A SERRE PRESENTATION FOR THE QUANTUM GROUPS

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Abstract. Let \((U, U')\) be a quasi-split quantum symmetric pair of arbitrary Kac-Moody type, where “quasi-split” means the corresponding Satake diagram contains no black node. We give a presentation of the quantum group \(U'\) with explicit \(q\)-Serre relations. The verification of new \(q\)-Serre relations is reduced to some new \(q\)-binomial identities. Consequently, \(U'\) is shown to admit a bar involution under suitable conditions on the parameters.

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1. Introduction

1.1. Let \(U\) be a Drinfeld-Jimbo quantum group with Chevalley generators \(E_i, F_i, K_i^{\pm 1}\), for \(i \in I\). It is a \(q\)-deformation of the universal enveloping algebra of a symmetrizable Kac-Moody algebra with a Serre presentation. In terms of divided powers \(F_i^{(n)} = F_i^n/[n]_q\) \(i\) (cf. [Lu93]; see (2.1) and its subsequent paragraph for notation \([n]_q\)), the \(q\)-Serre relations among \(F_i\)'s can be written in a compact form as follows: for \(i \neq j \in I\),

\[
\sum_{n=0}^{1-a_{ij}} (-1)^n F_i^{(n)} F_j F_i^{(1-a_{ij}-n)} = 0.
\]

(1.1)

The quantum group \(U\) is a Hopf algebra with a comultiplication which is denoted by \(\Delta\).

Quantum symmetric pairs (QSP for short), \((U, U')\), are deformations of symmetric pairs which are defined using Satake diagrams as the input, and \(U'\) satisfies the coideal subalgebra property \(\Delta : U' \to U' \otimes U\). The theory of QSP was systematically studied by Letzter for \(U\) of finite type (cf. [Le99, Le02] for historical remarks and references therein). The QSP of Kac-Moody type was subsequently developed by Kolb [Ko14], unifying various special cases beyond finite type considered in the literature, some of which we mention below. We remark
that the algebra $U^t = U^t_{\zeta, \kappa}$ actually depends on a number of parameters $\zeta = (\zeta_i)_{i \in I}, \kappa = (\kappa_i)_{i \in I}$; see (2.19). For example, some main generators of $U^t$ are of the form, cf. (2.18):

$$B_i = F_i + \zeta_i E_i \tilde{K}_i^{-1} + \kappa_i \tilde{K}_i^{-1}, \quad \text{for } i \in I,$$

where the definitions of $\tilde{K}_i$ and the involution $\tau$ can be found in (2.6) and (2.16) respectively. It has become increasingly clear in recent years (cf. [BW18a, BK15, BK19, BW18b] and the references therein) that the algebras $U^t$ on their own are of fundamental importance, and we shall refer to them as the quantum groups.

Borrowing terminologies from real Lie groups, we shall call a quantum symmetric pair and an quantum group quasi-split (and respectively, split) if the underlying Satake diagram contains no black node (respectively, with the trivial involution in the Satake diagram). In other words, these are the quantum groups associated to the Chevalley involution $\omega$, coupled with a diagram involution $\tau$ (which is allowed to be the identity). Examples of the split quantum groups were considered in the literature (cf., e.g., [T93, BaK05]) and they are also known as generalized $q$-Onsager algebras, cf. [BaB10]. We refer to [Ko14, Introduction, (1)] for more detailed historical remarks. A quasi-split quantum group depends only on the generalized Cartan matrix and a diagram involution $\tau$.

Obtaining a nice presentation of the quantum group $U^t$ is a fundamental problem, and it has useful applications. For example, to construct the bar involution on a general quantum group $U^t$ as predicted in [BW18a], one would need to have a precise presentation to see clearly what constraints on the parameters should be satisfied [BK15]. The bar involution on $U^t$ is a basic ingredient for the canonical basis [BW18b].

A presentation for a general $U^t$ of finite type was given by Letzter [Le02, Le03]. Some less precise presentation for a general $U^t$ of Kac-Moody type (where some Serre type relations were not explicit) was known earlier [Ko14]; under the assumption that the Cartan integers $|a_{ij}| \leq 3$, all the Serre type relations were found explicitly in terms of monomials in $B_i, B_j$ [Le03, Ko14, BK15, BK19], even though some of the formulas become complicated quickly as $|a_{ij}|$ increases; see the formulas (3.11) in the quasi-split setting. A new and more conceptual approach is called for in order to reorganize and go beyond the known cases.

1.2. The main result of this paper is a Serre presentation with precise and uniform relations for the quasi-split quantum groups of arbitrary Kac-Moody type with general parameters; see Theorems 3.1 and 3.3. One may view this work as an application of canonical bases to the foundational questions for the quantum groups.

The key to our Serre presentation is the so-called new $t$Serre relations between $B_i$ and $B_j$ for $\tau i = i \neq j$. They are expressed in terms of the $t$divided powers $B_{i, \overline{p}},$ for any fixed $\overline{p} \in Z_2 = \{0, 1\}$, as follows:

$$\sum_{n=0}^{1-a_{ij}} (-1)^n B_{i, a_{ij} + \overline{p}} B_{j, (1-a_{ij}-n)} = 0.$$

(1.2)

Note that this relation formally takes the same form as the standard $q$-Serre relation (1.1). Actually the relation (1.2) holds for general (beyond quasi-split) quantum groups; cf. Remark 3.4. Let us explain the $t$divided powers.

For distinguished parameters $\zeta^i$, i.e., $\zeta^i = q^i$ for $i \in I$ such that $\tau i = i$ (cf. (3.14)), and $\kappa_j = 0$ for all $j \in I$, the $t$divided powers $B_{i, \overline{p}},$ for $\overline{p} \in Z_2$ and $m \geq 1$, are explicit...
polynomials in $B_i$ introduced in [BW18a, BeW18] which depend on a parity $\overline{p}$ (arising from the parities of the highest weights of highest weight $U$-modules when evaluated at the coroot $h_i$). The $i$-divided powers are canonical basis elements for (the modified form of) $U^i$ in the sense of [BW18b], but we will not need this fact here. For $U^i$ with general parameters, we define $i$-divided powers as suitable polynomials in $B_i$ which are obtained via some rescaling isomorphism from those associated to the distinguished parameters; see (3.2)–(3.3). We caution that the $i$-divided powers for more general parameters may not necessarily be canonical basis elements.

It is instructive for the reader to verify that our $i$-Serre relations (1.2) provide a uniform reformulation of the case-by-case complicated relations in (3.11) (due to [Ko14, BK19]) when $|a_{ij}| \leq 3$. For illustration we convert the $i$-Serre relation for $a_{ij} = -4$ into a formula (3.12) in terms of monomials in $B_i, B_j$, and compare it to [BaB10, (2.1)].

The Serre presentation of $U^i$ is valid in the specialization at $q = 1$, providing a presentation of the fixed point Lie subalgebra $g^\tau\omega$ of a symmetrizable Kac-Moody algebra $g$, where $\omega$ is the Chevalley involution. Independently, when $\tau = 1$ Stokman [St18] gives a presentation of $g^\omega$ and calls it a generalized Onsager algebra, where the (classical) $i$-Serre relations are determined recursively; in contrast our formula for the (classical) $i$-Serre is closed. One verifies that our $i$-Serre relations for $|a_{ij}| \leq 4$ specialize to his formulas; cf. Remark 3.3.

1.3. Our strategy of establishing the $i$-Serre relations (1.2) is as follows. As $U^i$ with arbitrary parameters is isomorphic to $U^i$ with distinguished parameters $\varsigma^\circ$ in (3.14) and $\kappa = 0$ (cf. [Le02, Ko14]), we are reduced to the case of $U^i$ with distinguished parameters. As $U^i$ is embedded in $U$, it suffices to verify the identity (1.2) in the quantum group $U$, or alternatively, in the modified form $\hat{U}$.

To that end, the explicit expansion formulas of $i$-divided powers in terms of PBW basis of $\hat{U}$ given in [BeW18] play a crucial role. The identity (1.2) is reduced to the assertion of various coefficients in the PBW basis expansion of the left-hand side of (1.2) are zero. After reorganizations of the computations, the vanishings of all these coefficients somewhat miraculously reduce to a universal $q$-binomial identity

$$T(w, u, \ell) = 0$$

in 3 integral variables $w, u, \ell$; see (3.18) for the definition of $T(w, u, \ell)$.

It turns out to be difficult to prove this $q$-binomial identity directly (the reader is encouraged to try his hand on this). We find a way around by first generalizing the $q$-identity to some $q$-identity involving 3 additional variables, such as, for $\ell > 0$,

$$G(w, u, \ell; p_0, p_1, p_2) = 0.$$  

See (5.1) for definition of $G$. The additional variables allow us to write down simple recursive relations, and then the (generalized) $q$-identity follows.

1.4. Let us indicate several applications of the main results of this paper.

The $i$-divided powers and $i$-Serre relations can be used to describe the higher order $i$-Serre relations and construct the braid group actions on $U^i$; for the quantum group $U$ this was done in [Lu93]. This will be treated in a sequel to this paper [CLW19].

The bar involution on a quasi-split $i$-quantum group $U^i$ is an indispensable ingredient for the $i$-canonical basis of the modified form of $U^i$ developed in [BW18c].
The elegant \( \tilde{\text{S}} \)Serre relations (1.2) were motivated by and have in turn suggested a potential categorical interpretation à la Khovanov-Lauda; they are expected to play a fundamental role in the categorification of the quasi-split (modified) \( \tilde{\text{S}} \)quantum groups.

Having the Serre presentations of quasi-split \( \tilde{\text{S}} \)quantum groups available, one may hope that explicit \( \tilde{\text{S}} \)Serre relations for general \( \tilde{\text{S}} \)quantum groups \( U^q \) will be eventually written down in terms of \( \tilde{\text{S}} \)canonical bases. To that end, more formulas for \( \tilde{\text{S}} \)canonical bases need to be computed first, which are very challenging as the divided powers above already indicate.

1.5. The paper is organized as follows. In Section 2 we review and fix notations for quantum groups and quantum symmetric pairs. In Section 3, we formulate our main results and the main steps of their proofs. The Serre presentation of \( U^q \) can be found in Theorem 3.1. The \( q \)-binomial identity which is used to derive the \( \tilde{\text{S}} \)Serre relations is stated as Theorem 3.6. The bar involution on \( U^q \) with suitable conditions on parameters specified is formulated as Proposition 3.7.

Section 4, Section 5, and Appendix A form the technical parts of the paper. In Section 4, we reduce the proof of the \( \tilde{\text{S}} \)Serre relations to the \( q \)-binomial identity \( T(w, u, \ell) = 0 \); some additional reduction steps are collected in Appendix A. In Section 5, we formulate and prove a generalization of the \( q \)-identity \( T(w, u, \ell) = 0 \); in particular, this identity follows.

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2. The preliminaries

2.1. Quantum groups. We recall the definitions of Cartan datum and root datum from [Lu93, 1.1.1, 2.2.1]. A Cartan datum is a pair \((I, \cdot)\) consisting of a finite set \( I \) and a symmetric bilinear form \( \nu, \nu' \mapsto \nu \cdot \nu' \) on the free abelian group \( \mathbb{Z}[I] \) such that

(a) \( i \cdot i \in \{2, 4, 6, \ldots\} \) for any \( i \in I \);  
(b) \( 2 \frac{i^2}{i^2} \in \{0, -1, -2, \ldots\} \) for any \( i \neq j \) in \( I \).

A root datum of type \((I, \cdot)\) consists of

(a) two finitely generated free abelian groups \( Y, X \) and a perfect bilinear pairing \( \langle \cdot, \cdot \rangle : Y \times X \to \mathbb{Z} \);  
(b) an embedding \( I \subset X \) \((i \mapsto \alpha_i)\) and an embedding \( I \subset Y \) \((i \mapsto h_i)\) such that \( \langle h_i, \alpha_j \rangle = 2 \frac{i^2}{i^2} \) for all \( i, j \in I \).

The matrix \( A := (a_{ij}) := (\langle h_i, \alpha_j \rangle) \) is a symmetrizable generalised Cartan matrix. Let

\[ D = \text{diag}(\epsilon_i \mid i \in I), \quad \text{where} \quad \epsilon_i = \frac{i \cdot i}{2} \quad (\forall i \in I). \]

Then \( DA \) is symmetric. We shall assume that the root datum defined above is \( X \)-regular and \( Y \)-regular; that is, \( \{\alpha_i \mid i \in I\} \) is linearly independent in \( X \) and \( \{h_i \mid i \in I\} \) is linearly independent in \( Y \).
Let \( q \) be an indeterminate, and denote
\[
q_i := q^{\epsilon_i}, \quad \forall i \in I.
\]
For \( n, d, m \in \mathbb{Z} \) with \( m \geq 0 \), we denote the \( q \)-integers and \( q \)-binomial coefficients as
\[
[n] = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [m]! = [1][2] \cdots [m],
\]
\[
\binom{n}{d} = \begin{cases} \frac{[n][n-1] \cdots [n-d+1]}{[d]!}, & \text{if } d \geq 0, \\ 0, & \text{if } d < 0. \end{cases}
\]
We denote by \([n]_{q_i}, [m]_{q_i}^{\dagger}\) and \(\binom{n}{m}_{q_i}^{\dagger}\) the variants of \([n], [m]!, \) and \(\binom{n}{m}\) with \( q \) replaced by \( q_i \).

For any \( i \neq j \in I \), define the following polynomial in two (noncommutative) variables
\[
S_{ij}(x, y) = \sum_{n=0}^{1-a_{ij}} (-1)^n \binom{1-a_{ij}}{n}_{q_i} \ x^n y^{1-a_{ij} - n}.
\]

Let \( \mathbb{K} \) be a field of characteristic 0. Assume that a root datum \((Y, X, \langle \cdot, \cdot \rangle, \ldots)\) of type \((I, \cdot)\) is given. The quantum group \( U \) is the associative \( \mathbb{K}(q) \)-algebra with generators \( E_i, F_i, K_h \) for all \( i, j \in I \) and \( h \in Y \) subject to the following relations:
\[
\begin{align*}
(2.3) & \quad K_0 = 1, \quad K_h K_{h'} = K_{h+h'}, \quad \forall h, h' \in Y. \\
(2.4) & \quad K_h E_i = q^{(h, \alpha_i)} E_i K_h, \quad \forall i \in I, h \in Y. \\
(2.5) & \quad K_h F_i = q^{-(h, \alpha_i)} F_i K_h, \quad \forall i \in I, h \in Y. \\
(2.6) & \quad [E_i, F_j] = \delta_{ij} \widetilde{K}_i - \widetilde{K}_i^{-1}, \quad \text{where } \widetilde{K}_i = K_h^{q_i}, \quad \forall i \in I. \\
(2.7) & \quad (q\text{-Serre relations}) \quad S_{ij}(E_i, E_j) = 0 = S_{ij}(F_i, F_j), \quad \forall i \neq j \in I.
\end{align*}
\]

Let
\[
F_i^{(n)} = F_i^n/[n]_{q_i}^{\dagger}, \quad E_i^{(n)} = E_i^n/[n]_{q_i}^{\dagger}, \quad \text{for } n \geq 1 \text{ and } i \geq 1.
\]
Then the \( q \)-Serre relations \((2.7)\) above can be rewritten as follows: for \( i \neq j \in I \),
\[
\sum_{n=0}^{1-a_{ij}} (-1)^n E_i^{(n)} E_j E_i^{(1-a_{ij} - n)} = 0,
\]
\[
\sum_{n=0}^{1-a_{ij}} (-1)^n F_i^{(n)} F_j F_i^{(1-a_{ij} - n)} = 0.
\]

Denote by \( \omega \) the Chevalley involution, which is the \( \mathbb{K}(q) \)-algebra automorphism of \( U \) sending
\[
\omega(E_i) = -F_i, \quad \omega(F_i) = -E_i, \quad \omega(K_h) = K_{-h}.
\]
The following lemma is a higher rank generalization of the involution \( \varpi \) on \( U_q(\mathfrak{sl}_2) \) defined in [BeW18, Remark 2.3].
Lemma 2.1. There exists an involution \( \varpi \) on the \( \mathbb{K} \)-algebra \( U \) which sends
\[
\varpi : E_i \mapsto q_i^{-1} F_i \tilde{K}_i, \quad F_i \mapsto q_i^{-1} E_i \tilde{K}_i^{-1}, \quad K_\mu \mapsto K_\mu, \quad q \mapsto q^{-1}.
\]
for any \( i \in I, \mu \in Y \).

Proof. Knowing that the rank one relations are preserved, we see quickly that \( \varpi \) preserves the defining relations (2.3)–(2.6) for \( U \). It remains to show that \( \varpi \) preserves the \( q \)-Serre relations (2.7): \( S_{ij}(q_i^{-1} F_i \tilde{K}_i, q_j^{-1} F_j \tilde{K}_j) = 0 \) and \( S_{ij}(q_i^{-1} E_i \tilde{K}_i^{-1}, q_j^{-1} E_j \tilde{K}_j^{-1}) = 0 \), for \( i \neq j \in I \).

This is known, and for the sake of completeness let us include a short argument:
\[
S_{ij}(q_i^{-1} F_i \tilde{K}_i, q_j^{-1} F_j \tilde{K}_j) = q_i^{a_{ij}-1} q_j^{-1} \sum_{n=0}^{1-a_{ij}} (-1)^n \left( \frac{1-a_{ij}}{n} \right) (F_i \tilde{K}_i)^n (F_j \tilde{K}_j) (F_i \tilde{K}_i)^{1-a_{ij}-n} = q_i^{a_{ij}-1} q_j^{-1} S_{ij}(F_i, F_j) \tilde{K}_i^{1-a_{ij}} \tilde{K}_j = 0.
\]
The other relation is entirely similar. \( \square \)

Let \( U^+, U^- \) and \( U^0 \) be the subalgebra of \( U \) generated by \( \{ E_i \mid i \in I \} \), \( \{ F_i \mid i \in I \} \) and \( \{ K_\mu \mid h \in Y \} \) respectively.

2.2. The algebra \( \bar{U} \). Recall [Lu93, 23.1] that the modified form of \( U \), denoted by \( \bar{U} \), is a \( \mathbb{K}(q) \)-algebra (without 1) generated by \( 1_\lambda, E_i 1_\lambda, F_i 1_\lambda \), for \( i \in I, \lambda \in X \), where \( 1_\lambda \) are orthogonal idempotents. Let \( \mathcal{A} = \mathbb{Z}[q, q^{-1}] \). There is an \( \mathcal{A} \)-subalgebra \( \mathcal{A} \bar{U} \) generated by \( E_i^{(n)} 1_\lambda, F_i^{(n)} 1_\lambda \) for \( i \in I \) and \( n \geq 0 \) and \( \lambda \in X \). Note that \( \bar{U} \) is naturally a \( U \)-bimodule [Lu93, 23.1.3], and in particular we have
\[
K_h 1_\lambda = 1_\lambda K_h = q^{(h, \lambda)} 1_\lambda, \quad \forall h \in Y.
\]

We have the mod 2 homomorphism \( \mathbb{Z} \to \mathbb{Z}_2, k \mapsto \bar{k} \), where \( \mathbb{Z}_2 = \{ 0, 1 \} \). Let us fix an \( i \in I \). Define
\[
\bar{U}_{i,0} := \bigoplus_{\lambda : \langle h_i, \lambda \rangle \in 2\mathbb{Z}} \bar{U} 1_\lambda, \quad \bar{U}_{i,1} := \bigoplus_{\lambda : \langle h_i, \lambda \rangle \in 1+2\mathbb{Z}} \bar{U} 1_\lambda.
\]
Then \( \bar{U} = \bar{U}_{i,0} \oplus \bar{U}_{i,1} \). Similarly, letting \( \mathcal{A} \bar{U}_{i,0} = \mathcal{A} \bar{U}_{i,0} \cap \mathcal{A} \bar{U} \) and \( \mathcal{A} \bar{U}_{i,1} = \mathcal{A} \bar{U}_{i,1} \cap \mathcal{A} \bar{U} \), we have \( \mathcal{A} \bar{U} = \mathcal{A} \bar{U}_{i,0} \oplus \mathcal{A} \bar{U}_{i,1} \).

For our later use, with \( i \in I \) fixed once for all, we need to keep track of the precise value \( \langle h_i, \lambda \rangle \) in an idempotent \( 1_\lambda \) but do not need to know which specific weights \( \lambda \) are used. Thus it is convenient to introduce the following generic notation
\[
1^*_m = 1^*_{i,m}, \quad \text{for } m \in \mathbb{Z},
\]
to denote an idempotent $1_\lambda$ for some $\lambda \in X$ such that $m = \langle h_i, \lambda \rangle$. In this notation, the
identities in [Lu93, 23.1.3] can be written as follows: for any $m \in \mathbb{Z}$, $a, b \in \mathbb{Z}_{\geq 0}$, and $i \neq j \in I$,
\begin{equation}
E_i^{(a)} 1_{i,m}^{*} = 1_{i,m+2a} E_i^{(a)}; \quad F_i^{(a)} 1_{i,m}^{*} = 1_{i,m-2a} F_i^{(a)}; \tag{2.12}
\end{equation}
\begin{equation}
E_j 1_{i,m}^{*} = 1_{i,m+a_j} E_j; \quad F_j 1_{i,m}^{*} = 1_{i,m-a_j} F_j; \tag{2.13}
\end{equation}
\begin{equation}
F_i^{(a)} E_i^{(b)} 1_{i,m}^{*} = \min\{a,b\} \sum_{j=0}^{\min\{a,b\}} \frac{a-b-m}{j} q_i \langle h_k, \tau_i \rangle E_i^{(b-j)} F_i^{(a-j)} 1_{i,m}^{*}; \tag{2.14}
\end{equation}
\begin{equation}
E_i^{(a)} F_i^{(b)} 1_{i,m}^{*} = \min\{a,b\} \sum_{j=0}^{\min\{a,b\}} \frac{a-b+m}{j} q_i \langle h_k, \tau_i \rangle E_i^{(b-j)} F_i^{(a-j)} 1_{i,m}^{*}. \tag{2.15}
\end{equation}

From now on, we shall always drop the index $i$ to write the idempotents as $1_{m}^{*}$.

Remark 2.1. If $u \in U$ satisfies $u 1_{2k-1}^{*} = 0$ for all possible idempotents $1_{2k-1}^{*}$ with $k \in \mathbb{Z}$ (or respectively, $u 1_{2k}^{*} = 0$ for all possible $1_{2k}^{*}$ with $k \in \mathbb{Z}$), then $u = 0$.

2.3. The quantum group $U^\tau$. Let $(Y, X, \langle \cdot, \cdot \rangle, \cdots)$ be a root datum of type $(I, \cdot)$. We call a permutation $\tau$ of the set $I$ an involution of the Cartan datum $(I, \cdot)$ if $\tau^2 = \text{id}$ and $\tau i \cdot \tau j = i \cdot j$ for $i, j \in I$. Note we allow $\tau = \text{id}$. We shall always assume that $\tau$ extends to an involution on $X$ and an involution on $Y$ (also denoted by $\tau$), respectively, such that the perfect bilinear pairing is invariant under the involution $\tau$. The permutation $\tau$ of $I$ induces an $\mathbb{k}(q)$-algebra automorphism of $U$, defined by
\begin{equation}
\tau : E_i \mapsto E_{\tau i}, \quad F_i \mapsto F_{\tau i}, \quad K_h \mapsto K_{\tau h}, \quad \forall i \in I, h \in Y. \tag{2.16}
\end{equation}

Define
\begin{equation}
Y^\tau = \{h \in Y \mid \tau(h) = -h\}. \tag{2.17}
\end{equation}

In this paper we will only consider a subclass of quantum symmetric pairs defined in [Le99, Ko14] (which correspond to Satake diagrams without black nodes).

Definition 2.2 ([Le99, BaB10, Ko14]). The quasi-split quantum group, denoted by $U_{\zeta, \kappa}^\tau$ or $U^\tau$, is the $\mathbb{k}(q)$-subalgebra of $U$ generated by
\begin{equation}
B_i := F_i + \zeta_i E_{\tau i} K_i^{-1} + \kappa_i K_i^{-1} (i \in I), \quad K_\mu (\mu \in Y^\tau). \tag{2.18}
\end{equation}

Here the parameters
\begin{equation}
\zeta = (\zeta_i)_{i \in I} \in (\mathbb{k}(q)^\times)^I, \quad \kappa = (\kappa_i)_{i \in I} \in \mathbb{k}(q)^I \tag{2.19}
\end{equation}
are assumed to satisfy Conditions (2.20)–(2.21) below:
\begin{equation}
\kappa_i = 0 \text{ unless } \tau i = i \text{ and } \langle h_k, \alpha_i \rangle \in 2\mathbb{Z} \forall k = \tau(k); \tag{2.20}
\end{equation}
\begin{equation}
\zeta_i = \zeta_{\tau i} \text{ if } a_{i,\tau i} = 0. \tag{2.21}
\end{equation}

The conditions on the parameters ensure that $U^\tau$ has the expected size.

- The pair $(U, U^\tau)$ forms a quantum symmetric pair (QSP) [Le99, Ko14], as its $q \mapsto 1$ limit is the classical symmetric pair and $U^\tau$ is a (right) coideal subalgebra of $U$, i.e., $\Delta : U^\tau \rightarrow U^\tau \otimes U$.
- The quantum group $U^\tau$ is also called a quantum symmetric pair coideal subalgebra in some papers.
We refer to this subclass of QSP \((\mathbf{U}, \mathbf{U}')\) or \(\mathbf{U}'\) in Definition 2.2 as quasi-split, which correspond to Satake diagrams without black nodes.

We call QSP \((\mathbf{U}, \mathbf{U}')\) or \(\mathbf{U}'\) above split if in addition \(\tau = \text{id}\), borrowing terminologies from the literature of real groups. They are also known as generalized \(q\)-Onsager algebras, cf. [BaB10]. In case when \(\mathbf{U}\) is the quantum affine \(\mathfrak{sl}_2\), \(\mathbf{U}'\) was known as \(q\)-Onsager algebras; cf. [T93, BaK05]. Note that split \(\mathbf{U}'\) is generated only by \(B_i\) \((i \in I)\).

3. A Serre presentation of \(\mathbf{U}'\) and a \(q\)-binomial identity

3.1. \textbf{Divided powers.} Assume that a root datum \((\mathbf{Y}, \mathbf{X}, \langle \cdot, \cdot \rangle, \cdots )\) is given. Let \(\mathbf{U}' = \mathbf{U}'_{\varsigma, \kappa}\) be an quantum group with parameters \((\varsigma, \kappa)\); cf. §2.3.

For \(i \in I\) with \(\tau i \neq i\), imitating Lusztig’s divided powers, we define the divided power of \(B_i\) to be (cf. [BW18a, (2.2)])

\[
B_i^{(m)} := \frac{B_i^m}{[m]_{q_i}}, \quad \forall m \geq 0, \quad (\text{if } i \neq \tau i).
\]

(3.1)

For \(i \in I\) with \(\tau i = i\), generalizing [BW18a], we define the divided powers of \(B_i\) to be

\[
B_i^{(m)} = 1 \quad \text{if } m = 2k + 1,
\]

\[
B_i^{(m)} = \prod_{j=1}^{k} (B_i^2 - q_i \varsigma_i [2j - 1]_{q_i}^2) \quad \text{if } m = 2k;
\]

(3.2)

\[
B_i^{(m)} = 1 \quad \text{if } m = 2k + 1,
\]

\[
B_i^{(m)} = \prod_{j=1}^{k} (B_i^2 - q_i \varsigma_i [2j - 2]_{q_i}^2) \quad \text{if } m = 2k.
\]

(3.3)

In case when the parameter \(\varsigma_i = q_i^{-1}\), the formulas (3.2)–(3.3) first appeared in [BW18a, Conjecture 4.13] (where \(\kappa_i = 1\)) and were then studied in depth in [BeW18] (where \(\kappa_i = 0, 1\)). We shall see the formulas (3.2)–(3.3) for general parameter \(\varsigma_i\) arise from some rescaling isomorphism.

3.2. \textbf{A Serre presentation of }\(\mathbf{U}'\). Denote

\[ (a; x)_0 = 1, \quad (a; x)_n = (1 - a)(1 - ax) \cdots (1 - ax^{n-1}), \quad \forall n \geq 1. \]

Now we state our first main result. Let us fix \(\overline{\mathbf{p}}_i \in \mathbb{Z}_2\) for each \(i \in I\).
**Theorem 3.1.** The $\mathbb{K}(q)$-algebra $U^i_{\kappa}$ has a presentation with generators $B_i$ ($i \in I$), $K_{\mu}$ ($\mu \in Y^i$) and the relations (3.4)–(3.9) below: for $\mu, \mu' \in Y^i$ and $i \neq j \in I$,

\begin{align}
(3.4) & \quad K_{\mu}K_{-\mu} = 1, \quad K_{\mu}K_{\mu'} = K_{\mu+\mu'}, \\
(3.5) & \quad K_{\mu}B_i - q_i^{-\langle \mu, \alpha_i \rangle} B_i K_{\mu} = 0, \\
(3.6) & \quad B_i B_j - B_j B_i = 0, \quad \text{if } a_{ij} = 0 \text{ and } \tau i \neq j, \\
(3.7) & \quad \sum_{n=0}^{1-a_{ij}} (-1)^n B_i^{(n)} B_j B_i^{(1-a_{ij}-n)} = 0, \quad \text{if } j \neq \tau i \neq i, \\
(3.8) & \quad \sum_{n=0}^{1-a_{i,\tau i}} (-1)^{n+a_{i,\tau i}} B_i^{(n)} B_{\tau i} B_i^{(1-a_{i,\tau i}-n)} = \frac{1}{q_i - q_i^{-1}} \\
& \quad \cdot \left( q_i^{a_{i,\tau i}} (q_i^{-2}; q_i^{-2})_{-a_{i,\tau i}} \bar{K}_i \bar{K}_i^{-1} - (q_i^2; q_i^2)_{-a_{i,\tau i}} s_i B_i^{(-a_{i,\tau i})} \bar{K}_i \bar{K}_i^{-1} \right), \quad \text{if } \tau i \neq i, \\
(3.9) & \quad \sum_{n=0}^{1-a_{ij}} (-1)^n B_i^{(n)} B_j B_i^{(1-a_{ij}-n)} = 0, \quad \text{if } \tau i = i.
\end{align}

(This presentation will be called a Serre presentation of $U^i$.)

A proof of Theorem 3.1 will be presented at the end of this section, §3.6.

**Remark 3.1.** There is a presentation of $U^i$ in [Le03, Theorem 7.4] for finite type and a less precise one in [Ko14, Theorem 7.1] for Kac-Moody type, where relations (3.8)-(3.9) were replaced by some implicit identities of the form

\[\sum_{n=0}^{1-a_{ij}} (-1)^n \left(1 - \frac{a_{ij}}{n}\right) B_i^n B_j B_i^{1-a_{ij}-n} = C_{ij}\]

where $C_{ij}$ are some suitable unspecified lower terms, for $j \neq i \in I$ with $\tau i = i$; see [BK15, Theorem 3.6] for an update, which establishes the explicit relation (3.8). It follows by [Ko14, Theorem 7.1] that a presentation of $U^i$ in terms of $B_i$ and $K_{\mu}$ is independent of the parameters $\kappa_i$ ($i \in I$).

The $i$Serre relation (3.9) is a main novelty of this paper.

**Remark 3.2.** Under the assumption

\[a_{ij} \in \{0, -1, -2, -3\}, \quad \forall i \neq j \in I,\]

a Serre presentation of $U^i$ of Kac-Moody type has been given in [Ko14, Theorems 7.4, 7.8] (for $a_{ij} \in \{0, -1, -2\}$) and [BK15, Theorem 3.7] (for $a_{ij} = -3$), where the $i$Serre relation (3.9) was replaced by the following relations (some sign typos in [BK15, Theorem 3.7] when
\[a_{ij} = -3\] are corrected here):

\[
\sum_{n=0}^{1-a_{ij}} (-1)^n \left[ \frac{1 - a_{ij}}{n} \right] q_i^{n} B_j B_i^{1-a_{ij}-n}
\]

\[
\begin{cases}
q_i s_i B_j, & \text{if } a_{ij} = -1; \\
-2q_i^2 q_i s_i (B_j B_i - B_i B_j), & \text{if } a_{ij} = -2; \\
-2q_i ([2]_{q_i} [4]_{q_i} + q_i^2 + q_i^{-2}) q_i s_i B_j B_i \\
+ ([3]_{q_i}^2 + 1) q_i s_i (B_i^2 B_j + B_j B_i^2) - [3]_{q_i}^2 (q_i s_i)^2 B_j, & \text{if } a_{ij} = -3.
\end{cases}
\]

We leave it to the reader to convert these complicated formulas to (3.9), in 2 different forms with \(\bar{p}_i \in \{0, 1\}\), hence verifying Corollary 3.2 below directly in these cases.

For \(a_{ij} = -4\), the \(\iota\)Serre relation (3.9) can be converted to

\[
\sum_{n=0}^{5} (-1)^n \left[ \frac{5}{n} \right] q_i^{n} B_i^n B_j B_i^{5-n} = -[2]_{q_i}^2 (1 + [2]_{q_i}^2) q_i s_i (B_i^3 B_j - B_j B_i^3)
\]

\[+ [2]_{q_i}^2 [5]_{q_i} [3]_{q_i} q_i s_i (B_i^2 B_j B_i - B_i B_j B_i^2)
\]

\[+ [2]_{q_i}^2 [4]_{q_i} (q_i s_i)^2 (B_i B_j - B_j B_i).
\]

This formula is compatible with [BaB10, (2.1)], if the scalars \(\rho_{ij}^0, \rho_{ij}^1\) therein are chosen to be \(\rho_{ij}^0 = -2q_i^2 (1 + [2]_{q_i}^2) q_i s_i\), and \(\rho_{ij}^1 = [2]_{q_i}^2 [4]_{q_i} (q_i s_i)^2\).

**Remark 3.3.** Let \(g\) denote the Kac-Moody algebra associated to the generalized Cartan matrix \(A = (a_{ij})\). Theorem 3.1 specializes at \(q = 1\) to a presentation of the fixed point Lie subalgebra \(g^{\omega}\) of \(g\), where \(\omega\) is the Chevalley involution and \(\tau\) is a Dynkin diagram involution. When \(\tau = 1\), a presentation for \(g^{\omega}\) (called a generalized Onsager algebra) was independently obtained in a recent paper [St18], where the \(\iota\)Serre relations are given by some recursive formulas. One verifies directly that the above formulas for \(|a_{ij}| = 3, 4\) specialize at \(q = 1\) to [St18, (2.10)]. These Serre-type formulas (at \(q = 1\)) must be a priori compatible by a uniqueness argument similar to the proof of Corollary 3.2 below.

**Remark 3.4.** The new \(\iota\)Serre relations (3.9) remain valid for general \(\iota\)quantum groups which are not quasi-split. More precisely they are valid under the assumption \(\tau_i = i, w_i \iota = i\), but allowing possibly \(w_i \iota \neq i\); in this case, \(B_j = F_j + \zeta_{T_w E_{rj}} K_j^{-1} + \kappa_i K_j^{-1};\) see [BW18b, Definition 3.5] for notations. See Remark 4.1 for an outline of a proof.

**Corollary 3.2.** For \(j \neq i \in I\) with \(\tau_i = i\), we have

\[
\sum_{n=0}^{1-a_{ij}} (-1)^n B_i^{(n)} B_j B_i^{1-a_{ij}-n} = \sum_{n=0}^{1-a_{ij}} (-1)^n B_i^{(n)} B_j B_i^{1-a_{ij}-n}
\]

as polynomials in non-commutative variables \(B_i\) and \(B_j\).

**Proof.** Let us denote the LHS and RHS of (3.13) by \(S_{ij,0}\) and \(S_{ij,1}\), respectively. Note as polynomials in \(B_i, B_j, [1 - a_{ij}]_{q_i} S_{ij,p}\) (for \(p = 0, 1\)) is of the form \(S_{ij}(B_i, B_j) - C_{ij,p}\) (see (2.2), (3.2)–(3.3) for notations), where \(C_{ij,p}\) are some polynomials in \(B_i, B_j\) over \(U^{i,0}\) of degree lower than \(\deg S_{ij}(B_i, B_j)\); here \(U^{i,0}\) denotes the \(K(q)\)-subalgebra of \(U^i\) generated by \(K_\mu (\mu \in \Lambda^i).\)
Hence \([1 - a_{ij}]^{[p]}(S_{ij,0} - S_{ij,1}) = C_{ij,0} - C_{ij,1}\) is a polynomial in \(B_i, B_j\) over \(U^{r,0}\) of degree \(< 2 - a_{ij}\). By Theorem 3.1, \(S_{ij,p} = 0\), for \(p = 0, 1\), are relations in \(U^r\), and so is \(C_{ij,0} - C_{ij,1} = 0\). Recall from [Ko14, §7] that \(U^r\) has a filtration (roughly speaking by regarding \(F_i\) as the highest term of \(B_i\) (2.18)), whose associated graded is \(U^-\) (over \(U^{r,0}\)). If \(C_{ij,0} - C_{ij,1}\) were a nonzero polynomial in \(B_i, B_j\), then the relation \(C_{ij,0} - C_{ij,1} = 0\) in \(U^r\) would descend to a nontrivial relation in the associated graded between \(F_i, F_j\) of degree below \(\deg S_{ij}(F_i, F_j)\), a contradiction. So as a polynomial in \(B_i, B_j\) we have \(C_{ij,0} = C_{ij,1}\), and hence, \(S_{ij,0} = S_{ij,1}\). □

Recall a quasi-split quantum group \(U^r\) is split if \(\tau = \text{id}\). The Serre presentation for split \(U^r\) takes an especially simple form, which we record here.

**Theorem 3.3.** Fix \(\overline{p}_i \in \mathbb{Z}_2\), for each \(i \in I\). Then the split quantum group \(U^r\) has a Serre presentation with generators \(B_i \ (i \in I)\) and relations

\[
\sum_{n=0}^{1-a_{ij}} (-1)^n B_i^{(n)} B_j^{(1-a_{ij}-n)} = 0.
\]

Moreover, \(U^r\) admits a \(\mathbb{K}(q)\)-algebra anti-involution \(\sigma\) which sends \(B_i \mapsto B_i\) for all \(i \in I\).

**Proof.** Follows from Theorem 3.1 by noting that \(Y^r = \emptyset\) and \(\tau i = i\) for all \(i \in I\). □

### 3.3 Change of parameters.

By [Ko14, Theorem 7.1] (also cf. Remark 3.1), a presentation of the \(\mathbb{K}(q)\)-algebra \(U^r_{\varsigma,\kappa}\) is independent of the parameters \(\kappa_i\). It is also well known that the \(\mathbb{K}(q)\)-algebra \(U^r_{\varsigma,\kappa}\) (up to some field extension) is isomorphic to \(U^r_{\varsigma,0}\) for some distinguished parameters \(\varsigma^\circ\), i.e., \(\varsigma^\circ = q_i^{-1}\) for all \(i \in I\) such that \(\tau i = i\) (cf. [Le02], [Ko14, Proposition 9.2]). Let us formulate this precisely for later use.

For given parameters \(\varsigma\) satisfy (2.21), let \(\varsigma^\circ\) be the associated distinguished parameters such that \(\varsigma^\circ_i = \varsigma_i\) if \(\tau i \neq i\), and

\[
\varsigma^\circ_i = q_i^{-1}, \quad \text{if} \ \tau i = i.
\]

Let \(U^r_{\varsigma^\circ,0}\) be the quantum group with the parameters \(\varsigma^\circ\) and \(\kappa_i = 0\) for all \(i \in I\). Let \(\mathbb{F} = \mathbb{K}(q)(a_i \mid i \in I\text{ such that } \tau i = i)\) be a field extension of \(\mathbb{K}(q)\), where

\[
a_i = \sqrt{q_i s_i}, \quad \forall i \in I \text{ such that } \tau i = i.
\]

Denote by \(\mathbb{F} U^r_{\varsigma^\circ,\kappa} = \mathbb{F} \otimes_{\mathbb{K}(q)} U^r_{\varsigma,\kappa}\) the \(\mathbb{F}\)-algebra obtained by a base change.

**Proposition 3.4.** There exists an isomorphism of \(\mathbb{F}\)-algebras

\[
\phi_i : \mathbb{F} U^r_{\varsigma^\circ,0} \rightarrow \mathbb{F} U^r_{\varsigma,\kappa},
\]

\[
B_i \mapsto \begin{cases} B_i, & \text{if } \tau i \neq i; \\ a_i^{-1} B_i, & \text{if } \tau i = i; \end{cases} \quad K_\mu \mapsto K_\mu, \quad (\forall i \in I, \mu \in Y^r).
\]

**Proof.** Note that \(B_i\) in \(U^r_{\varsigma^\circ,0}\) and \(U^r_{\varsigma,\kappa}\) have different expressions under their respective embeddings into \(U^r\).

We consider the following (rescaling) automorphism of the \(\mathbb{F}\)-algebra \(\mathbb{F} U := \mathbb{F} \otimes_{\mathbb{K}(q)} U\) such that

\[
\phi_u : \mathbb{F} U \rightarrow \mathbb{F} U,
\]

\[
E_i \mapsto \begin{cases} E_i, & \text{if } \tau i \neq i; \\ a_i E_i, & \text{if } \tau i = i; \end{cases} \quad F_i \mapsto \begin{cases} F_i, & \text{if } \tau i \neq i; \\ a_i^{-1} F_i, & \text{if } \tau i = i; \end{cases} \quad K_\mu \mapsto K_\mu, \quad (\forall i \in I, \mu \in Y).}
\]
A direct computation shows that the automorphism $\phi_u$ on $\mathcal{F}U$ restricts to an $\mathcal{F}$-algebra isomorphism

$$\phi_1: \mathcal{F}U_{\varsigma,0} \rightarrow \mathcal{F}U_{\varsigma,0},$$

$$B_i \mapsto \left\{ \begin{array}{ll}
B_i, & \text{if } \tau_i \neq i; \\
\frac{a_i^{-1}}{a_i}B_i, & \text{if } \tau_i = i;
\end{array} \right.
K_\mu \mapsto K_\mu, \quad (\forall i \in I, \mu \in Y^*).$$

By [Ko14, Theorem 7.1], there is an $\mathcal{F}$-algebra isomorphism $\mathcal{F}U_{\varsigma,0} \cong \mathcal{F}U_{\varsigma,\kappa}$ which matches the corresponding generators $B_i, K_\mu$.

The isomorphism in the proposition follows by composing these two isomorphisms. □

3.4. Reduction to a $q$-binomial identity. For

$$(3.17) \quad w \in \mathbb{Z}, \quad u, \ell \in \mathbb{Z}_{\geq 0}, \text{ with } u, \ell \text{ not both 0},$$

we define

$$(3.18) \quad T(w, u, \ell)$$

$$= \sum_{c,e,r \geq 0} \sum_{t=0}^\ell \frac{q^{-t(\ell+u-1)+(\ell+u)(c-e)}}{c^e + r + u} \left[ \begin{array}{c} \ell \\
q^t \end{array} \right] q \left[ \begin{array}{c} w + t - \ell \\
r \end{array} \right] q \left[ \begin{array}{c} u - 1 + \frac{w+t-r}{2} \\
c \end{array} \right] q^2 \left[ \begin{array}{c} \frac{w+t-r}{2} - \ell \\
e \end{array} \right] q^2$$

$$- \sum_{c,e,r \geq 0} \sum_{t=0}^\ell \frac{q^{-t(\ell+u-1)+(\ell+u)(c-e)}}{c^e + r + u} \left[ \begin{array}{c} \ell \\
q^t \end{array} \right] q \left[ \begin{array}{c} w + t - \ell \\
r \end{array} \right] q \left[ \begin{array}{c} u + \frac{w+t-r-1}{2} \\
c \end{array} \right] q^2 \left[ \begin{array}{c} \frac{w+t-r-1}{2} - \ell \\
e \end{array} \right] q^2.$$

Proposition 3.5. If $T(w, u, \ell) = 0$ for all integers $w, u, \ell$ as in (3.17), then the $\iota$Serre relations (3.9) hold in the $\iota$quantum group $U_{\varsigma,0}^\iota$.

The proof of Proposition 3.5 will be given in Section 4 and Appendix A.

3.5. A $q$-binomial identity. The following is another main result of this paper, which will be generalized and proved in Section 5.

Theorem 3.6. The identity $T(w, u, \ell) = 0$ holds, for all integers $w, u, \ell$ as in (3.17).

3.6. Proof of Theorem 3.1. We assume the validity of Proposition 3.5 and Theorem 3.6.

First, we consider the $\iota$quantum group with distinguished parameters $\varsigma^\iota, U_{\varsigma,0}^\iota$. By the earlier works [Le02, Ko14, BK15] as explained in Remark 3.1, it remains to prove the $\iota$Serre relations (3.9) (with distinguished parameters $\varsigma^\iota$). Indeed, the $\iota$Serre relations (3.9) follow by combining Proposition 3.5 and Theorem 3.6.

By a direct computation, the $\iota$Serre relations for $U_{\varsigma,0}^\iota$ is transformed into the $\iota$Serre relations (3.9) for $U_{\varsigma,\kappa}^\iota$ with general parameters by the isomorphism $\phi$ in Proposition 3.4. Applying Remark 3.1 again, we have completed the proof of Theorem 3.1. □
3.7. Bar involution on $U^{i}$.

**Proposition 3.7.** Assume the parameters $\varsigma_{i}$, for $i \in I$, satisfy the conditions (a)-(c):

(a) $\varsigma_{i}q_{i} = \varsigma_{i}$, if $\tau i = i$ and $a_{ij} \neq 0$ for some $j \in I \setminus \{i\}$;

(b) $\varsigma_{i} = \varsigma_{i}$, if $\tau i \neq i$ and $a_{ij,\tau i} = 0$;

(c) $\varsigma_{i}q_{i} = q_{i}^{-a_{i,\tau i}}\varsigma_{i}$, if $\tau i \neq i$ and $a_{i,\tau i} \neq 0$.

Then there exists a $K$-algebra automorphism $\overline{\cdot}$: $U^{i} \to U^{i}$ (called a bar involution) such that

$$ \overline{q} = q^{-1}, \quad \overline{K}_{\mu} = K_{\mu}^{-1}, \quad \overline{B}_{i} = B_{i}, \quad \forall \mu \in Y^{i}, i \in I. $$

**Proof.** Under the assumptions, the divided powers $B_{1}^{(n)}$ in (3.1) and $B_{1,\overline{\mu}}^{(n)}$, for $\overline{\mu} \in \mathbb{Z}_{2}$, in (3.2)-(3.3) are clearly bar invariant. If follows by inspection that all the explicit defining relations for $U^{i}$ in (3.4)-(3.9) are bar invariant. □

**Remark 3.5.** One could further check that Conditions (a)-(c) in Proposition 3.7 are necessary for the existence of the bar involution as well. Under the constraint (3.10) on the Cartan matrix $A = (a_{ij})$, Proposition 3.7 and the necessity of the conditions on parameters were known earlier in [BK15].

4. Reduction of $i$Serre relations to a $q$-identity

This section is devoted to a proof of Proposition 3.5. As observed above, by the isomorphism $\phi$ in Proposition 3.4, the $i$-Serre relations for $U^{i}_{\varsigma,0}$ with distinguished parameters $\varsigma^{i}$ is transformed into the $i$-Serre relations (3.9) for $U^{i}_{\varsigma^{i},k}$ with general parameters. Hence we can and shall work the $i$quantum groups with distinguished parameters $\varsigma^{i}$, $U^{i} = U^{i}_{\varsigma^{i},0}$, in this section on reduction of the $i$-Serre relations.

4.1. Reduction by equivalence.

**Lemma 4.1.** For any $i \in I$ such that $\tau i = i$ and each $\overline{\mu} \in \mathbb{Z}_{2}$, then the following 2 identities in $U$ are equivalent: for $j \neq i \in I$,

$$ \sum_{n=0}^{1-a_{ij}} (-1)^{n}B_{1,\overline{\mu} + j}^{(n)}F_{j}B_{1,\overline{\mu}}^{(1-a_{ij}-n)} = 0, $$

(4.1)

$$ \sum_{n=0}^{1-a_{ij}} (-1)^{n}B_{1,\overline{\mu} + j}^{(n)}E_{\tau j}K_{j}^{-1}B_{1,\overline{\mu}}^{(1-a_{ij}-n)} = 0. $$

(4.2)

**Proof.** Recall the involution $\varpi$ from Lemma 2.1 and the involution $\tau$ of $U$ from (2.16). Assume the identity (4.1) holds. By definition, we have $\tau \circ \varpi(F_{i} + q_{i}^{-1}E_{i}K_{i}^{-1}) = B_{i}$ as $\tau i = i$. It then follows by definition of the divided powers (3.2)-(3.3) that $\tau \circ \varpi(B_{i}^{(n)}|_{\overline{\mu}}) = B_{i}^{(n)}|_{\overline{\mu}}$ for any $n, \overline{\mu}$. Hence applying $\tau \circ \varpi$ to (4.1) gives us

$$ \sum_{n=0}^{1-a_{ij}} (-1)^{n}B_{1,\overline{\mu} + j}^{(n)}E_{\tau j}K_{\tau j}^{-1}B_{1,\overline{\mu}}^{(1-a_{ij}-n)} = 0. $$

(4.2)

Then (4.2) follows by multiplying the above identity on the right by $K_{\tau j}^{-1}B_{i} = B_{i}K_{\tau j}^{-1}K_{j}$. Similarly by applying $\tau \circ \varpi$ to the identity (4.2), we can show that (4.2) implies (4.1). □
Since we have $B_j = F_j + \varepsilon_j E_{\tau j} \tilde{K}_j^{-1}$, the $t$Serre relation (3.9) in $U^t$ follows from the identities (4.1)–(4.2), and it suffices to prove (4.1) by Lemma 4.1.

As (4.1) is a statement in a rank 2 quantum group, for simplicity of notations, we further set $i = 1$ and $j = 2$ in the remainder of this section.

Remark 4.1. When we deal with general $t$ quantum groups as in Remark 3.4, a variant of Lemma 4.1 remains valid when we use a variant of the identity (4.2) where $E_{\tau j}$ is replaced by $T_{w^*}(E_{\tau j})$. Hence in this case, the $t$Serre relation (3.9) in $U^t$ follows again from the identity (4.1) alone (which we establish in this paper).

4.2. Expansion formula of the $t$ divided powers. Fix $i = 1$ and $j = 2$. Recall from (2.11) the notation for the idempotents $1^*_\lambda$, for $\lambda \in \mathbb{Z}$. The following expansion formulas will play a crucial role in proving (4.1).

Lemma 4.2. [BeW18, Propositions 2.8, 3.5] For $m \geq 1$ and $\lambda \in \mathbb{Z}$, we have

\begin{align}
B^{(2m)}_{1,0} 1^{*}_{2\lambda} &= \sum_{c=0}^{m} \sum_{a=0}^{2m-2c} q_1^{2(a+c)(m-a-\lambda)-2ac-(2c+1)} \left[ m - c - a - \lambda \right]_{q_1^2} E^{(a)}_1 F^{(2m-2c-a)}_1 1^{*}_{2\lambda}, \\
B^{(2m-1)}_{1,0} 1^{*}_{2\lambda} &= \sum_{c=0}^{m-1} \sum_{a=0}^{2m-1-2c} q_1^{2(a+c)(m-a-\lambda)-2ac-a-(2c+1)} \left[ m - c - a - \lambda - 1 \right]_{q_1^2} E^{(a)}_1 F^{(2m-1-2c-a)}_1 1^{*}_{2\lambda}, \\
B^{(2m)}_{1,1} 1^{*}_{2\lambda-1} &= \sum_{c=0}^{m} \sum_{a=0}^{2m-2c} q_1^{2(a+c)(m-a-\lambda)-2ac+a-(2c)} \left[ m - c - a - \lambda \right]_{q_1^2} E^{(a)}_1 F^{(2m-2c-a)}_1 1^{*}_{2\lambda-1}, \\
B^{(2m+1)}_{1,1} 1^{*}_{2\lambda-1} &= \sum_{c=0}^{m} \sum_{a=0}^{2m+1-2c} q_1^{2(a+c)(m-a-\lambda)-2ac+2a-(2c)} \left[ m - c - a - \lambda + 1 \right]_{q_1^2} E^{(a)}_1 F^{(2m+1-2c-a)}_1 1^{*}_{2\lambda-1}.
\end{align}

In particular, we have $B^{(n)}_{1,0} 1^{*}_{2\lambda} \in \mathcal{A} \hat{U}_{1,0}$ and $B^{(n)}_{1,1} 1^{*}_{2\lambda-1} \in \mathcal{A} \hat{U}_{1,1}$ for all $n \in \mathbb{N}$. 
The identity (4.1) is equivalent to the following 4 relations (4.7)–(4.10):

\[(4.7) \quad \sum_{n=0}^{2m+1} (-1)^n B^{(n)}_{1,0} F_2 B^{(2m+1-n)}_{1,0} = 0, \quad \text{if} \quad a_{12} = 2m \in 2\mathbb{N};\]

\[(4.8) \quad \sum_{n=0}^{2m+1} (-1)^n B^{(n)}_{1,1} F_2 B^{(2m+1-n)}_{1,1} = 0, \quad \text{if} \quad a_{12} = 2m \in 2\mathbb{N};\]

\[(4.9) \quad \sum_{n=0}^{2m} (-1)^n B^{(n)}_{1,1} F_2 B^{(2m-n)}_{1,0} = 0, \quad \text{if} \quad a_{12} = 2m - 1 \in 2\mathbb{N} + 1;\]

\[(4.10) \quad \sum_{n=0}^{2m} (-1)^n B^{(n)}_{1,0} F_2 B^{(2m-n)}_{1,1} = 0, \quad \text{if} \quad a_{12} = 2m - 1 \in 2\mathbb{N} + 1.\]

The necessity of applying different formulas in Lemma 4.2 forces us to divide the proof of the identity (4.1) into the 4 cases (4.7)–(4.10).

In the remainder of this section, we reduce the proof of the identity (4.7) to the \(q\)-binomial identity \(T(w, u, \ell) = 0\) in Theorem 3.6; similar reductions of the other relations (4.8)–(4.10) to the same identity are given in Appendix A.

### 4.3. Computing \(\mathfrak{B}\)erre in \(\hat{\mathfrak{U}}\).

Let \(a_{12} = -2m\). We shall use (4.3)–(4.4) to rewrite the element

\[(4.11) \quad \sum_{n=0}^{2m+1} (-1)^n B^{(n)}_{1,0} F_2 B^{(2m+1-n)}_{1,0} 1_{2\lambda}^* \in \hat{\mathfrak{U}}\]

for any \(\lambda \in \mathbb{Z}\) in terms of monomial basis in \(E_1, F_1, F_2\).

**Case I: n is even.** It follows from (4.4) that

\[B^{(2m+1-n)}_{1,0} 1_{2\lambda}^* = \sum_{c=0}^{m-\frac{n}{2}} \sum_{a=0}^{2m+1-n-2c} q_1^{(a+c)(2m+2-n-2a-2\lambda)-2ac-a-c(2c+1)} \left[ m - \frac{n}{2} - c - a - \lambda \right] q_1^a E_1^{(a)} F_1^{(2m+1-n-2c-a)} 1_{2\lambda}^*.\]

Since \(a_{12} = -2m\), by (2.12)–(2.13) we have \(F_2 1_{\lambda}^* = 1_{\lambda+2m}^* F_2\), and hence

\[F_2 E_1^{(a)} F_1^{(2m+1-n-2c-a)} 1_{2\lambda}^* = 1_{2(\lambda+2a-n-1+2c)}^* F_2 E_1^{(a)} F_1^{(2m+1-n-2c-a)}.\]
Furthermore, by using (4.3), we have

\[
B_{1,0}^{(n)} 1_{2(\lambda+2a-m-1+n+2c)}^* = \sum_{e=0}^{\lceil \frac{n}{2} \rceil} \sum_{d=0}^{n-2e} q_1^{2(d+e)(\frac{n}{2} - d - \lambda - 2a + n - 2c - 2d - e(2e+1))}
\cdot \left[ \frac{n}{2} - e - d - \lambda - 2a - n - 2c + m + 1 \right] \frac{E_1^{(d)} F_1^{(n-2c-d)}}{q_1^e} 1_{2(\lambda+2a-m-1+n+2c)}^*
\]

Hence combining the above 3 computations gives us

\[
(4.12) \quad B_{1,0}^{(n)} F_2 B_{1,0}^{(2m+1-n)} 1_{2\lambda}^* = \sum_{e=0}^{\lceil \frac{n}{2} \rceil} \sum_{d=0}^{n-2e} \sum_{c=0}^{m+1-n-2c} \sum_{a=0}^{2m+1-n-2c} q_1^{(a+c+d+e)(2m+1-n-2\lambda-2a-2c-2d-2e)+d}
\cdot \left[ m + 1 - e - d - \lambda - 2a - \frac{n}{2} - 2c \right] \frac{E_1^{(d)} F_1^{(n-2c-d)} F_2 E_1^{(a)} F_1^{(2m+1-n-2c-a)}}{q_1^e} 1_{2\lambda}^*. 
\]

Next, we move the divided powers of \( E_1 \) in the middle to the left. Using (2.12)–(2.13) we have

\[
F_2 F_1^{(2m+1-n-2c-a)} 1_{2\lambda}^* = 1_{2(\lambda+n+2c+a-m-1)}^* F_2 F_1^{(2m+1-n-2c-a)}; 
\]

Using (2.14) we have

\[
F_1^{(n-2c-d)} E_1^{(a)} 1_{2(\lambda+n+2c+a-m-1)}^* = \sum_{r=0}^{\min(a,n-2c-d)} \left[ n - 2c - d - a - 2(\lambda + n + 2c + a - m - 1) \right] \frac{E_1^{(a-r)} F_1^{(n-2c-d-r)}}{q_1^r} 1_{2(\lambda+n+2c+a-m-1)}^* 
\]

\[
= \sum_{r=0}^{\min(a,n-2c-d)} \left[ 2m + 2 - 2c - d - 3a - 2\lambda - 4c - n \right] \frac{E_1^{(a-r)} F_1^{(n-2c-d-r)}}{q_1^r} 1_{2(\lambda+n+2c+a-m-1)}^* .
\]
Plugging these new formulas into (4.12), we obtain

\begin{align*}
(4.13) \quad \sum_{n=0,2|n}^{2m+1} B_{1,0}^{(n)} F_2 D_{1,0}^{(2m+1-n)} 1^*_{2\lambda} \\
= \sum_{n=0,2|n}^{2m+1} \sum_{c=0}^{m-\frac{n}{2}} \sum_{e=0}^{\frac{n}{2}} \sum_{a=0}^{2m+1-n-2c} \sum_{d=0}^{n-2e} \sum_{r=0}^{a \min \{a,n-2e-d\}} q_1^{(a+c+d+e)(2m+1-n-2\lambda-2a-2c-2d-2e)+d} \\
\cdot \left[ a + d - r \atop d \right] q_1^{2m + 2 - 2e - d - 3a - 2\lambda - 4c - n} \left[ m - \frac{n}{2} - c - a - \lambda \atop c \right] q_1^{m + 1 - e - d - \lambda - 2a - \frac{n}{2} - 2c} E_1^{(a+d-r)} F_1^{(n-2e-d-r)} F_2 F_1^{(2m+1-n-2c-a)} 1^*_{2\lambda}.
\end{align*}

Case II: \( n \) is odd. Similarly, by (4.3) we have

\begin{align*}
B_{1,0}^{(2m+1-n)} 1^*_{2\lambda} = \sum_{c=0}^{m+\frac{1}{2}} \sum_{a=0}^{2m+1-n-2c} q_1^{(a+c)(2m+1-n-2a-2\lambda)-2ac-c(2c+1)} \\
\cdot \left[ m + \frac{1-n}{2} - c - a - \lambda \atop c \right] q_1^E_1^{(a)} F_1^{(2m+1-n-2c-a)} 1^*_{2\lambda}.
\end{align*}

Using (4.4) we have

\begin{align*}
B_{1,0}^{(n)} 1^*_{2(\lambda+2a-m-1+n+2c)} \\
= \sum_{e=0}^{n-2e} \sum_{d=0}^{2(d+e)(\frac{n+1}{2}-d-\lambda-2a+m+1-n-2c)-2de-d-e(2e+1)} q_1^{2(d+e)(\frac{n+1}{2}-d-\lambda-2a+m+1-n-2c)-2de-d-e(2e+1)} \\
\cdot \left[ \frac{n+1}{2} - e - d - \lambda - 2a - n - 2c + m \atop e \right] q_1^E_1^{(d)} F_1^{(n-2e-d)} 1^*_{2(\lambda+2a-m-1+n+2c)} \\
= \sum_{e=0}^{n-2e} \sum_{d=0}^{2(d+e)(m+\frac{3}{2}-d-\lambda-2a-\frac{n}{2}-2c)-2de-d-e(2e+1)} q_1^{2(d+e)(m+\frac{3}{2}-d-\lambda-2a-\frac{n}{2}-2c)-2de-d-e(2e+1)} \\
\cdot \left[ m + \frac{1}{2} - e - d - \lambda - 2a - \frac{n}{2} - 2c \atop e \right] q_1^E_1^{(d)} F_1^{(n-2e-d)} 1^*_{2(\lambda+2a+n+2c-m-1)}.
\end{align*}
Combining the above two formulas and simplifying the resulting expression, we obtain the following equality:

\[(4.14)\]

\[
\sum_{n=1,2|n}^{2m+1} B_{1,0}^{(n)} F_2 B_{1,0}^{(2m+1-n)} 1_{2\lambda}^{*}
\]

\[
= \sum_{n=1,2|n}^{2m+1} \sum_{c=0}^{m} \sum_{e=0}^{2m+1-n-22} \sum_{a=0}^{n} \sum_{d=0}^{n} \sum_{r=0}^{n} q_1^{(a+c+d+e)(2m+n-2\lambda-2a-2c-2d-2e)} n a - 2c
\]

\[
\cdot \left[ a + d - r \right] \left[ m + 1 - \frac{m}{2} - 2c - d - \lambda - 2a - 2c \right] \left[ m + 1 - \frac{m}{2} - c - a - \lambda \right] q_1^2
\]

Therefore, by combining the computations \((4.13)-(4.14)\) which depend on the parity of \(n\) above, we obtain the following formula for \((4.11)\):

\[(4.15)\]

\[
\sum_{n=0}^{2m+1} (-1)^n B_{1,0}^{(n)} F_2 B_{1,0}^{(2m+1-n)} 1_{2\lambda}^{*} =
\]

\[
\sum_{n=0,2|n}^{2m} \sum_{c=0}^{m-n/2} \sum_{e=0}^{m-n-2c} \sum_{a=0}^{n} \sum_{d=0}^{n} \sum_{r=0}^{n} q_1^{(a+c+d+e)(2m+1-n-2\lambda-2a-2c-2d-2e)+d}
\]

\[
\cdot \left[ a + d - r \right] \left[ 2m + 2 - n - 2\lambda - 2e - d - 3a - 4c \right] \left[ m + 1 - \frac{m}{2} - \lambda - c - a \right] q_1^2
\]

\[
\cdot \left[ m + 1 - \frac{m}{2} - \lambda - e - d - 2a - 2c \right] F_1^{(a+d-r)} F_1^{(n-2\lambda-d-r)} F_2 F_1^{(2m+1-n-2c-a)} 1_{2\lambda}^{*}
\]

\[
- \sum_{n=1,2|n}^{2m+1} \sum_{c=0}^{m} \sum_{e=0}^{2m+1-n-22} \sum_{a=0}^{n} \sum_{d=0}^{n} \sum_{r=0}^{n} q_1^{(a+c+d+e)(2m+2-n-2\lambda-2a-2c-2d-2e)} - a - 2c
\]

\[
\cdot \left[ a + d - r \right] \left[ 2m + 2 - n - 2\lambda - 2e - d - 3a - 4c \right] \left[ m + 1 - \frac{m}{2} - \lambda - c - a \right] q_1^2
\]

\[
\cdot \left[ m + 1 - \frac{m}{2} - \lambda - e - d - 2a - 2c \right] F_1^{(a+d-r)} F_1^{(n-2\lambda-d-r)} F_2 F_1^{(2m+1-n-2c-a)} 1_{2\lambda}^{*}
\]

Observe the monomials on the right-hand side of the equation \((4.15)\) above are of the form \(E_1^{(\ell)} F_1^{(y)} F_2 F_1^{(2m+1-\ell-y-2u)} 1_{2\lambda}^{*}\), for \(\ell, y, u \in \mathbb{N}\), so let us change variables to allow us to collect the like terms together. Set

\[
\ell = a + d - r, \quad y = n - 2e - d - r.
\]
Noting that \( n + 2c + a - \ell - y \) is even, we can write
\[
n + 2c + a - \ell - y = 2u,
\]
for some \( u \in \mathbb{Z} \).

Then we have \( 2m + 1 - n - 2c - a = 2m + 1 - \ell - y - 2u, \ a = \ell + y + 2u - 2c - n, \)
d\( = n + c - e - u - y, \) and
\[
u = c + e + r.
\]

If \( u = 0 \) and \( \ell = 0, \) then \( e = c = r = a = d = 0. \) In this case, collecting the corresponding monomials in (4.15) together gives us \( \sum_{n=0}^{2m+1} (-1)^n F_1^{(n)} F_2 F_1^{(2m+1-n)} 1^*_{2\lambda}, \) which equals 0 by the \( q \)-Serre relation (2.8).

Now assume \( u, \ell \in \mathbb{N}, \) not both 0. Then the monomial \( E_1^{(\ell)} F_1^{(u)} F_2 F_1^{(2m+1-\ell-y-2u)} 1^*_{2\lambda} \) in (4.15) has coefficient given by \( q_1^{((\ell+u)(2m+1-2\lambda-2\ell-3u-y))} S(y, u, \ell, \lambda), \) where
\[
S(y, u, \ell, \lambda) := \sum_{n=0}^{2m} \sum_{c, e, r \geq 0} q_1^{(u+y-n)(\ell+u-1)+c-e}
\]
\[
\left( \begin{array}{c}
\ell \\
-u-y-e+c+n
\end{array} \right)_{q_1} \left( \begin{array}{c}
2m + 2 - 2\lambda - 5u - 3\ell - 2y - e + c + n \\
r
\end{array} \right)_{q_1}
\]
\[
\cdot \left( \begin{array}{c}
m - \lambda - 2u - \ell - y + c + \frac{n}{2} \\
c
\end{array} \right)_{q_1^2} \left( \begin{array}{c}
m + 1 - \lambda - 2\ell - y - 3u + \frac{n}{2} + c \\
e
\end{array} \right)_{q_1^2}
\]
\[
- \sum_{n=1,2m} q_1^{(u+y-n)(\ell+u-1)}
\]
\[
\left( \begin{array}{c}
\ell \\
-u-y-e+c+n
\end{array} \right)_{q_1} \left( \begin{array}{c}
2m + 2 - 2\lambda - 5u - 3\ell - 2y - e + c + n \\
r
\end{array} \right)_{q_1}
\]
\[
\cdot \left( \begin{array}{c}
m - \lambda - 2u - \ell - y + c + \frac{n+1}{2} \\
c
\end{array} \right)_{q_1^2} \left( \begin{array}{c}
m - \lambda - 2\ell - y - 3u + \frac{n+1}{2} + c \\
e
\end{array} \right)_{q_1^2}.
\]

Rewriting the identity (4.15) using (4.16) and its preceding discussions, we have proved the following.

**Proposition 4.3.** We have
\[
(4.17) \quad \sum_{n=0}^{2m+1} (-1)^n B_{1,0}^{(n)} F_2 B_{1,0}^{(2m+1-n)} 1^*_{2\lambda} = \sum_{\ell, y, u \geq 0, u \leq \ell+1} q_1^{(\ell+u)(2m+1-2\lambda-2\ell-3u-y)} S(y, u, \ell, \lambda) E_1^{(\ell)} F_1^{(u)} F_2 F_1^{(2m+1-\ell-y-2u)} 1^*_{2\lambda}.
\]

4.4. **Proof of Proposition 3.5.** Using new variables \( t := -u - y - e + c + n \) and \( w := 2m + 2 - 2\lambda - 2\ell - 4u - y, \) we have
\[
(4.18) \quad S(y, u, \ell, \lambda) = T(w, u, \ell)|_{q \rightarrow q_1},
\]
by a direct calculation; cf. (3.18) for notation \( T(w, u, \ell) \) and (4.16) for notation \( S(y, u, \ell, \lambda). \)
By assumption, \( T(w, u, \ell) = 0 \) for any \( w \in \mathbb{Z} \) and \( u, \ell \in \mathbb{N} \) with \( u, \ell \) not both 0. Hence by (4.18) we have \( S(y, u, \ell, \lambda) = 0 \), and then by (4.17) we obtain

\[
\sum_{n=0}^{2m+1} (-1)^n P_{1,0}^{(n)} F_2 B_{1,0}^{(2m+1-n)} 1_{2\lambda}^* = 0, \quad \forall \lambda \in \mathbb{Z}.
\]

Thanks to Remark 2.1, this proves the identity (4.7).

Similar reductions of the identities (4.8)–(4.10) to the \( q \)-binomial identity \( T(w, u, \ell) = 0 \) in Theorem 3.6 can be found in Appendix A.

Being equivalent to the 4 identities (4.7)–(4.10), the identity (4.1) follows. Then by the reduction in §4.1, the Serre relation (3.9) holds. Proposition 3.5 is proved. \( \square \)

5. A \( q \)-binomial identity and generalization

The section is devoted to a proof of Theorem 3.6. We will first generalize \( T(w, u, \ell) \) to a function \( G \) which involves several new variables, and then establish various recursive relations for \( G \) to show some generalized identities involving \( G \).

5.1. Function \( G \) and its recursions. For \( w, p_0, p_1, p_2 \in \mathbb{Z} \) and \( u, \ell \in \mathbb{Z}_{\geq 0} \), we define

\[
G(w, u, \ell; p_0, p_1, p_2) := (-1)^w q^{w^2 - wu + \ell u} \cdot \left\{ \sum_{c,e,r \geq 0} \sum_{t=0}^{\ell} q^{-t(\ell+u-1)-(u(c+e)+2c+r)p_0+2c+2ep_2} \cdot \left[ \frac{t}{q} \right] \left[ \frac{w+t+p_0}{r} \right] \left[ \frac{w+t+p_1}{2} \right] \left[ \frac{w+t+p_2}{2} \right] \cdot \sum_{c,e,r \geq 0} \sum_{t=0}^{\ell} q^{-t(\ell+u-1)-(u-1)(c+e)+r+p_0+2c+2ep_2} \cdot \left[ \frac{t}{q} \right] \left[ \frac{w+t+p_0}{r} \right] \left[ \frac{w+t+p_1}{2} \right] \left[ \frac{w+t+p_2}{2} \right] \right\}.
\]

The following relation between \( T \) and \( G \) follows by definitions in (3.18) and (5.1):

\[
T(w, u, \ell) = (-1)^w q^{wu-u^2} G(w, u, \ell; -\ell, u-1, -\ell).
\]
Lemma 5.1. For any $w, p_0, p_1, p_2, k \in \mathbb{Z}$ and $u, \ell \in \mathbb{Z}_{\geq 0}$, we have the following recursive relations:

\begin{align*}
(5.3) \quad G(w + 1, u, \ell; p_0, p_1, p_2) &= q^{-2u}G(w, u, \ell; p_0, p_1, p_2 + 1) - q^{2p_0 + \ell}G(w, u - 1, \ell; p_0, p_1, p_2); \\
(5.4) \quad G(w, u, \ell; p_0, p_1 + 1, p_2) &= G(w, u, \ell; p_0, p_1, p_2) + q^{4p_1 + \ell + 4}G(w, u - 1, \ell; p_0, p_1, p_2); \\
(5.5) \quad G(w, u, \ell; p_0, p_1, p_2 + 1) &= G(w, u, \ell; p_0, p_1, p_2) + q^{4p_2 + \ell + 2}G(w, u - 1, \ell; p_0, p_1, p_2); \\
(5.6) \quad G(w, u, \ell + 1; p_0, p_1, p_2) &= q^uG(w, u, \ell; p_0, p_1, p_2) - q^{u + 2}\ell G(w + 1, u, \ell; p_0, p_1, p_2); \\
(5.7) \quad G(w, u, \ell; p_0, p_1, p_2) &= q^{4k}\ell G(w + 2k, u, \ell; p_0 - 2k, p_1 - k, p_2 - k); \\
(5.8) \quad G(w + 1, u, \ell; p_0, p_1, p_2) &= q^{-2u}G(w, u, \ell; p_0 + 1, p_2, p_1 + 1).
\end{align*}

Proof. We provide a detailed argument for (5.3). Applying the $q$-binomial identity

\begin{equation}
(5.9) \quad \binom{m}{t}_q = q^{-t} \binom{m - 1}{t}_q + q^{m-t} \binom{m - 1}{t-1}_q
\end{equation}

to the second $q$-binomial in each summand of $G(w + 1, u, \ell; p_0, p_1, p_2)$ (obtained from (5.1) with $w \rightarrow w + 1$), we have

\[ G(w + 1, u, \ell; p_0, p_1, p_2) = S_1 + S_2. \]

Here

\begin{align*}
S_1 & \quad = \quad (-1)^{w+1}q^{w^2-(w+1)u+\ell u} \\
& \quad \left\{ \sum_{c,e,r \geq 0} \sum_{c+e+r = u} \sum_{t=0}^\ell q^{-t(\ell+u-1)-(u(c+e)+2r+2c+2p_0+2cp_1+2p_2)} \\
& \quad \cdot \sum_{c,e,r \geq 0} \sum_{c+e+r = u} \sum_{t=0}^\ell q^{-t(\ell+u-1)-(u-1)(c+e)+r+2p_0+2p_1+2p_2} \\
& \quad \cdot \binom{\ell}{t}_q \binom{w+t+p_0}{r}_q \binom{\ell}{t}_q + 1 \right\} q^{[\frac{w+t-r-1}{2}]} + p_1 + 1 \right\} q^2 \left[ \frac{w+t-r-1}{2} + p_2 \right] \right\}.
\end{align*}
and
\[
S_2 := (-1)^{w+1} q^{u^2-(w+1)u+\ell u}
\]
\[
= \left\{ \sum_{c, e, r \geq 0} \sum_{t=0}^\ell q^{-\ell(t+u-1)-u(c+e)+2c+rp_0+2ep_1+w+1+t+p_0-r} \cdot \left[ \binom{\ell}{t} \left[ \frac{w+t+p_0}{r-1} \right] q^{\left[ \frac{w+t-(r-1)}{2} \right] c} + p_1 \right] \right. 
\]
\[
- \left. \sum_{c, e, r \geq 0} \sum_{t=0}^\ell q^{-\ell(t+u-1)-(u-1)(c+e)+rp_0+2ep_1+w+1+t+p_0-r} \cdot \left[ \binom{\ell}{t} \left[ \frac{w+t+p_0}{r-1} \right] q^{\left[ \frac{w+t-(r-1)1}{2} \right] e} + p_2 \right] \right\}.
\]

By permuting the variables \(c, e\), we obtain
\[S_1 = q^{-2u} G(w, u, \ell; p_0, p_2, p_1 + 1).\]

By a change of variables \(r \mapsto r + 1\), we have
\[S_2 = -q^{2p_0+\ell} G(w, u - 1, \ell; p_0, p_1, p_2).
\]

Then (5.3) follows by summing up \(S_1\) and \(S_2\) above.

The recursions (5.4)–(5.5) are proved similarly to (5.3).

The identity (5.6) can be proved similarly by the following \(q\)-binomial identity:
\[
\left[ \binom{m}{t} \right] = q^{t} \left[ \binom{m-1}{t} \right] + q^{-m+t} \left[ \binom{m-1}{t-1} \right].
\]

Finally, (5.7)-(5.8) can be easily verified directly. \qed

5.2. Specializations of the function \(G\). For \(p_1, p_2 \in \mathbb{Z}\) and \(u \in \mathbb{Z}_{\geq 0}\), we define
\[
H(u; p_1, p_2) := \sum_{c, e \geq 0} q^{c+2cp_1+2ep_2} \left[ \begin{array}{c} p_1 \\ c \end{array} \right] \left[ \begin{array}{c} p_2 \\ e \end{array} \right] q^u.
\]

Note that \(H(u; p_1, p_2) = G(0, u, 0; 0, p_1, p_2)\), a specialization of \(G\) defined in (5.1).

**Lemma 5.2.** For any \(p_1, p_2 \in \mathbb{Z}\) and \(u \in \mathbb{Z}_{>0}\), we have
\[
H(u; p_2, p_1 + 1) = q^{2u}(H(u; p_1, p_2) + H(u - 1; p_1, p_2));
\]
\[
H(u; p_1 + 1, p_2) = H(u; p_1, p_2) + q^{4(p_1+1)} H(u - 1; p_1, p_2);
\]
\[
H(u; p_1, p_2 + 1) = H(u; p_1, p_2) + q^{4p_2+2} H(u - 1; p_1, p_2).
\]

**Proof.** Note that (5.12) follows from (5.10), and (5.13)–(5.14) follow from (5.9). \qed
Define 
\begin{align}
(5.15) \quad G_0(w, u; p_0, p_1, p_2) & := G(w, u, 0; p_0, p_1, p_2), \\
(5.16) \quad G_{00}(w, u; p_1, p_2) & := G(w, u, 0; 0, p_1, p_2).
\end{align}

Observe by definition that \( H(u; p_1, p_2) = G_{00}(0, u; p_1, p_2). \)

**Proposition 5.3.** For any \( w, p_1, p_2 \in \mathbb{Z} \) and \( u \in \mathbb{Z}_{\geq 0}, \) \( G_{00}(w, u; p_1, p_2) \) is independent of \( w; \) that is, \( G_{00}(w, u; p_1, p_2) = H(u; p_1, p_2). \)

**Proof.** First assume \( w \geq 0. \) We prove the identity by induction on \( w. \) For \( w = 0, \) the identity follows by definitions. Using (5.3), (5.12) and the induction hypothesis, we have by definition of \( G_{00} \) in (5.16) that
\[
G_{00}(w + 1, u; p_1, p_2) = q^{-2u}G_{00}(w, u; p_2, p_1 + 1) - G_{00}(w, u - 1; p_1, p_2)
= q^{-2u}H(u; p_2, p_1 + 1) - H(u - 1; p_1, p_2)
= H(u; p_1, p_2).
\]

Viewing \( p_1, p_2, u \) as fixed, we regard the identity in the proposition as an identity involving rational functions in 2 variables \( q, q^w. \) Since this identity holds for all \( w \geq 0, \) it must hold as a formal identity in the 2 variables, and hence as an identity in \( q, \) for arbitrary \( w \in \mathbb{Z}. \)

The following corollary is immediate by setting \( p_2 = 0 \) in Proposition 5.3. Recall the definition of \( G_{00} \) in (5.16).

**Corollary 5.4.** The identity \( G_{00}(w, u; p_1, 0) = H(u; p_1, 0) \) holds; that is,
\begin{equation}
(5.17) \quad (-1)^w q^{w^2-wu} \sum_{c + c + r = u \atop 2(w-r)} q^{-u(c+e)+2c+2c_1} \left( \begin{array}{c}
w \\ r \end{array} \right) \frac{w-r}{2} + p_1 \left( \begin{array}{c}
w-r/2 \\ e \end{array} \right) q^2 = \sum_{c + c + r = u \atop 2(w-r)} q^{-(u-1)(c+e)+2c_1} \left( \begin{array}{c}
w \\ r \end{array} \right) \left( 1 + \frac{w-r-1}{c} + p_1 \right) \left( \begin{array}{c}
w-r-1/2 \\ e \end{array} \right) q^2.
\end{equation}

Recall \( T(w, u, \ell) \) from (3.18).

**Corollary 5.5.** We have \( T(w, u, 0) = 0, \) for any \( w \in \mathbb{Z}, u \in \mathbb{Z}_{>0}. \)

**Proof.** It follow by the identity (5.2) that
\[
T(w, u, 0) = (-1)^w q^{wu-u^2}G_{00}(w, u; u - 1, 0) = (-1)^w q^{wu+u^2} \left( \begin{array}{c}
u-1 \\ u \end{array} \right) q^2 = 0,
\]
where the second equality above uses (5.17).

5.3. **A multi-variable identity.** We now prove the main result of this section.

**Theorem 5.6.** The following identity holds, for any \( w, p_0, p_1, p_2 \in \mathbb{Z}, \ell \in \mathbb{Z}_{>0} \) and \( u \in \mathbb{Z}_{\geq 0}: \)
\[
G(w, u, \ell; p_0, p_1, p_2) = 0.
\]
Proof. By the recursion (5.6) on \( \ell \), it suffices to prove the desired identity at \( \ell = 1 \). The proof is divided into the following two cases. Recall the definitions of \( G_0 \) in (5.15) and \( G_{00} \) in (5.16).

Case I: \( p_0 \) is even. From the recursive relations (5.6)–(5.8) and Proposition 5.3 we have:

\[
G(w, u, 1; p_0, p_1, p_2) = q^u G_0(w, u; p_0, p_1, p_2) - q^u G_0(w + 1, u; p_0, p_1, p_2)
\]

\[
= q^{u+2p_0u} G_{00}(w + p_0, u; p_1 - \frac{p_0}{2}, p_2 - \frac{p_0}{2}) - q^{u+2p_0u} G_{00}(w + p_0 + 1, u; p_1 - \frac{p_0}{2}, p_2 - \frac{p_0}{2})
\]

\[
= q^{u+2p_0u} H(u; p_1 - \frac{p_0}{2}, p_2 - \frac{p_0}{2}) - q^{u+2p_0u} H(u; p_1 - \frac{p_0}{2}, p_2 - \frac{p_0}{2}) = 0.
\]

Case II: \( p_0 \) is odd. Similarly, we have

\[
G(w, u, 1; p_0, p_1, p_2) = q^u G_0(w, u; p_0, p_1, p_2) - q^u G_0(w + 1, u; p_0, p_1, p_2)
\]

\[
= q^{u+2(p_0-1)u} G_0(w + p_0 - 1, u; 1, p_1 - \frac{p_0 - 1}{2}, p_2 - \frac{p_0 - 1}{2})
\]

\[
- q^{u+2(p_0-1)u} G_0(w + p_0 - 1, u; 1, p_1 - \frac{p_0 - 1}{2}, p_2 - \frac{p_0 - 1}{2})
\]

\[
= q^{2p_0u+u} G_{00}(w + p_0, u; p_2 - \frac{p_0 + 1}{2}, p_1 - \frac{p_0 - 1}{2})
\]

\[
- q^{2p_0u+u} G_{00}(w + p_0 + 1, u; p_2 - \frac{p_0 + 1}{2}, p_1 - \frac{p_0 - 1}{2})
\]

\[
= q^{2p_0u+u} (H(u; p_2 - \frac{p_0 + 1}{2}, p_1 - \frac{p_0 - 1}{2}) - H(u; p_2 - \frac{p_0 + 1}{2}, p_1 - \frac{p_0 - 1}{2})) = 0.
\]

The theorem is proved. \( \square \)

5.4. Proof of Theorem 3.6. Let \( w \in \mathbb{Z}, u \in \mathbb{N} \). It follows from the identity (5.2) and Theorem 5.6 that

\[
T(w, u, \ell) = (-1)^w q^{wu-u^2} G(w, u, \ell; -\ell, u - 1, -\ell) = 0, \quad \forall \ell \in \mathbb{Z}_{>0}.
\]

Together with \( T(w, u, 0) = 0 \) (for \( u > 0 \)) from Corollary 5.5, this proves Theorem 3.6. \( \square \)

Appendix A. More reductions

In this appendix, we provide details on the proofs of the identities (4.8)–(4.10), which are modeled on the proof of (4.7).

A.1. Proof of the identity (4.8). Recall \( a_{12} = -2m \). Thanks to Remark 2.1, in order to prove the identity (4.8), it suffices to prove

\[
(A.1) \quad \sum_{n=0}^{2m+1} (-1)^n B_{1,1}^{(n)} F_2 B_{1,1}^{(2m+1-n)} 1_{2\lambda-1} = 0, \quad \forall \lambda \in \mathbb{Z}.
\]
Similar to (4.15), we can show that

\[ \text{(A.2)} \]
\[
\sum_{n=0}^{2m+1} (-1)^n B_{1,1}^{(n)} F_2 B_{1,1}^{(2m+1-n)} I_{2\lambda-1}^* \\
= \left\{ \sum_{n=0,2|n} \sum_{c=0}^{m-n/2} \sum_{e=0}^{n/2} \sum_{a=0}^{2m+1-n-2c-n-2e \min\{a,n-2e-d\}} \sum_{d=0}^{2m+1-n-2c-n-2e \min\{a,n-2e-d\}} \sum_{r=0}^{a+c+d+e} q_1^{(a+c+d+e)(2m+2-n-2\lambda-2a-2c-2d-2e)-c+d+e}
\right. \\
\cdot \left[ a + d - r \right] q_1 \left[ 2m + 3 - n - 2\lambda - 2e - d - 3a - 4c \right] q_1 \left[ m - \frac{n}{2} - \lambda - c - a + 1 \right] q_1^2 \\
\cdot \left[ m + 1 - \frac{n}{2} - \lambda - e - d - 2a - 2c \right] e q_1^2 \\
\left. - \sum_{n=1,2|n} \sum_{c=0}^{m+1-n/2} \sum_{e=0}^{n/2} \sum_{a=0}^{2m+1-n-2c-n-2e \min\{a,n-2e-d\}} \sum_{d=0}^{2m+1-n-2c-n-2e \min\{a,n-2e-d\}} \sum_{r=0}^{a+c+d+e} q_1^{(a+c+d+e)(2m+2-n-2\lambda-2a-2c-2d-2e)+d}
\right. \\
\cdot \left[ a + d - r \right] q_1 \left[ 2m + 3 - n - 2\lambda - 2e - d - 3a - 4c \right] q_1 \left[ m + \frac{1-n}{2} - \lambda - c - a \right] q_1^2 \\
\cdot \left[ m + \frac{1-n}{2} - \lambda - e - d - 2a - 2c \right] e q_1^2 \\
\left. \right\} E_1^{(a+d-r)} F_1^{(n-2e-d-r)} F_2 F_1^{(2m+1-n-2c-a)} I_{2\lambda-1}^*.
\]

We introduce variables $\ell = a + d - r$, and $y = n - 2e - d - r$. As $n + 2c + a - \ell - y$ is even, set $n + 2c + a - \ell - y = 2u$ for $u \in \mathbb{Z}$. Then we have $a = \ell + y + 2u - 2c - n$, $d = n + c - e - u - y$, and $u = e + c + r$. Observe the monomials on the right-hand side of the equation (A.2) above are of the form $E_1^{(\ell)} F_1^{(y)} F_2 F_1^{(2m+1-\ell-y-2a)} I_{2\lambda-1}^*$, for $\ell, y, u \in \mathbb{N}$.

If $u = 0$ and $\ell = 0$, then $e = c = r = a = d = 0$, collecting the corresponding monomials in (A.2) gives us $\sum_{n=0}^{2m+1} (-1)^n F_1^{(n)} F_2 F_1^{(2m+1-n)} I_{2\lambda-1}^* = 0$, by the q-Serre relation (2.8).

If $u$ and $\ell$ are not both 0, the monomial $E_1^{(\ell)} F_1^{(y)} F_2 F_1^{(2m+1-\ell-y-2a)} I_{2\lambda-1}^*$ has coefficient given by $q_1^{(\ell+u)(2m+2-2\lambda-2\ell-3u-y)} S'(y, u, \ell, \lambda)$, where

\[ (A.3) \quad S'(y, u, \ell, \lambda) = \sum_{n=0,2|n} \sum_{c,e,r \geq 0} q_1^{(u+y-n)(\ell+u-1)} \]
A.1. Proof of the identity \((4.9)\). Let \(a_{12} = 1 - 2m\). To prove the identity \((4.9)\), it suffices to prove that

\[
(A.4) \quad \sum_{n=0}^{2m} (-1)^n B_{1,1}^{(n)} F_2 B_{1,0}^{(2m-n)} 1_{2\lambda}^* = 0, \quad \forall \lambda \in \mathbb{Z}.
\]
Similar to (4.15), by computations we obtain

\[ (A.5) \quad \sum_{n=0}^{2m} (-1)^n B_{1,1}^{(n)} F_2 B_{1,0}^{(2m-n)} 1_{2\lambda}^* = \sum_{n=0,2|n}^{2m} \sum_{c=0}^{m-n-2c} \sum_{e=0}^{a-n-2c} \sum_{a=0}^{2m-n-2c-n-2e} \sum_{d=0}^{c} q_1^{(a+c+d+e)(2m-n-2\lambda-2a-2c-2d-2e)-c+d+e} \]

\[ \cdot \left[ \frac{a+d-r}{d} q_1^{2m-2\lambda-3a-4c-d-2e-n+1} \right] q_1^{m-\frac{n}{2} - c - a - \lambda} \]

\[ \cdot \left[ m - \lambda - e - d - 2a - 2c - \frac{n}{2} \right] q_1^2 \]

\[ - \sum_{n=1,2|n}^{2m} \sum_{c=0}^{m-n-2c-2e} \sum_{e=0}^{a-n-2c} \sum_{a=0}^{2m-n-2c-n-2e} \sum_{d=0}^{c} q_1^{(a+c+d+e)(2m-n-2\lambda-2a-2c-2d-2e)+d} \]

\[ \cdot \left[ \frac{a+d-r}{d} q_1^{2m-2\lambda-3a-4c-d-2e-n+1} \right] q_1^{m+\frac{n}{2} - c - a - \lambda - 1} \]

\[ \cdot \left[ m - \lambda - e - d - 2a - 2c - \frac{n+1}{2} + 1 \right] q_1^2 \]

\[ \cdot \left\{ E_{1}^{(a+d-r)} F_1^{(n-2e-d-r)} F_2 F_1^{(2m-n-2\lambda-c)} 1_{2\lambda}^* \right\} \]

Introduce new variables \( \ell = a+d-r, y = n-2e-d-r, \) and \( 2u = n+2c+a-\ell-y. \) Then we have \( a = \ell+y+2u-2c-n, d = n+c-e-u-y, \) and \( r = u-e-c. \) Observe the monomials on the right-hand side of the equation (A.5) above are of the form \( E_{1}^{(\ell)} F_1^{(y)} F_2 F_1^{(2m-\ell-y-2u)} 1_{2\lambda}^* \), for \( \ell, y, u \in \mathbb{N}. \)

For \( u, \ell \in \mathbb{N}, \) not both 0, the monomial \( E_{1}^{(\ell)} F_1^{(y)} F_2 F_1^{(2m-\ell-y-2u)} 1_{2\lambda}^* \) has coefficient given by \( q_1^{(\ell+u)(2m-2\lambda-2\ell-3a-y)} S''(u, \ell, \lambda), \) where

\[ (A.6) \quad S''(y, u, \ell, \lambda) = \sum_{n=0,2|n}^{2m} \sum_{c,e,r \geq 0} q_1^{-(n+u+y)(\ell+u-1)} \]

\[ \cdot \left[ \frac{\ell}{c} q_1^{2m+1-2\lambda-5u-3\ell-2y-e+c+n} \right] q_1^{u-e-c} \]

\[ \cdot \left[ m - \lambda - \ell - y - 2u + c + \frac{n}{2} \right] q_1^2 \]

\[ \cdot \left[ m - \lambda - 2\ell - y - 3u + \frac{n+1}{2} + c \right] q_1^2 \]

\[ - \sum_{n=1,2|n}^{2m} \sum_{c,e,r \geq 0} q_1^{-(n+u+y)(\ell+u-1)+c-e} \]

\[ \cdot \left[ \frac{\ell}{c} q_1^{2m+1-2\lambda-5u-3\ell-2y-e+c+n} \right] q_1^{u-e-c} \]

\[ \cdot \left[ m - \lambda - \ell - y - 2u + c + \frac{n+1}{2} - 1 \right] q_1^2 \]

\[ \cdot \left[ m - \lambda - 2\ell - y - 3u + \frac{n+1}{2} + c \right] q_1^2 \].
Using new variables $t = -u - y - e + c + n$ and $w = 2m + 1 - 2\lambda - 2\ell - 4u - y$, we can show that $S''(y, u, \ell, \lambda) = -T(w, u, \ell)|_{q \to q_1}$ as defined in (3.18), and (A.4) follows. The identity (4.9) is proved.

A.3. Proof of the identity (4.10). Let $a_{12} = 1 - 2m$. To prove the identity (4.10), it suffices to prove that

(A.7) \[ \sum_{n=0}^{2m} (-1)^n B^{(n)}_{1,0} F_2 B^{(2m-n)}_{1,1} 1^*_{2\lambda-1} = 0, \quad \forall \lambda \in \mathbb{Z}. \]

Similar to (4.15), by computations we can show

(A.8) \[ \sum_{n=0}^{2m} (-1)^n B^{(n)}_{1,0} F_2 B^{(2m-n)}_{1,1} 1^*_{2\lambda-1} \]

\[ = \left\{ \sum_{n=0,2|n} \sum_{c=0}^{m-\frac{n}{2}} \sum_{e=0}^{\frac{n}{2}} \sum_{a=0}^{2m-n-2c-n-2e \min\{a,n-2e-d\}} \sum_{d=0}^{e} \sum_{r=0}^{d} q_1^{(a+c+d+e)(2m+1-n-2\lambda-2a-2c-2d-2e)+d} \right. \]

\[ \cdot \left[ \frac{a + d - r}{d} \right] q_1 \left[ \frac{2m + 2 - 2\lambda - 3a - 4c - d - 2e - n}{r} \right] q_1 \left[ \frac{m - \frac{n}{2} - c - a - \lambda}{c} \right] q_1^2 \]

\[ \cdot \left[ \frac{m + 1 - \lambda - e - d - 2a - 2c - \frac{n-1}{2}}{q_1^2} \right] \]

\[ - \left\{ \sum_{n=0,2|n} \sum_{c=0}^{m-\frac{n+1}{2}} \sum_{e=0}^{\frac{n+1}{2}} \sum_{a=0}^{2m-n-2c-n-2e \min\{a,n-2e-d\}} \sum_{d=0}^{e} \sum_{r=0}^{d} q_1^{(a+c+d+e)(2m+1-n-2\lambda-2a-2c-2d-2e)+d+e-c} \right. \]

\[ \cdot \left[ \frac{a + d - r}{d} \right] q_1 \left[ \frac{2m + 2 - 2\lambda - 3a - 4c - d - 2e - n}{r} \right] q_1 \left[ \frac{m - \frac{n+1}{2} - c - a - \lambda}{c} \right] q_1^2 \]

\[ \cdot \left[ \frac{m - \lambda - e - d - 2a - 2c - \frac{n-1}{2}}{q_1^2} \right] \}

\[ \left. \left( E_1^{(a+d-r)} F_1^{(n-2e-d-r)} F_2 F_1^{(2m-n-2c-a)} 1^*_{2\lambda-1} \right) \right. \]

We change variables $\ell = a + d - j$, $y = i - 2e - d - j$, $2m - i - 2c - a = 2m - \ell - y - 2u$. Then we have $a = \ell + y + 2u - 2c - i$, $d = i + c - e - u - y$, $j = u - e - c$. Observe the monomials on the right-hand side of the equation (A.8) above are of the form $E_1^{(\ell)} F_1^{(y)} F_2 F_1^{(2m-\ell-y-2u)} 1^*_{2\lambda}$, for $\ell, y, u \in \mathbb{N}$. For $u, \ell \in \mathbb{N}$, not both 0, the monomial $E_1^{(\ell)} F_1^{(y)} F_2 F_1^{(2m-\ell-y-2u)} 1^*_{2\lambda-1}$ is
\( q_1^{(\ell+u)(2m+1-2\ell-3u-y)} S^{m}(y, u, \ell, \lambda) \), where

\[
\begin{align*}
S^{m}(y, u, \ell, \lambda) &= \sum_{n=0,2}^{2m} \sum_{c,e,r \geq 0 \atop c+e+r = u} q_1^{-(n+u+y)(\ell+u-1)+c-e} \\
&\cdot \left[ \begin{array}{c}
\ell \\
-\ell - y - e + c + n \\
\end{array} \right]_{q_1^r} \left[ \begin{array}{c}
2m + 2 - 2\ell - 5u - 3\ell - 2y - e + c + n \\
r \\
\end{array} \right]_{q_1^r} \\
&\cdot \left[ \begin{array}{c}
m - \lambda - \ell - y - 2u + c + \frac{n+1}{2} \\
\frac{n+1}{2} + c \\
\end{array} \right]_{q_1^r} \\
&- \sum_{n=1,2}^{2m} \sum_{c,e,r \geq 0 \atop c+e+r = u} q_1^{-(n+u+y)(\ell+u-1)} \\
&\cdot \left[ \begin{array}{c}
2m + 2 - 2\lambda - 5u - 3\ell - 2y - e + c + n \\
r \\
\end{array} \right]_{q_1^r} \left[ \begin{array}{c}
m - \lambda - \ell - y - 2u + c + \frac{n+1}{2} \\
\frac{n+1}{2} + c \\
\end{array} \right]_{q_1^r} \\
&\cdot \left[ \begin{array}{c}
m - \lambda - 2\ell - y - 3u + \frac{n+1}{2} + c \\
e \\
\end{array} \right]_{q_1^r} 
\end{align*}
\]

It is easy to note that \( S^{m}(y, u, \ell, \lambda) = q_1^{-(2\ell+2y)(\ell+u-1)} S(y, u, \ell, \lambda) \) (see (4.16)). Hence (A.7) follows from (4.18) and Theorem 3.6. The identity (4.10) is proved.

Summarizing, in this appendix we have completed the proofs of the identities (4.8)–(4.10).

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