A WEIGHTED MINIMUM GRADIENT PROBLEM WITH COMPLETE ELECTRODE MODEL BOUNDARY CONDITIONS FOR CONDUCTIVITY IMAGING

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Abstract. We consider the inverse problem of recovering an isotropic electrical conductivity from interior knowledge of the magnitude of one current density field generated by applying current on a set of electrodes. The required interior data can be obtained by means of MRI measurements. On the boundary we only require knowledge of the electrodes, their impedances, and the corresponding average input currents. From the mathematical point of view, this practical question leads us to consider a new weighted minimum gradient problem for functions satisfying the boundary conditions coming from the Complete Electrode Model of Somersalo, Cheney and Isaacson. This variational problem has non-unique solutions. The surprising discovery is that the physical data is still sufficient to determine the geometry of the level sets of the minimizers. In particular, we obtain an interesting phase retrieval result: knowledge of the input current at the boundary allows determination of the full current vector field from its magnitude. We characterize the non-uniqueness in the variational problem. We also show that additional measurements of the voltage potential along one curve joining the electrodes yield unique determination of the conductivity. A nonlinear algorithm is proposed and implemented to illustrate the theoretical results.

Key words. minimum gradient, conductivity imaging, complete electrode model, current density impedance imaging, minimal surfaces, magnetic resonance electrical impedance tomography, current density impedance imaging

AMS subject classifications. 35R30, 35J60, 31A25, 62P10

1. Introduction. We consider the inverse problem of reconstructing an inhomogeneous isotropic electrical conductivity $\sigma$ in a domain $\Omega \subset \mathbb{R}^d$, $d \geq 2$, from interior knowledge of the magnitude $a$ of one current density field and of corresponding boundary data.

Most of the existing results on this problem (see a brief survey of previous work at the end of this introduction) consider Dirichlet boundary conditions. In this paper we study boundary conditions which model what can actually be measured in practical experiments. We work with the beautiful Complete Electrode Model (CEM) originally introduced in [20] and shown to best describe the physical data: Let $e_k \subset \partial \Omega$ denote the surface electrode of impedance $z_k$ through which one injects a net current $I_k$, $k = 0, \ldots, N$. The CEM assumes the voltage potential $u$ inside and the constant voltages $U_k$’s on the surface of the electrodes distribute according to the boundary

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value problem
\[ \nabla \cdot \sigma \nabla u = 0, \quad \text{in } \Omega, \]  
\[ u + z_k \sigma \frac{\partial u}{\partial \nu} = U_k \quad \text{on } e_k, \quad \text{for } k = 0, \ldots, N, \]  
\[ \int_{e_k} \sigma \frac{\partial u}{\partial \nu} ds = I_k, \quad \text{for } k = 0, \ldots, N, \]  
\[ \frac{\partial u}{\partial \nu} = 0, \quad \text{on } \partial \Omega \setminus \bigcup_{k=0}^{N} e_k, \]  
where \( \nu \) is the outer unit normal. For brevity, we refer to the problem (1.1), (1.2), (1.3), and (1.4) as to the forward problem.

If a solution exists, an integration of (1.1) over \( \Omega \) together with (1.3) and (1.4) show that
\[ \sum_{k=0}^{N} I_k = 0 \]  
is necessary. Physically, the zero sum of the boundary currents account for the absence of sources/sinks of charges. The constants \( U_k \) appearing in (1.2) represent unknown voltages on the surface of the electrodes, and the difference from the traces \( u|_{e_k} \) of the interior voltage potential governs the flux of the current through the skin to the electrode. For conductivities of real part bounded away from zero and infinity, the problem has a unique solution \( (u; \langle U_0, \ldots, U_N \rangle) \in H^1(\Omega) \times \mathbb{C}^{N+1} \) up to an additive constant, as shown in [20].

In the inverse problem considered here \( \sigma \) is unknown, but assumed real valued and satisfying
\[ \epsilon \leq \sigma(x) \leq \epsilon^{-1}, \quad \text{a.e. in } \Omega, \]  
for some \( \epsilon > 0 \). We also limit our study to (real valued) positive surface electrode impedances, although we allow these, in general, to be inhomogeneous functions of position on the electrodes. More precisely we assume for each \( k = 0, \ldots, N \) that
\[ \epsilon \leq z_k(x) \leq \epsilon^{-1}, \quad \mathcal{H}^{n-1} \quad \text{a.e. on } e_k, \]  
where \( \mathcal{H}^{n-1} \) is the \((n-1)\)-Hausdorff measure, and \( \epsilon > 0 \), which without loss of generality can be chosen the same as above.

We normalize a constant by imposing the electrode voltages \( U = \langle U_0, \ldots, U_N \rangle \) to lie in the hyperplane
\[ \Pi := \{ U \in \mathbb{R}^{N+1} : U_0 + \ldots + U_N = 0 \}. \]  

We seek to determine \( \sigma \) given the magnitude \( a \) of one current density field inside \( \Omega \),
\[ a = |\sigma \nabla u|, \]  
where \((u, U) \in H^1(\Omega) \times \Pi \) is the solution of the forward problem. The electrodes \( e_k \subset \partial \Omega \) and their impedances \( z_k \), for \( k = 0, \ldots, N \), are known and assumed fixed. The boundary data consist of the known currents satisfying (1.5).
We note that, in practice, interior measurements of all three components of the current density \( J \) can be obtained from three magnetic resonance scans involving two rotations of the object [19]. However recent engineering advances in ultra-low field magnetic resonance may be used to recover \( J \) without rotating the object [18]. We hope that the results presented here may lead to further experimental progress on easier ways to measure directly just the magnitude of the current.

We start by remarking that there is non-uniqueness in the inverse problem stated above, as can be seen in the following example: Let \( \Omega = (0,1) \times (0,1) \) be the unit square. We inject the current \( I_1 = 1 \) through the top electrode \( e_1 = \{(x,1) : 0 \leq x \leq 1\} \) of impedance \( z_1 > 0 \), “extract” the current \( I_0 = -1 \) through the bottom electrode \( e_0 = \{(0,x) : 0 \leq x \leq 1\} \) of impedance \( z_0 = z_1 + 1 \), and measure the magnitude \( a \equiv 1 \) of the current density field in \( \Omega \). Then, for every \( \varphi : [0,1] \rightarrow [\varphi(0),\varphi(1)] \) an increasing Lipschitz continuous function, satisfying \( \varphi(0) + \varphi(1) = 1 \), the function \( u_\varphi(x,y) := \varphi(y) \) solves the forward problem (1.1), (1.2), (1.3), and (1.4) corresponding to a conductivity \( \sigma_\varphi(x,y) = 1/\varphi'(y) \), yet the magnitudes of the corresponding current densities yield the same interior measurements \( \sigma|\nabla u| = \sigma_\varphi|\nabla u_\varphi| \equiv 1 \).

More generally, if \( (u,U) \in H^1(\Omega) \times \Pi \) is the solution of the forward problem for some \( \sigma \), let \( \varphi \in Lip(u(\overline{\Omega})) \) be any Lipschitz-continuous increasing function of one variable, such that \( \varphi(t) = t + c_k \) whenever \( t \in u(e_k) \), for each \( k = 0, ..., N \), and constants \( c_k \) satisfying \( \sum_{k=0}^N c_k = 0 \). One can easily verify that the function

\[
u_\varphi := \varphi \circ u
\]

solves the forward problem with the conductivity

\[
\sigma_\varphi := \frac{\sigma}{\varphi' \circ u}, \tag{1.11}
\]

where \( \sigma|\nabla u| = \sigma_\varphi|\nabla u_\varphi| \).

For Hölder-continuous conductivities, in Theorem 3.1 we prove that (1.10), (1.11) is the only way non-uniqueness occurs. As in [14] we formulate the problem in terms of a weighted minimum gradient problem. Here we need a functional which accounts for boundary conditions coming from the Complete Electrode Model. We first show that the solution \( (u,U) \in H^1(\Omega) \times \Pi \) of the forward problem is a global minimizer of the functional

\[
G_\alpha(v,V) = \int_\Omega a|\nabla v|dx + \sum_{k=0}^N \int_{e_k} \frac{1}{2\varepsilon_k} (v - V_k)^2 ds - \sum_{k=0}^N I_k V_k, \tag{1.12}
\]

over \( H^1(\Omega) \times \Pi \), with \( \alpha = \sigma|\nabla u| \) as in (1.9). To understand the motivation for this definition, see our treatment of the forward problem in the Appendix.

Our analysis of the minimization problem for \( G_\alpha \) (which only involves the data in the inverse problem) will allow us to characterize the non-uniqueness described above. The surprising discovery here is that, given the positions and impedences of the electrodes, knowledge of the magnitude of one current density field and of the corresponding average applied currents (just one number in the case of two electrodes!) is still sufficient to determine the geometry of the equipotential sets. This is different from the Dirichlet case [14], where full knowledge of the trace of the level sets on the boundary (read off from the data) was used. Furthermore, we remark that, since we recover the direction \( \overrightarrow{N} \) (including orientation) of the electric field \( \nabla u \), we also obtain an interesting phase retrieval result: the full current density vector field \( J = a \overrightarrow{N} \) in
Ω is recovered from its magnitude \( a \), and knowledge of the input currents \( I_k \) on the surface electrodes (even if the conductivity is not uniquely determined).

Uniqueness can be restored by additional measurements of the voltage potential on one curve connecting the electrodes; see Theorem 3.4. This additional data involves only a one dimensional subset of boundary measurements, much less than the boundary measurements needed in the Dirichlet problem.

To illustrate the theoretical results, we propose an iterative algorithm and perform a numerical experiment, see Section 4. Similar to the algorithm in [14] we decrease the functional \( G_a \) on a sequences of solutions of the forward problems for updated conductivities.

Conductivity imaging using the interior knowledge of the magnitude of current densities was first introduced in [4]. The examples of non-existence and non-uniqueness for the ensuing Neumann problem lead the authors of [4] to consider the magnitudes of two currents. The possibility of conductivity imaging via the magnitude of just one current density field was shown in [13] via the Cauchy problem, and in [14, 16] via a minimum gradient problem with Dirichlet boundary conditions. Existence and uniqueness of such weighted gradient problems was studied in [2]. Extensions to the case of inclusions with zero or infinite conductivity were obtained in [11, 12]. A structural stability result for the minimization problem can be found in [17]. Reconstruction algorithms based on the minimization problem were proposed in [14] and [10], and based on level set methods in [13, 14, 22]. A local Hölder-continuous dependence of \( \sigma \) on \( |J| \) (for unperturbed Dirichlet data) has been recently established in [9]. Among the works which either use the interior knowledge of full \( J \) or multiple measurements of current magnitudes to determine an electrical conductivity we mention [24, 4, 6, 7, 3, 8, 5].

2. A weighted minimum gradient problem for the CEM boundary conditions. In this section we show that the solution of the forward problem is a global minimizer of the functional \( G_a \) in (1.12) over \( H^1(\Omega) \times \Pi \).

Proposition 2.1. Let \( \Omega \subset \mathbb{R}^n \), \( n \geq 2 \), be a bounded domain with Lipschitz boundary. Let \( \sigma \) satisfy (1.6) and, for \( k = 0, ..., N \), let \( e_k \) be disjoint subsets of the boundary of positive \((n - 1)\)-Hausdorff measure, with impedances \( z_k \) satisfying (1.7), and \( I_k \) satisfy (1.5). Let \( (u, U) \in H^1(\Omega) \times \Pi \) be the unique solution of the forward problem (1.1), (1.2), (1.3), (1.4). If \( a := \sigma|\nabla u| \), then

\[
G_a(v, V) \geq G_a(u, U), \quad \forall (v, V) \in H^1(\Omega) \times \Pi.
\]

Proof. For any \( (v, V) \in H^1(\Omega) \times \Pi \), we have the inequality

\[
G_a(v, V) \geq G_a(u, U), \quad \forall (v, V) \in H^1(\Omega) \times \Pi.
\]
A weighted minimum gradient problem in conductivity imaging

\[ G_a(v, V) = \int_{\Omega} \sigma |\nabla u| |\nabla v| dx + \frac{1}{2} \sum_{k=0}^{N} \int_{e_k} \frac{1}{z_k} (v - V_k)^2 ds - \sum_{k=0}^{N} I_k V_k \]

\[ \geq \int_{\Omega} \sigma \nabla u \cdot \nabla v dx + \sum_{k=0}^{N} \int_{e_k} \left[ \frac{1}{2z_k} (v - V_k)^2 - V_k \frac{\partial u}{\partial \nu} \right] ds \]

\[ = \int_{\partial \Omega} v \sigma \frac{\partial u}{\partial \nu} ds + \sum_{k=0}^{N} \int_{e_k} \left[ \frac{1}{2z_k} (v - V_k)^2 - V_k \sigma \frac{\partial u}{\partial \nu} \right] ds \]

\[ = \sum_{k=0}^{N} \int_{e_k} \frac{1}{z_k} \left[ (v - V_k)(U_k - u) + \frac{1}{2} (v - V_k)^2 \right] ds, \quad (2.2) \]

where the first equality uses (1.9), the next line uses (1.3) and the Cauchy-Schwarz inequality, the next equality uses (1.1) and the divergence theorem, the third equality uses (1.4), and the last equality uses (1.2).

In particular, when \( v = u \), the inequality in the estimate (2.2) holds with equality yielding

\[ G_a(u, V) = \sum_{k=0}^{N} \int_{e_k} \frac{1}{z_k} \left[ \frac{1}{2} (u - V_k)^2 - (u - V_k)(u - U_k)^2 \right] ds, \quad (2.3) \]

and

\[ G_a(u, U) = -\frac{1}{2} \sum_{k=0}^{N} \int_{e_k} \frac{1}{2z_k} (u - U_k)^2 ds. \quad (2.4) \]

The global minimizing property (2.1) then follows from (2.2) and (2.4) using the pointwise inequality

\[ \frac{1}{2} (v - V_k)^2 - (v - V_k)(u - U_k) \geq -\frac{1}{2} (u - U_k)^2. \quad (2.5) \]

Note how each of the CEM conditions (1.1), (1.2), (1.3), (1.4) was used in the above proof.

3. Characterization of non-uniqueness and applications. In this section we state and prove our main result and its consequences to the conductivity imaging problem.

Theorem 3.1. Let \( \Omega \subset \mathbb{R}^d \), \( d \geq 2 \) be a bounded, connected \( C^{1,\alpha} \)-domain, for some \( 0 < \alpha < 1 \), and let \( e_k \), \( k = 0, \ldots, N \), be disjoint subsets of the boundary of positive \((n-1)\)-Hausdorff measure. Assume that the corresponding impedances \( z_k \) satisfy (1.7), and that the given currents \( I_k \) are such that (1.5) holds. Let \( (u, U), (v, V) \in H^1(\Omega) \times \Pi \), be the solutions of the forward problem (1.1), (1.2), (1.3), and (1.4) corresponding to unknown conductivities \( \sigma, \tilde{\sigma} \in C^a(\Omega) \) satisfying (1.6). Assume that

\[ \sigma |\nabla u| = \tilde{\sigma} |\nabla v| > 0 \ a.e. \ in \ \Omega. \quad (3.1) \]
Then there exists \( \varphi \in C^1(u(\Omega)) \), with \( \varphi'(t) > 0 \) a.e. in \( \Omega \), such that

\[
v = \varphi \circ u, \quad \text{in } \Omega,
\]

and

\[
\tilde{\sigma} = \frac{\sigma}{\varphi' \circ u}, \quad \text{a.e. in } \Omega.
\]

Moreover, for each \( k = 0, \ldots, N \) and \( t \in v(e_k) \) (the range of \( v \) on the electrode), we have

\[
\varphi(t) = t + (U_k - V_k).
\]

Proof. From the interior elliptic regularity we have \( u, v \in C^{1,\alpha}(\Omega) \) (e.g., [1, Theorem 8.34]).

According to Proposition 2.1 \((u, U)\) and \((v, V)\) are both minimizers of \( G_a \), and thus

\[
G_a(u, U) = G_a(v, V).
\]

In particular the inequalities (2.2) and (2.5) hold with equality to yield

\[
u|_{e_k} - U_k = v|_{e_k} - V_k, \quad \text{a.e. on } e_k, \quad k = 0, 1, \ldots, N,
\]

and

\[
\nabla u \cdot \nabla v = |\nabla u| \cdot |\nabla v|, \quad \text{a.e. on } \Omega.
\]

Let \( S := \{x \in \Omega : |\nabla u(x)| = 0\} \cup \{x \in \Omega : |\nabla v(x)| = 0\} \). Since \( u, v \in C^1(\Omega) \), \( S \) is closed and by hypothesis (3.1) (Lebesgue-) negligible in \( \Omega \), so that the measure \(|\Omega| = |\Omega \setminus S| = |\Omega \setminus S|\). It follows that \( \Omega \setminus S \) is dense in \( \Omega \); otherwise, some \( x_0 \) together with a neighborhood \( B \) (of positive measure) lie in \( \Omega \setminus \Omega \setminus S \), thus \( B \cap \Omega \setminus S = \emptyset \).

Therefore the strict inequality \(|\Omega \setminus S| < |\Omega|\) holds; this would be a contradiction.

Since both gradients are continuous, in view of (3.7) they must be parallel whenever one of them is nonzero, in particular

\[
\nabla v = \mu \nabla u,
\]

for some \( \mu \in C(\Omega \setminus S) \) with \( \mu > 0 \) in \( \Omega \setminus S \). Moreover, since \( \Omega \setminus S \) is dense, \( \mu \) extends by continuity to the whole domain \( \Omega \).

A differentiation in the direction tangential to a regular level set together with (3.8) yield that \( u \) and \( v \) are constant on the connected components within \( \Omega \setminus S \) of each other’s level sets. For each \( t \in u(\Omega \setminus S) \) and \( L_t \) a connected component of the level set \( \{x \in \Omega \setminus S : u(x) = t\} \), we define

\[
\varphi(t) := v|_{L_t}
\]

Since \( L_t \) is also a level set for \( v \), the function \( \varphi \) is well defined by (3.9) on the range \( v(\Omega \setminus S) \), and the relation \( v(x) = \varphi(u(x)) \) holds on \( \Omega \setminus S \). Since \( v(x) \) is continuous and \( \Omega \setminus S \) is dense in \( \Omega \), the equality extends by continuity to \( \Omega \), thus proving (3.2).
Moreover, since \( v \in C^1(\Omega) \), at each point \( x \in \Omega \setminus S \), \( v \) is differentiable in the direction of \( \nabla v(x) \) (which does not vanish), and therefore \( \varphi \) is differentiable at \( t = u(x) \), and

\[
\nabla v(x) = \varphi'(u(x))\nabla u(x). \tag{3.10}
\]

Now (3.8) implies \( \varphi'(u(x)) = \mu(x) \), for all \( x \in \Omega \setminus S \). Since \( \mu \in C(\Omega) \), so is \( \varphi' \) on the range \( u(\Omega) \). Moreover, \( \varphi' \geq 0 \), with strict inequality on \( u(\Omega \setminus S) \).

By using (3.10) in (3.1) we obtain

\[
\sigma(x)|\nabla u(x)| = \tilde{\sigma}(x)\varphi'(u(x))|\nabla u(x)|, \quad \forall x \in \Omega \setminus S.
\]

Since \( |\nabla u| > 0 \) on \( \Omega \setminus S \), the relation (3.3) follows on \( \Omega \setminus S \).

By combining (3.6) with (3.2), for each \( t \in v(e_k) \) (a voltage on \( e_k \)) we obtain (3.4).

\[
\square
\]

The above theorem also shows that knowledge of the input currents at the boundary is sufficient to determine the full current density from measurements of its magnitude in the interior (even when the conductivity is not determined uniquely). We state separately this “phase retrieval” result.

**Corollary 3.2.** Let \( \Omega \subset \mathbb{R}^d \), \( d \geq 2 \) be a bounded connected \( C^{1,\alpha} \)-domain, for some \( 0 < \alpha < 1 \). Let \( I_k \) be known currents on the electrodes \( e_k \subset \partial \Omega \) of impedances \( z_k \), \( k = 0, \ldots, N \), which satisfy (1.5). Let \( J := \sigma \nabla u \) and \( \tilde{J} := \tilde{\sigma} \nabla v \), where \( (u, U) \), \( (v, V) \in H^1(\Omega) \times \Pi \) are the solutions of the forward problem (1.1), (1.2), (1.3), and (1.4) corresponding to some unknown \( \sigma, \tilde{\sigma} \in C^\alpha(\Omega) \) satisfying (1.6). If

\[
|J| = |	ilde{J}| > 0 \text{ a.e. in } \Omega, \tag{3.11}
\]

then

\[
J = \tilde{J} \text{ in } \Omega. \tag{3.12}
\]

The proof is immediate from (3.3) and (3.10):

\[
\tilde{J} = \tilde{\sigma} \nabla v = \frac{\sigma}{\varphi' \circ u} \nabla v = \sigma \nabla u = J, \quad \text{in } \Omega \setminus S.
\]

Since \( J, \tilde{J} \) are continuous in \( \Omega \), and \( \Omega \setminus S \) is dense in \( \Omega \), the result follows.

In order to determine the conductivity we identify next some additional data which will give \( \varphi' \equiv 1 \). It will suffice to assume that the voltage potential \( u \) along a curve on the boundary joining the electrodes is known up to a constant. The main idea of the proof will be to show that the range of \( u \) on the union of this curve and the electrodes is the same as the range of \( u \) in \( \overline{\Omega} \). We formulate this separately, as a maximum principle for the Complete Electrode Model.

**Proposition 3.3.** (Maximum principle for CEM) Let \( \Omega, \sigma, e_k' s, z_k' s, \) and \( I_k' s \), for \( k = 0, \ldots, N \) be as in Theorem 3.1, and let \( u \) be a solution of the forward problem. Then \( u \) achieves its minimum and maximum on the electrodes \( e_0 \cup \ldots \cup e_N \). Moreover, if \( \Gamma \subset \partial \Omega \) is a curve connecting the electrodes, then the range of \( u \) over \( \Gamma \cup e_0 \cup \ldots \cup e_N \) coincides with the range of \( u \) over \( \overline{\Omega} \).

**Proof.** By the weak maximum principle, the maximum \( M \) and minimum \( m \) of \( u \) over \( \overline{\Omega} \) occur on the boundary. By Hopf’s strong maximum principle, at a point of maximum, say \( x_0 \in \partial \Omega \), the normal derivative \( \frac{\partial u}{\partial n}(x_0) \) must be strictly positive. From
the boundary condition (1.4) we then deduce \( x_0 \in e_0 \cup \ldots \cup e_N \). The same argument applies to a point of minimum, where the normal derivative is strictly negative.

Since \( \Gamma \cup e_0 \cup \ldots \cup e_N \) is connected, \( u(\Gamma \cup e_0 \cup \ldots \cup e_N) \) is an interval. Since the maxima and minima occur on the electrodes \( e_0 \cup \ldots \cup e_N \), then the range \( u(\Gamma \cup e_0 \cup \ldots \cup e_N) = [m, M] = u(\Omega) \).

**Theorem 3.4 (Unique determination).** Let \( \Omega \subset \mathbb{R}^d, \ d \geq 2 \) be a bounded, connected \( C^{1,\alpha} \)-domain, for some \( 0 < \alpha < 1 \). Let \( e_k \subset \partial \Omega \) denote the electrode with impedance \( z_k \) satisfying (1.7), for \( k = 0, \ldots, N \) and \( \Gamma \) be a curve on the boundary connecting the electrodes. For currents \( I_k \) which satisfy (1.5), let \( (u,U), (\tilde{v},V) \in H^1(\Omega) \times \Pi \) be the solutions of the forward problem (1.1), (1.2), (1.3), and (1.4) corresponding to unknown conductivities \( \sigma, \tilde{\sigma} \in C^\alpha(\Omega) \) satisfying (1.6).

Assume that

\[
\sigma |\nabla u| = \tilde{\sigma} |\nabla \tilde{u}| > 0, \ a.e. \ in \ \Omega, \tag{3.13}
\]

\[
u|\Gamma = \tilde{u}|\Gamma + C, \tag{3.14}
\]

for some constant \( C \). Then

\[
u = \tilde{u} + C \ \text{in} \ \overline{\Omega}, \tag{3.15}
\]

\[
\sigma = \tilde{\sigma} \ \text{in} \ \Omega. \tag{3.16}
\]

**Proof.** By applying the characterization of possible non-uniqueness in Theorem 3.1 we have that

\[
u(x) = \varphi \circ \tilde{u}(x), \ \text{in} \ \Omega, \tag{3.17}
\]

and, for \( t \) a value in the range \( \tilde{u}(e_k) \), \( k = 0, \ldots, N \),

\[
\varphi(t) = t + (U_k - V_k). \tag{3.18}
\]

For \( t \in \tilde{u}(\Gamma) \) a measured voltage value on \( \Gamma \), by (3.2) we have that

\[
\varphi(t) = t + C. \tag{3.19}
\]

From the continuity of \( \tilde{u} \) on \( \overline{\Omega} \), in particular at a contact point on \( \Gamma \cap e_k \), we obtain

\[
U_k - V_k = C, \ \forall k = 0, \ldots, N, \tag{3.20}
\]

thus (3.18) holds for voltages \( t \) on \( \Gamma \cup e_0 \cup \ldots \cup e_N \).

Proposition 3.3 showed that the range of \( \tilde{u} \) on \( \Gamma \cup e_0 \cup e_1 \) is the same as the range on \( \overline{\Omega} \). We conclude that

\[
\varphi(t) = t + C, \ \forall t \in \tilde{u}(\overline{\Omega}). \tag{3.21}
\]

From (3.17) and (3.20) we have that (3.15) holds in \( \overline{\Omega} \). In particular \( \nabla u = \nabla \tilde{u} \) in \( \Omega \) and, by (3.13), \( \sigma = \tilde{\sigma} \) in \( \Omega. \)
4. A minimization algorithm for the weighted gradient functional with CEM boundary constraints. In this section we propose an iterative algorithm which minimizes the functional $G_a$ in (1.12). It is the analogue of an algorithm in [14] adapted to the CEM boundary conditions.

The following lemma is key to constructing a minimizing sequence for the functional $G_a$.

**Lemma 4.1.** Assume that $v \in H^1(\Omega)$ satisfies

$$\epsilon \leq \frac{a}{|\nabla v|} \leq \frac{1}{\epsilon},$$

(4.1)

for some $\epsilon > 0$, and let $(u, U) \in H^1(\Omega) \times \Pi$ be the unique solution to the forward problem for $\sigma := a/|\nabla v|$. Then

$$G_a(u, U) \leq G_a(v, V), \quad \text{for all } V \in \Pi.$$  

(4.2)

Moreover, if equality holds in (4.2) then $(u, U) = (v, V)$.

**Proof.** Let $V \in \Pi$ be arbitrary. Since $(u, U)$ is a global minimizer of $F_\sigma$ as in (A.11) with $\sigma = a/|\nabla v|$ as shown in Theorem A.4, we have the inequality:

$$G_a(v, V) = \int_\Omega a|\nabla v|dx + \frac{1}{2} \sum_{k=0}^N \left[ \int_{\epsilon_k} (v - V_k)^2 ds - 2I_k V_k \right]$$

$$= \frac{1}{2} \int_\Omega a|\nabla v|dx + F_\sigma(v, V)$$

$$\geq \frac{1}{2} \int_\Omega a|\nabla v|dx + F_\sigma(u, U).$$

(4.3)

Writing

$$\int_\Omega a|\nabla u|dx = \int_\Omega \left[ \frac{a}{|\nabla v|} \right]^{\frac{1}{2}} |\nabla v| \left[ \frac{a}{|\nabla v|} \right]^{\frac{1}{2}} |\nabla u|dx$$

$$\leq \left( \int_\Omega \frac{a}{|\nabla v|} |\nabla v|^2 dx \right)^{\frac{1}{2}} \left( \int_\Omega \frac{a}{|\nabla v|} |\nabla u|^2 dx \right)^{\frac{1}{2}}$$

$$\leq \frac{1}{2} \int_\Omega a|\nabla v|dx + \frac{1}{2} \int_\Omega \frac{a}{|\nabla v|} |\nabla u|^2 dx,$$

we also obtain

$$G_a(u, U) = \int_\Omega a|\nabla u|dx + \frac{1}{2} \sum_{k=0}^N \left[ \int_{\epsilon_k} (u - U_k)^2 ds - 2I_k U_k \right]$$

$$\leq \frac{1}{2} \int_\Omega a|\nabla v|dx + \frac{1}{2} \int_\Omega \frac{a}{|\nabla v|} |\nabla u|^2 dx + \frac{1}{2} \sum_{k=0}^N \left[ \int_{\epsilon_k} (u - U_k)^2 ds - 2I_k U_k \right]$$

$$= \frac{1}{2} \int_\Omega a|\nabla v|dx + F_\sigma(u, U).$$

(4.4)

From (4.3) and (4.4) we conclude (4.2). Moreover, if the equality holds in (4.2) then equality holds in (4.3), and thus

$$F_\sigma(u, U) = F_\sigma(v, V).$$

(4.5)
Since \((u, U)\) is a solution to the forward problem (for \(\sigma = a/|\nabla v|\)) it is also a global minimizer of \(F_{a/|\nabla v|}\) over \(H^1(\Omega) \times \Pi\). But (4.5) shows that \((v, V)\) is also a global minimizer for \(F_{a/|\nabla v|}\). Now the uniqueness of the global minimizers in Theorem A.4 (for \(\sigma = a/|\nabla v|\)) yields \((u, U) = (v, V)\).

**Algorithm:** We assume the magnitude \(a\) of the current density satisfies
\[
\operatorname{essinf}(a) > 0. \tag{4.6}
\]

Let \(\epsilon > 0\) be the lower bound in (1.6), and \(\delta > 0\) a measure of error to be used in the stopping criteria.

- Step 1: Solve (1.1, 1.2, 1.3) and (1.4) for \(\sigma = 1\), and let \(u_0\) be its unique solution. Define
\[
\sigma_1 := \min \left\{ \max \left\{ \frac{a}{|\nabla u_0|}, \epsilon \right\}, \frac{1}{\epsilon} \right\};
\]
- Step 2: For \(\sigma_n\) given: Solve (1.1, 1.2, 1.3) and (1.4) for the unique solution \(u_n\);
- Step 3: If
\[
\|\nabla u_n - \nabla u_{n-1}\|_{C(\Pi)} > \delta \frac{\epsilon}{\operatorname{essinf} a},
\]
then define
\[
\sigma_{n+1} := \min \left\{ \max \left\{ \frac{a}{|\nabla u_n|}, \epsilon \right\}, \frac{1}{\epsilon} \right\} \tag{4.7}
\]
and repeat Step 2;
- Else STOP.

5. **Numerical Implementations.** We illustrate the theoretical results on a numerical simulation in two dimensions.

**5.1. An algorithm for the forward problem.** Given a current pattern \(I \in \Pi\) and a set of surface electrodes \(e_k\) with impedances \(z_k\) (taken to be constant) for \(k = 0, 1, \ldots, N\) satisfying (1.7), our iterative algorithm consists in solving the forward problem (1.1), (1.2), (1.3) and (1.4) for an updated conductivity at each step.

A piecewise linear (spline) approximation of the solution to the forward problem is sought on an uniform triangulation of the unit box \([0, 1] \times [0, 1]\) as shown in figure 5.1.

For a (square) number \(m\) of grid nodes, let \(T_l\) be the set of planes supported in the \(l\)-th triangle \(\Delta_l\), for \(l = 1, 2, \ldots, 2(\sqrt{m} - 1)^2\). More precisely, when \(l = odd\),
\[
T_l = \left\{ 1 - \frac{1}{h}(x - x_{k_1}) - \frac{1}{h}(y - y_{k_1}), \frac{1}{h}(x - x_{k_1}), \frac{1}{h}(y - y_{k_1}) \right\}, \ (x, y) \in \Delta_l,
\]
where \((x_{k_1}, y_{k_1})\) is the southwest grid point of the square in which \(\Delta_l\) is inscribed, and \(h\) is the length of the side of the square.

When \(l = even\),
\[
T_l = \left\{ 1 + \frac{1}{h}(x - x_{r_1}) + \frac{1}{h}(y - y_{r_1}), -\frac{1}{h}(y - y_{r_1}), -\frac{1}{h}(x - x_{r_1}) \right\}, \ (x, y) \in \Delta_l,
\]
where \((x_{r1}, y_{r1})\) is the northeast grid point of the square in which \(\Delta_l\) is inscribed, and \(h\) is the length of the side of the square. For example, in Figure 5.1 the triangle \(\Delta_{11}\) lies in a square whose southwest grid point position is \((x_7, y_7)\) and the northeast grid point location is \((x_{12}, y_{12})\).

We seek an approximation to the solution of the forward problem (1.1), (1.2), (1.3), and (1.4) in the form

\[
u(x, y) \approx \sum_{j=1}^{m} u_j \psi_j(x, y) , \tag{5.1}\]

where \(\psi_j\) is the sum over those planes \(T_l\)'s, that are adjacent to the \(j\)-th node in the unit box, see figure 5.1. By substituting (5.1) into (A.12), and by selecting \(v = \psi_j\), for \(j = 1, \ldots, m\), and \(V \equiv 0\), we get the set of equations

\[
\int_\Omega \sigma \nabla u \cdot \nabla \psi_j \, dx \, dy + \sum_{k=0}^{N} \frac{1}{z_k} \int_{c_k} (u - U_k) \psi_j \, ds = 0, \quad \forall j = 1, \ldots, m , \tag{5.2}\]

which is augmented with the second set of equations

\[
- \sum_{k=0}^{N} \frac{1}{z_k} \int_{c_k} (u - U_k) V_k^j \, ds = \sum_{k=0}^{N} I_k V_k^j , \quad \forall j = 0, \ldots, N - 1 \tag{5.3}\]

obtained by setting \(v \equiv 0\) and \(V_k^j = 1\) whenever \(k = j\) for \(k = 0, \ldots, N - 1\), and \(V^j_{N} = -1\) for \(j = 0, \ldots, N - 1\).

Note that forming \(V\) in this fashion is equivalent to choosing for each \(j = 0, 1, \ldots, N - 1\) the \(j\)-th vector for the \(j\)-th equation in (5.3) from the set

\[
\left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ -1 \end{bmatrix} \right\} , \tag{5.4}\]
which is a basis for $\Pi$, and then setting

$$U_N = -\sum_{k=0}^{N-1} U_k.$$  

The values of $\{u_1, u_2, \ldots, u_m, U_0, U_1, \ldots, U_{N-1}\}$ are then solutions to the linear system

$$\begin{bmatrix} \Lambda & \Psi \\ \Psi^T & \Upsilon \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \\ U_0 \\ U_1 \\ \vdots \\ U_{N-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ I_0 - I_N \\ I_1 - I_N \\ \vdots \\ I_{N-1} - I_N \end{bmatrix},$$  

(5.4)

where the entries of $\Lambda$ are

$$\Lambda_{i,j} = \int_{\Omega} \sigma \nabla \psi_j \cdot \nabla \psi_i dxdy + \sum_{k=0}^{N-1} \frac{1}{z_k} \int_{\partial \epsilon_k} \psi_j \psi_i ds, \quad i,j = 1,2,\ldots,m,$$

the entries of $\Psi$ are

$$\Psi_{i,j} = \frac{1}{z_N} \int_{\partial \epsilon_N} \psi_i ds - \frac{1}{z_k} \int_{\partial \epsilon_k} \psi_i ds, \quad i = 1,2,\ldots,m, \text{ & } k = 0,\ldots,N-1,$$

and

$$\Upsilon = \begin{bmatrix} \frac{|e_0|}{z_0} + \frac{|e_N|}{z_N} \\ \frac{|e_0|}{z_0} + \frac{|e_1|}{z_1} \\ \vdots \\ \frac{|e_0|}{z_0} + \frac{|e_N|}{z_N} \\ \frac{|e_1|}{z_1} & \frac{|e_N|}{z_N} \\ \vdots & \vdots \\ \frac{|e_{N-2}|}{z_{N-2}} & \frac{|e_{N-1}|}{z_{N-1}} + \frac{|e_N|}{z_N} \end{bmatrix}.$$

In the matrix above $|e_j|$ denotes the surface area of the electrode, for $j = 0,\ldots,N$.

For other numerical schemes for solving the forward problem we refer to [23].

5.2. Simulating the interior data. We consider a simulated planar conductivity $\sigma$ which models the cross section of a torso embedded in the unit box $[0,1] \times [0,1]$; see Figure 5.2 on the left. The values of the conductivity range from $1.0 \ S/m$ to $1.8 \ S/m$.

Two currents $-I_0 = I_1 = 3 \ mA$ are respectively injected/extracted through the electrodes

$$e_0 = \{(x,y) \in [0,1] \times [0,1] : y = 0\} \quad \text{and} \quad e_1 = \{(x,y) \in [0,1] \times [0,1] : y = 1\}$$

of equal impedances $z_0 = z_1 = 8.3 \ m\Omega \cdot m^2$.

For the given $\sigma$ we solve the forward problem (1.1), (1.2), (1.3), (1.4) for $(u, U)$. The interior data of the magnitude $a$ of the current density field is defined by $a := \sigma |\nabla u|$; see Figure 5.2 on the right.
5.3. Numerical reconstruction of a simulated conductivity. Knowing the injected currents $I_0$ and $I_1$, the electrode impedances $z_0$ and $z_1$, and the corresponding magnitude $a$ of the current density we find an approximate minimizer of $G_a$ via the iterative algorithm in section 4. The iterations start with the guess $\sigma_0 \equiv 1$. An approximate solution $v$ is computed on a $90 \times 90$ grid. The stopping criterion (4.7) for this experiment used $\delta = 10^{-7}$, and was attained with 320 iterations. A pseudo-conductivity $\sigma_v := a/|\nabla v|$ in Figure 5.3 can be computed using this minimizer $v$.

From Theorem 3.1 we know that the correct voltage potential is related to $v$ via the scaling $u(x) = \varphi(v(x))$, for some unknown scaling $\varphi$. To apply Theorem 3.4 we use the additional measurement $u|_{\Gamma}$ on the curve $\Gamma = \{(1, y) : 0 \leq y \leq 1\}$. Since $\varphi(v) = u|_{\Gamma}(v)$ (see Figure 5.4 on the left), an application of the chain rule recovers the conductivity $\sigma$ by

$$\sigma(x) = \frac{1}{\varphi'(v(x))}\sigma_v(x), \quad x \in \Omega.$$ 

In Figure 5.3 the reconstructed conductivity $\sigma$ is shown on the right against the exact conductivity on the left. The $L_2$ error of the reconstruction is 0.04.
Acknowledgments. The work of A. Tamasan has been supported by the NSF Grant DMS-1312883, as was that of J. Veras as part of his Ph.D. research at the University of Central Florida. The work of A. Nachman has been supported by the NSERC Discovery Grant 250240.
Appendix A. A minimization approach for the Complete Electrode Model.

In this appendix we show solvability of the forward problem for the Complete Electrode Model of [20] by recasting it into a minimization problem. While this approach is less general than the one given in [20] (we assume a real valued conductivity and positive electrode impedances), it explains how we are led to introduce the functional (1.12) in the solution of the inverse problem.

Let \( H^1(\Omega) \) be the space of functions which together with their gradients lie in \( L^2(\Omega) \), and \( \Pi \) be the hyperplane in (1.8). We seek weak solutions to (1.1), (1.2) (1.3), (1.4), and (1.5) in the Hilbert space \( H^1(\Omega) \times \Pi \), endowed with the product

\[
\langle (u, U), (v, V) \rangle := \int_\Omega u v dx + \int_\Omega \nabla u \cdot \nabla v dx + \sum_{k=0}^N U_k V_k,
\]

and the induced norm

\[
\| (u, U) \| := \langle (u, U), (u, U) \rangle^{1/2}.
\] (A.1)

We’ll need the following variant of the Poicaré inequality, suitable for the complete electrode model.

**Proposition A.1.** Let \( \Omega \subset \mathbb{R}^n \), \( n \geq 2 \), be an open, connected, bounded domain with Lipschitz boundary \( \partial\Omega \), and \( \Pi \) be the hyperplane in (1.8). For \( k = 0, ..., N \), let \( e_k \subset \partial\Omega \) be disjoint subsets of the boundary of positive \( (n-1) \)-Hausdorff measure: \( |e_k| > 0 \).

There exists a constant \( C > 0 \), dependent only on \( \Omega \) and the \( e_k \)'s, such that for all \( u \in H^1(\Omega) \) and all \( U = (U_0, ..., U_N) \in \Pi \), we have

\[
\int_\Omega u^2 dx + \sum_{k=0}^N U_k^2 \leq C \left( \int_\Omega |\nabla u|^2 dx + \sum_{k=0}^N \int_{e_k} (u - U_k)^2 ds \right).
\] (A.2)

**Proof.** We will show that

\[
\inf_{(u, U) \in H^1(\Omega) \times \Pi} \int_\Omega |\nabla u|^2 dx + \sum_{k=0}^N \int_{e_k} (u - U_k)^2 ds =: \kappa > 0.
\] (A.3)

We reason by contradiction: Assume the infimum in (A.3) is zero. Without loss of generality (else normalize to 1), there exists a sequence \( \{(u_n, U^n)\} \) in the unit sphere of \( H^1(\Omega) \times \Pi \), \( \|(u_n, U^n)\| = 1 \), and such that

\[
0 = \lim_{n \to \infty} \int_\Omega |\nabla u_n|^2 dx, \quad (A.4)
\]

\[
0 = \lim_{n \to \infty} \int_{e_k} (u_n - U_k^n)^2, \quad \text{for } k = 0, ..., N. \quad (A.5)
\]

Due to the compactness of the unit sphere in \( \Pi \) and of the weakly compactness of the unit sphere in \( H^1(\Omega) \) it follows that there exists some \( (u_*, U^*) \in H^1(\Omega) \times \Pi \) with

\[
\|(u_*, U^*)\| = 1,
\] (A.6)
such that, on a subsequence (relabeled for simplicity),
\[ u_n \rightharpoonup u^* \text{ in } H^1(\Omega), \quad (A.7) \]
\[ U^n \to U^* \text{ in } \Pi, \quad \text{as } n \to \infty. \quad (A.8) \]

Since the sequence \( \{u_n\} \) is bounded in \( H^1(\Omega) \), the trace theorem implies that \( u_n|_{e_k} \) is (uniformly in \( n \)) bounded in \( H^{1/2}(e_k) \), hence also in \( L^1(e_k) \), for each \( k = 0, \ldots, N \).

Using (A.5) and (A.8) in
\[
\int_{e_k} (u_n - U^*_k)^2 ds = \int_{e_k} (u_n - U^n_k)^2 ds + 2(U^n_k - U^*_k) \int_{e_k} u_n ds
+ |e_k| \left( (U^*_k)^2 - (U^n_k)^2 \right),
\]
we obtain \( u_n|_{e_k} \to U^*_k \) in \( L^2(e_k) \). Since \( u_n|_{e_k} \rightharpoonup u^*|_{e_k} \), we conclude that
\[ u^*|_{e_k} = U^*_k \text{ for each } k = 0, \ldots, N. \quad (A.9) \]

Now using (A.4) and (A.7)
\[
0 \leq \int_{e_k} |\nabla (u_n - u_*)|^2 dx = \int_{e_k} |\nabla u_n|^2 dx - 2 \int_{e_k} \nabla u_n \cdot \nabla u_* + \int_{e_k} |\nabla u_*|^2 dx
\rightarrow - \int_{e_k} |\nabla u_*|^2 dx, \quad \text{as } n \to \infty,
\]
and, since \( \Omega \) is connected,
\[ u_* \equiv \text{const. in } \overline{\Omega}. \quad (A.10) \]

From (A.9) and (A.10) we conclude that \( u_* \) restricts to the same constant on each electrode, and thus \( U^*_0 = U^*_1 = \ldots = U^*_N = u_* \). Since \( U^* \subset \Pi \), we must have \( U^* = (0, \ldots, 0) \) and then \( u_* \equiv 0 \), thus contradicting (A.6).

\[ \square \]

**Proposition A.2.** Let \( \Omega, \Pi, \) and \( e_k \subset \partial \Omega, k = 0, \ldots, N \) be as in Proposition A.1. For \( z_k \) satisfying (1.7), \( \sigma \) satisfying (1.6), and \( I = (I_0, \ldots, I_N) \in \mathbb{R}^{N+1} \), let us consider the quadratic functional \( F_\sigma : H^1(\Omega) \times \Pi \to \mathbb{R} \) defined by
\[
F_\sigma(u, U) := \frac{1}{2} \int_{\partial \Omega} \sigma |\nabla u|^2 dx + \frac{1}{2} \sum_{k=0}^{N} \int_{e_k} \frac{1}{z_k} |u - U_k|^2 ds - \sum_{k=0}^{N} I_k U_k. \quad (A.11) \]

Then
(i) \( F_\sigma \) is strictly convex
(ii) \( F_\sigma \) is Gateaux differentiable in \( H^1(\Omega) \times \Pi \), and the derivative at \( (u, U) \) in the direction \( (v, V) \) is given by
\[
\langle DF_\sigma(u, U); (v, V) \rangle = \int_{\Omega} \sigma \nabla u \cdot \nabla v dx + \sum_{k=0}^{N} \int_{e_k} \frac{1}{z_k} (u - U_k)(v - V_k) ds
- \sum_{k=0}^{N} I_k V_k \quad (A.12)
\]
(iii) \( F_\sigma \) is coercive, more precisely,

\[
F_\sigma(u, U) \geq \frac{c}{2} \|(u, U)\| - \frac{1}{2c} \sum_{k=0}^{N} I_k^2,
\]

for some constant \( c > 0 \) dependent on the lower bound \( \epsilon \) in (1.6) and (1.7), and \( \kappa \) in (A.3).

**Proof.** (i) The functional has two quadratic terms, each strictly convex, and one linear term, hence the sum is strictly convex. (ii) The Gateaux differentiability and the formula (A.12) follow directly from the definition of \( F_\sigma \).

(iii) Proposition A.1 above shows that

\[
F_\sigma(u, U) \geq c \|(u, U)\|^2 - \sum_{k=0}^{N} I_k U_k,
\]

where \( c = \frac{2\epsilon}{\kappa} > 0 \). By completing the square one obtains

\[
F_\sigma(u, U) \geq c \|u\|^2_{H^1(\Omega)} + c \sum_{k=0}^{N} \left( U_k - \frac{I_k}{2c} \right)^2 - \frac{1}{4c} \sum_{k=0}^{N} I_k^2
\]

\[
\geq c \|u\|^2_{H^1(\Omega)} + c \sum_{k=0}^{N} \left( \frac{1}{2} U_k^2 - \frac{I_k^2}{4c^2} \right) - \frac{1}{4c} \sum_{k=0}^{N} I_k^2
\]

\[
\geq \frac{c}{2} \|(u, U)\|^2 - \frac{1}{2c} \sum_{k=0}^{N} I_k^2
\]

The proposition below revisits [20, Proposition 3.1.] and separates the role of the conservation of charge condition (1.5). This becomes important in our minimization approach, where we shall see that \( F_\sigma \) has a unique minimizer independently of the condition of (1.5) being satisfied. However, it is only for currents satisfying (1.5), that the minimizer satisfies (1.3). This result does not use the reality of \( \sigma \) and of \( z_k \)'s.

Recall that the Gateaux derivative of \( DF_\sigma \) is given in (A.12).

**Proposition A.3.** Let \( \Omega, \Pi, \epsilon_k \subset \partial \Omega, z_k, k = 0, \ldots, N, \) and \( \sigma \) be as in Proposition A.2.

(i) If \((u, U) \in H^1(\Omega) \times \Pi\) is a weak solution to (1.1), (1.2), (1.3) and (1.4), then (1.5) holds and

\[
\langle DF_\sigma(u, U); (v, V) \rangle = 0, \quad \forall (v, V) \in H^1(\Omega) \times \Pi.
\]

(ii) If \((u, U) \in H^1(\Omega) \times \Pi\) satisfies (A.14), then it solves (1.1), (1.2) and (1.4). In addition, if \( I_k \)'s satisfy (1.5), then (1.3) also holds.

**Proof.** (i) Follows from a direct calculation and Green’s formula.

(ii) Assume that (A.14) holds.

By choosing \( v \in H^1_0(\Omega) \) arbitrary and \( V = \overrightarrow{0} \) in (A.14) we see that

\[
\int_{\Omega} \sigma \nabla u \cdot \nabla v dx = 0.
\]

Thus \( u \in H^1(\Omega) \) is a weak solution of (1.1).
For each fixed $k = 0, 1, ..., N$ keep $V = \vec{0}$ as above, but now choose $v \in H^1(\Omega)$ arbitrary with $v|_{\partial\Omega\setminus e_k} = 0$. A straightforward calculation starting from (A.14) shows that

$$
\int_{e_k} \frac{1}{z_k} \left( u - U_k + z_k \sigma \frac{\partial u}{\partial \nu} \right) v ds = 0.
$$

Since $v|_{e_k}$ were arbitrary (1.2) follows.

Now choose $V = \vec{0}$ as above but $v \in H^1(\Omega)$ arbitrary with $v|_{\partial\Omega \setminus e_k} = 0$. A straightforward calculation starting from (A.14) shows that

$$
\int_{\partial e_k} \sigma \frac{\partial u}{\partial \nu} v ds = 0.
$$

Since the trace of $v$ is arbitrary off the electrodes (1.4) holds.

Finally, for an arbitrary $V \in \Pi$ choose $v \in H^1(\Omega)$ with the trace $v|_{e_k} = 0$ on each $e_k$, $k = 0, ..., N$ and $v = 0$ off the electrodes. By using the already established relations (1.1), (1.2), (1.4) and Green’s formula in (A.14) we obtain

$$
\sum_{k=0}^{N} V_k \left( \int_{e_k} \sigma \frac{\partial u}{\partial \nu} ds - I_k \right) = 0.
$$

On the one hand, by introducing the notation $\vec{\alpha} := \langle \alpha_0, ..., \alpha_N \rangle$ with

$$
\alpha_k := \int_{e_k} \sigma \frac{\partial u}{\partial \nu} ds - I_k, \quad k = 0, ..., N,
$$

we just showed that $\vec{\alpha} \perp \Pi$. Note that so far we have not used the conservation of charge condition (1.5).

On the other hand, by using (1.4), (1.5), and (1.1) in the divergence formula, we have

$$
\sum_{k=0}^{N} \alpha_k = \int_{\partial\Omega} \sigma \frac{\partial u}{\partial \nu} ds = \int_{\Omega} \nabla \cdot \sigma \nabla u dx = 0,
$$

which yields $\vec{\alpha} \in \Pi$. Therefore $\vec{\alpha} \in \Pi^\perp \cap \Pi = \vec{0}$, and (1.3) holds.

The following result establishes existence and uniqueness of the weak solution to the forward CEM problem; contrast with the proof of Theorem 3.3 in [20].

**Theorem A.4.** Let $\Omega$, $\Pi$, $e_k \subset \partial \Omega$, $z_k$, for $k = 0, ..., N$, and $\sigma$ be as in Proposition A.2. Let $F_\sigma : H^1(\Omega) \times \Pi \rightarrow \mathbb{R}$ be defined in (A.11).

(i) Then $F_\sigma$ has a unique minimizer $(u, U) \in H^1(\Omega) \times \Pi$. If, in addition, the injected currents $I_k$’s satisfy (1.5) the minimizer is the weak solution of the problem (1.1), (1.2), (1.3), and (1.4).

(ii) If the problem (1.1), (1.2), (1.3), (1.4) has a solution, then it is a minimizer of $F_\sigma$ in the whole space $H^1(\Omega) \times \Pi$ and hence unique. Moreover, the current $I_k$’s satisfy (1.5).

**Proof.** (i) Let

$$
d = \inf_{H^1(\Omega) \times \Pi} F_\sigma(u, U),
$$

where
and consider a minimizing sequence \( \{(u_n, U^n)\} \) in \( H^1(\Omega) \times \Pi \),
\[
d \leq F_\sigma(u_n, U^n) \leq d + \frac{1}{n}. \tag{A.15}
\]
Since \( \inf F_\sigma \geq -\frac{1}{4c} \sum_{k=0}^{N} I^2_k \) we have \( d \neq -\infty \). Following (A.13),
\[
\lim_{\| (u, U) \| \to \infty} F_\sigma(u, U) = \infty.
\]
Thus the minimizing sequence must be bounded, hence weakly compact. In particular, for a subsequence (relabeled for simplicity) there is some \( (u^*, U^*) \in H^1(\Omega) \times \Pi \), such that
\[
u_n \rightharpoonup u^* \text{ in } H^1(\Omega), \quad U_n \to U^* \text{ in } \Pi, \text{ as } n \to \infty. \tag{A.16}
\]
On the other hand since \( F_\sigma \) is convex, and Gateaux differentiable at \( (u^*, U^*) \) in the direction \( (u_n - u^*, U_n - U^*) \), we have
\[
F_\sigma(u_n, U^n) \geq F_\sigma(u^*, U^*) + \langle DF_\sigma(u^*, U^*); (u_n - u^*, U_n - U^*) \rangle. \tag{A.17}
\]
We take the limit as \( n \to \infty \). The weak convergence in (A.16) yields
\[
\langle DF_\sigma(u^*, U^*), (u_n - u^*, U_n - U^*) \rangle \to 0.
\]
Thus \( d \geq F_\sigma(u^*, U^*) \geq d \) which shows that \( (u^*, U^*) \) is a global minimizer. Strict convexity of \( F_\sigma \) implies it is unique. At the minimum \( (u^*, U^*) \) the Euler-Lagrange equations (A.14) are satisfied. An application of Proposition A.3 part (ii) shows that \( (u^*, U^*) \) is a weak solution to the forward problem.

(ii) Proposition A.3 part (i) shows that \( (u^*, U^*) \) solves the Euler-Lagrange equations, and due to the convexity it is a minimizer of \( F_\sigma \). Due to the strict convexity of the functional the minimizer is unique, hence the weak solution is unique. \( \Box \)

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