Canonical Quantization of Two Dimensional Gauge Fields

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Abstract

$SU(N)$ gauge fields on a cylindrical spacetime are canonically quantized via two routes revealing almost equivalent but different quantizations. After removing all continuous gauge degrees of freedom, the canonical coordinate $A_\mu$ (in the Cartan subalgebra $\mathfrak{h}$) is quantized. The compact route, as in lattice gauge theory, quantizes the Wilson loop $W$, projecting out gauge invariant wavefunctions on the group manifold $G$. After a Casimir energy related to the curvature of $SU(N)$ is added to the compact spectrum, it is seen to be a subset of the non-compact spectrum. States of the two quantizations with corresponding energy are shifted relative each other, such that the ground state on $G$, $\chi_0(W)$, is the first excited state $\Psi_1(A_\mu)$ on $\mathfrak{h}$. The ground state $\Psi_0(A_\mu)$ does not appear in the character spectrum as its lift is not globally defined on $G$. Implications for lattice gauge theory and the sum over maps representation of two dimensional QCD are discussed.
I. Introduction

Gauge theories are by nature overdetermined systems in which many different field configurations are in fact physically equivalent, thus the primary issue to be addressed in their quantization is the treatment of the field’s excessive degrees of freedom and the identification of the equivalence classes under gauge transformations.

Discretely formulating a gauge theory on the lattice as done by Wilson [1], introduces a novel solution to the gauge fixing problem, that is to say, it becomes a finite problem and can be addressed implicitly at the expense of exploring many extra dimensions in phase space. In the continuum the set of all allowed gauge transformations is an infinite dimensional space, the volume of which is delicately factored out of the partition function by the Fadeev-Popov Jacobian. On the lattice, the volume of gauge space is finite, and gauge degrees of freedom can be left in the path integral measure. They are divided out by the normalization when only gauge invariant quantities are computed.

The Wilson formulation of lattice gauge theory of course alters the form of the theory in a further very significant way; not only does it make the configuration space finite dimensional, but also compact. The (non-compact) gauge field $A_\mu$ becomes the (compact) link $U_\mu$,

\[ A_\mu(x) \in \mathfrak{g} \rightarrow U_\mu(x) = \exp\{i \int_x^{x+a} dy \cdot A_\mu(y)\} \in G \] (1.1)
where $G$ is the gauge group, and $\mathfrak{g}$ its Lie algebra. In the classical continuum limit as the lattice spacing $a$ goes to zero, this distinction should disappear since the group manifold $G$ looks more and more like its tangent space $\mathfrak{g}$ as the link $U_\mu$ is limited to a small region near the identity. As we shall discover however, the topological distinction remains for quantities which depend on the global structure of phase space such as the zero modes of the Hamiltonian.

We demonstrate this in two spacetime dimensions since there we can completely eliminate the gauge degrees of freedom and explicitly solve the model in full, a luxury not available in higher dimensions. The results may be applicable to compact theories in higher dimensions, since they depend only on the relation between differential operators under the compactification of the Lie algebra $\mathfrak{g}$ to $G$ via the exponential map, however the Hamiltonian in a higher dimensional theory is much more complicated, as is the gauge orbit structure. Furthermore the physical degrees of freedom are not all radial coordinates of $\mathfrak{g}$ or $G$ in higher dimensions, an aspect which is crucial here.

We will use the canonical formalism, since our focus is the Hamiltonian as a differential operator. It is well known that planar Yang-Mill is trivial, since the gauge fields can be transformed to zero throughout spacetime. Another reason planar Yang-Mills is trivial is that two dimensional Yang-Mills (2DYM) is a form of topological field theory [2,4,5,6,7], whose excitations depend only on the topology of the underlying spacetime. Thus we use a cylindrical spacetime, the only topologically non-trivial two dimensional manifold, with canonical time.

Investigations of gauge fields on arbitrary Riemann surfaces have begun in earnest and various facets were recently collected in two very probing works by Witten [2,3]. The first of these papers examines 2DYM in several ways: from the explicit lattice formulation, as the limiting cases of Chern-Simons and conformal field theories, and its relation to the theory of Reidmeister torsion. In the second paper, Witten re-examines 2DYM by generalizing the Duistermaat-Heckman integration formula [8] to a non-Abelian form, yielding
the Yang-Mills partition function as a polynomial in the coupling constant, ie. as a sum over critical points of phase space using the action as a Morse functional.

The focus of the present paper is a minute difference between these papers already addressed in [3], namely that the spectrum in [2], obtained in the lattice formulation of 2DYM is half the quadratic Casimir $C_2/2$ of $SU(N)$, whereas in [3] it is $C_2/2 + t$, where $t$ is a “lower order Casimir” to be determined by the regularization of the theory and its connection to the equivalent topological $BF$ theory.

This difference in spectra was previously seen in two canonical continuum solutions of $SU(N)$ on a cylinder, by Rajeev [9] where the quantization is done on the group manifold, and by Hosotani and the author [10], where the quantization is done in the algebra. We will see that the difference corresponds exactly to Witten’s $t$ parameter, and is related to the mapping of the Hamiltonian as a differential operator from the algebra to the group, via the exponential map. More interesting than this constant rescaling of vacuum energy, is a corresponding shift of states between the two quantizations such that the ground states are different. Thus we arrive at two inequivalent quantizations of the theory, whose excitations are in one-to-one correspondence, except for an extra set of lower energy states which appear in the non-compact quantization.

Recently Gross and Taylor [11] and Minahan [12], have shown that the partition function of 2DYM (as $N \to \infty$) is given by the sum over maps from compact genus $g$ worldsheets to the two dimensional target spacetime, renewing interest in the $1/N$ and string representations of QCD [13, 14]. An interesting feature of this expansion of the partition function, relating representation theory of Lie groups to the classification of maps of surfaces, is that degenerate maps in which the worldsheet is mapped to a single point of the target space are, somewhat mysteriously, absent.

In the canonical approach the nature of the excitations as Fourier modes on the maximal torus of $G$ is made clear and we find that the two different quantizations are essentially even and odd choices for these modes. We will see that compact quantization
naturally picks out the odd modes, precluding a constant wavefunction on the group algebra. We make conjectures about the relationship between this state and the zero winding maps, and about the general analogies between the canonical Fourier states and smooth maps in the conclusions.

The paper is organized as follows: in the next section we examine the general features of the gauge orbit and phase space of the theory, identifying explicitly the configuration space and its topology. We find that the residual Gribov copies play a crucial role in creating this topology [15,16]. We then study the theory canonically in the diagonal-Coulomb gauge, computing the Hamiltonian, and quantizing the theory in the group algebra.

We review the lattice method of quantization on the group manifold, and compare the results. This comparison leads us to the relationship between Laplacians on group manifolds and their algebræ. With this relationship in hand we examine the mapping of states in the two quantizations and see that the curvature of $G$ induces a shift between corresponding states. Finally we discuss the implications of inequivalent quantizations of this theory.

II. The Structure of Phase Space

Before gauge fixing and quantizing the specific model, we review the classical phase space structure of Yang-Mills fields momentarily to put various features in context [17].

From the Yang-Mills Lagrangian $\mathcal{L} = \frac{1}{2} \text{Tr} F_{\mu\nu} F^{\mu\nu}$ we have the canonical momenta to $A_\mu^a$

$$\frac{\delta \mathcal{L}}{\delta \partial_0 A_\mu} \equiv \Pi_\mu^a = F_{0\mu}^a. \quad (2.1)$$

In the unreduced phase space $\Gamma$, $\Pi_0^a$ vanishes identically providing $N^2 - 1$ functional constraints

$$\phi_p [A_\mu^a(x), \Pi_\mu^a(x)] = \Pi_0^a = 0 \quad (2.2)$$
and defining the primary constraint surface $\Gamma_c$. Further reductions of this surface come from the secondary constraints related to the gauge symmetries.

The Hamiltonian (setting $\Pi_0 = 0$ explicitly) is

$$H = \text{Tr} \int dx \left[ \Pi^1 \dot{A}_1 - \frac{1}{2} \Pi^1 \Pi_1 \right]. \quad (2.3)$$

From $\Pi_1 \equiv E = \dot{A}_1 - \partial_1 A_0 + ig[A_0, A_1]$, we have the Hamiltonian in terms of the coordinates $(A_\mu^a(x), \Pi_\mu^a(x))$ on $\Gamma_c$

$$H = \text{Tr} \int dx \left[ \frac{1}{2} E^2 - A_0 D \cdot E \right] \quad (2.4)$$

which shows $A_0^a$ to be a non-dynamical Lagrange multiplier for Gauss’ law

$$D \cdot E = \partial_1 E^1 + ig[A_1, E^1] = \phi_s^{(a)} \approx 0, \quad (2.5)$$

which are secondary constraints stemming from the fact that the primary constraints above must be time independent, $\{\Pi_0, H\} = D \cdot E \approx 0$. We now write the extended Hamiltonian, including the constraints and their multipliers,

$$H_{ext} = H + \text{Tr} \int dx \left[ \omega_p(x) \phi_p + \omega_s(x) \phi_s(x) \right] \quad (2.6)$$

$$= \text{Tr} \int dx \left[ \frac{1}{2} E^2 + \omega_p \Pi_0 + (\omega_s - A_0) D \cdot E \right].$$

Further we see that $\phi_p$ and $\phi_s$ generate infinitesimal gauge transformations of the coordinates $A_\mu$ on the surface of constraint $\Gamma_c$, since

$$\delta_\Omega A_\mu^a(x) = \int d^2y \left[ \omega_p^b(y) \{ A_\mu^a(x), \Pi_0^b(y) \} + \omega_s^b(y) \{ A_\mu^a(x), D_1 \Pi_1^b(y) \} \right] \quad (2.7)$$

$$= D_\mu \Omega^a(x)$$

where $\omega_p^a(x) = D_0 \Omega^a(x)$; $D_1 \omega_s^a(x) = -D_1 \Omega^a(x)$. Thus starting from a single configuration $(A_\mu, \Pi_\mu)$ on $\Gamma_c$, and evolving in time under two Hamiltonians $H_{ext}(\omega')$ and $H_{ext}(\omega'')$, we arrive at two different configurations $(A'_\mu, \Pi'_\mu)$ and $(A''_\mu, \Pi''_\mu)$, which are equivalent up to a gauge transformation.
We thus have a fibration of $\Gamma_c$ by the gauge orbits, i.e. points which are related to each other by gauge transformations

$$A'_\mu = \Omega A_\mu \Omega^\dagger - \frac{i}{g} \Omega \partial_\mu \Omega^\dagger \quad (2.8)$$

which are identified. The equivalence classes of these points (the different orbits), classify the reduced phase space $\Gamma_r$ which are the true independent degrees of freedom which we wish to quantize.

Usually phase space is the cotangent bundle $T^*Q$, of the configuration manifold $Q$, so that identifying $Q$, $Q_c$, and the reduced space $Q_r$ is the essential matter and with luck, $T^*Q_r$ just comes along for the ride. In our case $Q_r$ turns out to be an orbifold so that $T^*Q_r$ is not defined at every point, however we can quantize on the smooth manifold covering $Q_r$ and implement the orbifold identifications in the Hilbert space of the quantum theory.

The structure of $Q_c$ in a gauge theory is that of a fiber bundle

$$\mathcal{G} \rightarrow Q_c \xrightarrow{\pi} Q_r \quad (2.9)$$

where $Q_c$ is the space of connections of an $SU(N)$ bundle over the spatial manifold $X$, $\mathcal{G}$ is the group of allowed gauge transformations on $X$, and $Q_r$ is the set of equivalence classes of points in $Q_c$ under the action of $\mathcal{G}$. In the two dimensional model, $Q_c$ is labeled by the fields $A_1(x)$ which are maps of the circle $S^1$ into the Lie algebra $g$ of $G$

$$Q_c = Lg = \{A_1 : S^1 \rightarrow g\} \quad (2.10)$$

As such they are coordinates of a well known manifold $Lg$, the Lie algebra of the loop group $LG$ [18]. Furthermore, $\mathcal{G}$, the space of gauge transformations of $A_1(x)$, is isomorphic to the loop group $LG$ itself,

$$\mathcal{G} = LG = \{\Omega(x) : S^1 \rightarrow G\} \quad (2.11)$$
To classify the orbits of $LG$ in the space $Lg$, ie. the physical configuration space $Q_r$, consider the following representation of $Lg$ in which each element is almost “pure gauge”. Let $0 < x \leq 2\pi$ be the coordinate on $S^1$. To each element $A \in Lg$, there is an associated $G$-valued function $f : \mathbb{R} \to G$ which is the integral curve of the vector field $A(x)$ on $G$, satisfying

$$i\partial_x ff^\dagger = A.$$  \hfill (2.12)

Take the boundary condition $f(0) = 1$. Since $A(x + 2\pi) = A(x)$, $f$ is periodic up to a constant element of $G$

$$f(x + 2\pi) = f(x)W_A.$$  \hfill (2.13)

Under the action of $\Omega(x) \in LG$, $A$ transforms to

$$\tilde{A} = \Omega \cdot A = \Omega A\Omega^\dagger + i\partial_x \Omega \Omega^\dagger$$  \hfill (2.14)

hence $f$ goes into $\tilde{f}$

$$\tilde{f}(x) = \Omega(x)f(x)\Omega^\dagger(0).$$  \hfill (2.15)

Notice the trailing factor of $\Omega^\dagger(0)$ necessary to maintain $\tilde{f}(0) = 1$. The quasi-periodicity (2.13) of $f$, with $\Omega(2\pi) = \Omega(0)$ implies that

$$W_A \to \tilde{W}_A = \Omega(0)W_A\Omega^\dagger(0).$$  \hfill (2.16)

We thus have a homomorphism between $Lg$ and the space of maps $\{f : f(x+2\pi) = f(x)W\}$ for some $W \in G$, and can label every $A \in Lg$ by $(f_A(x), W_A)$.

Now for arbitrary $f(x)$ and $\tilde{f}(x)$ we can find a gauge transformation

$$\gamma = \tilde{f}\gamma(0)f^\dagger$$  \hfill (2.17)

in $LG$ (note that $\gamma(0)$ remains undetermined), such that

$$\gamma \cdot (f(x), W) \to (\tilde{f}(x), \tilde{W})$$  \hfill (2.18)
only when \( \widetilde{W} \) and \( W \) lie in the same conjugacy class of \( G \); then there will be a \( \gamma(0) \) which can take \( W \) into \( \widetilde{W} \). Thus, the manifold of physically distinct configurations, the orbits of \( LG \) in \( Lg \), is isomorphic to the space of conjugacy classes of \( G \) [9,10,16,19].

This space is an orbifold made by identifying points in the maximal Abelian subgroup of \( G \) (the maximal torus \( T_G \)), under the action of the Weyl group \( W_G \), a discrete set of transformations which permute the diagonal elements. These correspond to reflections in the hyperplanes through the origin and perpendicular to the roots of \( G \), hence the Weyl group is isomorphic to the permutation group of \( N \) elements, \( S_N \). Pure Yang-Mills theory is furthermore invariant under constant gauge transformations which lie in the center of the gauge group, hence the gauge symmetry is actually \( SU(N)/\mathbb{Z}_N \). The effect of the \( \mathbb{Z}_N \) symmetry on \( T_G \) is rather mild and simply changes its periodicity to \( 2\pi/N \) as opposed to \( 2\pi \) in an \( SU(N) \) theory. Therefore we have an exact identification of the topology of the gauge orbit space of \( SU(N) \) Yang-Mills theory on a cylinder as the orbifold \( T_G/S_N \sim T^{N-1}/S_N \).

### III. Quantization

#### A. Non-compact formulation

In this section we quantize 2DYM on the group algebra \( g \) where the theory is originally defined in terms of \( A_\mu \). For concreteness we specify the model as given by the Lagrangian and gauge fields defined on a cylindrical spacetime of circumference \( L \) as follows:

\[
\mathcal{L} = -\frac{1}{2} \text{Tr} F_{\mu\nu} F^{\mu\nu}
\]

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]
\]

\[
A_\mu(x, t) = A_\mu(x + L, t) = A_\mu^a \lambda_a; \quad \text{su}(N) = \text{Span}\{\lambda_a\}
\]

Note that the periodic boundary condition for the gauge field may be taken without loss of generality on a cylinder since the \( SU(N) \) bundle over \( S^1 \) is trivial [20]. Under gauge transformations (2.8), periodicity remains intact provided the gauge transformation satisfies the
condition
\[ \Omega(x + L, t) = \mathbb{Z}_n \Omega(x, t) \]  
(3.2)

where \( \mathbb{Z}_n \) is any element in the center of \( SU(N) \).

As mentioned above we use the diagonal-Coulomb gauge \([10]\) defined as
\[ \partial_1 A_1 = 0 \]
\[ A_1(t)_{ij} = \frac{1}{gL} \theta_i(t) \delta_{ij} \]  
(3.3)

which provides coordinates directly on the torus covering the reduced configuration space \( T^{N-1}/S_N \). It is easy to see that the gauge transformation
\[ \Omega(x, t) = \Lambda(t) \mathcal{W}(x, t) V^\dagger(x, t) \]  
(3.4)

where
\[ V(x, t) = \mathcal{P} \exp \left(-ig \int_0^x dy A_1(y, t) \right) \]
\[ \mathcal{W}(x, t) = \exp[-iB(t)/L], \quad \exp[-iB(t)] = V(L, t) \]  
(3.5)

\( \Lambda(t) \in G : ABA^\dagger = \begin{pmatrix} \theta_1(t) & & \\ & \ddots & \\ & & \theta_N(t) \end{pmatrix} \equiv \Theta(t), \quad \sum_{i=1}^N \theta_i(t) = 0 \) takes \( A_1(x, t) \) into \( (1/gL) \Theta(t) \). We use here a Weyl parameterization of the Cartan subalgebra \( \mathfrak{h} \) as it illuminates the toroidal structure in a simple fashion. We will turn to other bases later. In terms of the previous loop group decomposition in eq. (2.17), \( V(x, t) = f_{A_1}, \mathcal{W}(x, t) = f_{B}, \) and \( \Lambda = \gamma(0) \):
\[ \Omega = \Lambda f_B f_{A_1 \dagger} = \Lambda f_B A^\dagger f_{A_1 \dagger} = f_{\Theta(0)} f_{A_1 \dagger} = \gamma \]  
(3.6)

Although the diagonal-Coulomb gauge eliminates the continuous gauge degrees of freedom, there still remain a set of discrete transformations. These are implemented by gauge functions satisfying the boundary conditions of eq. (3.2):
\[ \Omega(t, x)_{jk} = \delta_{jk} \exp \left\{ i \left( \omega_j(t) + \frac{2\pi n_j x}{L} + \frac{2\pi m x}{L} \right) \right\} \]
\[ \sum_{j=1}^N \omega_j(t) = 0 \pmod{2\pi}; \quad n_j \in \mathbb{Z}, \quad \sum_{j=1}^N n_j = 0 \pmod{2\pi}, \]  
(3.7)

\[ m = \frac{\ell}{N}, \quad (\ell = 1, \cdots, N-1), \]
and the $N! - 1$ constant gauge transformations of the form

$$\Omega = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ & & \ddots \\ & & & 1 \end{pmatrix}. \quad (3.8)$$

which with the identity form the Weyl group of reflections $S$. Under (3.7)

$$\theta_j(t) \rightarrow \theta_j(t) + 2\pi n_j + 2\pi m, \quad (3.9)$$

while under (3.8)

$$\theta_i(t) \leftrightarrow \theta_j(t). \quad (3.10)$$

The $n_i$ term of the transformation (3.9) of represents the $SU(N)$ periodicity of the maximal torus while the second term reflects the $\mathbb{Z}_N$ symmetry. These residual (Gribov) gauge transformations make the identifications needed to compactify the configuration space $Θ (\sim \mathbb{R}^{N-1})$ into the maximal torus of $SU(N)$ and thus are crucial in constructing the correct topology of $Q_r$ [15,16].

The transformations (3.10) produce the identifications which give the configuration space it’s orbifold structure, with the orbifold singularities occuring on hyperplanes where $\theta_i = \theta_j$ for any $i, j$. The freedom represented by the $\omega_j(t)$’s is used to eliminate the $A_0$ components to which we now turn.

The Euler-Lagrange equations of motion for this model are

$$\partial_0 E + ig[A_0, E] = 0$$
$$\partial_1 E + ig[A_1, E] = 0. \quad (3.11)$$

Since $A_0$ is periodic in $x$

$$A_0(t, x)_{ij} = \sum_n a_n(t)_{ij} e^{2\pi inx/L}$$
$$E(t, x)_{ij} = \delta_{ij} \frac{1}{gL} \dot{\theta}_j - \frac{i}{L} \sum_n (2\pi n + \theta_i - \theta_j) a_n(t)_{ij} e^{2\pi inx/L}. \quad (3.12)$$
Due to the second equation of (3.11) and the gauge condition (3.3)

$$D^2_1 E = D^2_1 (\Theta) a_{n.ij} = (2\pi n + \theta_i - \theta_j)^2 a_n(t)_{ij} = 0.$$ (3.13)

Thus for general $\theta_i$'s, (3.13) shows that $a_n(t)_{ij} = 0$ ($i \neq j$), and the diagonal components can be gauged away by choosing the $\omega_i(t)$'s in (3.7) appropriately. In general some of $a_n(t)_{ij}$ ($i \neq j$) may be nonvanishing when $A_1$ lies on an orbifold singularity in the configuration space. However eq. (3.13) shows that these isolated configurations do not contribute to $E(t,x)$. In other words, all configurations are gauge equivalent to $A_0 = 0$. Path integral considerations leading to the same form were given in [10]. See also [21].

In this form the theory is readily quantized since the $\theta_i$'s are simply the coordinates of a point particle on the maximal torus and the whole model reduces to a quantum mechanics problem, albeit up to issues about quantization on orbifolds. In terms of the coordinates on $T_{SU(N)}$, the Lagrangian is

$$L = \int dx L = \frac{1}{g^2 L} \sum_{k=1}^{N} \dot{\theta}_k^2 = \frac{1}{g^2 L} \left( \sum_{k=1}^{N-1} \dot{\theta}_k^2 + (\sum_{k=1}^{N-1} \dot{\theta}_k)^2 \right).$$ (3.14)

Conjugate momenta to the $\theta_i$ are

$$\pi_i = \frac{2}{g^2 L} \left( \dot{\theta}_i + \sum_{j=1}^{N-1} \dot{\theta}_j \right) \equiv -i \frac{\partial}{\partial \theta_i}$$ (3.15)

yielding the Hamiltonian

$$H = -\frac{g^2 L}{4} \left\{ \sum_{j=1}^{N-1} \frac{\partial^2}{\partial \theta_j^2} - \frac{1}{N} \left( \sum_{j=1}^{N-1} \frac{\partial}{\partial \theta_j} \right) \right\}.$$ (3.16)

As shown above the configuration space of this model is an orbifold. In such cases one usually quantizes on the simplest covering space of the orbifold, demanding strict invariance of the wavefunction under the discrete symmetries [22]. In our case the wavefunction $\Psi(\theta_1, \ldots, \theta_{N-1})$ will be a function on $T_{SU(N)}$, periodic in each $\theta_i$ with period $2\pi$, and symmetric under the Weyl reflections eq. (3.8). The wave function acquires a
phase $e^{i2\pi/N}$ to some power under a $\mathbb{Z}_N$ transformation as is customary. This choice of periodicity makes the correspondence with wavefunctions on the group manifold much more straightforward, as well as simplifying the changes when fermions are added. Readers whose needs require periodic wavefunctions under $\mathbb{Z}_N$ can easily restrict the spectrum to the $N$th excitations, just as restricting to integer $j$ states does for $SU(2) \to SO(3)$ representations.

In general we can construct wavefunctions from properly symmetrized sums of phases,

$$\Psi_{\{n\}}(\theta) = \sum_{\{n\}} c(n) e^{i(n_1 \theta_1 + \cdots + n_{N-1} \theta_{N-1})}$$

where $c(n) = c(n_1, \cdots, n_{N-1})$ and the $n$’s are some integers. Since the wavefunction must be symmetric under the interchange of $\theta_j$ and $\theta_k$ ($j, k = 1, \cdots, N$), the $c(n)$’s must satisfy

$$c(n_1, n_2, \ldots, n_{N-1}) = c(n_1, \ldots, n_j \leftrightarrow n_k, \ldots, n_{N-1}) ,$$

$$c(n) = c(m), \quad n_k = \begin{cases} m_k - m_j, & \text{for } k \neq j \\ -m_j, & \text{for } k = j \end{cases}. \quad (3.18)$$

Wavefunctions which satisfy (3.18) are linear combinations of

$$\Psi_n(\theta) = \sum_{\theta} e^{in_1 \bar{\theta}_1 + \cdots + in_{N-1} \bar{\theta}_{N-1}}. \quad (3.19)$$

Here $(\bar{\theta}_1, \cdots, \bar{\theta}_{N-1})$ are $N - 1$ representatives made out of $\theta_k$ ($k = 1, \cdots, N$) where $\theta_N = -\sum_{j=1}^{N-1} \theta_j$. The summation over $\bar{\theta}_j$ extends over all symmetric combinations in the indices $k(=1, \cdots, N)$ of $\theta_k$. The energy spectrum is given by a set of integers $\{n_1, \cdots, n_{N-1}\}$:

$$E_{\{n\}} = \frac{g^2 L}{4} \left\{ \sum_{j} n_j^2 - \frac{1}{N} \left( \sum_{j} n_j \right)^2 \right\}. \quad (3.20)$$

The form of the wavefunction in (3.17) is rather specific to the representation of $\Theta(t)$ in this Weyl basis, but we see that the exponentials appearing in the sum (3.17) are simply Fourier modes a complex function on the lattice defining $T^{N-1}$. To extract a wavefunction for a general basis, note that this lattice is defined by

$$\Gamma(n_1, n_2, \ldots, n_{N-1}) = \prod_{i=1}^{N-1} z_i^{n_i} \quad (3.21)$$
where \( n_i \in \mathbb{Z} \), \( z_i = e^{i h_i} \) and the \( h_i \) are the (suitably normalized) generators of the Cartan subalgebra in any basis. \( \Gamma \) is an element of the “twist matrix” which labels the possible t’Hooft fluxes [23,24,25] of a non-Abelian gauge field on a spatial torus with boundary conditions \( A_\mu(x + L_\nu) = U_\nu A_\mu(x) U_\nu^\dagger \) via the cocycle condition

\[
U_\mu(x_\nu + L_\nu) U_\nu(x_\mu) U_\mu(x_\nu)^\dagger U_\nu(x_\mu + L_\mu)^\dagger = \Lambda_{\mu\nu}(n_1, n_2, \ldots, n_{N-1}).
\] (3.22)

A Fourier mode on this lattice is then

\[
\Psi_{\{n\}}(y) = \prod_{i=1}^{N-1} e^{i n_i y_i}
\] (3.23)

in terms of general coordinates \( y_i \) of the maximal torus; \( T_{SU(N)} = \exp\{\sum_i y_i h_i\} \). Having these we construct a wavefunction by taking the symmetric sum of these modes under Weyl transformations

\[
\Psi_{\text{sym}}^n(y) = \sum_{\omega_{\text{w}} y_i} \Psi_n(S_\omega y_i)
\] (3.24)

in which \( S_\omega \) implements the Weyl transformations.

As examples consider the groups \( SU(2) \) and \( SU(3) \). We have straightforwardly from (3.16), for \( SU(2) \):

\[
A_1 = \frac{1}{gL} \begin{pmatrix} \theta & -\theta \\ \theta & -\theta \end{pmatrix}, \quad H = -\frac{g^2 L}{8} \frac{\partial^2}{\partial \theta^2} \quad E_n = \frac{g^2 L}{8} n^2 \quad \Psi_n(\theta) = \sqrt{\frac{2}{\pi}} \cos(n\theta)
\] (3.25)

where \( n = 0, 1, 2, \ldots \)

which is simply the quantum mechanics of a particle on a circle, whose wavefunctions are symmetric about \( \theta = 0 \).

The case of \( SU(3) \) has a bit more internal structure. In the Weyl basis, using (3.17) and (3.18)

\[
A_1 = \frac{1}{gL} \begin{pmatrix} \theta_1 & \theta_2 \\ -\theta_1 & -\theta_2 \end{pmatrix}, \quad H = -\frac{g^2 L}{6} \left( \frac{\partial^2}{\partial \theta_1^2} - \frac{\partial}{\partial \theta_1} \frac{\partial}{\partial \theta_2} + \frac{\partial^2}{\partial \theta_2^2} \right)
\] (3.26)
SU(3) states in this basis have energy

\[ E_{n_1 n_2} = \frac{g^2 L}{6} (n_1^2 - n_1 n_2 + n_2^2) = \frac{g^2 L}{6} \varepsilon_{n_1 n_2}. \]  

(3.27)

Wavefunctions for some low-lying states are given in [10].

To illuminate the twist matrix discussion using a different basis, let

\[ h_1 = \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad h_2 = \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \]  

(3.28)

The expansion \( \Theta = \sum_i y_i h_i \) gives the relations

\[ y_1 = \frac{1}{2} (\theta_1 - \theta_2) \]

\[ y_2 = \frac{\sqrt{3}}{2} (\theta_1 + \theta_2) \]  

(3.29)

and the Hamiltonian (3.16) transforms to

\[ H = -\frac{g^2 L}{8} \left( \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2} \right) \equiv -\frac{g^2 L}{8} \Delta_h \]  

(3.30)

which is the Hamiltonian obtained in [26], up to normalization. To use Fourier modes on the twist lattice \( \Gamma \) as wavefunctions, recall that a general element of the twist matrix (3.22) for SU(3) in the Gell-Mann basis (3.28) is [25]

\[
\Gamma(m_1, m_2) = (z_1^{m_1} z_2^{m_2}) = \exp \left\{ 2\pi i \begin{pmatrix} m_1 + 2m_2 & 0 & 0 \\ 0 & m_1 - m_2 & 0 \\ 0 & 0 & -2m_1 - m_2 \end{pmatrix} \right\} 
\]  

\[
= \exp \left\{ 2\pi i \left[ \frac{3}{2} m_2 h_1 + \frac{\sqrt{3}}{2} (2m_1 + m_2) h_2 \right] \right\} 
\]  

(3.31)

where \( m_1, m_2 \in \mathbb{Z} \). A Fourier mode on \( \Lambda \) in this basis is then

\[ \Psi_{m_1 m_2}(y_1, y_2) = \exp \left\{ im_2 y_1 + i \frac{1}{\sqrt{3}} (2m_1 + m_2) y_2 \right\}. \]  

(3.32)

Applying the Hamiltonian (3.30) yields the energy

\[ E_{m_1, m_2} = \frac{g^2 L}{6} (m_1^2 + m_1 m_2 + m_2^2) \]  

(3.33)
in exact agreement with eq. (3.27). Since \( m_1, m_2 \) above and \( n_1, n_2 \) in (3.20) range over both positive and negative integers, there is some state such that \( E_{n_1,n_2} = E_{m_1,m_2} \) (or simply use \( \theta_2 \rightarrow -\theta_2 \)). The difference in the energy for \( n_1 = m_1, n_2 = m_2 \) is due to the fact that the bases defining the respective maximal tori have different fundamental Weyl chambers.

**B. Compact Formulation**

An alternative canonical quantization of this system was given by Rajeev [9] in the continuum, and is well known to lattice gauge theorists [27,28,29,30]. In this approach one does not quantize the coordinates \( A_1 \), but instead maps the degrees of freedom onto those of the Wilson loop \( W \), i.e. the system is viewed as the quantum mechanics of a particle moving on the gauge group manifold [31]. In lattice gauge theory this is the definition of the theory from the start, and the unreduced configuration space \( Q_c \) is just a large product of group manifolds, \( G \times G \times \ldots \times G \). After mapping the temporally gauge fixed coordinates to the Wilson loop, Rajeev postulated the Laplace-Beltrami operator as the Hamiltonian. In the lattice approach the same operator emerges without gauge fixing and we review it’s extraction and the natural projection onto the gauge invariant sector of Hilbert space.

On the lattice, the Hamiltonian is extracted from the partition function via the transfer matrix \( T \),

\[
Z = \int DU_\mu e^{S(U)} = \text{Tr} \ T^q \\
= \prod_{i=0}^{q-1} \prod_{x,\mu} \int dU_{x,\mu}(i) \langle U_{x,\mu}(i+1) | \hat{T} | U_{x,\mu}(i) \rangle
\]  

(3.34)

where \( q \) is the number of time slices in the lattice.

The gauge freedom can be dealt with by first gauge fixing to the reduced configuration space, or by projecting out gauge invariant wavefunctions in Hilbert space after quantization of the unreduced configuration space. To gauge fix a time slice of \( n \) sites,
since a link transforms as $\Omega \cdot U_\mu(x) \to G^\dagger(x)U_\mu(x)G(x + \hat{\mu})$, the gauge transformation, analogous to eq. (3.4)

$$G_x = \left(\prod_{y=1}^{x-1} U_1(y)\right)^\dagger \Lambda \omega^{x-1} \tag{3.35}$$

takes all spatial links into the Coulomb gauge $U_1(x) = \omega$, where $\omega$ is the $n$th root of the Wilson loop, $\omega^n = W = \prod_{x=1}^n U_1(x)$, and $\Lambda$ is arbitrary (and can be used to diagonalize $W$).

Without gauge fixing, we proceed by noting that the transfer matrix generally splits into potential and kinetic operators which are functions of spatial coordinates and momenta

$$\hat{T} = e^{-\frac{1}{2}V(x)} e^{K(\pi)} e^{-\frac{1}{2}V(x)} \tag{3.36}$$

$V(x)$ is a function of plaquettes with only space-like links, of which there are none on a two dimensional lattice. Using (3.34) and the lattice action $S = -2/(g^2 a_x^2 a_t^2) \sum_x \text{ReTr}[1 - U_0]$, we have the matrix elements of $\hat{T}$

$$\langle U_1' | \hat{T} | U_1 \rangle = \prod_x \int dU_0(x) \exp \left\{ - \frac{2}{g^2 a_t} \text{ReTr}[1 - U_1(x)U_0(x + a_x)U_1'(x)^\dagger U_0(x)^\dagger]\right\} \tag{3.37}$$

between time-like separated states in the coordinate basis $|U_1\rangle$. We further see that $\hat{T}$ factorizes into a kinetic operator $T_K$ and a projection operator $P_\Omega$, which projects out gauge invariant states from the Hilbert space $|U_1\rangle$.

Let $\hat{\Omega}(U_0)|U_1\rangle = |U_1^\Omega\rangle$ where $U_1(x)^\Omega = U_0(x)U_1(x)U_0(x + a_x)$, then $\hat{T} = \hat{P}_\Omega \hat{T}_K$ where

$$\hat{P}_\Omega = \prod_x \int dG(x) \hat{\Omega}(G) \tag{3.38}$$

and

$$\langle U_1' | \hat{T}_K | U_1 \rangle = \prod_x \exp \left\{ - \frac{2}{g^2 a_t} \text{ReTr}[1 - U_1(x)U_1'(x)^\dagger]\right\} \tag{3.39}$$

It is instructive to observe the action of the projection operator $\hat{P}_\Omega$ on wavefunctions $\langle U | \Psi \rangle = \Psi(U)$. In general

$$\Psi(U) = \prod_x \sum_{j_x m_x n_x} \lambda_{j_x m_x n_x} D_{m_x n_x}^{j_x}[U_1(x)] \tag{3.40}$$
using the completeness of the unitary irreducible representations $\mathcal{D}^j_{\alpha\beta}[G]$. Then (suppressing the $x$ dependence of the indices)

$$\Psi(U^G) = \prod_x \sum_{jmn,ab} \lambda_{jmn}\mathcal{D}^j_{ma}[G(x)]\mathcal{D}^j_{ab}[U_1(x)]\mathcal{D}^j_{bn}[G(x + a_x)] \quad (3.41)$$

From the orthogonality of the irreducible representations we have

$$\langle U|\hat{P}_\Omega|\Psi \rangle = \prod_x \int dG(x)\Psi(U^G) = \sum_{ji} \lambda^i_j\mathcal{D}^i_{ii}(\prod_x U_1(x)) = \sum_j \lambda_j\chi_j(W) \quad (3.42)$$

so that $\hat{P}_\Omega$ projects out the wavefunction as a series in conjugation invariant characters of the Wilson loop as expected.

The Hamiltonian then follows from (3.39)

$$\langle W'|\hat{T}_K|W \rangle = \exp \left\{ -\frac{2}{g^2} a_x \text{ReTr}[1 - WW'^{\dagger}] \right\}$$

$$= \int d\Gamma \exp \left\{ i\gamma \cdot X \right\} \exp \left\{ -\frac{2}{g^2} a_x \text{ReTr}[1 - \Gamma] \right\} \quad (3.43)$$

where $\Gamma = \exp\{i\gamma \cdot \lambda\}$ and the $X_i$ are differential operators generating (left) translations on $G$. As $a_t \to 0$, saddle point integration of (3.43) gives ($T \sim e^{-a_t H}$)

$$H = -\frac{1}{2} \frac{g^2}{a_x} X_1^2 \quad (3.44)$$

showing the Hamiltonian to be proportional to the quadratic Casimir operator of $G$.

Looking again at well known examples, the $X_i$ for $SU(2)$ can be written in Euler angles $W = W(\theta, \psi, \phi)$ say [32], as

$$X_1 = \cos \psi \frac{\partial}{\partial \theta} + \frac{\sin \psi}{\sin \theta} \frac{\partial}{\partial \phi} - \cot \theta \frac{\sin \psi}{\partial \psi}$$

$$X_2 = -\sin \psi \frac{\partial}{\partial \theta} + \frac{\cos \psi}{\sin \theta} \frac{\partial}{\partial \phi} - \cot \theta \frac{\cos \psi}{\partial \psi}$$

$$X_3 = \frac{\partial}{\partial \phi} \quad (3.45)$$
so that $X^2$ is indeed the Laplace-Beltrami operator on $SU(2)$. However since $P_{11}$ projects out only radial wavefunctions $\chi_j(\theta)$, we have

$$H = -\frac{g^2 L}{2} \left\{ \frac{\partial^2}{\partial \theta^2} + \cot \theta \frac{\partial}{\partial \theta} \right\} \equiv -\frac{1}{2} \Delta_{LB} H \quad \chi_j(\theta) = \frac{1}{\sqrt{2j+1}} \sin[(j + \frac{1}{2})\theta]$$

$$E_j = \frac{g^2 L}{2} j(j + 1), \quad \text{where} \quad j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$$

The Casimir spectrum appearing above, $E_j \propto c_{SU(2)}^2$ is in conflict though with that obtained in the previous section, $E_n \propto n^2$ eq. (3.25), and a similar mismatch appears for $SU(3)$; compare the non-compact spectrum eq. (3.33)

$$\varepsilon_{m_1 m_2} = \frac{1}{6} (m_1^2 + m_1 m_2 + m_2^2)$$

with the quadratic Casimir of $SU(3)$

$$c_{SU(3)}^2(\lambda, \mu) = \frac{1}{3} (\lambda^2 + \mu^2 + \lambda \mu) + \lambda + \mu$$

The discrepancy persists for all $N$, thus it appears that the two methods of quantization are in conflict [10].

**IV. Spectral Equivalence**

The previous section leaves us with the unpleasant result that quantizing the theory in the group algebra $\mathfrak{g}$ yields a different spectrum than quantizing on the group manifold $G$. However the essential feature, which emerged in both analyses, is that the Hamiltonians are the radial parts of Laplacians acting on their respective spaces. In fact, the projection of the radial parts of differential operators on Lie groups and symmetric spaces has been previously accounted by Berezin and Helgason [33,34], among others. This projection has been used fruitfully by Dowker and Schulman [31,35] for instance, in computing the propagator of a particle on a group manifold, allowing them to diagonalize the radial Laplacian in the geodesic equation on $G$. 19
For Lie groups, the radial part $\Delta|_H$, of a Laplacian $\Delta$, is a differential operator over the Cartan subgroup $H \subset G$, invariant under automorphisms of $H$:

$$h \rightarrow g^* h g \in H.$$  \hspace{1cm} (4.1)

The adjoint action of $g \in G$ is such that these motions form a finite set, the Weyl group $S$. Much of the structure of $\Delta|_H$ is determined by its invariance under $S$. The exponential map, $\exp : t \in \mathfrak{g} \rightarrow g(t) = \exp(it) \in G$ gives the canonical way of getting from the algebra $\mathfrak{g}$ to the group $G$. Then, given a radial differential operator, $\Delta_h = P(\frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2}, \ldots, \frac{\partial}{\partial t_N})$, polynomial in $\frac{\partial}{\partial t_i}$ on $\mathfrak{h}$, and invariant under (4.1), we have the corresponding radial Laplacian on $H$

$$\Delta_H(P) = \frac{1}{J(t)} P\left(\frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2}, \ldots, \frac{\partial}{\partial t_N}\right) J(t)$$

where $J(t) = \prod_{\alpha_i^+} \sin \left(\frac{\langle \alpha_i, t \alpha_i \rangle}{2}\right)$.  \hspace{1cm} (4.2)

$\alpha_i^+$ are the positive roots of $\mathfrak{g}$ and $J(t)$ enjoys the relationship: $J^4 = \det g_{\mu\nu} \equiv |g|$ with the metric on $G$, hence the Haar measure is $\prod_{\alpha^+} dt \alpha J^2(t)$.

Evidently for

$$\Delta_h \equiv \frac{\partial^2}{\partial t_1^2} + \frac{\partial^2}{\partial t_2^2} + \ldots + \frac{\partial^2}{\partial t_{N-1}^2} = \sum_{i \in \text{Cartan}} \frac{\partial^2}{\partial t_i^2}$$ \hspace{1cm} (4.3)

we obtain

$$\Delta_H(\Delta_h) = \Delta_{LB}|_r - R^2$$

$$\Delta_{LB} = \frac{1}{\sqrt{|g|}} \partial_\mu (\sqrt{|g|} g^{\mu\nu} \partial_\nu)$$ \hspace{1cm} (4.4)

where $\Delta_{LB}|_r$ is the radial part of the Laplace-Beltrami operator and $R$ is proportional to the scalar curvature of $G$. For compact Lie groups $R$ is given by a familiar algebraic quantity

$$R = \frac{1}{2} \sum_{\alpha^+} \alpha_i.$$ \hspace{1cm} (4.5)

It should be remembered that the square of $R$ and other scalar products involving the roots $\alpha_i$ are produced with the inner product on the root space via Killing form

$$\langle a(h), b(h) \rangle = \text{Tr} \ h_a h_b,$$ \hspace{1cm} (4.6)
where \( a(h), b(h) \) are linear functionals of \( h \in \mathfrak{h} \) [36].

Here we see the source of the discrepancy in the spectra obtained in the previous section. Since \( \Delta_h \) is a differential operator on the flat manifold \( \mathfrak{h} \), to represent it faithfully on \( G \) we must subtract the curvature induced by the exponential map when projecting \( \Delta_h \) to \( G \). Thus the energy eigenvalues are shifted up by the “Casimir” energy \( R^2 \).

In the non-compact \( SU(2) \) example, eq. (3.25), neglecting now the factors of \( g^2L \)

\[
H\Psi_n \sim -\frac{1}{8} \Delta_h \Psi_n = \frac{1}{2} \left[ \frac{n^2}{4} \right] \Psi_n = \frac{1}{2} (j)^2 \Psi_n. \tag{4.7}
\]

Using eq. (4.4) to obtain the projected compact Hamiltonian, \( \Delta_h \xrightarrow{\text{exp}} \Delta_H : \)

\[
H\chi_{\lambda} \sim -\frac{1}{2} \Delta_H \chi_{\lambda} = -\frac{1}{2} (\Delta_{LB} - R^2) \chi_{\lambda} = \frac{1}{2} \left[ \lambda(\lambda + 1) + \frac{1}{4} \right] \chi_{\lambda} = \frac{1}{2} \left( \lambda + \frac{1}{2} \right)^2 \chi_{\lambda} = \frac{1}{2} (j')^2 \chi_{\lambda}, \tag{4.8}
\]

using \( \left( \frac{1}{2} \sum_{\alpha} \alpha_i \right)^2 = \frac{1}{4}, \) for \( SU(2) \). Thus including the curvature term gives exact agreement of the spectra for \( j > 0 \). While comforting, this in itself is a rather meager profit, being just an overall constant shift in the energy, however we also observe a very interesting shift in the correspondence between states.

We index states \( \Psi_j \) on the algebra \( \mathfrak{g} \) now by \( j = 0, \frac{1}{2}, 1, \ldots = \frac{n}{2} \), while on the group manifold \( G \) the state of corresponding energy to \( j \), is \( j' = \lambda + R \) with wavefunction \( \chi_{\lambda} \).

Hence we have the correspondence

\[
\begin{align*}
on \ \ mathfrak{g} = \mathfrak{su}(2) : & \quad j \leftrightarrow j - R \\
on G = SU(2) : & \quad \Psi_j \leftrightarrow \chi_{j-R} \\
-\frac{1}{2} \Delta_h \Psi_j = \frac{1}{2} (j)^2 \Psi_j & \iff -\frac{1}{2} \Delta_H \chi_{j-R} = -\frac{1}{2} (\Delta_{LB} - R^2) \chi_{j-R} \\
& = \frac{1}{2} (j')^2 \chi_{j-R} \\
E_{\Psi_j} = \frac{j^2}{2} & = \frac{c_2(j - R) + R^2}{2} = E_{\chi_{j-R}}.
\end{align*}
\tag{4.9}
\]
An important consequence is that the ground states, $\Psi_0$ and $\chi_0$ of these two quantizations are different, since $\chi_0 \leftrightarrow \Psi_{\frac{1}{2}}$. The ground state of $\mathfrak{g}$, $\Psi_0$, corresponds to the unusual $\chi_{-\frac{1}{2}}$ state to which we will return in a moment. This structure is shown in Figure 1.

![Figure 1](image)

and the shift of ground state for $SU(2)$.

Generalizing to $SU(3)$ and beyond, recall that [36]

$$R_{SU(3)} = \frac{1}{2} \sum_{\alpha^+} \alpha_i = \frac{1}{2} \alpha_1 + \frac{1}{2} \alpha_2 + \frac{1}{2} (\alpha_1 + \alpha_2) = \alpha_1 + \alpha_2$$  \hspace{1cm} (4.10)

An $SU(3)$ representation is labeled by a (highest) weight in the fundamental Weyl chamber of the weight lattice and can be written

$$\Lambda(\lambda, \mu) = \frac{1}{3} (2\lambda + \mu) \alpha_1 + \frac{1}{3} (\lambda + 2\mu) \alpha_2$$  \hspace{1cm} (4.11)

We normalize the root lattice such that

$$< \alpha_1, \alpha_1 > = < \alpha_2, \alpha_2 > = 1$$

$$< \alpha_1, \alpha_2 > = -\frac{1}{2}.$$  \hspace{1cm} (4.12)
The highest weights of representations $\Lambda(\lambda, \mu)$ in the fundamental Weyl chamber of $SU(3)$. The $3$, $3^*$, and $8$ are displayed.

For $SU(2)$ we saw,

$$E_j^\Psi = \frac{1}{2} (j + \frac{1}{2})^2 = \frac{1}{2} (j \alpha + R)^2 = E_j^\chi = \frac{1}{2} (c_2(j) + R^2)$$ (4.13)

Then using eq. (3.33) and $\Lambda(\lambda, \mu) + R = \Lambda(\lambda + 1, \mu + 1)$ we find the same behavior for $SU(3)$

$$E_{\Lambda+R}^\Psi = \frac{1}{6} [(\lambda + 1)^2 + (\lambda + 1)(\mu + 1) + (\mu + 1)^2]$$

$$= \frac{1}{6} (\lambda^2 + \lambda \mu + \mu^2) + \frac{1}{2} (\lambda + \mu) + \frac{1}{2}$$ (4.14)

$$= \frac{1}{2} c_2^{SU(3)}(\lambda, \mu) + R^2$$.

The displacement of the Weyl chamber boundary states which are missing in compact quantization is shown in Figure 3.
The shift of lowest energy states from non-compact to compact quantization for $SU(3)$

Figure 3

Clearly this structure maintains for all $G$, since the quadratic Casimir of the state labeled by $\Lambda$ is given by

$$c^G_2(\Lambda) = \langle \Lambda + R, \Lambda + R \rangle - \langle R, R \rangle$$

so that $\frac{1}{2}(c^G_2 + R^2)$ always yields the energy $\frac{1}{2} < \Lambda + R, \Lambda + R >$ for the Fourier mode $\Psi_{\Lambda+R}$ on $g$. Compare for example, $\frac{1}{2} < \Lambda, \Lambda >$ with the non-compact spectrum of $SU(3)$ in eq. (3.33):

$$\frac{1}{2} < \Lambda, \Lambda > = \left( \frac{1}{3}(2\lambda + \mu)\alpha_1 + \frac{1}{3}(\lambda + 2\mu)\alpha_2 \right)^2$$

$$= \frac{1}{6} (\lambda^2 + \lambda\mu + \mu^2)$$

V. State Mapping

In the previous section we saw that the spectral decompositions of $\Delta_L$ and $\Delta_H$ are identified, revealing a shift in the corresponding wavefunctions, induced by the curvature of $G$. We now examine these wavefunctions more closely.
The $SU(2)$ Hamiltonian on the algebra $\mathfrak{g}$, was simply proportional to $\frac{\partial^2}{\partial \theta^2}$ with solutions $\Psi_n = \cos(n\theta), \sin(n\theta)$: the odd and even solutions under the Weyl reflection $S: \theta \rightarrow -\theta$. Invariance selects $\Psi_n^+ = \cos(n\theta)$.

On $G$, the wavefunction $\chi_\lambda$ is also Weyl invariant

$$\chi_\lambda^+(\theta) = \frac{\sin[(\lambda + \frac{1}{2})\theta]}{J(\theta)} = \frac{\sin[(\lambda + \frac{1}{2})\theta]}{\sin(\frac{\theta}{2})} \quad (5.1)$$

since both numerator and denominator are odd. The numerator corresponds, through eq. (4.2), to the odd wavefunction on $\mathfrak{g}$: $J(\theta)\chi_\lambda = \Psi_{\lambda+R}^- = \sin[(\lambda + \frac{1}{2})\theta]$ (recall that $\theta \rightarrow \frac{\theta}{2}$ from $\mathfrak{g} \rightarrow G$ in our notation). In passing, notice that the Weyl-odd function, using the even eigenfunction on $\mathfrak{g}$

$$\chi_\lambda^- = \frac{\cos[(\lambda + \frac{1}{2})\theta]}{J(\theta)} \quad (5.2)$$

is also an eigenfunction of $\Delta_{LB}$ with the same Casimir eigenvalue $-\lambda(\lambda + 1)$. However it cannot be globally defined as a function on $SU(2)$, since it is impossible to have an odd function of polar coordinates, which is equatorially constant.

This parity shifting between wave functions on $\mathfrak{g}$ and $G$ is a general feature of all semi-simple Lie groups [33]. The denominator appearing in the Weyl character formula is always odd, due to its geometrical nature. In general it is defined as

$$J(t_1, t_2, \ldots, t_n) = \prod_{\alpha^+} \left\{ e^{i\frac{\pi}{2} \langle \alpha, t \rangle} - e^{-i\frac{\pi}{2} \langle \alpha, t \rangle} \right\}
= \sum_\omega (\det S_\omega) e^{i<R, S_\omega t>} \quad t = \sum_i t_i \alpha_i \quad (5.3)$$

Functions such as $J(t)$, transforming as $S_\omega J(t) = (\det S_\omega) J(t)$ are called alternating. The Weyl character formula

$$\chi_{\Lambda}(t_1, \ldots, t_n) = \sum_\omega (\det S_\omega) e^{i<\Lambda + R, S_\omega t>} \quad (5.4)$$

uses an alternating numerator, the odd Fourier mode $\Lambda + R$, divided by $J(t)$ to produce an invariant character. In contrast, the general form of an invariant wavefunction on $\mathfrak{g}$ is

$$\Psi_{\Lambda}(t_1, \ldots, t_n) = \sum_\omega e^{i<\Lambda, S_\omega t>} \quad (5.5)$$
using the even mode \( \Lambda \).

What then of the state corresponding to \( \Psi^+_0 \) on \( \mathfrak{g} \) when projected to \( G \)? Returning to \( SU(2) \) we find in fact, that the Weyl-odd state, \( \chi^-_{-\frac{1}{2}} \), corresponding to \( \Psi_0 \) is an eigenfunction of \( \Delta_{LB} \) with eigenvalue \( R^2 \)

\[
\chi^-_{-\frac{1}{2}} = \frac{1}{J(\theta)} = \frac{1}{\sin(\frac{\theta}{2})} \\
\Delta_{LB}\chi^-_{-\frac{1}{2}} = \frac{1}{4}\chi^-_{-\frac{1}{2}} = R^2\chi^-_{-\frac{1}{2}} \\
\implies \Delta_H\chi^-_{-\frac{1}{2}} = -(\Delta_{LB} - R^2)\chi^-_{-\frac{1}{2}} = 0
\]  

(5.6)

so that the projected zero mode, \( \Psi_0 \rightarrow \chi^-_{-\frac{1}{2}} \), is a zero mode of the projected Hamiltonian \( \Delta_h \rightarrow \Delta_H \), but is not however, a globally defined function on \( G \).

It is possible to construct a Weyl-even zero mode, since both 1 and \( \theta \) belong to the kernel of \( \Delta_h \).

\[
\chi^+_{-\frac{1}{2}} = \frac{\theta}{\sin(\frac{\theta}{2})}
\]  

(5.7)

also satisfies \( \Delta_H\chi^+_{-\frac{1}{2}} = 0 \). This state is apparently related to Ruse’ invariant \( \rho \) of a harmonic manifold [37,38], which is defined by

\[
\rho(x_0, x) = \frac{\sqrt{|g_0||g|}}{J(x_0, x)}, \quad J(x_0, x) = \frac{\partial^2 \Omega}{\partial x_0 \partial x}, \quad \Omega = \frac{1}{2}g_{\mu\nu}x_0^\mu x_0^\nu = \frac{1}{2}s^2
\]  

(5.8)

For spaces of constant curvature

\[
R_{ijkl} = \kappa(g_{ik}g_{jl} - g_{il}g_{jk}) \\
\rho(s) = \frac{\sin^2(\sqrt{\kappa}s)}{\kappa s^2} \\
\Delta_{LB} \rho^{-\frac{1}{2}} = \kappa \rho^{-\frac{1}{2}}
\]  

(5.9)

This state does not belong to the Hilbert space of \( \Delta_H \), although it is square integrable on \( SU(2) \). While having the correct Weyl symmetries, it is not periodic in \( \theta \). Dowker has dubbed this state the ”zero representation”, in which all group elements are represented by \( 0 \) [38]. The dimension of a representation \( \Lambda \) is

\[
\dim \Lambda = \sum_{\alpha^+) \frac{<\alpha_i, \Lambda + R>}{<\alpha_i, R>}
\]  

(5.10)
Hence, using the correspondence found above between the states $\Lambda_g \leftrightarrow \Lambda_G - R$, we see that any state $\Psi_\Lambda$ which lies on the boundary of a Weyl chamber will project to such a zero-dimensional character in this sense, and thus not appear in the compact spectrum. In non-compact quantization these states have a constant (zero mode) wavefunction along some sub-torus of the maximal torus.

VI. Conclusions

We have seen that straightforward canonical quantization of two dimensional Yang-Mills theory leads to two definitions of the quantum theory, depending on the topology we allow for the configuration space. On the one hand we can quantize the field $A_\mu$, as a quantum theory on the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ subject to the periodicities of the maximal torus. Thus the configuration space is $\mathfrak{h}/\mathbb{Z}^{N-1} = T^{N-1}$ (up to the identification of Weyl reflections). On the other hand, we can map the gauge field to the group manifold $G$ itself and quantize the system there, restricting the wavefunctions to class functions of $G$, ie. they are again only functions of the coordinates of the maximal torus $T^{N-1} \subset G$.

Two slight subtleties in relating the quantizations are that $G$ has constant curvature while $\mathfrak{g}$ is flat so that, relative to $\mathfrak{g}$, the spectrum on $G$ is shifted by a constant “Casimir energy” (in a conspiracy of terms) as usually happens upon compactification. More importantly the wavefunctions in compact quantization must be globally defined functions on $G$, whereas non-compact wavefunctions live only on the maximal torus.

Keeping track of the correspondence of states, we found that the zero modes of the non-compact Hamiltonian disappear when the Hamiltonian is mapped to $G$, due to the new topology confronting wavefunctions. By zero mode here is meant any wavefunction $\Psi_{n_1n_2...0...n_{N-1}}(\theta_i)$ which is constant around one of the circles of the maximal torus, so that it is a zero mode of the Laplacian of that circle. In the root plane these states lie on the boundaries of the Weyl chamber. This effect is similar to the reason the zero
mode of the Dirac operator is absent under compactification of $\mathbb{R}^2$ to $S^2$ for instance [39], where $(i\gamma^\mu \partial_\mu)^2 = -\nabla^2 + \frac{1}{4}R$, inducing the same type of shift as in (4.4). Thus the primary observation is that the two quantizations have different ground states. Two areas of relevance for these observations certainly come to mind, namely: lattice gauge theory and the recently discovered sum over maps representation of the 2D-QCD partition function [11,12].

Clearly the compact lattice formulation of 2DYM will reproduce the compact quantization, more or less by definition. However the fact that the continuum theory has states of lower energy which cannot be seen even in the exact solution and classical continuum limit of the lattice model is rather unexpected. In the canonical formalism we see that the topology of the configuration space is experienced by the wavefunctions, which of course are global objects. As usual, this topology is most severely felt by the zero modes of the Laplacian.

Intricately woven into the above analysis is gauge fixing, which allows us to identify exactly the coordinates of the physical configuration space and quantize only these. On the lattice, without gauge fixing, wavefunctions live on the unreduced configuration space $Q = G^{n_{\text{links}}}$, hence all degrees of freedom are quantized. Perhaps by a strong form of gauge fixing we can restrict the quantization on the lattice to the physical configuration space so as to reproduce the non-compact results.

Strictly speaking we have analyzed here essentially the one plaquette model for the dynamics of a single link [28,42]. If these two inequivalent compact and non-compact quantizations persist to higher dimensional gauge theories, the implications could be very far reaching, however this is rather difficult to analyze since the Hamiltonian is then not just a differential operator in the radial coordinates of the maximal torus. For a non-radial Hamiltonian mapping the Lie algebra to $G$ is often necessitated in order to make residual gauge identifications of configuration space possible. This is the case even in two dimensions for instance. Using only the Coulomb gauge condition $\partial_1 A_1 = 0$ without diagonalizing
produces a complicated non-radial Hamiltonian with residual gauge symmetries. Similar shifts in spectra have been encountered upon interchanging the order of gauge fixing and quantization in simple models \[40\] thus the study of the gauge fixing — quantization process, while at the same time compactifying the configuration space as in lattice gauge theory demands further study. A recent BRST quantization of 2DYM has revealed more surprises as an anomaly develops in the Kac-Moody algebra of the constraints for certain polarizations of phase space \[41\].

As mentioned above, it was recently shown that the partition function of 2DYM on a Riemann surface, \( M_G \) of genus \( G \), computed in the continuum limit of the compact lattice formulation and expanded in \( \frac{1}{N} \) can be represented as \( N \to \infty \), as the sum over homotopicly distinct maps from a Riemannian worldsheet, \( W_g \) of genus \( g \), to the target space \( M_G \).

\[
Z(M, e) = \int D\!A_\mu \exp \left\{ -\frac{1}{2} \int_M d^2 x \sqrt{g} \text{Tr} \ F_{\mu\nu} F^{\mu\nu} \right\} = \sum_{\text{reps } \Lambda} d^{2-2G}_{\Lambda} \exp \left\{ -\frac{1}{2} e^2 A c_2(\Lambda) \right\}
\]

\[
\sum_{g=0}^{\infty} \sum_{n=1}^{\infty} \sum_{i} N^{2-2g} \omega_{g,G}^{n,i} (e^2 NA)^i \exp \left[ -n \frac{e^2 A}{2N} \right]
\]

where \( e \) is the coupling constant of the Yang-Mills fields, and \( \omega_{g,G}^{n,i} \) counts the number of topologically distinct, smooth maps from \( W_g \) to \( M_G \), with winding \( n \), and \( i \) branch points.

Two very interesting features of this string theory representation are the following. First, no degenerate maps occur in which the worldsheet fails to cover the target space at least once, ie. there are no maps of the entire worldsheet to a point or a Wilson loop on \( M_G \). Furthermore only one smooth map per topological class is summed, as opposed to the usual string path integral in which all possible maps are integrated over, including folded maps in the same homotopy class.

The canonical treatment above reveals that the quantum states of 2DYM are essentially the Fourier modes on the maximal torus of \( G \), with non-compact quantization picking the even cosine modes, while compact quantization chooses the odd sine modes.
and divides them by the Weyl denominator \( J(t) \). As such, these Fourier modes provide one representative map from some sub-torus of the maximal torus to the cylindrical spacetime, for each winding number \( n \). This hints that the worldsheets are really sub-tori of the maximal torus. In terms of this observation the missing states in the compact quantization which are constant Fourier modes along some sub-torus, would be the zero winding maps absent in the string representation of 2DYM.

Expressed differently, the above analysis shows that (at least for genus \( G = 1 \)) the partition function (6.1) should be modified to

\[
Z = \sum_{\text{reps } \Lambda}^\infty d_\Lambda^{2-2G} e^{-\frac{1}{2}e^2A[c_2(\Lambda)+R^2]} \tag{6.2}
\]

picking up an overall factor \( e^{-\frac{1}{2}e^2AR^2} \) due to compact quantization. While this is a trivial factor, the real issue is the range of the sum over representations. To express the quantization of \( A_\mu \) via compact quantization, we must include the “zero representations”, \( \chi_{-R} \), which are present in non-compact quantization and include the ground state of the theory defined by \( \mathcal{L} = \frac{1}{2} \text{Tr} \int dx^2 F_{\mu\nu}^2 \).

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