \rho^\varepsilon |\nabla \rho^\varepsilon|^2 - \Delta \rho^\varepsilon \right)$$

$$+ \partial_{\rho} \rho^\varepsilon - \partial_{\rho} \rho^\varepsilon \partial_{\rho} \rho^\varepsilon + \nabla \rho^\varepsilon \cdot (\nabla \rho^\varepsilon - \partial_{\rho} \rho^\varepsilon \nabla \rho^\varepsilon) \right)$$

$$+ \varepsilon^2 \left( W^\varepsilon \eta_{C^*,+} + W^- \eta_{C^*,-} \right). \quad (A.30)$$

Here the equations only have to hold in

$$S^\varepsilon := \{ (\rho,x,t) \in \mathbb{R} \times \Gamma(2\delta) : \rho = \frac{d\tau(x,t)}{\varepsilon} - h^\varepsilon(S(x,t),t) \}.$$ 

But in the following we consider them as ordinary differential equations in $\rho \in \mathbb{R}$, where $(x,t) \in \Gamma(2\delta)$ are seen as fixed parameters. Thus we assume from now on that (A.23)-(A.30) are fulfilled in $\mathbb{R} \times \Gamma(2\delta)$. The term $W^\varepsilon$ (cf. (A.18)) are used here in order to ensure the exponential decay of the right hand sides; in this context $C_S > 0$ is a constant which will be determined later on (see Remark [A.3]). We assume that the auxiliary functions have expansions of the form

$$u^\varepsilon(x,t) \approx \sum_{k \geq 0} u_k(x,t) \varepsilon^k, \quad g^\varepsilon(x,t) \approx \sum_{k \geq 0} g_k(x,t) \varepsilon^k, \quad f^\varepsilon(x,t) \approx \sum_{k \geq 0} f_k(x,t) \varepsilon^k, \quad (A.31)$$

for $(x,t) \in \Gamma(2\delta)$.

The following lemma comes from Lemma 2.6.2 in [38].

**Lemma A.2** Let $U \subset \mathbb{R}^n$ be an open subset and let $A : \mathbb{R} \times U \rightarrow \mathbb{R}, (\rho,x) \rightarrow A(\rho,x)$ be given and smooth. Assume that there exist $A^\varepsilon(x)$ such that the decay property $A(\pm \rho, x) - A^\varepsilon(x) = O(e^{-\rho})$ as $\rho \rightarrow \infty$ is fulfilled. Then for all $x \in U$ the system

$$w_{\rho \rho}(\rho,x) - f'(\theta_\rho(\rho))w(\rho,x) = A(\rho,x), \quad \rho \in \mathbb{R},$$

$$w(0,x) = 0, \quad w(\cdot,x) \in L^\infty(\mathbb{R}) \quad (A.32)$$
has a solution if and only if
\[ \int_{\mathbb{R}} A(\rho, x) \theta_0(\rho) d\rho = 0. \tag{A.33} \]
In addition, if the solution exists, then it is unique and satisfies for all \( x \in U \)
\[ D_x^l \left( w(\pm \rho, x) + \frac{A^\pm(x)}{f'(\pm 1)} \right) = O(e^{-\alpha \rho}) \text{ as } \rho \to \infty, \ l = 0, 1, 2. \tag{A.34} \]
Furthermore, if \( A(\rho, x) \) satisfies for all \( x \in U \)
\[ D_x^m D_\rho^l \left( A(\pm \rho, x) - A^\pm(x) \right) = O(e^{-\alpha \rho}), \text{ as } \rho \to \infty \]
for all \( m \in \{0, \cdots, M\} \) and \( l \in \{0, \cdots, L\} \), then
\[ D_x^m D_\rho^l \left( w(\pm \rho, x) + \frac{A^\pm(x)}{f'(\pm 1)} \right) = O(e^{-\alpha \rho}) \text{ as } \rho \to \infty \tag{A.35} \]
for all \( m \in \{0, \cdots, M\} \) and \( l \in \{0, \cdots, L\} \).

To proceed we show a modified version of Lemma 2.6.3 in [38].

**Lemma A.3** Let \( U \subset \mathbb{R}^n \) be an open subset and let \( B: \mathbb{R} \times U \to \mathbb{R}, (\rho, x) \mapsto B(\rho, x) \) be given and smooth. Assume that for all \( x \in U \) the decay property \( B(\pm \rho, x) = O(e^{-\alpha \rho}) \) as \( \rho \to \infty \) is fulfilled. Then for each \( x \in U \) the problem
\[ \partial_\rho (\nu(\theta_0) \partial_\rho w(\rho, x)) = B(\rho, x), \quad \rho \in \mathbb{R}, \tag{A.36} \]
has a solution \( w(\cdot, x) \in C^2(\mathbb{R}) \cap L^\infty(\mathbb{R}) \) if and only if
\[ \int_{\mathbb{R}} B(\rho, x) d\rho = 0. \tag{A.37} \]
Furthermore, if \( w_*(\rho, x) \) is such a solution, then all the solutions can be written as
\[ w(\rho, x) = w_*(\rho, x) + c(x), \quad \rho \in \mathbb{R}, x \in U, \tag{A.38} \]
where \( c: U \to \mathbb{R} \) is an arbitrary function.

In particular, if \( \tag{A.37} \) holds,
\[ w_*(\rho, x) = \int_0^\rho \frac{1}{\nu(\theta_0)} \int_{-\infty}^t B(s, x) ds dr, \quad \rho \in \mathbb{R}, x \in U, \tag{A.39} \]
is a solution. Additionally, if \( \int_{\mathbb{R}} B(\rho, x) d\rho = 0 \) for all \( x \in U \) and there exist \( M, L \in \mathbb{N} \) such that
\[ D_x^m D_\rho^l B(\pm \rho, x) = O(e^{-\alpha \rho}) \text{ as } \rho \to +\infty \tag{A.40} \]
for all \( m \in \{0, \cdots, M\} \) and \( l \in \{0, \cdots, L\} \) then there exist functions \( w^+(x) \) and \( w^-(x) \) such that
\[ D_x^m D_\rho^l \left( w(\pm \rho, x) - w^\pm(x) \right) = O(e^{-\alpha \rho}) \text{ as } \rho \to +\infty \tag{A.41} \]
for all \( m \in \{0, \cdots, M\} \) and \( l \in \{0, \cdots, L + 2\} \).

**Proof:** Assume the problem has a solution \( w(\cdot, x) \in C^2(\mathbb{R}) \cap L^\infty(\mathbb{R}) \). Along the same line of the proof in Lemma 2.6.3 in [38] we can prove
\[ \lim_{\rho \to \pm \infty} \partial_\rho w(\rho, x) = 0. \]
Integrating \( \tag{A.36} \) leads to \( \tag{A.37} \) which implies that the solution \( w(\cdot, x) \) is independent of \( \rho \) in the case of \( B = 0 \). Then we have \( \tag{A.38} \). Moreover if \( \tag{A.37} \) holds, it is easy to check \( w_*(\rho, x) \)
defined in (A.39) is a solution and satisfies (A.41) if (A.30) holds.

**Matching \( \varepsilon^k \)-order terms:** Firstly, by matching zero-order term in (A.39), the matching conditions (A.22) and (A.27) we get \( c_0 = \theta_0 \) satisfying (1.13)-(1.14). Matching the \( \varepsilon^k \) \( (k = 0, 1) \)-order terms, we derive the following ordinary differential equations in \( \rho \):

\[
\partial_\rho \left( \nu(\theta_0(\rho)) \left( \partial_\rho v_0 - u_0 d_1 \eta'(\rho) \right) \right) = \left( \theta'_0(\rho) \theta_0^2(\rho) + \partial_{\rho p-1} \right) \nabla d_1, \quad \text{(A.42)}
\]

and

\[
\partial_\rho \left( \nu(\theta_0(\rho)) \left( \partial_\rho v_1 + (u_0 h_1 - u_1 d_1) \eta'(\rho) \right) \right) = \left( \theta'_0(\rho) \partial_{\rho c_1}^2 + \theta'_0(\rho) \partial_{\rho c_1} + \partial_{\rho p_0} \right) \nabla d_1 + \partial_{\rho p_0} v_0 d_1 \partial_\rho \partial_\rho \partial_\rho d_1 + v_0 \cdot \nabla d_1 \partial_\rho \partial_\rho d_1 - \partial_\rho \left( 2\nu(\theta_0(\rho)) Dv_0 \right) \cdot \nabla d_1 - \partial_\rho \left( \nu(\theta_0(\rho)) D_0 d_1 \right) \nabla d_1 - \partial_{\rho p_0} \nu(\theta_0(\rho)) \nabla d_1 + \nu(\theta_0(\rho)) \left( \eta'(\rho)' \right) u_0 \partial_\rho v_0 + \left( \partial_{\rho p-1} - \partial_\rho \left( \frac{(\theta'_0(\rho))^2}{2} \right) \right) \nabla^2 h_1 + h_0,
\]

(A.43)

together with

\[
(\partial_\rho v_0 - u_0 d_1 \eta'(\rho)) \cdot \nabla d_1 = 0, \quad \text{(A.44)}
\]

\[
\partial_{\rho c_1}^2 - f'(\theta_0(c_1)) = \theta'_0(\rho) \partial_{\rho d_1} + v_0 \cdot \nabla d_1 - \Delta d_1 + \theta'_0(\rho) g_0 d_1. \quad \text{(A.45)}
\]

Matching the \( \varepsilon^k \)-order terms for \( k \geq 2 \), we obtain the following ordinary differential equations in \( \rho \):

\[
\partial_\rho \left( \nu(\theta_0(\rho)) \left( \partial_\rho v_k + (u_0 h_k - u_k d_1) \eta'(\rho) \right) \right) = \left( \partial_{\rho p-1} - \partial_\rho \left( \frac{(\theta'_0(\rho))^2}{2} \right) \right) \nabla^2 h_k + \nabla^{k-1}, \quad \text{(A.46)}
\]

\[
\left( \partial_\rho v_k + (u_0 h_k - u_k d_1) \eta'(\rho) \right) \cdot \nabla d_1 = \nabla^k h_k \cdot \left( \partial_\rho v_0 - u_0 d_1 \eta' \right), \quad \text{(A.47)}
\]

\[
\partial_{\rho c_k} \partial_\rho v_k - f'(\theta_0(c)) = B^{k-1}, \quad \text{(A.48)}
\]

where

\[
V^{k-1} = -\partial_\rho (v_1 \rho_1 v_{k-1}) + \partial_\rho p_{k-1} \nabla d_1 + \partial_\rho p_{0} \nabla^2 h_{k-1} + \partial_\rho v_{k-1} \partial_\rho d_1 + v_0 \cdot \nabla d_1 \partial_\rho v_{k-1} + v_{k-1} \cdot \nabla d_1 \partial_\rho v_0 - \partial_\rho \left( 2\nu(\theta_0(\rho)) Dv_{k-1} \right) \cdot \nabla d_1 - \partial_\rho \left( \nu(\theta_0(\rho)) \nabla d_1 v_{k-1} \right) - \Delta d_1 + \nabla^{k-1} + \overline{V}_{k-1}^{1},
\]

\[
\overline{V}_{k-1}^{1} = -\partial_\rho (v_0 \rho_0 v_{k-1} \cdot \nabla^2 h_{k-1} - \partial_\rho v_0 \partial_\rho^2 h_{k-1} - \partial_\rho p_0 \nabla^{k-1} h_{k-1} - \partial_\rho (v_0 \rho_0 \nabla d_1 v_{k-1} - \nabla^{k-1} h_{k-1}) \cdot \nabla d_1 v_{k-1} - \partial_\rho (2\nu_0 Dv_0) + \nabla^{k-1} h_{k-1} \cdot \partial_\rho (v_0 \nabla d_1) - (\nu(\theta_0) \eta'(\rho)) u_1 h_{k-1} - \eta'(\rho) h_{k-1},
\]

and

\[
A^{k-1} = \sum_{l=1}^{k-1} \partial_\rho u_l \cdot \nabla^{k-1} h_{k-1} - \nabla v_{k-1} - \eta'(\rho) \rho v_{k-1} \cdot \nabla d_1 - \eta'(\rho) d_1 \sum_{l=1}^{k-1} u_l \cdot \nabla^{k-1} h_{k-1}
\]

\[
- \eta'(\rho) 
\sum_{l=1}^{k-2} u_l \cdot \nabla d_1 v_{k-1} - \eta'(\rho) \rho \sum_{l=0}^{k-2} u_l \cdot \nabla^{k-1} h_{k-1-l}
\]

\[
+ \eta'(\rho) \sum_{l=0}^{k-2} h_{k-1} \sum_{m=0}^{k-2-l} u_m \cdot \nabla^{k-1-l-m} h_{k-1-m},
\]

and

\[
B^{k-1} = \theta_0(\nu) \left( \nabla^{k-1} h_{k-1} - \partial_\rho^2 h_{k-1} - v_0 \cdot \nabla^{k-1} h_{k-1} \right) - 2\theta_0(\nu) \nabla^{k-1} h_1 \cdot \nabla^{k-1} h_{k-1} - \theta_0(\rho) g_0 h_{k-1}
\]

\[
+ \theta_0(\rho) g_{k-1} d_1 + \theta_0(\rho) v_{k-1} \cdot \nabla d_1 + \overline{B}_{k-1}^{1},
\]

here \( \overline{V}_{k-1}^{1} \) and \( \overline{B}_{k-1}^{1} \) consist of “known terms” in the induction argument, which have an exponential decay as \( |\rho| \to \infty \).
Remark A.4 In the case \( k = 2 \) one easily calculates that \( \tilde{B}^1 = -\theta_0'(\rho)p_0 \) provided that \( c_1 \equiv 0 \), which will be seen in Remark A.6 below.

We note that all functions with negative index, except \( p_{-1} \), are supposed to be zero. If the upper limit of the summations is less than the lower limit, the sum has to be understood as zero. Furthermore, we remark that \( f_{k-1} = f_{k-1}(c_0, \ldots, c_{k-1}) \) and \( \nu_i = \nu_i(c_0, \ldots, c_i) \) are terms from the Taylor expansion (A.21) and we use the convention \( f_0(c_0) = 0 \).

A.3 Existence of the Expansion Terms

A.3.1 Solving zero order terms and derivation of jump conditions (1.8)–(1.10)

It follows from (A.42) and (A.44) that

\[
\int_{\Gamma} \rho v_0 \cdot \nabla \frac{\partial \rho}{\partial \rho} d\Gamma = \theta_0'(\rho)\theta_0''(\rho) + \theta_0 p_{-1} = 0.
\] (A.49)

Therefore one has

\[
p_{-1}(\rho, x, t) = -\frac{(\theta_0(\rho))^2}{2} \tag{A.50}
\]

and

\[
v_0(\rho, x, z) = \nabla v_0(x, t) + \rho_0(x, t) d\Gamma(x, t) \left( \eta(\rho) - \frac{1}{2} \right) \tag{A.51}
\]

for all \((x, t) \in \Gamma(3\delta), \rho \in \mathbb{R}\), and some function \( \nabla v_0(x, t) \).

We then get

\[
v_0^+ = \nabla v_0 + \frac{1}{2} \rho_0 d\Gamma, \quad v_0^- = \nabla v_0 - \frac{1}{2} \rho_0 d\Gamma \quad \text{in } \Gamma(3\delta) \tag{A.52}
\]

and then

\[
\nabla v_0 = \frac{1}{2}(v_0^+ + v_0^-), \quad \rho_0 d\Gamma = v_0^+ - v_0^- \quad \text{in } \Gamma(3\delta), \tag{A.53}
\]

which immediately imply that (1.9) and

\[
v_0(\rho, x, t) = v_0^+(x, t)\eta(\rho) + v_0^-(x, t)(1 - \eta(\rho)) \tag{A.54}
\]

and

\[
u_i = \begin{cases} 
\frac{v_i^+ - v_i^-}{\sigma_i} & \text{for } (x, t) \in \Gamma(2\delta) \setminus \Gamma, \\
\n \cdot \nabla (v_0^+ - v_0^-) & \text{for } (x, t) \in \Gamma. 
\end{cases} \tag{A.55}
\]

To proceed we show

Proposition A.5 On \( \Gamma \) there hold

\[
\int_{\mathbb{R}} v_0 \cdot \nabla d\Gamma \partial \rho v_0 d\rho = 0, \tag{A.56}
\]

\[
\int_{\mathbb{R}} \text{div} \left( 2\nu(\theta_0)D_dv_0 \right) d\rho = \sigma_n \int_{\mathbb{R}} \overline{\text{div}}v_0 d\rho = \sigma_n u_0, \tag{A.57}
\]

\[
\int_{\mathbb{R}} k_0 d\rho = -\sigma H n, \tag{A.58}
\]

\[
u_i \cdot n = 0, \tag{A.59}
\]

here \( \sigma_n = \int_{\mathbb{R}} \nu(\theta_0)\eta'(\rho) d\rho \) and

\[
\text{div}v_0 = (v_0^+ - v_0^-) \cdot \nabla^2 d\Gamma + (v_0^+ - v_0^-)\Delta d\Gamma + (\nabla d\Gamma \cdot \nabla)(v_0^+ - v_0^-). 
\]
Proof: By direct computation one has
\[
v_0 \cdot \nabla d_\tau \partial_\tau v_0 = (v_0^+ \eta' + v_0^- (1 - \eta)) \cdot \nabla d_\tau (v_0^+ - v_0^-),
\] (A.60)
\[\text{div} \left( 2(v_0^+ \eta') D_\tau v_0 \right) = (v_0^+ \eta') \text{div} \left( \left( v_0^+ - v_0^- \right) \nabla d_\tau + \left( (v_0^+ - v_0^-) \nabla d_\tau \right)^2 \right).
\]

Then we easily find that (A.56)-(A.58). Moreover, noting that
\[\theta_0 = -\theta_0' (\rho) \nabla^T h_0 + (\theta_0' (\rho))^2 \Delta d_\tau \nabla d_\tau,
\] (A.61)
we get
\[0 = \text{div} v_0^\pm = (n \otimes n : \nabla v_0^\pm) + \text{div}_\tau v_0^\pm,
\]
we get
\[u_0 \cdot n = [n \otimes n : \nabla v_0^\pm] = -[\text{div}_\tau v_0^\pm] = 0.
\] (A.63)

Applying Lemma A.3 to (A.43) we find
\[2(v_0^+ Dv_0^+) n - [p_0^+] n = \sigma \Delta d_\tau n
\] (A.65)
i.e., we get (A.43) and then
\[l_0 = \left( \rho_0^+ - \rho_0^- - \sigma \Delta d_\tau \right) \nabla d_\tau + (v_0^+ - v_0^-) \partial_\tau d_\tau + \frac{1}{2} (v_0^+ + v_0^-) \cdot \nabla d_\tau (v_0^+ - v_0^-)
\]
\[+ 2(v_0^+ Dv_0^+) - \nu^+ Dv_0^- \cdot \nabla d_\tau + \sigma_0 (\text{div} v_0 - u_0)
\]
\[\text{div}_\tau v_0 + l_0 d_\tau + \sigma_0 u_0 = 0.
\] (A.64)

Using (A.56)-(A.58) we can get
\[2(v_0^+ Dv_0^+) n - [p_0^+] n = \sigma \Delta d_\tau n
\]
\[\text{div}_\tau v_0 + l_0 d_\tau + \sigma_0 u_0 = 0.
\] (A.61)

Applying Lemma A.2 to (A.45) we see that the compatibility condition is equivalent to (1.10). To ensure \(c_1 \equiv 0\) in (A.45), we define
\[g_0 = \begin{cases} \frac{\Delta d_\tau - \text{div} \nabla d_\tau - \partial_\tau d_\tau}{d_\tau} & \text{for } (x,t) \in \Gamma(2\delta) \backslash \Gamma, \\
 \cdot \nabla (\Delta d_\tau - v_0 \cdot \nabla d_\tau - \partial_\tau d_\tau) & \text{for } (x,t) \in \Gamma.
\end{cases}
\] (A.66)

Remark A.6 For this choice of \(g_0\) we see that the right-hand side of (A.45) vanishes. Therefore \(c_1 \equiv 0\).

Furthermore one has
\[v_0^+ |_{\partial \Omega} = 0 \quad \text{on } \partial \Omega.
\] (A.67)

In summary we can derive that \((v_0^+ \cdot \rho_0^+)\) together with \((\Gamma_t)_{t \in [0, T_0]}\) solve the sharp interface limit system (1.9)-(1.10). Moreover,
Lemma A.7 (The zeroth order terms)
Let \((\mathbf{v}^\pm, p^\pm)\) be the extension of \((\mathbf{v}_0^\pm, p_0^\pm)\) satisfying the sharp interface limit system to \(\Omega^\pm \cup \Gamma(2\delta)\) as in Remark A.7. We define the terms of the outer expansion \((c_0^\pm, \mathbf{v}_0^\pm, p_0^\pm)\) for \((x,t) \in \Omega^\pm \cup \Gamma(2\delta; T)\) as
\[
c_0^\pm(x,t) = \pm 1, \quad \mathbf{v}_0^\pm(x,t) = \mathbf{v}^\pm(x,t), \quad p_0^\pm(x,t) = p^\pm(x,t), \quad p_{-1}^\pm(x,t) = 0, \tag{A.68}
\]
the terms of the inner expansion \((c_0, \mathbf{v}_0)\) as
\[
c_0(\rho, x, t) = \theta_0(\rho), \tag{A.69}
\]
\[
\mathbf{v}_0(\rho, x, t) = \mathbf{v}_0^+(x, t) \eta(\rho) + \mathbf{v}_0^-(x, t) (1 - \eta(\rho)), \tag{A.70}
\]
for all \((\rho, x,t) \in \mathbb{R} \times \Gamma(2\delta)\). Then there are smooth and bounded \(\theta_0 : \Gamma(2\delta) \to \mathbb{R}\), and \(u_0, l_0 : \Gamma(2\delta) \to \mathbb{R}^2\) such that the compatibility conditions (A.33) in \(\Omega(2\delta)\) are satisfied.

A.3.2 Determination of the Higher Order Terms
In this part we firstly aim to derive the equations related to higher order terms and then give the detailed induction argument in Lemma A.9.

Applying Lemma A.2 to (A.48) we can get \(c_k\). The compatibility condition (A.39) for (A.48) is equivalent to
\[
\partial_t^k h_{k-1} + \mathbf{v}_0 \cdot \nabla^k h_{k-1} - \Delta^k h_{k-1} + \sigma_0^{-1} \int_\mathbb{R} \nabla dr \cdot \mathbf{v}_{k-1}(\theta_0(\rho))^2 d\rho + g_0 h_{k-1} - g_{k-1} dt
= \sigma_0^{-1} \int_\mathbb{R} \vec{B}^{k-1} \theta_0(\rho) d\rho, \tag{A.71}
\]
where \(\sigma_0 = \int_\mathbb{R} (\theta_0(\rho))^2 d\rho\). Then one has
\[
\partial_t^k h_{k-1} + \mathbf{v}_0 \cdot \nabla^k h_{k-1} - \Delta^k h_{k-1} + \sigma_0^{-1} \int_\mathbb{R} \nabla dr \cdot \mathbf{v}_{k-1}(\theta_0(\rho))^2 d\rho + g_0 h_{k-1}
= \sigma_0^{-1} \int_\mathbb{R} \vec{B}^{k-1} \theta_0(\rho) d\rho \quad \text{on} \quad \Gamma \tag{A.72}
\]
and takes
\[
g_{k-1} = \frac{\partial_t^k h_{k-1} + \mathbf{v}_0 \cdot \nabla^k h_{k-1} - \Delta^k h_{k-1} + \sigma_0^{-1} \int_\mathbb{R} \nabla dr \cdot \mathbf{v}_{k-1}(\theta_0(\rho))^2 d\rho + g_0 h_{k-1}}{d\Gamma}
- \int_\mathbb{R} \vec{B}^{k-1} \theta_0(\rho) d\rho \quad \text{for} \quad (x,t) \in \Gamma(2\delta) \setminus \Gamma \tag{A.73}
\]
and
\[
g_{k-1} = n \cdot \nabla \left( \frac{\partial_t^k h_{k-1} + \mathbf{v}_0 \cdot \nabla^k h_{k-1} - \Delta^k h_{k-1} + \sigma_0^{-1} \int_\mathbb{R} \nabla dr \cdot \mathbf{v}_{k-1}(\theta_0(\rho))^2 d\rho + g_0 h_{k-1}}{d\Gamma} \right)
- \int_\mathbb{R} \vec{B}^{k-1} \theta_0(\rho) d\rho \quad \text{for} \quad (x,t) \in \Gamma \tag{A.74}
\]
such that the compatibility condition (A.33) in \(\Gamma(2\delta) \setminus \Gamma\) holds and \(g_{k-1}\) is smooth. The compatibility condition (A.39) for (A.40) is equivalent to
\[
(p_{k-1}^- - p_{k-1}^+) \nabla dr + (\mathbf{v}_{k-1}^+ - \mathbf{v}_{k-1}^-) \partial_t dr - (2\nu^+ D\mathbf{v}_{k-1}^+ - 2\nu^- D\mathbf{v}_{k-1}^-) \nabla dr + l_{k-1} \nabla dr + \int_\mathbb{R} \left( \mathbf{v}_0 \cdot \nabla dr \partial_t \mathbf{v}_{k-1} + \mathbf{v}_{k-1} \cdot \nabla dr \partial_\rho \mathbf{v}_0 - \text{div} \left( 2\nu(\theta_0(\rho)) D\mathbf{v}_{k-1} \right) \right) d\rho
= \sigma (\Delta^k h_{k-1} \nabla dr + \Delta^k h_{k-1}) - \int_\mathbb{R} \vec{V}_{k-1} \nabla dr - \int_\mathbb{R} \vec{V}_{k-1} \nabla dr. \tag{A.75}
\]
A.3 Existence of the Expansion Terms

If it is satisfied, the solution to (A.46) is given by

\[ u \in \Gamma, \] where we have used \( u \) which satisfies the inner-outer matching conditions (A.22).

Inserting (A.81) into (A.75) implies that

\[ k \int_0^\infty \mathbf{v} \cdot \mathbf{w} \cdot d\mathbf{r} \]

where \( \sigma \) is defined by

\[ \int_0^\infty \mathbf{v} \cdot \mathbf{w} \cdot d\mathbf{r} \]

By induction one has

\[ u \in \Gamma, \]

where \( u \) will be defined by

\[ u \in \Gamma, \]

Furthermore it follows from (A.70) that \( u \) will be defined by

\[ u \in \Gamma, \]

and then

\[ u \in \Gamma, \]

which satisfies the inner-outer matching conditions (A.22).

By induction one has

\[ u \in \Gamma, \]

Using (A.77) (with \( k - 1 \) instead of \( k \)) and (A.81) in (A.71) leads to

\[ \partial_t^2 h_{k-1} + v_0 \cdot \nabla h_{k-1} - \Delta x h_{k-1} + \sigma_0^{-1} \sigma_0 \mathbf{n} \cdot \mathbf{v}_{k-1} + (1 - \sigma_0^{-1} \sigma_2) \mathbf{n} \cdot \mathbf{v}_{k-1} + g h_{k-1} \]

where \( \sigma_2 = \int_0^{\eta(\theta_0)} \mathbf{v} \cdot \mathbf{w} \cdot d\mathbf{r} \) and

\[ w_{k-2}(x, t) = \int_0^\infty (1 - \eta(\theta_0))(\theta_0(\rho))^2 \int_0^\infty \mathbf{v} \cdot \mathbf{w} \cdot d\mathbf{r} \]

Inserting (A.81) into (A.75) implies that

\[ \mathbf{j}_{k-1} + l_{k-1} \mathbf{d} \mathbf{r} = - \int_0^{\mathbf{v}_{k-1}} d\mathbf{r} - \int_0^{\mathbf{v}_{k-1}} d\mathbf{r}. \]
where
\[ J_{k-1} = (p_{k-1}^+ - p_{k-1}^-) \nabla dr + (v_{k-1}^+ - v_{k-1}^-) \partial dr - (2\nu^+ Dv_{k-1}^+ - 2\nu^- Dv_{k-1}^-) \nabla dr \]
\[ + \frac{1}{2} (v_{k-1}^+ - v_{k-1}^-) \nabla dr (v_{k-1}^+ - v_{k-1}^-) + \frac{1}{2} (v_{k-1}^+ - v_{k-1}^-) \nabla dr (v_{k-1}^+ - v_{k-1}^-) - \sigma (\Delta^r h_{k-1} \nabla dr + \Delta dr \nabla^r h_{k-1}) \]
\[ - \sigma (v_{k-1}^+ - v_{k-1}^-) \cdot \nabla^2 dr - \sigma \eta (v_{k-1}^+ - v_{k-1}^-) \Delta dr - \sigma \eta \nabla (v_{k-1}^+ - v_{k-1}^-) \cdot \nabla dr. \quad (A.84) \]

Noting that
\[ \int_R \tilde{V}_1^{k-1} d\rho = -\frac{1}{2} (v_0^+ - v_0^-) (v_0^+ + v_0^-) \cdot \nabla^r h_{k-1} - (v_0^+ - v_0^-) \partial_x^r h_{k-1} - (p_0^+ - p_0^-) \nabla^r h_{k-1} \]
\[ + 2(\nu^+ Dv_0^+ - \nu^- Dv_0^-) \cdot \nabla^r h_{k-1} - l_0 h_{k-1}, \]
we get
\[ [2\nu^+ Dv_{k-1}^+ - p_{k-1}^+] \mathbf{n} r + [v_{k-1}^-] [\partial dr - \sigma_\eta \Delta dr] - \sigma_\eta [\nabla_{k-1}^+] \cdot \nabla^2 dr - \sigma_\eta [\nabla_{k-1}^+] \cdot \nabla dr \]
\[ = b_{k-1} - [2\nu^+ Dv_{k-1}^+ - p_{\tilde{G}}] \mathbf{n} r \nabla^r h_{k-1} - \sigma (\Delta^r h_{k-1} \nabla dr + \Delta dr \nabla^r h_{k-1}) \text{ on } \Gamma \quad (A.85) \]
and
\[ l_{k-1} = \begin{cases} b_{k-1} - J_{k-1} - f_R \tilde{V}_2^{k-1} d\rho, & \text{for } (x, t) \in \Gamma(2\delta) \setminus \Gamma, \\ n \cdot \nabla (b_{k-1} - J_{k-1} - f_R \tilde{V}_1^{k-1} d\rho), & \text{for } (x, t) \in \Gamma, \end{cases} \quad (A.86) \]
where
\[ b_{k-1} = -\int_R \tilde{V}_2^{k-1} d\rho. \]

Thanks to \( (A.81) \) one gets \( \partial_x v_0 = u_0 d_t \eta' \). Then the equation \( (A.87) \) becomes
\[ (\partial_x v_k + (u_0 h_k - u_0 d_t) \eta'(\rho)) \cdot \nabla dr = A^{k-1}. \quad (A.87) \]

Multiplying \( (A.46) \) by \( \nabla dr \) and utilizing \( (A.87) \) one has the equation for \( p_{k-1} \)
\[ \partial_x (p_{k-1} - \nu \partial_x v_{k-1} \cdot \nabla dr + v_{k-1} \cdot \nabla dr \partial_x dr + v_{k-1} \cdot \nabla dr v_0 \cdot \nabla dr \]
\[ - (2\nu(\theta(\rho))) Dv_{k-1} \cdot \nabla dr \partial_x dr + \int_{-\infty}^0 \text{ div } (2\nu(\theta(\rho)) D_x v_{k-1}(r, \cdot)) \cdot \nabla dr dr \]
\[ + \nu(\theta(\rho)) \partial_x v_{k-1} + l_{k-1} \partial_x \eta(\rho) \cdot \nabla dr - \nu(\theta(\rho)) A^{k-1} \]
\[ + \int_{-\infty}^0 (\tilde{V}_1^{k-1}(r, \cdot) + \tilde{V}_2^{k-1}(r, \cdot)) \cdot \nabla dr dr = 0. \quad (A.88) \]

Then we deduce
\[ p_{k-1} = \nu \partial_x v_{k-1} \cdot \nabla dr + \nu(\theta(\rho)) A^{k-1} - \nu(\theta(\rho)) \text{ div } v_{k-1} \]
\[ + (v_{k-1}^- + v_{k-1}^- (1 - \eta)) \cdot \nabla dr \partial_x dr - v_{k-1} \cdot \nabla dr \partial_x dr \]
\[ + (2\nu(\theta(\rho))) Dv_{k-1} \cdot \nabla dr \partial_x dr - \left( 2\nu^+ Dv_{k-1}^+ + 2\nu^- Dv_{k-1}^- \eta + 2\nu^- Dv_{k-1}^- (1 - \eta) : \nabla dr \otimes \nabla dr \right) \]
\[ + \left( \theta(\rho) \int_{-\infty}^0 \text{ div } (2\nu(\theta(\rho)) D_x v_{k-1}) \cdot \nabla dr dr + (1 - \eta) \int_{-\infty}^0 \text{ div } (2\nu(\theta(\rho)) D_x v_{k-1}) \cdot \nabla dr dr \right) \]
\[ + \left( \theta(\rho) \int_{-\infty}^0 (\tilde{V}_1^{k-1} + \tilde{V}_2^{k-1})(r, \cdot) \cdot \nabla dr dr - (1 - \eta) \int_{-\infty}^0 (\tilde{V}_1^{k-1} + \tilde{V}_2^{k-1})(r, \cdot) \cdot \nabla dr dr \right) \]
\[ + p_{k-1}^- \eta + p_{k-1}^- (1 - \eta) \quad (A.89) \]
which satisfies the inner-outer matching conditions \( (A.22) \). Here we have used \( (A.86) \).

Furthermore one has
\[ v_0^- |_{\partial\Omega} = 0, \quad \text{on } \partial\Omega. \quad (A.90) \]
A.3 Existence of the Expansion Terms

We note that the terms $W$ in (A.28) are not multiplied by terms of the kind $(d_r - \varepsilon (\rho + h^r))$. Therefore we have to make sure they vanish on the set $S'$. This is done by choosing the constant $C_S > 0$ in a suitable way. More precisely, we choose

$$C_S := \|h_1\|_{C^0(\tilde{\Omega} \times (0,T_0))} + 2$$

and assume that $\varepsilon > 0$ is so small that

$$\left| \sum_{k \geq 1} \varepsilon^k h_{k+1}(S(x,t),t) \right| \leq 1.$$  \hfill (A.91)

It turns out that $h_1$ does not depend on the term $\varepsilon^2 (W^+ \eta^{C_S,+} + W^- \eta^{C_S,-})$. Therefore this choice of $C_S$ does not cause problems. Then it is possible to show as in [13] Remark 4.2 (2) that for $\rho = \frac{d_r \varepsilon (1)}{\varepsilon (1)} - h^r (S(x,t),t)$ and $(x,t) \in \Gamma'(2\delta)$ such that $d_r(x,t) \geq 0$ it follows $\rho \geq -C_S + 1$. Thus, $\eta^{C_S,-}(\rho) = 0$ and since $(x,t) \in \Omega'$ we have $W^+(x,t) = 0$ and therefore

$$\varepsilon^2 \left( W^+ \eta^{C_S,+} + W^- \eta^{C_S,-} \right) = 0.$$  

An analogous argument shows this for $d_r(x,t) < 0$.

Lemma A.9 (The $k$-th order terms)

Let $k \in \{1, \ldots, N+2\}$ be given. Then there are smooth functions $v_k, v_k^\pm, u_{k-1}, l_{k-1}, c_k, c_k^+, g_{k-1}, h_k, p_{k-1}, \tilde{p}_k^\pm, p_{-1} = -\left(\theta_0'(\rho)\right)^2$

which are bounded on their respective domains, such that for the $k$-th order the outer equations (A.39), (A.42), (A.43), the inner equations (A.40), (A.47), (A.48), the inner-outer matching conditions (A.22) are satisfied. Moreover, $(v_k^\pm, p_k^\pm, h_k)$ satisfies

$$\partial_t v_k^\pm - \nu(\pm 1) \Delta v_k^\pm + \nabla p_k^\pm = -\sum_{j=0}^k v_{k-j}^\pm \cdot \nabla v_j^\pm \quad \text{in } \Omega^2(t),$$  \hfill (A.92)

$$\text{div } v_k^\pm = 0 \quad \text{in } \Omega^2(t),$$  \hfill (A.93)

$$\|v_k\| \cdot n = \tilde{a}_{k-1} \quad \text{on } \Gamma_t,$$  \hfill (A.94)

$$\|v_k\| \cdot \tau = \tilde{a}_{k-1} - u_0 \cdot \tau h_k \circ X_0^{-1} \quad \text{on } \Gamma_t,$$  \hfill (A.95)

$$\|v_k^2 \| \cdot n = \tilde{a}_{k-1} - u_0 \cdot \tau h_k \circ X_0^{-1} \quad \text{on } \Gamma_t,$$  \hfill (A.96)

$$\partial_t h_k + v_0 \cdot X_0 \cdot \partial_X h_k - \Delta_r h_k + \left(\sigma_0' \sigma_0 - 1 \sigma_0^2 \right) n \cdot v_k^+ \cdot X_0$$

$$+ g_0 h_k = \frac{1}{\sigma_0} \int_{S} \tilde{B} \theta_0'(\rho) d\rho - \frac{1}{\sigma_0} w_{k-1} \circ X_0 \quad \text{on } \Gamma_1,$$  \hfill (A.97)

$$v_k = 0 \quad \text{on } \partial \Omega$$  \hfill (A.98)

for every $t \in (0,T_0)$. Here (A.90) and (A.94) come from (A.80) and (A.82) (with $k-1$ instead of $k$). Additionally, it holds $h_{k+1}(s,0) = 0$ for all $s \in T^1$. Here $v_k^\pm, c_k^\pm$ and $p_k^\pm$ are considered to be extended onto $\Omega^2 \cup \Gamma'(2\delta)$ as in Remark A.7.

Proof: We mainly give the induction procedure. For the induction step we assume that

$$\{v_i, v_i^\pm, u_{i-1}, l_{i-1}, c_i, c_i^+, g_{i-1}, h_i, p_{i-1}, p_i^\pm\}$$

are known for $0 \leq i < k - 1$ and the matching conditions (A.22) for $\varphi = c_i, v_i, p_i$ with $0 \leq i \leq k - 1$ hold. Next we obtain the terms for $i = k$ by the following four steps.

Step 1: Noting that $\tilde{B}^{k-1}, \tilde{V}_1^{k-1}, \tilde{V}_2^{k-1}, \tilde{W}^{k-2}$ are known and then $l_{k-1}$ is known by (A.80). Collecting (A.30) and (A.31) we have $c_k^3 = 0$ (for $k \geq 1$).

Step 2: We have seen that the compatibility condition for (A.38) is equivalent to the evolution equation (A.71) for $h_{k-1}$ on $T^1 \times (0,T_0)$, which is satisfied by assumption. Then we can get
c_k, which satisfies the inner-outer matching conditions (A.22), by solving the equation (A.48) and g_{k-1} by (A.73)-(A.74). Moreover, since h_{k-1} is known, u_{k-1} can be obtained from (A.70) (with k − 1 instead of k) and A^{k-1} is known.

Step 3: We then get p_{k-1} by (A.80). Therefore V^{k-1} and W^{k-1} are known.

Step 4: According to the procedure of the asymptotic expansion we have the closed system (A.92)-(A.98) for (v_k, p_k, h_k). The details to solve the system will be given in the following Theorem A.10. We then get v_k by (A.80).

Hence we completed the proof of this lemma.$\blacksquare$

Proof of Theorem 3.2 From the construction one can verify the statements of Theorem 3.2 in the same way as the proof of [12, Theorem 3.2]. In the present situation the estimates are even much simpler since no coefficients, which depend on $\varepsilon$ and are controlled in $\varepsilon$ only in a limited way (as the $(M - \frac{3}{2})$-order terms in [7]), do not occur.$\blacksquare$

THEOREM A.10 Let $T \in (0, \infty)$, $u_0 : \Omega \times [0, T_0] \to \mathbb{R}^2$ and $u : \Omega^\varepsilon \to \mathbb{R}^2$ be smooth, and

$$f \in L^2(\Omega \times (0, T))^2,$$

$$g \in L^2(0, T; H^1(\Omega \setminus \Gamma)),$$

$$a_1 \in L^2(0, T; H^2(\Gamma_1)) \cap H^2(0, T; L^2(\Gamma_1)),$$

$$b \in L^2(0, T; H^2(\Gamma_1))^2 \cap H^2(0, T; L^2(\Gamma_1))^2,$$

$$w \in L^2(0, T; H^2(T)^2) \cap H^2(0, T; L^2(\partial \Omega))^4,$$

$$v_0 \in H^1(\Omega \setminus \Gamma(0))^2, \quad h_0 \in H^2(T)^1$$

satisfy

$$\int_{\Omega \times (t)} g \, dx = - \int_{\Gamma} a_1 \cdot n_{\Gamma} \, d\mathcal{H}^1(p) + \int_{\partial \Omega} a_2 \cdot n_{\partial \Omega} \, d\mathcal{H}^1(p)$$

(A.99)

for almost all $t \in (0, T)$, that $g = \text{div } R$ for some $R \in H^1(0, T; L^2(\Omega))^2 \cap L^2(0, T; H^2(\Omega \setminus \Gamma(t))^2)$, $\text{div } v_{0|t=0} = g|_{t=0}$ and $[v_0] = a_1|_{t=0}$, $v_0|_{\partial \Omega} = a_2|_{t=0}$. Then there are unique

$$v \in L^2(0, T; H^2(\Omega \setminus \Gamma))^2 \cap H^1(0, T; L^2(\Omega))^2, \quad p \in L^2(0, T; H^0(\Omega \setminus \Gamma_1))$$

$$h \in L^2(0, T; H^2(T)^1) \cap H^1(0, T; H^2(T)^1)$$

solving

$$\partial_t v^\pm + u^\pm \cdot \nabla v^\pm + v^\pm \cdot \nabla u^\pm - \nu(\pm \varepsilon) \Delta v^\pm + \nabla p^\pm = f, \quad \text{in } \Omega^\varepsilon(t), t \in (0, T),$$

$$\text{div } v^\pm = g, \quad \text{in } \Omega^\varepsilon(t), t \in (0, T),$$

$$[v] \cdot n = a_1 \cdot n, \quad \text{on } \Gamma, t \in (0, T),$$

$$[v] \cdot \tau = a_1 \cdot \tau - u_0 \cdot \tau h, \quad \text{on } \Gamma, t \in (0, T),$$

$$[2\nu Dv - p] n_{\Gamma} + [v] \left( \partial_t \sigma_1 - \sigma_{\epsilon} \Delta t \right)$$

(A.100)

$$- \sigma_0 [v] \cdot \nabla^2 t - \sigma_0 [\nabla v] \cdot \nabla t = b, \quad \text{on } \Gamma, t \in (0, T),$$

$$\partial_t h + v_0 \circ X_0 \cdot \nabla_t h - \Delta h + \sigma_0^{-1} \sigma_2 n \cdot v^+ \circ X_0$$

(A.105)

$$+(1 - \sigma_0^{-1} \sigma_2) n \cdot v^- \circ X_0 + \sigma_0^{-1} \sigma_1 g_0 h = w, \quad \text{on } T^1 \times (0, T),$$

$$v^- = a, \quad \text{on } \partial \Omega \times (0, T),$$

$$v^\pm|_{t=0} = v^\pm_0, \quad \text{in } \Omega,$$

(A.108)

$$h|_{t=0} = h_0, \quad \text{on } T^1.$$

Proof: First of all, we can easily reduce to the case $a_1 \equiv 0$ and $a_2$ by subtracting a suitable extension.

Step 1: Existence for $T = T_1 > 0$ sufficiently small: In this case the proof can be done in the same way as the proof of [12, Theorem 3.2]. In the present case the proof is even simpler
and one simply has to omit the terms related to the convective term $u \cdot \nabla_h u$ and the (given) surface tension term $\sigma \tilde{H}_h$.

**Step 2: Existence for general $T > 0$:** Since the system is linear we get that the existence time $T_1 > 0$ is independent of the norms of the data. Moreover, we obtain that for any $t_0 \in (0, T)$ there is some $T_1(t_0) > t_0$ such that the system has a unique solution for $t \in (t_0, T_1(t_0))$ for a given initial value $v_1|_{t=t_0} = \check{v}_0$ at $t = -1$. Because of the compactness of $[0, T]$, we can concatenate these solutions and obtain a solution on $(0, T)$.

**Remark A.11** With the aid of Theorem 3.2 one can obtain a smooth solution of \( (A.100)-(A.108) \) for smooth $f$ such that these functions vanish in $\left[ -1, T_0 \right]$ such that the system has a unique solution for $t \in \left( t_0, T_1(t_0) \right)$ for a given initial value $v_1|_{t=t_0} = \check{v}_0$ at $t = -1$. Because of the compactness of $[0, T]$, we can concatenate these solutions and obtain a solution on $(0, T)$.

**Acknowledgments**

M. Fei was partially supported by NSF of China under Grant No.11871075 and 11971357. The support is gratefully acknowledged. Moreover, we thank Maxilian Moser for several helpful comments on a previous version of this contribution and Yuning Liu for helpful discussions on this topic. Moreover, we are grateful to the anonymous referees for their careful reading of the manuscript and many valuable comments to improve the presentation.

**References**

[1] H. Abels. On a diffuse interface model for two-phase flows of viscous, incompressible fluids with matched densities. *Arch. Rat. Mech. Anal.*, 194(2):463–506, 2009.

[2] H. Abels. (Non-)convergence of solutions of the convective Allen-Cahn equation. *Partial Differ. Equ. Appl.*, 3(1), 2022.

[3] H. Abels, H. Garcke, and G. Grün. Thermodynamically consistent, frame indifferent diffuse interface models for incompressible two-phase flows with different densities. *Math. Models Methods Appl. Sci.*, 22(3):1150013 (40 pages), 2012.

[4] H. Abels and D. Lengeler. On sharp interface limits for diffuse interface models for two-phase flows. *Interfaces Free Bound.*, 16(3):395–418, 2014.

[5] H. Abels and Y. Liu. Sharp interface limit for a Stokes/Allen-Cahn system. *Arch. Ration. Mech. Anal.*, 229(1):417–502, 2018.

[6] H. Abels and A. Marquardt. Sharp interface limit of a Stokes/Cahn-Hilliard system, Part I: Convergence result. *Interfaces Free Bound.*, 23(3):353–402, 2021.

[7] H. Abels and A. Marquardt. Sharp interface limit of a Stokes/Cahn-Hilliard system, Part II: Approximate solutions. *J. Math. Fluid Mech.*, 23(2):Paper No. 38, 48, 2021.

[8] H. Abels and M. Moser. Well-posedness of a Navier-Stokes/mean curvature flow system. In *Mathematical analysis in fluid mechanics—selected recent results*, volume 710 of *Contemp. Math.*, pages 1–23. Amer. Math. Soc., [Providence], RI, [2018] ©2018.

[9] H. Abels and M. Moser. Convergence of the Allen-Cahn equation to the mean curvature flow with 90°-contact angle in 2D. *Interfaces Free Bound.*, 21(3):313–365, 2019.
[10] H. Abels and M. Moser. Convergence of the Allen-Cahn equation with a nonlinear robin boundary condition to mean curvature flow with contact angle close to 90°. *SIAM J. Math. Anal.*, 54(1):114-172, 2022.

[11] H. Abels and M. Röger. Existence of weak solutions for a non-classical sharp interface model for a two-phase flow of viscous, incompressible fluids. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 26:2403–2424, 2009.

[12] H. Abels and M. Wilke. Well-posedness and qualitative behaviour of solutions for a two-phase Navier-Stokes-Mullins-Sekerka system. *Interfaces Free Bound.*, 15(1):39–75, 2013.

[13] N. D. Alikakos, P. W. Bates, and X. Chen. Convergence of the Cahn-Hilliard equation to the Hele-Shaw model. *Arch. Rational Mech. Anal.*, 128(2):165–205, 1994.

[14] F. Boyer. Mathematical study of multi-phase flow under shear through order parameter formulation. *Asymptot. Anal.*, 20(2):175–212, 1999.

[15] J. W. Cahn and J. E. Hilliard. Free energy of a nonuniform system. I. Interfacial energy. *J. Chem. Phys.*, 28, No. 2:258–267, 1958.

[16] X. Chen, D. Hilhorst, and E. Logak. Mass conserving Allen-Cahn equation and volume preserving mean curvature flow. *Interfaces Free Bound.*, 12(4):527–549, 2010.

[17] P. De Mottoni and M. Schatzman. Geometrical evolution of developed interfaces. *Trans. Amer. Math. Soc.*, 347(5):1533–1589, 1995.

[18] L. C. Evans, H. M. Soner, and P. E. Souganidis. Phase transitions and generalized motion by mean curvature. *Comm. Pure Appl. Math.*, 45:1097–1123, 1992.

[19] M. Fei. Global sharp interface limit of the Hele-Shaw-Cahn-Hilliard system. *Math. Methods Appl. Sci.*, 40(3):833–852, 2017.

[20] J. Fischer, T. Laux, T. M. Simon. Convergence rates of the Allen-Cahn equation to mean curvature flow: A short proof based on relative entropies. *SIAM J. Math. Anal.*, 52(6):6222–6233, 2020.

[21] C. G. Gal and M. Grasselli. Longtime behavior for a model of homogeneous incompressible two-phase flows. *Discrete Contin. Dyn. Syst.*, 28(1):1–39, 2010.

[22] A. Giorgini, M. Grasselli, and H. Wu. Diffuse interface models for incompressible binary fluids and the mass-conserving Allen-Cahn approximation. *J. Funct. Anal.*, 283(9):109631, 2022.

[23] Y. Giga. Analyticity of the semigroup generated by the stokes operator in $L^p$ spaces. *Math. Z.*, 178:297–329, 1981.

[24] M. E. Gurtin, D. Polignone, and J. Viñals. Two-phase binary fluids and immiscible fluids described by an order parameter. *Math. Models Methods Appl. Sci.*, 6(6):815–831, 1996.

[25] S. Hensel, Y. Liu. The sharp interface limit of a Navier-Stokes/Allen–Cahn system with constant mobility: Convergence rates by a relative energy approach. *Preprint*, arXiv:2201.09423, 2022.

[26] P. Hohenberg and B. Halperin. Theory of dynamic critical phenomena. *Rev. Mod. Phys.*, 49:435–479, 1977.

[27] T. Ilmanen. Convergence of the Allen-Cahn equation to Brakke’s motion by mean curvature. *J. Differential Geom.*, 38:417–461, 1993.

[28] J. Jiang, Y. Li, and C. Liu. Two-phase incompressible flows with variable density: an energetic variational approach. *Discrete Contin. Dyn. Syst.*, 37(6):3243–3284, 2017.

[29] S. Jiang, X. Su, F. Xie. Remarks on Sharp Interface Limit for an Incompressible Navier-Stokes and Allen-Cahn Coupled System. *Preprint*, arXiv:2205.01501, 2022.

[30] T. Kagaya. Convergence of the Allen-Cahn equation with a zero Neumann boundary condition on non-convex domains. *Math. Ann.*, 373:1485–1528, 2019.

[31] M. Katsoulakis, G. T. Kokkolaris, F. Reitich. Generalized Motion by Mean Curvature with Neumann Conditions and the Allen-Cahn Model for Phase Transitions. *The Journal of Geometric Analysis* 5(2):255–279, 1995.
[32] T. Laux, T. M. Simon. Convergence of the Allen-Cahn Equation to Multiphase Mean Curvature Flow. Comm. Pure Appl. Math. 71(8):1493–1714, 2018.

[33] H.-G. Lee, J. S. Lowengrub, and J. Goodman. Modeling pinchoff and reconnection in a Hele-Shaw cell. I. The models and their calibration. Phys. Fluids, 14(2):492–513, 2002.

[34] C. Liu, N. Sato, and Y. Tonegawa. Two-phase flow problem coupled with mean curvature flow. Interfaces Free Bound., 14(2):185–203, 2012.

[35] C. Liu and J. Shen. A phase field model for the mixture of two incompressible fluids and its approximation by a Fourier-spectral method. Phys. D, 179(3-4):211–228, 2003.

[36] A. Marquardt. Sharp Interface Limit for a Stokes / Cahn-Hilliard System. PhD thesis, University Regensburg, urn:nbn:de:bvb:355-epub-384308, 2019.

[37] M. Mizuno, Y. Tonegawa. Convergence of the Allen-Cahn equation with Neumann boundary conditions. SIAM J. Math. Anal. 47(3):1906–1932, 2015.

[38] S. Schaubek. Sharp interface limits for diffuse interface models. PhD thesis, University Regensburg, urn:nbn:de:bvb:355-epub-294622, 2014.

[39] K. Schumacher. The instationary Navier-Stokes equations in weighted Bessel-potential spaces. J. Math. Fluid Mech., 11(4):552–571, 2009.

[40] E. M. Stein. Singular Integrals and Differentiability Properties of Functions. Princeton Hall Press, Princeton, New Jersey, 1970.

[41] X. Wang and Z. Zhang. Well-posedness of the Hele-Shaw-Cahn-Hilliard system. Ann. Inst. H. Poincaré Anal. Non Linéaire, 30(3):367–384, 2013.

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