Large spin-orbit effects in small interacting quantum dots

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We consider small ballistic quantum dots weakly coupled to the leads in the chaotic regime and look for significant spin-orbit effects. We find that these effects can become quite prominent in the vicinity of degeneracies of many-body energies. We illustrate the idea by considering a case where the intrinsic exchange term $-JS^2$ brings singlet and triplet many-body states near each other, while an externally tunable Zeeman term then closes the gap between the singlet and the one of the triplet states (with spin projection parallel the external field). Near this degeneracy, the spin-orbit coupling leads to a striking temperature dependence of the conductance, with observable effects of order unity at temperatures lower than the strength of the spin-orbit coupling. Under favorable circumstances, spelled out in the paper, these order unity effects in the conductance persist to temperatures much higher than the spin-orbit coupling strength. Our conclusions are unaffected by the presence of non-universal perturbations. We suggest a class of experiments to explore this regime.

I. INTRODUCTION

Electrons in planar quantum dots experience spin-orbit interactions due to intrinsic (Dresselhaus$^4$) terms and due to the electric field arising from the confinement to two dimensions (the Rashba term$^2$). Their strength is measured by the parameter $L/\lambda$, where $L$ is the dot size and $\lambda$ is a spin-orbit scattering length. (We do not worry in this paper about the two different scattering lengths for the two terms.) Spin-orbit couplings have been thoroughly investigated in noninteracting quantum dots$^{3-9}$, and more recently, in interacting dots as well in certain special regimes$^{10-12}$. Spin-orbit terms violate spin conservation and appear to move the symmetry class from orthogonal to symplectic to first order in $L/\lambda$. Aleiner and Fal’ko (AF)$^4$ used a judicious unitary transformation to show that despite appearances the spin-orbit interaction enters the hamiltonian only to order $(L/\lambda)^2$ and beyond. They also showed that to order $(L/\lambda)^2$ (but not beyond) a new quantity $\sigma_z^{AF}$ (similar algebraically to the usual $\sigma_z$ but distinct from it) is conserved. The crossover to the new symmetry class occurs at a scale $\epsilon^{AF} \simeq (L/\lambda)^4g\delta$, and the violation of $\sigma_z^{AF}$ occurs only at the scale$^4$ $(L/\lambda)^6g\delta$, where $\delta$ is the mean single-particle level spacing, $g\delta$ is the Thouless energy, and $g$ is the Thouless number. Thus in a small dot with $L \simeq 200$ nm and $\lambda \simeq 2000 - 3000$ nm, spin-orbit effects are expected to be extremely small.

Here we ask how these considerations, derived for non-interacting particles, are modified if the exchange interaction $(-JS^2)$ of the Universal Hamiltonian$^{14,15}$ is included. Recall that the Universal Hamiltonian is the correct low-energy fixed point$^{16}$ deep inside the Thouless band for sufficiently weak interactions$^{17}$. We find that in the presence of the exchange coupling $J$ the spin-orbit effects really are of order $L/\lambda$ in the hamiltonian and that in the proximity of many-body degeneracies (induced largely by $J$ and aided by a Zeeman field which also serves as a knob for dialing through the transition), effects linear in $L/\lambda$ may be found in energies, and most strikingly, effects of order unity may generically be found in the conductances (provided $T$ is low enough). This paper focuses on calculating these effects, and proposing experiments to measure them. The optimal situation is in dots where low temperatures (in units of level spacing) may be attained without too much reduction of $L/\lambda$. Currently, for dots of linear size $L \simeq 200$ nm, the typical mean level spacing is $\delta \simeq 70 - 100\mu eV$ and temperatures of $T \simeq 100$ are achievable. The effects under study will be more readily detected if $T/\delta$ is lower by a factor of 5 or more.

In Section II we consider the Universal Hamiltonian plus a spin-orbit term. We review some previous work and elucidate how and when various powers of $L/\lambda$ appear in observables and how the presence of a $J \neq 0$ affects this. We then turn to our main result, which is to show that in certain conductance measurements there is a good chance of seeing spin-orbit effects of order unity. We estimate the effects in terms of a single parameter and propose experiments to reveal them and to interpret the data. Discussions and conclusions follow in Section III.
II. LOW TEMPERATURE PHYSICS

Our starting point is the hamiltonian
\[ H = H_U + H_{so} + H_Z. \]  
(1)

The Universal Hamiltonian\textsuperscript{14} for the case of orthogonal symmetry is
\[ H_U = \sum_{\mu s} \epsilon_{\mu s} \sigma_{\mu s} + \frac{U}{2} N^2 - J S^2. \]  
(2)

Here \( \sigma_{\mu s} \) creates an electron in a single-particle state \( \mu \) of energy \( \epsilon_{\mu s} \) and spin projection \( s_z = s/2 \). The charging energy is \( U, N \) is the particle number, and \( S \) the total spin. The essence of the spin-orbit hamiltonian is captured in first-quantization by the expression
\[ H_{so} = \frac{z \cdot \sigma \times p}{2m \lambda}. \]  
(3)

where \( z \) is a unit vector along the \( z \)-axis. For simplicity of discussion we keep the Rashba\textsuperscript{1} term only. The Zeeman term due to a parallel field (chosen to be along the \( x \)-axis) is \( H_Z = -E_Z S_x \).

We will be dealing with the case when the spin-orbit coupling is very weak, in the sense that the Aleiner-Fal’ko crossover scale \((\frac{1}{\lambda})^4 q_0 \ll \delta\), which is applicable to small dots. The opposite regime for large dots leads to quite different physics\textsuperscript{11,12}, which we will comment on in the conclusion.

Before plunging into details we outline our strategy. Imagine that there is no spin-orbit coupling, that we are at very low temperatures \( T \approx 0 \), and are sitting at the Coulomb Blockade\textsuperscript{16} conductance maximum obtained by tuning the gate voltage \( V_y \) to make an \( N \)-body ground state with \( (S = 0, S_z = 0) \equiv (0, 0) \) degenerate with an \( N+1 \)-body ground state with \( (S = \frac{1}{2}, S_z = \frac{1}{2}) \equiv (\frac{1}{2}, \frac{1}{2}) \). Now imagine turning on the Zeeman term \( E_Z \) to cause a ground state transition in the \( N \)-body sector so that \( (S = 1, S_z = 1) \equiv (1, 1) \) beats \( (0, 0) \). The \(-JS^2\) term lowers \( E_Z \), the point where all three states-(0, 0), (1, 1), \((\frac{1}{2}, \frac{1}{2}) \)- become degenerate, by bringing down all three triplet states. At \( E_Z \) the conductance will have a different value as compared to either side where only two states are degenerate. For example (as will be seen in detail below) in the case where all tunnelling widths are equal at both point contacts, the conductance at \( E_Z \) will be \( \frac{1}{3} \) of its value on either side.

How does a spin-orbit interaction modify this expectation? By eliminating the level crossing, and yielding a unique ground state in the \( N \)-body sector, it will, in the illustrative example being considered, kill the bump entirely. This is an example of the order unity effect on the conductance due the spin-orbit interaction. If we now heat up the system so \( T \approx \alpha \) where \( \alpha \) defines the scale of the spin-orbit interaction, the bump in conductance, rounded by the elevated temperature, will reemerge, since the two states that avoided each other will contribute more or less equally. When we get down to details we will find that while the above mentioned behavior can indeed occur (provided all tunnelling rates are equal at both leads), so can many others, some even more dramatic, the actual behavior being sensitively dependent on the wave functions at the leads. How then are we to predict what to expect? We shall see that with a few additional conductance measurements one can obtain the relevant information on these wave functions and know in advance what to expect of a given conductance peak in a given dot.

A. Essential background

Here we review some recent results on the powers of the spin-orbit coupling that should appear in noninteracting dots.

Let us begin with a simple non-interacting hamiltonian:
\[ H = \frac{p^2}{2m} + V(x, y) + \frac{z \cdot \sigma \times p}{2m \lambda} = H_0 + H_{so} \]  
(4)

where \( V \) is the confining potential. The spin-orbit term, considered as a perturbation, is of order \( 1/\lambda \) and we consider its impact on single-particle levels described by time-reversal invariant real wave functions. We will refer to powers of \( 1/\lambda \) as powers of \( L/\lambda \) since in the end the \( L \) in the numerator will invariably appear in physical quantities. For future reference, we define the orthogonal basis of \( H_0 \)
\[ H_0 = \frac{p^2}{2m} + V(x, y) \]  
(5)

Going over to second-quantized notation we can express \( H_0 \) in its own orthogonal eigenbasis in the many-body Fock space as
\[ H_0 = \sum_{\mu,s} \epsilon_{\mu s} \sigma_{\mu s} \]  
(6)

and the total spin operator as \( S = \frac{1}{2} \sum \sigma_{\mu s} \sigma^*_{\mu s} \).

Because \( H_{so} \) has no diagonal matrix elements in the basis of the unperturbed hamiltonian, we expect this term to appear to order \( (\frac{1}{\lambda})^2 \) in the energy shift of levels. It was pointed out by AF that this is illusory\textsuperscript{4}. Upon completing squares they bring the kinetic term to a form
\[ \frac{(p + \frac{z \cdot \sigma \lambda}{2m})^2}{2m} - \frac{1}{4m \lambda^2} \]  
(7)

when it becomes apparent that the particle is coupled to a spatially constant nonabelian vector potential. Using the gauge transformation
II LOW TEMPERATURE PHYSICS

\[ U = \exp \left[ \frac{iz \cdot r \times \sigma}{2\lambda} \right] \]  

(8)

and expanding it in inverse powers of \( \frac{1}{\lambda} \), they transform to new variables in terms of which the kinetic term becomes

\[ H_{AF} = \left( p_{AF} - \frac{1}{2} \sigma_{z}^{AF} a_{\perp} \right)^{2} - \frac{1}{4m\lambda^{2}} \]  

(9)

where

\[ a_{\perp} = \frac{r \times z}{2\lambda^{2}}. \]  

(10)

Terms of order \( (\frac{1}{\lambda})^{3} \) and higher are not shown.

Note that \( \sigma_{z}^{AF} \) and \( p_{AF} \) are not the same as \( \sigma_{z} \) and \( p \), while \( r \) remains untransformed because it commutes with \( U \). We can rewrite \( H_{AF} \) as an unperturbed orthogonal Hamiltonian \( H_{0,AF} \) (the first two terms in the following equation) perturbed by spin-orbit terms

\[ H_{AF} = \frac{p_{AF}^{2}}{2m} + V(r) + H_{so,AF} \]  

(11)

Going over to second quantization, we express \( H_{0,AF} \) in its orthogonal eigenbasis

\[ H_{AF} = \sum_{\mu,t} \epsilon_{\mu,AF} d_{\mu,t}^{\dagger} d_{\mu,t} + H_{so,AF} \]  

(12)

where we have denoted the new fermionic operators \( d, d^{\dagger} \) to emphasize their difference from the earlier \( c, c^{\dagger} \). We now define a total spin operator in the \( H_{0,AF} \) eigenbasis as \( S_{AF} = \frac{1}{2} \sum_{\mu,t} d_{\mu,t}^{\dagger} \sigma_{\mu,t} d_{\mu,t} \). Clearly, \( S_{AF} \) is not the same operator as \( S \). In fact,

\[ S = U^{\dagger} S_{AF} U = S_{AF} + \delta S_{AF} \]  

(13)

which shows that \( S_{AF} \) is some complicated combination of the original spin and orbital degrees of freedom.

Note that the vector potential is not fully gauged away. This must be so since a spatially constant vector potential can have a nonzero field strength, which in our case is \( f_{\mu\nu} = [A_{\mu}, A_{\nu}] \approx s_{z}/\lambda^{2} \). This is why in the end the particle is coupled to a magnetic field of size \( (\frac{1}{\lambda})^{3} \) and a sign which depends on \( s_{z} \). This coupling will contribute to energies or rates to order \( (\frac{1}{\lambda})^{4} \). It is also clear that \( s_{z} \) in the new AF basis (to be referred to as \( \sigma_{z}^{AF} \)) is conserved by the \( a_{\perp} \) term and violated only by higher order terms.

It is instructive to ask how this correct physics would manifest itself if one did not make the AF transformation and worked perturbatively in the original orthogonal basis. The heart of the equivalence between the two bases is the operator identity

\[ \frac{p}{m} = \frac{i}{\hbar} [H_{0}, r] \]  

(14)

which means that

\[ \langle \mu | p | \nu \rangle = \frac{im}{\hbar} (\epsilon_{\mu} - \epsilon_{\nu}) \langle \mu | r | \nu \rangle \]  

(15)

This identity was emphasized and exploited by Halperin et al \( a^{\dagger} \) in their analysis of the interplay between spin-orbit and Zeeman couplings. To see explicitly how this enters our discussion, consider energy level shifts of order \( (\frac{1}{\lambda})^{4} \) in the orthogonal basis. One finds for the second order energy shift in the single-particle state \( \mu \) of either spin:

\[ \Delta E^{(2)}_{\mu} = \frac{1}{4m^{2}\lambda^{2}} \sum_{\nu \neq \mu} \frac{\langle \mu | p | \nu \rangle \cdot \langle \nu | p | \mu \rangle}{\epsilon_{\mu} - \epsilon_{\nu}} \]  

(16)

\[ = \frac{1}{4m^{2}\lambda^{2}} \frac{im}{\hbar} \sum_{\nu \neq \mu} \langle \mu | p | \nu \rangle \cdot \langle \nu | [H_{0}, r] | \nu \rangle \]  

(17)

\[ = -\frac{1}{4m^{2}\lambda^{2}} \frac{im}{\hbar} \langle \mu | p \cdot r | \mu \rangle \]  

(18)

\[ = -\frac{1}{4m^{2}\lambda^{2}} \frac{im}{\hbar} \langle \mu | p \cdot r | \mu \rangle \]  

(19)

where we have used the time-reversal\( ^{*} \) invariance of the orthogonal states \( | \mu \rangle \) to set \( \langle \mu | p | \mu \rangle = 0 \). Using time-reversal invariance again we get

\[ \langle \mu | p \cdot r | \mu \rangle = -\langle \mu | r \cdot p | \mu \rangle = -\frac{1}{2} \langle \mu | [p, r] | \mu \rangle \]  

(20)

ending up with a shift \( -\frac{1}{4m^{2}\lambda^{2}} \) which is the state-independent shift obtained by completing squares.

At fourth-order \( (\frac{1}{\lambda})^{4} \) one will find corrections that are state-dependent, in agreement with what is expected in the AF basis based on \( a_{\perp} \). Note that to get the answer correctly one needs completeness. Had the second-order calculation been done in a truncated space of low energy orbitals (say near the Fermi energy) a nonexistent state-dependent shift would have been obtained to order \( (\frac{1}{\lambda})^{2} \).

We mention this since we will perform some calculations with such truncations and will need to be careful.

All the above was for the case when the spin-orbit coupling was constant over the sample. However, as pointed out by Brouwer, Cremers, and Halperin\( ^{6} \), since the Rashba term depends on the confining electric field felt by the 2DEG, which in turn depends on the density, one can fabricate additional gates over the sample to make the spin-orbit coupling inhomogeneous. In this case, the spin-orbit matrix element has a typical size

\( \psi^{*} | \psi \rangle \) and its time-reversed state by \( | \psi^{*} \rangle \), and if one denotes the time-reversed operator corresponding to \( \hat{O} \) by \( \hat{O}^{*} \), then we have \( \langle \psi^{*} | \hat{O} | \psi \rangle = \langle \psi^{*} | \hat{O}^{*} | \psi \rangle \).

Also note that \( r^{*} = -r \), and \( p^{*} = -p \).
Let us now examine the effect of adding a \(-JS^2\) term, which is shown to be the correct starting point in the appendix. The result is best seen in the AF basis. If we add a \(-JS^2\) term to the Hamiltonian of Eq. (4) in the original orthogonal basis, following Eq. (13) we end up with \(-J(S_{AF} + \delta S_{AF})^2\) where \(\delta S_{AF}\) is the leading correction due to the action of \(U\). The leading term \(-J(S_{AF})^2\) commutes with \(\sigma^zF\) but the corrections do not. They will typically produce effects of order \((\frac{1}{N})^2\), (say in relaxation of \(S^2_F\) ) but the effects can be linear in \((\frac{1}{N})\) at and near degeneracies in many-body states as will become clear soon.

The effect of \(J\) may also be seen in the orthogonal basis, but not so easily. If we turn on a \(J\) and resort to many-body perturbation theory to order \((\frac{1}{N})^2\), \(J\) will enter energy denominators, in a calculation similar to the one leading to Eq. (19). If we set \(J = 0\) in any of these results, we must recover the free-particle case. Expanding the denominators in powers of \(J/(\epsilon_\mu - \epsilon_\nu)\) will generate nontrivial effects proportional to powers of \(J\). These will match the results described above in the AF basis.

We now turn to our main purpose.

B. Very low temperatures

We will begin by defining the states of interest in the extremely low energy sector, \(\Gamma \ll T < \alpha < \delta\), where \(\Gamma\) is a typical single-particle level width due to coupling to the leads. This will allow us to get a firm grasp of all the relevant effects. Later we will incorporate the effects of \(T \approx \alpha < \delta\). Let us first turn off \(H_{\phi_0}\) and \(E_2\) in Eqn. (1). Consider the \(N\) particle state. Figure 1 shows the lowest energy state with \((S,S) = (0,0)\). It has two particles in an orbital \(\phi_0\) at the Fermi energy (set to zero). The filled sea underneath can be ignored. The energy of this state is

\[
E_{0,0} = 2 \times 0 + \frac{U}{2} N^2 - 0 \times J. \tag{21}
\]

We could also form an \(S = 1\) state, symmetric in spin and antisymmetric between the orbital \(\phi_0\) and the one just above it with a spacing \(\delta\), labelled \(\phi_1\). The energy of these states is

\[
E_{S=1,S_z=\pm1,0} = \delta + \frac{U}{2} N^2 - 2J. \tag{22}
\]

Note that \(\delta - 2J\), the energy difference between singlet and triplet, can be much smaller than \(\delta\). This is why we keep the orbital \(\phi_1\) but not higher ones. For now, when we do not have the Zeeman term, we will ignore the triplet state, assuming \(\delta - 2J \gg T\).

\[
\begin{array}{c}
\text{FIG. 1. Schematic of orbitals and four of the low energy states (labelled by (S, S_z)) that play a key role at } \\
\Gamma \ll T \ll \alpha \ll \delta. \text{ The arrows denote spins.}
\end{array}
\]

Now for the \(N + 1\) particle state. The extra particle will go to the orbital state \(\phi_1\) with spin up or down, the two states being degenerate. The energy of these states is

\[
E_{1,\pm \frac{1}{2}} = \delta + \frac{U}{2} (N + 1)^2 - \frac{3}{4} J. \tag{23}
\]

By varying a gate voltage \(V_0\) we can make the \(N\) and \(N + 1\) particle states degenerate so the system conducts. Assume this has been done. Let us calculate the conductance of this peak, using simple ideas. Later we turn to a master formula, based on rate equations\(^{20,21}\), that covers any situation where \(T > \Gamma\).

We want an electron to jump in from the left with \(s_x\)-independent width \(\Gamma_{0,\frac{1}{2}}\) and hop out of the right with \(s_x\)-independent width \(\Gamma_{0,\frac{1}{2}}\). The subscript on \(\Gamma\) labels the \(S\) values of the \(N\) and \(N + 1\) particle states degenerate at this peak. The dot has to be in the singlet \(N\)-particle state (and not in either of the doublet \(N+1\)-particle states of the same energy) to receive this electron.

Since all three states are degenerate at this peak, the probability for this is 1/3. The reduced conductance \(\bar{G}\) (which has a trivial temperature dependence removed) is given by

\[
\bar{G}_{0 \rightarrow \frac{1}{2}}^0 \equiv \frac{2 \hbar kT}{e^2} G = \frac{2 \Gamma_{0,\frac{1}{2}} L \Gamma_{0,\frac{1}{2}} R}{3 \Gamma_{0,\frac{1}{2}} + \Gamma_{0,\frac{1}{2}} R}. \tag{24}
\]

The logic behind this definition of \(\bar{G}\) will be apparent later when we do a full-blown calculation of conductance.) The subscript on \(\bar{G}\) stands for the two states that are degenerate at this peak and the superscript reminds us that the Zeeman term is absent: \(E_Z = 0\). The factor of 2 comes from sum over spins. The dependence on \(\Gamma\)’s maybe understood as follows. The left and right contacts are like \(\text{resistors}\) in series while the \(\Gamma\)’s represent their \(\text{conductances}\). Thus we must add their reciprocals to obtain the reciprocal of the effective conductance.
II LOW TEMPERATURE PHYSICS

Since we want the electron to hop on the upper orbital described by $\phi_1$,
\[
\Gamma_{0 \rightarrow \frac{1}{2}}^{L/R} = \Gamma_{L/R} |\phi_1(L/R)|^2
\]
where the arguments $L/R$ refer to the coordinates of the two leads within the dot and $\Gamma_{L/R}$ represents an overall pre-factor for each contact. For now let us lump the overall scale $\Gamma_{L/R}$ into $\phi_1(L/R)^2$ and write
\[
\tilde{G}_{0 \rightarrow \frac{1}{2}}^0 \equiv \tilde{G}^0 = \frac{2}{3} |\phi_1(L)|^2 \cdot |\phi_1(R)|^2
\]
Later we will return to Eqn. (25) that explicitly displays the tunnelling rate as a product of two factors, one that depends on the point contact and another that depends on the dot wave function.

Let us now add a Zeeman term $E_Z$ and suitably adjust $V_g$ to follow this peak (which has been done experimentally). Once $E_Z >> T$, only one of the spin states, with $S_z = \pm \frac{1}{2}$ is allowed in the dot. The reduced conductance now becomes
\[
\tilde{G}_{0 \rightarrow \frac{1}{2}}^1 \equiv \tilde{G}^1 = \frac{1}{2} |\phi_1(L)|^2 \cdot |\phi_1(R)|^2
\]
where the superscript on $\tilde{G}$ signifies that the Zeeman term has a value strong enough to essentially filter just one of the spin-$\frac{1}{2}$ states, but not so large the spin-1 state $(1, 1)$ is competitive with $(0, 0)$. Compared to $G^0$, the pre-factor $\frac{1}{2}$ has become $\frac{1}{2}$ since we have just one spin channel (not two as earlier) and two degenerate many-body states (not three as earlier).

The reduced conductance $\tilde{G}$ has dropped to a value equal to $3/4$ of the initial maximum:
\[
\tilde{G}^1 = \frac{3}{4} \tilde{G}^0.
\]
The situation is sketched in Figure (2).

Let us crank up the Zeeman term further till the spin-1 state $(1, 1)$ meets the $(\frac{1}{2}, \frac{1}{2})$ state and we again have three degenerate states. At this point denoted by a star, the conductance is given by
\[
\tilde{G}^* = \frac{2}{3} (\tilde{G}^1 + \tilde{G}^2)
\]
where $\tilde{G}_2$ is the conductance past this point where the spin-1 state $(1, 1)$ is the ground state:
\[
\tilde{G}^2 \equiv \tilde{G}_{1 \rightarrow \frac{1}{2}}^2 = \frac{1}{2} |\phi_0(L)|^2 \cdot |\phi_0(R)|^2
\]
Note that since the extra fermion is placed in the lower orbital, only $\phi_0$ enters $\tilde{G}_2$.

To get the above result we need only realize that at the triple degeneracy the two N-particle states each occur with probability $1/3$ and the conductance via each is proportional to $\tilde{G}_1$ or $\tilde{G}_2$.

\[\text{FIG. 2. Schematic of } \tilde{G} \text{ as function of } E_Z \text{ at zero spin-orbit coupling (}\alpha = 0\text{) for very low } T. \text{ Some rounding due to } T > 0 \text{ is assumed. The initial drop from } \tilde{G}^0 \text{ is due to suppression of one spin orientation. The rise near } E_Z \text{ is due to the triple degeneracy.} \]

The behavior near $E_Z^*$ depends on $\tilde{G}^1$ and $\tilde{G}^2$. For instance if $\tilde{G}^1 = \tilde{G}^2 = \tilde{G}$,
\[
\tilde{G}^* = \frac{4}{3} \tilde{G}.
\]
In this case $\tilde{G}$ first drops from $\tilde{G}^0$ to $\frac{3}{4}$ of its value to $\tilde{G}^1$, rises to $\tilde{G}^0$ and drops to $\tilde{G}^2$. If however $\tilde{G}^1$ is less than $\tilde{G}^2/2$, the rise will be monotonic with no bump. Likewise if $\tilde{G}^2$ is less than $\tilde{G}^1/2$, the fall will be monotonic. Figure (2) describes the case $\tilde{G}^1 = \tilde{G}^2$.

For now let us focus on this case $\tilde{G}^1 = \tilde{G}^2$, to get a feeling for our approach, which is to ask how the above mentioned behavior of $\tilde{G}(E_Z)$ is altered by a spin-orbit term. The general form of this term is
\[
H_s = \sum_{\mu \nu} A_{\mu \nu} \left[ c_{\mu \uparrow}^\dagger c_{\nu \downarrow} + c_{\nu \uparrow}^\dagger c_{\mu \downarrow} \right] + i B_{\mu \nu} \left[ c_{\mu \uparrow}^\dagger c_{\nu \uparrow} - c_{\mu \downarrow}^\dagger c_{\nu \downarrow} \right]
\]
where $\mu$ and $\nu$ labels single-particle orbital states, $\uparrow$ and $\downarrow$ the direction of spin along $x$ and $A_{\mu \nu}$ or $B_{\mu \nu}$ are real antisymmetric matrices:
\[
A^T = -A \quad B^T = -B. \quad (33)
\]

The typical size of $A_{\mu \nu}$ or $B_{\mu \nu}$ can be estimated to be $(\frac{e}{a}) \delta / \sqrt{\bar{g}}$ for the case of homogeneous spin-orbit coupling, and $(\frac{e}{a}) \delta \sqrt{\bar{g}}$ for the case of inhomogeneous spin-orbit coupling. For a typical small quantum dot of size 200nm with $g \approx 10$, with the spin-orbit scattering length being $2 \mu m$, we find the spin-orbit matrix element to be typically $0.03 \delta$ for the case of homogeneous spin-orbit coupling, and $0.3 \delta$ for inhomogeneous spin-orbit coupling. These estimates should be considered crude, since the expressions for various matrix elements have prefactors of order unity which depend on the geometry.
Let us now focus on the very low energy sector spanned by \( (0,0), (1,1) \) and \( (\frac{1}{2}, \frac{1}{2}) \). Since \( H_{so} \) conserves particle number, it does not mix the \( S = \frac{1}{2} \) states with \( S = 0,1 \) and neither does it mix the two \( S = \frac{1}{2} \) states with each other\(^1\). In the \( S = 0,1 \) sector spanned by \( |0,0\rangle \) and \( |1,1\rangle \) the effective hamiltonian matrix assumes the form

\[
h_{so} = \begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix}
\]

(34)

where the single parameter \( \alpha \equiv A_{01} \) (see Eq.(32)) characterizes the spin-orbit interaction.

Note also that spin-rotation-invariant interactions beyond the Universal Hamiltonian, such as Landau interactions, will appear in our basis as diagonal corrections to the already random energies.

This operator clearly leads to an avoided crossing and selects a unique ground state in this sector. At the putative degeneracy point the ground state with \( \alpha \neq 0 \), denoted by \( |\alpha, *\rangle \) is unique and given by the antisymmetric combination:

\[
|\alpha, *\rangle = \sqrt{\frac{T}{2}} [ |0,0\rangle - |1,1\rangle ]
\]

(35)

The reduced conductance at very low temperature is

\[
\tilde{G}^*(\alpha) = \frac{1}{\Gamma_{4,\alpha}} \left[ |\phi_0(L)|^2 + |\phi_0(R)|^2 - |\phi_1(R)|^2 - |\phi_1(L)|^2 \right]
\]

\[
\Gamma_{4,\alpha} = \left[ \frac{1}{\Gamma_{4,0}} \left( \frac{1}{2} |\phi_0^\ast(L)|^2 + \frac{1}{2} |\phi_0^\ast(R)|^2 \right) - \frac{1}{2} |\phi_1^\ast(L)|^2 - \frac{1}{2} |\phi_1^\ast(R)|^2 \right]
\]

(36)

To derive this result we need to remember that there is now a probability of \( \frac{1}{2} \) for the dot to be in the N-particle state and that the widths are given, for example by

\[
\Gamma_{4,\alpha} = \sum_{\sigma = \pm} \frac{1}{2} \left( \frac{1}{2} |\phi_0^\ast(L)|^2 + \frac{1}{2} |\phi_0^\ast(R)|^2 \right) - \frac{1}{2} |\phi_1^\ast(L)|^2 - \frac{1}{2} |\phi_1^\ast(R)|^2
\]

(37)

and that in each of the two terms only one value of \( \sigma \) contributes.

If we are looking for the effects of \( \alpha \), we need to compare \( \tilde{G} \) with and without \( \alpha \). The answer depends on the wave functions via \( |\phi_{0,1}(L/R)|^2 \).

Again consider the simple case where all are wavefunctions are equal at both leads. It is readily seen that

\[
\tilde{G}^*(\alpha) = \tilde{G}^1 = \tilde{G}^2,
\]

(38)

i.e., there is no second bump at the would-be level crossing. This is an example of an order unity change in the conductance due to a small spin-orbit perturbation. However, one has to be at a temperature lower than \( \alpha \) to see it.

Since the first bump is there at \( E_Z = 0 \), the absence of the second should be an unambiguous signal of \( \alpha \). Furthermore as \( T \) is raised beyond \( \alpha \) (something we will consider in detail later) the bump should disappear (though rounded) since the symmetric combination of \( |0,0\rangle \) and \( |1,1\rangle \) (that got repelled upwards) also starts contributing to \( \tilde{G} \). The requisite \( T \) will also tell us how large \( \alpha \) is. While this is happening the bump at \( E_Z = 0 \) should be disappearing due to thermal effects.

This happy line of thought ends when we realize that in general the four quantities \( |\phi_{0,1}(L/R)|^2 \) at the leads are not all equal and that the ratio \( \tilde{G}^*(\alpha)/\tilde{G}^\ast \) is very sensitive to their values. We emphasize that even if we find experimentally that \( \tilde{G}^1 = \tilde{G}^2 \) we cannot say anything definite. For example \( \tilde{G}^1 = \tilde{G}^2 \) can also be realized if

\[
\begin{align*}
|\phi_1(R)|^2 &= |\phi_0(L)|^2 = A \quad (39) \\
|\phi_0(R)|^2 &= |\phi_1(L)|^2 = B. \quad (40)
\end{align*}
\]

In this case we find

\[
\tilde{G}^1_{\alpha} = \tilde{G}^2_{\alpha} = \frac{1}{4}(\sqrt{A}B + \sqrt{B}A)^2 \geq 1. \quad (41)
\]

Thus the ratio can be any number \( \geq 1 \) with large ratios resulting if either \( A \) or \( B \) is small. This behavior of \( \tilde{G}^1_{\alpha} \) can be understood by examining Eqn. (36). It is seen that if tunnelling amplitude is weak at one end and strong at the other for one wave function and the opposite is true for the other, \( \tilde{G}^1_{\alpha} \) is able to feed off the bigger amplitude at both ends, unlike \( \tilde{G}^1 \) or \( \tilde{G}^2 \) which necessarily have a weak amplitude at one end.

Thus if we are to say with confidence what effect \( \alpha \) will have on the ratio \( \tilde{G}^*(\alpha)/\tilde{G}^\ast \) at \( E_Z^\ast \), we need to extract the four numbers \( |\phi_{0,1}(L/R)|^2 \) in a given dot on top of a given peak. Here is our proposal.

Recall from Eqn. (25) that

\[
\begin{align*}
\Gamma_{L/R}^{\alpha} &= \Gamma_{L/R}|\phi_0(L/R)|^2 \quad (42) \\
\Gamma_{L/R}^{\alpha} &= \Gamma_{L/R}|\phi_0(L/R)|^2, \quad (43)
\end{align*}
\]

where \( \Gamma_{L/R} \) are overall prefactors that control tunnelling rates at the left and right contacts and are determined by geometry, barrier heights and so on, and independent of the wave functions. Let us take the given dot and manually suppress tunnelling at the right contact by raising the barrier there, so that \( \Gamma^R << \Gamma^L \). The conductances assume the values \( \tilde{G}^{1,2}(R) \) given by

\[
\begin{align*}
\tilde{G}^1(R) &= \frac{1}{2}\Gamma^R|\phi_1(R)|^2 \quad (44) \\
\tilde{G}^2(R) &= \frac{1}{2}\Gamma^R|\phi_0(R)|^2. \quad (45)
\end{align*}
\]

\(^1\)In the case \( S = \frac{1}{2} \) there would be an effect. Also, in metallic grains the situation is more complicated\(^{10}\)
Their ratio gives us
\[ r = \frac{\vec{G}^1(R)}{G^2(R)} = \frac{|\phi_1(R)|^2}{|\phi_0(R)|^2} \] (46)

Likewise we can set \( \Gamma^L \ll \Gamma^R \) and measure
\[ l = \frac{\vec{G}^1(L)}{G^2(L)} = \frac{|\phi_1(L)|^2}{|\phi_0(L)|^2} \] (47)

Armed with these four numbers: \((\vec{G}^1, \vec{G}^2, r,l)\) all measured at very low temperatures, we can solve for \(|\phi_{0/1}(L/R)|^2\):
\[ |\phi_1(L)|^2 = \frac{2\vec{G}^1 G^2 (l - r)}{G^1 - rG^2} \] (48)
\[ |\phi_1(R)|^2 = \frac{2\vec{G}^1 G^2 (r - l)}{G^1 - rG^2} \] (49)
\[ |\phi_0(L)|^2 = \frac{|\phi_1(L)|^2}{l} \] (50)
\[ |\phi_0(R)|^2 = \frac{|\phi_1(R)|^2}{r} \] (51)

While the right hand sides of the first two equations are not positive definite (and can be negative if \(\vec{G}^1\) and \(G^2\) are assigned arbitrary values) it can be shown that if we input any values for \(G\) that come from a real measurement, i.e., given by Eqs. (27,30) this will not happen. We will illustrate this point later. Note also our assumption that the wave functions in question are unaffected by what we do to alter the tunnelling rates.

We can now express the conductances of interest in terms of knowns:
\[ G^* = \frac{2}{3}(\vec{G}^1 + G^2) \] (52)
\[ \bar{G}^*(\alpha) = \frac{G^1 G^2 (1 + r)(1 + l)}{2(G^1 + rG^2)} \] (53)

It is now easy to specialize to any case we want. If for example \(\vec{G}^1 = G^2 = G\) and \(l = r = 1\),
\[ \bar{G}^* = \frac{4}{3} \bar{G}^*(\alpha). \] (54)

Before concluding this analysis of the \(T \ll \alpha\) case, we must ask what the \((\frac{3}{2}, \frac{3}{2})\) level is doing in the meantime. In particular we do not want it to cross the \((\frac{1}{2}, \frac{1}{2})\) state before \((0,0)\) crosses \((1,1)\), or better still before \(\vec{G}\) settles down to \(G^2\). To avert this we need at least
\[ \delta' > \delta + J \] (55)
where \(\delta'\) is the energy gap between the orbital \(\phi_1\) and the one just above it, \(\phi_2\). Since for dots with \(r_s \simeq 1\) \(J/\delta \simeq 3^{15}\), we would be fine in a system where \(\delta \simeq .7\delta\) and \(\delta' \simeq 1.3\delta\). If however the gap at the Fermi energy is larger than the one above it, we should expect the \((\frac{1}{2}, \frac{1}{2}) \rightarrow (\frac{3}{2}, \frac{3}{2})\) transition to precede the \((0,0) \rightarrow (1,1)\) transition. The corresponding calculations are simply variants of the above.

To test the picture described above we need to know where to look for putative degeneracies. Consider the peak associated with the \((0,0) - (\frac{1}{2}, \frac{1}{2})\) degeneracy. The gate voltage \(V_g\) (with leverages suppressed) at which this occurs is
\[ V_g(0,0)-(\frac{1}{2}, \frac{1}{2}) (E_Z) = E_{1,1} - E_{0,0} = \frac{U}{2}(2N + 1) - \frac{J}{4}E_Z. \] (56)

As a function of \(E_Z\) it has a slope of \(-\frac{1}{2}\). On the other hand when \((1,1)\) wins we have
\[ V_g^{(1,1)}-(\frac{1}{2}, \frac{1}{2}) (E_Z) = E_{1,1} - E_{1,1} + \frac{U}{2}(2N + 1) - \frac{5}{4}J + \frac{E_Z}{2}. \] (57)

which has a slope of \(+\frac{1}{2}\) as a function of \(E_Z\).

At the point \(E_Z = \delta' - 2J\), the slope changes from \(-\frac{1}{2}\) to \(0\), as shown in Figure (3).\(^{19}\)

If we turn on \(\alpha\), we need to find \(\epsilon_0(\alpha)\), the ground state of the energy matrix in the \((0,0) - (1,1)\) sector:
\[ \begin{pmatrix} 0 & \alpha \\ \alpha & \delta - 2J - E_Z \end{pmatrix} \] (58)
in terms of which the gate voltage to stay on the peak is
\[ V_g^{(0,0)-(\frac{1}{2}, \frac{1}{2})} = \frac{U}{2}(2N + 1) - \frac{3J}{4} - \frac{E_Z}{2} - \epsilon_0(\alpha). \] (59)

Due to the avoided crossing, \(\epsilon_0(\alpha)\) is lower by an amount \(\alpha\) and \(V_g\) is higher by an amount \(\alpha\) at \(E_Z\). A sketch is given in Figure (3). Note that the results are most reliable near \(E_Z\) where things depend on \(\alpha\). As we move away, results depend on \(\alpha^2\) and errors due to truncation of the orbital space will enter. These are expected to be small for \(\delta - 2J \ll \delta\), but we will not reproduce the arguments here.

C. Higher Temperatures: \(\Gamma \ll T \simeq \delta - 2J \ll \delta\)

Now we consider a more detailed study of temperatures comparable to \(\alpha\) but still considerably smaller than \(\delta\). Thus we want to consider just six states: two at \(S = \frac{1}{2}\) and four at \(S = 0,1\). In this larger space another parameter \(\beta \equiv B_{01}\), associated with the matrix \(B_{0\mu}\) of Eq. (32) enters. This matrix element mixes the states \((S, S_z) = (0,0)\) and \((S, S_z) = (1,0)\). Since \((1,0)\) is split by about \(\delta - 2J\) near the triple crossing, we can estimate its effects to be of order \(\beta^2/(\delta - 2J)\) which is very small in the case of interest to us. We will therefore
non-equilibrium probabilities in jumps into the dot and \( \Psi \) are functions defined by body transition \( i \).

Initially ignore it in the analysis. Later we show a graph representative of its insignificance.

As mentioned earlier, the cost of populating \( \phi_1 \) with \( S = 1 \) is not \( \delta \) but \( \delta - 2J \), and we assume \( T \) is such that many-body states involving \( \phi_0 \) and \( \phi_1 \) are accessible but not those involving higher orbitals.

The proper formalism for a finite \( T \) computation involves rate equations, developed by Beenakker\(^{20} \) for a model with charging energy alone, and extended to the Universal Hamiltonian case by Alhassid, Rupp, Kaminski, and Glazman\(^{21} \). We follow their notation here. The formula for conductance is\(^{1} \)

\[
G = \frac{e^2}{\hbar kT} \sum_{ij} \tilde{P}_i^{(N)} f_{ij} \left( \Psi_i^{(N+1)} - \Psi_i^{(N)} + \eta_L \right) \Gamma_{ij}^L
\]

where (i) \( i \) and \( j \) label many-body states with \( N \) and \( N + 1 \) particles with energies \( \epsilon_j^{(N+1)} \) and \( \epsilon_i^{(N)} \), (ii) \( f_{ij} = \epsilon_i - \epsilon_j^{(N+1)} - \epsilon_F - eV_g \) (where \( \epsilon_F \) is the Fermi energy, chosen equal to zero and \( V_g \) is the effective gate voltage), (iii) \( \tilde{P}_i^{(N)} \) is the equilibrium probability of being in state \( i \), (iv) \( f_{ij} = f(\epsilon_{ij}) \) is the Fermi function, with \( f(0) = \frac{1}{2} \), (v) \( \eta_L \) is the fraction of the bias voltage dropped at the left contact with \( \eta_L + \eta_R = 1 \), (vi) \( \Gamma_{ij}^L \) is the width for many-body transition \( i \to j \) at the left contact when an electron jumps into the dot and (vii) \( \Psi \) are functions defined by non-equilibrium probabilities in \( P_i^{(N)} \) and \( P_j^{(N+1)} \) to be in states \( i \) and \( j \) respectively in the presence of a bias voltage \( V \).

\[
P_i^{(N)} = \tilde{P}_i^{(N)} \left[ 1 + eV \beta \Psi_i^{(N)} \right]
\]

\[
P_j^{(N+1)} = \tilde{P}_j^{(N+1)} \left[ 1 + eV \beta \Psi_j^{(N+1)} \right].
\]

The values of \( \Psi \)'s are determined by demanding that the \( P \)'s constitute a time-independent solution to the rate equations and their normalization condition. The resulting equations are

\[
\sum_j f_{ij} \left[ (\Gamma_{ij}^L + \Gamma_{ij}^R) (\Psi_j^{(N+1)} - \Psi_i^{(N)}) + (\eta_L \Gamma_{ij}^L - \eta_R \Gamma_{ij}^R) \right] = 0 \quad \forall i \tag{63}
\]

\[
\sum_i (1 - f_{ij}) \left[ (\Gamma_{ij}^L + \Gamma_{ij}^R) (\Psi_j^{(N+1)} - \Psi_i^{(N)}) + (\eta_L \Gamma_{ij}^L - \eta_R \Gamma_{ij}^R) \right] = 0 \quad \forall j \tag{64}
\]

\[
\sum_i \tilde{P}_i^{(N)} \Psi_i^{(N)} + \sum_j \tilde{P}_j^{(N+1)} \Psi_j^{(N+1)} = 0. \tag{65}
\]

Several comments are needed here\(^{20,21} \). While there seem to be \( i + j + 1 \) equations for the \( \Psi \)'s, one of the first \( i + j \) equations is redundant, which is where the last equation, from normalization, comes to the rescue. The result, Eqn. (60), which seems to depend on \( \eta \), in fact does not. Likewise the answer which seems to depend on the left lead in an asymmetric manner, is not. (There is an equally asymmetric expression in terms of the right contact which gives the same result\(^{20,21} \).) All this is exemplified by the case where Eqns. (63, 64) hold for each term in the sum, i.e.,

\[
\Psi_i^{(N)} - \Psi_j^{(N+1)} = \frac{\eta_L \Gamma_{ij}^L - \eta_R \Gamma_{ij}^R}{\Gamma_{ij}^L + \Gamma_{ij}^R} \tag{66}
\]

in which case the expression Eqn. (60), for \( G \) reduces to

\[
G = \frac{e^2}{\hbar kT} \sum_{ij} \tilde{P}_i^{(N)} f_{ij} \frac{\Gamma_{ij}^L \Gamma_{ij}^R}{\Gamma_{ij}^L + \Gamma_{ij}^R}. \tag{67}
\]

Note that in the very-low-temperature case we considered with just a total of three states, Eqns. (66) and (67) are applicable.\(^5 \) Our heuristic conductance calculations (and the definition of \( G \) in terms of \( G \)) followed from above and the fact that \( f(0) = \frac{1}{2} \). However, in our numerical calculations at temperatures comparable to \( \alpha \) they do not hold and we solved Eqns.(63,64,65) numerically.

We now display the result of solving the above equations for elevated temperatures using six states: two at

\(^1\)In all cases we considered, either \( j \) had just one value and \( i \) had two or \textit{vice versa}. Suppose \( j \) had just one value. For the one redundant equation we are allowed to ignore, we choose the middle equation (that sums over \( i \)). The first equation then gives Eqn. (66) for each \( i \).
S = \frac{1}{4}$ and four in the $S = 0, 1$ sector. Since the graphs are functions of four variables, $G^1, G^2, r, l$ we need to decide how to sample them here. We have found that there are four main classes and give one from each.

In Figures (4, 5) we consider the case $G^1 = G^2$. Recall that in this case, in the absence of spin-orbit coupling ($\alpha = 0$) there is a bump at $E_Z$ in the reduced conductance $G^*$. This bump will broaden with increasing $T$. When we turn on $\alpha$ there are two cases. If the ratios of Eqs. (46, 47) satisfy $r \ll l$ there will be no bump in $G^*(\alpha)$ for $T \ll \alpha$ and as we cross $T > \alpha$, the $\alpha \neq 0$ will look more and more like $\alpha = 0$, as shown in Figure (4). If however $r \ll 1$ and $l > 1$, the case $\alpha \neq 0$ will have a bump at $E_Z$ even for $T \ll \alpha$ that will grow in height as $T$ increases as shown in Figure (5). This bump will be robust even as the one at $E_Z = 0$ is smoothed out by temperature.

Figures (6) and (7) consider the case of considerably unequal $G$'s. We choose for definiteness $G^1 = 2G^2$. The first case with $l = 2.5$ and $r = 1.5$ does not have a bump in $G^*$, or $G^*(\alpha)$ at low and high $T$, as shown in Figure (6). However if $l$ are $r$ are different enough from each other (we have chosen $l = 4.0, r = 0.25$) there is a robust bump in $G^*(\alpha)$ that grows with increasing $T$ as shown in Figure (7).

In summary, the most dramatic manifestations of $\alpha$ appear when the $r$ and $l$ are quite different in size. It is also interesting that the effect of $\alpha$ is felt at values of $T$ considerably higher than $\alpha$. This is because, as explained in the paragraph below Eq. (41), both the states at the avoided level crossing can and do take advantage of large level widths.

Finally we consider the effect of adding the term $\beta$ in the spin-orbit interaction. A typical graph is found in Figure (8) where even a $\beta$ twice as big as $\alpha$ is seen to make little difference.

** In choosing parameters for these graphs one cannot choose all the $G^1, G^2, r$ and $l$ arbitrarily. This is because even though every choice of the four $|\phi|^2$ will lead to positive values for $G^1, G^2, r$ and $l$, the converse is not true. Not every choice of $G^1, G^2, r$ and $l$ has an inverse image with all $|\phi|^2$ positive.

To keep the left hand sides of Eqs. (48, 49) positive we must either satisfy $l > r$, and $r > \frac{G^1}{G^2} < l$, as in our choices above or a similar set of restriction for $r < l$. This is not a concern in extracting $|\phi|^2$ from the experimental $G^1, G^2, r$ and $l$, they will automatically generate the sensible positive underlying values of $|\phi|^2$.

**D. Experimental checklist**

Our paper began by looking for spin-orbit effects by starting with some input quantities and calculating experimentally measured numbers in terms of them. However, these input numbers, unlike the electron mass or charge, involved wave functions at the leads, energy gaps and so on, and are not known a priori and the response of the system to turning on $\alpha$ is very sensitive to these quantities.

We suggest a possible road map below to deal with this, being fully aware that our experimental colleagues will ultimately devise a better solution.
III. DISCUSSION AND SUMMARY

![Graph 6](image6.png)

**FIG. 6.** Plot of $\bar{G}$ versus $E_z$ for the case $\bar{G}^1 = 2\bar{G}^2$ and $r = 1.5, l = 2.5$. There is no prominent bump with or without $\alpha$.

**FIG. 7.** Plot of $\bar{G}$ versus $E_z$ for the case $\bar{G}^1 = 2\bar{G}^2$ and $l = 4, r = \frac{3}{4}$. There is no bump at $\alpha = 0$ (dotted lines), but there is one with $\alpha \neq 0$ (solid lines) which grows with $T$.

- Go to the lowest temperature possible.

- By studying the $E_z$ dependence of $V_g^{22}$ make sure the conductance peak describes the degeneracy of $(0, 0)$ and $(\frac{1}{2}, \frac{1}{2})$ which changes over to the degeneracy of $(1, 1)$ and $(\frac{1}{2}, \frac{1}{2})$. Since $\alpha \neq 0$, the rounded graph of Figure (3) will be seen. Since $T << \alpha$, the rounding is not due to thermal effects.††

- By retracing the asymptotically linear segments in $V_g$ locate $E_z^*$.

- Since $E_z^* = \delta - 2J$, find $\delta$ if $J$ is known or use its average value of $\alpha$.

- Estimate the size of $\alpha$ by comparing the difference of $V_g$’s between the two graphs at the cusp in Figure (3).

- Measure $\bar{G}(E_z)$ and see if it looks like Figure (2). If so measure $\bar{G}^1$ and $\bar{G}^2$.

- By suppressing the overall tunnelling rate at the $R$ (right) and $L$ (left) ends, (with a controllable barrier) measure $r$ and $l$, corresponding to the ratios of conductances. Bring tunnelling rates back to old value in the dot.

- At $E_z^*$ compare the predictions for $G^*(\alpha)$ the value with $\alpha \neq 0$ to the value without spin-orbit coupling, namely $\bar{G}^* = \frac{2}{3}(\bar{G}^1 + \bar{G}^2)$.

- Raise $T$ and compare to theoretical predictions from the rate equations. All the parameters needed $-|\phi_0/1(R/L)|^2, \alpha$, and $\delta$ – are known by this point.

- Try to raise the value of $\alpha$ by turning on inhomogeneities in the spin-orbit interaction using additional gates over the dot, following Brouwer et al.⁹.

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††A caveat here is that the parallel magnetic field also has the effect of enhancing the confinement of the 2DEG in the $z$ direction²³,²², leading to a quadratic dependence in all gate voltages. This will have to subtracted before our suggestion can be implemented.

—

Spin-orbit couplings in quantum dots have been studied extensively in the noninteracting limit in recent years⁸–⁹. In systems with homogeneous spin-orbit couplings, there is a new symmetry⁴ which emerges at the Aleiner-Fal’ko (AF) scale $c^{AF} \simeq (\frac{1}{4})^g \delta$, corresponding to a $s_z$ symmetry in a unitarily transformed basis⁴. This symmetry is broken at a scale $(\frac{1}{4})^g \delta$. For small quantum dots with $(\frac{1}{4}) \simeq 0.1$ and $g \simeq 10$, one might therefore expect spin-orbit effects to be utterly negligible.

Here we considered the problem when a robust exchange term $-JS^2$ of the Universal Hamiltonian is present. There are order unity effects on the conductance at and near special degeneracy points, where two different many-body ground states with equal particle number but different total spin have the same energy. These effects have unique signatures that can be produced only by the spin-orbit interaction. We have investigated the circumstances in which spin-orbit effects have the best prospect of making their presence felt, and proposed ways
III DISCUSSION AND SUMMARY

![Graph](image)

FIG. 8. Plot of $\bar{G}$ versus $E_Z$ for the case $G^1 = G^2$ and $l = 4, r = \frac{1}{2}$. Both are at $\alpha = 0.02$ but one has $\beta = 0$ and the other $\beta = 2\alpha = 0.04$. Clearly $\beta$ makes little difference.

to measure the effect and compare it to theory. The ideal venue for studying the phenomenon discussed here is in dots where $J$ has already brought the $S = 1$ state close to the $S = 0$ state and one tunes through the degeneracy by a Zeeman coupling. Our results are robust under the addition of non-universal Landau type interactions.

Our calculations, done in the orthogonal basis, showed that in the presence of $J$ spin-orbit terms enter the Hamiltonian to order $(\frac{\lambda}{g})$. We also saw how this phenomenon would appear in the AF basis, in which the spin-orbit interaction has been subsumed in the very choice of basis and a conserved quantity $\sigma_{z,AF}^z$ emerges. In this basis the culprit is the $-JS^2$ term which does not commute with $H_{AF}$, the non-interacting AF Hamiltonian. Upon unitary transformation, one finds $-JS^2 = -J(S_{AF}^z + \delta S_{AF}^z)^2$, where $\delta S_{AF}^z$ (which does not commute with $\sigma_{z,AF}^z$) is order $(\frac{\lambda}{g})/\sqrt{g}$ for homogeneous spin-orbit coupling. We have verified, by doing a calculation in the AF basis, that with the nonzero $J$, the AF degeneracy is indeed lifted and the spectrum coincides with what was found in the orthogonal basis.

Although the spin-orbit term appears to a lower order in $\frac{\lambda}{g}$ in the presence of $J$ than in the AF basis without $J$, its parametric dependence on the Thouless number $g$ is very different for homogeneous spin-orbit couplings. Whereas in the AF basis a typical matrix element of the spin-orbit coupling is $(\frac{\lambda}{g})^2/\sqrt{g}$, in the original basis its typical size is $(\frac{\lambda}{g})/\sqrt{g}$. The reason is evident from Eq. (14). In the original basis, the typical matrix elements of spin-orbit coupling averaged over the entire Thouless band are order $(\frac{\lambda}{g})/\sqrt{g}$, but the matrix elements between levels separated by $\delta$ are of size $(\frac{\lambda}{g})/\sqrt{g}$. In the AF basis, the spin-orbit matrix elements are those of $(r \times p_{AF})$ which does not suffer from this suppression. This is why Brouwer et al suggested that one could boost the spin-orbit term by making its coefficient $1/\lambda$ space dependent. In this case the matrix elements of the spin-orbit coupling, even between neighboring states, would go parametrically as $(\frac{\lambda}{g})/\sqrt{g}$.

Throughout this paper we have restricted ourselves to low temperatures $T \ll \delta$, since that allows us to deal with only a few many-body states. Let us briefly consider the effect of raising $T$ to be of order $\delta$ or higher, and considering the effect on the conductance peak height distribution. Excited states are populated at higher $T$, and these will suffer stronger spin-orbit effects, since the typical matrix elements of the spin-orbit coupling increase with energy (Eq. (15)). In parallel with this increase, states of higher spin suffer larger spin-orbit effects, even holding the matrix element constant. It will be interesting to ask how this affects the peak height distribution at moderate temperatures, and in particular if it brings the theory into better accord with experiments.

We have concentrated here on small quantum dots in which $\epsilon_{AF} \ll \delta$, but where $(\frac{\lambda}{g})/\sqrt{g}$ is still accessible at low temperatures. Our philosophy has been to keep $g$ constant, and perturb in $(\frac{\lambda}{g})$.

For conceptual completeness, let us consider the other limit, when $\epsilon_{AF} = (\frac{\lambda}{g})^4 g \delta \gg \delta$. This regime would be accessible for very large dots. For noninteracting systems Aleiner, Fal’ko and co-workers have made predictions for conductance averages and fluctuations in open dots, which have been confirmed experimentally.

For closed dots with interacting electrons, one expects that at energies very low compared to the AF scale $\epsilon_{AF}$, the Universal Hamiltonian has only Ising exchange interactions, since only $S_{AF}^z$ is conserved (see appendix). As one increases the energy, one expects to see universal crossover effects in, for example, the spectral density of transverse spin excitations. At energies much larger than $\epsilon_{AF}$ orthogonal symmetry is restored. Both these calculations started with the interaction term $-J(S_{AF}^z)^2$, which, as we know, differs from the correct starting point (see appendix), $-JS^2$. However, the difference, which is at least first order in $S_{AF}$, vanishes as $g \to \infty$. In this limit, the previous analyses are correct.

Finally, let us briefly touch upon the Stoner transition in the presence of a small spin-orbit coupling. In the orthogonal class, one sees a series of metamagnetic transitions with states of successively higher conserved spin as $J \to \delta$, with the bulk Stoner transition being their accumulation point. In the presence of spin-orbit interactions, for any finite $g$ these steps in $S$ are rounded, and the transition will become a regular second-order quantum phase transition, where the order parameter is not conserved (as we have seen above, even $S_{AF}^z$ is not conserved for finite $g$). On the other hand, if one works in the universal limit $g \to \infty$ while keeping $\epsilon_{AF}/\delta$ fixed, there will be a sequence of metamagnetic transi-
tions between states with increasing conserved $S^z_{AB}$. For very large but finite $g$ one can expect the steps of $S^z_{AB}$ to be rounded and merge into a smooth curve as $J \to \delta$.

We hope to present a full analysis of the different regimes in a future publication.

To conclude, we have found that in the presence of a robust exchange interaction $J$, one can, by tuning the Zeeman coupling, readily expose order-unity effects of the spin-orbit coupling on the conductance at and near a many-body degeneracy. The effects have signatures that allow one to trace them back to the spin-orbit coupling. To access these effects experimentally one needs to reduce the temperature to be of the order of the spin-orbit coupling strength. Our results are unaffected by a modest amount of spin-conserving non-universal interactions. We have offered a proposal for a sequence of experiments between states with increasing conserved $S^z_{AB}$.

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V. APPENDIX A

Here we show that adding the $-JS^2$ is the correct effective theory at the Thouless scale $E_T$ as long as $\epsilon_{AF} \ll E_T$, and discuss the possible low-energy theories for energies much less than $\epsilon_{AF}$.

Our reasoning is based on renormalization group (RG), which is an unbiased method for revealing the relevant and irrelevant terms, and is as follows: As is well-known, fermionic RG in the clean system leads to Landau Fermi liquid theory on a scale $E_{FLT}$ much smaller than the Fermi energy. Since $E_T$ can be made as small as we want by increasing the size $L$, we can always achieve the condition $E_T \ll E_{FLT}$. Therefore the correct starting point at the Thouless scale is a Fermi liquid theory. This theory is parameterized by Landau interaction parameters $u_m$, $m = 0, 1, \ldots$ in the spin singlet and triplet channels. For $m = 0$ the Landau parameters are (up to a factor of $\delta$) none other than the charging and spin exchange energies. Now consider carrying out RG in the original disordered basis with both the two-body Fermi liquid interaction and in addition a spin-orbit coupling. Without the spin-orbit coupling, this has been done by one of the authors and Harsh Mathur. The results are: (i) Any two-body interaction which is a conserved quantity squared will not flow. This includes the charging interaction and the $-JS^2$ Stoner interaction in the Universal Hamiltonian. (ii) Any other two-body interaction will flow, and flow to zero at weak coupling. This means that one can neglect the non-$s$-wave Landau interactions since they are irrelevant, and allows us to start with the universal Hamiltonian $-JS^2$ coupling at the Thouless energy. This argument also shows that higher channel spin-rotation-invariant Landau interactions do not affect the results of this paper, as long as they are weak.

Coming now to the added spin-orbit coupling, this will grow in RG as the cutoff scale decreases, indicating that it is relevant for low-energy physics. At some cutoff scale the size of the spin-orbit coupling will rival that of the cutoff. This scale should be identified as $\epsilon_{AF}$. Below this scale, one cannot treat the spin-orbit coupling perturbatively, but should start with exact states which incorporate it. In the present paper, we are working in the case $\epsilon_{AF} \ll \delta$, allowing us to treat the spin-orbit coupling perturbatively.

For the case $\epsilon_{AF} \gg \delta$, neglecting the terms of order $(\delta)^2 \epsilon_{AF}$ which break this $S^z_{AB}$ symmetry, there is a separate universal Hamiltonian regime at energies much smaller than $\epsilon_{AF}$. This regime was identified by Alhassid and Rupp and has only that part of the interaction which commutes with the kinetic term $-Jz S^z_{AB}$. One can obtain the same result by RG, this time diagonalizing the kinetic term with spin-orbit, and realizing that since the in-plane parts of the original Stoner interaction are not conserved, they do flow, and flow moreover to zero leaving behind the $-Jz S^z_{AB}$.
V  APPENDIX A

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