Tight bounds on the convergence of noisy random circuits to uniform

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Abstract

We study the properties of output distributions of noisy, random circuits. We obtain upper and lower bounds on the expected distance of the output distribution from the uniform distribution. These bounds are tight with respect to the dependence on circuit depth. Our proof techniques also allow us to make statements about the presence or absence of anticoncentration for both noisy and noiseless circuits. We uncover a number of interesting consequences for hardness proofs of sampling schemes that aim to show a quantum computational advantage over classical computation. Specifically, we discuss recent barrier results for depth-agnostic and/or noise-agnostic proof techniques. We show that in certain depth regimes, noise-agnostic proof techniques might still work in order to prove an often-conjectured claim in the literature on quantum computational advantage, contrary to what was thought prior to this work.

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1 Introduction

Noise is an unavoidable part of any quantum computing experiment today. The importance of considering the limitations of noisy quantum computers is most palpable in the coinage and the popularity of the term noisy, intermediate-scale quantum (NISQ) computers [Pre18]. Because of these limitations, the study of quantum algorithms and their robustness to noise is a problem at the forefront of quantum information science today. With regard to the aim of outperforming classical computers at solving computational problems, two obstacles are noise and limited system size. There is a tradeoff between the two, since it is generally challenging to both realize a quantum computation with large enough number of qubits and low enough error rates. This is the reason why recent demonstrations of “quantum computational supremacy” [Aru19; Zho+20; Zho+21; Wu+21] have been rightly hailed as exciting developments. It is of prime importance in the field of quantum information today to study the tradeoff between system size and noise from the viewpoint of whether a given experiment is indeed efficiently simulable on a classical computer or not.

Separately, in recent years, the study of random quantum circuits has seen renewed vigor, because of their ability to model chaotic [BF12; BF15] and complex [Bra+21; Haf+21] quantum dynamics and the ability to numerically study certain universal properties of simulable subclasses of random circuits [CNPS19; LCF18]. Indeed, random circuits and random ans"atze sometimes inform the modeling of near-term variational quantum algorithms, an example being the barren-plateau problem [McC+18; Wan+21b].

In this work, we study various properties of noisy random circuits relating to their rate of convergence to the uniform distribution. Specifically, we study spatially local circuits of depth \(d\) on \(n\) qubits with Haar-random gates and local Pauli noise. The precise rate of convergence to the identity is a question of much significance in the complexity of random circuit sampling [GD18; BFLL21], the theory of benchmarking noisy circuits [ABIN96; Aha00; EAZ05; Boi+18; BSN17; Liu+21], and the investigation of near-term algorithms [SG21; Wan+21b; Wan+21a]. We prove upper and lower bounds on the expected total variation distance \(\delta\) of the output distribution (when measuring in the computational basis) to the uniform distribution, which take the form \(\delta \sim \exp[-\Theta(d)]\). These bounds are tight with respect to the scaling with \(d\). We also study a property known as anticoncentration in noisy and noiseless random circuits. Anticoncentration is a measure of “flatness” of the output distribution, which is why our results on the closeness to the uniform distribution inform anticoncentration properties.

We first briefly describe our main results and their consequences.

- We prove a lower bound on the expected total variation distance of the output distribution from the uniform distribution for any general local noise, denoted \(\mathbb{E}_B[\delta]\) (Theorem 1). This takes the form \(\mathbb{E}_B[\delta] \geq \exp[-O(d)]\).
- We prove an upper bound on the above quantity for the case of a stochastic Pauli noise channel we call heralded dephasing (Theorem 2). The upper bound takes the form \(\mathbb{E}_B[\delta] \leq \text{poly}(n) \exp[-\Omega(d)]\).
- We also study anticoncentration properties of noisy and noiseless random circuits. We show that at sublogarithmic depth, there is a severe lack of anticoncentration, strengthening the results of Dalzell, Hunter-Jones, and Brandao [DHB20] (Theorem 3).
- We complement the above result by showing that noisy random circuits do anticoncentrate at higher depth, since they anticoncentrate at least as fast as noiseless random circuits (Theorem 4).
- As a result of independent interest, we develop a mapping between noisy random circuits and a model in statistical mechanics in the context of proving our results. This model builds upon tools invented in the study of random circuits [NRVH17; NVH18; Hun19; DHB20].

As a consequence of the lower bound result (Theorem 1), we disprove the hypothesis that the total variation distance to uniformity follows \(\mathbb{E}_B[\delta] \leq \exp[-\Theta(nd)]\) in the asymptotic limit. This scaling was numerically observed at small system sizes [BSN17; Boi+18]. The tightness of current techniques for proving average-case hardness of approximating output probabilities depends upon the validity of this hypothesis [BFLL21]. Thus, our result indicates that current techniques may be improved to allow for larger robustness of average-case hardness results.

Theorem 3 also helps understand a barrier result of Napp et al. [Nap+19] on the average-case easiness of computing output probabilities at constant depth. Our theorem shows that for shallow (sublogarithmic) depth,
a trivial algorithm to approximate output probabilities achieves additive error $\leq 2^{-n}$ with high probability. This recovers one of the results of Napp et al. and extends it from constant depth to any sublogarithmic depth.

Finally, we comment on the relation of our work to recently obtained results in Ref. [DHB21], where bounds on the rate of convergence to the uniform distribution for noisy random circuits were obtained. That work considers a low noise limit where the local probability of error in each circuit location (denoted as $\epsilon$) satisfies the scaling $\epsilon(n)n \rightarrow 0$ as the number of qubits $n$ tends to infinity. Under these conditions, they are able to recover the scaling $\mathbb{E}_U[\delta] \leq \exp[-\Theta(nd)]$ observed in the small-scale numerics mentioned above. In contrast, we consider a more physically natural scaling limit where the noise rate stays constant in the large $n$ limit. In this case, the analysis of Ref. [DHB21] breaks down. On the other hand, our bounds on the convergence rate to uniformity apply in the limit of vanishing noise rates, but they become uninformative. Thus, the two sets of results provide complementary and, apparently, largely non-overlapping insight into the behavior of noisy quantum circuits on finite systems.

2 Consequences

2.1 Barriers on proof techniques in the complexity theory of random circuit sampling

We now elaborate more on the connection between the hardness of sampling and the hardness of computing output probabilities for random quantum circuits. There has been an effort in the literature to prove, under a reasonable complexity assumption, that approximately sampling from the output distribution of random quantum circuits is classically hard. In order to achieve this task, it suffices to prove that approximating an output probability $p_{00...0} := |\langle 00...0 | U | 00...0 \rangle|^2$ of an $n$-qubit random quantum circuit $U$ is hard on average [AA13; BM16]. More specifically, proving that $p_{00...0}$ is $\#P$-hard to compute to within additive error $2^{-n}/\text{poly}(n)$ would give the desired claim that approximately sampling from the target distribution to within a small additive error is classically hard.

The state-of-the-art results [BFLL21; KMM21] on the average-case hardness of computing output probabilities of quantum circuits come close to proving the desired result in a certain sense. The “closeness” is measured in terms of the largest imprecision, or the additive error, to which computing the output probability $p_{00...0}$ of a random circuit is still hard on average. The state-of-the-art results prove that computing $p_{00...0}$ is hard to within a smaller imprecision of $2^{-\Theta(nd\log d)}$, almost matching the required imprecision of $2^{-n}/\text{poly}(n)$ when $d$ is a constant. These results improve upon prior results [BFNV19; Mov19] that proved hardness with imprecision $2^{-\text{poly}(n)}$. Such results are often viewed as evidence for the conjecture that $p_{00...0}$ is hard to compute on average to a much larger imprecision of $2^{-n}/\text{poly}(n)$.

2.1.1 Shallow depth random circuits

In Ref. [Nap+19], the authors gave important no-go results for proving the desired result, i.e. the average-case hardness of computing $p_{00...0}$ to an imprecision $2^{-n}/\text{poly}(n)$ for a specific class of constant-depth random circuits. Specifically, they showed that it is in fact classically easy to compute $p_{00...0}$ to within this much imprecision, even when previous techniques implied that computing $p_{00...0}$ to within a much smaller imprecision of $2^{-\text{poly}(n)}$ is average-case hard. Therefore, these results mean that one cannot, in general, view the hardness of computing $p_{00...0}$ to a smaller imprecision as evidence for the hardness of computing $p_{00...0}$ to a larger imprecision. In other words, these results constitute a barrier for any technique purporting to prove the desired average-case hardness result for general quantum circuits. Any such technique must necessarily be sensitive to the depth of the circuit, otherwise it would work for constant-depth circuits of the sort studied in Ref. [Nap+19] and contradict their easiness results. Barrier results such as this are useful because they rule out certain proof techniques and guide the search for a proof technique resistant to these barriers. In this case, the barrier result informs us about depth-sensitivity of a proof technique.

Our result, specifically Theorem 3, casts more light on this barrier result. The theorem states that for any sublogarithmic depth $d = o(\log n)$, most output probabilities are at most $2^{-n} \times 2^{-\Theta(n)} = o(2^{-n})$. This indicates a severe lack of anticoncentration, a property cited as crucial for output distributions to be classically hard to simulate [AA13; HM17; HM18] in the literature on quantum advantage via sampling tasks. Because of Theorem 3, a trivial algorithm that always guesses “0” as the output probability turns out to work to within imprecision $\leq 2^{-n}$ with high probability, as desired. Therefore, we conclude that any technique to show the conjecture on average-case hardness to within imprecision $2^{-\Theta(n)}$ must not work at depth $d = o(\log n)$. Since we show that a trivial algorithm suffices to exhibit a barrier in this case, one may ask whether it suffices for a proof technique to only
beats such trivial algorithms. If true, this would greatly simplify the search for proof techniques that are resistant to these barriers.

2.1.2 Noisy random circuits

A second barrier was identified by Bouland et al. [BFLL21] concerning the issue of noise. The authors showed that existing hardness proof techniques were applicable to noisy random circuits as well, to yield hardness of computing \( p_{00...0} \) to small imprecision. Contrastingly, for the slightly larger imprecision of \( 2^{-n} / \text{poly}(n) \), it is known to be easy to compute output probabilities since the output distribution in the presence of noise is believed to converge rapidly to the uniform distribution [ABIN96; Boi+18; BSN17]. More specifically, it is known that for local depolarizing noise of strength \( \gamma \), the total variation distance between the output distribution \( D \) and the uniform distribution \( U \) obeys \( \| D - U \|_{\text{TVD}} \leq 2^{-\Theta(\gamma d)} \) for a depth \( d \) circuit [ABIN96]. In fact, it is surmised that the actual rate of convergence is faster and given by \( 2^{-\Theta(\gamma nd)} \) [Boi+18; BSN17]. Assuming this faster rate of convergence, the quantity \( 1/2^n \) is close enough to \( p_{00...0} : |p_{00...0} - 1/2^n| \leq 2^{-\Theta(n)} \) and a trivial algorithm, namely one that always guesses \( "1/2^n" \), successfully computes \( p_{00...0} \) to within the required imprecision. Therefore, the results of Bouland et al. exhibit another barrier for hardness proof techniques purporting to work with higher imprecision—these techniques must distinguish between noiseless and noisy random circuits.

Bouland et al. also showed, using their results on noisy circuits, that existing noise-agnostic techniques for proving average-case hardness are almost tight. The logic is that current techniques prove average-case hardness for imprecision \( 2^{-\Omega(nd \log nd)} \) or smaller, whereas the hypothesized rate of convergence of noisy circuits to uniform would mean that it is average-case easy to approximate output probabilities to within imprecision \( 2^{-O(nd)} \) or larger.

Our result, namely Theorem 1, shows that the faster rate of convergence \( 2^{-O(nd)} \) does not hold in the asymptotic limit. This implies that the above argument on tightness of current techniques needs to be refined. Since we show that the distance to uniform behaves as \( 2^{-\Theta(d)} \), there is some scope for improving current noise-agnostic techniques in the regime of depth \( d = o(n) \). However, at larger depths \( d \geq \Omega(n) \), the barrier result identified by Bouland et al. remains.

2.1.3 Summary of barrier results

Owing to this work, we have now identified a map of parameter regimes where a quantum advantage may be obtained over classical computers and whether depth- and noise-agnostic techniques can be used to prove the conjecture on average-case hardness of computing output probabilities to within imprecision at most \( O(2^{-n}) \). The limitations are explored via two trivial algorithms for computing output probabilities—one always outputting \( 1/2^n \) and the other always outputting 0.

On account of the convergence to uniformity, if the depth scales larger than \( an \) for some constant \( a \), the first trivial algorithm of outputting \( 1/2^n \) works well to imprecision \( 2^{-\Omega(d)} \) and a noise-agnostic technique will not work to prove the conjecture. Therefore, the depth must scale at most as \( bn \) for any constant \( b < a \). Moreover, we have also established that the depth cannot be sublogarithmic since otherwise there is a severe lack of anticoncentration, leading to the second trivial algorithm of outputting 0 working well with high probability. Therefore, noise-agnostic techniques can only work in the regime \( d = \Omega(\log n), d = O(n) \). In fact, in this regime, the possibility of being able to prove hardness results for both noisy and noiseless circuits to imprecision \( 2^{-O(n)} \) is not ruled out (see Fig. 1).

2.2 Benchmarking noise using random circuits

Sampling from random quantum circuits is a leading proposal for demonstrating a quantum computational advantage over classical computers [Aru19; Boi+18; AC17; BFNV19; AG20]. Part of the reason behind the strength of this proposal is the success of the linear cross-entropy measure as a predictor of fidelity [Boi+18; Liu+21; Cho+21]. This is believed to be a major advantage of RCS-based schemes over schemes based on other quantum sampling problems such as boson sampling and Gaussian boson sampling. In fact, benchmarking with random circuit sampling (RCS) can be more scalable to experimentally perform than randomized benchmarking [Liu+21]. The effectiveness of measures such as cross-entropy at reflecting the fidelity is crucially related to how the noisy distribution behaves and how close it is to the identity. Since we give tight results on the scaling of the total variation distance of the output distribution from the uniform distribution with respect to the circuit depth, our results justify the applicability of RCS-based benchmarking.
Figure 1: Status of hardness of approximating random output probabilities to within $2^{-\Theta(n)}$ imprecision, (a) before and (b) after our work, for noisy and noiseless circuits at various depths. The shaded regions correspond to the regions where the problem is known to be easy. The dark blue arrows imply that a technique to show hardness via a noise-agnostic or depth-agnostic technique will fail because of the easy region, while the green arrow indicates that such a technique is not ruled out. The works next to the arrow refer to the works that discuss the presence or absence of a barrier. In (a), the entire shaded region for noisy circuits follows from the assumption that noisy circuits converge to uniform at a rate $2^{-\Theta(nd)}$. Since we disprove this possibility in Theorem 1, this region is smaller in (b) and allows for a noise-agnostic technique in the regime of $\Theta(\log n) \leq d \leq \Theta(n)$. At large depths, noisy circuits continue to be easy due to their convergence to uniform, as strengthened by Theorem 2. We also extend the easy region at shallow depths from constant to $o(\log n)$ by virtue of Theorem 3. Finally, Theorem 4 implies that the trivial algorithm of outputting $0$ for noisy circuits stops working after depth $\Omega(\log n)$.

2.3 Near-term algorithms

Our results also have important consequences for near-term variational quantum algorithms that can be modeled by random circuits. Variational algorithms with parametrized circuits suffer from the problem of “barren plateaus”, which affects trainability of the circuit due to the vanishing of gradients in the cost function landscape. Previous work on noise-induced barren plateaus [Wan+21b] has shown that the gradient of local cost functions vanishes as $\text{poly}(n) \exp[-\Omega(d)]$ in the presence of noise. One possible workaround to avoid the barren-plateau problem would be to use cost functions that are not (sums of) local observables and instead rely on postprocessing the entire data at the output distribution. However, since we give an information-theoretic proof that the output distribution is close to the uniform one, our results (specifically Theorem 2) imply that even these strategies cannot ameliorate the problem.

More optimistically, at short depths $d = O(\log n)$, Theorem 1 implies that there is enough information content in the output distribution for circuit trainability. We thus avoid a pessimistic conclusion that would have followed from the stronger hypothesis on the convergence to uniformity.

3 Prior work

The foundational work by Aharonov et al. [ABIN96] dealt with the convergence of arbitrary circuits undergoing depolarizing noise to the maximally mixed state. They showed that the entropy of the system reaches its maximal value exponentially fast with $d$. They concluded that noisy circuits (without error correction) are essentially “worthless” after logarithmic depth $d = \Omega(\log n)$. Translated to our setting, their proof techniques imply that the variation distance to uniform $\delta$ satisfies $\delta \leq 2^{-\Omega(d)}$. Depolarizing noise is also studied in more detail in [MSW16; HRF20]. Ben-Or, Gottesman, and Hassidim [BGH13] generalized the work of Aharonov et al. to other forms of Pauli noise and observed that the time after which noisy circuits are worthless depends on the class of channels.

More recently, Gao and Duan [GD18] studied the case of random circuits with a more generalized form of Pauli noise and showed an upper bound on the distance to the uniform distribution of the form $\mathbb{E}[\delta] = \exp[-\Omega(d)]$. This result, however, does not work for the case of dephasing noise. In fact, the upper bound for dephasing noise
is a constant independent of $n$ and $d$, which is not informative. Moreover, the result explicitly assumes a property of random quantum circuits known as anticoncentration.

Note that for dephasing noise, it is not possible to prove a general upper bound on $\delta$ that works for arbitrary circuits. This is because a state in the computational basis is unaffected by dephasing, and hence there are instances where the state is always in the computational basis throughout the evolution and the output distribution is unaffected by the noise channels.

Prior work on anticoncentration has mostly been via second-moment bounds on the output probabilities \cite{BMS16; HM18; HBSE18; DHB20}, which can be proved via the design property of random circuits \cite{HL09; BHH16; HM18}. An exception is the case of the one-clean qubit model (DQC1) \cite{Mor17}. A tool to analyze random circuits that has gained popularity of late is that of mapping to models in statistical mechanics. This has been successfully used in prior work; see for example, \cite{NRVH17; NVH18; Hun19; DHB20}, among others.

## 4 Definitions

We define a depth-$d$ noisy circuit on $n$ qubits as a sequence of quantum channels

$$
\mathcal{N}_d(\rho) = (\mathcal{E}_d \circ \mathcal{C}_d \cdots \mathcal{E}_1 \circ \mathcal{C}_1)(\rho),
$$

where $\mathcal{C}_m(\rho) = U_m \rho U_m^\dagger$ is a unitary operation on $n$ qubits and $\mathcal{E}_m(\rho) = \sum_{E \in \mathcal{P}} p_m(E) E \rho E^\dagger$ is a Pauli error channel, i.e., $\mathcal{P}$ is the Pauli group on $n$ qubits and $p_m(E)$ is a probability distribution over the Pauli group elements. A noisy Haar random circuit is a noisy circuit for which each $U_m$ is decomposable into tensor products of two-qubit Haar random gates. The local noise rate in Pauli sector $\mu$ at depth $m$ on site $i$ is the marginal distribution $q_{\mu i} = \sum_{E \in \mathcal{P}, E_i = \mu} p_m(E)$.

We consider circuit architectures where, for simplicity, the gates are applied in parallel. More formally, we define a parallel architecture as one for which $n$ is an even number and every qubit is involved in a two-qubit gate at every unit of depth. Our results can be easily extended to more general gate layering strategies with a suitable redefinition of the depth. For a given site $i$, we define $n_m(i)$ as the set of neighbors of $i$, which are the sites involved in a two-qubit gate at circuit layer $m$ with $i$ including $i$ itself. We extend $n_m$ to all subsets of sites $A \subset \{1, \ldots, n\}$ through the composition rule $n_m(A \cup B) = n_m(A) \cup n_m(B)$. We define the lightcone $L_d(i)$ of site $i$ at depth $d$ as the set

$$
L_d(i) = n_d \circ \cdots \circ n_1(i).
$$

In general, we have the bound $|L_d(i)| \leq 2^d$.

We denote the above ensemble of noisy brickwork circuits $\mathcal{B}$, and our results are generally stated in terms of expectation values over this ensemble, denoted $E_{\mathcal{B}}$. We also denote $\delta = \|D - U\|_{\text{TVD}}$, where $U$ is the uniform distribution over $n$-bit strings and “TVD” denotes total variation distance.

## 5 Results

### 5.1 Lower bound on the distance to the uniform distribution

**Theorem 1.** For a noisy Haar-random circuit on any parallel circuit architecture with a uniform upper bound on the local noise rate $q_{\mu i} \leq q_\mu$ for all Pauli noise sectors $\mu$, $m$ and $i$, we have the lower bound

$$
E_{\mathcal{B}}[\delta] \geq \frac{(1 - 2b)^{2d}}{4 \cdot 30^d},
$$

where $b = \min[q_x + q_y, q_z, q_x + q_z]$.

**Proof.** Let $p_x$ denote the probability of observing the bitstring $x$ at the output. Given a region $A \subseteq \{1, 2, \ldots, n\}$ and its complement $\tilde{A}$ with $n_A$ and $n - n_A$ qubits, respectively, we make use of the identity $p_x = \sum c_{x_A}^d c_{x_{\tilde{A}}}^d$ for some set of complex numbers $c_{x_A}^d$ and $c_{x_{\tilde{A}}}^d$. Here, $x_B$ is the value of $x$ restricted to the sites in $B$. Now, by the definition of
the total variation distance,

\[ \delta = \frac{1}{2} \sum_{x_A, x_A'} \left| \sum_a c_{x_A}^a c_{x_A'}^a - \frac{1}{2^n} \right| \quad (4) \]

\[ \geq \frac{1}{2} \sum_{x_A} \left| \frac{1}{2^n} \right| \quad (5) \]

\[ = \frac{1}{2} \sum_{x_A} |p_{x_A} - \frac{1}{2^n}|. \quad (6) \]

Taking \( A \) equal to one site on site 1 results in the identity

\[ \delta \geq \frac{1}{2} \left( |p_0 - \frac{1}{2}| + |p_1 - \frac{1}{2}| \right), \quad (7) \]

where \( p_0 \) and \( p_1 \) are the probabilities of observing a 0 and a 1 on the first qubit at the (noisy) output distribution, respectively. Now since \( p_1 = 1 - p_0 \), this in turn gives

\[ \delta \geq \left| p_0 - \frac{1}{2} \right|. \quad (8) \]

Now since the above is a quantity in \([0, 1]\), it satisfies \( |p_0 - \frac{1}{2}| \geq |p_0 - \frac{1}{2}|^2 \). We now lower bound the expectation value of the latter quantity over brickwork circuits, giving us \( \mathbb{E}_B[\delta] \geq \mathbb{E}_B[p_0^2 - p_0 + 1/4] \). The expectation value \( \mathbb{E}_B \) on the right hand side (RHS), which is over brickwork circuits with local Haar-random gates, can be replaced by an expectation value over Clifford circuits \( \mathbb{E}_C \) because of the 2-design property of Clifford group. Consider the quantity \( \mathbb{E}_C[p_0^2 - p_0 + 1/4] \), which we reexpress and lower bound as follows:

\[ \mathbb{E}_C[p_0^2 - p_0 + 1/4] = \frac{1}{4} - \mathbb{E}_C[p_0(1 - p_0)]. \quad (9) \]

We can upper bound \( \mathbb{E}_C[p_0(1 - p_0)] \) by observing that for the Clifford group, the output probabilities are rational numbers of the form \( a/2^k \), where \( a \in \{0, 1, \ldots, 2^k\} \). The precise details do not matter—we merely use the fact that \( p_0 \) takes on values in a discrete set \( S \).

\[ \mathbb{E}_C[p_0(1 - p_0)] = \sum_{p \in S} p \cdot (1 - p) \Pr_C(p_0 = p) \quad (10) \]

Our overall strategy is as follows. First, observe that since the quantity \( p(1 - p) \) takes the maximum value 1/4 in the interval \( p \in [0, 1] \), a crude upper bound for Eq. (10) is simply \( \frac{1}{4} \sum_d \Pr_C(p_0 = p) = \frac{1}{4} \). This is a useless bound since it results in \( \mathbb{E}_B[\delta'] \geq 0 \). We refine this useless bound slightly by observing that at least some instances of the Clifford ensemble lead to a value of \( p \) bounded away from 1/2, implying that \( p(1 - p) \) is bounded away from 1/4. This will result in a better upper bound on \( \mathbb{E}_C[p_0(1 - p_0)] \), which will translate into a better lower bound on \( \mathbb{E}_B[\delta] \).

We split the sum in Eq. (10) into two parts: those with \( p < \frac{1}{2} + \epsilon \) and those with \( p \geq \frac{1}{2} + \epsilon \):

\[ \mathbb{E}_C[p_0(1 - p_0)] = \sum_{p \in S, p < \frac{1}{2} + \epsilon} p \cdot (1 - p) \Pr_C(p_0 = p) + \sum_{p \in S, p \geq \frac{1}{2} + \epsilon} p \cdot (1 - p) \Pr_C(p_0 = p) \quad (11) \]

\[ \leq \sum_{p \in S, p < \frac{1}{2} + \epsilon} \frac{1}{4} \Pr_C(p_0 = p) + \sum_{p \in S, p \geq \frac{1}{2} + \epsilon} \left( \frac{1}{4} - \epsilon^2 \right) \Pr_C(p_0 = p). \quad (12) \]

In the above we use the fact that if \( p \geq 1/2 + \epsilon \), then \( p(1 - p) \leq 1/4 - \epsilon^2 \). Continuing, we get

\[ \mathbb{E}_C[p_0(1 - p_0)] \leq \left( \frac{1}{4} - \epsilon^2 \right) \Pr_C \left( p_0 \geq \frac{1}{2} + \epsilon \right) + \frac{1}{4} \left( 1 - \Pr_C \left( p_0 \geq \frac{1}{2} + \epsilon \right) \right) \]

\[ = \frac{1}{4} - \epsilon^2 \Pr_C \left( p_0 \geq \frac{1}{2} + \epsilon \right). \quad (13) \]
It only remains to lower-bound $\Pr_C(p_0 \geq 1/2 + \epsilon)$. For this we observe that we can take an extreme case over circuits satisfying a certain property. As long as these circuits result in a final state with $p_0 \geq 1/2 + \epsilon$ and the likelihood of applying these (Clifford) circuits is large enough, we are done. The extreme case we choose is simple: it consists of Clifford circuits where all of the $d$ two-qubit Clifford gates that touch the first qubit map the Pauli operator $Z_1$ to $Z_1$. In particular, this means that the first qubit is never entangled with any other qubit in the system and the unitary dynamics does not change any mixture of the $|0\rangle|0\rangle$ and $|1\rangle|1\rangle$ states. Effectively, the only evolution acting upon it is the noise channel after every layer. These are rare events, however we will prove that the combination of parameters results in a small lower bound on $\Pr_C(p_0 \geq 1/2 + \epsilon)$.

We will use Lemma 1, which analyzes this case and shows that for a single-qubit evolution under only Pauli noise, $(p_0 - 1/2) = (1 - 2(q_x + q_y))^d/2$. This means that we can take $\epsilon = (1 - 2(q_x + q_y))^d/2$. Furthermore, the likelihood of observing this extreme event is lower-bounded away from 0. In each layer, the probability of applying a Clifford circuit with the property above is at least $1/30^d$. As a result, we obtain $\Pr_C(p_0 \geq 1/2 + \epsilon) \geq 1/30^d$ for $\epsilon = (1 - 2(q_x + q_y))^d/2$.

Wrapping everything up, this results in

$$E_C[p_0(1 - p_0)] \leq 1/4 - \frac{(1 - 2(q_x + q_y))^d}{4 \cdot 30^d}$$ \hspace{1cm} (15)

$$\implies E_C[p_0^2 - p_0 + 1/4] = E_E[p_0^2 - p_0 + 1/4] \geq \frac{(1 - 2(q_x + q_y))^d}{4 \cdot 30^d}$$ \hspace{1cm} (16)

$$\implies E_E[\delta] \geq \frac{(1 - 2(q_x + q_y))^d}{4 \cdot 30^d}.$$ \hspace{1cm} (17)

A comment is in order. First, note that the dependence on $q_x + q_y$ can also be written as $q - q_z$. For the extreme event we have considered, it is understandable that dephasing noise (where $q = q_z$) does not affect the quantity $p_0$. For these cases, we can apply a different extreme event, where the first gate act as Hadamards or Hadamards plus a phase gate $S$, the intermediate gates map $X_1$ to $X_1$ or $Y_1$ to $Y_1$, respectively, and the last gate inverts the first. By symmetry, the previous analysis holds for these events as well. We can combine everything to give the slightly better bound

$$E_E[\delta] \geq \frac{(1 - 2b)^d}{4 \cdot 30^d},$$ \hspace{1cm} (18)

where $b = \min[q_x + q_y, q_y + q_z, q_x + q_z] = q - \max[q_x, q_y, q_z]$.

We also note that for the case of perfect depolarizing noise on every qubit, we have $q_x = q_y = q_z = 1/4$. This gives the trivial bound $E_E[\delta] \geq 0$, as it should, because perfect depolarizing noise immediately gives the identity operator on every qubit and the distance to the uniform distribution is exactly 0.

**Lemma 1.** Consider a single qubit starting in the state $\rho = |0\rangle|0\rangle$. After $d$ applications of the channel $\mathcal{E}(\rho) = (1 - q)\rho + q_x X\rho X + q_y Y\rho Y + q_z Z\rho Z$ (where $q = q_x + q_y + q_z$), the resulting state $\mathcal{E}^d(\rho)$ obeys

$$p_0 := \text{Tr} \left[ \frac{1 + Z}{2} \mathcal{E}^d(\rho) \right] = \frac{1}{2} + \frac{1}{2} (1 - 2(q_x + q_y))^d.$$ \hspace{1cm} (19)

**Proof.** Suppose we have the initial state $|0\rangle|0\rangle$. One application of the noise channel takes it to the state $(1 - q + q_z)|0\rangle|0\rangle + (q_x + q_y)|1\rangle|1\rangle$. Upon further applications for $d$ layers in total, we observe that the coefficient of the term $|0\rangle|0\rangle$ will correspond to terms in the binomial series. We pick up a factor $(1 - q + q_z)$ for every operation that preserves $|0\rangle|0\rangle$ and a factor $(q_x + q_y)$ for every term that flips the qubit $|0\rangle|0\rangle \leftrightarrow |1\rangle|1\rangle$. Collecting all the terms, the state would be

$$\mathcal{E}^d(\rho) = \sum_{k=0; \text{k even}}^d (1 - q + q_z)^{d-k}(q_x + q_y)^k \binom{d}{k} |0\rangle|0\rangle + \sum_{k=1; \text{k odd}}^d (1 - q + q_z)^{d-k}(q_x + q_y)^k \binom{d}{k} |1\rangle|1\rangle.$$ \hspace{1cm} (20)

It can be checked that this is normalized since $\text{Tr} \left[ \mathcal{E}^d(\rho) \right] = (1 - q + q_z + q_x + q_y)^d = 1^d = 1$. Now $\text{Tr} \left[ \frac{1 + Z}{2} \mathcal{E}^d(\rho) \right]$
is given by

$$\text{Tr} \left[ \frac{1}{2} Z E^d (p) \right] = \sum_{k=0, k \text{ even}}^{d} (1 - q + q_x)^{d-k} (q_x + q_y)^k \binom{d}{k}$$

(21)

$$= \sum_{k=0}^{d} (1 - a)^{d-k} a^k \binom{d}{k} \times \frac{1 + (-1)^k}{2},$$

(22)

where \( a = q_x + q_y \). Splitting the sum into two gives us

$$\frac{1}{2} \sum_{k=0}^{d} (1 - a)^{d-k} a^k \binom{d}{k} + \frac{1}{2} \sum_{k=0}^{d} (1 - a)^{d-k} (-a)^k \binom{d}{k}$$

(23)

$$= \frac{1}{2} \left( 1^d + (1 - 2a)^d \right),$$

(24)

proving the lemma.

\( \square \)

5.2 Upper bound on the distance to uniform

In this section, we will study a noise model where a random set of sites are selected after each layer of the circuit independently with probability \( p \). At each site \( i \) in this random subset, a local dephasing channel \( \mathcal{E}_i \) is applied with dephasing parameter \( q \), where

$$\mathcal{E}_i (p) = (1 - q) p + q Z_i p Z_i.$$  

(25)

In the limit \( p \to 1 \), this becomes a standard local dephasing model with parameter \( q \), while \( q \to 1/2 \) is equivalent to a model where a random set of sites are measured at rate \( p \) in the \( Z \)-basis, but without keeping track of the measurement outcomes. For a noisy Haar random circuit with this noise model, we can prove an upper bound on the circuit-averaged total variation distance \( \delta \) that is independent of the circuit architecture:

**Theorem 2.** For a noisy Haar random circuit on any parallel circuit architecture with heralded dephasing noise at rate \( p \) with the dephasing parameter \( q \), we have the upper bound

$$\mathbb{E}_B [\delta] < \left( \frac{3}{2} \right)^{2/3} n^{1/3} e^{-\gamma \delta d/3},$$

(26)

where \( \gamma = 8q(1 - q)/3 \).

**Proof.** We start from the observation that the total variation distance is related to the KL-divergence (the classical relative entropy) by the relation \( \delta (P, Q) \leq \sqrt{D_{KL} (P \| Q)} / 2 \). The KL-divergence with respect to a uniform distribution is given by \( n \log 2 - H (P) \), where \( H (P) \) is the Shannon entropy. Let \( D \) denote the distribution of measurements for a circuit. This gives us the following chain of inequality for variation distance to the uniform.

$$2\delta (D, U)^2 \leq D_{KL} (D \| U) = \sum_x -p_x \log (1/2^n)) - H (D) = n \log (2) - H (D) \leq n \log (2) - H (D_2 (D),$$

(27)

where the last inequality is given by the fact that second Renyi entropy \( H_2 (D) = - \log (\sum_x p_x^2) \) is less than or equal than the von-Neumann entropy \( H (D) = H_{\alpha = 1} (D) = - \sum_x p_x \log p_x \). We can now use the Chebyshev inequality to bound the unitary-averaged TVD.

$$\mathbb{E}_B (\delta) = \int_{0}^{c} \text{d} \sigma \text{Pr}_B (\delta = \sigma) + \int_{c}^{1} \text{d} \sigma \text{Pr}_B (\delta = \sigma)$$

(28)

$$\leq e + \text{Pr}_B (\delta > e) \leq e + \frac{\mathbb{E}_B (\delta^2)}{e^2}. \quad (29)$$

If \( \mathbb{E}_B (\delta^2) \) decays exponentially or faster with depth, i.e., \( e^{-\gamma d} \), we can take \( e = e^{-\gamma d/3} \) to ensure that \( \mathbb{E}_B (\delta) \) decays exponentially with \( e^{-\gamma d} \). To show that the second moment of TVD must indeed decay exponentially with \( d \), we calculate the expectation of Eq. (27):

$$\mathbb{E}_B [2\delta (D, U)^2] \leq n \log 2 - \mathbb{E}_B (H_2 (D)) = n \log 2 + \mathbb{E}_B \left[ \log \left( \sum_x p_x^2 \right) \right] \leq n \log 2 + \log \mathbb{E}_B \left[ \sum_x p_x^2 \right].$$

(30)
In the last inequality, we have used Jensen’s inequality for concave functions \( (E_B(f(X)) \leq f(E(X)) \). The term inside the expectation function, \( \sum_i p^2_{xi} \), is the collision probability. From Lemma 2, we have that the expectation of collision probability is upper-bounded by \( 2^{-n}(1 + e^{-q pd /3})^n \), where \( \gamma = 8q(1-q)/3 \). With this, we have

\[
\mathbb{E}_B[2\delta(D,U)^2] \leq n \log 2 + \log \left[ 2^{-n} \left( 1 + \frac{1}{3} e^{-\gamma pd} \right)^n \right] = \log \left( 1 + \frac{1}{3} e^{-\gamma pd} \right)^n \leq \log \left( e^{-\gamma pd /3} \right)^n = \frac{n}{3} e^{-\gamma pd}
\]

Thus, we have that the second comment of the TVD decays exponentially in circuit depth. Using this inequality, we minimize the right hand side of Eq. (29) with respect to \( \epsilon \) to get the desired bound

\[
\mathbb{E}_B(\delta) \leq \left( \frac{3}{2} \right)^{2/3} n^{1/3} e^{-\gamma pd /3}. \tag{32}
\]

To complete the proof, it remains to prove the following lemma:

**Lemma 2.** For a noisy Haar random circuit on any parallel circuit architecture with heralded dephasing noise at rate \( p \) with the dephasing parameter \( q \), we have the upper bound on the collision probability

\[
\mathbb{E}_B[Z] = \mathbb{E}_B\left[ \sum_i p^2_{xi} \right] \leq 2^{-n} (1 + e^{-a pd /3})^n
\]

where \( a = 8q(1-q)/3 \).

To prove this bound we make use of the statistical mechanics mapping method developed by Dalzell, Hunter-Jones, and Brandão [DHB20]. The proof of Lemma 2 can be found in Appendix A.

### 5.3 No-go for anticoncentration at low depth

In this section, we study the properties of quantum circuit dynamics at sublogarithmic depth, which is defined as a limit where we fix \( 0 \leq a < 1 \) and scale depth as \( d = O((\log n)^a) \) while taking \( n \to \infty \). At this depth, there is still a notion of locality in the circuit because the lightcone \( L_d(i) \) of each site cannot extend across the whole system in the large-\( n \) limit. We will prove that sampling from Haar random circuits on any parallel circuit architecture at sublogarithmic depth leads to a poorly anticoncentrated output distribution. First, we give a formal definition of anticoncentration for a random circuit ensemble.

**Definition 1.** A family of random circuit ensembles is anticoncentrated if it satisfies the following limiting formula for every constant \( \alpha \in (0,1/2) \)

\[
\lim_{n \to \infty} \Pr \left[ \left| \langle \psi \mid U \mid \psi \rangle \right|^2 \geq \frac{\alpha}{2^n} \right] > 0. \tag{34}
\]

We say that a random circuit ensemble is poorly anticoncentrated if this condition fails.

**Theorem 3.** Consider a Haar random circuit ensemble on any parallel architecture at sublogarithmic depth \( d \geq 1 \), then we have the identity

\[
\lim_{n \to \infty} \Pr_B \left[ \left| \langle \psi \mid U \mid \psi \rangle \right|^2 \geq \frac{1}{2^n e^{a d / (8 - 30 a)} } \right] = 0. \tag{35}
\]

**Proof.** The strategy of the proof will be to show that a bound on the average logarithm

\[
-\frac{1}{n} \log \left| \langle \psi \mid U \mid \psi \rangle \right|^2,
\]

is sufficiently concentrated about its mean, which can be thought of as the first step in proving a central limit theorem-like behavior for this quantity. To see how this arises, we note that the output probability \( \left| \langle \psi \mid U \mid \psi \rangle \right|^2 \) is equal to the quantum mixed state fidelity \( F(\mid 0 \rangle, U \mid 0 \rangle) \) and satisfies the bound

\[
- \log \left| \langle \psi \mid U \mid \psi \rangle \right|^2 = - \log F(\mid 0 \rangle, U \mid 0 \rangle) \geq - \log F(\mid 0 \rangle, \rho_1 \otimes \cdots \otimes \rho_n) = - \sum_{i=1}^n \log \langle \psi \mid \rho_i \rangle,
\]

\[
-a \sum_{i=1}^n \log \langle \psi \mid \rho_i \rangle + \frac{1}{2^n} e^{a d / (8 - 30 a)},
\]

where \( a = 8q(1-q)/3 \).
where $\rho_i = \text{tr}_{[i]} [U |0\rangle \langle 0| U^\dagger]$ is the local density matrix on site $i$. The inequality follows from the monotonicity of the mixed state fidelity under quantum channels $F(p, \sigma) \leq F(E(p), E(\sigma)) \leq 1$.

For each site $i$, we let $U_{\text{loc}}(i)$ be the unitary circuit obtained by removing all quantum gates outside the lightcone $L_d(i)$ of site $i$. Then, we have the identity

$$p_{i\bar{0}} \equiv \langle 0| \rho_i |0 \rangle = \frac{1 - \langle Z_i \rangle}{2},$$

$$\langle Z_i \rangle = \langle 0| U^\dagger Z_i U |0 \rangle = \langle 0| U_{\text{loc}}^\dagger (i) Z_i U_{\text{loc}}(i) |0 \rangle.$$  

We define the mean and higher moments as

$$\mu_i = -\mathbb{E}_{B} \log p_{i\bar{0}},$$

$$s_n^k = \mathbb{E}_{B} \left| \sum_i (\log p_{i\bar{0}} + \mu_i) \right|^k.$$  

To prove the mean is finite, we make use of a decomposition of the probability of obtaining $p_{i\bar{0}} = p$

$$P_i(p) = \int_0^1 dp_x \int_{A_{Pp}}^{[0]} dp_{\sigma_x}$$

where $\sigma_d$ is the circuit ensemble for all gates up to depth $d$, $dp_d$ is the measure over these random circuits, $A_{Pp}^{[0]} = \{ U \in \sigma_d : \langle 0| U^\dagger Z_i U |0 \rangle = 1 - 2p, \langle 0| U^\dagger X_i U |0 \rangle = 1 - 2p_x \}$. $P_i(p)$ has the special property that it monotonically increases with $p$ in the interval $[0, 1/2]$. This monotonicity property follows from the fact that any $U \in A_{Pp}^{[0]}$ can be rotated by a $U$-independent single-site gate on site $i \ U_i$ to obtain a $U' = U_i U \in A_{Pp}^{[i]}$, for $p \leq q \leq 1 - p$. Furthermore, since these single-site gates are drawn from the Haar measure in our random circuit ensemble, the weight of this rotated set with respect to $dp_{\sigma_d}$ is equal to the weight of $A_{Pp}^{[i]}$, thus $P_i(p) \leq P_i(q)$. Note, since $P_i(p)$ is an increasing function on $[0, 1/2]$, according to a standard result in analysis it has at most a countable number of discontinuities in this region.

Now, taking any $0 < \epsilon < 1/2$ that is not at a discontinuity of $P_i(p)$ for any $i$, we find a uniform upper bound on $\mu_i$ that just follows from this monotonicity property

$$\mu_i = -\int_0^1 dp P_i(p) \log p \leq -P_i(\epsilon) \int_0^\epsilon dp \log p - \log \epsilon$$

$$= P_i(\epsilon)(1 - \log \epsilon) - \log \epsilon$$

$$\leq \frac{\epsilon}{1 - 2\epsilon} - \frac{1 - \epsilon}{1 - 2\epsilon} \log \epsilon,$$

where we used the fact that $P_i(\epsilon) \leq \frac{1}{1 - 2\epsilon} \int_0^\epsilon dp P_i(p) \leq \frac{1}{1 - 2\epsilon}$. This upper bound is minimized near $\epsilon = 0.189$, from which we find $\mu_i \leq 2.48$. A nearly identical argument with $0 < \epsilon < 1/2$ away from any discontinuity in $P_i(p)$ for any $i$ shows that the variance and all higher moments ($n > 2$),

$$\sigma_i^2 = \int_0^1 dp P_i(p)(\log p)^2 - \mu_i^2 \leq \frac{1}{1 - 2\epsilon} \int_0^\epsilon dp (\log p)^2 + (\log \epsilon)^2,$$

$$\mathbb{E}_{B} |\log p_{i\bar{0}} + \mu_i|^n = \int_0^1 dp P_i(p)|\log p + \mu_i|^n \leq \frac{1}{1 - 2\epsilon} \int_0^\epsilon dp |\log p + \mu_i|^n + | \log \epsilon + \mu_i|^n,$$

also have uniform upper bounds that are independent of $i$.

This allows us to bound the second moment of the sum of the logarithm of probabilities as

$$s^2 = \frac{1}{n} s_n^2 = \frac{1}{n} \sum_i \sum_j |\log p_{i\bar{0}} \log p_{j\bar{0}} - \mu_i \mu_j| \leq \max_i \max_{j \in L_d(i)} |[L_d(i)]| \sigma_i \sigma_j$$.

From which it follows that $s^2 \leq F(d)$ for a function $F(d)$ with at most an exponential growth rate $2^d$. From this result, we can obtain the bound

$$\Pr \left( \frac{1}{n} \sum_{i=1}^n |\log p_{i\bar{0}} + \mu_i|^2 \geq \epsilon \right) = \Pr \left( \frac{1}{n} \sum_{i=1}^n |\log p_{i\bar{0}} + \mu_i| \geq \sqrt{\frac{\epsilon}{n}} \right) \leq \frac{F(d)}{\epsilon}.$$
Making use of Theorem 1 for noiseless circuits and the identity \(- \log(1 + x) \geq -x + x^2/4\), we can also prove the uniform lower bound

\[
\mu_i \geq \log 2 + e^{-ad}/4,
\]

where \(a = \log 30\).

We are now ready to prove the main bound in the theorem. From Eq. (37), Eq. (50), and Eq. (49), it follows that

\[
\Pr\left( \sum |\langle 0 | U | 0 \rangle|^2 \geq \frac{1}{2^{ne^{(1/830^d)}}} \right) = \Pr\left( - \sum \log p_{i0} \leq n(\log 2 + e^{-ad}/8) \right) \leq \Pr\left( - \sum (\log p_{i0} + \mu_i) \leq n(\log 2 + e^{-ad}/8) - \sum \mu_i \right) \leq \Pr\left( \sum (\log p_{i0} + \mu_i) \geq ne^{-ad}/8 \right) \leq 64F(d)/ne^{-2ad},
\]

where we took \(\epsilon = ne^{-2ad}/64\) in applying the Markov bound from Eq. (49). This upper bound decays to zero with \(n\) for any sublogarithmic depth \(d\). Therefore, we arrive at the result

\[
\lim_{n \to \infty} \Pr\left( \sum |\langle 0 | U | 0 \rangle|^2 \geq \frac{1}{2^{ne^{(1/830^d)}}} \right) = 0.
\]

for any sublogarithmic depth \(d\).

The proof for the noiseless case follows directly from Theorem 3 and the fact that \(e^{-n/(830^d)}\) converges to 0 with \(n\) for any sublogarithmic depth \(d\). In the case of a finite noise rate, the factor \(e^{-n/(830^d)}\) can be replaced by \(e^{-ne^{-ad}/8}\) with \(a = -2\log(1 - 2b) + \log(30)\), where \(b = \min[q_x + q_y, q_y + q_z, q_z + q_x]\) and \(q_\mu\) is a uniform upper bound on the local noise rate in Pauli sector \(\mu\) for all sites \(i\) and circuit layers \(m\).

5.4 Anticoncentration at large enough depth

**Theorem 4.** Random circuits with noise (of any kind) anticoncentrate at least as fast as those without noise. More specifically, we show that the collision probability decreases when adding noise to a noiseless Clifford circuit, i.e.,

\[
Z(U, \mathcal{E}) \leq Z(U, \mathbb{I}).
\]

**Proof.** The defining properties of Clifford circuits is that they send Pauli group elements to Pauli group elements. Another result we need is that the composition of two Pauli noise channels is also a Pauli noise channel. We can use these two facts to move all the noise channels past the Clifford gates to arrive at the expression

\[
\mathcal{N}_{d}(|0 \rangle \langle 0 |) = (\mathcal{E}_U \circ \mathcal{C})(|0 \rangle \langle 0 |),
\]

where \(\mathcal{E}_U\) is a new Pauli noise channel with a modified distribution \(q(E)\) and \(\mathcal{C} = C_d \circ \cdots \circ C_1\) is the noiseless Clifford circuit. Now we are left to bound the expression

\[
Z(U, \mathcal{E}) = \sum x \left| \sum_{E \in \mathcal{P}} q(E) \langle x | E \prod_{i=1}^{n} \frac{I + g_i}{2} E^\dagger | x \rangle \right|^2 = \sum x \left| \sum_{g} q_g \langle x | \prod_{i=1}^{n} \frac{I + (-1)^{\bar{g} i} g_i}{2} | x \rangle \right|^2,
\]

where \(g_i = -U_d \cdots U_i Z_i U_i^\dagger \cdots U_d^\dagger\) are stabilizer generators for the evolved initial state under the noiseless circuit and \(Z_i\) is the Pauli \(z\)-operator on site \(i\). In the second equality, we have organized the sum into syndrome classes.

\[\]
defined by the anticommutation pattern $\vec{s}$ of the Pauli group elements $E$ with the stabilizer generating set $\{g_i\}$ using the definition

$$q_{\vec{s}} = \sum_{E \in \mathbb{P} \text{ s.t. } (-1)^{\vec{s}} E = [E, g_i]} q(E). \quad (61)$$

Here, $[[A, B]] = \text{tr}[ABA^{-1}B^{-1}] / 2^n$ is the scalar commutator and $(-1)^{\vec{s}} = [[E, g_i]]$. To bound Eq. (60), we first use properties of stabilizer states to evaluate the measurement probabilities as

$$\langle x | \prod_{i=1}^{n} \frac{1}{2} \left[ 1 + (-1)^{s_i} g_i \right] | x \rangle = p_{\text{max}}(U, \mathbb{I}) f(x, \vec{s}, U), \quad (62)$$

where $p_{\text{max}}(U, \mathbb{I}) = \max_x p_x(U, \mathbb{I})$ and $f(x, \vec{s}, U)$ is either 0 or 1 and satisfies the sum rule $\sum_x f(x, \vec{s}, U) = p_{\text{max}}^{-1}(U, \mathbb{I})$ for every $\vec{s}$. As a result,

$$Z(U, \mathcal{E}) = p_{\text{max}}^2(U, \mathbb{I}) \sum_{\vec{s}, \vec{t}} q_{\vec{s}} q_{\vec{t}} \sum_x f(x, \vec{s}, \vec{t}, U) \quad (63)$$

$$\leq p_{\text{max}}^2(U, \mathbb{I}) \sum_{\vec{s}, \vec{t}} q_{\vec{s}} q_{\vec{t}} \sum_x f(x, \vec{s}, U) \quad (64)$$

$$= p_{\text{max}}(U, \mathbb{I}) = Z(U, \mathbb{I}), \quad (65)$$

where we used the fact that $f(x, \vec{s}, U)f(x, \vec{t}, U) \leq f(x, \vec{s}, U)$ and $\sum_{\vec{s}} q_{\vec{s}} = 1$. \hfill \Box

Using the two-design property of the Clifford ensemble, this result directly implies the following corollary:

**Corollary 1.** The output probability distribution for a Haar random circuit with noise of depth greater than $O(\log n)$ on a general circuit architecture is anticoncentrated.

This result follows from the inequalities

$$2^{-n} \leq \mathbb{E}_B Z(U, \mathcal{E}) \leq \mathbb{E}_C Z(U, \mathcal{E}) \leq \mathbb{E}_C Z(U, \mathbb{I}) = \mathbb{E}_B Z(U, \mathbb{I}), \quad (66)$$

and the fact that $\mathbb{E}_B Z(U, \mathbb{I}) \rightarrow 2 / (2^n + 1)$ after depth $O(\log n)$.

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**A Proof details of Lemma 2**

Here, we introduce the concepts that go into proving Lemma 2.

If $X$ is a random-variable denoting the measurement outcome of $n$ qubits after a unitary $U$ the collision probability of the random-circuit architecture is defined as

$$Z = \mathbb{E}_U \left[ \sum_{x \in \{0,1\}^n} \Pr(X = x)^2 \right] = \mathbb{E}_U \left[ \sum_{x \in \{0,1\}^n} p_U(x)^2 \right], \quad (67)$$
where \( p_U(x) \) is the probability that the measurement, after evolution by \( U \), is \( x \). If there is at least one gate for each qubit, all measurement outcomes are equally random and, thus, there is a symmetry over them.

\[
Z = \mathbb{E}_U \left[ \sum_{x \in \{0,1\}^n} p_U(x)^2 \right] = 2^n \mathbb{E}_U \left[ p_U(1^n)^2 \right] \tag{68}
\]

Assuming that the input state is also \( 1^n \), the probability of measuring \( 1^n \) after the circuit is given by \( \text{Tr}\left( \left| 1^n \right\rangle \langle 1^n \mid U \left| 1^n \right\rangle \langle 1^n \mid U^\dagger \right) \).

To get the second moment of the probability distribution, we consider two copies of the circuit acting on two copies of the input state. Since the \( \text{Tr}(A \otimes B) = \text{Tr}(A) \text{Tr}(B) \), we get

\[
Z = 2^n \mathbb{E}_U \left[ p_U(1^n)^2 \right] = 2^n \mathbb{E}_U \text{Tr}\left((\left| 1^n \right\rangle \langle 1^n \mid U \left| 1^n \right\rangle \langle 1^n \mid U^\dagger)^{\otimes 2} \right)
 = 2^n \text{Tr}\left[ \left( \left| 1^n \right\rangle \langle 1^n \mid \otimes 2 \mathbb{E}_U \left[ \left| 1^n \right\rangle \langle 1^n \mid \otimes 2 \left( U^\dagger \right)^{\otimes 2} \right] \right) \right] \tag{69}
\]

For convenience, we denote the Haar-averaged channel, over \( k \) qubits, as \( M_{U_k} \).

\[
M_{U_k}[\rho] = \mathbb{E}_{U_k} \left[ U_k^{\otimes 2} \rho (U_k^\dagger)^{\otimes 2} \right], \tag{70}
\]

where \( U_k \) is an unitary acting over \( k \) qubits. To study noisy evolution, we define a dephasing noise of strength \( q \) with the dephasing parameter \( q \), we have the upper bound on the expected collision probability

\[
\mathbb{E}_B[Z] = \mathbb{E}_B \left[ \sum_x p_x^2 \right] \leq 2^{-n} (1 + e^{-\gamma pd}/3)^n, \tag{72}
\]

where \( \gamma = 8q(1 - q)/3 \).

**Lemma 3.** [Lemma 2 restated.] For a noisy Haar random circuit of depth \( d \) on any parallel circuit architecture with heralded dephasing noise at rate \( r \) with the dephasing parameter \( q \), we have the upper bound on the expected collision probability

\[
\mathbb{E}_B[Z] = \mathbb{E}_B \left[ \sum_x p_x^2 \right] \leq 2^{-n} \left( 1 + e^{-\gamma pd}/3 \right)^n, \tag{72}
\]

where \( \gamma = 8q(1 - q)/3 \).

**Proof.** We show in Lemma 5 that given a noisy random circuit, \( C \), with heralded dephasing, there exists another circuit \( C' \) composed solely of single-qubit gates, heralded dephasing events and SWAP gates, with a collision probability \( Z_{C'} \) greater than the collision probability of the original circuit, \( Z_C \). We can add a network of SWAP gate to \( C' \) to return all qubits to their original position. Adding SWAP gates does not change the collision probability, since swap gates only permute the support of the final probability distribution.

We can follow the path of a qubit, count the total number of dephasing events in that path and merge/split single-qubit gates. Since we are working with a brick-work architecture, there is never a case of consecutive dephasing events. Let \( t_i \) be the number of dephasing events on the path of qubit \( i \).

Using Lemma 4, the final state is given by

\[
\bigotimes_{i=1}^n \left( M_{U_i} \circ \mathcal{E} \circ \cdots \circ \mathcal{E} \circ M_{U_1} \right) \left[ \left| 1^n \right\rangle \langle 1^n \right] = \bigotimes_{i=1}^n \left[ \frac{1}{12} (3 - \beta^i) I + \frac{1}{6} \beta^i S \right], \tag{73}
\]

where \( \beta = 1 - 8q(1 - q)/3 \). Using (69) and noting that \( \text{Tr}\left( \left| 1 \right\rangle \langle 1 \mid^{\otimes 2} \right) = \text{Tr}\left( S \left| 1 \right\rangle \langle 1 \mid^{\otimes 2} \right) = 1 \), the collision probability equals

\[
Z_{C'} = 2^n \prod_{i=1}^n \left[ \frac{1}{12} (3 - \beta^i) + \frac{1}{6} \beta^i \right] = 2^n \prod_{i=1}^n \frac{1}{2^2} \left[ 1 + \frac{1}{3} \beta^i \right] = \frac{1}{2^n} \prod_{i=1}^n \left[ 1 + \frac{1}{3} \beta^i \right] \tag{74}
\]

We have the identity \( (1 - x) \leq e^{-x} \) for all real \( x \). Therefore \( \beta^i = (1 - 8q(1 - q)/3)^{t_i} \leq e^{-\gamma i} \) where we have defined \( \gamma = 8q(1 - q)/3 \).

\[
Z_{C'} \leq \frac{1}{2^n} \prod_{i=1}^n \left[ 1 + \frac{1}{3} e^{-\gamma t_i} \right] \tag{75}
\]

Taking \( t_{\text{min}} = \min_i t_i \), we can upper bound this by

\[
Z_{C'} \leq \frac{1}{2^n} \left( 1 + \frac{1}{3} e^{-\gamma t_{\text{min}}} \right)^n. \tag{76}
\]
Lemma 4. Consider a random circuit consisting of $n$ dephasing error channels of strength $q$ sandwiched between $n + 1$ single-qubit Haar gates (denoted by $U_l$). When this circuit acts on two copies of a single-qubit, the circuit-averaged state is given by

$$M_{U_l} \circ \mathcal{E} \circ \cdots \circ \mathcal{E} \circ M_{U_l} |1^n \rangle \langle 1^n| \otimes 2 = \frac{1}{12} (3 - \beta^n) I + \frac{1}{6} \beta^n S,$$  \hspace{1cm} (77)

where, $\beta = 1 - \frac{2}{3} q(1 - q)$, and $I$ and $S$ are the $4 \times 4$ identity and SWAP matrices, respectively.

Proof. We first make an observation that $M_{U_l}[\rho] = M_{U_l} \circ M_{U_l} [\rho]$, that is, one can split a Haar-random gate into two Haar-random gates without changing the statistics. In the circuit described above, leaving the two terminal unitary gates intact, we split all inner gates into two. This lets us treat the circuit as a repeating sequence of $n$ units of the composite channel $M_{U_l,e} = M_{U_l} \circ \mathcal{E} \circ M_{U_l}$.

From [DHB20], we know the following:

$$M_{U_l}[\sigma] = \frac{1}{3} \left( \Tr(\sigma) - 2^{-1} \Tr(\sigma S) \right) I + \frac{1}{3} \left( \Tr(\sigma S) - 2^{-1} \Tr(\sigma) \right) S.$$  \hspace{1cm} (78)

If we follow this gate by a dephasing error channel, we get

$$\mathcal{E} \circ M_{U_l}[\sigma] = \frac{1}{3} \left( \Tr(\sigma) - 2^{-1} \Tr(\sigma S) \right) \mathcal{E}[I] + \frac{1}{3} \left( \Tr(\sigma S) - 2^{-1} \Tr(\sigma) \right) \mathcal{E}[S]
= \frac{1}{3} \left( \Tr(\sigma) - 2^{-1} \Tr(\sigma S) \right) I + \frac{1}{3} \left( \Tr(\sigma S) - 2^{-1} \Tr(\sigma) \right) ((1 - q)^2 S + 2q(1 - q)(ZI)S(ZI) + q^2 ZZ) S(ZZ)
= \frac{1}{3} \left( \Tr(\sigma) - 2^{-1} \Tr(\sigma S) \right) I + \frac{1}{3} \left( \Tr(\sigma S) - 2^{-1} \Tr(\sigma) \right) \left( ((1 - q)^2 + q^2) S + 2q(1 - q)(ZI)S(ZI) \right).$$

We follow this channel by another single-qubit random gate to finish the composite block. First we observe that $M_{U_l}[I] = I$ and $M_{U_l}[S] = S$. Similarly using (78) together with the fact that $\Tr[S] = 2$, $\Tr[(IZ)(IZ)] = 0$,

$$M_{U_l}[(IZ)(IZ)] = \frac{2}{3} I - \frac{1}{3} S.$$  \hspace{1cm} (79)

The composite channel thus gives

$$\mathcal{M}_{U_l,e}[\sigma] = \frac{1}{3} \left( \Tr(\sigma) - 2^{-1} \Tr(\sigma S) \right) M_{U_l} [I] + \frac{1}{3} \left( \Tr(\sigma S) - 2^{-1} \Tr(\sigma) \right) \left( ((1 - q)^2 + q^2) M_{U_l} [S] + 2q(1 - q) M_{U_l} [(ZI)(ZI)] \right)
= \frac{1}{3} \left( \Tr(\sigma) - 2^{-1} \Tr(\sigma S) \right) I + \frac{1}{3} \left( \Tr(\sigma S) - 2^{-1} \Tr(\sigma) \right) \left( ((1 - q)^2 + q^2) S + 2q(1 - q) \left( \frac{2}{3} I - \frac{1}{3} S \right) \right)
= \frac{1}{3} \left( \Tr(\sigma) - 2^{-1} \Tr(\sigma S) + \frac{4}{3} q(1 - q) \left( \Tr(\sigma S) - 2^{-1} \Tr(\sigma) \right) \right) I + \frac{1}{3} \left( \Tr(\sigma S) - 2^{-1} \Tr(\sigma) \right) \left( 1 - \frac{8}{3} q(1 - q) \right) S.
$$  \hspace{1cm} (80)

The composite sum returns a state of the form $a I + b S$. Acting on this state with other composite block only changes the coefficients $a$ and $b$. In fact, we can work out exactly how $a$ and $b$ change after each block. Knowing that $\Tr(aI + bS) = 4a + 2b$ and $\Tr(S(aI + bS)) = 2a + 4b$, we get

$$\mathcal{M}_{U_l,e} [aI + bS] = \frac{1}{3} (3a + a(3b)) I + \frac{1}{3} (3b) b S = (a + ab) I + (\beta b) S.$$  \hspace{1cm} (82)

The first composite acts on the state $|1 \rangle \langle 1| \otimes 2$. Knowing that $\Tr[ |1 \rangle \langle 1| \otimes 2] = \Tr[S |1 \rangle \langle 1| \otimes 2] = 1$, and using (81),

$$\mathcal{M}_{U_l,e} [ |1 \rangle \langle 1| \otimes 2] = \frac{1}{3} \left( 1 - 2^{-1} + a \left( 1 - 2^{-1} \right) \right) I + \frac{1}{3} \left( 1 - 2^{-1} \right) b S = \frac{1}{6} (1 + a) I + \frac{1}{6} b S.$$
We take this state and apply another $n - 1$ composite channels (since there are $n$ in total). We can calculate the final state recursively using (82)

$$\tilde{M}_{U_{i, \ell}} \circ \ldots \circ \tilde{M}_{U_{i, \ell}} [1] \langle 1^{\otimes 2} \rangle = \frac{1}{6} \left[ 1 + x + x \sum_{i=1}^{n-1} \beta_i \right] I + \frac{1}{6} \beta^n S = \frac{1}{12} (3 - \beta^n). I + \frac{1}{6} \beta^n . S \quad (83)$$

This proves the lemma.

We now need a way to bound the collision probability of a general random circuit which will have two-qubit gates. We will show that you can replace all two-qubit gates with noisy single-qubit gates and the collision probability of the newer circuit will be larger than the original circuit.

**Lemma 5.** Consider a random-quantum circuit, $C$, with Haar-random two-qubit gates and heralded dephasing noise. There is a procedure to obtain another circuit $C'$, composed solely of noisy single-qubit channels and SWAP gates, with a higher collision probability than $C$. $Z_C \leq Z_{C'}$

**Proof.** When the noise is heralded, the circuit consists of three kind of two-qubit gates

1. where the two-qubit gate is noiseless.
2. where one of the two outgoing legs of the gate undergoes dephasing.
3. where both outgoing legs undergo dephasing.

For cases where we have single-qubit dephasing, we augment the circuit with a random single-qubit gate after the dephasing. This is allowed because we can always absorb this single-qubit gate into the next gate acting on that qubit. Following this augmentation, we have three kinds of two-qubit gates

1. Type A, where the two-qubit gate is noiseless.
2. Type B, where one of the two outgoing legs of the gate undergoes dephasing, followed by a single-qubit random gate.
3. Type C, where both outgoing legs undergo dephasing followed by single-qubit gates on both legs.

We will analyze each of these types separately. In [DHB20], it was shown that the two-copies $n$-qubit state, acted on by a random circuit, can be represented a linear combination of length-$q$ubit. Following this augmentation, we have three kinds of two-qubit gates

$$\tilde{M}_{U_{2}}[\tilde{\gamma}] = M_{U_{2}} \left[ \bigotimes_{a=1}^{n} \tilde{\gamma}_{a} \right] = \sum_{\vec{v} \in \{I,S\}} M_{U_{2}}^{\tilde{\gamma}, \vec{v}} \bigotimes_{b=1}^{n} v_{b} = \sum_{\vec{v} \in \{I,S\}} M_{U_{2}}^{\tilde{\gamma}, \vec{v}}, \quad (84)$$

where $M_{U_{2}}^{\tilde{\gamma}, \vec{v}}$ are matrix elements determined by qubit locations $i, j$

$$M_{U_{2}}^{\tilde{\gamma}, \vec{v}} = \begin{cases} 1 & \text{if } \gamma_{i} = \gamma_{j} \text{ and } \tilde{\gamma} = \vec{v}, \\ 2/5 & \text{if } \gamma_{i} \neq \gamma_{j} \text{ and } v_{i} = v_{j} \text{ and } \gamma_{k} = v_{k} \forall k \in [n]/[i, j], \\ 0 & \text{otherwise}. \end{cases} \quad (85)$$

Therefore, a state can, thus, be represented as a linear combination of trajectories of the configuration strings, with each trajectory weighted according to (85). Furthermore, since $\text{Tr} (\tilde{\gamma} \langle 1^{n} | \langle 1^{n} |^{\otimes 2} \rangle)$ for each $\gamma \in \{I, S\}^{n}$, the collision probability can be, similarly, written as a sum over weighted trajectories. More precisely, for a circuit with $s$ gates,

$$Z \propto \sum_{\gamma \in \{I, S\}^{2s}} \prod_{t=1}^{s+1} M_{U_{2}}^{\tilde{\gamma}_{t}, \tilde{\gamma}_{t+1}} = \sum_{\gamma \in \{I, S\}^{2s}} \prod_{t=1}^{s+1} \text{wt}(\gamma). \quad (86)$$

Consider the collision-probability of a circuit after applying the $s$th gate. Now we add one more gate of Type $A$, $B$ or $C$ on this circuit. Since all three types are two-qubit gates, we let $[i, j]$ denote the qubits the gate acts on. We can isolate the qubits $[i, j]$ from the decomposition in (86) as follows

$$Z_{s} = C \left[ \sum_{\gamma \in \{I, S\}^{2s}} \text{wt}(\gamma) + \sum_{\gamma \in \{I, S\}^{2s}} \text{wt}(\gamma) + \sum_{\gamma \in \{I, S\}^{2s}} \text{wt}(\gamma) + \sum_{\gamma \in \{I, S\}^{2s}} \text{wt}(\gamma) \right]. \quad (87)$$
The collision probability of the modified circuit is given by

\[ M_{U_2}[II] = II \quad M_{U_2}[SS] = SS \quad M_{U_2}[IS, SI] = \frac{2}{3}(II + SS). \] (88)

The trajectories for which \( \vec{\gamma}_{ij} \in \{II, SS\} \) have their weights unchanged. The trajectories for which \( \vec{\gamma}_{ij} \in \{IS, SI\} \) have their weights changed by 4/5 (the trajectory splits two ways, each weighted by 2/5).

\[ Z_{s+1} = \frac{1}{3^n} \left[ \sum_{\vec{\gamma} \in \{LS\}^{n+1}} \text{wt}(\gamma) + \frac{4}{5} \sum_{\vec{\gamma} \in \{IS\}^{n+1}} \text{wt}(\gamma) \right] + \frac{2}{3}(II + SS). \] (89)

If, instead, we had added two noiseless single-qubit gates, all the trajectories would have their retained their original weights since \( M_{U_1}[I] = I \) and \( M_{U_1}[S] = S \). Therefore,

\[ Z'_{s+1} = Z_s \Rightarrow Z'_{s+1} > Z_{s+1}. \] (90)

**Type A:** When we add a noiseless two-qubit gate, the bit-strings transform according to (85). Zooming on qubits \( i \) and \( j \), the trajectories evolve as the following:

\[ M_{U_1}[II] = II \quad M_{U_1}[SS] = SS \quad M_{U_1}[IS, SI] = \frac{2}{3}(II + SS). \]

The dephasing is also followed by a single-qubit random gate. To simplify things, we first understand the effect of the channel \( M_{U_1} \otimes \mathcal{E} \) on \( I \) and \( S \). Of course, \( M_{U_1} \circ \mathcal{E}[I] = I \), since neither the error nor the random gate has any effect on the identity matrix. However, for \( S \), we get

\[ M_{U_1}^{(i)} \circ \mathcal{E}^{(i)}[S] = M_{U_1}[(1 - q)^2 S + q(1 - q)IZS(IZ)] \]

\[ = aI + \beta S. \] (91)

with \( a = 4q(1 - q)/3, \beta = 1 - 8q(1 - q)/3 \). Without losing generality, we assume that the dephasing happens on gate \( i \), and the dephasing channel is denoted by \( \mathcal{E} \). We now tabulate the effect of this composite channel

\[ M_{U_1}^{(i)} \circ \mathcal{E}^{(i)} \circ M_{U_2}[II] = II \]

\[ M_{U_1}^{(i)} \circ \mathcal{E}^{(i)} \circ M_{U_2}[SS] = aIS + \beta SS \]

\[ M_{U_1}^{(i)} \circ \mathcal{E}^{(i)} \circ M_{U_2}[IS, SI] = M_{U_1}^{(i)} \left[ \frac{2}{3}(II + SS) \right] = \frac{2}{3}(II + aIS + \beta SS). \]

The collision probability of the new circuit is given by

\[ Z_{s+1} = \frac{1}{3^n} \left[ \sum_{\vec{\gamma} \in \{II\}^{n+1}} \text{wt}(\gamma) + \frac{2}{3}(1 + \alpha + \beta) \left( \sum_{\vec{\gamma} \in \{IS\}^{n+1}} \text{wt}(\gamma) + \sum_{\vec{\gamma} \in \{SI\}^{n+1}} \text{wt}(\gamma) \right) \right]. \] (93)

For the modified circuit, we replace the noisy two-qubit gate with two single qubit gates. And, with probability 1/2, we apply a SWAP gate immediately before the two-qubit gate.

\[ M_{U_1}^{(i)} \circ \mathcal{E}^{(i)} \circ M_{U_2} \rightarrow \frac{1}{2} M_{U_1}^{(i)} \circ \mathcal{E}^{(i)} \circ M_{U_1}^{(i)} + \frac{1}{2} M_{U_1}^{(i)} \circ \mathcal{E}^{(i)} \circ M_{U_1}^{(i)} \circ \text{SWAP} = \frac{1}{2} M_{U_1}^{(i)} \circ \mathcal{E}^{(i)} \circ M_{U_1}^{(i)} + \frac{1}{2} M_{U_1}^{(i)} \circ \mathcal{E}^{(i)} \circ M_{U_1}^{(i)}. \] (94)

Under this new composite channel, the bitstrings evolve as follows:

\[ II \rightarrow II \quad SS \rightarrow \beta SS + \frac{a}{2}(IS + SI) \quad IS \rightarrow \frac{1}{2}(IS + aII + SI) \quad SI \rightarrow \frac{1}{2}(aII + \beta SI + SI). \] (95)

The collision probability of the modified circuit is given by

\[ Z'_{s+1} = \frac{1}{3^n} \left[ \sum_{\vec{\gamma} \in \{II\}^{n+1}} \text{wt}(\gamma) + \frac{1}{2}(1 + \alpha + \beta) \left( \sum_{\vec{\gamma} \in \{IS\}^{n+1}} \text{wt}(\gamma) + \sum_{\vec{\gamma} \in \{SI\}^{n+1}} \text{wt}(\gamma) \right) \right]. \] (96)

Since 2/5 < 1/2, we have \( Z'_{s+1} > Z_{s+1} \).
The collision probability is thus given by
\[ \sum_{\tilde{r}_{ij} = II} \operatorname{wt}(\gamma) + \frac{2}{5} (\alpha + \beta)^2 \left( \sum_{\tilde{r}_{ij} = IS} \operatorname{wt}(\gamma) + \sum_{\tilde{r}_{ij} = SI} \operatorname{wt}(\gamma) \right) + (\alpha + \beta)^2 \sum_{\tilde{r}_{ij} = SS} \operatorname{wt}(\gamma) \].

If instead, we replace the two-qubit gate with two single-qubit gates. We get channels of the form $M^{(i)}_U \circ \mathcal{E}^{(i)} \circ M^{(j)}_U$, which is same as the composite channel in Lemma 4 applied to both qubits. The states evolve as:
\[ II \rightarrow II \quad SS \rightarrow \alpha^2 II + \alpha \beta IS + \alpha \beta SI + \beta^2 SS \quad IS \rightarrow \frac{2}{3} (\alpha^2 II + \alpha \beta IS + \alpha \beta SI + \beta^2). \] (97)

The collision probability is thus given by
\[ Z_{s+1} = \frac{1}{3^n} \left[ \sum_{\tilde{r}_{ij} = II} \operatorname{wt}(\gamma) + \frac{2}{5} (\alpha + \beta)^2 \left( \sum_{\tilde{r}_{ij} = IS} \operatorname{wt}(\gamma) + \sum_{\tilde{r}_{ij} = SI} \operatorname{wt}(\gamma) \right) + (\alpha + \beta)^2 \sum_{\tilde{r}_{ij} = SS} \operatorname{wt}(\gamma) \right]. \] (98)

Since, $(\alpha + \beta) = 1 + 4q(1 - q)/3$, we have that $(2/5)(\alpha + \beta) < 1$, and therefore $Z'_{s+1} > Z_s$.

Starting from the input state, we can use the replacement procedure discussed above to constructively build a new circuit, composed solely of single-qubit gates and SWAP gates, that has a higher collision probability. This concludes the lemma.

\[ \square \]

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