Uniqueness for a hyperbolic inverse problem with angular control on the coefficients

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Abstract

Suppose \( q_i(x), \ i = 1,2 \) are smooth functions on \( \mathbb{R}^3 \) and \( U_i(x,t) \) the solutions of the initial value problem

\[
\begin{align*}
\partial_t^2 U_i - \Delta U_i - q_i(x)U_i &= \delta(x,t), \quad (x,t) \in \mathbb{R}^3 \times \mathbb{R} \\
U_i(x,t) &= 0, \quad \text{for } t < 0.
\end{align*}
\]

Pick \( R,T \) so that \( 0 < R < T \) and let \( C \) be the vertical cylinder \( \{(x,t) : |x| = R, \ R \leq t \leq T\} \). We show that if \( (U_1, U_{1r}) = (U_2, U_{2r}) \) on \( C \) then \( q_1 = q_2 \) on the annular region \( R \leq |x| \leq (R+T)/2 \) provided there is a \( \gamma > 0 \), independent of \( r \), so that

\[
\int_{|x|=r} |\Delta_S(q_1 - q_2)|^2 \, dS_x \leq \gamma \int_{|x|=r} |q_1 - q_2|^2 \, dS_x, \quad \forall r \in [R,(R+T)/2].
\]

Here \( \Delta_S \) is the spherical Laplacian on \( |x| = r \).

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1 Introduction

Our goal is the study of a formally determined inverse problem for a hyperbolic PDE. Consider an acoustic medium, occupying the region $\mathbb{R}^3$, excited by an impulsive point source and the response of the medium is measured for a certain time period at receivers placed on a sphere surrounding the source. We study the question of recovering the acoustic property of the medium from this measurement.

Let $q(x)$ be a smooth function on $\mathbb{R}^3$ and $U(x,t)$ the solution of the initial value problem
\begin{align*}
U_{tt} - \Delta U - q(x)U &= 8\pi\delta(x,t), \quad (x,t) \in \mathbb{R}^3 \times \mathbb{R}, \quad (1.1) \\
U &= 0, \quad t < 0. \quad (1.2)
\end{align*}

Using the progressing wave expansion one may show that
\begin{equation}
U(x,t) = 2\frac{\delta(t - |x|)}{|x|} + u(x,t)H(t - |x|), \quad (1.3)
\end{equation}
where $u(x,t)$ is the solution of the Goursat problem
\begin{align*}
&u_{tt} - \Delta u - q(x)u = 0, \quad (x,t) \in \mathbb{R}^3, \ t \geq |x|, \quad (1.4) \\
&u(x,|x|) = \int_0^1 q(\sigma x) d\sigma. \quad (1.5)
\end{align*}

The well posedness of the above Goursat problem is proved in [9] and improved in [11], though the result is not optimal; [9] has suggestions for obtaining better results and we will address them elsewhere. For completeness we restate the well posedness result.

**Theorem 1.1** (See [9] and [11]). Suppose $\rho > 0$, and $q$ is a $C^8$ function on the ball $|x| \leq \rho$; then (1.4), (1.5) has a unique $C^2$ solution on the double conical region \( \{(x,t) \in \mathbb{R}^3 \times \mathbb{R} : |x| \leq \rho, \ |x| \leq t \leq 2\rho - |x|\} \). Further, the $C^2$ norm of $u$, on this double conical region, approaches zero if the $C^8$ norm of $q$, on $|x| \leq \rho$, approaches zero. Also, if $q$ is smooth then so is $u$.

Below $P \preceq Q$ will mean that $P \leq CQ$ for some constant $C$. Let $S$ denote the unit sphere centered at the origin. For any $0 < R < T$, we define (see Figure 1) the annular region
\[ A := \{ x \in \mathbb{R}^3 : R \leq |x| \leq (R + T)/2 \}, \]
the space-time cylinder
\[ C = \{ (x,t) \in \mathbb{R}^3 \times \mathbb{R} : |x| = R, \ R \leq t \leq T \}, \]
and
\[ K := \{ (x,t) \in \mathbb{R}^3 \times \mathbb{R} : R \leq |x| \leq (R + T)/2, \ |x| \leq t \leq R + T - |x| \}, \]
a region bounded by $C$ and two light cones.
In our model the source is at the origin, the receivers are on the sphere $|x| = R$ and the signals are measured up to time $T$. Hence we define the forward map

$$F : q \mapsto (u|_C, u_r|_C)$$

and our goal is to study the injectivity and the inversion of $F$. From the domain of dependence property of solutions of hyperbolic PDEs, it is clear that $F(q)$ is unaffected by changes in $q$ in the region $|x| \geq (R + T)/2$. Hence the best we can hope to do is recover $q$ on the ball $|x| \leq (R + T)/2$.

If $q$ is spherically symmetric then the problem reduces to an inverse problem for the one dimensional wave equation. In this case, recovering $q$ on the region $R \leq |x| \leq (R + T)/2$, from $F(q)$, is done by the downward continuation method or the layer stripping method - see [16] and other references there. However, even in the spherically symmetric case (i.e. the one dimensional case), recovering $q$ on $|x| \leq R$, from $F(q)$ is more difficult since the downward continuation scheme is not directly applicable. It is believed that uniqueness does not hold for this inverse problem if $T < 3R$ though explicit examples have not been constructed. If $T \geq 3R$, the question of recovering $q$ on $|x| \leq R$ from $(u, u_r)|_C$ was resolved by connecting this problem to one where the downward continuation method is applicable - see [8] and the references there. So it seems that in the general $q$ case, recovering $q$ over the region $|x| \leq R$ will be harder than recovering $q$ over the region $R \leq |x| \leq (R + T)/2$.

Our main result concerns the problem of recovering $q$ on $R \leq |x| \leq (R + T)/2$ from $(u, u_r)|_C$. The downward continuation method does not apply directly in higher space dimensions since the time-like Cauchy problem for hyperbolic PDEs is ill-posed in higher space dimensions. Further, an analysis of the linearized problem shows that there could be singularities in $q$ in certain directions, that is points in the wave front set of $q$, so that a signal emanating from the origin is reflected by this singularity in $q$, and the reflected signal never reaches the sphere $|x| = R$ where the receivers are located - see Figure 2. Hence there should not be any stability for this inverse problem, unless we restrict $q$ to a class of functions where singularities in $q$ of the above type are controlled. In [13], Sacks and Symes adapted the downward continuation method to apply to a slightly different inverse
problem, with an impulsive planar source \( \delta(z - t) \), with data measured on the hypersurface \( z = 0 \), where \( x = (y, z) \) with \( y \in \mathbb{R}^2 \) and \( z \in \mathbb{R} \). They proved uniqueness for the linearized inverse problem when the unknown coefficient was restricted to the class of functions whose derivatives in the \( y \) direction were controlled by derivatives in the \( z \) direction. Later Romanov showed the inversion methods for one dimensional problems could be used for the existence and reconstruction for the nonlinear version of the Sacks and Symes inverse problem provided \( q(y, z) \) lies in the class of functions which are analytic in \( y \) in a certain sense, that is strong restrictions are placed on the changes in \( q \) in the \( y \) direction - see \cite{10} for details. We apply the technique in \cite{13} to the uniqueness question for the problem of recovering \( q \) on \( R \leq |x| \leq (R + T)/2 \) from \( (u, u_r)|_C \); we will have to impose restrictions on the angular derivatives of \( q \).

Figure 2: Reflection by a singularity in \( q \)

For any \( x \in \mathbb{R}^3 \) we define \( r = |x| \) and for \( x \neq 0 \) we define \( \theta = x/r \in S \); hence \( x = r\theta \). Define the radial vector field \( \partial_r = r^{-1}x \cdot \nabla \) and, for \( 1 \leq i < j \leq 3 \), the angular vector fields \( \Omega_{ij} = x_i \partial_j - x_j \partial_i \).

**Definition 1.2.** Given \( \gamma > 0 \), we define \( Q_\gamma(R, T) \) to be the set of all \( C^2 \) functions \( q(x) \) on the ball \( |x| \leq (R + T)/2 \) with

\[
\|p\|_{H^2(S_r)} + \|\partial_{r} p\|_{H^1(S_r)} \leq \gamma \( \|p\|_{H^1(S_r)} + \|\partial_r p\|_{L^2(S_r)} \) \quad \forall r \in [R, (R + T)/2]
\]

where \( p(x) = \int_0^{|x|} q(\sigma x/|x|) d\sigma \) and \( S_r \) is the sphere \( |x| = r \).

So if \( q \) is a smooth function on \( |x| \leq (R + T)/2 \) with \( \|p\|_{H^1(S_r)} + \|\partial_r p\|_{L^2(S_r)} \) nonzero for every \( r \in [R, (R + T)/2] \) then \( q \in Q_\gamma \) where

\[
\gamma = \max_{r \in [R, (R + T)/2]} \frac{\|p\|_{H^2(S_r)} + \|\partial_r p\|_{H^1(S_r)} }{\|p\|_{H^1(S_r)} + \|\partial_r p\|_{L^2(S_r)}}
\]

Noting that \( \partial_r p = q \), using Garding’s inequality on a sphere\(^1\), one may show that \( q \in Q_{\gamma^*} \) for some \( \gamma^* > 0 \) if there is a \( \gamma > 0 \) so that

\[
\|\Delta_S q\|_{L^2(S_r)} \leq \gamma \|q\|_{L^2(S_r)}, \quad \forall r \in [R, (R + T)/2]
\]

\(^1\)The Euclidean version is (6.8) on page 66 of \cite{4}. Using a partition of unity argument and the Euclidean version, one may show that \( \|q\|_{H^2(S_r)} \leq C_r \|\Delta_S q\|_{L^2(S_r)} \) with \( C_r \) bounded if \( r \) is in a closed interval not containing 0.
where $\Delta_S$ is the Laplacian on $S_r$. In particular, if $q$ is a finite linear combination of the spherical harmonics with coefficients dependent on $r$ then $q \in Q_\gamma(R,T)$ for some $\gamma > 0$.

In section 2 we prove the following injectivity result using the ideas in [13].

**Theorem 1.3.** Suppose $0 < R < T$ and $q_1, q_2$ are $C^8$ functions on $\mathbb{R}^3$. If $F(q_1) = F(q_2)$ and $q_1 - q_2 \in Q_\gamma(R,T)$ for some $\gamma > 0$ then $q_1 = q_2$ on $R \leq |x| \leq (R + T)/2$.

One may tackle the problem dealt with in Theorem 1.3 using Carleman estimates also and one obtains a result which is stronger in some aspects and weaker in others. Using Carleman estimates one can prove uniqueness under slightly less stringent conditions on $q$ - one needs controls on the $L^2$ norms of only the first order angular derivatives of $p$ in terms of the $L^2$ norm of $p$, instead of on the second order angular derivatives required in Theorem 1.3. However, the price one pays is that the $\gamma$ cannot be arbitrary but is determined by $R, T$; further $R$ cannot be arbitrary, but must satisfy $R > T/2$ and uniqueness is proved only for the values of $q$ in an annular region $R \leq |x| \leq R^*$ for some $R^* < (R + T)/2$. This work will appear elsewhere.

From Theorem 1.3 we can easily derive the following interesting corollary.

**Corollary 1.4.** Suppose $0 < T$ and $q_1, q_2$ are smooth functions on $\mathbb{R}^3$ which vanish in a neighborhood of the origin. If $u_1$ and $u_2$ agree to infinite order on the line $\{(x = 0, t) : 0 \leq t \leq T\}$ and $q_1 - q_2 \in Q_\gamma(0,T)$ for some $\gamma > 0$, then $q_1 = q_2$ on $|x| \leq T$.

We give a short proof of the corollary. If $q_1 = q_2 = 0$ in some small neighborhood of the origin then the difference $u = u_1 - u_2$ satisfies the standard homogeneous wave equation in a semi-cylindrical region

$$\{(x,t) \in \mathbb{R}^3 \times R : |x| \leq \delta, \ |x| \leq t \leq T - |x|\},$$

for some $\delta > 0$. Now, from the hypothesis, we have $u$ is zero to infinite order on the segment of the $t$ axis consisting of $0 \leq t \leq T$. Then by Lebeau’s unique continuation result in [5] we have $u = 0$ in the semi-cylindrical region given in (1.6). Hence $u$ and $u_r$ are zero on the cylinder

$$\{(x,t) \in \mathbb{R}^3 \times R : |x| = \delta, \ \delta \leq t \leq T - \delta\}.$$

The corollary follows from Theorem 1.3 if the $R$ and $T$ in Theorem 1.3 are taken to be $\delta$ and $T - \delta$ respectively.

We also have a uniqueness result for the linearized version of the inverse problem considered in Theorem 1.3; the result is for a linearization about a radial background.

**Theorem 1.5.** Suppose $q_0(r)$ is a function on $[0, \infty)$ so that $q_0(|x|)$ is a smooth function on $\mathbb{R}^3$; further suppose $u_0(r,t)$ is the solution of (1.4), (1.5) when $q(x)$ is replaced by $q_0(|x|)$. Let $q(x)$ be
a smooth function on $\mathbb{R}^3$ and $u(x,t)$ the solution of the Goursat problem

\[ u_{tt} - \Delta u - q_0 u = qu_t, \quad t \geq |x|, \tag{1.7} \]

\[ u(x, |x|) = \int_0^1 q(\sigma x) \, d\sigma. \tag{1.8} \]

If $(u, u_r)|_C = 0$ then $q = 0$ on the region $R \leq |x| \leq (R + T)/2$.

This theorem holds with less regular $q_b$ and $q$; what is needed is enough regularity so that the spherical harmonic expansions of $q$, $q_b$ and $u_b$ converge in the $C^2$ norm.

We next focus on the problem of recovering $q$ on the region $|x| \leq R$ from $(u, u_r)|_C$ when $T \geq 3R$. The linearized problem about the $q = 0$ background, consisting of recovering $q$ from $(u, u_r)|_C$, where $u(x,t)$ is the solution of the Goursat problem

\[ u_{tt} - \Box u = 0, \quad t \geq |x|, \]

\[ u(x, |x|) = \int_0^1 q(\sigma x) \, d\sigma. \]

As observed by Romanov, since $T \geq 3R$, we may recover $q$ from $(u, u_r)|_C$ fairly quickly. In fact, from Kirchhoff’s formula (see [2]) expressing the solution of the wave equation in terms of the Cauchy data on $C$, we have

\[ u(x, t) = \int_{|y-x|=R} \frac{u_r(y, t + |x-y|)}{|x-y|} + \left( \frac{u(y, t + |x-y|)}{|x-y|^2} + \frac{u_t(y, t + |x-y|)}{|x-y|} \right) \frac{(y-x) \cdot y}{|x-y|} \, dS_y, \]

for all $(x,t)$ with $|x| \leq t \leq R$ - see Figure 3. In particular we can express $u(x, |x|)$ in terms of $(u, u_t, u_r)|_C$ and hence we can recover $q$.

![Figure 3: Kirchhoff’s Formula](image)

For the original nonlinear inverse problem we show a partial uniqueness and stability result when one of the $q$ is small.
**Theorem 1.6.** Suppose $0 < 3R < T$, $M > 0$ and $q_i$, $i = 1, 2$ are $C^8$ functions on $|x| \leq (R + T)/2$ with $\|q_i\|_\infty \leq M$. Let $u_i$ be the unique solution of (1.4), (1.5) with $q$ replaced by $q_i$; then there is a constant $\delta > 0$ depending only on $R, T$ and $M$ so that if $\|q_2\|_\infty \leq \delta$ then

$$\int_{|x| \leq R} |q_1 - q_2|^2 \, dx \approx \int_C |u_1 - u_2|^2 + |\nabla (u_1 - u_2)|^2 + |(u_1 - u_2)_t|^2 \, dS_{x,t};$$

the constant in (1.9) depending only on $R, T, M$.

A weaker form of this result, requiring that $\|q_1\| \leq \delta$ also, was given in [7]; a result similar to this weaker version was also derived in [12]. Later it was observed in [6], for a similar type of problem, that the above proofs go through without the extra assumption that $\|q_1\| \leq \delta$. We give this short proof of Theorem 1.6, in section 4. However, the original nonlinear inverse problem remains unsolved.

2 Proof of Theorem 1.3

2.1 Preliminary observations

We need the following observations in the proof. For the angular vector fields we have $[\Omega_{ij}, \partial_r] = 0$, and $[\Omega_{ij}, \Omega_{kl}] = 0$ if $\{i, j\} = \{k, l\}$ but $[\Omega_{ij}, \Omega_{ik}] = \Omega_{kj}$. Also $|\nabla f|^2 = f^2_r + r^{-2} \sum_{i<j} (\Omega_{ij} f)^2$ and if we define $\Omega = \sum_{i<j} \Omega_{ij}^2$, then $\Delta = \partial_r^2 + 2r^{-1} \partial_r + r^{-2} \Omega$ and $[\Omega_{ij}, \Delta] = 0$. Also, for any $i \neq j$, since $\Omega_{ij} f = x_i \partial_j f - x_j \partial_i f = \partial_j (x_i f) - \partial_i (x_j f)$ and $x_j x_i - x_i x_j = 0$, by the divergence theorem, for any $0 < R_1 < R_2$ we have

$$\int_{R_1 \leq |x| \leq R_2} \Omega_{ij} f \, dx = 0. \quad (2.1)$$

Applying (2.1) to the zeroth order homogeneous extension of a function $f$ on $S$, we conclude that for $C^1$ functions $f, g$ on $S$

$$\int_S \Omega_{ij} f \, dS = 0, \quad \int_S f \Omega_{ij} g \, dS = -\int_S g \Omega_{ij} f \, dS. \quad (2.2)$$

For $i = 1, 2$ let $u_i$ be the solution of (1.4), (1.5) when $q = q_i$. Define $v_i(x, t) = ru_i(x, t)$, $p_i(x) = \int_0^1 q_i(\sigma x) \, d\sigma = \int_0^1 q_i(\sigma \theta) \, d\sigma$. Define $v = v_1 - v_2$, $q = q_1 - q_2$ and $p = p_1 - p_2$. Then we have

$$v_{tt} - v_{rr} - \frac{1}{r^2} \Omega v = q_i v = q v_2, \quad t \geq |x| \quad (2.3)$$
$$v(x, |x|) = p(x). \quad (2.4)$$
We are given that \((v, v_r)\) are zero on \(C\) and we have to show that \(q = 0\) on \(R \leq |x| \leq (R + T)/2\).

Note that since \(v = 0\) on \(C\), we have \(p(x) = v(x, |x|) = 0\) on \(|x| = R\) and hence for \(|x| \geq R\) we have \(p(x) = \int_{R}^{x} q(\sigma \theta) \, d\sigma\) and hence to prove the theorem it will be enough to show that \(p(x) = 0\) on \(R \leq |x| \leq (R + T)/2\).

We will attempt to carry out the proof which works in the one dimensional case. The limitations of this method when applied to the three dimensional case force the restrictions on \(q\) in the statement of Theorem 1.3. In the one dimensional case the angular terms are missing from (2.3) so the roles of \(r, t\) are reversible and one has sideways energy estimates which allow us to estimate the \(H^1\) norm of \(v\) on \(t = |x|\) in terms of the norm of \(v, v_r\) on \(r = R\) and the \(L^2\) norm of the RHS of (2.3). The \(H^1\) norm of \(v\) on \(t = |x|\) dominates the \(L^2\) norm of \(q\) on \(A\) and the \(L^2\) norm of the RHS of (2.3) is dominated by \(T - R\) times the \(L^2\) norm of \(q\) on \(A\). So if \(T - R\) is small enough we obtain \(q = 0\) on \(A\); then one combines a unique continuation argument with a repeated application of the above to prove that \(q = 0\) on \(A\) no matter what the \(T\).

In the multidimensional case the above argument breaks down because of the angular Laplacian in (2.3); all other parts of the argument work as in the one dimensional case. To carry out the above procedure we will need two estimates. The first is a standard energy estimate for the wave equation and the second is an imitation of a sideways energy estimate for a one dimensional wave equation in \(r, t\) where the roles of \(r\) and \(t\) are reversed.

### 2.2 Energy identities

For each \(\rho \in [R, (R + T)/2]\), define (see Figure) the sub-region

\[
K_\rho := \{(x, t) \in \mathbb{R}^3 \times \mathbb{R} : R \leq |x| \leq \rho, \, |x| \leq t \leq R + T - |x|\},
\]

the annular region

\[
A_\rho := \{x \in \mathbb{R}^3 : R \leq |x| \leq \rho\},
\]

the vertical cylinder

\[
C_\rho := \{(x, t) \in \mathbb{R}^3 \times \mathbb{R} : |x| = \rho, \, \rho \leq t \leq R + T - r\},
\]

and for any function \(w(x, t)\) let \(\bar{w}\) and \(\bar{\bar{w}}\) be the the restrictions of \(w\) to the lower and upper characteristic cones, that is

\[
\bar{w}(x) = w(x, |x|), \quad \bar{\bar{w}}(x) = w(x, R + T - |x|).
\]

We derive some relations which lead to the estimates we need. These relations are either the standard energy identity or a sideways version of it. Suppose \(w(x, t)\) satisfies

\[
w_{tt} - w_{rr} - \frac{1}{r^2} \Omega w = F(x, t), \quad (x, t) \in K.
\]
Define the “sideways” energy (we will assume a sum over $1 \leq i < j \leq 3$)

$$J(\rho) := \int_{C_\rho} r^{-2}(w^2 + w_r^2 + |\nabla w|^2) dS_{x,t} = \int_{C_\rho} r^{-2}(w_t^2 + w_r^2 + w^2 + r^{-2}(\Omega_{ij}w)^2) dS_{x,t}$$

$$= \int_{R+T-\rho} \int_{S} (w_t^2 + w_r^2 + w^2 + r^{-2}(\Omega_{ij}w)^2)(\rho \theta, t) d\theta dt.$$

Multiplying the identity

$$2w_r(w_{tt} - w_{rr} - r^{-2}\Omega w - w) - 4r^{-2}\Omega_{ij}w_r\Omega_{ij}w + 2r^{-3}(\Omega_{ij}w)^2$$

$$= -(w_t^2 + w_r^2 + r^{-2}(\Omega_{ij}w)^2 + w^2) + 2(w_r w_t)_t - 2\Omega_{ij}(r^{-2}w_r\Omega_{ij}w)$$  \hspace{1cm} (2.5)$$

by $r^{-2}$, integrating over the region $K_\rho$, using (2.2) and Stokes’s theorem on a region in the $r,t$
plane, we obtain
\[
\int_{K_\rho} r^{-2} \left(2w_r (F - w) - 4r^{-2} \Omega_{ij} \omega r \Omega_{ij} w + 2r^{-3} (\Omega_{ij} w)^2 \right) \, dx \, dt
\]
\[
= \int_S \int_{R} \int_{r}^{R+T-r} - (w_t^2 + w_r^2 + r^{-2} (\Omega_{ij} w)^2 + w^2)_r + 2(w_r w_t) dt \, dr \, d\theta
\]
\[
= \int_S \int_{R} \int_{r}^{R+T-\rho} (w_t^2 + w_r^2 + r^{-2} (\Omega_{ij} w)^2 + w^2)(R\theta, t) dt \, d\theta
\]
\[
- \int_S \int_{\rho} \int_{r}^{R+T-\rho} (w_t^2 + w_r^2 + r^{-2} (\Omega_{ij} w)^2 + w^2)(\rho \theta, t) dt \, d\theta
\]
\[
- \int_S \int_{R} \int_{r}^{R+T-r} \int_{R}^{R+T-r} (w_t^2 + w_r^2 + r^{-2} (\Omega_{ij} w)^2 + w^2 - 2w_r w_t)(r\theta, R + T - r) dr \, d\theta
\]
\[
- \int_S \int_{R} \int_{r}^{R+T-r} \int_{R}^{R+T-r} (w_t^2 + w_r^2 + r^{-2} (\Omega_{ij} w)^2 + w^2 + 2w_r w_t)(r\theta, r) dr \, d\theta
\]
\[
= J(R) - J(\rho) - \int_{A_\rho} r^{-2} (\bar{\omega}_t^2 + r^{-2}(\Omega_{ij} \bar{\omega})^2 + \bar{\omega}^2)(x) \, dx
\]
\[
- \int_{A_\rho} r^{-2} (\bar{\omega}_r^2 + r^{-2}(\Omega_{ij} \bar{\omega})^2 + \bar{\omega}^2)(x) \, dx.
\]
Hence
\[
J(\rho) + \int_{A_\rho} r^{-2}(|\nabla \bar{\omega}|^2 + \bar{\omega}^2)(x) \, dx + \int_{A_\rho} r^{-2}(|\nabla \bar{\omega}|^2 + \bar{\omega}^2)(x) \, dx + \int_{K_\rho} 2r^{-5} (\Omega_{ij} w)^2 \, dx \, dt
\]
\[
= J(R) + \int_{K_\rho} r^{-2} \left(2ww_r + 4r^{-2} \Omega_{ij} w_r \Omega_{ij} w - 2Fw_r \right) \, dx \, dt, \quad R \leq \rho \leq \frac{R+T}{2}. \quad (2.6)
\]
This is the sideways energy identity we need.

Next we derive the standard energy identity for the wave equation. For any \(s \in [R, T]\), define (see Figure 5) the domain
\[
K^s = K \cap \{ (x, t) \in \mathbb{R}^3 \times \mathbb{R} : R \leq t \leq s \},
\]

![Figure 5: Standard energy estimate](image-url)
$H^s$ the horizontal disk obtained by intersecting $K$ with the plane $t = s$, that is

$$H^s = K \cap \{t = s\},$$

whose projection onto the plane $t = 0$ is the annular region

$$A^s := \{x \in \mathbb{R}^3 : R \leq |x| \leq \min(s, R + T - s)\}.$$

Next, we define the “energy at time $s$” for every $s \in [R, T]$ - the definition depends on $s \leq (R + T)/2$ or not because the geometry changes - see Figure [5]. For $s \in [R, (R + T)/2]$, we define (summation over $1 \leq k < l \leq 3$)

$$E(s) := \int_{A^s} r^{-2}(w^2 + w_t^2 + w_r^2 + r^{-2}(\Omega_{kl}w)^2)(x, s) \, dx$$

and for $s \in [(R + T)/2, T]$ we define

$$E(s) := \int_{A^s} r^{-2}(w^2 + w_t^2 + |\nabla w|^2)(x, s) \, dx + \int_{R+T-s \leq |x| \leq (R+T)/2} r^{-2}(\bar{w}(x)^2 + |(\nabla \bar{w})(x)|^2) \, dx.$$

First take $s \leq (R + T)/2$; multiplying the identity

$$2w_t(w_{tt} - w_{rr} - r^{-2}\Omega w + w) = (w^2 + w_t^2 + w_r^2 + r^{-2}(\Omega_{kl}w)^2)_t - 2(w_t w_r)_r - 2r^{-2}\Omega_{kl}(w_t \Omega_{kl}w) \quad (2.7)$$

by $r^{-2}$, integrating over the region $K^s$, and using (2.1), we obtain

$$\int_{K^s} 2r^{-2}w_t(F + w) \, dx \, dt = E(s) + 2 \int_{R}^{s} \int_{|x| = R} r^{-2}(w_t w_r)(x, s) \, dS_x \, dt$$

$$- \int_{A^s} r^{-2}(w^2 + w_t^2 + w_r^2 + 2w_t w_r + r^{-2}(\Omega_{kl}w)^2)(x, |x|) \, dx$$

$$= E(s) + 2 \int_{R}^{s} \int_{|x| = R} r^{-2}(w_t w_r)(x, s) \, dS_x \, dt - \int_{A^s} r^{-2}(\bar{w}(x)^2 + |(\nabla \bar{w})(x)|^2) \, dx.$$

Next take $s \in [(R + T)/2, T]$; multiplying (2.7) by $r^{-2}$, integrating over the region $K^s$, using (2.1)
we obtain
\[ \int_{K^*} 2r^{-2}w_t(F + w) \, dx \, dt \]
\[ = \int_{H^*} r^{-2}(w^2 + w_t^2 + |\nabla w|^2) \, dx \]
\[ + \int_{R+T-s \leq |x| \leq (R+T)/2} r^{-2}(w^2 + w_t^2 + w_r^2 - 2w_tw_r + r^{-2}(\Omega_{kl}w)^2)(x, R + T - |x|) \, dx \]
\[ - \int_A r^{-2}(w_t^2 + w_r^2 + 2w_tw_r + r^{-2}(\Omega_{kl}w)^2)(x, |x|) \, dx + 2 \int_{R}^{R} \int_{|x|=R} r^{-2}(w_tw_r)(x, s) \, dS_x \, dt \]
\[ = \int_{H^*} r^{-2}(w^2 + w_t^2 + |\nabla w|^2) \, dx + \int_{R+T-s \leq |x| \leq (R+T)/2} r^{-2}(\bar{w}(x)^2 + |(\nabla \bar{w})(x)|^2) \, dx \]
\[ - \int_A r^{-2}(\bar{w}(x)^2 + |(\nabla \bar{w})(x)|^2) \, dx + 2 \int_{R}^{R} \int_{|x|=R} r^{-2}(w_tw_r)(x, t) \, dS_x \, dt \]
\[ = E(s) - \int_A r^{-2}(\bar{w}(x)^2 + |(\nabla \bar{w})(x)|^2) \, dx + 2 \int_{K^*} \int_{|x|=R} r^{-2}(w_tw_r)(x, t) \, dS_x \, dt. \]

Hence, in either case, that is for any \( s \in [R, T] \), we have
\[ E(s) \leq \int_A r^{-2}(\bar{w}(x)^2 + |(\nabla \bar{w})(x)|^2) \, dx + 2 \int_{K^*} r^{-2}w_t(F + w) \, dx \, dt + \int_{C} r^{-2}(w_t^2 + w_r^2) \, dS_x \, dt. \]

\( (2.8) \)

### 2.3 Uniqueness

We now show that if \( v \) and \( v_r \) are zero on \( C \) then \( q = 0 \) on \( A \). We apply (2.6) to \( v = v_1 - v_2 \); note that \( F = v_{tt} - v_{rr} - r^{-2}\Omega v = q_1 v + qv_2 \) and \( J(R) = 0 \) because the Cauchy data of \( v \) is zero on \( C \). Hence
\[ J(\rho) + \int_{A_\rho} r^{-2}(\bar{v}^2 + |\nabla \bar{v}|^2) \leq \int_{K_\rho} r^{-2}(\bar{v}v_1 + 4r^{-2}\Omega_{ij}v_1\Omega_{ij}v - 2v_r(q_1 v + qv_2) \]
\[ \leq \int_{K_\rho} r^{-2}(v_r^2 + v_t^2 + 2r^{-2}(\Omega_{ij}v)^2 + q^2 + r^{-2}(\Omega_{ij}v_r)^2) \]
\[ = \int_{R}^{R} J(r) \, dr + \int_{A_\rho} r^{-2}q^2(x) \left( \int_{R}^{R+T-r} \, dt \right) \, dx + \int_{K_\rho} r^{-4}(\Omega_{ij}v_r)^2 \]
\[ \leq \int_{R}^{R} J(r) \, dr + (T - R) \int_{A} r^{-2}q^2(x) \, dx + \int_{K} r^{-4}(\Omega_{ij}v_r)^2 \]

with the constant associated to \( \leq \) being \( c_1 = 4 \max(1, \|q_1\|_{L^\infty(A)}, \|v_2\|_{L^\infty(K)}) \). Hence, by Gronwall’s inequality
\[ J(\rho) + \int_{A_\rho} r^{-2}(|\nabla \bar{v}|^2 + p^2) \leq (T - R) \int_{A} r^{-2}q^2(x) \, dx + \int_{K} r^{-4}(\Omega_{ij}v_r)^2, \quad R \leq \rho \leq \frac{R+T}{2}, \]

\( (2.9) \)
with the constant being $c_2 = c_1 e^{c_1(T-R)}$. In particular
\[
J(\rho) \leq (T-R) \int_A r^{-2} q^2(x) \, dx + \int_K r^{-4}(\Omega_{ij} v_r)^2, \quad R \leq \rho \leq \frac{R+T}{2}, \tag{2.10}
\]
and taking $\rho = (R+T)/2$ in (2.9) we have
\[
\int_A r^{-2}(|\nabla p|^2 + p^2) \leq (T-R) \int_A r^{-2} q^2(x) \, dx + \int_K r^{-4}(\Omega_{ij} v_r)^2 \tag{2.11}
\]
with the constant $c_2$. Integrating (2.10) w.r.t $\rho$ over $[R, (R+T)/2]$ we obtain
\[
\int_K r^{-2}(v^2 + v_t^2 + |\nabla v|^2) \leq (T-R)^2 \int_A r^{-2} q^2(x) \, dx + (T-R) \int_K r^{-4}(\Omega_{ij} v_r)^2. \tag{2.12}
\]
So we can combine (2.11), (2.12) into
\[
\int_K r^{-2}(v^2 + v_t^2 + |\nabla v|^2) + \int_A r^{-2}(p^2 + |\nabla p|^2) \leq (T-R) \int_A r^{-2} q^2 + \int_K r^{-4}(\Omega_{ij} v_r)^2 \tag{2.13}
\]
with the constant being $c_3 = (1+T-R)c_2$.

The equation (2.13) would have been enough to prove Theorem 1 in the one dimensional case, because $|\nabla p|^2 \geq p_t^2 = q^2$ and the last term in (2.13) would not be there. Then by taking $T-R$ small enough we could have absorbed the second term on the RHS of (2.13) into the LHS and we would have proved the theorem for $T$ close to $R$. Then a unique continuation argument would prove the theorem for all $T > R$. However, in the three dimensional case we do have the last term in (2.13) which cannot be absorbed in the LHS because it involves second order derivatives of $v$ - we will estimate it in terms of $p$ using the standard energy estimate for the wave operator.

Fix an $i, j$ pair with $i < j$. We apply (2.8) to the function $w = \Omega_{ij} v$, noting that $\Omega_{ij}$ commutes with $\Omega$. Note that from (2.3) and (2.4) we have
\[
w_{tt} - w_{rr} - \frac{1}{r^2} \Omega w = F
\]
with
\[
F(x, t) := q_1 w + (\Omega_{ij} q_1) v + (\Omega_{ij} q) v_2 + q\Omega_{ij} v_2. \tag{2.14}
\]
and
\[
\bar{w}(x, |x|) = (\Omega_{ij} p)(x). \tag{2.15}
\]
Further, since the Cauchy data of $v$ is zero on $C$, so the Cauchy data of $w$ is zero on $C$. Hence from (2.8) we have
\[
E(s) \leq \int_A r^{-2}((\Omega_{ij} p)^2 + |\nabla \Omega_{ij} p|^2) + \int_K r^{-2}(w_t^2 + w_r^2 + F^2) \leq \int_A r^{-2}((\Omega_{ij} p)^2 + |\nabla \Omega_{ij} p|^2) + \int_K r^{-2}(w_t^2 + w_r^2 + v^2 + q^2 + (\Omega_{ij} q)^2) \leq \int_R E(t) \, dt + \int_A r^{-2}(p^2 + |\nabla p|^2 + |\nabla \Omega_{ij} p|^2) + \int_K r^{-2}v^2
\]

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with the constant being \( c_4 = 2 \max(1, (R + T)^2, \|q_1\|_\infty, \|\Omega_{ij}q_1\|_\infty, \|v_2\|_\infty) \). So from Gronwall’s inequality we have

\[
E(s) \leq \int_A r^{-2}(p^2 + |\nabla p|^2 + |\nabla \Omega_{ij}p|^2) + \int_K r^{-2}v^2, \quad R \leq s \leq T
\]

with the constant being \( c_5 = c_4 e^{c_4(T - R)} \). Integrating this w.r.t \( s \) over the interval \([R, T]\) we obtain

\[
\int_K r^{-2}(w^2 + w_t^2 + |\nabla w|^2) \leq c_5(T - R) \left( \int_A r^{-2}(p^2 + |\nabla p|^2 + |\nabla \Omega_{ij}p|^2) + \int_K r^{-2}v^2 \right);
\]

hence, since \( w = \Omega_{ij}v \),

\[
\int_K r^{-4}(\Omega_{ij}v_r)^2 \leq c_5R^{-2}(T - R) \left( \int_A r^{-2}(p^2 + |\nabla p|^2 + |\nabla \Omega_{ij}p|^2) + \int_K r^{-2}v^2 \right). \tag{2.16}
\]

Using this in (2.13), we have

\[
\int_K r^{-2}v^2 + \int_A r^{-2}(p^2 + |\nabla p|^2) \leq (T - R) \int_A r^{-2}(p^2 + |\nabla p|^2 + |\nabla \Omega_{ij}p|^2) + (T - R) \int_K r^{-2}v^2 \tag{2.17}
\]

with the constant \( c_6 = \max(c_3, c_3c_5R^{-2}) \). However, \( q \) is in \( Q_\gamma \) so

\[
\int_A r^{-2}|\nabla (\Omega_{ij}p)(x)|^2 dx = \int_R^{(R+T)/2} \int_{\theta=1}^{(R+T)/2} (\nabla \Omega_{ij}p)(r \theta)^2 d\theta dr
\]

\[
\leq \gamma \int_R^{(R+T)/2} r^2 \int_{\theta=1}^{(R+T)/2} (p^2 + |\nabla p|^2)(r \theta) d\theta dr
\]

\[
\leq \gamma(R + T)^2 \int_A r^{-2}(p^2 + |\nabla p|^2).
\]

Using this in (2.17), we see that \( p = 0 \) on \( A \) if \( T - R \) is small enough - depending on \( \gamma, c_6 \) and \( R + T \). Now \( v(x, |x|) = p(x) \) and \( v = 0 \) on \( |x| = R \) so \( p = 0 \) on \( |x| = R \), that is \( \int_0^R q(\sigma \theta) d\sigma = 0 \) for all unit vectors \( \theta \). Hence

\[
\int_R^r q(\sigma \theta) d\sigma = 0, \quad R \leq r \leq T
\]

which implies \( q(x) = 0 \) when \( R \leq |x| \leq T \), provided \( T - R \) is small enough.

Actually, adjusting the height of the downward pointing cone, what we have shown is the following: there is a \( \delta > 0 \) dependent only on \( \gamma, R, T, \|q_1\|_{C^1(A)}, \|v_2\|_{C^1(K)} \), so that if, for some \( R^* \in [R, (R + T)/2] \), \( v \) and \( v_r \) are zero on the cylinder

\[
\{(x, t) : |x| = R^*, \ R^* \leq t \leq R^* + 2\delta\},
\]

then \( q = 0 \) on \( R^* \leq |x| \leq R^* + \delta \), with the obvious modification in the assertion if \( R^* + \delta > (R + T)/2 \). We use this observation to prove that \( q = 0 \) for any \( R, T \).
Since \( v \) and \( v_r \) are zero on \( C \), then from the above claim, we have \( q = 0 \) on \( R \leq |x| \leq R + \delta \). Let \( u = u_1 - u_2 \) where \( u_1, u_2 \) are solutions to (1.4), (1.5) for \( q = q_1, q_2 \). Then, \( u \) satisfies the homogeneous equation
\[
 u_{tt} - \Delta u - q_1 u = 0
\]
over the region \( K_\rho \) where \( \rho = R + \delta \). Now \( u \) and \( u_r \) are zero on \( C \), and \( q_1 \) is independent of \( t \), so by the Robbiano-Tataru unique continuation theorem (see Theorem 3.16 in [KKL01]) we have \( u = 0 \) in the region \( K_\rho \); in particular \( u \) and \( u_r \) are zero on \( C_\rho \) and hence \( v, v_r \) are zero on \( C_\rho \). Now repeat the above argument, except \( R \) is replaced by \( R + \delta \); this argument repeated will complete the proof of Theorem 1.3.

3 Proof of Theorem 1.5

Let \( \{\phi_n(x)\}_{n=1}^\infty \) be a sequence of homogeneous harmonic polynomials on \( \mathbb{R}^3 \) so that their restrictions to the unit sphere \( S \) form an orthonormal basis on \( L^2(S) \) - see Chapter 4 of [15]. Let \( k(n) \) be the degree of homogeneity of \( \phi_n \). Then \( q(x) \) and \( u(x, t) \) have spherical harmonic decompositions in \( L^2(S) \) given by
\[
 q(r\theta) = \sum_{n=1}^\infty q_n(r)r^{k(n)}\phi_n(\theta), \quad u(r\theta, t) = \sum_{n=1}^\infty u_n(r, t)r^{k(n)}\phi_n(\theta)
\]
where
\[
 r^{k(n)}q_n(r) = \int_{|\theta|=1} q(r\theta) \phi_n(\theta) d\theta, \quad r^{k(n)}u_n(r, t) = \int_{|\theta|=1} u(r\theta, t) \phi_n(\theta) d\theta.
\]
Since \( u \) and \( q \) are smooth, we may show\(^2\) that \( q_n(r) \) and \( u_n(r, t) \) decay as \( n^{-p} \) for large \( n \) for any positive integer \( p \), uniformly in \( r, t \). Hence the series also converge in the \( C^2 \) norm.

To prove the theorem, it will be enough to prove that \( q_n(r) = 0 \) on \( R \leq r \leq (R + T)/2 \) for all \( n \geq 1 \). One may show that for sufficiently regular \( f \) (see page 1235 of [1])
\[
 \Delta \left( f(r, t)r^{k(n)}\phi_n(\theta) \right) = r^{k(n)}\phi_n(\theta)\left(f_{tt} - f_{rr} - \frac{2k(n) - 2}{r}f_r \right)
\]
hence, using (1.7), (1.8), the \( u_n(r, t) \) are solutions of the one dimensional Goursat problems
\[
 \partial_t^2 u_n - \partial_r^2 u_n + \frac{2k(n) - 2}{r} \partial_r u_n - q_n u_n = q_n u_b, \quad t \geq |r|, \quad u_n(r, |r|) = \int_0^{|r|} \sigma^{k(n)} q_n(\sigma) d\sigma.
\]
The hypothesis of the theorem implies that \( u_n(R, t) \) and \( (\partial_t u_n)(R, t) \) are zero for \( R \leq t \leq T \). So repeating the standard argument for one dimensional hyperbolic inverse problems with reflection data, as in [17], or repeating just the sideways energy argument in the proof of Theorem 1.3 without the complication of the angular terms, one may show that \( q_n(r) = 0 \) for \( R \leq r \leq (R + T)/2 \).

\(^2\)Use the definition of \( q_n \) and \( u_n \), observe that the \( \phi_n(\theta) \) are eigenvalues of the spherical Laplacian, and use the Divergence Theorem on \( S \) to transfer the Laplacian from the \( \phi_n \) to \( q \) or \( u \) - see Theorems 2 and 4 in [14].
4 Proof of Theorem 1.6

Let (see Figure 6) $B$ denote the origin centered ball of radius $R$ in $\mathbb{R}^3$, $D$ the region

$$D := \{(x,t) \in \mathbb{R}^3 \times \mathbb{R} : |x| \leq R, |x| \leq t \leq T\},$$

and as before $C$ the cylinder

$$C := \{(x,t) : |x| = R, R \leq t \leq T\}.$$

Let $u_i, i = 1, 2$ be the solutions of (1.4), (1.5) when $q = q_i$; define $q = q_1 - q_2$ and $u = u_1 - u_2$. Then $u$ satisfies

$$u_{tt} - \Delta u - q_1 u = qu_2, \quad (x,t) \in D \quad (4.1)$$

$$u(x,|x|) = \int_0^1 q(\sigma x) \, d\sigma. \quad (4.2)$$

Then, restricting attention to the cylindrical region $B \times [R, T]$, from [3] we have the following stability estimate for the time-like Cauchy problem (note $T > 3R$): there is a constant $C_1$ dependent only on $M,R,T$ so that

$$\|u(\cdot,t)\|_{H^1(B)}^2 + \|u_t(\cdot,t)\|_{L^2(B)}^2 \leq C_1 \left( \|q u_2\|_{L^2(B \times [R,T])}^2 + \|u\|_{H^1(C)}^2 + \|u_r\|_{L^2(C)}^2 \right), \quad R \leq t \leq T. \quad (4.3)$$

Next, if we multiply (4.1) by $u_t$ and use the techniques for standard energy estimates (backward in time) on the region $|x| \leq t \leq R$, we obtain

$$\int_B |\bar{u}(x)|^2 + |\nabla \bar{u}(x)|^2 \, dx \leq C_2 \left( \int_{|x| \leq R} |q u_2|^2 \, dx \, dt + \|u(\cdot,R)\|_{H^1(B)}^2 + \|u_t(\cdot,R)\|_{L^2(B)}^2 \right) \quad (4.4)$$

where $\bar{u}(x) = u(x,|x|)$ and $C_2$ depends only on $M,R$. Hence, combining (4.3), (4.4) we obtain

$$\int_B |\bar{u}(x)|^2 + |\nabla \bar{u}(x)|^2 \, dx \leq C_3 \left( \|q u_2\|_{L^2(D)}^2 + \|u\|_{H^1(C)}^2 + \|u_r\|_{L^2(C)}^2 \right) \quad (4.5)$$
where $C_3$ depends only on $R, T, M$. Now $r \bar{u}(x) = \int_0^r q(s\theta) ds$, hence $q(x) = (r\bar{u})_r = \bar{u} + r\bar{u}_r$. So
\[ q^2 \leq 2(\bar{u}^2 + r^2 \bar{u}_r^2) \leq 2 \max(1, R^2)(\bar{u}^2 + \bar{u}_r^2) \leq 2 \max(1, R^2)(\bar{u}^2 + |\nabla \bar{u}|^2), \]
and
\[ \|q\|_{L^2(B)}^2 \leq C_4 \left( \|qu_2\|_{L^2(D)}^2 + \|u\|_{H^1(C)}^2 + \|u_r\|_{L^2(C)}^2 \right) \tag{4.6} \]
with $C_4$ dependent only on $R, T, M$. Finally, using Theorem 1.1 we have
\[ \|qu_2\|_{L^2(D)} \leq \|u_2\|_{L^\infty(D)} \|q\|_{L^2(D)} \leq \mathcal{N}(T, \|q_2\|_\infty) \|q\|_{L^2(D)} \]
where the $\|q_2\|_\infty$ norm is over the region $|x| \leq (R+T)/2$. Since $\mathcal{N}(T, \|q_2\|_\infty)$ goes to zero as $\|q_2\|_\infty$ approaches 0, we can choose a $\delta > 0$ so that
\[ C_4 \mathcal{N}(T, \|q_2\|_\infty) < \frac{1}{2} \]
if $\|q_2\|_\infty \leq \delta$; note that this $\delta$ will depend only on $R, T, M$. Using this in (4.6), we conclude that if $\|q_2\|_\infty \leq \delta$ then
\[ \|q\|_{L^2(B)}^2 \leq C_5 \left( \|u\|_{H^1(C)}^2 + \|u_r\|_{L^2(C)}^2 \right) \tag{4.7} \]
with $C_5$ dependent only on $R, T, M$.

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