Many-body coherence and entanglement from randomized correlation measurements

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We show that $k$-point correlation measurements on output of a non-interacting, multimode, random unitary allow to quantify the $k$-particle coherence of $N \geq k$ identical (bosonic or fermionic) particles. For separable many-particle input states, we establish an on average strictly monotonic relationship between such $k$-particle coherences, the interference contrast in the experimentally accessible counting statistics and the degree of the particles’ mutual distinguishability, as controlled by their internal degrees of freedom. Non-separability on input can be unveiled by comparison of correlation measurements of different orders.

Systems of identical particles exhibit a rich phenomenology rooted in many-body interference (MBI) [1–13]. Since MBI arises from the inability to assign individual labels to the particles, any distinguishing information carried by unobserved degrees of freedom (dof) of the particles results in a suppression of MBI [14–19] and in a reduction of the complexity of associated measurement records. Mapping out the corresponding quantum-to-classical transition is crucial to understand the dynamics and decoherence of many-body quantum systems and, a fortiori, to conceive and assess quantum protocols for communication, metrology or computation with identical particles, such as the boson sampler, which heavily rely on MBI [7, 20–28].

An evaluation of the capacity of many-body states to display interference is challenging in itself. Beyond systems of few particles and modes, the sheer number of output events makes it impossible to access the full particle-counting statistics experimentally. A natural strategy then is to detect signatures of MBI in specific event probabilities or in low-order correlations, without probing the full output statistics. It is known that such type of measurements upon transmission across highly symmetric or random scatterers convey non-trivial information on the injected quantum states [3, 10, 29–34], with applications, e.g., in the inference of a many-particle state’s purity [35, 36] or the characterization of multi-partite entanglement [37, 38].

In [7] (see also [39] for a similar strategy), a certification scheme of boson sampling was proposed, which builds on the statistical analysis of two-point intensity correlators at the output of randomly sampled linear optical networks. It was shown that these contain sufficient information to discriminate between states of distinguishable particles and those of perfectly indistinguishable fermions and bosons, as well as to sense salient features of the underlying dynamics [7, 40–43]. In particular, for partially distinguishable particles, these two-point correlators give an indication of the state’s level of distinguishability [40].

This raises two questions: Which is the specific property of the quantum state that correlation measurements in randomly chosen bases probe? And under which conditions are low-order correlators, which are only sensitive to few-particle interference processes, sufficient to faithfully characterize the distinctive coherence properties of a many-body state?

We answer these questions by formulating a theory of randomized density correlation measurements in non-interacting bosonic or fermionic many-body systems, where the particles can be partially distinguished through an internal dof. We show that $k$-point correlation measurements in randomly chosen single-particle modes give access to the unbiased average over all matrix elements of the $k$-particle ($kP$) reduced density matrix of the initial state, and, therefore, provide experimentally accessible measures of $kP$ coherence. Furthermore, we identify the absence of mode entanglement in the input state as a sufficient criterion for all these correlators to parametrize the same, on average strictly monotonic transition from states of indistinguishable fermions to indistinguishable bosons, via the intermediate case of distinguishable particles. In this case, low-order interference processes probed in second-order correlators [7] contain enough information to faithfully characterize the MBI contributions to correlation measurements of any order. On the other hand, this provides a theoretical foundation for the statistical benchmark of [7, 40, 42]. On the other, it identifies non-classical correlations of the many-body input state as the source of richer MBI phenomena during the dynamics.

**Partially distinguishable particles**— To describe states of many partially distinguishable particles, we start from a single-particle (1P) Hilbert space erected upon both external and internal dof: $\mathcal{H} = \mathcal{H}_{\text{ext}} \otimes \mathcal{H}_{\text{int}}$ [27, 44–46]. In contrast to the former, we assume that the latter are neither affected by the dynamics, nor interrogated by measurement. Nevertheless, the internal dof may (partially) distinguish the particles and, therefore, alter MBI in the external dynamics. For example, in a photonic interference experiment, external basis states are typically the input modes of a multi-port interferometer, while polarization and/or arrival-time
provide the internal dof. In a cold-atom experiment, the Wannier orbitals of an optical lattice and the electronic levels of the atoms can establish external and internal states, respectively. The distinction between external and internal dof and the associated notion of partial distinguishability are, therefore, not absolute, but determined by the experimental setup and the measurement protocol under consideration.

Many-particle states are represented in the bosonic or fermionic Fock space $\mathcal{F}[\mathcal{H}]$, whose NP sector is, respectively, the symmetric or anti-symmetric subspace of $\mathcal{H}^{\otimes N}$. This space can further be decomposed into the tensor product of Fock spaces built upon the internal dof and each associated with one of the orthogonal external modes $p$ belonging to a basis $\mathcal{B}_{\text{ext}}$ of $\mathcal{H}_{\text{ext}}$: $\mathcal{F}[\mathcal{H}] \simeq \bigotimes_{p \in \mathcal{B}_{\text{ext}}} \mathcal{F}_p[\mathcal{H}_{\text{int}}]$. A state is separable in the external modes, in short externally separable, if it is separable according to this partition. Otherwise, it is externally entangled [47].

Many-particle states are created from the vacuum $|0\rangle$ by multiple application of the (bosonic or fermionic) creation operators $\hat{a}^\dagger_{p\alpha}$, where Greek indices $\alpha \in \mathcal{B}_{\text{int}}$ label the orthogonal internal basis modes. For simplicity, we take $\mathcal{H}_{\text{ext}}$ and $\mathcal{H}_{\text{int}}$ to be finite-dimensional and set $d = \dim \mathcal{H}_{\text{ext}}$. The external number operator $\hat{N}_p = \sum_{\alpha \in \mathcal{B}_{\text{int}}} \hat{a}_{p\alpha}^\dagger \hat{a}_{p\alpha}$ counts the number of particles in mode $p \in \mathcal{B}_{\text{ext}}$, irrespective of their internal state, and we assume faithful input state preparation with the corresponding eigenvalue $N_p \in \{0,1\}$. Such setting allows for a particularly transparent analysis of the interdependence of partial distinguishability, coherence and entanglement [48]. More general scenarios will be addressed elsewhere.

Mode correlations and reduced states — For an NP input state $\rho$, we consider a non-interacting dynamics of the external modes (e.g., a linear interferometer), such that the evolution operator $U$ acts as $U \hat{a}_{p\alpha}^\dagger U = \sum_{\alpha \in \mathcal{B}_{\text{int}}} U_{p\alpha} \hat{a}_{p\alpha}$, with $U$ a unitary transformation of $\mathcal{H}_{\text{ext}}$. We then assess (many-body) interference through measurements of many-body observables that are blind to the internal dof [49]. Typical examples are density correlation measurements between $k \leq N$ modes

$$\text{tr} \left[ \rho U \hat{N}_p \ldots \hat{N}_puU^\dagger \right] = \sum_{m,n \in \mathcal{B}_{\text{int}}} \prod_{i=1}^k U_{p_i,m_i} U_{p_i,n_i} \sum_{\alpha \in \mathcal{B}_{\text{int}}} \text{tr} \left[ \rho \hat{a}_{\alpha m}^\dagger \hat{a}_{\alpha n} \right]$$

with orthogonal $p_i \in \mathcal{B}_{\text{ext}}$, $\hat{a}_{\alpha m}^\dagger = \hat{a}_{m_1}^\dagger \ldots \hat{a}_{m_k}^\dagger$, and $\hat{a}_{\alpha m} = (\hat{a}_{\alpha m}^\dagger)^\dagger$ for multi-indices $m = (m_1 \ldots m_k) \in \mathcal{B}_{\text{int}}^k$, $\alpha = (\alpha_1 \ldots \alpha_k) \in \mathcal{B}_{\text{int}}$. The external k-th order correlation functions $\sum_{\alpha \in \mathcal{B}_{\text{int}}} \text{tr} [\rho \hat{a}_{\alpha m}^\dagger \hat{a}_{\alpha n}]$ appear already in [50], in the study of coherence in many-body systems. They can be identified [48] with the matrix elements $\langle m | \rho_{\text{ext}}^{(k)} | m \rangle = \sum_{\alpha \in \mathcal{B}_{\text{int}}^k} \text{tr} [\rho \hat{a}_{\alpha m}^\dagger \hat{a}_{\alpha n}] (N-k)!/N!$ of the external kP reduced density operator $\rho_{\text{ext}}^{(k)}$ in the un-symmetrized (first quantization) product basis $|m\rangle = |m_1 \ldots m_k\rangle$ of $\mathcal{H}_{\text{ext}}^\otimes k$. More precisely, $\rho_{\text{ext}}^{(k)}$ is obtained from $\rho$ by embedding the NP Fock sector into $\mathcal{H}^{\otimes N} \simeq \mathcal{H}_{\text{ext}}\otimes \mathcal{H}_{\text{int}}$ and performing the partial trace operations $\mathcal{H}_{\text{ext}}\otimes \mathcal{H}_{\text{int}} \xrightarrow{\text{tr}_{\text{int}}} \mathcal{H}_{\text{ext}}\otimes \mathcal{H}_{\text{int}} \xrightarrow{\text{tr}_{\text{ext}}} \mathcal{H}_{\text{ext}}^\otimes k$. Note that these traces commute [48, 51].

We call the off-diagonal elements $\langle m | \rho_{\text{ext}}^{(k)} | m \rangle$, $m \neq n$, the kP coherences. In Eq. (1), these come with weights defined by the specific unitary $U$. We show below that randomly chosen unitaries $U$, in combination with a suitable truncation scheme of the observable, realize (on average) an unbiased sampling of the $\langle m | \rho_{\text{ext}}^{(k)} | m \rangle$. This gives direct experimental access to the coherence of the initial state, as quantified by the cumulative measures

$$\mathcal{W}^{(k)} = \sum_{m,n \in \mathcal{B}_{\text{ext}}} \langle m | \rho_{\text{ext}}^{(k)} | m \rangle,$$

which we term the kP mean coherence. Hermiticity and positivity of $\rho_{\text{ext}}^{(k)}$ ensures that $\mathcal{W}^{(k)}$ is real and positive.

External separability and coherence — Let us first consider pure, externally separable states with $N_p \in \{0,1\}$. These are exactly the states where each particle carries an individual pure internal state $|\phi_{\alpha}\rangle \in \mathcal{H}_{\text{int}}$, $i = 1, \ldots, N$ [48]. In this case, the non-zero matrix elements of $\rho_{\text{ext}}^{(k)}$ stem from multi-indices $m, n$ that are connected via a unique permutation $\pi \in S_k$ in the symmetric group of k elements: $(m_1 \ldots m_k) = (\pi^{-1}(1), \ldots, \pi^{-1}(k))$. They are given by products of overlaps of internal 1P states $\langle m | \rho_{\text{ext}}^{(k)} | m \rangle \propto \text{sgn}(\pi) \prod_{i=1}^k \langle \phi_{m_i}, \phi_{m_{\pi^{-1}(i)}} \rangle$, with $\text{sgn}(\pi)$ the signature of $\pi$ for fermions, and one for bosons.

In the “classical limit” of perfectly distinguishable particles, i.e. all particles have mutually orthogonal internal states, the reduced density matrices $\rho_{\text{ext}}^{(k)}$ are diagonal with $\mathcal{W}^{(k)} = \text{tr} \rho_{\text{ext}}^{(k)} = 1$, at any order $k$. Hence, in Eq. (1) only diagonal terms ($m = n$) contribute. In turn, any deviation of $\mathcal{W}^{(k)}$ from one signals the existence of coherence in $\rho_{\text{ext}}^{(k)}$, giving rise to (off-diagonal) kP interference contributions in Eq. (1). This extends the conventional interpretation of interference to the many-body setting. Indistinguishable bosons exhibit the maximum value of $\mathcal{W}^{(k)} = k!$, because all matrix elements are positive and equal. For indistinguishable fermions, each matrix element in $\mathcal{W}^{(k)}$ is canceled by another one (due to the factor $\text{sgn}(\pi)$), resulting in $\mathcal{W}^{(k)} = 0$.

More specifically, 2P coherences are given by $\langle m, n | \rho_{\text{ext}}^{(k)} | m, n \rangle \propto \pm \langle \phi_{m}, \phi_{n} \rangle^2$ (+ for bosons and − for fermions). This justifies the direct physical interpretation of $\mathcal{W}^{(2)}$ in terms of the particles’ distinguishability as controlled by the overlaps of their internal states. A numerical analysis shows that higher order mean coherences $\mathcal{W}^{(k)}$ depend on average strictly monotonically on $\mathcal{W}^{(2)}$ for the considered class of states. In Fig. 1, we present scatter plots of $\mathcal{W}^{(k)}$, $k > 2$, against $\mathcal{W}^{(2)}$ for states of
seven particles, each with a randomly sampled (pure) internal state \( |\phi_i^i\rangle \) in \( \mathcal{H}_{\text{int}} \), \( i = 1, \ldots , 7 \). The sampling of internal states is designed to cover the range of \( \mathcal{W}^{(2)} \) in [0, 2] as uniformly as possible. The precise procedure is supplemented [52], occupying seven orthogonal external modes. We observe an, on average, strictly monotonic dependence of \( \mathcal{W}^{(k)} \) on \( \mathcal{W}^{(2)} \) along the entire transition between indistinguishable fermions \( \mathcal{W}^{(2)} = 0 \) and bosons \( \mathcal{W}^{(2)} = 2 \). All measures unambiguously discriminate fermions \( \mathcal{W}^{(k)} < 1 \), distinguishable particles \( \mathcal{W}^{(k)} = 1 \), and bosons \( \mathcal{W}^{(k)} > 1 \).

FIG. 1. Correlation between \( \mathcal{W}^{(k)} \), \( k = 3, \ldots , 7 \), (log scale) and \( \mathcal{W}^{(2)} \), see eq. (2), for each of 1000 fermionic and bosonic externally separable 7P states with random internal pure states \( |\phi\rangle \) for each particle (the detailed sampling procedure is supplemented [52]), occupying seven orthogonal external modes. We observe an, on average, strictly monotonic dependence of \( \mathcal{W}^{(k)} \) on \( \mathcal{W}^{(2)} \) along the entire transition between indistinguishable fermions \( \mathcal{W}^{(2)} = 0 \) and bosons \( \mathcal{W}^{(2)} = 2 \). All measures unambiguously discriminate fermions \( \mathcal{W}^{(k)} < 1 \), distinguishable particles \( \mathcal{W}^{(k)} = 1 \), and bosons \( \mathcal{W}^{(k)} > 1 \).

The K\( P\) coherence measures defined in Eq. (2) are related to quantities already considered in the literature: In [27], the sums over all absolute values, respectively absolute square values, of matrix-elements of the full external NP state \( \rho_{\text{ext}} \) are considered as a measure of partial distinguishability. Because of the absolute values, these measures erase the difference between bosonic and fermionic statistics and cannot be directly accessed experimentally, in contrast to \( \mathcal{W}^{(k)} \). As we detail in the supplemental material [52], the degree of indistinguishability \( I \) introduced in [49, 54], to quantify partial distinguishability of bosonic Fock states, is proportional to \( \mathcal{W}^{(2)} - 1 \). Moreover, \( \mathcal{W}^{(N)} \prod_{m \in \mathcal{B}_{\text{ext}}} N_m! / N! \) measures the projection of \( \rho_{\text{ext}} \) on the symmetric subspace of \( \mathcal{H}_{\text{sym}}^N \), a quantity considered in [55] and [27, 56], which also coincides with the degree of interference [14]. Our present definition of \( \mathcal{W}^{(k)} \) links these quantities to the coherence of the reduced states \( \rho^{(k)}_{\text{ext}} \) and, therefore, allows a direct interpretation in terms of K\( P\) interference.

Connected correlators and random matrix average—For an arbitrary input state \( \rho \), the low-order interference terms in (1) typically dominate the expectation value. To enhance higher order contributions, we truncate lower orders by employing the connected, or truncated, \( k\)-point correlators, recursively defined as

\[
C_p^{(k)} = \langle U^\dagger \hat{N}_{p_1} \ldots \hat{N}_{p_k} U \rangle - \sum_{P \subset P} \prod_{p \in P} C_i^{(|q|)},
\]

where the sum runs over all non-trivial partitions \( P \) of modes \( p = \{p_1, \ldots , p_k\} \) into disjoint subsets \( q \) of length \( |q| \), each being associated with a possible factorization of the correlators. For example, for \( k = 2 \),

\[
C_{p_2}^{(2)} = \langle \hat{N}_{p_1} \hat{N}_{p_2} \rangle - \langle \hat{N}_{p_1} \rangle \langle \hat{N}_{p_2} \rangle
\]

is the covariance. Connected correlators are commonly used in various fields of physics (notably also in the theoretical analysis of many-body quantum systems [57]) and mathematics, where they are also known as joint cumulants.

By choosing \( U \) at random from the Haar measure [58] on the unitary group \( U(d) \), we conceive a correlation measurement in randomly chosen external modes. The average connected correlator [cf. eqs. (1),(3)] can be evaluated by formally integrating over the unitary group using the following identity (for orthogonal \( p_i \)) [59]:

\[
\prod_{i=1}^k U_{p_im_i} U^*_{p_in_i} = \sum_{\pi \in S_k} \text{Wg}_d(\pi) \prod_{i=1}^k \delta_{m_\pi(i), n_i}.
\]

The overline indicates the Haar integration and \( \text{Wg}_d \) is the Weingarten function [59, 60], which only depends on the cyclic structure of the permutation \( \pi \in S_k \) and on the external dimension \( d \). For \( k = 2, 3 \) and unit filling factor, i.e. \( N = d \), the truncation (3) of the correlators and the Haar average (4) cooperate in exactly the right way to ensure that all matrix elements of \( \rho^{(k)}_{\text{ext}} \) are uniformly weighted [48]:

\[
C_{p_1p_2}^{(2)} = \frac{\mathcal{W}^{(2)}}{N+1}, \quad C_{p_1p_2p_3}^{(3)} = \frac{2\mathcal{W}^{(3)}}{(N+1)(N+2)}.
\]

Note that this result holds for arbitrary, possibly externally entangled, input states. However, the interpretation of \( \mathcal{W}^{(k)} \) as a partial distinguishability measure is only valid in the case of externally separable input states, as discussed above. Relaxing the assumption \( N = d \) leads to a similar expression, where the various matrix elements of \( \rho^{(k)}_{\text{ext}} \) acquire different weights depending on \( d \) and \( N \), as we will show in detail elsewhere.

The linear relations (5) do not exactly hold at higher correlation orders. However, as we show by numerical
simulations, uniform sampling of the matrix elements of \( \rho^{(k)}_{ext} \) in the input basis – through the introduced scheme of truncated randomized correlations – is observed, to very good approximation, also for \( k > 3 \). In Fig. 2, we show that for 6P input states (sampled according to the same procedure as for Fig. 1 [52]), a tight relation persists between \( C^{(k)}_{p_1, \ldots, p_k} \) and \( \mathcal{W}^{(k)} \) [cf. eqs. (2),(3)] for \( k = 4, 5 \). Indeed, we observe an almost linear relationship between the two quantities over the entire range between indistinguishable fermions and bosons, garnished by small, but systematic, deviations from linearity. Averaging \( k \)-point correlators over randomly sampled unitaries \( U \) thus yields a valid estimate for the corresponding \( \mathcal{W}^{(k)} \). This approach is especially promising in reconﬁgurable linear optical networks [61–63]. Furthermore, Fig. 2 shows that replacing the random matrix integration by an average over all connected correlators \( C^{(k)}_{p_1, \ldots, p_k} \) of \( k \) out of \( d \) output modes for a single random unitary leads to a similar linear relation. Note, in this case, that while the resulting slope of \( C^{(k)}_p \) vs. \( \mathcal{W}^{(k)} \) depends on the specific unitary, the deviations from linearity are centered on the predictions from the Haar integration. Indeed, the equivalence of mode average and random matrix prediction is reasonable for large systems: Then, the matrix elements of a (sub-matrix of a) random unitary \( U \) are approximately i.i.d. Gaussian and, hence, the mode average approximately realizes a sample mean of the true distribution, which converges for large samples by the law of large numbers. This allows to estimate \( \mathcal{W}^{(k)} \) in experimental situations where sampling many random Haar unitaries is not possible.

In [7, 40, 42, 43] a statistical analysis of the moments of the distribution of connected two-point correlators was put forward as a certiﬁcation tool for MBI. It is clear from (1,5) that such a protocol does not access the coherence of \( \rho^{(k)}_{ext} \) for \( k > 2 \). However, Fig. 1 and the subsequent discussion shows that for the considered externally separable states, higher order coherence depends monotonically on the 2P coherences. Therefore, knowledge of \( \mathcal{W}^{(2)} \) is sufﬁcient to assess the mean coherence of the full NP state. This, however, is no longer valid for more general, entangled input states, as we will show now.

**External entanglement**— For externally entangled states, second order coherences of \( \rho^{(2)}_{ext} \) do not convey unambiguous information about higher order coherence, as we now demonstrate by example. Take orthogonal external and internal modes \( p, q, r \) and \( \alpha, \beta, \gamma, \) respectively. The entangled 2P state

\[
|\psi_2\rangle = \frac{1}{\sqrt{2}} \left( \hat{a}_{p\alpha}^{\dagger}\hat{a}_{q\beta}^{\dagger} - \hat{a}_{p\beta}^{\dagger}\hat{a}_{q\alpha}^{\dagger} \right) |0\rangle
\]

(6)

has \( \mathcal{W}^{(2)} = 0 \) for bosons and \( \mathcal{W}^{(2)} = 2 \) for fermions, i.e. the exact opposite of what is obtained for an externally separable state of indistinguishable particles (recall Fig. 2). Such swapping of quantum statistics induced by entanglement has, e.g., been discussed in [64–66]. A further example is given by the entangled 3P state

\[
|\psi_3\rangle = \frac{1}{\sqrt{3}} \left( \hat{a}_{p\alpha}^{\dagger}\hat{a}_{q\beta}^{\dagger}\hat{a}_{r\gamma}^{\dagger} + \hat{a}_{p\beta}^{\dagger}\hat{a}_{q\alpha}^{\dagger}\hat{a}_{r\gamma}^{\dagger} + \hat{a}_{p\gamma}^{\dagger}\hat{a}_{q\beta}^{\dagger}\hat{a}_{r\alpha}^{\dagger} \right) |0\rangle
\]

(7)

with \( \mathcal{W}^{(2)} = 1 \), but \( \mathcal{W}^{(3)} = 3 \), for both bosons and fermions, which contradicts the outlined monotonic dependence in Fig. 1 for externally separable states. Non-classical correlations as those inscribed into \( |\psi_3\rangle \) result in pure 3P interference: All 2P coherences \( \langle m, n | \rho^{(2)}_{ext} | n, m \rangle \), \( m \neq n \) vanish, such that any two-point correlation measurement, in fact any 2P observable as deﬁned in [49, 54], must yield a classical result, while an arbitrary three-point correlation measurement will unveil the presence of coherence in \( |\psi_3\rangle \). Note that the phenomenon of pure kP interference can be extended to higher orders by a suitable generalization of the state \( |\phi_3\rangle \), and that states with a similar cyclic structure are...
employed in [67] to define the notion of genuine k-partite indistinguishability.

Both $W^{(2)}$ and $W^{(3)}$ are experimentally directly accessible through the introduced protocol of randomized correlation measurements (5), such that a comparison of $W^{(3)}$ to the value predicted by Fig. 1 for externally separable states provides an experimental test of external mode entanglement. Hence, randomized correlation measurements promise diagnostic power to assess the multi-partite entanglement properties of the input state, in its external dof, with benign scaling properties of the experimental protocol.

**Conclusion —** We can now answer our initially formulated questions: $k$-point correlation measurements in randomized bases give (to very good approximation) direct access to the input state’s $kP$ mean coherence $W^{(k)}$ (Fig. 2). For externally separable input states, all $W^{(k)}$ describe the same, on average strictly monotonic transition from indistinguishable fermions to bosons, via the intermediate case of distinguishable particles (Fig. 1). This validates certification schemes of MBI, such as [7, 40, 42, 43], building on second-order correlation measurements (5), such that a comparison to the value predicted by Fig. 1 for externally separable input states provides an experimental test of external mode entanglement properties of the input state, in its external dof, with benign scaling properties of the experimental protocol.

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It was shown to correlate with the time-average reduced state evolving from the initial Fock state.

\[ I = \sum_{m \neq n \in \mathcal{B}_{\text{ext}}} N_{mn}N_{nm} / \sum_{m \neq n \in \mathcal{B}_{\text{ext}}} N_mN_n \]

This reduction, first, to only external dof, and, second, to a subset of particles, is sensitive to non-classical correlations between the external and internal dof, as well as between the particles. Such correlations are, however, to be sharply distinguished from external mode-entanglement which we here focus on.

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| [51]      | This reduction, first, to only external dof, and, second, to a subset of particles, is sensitive to non-classical correlations between the external and internal dof, as well as between the particles. Such correlations are, however, to be sharply distinguished from external mode-entanglement which we here focus on. |
| [52]      | Supplemental Material |

**Supplemental Material**

**RELATION OF THE MEAN COHERENCE TO QUANTITIES DEFINED IN THE LITERATURE**

For Fock states (i.e. eigenstates of all number operators \( N_p = \hat{a}_p^\dagger \hat{a}_p \)), the degree of indistinguishability

\[ I = \sum_{m \neq n \in \mathcal{B}_{\text{ext}}} N_{mn}N_{nm} / \sum_{m \neq n \in \mathcal{B}_{\text{ext}}} N_mN_n \]

was introduced in [49, 54] to quantify partial distinguishability in multi-component bosonic systems. In particular, \( I \) was shown to correlate with the time-average of the density variances \( \langle N_m^2(t) \rangle - \langle N_m(t) \rangle^2 \), which probe the 2P reduced state evolving from the initial Fock state. The degree of indistinguishability is related to the 2P mean coherence for arbitrary definite external mode occupations \( N_p \in \mathbb{N} \), by

\[ I = N(N - 1)(W^{(2)} - 1) / \sum_{m \neq n \in \mathcal{B}_{\text{ext}}} N_mN_n . \]

For an NP state \( \rho \), the reduced external state \( \rho_{\text{ext}}^{(N)} = \rho_{\text{ext}}^{(N)} \) coincides with \( 1/N! \) times the \( J \) matrix introduced in [15, 24, 55] if ideal detectors are assumed. Actually, the author of [15] writes “Note that quantum coherence in [15, 24, 55] if ideal detectors are assumed. Actually, the author of [15] writes “Note that quantum coherence

\[ p_s = \text{tr}(\rho_{\text{ext}}P_S) , \]

where \( P_S = \frac{1}{N!} \sum_{\pi \in S_N} \pi \) is the symmetrizer and \( \pi \) acts on \( m \in \mathcal{P}_{\text{ext}}^{(N)} \) as \( \pi [m] = |m_{\pi^{-1}(1)}, \ldots, m_{\pi^{-1}(N)} \rangle \). This quantity is proportional to the NP mean coherence, with

\[ p_s = W^{(N)} \prod_{m \in \mathcal{B}_{\text{ext}}} N_p!/N! . \]

For particles with individual pure internal states \( |\phi_i\rangle \), this is also equal to \( 1/N! \) times the permanent of the
distinguishability matrix $\mathcal{S} = ((\phi_i|\phi_j))_{i,j}$ introduced in [14].

**SAMPLING OF INTERNAL STATES**

To map out the full transition from indistinguishable fermions to bosons via the intermediate case of distinguishable particles, in terms of the $kP$ mean coherences $W^{(k)}$ as uniformly as possible, we use the following two-step sampling procedure of pure internal states for each of the particles: The dimension of the internal Hilbert space has to be taken larger than the particle number. To map out the neighborhood of indistinguishable particles, we start from a unit vector $|e\rangle \in \mathcal{H}_{\text{int}}$ and add a perturbation $|f_i\rangle$, with the real and imaginary parts of the components of $|f_i\rangle$ sampled from a normal distribution with zero mean and variance $\epsilon$. By choosing $\epsilon$ sufficiently small, the resulting internal states $|\phi_i\rangle = |e\rangle + |f_i\rangle$, after normalization, are almost parallel. The larger $\epsilon$ gets, the smaller the relative contribution of the constant vector $|e\rangle$ becomes, after renormalization, and we sample the unit sphere in $\mathcal{H}_{\text{int}}$ almost uniformly. As a second step, we sample the neighborhood of perfectly distinguishable particles by choosing orthogonal unit vectors $|e_i\rangle \in \mathcal{H}_{\text{int}}$ for each particle, perturbed by vectors $|f_i\rangle$ sampled as before with normally distributed components in $\mathbb{C}$, followed by renormalization. As before, for large $\epsilon$ the contributions from the constant vectors $|e_i\rangle$ in $|\phi_i\rangle = |e_i\rangle + |f_i\rangle$ vanish after renormalization, and we approach uniform sampling of the unit sphere in $\mathcal{H}_{\text{int}}$. For sufficiently small $\epsilon$, we generate states $|\phi_i\rangle$ in the vicinity of perfect distinguishability.