Harmony of the primes: harmonic numbers and the prime counting function

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Abstract

We provide approximations to the prime counting function by various discretized versions of the logarithmic integral function, expressed solely in terms of the harmonic numbers. We demonstrate with explicit error bounds that these approximations are at least as good as the logarithmic integral approximation. As a corollary, we provide some reformulations of the Riemann hypothesis in terms of the prime counting function and the harmonic numbers.

Keywords: prime counting function, Riemann hypothesis, Riemann zeta function, harmonic numbers.

MSC: 11N05, 11M26

1 Introduction

This paper concerns the function \( \pi : \mathbb{R}_{>0} \rightarrow \mathbb{R} \) that for any \( x > 0 \) counts the number of primes less than or equal to \( x \):

\[ \pi(x) = \# \{ p \leq x : p \text{ is prime} \}, \quad x > 0. \]

The function \( \pi(x) \) is known as the prime counting function. We call the related function \( p : \mathbb{R}_{>0} \rightarrow \mathbb{R} \) defined by

\[ p(x) = \frac{\pi(x)}{x}, \quad x > 0, \]

the prime density function. The celebrated prime number theorem, proved independently by de la Vallée Poussin \[3\] and Hadamard \[8\] in 1896, states that

\[ \pi(x) \sim \frac{x}{\log x} \quad (x \to \infty), \]

or, equivalently,

\[ \lim_{x \to \infty} x^{p(x)} = e \quad (x \to \infty), \]

where \( \log x \) is the natural logarithm. It is known, however, that the logarithmic integral function

\[ \text{li}(x) = \int_0^x \frac{dt}{\log t}, \quad x > 0, \]

provides a better approximation to \( \pi(x) \) than any algebraic function of \( \log x \), where the integral assumes the Cauchy principal value \( \lim_{\varepsilon \to 0^+} \left( \int_0^{1-\varepsilon} \frac{dt}{\log t} + \int_{1+\varepsilon}^x \frac{dt}{\log t} \right) \). The prime number theorem with error term, proved by de la Vallée Poussin in 1899 \[4\], states that

\[ \pi(x) - \text{li}(x) = O \left( xe^{-C \sqrt{\log x}} \right) \quad (x \to \infty) \]
for some constant $C > 0$, which is easily seen to imply

$$\pi(x) - \text{li}(x) = O_k \left( \frac{x}{(\log x)^k} \right) \quad (x \to \infty)$$

for all $k > 0$, where we write $O_{t_1 \ldots t_n}$ if the $O$ constant depends on parameters $t_1, \ldots, t_n$. De la Vallée Poussin’s result has since been improved to

$$\pi(x) - \text{li}(x) = O \left( x e^{-A(\log x)^{3/5}(\log \log x)^{-1/5}} \right) \quad (x \to \infty),$$

where $A = 0.2098$ [7], which is the strongest known $O$ bound on $\pi(x) - \text{li}(x)$ to date.

Proofs of all known bounds on the error $\pi(x) - \text{li}(x)$ are based on Riemann’s explicit formula for $\pi(x)$ in terms of the zeros of the Riemann zeta function $\zeta(s)$ and rather sophisticated methods for verifying zero-free regions of $\zeta(s)$ in the critical strip $0 \leq \Re s \leq 1$. The celebrated Riemann hypothesis states that all such zeros lie on the line $\Re s = \frac{1}{2}$. As is now well known, von Koch proved in 1901 [9] that the Riemann hypothesis is equivalent to

$$\pi(x) - \text{li}(x) = O(\sqrt{x} \log x) \quad (x \to \infty),$$

which to date is the strongest bound on the error $\pi(x) - \text{li}(x)$ that is widely conjectured to hold. It is known [14], more generally, that if

$$\delta = \sup \{ \Re(s) : s \in \mathbb{C}, \zeta(s) = 0 \}$$

denotes the supremum of the real parts of the zeros of $\zeta(s)$, then $\frac{1}{2} \leq \delta \leq 1$, and $\delta$ is the least $\alpha \in \mathbb{R}$ such that

$$\pi(x) - \text{li}(x) = O(x^{\alpha} \log x) \quad (x \to \infty), \quad (1.1)$$

and $\delta$ is also the infimum of all $\alpha \in \mathbb{R}$ such that

$$\pi(x) - \text{li}(x) = O(x^{\alpha}) \quad (x \to \infty). \quad (1.2)$$

Moreover, the Riemann hypothesis is equivalent to $\delta = \frac{1}{2}$.

For every nonnegative integer $n$, let

$$H_n = \sum_{k=1}^{n} \frac{1}{k}$$

denote the $n$th harmonic number. The summatory function of a function $f(x)$ is the function $\sum_{k=1}^{n} f(n)$. Thus, the function $H_n$ is the summatory function of $x^\frac{1}{2}$. Summatory functions are discrete integrals in the sense that $\sum_{k=1}^{n} f(n) = \int_{1}^{n} f(x) \, dx$ for all integers $n \geq N$, where $\nu$ is the unique discrete measure (with respect to Lebesgue measure) that is supported on $\mathbb{Z}$ with all weights equal to 1. Thus, the $n$th harmonic number $H_n = \int_{1}^{n} \frac{1}{x} \, dx$ is a discrete integral of $\frac{1}{x}$ and is in this sense a “discrete natural logarithm.” Not unexpectedly, one has

$$H_n \sim \int_{1}^{n} \frac{dx}{x} = \log n \quad (n \to \infty),$$

and, more precisely, the limit

$$\lim_{n \to \infty} \frac{H_n - \log n}{n} = \gamma = 0.5772156649015328606651 \ldots$$

is finite. This important constant, known as the Euler-Mascheroni constant, is a precise measure of the discrepancy between the natural logarithm and the “discrete natural logarithm.”

Because $H_n \sim \log n \quad (n \to \infty)$, the prime number theorem is equivalent to

$$\pi(n) \sim \frac{n}{H_n} \quad (n \to \infty),$$

where $\frac{n}{H_n}$ is also the harmonic mean of the integers $1, 2, 3, \ldots, n$. This simple observation, alongside an inequality equivalent to the Riemann hypothesis involving the sum of divisors function and the harmonic
numbers discovered by J. Lagarias, described below, led me to wonder if the harmonic numbers could be used to provide approximations to \( \pi(x) \) that are better than \( \frac{\pi(x)}{x} \)—or, ideally, even as good as \( \text{li}(n) \). The former problem in part inspired the paper [5], where I provided various asymptotic expansions of the prime counting function, including several involving the harmonic numbers, such as the (divergent) asymptotic continued fraction expansion

\[
p(e^{\gamma} n) \sim \frac{1/H_n}{1 - \frac{1}{H_n}} - \frac{1/2H_n}{1 - \frac{1}{2H_n}} - \frac{1/3H_n}{1 - \frac{1}{3H_n}} - \cdots \quad (n \to \infty),
\]

where

\[
e^{\gamma} = \lim_{n \to \infty} \frac{e^{H_n}}{n} = 1.7810724179901979852365\ldots,
\]

and also where

\[
e^{\gamma} = \lim_{x \to \infty} \frac{\pi(x)}{\prod_{p \leq x} \left(1 - \frac{1}{p}\right)}
\]

due to the prime number theorem and the third of Mertens’ famous three theorems of 1874. Such results indicate that the harmonic numbers, just like natural logarithm function and the constants \( e \), \( \gamma \), and \( e^{\gamma} \), encode information about the primes.

Regarding the remarkable constant \( e^{\gamma} \), in 1984, G. Robin proved [15] that the Riemann hypothesis holds if and only if

\[
\sum_{d \mid n} d \leq e^{\gamma} n \log \log n, \quad \forall n \geq 5041.
\]

Since by a 1913 result of Grönwall and Mertens’ second theorem one also has

\[
e^{\gamma} = \lim_{n \to \infty} \frac{\sum_{d \mid n} d}{n \log \log n} = \limsup_{n \to \infty} \frac{\sum_{d \mid n} \frac{1}{d}}{\sum_{p \leq n} \frac{1}{p}},
\]

the constant \( e^{\gamma} \) in Robin’s equivalence the best possible. In 2000, J. Lagarias used Robin’s result to show [10] that the Riemann hypothesis holds if and only if

\[
\sum_{d \mid n} d < H_n + e^{H_n} \log H_n, \quad \forall n > 1,
\]

if and only if

\[
\sum_{d \mid n} d < e^{H_n} \log H_n, \quad \forall n > 60.
\]

Lagarias’ inequalities are closely related to Robin’s because, by asymptotics noted earlier, one has

\[
e^{\gamma} n \log \log n \sim e^{H_n} \log H_n \sim H_n + e^{H_n} \log H_n \quad (n \to \infty).
\]

I gather from reading [10] that such observations are what led Lagarias to consider his inequalities.

Although they are “elementary,” the three reformulations of the Riemann hypothesis above concern the sum of divisors function rather than the prime counting function. In this paper, we provide several reformulations of the Riemann hypothesis that are expressed solely in terms of the harmonic numbers and the prime counting function. We show, for example, that the Riemann hypothesis holds if and only if

\[
\pi(e^{\gamma} n) = e^{\gamma} \sum_{k=1}^{n} \frac{1}{H_k} + O \left( \sqrt{n} H_n \right) \quad (n \to \infty),
\]
if and only if
\[ p(e^n) = \frac{1}{n} \sum_{k=1}^{n} \frac{1}{H_k} + O \left( \frac{H_n}{\sqrt{n}} \right) \quad (n \to \infty), \]
if and only if
\[ \left| p(e^n) - \frac{1}{n} \sum_{k=1}^{n-1} \frac{1}{H_k} \right| < \frac{1}{8\pi e^{\gamma/2}} \sqrt{\frac{n}{\sqrt{n}}} + \frac{0.4986013304}{n}, \quad \forall n \geq 803, \]
if and only if
\[ \left| p(e^n) - \frac{1}{n} \sum_{k=2}^{n-1} \frac{1}{H_k} \right| < \frac{1}{8\pi e^{\gamma/2}} \sqrt{\frac{n}{\sqrt{n}}} + \frac{1 + 0.4986013304}{n}, \quad \forall n \geq 1. \]

Moreover, any choice of larger constants still yields a Riemann hypothesis equivalent, so, for example, since
\[ \frac{1}{8\pi e^{\gamma/2}} = \frac{1}{33.54135851775...} < \frac{1}{33}, \]
the Riemann hypothesis is also equivalent to
\[ \left| p(e^n) - \frac{1}{n} \sum_{k=2}^{n-1} \frac{1}{H_k} \right| < \frac{1}{33} \sqrt{\frac{n}{\sqrt{n}}} + \frac{3}{2n}, \quad \forall n \geq 1. \]

Such a reformulation of the Riemann hypothesis is noteworthy because it makes no mention of transcendental functions (like \( e^x \), \( \log x \), and \( \text{li}(x) \)), and the only numbers in the given inequality that may not be rational are \( e^{\gamma} \) and \( \sqrt{n} \). Moreover, \( \pi(n) \) can be computed for decent-sized values of \( n \)—currently the largest known value is for \( n = 10^{27} \)—, while also one has
\[ H_n = \gamma + \Psi(n + 1), \quad \forall n \geq 0, \]
where
\[ \Psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} \]
is the logarithmic derivative of the gamma function, known as the \textit{digamma function}, which allows \( H_n \) to be computed easily and accurately for large \( n \). (Note also that \( \gamma = -\Psi(1) = -\Gamma'(1) \).)

The third and fourth of our Riemann equivalents above are made possible by a well-known reformulation due to L. Schoenfeld [16]: the Riemann hypothesis holds if and only if
\[ |\pi(x) - \text{li}(x)| < \frac{1}{8\pi} \sqrt{x} \log x, \quad \forall x \geq 2657. \]

The constant 0.4986013304 in those two Riemann equivalents can be replaced with any other upper bound of the limit
\[ \kappa = \lim_{n \to \infty} \left( \frac{\text{li}(e^n)}{e^n} - \sum_{k=1}^{n} \frac{1}{H_k} \right) \approx 0.4986. \]

The limit \( \kappa \) exists because the sequence \( \frac{\text{li}(e^n)}{e^n} - \sum_{k=2}^{n-1} \frac{1}{H_k} \) is positive, monotonically increasing, and bounded above, and therefore bounded above by \( \kappa \). Our results allow us to compute upper and lower bounds of constants like \( \kappa \) to any desired degree of accuracy, so, for example, one has
\[ 0.4985987518 < \kappa < 0.4986013304. \]

Moreover, since \( \text{li}(x) > \pi(x) + \frac{1.49}{x} \) for all \( x \geq 6 \) for which the value of \( \pi(x) \) is known, the sum \( \sum_{k=2}^{n-1} \frac{1}{H_k} \) is closer to \( \frac{\text{li}(e^n)}{e^n} \) than is \( \frac{\text{li}(e^n)}{e^n} \) for all \( n \geq 6 \) for which the value of \( \pi(e^n) \) can be computed by current means (while the difference between the two approximations of \( \pi(e^n) \) is less than \( \kappa \) for all \( n \)). Thus, approximations of the prime counting function using harmonic numbers can indeed be worthy rivals of the standard logarithmic integral approximation.
Fundamentally, our results in this paper arose from the realization that for any \( t \in \mathbb{R} \) one can approximate \( \text{li}(e^t x) \) with a “discrete logarithmic integral,” namely, a discrete integral of \( \frac{1}{H_k} \), or, more generally of \( \frac{1}{H_k - \gamma + t} \) for any fixed \( t \), which ultimately we show can approximate \( \frac{\pi(e^t x)}{e^t} \) at least as well as \( \frac{\text{li}(e^t x)}{e^t} \). More precisely, in Section 3, we prove the following theorem.

**Theorem 1.1.** For all \( t \in \mathbb{R} \), one has

\[
\frac{\text{li}(e^t n)}{e^t} = \sum_{k=\lceil \mu e^{-t} \rceil}^{n-1} \frac{1}{H_k - \gamma + t} + \beta_n(t) = \sum_{k=\lceil \mu e^{-t} \rceil}^{n-1} \frac{1}{H_k - \gamma + t} + \beta(t) - \frac{1 + o(1)}{12n(\log n)^2} (n \to \infty)
\]

for unique error functions \( \beta_n(t) > 0 \) and \( \beta(t) > 0 \) with \( \beta(t) = \lim_{n \to \infty} \beta_n(t) \), where

\( \mu = 1.45130923488381050\ldots \)

is the unique positive real root of \( \text{li}(x) \) (called the Ramanujan–Soldner constant).

We also prove explicit bounds on \( \beta_n(t) \) and \( \beta(t) \) in terms of \( t \), and we show, for example, that

\[
\beta(t) < \text{li}'(\mu) = \limsup_{t \to -\infty} \beta(t),
\]

\[
\beta(t) = \frac{1}{t} + O\left(\frac{1}{t^2}\right) \quad (t \to \infty),
\]

and

\[
\beta(t) = \text{li}(e^t \lceil \mu e^{-t} \rceil) - \frac{\text{li}''(\mu)}{12} e^t (1 + o(1)) \quad (t \to -\infty),
\]

where

\[
\text{li}'(\mu) = \frac{1}{\log \mu} = 2.6845103508207076525\ldots
\]

and

\[
-\frac{\text{li}''(\mu)}{12} = \frac{1}{12\mu(\log \mu)^2} = \frac{1}{2.4167347864038834213\ldots}.
\]

Note that the constant \( \kappa \approx 0.4986 \) introduced earlier is precisely \( \beta(\gamma) \). In Section 4, we use Theorem 1.1 to make precise the approximation

\[
\frac{\pi(e^t n)}{e^t} \approx \sum_{k=\lceil \mu e^{-t} \rceil}^{n-1} \frac{1}{H_k - \gamma + t}, \quad n \geq \lceil \mu e^{-t} \rceil,
\]

from which we derive our Riemann hypothesis equivalents. The motivation behind Theorem 1.1 is that, analogous to the integral representation

\[
\int_{\mu e^{-t}}^{x} \frac{du}{t + \log u} = \frac{\text{li}(e^t x)}{e^t}
\]

of \( \frac{\text{li}(e^t x)}{e^t} \), the sum \( \sum_{\mu e^{-t} \leq k < x} \frac{1}{H_k - \gamma + t} \) can be represented as the discrete integral

\[
\int_{\mu e^{-t}}^{x+0^{-}} \frac{dv(u)}{t + H_u - \gamma} = \sum_{\mu e^{-t} \leq k < x} \frac{1}{H_k - \gamma + t},
\]

Our approximation to \( \frac{\pi(e^t n)}{e^t} \) above is therefore a doubly discretized version of the logarithmic integral.

Rather than just provide various reformulations of the Riemann hypothesis, we find it useful to generalize such reformulations to unconditional results that are expressed in terms of the supremum of the real parts of the zeros of the Riemann zeta function. The quintessential example of such a generalization is provided
by Eq. (1.1). The motivation for such generalizations is the fact that the Riemann hypothesis, for all we
know, could be false. Although it has substantial numerical and heuristic evidence, and although its rightful
analogue for function fields was proved to be true, the Riemann hypothesis has stubbornly resisted proof
since its original formulation by Riemann in 1859. Moreover, other conjectures about the prime counting
function that once had very strong numerical evidence, some of which were considered plausible even by the
likes of Gauss and Riemann, were later shown to be false. Quite famously, for example, Littlewood proved
in 1914 \[ \pi(x) = \text{li}(x) + \Omega \pm \left( \frac{\sqrt{x} \log \log x}{\log x} \right) \quad (x \to \infty), \] (1.3)
from which it follows that the set of all \( x \geq 2 \) such that \( \text{li}(x) < \pi(x) \) is both nonempty and unbounded.
However, no such value of \( x \) is explicitly known, and the smallest such \( x \) is currently expected to be about
\( 1.3971 \times 10^{316} \). More recently, it was shown that the largest positive zero of Riemann’s approximation
\( R(x) = \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \text{li}(x^{1/k}) \) to the prime counting function is approximately \( 1.8286 \times 10^{-14828} \). Results
like these demonstrate that the proper measure of the “size” of a prime \( p \) is not \( p \), but \( \log p \), and for some
purposes even \( \log \log \log p \). Thus, conjectures related to the prime numbers based solely upon numerical
evidence are fraught with peril.

I would like to thank Sean Lubner and Daniel Brice for helping me check the inequalities in Corollaries
4.7 through 4.14 for small values of \( n \) using Python.

## 2 Approximating \( \log x \) with harmonic numbers

In this section, we list some properties of the harmonic numbers that form the basis for our results. Many
of these are known, but some are possibly new.

From the functional equation
\[
\Gamma(z + 1) = z \Gamma(z), \quad z \in \mathbb{C}\{0, -1, -2, -3, \ldots \}
\]
for the gamma function follows, by logarithmic differentiation, the functional equation
\[
\Psi(z + 1) = \frac{1}{z} + \Psi(z), \quad z \in \mathbb{C}\{0, -1, -2, -3, \ldots \}
\]
for the digamma function. Since \( \Psi(1) = -\gamma \) and \( H_0 = 0 \), it follows that the harmonic numbers \( H_n \) are
interpolated by the complex function
\[
H_z = \Psi(z + 1) + \gamma = \sum_{k=1}^{\infty} \left( \frac{1}{k} - \frac{1}{z + k} \right) = \lim_{n \to \infty} \left( H_n - \sum_{k=1}^{n} \frac{1}{z + k} \right), \quad z \in \mathbb{C}\{-1, -2, -3, \ldots \}. \quad (2.1)
\]
It is known that
\[
H_z - \gamma - \log z \sim \frac{1}{2z} \quad (z \to \infty),
\]
and, more generally, by the Euler-Maclaurin formula, that one has the (divergent) asymptotic expansion
\[
H_z - \gamma - \log z - \frac{1}{2z} \sim \sum_{k=1}^{\infty} \frac{-B_{2k}}{2kz^{2k}} \quad (z \to \infty),
\]
where \( B_k \) is the \( k \)th Bernoulli number. From this well-known expansion follows the asymptotic expansion
\[
H_{z - 1/2} - \gamma - \log z \sim \sum_{k=1}^{\infty} \frac{(1 - 2^{-2k+1})B_{2k}}{2kz^{2k}} \quad (z \to \infty).
\]
From the latter expansion and \[1\] Theorem 8, one can show that
\[
\sum_{k=1}^{n+1} \frac{(1 - 2^{-2k+1})B_{2k}}{2kz^{2k}} < H_{z - 1/2} - \gamma - \log x < \sum_{k=1}^{n} \frac{(1 - 2^{-2k+1})B_{2k}}{2kx^{2k}}
\]

for all $x > 0$ and all odd positive integers $n$. Thus, for example, one has

$$\frac{1}{24(x+1/2)^2} - \frac{7}{960(x+1/2)^4} < H_x - \gamma - \log(x+1/2) < \frac{1}{24(x+1/2)^2}, \quad \forall x > -1/2.$$  

and therefore

$$\frac{1}{24(x+1)^2} < H_x - \gamma - \log(x+1/2) < \frac{1}{24(x+1/2)^2}, \quad \forall x > -1/2. \quad (2.2)$$

In particular, $\log(x+1/2) + \gamma$ is an excellent approximation for $H_x$, and, correspondingly, $H_{x-1/2} - \gamma$ is an excellent approximation for $\log x$. Since

$$H_x - \gamma - \log x \sim \frac{1}{2x} (x \to \infty),$$

while

$$H_x - \gamma - \log(x+1/2) \sim \frac{1}{24x^2} (x \to \infty),$$

the advantage gained by shifting the log by $1/2$ is clear. This makes sense heuristically because of the demonstrable advantage, for monotonic functions, of the midpoint rule over left-hand or right-hand Riemann sums. See Figures 1 and 2 for a graphical comparison of the approximations above.

Figure 1: Graphs of $H_x > \log(x+1/2) + \gamma > \log x + \gamma$ on $(-1/2, 2]$  

Figure 2: Graphs of $\frac{1}{24(x+1)^2}, H_x - \gamma - \log(x+1/2), \frac{1}{24(x+1/2)^2}, \frac{1}{24(x+1)^2}, H_x - \gamma - \log x, \frac{1}{x^2}$ on $(-1/2, 2]$, ordered from smallest to largest on $[0, \infty)$

By the functional equation

$$H_z = \frac{1}{z} + H_{z-1}, \quad z \in \mathbb{C}\{0, -1, -2, -3, \ldots\}$$

for $H_x$, Eq. (2.2) generalizes as follows.
Proposition 2.1. For all $x \in \mathbb{R}\backslash\{-1,-2,-3,\ldots\}$ and all integers $n > x - \frac{1}{2}$, one has
\[
\frac{1}{24(x + n + 1)^2} \leq (H_x - \gamma) - \left(\log(x + n + 1/2) - \sum_{k=1}^{n} \frac{1}{x + k}\right) \leq \frac{1}{24(x + n + 1/2)^2}.
\]

Consequently, for all $x \in \mathbb{R}\backslash\{-1,-2,-3,\ldots\}$ one has
\[
\Psi(x + 1) = H_x - \gamma = \log(x + n + 1/2) - \sum_{k=1}^{n} \frac{1}{x + k} + \frac{1 + o(1)}{24n^2} \ (n \to \infty)
\]
and therefore
\[
\Psi(x + 1) = H_x - \gamma = \lim_{n \to \infty} \left(\log(x + n + 1/2) - \sum_{k=1}^{n} \frac{1}{x + k}\right).
\]

Of course, Eq. (2.2) is just Proposition 2.1 with $n = 0$ and $x > -\frac{1}{2}$. For $n = 1$ we obtain the estimate
\[
\frac{1}{24(x + 2)^2} \leq (H_x - \gamma) - \left(\log(x + 3/2) - \frac{1}{x + 1}\right) \leq \frac{1}{24(x + 3/2)^2},
\]
which is a substantial improvement on Eq. (2.2) for small positive $x$: see Figure 3. In general, the estimate
\[
H_x - \gamma \approx \log(x + n + 1/2) - \sum_{k=1}^{n} \frac{1}{x + k}, \quad x \in \mathbb{R}\backslash\{-1,-2,-3,\ldots\}, \quad n > -x - \frac{1}{2},
\]
for $x > -\frac{1}{2}$ and $n \geq 1$, is closer to $H_x - \gamma$ than is log$(x + 1/2)$ by exactly log$(1 + \frac{n}{x + 1/2}) - \sum_{k=1}^{n} \frac{1}{x + k}$, a positive quantity that satisfies
\[
\frac{1}{12(x + 1)^3} < \log\left(1 + \frac{n}{x + 1/2}\right) - \sum_{k=1}^{n} \frac{1}{x + k} < \frac{n}{12(x + 1/2)^3}
\]
and therefore
\[
\log\left(1 + \frac{n}{x + 1/2}\right) - \sum_{k=1}^{n} \frac{1}{x + k} \sim \frac{n}{12x^3} \ (x \to \infty).
\]
Thus, for example, log$(x + 3/2) - \frac{1}{x + 1}$ is closer to $H_x - \gamma$ than is log$(x + 1/2)$, by exactly log$(x + 3/2) - \log(x + 1/2) - \frac{1}{x + 1}$, which is positive and asymptotic to $\frac{1}{12x^3}$ as $x \to \infty$. Increasing $n$ in Proposition 2.1 or Eq. (2.3) provides substantial benefit only for $x$ not too much greater than $-n - 1/2$. An obvious advantage of these estimates over Eq. (2.1) is the fact that they come equipped with both upper and lower bounds for the error that both approach 0 as $n \to \infty$. Fortunately, however, one may also equip Eq. (2.1) with error bounds, as follows.

Proposition 2.2. Let $x > 0$, and let $N$ be a positive integer.
1. The sequence
\[
H_{x,N,n} = \sum_{k=N}^{n} \left(\frac{1}{k} - \frac{1}{x + k}\right), \quad n > N
\]
is positive, increasing, and satisfies
\[
\left(\log\left(1 + \frac{x}{u}\right)\right)_{n+1}^{N} < H_{x,N,n} < \left(\log\left(1 + \frac{x}{u}\right)\right)_{n+1/2}^{N-1/2}
\]
for all $n > N$. 

Figure 3: Graphs of the errors $(H_x - \gamma) - \left(\log(x + 3/2) - \frac{1}{x + 1}\right) < (H_x - \gamma) - \log(x + 1/2)$ on $(-1/2, 2]$

2. The limit

$$H_{x,N} := \sum_{k=N}^{\infty} \left( \frac{1}{k} - \frac{1}{x + k} \right) = H_x - \sum_{n=1}^{N-1} \left( \frac{1}{k} - \frac{1}{x + k} \right)$$

exists.

3. For all nonnegative integers $n$, one has

$$\frac{x}{n + 1 + x/2} < \log \left(1 + \frac{x}{n + 1}\right) \leq H_x - \sum_{k=1}^{n} \left( \frac{1}{k} - \frac{1}{x + k} \right) \leq \log \left(1 + \frac{x}{n + 1/2}\right) < \frac{x}{n + 1/2},$$

where also

$$\log \left(1 + \frac{x}{n + 1/2}\right) \sim \log \left(1 + \frac{x}{n + 1}\right) \sim \frac{x}{n} \ (n \to \infty).$$

Consequently, one has

$$H_x = \sum_{k=1}^{n} \left( \frac{1}{k} - \frac{1}{x + k} \right) + \frac{x + o(1)}{n} \ (n \to \infty).$$

Proof. The function $\frac{1}{u(x+u)}$ of $u \in (0, \infty)$ is positive, decreasing, and concave up. Therefore one has

$$H_{x,N,n} = x \sum_{k=N}^{n} \frac{1}{k(x+k)} < x \int_{N-1/2}^{n+1/2} \frac{du}{u(x+u)} = \left( \log \left(1 + \frac{x}{n+1}\right) \right)^{N-1/2}$$

and

$$H_{x,N,n} = x \sum_{k=N}^{n} \frac{1}{k(x+k)} > x \int_{N-1/2}^{n+1} \frac{du}{u(x+u)} = \left( \log \left(1 + \frac{x}{n+1}\right) \right)^{N}.$$

This proves statement (1). Since the sequence $H_{x,N,n}$ for $n > N$ is increasing and bounded above, it converges to a limit as $n \to \infty$, namely, to

$$H_{x,N} := \sum_{k=N}^{\infty} \left( \frac{1}{k} - \frac{1}{x + k} \right) = H_x - \sum_{n=1}^{N-1} \left( \frac{1}{k} - \frac{1}{x + k} \right).$$

This proves statement (2). Using the bounds in statement (1) and taking the limit as $n \to \infty$, we see that

$$\log \left(1 + \frac{x}{N}\right) \leq H_{x,N} \leq \log \left(1 + \frac{x}{N^{1/2}}\right)$$

Statement (3) follows by letting $n = N - 1$. □
By Propositions 2.1 and 2.2, the approximation of $H_x$ given in the former proposition is much better than that given in the latter and in Eq. (2.1). For more precise estimates of $H_x$ and $\Psi(x)$, see [1].

Although we do not require Proposition 2.2 in later sections, the simple method used in the proof is used twice in the next section in order to approximate $\text{li}(x)$ using the harmonic numbers.

3 Approximating $\text{li}(x)$ with harmonic numbers

Let

$$\mu = 1.451369234883381050283968485892027449\ldots$$

denote the Ramanujan–Soldner constant, which by definition is the unique positive zero of $\text{li}(x)$, or equivalently the unique positive real number $\mu$ such that

$$\text{li}(x) = \int_{\mu}^{x} \frac{dx}{\log x}, \quad \forall x > 1.$$

In this section, we prove Theorem 1.1, which we use in Section 4 to make precise the approximation

$$\frac{\pi(e^t n)}{e^t} \approx \sum_{r \leq k < x} \frac{1}{H_k - \gamma + t}, \quad \forall n \geq \lceil \mu e^{-t} \rceil,$$

from which we derive several Riemann hypothesis equivalents.

The following definitions summarize some of the notation that is used in this section, in descending order of relevance. Existence of all of the limits involved are proved within.

**Definition 3.1.** Let $t, r \in \mathbb{R}$ with $r > e^{-t}$. Let

$$\beta_x(t, r) = \frac{\text{li}(e^t x)}{e^t} - \sum_{r \leq k < x} \frac{1}{H_k - \gamma + t}$$

for all $x \geq r$, and let

$$\beta(t, r) = \lim_{n \to \infty} \beta_n(t, r) = \lim_{x \to \infty} \beta_x(t, r).$$

**Definition 3.2.** Let $N$ be a positive integer, and let $t \in \mathbb{R}$ with $t > -\log N$.

1. Let

$$\theta_n(t, N) = \lim_{n \to \infty} \left( \int_{N}^{n} \frac{dx}{t + \log x} - \sum_{k=N}^{n-1} \frac{1}{H_k - \gamma + t} \right)$$

for all $n \geq N$, and let

$$\theta(t, N) = \lim_{n \to \infty} \theta_n(t, N) = \lim_{n \to \infty} \left( \int_{N}^{n} \frac{dx}{t + \log x} - \sum_{k=N}^{n-1} \frac{1}{H_k - \gamma + t} \right).$$

2. Let

$$\eta_n(t, N) = \sum_{k=N}^{n} \left( \frac{1}{t + \log(k + 1/2)} - \frac{1}{H_n - \gamma + t} \right),$$

for all $n \geq N$, and let

$$\eta(t, N) = \lim_{n \to \infty} \eta_n(t, N) = \sum_{k=N}^{\infty} \left( \frac{1}{t + \log(k + 1/2)} - \frac{1}{H_n - \gamma + t} \right).$$

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3. Let

\[ \delta_n(t, N) = \int_N^n \frac{dx}{t + \log x} - \frac{1}{\log(k + 1/2) + t} \sum_{k=N}^{n-1} \frac{1}{\log(k + 1/2) + t} \]

for all \( n \geq N \), and let

\[ \delta(t, N) = \lim_{n \to \infty} \delta_n(t, N) = \lim_{n \to \infty} \left( \int_N^n \frac{dx}{t + \log x} - \sum_{k=N}^{n-1} \frac{1}{\log(k + 1/2) + t} \right). \]

Our first goal in this section is to prove the following proposition, from which Theorem 1.1 follows. It is an analogue of Proposition 2.2 for the sequence \( \theta_n(t, N) \) and its limit \( \theta(t, N) \).

**Proposition 3.3.** Let \( N \) be a positive integer, and let \( t > -\log N \).

1. The sequence

\[ \theta_n(t, N) = \int_N^n \frac{dx}{t + \log x} - \sum_{k=N}^{n-1} \frac{1}{H_k - \gamma + t} = \eta_{n-1}(t, N) + \delta_{n-1}(t, N), \quad n > N \]

is positive, increasing, strictly bounded from above by the increasing sequence

\[ \int_N^n \frac{dx}{24x^2(t + \log x)^2} + \left( \frac{1}{24x(t + \log x)^2} \right)^{N+1/2} \left|_{n-1/2} \right. + \frac{1}{24N^2(t + \log N)^2} + \frac{1}{12N^2(t + \log N)^3}, \quad n > N \]

and strictly bounded from below by the increasing sequence

\[ \int_{N+1}^{n+1} \frac{dx}{24x^2(t + \log x)^2} + \left( \frac{1}{24x(t + \log x)^2} \right)^{N+1} \left|_{n+1} \right. , \quad n > N. \]

2. The limit

\[ \theta(t, N) = \lim_{n \to \infty} \theta_n(t, N) = \lim_{n \to \infty} \left( \int_N^n \frac{dx}{t + \log x} - \sum_{k=N}^{n-1} \frac{1}{H_k - \gamma + t} \right) = \eta(t, N) + \delta(t, N) > 0 \]

exists.

3. For all \( n \geq N \), one has

\[ \int_{n+1}^{\infty} \frac{dx}{24x^2(t + \log x)^2} + \frac{1}{24(n + 1)(t + \log(n + 1))^2} \leq \theta(t, N) - \theta_n(t, N) = \theta(t, n) \]

\[ \leq \int_n^{\infty} \frac{dx}{24x^2(t + \log x)^2} + \frac{1}{24(n + 1/2)(t + \log(n + 1/2))^2} + \frac{1}{24n^2(t + \log n)^2} + \frac{1}{12n^2(t + \log n)^3}, \]

and, if \( t > -\log(n - 1/2) \) (or \( n \geq N + 1 \)), also

\[ \int_{n+1}^{\infty} \frac{dx}{24x^2(t + \log x)^2} + \frac{1}{24(n + 1)(t + \log(n + 1))^2} \leq \theta(t, N) - \theta_n(t, N) \]

\[ \leq \int_n^{\infty} \frac{dx}{24x^2(t + \log x)^2} + \frac{1}{24(n - 1/2)(t + \log(n - 1/2))^2}. \]
4. One has

\[ \theta(t, N) = \theta_n(t, N) + \frac{1 + o(1)}{12n(\log n)^2} \ (n \to \infty) \]

and

\[ \theta(t, N) = O \left( \frac{1}{N^2} \right) \ (t \to \infty), \]

where the \( O \) constant does not depend on \( N \).

We divide the proof of the proposition into two main steps. First, we prove the following analogue of Proposition 2.2 for the sequence \( \eta_n(t, N) \) and its limit \( \eta(t, N) \). The method of proof is the same as that for Proposition 2.2.

**Proposition 3.4.** Let \( N \) be a positive integer, and let \( t > -\log N \).

1. The sequence

\[ \eta_n(t, N) = \sum_{k=N}^n \frac{1}{t + \log(k + 1/2)} - \sum_{k=N}^n \frac{1}{H_k - \gamma + t}, \quad n \geq N \]

is increasing and satisfies

\[ 0 < \frac{1}{24} \int_{N+1}^{n+2} \frac{dx}{x^2(t + \log x)^2} < \eta_n(t, N) < \frac{1}{24} \int_N^{n+1} \frac{dx}{x^2(t + \log x)^2} \]

for all \( n \geq N \) (and equalities hold for \( n = N - 1 \)).

2. The limit

\[ \eta(t, N) = \lim_{n \to \infty} \eta_n(t, N) = \sum_{k=N}^\infty \left( \frac{1}{t + \log(k + 1/2)} - \frac{1}{H_k - \gamma + t} \right) > 0 \]

exists.

3. For all \( n \geq N \), one has

\[ \frac{1}{24} \int_{n+1}^{\infty} \frac{dx}{x^2(t + \log x)^2} \leq \eta(t, N) - \eta_{n-1}(t, N) = \eta(t, n) \leq \frac{1}{24} \int_n^{\infty} \frac{dx}{x^2(t + \log x)^2}. \]

**Proof.** By Eq. (2.2), for all \( k > e^{-t} \), hence for all \( k \geq N \), one has

\[ 0 < \frac{1}{t + \log(k + 1/2)} - \frac{1}{H_k - \gamma + t} = \frac{H_k - \gamma - \log(k + 1/2)}{(t + \log(k + 1/2))(H_k - \gamma + t)} \]

\[ < \frac{1}{24(k + 1/2)^2(t + \log(k + 1/2))^2}, \]

and therefore

\[ \sum_{k=N}^n \frac{1}{t + \log(k + 1/2)} - \sum_{k=N}^n \frac{1}{H_k - \gamma + t} < \frac{1}{24} \sum_{k=N}^n \frac{1}{(k + 1/2)^2(t + \log(k + 1/2))^2} \]

\[ < \frac{1}{24} \int_N^{n+1} \frac{1}{x^2(t + \log x)^2} \ dx. \]

Likewise, for all \( k > e^{-t} \), one has

\[ \frac{1}{t + \log(k + 1/2)} - \frac{1}{H_k - \gamma + t} = \frac{H_k - \gamma - \log(k + 1/2)}{(t + \log(k + 1/2))(H_k - \gamma + t)} \]

\[ > \frac{1}{24(k + 1)^2(t + \log(k + 1))^2}, \]

\[ > \frac{1}{24(k + 1/2)^2(t + \log(k + 1/2))^2}. \]
and therefore
\[
\sum_{k=N}^{n} \left( \frac{1}{t + \log(k + 1/2)} - \frac{1}{H_k - \gamma + t} \right) > \sum_{k=N}^{n} \frac{1}{24(k+1)^2(t + \log(k+1))^2}
\]
\[
= \sum_{k=N+1}^{n+1} \frac{1}{24k^2(t + \log k)^2}
\]
\[
> \int_{N+1}^{n+2} \frac{dx}{24x^2(t + \log x)^2}.
\]

This proves statement (1). Since the sequence \(\eta_n(t, N)\) for \(n \geq N\) is increasing and bounded above, it converges to a limit as \(n \to \infty\). This proves statement (2). Using the bounds in statement (1) and taking the limit as \(n \to \infty\), we see that
\[
\frac{1}{24} \int_{N+1}^{\infty} \frac{dx}{x^2(t + \log x)^2} \leq \eta(t, N) \leq \frac{1}{24} \int_{N}^{\infty} \frac{dx}{x^2(t + \log x)^2}.
\]

Finally, statement (3) follows from the inequalities above and the fact that
\[
\eta(t, N) - \eta_{n-1}(t, N) = \eta(t, n)
\]
for all \(n \geq N\). \(\Box\)

Note that statement (3) gives sharp upper and lower bounds
\[
\eta_{n-1}(t, N) < \eta_{n-1}(t, N) + \frac{1}{24} \int_{n+1}^{\infty} \frac{dx}{x^2(t + \log x)^2} \leq \eta(t, N) \leq \eta_{n-1}(t, N) + \frac{1}{24} \int_{n}^{\infty} \frac{dx}{x^2(t + \log x)^2}.
\]
of \(\eta(t, N)\) in that given inequalities allow one to approximate \(\eta(t, N)\) to an arbitrary degree of accuracy by taking \(n \geq N\) sufficiently large. Moreover, the integrals above can be expressed in terms of the logarithmic integral function and also bounded from above and below, as follows.

Lemma 3.5. Let \(t, r \in \mathbb{R}\) with \(t > -\log r\). One has
\[
\int_{r}^{\infty} \frac{dx}{x^2(\log x + t)^2} = e^t \text{li} \left( \frac{1}{re^t} \right) + \frac{1}{r(t + \log r)} > 0.
\]
Moreover, for every even positive integer \(n\), one has
\[
\sum_{k=2}^{n+1} \frac{(-1)^k(k-1)!}{r(t + \log r)^k} < \int_{r}^{\infty} \frac{dx}{x^2(\log x + t)^2} < \sum_{k=2}^{n} \frac{(-1)^k(k-1)!}{r(t + \log r)^k}.
\]
In particular, one has
\[
\frac{1}{r(t + \log r)^2} - \frac{2}{r(t + \log r)^3} < \int_{r}^{\infty} \frac{dx}{x^2(\log x + t)^2} < \frac{1}{r(t + \log r)^2}
\]
and
\[
\int_{r}^{\infty} \frac{dx}{x^2(\log x + t)^2} = \frac{1}{r(t + \log r)^2} + \frac{-2 + o(1)}{r(t + \log r)^3} \quad (r \to \infty).
\]

Proof. The exact expression for the integral is easily verified by integration by parts. Since it is known that
\[
-\sum_{k=0}^{n} \frac{(-1)^k k! x}{(\log x)^{k+1}} < \text{li}(x) < -\sum_{k=0}^{n-1} \frac{(-1)^k k! x}{(\log x)^{k+1}}
\]
for all $0 < x < 1$ and all even positive integers $n$, letting $x = \frac{1}{e^r}$, we see that

$$- \sum_{k=0}^{n} \frac{(-1)^k k!}{r(t + \log r)^{k+1}} < e^t \left( \frac{1}{e^r} \right) < - \sum_{k=0}^{n-1} \frac{(-1)^k k!}{r(t + \log r)^{k+1}},$$

whence

$$- \sum_{k=1}^{n} \frac{(-1)^k k!}{r(t + \log r)^{k+1}} < e^t \left( \frac{1}{e^r} \right) + \frac{1}{r(t + \log r)} < - \sum_{k=1}^{n-1} \frac{(-1)^k k!}{r(t + \log r)^{k+1}}.$$

The proposition follows.

Note that the lower bounds in the lemma above are positive only for $t \gg - \log r$.

As a corollary of Proposition 3.4 and the lemma above, we obtain the following.

**Corollary 3.6.** Let $N$ be a positive integer, and let $t > - \log N$. For all $n \geq N$, one has

$$\eta_{n-1}(t, N) + \frac{1}{24(n+1)(t + \log(n+1))^2} - \frac{1}{12(n+1)(\log(n+1) + t)^3} < \eta(t, N) < \eta_{n-1}(t, N) + \frac{1}{24n(t + \log n)^2}.$$

Consequently, one also has

$$\eta(t, N) = \eta_{n-1}(t, N) + \frac{1 + o(1)}{24n(\log n)^2} (n \to \infty)$$

and

$$\eta(t, N) = O \left( \frac{1}{N^2} \right) (t \to \infty),$$

where the $O$ constant does not depend on $N$.

The second and final step in the proof of Proposition 3.3 is to prove the following analogue of Propositions 2.2 and 3.4 for the sequence $\delta_n(t, N)$ and its limit $\delta(t, N)$.

**Proposition 3.7.** Let $N$ be a positive integer, and let $t > - \log N$.

1. The sequence

$$\delta_n(t, N) = \int_{N}^{n+1} \frac{dx}{t + \log x} - \sum_{k=N}^{n} \frac{1}{t + \log(k + 1/2)}, \quad n \geq N$$

is increasing and for all $n \geq N$ satisfies

$$\left( \frac{1}{24x(t + \log x)^2} \right)|_{n+2}^{N+1} < \delta_n(t, N) < \left( \frac{1}{24x(t + \log x)^2} \right)|_{n+1/2}^{N+1/2} + \frac{1}{24N^2(t + \log N)^2} + \frac{1}{12N^2(t + \log N)^3}$$

and, if $t > - \log(N - 1/2)$, also

$$\left( \frac{1}{24x(t + \log x)^2} \right)|_{n+2}^{N+1} < \delta_n(t, N) < \left( \frac{1}{24x(t + \log x)^2} \right)|_{n+1/2}^{N-1/2}.$$

2. The limit

$$\delta(t, N) = \lim_{n \to \infty} \delta_n(t, N) = \lim_{n \to \infty} \left( \int_{N}^{n+1} \frac{dx}{t + \log x} - \sum_{k=N}^{n} \frac{1}{t + \log(k + 1/2)} \right) > 0$$

exists.
3. For all \( n \geq N \), one has
\[
\frac{1}{24(n+1)(t + \log(n+1))^2} \leq \delta(t, N) - \delta_{n-1}(t, N) = \delta(t, n)
\]
\[
\leq \frac{1}{24(n+1/2)(t + \log(n+1/2))^2} + \frac{1}{24n^2(t + \log n)^2} + \frac{1}{12n^2(t + \log n)^3},
\]
and, if \( t \geq -\log(n - 1/2) \) (or \( n > N \)), also
\[
\frac{1}{24(n+1)(t + \log(n+1))^2} \leq \delta(t, N) - \delta_{n-1}(t, N) \leq \frac{1}{24(n - 1/2)(t + \log(n - 1/2))^2}.
\]
Consequently, one also has
\[
\delta(t, N) = \delta_{n-1}(t, N) + \frac{1 + o(1)}{24n(\log n)^2} \quad (n \to \infty)
\]
and
\[
\delta(t, N) = O\left(\frac{1}{N^2}\right) \quad (t \to \infty),
\]
where the \( O \) constant does not depend on \( N \).

Proof. Let \( f(x) = \frac{1}{x + \log x} \), which is positive, decreasing, and concave up on \([N, \infty)\). Then, also, \( f'(x) = -\frac{1}{x(x + \log x)^2} \) is negative, increasing, and concave down on \([N, \infty)\), while \( f''(x) = \frac{1}{x(x + \log x)^3} + \frac{2}{x^2(x + \log x)^2} \) is positive, decreasing, and concave up on \([N, \infty)\). In particular, by the well-known expression for the error from the midpoint rule, one has
\[
0 < \frac{1}{24} f''(k + 1) < \int_k^{k+1} \frac{dx}{t + \log x} - \frac{1}{\log(k + 1/2) + t} < \frac{1}{24} f''(k)
\]
and therefore
\[
\frac{1}{24} \sum_{k=N}^{n} f''(k + 1) < \int_N^{n+1} \frac{dx}{t + \log x} - \sum_{k=N}^{n} \frac{1}{\log(k + 1/2) + t} < \frac{1}{24} \sum_{k=N}^{n} f''(k).
\]
Moreover, one has
\[
\sum_{k=N}^{n} f''(k) = f''(N) + \sum_{k=N+1}^{n} f''(k)
\]
\[
< f''(N) + \int_{N+1/2}^{n+1/2} f''(x) \, dx
\]
\[
= f''(N) + f'(n + 1/2) - f'(N + 1/2),
\]
while
\[
\sum_{k=N}^{n} f''(k + 1) = \sum_{k=N+1}^{n+1} f''(k)
\]
\[
> \int_{N+1}^{n+2} f''(x) \, dx
\]
\[
= f'(n + 2) - f'(N + 1).
\]
Furthermore, if \( t > -\log(N - 1/2) \), then also
\[
\sum_{k=N}^{n} f''(k) < \int_{N-1/2}^{n-1/2} f''(x) \, dx
\]
\[
= f'(n + 1/2) - f'(N - 1/2).
\]
This proves statement (1), from which (2) and (3) follow.
Proposition 3.3 now follows immediately from Propositions 3.4 and 3.7. As a corollary of the proposition, we obtain the following.

**Corollary 3.8.** Let $N$ be a positive integer, and let $t > -\log N$. For all $n \geq N$, one has

$$\frac{1}{24n(t + \log n)^2} + \frac{1}{24(n + 1/2)(t + \log(n + 1/2))^2} + \frac{1}{24n^2(t + \log n)^2} + \frac{1}{12n^2(t + \log n)^3},$$

and, if $t > -\log(n - 1/2)$ (or $n \geq N + 1$), also

$$\frac{1}{24n(t + \log n)^2} + \frac{1}{24(n - 1/2)(t + \log(n - 1/2))^2}.$$

Because the sums $\sum_{n=N}^{\infty} \frac{1}{n(t + \log n)^2}$, $\sum_{n=N}^{\infty} \frac{1}{n^2(t + \log n)^2}$, and $\sum_{n=N}^{\infty} \frac{1}{n^3(t + \log n)^2}$ converge for all $t \in (-\log N, \infty)$ and are bounded on compact subsets of $(-\log N, \infty)$, the corollary implies that the convergence of $\theta_n(t, N)$ to $\theta(t, N)$ is uniform on compact subsets of $(-\log N, \infty)$. Therefore, the function $\theta(t, N)$ is continuous on $(-\log N, \infty)$. In fact, the following argument shows that $\theta(t, N)$ is differentiable, with negative derivative, on $(-\log N, \infty)$. First, note that

$$\theta_n(t, N) = \int_{N}^{n} \frac{dx}{t + \log x} - \sum_{k=N}^{n-1} \frac{1}{H_k - \gamma + t} = \frac{\text{li}(e^t n) - \text{li}(e^t N)}{e^t} - \sum_{k=N}^{n-1} \frac{1}{H_k - \gamma + t},$$

for all $n > N$. It follows that

$$\theta'_n(t, N) = -\left(\frac{\text{li}(e^t n)}{e^t} - \frac{x}{t + \log x}\right)\bigg|_{x=N}^{x=n} + \sum_{k=N}^{n-1} \frac{1}{(H_k - \gamma + t)^2} = -\int_{N}^{n} \frac{dx}{(t + \log x)^2} + \sum_{k=N}^{n-1} \frac{1}{(H_k - \gamma + t)^2}.$$

A straightforward repetition of our argument for $\theta_n(t, N)$ shows that, just as one has $\theta_n(t, N) > 0$ because

$$\int_{N}^{n} \frac{dx}{(t + \log x)^2} > \sum_{k=N}^{n-1} \frac{1}{(\log(k + 1/2) + t)^2} > \sum_{k=N}^{n-1} \frac{1}{H_k - \gamma + t},$$

one has $\theta'_n(t, N) < 0$ because

$$\int_{N}^{n} \frac{dx}{(t + \log x)^2} > \sum_{k=N}^{n-1} \frac{1}{(\log(k + 1/2) + t)^2} > \sum_{k=N}^{n-1} \frac{1}{(H_k - \gamma + t)^2}.$$}

One can then bound the error terms as with $\theta(t, N)$ and show that $\theta'_n(t, N)$ converges uniformly on compact subsets of $(-\log N, \infty)$ to a limit

$$f(t, N) = \lim_{n \to \infty} \theta'_n(t, N)$$

that is negative for all $t$. It follows that $\theta(t, N)$ is differentiable on $(-\log N, \infty)$ with derivative $f(t, N)$.

A real function $f(x)$ on an interval $I \subseteq \mathbb{R}$ is said to be **strictly totally monotone on $I$** if $f(x)$ is continuous on $I$, infinitely differentiable on the interior of $I$, and satisfies

$$(-1)^k \frac{d^k}{dx^k} f(x) > 0$$

for all $x$ in the interior of $I$ and for all nonnegative integers $k$. (In particular, such a function is positive, strictly decreasing, and concave up on $I$.) For example, the function $\frac{1}{x^\alpha}$ is strictly totally monotone on $(0, \infty)$ for any $\alpha > 0$, as is the function $-\log x$. A generalization of the argument above shows the following.
Proposition 3.9. Let $N$ be a positive integer. The function $\theta(t, N)$ strictly totally monotone on $(-\log N, \infty)$ with

$$(-1)^k \frac{d^k}{dt^k} \theta(t, N) = (-1)^k \lim_{n \to \infty} \frac{d^k}{dt^k} \theta_n(t, N) = k! \lim_{n \to \infty} \left( \int_t^n \frac{dx}{(t + \log x)^{k+1}} - \sum_{k=N}^{n-1} \frac{1}{(H_k - \gamma + t)^{k+1}} \right) > 0,$$

for all nonnegative integers $k$ and all $t \in (-\log N, \infty)$.

Similar statements hold for the functions $\eta(t, N)$ and $\delta(t, N)$.

Now, let $t, r \in \mathbb{R}$ with $r > e^t$. Recall from Definition 3.1 the definitions of $\beta_x(t, r)$ and $\beta(t, r)$. Since

$$\beta(t, r) = \beta(t, [r]) = \frac{\ln(e^t [r])}{e^t} + \theta(t, [r])$$

and

$$\beta(t, r) - \beta_n(t, r) = \beta(t, [r]) - \beta_n(t, [r]) = \theta(t, [r]) - \theta_n(t, [r]) = \theta(t, n)$$

for all $n \geq [r]$, by Proposition 3.3 one has the following.

Corollary 3.10. Let $t, r \in \mathbb{R}$ with $r > e^t$. For all $n \geq [r]$, one has

$$\int_{n+1}^\infty \frac{dx}{24x^2(t + \log x)^2} + \frac{1}{24(n+1)(t + \log(n+1))^2} \leq \beta(t, r) - \beta_n(t, r)$$

$$\leq \int_{n}^\infty \frac{dx}{24x^2(t + \log x)^2} + \frac{1}{24(n+1/2)(t + \log(n+1/2))^2} + \frac{1}{24n^2(t + \log n)^2} + \frac{1}{12n^2(t + \log n)^3},$$

and, if $t > -\log(n - 1/2)$ (or $n \geq [r] + 1$), also

$$\int_{n+1}^\infty \frac{dx}{24x^2(t + \log x)^2} + \frac{1}{24(n+1)(t + \log(n+1))^2} \leq \beta(t, r) - \beta_n(t, r)$$

$$\leq \frac{1}{24} \int_{n}^\infty \frac{dx}{x^2(t + \log x)^2} + \frac{1}{24(n-1/2)(t + \log(n-1/2))^2}.$$}

Moreover, one has

$$\beta(t, r) = \beta_n(t, r) + \frac{1 + o(1)}{12n(\log n)^2} \quad (n \to \infty)$$

and

$$\beta(t, r) = \frac{\ln(e^t [r])}{e^t} + \theta(t, [r]) = \frac{\ln(e^t [r])}{e^t} + O \left( \frac{1}{[r]t^2} \right) = \frac{[r]}{t} + O \left( \frac{1}{[r]t^2} \right) \quad (t \to \infty),$$

and therefore also

$$\beta(t, r) \sim \frac{\ln(e^t [r])}{e^t} \sim \frac{[r]}{t} \quad (t \to \infty).$$

Note that the corollary above implies Theorem 1.1 of the introduction.

Again, let $t, r \in \mathbb{R}$ with $r > e^t$. Then

$$\frac{\ln(e^t n)}{e^t} = \sum_{r \leq k < n} \frac{1}{H_k - \gamma + t} + \beta(t, r) - (\beta(t, r) - \beta_n(t, r)),$$

so that, by Corollary 3.10, the quantity $\beta(t, r) - \beta_n(t, r)$ is small for $n \geq r$, one has

$$\frac{\ln(e^t n)}{e^t} \approx \sum_{r \leq k < n} \frac{1}{H_k - \gamma + t} + \beta(t, r), \quad \forall n \geq r$$

Note: The last part of the corollary and the final equation are not fully clear due to overlapping and missing symbols.
in a sense made precise by the corollary. Thus it behooves us to choose \( r = r_t \) in terms of \( t \) so that the absolute value of the quantity

\[
\beta(t, r) = \beta(t, [r]) = \frac{\text{li}(e^t[r])}{e^t} + \theta(t, [r])
\]

minimized. Since \( \theta(t, [r]) \) is always nonnegative, by far the dominant and more unpredictable term in the expression above is

\[
\frac{\text{li}(e^t[r])}{e^t} = \beta_{[r]}(t, [r]).
\]

At least the nonnegativity of \( \beta(t, r) \) can be guaranteed as long \( \text{li}(e^t[r]) \) is nonnegative. So it would be prudent to minimize that term subject to the constraint that \( \text{li}(e^t[r]) \) be nonnegative.

This can be achieved by employing the Ramanujan–Soldner constant \( \mu \). Clearly \( \text{li}(e^t[r]) \) is nonegative if and only if \( e^t[r] \geq \mu \), and then the term is minimized for any \( r \) with \( [r] = [\mu e^{-t}] \), e.g., for

\[
r_t = \mu e^{-t},
\]

where \( r_t \) is uniquely determined by the equation

\[
\text{li}(r_t e^t) = 0,
\]

Let

\[
R_t = [r_t] = [\mu e^{-t}].
\]

Since \( \text{li}(x) \) is increasing for \( x > 1 \), one has

\[
0 = \frac{\text{li}(e^t r_t)}{e^t} \leq \frac{\text{li}(e^t R_t)}{e^t} < \frac{\text{li}(e^t (r_t + 1))}{e^t} = \frac{\text{li}(\mu + e^t)}{e^t}.
\]

Moreover, the function \( \frac{\text{li}(\mu + e^t)}{e^t} \) is a strictly decreasing function of \( t \) with

\[
\lim_{t \to \infty} \frac{\text{li}(\mu + e^t)}{e^t} = 0
\]

and

\[
\lim_{t \to -\infty} \frac{\text{li}(\mu + e^t)}{e^t} = \lim_{s \to 0} \frac{\text{li}(\mu + s) - \text{li}(\mu)}{s} = \text{li}^\prime(\mu) = \frac{1}{\log \mu} = 2.6845103508207076525 \ldots.
\]

Consequently, one has

\[
0 \leq \frac{\text{li}(e^t R_t)}{e^t} < \frac{\text{li}(\mu + e^t)}{e^t} < \frac{1}{\log \mu}
\]

for all \( t \in \mathbb{R} \). Thus, the function \( \beta(t, r_t) \) is positive and bounded above, for all \( t \in \mathbb{R} \). More precise bounds on the main term \( \frac{\text{li}(e^t R_t)}{e^t} \) of \( \beta(t, r_t) \) can be obtained from the following lemma, which follows readily from the fact that the integrand \( \frac{1}{\log u} \text{li}(x) = \int_0^x \frac{du}{\log u} \) is positive, decreasing, and concave up on \((1, \infty)\).

**Lemma 3.11.** One has the following.

1. For all \( y > x > 1 \), one has

\[
0 < \frac{y - x}{\log \left( \frac{x + y}{2} \right)} < \text{li}(y) - \text{li}(x) < \frac{y - x}{\log x}.
\]

2. For all \( y > x > 0 \) and all \( t > -\log x \), one has

\[
0 < \frac{y - x}{t + \log \left( \frac{x + y}{2} \right)} < \frac{\text{li}(e^t y)}{e^t} - \frac{\text{li}(e^t x)}{e^t} < \frac{y - x}{t + \log x}.
\]
Corollary 3.12. Let $t \in \mathbb{R}$, let $r_t = \mu e^{-t}$, and let $R_t = \lceil r_t \rceil$. One has

$$0 \leq \frac{\log(e^t R_t)}{e^t} = \beta_R(t, R_t) < \frac{\log(\mu + e^t)}{e^t} < \frac{1}{\log \mu}$$

and

$$0 \leq \frac{R_t - r_t}{t + \log R_t} \leq \frac{R_t - r_t}{t + \log \left( \frac{R_t + r_t}{2} \right)} = \frac{\log(e^t R_t)}{e^t} \leq \frac{R_t - r_t}{\log \mu} < \frac{1}{\log \mu}.$$

The discussion above motivates the following definition.

Definition 3.13. Let $t \in \mathbb{R}$.

1. Let $r_t = \mu e^{-t}$.

   Equivalently, $r_t$ is the unique $r \in \mathbb{R}$ such that $\log(re^t) = 0$.

2. Let $R_t = \lceil r_t \rceil = \lceil \mu e^{-t} \rceil$.

   Equivalently, $R_t$ is the unique positive integer $N$ such that $t \in [\log \mu - \log N, \log \mu - \log(N - 1))$ (where we set $-\log 0 = \infty$, so that $R_t = 1$ if and only if $t \in [\log \mu, \infty)$).

3. Let

   $$\beta_x(t) = \beta(t, r_t) = \frac{\log(e^t x)}{e^t} - \sum_{R_t \leq k < x} \frac{1}{H_k - \gamma + t}$$

   for all $x \geq r_t$, and let

   $$\beta(t) = \lim_{n \to \infty} \beta_n(t) = \lim_{x \to \infty} \beta_x(t) = \beta(t, r_t).$$

A graph of the function $R_t$ is provided in Figure 4. Graphs of the function $\beta(t, r_t) = \frac{\log(e^t R_t)}{e^t} \approx \beta(t)$, alongside graphs of the upper and lower bounds given in Corollary 3.12 are provided in Figures 5 and 6.

Applying Corollary 3.10 to $r = r_t = \mu e^{-t}$, we obtain the following.

Corollary 3.14. Let $t \in \mathbb{R}$. For all $n \geq R_t = \lceil \mu e^{-t} \rceil$, one has

$$\int_{n+1}^{\infty} \frac{dx}{24x^2(t + \log x)^2} + \frac{1}{24(n+1)(t + \log(n+1))^2} \leq \beta(t) - \beta_n(t)$$

$$\leq \int_{n}^{\infty} \frac{dx}{24x^2(t + \log x)^2} + \frac{1}{24(n + \frac{1}{2})(t + \log(n + \frac{1}{2}))^2} + \frac{1}{24n^2(t + \log n)^2} + \frac{1}{12n^2(t + \log n)^3},$$

Figure 4: Graph of $R_t$ on $[-3, 3]$
Figure 5: Approximate graph of $\frac{\ln(e^t R_t)}{e^t}$ and its upper bound $\frac{\ln(\mu+e^t)}{e^t} < \frac{1}{\log \mu}$ on $[-2, 5]$

Figure 6: Approximate graph of $\frac{\ln(e^t R_t)}{e^t}$, its upper bound $R_t - \frac{r_t}{\log \mu}$, and its lower bound $R_t - \frac{r_t}{t + \log R_t}$, on $[-2, 5]$

and, if $t > -\log(n - 1/2)$ (or $n \geq \lceil r \rceil + 1$), also

$$\int_{n+1}^{\infty} \frac{dx}{x^2(t + \log x)^2} + \frac{1}{24(n + 1)(t + \log(n + 1))^2} \leq \beta(t) - \beta_n(t)$$

$$\leq \frac{1}{24} \int_{n}^{\infty} \frac{dx}{x^2(t + \log x)^2} + \frac{1}{24(n - 1/2)(t + \log(n - 1/2))^2}. $$

Moreover, one has

$$\beta(t) = \beta_n(t) + \frac{1 + o(1)}{12n(\log n)^2} \quad (n \to \infty).$$

In particular, letting $n = R_t$ (which is the smallest value of $n$ allowed in the corollary), one has

$$\int_{R_t+1}^{\infty} \frac{dx}{24x^2(t + \log x)^2} + \frac{1}{24R_t(t + \log(R_t + 1))^2} \leq \beta(t) - \beta_{R_t}(t) = \beta(t) - \frac{\ln(e^t R_t)}{e^t}$$

$$\leq \int_{R_t}^{\infty} \frac{dx}{24x^2(t + \log x)^2} + \frac{1}{24R_t(t + \log(R_t + 1))^2} + \frac{1}{24R_t^2(t + \log R_t)^2} + \frac{1}{12R_t^2(t + \log R_t)^3}. $$

Table 1 shows the upper and lower bounds for $\beta(t)$, for all integers $t \in [-15, 15]$, provided by the inequalities above, and compares them with approximate values of $\beta(t)$ that we computed using WolframAlpha by taking as many terms of the limit formula for $\beta(t)$ as the software would allow. In these examples, for $-15 \leq t \leq -7$, the bounds provided above are better than the direct estimates of $\beta(t)$ that we could compute using WolframAlpha. The estimates in Table 1 are the coarsest bounds provided by Corollary 3.14 and can be improved by increasing $n$. In Table 2, the bounds in Corollary 3.14 are computed with $n = 50$ for
Table 1: Upper and lower bounds of $\beta(t)$ computed with $n = R_t = \lceil e^{-t} \mu \rceil$

| $t$  | $\beta(t) <$ | $\beta(t) \approx$ | $\beta(t) >$ | $\frac{\text{li}(e^t R_t)}{e^t} \approx R_t$ |
|------|---------------|---------------------|---------------|---------------------------------------------|
| 15   | 0.07236490    | 0.07220             | 0.07203360    | 0.07187354 1                               |
| 14   | 0.07805640    | 0.07786             | 0.07767424    | 0.07749225 1                               |
| 13   | 0.08473348    | 0.08450             | 0.08428780    | 0.08407993 1                               |
| 12   | 0.09268293    | 0.09241             | 0.09215649    | 0.09191455 1                               |
| 11   | 0.1023178     | 0.10199             | 0.1016865     | 0.1014028 1                                |
| 10   | 0.1142551     | 0.11385             | 0.1134844     | 0.1131470 1                                |
| 9    | 0.1295422     | 0.12895             | 0.1284922     | 0.1280844 1                                |
| 8    | 0.1494680     | 0.14880             | 0.1482342     | 0.1477310 1                                |
| 7    | 0.1769053     | 0.17601             | 0.1752663     | 0.1746297 1                                |
| 6    | 0.2162592     | 0.21500             | 0.2139786     | 0.2131473 1                                |
| 5    | 0.2752827     | 0.27337             | 0.2718986     | 0.2707663 1                                |
| 4    | 0.3667085     | 0.36349             | 0.3611866     | 0.3595520 1                                |
| 3    | 0.5076564     | 0.50123             | 0.4971471     | 0.4945764 1                                |
| 2    | 0.7018947     | 0.68422             | 0.6751323     | 0.6704827 1                                |
| 1    | 0.8530561     | 0.74190             | 0.7082072     | 0.691749 1                                 |
| 0    | 1.1635319     | 1.09564             | 1.0615462     | 1.0451638 2                                |
| −1   | 0.3044511     | 0.22299             | 0.1512070     | 0.143674 4                                 |
| −2   | 0.7531859     | 0.74512             | 0.7346875     | 0.7160222 11                               |
| −3   | 2.054409      | 2.0450              | 2.027692      | 2.1938815 30                               |
| −4   | 2.0135694     | 2.01342             | 2.013210      | 2.0090540 80                               |
| −5   | 1.6003036     | 1.60028             | 1.6002378     | 1.5986089 216                               |
| −6   | 1.2766878     | 1.27667             | 1.2766786     | 1.2760617 586                               |
| −7   | 1.0210154     | 1.02099             | 1.0210141     | 1.0207849 1592                              |
| −8   | 1.4207283     | 1.42068             | 1.4207280     | 1.4206435 4327                              |
| −9   | 1.1631179     | 1.16309             | 1.1631178     | 1.1630867 11761                             |
| −10  | 1.2488004     | 1.24879             | 1.2488003     | 1.2487890 31969                             |
| −11  | 1.3764747     | 1.37647             | 1.3764746     | 1.3764704 86900                             |
| −12  | 1.8869487     | 1.88694             | 1.8869486     | 1.88694713 236218                           |
| −13  | 2.1844846     | 2.18448             | 2.1844845     | 2.18448397 642106                           |
| −14  | 0.3764374     | 0.37643             | 0.37643373    | 0.3764352 1745423                           |
| −15  | 2.03296502    | 2.03296             | 2.03296527    | 2.032964950 474552                           |

Ten special values of $\beta(t)$, including those for $t = 0, 1, -1, \gamma, \gamma + 1, \log 2$, and $-\log 2$, which are of particular interest in the next section.

Tables 1 and 2 also provide approximate values for the coarsest of all of our lower bounds of $\beta(t)$, namely,

$$\beta_{R_t}(t) = \frac{\text{li}(e^t R_t)}{e^t}. $$

In particular, one can see that

$$\beta(t) \approx \frac{\text{li}(e^t R_t)}{e^t}, \quad |t| \gg 0. $$

This is made precise by the following proposition.

**Proposition 3.15.** Let $t \in \mathbb{R}$.

1. One has

$$R_t \sim \mu e^{-t} \quad (t \to -\infty),$$

$$t + \log R_t \geq \log \mu,$$
Table 2: Upper and lower bounds of $\beta(t)$ computed with $n = 50$

| $t$ | $\beta(t) <$ | $\beta(t) >$ | $\frac{\text{li}(e^t R_t)}{e^t}$ | $R_t$ |
|-----|-------------|--------------|-----------------|-----|
| 2   | 0.6842208546 | 0.68421944246 | $\text{li}(e^t) / e^t \approx 0.6704827098$ | 1   |
| $\gamma + 1 \approx 1.5772156649$ | 0.7509547014 | 0.7509261228 | $\text{li}(e^t + 1) / e^{t+1} \approx 0.7301968559$ | 1   |
| $\log \alpha \approx 1.3471552511$ | 0.769524794 | 0.7695229079 | $\text{li}(\alpha) / \alpha \approx 0.742304916$ | 1   |
| 1   | 0.7418976158  | 0.7418955006 | $\text{li}(e) / e \approx 0.697148832$ | 1   |
| $\log 2 \approx 0.6931471806$ | 0.6026096358 | 0.6026071971 | $\text{li}(2) / 2 \approx 0.522581901$ | 1   |
| $\gamma \approx 0.5772156649$ | 0.4986013304 | 0.4985987518 | 1   |
| $\log \mu \approx 0.3725074108$ | 0.195255336 | 0.1952526746 | 0   |
| 0   | 1.0956456993  | 1.0956421994 | $\text{li}(2) \approx 1.045163781$ | 2   |
| $-\log 2 \approx -0.6931471806$ | 0.341732460  | 0.3417318184 | 2   |
| $-1$ | 0.2229882714  | 0.2229814526 | $e\text{li}(4/e) \approx 0.1443674107$ | 4   |

and

$$\limsup_{t \to -\infty} \frac{\text{li}(e^t R_t)}{e^t} = \text{li}'(\mu) = \frac{1}{\log \mu} = \lim_{t \to -\infty} \frac{1}{t + \log R_t} = 2.68451035082070765250 \ldots$$

2. One has

$$\frac{1}{12(R_t + 1)(t + \log(R_t + 1))^2} - \frac{1}{12(R_t + 1)^2(t + \log(R_t + 1))^2} = \beta(t) - \frac{\text{li}(e^t R_t)}{e^t}$$

and therefore

$$\beta(t) = \frac{\text{li}(e^t R_t)}{e^t} + \frac{1 + o(1)}{12(\log \mu)^2 R_t} \quad (t \to -\infty)$$

and therefore

$$\beta(t) = \frac{\text{li}(e^t R_t)}{e^t} + \frac{(1 + o(1))e^t}{12\mu(\log \mu)^2} = \frac{\text{li}(e^t R_t)}{e^t} - \left( \frac{\text{li}''(\mu)}{12} + o(1) \right) e^t \quad (t \to -\infty)$$

Consequently, one has

$$\beta(t) - \frac{\text{li}(e^t R_t)}{e^t} \sim \frac{e^t}{12\mu(\log \mu)^2} \quad (t \to -\infty).$$

3. One has

$$\beta(t) = \frac{\text{li}(e^t R_t)}{e^t} + \frac{1 + o(1)}{12(\log \mu)^2 R_t} \quad (t \to -\infty)$$

4. On $[\log \mu, \infty)$, the upper bound

$$\frac{\text{li}(e^t)}{e^t} + \frac{1}{24} e^t \text{li} \left( \frac{1}{e^t} \right) + \frac{1}{24t} + \frac{1}{24t^2} + \frac{1}{36(t + \log(3/2))^2} + \frac{1}{12t^3} > \beta(t)$$

of $\beta(t)$ is decreasing, and therefore

$$\beta(t) < \frac{1}{24} \mu \text{li} \left( \frac{1}{\mu} \right) + \frac{1}{24 \log \mu} + \frac{1}{24(\log \mu)^2} + \frac{1}{36(\log(3/2))^2} + \frac{1}{12(\log \mu)^3} = 2.0248039777 \ldots$$
5. On \([\log \mu, \infty)\), one has

\[
\frac{1}{32(t + \log 2)^2} - \frac{1}{24(t + \log 2)^3} < \beta(t) - \frac{\text{li}(e^t)}{e^t} < \frac{1}{9t^2} + \frac{1}{12t^3}
\]

and therefore

\[
\beta(t) = \frac{\text{li}(e^t)}{e^t} + O \left( \frac{1}{t^2} \right) = \frac{1}{t} + O \left( \frac{1}{t^2} \right) \quad (t \to \infty)
\]

and

\[
\beta(t) \sim \frac{\text{li}(e^t)}{e^t} \sim \frac{1}{t} \quad (t \to \infty).
\]

6. One has

\[
\beta(t) < \limsup_{u \to -\infty} \beta(u) = \text{li}'(\mu) = \frac{1}{\log \mu}.
\]

Proof. Statement (1) is an easy consequence of Corollary 3.12, and statement (2) follows from Corollary 3.14 and Lemma 3.5. Statement (3) follows immediately from (1) and (2). Statement (4) follows from Corollary 3.14 and some straightforward calculus, namely, that the given upper bound has negative derivative on \([\log \mu, \infty)\). Statement (5) follows from (4) and Lemma 3.5.

Finally, we prove statement (6). By statement (4), we know that

\[
\beta(t) < 2.0248039777 \ldots < \frac{1}{\log \mu}
\]
on \([\log \mu, \infty)\). By Corollary 3.12 one has

\[
\frac{\text{li}(e^t) R_t}{e^t} < \frac{\text{li}(\mu + e^t)}{e^t}.
\]

Therefore, by statement (2), one has

\[
\beta(t) < \frac{\text{li}(\mu + e^t)}{e^t} + \frac{e^t}{12\mu(\log \mu)^2} + \frac{e^{2t}}{12\mu^2(\log \mu)^2} \left( \frac{1}{2} + \frac{1}{\log \mu} \right).
\]

Moreover, the upper bound of \(\beta(t)\) above is decreasing on \((\infty, -1]\) with limit \(\frac{1}{\log \mu}\) as \(t \to -\infty\), and it is less than \(\frac{1}{\log \mu}\) also on \([-1, \log(\mu/2)]\). Therefore one has \(\beta(t) < \frac{1}{\log \mu}\) on \((-\infty, \log(\mu/2)]\). Moreover, on \([\log(\mu/2), \log \mu)\), one has \(R_t = 2\) and therefore, by Corollary 3.14

\[
\beta(t) < \frac{\text{li}(2e^t)}{e^t} + \frac{1}{24(t + \log 2)^2} + \frac{1}{60(t + \log(5/2))^2} + \frac{1}{96(t + \log 2)^2} + \frac{1}{48(t + \log 2)^3}.
\]

Finally, the upper bound of \(\beta(t)\) above is maximized on \([\log(\mu/2), \log \mu)\] at the endpoint \(\log \mu\), and therefore

\[
\beta(t) < \frac{\text{li}(2\mu)}{\mu} + \frac{1}{24(2\log(2\mu))^2} + \frac{1}{60(2\log(5/2))^2} + \frac{1}{96(2\log(2))^2} + \frac{1}{48(2\log(2))^3} = 1.5019080010 \ldots < \frac{1}{\log \mu}
\]
on \([\log(\mu/2), \log \mu)\). Thus, we have shown that \(\beta(t) < \frac{1}{\log \mu}\) for all \(t \in \mathbb{R}\).

The analysis above explains why the estimate \(\beta(t) \approx \frac{\text{li}(e^t) R_t}{e^t}\) is much better for negative \(t\) than for positive \(t\). To obtain better estimates of \(\beta(t)\) for any \(t\) (positive or negative), one can take larger values of \(n\) in Corollary 3.14. For large negative values of \(t\) this is achieved automatically, since \(R_t \sim \mu e^{-t} \quad (t \to \infty)\) grows exponentially as \(t \to -\infty\). Therefore, for large negative \(t\), increasing \(n\) slightly won’t improve the approximation much, but for positive \(t\), increasing \(n\) slightly can make a big difference.
4 Approximating $\pi(x)$ with harmonic numbers

Throughout this section, 
$$\delta = \sup \{ \Re(s) : s \in \mathbb{C}, \zeta(s) = 0 \}$$
denotes the supremum of the real parts of the zeros of the Riemann zeta function $\zeta(s)$. The only known bounds for $\delta$ to date are 
$$\frac{1}{2} \leq \delta \leq 1.$$ 
Moreover, the Riemann hypothesis is equivalent to $\delta = \frac{1}{2}$. As noted in the introduction, it is known [14] that $\delta$ is the least $\alpha \in \mathbb{R}$ such that 
$$\pi(x) - \text{li}(x) = O(x^{\alpha} \log x) \quad (x \to \infty).$$
It follows that the Riemann hypothesis is equivalent to 
$$\pi(x) - \text{li}(x) = O(\sqrt{x} \log x) \quad (x \to \infty).$$

From the following result of Montgomery and Vaughan [13], we deduce an analogue of Lemma 3.11 for the prime counting function.

**Theorem 4.1** ([13]). For all $y > x > 0$ with $y - x > 1$, one has 
$$0 \leq \pi(y) - \pi(x) < \frac{2(y - x)}{\log(y - x)}.$$

**Corollary 4.2.** Let $N$ be a positive integer. One has the following.
1. For all $y > x > 0$ and all $t > -\log(y - x)$, one has 
$$0 \leq \frac{\pi(e^t y)}{e^t} - \frac{\pi(e^t x)}{e^t} < \frac{2(y - x)}{t + \log(y - x)}.$$
2. For all $x > 0$ and all $t > -\log N$, one has 
$$0 \leq \frac{\pi(e^t (x + N))}{e^t} - \frac{\pi(e^t x)}{e^t} < \frac{2N}{t + \log N}.$$
3. For all $y > x > 0$ with $y - x \leq N$ and all $t > -\log N$, one has 
$$0 \leq \frac{\pi(e^t y)}{e^t} - \frac{\pi(e^t x)}{e^t} \leq \frac{\frac{\pi(e^t (x + N))}{e^t}}{e^t} - \frac{\pi(e^t x)}{e^t} < \frac{2N}{t + \log N}.$$
4. For all $x > 1$ and all $t > -\log N$, one has 
$$0 \leq \frac{\pi(e^t x)}{e^t} - \frac{\pi(e^t |x|)}{e^t} < \frac{2N}{t + \log N}.$$

The following theorem and corollaries describe the relevance of the functions $\beta(t, r)$ and $\beta(t)$ to the prime counting function.

**Theorem 4.3.** Let $\delta$ denote the infimum of the real parts of the zeros of the Riemann zeta function. Suppose that $M > 0$, $C \geq \mu$, and $\alpha$ are constants such that 
$$|\pi(x) - \text{li}(x)| \leq Mx^{\alpha} \log x$$
for all $x \geq C$. Then $\alpha \geq \delta$ and, for all $t, r, \lambda \in \mathbb{R}$ with $|r| \geq \mu e^{-t}$ and $\lambda \geq \beta(t, r)$, and for all integers $n \geq Ce^{-t}$, one has the following.
1. \[ \pi(e^n) - e^t \sum_{r \leq k < n} \frac{1}{H_k - \gamma + t} \leq M e^{\alpha t} n^\alpha (t + \log n) + e^t \lambda. \]

2. \[ \pi(e^n) - e^t \sum_{r \leq k < n} \frac{1}{H_k - \gamma + t} < M e^{\alpha t} n^\alpha (H_n - \gamma + t) + e^t \lambda. \]

3. \[ p(e^n) - \frac{1}{n} \sum_{r \leq k < n} \frac{1}{H_k - \gamma + t} < \frac{M}{e^{(1-\alpha)t}} n^{1-\alpha} + \frac{\lambda}{n}. \]

4. \[ p(e^n) - \frac{1}{n} \sum_{r \leq k < n} \frac{1}{H_k - \gamma + t} < \frac{M}{e^{(1-\alpha)t}} n^{1-\alpha} + \frac{\lambda}{n}. \]

Conversely, if any of the conditions above hold for all \( n \geq C \) for some constants \( M > 0, C \geq \mu, \lambda > 0, \) and \( t, r, \alpha \in \mathbb{R}, \) then \( \alpha \geq \delta, \) so there exist constants \( M' > 0 \) and \( C' \geq \mu \) such that

\[ |\pi(x) - \text{li}(x)| \leq M' x^\alpha \log x \]

for all \( x \geq C'. \)

**Proof.** Let \( \alpha, M, C, t, r, \) and \( \lambda \) be given as in the forward hypothesis. For all \( n \geq \lceil r \rceil \geq \mu e^{-t}, \) one has

\[ \frac{\text{li}(e^n)}{e^t} - \sum_{r \leq k < n} \frac{1}{H_k - \gamma + t} = \beta_n(t, r) = \frac{\text{li}(e^{\lceil r \rceil})}{e^t} + \theta_n(t, \lceil r \rceil), \]

where also

\[ 0 = \frac{\text{li}(\mu)}{e^t} \leq \frac{\text{li}(e^{\lceil r \rceil})}{e^t} \leq \frac{\beta_n(t, r)}{e^t}, \]

and therefore

\[ 0 \leq \frac{\text{li}(e^n)}{e^t} - \sum_{r \leq k < n} \frac{1}{H_k - \gamma + t} < \beta(t, r), \]

Moreover, by hypothesis and the obvious change of variables one has

\[ \left| \frac{\pi(e^n)}{e^t} - \frac{\text{li}(e^n)}{e^t} \right| \leq \frac{M}{e^{(1-\alpha)t}} n^{\alpha} (t + \log n) \]

for all \( n \geq Ce^t \geq \mu e^{-t}. \) Therefore, by the triangle inequality, one has

\[ \left| \frac{\pi(e^n)}{e^t} - \sum_{r \leq k < n} \frac{1}{H_k - \gamma + t} \right| \leq \frac{M}{e^{(1-\alpha)t}} n^{\alpha} (t + \log n) + \beta(t, r) \]

for all \( n \geq Ce^t. \) The forward direction of the theorem follows.

Conversely, suppose that \( \alpha, M, C, t, r, \) and \( \lambda \) are constants satisfying the reverse hypothesis. We wish to show that \( \alpha \geq \delta. \) We may suppose without loss of generality that \( \alpha > 0. \) Now, by hypothesis one has

\[ \frac{\pi(e^n)}{e^t} - \sum_{r \leq k < n} \frac{1}{H_k - \gamma + t} = O(n^\alpha \log n) \quad (n \to \infty), \]

where the \( O \) constant depends on the given constants. Moreover, by Corollary 3.10 one has

\[ \frac{\text{li}(e^n)}{e^t} - \sum_{r \leq k < n} \frac{1}{H_k - \gamma + t} = O(1) \quad (n \to \infty), \]
and therefore
\[
\frac{\pi(e^n)}{e} - \frac{\text{li}(e^n)}{e} = O(n^\alpha \log n) \ (n \to \infty).
\]

We wish to show that we can replace the discrete variable \(n\) in the above estimate with a continuous variable \(x\). Choose any positive integer \(N > e^{-t}\), so that \(t > -\log N\). Then, by Corollary 4.2, for all \(x > 1\) one has
\[
0 \leq \frac{\pi(e^t x)}{e^t} - \frac{\pi(e^t |x|)}{e^t} < \frac{2N}{t + \log N},
\]
so that
\[
\frac{\pi(e^t x)}{e^t} - \frac{\pi(e^t |x|)}{e^t} = O(1) \ (x \to \infty).
\]

Similarly, by Lemma 3.11, one has
\[
\frac{\text{li}(e^t x)}{e^t} - \frac{\text{li}(e^t |x|)}{e^t} = O(1) \ (x \to \infty).
\]

Therefore, one has
\[
\frac{\pi(e^t x)}{e^t} - \frac{\text{li}(e^t x)}{e^t} = \left( \frac{\pi(e^t |x|)}{e^t} - \frac{\text{li}(e^t |x|)}{e^t} \right) + \left( \frac{\pi(e^t x)}{e^t} - \frac{\pi(e^t |x|)}{e^t} \right) - \left( \frac{\text{li}(e^t x)}{e^t} - \frac{\text{li}(e^t |x|)}{e^t} \right)
\]
\[
= O(|x|^\alpha \log |x|) \ (x \to \infty)
\]
\[
= O(x^\alpha \log x) \ (x \to \infty).
\]

It follows, then, that
\[
\pi(x) - \text{li}(x) = O(x^\alpha \log x) \ (x \to \infty),
\]
and therefore \(\alpha \geq \delta\).

The special case where \(r = \mu e^{-t}\) yields the following.

**Corollary 4.4.** Let \(\delta\) denote the infimum of the real parts of the zeros of the Riemann zeta function. Suppose that \(M > 0\), \(C \geq \mu\), and \(\alpha\) are constants such that
\[
|\pi(x) - \text{li}(x)| \leq Mx^\alpha \log x
\]
for all \(x \geq C\). Then \(\alpha \geq \delta\) and, for all \(t, \lambda \in \mathbb{R}\) with \(\lambda \geq \beta(t)\) (e.g., \(\lambda = \frac{1}{\log \mu}\)), and for all integers \(n \geq Ce^{-t}\), one has the following.

1. \[
\left| \pi(e^n) - e^n \sum_{k=\lfloor \mu e^{-t} \rfloor}^{n-1} \frac{1}{H_k - \gamma + t} \right| < Me^{\alpha t}n^\alpha(t + \log n) + e^t \lambda.
\]
2. \[
\left| \pi(e^n) - e^n \sum_{k=\lfloor \mu e^{-t} \rfloor}^{n-1} \frac{1}{H_k - \gamma + t} \right| < Me^{\alpha t}n^\alpha(H_n - \gamma + t) + e^t \lambda.
\]
3. \[
\left| p(e^n) - \frac{1}{n} \sum_{k=\lfloor \mu e^{-t} \rfloor}^{n-1} \frac{1}{H_k - \gamma + t} \right| < M \frac{t + \log n}{e^{(1-\alpha)t}} \frac{1}{n^{1-\alpha}} + \frac{\lambda}{n}.
\]
4. \[
\left| p(e^n) - \frac{1}{n} \sum_{k=\lfloor \mu e^{-t} \rfloor}^{n-1} \frac{1}{H_k - \gamma + t} \right| < M \frac{H_n - \gamma + t}{e^{(1-\alpha)t}} \frac{1}{n^{1-\alpha}} + \frac{\lambda}{n}.
\]

Conversely, if any of the conditions above hold for all \(n \geq C\) for some constants \(M > 0\), \(C \geq \mu\), \(\lambda > 0\), and \(t, r, \alpha \in \mathbb{R}\), then \(\alpha \geq \delta\), so there exist constants \(M' > 0\) and \(C' \geq \mu\) such that
\[
|\pi(x) - \text{li}(x)| \leq M' x^\alpha \log x
\]
for all \(x \geq C'\).
In 1976 [16], L. Schoenfeld proved that the Riemann hypothesis is equivalent to

$$|\pi(x) - \text{li}(x)| < \frac{1}{8\pi} \sqrt{x} \log x, \quad \forall x \geq 2657.$$  

From Schoenfeld’s result and Theorem 4.3 and its corollary, we can relate our results to the Riemann hypothesis as follows.

**Theorem 4.5.** Suppose that the Riemann hypothesis implies that

$$|\pi(x) - \text{li}(x)| \leq M \sqrt{x} \log x$$

for all $x \geq C$, for constants $M > 0$ and $C \geq \mu$. For example, this implication holds for $M = \frac{1}{8\pi}$ and $C = 2657$. Let $t, r, \lambda \in \mathbb{R}$ with $r > e^{-t}$ and $\lambda \geq \beta(t, r)$. Then the Riemann hypothesis holds if and only if any of the following conditions hold for all $n \geq Ce^{-t}$.

1. $$\left| \pi(e^n) - e^t \sum_{r \leq k < n} \frac{1}{H_k - \gamma + t} \right| \leq M e^{t/2} \sqrt{n(t + \log n)} + \lambda e^t.$$  
2. $$\left| \pi(e^n) - e^t \sum_{r \leq k < n} \frac{1}{H_k - \gamma + t} \right| \leq M e^{t/2} \sqrt{n(H_n - \gamma + t)} + \lambda e^t.$$  
3. $$\left| p(e^n) - \frac{1}{n} \sum_{r \leq k < n} \frac{1}{H_k - \gamma + t} \right| \leq \frac{M t + \log n}{e^{t/2}} \frac{1}{\sqrt{n}} + \frac{\lambda}{e^t}.$$  
4. $$\left| p(e^n) - \frac{1}{n} \sum_{r \leq k < n} \frac{1}{H_k - \gamma + t} \right| \leq \frac{M H_n - \gamma + t}{e^{t/2}} \frac{1}{\sqrt{n}} + \frac{\lambda}{e^t}.$$  

**Corollary 4.6.** Suppose that the Riemann hypothesis implies that

$$|\pi(x) - \text{li}(x)| \leq M \sqrt{x} \log x$$

for all $x \geq C$, for constants $M > 0$ and $C \geq \mu$. For example, this implication holds for $M = \frac{1}{8\pi}$ and $C = 2657$. Let $t, \lambda \in \mathbb{R}$ with $\lambda \geq \beta(t)$ (e.g., $\lambda = \frac{1}{\log n}$). Then the Riemann hypothesis holds if and only if any of the following conditions hold for all $n \geq Ce^{-t}$.

1. $$\left| \pi(e^n) - e^t \sum_{r \leq k < n} \frac{1}{H_k - \gamma + t} \right| \leq M e^{t/2} \sqrt{n(t + \log n)} + \lambda e^t.$$  
2. $$\left| \pi(e^n) - e^t \sum_{r \leq k < n} \frac{1}{H_k - \gamma + t} \right| \leq M e^{t/2} \sqrt{n(H_n - \gamma + t)} + \lambda e^t.$$  
3. $$\left| p(e^n) - \frac{1}{n} \sum_{r \leq k < n} \frac{1}{H_k - \gamma + t} \right| \leq \frac{M t + \log n}{e^{t/2}} \frac{1}{\sqrt{n}} + \frac{\lambda}{e^t}.$$  
4. $$\left| p(e^n) - \frac{1}{n} \sum_{r \leq k < n} \frac{1}{H_k - \gamma + t} \right| \leq \frac{M H_n - \gamma + t}{e^{t/2}} \frac{1}{\sqrt{n}} + \frac{\lambda}{e^t}.$$  

For example, letting $t = \gamma$, and employing the upper and lower bounds for $\beta(\gamma)$ provided in Table 2, we obtain the following.
Corollary 4.7. Let $\lambda \geq \beta(\gamma)$, e.g., $\lambda = 0.4986013304$. The Riemann hypothesis holds if and only if any of the following equivalent conditions hold for all integers $n \geq 803$.

1. $|\pi(e^n) - e^n \sum_{k=1}^{n-1} \frac{1}{H_k}| \leq \frac{e^{\gamma/2}}{8\pi} \sqrt{n} \log n + \gamma + \lambda e^\gamma.$

2. $|\pi(e^n) - e^n \sum_{k=1}^{n-1} \frac{1}{H_k}| \leq \frac{e^{\gamma/2}}{8\pi} \sqrt{n} H_n + \lambda e^\gamma.$

3. $p(e^n) - \frac{1}{n} \sum_{k=1}^{n-1} \frac{1}{H_k} \leq \frac{1}{8\pi e^{\gamma/2} \sqrt{n}} \log n + \gamma + \lambda.$

4. $p(e^n) - \frac{1}{n} \sum_{k=1}^{n-1} \frac{1}{H_k} \leq \frac{1}{8\pi e^{\gamma/2} \sqrt{n}} \frac{H_n + \lambda}{n}.$

Proof. By Corollary 4.6, the Riemann hypothesis holds if and only if any of the equivalent conditions hold for all integers $n \geq 2657 e^{-\gamma} > 1491$, and the given inequalities can be verified directly to hold for all $803 \leq n \leq 1491$, even for the lower bound $\lambda = 0.49859$ of $\beta(\gamma)$.

Corollary 4.8. Let $\lambda \geq \beta(\gamma) + 1 = \beta(\gamma, \mu e^{-\gamma} + 1)$, e.g. $\lambda = 1.4986013304$. The Riemann hypothesis holds if and only if any of the following equivalent conditions hold for all positive integers $n$.

1. $|\pi(e^n) - e^n \sum_{k=2}^{n-1} \frac{1}{H_k}| \leq \frac{e^{\gamma/2}}{8\pi} \sqrt{n} \log n + \gamma + \lambda e^\gamma.$

2. $|\pi(e^n) - e^n \sum_{k=2}^{n-1} \frac{1}{H_k}| \leq \frac{e^{\gamma/2}}{8\pi} \sqrt{n} H_n + \lambda e^\gamma.$

3. $p(e^n) - \frac{1}{n} \sum_{k=2}^{n-1} \frac{1}{H_k} \leq \frac{1}{8\pi e^{\gamma/2} \sqrt{n}} \log n + \gamma + \lambda.$

4. $p(e^n) - \frac{1}{n} \sum_{k=2}^{n-1} \frac{1}{H_k} \leq \frac{1}{8\pi e^{\gamma/2} \sqrt{n}} \frac{H_n + \lambda}{n}.$

Proof. By Theorem 4.5, the Riemann hypothesis holds if and only if any of the equivalent conditions hold for all $n \geq 2657 e^{-\gamma} > 1491$, and the given inequalities can be verified directly to hold for all $1 \leq n \leq 1491$, even for the lower bound $\lambda = 1.49859$ of $\beta(\gamma) + 1$.

Similarly, letting $t$ equal $\gamma + 1$, $0$, $\log 2$, $-\log 2$, $1$, and $-1$, respectively, we obtain the following.

Corollary 4.9. Let $\lambda \geq \beta(\gamma + 1)$, e.g. $\lambda = 0.7509547014$. The Riemann hypothesis holds if and only if any of the following equivalent conditions hold for all positive integers $n$.

1. $|\pi(e^{\gamma+1} n) - e^{\gamma+1} \sum_{k=1}^{n-1} \frac{1}{H_k + 1}| \leq \frac{e^{(\gamma+1)/2}}{8\pi} \sqrt{n} \log n + \gamma + 1 + \lambda e^{\gamma+1}.$

2. $|\pi(e^{\gamma+1} n) - e^{\gamma+1} \sum_{k=1}^{n-1} \frac{1}{H_k + 1}| \leq \frac{e^{(\gamma+1)/2}}{8\pi} \sqrt{n} (H_n + 1) + \lambda e^{\gamma+1}.$

3. $p(e^{\gamma+1} n) - \frac{1}{n} \sum_{k=1}^{n-1} \frac{1}{H_k + 1} \leq \frac{1}{8\pi e^{(\gamma+1)/2} \sqrt{n}} \log n + \gamma + 1 + \lambda.$
4. \( p(e^{\gamma+1}n) - \frac{1}{n} \sum_{k=1}^{n-1} \frac{1}{H_k + 1} \leq \frac{1}{8\pi e(\gamma+1)/2} \frac{H_n + 1}{\sqrt{n}} + \frac{\lambda}{n} \).

Proof. By Corollary 4.6, the Riemann hypothesis holds if and only if any of the equivalent conditions hold for all \( n \geq 2657e^{-\gamma-1} > 548 \), and the given inequalities can be verified directly to hold for all \( 1 \leq n \leq 548 \), even for the lower bound \( \lambda = 0.750926 \) of \( \beta(\gamma) \).

**Corollary 4.10.** Let \( \lambda \geq \beta(0) \), e.g., \( \lambda = 1.0956456993 \). The Riemann hypothesis holds if and only if any of the following equivalent conditions hold for all positive integers \( n \geq 1427 \).

1. \( \pi(n) - \sum_{k=2}^{n-1} \frac{1}{H_k - \gamma} < \frac{1}{8\pi \sqrt{n}} \log n + \lambda \).
2. \( \pi(n) - \sum_{k=2}^{n-1} \frac{1}{H_k - \gamma} < \frac{1}{8\pi \sqrt{n}} (H_n - \gamma) + \lambda \).
3. \( p(n) - \frac{1}{n} \sum_{k=2}^{n-1} \frac{1}{H_k - \gamma} < \frac{1}{8\pi \sqrt{n}} \log n + \frac{\lambda}{n} \).
4. \( p(n) - \frac{1}{n} \sum_{k=2}^{n-1} \frac{1}{H_k - \gamma} < \frac{1}{8\pi \sqrt{n}} \frac{H_n - \gamma}{\sqrt{2n}} + \frac{\lambda}{n} \).

Proof. By Corollary 4.6, the Riemann hypothesis holds if and only if any of the equivalent conditions hold for all \( n \geq 2657 \), and the given inequalities can be verified directly to hold for all \( 1427 \leq n \leq 2657 \), even for the lower bound \( \lambda = 1.09564 \) of \( \beta(\gamma) \).

**Corollary 4.11.** Let \( \lambda \geq \beta(\log 2) \), e.g., \( \lambda = 0.6026096358 \). The Riemann hypothesis holds if and only if any of the following equivalent conditions hold for all positive integers \( n \geq 714 \).

1. \( \pi(2n) - 2 \sum_{k=1}^{n-1} \frac{1}{H_k - \gamma + \log 2} < \frac{1}{8\pi \sqrt{2n}} \log(2n) + 2\lambda \).
2. \( \pi(2n) - 2 \sum_{k=1}^{n-1} \frac{1}{H_k - \gamma + \log 2} < \frac{1}{8\pi \sqrt{2n}} (H_n - \gamma + \log 2) + 2\lambda \).
3. \( p(2n) - \frac{1}{n} \sum_{k=1}^{n-1} \frac{1}{H_k - \gamma + \log 2} < \frac{1}{8\pi \sqrt{2n}} \log(2n) + \frac{\lambda}{n} \).
4. \( p(2n) - \frac{1}{n} \sum_{k=1}^{n-1} \frac{1}{H_k - \gamma + \log 2} < \frac{1}{8\pi \sqrt{2n}} \frac{H_n - \gamma + \log 2}{\sqrt{2n}} + \frac{\lambda}{n} \).

Proof. By Corollary 4.6, the Riemann hypothesis holds if and only if any of the equivalent conditions hold for all \( n \geq 2657/2 > 1328 \), and the given inequalities can be verified directly to hold for all \( 714 \leq n \leq 1328 \), even for the lower bound \( \lambda = 0.602607 \) of \( \beta(\gamma) \).

**Corollary 4.12.** Let \( \lambda \geq \beta(-\log 2) \), e.g., \( \lambda = 0.3417372460 \). The Riemann hypothesis holds if and only if any of the following equivalent conditions hold for all positive integers \( n \geq 5314 \).

1. \( \pi(n/2) - \frac{1}{2} \sum_{k=3}^{n-1} \frac{1}{H_k - \gamma - \log 2} < \frac{1}{8\pi \sqrt{n/2}} \log(n/2) + \lambda/2 \).
2. \( \pi(n/2) - \frac{1}{2} \sum_{k=3}^{n-1} \frac{1}{H_k - \gamma - \log 2} < \frac{1}{8\pi \sqrt{n/2}} (H_n - \gamma - \log 2) + \lambda/2 \).
3. \[ p(n/2) - \frac{1}{n} \sum_{k=3}^{n-1} \frac{1}{H_k - \gamma - \log 2} \leq \frac{1}{8\pi} \frac{\log(n/2)}{\sqrt{n/2}} + \frac{\lambda}{n}. \]

4. \[ p(n/2) - \frac{1}{n} \sum_{k=3}^{n-1} \frac{1}{H_k - \gamma - \log 2} \leq \frac{1}{8\pi} \frac{H_n - \gamma - \log 2}{\sqrt{n/2}} + \frac{\lambda}{n}. \]

**Corollary 4.13.** Let \( \lambda \geq \beta(1) \), e.g., \( \lambda = 0.7418976158 \). The Riemann hypothesis holds if and only if any of the following equivalent conditions hold for all positive integers \( n \neq 82 \).

1. \[ \pi(en) - e \sum_{k=1}^{n-1} \frac{1}{H_k - \gamma + 1} \leq \frac{1}{8\pi} \sqrt{en} \log(en) + e\lambda. \]

2. \[ \pi(en) - e \sum_{k=1}^{n-1} \frac{1}{H_k - \gamma + 1} \leq \frac{1}{8\pi} \sqrt{en} (H_n - \gamma + 1) + e\lambda. \]

3. \[ p(en) - \frac{1}{n} \sum_{k=1}^{n-1} \frac{1}{H_k - \gamma + 1} \leq \frac{1}{8\pi} \sqrt{en} + \frac{\lambda}{n}. \]

4. \[ p(en) - \frac{1}{n} \sum_{k=1}^{n-1} \frac{1}{H_k - \gamma + 1} \leq \frac{1}{8\pi} \frac{H_n - \gamma + 1}{\sqrt{en}} + \frac{\lambda}{n}. \]

**Proof.** By Corollary 4.10, the Riemann hypothesis holds if and only if any of the equivalent conditions hold for all \( n \geq 2657e^{-1} > 977 \), and the given inequalities can be verified directly to hold for all \( 1 \leq n \leq 977 \) with \( n \neq 82 \), even for the lower bound \( \lambda = 0.741895 \) of \( \beta(\gamma) \).

**Corollary 4.14.** Let \( \lambda \geq \beta(-1) \), e.g., \( \lambda = 0.2229882714 \). The Riemann hypothesis holds if and only if any of the following equivalent conditions hold for all positive integers \( n \geq 7223 \).

1. \[ \pi(n/e) - \frac{1}{e} \sum_{k=4}^{n-1} \frac{1}{H_k - \gamma - 1} \leq \frac{1}{8\pi} \sqrt{n/e} \log(n/e) + \lambda/e. \]

2. \[ \pi(n/e) - \frac{1}{e} \sum_{k=4}^{n-1} \frac{1}{H_k - \gamma - 1} \leq \frac{1}{8\pi} \sqrt{n/e} (H_n - \gamma + 1) + \lambda/e. \]

3. \[ p(n/e) - \frac{1}{n} \sum_{k=4}^{n-1} \frac{1}{H_k - \gamma - 1} \leq \frac{1}{8\pi} \frac{\log(n/e)}{\sqrt{n/e}} + \frac{\lambda}{n}. \]

4. \[ p(n/e) - \frac{1}{n} \sum_{k=4}^{n-1} \frac{1}{H_k - \gamma - 1} \leq \frac{1}{8\pi} \frac{H_n - \gamma + 1}{\sqrt{n/e}} + \frac{\lambda}{n}. \]

Our results specialize to \( O \) bounds as follows.

**Theorem 4.15.** Let \( \delta \) be the supremum of the real parts of the zeros of the Riemann zeta function, let \( N \) be a positive integer, and let \( t \in \mathbb{R} \) so that \( t \neq \gamma - H_n \) for all \( n \geq N \) (which holds if \( t > \gamma - H_N \)). Then \( \delta \) is the smallest real number \( \alpha \) such that any of the following equivalent \( O \) bounds hold.

1. \[ \pi(e^t x) = e^t \sum_{N \leq k \leq x-1} \frac{1}{H_k - \gamma + t} + O\left(x^\alpha H_{x^\alpha}\right) \quad (x \to \infty). \]

2. \[ \pi(e^t n) = e^t \sum_{k=N}^{n-1} \frac{1}{H_k - \gamma + t} + O\left(n^\alpha H_n\right) \quad (n \to \infty). \]

3. \[ p(e^t x) = \frac{1}{x} \sum_{N \leq k \leq x-1} \frac{1}{H_k - \gamma + t} + O\left(H_{x^{1/\alpha}}\right) \quad (x \to \infty). \]
4. \( p(e^n) = \frac{1}{n} \sum_{k=N}^{n-1} \frac{1}{H_k - \gamma + t} + O \left( \frac{H_x}{x^{1-\alpha}} \right) \) \((n \to \infty)\).

Note that the hypothesis on \( t \) in the theorem is present only to guarantee that none of the denominators in the given sums vanish. Note also that the \( O \) constants depend on \( N \) and \( t \).

**Corollary 4.16.** Let \( N \) be a positive integer, and let \( t \in \mathbb{R} \) so that \( t \neq \gamma - H_n \) for all \( n \geq N \) (which holds if \( t > \gamma - H_N \)). Each of the following statements is equivalent to the Riemann hypothesis.

1. \( \pi(e^t \cdot x) = e^t \sum_{N \leq k \leq x-1} \frac{1}{H_k - \gamma + t} + O \left( \sqrt{\pi H_x} \right) \) \((x \to \infty)\).

2. \( \pi(e^n) = e^n \sum_{k=N}^{n-1} \frac{1}{H_k - \gamma + t} + O \left( \frac{H_n}{\sqrt{n}} \right) \) \((n \to \infty)\).

3. \( p(e^t \cdot x) = \frac{1}{x} \sum_{N \leq k \leq x-1} \frac{1}{H_k - \gamma + t} + O \left( \frac{H_x}{\sqrt{x}} \right) \) \((x \to \infty)\).

4. \( p(e^n) = \frac{1}{n} \sum_{k=N}^{n-1} \frac{1}{H_k - \gamma + t} + O \left( \frac{H_n}{\sqrt{n}} \right) \) \((n \to \infty)\).

**Corollary 4.17.** Each of the following statements is equivalent to the Riemann hypothesis.

1. \( p(e^t) = \frac{1}{x} \sum_{1 \leq k \leq x-1} \frac{1}{H_k} + O \left( \frac{H_x}{\sqrt{x}} \right) \) \((x \to \infty)\).

2. \( p(e^n) = \frac{1}{n} \sum_{n=1}^{n-1} \frac{1}{H_k} + O \left( \frac{H_n}{\sqrt{n}} \right) \) \((n \to \infty)\).

3. \( p(x) = \frac{1}{x} \sum_{1 \leq k \leq x-1} \frac{1}{H_k - \gamma} + O \left( \frac{H_x}{\sqrt{x}} \right) \) \((x \to \infty)\).

4. \( p(n) = \frac{1}{n} \sum_{k=1}^{n-1} \frac{1}{H_k - \gamma} + O \left( \frac{H_n}{\sqrt{n}} \right) \) \((n \to \infty)\).

For any positive real numbers \( x_1, x_2, \ldots, x_n \) and any \( s \in \mathbb{C} \setminus \{0\} \), let

\[
M_s(x_1, x_2, \ldots, x_n) = \left( \frac{x_1^s + x_2^s + \cdots + x_n^s}{n} \right)^{1/s}
\]

denote the \( s \)-power mean of \( x_1, x_2, \ldots, x_n \), so that \( M_{-1}(x_1, x_2, \ldots, x_n) = \frac{1}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}} \) denotes the harmonic mean of \( x_1, x_2, \ldots, x_n \). Noting that

\[
\frac{1}{H_n - \gamma + t} = o(1) \quad (n \to \infty),
\]

one can replace the upper limits \( x - 1 \) and \( n - 1 \) of the sums in the results above with \( x \) and \( n \), respectively. Thus, we have the following.

**Corollary 4.18.** Let \( \delta \) be the supremum of the real parts of the zeros of the Riemann zeta function, let \( N \) be a positive integer, and let \( t > -1 \). Then \( \delta \) is the smallest real number \( \alpha \) such that any of the following equivalent \( O \) bounds hold.
1. \( p(e^{t+\gamma}x) = \frac{1}{M_{-1}(H_1 + t, H_2 + t, \ldots, H_{[x]} + t)} + O\left(\frac{H_n}{x^{1-\alpha}}\right) \) \((x \to \infty)\).

2. \( p(e^{t+\gamma}n) = \frac{1}{M_{-1}(H_1 + t, H_2 + t, \ldots, H_n + t)} + O\left(\frac{H_n}{n^{1-\alpha}}\right) \) \((n \to \infty)\).

3. \( \frac{1}{p(e^{t+\gamma}x)} = M_{-1}(H_1 + t, H_2 + t, \ldots, H_{[x]} + t) + O\left(\frac{H_n^2}{x^{1-\alpha}}\right) \) \((x \to \infty)\).

4. \( \frac{1}{p(e^{t+\gamma}n)} = M_{-1}(H_1 + t, H_2 + t, \ldots, H_n + t) + O\left(\frac{H_n^3}{n^{1-\alpha}}\right) \) \((n \to \infty)\).

Corollary 4.19. Let \( N \) be a positive integer, and let \( t > -1 \). Each of the following statements is equivalent to the Riemann hypothesis.

1. \( p(e^{t+\gamma}x) = \frac{1}{M_{-1}(H_1 + t, H_2 + t, \ldots, H_{[x]} + t)} + O\left(\frac{H_n}{\sqrt{x}}\right) \) \((x \to \infty)\).

2. \( p(e^{t+\gamma}n) = \frac{1}{M_{-1}(H_1 + t, H_2 + t, \ldots, H_n + t)} + O\left(\frac{H_n}{\sqrt{n}}\right) \) \((n \to \infty)\).

3. \( \frac{1}{p(e^{t+\gamma}x)} = M_{-1}(H_1 + t, H_2 + t, \ldots, H_{[x]} + t) + O\left(\frac{H_n}{\sqrt{x}}\right) \) \((x \to \infty)\).

4. \( \frac{1}{p(e^{t+\gamma}n)} = M_{-1}(H_1 + t, H_2 + t, \ldots, H_n + t) + O\left(\frac{H_n}{\sqrt{n}}\right) \) \((n \to \infty)\).

Corollary 4.20. Each of the following statements is equivalent to the Riemann hypothesis.

1. \( p(e^\gamma n) = \frac{1}{M_{-1}(H_1, H_2, \ldots, H_n)} + O\left(\frac{H_n}{\sqrt{n}}\right) \) \((n \to \infty)\).

2. \( \frac{1}{p(e^\gamma n)} = M_{-1}(H_1, H_2, \ldots, H_n) + O\left(\frac{H_n}{\sqrt{n}}\right) \) \((n \to \infty)\).

3. \( p(n) = \frac{1}{M_{-1}(H_1 - \gamma, H_2 - \gamma, \ldots, H_n - \gamma)} + O\left(\frac{H_n}{\sqrt{n}}\right) \) \((n \to \infty)\).

4. \( \frac{1}{p(n)} = M_{-1}(H_1 - \gamma, H_2 - \gamma, \ldots, H_n - \gamma) + O\left(\frac{H_n}{\sqrt{n}}\right) \) \((n \to \infty)\).

A famous 1979 conjecture of Montgomery [12] implies a certain conjectural strengthening of the Riemann hypothesis, namely, [6] Conjecture 5.1, which states that

\[ \pi(x) = \text{li}(x) + O\left(\sqrt{x} (\log x)^{-1+\varepsilon}\right) \]

for all \( \varepsilon > 0 \). If the conjecture above holds, then the exponents in \( H_n \) in all of the conjectural \( O \) Riemann equivalents above, namely, those in Corollaries 4.16, 4.17, 4.19, and 4.20, can be decreased by \( 2 - \varepsilon \) for any \( \varepsilon > 0 \) (but they cannot be decreased by 2 due to Littlewood’s result Eq. (1.3)). Thus, the conjecture above, for example, is equivalent to the following.

Conjecture 4.21. For all \( t > -1 \) and all \( \varepsilon > 0 \), one has

\[ p(e^{t+\gamma}n) = \frac{1}{M_{-1}(H_1 + t, H_2 + t, \ldots, H_n + t)} + O_{t,\varepsilon}\left(\frac{H_n^{1+\varepsilon}}{\sqrt{n}}\right) \] \((n \to \infty)\)

or, equivalently,

\[ \frac{1}{p(e^{t+\gamma}n)} = M_{-1}(H_1 + t, H_2 + t, \ldots, H_n + t) + O_{t,\varepsilon}\left(\frac{H_n^{1+\varepsilon}}{\sqrt{n}}\right) \] \((n \to \infty)\).
The two conjectural $O$ bounds in the conjecture above are false for $\varepsilon = 0$. Indeed, from Littlewood’s result Eq. 1.3 of [11] it follows, for example, that
\[
\begin{align*}
p(e^{t+\gamma}n) &= \frac{1}{M_{-1}(H_1 + t, H_2 + t, \ldots, H_n + t)} + \Omega\left(\frac{H^1_{n \log \log H_n}}{\sqrt{n}}\right) (n \to \infty) \\
\frac{1}{p(e^{t+\gamma}n)} &= M_{-1}(H_1 + t, H_2 + t, \ldots, H_n + t) + \Omega\left(\frac{H_n \log \log H_n}{\sqrt{n}}\right) (n \to \infty).
\end{align*}
\]

5 Monotonicity properties of the error term $\beta(t)$

In this final section, we examine the intervals of increase and decrease of the function $\beta(t)$.

By Proposition 3.9, the function $\theta(t, 1)$ is strictly totally monotone on the interval $(0, \infty)$. Since $\theta(t, 1) = \beta(t) - \frac{\li(x)}{e^t}$ on $[\log \mu, \infty)$, it follows that $\beta(t) - \frac{\li(x)}{e^t}$ is strictly totally monotone on the interval $[\log \mu, \infty)$. However, the function $\frac{\li(x)}{e^t}$ is not. Let
\[
\alpha = 3.84646771704685632679869457035315613 \ldots
\]
denote the unique zero of $\frac{d\ li(x)}{dx}$, which is also the unique solution to the equation $\li(x) = \frac{x}{\log x}$. Alternatively,
\[
\log \alpha = 1.347155251069168225 \ldots
\]
is the unique zero of $\frac{d\ li(e^x)}{dx}$ and is the unique solution to the equation $x \li(e^x) = e^x$. The function $\frac{\li(x)}{x}$ is strictly increasing on $(1, \alpha)$ and strictly decreasing on $[\alpha, \infty)$. Likewise, the function $\frac{\li(x)}{e^x}$ is strictly increasing on $(0, \log \alpha]$ and strictly decreasing on $[\log \alpha, \infty)$. The first statement of the following proposition follows from these observations.

**Proposition 5.1.** Let $t \in \mathbb{R}$. One has the following.

1. Suppose that $t \geq \log \mu$, that is, that $R_t = 1$. The function $\beta(t) - \frac{\li(e^t)}{e^t}$ is strictly totally monotone on the interval $[\log \mu, \infty)$, and the function $\beta(t)$ is strictly decreasing on the interval $[\log \alpha, \infty)$. Consequently, one has
\[
\beta(t) \leq \beta(\log \alpha) < 0.7695247294
\]
on $[\log \alpha, \infty)$.

2. If $t < \log \mu$, that is, if $R_t \geq 2$, then the function $\beta(t) - \frac{\li(e^{R_t})}{e^t}$ is strictly totally monotone, and the function $\frac{\li(e^{R_t})}{e^t}$ is strictly increasing and concave down, on $[\log(\mu/R_t), (\log(\mu/R_t) - 1))]$.

3. If $t < \log(\mu/2)$, that is, if $R_t \geq 3$, then $\beta(t)$ is strictly increasing on $[\log(\mu/R_t), (\log(\mu/(R_t - 1))]$.

4. $\beta(t)$ is strictly increasing on $[\log(\mu/2), \log(\alpha/3)]$, on which $R_t = 2$.

**Proof.** We have already proved statement (1), so we may suppose $t < \log \mu$ and $R_t \geq 2$. For $t \in I = [\log(\mu/R_t), (\log(\mu/(R_t - 1))]$, the function $R_t = N$ is constant. Therefore, by Proposition 3.9, the function
\[
\beta(t) - \frac{\li(e^{R_t})}{e^t} = \theta(t, N)
\]
is strictly totally monotone on $I$. Moreover, since
\[
e^tN \leq \mu N/(N - 1) \leq 2\mu < \alpha
\]
on $I$, the function $\frac{\li(e^{R_t})}{e^t} = \frac{\li(e^{R_t}N)}{e^t}$ is strictly increasing on $I$. Furthermore, the derivative
\[
\frac{d\ li(e^tN)}{dt} e^t = \frac{N}{t + \log N} - \frac{\li(e^tN)}{e^t}
\]

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is decreasing on $I$, so that $\frac{\text{li}(e^t x)}{e^t x}$ is concave down on $I$. Again by Proposition 3.9 one has

$$\beta'(t) = \frac{d}{dt} \frac{\text{li}(e^t)}{e^t} + \frac{d}{dt} \theta(t, N)$$

$$= \lim_{n \to \infty} \left( \frac{d}{dt} \frac{\text{li}(e^t)}{e^t} - \int_{N}^{n} \frac{dx}{x + \log x} + \sum_{k=N}^{n-1} \frac{1}{(H_k - \gamma + t)^2} \right)$$

$$= \lim_{n \to \infty} \left( \frac{d}{dt} \frac{\text{li}(e^t n)}{e^t n} + \sum_{k=N}^{n-1} \frac{1}{(H_k - \gamma + t)^2} \right)$$

$$\geq \lim_{n \to \infty} \left( \frac{d}{dt} \frac{\text{li}(e^t n)}{e^t n} + \sum_{k=N+1}^{n} \frac{1}{(\log k + \gamma + t)^2} \right)$$

$$= \lim_{n \to \infty} \left( \frac{d}{dt} \frac{\text{li}(e^t n)}{e^t n} + \int_{N+1}^{n+1} \frac{dx}{(\log x + \gamma + t)^2} \right)$$

$$= \lim_{n \to \infty} \left( -n \int_{0}^{1} (\log x + \gamma + t)^2 + \int_{N+1}^{n+1} \frac{dx}{(\log x + \gamma + t)^2} \right)$$

$$= \lim_{n \to \infty} \left( \int_{N+1}^{n+1} \frac{dx}{(\log x + \gamma + t)^2} + \int_{n+1}^{n} \frac{dx}{(\log x + \gamma + t)^2} \right)$$

$$= \int_{R_{t}+1}^{\alpha e^{-t}} \frac{dx}{(\log x + \gamma + t)^2}.$$ 

Moreover, one has $\alpha e^{-t} > R_{t} + 1$ provided that $R_{t} \geq 3$ since

$$t < \log(\mu/(R_{t} - 1)) < \log(\alpha/(R_{t} + 1))$$

if $R_{t} \geq 3$. Therefore $\beta'(t) \geq \int_{N+1}^{n+1} \frac{dx}{(\log x + \gamma + t)^2} > 0$ if $N \geq 3$. Finally, if $R_{t} = 2$, then $\alpha e^{-t} > R_{t} + 1 = 3$ provided that $t < \log(\alpha/3)$, so that $\beta'(t) > 0$ on $[\log(\mu/2), \log(\alpha/3)]$. \hfill \Box

Thus, $\beta(t)$ is strictly increasing on $[log(\mu/R_t), log(\mu/(R_t - 1))]$ as long as $R_{t} \geq 3$. The cases $R_{t} = 1$ and $R_{t} = 2$, that is, the intervals $[\log \mu, \infty)$ and $[\log(\mu/2), \log \mu]$, are still somewhat of a mystery, since we only know that $\beta(t)$ is strictly decreasing on $[\log(\mu, \infty) \subseteq [\log \mu, \infty)$ and $\beta(t)$ is strictly increasing on $[\log(\mu/2), \log(\alpha/3)] \subseteq [\log(\mu/2), \log \mu)$. The only remaining intervals to examine, then, are $[\log(\mu, \log \alpha)$ and $[\log(\alpha/3), \log \mu)$.

Let us examine the first interval. Since $\frac{\text{li}(e^t x)}{e^t x}$ is a reasonable approximation (and lower bound) of $\beta(t)$ on $[\log \mu, \infty)$, one might expect that there exists a constant $c \geq \log \mu$ such that $\beta(t)$ is increasing on $[\log \mu, c]$ and decreasing on $[c, \infty)$. This expectation is realized if the following two conjectures hold: (1) for all positive integers $n$, function

$$\beta_{n}(t) = \frac{\text{li}(ne^t)}{e^t} - \sum_{k=1}^{n-1} \frac{1}{H_k - \gamma + t}$$

has a unique local maximum on $[\log \mu, \infty)$ at some $\rho_{n} \in (\log \mu, \log \alpha)$, and (2) the limit

$$\rho = \lim_{n \to \infty} \rho_{n}$$

exists. Numerical evidence leads one to suspect further that: (3) the $\rho_{n}$ are bounded below by $\rho$, and (4) the $\rho_{n}$ are strictly decreasing as $n \to \infty$ (which, together with (1), would imply (2)). Suppose, for the sake of argument, that conjectures (1) and (2) are true. Let $\varepsilon > 0$. Then the $\beta_{n}(t)$ are decreasing on $[\rho + \varepsilon, \infty)$ for sufficiently large $n$, whence $\beta(t) = \lim_{n \to \infty} \beta_{n}(t)$ is also decreasing on $[\rho + \varepsilon, \infty)$. At the same time, the $\beta_{n}(t)$ are increasing on $[\log \mu, \rho - \varepsilon]$ for sufficiently large $n$, so that $\beta(t)$ is increasing on $[\log \mu, \rho - \varepsilon]$. Therefore, if
Table 3: Local maximum of $\beta_n(t)$ on $[\log \mu, \infty)$ attained at $t = \rho_n$

| $n$ | $\rho_n$ |
|-----|----------|
| 1   | $\log \alpha \approx 1.347155$ |
| 2   | $\approx 1.29475$ |
| 3   | $\approx 1.28724$ |
| 4   | $\approx 1.28489$ |
| 5   | $\approx 1.28386$ |
| 6   | $\approx 1.28331$ |
| 7   | $\approx 1.28298$ |
| 8   | $\approx 1.28277$ |
| 9   | $\approx 1.28262$ |
| 4000| $\approx 1.28202$ |
| 5000| $\approx 1.28202$ |

Table 4: Upper and lower bounds of $\beta(t)$ computed with $n = 100$ and approximations with $n = 1000$

| $t$  | $\beta(t) <$ | $\beta(t) \approx$ | $\beta(t) >$ |
|------|--------------|----------------|---------------|
| 1.274| 0.770653     | 0.770651       | 0.770639      |
| 1.280| 0.770670     | 0.770668       | 0.770656      |
| 1.281| 0.770671     | 0.770669       | 0.770657      |
| 1.282| 0.770671     | 0.770669       | 0.770657      |
| 1.283| 0.770671     | 0.770669       | 0.770657      |
| 1.284| 0.770670     | 0.770668       | 0.770656      |
| 1.285| 0.770669     | 0.770667       | 0.770655      |
| 1.290| 0.770653     | 0.770651       | 0.770663      |

Conjectures (1) and (2) are true, then $\beta(t)$ is increasing on $[\log \mu, \rho]$ and decreasing on $[\rho, \infty)$, and therefore $\beta(t)$ attains a local maximum at $t = \rho$. Table 3 lists approximate values of $\rho_n$ for $n = 1, 2, 3, \ldots, 10$, where $\beta_n(t)$ attains a unique local maximum at the given values of $t = \rho_n$, and also for $n = 4000$ and $n = 5000$, where $\beta_n(t)$ attains at least one local maximum at $t \approx 1.28202$. Thus, from the computations in Table 3, it appears that $\rho \approx 1.28202$ exists. A separate calculation, shown in Table 4, shows that indeed $\beta(t)$ attains at least one local maximum value of approximately 0.77067 at some $t$ near 1.282. More precisely, from the calculations in Table 4 one has

$$\beta(1.274) < 0.770653 < 0.770657 < \beta(1.282)$$

and

$$\beta(1.290) < 0.770653 < 0.770657 < \beta(1.282),$$

and therefore, since $\beta(t)$ is differentiable on $(\log \mu, \infty)$, the function $\beta(t)$ must attain at least one local maximum value at some $t = \rho$ satisfying

$$1.274 < \rho < 1.290.$$  

A similar analysis of $\beta(t)$ on the interval $[\log(\mu/2), \log \mu]$ suggests that $\beta(t)$ is strictly increasing on the entire interval, not just on $[\log(\mu/2), \log(\alpha/3)]$. Thus we make the following conjecture.

**Conjecture 5.2.** The function $\beta(t)$ attains a unique local maximum on $[\log \mu, \infty)$. Equivalently, there exists a constant $\rho > \log \mu$, where $\rho \approx 1.28202$, such that $\beta(t)$ is strictly increasing on $[\log \mu, \rho]$ and strictly decreasing on $[\rho, \infty)$. Moreover, the function $\beta(t)$ is strictly increasing on $[\log(\mu/2), \log \mu]$.
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