Novel critical phenomena in compressible polar active fluids: A functional renormalization group approach

Patrick Jentsch and Chiu Fan Lee

Department of Bioengineering, Imperial College London, South Kensington Campus, London SW7 2AZ, U.K.

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Active matter is not only relevant to living matter and diverse nonequilibrium systems, but also constitutes a fertile ground for novel physics. Indeed, dynamic renormalization group (DRG) analyses have uncovered many new universality classes (UCs) in polar active fluids - an archetype of active matter systems. However, due to the inherent technical difficulties in the DRG methodology, almost all previous studies have been restricted to polar active fluids in the incompressible or infinitely compressible (i.e., Malthusian) limits, and, when the $$\epsilon$$-expansion was used in conjunction, to the one-loop level. Here, we use functional renormalization group methods to bypass some of these difficulties and unveil for the first time novel critical behavior in compressible polar active fluids, and calculate the corresponding critical exponents beyond the one-loop level. Specifically, focusing on a multicritical region of the system, we find three novel UCs and quantify their associate scaling behavior near the upper critical dimension $$d_c = 6$$.

Active matter refers to many-body systems in which the microscopic constituents can exert forces or stresses on their surroundings, and as such detailed balance is broken at the microscopic level [11]. However, even if the microscopic dynamics are fundamentally different from more traditional systems considered in physics, it remains unclear whether novel behavior will emerge in the hydrodynamic limits (i.e., the long time and large distance limits [3]). One unambiguous way to settle this question is to identify whether the system’s dynamical and temporal statistics are governed by a new universality class (UC), typically characterized by a set of scaling exponents [4–6]. These exponents can in principle be determined using either simulation or renormalization group (RG) methods. However, simulation studies can be severely plagued by finite-size effects (e.g., two recent controversies concern the scaling behavior of active polymer networks [7, 8] and critical motility-induced phase separation [9, 10]). Therefore, RG analyses remain as of today the gold standard in the categorization of dynamical systems into distinct UCs. Indeed, for polar active fluids (PAFs) [12, 13], an archetype of active matter systems, the use of dynamic renormalization group (DRG) [14] analyses have led to, on one hand, surprising realizations that certain types of PAFs are no different from thermal systems in the hydrodynamic limit [15, 16], and on the other hand discoveries of diverse novel phases [13, 14, 18, 25] and critical phenomena [26, 27]. However, due to the inherent technical difficulties in DRG methods, all of these studies have been restricted to PAFs in the incompressible or infinitely compressible (i.e., Malthusian) limits except for rare exceptions [19, 20]. Further, when a DRG analysis was used in conjunction with the $$\epsilon$$-expansion method, which was typically the case, it has always been restricted to the one-loop level.

To bypass some of these technical difficulties within the DRG methodology, we apply for the first time a functional renormalization group (FRG) [30, 37] analysis on compressible PAFs. FRG analyses are intrinsically non-perturbative and are based on an exact RG flow equation to which approximate solutions can be readily obtained numerically. Recent successes in the applications of FRG include the elucidation of scaling behavior in, e.g., critical X-component ferromagnets [33], reaction-diffusion systems [39, 43], the Kardar-Parisi-Zhang model [44, 46], and turbulence [47, 49], as well as non-universal observ-
ables far from scaling regimes [50, 51]. Using FRG, we uncover here three novel nonequilibrium UCs by studying a multicritical region of dry compressible PAFs, and quantify the associate scaling behaviors beyond the one-loop level.

Model & critical phenomena.—The hydrodynamic equations of motion (EOM) of a system can generally be derived by considering the system’s symmetry and conservation laws alone. In the case of generic dry compressible PAFs, the hydrodynamic EOM are called the Toner-Tu equations [13, 14, 52]. Expressed in terms of the particle mass density field $\rho$ and the momentum density field $\mathbf{g}$, the Toner-Tu equations consist first of the continuity equation,

$$\partial_t \rho + \nabla \cdot \mathbf{g} = 0 ,$$

(1)

and then the EOM of $\mathbf{g}$,

$$\partial_t \mathbf{g} + \lambda_1 \nabla(|\mathbf{g}|^2) + \lambda_2 \mathbf{g} \cdot \nabla \mathbf{g} + \lambda_3 \mathbf{g} \nabla \cdot \mathbf{g} = \mu_1 \nabla^2 \mathbf{g} + \mu_2 \nabla (\nabla \cdot \mathbf{g}) - \alpha \mathbf{g} - \beta |\mathbf{g}|^2 \mathbf{g} - \kappa \nabla \rho + ... + \mathbf{f} ,$$

(2)

where the ellipsis represents the omitted higher-order terms (e.g., terms of higher-order spatial derivatives in $\rho$). In the above EOM, all coefficients are functions of $\rho$ and $|\mathbf{g}|$; and their spatial derivatives, and the noise term $\mathbf{f}(\mathbf{r}, t)$ are zero mean Gaussian white noises of the form

$$\langle f_s(\mathbf{r}, t) f_s(\mathbf{r}', t') \rangle = 2D \delta_{ij} \delta^d(\mathbf{r} - \mathbf{r}') \delta(t - t') .$$

(3)

Compressible PAFs admit a complex phase diagram: diverse phase transitions and phase co-existences can occur (Fig. 1). In particular, expressing $\alpha$ and $\kappa$ in Eq. (2) as

$$\alpha = \sum_{n \geq 0} \alpha_n \delta \rho^n , \quad \kappa = \sum_{n \geq 0} \kappa_n \delta \rho^n ,$$

(4)

where $\delta \rho = \rho - \rho_0$ with $\rho_0$ being the average particle density in the system, two ordered phases (with distinct densities) can co-exist if $\alpha_0 > 0$ and $\kappa_0 < 0$ (blue region in Fig. 1[a]) [10], while an ordered phase can co-exist with a disordered phase if $\kappa_0 > 0$ and $\alpha_0 < 0$ (green region) [25]. Further, the system can become critical upon fine-tuning: if $\alpha_0 > 0$ and $\kappa_0 = \kappa_1 = 0$, the resulting critical behavior belongs to the Ising universality class (UC) (blue triangle) [10], while if $\alpha_0 = \alpha_1 = 0$ and $\kappa_0 > 0$, the associate critical behavior corresponds to a yet to be characterized UC (yellow inverted triangle) [28]. Recently, a third type of critical behavior was identified [29], which corresponds to the merging of these two distinct critical points by simultaneously fine-tuning $\alpha_0$, $\alpha_1$, $\kappa_0$ and $\kappa_1$ to zero (red circle in Fig. 1[a]). The universal behavior of this new multicritical point (MCP) is the focus of this Letter.

Linear regime.—Around this MCP, $|\mathbf{g}| \approx 0$ and the linearized EOM are thus

$$\partial_t \rho = - \nabla \cdot \mathbf{g} ,$$

(5a)

$$\partial_t \mathbf{g} = \mu_1 \nabla^2 \mathbf{g} + \mu_2 \nabla (\nabla \cdot \mathbf{g}) + \zeta \nabla^2 \nabla \rho + \mathbf{f} ,$$

(5b)

where we have introduced the term characterized by $\zeta$ since, when $\kappa_0$ is fine-tuned to zero, this term is now the leading order term linear in $\rho$. In the above, we have redefined $\rho$ to be $\delta \rho$ to ease notation, and we will continue to do so from now on.

Upon rescaling time, lengths, and fields as

$$\mathbf{r} \rightarrow r^\ell , \quad t \rightarrow t e^{\chi_\rho \ell} , \quad \rho \rightarrow \rho e^{\chi_\rho \ell} , \quad \mathbf{g} \rightarrow g e^{\chi_g \ell} ,$$

(6)

we find that the linearized EOM are preserved if [53]

$$z^{\text{lin}} = 2 , \quad \chi_\rho^{\text{lin}} = \frac{4 - d}{2} , \quad \chi_g^{\text{lin}} = \frac{2 - d}{2} .$$

(7)

In particular, the correlation functions of the system at this multicritical point behave as follows:

$$C_\rho(\mathbf{r}, t) = \langle \rho(\mathbf{r}, t) \rho(0, 0) \rangle = r^{2 \chi_\rho^{\text{lin}}} S_{\rho \rho} \left( \frac{t}{r^{z_{\text{lin}}}} \right) ,$$

(8a)

$$C_g(\mathbf{r}, t) = \langle \mathbf{g}(\mathbf{r}, t) \cdot \mathbf{g}(0, 0) \rangle = r^{2 \chi_g^{\text{lin}}} S_{gg} \left( \frac{t}{r^{z_{\text{lin}}}} \right) ,$$

(8b)

where $r = |\mathbf{r}|$, and the $S$’s are two universal scaling functions. The scaling exponents [7] obtained in this linear theory are expected to be exact when the spatial dimension $d$ is high enough. We will now use these exponents to gauge the importance of various nonlinearities in the EOM [2] as $d$ becomes small.

Nonlinear regime.—We now turn to the full EOM of $\mathbf{g}$ [2]. As $d$ decreases from, say infinity, the nonlinear terms that first become relevant (and are not fine-tuned to zero), i.e., terms that diverge as $\ell \rightarrow \infty$, are

$$\alpha_2 \rho^2 \mathbf{g} \quad \text{and} \quad \kappa_2 \rho^2 \nabla \rho ,$$

(9)

which happens at the upper critical dimension $d_c = 6$. These non-linear terms, together with the linear terms, support the following symmetry:

$$\rho \rightarrow - \rho \quad \text{and} \quad \mathbf{g} \rightarrow - \mathbf{g} ,$$

(10)

which emerges around the MCP. One can therefore simplify the consideration by restricting to the subspace of EOM compatible with this symmetry: $\alpha_1$ and $\kappa_1$ are
vanishing and will not be generated under RG transformations. Just below six dimensions, the universal hydrodynamical EOM [2] is therefore

\[
\partial_t g = \mu_1 \nabla^2 g + \mu_2 \nabla(\nabla \cdot g) - \alpha_0 g - \kappa_0 \nabla \rho + f \\
- \alpha_2 \rho^2 g - \kappa_2 \rho^2 \nabla \rho + \zeta \nabla^2 \nabla \rho .
\]

(11)

Note that the signs of these nonlinear terms (with \(\alpha_2, \kappa_2 > 0\)) are chosen for the sake of stability. By the same token, the term \(\zeta \nabla^2 \nabla \rho\) is introduced so that the case of \(\kappa_0 < 0\) can be considered. In fact, this term is marginal according to our linear theory and is therefore required in our discussion.

Traditionally, a DRG analysis together with the \(\epsilon\)-expansion method would now be applied. However, the fact that all nonlinear terms are cubic in nature rules out any graphical renormalizations to the \(\mu\)'s at the one-loop level. As such, there can be no corrections to the scaling exponents from the linear theory [7], again at the one-loop level [53]. To go beyond one-loop, we will now use the FRG formalism to tackle this problem.

**FRG analysis.**—Our FRG analysis is based on the so-called Wetterich equation [20, 32]:

\[
\partial_k \Gamma_k = \frac{1}{2} \text{Tr} \left[ (R_k^{(2)} + R_k)^{-1} \partial_k R_k \right], \tag{12}
\]

where \(\Gamma_k\) is the \(k\)-dependent effective average action, with \(k\) being the inverse length scale up to which fluctuations have been averaged out. The functional \(\Gamma_k\) interpolates from the microscopic action \(\Gamma_\Lambda\) to the macroscopic effective average action \(\Gamma_0\), which provides the hydrodynamic EOM of the average fields, with all fluctuations incorporated. The gradual incorporation of fluctuations as \(k \to 0\) is facilitated by the “regulator” \(R_k\), which serves to suppress fluctuations of length scales greater than \(k^{-1}\). The regulator can be chosen arbitrarily as long as \(R_k \approx \infty\) and \(R_0 = 0\) to ensure the correct boundary conditions for \(\Gamma_k\). Further, \(\Gamma_k^{(2)}\) in Eq. (12) denotes the field dependent matrix of the second functional derivatives of \(\Gamma_k\) (i.e., entries are of the form \(\delta^2 \Gamma_k / (\delta g \delta \rho)\), etc), and \(\text{Tr}\) stands for the matrix trace over internal indices and integration over the internal wave vector and frequency.

While the Wetterich equation (12) is in principle exact, the actual implementation of the RG flow relies on restricting the functional \(\Gamma_k\) to a manageable form. Here, we will take \(\Gamma_k\) to be the functional obtained from the EOM (5,11) via the Martin-Siggia-Rose-de Dominicis-Jansen formalism [54, 56]:

\[
\Gamma_k[g, g, \tilde{\rho}, \rho] = \int \frac{d^d \rho}{\mathbb{R}} \left\{ \tilde{\rho} (\partial_t \rho + \nabla \cdot g) - D|g|^2 \\
+ \tilde{g} \left[ \gamma \partial_t g - \mu_1 \nabla^2 g - \mu_2 \nabla(\nabla \cdot g) + \alpha_0 g \\
+ \kappa_0 \nabla \rho + \alpha_2 \rho^2 g + \kappa_2 \rho^2 \nabla \rho - \zeta \nabla^2 \nabla \rho \right] \right\}, \tag{13}
\]

where \(f_\mu = \int d^d x \rho d t\), and all coefficients above (\(\mu_1, \mu_2, \alpha_0, \kappa_0, \kappa_2, \zeta\), etc) are now \(k\) dependent. We have also introduced a \(k\)-dependent coefficient \(\gamma\) for the time derivative of \(g\), which will be renormalized. The response fields introduced by the formalism are denoted by \(\tilde{g}\) and \(\tilde{\rho}\). The density ‘sector’ of \(\Gamma_k\), i.e., terms proportional to \(\tilde{\rho}\) in (13), does not renormalize due to an extended symmetry [57] which we discuss in [53]. From our linear theory, we know that this form of \(\Gamma_k\) is sufficient only around the critical dimension \(d_c = 6\). As a result, we expect that the validity of our quantitative predictions is limited to around \(d_c\). Therefore, we will express our results as corrections to the linear theory in terms of \(\epsilon = d_c - d\).

Besides the form of the average action, another key ingredient of the FRG is the regulator, which we choose to be, in spatio-temporally Fourier transformed space,

\[
R_k(q, \tilde{p}) = (2\pi)^{d+1} \delta^{d+1} (q + \tilde{p}) \times \begin{pmatrix}
0 & i d A_k(q^2) & 0 & i q B_k(q^2) \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
- i q B_k(q^2) & 0 & 0 & 0
\end{pmatrix},
\]

(14)

where \(q \equiv (q, \omega)\), \(i d\) is the \(d\)-dimensional unit matrix and the ordering of the matrix entries is: \((g, \tilde{g}, \rho, \tilde{\rho})\). The choice of a time-independent regulator is common for dynamical systems [37]. Also, this matrix form does not regulate the density sector directly, but rather introduces a \(k\) dependent “pressure term” in the momentum field sector. This regularization sufficiently cuts off large and small scale fluctuations while respecting the extended symmetry mentioned above, ensuring mass conservation, even in the regulated theory. We also defined the following in Eq. (14):

\[
A_k(q^2) = \mu_{1,k} k^2 m(q^2/k^2), \tag{15a}
B_k(q^2) = \zeta k^2 m(q^2/k^2), \tag{15b}
\]

\(m(y) = ay\),

where we write the \(k\)-dependence of the couplings explicitly, \(\mu_{1,k} = \mu_{1,1} + \mu_{2,k}\) and \(a\) is an arbitrary positive constant. In principle, all results obtained should be independent of the regulator choice, however, truncating the form of \(\Gamma_k\) usually introduces some form of regulator dependence. This dependence can be judged by the \(a\)-dependence of the critical exponents. It turns out that for an algebraic regulator as in Eq. (15), the critical exponents are independent of \(a\) [53, 58], thus in compliance with the principle of minimal sensitivity [60]. Nevertheless, we have also verified our results quantitatively using a different regulator [53].

With the forms of \(\Gamma_k\) and \(R_k\) defined, one can then use the Wetterich equation (12) to project a set of coupled ordinary differential equations (ODEs), one for each coefficient in the functional (13) (see Ref. [52] for details). The key ingredient of our approach is to project the flow...
The coefficient state of density deviation $k$ is motivated by the physical argument that when the system is projected as

\[ \rho_{\text{unif}} = \sqrt{|\alpha_0/\alpha_2|} \text{ if } \alpha_0 > 0 \text{ or } \sqrt{|\alpha_0/\kappa_2|} \text{ otherwise.} \]

This is motivated by the physical argument that when the system is globally in a state of phase separation, locally, in subsystems of size $k^{-1}$, the system is in a homogeneous state of density deviation $\rho = \pm \rho_{\text{unif}}$. As an example, the coefficient $\mu_{1,k}$ is projected as

\[ \mu_{1,k} = \frac{1}{2VT} \left. \int \frac{d^2 q}{d_i^2} \frac{d^2 \Gamma_k}{\delta g(q) \delta g(-q)} \right|_{\rho = \rho_{\text{unif}}}. \]

where $VT$ is the spatio-temporal volume, $q = |q|$ and $P_{ij}(q) \equiv \delta_{ij} - q_i q_j / q^2$ is the projector transverse to $q$. For instance, this strategy has been successfully applied to the equilibrium Ising model in Ref. [35].

The coupled ODEs derived from the Wetterich equation [12] by applying these projections constitute the RG flow equations that describe the coarse-graining of our system around the multicritical region.

The projections at nonuniform density lead to a non-trivial renormalization of the diffusion coefficients of order $\epsilon^2$, which in the DRG formalism can only be obtained by performing a two-loop calculation. The FRG formalism therefore can account for DRG two-loop effects.

**Novel RG fixed points.**—We now determine the RG flows by numerically integrating the FRG-derived integral-differential equations. In a typical perturbative DRG calculation to one-loop order, one would find that the flow equations for the non-linear couplings $\kappa_2$ and $\alpha_2$ decouple from the relevant couplings $\kappa_0$ and $\alpha_0$. Their fixed point (FP) values can therefore be easily obtained even in a numerical calculation since the FP in this subspace is attractive.

In our FRG approach, however, the flow equations for the diffusion coefficients $\mu_1$, $\mu_{||}$ and $\zeta$, which are non-linear couplings depend on, are directly proportional to the relevant couplings. Therefore, one has to solve all flow equations simultaneously. This is problematic since the relevant couplings diverge from the FP. To tackle this problem in an FRG calculation, one typically invokes the shooting method [33, 35] to fine-tune the relevant parameters, which however becomes difficult when there are many parameters to fine-tune. Here, we instead simply invert the sign of the relevant flow equations. This trick manifestly leaves the locations of the FPs invariant, but changes their stability [53]. The flow equations, therefore, fine-tune themselves. Once the fixed point solution is found, the original signs can be restored to obtain the critical exponents. This method can also be extended to explore other unstable FPs by inverting additional flow equations.

With the help of this method, we find a total of four FPs (Fig. 2): one stable FP that governs generically the universal critical behavior of the MCP under consideration (denoted by the red circle), and three other unstable FPs (yellow pentagon, green square, and blue diamond). Their resulting values for the critical exponents of these FPs, expressed in terms of $\epsilon = (d_c - d)$, are shown in Table I. For instance, the scaling exponents describing the system’s correlation functions [3] are indicated. In addition to these critical exponents, we can also provide quantitative predictions on two universal amplitude ratios [53]: $\mu \equiv \mu_1/\mu_{||}$ and $\zeta \equiv \gamma \zeta/\mu_{||}$, whose universal values are shown in Table I.

Further critical exponents, such as those that quantify how the correlation length scales as one moves away from the MCP, are discussed in [53].

**Summary & Outlook.**—We have used the functional

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**TABLE I. Critical exponents and universal amplitude ratios, expressed as an $\epsilon$-expansion from the upper dimension $d_c = 6$, for the four distinct fixed points. When no value for the amplitude ratios is given, it is not universal and can take arbitrary values.**

| FP | $z - 2$ | $\chi_0 + (d - 2)/2$ | $\chi_\rho + (d - 4)/2$ | $\mu$ | $\zeta$ |
|----|---------|----------------------|----------------------|-------|-------|
| 1  | 0       | 0                    | 0                    | 0     | 0     |
| 2  | 0       | 0                    | 0                    | 0     | 1.43  |
| 3  | 0.011$c^2$ | 0.022$c^2$ | 0.033$c^2$ | 1.45  | 1.37  |
| 4  | 0.084$c^2$ | -0.013$c^2$ | 0.112$c^2$ | 0.31  | 0     |
renormalization group (FRG) to elucidate the universal behavior of a multicritical point in a generic dry, compressible, polar active fluid model. Our achievements are three folds: (1) the discovery of three novel universality classes, two of them being demonstrably out of equilibrium [53], (2) the first analytical elucidation of critical behavior for compressible active fluids, and (3) the first application of FRG on active matter systems beyond the equivalence of the perturbative one-loop level. Interesting future directions include the applications of FRG to other active matter systems to obtain quantitative results beyond the traditional one-loop limits, and to tackle well-known open questions in the field, such as: what is the universal behavior of the homogeneous ordered phase of the Toner-Tu model? [52, 60]?
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