SINGLE-ANGLE RADON SAMPLES BASED RECONSTRUCTION
OF FUNCTIONS IN REFINABLE SHIFT-INVARIANT SPACE

YOUFA LI, SHENGLI FAN, AND YANFENG HUANG

ABSTRACT. The traditional approaches to computerized tomography (CT) depend
on the samples of Radon transform at multiple angles. In optics, the real time
imaging requires the reconstruction of an object by the samples of Radon transform
at a single angle (SA). Driven by this and motivated by the connection between Bin
Han’s construction of wavelet frames (e.g [13]) and Radon transform, in refinable
shift-invariant spaces (SISs) we investigate the SA-Radon sample based reconstruc-
tion problem. We have two main theorems. The fist main theorem states that, any
compactly supported function in a SIS generated by a general refinable function can
be determined by its Radon samples at an appropriate angle. Motivated by the ex-
tensive application of positive definite (PD) functions to interpolation of scattered
data, we also investigate the SA reconstruction problem in a class of (refinable)
box-spline generated SISs. Thanks to the PD property of the Radon transform of
such spline, our second main theorem states that, the reconstruction of compactly
supported functions in these spline generated SISs can be achieved by the samples
of Radon transform at almost every angle. Numerical simulation is conducted to
check the result.

1. INTRODUCTION

1.1. CT and Radon transform. We start with the X-ray computerized tomo-
graphy (CT) on \( \mathbb{R}^2 \). Its core mathematics includes the Radon transform and its inversion.
For a function \( f \in L^2(\mathbb{R}^2) \), the Radon transform, w.r.t a fixed angle \( \theta \in [0, 2\pi) \), at \( t \in \mathbb{R} \) is defined as the integral of \( f \) along the line \( (x, y) = t(\cos \theta, \sin \theta) + s(-\sin \theta, \cos \theta) \):

\[
(1.1) \quad \int_{-\infty}^{\infty} f(t \cos \theta - s \sin \theta, t \sin \theta + s \cos \theta) ds.
\]

Date: March 8, 2021.

2010 Mathematics Subject Classification. Primary 42C40; 65T60; 94A20.

Key words and phrases. computerized tomography, single-angle Radon sample, refinable shift-
invartiant spaces, box-spline, positive definite function.

Youfa Li is partially supported by Natural Science Foundation of China (Nos: 61961003, 61561006,
11501132), Natural Science Foundation of Guangxi (Nos: 2018JJA110110, 2016GXNSFAA380049)
and the talent project of Education Department of Guangxi Government for Young-Middle-Aged
backbone teachers.
Denote $P = [\cos \theta, \sin \theta]$. For simplicity, following B. Han [13] the Radon transform in (1.1) is denoted as

$$Pf(t) = \int_{-\infty}^{\infty} f(t \cos \theta - s \sin \theta, t \sin \theta + s \cos \theta) ds.$$  

The Fourier transform of $Pf$ can be expressed as

$$\hat{P}f(\xi) = \hat{f}(P^T \xi) = \hat{f}(\xi \cos \theta, \xi \sin \theta), \quad \xi \in \mathbb{R},$$  

where for any function $g \in L^p(\mathbb{R}^D)$ its Fourier transform at $\gamma \in \mathbb{R}^D$ is defined as $\hat{g}(\gamma) := \int_{\mathbb{R}^D} g(x)e^{-ix \cdot \gamma} dx$. It follows from (1.2) that $\hat{P}f$ is essentially the projection of $\hat{f}$ onto the subspace (slice) $\{P^T \xi : \xi \in \mathbb{R}\}$. Correspondingly, from now on $P$ is called the projection vector. On the other hand, $f$ can be reconstructed via the so called inverse Radon transform (c.f. [29]):

$$f(x, y) = \frac{1}{4\pi^2} \int_{0}^{2\pi} \int_{0}^{\infty} \hat{f}(P^T \xi)e^{i\xi(x\cos \theta + y\sin \theta)} \xi d\xi d\theta,$$

with $P = [\cos \theta, \sin \theta]$.

1.2. Traditional approaches conducted by Radon transforms at multiple angles. Theoretically, (1.3) implies that the reconstruction of $f$ requires the projections $\hat{f}(P^T \xi)$ for all angle $\theta$. In practice, however, what we can observe are the samples of limited projections. Therefore, the essential problem of CT (also referred to as the limited angle problem (LAP)) is to construct the approximation to $f$ by the samples of finitely many projections. The most classical approximation is the filtered backprojection (FBP) (c.f. [19, 10]) derived from (1.3). In principle, FBP requires $f$ to be bandlimited and needs the Fourier samples. If FBP is applied to non-bandlimited functions, then the low-pass filtering (to cut off the high-frequency component) and linear interpolation are commonly necessary. They may introduce serious errors (c.f. [35, 30]).

Some recent alternatives to FBP have been introduced (e.g. [8, 21, 26, 28, 35]). Unlike FBP, they are conducted by the samples of Radon transforms. For example, based on the orthogonal polynomial system, Xu [35] established the approach to CT. Entezari, Nilchian and Unser [8] and McCann and Unser [28] established a spline-based reconstruction. Note that the samples required for the above approaches are derived from Radon transforms at multiple angles.

Contrary to the traditional approaches requiring multiple angles, our purpose is to reconstruct a function by the samples of its Radon transform at a single angle (SA). In what follows, we introduce the requirement for such a SA-based approach in optical imaging.
1.3. **SA-based reconstruction is required for real time imaging.** Optical imaging has been widely used in observing biological objects, such as blood cells (thin objects) and bones (thick objects). The thin objects are commonly imaged directly by refractive-index distributions, which is achieved by holographic tomography (HT) ([22]). However, for imaging thick objects, CT is usually employed.

CT commonly requires samples (measurements) of the light fields penetrating through the object from different angles (views). To do so, the object needs to be rotated by a rotation motor ([36]) or the illumination needs to be scanned by a beam steering device, which not only causes instability for the imaging system, but makes the system bulky ([3, 27]). More importantly, limited by the time of recording fields, rotating objects or scanning illuminations becomes not suitable for real-time imaging, especially for observing fast dynamic events ([27]). Now a natural imaging problem is:

\[(1.4) \text{ can CT be achieved by the samples of SA-Radon transform?}\]

Most recently, R. Horisaki, K. Fujii, and J. Tanida [27] established a SA method for HT by inserting a diffuser. However, to our best of knowledge, (1.4) (the SA problem for CT) is very challenging and remains less explored. The purpose of this paper is to investigate it from the mathematical perspective. In what follows, we organize the problem in the present paper.

1.4. **The mathematical problem of SA-based reconstruction in refinable SISs.** Recall that the refinable shift-invariant space (SIS) is a type of function space that is widely used in signal processing and time-frequency analysis (e.g. [1, 9, 31, 32, 33]). On the other hand, motivated by the connection between Bin Han’s construction of wavelet frames (c.f. [13, 14, 15, 17] and to be introduced in subsection 2.1) and Radon transform (1.2), we will investigate the following problem:

\[(1.5) \text{ Q : Can one reconstruct a function in a refinable SIS by its SA-Randon (transform) samples?}\]

As the problem (1.4) in optics, the above problem is still less explored in mathematics. It might be "stereotyped" by the inverse Radon transform (1.3) which requires the Fourier samples of Radon transform at multiple angles.

1.5. **Our contributions.** Our main results are organized in Theorems 3.1 and 4.3. In Theorem 3.1 we prove that, any compactly supported function from a general refinable SIS can be exactly reconstructed by its Radon samples at the appropriate angle. For a class of refinable box-splines, we prove that their Radon transforms at any angle are positive definite (PD). And Theorem 4.3 states that for almost every angle $\theta \in [0, 2\pi)$, any compactly supported function in such box-splines generated SISs can be reconstructed by its Radon (transform at angle $\theta$) samples.
2. Preliminary

2.1. Connection between Han’s construction of wavelet frame and Radon transform (1.2). B. Han et al. [13, 14, 15, 17] and [16, Section 7.1.3] provided the projection method for constructing wavelet frames. Specifically, the wavelet frames \{\psi_1, \ldots, \psi_L\} \subseteq L^2(\mathbb{R}^2) are first constructed, then the ones \{\tilde{P}\psi_1, \ldots, \tilde{P}\psi_L\} \subseteq L^2(\mathbb{R}) defined by

\[
\tilde{P}\psi_l(\xi) = \hat{\psi}_l(\tilde{P}^T \xi)
\]

can have desirable properties if choosing an appropriate vector \(\tilde{P} \in \mathbb{Z}^{1 \times 2}\). Note that both (1.2) and (2.1) take the identical form. Moreover, since \(\hat{\tilde{P}}\psi_l = \hat{\psi}_l(\|\tilde{P}\|_2 \tilde{P}^T \xi)\) then \(\tilde{P}\psi_l\) is essentially the composition of Radon transform (projection vector: \(\tilde{P}^T \xi\)) and the dilation operation (dilation factor: \(\|\tilde{P}\|_2^{-1}\)).

2.2. On the support of \(\tilde{P}f\) defined via (2.1). For a function \(f \in L^1(\mathbb{R}^2)\) and a vector \(\tilde{P} \in \mathbb{R}^{1 \times 2}\), motivated by [13, 16] we next address the relationship between \(\tilde{P}f\) and \(f\) in the spatial domain. Denote the singular value decomposition (SVD) of \(\tilde{P}\) by \(\tilde{P} = \Sigma V^T\), where \(V\) is a \(2 \times 2\) real-valued unitary matrix, and \(\Sigma = [\sigma, 0] \) with \(\sigma = \|P\|_2\). Now it follows from [13, 16] that \(\tilde{P}f = \Sigma(V^Tf)\), where \(V^Tf(x) = f((V^T)^{-1}x)\) with \(x = (x_1, x_2)^T \in \mathbb{R}^2\), and for any \(g\) on \(\mathbb{R}^2\) the function \(\Sigma g\) is defined by

\[
\Sigma g(x_1) = \sigma^{-1} \int_{\mathbb{R}} g(\sigma^{-1}x_1, x_2)dx_2.
\]

It is easy to check that if \(f \in L^1(\mathbb{R}^2)\) then \(\tilde{P}f \in L^1(\mathbb{R})\). Moreover, the following lemma states that if \(f \in L^2(\mathbb{R}^2)\) is compactly supported, then \(\tilde{P}f\) still lies in \(L^2(\mathbb{R})\) and is also compactly supported.

**Lemma 2.1.** Suppose that \(f \in L^2(\mathbb{R}^2)\) is compactly supported on \([a_1, b_1] \times [a_2, b_2]\). For \(0 \neq \tilde{P} \in \mathbb{R}^{1 \times 2}\), define \(\tilde{P}f\) via (2.1). Then

\[
\text{supp}(\tilde{P}f) \subseteq [-\sqrt{2}\|\tilde{P}\|_2 \max\{|b_i|, |a_i| : i = 1, 2\}, \sqrt{2}\|\tilde{P}\|_2 \max\{|b_i|, |a_i| : i = 1, 2\}]
\]

and \(\tilde{P}f \in L^2(\mathbb{R})\).
Proof. Denote $V^T f$ by $g$. Then for any $x_1 \in \mathbb{R}$,

$$
\tilde{P} f(x_1) = \sigma^{-1} \int_{\mathbb{R}} g(\sigma^{-1} x_1, x_2) dx_2 = \sigma^{-1} \int_{\mathbb{R}} [V^T f](\sigma^{-1} x_1, x_2) dx_2 = \sigma^{-1} \int_{\mathbb{R}} f(V(\sigma^{-1} x_1, x_2)^T) dx_2,
$$

(2.4)

where $\sigma = \|\tilde{P}\|_2$, the second and last equalities are derived from (2.2) and $V$ being a real-valued unitary matrix. It follows from supp$(f) \subseteq [a_1, b_1] \times [a_2, b_2]$ that for any $x \in \text{supp}(f)$, we have $\|x\|_2 \leq \sqrt{2} \max\{|b_i|, |a_i| : i = 1, 2\}$. Now by (2.4) and $V$ being a unitary matrix it is easy to check that (2.3) holds. Define $G(x_1, x_2) := f(V(x_1, x_2)^T)$. For any $(x_1, x_2)^T \in \text{supp}(G)$, through the similar analysis as above we have

$$
|x_2| \leq \sqrt{2} \max\{|b_i|, |a_i| : i = 1, 2\}.
$$

Moreover,

$$
\|\tilde{P} f\|_{L^2}^2 = \|\Sigma(V^T f)\|_{L^2}^2 = \frac{1}{\sigma^2} \int_{\mathbb{R}} \int_{\mathbb{R}} |[V^T f](\sigma^{-1} x_1, x_2)|^2 dx_2 dx_1
$$
$$
\leq \frac{\sqrt{2} \max\{|b_i|, |a_i| : i = 1, 2\}}{\sigma^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \|f(V(\sigma^{-1} x_1, x_2))^T\|^2 dx_2 dx_1
$$
$$
\leq \frac{\sqrt{2} \max\{|b_i|, |a_i| : i = 1, 2\}}{\sigma} \|f\|_{L^2}^2 < \infty,
$$

(2.6)

where the first inequality is derived from (2.5) and the Cauchy-Swart inequality. The proof is concluded. \hfill \Box

2.3. (Quasi) shift-invariant space. The space of square summable sequences on $\mathbb{Z}^D$ is defined as $\ell^2(\mathbb{Z}^D) := \{(c_k)_{k \in \mathbb{Z}^D} : \|\{c_k\}_{k \in \mathbb{Z}^D}\|_{\ell^2} = \left(\sum_{k \in \mathbb{Z}^D} |c_k|^2\right)^{1/2} < \infty\}$. For a generator $\phi \in L^2(\mathbb{R}^D)$ with $D \geq 1$, the associated shift-invariant space (SIS) $V(\phi, \mathbb{Z}^D)$ is defined as

$$
V(\phi, \mathbb{Z}^D) = \left\{ \sum_{k \in \mathbb{Z}^D} c_k \phi(\cdot - k) : c_k \in \mathbb{Z}^D, c_k \in \ell^2(\mathbb{Z}^D) \right\}.
$$

(2.7)

For the stable recovery of functions in $V(\phi, \mathbb{Z}^D)$, it is commonly required that $\{\phi(\cdot - k) : k \in \mathbb{Z}^D\}$ is a Riesz basis for $V(\phi, \mathbb{Z}^D)$ (c.f. [1, 2, 23]), namely, there exist $0 < C_1 \leq C_2 < \infty$ such that

$$
C_1 \|\{c_k\}_{k \in \mathbb{Z}^D}\|_{\ell^2}^2 \leq \|\sum_{k \in \mathbb{Z}^D} c_k \phi(\cdot - k)\|_{L^2}^2 \leq C_2 \|c_k\|_{L^2}^2,
$$

(2.8)
Here $C_1$ and $C_2$ are referred to as the Riesz lower and upper bounds of $\{\phi(\cdot - k) : k \in \mathbb{Z}^D\}$. For a generator $\varphi \in L^2(\mathbb{R}^D)$ and the shift set $\mathcal{X} = \{x_j\}_{j \in \mathbb{Z}^D} \subseteq \mathbb{R}^D$, the associated quasi shift-invariant space (QSIS) is defined as

$$V(\varphi, \mathcal{X}) := \left\{ \sum_{k \in \mathbb{Z}^D} c_k \varphi(\cdot - x_j) : \{c_k\}_{k \in \mathbb{Z}^D} \in l^2(\mathbb{Z}^D) \right\}. \tag{2.9}$$

If $\mathcal{X} = \mathbb{Z}$ then $V(\varphi, \mathcal{X})$ degenerates to be an SIS. As implied in [12], the recovery theory for the QSIS ($\mathcal{X} \neq \mathbb{Z}$) is not the trivial generalization of that for the SIS. For example, the generator for the QSIS needs to be a positive definite function (the definition is postponed to section 4) such that the recovery can be achieved (c.f. [12, section 3.1(A1)]). Readers can refer to [4, 5, 11, 12] for the recent recovery results in QSISs (generated from some interesting generators such as the sinc and Gaussian functions).

### 2.4. On the Sobolev smoothness of a function.

For any $\varsigma \in \mathbb{R}$, the Sobolev space $H^\varsigma(\mathbb{R}^D)$ is defined as

$$H^\varsigma(\mathbb{R}^D) := \left\{ f : \int_{\mathbb{R}^D} |\hat{f}(\xi)|^2(1 + \|\xi\|^2)^\varsigma d\xi < \infty \right\}. \tag{2.10}$$

The deduced norm is defined by

$$\|f\|_{H^\varsigma(\mathbb{R}^D)} := \frac{1}{(2\pi)^{D/2}} \left( \int_{\mathbb{R}^D} |\hat{f}(\xi)|^2(1 + \|\xi\|^2)^\varsigma d\xi \right)^{1/2}, \quad \forall f \in H^\varsigma(\mathbb{R}^D).$$

And the Sobolev smoothness of $f$ is defined as $\nu_2(f) := \sup\{\varsigma : f \in H^\varsigma(\mathbb{R}^D)\}$. For $\varsigma > D/2$, by [31, Chapter 9.1] or [24, section 1] the functions in $H^\varsigma(\mathbb{R}^D)$ are continuous.

### 3. SA-Radon samples based reconstruction for compactly supported functions in general refinable SISs

Some definitions and denotations are necessary for our discussion. For a SIS $V(\phi, \mathbb{Z}^D)$ and a compact set $E \subseteq \mathbb{R}^D$, define

$$V_E(\phi, \mathbb{Z}^D) := \{ f \in V(\phi, \mathbb{Z}^D) : \text{supp}(f) \subseteq E \}.$$ 

Denote by $[x]$ ([x]) the largest (smallest) number that is not larger (not smaller) than $x$. The cardinality of a set $G$ is denoted by $\#G$. A function $\phi \in L^2(\mathbb{R}^D)$ is refinable if $\hat{\phi}(2\xi) = \hat{a}(\xi)\hat{\phi}(\xi)$ where $\hat{a}(\xi)$ is a $2\pi\mathbb{Z}^D$-periodic trigonometric polynomial (c.f. [6, 16, 25]). In what follows we establish the first main theorem in this paper.
3.1. The first main theorem. The following is the first main result in this paper.

**Theorem 3.1.** Suppose that $\phi \in L^2(\mathbb{R}^2)$ is refinable and continuous such that $\text{supp}(\phi) \subseteq [N_1, M_1] \times [N_2, M_2]$. Moreover, suppose that the projection vector $P \in \mathbb{R}^{1 \times 2}$ and there exists $\gamma \in \mathbb{R}$ such that $\gamma P \in \mathbb{Z}^{1 \times 2}$ and $\{\gamma P x : x \in \mathbb{Z}^2\} = \mathbb{Z}$. Then for almost all discrete set $X \subseteq (0, 1)$ such that $2 \leq \#X < \infty$, any $f \in V_E(\phi, \mathbb{Z}^2)$ with $E = [a_1, b_1] \times [a_2, b_2]$ can be determined by the samples of its SA-Radon transform $Pf$ at

\begin{equation}
(X/\gamma + \mathbb{Z}/\gamma) \cap \left[-\sqrt{2} \max\{|N_i|, |M_i| : i = 1, 2\} + K_{\text{min}}, \sqrt{2} \max\{|N_i|, |M_i| : i = 1, 2\} + K_{\text{max}}\right],
\end{equation}

provided that the map $P_E : \hat{E} \rightarrow \mathbb{Z}$ defined by $P_E(x) = P x$ is an injection, where

\begin{equation}
\hat{E} = \left\{\left[[a_1 - M_1], [b_1 - N_1]\right] \times \left[[a_2 - M_2], [b_2 - N_2]\right]\right\} \cap \mathbb{Z}^2,
\end{equation}

$K_{\text{min}} = \min\{\gamma P k : k \in \hat{E}\}$ and $K_{\text{max}} = \max\{\gamma P k : k \in \hat{E}\}$.

**Proof.** The proof is given in subsection 3.3. $\Box$

**Remark 3.1.** In Theorem 3.1, we assume that the functions to be reconstructed are compactly supported and their supports are contained in a known rectangle. Such an assumption is reasonable for CT (c.f. [35]).

3.2. Design of feasible projection vector. In Theorem 3.1 the projection vector $P$ is required to satisfy the following two conditions:

(A1) : the map $P_E : \hat{E} \rightarrow \mathbb{Z}$ defined by $P_E(x) = P x$ is injective.

(A2) : there exists $0 \neq \gamma \in \mathbb{R}$ such that $\{\gamma P x : x \in \mathbb{Z}^2\} = \mathbb{Z}$.

In what follows, we establish a choice of $P$ such that (A1) and (A2) hold.

**Proposition 3.2.** Define $\hat{E}^- := \{k - \hat{k} : k, \hat{k} \in \hat{E}\}$. For $j = 1, 2$, denote $\hat{E}_{j,\text{max}}^- := \{|(k, e_j) : k \in \hat{E}^-\}$, where $e_j \in \mathbb{R}^2$ is the $j$th unit coordinate vector with the only nonzero entry 1 at the $j$th entry. Choose $\hat{P} = [\hat{P}_1, \hat{P}_2]$ recursively by $\hat{P}_1 = 1$, $\hat{P}_2 = \hat{E}_{1,\text{max}}^- + \hat{P}_1$. Then $P = \frac{\hat{P}}{||\hat{P}||_2}$ satisfies (A1) and (A2).

**Proof.** It is easy to prove that (A1) holds if and only if $\langle \hat{P}, k \rangle \neq 0$ for any fixed $0 \neq k = (k_1, k_2) \in \hat{E}^-$. Denote $J_k = \max\{j : k_j \neq 0\}$. If $J_k = 1$, then clearly $|\langle \hat{P}, k \rangle| \geq 1$. If $J_k = 2$, then

$|\langle \hat{P}, k \rangle| \geq |\hat{P}_2 k_2| - |\hat{P}_1 k_1| = |\hat{P}_2 k_2| - |k_1| \geq |\hat{P}_2| - |k_1| \geq |\hat{P}_2| - \hat{E}_{1,\text{max}}^- = 1.$

Then (A1) holds. Since $\langle \hat{P}, e_1 \rangle = \hat{P}_1 = 1$ then (A2) holds with $\gamma = ||\hat{P}||_2$. $\Box$

3.3. Proof of Theorem 3.1.
3.3.1. **Two lemmas.** In the following lemma, we establish a necessary and sufficient condition on the generator Radon transform $P\phi$, such that any compactly supported $f \in V(\phi, \mathbb{Z}^2)$ can be determined by its SA-Radon transform $Pf$.

**Lemma 3.3.** Suppose that $\phi \in L^2(\mathbb{R}^2)$ is refinable such that $\text{supp}(\phi) \subseteq [N_1, M_1] \times [N_2, M_2]$. Then any $f \in V_E(\phi, \mathbb{Z}^2)$ can be determined by its Radon transform $Pf$ if and only if \{\$P\phi(\cdot - Pk) : k \in \tilde{E}\$\} is a basis for $\text{span}\{P\phi(\cdot - Pk) : k \in \tilde{E}\}$, where $E = [a_1, b_1] \times [a_2, b_2]$ and

$$\tilde{E} = \left\{\left([a_1 - M_1], [b_1 - N_1]\right) \times \left([a_2 - M_2], [b_2 - N_2]\right)\right\} \cap \mathbb{Z}^2. \quad (3.3)$$

**Proof.** Necessity: For $f \in V_E(\phi, \mathbb{Z}^2)$, it follows from $\text{supp}(f) \subseteq E$ that

$$f = \sum_{k_1 = [a_1 - M_1]}^{(b_1 - N_1]} \sum_{k_2 = [a_2 - M_2]}^{(b_2 - N_2]} c_{k_1,k_2}\phi(\cdot - k_1, \cdot - k_2). \quad (3.4)$$

Then for any $\xi \in \mathbb{R}$, we have

$$\hat{P}f(\xi) = \hat{f}(P^T\xi) = \sum_{k_1 = [a_1 - M_1]}^{(b_1 - N_1]} \sum_{k_2 = [a_2 - M_2]}^{(b_2 - N_2)} c_{k_1,k_2} e^{-iK\cdot P^T\xi} \hat{\phi}(P^T\xi) = \sum_{k_1 = [a_1 - M_1]}^{(b_1 - N_1]} \sum_{k_2 = [a_2 - M_2]}^{(b_2 - N_2)} c_{k_1,k_2} e^{-iP^T\xi} \hat{\phi}(\xi), \quad (3.5)$$

where $k = (k_1, k_2)^T$. That is,

$$Pf = \sum_{k_1 = [a_1 - M_1]}^{[b_1 - N_1]} \sum_{k_2 = [a_2 - M_2]}^{[b_2 - N_2]} c_{k_1,k_2} P\phi(\cdot - Pk) \in \text{span}\{P\phi(\cdot - Pk) : k \in \tilde{E}\}. \quad (3.6)$$

If \{\$P\phi(\cdot - Pk) : k \in \tilde{E}\$\} is a basis for $\text{span}\{P\phi(\cdot - Pk) : k \in \tilde{E}\}$, then the map $P_E : \tilde{E} \rightarrow P \mathcal{X}$, $x \in \tilde{E}$ is an injection. For convenient narration we denote $\{Px : x \in \tilde{E}\}$ by $A$. Then there exists uniquely $\{\tilde{c}_j\}_{j \in A}$ such that $Pf = \sum_{j \in A} \tilde{c}_j P\phi(\cdot - j)$. Comparing (3.4) and (3.6), we have

$$f = \sum_{j \in A} \tilde{c}_j P\phi(\cdot - j).$$

Sufficiency: Without loss of generality, suppose that the dimension $\text{dim}(\text{span}\{P\phi(\cdot - Pk) : k \in \tilde{E}\}) \geq 1$. If $\{P\phi(\cdot - Pk) : k \in \tilde{E}\}$ is not a basis for $\text{span}\{P\phi(\cdot - Pk) : k \in \tilde{E}\}$, then there exists a nonzero sequence $\{c_k\}_{k \in \tilde{E}}$ such that $0 = \sum_{k \in \tilde{E}} c_k P\phi(\cdot - Pk)$. Therefore, $0 \neq \tilde{f} := \sum_{k \in \tilde{E}} c_k \phi(\cdot - k)$ is not distinguishable from $g \equiv 0 \in V_E(\phi, \mathbb{Z}^D)$ since they have the same projection. This leads to a contradiction. \quad \Box

**Lemma 3.4.** Suppose that $\phi \in L^2(\mathbb{R}^2)$ is refinable and $\tilde{P} \in \mathbb{Z}^{1 \times 2}$ such that $\{\$\tilde{P}x : x \in \mathbb{Z}^2\$\} = \mathbb{Z}$. Then $\tilde{P}\phi$ defined by $\hat{\tilde{P}}\phi(\xi) = \hat{\phi}(\tilde{P}^T\xi)$ is compactly supported and refinable in $L^2(\mathbb{R})$. 


Proof. By Lemma 2.1, \( \tilde{P}\phi \) is compactly supported. By [16, Theorem 7.1.9], if \( \tilde{P}^T(Z \setminus [2Z]) \subseteq 2Z^2 \) then \( \tilde{P}\phi \) is still refinable such that \( \{\tilde{P}\phi(\cdot - k)\}_{k \in Z} \) is linearly independent. Since \( \{\tilde{P}x : x \in Z^2\} = Z \), then it follows from [14, Proposition 4.2] that \( \tilde{P}^T(Z \setminus [2Z]) \subseteq 2Z^2 \) is satisfied. Now the proof can be easily concluded. \( \square \)

3.3.2. Proof of Theorem 3.1. Denote \( \tilde{P} = \gamma P \). It is straightforward to check that \( Pf(x) = \gamma \tilde{P}f(\gamma x) \). As in (3.6), we have

\[
(3.7) \quad \tilde{P}f = \sum_{k_1=|a_1-M_1|}^{b_1-N_1} \sum_{k_2=|a_2-M_2|}^{b_2-N_2} c_{k_1,k_2} \tilde{P}\phi(\cdot - \tilde{P}k) \in \text{span}\{\tilde{P}\phi(\cdot - \tilde{P}k) : k \in \tilde{E}\}.
\]

It follows from Lemma 3.4 that \( \tilde{P}\phi \) is compactly supported and refinable in \( L^2(\mathbb{R}) \). Moreover, by [15, Lemma 2.4] we have \( \nu_2(\tilde{P}\phi) \geq \nu_2(\phi) \geq 1 \), which together with [31, Chapter 9.1] (or [24, section 1]) leads to that \( \tilde{P}\phi \) is continuous. On the other hand, since \( \{\tilde{P}x : x \in Z^2\} = Z \) then \( \tilde{P}f \in V(\tilde{P}\phi, Z) \). Now it follows from [32, section 4] that \( \tilde{P}f \) (or the sequence \( \{c_{k_1,k_2}\} \) in (3.7)) can determined by its samples at \( X + Z \). Since the map \( P_E : \tilde{E} \rightarrow P_x, x \in \tilde{E} \) is injective, then so is the map \( \tilde{P}_E : \tilde{E} \rightarrow \tilde{P}x, x \in \tilde{E} \). Now by (3.7) and Lemma 3.4, \( Pf \) can be rewritten as

\[
(3.8) \quad Pf = \sum_{j \in \Lambda} \tilde{c}_j \tilde{P}\phi(\cdot - j),
\]

where \( \Lambda = \{\tilde{P}x : x \in \tilde{E}\} \) and \( \{\tilde{c}_j : j \in \Lambda\} \) is a rearrangement of \( \{c_{k_1,k_2}\} \). Denote \( K_{\min} = \min\{\tilde{P}k : k \in \tilde{E}\} \) and \( K_{\max} = \max\{\tilde{P}k : k \in \tilde{E}\} \). Then

\[
(3.9) \quad \Lambda \subseteq \{K_{\min}, K_{\min} + 1, \ldots, K_{\max}\}.
\]

By Lemma 2.1, we have

\[
(3.10) \quad \text{supp}(\tilde{P}\phi) \subseteq [-\sqrt{2}\gamma|\max\{|N_i|, |M_i| : i = 1, 2\}|, \sqrt{2}\gamma|\max\{|N_i|, |M_i| : i = 1, 2\}].
\]

Now it follows from (3.8), (3.9) and (3.10) that \( \tilde{P}f \) can be determined by its samples at

\[
(3.11) \quad (X + Z) \cap \left[ -\sqrt{2}\gamma|\max\{|N_i|, |M_i| : i = 1, 2\} + K_{\min}, \sqrt{2}\gamma|\max\{|N_i|, |M_i| : i = 1, 2\} + K_{\max}\right].
\]

Equivalently, \( Pf \) can be determined by its samples at

\[
(3.12) \quad (X/\gamma + Z/\gamma) \cap \left[ -\sqrt{2}\max\{|N_i|, |M_i| : i = 1, 2\} + K_{\min}, \sqrt{2}\max\{|N_i|, |M_i| : i = 1, 2\} + K_{\max}\right].
\]

The proof is concluded by Lemma 3.3.
4. RECONSTRUCTION OF FUNCTIONS IN SISs GENERATED FROM A CLASS OF REFINABLE SPLINES BY SA-RADON SAMPLES

In this section, we are interested in the SA problem (1.5) for compactly supported functions in the SISs generated by a class of refinable box-splines. The main result will be organized in Theorem 4.3. Due to the positive definite (PD) property of the Radon transform at any angle of such splines, the theorem enjoys great angle flexibility. In what follows, we introduce the PD functions and the determination of functions in their generated QSISs.

4.1. PD functions and the determination of functions in the generated QSISs. We start with the definition of a PD function which is extensively applied to interpolation of scattered data (c.f. [20, 34]).

**Definition 4.1.** A function \( \varphi : \mathbb{R}^D \rightarrow \mathbb{C} \) is PD if, for all \( N \in \mathbb{N} \), all sets \( X = \{x_1, x_2, \ldots, x_N\} \subseteq \mathbb{R}^D \) of pairwise distinct elements, and all \( 0 \neq \alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{C}^N \), the quadratic form

\[
\sum_{j=1}^{N} \sum_{k=1}^{N} \alpha_j \overline{\alpha_k} \varphi(x_j - x_k) > 0.
\] (4.1)

The following proposition concerns on the determination of functions by the PD property. It can be easily proved by (4.1).

**Proposition 4.1.** If \( \varphi : \mathbb{R}^D \rightarrow \mathbb{C} \) is PD, then \( \varphi(0) > 0 \), and for any set \( X = \{x_1, x_2, \ldots, x_N\} \subseteq \mathbb{R}^D \), the system \( \{\varphi(\cdot - x_k)\}_{k=1}^{N} \) is linearly independent. Moreover, any function \( f = \sum_{k=1}^{N} \alpha_k \varphi(\cdot - x_k) \) can be determined by its samples at \( X \).

Next we introduce why we require the PD property for the SA problem (1.5). Recall that in the proof of Theorem 3.1, the reconstruction of \( f \in V(\phi, \mathbb{Z}) \) depends on that of its Radon transform \( Pf \). For the projection vector \( P \), as in section 3.2 (A1) it is required that there exists \( \gamma \in \mathbb{R} \) such that \( \tilde{P} := \gamma P \in \mathbb{Z}^{1 \times 2} \). The condition \( \tilde{P} \in \mathbb{Z}^{1 \times 2} \) guarantees that \( \tilde{P} \phi \) is still refinable and \( \tilde{P} f \) lies in the SIS \( V(\tilde{P} \phi, \mathbb{Z}) \). By the relationship \( Pf(x) = \gamma \tilde{P} f(\gamma x) \), we can reconstruct \( Pf \) by the sampling method in the SIS \( V(\tilde{P} \phi, \mathbb{Z}) \). For a general refinable function \( \phi \in L^2(\mathbb{R}^2) \), however, if \( P \) does not satisfy (A1) then Theorem 3.1 probably fails. In this section we will investigate the SA problem (1.5) for the SISs generated by a class of refinable box-splines. It will be clear in Theorem 4.3 that (A1) is not necessary for such refinable functions since their Radon transforms at any angle are PD, which admits the reconstruction in the QSISs by Proposition 4.1. Stated another way, the reconstruction will have great flexibility for choosing the angle of Radon transform.

4.2. The second main theorem. We first introduce a class of refinable box-splines which admit the PD property of their Radon transforms at any angle. The \( m \)th
cardinal B-spline $B_m$ is defined by $B_m := \chi_{(0,1]} \ast \cdots \ast \chi_{(0,1]}$, where $m \in \mathbb{N}$, $\chi_{(0,1]}$ is the characteristic function of $(0,1]$, and $\ast$ is the convolution. Through the simple calculation we have $\text{supp}(B_m) = [0, m]$, and $\hat{B}_m(\xi) = e^{-im\xi/2[\sin \xi/2]^m}$ by which it is easy to check that $B_m$ is refinable (c.f. [6]). Define $\varphi_{2n} = B_{2n}(\cdot + n)$, $n \in \mathbb{N}$. Then $\hat{\varphi}_{2n}(\xi) = [\sin \xi/2]^{2n}$. Moreover, through the tensor product we define $\phi_B : \mathbb{R}^2 \to \mathbb{R}$ by

\begin{equation}
\phi_B(x_1, x_2) = \prod_{k=1}^{2} \varphi_{2n_k}(x_k), \quad n_k \in \mathbb{N}.
\end{equation}

Clearly,

\begin{equation}
\hat{\phi}_B(\xi_1, \xi_2) = \prod_{k=1}^{2} \left[\frac{\sin \xi_k/2}{\xi_k/2}\right]^{2n_k}
\end{equation}

and $\phi_B$ is still refinable. Moreover, $\text{supp}(\phi_B) = [-n_1, n_1] \times [-n_2, n_2]$.

**Proposition 4.2.** Let $\phi_B$ be as in (4.2). Then for any projection vector $0 \neq P \in \mathbb{R}^{1 \times 2}$, the Radon transform $P\phi_B$ is PD and even.

**Proof.** The proof is given in section 4.3. \qed

In what follows, we establish the main result in this section.

**Theorem 4.3.** Suppose that $\phi_B \in L^2(\mathbb{R}^2)$ is defined in (4.2), and denote $\text{supp}(\phi_B) = [N_1, M_1] \times [N_2, M_2]$. Let $E = [a_1, b_1] \times [a_2, b_2]$ and

\begin{equation}
\hat{E} = \left\{\left[|a_1 - M_1|, |b_1 - N_1|\right] \times \left[|a_2 - M_2|, |b_2 - N_2|\right]\right\} \cap \mathbb{Z}^2.
\end{equation}

Then for almost all projection vector $P \in \mathcal{B} := \{x \in \mathbb{R}^{1 \times 2} : \|x\|_2 = 1\}$, any $f \in V(\phi_B, \mathbb{Z}^2)$ such that $\text{supp}(f) \subseteq E$ can be determined by its SA-Radon samples $\{Pf(Pk) : k \in \hat{E}\}$.

**Proof.** The proof is given in section 4.4. \qed

**Remark 4.1.** Unlike Theorem 3.1, the projection vector $P$ in Theorem 4.3 does not necessarily satisfy the conditions $A_1$ and $A_2$ in section 3.2.

4.3. **The proof of Proposition 4.2.** The following lemma is derived from [34, section 6.2].

**Lemma 4.4.** Suppose that $\varphi \in C(\mathbb{R}) \cap L^1(\mathbb{R})$ such that $\widehat{\varphi} \in L^1(\mathbb{R})$. Then for any set $\{x_1, \ldots, x_n\}$ of pairwise distinct points on $\mathbb{R}$, it holds that

\begin{equation}
\sum_{k=1}^{n} \sum_{l=1}^{n} c_k \overline{c_l} \varphi(x_k - x_l) = \int_{\mathbb{R}} \overline{\widehat{\varphi}(\xi)} \sum_{k=1}^{n} c_k e^{-ix_k \xi} d\xi,
\end{equation}

where $\{c_1, \ldots, c_n\} \subseteq \mathbb{C}$. 
Based on Lemma 4.4 we establish the following lemma.

**Lemma 4.5.** Suppose that \( \varphi \in C(\mathbb{R}) \) is even and real-valued such that \( \hat{\varphi} \in L^1(\mathbb{R}) \). If \( \hat{\varphi}(\xi) \geq 0 \) for any \( \xi \in \mathbb{R} \) and there exists \( \zeta > 0 \) such that \( \hat{\varphi}(\xi) > 0 \) on \( [-\zeta, \zeta] \), then the matrix

\[
A_{x_1,\ldots,x_n} = \begin{bmatrix}
\varphi(0) & \varphi(x_1 - x_2) & \cdots & \varphi(x_1 - x_n) \\
\varphi(x_2 - x_1) & \varphi(0) & \cdots & \varphi(x_2 - x_n) \\
\vdots & \vdots & \ddots & \vdots \\
\varphi(x_n - x_1) & \varphi(x_n - x_2) & \cdots & \varphi(0)
\end{bmatrix}
\]

is positive definite for any set \( X = \{x_1, x_2, \ldots, x_n\} \subseteq \mathbb{R} \) of pairwise distinct points. More precisely, there exists a positive constant \( \Lambda \) (being dependent on \( \varphi, X \) and \( \zeta \)) such that for any vector \( 0 \neq (c_1, \ldots, c_n) \in \mathbb{C}^n \), it holds

\[
(c_1, \ldots, c_n)A_{x_1,\ldots,x_n}(c_1, \ldots, c_n)^* = \sum_{k=1}^{n} \sum_{l=1}^{n} c_k \bar{c}_l \varphi(x_k - x_l) \geq \Lambda \sum_{j=1}^{n} |c_j|^2,
\]

where \((c_1, \ldots, c_n)^*\) is the transpose and complex conjugate of \((c_1, \ldots, c_n)\).

**Proof.** The proof is given in section 7.

**Proof of Proposition 4.2:** Denote \( P = [P_1, P_2] \). By [6, Theorem 4.3], it is easy to prove that \( \varphi_{2n} \) mentioned in (4.2) is even, real-valued and compactly supported. Moreover, by (1.2), for any \( \xi \in \mathbb{R} \) we have

\[
\hat{P} \hat{\phi}_B(\xi) = \hat{\phi}_B(P^T \xi) = \prod_{k=1}^{2} \left[ \sin \frac{P_k \xi/2}{P_k \xi/2} \right]^{2n_k}.
\]

Clearly, \( \hat{P} \hat{\phi}_B(\xi) \geq 0 \). Furthermore, \( \hat{P} \hat{\phi}_B(\xi) > 0 \) whenever \( |\xi| < \min\{ |\pi| : k \in \text{supp}(P) \} \). Here \( k \in \text{supp}(P) \) means that \( P_k \neq 0 \). Now by Lemma 4.5, \( P \phi_B \) is PD.

4.4. **Proof of Theorem 4.3.** It follows from Proposition 4.2 that for any \( 0 \neq P \in \mathbb{R}^{1 \times D} \), the projection \( P \phi_B \) is PD. As in (3.4) we have

\[
f(x_1, x_2) = \sum_{k_1=\lfloor a_1 - M_1 \rfloor}^{\lfloor b_1 - N_1 \rfloor} \sum_{k_2=\lfloor a_2 - M_2 \rfloor}^{\lfloor b_2 - N_2 \rfloor} c_{k_1, k_2} \phi_B(x_1 - k_1, x_2 - k_2).
\]

As in the proof of Theorem 3.1, define the map \( P_E : \hat{E} \rightarrow \mathbb{R} \) by \( P_E(x) = P x, x \in \hat{E} \). By (3.6) we can denote \( P f \) as \( \sum_{k=1}^{\# \Lambda} \hat{c}_{j_k} [P \phi_B](-j_k) \) where \( \Lambda = \mathcal{R}(P_E) = \{j_1, \ldots, j_{\# \Lambda} \} \). Therefore,

\[
[P f(j_1), \ldots, P f(j_{\# \Lambda})]^T = A_{j_1,\ldots,j_{\# \Lambda}} [\hat{c}_{j_1}, \ldots, \hat{c}_{j_{\# \Lambda}}]^T,
\]

where \( A_{j_1,\ldots,j_{\# \Lambda}} \) is defined by (4.6) with \( \varphi \) and \( \{x_1, x_2, \ldots, x_N \} \) therein replaced by \( P \phi_B \) and \( \{j_1, \ldots, j_{\# \Lambda} \} \), respectively. In what follows, we prove that for a almost all
projection vector \( P \in B \), the map \( P_E : \tilde{E} \rightarrow \mathcal{R}(\tilde{E}) \) is a bijection. For convenient narration denote \( \tilde{E} = \{ e_1, \ldots, e_{\#E} \} \). For any \( k, j \in \{ 1, \ldots, \#E \} \), it is easy to check that the Lebesgue measure \( m\{ \tilde{P} \in \mathbb{R}^{1 \times 2} : \tilde{P}(e_k - e_j) = 0 \} = 0 \). Then \( m\left( \bigcup_{k=1}^{\#E}\{ \tilde{P} \in \mathbb{R}^{1 \times 2} : \tilde{P}(e_k - e_j) = 0 \} \right) = 0 \), by which it is easy to prove that

\[
(4.11) \quad m\left\{ \tilde{P} \in \mathbb{R}^{1 \times 2} : \text{such that } \tilde{P}(e_k - e_j) = 0 \text{ for some } k \in \{ 1, \ldots, \#E \} \right\} = 0.
\]

Clearly, (4.11) holds with \( \tilde{P} \) replaced by \( P \). Stated another way, \( P_E \) is injective. By Proposition 4.2, \( P\phi \) is PD. Then it follows from Proposition 4.1 that \( f \) can be determined by the SA-Radon samples \( \{ Pf(Pk) : k \in \tilde{E} \} \). The proof is concluded. \( \square \)

5. Numerical Simulation

Let \( \phi_B(x_1, x_2) = \varphi_2(x_1)\varphi_2(x_2) \) be defined by (4.2). For a projection vector \( P = [\cos \theta, \sin \theta] \) such that \( \tan \theta \geq 2 \), through direct calculation it follows from (2.4) that the Radon transform

\[
(5.1) \quad P\phi_B(x, \theta) = \begin{cases}
\frac{(\tan \theta - \frac{x}{\cos \theta}) \left[ (\frac{x}{\cos \theta} - \tan \theta - \frac{3}{2})^2 + \frac{1}{4} \right]}{6 \cos \theta \tan^2 \theta}, & x \in (\sin \theta, \sin \theta + \cos \theta) \\
\frac{3(\frac{x}{\cos \theta} - \tan \theta)^2 + (\frac{x}{\cos \theta} - \tan \theta)^3 + 3\tan \theta - \frac{3x}{\cos \theta} + 1}{6 \cos \theta \tan^2 \theta}, & x \in (\sin \theta - \cos \theta, \sin \theta] \\
\frac{\tan \theta - \frac{x}{\cos \theta}}{\cos \theta \tan^2 \theta}, & x \in [\cos \theta, \sin \theta - \cos \theta] \\
\frac{\frac{x}{\cos \theta} (2x^2 - 6\tan \theta + 6\tan \theta - \frac{3x}{\cos \theta} + 3) + 6\tan \theta - \frac{3x}{\cos \theta} - 2}{6 \cos \theta \tan^2 \theta}, & x \in (0, \cos \theta) \\
\frac{6 \tan \theta + \frac{6x}{\cos \theta} - 2(\frac{x}{\cos \theta} + 1)^3}{6 \cos \theta \tan^2 \theta}, & x \in (- \cos \theta, 0] \\
\frac{\tan \theta + \frac{x}{\cos \theta}}{\cos \theta \tan^2 \theta}, & x \in [- \sin \theta + \cos \theta, - \cos \theta] \\
\frac{3\tan \theta + \frac{3x}{\cos \theta} + 1 + (3 - \tan \theta - x) \left( \tan \theta + \frac{x}{\cos \theta} \right)^2}{6 \cos \theta \tan^2 \theta}, & x \in [- \sin \theta, - \sin \theta + \cos \theta] \\
\frac{\left( \tan \theta + \frac{x}{\cos \theta} + 1 \right)^3}{6 \cos \theta \tan^2 \theta}, & x \in (- \sin \theta - \cos \theta, - \sin \theta) \\
0, & \text{else.}
\end{cases}
\]
Figure 5.1. Left: the plot of $\phi_B$. Right: the plot of Radon transform $P\phi_B(x)$ with $P = [\cos(1.2208), \sin(1.2208)]$.

Figure 5.2. (a) The figure of $f(x_1, x_2)$; (b) The figure of Radon transform $Pf$; (c) The reconstruction version $\tilde{f}(x_1, x_2)$ when SNR=30; (d) The reconstruction version $\tilde{f}(x_1, x_2)$ when SNR=35; (e) The reconstruction version $\tilde{f}(x_1, x_2)$ when SNR=40; (f) The reconstruction version $\tilde{f}(x_1, x_2)$ when SNR=45; (g) The reconstruction version $\tilde{f}(x_1, x_2)$ when SNR=50; (h) The reconstruction version $\tilde{f}(x_1, x_2)$ when SNR=55.
Without bias, we choose a function

\[ f = \sum_{k_1=0}^{4} \sum_{k_2=0}^{4} C_{k_1,k_2} \phi_B(x_1 - k_1, x_2 - k_2) \in V(\phi_B) \]

being supported on \([0, 6]^2\). Randomly choosing an angle \(\theta\) from \((\arctan 2, \pi/2)\), it follows from Theorem 4.3 that with probability 1, \(f\) can be determined by its SP Radon samples \(\{Pf(Pk) : k \in \hat{E}\}\) with \(\hat{E} = [0, 4]^2 \cap Z^2\). For simplicity, by the lexicographic order \(\hat{E}_N\) is denoted as \(\{e_1, \ldots, e_{25}\}\). Then \(f\) is determined through the linear equation system

\[ A(C_{e_1}, \ldots, C_{e_{25}})^T = (Pf(Pe_1), \ldots, Pf(Pe_{25}))^T, \]

where \(A = (P\phi_B(Pe_j - Pe_l))_{j,l=1}^{25}\). In this simulation we choose \(\theta = 1.2208\) and \(P = [\cos(1.2208), \sin(1.2208)]\). The graphs of \(\phi_B\) and \(P\phi_B(x)\) are illustrated in Figure 5.2. We add the Gaussian noise \(\epsilon \sim N(0, \sigma^2)\) to the samples on the right-hand side of (5.3), and what we observed is \(Pf(Pe_l) = Pf(Pe_l) + \epsilon\). Here the variance \(\sigma^2\) is chosen such that the desired signal to noise ratio (SNR) is expressed by

\[ \text{SNR} = 10 \log_{10} \left( \frac{\|\{Pf(Pe_l) : l=1, \ldots, 25\}\|_2^2}{25 \sigma^2} \right). \]

We consider the Tikhonov regularization problem

\[ \arg \min_C \|AC - (Pf(Pe_1), \ldots, Pf(Pe_{25}))^T\|_2 + \alpha \|C\|_2, \]

where the regularization parameter \(\alpha\) satisfies the requirement in [7, Theorem 2.5]. In this simulation, \(\alpha\) is chosen as in Table 1. The solution to (5.5) is

\[ \hat{C} = (\alpha I + A^*A)^{-1}A^*(Pf(Pe_1), \ldots, Pf(Pe_{25}))^T, \]

where \(I\) is the identity matrix. For every SNR \(\in \{30, 35, 40, 45, 50, 55\}\), conduct (5.6) for 1000 trials and the average of the 1000 outputs is denoted by \(\hat{\tilde{f}}\). Construct \(\tilde{f}\) through (5.2) with \((C_{e_1}, \ldots, C_{e_{25}})^T\) being replaced by \(\hat{\tilde{f}}\). And the reconstruction error is defined as

\[ \text{error} = \left[ \sum_{k=0}^{80} \sum_{l=0}^{80} \left( \tilde{f}(-2 + 0.1k, -2 + 0.1l) - f(-2 + 0.1k, -2 + 0.1l) \right)^2 \right]^{1/2}, \]

\[ \left[ \sum_{k=0}^{80} \sum_{l=0}^{80} \left( f(-2 + 0.1k, -2 + 0.1l) \right)^2 \right]^{1/2}. \]

The average errors corresponding to different choices of SNR are recorded in Table 1.
6. Conclusion

We establish the single-angle (SA) Radon samples based reconstruction for the compactly supported functions in refinable shift-invariant spaces (SIS). We prove that any compactly supported function in a general refinable SIS can be reconstructed by its Radon samples at an appropriate angle. We also investigate the SA reconstruction problem in a class of SISs generated by a class of refinable box-splines. It is proved that any compactly supported function in the SISs generated by such box-splines, can be reconstructed by the SA Radon samples at almost every angle in $[0, 2\pi)$.

7. Appendix: Proof of Lemma 4.5

Since $\varphi$ is even and real-valued, so is $\hat{\varphi}$. Consequently, $A_{x_1, \ldots, x_n}$ is symmetric. Now we choose $0 < \tilde{\zeta} \leq \zeta$ such that $\hat{\varphi}(\xi) > 0$ on $[-\tilde{\zeta}, \tilde{\zeta}]$. Then for any $0 \neq (c_1, \ldots, c_n) \in \mathbb{C}^n$, it follows from Lemma 4.4 that

$$\sum_{k=1}^{n} \sum_{l=1}^{n} c_k \bar{c}_l \varphi(x_k - x_l) = (c_1, \ldots, c_n) A_{x_1, \ldots, x_n} (c_1, \ldots, c_n)^*$$

$$= \int_{\mathbb{R}} \hat{\varphi}(\xi) \left| \sum_{k=1}^{n} c_k e^{-ix_k \xi} \right|^2 d\xi$$

$$\geq \int_{-\tilde{\zeta}/2}^{\tilde{\zeta}/2} \hat{\varphi}(\xi) \left| \sum_{k=1}^{n} c_k e^{-ix_k \xi} \right|^2 d\xi,$$

where in the inequality we use $\hat{\varphi}(\xi) \geq 0$ for any $\xi \in \mathbb{R}$. Since $x_k \neq x_j$, it follows from [34, Lemma 6.7] that $\{e^{-ix_k \xi}\}_{k=1}^{n}$ is linear independent in the space $L^2([-\tilde{\zeta}/2, \tilde{\zeta}/2])$. Consequently, $\{e^{-ix_k \xi}\}_{k=1}^{n}$ is a Riesz basis for its finitely spanning space $\{\sum_{k=1}^{n} c_k e^{-ix_k \xi} : c_k \in \mathbb{C}\} \subseteq L^2([-\tilde{\zeta}/2, \tilde{\zeta}/2])$, and there exists $\lambda_{X, \tilde{\zeta}} > 0$ (depending on $X$ and $\tilde{\zeta}$) such that

$$\int_{-\tilde{\zeta}/2}^{\tilde{\zeta}/2} \left| \sum_{k=1}^{n} c_k e^{-ix_k \xi} \right|^2 d\xi \geq \lambda_{X, \tilde{\zeta}} \|\{c_k\}_{k=1}^{n}\|_2^2.$$

On the other hand, since $\varphi \in C(\mathbb{R})$ is compactly supported, then $\hat{\varphi} \in C^\infty(\mathbb{R})$. Therefore, $\min\{\hat{\varphi}(\xi) : \xi \in [-\zeta, \zeta]\}$ exists and is positive. Consequently, combining

| SNR | 30  | 35  | 40  | 45  | 50  | 55  |
|-----|-----|-----|-----|-----|-----|-----|
| $\alpha$ | 2e-03 | 2.5e-04 | 3.5e-05 | 2e-06 | 1.5e-07 | 9.4e-08 |
| error | 0.4255 | 0.2471 | 0.0924 | 0.0480 | 0.0290 | 0.0087 |

Table 1. The error vs SNR.
(7.1) and (7.2) we have
\[\sum_{k=1}^{n} \sum_{l=1}^{n} c_k \bar{c}_l \varphi(x_k - x_l) = (c_1, \ldots, c_n) A_{x_1,\ldots,x_n}(c_1, \ldots, c_n)^* \]
\[\geq \lambda X,\tilde{\zeta} \|\{c_k\}_{k=1}^{n}\|_2^2 > 0.\]

Stated another way, the matrix \(A_{x_1,\ldots,x_n}\) is positive definite. Therefore, by (7.3) we have
\[\sum_{k=1}^{n} \sum_{l=1}^{n} c_k \bar{c}_l \varphi(x_k - x_l) \geq \sup_{0<\tilde{\zeta} \leq \zeta} \left\{\lambda X,\tilde{\zeta} \min\{\hat{\varphi}(\xi) : \xi \in U(0,\tilde{\zeta})\}\right\} \sum_{j=1}^{n} |c_j|^2.

Now we choose \(0 < \Lambda := \sup_{\tilde{\zeta}} \left\{\lambda X,\tilde{\zeta} \min\{\hat{\varphi}(\xi) : \xi \in U(0,\tilde{\zeta})\}\right\} < \infty\) to conclude the proof.

References

[1] A. Aldroubi, K. Gröchenig, Nonuniform sampling and reconstruction in shift-invariant spaces, *SIAM Rev.*, 43, 585-620, 2001.
[2] A. Aldroubi, Q. Sun, W.S. Tang, Convolution, average sampling, and a Calderon resolution of the identity for shift-invariant spaces, *J. Fourier Anal. Appl.*, 22, 215-244, 2005.
[3] N. Antipa, G. Kuo, R. Heckel, B. Mildenhall, E. Bostan, R. Ng, L. Waller, DiffuserCam: lensless single-exposure 3D imaging, *Optica*, 5, 1-9, 2018.
[4] N.D. Atreas, On a class of non-uniform average sampling expansions and partial reconstruction in subspaces of \(L^2(\mathbb{R})\), *Adv. Comput. Math.*, 36(1), 21-38, 2012.
[5] L. de Carli, P. Vellucci, \(p\)-Riesz bases in quasi shift invariant spaces, *arXiv:1710.00702*, Preprint.
[6] C. K. Chui, An Introduction to Wavelets, *Academic Press*, 1992.
[7] H. W. Engl, M. Hanke, and A. Neubauer, Regularization of Inverse Problems, *Mathematics and Its Applications*, 375, Kluwer, Dordrecht, 1996.
[8] A. Entezari, M. Nilchian, and M. Unser, A box spline calculus for the discretization of computed tomography reconstruction problems, *IEEE Transactions on Medical Imaging*, 31, 1532-1541, 2012.
[9] R. S. Feris, Volker Krueger, R. M. C. Junior, A wavelet subspace method for real-time face tracking, *Real-Time Imaging*, 10, 339-350, 2002.
[10] S. Fan, S. S-Dryden, J. Zhao, S. Gausmann, A. Schülzgen, G. Li, B. E. A. Saleh, Optical fiber refractive index profiling by iterative optical diffraction tomography, *J. Lightwave Technol.*, 36(24), 5754-5763, 2018.
[11] K. Gröchenig, J. Stöckler, Gabor frames and totally positive functions, *Duke Math. J.*, 162(6), 1003-1031, 2013.
[12] K. Hamm, J. Ledford, On the structure and interpolation properties of quasi shift-invariant spaces, *Journal of Functional Analysis*, 274, 1959-1992, 2018.
[13] B. Han, The projection method for multidimensional framelet and wavelet analysis, *Math. Model. Nat. Phennom.*, 7(2), 32-59, 2012.
[14] B. Han, Construction of wavelets and framelets by the projection method, *Int. J. Appl. Math. Appl.*, 1(1), 1-40, 2008.
[15] B. Han, Projectable multidimensional refinable functions and biorthogonal wavelets, *Appl. Comput. Harmon. Anal.*, 13, 89-102, 2002.

[16] B. Han, Framelets and wavelets: Algorithms, analysis, and applications, *Applied and Numerical Harmonic Analysis*, Birkhäuser/Springer, Cham, 2017. xxxiii +724 pp.

[17] B. Han, T. Li, X. Zhuang, Directional compactly supported box spline tight framelets with simple geometric structure, *Applied Mathematics Letters*, 91, 213-219, 2019.

[18] B. Han, Vector cascade algorithms and refinable function vectors in Sobolev spaces, *Journal of Approximation Theory*, 124(1), 44-88, 2003.

[19] A. C. Kak, M. Slaney, Principles of Computerized Tomographic Imaging, *SIAM, Philadelphia, PA*, 2001.

[20] N. Karimi, S. Kazem, D. Ahmadian, H. Adibi, L.V. Ballestra, On a generalized Gaussian radial basis function: Analysis and applications, *Engineering Analysis with Boundary Elements*, 112, 46-57, 2020.

[21] G. Kerkyacharian, G. Kyriazis, E. L. Pennec, P. Petrushev, D. Picard, Inversion of noisy Radon transform by SVD based needlets, *Applied and Computational Harmonic Analysis*, 28, 24-45, 2010.

[22] M. K. Kim, Principles and techniques of digital holographic microscopy, *SPIE Reviews*, 1, 1-51, 2010.

[23] Y. X. Li, J. Wen, J. Xian, Reconstruction from convolution random sampling in local shift invariant spaces, *Inverse Problems*, 35(12), 125008, 2019.

[24] Y. Li, D. Han, S. Yang, G. Huang, Nonuniform sampling and approximation in Sobolev space from the perturbation of framelet system, *SCIENCE CHINA Mathematics*, 64, 351-372, 2021.

[25] Y. Li, S. Yang, D. Yuan, Bessel multwavelet sequences and dual multiframelets in Sobolev spaces, *Advances in Computational Mathematics*, 38: 491-529, 2013.

[26] X. Lu, Y. Sun, G. Bai, Adaptive wavelet-Galerkin methods for limited angle tomography, *Image and Vision Computing*, 28, 696-703, 2010.

[27] R. Horisaki, K. Fuji, and J. Tanida, Diffusion-based single-shot diffraction tomography, *Optics Letters*, 44, 1964-1967, 2019.

[28] M. T. McCann, and M. Unser, High-quality parallel-ray X-ray CT back projection using optimized interpolation, *IEEE Transactions on Image Processing*, 26, 4639-4647, 2017.

[29] F. Natterer, The mathematics of computerized tomography, in: *Classics Appl. Math.*, vol. 32, *SIAM, Philadelphia, PA*, 2001.

[30] F. Natterer, F. Wübbeling, Mathematical Methods in Image Reconstruction, *SIAM, Philadelphia, PA*, 2001.

[31] M. Stéphane, A Wavelet Tour of Signal Processing, *Elsevier Inc.*, 2009.

[32] Q. Sun, Local reconstruction for sampling in shift-invariant spaces, *Advances in Computational Mathematics*, 32, 335-352, 2010.

[33] Y. Tian, Y. Z. Li, Subspace dual super wavelet and Gabor frames, *SCIENCE CHINA Mathematics*, 60, 2429-2446, 2017.

[34] H. Wendland, Scattered data approximation, *Cambridge University Press*, 17, 2004.

[35] Y. Xu, A new approach to the reconstruction of images from Radon projections, *Advances in Applied Mathematics*, 36, 388-460, 2006.

[36] A. D. Yablon, Multi-wavelength optical fiber refractive index profiling by spatially resolved fourier transform spectroscopy, *Journal of Lightwave Technology*, 28, 360-364, 2010.
College of Mathematics and Information Science, Guangxi University, Nanning, China

Email address: youfalee@hotmail.com

CREOL College of Optics & Photonics, University of Central Florida, Orlando, FL 32816

Email address: shengli.fan@knights.ucf.edu

College of Mathematics and Information Science, Guangxi University, Nanning, China

Email address: hyfbqy@163.com