SYMPLECTIC STRUCTURE PERTURBATIONS AND CONTINUITY OF SYMPLECTIC INVARIANTS

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ABSTRACT. This paper studies how some symplectic invariants which are born from Hamiltonian Floer theory (e.g. spectral invariant, boundary depth, (partial) symplectic quasi-state) change with respect to symplectic structure perturbations, i.e., new symplectic structures perturbed from a known symplectic structure. This paper can be roughly divided into two parts. In the first part, we will prove a family of energy estimation inequalities which control the shifts of action functional in the Hamiltonian Floer theory. This directly implies an affirmative conclusion on continuity of spectral invariant and boundary depth in several important cases, for instance, the symplectic surface $\Sigma_{g>1}$ or closed symplectic manifold $M$ with $\dim_K H_2(M, \mathbb{K}) = 1$. This follows by an application on the rigidity of subsets on symplectic manifolds in terms of heavy or superheavy. In the second part, we generalize the construction in the first part to any symplectic manifold. Specifically, in order to deal with the change of Novikov ring due to the perturbations, we will construct a (local) family of variant Floer chain complexes over a variant Novikov ring and study its homologies, which takes its inspiration from Ono’s construction in [Ono05]. We will also prove, in this set-up, a new family of spectral invariant called $t$-spectral invariant is upper semicontinuous. This has applications on a quasi-embedding from $\mathbb{R}^\omega$ to $\text{Ham}(M, \omega)$ under a certain dynamical condition imitating the main result from [Ush13] and several continuity properties of Hofer-Zehnder capacity and spectral capacity. Finally, $t$-boundary depth (defined over the local family of variant Floer chain complexes above) is defined and briefly discussed but its continuity property is unknown.

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1. Introduction

During the process of Floer’s method on solving Arnold’s conjecture (see [Fl89] or [HS95]), Floer chain complex is constructed. Fixing a symplectic manifold \( (M, \omega) \), from its construction (which will be briefly explained in Section 2), Floer chain complex symbolically depends on the following three parameters. One is an almost complex structure \( J : TM \to TM \) (such that \( J^2 = -\mathbb{I} \)), one is a Hamiltonian function \( H \in C^\infty(S^1 \times M) \) and one is a symplectic form \( \omega \in \Omega_2(M) \) (collection of all closed 2-forms on \( M \)). Let us denote it as \( (CF_\ast(M, J, H, \omega), \partial_{J,H,\omega}) \). Meanwhile, with the help of action functional (see (8)), Floer chain complex is actually a filtered chain complex which is filtered by \( \mathbb{R} \). Sometimes when we are interested in a certain filtered (or truncated) Floer chain (sub)complex, we usually denote it as \( (CF^\ast_\ast(M, J, H, \omega), \partial_{J,H,\omega}) \)

\( \lambda \) for any preferred \( \lambda \in \mathbb{R} \). It is natural to ask the relation between two filtered Floer chain complexes if they are constructed from different almost complex structures or different Hamiltonians. Fortunately, this relation has been well-known nowadays due to the construction of (Floer) continuation map between these two chain complexes (see [SZ93]). Moreover, this relation builds an interesting bridge between Floer theory and Hofer geometry (see [Pol01]). We summarize it as follows,

**Proposition 1.1.** (Proposition 5.1 in [Ush13]) Given two pairs of almost complex structures and Hamiltonians \((J_0, H_0)\) and \((J_1, H_1)\) such that the corresponding Floer chain complexes are well-defined, for each \( \lambda \in \mathbb{R} \), there exist

- a chain map

\[
\Phi : (CF^\lambda_\ast(M, J_0, H_0, \omega), \partial_{J_0,H_0,\omega}) \to (CF^\lambda_\ast(M, J_1, H_1, \omega), \partial_{J_1,H_1,\omega}),
\]

- a chain map

\[
\Psi : (CF^\lambda_\ast(M, J_1, H_1, \omega), \partial_{J_1,H_1,\omega}) \to (CF^\lambda_\ast(M, J_0, H_0, \omega), \partial_{J_0,H_0,\omega}),
\]

- two homotopy maps, where \( i = 0, 1 \),

\[
K_i : (CF^\lambda_\ast(M, J_i, H_i, \omega), \partial_{J_i,H_i,\omega}) \to (CF^{\lambda + \int_0^1 (\max_M (H_1 - H_0) - \min_M (H_1 - H_0)) dt}_\ast(M, J_i, H_i, \omega), \partial_{J_i,H_i,\omega})
\]

such that they satisfy the following homotopy relations

\[
\Psi \circ \Phi - \mathbb{I} = \partial_{J_0,H_0,\omega} \circ K_0 + K_0 \circ \partial_{J_0,H_0,\omega},
\]

and

\[
\Phi \circ \Psi - \mathbb{I} = \partial_{J_1,H_1,\omega} \circ K_1 + K_1 \circ \partial_{J_1,H_1,\omega}.
\]

The Hofer norm of \( H \in C^\infty(\mathbb{R}/\mathbb{Z} \times M) \) is defined as \( \|H\|_H := \int_0^1 (\max_M H - \min_M H) \) \( dt \), so Proposition 1.1 can be restated as \( (CF_\ast(M, J_0, H_0, \omega), \partial_{J_0,H_0,\omega}) \) and \( (CF_\ast(M, J_1, H_1, \omega), \partial_{J_1,H_1,\omega}) \) are \( \|H_1 - H_0\|_H \)-quasiequivalent (see Definition 3.7 in [Ush13]). One perspective of viewing
Floer theory is from various constructions of symplectic invariants based on the Floer chain complex or Floer homology. For instance, people have been interested in the following three ones which encode certain homological information (in filtered homology group) as well as its application in the study of dynamics, spectral invariant \( \rho(a,H) \) (see [Vit92], [Sch00], [Oh05] and [Ush08]), boundary depth \( \beta(\phi) \) (see [Ush13] and [UZ15]) and (partial) symplectic quasi-state \( \zeta_a(H) \) (see [EP08] or [EP09]) \(^1\). Their explicit definitions will be given in Section 2. It can be proved that all three of them are independent of almost complex structure. Interestingly, all three of them satisfy the following 1-Lipschitz type propositions in terms of Hamiltonian functions or Hamiltonian diffeomorphisms. Specifically,

**Theorem 1.2.** (1-Lipschitz proposition)

(a) ((5) in Theorem 1 in [Oh05]) For any fixed \( a \in QH_s(M,\omega) \), \( \rho(a,\cdot) \) is 1-Lipschitz with respect to the Hofer norm on \( C^\infty(\mathbb{R}/\mathbb{Z} \times M) \): for any \( H, G \in C^\infty(\mathbb{R}/\mathbb{Z} \times M) \), we have

\[
|\rho(a,H) - \rho(a,G)| \leq ||H - G||_H.
\]

(b) ((iii) in Theorem 1.4 in [Ush13]) \( \beta(\cdot) \) is 1-Lipschitz with respect to the Hofer norm on \( \text{Ham}(M,\omega) \) (set of all the Hamiltonian diffeomorphisms) which is defined as \( ||\phi||_H = \inf_{H \in C^\infty(\mathbb{R}/\mathbb{Z} \times M), \phi = \phi_H} ||H||_H \): for any \( \phi, \psi \in \text{Ham}(M,\omega) \), we have

\[
|\beta(\phi) - \beta(\psi)| \leq ||\phi^{-1}\psi||_H.
\]

(c) (Theorem 3.2 in [Ent14]) Suppose \( a \in QH_s(M,\omega) \) is any idempotent element. \( \zeta_a(\cdot) \) is 1-Lipschitz with respect to the uniform norm: for any \( H, G \in C^\infty(M) \),

\[
\min_M (H - G) \leq \zeta_a(H) - \zeta_a(G) \leq \max_M (H - G).
\]

**Remark 1.3.** Recently, in [UZ15], persistent homology provides a different perspective to view Floer homology or Floer chain complex. This idea can generalize the philosophy of constructing some symplectic invariants from the language of barcodes (see Theorem A and B in [UZ15]). For instance, there is a uniform way to view spectral invariant \( \rho(a,\cdot) \) and and boundary depth \( \beta(\cdot) \). In particular, the later is one of the members in a family of analogous construction called generalized boundary depths, which have been used in [Zha16] to generalize the main result from [PS14].

In the spirit of perturbation (in \( C^0 \)-sense) of Hamiltonian functions above, it is natural to ask how these symplectic invariants change when we perturb the symplectic forms (not necessarily in the same cohomology class). Suppose, once and for all, we start from \((M,\omega_0)\) with a fixed initial symplectic structure \( \omega_0 \). Likewise in the Hamiltonian case, when \( \omega_0 \) is perturbed to another symplectic form \( \omega_1 \), the corresponding Hamiltonian dynamics will change. However, Moser’s trick can be used to simply the set-up. First, let \( \text{Per}(\omega_0, H) \) be the collection of all the (geometric) distinct Hamiltonian 1-periodic orbits (of Hamiltonian system in terms of symplectic form \( \omega_0 \) and Hamiltonian \( H \)) and \( x(t) \) is a generic element in \( \text{Per}(\omega_0, H) \). In Section 3, we can prove

**Proposition 1.4.** For sufficiently small perturbation of symplectic form \( \omega_0 \), say \( \omega_1 \), there exists a \( C^0 \)-small diffeomorphism \( \phi \in \text{Diff}(M) \) such that \( \phi^* \omega_1 \) is symplectic and \( \phi^* \omega_1 - \omega_0 \) vanishes in a neighborhood of each \( x(t) \) in \( \text{Per}(\omega_0, H) \).

\(^1\)Through the paper, notations of these invariants might change slightly according to different situations because sometimes we want to emphasize their dependence on the symplectic forms.
This proposition will be restated and proved in a more precise way. See Proposition 3.3. Therefore, instead of comparing \((\text{CF}_*(M, J, H, \omega_0), \partial_{J,H,\omega_0})\) with \((\text{CF}_*(M, J, H, \omega_1), \partial_{J,H,\omega_1})\) directly, we can insert two intermediate steps as follows,

\[
\begin{align*}
\text{CF}_*(M, J, H, \omega_0) & \xrightarrow{(1)} \text{CF}_*(M, J, H, \omega_1) \\
& \xrightarrow{(2)} \text{CF}_*(M, J, H, \phi^*\omega_1) \\
& \xrightarrow{(3)} \text{CF}_*(M, \phi^*J, \phi^*H, \phi^*\omega_1)
\end{align*}
\]

Tracing how any symplectic invariant change along the diagram above, its value is preserved in (2) because, by definition (or invariant proposition), it is invariant under any symplectomorphism. By Theorem 1.2 and the fact on the independence of almost complex structure, (3) only results in a 1-Lipschitz deviation on the symplectic invariant due to (small) perturbation of \(J\) and \(H\). Therefore, the original comparison (1) can be replaced by (4) if we forgive the small deviation from (3). In other words, we will only consider the perturbation in the form of

\[
\omega_1 = \omega_0 + \alpha
\]

where \(\alpha\) vanishes near each Hamiltonian orbit and sufficiently small in a certain norm (explained at beginning of Section 3). Denote a set of all such \(\alpha\) by \(\Omega\). Because our manifold \(M\) is closed, it is not hard to see \(\text{Per}(\omega_0, H) = \text{Per}(\omega_0 + \alpha, H)\) for any \(\alpha \in \Omega\) (see Corollary 3.4). Therefore, the generators of the original Floer chain complex constructed from data \((M, J, H, \omega_0)\) and the Floer chain complex constructed from perturbed data \((M, J, H, \omega_0 + \alpha)\) are identical. One of the differences is then from the boundary operator (different symplectic forms will give different perturbed Cauchy-Riemann equations). Meanwhile, what might be overlooked at first sight is, in general, Floer chain complex is constructed as a (free) module over a Novikov ring in order to keep tracking the energy. However, by definition (see (28)) where people very often use, Novikov rings are defined with respect to symplectic forms, so change of symplectic forms will result in many issues. This will be explained more explicitly later in this section. However, a special case that we can ignore the change of Novikov ring is when \(M\) is aspherical, i.e., \(\pi_2(M) = 0\), then in this case any Novikov ring is trivial so we can just focus on the homology or chain complex (with coefficients in a field).

Recall in Proposition 1.1 or subsection 2.2, comparison between two Floer chain complexes can be studied by constructing continuation maps \(\Phi\) and \(\Psi\) which fundamentally depends on an energy estimation (see a computation in subsection 2.2) where in the standard case, the perturbation is from pair \((J_-, H_-)\) to \((J_+, H_+)\). Here the perturbation is from \(\omega_0 = \omega\) to \(\omega_1 = \omega + \alpha\) and we can get a similar energy estimation inequality as follows in Section 4.

**Proposition 1.5.** Suppose \(u : \mathbb{R} \times S^1 \to M\) is a Floer connecting trajectory from \([\gamma_-, w_-]\) to \([\gamma_+, w_+]\) (i.e, satisfying (a), (b) and (c) in Section 4) with energy \(E < \infty\). We have the following energy estimation between action functionals,

\[
(2) \quad -(1 + C')E - \int_{D^2} (w_-)^*\alpha \leq \mathcal{A}_{\omega_1}([\gamma_+, w_+]) - \mathcal{A}_{\omega_0}([\gamma_-, w_-]) \leq -(1 - C')E - \int_{D^2} (w_-)^*\alpha
\]

for some constant \(C' = |\alpha| \left(\frac{C}{N} + 1\right)\) where \(C\) and \(N\) are independent of connecting trajectory.

The proof of Proposition 1.5 gives a basic model which can be generalized to variant versions of energy estimation serving different purposes in the later part of the paper. The main variation comes from two types. One is the variation of Definition 4.1 arising from different homotopies.
of symplectic forms. The other is the variation of valuations by action functionals on the two (asymptotic) ends of connecting trajectory. For specific classifications and their corresponding statement, see subsection 4.2.

Compared with the standard energy estimation of deformation of Hamiltonians, Proposition 1.5 does not give a uniform upper bound for a general manifold since different capped Hamiltonian orbits differed from sphere class will probably cause term \[ \int_{D^2} (w_-) \wedge \alpha \] getting unexpected large. This is one of the main reasons that some of the arguments are more subtle compared with the classical case. However, again, in certain special cases, for instance, when \( M \) is aspherical or \( \alpha \) is an exact form, this estimation does give a uniform upper bound. Therefore, we can prove

**Theorem 1.6.** Suppose \( M \) is aspherical and any \( \alpha \in \Omega \) or suppose any symplectic manifold \( (M, \omega_0) \) and perturbation \( \alpha \) is exact \(^2\). Then spectral invariant and boundary depth are continuous with respect to the perturbation of symplectic forms. Specifically, there exists a constant \( C \) (independent of cappings) such that

\[
\begin{align*}
\text{• for any } & a \in QH_*(M, \omega_0), \\
|\rho(H, a; \omega_0) - \rho(H, a; \omega_0 + \alpha)| & \leq C|\alpha|; \\
\text{• for any } & \phi \in \text{Ham}(M, \omega_0), \\
|\beta(\omega_0, \phi) - \beta(\omega_0 + \alpha, \phi)| & \leq C|\alpha|.
\end{align*}
\]

The proof will be given in Section 5, following the same idea of Proposition 1.1 constructing Floer continuation maps and homotopy maps. Notice the second case actually reduces this continuity question to the consideration only on the level of cohomology classes represented by symplectic forms. For the first case, the standard example it covers is surface \( (\Sigma_g, g \geq 1) \).

On the other hand, we want to emphasize that in the situation of Theorem 1.6, there is no ambiguity of the choice of class \( a \in QH_*(M, \omega_0) \) when comparing spectral invariants since in both cases, \( QH_*(M, \omega_0) = QH_*(M, \omega_0 + \alpha) = H_*(M, \mathcal{K}) \). In general, for any quantum homology, there always exists a well-defined element - the fundamental class, denoted as \([M]\).

The following result can be regarded as a corollary of Theorem 1.6 because they share almost the same method for proofs.

**Theorem 1.7.** Suppose \( (M, \omega_0) \) satisfies \( \dim_H H^2(M; \mathcal{K}) = 1 \), then spectral invariant (of fundamental class) and boundary depth are continuous with respect to any perturbation \( \alpha \in \Omega \). Specifically, there exists a constant \( C \) such that

\[
\begin{align*}
\text{• for fundamental class } & [M], \\
|\rho(H, [M]; \omega_0) - \rho(H, [M]; \omega_0 + \alpha)| & \leq C|\alpha|; \\
\text{• for any } & \phi \in \text{Ham}(M, \omega_0), \\
|\beta(\omega_0, \phi) - \beta(\omega_0 + \alpha, \phi)| & \leq C|\alpha|.
\end{align*}
\]

Notice this covers several important cases, for instance, \( S^2 \) and \( \mathbb{C}P^n \) for any \( n \in \mathbb{N} \).

Application of Theorem 1.6 reproves (from a different perspective) several recent results on some interesting rigidity properties of subsets on the surface \( \Sigma_g \) with \( g \geq 1 \). We need a definition first.

\(^2\text{Moser's trick can easily and directly imply the same conclusion when } \alpha \text{ is exact.}\)
Definition 1.8. (Definition 2.9 in [Kaw14]) For a fixed symplectic manifold \((M, \omega)\) and an element \(a \in QH_\ast(M, \omega)\), a subset \(U \subset M\) satisfies bounded spectrum condition (with respect to \(a\)) if there exists a constant \(K > 0\) such that

\[ \rho(H, a; \omega) \leq K \]

for any Hamiltonian function \(H \in C^\infty(\mathbb{R}/\mathbb{Z} \times U)\) i.e., supported in \(\mathbb{R}/\mathbb{Z} \times U\).

Theorem 1.9. For any symplectic surface \((\Sigma_{g \geq 1}, \omega)\), any disjoint union of simply connected open subset satisfies bounded spectrum condition with respect to any \(a \in QH_\ast(\Sigma_{g \geq 1}, \omega)\).

The proof of this is given in Section 6 which takes its inspiration from Ostrover’s trick (see [Ost03]). This can quickly imply the following result.

Corollary 1.10. Suppose we have a symplectic surface \((\Sigma_{g \geq 1}, \omega)\).

(a) For a closed subset \(X \subset \Sigma_g\), if \(\Sigma_{g \geq 1} \setminus X\) is a disjoint union of simply connected regions, then it is \(a\)-superheavy for any \(a \in QH_\ast(\Sigma_{g \geq 1}, \omega)\).

(b) If a closed subset \(X\) is contained in a disk, then \(X\) is not \(a\)-heavy for any \(a \in QH_\ast(\Sigma_{g \geq 1}, \omega)\).

Notice that (a) includes the case that \(\bigvee_{i=1}^{2g} S^1 \hookrightarrow \Sigma_g\) (Example 4.8 in [Ish15] which generalizes the main result from [Kaw14]). (b) provides a topological obstruction for a closed subset being \(a\)-heavy. Necessary and sufficient conditions for being \(a\)-heavy and \(a\)-superheavy on a symplectic surface are given by Proposition 6 in [HRS14].

In general, when considering any perturbation of symplectic form, as mentioned above, we need to take care of the change of quantum homologies (or equivalently the Novikov rings with respect to different symplectic forms for Floer chain complexes). On the one hand, by Theorem 1.6, comparison between \(\omega_0\) and \(\omega_0 + \alpha\) (for \(\alpha \in \Omega\)) can be reduced to the cohomology level, that is, \([\omega_0]\) and \([\omega_0 + \alpha]\) in \(H^2(M, \mathcal{X})\), which has the advantage that we are working within a finite-dimensional vector space over \(\mathcal{X}\). Moreover, we observe \([\omega_0 + \alpha]\) arising from a (small) perturbation lies in a finite-dimensional convex polygon \(\Delta \subset H^2(M, \mathcal{X})\) around \([\omega_0]\). In particular, after fixing the vertices of \(\Delta\), say \([\omega_1, \ldots, [\omega_m]]\) (where \(m \geq \dim_{\mathcal{X}} H^2(M, \mathcal{X})\)), for any \(\alpha \in \Omega\), there exist nonnegative numbers \(t_0, t_1, \ldots, t_m\) with \(\sum_{i=0}^m t_i = 1\) such that

\[ [\omega + \alpha] = t_0 [\omega_0] + t_1 [\omega_1] + \ldots + t_m [\omega_m]. \]

Here each \(\omega_i\) is chosen such that it satisfies the condition of Proposition 1.4 and also make constant \(C'\) in Proposition 1.5 no bigger than 1 (or briefly each \(\omega_i\) lies in a sufficiently small neighborhood of \(\omega_0\) in \(\Omega_2(M)\)). Apparently, convex combination as in (3) can be generated inductively, therefore, for brevity, we will only consider the case when \(m = 1\) for most of the rest of the paper. Specifically, \([\omega_0 + \alpha]\) lies in the line segment connecting \([\omega_0]\) and \([\omega_1]\) where \([\omega_0]\) and \([\omega_1]\) are not co-linear, so \(\dim_{\mathcal{X}} H^2(M, \mathcal{X}) \geq 2\). The motivation of all this consideration is to define a Floer style chain complexes over a common coefficient ring under the situation of perturbation. Here let us list all the Novikov rings that will appear in this paper.

- \(\Lambda^{\mathcal{X}, \Gamma, \omega}\) denotes the most often used Novikov rings (actually a field) where exponentials of each element are in a subgroup \(\Gamma_0 \leq \mathbb{R}\);
- \(\Lambda_\omega\) denotes the extended version of Novikov ring whose coefficients are in a group ring of \(\ker(\omega)\), see [Ush08] or beginning of Section 7;
- \(\Lambda_{[0,1]}\) denotes a Novikov ring with multi-finiteness condition (see Definition 7.3) which captures the Novikov-finiteness conditions for all \(t \in [0,1]\) (which is equivalent to the satisfaction of Novikov-finiteness condition only for \(\omega_0\) and \(\omega_1\) by Lemma 7.4).
Here we remark that except $\Lambda_{[0,1]}$ being a integral domain (see Lemma 7.1), not so much algebraic structure of $\Lambda_{[0,1]}$ is known at present, which gives rise to the main obstruction when applying some classical results. The algebraic relations of these three rings are: $\Lambda^{\mathcal{X},\mathcal{F}_\omega}$ is a $\Lambda_{\omega}$-module and $\Lambda_{\omega}$ is a $\Lambda_{[0,1]}$-module. Now, over $\Lambda_{[0,1]}$,

On this new type of Novikov ring $\Lambda_{[0,1]}$, we can construct a variant Floer chain complex which satisfies finiteness conditions for action functional $\mathcal{A}_{\omega_t}$ for each $t \in [0,1]$ (hence, by Lemma 7.4, sufficient for $\mathcal{A}_{\omega_0}$ and $\mathcal{A}_{\omega_1}$). Specifically, we have

**Theorem 1.11.** There exists a family of well-defined Floer chain complexes parametrized by $s \in [0,1]$, denoted as

\[
(4) \quad \{\left((CF_{[0,1]})_s, \partial_s, \ell_s\right)\}_{0 \leq s \leq 1}
\]

where each $s$-slice is a Floer chain complex (free) over $\Lambda_{[0,1]}$. Moreover, for any $s, t \in [0,1]$, $((CF_{[0,1]})_{s}, \partial_{s})$ and $((CF_{[0,1]})_{s}, \partial_{t})$ are chain homotopy equivalent.

This is proved in subsection 7.2. The well-definedness of boundary operator and chain maps over $\Lambda_{[0,1]}$ is the main non-trivial part to be proved, which uses variant energy estimations from subsection 4.2. Therefore, passing to the homology, their homologies are well-defined and independent of the choice of boundary operator (or isomorphic to each other and isomorphic to $(QH_{[0,1]})_s := H_s(M; \mathcal{X}) \otimes \Lambda_{[0,1]}$).

**Remark 1.12.** Here, we put a quick remark that the structure from (4) forms a standard structure of persistence module parametrized by $s \in [0,1]$ (if we forget about filtration function $\ell_s$ for each $s$-slice Floer chain complex). More interestingly, exactly due to $\ell_s$ for each $s \in [0,1]$, (4) also forms a 2-D persistence module, see [CZ09]. Continuity question studied in this paper might be transferred into a stability problem of higher dimensional persistence module.

Meanwhile, due to the algebraic relations between different Novikov rings as above, we can extend the coefficient for multiple times to get back to the Floer complexes people often work on. In summary, we have the following two diagrams.

(a) On the chain complex level,

\[
(5) \quad \begin{array}{c}
\xymatrix{ & \left(CF_{[0,1]}, \partial_0\right) \ar[r]^{\ell_0} & \left(CF_{[0,1]}, \partial_0\right) \ar[r]^{R_0} & \left(CF(M, H, J, \omega_0), \partial_0\right) \\
(CM_{[0,1]}, \partial_{Morse}) \ar[ru]^{PSS_0} & (CF_{[0,1]}, \partial_t) \ar[u]^{\Phi_{0,t}} & (CF_{[0,1]}, \partial_t) \ar[u] \ar[ru]^{PSS_t} \\
& \left(CF_{[0,1]}, \partial_1\right) \ar[r]^{\ell_1} & \left(CF_{[0,1]}, \partial_1\right) \ar[r]^{R_1} & \left(CF(M, H, J, \omega_1), \partial_1\right) \\
& \Lambda_{[0,1]}\text{-module} & \Lambda_{[0,1]}\text{-module} & \Lambda_{[0,1]}\text{-module}
\end{array}
\]

where $\widetilde{CF}_t = CF_{[0,1]} \otimes_{\Lambda_{[0,1]}} \Lambda_{\omega_t}$ as a free $\Lambda_{\omega_t}$-module for $t \in [0,1]$ and $CF(M, H, J, \omega_0) = \widetilde{CF}_0 \otimes_{\Lambda_{\omega_0}} \Lambda^{\mathcal{X},\mathcal{F}_{\omega_0}}$ as a free $\Lambda^{\mathcal{X},\mathcal{F}_{\omega_0}}$-module.
(b) On the homology level,

\[
\begin{align*}
HF_{[0,1],0} & \xrightarrow{(t_0)_*} \widetilde{HF}_0 \xrightarrow{(R_0)_*} HF_{\omega_0} \\
QH_{[0,1]} & \xrightarrow{(PSS)_*} HF_{[0,1],t} \xrightarrow{(\Phi_0)_*} \widetilde{HF}_t \\
HF_{[0,1],1} & \xrightarrow{(t_1)_*} \widetilde{HF}_1 \xrightarrow{(R_1)_*} HF_{\omega_1}
\end{align*}
\]

\[\Lambda_{[0,1]}\text{-module}\]

where \(QH_{[0,1]} = H(M; \mathcal{X}) \otimes \Lambda_{[0,1]}, HF_{[0,1],t}\) is the homology of Floer chain complex \(((CF_{[0,1]})_*, \partial_t)\), \(\widetilde{HF}_t\) is the homology of Floer chain complex \(((\widetilde{CF}_t)_*, \partial_t)\) for \(t \in [0,1]\) and \(HF_{\omega_i}\) is the homology of Floer chain complex \((CF(M,H,J,\omega_i)_*, \partial_t)\) for \(i = 0,1\).

This construction (of variant Floer chain complexes as above) takes its inspiration from the main part of [Ono05]. We want to emphasize that in general, we are not able to conclude any quantitative relation between \(((CF_{[0,1]})_*, \partial_t, \ell_t)\) and \(((CF_{[0,1]})_*, \partial_t, \ell_t)\) in terms of filtrations which is mainly due to the non-uniform upper bound in general from energy estimation in Proposition 1.5.

Next, with the set-up above, we can study the continuity problem of symplectic invariants when \(t\) approaches to 0.

(A) **Spectral invariant:** as for each \(t\)-slice Floer chain complex \(((CF_{[0,1]})_*, \partial_t)\), there is a filtration function \(\ell_t\) defined with the help of action functional \(\mathcal{A}_{\omega_t}\) (where \(H\) is fixed for all \(t \in [0,1]\)). Similar to the definition of spectral invariant in the classical case in [Ush08], we can define \(t\)-spectral invariant \(\rho_t : QH_{[0,1]} \times C^\infty(\mathbb{R}/\mathbb{Z} \times M) \to \mathbb{R} \cup \{-\infty\}\) (see Definition 8.1). Here we want to emphasize in general the continuity problem on comparing spectral invariants will make sense only if we choose \(a \in QH_{[0,1]}\). Therefore, in terms of computation, we will go through the following steps,

\[a(\in QH_{[0,1]} ) \to \rho_t(a,H) \to \tilde{\rho}_t(a,H) \to \rho(H,a;\omega_t)\]

where \(\tilde{\rho}_t\) is the spectral invariant (on \(\widetilde{HF}_t\) over \(\Lambda_{\omega_t}\)) defined in [Ush08] with respect to symplectic form \(\omega_t\) and \(\rho(H,a;\omega_t)\) is the most often used spectral invariant (on \(HF_{\omega_t}\) over field \(\Lambda_{\mathbb{X},I_{\omega_t}}\) ) with respect to symplectic form \(\omega_t\). It is readily to see \(\tilde{\rho}_t(a,H) = \rho(H,a;\omega_t)\) for \(t \in [0,1]\). First, we can prove

**Proposition 1.13.** Let \(t \in [0,1]\) and \(H \in C^\infty(\mathbb{R}/\mathbb{Z} \times M)\).

1. **(finiteness)** For any nonzero \(a \in QH_{[0,1]}\), \(\rho_t(a,H) > -\infty\).
2. **(realization)** For any \(a \in QH_{[0,1]}\), there exists an \(\alpha_t \in CF_{[0,1]}\) such that

\[\rho_t(a,H) = \ell_t(\alpha_t)\]

3. **(extension)** For any \(a \in QH_{[0,1]}\), \(\rho_t(a,H) = \tilde{\rho}_t(a,H)\).
By extension property (3) in Proposition 1.13, we can reduce the comparison between \( \tilde{\rho}_t \) to the comparison just between \( \rho_t \). Question of continuity of spectral invariant is answered by the following theorem.

**Theorem 1.14.** For a fixed \( a \in QH_{[0,1]} \) and \( H \in C^\infty(\mathbb{R}/\mathbb{Z} \times M) \), \( \rho_t(a,H) \) is upper semicontinuous at \( t = 0 \), i.e., for any given \( \epsilon > 0 \), there exists \( \delta > 0 \) (depending on \( a, H \) and \( \epsilon \)) such that for any \( t \in [0,\delta] \), \( \rho_t(a,H) - \rho_0(a,H) \leq \epsilon \). In general, for any fixed \( a \in QH_\Delta \) (where \( \Delta \) is a finitely dimensional open polygon near \( [\omega_0] \) in \( H^2(M;\mathcal{X}) \)), function \( a \rightarrow \rho(H,a;\omega_0 + a) \) is upper semicontinuous for \( a \in \Omega \).

Proofs of both Proposition 1.13 and Theorem 1.14 are given in Section 8. There is an obvious question on the lower semicontinuity. Note that the method we provide here can not conclude any positive or negative result for lower semicontinuity (see Remark 8.5). Moreover, we notice that realization proposition (2) can be viewed from a more general perspective by improving “fixed point theorem”, Lemma 2.1 and “best approximation theorem”, Lemma 2.4, in [Ush08]. The key step is an algebraic reconstruction of \( \Lambda_{[0,1]} \) and an application of valuation of rank 2. See Appendix, Section 10.

**(B) Boundary depth:** boundary depth is defined without specifying any homology class in quantum homology, therefore, on the one hand, we can directly compare two boundary depth without passing to the variant Floer chain complexes defined as above. However, as mentioned above, we can not conclude any quantitative relation between two Floer chain complexes with respect to different symplectic forms, which results in a failure when applying classical results, for instance, Proposition 3.8 in [Ush13]. On the other hand, imitating the way we study on the spectral invariant starting from \( (\text{CF}[0,1],\partial_t) \), \( t \)-boundary depth \( \beta_t \) similar to \( t \)-spectral invariant could be defined by the original definition of boundary depth appearing in [Ush11]. Specifically,

**Definition 1.15.** Define \( t \)-boundary depth as

\[
\beta_t(\phi) = \inf \left\{ \beta \in \mathbb{R}_{\geq 0} \mid \text{for any } \lambda \in \mathbb{R}, \partial_t \left( \text{CF}[0,1] \right) \cap \left( \text{CF}[0,1]^{\lambda+\beta} \right) \subset \partial_t \left( \text{CF}[0,1]^{\lambda+\beta} \right) \right\}
\]

where the filtered \( \text{CF}[0,1]^{\lambda} \) is defined by using filtration function \( \ell_t \).

Notice for any \( t \in [0,1] \), \( t \)-boundary depth \( \beta_t(\phi) \) is bounded from above by the Hofer norm of Hamiltonian function \( ||H||_H \) (by Corollary 3.5 in [Ush11]). However the questions on realization and extension (directly to be over Novikov field \( \Lambda^{X,R,W} \)) are naturally raised but our method in this paper seems not capable to answer this question either affirmatively or negatively. For continuity of \( t \)-boundary depth, we expect higher-dimensional persistent homology theory can provide a tool to answer this question.

**(C) (Partial) symplectic quasi-state:** the method we provide in this paper can not apply to the discussion of continuity of (partial) symplectic quasi-state because we insist that our Hamiltonian function \( H \) should be fixed when symplectic form is perturbed (or only finitely many Hamiltonian functions are involved). However, in general, we should not expect any continuity result for (partial) symplectic quasi-state. We illustrate this by the following easy example.

**Example 1.16.** Let \( M = S^2 \) with the standard area form \( \omega_{st} \) located in the standard \( xyz \)-coordinate where the center of \( S^2 \) is the origin \((0,0,0)\). With this symplectic form, the standard equation \( L := \{(x,y,z) \mid z = 0\} \) is a heavy subset (actually superheavy because it is a stem with
respect to the standard height function on \( S^2 \). For any given \( \varepsilon > 0 \), let

\[
S_\varepsilon := \{(x, y, z) \in S^2 | z \geq -\varepsilon\}.
\]

Take \( H \) to be a time-independent Hamiltonian (not necessarily smooth \(^3\)) supported on \( S_\varepsilon \) such that \( H|_L = 1 \) and \( \omega_{st}(\text{supp}(H)) = \frac{1}{2}\omega_{st}(S^2) \). Let our perturbation 2-form \( \alpha \) to be positively supported on \( S^2 \setminus S_\varepsilon \). For any class \( a \in \text{QH}_s(S^2) \), for any \( \delta > 0 \), on the one hand, by definition of heavy subset,

\[
\zeta_{a,H}(\omega_{st}) \geq \inf_L H = 1;
\]

while on the other hand,

\[
\zeta_{a,H}(\omega_{st} + \delta \alpha) = 0
\]

because \((\omega_{st} + \delta \alpha)(\text{supp}(H)) < \frac{1}{2}(\omega_{st} + \delta \alpha)(S^2) \) and then support of \( H \) is displaceable in \((S^2, \omega_{st} + \delta \alpha)\).

**Remark 1.17.** Notice for the example above, our choice of perturbation is quite special. It is easy to see there are plenty of other directions that we can perturb so that quasi-state is invariant (so, in particular, change continuously). In general, it might be interesting to systematically study in which way we perturb so that quasi-state is able to satisfy the continuity.

In Section 9, we give two applications of Theorem 1.14. First, we claim there exists a quasi-embedding from \( (\mathbb{R}^\infty, || \cdot ||_\infty) \rightarrow (\text{Ham}(M, \omega), || \cdot ||_H) \) under a certain topological/dynamical condition on manifold \( M \). It is well-known that under the Hofer norm, diameter of \( \text{Ham}(M, \omega) \) is infinity (see [Ost03]). Later, with the advent of main result in [Ush13], it improves Ostrover’s method getting a quasi-embedding from \( (\mathbb{R}^\infty, || \cdot ||_\infty) \rightarrow (\text{Ham}(M, \omega), || \cdot ||_H) \) under the following condition.

**Condition 1.18.** \((M, \omega)\) admits an autonomous Hamiltonian function \( H : M \rightarrow \mathbb{R} \) which has no nonconstant contractible Hamiltonian orbit.

There are two other conditions closely related with Condition 1.18. One is stronger than Condition 1.18 (see Definition 9.1 of aperiodic symplectic form or Definition 1.3 in [Ush12]) and the other is weaker than Condition 1.18 under which spectral invariant is rather easy to compute (see Proposition 9.2 or Proposition 4.1 in [Ush10]). Together, we can prove

**Theorem 1.19.** Suppose \( M \) admits a symplectic structure \( \omega \) which can be approximated by aperiodic symplectic forms (under the same Hamiltonian function). Then \((M, \omega)\) has a quasi-embedding \( \Phi : \mathbb{R}^\infty \rightarrow \text{Ham}(M, \omega) \) such that for every \( \vec{v}, \vec{w} \in \mathbb{R}^\infty \), we have

\[
||\vec{v} - \vec{w}||_{\ell_\infty} \leq d_H(\Phi(\vec{v}), \Phi(\vec{w})) \leq \text{osc}(\vec{v} - \vec{w}),
\]

where (if \( \vec{a} = (a_1, a_2, ...) \)), \( \ell_\infty(\vec{a}) = \max_i |a_i| \) and \( \text{osc}(\vec{a}) = \max_{i,j} |a_i - a_j| \).

Notice the main theorem of [Ush12], Theorem 1.6, points out there are rather diverse classes of symplectic manifolds (especially in dimension four) who satisfy the assumption of Theorem 1.19. The second one is an application on the continuity problem on some capacities. Here we will mainly focus on Hofer-Zehnder capacity \( c_{HZ}^{\omega}(A) \) (see Definition 9.4 or subsection 1.2 in [FGS05]) and spectral capacity \( c_{\rho}^{\omega}(A) \) (see Definition 9.5 or in general by action selector in Definition 1.6 in [FGS05]). Here we put symplectic form \( \omega \) in both notations to emphasize their

\(^3\)By Lipschitz property of partial symplectic quasi-state, we can extend the definition of (sub)heavy subset by using just continuous functions, see Section 4 in [EP08].
dependence on symplectic forms. It is well-known as energy-capacity inequality (see Theorem 1 in [FGS05]) that

\[
\inf c_{HZ}^{\omega}(A) \leq \inf c_{\rho}^{\omega}(A) \leq \inf e^{\omega}(A)
\]

where \( e^{\omega}(A) \) is defined as \( e^{\omega}(A) = \inf \{ ||H||_{H} \mid \phi_{H}^{1}(A) \cap A = \emptyset \} \) under the symplectic form \( \omega \). Since they all depend on the symplectic form, it is natural to ask how they change when symplectic form is perturbed. In general, we should not expect the continuity of change of displacement energy \( e^{\omega}(A) \). Here we give an example essentially in the same spirit of Example 1.16 above.

**Example 1.20.** Take \( (M = S^{2}, \omega_{0} = \omega_{S^{2}}) \) and let \( A \) be the open upper hemisphere. Then subset \( A \) is displaceable (under \( \omega_{0} \)) by definition and \( e^{\omega_{0}}(A) = \text{Area}_{\omega_{0}}(A) = 2\pi \). Take a sequence of symplectic forms \( \omega_{n} \to \omega_{0} \) by \( \omega_{n} = \omega_{0} + \alpha_{n} \), where each \( \alpha_{n} \) is positively supported over an nonempty open subset of \( A \) and negatively supported over an nonempty open subset of \( M \setminus \bar{A} \) (such that the total area of \( M \) remains the same). Then for each \( n \), \( A \) is not displaceable (under \( \omega_{n} \) by area consideration). So by definition \( e^{\omega_{n}}(A) = \infty \). Hence \( e^{\omega}(A) \) is not upper semicontinuous at \( \omega_{0} \).

However, we have the following series of “boundedness” conclusion in terms of perturbation of these capacities. If \( \omega \) is a perturbation of \( \omega_{0} \), then we call \( e^{\omega}(A) \) a perturbation of \( e^{\omega_{0}}(A) \) and similar to \( c_{\rho}^{\omega}(A) \) and \( c_{HZ}^{\omega}(A) \).

**Theorem 1.21.** Fix \( (M, \omega_{0}) \) and \( A \) as a subset of \( M \).

1. If \( M \) is aspherical or \( \dim_{\mathcal{C}} H^{2}(M; \mathcal{C}) = 1 \) (for instance, \( S^{2} \) or \( \mathbb{C}P^{n} \)), there exists a lower bound (possibly infinitely) for any small perturbation \( e^{\omega}(A) \). Specifically, for any \( \epsilon > 0 \), there exists a neighborhood \( \Omega_{\omega_{0}} \) of \( \omega_{0} \) such that for any \( \omega \in \Omega_{\omega_{0}} \), \( e^{\omega}(A) \geq c_{\rho}^{\omega_{0}}(A) - \epsilon \).
2. For any \( M \), there exists an upper bound for any small perturbation \( c_{\rho}^{\omega}(A) \). Specifically, for any \( \epsilon > 0 \), there exists a neighborhood \( \Omega_{\omega_{0}} \) of \( \omega_{0} \) such that for any \( \omega \in \Omega_{\omega_{0}} \), \( c_{\rho}^{\omega}(A) \leq e^{\omega_{0}}(A) + \epsilon \).
3. For any \( M \), there exists an upper bound for any small perturbation \( c_{HZ}^{\omega}(A) \). Specifically, for any \( \epsilon > 0 \), there exists a neighborhood \( \Omega_{\omega_{0}} \) of \( \omega_{0} \) such that for any \( \omega \in \Omega_{\omega_{0}} \), \( c_{HZ}^{\omega}(A) \leq c_{\rho}^{\omega_{0}}(A) + \epsilon \) (so \( c_{HZ}^{\omega}(A) \leq e^{\omega_{0}}(A) + \epsilon \)).

Notice by Example 1.20, \( e^{\omega}(A) \) might blow up in general, so (compared with \( e^{\omega}(A) + \epsilon \)) \( e^{\omega_{0}}(A) + \epsilon \) is a better upper bound of \( c_{\rho}^{\omega}(A) \) and \( c_{HZ}^{\omega}(A) \) from (2) and (3) of Theorem 1.21.

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\footnote{Some definition of displacement energy is defined only for a compact subset \( K \). If \( A \) is open, then \( e(A) = \sup \{ e^{\omega}(K) \mid K \text{ is a compact subset of } A \} \).}
2. Preliminary

2.1. Construction of Floer chain complex. In this subsection, we will briefly review the construction of Hamiltonian Floer homology for the most often used case. In the later part of this paper, different but similar construction will be carried out for variant version of Hamiltonian Floer homology.

On a closed symplectic manifold \( (M, \omega) \), given a smooth function \( H \in C^\infty(\mathbb{R}/\mathbb{Z} \times M) \), Hamiltonian system comes from the following differential equation

\[
\frac{d\phi_t}{dt} = X_H \circ \phi_t \quad \text{where} \quad \omega(\cdot, X_H) = d(H(\cdot, \cdot)).
\]

Fixed point of time-1 map \( \phi_H^1 \) is corresponding to a loop \( \gamma : \mathbb{R}/\mathbb{Z} \to M \) such that \( \gamma(t) = \phi_H^t(\gamma(0)) \). \( H \) is called non-degenerate if for each such loop \( \gamma \),

\[
(d\phi_H^1)_{\gamma(0)} : T_{\gamma(0)}M \to T_{\gamma(0)}M
\]

has all eigenvalues distinct from 1. For our purpose, we will only consider all the contractible loops. For each contractible loop \( \gamma(t) \), we can view it as a boundary of an embedded disk \( u : D^2 \to M \) such that \( u|_{S^1} = \gamma(t) \). Denote collection of all such capped loops by \( \mathcal{L}(M) \) and now we consider a covering space of \( \mathcal{L}(X) \), denoted as \( \overline{\mathcal{L}(X)} \) constructed by

\[
\overline{\mathcal{L}(X)} = \left\{ \text{equivalent class } [\gamma, u] \mid \right. \begin{align*}
(\gamma, u) \text{ is equivalent to } (\tau, v) \iff \\
\gamma(t) = \tau(t) \text{ and } [u#(-v)] \text{ is homologically trivial}
\end{align*} \left. \right\}.
\]

For each \( [\gamma, u] \in \overline{\mathcal{L}(X)} \), there are two functions associated to it. One is action functional \( \mathcal{A}_H : \overline{\mathcal{L}(X)} \to \mathbb{R} \) defined as

\[
(8) \quad \mathcal{A}_H([\gamma, u]) = -\int_{D^2} u^*\omega + \int_0^1 H(t, \gamma(t))dt.
\]

The other one is Conley-Zehnder index \( \mu_{CZ} : \overline{\mathcal{L}(X)} \to \mathbb{Z} \) defined by counting rotation of linearization \( d\phi_H^t \) on along \( \gamma(t) \) with the help of trivialization induced by \( u \). Its explicit definition can be referred to [RS93]. Note that action functional \( \mathcal{A}_H \) and Conley-Zehnder index \( \mu_{CZ} \) are both well-defined.

Floer chain complex consists of two components. One is vector space \( CF_*(M, J, H, \omega) \) and the other is boundary operator \( \partial \). For the first one, \( CF_*(M, J, H, \omega) \) is first a graded vector space over some ground field \( \mathcal{K} \) consisting of chains

\[
\left\{ \left. \sum_{[\gamma, u] \in \mathcal{L}(X), \mu_{CZ}([\gamma, u]) = n} a_{[\gamma, u]}[\gamma, u] \in \mathcal{K}, \forall \mathcal{C} \in \mathbb{R} \right| \#\{[\gamma, u] \mid a_{[\gamma, u]} \neq 0, \mathcal{A}_H([\gamma, u]) > C\} < \infty \right\}.
\]

As it is well known that different choice of capping might result in infinite dimension issue, [HS95] suggests to consider an extended coefficient called Novikov field (or in general Novikov ring), denoted as \( \Lambda^{\mathcal{K}, \mathcal{I}} \). There are variant definitions of \( \Lambda^{\mathcal{K}, \mathcal{I}} \) for different purposes. For
example, for most often use, $\Lambda^{\mathcal{X}, \Gamma}$ is defined as
\begin{equation}
\Lambda^{\mathcal{X}, \Gamma} = \left\{ \sum_{g \in \Gamma} a_g T^g \left| a_g \in \mathcal{X}, (\forall C \in \mathbb{R})(\# \{ g \mid a_g \neq 0, g < C \} < \infty) \right. \right\}
\end{equation}
where $\Gamma = \{ \text{Im}[\omega] : H_2(M; \mathbb{Z})/\text{Tor} \to \mathbb{R} \} \leq \mathbb{R}$ and $T$ is a formal symbol. Meanwhile, action functional $\mathcal{A}_H$ gives rise to a filtration on $CF_*(M, J, H, \omega)$ defined by
\begin{equation}
\ell_H \left( \sum_{[\gamma, u]} a_{[\gamma, u]} \right) = \max \{ \mathcal{A}_H([\gamma, u]) \mid a_{[\gamma, u]} \neq 0 \}.
\end{equation}
Note that field $\Lambda^{\mathcal{X}, \Gamma}$ itself defined above has a well-defined valuation by $\nu \left( \sum_{g \in \Gamma} a_g T^g \right) = \min \{ g \in \mathbb{R} \mid a_g \neq 0 \}$. Then for any $\lambda \in \Lambda^{\mathcal{X}, \Gamma}$ and $c \in CF_*(M, J, H, \omega)$, $\ell_H(\lambda c) = \ell_H(c) - \nu(\lambda)$. For the second, Floer boundary operator $\partial : CF_*(M, J, H, \omega) \to CF_{*-1}(M, J, H, \omega)$ is defined, roughly speaking, by counting the solution (modulo differential equation (as a formal negative gradient flow of $\nu$).

The key step of proving Proposition 1.1 (which also easily implies Lipschitz continuity property of both spectral invariant and boundary depth) is constructing (Floer) continuation map. Specifically, for different pairs $(J_-, H_-)$ and $(J_+, H_+)$, considering a homotopy $(\mathcal{H}, \mathcal{X})$ where $\mathcal{H} = \mathcal{H}_s = \mathcal{X}(s, t, x) : \mathbb{R} \times \mathbb{R} \times \mathbb{Z} \times M \to \mathbb{R}$ between $H_-(t, x)$ and $H_+(t, x)$ in the form of
\begin{equation}
\mathcal{X}(s, t, x) = (1 - \alpha(s))H_-(t, x) + \alpha(s)H_+(t, x),
\end{equation}
where $\alpha(s)$ is a cut-off function, i.e., $\alpha(s) = 0$ for $s \in (-\infty, 0]$, $\alpha(s) = 1$ for $s \in [1, \infty)$ and $\alpha'(s) > 0$ for $s \in (0, 1)$; $\mathcal{H}$ is a homotopy (compatible with $\omega$) between $J_-$ and $J_+$. The latter is called quantum homology denoted as $QH_*(M, \omega)$ and the classical way to prove this isomorphism in the above theorem is by constructing PSS-map, denoted as $PSS_*$, see [PSS96].
by a cut-off function, too. Then similar to boundary operator, construction of continuation map \( \Phi \) is, roughly speaking, by counting solution \( u(s, t) : \mathbb{R} \times \mathbb{R} / \mathbb{Z} \to M \) of a parametrized pseudoholomorphic equation
\[
\frac{\partial u}{\partial s} + \mathcal{J}_s(u(s, t)) \left( \frac{\partial u}{\partial t} - X_{\mathcal{H}}(t, u(s, t)) \right) = 0
\]
such that
- \( u \) has finite energy \( E(u) = \int_{\mathbb{R} \times \mathbb{R} / \mathbb{Z}} \left| \frac{\partial u}{\partial s} \right|^2 \, dt \, ds \);
- \( u \) has asymptotic condition \( u(s, \cdot) \to \gamma_{\pm}(\cdot) \) as \( s \to \pm \infty \);
- \( \mu_{\text{CF}}([\gamma_{-}, w_{-}]) - \mu_{\text{CF}}([\gamma_{+}, w_{+}]) = 0 \) and \( [\gamma_{+}, w_{+}] = [\gamma_{+}, w_{-} # u] \).

There are two well-know facts. One is that \( \Phi \) is a chain map (by Floer gluing argument, see [Sal97]); the other is for any other homotopy \( (\mathcal{J}', \mathcal{H}') \), the associated chain map \( \Phi' \) is chain homotopic to \( \Phi \) (and the way to prove this is to consider a homotopy of homotopies and then back to another parametrized pseudoholomorphic equation as in (12)). Here we give some details on the shift of filtration. The standard computation goes as follows.

\[
\mathcal{A}_{\mathcal{H}}([\gamma_{+}, w_{+}]) - \mathcal{A}_{\mathcal{H}}([\gamma_{-}, w_{-}]) = \int_{-\infty}^{\infty} \frac{d}{ds} \mathcal{A}_{\mathcal{H}}([u(s, \cdot), (w_{-} # u)(s, \cdot)]) \, ds
\]
\[
= -E(u) + \int_{-\infty}^{1} \alpha'(s)(H_{+} - H_{-})(t, u(s, t)) \, dt \, ds
\]
\[
\leq -E(u) + \int_{0}^{1} \max_{\mathcal{M}}(H_{+} - H_{-})(t, u(s, t)) \, dt
\]
\[
\leq \int_{0}^{1} \max_{\mathcal{M}}(H_{+} - H_{-}) \, dt.
\]

Our energy estimation in Section 4 will follow the same theme of this computation.

### 2.3. Some symplectic invariants.

#### 2.3.1. Spectral invariant.

The filtration \( \ell \) defined in (10) descends to \( HF_*(M) \) by a measurement called spectral invariant,

**Definition 2.2.** For any \( a \in QH_*(M, \omega) \),
\[
\rho(a, J, H) = \inf \{ \ell_{\mathcal{H}}(a) | a \in CF_*(M, J, H, \omega) \text{ s.t. } [a] = PSS_*(a) \}.
\]

It is easy to see spectral invariant is independent of almost complex structure, so denoted as \( \rho(H, a) \). In the paper, we will also denote spectral invariant by \( \rho(H, a; \omega) \) or \( \rho_{t}(a, H) \) if we emphasize its dependence on symplectic form \( \omega \) or \( \omega_t \) for \( t \in [0, 1] \). Also different notations also reflects different types of spectral invariants that we use under variant circumstances. In general, spectral invariant enjoys many good properties. The following one is the one that will be used later.

**Theorem 2.3.** (Theorem 1.4 in [Ush08]) For any \( a \in QH_*(M) \), there exists some \( a \in CF_*(M, J, H, \omega) \) such that \([a] = PSS_*(a)\) and \( \rho(H, a) = \ell_{\mathcal{H}}(a) \). In particular, denote \( \text{Spec}(H) := \{ \ell_{\mathcal{H}}(a) | a \in CF_*(M, J, H, \omega) \} \), then \( \rho(a, H) \in \text{Spec}(H) \).

This is called **spectrality property** (or realization property). Moreover, it is well-known that \( \text{Spec}(H) \), as a subset of \( \mathbb{R} \), is a measure zero subset.
2.3.2. Boundary depth. Boundary depth is defined on the chain complex level (and in general can be defined for any filtered chain complexes or Floer-type chain complexes).

**Definition 2.4.** For Floer chain complex \( (CF_\ast(M,J,H,\omega),\partial) \),

\[
\beta(J,H) = \sup_{x \in \text{Im} \partial} \inf \{ \ell(y) - \ell(x) | \partial y = x \}.
\]

By Remark 3.3 in [Ush11] and Cor 5.4 in [Ush13], we know boundary depth is independent of almost complex structure, so denoted as \( \beta(H) \) and it is also well-defined on \( \text{Ham}(M,\omega) \) (see Corollary 5.4 in [Ush13]), so denoted further as \( \beta(\phi) \). Again, we will also denote \( \beta(\phi, \omega) \) or \( \beta_i(\phi) \) if we emphasize its dependence on symplectic form \( \omega \) or \( \omega_t \) for \( t \in [0,1] \). Similar to spectral invariant, boundary depth also enjoys many good properties. In particular, similar to spectral invariant, it also satisfies “realization proposition”, that is,

**Theorem 2.5.** *(Theorem 7.4 in [Ush13])* For any Hamiltonian Floer chain complex associated with Hamiltonian diffeomorphism \( \phi = \phi_H \), there exists a \( y \in CF_\ast(M,J,H,\omega) \) such that \( \beta(\phi) = \ell(y) - \ell(\partial y) \). In particular, \( \beta \in \text{Spec}^H(M) := \{ s - t | s, t \in \text{Spec}(H) \} \).

The main property that we will use later is

**Theorem 2.6.** *(Proposition 3.8 in [Ush13])* If two filtered chain complexes \( (C_1, \partial_1, \ell_1) \) and \( (C_2, \partial_2, \ell_2) \) are \( c \)-quasiequivalent, then \( |\beta_1 - \beta_2| \leq c \) where \( \beta_i \) are boundary depth of \( (C_i, \partial_i, \ell_i) \).

2.3.3. (Partial) symplectic quasi-state. In general, any stable (satisfying some stability property, see [EP03], [EP08]) homogenous quasi-morphism can induce a quasi-state, i.e., a functional \( \zeta : C^\infty(M) \to \mathbb{R} \) satisfying

- If \( \{ F, G \} = 0 \), then for any \( a \in \mathbb{R}, \zeta(H + aG) = \zeta(H) + a\zeta(G) \).
- If \( H \leq G \), then \( \zeta(H) \leq \zeta(K) \).
- \( \zeta(1) = 1 \).

In particular, if we use spectral quasi-morphism which constructed from spectral invariant as suggested in [EP03], we can get a (partial) symplectic quasi-state, that is

\[
\zeta_a(H) = \lim_{k \to \infty} \frac{\rho(a, kH)}{k}
\]

where \( a \in \text{QH}_\ast(M) \) and \( H \in C^\infty(M) \).\(^5\) Symplectic quasi-state is a powerful tool to systematically study the rigidity of intersection property of different subsets in a symplectic manifold. Its closely related concepts are heavy subset and superheavy subset.

**Definition 2.7.** *(Proposition 4.1 in [EP09])* We say a closed subset \( X \subset M \) a-heavy (for some \( a \in \text{QH}_\ast(M,\omega) \)), if \( \zeta_a(H) = 0 \) for \( H \leq 0 \) and \( H_X = 0 \); and a-superheavy, if \( \zeta_a(H) = 0 \) for \( H \geq 0 \) and \( H_X = 0 \).

Note that the original definition of heavy subset and superheavy subset in [EP09] are the followings.

**Definition 2.8.** We say a closed subset \( X \subset M \) a-heavy (for some \( a \in \text{QH}_\ast(M,\omega) \)), if \( \zeta_a(H) \geq \inf_X H \) for all \( H \in C^\infty(M) \) and a-superheavy, if \( \zeta_a(H) \leq \sup_X H \) for all \( H \in C^\infty(M) \).

Note that, in particular, \( X \) is not a-heavy (so not a-superheavy) if we can find some Hamiltonian function \( H \) such that \( \zeta_a(H) < \inf_X H \). Both Definition 2.7 and Definition 2.8 are used in Section 6.

\(^5\)It will be a symplectic quasi-state if, for example, \( \text{QH}_\ast(M,\omega) \) is field-splitting and \( a \) is a unity of the field factor.
3. Algebraic set-up; proof of Proposition 1.4

We will restrict our choice of $\alpha$ advertised in the form (1) by the following three steps.

\[(\Omega_0, \| \cdot \|_k) \xrightarrow{\text{(i)}} \left( (\Omega_0)_{\mathcal{E}} , \| \cdot \|_{\mathcal{E}} \right) \xrightarrow{\text{(i)}} (\Omega, \| \cdot \|_{\mathcal{E}})\]

(i) $(\Omega_0, \| \cdot \|_k)$. For a fixed symplectic form $\omega_0$, we will first consider the following subspace of smooth closed 2-forms,

$$\Omega_0 = \left\{ \text{closed 2-form } \alpha \mid \alpha \text{ vanishes in a neighborhood of each } x(t) \in \text{Per}(\omega_0, H) \right\}.$$  

Recall that we can put a $k$-norm on the set of all closed 2-form, for each $k \in \mathbb{Z}$. Choose, once and for all, (finite) open cover of manifold $M$, say $\{(B_i, \phi_i)\}_{i=1}^m$ where each $\phi_i : B_i \to B(0,1)$ is a diffeomorphism onto the unit ball in $\mathbb{R}^{2n}$ and in each $B_i$, 2-form $\alpha|_{B_i}$ can be written as (with the help of coordinate $(x_i)^{2n}_{i=1}$ of $\mathbb{R}^{2n}$)

$$(\phi_i^{-1})^*(\alpha|_{B_i}) = \sum_{(s,t)\in\{1,...,2n\}^2 \times \{1,...,2n\}} f_{i,s,t} dx_s \wedge dx_t.$$  

Note that lower-indices here only indicate the dependence of parameters, not partial derivative! Then define $k$-norm $\| \cdot \|_k$ of a closed 2-form $\alpha$ in the following way. First, for any $i \in \{1,...,m\}$, define

$$\|\alpha|_{B_i}\|_k = \max_{(s,t)\in\{1,...,2n\}^2 \times \{1,...,2n\}} \sup_{B(0,1)} |d^k(f_{i,s,t})|$$

where symbol $|d^k f|$ takes the sup of function $|d^\gamma f|$ (over the corresponding domain) over all the multi-indices $\gamma$ of length $k$. Then define

$$\|\alpha\|_k = \max_{i \in \{1,...,m\}} \|\alpha|_{B_i}\|_k.$$  

With $k$-norm (for any $k \in \mathbb{Z}$), $(\Omega_0, \| \cdot \|_k)$ is a normed vector space.

(ii) $(\Omega_0)_{\mathcal{E}}, \| \cdot \|_{\mathcal{E}}$. For transversality of moduli space, we need to modify $k$-norm so that we can have a complete normed vector space.

Definition 3.1. Choosing a certain sequence of positive real numbers $\mathcal{E} = (\epsilon_k)_{k \geq 0}$, define $\mathcal{E}$-norm as

$$\|\alpha\|_{\mathcal{E}} = \sum_{k \geq 0} \epsilon_k \|\alpha\|_k.$$  

It is easy to check that the subspace $(\Omega_0)_{\mathcal{E}} = \{ \alpha \in \Omega_0 \mid \|\alpha\|_{\mathcal{E}} < \infty \}$ is a complete normed vector space with $\mathcal{E}$-norm. Moreover, we can choose $\mathcal{E}$ such that $(\Omega_0)_{\mathcal{E}}$ is dense (in terms of $C^k$-topology for any $k \geq 1$) in $\Omega_0$ (for a similar construction, see Section 8.3 in [AD14]). This can be done basically by separability property of $C^\infty(M)$ (or Stone-Weierstrass Theorem) when $M$ is compact and Hausdorff.

(iii) $(\Omega, \| \cdot \|_{\mathcal{E}})$. By definition, $\omega_0$ is non-degenerate, which is an open condition. Therefore, by adding a sufficiently small (in the sense of $\| \cdot \|_{\mathcal{E}}$) closed 2-form $\alpha$, $\omega_0 + \alpha$ is also non-degenerate, so symplectic. Therefore we restrict $\alpha$ to be chosen always from the following ball, denoted as $\Omega$, in $(\Omega_0)_{\mathcal{E}}$ with a certain radius $\delta_* > 0$ (depending on $\omega_0$) which is small enough so that any deformation is still symplectic.

$$\Omega = \{ \alpha \in (\Omega_0)_{\mathcal{E}} \mid \|\alpha\|_{\mathcal{E}} \leq \delta_* \}.$$  

\footnote{Depending on the manifold $M$, sometimes $\alpha$ can go very far, for instance, on symplectic surface.}
Since any closed subspace of a complete space is also complete, we know \((\Omega, || \cdot ||_\omega)\) is complete.

Remark 3.2. For simplicity of notation, we will just use \(| \cdot |\) to denote \(\bar{\omega}\)-norm \(| \cdot |_\omega\). On the other hand, once providing a norm \(| \cdot |\) to measure a closed 2-form, we can define a semi-norm on the cohomology class represented by this 2-form. Specifically, for \(c \in H^2(M; \mathcal{X}) (\cong H^2_{dR}(M; \mathcal{X}))\), define

\[|c|_h = \inf\{|[\alpha] | [\alpha] = c\}\]

Note that by definition, it is always true \(|[\alpha]|_h \leq |\alpha|\). Meanwhile, by definition, for any given \(\delta > 0\), there exists some exact form \(d\gamma\) (depending on \(\delta\)) such that

(14) \nline|\alpha + d\gamma| \leq |[\alpha]|_h + \delta.

Using this language, Proposition 1.4 can be restated in the following more precise statement.

Proposition 3.3. Let \(\omega_0\) be a fixed symplectic form. If a perturbed (symplectic) form \(\omega_1\) satisfies that \(|\omega_1 - \omega_0|\) is sufficiently small, then there exists a diffeomorphism \(\phi \in \text{Diff}(M)\) such that \(\phi^* \omega_1 - \omega_0 \in \Omega\) and \(d_{\text{\text{c}}}(\phi, \mathbb{I}) \leq C|\omega_0 - \omega_1|\) for some constant \(C\) (not depending on \(\omega_1\)).

Proof. Let \(U_i\) be some neighborhood of \(x_i(t)\) for each \(x_i(t) \in \text{Per}(\omega_0, H)\). First, considering the following long exact sequence

\[
\cdots \rightarrow H^2_{dR}(M, \cup_i U_i; \mathbb{R}) \xrightarrow{\iota_*} H^2_{dR}(M; \mathbb{R}) \rightarrow H^2_{dR}(\cup_i U_i; \mathbb{R}) \rightarrow \cdots.
\]

Each \(U_i \approx S^1 \times D^{n-1}\), so \(H^2_{dR}(\cup_i U_i; \mathbb{R}) = 0\). Suppose

\[H^2_{dR}(M; \mathbb{R}) = \bigoplus_{j=1}^m \mathbb{R} \cdot \langle c_j \rangle.
\]

Then \(H^2_{dR}(\cup_i U_i; \mathbb{R}) = 0\) can imply that for any basis element \(c_j\), there exists some nonzero \(b_j \in H^2_{dR}(M, \cup_i U_i; \mathbb{R})\) such that \(\iota_*(b_j) = c_j\) where \(b_j = [\alpha_j]\) for some \(\alpha_j \in \Omega^2(M, \cup_i U_i)\). By definition, \(\alpha_j\) vanishes in every \(U_i\) and \(d\alpha_j = 0\). Meanwhile, we know

\[[\omega_1] - [\omega_0] = \sum_{j=1}^m t_j c_j = \sum_{j=1}^m t_j \iota_*(\alpha_j)\] for some \((t_1, \ldots, t_m) \in \mathbb{R}^m\).

Therefore, \([\omega_1 - \omega_0] = [\sum_{j=1}^m t_j \iota(\alpha_j)]\). On the one hand, we know \(|[\omega_1 - \omega_0]|_h \leq |\omega_1 - \omega_0|\).

On the other hand, by (14), for any \(\delta > 0\), we can choose

\[\alpha := \sum_{j=1}^m t_j \iota(\alpha_j) + \text{exact form}\]

for some exact form such that

\[|\alpha| \leq |[\alpha]|_h + \delta = |[\omega_1 - \omega_0]|_h + \delta \leq |\omega_1 - \omega_0| + \delta.
\]

We can choose \(\delta\) such that \(|\omega_1 - \omega_0| + \delta \leq C_1|\omega_1 - \omega_0|\) for some preferred constant \(C_1 > 1\), so

(15) \nline|\alpha| \leq C_1|\omega_1 - \omega_0|.

Moreover, by the relation from long exact sequence, any exact form of \(M\) (so representing \(0 \in H^2_{dR}(M; \mathbb{R})\)) is equal to \(d\gamma\) for some \(\gamma \in \Omega^2(M, \cup_i U_i)\). Therefore, \(\alpha\) chosen above vanishes near every Hamiltonian orbit. Hence when \(|\omega_1 - \omega_0|\) is sufficiently small (say, less than \(\frac{\delta}{2C_1}\)), we get that \(\alpha \in \Omega\).
Next, consider the (line) homotopy $h_t = (1 - t)(\omega_0 + \alpha) + t\omega_1$ where $t \in [0, 1]$. Note that
\[
\left[ \frac{dh_t}{dt} \right] = [\omega_1 - \omega_0 - \alpha] = [\omega_1 - \omega_0] - [\alpha] = [\alpha] - [\alpha] = 0.
\]
Therefore, $h_t$ represents the same cohomology class for each $t \in [0, 1]$, then by Moser's trick, there exists some $\phi \in \text{Diff}(M)$ such that
\[
\phi^* \omega_1 = \omega_0 + \alpha.
\]
Specifically, $\phi$ is the time-one map of flow $\phi_t$ from solving the differential equation $d\phi_t/dt = X_1(\phi_t)$, where $X_1$ is vector field from the relation
\[
h_t(X_t, -) = -\tau \quad \text{and} \quad d\tau = \omega_1 - (\omega_0 + \alpha).
\]
By triangle inequality, we know
\[
|d\tau| \leq |\omega_1 - \omega_0| + |\alpha| \leq C_2|\omega_1 - \omega_0|
\]
for some constant $C_2 (= 1 + C_1)$. Therefore we get $|X_t| \leq C_3|\omega_1 - \omega_0|$ over $M$,\(^7\) which implies
\[
d_{C^0}(\phi, 1) = \sup_{x \in M} \text{dist}(\phi(x), x) \leq C_4|\omega_1 - \omega_0|
\]
for some constant $C_3$ and $C_4$. Here $C_3$ and $C_4$ involve integral of form and vector field along the manifold $M$, therefore, since $M$ is closed, both constants are finite and only depend on $M$. \(\square\)

Moreover, because of vanishing property of $\alpha$, we have the following proposition.

**Corollary 3.4.** $\text{Per}(\omega_0, H) = \text{Per}(\omega_0 + \alpha, H)$ for any $\alpha \in \Omega$.

**Proof.** By the defining property of $\alpha \in \Omega$, each $x_i(t) \in \text{Per}(\omega_0, H)$ is also an element in $\text{Per}(\omega_0 + \alpha, H)$. Therefore, $\text{Per}(\omega_0 + \alpha, H)$ has at least as many Hamiltonian periodic orbits as $\text{Per}(\omega_0, H)$ has. Next, we claim that there is no other orbit for $\text{Per}(\omega_0 + \alpha, H)$ outside $\bigcup_i U_i$, where $U_i$ is some neighborhood of $x_i(t)$. Without loss of generality, we assume $\text{Per}(\omega_0, H)$ consists of only one element $x(t)$ and its open neighborhood is denoted as $U$.

By contradiction, suppose there is a sequence of symplectic form $\{\omega_{1/n}\}$ ($n$ starts from 1) approaching to $\omega_0$ and for each $\omega_{1/n}$, there exists some $z_n(t)$ such that $z_n(t) \in \text{Per}(\omega_{1/n}, H)$ and $z_n(t) \in M \setminus U$. On the one hand, since $M$ is compact, $z_n(t)$ is bounded. On the other hand, since $M$ is compact again, Hamiltonian vector field (in terms of symplectic form $\omega_{1/n}$) $X_{H\omega_{1/n}}^t$ is uniformly bounded, which implies that $z_n(t)$ is equicontinuous. Therefore, by Arzela-Ascoli theorem, passing to a subsequence, $z_n(t)$ approaches to some $z_0(t)$ in $M \setminus U$ (because it is closed). Finally, we claim that $z_0(t)$ is a Hamiltonian orbit in terms of $\omega_0$, which contradicts our hypothesis (that there is only one such Hamiltonian orbit). In fact, we only need to check
\[
z_0(t) - z_0(0) = \int_0^t X_{H\omega_0}^\tau(z_0(\tau))d\tau
\]
\(^7\)Here we use norm on vector field as the dual norm of $\varepsilon$-norm defined earlier on the space of differential 1-form.
4.1. Proof of Proposition 1.5. In the spirit of proof of the invariant property in Floer theory, we will consider the following parametrized Floer operator. Take a smooth cut-off function $\kappa(s)$ such that for $s \in [-\infty, 0]$, $\kappa(s) = 0$, $s \in [1, \infty]$, $\kappa(s) = 1$ and $\kappa'(s) > 0$. Define interpolating homotopy between $\omega_0$ and a perturbed $\omega_1 := \omega_0 + \alpha$ by $\omega_s = \omega_0 + \kappa(s)\alpha$ for some $\alpha \in \Omega$ (therefore, for any $s \in \mathbb{R}$, $\omega_s$ is also a symplectic form).
Definition 4.1. A parametrized Floer operator $\mathcal{F}^s$ is defined as

$$\mathcal{F}^s = \frac{\partial}{\partial s} + J \left( \frac{\partial}{\partial t} - X_H^{\omega_0} \right)$$

where $X_H^{\omega_0} = J \text{grad}_{\omega_0} H$ and grad$_{\omega_0} H$ is the gradient of $H$ with respect to the metric $g_s(v, w) = \frac{1}{2}(\omega_s(v, Jw) + \omega_s(w, Jv))$.

Note that this operator coincides with standard Floer operator defined by using $\omega_0$ when $s \in [-\infty, 0]$ and standard Floer operator defined by using $\omega_1$ when $s \in [1, \infty]$. Moreover, we will call a $u : S^1 \times \mathbb{R} \to M$ a Floer connecting trajectory if $u$ satisfies $\mathcal{F}^s(u) = 0$.

Now we will prove a similar result for energy estimation as in subsection 2.2.

Remark 4.2. Any $s$-independent solution (corresponding to the zero energy connecting trajectory from $[x, w]$ to itself) actually attains the upper bound in the inequality (2) because $E = 0$ and

$$\mathcal{A}^{\omega_1}([x, w]) - \mathcal{A}^{\omega_0}([x, w]) = -\int_{D^2} w^* a,$$

that is, the upper bound and lower bound in Proposition 1.5 are sharp.

Proof. (Proof of Proposition 1.5) We have the following computation.

$$\mathcal{A}^{\omega_1}([\gamma_+, w_+]) - \mathcal{A}^{\omega_0}([\gamma_-, w_-]) = \int_{-\infty}^{\infty} \frac{d}{ds} \mathcal{A}^{\omega_0}([u(s, \cdot), (w_\# u)(s, \cdot)]) ds$$

$$= -E(u) - \int_{-\infty}^{\infty} \kappa'(s) \int_{D^2} ((w_\# u)(s, \cdot))^* a ds.$$
Denote $w_s = (w_- # u)(s, t)$. For the second term, we have, integral by part,
\[
\int_{-\infty}^{\infty} \kappa'(s) \int_{D^2} w_s^* \alpha \, ds = \int_{-\infty}^{\infty} \frac{d}{ds} \left( \kappa(s) \int_{D^2} w_s^* \alpha \right) \, ds - \int_{-\infty}^{\infty} \kappa(s) \left( \frac{d}{ds} \int_{D^2} w_s^* \alpha \right) \, ds
\]
\[
= \int_{D^2} (w_-)^* \alpha + \int_{-\infty}^{\infty} \int_0^1 (1 - \kappa(s)) \alpha \left( \frac{\partial u}{\partial s}, \frac{\partial u}{\partial t} \right) \, dt \, ds.
\]

Notice the last term satisfies the following inequality,
\[
- \int_{-\infty}^{\infty} \int_0^1 \left| \alpha \left( \frac{\partial u}{\partial s}, \frac{\partial u}{\partial t} \right) \right| \, dt \, ds \leq \int_0^1 (1 - \kappa(s)) \alpha \left( \frac{\partial u}{\partial s}, \frac{\partial u}{\partial t} \right) \, dt \, ds \leq \int_{-\infty}^{\infty} \int_0^1 \left| \alpha \left( \frac{\partial u}{\partial s}, \frac{\partial u}{\partial t} \right) \right| \, dt \, ds.
\]

Now we claim that there exist some constant $C$ and $N$, independent of connecting trajectory $u$ such that
\[
\int_{-\infty}^{\infty} \int_0^1 \left| \alpha \left( \frac{\partial u}{\partial s}, \frac{\partial u}{\partial t} \right) \right| \, dt \, ds \leq |\alpha| \cdot C \cdot \frac{E}{\sqrt{N}} + |\alpha| \cdot E.
\]

In fact, using the relation $X_{\omega_H}^o + J(u) \frac{\partial u}{\partial s} = \frac{\partial u}{\partial t}$, we rewrite what we want to estimate as
\[
\int_{-\infty}^{\infty} \int_0^1 \left| \alpha \left( \frac{\partial u}{\partial s}, X_{\omega_H}^o \right) + J(u) \frac{\partial u}{\partial s} \right| \, dt \, ds \leq \int_{-\infty}^{\infty} \int_0^1 \left| \alpha \left( \frac{\partial u}{\partial s}, X_{\omega_H}^o \right) \right| \, dt \, ds
\]
\[
+ \int_{-\infty}^{\infty} \int_0^1 \left| \alpha \left( \frac{\partial u}{\partial s}, J(u) \frac{\partial u}{\partial s} \right) \right| \, dt \, ds.
\]

The second term is bounded from above by $|\alpha| \cdot E$. For the first term, by asymptotic property of $u$ and definition of $\alpha$ (that vanishes near orbits $\gamma_-(t)$ and $\gamma_+(t)$), we know there exists some $s_u \in \mathbb{R}$ (depending on $u$) such that $\alpha = 0$ when $s \in [-\infty, -s_u] \cup [s_u, +\infty]$. So
\[
\int_{-\infty}^{\infty} \int_0^1 \left| \alpha \left( \frac{\partial u}{\partial s}, X_{\omega_H}^o \right) \right| \, dt \, ds = \int_{-s_u}^{s_u} \int_0^1 \left| \alpha \left( \frac{\partial u}{\partial s}, X_{\omega_H}^o \right) \right| \, dt \, ds
\]

Meanwhile, by Lemma 5.2 in [LO95], there exists some constant positive $N$ (independent of $u$) such that for any $s_n \in [-s_u, s_u]$ (enlarge $s_u$ if necessary),
\[
\left| \frac{\partial u(s, t)}{\partial s} \right|_{s=s_n}^2 = \int_0^1 \left| \frac{\partial u}{\partial s} \right|_{s=s_n}^2 \, dt \geq N.
\]

In other words, if we set
\[
\mathcal{G} = \left\{ s_n \in [-\infty, \infty] \mid \left| \frac{\partial u(s, t)}{\partial s} \right|_{s=s_n}^2 \geq N \right\}
\]
then $[-s_u, s_u] \subset \mathcal{G}$. So for (20), we can improve it to be integrated over $\mathbb{R}/\mathbb{Z} \times \mathcal{G}$. It will not change the value of the integral by the vanishing property of $\alpha$. Therefore, we have
\[
\int_{-\infty}^{\infty} \int_0^1 \left| \alpha \left( \frac{\partial u}{\partial s}, X_{\omega_H}^o \right) \right| \, dt \, ds = \int_{\mathcal{G}} \int_0^1 \left| \alpha \left( \frac{\partial u}{\partial s}, X_{\omega_H}^o \right) \right| \, dt \, ds
\]
\[
\leq |\alpha| \cdot C \int_{\mathcal{G}} \int_0^1 \left| \frac{\partial u}{\partial s} \right|_{g, s} \, dt \, ds
\]
where $C$ is an upper bound of uniform norm of vector field $X^\omega_{|I}$ for any $s \in [0,1]$ on closed manifold $M$. On the other hand, due to the energy constraint, (Lebegue) measure of $\mathcal{A}$ satisfies $\mu(\mathcal{A}) \leq E/N$. Applying Cauchy-Schwarts inequality, we get

$$\left( \int_\mathcal{A} \int_0^1 \left| \frac{\partial u}{\partial s} \right|^2 ds \right)^2 \leq \left( \int_\mathcal{A} \int_0^1 1^2 ds \right) \cdot \left( \int_\mathcal{A} \int_0^1 \left| \frac{\partial u}{\partial s} \right|^2 ds \right) \leq \frac{E^2}{N}.$$ 

Together, we get the desired conclusion. \hfill \Box

4.2. **Variant energy estimations.** As advertised in the introduction, there will be a brief classification, namely, Type I and Type II. The former one comes from the variation of homotopies, which consists of three sub-cases. The latter one comes from variation of valuations of action functionals, which consists of two sub-cases.

**Type I.** (i) Interpolation homotopy from $\omega_1$ to $\omega_0$.

This homotopy is constructed from reserving the time $s$ to $-s$ (so the corresponding $\mathcal{A}^\omega_{\omega_0}$ and $\mathcal{A}^\omega_{\omega_1}$). Moreover, we also switch the asymptotic condition and homotopy condition, that is,

- (b') $\lim_{s \to -\infty} u(s, t) = \gamma_+(t)$ and $\lim_{s \to -\infty} u(s, t) = \gamma_-(t)$.
- (c') $[\gamma_-, w_-] = [\gamma_+, w_+ \# u]$.

Then by the same argument as in Proposition 1.5,

**Corollary 4.3.** Suppose $u : \mathbb{R} \times S^1 \to M$ is a Floer connecting trajectory from $[\gamma_+, w_+]$ to $[\gamma_-, w_-]$ (i.e, satisfying (a) in Section 4, (b') and (c') as above) with energy $E < \infty$. We have the following energy estimation between action functionals,

$$(21) \quad -(1 + C')E + \int_{D^2} (w_-)^* \alpha \leq \mathcal{A}^\omega_{\omega_0}([\gamma_-, w_-]) - \mathcal{A}^\omega_{\omega_0}([\gamma_+, w_+]) \leq -(1 + C')E + \int_{D^2} (w_-)^* \alpha$$

for some constant $C'$ in Proposition 1.5.

**Type I.** (ii) Fix a $\omega_1 := \omega_0 + \alpha$ for some $\alpha \in \Omega$. Denote $\omega_s = (1 - s)\omega_0 + s\omega_1 = \omega_0 + s\alpha$ for any $s \in [0,1]$. Interpolation homotopy from $\omega_s$ to $\omega_t$ for any $s, t \in [0,1]$.

Let $s < t$ (for $t < s$, by the discussion of Type I (i) above, it is easy to see how to modify the following Corollary 4.4 to get the corresponding result). Similar to Proposition 1.5,

**Corollary 4.4.** Suppose $u : \mathbb{R} \times S^1 \to M$ is a Floer connecting trajectory from $[\gamma_-, w_-]$ to $[\gamma_+, w_+]$ (i.e, satisfying (a), (b) and (c) in Section 4) with energy $E < \infty$. We have the following energy estimation between action functionals,

$$(22) \quad -(1 + C_{s,t})E + (s-t) \int_{D^2} (w_-)^* \alpha \leq \mathcal{A}^\omega_{\omega_1}([\gamma_-, w_-]) - \mathcal{A}^\omega_{\omega_1}([\gamma_+, w_+]) \leq -(1 + C_{s,t})E + (s-t) \int_{D^2} (w_-)^* \alpha$$

for some constant $C_{s,t} = |(t-s)|C'$ for $C'$ in Proposition 1.5 (so still independent of connecting trajectory).

Note that when $t$ approaches to $s$, (22) approaches to the standard energy computation:

$$\mathcal{A}^\omega_{\omega_0}([\gamma_-, w_-]) - \mathcal{A}^\omega_{\omega_0}([\gamma_+, w_+]) = -E.$$

**Type I.** (iii) Consider “symmetric” interpolation between $\omega_0$ (or in general $\omega_s$ for any $s \in [0,1]$) and itself.
This is constructed by a cut-off function $\kappa(s)$ chosen as, for a fixed $R \in \mathbb{R}$, when $s \in [-\infty, -R] \cup [R, \infty]$, $\kappa(s) = 0$ and when $s \in [-R+1, R+1]$, $\kappa(s) = 1$. Moreover, when $s \in [-(R+1), -R]$, $\kappa'(s) > 0$ and when $s \in [R, R+1]$, $\kappa'(s) < 0$. We have an energy estimation (stated only for $\omega_0$ which is easy to be adapted to $\omega_s$ for any $s \in [0, 1]$)

**Corollary 4.5.** Suppose $u : \mathbb{R} \times S^1 \rightarrow M$ is a Floer connecting trajectory from $[\gamma_-, w_-]$ to $[\gamma_+, w_+]$ (i.e., satisfying (a), (b), (c) in Section 4) with energy $E < \infty$. We have the following energy estimation between action functionals,

$$(23) \quad -(1 + C')E \leq \mathcal{A}_{\omega_0}([\gamma_-, w_-]) - \mathcal{A}_{\omega_0}([\gamma_+, w_+]) \leq -(1 - C')E$$

for some constant $C'$ in Proposition 1.5.

**Proof.** By the same start as in Proposition 1.5,

$$\mathcal{A}_{\omega_0}([\gamma_+, w_+]) - \mathcal{A}_{\omega_0}([\gamma_-, w_-]) = \int_{-\infty}^{\infty} \frac{d}{ds} \mathcal{A}_{\omega_0}([u(s, \cdot), (w_- \# u)(s, t)])ds$$

$$= -E - \int_{-\infty}^{\infty} \kappa'(s) \int_{D^2} [(w_- \# u)(s, t)]^* \alpha ds$$

$$= -E - \int_{-\infty}^{\infty} \kappa'(s) \int_{D^2} w_s^* \alpha ds \quad \text{where} \quad w_s(t) := (w_- \# u)(s, t)$$

$$= -E + \int_{-\infty}^{\infty} \int_{0}^{1} \kappa(s) \alpha \left( \frac{\partial u}{\partial s}, \frac{\partial u}{\partial t} \right) dtds.$$

Moreover,

$$-C'E \leq \int_{-\infty}^{\infty} \int_{0}^{1} \kappa(s) \alpha \left( \frac{\partial u}{\partial s}, \frac{\partial u}{\partial t} \right) dtds \leq C'E$$

by the claim (19). Notice that we don’t have the “unfriendly” term $\int_{D^2} w_s^* \alpha$ because the cut-off function $\kappa(s)$ is symmetric on both ends ($s = \infty$ and $s = -\infty$), which leaves no extra terms after integration by part. \hfill \Box

**Type II.** (i) Homotopy from $\omega_0$ to $\omega_1$ but evaluated at ends both by $\mathcal{A}_{\omega_0}$ or both by $\mathcal{A}_{\omega_1}$.

In the same spirit of energy estimation from Proposition 1.5, we can evaluation action on $[\gamma_+, w_+]$ by also using the same action functional. This looks a bit unnatural because it is incompatible with the perturbation of the symplectic forms, but this will be used later in the Section 5 for the construction of Floer chain map, see Proposition 7.10. It turns out we can get a similar energy estimation,

**Corollary 4.6.** Suppose $u : \mathbb{R} \times S^1 \rightarrow M$ is a Floer connecting trajectory from $[\gamma_-, w_-]$ to $[\gamma_+, w_+]$ (i.e., satisfying (a), (b) and (c) in Section 4) with energy $E < \infty$. We have the following energy estimation between action functionals,

$$-E \leq \mathcal{A}_{\omega_0}([\gamma_+, w_+]) - \mathcal{A}_{\omega_0}([\gamma_-, w_-]) \leq -(1 - C'')E$$

and

$$-E \leq \mathcal{A}_{\omega_1}([\gamma_+, w_+]) - \mathcal{A}_{\omega_1}([\gamma_-, w_-]) \leq -(1 - C'')E$$

for some constant $C'' = 2C'$ (independent of connecting orbit) where $C'$ is the constant in Proposition 1.5.
\textbf{Proof.} We will only prove the case of }$\omega_0$. It is exactly the same for the case of }$\omega_1$. First, note that
\[
\mathcal{A}_{\omega_0}([\gamma_+, w_+]) = \mathcal{A}_{\omega_0}([\gamma_-, w_-]) - \int_{D^2} w^*_+\alpha.
\]
Therefore, by energy estimation from Proposition 1.5, we have
\[
\mathcal{A}_{\omega_0}([\gamma_+, w_+]) - \mathcal{A}_{\omega_0}([\gamma_-, w_-]) \leq -(1 - C')E + \int_{D^2} w^*_+\alpha - \int_{D^2} w^*_-\alpha
\]
\[= -(1 - C')E + \int_{\mathbb{R} \times S^1} u^*\alpha
\]
\[= -(1 - C')E + \int_{\mathbb{R}} \int_{0}^{1} a\left(\frac{\partial u}{\partial s}, \frac{\partial u}{\partial t}\right) dt ds
\]
where the second line comes from the homological condition that }$w_+ \sim w_- \# u$. Moreover,
\[
-C'E \leq \int_{-\infty}^{\infty} \int_{0}^{1} a\left(\frac{\partial u}{\partial s}, \frac{\partial u}{\partial t}\right) dt ds \leq C'E
\]
by the claim (19). Together we get the desired conclusion. \hfill \Box

This can be easily generalized to the case between }$\omega_s$ and }$\omega_t$ for any }$s, t \in [0, 1]$ when }$\omega_s$ (or }$\omega_t$) moves along the line segment connecting }$\omega_0$ and }$\omega_1$ as in Type I (ii). The result will be a similar to Corollary 4.6 based on Corollary 4.4.

\textbf{Type II. (ii) Unperturbed Floer operator }$\mathfrak{F}^s$ \textit{but evaluated at ends both by }$\omega_t$ \textit{for any }$s, t \in [0, 1]$ \textit{when }$\omega_s$ (or }$\omega_t$ \textit{moves along the line segment connecting }$\omega_0$ and }$\omega_1$.

This is another type of energy estimation (still appearing unnatural) that will be used in Section 7 for checking a certain Floer boundary map is well-defined, see Proposition 7.8. As explained in the introduction, if connection trajectory }$u(s, t)$ satisfies unperturbed Floer equation }$\mathfrak{F}^s$ (with a required asymptotic condition), then when }$t = s$, \(\mathcal{A}_{\omega_s}([\gamma_+, w_+]) - \mathcal{A}_{\omega_s}([\gamma_-, w_-]) = -E(u)\). For the general case of }$t \in [0, 1]$,

\textbf{Corollary 4.7.} Suppose }$u : \mathbb{R} \times S^1 \to M$ is a Floer connecting trajectory from }$[\gamma_-, w_-]$ to }$[\gamma_+, w_+]$ (i.e., satisfying (b) and (c) in Section 4 and unperturbed Floer operator }$\mathfrak{F}^s$ for a fixed }$s \in [0, 1]$ with energy }$E < \infty$. We have the following energy estimation between action functionals, for any }$t \in [0, 1]$,
\[
-(1 + C'_{s,t})E \leq \mathcal{A}_{\omega_s}([\gamma_+, w_+]) - \mathcal{A}_{\omega_s}([\gamma_-, w_-]) \leq -(1 - C'_{s,t})E
\]
for some constant }$C'_{s,t} = |s - t|C'$ for constant }$C'$ in Proposition 1.5 with constant }$C$ in the expression of }$C'$ possibly replaced by another constant still independent of connecting trajectory.

\textbf{Proof.} Similar to the proof of Corollary 4.6, we have
\[
\mathcal{A}_{\omega_s}([\gamma_+, w_+]) - \mathcal{A}_{\omega_s}([\gamma_-, w_-]) = -E + (s - t) \int_{S^1 \times \mathbb{R}} u^*\alpha
\]
by homological condition }$w^*_+ \sim w^*_- \# u$. On the other hand, by almost exactly the same argument proving the claim (19), it is readily to see
\[
\int_{S^1 \times \mathbb{R}} u^*\alpha = \int_{-\infty}^{\infty} \int_{0}^{1} a\left(\frac{\partial u}{\partial s}, \frac{\partial u}{\partial t}\right) dt ds \leq |a| \cdot C \cdot \frac{E}{\sqrt{N}} + |a| \cdot E
\]
here \( C \) is the uniform bound of vector field \( X^{\omega} \) on closed manifold \( M \) for this fixed \( s \). Therefore we get the conclusion.

**Remark 4.8.** When applying any version of energy estimation above, we will also assume \( |\alpha| \) is small enough so that \( 1 - C' \) is always positive (hence \( 1 - C'_{s,t} \) and \( 1 - C'_{s'} \) are both positive for any \( s, t \in [0, 1] \)).

The following result can be regarded as a corollary of Corollary 4.7 which plays an important role in the proof of the main result in Appendix.

**Corollary 4.9.** Fix capped Hamiltonian orbits \( [x, w] \) and \( [y, v] \). Suppose a sequence of spheres \( \{A_n\}_{n=1}^\infty \) in \( \pi_2(M) \) is introduced by a sequence of Floer connecting trajectories \( \{u_n\}_{n=1}^\infty \) connecting \( [x, w] \) and \( [y, v] \) satisfying unperturbed Floer operator \( \mathcal{F} \) for a fixed \( s \in [0, 1] \) and (b), (c) in Section 4, i.e.,

\[
[x, w \# u_n] = [y, v \# A_n] \quad \text{(or denoted as } T^{A_n}[y, v]).
\]

Then for any \( t \in [0, 1] \),

\[
\int_{S^2} A^*_n \omega_t \to \pm \infty \quad \text{if and only if} \quad \int_{S^2} A^*_n \omega_s \to \pm \infty.
\]

**Proof.** Denote

\[
E_n := E(u_n) = \mathcal{A}_{\omega_s}([x, w]) - \mathcal{A}_{\omega_t}(T^{A}[y, v]) = \int_{S^2} A^*_n \omega_s + N_1
\]

where \( N_1 = \mathcal{A}_{\omega_s}([x, w]) - \int_0^1 H(y(t), t) dt \). By Corollary 4.7,

\[
(1 + C'_{s,t}) E_n \geq \mathcal{A}_{\omega_t}([x, w]) - \mathcal{A}_{\omega_s}([y, v]) \geq (1 - C'_{s,t}) E_n
\]

which is equivalent to

\[
(1 + C'_{s,t}) E_n \geq \int_{S^2} A^*_n \omega_t + N_2 \geq (1 - C'_{s,t}) E_n
\]

where \( N_2 = \mathcal{A}_{\omega_s}([x, w]) - \int_0^1 H(x(t), t) dt \). So

\[
(1 + C'_{s,t}) \left( \int_{S^2} A^*_n \omega_s + N_1 \right) - N_2 \geq \int_{S^2} A^*_n \omega_t \geq (1 - C'_{s,t}) \left( \int_{S^2} A^*_n \omega_s + N_1 \right) - N_2.
\]

Since \( 1 - C'_{s,t} \) and \( 1 + C'_{s,t} \) are positive, we get the conclusion. \( \square \)

5. Continuity results; proofs of Theorem 1.6, 1.7

The proof of Theorem 1.6 follows from a quantitative comparison between Floer chain complexes (over \( \mathcal{X} \)) \( \left( CF_s(M, J, H, \omega_0, \partial_{\omega_0}) \right) \) and \( \left( CF_s(M, J, H, \omega_0 + \alpha, \partial_{\omega_0 + \alpha}) \right) \) for any \( \alpha \in \Omega \). Specifically,

**Lemma 5.1.** Under the hypothesis of Theorem 1.6, for any \( \alpha \in \Omega \), there exists some constant \( S(\alpha) \) (depending on \( \alpha \)) such that \( \left( CF_s(M, J, H, \omega_0), \partial_{\omega_0} \right) \) and \( \left( CF_s(M, J, H, \omega_0 + \alpha), \partial_{\omega_0 + \alpha} \right) \) are \( S(\alpha) \)-quasiequivalent (c.f. Proposition 1.1).
Proof. Case I (when $M$ is aspherical). Fix a basis of capped Hamiltonian orbits (over $\mathcal{X}$) of $CF_*(M,J,H,\omega_0)$ (so also a graded basis for $CF_*(M,J,H,\omega_0 + \alpha)$ by Corollary 3.4 and the discussion following it), say, $\{x_1,w_1\}, \ldots, \{x_n,w_n\}$. By definition (see Definition 3.7 in [Ush13]) of quasi-equivalence, we need to find a quadruple $(\Phi, \Psi, K_0, K_\alpha)$ as in Proposition 1.1 and satisfies certain filtration shifts. First, we construct map $\Phi : CF_*(M,J,H,\omega_0) \to CF_*(M,J,H,\omega_0 + \alpha)$ by

$$\Phi([x_i,w_i]) = \sum_{j \in \{1, \ldots, n\}} n([x_i,w_i],[x_j,w_j])[x_j,w_j]$$

where number $n([x_i,w_i],[x_j,w_j])$ is defined by counting the solution of (17), with a certain asymptotic condition and with condition on indices. Specifically, using the language of moduli space, we are considering the following moduli space

$$\mathcal{M}([x_i,w_i],[x_j,w_j],\omega_s;H,J) = \left\{ u : \mathbb{R} \times S^1 \to M \left| \begin{array}{l}
\lim_{s \to -\infty} u(s,t) = x_i(t), \\
\lim_{s \to \infty} u(s,t) = x_j(t), \\
E(u) < \infty, \mathcal{F}(u) = 0, \\
w_i \# u \text{ is homologous } w_j, \\
\mu_{CZ}([x_i,w_i]) = \mu_{CZ}([x_j,w_j]) 
\end{array} \right\}. \right.$$ 

By standard Floer theory, $\mathcal{M}([x_i,w_i],[x_j,w_j],\omega_s;H,J)$ is a zero-dimensional (from index condition) compact manifold, therefore, $n([x_i,w_i],[x_j,w_j])$ is a finite number. Because there are only finitely many Hamiltonian orbits, the sum in the expression (24) is a finite sum. Therefore, convergence (in Novikov sense) holds automatically. Moreover, this map is a chain map, which comes from standard gluing argument. Similarly, we can define $\Psi : CF_k(M,J,H,\omega_0 + \alpha) \to CF_k(M,J,H,\omega_0)$ and it is also a chain map. Now we trace the change of filtrations. Any Floer connecting trajectory $u$ between $[x_i,w_i]$ and $[x_j,w_j]$ from moduli space above gives rise to an inequality by Theorem 1.5, that is

$$\mathcal{A}_{\omega_0 + \alpha}([x_j,w_j]) - \mathcal{A}_{\omega_0}([x_i,w_i]) \leq -(1 - C')E(u) - \int_{D^2} w_i^* \alpha \leq -\int_{D^2} w_i^* \alpha.$$

By definition of filtration function (10), for $\Phi(c)$ where $c$ is a chain from $CF_*(M,J,H,\omega_0)$, there exists some $j_0 \in \{1, \ldots, n\}$ depending on the perturbation $\alpha$ such that

$$\ell_{\omega_0 + \alpha}(\Phi(c)) = \mathcal{A}_{\omega_0 + \alpha}([x_{j_0},w_{j_0}]).$$

Meanwhile, there exists some $i_0 \in \{1, \ldots, n\}$ such that $[x_{i_0},w_{i_0}]$ (as a generator of chain $c$) linked to $[x_{j_0},w_{j_0}]$ by some Floer connecting trajectory $u$ in the moduli space above. Therefore, we know

$$\mathcal{A}_{\omega_0 + \alpha}([x_{i_0},w_{i_0}]) - \mathcal{A}_{\omega_0}([x_{i_0},w_{i_0}]) \leq -\int_{D^2} w_i^* \alpha.$$

Meanwhile, by (10) again, $\ell_{\omega_0}(c) \geq \mathcal{A}_{\omega_0}([x_{i_0},w_{i_0}])$. Therefore,

$$\ell_{\omega_0 + \alpha}(\Phi(c)) - \ell_{\omega_0}(c) \leq \mathcal{A}_{\omega_0 + \alpha}([x_{j_0},w_{j_0}]) - \mathcal{A}_{\omega_0}([x_{i_0},w_{i_0}]).$$

Let $s_1(\alpha) = \max\{ -\int_{D^2} w_i^* \alpha \} = -\min\{ \int_{D^2} w_i^* \alpha \}$, independent of connecting trajectory, so for any $\lambda \in \mathbb{R}$, we have a well-defined chain map $\Phi : CF_*(M,J,H,\omega_0) \to CF_*(M,J,H,\omega_0 + \alpha)$. A similar argument together with Corollary 4.3 results in another constant $s_2(\alpha) = \max\{ \int_{D^2} w_i^* \alpha \}$ and a chain map $\Psi : CF^\lambda_*(M,J,H,\omega + \alpha) \to CF^\lambda_{\omega_0 + \alpha}(M,J,H,\omega)$ for every $\lambda \in \mathbb{R}$. Next, since
\( \Psi \circ \Phi \) is (Floer) homotopic to \( \mathbb{I} \) (induced by two different homotopies between \( \omega_0 \) and itself) on \( CF_*(M,J,H,\omega_0) \), there exists a map \( K_0 : CF_*(M,J,H,\omega_0) \to CF_{*+1}(M,J,H,\omega_0) \) as
\[
K_0([x_i,w_j]) = \sum_{j=1}^{\infty} n([x_i,w_j],[x_j,w_j]) [x_j,w_j]
\]
where number \( n([x_i,w_j],[x_j,w_j]) \) is defined by counting the solution of (17) with the Type 1 (iii) homotopy between \( \omega_0 \) and itself, asymptotic condition and with condition on indices. Specifically, using the language of moduli space, we are considering the following moduli space
\[
M([x_i,w_j],[x_j,w_j],\omega_s;H,J) = \left\{ u : \mathbb{R} \times S^1 \to M \middle| \begin{array}{c}
\lim_{s \to -\infty} u(s,t) = x_i(t), \\
\lim_{s \to +\infty} u(s,t) = x_j(t), \\
E(u) < \infty, \mathcal{G}_{\omega_s}(u) = 0, \\
w_i \# u \text{ is homologous } w_j, \\
\mu_{\mathcal{CZ}}([x_i,w_i]) + 1 = \mu_{\mathcal{CZ}}([x_j,w_j])
\end{array} \right\}.
\]
Again, by standard Floer theory, this moduli space is a compact zero-dimensional space, so finite. Moreover, by the same reason, the sum in (27) is a finite sum so it automatically satisfies Novikov finiteness condition. Again, tracing the change of filtration, for some chain \( c \in CF_*(M,J,H,\omega_0) \), suppose \( \ell_{\omega_0}(K_0(c)) = \mathcal{A}_{\omega_0}([x_q,0,w_q]) \) for some \( q_0 \). There exists some \( [x_{p_0},w_{p_0}] \) as a generator of \( c \) linking to \( [x_q,0,w_q] \) by some Floer connecting trajectory in the corresponding moduli space. By Corollary 4.5, \( \mathcal{A}_{\omega_0}([x_q,0,w_q]) \leq \mathcal{A}_{\omega_0}([x_{p_0},w_{p_0}]) \), which implies
\[
\ell_{\omega_0}(K_0(c)) - \ell_{\omega_0}(0) \leq \mathcal{A}_{\omega_0}([x_{q_0},w_{q_0}]) - \mathcal{A}_{\omega_0}([x_{p_0},w_{p_0}]) \leq 0.
\]
Therefore, \( K_0 : CF_*^*(M,J,H,\omega_0) \to CF_{*+1}^*(M,J,H,\omega_0) \to CF_{*+1}^{\lambda+s_1(\alpha)+s_2(\alpha)}(M,J,H,\omega_0) \) (because \( s_1(\alpha) + s_2(\alpha) \geq 0 \)). Similar argument results in a map \( K_0 : CF_*^*(M,J,H,\omega_0 + \alpha) \to CF_{*+1}^{\lambda+s_1(\alpha)+s_2(\alpha)}(M,J,H,\omega_0 + \alpha) \) for any \( \lambda \in \mathbb{R} \). To get the desired conclusion, only take \( S(\alpha) = s_1(\alpha) + s_2(\alpha) \).

**Case II** (when \( \alpha \) is exact and \( M \) is an arbitrary symplectic manifold). For each Hamiltonian orbit \( x_i \), there might be infinitely many cappings but they are differed by element in \( \pi_2(M) \). Therefore, for \( [x_i,w] \) with any capping \( w \),
\[
\int_{D^2} w^* \alpha = \int_{D^2} w^*_i \alpha + \int_{S^2} w^*_i \alpha = \int_{D^2} w^*_i \alpha
\]
by Stoke's theorem since \( \alpha \) is exact. The entire argument above for Case I goes through. In particular, it provides a uniform upper bounds for the shifts of filtration which are \( s_1(\alpha), s_2(\alpha) \) and \( S(\alpha) = s_1(\alpha) + s_2(\alpha) \).

**Proof.** (Proof of Theorem 1.6) Lemma 5.1 together with Theorem 2.6 gives the continuity result of boundary depth, that is
\[
|\beta(\omega_0,\phi) - \beta(\omega_0 + \alpha,\phi)| \leq S(\alpha).
\]
For spectral invariant, by the same idea of proof of (iii) in Theorem 3.1 in [PSS96], we have the following commutative diagram

\[
\begin{array}{ccc}
\Psi_{\alpha} & \xrightarrow{QH_*(M;\mathcal{X})} & \Psi_{\alpha+\beta} \\
\Phi_0 & \xrightarrow{HF_*(M,J,H,\omega_0)} & \Phi_{\alpha} \\
\end{array}
\]
that is \( PSS^{\omega_0+\alpha} = \Phi_* PSS^{\omega_0} \) where \( \Phi \) is the chain map constructed from Lemma 5.1. By Theorem 2.3, there exists some optimization element \( x \in CF_*(M,J,H,\omega_0) \) such that \( \rho(a,H,\omega_0) = \ell_{\omega_0}(x) \), where \([x] = PSS^{\omega_0}(a)\). So
\[
[\Phi(x)] = [\Phi([x])] = \Phi_*(PSS^{\omega_0}(a)) = PSS^{\omega_0+\alpha}(a).
\]
Moreover,
\[
\rho(a,H,\omega_0 + \alpha) - \rho(a,H,\omega_0) \leq \ell_{\omega_0+\alpha}(\Phi(x)) - \ell_{\omega_0}(x) \leq S(\alpha).
\]
Switch the role of \( \omega_0 \) and \( \omega_0 + \alpha \), we will get the other direction. Therefore,
\[
|\rho(a,H,\omega_0) - \rho(a,H,\omega_0 + \alpha)| \leq S(\alpha).
\]
Last but not least, by definition of \( S(\alpha) \) in Lemma 5.1 \((S(\alpha) = \max_i \int_{D^2} w_i^* \alpha - \min_i \int_{D^2} w_i^* \alpha)\), there exists a constant \( C := 2 \max_i (\text{Area}(w_i)) \) such that \( S(\alpha) \leq C|\alpha| \).

**Proof.** (Proof of Theorem 1.7) By hypothesis on dimension of \( H^2(M; \mathbb{R}) \), \([\alpha] \) and \([\omega_0] \) are co-linear, so we can assume
\[
[\alpha] = \epsilon[\omega_0]
\]
for some (sufficiently) small \( \epsilon \geq 0 \). Specifically, \( \epsilon = \frac{|\alpha|}{\|\omega_0\|} \leq \frac{|\alpha|}{\|\omega_0\|} \). The key observation is that for parameters, \( J, H \) and \( \omega \), if we rescale them to be \((1+\epsilon)J, (1+\epsilon)H \) and \((1+\epsilon)\omega \), then
\[
(1 + \epsilon)\beta(\omega_0, \phi_H) = \beta((1 + \epsilon)\omega_0, \phi_{(1+\epsilon)H})
\]
and
\[
(1 + \epsilon)\rho([M], H, \omega_0) = \rho([M], (1 + \epsilon)H, (1 + \epsilon)\omega_0).^9
\]
Hence
\[
|\beta(\omega_0, \phi_H) - \beta(\omega_0 + \alpha, \phi_H)| \leq |\beta(\omega_0, \phi_H) - \beta((1 + \epsilon)\omega_0, \phi_{(1+\epsilon)H})|
+ |\beta((1 + \epsilon)\omega_0, \phi_{(1+\epsilon)H}) - \beta((1 + \epsilon)\omega_0, \phi_H)|
+ |\beta((1 + \epsilon)\omega_0, \phi_H) - \beta(\omega_0 + \alpha, \phi_H)|
\leq \epsilon \beta(\omega_0, \phi_H) + \epsilon \|H\||\alpha| + C|\epsilon \omega_0 - \alpha|
\leq C'|\alpha|
\]
where \( C' = \frac{\beta(\omega_0, \phi_H)}{|\omega_0|} + \frac{\epsilon}{|\omega_0|} + \frac{C}{|\omega_0|} \) + \( C \) and the third term of the second last line of the computation above comes from Theorem 1.6. Similarly, for spectral invariant,
\[
|\rho([M], H, \omega_0) - \rho([M], H, \omega_0 + \alpha)| \leq |\rho([M], H, \omega_0) - \rho([M], (1 + \epsilon)H, (1 + \epsilon)\omega_0)|
+ |\rho([M], (1 + \epsilon)H, (1 + \epsilon)\omega_0) - \rho([M], H, (1 + \epsilon)\omega_0)|
+ |\rho([M], H, (1 + \epsilon)\omega_0) - \rho([M], H, \omega_0 + \alpha)|
\leq \epsilon \rho([M], H, \omega_0) + \epsilon \|H\||\alpha| + C|\epsilon \omega_0 - \alpha|
\leq C''|\alpha|
\]
where \( C'' = \frac{\rho([M], H, \omega_0)}{|\omega_0|} + \frac{\epsilon}{|\omega_0|} + \frac{C}{|\omega_0|} \) + \( C \). Thus we get the desired conclusion. \( \square \)

^9To be more precise, \([M] \) on the right hand side should be the fundamental class of \( QH_*(M, (1 + \epsilon)\omega_0) \) which is \((1 + \epsilon)[M] \).
Remark 5.2. (i) Under the hypothesis of Theorem 1.6, when $\{\omega_s\}_{s \in [0,1]}$ moves along the line segment between $\omega_0$ and $\omega_1 = \omega_0 + \alpha$, the same argument of Lemma 5.1 can be generalized to the following result by Corollary 4.4: for any $s, t \in [0,1]$, there exist some constant $S_{s,t}(\alpha)$ such that $(CF_*(M,J,H,\omega_s),\partial\omega_s)$ and $(CF_*(M,J,H,\omega_t),\partial\omega_t)$ are $S_{s,t}(\alpha)$-quasiequivalent. Moreover, $S_{s,t}(\alpha) \leq |s-t| \cdot C|\alpha|$ for some constant $C$.

(ii) From the perspective of persistent homology from [UZ15], we can associate both Floer-type complexes $\mathcal{C}_0 = (CF_*(M,J,H,\omega_0),\partial_0)$ and $\mathcal{C}_\alpha = (CF_*(M,J,H,\omega_0),\partial_{\omega_0 + \alpha})$ their barcodes, denoted as $\mathcal{B}_0$ and $\mathcal{B}_\alpha$, respectively. By Lemma 5.1, their quasi-equivalence distance (see Definition 8.1 in [UZ15])

$$d_Q(\mathcal{C}_0, \mathcal{C}_\alpha) \leq \frac{S(\alpha)}{2}.$$ 

Therefore, by Stability Theorem in [UZ15], we know (where $d_B$ is bottleneck distance, see Definition 8.14 in [UZ15]),

$$d_B(\mathcal{C}_0, \mathcal{C}_\alpha) \leq 2d_Q(\mathcal{C}_0, \mathcal{C}_\alpha) \leq S(\alpha) \leq C|\alpha|.$$ 

In particular, length of each bar (where boundary depth is the length of the longest finite length bar) converges to 0 when $\alpha \to 0$. To some extent, this can be regarded as a generalization of Theorem 1.6 (also see the relations between barcode and spectral invariant in subsection 6.1 in [UZ15]).

6. Application; proof of Theorem 1.9

Proof. (Proof of Theorem 1.9) Let $U \subset \Sigma$ be a disjoint union and $H \in C^\infty(\mathbb{R}/\mathbb{Z} \times U)$. Pick an open ball $V \subset \Sigma \setminus \bar{U}$ and a closed 2-form $\alpha$ positively supported (only) in $V$. Considering the following perturbation (of symplectic form)

$$\omega_s = \omega_0 + \kappa(s)\alpha$$

for some $\kappa(s)$ such that $\omega_1$ is sufficiently larger than $\omega_0$. Note that since we are on symplectic surface and by the definition of $\alpha$, there is no problem to rescale $\alpha$ to be arbitrary large so that $\omega_s$ is still symplectic. Then in $(\Sigma, \omega_1)$, $U$ can be viewed as a disjoint union of topological balls with total area smaller than half of the total area of $\Sigma$. Then it is displaceable. By well-known energy-capacity inequality,

$$\rho(a,H,\omega_1) \leq e(U)$$

where $e(U)$ denotes the displacement energy of $U$. Moreover since $\alpha$ is supported in $V$ and $H(t,\cdot)$ is supported in $U$ for every $t \in \mathbb{R}/\mathbb{Z}$, there is no dynamics outside $U$. Therefore,

$$\rho(a,H,\omega_s) \in \text{Spec}(H,\omega_0) \quad \text{where} \quad s \in [0,1].$$

Recall that $\text{Spec}(H,\omega_0)$ has measure zero (actually, it is just a finite set of $\mathbb{R}$) and Theorem 1.6 implies $\{\rho(a,H,\omega_s)\}_{0 \leq s \leq 1}$ is a continuous path 10. So $\rho(a,H,\omega_s)$ is constant for all $s \in [0,1]$. Therefore, $\rho(a,H,\omega_0) \leq e(U)$. This is the desired conclusion where $E = e(U)$ 11.

10In fact, we need some extra care here that Theorem 1.6 only applies locally, so we need to cover this path by finitely many segments. In general, we can not always do this because for perturbation, there are two requirements we need to satisfy. One is the size of neighborhood of perturbation, that is $\delta'$ in (iii) at the beginning of Section 3, which depends on each initial symplectic form and the other is the requirement that $C' < 1$ from Proposition 1.5 which does not depend on the initial symplectic form. However, as we are in a symplectic surface and by the definition of $\alpha$, we only need to care about the latter one. It allows us to choose a uniform size of neighborhood of symplectic forms (so only need finitely many ones by compactness property) to cover this path.

11For $\Sigma = S^2$ case, this argument does not apply since function $\rho([M],H,\omega_0)$ is not well-defined (from $\mathbb{R} \to \text{Spec}(H,\omega_0)$) due to the fact that $[M]$ lies in different quantum homology when $\omega_s$ changes.
Proof. (Proof of Corollary 1.10) For (a), this is almost immediate from Definition 2.7. For any $H \geq 0$ with $H|_X = 0$, it is supported in $\Sigma_g \setminus X$ which is a disjoint union of simply connected regions. By Theorem 1.9, $\rho(a, H, \omega) \leq E$ for some $E \geq 0$, so is $\rho(a, kH, \omega)$ for any $k \in \mathbb{N}$. Meanwhile, by definition of (partial) symplectic quasi-state, we know $\zeta_\omega(H) = 0$. Therefore, $X$ is $a$-superheavy. For (b), let $U$ be the simply connected region that $X$ lie in. By Theorem 1.9 and argument above, for any $H$ supported in the $U$, $\zeta_\omega(H) = 0$. Meanwhile, we can choose $H$ such that $H(x) \geq \delta > 0$ for every $x \in X$. Therefore, it contradicts the Definition 2.8. \hfill \square

7. Variant Floer chain complexes

7.1. Novikov ring with multi-finiteness condition. Recall the extended version (compared with (9)) of Novikov ring considered in [Ush08] is

$$\Lambda_\omega = \left\{ \sum_{A \in H^2_\omega(M)} a_A T^A \bigg| a_A \in \mathcal{X}, (\forall C \in \mathbb{R}) (\# \{a_A \neq 0 \mid [\omega](A) \leq C \} < \infty) \right\}$$

where $H^2_\omega(M)$ is the image of $\pi_2(M)$ in $H_3(M; \mathbb{Z})/\text{Tor}$ under Hurewicz map $\iota : \pi_2(M) \to H_3(M; \mathbb{Z})$. By the following exact sequence,

$$0 \to \ker \omega \to H^2_\omega(M) \xrightarrow{\omega} \Gamma_\omega \to 0,$$

where $\Gamma_\omega = \text{Im}(\omega)$, we can write any element $x = \sum_{A \in H^2_\omega(M)} a_A T^A \in \Lambda_\omega$ as

$$x = \sum_{g \in \Gamma_\omega} a_g T^g \quad \text{where} \quad a_g \in \mathcal{X}[\ker(\omega)].$$

Therefore, $\Lambda_\omega$ can be always identified with

$$\Lambda_\omega = \left\{ \sum_{g \in \Gamma_\omega} a_g T^g \bigg| a_g \in \mathcal{X}[\ker(\omega)], (\forall C \in \mathbb{R}) (\# \{a_g \neq 0 \mid g \leq C \} < \infty) \right\}.$$

Note that in general, $\mathcal{X}[\ker(\omega)]$ is not necessarily a PID, therefore, by Theorem 4.2 in [HS95], $\Lambda_\omega$ is not always a PID. However, since $\mathcal{X}$ is Noetherian, $\mathcal{X}[\ker(\omega)]$ is Noetherian. Compared with the most often used Novikov field $\Lambda^{\mathcal{X}, \Gamma}$ in (9), there is a natural homomorphism $R_\omega : \Lambda_\omega \to \Lambda^{\mathcal{X}, \Gamma}$ by

$$\sum_{g \in \Gamma_\omega} a_g T^g \xrightarrow{R_\omega} \sum_{g \in \Gamma_\omega} [a_g] T^g, \quad \text{where} \quad [a_g] \in \mathcal{X}.$$ 

Specifically, if $a_g = \sum_h a_{g,h} S^h$, then $[a_g] = \sum h a_{g,h}$. In other words, we uniformly weight any $h \in \ker(\omega)$ by value zero. Therefore, $\Lambda^{\mathcal{X}, \Gamma}$ can be regarded as a $\Lambda_\omega$-module. The following property of $\Lambda_\omega$ will be useful later.

**Lemma 7.1.** $\Lambda_\omega$ is an integral domain.

\footnote{Note that the same argument can also work for the conclusion being $a$-heavy. To distinguish $a$-heavy and $a$-superheavy subsets on $\Sigma_g$, we need more subtle topological discussion. See Proposition 6 in [HRS14].}
Proof. First, because group \( \ker \omega \) is a subgroup of \( H^2(M) \) which is torsion-free and abelian, \( \ker(\omega) \) is also torsion-free and abelian. By Proposition 3.1 in [Bar14], group \( \mathcal{X}[\ker(\omega)] \) is an integral domain. Then taking two non-zero elements \( \lambda_1 \) and \( \lambda_2 \) in \( \Lambda_\omega \), we can write them as, for \( i = 1, 2 \),

\[
\lambda_i = \sum_{g_{ij} \in \Gamma_\omega} a_{g_{ij}} T^{g_{ij}} \quad \text{where} \quad a_{g_{ij}} \in \mathcal{X}[\ker(\omega)].
\]

By finiteness condition, since \( \lambda_i \neq 0 \), there exists a smallest power. Denote the smallest powers by \( g_{1j} \) for \( \lambda_1 \) (for some \( j_1 \)) and \( g_{2j_2} \) for \( \lambda_2 \) (by some \( j_2 \)). The corresponding coefficients in \( \mathcal{X}[\ker(\omega)] \) are \( a_{g_{1j_1}} \) and \( a_{g_{2j_2}} \) which in particular, non-zero. Then by definition of integral domain,

\[
a_{g_{1j_1}} \cdot a_{g_{2j_2}} \neq 0 \quad \Rightarrow \quad \lambda_1 \cdot \lambda_2 \neq 0.
\]

Therefore, \( \Lambda_\omega \) is an integral domain. \( \square \)

Remark 7.2. The issue whether a group ring is an integral domain or not exactly when group is torsion-free comes from the famous Kaplansky zero-divisor conjecture. Fortunately, here the group we are considering in the Lemma 7.1 above is abelian. Conjecture has affirmative answer in this case.

As advertised in the introduction, what we are really interested in is

\[
[\omega_t] = (1 - t)[\omega_0] + t[\omega_1]
\]

for some fixed \( \omega_1 = \omega_0 + \alpha \) (where \( \alpha \in \Omega \)). Here we will require \([\omega_0]\) and \([\omega_1]\) are linearly independent (over \( \mathcal{X} \)) in \( H^2(M; \mathcal{X}) \) (Otherwise, it will be reduced to the rescaling case as studied in the Section 6). This automatically requires \( \dim_{\mathcal{X}} H^2(M, \mathcal{X}) \geq 2 \).

Definition 7.3. We call \( \Lambda_{[0,1]} \) constructed below a Novikov ring with multi-finiteness condition, (33)

\[
\Lambda_{[0,1]} = \left\{ \sum_{A \in H^2(M)} a_A T^A \middle| a_A \in \mathcal{X}, (\forall C \in \mathbb{R})(\forall t \in [0, 1]) (\# \{a_A \neq 0 | [\omega_t](A) \leq C \} < \infty) \right\}.
\]

By this definition, \( \Lambda_{[0,1]} = \bigcap_{t \in [0,1]} \Lambda_{\omega_t} \). Also notice \( \Lambda_{[0,1]} \) is non-empty since every finite length series (so polynomial) lies inside. Meanwhile, by the following lemma, we can simplify the structure of \( \Lambda_{[0,1]} \). Specifically,

Lemma 7.4. \( \Lambda_{[0,1]} = \Lambda_{\omega_0} \cap \Lambda_{\omega_1} \). In particular, \( \Lambda_{[0,1]} \) is an integral domain.

Proof. The direction of inclusion \( \Lambda_{[0,1]} \subseteq \Lambda_{\omega_0} \cap \Lambda_{\omega_1} \) is trivial since the finiteness condition in (33) is particularly valid for \( t = 0 \) and \( t = 1 \). Now we will prove the other inclusion. Take any \( x \in \Lambda_{\omega_0} \cap \Lambda_{\omega_1} \), say \( x = \sum_{A \in H^2(M)} a_A T^A \) satisfying finiteness conditions for both \( \omega_0 \) and \( \omega_1 \), then for any \( C \in \mathbb{R} \) and for any \( t \in (0, 1) \), \([\omega_t](A) = (1 - t)[\omega_0](A) + t[\omega_1](A) \leq C \) implies either \((1 - t)[\omega_0](A) \leq C/2\) or \( t[\omega_1](A) \leq C/2 \). Therefore,

\[
[\omega_0](A) \leq \frac{C}{2(1 - t)} \quad \text{or} \quad [\omega_1](A) \leq \frac{C}{2t}.
\]

Defining property of element \( x \) implies that in either case, there are only finitely many \( A \)'s. Therefore, \( x \) also satisfies the finiteness condition in (33). The last conclusion comes from Lemma 7.1 because intersection of two integral domains is also an integral domain. \( \square \)
Example 7.5. (a) A typical element $x$ that is in $\Lambda_{\omega_0}$ but not in $\Lambda_{[0,1]}$ is

$$x = \sum_{n=0}^{\infty} T^n A, \text{ where } A \in \ker(\omega_1) \setminus \ker(\omega_0).$$

So in general, $\Lambda_{[0,1]}$ is strictly contained in $\Lambda_{\omega_t}$ for any $t \in [0,1]$.

(b) The following computation shows that $\Lambda_{[0,1]}$ does not act on $\Lambda_{\omega_0} \setminus \Lambda_{[0,1]}$. Take $x$ as in (a),

$$(1 - T^A)x = (1 - T^A) \left( \sum_{n=0}^{\infty} T^n A \right) = 1 \in \Lambda_{[0,1]}.$$

This is sharp contrast with the observation in the proof of Theorem 2.5 in [Ush08] that for any proper subgroup $G \leq \Gamma_{\omega_t}$, $\Lambda^{\mathcal{X},G}$ acts on $\Lambda^{\mathcal{X},\Gamma_{\omega_t}}$.\text{G}.$$

Remark 7.6. Definition 7.3 can be easily generalized to higher dimensional construction. Given $n+1$ distinct symplectic forms $\{\omega_i\}_{i=0}^{n}$ such that $\{[\omega_i]\}_{i=0}^{n}$ are in a generic position in $H^2(M; \mathbb{R})$. (This requires $b_2(M)$ to be sufficiently large). Define

$$\Delta^n_{[\omega_i]} = \left\{ [\omega_\vec{t}] = t_0[\omega_0] + ... + t_n[\omega_n] \mid \vec{t} = (t_0, ..., t_n) \in \Delta^n = \{(t_0, ..., t_n) \in [0,1]^{n+1} \mid t_0 + ... + t_n = 1\} \right\}.$$

Then define

(34)

$$\Lambda_{\Delta^n} = \left\{ \sum_{A \in H^2_0(M)} a_A T^A \mid a_A \in \mathcal{X}, (\forall C \in \mathbb{R})(\forall [\omega_\vec{t}] \in \Delta^n_{[\omega_i]})(\# \{a_A \neq 0 \mid [\omega_\vec{t}](A) \leq C < \infty\} \right\}.$$\text{C} \right\}.$$

In particular, $\Lambda_{\Delta^1} = \Lambda_{[0,1]}$. With the same argument, Lemma (7.4) can be generalized to the following one

Lemma 7.7. $\Lambda_{\Delta^n} = \Lambda_{\omega_0} \cap ... \cap \Lambda_{\omega_n}.$

For simplicity, for the rest of the paper, we will only work on the case of two different symplectic forms in generic position.

Because of Lemma 7.4, there are natural homomorphisms - inclusion maps $i_0 : \Lambda_{[0,1]} \to \Lambda_{\omega_0}$ and $i_1 : \Lambda_{[0,1]} \to \Lambda_{\omega_1}$. Therefore, we can regard both $\Lambda_{\omega_0}$ and $\Lambda_{\omega_1}$ as $\Lambda_{[0,1]}$-modules. Together with what has been discussed above, we have the following picture of extending the coefficients,
Therefore, for quantum homology, there are variant versions,

\[
\begin{align*}
\text{QH}_0 & \rightarrow \text{QH}_0 \\
\text{QH}_{[0,1]} & \rightarrow \text{QH}_{[0,1]} \\
\text{QH}_1 & \rightarrow \text{QH}_1
\end{align*}
\]

here the notations are

- \(\text{QH}_{[0,1]} = H_n(M; \mathcal{K}) \otimes_{\mathcal{K}} \Lambda_{[0,1]};\)
- \(\text{QH}_i = \text{QH}_{[0,1]} \otimes_{\Lambda_{[0,1]}} \Lambda_{\omega_i}\) for \(i = 0, 1;\)
- \(\text{QH}_i = \text{QH}_1 \otimes_{\Lambda_{\omega_i}} \Lambda^{\omega_i, T = \infty}\) for \(i = 0, 1.\)

Similar to the treatment at beginning of this section on the extended version Novikov ring \(\Lambda_{\omega}\) from [Ush08], we can consider the following short exact sequence,

\[
0 \rightarrow \ker(\omega_0) \cap \ker(\omega_1) \rightarrow H^S_2(M) \rightarrow \Gamma_{\omega_0} \times \Gamma_{\omega_1} \rightarrow 0,
\]

which allows us to identify any \(x = \sum_{A \in H^S_2(M)} a_A T^A \in \Lambda_{[0,1]}^n\) with

\[
(35) \quad x = \sum_{(g_0, g_1) \in \Gamma_0 \times \Gamma_1} a_{(g_0, g_1)} T^{(g_0, g_1)} \text{ where } a_{(g_0, g_1)} \in (\mathcal{K}[\ker(\omega_0) \cap \ker(\omega_1)])^n.
\]

Therefore, each \(x\) can be identified with a set of points on the \(g_0g_1\)-plane. Moreover, by multi-finiteness condition, this set is discrete in \(\mathbb{R}^2\).

### 7.2. Floer chain complex with multi-finiteness condition.

Similar to the construction of Floer chain complex, in this subsection, we will construct a variant Floer-Novikov chain complex, denoted as \((CF_{[0,1]}^*)\), over \(\Lambda_{[0,1]}\). First of all, as a (graded) finitely generated free \(\Lambda_{[0,1]}\)-module,

\[
CF_{[0,1]} = \bigoplus_{i=1}^{n} \Lambda_{[0,1]} \langle \{x_i, w_i\} \rangle \simeq \bigoplus_{i=1}^{n} \Lambda_{[0,1]}
\]

where \(n\) is the number of contractible Hamiltonian periodic orbit with CZ-index equal to \(k\) and \(w_i\) is a prior fixed capping Hamiltonian orbit \(x_i\). Corollary 3.4 and the discussion following it confirms that, up to a shift of degree by some \(A \in H^S_2(M)\), we can assume that \(CF_k(M, J, H, \omega_0)\) and \(CF_k(M, J, H, \omega_1)\) have the same generators (as capped orbits). Moreover, by Lemma 7.4, for each degree \(k \in \mathbb{Z}\),

\[
(36) \quad (CF_{[0,1]}^*)_k = (\hat{CF}_0)_k \cap (\hat{CF}_1)_k = \bigcap_{i \in [0,1]} (\hat{CF}_t)_k
\]

here we view \(\hat{CF}_t\) as a (free) \(\Lambda_{\omega_t}\)-module for \(t \in [0,1]\). More specifically,

\[
(37) \quad (CF_{[0,1]}^*)_k = \left\{ \sum_{[x, w] \in \text{basis}} \left( \sum_{A \in H^S_2(M)} a_A T^A \right) [x, w] \middle| \begin{array}{c}
a_A \in \mathcal{K}, (\forall C \in \mathbb{R}) (t = 0, 1) \\
\# \{a_A \neq 0 \mid [\omega_t](A) < C < \infty\}
\end{array} \right\}
\]
In order to form a chain complex, we need to choose a proper boundary operator on \((CF_{[0,1]}_s)\). Actually, for any \(s \in [0,1]\), we can choose Floer boundary operator \(\partial_{J,H,\omega_s}\), simply denoted as \(\partial_s\). In other words, we have

**Proposition 7.8.** For any \(s \in [0,1]\), \(\partial_s\) is well-defined on \((CF_{[0,1]}_s)\) and satisfies \(\partial_s^2 = 0\).

**Proof.** \(\partial^2 = 0\) because it is well-defined as a boundary operator for the Floer chain complex \((CF_s(M,J,H,\omega_s),\partial_s)\). In order to show \(\partial_s\) is well-defined on \((CF_{[0,1]}_s)\), by (37), we need to show for any \(t \in [0,1]\), the output of \(\partial_s\) satisfies the finiteness condition in terms of \(\omega_t\). Suppose, for basis element \([x,w]\),

\[
\partial_s([x,w]) = \sum_{[y,v] \in \text{basis}_{A \in H_2^0(M)}} n_A T^A[y,v].
\]

By definition, there exists a Floer connecting trajectory \(u\) linking \([x,w]\) and \([y,v+A]\) satisfying unperturbed Floer operator \(\mathcal{F}\) as Type II (ii) above. Therefore, by Corollary 4.7, we know

\[
\mathcal{A}_{\omega_t}(T^A[y,v]) - \mathcal{A}_{\omega_t}([x,w]) \leq -(1 - C_{s,t}') E(u)
\]

for some constant \(C_{s,t}' = |s-t|C'\). By definition of action functional, (39) can be rewritten as

\[
-\int_{S^2} A^s \omega_t - N_2 \leq -(1 - C_{s,t}') E(u)
\]

where \(N_2 = \mathcal{A}_{\omega_t}([x,w]) - \mathcal{A}_{\omega_t}([y,v])\) (independent of sphere class \(A\)). So since \(1 - C_{s,t}' > 0\),

\[
E(u) \leq \frac{1}{1 - C_{s,t}'} \left( \int_{S^2} A^s \omega_t + N_2 \right).
\]

If \(\int_{S^2} A^s \omega_t < \lambda\) for any \(\lambda \in \mathbb{R}\), then

\[
E(u) \leq \frac{\lambda + N_2}{1 - C_{s,t}'} < \infty,
\]

which, by finiteness condition of \(\omega_s\) (or directly by Gromov compactness theorem), there are only finitely many sphere class \(A\). Hence, (38) also satisfies finiteness condition of \(\omega_t\). \(\square\)

**Remark 7.9.** For each \((CF_{[0,1]}_s, \partial_s)\), we can associate a filtration function \(\ell_{\omega_t}\), denoted as \(\ell_s\), just by using action functional \(\mathcal{A}_{\omega_t}\) (as for the Floer chain complex \((CF_s(M,J,H,\omega_s))\)). So Proposition 7.8 says there exists a family of variant version of Floer chain complexes \(\{(CF_{[0,1]}_s, \partial_s, \ell_s)\}_{s \in [0,1]}\) that is parametrized by \([0,1]\). It is important to use the action functional corresponding to the same indexed boundary operator so that this boundary operator strictly decreases the corresponding filtration. In general, near \(\omega_0\), the same argument shows there exists a family of variant version of Floer chain complexes that is parametrized by a convex polygon. In a recent paper [Le15], its Theorem 3.12 provides a similar (but from essentially different background) construction of a family of Floer style chain complexes.

In order to prove Theorem 1.11, we also need the following proposition that compares \((CF_{[0,1]}_s, \partial_s, \ell_s)\) and \((CF_{[0,1]}_t, \partial_t, \ell_t)\) for any \(s, t \in [0,1]\), which is proved by imitating the proof of Lemma 5.1.

**Proposition 7.10.** For any \(s, t \in [0,1]\), \((CF_{[0,1]}_s, \partial_s, \ell_s)\) and \((CF_{[0,1]}_t, \partial_t, \ell_t)\) are chain homotopic equivalent.
**Proof.** We need to find a quadruple \((\Phi_{t,s}, \Phi_{t,s}, K_s, K_t)\) such that \(\Phi_{s,t}\) and \(\Phi_{t,s}\) are chain maps between \((\text{CF}_{[0,1]}, \partial_s, \partial_t)\) and \((\text{CF}_{[0,1]}, \partial_t, \partial_s)\) and \(K_s\) and \(K_t\) are homotopies. First, for any basis element \([x, w]\), define

\[
\Phi_{s,t}(x, w) = \sum_{[y, v] \in \text{basis}} \sum_{\mu_{cz}(x, w) = \mu_{cz}(y, v)} n_{A} T^{A}[y, v]
\]

where \(n_{A}\) counts the number of Floer connecting trajectory \(u: \mathbb{R}^2 \times S^1 \to M\) satisfying (a), (b) (with the same index) and (c) in Section 4 for the perturbation of Type I (ii) above. We know \(\Phi_{s,t}\) is a chain map by the standard gluing argument. In order to show \(\Phi_{s,t}\) acts on \((\text{CF}_{[0,1]}),\) we need to check the output of \(\Phi_{s,t}\) satisfies finiteness condition of \(\omega\) for any \(r \in [0, 1]\). Without loss of generality, assume \(s < t\). By Corollary 4.4,

\[
\mathcal{A}_{\omega_t}(T^{A}[y, v]) - \mathcal{A}_{\omega_s}([x, w]) \leq -(1 - C_{s,t}) E(u) + (s - t) \int_{D^2} w^* \alpha.
\]

Meanwhile, evaluate \(T^{A}[y, v]\) by \(\mathcal{A}_{\omega_s}\), so similar to Corollary 4.6,

\[
\mathcal{A}_{\omega_s}(T^{A}[y, v]) = \mathcal{A}_{\omega_s}(T^{A}[y, v]) + (r - t) \int_{S^1 \times \mathbb{R}} u^* \alpha + (r - t) \int_{D^2} w^* \alpha.
\]

So

\[
\mathcal{A}_{\omega_t}(T^{A}[y, v]) - \mathcal{A}_{\omega_s}([x, w]) \leq -(1 - C_{s,t}) E(u) + (t - r) \int_{S^1 \times \mathbb{R}} u^* \alpha + (s - r) \int_{D^2} w^* \alpha
\]

\[
\leq -(1 - C_{s,t}) E(u) + (t - r) C' \cdot E(u) + (s - r) \int_{D^2} w^* \alpha
\]

\[
\leq -(1 - C_{s,r}) E(u) + (s - r) \int_{D^2} w^* \alpha.
\]

Also \(\mathcal{A}_{\omega_s}(T^{A}[y, v]) - \mathcal{A}_{\omega_s}([x, w]) = -\int_{S^2} A^* \omega_r - N_3\) where \(N_3 = \mathcal{A}_{\omega_s}([x, w]) - \mathcal{A}_{\omega_s}([y, v])\) (independent of sphere class \(A\)). Therefore,

\[
E(u) \leq \frac{1}{1 - C_{s,r}} \left( \int_{S^2} A^* \omega_r + N_3 + (s - r) \int_{D^2} w^* \alpha \right).
\]

If \(\int_{S^2} A^* \omega_r \leq \lambda\) for any given \(\lambda \in \mathbb{R}\), then since \(1 - C_{s,r} > 0\) (for any \(r \in [0, 1]\)),

\[
E(u) \leq \frac{\lambda + N_3 + (s - r) \int_{D^2} w^* \alpha}{1 - C_{s,r}} < \infty
\]

which, by Gromov compactness theorem, there are only finitely many sphere class \(A\). Hence, (40) also satisfies finiteness condition of \(\omega_r\). Symmetrically, we can define \(\Phi_{t,s}\) and it is a well-defined chain map.

Second, since \(\Phi_{s,t} \circ \Phi_{t,s}\) is (Floer) homotopic to \(I\) on \((\text{CF}_{[0,1]}), \partial_1\), by standard Floer theory, there exists a homotopy \(K_s\) where

\[
K_s([x, w]) = \sum_{[y, v] \in \text{basis}} \sum_{\mu_{cz}(x, w) + 1 = \mu_{cz}(y, v)} n_{A} T^{A}[y, v]
\]
where $n_A$ counts the number of Floer connecting trajectory $u : \mathbb{R}^2 \times S^1 \to M$ satisfying (a), (b) (with index shifted up by 1) and (c) in Section 4 for the perturbation of Type I (iii) above. Again, in order to show $K_s$ acts on $(CF_{[0,1]}),$ we need to check the output of $K_s$ satisfies finiteness condition for any $r \in [0, 1]$. By Corollary 4.5,

$$A_{\omega_s}(T^A[y, v]) - \mathcal{A}_{\omega_s}([x, w]) \leq -(1 - C_{s,t})E(u)$$

Evaluated $T^A[y, v]$ by $\mathcal{A}_{\omega_s}$, so similar to Corollary 4.6,

$$\mathcal{A}_{\omega_s}(T^A[y, v]) = \mathcal{A}_{\omega_s}(T^A[y, v]) + (r - s) \int_{S^1 \times \mathbb{R}} u^* \alpha + (r - s) \int_{D^2} w^* \alpha.$$  

So similar computation as above, we get

$$\mathcal{A}_{\omega_s}(T^A[y, v]) - \mathcal{A}_{\omega_s}([x, w]) \leq -(1 - C_{s,t})E(u) + (s - r) \int_{D^2} w^* \alpha.$$  

Then the same argument as above implies the finiteness condition of $\omega_r$. Thus we get the conclusion. \qed

Remark 7.11. Actually continuation map $\Phi_{s,t}$ is invertible for any $s, t \in [0, 1]$. Indeed, by the construction of $K_s$ and its filtration change, for any chain $c \in (CF_{[0,1]}, \partial_s),$ for any $t > s$,

$$(\Phi_{t,s} \circ \Phi_{s,t}) (c) = c + \{\text{strictly lower filtration terms}\}.$$

In other words, $\Phi_{t,s} \circ \Phi_{s,t} = I - B_c$ with some operator $B_c = \partial s K_s + K_s \partial_s$ which strictly lowers the filtration. Then $\sum_k B_c^k$ is a well-defined operator on $(CF_{[0,1]}, \partial_s)$. Therefore, $(\sum_k B_c^k) \circ \Phi_{t,s}$ form a left inverse of $\Phi_{s,t}$. Similarly, for $K_t$, the associated $\sum_k B_t^k$ is also a well-defined operator on $(CF_{[0,1]}, \partial_t)$. So $\Phi_{t,s} (\sum_k B_c^k)$ is a right inverse of $\Phi_{s,t}$. Moreover, since for each degree, the number of basis elements are the same, $\Phi_{s,t}$ is a square matrix. Hence left inverse is equal to the right inverse. Notice this inverse is also a chain map because $\partial_s B_c = B_c \circ \partial_s$.

Proof. (Proof of Theorem 1.11). Proposition 7.8 and Proposition 7.10. \qed

7.3. Revised Floer homology. By Proposition 7.10, we have the diagram (6) in the introduction on the homology level where all the arrows between $HF_{[0,1]}$'s are isomorphisms. So if no other information (for instance, the valuation function which depends on $t$ mentioned later) is needed, we simply denote $HF_{[0,1]}$ as $HF_{[0,1]}$. Moreover, for any $t \in [0, 1]$,

$$(\Phi_{0,t})_* \circ (PSS)_* = (PSS)_*.$$  

Once we have these variant Floer chain complexes, a natural question is how the associated Floer homologies change when we extend the coefficients in each step. First, Universal Coefficient Theorem (see Corollary 7.56 (ii) and Theorem 7.15 in [Rot09]) says, for each degree $k \in \mathbb{Z}$, we have the following splitting,

$$H_k(\tilde{CF}; \Lambda_{\omega_s}) \simeq H_k(CF_{[0,1]}; \Lambda_{[0,1]}) \otimes \Lambda_{[0,1]} \otimes \bigoplus \text{Tor}_{[0,1]}^k(CF_{[0,1]}, \Lambda_{\omega_s})$$

where $\text{Tor}_{[0,1]}^k(CF_{[0,1]}, \Lambda_{\omega_s})$ is a torsion module over $\Lambda_{[0,1]}$. On the other hand, it is not easy to see the (algebraic) relation between $\Lambda_{[0,1]}$ and $\Lambda_{\omega_s}$ if we try to apply some well-known fact such as a module over a PID is flat if and only if it is torsion-free (by Lemma 7.1, we only know $\Lambda_{[0,1]}$ is a domain). However, we still have the following property claiming that torsion part actually vanishes mainly due to the PSS-map who transfers discussion back to the Morse homology.
Proposition 7.12. For any $k \in \mathbb{Z}$,
\[ H_k(CF_i; \Lambda_{\omega_i}) \cong H_k(CF_{[0,1]}; \Lambda_{[0,1]} \otimes \Lambda_{\omega_i}) \]
for $i = 0, 1$ or simply $\overline{HF}_i \cong HF_{[0,1]} \otimes \Lambda_{[0,1]} \otimes \Lambda_{\omega_i}$. In particular
\[ \text{rank}_{\Lambda_{\omega_i}} H_k(CF_0; \Lambda_{\omega_0}) = \text{rank}_{\Lambda_{\omega_i}} H_k(\overline{CF}_1; \Lambda_{\omega_i}) \].

Proof. First, following the same idea of subsection 6.1 and 6.2, starting from the Morse chain complex over $\mathbb{Z}$, denoted as $CM$, we can construct $CM_{[0,1]} = CM \otimes_{\mathbb{Z}} \Lambda_{[0,1]}$. By Universal Coefficient Theorem, we have
\[ H_k(CM_{[0,1]}; \Lambda_{[0,1]}) \cong H_k(CM) \otimes_{\mathbb{Z}} \Lambda_{[0,1]} \bigoplus \text{Tor}^Z(H_{k-1}(CM_{[0,1]}), \Lambda_{[0,1]}) \]
where $H_*(CM) := H_*(CM; \mathbb{Z})$. Since $\mathbb{Z}$ is a PID, then by the same argument as in Lemma 7.1, we know $\Lambda_{[0,1]}$ is an integral domain. So it is torsion-free (as a $\mathbb{Z}$-module), which implies flatness. Therefore, Tor functor vanishes, that is,
\[ H_*(CM) \otimes_{\mathbb{Z}} \Lambda_{[0,1]} \cong H_*(CM_{[0,1]}; \Lambda_{[0,1]}) \].

From the same argument, we have
\[ H_*(CM) \otimes_{\Lambda_{\omega_i}} \Lambda_{\omega} \cong H_*(CM \otimes_{\Lambda_{[0,1]}} \Lambda_{\omega_i}; \Lambda_{\omega_i}) \].

Together, we get
\[ H_*(CM) \otimes_{\Lambda_{[0,1]}} \Lambda_{\omega_i} = \left( H_*(CM) \otimes_{\Lambda_{[0,1]}} \Lambda_{\omega_i} \right) \otimes_{\Lambda_{[0,1]}} \Lambda_{\omega_i} \cong H_*(CM_{[0,1]}; \Lambda_{[0,1]} \otimes \Lambda_{\omega_i}) \].

But on the other hand, similarly to (42), we have
\[ H_*(CM_i; \Lambda_{\omega_i}) \cong H_*(CM_{[0,1]}; \Lambda_{[0,1]} \otimes \Lambda_{\omega_i}) \bigoplus \text{Tor}^A(H_{*}(CM_{[0,1]}), \Lambda_{\omega_i}) \]
where $CM_i = CM_{[0,1]} \otimes_{\Lambda_{[0,1]}} \Lambda_{\omega_i}$. Consider the following commutative diagram
\[
\begin{array}{ccc}
H_*(CM) & \otimes_{\Lambda_{[0,1]}} & \Lambda_{\omega_i} \\
\downarrow & \downarrow & \downarrow \\
H_*(CM_{[0,1]}; \Lambda_{[0,1]} \otimes \Lambda_{\omega_i}) & \cong & H_*(CM_{[0,1]}; \Lambda_{[0,1]} \otimes \Lambda_{\omega_i}) \\
\downarrow & \downarrow & \downarrow \\
H_*(CF_{[0,1]}; \Lambda_{[0,1]} \otimes \Lambda_{\omega_i}) & \cong & H_*(CF_{[0,1]}; \Lambda_{[0,1]} \otimes \Lambda_{\omega_i})
\end{array}
\]

In this diagram,
- $f$ is an identity map because $CM_i = CM \otimes_{\Lambda_{\omega_i}}$;
- $q$ is an isomorphism because of (44);
- $g$ and $h$ are PSS-maps (see [PSS96]), so isomorphisms;
- $j$ is an identity map because of composition of extension of coefficients;
- $i$ is an isomorphism because of (43).

Therefore, $t$ being an isomorphism implies that $s$ is an isomorphism, which implies $i$ is an isomorphism. So $p$ is an isomorphism. \qed
Moreover, since $\Lambda^{\times,T_{\omega_1}}$ is a field (which implies torsion always vanishes), so

**Corollary 7.13.** For any $k \in \mathbb{Z}$,

$$H_k(CF(M,J,H,\omega_1);\Lambda^{\times,T_{\omega_1}}) \simeq H_k(\widetilde{CF}_i;\Lambda_{\omega}) \otimes_{\Lambda_{\omega}} \Lambda^{\times,T_{\omega_1}}$$

for $i = 0,1$ for simply $HF_{\omega_1} = \widetilde{HF}_{[0,1]} \otimes_{\Lambda_{[0,1]}} \Lambda_{\omega_1}$. In particular (Arnold conjecture from Floer’s theory, see Theorem 2.1)

$$\text{rank}_{\Lambda^{\times,T_{\omega_1}}}(HF_{\omega_0})_k = \text{rank}_{\Lambda^{\times,T_{\omega_1}}}(HF_{\omega_1})_k.$$ 

**8. Variant spectral invariants; proof of Theorem 1.14**

Fix any homology class $a \in QH_{[0,1]}$. Using different $(PSS)_a$, we will get homology class $(PSS)_a(a)$ in $HF_{[0,1],\epsilon}$. As mentioned in the introduction, $(CF_{[0,1]},\partial_\epsilon,\ell_\epsilon)$ is a filtered chain complex with respect to the symplectic form $\omega_\epsilon$.

**Definition 8.1.** We call the following value $t$-spectral invariant

$$\rho_t(a,H) = \inf(\ell_\epsilon(\sigma_t) | [\sigma_t] = (PSS)_t(a))$$

where $\sigma_t \in (CF_{[0,1]},\partial_\epsilon)$.

**Proof.** (Proof of Proposition 1.13) (1): Regard $a(= a \otimes 1)$ as an element in $\widetilde{QH}_\epsilon$, still non-zero. Then by Theorem 1.3 in [Ush08], we know

$$\rho_\epsilon(a,H) > -\infty.$$ 

On the other hand, for any $\epsilon > 0$, there exists some $\alpha_t \in (CF_{[0,1],\epsilon})$ represents $[\alpha_t] = (PSS)_t(a)$ such that

$$\ell_\epsilon(\alpha_t) \leq \rho_\epsilon(a,H) + \epsilon.$$ 

Then in $\widetilde{CF}_\epsilon$, $\alpha_t(= \alpha_t \otimes 1)$ also represents $(PSS)_t(a)$. By definition, $\rho_\epsilon(a,H) \leq \ell_\epsilon(\alpha_t) \leq \rho_\epsilon(a,H) + \epsilon$. So $\rho_\epsilon(a,H) > -\infty$. Therefore, we get the conclusion (1).

(2) and (3): The same argument works for $t$-spectral invariant for any $t \in [0,1]$, so we will only prove the case when $t = 0$. Since $a$ is a non-zero element in $\widetilde{QH}_0$, (up to PSS-map) represented by $a(n)$ (arbitrary) cycle $a \in CF_{[0,1]}$, Theorem 1.4 in [Ush08] says there exists an optimal boundary $\partial_0\tilde{y}$ such that

$$\rho_0(a,H) = \ell_0(\alpha - \partial_0\tilde{y})$$

where $\tilde{y} \in \widetilde{CF}_0$, but not necessarily in $CF_{[0,1]}$ at present. Meanwhile, we observe that $\tilde{y}$’s function is used to kill the peak of $\alpha$ and then introduce (in most possibility, strictly) lower filtration terms. If we denote the part of $\tilde{y}$ containing all the terms in the chain $\tilde{y}$ which have filtration lower than $\rho_0(a)$ (whose value is finite by (1) proved earlier) by $y_s$, we know $\tilde{y} - y_s$ serves the same function as $\tilde{y}$, killing the peak of $\alpha$ without changing the linking behavior above the filtration level $\rho_0(a,H)$, because boundary operator strictly decreases the filtration level. Moreover,

$$\ell_0(\alpha - \partial_0(\tilde{y} - y_s)) = \ell_0(\alpha - \partial_0\tilde{y} + \partial_0y_s) = \ell_0(\alpha - \partial_0\tilde{y}) = \rho_0(a,H).$$

On the other hand, $\tilde{y} - y_s$, by finiteness condition, has only finitely many terms, so contained in $CF_{[0,1]}$ by definition. By Proposition 7.8, $\partial_0(\tilde{y} - y_s)$ is also in $CF_{[0,1]}$, which implies $\alpha -$
Theorem 8.1 also represents class $(PSS_0)_a$. By definition of 0-spectral invariant in Definition 8.1, \( \ell_0(a - \partial \tilde{y} - y_*) \geq \rho_0(a) \). Hence, together, we get
\[
\tilde{\rho}_0(a, H) = \ell_0(a - \partial \tilde{y} - y_*) \geq \rho_0(a, H) \geq \tilde{\rho}_0(a, H).
\]

So they are all equal to each other, which proves both (2) and (3).

Before giving the proof of proposition 1.14, we will start from the following lemma on the continuity of filtration function.

**Lemma 8.2.** For any fixed chain \( c \in CF_{[0,1]} \), \( \ell_t(c) \) is (pointwise) continuous at any \( t \in [0,1] \).

**Proof.** We will only prove the case when \( t = 0 \). For other \( t \in (0,1] \), the proof is exactly the same. First, since \( CF_{[0,1]} \) is a free module over \( \Lambda_{[0,1]} \) (of rank \( n \)), we can identify \( c \) as a vector (or \( n \)-tuple) \( x \) in \( \Lambda^*_n \). Moreover, by (35), we can write
\[
x = \sum \tilde{a}_{(g_0,g_1)} T^{(g_0,g_1)} \quad \text{where} \quad \tilde{a}_{(g_0,g_1)} \in (\mathbb{Q}[\ker(\omega_0) \cap \ker(\omega_1)])^n.
\]

So \( x \) can be identified further to be a set of points on the \( g_0g_1 \)-plane. Without loss of generality, by the finiteness condition of both \( \omega_0 \) and \( \omega_1 \), up to a uniform shift on both indices, we can assume all the points are in the first quadrant.

Second, by definition, \( \omega_t = (1 - t)\omega_0 + t\omega_1 \). Let \( t := \frac{1}{1+\lambda} \) for some non-negative \( \lambda \) and define
\[
\omega_\lambda = \lambda\omega_0 + \omega_1.
\]

Note that \( \omega_\lambda \) obtains its minimal value (over a set of homological sphere) if and only if \( \omega_\lambda \) attains its minimal value. One way of viewing \( \ell_t \) is through the perturbation of valuation function \( \tilde{\nu}_t \). Specifically, for any \( x \in CF_{[0,1]} \),
\[
\tilde{\nu}_t(c) = \min \left\{ \int_{S^2} A^* \omega_t \middle| A \text{ is an exponent of } x \right\},
\]

and then
\[
\ell_t(c) = -\tilde{\nu}_t(c) + p_t(c),
\]

where \( p_t(c) \) comes from Hamiltonian actions on orbits (of generator of \( c \)) and symplectic area of (fixed) cappings of basis elements. As \( p_t(c) \) eventually goes to \( p_0(c) \) when \( t \to 0 \), it is sufficient to only focus on \( \tilde{\nu}_t(c) \) when studying continuity of \( \ell_t(c) \). Actually, as mentioned above, we will focus on \( \lambda \lambda \nu_t(c) \), that is
\[
\tilde{\nu}_\lambda(c) := \min \left\{ \int_{S^2} A^* \omega_\lambda \middle| A \text{ is an exponent of } x \right\},
\]

Once rephrased in this way, it suggests a geometric way to view the value \( \tilde{\nu}_\lambda(c) \): for any \( \lambda \geq 0 \) and for any point \( (g_0, g_1) \), draw a line passing through \( (g_0, g_1) \) with slope \( -\lambda \), that is
\[
y = -\lambda(x - g_0) + g_1.
\]

Then the minimal \( y \)-intercept is just the value \( \tilde{\nu}_\lambda(c) \). The nontrivial part is that the optimal point \( (g_0, g_1) \) who attains the minimal \( y \)-intercept might change along the change of \( \lambda \). However, we claim that when \( \lambda >> 0 \), there exists a point \( (g_0^*, g_1^*) \) who serves as the optimal choice for all sufficiently large \( \lambda \). The key observation is that for any point \( P = (g_0, g_1) \) attaining the value \( \tilde{\nu}_\lambda(c) \) for some \( \lambda \), it fails to attain the value \( \tilde{\nu}_\eta(c) \) for any \( \eta > \lambda \) if there exists another...
point Q in the region enclosed by y-axis, the line (47) passing through \((g_0, g_1)\) with slope \(\lambda\) and the line (47) passing through \((g_0, g_1)\) with slope \(\eta\). When \(\lambda \to \infty\), the width of this closed region goes to zero. Therefore, by discreteness of our points, the choice of optimal point will be eventually stable.

Hence,
\[
\bar{v}_t(c) = \frac{\lambda g_0^* + g_1^*}{1 + \lambda} \xrightarrow{\lambda \to \infty} g_0^* = v_0(c).
\]

The following proposition with highly non-trivial proof is the key step towards the proof of Theorem 1.14. A similar result (from different set-up) in this type is the Proposition 8.4 in [Oh09].

**Proposition 8.3.** For any chain \(c \in (CF_{[0,1]}, \partial_0)\), the function
\[
t \mapsto \ell_t(\Phi_{0,t}(c))
\]
is continuous at \(t = 0\) where \(\Phi_{0,t}\) is the chain map defined in (40) \(^{13}\).

**Proof.** Upper semicontinuity at \(t = 0\). Suppose not. There exists a constant \(\epsilon_0 > 0\) and a sequence \(t_n \to 0\) such that
\[
(48) \quad \ell_{t_n}(\Phi_{0,t_n}(c)) - \ell_0(c) \geq \epsilon_0.
\]
We have seen trivial solution (s-independent) satisfies perturbed Floer operator, so
\[
\Phi_{0,t_n}(c) = c + x_n
\]
where each \(x_n\) is linked with \(c\) by non-trivial Floer connecting trajectories in some way. By triangle inequality of \(\ell_{t_n}\), we know
\[
\ell_{t_n}(\Phi_{0,t_n}(c)) = \ell_{t_n}(c + x_n) \leq \max\{\ell_{t_n}(c), \ell_{t_n}(x_n)\}.
\]
Then (48) implies
\[
(49) \quad \max\{\ell_{t_n}(c) - \ell_0(c), \ell_{t_n}(x_n) - \ell_0(c)\} \geq \epsilon_0 > 0.
\]
By Lemma 8.2, the first term in (49) will be smaller than \(\epsilon_0\) when \(n\) is sufficiently large. Therefore, (49) is possible only for \(\ell_{t_n}(x_n) - \ell_0(c) \geq \epsilon_0\).

Now we will carefully study the linking property between \(x_n\) and \(c\). First, note that there are only finitely many basis elements, by passing to a subsequence, we can assume for each \(n\), one generator of peak of \(x_n\) with respect to \(\ell_{t_n}\) is in the form of \(T^A_n[y, v]\) where \([y, v]\) is a basis element, i.e., \(\ell_{t_n}(x_n) = \kappa_{\omega_{t_n}}(T^A_n[y, v])\). By definition, there exists some (sub)chain of \(c\) linking with \(T^A_n[y, v]\). Again, since there are only finitely many basis generators, by passing to a (sub)subsequence, we can assume
\[
T^B_n[x, w] \xrightarrow{\text{linking}} T^A_n[y, v]
\]
for some basis element \([x, w]\). In particular, \(\{T^B_n[x, w]\}_n\) are chain elements of \(c\) for each \(n\). Now we have

\(^{13}\)Actually \(t = 0\) is not special at all. The same argument can prove that this function is pointwise continuous at any point \(t \in [0,1]\) (still mainly by Lemma 8.2). The only difference is, instead of using Proposition 1.5, we need to use a more general conclusion of energy estimation as Type I (ii). Another main difference is the key step (51) where for general \(t \in [0,1]\), the finiteness condition of \(\omega_t\) (due to (36) as \(c\) is chosen from \(CF_{[0,1]}\)) implies that
\[
\int_{\omega_t} \gamma^t_B \to \infty.
\]
Claim 8.4. \( B_n \) satisfies \( |\int_{S^2} B_n^* \alpha| \to \infty \), where \( \alpha = \omega_1 - \omega_0 \).

In fact, since \( \ell_0(c) \geq \mathcal{A}_{\omega_0}(T^{B_n}[x,w]) \), we have, by Proposition 1.5,

\[
\ell_{t_n}(x_n) - \ell_0(c) \leq \mathcal{A}_{\omega_{t_n}}(T^{A_n}[y,v]) - \mathcal{A}_{\omega_0}(T^{B_n}[x,w]) \leq -t_n \int_{D^2} (w \# B_n)^* \alpha
\]

Therefore, if \( |\int_{S^2} B_n^* \alpha| \) is bounded, then when \( t_n \) is sufficiently close to 0, this will violate (49). In particular, \( B_n \) is not stable. Here stable means it is equal to some fixed homotopy class when \( n \) is sufficiently large. Finiteness condition (of chain \( c \)) of \( \omega_0 \) implies,

\[
\int_{S^2} B_n^* \omega_0 \to \infty.
\]

Back to the continuation chain map, we know

\[
\Phi_{0,t_n}(T^{B_n}[x,w]) = T^{B_n}[x,w] + T^{A_n}[y,v] + \ldots.
\]

Meanwhile, by \( A_{[0,1]} \)-linearity, for any \( m \in \mathbb{N} \),

\[
\Phi_{0,t_n}(T^{B_m}[x,w]) = T^{B_m}[x,w] + T^{A_n+B_m-B_n}[y,v] + \ldots.
\]

Since \( T^{B_m}[x,w] \) is a generator of chain \( c \), \( T^{A_n+B_m-B_n}[y,v] \) will be a generator of chain \( x_n \) too (possibly being cancelled). However, since \( T^{A_n}[y,v] \) is a generator of the peak of \( x_n \), we know, for any \( m \),

\[
\int_{S^2} (A_n + B_m - B_n)^* \omega_{t_n} \geq \int_{S^2} A_n^* \omega_{t_n} \Rightarrow \int_{S^2} B_m^* \omega_{t_n} \geq \int_{S^2} B_n^* \omega_{t_n}.
\]

Rewrite

\[
\int_{D^2} B_m^* \omega_{t_n} = \int_{S^2} B_m^* \omega_0 + t_n \int_{S^2} B_m^* \alpha := a_m + t_n b_m,
\]

where \( a_m = \int_{S^2} B_m^* \omega_0 \) and \( b_m = \int_{S^2} B_m^* \alpha \). Moreover, denote \( c_n = \int_{S^2} B_n^* \omega_{t_n} (= a_n + t_n b_n) \). Then (52) says,

\[
a_m + t_n b_m \geq c_n.
\]

Switch the index \( m \) and \( n \), we get

\[
a_n + t_m b_n \geq c_m.
\]

But \( b_m = \frac{c_n-a_n}{t_m} \) and \( b_n = \frac{c_n-a_n}{t_n} \). So inequalities above are

\[
a_m + \frac{t_n}{t_m}(c_n - a_m) \geq c_n \quad \text{and} \quad a_n + \frac{t_m}{t_n}(c_n - a_n) \geq c_m.
\]

Solve \( c_m \) from the first inequality and then together with the second inequality, we get

\[
a_n + \frac{t_m}{t_n}(c_n - a_n) \geq a_m + \frac{t_m}{t_n}(c_n - a_n) \Leftrightarrow \left( \frac{t_m}{t_n} - 1 \right)(a_m - a_n) \geq 0.
\]

By (51), we know when \( m >> n \), \( a_m > a_n \) (strictly positive because of (51)). This implies \( t_m > t_n \) which is a contradiction because \( t_n \) converge to 0 (so when \( m >> n \), \( t_m < t_n \)).

Lower semicontinuity at \( t = 0 \). Suppose not. Then there exists some \( \epsilon_0 \geq 0 \) and a sequence \( t_n \to 0 \) such that

\[
\ell_0(c) - \ell_{t_n}(\Phi_{0,t_n}(c)) \geq \epsilon_0 > 0 \quad \text{for all} \ n.
\]
First notice that, by proof of Lemma 8.2, peak of \( c \) under \( \ell_{t_n} \) is stable (so equals to peak of \( c \) under \( \ell_0 \)) when \( t_n \) is sufficiently close to 0. We denote this by \( c_s \). Then there exists a subsequence of \( \{t_n\} \) such that \( c_s \) is \textit{completely cancelled} and only strictly lower filtration terms are left or introduced. Indeed, otherwise, \( \ell_{t_n} (\Phi_{0,t_n}(c)) \geq \ell_{t_n}(c_s) \). So we get a contradiction by
\[
\ell_{0}(c_s) - \ell_{t_n}(c) = \ell_{0}(c) - \ell_{t_n}(c) \geq \epsilon_0 > 0
\]
which \( t_n \) is sufficiently close to 0. Still denote this subsequence by \( \{t_n\} \), by passing to a subsequence again if necessary, there exists \( T^{\mathcal{A}}[y,v] \) as a generator of \( c_s \) (since peak has only finitely many generators) and a sequence \( \{T^{B_n}[x,w]\} \) as generators of chain \( c \) such that
\[
T^{B_n}[x,w] \xrightarrow{\text{linking}} T^{\mathcal{A}}[y,v].
\]
The key observation here is \( T^{\mathcal{A}}[y,v] \) and \( \{T^{B_n}[x,w]\} \) are all generators of the same chain \( c \) (as \( c_s \) is a subchain of \( c \)). Moreover, sequence \( \{T^{B_n}[x,w]\} \) can possibly contain only finitely many terms from \( c_s \). In fact, if not, there exists a subsequence \( \{t_i\} \) such that \( \{T^{B_n}[x,w]\} \) are all from \( c_s \). Because there are only finitely many generators for \( c_s \), \( B_{t_i} \) will be eventually stable to be \( B_{s} \in \pi_2(M) \) when \( i \) is sufficiently large. Then on the one hand, \( \mathcal{A}_{\omega_0}(T^{B_n}[x,w]) = \mathcal{A}_{\omega_0}(T^{\mathcal{A}}[y,v]) \) (because both generators are from peak \( c_s \)), while on the other hand, \( \ell_{t_{n_i}} (\Phi_{t_{n_i}}(c)) \geq \mathcal{A}_{\omega_{t_{n_i}}}(T^{\mathcal{A}}[y,v]) \). Together, we get a contradiction,
\[
\mathcal{A}_{\omega_0}(T^{\mathcal{A}}[y,v]) - \mathcal{A}_{\omega_{t_{n_i}}}(T^{\mathcal{A}}[y,v]) \geq \ell_{0}(c) - \ell_{t_{n_i}} (\Phi_{t_{n_i}}(c)) \geq \epsilon_0 > 0.
\]
Therefore, by discreteness of finiteness condition of \( \omega_0 \), there exists a fixed filtration gap \( \kappa > 0 \) (depending on \( c \) and independent of \( n \)) between \( \mathcal{A}_{\omega_0}(T^{\mathcal{A}}[y,v]) \) and any \( \mathcal{A}_{\omega_0}(T^{B_n}[x,w]) \). Specifically, since \( T^{\mathcal{A}}[y,v] \) belongs to the peak,
\[
\mathcal{A}_{\omega_0}(T^{\mathcal{A}}[y,v]) - \mathcal{A}_{\omega_0}(T^{B_n}[x,w]) \geq \kappa > 0 \text{ for any sufficiently large } n,
\]
which implies
\[
(55) \quad \mathcal{A}_{\omega_0}(T^{\mathcal{A}}[y,v]) - \mathcal{A}_{\omega_0}(T^{B_n}[x,w]) \geq \frac{\kappa}{2} > 0
\]
when \( t_n \) is sufficiently close to 0. Now we get the Claim 8.4 again. In fact, if not, by Proposition 1.5,
\[
\mathcal{A}_{\omega_0}(T^{\mathcal{A}}[y,v]) - \mathcal{A}_{\omega_0}(T^{B_n}[x,w]) \leq -t_n \int_{D^2} (\#B_n)^{\alpha} \to 0
\]
as \( t_n \) is sufficiently close to 0, which contradicts (55). So \( \{B_{t_n}\} \) is not stable. By discreteness from finiteness condition of \( \omega_0 \) again, \( \int_{D^2} B_n^{\alpha} \to \infty \). Then The rest of the argument goes exactly the same as the part after (51) of the proof of upper semicontinuity as above. Thus we get the conclusion.

\textit{Proof.} (Proof of Theorem 1.14) By realization property (2) in Proposition 1.13, we know there exists some \( c \in (CF_{[0,1]}^{0}, \bar{\partial}_0) \) such that \( \ell_0(c) = \rho_0(a,H) \) where \( [c] = (PSS_0)_a(a) \). On the other hand, \( \Phi_{0,t}(c) \) represents
\[
[\Phi_{0,t}(c)] = (\Phi_{0,t})_a(c) = (\Phi_{0,t})_a((PSS_0)_a(a)) = (PSS_0)_a(a).
\]
By definition, we know \( \rho_1(a,H) \leq \ell_t(\Phi_{0,t}(c)) \). So
\[
\rho_1(a,H) - \rho_0(a,H) \leq \ell_t(\Phi_{0,t}(c)) - \ell_0(c).
\]
Upper semicontinuity from Proposition 8.3 implies the upper semicontinuity of \( \rho_1(a,H) \). Moreover, the last conclusion comes from the upper semicontinuity over \([0,1]\) that is just proved,
Remark 7.3 and finite steps of convex combination to generate Novikov ring in general in the introduction. □

Remark 8.5. There is an obvious question concerning the lower semicontinuity of $t$-spectral invariant. An trial of imitating the proof of Theorem 8.3 in [Oh09] was carried out but some details could not go through deeply due to the non-uniform upper bound from the energy estimation from Proposition 1.5. Meanwhile, another perspective from realization proposition, (2) in Proposition 1.13, generates an optimal cycle $\alpha_t$ such that $[\alpha_t] = (PSS_t)_*(a)$ and $t$-spectral invariant $\rho_t(a) = \ell_t(\alpha_t)$, equipped with Floer homotopy relation, we know $\ell_t((\partial_t \circ K_t)(\alpha_t)) < \ell(K_t(\alpha_t)) \leq \ell_t(\alpha_t)$ by Corollary 4.5. Therefore, $(\Phi_{0,t} \circ \Phi_{t,0})(\alpha_t)$ is also an optimal cycle that realizes the $t$-spectral invariant. On the other hand, cycle $\Phi_{t,0}(\alpha_t)$ also represents $(PSS_0)_*(a)$, which implies, for each $t \in [0,1]$, there exists a $\gamma_0(t) \in (CF_{[0,1]}; \partial_0)$ such that $\Phi_{t,0}(\alpha_t) = \alpha_0 + \partial_0 \gamma_0(t)$. Since $\alpha_t$ is quite arbitrary (that does not encode information on any nearby $a_{t \pm \delta}$), we are lack of information on $\gamma_0(t)$. However, we give the following claim (interested reader can follow the method of proof of Theorem 8.3 in [Oh09] and the lower semicontinuity from Proposition 8.3 above to give its proof).

Claim 8.6. If there exists a family $\{\alpha_t\}_{t \in [0,1]}$ above such that there exists a $C$ (independent of $t$) satisfying $\ell_0(\gamma_0(t)) \leq C$ for any $t \in [0,1]$, then $\rho_t(a,H)$ is lower semicontinuous.

9. Application on quasi-embedding and capacity

9.1. Proof of Theorem 1.19.

Definition 9.1. (see Definition 1.3 in [Ush12]) We call $M$ admits an aperiodic symplectic form if there exists a symplectic form $\omega$ such that $(M,\omega)$ admits an autonomous Hamiltonian function $H$, not everywhere locally constant, such that its Hamiltonian flow has no nonconstant periodic orbit.

Note that this definition is stronger than the assumption on Theorem 1.1 in [Ush13] (Condition 1.18). For spectral invariant, it is well-known that its computation is difficult in general. However, the following theorem will be quite helpful in the later proof.

Proposition 9.2. (see Proposition 4.1 in [Ush10]) Let $(M,\omega)$ be a symplectic manifold. If $H$ is an autonomous Hamiltonian function on $M$ such that its Hamiltonian flow has no nonconstant contractible periodic orbit with period at most 1, then

$$\rho(H,[M],\omega) = -\min_M H.$$

Notice that the condition in this proposition is weaker than the assumption of Theorem 1.1 in [Ush13]. In [Ost03], for any symplectic manifold $(M,\omega)$, it proves that diameter (under Hofer norm) of $\Ham(M,\omega)$ is infinite by using, roughly speaking, a bump function on a displaceable subset. Theorem 1.19 shows that under a certain condition (which again covers a variety of symplectic manifolds, especially in dimension four) the diameter of $\Ham(M,\omega)$ goes to infinity in uncountably many directions. The proof of Theorem 1.19 takes its inspiration from the proof of Theorem 1.1 in [Ush13].

Proof. (Proof of Theorem 1.19) First, we note that if $\omega$ is already aperiodic, then in particular, $(M,\omega)$ admits an autonomous Hamiltonian $H$ such that its Hamiltonian flow has no nonconstant contractible periodic orbit. Then by compactness of $M$ and Sard’s theorem, there exists a non-trivial closed interval $[a, b]$ (assuming to be $[0,1]$) such that every $c \in [0,1]$ is a regular value. Now take a function $g : \mathbb{R} \to [0,1]$ such that its support is in $(0,1)$, $\max g = 1$ and
its only local minima has its value 0. Now for each \( \vec{v} = (v_1, v_2, \ldots) \in \mathbb{R}^\infty \), define \( f_{\vec{v}} : \mathbb{R} \to \mathbb{R} \) constructed as

\[
f_{\vec{v}}(s) = \sum_{i=1}^{n} g \left( 2^i (s - (1 - 2^{1-i})) \right).
\]

The embedding \( \Phi : \mathbb{R}^\infty \to \text{Ham}(M, \omega) \) is constructed as \( \Phi(\vec{v}) = [\phi_{f_{\vec{v}} \circ H}] \). Then for any non-zero \( \vec{v} \in \mathbb{R}^\infty \), \( X_{f_{\vec{v}} \circ H} = f_{\vec{v}}(H) \cdot X_H \), which implies \( \Phi \) is an homomorphism. Therefore,

\[
d_H(\Phi(\vec{v}), \Phi(\vec{w})) = d_H([\phi_{f_{\vec{v}} \circ H}], [\phi_{f_{\vec{w}} \circ H}])
\]

\[
= d_H([\phi_{f_{\vec{v}} \circ H}], 1)
\]

\[
\leq ||f_{\vec{v} - \vec{w}} \circ H||_H
\]

\[
= \max(f_{\vec{v}} \circ H) - \min(f_{\vec{w}} \circ H) = \text{osc}(\vec{v} - \vec{w}).
\]

Note that this computation is true for any \( \omega \) without assuming \( \omega \) being aperiodic. On the other hand, \( f_{\vec{v}} \circ H \) also satisfies the condition that it has no nonconstant contractible periodic orbit. In particular, it has no nonconstant contractible periodic orbit with period at most 1. By Theorem 1.14, we know

\[
\rho(f_{\vec{v} - \vec{w}} \circ H, [M]; \omega) = -\min_M f_{\vec{v} - \vec{w}} \circ H = \max_i (w_i - v_i).
\]

and

\[
\rho(f_{\vec{v} - \vec{w}} \circ H, [M]; \omega) = -\min_M f_{\vec{v} - \vec{w}} \circ H = \max_i (v_i - w_i).
\]

Hence

\[
d_H(\Phi(\vec{v}), \Phi(\vec{w})) \geq \max\{\max_i (w_i - v_i), \max_i (v_i - w_i)\} = ||\vec{v} - \vec{w}||_{\ell_\infty}
\]

where the first inequality comes from the well-known result that spectral invariant is bounded from above by the Hofer norm. Next, if \( \omega \) is not aperiodic, then take a sequence of aperiodic symplectic forms \( \omega_n \to \omega \). By Theorem 1.14, we know for any given \( \epsilon > 0 \), there exists a \( N \in \mathbb{N} \) such that whenever \( n \geq N \), we have

\[
\max_i (w_i - v_i) = \rho(f_{\vec{v} - \vec{w}} \circ H, [M]; \omega_n) \leq \rho(f_{\vec{v} - \vec{w}} \circ H, [M]; \omega) + \epsilon
\]

and

\[
\max_i (v_i - w_i) = \rho(f_{\vec{v} - \vec{w}} \circ H, [M]; \omega_n) \leq \rho(f_{\vec{v} - \vec{w}} \circ H, [M]; \omega) + \epsilon
\]

where the first equalities above come from the computation as above when symplectic form is aperiodic. Hence,

\[
d_H(\Phi(\vec{v}), \Phi(\vec{w})) + \epsilon \geq \max\{\max_i (w_i - v_i), \max_i (v_i - w_i)\} = ||\vec{v} - \vec{w}||_{\ell_\infty}.
\]

Since this result is true for any \( \epsilon > 0 \), we get the conclusion. \( \square \)

Remark 9.3. Here we put a remark that if the continuity result of boundary depth is affirmative (especially lower semicontinuity), the an almost the same argument as in the proof of Theorem 1.19 can imply if \( M \) admits a symplectic form that can be approximated by a sequence of aperiodic symplectic forms, then there is an quasi-isomorphic embedding from \( \mathbb{R}^\infty \) into \( \text{Ham}(M, \omega) \) as in Theorem 1.1 in [Ush13].
9.2. **Proof of Theorem 1.21.** Recall the definitions of $c^\omega_{HZ}(A)$ and $c^\omega_\rho(A)$.

**Definition 9.4.** Hofer-Zehnder capacity under $\omega$, $c^\omega_{HZ}(A)$, is defined as

$$c^\omega_{HZ}(A) = \sup\{\max H | H \in \mathcal{H}(A) \text{ and } \text{HZ-admissible}\}$$

where $\mathcal{H}(A)$ (involving only topological condition) contains all the autonomous function on $M$ with compact support in $A$ and $H^{-1}(0)$ and $H^{-1}(\max H) \text{ contains } H$ nonempty open sets (roughly speaking it is like bump function over subset $A$) and HZ-admissible means $H$ has it Hamiltonian flow (depending on $\omega$) containing no nonconstant period orbit of period at most 1 (roughly speaking it excludes those fast orbits).

**Definition 9.5.** Spectral capacity under $\omega$, $c^\omega_\rho(A)$, is defined as

$$c^\omega_\rho(A) = \sup\{\rho(H, [M]; \omega) | H \in C^\infty(\mathbb{R}/\mathbb{Z} \times A)\}.$$

**Proof.** (Proof of Theorem 1.21) (1) Under this condition, by Theorem 1.6 and 1.7, in some small neighborhood $\Omega_{\omega_0}$ of $\omega_0$, $\omega(\in \Omega_{\omega_0}) \rightarrow \rho(H, [M]; \omega)$ is continuous, so in particular, lower semicontinuous. For any given $\epsilon > 0$, there exists a neighborhood of $\omega_0$, still denoted as $\Omega_{\omega_0}$, such that for any $\omega \in \Omega_{\omega_0}$ and any fixed $H$, we have $\rho(H, [M]; \omega) - \rho(H, [M]; \omega_0) \geq -\epsilon$. Therefore,

$$c^\omega_\rho(A) - c^\omega_{\rho_0}(A) \geq \rho(H_1, [M]; \omega) - \rho(H_1, [M]; \omega_0) \geq -\epsilon$$

for some $H_1 \in C^\infty(\mathbb{R}/\mathbb{Z} \times A)$. In other words, $c^\omega_\rho(A)$ changes in a lower semicontinuous way.

Therefore, by (7),

$$e^\omega(A) \geq c^\omega_\rho(A) \geq c^\omega_{\rho_0}(A) - \epsilon. \text{  \textcircled{14}}$$

(2) By Theorem 1.14, for any given $\epsilon > 0$, there exists a neighborhood $\Omega_{\omega_0}$ of $\omega_0$ such that for any $\omega \in \Omega_{\omega_0}$ and any fixed $H$, we have $\rho(H, [M]; \omega) - \rho(H, [M]; \omega_0) \leq \epsilon$. Therefore,

$$c^\omega_\rho(A) - c^\omega_{\rho_0}(A) \leq \rho(H_2, [M]; \omega) - \rho(H_2, [M]; \omega_0) \leq \epsilon$$

for some $H_2 \in C^\infty(\mathbb{R}/\mathbb{Z} \times A)$. In other words, $c^\omega_\rho(A)$ changes in an upper semicontinuous way.

Therefore, by (7),

$$c^\omega_\rho(A) \leq c^\omega_{\rho_0}(A) + \epsilon \leq e^\omega_\rho(A) + \epsilon.$$  

(3) Similar to the argument of (2), for any given $\epsilon > 0$, there exists a neighborhood $\Omega_{\omega_0}$ of $\omega_0$ such that for any $\omega \in \Omega_{\omega_0}$, by (7),

$$c^\omega_{HZ}(A) \leq c^\omega_{\rho}(A) \leq c^\omega_{\rho_0}(A) + \epsilon.$$

□

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\text{\textcircled{14}}We can also use the another capacity $c^\omega_\rho(A)$ defined by boundary depth, see Corollary 5.12 in [Ush13], and by lower semicontinuity of boundary depth from Theorem 1.6 and 1.7, we can also prove that $e^\omega(A)$ is bounded from below by $\frac{1}{2} c^\omega_\rho(A)$.\[\]
10. Appendix

In this appendix, we will provide a version where “fixed point theorem” - Lemma 2.1 in [Ush08] is almost true for valuation of rank 2 (or in general, of rank $n$). First of all, we will introduce such valuation which has its motivation from the geometric interpretation of elements in $\Lambda_{[0,1]}$ from (35). In other words, we can identify $\Lambda_{[0,1]}$ defined in Definition 7.3 with $\Lambda^2_k(G)$ where $G$ is a subgroup of $\mathbb{R}^2$ and $K = \mathcal{K} [\ker(\omega_0) \cap \ker(\omega_1)]$. Note that $K$ is Noetherian (because $\ker(\omega_0) \cap \ker(\omega_1)$ is a finitely generated group).

**Definition 10.1.** Define a function called valuation of rank 2 $\tilde{\nu} : \Lambda^2_{[0,1]}(= \Lambda^2_k(G)) \to \mathbb{R}^2$ by

$$\tilde{\nu}(x) = (\tilde{\nu}_0(x), \tilde{\nu}_1(x))$$

where practically $\tilde{\nu}_0$ is defined in terms of $\omega_0$ and $\tilde{\nu}_1$ is defined in terms of $\omega_1$.

It is easy to check that this is indeed a valuation and in particular, $x = 0$ if and only if $\tilde{\nu}(x) = (\infty, \infty)$. In order to compare two elements (via this valuation of rank 2), here we use lexicographical order, that is,

$$\begin{align*}
(a, b) &\geq (c, d) \text{ if and only if } \\
&\quad \begin{cases} a > c \\
\quad \quad \text{or} \quad \quad \text{a = c and b} \geq d. \end{cases}
\end{align*}$$

(57)

We can define $(a, b) > (c, d)$ to be $(a, b) \geq (c, d)$ excluding the case that $(a, b) = (c, d)$ (that is, $a = c$ and $b = d$). This order has the following easy property when we reduce rank-2 order to rank-1 order.

**Lemma 10.2.** For given sets $A_1$ and $A_2$, if $(a, b) = \max\{(x, y) | x \in A_1$ and $y \in A_2\}$ then $a = \max\{x | x \in A_1\}$.

**Proof.** Suppose not, then there exists a $x_* \in A_1$ such that $x_* > a$, which implies for any $y \in A_2$, $(x_*, y) > (a, b)$. Contradiction. \qed

**Example 10.3.** $\Lambda_{[0,1], \geq 0} = \{(g_0, g_1) | g_0 > 0$ or $(0, g_1)$ with $g_1 \geq 0\}$. It is similar to define $V_{\geq 0}$ for any submodule of $V \leq \Lambda^2_{[0,1]}$.

**Example 10.4.** $\Lambda_{[0,1], > 0} = \{(g_0, g_1) | g_0 > 0$ or $(0, g_1)$ with $g_1 > 0\}$. It is similar to define $V_{> 0}$ for any submodule of $V \leq \Lambda^2_{[0,1]}$.

From these two examples above, $\Lambda_{[0,1], \geq 0}/\Lambda_{[0,1], > 0} \simeq K$ which corresponds to the coefficient with the exponential $\tilde{\theta} = (0, 0)$. Moreover, for any $x \in \Lambda^2_{[0,1]}$, we can shift it by multiplying $T^{-\tilde{\nu}(x)}$ so that $x' = T^{-\tilde{\nu}(x)}x \in \Lambda^2_{[0,1], \geq 0}$ and $\tilde{\nu}(x) = \tilde{\theta}$. Moreover, for any submodule $V \leq \Lambda^2_{[0,1]}$, similar to the argument in Page 8 of [Ush08], $\tilde{V} := V_{\geq 0}/V_{> 0}$ is a submodule of $K^n$, therefore, finitely generated over $K$ (because here $K$ is also Noetherian).

We need to emphasize here that once we want to generalize the Lemma 2.1 [Ush08] when using valuation $\tilde{\nu}$ defined above, some part of the argument can go wrong if we don’t put any (geometric) condition. This basically arise from the following phenomenon. Suppose $\{x_i\}_i$ is a monotone increasing sequence of valuation of rank 2 in a discrete subset of $\mathbb{R}^2$, the following three cases possibly happen as $i \to \infty$,

1. $\tilde{\nu}(x_i) \to (a, b)$ for finite $a$ and finite $b$;
2. $\tilde{\nu}(x_i) \to (a, \infty)$ for finite $a$;
3. $\tilde{\nu}(x_i) \to (\infty, \infty)$. 
The divergence case (2) above is the essential difference between the valuation of rank 2 and valuation of rank 1 used in [Ush08]. The problem this case causes is that the element \( u \) constructed in Lemma 2.1 in [Ush08] is not a valid element in \( \Lambda_{[0,1]} \) (more specifically, it violates the finiteness condition of \( \omega_0 \)). The following example gives an explicit construction that this case indeed happens (algebraically).

**Example 10.5.** (c.f. Example 7.5) Let \( V = \Lambda_{[0,1]} \). Let \( U = \text{span}_{\Lambda_{[0,1]}} \{1 - T^B\} \subseteq V \) where \( \omega_0(B) = 0 \) but \( \omega_1(B) = 1 \) (so \( T^B \) can be identified with \( T^{(0,1)} \)). Now fix a sequence of elements in \( U \) as

\[
v^{(j)} = T^jB - T^{(j+1)B}.
\]

By our assignment of valuations, \( \bar{v}(v^{(j)}) = (0, j + 1) \rightarrow (0, \infty) \) as \( j \rightarrow \infty \).

However, this example can never happen in Floer theory. More specifically, \( 1 - T^B \) constructed above can never be a Floer boundary operator. This can be viewed from many perspectives in terms of several proved conclusions in this paper. For simplicity we will only consider the following two-term chain complex (over \( \Lambda_{[0,1]} \)):

\[
\cdots \rightarrow 0 \rightarrow V \xrightarrow{1 - T^B} V \rightarrow 0 \rightarrow \cdots.
\]

(1) **From perspective of spectral invariant.** Take \( T^B \in V \). Note that \( T^B \notin U \) (otherwise \( \sum_{n} T^B \) will be a valid element in \( V \), contradiction) and nonzero. Then it is easy to see

\[
\sup_{u \in U} v_1(T^B - u) = \infty \text{ where we can take the sequence } \{\sum_{j=1}^{n} v^{(j)}\}_n.
\]

This violates the finiteness conclusion of 1-spectral invariant from (1) in Proposition 1.13 when we view \( \ell_1 \) as a perturbed \( -v_1 \).

(2) **From perspective of homology.** Extending the coefficient to \( \Lambda_{\omega_0} \) and \( \Lambda_{\omega_1} \) respectively, we get

\[
\cdots \rightarrow 0 \rightarrow \Lambda_{\omega_0} \xrightarrow{(1 - T^B)\otimes \mathbb{R}} \Lambda_{\omega_0} \rightarrow 0 \rightarrow \cdots
\]

and

\[
\cdots \rightarrow 0 \rightarrow \Lambda_{\omega_1} \xrightarrow{(1 - T^B)\otimes \mathbb{R}} \Lambda_{\omega_1} \rightarrow 0 \rightarrow \cdots.
\]

Over \( \Lambda_{\omega_0} \), \( 1 - T^B \) is not invertible while over \( \Lambda_{\omega_1} \), \( 1 - T^B \) is indeed invertible (with inverse \( \sum_i T^i B \)). Therefore, the corresponding homologies have different ranks, which violates the conclusion of Proposition 7.12.

(3) **From perspective of energy.** The existence of family of boundary elements

\[
\left\{ \sum_{j=1}^{n} v^{(j)} \right\}_n \rightarrow \{T^B - T^{(n+1)B}\}_n
\]

implies (after fixing basis elements of Floer chain complexes), there are Floer connecting trajectories introducing homotopy classes \( A_n = (n + 1)B \). However, our assignment of valuations gives rise to the situation that \( \int_{S^2} A^*_n \omega_0 = 0 \) but \( \int_{S^2} A^*_n \omega_1 \rightarrow \infty \). Notice that this violates the conclusion from Corollary 4.9 (for \( s = 0 \) and \( t = 1 \)).

In general, we give the following definition.

**Definition 10.6.** We say a \( \Lambda_{[0,1]} \)-linear map \( T : \Lambda_{[0,1]}^n \rightarrow \Lambda_{[0,1]}^m \) satisfies the Floer divergence condition if for any given sequence \( \{a_n\}_n \subset \text{Im}(T) \), \( \bar{v}_s(a_n) \rightarrow \pm \infty \) if and only if \( \bar{v}_t(a_n) \rightarrow \pm \infty \) for any \( s, t \in [0, 1] \).
By Corollary 4.9, we know

**Lemma 10.7.** Any Floer boundary operator \( \partial_t : (CF_{[0,1]}^n)_{+1} \rightarrow (CF_{[0,1]}^n)_{+0} \) satisfies the Floer divergence condition for any \( t \in [0,1] \).

Recall that Lemma 2.3 and Lemma 2.4 in [Ush08] can be formally derived from “fixed point theorem” - Lemma 2.1 in [Ush08]. Therefore the following is the main result in this section which is a version of “fixed point theorem” under valuation of rank 2.

**Proposition 10.8.** Suppose \( T : \Lambda^m_{[0,1]} \rightarrow \Lambda^n_{[0,1]} \) satisfies the Floer divergence condition. Let \( U = \text{Im}(T) \) spanned by \( \{u_1, \ldots, u_k\} \) where we can assume \( \bar{v}(u_i) = 0 \) for all \( 1 \leq i \leq k \). Let \( V \leq \Lambda^m_{[0,1]} \) be a \( \Lambda_{[0,1]} \)-submodule containing \( U \). Suppose \( \phi : V \rightarrow V \) is any function such that

1. For all \( v \in V \), either \( \phi(v) = v \) or \( \overline{\phi(v)} > \overline{v} \);
2. If \( \phi(v) \neq v \), then \( v - \phi(v) \in \text{span}_k \{T\overline{v(u_1)}, \ldots, T\overline{v(u_k)}\} \).

Then for every \( v \in V \), there exists a \( u \in U \) such that \( \phi(v - u) = v - u \) and either \( u = 0 \) or else \( \overline{v(u)} = \overline{v(v)} \), \( T^{-\overline{v(u)}} u \in \text{span}_{\Lambda_{[0,1]}} \{u_1, \ldots, u_k\} \).

The proof of this proposition is almost the same as the proof of Lemma 2.1 in [Ush08]. For reader’s convenience, we give the sketch of the proof.

**Proof.** (Proof of Proposition 10.8) First, we define two sequences recording the information from iterations of \( \phi \).

1. \( v^{(i)} = \phi(v^{(i-1)}) \) and \( v^{(0)} = v \) (iteration sequence);
2. \( w^{(i)} = v^{(i-1)} - \phi(v^{(i-1)}) \) (self-distance sequence).

Also denote, for any subset \( S \) of \( \Lambda^m_{[0,1]} \) or \( \Lambda^n_{[0,1]} \),

\[
N(S) = \left\{ \bar{g} \in \mathbb{R}^2 \mid \exists a = \left( \sum a_{\bar{g}} T\bar{g} \right) \in S \text{ s.t. } a_{\bar{g}} \neq 0 \right\}.
\]

Second, by our assumption, the set

\[
Z := N(\{v\}) + L(\{u_1, \ldots, u_k\}) \subset \mathbb{R}^2
\]

is a discrete subset of \( \mathbb{R}^2 \) where \( L(\{u_1, \ldots, u_k\}) \) is the collection of all nonnegative-integer linear combinations of elements from \( N(\{u_1, \ldots, u_k\}) \). Moreover, \( \{\bar{v}(v^{(i)})\}_i \) is a monotone increasing sequence in \( Z \). Since \( T \) satisfies Floer divergence condition (in particular, \( s = 0 \) and \( t = 1 \)), we know

either \( \overline{v(v^{(i)})} \rightarrow (a, b) \) (with finite \( a \) and \( b \)) or \( \overline{v(v^{(i)})} \rightarrow (\infty, \infty) \).

Third, for the first case, the desired \( u = \sum_{i=1}^N w^{(i)} \) (for a sufficiently large \( N \)); while for the second case, \( \sum_{i=1}^\infty w^{(i)} \) defines a valid element in \( \text{span}_k \{T\overline{v(u_1)}, \ldots, T\overline{v(u_k)}\} \), which forces \( u = v \) (so \( v \in \text{Im}(T) \)). \( \square \)

Then imitating Lemma 2.4 in [Ush08]), here is our “best approximation theorem” under valuation of rank 2.

**Theorem 10.9.** Suppose \( T : \Lambda^m_{[0,1]} \rightarrow \Lambda^n_{[0,1]} \) satisfies Floer divergence condition. Let \( U = \text{Im}(T) \) and \( w \in \Lambda^m_{[0,1]} \). Then there is a \( u \in U \) such that

\[
\overline{v}(w - u) = \sup_{v \in U} \overline{v}(w - v) \text{ and either } u = 0 \text{ or } \overline{v}(u) = \overline{v}(w).
\]
Notice when choosing $T = \partial_0$, Theorem 10.9 together with Lemma 10.2 reproofs the realization property of 0-spectral invariant from (2) in Proposition 1.13. In general, this method can also prove the realization property of any $t$-spectral invariant for $t \in [0, 1]$. In order to do this, we need to generalize lexicographical order to $t$-weighted lexicographical order defined as follows.

**Definition 10.10.** For a given $t \in [0, 1]$, we define $t$-weighted lexicographical order, $\succeq_t$, by,

$$
(a, b) \succeq_t (c, d) \text{ if and only if } \begin{cases} (1-t)a + tb > (1-t)c + td \\
(1-t)a + tb = (1-t)c + td \text{ and } b \geq d.
\end{cases}
$$

Also we define $(a, b) >_t (c, d)$ is the same as $(a, b) \succeq_t (c, d)$ excluding the case $(a, b) = (c, d)$, that is $a = c$ and $b = d$.

It is not hard to see that everything we did earlier, especially the Floer divergence condition (due to Corollary 4.9) in this section can go through with this new order. So we get a $t$-weighted best approximation theorem which has exactly the same statement as Theorem 10.9 but with new order. Again, by a corresponding $t$-weighted version of Lemma 10.2, we fully reproof (2) in Proposition 1.13.

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