Superconformal Field Theory with Boundary:
Fermionic Model

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Abstract

Fermionic model of Superconformal field theory with boundary is considered. There were written the "boundary" Ward Identity for this theory and also constructed boundary states for fermionic and spin models. For this model were derived "bootstrap" equations for boundary structure constants.

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1 Introduction

Superconformal invariant theories with boundary undoubtedly have a great interest themselves as well as in their connection with open superstring theories [1]. There are other reasons for studying Superconformal Field Theory on manifolds with boundary. They relate to the connection with statistical mechanics [2].

In introduction we will give only a brief review of necessary facts from superconformal field theory. Background on superconformal field theory can be found in refs. [3],[4].

In complex coordinates $\hat{z} = (z, \theta)$ two-dimensional superconformal transformations are:

\[
\begin{align*}
\delta \theta &= \varepsilon + \frac{i}{2} \theta u_z; \\
\delta \bar{\theta} &= \bar{\varepsilon} + \frac{i}{2} \bar{\theta} \bar{u} \bar{z}; \\
\delta z &= u + \theta \varepsilon; \\
\delta \bar{z} &= \bar{u} + \bar{\theta} \bar{\varepsilon}
\end{align*}
\]

and these supertransformations are generated by

\[
\begin{align*}
W(z, \theta) &= \frac{1}{2} S(z) + \theta T(z); \\
\bar{W}(\bar{z}, \bar{\theta}) &= \frac{1}{2} \bar{S}(\bar{z}) + \bar{\theta} \bar{T}(\bar{z}).
\end{align*}
\]

Currents $T(z)$ and $S(z)$ are generators of holomorphic conformal and supersymmetric transformations respectively. Coefficients of the Laurent expansion of $T(z)$ and $S(z)$ form the superconformal algebra:

\[
\begin{align*}
[L_m, L_n] &= (m - n)L_{m+n} + \frac{c}{8}(m^2 - 1)\delta_{m+n} \\
\{S_r, S_s\} &= 2L_{r+s} + \frac{c}{4}(r^2 - 1)\delta_{r+s} \\
[L_m, S_r] &= (\frac{m}{2} - r)S_{m+r}
\end{align*}
\]

where $r$ runs over half-integer values (Neveu-Schwarz–NS) or integer values (Ramond–R). One can construct irreducible representation of this algebra in each sector from primary states. To each highest weight state of NS-algebra:

\[
\begin{align*}
L_n | h \rangle &= 0 \quad n > 0 \\
S_r | h \rangle &= 0 \quad r > 0 \\
L_0 | h \rangle &= h | h \rangle
\end{align*}
\]

corresponds primary superfield $V_h(z, \theta) = \phi(z) + \theta \psi(z)$:

\[
|h \rangle = \phi(0) | 0 \rangle; \quad S_{-1/2} | h \rangle = \psi(0) | 0 \rangle;
\]

In the Ramond sector superconformal current has zero mode, which form two dimensional Clifford Algebra with the Fermion Number Operator $\Gamma = (-)^F$, commuting with the $L_0$. As a result, we have double degeneration of the
ground state \[^3\]. In this space we can choose the following orthogonal basis
\(|h^+\rangle = R^+_h(0) \ket{0}, |h^-\rangle = R^-_h(0) \ket{0}\) (where \(R^\pm\)-Ramond spin fields):

\(|h^-\rangle = S_0 |h^+\rangle \quad (5)

where \(|h^+\rangle\) and \(|h^-\rangle\) are eigenvectors of operator \((-)^F\) with eigenvalues +1 and −1 respectively having the same conformal weight \(h\). Using commutation relations \([2]\) we can obtain:

\(S_0 |h^-\rangle = S^2_0 |h^+\rangle = (L_0 - \frac{c}{16}) |h^+\rangle = (h - \frac{c}{16}) |h^+\rangle \quad (6)

Thus, if one normalizes \(|h^+\rangle\) as, \(\langle h^+ | h^+ \rangle = 1\), then from \([3]\) it follows, that \(\langle h^- | h^- \rangle = h - \frac{c}{16}\). In case, when \(h \neq \frac{c}{16}\), it can be chosen basis \(|h^{\pm}\rangle\) such, that

\(S_0 |h^{\pm}\rangle = \sqrt{h - \frac{c}{16}} |h^{\mp}\rangle\) and which is orthonormal. In further we will use the basis \([3]\). Let us note, that if \(h = \frac{c}{16}\), then \(|h^-\rangle\) becomes 0-vector and decouples from the representation of algebra. Hence chiral symmetry of the ground state is destroyed and the global supersymmetry is restored.

In the general superconformal theory the full operator algebra of \(NS\) superfields and \(R^\pm\) spin fields is nonlocal, since the spin fields have double-valued OPE with respect to the fermionic operators of superfields \([3]\). Besides, OPE of the spin fields of the same chirality are local, while OPE of spin fields of opposite chirality are nonlocal, since their OPE contain fermionic fields. There are two possibility for projecting onto a local set of fields. First one, keeping only the \(NS\)-sector giving the usual algebra of superfields, a fermionic model. The second one, we can get a local field theory the ”spin model” restricting in superconformal field theory by \(\Gamma = 1\) sector. In the framework of this paper we are restricted only for fermionic model. Boundary ”spin model” are going to consider in the following papers.

2 Boundary Ward Identity

After this brief review of ”bulk” theory, let us consider theory with boundary defined on upper half plane \([\tilde{z}]\). The variation of superfields under infinitesimal superconformal transformation is given

\(\delta_v \Phi(\tilde{z}) = \frac{1}{2\pi i} \oint_C dzd\theta v(\tilde{z})W(\tilde{z})\Phi(\tilde{z}) = \frac{1}{2\pi i} \oint_C dz(\varepsilon S + uT)\Phi(\tilde{z}) \quad (7)\)
where \( v(\hat{z}) = u(z) + 2\theta \varepsilon(z) \) is infinitesimal parameter. A superfield \( \Phi(\hat{z}) \) is a primary superconformal field if it obeys

\[
\delta_v \Phi = v \partial_\hat{z} \Phi + \frac{1}{2} (Dv) D\Phi + \Delta_\Phi (\partial_\hat{z} v) \Phi
\]

(8)

where \( D = \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial z} \). Using these relations one can derive holomorphic Ward Identity in superconformal field theory:

\[
\langle T(z) \Phi_1(\hat{z}_1, \bar{z}_1), ..., \Phi_n(\hat{z}_n, \bar{z}_n) \rangle = \left[ \sum_i \frac{1}{(z - z_i) d z_i} + \sum_i \frac{\Delta_i}{(z - z_i)^2} \right] \langle \Phi_1, ..., \Phi_n \rangle
\]

(9)

where \( \hat{\Delta} = \Delta_\Phi + \frac{1}{2} \theta \frac{\partial}{\partial \theta} \) and

\[
\langle S(z) \Phi_1, ..., \Phi_n \rangle = \left[ -2 \sum_i \frac{\Delta_i \theta_i}{(z - z_i)^2} + \sum_i \frac{1}{z - z_i} \left( \frac{\partial}{\partial \theta_i} - \theta_i \frac{\partial}{\partial z_i} \right) \right] \langle \Phi_1, ..., \Phi_n \rangle
\]

(10)

In the same way one can find anti–holomorphic Ward Identity.

It is easy to see, that the requirement of preservation of the geometry gives strong limitations on parameters of superconformal transformation \( \varepsilon \) and \( u \). One can see that the coefficients of expansion \( u \) and \( \varepsilon \) must be real. Therefore holomorphical and anti–holomorphical transformations are not independent, and in relation (7) we should consider \( \delta_v + \delta_\bar{v} \). For let us make analytical continuation of \( T \) and \( S \) on to lower half plane.

\[
T(z) = \bar{T}(z) \\
S(z) = \bar{S}(z) \quad \text{for} \quad \text{Im}z < 0
\]

(11)

It means that now we have only one algebra (2) in opposite to ”bulk” theory, there were two, holomorphical and anti–holomorphical algebras, which is consistent with the fact, that in theory with boundary we have only one set of coefficient in expansion of parameters \( \varepsilon, u \). Using (9) and (11) we get:

\[
(\delta_v + \delta_\bar{v}) \Phi(\hat{z}, \bar{z}) = \frac{1}{2\pi i} (\oint_{C_+}(\varepsilon S + uT)\Phi dz - \oint_{C_+}(\varepsilon S + \bar{u}\bar{T})\Phi d\bar{z})
\]

\[
= \frac{1}{2\pi i} \oint_{C_+ \cup C_-}(\varepsilon S + uT)\Phi dz
\]

(12)

where contour \( C_+ \cup C_- \) contains all points \((z_1, \bar{z}_1, ..., z_n, \bar{z}_n)\). >From (12) follows, that, in contrast to ”bulk” theory where \( T(z) \) and \( S(z) \) acts only on \((z_1, ..., z_n)\), in theory with boundary the action of \( T(z) \) and \( S(z) \) is extended...
to \((z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n)\) and therefore in the relations of the type (9), (10) the doubling of terms on the right hand sides takes place due to terms with \(z_i' = \bar{z}_i\).

### 3 Boundary States

Further we will consider boundary state problem in superconformal field theory with boundary. We will deal with theories defined on the upper half plane or strip, which one can also interpretate as a world sheet of an open superstring. Mapping of the upper half plane on to strip is given by the conformal trasformation \(z = e^{\tau + i\sigma}\), where \((\tau, \sigma)\) are coordinates on strip.

In general superconformal field theory with boundary, the unique requirement on boundary condition is the superconformal invariance:

\[
T(z = e^t) = \bar{T}(\bar{z} = e^\bar{t}) \\
S(z = e^t) = \bar{S}(\bar{z} = e^\bar{t}) \\
S(z = e^t) = \bar{S}(\bar{z} = e^\bar{t})
\]

\[NS\text{-sector}\]

\[
T(z = e^{t+i\pi}) = \bar{T}(\bar{z} = e^{\bar{t}-i\pi}) \\
S(z = e^{t+i\pi}) = \bar{S}(\bar{z} = e^{\bar{t}-i\pi}) \\
S(z = e^{t+i\pi}) = -\bar{S}(\bar{z} = e^{\bar{t}-i\pi})
\]

\[R\text{-sector}\]

\[\text{It is well known that there is an isomorphism between the space of conformal invariant boundary conditions and the space of boundary states. Indeed, if one compactifies } t \text{ by mod } 2\pi Im\tau \text{ (}\tau\text{ is purely imaginary) (in this way we obtain the theory defined on cylinder with radius } Im\tau)\text{, then partition function with boundary conditions } \alpha \text{ and } \beta \text{ at the ends of cylinder can be written as follows:}
\]

\[
Z_{\alpha\beta}^{NS} = Tr e^{2\pi iH_{open}}
\]

\[\text{From the other side, the same partition function can be considered as a propagation of closed superstring on } \sigma \text{ direction between states } \langle \alpha |, | \beta \rangle,
\]

\[
Z_{\alpha\beta} = \langle \alpha | e^{-\pi H_{cyl}} | \beta \rangle = \langle \alpha | e^{-\pi \frac{n}{2m} (L_{0}+\bar{L}_{0})} | \beta \rangle
\]

where \(H_{cyl}\) is the Hamiltonian for closed superstring, \(L_{0}^{cyl}, \bar{L}_{0}^{cyl}\) are generators of Virasoro and \(| \alpha \rangle, | \beta \rangle\) satisfy conditions (13), which can be rewritten as

\[
T^{cyl}(\zeta) = \bar{T}^{cyl}(\bar{\zeta})|_{\zeta = e^{-it}} \\
S^{cyl}(\zeta) = \bar{S}^{cyl}(\bar{\zeta})|_{\zeta = e^{-it}} \\
S^{cyl}(\zeta) = -i\bar{S}^{cyl}(\bar{\zeta})|_{\zeta = e^{-it}} \\
S^{cyl}(\zeta) = i\bar{S}^{cyl}(\bar{\zeta})|_{\zeta = e^{-it}}
\]

\[NS\text{,} \ R\]

\[
\text{(16)}
\]
where $\zeta = e^{-i(t+\sigma)}$. One can rewrite conditions (16), in the form:

$$
\begin{align*}
(L_n - L_{-n}) \mid B \rangle &= 0 \\
(S_r + iS_{-r}) \mid B \rangle &= 0
\end{align*}
$$

(17)

where $r \in Z$ or $r \in Z + \frac{1}{2}$. One of the basic aims of this paper is to find solutions (17) in each irreducible representation of superconformal algebra. It is well known that an irreducible representation contains four sectors:

$$(NS + R) \otimes (NS + R) = NS \otimes NS + NS \otimes R + R \otimes NS + R \otimes R. \quad (18)$$

and it is easy to see, that equation (16) and consequently (17) does not have nontrivial solutions in cross sectors $NS \otimes R$ and $R \otimes NS$ and nontrivial solutions exist only in $R \otimes R$ and $NS \otimes NS$ sectors.

$$
\mid B \rangle = \mid NS \rangle \otimes \mid NS \rangle + \mid R \rangle \otimes \mid R \rangle
$$

(19)

In order to solve conditions (17) in $NS$ sector let us consider the following anzats

$$
\mid h \rangle_B = \sum_{\mid N \rangle} \mid h, N \rangle \otimes U_{NS} \mid h, N \rangle
$$

(20)

where $U_{NS}$ is an anti-unitary operator, satisfying the following condition:

$$
\begin{align*}
L_n U_{NS} &= U_{NS} L_n \\
U_{NS} S_r &= -i S_r U_{NS} (-)^F \\
U_{NS} \mid h \rangle &= \mid h \rangle
\end{align*}
$$

(21)

From (17) one can derive analytical expression for $U_{NS}$

$$
U_{NS} \mid h, N \rangle = \frac{1 - i}{2} (1 + i(-)^F) \mid h, N \rangle
$$

(22)

Let us show, that (20) satisfies to conditions (17). For this purpose we show, that for any $\mid i \rangle \otimes \langle j \mid$, where $\langle i \mid \otimes \langle j \mid$ basic vector, $\langle i \mid \otimes \langle j \mid (S_r + i S_{-r}) \mid h \rangle = 0$ and $\langle i \mid \otimes \langle j \mid (L_n - L_{-n}) \mid h \rangle = 0$.

Indeed, using antiunitarity of $U_{NS}$ and condition (21) we have:

$$
\begin{align*}
\sum_n \langle i \mid \otimes \langle j \mid (S_r + i S_{-r}) \mid n \rangle \otimes U \mid n \rangle &= 0 \\
\sum_n \langle j \mid S_r \mid n \rangle \langle i \mid U n \rangle + i \sum_n \langle i \mid S_{-r} \mid U n \rangle \langle j \mid n \rangle (-)^F &= 0 \\
\sum_n \langle j \mid S_r \mid n \rangle \langle n \mid i U^+ \rangle - \langle i \mid U S_{-r} (-)^F \rangle \langle j \mid (-)^F &= 0 \\
\langle j \mid S_r \mid U^+ i \rangle - \langle j \mid S_r \mid U^+ i \rangle &= 0
\end{align*}
$$

(23)
By the same way we can show, that the first equation in (17) is also satisfied.
It is more interesting to study Ramond sector. At the beginning let us consider the case $h \neq \frac{c}{16}$. There are two ground states in the theory $| h^+ \rangle$ and $| h^- \rangle$. As in the case above let us use the same ansatz (20) to solve (17),

$$| h^\pm \rangle_B = \sum_{|N\rangle} | h^\pm, N \rangle \otimes U_R | h^\pm, N \rangle$$  \hspace{1cm} (24)

where $U_R$ is anti-unitary operator, satisfying to condition:

$$L_n U_R = U_R L_n$$
$$U_R S_r = -i S_r U_R (-)^F$$  \hspace{1cm} (25)

Since the ground state is now non–trivial, we have freedom in a definition of the action $U_R$ on this space. And we have the only one restriction on $U_R$:

$$(U_R S_0 + i S_0 U_R (-)^F) | h^\pm \rangle = 0$$  \hspace{1cm} (26)

In representation, where

$$| h^+ \rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } | h^- \rangle = \begin{pmatrix} 0 \\ \sqrt{h - \frac{c}{16}} \end{pmatrix}$$

$S_0$ and $(-)^F$ can be represented as

$$S_0 = \sqrt{h - \frac{c}{16}} \sigma_x; \quad (-)^F = \sigma_z$$  \hspace{1cm} (27)

where $\sigma_x$ and $\sigma_z$ are Pauli matrixes. Using (26) and representation (27), we get:

$$U_R = \begin{pmatrix} a & -ic \\ c & -ia \end{pmatrix}$$  \hspace{1cm} (28)

where $a$ and $c$ satisfy anti-unitary condition: $aa^* + cc^* = 1$ and $ac^* + a^*c = 0$. Thus, we get, that in opposite to Neveu-Schwarz sector in Ramond sector $U_R$ is not determined uniqely. It is interesting to note, that if $h = \frac{c}{16}$, the uniqueness of $U_R$ is recovered.

The partition function (14) of the theory defined on compactified cylinder can be expressed as a linear combination of characters since instead of holomorphic and atiholomorphic algebras (in the bulk) now there is just one algebra:

$$Z^{NS}_{\alpha\beta} = \sum n^h_{\alpha\beta} \chi^{NS}_h(q)$$  \hspace{1cm} (29)
\[ \chi^{NS}(q) = q^{-c/24} \text{Tr} q^{L_0} \]

is the character of the superconformal algebras in NS-sector. By non-negative integer \( n_{\alpha \beta}^h \) denoted the number of times that representation \( h \) occurs in the spectrum of \( H_{\alpha \beta}^{open} \). The character formulas for the NS and R algebra have been derived by Goddard, Kent and Olive [8] and by Kac and Wakimoto [9] and under the modular transformation \( \tau \to -1/\tau \) the character for the fermionic model transform linearly [10],

\[ \chi^h_{NS}(q) = \sum S^h_{h'} \chi^h_{NS}(\tilde{q}) \]  

which leads to

\[ Z_{\alpha \beta}^{NS} = \sum n_{\alpha \beta}^h S^h_{h'} \chi^h_{NS}(\tilde{q}) \]  

where \( \tilde{q} = e^{-2\pi i/\tau} \). In order to have complete set of boundary states defined by equation (20), we have to consider diagonal bulk theory. Following to Kastor [10] there are different superconformal theories corresponding to different modular invariant combination of characters

\[ Z_{NS,R} = \sum_{nm,kl} F_{nm,kl} N_{nm,kl} \chi_{nm}(q) \bar{\chi}_{kl}(\bar{q}) \]  

here the factor \( F \) is equal to 2 for the nonsupersymmetric R highest weight states, which one twofold degenerated, and is equal to 1 otherwise. \( N_{nm,kl} \) is the number of highest weight states \((h_{nm}, \tilde{h}_{kl})\) in the theory which one obeys to following sum rule for NS,

\[ \sum N_{nm,kl} \sin \pi n \frac{m'}{p} \sin \pi m \frac{m'}{p+2} \sin \pi k \frac{k'}{p} \sin \pi l \frac{l'}{p+2} = \frac{1}{16} p(p+2) N_{n'm',k'l'} \]  

There are at least two series of solutions to the above sum rule. One of these the diagonal (or scalar) solution of the superconformal sum rule is given by \( N_{nm,kl} = \delta_{nk} \delta_{ml} \) in NS sector. So, if there are complete set in the space of boundary states we can therefore write

\[ \langle \alpha | \beta \rangle = \langle \alpha | h \rangle \langle h | \beta \rangle \]  

Using these representations we can rewrite (15)

\[ Z_{\alpha \beta}^{NS} = \sum \langle \alpha | h \rangle \langle h | \beta \rangle \chi^h_{NS}(\tilde{q}) \]  

8
and comparing with the (31) we get

$$\sum_{h'} S_{h'}^{h} n_{\alpha \beta} = \langle \alpha | h \rangle \langle h | \beta \rangle$$

(36)

This equation for fermionic model is the same which one found Cardy [6] for conformal theory. As result of this equation boundary states $| \tilde{h} \rangle$ can be read

$$| \tilde{h} \rangle = \sum_{h'} \frac{S_{h'}^{h}}{(S_{h'}^{h})^{1/2}} | h' \rangle$$

(37)

These states have property that $n_{0 \tilde{h}}^{h} = \delta_{h'}^{h}$ which means the representation $h'$ appears in the spectrum of $H_{0\tilde{h}'}$.

4. One and three point boundary correlation functions.

In the superstring theories we generally are interested in calculation of scattering amplitudes with both open and closed strings in the initial and final states. A string diagram with external open and closed string can be conformally mapped to the upper half plane. After this mapping the external open string are represented by vertex operators at finite points on the boundary, while the closed strings are represented by vertex operators at finite points on the upper half plane. All of this means that for construction open and closed superstring theories we are really interested in superconformal field theory with boundary (SCFT on half plane). One of the interesting question is how in the intermediate channel of string diagram (with external open and closed strings) closed string vertex can be expressed by open string vertex operators with given type of boundary condition. In a superconformal field theory (in which the boundary conditions do not break the superconformal symmetry) this can be represented as short distance expansion of bulk vertex operators near a boundary. If we will write full bulk Neveu-Schwarz superfield,

$$\Phi(\hat{z}, \bar{z}) = \phi(z, \bar{z}) + \theta \Psi(z, \bar{z}) + \bar{\theta} \bar{\Psi}(z, \bar{z}) + \theta \bar{\theta} F(z, \bar{z})$$

(38)

where

$$\Psi(z, \bar{z}) = S_{-1/2} \phi(z, \bar{z});$$
$$\bar{\Psi}(z, \bar{z}) = \bar{S}_{-1/2} \phi(z, \bar{z});$$
$$F(z, \bar{z}) = S_{-1/2} \bar{S}_{-1/2} \phi(z, \bar{z})$$

(39)
then we can write short distance expansion for $\phi(z, \bar{z})$ near boundary as

$$
\phi(z, \bar{z}) = \sum_i (z - \bar{z})^{\Delta_{\phi_i} - \Delta_\phi} C^{B}_{\phi B}[\phi^B_i(x)] 
$$

$$
\Psi(z, \bar{z}) = \sum_r (z - \bar{z})^{\Delta_{\Psi_r} - \Delta_\Psi} C^{B}_{\Psi B}[\Psi^B_r(x)] 
$$

here $[\phi^B(x)], [\Psi_B(x)]$–are conformal class of $\phi^B, \Psi^B$ boundary vertex operators and $C^{B}_{\phi B}, C^{B}_{\Psi B}$ –are boundary structure constants of theory. Now we are interested to obtain these boundary structure constants. First of all let’s note that for identity boundary operator corresponding structure constant is equal to constant factor of one point boundary correlation function. One point boundary correlation (with boundary condition labelled by B) of NS superfield with corresponding to superconformal invariance and boundary Ward identity can be written

$$
\langle \Phi(\hat{z}, \bar{z}) \rangle_B = \frac{A^B_{\Phi}}{(z - \bar{z} - \theta \bar{\theta})^{\Delta}} 
$$

where $A^B_{\Phi} = C^B_{\Phi I}$. For components of superfield we can obtain the relations

$$
\langle \phi(z, \bar{z}) \rangle_B = \frac{A^B_{\phi}}{(z - \bar{z})^{\Delta}}; \quad \langle F(z, \bar{z}) \rangle_B = \frac{\Delta A^B_{\phi}}{(z - \bar{z})^{\Delta + 1}} 
$$

$$
\langle \Psi(z, \bar{z}) \rangle_B = 0; \quad \langle \bar{\Psi}(z, \bar{z}) \rangle_B = 0 
$$

Thus, according to the simple definition of $A^B_{\phi}$ [6],

$$
A^B_{\phi} = \frac{\langle \phi | B \rangle}{\langle 0 | B \rangle} 
$$

and using the superconformal physical boundary states [37] we find

$$
A^h_{\phi} = \frac{(S^0_0)^{1/2}}{S^0_h} \frac{S^\phi_h}{(S^\phi_0)^{1/2}} 
$$

To determine the boundary structure constants $C^{B}_{\phi \phi B}, C^{B}_{\Psi \Psi B}$ we need some dynamical principle. Associativity of the boundary operator algebra imposes...
global constraints on correlation function as usual. For this, let’s consider 2-point functions,

\[ \langle \phi_i(z_1, \bar{z}_1) \phi_j(z_2, \bar{z}_2) \rangle_B; \quad \langle \Psi_r(z_1, \bar{z}_1) \Psi_\sigma(z_2, \bar{z}_2) \rangle_B \] (47)
in two channels. We can evaluate these correlation functions using bulk OPE taking \( z_1 \to z_2, \bar{z}_1 \to \bar{z}_2 \) and can alternatively be evaluated using boundary OPE by taking \( z_1 \to \bar{z}_1, z_2 \to \bar{z}_2 \). Associativity of the operator algebra implies that correlation function of these two channels should give the same result (crossing symmetry),

\[ \sum_k C^B_{\phi_i \phi_k} C^B_{\phi_j \phi_k} F^k_{ij}(1 - \eta) = \sum_m C_{ijm} A^B_{\phi_m} F^m_{ij}(\eta) \] (48)

\[ \sum_{\rho} C^B_{\Psi_r \Psi_\rho} C^B_{\Psi_\sigma \Psi_\rho} F^\rho_{r\sigma}(1 - \eta) = \sum_m C_{r\sigma m} A^B_{\phi_m} F^m_{r\sigma}(\eta) \] (49)

here \( \eta = |z_1 - z_2|^2 / |z_1 - \bar{z}_2|^2 \) is cross-ratios, \( F^k_{ij}(\eta) \), \( C_{ijm} \) are conformal blocks and bulk structure constants respectively. The conformal blocks are solutions of differential equations. According to different basis of differential equations the solutions are expressed by each other linearly [11],

\[ F^k_{ij}(\eta) = \sum \alpha^{k,pl}_{ij,m} F^m_{pl}(1 - \eta) \] (50)

Inserting to the equation (48-49) we get

\[ C^B_{\phi_i \phi_k} C^B_{\phi_j \phi_k} = \sum_m C_{ijm} A^B_{\phi_m} \alpha^{k,ij}_{ij,m} \] (51)

\[ C^B_{\Psi_r \Psi_\rho} C^B_{\Psi_\sigma \Psi_\rho} = \sum_m C_{r\sigma m} A^B_{\phi_m} \alpha^{\rho,r\sigma}_{r\sigma,m} \] (52)

In second "bootstrap" equation we see that right hand side is nonzero due to \( A^B_\phi \). So, all boundary structure constants are expressed via well known bulk quantities. Finally, we note that an examination for "spin model" will be given elsewhere.

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