QUASICONVEX SUBGROUPS OF $F_m \times \mathbb{Z}^n$ ARE CONVEX

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Abstract. We show that with respect to the usual action of $F_m \times \mathbb{Z}^n$ on $Tree \times \mathbb{R}^n$, every quasiconvex subgroup of $F_m \times \mathbb{Z}^n$ is convex.

1. Introduction and Basic Notions

The motivation for the work in this paper comes from the following remark found in the introductory section of [3]: "...it is currently unknown whether a quasiconvex subgroup of a CAT(0) group is itself CAT(0)." At the time of first being acquainted with the problem, it seemed to me almost absurd that the answer to such a basic question in the theory of CAT(0) groups was not yet known. By comparison, the corresponding statement in the theory of hyperbolic groups has long been known to be true.

Let us begin by recalling some basic definitions:

Definition 1.1. Let $X$ be a (uniquely) geodesic metric space and let $Y \subseteq X$ be a subspace. We shall say that $Y$ is $\nu$-quasiconvex if there exists $\nu > 0$ such that $[x, y] \subseteq N_\nu(Y)$, for all $x, y \in Y$.

Quasiconvexity is a generalization, or more precisely, a "quasification" of the notion of convexity in a geodesic space. Recall that a subspace is called convex if geodesics joining points in that subspace are completely contained in it. By contrast, the notion of quasiconvexity allows for some wiggle room: a geodesic joining points in $Y$ need not be contained in $Y$ itself but is rather allowed to travel in a fixed bounded neighborhood of $Y$ instead.

Definition 1.2. Let $G$ be a CAT(0) group acting geometrically on the CAT(0) space $X$, let $H \subseteq G$ be a subgroup, and let $x_0 \in X$ be a basepoint. The subgroup $H$ is called $\nu$-quasiconvex if the group orbit $Hx_0$ is a $\nu$-quasiconvex subspace of $X$.

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Often, we make no mention of the quasiconvexity constant $\nu$ and simply say that $H$ is a quasiconvex subgroup. While the value of the constant $\nu$ in Definition 1.2 may depend on the choice of the basepoint $x_0$, whether $H$ is quasiconvex or not does not depend on this choice.

It is important to note here that the notion of quasiconvexity in CAT(0) groups, unlike its counterpart in hyperbolic groups, depends on the choice of action. As the following example found in [3] shows, considering two actions of $G = F_2 \times \mathbb{Z}$ on the same CAT(0) space, we can arrange a subgroup of $G$ to be quasiconvex with respect to one action but not the other.

**Example 1.3.** We consider the group $G = F_2 \times \mathbb{Z} = \langle a, b \rangle \times \langle t \rangle$. Let $X$ be the universal cover of the presentation 2-complex of $G$ with the usual action of $G$. If all the edges in $X$ have length equal to 1, it is easy to see that the subgroup $H = \langle a, b \rangle$ is $\nu$-quasiconvex for any $\nu > \frac{1}{2}$. Note that the subgroup $K = \langle a, bt \rangle$ is not quasiconvex, otherwise the subgroup $H \cap K$ would also be quasiconvex, as the intersection of two quasiconvex subgroups is quasiconvex. However, the subgroup $H \cap K$ is the subgroup of $H$ which consists of elements for which the sum of the powers of the generator $b$ in a reduced word equals 0. Since quasiconvex $\Rightarrow$ finitely generated, $H \cap K$ is not quasiconvex as it is not finitely generated.

Now, consider the automorphism $\phi$ of $G$ defined by $a \mapsto a$, $b \mapsto bt$, $t \mapsto t$, which sends $H$ to $K$. If we compose the action $\rho : G \to \text{Isom}(X)$ of $G$ on $X$ described above with $\phi$, we obtain a new action $\rho \circ \phi : G \to \text{Isom}(X)$, such that $\rho$ restricted to $K$ is the same as $\rho \circ \phi$ restricted to $H$. Thus, $H$ is not quasiconvex with respect to this new action.

In order to show that a quasiconvex subgroup $H$ of the CAT(0) group $G$ is CAT(0), we need to exhibit a CAT(0) space $Y$ and a geometric action of $H$ on $Y$. The reader may at this point rightfully ask the question, why can we not take the convex hull of the $H$-orbit of some point $x_0 \in X$? The convex hull is a convex, $H$-invariant subspace of a CAT(0) space, and the action of $H$ on it is proper, as the action of $G$ on $X$ is proper. The problem is this: while the action of $G$ on $X$ was assumed to be cocompact, it is not at all obvious, and perhaps not even true in general, that quasiconvexity of $H$ in $G$ should imply cocompactness of the action of $H$ on $\text{conv}(Hx_0)$. Of course, even if one were to find a counterexample, that is an example where the induced action of $H$ on $\text{conv}(Hx_0)$ is not cocompact, this would not necessarily mean that $H$ is not a CAT(0) group because there is still the possibility that $H$ may act geometrically on some other CAT(0) space. As far as we know, the general question of whether quasiconvexity implies CAT(0) is still wide open. In this chapter, we show that what should be true is indeed true in one special case,
namely, we prove the following:

**Theorem:** Let $H$ be a quasiconvex subgroup of $G = F_m \times \mathbb{Z}^n$, and let $X$ be the product of the regular $2m$-valent tree with $\mathbb{R}^n$ with the usual action of $G$. Then the action of $H$ on the convex hull of any orbit $Hx_0$ is cocompact.

If $G$ is a CAT(0) group acting geometrically on the CAT(0) space $X$, and there exists a closed convex $H$-invariant subset of $X$ on which $H$ acts cocompactly, $H$ is called convex.

With this terminology our theorem becomes: Any quasiconvex subgroup of $G = F_m \times \mathbb{Z}^n$ is convex with respect to the usual action of $G$ on $\text{Tree} \times \mathbb{R}^n$.

In the course of proving this result, we introduce a technique for analyzing the convex hull in certain CAT(0) spaces and also prove that the quasiconvex subgroups of $F_m \times \mathbb{Z}^n$ are precisely those which are virtually of the form $A \times B$ where $A \leq F_m$ is finitely generated and $B \leq \mathbb{Z}^n$.

## 2. Convex Hulls and Quasiconvex Subgroups

As we mentioned in the previous section, every convex subspace of a CAT(0) space is obviously itself a CAT(0) space with the induced metric. An idea which dates back to Minkowski and Brunn, and which we independently rediscovered, is to construct $\text{conv}(Y)$ by means of a sequential process as follows: For $S \subseteq X$, we define $\text{conv}^1(S)$ to be the union of all geodesic segments having both endpoints in $S$, or symbolically $\text{conv}^1(S) = \bigcup_{s_1, s_2 \in S} [s_1, s_2]$. Now, we set $\text{conv}^0(Y) = Y$ and define recursively $\text{conv}^i(Y) = \text{conv}^1(\text{conv}^{i-1}(Y))$. This process of "convexification" results in an ascending sequence of subsets of $X$: $Y = \text{conv}^0(Y) \subseteq \text{conv}^1(Y) \subseteq \ldots \subseteq \text{conv}^i(Y) \subseteq \ldots \subseteq \text{conv}(Y) \subseteq X$, each of which gets closer to the convex hull of $Y$ in the following sense:

**Lemma 2.1.** Let $X$ be a uniquely geodesic space and $Y$ a subspace, then $\text{conv}(Y) = \bigcup_{i=0}^{\infty} \text{conv}^i(Y)$.

**Proof.** Obviously, $\text{conv}^i(Y) \subseteq \text{conv}(Y)$ for all $i$. It is also clear that $\bigcup_{i=0}^{\infty} \text{conv}^i(Y)$ is convex, hence it equals $\text{conv}(Y)$. □

Lemma 2.1 was the starting point for our investigation of convex hulls. Its proof is not difficult and we realized that the result was already attributed to Hermann Brunn in the case when $X$ is a vector space only after we had proved the results on convexity in polygonal complexes below.

**Remark:** A set $Y$ is $\nu$-quasiconvex if and only if $\text{conv}^1(Y) \subseteq N_\nu(Y)$. 
We now relate the foregoing discussion on convexity with non-positive curvature. In CAT(0) spaces, as a consequence of the convexity of the metric, one has control over the growth of the sizes of the sets \( \text{conv}^i(Y) \) as the following result shows:

**Lemma 2.2.** Let \( X \) be a CAT(0) space and let \( Y \subseteq X \) be a \( \nu \)-quasiconvex subset of \( X \). Then, \( \text{conv}^i(Y) \subseteq N_{\nu}(Y) \).

**Proof.** In this proof we assume that all geodesics are parametrized proportional to arc length and we proceed by induction on \( i \). The starting step \( i = 1 \) is handled by the remark above. For the inductive step, suppose that \( \text{conv}^{i-1}(Y) \subseteq N_{(i-1)\nu}(Y) \) and let \( x \in \text{conv}^i(Y) \). Then, \( x \in [x_1, x_2] \), where \( x_1, x_2 \in \text{conv}^{i-1}(Y) \subseteq N_{(i-1)\nu}(Y) \). Let \( x'_1, x'_2 \in Y \) be such that \( d(x_j, x'_j) < (i-1)\nu \), for \( j = 1, 2 \). By convexity of the CAT(0) metric, \( d([x_1, x_2](t), [x'_1, x'_2](t)) \leq (1-t)d(x_1, x'_1) + td(x_2, x'_2) < (i-1)\nu \). This shows that \( d(x, \text{conv}^i(Y)) < (i-1)\nu \) and thus we conclude that \( x \in N_{\nu}(Y) \), as desired. \( \square \)

The number \( k = \inf \{ i : \text{conv}^i(Y) = \text{conv}(Y) \} \) is called the Brunn number. Brunn gave a lower and an upper bound for \( k \) in finite-dimensional vector spaces, see [2]. In view of Lemma 2.2, a uniform bound for the Brunn number over all subsets of a CAT(0) space has important implications from the point of view of cocompactness of group actions. Before we proceed to give a bound for the Brunn number for Euclidean spaces, we make the following important observation: In analyzing the convex hull of an arbitrary set using Lemma 2.1 it suffices to only consider finite sets, which are substantially easier to work with.

**Lemma 2.3.** Let \( X \) be a geodesic space, let \( Y \subseteq X \), and let \( i \in \mathbb{N}^+ \). If \( \text{conv}^i(S) = \text{conv}(S) \) for every finite subset \( S \subseteq Y \), then \( \text{conv}^i(Y) \) is convex.

**Proof.** Let \( x, y \in \text{conv}^i(Y) \). Then, we can find \( a_1, a_2, b_1, b_2 \in \text{conv}^{i-1}(Y) \) such that \( x \in [a_1, a_2] \) and \( y \in [b_1, b_2] \). Similarly, we can find \( c_1, c_2, d_1, d_2 \in \text{conv}^{i-2}(Y) \) such that \( a_1 \in [c_1, c_2] \) and \( a_2 \in [d_1, d_2] \), etc. Proceeding recursively, we see that we can find points \( x_1, \ldots, x_m \in Y \) such that \( x, y \in \text{conv}^i([x_1, \ldots, x_m]) = \text{conv}([x_1, \ldots, x_m]) \). Hence, \( [x, y] \subseteq \text{conv}^i([x_1, \ldots, x_m]) \subseteq \text{conv}^i(Y) \). \( \square \)

As we were unable to procure Brunn’s original paper, we present our own proof of the intuitively obvious fact that the Brunn number of any subset of \( \mathbb{R}^n \) is less or equal to \( n \). Our proof uses Caratheodory’s theorem which we recall below.
2.1. Convex Hulls in Euclidean Spaces.

**Theorem 2.4.** (Caratheodory) If $E$ is a vector space of dimension $d$, then, for every subset $X$ of $E$, every element in the convex hull $\text{conv}(X)$ is an affine convex combination of $d + 1$ elements of $X$.

*Proof.* See Proposition 5.2.3 in [4]. □

**Lemma 2.5.** For any finite set $S \subseteq \mathbb{R}^n$, $\text{conv}^n(S) = \text{conv}(S)$.

*Proof.* By Caratheodory’s theorem, $\text{conv}(S) = \bigcup \text{conv} \{s_1, ..., s_{n+1}\}$, where the union is taken over all $s_1, ..., s_{n+1} \in S$. Therefore, it suffices to show that for any set $\{s_1, ..., s_{n+1}\} \subseteq \mathbb{R}^n$, $\text{conv}^n(\{s_1, ..., s_{n+1}\}) = \text{conv}(\{s_1, ..., s_{n+1}\})$. Consider the points $e_1, ..., e_{n+1} \in \mathbb{R}^{n+1}$. Their convex hull is the standard $n$-simplex $\Delta_n$ in $\mathbb{R}^{n+1}$. Suppose $\text{conv}^{i-1}(\{e_1, ..., e_{n+1}\})$ contains all the $(i-1)$-faces of $\Delta_n$, then $\text{conv}^i(\{e_1, ..., e_{n+1}\})$ contains all joins of the form $\text{join}(F, e_j)$, $1 \leq j \leq n+1$, where $F$ is an $(i-1)$-face. But all of the $i$-faces are joins of this form. By induction, $\text{conv}^i(\{e_1, ..., e_{n+1}\})$ contains all the $i$-faces. Hence, $\text{conv}^n(\{e_1, ..., e_{n+1}\})$ contains and therefore equals $\Delta_n$. Now, let $\phi$ be the affine map which sends $e_i$ to $s_i$. This map sends lines to lines, therefore $\phi(\text{conv}^i(\{e_1, ..., e_{n+1}\})) \subseteq \text{conv}^i(\{s_1, ..., s_{n+1}\})$. Now, $\phi(\text{conv}^n(\{e_1, ..., e_{n+1}\}))$ is convex, contains $\{s_1, ..., s_{n+1}\}$, and is contained in $\text{conv}^n(\{s_1, ..., s_{n+1}\})$. Therefore, \[ \text{conv}(\{s_1, ..., s_{n+1}\}) = \text{conv}^n(\{s_1, ..., s_{n+1}\}), \] as desired. □

Combining Lemma 2.3 and Lemma 2.5 we obtain the desired bound on the Brunn number:

**Corollary 2.6.** For any subset $Y \subseteq \mathbb{R}^n$, $\text{conv}^n(Y) = \text{conv}(Y)$. Therefore, $k \leq n$ for any subset of $\mathbb{R}^n$.

As a straightforward application of Corollary 2.6 we have:

**Corollary 2.7.** In any uniquely geodesic Hilbert geometry, the Brunn number of any subset is bounded above by the dimension of the underlying Euclidean space.

*Proof.* Each affine segment in the underlying Euclidean space is a geodesic for the Hilbert metric. Since the Hilbert geometry we consider is uniquely geodesic, Corollary 2.6 immediately applies. □

Unfortunately, obtaining a bound on the Brunn number in an arbitrary CAT(0) space in the absence of any restrictions on the subsets in consideration is impossible. We are, however, able to bound the Brunn number in certain "planar" piecewise Euclidean CAT(0) polygonal complexes.
2.2. Convex Hulls in CAT(0) Planes.

Definition 2.8. A CAT(0) plane is a simply connected piecewise Euclidean polygonal complex $X$ with $\text{Shapes}(X)$ finite, such that $Lk(v)$ is isometric to a circle of length $\geq 2\pi$ for every vertex $v \in X$.

We immediately note that in view of the Gromov Link Condition, a CAT(0) plane is a CAT(0) metric space. We prove that the Brunn number of any subset of a CAT(0) plane is $\leq 2$. The proof of this intuitively "obvious" result turned out to be surprisingly technical. Our proof employs a local-to-global technique which makes use of the Cartan-Hadamard theorem. The idea is that under mild hypotheses, convexity on the small scale implies global convexity. Before we are able to state and prove the promised results, we need to transpose some familiar definitions to the "small scale":

Definition 2.9. Let $(X, d)$ be a metric space.

1. The metric $d$ is said to be locally convex if every point in $X$ has a neighborhood in which the induced metric is convex.
2. The metric space is said to be locally CAT(0) if every point in $X$ has a convex neighborhood $U$ with the property that $(U, d)$ is a CAT(0) metric space.
3. Let $f : X \to Y$ be a map between two metric spaces. We shall say that $f$ is a local isometry if every point $x \in X$ has a neighborhood $U$, such that $f$ restricted to $U$ is an isometric embedding.

Theorem 2.10. (Cartan-Hadamard) Let $(X, d)$ be a complete connected metric space. If the metric on $X$ is locally convex, the induced length metric on the universal covering $\tilde{X}$ is convex. In particular, there is a unique geodesic segment joining each pair of points in $\tilde{X}$. Further, if $X$ is a locally CAT(0) space, then $\tilde{X}$ with the induced length metric is a CAT(0) space and the covering map $p : \tilde{X} \to X$ is a local isometry.

Proof. See Theorem 4.1 in [1].

In view of the Cartan-Hadamard theorem above, we shall say that a subset $Y$ of the CAT(0) space $X$ is locally convex if every point of $Y$ has a convex neighborhood $U \subseteq X$ such that $U \cap Y$ is convex.

Proposition 2.11. Let $X$ be a CAT(0) space. Then, $\text{conv}^n(Y) = \text{conv}(Y)$ for all $Y \subseteq X$ if and only if $\text{conv}^n(S)$ is locally convex for every finite subset $S \subseteq X$.

Proof. First, we show that for any compact subset $S \subseteq X$, $\text{conv}^i(S)$ is compact for every $i$. The proof is by induction on $i$, the case $i = 0$ being trivial. Note that we have an obvious map $\varphi : \text{conv}^{i-1}(S) \times \text{conv}^{i-1}(S) \times$
Let $I \to \operatorname{conv}^i(S)$ given by $\varphi(x,y,t) = [x,y](t)$. Since in any CAT(0) space geodesics vary continuously with endpoints, the map $\varphi$ is a continuous surjection which maps the compact set $\operatorname{conv}^{i-1}(S) \times \operatorname{conv}^{i-1}(S) \times I$ to $\operatorname{conv}^i(S)$ thus proving our claim.

Now, suppose that $\operatorname{conv}^n(S)$ is locally convex for every finite subset $S \subseteq X$. Since $\operatorname{conv}^n(S)$ is a compact and therefore complete, connected, and locally convex subset of a CAT(0) space, Theorem 2.10 tells us that the universal cover $\widetilde{\operatorname{conv}^n(S)}$ endowed with the length metric is a CAT(0) space, and that the covering map $p : \operatorname{conv}^n(S) \to \widetilde{\operatorname{conv}^n(S)}$ is a local isometry. Let $x, y \in \operatorname{conv}^n(S)$ and choose any $\tilde{x} \in p^{-1}(x)$, and $\tilde{y} \in p^{-1}(y)$. Let $\alpha(t)$ be the unique geodesic in $\operatorname{conv}^n(S)$ joining $\tilde{x}$ to $\tilde{y}$. Then, because $\operatorname{conv}^n(S)$ is compact, a simple argument using the Lebesgue covering lemma shows that $p \circ \alpha(t)$ is a local geodesic in $X$ joining $x$ to $y$. Since in a CAT(0) space any local geodesic is a geodesic, see Proposition 1.4 in [1], we see that $p \circ \alpha$ is the geodesic in $X$ joining $x$ to $y$. But the image of $p \circ \alpha$ is contained in $\operatorname{conv}^n(S)$. This shows that $\operatorname{conv}^n(S)$ is convex, and Lemma 2.3 yields the desired conclusion.

In the course of proving Proposition 2.14 we will encounter the phenomenon of bifurcating geodesics. We shall call geodesics which coincide up to a point of bifurcation or divergence, bifurcating geodesics. The key fact we shall need is that in a CAT(0) plane, at the point of bifurcation, the geodesic which splits can be extended in an infinite number of ways. To prove this result, we need the following reformulation of the CAT(0) condition:

**Lemma 2.12.** Let $X$ be a geodesic metric space. Then, $X$ is a CAT(0) space if and only if the Alexandrov angle between the sides of any geodesic triangle in $X$ with distinct vertices is no greater than the angle between the corresponding sides of its comparison triangle in $E^2$.

**Proof.** See Proposition 1.7(4) in [1].

**Lemma 2.13.** Let $X$ be a CAT(0) plane and let $a_1, a_2, b, q \in X$ be such that $[a_1,b] \cap [a_2,b] = [q,b]$. Then, for any $c \in [a_1,a_2]$, the concatenation of the geodesic segment $[c,q]$ and $[q,b]$ is a geodesic segment.

**Proof.** The proof is essentially the observation that for any $c \in [a_1,a_2]$, the distance in $\operatorname{Lk}(q)$ between the directions determined by $[c,q]$ and $[q,b]$ is at least $\pi$, hence the Alexandrov angle $\angle_q(c,b) = \pi$. By Lemma 2.12 $\angle_q(c,b) \leq \angle (\tau,\overrightarrow{0})$, hence $\angle (\tau,\overrightarrow{0}) = \pi$. Now, we conclude that $d(\tau,\overrightarrow{0}) = d(\tau,q) + d(q,b)$. Therefore, $d(c,b) = d(c,q) + d(q,b)$ thus showing that the concatenation of $[c,q]$ and $[q,b]$ is indeed a geodesic segment.
We are now ready to show that the Brunn number for any subset of a CAT(0) plane is at most equal to 2.

**Proposition 2.14.** Let $X$ be a CAT(0) plane. Then the Brunn number of any subset $Y \subseteq X$ is at most 2.

**Proof.** In view of Proposition 2.11, we only need to show that for every finite subset $S \subseteq X$, $\text{conv}^2(S)$ is locally convex at every point $p \in \text{conv}^2(S)$. Suppose to the contrary that $\text{conv}^2(S)$ is not locally convex at some $p \in \text{conv}^2(S)$. Then, given any convex neighborhood $U$ of $p$ in $X$, since $\text{conv}^2(S)$ is compact, we can find a geodesic $\gamma : [0,1] \rightarrow U$ such that $\gamma(0), \gamma(1) \in \text{conv}^2(S)$ and $\gamma(t) \notin \text{conv}^2(S)$ for all $0 < t < 1$. Let $\gamma(0) = x_0$ and note that we may have $x_0 = p$. Then, there exist points $x_1, x_2, y_1, y_2 \in S$ such that $x_0 \in [x_1, y_1](t_1), [x_2, y_2](t_2)$ for some $t_1, t_2 \geq 0$. Consider the 1-parameter family of geodesics $t' \mapsto [x_1, y_1](t'), [x_2, y_2](t_2)$. If $x_0 \in [x_1, y_1](t'), [x_2, y_2](t_2)$ for all $t' \geq t_1$ (or all $t' \leq t_1$), then $x_0$ lies on a geodesic having one endpoint $s$ in $S$. In this case we consider the family $t' \mapsto [s, x_2, y_2](t')$. If $x_0 \in [s, x_2, y_2](t')$ for all $t' \geq t_2$ (or all $t' \leq t_2$), then $x_0 \in \text{conv}^1(S)$, which is a contradiction. Therefore, without loss of generality, we may assume that $x_0 \notin [x_1, y_1](t'), [x_2, y_2](t_2)$ for some values of $t'$ both greater than and less than $t_1$. As $X$ has no free edges, $\gamma$ may be extended to $\gamma : [-\epsilon,1] \rightarrow X$ such that this extension of $\gamma$ crosses $[x_1, y_1](t_1), [x_2, y_2](t_2)$ at $x_0$. Because $\gamma(t) \notin \text{conv}^2(S)$ for $t > 0$ and because geodesics vary continuously with endpoints, we can find $t'_0 < t_1 < t''_0$ and $-\epsilon < -\epsilon' < 0$ such that $[x_1, y_1](t'_0), [x_2, y_2](t_2)$ and $[x_1, y_1](t''_0), [x_2, y_2](t_2)$ intersect at $\gamma(-\epsilon')$, and such that neither of these geodesic segments passes through $x_0$. Because of uniqueness of geodesics in $X$, the segments $[x_1, y_1](t'_0), [x_2, y_2](t_2)$ and $[x_1, y_1](t''_0), [x_2, y_2](t_2)$ must coincide up to some point $q$ which is their point of bifurcation. Then, Lemma 2.13 shows that for every $t_1 < t < t_2$, the concatenation of the geodesic segments $[x_2, y_2](t_2), q$ and $q, [x_1, y_1](t)$ is a geodesic. Again by uniqueness of geodesics, this concatenation must be the geodesic segment $[x_1, y_1](t_1), [x_2, y_2](t_2)$, and we conclude that $x_0 \in [x_1, y_1](t'_0), [x_2, y_2](t_2)$ which is a contradiction. 

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3. **Free $\times$ Free-abelian Groups**

Throughout this section, $G$ will be the group $F_m \times \mathbb{Z}^n$ and $X$ will stand for the product of the regular $2m$-valent tree $T_{2m}$ and $\mathbb{R}^n$. The action of $G$ on $X$ is the product action where $F_{2m}$ acts as the group of deck transformations on the universal cover of the m-rose, $T'_{2m}$, and $\mathbb{Z}^n$ acts by translation on $\mathbb{R}^n$. 
Lemma 3.1. Let $H = \{f_1 z_1, ..., f_s z_s\}$, $f_i \in F_m$, $z_i \in \mathbb{Z}^n$ be a quasiconvex subgroup of $G$ such that not all of the $f_i$ have the same axis of translation in $T_{2m}$. Then, there exist positive integers $k_1, ..., k_s$ such that $H$ contains the subgroup $A = \langle z_1^{k_1}, ..., z_s^{k_s} \rangle$.

Proof. Let $1 \leq i \leq s$, let $j$ be such that $f_j$ and $f_i$ have different axes of translation, and let $l$ be a positive integer. Find an axis of translation for $f_j z_i$ whose projection to the Euclidean factor passes through $0 \in \mathbb{R}^n$. This can always be done by translating the Euclidean component of any given axis for $f_j z_i$. Let $x_0$ be a point on the chosen axis of translation for $f_j z_i$ such that $pr_{\mathbb{R}^n}(x_0) = 0$. Consider the sequences of points $(f_j z_i)^l x_0$ and $(f_j z_j)^l x_0$. Because $f_j$ and $f_i$ have different axes, there is a vertex $v$ in $T_{2m}$ such that for every $l$, the geodesic segment $[(f_j z_i)^l x_0, (f_j z_j)(f_j z_i)^l x_0]$ passes through the flat $\{ v \} \times \mathbb{R}^n$. Let $y_l$ denote the point of intersection of $\{ v \} \times \mathbb{R}^n$ and $[(f_j z_i)^l x_0, (f_j z_j)(f_j z_i)^l x_0]$. The orthogonal projection of this geodesic segment to the flat $\{ v \} \times \mathbb{R}^n \cong \mathbb{R}^n$ is the geodesic segment between $z_i^l$ and $z_i^l + z_j$. Since the geodesic $[(f_j z_i)^l x_0, (f_j z_j)(f_j z_i)^l x_0]$ intersects its orthogonal projection in the point $y_l$, we have $d(y_l, (v, z_i^l)) \leq \| z_j \|$. The orbit $H x_0$ is quasiconvex, hence there is $\nu > 0$ and $h_i \in H$ such that $d(h_i x_0, y_l) < \nu$. Then, $d(h_i x_0, z_i^l x_0) \leq d(h_i x_0, y_l) + d(y_l, z_i^l (v, 0)) + d(z_i^l (v, 0), z_i^l x_0) < \nu + \| z_j \| + d(x_0, (v, 0))$. If $\tau = \nu + \| z_j \| + d(x_0, (v, 0))$, then $B_\tau(h_i x_0) \cap B_\tau(z_i^l x_0) \neq \emptyset$, or $B_\tau(h_i^{-1} z_i^l x_0) \cap B_\tau(x_0) \neq \emptyset$, for all $l$. Because the action of $G$ is proper, $h_i^{-1} z_i^l = g \in G$ for infinitely many values of $l$. Then, for some $k,l$ we have $z_i^{l-k} = h_i^{-1} h_i h_i^{-1} H$. Setting $k_i = l - k$, we obtain $z_i^{k_i} \in H$. \hfill $\square$

Let $p : T_{2m} \times \mathbb{Z}^n \to T_{2m}$ be the projection onto the first factor, and let $V$ denote the real span of the vectors $z_i^{k_1}, ..., z_i^{k_s}$. We then, have the following:

Lemma 3.2. With the same notation as in Lemma 3.1, the convex hull of $H x_0$ equals $conv(p(H x_0)) \times V$.

Proof. First, we note that the projection maps $p, pr_{\mathbb{R}^n}$ commute with the operation of forming the convex hull. That is, $p(conv(H x_0)) = conv(p(H x_0))$, and similarly for $pr_{\mathbb{R}^n}$. Let us show this for the projection map $p$. We begin by making the observation that $p(conv^t(S)) = conv^t(p(S))$ for any set $S$, since $p$ maps the geodesic segment connecting two points to the geodesic segment connecting their images. Therefore, we have $p(conv(H x_0)) = p \left( \bigcup_i conv^t(H x_0) \right) = \bigcup_i p(conv^t(H x_0)) = \bigcup_i conv^t(p(H x_0)) = conv(p(H x_0))$. \hfill $\square$

Now, we proceed with the proof of the lemma.
Proof. Without loss of generality, we may assume that \( p_{\mathbb{R}_+}(x_0) = 0 \). Clearly, \( \text{conv}(Hx_0) \subseteq p(\text{conv}(Hx_0)) \times p_{\mathbb{R}_+}(\text{conv}(Hx_0)) \), which after commuting the projection maps past \( \text{conv} \) gives us the desired inclusion.

\( \supseteq \): Let \( x \in \text{conv}(p(Hx_0)) \times V \). Let \( y \in \text{conv}(Hx_0) \) be such that \( p(y) = p(x) \). Note that because \( H \) contains powers of the Euclidean translations \( z_1, \ldots, z_k \), the projection of the convex hull of the orbit \( Hx_0 \) to the Euclidean factor will equal \( V \). Also, \( \text{conv}(Hx_0) \supseteq V \cdot y \), as \( \text{conv}(Hx_0) \) is stable under the action of \( V \) by translations on the second factor. Hence, we can write \( x = w \cdot y \), for some \( w \in V \), so that \( x \in \text{conv}(Hx_0) \).

**Lemma 3.3.** Let \( H \) be as in Lemma 3.2. Then, the group \( H \) acts cocompactly on its convex hull. In particular, the Brunn number of the orbit \( Hx_0 \) is bounded above by \( 1 + \dim(V) \).

**Proof.** Note that we can write \( p(\text{conv}(Hx_0)) = \text{conv}(p(Hx_0)) \) as a union of biinfinite geodesic rays \( \gamma \), such that any point on \( \gamma \) lies between two points in \( p(Hx_0) \). Then, \( \text{conv}(Hx_0) = \bigcup \gamma \times V \), and \( \gamma \times V \cong \mathbb{R}^{1+\dim(V)} \).

Note that \( \gamma \times V \) contains the lattices \( p(hx_0) \times \mathbb{Z} \cdot \langle z_1^{k_1}, \ldots, z_s^{k_s} \rangle \), where \( h \in H \). Because of the assumption that any point on \( \gamma \) lies between two points \( p(h_1), p(h_2) \in p(Hx_0) \), the convex hull of these lattices is all of \( \gamma \times V \), and by Corollary 2.4 \( \text{conv}^{1+\dim(V)} \left( \bigcup (p(hx_0) \times \mathbb{Z} \cdot \langle z_1^{k_1}, \ldots, z_s^{k_s} \rangle) \right) = \gamma \times V \). Finally, \( \text{conv}^{1+\dim(V)}(Hx_0) \supseteq \text{conv}^{1+\dim(V)} \left( \bigcup (p(hx_0) \times \mathbb{Z} \cdot \langle z_1^{k_1}, \ldots, z_s^{k_s} \rangle) \right) = \text{conv}(Hx_0) \).

Combining Lemmas 3.1-3.3, we obtain:

**Theorem 3.4.** Any quasiconvex subgroup of \( F_m \times \mathbb{Z}^n \) acts cocompactly on the convex hull of any of its orbits.

**Proof.** Lemmas 3.1-3.3 take care of the case when for each \( i \) there is \( j \) such that \( f_i \) and \( f_j \) have different axes of translation. If \( H = \langle f_1^{k_1}, \ldots, f_s^{k_s}, z_s \rangle \), \( f \in F_m \), then the orbit \( Hx_0 \) is contained in a single flat \( a_f \times V \) isometric to \( \mathbb{R}^{1+\dim(V)} \), where \( a_f \) is a common axis for all \( f^{k_i} \), and \( x_0 \) is on a common axis for all the \( f_i z_i \). Hence, \( \text{conv}(Hx_0) = \text{conv}^{1+\dim(V)}(Hx_0) \), which shows cocompactness of the action of \( H \). In either of the cases \( H = \langle f_1, \ldots, f_s \rangle \) or \( H = \langle z_1, \ldots, z_s \rangle \), the conclusion is again trivially true. In the former case \( \text{conv}(Hx_0) = \text{conv}^v(Hx_0) \), while in the latter \( \text{conv}(Hx_0) = \text{conv}^s(Hx_0) \).

In the course of proving the theorem, we have the essential ingredients for the following corollary:
Corollary 3.5. If $H$ is a quasiconvex subgroup of $F_\mathbf{m} \times \mathbb{Z}^n$, whose image in $F_\mathbf{m}$ under the natural projection $p : F_\mathbf{m} \times \mathbb{Z}^n \to F_\mathbf{m}$ has rank greater than 1, then $H$ is virtually of the form $A \times B$, where $A \leq F_\mathbf{m}$ and $B \leq \mathbb{Z}^n$.

Proof. Let $H$ be as in Lemma 3.1. The proof of Lemma 3.1 shows that for $g = fz \in H$, there exists $s$ such that $z^s \in H$, and hence $f^s \in H$. Let $A = \mathbb{Z}^n \cap H$, $F = F_\mathbf{m} \cap H$. Then, $g^s \in AF$. On the other hand, $[H, H] \subseteq F$, and also $AF$ is normal in $H$. Hence, we see that $H/AF$ is a finitely generated, torsion, abelian group, and is therefore finite. If $H = \langle f^{k_1}z_1, ..., f^{k_s}z_s \rangle$, or $H = \langle z_1, ..., z_s \rangle$, then $H$ is already free abelian. \qed

On the other hand, it is easy to show that any subgroup which is virtually of the form $A \times B \subseteq F_\mathbf{m} \times \mathbb{Z}^n$, where $A \subseteq F_\mathbf{m}$ is finitely generated, and $B \subseteq \mathbb{Z}^n$, is quasiconvex. A simple analysis along the lines of the proof of Lemma 3.3 shows that the convex hull of $(A \times B)x_0$ is contained in a finite neighborhood of the lattice $(A \times B)x_0$. Thus, we have:

Theorem 3.6. Let $H$ be a subgroup of $F_\mathbf{m} \times \mathbb{Z}^n$. If $H$ is not virtually cyclic, then $H$ is quasiconvex with respect to the standard action of $G$ on $T_{2\mathbf{m}} \times \mathbb{R}^n$ if and only if $H$ is virtually of the form $A \times B$, where $A \subseteq F_\mathbf{m}$ is finitely generated, and $B \subseteq \mathbb{Z}^n$.

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