Basic-deformed quantum mechanics

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Starting on the basis of $q$-symmetric oscillator algebra and on the associate $q$-calculus properties, we study a deformed quantum mechanics defined in the framework of the basic square-integrable wave functions space. In this context, we introduce a deformed Schrödinger equation, which satisfies the main quantum mechanics assumptions and admits, in the free case, plane wave functions that can be expressed in terms of the $q$-deformed exponential, originally introduced in the framework of the basic-hypergeometric functions.

I. INTRODUCTION

Quantum algebras are deformed version of the Lie algebras, to which they reduce when the deformation parameter $q$ is set equal to unity. Their use in physics became popular from the works of Biedenharn [1] and Macfarlane [2], and with the introduction of the $q$-deformed harmonic oscillator based on the construction of $SU_q(2)$ algebra of $q$-deformed commutation or anticommutation relations between creation and annihilation operators. Initially used for solving the quantum Yang-Baxter equation, quantum algebra have subsequently found several applications in different physical fields such as cosmic strings and black holes [3], conformal quantum mechanics [4], nuclear and high energy physics [5, 6], fractional quantum Hall effect and high-$T_c$ superconductors [7]. At the same time, it was clear that the $q$-calculus, originally introduced in the study of the basic hypergeometric series [8, 9, 10], plays a central role in the representation of the quantum groups with a deep physical meaning and not merely a mathematical exercise.

In Ref.s [11, 12], it was shown that a natural realization of quantum thermostatistics of $q$-deformed bosons and fermions can be built on the formalism of $q$-calculus. In Ref.s [13, 14], a $q$-deformed Poisson bracket, invariant under the action of the $q$-symplectic group, has been derived and a classical $q$-deformed thermostatistics has been proposed in Ref. [15]. It is further remarkable to point out that relativistic mechanics in the two-dimensional noncommutative Minkowski space-time has been introduced in Ref. [16]. $q$-deformed Hamilton equations of motion have been studied in Ref. [17], while, in Ref. [18], $q$-deformed classical mechanics has been derived by using the variational technique on a $q$-deformed Lagrangian.

Furthermore, it has been shown that it is possible to obtain a "coordinate" realization of the Fock space of the $q$-oscillators by using the Jackson derivative (JD) [19, 20, 21]. In Ref.[21], the quantum Weyl-Heisenberg algebra is studied in the frame of the Fock-Bargmann representation and is incorporated into the theory of entire analytic functions allowing a deeper mathematical understanding of squeezed states, relation between coherent states, lattice quantum mechanics and Block functions. Moreover, in the recent past, a tentative to constructing a classical counterpart to the quantum group and $q$-deformed dynamics has been investigated [22]. In this context, it appears important to outline that in Ref.s [23, 24, 25] a very general approach, based on Bargmann representation in connection to coherent states, has been introduced.

It is remarkable to observe that $q$-calculus is very well suited for to describe fractal and multifractal systems. As soon as the system exhibits a discrete-scale invariance, the natural tool is provided by JD and $q$-integral in the framework of the basic hypergeometric functions, which constitute the natural generalization of the regular derivative and integral for discretely self-similar systems [26]. It is also relevant to outline that the statistical origin of such $q$-deformation lies in the modification, relative to the standard case, of number of states $W$ of the system corresponding to the set of occupational number $n_j$ [11]. In literature, other statistical generalization are present, such as the so-called nonextensive thermostatistics or superstatistics with a completely different origin [27, 28, 29].

In the past, the study of generalized linear and non-linear Schrödinger equations has attracted a lot of interest because of many collective effects in quantum many-body models can be described by means of effective theories with generalized one-particle Schrödinger equation [30, 31, 32, 33]. In Ref. [31] a deformation of the canonical commutation relations, based on quantum group arguments, has been studied and in Ref. [35] the free motion of $q$-deformed quantum particle has been investigated. Always in the framework of the $q$-Heisenberg algebra, $q$-deformed Schrödinger equations have been proposed [36, 37, 38].

Although the investigated quantum dynamics is based on noncommutative differential structure on configuration space, we believe that a fully consistent $q$-deformed formalism of the quantum dynamics, starting from the properties of $q$-calculus and basic-hypergeometric functions, has been still lacking. Furthermore, it is known in literature that in the framework $q$ non-symmetric deformed algebra, the momentum operator, associate to the JD, is not Hermitian in the common sense and a complicate combination of JD must be postulated to introduce a $q$-deformed quantum mechanics [39, 40, 41].
In this paper, working in the context of the $q \leftrightarrow q^{-1}$ symmetric algebra, we study a generalized deformed quantum mechanics consistently with the prescriptions of the $q$-calculus. At this scope, we introduce a scalar product in a basic square-integrable wave space and an associate definition of basic-adjoint and basic-Hermitian operators. In this framework, we introduce a $q$-deformed Schrödinger equation which satisfies the principal assumptions of the quantum mechanics.

As mentioned before, we like to outline that in literature exist several examples of concrete functions representation of deformed quantum mechanics, that are different from our approach, based on $q$-calculus and basic hypergeometric functions. Besides the already quoted papers, very recently coherent states for $q$-deformed quantum mechanics on a circle have been rigorously investigated [42].

The paper is organized as follows. In Sec. 2, we review the main features, useful in the present investigation, of $q$-deformed oscillator algebra and the principal properties of $q$-calculus and basic hypergeometric elementary functions. In Sec. 3, we introduce the principal operator properties in the so-called basic square-integrable space, well known in mathematical literature of the basic hypergeometric functions. These properties allow us to introduce, in Sec. 4, the mean value of a dynamical variable and a generalized basic-deformed Schrödinger equation which admits, in the free case, plane wave functions that can be expressed in terms of the basic-hypergeometric exponential function. A brief conclusion is reported in Sec. 5.

II. DEFORMED OSCILLATOR ALGEBRA, $q$-CALCULUS AND BASIC ELEMENTARY FUNCTIONS

We shall review the principal relations of $q$-oscillators defined by the $q$-Heisenberg algebra of creation and annihilation operators introduced by Biedenharn and McFarlane [1, 2], derivable through a map from SU$_q$(2). Furthermore, we will review the main features of the strictly connected $q$-calculus and basic-deformed elementary functions useful in the present investigation. Although most of the collected material and results reported in this Section are not original, this review constitutes the basis of the following study and represents a bridge between physical and mathematical concepts known in literature.

The symmetric deformed algebra is determined by the following commutation relations for $a$, $a^\dagger$ and the number operator $N$, thus (for simplicity we omit the particle index)

\[
[a, a] = [a^\dagger, a^\dagger] = 0, \quad aa^\dagger - qa^\dagger a = q^{-N},
\]

\[
[N, a] = a^\dagger, \quad [N, a^\dagger] = -a.
\]

The $q$-Fock space spanned by the orthonormalized eigenstates $|n\rangle$ is constructed according to

\[
|n\rangle = \frac{(a^\dagger)^n}{\sqrt{[n]!}}|0\rangle, \quad a|0\rangle = 0,
\]

where the $q$-basic factorial is defined as

\[
[n]! = [n][n-1] \cdots [1],
\]

and the so-called $q \leftrightarrow q^{-1}$ symmetric basic number $[x]$ is defined in terms of the $q$-deformation parameter

\[
[x] = \frac{q^x - q^{-x}}{q - q^{-1}}.
\]

In the limit $q \to 1$, the basic number $[x]$ reduces to the ordinary number $x$ and all the above relations reduce to the standard boson relations.

The actions of $a$, $a^\dagger$ on the Fock state $|n\rangle$ are given by

\[
(a^\dagger|n\rangle = [n+1]^{1/2}|n+1\rangle,
\]

\[
a|n\rangle = [n]^{1/2}|n-1\rangle,
\]

\[
N|n\rangle = n|n\rangle.
\]

From the above relations, it follows that $a^\dagger a = [N]$, $aa^\dagger = [N+1]$.

The transformation from Fock observable to the configuration space (Bargmann holomorphic representation) may be accomplished by the replacement [19, 20].
\[ a^t \rightarrow x, \quad a \rightarrow D_x, \]  

where \( D_x \) is the Jackson derivative (JD) \[8\]

\[
D_x = \frac{D_x - (D_x)^{-1}}{(q - q^{-1}) x},
\]

and

\[
D_x = q^x \partial_x,
\]

is the dilatation operator. Its action on an arbitrary real function \( f(x) \) is given by

\[
D_x f(x) = \frac{f(q x) - f(q^{-1} x)}{(q - q^{-1}) x}.
\]

In contrast to the usual derivative, which measures the rate of change of the function in terms of an incremental translation of its argument, the JD measures its rate of change with respect to a dilatation of its argument by a factor of \( q \). The JD satisfies some simple properties which will be useful in the following. For instance, its action on a monomial \( f(x) = x^n \), where \( n \geq 0 \), is given by

\[
D_x (a x^n) = a [n] x^{n-1},
\]

where \( a \) is a real constant.

Moreover, it is easy to verify the following basic-version of the Leibnitz rule

\[
D_x \left( f(x) g(x) \right) = D_x f(x) g(q^{-1} x) + f(q x) D_x g(x),
\]

\[
= D_x f(x) g(q x) + f(q^{-1} x) D_x g(x).
\]

In addition the following property holds

\[
D_{ax} f(x) = \frac{1}{a} D_x f(x).
\]

Consistently with the \( q \)-deformed theory, the standard integral must be generalized to the basic-integral defined, for \( 0 < q < 1 \) in the interval \([0, a]\), as \[8, 9\]

\[
\int_0^a f(x) d_q x = a (q^{-1} - q) \sum_{n=0}^{\infty} q^{2n+1} f(q^{2n+1} a),
\]

while in the interval \([0, \infty]\)

\[
\int_0^\infty f(x) d_q x = (q^{-1} - q) \sum_{n=-\infty}^{\infty} q^{2n+1} f(q^{2n+1}).
\]

The indefinite \( q \)-integral is defined as

\[
\int f(x) d_q x = (q^{-1} - q) \sum_{n=0}^{\infty} q^{2n+1} x f(q^{2n+1} x) + \text{constant}.
\]

One can easily see that the basic-integral approaches the Riemann integral as \( q \rightarrow 1 \) and also that \( q \)-differentiation and \( q \)-integration are inverse to each other, thus

\[
D_x \int_0^x f(t) d_q t = f(x), \quad \int_0^a D_x f(x) d_q x = f(a) - f(0).
\]
where the second identity occurs when the function $f(x)$ is $q$-regular at zero, i.e.

$$\lim_{n \to \infty} f(xq^n) = f(0).$$  \hspace{1cm} (20)$$

By using the deformed Leibnitz rule of Eq.(14), analogue formulas for integration by parts may easily be deduced as

$$\int_0^a f(qx) \frac{d}{dq} g(x) \, dq \, dx = f(x) \frac{d}{dx} g(x) \bigg|_{x=0}^{x=a} - \int_0^a \frac{d}{dx} f(x) \frac{d}{dq} g(q^{-1}x) \, dq \, dx, \hspace{1cm} (21)$$

$$\int_0^a f(q^{-1}x) \frac{d}{dq} g(x) \, dq \, dx = f(x) \frac{d}{dx} g(x) \bigg|_{x=0}^{x=a} - \int_0^a \frac{d}{dx} f(x) \frac{d}{dq} g(qx) \, dq \, dx. \hspace{1cm} (22)$$

For the following developments it appears very relevant to observe that the above integration by parts can be also expressed, for example, as

$$\int_0^a f(x) \frac{d}{dx} g(x) \, dx = f(x) \frac{d}{dx} g(x) \bigg|_{x=0}^{x=a} - \int_0^a \frac{d}{dx} f(x) \frac{d}{dx} g(x) \, dx, \hspace{1cm} (23)$$

or, equivalently,

$$\int_0^a f(x) \frac{d}{dx} g(x) \, dx = f(q^{-1}x) \frac{d}{dx} g(x) \bigg|_{x=0}^{x=a} - \int_0^a \frac{d}{dx} f(q^{-1}x) \frac{d}{dx} g(x) \, dx, \hspace{1cm} (24)$$

where we have rewritten the $q$-Leibnitz rule as

$$\frac{d}{dx} \left( f(qx) g(x) \right) = \frac{d}{dx} f(qx) g(x) + f(x) \frac{d}{dx} g(x), \hspace{1cm} (25)$$

or

$$\frac{d}{dx} \left( f(q^{-1}x) g(x) \right) = \frac{d}{dx} f(q^{-1}x) g(x) + f(x) \frac{d}{dx} g(x). \hspace{1cm} (26)$$

From the above relations, as also pointed out in Ref.\cite{26,43}, it appears evident that JD and $q$-calculus provides a custom made formalism in which to express scaling relations. When $x$ is taken as the distance from a critical point, JD thus quantifies the discrete self-similarity of the function $f(x)$ in the vicinity of the critical point and can be identified with the generator of fractal and multifractal sets with discrete dilatation symmetries.

Let us now introduce the following $q$-deformed exponential function defined by the series

$$E_q(x) = \sum_{k=0}^{\infty} \frac{x^k}{[k]_q!} = 1 + x + \frac{x^2}{[2]_q!} + \frac{x^3}{[3]_q!} + \cdots . \hspace{1cm} (27)$$

The function \cite{27}, reducing to the ordinary exponential function in the $q \to 1$ limit, defines the basic-exponential, well known in the literature since a long time ago, originally introduced in the study of basic hypergeometric series \cite{3,10}.

In this context, it is relevant to observe that the basic-exponential can be also written as

$$E_q(x) = \sum_{n=0}^{\infty} \frac{(1-q)^n x^n}{(q;q)_n} q^{n(n-1)/2}, \hspace{1cm} (28)$$

where we have introduced the $q$-shifted factorial, defined by

$$(a; q)_0 = 1, \quad (a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k), \quad (a; q)_{\infty} = \prod_{k=0}^{\infty} (1 - aq^k), \hspace{1cm} (29)$$

and we have used the identity

$$\frac{(1-q)^n}{(q;q)_n} q^{n(n-1)/2} = \frac{1}{[n]_q!}. \hspace{1cm} (30)$$

The right hand expression of Eq.(28) occurs principally in mathematical literature, being the $q$-shifted factorial commonly used in the framework of the basic hypergeometric functions \cite{10}.
Among many properties, it is important to recall the following relation \[9\]

$$D_x E_q(a x) = a E_q(a x) ,$$

(31)

and its dual

$$\int_0^x E_q(a y) \, dy = \frac{1}{a} \left[ E_q(a x) - 1 \right] .$$

(32)

where \(a\) is a real number different from zero. It is important to point out that Eqs. (31) and (32) are two important properties of the basic-exponential which turns out to be not true if we employ the ordinary derivative or integral.

Beside to the \(q\)-deformed exponential, it is natural to introduce the basic-deformed trigonometric functions as \[9\]

$$E_q(ix) = C_q(x) + i S_q(x) .$$

(33)

As a consequence, we have

$$S_q(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)_q!} ,$$

(34)

$$C_q(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2n_q!} ,$$

(35)

and it easy to see that

$$S_q(x) = \frac{E_q(ix) - E_q(-ix)}{2i} ,$$

(36)

$$C_q(x) = \frac{E_q(ix) + E_q(-ix)}{2} .$$

(37)

Analogously to Eq. (28), the basic-trigonometric functions can be cast in term of the \(q\)-shifted factorial as

$$S_q(x) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n+1/2)}}{(1 - q)(q^2; q^2)_n} x^{2n+1} ,$$

(38)

$$C_q(x) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n-1/2)}}{(q, q^2; q^2)_n} x^{2n} ,$$

(39)

where the notation \((a; b; q)_n\) used before means: \((a; q)_n(b; q)_n\), and we have used the identities

$$\frac{(1 - q)^{2n+1}}{(1 - q)(q^2; q^2)_n} q^{n(n+1/2)} = \frac{1}{[2n + 1]_q!} ,$$

(40)

$$\frac{(1 - q)^{2n}}{(q, q^2; q^2)_n} q^{n(n-1/2)} = \frac{1}{[2n]_q!} .$$

(41)

Instead of the familiar identity \(\sin^2 x + \cos^2 x = 1\), one has \[45\]

$$S_q(q^{-1} x) S_q(x) + C_q(q^{-1} x) C_q(x) = 1 .$$

(42)

Furthermore,

$$D_x S_q(a x) = a C_q(a x) ,$$

(43)

$$D_x C_q(a x) = -a S_q(a x) ,$$

(44)

therefore, one can also easily see that \(S_q(a x)\) and \(C_q(a x)\) are linearly independent solutions of the \(q\)-differential equation

$$D_x^2 u(x) + a^2 u(x) = 0 .$$

(45)

The above introduced basic-trigonometric functions can be expressed in terms of the basic-hypergeometric functions and of the so-called Hahn-Exton \(q\)-Bessel function \[14\, 44\]. Furthermore, it is very relevant to observe that in Ref. 13 it has been shown that \(S_q(x)\) and \(C_q(x)\) exhibit an analogue \(q\)-orthogonality relation and can be considered Fourier expansions in these functions. Finally, it should also be noticed that \(q\)-deformed polynomials, such as \(q\)-deformed Hermite and \(q\)-deformed Laguerre polynomials have been defined in literature in the framework of the basic-hypergeometric functions \[40\, 47\, 49\].
III. OPERATOR PROPERTIES IN BASIC SQUARE-INTEGRABLE SPACE

On the basis of the properties of the previous Section, we are ready to introduce the main ingredients to develop a consistent deformed quantum mechanics in the framework of $q$-calculus and basic-hypergeometric functions.

Let $L^2_2$ be the basic square-integrable space of all complex functions defined in $(-\infty, \infty)$ such that
$$
\|\psi_q\| = \left( \int_{-\infty}^{\infty} |\psi_q(x)|^2 \, dq\,x \right)^{1/2} < \infty.
$$
(46)

The space $L^2_2$ defined above, originally introduced in the literature of the basic hypergeometric functions [9, 48, 49], is a linear space. If $\psi_q$ and $\phi_q$ are two basic square-integrable functions, any linear combinations $\alpha\psi_q + \beta\phi_q$, where $\alpha$ and $\beta$ are arbitrarily chosen complex numbers, are also basic square-integrable functions. Moreover, $L^2_2$ is a separable Hilbert space with the inner (scalar) product
$$
\langle \phi, \psi \rangle_q = \int \phi^*_q(x) \psi_q(x) \, dq\,x.
$$
(47)

A simple example of orthonormal basis of $L^2_2$ is (for $0 < q < 1$, in the symmetric case) [49]
$$
\varphi_n(x) = \begin{cases} 
1 / \sqrt{x(q^2 - q)}, & \text{if } x = q^{2n+1}, \\
0, & \text{otherwise},
\end{cases}
$$
(48)

with $n = \ldots, -2, -1, 0, 1, 2, \ldots$. For completeness, let us mention that there exist representations of the $q$-oscillator algebra [11] where the continuous $q$-Hermite polynomials play the role of vector basis [46, 48, 50]. Such orthogonal polynomials can be expressed in terms of the basic-hypergeometric functions and reduce, in the limit $q \to 1$, to the standard Hermite polynomials.

The above scalar product (47) is linear with respect to $\psi_q$, the norm of a function $\psi_q$ is a real, non-negative number:
$$
\langle \psi, \psi \rangle_q \geq 0 \text{ and } \langle \phi, \psi \rangle_q = \langle \psi, \phi \rangle_q^*.
$$
(49)

Analogously to the undeformed case, it is easy to see that from the above properties of the basic-scalar product follows the $q$-Schwarz inequality
$$
|\langle \phi, \psi \rangle_q|^2 \leq \langle \phi, \phi \rangle_q \langle \psi, \psi \rangle_q.
$$
(50)

Consistently with the above definitions, the basic-adjoint of an operator $\hat{A}_q$ is defined by means of the relation
$$
\langle \psi, \hat{A}^*_q \phi \rangle_q = \langle \phi, \hat{A}_q \psi \rangle_q^*,
$$
(51)

and, by definition, a linear operator is basic-Hermitian if it is its own basic-adjoint. More explicitly, an operator $\hat{A}_q$ is basic-Hermitian if for any two states $\varphi_q$ and $\psi_q$ we have
$$
\langle \phi, \hat{A}_q \psi \rangle_q = \langle \hat{A}_q \phi, \psi \rangle_q.
$$
(52)

On the basis of the results of the previous Section, it appears natural to associate the operators corresponding to the position coordinate and the momentum of a particle as follows
$$
\hat{x}_q = x, \quad \hat{p}_q = -i\hbar \hat{D}_x.
$$
(53)

At this point, it becomes a crucial question if the introduced deformed momentum operator is basic-Hermitian. Let us therefore consider the following product
$$
\langle \phi, \hat{D}_x \psi \rangle_q = \int \phi^*_q(x) \hat{D}_x \psi_q(x) \, dq\,x.
$$
(54)
By considering the integration by parts of Eq. (23) and imposing that the functions \( \varphi_q(x) \) and \( \psi_q(x) \) go to zero in the limit of \( x \to \pm \infty \), we have

\[
\langle \varphi, D_x \psi \rangle_q = -\int D_x \varphi_q^*(qx) \psi_q(qx) \, dq x
\]

\[
= -\int D_{qx} \varphi_q^*(qx) \psi_q(qx) \, dq x
\]

\[
= -\int D_x \varphi_q^*(y) \psi_q(y) \, dq y ,
\]

where in the second equivalence we have used the property (15) and in the last equivalence we have changed the integration variable \( y = qx \). Therefore, we have that the momentum operator, introduced in Eq. (53), is a basic-Hermitian operator. Equivalently,

\[
\langle \varphi, \hat{p}_q \psi \rangle_q = \langle \hat{p}_q \varphi, \psi \rangle_q .
\]

Let us remark that this important property is a peculiarity of the \( q \to q^{-1} \) symmetric basic-framework defined in Eqs. (1), (5) and (9). Conversely, it is known that, in the non-symmetric framework, the momentum operator defined in term of the JD, like in Eq. (53), is not Hermitian [14, 39, 40, 41].

IV. OBSERVABLE AND BASIC-DEFORMED SCHRÖDINGER EQUATION

On the basis of the above properties, we have now the recipe to generalize the definition of an observable and to introduce a basic-deformed Schrödinger equation in the framework of \( q \)-deformed theory. Consistently with the standard quantum mechanics, we have two fundamental postulates: 1) with the dynamical variable \( A(x,p) \) is associated the linear operator \( \hat{A}_q(x,-i\hbar D_x) \); 2) the mean value of this dynamical variable, when the system is in the (normalized) state \( \psi_q \), is

\[
\langle \hat{A} \rangle_q = \int \psi_q^* \hat{A}_q \psi_q \, dq x .
\]

Observeable are real quantities, hence the expectation value (57) must be real for any state \( \psi_q \):

\[
\int \psi_q^* \hat{A}_q \psi_q \, dq x = \int (\hat{A}_q^\dagger \psi_q^*) \psi_q \, dq x ,
\]

therefore, on the basis of Eq. (52), observable must be represented by basic-Hermitian operators.

If we require that there is a state \( \psi_q \) for which the result of measuring the observable \( A \) is unique, in other words that the fluctuations

\[
(\Delta A_q)^2 = \int \psi_q^* (\hat{A}_q - \langle \hat{A} \rangle_q)^2 \psi_q \, dq x ,
\]

must vanish, we obtain the following basic-eigenvalue equation of a basic-Hermitian operator \( \hat{A}_q \) with eigenvalue \( a \)

\[
\hat{A}_q \psi_q = a \psi_q .
\]

As a consequence, the eigenvalues of a basic-Hermitian operator are real because \( \langle \hat{A} \rangle_q \) is real for any state; in particular for an eigenstate with the eigenvalue \( a \) for which \( \langle \hat{A} \rangle_q = a \).

Furthermore, as in the undeformed case, two eigenfunctions \( \psi_{q,1} \) and \( \psi_{q,2} \) of the basic-Hermitian operator \( \hat{A}_q \), corresponding to different eigenvalues \( a_1 \) and \( a_2 \), are orthogonal. We can always normalize the eigenfunction, therefore we can chose all the eigenvalues of a basic-Hermitian operator orthonormal, i.e.

\[
\int \psi_{q,n}^* \psi_{q,m} \, dq x = \delta_{n,m} .
\]

Consequently, two eigenfunctions \( \psi_{q,1} \) and \( \psi_{q,2} \) belonging to different eigenvalues are linearly independent.

It easy to see that, adapting step by step the undeformed case to the introduced \( q \)-deformed framework, the totality of the linearly independent eigenfunctions \( \{ \psi_{q,n} \} \) of basic-Hermitian operator \( \hat{A}_q \) form a complete (orthonormal) set
in the space of the basic square-integrable wave functions. In other words, if \( \psi_q \) is any state of a system, then it can be expanded in terms of the eigenfunctions (with a discrete spectrum) of the corresponding basic-Hermitian operator \( \hat{A}_q \) associate to the observable:

\[
\psi_q = \sum_n c_{q,n} \psi_{q,n},
\]

where

\[
c_{q,n} = \int \psi_{q,n}^{*} \psi_q \, dq \, x.
\]

The above expansion allows us, as usual, to write the expectation value of \( \hat{A}_q \) in the normalized state \( \psi_q \) as

\[
\langle \hat{A} \rangle_q = \int \psi_q^{*} \hat{A}_q \psi_q \, dq \, x = \sum_n |c_{q,n}|^2 a_n,
\]

where \( \{a_n\} \) are the set of eigenvalues (assumed, for simplicity, discrete and non-degenerate) and the normalization condition of the wave function can be written in the form

\[
\sum_n |c_{q,n}|^2 = 1.
\]

Following the prescriptions at the beginning of this Section, we are able to introduce the basic-Hamiltonian operator as

\[
\hat{H}_q = -\frac{\hbar^2}{2m} D_x^2 + V_q(x).
\]

The above Hamiltonian is basic-Hermitian and the time development of a system will be given by the following basic-deformed Schrödinger equation

\[
i\hbar \frac{\partial \psi_q(x,t)}{\partial t} = \hat{H}_q \psi_q(x,t).
\]

The above equation implies a consistent conservation in time of the probability density. In fact, by taking the complex conjugation of Eq.(67), summing and integrating term by term the two equations, we get

\[
i\hbar \frac{\partial}{\partial t} \int \psi_q^{*} \psi_q \, dq \, x = \int \left[ \psi_q^{*} (\hat{H}_q \psi_q) \right. - \left. (\hat{H}_q \psi_q^{*}) \psi_q \right] \, dq \, x = 0,
\]

where the last equivalence follows from the fact that the Hamiltonian operator is basic-Hermitian.

The Eq.(67) admits factorized solution \( \psi_q(x,t) = \phi(t) \varphi_q(x) \), where \( \phi(t) \) satisfies to the equation

\[
i\hbar \frac{d\phi(t)}{dt} = E \phi(t),
\]

with the standard (undeformed) solution

\[
\phi(t) = \exp \left( -\frac{i}{\hbar} E t \right),
\]

while \( \varphi_q(x) \) is the solution of time-independent basic-Schrödinger equation

\[
\hat{H}_q \varphi_q(x) = E \varphi_q(x).
\]

In one dimensional case, for a free particle (\( V_q = 0 \)) described by the wave function \( \varphi_q^f(x) \), Eq.(71) becomes

\[
D_x^2 \varphi_q^f(x) + k^2 \varphi_q^f(x) = 0,
\]

where \( k = \sqrt{2mE/\hbar^2} \). The previous equation is equivalent to Eq.(45) of Section II and, therefore, the solution can be written in term of the basic-exponential

\[
\varphi_q^f(x) = N E_q(ikx),
\]
or, equivalently, it can be expressed in term of a linear combination of the basic-trigonometric functions $S_q(kx)$ and $C_q(kx)$, introduced in Eqs. (34) and (35). The above equation generalizes the plane wave functions in our framework of the $q$-calculus and basic hypergeometric functions. In this context, as already remembered in the Introduction, it is remarkable to note that the quantum dynamics of free Hamiltonian and the properties of plane waves, eigenfunctions of the $q$-deformed momentum operator, were also studied in Refs. (34, 35) in a different approach.

Moreover, by using the time-independent basic-Schrödinger equation (71), it is possible to study more complex examples, such as the presence of the harmonic oscillator potential. In this case, the eigenfunctions will be related to the $q \leftrightarrow q^{-1}$ symmetric $q$-Hermite polynomials (46, 48). This important item will be extensively studied in future investigations.

**V. CONCLUSION**

On the basis of the $q \leftrightarrow q^{-1}$ symmetric calculus, originally introduced in the framework of the basic-hypergeometric functions, we have introduced a generalized linear Schrödinger equation which involves a $q$-deformed Hamiltonian that is a basic-Hermitian operator in the space of basic square-integral wave functions. Although a complete description of the introduced quantum dynamical equations lies out the scope of this paper, we think that the results derived here appear to provide a starting point to obtain a deeper insight into a full consistent basic-deformed quantum mechanics in the framework of the $q$-calculus and basic hypergeometric functions.

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