On the Stokes System in Cylindrical Domains

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Abstract. The existence of solutions to some initial-boundary value problem for the Stokes system is proved. The result is shown in Sobolev–Slobodetskii spaces such that the velocity belongs to $W^{2+\sigma,1+\sigma/2}_r(\Omega^T)$ and the gradient of pressure to $W^{\sigma,\sigma/2}_r(\Omega^T)$, where $r \in (1,\infty)$, $\sigma \in (0,1)$, $\Omega^T = \Omega \times (0,T)$. These are special Besov spaces: $B^{2+\sigma,1+\sigma/2}_r(\Omega^T)$ and $B^{\sigma,\sigma/2}_r(\Omega^T)$, respectively. The existence is proved by the technique of regularizer.

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1. Introduction

We consider the following initial-boundary value problem for the Stokes system in a cylindrical domain $\Omega \subset \mathbb{R}^3$,

\begin{align*}
  v_t - \nu \Delta v + \nabla p &= f & \text{in } \Omega \times (0,T) \equiv \Omega^T, \\
  \text{div} v &= g & \text{in } \Omega^T, \\
  \bar{n} \cdot D(v) \cdot \bar{\tau}_\alpha &= b_\alpha, & \alpha = 1,2, & \text{on } S \times (0,T) \equiv S^T, \\
  v \cdot \bar{n} &= b_3 & \text{on } S^T, \\
  v|_{t=0} &= v_0 & \text{in } \Omega,
\end{align*}

where $S = S_1 \cup S_2$. Introducing the Cartesian system of coordinates $x = (x_1,x_2,x_3)$ we assume that $\Omega$ and $S_1$ are parallel to the $x_3$-axis and $S_2$ is perpendicular to it. By $\bar{n}$ we denote the unit outward vector normal to $S$ and $\bar{\tau}_1, \bar{\tau}_2$ are tangent to $S$.

The boundaries $S_1$ and $S_2$ of cylindrical domain $\Omega$ meet along curves $L_1$ and $L_2$ under the angle $\pi/2$. This means that the considered domain $\Omega$ is geometrically complicated. Therefore, we prove existence of solutions to problem (1.1) by the technique of regularizer. For this we need a complex partition of unity.

The most difficult local problem are problems near edges $L_\alpha$, $\alpha = 1,2$. This can be also transformed to a problem in the half space.

The main result is the following...
Theorem 1.1. Assume that \( f \in W^r,σ/2(Ω^T) \), \( v_0 \in W^{2+r−σ,2/r}(Ω) \), \( g \in W_{r}^{1+\sigma,1/2+\sigma/2}(Ω^T) \), \( b_α \in W_{1+r/2}^{1+\sigma−1/2,\sigma/2−1/2}(S_1^T) \), \( α = 1,2 \), \( S_1 \in C^3 \) and \( b_3 \in W_{r}^{2+\sigma−1/2,1+\sigma/2−1/2}(S_1^T) \), where \( r \in (1,∞) \), \( σ \in (0,1), i = 1,2 \). Assume that there exists such \( χ \) that \( g = \text{div } χ \) with \( χ,χ_t \in W_{r}^{σ,σ/2}(Ω^T) \). Assume some compatibility conditions (see Remark 1.2).

Then there exists a solution to problem (1.1) such that \( v \in W_{r}^{2+σ,1/2+σ/2}(Ω^T) \), \( \nabla p \in W_{r}^{σ,σ/2}(Ω^T) \), and the estimate holds

\[
\begin{aligned}
\|v\|_{W_{r}^{2+σ,1/2+σ/2}(Ω^T)} + \|\nabla p\|_{W_{r}^{σ,σ/2}(Ω^T)} &\leq c \left( \|f\|_{W_{r}^{σ,σ/2}(Ω^T)} + \|g\|_{W_{r}^{1+\sigma,1/2+\sigma/2}(Ω^T)} + \|\chi_t\|_{W_{r}^{σ,σ/2}(Ω^T)} \\
&+ \|\Delta χ\|_{W_{r}^{σ,σ/2}(Ω^T)} + \sum_{i=1}^{2} \left( \sum_{α=1}^{2} \|b_α\|_{W_{1+r/2}^{1+\sigma−1/2,\sigma/2−1/2}(S_1^T)} + \|b_3\|_{W_{r}^{2+\sigma−1/2,1+\sigma/2−1/2}(S_1^T)} \right) \right),
\end{aligned}
\]

where \( c \) does not depend on \( v \) neither \( p \).

Remark 1.2. Compatibility conditions

\[
\begin{aligned}
\bar{n} \cdot \mathbb{D}(v_0) \cdot \bar{τ}_α |_{S_1} &= b_α |_{S_1}, α = 1,2, &\text{ in }& W_{r}^{1+\sigma−3/r}(S_1), \\
v \cdot \bar{n} |_{S} &= b_3 |_{t=0}, &\text{ in }& W_{2+r}^{2+σ−3/r}(S), \\
\text{div } v_0 &= g |_{t=0}, &\text{ in }& W_{r}^{1+\sigma−2/r}(Ω), \\
\text{div } (v_0 − χ |_{t=0}) &= 0, &\text{ in }& W_{r}^{1+\sigma−2/r}(Ω).
\end{aligned}
\]

In this paper, we prove the existence of solutions and we establish appropriate estimates for solutions to the Stokes system in a cylindrical domain and in Besov spaces. The proof bases on the classical approach initiated by Agmon-Douglis-Nirenberg [2] and by Solonnikov in [37,38] and Ladyzhenskaya-Solonnikov-Ural’tseva in [19, Ch. 4]. We sketch the idea of the proof as follows. By an appropriate partition of unity the problem in the whole domain \( Ω \) is localized to the following problems:

1. near an interior point of \( Ω \),
2. near a point either of \( S_1 \) or of \( S_2 \),
3. near a point of the edge, so a point either of \( L_1 \) or of \( L_2 \).

Solving the above local problems we prove the existence of solutions to the Stokes system in \( Ω \) by applying the technique of regularizer (see [2,37,38], [19, Ch.4]).

The problem 1. is the problem for the Stokes system in the whole \( \mathbb{R}^3 \times \mathbb{R}_+ \). We solve it explicitly using the Fourier-Laplace transform. Then we derive an estimate for solutions in Besov spaces by using the Nikolskii-Triebel definition of Besov spaces (see [21,39]). The problem 2. is a problem in the half-space \( \mathbb{R}_+^3 \times \mathbb{R}_+ \), \( \mathbb{R}^3_+ = \{ x \in \mathbb{R}^3 : x_3 > 0 \} \) with corresponding boundary conditions on the plane \( x_3 = 0 \). After applying the Laplace transform with respect to time and the Fourier transform with respect to tangent directions \( x_1, x_2 \) we obtain a system of ordinary differential equations with respect to \( x_3 \) with boundary conditions on \( x_3 = 0 \) and with solutions vanishing for \( x_3 \to \infty \). We solve the equations explicitly and then we estimate solutions using the definition of Besov spaces (see [21,39]). The problem 3. is much more complicated because it must be transformed to a problem in the half-space and then treated respectively, what is executed.

In this paper we restrict our considerations to special Besov spaces \( B_{p,q}^{2+s,1+s/2} \) which are called the Sobolev–Slobodetskii spaces and denoted by \( W_{2+s,1+s/2} \). It seems not difficult to extend the results of this paper to general Besov spaces \( B_{p,q}^{2+s,1+s/2} \) or to anisotropic Sobolev–Slobodetskii spaces \( W_{2+s,1+s/2} \).

The existence of solutions to problem (1.1) for sufficiently smooth boundary is proved in [43] in Besov spaces such that \( v \in B_{p,q}^{σ,σ/2}(Ω^T) \), \( \nabla p \in B_{p,q}^{σ,σ/2}(Ω^T) \), \( σ \in \mathbb{R}_+, p,q \in (1,∞) \). However, in [43] the
classical fundamental approach as solvability of problems in the whole space, in the half space and finally the existence in a bounded domain by the technique of regularizer are not used.

In [42] the existence of solutions to the initial boundary value problem for the heat equation is proved in Besov spaces using the Triebel definition of Besov spaces (see [39, Def. 2.3.1]) and using the techniques developed by Triebel in [39, Section 2.3.6]. Continuing the approach from [42] we prove the existence of solutions to problem (1.1) in the cylindrical domain $\Omega$.

We mention that the existence of solutions to the nonstationary Stokes system in domains with smooth boundaries was extensively studied by many authors (see [3,34–36]). In these papers the anisotropic Sobolev and Hölder spaces were applied.

Besov spaces were used in some approach to Navier–Stokes equations. We mention some of these results below.

In [5], the author proves analyticity of solutions of the Stokes operator in Besov space $B^0_{2,q}(\mathbb{R}^n_+)$. First, it is shown that the semigroup obtained for the Stokes system on $\mathbb{R}^n$ is analytic in Besov spaces in time. Then, the asymptotic behavior of solutions for the nonstationary Navier–Stokes equations is estimated and the decay rate in the Besov space is studied.

The paper [8] is devoted to the boundary value problem for the incompressible inhomogeneous Navier–Stokes equations in the half-space in the case of small data with critical regularity. The result states that in dimension $n \geq 3$, if the initial density is close to a positive constant in $L_\infty \cap W^1_1(\mathbb{R}^n_+)$ and the initial velocity is small with respect to the viscosity in the homogeneous Besov space $\dot{B}^0_{n,1}(\mathbb{R}^n_+)$ then the equations have a unique global solution. The proof strongly relies on maximal regularity estimates for the Stokes system in the half-space in $L_1(0,T; \dot{B}^0_{n,1}(\mathbb{R}^n_+))$. Namely, it was necessary to obtain time-independent maximal estimates for the linearized velocity equation, i.e. the evolutionary Stokes system. They obtained standard estimates in Lebesgue spaces for the transport equation for the density and estimates in homogeneous Besov spaces for the Stokes system and in order to bound nonlinear terms in the linearized problem, bilinear estimates in Besov spaces have been used.

In [25], the author studied global well-posedness for the nonhomogeneous Navier–Stokes equations on $\mathbb{R}^n, n \geq 2$, with initial velocity in endpoint critical Besov spaces $B^{-1+n/q}_{q,\infty}(\mathbb{R}^n), n \leq q < 2n$, and merely bounded initial density with a positive lower bound. He considered a multiplication property of $L_\infty$-functions in some Bessel potential and Besov spaces. Based on it and on maximal regularity of the Stokes operator in little Nicolskii spaces, it was shown solvability for the momentum equations with fixed bounded density. The proof for existence of a solution to the nonhomogeneous Navier–Stokes equations was done via an iterative scheme when $B^{-1+n/q}_{q,\infty}$-norm of initial velocity and relative variation of initial density are small, while uniqueness of a solution was proved via a Lagrangian approach when initial velocity belongs to $B^{-1+n/q}_{r,\infty}(\mathbb{R}^n) \cap B^{-1+n/q}_{q,\infty}(\mathbb{R}^n)$ for slightly larger $r > q$.

In [26], a local in time solution was constructed for the Cauchy problem of the $n$-dimensional Navier–Stokes equations when the initial velocity belongs to Besov spaces of nonpositive order. The space contains $L_\infty$ in some exponents, so the solution may not decay at space infinity. In order to use iteration scheme it was necessary to establish the Hölder type inequality for estimating bilinear term by dividing the sum of Besov norm with respect to levels of frequency. Moreover, by regularizing effect solutions belong to $L_\infty$ for any positive time.

In [22], authors considered global well-posedness of the Cauchy problem of the incompressible Navier–Stokes equations under the Lagrangian coordinates in scaling critical Besov spaces. They proved the system is globally well-posed in the homogeneous Besov space $\dot{B}^{-1+n/p}_{p,1}(\mathbb{R}^n)$ with $1 \leq p \leq \infty$. They improved the former result, restricted for $1 \leq p \leq 2n$ and the main reason why the well-posedness space was enlarged is that the quasi-linear part of the system has a special feature called a multiple divergence structure and the bilinear estimate for the nonlinear terms are improved by such a structure. The result indicates that the Navier–Stokes equations can be transferred from the Eulerian coordinates to the Lagrangian coordinates even for the solution in the limiting critical Besov spaces.

In [15], authors got the full regularity of weak solutions to Navier–Stokes equations under some assumptions on the velocity $u$, namely, if $u \in L^2(0,T; \dot{B}^{0}_{\infty,\infty}(\mathbb{R}^3))$, then the solution $u$ is regular. By the definition
of Besov space and Bernstein inequality, the condition is equivalent to: \( \nabla u \in L^2(0, T; \dot{B}_{\infty, \infty}^{-1}(\mathbb{R}^3)) \). On the other hand, in [10], if \( \nabla u_k \in L^{8/3}(0, T; \dot{B}_{\infty, \infty}^{-1}(\mathbb{R}^3)) \) or \( \nabla u_3 \in L^{5-2s/3}(0, T; \dot{B}_{\infty, \infty}^{-s}(\mathbb{R}^3)) \), \( 0 < s < 1 \), where \( \nabla u_k \) denotes horizontal gradient components, then \( u \) is a regular solution.

In [16], the local existence theorem for the Navier–Stokes equations in \( \mathbb{R}^n, n \geq 2 \), with the initial data in \( B_{\infty, \infty}^0 \) containing functions that do not decay at infinity, was proved. Moreover, authors established the extension criterion on local solutions in terms of the vorticity in the homogeneous Besov space \( \dot{B}_{\infty, \infty}^0 \).

In [7], authors established the space-time estimates in the Besov spaces of the solution to the Navier–Stokes equations in \( \mathbb{R}^n, n \geq 3 \). As an application, they improved some known results about the regularity criterion of weak solutions and the blow-up criterion of smooth solutions. Instead of the logarithmic Sobolev inequality, the main tools were the frequency localization and the Littlewood–Paley trichotomy decomposition, as a basic way to analyze bilinear expressions.

In [17], authors show the existence theorem of global mild solutions to Navier–Stokes equations in the whole space \( \mathbb{R}^n \) with small initial data and external forces in the time-weighted Besov space which is an invariant space under the change of scaling. The result on local existence of solutions for large data is also discussed. The method is based on the \( L^p - L^q \) estimate of the Stokes equations in Besov spaces. Using various estimates for the Stokes semi-group in the homogeneous Besov space and based on the paraproduct formula, they established a bilinear estimate related to the nonlinear term of Navier–Stokes equation to apply the implicit function theorem and show the local existence.

In [14], the local existence, uniqueness and regularity of solutions of the initial-value problem for non-stationary Navier–Stokes equations were studied via abstract Besov spaces. Authors could prove an estimate of semigroups in abstract Besov spaces instead of fractional powers. In the paper, the domain is a bounded domain in \( \mathbb{R}^n \), a half-space of \( \mathbb{R}^n \) with \( n \geq 2 \), or an exterior domain in \( \mathbb{R}^n \) with \( n \geq 3 \), and the boundary is smooth.

Recently appeared a new technique of dealing with initial-boundary value problems to parabolic systems as well as to the Stokes system. The new approach relies on the Fourier-multiplier theorem, \( \mathcal{R} \)–boundedness, the maximal regularity and the semigroup technique. The theory has been described by Amann [4], Weis [41] and Denk et al. [9]. Applications of this theory needs many efforts and deep understanding of its construction.

Applying these ideas and very general techniques of functional analysis, namely the Fourier multiplier, \( \mathcal{R} \)–boundedness and the maximal regularity, Y. Shibata was able to solve some problems for the Navier–Stokes equations, see [11,20,23,24,27–32]. It is important that Y. Shibata and his co-authors could derive \( L_p - L_q \) maximal regularity and some time-weighted estimates which make possible to prove global existence of solutions to many problems to the Navier–Stokes equations.

The remaining part of the paper is divided into the following parts.

* In Sect. 2 we introduce all necessary notations, definitions and auxiliary results.
* In Sect. 3 we consider the Stokes system in the whole space.
* In Sect. 4 the Stokes system in the half-space is examined.
* In Sect. 5 the existence of solutions in the cylindrical domain is proved with the technique of regularizer.
* Finally, in Sect. 6 solvability of the Stokes system in Sobolev spaces is discussed.

2. Notation and Preliminaries

In this Section we begin the part of this paper devoted to the problem of existence of solutions to the Stokes system in anisotropic Besov spaces. Let \( \mathbb{R}^3 \) be a three-dimensional real Euclidean space. Let \( x = (x_1, x_2, x_3) \in \mathbb{R}^3 \) be the system of Cartesian coordinates.

Throughout the paper we use the notation: \( x' = (x_1, x_2) \), \( \bar{x} = (x_1, x_2, x_3, t) \equiv (x, t) \), \( \bar{x}' = (x_1, x_2, t) \equiv (x', t) \). For \( \bar{x} \in \mathbb{R}^4 \) we introduce the anisotropic distance from the origin of coordinates,
|\bar{x}|_a = \left(|t| + \sum_{i=1}^{3} |x_i|^2 \right)^{1/2}

and, similarly for \( \bar{x}' \in \mathbb{R}^3 \), we have

|\bar{x}'|_a = \left(|t| + \sum_{i=1}^{2} |x_i|^2 \right)^{1/2}.

Let \( S(\mathbb{R}^n) \) and \( S'(\mathbb{R}^n) \) be the Schwartz space and the space of tempered distributions on \( \mathbb{R}^n \), respectively.

**Definition 2.1.** By \( \Phi_a(\mathbb{R}^n) \) we denote the collection of all systems \( \varphi = \{ \varphi_k(\bar{x}) \}_{k=0}^\infty \subset S(\mathbb{R}^4) \) with the following properties

1° \( \text{supp} \varphi_0 \subset \{ \bar{x}: |\bar{x}|_a \leq 2 \} \);
2° \( \text{supp} \varphi_k \subset \{ \bar{x}: 2^{k-1} \leq |\bar{x}|_a \leq 2^{k+1} \}, k = 1, 2, \ldots \);
3° for every multi-index \( \bar{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \alpha_0) \), there exists a positive number \( c_{\bar{\alpha}} \) such that

\[ 2^{k(2\alpha_0 + \sum_{i=1}^{3} \alpha_i)} |D^{\alpha} \varphi_k(\bar{x})| \leq c_{\bar{\alpha}} \]

for all \( k \in \mathbb{N} \cup \{0\} \equiv \mathbb{N}_0 \) and all \( \bar{x} \in \mathbb{R}^4 \), where

\[ D^{\alpha} u = \partial_t^{\alpha_0} \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3} u \]

and \( |\bar{\alpha}| = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3, \alpha_j \in \mathbb{N}_0, j \in \{0, 1, 2, 3\} \);

4° \( \sum_{k=0}^{\infty} \varphi_k(\bar{x}) = 1 \) for all \( \bar{x} \in \mathbb{R}^4 \).

A similar definition can be introduced for functions \( \varphi_k \) depending on \( \bar{x}' \).

**Definition 2.2.** Introduce the following Fourier transforms. Let \( f \in S'(\mathbb{R}^4) \). Then

\[
(Ff)(\xi, \xi_0) = \int_{\mathbb{R}^2 \times \mathbb{R}} e^{-i(t\xi_0 + x \cdot \xi)} f(x, t) dx dt
\]

and

\[
(F^{-1}f)(x, t) = \frac{1}{(2\pi)^4} \int_{\mathbb{R}^3 \times \mathbb{R}} e^{i(t\xi_0 + x \cdot \xi)} f(\xi, \xi_0) d\xi d\xi_0,
\]

where \( x \cdot \xi = x_1\xi_1 + x_2\xi_2 + x_3\xi_3 \), \( d\xi = d\xi_1d\xi_2d\xi_3 \).

Then the corresponding Laplace-Fourier transform has the form

\[
(\mathcal{L}f)(\xi, \xi_0, \gamma) = (Ff_{\gamma})(\xi, \xi_0)
\]

where

\[
f_{\gamma} = \begin{cases} 
    e^{-\gamma t} f & \text{for } t > 0, \\
    0 & \text{for } t < 0.
\end{cases}
\]

Let \( f \in S'(\mathbb{R}^3) \). Then

\[
(F_1f)(\xi', \xi_0) = \int_{\mathbb{R}^2 \times \mathbb{R}} e^{-i(t\xi_0 + x' \cdot \xi')} f(x', t) dx' dt
\]

and

\[
(F^{-1}_1f)(x', t) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^2 \times \mathbb{R}} e^{i(t\xi_0 + x' \cdot \xi')} f(\xi', \xi_0) d\xi' d\xi_0,
\]

where \( x' \cdot \xi' = x_1\xi_1 + x_2\xi_2, d\xi' = d\xi_1d\xi_2, dx' = dx_1dx_2 \). Hence

\[
(\mathcal{L}_1f)(\xi', \xi_0, \gamma) = F_1(f_{\gamma})(\xi', \xi_0).
\]
Now we are going to define anisotropic Besov spaces
\[ B_{p,q}^{\sigma,\sigma/2}(\mathbb{R}^3 \times \mathbb{R}) , \quad B_{p,q}^{\sigma,\sigma/2}(\mathbb{R}^2 \times \mathbb{R}) , \quad B_{p,q}^{\sigma,\sigma/2}(\mathbb{R}^3_+ \times \mathbb{R}) , \]
where \( \mathbb{R}^3_+ = \{ x \in \mathbb{R}^3 : x_3 > 0 \} .

To this end, for \( f \in S'(\mathbb{R}^4) \) we introduce the Fourier transform
\[
(F_2 f)(\xi', x_3, \xi_0) = \int_{\mathbb{R}^2 \times \mathbb{R}} e^{-i(t \xi_0 + x' \cdot \xi')} f(x', x_3, t) dx' dt .
\]

Then the corresponding Laplace-Fourier transform has the form
\[
(L_2 f)(\xi', x_3, \xi_0, \gamma) = F_2(f_\gamma)(\xi', x_3, \xi_0) .
\]

**Definition 2.3** (see [42], [39, Sect. 2.3.1]). Let \( \sigma \in \mathbb{R}_+ = (0, \infty) \) and \( p, q \in [1, \infty] . \) The anisotropic Besov space \( B_{p,q,\gamma}^{\sigma,\sigma/2}(\mathbb{R}^3 \times \mathbb{R}) \) is the space of functions \( u = u(x,t) \in S'(\mathbb{R}^4) \) with the finite norm
\[
\| u \|_{B_{p,q,\gamma}^{\sigma,\sigma/2}(\mathbb{R}^3 \times \mathbb{R})} = \left[ \sum_{k=0}^{\infty} \left( \int_{\mathbb{R}^4 \times \mathbb{R}} |2^{\sigma k}(F_1^{-1}(\varphi_k F_1 u_\gamma))(x,t)|^p dx dt \right)^{q/p} \right]^{1/q} ,
\]
where the Fourier transform \( F \) is defined above and \( \varphi_k \in \Phi_\alpha(\mathbb{R}^4) \) is introduced in Definition 2.1.

**Definition 2.4** (see [42], [39, Sect. 2.3.1]). Let \( p, q \in [1, \infty] , \sigma \in \mathbb{R}_+ . \) The anisotropic Besov space \( B_{p,q,\gamma}^{\sigma,\sigma/2}(\mathbb{R}^2 \times \mathbb{R}) \) is the space of functions \( u = u(x',t) \in S'(\mathbb{R}^3) \) with the finite norm
\[
\| u \|_{B_{p,q,\gamma}^{\sigma,\sigma/2}(\mathbb{R}^2 \times \mathbb{R})} = \left[ \sum_{k=0}^{\infty} \left( \int_{\mathbb{R}^4 \times \mathbb{R}} |2^{\sigma k}(F_1^{-1}(\varphi_k F_1 u_\gamma))(x',t)|^p dx' dt \right)^{q/p} \right]^{1/q} ,
\]
where \( F_2 \) and \( \varphi_k, k \in \mathbb{N}_0 , \) are defined above.

**Definition 2.5** (Besov spaces defined on \( \mathbb{R}_+^3 \times \mathbb{R} \)). Let \( p, q \in [1, \infty] \) and \( \sigma \in \mathbb{R}_+ . \) The anisotropic Besov space \( B_{p,q,\gamma}^{\sigma,\sigma/2}(\mathbb{R}_+^3 \times \mathbb{R}) \) is the space of functions \( u = u(x,t) \in S'(\mathbb{R}_+^4) \) with the finite norm
\[
\| u \|_{B_{p,q,\gamma}^{\sigma,\sigma/2}(\mathbb{R}_+^3 \times \mathbb{R})} = \| u \|_{L_\sigma(\mathbb{R}_+^3 \times \mathbb{R})} + \sum_{k=0}^{\infty} \left( \sum_{j \leq \sigma} \int_{\mathbb{R}_+^3 \times \mathbb{R}} |2^{(\sigma-j)k}(F_1^{-1}(\varphi_k F_2 \partial_{x_3}^j u_\gamma))(x,t)|^p dx dt \right)^{q/p} \right]^{1/q} ,
\]
where \( |\sigma| \) is the integer part of \( \sigma . \)

**Remark 2.6.** By \( B_{p,q,\gamma}^{\sigma,\sigma/2} \) we denote such Besov space that in the definition of \( B_{p,q}^{\sigma,\sigma/2} \) the Fourier transform is replaced by the Fourier-Laplace transform.

**Lemma 2.7** (see Ch. 4, Sect. 18 [6]). Let \( f \in B_{p,q}^{\sigma,\sigma/2}(\mathbb{R}_+^3 \times \mathbb{R}) , \sigma \in \mathbb{R}_+ , p, q \in (1, \infty) . \) Then there exists an extension of \( f \) onto \( \mathbb{R}^3 \times \mathbb{R} \) denoted by \( f' \) such that \( f'|_{\mathbb{R}_+^3 \times \mathbb{R}} = f \) and
\[
\| f' \|_{B_{p,q}^{\sigma,\sigma/2}(\mathbb{R}^3 \times \mathbb{R})} \leq c \| f \|_{B_{p,q}^{\sigma,\sigma/2}(\mathbb{R}_+^3 \times \mathbb{R})} ,
\]
where \( c \) does not depend on \( f . \)
Lemma 2.8 (see [21]). Let \( f = f(x,t) \in B^p_{\rho,q} \sigma/2 (\Omega \times \mathbb{R}_+) \), \( \sigma \in \mathbb{R}_+ \), \( \sigma > 2/p \), \( p,q \in (1,\infty) \), \( x \in \Omega \), \( t \in \mathbb{R}_+ \), where \( \Omega \) stands for either \( \mathbb{R}^3_+ \) or \( \mathbb{R}^3 \). Then \( f|_{t=0} = \varphi \in B^p_{\rho,q} \sigma/2 (\Omega) \) and

\[
\|\varphi\|_{B^p_{\rho,q} \sigma/2 (\Omega)} \leq c \|f\|_{B^p_{\rho,q} \sigma/2 (\Omega \times \mathbb{R}_+)}
\]

where \( c \) does not depend on \( f \).

Lemma 2.9 (see [21]). Let \( \varphi^{(k)} \in B^p_{\rho,q} \sigma/2 - 2k (\Omega) \), where \( \Omega \) stands for either \( \mathbb{R}^3_+ \) or \( \mathbb{R}^3 \), and

\[
k = 0, 1, \ldots, l, \quad l = \left\{ \begin{array}{ll}
\sigma/2 - 1/p - 1 & \text{if } \frac{\sigma}{2} - \frac{1}{p} \notin \mathbb{N}, \\
\left[ \frac{\sigma}{2} - \frac{1}{p} \right] & \text{if } \frac{\sigma}{2} - \frac{1}{p} \in \mathbb{N},
\end{array} \right.
\]

Then there exists a function \( f = f(x,t) \in B^p_{\rho,q} \sigma/2 (\Omega \times \mathbb{R}_+) \) such that \( \partial_t^k f|_{t=0} = \varphi^{(k)} \), \( k = 0, 1, \ldots, l \), and

\[
\|f\|_{B^p_{\rho,q} \sigma/2 (\Omega \times \mathbb{R}_+)} \leq c \sum_{k=0}^l \|\varphi^{(k)}\|_{B^p_{\rho,q} \sigma/2 - 2k (\Omega)},
\]

where \( c \) does not depend on \( f \).

Let \( \Omega \) be either \( \mathbb{R}^3 \) or \( \mathbb{R}^3_+ \). In Definitions 2.4 and 2.5 the Besov spaces are defined through Fourier transforms. Applying differences we can define the Besov space in more classical way (see papers [6], [21] of Besov and Nikolskii).

To define the spaces we introduce spacial differences. Let \( x, z \in \Omega \), \( t \in \mathbb{R} \). Then for \( N \geq m > 1 \) we set

\[
\Delta_i (t) f(x) = f(x + t e_i) - f(x),
\]

\[
\Delta_i^m (t) f(x) = \Delta_i (t) [\Delta_i^{m-1} (t) f(x)] = \sum_{j=0}^m (-1)^{m-j} c_{jm} f(x + j t e_i)
\]

\[
\Delta(z) f(x) = f(x + z) - f(x),
\]

\[
\Delta^m (z) f(x) = \Delta(z) [\Delta^{m-1} (z) f(x)] = \sum_{j=0}^m (-1)^{m-j} c_{jm} f(x + j z),
\]

where \( c_{jm} = \binom{m}{j} \) and \( e_i \) is the unit vector directed along the axis \( x_i \).

Definition 2.10 (see [6,21]). The Besov space \( B^p_{\rho,q} (\Omega) \), where \( \Omega \) stands either for \( \mathbb{R}^3 \) or \( \mathbb{R}^3_+ \), is a space of functions \( u = u(x) \) with the finite norm

\[
\|u\|_{B^p_{\rho,q} (\Omega)} = \|u\|_{L^p(\Omega)} + \sum_{i=1}^3 \left( \int_0^h \frac{\|\Delta_i^m (h, \Omega) \partial_{x_i}^k u\|_{L^q(\Omega)}^q}{h^{1+q(\sigma-k)}} dh \right)^{1/q},
\]

where \( x = (x_1,x_2,x_3) \), \( m > \sigma - k \), \( k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \), \( \sigma \in \mathbb{R}_+ \) and \( \sigma > k \).

Moreover, we assumed that

\[
\Delta_i^m (h, \mathbb{R}^3_+) u(x) = \begin{cases} \\
\Delta_i^m (h) u(x) \quad \text{if } [x, x + h e_i] \in \mathbb{R}^3_+,
0 \quad \text{otherwise}.
\end{cases}
\]

(2.2)

Now, let

\[
\Delta_i^{m_0} (h) u(t) = \sum_{j=0}^{m_0} (-1)^{m_0-j} c_{jm_0} u(t + j h)
\]

(2.3)
and
\[ \Delta_{m_0}^{m_0}(h, (\tau, T))u(t) = \begin{cases} \Delta_{m_0}^{m_0}(h)u(t) & \text{if } \lfloor t, t + m_0h \rfloor \subset (\tau, T), \\ 0 & \text{otherwise}, \end{cases} \]

where \( m_0 \in \mathbb{N} \) and \( c_{jm_0} \) is defined above.

**Definition 2.11** (see [6, 21]). The Besov space \( B_{p,q}^{\sigma,\sigma/2}(\Omega \times (\tau, T)) \), where \(-\infty < \tau < T < \infty \) and \( \Omega \) is an open set in \( \mathbb{R}^3 \), is a space of functions \( u = u(x,t) \) with the finite norm
\[
\| u \|_{B_{p,q}^{\sigma,\sigma/2}(\Omega \times (\tau, T))} = \| u \|_{L_p(\Omega \times (\tau, T))} + \sum_{i=1}^3 \left( \int_0^{h_0} \| \Delta_i^{m_0}(h, \Omega) \partial_{x_i}^{[\sigma]} u \|_{L_p(\Omega \times (\tau, T))}^{q} \right)^{1/q} dh + \left( \int_0^{h_0} \| \Delta_i^{m_0}(h, (\tau, T)) \partial_{x_i}^{[\sigma/2]} u \|_{L_p(\Omega \times (\tau, T))}^{q} \right)^{1/q} dh,
\]

where we used formulas (2.2) with \( \Omega \) instead of \( \mathbb{R}^3 \) and (2.3), \( h_0 > 0, m > \sigma - k, m_0 > \sigma/2 - k_0, k, k_0 \in \mathbb{N}_0, \sigma \in \mathbb{R}_+, \sigma > k \) and \( \sigma/2 > k_0 \).

From [12] and Theorem 18.2 from [6, Ch. 4, Sect. 18] we have

**Lemma 2.12.** Norms of spaces \( B_{p,q}^{\sigma,\sigma/2}(\Omega \times (\tau, T)) \) are equivalent for any open set in \( \Omega \times (\tau, T) \) and for any \( m, k \) and \( m_0, k_0 \) satisfying conditions \( m + k > \sigma > k > 0, m_0 + k_0 > \sigma/2 > k_0 > 0 \).

**Lemma 2.12** implies that the norm of \( B_{p,q}^{\sigma,\sigma/2}(\Omega \times (\tau, T)) \) from Definition 2.11 is equivalent to the following:
\[
\| u \|_{B_{p,q}^{\sigma,\sigma/2}(\Omega \times (\tau, T))} = \| u \|_{L_p(\Omega \times (\tau, T))} + \sum_{i=1}^3 \left( \int_0^{h_0} \| \Delta_i^{m_0}(h, \Omega) \partial_{x_i}^{[\sigma]} u \|_{L_p(\Omega \times (\tau, T))}^{q} \right)^{1/q} dh + \left( \int_0^{h_0} \| \Delta_i^{m_0}(h, (\tau, T)) \partial_{x_i}^{[\sigma/2]} u \|_{L_p(\Omega \times (\tau, T))}^{q} \right)^{1/q} dh.
\]

From Lemma 7.44 from [1] we have

**Lemma 2.13.** Let \( p = q \). Then the norm (2.4) is equivalent to the following:
\[
\| u \|_{B_{p,p}^{\sigma,\sigma/2}(\Omega \times (\tau, T))} = \| u \|_{L_p(\Omega \times (\tau, T))} + \left( \int_\tau^T \int_\Omega \int_\Omega dx' \int dx'' \int_\Omega dx'' \left( \frac{D_x^{[\sigma]} u(x', t') - D_x^{[\sigma]} u(x'', t')}{|x' - x''|^{n+p(\sigma-\sigma)} + 1} \right)^{1/p} \right) + \left( \int_\Omega \int_\Omega \int_\Omega dx' \int dx'' \int dx''' \left( \frac{\partial_{x_i}^{[\sigma/2]} u(x, t') - \partial_{x_i}^{[\sigma/2]} u(x, t'')}{|t' - t''|^{1+p(\sigma/2-\sigma/2)} + 1} \right)^{1/p} \right),
\]

where \( D_x^{[\sigma]} = \partial_{x_1}^{\sigma_1} \partial_{x_2}^{\sigma_2} \partial_{x_3}^{\sigma_3}, \sigma_i \in \mathbb{N}_0, i = 1, 2, 3, \sigma_1 + \sigma_2 + \sigma_3 = [\sigma] \).

The norm (2.5) for \( \sigma \not\in \mathbb{N} \) is denoted also by \( \| u \|_{W_p^{\sigma,\sigma/2}(\Omega \times (\tau, T))} \), where \( W_p^{\sigma,\sigma/2}(\Omega \times (\tau, T)) \) is called the Sobolev–Slobodetskii space.

**Lemma 2.14.** Let \( f \in W_p^{\sigma,\sigma/2}(\Omega \times (0, T)), \sigma \not\in \mathbb{N}, p \in (1, \infty) \). Let
\[
f' = \begin{cases} f & \text{for } t > 0, \\ 0 & \text{for } t < 0. \end{cases}
\]
Then

\[
\|f'\|_{W^\sigma,\sigma/2_p(\Omega \times (-\infty, T))} = \|f\|_{W^\sigma,\sigma/2_p(\Omega \times (0, T))} + c \left( \int_0^T \left( \int_0^T \frac{|\partial_t f|^p}{|t^p|^{(\sigma/2) - (\sigma/2)}} \right) \right)^{1/p}.
\]  

(2.6)

If \( p(\sigma - [\sigma/2]) \geq 1 \) then \( \partial^j_t f|_{t=0} = 0, j \leq [\sigma/2] \).

This is a compatibility condition.

**Proof.** We have

\[
\|f'\|_{W^\sigma,\sigma/2_p(\Omega \times (-\infty, T))} = \|f\|_{L_p(\Omega \times (-\infty, T))} + \left( \int_0^T \left( \int_0^T \int_\Omega dx' dx'' \frac{|D_x f(x', t) - D_x f(x'', t)|^p}{|x' - x''|^{3 + p(\sigma - \sigma)}}, t \right) \right)^{1/p}
\]

\[
+ \left( \int_\Omega dx \int_0^T dt' \int_0^T \int_\Omega dx'' \frac{|\partial_t f(x', t') - \partial_t f(x', t'')|^p}{|t' - t''|^{1 + p(\sigma/2 - \sigma/2)}} \right)^{1/p}
\]

\[
\equiv I_1 + I_2 + I_3.
\]

In view of the definition of \( f' \) we have

\[ I_1 = \|f\|_{L_p(\Omega \times (0, T))}, \]

\[ I_2 = \left( \int_0^T \left( \int_\Omega dx' dx'' \frac{|D_x f(x', t) - D_x f(x'', t)|^p}{|x' - x''|^{3 + p(\sigma - \sigma)}}, t \right) \right)^{1/p} \]

and

\[ I_3^p = \int_\Omega dx \int_0^T dt' \int_0^T \int_\Omega dx'' \frac{|\partial_t f(x', t') - \partial_t f(x', t'')|^p}{|t' - t''|^{1 + p(\sigma/2 - \sigma/2)}} \]

\[ + \int_\Omega dx \int_0^T dt' \int_0^T \int_\Omega dx'' \frac{|\partial_t f(x', t')|^p}{|t'|^{p(\sigma/2 - \sigma/2)}} \equiv I_{31}^p + I_{32}^p. \]

Integrating with respect to \( t'' \) in \( I_{32} \) yields

\[ I_{32}^p = c \int_\Omega dx \int_0^T dt' \frac{|\partial_t f(x, t')|^p}{|t'|^{p(\sigma/2 - \sigma/2)}}. \]

Hence (2.6) holds.

If \( p(\sigma - [\sigma/2]) \geq 1 \) then the condition \( I_{32} < \infty \) implies that

\[ \|\partial_t f(x, t)\|_{L_p(\Omega)}|_{t=0} = 0. \]

Consequently, conditions \( I_{32} < \infty \) and \( p(\sigma - [\sigma/2]) \geq 1 \) imply that \( \|\partial_t f(x, t)\|_{L_p(\Omega)} \) must converge to zero sufficiently fast as \( t \) goes to zero. This ends the proof.

**Lemma 2.15.** Let \( f \in W^\sigma,\sigma/2_p(\Omega^T), \sigma \not\in \mathbb{N} \). Then

\[
\left( \int_0^T \left( \int_\Omega \frac{|\partial_t f(x, t)|^p}{|t^p|^{(\sigma/2) - (\sigma/2)}} dx dt \right) \right)^{1/p} \leq c \|f\|_{W^\sigma,\sigma/2_p(\Omega^T)},
\]

(2.7)

where \( c \) does not depend on \( f \).
Proof. Let $\sigma/2 - [\sigma/2] < 1/p$. Then Lemma 2 from [33] implies

$$
\left(\int_0^\infty \int_\Omega \left| \frac{\partial|^{\sigma/2}}{p(p(\sigma/2-[\sigma/2]))} f(x,t) \right|^p \ dx \, dt \right)^{1/p} \leq c \|f\|_{L_p(\Omega;W^{\sigma/2-[\sigma/2]}_{p}[0,\infty))}
$$

(2.8)

where an appropriate extension with respect to time from $(0,T)$ to $(0,\infty)$ was used.

Let $1/p < \sigma/2 - [\sigma/2] < 1 + 1/p$ and let $\partial_l|^{\sigma/2} f|_{l=0} = 0$. Then Lemma 2 from [33] implies (2.8) as well. This concludes the proof.

**Definition 2.16** (see [39, Sect. 2.3.5]). Let $L > 0$ be a given natural number. By $A_{aL}(\mathbb{R}^4)$ we denote the collection $\varphi = \{\varphi_j(\bar{x})\}_{j=0}^\infty \in S(\mathbb{R}^4)$ of functions with compact supports such that

$$
C(\varphi) = \sup_{\bar{x} \in \mathbb{R}^4} \left| \varphi(\bar{x}) \right|^L_{a} \sum_{|\alpha| \leq L} \left| D^{\alpha}_\bar{x} \varphi(\bar{x}) \right|
$$

$$
+ \sup_{\bar{x} \in \mathbb{R}^4 \setminus \{0\}} \left( |\varphi(\bar{x})| + |\varphi(\bar{x})|^L \right) \sum_{|\alpha| \leq L} \left| D^{\alpha}_\bar{x} \varphi_j(2^j \bar{x}, 2^j \bar{x} \bar{y}) \right| < \infty,
$$

where $D^{\alpha}_\bar{x} = \partial^{\alpha_0}_{x_0} \partial^{\alpha_1}_{x_1} \partial^{\alpha_2}_{x_2} \partial^{\alpha_3}_{x_3}$ and $\sum_{i=0}^4 \alpha_i = |\alpha|$.

**Definition 2.17** (Partition of unity) (see Ch.4, Sect. 4)[19]. Let $\Omega$ be the cylindrical domain with boundary $S = S_1 \cup S_2$; we defined previously. We assume that $S_2$ is flat and $S_1$, $S_2$ meet along a curve $L = S_1 \cap S_2$ under the angle $\pi/2$. The boundary $S_1$ must be sufficiently smooth so that at each point of $S_1$ there must exist a tangent plane.

Let $n(\xi_k)$, $k \in \mathfrak{N}_1$, be the unit outward vector normal to $S_1$ at the point $\xi_k$.

A Cartesian coordinate system $y = (y_1, y_2, y_3)$ with origin at $\xi_k$, $k \in \mathfrak{N}_1$, and the $y_3$ axis directed along $n(\xi_k)$, is usually called a local coordinate system. Similarly, we can introduce a local coordinate system with origin at $\xi_k \in S_2$, $k \in \mathfrak{N}_2$ and at $\xi_k \in L_i$, $k \in \mathfrak{N}_3$, where $L_i = S_1 \cap S_2(a_i)$, $i = 1, 2$.

Assuming that $\Omega$ is defined by a global Cartesian system $x = (x_1, x_2, x_3)$ we can transform it to the local Cartesian system with origin at $\xi_k$, $k \in \mathfrak{N}_1 \cup \mathfrak{N}_2 \cup \mathfrak{N}_3$, by some composition of a translation and a rotation. We denote the mapping by $Y_k$. Restrict our definition to $S_1$ part of the boundary. It is assumed that there exists a number $d > 0$ such that, in a sphere of radius $d$ with center at any point $\xi_k \in S_1$, $k \in \mathfrak{N}_1$, the surface $S_1$ is given in a local system at the point $\xi_k$ by the equation

$$
y_3 = F_k(y'), \quad y' = (y_1, y_2), \quad k \in \mathfrak{N}_1,
$$

(2.9)

where $F_k$ is a single-valued function.

We will say that $S_1 \in C^m$ if $F_k(y') \in C^m(K_0)$, $k \in \mathfrak{N}_1$, for any $\xi_k \in S_1$, where $K_0$ is the ball $|y'| \leq d/2$ and if the norms $\|F_k\|_{C^m(K_0)}$ are bounded by a common constant.

We require that at least $S_1 \in C^{1+\alpha}$, $\alpha \in (0,1]$. Under this assumption, in a neighborhood of $\xi_k$ we have the inequality

$$
\left| \frac{\partial F_k}{\partial y_\beta} \right| \leq c |y'|^\alpha, \quad \beta = 1, 2.
$$

(2.10)

We assume that it is possible to construct in domain $\Omega$ for any $\lambda > 0$, no matter how small, a finite number of subdomains $\omega(k)$ and $\Omega(k)$ possessing the following properties

1. $\omega(k) \subset \Omega(k) \subset \Omega$, $\bigcup_k \omega(k) = \bigcup_k \omega(k) = \Omega$.
2. For any point $x \in \Omega$ there exists an $\omega(k)$ such that $x \in \omega(k)$ and the distance from to $\Omega \setminus \Omega(k)$ is not less than $d/\lambda$.
3. There exists a number $N_0$, not depending on $\lambda$, such that the intersection of any $N_0 + 1$ distinct sets $\Omega(k)$ (and consequently any $N_0 + 1$ distinct sets $\omega(k)$) is empty.
4. Sets $\omega^{(k)}$ and $\Omega^{(k)}$, $k \in \mathfrak{M}$, are separated from the boundary $S = S_1 \cup S_2$ by a positive distance. We assume that they are 3-dimensional cubes with common center $\xi_k \in \Omega$ whose linear dimensions are equal $\beta \lambda$ and $2\beta \lambda$ ($\beta > 0$), respectively.

The sets $\omega^{(k)}$ and $\Omega^{(k)}$, $k \in \mathfrak{M}_1 \cup \mathfrak{M}_2$, are defined in local coordinates at a point $\xi_k \in S$ by the inequalities

$$|y_\alpha| < \lambda/2, \quad \alpha = 1, 2, \quad 0 < y_3 - F_k(y') < \lambda,$$

$$|y_\alpha| < \lambda, \quad \alpha = 1, 2, \quad 0 < y_3 - F_k(y') < 2\lambda.$$

Consider $\omega^{(k)}$ and $\Omega^{(k)}$, $k \in \mathfrak{M}_3$. Then we introduce a local system of coordinates with the origin at $\xi_k \in L_i, i = 1, 2$, such that $S_2$ is perpendicular to $y_3$ and $S_1$ is described by $y_1 = F_k(y_2, y_3)$.

Therefore, $\omega^{(k)}$ and $\Omega^{(k)}$, $k \in \mathfrak{M}_3$, are defined by

$$|y_\beta| < \lambda/2, \quad \beta = 2, 3, \quad 0 < y_1 - F_k(y_2, y_3) < \lambda,$$

$$|y_\beta| < \lambda, \quad \beta = 2, 3, \quad 0 < y_1 - F_k(y_2, y_3) < 2\lambda.$$

The change of variables

$$z_\alpha = y_\alpha, \quad z_3 = y_3 - F_k(y'), \quad k \in \mathfrak{M}_1 \quad (2.11)$$

transforms domains $\omega^{(k)}$, $\Omega^{(k)}$, $k \in \mathfrak{M}_1$, into the cubes

$$|z_\alpha| < \lambda/2, \quad \alpha = 1, 2, \quad 0 < z_3 < \lambda,$$

$$|z_\alpha| < \lambda, \quad \alpha = 1, 2, \quad 0 < z_3 < 2\lambda.$$

Since $S_2$ is flat, coordinates $x, y$ and $z$ can be taken as the same.

For $k \in \mathfrak{M}_3$, the change of variables has the form

$$z_\alpha = y_\alpha, \quad \alpha = 2, 3, \quad z_1 = y_1 - F_k(y_2, y_3). \quad (2.12)$$

We introduce functions $\zeta^{(k)}(x)$ having the properties

$$0 \leq \zeta^{(k)}(x) \leq 1, \quad |D^s \zeta^{(k)}| \leq c_s/\lambda^s, \quad \zeta^{(k)}(x) = \begin{cases} 1 & \text{for } x \in \omega^{(k)} \\ 0 & \text{for } x \in \Omega \setminus \Omega^{(k)} \end{cases}.$$

By virtue of property 3 of the domains $\Omega^{(k)}$,

$$1 \leq \sum_k (\zeta^{(k)})^2 \leq N_0$$

and hence the functions

$$\eta^{(k)}(x) = \frac{\zeta^{(k)}(x)}{\sum_j \zeta^{(j)}(x)}$$

possess the following properties:

$$\eta^{(k)} = 0 \quad \text{in } \Omega \setminus \Omega^{(k)}, \quad |D^s \eta^{(k)}(x)| \leq c_s\lambda^s$$

and moreover,

$$\sum_k \eta^{(k)}(x)\zeta^{(k)}(x) = 1.$$

Remark 2.18. The norms of Besov spaces described by Fourier transforms and the norms defined by differences are equivalent (see [4,18,21,39,40]).

Lemma 2.19. Let $e_0(x_3) = e^{-s_3 x_3}$, $e_1(x_3) = \frac{e^{-s_3 x_3} - e^{-|\xi|x_3}}{s - \xi}$, $e_2(x_3) = e^{-|\xi|x_3}$, $|\xi| = \sqrt{\xi_1^2 + \xi_2^2}$, $s = s + \gamma$, $\xi = (\xi_1, \xi_2)$, $s = \gamma + i\xi_0, \xi \in \mathbb{R}^2, \xi_0 \in \mathbb{R}, \text{Re } s = \gamma > 0$. Let $j \in \mathbb{N} \cup \{0\} \equiv \mathbb{N}_0$, $\partial_{\xi}^{k'} = \partial_{\xi_1}^{k_1} \partial_{\xi_2}^{k_2}, k' = k_1 + k_2, \alpha \in (0, 1)$. 
Then for \( e_0(x_3) \) we have

\[
\sum_{2k+k' \leq 2} \int_0^\infty |\frac{\partial^j}{\partial x_3^j} \frac{\partial^{k'}}{\partial \xi_0^{k'}} e_0|^p dx_3 \leq c \sum_{2k+k' \leq 2} |\tau|^{p(j+p(2k+k')-1)}, \tag{2.13}
\]

\[
\sum_{j+2k+k' \leq 2} \int_0^\infty \int_0^\infty |\frac{\partial^j}{\partial x_3^j} \frac{\partial^{k'}}{\partial \xi_0^{k'}} e_0(x_3 + z) - \frac{\partial^j}{\partial x_3^j} \frac{\partial^{k'}}{\partial \xi_0^{k'}} e_0(x_3)|^p dx_3 dz \leq c \sum_{j+2k+k' \leq 2} |\tau|^{p(j+\kappa)-p(2k+k')-1}. \tag{2.14}
\]

Next \( e_1(x_3) \) satisfies the estimates

\[
\sum_{j+2k+k' \leq 2} \int_0^\infty |\frac{\partial^j}{\partial x_3^j} \frac{\partial^{k'}}{\partial \xi_0^{k'}} e_1(x_3)|^p dx_3 dz \leq c \sum_{j+2k+k' \leq 2} |\tau|^{p(j+p(2k+k')-1) + |\xi|^{p(2k+k')-1}}. \tag{2.15}
\]

\[
\sum_{j+2k+k' \leq 2} \int_0^\infty \int_0^\infty |\frac{\partial^j}{\partial x_3^j} \frac{\partial^{k'}}{\partial \xi_0^{k'}} e_1(x_3 + z) - \frac{\partial^j}{\partial x_3^j} \frac{\partial^{k'}}{\partial \xi_0^{k'}} e_1(x_3)|^p dx_3 dz \leq c \sum_{j+2k+k' \leq 2} |\tau|^{p(j+\kappa)-p(2k+k')-1} + |\xi|^{p(2k+k')-1}. \tag{2.16}
\]

Finally, \( e_2 = e_0 - (\tau - |\xi|)e_1 \).

**Proof.** First we prove (2.13). Let \( k = k' = j = 0 \). Then

\[
\int_0^\infty |e_0(x_3)|^p dx_3 = \int_0^\infty e^{-p\text{Re} \tau x_3} dx_3 \leq \frac{1}{p\text{Re} \tau} \leq \frac{\sqrt{2}}{p}|\tau|^{-1}, \tag{2.17}
\]

where we have used the fact that \( \text{Arg} \tau \in (-\pi/4, \pi/4) \). Thus \( |\tau| \leq \sqrt{2}\text{Re} \tau \). Let \( k = k' = 0 \) and \( j \in \mathbb{N} \). Then

\[
\frac{\partial^j}{\partial x_3^j} e_0 = (-1)^j \frac{\partial^j}{\partial x_3^j} e^{-\tau x_3} \tag{2.18}
\]

and

\[
\int_0^\infty |\frac{\partial^j}{\partial x_3^j} e_0|^p dx_3 \leq |\tau|^{p(j+p-1)} \int_0^\infty |e_0(x_3)|^p dx \leq c|\tau|^{p(j-1)}. \tag{2.19}
\]

Let \( k = 1 \). Then

\[
\frac{\partial}{\partial \xi_0} \frac{\partial^j}{\partial x_3^j} e_0 = (c_1 \tau^{j-1} + c_2 \tau^j x_3) \tau \xi_0 e^{-\tau x_3}. \tag{2.20}
\]

Since \( |\tau \xi_0| \leq c/|\tau| \) and

\[
\sup_{x_3} x_3^p e^{-(p/2)\text{Re} \tau x_3} \leq c/|\tau|^p, \tag{2.21}
\]

we obtain

\[
\int_0^\infty |\frac{\partial}{\partial \xi_0} \frac{\partial^j}{\partial x_3^j} e_0|^p dx_3 \leq c|\tau|^{p(j-2p)} \int_0^\infty e^{-(p/2)\text{Re} \tau x_3} dx_3 \leq c|\tau|^{p(j-2p-1)}. \tag{2.22}
\]
Let \( k' = 2 \). Since \(|\partial_\xi \tau| \leq c, |\partial_\xi^2 \tau| \leq c/|\tau|\) we obtain
\[
|\partial_\xi^2 \partial_3^j e_0| \leq (c_1|\tau|^{-2} + c_2|\tau|^{-1}x_3 + c_3|\tau|^j x_3^2)e^{-Re\tau x_3}.
\]
Using (2.21) and
\[
\sup_{x_3} x_3^{2p} e^{-(p/2)Re\tau x_3} \leq c/|\tau|^{2p}
\]
we obtain
\[
\int_0^\infty |\partial_\xi^2 \partial_3^j e_0|^p dx_3 \leq c|\tau|^{p-2p-1}.
\] (2.24)

Estimates (2.19), (2.22) and (2.24) imply (2.13).

Next we prove (2.14). For \( k = k' = 0 \) and \( j \in \mathbb{N}_0 \) we have
\[
\int_0^\infty \int_0^\infty \frac{|\partial_\xi \partial_3 \epsilon_j(x_3 + z) - \partial_\xi \partial_3 \epsilon_j(x_3)|^p}{z^{1+\kappa\tau}} dx_3 dz
\]
\[
\leq c \int_0^\infty \int_0^\infty |\tau|^{pj} \frac{|e^{-\tau(x_3 + z)} - e^{-\tau x_3}|^p}{z^{1+\kappa\tau}} dx_3 dz
\]
\[
\leq c \int_0^\infty |\tau|^{p} e^{-Re\tau x_3} dx_3 \int_0^\infty \frac{|e^{-\tau z} - 1|^p}{z^{1+\kappa\tau}} dz
\]
\[
\leq c|\tau|^{p(j+\kappa) - 1}.
\]

Finally, we prove (2.14) for \( k = 1, k' = 0, j \in \mathbb{N}_0 \). Then we get
\[
\int_0^\infty \int_0^\infty \frac{|\partial_{\xi_0} \partial_3 \epsilon_j(x_3 + z) - \partial_{\xi_0} \partial_3 \epsilon_j(x_3)|^p}{z^{1+\kappa\tau}} dx_3 dz \equiv I_1.
\]

Using that
\[
\partial_{\xi_0} (\tau^{j} e^{-\tau x_3}) = (j\tau^{j-1} - \tau^j x_3)\partial_{\xi_0} e^{-\tau x_3}
\]
and
\[
\partial_{\xi_0} (\tau^{j} e^{-\tau x_3}) = (j\tau^{j-1} - \tau^j x_3)\partial_{\xi_0} e^{-\tau x_3}
\]
we obtain
\[
\partial_{\xi_0} (\tau^{j} e^{-\tau x_3}) - \partial_{\xi_0} (\tau^{j} e^{-\tau x_3}) = -\tau^j z e^{-\tau x_3} \partial_{\xi_0} e^{-\tau x_3} + (j\tau^{j-1} - \tau^j x_3)(e^{-\tau x_3} - e^{-\tau x_3}) \partial_{\xi_0} e^{-\tau x_3}.
\]

Using that \(|\tau_{\xi_0}| \leq c/|\tau|\) we obtain
\[
I_1 \leq c|\tau|^{p(j-1)} \int_0^\infty \int_0^\infty \frac{|\tau|^{p} e^{-pRe\tau x_3} |z|^{1+\kappa\tau}}{z^{1+\kappa\tau}} dx_3 dz
\]
\[
+ c|\tau|^{p(j-2)} \int_0^\infty \int_0^\infty \frac{|e^{-\tau x_3 + z} - e^{-\tau x_3}|^p}{z^{1+\kappa\tau}} dx_3 dz
\]
\[
+ c|\tau|^{p(j-1)} \int_0^\infty \int_0^\infty x_3^p |e^{-\tau x_3 + z} - e^{-\tau x_3}|^p dx_3 dz
\]
\[
\equiv J_1 + J_2 + J_3.
\]
To estimate $J_1$ we use
\[
\int_0^\infty \int_0^\infty \frac{z p^\frac{1}{r} e^{-\frac{p}{r} \Re \tau (x_3 + z)}}{z^{1+rp}} \, dz \, dx_3 = \int_0^\infty e^{-p \Re \tau x_3} \, dx_3 \int_0^\infty \frac{z p^\frac{1}{r} e^{-\frac{p}{r} \Re \tau z}}{z^{1+rp}} \, dz.
\]
\[
= \frac{1}{p \Re \tau} \int_0^\infty \frac{y p e^{-py}}{y^{1+rp}} \, dy (\Re \tau)^{\frac{1}{r} - p} \leq c |\tau|^{\frac{1}{r} - p - 1}.
\]

To estimate $J_2$ we need
\[
\int_0^\infty \int_0^\infty \frac{|e^{-\tau (x_3 + z)} - e^{-\tau x_3}|^p}{z^{1+rp}} \, dx_3 \, dz = \int_0^\infty e^{-p \Re \tau x_3} \, dx_3 \cdot \int_0^\infty \frac{|e^{-\tau z} - 1|^p}{y^{1+rp}} \, dy \leq c |\tau|^{\frac{1}{r} - 1}.
\]

Finally, we examine $J_3$. Then we have
\[
\int_0^\infty \int_0^\infty \frac{x_3^p|e^{-\tau (x_3 + z)} - e^{-\tau x_3}|}{z^{1+rp}} \, dx_3 \, dz = \int_0^\infty x_3^p e^{-p \Re \tau x_3} \, dx_3 \int_0^\infty \frac{|e^{-\tau z} - 1|^p}{z^{1+rp}} \, dz \equiv J_3^1,
\]
where the first integral is bounded by
\[
\sup_{x_3} x_3^p e^{-(p/2) \Re \tau x_3} \int_0^\infty e^{-p \Re \tau x_3} \, dx_3 \leq c |\Re \tau|^{-p - 1}.
\]

Hence
\[
J_3^1 \leq c |\tau|^{\frac{1}{r} - p - 1}.
\]

Summarizing, we proved (2.14) in the case $k = 1, j \in \mathbb{N}_0$. A corresponding estimate holds for $k' = 2, j \in \mathbb{N}_0$.

Next, we show (2.15). We consider $e_1(x_3)$ as the convolution
\[
e_1(x_3) = - \int_0^{x_3} e^{-|\xi| (x_3 - y)} e^{-\tau y} \, dy.
\]

Hence, by the Young inequality, we obtain
\[
\int_0^\infty |e_1(x_3)|^p \, dx_3 \leq \left( \int_0^\infty e^{-p \Re \tau y} \, dy \right)^p \int_0^\infty e^{-p |\xi| x_3} \, dx_3 \leq \frac{c}{|\tau|^p |\xi|}.
\]

We calculate
\[
\partial_{\xi_0} e_1(x_3) = \int_0^{x_3} e^{-|\xi| (x_3 - y)} ye^{-\tau y} \, dy.
\]

Using $|\partial_{\xi_0} \tau| \leq c / |\tau|$ and the Young inequality yield
\[
\int_0^\infty |\partial_{\xi_0} e_1(x_3)|^p \, dx_3 \leq \frac{c}{|\tau|^3 |\xi|}.
\]
In view of the relation
\[
\frac{d^3 e_1(x_3)}{dx_3^3} = (-|\xi|)^j e_1(x_3) + (-1)^j \tau^{j-1} + \tau^{j-2}|\xi| + \cdots + |\xi|^{j-1} e_0(x_3)
\]  \hspace{1cm} (2.27)
and the estimate
\[
\int_0^\infty \left| \frac{d^j}{dx_3^j} e_0(x_3) \right|^p dx_3 \leq c|\tau|^{pj-1}
\]
we obtain
\[
\int_0^\infty \left| \frac{d^j}{dx_3^j} e_1(x_3) \right|^p dx_3 \leq c|\xi|^{pj} \int_0^\infty |e_1(x_3)|^p dx_3 + c(|\tau|^{p(j-1)} + |\tau|^{p(j-2)}|\xi|^p + \cdots + |\xi|^{p(j-1)}) \int_0^\infty |e_0(x_3)|^p dx_3 \equiv I_1.
\]  \hspace{1cm} (2.28)
Applying (2.17) and (2.26) we have
\[
I_1 \leq c|\xi|^{pj} \frac{1}{|\tau|^p} + c \left( |\tau|^{p(j-1)} + |\tau|^{p(j-2)}|\xi|^p + \cdots + |\xi|^{p(j-1)} \right) \frac{1}{|\tau|}
\]
\[
= c \left( \frac{|\tau|^{pj-1}}{|\tau|^p} + \frac{|\xi|^{pj-1}}{|\tau|^p} \right) + c \left( |\tau|^{p(j-2)}|\xi|^p + \cdots + |\xi|^{p(j-1)} \right) \frac{1}{|\tau|} \equiv I_2.
\]
Now we estimate terms under the square bracket in $I_2$. The first term equals
\[
\frac{|\tau|^{pj-1}}{|\tau|^p} \cdot |\xi|^p \equiv I_2^1
\]
Since $\tau^2 = \gamma + i\xi_0 + \xi^2$, $\gamma > 0$, we have $|\tau|^2 = \gamma + \xi^2 + \xi_0^2 \geq |\xi|^2$. Using this in $I_2^1$ yields
\[
I_2^1 \leq c \frac{|\tau|^{pj-1}}{|\tau|^p}.
\]
Consider the last term under the square bracket in $I_2$. Applying $|\xi| \leq |\tau|$ and $j > 1$, we obtain
\[
\frac{|\xi|^{p(j-1)}}{|\tau|} \leq \frac{|\tau|^{pj-p}}{|\tau|} \leq \frac{|\tau|^{pj-1}}{|\tau|^p}.
\]
Next, we can incorporate the estimates in $I_2$ to conclude (2.28) in the form
\[
\int_0^\infty \left| \frac{d^j}{dx_3^j} e_1(x_3) \right|^p dx_3 \leq c \frac{|\tau|^{pj-1} + |\xi|^{pj-1}}{|\tau|^p}.
\]  \hspace{1cm} (2.29)
Consider
\[
\int_0^\infty |\partial_{x_0} e_1(x_3)|^p dx_3 = \int_0^\infty |x_3|^3 e^{-|\xi| (x_3 - y)} dy |d^3 x_3| dx_3
\]
\[
\leq \frac{c}{|\tau|^p} \sup_{y \in \mathbb{R}_+} (ye^{-R y / 2})^p \int_0^\infty |d^3 x_3| dy |d^3 x_3|
\]
\[
\leq \frac{c}{|\tau|^{3p}} \equiv I_3,
\]
where we used (2.26), $|\tau, x_0| \leq \frac{c}{|\tau|}$ and $\sup_{y \in \mathbb{R}_+} (ye^{-R y / 2}) \leq \frac{c}{|\tau|}$. For $|\tau| \leq 2|\xi|$ we have
\[
I_3 \leq c \frac{|\tau|^{-2p-1}}{|\tau|^p}
\]
and for $|\tau| \geq 2|\xi|$ we obtain

$$I_3 = \frac{c}{|\tau|^{2p}|\tau|^p|\xi|} \leq c \frac{|\xi|^{-2p-1}}{|\tau|^p}. $$

Hence, (2.30) implies the inequality

$$\int_0^\infty |\partial_{\xi_0} e_1(x_3)|^p dx_3 \leq c \frac{|\tau|^{-2p-1} + |\xi|^{-2p-1}}{|\tau|^p}.$$  \hspace{1cm} (2.31)

Therefore, (2.31) implies (2.15) for $j = 0, k = 1, k' = 0$. Continuing the considerations, analogously we prove (2.15) for other parameters.

Finally, we need to show (2.16). First we consider the expression

$$J \equiv \int_0^\infty \int_0^\infty \frac{|e_1(x_3) - e_1(x_3)|^p}{z^{1+p\kappa}} dx_3 dz.$$  \hspace{1cm} (2.32)

To prove (2.16), we have

$$J \leq \int_0^\infty \int_0^\infty \frac{|e^{-\tau x_3} e_1(z)|^p}{z^{1+p\kappa}} dx_3 dz + \int_0^\infty \int_0^\infty \frac{|e_1(x_3)(e^{-|\xi|z} - 1)|^p}{z^{1+p\kappa}} dx_3 dz$$

$$\leq \frac{c}{|\tau|} \int_0^\infty \frac{|e_1(z)|^p}{z^{1+p\kappa}} dz + \frac{c}{|\tau|^p|\xi|} \int_0^\infty \frac{|(e^{-|\xi|z} - 1)|^p}{z^{1+p\kappa}} dz$$

where we used (2.17) and (2.26).

In order to estimate $J_1$ we will need (see [34], page 158).

$$|e_1(z)| \leq \frac{e^{-|\xi|z}}{|\tau - |\xi| |} e^{-(|\xi| - 1)z} \leq |z| e^{-|\xi| z}.$$  \hspace{1cm} (2.34)

In view of (2.34) we have

$$J_1 \leq \frac{c}{|\tau|} \int_0^\infty \frac{|z|^{p-1-p\kappa} e^{-p|\xi|z}}{z^{1+p\kappa}} dz$$

$$= c \frac{|\xi|^{p(p-1)}}{|\tau|} \int_0^\infty y^{p(1-\kappa)-1} e^{-py} dy \equiv J_1' \equiv J_1.$$  \hspace{1cm} (2.35)

where the integral in $J_1'$ can be written in the form

$$\int_0^1 y^{p(1-\kappa)-1} e^{-py} dy + \int_1^\infty y^{p(1-\kappa)-1} e^{-py} dy \equiv I_1 + I_2.$$  \hspace{1cm} (2.36)

For $\kappa \in (0, 1)$ we conclude

$$I_1 \leq \int_0^1 y^{p(1-\kappa)-1} dy < \infty.$$  \hspace{1cm} (2.37)

In order to consider $I_2$ we take into account two cases. For $p(1-\kappa) - 1 < 0$ holds

$$I_2 \leq \int_1^\infty e^{-py} dy < \infty.$$

and for $p(1-\kappa) - 1 > 0$ we obtain

$$I_2 = \int_1^\infty y^{p(1-\kappa)-1} e^{-py/2} e^{-py/2} dy \leq \sup_{y \in \mathbb{R}_+} (y^{p(1-\kappa)-1} e^{-py/2}) \int_1^\infty e^{-py/2} dy < \infty.$$
Therefore,
\[ J_1 \leq \frac{c}{|\tau|} |\xi|^{p\kappa - p} \tag{2.35} \]

Now, we want to estimate \( J_1 \) in a different way. We are going to apply
\[ |e_1(z)| = \left| \frac{e^{-\tau z} - e^{-|\xi|z}}{|\tau - |\xi||} \right| = \left| \frac{e^{-\tau z} - 1 - (e^{-|\xi|z} - 1)}{|\tau - |\xi||} \right| \leq \frac{|e^{-\tau z} - 1| + |e^{-|\xi|z} - 1|}{|\tau - |\xi||} \]

Then
\[ J_1 \leq \frac{c}{|\tau||\tau - |\xi||^p} \int_0^\infty \left( \frac{|e^{-\tau z} - 1|^p + |e^{-|\xi|z} - 1|^p}{z^{1+p\kappa}} \right) dz \equiv J_1^2. \]

We look closer at the first integral in \( J_1^2 \) and calculate
\[ \int_0^\infty \frac{|e^{-\tau z} - 1|^p}{z^{1+p\kappa}} dz = |\tau|^{p\kappa} \int_0^\infty \frac{|e^{-\tau y} - 1|^p}{y^{1+p\kappa}} dy \equiv I, \]

where the integral in \( I \) can be split into
\[ \int_1^\infty \frac{|e^{-\tau y} - 1|^p}{y^{1+p\kappa}} dy + \int_1^\infty \frac{|e^{-\tau y} - 1|^p}{y^{1+p\kappa}} dy \equiv I_1 + I_2, \]

where
\[ I_1 \leq c \int_0^1 y^{p(1-\kappa)-1} e^y dy < \infty \]

and
\[ I_2 \leq \int_1^\infty \frac{dy}{y^{1+p\kappa}} < \infty. \]

Similarly, the second integral in \( J_1^2 \) can be bounded by
\[ \frac{c}{|\tau||\xi|^{p\kappa}}. \]

Therefore,
\[ J_1 \leq \frac{c}{|\tau||\tau - |\xi||^p} (|\tau|^{p\kappa} + |\xi|^{p\kappa}). \tag{2.36} \]

In order to estimate the right hand side of (2.35) (i.e. the bound on \( J_1 \)) we set \(|\tau| \leq 2|\xi|\). Then
\[ \frac{1}{|\tau|} |\xi|^{p\kappa - 1} |\xi|^{-(p-1)} \leq c \frac{|\xi|^{p\kappa - 1}}{|\tau|^p}. \tag{2.37} \]

Consequently, we estimate the right hand side of (2.36) (the other bound on \( J_1 \)) in the case \(|\tau| \geq 2|\xi|\), which yields
\[ \frac{c}{|\tau|^{p+1}} \left( |\tau|^{p\kappa} + |\xi|^{p\kappa} \right) \leq c \frac{|\tau|^{p\kappa - 1}}{|\tau|^p} + c \frac{|\xi|^{p\kappa}}{|\tau|^{p+1}} \leq c \frac{|\tau|^{p\kappa - 1}}{|\tau|^p} + c \frac{|\xi|^{p\kappa - 1}}{|\tau|^p}, \tag{2.38} \]

where in the second inequality we applied \( \frac{1}{|\tau|} \leq \frac{1}{2 |\xi|} \).

From (2.37) and (2.38) we conclude
\[ J_1 \leq c \frac{|\tau|^{p\kappa - 1} + |\xi|^{p\kappa - 1}}{|\tau|^p}. \tag{2.39} \]
After changing variables in the integral in $J_2$ we infer
\[ J_2 = c \frac{|\xi|^{p-1}}{|\tau|^p} \int_0^\infty \frac{|e^{-y} - 1|^p}{y^{1+p\epsilon}} dy \leq c \frac{|\xi|^{p-1}}{|\tau|^p}. \]

Using the above estimates in (2.32) yields
\[ \int_0^\infty \int_0^\infty \frac{|e_1(x_3 + z) - e_1(x_3)|^p}{z^{1+p\epsilon}} dx_3 dz \leq c \frac{|\tau|^{p\epsilon - 1} + |\xi|^{p\epsilon - 1}}{|\tau|^p}. \tag{2.40} \]

This implies (2.16) for $j = 0, k = 0, k' = 0$.

Our next step is to show (2.16) for $j \geq 1, k = 0, k' = 0$. Employing (2.27) we have
\[
\int_0^\infty \int_0^\infty \frac{|d^j_{x_3} e_1(x_3 + z) - d^j_{x_3} e_1(x_3)|^p}{z^{1+p\epsilon}} dx_3 dz \\
\leq c |\xi|^p \int_0^\infty \int_0^\infty \frac{|e_1(x_3 + z) - e_1(x_3)|^p}{z^{1+p\epsilon}} dx_3 dz \\
+ c \left( |\tau|^{p(j-1)} + |\tau|^{p(j-2)} |\xi|^p + \cdots + |\xi|^{p(j-1)} \right) \cdot \int_0^\infty \int_0^\infty \frac{|e_0(x_3 + z) - e_0(x_3)|^p}{z^{1+p\epsilon}} dx_3 dz = L.
\]

Applying that
\[
\int_0^\infty \int_0^\infty \frac{|e_0(x_3 + z) - e_0(x_3)|^p}{z^{1+p\epsilon}} dx_3 dz \leq c |\tau|^{p\epsilon - 1}
\]
and
\[
\int_0^\infty \int_0^\infty \frac{|e_1(x_3 + z) - e_1(x_3)|^p}{z^{1+p\epsilon}} dx_3 dz \leq \frac{|\tau|^{p\epsilon - 1} + |\xi|^{p\epsilon - 1}}{|\tau|^p}
\]

we infer
\[
L \leq c |\xi|^p \frac{|\tau|^{p\epsilon - 1} + |\xi|^{p\epsilon - 1}}{|\tau|^p} \\
+ c \left( |\tau|^{p(j-1)} + |\tau|^{p(j-2)} |\xi|^p + \cdots + |\xi|^{p(j-1)} \right) |\tau|^{p\epsilon - 1} \\
= c \left( |\tau|^{p\epsilon - 1} |\xi|^p + |\xi|^{p(j+\epsilon)-1} + |\tau|^{p(j-1)+p\epsilon - 1} + |\tau|^{p(j-2)+p\epsilon - 1}|\xi|^p + \cdots + |\xi|^{p(j-1)} |\tau|^{p\epsilon - 1} \right) \equiv L_1.
\]

Using that $|\xi| \leq |\tau|$, we have
\[
L_1 \leq c \left( \frac{|\tau|^{p(j+\epsilon)-1}}{|\tau|^p} + \frac{|\xi|^{p(j+\epsilon)-1}}{|\tau|^p} \right).
\]

This implies (2.16) for $j \geq 1, k = 0, k' = 0$. Continuing the considerations we can prove in analogous way (2.16) for $j \in \mathbb{N}, k \leq 1, k' \leq 2$.

This ends the proof. □
3. The Stokes System in the Whole Space

We consider the following Stokes system
\[
\begin{align*}
v_t - \nu \Delta v + \nabla p &= f \quad \text{in } \mathbb{R}^3 \times \mathbb{R}_+ , \\
div v &= g \quad \text{in } \mathbb{R}^3 \times \mathbb{R}_+ , \\
v|_{t=0} &= v_0 \quad \text{in } \mathbb{R}^3 .
\end{align*}
\]  
(3.1)

We are looking for solutions to (3.1) under the following assumptions
\[
\begin{align*}
f &\in W^{\sigma,\sigma/2} p (\mathbb{R}^3 \times \mathbb{R}_+ ) , \\
g &\in W^{\sigma+1,\sigma/2+1/2} p (\mathbb{R}^3 \times \mathbb{R}_+ ) , \\
v_0 &\in W^{\sigma+2-2/p} p (\mathbb{R}^3 ) ,
\end{align*}
\]  
(3.2)

where \( p \in (1, \infty) , \sigma \not\in \mathbb{N} \).

Since \( v_0 \in W^{\sigma+2-2/p} p (\mathbb{R}^3 ) \) there exists the time extension \( \tilde{v}_0 \in W^{\sigma+2,\sigma/2+1} p (\mathbb{R}^3 \times \mathbb{R}_+ ) \) such that
\[
\tilde{v}_0|_{t=0} = v_0 
\]  
(3.3)

and
\[
\|\tilde{v}_0\|_{W^{\sigma+2,\sigma/2+1} p (\mathbb{R}^3 \times \mathbb{R}_+ )} \leq c \|v_0\|_{W^{\sigma+2-2/p} p (\mathbb{R}^3 )} ,
\]  
(3.4)

where \( c \) does not depend on \( v_0 \).

Having the extension \( \tilde{v}_0 \) we introduce the new function
\[
\tilde{v} = v - \tilde{v}_0
\]  
(3.5)

such that \((\tilde{v}, p)\) is a solution to the Stokes system with vanishing initial data
\[
\begin{align*}
\tilde{v}_t - \nu \Delta \tilde{v} + \nabla p &= f - \tilde{v}_{0,t} + \nu \Delta \tilde{v}_0 = \tilde{f} , \\
div \tilde{v} &= g - \text{div } \tilde{v}_0 = \tilde{g} \\
\tilde{v}|_{t=0} &= 0 .
\end{align*}
\]  
(3.6)

Using Lemmas 2.14 and 2.15 we can extend functions \( \tilde{f} \) and \( \tilde{g} \) by zero for \( t < 0 \). We denote the extended functions by
\[
\begin{align*}
f' &= \begin{cases} 
\tilde{f} & \text{for } t > 0 , \\
0 & \text{for } t < 0 ,
\end{cases} \\
g' &= \begin{cases} 
\tilde{g} & \text{for } t > 0 , \\
0 & \text{for } t < 0 .
\end{cases}
\end{align*}
\]

The extensions are possible if the following compatibility conditions hold
\[
\begin{align*}
\partial_t^i \tilde{f}|_{t=0} &= 0 \quad \text{for } i \leq \lfloor \sigma/2 \rfloor , \\
\partial_t^i \tilde{g}|_{t=0} &= 0 \quad \text{for } i \leq \lfloor \sigma/2 + 1/2 \rfloor .
\end{align*}
\]  
(3.7)

Employing the extensions in (3.6) we get the problem
\[
\begin{align*}
v'_t - \nu \Delta v' + \nabla p' &= f' \quad \text{in } \mathbb{R}^3 \times \mathbb{R} , \\
div v' &= g' \quad \text{in } \mathbb{R}^3 \times \mathbb{R} .
\end{align*}
\]  
(3.8)

Let \( E(x) = c|x|^{-1} \) be the fundamental solution to the Laplace equation. Then any solution to the equation
\[
\Delta \varphi = g'
\]
has the form
\[
\varphi = E \ast g' ,
\]  
(3.9)
where $\star$ means convolution.

Introducing the new functions

$$v'' = v' - \nabla \varphi, \quad p'' = p', \quad f'' = f' - \nabla \varphi + \nu \Delta \nabla \varphi$$

(3.10)

we replace $v''$, $p''$, $f''$ by $v$, $p$, $f$. Let $g = \text{div} \chi$. Then $\tilde{g} = \text{div} (\chi - \tilde{v}_0)$ and $g' = \text{div} (\chi' - \tilde{v}_0')$. Moreover, (3.9) yields $\nabla \varphi = \nabla^2 E \star (\chi' - \tilde{v}_0')$. Hence,

$$\|\nabla \varphi\|_{W^{\sigma/2}_r(\mathbb{R}^3 \times \mathbb{R}^+)} \leq c(\|\chi\|_{W^{\sigma/2}_r(\mathbb{R}^3 \times \mathbb{R}^+)} + \|\tilde{v}_0\|_{W^{2+\sigma/2}_r(\mathbb{R}^3)})$$

and

$$\|\Delta \nabla \varphi\|_{W^{\sigma/2}_r(\mathbb{R}^3 \times \mathbb{R}^+)} \leq c(\|\Delta \chi\|_{W^{\sigma/2}_r(\mathbb{R}^3 \times \mathbb{R}^+)} + \|\tilde{v}_0\|_{W^{2+\sigma/2}_r(\mathbb{R}^3)}).$$

In view of the above remarks

$$\|f''\|_{W^{\sigma/2}_r(\mathbb{R}^3 \times \mathbb{R}^+)} \leq c(\|f\|_{W^{\sigma/2}_r(\mathbb{R}^3 \times \mathbb{R}^+)} + \|\chi\|_{W^{\sigma/2}_r(\mathbb{R}^3 \times \mathbb{R}^+)} + \|\Delta \chi\|_{W^{\sigma/2}_r(\mathbb{R}^3 \times \mathbb{R}^+)} + \|\tilde{v}_0\|_{W^{2+\sigma/2}_r(\mathbb{R}^3)}).$$

Then we see that $(v, p)$ is a solution to the problem

$$v_t - \nu \Delta v + \nabla p = f' - \nabla \varphi + \nu \Delta \nabla \varphi \equiv f,$$

$$\text{div} \, v = 0.$$ (3.11)

To solve (3.11) we use the Fourier and the Fourier-Laplace transforms.

First we recall Definition 2.2 for the Fourier transform and its inverse

$$\hat{f}(\xi, \xi_0) \equiv (Ff)(\xi, \xi_0) = \int_{\mathbb{R}^4} e^{-i\xi_0 t - i\xi \cdot x} f(x, t) dx dt$$

(3.12)

$$\text{(3.12)}$$

$$(F^{-1} f)(x, t) = \frac{1}{(2\pi)^4} \int_{\mathbb{R}^4} e^{i\xi_0 t + i\xi \cdot x} f(\xi, \xi_0) d\xi d\xi_0,$$

where $x \cdot \xi = x_1 \xi_1 + x_2 \xi_2 + x_3 \xi_3$.

In the same Definition, the Fourier-Laplace transform and its inverse have been given with

$$(F_2 f)(\xi, \tau) \equiv \hat{f}(\xi, s) \equiv (Ff_\gamma)(\xi, s) = \int_{\mathbb{R}^4} e^{-st - i\xi \cdot x} f(x, t) dx dt,$$

$$\text{(3.13)}$$

$$(F_2^{-1} f)(x, t) = \frac{1}{(2\pi)^4} \int_{\mathbb{R}^4} e^{st + i\xi \cdot x} f(\xi, s) d\xi d\xi_0,$$

where $s = \gamma + i\xi_0, \tau = \sqrt{s + |\xi|^2}, 0 < \gamma \in \mathbb{R}_+$. Comparing (3.12) and (3.13) we have

$$(F_2 f)(\xi, s) = (F_2 e^{-\gamma t} f)(\xi, \xi_0),$$

$$\text{(3.14)}$$

$$(F_2^{-1} f)(x, t) = e^{\gamma t} (F^{-1} f)(x, t).$$

Applying the Fourier-Laplace transform to (3.11) yields

$$\tau + |\xi|^2 \hat{v}(\xi, s) + i\xi \hat{p}(\xi, s) = \hat{f}(\xi, s),$$

$$i\xi \cdot \hat{v}(\xi, s) = 0.$$ (3.15)

Solving (3.15) we get

$$\hat{v}(\xi, s) = \frac{P(\xi) \hat{f}(\xi, s)}{\tau + |\xi|^2}, \quad \hat{p}(\xi, s) = \frac{\xi \cdot \hat{f}(\xi, s)}{i|\xi|^2},$$

(3.16)

where $P(\xi) = \{\delta_{jk} - \xi_j \xi_k / |\xi|^2\}_{j,k=1,2,3}$. 


Lemma 3.1. Assume that $p \in (1, \infty)$, $\sigma \in \mathbb{R}_+$ and $f \in B_{p,p}^{\sigma/2}((\mathbb{R}^3 \times \mathbb{R})$. Then there exists a unique solution to problem (3.11) such that $v \in B_{p,p}^{\sigma/2,\sigma/2+1}((\mathbb{R}^3 \times \mathbb{R}), \nabla p \in B_{p,p}^{\sigma/2}((\mathbb{R}^3 \times \mathbb{R})$ and
\[
\|v\|_{B_{p,p}^{\sigma/2,\sigma/2+1}((\mathbb{R}^3 \times \mathbb{R})} \leq c \|f\|_{B_{p,p}^{\sigma/2}((\mathbb{R}^3 \times \mathbb{R})},
\]
\[
\|\nabla p\|_{B_{p,p}^{\sigma/2}((\mathbb{R}^3 \times \mathbb{R})} \leq c \|f\|_{B_{p,p}^{\sigma/2}((\mathbb{R}^3 \times \mathbb{R})}. \tag{3.17}
\]

The existence and uniqueness of solutions follow from (3.16).

Proof. First we consider
\[
\|v\|^\infty_{B_{p,p}^{\sigma/2,\sigma/2+1}((\mathbb{R}^3 \times \mathbb{R})} = \sum_{k=0}^{\infty} \left( \int_{\mathbb{R}^4} 2^{(\sigma+2)k} F^{-1}_2 \varphi_k \frac{P(\xi,\tau)}{\tau + |\xi|^2} \right)^{1/p} d\xi d\tau
\]

Continuing, we have
\[
I_1 = \sum_{k=0}^{\infty} \left( \int_{\mathbb{R}^4} 2^{(\sigma+2)k} F^{-1}_2 \psi_l \frac{P(\xi,\tau)}{\tau + |\xi|^2} \varphi_k \right)^{1/p} d\xi d\tau
\]

where $x = (x,t)$.

We recall the formula
\[
(F^{-1}_2 \psi_l \frac{P(\xi,\tau)}{\tau + |\xi|^2} F^{-1}_2 \varphi_k)(\tilde{y}) = F^{-1}_2 \left( F^{-1}_2 \psi_l \frac{P(\xi,\tau)}{\tau + |\xi|^2} F^{-1}_2 \varphi_k \right)(\tilde{y})
\]

Moreover, we have the notation $f_i(2^k,\cdot,2^{2l},i = 1,2$, where $\cdot$ replaces the corresponding argument in convolution. Then we have (see Lemma 3.4)
\[
[F^{-1}_2(f_i(2^k,\cdot,2^{2l},i) * F^{-1}_2(f_2(2^k,\cdot,2^{2l},i))(\tilde{y}) = 2^{-5l}(F^{-1}_2 f_i * F^{-1}_2 f_2)(2^{-1}\tilde{y}), \tag{3.19}
\]

where $\tilde{y} = (y,2^{-l}y_0)$.
With the help of (3.18) and (3.19), assuming that \( f_1 = \psi_1 \frac{P(\xi)}{\tau + \xi^2} \), \( f_2 = \varphi_k \) the expression \( I_1 \) takes the form

\[
I_1 = \sum_{k=0}^{\infty} \left( \int_{\mathbb{R}^4} \left[ \sum_{l=0}^{\infty} 2^{(\sigma+2)k} 2^{-5l} \int_{\mathbb{R}^4} d\bar{y} \left( F_{2}^{-1} \psi_l \frac{P(\xi)}{\tau + \xi^2} \right) \ast \left( 2^{-l} \bar{y} \right) (F_{2}^{-1} \varphi_l F_2 f)(\bar{x} - 2^{-l} \bar{y}) \right]^{p} \right)^{1/p}.
\]

In view of (3.19), we obtain

\[
I_1 = \sum_{k=0}^{\infty} \left( \int_{\mathbb{R}^4} \left[ \sum_{l=0}^{\infty} 2^{(\sigma+2)k} \int_{\mathbb{R}^4} d\bar{y} \left( F_{2}^{-1} \psi_l \frac{P(\xi)}{\tau + \xi^2} \right) (2^l, 2^{2l}) \right) \ast \left( F_{2}^{-1} \varphi_k \right) (2^l, 2^{2l}) \right] (\bar{y}) \left( \int_{\mathbb{R}^4} d\bar{x} |F_{2}^{-1} \varphi_l F_2 f((\bar{x} - \bar{y})) \right)^{p} \right)^{1/p} \equiv I_2.
\]

Applying the Minkowski inequality with respect to \( \bar{x} \) yields

\[
I_1 \leq \sum_{k=0}^{\infty} \left( \sum_{l=0}^{\infty} 2^{(\sigma+2)k} \int_{\mathbb{R}^4} d\bar{y} \left[ F_{2}^{-1} \psi_l \frac{P(\xi)}{\tau + \xi^2} \right] (2^l, 2^{2l}) \right) \ast \left( F_{2}^{-1} \varphi_k \right) (2^l, 2^{2l}) \right] (\bar{y}) \left( \int_{\mathbb{R}^4} d\bar{x} |F_{2}^{-1} \varphi_l F_2 f((\bar{x} - \bar{y})) \right)^{p} \right)^{1/p} \equiv I_2.
\]

Changing variables \( \bar{z} = \bar{x} - \bar{y} \) in the integral above with respect to \( \bar{x} \) implies

\[
I_2 = \sum_{k=0}^{\infty} \left( \sum_{l=0}^{\infty} 2^{(\sigma+2)k} \int_{\mathbb{R}^4} d\bar{y} \left[ F_{2}^{-1} \psi_l \frac{P(\xi)}{\tau + \xi^2} \right] (2^l, 2^{2l}) \right) \ast \left( F_{2}^{-1} \varphi_k \right) (2^l, 2^{2l}) \right] (\bar{y}) \left( \int_{\mathbb{R}^4} d\bar{z} |F_{2}^{-1} \varphi_l F_2 f((\bar{z})) \right)^{p} \right)^{1/p} \equiv I_3.
\]

Using the Hölder inequality in the integral with respect to \( \bar{y} \) and replacing \( \bar{z} \) by \( \bar{x} \), we obtain

\[
I_2 \leq \sum_{k=0}^{\infty} \left( \sum_{l=0}^{\infty} 2^{(\sigma+2)k} \int_{\mathbb{R}^4} d\bar{y} \left[ F_{2}^{-1} \psi_l \frac{P(\xi)}{\tau + \xi^2} \right] (2^l, 2^{2l}) \ast \left( F_{2}^{-1} \varphi_k \right) (2^l, 2^{2l}) \right] (\bar{y}) \left( \int_{\mathbb{R}^4} d\bar{x} |F_{2}^{-1} \varphi_l F_2 f(\bar{x}) \right)^{p} \right)^{1/p} \equiv I_3.
\]
For the power \( d > 4 \) we get
\[
I_3 \leq c \sum_{k=0}^{\infty} \left( \sum_{l=0}^{\infty} 2^{(\sigma+2)k} \left( \int_{\mathbb{R}^4} d\bar{y} \right) \left| \left( F_{\frac{2}{\tau}}^{-1} \psi_l \frac{P(\xi)}{\tau + \xi^2} \right)(2^l, 2^{2l}) \right| \right) \cdot \left( F_{\frac{2}{\tau}}^{-1} \varphi_k (2^l, 2^{2l}) \right) (2^l, 2^{2l})
\]
\[
\cdot \left( \int_{\mathbb{R}^4} d\bar{x} \left| \left( F_{\frac{2}{\tau}}^{-1} \varphi_l F_2 f \right)(\bar{x}) \right|^p \right)^{1/p} = I_4.
\]

By the Parseval identity, we have
\[
I_4 = c \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} 2^{(\sigma+2)(k-l)+2(\sigma+2)} \left( \int_{\mathbb{R}^4} d\bar{y} \right) \left| \left( \psi_l \frac{P(\xi)}{\tau + \xi^2} \right)(2^l, 2^{2l}) \right| \cdot \varphi_k (2^l, 2^{2l}) \left| W_{2^{d/2}, 2^{d/2}} \right| \left( \int_{\mathbb{R}^4} d\bar{x} \left| \left( F_{\frac{2}{\tau}}^{-1} \varphi_l F_2 f \right)(\bar{x}) \right|^p \right)^{1/p}.
\]

In view of Lemma 3.2 (see below), we have
\[
I_4 \leq c \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} 2^{(\sigma+2+2d+5-L)} \left( \sum_{k=0}^{L} 2^{2(\sigma+1)} \left| P_{\frac{2}{\tau}}^{-1} \varphi_l F_2 f \right|_{L_p(\mathbb{R}^4)} \right)^{1/p} \equiv I_5.
\] (3.20)

For \( L \) sufficiently large, we obtain
\[
I_5 \leq c \sum_{l=0}^{\infty} 2^{2l} \left| P_{\frac{2}{\tau}}^{-1} \varphi_l F_2 f \right|_{L_p(\mathbb{R}^4)}.
\]

Hence (3.17) is proved. Similarly we show (3.17)\_2. This concludes the proof. \( \square \)

To prove (3.20) we need

**Lemma 3.2.** Let \( d \) be an even number such that \( d > 4 \). Then
\[
I \equiv \left\| \psi_l \left( \frac{P(\xi)}{\tau + \xi^2} \right)^{1/2} \varphi_k (2^l, 2^{2l}) \right\|_{W_{2^{d/2}, 2^{d/2}}(\mathbb{R}^4)} \leq c 2^{-2d(2d+5-L)(l-k)},
\] (3.21)

where \( P(\xi) = \left\{ \delta_{jk} - \xi_j \xi_k / |\xi|^2 \right\} \). for \( l = 0 \) the constant \( c \) depends on \( \gamma \) and \( L > 0 \) may be assumed sufficiently large.

**Proof.** To estimate \( I \), we express it explicitly as
\[
I = \sum_{s_1+2s_2+l \leq d} \left( \int_{\mathbb{R}^4} d\bar{y} \right) \left| \left( \partial_{\xi^l} \partial_{\xi^0}^s \psi_l \right)(2^l, 2^{2l}) \left( \partial_{\xi^0}^s \partial_{\xi^0}^2 \frac{P(\xi)}{\tau + \xi^2} \right)(2^l, 2^{2l}) \right| \left( \partial_{\xi^l} \partial_{\xi^0}^s \varphi_k (2^l, 2^{2l}) \right) \left|_{L_2(\mathbb{R}^4)} \right|.
\]

Since \( \psi_l(2^l, 2^{2l}) \subset \{ \xi : |\xi|_a \leq 4 \} \) for \( l = 0 \), \( \psi_l(2^l, 2^{2l}) \subset \{ \xi : 1/4 \leq |\xi|_a \leq 4 \} \) for \( l \neq 0 \), and
\[
|\partial_{\xi^l} \partial_{\xi^0}^s \psi_l (2^l, 2^{2l})| \leq c \quad \text{for} \quad \xi \in A,
\] (3.22)

where
\[
A = \{ \xi : |\xi|_a \leq 4 \} \quad \text{for} \quad l = 0,
A = \{ \xi : 1/4 \leq |\xi|_a \leq 4 \} \quad \text{for} \quad l \neq 0.
\]
Then we have

\[ I \leq c \left\| \sum_{l=2}^{d} \left( \frac{\partial^l \varphi}{\partial \xi^l} \frac{P(\xi)}{\tau + \xi^2} \right)(2^l, 2^{2l}) \right\|_{L_2(A)}. \]

Next, we obtain

\[ \left\| \left( \frac{\partial^l \varphi}{\partial \xi^l} \frac{P(\xi)}{\tau + \xi^2} \right)(2^l, 2^{2l}) \right\|_{L_4(A)} \leq c(\gamma)2^{-2l} \]

and finally, we estimate

\[ \left\| \frac{\partial^l \varphi}{\partial \xi^l} \varphi_k(2^l, 2^{2l}) \right\|_{L_4(A)} = \left\| \frac{\partial^l \varphi}{\partial \xi^l} \right\|_{L_4(A)} \]

\[ \leq c2^{(d+2)[l-k]} \left\| \frac{\partial^l \varphi}{\partial \xi^l} \varphi_k(2^k, 2^{2k}) \right\|_{L_4(B)} \]

\[ \leq c2^{(d+4-L)[l-k]}, \]

where

\[ B = \{ y : 2^{l-k} \leq |y|_a \leq 2^{l-k+2} \} \quad \text{for} \quad l \neq 0, \]

\[ B = \{ y : |y|_a \leq 2^{1-k} \} \quad \text{for} \quad l = 0, \quad L > 0, \]

\( L \) is a chosen number, and we have been used the fact that \( \{ \varphi_k \} \in A_{aL}(\mathbb{R}^4) \) (see Definition 2.16) and we change variables \( y_0 = 2^{(l-k)}\xi_0, y = 2^{-l-k}\xi \). Hence (3.20) holds. This concludes the proof. \( \square \)

**Theorem 3.3.** Assume that \( f \in W^p,\sigma/2(\mathbb{R}^3 \times \mathbb{R}^+), g \in W^{p,1+\sigma/2+1/2}(\mathbb{R}^3 \times \mathbb{R}^+), v_0 \in W^{p+2-2/p}(\mathbb{R}^3), p \in (1, \infty), \sigma \notin \mathbb{N}. \)

Then there exists a solution to problem (3.1) such that \( v \in W^{p,1+\sigma/2}(\mathbb{R}^3 \times \mathbb{R}^+), \nabla p \in W^{p,\sigma/2}(\mathbb{R}^3 \times \mathbb{R}^+) \) and the estimate holds

\[ \|v\|_{W^{p,1+\sigma/2}(\mathbb{R}^3 \times \mathbb{R}^+)} + \|\nabla p\|_{W^{p,\sigma/2}(\mathbb{R}^3 \times \mathbb{R}^+)} \leq c(\|f\|_{W^{p,\sigma/2}(\mathbb{R}^3 \times \mathbb{R}^+)} + \|g\|_{W^{p+1,\sigma/2+1/2}(\mathbb{R}^3 \times \mathbb{R}^+)} \]

\[ + \|v_0\|_{W^{p+2-2/p}(\mathbb{R}^3)}, \quad (3.23) \]

where \( c \) does not depend on \( v \) and \( p \).

**Proof.** By Lemma 3.1 and (3.10) we have

\[ \|v''\|_{W^{p,1+\sigma/2}(\mathbb{R}^4)} \leq c\|f''\|_{W^{p,\sigma/2}(\mathbb{R}^4)}. \]

From (3.10), it follows

\[ \|v' - \nabla \varphi\|_{W^{p,1+\sigma/2}(\mathbb{R}^4)} \leq c\|f' - \nabla \varphi + \nu \Delta \varphi\|_{W^{p,\sigma/2}(\mathbb{R}^4)}. \]

Simplifying the above inequality yields

\[ \|v'\|_{W^{p,1+\sigma/2}(\mathbb{R}^4)} \leq c\|f'\|_{W^{p,\sigma/2}(\mathbb{R}^4)} + \|\nabla \varphi\|_{W^{p,1+\sigma/2}(\mathbb{R}^4)} \]

\[ + \|\nabla \varphi\|_{W^{p,\sigma/2}(\mathbb{R}^4)} + \|\Delta \varphi\|_{W^{p,\sigma/2}(\mathbb{R}^4)}. \quad (3.24) \]

Next (3.9) implies that

\[ \varphi = \Delta^{-1}g'. \]

Hence

\[ \|\nabla \Delta^{-1}g'\|_{W^{p,1+\sigma/2}(\mathbb{R}^4)} \leq c\|g'\|_{W^{p,1+\sigma/2}(\mathbb{R}^4)}; \]

\[ \|\nabla \Delta^{-1}g'\|_{W^{p,\sigma/2}(\mathbb{R}^4)} \leq c\|g'\|_{W^{p,1+\sigma/2}(\mathbb{R}^4)}; \]

\[ \|\Delta \Delta^{-1}g'\|_{W^{p,\sigma/2}(\mathbb{R}^4)} \leq c\|g'\|_{W^{p,\sigma/2}(\mathbb{R}^4)}. \quad (3.25) \]
Using (3.25) in (3.24) gives
\[ \|v'\|_{W^{2+\sigma,1+\sigma/2}_p(\mathbb{R}^4)} \leq c\|f'\|_{W^{\sigma,1/2}_p(\mathbb{R}^4)} + c\|g'\|_{W^{4+\sigma,1/2+\sigma/2}_p(\mathbb{R}^4)}. \] (3.26)

In view of relation between \(v', f', g\) and \(\bar{v}, \bar{f}, \bar{g}\), (3.27) implies
\[ \|\bar{v}\|_{W^{2+\sigma,1+\sigma/2}_p(\mathbb{R}^4)} \leq c(\|\bar{f}\|_{W^{\sigma,1/2}_p(\mathbb{R}^4)} + \|\bar{g}\|_{W^{4+\sigma,1/2+\sigma/2}_p(\mathbb{R}^4)}). \] (3.27)

Finally, extension (3.3) gives
\[
\|v\|_{W^{2+\sigma,1+\sigma/2}_p(\mathbb{R}^3 \times \mathbb{R}^+)} \leq c\|f\|_{W^{\sigma,1/2}_p(\mathbb{R}^3 \times \mathbb{R}^+)} + c\|g\|_{W^{4+\sigma,1/2+\sigma/2}_p(\mathbb{R}^3 \times \mathbb{R}^+)} + c\|v_0\|_{W^{2+\sigma,2\sigma}_p(\mathbb{R}^3)}.
\] (3.28)

\[ \square \]

**Lemma 3.4.** Let \(l \in \mathbb{N}\), \(\tilde{y} = (y, y_0)\), \(y \in \mathbb{R}^n\) and \(\tilde{y} = (y, 2^{-l}y_0)\). Then the following equality holds
\[
[F^{-1}_2(f_1(2^l \cdot, 2^l \cdot)) * F^{-1}_2(f_2(2^l \cdot, 2^l \cdot))](\tilde{y}) = 2^{-(n+2)l}F^{-1}_2(F^{-1}_2f_1 * F^{-1}_2f_2)(2^{-l}\tilde{y}).
\] (3.29)

**Proof.** Using the definition of convolution we have
\[
I_1 = \int_{\mathbb{R}^{n+1}} d\tilde{z} \int_{\mathbb{R}^{n+1}} e^{i(y - z)\bar{\eta}} f_1(2^l \eta, 2^l \eta_0) d\bar{\eta}.
\]
\[
\cdot \int_{\mathbb{R}^{n+1}} e^{i\tilde{z}\bar{\vartheta}} f_2(2^l \vartheta, 2^l \vartheta_0) d\bar{\vartheta},
\]
where \(\tilde{z} = (z, z_0)\), \(z \in \mathbb{R}^n\), \(z_0 \in \mathbb{R}\).

From the definition of the Fourier transform it follows
\[
I_1 = \int_{\mathbb{R}^{n+1}} d\tilde{z} \int_{\mathbb{R}^{n+1}} e^{i(y_0 - z_0)2^{-2l}\eta_0' + i(y - z)2^{-l}\eta'} f_1(\eta', \eta_0') d\eta' 2^{-(n+2)l}.
\]
\[
\cdot \int_{\mathbb{R}^{n+1}} e^{i\tilde{z}\vartheta_0' + i\vartheta_02^{-l}\eta_0'} f_2(\vartheta', \vartheta_0') d\vartheta' e^{-(n+2)l}
\]
\[
= 2^{-(n+2)l} \int_{\mathbb{R}^{n+1}} d\tilde{z} \int_{\mathbb{R}^{n+1}} e^{i(2^{-2l}y_0 - 2^{-2l}z_0)\eta_0' + i(2^{-l}y - 2^{-l}z)\eta'} f_1(\eta', \eta_0') d\eta' 2^{-(n+2)l}.
\]
\[
\cdot \int_{\mathbb{R}^{n+1}} e^{i\tilde{z}\vartheta_0' + i\vartheta_02^{-l}z_02^{-l}\eta_0'} f_2(\vartheta', \vartheta_0') d\vartheta'.
\]

Setting up more variables
\[ z_0' = 2^{-2l}z_0, \quad z = 2^{-l}z, \quad \tilde{z}' = (z', z_0') \]
we have $d\tilde{z} = 2^{(2+n)}dz$. Then

$$I_1 = 2^{-(n+2)} \int_{\mathbb{R}^{n+1}} dz' \int_{\mathbb{R}^{n+1}} e^{i(2^{-2}y_0 - z_0')\eta_0 + i(2^{-l}y - z')\eta'} \cdot f_1(\eta', \eta_0) d\eta' \int_{\mathbb{R}^{n+1}} e^{i\epsilon \delta_0 + i\epsilon \eta'} f_2(\delta', \eta_0') d\delta' .$$

Finally, we denote

$$\tilde{y} = (y, 2^{-l}y_0) .$$

Then $I_1$ takes the form

$$I_1 = 2^{-(n+2)}(F_2^{-1}f_1 * F_2^{-1}f_2)(2^{-l}\tilde{y}) .$$

This ends the proof.

\[ \square \]

4. The Stokes System in the Half-Space

In this Section we prove the existence of solutions to the Stokes system with slip boundary conditions in the half space. Let $\mathbb{R}^3_+ = \{ x \in \mathbb{R}^3 : x_3 > 0 \}$. Then the Stokes system has the form

$$\begin{align*}
v_t - \nu \Delta v + \nabla p &= f \quad \text{in} \quad \mathbb{R}^3_+ \times \mathbb{R}_+, \\
\text{div} v &= g \quad \text{in} \quad \mathbb{R}^3_+ \times \mathbb{R}_+, \\
v_{3,x_1} + v_{1,x_3} &= b_1 \quad \text{for} \quad x_3 = 0 \quad \text{in} \quad \mathbb{R}^2 \times \mathbb{R}_+, \\
v_{3,x_2} + v_{2,x_3} &= b_2 \quad \text{for} \quad x_3 = 0 \quad \text{in} \quad \mathbb{R}^2 \times \mathbb{R}_+, \\
v_3 &= b_3 \quad \text{for} \quad x_3 = 0 \quad \text{in} \quad \mathbb{R}^2 \times \mathbb{R}_+, \\
v|_{t=0} &= v_0 \quad \text{in} \quad \mathbb{R}^3_+.
\end{align*} \tag{4.1}$$

We are looking for solutions to (4.1) under the following assumptions

$$\begin{align*}
f &\in W_p^{\sigma, \sigma/2}(\mathbb{R}^3_+ \times \mathbb{R}_+), \\
g &\in W_p^{\sigma + 1, \sigma/2+1/2}(\mathbb{R}^3_+ \times \mathbb{R}_+), \\
b_\alpha &\in W_p^{\sigma+1-1/p, \sigma/2+1/2-1/2p}(\mathbb{R}^2 \times \mathbb{R}_+), \quad \alpha = 1, 2, \\
b_3 &\in W_p^{\sigma+2-1/p, \sigma/2+1-1/2p}(\mathbb{R}^2 \times \mathbb{R}_+), \\
v_0 &\in W_p^{\sigma+2-2/p}(\mathbb{R}^3_+),
\end{align*} \tag{4.2}$$

where $p \in (1, \infty), \sigma \notin \mathbb{N}$.

Since $v_0 \in W_p^{\sigma+2-2/p}(\mathbb{R}^3_+)$ there exists the time extension $\tilde{v}_0 \in W_p^{\sigma+2, \sigma/2+1}(\mathbb{R}^3_+ \times \mathbb{R}_+)$ such that

$$\tilde{v}_0|_{t=0} = v_0 \quad \text{in} \quad \mathbb{R}^3_+ \times \mathbb{R}_+ \tag{4.3}$$

and

$$||\tilde{v}_0||_{W_p^{\sigma+2, \sigma/2+1}(\mathbb{R}^3_+ \times \mathbb{R}_+)} \leq c||v_0||_{W_p^{\sigma+2-2/p}(\mathbb{R}^3_+)} \tag{4.4}$$

where $c$ does not depend on $v_0$.

Having extension $\tilde{v}_0$ we can introduce the function

$$\tilde{v} = v - \tilde{v}_0 \quad \text{in} \quad \mathbb{R}^3_+ \times \mathbb{R}_+ \tag{4.5}$$
such that \((\tilde{v}, p)\) is a solution to the Stokes system with vanishing initial data

\[
\begin{align*}
\tilde{v}_t - \nu \Delta \tilde{v} + \nabla p &= f - \tilde{v}_{0,t} + \nu \Delta \tilde{v}_0 \equiv \tilde{f}, \\
\text{div } \tilde{v} &= g - \text{div } \tilde{v}_0 \equiv \tilde{g}, \\
\tilde{v}_0 &= b_0 = \tilde{v}_{30} \\
\tilde{v}_3 &= b_3 = \tilde{v}_{30} \equiv \tilde{b}_3,
\end{align*}
\]

\text{(4.6)}

In view of Lemmas 2.14 and 2.15 we can extend functions \(\tilde{f}, \tilde{g}, \tilde{b} = (\tilde{b}_1, \tilde{b}_2, \tilde{b}_3)\) by zero for \(t < 0\) in the same classes assuming the compatibility conditions

\[
\begin{align*}
\partial_i^t \tilde{f}|_{t=0} &= 0 \quad \text{for } i \leq [\sigma/2], \\
\partial_i^t \tilde{g}|_{t=0} &= 0 \quad \text{for } i \leq [\sigma/2 + 1/2], \\
\partial_i \tilde{b}_\alpha|_{t=0} &= 0 \quad \text{for } i \leq \left[\frac{\sigma}{2} + \frac{1}{2} \right], \quad \alpha = 1, 2, \\
\partial_i^t \tilde{b}_3|_{t=0} &= 0 \quad \text{for } i \leq \left[\frac{\sigma}{2} + 1 - \frac{1}{p}\right].
\end{align*}
\]

\text{(4.7)}

Denote extended functions by \(f', g', b',\) respectively. Then problem \((4.6)\) for extended functions takes the form

\[
\begin{align*}
v'_t - \nu \Delta v' + \nabla p' &= f', \quad \text{in } \mathbb{R}_+^3 \times \mathbb{R}, \\
\text{div } v' &= g', \quad \text{in } \mathbb{R}_+^3 \times \mathbb{R}, \\
v_{3,x_\alpha} + v'_{\alpha,x_3} &= b'_\alpha, \quad \alpha = 1, 2, \quad \text{for } x_3 = 0, \quad \text{in } \mathbb{R}_+^2 \times \mathbb{R}, \\
v'_3 &= b_3' \quad \text{for } x_3 = 0, \quad \text{in } \mathbb{R}_+^2 \times \mathbb{R}.
\end{align*}
\]

\text{(4.8)}

Consider the Neumann problem

\[
\begin{align*}
\Delta \varphi' &= g' \quad \text{in } \mathbb{R}_+^3, \\
\frac{\partial}{\partial x_3} \varphi'|_{x_3=0} &= 0 \quad \text{in } \mathbb{R}^2
\end{align*}
\]

\text{(4.9)}

Introducing the function

\[
v'' = v' - \nabla \varphi'
\]

\text{(4.10)}

we see that \((v'', p')\) is a solution to the problem

\[
\begin{align*}
v''_t - \nu \Delta v'' + \nabla p' &= f' - \nabla \varphi'_t + \nu \Delta \varphi' \equiv f'', \\
\text{div } v'' &= 0, \\
v''_{3,x_\alpha} + v''_{\alpha,x_3} &= b'_\alpha - 2 \varphi'_{x_\alpha x_3} \equiv b''_\alpha, \quad \alpha = 1, 2, \quad x_3 = 0, \\
v''_3 &= b_3'' \equiv b_3', \quad x_3 = 0.
\end{align*}
\]

\text{(4.11)}

Let \(g = \text{div } \chi.\) Then \(\tilde{g} = \text{div } (\chi - \tilde{v}_0), (\chi - \tilde{v}_0)\tilde{n}|_{x_3=0} = 0\) and \(g' = \text{div } (\chi' - \tilde{v}'_0),\) where \(\chi', \tilde{v}'_0\) are extensions on \(t < 0.\) Then \((4.9)\) yields \(\varphi' = G \ast \text{div } (\chi' - \tilde{v}'_0)\) and \(G\) is the Green function to the Neumann problem \((4.9).\) Hence

\[
\begin{align*}
\|\nabla \varphi'\|_{W^{\sigma/2,\sigma/2}(\mathbb{R}_+^3 \times \mathbb{R})} + \|\Delta \varphi'\|_{W^{\sigma/2,\sigma/2}(\mathbb{R}_+^3 \times \mathbb{R})} \\
\leq c(\|\chi\|_{W^{\sigma,\sigma/2}(\mathbb{R}_+^3 \times \mathbb{R}^+)} + \|\Delta \chi\|_{W^{\sigma,\sigma/2}(\mathbb{R}_+^3 \times \mathbb{R}^+)} + \|v_0\|_{W^{2\sigma/2-2,\sigma/2}(\mathbb{R}_+^3)}).
\end{align*}
\]
Then
\[
\|f''\|_{W^{\sigma/2}_p(\mathbb{R}^3_+ \times \mathbb{R}^+)} \leq c(\|f\|_{W^{\sigma/2}_p(\mathbb{R}^3_+ \times \mathbb{R}^+}) + \|\chi\|_{W^{\sigma/2}_p(\mathbb{R}^3_+ \times \mathbb{R}^+)} + \|\Delta \chi\|_{W^{\sigma/2}_p(\mathbb{R}^3_+ \times \mathbb{R}^+)} + \|v_0\|_{W^{2+\sigma-2/\tau}_p(\mathbb{R}^3_+)}).
\]

Consider the Stokes system
\[
v''_t - \nu \Delta v'' + \nabla p' = f'' \quad \text{in} \quad \mathbb{R}^3_+ \times \mathbb{R},
\]
\[
\text{div } v'' = 0 \quad \text{in} \quad \mathbb{R}^3_+ \times \mathbb{R}.
\]

(4.12)

To apply the results from Sect. 3 we have to extend problem (4.12) to the problem in the whole space.

Since \( f'' \in W^\sigma_{p,\sigma/2}(\mathbb{R}^3_+ \times \mathbb{R}), \sigma \notin \mathbb{N} \), we have to extend \( f'' \) by zero for \( x_3 < 0 \). For this purpose we examine the norm
\[
\|f''\|_{W^\sigma_{p,\sigma/2}(\mathbb{R}^3_+ \times \mathbb{R})} = \|f''\|_{L^p(\mathbb{R}^3_+ \times \mathbb{R})}
\]
\[
+ \left( \int_0^\infty \int_{\mathbb{R}^3_+} \int_0^\infty \frac{|D^{[\sigma]} f''(x'', x_3, t) - D^{[\sigma]} f(x'', x_3, t)|^p}{|x''|^{3+|\sigma|}} dx_3 \right)^{1/p}
\]
\[
+ \left( \int_0^\infty \int_{\mathbb{R}^3_+} \int_0^\infty \frac{|\partial^{[\sigma]}_{x_3} f''(x'', x_3, t) - \partial^{[\sigma]}_{x_3} f(x'', x_3, t)|^p}{|x''|^{3+|\sigma|}} dx_3 \right)^{1/p}
\]
\[
+ \left( \int_{\mathbb{R}^3_+} \int_{\mathbb{R}^3_+} \int_{\mathbb{R}^3_+} \frac{|\partial^{[\sigma/2]}_{x''} f''(x, t') - \partial^{[\sigma/2]}_{x''} f''(x, t'')|^p}{|t' - t''|^{3+|\sigma/2|}} dt dt' dx'' \right)^{1/p}
\]
\[
= I_1 + I_2 + I_3 + I_4.
\]

Let \( \tilde{f}'' = \begin{cases} f'' & \text{for } x_3 > 0, \\ 0 & \text{for } x_3 < 0. \end{cases} \)

It is clear that terms \( I_1, I_2, I_4 \) holds also for \( \tilde{f}'' \). To express \( I_3 \) for \( \tilde{f}'' \) we need that
\[
\left( \int_0^\infty \int_{\mathbb{R}^3_+} \int_0^\infty \frac{|\partial^{[\sigma]}_{x_3} f''|^p}{x_3^{3+|\sigma|}} dx_3 dt \right)^{1/p}
\]
(4.13)
is finite. Similarly as in Lemma 2.15 the term (4.13) is bounded by
\[
\|f''\|_{W^\sigma_{p,\sigma/2}(\mathbb{R}^3 \times \mathbb{R})}.
\]

Then we can transform (4.12) to the form
\[
v''_t - \nu \Delta v'' + \nabla \tilde{p}'' = \tilde{f}'' \quad \text{in} \quad \mathbb{R}^4
\]
\[
\text{div } v'' = 0
\]
(4.14)

From Theorem 3.3 we have the existence of solutions to problem (4.14) and the estimate
\[
\|\tilde{v}''\|_{W^{2+\sigma/2+1}_{p,\sigma/2+1}(\mathbb{R}^4)} + \|\nabla \tilde{p}''\|_{W^{\sigma/2}_{p,\sigma/2}(\mathbb{R}^4)}
\]
\[
\leq c\|f''\|_{W^\sigma_{p,\sigma/2}(\mathbb{R}^3 \times \mathbb{R})}.
\]

(4.15)

Introducing new functions
\[
u = v'' - \tilde{v}'', \quad q = p' - \tilde{p}'
\]
(4.16)
we see that \((u, q)\) is a solution to the problem
\[
\begin{align*}
\frac{\partial u}{\partial t} - \nu \Delta u + \nabla q & = 0 \\
\text{div } u & = 0 \\
u u_{3,x_\alpha} + u_{\alpha,x_3} = b_\alpha'' - \bar{v}_{3,x_\alpha}'' - \bar{v}_{\alpha,x_3}'' \equiv \tau_d, \quad \alpha = 1, 2, \\
u u_3 = b_3'' - \bar{v}_3'' \equiv \tau_d.
\end{align*}
\tag{4.17}
\]
To solve problem (4.17) we use the Fourier-Laplace transform
\[
\tilde{u}(\xi, x_3, s) = (F_2 u)(\xi, x_3, s) = \int_{\mathbb{R}^2} e^{-st} e^{-ix' \cdot \xi} u(x, t) d x' dt,
\tag{4.18}
\]
where \(s = \gamma + i \xi_0, \gamma \in \mathbb{R}^+, \xi_0 \in \mathbb{R}, \xi = (\xi_1, \xi_2), x' = (x_1, x_2), x = (x_1, x_2, x_3), x' \cdot \xi = x_1 \xi_1 + x_2 \xi_2.
\]
Applying the Fourier-Laplace transform (4.18) to system (4.17) yields
\[
\begin{align*}
\nu\left(\frac{d^2}{dx_3^2} + \tau^2\right) \tilde{u}_\alpha + i \xi_\alpha \tilde{q} & = 0, \quad \alpha = 1, 2, \\
\nu\left(\frac{d^2}{dx_3^2} + \tau^2\right) \tilde{u}_3 + \frac{d \tilde{q}}{dx_3} & = 0, \\
i \xi_1 \tilde{u}_3 + i \xi_2 \tilde{u}_2 + \frac{d \tilde{u}_3}{dx_3} & = 0, \\
\tilde{u} & \rightarrow 0, \quad \tilde{q} \rightarrow 0 \quad \text{as} \quad x_3 \rightarrow +\infty,
\end{align*}
\tag{4.19}
\]
where \(\tau^2 = \frac{\xi_1^2}{\nu} + \xi_2^2, \xi^2 = \xi_1^2 + \xi_2^2, \text{arg} \tau \in (-\pi/4, \pi/4)\) and conditions (4.19)\(\text{d}\) are called the Shapiro-Lopatinskii conditions.

Solutions of (4.19) have the form (see [34])
\[
\begin{align*}
\tilde{u} & = \Phi(\xi, s)e^{-\tau x_3 + \varphi(\xi, s)(i \xi_1, i \xi_2, -|\xi|)e^{-|\xi| x_3}}, \\
\tilde{q} & = -s \varphi(\xi, s)e^{-|\xi| x_3},
\end{align*}
\tag{4.20}
\]
where \(\Phi(\xi, s) = (\Phi_1, \Phi_2, (i \xi_1 \Phi_1 + i \xi_2 \Phi_2)/\tau)\) and \(\Phi_1, \Phi_2, \varphi\) are arbitrary parameters which must be calculated from boundary conditions (4.17)\(\text{d}\).

Applying the Fourier-Laplace transform to (4.17)\(\text{d}\) yields
\[
\begin{align*}
i \xi_1 \tilde{u}_3 + \tilde{u}_{1,x_3} & = \tilde{d}_1, \\
i \xi_2 \tilde{u}_3 + \tilde{u}_{2,x_3} & = \tilde{d}_2, \\
\tilde{u}_3 & = \tilde{d}_3.
\end{align*}
\tag{4.21}
\]
Using (4.20) in (4.21) yields
\[
\begin{align*}
(\tau^2 + \xi_1^2)\Phi_1 + \xi_1 \xi_2 \Phi_2 + 2i \xi_1 |\xi| \tau \varphi & = -\tau \tilde{d}_1, \\
\xi_1 \xi_2 \Phi_1 + (\tau^2 + \xi_2^2)\Phi + 2i \xi_2 |\xi| \tau \varphi & = -\tau \tilde{d}_2, \\
i \xi_1 \Phi_1 + i \xi_2 \Phi_2 - |\xi| \tau \varphi & = \tau \tilde{d}_3.
\end{align*}
\tag{4.22}
\]
Eliminating \(\varphi\) from (4.22) gives
\[
\begin{align*}
(\tau^2 + \xi_1^2)\Phi_1 - \xi_1 \xi_2 \Phi_2 & = 2i \xi_1 \tau \tilde{d}_4 - \tau \tilde{d}_1 \equiv \tilde{B}_1, \\
- \xi_1 \xi_2 \Phi_1 + (\tau^2 + \xi_2^2)\Phi_2 & = 2i \xi_2 \tau \tilde{d}_3 - \tau \tilde{d}_2 \equiv \tilde{B}_2, \\
|\xi| \tau \varphi & = i(\xi_1 \Phi_1 + \xi_2 \Phi_2) - \tau \tilde{d}_3.
\end{align*}
\tag{4.23}
\]
From (4.23)$_{1,2}$ we calculate
\[
\Phi_1 = \frac{(\tau^2 - \xi_2^2)\tilde{B}_1 + \xi_1 \xi_2 \tilde{B}_2}{D},
\]
\[
\Phi_2 = \frac{\xi_1 \xi_2 \tilde{B}_1 + (\tau^2 - \xi_1^2)\tilde{B}_2}{D},
\]
where \(D = \tau^2(\tau^2 - \xi_2^2) = \xi^2\tau^2\) because \(\tau^2 = \frac{\xi}{\nu} + \xi^2\).

Using (4.24) in (4.23)$_3$ yields
\[
\varphi = \frac{i(\xi_1 \tilde{B}_1 + \xi_2 \tilde{B}_2)}{|\xi|\tau s/\nu} - \frac{1}{|\xi|} \tilde{d}_3 e^{-|\xi|}.
\]

Using the expressions of \(\Phi_1, \Phi_2\) and \(\varphi\) in (4.20) we obtain
\[
\tilde{u}_1 = \frac{(\tau^2 - \xi_2^2)}{\tau^2 s/\nu} \tilde{B}_1 + \xi_1 \xi_2 \tilde{B}_2 e^{-\tau x_3} - \frac{\xi_1^2}{|\xi|\tau s/\nu} e^{-|\xi| x_3} - \frac{i \xi_1}{|\xi|} \tilde{d}_3 e^{-|\xi| x_3}
\]
\[
+ \left[ \frac{\tau^2 - \xi_2^2}{\tau^2 s/\nu} e^{-\tau x_3} - \frac{\xi_1^2}{|\xi|\tau s/\nu} e^{-|\xi| x_3} \right] \tilde{B}_1
\]
\[
+ \left[ \frac{\xi_1 \xi_2}{\tau^2 s/\nu} e^{-\tau x_3} - \frac{\xi_1 \xi_2}{|\xi|\tau s/\nu} e^{-|\xi| x_3} \right] \tilde{B}_2 - \frac{i \xi_1}{|\xi|} \tilde{d}_3 e^{-|\xi| x_3}.
\]

Consider the coefficient next to \(\tilde{B}_1\). We write it in the form
\[
\frac{\tau^2 - \xi_2^2}{\tau^2 s/\nu} e^{-\tau x_3} + \xi_1 \left( \frac{1}{\tau^2 s/\nu} e^{-\tau x_3} - \frac{1}{|\xi|\tau s/\nu} e^{-|\xi| x_3} \right)
\]
\[
= \frac{1}{\tau^2} e^{-\tau x_3} + \xi_1 \left( \frac{1}{\tau^2 s/\nu} - \frac{1}{|\xi|\tau s/\nu} \right) e^{-\tau x_3} + \frac{\xi_1^2}{|\xi|\tau s/\nu} (e^{-\tau x_3} - e^{-|\xi| x_3})
\]
\[
= \frac{1}{\tau^2} e^{-\tau x_3} + \frac{\xi_1^2}{\tau} \frac{|\xi| - \tau}{|\xi|\tau s/\nu} e^{-\tau x_3} + \frac{\xi_1^2 (\tau - |\xi|)}{|\xi|\tau s/\nu} e^{-\tau x_3} - e^{-|\xi| x_3}
\]
\[
= \frac{1}{\tau^2} e^{-\tau x_3} - \frac{\xi_1^2}{\tau^2 |\xi| (\tau + |\xi|)} e^{-\tau x_3} + \frac{\xi_1^2}{\tau |\xi| (\tau + |\xi|)} e^{-\tau x_3} - e^{-|\xi| x_3}
\]

Consider the coefficient next to \(\tilde{B}_2\). We express it in the form
\[
\xi_1 \xi_2 \left[ \left( \frac{1}{\tau^2 s/\nu} - \frac{1}{|\xi|\tau s/\nu} \right) e^{-\tau x_3} + \frac{\tau - |\xi|}{|\xi|\tau s/\nu} e^{-\tau x_3} - e^{-|\xi| x_3} \right]
\]
\[
= \xi_1 \xi_2 \left[ \frac{1}{\tau s/\nu} \frac{|\xi| - \tau}{|\xi|\tau s/\nu} e^{-\tau x_3} + \frac{1}{|\xi|\tau s/\nu} e^{-\tau x_3} - e^{-|\xi| x_3} \right]
\]
\[
= \xi_1 \xi_2 \left[ - \frac{1}{\tau^2 |\xi| (\tau + |\xi|)} e^{-\tau x_3} + \frac{1}{|\xi|\tau (\tau + |\xi|)} e^{-\tau x_3} - e^{-|\xi| x_3} \right].
\]

Introduce the notation
\[
e_0(x_3) = e^{-\tau x_3}, \quad e_1(x_3) = \frac{e^{-\tau x_3} - e^{-|\xi| x_3}}{\tau - |\xi|}, \quad e_2(x_3) = e^{-|\xi| x_3}.
\]

Then \(\tilde{u}_1\) takes the form
\[
\tilde{u}_1 = \left[ \frac{1}{\tau^2 \epsilon_0} - \frac{\xi_1^2}{\tau^2 |\xi| (\tau + |\xi|)} \frac{\epsilon_0}{\tau - |\xi|} + \frac{\xi_1^2}{|\xi| \tau (\tau + |\xi|)} \frac{\epsilon_1}{\tau - |\xi|} \right] \tilde{B}_1
\]
\[
+ \xi_1 \xi_2 \left[ - \frac{1}{\tau^2 |\xi| (\tau + |\xi|)} \epsilon_0 + \frac{1}{|\xi| \tau (\tau + |\xi|)} \epsilon_1 \right] \tilde{B}_2 - \frac{i \xi_1}{|\xi|} \tilde{d}_3 \epsilon_2.
\]
Similarly,
\[
\tilde{u}_2 = \left( -\frac{\xi_1 \xi_2}{\tau^2 |\xi| (\tau + |\xi|)} e_0 + \frac{\xi_1}{\tau |\xi| (\tau + |\xi|)} e_1 \right) \tilde{B}_1 \\
+ \left( \frac{1}{\tau^2} e_0 - \frac{\xi_2}{\tau^2 |\xi| (\tau + |\xi|)} e_0 + \frac{\xi_2}{\tau |\xi| (\tau + |\xi|)} e_1 \right) \tilde{B}_2 - \frac{i \xi_2}{|\xi|} \tilde{d}_3 e_2. 
\] (4.27)

Next, we consider
\[
\tilde{u}_3 = (i \xi_1 \Phi_1 + i \xi_2 \Phi_2) / \tau e^{-\tau x_3} - \varphi |\xi| e^{-|\xi| x_3} \\
= \left[ \frac{i \xi_1 (\tau^2 - \xi_2^2) \tilde{B}_1 + \xi_1 \xi_2 \tilde{B}_2}{\tau^2 s/\nu} + \frac{i \xi_2 \xi_1 \xi_2 \tilde{B}_1 + (\tau^2 - \xi_2^2) \tilde{B}_2}{\tau^2 s/\nu} \right] e^{-\tau x_3} \\
- \frac{|\xi| i(\xi_1 \tilde{B}_1 + \xi_2 \tilde{B}_2)}{|\xi| s/\nu} e^{-|\xi| x_3} + \tilde{d}_3 e^{-|\xi| x_3} \\
= \left[ \frac{i \xi_1 (\tau^2 - \xi_2^2) + i \xi_2 \xi_1 \xi_2}{\tau^2 s/\nu} e^{-\tau x_3} - \frac{i \xi_1}{\tau^2 s/\nu} e^{-|\xi| x_3} \right] \tilde{B}_1 \\
+ \left[ \frac{i \xi_2 (\tau^2 - \xi_2^2)}{\tau^2 s/\nu} + \frac{i \xi_2 \xi_1 \xi_2}{\tau^2 s/\nu} e^{-\tau x_3} - \frac{i \xi_2}{\tau^2 s/\nu} e^{-|\xi| x_3} \right] \tilde{B}_2 \\
+ \tilde{d}_3 e^{-|\xi| x_3} \\
\tag{4.28}
\]

Finally, we calculate
\[
\tilde{q} = -\frac{i \nu (\xi_1 \tilde{B}_1 + \xi_2 \tilde{B}_2)}{|\xi| s/\nu} e^{-|\xi| x_3} + \frac{s}{|\xi|} \tilde{d}_3 e^{-|\xi| x_3}. 
\] (4.29)

Using the form of \( \tilde{B}_1, \tilde{B}_2 \) and expressing \( e_2 \) in terms of \( e_0 \) and \( e_1 \) we have
\[
\tilde{u}_1 = \left[ \frac{1}{\tau^2} e_0 - \frac{\xi_1^2}{\tau^2 |\xi| (\tau + |\xi|)} e_0 + \frac{\xi_1}{\tau |\xi| (\tau + |\xi|)} e_1 \right] (2 i \xi_1 \tau \tilde{d}_3 - \tau \tilde{d}_1) \\
+ \xi_1 \xi_2 \left( - \frac{1}{\tau^2 |\xi| (\tau + |\xi|)} e_0 + \frac{1}{|\xi| (\tau + |\xi|)} e_1 \right) (2 i \xi_2 \tau \tilde{d}_3 - \tau \tilde{d}_2) \\
- \frac{i \xi_1}{|\xi|} \tilde{d}_3 (e_0 - (\tau - |\xi|) e_1) \\
= -\frac{\xi_1^2}{|\xi| (\tau + |\xi|)} e_1 \tilde{d}_1 - \frac{\xi_1 \xi_2}{|\xi| (\tau + |\xi|)} e_1 \tilde{d}_2 + i \xi_1 \frac{|\xi|^2 + \tau^2}{|\xi| (\tau + |\xi|)} e_1 \tilde{d}_3 \\
+ \left( - \frac{1}{\tau} + \frac{\xi_1^2}{\tau |\xi| (\tau + |\xi|)} \right) e_0 \tilde{d}_1 + \frac{\xi_1 \xi_2}{\tau |\xi| (\tau + |\xi|)} e_0 \tilde{d}_2 + i \xi_1 (|\xi| - \tau) e_0 \tilde{d}_3 \\
\equiv \sum_{r=1}^{3} (g_1, e_1 \tilde{d}_r + h_1, e_0 \tilde{d}_r). 
\] (4.30)
Next, we calculate
\[
\tilde{u}_2 = \left( -\frac{\xi_1\xi_2}{\tau^2|\xi|(\tau + |\xi|)} e_0 + \frac{\xi_1\xi_2}{\tau|\xi|(\tau + |\xi|)} e_1 \right) (2i\xi_1\tau \tilde{d}_3 - \tau \tilde{d}_1) \\
+ \left( -\frac{\xi_2^2}{\tau^2|\xi|(\tau + |\xi|)} e_0 + \frac{\xi_2}{\tau|\xi|(\tau + |\xi|)} e_1 \right) (2i\xi_2\tau \tilde{d}_3 - \tau \tilde{d}_2) \\
- \frac{i\xi_2}{|\xi|} \tilde{d}_3 (e_0 - (\tau - |\xi|) e_1)
\]
\[
= -\frac{\xi_1\xi_2}{|\xi|(\tau + |\xi|)} e_1 \tilde{d}_1 - \frac{\xi_2^2}{|\xi|(\tau + |\xi|)} e_1 \tilde{d}_2 + i\xi_2 \frac{|\xi|^2 + \tau^2}{|\xi|(\tau + |\xi|)} e_1 \tilde{d}_3
\]
\[
- \frac{\xi_1\xi_2}{\tau|\xi|(\tau + |\xi|)} e_0 \tilde{d}_1 + \left( -\frac{1}{\tau} + \frac{\xi_2^2}{\tau|\xi|(\tau + |\xi|)} \right) e_0 \tilde{d}_2 \\
+ i\xi_2 (|\xi| - \tau) \frac{\tilde{d}_3}{|\xi|(\tau + |\xi|)} e_0 \tilde{d}_3
\]
\[
\equiv \sum_{r=1}^3 (g_{2r} e_1 \tilde{d}_r + h_{2r} e_0 \tilde{d}_r).
\]
From (4.28) we have
\[
\tilde{u}_3 = \frac{i\xi_1}{\tau(\tau + |\xi|)} e_1 (2i\xi_1\tau \tilde{d}_3 - \tau \tilde{d}_1) \\
+ \frac{i\xi_2}{\tau(\tau + |\xi|)} e_1 (2i\xi_2\tau \tilde{d}_3 - \tau \tilde{d}_2) + \tilde{d}_3 (e_0 - (\tau - |\xi|) e_1)
\]
\[
= -\frac{i\xi_1}{\tau + |\xi|} e_1 \tilde{d}_1 - \frac{i\xi_2}{\tau + |\xi|} e_1 \tilde{d}_2 - \frac{\tau^2 + |\xi|^2}{\tau + |\xi|} e_1 \tilde{d}_3 + e_0 \tilde{d}_3
\]
\[
\equiv \sum_{r=1}^3 (g_{3r} e_1 \tilde{d}_r + g_{3r} e_0 \tilde{d}_r).
\]
Finally, (4.29) yields
\[
\tilde{q} = \left[ -\frac{i\nu \xi_1}{|\xi|} (2i\xi_1\tau \tilde{d}_3 - \tau \tilde{d}_1) - \frac{i\nu \xi_2}{|\xi|} (2i\xi_2\tau \tilde{d}_3 - \tau \tilde{d}_2) \\
+ \frac{s}{|\xi|} \tilde{d}_3 (e_0 - (\tau - |\xi|) e_1) \right] e_0 \\
= -\frac{i\nu \xi_1 (\tau - |\xi|)}{|\xi|} e_1 \tilde{d}_1 - \frac{i\nu \xi_2 (\tau - |\xi|)}{|\xi|} e_1 \tilde{d}_2 \\
+ \frac{\nu (\tau^2 + |\xi|^2) (\tau - |\xi|)}{|\xi|} e_1 \tilde{d}_3 + i\nu \xi_1 e_0 \tilde{d}_1 + i\nu \xi_2 e_0 \tilde{d}_2 \\
- \frac{\nu (\tau^2 + |\xi|^2)}{|\xi|} e_0 \tilde{d}_3
\]
\[
\equiv \sum_{r=1}^3 (g_{4r} e_1 \tilde{d}_r + h_{4r} e_0 \tilde{d}_r).
\]

Lemma 4.1. Let \(\tau = \sqrt{\tau + i\xi_0 + |\xi|^2}, |\xi|^2 = \xi_1^2 + \xi_2^2\). Then the following estimates hold:
1. \(|\partial_{\xi_0} g_{mr}| \leq \frac{c}{|\xi|^2}, |\partial_{\xi_1}^2 g_{mr}| \leq \frac{c}{|\xi|^3}, m = 1, 2, 3, r = 1, 2.
2. \(|\partial_{\xi_1} g_{mr}| \leq c/|\xi|, |\partial_{\xi_2}^2 g_{mr}| \leq c/|\xi| |\xi|, |\partial_{\xi_1}^3 g_{mr}| \leq \frac{c}{|\xi|^3 |\xi|^2} + \frac{c}{|\xi|^4 |\xi|^2}, |\partial_{\xi_1}^3 g_{mr}| \leq \frac{c}{|\xi|^3 |\xi|^2} + \frac{c}{|\xi|^4 |\xi|^2} + \frac{c}{|\xi|^5 |\xi|^3}, m = 1, 2, 3, r = 1, 2.
(3) $|\partial_{\xi_0} g_{m r}| \leq c/|\tau|$, $|\partial_{\xi_0} g_{m r}| \leq c/|\tau|^3$, $r = 3$.
(4) $|\partial_{\xi} g_{m 3}| \leq c + c|\tau|/|\xi|$, $|\partial_{\xi} g_{m 3}| \leq c/|\tau| + c/|\xi| + c/|\tau|^2$, $|\partial_{\xi} g_{m 3}| \leq c/|\xi|(|\tau| + |\xi|) + c/|\tau| |\xi| + |\xi|)^2$, $|\partial_{\xi} g_{m 3}| \leq c/|\xi|(|\tau| + |\xi|) + c/|\tau| |\xi| + |\xi|)^2$.
(5) $|h_{m r, \xi_0}| \leq c/|\tau|^3$, $|h_{m r, \xi_0}| \leq c/|\tau|^5$, $m = 1, 2, 3$, $r = 1, 2$.
(6) $|\partial_{\xi} h_{m r}| \leq c/|\tau|^2$, $|\partial_{\xi} h_{m r}| \leq c/|\tau|^3 + c/|\tau|^2 |\xi|$, $|\partial_{\xi} h_{m r}| \leq c/|\tau|^4 + c/|\tau|^4 |\xi| + c/|\tau|^3 |\xi|$, $|\partial_{\xi} h_{m r}| \leq c/|\tau|^5 + c/|\tau|^5 |\xi| + c/|\tau|^4 |\xi| + c/|\tau|^3 |\xi|$, $m = 1, 2, 3$, $r = 1, 2$.
(7) $|h_{m 3, \xi_0}| \leq c/|\tau|^2$, $|h_{m 3, \xi_0}| \leq c/|\tau|^4$, $m = 1, 2$, $|\partial_{\xi} h_{m 3}| \leq c/|\tau| + c/|\xi|$, $|\partial_{\xi} h_{m 3}| \leq c/|\tau|^2 + c/|\xi|^2$, $|\partial_{\xi} h_{m 3}| \leq c/|\tau|^3 + c/|\xi|^3$, $m = 1, 2, 3$.

Proof. Consider $g_{11}$. Then

$$|g_{11, \xi_0}| = \left| \frac{\xi_1^2}{|\xi|(|\tau| + |\xi|)^2} \frac{1}{|\tau|} \right| \leq \frac{c}{(|\tau|^3 + |\xi|^3)} \frac{1}{|\tau|^2} \leq \frac{c}{|\tau|^2}.$$ 

$$|g_{11, \xi_0}| = \left| \frac{\xi_1^2}{|\xi|(|\tau| + |\xi|)^3} \frac{1}{|\tau|^2} + \frac{\xi_1^2}{|\xi|(|\tau| + |\xi|)^3} \frac{1}{|\tau|^3} \right| \leq \frac{c}{(|\tau|^3 + |\xi|^3)} \frac{1}{|\tau|^2} + \frac{c}{(|\tau|^3 + |\xi|^3)} \frac{1}{|\tau|^3} \leq \frac{c}{(|\tau|^3 + |\xi|^3)} \frac{1}{|\tau|^2} + \frac{c}{(|\tau|^3 + |\xi|^3)} \frac{1}{|\tau|^3} \leq \frac{c}{|\tau|^4}.$$ 

The same estimates can be proved for $g_{m r}$, $m = 1, 2, 3$, $r = 1, 2$. Hence (1) holds.

In order to show (2) we consider as the example $\partial_{\xi} g_{11}$. First, we recall that

$$g_{11} = \frac{\xi_1^2}{|\xi|(|\tau| + |\xi|)}.$$ 

We calculate

$$\partial_{\xi} g_{11} = \partial_{\xi} \frac{\xi_1^2}{|\xi|(|\tau| + |\xi|)} = \frac{2\xi_1}{|\xi|(|\tau| + |\xi|)} + \frac{\xi_1^2 \partial_{\xi} |\xi|}{|\xi|^2(|\tau| + |\xi|)} + \frac{\xi_1^2 \partial_{\xi} (|\tau| + |\xi|)}{|\xi|(|\tau| + |\xi|)^2}.$$ 

Consequently, we carefully analyze each term. Thus, applying $Re \tau \geq \frac{|\tau|}{c}$:

$$|I_1| \leq \frac{c}{|\tau| + |\xi|} = \frac{c|Re \tau + |\xi| - iIm \tau|}{|\tau| + |\xi|} \leq \frac{|Re \tau + |\xi| + iIm \tau|(Re \tau + |\xi| - iIm \tau)}{|Re \tau + |\tau| + |\xi|} \leq \frac{(|Re \tau + |\xi||^2 + |Im \tau|^2)^2}{|Re \tau|^2 + (Im \tau)^2 + 2Re \tau|\xi| + |\tau|^2} \leq \frac{1}{|\tau| + |\xi|} \frac{1}{\sqrt{|\tau|^2 + |\xi|^2}} \leq \frac{c}{\sqrt{|\tau|^2 + |\xi|^2}} \leq \frac{c}{|\tau|}.$$
Next, we estimate
\[ |I_2| \leq \frac{1}{|\tau + |\xi||} \leq \frac{c}{|\tau|} \]
where we used that
\[ |\partial_\xi |\xi| | \leq \frac{\xi}{\sqrt{\xi^2}} \leq c. \]

Finally, for \( I_3 \) we have
\[ |I_3| \leq \frac{|\xi|}{(\tau + |\xi|)^2} (|\partial_\xi \tau| + |\partial_\xi |\xi|) \leq c \frac{|\xi|}{(\tau + |\xi|)^2} \left( \frac{|\xi|}{|\tau|} + 1 \right) \]
\[ \leq \frac{c}{|\tau|} + \frac{c}{|\tau + |\xi||} \leq \frac{c}{|\tau|}, \]
where we applied
\[ \partial_\xi \tau = \frac{\xi}{\sqrt{s + i \xi_0 + \xi^2}}, \quad |\partial_\xi \tau| \leq \frac{|\xi|}{|\tau|}. \]

Summarizing, we conclude
\[ |\partial_\xi g_{11}| \leq \frac{c}{|\tau|}. \]

Similarly, we obtain
\[ |\partial_\xi^2 g_{11}| = \partial_\xi^2 \frac{\xi_1^2}{|\xi||\tau + |\xi||} = \frac{1}{|\xi||\tau + |\xi||} + \left( \partial_\xi^2 \frac{1}{|\xi||\tau + |\xi||} \right) \frac{\xi_1^2}{|\xi||\tau + |\xi||} + \frac{\xi_1^2}{|\xi|} \frac{1}{|\tau + |\xi||} \]
\[ + \xi_1 \left( \partial_\xi^2 \frac{1}{|\xi|} \right) \frac{1}{|\tau + |\xi||} + \frac{\xi_1}{|\xi|} \partial_\xi \frac{1}{|\tau + |\xi||} + \xi_1^2 \left( \partial_\xi \frac{1}{|\xi|} \partial_\xi \frac{1}{|\tau + |\xi||} \right) \]
\[ \equiv J_1 + \cdots + J_6, \]

Hence
\[ |J_1| \leq \frac{c}{|\xi||\tau + |\xi||} \leq \frac{c}{|\xi||\tau|}, \]
\[ |J_2| \leq \frac{c}{|\tau + |\xi||} \frac{1}{|\xi|} \leq \frac{c}{|\xi||\tau + |\xi||} \leq \frac{c}{|\xi|}, \]
\[ |J_3| \leq \frac{c}{|\tau + |\xi||^3} \leq \frac{c}{|\tau + |\xi||} \leq \frac{c}{|\xi||\tau + |\xi||} \leq \frac{c}{|\xi||\tau|}, \]

and we estimate analogously \( J_4, J_5, J_6 \) to get that
\[ |\partial_\xi^2 g_{11}| \leq \frac{c}{|\xi||\tau|}. \]

We can derive as well
\[ |\partial_\xi^3 g_{11}| \leq \frac{c}{|\xi||\tau|^2} + \frac{c}{|\xi||\tau|^2}, \]
\[ |\partial_\xi^4 g_{11}| \leq \frac{c}{|\xi|^2|\tau|} + \frac{c}{|\xi|^2|\tau|^2} + \frac{c}{|\xi|^3|\tau|^3}. \]

The same estimates can be proved for \( g_{mr}, m = 1, 2, 3, r = 1, 2. \) Hence (2) holds.
Consider \( g_{13} \). Then

\[
|\partial_{\xi_0}^1 g_{13}| \leq c \frac{\xi_1}{|\xi|(|\tau + |\xi|)} + c \frac{\xi_1(|\xi|^2 + \tau^2) 1}{|\xi|(|\tau + |\xi|)^2} \frac{1}{|\tau|} \\
\leq c \frac{1}{|\tau| + |\xi|} + c \frac{|\xi|^2 + |\tau|^2 + 1}{(|\tau + |\xi|)^2} \frac{1}{|\tau|} \leq c \frac{1}{|\tau|^3}.
\]

Next, we have

\[
|\partial_{\xi_0}^2 g_{13}| \leq c + c \frac{|\tau|}{|\xi|} \frac{1}{|\tau|^3}, \quad |\partial_{\xi_0}^2 g_{13}| \leq c \frac{c}{|\tau|} \frac{1}{|\xi|^2},
\]

\[
|\partial_{\xi}^1 g_{13}| \leq \frac{|\xi|(|\tau + |\xi|)}{|\xi|(|\tau + |\xi|)} + \frac{(|\tau + |\xi|)^2}{(|\tau + |\xi|)^2},
\]

\[
|\partial_{\xi}^2 g_{13}| \leq \frac{c}{|\tau|^2|\xi|(|\tau + |\xi|)} + \frac{c}{|\tau|^2|\xi|(|\tau + |\xi|)^2}.
\]

The same estimates can be derived for \( g_{m3}, m = 2, 3 \). Hence (3) and (4) hold.

Consider \( h_{11} \). Qualitatively, we have

\[
|h_{11,\xi_0}| \leq c \frac{c}{|\tau|^3} + c \frac{\xi_1^2}{|\tau|^3|\xi|(|\tau + |\xi|)} + c \frac{\xi_1^2}{|\tau|^3|\xi|(|\tau + |\xi|)^2} \leq c/|\tau|^3,
\]

\[
|h_{11,\xi_0|\xi_0}| \leq c/|\tau|^3.
\]

Next

\[
|h_{11,\xi}| \leq c \frac{c}{|\tau|^2} + c \frac{\xi_1}{|\tau|(|\tau + |\xi|)} + c \frac{c}{|\tau|^2|\xi|(|\tau + |\xi|)} \cdots \leq c/|\tau|^2,
\]

\[
|\partial_{\xi}^2 h_{11}| \leq c \frac{c}{|\tau|^3} + c \frac{c}{|\tau|^3|\xi|(|\tau + |\xi|)} + c \frac{c}{|\tau|^3|\xi|(|\tau + |\xi|)^2} \cdots
\]

\[
\leq c/|\tau|^3 + c \frac{1}{|\tau|^2|\xi|} + c \frac{1}{|\tau|^3|\xi|}.
\]

\[
|\partial_{\xi}^2 h_{11}| \leq c/|\tau|^4 + c \frac{c}{|\tau|^2|\xi|(|\tau + |\xi|)} + c \frac{c}{|\tau|^2|\xi|(|\tau + |\xi|)^2} + \cdots
\]

\[
\leq c/|\tau|^4 + c \frac{1}{|\tau|^2|\xi|^2} + c \frac{1}{|\tau|^3|\xi|}.
\]

\[
|\partial_{\xi}^2 h_{11}| \leq c/|\tau|^5 + c \frac{c}{|\tau|^3|\xi|(|\tau + |\xi|)} + c \frac{c}{|\tau|^3|\xi|(|\tau + |\xi|)^2} + \cdots
\]

\[
\leq c/|\tau|^5 + c \frac{1}{|\tau|^2|\xi|^3} + c \frac{1}{|\tau|^4|\xi|} + c \frac{1}{|\tau|^3|\xi|^2}.
\]

Similar estimates hold for \( h_{mr}, m = 1, 2, 3, r = 1, 2 \). Hence (5) and (6) hold.
Finally, we examine \( h_{13} \). Then we obtain
\[
|h_{13, \xi_0}| \leq c/|\tau|^2, \quad |h_{13, \xi_0 \xi_0}| \leq c/|\tau|^4, \\
|\partial_\xi h_{13}| \leq c/|\tau + |\xi|| + c|\tau|/|\xi||(|\tau| + |\xi|)| \leq c/|\tau + |\xi||, \\
|\partial_\xi^2 h_{13}| \leq c/|\xi||(|\tau| + |\xi|)| + c|\xi| + |\tau|/|\xi|^2(|\xi| + |\tau|)| \leq c/|\xi|^2 + |\tau|^2, \\
|\partial_\xi^3 h_{13}| \leq c/|\xi|^2(|\tau| + |\xi|)| + c|\xi|(|\tau| + |\xi|)|^2 + \cdots \\
|\partial_\xi^4 h_{13}| \leq c/|\xi|^3|\tau| + c/|\xi^2|\tau| + c/|\xi||\tau|^3. 
\]
The same estimates hold for \( h_{23} \). We do not need to estimate \( h_{33} \) because \( h_{33} = 1 \). Hence (7) and (8) are proved. Estimates for \( g_{4r}, h_{4r}, r = 1, 2, 3 \), can be derived similarly. This concludes the proof of Lemma 4.1.

\[\square\]

**Remark 4.2.** From (4.30)–(4.33) the solution to (4.19) is expressed in the form
\[
F_2u_m \equiv \tilde{u}_m = \sum_{r=1}^{3} (g_{mr}e_1 \tilde{d}_r + h_{mr}e_0 \tilde{d}_r), \\
F_2q \equiv \tilde{q} = \sum_{r=1}^{3} (g_{4r}e_1 \tilde{d}_r + h_{4r}e_0 \tilde{d}_r), 
\]
where \( m = 1, 2, 3 \) and \( g_{mr}, h_{mr}, g_{4r}, h_{4r}, r = 1, 2, 3 \) are defined by formulas (4.30)–(4.33).

**Theorem 4.3.** Let \( p, q \in (1, \infty), \sigma \in (0, 1) \), \( d_k \in B_{p,q,\gamma}^{1+\sigma-1/p,1/2-\sigma/2-1/2p}(\mathbb{R}^2 \times \mathbb{R}^+), k = 1, 2, d_3 \in B_{p,q,\gamma}^{2+\sigma-1/p,1+\sigma/2-1/2p}(\mathbb{R}^2 \times \mathbb{R}^+) \).

Then there exists a solution to problem (4.17) such that \( u \in B_{p,q,\gamma}^{2+\sigma,1+\sigma/2}(\mathbb{R}^3_+ \times \mathbb{R}^+) \), \( \nabla q \in B_{p,q,\gamma}^{\sigma,\sigma/2}(\mathbb{R}^3_+ \times \mathbb{R}^+) \) and
\[
\|u\|_{B_{p,q,\gamma}^{2+\sigma,1+\sigma/2}(\mathbb{R}^3_+ \times \mathbb{R}^+)} \leq cI, \quad \|\nabla q\|_{B_{p,q,\gamma}^{\sigma,\sigma/2}(\mathbb{R}^3_+ \times \mathbb{R}^+)} \leq cI, 
\]
where
\[
I = \sum_{k=1}^{2} \|d_k\|_{B_{p,q,\gamma}^{1+\sigma-1/p,1+\sigma-1/p}(\mathbb{R}^2 \times \mathbb{R}^+)} + \|d_3\|_{B_{p,q,\gamma}^{2+\sigma-1/p,2+\sigma-1/p}(\mathbb{R}^2 \times \mathbb{R}^+)}. 
\]

To prove estimates (4.35) and (4.36) we recall Definition 2.5 to describe the norms from the l.h.s. of (4.35) and (4.36). We restrict our considerations to prove estimate (4.35) only.
Hence, we have
\[
\|u\|_{B^{2+\sigma,1+\sigma/2}_p(R^3_+ \times \mathbb{R})} = \sum_{m=1}^{3} \|u_{m\gamma}\|_{B^{2+\sigma,1+\sigma/2}_p(R^3_+ \times \mathbb{R})} = \sum_{m=1}^{3} \|u_{m\gamma}\|_{L_p(R^3_+ \times \mathbb{R})} \\
+ \sum_{m=1}^{3} \left[ \sum_{k=0}^{\infty} \left( \sum_{j=2}^{\infty} \int_{\mathbb{R}^3_+} |2(2+\sigma-j)k F^{-1}_2 \varphi_k F_2 \partial_{x_3}^j u_{m\gamma}(x,t)|^{q/p} dx dt \right)^{1/q} \right]
\]
\[
+ \sum_{m=1}^{3} \left[ \sum_{k=0}^{\infty} \left( \int_{\mathbb{R}^3_+} \int_{\mathbb{R}^2} \int \frac{|F^{-1}_2 \varphi_k F_2 (\partial_{x_3}^j u_{m\gamma}(\bar{x}', x_3 + z) - \partial_{x_3}^j u_{m\gamma}(\bar{x}', x_3))|^p}{z^{1+\sigma}} \frac{dz}{q/p} \right)^{1/q} \right]
\]
\[
\equiv \sum_{m=1}^{3} (\|u_{m\gamma}\|_{L_p(R^3_+ \times \mathbb{R})} + \|u_m\|_{1, B^{2+\sigma,1+\sigma/2}_p(R^3_+ \times \mathbb{R})} + \|u_m\|_{2, B^{2+\sigma,1+\sigma/2}_p(R^3_+ \times \mathbb{R})}),
\]
where \(\bar{x}' = (x_1, x_2, t)\).

**Lemma 4.4.** Let the assumptions of Theorem 4.3 be satisfied. Then
\[
\|u\|_{1, B^{2+\sigma,1+\sigma/2}_p(R^3_+ \times \mathbb{R})} = \sum_{m=1}^{3} \|u_m\|_{1, B^{2+\sigma,1+\sigma/2}_p(R^3_+ \times \mathbb{R})} \leq c \left( \sum_{k=1}^{2} \|d_k\|_{B^{2+\sigma-1/p, (1+\sigma-1/p)/2}_p(R^3_+ \times \mathbb{R})} \right)
\]
\[
+ \|d_3\|_{B^{2+\sigma-1/p, (2+\sigma-1/p)/2}_p(R^3_+ \times \mathbb{R})}.
\]

**Proof.** Using (4.34) we have
\[
\|u_m\|_{1, B^{2+\sigma,1+\sigma/2}_p(R^3_+ \times \mathbb{R})}
\]
\[
= \sum_{k=0}^{\infty} \left( \sum_{j=2}^{\infty} \int_{\mathbb{R}^3_+} \left| 2(2+\sigma-j)k F^{-1}_2 \varphi_k \sum_{r=1}^{3} (g_{mr} \partial_{x_3}^j e_1
\]
\[
+ h_{mr} \partial_{x_3}^j e_0) d_{r\gamma}) \right|^p \frac{dx dt}{q/p} \right)^{1/q} \right]
\]
\[
\leq c \left( \sum_{k=0}^{\infty} \sum_{j=2}^{\infty} \sum_{r=1}^{3} I^q_{1kj\gamma} \right)^{1/q},
\]
where
\[
I_{1kj\gamma} = \left( \int_{\mathbb{R}^3_+} \left| 2(2+\sigma-j)k F^{-1}_2 \varphi_k (g_{mr} \partial_{x_3}^j e_1
\]
\[
+ h_{mr} \partial_{x_3}^j e_0) F_2 d_{r\gamma}) \right|^p dx dt \right)^{1/p}.
\]
Introduce the family of functions \( \{\psi_j(\bar{\xi})\} \), \( \bar{\xi} = (\xi, \xi_0), \xi = (\xi_1, \xi_2) \) such that \( \text{supp} \psi_0 \subset \{\bar{\xi} : |\bar{\xi}|_a \leq 4\} \), \( \text{supp} \psi_j \subset \{\bar{\xi} : 2^{j-2} \leq |\bar{\xi}|_a \leq 2^{j+2}\} \) and \( \psi_j(\bar{\xi}) = 1 \) for \( \bar{\xi} \in \text{supp} \varphi_j \). Then

\[
I_{1kj\gamma} = \left( \int_{\mathbb{R}^3 \times \mathbb{R}} \left| \sum_{l=0}^{\infty} 2^{(2+\sigma-j)k} F_2^{-1} \psi_l(g_{mr} \partial_{x_3} e_1)ight|^p d\bar{x} d\bar{x}_3 \right)^{1/p}
\]

\[
+ h_{mr} \partial_{x_3} e_0 \varphi_k \varphi_{l} F_2 d_{r\gamma} \left( \bar{x}, x_3 \right) \left| F_2^{-1} \psi_l(g_{mr} \partial_{x_3} e_1 + h_{mr} \partial_{x_3} e_0) \right|^p \right)^{1/p}
\]

\[
= \left( \int_{\mathbb{R}^3 \times \mathbb{R}^+} \left| \sum_{l=0}^{\infty} 2^{(2+\sigma-j)k} (F_2^{-1} \psi_l(g_{mr} \partial_{x_3} e_1 + h_{mr} \partial_{x_3} e_0)) \right|^p d\bar{x} d\bar{x}_3 \right)^{1/p}
\]

\[
\cdot F_2 F_2^{-1} \varphi_k \varphi_{l} F_2 d_{r\gamma} \left( \bar{x} - \bar{y} \right) \left| F_2^{-1} \psi_{l} \left( \bar{y}, x_3 \right) \right|^p \right)^{1/p},
\]

where \( \bar{x} = (x', t) = (x', x_0), x' = (x_1, x_2) \in \mathbb{R}^2 \).

Continuing, we can rewrite \( I_{1kj\gamma} \) as

\[
I_{1kj\gamma} \left( \int_{\mathbb{R}^3 \times \mathbb{R}^+} \left| \sum_{l=0}^{\infty} 2^{(2+\sigma-j)k} (F_2^{-1} \psi_l(g_{mr} \partial_{x_3} e_1 + h_{mr} \partial_{x_3} e_0)) \right|^p d\bar{x} d\bar{x}_3 \right)^{1/p}
\]

\[
\cdot F_2 F_2^{-1} \varphi_k \varphi_{l} F_2 d_{r\gamma} \left( \bar{x} - \bar{y} \right) \left| F_2^{-1} \psi_{l} \left( \bar{y}, x_3 \right) \right|^p \right)^{1/p},
\]

where \( \bar{y} = (y', y), y' = (y_1, y_2) \in \mathbb{R}^2 \).

Continuing, we have

\[
I_{1kj\gamma} = \left( \int_{\mathbb{R}^3 \times \mathbb{R}^+} \left| \sum_{l=0}^{\infty} 2^{(2+\sigma-j)k} (F_2^{-1} \psi_l(g_{mr} \partial_{x_3} e_1 + h_{mr} \partial_{x_3} e_0)) \right|^p d\bar{x} d\bar{x}_3 \right)^{1/p}
\]

\[
\cdot F_2 F_2^{-1} \varphi_k \varphi_{l} F_2 d_{r\gamma} \left( \bar{x} - \bar{y} \right) \left| F_2^{-1} \psi_{l} \left( \bar{y}, x_3 \right) \right|^p \right)^{1/p},
\]

From Lemma 3.4 we have

\[
[F_2^{-1} (f(2^{l'}, 2^{2l}), g(2^{l'}, 2^{2l}))](\bar{y}) = 2^{-4l'}(F_2^{-1} f \ast F_2^{-1} g)(2^{-1}\bar{y}), \tag{4.39}
\]

where \( f = \psi_l(g_{mr} \partial_{x_3} e_1 + h_{mr} \partial_{x_3} e_0), g = \varphi_k, \bar{y} = (y', 2^{-l}y_0), y' = (y_1, y_2) \).

Moreover, we used in (4.38) the notation

\[
h(\xi', \xi_0) = \tilde{h}(\xi', \gamma + i\xi_0), \quad h \in \{f, g\}.
\]
Using the change of variables $y_i = 2^{-i} \omega_i$, $i = 1, 2$, $y_0 = 2^{-2} \omega_0$ and the notation $\bar{\omega} = (\omega', 2^{-l} \omega_0)$, $\omega' = (\omega_1, \omega_2)$, $\bar{\omega} = (\omega', \omega_0)$ we obtain

\[
I_{1kj\gamma} = \left[ \int_{\mathbb{R}^3 \times \mathbb{R}_+} \left| \sum_{l=0}^\infty 2^{(2+\sigma-j)k} 2^{-4l} \int_{\mathbb{R}^3} d\bar{\omega} [F_{2}^{-1}(\psi_{l}(g_{mr}\partial_{x_3}^j e_1 \\
+ h_{mr}\partial_{x_3}^j e_0)) \ast F_{2}^{-1}(\varphi_k(2^{-l}\bar{\omega}, x_3))(F_{2}^{-1}(\varphi_l F_2 d_{r\gamma})(\bar{x} - 2^{-l}\bar{\omega})]\right] d\bar{x} d\bar{x}_3 \right]^{1/p}.
\]

Then (4.39) yields

\[
I_{1kj\gamma} = \left\{ \int_{\mathbb{R}^3 \times \mathbb{R}_+} d\bar{x} \left| \sum_{l=0}^\infty 2^{(2+\sigma-j)k} \int_{\mathbb{R}^3} d\bar{\omega} [F_{2}^{-1}(\psi_{l}(g_{mr}\partial_{x_3}^j e_1 \\
+ h_{mr}\partial_{x_3}^j e_0)) \ast F_{2}^{-1}(\varphi_k(2^{-l}, 2^{2l}))(\bar{\omega}, x_3)](F_{2}^{-1}(\varphi_l F_2 d_{r\gamma})(\bar{x} - \bar{\omega})]\right| \right\}^{1/p}.
\]

Next, the Minkowski inequality with respect to $\bar{x}$ gives

\[
I_{1kj\gamma} \leq \left[ \int_{\mathbb{R}^3} d\bar{x} \left| \sum_{l=0}^\infty 2^{(2+\sigma-j)k} \int_{\mathbb{R}^3} d\bar{\omega} [F_{2}^{-1}(\psi_{l}(g_{mr}\partial_{x_3}^j e_1 \\
+ h_{mr}\partial_{x_3}^j e_0))(2^{-l}, 2^{2l}))(\varphi_k(2^{-l}, 2^{2l}))(\bar{\omega}, x_3)](F_{2}^{-1}(\varphi_l F_2 d_{r\gamma})(\bar{x} - \bar{\omega})]^p \right| \right]^{1/p} \equiv I_{1kj\gamma}^1.
\]

The change of variables $\bar{z} = \bar{x} - \bar{\omega}$ in the integral with respect to $\bar{x}$ implies

\[
I_{1kj\gamma}^1 = \left\{ \int_{\mathbb{R}^3} d\bar{z} \left| \sum_{l=0}^\infty 2^{(2+\sigma-j)k} \int_{\mathbb{R}^3} d\bar{\omega} [F_{2}^{-1}(\psi_{l}(g_{mr}\partial_{x_3}^j e_1 \\
+ h_{mr}\partial_{x_3}^j e_0))(2^{-l}, 2^{2l}))(\varphi_k(2^{-l}, 2^{2l}))(\bar{\omega}, x_3)](F_{2}^{-1}(\varphi_l F_2 d_{r\gamma})(\bar{z})]^p \right| \right\}^{1/p},
\]

where $\bar{z} = (\bar{z}', \bar{z}_0)$, $\bar{z}' = (\bar{z}_1, \bar{z}_2)$ and we used the fact that $F_2 d_{r\gamma} = F_1 d_{r\gamma}$. 

Therefore, applying the Hölder inequality in the integral with respect to \( \bar{\omega} \) and replacing \( \bar{\omega} \) by \( \bar{y} \) and \( \bar{z} \) by \( \bar{x} \) we obtain

\[
I_{1kjr\gamma}^1 \leq \left\{ \int d\bar{x}_3 \left| \sum_{l=0}^{\infty} 2^{(2+\sigma-j)l} \left( \int d\bar{y} \frac{1}{(1 + |\bar{y}|_a^2)} \right)^{1/2} \cdot \left( \int d\bar{y} |F_2^{-1}(\psi_l(g_{mr}\partial_{x_3}^j e_1 + h_{mr}\partial_{x_3}^j e_0)(2^l, 2^{2l} \cdot) \ast F_2^{-1}(\varphi_k(2^l, 2^{2l} \cdot))(\bar{y}, x_3)(1 + |\bar{y}|_a^2)^{1/2} \right)^{1/2} \cdot \left( \int d\bar{x}_3 |F_2^{-1}\varphi_l F_1 d_{r\gamma}(\bar{x})|^p \right)^{1/p} \right) \right\}^{1/p}.
\]

Using that

\[
\left( \int d\bar{y} \frac{1}{(1 + |\bar{y}|_a^2)} \right)^{1/2} \leq c
\]

and the Parseval identity we have

\[
I_{1kjr\gamma}^2 \leq c \left\{ \int d\bar{x}_3 \left| \sum_{l=0}^{\infty} 2^{(2+\sigma-j)(k-l)} 2^{(2+\sigma-j)l} \cdot \left( \int d\bar{y} |F_2^{-1}(\psi_l(g_{mr}\partial_{x_3}^j e_1 + h_{mr}\partial_{x_3}^j e_0)(2^l, x_3, 2^{2l} \cdot) \varphi_k(2^l, 2^{2l} \cdot)\|W_{4,2}^2(\mathbb{R}^3)\|^{1/p} \right) \right) \right\}^{1/p}.
\]

\[
\leq c \sum_{l=0}^{\infty} 2^{(2+\sigma-j)(k-l)} 2^{(2+\sigma-j)l} \left( \int d\bar{x}_3 |\psi_l(g_{mr}\partial_{x_3}^j e_1 + h_{mr}\partial_{x_3}^j e_0)(2^l, x_3, 2^{2l} \cdot)\varphi_k(2^l, 2^{2l} \cdot)\|W_{4,2}^2(\mathbb{R}^3)\|^{1/p} \right) \left( \int d\bar{x}_3 |F_2^{-1}\varphi_l F_1 d_{r\gamma}(\bar{x})|^p \right)^{1/p} \equiv I_{1kjr\gamma}^3.
\]

Using (4.41) (see Lemma 4.5 below) we get

\[
I_{1kjr\gamma}^3 \leq c \sum_{l=0}^{\infty} 2^{(\delta-j-l)(k-l)} 2^{(\delta-j-l-1/p-r)} \cdot \|F_2^{-1}\varphi_l F_1 d_{r\gamma}\|_{L_p(\mathbb{R}^3)},
\]

where \( \delta = 2 + \sigma + 8 \), \( c_r = 1 \) for \( r = 1, 2 \), \( c_r = 0 \) for \( r = 3 \).
Therefore, by the Hölder inequality

\[ \|u_m\|_{1, B^{2+\sigma, 1+\sigma/2}_{p,q,R_3}} \leq c \left( \sum_{k=0}^{\infty} \sum_{r=1}^{3} \sum_{l=0}^{\infty} \right) \left[ 2^{(\delta - L)[l-k]} l^{2+\sigma - 1/p - c_r} \| F_1^{-1} \varphi_l F_1 d_{r\gamma} \|_{L_p(R^3)} \right]^{q} \]

\[ \leq c \left( \sum_{k=0}^{\infty} \sum_{r=1}^{3} \sum_{l=0}^{\infty} 2^{(\delta + \varepsilon - L)[l-k]} l^{2+\sigma - 1/p - c_r} \right) \]

where \( \varepsilon > 0 \) is arbitrary small. Assuming that \( L > \delta + \varepsilon \) we obtain (4.37). This ends the proof. \( \Box \)

To prove (4.40) we need

**Lemma 4.5.** We have

\[ J_{mr} = \left( \int_{R_+} dx_3 \| \psi_1 (g_{mr} \partial_{x_3}^2 e_1 + h_{mr} \partial_{x_3}^0 e_0) \|_{L^2(R)} \right)^{1/p} \leq c 2^{(j-1/p - c_r)} \| \gamma - L \| \| l - k \| , \]

where \( c \) depends on \( \gamma \) for \( l = 0; c_r = 1 \) for \( r = 1, 2; c_r = 0 \) for \( r = 3; m = 1, 2, 3; L > 0 \) can be chosen sufficiently large.

**Proof.** Introduce the notation \( \partial_{\xi_1}^{s_i} = \partial_{\xi_1}^{s_i} \partial_{\xi_0}^{s_i} \), where \( s_i = |s_i|' + 2s_i \), \( s_i = (s_{i1}, s_{i2}) \). Then \( J_{mr} \) can be written as

\[ J_{mr} = \left( \int_{R_+} dx_3 \sum_{s_i} \| \partial_{\xi_1}^{s_i} \psi_l \|_{L^2(R^3)} \right)^{1/p} \]

\[ \cdot \varphi_k (2^l, 2^{2l}) \right) \]

where

\[ \partial_{\xi}^{s} = \partial_{\xi_1}^{s_1} \partial_{\xi_2}^{s_2}, \quad |\sigma| = s_1 + s_2. \]

Applying the Minkowski inequality we get

\[ J_{mr} \leq c \sum_{s_i} \| \partial_{\xi_1}^{s_i} \psi_l \|_{L^2(R^3)} \]

From the properties of \( \psi_l \) it follows that

\[ \text{supp } \psi_l (2^l, 2^{2l}) = \{ \xi : |\xi|_a \leq 4 \} \equiv A \quad \text{for } l = 0, \]

\[ \text{supp } \psi_l (2^l, 2^{2l}) = \{ \xi : 1/4 \leq |\xi|_a \leq 4 \} \equiv A \quad \text{for } l \neq 0. \]
Therefore, we obtain
\[
|\partial^{s_{1}}_{\xi} \psi_l(2^{l'}2^{2l})| \leq c \quad \text{for} \quad \xi \in A. \tag{4.42}
\]

Therefore, we obtain
\[
J^1_{mr} \leq c \sum_{s' + 2s \leq 4} (\|\partial^{s_{2}}_{\xi} g_{mr}(2^{l'}, 2^{2l'})
\cdot \|\partial^{s_{3}}_{\xi} \partial_{\xi} e_1\|_{L_p(\mathbb{R}^+)}(2^{l'}, 2^{2l'})(\partial^{s_{4}}_{\xi} \varphi_k)(2^{l'}, 2^{2l'})\|_{L_2(A)}
+ \|\partial^{s_{3}}_{\xi} h_{mr}(2^{l'}, 2^{2l'})\|_{L_2(A)} \|\partial^{s_{2}}_{\xi} \partial_{\xi} e_0\|_{L_p(\mathbb{R}^+)}(2^{l'}, 2^{2l'})
\cdot (\partial^{s_{4}}_{\xi} \varphi_k)(2^{l'}, 2^{2l'})\|_{L_2(A)}).
\tag{4.43}
\]

Recall that \(\tau^2 = \gamma + 2^{2l} \xi'^2 + 2^{2l_1} \xi_0, \gamma \leq 2^{2l} \). Then
\[
c_1 2^l \leq |\tau| \leq c_2 2^l \tag{4.44}
\]
and from Lemma 4.1 we have
\[
c_1 2^l \leq |\xi| \leq c_2 2^l
\]
for \(\xi \in A.\)

Then
\[
|\partial^{s_{2}}_{\xi} g_{mr}(2^{l'}, 2^{2l'})| \leq c, \quad r = 1, 2,
|\partial^{s_{2}}_{\xi} g_{m_3}(2^{l'}, 2^{2l'})| \leq c_2^l,
|\partial^{s_{2}}_{\xi} h_{mr}(2^{l'}, 2^{2l'})| \leq c_2^l, \quad r = 1, 2,
|\partial^{s_{2}}_{\xi} h_{m_3}(2^{l'}, 2^{2l'})| \leq c.
\tag{4.45}
\]

From (2.13) and (2.15) of Lemma 2.19 we have
\[
\|\partial^{j} j \partial^{s_{3}}_{\xi} \partial_{\xi} e_1\|_{L_p(\mathbb{R}^+)}(2^{l'}, 2^{2l'}) \leq c_2^{l(j/p - 1)},
\|\partial^{j} j \partial^{s_{3}}_{\xi} \partial_{\xi} e_0\|_{L_p(\mathbb{R}^+)}(2^{l'}, 2^{2l'}) \leq c_2^{l(j/p - 1)}.
\tag{4.46}
\]

Since \{\varphi_k\} \in A_{cl}(\mathbb{R}^4) \ (\text{see Definition 2.16}) we have
\[
\|\partial^{s_{4}}_{\xi} \varphi_k(2^{l'}, 2^{2l'})\|_{L_2(A)} = \|\partial^{s_{4}}_{\xi} \varphi_k(2^{k}2^{l(k-1)}, 2^{k}2^{2(l-k)})\|_{L_2(A)}
\leq c_2^{2l(l-k)}\|\partial^{s_{4}}_{\xi} \varphi_k(2^{k}, 2^{k})\|_{L_2(B)}
\leq c_2^{2(8-L)(l-k)},
\tag{4.47}
\]

where \(L\) is chosen sufficiently large,
\[
B = \{\bar{y} : 2^{l-k} - 2 \leq |\bar{y}| \leq 2^{l-k+2}\} \quad \text{for} \quad l \neq 0,
B = \{\bar{y} : |\bar{y}| \leq 2^{l-k+2}\} \quad \text{for} \quad l = 0
\]
and we have used the change of variables
\[
y_0 = 2^{2(l-k)} \xi_0, \quad y' = 2^{(l-k)} \xi, \quad \xi_0 = (\xi_1, \xi_2), \quad y' = (y_1, y_2).
\]

In view of the above estimates (4.41) holds. This ends the proof of Lemma 4.5. \(\square\)

Now, we shall derive the estimate for
\[
\|u\|_{2, B_{p,q}^{2+\gamma+1/r}(\mathbb{R}^4 \times \mathbb{R})} = \left[ \sum_{k=0}^{\infty} \left( \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} \frac{|F_{2}^{-1} \varphi_k F_{2} U_{F}(\bar{x}, x, z)|^{p}}{|z|^{1+p\gamma}} d\bar{x} dx_3 dz \right)^{q/p} \right]^{1/q},
\]
where \( U = \partial_{x_3}^2 u_\gamma (\bar{x}, x_3 + z) - \partial_{x_3}^2 u(\bar{x}, x_3) \).

**Lemma 4.6.** Let the assumptions of Theorem 4.3 be satisfied. Then

\[
\| u \|_{2, B^{2+\sigma, 1+\sigma/2}_{p, q, \gamma} (\mathbb{R}_1^2 \times \mathbb{R})} 
\leq c \left( \sum_{k=1}^{2} \| d_k \|_{B^{1+\sigma-1/p, (1+\sigma-1/p)/2}_{p, q, \gamma} (\mathbb{R}^2 \times \mathbb{R}_+)} + \| d_3 \|_{B^{2+\sigma-1/p, (2+\sigma-1/p)/2}_{p, q, \gamma} (\mathbb{R}^2 \times \mathbb{R})} \right).
\]

(4.48)

**Proof.** Since

\[
F_2 U_m = \sum_{r=1}^{3} (g_{mr} \partial_{x_3}^2 E_1 + h_{mr} \partial_{x_3}^2 E_0) \tilde{d}_r,
\]

where

\[
E_1 = e_1 (\bar{\xi}, x_3 + z) - e_1 (\xi, x_3),
\]

\[
E_0 = e_0 (\bar{\xi}, x_3 + z) - e_0 (\xi, x_3)
\]

we have

\[
\| u \|_{2, B^{2+\sigma, 1+\sigma/2}_{p, q, \gamma} (\mathbb{R}_1^2 \times \mathbb{R})} = \left[ \sum_{k=0}^{\infty} \sum_{m=1}^{3} \left( \int \int_{\mathbb{R}_1^2 \times \mathbb{R}_+} \left| F_2^{-1} \varphi_k \sum_{r=1}^{3} (g_{mr} \partial_{x_3}^2 E_1 + h_{mr} \partial_{x_3}^2 E_0) \tilde{d}_r \right|^p dx dz dt \right)^{q/p} \right]^{1/q}.
\]

(4.48)

where

\[
I_{2kmr\gamma} = \int \int_{\mathbb{R}_+ \times \mathbb{R}_+} \left| F_2^{-1} \varphi_k (g_{mr} \partial_{x_3}^2 E_1 + h_{mr} \partial_{x_3}^2 E_0) \tilde{d}_r \right|^p d\bar{x} d\bar{x}_3 dz,
\]

where \( \bar{x} = (x', t), x' = (x_1, x_2) \).

Introducing the same family of functions \( \{ \psi_k (\bar{x}) \} \) as in the proof of Lemma 4.4 we get

\[
I_{2kmr\gamma} = \left( \int \int_{\mathbb{R}_+ \times \mathbb{R}_+} \left| \sum_{l=0}^{\infty} \left| F_2^{-1} \psi_l (g_{mr} \partial_{x_3}^2 E_1 + h_{mr} \partial_{x_3}^2 E_0) \right| \right| d\bar{x} d\bar{x}_3 dz \right)^{1/p}.
\]

(4.50)
Using the formula

\[
F_2^{-1}(f F_2^{-1} g) = F_2^{-1}(F_2 F_2^{-1} f) F_2^{-1} g
\]

where \( f = \psi_l(g_{mr} \partial_{x_3}^2 E_1 + h_{mr} \partial_{x_3}^2 E_0) \), \( g = \varphi_k \) we obtain

\[
I_{2kmr} = \int_{R^3} \int_{R^3} \sum_{l=0}^{\infty} 2^{-4l} \int_{R^3} d\omega \frac{|F_2^{-1}\psi_l(g_{mr} \partial_{x_3}^2 E_1 + h_{mr} \partial_{x_3}^2 E_0)\ast F_2^{-1}\varphi_k(2^{-l} \omega, x, z)(F_2^{-1}\varphi_l F_2 d_{r\gamma})(\bar{x} - 2^{-l}\bar{\omega})|^p}{z^{1/p + \sigma}} d\bar{x} d\bar{z} d\bar{\omega}
\]

where we used the change of variables \( y_i = 2^{-l} \omega_i, \ i = 1, 2, \ y_0 = 2^{-2l} \omega_0 \) and the notation \( \bar{\omega} = (\omega', 2^{-l} \omega_0) \), \( \omega' = (\omega_1, \omega_2) \), \( \bar{\omega} = (\omega', \omega_0) \).

Next, formula (4.49) yields

\[
I_{2kmr} = \int_{R^3} \int_{R^3} \sum_{l=0}^{\infty} 2^{-4l} \int_{R^3} d\omega \cdot \{ |F_2^{-1}\psi_l(g_{mr} \partial_{x_3}^2 E_1 + h_{mr} \partial_{x_3}^2 E_0)\ast F_2^{-1}\varphi_k(2^{-l} \omega, x, z)(F_2^{-1}\varphi_l F_2 d_{r\gamma})(\bar{x} - \bar{\omega})|^p \}^{1/p}
\]

\[
\leq \left\{ \int_{R^3} \int_{R^3} \sum_{l=0}^{\infty} 2^{-4l} \int_{R^3} d\omega \{ |F_2^{-1}\psi_l(g_{mr} \partial_{x_3}^2 E_1 + h_{mr} \partial_{x_3}^2 E_0)| (2^{-l} \omega, x, z) \cdot (F_2^{-1}\varphi_k(2^{-l} \omega, x, z)) \cdot \left( \int_{R^3} |F_2^{-1}\varphi_l F_2 d_{r\gamma}(\bar{x} - \bar{\omega})|^p \right)^{1/p} \} \right\}^{1/p} \equiv I_{2kmr}^1,
\]

where we also used the Minkowski inequality.

Applying the change of variables \( \bar{\zeta} = \bar{x} - \bar{\omega} \) in the integral with respect to \( \bar{x} \) gives

\[
I_{2kmr}^1 = \left\{ \int_{R^3} \int_{R^3} \sum_{l=0}^{\infty} 2^{-4l} \int_{R^3} d\omega \{ |F_2^{-1}\psi_l(g_{mr} \partial_{x_3}^2 E_1 + h_{mr} \partial_{x_3}^2 E_0)| (2^{-l} \omega, x, z) \cdot (F_2^{-1}\varphi_k(2^{-l} \omega, x, z)) \cdot \left( \int_{R^3} |F_2^{-1}\varphi_l F_2 d_{r\gamma}(\bar{\zeta})|^p \right)^{1/p} \} \right\}^{1/p},
\]

where \( \bar{\zeta} = (\zeta', \zeta_0) \), \( \zeta' = (\zeta_1, \zeta_2) \).
Using the Hölder inequality in the integral with respect to $\tilde{\omega}$ and replacing $\tilde{\omega}$ by $\tilde{y}$ and $\zeta$ by $\tilde{x}$ we get

$$I_{2kmr\gamma}^1 \leq \left\{ \int_{\mathbb{R}_+^3} dx_3 \int_{\mathbb{R}_+^2} dz \left| \sum_{l=0}^{\infty} \left( \int d\tilde{y} \frac{1}{(1 + |\tilde{y}|^4)} \right)^{1/2} \right. \right.$$  

$$\cdot \left( \int d\tilde{y} \{ [F_2^{-1} \psi_l(g_{mr} \partial_{x_3}^2 E_1 + h_{mr} \partial_{x_3}^2 E_0)](2^l, 2^{2l}) \} \right)^{1/p}$$  

$$\cdot \| F_1^{-1} \varphi_l F_1^{-1} d_{r\gamma} \|_{L_p(\mathbb{R}^3)} \equiv I_{2kmr\gamma}^2.$$

Using that

$$\left( \int d\tilde{y} \frac{1}{(1 + |\tilde{y}|^4)} \right)^{1/2} \leq c$$

and applying the Parseval identity, we obtain

$$I_{2kmr\gamma}^2 \leq c \left( \int_{\mathbb{R}_+^3} dx_3 \int_{\mathbb{R}_+^2} dz \left| \sum_{l=0}^{\infty} \left[ \left\| \psi_l(g_{mr} \partial_{x_3}^2 E_1 \right. \right.$$  

$$\left. + h_{mr} \partial_{x_3}^2 E_0) \right\|_{W_{2}^{4,2}(\mathbb{R}^3)} \right)^{1/p} \right.$$  

$$\cdot \| F_1^{-1} \varphi_l F_1^{-1} d_{r\gamma} \|_{L_p(\mathbb{R}^3)} \equiv I_{2kmr\gamma}^3.$$

Lemma 4.7 below implies

$$I_{2kmr\gamma}^3 \leq c \sum_{l=0}^{\infty} 2^{(2+\sigma-1/p-c_r)q(8-L_1)|l-k|} \| F_1^{-1} \varphi_l F_1^{-1} d_{r\gamma} \|_{L_p(\mathbb{R}^3)},$$

where $c_r = 1$ for $r = 1, 2$, $c_r = 0$ for $r = 3$.

Hence,

$$\| u_m \|_{2, B_{p, q}^{2+\sigma, 1+\sigma/2}(\mathbb{R}_+^* \times \mathbb{R})} \leq c \left( \sum_{k=0}^{\infty} \sum_{r=1}^{\infty} \sum_{l=0}^{\infty} 2^{(2+\sigma-1/p-c_r)q(8+\varepsilon-L_1)|l-k|} \| F_1^{-1} \varphi_l F_1^{-1} d_{r\gamma} \|_{L_p(\mathbb{R}^3)} \right)^{1/q},$$

where $\varepsilon > 0$ is arbitrary small. Let $L_1 > 8 + \varepsilon$. Then (4.48) holds. This ends the proof of Lemma 4.6. $\square$

Now, we prove

**Lemma 4.7.** The following inequality holds

$$K_{mr} \equiv \left( \int_{\mathbb{R}_+^3} dx_3 \int_{\mathbb{R}_+^2} dz \left[ |\psi_l(g_{mr} \partial_{x_3}^2 E_1 + h_{mr} \partial_{x_3}^2 E_0) | \right.$$  

$$\cdot (2^l, 2^{2l}, x_3, z) \varphi_k(2^l, 2^{2l}) \|_{W_{2}^{4,2}(\mathbb{R}^2 \times \mathbb{R})} \right)^{1/p} \right.$$  

$$\leq c 2^{(2+\sigma-1/p-c_r)q(8-L_1)|l-k|},$$

where for $l = 0$ the constant $c$ depends on $\gamma$, $L_1$ can be chosen sufficiently large.
Proof. We can write $K_{mr}$ in the form

$$K_{mr} = \left( \int d x_3 \sum_{s_i = 1}^{s_i + 2s_i} \left\| (\partial_{\xi}^{s_i} \partial_{\xi_0}^{s_i} \psi_l)(2^l, 2^{2l}) \right\| \right) \cdot (\partial_{\xi}^{s_i} \partial_{\xi_0}^{s_i} g_{mr} \partial_{\xi}^{s_i} \partial_{\xi_0}^{s_i} \partial_{\xi_3} E_1 + \partial_{\xi}^{s_i} \partial_{\xi_0}^{s_i} h_{mr} \partial_{\xi}^{s_i} \partial_{\xi_0}^{s_i} \partial_{\xi_3} E_0)(2^l, x_3, 2^{2l}) \cdot (\partial_{\xi}^{s_i} \partial_{\xi_0}^{s_i} \varphi_k(2^l, 2^{2l})\|_{L^2(\mathbb{R}^3)}^{1/p}) \right).$$

Using the notation

$$\partial_{\xi}^{s_i} = \partial_{\xi}^{s_i}, \quad \bar{s}_i = |s_i| + 2s_i, \quad i = 1, 2, 3, 4$$

and the Minkowski inequality we obtain

$$K_{mr} \leq c \sum_{s_i = 1}^{s_i + 2s_i} \left[ \left\| (\partial_{\xi}^{s_i} \psi_l)(2^l, 2^{2l})(\partial_{\xi}^{s_i} g_{mr})(2^l, 2^{2l}) \right\| \right] \left\| \partial_{\xi}^{s_i} \partial_{\xi_3} E_1 \right\|_{L^p(\mathbb{R}^3)} \left\| \partial_{\xi}^{s_i} \partial_{\xi_0}^{s_i} \partial_{\xi_3} E_1 \right\|_{L^2(\mathbb{R}^3)} + \left\| (\partial_{\xi}^{s_i} \psi_l)(2^l, 2^{2l})(\partial_{\xi}^{s_i} h_{mr})(2^l, 2^{2l}) \right\| \left\| \partial_{\xi}^{s_i} \partial_{\xi_3} E_0 \right\|_{L^p(\mathbb{R}^3)} \left\| \partial_{\xi}^{s_i} \partial_{\xi_0}^{s_i} \partial_{\xi_3} E_0 \right\|_{L^2(\mathbb{R}^3)} \left\| \partial_{\xi}^{s_i} \partial_{\xi_0}^{s_i} \varphi_k(2^l, 2^{2l})\|_{L^2(\mathbb{R}^3)}^{1/p} \right).$$

Using that

$$\left\| \partial_{\xi}^{s_i} \psi_l(2^l, 2^{2l}) \right\| \leq c \quad \text{for } \bar{\xi} \in A,$$

$$\left\| \partial_{\xi}^{s_i} \partial_{\xi_3} E_1 \right\|_{L^p(\mathbb{R}^3)} \leq c 2^{l(\sigma - 1/p - 1)} \quad \text{for } \bar{\xi} \in A, \quad l \neq 0,$$

$$\left\| \partial_{\xi}^{s_i} \partial_{\xi_3} E_0 \right\|_{L^p(\mathbb{R}^3)} \leq c 2^{l(\sigma - 1/p)} \quad \text{for } \bar{\xi} \in A, \quad l \neq 0,$$

$$\sum_{i=0}^{1} \left\| \partial_{\xi}^{s_i} \partial_{\xi_0}^{s_i} \partial_{\xi_3} E_1 \right\|_{L^p(\mathbb{R}^3)} \leq c(\gamma) \quad \text{for } \bar{\xi} \in A, \quad l = 0,$$

where $A$ is defined in Lemma 4.5.

By the above estimates, inequality (4.50) follows. This ends the proof. \qed

The above lemmas imply Theorem 4.3.

In view of Theorem 4.3 and properties of the transformation from problem (4.1) to (4.17) we have.

Theorem 4.8. Let $p \in (1, \infty)$, $\sigma \in (0, 1)$. Assume that

$$f \in B_{p,p}^{\sigma, \sigma/2}(\mathbb{R}^3 \times \mathbb{R}^+), b_\alpha \in B_{p,p}^{1-\sigma/2-1/p, \alpha-1/p}(2 \times \mathbb{R}^3), b_\alpha \in B_{p,p}^{2+\sigma-1/p, \alpha-1/2-1/2p}(2 \times \mathbb{R}^3),$$

$$v_0 \in B_{p,p}^{2+\sigma-2/p, 1}(\mathbb{R}^3).$$

Then there exists a solution to problem (4.1) such that
\(v \in B_{p,p}^{2+\sigma,1+\sigma/2}(\mathbb{R}^3_+ \times \mathbb{R}_+)\), \(\nabla p \in B_{p,p}^{\sigma,\sigma/2}(\mathbb{R}^3_+ \times \mathbb{R}_+)\) and

\[
\|v\|_{B_{p,p}^{2+\sigma,1+\sigma/2}(\mathbb{R}^3_+ \times \mathbb{R}_+)} + \|\nabla p\|_{B_{p,p}^{\sigma,\sigma/2}(\mathbb{R}^3_+ \times \mathbb{R}_+)} \\
\leq \left(c(\|f\|_{B_{p,p}^{\sigma,\sigma/2}(\mathbb{R}^3_+ \times \mathbb{R}_+)} + \|g\|_{B_{p,p}^{1+\sigma,1/2+\sigma/2}(\mathbb{R}^3_+ \times \mathbb{R}_+)} + \sum_{\alpha=1}^{2}\|b_\alpha\|_{B_{p,p}^{1+\sigma-1/p,1+\sigma/2-1/2p}(\mathbb{R}^2 \times \mathbb{R}_+)}} + \|b_3\|_{B_{p,p}^{2+\sigma-1/p,1+\sigma/2-1/2p}(\mathbb{R}^3_+ \times \mathbb{R}_+)} + \|v_0\|_{B_{p,p}^{2+\sigma-2/p}(\mathbb{R}^3_+)} \right),
\]

where \(c\) does not depend on \(v\) neither on \(p\).

### 4.1. The Stokes System in Neighborhoods of Edges

Finally, we want to solve the Stokes system in a neighborhood \(\Omega(\xi)\) of a point \(\xi \in L_i, i = 1, 2\). Along the edge \(L_i, i = 1, 2\), \(S_2\) meets \(S_1\) under angle \(\pi/2\). Let \(\zeta\) be a smooth function from the partition of unity such that

\[
\text{supp} \zeta = \Omega(\xi).
\]

Introduce a local system of coordinates \((x_1, x_2, x_3)\) with origin at \(\xi\) such that \(S_2 \cap \Omega(\xi)\) is described by \(x_3 = 0\). Next we transform \(S_1 \cap \Omega(\xi)\) to the plane \(x_1 = 0\) by making an appropriate extension. Then \(L_i, i = 1, 2\), becomes a straight line \(x_1 = 0, x_3 = 0\) so it is the \(x_2\)-axis.

Therefore, the transformed Stokes system takes the form

\[
v_t - \nu \Delta v + \nabla p = f, \\
\text{div } v = 0
\]

in the dihedral angle \(\pi/2\) located between planes \(x_1 = 0\) and \(x_3 = 0\), denoted by \(D_{\pi/2} \times \mathbb{R}_+\). On the plane \(x_3 = 0\) we have the boundary conditions

\[
v_3 = b'_3, \\
v_{3,x_1} + v_{1,x_3} = b'_1, \\
v_{3,x_2} + v_{2,x_3} = b'_2
\]

and on \(x_1 = 0\) we have

\[
v_1 = b''_1, \\
v_{1,x_2} + v_{2,x_1} = b''_2, \\
v_{1,x_3} + v_{3,x_1} = b''_3
\]

Boundary conditions (4.54) can be expressed in the form

\[
v_1 = b''_1, \\
v_{2,x_1} = b''_2 - b'_{1,x_2}, \\
v_{3,x_1} = b''_3 - b'_{1,x_3}
\]

Hence (4.55)\(_1\) is the Dirichlet boundary condition and (4.55)\(_2,3\) are the Neumann boundary conditions.

We transform solutions to problem (4.52), (4.53), (4.54) in such a way that the boundary conditions (4.55) become homogeneous. Then the solutions are extended by reflection on \(x_1 < 0\). Thus we obtain problem (4.52), (4.53) in the half space \(x_3 > 0\).

Then Theorem 4.3 is also valid in this case.
5. The Stokes System in the Cylindrical Domain $\Omega$

We consider the following initial-boundary value problem for the Stokes system

$$
\begin{align*}
    v_t - \nu \Delta v + \nabla p &= f_0 & \text{in } \Omega \times (0, \tau), \\
    \text{div } v &= g_0 & \text{in } \Omega \times (0, \tau), \\
    \bar{n} \cdot \mathbb{D}(v) \cdot \bar{\tau}_\alpha &= b_{0\alpha}, & \alpha = 1, 2, \text{ on } S \times (0, \tau), \\
    v \cdot \bar{n} &= b_{03} & \text{on } S \times (0, \tau), \\
    v|_{t=0} &= v_0 & \text{in } \Omega,
\end{align*}
$$

(5.1)

where $S = S_1 \cup S_2$, $\bar{n}$ is the unit outward normal vector to $S$ and $\bar{\tau}_1$, $\bar{\tau}_2$ are tangent vectors to $S$.

Let $\bar{v}_0$ be the time extension of $v_0$ such that

$$
\bar{v}_0|_{t=0} = v_0
$$

(5.2)

In view of (4.5) we can introduce the new function

$$
u = v - \bar{v}_0
$$

(5.3)

and $(u, p)$ is a solution to the following initial-boundary value problem with vanishing initial data

$$
\begin{align*}
    u_t - \nu \Delta u + \nabla p &= f_0 - \bar{v}_0t + \nu \Delta \bar{v}_0 \equiv f & \text{in } \Omega \times (0, \tau) \equiv \Omega', \\
    \text{div } u &= g_0 - \text{div } \bar{v}_0 \equiv g & \text{in } \Omega \times (0, \tau) \equiv \Omega', \\
    B_\alpha(u) &\equiv \bar{n} \cdot \mathbb{D}(u) \cdot \bar{\tau}_\alpha = b_{0\alpha} - \bar{n} \cdot \mathbb{D}(\bar{v}_0) \cdot \bar{\tau}_\alpha \equiv b_\alpha, & \alpha = 1, 2, \\
    B_3(u) &\equiv u \cdot \bar{n} = b_{03} - \bar{v} \cdot \bar{n} \equiv b_3 & \text{on } S \times (0, \tau) \equiv S',
\end{align*}
$$

(5.4)

where $b_\alpha(u)$, $\alpha = 1, 2$, is the Neumann boundary condition and $B_3(u)$ the Dirichlet condition.

Using the technique of regularizer we prove

**Lemma 5.1.** Let $p \in (1, \infty)$, $\sigma \in \mathbb{N}$. Let $f \in W_p^{\sigma,\sigma/2}(\Omega')$, $g \in W_p^{1+\sigma,1/2+\sigma/2}(\Omega')$, $b_\alpha \in W_p^{1+\sigma-1/p,1/2+\sigma/2-1/2p}(S')$, $\alpha = 1, 2$, $b_3 \in W_p^{2+\sigma-1/p,1+\sigma/2-1/2p}(S')$.

Then there exists a solution to problem (5.4) such that $u \in W_p^{2+\sigma,1+\sigma/2}(\Omega')$, $\nabla p \in W_p^{\sigma,\sigma/2}(\Omega')$, where $\tau$ is sufficiently small, and there exists a constant $c$ independent of $u$, $p$ such that

$$
\begin{align*}
    \|u\|_{W_p^{2+\sigma,1+\sigma/2}(\Omega')} + \|\nabla p\|_{W_p^{\sigma,\sigma/2}(\Omega')}
    &\leq c \left( \|f\|_{W_p^{\sigma,\sigma/2}(\Omega')} + \|g\|_{W_p^{1+\sigma,1/2+\sigma/2}(\Omega')} + \sum_{\alpha=1}^2 \|b_\alpha\|_{W_p^{1+\sigma-1/p,1/2+\sigma/2-1/2p}(S')} \right. \\
    &\quad \left. + \|b_3\|_{W_p^{2+\sigma-1/p,1+\sigma/2-1/2p}(S')} \right).
\end{align*}
$$

(5.5)

**Proof.** To prove the lemma we use the partition of unity introduced in Definition 2.17. We introduce the simplified notation

$$
\begin{align*}
    L(\partial_x, \partial_t) &= \begin{pmatrix}
        \partial_t - \nu \Delta, \\
        \text{div}
    \end{pmatrix}, \\
    B(\partial_x) &= \begin{pmatrix}
        \bar{n} \cdot \mathbb{D}(\cdot) \bar{\tau}_1 \\
        \bar{n} \cdot \mathbb{D}(\cdot) \bar{\tau}_2 \\
        \bar{n}
    \end{pmatrix},
\end{align*}
$$

where

$$
L(\partial_x, \partial_t)(u, p) = [L(\partial_x, \partial_t)(u)]^T = \begin{pmatrix}
    \partial_t u - \nu \Delta u + \nabla p \\
    \text{div } u
\end{pmatrix}^T
$$
Let $\bar{\kappa}$ which solves the Cauchy problem with vanishing initial data

$$L(\partial_x, \partial_t)(u^{(k)}, p^{(k)}) = (f^{(k)}(x, t), g^{(k)}(x, t)).$$

In view of Theorem 3.3 there exists an operator $R^{(k)}$ such that $(u^{(k)}, p^{(k)}) = R^{(k)}(f^{(k)}, g^{(k)})$ and

$$\|R^{(k)}(f^{(k)}, g^{(k)})\|_{W_p^{2+\sigma, 1+\sigma/2}(\mathbb{R}^3 \times (0, \tau)) \times W_p^{\sigma, \sigma/2}(\mathbb{R}^3 \times (0, \tau))} \leq c(\|f^{(k)}\|_{W_p^{\sigma, \sigma/2}(\mathbb{R}^3 \times (0, \tau))} + \|g^{(k)}\|_{W_p^{1+\sigma, 1/2+\sigma/2}(\mathbb{R}^3 \times (0, \tau))}). \tag{5.6}$$

Let $k \in N_1$. Then $\text{supp} \zeta^{(k)}$ is a neighborhood of a point $\xi \in S_1$ located at a positive distance from edges $L_1, L_2$.

Then after a transformation $x = x(z)$ which makes flat the part of $S_1$ equal to $S_1 \cap \text{supp} \zeta^{(k)}$, we consider the problem

$$L(\partial_x, \partial_t)(u^{(k)}, p^{(k)}) = (f^{(k)}(z, t), g^{(k)}(z, t)), \quad B(\partial_x u^{(k)})|_{x_3=0} = b^{(k)}(z, t), \quad x_3 = 0. \tag{5.7}$$

Let $R^{(k)}$, $k \in N_1$, present a solution to problem (5.7) by

$$(u^{(k)}, p^{(k)}) = R^{(k)}(f^{(k)}, g^{(k)}, b^{(k)}), \quad k \in N_1$$

and

$$\|R^{(k)}(f^{(k)}, g^{(k)})\|_{W_p^{2+\sigma, 1+\sigma/2}(\mathbb{R}^3 \times (0, \tau)) \times W_p^{\sigma, \sigma/2}(\mathbb{R}^3 \times (0, \tau))} \leq c(\|f^{(k)}\|_{W_p^{\sigma, \sigma/2}(\mathbb{R}^3 \times (0, \tau))} + \sum_{\alpha=1}^{2} \|b^{(k)}_{\alpha}\|_{W_p^{1+\sigma-1/p, 1/2+\sigma/2-1/2p}(\mathbb{R}^2 \times (0, \tau))} \tag{5.8}$$

Let $k \in N_2$. Then $\text{supp} \zeta^{(k)}$ is a neighborhood of a point $\xi \in S_2$ located at a positive distance from edges $L_{\alpha}, \alpha = 1, 2$. Since $S_2$ is flat we do not need to pass to variables $z$. Therefore the considered problem in $\text{supp} \zeta^{(k)}$, $k \in N_2$, can be formulated in the original coordinates $x$. Hence it has the form

$$L(\partial_x, \partial_t)(u^{(k)}, p^{(k)}) = (f^{(k)}(x, t), g^{(k)}(x, t)), \quad B(\partial_x u^{(k)})|_{x_3=0} = b^{(k)}(x', t), \tag{5.9}$$

where $x' = (x_1, x_2)$.

Let $R^{(k)}$, $k \in N_2$, present a solution to problem (5.9). This has the form

$$(u^{(k)}, p^{(k)}) = R^{(k)}(f^{(k)}, g^{(k)}, b^{(k)}), \quad k \in N_2$$
and
\[
\|u^{(k)}\|_{W^{2+\sigma,1+\sigma/2}_p(\mathbb{R}^3_+\times(0,\tau))} + \|\nabla p^{(k)}\|_{W^{\sigma,\sigma/2}_p(\mathbb{R}^3_+\times(0,\tau))} = \|R^{(k)}(f^{(k)},g^{(k)},b^{(k)})\|_{W^{2+\sigma,1+\sigma/2}_p(\mathbb{R}^3_+\times(0,\tau))} + \|p^{(k)}\|_{W^{\sigma,\sigma/2}_p(\mathbb{R}^3_+\times(0,\tau))} \leq c \left( \|f^{(k)}\|_{W^{\sigma,\sigma/2}_p(\mathbb{R}^3_+\times(0,\tau))} + \|g^{(k)}\|_{W^{1+\sigma,1/2+\sigma/2}_p(\mathbb{R}^3_+\times(0,\tau))} + \|b^{(k)}\|_{W^{2+\sigma-1/p,1/2+\sigma/2-1/2p}_p(\mathbb{R}^2\times(0,\tau))} \right).
\]

(5.10)

For \( k \in \mathcal{N}_3 \), \( \text{supp} \zeta^{(k)} \) is a neighborhood of a point of \( L_\alpha \), \( \alpha \in \{1,2\} \). In order to consider the Stokes system in this neighborhood we have to use a transformation which makes \( S_1 \cap \text{supp} \zeta^{(k)} \) flat. Then \( L_\alpha \), \( \alpha \in \{1,2\} \), becomes a straight line. Therefore, the considered domain becomes the dihedral angle of the magnitude \( \pi/2 \), where the one side is the plane \( x_3 = 0 \) derived by extension of \( S_2 \cap \text{supp} \zeta^{(k)} \) and the other side is the plane \( x_2 = 0 \) which is an extension of the flatten part of \( S_1 \cap \text{supp} \zeta^{(k)} \).

Now we formulate the problem in the case \( k \in \mathcal{N}_3 \). On the plane \( x_3 = 0 \) we have the boundary conditions
\[
u^{(k)}_{3,\alpha} + u^{(k)}_{\alpha,x_3} = b^{(k)}_{\alpha}, \quad \alpha = 1,2, \tag{5.11}
\]
and on the plane \( x_2 = 0 \) we have
\[
u^{(k)}_{2,\alpha} + u^{(k)}_{\alpha,x_2} = \bar{b}^{(k)}_{\alpha}, \quad \alpha = 1,2, \tag{5.12}
\]
where \( b^{(k)}_{\alpha}, \alpha = 1,2, \) are derived from \( b_{\alpha}, \alpha = 1,2, \) from (5.4) restricted to \( S_1 \cap \text{supp} \zeta^{(k)} \) and transformed by transformation which makes \( S_1 \cap \text{supp} \zeta^{(k)} \) flat.

We simplify boundary conditions (5.12) to the form
\[
u^{(k)}_{\alpha,x_2} = \bar{b}^{(k)}_{\alpha}, \quad \alpha = 1,3, \quad \text{on} \ x_2 = 0, \tag{5.13}
\]
\[
u^{(k)}_2 = 0 \quad \text{on} \ x_2 = 0.
\]

Now, we perform a transformation which makes the Neumann boundary conditions homogeneous. Then we can extend \( u^{(k)}, p^{(k)} \) by the reflection with respect to the plane \( x_2 = 0 \). Formally, the problem has the same form as problem (5.9) for \( k \in \mathcal{N}_2 \).

Let \( R^{(k)} \) solves the problem. Then we have
\[
(u^{(k)}, \nabla p^{(k)}) = R^{(k)}(f^{(k)},g^{(k)},b^{(k)}|_{S_1},b^{(k)}|_{S_2})
\]
and the estimate
\[
\|u^{(k)}\|_{W^{2+\sigma,1+\sigma/2}_p(\mathbb{R}^3_+\times(0,\tau))} + \|\nabla p^{(k)}\|_{W^{\sigma,\sigma/2}_p(\mathbb{R}^3_+\times(0,\tau))} \leq c \left( \|f^{(k)}\|_{W^{\sigma,\sigma/2}_p(\mathbb{R}^3_+\times(0,\tau))} + \|g^{(k)}\|_{W^{1+\sigma,1/2+\sigma/2}_p(\mathbb{R}^3_+\times(0,\tau))} + \|b^{(k)}\|_{W^{2+\sigma-1/p,1/2+\sigma/2-1/2p}_p(\mathbb{R}^2\times(0,\tau))} \right) + \|b^{(k)}_{\alpha}\|_{S_2 \cup W^{2+\sigma-1/p,1+\sigma/2-1/2p}_p(\mathbb{R}^2\times(0,\tau))} \right).
\]

(5.14)

Consequently, we construct an operator \( R \) called the regularizer.
Let
\[ h^{(k)} = (f^{(k)}, g^{(k)}) \quad \text{for} \quad k \in \mathcal{M}, \]
\[ h^{(k)} = (f^{(k)}, g^{(k)}, b^{(k)}) \quad \text{for} \quad k \in \mathcal{N}_1, \]
\[ h^{(k)} = (f^{(k)}, g^{(k)}, b^{(k)}) \quad \text{for} \quad k \in \mathcal{N}_2, \]
\[ h^{(k)} = (f^{(k)}, g^{(k)}, b^{(k)}, b^{(k)}) \quad \text{for} \quad k \in \mathcal{N}_3, \]
where \( b^{(k)} \) is defined on \( S_1 \cap \text{supp} \zeta^{(k)} \times (0, \tau) \) and \( b^{(k)} \) on \( S_2 \cap \text{supp} \zeta^{(k)} \times (0, \tau) \).

Next, we introduce
\[ \| h \|_{S_p^\sigma} = \sum_{k \in \mathcal{M} \cup \mathcal{N}_1 \cup \mathcal{N}_2 \cup \mathcal{N}_3} \left( \| f^{(k)} \|_{W_p^{\sigma, \sigma/2}(\mathbb{R}^3_+ \times (0, \tau))} + \| g^{(k)} \|_{W_p^{1+\sigma, 1/2+2\sigma/2}(\mathbb{R}^3_+ \times (0, \tau))} \right) \]
\[ + \sum_{k \in \mathcal{N}_1 \cup \mathcal{N}_3} \left( \sum_{\alpha=1}^2 \| b^{(k)}_{\alpha} \|_{W_p^{1+\sigma-1/p, 1/2+\sigma/2-1/2p}(\mathbb{R}^2_+ \times (0, \tau))} \right) \]
\[ + \sum_{k \in \mathcal{N}_2 \cup \mathcal{N}_3} \left( \sum_{\alpha=1}^2 \| b^{(k)}_{\alpha} \|_{W_p^{1+\sigma-1/p, 1+\sigma/2-1/2p}(\mathbb{R}^2_+ \times (0, \tau))} \right) \]
\[ + \sum_{k \in \mathcal{N}_3} \left( \sum_{\alpha=1}^2 \| b^{(k)}_{\alpha} \|_{W_p^{1+\sigma-1/p, 1+\sigma/2-1/2p}(\mathbb{R}^2_+ \times (0, \tau))} \right). \]

Then we define an operator \( R \) by
\[ Rh = \sum_k \eta^{(k)}(u^{(k)}, \nabla p^{(k)}) \equiv \sum_k \eta^{(k)} w^{(k)}, \]
where
\[ w^{(k)} = (u^{(k)}, \nabla p^{(k)}) = \begin{cases} R^{(k)}(f^{(k)}, g^{(k)}) & k \in \mathcal{M}, \\ Z_k R^{(k)}(Z_k^{-1} f^{(k)}, Z_k^{-1} g^{(k)}, Z_k^{-1} b^{(k)}) & k \in \mathcal{N}_1, \\ R^{(k)}(f^{(k)}, g^{(k)}, b^{(k)}) & k \in \mathcal{N}_2, \\ R^{(k)}(f^{(k)}, g^{(k)}, b^{(k)}, b^{(k)}) & k \in \mathcal{N}_3, \end{cases} \]
where \( Z_k \) is a map from coordinates \( z \), which makes \( S_1 \cap \text{supp} \zeta^{(k)} \) flat, to coordinates \( x \) and
\[ \| Rh \|_{B^{\sigma+2}_p} = \sum_k \left( \| u^{(k)} \|_{W_p^{2+\sigma, 1+\sigma/2}(\mathbb{R}^3_+ \times (0, \tau))} + \| \nabla p^{(k)} \|_{W_p^{\sigma/2}(\mathbb{R}^3_+ \times (0, \tau))} \right). \]

Following ideas of [19], Ch.4, we infer the following result:

**Lemma 5.2.** The operator \( R: S_p^\sigma \to B^{\sigma+2}_p \) is a bounded operator and
\[ \| Rh \|_{B^{\sigma+2}_p} \leq c \| h \|_{S_p^\sigma}. \tag{5.15} \]

Let \( w = (u, \nabla p) \) and let the considered problem be denoted by \( A \). Then our aim is to show the existence of operators \( T \) and \( W \) such that
\[ ARh = h + Th, \]
\[ RAw = w + Ww. \]
Hence, to show the existence of solutions to problem (5.4) we have to prove that \( \|T\| < 1, \|W\| \leq 1 \).

This will be shown for sufficiently small \( \lambda \) and \( \tau \).

Now we construct operator \( T \). The operator can be divided into two parts: \( T = (T_1, T_2) \), where

\[
LRh = h + T_1h, \\
BRh = h + T_2h.
\]

First we describe \( T_1 \),

\[
T_1h = \sum_{k \in M} (L\eta^{(k)}w^{(k)} - \eta^{(k)}Lw^{(k)}) + \sum_{k \in N_1} [(L\eta^{(k)}w^{(k)} - \eta^{(k)}Lw^{(k)}) \nonumber \\
+ \eta^{(k)}Z_k(L(\partial_z - \nabla F_k\partial_z, \partial_t) - L(\partial_z, \partial_t))Z_k^{-1}R(k)h^{(k)}] \\
+ \sum_{k \in N_2} (L\eta^{(k)}w^{(k)} - \eta^{(k)}Lw^{(k)}) \\
+ \sum_{k \in N_3} (L \circ \Phi_1^{(k)}\eta^{(k)}w^{(k)} - \eta^{(k)}L \circ \Phi_1^{(k)}w^{(k)}),
\]

where \( \Phi_1^{(k)} \) is a map which is a composition of the following transformations:

1. A neighborhood \( \text{supp} \zeta^{(k)} \) of a point \( \xi^{(k)} \in L_i, i = 1, 2 \), is transformed to the \( \pi/2 \)-dihedral angle with edge \( L_i \) and sides \( S_1^{(k)} \) and \( S_2^{(k)} \), where \( L_i \) is a straight line passing through \( \xi^{(k)} \), \( S_i^{(k)}, i = 1, 2 \), are planes derived by extension of \( S_2 \cap \text{supp} \zeta^{(k)} \) and \( S_1 \cap \text{supp} \zeta^{(k)} \). We have to emphasize that \( S_1 \cap \text{supp} \zeta^{(k)} \) can be transformed locally to a plane.
2. Nonhomogeneous Neumann and Dirichlet problems on \( S_1^{(k)} \) are transformed to homogeneous.
3. Finally, the considered localized problem on \( \text{supp} \zeta^{(k)} \) is reflected with respect to plane \( \tilde{S}_1^{(k)} \).

Next we construct \( T_2 \),

\[
T_2h = \sum_{k \in N_1} \left\{ \sum_{\alpha=1}^{3} \left[ (B_{\alpha}\eta^{(k)}w^{(k)} - \eta^{(k)}B_{\alpha}w^{(k)}) \nonumber \\
+ \eta^{(k)}(B_{\alpha}(x, t, \partial_z) - B_{\alpha}(\xi^{(k)}, 0, \partial_z))w^{(k)}|_{S_1^{(k)}} \right] \right. \\
+ \sum_{\alpha=1}^{2} \eta^{(k)}Z_k(B_{\alpha}(\xi^{(k)}, 0, \partial_z - \nabla F_k\partial_z, \partial_t) - B_{\alpha}(\xi^{(k)}, 0, \partial_z))R(k)h^{(k)} \right\} \\
+ \sum_{k \in N_2} \sum_{\alpha=1}^{3} [(B_{\alpha}\eta^{(k)}w^{(k)} - \eta^{(k)}B_{\alpha}w^{(k)}) \nonumber \\
+ \eta^{(k)}(B_{\alpha}(x, t, \partial_z) - B_{\alpha}(\xi^{(k)}, 0, \partial_z))w^{(k)}|_{S_1^{(k)}}] \\
+ \sum_{k \in N_3} \sum_{\alpha=1}^{3} (B_{\alpha} \circ \Phi_2^{(k)}\eta^{(k)}w^{(k)} - \eta^{(k)}B_{\alpha} \circ \Phi_2^{(k)}w^{(k)}),
\]

where \( \Phi_2^{(k)} \) is equal to map \( \Phi_1^{(k)} \) restricted to the boundary \( S \).

For \( \tau \) and \( \lambda \) sufficiently small the norm \( \|T\|_{S_p^\tau \rightarrow S_p^\tau} \) is less than 1.
Then, we construct an operator $W$

$$W w = \sum_{k \in \mathcal{M}} \eta^{(k)} R^{(k)}(\zeta^{(k)} L w - L \zeta^{(k)} w)$$

$$+ \sum_{k \in \mathcal{N}_1} \sum_{a=1}^{3} \eta^{(k)} Z_k R^{(k)}[Z_k^{-1} (\zeta^{(k)} L w - L \zeta^{(k)} w),$$

$$Z_k^{-1} (\zeta^{(k)} B_\alpha w - B_\alpha \zeta^{(k)} w) |_{S_1^{(k)}}]$$

$$+ \sum_{k \in \mathcal{N}_2} \sum_{a=1}^{3} \eta^{(k)} Z_k R^{(k)}[(L(\zeta^{(k)}, \partial_z - \nabla F_1 \partial z) - L(\zeta^{(k)}, \partial_z)) Z_k^{-1} \zeta^{(k)} w,$$

$$(B_\alpha(\zeta^{(k)}, \partial_z - \nabla F_1 \partial z) - B_\alpha(\zeta^{(k)}, \partial_z)) Z_k^{-1} \zeta^{(k)} w |_{S_1^{(k)}}]$$

$$+ \sum_{k \in \mathcal{N}_3} \sum_{a=1}^{3} \eta^{(k)} R^{(k)}[\zeta^{(k)} L w - L \zeta^{(k)} w,\)$$

$$(\zeta^{(k)} B_\alpha w - B_\alpha \zeta^{(k)} w) |_{S_2^{(k)}}]$$

$$+ \sum_{k \in \mathcal{N}_2} \sum_{a=1}^{3} \eta^{(k)} R^{(k)}[(L(x, t, \partial_x) - L(\zeta^{(k)}, 0, \partial_x)) \zeta^{(k)} w,$$

$$(B_\alpha(x, \partial_x) - B_\alpha(\zeta^{(k)}, \partial_x)) \zeta^{(k)} w |_{S_2^{(k)}]}$$

$$+ \sum_{k \in \mathcal{N}_3} \sum_{a=1}^{3} \eta^{(k)} R^{(k)}[(\zeta^{(k)} L \circ \Phi_1^{(k)} w - L \circ \Phi_1^{(k)} \zeta^{(k)} w),$$

$$(\zeta^{(k)} B_\alpha \circ \Phi_2^{(k)} w - B_\alpha \circ \Phi_2^{(k)} \zeta^{(k)} w) |_{S_1^{(k)} \cup S_2^{(k)}]}],$$

where transformations $\Phi_1^{(k)}$ and $\Phi_2^{(k)}$ are defined above.

For $\tau$ and $\lambda$ sufficiently small the norm

$$\|W\|_{\mathcal{B}_p^{r+2} \to \mathcal{B}_p^{r+2}}$$

is less than 1. This ends the proof of Lemma 5.1.

Repeating the proof of Lemma 5.1 we have

**Lemma 5.3.** Let $p \in (1, \infty)$, $\sigma \in \mathbb{N}_0 \equiv \mathbb{N} \cup \{0\}$. Let the assumptions of Lemma 5.1 hold. Then there exists a solution to problem (5.4) such that $u \in W_p^{2+\sigma, 1+\sigma/2}(\Omega^r)$, $\nabla p \in W_p^{\sigma, \sigma/2}(\Omega^r)$, where $\tau$ is sufficiently small and estimate (5.5) holds.

### 6. Existence in Sobolev Spaces

**Theorem 6.1.** Assume that $f \in L_r(\Omega^T)$, $g \in W_r^{1,1/2}(\Omega^T)$, $b_\alpha \in W_r^{1-1/r,1/2-1/2r}(S^T)$, $\alpha = 1, 2$, $b_3 \in W_r^{2-1/r,1/2-1/2r}(S^T)$, $v_0 \in W_r^{2-2/r}(\Omega)$, $r \in (1, \infty)$, $S = S_1 \cup S_2$, $S_1 \subset C^2$, $L \subset C^2$. Then there exists a solution to problem (1.1) such that

$$v \in W_r^{2,1}(\Omega^T), \quad \nabla p \in L_r(\Omega^T)$$
and
\[
\|v\|_{W^{2,1}_r(\Omega)} + \|\nabla p\|_{L_r(\Omega)} \leq c \left( \|f\|_{L_r(\Omega)} + \|g\|_{W^{1,1/2}_1(\Omega)} + \sum_{\alpha=1}^{2} \|b_\alpha\|_{W^{1-1/r, 1/2-1/2r}_r(S_T^\alpha)} + \|b_3\|_{W^{2-1/r, 1-1/2r}_r(S_T)} + \|v_0\|_{W^{2-2/r}_r(\Omega)} \right) .
\]

(6.1)

**Proof.** Using the partition of unity we consider localized problems from (1.1) near an interior point of \( \Omega \), near \( S_1, S_2 \) and near edges \( L_1, L_2 \). Existence of solutions of these problems and estimates in \( W^{2,1}_r \times L_r \) can be shown. Finally, the technique of regularizer (see Sect. 5) ends the proof of the theorem. 

\( \square \)

**Data Availability Statement** The datasets generated during and/or analysed during the current study are available from the corresponding author on reasonable request.

**Declarations**

**Conflict of interest** On behalf of all authors, the corresponding author states that there is no conflict of interest.

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