Predictive pion-quark BCS relation and Thornber-Feynman high-$T_c$ gap

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A pion-quark pairing temperature is defined by a BCS-like relation identified from a quark-level Goldberger-Treiman relation with a Nambu scalar mass “gap parameter” taken in the low-mass limit. This intuitive relation predicts the associated “experimental” lattice-gauge pairing temperature. The opposite high-mass limit predicts the sigma mass, and notably has a predictive analog in high-$T_c$ superconductivity in the stable nondispersive energy gap as defined by Thornber-Feynman polaron dynamics.
A predictive analog pion-quark BCS relation with a Nambu-mass gap parameter has a further parallel in the nondispersive high-$T_c$ superconductivity gap as defined from Thornber-Feynman (TF) polaron dynamics [1]. This BCS-like relation for the pion $\bar{q}q$ binding energy and pair-breaking “temperature”, as defined from a quark-level Goldberger-Treiman relation (GTR) in the low-mass limit [1], predicts this latest calculated temperature as follows. The GTR for the pion-nucleon coupling constant $g_{\pi NN}$ is $g_{\pi NN}=m_N g_A/f_\pi$ in terms of the nucleon mass $m_N$ (939 MeV), pion decay constant in the chiral pairing limit $f_\pi \sim 90$ MeV, and axial current form factor $g_A=1$ for structureless quarks in the constituent quark model [2]. For the constituent quark this relation becomes $g_{\pi qq}=m_q/f_\pi$ where $m_q \approx 939$ MeV/3=313 MeV and $g_A$ is unity for structureless quarks, so an “experimental” two-flavor quark-level GTR is

$$g_{\pi qq}=m_q/f_\pi \sim 313 \text{ MeV}/90 \text{ MeV}=3.5. \quad (1)$$

Noting that the righthand side of Eq. 1 numerically approximates the BCS ratio, we cast it into “BCS form” by multiplying the numerator and denominator by two, i.e.,

$$g_{\pi qq}=2m_q/2f_\pi \sim 3.5. \quad (2)$$

The numerator $2m_q$ is recognized as the scalar meson mass identified from the Nambu scalar mass (Eq. A-3) in the Appendix as an analog superconducting gap $2\Delta$. From this approximate BCS ratio $2\Delta/k_BT_c \sim 3.5$ in Eq. 2 for $T_c \sim 3-4$ K we identify the numerator ($2m_q$) with a pair binding energy $2\Delta$, where the constituent quark masses are largely taken up in the binding as with the pion [3]. This further suggests the BCS denominator $2f_\pi$ is an intuitive $\bar{q}q$ pairing or chiral
restoration “temperature” $T_c$, i.e., $T_c = 2f_r - 180$ MeV [4], which in fact is compatible
with the recent “experimental” value of $173 \pm 8$ MeV obtained from a computer
“lattice” calculation for two quark flavors [5]. Notably this intuitive $T_c$ determina-
tion from the quark-level GTR complements the earlier ones in Ref. 4 in the
framework of chiral symmetry breaking.

This Nambu scalar mass-pion BCS connection in the low-mass limit
\((2m_q \to 2\Delta, 2f_\pi \to T_c)\) is further supported by independent calculations of the BCS
ratio and $g_{\pi qq}$ as analytic expressions that are similar in both magnitude and
form. Namely for acoustical phonons the gap ratio becomes [1]

\[ \frac{2\Delta/k_B T_c}{2\pi e^{-\gamma}} \approx 3.528, \]  (3)

where $\gamma$ is the Euler constant (0.5772), and for the pion-quark coupling constant
with color number $N_c = 3$ we find [6,4]

\[ g_{\pi qq} = \frac{2\pi}{N_c^{1/2}} = \frac{2\pi}{3^{1/2}} = 3.628. \]  (4)

Note this numerical nearness of Eqs. 3 and 4 to the BCS ratio in Eqs. 1,2.

The above identification of the double quark mass $2m_q$ in Eq. 2 with the
pion $\bar{q}q$ pairing energy $2\Delta$ is reinforced by theoretical and experimental
determinations of the pion charge radius $r_\pi$ from a “fused” quark pair that accords
with the tight-binding massless Nambu-Goldstone pion implicit in the GTR, Eq. 1.
Hence $r_\pi$ is specified by a single quark mass suggestive of the fused pair as
\[ r_\pi = \frac{h c}{m_q} = 197.3 \text{ MeV-fm/313 MeV} = 0.63 \text{ fm (} h c = 197.3 \text{ MeV-fm)} \] [7] from the
“common sense” quark mass 313 MeV=$m_N/3$ as expected with three constituent
non-strange quarks in a nucleus. Similarly, vector meson dominance (VMD) [8]
with rho-meson mass 770 MeV specifies a pion charge radius $6^{1/2}(197.3 \text{ MeV-fm})/770 \text{ MeV} \approx 0.63 \text{ fm}$.

Not surprisingly, the numerical closeness of 3.528 and 3.628 in Eqs. 3 and 4 is not accidental. In both cases the energy scales with the momentum as $E=p$ (in units $c=1$), but for different reasons. For the acoustic-phonon BCS coupling in Eq. 3 the linear scaling is in the low-energy limit due to the large Fermi surface (FS), which follows from the relation $\Delta E = (p/m) \Delta p$ with constant of proportionality $p/m$ defined from energy-momentum changes $\Delta E$ and $\Delta p$. Hence $p/m$ is approximately constant on the FS because the changes $\Delta p/p$ are small. Whereas in Eq. 4 the linear scaling is due to tight binding, where the fused pion mass vanishes in the relativistic energy-mass relation $E = p^2 + m^2$ as the mass converts to binding energy, so $E$ is approximately $p$ in this (tight binding) limit.

The Nambu scalar mass gap parameter Eq. A-3 in the opposite high-mass limit predicts the sigma mass as shown in the Appendix. The further analog in the flatband high-$T_c$ superconductivity gap $2\Delta_{HTC}$ in this limit, paralleling the long-range low-$T_c$ gap parameter Eq. A-2 [1], is experimentally specified by the non-dispersive (massive) energy-constrained TF quasiparticle shift $E_{LO}$ (the longitudinal-optical (LO) phonon mode energy) [9]. This mobile, stable, gapped quasiparticle in paralleling the high-$T_c$ particle arises from nonadiabatic (conserved) internal polaron dynamics in such low-symmetry Frohlich media [10] as defined by a “universal” TF mobility [1] (e.g., in photoconducting alkali halides and related transition-metal oxides [12]). Accordingly the optimum, constrained high-$T_c$ gap analogous to Eq. A-2 is empirically
\[ 2\Delta_{HTC} = E_{LO}, \quad (5a) \]

where in the cuprates this gap as defined by angle-resolved photoemission spectroscopy (ARPES) data in fact equals the resonance-softened in-plane Cu-O mode energy \( E_{LO}^R \) in satisfying Eq. 5a directly or from combined cuprate data [1], so the above TF gap relation becomes

\[ 2\Delta_{HTC} = E_{LO}^R. \quad (5b) \]

In particular for optimum-doped \( \text{Bi}_2\text{Sr}_2\text{CaCu}_3\text{O}_8 \) (Bi-2212) this ARPES energy shift \( 2\Delta_{SC} \) in Fig. 1(a) is 74 meV (1 meV = \( 10^{-3} \) eV), where combining with resonance-softened \( \text{La}_{2-x}\text{Sr}_x\text{CuO}_4 \) (La-214) Cu-O mode data \( E_{LO}^R \approx 70 \) meV from Table I. Whereas for \( \text{YBa}_2\text{Cu}_3\text{O}_7 \) (Y-123) the ARPES gap Eq. 5b is predicted by this Y-123 data alone, with the shift \( 2\Delta_{HTC} \) and \( E_{LO}^R \) from Table I and Fig. 1(b) respectively 58 meV and 56 meV [20]. Notably Eq. 5b is further given by a directional resonance coupling of these TF dynamics to the ARPES distribution [1].

Moreover Eq. 5 in addition to a Nambu-sigma mass analog constitutes an empirical “slowing” relation, where the simple energy and coupling independence reflects nondispersive, nonlinear inelastic-Frohlich slowing, and the conservation or irreversibility derives from the system asymmetry [1]. Such energetic “bunching” from these nonlocal dynamics, in setting the (TF) polaron apart from that due to Feynman [10], is consistent with the short-range stable normal-state pairing. Similarly, such nonlinear extension of this Frohlich “action” [11b] facilitates its energetic sharing in the many-particle state.

Significantly an extra-high-\( T_c \) multi-layer cubic BCS cuprate possibly observed as a minority phase is defined from this empirical TF Nambu-mass
analog gap relation Eq. 5b for the non-cubic cuprates, and in particular for the BCS-like/cubic alkali-fullerene Rb$_2$CsCo$_{60}$ [21]. Hence in the latter where $T_c=40$ K the TF shift of 13 meV from Table I yields the expected near-BCS ratio,

$$2\Delta_{HTC}/T_c k_B = (13 \text{ meV}/40 \text{ K})(11.6 \text{ K/meV}) = 3.8,$$

(6a)

paralleling a low-$T_c$ Nambu-analog gap mode in NbSe$_2$ with gap ratio 3.4 in Eq. A-1. Whereas for the (non-cubic) cuprates non-BCS gap ratios are specified, viz., for Bi-2212 with $T_c=95$ K and $2\Delta_{HTC}=74$ meV experimentally from the TF-ARPES relation Eq. 5b, the ratio is

$$2\Delta_{HTC}/T_c k_B = (74 \text{ meV}/95 \text{ K})(11.6 \text{ K/meV}) = 9.0 $$

(6b)

($k_B=1$ meV/11.6 K). Similarly for Y-123 where $T_c=93$ K and $2\Delta_{HTC}=58$ meV again given by Eq. 5b a decreased ratio is specified as

$$2\Delta_{HTC}/T_c k_B = (58 \text{ meV}/93 \text{ K})(11.6 \text{ K/meV}) = 7.2,$$

(6c)

where this decrease would reflect the relative uniformity of the Y-123 structure. Notably for the purposes of our projected ideal BCS cuprate, the diversity in the optimum high-$T_c$ gap of these structures of 13 meV, 74 meV and 58 meV is

fundamentally specified by the TF shift Eq. 5. (Albeit the accompanying $T_c$ increases in the cuprates are offset by the large ratios reflecting non-BCS localization associated with the low lattice symmetry).

Added basis for this ideal BCS cuprate comes from such high-$T_c$ superconductivity associated with cubic symmetry in general as in optimum Rb$_2$CsC$_{60}$ above. Hence in Ba$_0.6$K$_{0.4}$BiO$_3$ with $2\Delta_{HTC}=70$ cm$^{-1}$ and $T_c=30$ K [22] (recall $\hbar c=197.3$ MeV-fm=0.01973 meV-cm) the gap ratio is

$$2\Delta_{HTC}/T_c k_B = (2\pi)(70 \text{ cm}^{-1}/30 \text{ K})(11.6 \text{ K/meV})(0.01973 \text{ meV-cm})=3.4,$$

(7a)
and in Rb$_3$C$_{60}$ with $\Delta_{SC}/k_B$=53 K, or $\Delta$=4.6 meV and $T_c$=29.4 K [23] the ratio is

$$2\Delta_{HTC}/T_c k_B = 2(4.6 \text{ meV}/29.4 \text{ K})(11.6 \text{ K/meV})=3.6. \quad (7b)$$

These Nambu-analog TF gap and BCS ratio specifications from Eqs. 5b, 6 and 7 provide an empirical basis for defining the ideal BCS cuprate, e.g., the gap specifications imply the TF shift is fundamental and therefore also holds for the record-high-$T_c$ 164 K multi-layer structure HgBa$_2$Ca$_2$Cu$_3$O$_8$ under applied hydrostatic pressure [24]. Hence taking the resonance LO-mode energy $E_{LO}^R$ in Eq. 5b in this case as $\sim$65 meV, as estimated from an average of the Y-123 and La-214 data, the gap ratio is logically reduced from 7-9 in Bi-2212 and Y-123 to

$$2\Delta_{HTC}/T_c k_B \sim (65 \text{ meV}/164 \text{ K})(11.6 \text{ K/meV})=4.6, \quad (8)$$

where this reduction is consistent with symmetry increases toward a cubic BCS structure from the added planes and hydrostatic pressure.

In Ref. 1 a $\sim$250 K infinite-layer cubic cuprate superconductor was projected from this predictive TF gap Eq. 5b plus BCS condition generally observed in cubic high-$T_c$ structures (cf. Eqs. 6a and 7). For such a Bi-2212 base structure this $T_c$ limit from Eq. 6b as a result of added planes and applied pressure is thus

$$T_c = 74 \text{ meV}(11.6 \text{ K/meV})/3.5=245 \text{ K}. \quad (9a)$$

Stated in reverse, the BCS ratio limit is reached for the ideal 245 K $T_c$ structure as

$$2\Delta_{HTC}/T_c k_B = (74 \text{ meV}/245 \text{ K})(11.6 \text{ K/meV})=3.5. \quad (9b)$$

Notably this $T_c$ is consistent with numerous observations of possible minority-phase superconductivity near this temperature.
Summarizing this Nambu-analog gap ratio progression toward the extra-high-$T_c$ structure, starting with the low-$T_c$ analog NbSe$_2$ with a ratio 3.4 in Eq. A-1, the sequence continues in the optimum high-$T_c$ structures with ratios 3.8, 9.0 and 7.2, all given by the TF gap Eq. 5. Continuing to higher $T_c$ a TF-lattice symmetry gap ratio pattern peaking at 9.0 emerges with increasing $T_c$ as 3.8, 9.0, 7.2, 4.6(?) $\rightarrow$ 3.5(?), with (?) denoting the TF-BCS-like projections.

In conclusion, the latest calculated pion-quark pairing temperature is predicted by an intuitive BCS relation identified from a quark-level Goldberger-Treiman relation with a Nambu scalar-mass gap parameter taken in the low-mass limit. Whereas in the high-mass limit this parameter predicts the sigma mass, and moreover has an analog in high-$T_c$ superconductivity in the stable localized energy gap as specified by the Thomber-Feynman (TF) polaron with a strongly-constrained shift from conserved dynamics paralleling the high-$T_c$ case. Notably an extra-high-$T_c$ BCS cuprate possibly observed as a minority phase is defined by a high-$T_c$ gap ratio sequence based on these empirical Nambu-analog TF gaps.

**Appendix: Particle physics analog of a low-$T_c$ gap mode**

We outline here how a massive “gap mode” appearing at $T_c$ near $2\Delta$ in the BCS superconductor NbSe$_2$ constitutes a particle physics analog leading in the low-mass limit to the pion pairing BCS relation Eq. 2. A near-BCS gap ratio is defined in NbSe$_2$ from $2\Delta=17$ cm$^{-1}$ (with $\omega=2\pi\nu$) and $T_c=7.2$ K [25],

$$2\Delta/T_c k_B = [(2\pi)17 \text{ cm}^{-1}/7.2 \text{ K}][0.01973 \text{ meV-cm}] [11.6 \text{ K/meV}] \approx 3.4, \quad (A-1)$$
suggesting a nominal global pairing over the Fermi surface. The particle physics connection of this gap mode follows from its identification by Littlewood and Varma with the Nambu scalar mass [26]. This mode arises from a coupling of the superconductivity/gap to an amplitude mode of the charge-density-wave (CDW) (created from optical-phonon induced oscillation of the CDW gap at a local zone face). Hence as a local paired bound state near $2\Delta$ with mass $M$, or

$$2\Delta=M,$$

(A-2)

it has an analog in the Nambu scalar mass [27] identified in Eq. 2, i.e.,

$$2m_q=m_s.$$

(A-3)

This “gap parameter” Eq. A-3 in the high-mass limit predicts the sigma meson mass $m_\sigma$, as also concluded in Ref. 28 from a chiral-breaking analog mode of Nambu and Jona-Lasinio [27], where $m_q$ is the constituent quark mass as verified experimentally by the scalar sigma with mass $m_\sigma=600$ MeV [29]. Independent determination of the quark mass $m_q$ of 325 MeV in the chiral limit from the linear $\sigma$ model [6] (or simply $m_N/3=313$ MeV from the 3 quarks in the nucleus) yields 626-650 MeV for $m_\sigma$. Equation A-3 expresses the dispersion-theoretic limit in the dispersion range with squared four-momentum $q^2=4m^2$ to 0, where the quark-anti-quark pair can be considered as “touching” with $q^2=(2m_q)^2$.

A BCS-particle physics analog of Eq. A-2 in the opposite zero-mass limit in the text gives the predictive BCS-like relation for the pion with the $\bar{q}q$ pair considered as “fused”, i.e., a tightly-bound pair at $q^2=m^2_\pi=0$, or at the VMD value $r_\pi=0.63$ fm.
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Optimum gap, $2\Delta$ (meV) | In-plane LO-phonon energy (meV) | Critical temperature, $T_c$ (K)
---|---|---
$\text{Bi}_2\text{Sr}_2\text{CaCu}_2\text{O}_8$ (Bi-2212) | 74 [12] | 80 [15] | 95
$\text{La}_{1.85}\text{Sr}_{0.15}\text{CuO}_4$ (La-214) | 15 | 83 [16] | 70 [16] | 40
$\text{YBa}_2\text{Cu}_3\text{O}_7$ (Y-123) | 58$^a$ [14] | 68$^a$ [17] | 56$^a$ [17] | 93
$\text{Rb}_2\text{CsC}_{60}$ | 12$^b$ | 13 [18] | 40 [19]

$^a$A-axis data in direction of reduced screening across charge stripes.
$^b$This gap with the observed $T_c$ of 40 K corresponds to the BCS gap ratio generally observed in these structures.

**Table I.** Optimum gap $2\Delta$ and non-resonance and resonance in-plane LO-phonon energies $E_{LO}$ and $E_{LO}^R$ in the (1,0,0) direction in Bi-2212, La-214 and Y-123 from ARPES and inelastic neutron scattering data, respectively. Note the $2\Delta-E_{LO}^R$ resonance match according to the TF polar coupling relation Eq. 5b from such resonance “softening” of the mid-zone Cu-O mode, and similarly in $\text{Rb}_2\text{CsC}_{60}$ according to Eq. 5a. The small gap in La-214 reflects the short-range order.
Figure 1

(a) ARPES distribution of Bi-2212 at ($\pi,0$) momentum location on the Fermi surface that defines the optimum energy gap $\Delta$ for the normal and superconducting states (from Ref. 13). (b) High-density a-axis superconducting peak defining the gap of Y-123 at ($\pi,0$), with the corresponding Bi-2212 peak (from Ref. 14).
