Quadrupolar power radiation by a binary system in de Sitter

Background

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Abstract

Cosmological observations over past couple of decades favor our universe with a tiny positive cosmological constant. Presence of cosmological constant not only imposes theoretical challenges in gravitational waves physics, it has also observational relevance. Inclusion of cosmological constant in linearized theory of gravitational waves modifies the power radiated quadrupole formula. There are two types of observations which can be impacted by the modified quadrupole formula. One is the orbital decay of an inspiraling binary and other is the modification of the waveform at the detector. Modelling a compact binary system in an elliptic orbit on de Sitter background we obtain quadrupolar power radiation and we also investigate its impact on orbital decay rate.

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I. INTRODUCTION

Einstein’s quadrupole formula was the first quantitative estimate of the power radiated in the form of gravitational radiation. It was derived in the strictly linearized approximation and it was also shown that to the leading order the emitted power due to gravitational radiation is proportional to square of third derivative of mass quadrupole moment of the source [1]. This power loss would cause binary system to shrink slowly. Such a secular change in orbital period of Hulse-Taylor binary pulsar was confirmed by observation to the accuracy of $10^{-3}$, thereby providing an indirect affirmation of gravitational waves [2–4]. Einstein’s theory also passed with flying colors in the direct observation of newly opened gravitational wave astronomy - gravitational wave template predicted by Einstein’s theory matches with observed signal [5–8].

All these theoretical frameworks assume a vanishing cosmological constant. However, by now cosmological observations (e.g. red shift of type Ia supernovae) have established that our universe favors a positive cosmological constant ($\Lambda$). Inclusion of cosmological constant not only posits theoretical challenges, it could also potentially provide an independent estimate of $\Lambda$ from current accuracy of observation. It is natural to ask how does presence of cosmological constant affect orbital decay of a binary system. Change in orbital decay also induces change in orbital phase which is sensitive to current gravitational wave detectors. The aim of this work is to get an estimate of quadrupolar power loss by an elliptic binary system in de Sitter background. This work is a part of AA’s master’s thesis [9] and the case of circular orbit has been discussed in JH’s PhD thesis [10].

While it is well recognized that there is no tensorial (which is both local and generally covariant) definition of stress tensor for gravitational field, it is possible to construct meaningful, quasi-local quantities to represent total energy/momentum in specific contexts. One of the earliest such proposals is by Isaacson tailored for the context in which there are two widely separated spatio-temporal scales. For sources which are rapidly varying (relative to the length scale set by cosmological constant), there is an identification of gravitational waves as ripple on a background within the so called ‘short wave approximation’. Let $L_B$ denote the length scale of variation of the background and $\lambda$ the length scale of the ripple with $\lambda \ll L_B$. In this context Isaacson defined an effective gravitational stress tensor for the ripples which is gauge invariant to leading order in the ratio of the two scales [11]. In a
recent paper [12] it has been discussed how Isaacson prescription can be adopted to compute quadrupolar power, radiated by a rapidly varying compact source in de Sitter background. Compact binary system is the natural testbed to apply the new quadrupole formula.

Value of cosmological constant being tiny, $10^{-29}$ gm/cc or $10^{-52}$ m$^{-2}$ in the geometrized unit with $G = 1 = c$, one can neglect $\Lambda$ in the vicinity of astrophysical sources or near ground-based gravitational wave detectors. Signature of $\Lambda$ may be visible over the vast distances of source-free regions in which gravitational waves propagate. To appreciate this statement, let us take the example of Schwarzschild-de Sitter space-time in static coordinates,

$$ds^2 = -\left(1 - \frac{2GM}{r} - \frac{\Lambda r^2}{3}\right)dt^2 + \left(1 - \frac{2GM}{r} - \frac{\Lambda r^2}{3}\right)^{-1}\frac{dr^2}{r^2} + r^2 d\Omega^2_2$$  

(1)

Near the source the mass term dominates while at the large distance repulsive nature of pure de Sitter potential comes into play. Hence, neglecting cosmological term near the source we assume that the source dynamics is governed by Newtonian potential and far away from the source we relate the energy lost due to gravitational radiation to that of quadrupolar energy flux in de Sitter background.

This paper is organized as follows. In section II, we recall the power radiated quadrupole formula from [12]. The modified quadrupole formula involves mass quadrupole moment as well as pressure quadrupole moment of source. As for non-relativistic (weakly stressed) compact source we can neglect the pressure term, in section III, we discuss the derivation of mass quadrupole moment of a point particle in de Sitter background. In section IV, we spell out the assumptions made to model an elliptic binary in de Sitter background. Given these assumptions, the quadrupolar power radiated by an elliptic binary system is presented in section V. This is a new result. The final section VI concludes with a summary and discussion. Some of the technical details are given in an appendix. We set $c = 1$ throughout.

II. PRELIMINARIES

Cosmological observations over the decades have indicated that our universe is undergoing an accelerated expansion, which is most simply modelled by a positive cosmological constant. Einstein equation with positive cosmological constant ($\Lambda > 0$) take the de Sitter solutions as background. To compute quadrupolar power radiation due to a elliptic binary system in
FIG. 1: The full square is the Penrose diagram of de Sitter space-time with generic point representing a 2-sphere. The blue line denotes trajectory of spatially compact source in de Sitter background. ABD part is the the future Poincaré patch, covered by the conformal chart \((\eta, r, \theta, \phi)\). The line AB denotes the future null infinity, \(J^+\) while the line AE denotes the cosmological horizon, \(H^+\). Two constant \(\eta\) space-like hypersurfaces are shown with \(\eta_2 > \eta_1\). The two constant \(r\), time-like hypersurfaces have \(r_2 > r_1\). The two dotted lines denote the out-going null rays emanating from \(\eta = \eta_1, \eta_2\) on the world line through the source. During the interval \((\eta_1, \eta_2)\) the source is ‘active’ i.e. varying rapidly enough to be in detectable range of frequencies. The region AED is the static patch admitting the time translational Killing vector, \(T\).

de Sitter background we follow the framework developed in [12]. In this section we review the work and give relevant expressions.

The de Sitter space-time defined as the hyperboloid in five dimensional Minkowski space-time, has a ‘global chart’ of coordinates, as shown in figure 1. To be definite, let us take the world-tube of the spatially compact source to be around the line AD for all times. In
this case source world-tube has future and past time-like infinity, denoted by $i^\pm$. Causal future of the compact source is only the future Poincaré patch (ABD) of full de Sitter. No observer whose world-line is confined to the past Poincaré patch (BCD) can detect the radiation emitted by the gravitating source. Therefore, to study gravitational radiation due to compact sources, it is sufficient to restrict oneself to future Poincaré Patch rather than full de Sitter space-time. There are two natural coordinate charts for the future Poincaré patch, i.e., a conformal chart: $(\eta, x^i)$ and a cosmological chart: $(t, x^i)$. In the conformal chart $(\eta, x^i)$ the background de Sitter metric takes the form,

$$ds^2 = a^2(\eta) \left[ -d\eta^2 + \sum_i (dx^i)^2 \right], \quad a(\eta) = -(H\eta)^{-1}, \quad \eta \in (-\infty, 0), \quad H := \sqrt{\frac{\Lambda}{3}}.$$  \hspace{1cm} (2)

While conformal coordinates are convenient in detailed calculations of gravitational perturbations, they are not suitable for taking the limit $\Lambda \to 0$. To take that limit we have to use proper time $t$, which is related to conformal time $\eta$ via $\eta := -H^{-1}e^{-Ht}$. In these coordinates $(t, x^i)$, line element becomes,

$$ds^2 = -dt^2 + e^{-2Ht} \sum_{i=2}^{4} (dx^i)^2$$  \hspace{1cm} (3)

Poincaré patch has seven dimensional symmetry group - 3 spatial rotations, 3 spatial translations and 1 time translation. For computing power radiated quadrupole formula in de Sitter background, it suffices to focus on the time translational Killing field. In order to find the Killing vector field of time translations we let $t \to t + \delta t$ in the above metric. This leads to

$$ds^2 \to -dt^2 + e^{2Ht}(1 + 2H \delta t) \sum_{i=2}^{4} (dx^i)^2$$  \hspace{1cm} (4)

To make the metric invariant under this time translation, the spatial coordinates, $x^i$, must be transformed as $x^i \to x^i - Hx^i\delta t$. In general, a Killing vector $\xi^\alpha$ is an infinitesimal generator of isometry, i.e., for $x^\alpha \to x^\alpha + \epsilon \xi^\alpha$, the metric remains invariant. Identifying $\epsilon = \delta t$, the Killing vector which generates time translation in the cosmological coordinate system is $T^\mu = (1, -Hx^i)$. We will work with this time-translational Killing field to compute quadrupolar radiation.

Under the ‘short wavelength approximation’ for rapidly varying compact source we visualize gravitational waves as high frequency ripple over a slowly varying background space-time. Let $L_B$ be the length scale variation of background and $\lambda$ be the length scale of ripple. In
the present context of de Sitter background, $L_B$ length scale is set by the cosmological constant, $L_B \sim \sqrt{\Lambda}$. To maintain a clear cut separation between background and ripple for all time, we demand that $\lambda \ll L_B$. In this context it is possible to define Isaacson effective gravitational stress tensor, $t_{\mu\nu}$ for the ripples. To obtain Isaacson stress tensor, one begins with an expansion of the form $g_{\mu\nu} = \bar{g}_{\mu\nu} + \epsilon h_{\mu\nu}$ and writes the Einstein equation in source free region as,

$$R_{\mu\nu}(\bar{g} + \epsilon h) = \Lambda(\bar{g}_{\mu\nu} + \epsilon h_{\mu\nu})$$

$$\therefore R^{(0)}_{\mu\nu}(\bar{g}) + \epsilon R^{(1)}_{\mu\nu}(\bar{g}, h) + \epsilon^2 R^{(2)}_{\mu\nu}(\bar{g}, h) = \Lambda(\bar{g}_{\mu\nu} + \epsilon h_{\mu\nu}) \quad (5)$$

Introduce an averaging over an intermediate scale $\ell$, $\lambda \ll \ell \ll L_B$, which satisfies the properties: (i) average of odd powers of $h$ vanishes and (ii) average of space-time divergence of tensors is sub-leading [13,14]. Taking the average of the above equation gives,

$$\langle R^{(0)}_{\mu\nu} \rangle + \epsilon^2 \langle R^{(2)}_{\mu\nu} \rangle = \Lambda \bar{g}_{\mu\nu}. \quad (6)$$

Notice that $R^{(2)}$, which is quadratic in $h$, can have $L_B$ - scale variations and hence non-zero average. Thus it incorporates back reaction of ripple on the background and modifies the background equation as,

$$8\pi t_{\mu\nu} = \bar{R}_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} \bar{R} + \Lambda \bar{g}_{\mu\nu} \quad (7)$$

This can be appropriately termed as ‘coarse-grained’ form of Einstein equation in source free region, where

$$t_{\mu\nu}(\bar{g}, h) := - \frac{\epsilon^2}{8\pi} \left[ \langle R^{(2)}_{\mu\nu} \rangle - \frac{1}{2} \bar{g}_{\mu\nu} \bar{g}^{\alpha\beta} \langle R^{(2)}_{\alpha\beta} \rangle \right] \quad (8)$$

is the effective stress-energy tensor of ripple. It should be noted that the stress tensor is symmetric and conserved with respect to background. Given a symmetric, conserved stress tensor and time-translational Killing vector of de Sitter background, one can construct conserved current

$$J^\mu_T := - t^\mu_{\nu} T^\nu.$$ 

From the conservation equation it follows,

$$0 = \int_V d^4x \sqrt{\bar{g}} \nabla_\mu J^\mu_T = \int_V d^4x \partial_\mu (\sqrt{\bar{g}} J^\mu_T) = \int_{\partial V} d\sigma J^\mu_T, \quad (9)$$

For linearized retarded solution in de Sitter background, computing the energy flux ($\int_{\Sigma} d\sigma \mu T^\mu_T$) integral across killing hypersurfaces, we obtain the power radiated quadrupole formula [12],

$$P = \frac{G}{8\pi} \int_{S^2} d^2s \langle Q^\mu_{ij} Q^{ij}_\mu \rangle \quad (10)$$
where, $Q_{ij} := [\mathcal{L}_T^3 Q_{ij} + 3H\mathcal{L}_T^2 Q_{ij} + 2H^2 \mathcal{L}_T Q_{ij} + H\mathcal{L}_T^2 \bar{Q}_{ij} + 3H^2 \mathcal{L}_T \bar{Q}_{ij} + 2H^3 \bar{Q}_{ij}] (t_{ret})$. $\mathcal{L}_T$ denotes Lie derivative with respect to time translational Killing vector. We would like to express this quantity in proper time coordinate $t$, of matter source. Hence, using $\mathcal{L}_T := \partial_t - 2H$ on $Q_{ij}$ and $\bar{Q}_{ij}$, we can express $Q_{ij}$ as,

$$Q_{ij} = [\partial_t^3 Q_{ij} - 3H \partial_t^2 Q_{ij} + 2H^2 \partial_t Q_{ij} + H \partial_t^2 \bar{Q}_{ij} - H^2 \partial_t \bar{Q}_{ij}] (t_{ret}).$$

(11)

$Q_{ij}$ and $\bar{Q}_{ij}$ are mass quadrupole moment and pressure quadrupole moment respectively (see below). $t_{ret}$ remains constant along out-going null rays and defined as $\eta - r = - \frac{1}{\Pi} e^{-Ht_{ret}}$.

The label $tt$ denotes algebraically projected part of tensor field and is defined as $Q_{ij}^{tt} := \Lambda_{ij}^{kl} Q_{kl}$. $\Lambda_{ij}^{kl}$ is the algebraic projection operator which projects spatial components of a tensor to a plane orthogonal to radial direction and makes it traceless. The projection operator is defined as $\Lambda_{ij}^{kl} := \frac{1}{2} (P_{k}^{i} P_{l}^{j} + P_{l}^{i} P_{k}^{j} - P_{ij}^{kl})$, with $P_{j}^{i} = \delta_{j}^{i} - \hat{x}_i \hat{x}_j$. A projected tensor fields satisfy spatial transversality and tracelessness (TT) condition to the leading order in $r^{-1}$ (for a detailed analysis between $tt$ vs. TT in the context of asymptotically flat space-time, see [16, 17]). Throughout the paper we use projected $tt$ field. Interested readers can take a look at [18] for derivation of power radiation by a circular binary system using TT part of $Q_{ij}$.

Our main result of the paper considers an elliptic orbit and we will see in section V, in the limit of circular orbit it reduces to the result of [18] (see eq. (25) of the reference).

Let us take a quick detour to the definition of moments in de Sitter background. To maintain coordinate invariance and moment integrals to be well-defined, the moment variables must be coordinate scalars. The natural choice is to write moment variables in terms of tetrad components. The conformal form of de Sitter metric in eq. (2) suggests a natural choice [19],

$$f_0^\alpha := -H\eta (1, \vec{0}) , \quad f_m^\alpha := -H\eta \delta_m^\alpha \quad \Leftrightarrow \quad f_0^\alpha := -H\eta \delta_0^\alpha$$

(12)

The corresponding components of the stress tensor are given by, $\rho := P_{00} = T_{\alpha\beta} f_0^\alpha f_0^\beta = H^2 \eta^2 T_{00} \delta_0^0 \delta_0^0, \quad P_{ij} := T_{\alpha\beta} f_0^\alpha f_j^\beta = H^2 \eta^2 T_{ij} \delta_i^0 \delta_j^0, \quad \pi := P_{ij} \delta_{ij}^0.$ The quadrupole

1 Any symmetric rank-2 tensor field can be decomposed as [15],

$$A_{ij} = \frac{1}{3} \delta_{ij} \delta_{kl} A_{kl} + (\partial_i \partial_j - \frac{1}{3} \delta_{ij} \nabla^2) B + \partial_i B_j^T + \partial_j B_i^T + A_{ij}^{TT}. $$

$A_{ij}^{TT}$ refers to the transverse-traceless part of the field, so that $\partial^i A_{ij}^{TT} = 0 = \delta^{ij} A_{ij}^{TT}$. 

7
moments of the two rotational scalars, $\rho, \pi$, are defined by integrating over the source distribution at $t = \text{const.}$ hypersurface,

\[ Q^{ij}(t) := \int_{\text{Source}(t)} d^3 x \, a^3(t) \rho(t, \vec{x}) \bar{x}^i \bar{x}^j , \]  
(13)

\[ \bar{Q}^{ij}(t) := \int_{\text{Source}(t)} d^3 x \, a^3(t) \pi(t, \vec{x}) \bar{x}^i \bar{x}^j . \]  
(14)

The determinant of the induced metric on $t = \text{const.}$ hypersurfaces is $a^3(t)$. The tetrad components of the moment variable are given by, $\bar{x}^i := f^i_\alpha x^\alpha = -(\eta H)^{-1} \delta^i_j x^j = a(t)x^i$. It should be noted that the tetrad components measure the physical distance and $d^3 x \, a^3(t) = d^3 \bar{x}$. The upshot is that moment variable is computed in a tetrad frame attached with source.

III. MASS QUADRUPOLE MOMENT OF A POINT PARTICLE IN DE SITTER

To proceed let us investigate quadrupole moment of a point particle of mass $m$ in de Sitter background. For a compact, non-relativistic source, we can neglect $\pi$ with respect to $\rho$ and it is sufficient to compute only mass quadrupole moment. Let a point particle of mass $m$ is moving on a worldline $\gamma$ in a curved space-time with metric $g_{\alpha\beta}$. Action functional of the particle is given by,

\[ S_p = -m \int_\gamma \sqrt{-g_{\mu\nu}} \, \dot{x}^\mu \dot{x}^\nu \, d\sigma \]  
(15)

where $\sigma$ is an arbitrary parameter along worldline of particle (which can be taken as proper time for convenience). We assume that the $m$ is small, so that perturbation $h_{\mu\nu}$ created by the particle can also be considered to be small. Now, as usual, decompose $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$. Background metric $\bar{g}_{\mu\nu}$ is independent of $m$ while the perturbation $h_{\mu\nu}$ contains dependence of $m$. To the leading order in $m$, particle’s energy-momentum tensor in background space-time is given by [20, 21],

\[ T^{\mu\nu} = \frac{m}{\sqrt{-\bar{g}}} \int_\gamma u^\mu \delta^4(x - x_p) \, d\tau \]  
(16)

where $u^\mu$ denotes particle’s four velocity in background space-time and $x_p$ is the position of the particle. In conformal chart $(\eta, x^i)$ of de Sitter background proper time, $d\tau = \sqrt{\bar{g}_{\mu\nu}} \, dx^\mu dx^\nu = a \sqrt{\eta_{\mu\nu}} \, dx^\mu dx^\nu = a \gamma^{-1} d\eta$, where $\gamma^{-1} = \sqrt{1 - \left(\frac{d\vec{x}}{d\eta}\right)^2}$. For computing
mass quadrupole moment, let us concentrate on $T^{00}$,

$$T^{00} = \frac{m}{a^2} \int \left( \frac{d\eta}{d\tau} \right)^2 \delta^4(x - x_p(\eta)) \frac{d\tau}{d\eta} d\eta$$  \hspace{1cm} (17)

$$= \frac{m}{a^5} \gamma \delta^3(\vec{x} - \vec{x}_p(\eta))$$  \hspace{1cm} (18)

$$\therefore T^{00} \approx \frac{m}{a^5} \delta^3(\vec{x} - \vec{x}_p(\eta))$$  \hspace{1cm} (19)

In the last line we have dropped the Lorentz factor $\gamma$ assuming the source is non-relativistic. As the tetrad frame (12) is defined in $(\eta, x^i)$ coordinates, we compute $\rho$ in conformal coordinates, after that we convert it in $(t, x^i)$ coordinates. Hence,

$$\rho := a^{-2}(\eta) T^{00} \delta^0_0 \delta^0_0 = a^{-2}(\eta) \frac{m}{a(\eta)} \delta^3(\vec{x} - \vec{x}_p(\eta)) \frac{d\eta}{d\tau} \delta^0_0 \delta^0_0 = \frac{m}{a^3(t)} \delta^3(\vec{x} - \vec{x}_p(t)) \frac{d\eta}{d\tau} \delta^0_0 \delta^0_0$$  \hspace{1cm} (20)

In the intermediate step we have used conformally flat metric in $(\eta, x^i)$ to lower the indices of $T^{00}$. Plugging this expression in eqn. (13), we obtain mass quadrupole moment of a point particle in de Sitter background,

$$Q^{ij}(t) = m \bar{x}^i_p \bar{x}^j_p .$$  \hspace{1cm} (21)

In the final expression we have suppressed the constant tetrad $\delta^0_0$. It should be noted that the mass quadrupole moment is expressed in terms of tetrad components. In terms of coordinates $(\eta, x^i)$, tetrad components $\bar{x}^i := a(\eta) \delta^i_0 x^j$, represent physical distance.

\section*{IV. SOURCE MODELLING IN DE SITTER BACKGROUND}

To model the binary system in de Sitter background, we follow the strategy of flat background \cite{22, 23}. In Minkowski space-time the ultimate destination of all $r = \text{const}.$ hypersurface is future timelike infinity, $i^+$. However in de Sitter background, $r = \text{const}.$ hypersurface intersects future null infinity, $\mathcal{J}^+$. Any two points on $\mathcal{J}^+$ are physically infinitely separated. Hence, to maintain finite separation between source components, compact source should be within the cosmological horizon and converges on $\mathcal{J}^+$ exactly at $i^+$ (see fig. 2). For example, a circular orbit in de Sitter background is represented by the $r_{ph} = \text{const.}$ curve.

We will further make the assumption that the conservative dynamics of binary is completely governed by Newtonian potential (presently we focus on special case of elliptic Keplerian orbit). These orbits are not necessarily the physical ones, nor do they follow the
geodesics of background geometry (in fact there is no closed geodesic in de Sitter or flat background). For a more realistic scenario, one should investigate motion of a test particle in Schwarzschild de Sitter background. In principle, one can define source moment in Schwarzschild de Sitter background and obtain the linearized field in terms of moments using the conservation of stress tensor. In the far zone this solution is expected to match with that of linearized field solution of de Sitter background.

The repulsive nature of de Sitter potential is taken to be negligible as far as source motion is concerned. We visualize orbital decay of binary system as an iterative process. To start with, say, there is only Newtonian approximation and we assume that the orbit is pre-assigned to the de Sitter background. There is no orbital decay then and the orbital parameters remain constant forever. Now to the first iteration (1 PN), we obtain the gravitational field using conservation of stress-energy tensor of matter source with respect to de Sitter background. As the gravitational radiation carries energy it causes the orbit to shrink. Hence, we equate the power lost by source to that power associated with gravitational waves.

We will consider a binary system in an elliptic Keplerian orbit. Therefore in the center of mass frame of binary, this system is equivalent to an effective one body problem with reduced mass \( \mu = \frac{m_1 m_2}{M} \) following an elliptic trajectory, where \( M = m_1 + m_2 \) is the total mass of the system. As source moment is defined in tetrad variable, we will attach the tetrad system of conformal chart to the center of mass of the system and assign \( r_{ph} = 0 \) to the focus of ellipse. We also want to express orbital parameters in terms of physical variables. As tetrad component measures physical distance, the orbital parameters should also be expressed in tetrad variable (this automatically incorporates the effect of scale factor in the orbit). In terms of orbital parameters, eccentricity \( \epsilon \) and semi-major axis \( R \), the equation of orbit in de Sitter background is given by,

\[
r_{ph} = \frac{R(1 - \epsilon^2)}{1 + \epsilon \cos \psi}
\]

(22)

\( r_{ph} \) denotes relative physical separation between two components of binary. As discussed earlier, moments should be defined in the tetrad frame. We choose the time direction of tetrad frame along the worldline (measures the proper time \( t \) of source and a triad frame \((\bar{x}, \bar{y}, \bar{z})\) is attached to the focus of the ellipse such that the orbit is restricted to \((\bar{x}, \bar{y})\) plane.
FIG. 2: In de Sitter background $r = \text{const.}$ hypersurfaces intersect $\mathcal{J}^+$. Red lines denote $r_{\text{ph}} := ar = \text{const.}$ surfaces, which are also Killing surfaces. AE denotes the cosmological horizon ($r_{\text{ph}} = H^{-1}$) of the source. The spatially compact source is well inside the cosmological horizon. Blue line is the worldline of source trajectory which follows an elliptic orbit in de Sitter background. A tetrad frame is attached to the focus of ellipse.

and is given by,

$$\bar{x}(t) = r_{\text{ph}} \cos \psi , \quad \bar{y}(t) = r_{\text{ph}} \sin \psi , \quad \bar{z}(t) = 0 \quad (23)$$

Hence, from eq. (13) the mass quadrupole moment of the binary system can be written in a matrix form,

$$Q^{ij} = \mu r_{\text{ph}}^2 \begin{pmatrix} \cos^2 \psi & \sin \psi \cos \psi & 0 \\ \sin \psi \cos \psi & \sin^2 \psi & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (24)$$

We restrict our analysis to the adiabatic limit, i.e., time scale for orbital parameters to change is much longer than orbital time period. For example, this is equivalent to saying $|\frac{\Delta E}{E}| \ll 1$, where $\Delta E$ denotes change in energy of the system over one period and $E$ is the total energy of the system. This can be seen easily from leading term of eq. (27), $|\frac{\Delta E}{E}| = |\frac{\langle P \rangle T_p}{E}| \sim \frac{\mu (\frac{GM}{R})^{5/2}}{E} \ll 1$. At this point, it is worthy to mention that to study
the secular change in orbital parameters one needs to investigate the change over a period, only instantaneous change does not help. For circular orbit instantaneous power does not have angular dependence (see eq. (A11)), hence average over one time period is same as the unaveraged answer. But for elliptic orbit instantaneous power has angular dependence, hence averaging plays a crucial role. In adiabatic approximation the orbital parameters, semi-major axis and eccentricity of elliptic orbit (or equivalently energy and angular momentum) are assumed to be constant of motion over one orbital time period. The system spends many periods near any point of its phase space trajectory. As the gravitational radiation carries both energy and angular momentum, the binary system undergoes secular changes, both in its semi major axis and eccentricity. For our purpose, in the next section, we concentrate on energy loss and consequent orbital decay rate (time derivative of period).

V. POWER RADIATION FROM INSPIRALING BINARY

At first we would like to compute power radiated by an elliptic inspiraling binary system due to new quadrupole formula in eq. (10). Now using Λ projection,

\[ Q_{ij}^t Q_{kl}^t := \Lambda_{ijkl} \Omega_{mn} Q_{mn} Q_{ijkl} \]

eq. (10) can be written as,

\[ P = \frac{G}{5} \langle Q_{ij} Q_{ij} - \frac{1}{3} Q^2 \rangle, \] (25)

where \( Q := \delta_{ij} Q_{ij} \). In deriving this expression we have used the identity for Λ projector,

\[ \int d^2 S \Lambda_{ijkl} = \frac{2\pi}{15} \left[ 11 \delta_i^i \delta_j^j - 4 \delta_i^j \delta_j^i + \delta_i^i \delta_j^j \right]. \]

For a weakly stressed system, as for Newtonian fluids, pressure \( \pi \) can be neglected compared to the energy density \( \rho \), so we can neglect the pressure quadrupole moment terms in eq. (11). Hence neglecting the pressure quadrupole moment terms, power radiated by binary system can be expressed as,

\[ P = \frac{G}{5} \langle Q_{ij} Q_{ij} - \frac{1}{3} Q^2 \rangle \]

\[ \approx \frac{32G^4 \mu^2 M^2}{5R^5} f(\epsilon) + \frac{8H^2 G^3 \mu^2 M^2}{R^2} g(\epsilon) + \frac{8H^4 G^2 \mu^2 M R}{5} h(\epsilon) \] (27)

where \( f(\epsilon), g(\epsilon), h(\epsilon) \) are explicitly given in (A15), (A16) and each of these functions equal to unity as \( \epsilon \rightarrow 0 \). Schematics of the quadrupolar power computation are given in appendix A. Order \( H \) and \( H^3 \) terms exactly vanish, after taking time average over orbital period. This can also be understood by following observation. When direction of time is reversed, orbit goes from anticlockwise to clockwise direction and radiated energy does not depend
on orientation of revolution. The order $H$ and $H^3$ terms in (A11) have been generated from odd number of time derivatives of moment and thus have odd parity under time reversal and therefore have to vanish. For $\Lambda \to 0$, this formula reduces to the usual formula for quadrupolar power radiation by binary system in an elliptic orbit [24].

To get a feel for order of magnitude, for simplicity let us take a look at power radiated by a circular orbit. Taking $\epsilon \to 0$ in the eq. (27) we obtain the expression for circular orbit,

$$P \approx \frac{32G^4\mu^2M^3}{5R^5} \left[ 1 + \frac{5}{4} \frac{H^2R^3}{GM} + \frac{1}{4} \frac{H^4R^6}{G^2M^2} \right]$$

This expression has the dimensionless expansion parameter $\frac{H^2R^3}{GM}$. As mentioned in [25], a compact binary that coalesces after passing through the last stable orbit is a powerful source of gravitational waves, we assume $R = r_{LSO} = 3R_S$. For simplicity assume $m_1 = m_2 = 1M_\odot$ and using Schwarzschild radius of sun, $R_S = 2GM_\odot = 3 \times 10^3$ meters,

$$\frac{H^2R^3}{GM} = \frac{2H^2R_S^3}{R_S} \times 27 = H^2R_S^3 \times 54 \approx 10^{-52} \times 10^6 \times 10^2 \approx 10^{-44}$$

It is customary and convenient to express (28) in terms of chirp mass and gravitational wave frequency. For circular orbit substituting $\omega_s = \sqrt{\frac{GM}{R^3}}$. Hence, power loss due to gravitational radiation from a circular binary orbit can be expressed as,

$$P \approx \frac{32G^4\mu^2R^4\omega_s^6}{5} \left( 1 + \frac{5}{4} H^2\omega_s^{-2} + \frac{1}{4} H^4\omega_s^{-4} \right)$$

$$= \frac{32}{5G} \left( \frac{GM_c \omega_{gw}}{2} \right)^{10/3} \left[ 1 + 5H^2\omega_{gw}^{-2} + 4H^4\omega_{gw}^{-4} \right]$$

where we introduce the chirp mass $M_c := \mu^{3/5}M^{2/5}$ and $\omega_{gw} = 2\omega_s$. This result matches with that of [18] (see eq. (25) of the reference), indicating that for energy flux computation by a circular binary, TT vs $tt$ may not matter in de Sitter background. \(^2\) Whether this result generalizes to elliptic orbit case also is still needed to be investigated. It is also clear from eq. (30) - correction terms are down by $H^2/\omega_{gw}^2$, i.e. for a typical rapidly varying source, $(\lambda/L_B)^2 \sim 10^{-42} - 10^{-44}$ (this reduced wavelength $\lambda$ (50 – 500 km) corresponds to frequency band $f \sim 10^2 - 10^3$ Hz in LIGO).

\(^2\) It is mentioned in [16, 17] for energy flux computation in asymptotically flat space-time, identification of $tt$ vs TT does not matter. Distinction between $tt$ and TT is crucial for angular momentum flux.
Now let us investigate an observable parameter $\dot{P}_b$ (time derivative of orbital period) for elliptic orbit. Orbital period $P_b$ is related to orbital energy as $P_b = \text{const.} \times (-E)^{3/2}$. Hence,

$$\frac{\dot{P}_b}{P_b} = -\frac{3}{2} \frac{\dot{E}}{E} \quad (31)$$

Now, assuming that the binary system is loosing energy entirely due to quadrupolar radiation, substituting eq. (27) in $\dot{E}$, we find

$$\dot{P}_b = -\frac{192\pi}{5} G^{5/3} \mu M^{2/3} \left( \frac{P_b}{2\pi} \right)^{-5/3} f(\epsilon) \times \left[ 1 + \frac{5}{4} H^2 \left( \frac{P_b}{2\pi} \right)^2 \frac{g(\epsilon)}{f(\epsilon)} + \frac{1}{4} H^4 \left( \frac{P_b}{2\pi} \right)^4 \frac{h(\epsilon)}{f(\epsilon)} \right], \quad (32)$$

where the average over an orbital period is understood. In the intermediate step we have also used $E = -\frac{GM^2}{2r}$ and $P_b = 2\pi \sqrt{\frac{R^3}{G M}}$. For Hulse-Taylor Pulsar the values of relevant parameters are: $m_1 = 1.4414 M_\odot, m_2 = 1.3867 M_\odot, \epsilon = 0.617338, P_b = 2.790698 \times 10^4 s$ [25]. Hence, for Hulse-Taylor binary first order correction for orbital decay rate,

$$\frac{5}{4} H^2 \left( \frac{P_b}{2\pi} \right)^2 \frac{g(\epsilon)}{f(\epsilon)} = \frac{5}{4} \times \frac{10^{-35}}{3} \times \left( \frac{P_b}{2\pi} \right)^2 \frac{g(\epsilon)}{f(\epsilon)} \sim 10^{-29} \quad (33)$$

Current accuracy level of the observation of orbital decay rate of Hulse-Taylor pulsar is at $10^{-3}$. Hence correction terms due to $\Lambda$ are utterly negligible in the current observational context.

VI. SUMMARY AND DISCUSSION

In this short paper we applied the quadrupole formula for the power loss from a binary system in de Sitter background. As noted earlier, the modified quadrupole formula can impact both direct and indirect observations of gravitational waves. While indirect observations track orbital decay rate of binary system, direct observation in gravitational wave astronomy is sensitive to change in orbital phase. This paper focuses on the former and concludes that impact of $\Lambda$ in orbital decay is negligible in the context of current accuracy of observations. From dimensional analysis one may argue that correction terms due to $\Lambda$ should be $\sqrt{\Lambda} \times \text{lengthscale}$. This correction may be relevant over cosmological distances, e.g. mega-parsec. There are two natural length scales - one is observational distance and another is source dimension. It should be noted that though linearized field expression depends on observational length scale, energy flux is independent of observational length scale. Hence, only available length scale is orbital length scale which enters into the expression via
definition of source quadrupole moments. A typical compact object has orbital extension
to the order of $10^6$ m. Hence, this crude analysis also suggests that the correction terms
should be of the order of $\sqrt{\Lambda} \times \text{orbital lengthscale} \sim 10^{-26} \times 10^6 = 10^{-20}$. In our case, the
correction terms are even smaller as order $H(\sqrt{\Lambda}/3)$ term drops out and leading correction
term is of the order $H^2$.

![Diagram](image)

**FIG. 3:** Blue line denotes trajectory of source in de Sitter background while the red lines are Killing
surfaces. The source is active between the time interval $t_1$ to $t_2$. As the energy propagation is
sharp, there is no energy flux across the out-going null surfaces. $t = \tilde{t}_1$ and $t = \tilde{t}_2$ denote two
out-going null rays. $\tilde{t}$ is defined via retarded time $\eta - r = -\frac{1}{H}e^{-Ht}$. $\tau$ is Killing parameter. It
is shown in [12] that between two outgoing null rays, the Killing interval $d\tau$ is same for different
Killing surfaces, it also equals to $dt$. Therefore, energy lost by source in the time interval $dt$ is
registered on $\mathcal{H}^+$ or $\mathcal{J}^+$ along the out-going null direction.

A question regarding validity of eq. (32) (where we related rate of energy lost by binary
to that of quadrupolar power of gravitational waves) arises due to non availability of Bondi-
type mass loss formula relating to flux in de Sitter background. Recall from [12], though linearized field in de Sitter background has a ‘tail’ term, in the energy propagation tail term does not contribute. This is because energy depends on time derivative of field, not only the field. In the process of taking derivative tail term cancels out exactly, leaving the sharp propagation. Hence, flux across out-going null hypersurface vanishes. Along the Killing trajectory \(d\eta/d\tau = -H\eta, dr/d\tau = -Hr\); therefore \(\eta - r = (\eta - r_\star)e^{-H\tau} = (-\frac{1}{H}\epsilon^{-H\bar{t}}) e^{-H\tau}\). Again from the definition of \(\bar{t}\), \(\eta - r = -\frac{1}{H}\epsilon^{-H\bar{t}}\). Hence \(\tau = \bar{t} - \bar{t}_\star \implies d\tau = d\bar{t}\). As \(\bar{t}\) is constant along null ray we can compute \(d\bar{t}\) on source trajectory which is \(dt\). Now if we assume that the source is loosing energy only via gravitational radiation, energy lost by the source in the time interval \(dt\) is exactly the energy flux across gravitational wave at the portion \(\mathcal{H}^+\) or \(\mathcal{J}^+\) bounded by two out-going null lines (as shown in fig. 3).

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Appendix A: Power radiated by an elliptic binary

As computation of radiated power needs derivative of moments, let us give the expressions for derivative of moments. Taking the quadrupolar mass moment tensor (24) for elliptic orbit, derivatives of different components are given by,

\[
\dot{Q}_{11} = \frac{-\alpha}{1 + \epsilon \cos \psi} \sin 2\psi, \tag{A1}
\]

\[
\dot{Q}_{12} = \frac{\alpha}{1 + \epsilon \cos \psi} (\epsilon \cos \psi + \cos 2\psi), \tag{A2}
\]

\[
\dot{Q}_{22} = \frac{2\alpha}{1 + \epsilon \cos \psi} (\epsilon \sin \psi + \sin \psi \cos \psi), \tag{A3}
\]

with \(\alpha = \mu \sqrt{GMR(1 - \epsilon^2)}\).
\[ \ddot{Q}_{11} = -\beta [2 \cos 2\psi(1 + \epsilon \cos \psi) + \epsilon \sin 2\psi \sin \psi], \quad (A4) \]
\[ \ddot{Q}_{12} = -2\beta [\sin 2\psi + \epsilon \sin (1 + \cos^2 \psi)], \quad (A5) \]
\[ \ddot{Q}_{22} = 2\beta [\cos 2\psi + \epsilon \cos (1 + \cos^2 \psi) + \epsilon^2], \quad \text{with } \beta = \frac{\mu GM}{R (1 - \epsilon^2)}. \quad (A6) \]

\[ \ddot{Q}_{11} = \gamma (1 + \epsilon \cos \psi)^2[2 \sin 2\psi + 3\epsilon \sin \psi \cos^2 \psi], \quad (A7) \]
\[ \ddot{Q}_{12} = \gamma (1 + \epsilon \cos \psi)^2[-2 \cos 2\psi + \epsilon \cos (1 - 3 \cos^2 \psi)], \quad (A8) \]
\[ \ddot{Q}_{22} = -\gamma (1 + \epsilon \cos \psi)^2[2 \sin 2\psi + \epsilon \sin (1 + 3 \cos^2 \psi)], \quad \text{with } \gamma = \frac{2\mu (GM)^{3/2}}{R^{5/2}(1 - \epsilon^2)^{3/2}}. \quad (A9) \]

The expression for radiated power is given by (25),
\[ \mathcal{P} = \frac{G}{5} \langle Q_{ij} Q_{ij} - \frac{1}{3} Q^2 \rangle := \langle P(\psi) \rangle \quad (A10) \]

Now we plug the derivatives of moments in computing unaveraged quantity \( P(\psi) \) and express our result order by order in \( H \).

\[ P(\psi) \approx \frac{8G^4}{15} \frac{\mu^2 M^3}{R^5(1 - \epsilon^2)^5} (1 + \epsilon \cos \psi)^4 \left[ 12(1 + \epsilon \cos \psi)^2 + \epsilon^2 \sin^2 \psi \right] \]
\[ + \frac{4G^{7/2}H}{5} \frac{\epsilon^2 \mu^2 M^{5/2}}{R^{7/2}(1 - \epsilon^2)^{7/2}} \sin (1 + \epsilon \cos \psi)^2 \left[ 18 + 13 \epsilon^2 + 40 \epsilon \cos \psi + 9 \epsilon^2 \cos 2\psi \right] \]
\[ + \frac{2G^3H^2}{15} \frac{\epsilon^2 \mu^2 M^2}{(1 - \epsilon^2)^2 R^2} \left[ 60 + 100 \epsilon^2 + 36 \epsilon^4 + \epsilon (180 + 113 \epsilon^2) \cos \psi + 116 \epsilon^2 \cos 2\psi + 19 \epsilon^3 \cos 3\psi \right] \]
\[ - \frac{32G^{5/2}H^3}{15} \frac{\mu^2 M^{3/2}}{\sqrt{R}(1 - \epsilon^2)} \frac{\sin (\epsilon + \cos \psi)}{1 + \epsilon \cos \psi} \]
\[ + \frac{4G^2H^4}{15} \frac{\mu^2 MR(1 - \epsilon^2)}{(1 + \epsilon \cos \psi)^2} \left[ 6 + 7 \epsilon^2 + 12 \epsilon \cos \psi - \epsilon^2 \cos 2\psi \right] \]

In the intermediate step we have neglected pressure quadrupole moment term in eq. (11) and used \( Q_{ij} \approx \partial_p Q_{ij} - 3H \partial_i Q_{ij} + 2H^2 \partial_i \partial_j Q_{ij} \). To get the expression for power we have to perform an averaging integral of (A11). An explicit averaging procedure is illustrated in the appendix of [12]. It permits to split four dimensional averaging integral into an integral over \( r_{ph} = \text{const.} \) hypersurface and a three dimensional flux integral. Averaging integral over \( r_{ph} = \text{const.} \) hypersurface and angular integral can be done explicitly, leaving the four dimensional averaging integral to a time-averaged quantity only. Hence average power over an orbital period is given by
\[ \mathcal{P} = \frac{1}{T} \int_0^T dt P(\psi) = \frac{\omega_\psi}{2\pi} \int_0^{2\pi} \frac{d\psi}{\psi} P(\psi) \]
\[ = \left( \frac{GM}{R^3} \right)^{1/2} \times \frac{1}{2\pi} \int_0^{2\pi} \frac{d\psi}{\psi} P(\psi) \quad (A12) \]
Using A11 we obtain,

\[
P = \frac{32G^4\mu^2 M^3}{5R^5} f(\epsilon) + \frac{8H^2G^2\mu^2 M^2}{R^2} g(\epsilon) + \frac{8H^4G^2\mu^2 MR}{5} h(\epsilon) \tag{A14}
\]

where,

\[
f(\epsilon) = (1 - \epsilon^2)^{-7/2} \left(1 + \frac{73}{24}\epsilon^2 + \frac{37}{96}\epsilon^4\right), \tag{A15}
\]

\[
g(\epsilon) = (1 - \epsilon^2)^{-1/2} \left[\frac{4 - \sqrt{1 - \epsilon^2}}{3}\right], \quad h(\epsilon) = 1 - \frac{\epsilon^2}{3}. \tag{A16}
\]

Deriving this expression we have used \(\dot{\psi} = \sqrt{\frac{GMR(1-\epsilon^2)}{r_{ph}^3}}\) for elliptic Keplerian orbit.

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