Radiative Leptonic Decays of the charged $B$ and $D$ Mesons Including Long-Distance Contribution

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In this work we study the radiative leptonic decays of $B^-$, $D^-$ and $D_s^-$ to $\gamma\ell\bar{\nu}$, including both the short-distance and long-distance contributions. The short-distance contribution is calculated by using the relativistic quark model, where the bound state wave function we used is that obtained in the relativistic potential model. The long-distance contribution is estimated by using vector meson dominance model. Smaller result for $D_{(s)} \rightarrow l\bar{\nu}\gamma$ is also obtained in Ref. [1] within the non-relativistic constituent quark model, which gives that the branching ratio of $D^- \rightarrow l\bar{\nu}\gamma$ is of the order of $10^{-6}$ and $D_s^- \rightarrow l\bar{\nu}\gamma$ of the order of $10^{-5}$. The problem of factorization in QCD for $B \rightarrow l\nu\gamma$ is studied in Ref. [3].

In this work, we study the radiative leptonic decays of the charged $B$, $D$ and $D_s$ mesons to $l\bar{\nu}\gamma$ including both the short and long-distance contributions. The short-distance contribution is considered at tree level. The wave function of the heavy meson used here is obtained in the relativistic potential model previously [6]. The long-distance contribution is estimated by using the idea of the vector meson dominance (VMD) [1][4][5] followed by the transition of the vector meson to a photon. We find that the long-distance contribution can enhance the decay rates seriously.

The remaining part of this paper is organized as follows. In Sec.II, we present the short-distance amplitude. In Sec.III, the long-distance contribution is considered. The numerical results and discussion are given in Sec.IV. Sec.V is a brief summary.

II The Short-Distance Contribution

We use $P$ to denote the pseudoscalar meson which is composed of a heavy anti-quark $\bar{Q}$ and a light quark $q$, such as $B$ and $D$ mesons. There are four Feynman diagrams contributing to the radiative decays $P^- \rightarrow l\nu\gamma$ at tree level, which are shown in Fig.1. However the contribution of Fig.1(d) is suppressed by a factor of $1/M_{\pi^0}^2$, it can be neglected for simplicity. The effective Hamiltonians corresponding to the other three diagrams in Fig.1 can be written as:

$$B(P \rightarrow l\bar{\nu}) = \frac{G_F^2|V_{Qq}|^2}{8\pi}\tau_P f_P m_l^2 m_P (1 - \frac{m_l^2}{m_P^2})^2,$$

where $G_F$ is the Fermi coupling constant, $V_{Qq}$ the Cabibbo-Kobayashi-Maskawa (CKM) matrix element, $\tau_P$ the life time of the meson $P$, and $m_P$ and $m_l$ the masses of the meson $P$ and lepton $l$, respectively. The decay rate is proportional to the lepton mass squared $m_l^2$ is the consequence of the helicity suppression. However, the presence of one photon in the final state can compensate the helicity suppression. As a result, the radiative leptonic decay can be as large as, or even larger than the pure-leptonic decay mode. It thus opens a window for detecting the dynamics of strong interaction in the heavy meson or studying the effect of strong interaction in the decay.

The radiative leptonic decay rates of the charged $B$ and $D$ mesons have been studied with various methods in the literature. In Ref. [1], $B$ and $D_s \rightarrow l\bar{\nu}\gamma$ are calculated in a non-relativistic quark model, the branching ratios of the order of $10^{-4}$ for $D_s \rightarrow l\bar{\nu}\gamma$ and $10^{-6}$ for $B \rightarrow l\bar{\nu}\gamma$ are found. In Ref. [2] with perturbative QCD approach, it is found that the branching ratio of $D_s^+ \rightarrow e^+\nu\gamma$ is of the order of $10^{-3}$ and $D^+ \rightarrow e^+\nu\gamma$ of the order of $10^{-4}$, while the branching ratio of $B^+ \rightarrow e^+\nu\gamma$ is at the order of $10^{-6}$. On the other hand, a smaller branching ratio is obtained for $D_{(s)} \rightarrow l\bar{\nu}\gamma$ within the light front quark model [3]. Smaller result for $D_{(s)} \rightarrow l\bar{\nu}\gamma$ is also obtained in Ref. [3] within the non-relativistic constituent quark model, which gives that the branching ratio of $D^- \rightarrow l\bar{\nu}\gamma$ is of the order of $10^{-6}$ and $D_s^- \rightarrow l\bar{\nu}\gamma$ of the order of $10^{-5}$. The problem of factorization in QCD for $B \rightarrow l\nu\gamma$ is studied in Ref. [3].

The numerical results and discussion are given in Sec.IV. Sec.V is a brief summary.
where $P_L$ is defined as $\gamma^\mu (1 - \gamma_5) q$, and $V_{Qq}$ represents for the CKM matrix elements. $Q_Q$ and $Q_q$ are the electric charges of the quarks $Q$ and $q$, respectively. $A$ is the electro-magnetic field.

$H_a$, $H_b$ and $H_c$ can be divided into two terms for convenience, according to the numerator of the fermion propagator. For example, $H_a$ can be written as

$$H_a = \frac{-ieQ_Q G_F V_{Qq}}{2\sqrt{2}} (M_{a1} + 2M_{a2})^\mu (lP_{L\bar{u}}\bar{v}),$$

where

$$M_{a1} = Q_A \frac{p_q}{p_\gamma p_Q} \gamma^\mu (1 - \gamma_5) q,
= -ie\alpha^\mu (A_{a1}) p_\gamma v_a + \frac{m^2_P}{p_\gamma p_Q} (A_{a2} - A_{a1}) (p_\gamma v_a),$$

$$M_{a2} = \frac{1}{2} Q_A \frac{m^2_P - p_q^2}{p_\gamma p_Q} \gamma^\mu (1 - \gamma_5) q = -A_{a1} v_a^\mu,$$

with

$$v_a^\mu = Q_A \frac{1}{p_\gamma p_Q} \gamma^\mu (1 - \gamma_5) q,$$

$$t_a^\mu = Q_A \frac{p_q^2}{p_Q p_\gamma} \gamma^\mu (1 - \gamma_5) q.$$  

The amplitude of the radiative leptonic decay can be obtained by inserting the operator of the effective Hamiltonian between the initial and final particle states. For example, the contribution of Fig. 1(a) is

$$A_a = <\gamma \bar{v} l | H_a | P > = \frac{-ieQ_Q G_F V_{Qq}}{2\sqrt{2}} (u_\gamma P_{L\bar{u}} v_\bar{l}) \times \chi (| M_{a1} | P > + 2 | \gamma | M_{a2} | P >)$$

$$\times (-ie\alpha^\mu (A_{a1}) p_\gamma < v_a | P > + p^\mu_\gamma \gamma_5 < v_a | P > - \epsilon^\mu_\gamma < v_a | P > - 2\epsilon^\mu_\gamma < v_a | t_a^\mu | P >).$$

The matrix elements $< v_a | P >$ and $< t_a^\mu | P >$ only depend on the momenta $p_P$ and $p_\gamma$. According to their Lorentz structure, they can be decomposed as a linear combination of two terms of $p_P$ and $p_\gamma$.

$$V_a^\mu = < v_a | P > = \frac{m_P}{p_P p_\gamma} (A_{a1} p_\gamma + A_{a2} p_P p_\gamma),$$

$$T_a^\mu = < t_a^\mu | P > = \frac{m_P}{p_P p_\gamma} (A_{a2} p_\gamma p_P + m^2_P (p_P p_\gamma + C_2 p_P p_\gamma)$$

$$+ \frac{m^2_P}{p_\gamma p_P} D_{a2} p_\gamma p_P^2 + E_{a2} m^2_P g_{a\mu}$$

$$+ F_{a2} \frac{m^2_P}{p_\gamma p_P} \gamma_\mu^a p_P p_\gamma)$$

where $A_{a1}$, $B_{a1}$, $A_{a2}$, $B_{a2}$, $C_{a2}$, $D_{a2}$, $E_{a2}$ and $F_{a2}$ are all dimensionless constants. The terms of $B_{a1}$, $C_{a2}$ and $D_{a2}$ do not contribute to the decay amplitude $A_a$ when substituting the above decomposition into eq.6. Therefore these terms can be dropped. The coefficients can be obtained by the treatment in the following. Multiplying $V_a^\mu$ with $p_\gamma$, we can obtain $A_{a1}$ as

$$A_{a1} = \frac{1}{m_P} < v_a | - \frac{1}{p_P p_\gamma} \gamma_\mu (1 - \gamma_5) q | P >.$$  

Similarly, multiplying $T_a^\mu$ with $p_\gamma$, we have

$$B_{a2} + E_{a2} = 0,$$

$$A_{a2} = \frac{1}{m_P} < v_a | - \frac{1}{p_P p_\gamma} \gamma_\mu (1 - \gamma_5) q | P >.$$  

Multiplying $T_a^\mu$ with $p_P p_\mu$, and $g_{a\mu}$, and using $B_{a2} = -E_{a2}$, one can get

$$E_{a2} = -B_{a2} = \frac{1}{2m_P} < v_a | (p_P p_\gamma) \gamma_\mu$$

$$- (p_P p_\gamma) \gamma_\mu (1 - \gamma_5) q | P >.$$
Finally $F_{a2}$ can be obtained by multiplying $T_{a}^{\alpha\mu}$ with $\varepsilon_{\alpha\beta\gamma\delta}p_{\gamma}^{\mu}p_{\delta}^{\beta}$:

$$F_{a2} = \frac{p_{\gamma}^{\mu}p_{\delta}^{\beta}}{2m_{P}^{2}} < 0 \mid Q_{\gamma}p_{\gamma}^{2}p_{\delta}^{\beta} (1 - \gamma_{5}) q \mid P > .$$ (11)

The amplitude $A_{b}$ can be treated in the same way with some coefficients defined as follows

$$< 0 \mid Q_{\gamma}p_{\gamma}^{2} (1 - \gamma_{5}) q \mid P > =$$

$$\frac{m_{P}^{2}}{p_{P} \cdot p_{\gamma}} \left( A_{b1}p_{P}^{\mu} + B_{b1} m_{P}^{2} p_{\gamma}^{\mu} \right),$$

$$< 0 \mid Q_{\gamma}p_{\gamma}^{2} (1 - \gamma_{5}) q \mid P > =$$

$$\frac{m_{P}^{2}}{p_{P} \cdot p_{\gamma}} \left( A_{b2}p_{P}^{\mu}p_{P}^{\nu} + \frac{m_{P}^{2}}{p_{P} \cdot p_{\gamma}} (B_{b2}p_{P}^{\mu}p_{\gamma}^{\nu} + C_{b2}p_{P}^{\mu}p_{\gamma}^{\nu}) + \frac{m_{P}^{4}}{(p_{P} \cdot p_{\gamma})^{2}} D_{b2}p_{P}^{\mu}p_{\nu}^{\gamma} + E_{b2}m_{P}^{2} g^{\mu\nu} \right).$$ (12)

Using the matrix element $A_{a3}m_{P}p_{P}^{\mu} = < 0 \mid Q_{\alpha}^{\mu} (1 - \gamma_{5}) q \mid P >$, the amplitude $A_{c}$ can be treated similarly. Finally, the total amplitude can be expressed as

$$A_{a} = \frac{ieG_{F}V_{Qq}}{2\sqrt{2}} \left\{ u_{l}p_{\mu}V_{l} \right\} \left( \frac{Q_{m}m_{P}A_{a1}}{p_{P} \cdot p_{\gamma}} + \frac{Q_{m}m_{P}A_{a2}}{p_{P} \cdot p_{\gamma}} + \frac{m_{P}A_{a2}}{p_{P} \cdot p_{\gamma}} \right)$$

$$- 2 \left( Q_{m}m_{P}A_{a1} + \frac{Q_{m}m_{P}A_{a2}}{p_{P} \cdot p_{\gamma}} + \frac{m_{P}A_{a2}}{p_{P} \cdot p_{\gamma}} \right) \left( \frac{Q_{Q}m_{P}A_{a1}}{p_{P} \cdot p_{\gamma}} + \frac{Q_{Q}m_{P}A_{a2}}{p_{P} \cdot p_{\gamma}} + \frac{m_{P}A_{a2}}{p_{P} \cdot p_{\gamma}} \right),$$

$$A_{b} = \frac{ieG_{F}V_{Qq}}{2\sqrt{2}} \left\{ u_{l}p_{\mu}V_{l} \right\} \left( \frac{Q_{m}m_{P}A_{b1}}{p_{P} \cdot p_{\gamma}} + \frac{Q_{m}m_{P}A_{b2}}{p_{P} \cdot p_{\gamma}} + \frac{m_{P}A_{b2}}{p_{P} \cdot p_{\gamma}} \right)$$

$$- 2 \left( Q_{m}m_{P}A_{b1} + \frac{Q_{m}m_{P}A_{b2}}{p_{P} \cdot p_{\gamma}} + \frac{m_{P}A_{b2}}{p_{P} \cdot p_{\gamma}} \right) \left( \frac{Q_{Q}m_{P}A_{b1}}{p_{P} \cdot p_{\gamma}} + \frac{Q_{Q}m_{P}A_{b2}}{p_{P} \cdot p_{\gamma}} + \frac{m_{P}A_{b2}}{p_{P} \cdot p_{\gamma}} \right),$$

$$A_{c} = \frac{ieG_{F}V_{Qq}}{2\sqrt{2}} \left\{ u_{l}p_{\mu}V_{l} \right\} \left( \frac{Q_{m}m_{P}A_{c1}}{p_{P} \cdot p_{\gamma}} + \frac{Q_{m}m_{P}A_{c2}}{p_{P} \cdot p_{\gamma}} + \frac{m_{P}A_{c2}}{p_{P} \cdot p_{\gamma}} \right)$$

$$- 2 \left( Q_{m}m_{P}A_{c1} + \frac{Q_{m}m_{P}A_{c2}}{p_{P} \cdot p_{\gamma}} + \frac{m_{P}A_{c2}}{p_{P} \cdot p_{\gamma}} \right) \left( \frac{Q_{Q}m_{P}A_{c1}}{p_{P} \cdot p_{\gamma}} + \frac{Q_{Q}m_{P}A_{c2}}{p_{P} \cdot p_{\gamma}} + \frac{m_{P}A_{c2}}{p_{P} \cdot p_{\gamma}} \right).$$ (13)

The above equations show that the contribution of each diagram in Fig. 11 is not gauge invariant separately, but the sum of them is indeed gauge invariant, which is given in the following

$$A_{a+b+c} = -ieG_{F}V_{Qq} \left\{ iV_{\alpha}^{\beta}m_{P}^{\gamma} \varepsilon_{e}^{\gamma}p_{P} \varepsilon_{\gamma}^{\alpha} \right\}$$

$$+ \left( p_{P} \cdot p_{\gamma} \right) \left( p_{P} \cdot p_{\gamma} \right) + 2 \left( p_{P} \cdot p_{\gamma} \right) \left( p_{P} \cdot p_{\gamma} \right) A_{a2}m_{P}^{2} p_{P}^{\mu}$$

$$\times \left( u_{l}p_{\mu}p_{P} \right).$$ (14)

This equation clearly shows that the sum of the contributions of all the diagrams in Fig. 11 is gauge invariant. In eq. (14) the factors $V$ and $A$ are

$$V = - \frac{Q_{Q}m_{P}A_{a1}}{p_{P} \cdot p_{\gamma}} - \frac{Q_{Q}m_{P}A_{a2}}{p_{P} \cdot p_{\gamma}} + \frac{m_{P}A_{a2}}{p_{P} \cdot p_{\gamma}} + \frac{2(Q_{Q}E_{a2} - Q_{Q}E_{a1}) m_{P}^{2}}{(p_{P} \cdot p_{\gamma})^{2}}$$

$$A = - \frac{Q_{Q}m_{P}A_{b1}}{p_{P} \cdot p_{\gamma}} - \frac{Q_{Q}m_{P}A_{b2}}{p_{P} \cdot p_{\gamma}} + \frac{m_{P}A_{b2}}{p_{P} \cdot p_{\gamma}} + \frac{2(Q_{Q}E_{b2} - Q_{Q}E_{b1}) m_{P}^{2}}{(p_{P} \cdot p_{\gamma})^{2}}.$$ (15)

Next we shall calculate the coefficients $A_{a1}, A_{a2}, E_{a1}$ and $F_{a2}$.

The pseudoscalar meson state can be written in terms of the quark-antiquark creation and annihilation operators

$$| P (P = 0) > = \frac{1}{\sqrt{3}} \sum_{i} \int d^{3}k \Psi_{0}(| k |)$$

$$\times \frac{1}{\sqrt{2}} \left( b_{q}^{+} (k, \gamma) d_{q}^{-} (-\bar{k}, \gamma) - b_{q}^{+} (k, \gamma) d_{q}^{-} (-\bar{k}, \gamma) \right) | 0 > .$$ (16)

where $i$ is the color index, the factor $1/\sqrt{3}$ the normalization factor for color indices, and $\bar{k}$ the 3-momentum of the quarks in the rest frame of the heavy meson. The wave function $\Psi_{0}(| k |)$ has been calculated in the relativistic potential model previously, the numerical solution of the wave function can be fitted in the exponential form

$$\Psi_{0}(| k |) = 4\pi \sqrt{3m_{P}e^{-\lambda| k |}}.$$ (17)

The numerical solutions of the parameter $\lambda$ for $D$, $D_{s}$ and $B$ mesons are quoted from Ref. 12 recently

$$\lambda_{D} = 3.4 GeV^{-1}, \quad \lambda_{D_{s}} = 3.2 GeV^{-1},$$

$$\lambda_{B} = 2.8 GeV^{-1}.$$ (18)

With the meson state given in eq. (16), the matrix element $< 0 \mid Q\Gamma q \mid P >$ can be calculated straightfor-
The vector's decay constant $f_V$ can be derived from the decay rate of $V \to e^+e^-$. After a short calculation, the decay constants can be related to the vector meson's leptonic decay widths

\[
\begin{align*}
 f_{\rho}^2 & = \frac{3m_{\rho}^3}{4\pi\alpha^2} \frac{2}{(Q_u - Q_d)^2} \Gamma_{\rho \to e^+e^-}, \\
 f_{\omega}^2 & = \frac{3m_{\omega}^3}{4\pi\alpha^2} \frac{2}{(Q_u + Q_d)^2} \Gamma_{\omega \to e^+e^-}, \\
 f_{\phi}^2 & = \frac{3m_{\phi}^3}{4\pi\alpha^2} \frac{1}{Q_s^2} \Gamma_{\phi \to e^+e^-},
\end{align*}
\]

where $\alpha$ is the electromagnetic fine structure constant, $Q_u$, $Q_d$, and $Q_s$ are the charges of the quarks $u$, $d$, and $s$, respectively.

The hadronic matrix element $<V | J^\mu | P>$ in eq. (20) can be decomposed according to its Lorentz structure as [13,14]

\[
\begin{align*}
 & <V | J_{V - A_\mu} | P > = \sum_{q=pp,pV} e^{\phi_V} \frac{2}{m_P + m_V} (m_P + m_V) A_0 (q^2) \\
 & \quad \times \left\{ e_{V} \mu \cdot q \frac{m_P + m_V}{p^2} V + i \frac{\varepsilon_V^\mu \cdot \varepsilon_V}{q^2} \frac{m_P + m_V}{p^2} \right\} + i \frac{\varepsilon_V^\mu \cdot \varepsilon_V}{q^2} \frac{2m_V q_\mu A_0 (q^2)}{q^2},
\end{align*}
\]

where $q = p_P - p_V$, and $V$, $A_0$, $A_1$, $A_2$ and $A_3$ are form factors.

To obtain the amplitude gauge invariant, we take the trick used in Ref. [12] in treating the long-distance contribution in the process $b \to s\gamma$ via the resonance $J/\Psi$. With the Lorentz decomposition of the hadronic matrix element in eq. (20), the product of the two matrix elements in eq. (20) can be calculated to be

\[
\begin{align*}
 & \varepsilon_{\gamma \alpha} < 0 | q \gamma^\alpha q | V < V | J_{V - A}^\mu | P > = \\
 & e_Q \phi_{qV} f_V \left\{ - \frac{2}{m_P + m_V} e^{\alpha\beta\mu\nu} p_{P\sigma} p_{V\rho} \varepsilon_{\gamma\alpha} V (q^2) \\
 & \quad - i A_1 (q^2) \left\{ \varepsilon_{\gamma\alpha} \frac{m_P + m_V}{p^2} - \frac{(m_P + m_V)(p_P - \varepsilon_{\gamma\alpha})}{p_P p_V} \right\} \\
 & \quad - i (p_P - \varepsilon_{\gamma}) \left\{ \frac{m_P + m_V}{p^2} A_1 (q^2) \\
 & \quad - \frac{(m_P + m_V)}{m_P + m_V} A_2 (q^2) + \frac{2m_V q^\mu}{q^2} (A_3 (q^2) - A_0 (q^2)) \right\} \right\}.
\end{align*}
\]

In the rest-frame of the heavy meson $P$, the product of the four-momentum of the meson $P$ and the polarization vector of the photon satisfies $p_P \cdot \varepsilon = 0$. Then the last term in the above equation can be dropped. With $p_V = p_P$,
we obtain
\[
A_{LD} = \sum_V \frac{eG_F V_{q\bar{q}} Q_V}{\sqrt{2}} u_i \gamma^\mu (1 - \gamma_5) v_{\bar{\nu}} \\
\times \left\{ iV_L V^\dagger \epsilon_{\alpha \beta \gamma} p_\gamma^\dagger \epsilon_{\alpha \beta} p_\gamma \\
+ A_{LD}^I [(p_{\gamma} - p_{\gamma'}) \epsilon^\dagger \gamma_5 - (p_{\gamma'} - p_{\gamma}) \epsilon^\dagger \gamma_5] \right\},
\]
(26)
where:
\[
V_{LD}^V = \frac{e^{i\phi_V} C_{q\bar{q}} f_V}{-m_{V}^2 + i m_{V} \Gamma_{V}} \frac{2}{m_{p} + m_{V}} V(q^2),
\]
\[
A_{LD}^I = \frac{e^{i\phi_V} C_{q\bar{q}} f_V}{-m_{V}^2 + i m_{V} \Gamma_{V}} \frac{m_{p} + m_{V}}{p_{\gamma} p_{\gamma'}} A_1(q^2).
\]
(27)

IV Numerical Result and Discussion

The numerical calculation is performed in the center-of-mass frame of the heavy meson, and the momentum parameters, the masses of the quarks are taken as
\[
m_q = m_\ell = 0.08 \text{GeV}, \quad m_\ell = 0.30 \text{GeV},
\]
\[
m_{b} = 4.98 \text{GeV}, \quad m_{c} = 1.54 \text{GeV},
\]
which are taken to be consistent with that used to derive the wave function of the heavy meson in the relativistic potential model in Ref. [6].

The electron and neutrinos are taken to be massless, the masses of the other leptons are taken from PDG [10]. The Cabibbo-Kobayashi-Maskawa (CKM) matrix elements are:
\[
V_{cd} = 0.2259, V_{cs} = 0.974, V_{ub} = 0.00389.
\]

The infrared divergence appears as the energy of the real photon goes to soft limit or the momentum of the photon is parallel to the momentum of a massless lepton. This divergence can be canceled when the decay width of the radiative leptonic decay is added with the pure leptonic decay width, in which one-loop radiative corrections are included [17]. The radiative leptonic decay with the energy of the photon lower than the experimental resolution can not be distinguished from the pure leptonic decay. Only photons with the energy larger than the experimental resolution can not be distinguished from the pure leptonic decay. The photon energy resolution can not be distinguished from the pure leptonic decay. Only photons with the energy larger than the experimental resolution can be distinguished. Therefore the radiative leptonic decay width depends on the photon energy resolution. The photon energy resolution can be a few MeV in experiment [18]. The dependence of the decay width on the resolution \(\Delta E_\gamma\) is shown in Table I and Fig. 3. For example, if taking \(\Delta E_\gamma = 10\) MeV, the decay width and branching ratio of \(D \rightarrow \gamma e\bar{\nu}_e\) are
\[
\Gamma(D \rightarrow \gamma e\bar{\nu}_e) = 1.98 \times 10^{-18} \text{GeV},
\]
\[
Br(D \rightarrow \gamma e\bar{\nu}_e) = 3.29 \times 10^{-6}.
\]
(28)

In the following all the numerical calculation is performed by taking the resolution \(\Delta E_\gamma = 10\) MeV.

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FIG. 3: Decay widths of \(D^- \rightarrow \gamma e\bar{\nu}\) as a function of \(\Delta E_\gamma\).

| \(\Delta E_\gamma\) (MeV) | \(\Gamma(D^- \rightarrow \gamma e\bar{\nu})\) (MeV) | \(\Delta E_\gamma\) (MeV) | \(\Gamma(D^- \rightarrow \gamma e\bar{\nu})\) (MeV) |
|---------------------------|---------------------------------|---------------------------|---------------------------------|
| 5                         | 2.30                           | 55                        | 1.25                           |
| 10                        | 1.98                           | 60                        | 1.12                           |
| 15                        | 1.80                           | 65                        | 1.19                           |
| 20                        | 1.67                           | 70                        | 1.16                           |
| 25                        | 1.57                           | 75                        | 1.14                           |
| 30                        | 1.49                           | 80                        | 1.12                           |
| 35                        | 1.43                           | 85                        | 1.10                           |
| 40                        | 1.38                           | 90                        | 1.08                           |
| 45                        | 1.33                           | 95                        | 1.06                           |
| 50                        | 1.29                           | 100                       | 1.04                           |

We also checked that the decay width is not sensitive to the cutoff of the angle between the momentum of the photon and the electron. We find taking different values for the angle cutoff \(\Delta \theta\), the result changes extremely slowly.

To show the contribution of each diagram in Fig. 1, we list each diagram’s contribution to the decay width in Table II. It is shown that, although the amplitude of the diagram with the photon emitted from the heavy quark is suppressed by a factor of the heavy quark mass in the propagator, which is like \(\frac{i}{p_{\gamma} - p_{\gamma'}}\), for the cases of \(D\) and \(D_s\) mesons, the mass of \(c\) quark is not large enough, so the contributions of diagrams \(a\), \(b\) and \(c\) are all at the same order, but for the case of \(B\) decay, the suppression is large, the contribution of diagram \(b\) dominates. It can also be shown that, the contributions of the diagrams in Fig. 1 interfere destructively, especially in the case of \(D\) and \(D_s\) mesons, this is consistent with Ref. [4].

We present the branching ratios with only the short-distance contributions for all the decay modes in TA-
Using the data on the decay rate of

tau is related to the decay constant defined in eq.(21).

For the short-distance contribution, the branching ratio is via \( \phi \) meson. \( BR_{LD2} \) is the branching ratios of long-distance contribution via \( \omega \) meson.

| Modes | \( BR_{SD} \) | \( BR_{LD1} \) | \( BR_{LD2} \) |
|-------|----------------|----------------|----------------|
| \( D_s \to \gamma \tau \bar{\nu}_\tau \) | \( 3.3 \times 10^{-6} \) | \( 7.5 \times 10^{-6} \) | \( 6.3 \times 10^{-6} \) |
| \( D_s \to \gamma \mu \bar{\nu}_\mu \) | \( 1.6 \times 10^{-5} \) | \( 7.3 \times 10^{-6} \) | \( 6.1 \times 10^{-6} \) |
| \( D_s \to \gamma \tau \bar{\nu}_\tau \) | \( 5.5 \times 10^{-9} \) | \( 9.1 \times 10^{-10} \) | \( 7.6 \times 10^{-10} \) |
| \( B \to \gamma \tau \bar{\nu}_\tau \) | \( 6.8 \times 10^{-6} \) | \( 1.0 \times 10^{-4} \) | - |
| \( B \to \gamma \mu \bar{\nu}_\mu \) | \( 1.9 \times 10^{-5} \) | \( 1.0 \times 10^{-4} \) | - |
| \( B \to \gamma \tau \bar{\nu}_\tau \) | \( 2.3 \times 10^{-6} \) | \( 6.5 \times 10^{-8} \) | - |
| \( B \to \gamma \mu \bar{\nu}_\mu \) | \( 2.3 \times 10^{-5} \) | \( 1.0 \times 10^{-4} \) | - |
| \( B \to \gamma \tau \bar{\nu}_\tau \) | \( 2.1 \times 10^{-6} \) | \( 3.8 \times 10^{-7} \) | \( 2.7 \times 10^{-7} \) |

The branching ratio for \( D_s \to \gamma \tau \bar{\nu}_\tau \) is very small, because the mass of \( \tau \) is very large, the phase space for this decay mode is too small.

The short-distance branching ratios obtained in this work is slightly smaller than the previous works[3]. More details about the quark momentum distribution are included in this work by using wave function obtained in the relativistic potential model.

To calculate the long-distance contribution, the transition amplitude \( V \to \gamma \) is needed. The transition amplitude is related to the decay constant defined in eq.(21).

Using the data on the decay rate of \( V \to l^+l^- \) given in PDG[10], the decay constants of the vector mesons can be extracted as

\[
\begin{align*}
&f_\rho = 0.169869 \text{GeV}, \\
&f_\omega = 0.154631 \text{GeV}, \\
&f_\phi = 0.231784 \text{GeV}.
\end{align*}
\]

The \( q^2 \) dependence of the form factors defined in the hadronic matrix element \( <V|J_{\mu}^p|A_i> \) are taken to be the usual pole approximation as

\[
V(q^2) = \frac{h_v}{1 - \frac{q^2}{M_V^2}}, \quad A_i(q^2) = \frac{h_{\alpha i}}{1 - \frac{q^2}{M_{A_i}^2}}.
\]

The parameters in the form factors for \( D \to V \) and \( B \to V \) transitions are quoted from Refs. [19] and [20], they are

\[
\begin{align*}
&h_{\nu_{V \to \rho}} = 1.0, \quad h_{\alpha 1_{D \to \rho}} = 0.5; \\
&M_{V_{D \to \rho}} = 2.5 \text{ GeV}; \quad M_{A_{1\rho}} = 2.5 \text{ GeV}; \\
&h_{\nu_{B \to \rho}} = 0.323, \quad M_{V_{B \to \rho}}^2 = 38.34 \text{ GeV}^2; \\
&h_{\alpha 1_{B \to \rho}} = 0.242, \quad M_{A_{1\rho}}^2 = 37.51 \text{ GeV}^2; \\
&h_{\nu_{B \to \omega}} = 0.293, \quad M_{V_{B \to \omega}} = 37.45 \text{ GeV}^2; \\
&h_{\alpha 1_{B \to \omega}} = 0.219, \quad M_{A_{1\omega}} = 37.01 \text{ GeV}^2.
\end{align*}
\]

For the form factor of \( D \to \omega \), we assume it is the same as that of \( D \to \rho \). The form factors of \( D_s \to \phi \) is taken from [21] as

\[
h_{\nu_{D_s \to \phi}} = 1.21, \quad h_{\alpha 1_{D_s \to \phi}} = 0.55, \quad M_{V_{D_s \to \phi}} = 2.08 \text{ GeV}.
\]

The form factor \( A_1 \) for \( D_s \to \phi \) transition is approximated as a constant because the \( q^2 \) dependence of \( A_1 \) is very weak [21].

With the parameters given above, the long-distance contributions to the decay width for each decay mode can be estimated, they are listed in Table III where the short-distance and long-distance contributions to the branching ratios of each decay mode are presented separately. It shows that for decays of \( D \) and \( D_s \) mesons, long-distance contributions are as large as or even larger than the short-distance contributions, while for the case of \( B \) decays, short-distance contributions dominate, long-distance contributions are roughly 4 ~ 5 times smaller than short-distance contributions.

To get the total decay width, including both the short and long-distance contributions, one has to know the relative phase between the long and short-distance amplitudes. Unfortunately we do not know the relative phases exactly up to now. We have to leave the relative phases as free parameters. To show how the decay width depends on the relative phases, we show the decay widths of \( B \), \( D \) and \( D_s \to \gamma \bar{\nu}_\tau \) decays in Fig. I as an example. In the case of \( D \to \gamma \bar{\nu}_\tau \) decay, because the long-distance contributions are as important as the short-distance contributions, the relative phases between the long and short-distance contributions can affect the decay widths considerably, the decay widths can change several times. For \( D_s \to \gamma \bar{\nu}_\tau \) decay, the long-distance contribution dominates (see Table III), the dependence of the total branching ratio on the relative phase is weak. While for \( B \to \gamma \bar{\nu}_\tau \) meson decay, the amplitudes of the short and long-distance contributions are at the same order, therefore the decay width still depends on the relative phase severely. For the case of the other decay modes, the dependence of the total branching ratios on the relative phases are illustrated numerically in Table IV. The situation is similar to the \( B \), \( D \) and \( D_s \to \gamma \bar{\nu}_\tau \) decay modes. The contribution of the long-distance physics is important, in general the branching ratio heavily depends on the relative phase between
FIG. 4: From up to down, they are decay widths (GeV) of $D^-$, $B^-$ and $D_s^-$ → $\gamma e\bar{\nu}_e$ decays, including both the short and long-distance contributions, as functions of $\phi_\rho$ and $\phi_\omega$ in the cases of $D^-$ and $B^-$ meson decays, and $\phi_\phi$ in the case of $D_s^-$ meson decay.

the long and short-distance contributions. Some decay modes can be greatly enhanced by the long-distance contributions. The branching ratio of $D_s^-$ → $\gamma e\bar{\nu}_e$ decay can be enhanced from the order 10$^{-6}$ to 10$^{-4}$.

V Summary

The radiative leptonic decays of $B$, $D$ and $D_s$ → $l\nu\gamma$ are studied in this work. The short-distance contribution is calculated by using the wave functions of the heavy mesons obtained in the relativistic potential model, more details about the quark-momentum distribution are included in this work. In addition to the short-distance contribution, the long-distance contribution is also estimated in the vector meson dominance model. The study shows that the long-distance contributions can seriously affect the decay rates. The branching ratio of $D_s^-$ → $\gamma e\bar{\nu}_e$ can be enhanced to the order of 10$^{-4}$, which should only be at the order of 10$^{-6}$ if only considering the short-distance contribution.

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TABLE IV: Total branching ratios of the radiative leptonic decays including both the short and long-distance contributions. In case of $D$ meson decays, for illustration, the relative phase $\phi = \phi_\omega$ and $\phi_\rho = 0$ are taken. While for $B$ meson decays, $\phi = \phi_\rho = \phi_\omega$ is taken, and in case of $D_s$ meson decays, $\phi$ is $\phi_\omega$.

| Modes   | $BR_{tot}(\phi = 0^\circ)$ | $BR_{tot}(\phi = 30^\circ)$ | $BR_{tot}(\phi = 60^\circ)$ | $BR_{tot}(\phi = 90^\circ)$ | $BR_{tot}(\phi = 120^\circ)$ | $BR_{tot}(\phi = 150^\circ)$ |
|---------|----------------------------|-------------------------------|-------------------------------|-------------------------------|-------------------------------|-------------------------------|
| $D \to \gamma e \bar{\nu}_e$ | $4.0 \times 10^{-6}$ | $6.0 \times 10^{-9}$ | $3.0 \times 10^{-9}$ | $4.1 \times 10^{-9}$ | $3.4 \times 10^{-9}$ | $1.5 \times 10^{-9}$ |
| $D \to \gamma \mu \bar{\nu}_\mu$ | $2.5 \times 10^{-5}$ | $3.1 \times 10^{-5}$ | $4.8 \times 10^{-5}$ | $5.9 \times 10^{-5}$ | $5.2 \times 10^{-5}$ | $3.5 \times 10^{-5}$ |
| $D \to \gamma \tau \bar{\nu}_\tau$ | $5.7 \times 10^{-9}$ | $7.5 \times 10^{-9}$ | $1.2 \times 10^{-8}$ | $1.4 \times 10^{-8}$ | $1.2 \times 10^{-8}$ | $8.0 \times 10^{-9}$ |
| $B \to \gamma e \bar{\nu}_e$ | $6.7 \times 10^{-6}$ | $8.9 \times 10^{-6}$ | $1.3 \times 10^{-5}$ | $1.4 \times 10^{-5}$ | $1.2 \times 10^{-5}$ | $8.2 \times 10^{-6}$ |
| $B \to \gamma \mu \bar{\nu}_\mu$ | $6.8 \times 10^{-6}$ | $8.9 \times 10^{-6}$ | $1.3 \times 10^{-5}$ | $1.4 \times 10^{-5}$ | $1.2 \times 10^{-5}$ | $8.3 \times 10^{-6}$ |
| $B \to \gamma \tau \bar{\nu}_\tau$ | $5.3 \times 10^{-6}$ | $6.7 \times 10^{-6}$ | $9.2 \times 10^{-6}$ | $1.0 \times 10^{-5}$ | $8.8 \times 10^{-6}$ | $6.3 \times 10^{-6}$ |
| $D_s \to \gamma e \bar{\nu}_e$ | $1.5 \times 10^{-4}$ | $1.3 \times 10^{-4}$ | $0.9 \times 10^{-4}$ | $0.7 \times 10^{-4}$ | $0.9 \times 10^{-4}$ | $1.3 \times 10^{-4}$ |
| $D_s \to \gamma \mu \bar{\nu}_\mu$ | $4.5 \times 10^{-4}$ | $4.4 \times 10^{-4}$ | $4.1 \times 10^{-4}$ | $3.9 \times 10^{-4}$ | $4.1 \times 10^{-4}$ | $4.4 \times 10^{-4}$ |
| $D_s \to \gamma \tau \bar{\nu}_\tau$ | $1.0 \times 10^{-6}$ | $1.0 \times 10^{-6}$ | $1.2 \times 10^{-6}$ | $1.3 \times 10^{-6}$ | $1.2 \times 10^{-6}$ | $1.0 \times 10^{-6}$ |