Degree Complexity of Matrix Inversion

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Abstract. For a \(q \times q\) matrix \(x = (x_{i,j})\) we let \(J(x) = (x_{i,j}^{-1})\) be the Hadamard inverse, which takes the reciprocal of the elements of \(x\). We let \(I(x) = (x_{i,j})^{-1}\) denote the matrix inverse, and we define \(K = I \circ J\) to be the birational map obtained from the composition of these two involutions. We consider the iterates \(K^n = K \circ \cdots \circ K\) and determine degree complexity of \(K\), which is the exponential rate of degree growth \(\delta(K) = \lim_{n \to \infty} (\deg(K^n))^{1/n}\) of the degrees of the iterates.

§0. Introduction

Let \(\mathcal{M}_q\) denote the space of \(q \times q\) matrices with coefficients in \(\mathbb{C}\), and let \(\mathbb{P}(\mathcal{M}_q)\) denote its projectivization. We consider two involutions on the space of matrices: \(J(x) = (x_{i,j}^{-1})\) takes the reciprocal of each entry of the matrix \(x = (x_{i,j})\), and \(I(x) = (x_{i,j})^{-1}\) denotes the matrix inverse. The composition \(K = I \circ J\) defines a birational map of \(\mathbb{P}(\mathcal{M}_q)\).

For a rational self-map \(f\) of projective space, we may define its \(n\)th iterate \(f^n = f \circ \cdots \circ f\), as well as the degree \(\deg(f^n)\). The degree complexity or dynamical degree is defined as

\[
\delta(f) := \lim_{n \to \infty} (\deg(f^n))^{1/n}.
\]

In general it is not easy to determine \(\delta(f)\), or even to make a good numerical estimate. Birational maps in dimension 2 were studied in [DF], where a technique was given that, in principle, can be used to determine \(\delta(f)\). This method, however, does not carry over to higher dimension. In the case of the map \(K_q\), the dimension of the space and the degree of the map both grow quadratically in \(q\), so it is difficult to write even a small composition \(K_q \circ \cdots \circ K_q\) explicitly. This paper is devoted to determining \(\delta(K_q)\).

Theorem. For \(q \geq 3\), \(\delta(K_q)\) is the largest root of the polynomial \(\lambda^2 - (q^2 - 4q + 2)\lambda + 1\).

The map \(K\) and the question of determining its dynamical degree have received attention because \(K\) may be interpreted as acting on the space of matrices of Boltzmann weights and as such represents a basic symmetry in certain problems of lattice statistical mechanics (see [BHM], [BM]). In fact there are many \(K\)-invariant subspaces \(T \subset \mathbb{P}(\mathcal{M}_q)\) (see, for instance, [AMV1] and [PAM]), and it is of interest to know the values of the restrictions \(\delta(K|_T)\). The first invariant subspaces that were considered are \(\mathcal{S}_q\), the space of symmetric matrices, and \(\mathcal{C}_q\), the cyclic (also called circulant) matrices. The value \(\delta(K|_{\mathcal{C}_q})\) was found in [BV], and another proof of this was given in [BK1]. Anglès d’Auriac, Maillard and Viallet [AMV2] developed numerical approaches to finding \(\delta\) and found approximate values of \(\delta(K_q)\) and \(\delta(K|_{\mathcal{S}_q})\) for \(q \leq 14\). A comparison of these values with the (known) values of \(\delta(K|_{\mathcal{C}_q})\) led them to conjecture that \(\delta(K|_{\mathcal{C}_q}) = \delta(K_q) = \delta(K|_{\mathcal{S}_q})\) for all \(q\).

The Theorem above proves the first of these conjectured equalities. We note that the second equality, \(\delta(K|_{\mathcal{S}_q}) = \delta(K_q)\), involves additional symmetry, which adds another layer of subtlety to the problem. An example where additional symmetry leads to additional complication has been seen already with the \(K\)-invariant space \(\mathcal{C}_q \cap \mathcal{S}_q\): the value
of $\delta(K_{C_q} \cap S_q)$ has been determined in [AMV2] (for prime $q$) and [BK2] (for general $q$), and in the general case it depends on $q$ in a rather involved way. The reason why the cyclic matrices were handled first was that $K|_{C_q}$ (see [BV]) and $K|_{C_q} \cap S_q$ (see [AMV2]) can be converted to maps of the form $L \circ J$ for certain linear $L$. In the case of $K|_{C_q}$, the associated map is “elementary” in the terminology of [BK1], whereas $K|_{C_q} \cap S_q$ exhibits more complicated singularities, i.e., blow-down/blow-up behavior.

In contrast, the present paper treats matrices in their general form, so our methods should be applicable to much wider classes of $K$-invariant subspaces. Our approach is to replace $P(M_q)$ by a birationally equivalent manifold $\pi : X \to P(M_q)$ and consider the induced birational map $K_X := \pi^{-1} \circ K \circ \pi$. A rational map $K_X$ induces a well-defined linear map $K_X^*$ on the cohomology group $H^{1,1}(X)$, and the exponential growth rate of degree is equal to the exponential growth rate of the induced maps on cohomology:

$$
\delta(K) = \lim_{n \to \infty} \left( ||K_X^n||_{H^{1,1}(X)} \right)^{1/n}.
$$

Our approach is to choose $X$ so that we can determine $(K_X^n)^*$ sufficiently well. A difficulty is that frequently $(K^*)^n \neq (K^n)^*$ on $H^{1,1}$. In the cases we consider, $H^{1,1}$, the cohomology group in (complex) codimension 1, is generated by the cohomology classes corresponding to complex hypersurfaces. So in order to find a suitable regularization $X$, we need to analyze the singularity of the blow-down behavior of $K$, which means that we analyze $K$ at the hypersurfaces $E$ with the property that $K(E)$ has codimension $\geq 2$.

Let us give the plan for this paper. In general, $\text{deg}(K \circ K) \leq \text{deg}(K)^2$, so $\delta(K) \leq \deg(K)$. On the other hand, $\delta$ decreases when we restrict to a linear subspace, so $\delta(K) \geq \delta(K|_{C_q})$. The paper [BV] shows that $\delta(K|_{C_q})$ is the largest root of the polynomial $\lambda^2 - (q^2 - 4q + 2)\lambda + 1$, so it will suffice to show that this number is also an upper bound for $\delta(K)$. In order to find the right upper bound on $\delta(K_q)$, we construct a blowup space $\pi : Z \to P(M_q)$. Such a blowup induces a birational map $K_Z$ of $Z$. Each birational map induces a linear mapping $K_Z^*$ on the Picard group $Pic(Z) \cong H^{1,1}(Z)$. A basic property is that $\delta(K_Z) \leq sp(K_Z^*)$, where $sp(K_Z^*)$ indicates the spectral radius, or modulus of the largest eigenvalue of $K_Z^*$. Thus the goal of this paper is to construct a space $Z$ such that the spectral radius of $K_Z^*$ is the number given in the Theorem.

§1. Basic properties of $I$, $J$, and $K$

For $1 \leq j \leq q - 1$, define $R_j$ as the set of matrices in $M_q$ of rank less than or equal to $j$. In $P(M_q)$, $R_1$ consists of matrices of rank exactly 1 since the zero matrix is not in $P(M_q)$. For $\lambda, \nu \in P^{q-1}$, let $\lambda \otimes \nu = (\lambda_i \nu_j) \in P(M_q)$ denote the outer vector product. The map

$$
P^{q-1} \times P^{q-1} \ni (\lambda, \nu) \mapsto \lambda \otimes \nu \in R_1 \subset P(M_q)
$$

is biholomorphic, and thus $R_1$ is a smooth submanifold.

We let $I : P(M_q) \to P(M_q)$ denote the birational involution given by matrix inversion $I(A) = A^{-1}$. We let $x_{[k,m]}$ denote the $(q - 1) \times (q - 1)$ sub-matrix of $(x_{i,j})$ which is obtained by deleting the $k$-th row and the $m$-th column. We recall the classic formula
$I(x) = (\det(x))^{-1} \hat{I}(x)$, where $\hat{I} = (\hat{I}_{i,j})$ is the homogeneous polynomial map of degree $q - 1$ given by the cofactor matrix

$$\hat{I}_{i,j}(x) = C_{j,i}(x) = (-1)^{i+j} \det(x_{[i,j]}).$$ \hfill (1.1)

Thus $\hat{I}$ is a homogeneous polynomial map which represents $I$ as a map on projective space. We see that $\hat{I}(x) = 0$ exactly when the determinants of all $(q - 1) \times (q - 1)$ minors of $x$ vanish.

We may always represent a rational map $f = [f_1 : \cdots : f_{q^2}]$ of projective space $\mathbb{P}^{q^2-1}$ in terms of homogeneous polynomials of the same degree and without common factor. We define the degree of $f$ to be the degree of $f_j$, and the indeterminacy locus is defined as $\mathcal{I}(f) = \{f_1 = \cdots = f_{q^2} = 0\}$. The indeterminacy locus represents the points where it is not possible to extend $f$, even as a continuous mapping. The indeterminacy locus always has codimension at least 2. In the case of the rational map $I$, the polynomials $C_{j,i}(x)$ have no common factor. Further, $\hat{I}(x) = 0$ exactly when $x \in R_{q-2}$, so it follows that the indeterminacy set is $\mathcal{I}(I) = R_{q-2}$.

We let $J : \mathbb{P}(\mathcal{M}_q) \to \mathbb{P}(\mathcal{M}_q)$ be the birational involution given by $J(x) = (J(x)_{i,j}) = (1/x_{i,j})$, which takes the reciprocal of all the entries. In the sequel, we will sometimes write $J(x) = \frac{1}{x}$. We may define

$$\hat{J}(x) = J(x)\Pi(x)$$ \hfill (1.2)

where $\Pi(x) = \prod x_{a,b}$ is the homogeneous polynomial of degree $q^2$ obtained by taking the product of all the entries $x_{a,b}$ of $x$, and $\hat{J}(x) = (\hat{J}_{i,j})$ is the matrix of homogeneous polynomials of degree $q^2 - 1$ such that $\hat{J}_{i,j} = \Pi_{(a,b) \neq (i,j)} x_{a,b}$ is the product of all the $x_{a,b}$ except $x_{i,j}$. Thus $\hat{J}$ is the projective representation of $J$ in terms of homogeneous polynomials.

We define $K = I \circ J$. On projective space the map $K$ is represented by the polynomial map (1.4) below. Since $I \circ J$ has degree $(q - 1)(q^2 - 1)$, we see from Proposition 1.1, that the entries of $\hat{I} \circ \hat{J}$ must have a common factor of degree $q^3 - 2q^2$.

When $V$ is a variety, we write $K(V) = W$ for the strict transform of $V$ under $K$, which is the same as the closure of $K(V - \mathcal{I}(K))$. We say that a hypersurface $V$ is exceptional if $K(V)$ has codimension at least 2. The map $I$ is a biholomorphic map from $\mathcal{M}_q - R_{q-1}$ to itself, so the only possible exceptional hypersurface for $I$ is $R_{q-1}$. We define

$$\Sigma_{i,j} = \{x = (x_{k,l}) \in \mathcal{M}_q : x_{i,j} = 0\}.$$ \hfill (1.3)

The map $J$ is a biholomorphic map of $\mathcal{M}_q - \bigcup_{i,j} \Sigma_{i,j}$ to itself, and the exceptional hypersurfaces are the $\Sigma_{i,j}$. Further, the indeterminacy locus is

$$\mathcal{I}(J) = \bigcup_{(a,b) \neq (c,d)} \Sigma_{a,b} \cap \Sigma_{c,d}.$$ 

**Proposition 1.1.** The degree of $K$ is $q^2 - q + 1$. Its representation $\hat{K} = (\hat{K}_{i,j})$ in terms of homogeneous polynomials is given by

$$\hat{K}_{i,j}(x) = C_{j,i} \left(1/x\right) \Pi(x)$$ \hfill (1.4)
where $C_{j,i}$ and II are as in (1.1) and (1.2).

Proof. Observe that $C_{j,i}(1/x)$ is independent of the variable $x_{j,i}$, while $\hat{K}(x)_{j,i}$ is not divisible by the variables $x_{k,\ell}$ with $k \neq j$ and $\ell \neq i$. Hence the greatest common divisor of all polynomials on the right hand side of (1.4) is 1. Thus the algebraic degree of $K$ is equal to the degree of $\hat{K}(x)_{i,j}$, which is $q^2 - q + 1$. \hfill \Box

§2. Construction of $R^1$

We will construct a complex manifold $\pi: \mathcal{Z} \to \mathbf{P}(\mathcal{M}_q)$ by performing a series of blowups. First we will blow up the spaces $R_1$ and $A_{i,j}$, $1 \leq i,j \leq q$. The exceptional (blowup) hypersurfaces will be denoted $R^1$ and $A^{i,j}$. Then we will blow up surfaces $B_{i,j} \subset A^{i,j}$, which will create exceptional hypersurfaces $B^{i,j}$. The precise nature of $\mathcal{Z}$ depends on the order in which the various blowups are performed. Different orders of blowup will produce different spaces $\mathcal{Z}$, but the identity map of $\mathbf{P}(\mathcal{M}_q)$ to itself induces a birational equivalence between the spaces, and this equivalence induces the identity map on $\text{Pic}(\mathcal{Z})$ (as well as on $H^{1,1}(\mathcal{Z})$). Any of these spaces $\mathcal{Z}$ yields an induced birational map $K_\mathcal{Z}$, and each $K_\mathcal{Z}$ induces essentially the same pullback map $K^1_\mathcal{Z}$ on $\text{Pic}(\mathcal{Z})$.

We start our discussion with $R_1$. Let $\pi_1: \mathcal{Z}_1 \to \mathbf{P}(\mathcal{M}_q)$ denote the blowup of $\mathbf{P}(\mathcal{M}_q)$ along $R_1$. We will give a coordinate chart for points of $\mathcal{Z}_1$ lying over a point $x^0 \in R_1$. Let us first make a general observation. Let $\rho_{\ell,m}$ denote the matrix operation which interchanges the $\ell$-th and $m$-th rows of a matrix $x \in \mathcal{M}_q$, and let $\gamma_{\ell,m}$ denote the interchange of the $\ell$-th and $m$-th columns. It is evident that $J$ commutes with both $\rho_{\ell,m}$ and $\gamma_{\ell,m}$, whereas we have $\rho_{\ell,m}(I(x)) = I(\gamma_{\ell,m}(x))$. Thus, for the purposes of looking at the induced map $K_{\mathcal{Z}_1}$, we may permute the coordinates of $(x_{i,j})$, and without loss of generality we may assume that the (1,1) entry of $x^0$ does not vanish. This means that we may assume that $x^0 = \lambda^0 \otimes \nu^0$ with $\lambda^0, \nu^0 \in U_1$, where $U_1 = \{z = (z_1, \ldots, z_q) \in \mathbb{C}^q : z_1 = 1\}$.

We write the standard affine coordinate charts for $\mathbf{P}(\mathcal{M}_q)$ as

$$W_{r,s} = \{x \in \mathcal{M}_q : x_{r,s} = 1\} \subset \mathbb{C}^{q^2},$$

where $1 \leq r, s \leq q$. Let us define $V$ to be the set of all matrices $x \in \mathcal{M}_q$ such that the first row and column vanish. Further, for $2 \leq k, \ell \leq q$, we define a subset of $V$:

$$V_{k,\ell} = \{x \in \mathcal{M}_q : x = \begin{pmatrix} 0 & 0 \\ 0 & x_{[1,1]} \end{pmatrix} \text{ and } x_{k,\ell} = 1\}.$$ (2.2)

Now we may represent a coordinate neighborhood of $\mathcal{Z}_1$ over $x^0$ as

$$\pi_1: \mathbb{C} \times U_1 \times U_1 \times V_{k,\ell} \to W_{1,1}, \quad \pi_1(s, \lambda, \nu, v) = \lambda \otimes \nu + sv.$$ (2.3)

Since $\lambda \otimes \nu$ has rank 1 and nonvanishing (1,1) entry, we see that $\pi_1(s, \lambda, \nu, v) \in R_1$ exactly when $s = 0$. Thus the points of $R^1$ which are in this coordinate neighborhood are given by $\{s = 0\}$. If $y \in \mathcal{M}_q$ is a matrix with $y_{k,\ell} \neq 0$, then we find $\pi_1^{-1}(y) = (s, \lambda, \nu, v)$, where

$$\tilde{y} = y/y_{k,\ell}, \quad s = y_{k,\ell}, \quad \lambda = \tilde{y}_{s,1}, \quad \nu = \tilde{y}_{1,s}, \quad v = s^{-1}(\tilde{y} - \lambda \otimes \nu).$$ (2.4)

We may write the induced map $K_{\mathcal{Z}_1} = \pi_1^{-1} \circ K \circ \pi_1$ in a neighborhood of $\mathcal{Z}_1$ by using the coordinate projections (2.3) and (2.4). This allows us to show that $K_{\mathcal{Z}_1}|_{R^1}$ has a relatively simple expression:
Proposition 2.1. We have $K_{Z_1}(R^1) = R_{q-1}$, so $R^1$ is not exceptional for $K_{Z_1}$. In fact for $z_0 = \pi_1(0, \lambda, \nu, v) \in R^1$,\[K_{Z_1}(z) = B \begin{pmatrix} 0 & 0 \\ 0 & I_{q-1}(v') \end{pmatrix} A \] (2.5) where $I_{q-1}$ denotes matrix inversion on $M_{q-1}$, and\[v' = \left( \frac{-v_{j,k}}{\lambda_j^2 \nu_k^2} \right)_{2 \leq j,k \leq q}, \quad A = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ -\lambda_2^{-1} & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ -\lambda_q^{-1} & \cdots & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -\nu_2^{-1} & \cdots & -\nu_q^{-1} \\ 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix}. \] (2.6)

Proof. Without loss of generality, we work at points $\lambda, \nu \in U_1$ such that $\lambda_j, \nu_k \neq 0$ for all $j, k$ and $V$ such that the $v'$ in (2.6) is invertible. Then\[J(\pi_1(s, \lambda, \nu, v)) = \frac{1}{\lambda \otimes \nu} + sv' + O(s^2) = \pi_1(s + O(s^2), \lambda^{-1}, \nu^{-1}, v' + O(s)). \] (2.7)

Observe that\[A \left( \frac{1}{\lambda \otimes \nu} \right) B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \] and\[s A v' B = \begin{pmatrix} 0 & 0 \\ 0 & sA_{[1,1]}v'B_{[1,1]} \end{pmatrix}. \]

Thus\[K_{Z_1}(z) = \pi_1^{-1} \circ I \circ J \circ \pi_1(z) = \pi_1^{-1} I \left( \frac{1}{\lambda \otimes \nu} + sv' + O(s^2) \right) = \pi_1^{-1} \left( B I \left( A \left( \frac{1}{\lambda \otimes \nu} + sv' + O(s^2) \right) B \right) A \right) = \pi_1^{-1} \left( B I \left( \frac{1}{\lambda \otimes \nu} + sv' + O(s^2) \right) A \right), \]
and the Proposition follows if we let $s \to 0.$

Now we will use the identities\[K_{Z_1} \circ J_{Z_1} = I_{Z_1}, \quad I_{Z_1} \circ K_{Z_1} = J_{Z_1}. \]

Proposition 2.2. We have $K_{Z_1}(JR_{q-1}) = R^1$, and thus $JR_{q-1}$ is not exceptional for $K_{Z_1}$.

Proof. For generic $s, \lambda, \nu, v,$ and $v'$ as in (2.6), we have (2.7) in the previous Proposition. Letting $s \to 0$, we see that these points are dense in $R^1$, and thus $J_{Z_1}R^1 = R^1$. Now\[K_{Z_1}(J(R_{q-1})) = I_{Z_1}(R_{q-1}) = I_{Z_1}(K_{Z_1}R^1) = J_{Z_1}(R^1) = R^1, \]
where the second equality in the first line follows from the previous Proposition.
§3. Construction of \( \mathcal{A}^{i,j} \)

We let \( A_{i,j} \) denote the set of \( q \times q \) matrices whose \( i \)-th row and \( j \)-th columns consist entirely of zeros. Let \( \pi_2 : \mathbb{Z}_2 \to \mathbb{P}(\mathcal{M}_q) \) denote the space obtained by blowing up along all of the the centers \( A_{i,j} \) for \( 1 \leq i, j \leq q \). As we discussed earlier, it will be immaterial for our purposes what order we do the blowups in. Let us fix our discussion on \((i, j) = (1, 1)\). The set \( A_{1,1} \) is equal to the set \( V \) which was introduced in the previous section. Let us use the notation

\[
U = U_{1,r} = \{ z \in \mathcal{M}_q : z = \begin{pmatrix} * & * \\ * & 0_{q-1} \end{pmatrix}, \ z_{1,r} = 1 \} 
\]

for the matrices which consist of zeros except for the first row and column, and which are normalized by the entry \( z_{1,r} \). With this notation and with \( W_{k,\ell}, V_{k,\ell} \) as in (2.1,2), we define the coordinate chart

\[
\pi_2 : C \times U \times V_{k,\ell} \to W_{k,\ell} \subset \mathcal{M}_q, \quad \pi_2(s, \zeta, v) = s\zeta + v = \begin{pmatrix} s\zeta & s\zeta \\ s\zeta & v \end{pmatrix}.
\]

Coordinate charts of this form give a covering of \( \mathcal{A}^{1,1} \), and \( \{ s = 0 \} \) defines the set \( \mathcal{A}^{1,1} \) within each coordinate chart. If \( x \in \mathcal{M}_q \), then we normalize to obtain \( \hat{x} := x/x_{k,\ell} \in W_{k,\ell} \), and

\[
\pi_2^{-1}(x) = (s, \zeta, v), \quad v = \hat{x}_{[1,1]}, \quad s = \hat{x}_{1,r}, \quad \zeta = (\hat{x} - v)/\hat{x}_{1,r}.
\]

We let \( K_{\mathbb{Z}_2} = \pi_2^{-1} \circ K \circ \pi_2 \) denote the induced birational map on \( \mathbb{Z}_2 \).

**Proposition 3.1.** For \( 1 \leq r, s \leq q \), \( K_{\mathbb{Z}_2}(\Sigma_{r,s}) = A^{s,r} \), and in particular \( \Sigma_{r,s} \) is not exceptional for \( K_{\mathbb{Z}_2} \).

**Proof.** As was noted at the beginning of the previous section, it is no loss of generality to assume \((r, s) = (1, 1)\) and \( 2 \leq k, \ell \leq q \). For generic \( x \in \mathcal{M}_q \), we may use \( \hat{K} \) from (1.4) and define \( y \) by

\[
\hat{K}(x) = \Pi(x) \left( C_{i,i}(\frac{1}{x}) \right) = y.
\]

We write \( \pi(\sigma, \zeta, v) = y \), and we next determine \( \sigma, \zeta \) and \( v \). Now let us use the notation \( s = x_{1,1} \), so \( \Pi(x) = s\Pi'(x) \), where \( \Pi' \) denotes the product of all \( x_{a,b} \) except \((a, b) = (1, 1)\). For \( 2 \leq i, j \leq q \), we have

\[
y_{i,j} = s\Pi'(x) \left( \frac{1}{s}a_{i,j}(x) + O(1) \right)
\]

with \( a_{i,j}(x) = (-1)^{i+j} \text{det}((1/x)_{[i,j],[1,1]}) \), which gives

\[
v_{i,j} = \tilde{y}_{i,j} = y_{i,j}/y_{k,\ell} = a_{i,j}(x) + O(s), \quad 2 \leq i, j \leq q.
\]

For generic \( x \), we may let \( s \to 0 \), and then the value of \( v \) approaches \( (a_{i,j}(x))/a_{k,\ell}(x) \) which by (1.4) is just \( K_{q-1}(x_{[1,1]}) \), normalized at the \((k, \ell)\) slot.
We find values of $y$ and $\pi_B$. We define the set $\Pi_B$. Let our attention on the case $(i, j)$. We may consider $A$ projection $\zeta$. Let $U'$ be as in (3.1), and set $(4.2)$. We define the (normalized) to be a coordinate chart in the fiber over a point of $\Sigma$. Let us use the following homogeneity property of $K$. An element of the first row of $y$ is given by $y_{1,j} = (-1)^{j+1} det(1/x_{[j,1]})$. If we expand this determinant into minors along the top row, we have

$$y_{1,j} = \sum_{2 \leq p \leq q} (-1)^{j+1+p} det \left((1/x_{[j,1]})_{[1,p]}\right) x_{1,p}^{-1}$$

We use the notation $y_{1,*}$ and $(1/x_{1,*})$ for the vectors $(y_{1,p})_{2 \leq p \leq q}$ and $(1/x_{1,p})_{2 \leq p \leq q}$. Thus we find $y_{1,*} = v(1/x_{1,*})$. It is evident that $y_{1,1} = det(1/x_{[1,1]})$.

Now we consider the range of $y$. An element of the first row and column of $y$ are dense in $Y_{k,\ell}$. Thus from (4.2), we see that $\pi_B$ is dense in $Y$. Thus $K_{\Sigma_2}(\Sigma_{1,1}) = A^{1,1}$.

§4. Construction of $B^{i,j}$

For $1 \leq i, j \leq q$, we let $U_{i,j} = \{\zeta \in M_q : \zeta_{[i,j]} = 0\}$ to be the set of matrices for which all entries are zero, except on the $i$-th row and $j$-th column. In the construction of $A^{i,j}$, we may consider $U_{i,j}$ (normalized) to be a coordinate chart in the fiber over a point of $A_{i,j}$. We define $B_{i,j} = \{(s, \zeta, v) \in A^{i,j} : s = 0, \zeta_{i,j} = 0\}$, which has codimension 2 in $Z_2$, and we let $\pi_3 : Z_3 \to Z_2$ be the new manifold obtained by blowing up all the sets $B_{i,j}$.

Let $K_{\Sigma_3}$ denote the induced birational map on $Z_3$. As we have seen before, we may focus our attention on the case $(i, j) = (1, 1)$. Let us use the $(s, \zeta, v)$ coordinate system (3.2) at $A^{1,1}$. Let $U$ be as in (3.1), and set $U' = \{\zeta \in U : \zeta_{1,1} = 0\}$. We define the coordinate projection

$$\pi_3 : \mathbb{C} \times \mathbb{C} \times U' \times V_{1,1} \to \mathbb{C} \times U \times V_{1,1}, \quad \pi(t, \tau, \xi, v) = (s, \zeta, v), \quad s = t, \zeta = (t\tau, \xi), v = v,$$  \hspace{1cm} (4.1)

where the notation $\zeta = (t\tau, \xi)$ means that $\zeta_{1,1} = t\tau$, and $\zeta_{a,b} = \xi_{a,b}$ for all $(a, b) \neq (1, 1)$. Thus $B^{1,1}$ is defined by the condition $t = 0$ in this coordinate chart. Composing the two coordinate projections, $Z_3 \to Z_2$ and $Z_2 \to M_q$, we have

$$\pi : (t, \tau, \xi, v) \mapsto \begin{pmatrix} t^2 \tau & t\xi \\ t\xi & v \end{pmatrix} = x.$$  \hspace{1cm} (4.2)

From (4.2), we see that $\pi^{-1}(x) = (t, \tau, \xi, v)$, where

$$\tilde{x} = x/x_{\ell,k}, \quad v = \tilde{x}_{[1,1]}, \quad t = \tilde{x}_{1,r}, \quad \tau = \tilde{x}_{1,1}/t^2, \quad \xi_{1,j} = x_{1,j}/x_{1,r}, \quad 2 \leq j \leq q.$$  \hspace{1cm} (4.3)

We will use the following homogeneity property of $K$. If $x \in M_q$, we let $\chi_t(x)$ denote the matrix obtained by multiplying the 1st row by $t$ and then the 1st column by $t$, so the (1,1) entry is multiplied by $t^2$. It follows that $\chi_t J_{\chi_t} = J$ and $\chi_t I_{\chi_t} = I$, so

$$K \begin{pmatrix} \tau & \xi \\ \xi & v \end{pmatrix} = \begin{pmatrix} \tau' & \xi' \\ \xi' & v' \end{pmatrix} \quad \text{implies} \quad K \begin{pmatrix} t^2 \tau & t\xi \\ t\xi & v \end{pmatrix} = \begin{pmatrix} t^2 \tau' & t\xi' \\ t\xi' & v' \end{pmatrix}.$$  \hspace{1cm} (4.4)
Proposition 4.1. For $1 \leq i, j \leq q$, we have $K_{Z_3}(B^{i,j}) = B^{i,i}$, and in particular, $B^{i,j}$ is not exceptional.

Proof. As before, we may assume that $(i,j) = (1,1)$. A point near $B^{1,1}$ may be represented in the coordinate chart (4.2) as $\pi(t,\tau,\xi,v) = \left( t^2\tau, \frac{t\xi}{\tau}, \frac{v}{v^2} \right) = x$. We define $\tau'$, $\xi'$, and $v'$ by the condition $K\left( \frac{\tau}{\xi}, \frac{\xi}{v} \right) = K(\tau', \xi', v')$, so $K(x)$ is given by the right hand side of (4.4). By (4.3), the coordinates $(t'', \tau'', \xi'', v'') = \pi^{-1}K(x)$ are

$$v'' = v/v_{k,\ell}, \ t'' = t\xi_{1,r}/v_{i,j}^{1,1}, \ \tau'' = \tau(v_{k,\ell}/v_{i,j}^{1,1})^2.$$ 

From this we see that $t'' \to 0$ as $t \to 0$, which means that $K_{Z_3}(B^{1,1}) \subset B^{1,1}$. And since $K$ is dominant on $P(M_q)$, we see that $K_{Z_3}(B^{1,1})$ is dense in $B^{1,1}$. \qed

Next we see how $A^{i,j}$ maps under $K_{Z_3}$. A point near $A^{1,1}$ may be written in coordinates (3.2) as $(s,\zeta,v)$. We write $K$ of this point in coordinates (4.1) as $(t,\tau,\xi,w)$.

Proposition 4.2. For $1 \leq i, j \leq q$, we have $K_{Z_3}(A^{i,j}) \subset B^{i,j}$. Further, $\frac{dt}{ds} \neq 0$ at generic points $(0,\zeta,v) \in A^{i,j}$.

Proof. Without loss of generality we assume $(i,j) = (1,1)$. Let us define $x$ and $y$ as

$$x = \pi_2(s,\zeta,v) = \left( \frac{s\zeta}{s\zeta}, \frac{s\zeta}{x} \right), \ \ y = \tilde{K}(x) = \Pi(x)C\left( \frac{1}{x} \right).$$ 

For $2 \leq h, m \leq q$ there are polynomials $a_{h,m}(\zeta,v)$ and $b_{h,m}(\zeta,v)$ such that

$$y_{1,1} = s^{2q-1}a_{1,1}(\zeta,v), \ y_{1,m} = s^{2q-2}a_{1,m}(\zeta,v), \ y_{h,m} = s^{2q-3}a_{h,m}(\zeta,v) + s^{2q-2}b_{h,m}(\zeta,v).$$

We have $t = sa_{1,r}/a_{k,\ell} + O(s^2)$, so $dt/ds \to a_{1,r}/a_{k,\ell}$ as $s \to 0$. Thus $dt/ds \neq 0$ at generic points of $A^{1,1} = \{s = 0\}$. By (4.3), we see that

$$(t,\tau,\xi,w) \to (0,a_{1,1}a_{k,\ell}/a_{1,r}^2, a_{1,s}/a_{1,r}, a_{[1,1]}/a_{k,\ell}) \in B^{1,1}$$ 

as $s \to 0$. \qed

§5. Picard Group Pic($Z$)

We write $Z = Z_3$ and recall that the Picard group Pic($Z$) is the set of divisors modulo linear equivalence. Pic($P(M_q)$) = $\langle H \rangle$ is generated by any hyperplane $H$. We will work with the following basis for Pic($Z$):

$$\{H, F^{1}, A^{i,j}, B^{i,j}, 1 \leq i, j \leq q\}. \quad (5.1)$$

Now consider the hypersurface $\Sigma_{i,j}$. Pulling this back under $\pi_1 : Z_1 \to P(M_q)$, we find

$$\pi_1^*\Sigma_{i,j} = H_{Z_1} = \Sigma_{i,j},$$
where $\Sigma_{i,j}$ on the right hand side denotes the strict transform $\pi^{-1}\Sigma_{i,j}$. The equality between the strict and total transforms follows because the indeterminacy locus $I(\pi^{-1}) = R_1$ is not contained in $\Sigma_{i,j}$. On the other hand, if we define

$$T_{i,j} := \{(a, b) : a = i \text{ or } b = j\} \quad (5.2),$$

then $\Sigma_{i,j}$ contains $A_{a,b}$ exactly when $(a, b) \in T_{i,j}$. Thus, pulling back under $\pi_2 : \mathcal{Z}_2 \rightarrow \mathcal{Z}_1$, we have

$$\pi_2^*\Sigma_{i,j} = H_{\mathcal{Z}_2} = \Sigma_{i,j} + \sum_{(a,b) \in T_{i,j}} A_{a,b}.$$

We will next pull this back under $\pi_3 : \mathcal{Z}_3 \rightarrow \mathcal{Z}_2$. For this, we note that $B_{a,b} \subset A_{a,b}$, and in addition $B_{i,j} \subset \Sigma_{i,j}$. Rearranging our answer, we have:

$$\Sigma_{i,j} = H_{\mathcal{Z}} - B^{i,j} - \sum_{(a,b) \in T_{i,j}} (A_{a,b} + B_{a,b}). \quad (5.3)$$

**Proposition 5.1.** The class of $JR_{q-1}$ in $\text{Pic}(\mathcal{Z})$ is given in the basis (5.1) by

$$JR_{q-1} = (q^2 - q)H - (q - 1)R^1 - (2q - 3)\sum_{a,b} A_{a,b} - (2q - 2)\sum_{a,b} B_{a,b}. \quad (5.4)$$

**Proof.** The polynomial $P(x) := \Pi(x)\det(\frac{1}{x})$, analogous to (1.4), is irreducible and has degree $q^2 - q$. Thus $JR_{q-1} = \{P = 0\} = (q^2 - q)H$ in $\text{Pic}(\mathbb{P}(\mathcal{M}_q))$. Now we pull this back under the coordinate projection $\pi_1$ in (2.3). That is, we evaluate $P(x)$ for $x = \pi_1(s, \lambda, \nu, v)$. For $s = 0$ and generic $\lambda, \nu$, and $v$, the entries of $x = \lambda \otimes \nu + sv$ are nonzero, so $\Pi(x) \neq 0$. We will show $\det(\frac{1}{x}) = \alpha s^{q-1} + \cdots$, where $\alpha \neq 0$ for generic $\lambda, \nu$, and $v$. By (2.7), we must evaluate $\det(M)$ with $M = \lambda^{-1} \otimes \nu^{-1} + sv' + O(s^2)$. Now we do elementary row and columns such as add $\lambda_j^{-1}\nu$ to the $j$th row, and we do not change the determinant. In this way, we see that $\det(M)$ is equal to $\det\left(\begin{array}{cc} 1 & 0 \\ 0 & sv' + O(s^2) \end{array}\right) = \alpha s^{q-1} + \cdots$. This means that

$$(q^2 - q)H = \pi_1^*(JR_{q-1}) = JR_{q-1} + (q - 1)R^1 \in \text{Pic}(\mathcal{Z}_1).$$

Now we bring this back to $\mathcal{Z}_2$ by pulling back under the projection $\pi_2$ defined in (3.2). In this case, we have $\Pi(\pi_2(s, \zeta, v)) = \alpha s^{2q-1} + \cdots$, where $\alpha = \alpha(\zeta, v) \neq 0$ for generic $\zeta$ and $v$. On the other hand, we have $\det\left(\begin{array}{cc} s^{-1}\zeta^{-1} & s^{-1}\zeta^{-1} \\ s^{-1}\zeta^{-1} & v^{-1} \end{array}\right) = s^{-2}\beta + s^{-1}\gamma + \cdots$, and $\beta(\zeta, v) \neq 0$ at generic points. Thus $P(\pi_2(s, \zeta, v)) = cs^{q-3}$, which gives the coefficient $2q - 3$ for each $A^{i,j}$:

$$(q^2 - q)H = JR_{q-1} + (q - 1)R^1 + (2q - 3)\sum_{i,j} A^{i,j} \in \text{Pic}(\mathcal{Z}_2).$$

Pulling back to $\mathcal{Z}_3$ is similar, except that $\Pi(\pi_3(t, \tau, \xi, v)) = \alpha t^{2q} + \cdots$. Thus we obtain the coefficient $2q - 2$ for $B^{i,j}$ in (5.4).
§6. The induced map $K^*_Z$ on $\text{Pic}(Z)$

We define the pullback map on functions by composition $K^*_Z \varphi := \varphi \circ K_Z$. We may apply $K^*_Z$ to local defining functions of a divisor, and since $K^*_Z$ is well defined off the indeterminacy locus, which has codimension $\geq 2$, $K^*_Z$ induces a well-defined pullback map on $\text{Pic}(Z)$.

**Proposition 6.1.** $K^*_Z$ maps the basis (5.1) according to:

$$
H \mapsto (q^2 - q + 1) H - (q - 2) \mathcal{R}^1 - \sum_{a,b} ((2q - 3) A^{a,b} - (2q - 2) B^{a,b})
$$

$$
\mathcal{R}^1 \mapsto (q^2 - q) H - (q - 1) \mathcal{R}^1 - \sum_{a,b} ((2q - 3) A^{a,b} - (2q - 2) B^{a,b})
$$

$$
A^{i,j} \mapsto H - B^{j,i} - \sum_{(a,b) \in T_{i,j}} (A^{a,b} + B^{a,b})
$$

$$
B^{i,j} \mapsto A^{i,j} + B^{j,i}
$$

**Proof.** Let us start with $\mathcal{R}^1$. By §2, $K_Z|_{JR_{q-1}}$ is dominant as a map to $\mathcal{R}^1$. Since $K_Z$ is birational, it is a local diffeomorphism at generic points of $JR_{q-1}$. Thus we have $K^*_Z(\mathcal{R}^1) = JR_{q-1}$, so the second line in (6.1) follows from Proposition 5.1.

Similarly, since $K_Z|_{\Sigma_{i,j}}$ is a dominant map to $A^{j,i}$, we have $K^*_Z(A^{i,j}) = \Sigma_{j,i}$, and the third line of (6.1) follows from (5.3).

In the case of $B^{i,j}$, we know from §4 that $K^*_Z^{-1} B^{i,j} = A^{j,i} \cup B^{j,i}$. Thus $K^*_Z B^{i,j} = \lambda A^{i,j} + \mu B^{j,i}$ for some integer weights $\lambda$ and $\mu$. Again, since $K_Z$ is birational, and $K_Z|_{B^{i,j}}$ is a dominant map to $B^{j,i}$, we have $\mu = 1$. Proposition 4.2 gives us $\lambda = 1$.

Finally, set $h(x) = \sum_{i,j} a_{i,j} x_{i,j}$, and let $H = \{h = 0\}$ be a hyperplane. The pullback is given by the class of $\{h \tilde{K}(x) = 0\} = \sum_{i,j} a_{i,j} \tilde{K}_{i,j}(x) = 0$, where $\tilde{K}$ is given by (1.4). Pulling back $h$ is similar to the situation in Proposition 5.1, where we pulled back the function $P(x)$. The difference is that instead of working with $\text{det}(\mathcal{J})$ we are working with all of the $(q-1) \times (q-1)$ minors. By Proposition 1.1, we have $K^*H = (q^2 - q + 1) H \in \text{Pic}(P(\mathcal{M}_q))$. Next we will move up to $Z_1$ by pulling back under $\pi_1$ and finding the multiplicity of $\mathcal{R}^1$. We consider $h \tilde{K}_1(s, \lambda, \nu, \upsilon)$, and we recall the matrix $M$ from the proof of Proposition 5.1. We see that each $(q-1) \times (q-1)$ minor of $M$ is either $O(s^{q-1})$ or $O(s^{q-2})$. Thus for a generic hyperplane, the order of vanishing is $q - 2$, so we have

$$(q^2 - q + 1) H = K^*H + (q - 2) \mathcal{R}^1 \in \text{Pic}(Z_1).$$

Next, to move up to $Z_2$, we look at the order of vanishing of $h \tilde{K}_2(s, \zeta, \upsilon)$ in $s$. Again $\Pi(\pi_2(s, \zeta, \upsilon)) = \alpha_s s^{2q-1} + \cdots$. The $(q-1) \times (q-1)$ minors of

$$
\begin{pmatrix}
s^{-1} \zeta^{-1} & s^{-1} \zeta^{-1} \\
1 & v^{-1}
\end{pmatrix}
$$

grow most quickly behave like $s^{-2} \beta + s^{-1} \gamma + \cdots$. Thus for generic coefficients $a_{i,j}$ we have vanishing to order $2q - 3$ in $s$, and so $2q - 3$ is the coefficient for each $A^{i,j}$ as we pull back to $\text{Pic}(Z_2)$. Coming up to $Z_3 = Z$, we pull back under $\pi_3$, and the calculation of the multiplicity of $B^{i,j}$ is similar. This gives the first line in (6.1).
Proposition 6.2. The characteristic polynomial of the transformation (6.1) is

\[ P(\lambda)Q(\lambda)q^{-1}(\lambda - 1)^{q^2-4q+2}(\lambda + 1)q^2-3q+2, \]

where \( P(\lambda) = \lambda^2 - (q^2 - 4q + 2)\lambda + 1 \) and \( Q = (\lambda^2 + 1)^2 - (q - 2)^2\lambda^2 \).

Proof. We will exhibit the invariant subspaces of \( \text{Pic}(\mathcal{Z}) \) which correspond to the various factors of the characteristic polynomial. First, we set \( A := \sum A^{k,\ell} \) and \( B := \sum B^{k,\ell} \), where we sum over all \( k \) and \( \ell \), and we set \( S_1 = \langle H, R^1, A, B \rangle \). By (6.1), \( S_1 \) is \( K_2^* \)-invariant, and the characteristic polynomial of \( K_2^*|_{S_1} \) is seen to be \( P(\lambda)(\lambda - 1)^2 \).

Next, if \( i < j \), then we set \( \alpha_{i,j} = A^{i,j} + A^{j,i} - (A^{i,j} + A^{j,i}) \), and similarly for \( \beta_{i,j} \), using the \( B^{k,\ell} \). Then by (6.1), \( S_{i,j} := \langle \alpha_{i,j}, \beta_{i,j} \rangle \) is invariant, and the characteristic polynomial of \( K_2^*|_{S_{i,j}} \) is \( (\lambda - 1)^2 \).

Similarly, if \( i < j < k \), we set \( \alpha_{i,j,k} = A^{i,j} + A^{i,k} - (A^{i,j} + A^{i,k}) \) and define \( \beta_{i,j,k} \) similarly. Then the 2-dimensional subspace \( S_{i,j,k} := \langle \alpha_{i,j,k}, \beta_{i,j,k} \rangle \) is invariant, and the characteristic polynomial of \( K_2^*|_{S_{i,j,k}} \) is \( (\lambda + 1)^2 \).

Finally, for each \( i \), we consider the row and column sums \( A_{r_i} = q \sum A^{i,j} - A \), \( A_{c_i} = q \sum A^{j,i} - A \), and we make the analogous definition for \( B_{r_i} \) and \( B_{c_i} \). The 4-dimensional subspace \( \langle A_{r_i}, A_{c_i}, B_{r_i}, B_{c_i} \rangle \) is invariant and yields the factor \( Q(\lambda) \). These invariant subspaces span \( \text{Pic}(\mathcal{Z}) \), and the product of these factors gives the characteristic polynomial stated above.

Proof of the Theorem. The spectral radius of \( K_2^* \) is the largest root of the characteristic polynomial, which is given in Proposition 6.2. By inspection, the largest root of the characteristic polynomial is the largest root of \( P(\lambda) \). The spectral radius of \( K_2^* \) is an upper bound for \( \delta(K) \). On the other hand, it was shown in [BV] that this same number is also a lower bound for \( \delta(K) \), so the Theorem is proved.

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