Another Counter-Example to Dirac’s Conjecture

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Abstract

Another counter-example to Dirac’s Conjecture is presented, which resembles the Cawley model but is so modified as to include second class constraints. The arbitrary function in the general solution to the defining equations of momenta satisfies a nonlinear differential equation. Dirac’s conjecture is examined for some solutions to the equation.

1 Introduction

The problem of Dirac’s conjecture has been sometimes argued by various authors since Dirac stated it[1]. One group of them agrees that it holds, while another group does not. The reason of the disagreement seems to be in the setting of the conjecture and the procedure leading to the canonical formalism.

The original statement by Dirac himself can be expressed as that every transformation generated by first class constraint, which we call Dirac transformation, maps a state to its physically equivalent one. According to an interpretation the above statement is equivalent to that the appropriate hamiltonian is the sum of the canonical hamiltonian and linear combination of all first class constraints including the secondary ones. Sugano et al.[2] argued that the above hamiltonian does not work in some models. However, in his canonical formalism, there seems to be no rigorous proof of the equivalence to the lagrangian formalism. On the other hand Deriglazov et al.[3] showed that there is an extended lagrangian, which is obtained by adding auxiliary variables and is classically equivalent to the original lagrangian, where the Dirac transformation of the original lagrangian is the physically equivalent transformation in the theory described by the extended lagrangian.

A system may be described by various lagrangians which are classically equivalent but different from each other in auxiliary or unphysical variables. The constraint structures of the various lagrangians are different and the validity of Dirac’s conjecture depends on the form of each lagrangian. Indeed the gauge symmetry itself is an artifact emerging from the degrees of freedom of the unphysical variables. Hence the meaning of Dirac’s conjecture in Deriglazovs’ formalism seems to be different from that of original Dirac’s one.

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The author of the present paper has proposed a canonical formalism of singular system\[4\] without extending the phase space, which is a straightforward generalization of that of non-singular system. The method is based on employing the general solution for the velocity variables, $u = \hat{U}(q, \pi)$, to the defining equations of momenta, $\pi = W \overset{\text{def}}{=} \partial L/\partial u$. The procedure of the method is logically clear compared with that of Lagrange multiplier by Dirac. The latter is mysterious according to Deriglazov\[5\].

For the model with only first class constraints the condition for the generating function of the Dirac transformation to be the physically equivalent one was given in \[4\]. The Cawley model\[6, 7\] is one of the models in which Dirac’s conjecture does not hold. In order to generalize the formalism of \[4\] to the models with second class constraints, we present here a model with such constraint, where Dirac’s conjecture does not hold. A new aspect which is absent in the model with only first class constraints is that the arbitrary functions appeared in the general solution to the defining equations of momenta are restricted.

2 Cawley model

In order to illustrate the canonical formalism of \[4\], let us start with the Cawley model\[6, 7\]. The action of the Cawley model\[6\] is

$$S_{\text{Cawley}} = \int d\tau L, \quad L = u^1 u^2 + \frac{1}{2} q^3(q^2)^2,$$

where $q^A, (A = 1, 2, 3)$ are the coordinate variables and $u^A, (A = 1, 2, 3)$ are the corresponding velocity variables. The action is invariant under the transformation

$$\delta q^A = \delta_1^A \epsilon + \delta_3^A \frac{d}{d\tau} \left( \frac{\dot{q}^2}{q^2} \right) \quad \delta u^A = \frac{d}{d\tau} (\delta q^A), \quad (A = 1, 2, 3)$$

where $\epsilon$ is an arbitrary infinitesimal parameter.

The Euler-Lagrange equations obtained by varying $q^A$ are

$$[\text{EL}]_1 = \dot{u}^2 = 0, \quad [\text{EL}]_2 = \dot{u}^1 - q^3 q^2 = 0, \quad [\text{EL}]_3 = q^2 = 0.$$

Since the lagrangian does not contain $u^3$, the time development of $u^3$ (and $q^2$) is not determined, so we put $\dot{u}^3 = c$, where $c$ is an arbitrary function of $q$’s and $u$’s. We regard $q^3$ as an unphysical variables. The consistency of the Euler-Lagrange equations requires

$$\ell_1 \overset{\text{def}}{=} q^2 = 0, \quad \ell_2 \overset{\text{def}}{=} u^2 = 0,$$

which are called lagrangian constraints\[8\], and should be satisfied in the initial condition for the differential equations (3). Equation $\ell_1 = 0$ is the third equation of (3) and is called first order lagrangian constraint, while $\ell_2 = 0$ is obtained by time derivative of $\ell_1 = 0$ and is called second order one.

Let us proceed to the canonical theory according to \[4\]. Denote the canonical conjugate of $q^A$ as $\pi_A$. Since the Hessian matrix, $M_{AB} = \partial W_B/\partial u^A$, where $W_B \overset{\text{def}}{=} \partial L/\partial u^B$, is singular, the defining equations of momenta, $\pi_A = W_A(q, u)$, have not unique solution for the velocity
variables. Hence we consider the general solution to the equation and denote it as \( u^A = \hat{U}^A(q, \pi) \). The following functions on the coordinate-velocity space play an important role:

\[
U_{pb}^A(p, u) \overset{\text{def}}{=} \hat{U}^A(q, W(q, u)).
\]  

(5)

We call a function \( A(q, u) \) the pull-back of a function \( \hat{A}(q, \pi) \) if \( A(q, u) = \hat{A}(q, W(q, u)) \), and denote \( A = A_{PB} \). For example, \( U_{pb}^A(p, u) \) are the pull-back of \( \hat{U}^A(q, \pi) \). The functions \( \hat{U}'s \) contain some arbitrary functions, which play the similar role as the Lagrange multipliers in the Dirac theory. In the present model they are \( \hat{U}^1 = \pi_2, \hat{U}^2 = \pi_1, \hat{U}^3 = \hat{v}(q, \pi) \), where \( \hat{v} \) is an arbitrary function of the canonical variables.

The Hamiltonian in the present formalism is defined by

\[
H = \pi_A \hat{U}^A - L(q, \hat{U}),
\]

(6)

and nothing is added[4]. In the usual approach the Hamiltonian is defined by using \( \dot{q}'s \) instead of \( \hat{U}'s \). But the meaning of \( \dot{q}'s \) is obscure, since we are arguing on the phase space. Kamimura[8] developed a generalized canonical formalism on the space spanned by \( (q, \dot{q}, \pi) \). He introduced the concept of generalized canonical quantity which plays a role of canonical variables in the usual formalism.

Now the primary constraint is \( \varphi \overset{\text{def}}{=} \pi_3 = 0 \). Hamiltonian of the present model is expressed as

\[
H = v\varphi + q^3\chi_1 + \pi_2\chi_2, \quad \chi_1 \overset{\text{def}}{=} -\frac{1}{2}(q^2)^2, \quad \chi_2 \overset{\text{def}}{=} \pi_1.
\]

(7)

The first order and the second order secondary constraints are \( \chi_1 = 0 \) and \( \chi_2 = 0 \), respectively. From the definition of the Hamiltonian we see

\[
\dot{\pi}_A = \frac{\partial H}{\partial q^A}, \quad \text{mod } \varphi,
\]

(8)

since we have \( \pi = W(q, \hat{U}(q, \pi)) \) on the constrained sub-space. An orbit O in the velocity-coordinate space is mapped by \( \Phi : (q, u) \mapsto (q, \pi = W(q, u)) \) to an orbit \( \hat{O} \) in the phase space. Since \( \pi = W(q, \hat{U}(q, \pi)) \) on \( \hat{O} \), by differentiating it with respect to \( \tau \) along the orbit we have

\[
\dot{\pi}_A = -\frac{\partial H}{\partial q^A} + \left[ \text{[EL]}_A + (q^B - u^B) \frac{\partial W_A}{\partial q^B} \right]_{u=\hat{U}}, \quad \text{mod } \varphi.
\]

(9)

At this stage, however, the solution orbit of the Euler-Lagrange equations in the velocity-coordinate space has no counterpart in the phase space which is obtained by a full set of canonical equations of motion. In the Dirac recipe they are obtained by the canonical variational principle which imposes the Hamiltonian action, \( \dot{q}^A\pi_A - H_T \), to be stationary, where \( H_T \) is the canonical Hamiltonian plus the Lagrange multiplier terms assuring the constraints. In the present method we do not adopt the Dirac recipe, and we have only the relations (8) and (9).

In order to get the canonical equations of motion, we need relations which express \( \dot{q}^A \) in terms of canonical variables. We determine the relation by imposing that the resulting
canonical equations are equivalent to the Euler-Lagrange equations. The correct choice turns out to be

\[ \dot{q}^A = \dot{U}^A(q, \pi). \] (10)

In fact from eqs.(8)-(10) we see that the canonical equations of motion, \( \dot{q}^A = \partial H / \partial \pi_A, \) \( \dot{\pi}_A = -\partial H / \partial q^A, \) are equivalent to \([\text{EL}]_A(U_{pb}) = \dot{q}^A - U^A_{pb} = 0.\) These equations are equivalent to the second order Euler-Lagrange equations obtained by eliminating \( u \)-variables. As for the secondary constraints, the pull-back of the \( k \)-th order secondary constraints are shown to be the \( k \)-th order lagrangian constraints where \( u \)'s are replaced by \( U_{pb} \)'s [4]. (The pull-back of the primary constraints are of course identity.) Thus the pull-backed theory from the canonical theory is completely described by the original lagrangian with the notational change \( u \to U_{pb}. \)

The canonical equation of motion for a function \( \hat{F}(q, \pi) \) is expressed as

\[ \frac{d}{d \tau} \hat{F} = \{ \hat{F}, H \}, \] (11)

where the Poisson bracket is defined by

\[ \{ \hat{F}, \hat{G} \} \overset{\text{def}}{=} \frac{\partial \hat{F}}{\partial q^A} \frac{\partial \hat{G}}{\partial \pi_A} - \frac{\partial \hat{F}}{\partial \pi_A} \frac{\partial \hat{G}}{\partial q^A}. \] (12)

Now let us examine Dirac’s conjecture. The transformation generated by a linear combination of first class constraints with arbitrary parameters is called Dirac transformation[4]. Dirac’s conjecture claims that Dirac transformation map a state to its physically equivalent one. The all of the Poisson brackets among \( \varphi, \chi_1 \) and \( \chi_2 \) vanish, and they constitute first class constraints. Hence the Dirac transformation is generated by

\[ Q = \hat{\epsilon}^0 \varphi + \hat{\epsilon}^i \chi_i, \quad (\text{sum over } i = 1, 2) \] (13)

where \( \hat{\epsilon}^n, (n = 0, 1, 2) \) are arbitrary infinitesimal quantities, and the transformation is

\[ \delta_Q q^A = \{ q^A, Q \} = \delta_1^A \hat{\epsilon}^2 + \delta_3^A \hat{\epsilon}^0 + \{ q^A, \hat{\epsilon}^1 \} \chi_1 + \{ q^A, \hat{\epsilon}^2 \} \chi_2, \] (14)

where \( \epsilon^n(q, u) \overset{\text{def}}{=} \epsilon^n(q, W(q, u)), (n = 0, 1, 2). \) If one chooses parameters, \( \hat{\epsilon} \)'s, which do not depend on \( \pi, \) then the pull-back transformation is

\[ \delta_D q^A = \delta_1^A \hat{\epsilon}^2 + \delta_3^A \hat{\epsilon}^0, \quad \delta u^A = \frac{d}{d \tau} (\delta q^A). \] (15)

Under the above variations the Euler-Lagrange equation \([\text{EL}]_2 = 0\) varies as

\[ \delta_D [\text{EL}]_2 = \epsilon^2 - \epsilon^0 q^2, \] (16)

which cannot vanish identically at the point where Euler-Lagrange equations and the lagrangian constraints hold. This means the breakdown of Dirac’s conjecture.
For a function $\hat{F}(q, \pi, \tau)$ let us denote

$$\hat{F} \stackrel{\text{def}}{=} \frac{\partial \hat{F}}{\partial \tau} + \{\hat{F}, H\}. \quad (17)$$

We can prove[4] that

$$\dot{F}(q, U_{pb}) = \hat{F}(q, W(q, u)) + [\text{EL}]_A(U_{pb}) \left[ \frac{\partial \hat{F}}{\partial \pi_A} \right]_{\text{PB}}, \quad (18)$$

where $F(q, u, \tau) = \hat{F}(q, W(q, u), \tau)$. Let us define the function $E(\tau)$ by

$$E(\tau) \stackrel{\text{def}}{=} F(q_{sol}, \dot{q}_{sol}),$$

where $q_{sol}$ is the solution to $[\text{EL}]_A = 0$. Then it satisfies $\dot{E}(\tau) = \hat{F}(q_{sol}, \dot{q}_{sol})$. If the generating function $Q$ of the Dirac transformation satisfies $Q = 0 \mod \varphi$, then the transformation is a gauge transformation. This can be proved by a general theorem[4], i.e., $Q = \delta_D L \bigg|_{u=\hat{U}} + \text{total derivative}$, where $\delta_D$ stands for the pull-back of the Dirac transformation. In the present model we have

$$Q = (\hat{\epsilon}^1 - \dot{\epsilon}) \chi_1 + (\hat{\epsilon}^2 - q^2 \hat{\epsilon}^1) \chi_2 \mod \varphi. \quad (19)$$

Hence the Dirac transformation is gauge transformation if $\hat{\epsilon} = \hat{\epsilon}^1$, $\hat{\epsilon}^1 = \hat{\epsilon}^2 / q^2$. The pull-backed transformation is eqs.(2) with $\epsilon(\tau) = \epsilon^2(q_{sol}, q_{sol})$, which keeps the action be invariant. Since $\dot{\epsilon}$ is an arbitrary function of $(q, \pi)$, the parameter $\epsilon(\tau)$ is sufficiently arbitrary. In fact the action is invariant for completely arbitrary $\epsilon(\tau)$.

### 3 A model with second class constraints

The Cawley model contains only the first class constraints. Let us examine whether model with second class constraints has the similar property or it needs modification of the general theory[4], where it assumed there is no second class constraint.

Consider the lagrangian

$$L = u^1 u^2 + q^3 \left( q^2 - \frac{1}{2} q^3 \right). \quad (20)$$

The action is invariant under the following transformation

$$\delta q^A = \delta^A_{\hat{\epsilon}} \epsilon - \delta^A_{\hat{\epsilon}^1} \frac{\dot{\epsilon} u^2}{q^2 - q^3}, \quad (21)$$

where $\epsilon$ is an arbitrary infinitesimal quantity. The Euler-Lagrange equations are

$$[\text{EL}]_1 \stackrel{\text{def}}{=} \dot{u}^2 = 0, \quad [\text{EL}]_2 \stackrel{\text{def}}{=} \dot{u}^1 - q^3 = 0, \quad [\text{EL}]_3 \stackrel{\text{def}}{=} -(q^2 - q^3) = 0. \quad (22)$$

Since the time development of $u^3$ is not determined by the Euler-Lagrange equations, $q^3$ is regarded as unphysical variable, and we set $\dot{u}^3 = c$, where $c$ is tentatively an arbitrary function of $(q, u)$'s.
The first and the second order lagrangian constraints are

\[ \ell_1 \equiv q^2 - q^3 = 0, \quad \ell_2 \equiv u^2 - u^3 = 0. \quad (23) \]

Instead of the third order lagrangian constraint we have the condition \( c = 0 \). It is important to note that the unphysical variable \( q^3 \) must be of the form \( q^3 = a\tau + b \), with constant \( a \) and \( b \), though the action has the gauge degrees of freedom. Otherwise the Euler-Lagrange equations have no solution. In most of the gauge model this is not the case, i.e., the origin of the gauge symmetry is the arbitrariness of the unphysical variables. It would be thought that the above curiosity comes from the fact that the gauge transformation is singular at the point satisfying \( \ell_1 = 0 \). However, in the Cawley model the unphysical variable \( q^3 \) is completely arbitrary, though the gauge transformation is singular at \( \ell_1 = 0 \). The above property in the present model is related to the existence of second class constraint as is shown in the canonical theory.

The Hessian matrix and \( W = \partial L/\partial u \) are

\[
M_{ij} = \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad W_1 = u^2, \quad W_2 = u^1, \quad W_3 = 0. \quad (24)
\]

The primary constraint is \( \varphi \equiv \pi_3 = 0 \), and the general solution for \( u \)'s to the defining equation of momenta, \( \pi = W \), is

\[
\hat{U}^1(q, \pi) = \pi_2, \quad \hat{U}^2(q, \pi) = \pi_1, \quad \hat{U}^3(q, \pi) = \hat{v}(q, \pi), \quad (25)
\]

where \( \hat{v} \) is an arbitrary function of the canonical variables. We see

\[
U_{\text{pb}}^{1,2} = u^{1,2}, \quad U_{\text{pb}}^3 = v(q, u), \quad (26)
\]

where \( v(q, u) \equiv \hat{v}(q, W(q, u)) \).

The hamiltonian is

\[
H = \hat{v}\varphi + \pi_2\pi_1 - q^3 \left( q^2 - \frac{1}{2}q^3 \right) = H_0 + \hat{v}\varphi - q^3\chi_1 + \pi_2\chi_2, \quad H_0 \equiv \hat{v}\pi_2 - \frac{1}{2}(q^3)^2, \quad (27)
\]

where

\[
\chi_1 \equiv q^2 - q^3, \quad \chi_2 \equiv \pi_1 - \hat{v}. \quad (28)
\]

The first and the second order secondary constraints are \( \chi_1 = 0 \) and \( \chi_2 = 0 \), respectively. There is no third order secondary constraints, but we have the condition for \( \hat{v} \):

\[
\hat{v}^\sim = 0 \mod (\varphi, \chi_1, \chi_2). \quad (29)
\]

The pull-back of the secondary constraints, \( \chi_1 = \chi_2 = 0 \), are the lagrangian constraints, \( \ell_1(U_{\text{pb}}) = \ell_2(U_{\text{pb}}) = 0 \), and that of the condition (29) corresponds to the condition \( c = 0 \) in the lagrangian formalism.
Introducing the new constraints by

\[ \phi_2 \overset{\text{def}}{=} \chi_2 - \{\chi_1, \dot{v}\} \varphi + \{\varphi, \dot{v}\} \chi_1 = 0, \]  

(30)

the Poisson brackets among \((\phi_0, \phi_1, \phi_2) = (\varphi, \chi_1, \phi_2)\) are

\[
\{\phi_n, \phi_m\} \overset{\text{def}}{=} \begin{pmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 0 \\
\end{pmatrix} \mod (\phi_0, \phi_1), \quad (n, m = 0, 1, 2).
\]  

(31)

Hence we have two second class constraints, \(\phi_0, \phi_1\) and one first class constraint, \(\phi_2\). The condition (29) comes from the fact that \(\chi_2\) contains \(\dot{v}\), which is a consequence of \(\{\phi_0, \phi_1\} = 1\).

In general the emergence of the condition for the arbitrary function in the general solution to the defining equations of momenta is a characteristic feature in the model with second class constraints.

Since we have only one first class constraint, \(\phi_2 = 0\), the Dirac transformation is generated by \(Q = \hat{\epsilon}\phi_2\), with arbitrary function \(\hat{\epsilon}(q, \pi)\). Let us obtain the condition for \(\hat{\epsilon}\) that the Dirac transformation is gauge transformation. The criterion is \(Q^\sim = 0 \mod \varphi\).

\[
\hat{\epsilon}^\sim = \hat{\epsilon} \left( \frac{\partial \dot{\hat{v}}}{\partial q^3} + c_2 \right), \quad c_1 + c_2 \frac{\partial \dot{\hat{v}}}{\partial q^3} + \frac{\partial \dot{\hat{v}}^\sim}{\partial q^3} = 0, \quad \mod \varphi,
\]  

(32)

where \(c_i\) are defined by

\[
\hat{v}^\sim \overset{\text{def}}{=} c_1 \chi_1 + c_2 \chi_2 \mod \varphi.
\]  

(33)

The pull-back of the Dirac transformation is

\[
\delta_D q^1 = \frac{\partial}{\partial u^2} (\epsilon F), \quad \delta_D q^2 = \frac{\partial}{\partial u^1} (\epsilon F), \quad \delta_D q^3 = \epsilon K + \epsilon^2 F,
\]  

(34)

where

\[
K \overset{\text{def}}{=} \frac{\partial \phi_2}{\partial \pi_3} \bigg|_{\text{PB}}, \quad F \overset{\text{def}}{=} \frac{\phi_2}{\text{PB}}.
\]  

(35)

A choice for the arbitrary function, \(\dot{v}\), partially determines not only the canonical formalism but restricts the lagrangian formalism. Hence, if one choses one of the general solutions, then the pull-back of it restricts the unphysical variables expressed in terms of \(U_{\text{pb}}\) in the lagrangian theory. This is illustrated below.

The condition (29) restricts the function form of \(\dot{v}\), and is written as

\[
\left\{ \dot{\hat{v}}, \pi_1 \pi_2 - q^3 \left( q^2 - \frac{1}{2} q^3 \right) \right\} + \{\dot{\hat{v}}, \pi_3\} \dot{v} = 0 \quad \mod (\varphi, \chi_1, \chi_2)
\]  

(36)

which is a non-linear differential equation for \(\dot{v}\). It seems difficult to obtain the general solution to (36), but we find some solutions:

\[
\dot{v} = 0, \quad \varphi, \quad \chi_1, \quad \pi_1, \quad \frac{(q^3)^2}{2\pi_2}.
\]  

(37)
For different choices of \( \hat{v} \), the gauge structure and the Dirac transformations become different. In the case \( \hat{v} = \pi_1 \) there is no first class constraint, and the gauge symmetries of the canonical formalism are absent. Let us examine the remaining four cases separately.

(1) \( \hat{v} = 0, \varphi, \chi_1 \):
In these cases \( \phi_2 = \pi_1, K = 0, F = u^2 \), and the pull-back of the Dirac transformation is
\[
\delta_D q^A = \delta^A_1 (\epsilon + u^2 \hat{\epsilon}^1) + \delta^A_2 u^2 \hat{\epsilon}^2 + \delta^A_3 \epsilon^3 u^2, \tag{38}
\]
where \( \epsilon^A = [\partial \hat{\epsilon}/\partial \pi_A]_{PB} \). For simplicity let us take \( \hat{\epsilon}^1 = \hat{\epsilon}^2 = 0 \), then the Euler-Lagrange equation, \( [EL]_2 = 0 \), varies under the transformation as
\[
\delta_D [EL]_2 = \dot{\epsilon} - \epsilon^3 u^2, \tag{39}
\]
which cannot vanish for any choice of the unphysical variable, \( q^3 \). Hence Dirac’s conjecture does not hold.

The second term in (32) is satisfied in these cases, and the condition for the Dirac transformation to be gauge transformation becomes \( \dot{\epsilon} = [EL]_A \epsilon^A \). In fact lagrangian varies under the Dirac transformation as
\[
\delta_D L = u^2 (\dot{\epsilon} - [EL]_A \epsilon^A) + T.D.. \tag{40}
\]
If one choose the parameter \( \epsilon(\tau) = \epsilon(q_{sol}, \dot{q}_{sol}) \) as in the case of the Cawley model, then we see \( \dot{\epsilon} = 0 \). This exhibits no gauge symmetry. Instead, we choose the parameter as \( \partial \epsilon/\partial u \equiv 0 \). Here \( \partial W_A/\partial u \equiv 0 \) is automatically satisfied since \( \partial W_A/\partial u^3 = 0 \). Then the condition (32) becomes \( \dot{\epsilon} = -(q^2 - q^3)\epsilon^3 \), and the gauge Dirac transformation is expressed as eq.(21).

(2) \( \hat{v} = (q^3)^2/2\pi_2 \):
In this case we have \( \hat{\epsilon}^\sim = 0 \) mod \( \varphi \). In eq.(27) we see \( H_0 = 0 \), so the hamiltonian itself is a linear combination of constraint functions. In fact, by direct calculations we have \( \phi_2 = H/\pi_2 \).

For simplicity let us express the Dirac transformation in the case of \( \partial \hat{\epsilon}/\partial \pi_A = 0 \):
\[
\delta_D q^1 = \epsilon, \quad \delta_D q^2 = \frac{\epsilon q^3 (2q^2 - q^3)}{2(u^1)^2}, \quad \delta_D q^3 = \epsilon \left( \frac{q^3}{u^1} \right)^2 + \frac{\epsilon^3 H_{PB}}{u^1}. \tag{40}
\]

The Euler-Lagrange equations, \( [EL]_1 = [EL]_2 = 0 \), vary as
\[
\delta_D [EL]_1 = \frac{d^2}{d\tau^2} (\delta_D q^2), \quad \delta_D [EL]_2 = \dot{\epsilon} - \delta_D q^3, \tag{41}
\]
which cannot vanish simultaneously for any choice of the unphysical variable, \( q^3 \). Hence Dirac’s conjecture does not hold.

The condition for the Dirac transformation to be gauge transformation is
\[
\dot{\epsilon} = [EL]_A \epsilon^A = \epsilon \frac{q^3}{u^1}. \tag{42}
\]

We set again \( \epsilon^{1,2} = 0 \), then we have \( \epsilon^3 = (\epsilon q^3 - \dot{\epsilon} u^1)/(u^1(q^2 - q^3)) \). Substituting it into (40), we have the gauge Dirac transformation under which the action is invariant.

Finally, let us comment on the relation of the second class constraints and the Dirac bracket. The Poisson brackets among \( \phi_0, \phi_1 \) are written as
\[
\{ \phi_i, \phi_j \} = A_{ij} \overset{\text{def}}{=} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (i, j = 0, 1). \tag{43}
\]
Using $A_{ij}$, the hamiltonian is expressed as

$$H = H' - \phi_i (A^{-1})^{ij} \{\phi_j, H'\}, \quad H' \overset{\text{def}}{=} -\frac{1}{2} (q^3)^2 + \pi_2 (\phi_2 + \dot{\nu}).$$

(44)

In the above expression the squares of the constraint functions are omitted, which are no effect on the canonical equations of motion, $\dot{F} = \{\hat{F}, H'\}_D$, here the Dirac bracket is defined by

$$\{\hat{F}, \hat{G}\}_D \overset{\text{def}}{=} \{\hat{F}, \hat{G}\} - \{\hat{F}, \phi_i\} (A^{-1})^{ij} \{\phi_j, G\}. \quad (45)$$

4 Concluding remarks

The arbitrary functions, $\hat{U}$’s, emerging in the general solution to the defining equations of momenta play a role similar as the Lagrange multipliers, $\lambda$’s, in the Dirac recipe, though the ways of the appearance of them are entirely different. $\lambda$’s are independent variables, and satisfy linear algebraic equations. The solution to the linear equations is a sum of a special solution and the general solution to a homogeneous equation. The former is a consequence of the presence of the second class constraints and the latter corresponds to the first class ones. In the present approach $\hat{U}$’s generally satisfy non-linear differential equations due to the presence of the second class constraints.

The Dirac recipe may be easy to treat compared with that of the present paper. However, logical validity of the method of Lagrange multiplier is obscure. In fact the method may contain contradiction in a simple model[9]. On the other hand the non-linear differential equations emerged in the presence of the second class constraints may cause a technical difficulty in constructing a general theory with such constraints.

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References

[1] P.A.M. Dirac, *Lectures on Quantum Mechanics*, (Belfer Graduate School of Science, 1964)

[2] R. Sugano and H. Kamo, Prog. Theor. Phys. 67 (1982), 1966.

[3] A.A. Deriglazov, J. Math. Phys. 50, 012907 (2009)

[4] T. Hori, arXiv:1812.08899 [math-ph].

[5] A.A. Deriglazov, *Classical Mechanics: Hamiltonian and Lagrangian Formalism*, (Springer, 2017), the word 'mysterious' appears in p.288.
[6] Cawley, Phys.Rev.Lett. 42(1979), 413.
[7] Cawley, Phys.Rev. D21(1980), 252.
[8] K. Kamimura, IL Nuovo Cimmento, 68B(1982), 33.
[9] A.Frenkel, Phys.Rev. D 21(1982), 2986.