We compare the conventional description of the interaction of matter with the four known forces in the standard model with an alternative Weyl description in which the chiral coupling is extended to include gravity. The two are indistinguishable at the low energy classical level of equations of motion, but there are subtle differences at the quantum level when nonvanishing torsion and the Adler-Bell-Jackiw anomaly is taken into account. The spin current and energy-momentum of the chiral theory then contain non-Hermitian terms which are not present in the conventional theory. In the chiral alternative, CPT invariance is not automatic because chirality supersedes Hermiticity but full Lorentz invariance holds. New fermion loop processes associated with the theory are discussed together with a perturbative regularization which explicitly maintains the chiral nature and local symmetries of the theory.

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I. INTRODUCTION

It is well-known that in the standard model, fermions interact chirally with internal gauge fields and can be described by a multiplet of 15 left-handed Weyls per generation coupled to the basic forces. In the absence of gravity, at least as far as perturbation theory is concerned, the rewriting of right-handed fields as left-handed ones is mere relabeling without physical consequence. The conventional coupling to gravity is Majorana since the Hermitian Weyl action contains both the left- and right-handed spin connections and can be expressed as a Majorana action in the absence of internal gauge fields. As shown in Ref. [1], in four dimensions, gravity can be described using spin connections of a single-handedness. When we bring in internal gauge fields, the difference is important, because the fermion multiplet belongs to a complex representation of the internal gauge group. It is not possible to express the complete action with all four forces in pure Majorana form. Whether we write some of the fields of one chirality as conjugate fields of the opposite chirality can produce couplings to spin connections of different handedness.

If fermions couple to only the left-handed spin connection, then what dynamics is to be prescribed for the left-handed spin connection without leading to spurious equations of motion for the right-handed spin connection? An answer that suggests itself is that when such a chiral Weyl theory of fermions is quantized, the one-loop effective action generated in background curved spacetimes inevitably result in counterterms. These will be compatible with the Weyl nature and symmetries of the theory if there are no anomalies. Thus the lowest order curvature counterterm which is the analog of the Einstein-Hilbert-Palatini action for the conventional case is therefore the natural candidate for the classical action. As we shall see, this leads naturally to the Samuel-Jacobson-Smolin action [2] for (anti-)self-dual Ashtekar variables [3]. Conversely, if such an action which involves not the full but only the left-handed spin connection is used, then fermions of only one chirality is allowed. An obvious question is whether such a theory differs in any detectable way from the conventional one [3, 4].

The Hermitian Weyl action in curved spacetimes requires the right-handed spin connection. A Weyl action with only the left-handed spin connection is therefore not Hermitian in general. However, the chiral Weyl theory is local Lorentz invariant because in four dimensions, the miracle is that the relevant \( SO(3,1) \) group is isomorphic to \( SO(3,C) \) and \( SL(2,C) \)
is the complexification of \(SU(2)\). In fact the (anti-)self-dual Ashtekar connection is the left-handed complexified \(SU(2)\) connection when the Lorentz group is gauged. While P and CP have been observed to be violated, there is a bias to believe that CPT is good. The CPT-theorem \(\text{[5]}\) says that a local, Lorentz-invariant quantum field theory with a Hermitian action \(S\) quantized according to the spin-statistics rule must be CPT-invariant. Under CPT, \(S \rightarrow S^\dagger\). The Hermitian qualification is essential to CPT being good and the chiral nature of the standard model suggests that it is a possible candidate for a breakdown of CPT invariance.\(^\text{[7]}\) As we shall see, violations of CPT are subtle and occur in regimes which have not been rigorously tested.

In what follows, we discuss the standard model with gravitational coupling in the context of a truly chiral Weyl theory \(\text{[4]}\) and detail how it differs from the conventional scheme. We prove its consistency with respect to all the local symmetries of the theory and provide a regularization scheme \(\text{[6]}\) which explicitly respects the chirality and local symmetries and permits the computation of fermion loops which include CPT-violating processes in the presence of torsion. We show that the energy-momentum tensor and spin current contain anti-Hermitian pieces which all involve the Adler-Bell-Jackiw (ABJ) current \(\text{[7]}\). The scaling behaviour of the theory also differs from the conventional case in that the trace of the energy-momentum tensor contains anti-Hermitian contributions involving the ABJ current and the ABJ anomaly.

II. WEYL, MAJORANA AND HERMITIAN ACTIONS

We start with the bare chiral (Weyl) fermion action which is

\[
S^- = \int_M d^4x \overline{\Psi}_L i \not{D} \Psi_L
= \int_M d^4x \overline{\Psi}_L e^{i \frac{1}{2} \not{D} - \frac{1}{2} \not{D}} e^{\frac{1}{2} \not{D}} \Psi_L
\]

\[
(1)
\]

\(^1\)The violations studied here are due to non-Hermiticity rather than to infinite number of basic fields, nonlocality, strings and extended objects or a breakdown of Lorentz invariance.
where \( i\mathcal{D} = \gamma^\mu (i\partial_\mu + W_{\mu a}T^a + \frac{i}{2}A_{\mu AB}\sigma^{AB}) \), and \( e \) denotes the determinant of the vierbein. \( W_{\mu a} \) is the internal gauge connection while \( A_{\mu AB} \) is the spin connection. The convention is
\[
\{\gamma^A, \gamma^B\} = 2\eta^{AB},
\]
with \( \eta^{AB} = \text{diag}(-1, +1, +1, +1) \). Lorentz indices are denoted by uppercase Latin indices while Greek indices are spacetime indices.

In the standard model, it is possible to relabel right-handed fields \( \Psi_{Ri} \) as left-handed ones \( \chi_{Li} \) through \( \Psi_{Ri} = C^4\chi_{Li}^T \) and rewrite all Weyl fermion fields in a multiplet \( \Psi_L \), of 15 Weyl fermions, which belongs to a complex representation of the gauge group.\(^2\) The arguments and results presented in this article are however not confined to just the \( SU(3) \times SU(2) \times U(1) \) group and 15 Weyls, but are also applicable to Grand Unified Theories \(^3\) coupled chirally to gravity.

Recall that if the gauge generators \( T^a \) satisfy
\[
[T^a, T^b] = if^{abc}T^c,
\]
then \((-T^a)^*\) satisfy the same Lie algebra.\(^4\) If there exists a \( U \) such that \( U^{-1}(-T^a)^*U = T^a \), then the representation is called real (pseudoreal) if \( U \) is symmetric (antisymmetric). Otherwise, the representation is termed complex.

The Hermitian conjugate of the Weyl action is
\[
(S^{-})^\dagger = \int_M d^4x e \left[ -i(\partial_\mu \Psi_L)\gamma^\mu \Psi_L + \Psi_L(i\frac{1}{2}A_{\mu AB}\sigma^{AB} + W_{\mu a}T^a)\gamma^\mu \Psi_L \right].
\]

Without gauge fields \( (W_{\mu a} = 0) \), the Hermitized action is expressible in Majorana form as
\[
S_{\text{Majorana}} = \frac{1}{2}(S^{-} + (S^{-})^\dagger) = \int_M d^4x \overline{\Psi}_M i\mathcal{D}\Psi_M,
\]
with
\[
\Psi_M = \frac{1}{\sqrt{2}}(\Psi_L + C^4\overline{\Psi}_L^T)
\]
\(^2\)\( i \) denotes the flavor/color and \( C_4 \) is the charge conjugation matrix in four dimensions with \( C^4_4 = C^{-1}_4 = C^1_4 = -C_4 \).

\(^3\)We adopt the convention of \((T^a)^\dagger = T^a\) and real structure constants.
\[ \Psi_M = \frac{1}{\sqrt{2}} (\Psi_L + \Psi_L^T C_4) \]  

(6)

being the Majorana spinors. When there are internal gauge couplings, it is still possible to write the Hermitized Weyl action in the Majorana form for real and pseudoreal representations since

\[ S'_{\text{Majorana}} = \int_M d^4 x \Psi_M i \partial \Psi_M' \]

\[ = \frac{1}{2} (S^- + (S^-)^\dagger) + \int_M d^4 x \Psi_L \gamma^\mu W_{\mu a} (U^{-1}(-T^a)^T U - T^a) \Psi_L \]  

(7)

with a different set of Majorana spinors

\[ \Psi_M' = \frac{1}{\sqrt{2}} (\Psi_L + U^{-1} C_4 \Psi_L^T), \]

\[ \Psi_M = \frac{1}{\sqrt{2}} (\Psi_L + \Psi_L^T C_4 U), \]  

(8)

and the choice of \( U \) for which \( U(-T^a)^T U^{-1} - T^a = 0 \) can be found. For complex representations however, the Hermitized action has a Majorana form only for gravity couplings.

We next examine the case in which the coupling of matter to all four known forces including gravity is entirely chiral or Weyl i.e. the couplings are described precisely by \( S^- [\Psi_L, \Psi_L, e_{\mu A}, A_{\mu AB}, W_{\mu a}] \) as in Eq. (1), rather than its Hermitian form. The quantity \( A_{\mu AB}^- = \frac{1}{2} (-i A_{AB} + \frac{1}{2} \epsilon_{ABCD} A_{CD}) \) is the anti-self-dual or left-handed spin connection, and only left-handed fermions appear in the action.\(^4\)

For clarity, we shall use the explicit chiral representation with

\[ \gamma^5 = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix}, \]

\(^4\)Note that in \((S^-)^\dagger\) in Eq. (4) the coupling projects out the right-handed spin connection while in \( S^- \) only the left-handed spin connection is required. So a Majorana fermion couples to both left- and right-handed spin connections. The same is true for a Weyl fermion in the Hermitized action. See for instance Ref. 4 for explicit self- and anti-self-dual decompositions of the spin connection and their couplings to two-component Weyl fermions.
\[ \gamma^A = \begin{pmatrix} 0 & i \tau^A \\ i \tau^A & 0 \end{pmatrix}. \]  

(9)

In the above, \( \tau^a = -\tau^a \) (\( a = 1, 2, 3 \)) are Pauli matrices, and \( \tau^0 = \tau^0 = -1 \). In the chiral representation,

\[ \frac{i}{2} A_{\mu AB} \sigma^{AB} P_L = \begin{bmatrix} 0 & 0 \\ 0 & A_{\mu a} \tau^a \end{bmatrix}. \]  

(10)

Note that

\[ A_{\mu a} \tau^a \frac{1}{2} = \left[ i A_{\mu 0a} - \frac{1}{2} \epsilon_{0a}^{bc} A_{\mu bc} \right] \tau^a \frac{1}{2} \]

\[ = -\frac{i}{4} A_{\mu AB} \tau^A \tau^B \]  

(11)

is also precisely the Ashtekar connection in the (anti-)self-dual formulation of gravity in four dimensions \([1, 2]\). In this regard, since right-handed spin connections do not appear at the fundamental level, couplings only to left-handed Weyl fermions are allowed \([4, 9]\).

In what follows, we shall suppose as in first order formulations, that the spin connection and vierbein are independent and that the torsion is not necessarily zero. Before isolating the pure and imaginary parts of the action, let us first summarize some basic relations involving torsion.

A GL(4,R) connection \( \Gamma^\alpha_{\mu \nu} \) may be introduced through

\[ \nabla_\mu E_\nu^A = \partial_\mu E_\nu^A + \Gamma^\alpha_{\nu \mu} E^A e^\alpha + A_{\mu AB} E_\nu^B = 0, \]  

(12)

and

\[ \nabla_\mu e_\nu^A = \partial_\mu e_\nu^A - \Gamma^\alpha_{\nu \mu} e^\alpha e_\nu^A + A_{\mu AB} e_\nu^B = 0. \]  

(13)

The covariant and contravariant metrics \( g_{\mu \nu} = e_\mu^A e_\nu^A \) and \( g^{\mu \nu} = E^\mu^A E^\nu_A \) therefore satisfy the metricity condition (which preserves lengths under parallel transport) with respect to the connection \( \Gamma^\alpha_{\mu \nu} \) i.e.

\[ \nabla_\mu g_{\alpha \beta} = \nabla_\mu g^{\alpha \beta} = 0. \]  

(14)

The connection \( \Gamma^\alpha_{\mu \nu} \) is not necessarily torsionless or symmetric. In fact,

\[ \Gamma^\alpha_{\mu \nu} = \frac{1}{2} \left( \Gamma^\alpha_{\mu \nu} - \Gamma^\alpha_{\nu \mu} \right) \]  

(15)
is related to the torsion

\[ T_A = \frac{1}{2} T_{A\mu \nu} d\varepsilon^\mu \wedge d\varepsilon^\nu = d\varepsilon_A + A_{AB} \wedge e^B \] (16)

by

\[ T_{A\mu \nu} = 2\Gamma_{[\nu\mu]}^\alpha e_{\alpha A}. \] (17)

In the presence of torsion, the Dirac matrices \( \gamma^\mu = E_A^\mu \gamma^A \) satisfy

\[ \partial_\mu \gamma^\mu + (\partial_\mu \ln e)\gamma^\mu + \frac{1}{2} A_{\mu AB}[\sigma^{AB}, \gamma^\mu] = 2\Gamma^\nu_{[\nu\mu]}\gamma^\mu \] (18)

with \( \partial_\mu \ln e = \Gamma^\nu_{\nu\mu} \) and \( \sigma^{AB} = \frac{1}{4}[\gamma^A, \gamma^B] \). However, if the torsion vanishes, then

\[ D_\mu \gamma^\mu f = e\gamma^\mu D_\mu f. \] (19)

In the quantum theory, to be compatible with the diffeomorphism-invariant measure [10]

\[ \prod_x D[\Psi_L(x)e^{\frac{1}{2}\varepsilon}(x)]D[e^{\frac{1}{2}\varepsilon}(x)\Psi_L(x)], \] (20)

in curved spacetimes, we shall chose densitized variables defined by

\[ \tilde{\Psi}_L \equiv \Psi_L e^{\frac{1}{2}\varepsilon}, \]

\[ \tilde{\Psi}_L \equiv e^{\frac{1}{2}\varepsilon}\Psi_L. \] (21)

This accounts for the second line of Eq. (1) in the Weyl action.

We may now decompose the Weyl action into explicitly Hermitian and anti-Hermitian pieces by writing

\[ S^- = \frac{1}{2}(S^- + (S^-)^\dagger) + \frac{1}{2}(S^- - (S^-)^\dagger), \] (22)

with the pure imaginary piece \( i\text{Im}(S^-) \) given by

\[ \frac{1}{2}(S^- - (S^-)^\dagger) = \int_M \left( d^4x \frac{i}{2} \partial_\mu (\tilde{\Psi}_L \gamma^\mu \tilde{\Psi}_L) - \frac{i}{4} \tilde{\Psi}_L \gamma^A \tilde{\Psi}_L e^{-1} \epsilon_{ABCD} T^B \wedge e^C \wedge e^D \right). \] (23)

We may note that in the presence of torsion, the second term is \textit{not a total divergence} and therefore gives rise to anti-Hermitian contributions to the
energy-momentum and spin current, but not the internal gauge current. We shall return to these later on in Section VII.

Recalling that $T_{\alpha \mu} = 2\Gamma_{\nu}^\alpha [\nu \mu]$, and defining $\Gamma_{\nu}^\mu \equiv B_\mu$, the imaginary part of the Weyl action is in fact

$$i \text{Im}(S^-) = \frac{1}{2}(S^- - (S^-)^\dagger)$$

$$= \frac{i}{2} \int_M d^4x \left[ \partial_\mu (\bar{\Psi}_L \gamma^\mu \Psi_L) - 2B_\mu \bar{\Psi}_L \gamma^\mu \Psi_L \right].$$  \hspace{1cm} (24)

It is interesting to realise that $B_\mu$ actually transforms as an Abelian gauge potential \[1\] under local scaling

$$B_\mu \rightarrow B_\mu + \frac{3}{2} \partial_\mu \alpha$$  \hspace{1cm} (25)

under local scaling

$$e_\mu A \rightarrow \exp(\alpha(x)) e_\mu A,$$

$$E^{A}_\mu \rightarrow \exp(-\alpha(x)) E^{A}_\mu,$$

$$\Psi_L \rightarrow \exp(-\frac{3}{2} \alpha(x)) \Psi_L, \quad \bar{\Psi}_L \rightarrow \exp(-\frac{3}{2} \alpha(x)) \bar{\Psi}_L,$$

$$A_{\mu AB} \rightarrow A_{\mu AB}, \quad W_{\mu a} \rightarrow W_{\mu a}. $$  \hspace{1cm} (26)

For the densitized variables, under scaling

$$\tilde{\Psi}_L \rightarrow \exp(\frac{1}{2} \alpha(x)) \tilde{\Psi}_L, \quad \tilde{\bar{\Psi}}_L \rightarrow \exp(\frac{1}{2} \alpha(x)) \tilde{\bar{\Psi}}_L.$$  \hspace{1cm} (27)

However, it is important to observe that the gauge group parametrized by $\exp(\frac{1}{2} \alpha(x))$ is noncompact rather than $U(1)$. This accounts for an imaginary term in the change of the action under local scaling which we shall address in Section VI. Moreover, here we do not assume that the underlying theory has this local scaling symmetry but rather possesses a global invariance under scaling. In fact, the Weyl fermion action is not invariant under such a local scaling but its Hermitized version is. As we shall see, this global symmetry is also broken in the quantum theory by regularization and is related to the conformal anomaly of the theory.
The ordinary divergence of the (densitized) ABJ or axial current

\[ J_0^\mu = - \bar{\Psi}_L \gamma^\mu \bar{\Psi}_L = - J^\mu \]  

is related to its covariant divergence and torsion through

\[
\partial_\mu (\bar{\Psi}_L \gamma^\mu \bar{\Psi}_L) = e [\partial_\mu (\bar{\Psi}_L \gamma^\mu \Psi_L) + \Gamma_{\mu}^{\nu} \bar{\Psi}_L \gamma^\nu \Psi_L] \\
= e \nabla_\mu (\bar{\Psi}_L \gamma^\mu \Psi_L) + 2e \Gamma_{[\nu\mu]}^\nu \bar{\Psi}_L \gamma^\mu \Psi_L. 
\]

Therefore, in terms of undensitized variables,

\[ \text{Im}(S^-) = \frac{i}{2} \int_M d^4 x e \nabla_\mu (\bar{\Psi}_L \gamma^\mu \Psi_L). \]  

We may note from Eq.(24) that the Abelian conformal gauge field is actually coupled to the fermion current \( J^\mu \) (hence the ABJ or axial current) which acts as the source for \( B_\mu \) when there is torsion.

### III. EQUATIONS OF MOTION

At first thought, we might suspect that replacing the Hermitized Weyl action by the non-Hermitized version would lead to a different Weyl equation and hence the theories would be inequivalent even classically when there is nonvanishing torsion. This might be expected because the two actions are not even related by just a boundary term when there is torsion. For instance, variation with respect to \( \bar{\Psi}_L \) for the Hermitized Weyl action \( \frac{1}{2} (S^- + S^-) \) yields the equation of motion (see also Ref. [11])

\[ ie\gamma^\mu (D_\mu + B_\mu) \Psi_L = 0 \]  

if we take into account the identity (18), whereas the Weyl action \( S^- \) gives

\[ ie\gamma^\mu D_\mu \Psi_L = 0. \]

However, the dynamics of the gravitational fields has so far not been taken into account. It is important to remember that in the Hermitized action, fermions couple to both left- and right-handed spin connection whereas

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\(^5\)All currents in this article are densitized tensors of weight 1.
in the Weyl action, only the left-handed spin connection is involved. In conventional theory, we introduce the Einstein-Hilbert-Palatini action which, together with the Hermitian fermion action, determine the classical dynamics of the vierbein and the full spin connection of both chiralities (or both dualities). For the non-Hermitized Weyl action we must therefore introduce the dynamics of gravitation in such a way that it involves only the left-handed or anti-self-dual spin connection. The simplest candidate for the gravitational part of such a theory is the Samuel-Jacobson-Smolin action \[3\]. Such an action is at least a reasonable candidate for low energy and classical gravitational dynamics. In fact, it can be argued that in any case the lowest order one-loop counterterm in the effective action when this non-Hermitized chiral fermion theory is quantized in curved background will be the anti-self-dual Samuel-Jacobson-Smolin action with cosmological constant. We shall next consider the theory specified by Eq. (1) augmented by this gravitational term.

What is remarkable is that the combined chiral Weyl and gravitational action gives precisely the same equations of motion for the vierbein, spin connection and fermions as the sum of the Einstein-Hilbert-Palatini and Hermitian Weyl actions. This is true despite the fact that these two versions of fundamental interactions are not even canonically related without further restrictions when there is nonvanishing torsion. To see this we may use the identity\[3]

\[
S^- + S^-_{JS} = -\frac{1}{16\pi G} \int_M \epsilon^A \wedge \epsilon^B \wedge \ast F_{AB} + \frac{2\lambda}{16\pi G} \int_M (\ast 1) + \frac{1}{2} (S^- + (S^-)^\dagger) + \frac{i}{2} \int_M d \left\{ \frac{1}{3!} (\epsilon_{ABCD} \Psi_L \gamma^A \Psi_L \epsilon^B \wedge \epsilon^C \wedge \epsilon^D) - \frac{1}{8\pi G} \epsilon^A \wedge T_A \right\} + \frac{i}{16\pi G} \int_M \Theta_A \wedge \Theta^A
\]

(33)

where

\[
\Theta_A = T_A + (2\pi G) \epsilon_{ABCD} \Psi_L \gamma^B \Psi_L \epsilon^C \wedge \epsilon^D,
\]

(34)

and \(\ast\) denotes the Hodge duality operator. The Samuel-Jacobson-Smolin

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\[6\] See for instance Ref. \[4\]. In comparing with Ref. \[4\], note that \(\Psi_L = \phi_L^\dagger \tau^A \phi_L\)

where in terms of left-handed two-component Weyl fermions, \(\Psi_L = \begin{bmatrix} 0 \\ \phi_L \end{bmatrix}\). The total action in Ref. \[4\] differs from the one used here by an overall sign.
which contains only $A_{AB}^- = \frac{1}{2}(-iA_{AB} + \frac{1}{2}\epsilon_{AC}{}^{CD}A_{CD})$ rather than the full spin connection $A_{AB}$ and the vierbein is

$$S_{SJS}^- = -\frac{i}{8\pi G}\int_M \Sigma^{-AB} \wedge F_{AB}^- + \frac{i\lambda}{3(16\pi G)}\int_M \Sigma^{-AB} \wedge \Sigma_{AB}^-$$

$$= -\frac{1}{16\pi G}\int_M \left\{e^A \wedge e^B \wedge *F_{AB} - 2\lambda(*)\right\} - \frac{i}{16\pi G}\int_M \left\{d(e^A \wedge T_A) - T_A \wedge T^A\right\}$$

(35)

with the anti-self-dual two-forms

$$F_{AB}^- = dA_{AB}^- + A_{AC}^- \wedge A_{BD}^-$$

$$= \frac{1}{2}(-iF_{AB} + \frac{1}{2}\epsilon_{AB}{}^{CD}F_{CD}),$$

$$\Sigma_{AB}^- = \frac{1}{2}(-ie_A \wedge e_B + \frac{1}{2}\epsilon_{AB}{}^{CD}e_C \wedge e_D).$$

(36)

$F_{AB}$ is the curvature of the full spin connection.

In Eq. (33), we observe that the first line, which is the Einstein-Hilbert-Palatini action plus the Hermitian fermion action, is the real part of the combined chiral action, while the imaginary part consists of a total divergence or boundary term and a term involving $\Theta$-squared. The equations of motion are such that when their real parts are satisfied, the imaginary parts give rise to no spurious or extra equations of motion. More precisely, on varying with respect to $A_{AB}^-$, we obtain

$$D^-\Sigma_{AB}^- = \frac{4\pi G}{3!} \Psi_L \gamma^C \sigma^{AB} \Psi_L \epsilon_{CDFG} e^D \wedge e^F \wedge e^G,$$

(37)

with unique solution

$$A_{AB}^- = \frac{1}{2}(-i\omega_{AB} + \frac{1}{2}\epsilon_{AB}{}^{CD}\omega_{CD}) - (2\pi G)\left\{J_{[A}e_{B]} + \frac{i}{2}\epsilon_{AB}{}^{CD}J_{[C}e_{D]}\right\},$$

(38)

where $\omega_{AB}$ is the torsionless $(de_A + \omega_{AB} \wedge e_B = 0)$ spin connection and $J_A = \bar{\Psi}_L \gamma_A \Psi_L$. Using the usual reality conditions that $\omega_{AB}, e_A$ and $J_A$ are real, we equate the real and imaginary parts of Eq. (38), and deduce that on-shell, $\Theta_A = 0$. Consequently, the nonboundary imaginary part of the action

---

The cosmological constant term is included here for completeness.
in Eq. (33) yields equations of motion which are automatically satisfied on-shell due to its dependence on $\Theta$-squared. Variations with respect to the vierbein and fermions must therefore produce the same set of equations of motion as from the Einstein-Hilbert-Palatini plus Hermitian Weyl action. In particular, note that $\Theta^A = 0$ leads to

$$T^A = -(2\pi G)\epsilon_{ABCD}\nabla_{L}^B \Psi_L e^C \wedge e^D,$$

and thus

$$B_\mu = -T_{A\mu} e^{\nu A}$$

vanishes when the on-shell value of torsion in Eq. (39) is substituted. This compensates for the discrepancies between Eq. (31) and Eq. (32) before the dynamics of the gravitational fields have been taken into account.

We may further work out from Eq. (33) that the complete imaginary part of the total action is

$$\text{Im}(S^- + S_{SJS}) = \frac{i}{2} \int_M (\partial_\mu J^\mu - 2B_\mu J^\mu) + \frac{i}{16\pi G} \int_M e^A \wedge e^B \wedge F_{AB},$$

The first two terms come from the fermionic Weyl action while the final term is the dual of the Einstein-Hilbert-Palatini action which is present in the Samuel-Jacobson-Smolin action.

**IV. REGULARIZATION**

In order to define the quantum field theory of fermions in background curved spacetimes, it is necessary to regularize divergent fermion loops. However, because chirality of the fermions and (anti-)self-duality of the spin connection are of the essence here, it is desirable to have a regularization which explicitly maintains not just the local symmetries but also the chirality and (anti-)self-dual coupling to gravity. To this end, various generic techniques such dimensional and zeta function regularizations fall short. Indeed to preserve explicit Lorentz and diffeomorphism invariance, spectator fields which do not couple to gravity are also not allowed. Furthermore, with only a single left-handed multiplet, Lorentz-invariant mass terms are Majorana in nature. The simple form of $\Psi_L^T C_A \Psi_L + H.c.$ is not invariant under internal symmetry transformations. With real representations, an invariant mass
term $m \Psi_L^T U C_4 \Psi_L + H.c.$ can be constructed from a single multiplet. However for complex representations, a gauge and Lorentz invariant mass term cannot be made out of a single multiplet. Thus in the standard model there can be no bare masses. This poses a challenge for invariant Pauli-Villars-Gupta regularization [12], even though the chiral fermions belong to an anomaly-free representation. A generalization of the method of Frolov and Slavnov using an infinite tower of anticommuting and commuting regulators which are doubled in internal space [13] has been shown to be a suitable regularization scheme [6]. This proposed regularization retains the chiral (Weyl) nature of the theory even to the extent of the coupling of matter to gravity in that no right-handed fields and right-handed spin connections are introduced.

In this Section, we recount the scheme and illustrate how it fits into the theory. The perturbative theory of the chiral fermion determinant is then defined through this regularization. A nonperturbative definition may also be possible if the “overlap” formulation for the chiral fermion determinant is suitably extended to curved spacetimes in such a manner that no right-handed spin connection and regulator fermions are introduced. In this regard, we do not advocate that doubling in internal gauge space is equivalent to doubling in external space through the introduction of right-handed regulator fermion fields [15]. While this may be true perturbatively for flat spacetime, to maintain local Lorentz invariance, the introduction of right-handed fermions must be accompanied by that of right-handed spin connections in curved spacetimes. The chiral coupling to gravity is thereby disturbed by such a regularization.

Doubling in internal gauge group space to form invariant masses for Pauli-Villars-Gupta regularization is achieved by including fermions which transform according to the $(-T^a)^*$ representation. An invariant mass term can be formed because under

$$\Psi_L^r \rightarrow e^{i\alpha_a T^a} \Psi_L^r, \quad \Psi_L^r \rightarrow e^{i\alpha_a (-T^a)^*} \Psi_L^r,$$

the combination $[(\Psi_L^r)^T C_4 \Psi_L^r + (\Psi_L^r)^T C_4 \Psi_L^r + H.c.]$ is invariant under internal gauge and Lorentz transformations. Introducing in the enlarged space the quantities

$$T^a \equiv \begin{pmatrix} (T^a)^* & 0 \\ 0 & T^a \end{pmatrix}, \quad \sigma^1 \equiv \begin{pmatrix} 0 & 1_d \\ 1_d & 0 \end{pmatrix}, \quad \sigma^3 \equiv \begin{pmatrix} 1_d & 0 \\ 0 & -1_d \end{pmatrix},$$

(42)

(43)
where \( d \) denotes the number of Weyls in the bare action, the original multiplet is projected as

\[
\begin{bmatrix}
0 \\
\Psi_L
\end{bmatrix} = \frac{1}{2}(12d - \sigma^3)\Psi_{L0}.
\]  

(44)

The mass terms for the regulator fermions,

\[
\Psi_{Lr} = \begin{bmatrix}
\Psi_{Lr}^+ \\
\Psi_{Lr}^-
\end{bmatrix},
\]  

(45)

can be written as \( m_r(\Psi_{Lr}^T \sigma^1 C_4 \Psi_{Lr} + H.c.) \). The doubled regulator fermion multiplets are to be coupled to the 2\(d\)-dimensional representation of the gauge connection, \( W_{\mu a} T^a \).

The \( \Psi_{Lr} \) fields are assumed to be anticommuting. Commuting doubled regulator fields \( \Phi_{Ls} \) are introduced in a similar manner. These have mass terms

\[
m_s \Phi_{Ls}^T (-i\sigma^2) C_4 \Phi_{Ls} = m_s \left[ -(\Phi_{Ls}^+)^T C_4 \Phi_{Ls}^- + (\Phi_{Ls}^-)^T C_4 \Phi_{Ls}^+ \right],
\]  

(46)

with

\[
-i\sigma^2 \equiv \begin{pmatrix} 0 & -1_d \\ 1_d & 0 \end{pmatrix} = \sigma^1 \sigma^3.
\]  

(47)

These invariant mass terms for the doubled anticommuting and commuting fields exist, because for the \( T^a \) representation, there is a symmetric (\( \sigma^1 \)) and an antisymmetric (\( -i\sigma^2 \)) matrix which satisfy

\[
(\sigma^1) T^a (\sigma^1)^{-1} = (-i\sigma^2) T^a (-i\sigma^2)^{-1} = (-T^a)^*.
\]  

(48)

Note that all the fields are left-handed.

The total regularized action which is explicitly gauge and Lorentz and, also diffeomorphism invariant is taken to be

\[
S_{F_{reg}} = \int d^4x \left[ \sum_{r=0,2,...} \left\{ \bar{\Psi}_{Lr} i\gamma^\mu \Psi_{Lr} + \frac{1}{2} m_r (\Psi_{Lr}^T \sigma^1 C_4 \Psi_{Lr} + \bar{\Psi}_{Lr} \sigma^1 C_4 \bar{\Psi}_{Lr}^T) \right\} \right.

- \left. \sum_{s=1,3,...} \left\{ \bar{\Phi}_{Ls} \sigma^3 i\gamma^\mu \Phi_{Ls} + \frac{1}{2} m_s (\Phi_{Ls}^T \sigma^1 C_4 \Phi_{Ls} + \bar{\Phi}_{Ls} C_4 \bar{\Phi}_{Ls}^T \sigma^1 \sigma^3) \right\} \right].
\]  

(49)

\[\text{We also allow all the fields to transform under general coordinate transformations. Here, we regularize only fermion loops in background fields, and do not address the question of the regularization of the gauge and gravitational fields. Gauge propagators may be regularized by other methods. Full quantum gravity effects are beyond the scope of this paper.}\]
The sums are over all even natural numbers for the anticommuting fields and over all odd natural numbers for the commuting fields. The usefulness of this convention will become apparent (see for instance Eq. (54).) With the exception of

$$\Psi_{L_0} = \frac{1}{2}(1 - \sigma^3)\Psi_{L_0} = \begin{bmatrix} 0 \\ \Psi_{L_0} \end{bmatrix},$$

(50)

which is the original and undoubled chiral massless \((m_0 = 0)\) fermion multiplet, all other anticommuting \(\Psi_{L_r}\) and commuting \(\Phi_{L_s}\) multiplets are generalized Pauli-Villars-Gupta regulator fields, doubled in internal space, and endowed with Majorana masses, which we take for definiteness to satisfy \(m_n = n\Lambda\). Due to the fact that all the multiplets are left-handed, there are no couplings to the right-handed spin connection which does not need to be introduced for the Weyl action.

As detailed in Ref. [6], the original gauge current coupled to \(W_{\mu a}\) which

$$J_{\mu a} = \frac{\delta S_F}{\delta W_{\mu a}} = \tilde{\Psi}_{L_0} \gamma^\mu T^a \frac{1}{2}(1 - \sigma^3)\tilde{\Psi}_{L_0},$$

(51)

is modified by the regulators to

$$J_{\mu a} = \tilde{\Psi}_{L_0} \gamma^\mu T^a \frac{1}{2}(1 - \sigma^3)\tilde{\Psi}_{L_0} + \sum_{r=2,4,...} \tilde{\Psi}_{L_r} \gamma^\mu T^a \tilde{\Psi}_{L_r} + \sum_{s=1,3,...} \tilde{\Phi}_{L_s} \gamma^\mu T^a \tilde{\Phi}_{L_s},$$

(52)

As with conventional Pauli-Villars-Gupta regularization, the regularized composite current operator is summarized by

$$\langle J_{\mu a}(x) \rangle_{reg} = \lim_{x \to y} Tr\{-\gamma^\mu(x)T^a \frac{1}{2}(1 - \sigma^3)\langle T\{\tilde{\Psi}_{L_0}(x)\tilde{\Psi}_{L_0}(y)\}\rangle + \sum_{r=2,4,...} \langle T\{\tilde{\Psi}_{L_r}(x)\tilde{\Psi}_{L_r}(y)\}\rangle + \sigma^3 \sum_{s=1,3,...} \langle T\{\tilde{\Phi}_{L_s}(x)\tilde{\Phi}_{L_s}(y)\}\rangle\}$$

(53)

with the trace running over Dirac and Yang-Mills indices. As a result

$$\langle J_{\mu a}(x) \rangle_{reg} = \lim_{x \to y} Tr\{\gamma^\mu(x)T^a P_L \frac{1}{2}(1 - \sigma^3)(i\slashed{\mathcal{D}}) \frac{1}{(i\slashed{\mathcal{D}})(i\slashed{\mathcal{D}})^\dagger} + \sum_{r=2,4,...} \langle i\slashed{\mathcal{D}} \rangle \frac{1}{r^2\Lambda^2 + (i\slashed{\mathcal{D}})(i\slashed{\mathcal{D}})^\dagger} - \sum_{s=1,3,...} \langle i\slashed{\mathcal{D}} \rangle \frac{1}{s^2\Lambda^2 + (i\slashed{\mathcal{D}})(i\slashed{\mathcal{D}})^\dagger}\delta(x - y)\}$$

15
\[
\lim_{x \to y} Tr \left\{ \gamma^\mu(x) T^a \frac{1}{2} P_L \left[ \frac{1}{i\not\mathcal{D}} \left( \sum_{n=-\infty}^{\infty} \frac{(-1)^n \not\mathcal{D} \not\mathcal{D}^\dagger}{n^2 \Lambda^2 + \not\mathcal{D} \not\mathcal{D}^\dagger} - \sigma^3 \right) \right] \delta(x-y) \right\} \\
\equiv \lim_{x \to y} Tr \left\{ \gamma^\mu(x) T^a \frac{1}{2} P_L \left[ \frac{1}{i\not\mathcal{D}} \left( f(\not\mathcal{D} \not\mathcal{D}^\dagger / \Lambda^2) - \sigma^3 \right) \right] \delta(x-y) \right\}.
\]

(54)

In the above \( n \) is summed over all integers, \( i\not\mathcal{D} \equiv e^{\pi i / 4} \not\mathcal{D} e^{-\pi i / 4} \) and \( P_L \equiv \frac{1}{2}(1 - \gamma^3) \).

The effect of the tower of regulators is to replace the divergent bare expression

\[
(J^{\mu a})_{\text{bare}} = \lim_{x \to y} Tr \left\{ \gamma^\mu(x) T^a P_L \left[ \frac{1}{i\not\mathcal{D}} \frac{1}{2} \left( 1 - \sigma^3 \right) \right] \delta(x-y) \right\},
\]

(55)

by

\[
(J^{\mu a})_{\text{reg}} = \lim_{x \to y} Tr \left\{ \gamma^\mu(x) T^a \frac{1}{2} P_L \left[ \frac{1}{i\not\mathcal{D}} \left( f(\not\mathcal{D} \not\mathcal{D}^\dagger / \Lambda^2) - \sigma^3 \right) \right] \delta(x-y) \right\}.
\]

(56)

This general feature of the effect of the tower shows up in all the regularized currents.

The regulator function

\[
f(z) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n z}{n^2 + z} = \frac{\pi \sqrt{z}}{\sinh(\pi \sqrt{z})}
\]

has the required properties [13, 15] to ensure convergence. For instance, it falls rapidly to zero as \( z \to \infty \). When the regulator masses are taken to \( \infty \), \( f(0) = 1 \). The \( \sigma^3 \) part of the current remains unmodified, essentially because the tower consists of regulators which are doubled in internal space and is “\( \sigma^3 \) neutral”. It, therefore, can regularize only the singlet part of the \( \frac{1}{2}(1 - \sigma^3) \) projection of the bare current. However, when the representation satisfies the anomaly cancellation conditions for perturbative chiral gauge [16] and mixed Lorentz-gauge anomalies [17] i.e.

\[
Tr \left( T^a \left\{ T^b, T^c \right\} \right) = 0
\]

(58)
and
\[ T_r(T^a) = 0 \] (59)
respectively, the \( \sigma^3 \) part gives rise to no further divergences and, the current is successfully regularized by the tower of regulators. That all this is true is explained in Ref. [9].

We turn next to the gravitational currents. The bare spin current coupled to \( A_{\mu AB} \) is
\[ J^{\mu AB} = \bar{\Psi}_L \gamma^\mu \frac{i}{2} \sigma^{AB} P_L \frac{1}{2} (1 - \sigma^3) \Psi_L. \] (60)

Similarly, the regularized expression is
\[ \langle J^{\mu AB} \rangle_{\text{reg}} = \lim_{x \to y} T_r \left\{ \gamma^\mu(x) \frac{i}{2} \sigma^{AB} P_L \left[ \frac{1}{2} \frac{1}{\mathcal{D}} \left( f(f(D/D^\dagger/\Lambda^2) - \sigma^3) \right) \right] \delta(x - y) \right\}. \] (61)

Various proposals for defining the energy momentum tensor have been suggested [10]. If the classical bare action is regarded as \( S^{-}[\bar{\Psi}_L, \Psi_L, e_{\mu A}, W_{\mu a}, A_{\mu AB}^{-}] \) in the first line of Eq.(1), then the energy momentum tensor \( \Theta_{\mu \nu} \) is obtained from
\[ e \Theta_{\mu \nu} = e_{\mu A} \frac{\delta S^{-}}{\delta e^\nu_A} = \bar{\Psi}_L \gamma_{\mu} iD_{\nu} \Psi_L - g_{\mu \nu} \mathcal{L}, \] (62)
where \( \mathcal{L} \) is the Lagrangian. On the other hand, if the variables \( \bar{\Psi}_L \) and \( \Psi_L \) are to be treated as independent integration variables as is suggested by the diffeomorphism-invariant measure (20), then the energy momentum tensor \( T_{\mu \nu} \) regarded as the source current for the vierbein is
\[ e T_{\mu \nu} = e_{\mu A} \frac{\delta S^{-}}{\delta e^\nu_A} \] (63)
with
\[ S^{-}[\bar{\Psi}_L, \Psi_L, e_{\mu A}, W_{\mu a}, A_{\mu AB}^{-}] = \int d^4 x \bar{\Psi}_L e^{\frac{1}{2} iD e^{-\frac{1}{2} \bar{\Psi}_L}} \]
\[ = \int d^4 x \bar{\Psi}_L E^\mu_A \gamma^A \left[ iD_{\mu} - \frac{i}{2} (\partial_{\mu} \ln e) \right] \Psi_L. \] (64)
The expression for the corresponding energy-momentum tensor is then

\[ eT_{\mu\nu} = \bar{\Psi}_L \gamma_\mu i \left(D_\nu - \frac{1}{2} \Gamma^\alpha_{\alpha\nu}\right) \Psi_L - \frac{i}{2} g_{\mu\nu} \partial_\alpha (\bar{\Psi}_L \gamma^\alpha \Psi_L). \]  

(65)

As a result, \( T_{\mu\nu} \) and \( \Theta_{\mu\nu} \) are related by

\[ eT_{\mu\nu} = e\Theta_{\mu\nu} + \frac{1}{2} g_{\mu\nu} \left( \bar{\Psi}_L \frac{\delta S^-}{\delta \bar{\Psi}_L} + \bar{\Psi}_L \frac{\delta S^-}{\delta \Psi_L} \right). \]  

(66)

The difference between the two is therefore not significant classically when the equations of motion can be imposed. However, at the quantum level, there can be subtleties [10]. Due to the choice of the densitized variables, all bare mass terms and, in particular, regulator bare mass terms, are independent of the vierbein and therefore do not contribute to \( T_{\mu\nu} \). The regularized expression consequently becomes

\[ \langle eT_{\mu\nu}\rangle_{\text{reg}} = \lim_{x \to y} Tr \left\{ \frac{\gamma_\mu i \left(D_\nu - \frac{1}{2} \Gamma^\alpha_{\alpha\nu}\right) \frac{1}{2} P_L \frac{1}{2} \left[f \left( \frac{\mathcal{D}\mathcal{D}^\dagger}{\Lambda^2} \right) - \sigma^3 \right] \delta(x - y) \right\} \right. 

- \left. \frac{i}{2} g_{\mu\nu} \langle \partial_\alpha J^\alpha \rangle_{\text{reg}}, \right. \]

(67)

where \(-J^\alpha = J_5^\alpha\) is the ABJ current.

Again, the \( \sigma^3 \) part of the energy-momentum tensor gives rise to no divergent fermion loops if conditions (58) and (59) hold. Hence the expression for the energy-momentum tensor is regularized for finite \( \Lambda \).

The regularized trace of the energy-momentum tensor is therefore

\[ \langle eT^\mu_{\mu}\rangle_{\text{reg}} = \lim_{x \to y} Tr \left\{ P_L \frac{1}{2} \left[f \left( \frac{\mathcal{D}\mathcal{D}^\dagger}{\Lambda^2} \right) - \sigma^3 \right] \delta(x - y) \right\} - 2i \langle \partial_\mu J^\mu \rangle. \]  

(68)

In our present discussion, we do not densitize the background variables and eschew use, for instance, of \( W_{Aa} \equiv e^{\frac{i}{2} E^a_A W_{\mu a}} \) instead of \( W_{\mu a} \). This choice would be useful if an explicitly diffeomorphism invariant measure \( \prod DW_{Aa} \) is required when the path integral formalism is to be applied to the quantization of the gauge fields [10].

The energy-momentum tensor should be symmetrized if it is to be regarded as the source of the metric. Here in the first order formulation, the

\[\text{In terms of variables which are not densitized, } T_{\mu\nu} = \bar{\Psi}_L \gamma_\mu i D_\nu \Psi_L - \frac{i}{2} g_{\mu\nu} \left[ \partial_\alpha (\bar{\Psi}_L \gamma^\alpha \Psi_L) + \Gamma^\alpha_{\beta\sigma} (\bar{\Psi}_L \gamma^\alpha \Psi_L) \right].\]
antisymmetric part of the energy-momentum tensor serves as the source of
the spin current (see also Eq.(84)). It is known that there are no perturbative
Lorentz anomalies in four dimensions [18]. This is verified by the explicitly
Lorentz-invariant regularization scheme proposed here and can indeed be
checked by the explicit verification of the consistency condition for the ab-
sence of Lorentz anomalies. This condition is derived later on in Section
V.

To summarize, all the currents carry left-handed projections and are
regularized by the scheme for finite Λ. Fermion loops are generated by the
multipoint correlation functions obtained by functional differentiation of the
regularized currents in Eqs. (56), (61) and (67). We may further note the
explicit role of the ABJ anomaly in Eqs. (67) and (68).

Under a chiral $\gamma^5$ rotation,
\[
\tilde{\Psi}_{L_r} \rightarrow e^{i\alpha\gamma^5} \tilde{\Psi}_{L_r} = e^{-i\alpha} \tilde{\Psi}_{L_r},
\]
and similarly for $\tilde{\Phi}_{L_s}$ and $\tilde{\Phi}_{L_s}$. Kinetic terms are invariant under this global
transformation, but mass terms are not. The bare massless action is invariant
under such a global transformation, and the associated ABJ or $\gamma^5$ current
\[
J_5^\mu = \tilde{\Psi}_{L_0} \gamma^5 \tilde{\Psi}_{L_0} = -\tilde{\Psi}_{L_0} \gamma^\mu \tilde{\Psi}_{L_0} = -J_F^\mu,
\]
is conserved classically, i.e., $\partial_\mu J_5^\mu = 0$. However, the bare quantum com-
posite current
\[
\langle J_5^\mu \rangle_{\text{bare}} = -\lim_{x \rightarrow y} Tr \left\{ \gamma^\mu(x) P_L \left[ \frac{1}{i\partial^\mu} \left( 1 - \sigma^3 \right) \right] \right\}
\]
is divergent. The regularized current is not necessarily conserved. In the
generalized Pauli-Villars-Gupta scheme, the mass terms of the regulators
break the symmetry explicitly. The expectation value of the regularized
ABJ current is
\[
\langle J_5^\mu(x) \rangle_{\text{reg}} = -\lim_{x \rightarrow y} Tr \left\{ \gamma^\mu(x) \frac{1}{2} (1 - \gamma^5) \frac{1}{i\partial^\mu} \left( f(\mathcal{D}^\dagger/\Lambda^2) - \sigma^3 \right) \delta(x - y) \right\}.
\]
The previous arguments concerning the unregulated $\sigma^3$ part are still valid.
Within this context, we have in effect regularized the ABJ current, and the
associated amplitudes can be computed explicitly.
The ABJ anomaly can be explicitly computed by taking the divergence of the expectation value of the regularized expression in Eq. (72) as

\[ \langle \partial_\mu J_5^\mu \rangle_{\text{reg}} = \partial_\mu \lim_{x \to y} Tr \left\{ -\gamma_5 \frac{1}{2} (1 - \gamma_5^5) \frac{1}{i\not{D}} \left( f(\not{D}\not{D}/\Lambda^2) - \sigma^3 \right) \delta(x - y) \right\}. \]

(73)

The trace can be evaluated by using the complete sets of eigenvectors, \( \{X_n\} \) and \( \{Y_n\} \), of the positive-semidefinite Hermitian operators with

\[ \not{D}\not{D}^\dagger X_n = \lambda_n^2 X_n, \]
\[ \not{D}^\dagger \not{D} Y_n = \lambda_n^2 Y_n. \]

(74)

Consequently,

\[ \langle \partial_\mu J_5^\mu \rangle_{\text{reg}} = \lim_{\Lambda \to \infty} \frac{i}{4} \sum_n \left[ Y_n^\dagger \gamma_5 f(\not{D}\not{D}/\Lambda^2) Y_n + X_n^\dagger \gamma_5 f(\not{D}\not{D}/\Lambda^2) X_n \right]. \]

(75)

For Euclidean signature, this works out to be

\[ \langle \partial_\mu J_5^\mu \rangle = \frac{i \times d}{768 \pi^2} F_{\alpha\beta a} \epsilon^{\alpha\beta\mu\nu} F_{\mu\nu}^{AB} + \frac{i}{32\pi^2} Tr(\epsilon^{\alpha\beta\mu\nu} G_{\alpha\beta a} T^a G_{\mu\nu b} T^b) \]

(76)

in the absence of torsion.\(^{\text{11}}\) In the above, \( G_{\mu
u a} \) and \( F_{\mu\nu AB} \) are, respectively, the curvatures of \( W_{\mu a} \) and \( A_{\mu AB} \). This gives the result which is one-half of the chiral anomaly of a vector theory. Because all the fields are Weyl, the factor we get for the gravitational part is also \( d \) rather than \( 2d \). This is in agreement with the fact that there are \( d \) Weyl fermions coupled to gravity in the bare action.

\(^{\text{11}}\)When there is torsion, an additional contribution which diverges as the regulator masses are taken to infinity is present. The associated counterterm will be discussed in Section VII.
V. EFFECTIVE ACTION AND CONSERVATION EQUATIONS

In order to derive the conservation equations associated with the local symmetries of the quantum theory, we may consider the generating function
\[ Z = \exp(-\Gamma_{\text{eff.}}[E^{\mu A}, A_{\mu AB}^-, W_{\mu a}]) \]
\[ = \int D\tilde{\Psi}_L D\Psi_L \exp(-S^{-}[\tilde{\Psi}_L, \tilde{\Psi}_L, E^{\mu A}, A_{\mu AB}^-, W_{\mu a}]). \]  
(77)

Under a change of integration variables,
\[ \tilde{\Psi}_L \rightarrow \tilde{\Psi}_L', \quad \tilde{\Psi}_L' \rightarrow \tilde{\Psi}_L', \]  
(78)
there is no change in the partition function if there are no anomalous Jacobians in the measure.\(^1\) Thus
\[ 0 = \delta \ln Z = -\int_M (\delta \tilde{\Psi}_L \frac{\delta S^{-}}{\delta \tilde{\Psi}_L} + \delta \tilde{\Psi}_L \frac{\delta S^{-}}{\delta \Psi_L}). \]  
(79)

But under simultaneous transformations
\[ E^{\mu A} \rightarrow E'^{\mu A} \]
\[ A_{\mu AB} \rightarrow A'_{\mu AB} \]
\[ W_{\mu a} \rightarrow W'_{\mu a} \]
\[ \tilde{\Psi}_L \rightarrow \tilde{\Psi}_L', \quad \tilde{\Psi}_L' \rightarrow \tilde{\Psi}_L' \]  
(80)
which correspond to symmetries of the action,
\[ \delta S^{-} = \int_M (\delta \tilde{\Psi}_L \frac{\delta S^{-}}{\delta \tilde{\Psi}_L} + \delta \tilde{\Psi}_L \frac{\delta S^{-}}{\delta \Psi_L}) + \delta E^{\mu A} \frac{\delta S^{-}}{\delta E^{\mu A}} + \delta A_{\mu AB} \frac{\delta S^{-}}{\delta A_{\mu AB}} + \delta W_{\mu a} \frac{\delta S^{-}}{\delta W_{\mu a}} = 0. \]  
(81)

\(^1\)The measure may be defined by expansion in terms of the complete sets of eigenvectors \(\{X_n\}\) and \(\{Y_n\}\) as in Eq.(74).
Therefore for such symmetry transformations

\[- \delta \Gamma_{\text{eff.}} = \delta \ln Z = \int_M \left( \langle \delta E_{\mu A} \frac{\delta S^-}{\delta E_{\mu A}} + \delta A_{\mu AB} \frac{\delta S^-}{\delta A_{\mu AB}} + \delta W_{\mu a} \frac{\delta S^-}{\delta W_{\mu a}} \rangle \right) \]

\[= - \int_M \left( \frac{\delta \Gamma_{\text{eff.}}}{\delta E_{\mu A}} \delta E_{\mu A} + \frac{\delta \Gamma_{\text{eff.}}}{\delta A_{\mu AB}} \delta A_{\mu AB} + \frac{\delta \Gamma_{\text{eff.}}}{\delta W_{\mu a}} \delta W_{\mu a} \right) \]

\[= - \int_M \left( \langle E^\nu A \epsilon T_{\nu \mu} \rangle \delta E_{\mu A} + \langle J^{\mu AB} \rangle \delta A_{\mu AB} + \langle J^{\mu a} \rangle \delta W_{\mu a} \right) \]

\[= 0, \quad (82) \]

since

\[\langle E^\mu A \epsilon T_{\mu \nu} \rangle = \frac{\delta \Gamma_{\text{eff.}}}{\delta E^\nu A}, \]

\[\langle J^{\mu AB} \rangle = \frac{\delta \Gamma_{\text{eff.}}}{\delta A_{\mu AB}}, \]

\[\langle J^{\mu a} \rangle = \frac{\delta \Gamma_{\text{eff.}}}{\delta W_{\mu a}}. \quad (83) \]

The resultant conservation equations for local gauge, Lorentz and diffeomorphism symmetries are, respectively,

\[\langle D_\mu J^{\mu a} \rangle = 0, \]

\[E^{[\mu A} E^{\nu B]} \langle \epsilon T_{\mu \nu} \rangle + \langle D_\mu J^{\mu AB} \rangle = 0 \quad (84)\]

and

\[- A^{-}_{\alpha AB} \left\{ E^{[\mu A} E^{\nu B]} \langle \epsilon T_{\mu \nu} \rangle + \langle D_\mu J^{\mu AB} \rangle \right\} - \langle W_{\alpha a} D_\mu J^{\mu a} \rangle \]

\[+ \langle \partial_\mu (\epsilon T^\mu \alpha) - \epsilon \Gamma_{\alpha \mu \nu} T^\mu \nu \rangle + \langle G_{\alpha a} J^{\mu a} \rangle + \langle F^{-}_{\alpha \mu AB} J^{\mu AB} \rangle = 0 \quad (85)\]

with $F^{-}_{\mu \nu AB}$ and $G_{\mu \nu a}$ being the respective curvatures of $A^{-}_{\mu AB}$ and $W_{\mu a}$.  

\[\text{22} \]
The first equation is just the condition for the gauge current to be conserved. In the second equation, note that the antisymmetric part of the energy-momentum tensor acts as the source of the spin current. The final equation is the complete expression for invariance under local infinitesimal diffeomorphisms in the first order formulation when there are also couplings to internal gauge fields. This expression is more involved the familiar condition

$$\langle \nabla_\mu T^\mu \alpha \rangle = 0 \quad (86)$$

for scalar fields when the gravitational coupling is only through the metric. The expression given here for fermion theories in the first order formulation agrees with that of Nieh and Yan [11]. Note that

$$\partial_\mu (eT^\mu \alpha) - e\Gamma^\nu_{\alpha \mu} T^\mu \nu = e(\nabla_\mu T^\mu \alpha + 2B_\mu T^\mu \alpha). \quad (87)$$

We may also observe that all the currents defined here are left-handed and therefore only the left-handed spin connection is projected in Eqs. (84) and (85).

Equipped with a regularization scheme, we can check explicitly that these equations for the expectation values which ensure the local symmetries of the theory are free of anomalies, are indeed satisfied. In particular, we can check that the equations for Lorentz and diffeomorphism invariance hold thus ensuring no inconsistencies or perturbative anomalies for the non-Hermitian Weyl theory studied here, despite the unfamiliar appearances of the ABJ current and ABJ anomaly in the imaginary parts of the spin current and energy-momentum tensor. This is in agreement with the fact that an explicit gauge as well as Lorentz and diffeomorphism invariant regularization scheme can be found for the theory [3].

In the next section we shall consider the case of global ABJ and scaling symmetries when there are anomalous Jacobians in the measure. Note that the Pauli-Villars regularization scheme discussed earlier explicitly preserves the local symmetries of the theory but also explicitly violate global scaling and $\gamma^5$ transformations since the regulator mass terms are not invariant under these. Thus the scheme also provides consistent computations of these anomalies for the Weyl theory.
VI. ANOMALOUS SYMMETRIES

In addition to the local gauge, Lorentz and diffeomorphism symmetries, the
bare action is also invariant under global $\gamma^5$ and scaling transformations.
Classically this results in the conservation of the ABJ current and the trace-
lessness of the energy-momentum tensor. However, quantum mechanically
these symmetries are violated by the regularization which breaks these sym-
metries explicitly.

In the path integral approach, the anomalies are related to the nontrivial
Jacobians of the fermion measure [19]. To obtain the relation between
the anomalous Jacobians and the expectation values of the currents, we re-
call that the partition function $Z$ is invariant under an arbitrary change of
fermion integration variables. However if there is an anomalous Jacobian in
the measure which transforms as

$$D\tilde{\Psi}_L D\tilde{\Psi}_L \rightarrow D\tilde{\Psi}'_L D\tilde{\Psi}'_L = \exp(A) D\tilde{\Psi}_L D\tilde{\Psi}_L,$$

then

$$Z = \int \exp(A) D\tilde{\Psi}_L D\tilde{\Psi}_L \exp(-S - [\tilde{\Psi}'_L, \tilde{\Psi}'_L, e_A, A^+_A, W_a]),$$

since the partition function is unaltered by a change of integration variables.

For infinitesimal transformations, we have

$$0 = \delta Z$$

$$= (A)Z - \int D\tilde{\Psi}_L D\tilde{\Psi}_L \int_M (\delta\tilde{\Psi}_L \frac{\delta S^-}{\delta\tilde{\Psi}_L} + \delta\tilde{\Psi}_L \frac{\delta S^-}{\delta\tilde{\Psi}_L}) \exp(-S - [\tilde{\Psi}_L, \tilde{\Psi}_L, e_A, A^+_A, W_a]).$$

(90)

This yields the sought-after relation

$$A = \langle \int_M (\delta\tilde{\Psi}_L \frac{\delta S^-}{\delta\tilde{\Psi}_L} + \delta\tilde{\Psi}_L \frac{\delta S^-}{\delta\tilde{\Psi}_L}) \rangle.$$  

(91)

In the case of $\gamma^5$ transformations,

$$\delta\tilde{\Psi}_L = i\alpha\tilde{\Psi}_L, \quad \delta\tilde{\Psi}_L = -i\alpha\tilde{\Psi}_L,$$

(92)
and

\[-i\int_M \alpha A^5 = \langle \int_M i\alpha (\overline{\Psi}_L \frac{\delta S}{\delta \overline{\Psi}_L} - \overline{\Psi}_L \frac{\delta S}{\delta \overline{\Psi}_L}) \rangle \]  (93)

with \( \mathcal{A} \) written as \(-i\int_M \alpha A^5\). The ABJ anomaly due to the nontrivial Jacobian is therefore

\[A^5 = -\langle i\partial_\mu (\overline{\Psi}_L \gamma^\mu \overline{\Psi}_L) \rangle.\]  (94)

Given a regularization scheme, we can actually evaluate the anomaly as the expectation value of the divergence of the regularized current.

In the case of scaling,

\[\delta \overline{\Psi}_L = \frac{1}{2} \alpha \overline{\Psi}_L, \quad \delta \overline{\Psi}_L = \frac{1}{2} \alpha \overline{\Psi}_L,\]  (95)

and writing \( \mathcal{A} = \int_M \alpha A \) yields

\[\int_M \alpha A = \frac{1}{2} \int_M \alpha (\overline{\Psi}_L \frac{\delta S}{\delta \overline{\Psi}_L} + \overline{\Psi}_L \frac{\delta S}{\delta \overline{\Psi}_L}) \) \]  (96)

Hence the anomaly associated with the nontrivial Jacobian under scaling is

\[A = \frac{1}{2} \left( \left( \overline{\Psi}_L e^{\frac{1}{2}i\psi} e^{-\frac{1}{2}i\psi} \overline{\Psi}_L - i\partial_\mu (\overline{\Psi}_L \gamma^\mu) \overline{\Psi}_L + \overline{\Psi}_L \gamma^\mu \left( \frac{i}{2} A_{\mu AB} \sigma^{AB} - \frac{i}{2} \partial_\mu \ln e + W_\mu a T^a \right) \overline{\Psi}_L \right) \right).\]  (97)

By comparing with the trace of Eq. (65), we see that the anomaly from the nontrivial Jacobian under scaling of the integration variables is related to the trace of the energy-momentum tensor (see also Eq. (107)) by

\[A = \langle e T^\mu_{\mu} + i \frac{3}{2} \partial_\mu J^\mu \rangle.\]  (98)

There is an additional term proportional to the divergence of the ABJ current which is nonvanishing quantum mechanically. This relation can also be deduced by considering the simultaneous local scaling transformations as in (26) and (27). Under such a scaling the effective action \( \Gamma_{\text{eff.}} \) changes by

\[\delta \Gamma_{\text{eff.}} = \int_M \langle e T\nu_\mu E^{\nu A} \rangle \delta E^\mu_A \]

\[= \int_M \langle e T\nu_\mu E^{\nu A} \rangle g^{\nu\alpha} e_\alpha^A \delta E^\mu_A \]

\[= -\frac{1}{4} \int_M \langle e T^\mu_{\mu} \rangle \delta (\ln e) \]
\begin{equation}
\langle e T^\mu_{\mu} \rangle = -\int_M \langle \alpha e T^\mu_{\mu} \rangle \tag{99}\end{equation}

since \( \delta \ln e = 4\alpha \) under scaling. But the Weyl action changes by

\begin{equation}
S^- \to S^- - i\frac{3}{2} \int_M (\partial_\mu \alpha) J^\mu \tag{100}\end{equation}

under local scale transformations. Thus, under simultaneous scale transformations we have

\[
\exp(-\Gamma_{\text{eff.}}[\exp(\alpha)e_A, A_{AB}, W_a]) = \int \exp(\int_M \alpha A) D\bar{\Psi}_L D\Psi_L \exp(-S^- - i\frac{3}{2} \int_M \alpha \partial_\mu J^\mu). \tag{101}\]

For infinitesimal transformations, the result is therefore

\[
\langle e T^\mu_{\mu} \rangle = -\frac{\delta \Gamma_{\text{eff.}}}{\delta \alpha} = A + i\frac{3}{2} A^5 \tag{102}\]

which is in complete agreement with the relation derived earlier. \( A \) can be computed from the regularized expressions of \( \langle e T^\mu_{\mu} \rangle \) and \( \langle \partial_\mu J^\mu \rangle \).

Had the Hermitized Weyl action been used, then under local scaling the action is invariant, and the result will be that the anomalous Jacobian agrees with the trace of the energy-momentum with no extra imaginary ABJ-anomaly contribution. Another way to understand this is to note that as far as scaling is concerned, the Weyl action differs from the Hermitized version by a scale-noninvariant term

\[
- i\int_M B_\mu J^\mu \rightarrow - i\int_M B_\mu J^\mu - i\frac{3}{2} \int_M (\partial_\mu \alpha) J^\mu \tag{103}\]

since \( B_\mu \) changes by \( \frac{3}{2} \partial_\mu \alpha \) while \( J^\mu \) is invariant under local scaling. Thus these differences in scaling behaviour from the Hermitized theory offer further physical avenues to test the validity of the Weyl theory.
VII. CPT VIOLATION

In order to couple fermions to the four forces in a completely chiral fashion without introducing the right-handed spin connection we do not Hermitize the Weyl action. The difference between the Weyl action and the Hermitian version, as has been addressed in Sections II and III, involves the divergence of the ABJ current and also torsion terms. This difference is subtle because classically, the ABJ current is conserved while torsion is also zero for most familiar background solutions in general relativity. Moreover, within the context of Section III, $B_\mu$ is zero on-shell. However, quantum mechanically, the ABJ anomaly exists and off-shell torsion in the presence of fermions cannot be guaranteed to vanish. As a result, among other things, the energy-momentum tensor and spin current presented here acquire imaginary terms (in Lorentzian signature spacetimes). These originate precisely from the non-Hermiticity of the Weyl Lagrangian whose anti-Hermitian part is not of the form of a ordinary divergence but has local contributions when there is torsion. Since the expectation value of the divergence of the ABJ current is not zero quantum mechanically, there are subtle violations of discrete symmetries due to the ABJ current and ABJ anomaly in the presence of topologically nontrivial gauge and gravitational instantons, and also non-vanishing torsion. It is easy to check that the imaginary part of the Weyl action is CP and CPT-odd since it is local and Lorentz invariant.\(^\text{13}\) Thus the Weyl action while obeying all the local gauge, Lorentz and diffeomorphism symmetries of the theory violates discrete symmetries such as CP and CPT in contrast with the Hermitized Weyl and Majorana theories.

In addition to possible nonperturbative violations due to instantons, the precise perturbative and local processes which are involved are contained in the imaginary parts of the spin current and the energy-momentum tensor and the fermion loops generated by them. All these processes involve torsion and the ABJ current and originate from the $B_\mu J^\mu$ coupling in the imaginary part of the action.

To isolate these processes we decompose the energy-momentum and spin current into their Hermitian and anti-Hermitian parts.

\[
e T_{\mu\nu} = \bar{\Psi}_L \gamma_\mu [iD_\mu - \frac{i}{2} (\partial_\mu \ln e)] \Psi_L - \frac{i}{2} g_{\mu\nu} \partial_\alpha J^\alpha
\]

\(^\text{13}\)Details can also be found in Refs. [3, 4].
\[ eT_{\mu\nu} = \frac{1}{2}(eT_{\mu\nu} + (eT_{\mu\nu})^\dagger) + i \text{Im}(eT_{\mu\nu}) \]  

where

\[ \frac{1}{2}(eT_{\mu\nu} + (eT_{\mu\nu})^\dagger) = \bar{\Psi}_L \gamma_\mu [iD_\nu - \frac{i}{2}(\partial_\nu \ln e)] \Psi_L \]

\[ -i(\partial_\nu \bar{\Psi}_L) \gamma_\mu \bar{\Psi}_L + \frac{i}{2} \bar{\Psi}_L [(\partial_\nu \ln e) + A_{\nu AB}\sigma^{AB}] \gamma_\mu \bar{\Psi}_L \]  

and

\[ i \text{Im}(eT_{\mu\nu}) = -\frac{i}{2}g_{\mu\nu}\partial_{\alpha}J^{\alpha} + \frac{i}{2} \left[ \partial_\nu J_\mu - \Gamma^{\alpha}_{\mu\nu}J_\alpha - \Gamma^{\alpha}_{\alpha\nu}J_\mu \right] \]  

with \( J_\mu = g_{\mu\nu} \bar{\Psi}_L \gamma^\nu \Psi_L \).

This implies that the trace of the energy-momentum tensor which is related to the conformal anomaly also picks up an imaginary term. The full expression is

\[ eT^\mu_\mu = \bar{\Psi}_L \gamma^\mu \left[ i \partial_\mu + \frac{i}{2} A_{\mu AB}\sigma^{AB} - \frac{i}{2} \partial_\mu \ln e + W_{\mu a} T^a \right] \Psi_L - 2i \partial_\mu(\bar{\Psi}_L \gamma^\mu \Psi_L) \]

\[ = \frac{1}{2}(\bar{\Psi}_L \gamma^\mu e^{\frac{1}{2}iD_\mu e^{-\frac{1}{2}iD_\mu}} - \frac{3}{2} \partial_\mu J^\mu - iB_\mu J^\mu). \]  

The spin current

\[ J^{\mu AB} = \frac{i}{2} \bar{\Psi}_L \gamma^\mu \sigma^{AB} \Psi_L \]  

has an anti-Hermitian part which is

\[ i \text{Im}(J^{\mu AB}) = \frac{i}{4} \bar{\Psi}_L \left[ \gamma^B e^{\nu A} - \gamma^A e^{\nu B} \right] \Psi_L \]

\[ = \frac{i}{4} \left[ e^{\nu B} e^{\mu A} - e^{\nu A} e^{\mu B} \right] J^{\nu} \]  

All these imaginary terms are not present in the conventional Hermitian theory. We see that there are CPT-violating processes from fermion
loops generated by the fermion (hence axial or ABJ) current coupled to $B_\mu$. As we have shown, this ABJ current, although anomalous, is regularized by the proposed scheme. In principle, the discussion provided here gives a self-consistent and self-contained method to compute these processes by computing the expectation values of the imaginary parts of the spin current and energy-momentum tensor in background fields.

We can estimate some of the effects of these new processes on the effective action. It can be argued that when the standard model chirally coupled to gravity is quantized in curved spacetimes with nonvanishing torsion, then the one-loop effective action will necessarily yield the Samuel-Jacobson-Smolin action with cosmological constant as the lowest order curvature counterterm just as the Hermitian Weyl action for spin $1/2$ particles must require an effective action which contains the cosmological and Einstein-Hilbert-Palatini actions as the lowest order counterterms if we include gravitational couplings of both chiralities. Thus a quantum field theory of Weyl fermions in background curved spacetimes coupled in this chiral manner must violate CPT to the lowest order in curvature by a term of the form $i/(16\pi G_{\text{renor}}) \int_M e^A \wedge e^B \wedge F_{AB}$ in the effective action.

We sketch the arguments of how and why for the truly Weyl theory such an imaginary term in addition to the familiar real counterterms appears in the effective action. As emphasized, the Weyl fermion action contains no coupling to the right-handed or self-dual spin connection $A^+_{\mu\nu} = \frac{1}{2}(i A_{\mu\nu} + \frac{1}{2} \epsilon_{\mu\nu} C^D A_{CD})$. So the usual Einstein-Hilbert-Palatini action which involves both $A^+_{\mu\nu}$ and $A^-_{\mu\nu}$ cannot occur as a counterterm in the effective action without modification. This is particularly clear if an explicitly chiral regularization which also involves only $A^-_{\mu\nu}$ but no $A^+_{\mu\nu}$ spin connection such the one advocated here is used. Moreover, whenever the spin connection makes its appearance in a counterterm, it must appear only in the anti-self-dual combination of $A^-_{\mu\nu}$. The lowest order curvature term invariant under all the local chiral symmetries of the theory is the Samuel-Jacobson-Smolin action which is the anti-self-dual projection of the Einstein-Hilbert-Palatini action. To be more explicit, we can also consider the CPT-violating terms in the fermionic Weyl action as in Eq. (24) and relate the counterterms to the imaginary pieces of the Samuel-Jacobson-Smolin action which will be generated by the imaginary Weyl action in addition to the usual

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14 Details of fermion loop calculations using the explicitly invariant chiral regularization will be presented elsewhere.

15 The cosmological constant term is real and appears as one of the usual counterterms generated by quantized fermions.
counterterms. For this purpose, we may note that

$$\frac{i}{16\pi G} \int_M e^A \wedge e^B \wedge F_{AB} = -\frac{i}{16\pi G} \int_M \left\{ d(e^A \wedge T_A) - T_A \wedge T^A \right\}. \quad (110)$$

The imaginary boundary term from the integral of ABJ-anomaly in Eq. (24) is related to the first boundary term above. $d(e^A \wedge T_A)$ is known as the Nieh-Yan four-form [20] and is actually an additional contribution to the ABJ-anomaly when the torsion is nonvanishing [21]. However, as it is apparent from dimensional arguments, the Nieh-Yan four-form appears in the ABJ anomaly multiplied by the square of the regulator mass if a Pauli-Villars-Gupta scheme, such as the one proposed here, is used i.e.

$$\partial_\mu \langle J^\mu \rangle \propto \Lambda^2 \ast d(e^A \wedge T_A)$$

$$+ \text{ (usual } \ast Tr(F \wedge F) \text{ and } \ast Tr(G \wedge G) \text{ terms)} + O(\Lambda^{-2}).$$

$$\quad (111)$$

Recall that $\Lambda$ is the regulator mass scale. The additional contribution to the ABJ anomaly when there is torsion diverges as the regulator mass is taken to infinity. Therefore the integral of the Nieh-Yan four-form will appear as a counterterm in the effective action of the Weyl theory. So when the torsion is nonvanishing, there is an additional boundary counterterm from the imaginary part of the Weyl action which corresponds to the boundary term in the Samuel-Jacobson-Smolin action and accounts for the first term in Eq.(110). The other imaginary term $i \int_M B_\mu J^\mu$ in the Weyl action generates fermion loops from the expansion of $\langle J^\mu \rangle$ which gives rise to torsion counterterms including the torsion-squared term which is the second term of Eq.(110) in the Samuel-Jacobson-Smolin action.

\footnote{It may be worth pointing out that with nonvanishing torsion, the complete list of counterterms for even the ordinary Hermitized theory is quite involved. See for instance Ref. [11].}
VIII. REMARKS

We have compared the conventional prescription for the interaction of matter and the four known forces with one which extends the chiral (Weyl) coupling to include gravity in four dimensions. In this regard, when fermions and the standard model are incorporated to reproduce the observed phenomena, the differences studied here can serve as tests of the attempts to quantize gravity with (anti)-self-dual variables without having to rely exclusively on predictions from the full quantum theory of gravity.

At the low energy classical level of equations of motion, the alternatives are indistinguishable. As far as local symmetries are concerned, there are no inconsistencies for the completely chiral prescription since there are no local gauge, Lorentz or diffeomorphism anomalies. Moreover, a regularization which preserves these invariances and the Weyl nature of the theory exists. In the chiral alternative, the Weyl nature supersedes Hermiticity and CPT invariance is not automatic. Off-shell, CPT is violated when the torsion component $B_\mu$ is nonvanishing. At the level of quantum field theory in curved spacetimes with nonvanishing torsion, there are detectable differences because the ABJ anomaly exists and because one-loop counterterms generated by the alternatives are different. These differences serve to characterize the chiral nature of the interaction of matter and the known forces at an even more fundamental level than current empirical data may suggest.

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References
A. Ashtekar, Phys. Rev. Lett. 57, 2244 (1986); Phys. Rev. D36, 1587(1986); Lectures on nonperturbative canonical gravity, (World Scientific, Singapore, 1991) and references therein.

J. Samuel, Pramåna J. Phys. 28, L429(1987); Class. Quantum Grav. 5, L123 (1988); T. Jacobson and L. Smolin, Phys. Lett. B196, 39 (1987), Class. Quantum Grav. 5, 583 (1988).

C. Soo, Phys. Rev. D 52, 3484 (1995).

L. N. Chang and C. Soo, Phys. Rev. D 53, 5682 (1996).

W. Pauli, Exclusion Principle, Lorentz Group and Reflection of Spacetime and Charge, in Niels Bohr and the Development of Physics edited by W. Pauli, L. Rosenfeld and V. Weisskopf (McGraw-Hill, New York, 1955) and references to earlier works of Lüders and Schwinger mentioned therein; G. Lüders, Ann. Phys. 2, 1 (1957); see also, for instance, Chapter 14 of Particle Physics and Introduction to Field Theory by T. D. Lee (Harwood Academic Publishers, 1981).

L. N. Chang and C. Soo, Phys. Rev. D 55, 2410 (1997).

S. L. Adler, Phys. Rev. 177, 2426 (1969); J. S. Bell and R. Jackiw, Nuovo Cimento 60A, 47 (1969); W. A. Bardeen, Phys. Rev. 184, 1848 (1969).

H. Georgi and S. L. Glashow, Phys. Rev. Lett. 32, 438 (1974); H. Georgi, in Particles and Fields - 1974, edited by C. E. Carlson, (AIP Conf. Proc. No. 23, New York, 1975); H. Fritzsch and P. Minkowski, Ann. Phys. 93, 193 (1975).

T. Jacobson, Class. Quantum Grav. 5, L143 (1988); A. Ashtekar, J. D. Romano and R. S. Tate, Phys. Rev. D 40, 2572 (1989); H. Kodama, Int. J. Mod. Phys. D1, 439 (1993).

K. Fujikawa, Phys. Rev. D29 285 (1984).

H. T. Nieh and M. L. Yan, Ann. Phys. 138, 237 (1982).

W. Pauli and F. Villars, Rev. Mod. Phys. 21, 434 (1949); S. N. Gupta, Proc. Phys. Soc. A66, 129 (1953).

S. A. Frolov and A. A. Slavnov, Phys. Lett. B296, 159 (1992).
[14] R. Narayanan and H. Neuberger, Phys. Lett. B302, 62 (1993); Phys. Rev. Lett. 71 3251 (1993); Nucl. Phys. B412, 574 (1994); B443, 305 (1995).

[15] K. Okuyama and H. Suzuki, hep-th/9603062; Phys. Lett. B382, 117 (1996).

[16] H. Georgi and S. Glashow, Phys. Rev. D6, 429 (1972); D. J. Gross and R. Jackiw, ibid. 6, 477 (1972); C. Bouchiat, J. Illiopoulos, and P. Meyer, Phys. Lett. 38B, 519 (1972).

[17] H. T. Nieh, Phys. Rev. Lett. 53, 2219 (1984); L. Alvarez-Gaume and E. Witten, Nucl. Phys. B234, 269 (1984); S. Yajima and T. Kimura, Phys. Lett. B173, 154 (1986).

[18] L. N. Chang and H. T. Nieh, Phys. Rev. Lett. 53, 21 (1984); also the second reference in [17].

[19] K. Fujikawa, Phys. Rev. D25 2584 (1982); D21, 2848 (1980); D 22 1499(E) (1980); D 23, 2262 (1981); Phys. Rev. Lett. 42 1195 (1979); ibid. 44, 1733 (1980).

[20] H. T. Nieh and M. L. Yan, J. Math. Phys. 23, 373 (1982).

[21] Two recent works on the relation of the Nieh-Yan four-form to the ABJ-anomaly are O. Chandia and J. Zanelli, hep-th/9702025; Y. N. Obukhov, E. W. Mielke, J. Budczie and F. W. Hehl, gr-qc/9702011.