CARBON SPOT PRICES IN EQUILIBRIUM FRAMEWORKS ASSOCIATED WITH CLIMATE CHANGE

ZHENZHEN WANG
School of Mathematics and Statistics, Guangdong University of Foreign Studies
Guangzhou, 510006, China

HAO DONG
Lingnan (University) College, Sun Yat-Sen University
Guangzhou, 510275, China

ZHEHAO HUANG*
Guangzhou Institute of International Finance, Guangzhou University
Guangzhou, 510405, China

(Communicated by Jinxia Zhu)

ABSTRACT. At present, it is believed that the best approach to mitigate global warming is the market-based formulation of carbon emission pricing. Thus, in this paper, we work on determining the carbon spot prices in a stochastic equilibrium framework associated with climate change. Two circumstances, differentiated by whether taking carbon trading in the market, are considered. We construct optimization problems and solve them by using dynamic programming principle. The Fourier transform and its properties are fully made use of to return the explicit formulas of carbon prices. In addition, some surprising but interesting properties of the carbon prices are also found. First, the carbon prices happen jumps at the end of the abatement period. Second, the return rates of carbon prices are completely dependent on the climate elements. Finally, we present some numeric results in response to our theoretical results.

1. Introduction. The increasingly growing greenhouse gas emission was leading to global climate warming. Consequently, rise in temperature caused negative impacts on the economic output (Burke et al. [5], Mardani et al. [32], Solana [42]). In keeping with the need to ensure the environmental quality and mitigate the global warming, governments have been on the lookout for possible ways to reduce greenhouse gas emissions. The Kyoto Protocol, announced by thirty-seven industrialized countries and entered into force in 2005, stabilized the greenhouse gas concentration in the atmosphere at an appropriate level and prevented severely climate change from damaging human welfare. The protocol imposed a higher burden on these industrialized countries because it committed them to an overall reduction of six main greenhouse gases emissions by a minimum of 5% between 2008 and 2012 (Arouri et
To lower abatement costs, the European Union (EU) established the Emission Trading Scheme (ETS) in the same year, which is nowadays the largest and most liquid greenhouse gas emission reduction mechanism in the world (Ellerman and Buchner [18]). The EU ETS has formed an effective operating paradigm for carbon emission trading around the world and accumulated a wealth of data and experience (Zhang and Wei [51]). Afterward, the Paris Agreement was reached in April 2016 with the intention of designing new mitigation ways for climate change after 2020 (Farouq et al. [20]). China performs her environmental responsibility by establishing a gradually mature and integrated national carbon market to meet the challenges of climate change. Eight pilot carbon markets have been set up in China to date, making effective progress in reducing greenhouse gas emissions.

It is widely believed that the best approach to reduce global warming is the market-based formulation of carbon emission pricing, which can calculate the emission cost by directly considering the economic efficiency (Klepper and Peterson [29], Nordhaus [35], Cai and Lontzek [6]). In light of this, it is particularly major to determine rational carbon prices. Methods for carbon market and carbon prices have merged in an endless stream for decades. With both theory and practice developing, more and more sophisticated methods appear at carbon price modeling with the widespread application of mathematical methods, statistic tools, and computer technology. These methods are generally considered to be classified into three types: econometric models, artificial intelligence models and equilibrium framework models. The econometric models are relatively traditional, including autoregressive moving integrated moving average (ARIMA), vector auto-regression (VAR), generalized autoregressive conditional heteroscedasticity (GARCH) and the like (Benz and Truck [4], Garcia-Martos et al. [22], Sanin et al. [40], Zheng et al. [52], Fu and Zheng [21]). Although these statistical and econometric methods are able to achieve effective results and perform commendable theoretical description based on statistical logic, their implementations are rooted in the linear hypothesis, which does not have adequate ability to uncover the nonlinear characteristics hidden in the time series. For the sake of solving this problem, artificial intelligence models sprung up, applied to model time series, which are the data-driving methods (Atsalakis [3]). Artificial intelligence models mainly cover artificial neural network (ANN) and support vector machine (SVM) (Patel et al. [37], Zhu et al. [54]). Nonetheless, given a large fluctuation towards the original carbon price sequences, the effect of a single artificial intelligence technology is not very outstanding. It is usually associated with the step of data preprocessing (Fan et al. [19], Rather et al. [39], Li [31], Wang et al. [47]). Artificial intelligence models indeed show their advantage in modelling carbon prices, but they cannot clearly reply to the question which quantitative formulas carbon prices should satisfy, because they do not disclose the inner links between carbon prices and some endogenous and exogenous variables. It is difficult for us to understand the formation mechanism of carbon prices. However, if considering carbon price modeling and forecasting in equilibrium frameworks with fully making use of mathematical tools, one can return some desired quantitative results. This is what we take our efforts to work on in this paper.

The literature that evaluates carbon prices under an equilibrium framework usually results in optimal equilibrium carbon prices. Seifert et al. [41] presented a tractable stochastic equilibrium model reflecting stylized features of the EU ETS and found that carbon prices do not have to follow any seasonal pattern. Abadie
and Chamorro [1], used a two-dimensional binomial lattice to derive the optimal investment rule. In particular, they obtained the trigger allowance prices above which it is optimal to install the capture unit immediately. Daskalakis et al. [14] claimed that the prohibition of banking of emission allowances between distinct phases of the EU ETS has significant implications in terms of future pricing, and developed an empirically and theoretically valid framework for pricing and hedging. By exploiting an arbitrage relationship, Çetin and Verschuere [9] derived the spot prices of carbon allowances according to filtering theory, given a forward contract whose price is exogenous to the model. Carmona et al. [7] showed that the economic mechanism of carbon price formation can be formulated in the framework of competitive stochastic equilibrium models, and that its solution reduces to an optimally stochastic control problem. Using this setup, they identified the main carbon price drivers and showed how stochastic control can be used to treat quantitative problems in carbon price risk management. Carmona and Hinz [8] gave a rigorous analysis of a simple risk-neutral reduced-form model for carbon future prices, demonstrated its calibration to historical data, and showed how to price European call options written on these contracts. Chesney and Taschini [13] modeled endogenously the price dynamics of emission permits under asymmetric information, allowing intertemporal banking and borrowing by a dynamic optimization method. Golosov et al. [23] analyzed a dynamic stochastic general-equilibrium (DSGE) model with an externality through climate change from using fossil energy. Their central result is a simple formula for the marginal externality damage of emissions. van der Ploeg and de Zeeuw [44] priced carbon by allowing for conventional marginal climate damages and decomposed the optimal carbon tax into catastrophe component and the conventional component. Hitzemann and Uhrig-Homburg [27] presented a stochastic equilibrium model for environmental markets that allows studying the characteristic properties of emission permit prices and characterized emission permits as highly nonlinear contingent claims on economy-wide emissions. Meanwhile, some empirical performances for price dynamics of emission permits were addressed as well (Hitzemann and Uhrig-Homburg [28]). Hambel et al. [26] studied a DSGE model involving climate change which allows for a systematic analysis of the social cost of carbon. They documented that climate uncertainty delivers a major contribution to the social cost of carbon.

In this paper, we determine carbon prices by stochastic equilibrium models associated with climate change. It is known that equilibrium prices not only reflect the carbon abatement cost, helping the central planner of society formulate optimally abatement strategies, but also provides a benchmark for prices in the carbon trading markets. Following Hambel et al. [26], our baseline model consists of three components, the carbon emission, the economic output and the climate change. Moreover, these three components form a loop and exhibit the nonlinear mechanism. Then we construct optimization problems on the baseline model, where the objectives of the central planner are to achieve carbon abatement with minimum cost. Meanwhile, two circumstances are considered, where the first is that the society has to assume penalty cost if the objective of carbon abatement fails to be achieved, and the other is that carbon emission trading is allowable to avoid penalty cost. The dynamic programming principle is applied to solve the constructed optimization problems, and in particular, the explicit formulas of carbon prices are derived by making use of the Fourier transform and its properties. The explicit formulas clearly disclose the intrinsic relationship between carbon prices and other endogenous or
exogenous variables, such that both sensitivity analysis and effect analysis can be taken. Further analysis on the derived explicit formulas is addressed to return more information and characteristics of the carbon prices.

The rest of the paper is organized as follows. In Section 2, we depict the three components of the baseline model, involving the economic model, the carbon emission model and the climate model. In Section 3, we construct the optimization problems in two circumstances, differentiated by whether considering carbon trading in response to a failure of carbon abatement. In Section 4, we solve the optimization problems and return the closed-form solutions. In Section 5, we take an analysis on the closed-form solutions of carbon prices, showing some interesting and surprising findings on their shapes and behaviors. In Section 6, we present some numeric results. In Section 7, we conclude the paper.

2. Baseline model setup. In this section, we introduce the baseline model, consisted of the economic model, the carbon emission model and the climate model. The carbon emission model keeps track of carbon dioxide in the atmosphere. The growing carbon dioxide mainly origins from anthropological shocks, while the non-man-made carbon dioxide is assumed natural sinking such as oceanic absorption. The central planner of society can control anthropological carbon emission by formulating effective abatement strategies. But these efforts are costly. The climate model depicts the change of average temperature of some area. Empirically, there exists a significantly positive relation between carbon dioxide concentration and the zone temperature, despite the noisy relation. The economic model describes the dynamics of economic output in a stylized production economy. In our benchmark settings, the climate warming has a damage on the economic output. The central planner of society can only indirectly mitigate this damage by carrying out carbon abatement. This is the link of the economic model to the carbon emission model. In this way, it completes the circle. The central planner of society would formulate an optimal abatement strategy, resulting in mitigating the climate warming and then promoting the economic output. In turn, the increasing output signifies letting out more carbon dioxide.

2.1. Economic model. The economic output of a society is usually measured by its GDP data. We use the geometric Brownian motion (lognormal distribution model) to model the dynamics of the GDP process

\[ dY_t = \mu Y_t dt + \sigma Y_t dW_t^Y, \]

where \( Y_t \) is the GDP at time \( t \), \( \mu \) is the return rate, \( \sigma \) is the volatility and \( W_t^Y \) is a standard Brownian motion. Chen [11, 12] discussed the rationality and limitation of using geometric Brownian motion for GDP modeling and pointed that there is little doubt about the validity of geometric Brownian motion in economic dynamics. Some existing literature that uses continuous and discrete geometric Brownian motions for GDP modeling can also be referred to Chamon and Mauro [10], Kruse et al. [30], Matei [33].

There are two approaches to model economic damages induced by climate warming. The first one is modeling damages as negative impacts on the growth rate of output, which is suggested by empirical evidence (Dell et al. [15, 16]). Another one is assuming that the current temperature directly affects the level of output (Nordhaus [34]). In this paper, it is more appropriate to adopt the growth rate impact in our modelling (Nordhaus and Sztorc [36], Weitzman [48]). Then the GDP
accumulates according to
\[ dY_t = \mu Y_t dt - \beta Z_t Y_t dt + \sigma Y_t dW^Y_t, \]
where the scaling parameter \( \beta \) is a positive constant that relates temperature increase \( Z \) to loss of economic growth.

2.2. Carbon emission model. We follow the STIRPAT (Stochastic Impacts by Regression on Population, Affluence and Technology) model, developed by Dietz and Rosa [17] and original from the IPAT model (Waggoner and Ausubel [46]). The STIRPAT model can be written as
\[ X = aP^bA^cT^d e, \]
(2.3)
where the variable \( X \) represents the environmental effect, measured by the carbon emissions in our paper, \( P \) represents the population, \( A \) represents the affluence, \( T \) represents the technology and \( e \) is the error with mean one. \( a, b, c, d \) are supposed to be positive constants. In practice, the technology \( T \) is usually difficult to measure through data. It is always involved into the error together with other influencing factors. Thus the STIRPAT model (2.3) is simplified into the form
\[ \log X = \log a + (b - c) \log P + c \log Y + \log \hat{e}, \]
(2.4)
where \( \log \hat{e} = f_0(t) dt + \sigma dW^X_t \),
(2.7)
where \( f_0(t) = \rho \left( P_m P_0 \right) e^{-\rho t} \) and we add the white noise for uncertainty in the development of the population. Then according to the simplified STIRPAT model (2.5), the carbon emission can be modeled as follows:
\[ dX_t = \alpha_1 dY_t + \alpha_2 dP_t + \alpha_3 dW^X_t = (\alpha_1 \mu Y_t - \alpha_1 \beta Z_t + \alpha_2 f_0(t))dt + \sigma dW_t, \]
(2.8)
where \( \sigma^2 = \alpha_2^2 \sigma^2_P + \alpha_3^2 \sigma^2_W + \sigma^2_Y \) and \( W^X_t \) is a standard Brownian motion independent of \( W^Y_t \) and \( W^P_t \).
2.3. Climate model. The climate model is the empirically well-documented logarithmic relationship between temperature and atmospheric carbon emissions (Hambel et al. [26])

\[ Z_t = \zeta \log X_t. \]  

(2.9)

The parameter \( \zeta \) relates the change in temperature to the change in carbon emission. Hambel et al. [26] added a self-exciting process to the dynamic change of \( Z_t \), since there is empirical evidence that the distribution of future temperature change is right-skewed (Tong and Liu [43]). One reason for this is that there might be delayed climate feedback loops triggered by the increase in the temperature. In our paper, we assume that this characteristic of temperature dynamics involving a self-exciting process is captured by setting \( \zeta = \zeta(t) \), i.e. the parameter \( \zeta \) is time-varying.

3. Optimization problems. In general, it is supposed that the central planner of society would like to achieve the abatement objective with a minimum abatement cost expenditure. The objective is to control the carbon emission below some given threshold at the deadline of some period. For instance, China has announced her objective of peak carbon emission in around 2030. The Global Energy Internet Development Cooperation Organization published a report about China’s carbon peak on March 18, 2021 in Beijing, forecasting that China’s total carbon emission will reach the peak with about 11000 million tons. Then it is rational to determine the emission threshold referring to this forecasted peak. On the other hand, for high contracting parties in the Kyoto Protocol, this emission threshold can be determined as the assigned amount units of carbon emission (carbon quota).

In light of the above analysis, the social abatement cost can be formulated as follows:

\[ C^u_t = \mathbb{E}_t \left[ \int_t^T C_1(s, u_s)ds + C_2(X_T) \right], \]

where the control variable \( u_t \) is the abatement strategy. The total abatement cost consists of two parts, where \( C_1 \) describes the abatement cost per unit of time, while \( C_2 \) describes the terminal cost at the end of the abatement period, which closely relates to the emission threshold \( X \). We set \( C_1(t, u_t) = c_1 u_t^2 \), where \( c_1 \) is the abatement cost coefficient. The underlying assumption is that the industry in the society has approximately linearly increasing aggregated marginal abatement cost (Seifert et al. [41], Yang and Liang [49]). The abatement strategy \( u_t \) is monotonically increased and concave function of \( C_1 \), which indicates that the central planner first uses the cheapest abatement opportunities when reducing carbon emission. The more they have to abate, the more expensive the remaining abatement opportunities will be. This is consistent with the principle of diminishing marginal effect. An aggregated marginal abatement cost curve for the EU can be seen in Viguier et al. [45]. A constant cost coefficient means that the abatement technology does not change during the considered time frame. This seems a reasonable assumption, as a technical breakthrough is difficult and usually needs a long enough time. With regard to the terminal cost \( C_2 \), it is determined in the following two circumstances.

Notice that we employ a zero discount rate such that we can deduce the explicit formulas of carbon prices in the following argument. Indeed, a discount rate will not play the important role in the formation of carbon prices.

3.1. Carbon abatement without carbon trading. In this circumstance, the society autonomously performs its environmental responsibility by carrying out carbon abatement projects without taking part in the carbon trading market, like some
developing and emerging countries, especially China. So the terminal cost $C_2$ is a penalty cost, involving all possible losses and expenditures should be paid for the failure to achieve the abatement objective. Thus, the penalty cost is set to be

$$C_2(X_T) = c_2(X_T - X)^+,$$

in this circumstance, where $c_2$ is the penalty cost coefficient. Similar carbon abatement model without considering carbon trading can be referred to the paper of Yang and Liang [49]. Then the corresponding optimization problem can be formulated as follow:

$$V(t, x; u_t) = \min_u \{C^n_t | X_t = x\}, \quad (3.1)$$

subject to

$$dX_t = (\alpha_1 \mu_Y - \alpha_1 \beta Z_t + \alpha_2 f_0(t) - u_t)X_t dt + \sigma X_t dW_t, \quad (3.2)$$

where $Z_t = \zeta(t) \log X_t$, the relationship between the average temperature and carbon emission.

3.2. **Carbon abatement with carbon trading.** In this circumstance, the society can purchase carbon emission permits through the carbon trading market due to the failure of controlling the terminal carbon emission below the emission threshold. Thus, the terminal cost $C_2$ is a purchase expenditure. Similar carbon abatement models considering carbon market trading can be referred to Guo and Liang [24, 25]. Indeed, the contracting parties in the *Kyoto Protocol* belong to this circumstance. They can meet their commitments by either reducing their carbon emissions or purchasing AAU (Assigned Amount Unit), CER (Certified Emission Reduction) or ERU (Emission Reduction Unit) in the trading market. Accordingly, the purchase cost is set to be

$$C_2(X_T) = Q_T(X_T - X)^+,$$

where $Q_T$ is the carbon market price at time $T$. The corresponding optimization problem can be formulated as follow:

$$V(t, x, q; u_t) = \min_u \{C^n_t | X_t = x, Q_t = q\}, \quad (3.3)$$

subject to the carbon emission (3.2) and the change of market price

$$dQ_t = \mu_Q Q_t dt + \sigma_Q Q_t dW_t^Q, \quad (3.4)$$

where the Brownian motion $W_t^Q$ is correlative to the Brownian motion $W_t$ in (3.2), satisfying $\rho dt = dW_t^Q \cdot dW_t$, $-1 \leq \rho \leq 1$.

4. **Closed-form solutions.** In this section, we solve the optimization problem (3.1) subject to condition (3.2), and the optimization problem (3.3) subject to conditions (3.2), (3.4) through the dynamic programming principle.

4.1. **Solution without carbon trading.** The following theorem presents the closed-form solution to the optimization problem (3.1) subject to condition (3.2).

**Theorem 4.1.** The closed-form solution to the optimization problem (3.1) subject to condition (3.2) is given as

$$u^*_t = \frac{\sigma^2 \int_\infty^{-\infty} \exp(-d(e^y - X)^+)w(t, \log x - y)(\log x - y + B(t))dy}{2A(t) \int_\infty^{-\infty} \exp(-d(e^y - X)^+)w(t, \log x - y)dy}, \quad (4.1)$$
and
\[ V(t, x; u_t^*) = 2c_1\sigma^2 \log \sqrt{2\pi} - 2c_1\sigma^2\alpha_1\beta \int_t^T \zeta(s)ds \]
\[ - 2c_1\sigma^2 \log \int_{-\infty}^{\infty} \exp(-d(e^y - \overline{X})^+)w(t, \log x - y)dy, \]

where \( d = c_2/(2c_1\sigma^2) \), and
\[ w(t, x) = \frac{1}{\sqrt{2A(t)}} \exp \left( -\frac{(x + B(t))^2}{4A(t)} \right), \]

and
\[ A(t) = \frac{\sigma^2}{2} \int_t^T \exp \left( 2\alpha_1\beta \int_t^r \zeta(s)ds \right)dr, \]

and
\[ B(t) = \int_t^T \left( \alpha_1\mu_Y + \alpha_2 f_0(r) - \frac{\sigma^2}{2} \right) \exp \left( \alpha_1\beta \int_t^r \zeta(s)ds \right)dr. \]

Proof. The Hamilton-Jacobi-Bellman (HJB) equation corresponding to the optimization problem is written as
\[ \frac{\partial V}{\partial t} + (g(t) - \alpha_1\beta \zeta(t) \log x) x \frac{\partial V}{\partial x} + \frac{\sigma^2}{2} x^2 \frac{\partial^2 V}{\partial x^2} + \inf_u \left\{ c_1 u_t^2 - u_t x \frac{\partial V}{\partial x} \right\} = 0, \]

(4.3)

where \( g(t) = \alpha_1\mu_Y + \alpha_2 f_0(t) \), with terminal condition
\[ V(T, x) = c_2(x - \overline{X})^+. \]

(4.4)

The optimal abatement strategy is given by
\[ u_t^* = x \frac{\partial}{\partial x} V(t, x). \]

(4.5)

Substituting (4.5) into (4.3) gives
\[ \frac{\partial V}{\partial t} + (g(t) - \alpha_1\beta \zeta(t) \log x) x \frac{\partial V}{\partial x} + \frac{\sigma^2}{2} x^2 \frac{\partial^2 V}{\partial x^2} - \frac{x^2}{4c_1} \left( \frac{\partial V}{\partial x} \right)^2 = 0, \]

(4.6)

Now we take a series of transforms to simplify (4.6). First applying the Cole-Hopf transform
\[ V(t, x) = -2c_1\sigma^2 \log U(t, x) \]

to (4.6), we get
\[ \frac{\partial U}{\partial t} + (g(t) - \alpha_1\beta \zeta(t) \log x) x \frac{\partial U}{\partial x} + \frac{\sigma^2}{2} x^2 \frac{\partial^2 U}{\partial x^2} = 0, \]

(4.7)

with terminal data
\[ U(T, x) = \exp(-d(x - \overline{X})^+), \]

where \( d = c_2/(2c_1\sigma^2) \). Then taking the coordinate transform \( \xi = \log x, \tau = T - t \) gives
\[ \frac{\partial U}{\partial \tau} = \frac{\sigma^2}{2} \frac{\partial^2 U}{\partial \xi^2} + \left( g(T - t) - \alpha_1\beta \zeta(T - \tau)\xi - \frac{\sigma^2}{2} \right) \frac{\partial U}{\partial \xi}, \]

(4.8)

with initial data
\[ U(0, \xi) = \exp(-d(e^\xi - \overline{X})^+). \]

(4.9)

We approximate the initial data (4.9) by the following function
\[ U_n(\xi) = \exp(-d(e^\xi - \overline{X})^+)\chi_n(\xi), \]
where function \( \chi_n(\xi) \) satisfies
\[
\chi_n(\xi) = 1 \text{ for } \xi \geq -n, \quad \chi_n(\xi) = 0 \text{ for } \xi < -n, \quad (4.10)
\]
for some sufficient large \( n \), i.e. the initial data \( U(0, \xi) \) is truncated at the negative infinity such that \( U_n(0, \xi) \in L(-\infty, \infty) \). Then taking a Fourier transform
\[
\hat{U}(\tau, \lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} U(\tau, \xi)e^{-i\lambda\xi}d\xi,
\]
in (4.8), whose inversion formula is given as
\[
U(\tau, \xi) = \lim_{N \to \infty} \frac{1}{\sqrt{2\pi}} \int_{-N}^{N} \hat{U}(\tau, \lambda)e^{i\lambda\xi}d\lambda,
\]
and using the following properties of Fourier transform
\[
\frac{\partial \hat{U}}{\partial \xi} = i\lambda \hat{U}, \quad \frac{\partial^2 \hat{U}}{\partial \xi^2} = -\lambda^2 \hat{U}, \quad \xi \frac{\partial \hat{U}}{\partial \xi} = i \frac{d}{d\lambda}(i\lambda \hat{U}) = -\frac{d}{d\lambda}(\lambda \hat{U}), \quad (4.11)
\]
there holds that
\[
\frac{\partial \hat{U}}{\partial \tau} = -\frac{\sigma^2}{2}\lambda^2 \hat{U} + \left( g(T - \tau) - \frac{\sigma^2}{2} \right)i\lambda \hat{U} + \alpha_1 \beta \zeta(T - \tau) \frac{\partial}{\partial \lambda}(\lambda \hat{U}).
\]
Letting
\[
\hat{U}(\tau, \lambda) = \exp \left( \alpha_1 \beta \int_{0}^{\tau} \zeta(T - s)ds \right) \phi(\tau, \lambda),
\]
then \( \phi(\tau, \lambda) \) satisfies
\[
\frac{\partial \phi}{\partial \tau} = -\frac{\sigma^2}{2} \hat{\lambda}^2 \phi + \left( g(T - \tau) - \frac{\sigma^2}{2} \right)i\lambda \phi + \alpha_1 \beta \zeta(T - \tau) \frac{\partial \phi}{\partial \lambda}.
\]
Taking a coordinate transform \( \ell = \log \lambda \) and letting \( \psi(\tau, \ell) = \phi(\tau, \lambda) \), then \( \psi(\tau, \ell) \) satisfies
\[
\frac{\partial \psi}{\partial \tau} = -\frac{\sigma^2}{2} e^{2\ell} \psi + \left( g(T - \tau) - \frac{\sigma^2}{2} \right)i e^{\ell} \psi + \alpha_1 \beta \zeta(T - \tau) \frac{\partial \psi}{\partial \ell}.
\]
Finally, taking the transform \( \psi(\tau, \ell) = \varphi(\tau, z) \), where \( z = \ell + \alpha_1 \beta \int_{0}^{\tau} \zeta(T - s)ds \), there holds that
\[
\frac{\partial \varphi}{\partial \tau} = -\frac{\sigma^2}{2} \exp \left( 2z - 2\alpha_1 \beta \int_{0}^{\tau} \zeta(T - s)ds \right) \varphi
\]
\[
+ \left( g(T - \tau) - \frac{\sigma^2}{2} \right)i \exp \left( z - \alpha_1 \beta \int_{0}^{\tau} \zeta(T - s)ds \right) \varphi,
\]
which is an ordinary differential equation and can be solved as
\[
\varphi(\tau, z) = \varphi_0(z) \exp(A(\tau, z) + iB(\tau, z)),
\]
where
\[
\varphi_0(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-d(e^\xi - X)^+ - ie^\xi) \chi_n(\xi)d\xi,
\]
and
\[
A(\tau, z) = -\frac{\sigma^2}{2} \int_{0}^{\tau} \exp \left( 2z - 2\alpha_1 \beta \int_{0}^{\tau} \zeta(T - s)ds \right) dr,
\]
and
\[
B(\tau, z) = \int_{0}^{\tau} \left( g(T - r) - \frac{\sigma^2}{2} \right) \exp \left( z - \alpha_1 \beta \int_{0}^{r} \zeta(T - s)ds \right) dr.
\]
Now we return to the original value function \( V(t, x) \) with a series of inverse transforms. Then the value function \( V(t, x) \) satisfies

\[
V(t, x) = -2c_1\sigma^2 \log U(t, x),
\]

where

\[
U(t, x) = \lim_{N \to \infty} \frac{1}{\sqrt{2\pi}} \int_{-N}^{N} \varphi_0(\lambda) \times \exp \left( -A(t)\lambda^2 + i(B(t) + \log x)\lambda + \alpha_1\beta \int_{t}^{T} \zeta(s)ds \right) d\lambda,
\]

and

\[
\varphi_0(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-d(e^\xi - \overline{X})^\tau - i\lambda\xi) \chi_n(\xi) d\xi,
\]

and

\[
A(t) = \frac{\sigma^2}{2} \int_{t}^{T} \exp \left( 2\alpha_1\beta \int_{t}^{s} \zeta(s)ds \right)dr,
\]

and

\[
B(t) = \int_{t}^{T} \left( g(r) - \frac{\sigma^2}{2} \right) \exp \left( \alpha_1\beta \int_{t}^{s} \zeta(s)ds \right) dr.
\]

Note that the Fourier transform of function \( e^{-\xi^2} \) is given as

\[
\hat{e}^{-\xi^2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\xi^2 - i\lambda\xi} d\xi = \frac{1}{\sqrt{2}} e^{-\lambda^2/4}.
\] (4.12)

Indeed, there holds that

\[
e^{-\xi^2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\xi^2 - i\lambda\xi} d\xi = \frac{2i}{\sqrt{2\pi}\lambda} \int_{-\infty}^{\infty} \xi e^{-\xi^2 - i\lambda\xi} d\xi = \frac{2i}{\lambda} \int_{-\infty}^{\infty} e^{-\xi^2} d\xi = \frac{2}{\lambda} d\lambda e^{-\xi^2}.
\] (4.13)

Then (4.12) is the solution of the equation (4.13), which implies that

\[
e^{-\xi^2/(4A(t))} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\xi^2/(4A(t)) - i\lambda\xi} d\xi = \sqrt{2A(t)} e^{-A(t)\lambda^2}.
\]

Then it follows that

\[
\lim_{N \to \infty} \frac{1}{\sqrt{2\pi}} \int_{-N}^{N} \exp(-A(t)\lambda^2 + i\lambda\xi) d\lambda = \frac{1}{\sqrt{2A(t)}} \exp \left( -\frac{\xi^2}{4A(t)} \right),
\]

which implies that

\[
\lim_{N \to \infty} \frac{1}{\sqrt{2\pi}} \int_{-N}^{N} \exp(-A(t)\lambda^2 + i\lambda B(t) + i\lambda\xi) d\lambda
\]

\[
= \lim_{N \to \infty} \frac{1}{\sqrt{2\pi}} \int_{-N}^{N} \exp(-A(t)\lambda^2 + i\lambda(\xi + B(t))) d\lambda
\]

\[= w(t, \xi),
\]

where

\[
w(t, \xi) = \frac{1}{\sqrt{2A(t)}} \exp \left( -\frac{(\xi + B(t))^2}{4A(t)} \right).
\]
Then by utilizing the property of Fourier transform, we have
\[
\lim_{N \to \infty} \frac{1}{\sqrt{2\pi}} \int_{-N}^{N} \varphi_0(\lambda) \exp(-A(t)\lambda^2 + iB(t)\lambda + i\lambda \xi) d\lambda \\
= \lim_{N \to \infty} \frac{1}{\sqrt{2\pi}} \int_{-N}^{N} U_n(\lambda) \overrightarrow{w(t, \lambda)} e^{i\lambda \xi} d\lambda \\
= \frac{1}{\sqrt{2\pi}} U_n * \overrightarrow{w(t, \xi)},
\]
where the convolution satisfies
\[
U_n * \overrightarrow{w(t, \xi)} = \int_{-\infty}^{\infty} U_n(y) \overrightarrow{w(t, \lambda)}(y) \lambda \overrightarrow{w(t, \lambda)}(y) d\lambda.
\]
Thus the function \(U(t, x)\) is rewritten as
\[
U(t, x) = \frac{1}{\sqrt{2\pi}} \exp \left( \alpha_1 \beta \int_{t}^{T} \zeta(s) ds \right) \int_{-\infty}^{\infty} U_n(y) \overrightarrow{w(t, \lambda)}(y) \lambda \overrightarrow{w(t, \lambda)}(y) d\lambda.
\]
Letting \(n \to \infty\) gives
\[
U(t, x) = \frac{1}{\sqrt{2\pi}} \exp \left( \alpha_1 \beta \int_{t}^{T} \zeta(s) ds \right) \int_{-\infty}^{\infty} \exp(-d(y - q\overrightarrow{X})^+) \overrightarrow{w(t, \lambda)}(y) \lambda \overrightarrow{w(t, \lambda)}(y) d\lambda,
\]
which is followed by
\[
V(t, x) = 2c_1 \sigma^2 \log \sqrt{2\pi} - 2c_1 \sigma^2 \alpha_1 \beta \int_{t}^{T} \zeta(s) ds \\
- 2c_1 \sigma^2 \log \int_{-\infty}^{\infty} \exp(-d(y - q\overrightarrow{X})^+) \overrightarrow{w(t, \lambda)}(y) \lambda \overrightarrow{w(t, \lambda)}(y) d\lambda.
\]
We can drive the formula for \(u^*_t\) by (4.5).

4.2. **Solution with carbon trading.** The following theorem presents the closed-form solution to the optimization problem (3.3) subject to conditions (3.2) and (3.4).

**Theorem 4.2.** The closed-form solution to the optimization problem (3.3) subject to conditions (3.2), (3.4) is given as
\[
\begin{align*}
u^*_t &= \frac{\Sigma \int_{-\infty}^{\infty} \exp(-d(y - q\overrightarrow{X})^+) \overrightarrow{w(t, \lambda)}(y) \lambda \overrightarrow{w(t, \lambda)}(y) d\lambda}{A(t) \int_{-\infty}^{\infty} \exp(-d(y - q\overrightarrow{X})^+) \overrightarrow{w(t, \lambda)}(y) \lambda \overrightarrow{w(t, \lambda)}(y) d\lambda}, \\
V(t, x, q; u^*_t) &= -4c_1 \Sigma \alpha_1 \beta \int_{t}^{T} \zeta(s) ds + 4c_1 \Sigma \log \sqrt{2\pi} \\
&\quad - 4c_1 \Sigma \log \int_{-\infty}^{\infty} \exp(-d(y - q\overrightarrow{X})^+) \overrightarrow{w(t, \lambda)}(y) \lambda \overrightarrow{w(t, \lambda)}(y) d\lambda,
\end{align*}
\]
where \(\Sigma = \sigma^2/2 + \sigma_{Q}^2/2 + \sigma \sigma_{p}, d = 1/4c_1 \Sigma\), and
\[
w(t, x, q) = \frac{1}{\sqrt{2A(t)}} \exp \left( -\frac{(x + B(t, q))^2}{4A(t)} \right),
\]
and
\[
A(t) = \Sigma \int_{t}^{T} \exp \left( 2\alpha_1 \beta \int_{t}^{s} \zeta(r) dr \right) ds,
\]
and

\[ B(t, q) = \int_t^T (\alpha_1 Y + \alpha_2 f_0(s) + \alpha_1 \beta \zeta(s) \log q + \mu_q - \Sigma) \exp \left( \alpha_1 \beta \int_t^s \zeta(r) \, dr \right) \, ds. \]

**Proof.** The HJB equation corresponding to the optimization problem can be written as

\[
\frac{\partial V}{\partial t} + (g(t) - \alpha_1 \beta \zeta(t) \log x) \frac{\partial V}{\partial x} + \frac{\sigma^2}{2} x^2 \frac{\partial^2 V}{\partial x^2} + \inf_u \left\{ c_1 u^2 - u \frac{\partial V}{\partial x} \right\} + \mu_q q \frac{\partial V}{\partial q} + \frac{\sigma^2 Q}{2} q^2 \frac{\partial^2 V}{\partial q^2} + \sigma \rho x q \frac{\partial^2 V}{\partial x \partial q} = 0,
\]

(4.16)

where \( g(t) = \alpha_1 Y + \alpha_2 f_0(t) \), with terminal condition

\[ V(T, x, q) = q(x - X)^+. \]

(4.17)

The optimal abatement strategy is given by

\[ u^*_t = \frac{x}{2c_1} \frac{\partial V}{\partial x}(t, x, q). \]

(4.18)

Substituting (4.18) into (4.16) gives

\[
\frac{\partial V}{\partial t} + (g(t) - \alpha_1 \beta \zeta(t) \log x) \frac{\partial V}{\partial x} + \frac{\sigma^2}{2} x^2 \frac{\partial^2 V}{\partial x^2} - \frac{x^2}{4c_1} \left( \frac{\partial V}{\partial x} \right)^2 + \mu_q q \frac{\partial V}{\partial q} + \frac{\sigma^2 Q}{2} q^2 \frac{\partial^2 V}{\partial q^2} + \sigma \rho x q \frac{\partial^2 V}{\partial x \partial q} = 0.
\]

(4.19)

Similarly, we also take a series of transformations to simplify (4.19). First take

\[ U(t, \xi) = V(t, x, q), \quad \xi = xq. \]

Then \( U(t, \xi) \) satisfies

\[
\frac{\partial U}{\partial t} + (\kappa(t, q) - \alpha_1 \beta \zeta(t) \log \xi) \frac{\partial U}{\partial \xi} + \xi \frac{\sigma^2}{2} \frac{\partial^2 U}{\partial \xi^2} - \frac{\xi}{4c_1} \left( \frac{\partial U}{\partial \xi} \right)^2 = 0,
\]

(4.20)

where \( \kappa(t, q) = g(t) + \alpha_1 \beta \zeta(t) \log q + \mu_q, \quad \Sigma = \sigma^2/2 + \frac{\sigma^2 Q}{2} + \sigma \rho q, \) with terminal condition

\[ U(T, \xi) = (\xi - qX)^+. \]

(4.21)

Taking the Cole-Hopf transformation

\[ U(t, \xi) = -4c_1 \Sigma \log \phi(t, \xi), \]

we get

\[
\frac{\partial \phi}{\partial t} + (\kappa(t, q) - \alpha_1 \beta \zeta(t) \log \xi) \frac{\partial \phi}{\partial \xi} + \xi \frac{\sigma^2}{2} \frac{\partial^2 \phi}{\partial \xi^2} = 0,
\]

with terminal condition

\[ \phi(T, \xi) = \exp(-d(\xi - qX)^+), \]

where \( d = 1/4c_1 \Sigma. \) Taking a coordinate transformation \( \tau = T - t, \ell = \log \xi, \) and \( \psi(\tau, \ell) = \phi(t, \xi), \) then there holds

\[
\frac{\partial \psi}{\partial \tau} = (\kappa(T - \tau, q) - \alpha_1 \beta \zeta(T - \tau) \ell - \Sigma) \frac{\partial \psi}{\partial \ell} + \xi \frac{\partial^2 \psi}{\partial \ell^2},
\]

(4.22)

with initial data

\[ \psi(0, \ell) = \exp(-d(\ell^q - qX)^+). \]
Approximate this initial data by the function
\[ \psi_n(t) = \exp(-d(e^t - qX)^+)\chi_n(t), \]
where \( \chi_n \) is the truncation function given in (4.10). Taking a Fourier transformation in (4.22) and using the properties (4.11), we have
\[ \frac{\partial \hat{\psi}}{\partial \tau} = (\kappa(T - \tau, q) - \Sigma) i \lambda \hat{\psi} - \Sigma \lambda^2 \hat{\psi} + \alpha_1 \beta \zeta(T - \tau) \frac{\partial}{\partial \lambda} (\lambda \hat{\psi}). \]
Letting
\[ \hat{\psi}(\tau, \lambda) = \exp \left( \alpha_1 \beta \int_0^\tau \zeta(T - s) ds \right) \varphi(\tau, \lambda), \]
then \( \varphi \) satisfies
\[ \frac{\partial \varphi}{\partial \tau} = (\kappa(T - \tau, q) - \Sigma) i \lambda \varphi - \Sigma \lambda^2 \varphi + \alpha_1 \beta \zeta(T - \tau) \frac{\partial \varphi}{\partial \lambda}. \]
Finally, taking the transformation \( \Theta(\tau, z) = \varphi(\tau, \lambda) \), where \( z = \log \lambda + \alpha_1 \beta \int_0^\tau \zeta(T - s) ds \), it follows that
\[ \frac{\partial \Theta}{\partial \tau} = (\kappa(T - \tau, q) - \Sigma) i \exp \left( z - \alpha_1 \beta \int_0^\tau \zeta(T - s) ds \right) \Theta \]
with initial data \( \Theta_0(z) = \hat{\psi}_n(e^z) \). Then (4.23) can be solved as
\[ \Theta(\tau, z) = \Theta_0(z) \exp(A(\tau, z) + B(\tau, z, q)i), \]
where
\[ A(\tau, z) = -\Sigma \int_0^\tau \exp \left( 2z - 2\alpha_1 \beta \int_0^r \zeta(T - s) ds \right) dr, \]
and
\[ B(\tau, z, q) = \int_0^\tau (\kappa(T - s, q) - \Sigma) \exp \left( z - \alpha_1 \beta \int_0^r \zeta(T - s) ds \right) dr. \]
Now return to the original value function \( V(t, x, q) \) with a series of inverse transformations. Using the fact that
\[ \lim_{N \to \infty} \frac{1}{\sqrt{2\pi}} \int_{-N}^N \hat{\psi}_n(\lambda) \exp(-A(t)\lambda^2 + iB(t, q)\lambda + i\lambda \ell) d\lambda = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi_n(y) w(t, \ell - y) dy, \]
where
\[ w(t, \ell, q) = \frac{1}{\sqrt{2A(t)}} \exp \left( - \frac{(\ell + B(t, q))^2}{4A(t)} \right), \]
and
\[ A(t) = \Sigma \int_t^T \exp \left( 2\alpha_1 \beta \int_t^r \zeta(r) dr \right) ds, \]
and
\[ B(t, q) = \int_t^T (\kappa(s, q) - \Sigma) \exp \left( \alpha_1 \beta \int_t^r \zeta(r) dr \right) ds, \]
it follows that
\[ \psi(t, \ell) = \exp \left( \alpha_1 \beta \int_t^T \zeta(s) ds \right) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \psi_n(y) w(t, \ell - y, q) dy, \]
and
\[ V(t, x, q) = -4c_1 \Sigma \alpha_1 \beta \int_t^T \zeta(s) ds + 4c_1 \Sigma \log \sqrt{2\pi} \\
- 4c_1 \Sigma \log \int_{-\infty}^\infty \exp(-d(e^y - qX)^+) w(t, \log xq - y, q) dy, \]
where letting \( n \to \infty \). We can drive the formula for \( u^*_t \) by (4.18).

\[ \square \]

5. Carbon price behaviors. We use the principle that the carbon prices equal the marginal abatement costs (Seifert et al. [41]), i.e.
\[ S(t, x) = 2c_1 u^*_t = x \frac{\partial}{\partial x} V(t, x) \]
and \( S(t, x, q) = 2c_1 u^*_t = x \frac{\partial}{\partial x} V(t, x, q) \) (5.1)
where \( S(t, x) \) is the carbon price in the circumstance without considering carbon trading, while \( S(t, x, q) \) is the carbon price in the circumstance with considering carbon trading. In light of the results in Theorem 4.1 and Theorem 4.2, it is direct to get the analytic formulas of the carbon prices \( S(t, x) \) and \( S(t, x, q) \).

In the following some arguments concerning the carbon price behaviors are presented.

Proposition 1. Both the terminal carbon prices \( S(T, x) \) and \( S(T, x, q) \) given by (5.1) happen a jump across the threshold \( \overline{X} \), i.e. there hold
\[ S(T, x) = c_2 x, \text{ if } x \geq \overline{X}; \quad S(T, x) = 0, \text{ if } x < \overline{X}, \quad (5.2) \]
and
\[ S(T, x, q) = qx, \text{ if } x \geq \overline{X}; \quad S(T, x, q) = 0, \text{ if } x < \overline{X}. \quad (5.3) \]

Proof. It is easy to check that the formula for the value function (4.2) also satisfies
\[ V(t, x) = -2c_1 \alpha_1 \beta \int_t^T \zeta(s) ds + 2c_1 \sigma^2 \log \sqrt{2\pi} \\
- 2c_1 \sigma^2 \log \int_{-\infty}^\infty \exp(-d(e^y - \overline{X})^+) w(t, \log x - y) dy \\
= -2c_1 \alpha_1 \beta \int_t^T \zeta(s) ds + 2c_1 \sigma^2 \log \sqrt{2\pi} \\
- 2c_1 \sigma^2 \log \int_{-\infty}^\infty \exp(-d(xe^{-y} - \overline{X})^+) w(t, y) dy, \]
which implies that the carbon price \( S(t, x) \) satisfies
\[ S(t, x) = \frac{c_2 x \int_{-\infty}^\infty \exp(d(xe^{-y} - \overline{X}) - y) w(t, y) dy}{\int_{-\infty}^\infty \exp(-d(xe^{-y} - \overline{X})^+) w(t, y) dy}. \quad (5.4) \]
As \( A(t), B(t) \to 0 \) as \( t \to T \), then there holds that
\[ w(t, y) \to \sqrt{2\pi} \delta_0(y) \text{ as } t \to T, \]
where \( \delta_0(y) \) denotes the Dirac function, whose density falls at zero. Taking \( t \to T \) in (5.4) gives
\[ S(T, x) = c_2 x \exp(d(x - \overline{X})^+ - d(x - \overline{X})) \chi(0 \in (\log x - \log \overline{X})). \]
A direct analysis leads to (5.2). Similarly, the formula for the value function (4.15) satisfies
\[
V(t, x, q) = -4c_1 \Sigma \alpha_1 \beta \int_t^T \zeta(s) ds + 4c_1 \Sigma \log \sqrt{2\pi}
- 4c_1 \Sigma \log \int_{-\infty}^{\infty} \exp(-d(e^y - qX)\bar{t})w(t, \log xq - y, q) dy
= -4c_1 \Sigma \alpha_1 \beta \int_t^T \zeta(s) ds + 4c_1 \Sigma \log \sqrt{2\pi}
- 4c_1 \Sigma \log \int_{-\infty}^{\infty} \exp(-d(xe^{-y} - qX)\bar{t})w(t, y, q) dy,
\]
which implies that the carbon price \( S(t, x, q) \) satisfies
\[
S(t, x, q) = \frac{q_x \int_{-\infty}^{\log x - \log X} \exp(-d(qxe^{-y} - qX) - y) w(t, y, q) dy}{\int_{-\infty}^{\infty} \exp(-d(qxe^{-y} - qX)\bar{t})w(t, y, q) dy}.
\]
Taking \( t \to T \) in (5.5) gives
\[
S(T, x, q) = q_x \exp(dq(x - X)\bar{t} - dq(x - X))\chi(0 \in (-\infty, \log x - \log X)).
\]
Similar analysis leads to (5.3).

On the one hand, it is easy to understand that if the terminal carbon emission is smaller than the threshold, the society does not need to invest any to meet the carbon abatement objective. Thus, it results in zero-carbon prices. On the other hand, if the terminal carbon emission is larger than the threshold, more expenditures are needed, either enhancing carbon abatement or purchasing carbon emission permits from the market. This indeed results in non-zero carbon prices. However, it is interesting and surprising to find that the terminal carbon prices only depend on the carbon emission, rather than the difference between the carbon emission and the threshold, such that the carbon prices happen jumps.

Using the closed-form solutions from Theorem 4.1 and Theorem 4.2, we can write down the Black-Scholes type equations for both the carbon prices like
\[
\frac{dS_t}{S_t} = \mu_S(t) dt + \sigma_S(t) dW_t,
\]
where \( \mu_S(t) \) is the expected return rate and \( \sigma_S(t) \) is the volatility. It is believed that (5.6) is more preferable to take some economic and financial analysis in practice. In particular, for the expected return rate \( \mu_S(t) \), we have the following proposition.

**Proposition 2.** The expected return rate \( \mu_S(t) \) given in (5.6) for both carbon prices derived in different circumstances satisfies
\[
\mu_S(t) = \alpha_1 \beta \zeta(t).
\]

**Proof.** For carbon price \( S_t = S(t, X_t) \) derived in the circumstance without considering carbon trading, by Itô’s lemma, there holds
\[
dS(t, X_t) = \frac{\partial}{\partial t} S(t, X_t) dt + \frac{\partial}{\partial x} S(t, X_t) dX_t + \frac{1}{2} \frac{\partial^2}{\partial x^2} S(t, X_t) (dX_t)^2
= \frac{\partial}{\partial t} S(t, X_t) dt + \frac{\partial}{\partial x} S(t, X_t)(\alpha_1 \mu_Y - \alpha_1 \beta Z_t + \alpha_2 f_0(t) - u^*_t) X_t dt
+ \sigma \frac{\partial}{\partial x} S(t, X_t) X_t dW_t + \frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} S(t, X_t) X_t^2 dt.
\]
where \( Z_t = \zeta(t) \log X_t, \) \( u_t^* = S(t, X_t)/2c_1. \) Differentiating (4.6) by \( x \) and then multiplying it by \( x \) gives
\[
\frac{x}{\partial t} \frac{\partial^2 V}{\partial t \partial x} - \alpha_1 \beta \zeta(t) x \frac{\partial V}{\partial x} + (g(t) - \alpha_1 \beta \zeta(t) \log x) x \left( \frac{\partial V}{\partial x} + x \frac{\partial^2 V}{\partial x^2} \right) \\
+ \frac{\sigma^2}{2} \sigma^2 \left( \frac{\partial^2 V}{\partial x^2} + x \frac{\partial^3 V}{\partial x^3} \right) - \frac{x^2}{2c_1} \frac{\partial V}{\partial x} + x \frac{\partial^2 V}{\partial x^2} = 0.
\]
Together with the fact
\[
\frac{\partial S}{\partial t} = x \frac{\partial S}{\partial t} \frac{\partial S}{\partial x} + \frac{\partial S}{\partial x} \frac{\partial^2 S}{\partial x^2} + \frac{\partial^2 S}{\partial x^2} = 2 \frac{\partial^2 S}{\partial S^2} + x \frac{\partial^3 S}{\partial x^3}, \tag{5.8}
\]
we derive the equation satisfied by the carbon price \( S(t, x) \) as follows
\[
\frac{\partial S}{\partial t} + \left( g(t) - \alpha_1 \beta \zeta(t) \log x - \frac{S}{2c_1} \right) x \frac{\partial S}{\partial x} + \frac{\sigma^2}{2} \sigma^2 \frac{\partial^2 S}{\partial x^2} = \alpha_1 \beta \zeta(t) S,
\]
which implies by (5.7) that the expected return rate \( \mu_S(t) \) in this circumstance should satisfy \( \mu_S(t) = \alpha_1 \beta \zeta(t). \)

For carbon price \( S_t = S(t, X_t, Q_t) \) in the circumstance with considering carbon trading, again by Itô’s lemma, there holds that
\[
dS(t, X_t, Q_t) = \frac{\partial}{\partial t} S(t, X_t, Q_t) dt + \frac{\partial}{\partial x} S(t, X_t, Q_t) dx_t + \frac{\partial}{\partial q} S(t, X_t, Q_t) dQ_t \\
+ \frac{1}{2} \frac{\partial^2}{\partial x^2} S(t, X_t, Q_t) (dx_t)^2 + \frac{1}{2} \frac{\partial^2}{\partial q^2} S(t, X_t, Q_t) (dQ_t)^2 \\
+ \frac{\partial}{\partial x} S(t, X_t, Q_t) dx_t dQ_t \\
= \frac{\partial}{\partial t} S(t, X_t, Q_t) dt \\
+ \frac{\partial}{\partial x} S(t, X_t, Q_t) (\alpha_1 \mu_Y - \alpha_1 \beta Z_t + \alpha_2 f_0(t) - u_t^*) X_t dt \\
+ \frac{\sigma}{\partial q} S(t, X_t, Q_t) X_t dW_t + \frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} S(t, X_t, Q_t) X_t^2 dt \\
+ \frac{\partial}{\partial q} S(t, X_t, Q_t) \mu Q_t dt + \frac{\partial}{\partial q} S(t, X_t, Q_t) \sigma Q_t dW_t \\
+ \frac{\sigma^2}{2} \frac{\partial^2}{\partial q^2} S(t, X_t, Q_t) Q_t^2 dt + \rho \sigma \frac{\partial}{\partial x} S(t, X_t, Q_t) X_t Q_t dt,
\tag{5.9}
\]
where \( u_t^* = S(t, X_t, Q_t)/2c_1. \) Differentiating (4.19) by \( x \) and then multiplying it by \( x \) gives
\[
x \frac{\partial^2 V}{\partial x^2} - \alpha_1 \beta \zeta(t) x \frac{\partial V}{\partial x} + (g(t) - \alpha_1 \beta \zeta(t) \log x) x \left( \frac{\partial V}{\partial x} + x \frac{\partial^2 V}{\partial x^2} \right) \\
+ \sigma^2 \sigma^2 \left( \frac{\partial^2 V}{\partial x^2} + x \frac{\partial^3 V}{\partial x^3} \right) - \frac{x^2}{2c_1} \frac{\partial V}{\partial x} + x \frac{\partial^2 V}{\partial x^2} = 0.
\]
Together with the fact (5.8) adding
\[
\frac{\partial S}{\partial q} = x \frac{\partial^2 V}{\partial x^2} \frac{\partial^2 S}{\partial q^2} = x \frac{\partial^3 V}{\partial x^3} \frac{\partial^2 S}{\partial q^2} = \frac{\partial^2 V}{\partial x^2} + x \frac{\partial^3 V}{\partial x^2 \partial q},
\]
we derive the equation satisfied by the carbon price $S(t, x, q)$ as follows
\[
\frac{\partial S}{\partial t} + \left( g(t) - \alpha_1 \beta \zeta(t) \log x - \frac{S}{2c_1} \right) \frac{\partial S}{\partial x} + \frac{\sigma^2}{2} x \frac{\partial^2 S}{\partial x^2} + \mu Q \frac{\partial S}{\partial q} + \frac{\sigma^2}{2} Q q \frac{\partial^2 S}{\partial q^2} + \sigma \rho x q \frac{\partial^2 S}{\partial x \partial q} = \alpha_1 \beta \zeta(t) S,
\]
which implies by (5.9) that the expected return rate $\mu_S(t)$ in this circumstance also satisfies $\mu_S(t) = \alpha_1 \beta \zeta(t)$.

It is surprising to find that the expected return rates $\mu_S(t)$ are completely determined by the climate factors, involving the damage coefficient $\beta$ of climate warming on the economic output and the process $\zeta(t)$, which relates carbon emission to the temperature. More specifically, the expected return rates $\mu_S(t)$ are simply linear on the product of these two parameters. With regard to the volatilities $\sigma_S(t)$, by (5.7) and (5.9), and using the closed-form solutions in Theorem 4.1 and Theorem 4.2, we can write down the explicit formulas for the volatilities $\sigma_S(t)$ through a trivial calculation. However, it results in intricate expressions, which cannot provide visualized information for readers.

6. Numeric results. In this section, we present some images of the derived carbon price formulas such that we can visually capture the dynamics of the carbon prices. We use quarterly data from the National Bureau of Statistics of China, Carbon Emission Accounts & Datasets and 166 surface meteorological stations in China to estimate the parameters in the economic model, the carbon emission model and the climate model respectively. Data preprocessing was taken. The economic output is measured by GDP, whose metric of per unit used in this section is hundred million RMB. The metric of each unit of carbon emission is million tons. Centigrade degree is used for the metric of temperature.

As the impact of population growth on the concentration of carbon dioxide should be slighter than that of human economic production, meanwhile, we focus more on the impact of the latter on the carbon emission, here it is appropriate to set the parameter $\alpha_2 = 0$. The estimated results of other parameters are as follows:
\[
\mu_Y = 0.088, \beta = 0.004, \sigma_Y = 0.0127, \alpha_1 = 0.42, \sigma_X = 0.0121, \zeta(t) = 0.01873.
\]

To determine the threshold of carbon emission $X$, according to the report on China’s carbon peak by 2030, published by the Global Energy Internet Development Cooperation Organization, it is forecasted that the total carbon emission of China will reach the peak around 2030 with about 110000 million tons. In light of this, we set the threshold $X = 2700$, as quarterly data used in this section. For setting the cost coefficients $c_1$ and $c_2$, on the one hand, referring to the method given in Zhou et al. [53], the abatement cost coefficient $c_1$ is estimated as $c_1 = 0.1$; on the other hand, it is difficult to determine the penalty cost coefficient $c_2$ in practice, so we set it to be $c_2 = 1$ for convenience, which means paying 100 RMB per excessive ton of emitted carbon dioxide. In the circumstance with carbon trading, we set the expected return rate and the volatility of carbon market price to be $\mu_Q = \sigma_Q = 0.01$. The correlation coefficient is set to be $\rho = 0.1$.

Figure 1 shows the surface of carbon price $S(t, x)$ in the circumstance without considering carbon trading, while Figure 2 shows some dynamic curves of carbon prices around the given threshold of carbon emission, which helps us identify more clearly the jumping behavior of carbon price across the threshold of carbon emission.
For an initial carbon emission lower than the threshold, the evaluated carbon price is concave first and then convex on time and convergent to zero when close to the deadline of the abatement period, while it is only convex on time for an initial emission larger than the threshold. This is consistent with the result given in Proposition 1. Figure 3 shows the surfaces of carbon prices dependent on time and initial carbon emission in the circumstance with considering carbon trading with respect to different carbon market prices (The metric is 100 RMB per ton of carbon emission.). The surfaces have the same topological structures. The jumping behaviors happen across the threshold of carbon emission as well, but the jumping ranges depend on the carbon market prices rather than the penalty cost. Figure 4 shows the surfaces of carbon price dependent on time and carbon market trading price with respect to different initial carbon emissions. The topological structures of these surfaces are differentiated by the threshold of carbon emission. Similarly, for an initial carbon emission lower than the threshold, the carbon price is concave first and then convex, and convergent to zero as time tends to the deadline of abatement period, while it is monotonously convex for an initial carbon emission larger than the threshold. Besides, it is not difficult to understand that the evaluated carbon price is increasing on the market trading price.

7. Conclusions. In this paper, we work on the equilibrium carbon prices, which are determined by optimization problems in stochastic equilibrium frameworks, consisted of three components, the economic model, the carbon emission model and the climate model. We solve the optimization problems and return the explicit formulas of carbon prices, which are preferable in engineering as they clearly exhibit their links to any considered control variables, such that we can propose some definite management strategies. In addition, some intrinsic properties of the carbon prices
Figure 2. Carbon price curves with respect to different carbon emissions in the circumstance without considering carbon trading.

are found, for instance, the prices happen jumps at the end of the abatement period and the expected return rates of carbon prices are completely dependent on the climate elements. Finally, we stress that our equilibrium carbon prices provide benchmarks for pricing carbon emission permits in a society.

REFERENCES

[1] L. M. Abadie and J. M. Chamorro, European CO$_2$ prices and carbon capture investments, *Energy Economics*, 30 (2008), 2992–3015.
[2] M. E. Arouri, F. Jawadi and D. K. Nguyen, Nonlinearities in carbon spot-futures price relationships during Phase II of the EU ETS, *Economic Modelling*, 29 (2012), 884–892.
[3] G. S. Atsalakis, Using computational intelligence to forecast carbon prices, *Applied Soft Computing*, 43 (2016), 107–116.
[4] E. Benz and S. Truck, Modeling the price dynamics of CO2 emission allowances, *Energy Economics*, 31 (2009), 4–15.
[5] M. Burke, S. M. Hsiang and E. Miguel, Global non-linear effect of temperature on economic production, *Nature*, 527 (2015), 235–239.
[6] Y. Cai and T. S. Lontzek, The social cost of carbon with economic and climate risks, *Journal of Political Economy*, 127 (2019), 2684–2734.
[7] R. Carmona, M. Fehr and J. Hinz, Optimal stochastic control and carbon price formation, *SIAM Journal on Control and Optimization*, 48 (2009), 2168–2190.
[8] R. Carmona and J. Hinz, Risk-neutral models for emission allowance prices and option valuation, *Management Science*, 57 (2011), 1453–1468.
[9] U. Çetin and M. Verschuere, Pricing and hedging in carbon emissions markets, *International Journal of Theoretical and Applied Finance*, 12 (2009), 949–967.
[10] M. Chamon and P. Mauro, Pricing growth-indexed bonds, *Journal of Banking and Finance*, 30 (2006), 3349–3366.
[11] P. Chen, Understanding economic complexity and coherence: Market crash, excess capacity, and technology wavelets, Working Paper, 2009.
Figure 3. Carbon price surfaces in the circumstance with considering carbon trading with respect to different carbon market prices.

[12] P. Chen, A biological perspective of macro dynamics and division of labor: Persistent cycles, disruptive technology, and the trade-off between stability and complexity, Working Paper, 2003.

[13] M. Chesney and L. Taschini, The endogenous price dynamics of emission allowances and an application to CO$_2$ option pricing, *Applied Mathematical Finance*, 19 (2012), 447–475.

[14] G. Daskalakis, D. Psychoyios and R. N. Markellos, Modeling CO2 emission allowance prices and derivatives: Evidence from the European trading scheme, *Journal of Banking & Finance*, 33 (2009), 1230–1241.

[15] M. Dell, B. F. Jones and B. A. Oleson, Temperature and income: Reconciling new cross-sectional and panel estimates, *American Economic Review*, 99 (2009), 198–204.

[16] M. Dell, B. F. Jones and B. A. Oleson, Temperature shocks and economic growth: Evidence from the last half century, *American Economic Journal: Macroeconomics*, 4 (2012), 66–95.

[17] T. Dietz and E. A. Rosa, Rethinking the environmental impacts of population, affluence and technology, *Human Ecology Review*, 1 (1994), 277–300.
Figure 4. Carbon price surfaces in the circumstance with considering carbon trading with respect to different carbon emissions.

[18] A. Ellerman and B. K. Buchner, The european union emissions trading scheme: Origins, allocation, and early results, Review of Environmental Economics and Policy, 1 (2007), 66–87.

[19] X. Fan, S. Li and L. Tian, Chaotic characteristic identification for carbon price and an multi-layer perceptron network prediction model, Expert Systems with Applications, 42 (2015), 3945–3952.

[20] I. S. Farouq, N. U. Sambo, A. U. Ahmad, A. H. Jakada and I. A. Danmaraya, Does financial globalization uncertainty affect CO2 emissions? Empirical evidence from some selected SSA countries, Quantitative Finance and Economics, 5 (2021), 247–263.

[21] Y. Fu and Z. Zheng, Volatility modeling and the asymmetric effect for China’s carbon trading pilot market, Physica A, 542 (2020), 123401.

[22] C. García-Martos, J. Rodríguez and M. J. Sánchez, Modelling and forecasting fossil fuels, CO2 and electricity prices and their volatilities, Applied Energy, 101 (2013), 363–375.
[23] M. Golosov, J. Hassler, P. Krusell and A. Tayvinski, Optimal taxes on fossil fuel in general equilibrium, *Econometrica*, 82 (2014), 41–88.
[24] H. Guo and J. Liang, An optimal control model for reducing and trading of carbon emissions, *Physica A: Statistical Mechanics and its Applications*, 446 (2016), 11–21.
[25] H. Guo and J. Liang, An optimal control model of carbon reduction and trading, *Mathematical Control and Related Fields*, 6 (2016), 535–550.
[26] C. Hambel, H. Kraft and E. Schwartz, Optimal carbon abatement in a stochastic equilibrium model with climate change, *European Economic Review*, 132 (2021), 103642.
[27] S. Hitzemann and M. Uhrig-Homburg, Equilibrium price dynamics of emission permits, *Journal of Financial and Quantitative Analysis*, 53 (2018), 1653–1678.
[28] S. Hitzemann and M. Uhrig-Homburg, Empirical performance of reduced-form models for emission permit prices, *Review of Derivatives Research*, 22 (2019), 389–418.
[29] G. Klepper and S. Peterson, Marginal abatement cost curves in general equilibrium: The influence of world energy prices, *Resource and Energy Economics*, 28 (2006), 1–23.
[30] S. Kruse, M. Meitner and M. Schröder, On the pricing of GDP-linked financial products, *Applied Financial Economics*, 15 (2005), 1125–1133.
[31] Y. Li, Forecasting Chinese carbon emissions based on a novel time series prediction method, *Energy Science & Engineering*, 8 (2020), 2274–2285.
[32] A. Mardani, D. Streimikiene, F. Cavallaro, N. Loganathan and M. Khoshnoudi, Carbon dioxide (CO2) emissions and economic growth: A systematic review of two decades of research from 1995 to 2017, *Science of the Total Environment*, 649 (2019), 31–49.
[33] I. Matei, Is financial development good for economic growth? Empirical insights from emerging European countries, *Quantitative Finance and Economics*, 4 (2020), 653–678.
[34] W. D. Nordhaus, *A Question of Balance: Weighing the Options on Global Warming Policies*, Yale University Press, New Haven, 2008.
[35] W. D. Nordhaus, Revisiting the social cost of carbon, *Proceedings of the National Academy of Sciences of the United States of America*, 114 (2017), 1518–1523.
[36] W. D. Nordhaus and P. Sztorc, DICE 2013R: Introduction and Users Manual, *Technical Report*, Yale University, 2013.
[37] J. Patel, S. Shah, P. Thakkar and K. Kotecha, Predicting stockmarket index using fusion of machine learning techniques, *Expert Systems with Applications*, 42 (2015), 2162–2172.
[38] K. Rana, S. R. Singh, N. Saxena and S. S. Sana, Growing items inventory model for carbon emission under the permissible delay in payment with partially backlogging, *Green Finance*, 3 (2021), 153–174.
[39] A. M. Rather, A. Agarwal and V. N. Sastry, Recurrent neural network and a hybrid model for prediction of stock returns, *Expert Systems with Applications*, 42 (2015), 3234–3241.
[40] M. E. Sanin, F. Violante and M. Mansanet-Bataller, Understanding volatility dynamics in the EU-ETS market, *Energy Policy*, 82 (2015), 321–331.
[41] J. Seifert, M. Uhrig-Homburg and M. Wagner, Dynamic behavior of CO2 spot prices, *Journal of Environmental Economics and Management*, 56 (2008), 180–194.
[42] J. Solana, Climate change litigation as financial risk, *Green Finance*, 2 (2020), 344–372.
[43] K. Z. Tong and A. Liu, Modeling temperature and pricing weather derivatives based on subordinate Ornstein-Uhlenbeck processes, *Green Finance*, 2 (2020), 1–19.
[44] F. van der Ploeg and A. de Zeeuw, Pricing carbon and adjusting capital to fend off climate catastrophes, *Environmental & Resource Economics*, 72 (2018), 29–50.
[45] L. L. Viguier, M. H. Babiker and J. M. Reilly, Carbon emissions and the Kyoto commitment in the European Union, *Report No.70, MIT Joint Program on the Science and Policy of Global Change*, 2001.
[46] P. E. Waggoner and J. H. Ausubel, A framework for sustainability science: A renovated IPAT identity, *Proceedings of the National Academy of Sciences of the United States of America*, 99 (2002), 7860–7865.
[47] J. Wang, X. Sun, Q. Cheng, and Q. Cui, An innovative random forest-based nonlinear ensemble paradigm of improved feature extraction and deep learning for carbon price forecasting, *Science of the Total Environment*, 762 (2021), 143099.
[48] M. L. Weitzman, GHG Targets and Insurance against Catastrophic Climate Damages, *Journal of Public Economic Theory*, 14 (2012), 221–244.
[49] X. Yang and J. Liang, Minimization of carbon abatement cost: modeling analysis and simulation, *Discrete and Continuous Dynamical System B*, 22 (2017), 2939–2969.
[50] E. Zagheni and F. C. Billari, A cost valuation model based on a stochastic representation of the IPAT equation, *Population & Environment*, **29** (2007), 68–82.
[51] Y. Zhang and Y. M. Wei, An overview of current research on EU ETS: Evidence from its operating mechanism and economic effect, *Applied Energy*, **87** (2010), 1804–1814.
[52] Z. Zheng, R. Xiao, H. Shi, G. Li and X. Zhou, Statistical regularities of Carbon emission trading market: Evidence from European Union allowances, *Physica A*, **426** (2015), 9–15.
[53] P. Zhou, L. Zhang, D. Q. Zhou and W. J. Xia, Modeling economic performance of interprovincial CO$_2$ emission reduction quota trading in China, *Applied Energy*, **112** (2013), 1518–1528.
[54] B. Zhu, X. Shi, J. Chevallier, P. Wang and Y. Wei, An adaptive multiscale ensemble learning paradigm for nonstationary and nonlinear energy price time series forecasting, *Journal of Forecasting*, **35** (2016), 633–651.

Received July 2021; revised October 2021; early access December 2021.

E-mail address: wangzhzh@gdufs.edu.cn
E-mail address: dongh26@mail.sysu.edu.cn
E-mail address: zhehao.h@gzhu.edu.cn