MAPS BETWEEN CERTAIN COMPLEX GRASSMANN MANIFOLDS

PRATEEP CHAKRABORTY AND PARAMESWARAN SANKARAN

Abstract. Let $k, l, m, n$ be positive integers such that $m - l > l > k$, $m - l > n - k \geq k$ and $m - l > 2k^2 - k - 1$. Let $G_k(\mathbb{C}^n)$ denote the Grassmann manifold of $k$-dimensional vector subspaces of $\mathbb{C}^n$. We show that any continuous map $f : G_l(\mathbb{C}^m) \to G_k(\mathbb{C}^n)$ is rationally null-homotopic. As an application, we show the existence of a point $A \in G_l(\mathbb{C}^m)$ such that the vector space $f(A)$ is contained in $A$; here $\mathbb{C}^n$ is regarded as a vector subspace of $\mathbb{C}^m \cong \mathbb{C}^n \oplus \mathbb{C}^{m-n}$.

1. Introduction

Let $U(n) \subset GL(n, \mathbb{C})$ denote the unitary group and let $G_k(\mathbb{C}^n)$ denote the homogeneous space $G_n,k = U(n)/U(k) \times U(n-k)$. The smooth manifold $G_k(\mathbb{C}^n)$ is the complex Grassmann manifold of $k$-dimensional vector subspaces of $\mathbb{C}^n$. It is simply connected and has the structure of a smooth projective variety of (complex) dimension is $k(n - k)$. To simplify notation we shall hereafter write $G_n,k$ to mean $G_k(\mathbb{C}^n)$ since we will only be concerned with complex Grassmann manifolds in this paper.

The purpose of this note is to prove the following theorem.

Theorem 1.1. Let $1 \leq k \leq \lfloor n/2 \rfloor$, $1 \leq l \leq \lfloor m/2 \rfloor$ and $k < l$, where $m,n$ are positive integers such that $m - l > n - k$. Suppose that $m - l \geq 2k^2 - k - 1$ or $1 \leq k \leq 3$. Then any homomorphism of graded rings $\phi : H^*(G_n,k; \mathbb{Z}) \to H^*(G_m,l; \mathbb{Z})$ vanishes in positive dimensions.

As a corollary to the above theorem we obtain the following result on the homotopy classification of maps between the complex Grassmann manifolds.

Theorem 1.2. Let $l,k,m,n$ be as in the above theorem. Then the set $[G_m,l,G_n,k]$ of homotopy classes of maps is finite and moreover each homotopy class is rationally null-homotopic.

As another application of Theorem 1.1 we obtain the following invariant subspace theorem. See [13] for an analogous result for real Grassmann manifolds. We shall regard $\mathbb{C}^n$ as a subspace of $\mathbb{C}^m$ consisting of vectors with last $m - n$ coordinates zero. Thus, if $y \in G_n,k$ and $x \in G_m,l$ it is meaningful to write $y \subset x$.

Theorem 1.3. Let $f : G_m,l \to G_n,k$ be any continuous map where $l,k,m,n$ are as in Theorem 1.1. Then there exists an element $x \in G_m,l$ such that $f(x) \subset x$. 

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We point out that the classification of self-maps of a complex Grassmann manifold has been studied in terms of their induced endomorphisms of the cohomology algebra by several authors. See [14], [2], [7], [9], [10]. Similar study of maps between two distinct (real) Grassmann manifolds seems to have been initiated in [11]. Sankaran and Sarkar [17] have studied the existence (or non-existence) of maps of non-zero degree between two different complex (resp. quaternionic) Grassmann manifolds of the same dimension. The same problem for oriented real Grassmann manifolds has been settled by Ramani and Sankaran [16].

Our methods are straightforward. To prove Theorem 1.1, we reduce the problem to one about endomorphism of the cohomology of a certain Grassmann manifold and appeal to a well-known result of Glover and Homer [7]. Theorem 1.2 is proved using a result due to Glover and Homer [8], namely, any map between any two complex Grassmann manifolds—indeed complex flag manifolds—is formal. Our approach to the proof of Theorem 1.3 is similar in spirit to that of [13, Theorem 1.1].

It has been conjectured that if \( \phi \) is any endomorphism of the graded \( \mathbb{Q} \)-algebra \( H^*(G_{n,k}; \mathbb{Q}) \) which is vanishes on \( H^2(G_{n,k}; \mathbb{Z}) \), then \( \phi \) vanishes in all positive degree. See [7]. Our proof shows that the conjecture implies the validity of Theorems 1.1 and 1.2 hold without the restriction \( m - l \geq 2k^2 - k - 1 \).

2. Proofs

The cohomology ring \( H^*(G_{n,k}; \mathbb{Z}) \) of \( G_{n,k} \) is well-known to be generated by the Chern classes \( c_i(\gamma_{n,k}) \in H^{2i}(G_{n,k}; \mathbb{Z}), 1 \leq i \leq k \), of the canonical complex \( k \)-plane bundle \( \gamma_{n,k} \). Indeed, the cohomology ring has a presentation

\[
H^*(G_{n,k}; \mathbb{Z}) = \mathbb{Z}[c_1, \ldots, c_k]/\langle h_{n-k+1}, \ldots, h_n \rangle
\]

as the quotient of the polynomial ring modulo the ideal generated by the elements \( h_j, n - k + 1 \leq j \leq n \), where \( |c_i| = 2i \); here \( h_r \) is defined as the \( 2r \)-th degree term in the expansion of \( (1 + c_1 + \cdots + c_k)^{-1} \). Under the above isomorphism \( c_i \) corresponds to the element \( c_i(\gamma_{n,k}) \in H^{2i}(G_{n,k}; \mathbb{Z}), 1 \leq i \leq k \). We shall denote by \( R_k \) the polynomial algebra \( \mathbb{Z}[c_1, \ldots, c_k] \) and

The following are well-known facts concerning the cohomology ring:

1. The cohomology group \( H^r(G_{n,k}; \mathbb{Z}) \) is a free abelian group. It is zero when \( r \) is odd. This follows from the fact that \( G_{n,k} \) admits a cell-structure with cells only in even dimensions. See, for example, [5].

2. The elements \( h_j, n - k + 1 \leq j \leq n \), form a regular sequence in the polynomial algebra \( R_k := \mathbb{Z}[c_1, \ldots, c_k] \) for any \( n \geq 2k \). That is, \( h_{n-k+1} \neq 0 \) and \( h_{n-k+r} \) is a not a zero divisor in \( R_k/\langle h_{n-k+1}, \ldots, h_{n-k+r-1} \rangle \), \( 2 \leq r \leq k \). See for example [1].

3. The element \( c_1^d \neq 0 \) where \( d = \dim_{\mathbb{C}} G_{n,k} = k(n - k) \). This follows immediately from the fact that \( G_{n,k} \) has the structure of a Kähler manifold with second Betti number
1. In fact it is known $H^{2d}(G_{n,k}; \mathbb{Z})$ is generated by the element $c_{k}^{n-k}$ and that $c_{k}^{2} = N c_{k}^{n-k}$ where $N = (d!1!2! \cdots (k-1)!)/(n-k)! \cdots (n-1)!$. See [4, §14].

(4) The natural imbedding $i : G_{n,k} \subset G_{n+1,k}$ and $j : G_{n,k} \subset G_{n+1,k+1}$, defined by the natural inclusion of $U(n)$ in $U(n+1)$, induce surjections $i^{*} : H^{*}(G_{n+1,k}; \mathbb{Z}) \to H^{*}(G_{n,k}; \mathbb{Z})$ and $j^{*} : H^{*}(G_{n+1,k+1}; \mathbb{Z}) \to H^{*}(G_{n,k}; \mathbb{Z})$ where $i^{*}(c_{r}(\gamma_{n+1,k})) \mapsto c_{r}(\gamma_{n,k}), 1 \leq r \leq k$ and $j^{*}(c_{r}(\gamma_{n+1,k+1})) = c_{r}(\gamma_{n,k})$ when $r \leq k$ and $j^{*}(c_{r}(\gamma_{n+1,k+1})) = 0$. The homomorphism $i^{*}$ induces isomorphisms in cohomology in dimensions up to $2(n-k)$ and $j^{*}$ induces isomorphisms in cohomology in dimensions up to $2k$.

**Proof of Theorem 1.1:** One has an inclusion $U(m-l+k) \subset U(m)$ where a matrix $X \in U(m-l+k)$ corresponds to the matrix in block diagonal form with diagonal blocks $X, I_{k-l}$. (Here $I_{k-l}$ denotes the identity matrix.) This induces an imbedding $G_{m-l+k,k} \subset G_{m,l}$. Similarly, since $m-l > n-k$, we have the inclusion $U(n) \subset U(m-1+l)$ which induces an imbedding $G_{n,k} \subset G_{m+l-k,k}$. These inclusions are merely compositions of appropriate inclusions considered in Fact (4) above. Let $\alpha : H^{*}(G_{m,l}; \mathbb{Z}) \to H^{*}(G_{m-l+k,k}; \mathbb{Z})$ and $\beta : H^{*}(G_{m-l+k,k}; \mathbb{Z}) \to H^{*}(G_{n,k}; \mathbb{Z})$ be the inclusion-induced homomorphisms. It follows from Fact (4) that $\beta(c_{i}(\gamma_{m-l+k,k})) = c_{i}(\gamma_{n,k}), i \leq k$. Also, $\alpha(c_{i}(\gamma_{m,l})) = c_{i}(\gamma_{m-l+k,k}), i \leq k$. Then we obtain an endomorphism of the graded ring $\alpha \circ \phi \circ \beta$ of $H^{*}(G_{m-l+k,k})$ where $\phi : H^{*}(G_{n,k}; \mathbb{Z}) \to H^{*}(G_{m,l}; \mathbb{Z})$ is any graded ring homomorphism. Note that our hypothesis on $k, l, m, n$ implies that $\dim G_{n,k} < \dim G_{m,l}$. Hence by Fact (3) above, $\phi(c_{1}(\gamma_{n,k})) = 0$. Therefore $\alpha \circ \phi \circ \beta(c_{1}(\gamma_{m-l+k,k})) = 0$. Tensoring with $\mathbb{Q}$ we obtain a graded $\mathbb{Q}$-algebra endomorphism $H^{*}(G_{m-l+k,k}; \mathbb{Q}) \xrightarrow{\alpha \circ \phi \circ \beta} H^{*}(G_{m-l+k,k}; \mathbb{Q})$ which vanishes in degree 2. Our hypothesis that $m-l \geq 2k^{2} - k - 1$ or $k \leq 3$ implies, by [7], that this endomorphism is zero in positive dimensions. \[\square\]

We remark that Theorem 1.1 and the above proof hold when the coefficient ring $\mathbb{Z}$ is replaced by any subring of $\mathbb{Q}$ through out. If $\phi$ is induced by a continuous map $f$, then $H^{*}(f; R)$ is zero for any commutative ring $R$.

Before taking up the proof of Theorem 1.2, we recall the relation between the homotopy class of a map and the homomorphism it induces in cohomology with rational coefficients. We assume familiarity with basic notions in the theory of rational homotopy as in [6]. (For a comprehensive treatment see [3].)

Let $X$ be any simply connected finite CW complex and let $X_{0}$ denote its rationalization. Thus $\widetilde{H}^{*}(X_{0}; \mathbb{Z}) \cong \widetilde{H}^{*}(X; \mathbb{Q})$. If $f : X \to Y$ is a continuous map of such spaces, then there exists a rationalization of $f$, namely a continuous map $f_{0} : X_{0} \to Y_{0}$ such that $f_{0}^{*} : \widetilde{H}^{*}(Y_{0}; \mathbb{Z}) \to \widetilde{H}^{*}(X_{0}; \mathbb{Z})$ is the same as $f^{*} : \widetilde{H}^{*}(Y; \mathbb{Q}) \to \widetilde{H}^{*}(X; \mathbb{Q})$. Denoting the minimal model of $X$ by $\mathcal{M}_{X}$, one has a bijection $[X_{0}, Y_{0}] \cong [\mathcal{M}_{Y}, \mathcal{M}_{X}], [h] \mapsto [\Phi_{h}]$ where on the left we have homotopy classes of continuous maps $X_{0} \to Y_{0}$ and on the right we have homotopy classes of differential graded commutative algebra homomorphisms of the minimal models $\mathcal{M}_{Y} \to \mathcal{M}_{X}$. In the case when $X = U(n)/(U(n_{1}) \times \cdots \times U(n_{r}))$ is a complex flag manifolds, one knows that $X$ is Kähler and hence is formal, that is, there
exists a morphism of differential graded commutative algebras $\rho_X : \mathcal{M}_X \to H^*(X; \mathbb{Q})$ which induces isomorphism in cohomology, where $H^*(X; \mathbb{Q})$ is endowed with the zero differential. Moreover, it is known that when both $X$ and $Y$ are complex flag manifolds, any continuous map $f : X \to Y$ is formal, that is, the homotopy class of the morphism $f_0 : X_0 \to Y_0$ is determined by the graded $\mathbb{Q}$-algebra homomorphisms $h^* : H^*(Y; \mathbb{Q}) \to H^*(X; \mathbb{Q})$. More precisely, we have the following result.

**Theorem 2.1.** ([8, Theorem 1.1]) Let $X, Y$ be complex flag manifolds. Then $[h] \mapsto H^*(h; \mathbb{Q})$ establishes an isomorphism from $[X_0, Y_0]$ to the set of graded $\mathbb{Q}$-algebra homomorphisms $H^*(Y; \mathbb{Q}) \to H^*(X; \mathbb{Q})$.

We now turn to the proof of Theorem 1.2.

**Proof of Theorem 1.2:** By Theorem 1.1 we know that any such $f^*$ is the trivial homomorphism (which is identity in degree zero and is zero in positive dimensions). By the above theorem $f_0$ is null-homotopic. This proves the second statement of Theorem 1.2. The first statement follows from the second since there exists, up to homotopy, at most finitely many continuous map $f : G_{m,l} \to G_{n,k}$ having the same rationalisation $f_0$. (See [19, §12].) This completes the proof of Theorem 1.2.

Next we turn to the proof of Theorem 1.3. In the following proof, we use cohomology with rational coefficients although one may use integer coefficients.

We shall write $M, N$ respectively for $G_{m,l}$ and $G_{n,k}$. Suppose that $1 \leq k < l$, $m - l \geq n - k$. As usual we assume that $2k, 2l \leq m$. Let $V \subset M \times N$ be the subspace $V := \{(x, y) \in M \times N \mid y \subset x\} \subset M \times N$. (Recall that $\mathbb{C}^n = \mathbb{C}^n \oplus 0 \subset \mathbb{C}^m$.) One has a map $\pi : V \to N$ that sends $(x, y) \in V$ to $y \in N$. This is the projection of a fibre bundle over $N$ with fibre space $G_{n-k,l-k}$. To see this, regard $V$ as a submanifold of the complex flag manifold $F = U(m)/U(k) \times U(l-k) \times U(m-l) = \{(A, B) \mid \dim_{\mathbb{C}} A = k, \dim_{\mathbb{C}} B = l-k, A \perp B, A, B \subset \mathbb{C}^n\}$ where a point $(x, y) \in V$ is identified with the point $(y, x') \in F$ where $x'$ is the orthogonal complement of $y$ in $x$ so that $x' \perp y$ and $x = x' + y$. The projection map $p : F \to G_{m,k}$, defined as $(A, B) \mapsto A \in G_{m,k}$, of the $G_{n-k,l-k}$-bundle $\theta$ over $G_{m,k}$ maps $V$ onto $G_{n,k} \subset G_{m,k}$. In fact $V = p^{-1}(G_{n,k})$ and so $\pi : V \to G_{n,k}$ is the projection of the bundle $\theta|_{G_{n,k}}$.

As usual we denote by $[N]$ the generator of the top cohomology group $H^{2k(n-k)}(N; \mathbb{Q})$.

**Lemma 2.2.** Let $c = \text{codim}_{M \times N} V = 2k(m-l)$. Let $v \in H^c(M \times N; \mathbb{Q})$ denote the cohomology class dual to $j : V \hookrightarrow M \times N$. Then $v \cup [N] \neq 0$ in $H^* (M \times N; \mathbb{Q})$.

**Proof.** The cohomology class $[N]$ is dual to the submanifold $i : M \hookrightarrow M \times N$ where $i(x) = (x, \mathbb{C}^k), x \in M$. First we shall show that $i(M)$ intersects $V$ transversely. Note that $i(M) \cap V = \{(x, \mathbb{C}^k) \mid \mathbb{C}^k \subset x \subset \mathbb{C}^n\} \cong G_{n-k,l-k}$, which is the fibre over the point $\mathbb{C}^k \in N$ of the bundle projection $\pi : V \to N$. Therefore $T_{i(x)}(M \times N)/T_{i(x)}(V \cap T_{i(x)}i(M)) \cong T_{\mathbb{C}^k}N$. Since $T_{i(x)}(M \times N)/T_{i(x)}M \cong T_{\mathbb{C}^k}N$, follows that $i(M)$ intersects $V$ transversely. Therefore $v \cup [N]$ is dual to the submanifold $V \cap i(M) \subset M \times N$. Since $V \cap i(M) \cong$
$G_{m-k,l-k} \subset G_{m,l} = M$ represents a non-zero homology class in $H_{2(l-k)(m-l)}(M; \mathbb{Q}) \cong H_{2(l-k)(m-l)}(M; \mathbb{Q}) \otimes H_0(N; \mathbb{Q}) \subset H_{2(l-k)(m-l)}(M \times N; \mathbb{Q})$, its Poincaré dual, which equals $v \cup [N]$, is a non-zero cohomology class in $H^{2d}(M \times N; \mathbb{Q})$ where $d = k(m-l)+k(n-k)$. □

Proof of Theorem 1.3: Consider the map $\phi := id \times f : M \times M \to M \times N$ defined as $\phi(x, y) = (x, f(y))$. Denote by $\delta : M \to M \times M$ the diagonal map.

We need to show that $\phi(\delta(M)) \cap V \neq \emptyset$.

Let $v \in H^*(M \times N; \mathbb{Q})$ denote the cohomology class dual to the manifold $V \subset M \times N$ and let $\Delta \in H^*(M \times M; \mathbb{Q})$ denote the diagonal class, i.e., the class dual to $\delta(M) \subset M \times M$. As is well-known $v$ is in the image of the inclusion-induced homomorphism $H^*(M \times N, M \times N \setminus V; \mathbb{Q}) \to H^*(M \times N; \mathbb{Q})$. (See for example [12, Chapter 11].) Using the naturality of cup-products and by considering the bilinear map $H^*(M \times N, M \times N \setminus \phi(\delta(M)); \mathbb{Q}) \otimes H^*(M \times N, M \times N \setminus V; \mathbb{Q}) \to H^*(M \times N, M \times N \setminus (V \cap \phi(\delta(M)))); \mathbb{Q})$ induced by the inclusion maps, it follows that if $V \cap \phi(\delta(M)) = \emptyset$, then $v \cup w = 0$ for any $w \in H^+(M \times N, M \times N \setminus \phi(\delta(M))); \mathbb{Q})$. (See [18, §6, Chapter 5].) In particular, this holds for the class $w$ that maps to the cohomology class $\alpha_f$ dual to the submanifold $\phi(\delta(M)) \hookrightarrow M \times N$ under the inclusion-induced map $H^{2k(n-k)}(M \times N, M \times N \setminus \phi(\delta(M)); \mathbb{Q}) \to H^{2k(n-k)}(M \times N; \mathbb{Q})$. Thus $v \cup \alpha_f = 0$.

On the other hand, $\mu_{M \times N} \cap \alpha_f = \phi_* (\delta_* (\mu_M))$. Our hypothesis on $k, l, m, n$ implies, by Theorem 1.1, that $\phi_*$ does not depend on $f$. In particular, taking $f = c$, the constant map sending $M$ to $\mathbb{C}^k \in N$, we obtain $\phi \circ \delta = i : M \hookrightarrow M \times N$ considered in the previous lemma. So $\phi_* (\delta_* (\mu_M)) = i_* (\mu_M)$ and we have $\alpha_f = [N]$. By the above lemma we have $v \cup \alpha_f = v \cup [N] \neq 0$, a contradiction. This completes the proof. □

We conclude this paper with the following remark. Suppose that $\dim(G_{n,k}) \leq \dim G_{m,l}$ and let $f : G_{m,l} \to G_{n,k}$ be a holomorphic map where we assume that $k \leq n/2, l \leq m/2$. When $\dim(G_{n,k}) = \dim G_{m,l}$ and $k > 1$, so that $G_{n,k}$ is not the projective space, it is was proved by Paranjape and Srinivas [15] that if $f$ is not a constant map, then $(n, k) = (m, l)$ and $f$ is an isomorphism of varieties.

Suppose that $\dim G_{n,k} < \dim G_{m,l}$. We claim that any holomorphic map $f : G_{m,l} \to G_{n,k}$ is a constant map. Indeed, the Picard group $Pic(G_{m,l})$ of $G_{m,l}$ of the isomorphism classes of complex (equivalenty algebraic or holomorphic) line bundles is isomorphic to $H^2(G_{m,l}; \mathbb{Z}) \cong \mathbb{Z}$ via the first Chern class. It is generated by the bundle $\xi_{m,l} := \det(\gamma_{m,l})$.

The dual bundle $\xi_{m,l}^\vee$ is a very ample bundle (or a positive line bundle in the sense of Kodaira). Note that any holomorphic map between non-singular complex projective manifolds is a morphism of algebraic varieties. Now our claim is a consequence of the following more general observation.

Lemma 2.3. Let $f : X \to Y$ be a morphism between two complex projective varieties where $Pic(X)$, group of isomorphism class of algebraic line bundles over $X$, is isomorphic to the infinite cyclic group. If $\dim X > \dim Y$, then $f$ is a constant morphism.
Proof. Suppose that $f$ is a non-constant morphism. Then there exists a projective curve $C \subset X$ such that $f|_C$ is a finite morphism. Let $\xi$ be a very ample line bundle over $Y$ and let $\eta = f^*(\xi)$. Since $\xi$ is very ample, it is generated by its (algebraic) sections and so it follows that $\eta$ is also generated by sections. Since $f|_C$ is a finite morphism, we see that $\eta|_C$ is ample, that is, some positive tensor power of $\eta|_C$ is very ample. In particular $\eta$ is not trivial. Denote by $\omega$ the ample generator of $\text{Pic}(X) \cong \mathbb{Z}$ and let $\eta = \omega^r$ for some $r$. Since $\eta$ is generated by its sections, we have $r \geq 0$. Since $\eta$ is non-trivial, $r \neq 0$. It follows that $r > 0$ and $\eta$ is ample.

On the other hand, since $\dim X > \dim Y$, some fibre $Z$ of $f$ is positive dimensional and the bundle $\eta|_Z$ is trivial. This is a contradiction since the restriction of an ample bundle to a positive dimensional subvariety is ample and non-trivial. $\square$

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The Institute of Mathematical Sciences, CIT Campus, Taramani, Chennai 600113, India

E-mail address: prateepc@imsc.res.in
E-mail address: sankaran@imsc.res.in