Gibbsianness of Fermion Random Point Fields

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Abstract

We consider fermion (or determinantal) random point fields on Euclidean space $\mathbb{R}^d$. Given a bounded, translation invariant, and positive definite integral operator $J$ on $L^2(\mathbb{R}^d)$, we introduce a determinantal interaction for a system of particles moving on $\mathbb{R}^d$ as follows: the $n$ points located at $x_1, \ldots, x_n \in \mathbb{R}^d$ have the potential energy given by

$$U^{(J)}(x_1, \ldots, x_n) := -\log \det(j(x_i - x_j))_{1 \leq i, j \leq n},$$

where $j(x - y)$ is the integral kernel function of the operator $J$. We show that the Gibbsian specification for this interaction is well-defined. When $J$ is of finite range in addition, and for $d \geq 2$ if the intensity is small enough, we show that the fermion random point field corresponding to the operator $J(I + J)^{-1}$ is a Gibbs measure admitted to the specification.

Keywords. Fermion random point fields, specification, Gibbs measure, interaction.

Running head. Gibbsianness of FRPF's

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1 Introduction

Fermion (or determinantal) random point fields (FRPF’s hereafter) are probability measures on the configuration space of particles (moving on discrete or continuum spaces) whose correlation functions are determined by determinants of matrices. See Section 2 for the definition. In many literature FRPF’s are investigated; the problem of existence, basic properties, ergodicity (for translationally invariant case), stochastic domination, and connection to other physical problems have been studied [2, 3, 6, 7, 8, 9, 15, 16, 19 and references therein].

The aim of this paper is to investigate the Gibbsianness of FRPF’s on continuum spaces and also to construct the suitable interactions. The Gibbsianness of FRPF’s on discrete spaces was first shown in [16] and then in [17] in different ways for suitable
FRPF's. Recently, Georgii and the present author studied the conditional intensity of FRPF's on continuum spaces and considered the Gibbsianness in a different way \[6\]. In this paper, we particularly focus on the Hamiltonian and Gibbsian specification to which certain FRPF's are admitted. It provides us with a new viewpoint for FRPF's. In addition, it has also other merits. First, there are not so many non-trivial examples of interactions for particle systems moving on continuum spaces for which the equilibrium measures (Gibbs measures) are proved to exist. The typical examples are the superstable interactions introduced by Ruelle \[13, 14\]. The Gibbsianness of FRPF's thus gives rise to another example of interactions. One more benefit comes from its applicability when one wants to construct some dynamics of particles for which given FRPF's are invariant. In \[17\] we have constructed the Glauber dynamics on the discrete space leaving a given FRPF invariant. In \[21\], we have constructed the Dirichlet forms and the associated diffusion processes on the configuration space of particles moving on continuum spaces for which certain FRPF's are invariant.

We briefly summarize the contents of this paper. Let \( J \) be a bounded, positive definite integral operator on \( L^2(\mathbb{R}^d) \). Suppose that its kernel function \( J(x, y), x, y \in \mathbb{R}^d \), is bounded, continuous, and translation invariant, i.e., there is a bounded and continuous function \( j(x) \) of positive type \[12\] such that
\[
J(x, y) = j(x - y). \tag{1.1}
\]

By using it we define the potential energy \( U^{(J)}(x_1, \cdots, x_n) \) of \( n \) particles located at \( x_1, \cdots, x_n \) by
\[
U^{(J)}(x_1, \cdots, x_n) := -\log \det(j(x_i - x_j))_{1 \leq i, j \leq n}. \tag{1.2}
\]

We show that the Gibbsian specification for the interaction is well-defined (Proposition 3.3). Furthermore, we show that if \( j(x) \) is of finite range, i.e., there is some \( R > 0 \) such that \( j(x) = 0 \) if \( |x| \geq R \) and, for \( d \geq 2 \), if the intensity \( j(0) \) is sufficiently small, the FRPF corresponding to the operator \( J(I + J)^{-1} \) is a Gibbs measure for the specification (Theorem 3.5).

This paper is organized as follows. In Section 2 we review the definition of FRPF's with basic properties. In Section 3 we define a Gibbsian specification and state the main results. Section 4 is devoted to the proofs. In Section 5 we discuss some possible improvements. In Appendix, we give examples of bounded and continuous functions of positive type which have finite ranges.
Gibbsianness of FRPF’s

2 Preliminaries

2.1 FRPF’s on Continuum Spaces

In this subsection we briefly recall the definition of FRPF’s. For a more complete survey on this field, we refer to the articles [2, 5, 7, 8, 15, 16, 19]. The state space for FRPF’s may be a very general separable Hausdorff space but in this paper we fix it to be $\mathbb{R}^d$. It is understood as a one particle space.

We denote by $\mathcal{N}$ the space of locally finite, integer-valued Radon measures on $\mathbb{R}^d$, equipped with the vague topology. We notice that an element (called a configuration) $\xi \in \mathcal{N}$ is expressible as

$$\xi = \sum_i k_i \delta_{x_i},$$

where each $k_i$ is a positive integer and $\delta_{x_i}$ is a Dirac measure, and distinct points $\{x_i\}$ form a countable set with at most finitely many $x_i$’s in any bounded Borel subset of $\mathbb{R}^d$. We recall that $\mathcal{N}$ is a Polish space, i.e., $\mathcal{N}$ can be given a metric so that it becomes a complete separable metric space. Moreover, the induced topology from that metric is equivalent to the vague topology [4, Corollary 7.1.IV and section A2.6]. The Borel $\sigma$-algebra $\mathcal{F}$ on $\mathcal{N}$ is equal to the smallest $\sigma$-algebra with respect to which the mappings

$$\xi \mapsto N_\Lambda(\xi) := \xi(\Lambda)$$

are measurable for any bounded Borel subset $\Lambda \subset \mathbb{R}^d$ [4, Corollary 7.1.VI]. For each Borel subset $\Delta \subset \mathbb{R}^d$, we let $\mathcal{F}_\Delta$ the $\sigma$-algebra on $\mathcal{N}$ generated by $N_\Lambda$’s where $\Lambda$ runs over all bounded Borel subsets of $\Delta$.

In the sequel a measurable subset of $\mathcal{N}$ will play a central role. Recall that $\xi \in \mathcal{N}$ is called simple if all the $k_i$’s are 1 in the representation (2.1). The space of all simple measures is denoted by $\Gamma$, which is a measurable subset of $\mathcal{N}$ [4, Proposition 7.1.III]. We will denote by $\mathcal{F}(\Gamma)$ resp. $\mathcal{F}(\Gamma)_\Delta$ the $\sigma$-algebras $\{A \cap \Gamma : A \in \mathcal{F}\}$ resp. $\{A \cap \Gamma : A \in \mathcal{F}_\Delta\}$ in $\Gamma$.

By a random point field (abbreviated RPF) we mean a triple $(\mathcal{N}, \mathcal{F}, \mu)$ where $\mu$ is a probability measure on $\mathcal{F}$. For simplicity we call such a measure $\mu$ itself as a RPF.

Definition 2.1 A locally integrable function $\rho_n : (\mathbb{R}^d)^n \rightarrow \mathbb{R}_+$ is called the $n$-point correlation function of a RPF $\mu$ if for any disjoint bounded Borel subsets $\Lambda_1, \cdots, \Lambda_m$ of $\mathbb{R}^d$ and $k_i \in \mathbb{Z}_+$, $i = 1, \cdots, m$, $\sum_{i=1}^m k_i = n$, the following identity holds:

$$\mathbb{E}_{\mu} \prod_{i=1}^m \frac{(N_{\Lambda_i})!}{(N_{\Lambda_i} - k_i)!} \int_{\Lambda_1^{k_1} \times \cdots \times \Lambda_m^{k_m}} \rho_n(x_1, \cdots, x_n) dx_1 \cdots dx_n,$$

where $\mathbb{E}_{\mu}$ denotes the expectation w.r.t. $\mu$ and $dx$ is the Lebesgue measure on $\mathbb{R}^d$. 
**Definition 2.2** A RPF is called fermion (or determinantal) if its \( n \)-point correlation functions are given by

\[
\rho_n(x_1, \cdots, x_n) = \det(K(x_i, x_j))_{1 \leq i, j \leq n}, \tag{2.4}
\]

where \( K(x, y), x, y \in \mathbb{R}^d \), denotes the integral kernel function of an integral operator \( K \) on \( L^2(\mathbb{R}^d) \).

For the existence of FRPF’s we state the following theorem from [13] (see also [9, 15]). We denote by \( I \) the identity operator on \( L^2(\mathbb{R}^d) \).

**Theorem 2.3** Hermitian locally trace class operator \( K \) on \( L^2(\mathbb{R}^d) \) defines a FRPF if and only if \( 0 \leq K \leq I \). If the corresponding FRPF exists it is unique.

In this paper we restrict ourselves to the Hermitian operators for the defining operator \( K \), but there are examples of FRPF’s with non-Hermitian operators [2]. We notice that from the determinantal nature of the correlation functions in (2.4), FRPF’s are in fact measures on \( (\Gamma, \mathcal{F}(\Gamma)) \). Below we summarize some basic properties of FRPF’s.

### 2.2 Basic Properties of FRPF’s

First we remark that any FRPF has a system of density distributions. Recall that the density distributions, or called the Janossy densities [11], of a RPF \( \mu \) are the measurable functions \( (\sigma^m_\Lambda) \), where \( m \in \mathbb{Z}_+ \) and \( \Lambda \) runs over all bounded Borel subsets of \( \mathbb{R}^d \), that satisfy following properties [11]:

(a) (Symmetry) \( \sigma^m_\Lambda(x_1, \cdots, x_m) = \sigma^m_\Lambda(x_{i_1}, \cdots, x_{i_m}) \) for every permutation \( (1, \cdots, m) \to (i_1, \cdots, i_m) \).

(b) (Normalization)

\[
\sum_{m=0}^{\infty} \frac{1}{m!} \int_{\Lambda^m} \sigma^m_\Lambda(x_1, \cdots, x_m)dx_1 \cdots dx_m = 1. \tag{2.5}
\]

(c) (Compatibility) If \( \Lambda \subset \Delta \), then

\[
\sigma^m_\Lambda(x_1, \cdots, x_m) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{(\Delta \setminus \Lambda)^n} \sigma^{m+n}_\Delta(x_1, \cdots, x_{m+n})dx_{m+1} \cdots dx_{m+n}. \tag{2.6}
\]

The relation between \( \mu \) and \( (\sigma^m_\Lambda) \) is given by the following properties: if \( f : \mathcal{N} \to \mathbb{R} \) is any measurable local (cylindrical) function, say \( \Lambda \)-local, then

\[
\int f(\xi)d\mu(\xi) = \sum_{m=0}^{\infty} \frac{1}{m!} \int_{\Lambda^m} f(x_1, \cdots, x_m)\sigma^m_\Lambda(x_1, \cdots, x_m)dx_1 \cdots dx_m. \tag{2.7}
\]
Moreover, the correlation functions of $\mu$ are then recovered from $(\sigma_{\Lambda}^n)$ by the following relation:

$$
\rho_n(x_1, \cdots, x_n) = \sum_{m=0}^{\infty} \frac{1}{m!} \int_{\Lambda^m} \sigma_{\Lambda}^{n+m}(x_1, \cdots, x_{n+m}) dx_{n+1} \cdots dx_{n+m}
$$

for $x_1, \cdots, x_n \in \Lambda$.

For each bounded Borel subset $\Lambda \subset \mathbb{R}^d$, we denote by $P_\Lambda$ the projection from $L^2(\mathbb{R}^d)$ onto $L^2(\Lambda)$. Let $\mu$ be any FRPF corresponding to an operator $K$ (see Theorem 2.3). Let $K_\Lambda := P_\Lambda K P_\Lambda$ be the restriction of $K$ to $L^2(\Lambda)$ and $K_\Lambda(x, y)$ its kernel function. That is, $K_\Lambda(x, y) = 1_\Lambda(x) K(x, y) 1_\Lambda(y)$ with $1_\Lambda$ being the characteristic function on the set $\Lambda$. The density functions of $\mu$ are given by

$$
\sigma_{\Lambda}^n(x_1, \cdots, x_m) = \det(I - K_\Lambda) \det(J_{[\Lambda]}(x_i, x_j))_{1 \leq i, j \leq m}, \quad x_i \in \Lambda, \ i = 1, \cdots, m,
$$

where $\det(I - K_\Lambda)$ is a Fredholm determinant and $J_{[\Lambda]} := K_\Lambda(I - K_\Lambda)^{-1}$. In the following remark we gather some basic facts about the density distributions for FRPF’s.

**Remark 2.4** (i) The formula (2.9) is well-defined even in the case $1 \in \text{spec } K_\Lambda$, the spectrum of $K_\Lambda$. See [15].

(ii) We recall that given a trace class operator $T$ on $L^2(\mathbb{R}^d)$ the Fredholm determinant of $I + T$ is given by

$$
\det(I + T) = \sum_{n=0}^{\infty} \text{Tr}(\wedge^n T),
$$

where $\wedge^n T$ is the $n$-th exterior product of $T$ and the following estimate holds:

$$
\| \wedge^n T \|_1 \leq \frac{1}{n!} \| T \|_1^n,
$$

where $\| \cdot \|_1$ is the trace norm. For the density distributions of FRPF’s we have the following relation:

$$
\mathbb{E}(1_{\{N_\Lambda = m\}}) = \frac{1}{m!} \int_{\Lambda^m} \sigma_{\Lambda}^n(x_1, \cdots, x_m) dx_1 \cdots dx_m
= \det(I - K_\Lambda) \frac{1}{m!} \int \det(J_{[\Lambda]}(x_i, x_j))_{1 \leq i, j \leq m} dx_1 \cdots dx_m
= \det(I - K_\Lambda) \text{Tr}(\wedge^m(J_{[\Lambda]})).
$$

The last equality follows from [18, Theorem 3.10] if the kernel function $J_{[\Lambda]}(x, y)$ is continuous. The general case follows from the argument of [19] (see the equation (1.26) of [19]).
(iii) If $\mu$ is a FRPF corresponding to an integral operator $K$, using the expression (2.9) we have the following Laplace transform of $\mu$ (cf. [15]): for $f \in C_0(\mathbb{R}^d)$,

$$\int \exp(-<f, \xi>)d\mu(\xi) = \det(I - K\psi_f),$$  \hspace{1cm} (2.13)

where $<f, \xi> := \sum_{x_i \in \xi} f(x_i)$ and $\psi_f(x) = 1 - \exp(-f(x))$ and $K\psi_f$ is the product of $K$ and the multiplication operator with function $\psi_f$, and the determinant is a Fredholm determinant.

3 Results

3.1 Determinantal Potentials and Gibbsian Specifications

In this subsection we introduce a particle system with an interaction which is given by determinants of matrices. The matrix components are given by the kernel function of an integral operator on $L^2(\mathbb{R}^d)$. We consider bounded linear operators $J$ on $L^2(\mathbb{R}^d)$ constructed in the following way:

**Assumption 3.1** The operator $J$ is defined as an integral operator on $L^2(\mathbb{R}^d)$ with integral kernel function

$$J(x, y) := j(x - y),$$  \hspace{1cm} (3.1)

where $j(\cdot) \in L^1(\mathbb{R}^d)$ is a (inverse) Fourier transform of a finite measure $d\rho$ on $\mathbb{R}^d$:

$$j(x) := (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ix\cdot t}d\rho(t).$$  \hspace{1cm} (3.2)

By Bochner's theorem and Young's inequality [12], the operators $J$ in Assumption 3.1 are bounded, positive definite linear operators on $L^2(\mathbb{R}^d)$.

We fix an operator $J$ satisfying the conditions in Assumption 3.1. For each integer $n \geq 0$ and $x_1, \cdots, x_n \in \mathbb{R}^d$, the potential energy $U^{(J)}(x_1, \cdots, x_n)$ of $n$ particles located at $x_1, \cdots, x_n$ is defined by

$$U^{(J)}(x_1, \cdots, x_n) := -\log \det(j(x_i - x_j))_{1 \leq i, j \leq n}. \hspace{1cm} (3.3)$$

Since the matrix $(j(x_i - x_j))_{1 \leq i, j \leq n}$ as an operator on $\mathbb{C}^n$ is positive definite the function $\det(j(x_i - x_j))_{1 \leq i, j \leq n}$ is nonnegative. For a convenience we set $-\log 0 \equiv +\infty$. Then $U^{(J)}(x_1, \cdots, x_n)$ is well-defined with values in $\mathbb{R} \cup \{+\infty\}$ (if $n = 0$, we set $U^{(J)} = 0$), in particular if some of two points $x_i$ and $x_j$ in $(x_1, \cdots, x_n)$ are the same, then $U^{(J)}(x_1, \cdots, x_n) = +\infty$, i.e., under this interaction, two or more particles can
not share a single point. It is obvious that \( U^{(J)} \) is translation invariant. Moreover, we notice that
\[
\det \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} = 0 \quad \text{whenever} \quad \det A = 0 \text{ or } \det C = 0 \quad (3.4)
\]
for any positive definite matrices \( \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \) (\( B^* \) denotes the adjoint matrix of \( B \)).

This follows from the following Fischer’s inequality:
\[
\det \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \leq \det A \det C. \quad (3.5)
\]

Thus, \( U^{(J)} \) is uniquely decomposed as
\[
U^{(J)}(x_1, \cdots, x_n) = \sum_k \sum_{1 \leq i_1 < \cdots < i_k \leq n} \Phi^{(J)}_k(x_{i_1}, \cdots, x_{i_k}) \quad (3.6)
\]
with some functions \( \Phi^{(J)}_k \). Notice that the \( k \)-body potential \( \Phi^{(J)}_k \) is invariant under permutation of its \( k \) arguments and under translations in \( \mathbb{R}^d \). We call the sequence \( (\Phi^{(J)}_k)_{k \geq 1} \) of \( k \)-body potentials the interaction determined by the operator \( J \).

We now construct a Gibbsian specification for the interaction determined by \( J \). For conveniences, we introduce the following notations.

**Notations:** Each element \( \xi \in \mathcal{N} \) will also be understood as a finite or countably infinite sequence \( \xi = (x_1, x_2, \cdots) \) in \( \mathbb{R}^d \) determined by the support of the measure \( \xi \) (see (2.1)). Of course, some of the components are repeated in general, but when \( \xi \in \Gamma \), all the components are distinct. Any set of finite points \( (x_1, \cdots, x_n) \in (\mathbb{R}^d)^n \) is denoted by \( x_n \). For any \( \Lambda \subset \mathbb{R}^d \), \( \xi_\Lambda \) represents a configuration on \( \Lambda \) or a restriction of a configuration \( \xi \in \mathcal{N} \) to the region \( \Lambda \), i.e., \( \xi_\Lambda = \xi \cap \Lambda \). \( \mathcal{N}_\Lambda \) denotes the set of all configurations \( \xi \) such that \( \xi = \xi_\Lambda \) and set \( \Gamma_\Lambda := \mathcal{N}_\Lambda \cap \Gamma \). If \( \Lambda_1 \) and \( \Lambda_2 \) are two disjoint subsets of \( \mathbb{R}^d \), by \( \xi_{\Lambda_1}, \xi_{\Lambda_2} \) we denote a configuration in \( \mathcal{N}_{\Lambda_1 \cup \Lambda_2} \) which coincides with \( \xi_{\Lambda_1} \) on \( \Lambda_1 \) and with \( \xi_{\Lambda_2} \) on \( \Lambda_2 \). Any bounded Borel subset \( \Lambda \subset \mathbb{R}^d \) is denoted by \( \Lambda \subset \subset \mathbb{R}^d \). For any function \( A(\cdot, \cdot) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C} \) and \( x_n = (x_1, \cdots, x_n) \in (\mathbb{R}^d)^n \), \( A(x_n, x_n) \) denotes the finite matrix
\[
A(x_n, x_n) := (A(x_i, x_j))_{1 \leq i, j \leq n}. \quad (3.7)
\]
Finally, for any \( \Lambda \subset \subset \mathbb{R}^d \) and a \( \Lambda \)-local function \( f \) we simplify the integration
\[
\sum_{n \geq 0} \frac{1}{n!} \int_{\Lambda^n} d\xi_n f(\xi_n) =: \int_{\Lambda} d\xi f(\xi), \quad (3.8)
\]
where \( d\xi_n \) denotes the Lebesgue measure on \( (\mathbb{R}^d)^n \).
We now consider the energy of a particle configuration in a bounded region with a given boundary condition. We will need to refine the boundary particles that would interact with some particles inside the region. We say that the system has an interaction range $R \in (0, +\infty]$ defined by

$$R := \inf\{R' \in \mathbb{R} : j(x) = 0 \text{ whenever } |x| \geq R'\}. \quad (3.9)$$

(We consider only the case $R > 0$.) Given a region $\Lambda \subset \mathbb{R}^d$ and a configuration $\xi = (x_i)_{i=1,2,\ldots} \in \mathcal{N}$, we define a subset $\xi_{\partial \Lambda} \subset \xi_{\Lambda^c}$ of boundary particles that interact with the particles inside $\Lambda$ as follows. In the case $R = +\infty$, we let $\xi_{\partial \Lambda} \equiv \xi_{\Lambda^c}$. In the case $R < \infty$, we say that a particle $x_i \in \xi_{\Lambda^c}$ interacts with particles in the region $\Lambda$ if there is a finite sequence $(x_{j_1}, \ldots, x_{j_k}) \subset \xi_{\Lambda^c}$ and a point $x_{j_0} \in \Lambda$ such that $x_{j_k} = x_i$ and $|x_{j_l} - x_{j_{l-1}}| < R$ for $l = 1, \ldots, k$. We define

$$\xi_{\partial \Lambda} := \{x_i \in \xi_{\Lambda^c} : x_i \text{ interacts with particles inside } \Lambda\}. \quad (3.10)$$

From the decomposition (3.6) we see that for any $\Lambda_1, \Lambda_2 \subset \mathbb{R}^d$ with $\Lambda_1 \cap \Lambda_2 = \emptyset$, and finite configurations $\xi_{\Lambda_1}$ and $\xi_{\Lambda_2}$, the mutual potential energy $W^{(J)}(\xi_{\Lambda_1}; \xi_{\Lambda_2})$ is well-defined to satisfy

$$U^{(J)}(\xi_{\Lambda_1 \cup \Lambda_2}) = U^{(J)}(\xi_{\Lambda_1}) + U^{(J)}(\xi_{\Lambda_2}) + W^{(J)}(\xi_{\Lambda_1}; \xi_{\Lambda_2}) \quad (3.11)$$

if $U^{(J)}(\xi_{\Lambda_1 \cup \Lambda_2}) < \infty$, and $W^{(J)}(\xi_{\Lambda_1}; \xi_{\Lambda_2}) = \infty$ if $U^{(J)}(\xi_{\Lambda_1 \cup \Lambda_2}) = \infty$. Now for each $\Lambda \subset \mathbb{R}^d$, $\zeta_\Lambda \in \mathcal{N}_\Lambda$, and $\xi \in \mathcal{N}$, we define the energy of the particle configuration $\zeta_\Lambda$ on $\Lambda$ with boundary condition $\xi$ by

$$H_\Lambda^{(J)}(\zeta_\Lambda; \xi) := \lim_{\Delta \uparrow \mathbb{R}^d} [U^{(J)}(\zeta_\Lambda) + W^{(J)}(\zeta_\Lambda; \hat{\xi}_{\Delta \setminus \Lambda})], \quad (3.12)$$

whenever the limit exists. Here $\hat{\xi}_{\Delta \setminus \Lambda}$ is defined by

$$\hat{\xi}_{\Delta \setminus \Lambda} := \xi_{\Delta \setminus \Lambda} \cap \xi_{\partial \Lambda}. \quad (3.13)$$

In Lemma 3.2 below we show that $H_\Lambda^{(J)}(\zeta_\Lambda; \xi)$ does exist for all $\zeta_\Lambda \in \Gamma_\Lambda$ and “physically possible” configurations $\xi \in \Gamma$. For that purpose we introduce the following events. For each $\Lambda \subset \mathbb{R}^d$, define a subset $\mathcal{R}_\Lambda \in \mathcal{F}_{\Lambda^c}$, which will represent the “possible” event in $\mathcal{F}_{\Lambda^c}$ (see [11, page 16]), as follows:

$$\mathcal{R}_\Lambda := \{\xi \in \mathcal{N} : \det(J(\hat{\xi}_\Delta, \hat{\xi}_\Delta)) \neq 0, \quad \forall \Delta \subset \subset \Lambda^c\}, \quad (3.14)$$

where as before $\hat{\xi}_\Delta := \xi_\Delta \cap \xi_{\partial \Lambda}$.

**Lemma 3.2** Suppose that $J$ is an integral operator on $L^2(\mathbb{R}^d)$ satisfying the conditions in Assumption 3.1. Then for any $\Lambda \subset \mathbb{R}^d$, $\zeta_\Lambda \in \Gamma_\Lambda$, and $\xi \in \mathcal{R}_\Lambda$, the function $H_\Lambda^{(J)}(\zeta_\Lambda; \xi)$ in (3.12) is well-defined.
We are now ready to define the Gibbsian specification. For a convenience we extend the function $H^{(J)}_\Lambda(\zeta;\xi)$ to the whole $\zeta_\Lambda \in \mathcal{N}_\Lambda$ and $\xi \in \mathcal{N}$. We set

\begin{equation}
H^{(J)}_\Lambda(\zeta;\xi) \equiv +\infty \text{ unless } \zeta_\Lambda \in \Gamma_\Lambda \text{ and } \xi \in \mathcal{R}_\Lambda.
\end{equation}

For each $\Lambda \subset \subset \mathbb{R}^d$, $\zeta_\Lambda \in \mathcal{N}_\Lambda$, and $\xi \in \mathcal{N}$, we define the function

\begin{equation}
\gamma^{(J)}_\Lambda(\zeta;\xi) := \begin{cases} 
\frac{1}{Z^{(J)}_\Lambda(\xi)} \exp[-H^{(J)}_\Lambda(\zeta;\xi)], & \text{if } \zeta_\Lambda \in \Gamma_\Lambda \text{ and } \xi \in \mathcal{R}_\Lambda, \\
0, & \text{otherwise}.
\end{cases}
\end{equation}

In the above $Z^{(J)}_\Lambda(\xi)$ is the partition function, i.e., a normalization constant defined by

\begin{equation}
Z^{(J)}_\Lambda(\xi) := \sum_{n \geq 0} \frac{1}{n!} \int_{\Lambda^d} d\mu_n \exp[-H^{(J)}_\Lambda(x;\xi)].
\end{equation}

We let $J_\Lambda$ denote the restriction of the operator $J$ to $L^2(\Lambda)$:

\begin{equation}
J_\Lambda := P_\Lambda J P_\Lambda.
\end{equation}

It turns out that for any $\xi \in \mathcal{R}_\Lambda$, $Z^{(J)}_\Lambda(\xi)$ is a finite number satisfying (see (4.14))

\begin{equation}
1 \leq Z^{(J)}_\Lambda(\xi) \leq \det(I + J_\Lambda),
\end{equation}

where $\det(I + J_\Lambda)$ is the Fredholm determinant of the operator $I + J_\Lambda$. Now for any bounded measurable function $f : \mathcal{N} \to \mathbb{R}$, $\Lambda \subset \subset \mathbb{R}^d$, and $\xi \in \mathcal{N}$, we define

\begin{equation}
\gamma^{(J)}_\Lambda(f|\xi) := \int_{\Lambda} d\xi_\Lambda \gamma^{(J)}_\Lambda(\zeta_\Lambda;\xi)f(\zeta_\Lambda \xi_{\Lambda^c}).
\end{equation}

We will prove that the system $(\gamma^{(J)}_\Lambda(\cdot|\cdot))_{\Lambda \subset \subset \mathbb{R}^d}$ defines a specification.

### 3.2 Gibbsianness of FRPF’s

We start by summarizing the construction in the last subsection.

**Proposition 3.3** Suppose that $J$ satisfies the conditions in Assumption 3.1. Then the system $(\gamma^{(J)}_\Lambda(\cdot|\cdot))_{\Lambda \subset \subset \mathbb{R}^d}$ given in (3.18) defines a specification with respect to $\mathcal{R} := (\mathcal{R}_\Lambda)_{\Lambda \subset \subset \mathbb{R}^d}$ (see [11, page 16]).

The main purpose of this paper is to characterize the Gibbs measures admitted to the specification $(\gamma^{(J)}_\Lambda(\cdot|\cdot))_{\Lambda \subset \subset \mathbb{R}^d}$ in the above. Recall that a probability measure $\mu$ on $(\mathcal{N}, \mathcal{F})$ is said to be admitted to $(\gamma^{(J)}_\Lambda(\cdot|\cdot))_{\Lambda \subset \subset \mathbb{R}^d}$, or to satisfy the DLR equations (see [5, 11]) if for any $\Lambda \subset \subset \mathbb{R}^d$ and bounded measurable function $f : \mathcal{N} \to \mathbb{R}$,

\begin{equation}
\int d\mu(\xi)f(\xi) = \int d\mu(\xi)\gamma^{(J)}_\Lambda(f|\xi).
\end{equation}
Suppose that $J$ is an integral operator as in Assumption 3.1. We define
\[ K^{(J)} := J(I + J)^{-1}. \] (3.22)

$K^{(J)}$ then is a locally trace class operator and satisfies $0 \leq K^{(J)} < I$. Therefore by Theorem 2.3 defines a FRPF which we denote by $\mu^{(J)}$. We conjecture that $\mu^{(J)}$ is a Gibbs measure for the specification in Proposition 3.3. Unfortunately, however, we couldn’t completely prove it. We impose further conditions on the operator $J$:

**Assumption 3.4** In addition to the conditions in Assumption 3.1, we assume that the finite measure $d\rho(t)$ in (3.2) has a density: $d\rho(t) = \hat{\varphi}(t) dt$, and $j(\cdot)$ is of finite range, i.e., there exists $0 < R < \infty$ such that
\[ j(x) = 0 \text{ if } |x| \geq R. \] (3.23)

In the Appendix, we provide with some examples of $j(\cdot)$ in Assumption 3.4. We call the finite number $J(0,0) \equiv j(0)$ the intensity of the system (in [9, page 112], the terminology “intensity” was used for the quantity $K^{(J)}(0,0)$, but the two are similar in nature). The following is a main result of this paper:

**Theorem 3.5** Suppose that $J$ is an integral operator on $L^2(\mathbb{R}^d)$ satisfying the conditions in Assumption 3.4. For $d \geq 2$, assume further that the intensity $j(0)$ is small enough. Then the corresponding FRPF $\mu^{(J)}$ is a Gibbs measure admitted to the specification $(\gamma^{(J)}_\Lambda)_{\Lambda \subset \subset \mathbb{R}^d}$ in Proposition 3.3.

We also introduce the activity of the system. We recall [13] that by an activity $z > 0$ of the system we mean that for any $\Lambda \subset \subset \mathbb{R}^d$, a grand canonical ensemble on $\sum_{n \geq 0} \Lambda^n$ is a measure with restriction to $\Lambda^n$ given by
\[ \frac{z^n}{n!} \exp[-U^{(J)}(x_1, \cdots, x_n)] dx_1 \cdots x_n. \] (3.24)
Analogously, for each $z > 0$ we define a new specification $(\gamma^{(J,z)}_\Lambda)_{\Lambda \subset \subset \mathbb{R}^d}$ by multiplying $z^{\left|\Lambda\right|}$, $\left|\Lambda\right|$ being the cardinality of $\Lambda$, in front of $\exp[-H^{(J)}(\zeta; \xi)]$ in (3.16) and suitably re-defining the partition function $Z^{(J,z)}_\Lambda(\xi)$ as
\[ Z^{(J,z)}_\Lambda(\xi) := \sum_{n \geq 0} \frac{z^n}{n!} \int_{\Lambda^n} dx_n \exp[-H^{(J)}(\zeta; \xi)]. \] (3.25)
We say that the system has an activity $z$.

**Corollary 3.6** Assume that $J$ is an integral operator satisfying the conditions in Assumption 3.4. If the activity $z > 0$ of the system is sufficiently small then the FRPF $\mu^{(z,J)}$ is a Gibbs measure for the specification $(\gamma^{(J,z)}_\Lambda)_{\Lambda \subset \subset \mathbb{R}^d}$. 
Proof: It is easily seen from (3.3) and (3.11)-(3.12) that
\[ H^\Lambda_{\Delta}(\zeta; \xi) = -\log |\zeta| + H^J_{\Delta}(\zeta; \xi). \] (3.26)
(See (4.2).) Thus the scaling property \( \gamma^{(J,\zeta)}_{\Lambda} = \gamma^{(z,\zeta)}_{\Lambda} \) holds, i.e., the specification \( (\gamma^{(J,\zeta)}_{\Lambda})_{\Lambda \subset \mathbb{R}^d} \) is the same as \( (\gamma^{(z,\zeta)}_{\Lambda})_{\Lambda \subset \mathbb{R}^d} \). Therefore, smallness of the activity implies smallness of the intensity \( zj(0) \) of the system distributed by \( \mu^{(z,\zeta)} \). The conclusion follows from Theorem 3.5. □

The proofs are provided in the next section.

4 Proofs

This section is devoted to the proofs of the results stated in the last section. In order to prove the Gibbsianness we will first observe that it is the case for the FRPF’s of compact supported operators. Then we prove that the FRPF’s of our concern are weak limits of such measures. We will apply these facts after approximating some bounded measurable functions by good bounded continuous functions.

4.1 Proof of Proposition 3.3

In this subsection we prove the construction of Gibbsian specification, Proposition 3.3. First we prove Lemma 3.2. Recall the notations \( H^\Lambda_{\Delta}(\zeta; \xi) \) and \( R^\Lambda_{\Delta} \), respectively in (3.12) and (3.14).

Proof of Lemma 3.2: Let \( \Lambda \subset \mathbb{R}^d \), \( \zeta \in \Gamma^\Lambda \), and \( \xi \in \mathcal{R}^\Lambda \). Without loss, we consider only the case of the interaction range \( R = +\infty \). (For the case of \( R < \infty \), we only need to use \( \hat{\xi} \) for \( \xi \) below.) For any bounded Borel set \( \Delta \supset \Lambda \) define
\[ H^J_{\Lambda,\Delta}(\zeta; \xi) := U^J(\zeta, \Delta) + W^J(\varphi(\zeta, \Delta \setminus \Lambda)). \] (4.1)
Since \( \xi \in \mathcal{R}^\Lambda \), \( \det J(\varphi^\Lambda_{\Delta \setminus \Lambda}) \neq 0 \). That is, \( U^J(\varphi^\Lambda_{\Delta \setminus \Lambda}) < \infty \) for all bounded \( \Delta \supset \Lambda \). Therefore, by (3.11) and (3.3) we see that
\[ H^J_{\Lambda,\Delta}(\zeta; \xi) = U^J(\zeta, \Delta \setminus \Lambda) - U^J(\varphi(\zeta, \Delta \setminus \Lambda)) \]
\[ = -\log \frac{\det J(\varphi(\zeta, \Delta \setminus \Lambda), \varphi(\zeta, \Delta \setminus \Lambda))}{\det J(\varphi(\zeta, \Delta \setminus \Lambda), \varphi(\zeta, \Delta \setminus \Lambda))}. \] (4.2)
If \( \det J(\varphi(\zeta, \Delta \setminus \Lambda), \varphi(\zeta, \Delta \setminus \Lambda)) = 0 \) for some \( \Delta \supset \Lambda \), then \( \det J(\varphi(\zeta, \Delta \setminus \Lambda), \varphi(\zeta, \Delta \setminus \Lambda)) = 0 \) for all \( \Delta' \supset \Delta \) (see (5.1)), and thus we are done. We suppose \( \det J(\varphi(\zeta, \Delta \setminus \Lambda), \varphi(\zeta, \Delta \setminus \Lambda)) \neq 0 \) for
all $\Delta \supset \Lambda$ (in particular, $J(\zeta_A\xi_{\Delta\setminus\Lambda},\zeta_A\xi_{\Delta\setminus\Lambda})$ and its submatrices are invertible). By an elementary manipulation on determinants of finite matrices we have the identity:

$$\frac{\det J(\zeta_A\xi_{\Delta\setminus\Lambda},\zeta_A\xi_{\Delta\setminus\Lambda})}{\det J(\xi_{\Delta\setminus\Lambda},\xi_{\Delta\setminus\Lambda})} = \det (J(\zeta_A,\zeta_A) - J(\zeta_A,\xi_{\Delta\setminus\Lambda})J(\xi_{\Delta\setminus\Lambda},\xi_{\Delta\setminus\Lambda})^{-1}J(\xi_{\Delta\setminus\Lambda},\zeta_A)), \tag{4.3}$$

where we have used the obvious notations, e.g., $J(\zeta_A,\xi_{\Delta\setminus\Lambda})$ is the matrix

$$J(\zeta_A,\xi_{\Delta\setminus\Lambda}) = (J(x_i,y_j))_{x_i \in \zeta_A; y_j \in \xi_{\Delta\setminus\Lambda}}. \tag{4.4}$$

Let $l^2(\zeta_A\xi_{\Delta\setminus\Lambda})$ be the (complex-valued) $l^2$-space with index set $\zeta_A\xi_{\Delta\setminus\Lambda}$. For any $\Delta \supset \Lambda$ let $Q_{\Lambda}$ be the projection operator on $l^2(\zeta_A\xi_{\Delta\setminus\Lambda})$ onto $l^2(\zeta_A\xi_{\Delta\setminus\Lambda})$. For a convenience we understand $J(\zeta_A\xi_{\Delta\setminus\Lambda},\zeta_A\xi_{\Delta\setminus\Lambda})$ as $Q_{\Lambda}J(\zeta_A\xi_{\Delta\setminus\Lambda},\zeta_A\xi_{\Delta\setminus\Lambda}) \sim Q_{\Delta}$ where $J(\zeta_A\xi_{\Delta\setminus\Lambda},\zeta_A\xi_{\Delta\setminus\Lambda}) \sim := J(\zeta_A\xi_{\Delta\setminus\Lambda},\zeta_A\xi_{\Delta\setminus\Lambda}) \oplus 1$ is a bounded linear operator on $l^2(\zeta_A\xi_{\Delta\setminus\Lambda}) \equiv l^2(\zeta_A\xi_{\Delta\setminus\Lambda}) \oplus l^2(\xi_{\Delta\setminus\Lambda})$ acting as an identity operator on $l^2(\xi_{\Delta\setminus\Lambda})$. We notice that the r.h.s. of (4.3) is equal to

$$\left(\det (Q_{\Lambda}J(\zeta_A\xi_{\Delta\setminus\Lambda},\zeta_A\xi_{\Delta\setminus\Lambda})^{-1}Q_{\Delta})\right)^{-1}. \tag{4.5}$$

On the other hand, for any bounded operator $T$ with bounded inverse $T^{-1}$ and any projection $P$, from the decomposition (see [10] page 18] and [6] for a proof)

$$PT^{-1}P = P(PTP)^{-1}P + PT^{-1}P^\perp (P^\perp T^{-1}P^\perp)^{-1}P^\perp T^{-1}P, \tag{4.6}$$

we get the inequality

$$PT^{-1}P \geq P(PTP)^{-1}P. \tag{4.7}$$

Therefore if $\Delta' \supset \Delta$, then by replacing $P = Q_{\Delta}$ and $T = J(\zeta_A\xi_{\Delta\setminus\Lambda},\zeta_A\xi_{\Delta\setminus\Lambda})$ in (4.7) we get

$$Q_{\Delta}J(\zeta_A\xi_{\Delta\setminus\Lambda},\zeta_A\xi_{\Delta\setminus\Lambda})^{-1}Q_{\Delta} \geq Q_{\Delta}(Q_{\Delta}J(\zeta_A\xi_{\Delta\setminus\Lambda},\zeta_A\xi_{\Delta\setminus\Lambda})Q_{\Delta})^{-1}Q_{\Delta} = Q_{\Delta}J(\zeta_A\xi_{\Delta\setminus\Lambda},\zeta_A\xi_{\Delta\setminus\Lambda})^{-1}Q_{\Delta} = J(\zeta_A\xi_{\Delta\setminus\Lambda},\zeta_A\xi_{\Delta\setminus\Lambda})^{-1}. \tag{4.8}$$

Applying $Q_{\Lambda}$ from left and right of both sides of (4.8) we see that

$$Q_{\Lambda}J(\zeta_A\xi_{\Delta\setminus\Lambda},\zeta_A\xi_{\Delta\setminus\Lambda})^{-1}Q_{\Lambda} \geq Q_{\Lambda}J(\zeta_A\xi_{\Delta\setminus\Lambda},\zeta_A\xi_{\Delta\setminus\Lambda})^{-1}Q_{\Lambda}, \quad \Delta' \supset \Delta, \tag{4.9}$$

as operators on $l^2(\zeta_A)$. Notice also that if $0 \leq A \leq B$ are two positive definite $n \times n$ matrices then

$$0 \leq \lambda_i^A(A) \leq \lambda_i^B(B), \quad i = 1, \cdots, n, \tag{4.10}$$

where $\lambda_i^A(A)$ ($\lambda_i^B(B)$, respectively), $i = 1, \cdots, n$, are the eigenvalues of $A$ (of $B$, respectively) ordered in decreasing order [11 Corollary III.2.3]. Therefore from (4.3),
and (4.9), the l.h.s. of (4.3) decreases as $\Delta$ increases. From this and monotonicity of logarithmic function it follows that the limit

$$H^{(J)}_{\Lambda}(\zeta; \xi) := \lim_{\Delta \uparrow \mathbb{R}^d} H^{(J)}_{\Lambda; \Delta}(\zeta; \xi)$$

exists. □

Now let us recall the definition of $\gamma^{(J)}_{\Lambda}(\zeta; \xi)$ in (3.16). For $\xi \in \mathcal{R}_\Lambda$, the partition function $Z^{(J)}_{\Lambda}(\xi)$ is defined by

$$Z^{(J)}_{\Lambda}(\xi) = 1 + \sum_{n \geq 1} \frac{1}{n!} \int_{\Lambda^n} dx_n \exp[-H^{(J)}_{\Lambda}(x_n; \xi)].$$

In the case $R = \infty$, from (4.11) and (4.2)-(4.3) we see that

$$\exp[-H^{(J)}_{\Lambda}(x_n; \xi)] = \lim_{\Delta \uparrow \mathbb{R}^d} \exp[-H^{(J)}_{\Lambda; \Delta}(x_n; \xi)]$$

$$= \lim_{\Delta \uparrow \mathbb{R}^d} \det(J(x_n, x_n; \xi, \xi) - J(x_n, \xi_{\Delta \setminus \Lambda})J(\xi_{\Delta \setminus \Lambda}, \xi_{\Delta \setminus \Lambda})^{-1}J(\xi_{\Delta \setminus \Lambda}, x_n))$$

$$\leq \det J(x_n, x_n).$$

Thus from (4.12)-(4.13) we have

$$1 \leq Z^{(J)}_{\Lambda}(\xi) \leq 1 + \sum_{n \geq 1} \frac{1}{n!} \int_{\Lambda^n} dx_n \det J(x_n, x_n) = \det(I + J_{\Lambda}).$$

In the last equality we have used a formula for the Fredholm determinant [18, Theorem 3.10]. In the case $R < \infty$, we just replace $\xi$ by $\tilde{\xi}$ (see (3.12)) in the above, and we get (4.14), too.

We now prove Proposition 3.3.

Proof of Proposition 3.3: Let us define for $\Lambda \subset \subset \mathbb{R}^d$, $A \in \mathcal{F}$, and $\xi \in \mathcal{N}$,

$$\gamma^{(J)}_{\Lambda}(A|\xi) := \gamma^{(J)}_{\Lambda}(1_A|\xi),$$

where $1_A$ is the characteristic function on the set $A$ (see (3.20)). We have to show that (see [11, page 16]):

(i) $\gamma^{(J)}_{\Lambda}(-|\xi)$ is a probability measure for each $\xi \in \mathcal{R}_\Lambda$, $\Lambda \subset \subset \mathbb{R}^d$;

(ii) $\gamma^{(J)}_{\Lambda}(A|\xi) = 0$ for all $\xi \notin \mathcal{R}_\Lambda$, $\Lambda \subset \subset \mathbb{R}^d$, $A \in \mathcal{F}$;

(iii) $\gamma^{(J)}_{\Lambda}(A|-)$ is $\mathcal{F}_{\Lambda^c}$-measurable for all $A \in \mathcal{F}$, $\Lambda \subset \subset \mathbb{R}^d$;

(iv) $\gamma^{(J)}_{\Lambda}(A|-) = 1_{A \cap \mathcal{R}_\Lambda}$ if $A \in \mathcal{F}_{\Lambda^c}$, $\Lambda \subset \subset \mathbb{R}^d$;

(v) $\gamma^{(J)}_{\Lambda}(A|\xi) := \int d\xi \gamma^{(J)}_{\Lambda}(d\xi|\xi)\gamma^{(J)}_{\Lambda}(A|\xi) = \gamma^{(J)}_{\Delta}(A|\xi)$ whenever $\Lambda \subset \Delta \subset \subset \mathbb{R}^d$.

From the definition, the properties (i)-(iv) are obvious. The proof of (v) is a routine, but a simple computation by noticing the product property of the measure:

$$\int_{\Delta} d\xi \Delta f(\xi) = \int_{\Lambda} d\xi \int_{\Delta \setminus \Lambda} d\xi \Delta \setminus \Lambda f(\xi \Delta \setminus \Lambda), \quad \Lambda \subset \Delta,$$

(4.16)
which is easily shown as follows:

\[
\oint_{\Lambda} d\zeta \oint_{\Delta \setminus \Lambda} d\zeta_{\Delta \setminus \Lambda} f(\zeta \zeta_{\Delta \setminus \Lambda}) \\
= \sum_{l \geq 0} \frac{1}{l!} \int_{\Lambda^l} d\bar{x}_l \sum_{m \geq 0} \frac{1}{m!} \int_{(\Delta \setminus \Lambda)^m} d\bar{y}_m f(\bar{x}_l \bar{y}_m) \\
= \sum_{n \geq 0} \frac{1}{n!} \int_{\Delta^n} d\bar{z}_n f(\bar{z}_n) \\
= \oint_{\Delta} d\zeta_{\Delta} f(\zeta_{\Delta}).
\]

In the second equality, we have put \(l + m = n\) and \(\bar{x}_l \bar{y}_m = \bar{z}_n\). The proof is completed. \(\square\)

### 4.2 Proof of Gibbsianness

In order to prove Theorem 3.5 we need some preparations. First we will observe that FRPF’s corresponding to compact supported operators are Gibbs measures.

Let \(J\) be an integral operator supported on a compact region in \(\mathbb{R}^d\). That is, there is a compact set \(\Delta_0 \subset \mathbb{R}^d\) such that the kernel function \(J(x, y)\) satisfies \(J(x, y) = 0\) unless \(x\) and \(y\) belong to \(\Delta_0\). We define \(\gamma^{(J)}_{\Lambda}(\bar{x}_n; \xi)\) as in (3.16) (with a suitable \(\mathcal{R}_\Lambda\), e.g., \(\mathcal{R}_\Lambda := \{\xi : \xi(\Lambda \cup \Delta_0)^c = \emptyset\}\)). It is obvious that \(\gamma^{(J)}_{\Lambda}(\bar{x}_n; \xi) = 0\) whenever \(\xi_{\Delta_0^c} \neq \emptyset\) and it is well-defined for all (finite) configurations \(\xi\) in \(\Delta_0\). Using this density function we define a specification \((\gamma^{(J)}_{\Lambda})_{\Lambda \subset \subset \mathbb{R}^d}\) through the formula (3.20). Let \(\mu^{(J)}\) be the FRPF corresponding to the operator \(K := J(I + J)^{-1}\). We notice that the operator \(K\) is also supported on \(\Delta_0\) and hence \(\mu^{(J)}\) is supported on the set \(\Gamma_{\Delta_0}\). The following is a key observation:

**Proposition 4.1** Let \(J\) be an integral operator with compact support. Define a specification \((\gamma^{(J)}_{\Lambda})_{\Lambda \subset \subset \mathbb{R}^d}\) and a FRPF \(\mu^{(J)}\) as above. Then \(\mu^{(J)}\) is a Gibbs measure for \((\gamma^{(J)}_{\Lambda})_{\Lambda \subset \subset \mathbb{R}^d}\).

**Proof:** For simplicity we omit all the superscripts from the notations. Let \(A \in \mathcal{F}\) and \(\Lambda \subset \subset \mathbb{R}^d\). We have to show that

\[
\mu(A) = \int d\mu(\xi) \gamma_{\Lambda}(A|\xi).
\]
Since $\gamma_\Lambda(A|\xi) = 0$ if $\xi_{\Delta^c} \neq \emptyset$, by using (2.10), (3.16), and (4.16) the r.h.s. equals to
\[
\int_{\Delta_0} d\xi_{\Delta_0} \sigma_{\Delta_0}(\xi_{\Delta_0}) \gamma_\Lambda(A|\xi_{\Delta_0})
= \det(I - K) \int_{\Delta_0} d\xi_{\Delta_0} \det J(\xi_{\Delta_0}, \xi_{\Delta_0})
\times \frac{1}{Z_\Lambda(\xi_{\Delta_0})} \int_\Lambda d\zeta_\Lambda \frac{\det J(\zeta_\Lambda \xi_{\Delta_0} \setminus \Lambda, \zeta_\Lambda \xi_{\Delta_0} \setminus \Lambda)}{\det J(\xi_{\Delta_0} \setminus \Lambda, \xi_{\Delta_0} \setminus \Lambda)} 1_A(\zeta_\Lambda \xi_{\Delta_0} \setminus \Lambda)
\]
\[
= \det(I - K) \int_{\Delta_0 \setminus \Lambda} d\xi_{\Delta_0} \frac{1}{Z_\Lambda(\xi_{\Delta_0})} \int_\Lambda d\xi_\Lambda \frac{\det J(\xi_{\Delta_0}, \xi_{\Delta_0})}{\det J(\xi_{\Delta_0} \setminus \Lambda, \xi_{\Delta_0} \setminus \Lambda)} \int_\Lambda d\zeta_\Lambda \det J(\zeta_\Lambda \xi_{\Delta_0} \setminus \Lambda, \zeta_\Lambda \xi_{\Delta_0} \setminus \Lambda) 1_A(\zeta_\Lambda \xi_{\Delta_0} \setminus \Lambda)
\]
\[
= \det(I - K) \int_{\Delta_0 \setminus \Lambda} d\xi_{\Delta_0} \int_\Lambda d\zeta_\Lambda \det J(\zeta_\Lambda \xi_{\Delta_0} \setminus \Lambda, \zeta_\Lambda \xi_{\Delta_0} \setminus \Lambda) 1_A(\zeta_\Lambda \xi_{\Delta_0} \setminus \Lambda)
= \mu(A).
\] (4.18)

In the first and last equalities we have used $K_{\Delta_0} = K$ and $J_{(\Delta_0)} = K_{\Delta_0}(I - K_{\Delta_0})^{-1} = J$. The fractions are set to be zero if the denominator equals to zero by the property (3.14). We have proven (4.17). □

Next we discuss some weak convergence of FRPF’s. The following may be well known.

**Lemma 4.2** Let $J$ be a bounded, positive definite, locally trace class integral operator. Let $(\Lambda_n)_{n \geq 1}$ be any increasing sequence of bounded Borel subsets of $\mathbb{R}^d$ such that $\cup_n \Lambda_n = \mathbb{R}^d$. Define $K_n := J_{\Lambda_n}(I + J_{\Lambda_n})^{-1}$. Then for any $\Lambda \subseteq \mathbb{R}^d$, $\|P_\Lambda K_n P_\Lambda - P_\Lambda K P_\Lambda\| \to 0$ as $n \to \infty$, where $K := J(I + J)^{-1}$.

**Proof:** Let $B := P_\Lambda J P_\Lambda$. Then $B$ is a trace class operator and $P_\Lambda K P_\Lambda \leq B$ and $P_\Lambda K_n P_\Lambda \leq P_\Lambda J_{\Lambda_n} P_\Lambda = B$ whenever $\Lambda_n \supset \Lambda$. Moreover, $P_\Lambda K_n P_\Lambda \to P_\Lambda K P_\Lambda$ weakly since $P_{\Lambda_n}$ converges strongly to the identity. Now we use Theorem 2.16 of [18] to complete the proof. □

**Lemma 4.3** (cf. [15] [17]) We suppose the same setting as in Lemma 4.2. Then the sequence of FRPF’s $\mu_n^{(j)}$ corresponding to $K_n$ converges weakly to the FRPF $\mu^{(j)}$ corresponding to $K$.

**Proof:** By using Lemma 4.2 the proof follows from [19] Theorem 5]. We provide however a proof here. We will show that for any bounded, measurable, and local functions $F: \Gamma \to \mathbb{R}$ (we emphasize again that FRPF’s are supported on $\Gamma$),
\[
\int F(\xi) d\mu_n^{(j)}(\xi) \to \int F(\xi) d\mu^{(j)}(\xi) \quad \text{as } n \to \infty.
\] (4.19)
For any $\Lambda \subset \subset \mathbb{R}^d$, $\Gamma_{\Lambda}$ is isomorphic to the space of disjoint sum $\sum_{n \geq 0} \widetilde{\Lambda}^n$, where $\widetilde{\Lambda}^n$ is the symmetric space of $n$ different points in $\Lambda$. In particular, it is a locally compact space. By Stone-Weierstrass theorem it is therefore enough to show (4.19) only for $F \in C_+$, where $C_+$ is defined by

$$C_+ := \{ F : \Gamma \to \mathbb{R} : F(\xi) = e^{-<f, \xi>} \text{ for some } 0 \leq f \in C_0(\mathbb{R}^d) \}. \quad (4.20)$$

For such an $F(\xi) = e^{-<f, \xi>}$, we have by (2.13)

$$\int d\mu_n^{(J)}(\xi) e^{-<f, \xi>} = \det(I - K_n \psi_f). \quad (4.21)$$

Since $0 \leq f \in C_0(\mathbb{R}^d)$, we have also $0 \leq \psi_f \in C_0(\mathbb{R}^d)$. Thus we can rewrite

$$\det(I - K_n \psi_f) = \det(I - \sqrt{\psi_f} K_n \sqrt{\psi_f}). \quad (4.22)$$

By Lemma 4.2, $\sqrt{\psi_f} K_n \sqrt{\psi_f}$ converges to $\sqrt{\psi_f} K \sqrt{\psi_f}$ in trace norm. Since $A \mapsto \det(I+A)$ is continuous in trace norm ([18, Theorem 3.4]), the r.h.s. of (4.22) converges to

$$\det(I - \sqrt{\psi_f} K \sqrt{\psi_f}) = \int d\mu^{(J)}(\xi) e^{-<f, \xi>}. \quad (4.23)$$

By [1] Proposition 9.1.VII, (4.19) is already equivalent to the weak convergence. The proof is completed. \qed

In order to prove the Gibbsianness of our $\mu^{(J)}$ in Theorem 3.5 we will use Proposition 4.1 and Lemma 4.3. First notice that instead of showing (4.17) it is enough to show that for all $\Lambda \subset \subset \mathbb{R}^d$ and any bounded and continuous function $f$

$$\int d\mu^{(J)}(\xi) f(\xi) = \int d\mu^{(J)}(\xi) \gamma^{(J)}(f|\xi). \quad (4.24)$$

Let $\mu^{(J)}_n$ be the FRPF’s weakly converging to $\mu^{(J)}$ as in Lemma 4.3. It turns out that (4.24) holds true for $\mu^{(J)}_n$‘s. For our goal we want to let $n$ tend to infinity. The difficulty in this step occurs because we do not know the continuity of the function $\xi \mapsto \gamma^{(J)}_A(f|\xi)$ in general. To overcome this difficulty we need to approximate the function $\gamma^{(J)}_A(f|\xi)$ by good continuous functions. For that purpose we rely on finiteness of the range and the non-percolating property of FRPF’s for small intensity.

Let $\xi = (x_i)_{i=1,2,\ldots} \in \mathcal{N}$ be a configuration. Let $R > 0$ be the number in Assumption 3.4. For each $i = 1, 2, \ldots$, we position a closed $d$-dimensional sphere $S_i$ of fixed radius $R$ with center $x_i$. We call two spheres $S_i$ and $S_j$ adjacent if $S_i \cap S_j \neq \emptyset$. We write $S_i \leftrightarrow S_j$ if there exists a sequence $S_{i_1}, S_{i_2}, \ldots, S_{i_k}$ of spheres such that $S_{i_1} = S_i$, $S_{i_k} = S_j$, and $S_{i_l}$ is adjacent to $S_{i_{l+1}}$ for $1 \leq l < k$. A cluster of spheres is a set $(S_i : i \in I)$ of spheres which is maximal with the property that $S_i \leftrightarrow S_j$ for all $i,j \in I$. The size of a cluster is the number of spheres belonging to it. The following was proved in [6, Corollary 3.5]:

...
**Theorem 4.4** Suppose that $J$ satisfies the conditions in Assumption 3.1. Then, there is a critical intensity $\alpha_c(d) > 0$ ($\alpha_c(1) = \infty$, in particular) such that if $j(0) < \alpha_c(d)$, then

$$\mu^{(J)}(\text{there is an infinite cluster}) = 0.$$  \hspace{1cm} (4.25)

We are now ready to prove the main result.

**Proof of Theorem 3.5:** Let $f : \mathcal{N} \to \mathbb{R}$ be any bounded and continuous function and $\Lambda \subset\subset \mathbb{R}^d$. We have to show \hspace{1cm} (4.24). Let $(\Lambda_n)_{n \geq 1}$ be any increasing sequence of bounded Borel subsets of $\mathbb{R}^d$ with $\cup_n \Lambda_n = \mathbb{R}^d$. We define $K_n := J_n(I + J_n)^{-1}$ and $J_n := K_n(I - K_n)^{-1} = J_n$. Let $\mu_n^{(J)}$ be the FRPF corresponding to $K_n$. Since $J_n$ is compactly supported we have by Proposition 4.1

$$\mu_n^{(J)}(f) = \int d\mu_n^{(J)}(\xi) \gamma^{(J_n)}(\xi)(f).$$ \hspace{1cm} (4.26)

We recall that $\gamma^{(J_n)}(f|\xi) = 0$ if $\xi \in \Lambda_n$ (i.e., $\Lambda_n = \emptyset$)

$$\gamma^{(J_n)}(f|\xi) = \gamma^{(J_n)}(f|\Lambda_n) = \frac{1}{Z^{(J_n)}(\xi_n)} \int d\xi \frac{\det J_n(\xi_n \setminus \Lambda_n, \xi_n \setminus \Lambda_n)}{\det J_n(\xi_n \setminus \Lambda_n, \Lambda_n \setminus \Lambda_n)} f(\xi_n \setminus \Lambda_n)$$

$$= \gamma^{(J_n)}(f|\Lambda_n) = \gamma^{(J_n)}(f|\xi).$$

In the third equality we have used the fact that $J_n(\eta_n, \eta_n) = J(\eta_n, \eta_n)$ for any $\eta_n \in \Gamma_n$. Thus \hspace{1cm} (4.26) is equivalent to the equation

$$\mu_n^{(J)}(f) = \int d\mu_n^{(J)}(\xi) \gamma^{(J)}(\xi)(f).$$ \hspace{1cm} (4.27)

For $d \geq 2$, we assume that $j(0)$ is sufficiently small that \hspace{1cm} (4.25) holds. We let

$$\mathcal{C}_\Lambda := \{\xi \in \mathcal{N} : \xi_{\Lambda^c} \text{ has an infinite cluster connected to } \Lambda\}$$ \hspace{1cm} (4.28)

and define

$$\mathcal{O}_\Lambda := \mathcal{R}_\Lambda \cap \mathcal{C}_\Lambda.$$ \hspace{1cm} (4.29)

A short consideration tells us that $\mathcal{O}_\Lambda$ is an $\mathcal{F}_{\Lambda^c}$-measurable, open subset of $\mathcal{N}$. In Lemma 4.6 below we will show that

$$\mu^{(J)}(\mathcal{O}_\Lambda) = 1.$$ \hspace{1cm} (4.30)

We also consider the clusters of open balls centered at the particles. Let $B_0$ be a big enough closed ball in $\mathbb{R}^d$ centered at the origin and such that $B_0 \supset \Lambda$ and $\text{dist}(\Lambda, B_0) > 2R + 1$, where $\text{dist}(\Lambda, B_0)$ denotes the distance between $\Lambda$ and $B_0$. Let $B_1 \subset B_2 \subset \cdots$
be any increasing sequence of closed balls in $\mathbb{R}^d$ centered at the origin such that $B_0 \subset B_1$ and $\cup_n B_n = \mathbb{R}^d$. For each $n = 1, 2, \cdots$, we let $R_n := R + 1/n$ and define

$$C_A^{(n)} := \{ \xi \in \mathcal{N}: \text{ clusters (of } \xi \text{) of } R_n\text{-open balls connected to } B_0 \text{ do not reach } B_c^\circ \}. $$ (4.31)

We let $\mathcal{D}_\Lambda \equiv \mathcal{O}_\Lambda$ and

$$\mathcal{D}_\Lambda^{(n)} := \{ \xi \in \mathcal{N}: \xi \in C_A^{(n)}, \det J(\xi_{\partial \Lambda}, \xi_{\partial \Lambda}) \geq 1/n \}. $$ (4.32)

We notice that $(\mathcal{D}_\Lambda^{(n)})_{n \geq 1}$ is a sequence of increasing closed subsets of $\mathcal{N}$ and it is not hard to see that

$$\bigcup_{n \geq 1} \mathcal{D}_\Lambda^{(n)} = \mathcal{O}_\Lambda. $$ (4.33)

Let $d$ be a metric that makes $\mathcal{N}$ a complete separable metric space and the metric topology is equivalent to the vague topology. For each $n = 1, 2, \cdots$, define a function $\chi_n : \mathcal{N} \to [0, 1]$ by

$$\chi_n(\xi) := \frac{d(\xi, \mathcal{D}_\Lambda)}{d(\xi, \mathcal{D}_\Lambda^{(n)}) + d(\xi, \mathcal{D}_\Lambda)}. $$ (4.34)

We notice that $\chi_n$ is continuous and $\chi_n = 1$ on $\mathcal{D}_\Lambda^{(n)}$ and $\chi_n = 0$ on $\mathcal{D}_\Lambda$, and by (4.33)

$$\lim_{n \to \infty} \chi_n \overset{\mathbb{P}}{\to} 1 \quad \mu^{(j)}\text{-a.e.} $$ (4.35)

Moreover, since $\mathcal{D}_\Lambda^{(n)}$ and $\mathcal{D}_\Lambda$ are both $\mathcal{F}_{\Lambda^c}$-measurable, the function $\chi_n$ is also $\mathcal{F}_{\Lambda^c}$-measurable. Observe that for each fixed $n = 1, 2, \cdots$ the function $\xi \mapsto \gamma^{(j)}_A(\chi_n f | \xi)$ is continuous. In fact, since $\chi_n$ is $\mathcal{F}_{\Lambda^c}$-measurable, we have

$$\gamma^{(j)}_A(\chi_n f | \xi) = \chi_n(\xi) \gamma^{(j)}_A(f | \xi) = \chi_n(\xi) \int_{\Lambda} d\zeta \gamma^{(j)}_A(\zeta; \xi) f(\zeta A^c). $$ (4.36)

For $\xi \in \mathcal{D}_\Lambda$, we have $\gamma^{(j)}_A(\chi_n f | \xi) = 0$ and $|\gamma^{(j)}_A(\chi_n f | \eta)| \leq \chi_n(\eta)\|f\|_\infty$, $\eta \in \mathcal{N}$. Thus $\gamma^{(j)}_A(\chi_n f | \cdot)$ is continuous at $\xi$. On the other hand, if $\xi \in \mathcal{O}_\Lambda$, then there is no infinite cluster (of closed $R$-spheres) of $A^c$ connected to $\Lambda$ and therefore we have by definition

$$\gamma^{(j)}_A(\zeta_\Lambda; \xi) = \gamma^{(j)}_A(\zeta_\Lambda; \xi_{\partial \Lambda}). $$ (4.37)

Furthermore, since $\det J(\xi_{\partial \Lambda}, \xi_{\partial \Lambda}) > 0$, we can find some bounded and, say, open set $\Delta_0 \subset \mathbb{R}^d$ and an open neighborhood $U$ of $\xi$ such that for each $\eta \in U$, $\eta_{\partial \Lambda} = \eta_{\Delta_0 \setminus \Lambda}$ and $\det J(\eta_{\partial \Lambda}, \eta_{\partial \Lambda}) > 0$, i.e.,

$$\gamma^{(j)}_A(\zeta_\Lambda; \eta) = \gamma^{(j)}_A(\zeta_\Lambda; \eta_{\Delta_0}) \text{ and } \det J(\eta_{\Delta_0 \setminus \Lambda}, \eta_{\Delta_0 \setminus \Lambda}) > 0, \eta \in U. $$ (4.38)
Since the function $\eta \mapsto \gamma^{(J)}_{\Lambda}(\zeta; \eta_{0})$ is continuous on $U$, the function $\eta \mapsto \gamma^{(J)}_{\Lambda}(\zeta|\eta)$ is continuous at $\xi$ and therefore so is the function $\gamma^{(J)}_{\Lambda}(\chi_{n}|\cdot)$. From (4.27) we have
\[
\mu^{(J)}_{m}(\chi_{n}f) = \int d\mu^{(J)}_{m}(\xi)\gamma^{(J)}_{\Lambda}(\chi_{n}f|\xi). \tag{4.39}
\]
Now both $\chi_{n}f$ and $\gamma^{(J)}_{\Lambda}(\chi_{n}f|\cdot)$ are bounded and continuous and $\mu^{(J)}_{m}$ converges weakly to $\mu^{(J)}$ as $m$ tends to infinity. By letting $m$ go to infinity in (4.39) we get
\[
\mu^{(J)}(\chi_{n}f) = \int d\mu^{(J)}(\xi)\gamma^{(J)}_{\Lambda}(\chi_{n}f|\xi). \tag{4.40}
\]
We let now $n$ tend to infinity and use (4.35) to get
\[
\mu^{(J)}(f) = \int d\mu^{(J)}(\xi)\gamma^{(J)}_{\Lambda}(f|\xi). \tag{4.41}
\]
The proof is completed. □

Finally, we provide with a proof (4.30). For that purpose we need the following property.

**Lemma 4.5** Let $J$ be an operator satisfying the conditions in Assumption 3.1 with $d\rho(t) = \tilde{\varphi}(t)dt$ for some $0 \leq \tilde{\varphi}(\cdot) \in L^{1}(\mathbb{R}^{d})$. Then for each $\Lambda \subset \subset \mathbb{R}^{d}$, the operator $J_{[\Lambda]}$ admits a continuous kernel function $J_{[\Lambda]}(x, y)$, $x, y \in \Lambda$, where $J_{[\Lambda]} := K_{\Lambda}(I - K_{\Lambda})^{-1}$ with $K := J(I + J)^{-1}$.

**Proof:** It is readily seen that $K$ has an integral kernel function $K(x, y) = k(x - y)$, where
\[
k(x) := (2\pi)^{-d} \int_{\mathbb{R}^{d}} e^{ix \cdot t} \frac{\tilde{\varphi}(t)}{1 + \tilde{\varphi}(t)} dt. \tag{4.42}
\]
In particular, $k(\cdot)$ is continuous. Now let $J'_{[\Lambda]}(x, y)$, $x, y \in \Lambda$, be any kernel function of the operator $J_{[\Lambda]}$, which exists because $J_{[\Lambda]}$ is a Hilbert-Schmidt operator. Since
\[
J_{[\Lambda]} = K_{\Lambda} + (K_{\Lambda})^{2} + K_{\Lambda}J_{[\Lambda]}K_{\Lambda},
\]
$J'_{[\Lambda]}(x, y)$ coincides for almost all $(x, y) \in \Lambda^{2}$ with
\[
J_{[\Lambda]}(x, y) := k(x - y) + \int_{\Lambda} k(x - u) k(u - y) du
+ \int_{\Lambda} \int_{\Lambda} k(x - u) J'_{[\Lambda]}(u, v) k(v - y) du dv.
\]
Since $k$ is continuous, it is easily checked that $J_{[\Lambda]}(x, y)$ is continuous, as required. □
Lemma 4.6 For each \( \Lambda \subset \mathbb{R}^d \), let \( O_\Lambda \) be defined as in (4.29). Then we have

\[
\mu^{(J)}(O_\Lambda) = 1. \tag{4.43}
\]

Proof: Since \( \mu^{(J)}(C_\Lambda) = 0 \) by (4.25), it is enough to show that

\[
\mu^{(J)}(R_\Lambda^c \cap C_\Lambda^c) = 0. \tag{4.44}
\]

We let \( \mathcal{G} \) be a countable class of bounded open subsets of \( \mathbb{R}^d \) such that for any compact subset \( C \subset \mathbb{R}^d \) and \( z \notin C \), we can find a \( G \in \mathcal{G} \) such that

\[
C \subset G \quad \text{and} \quad z \notin G. \tag{4.45}
\]

It is obvious that

\[
R_\Lambda^c \cap C_\Lambda^c \subset \bigcup_{G \in \mathcal{G}} \bigcup_{n \geq 1} N_{G,n}, \tag{4.46}
\]

where

\[
N_{G,n} := \{ \xi \in \mathcal{N} : \xi(G \setminus \Lambda) = n \quad \text{and} \quad \det (J(\xi G \setminus \Lambda), \xi G \setminus \Lambda) = 0 \}. \tag{4.47}
\]

On the other hand, by using the inequality (4.7), it is not hard to show the operator ordering: \( J_{[\Delta]} \preceq J_\Delta \) for all \( \Delta \subset \mathbb{R}^d \) (see [6, Lemma 4.1] for a proof). Moreover, since \( J(x,y) \) is continuous and \( J_{[\Delta]} \) also admits a continuous kernel \( J_{[\Delta]}(x,y) \) by Lemma 4.5, we have for any \( \Delta \subset \subset \mathbb{R}^d \) and \( n \geq 1 \), \( (J_{[\Delta]}(x_i,x_j))_{1 \leq i,j \leq n} \leq (J(x_i,x_j))_{1 \leq i,j \leq n} \) as operators on \( \mathbb{C}^n \), and thus

\[
\det(J_{[\Delta]}(x_i,x_j))_{1 \leq i,j \leq n} \leq \det(J(x_i,x_j))_{1 \leq i,j \leq n} \tag{4.48}
\]

for all \( (x_1, \cdots, x_n) \in \Delta^n \). We therefore get by (2.9), (4.44), and (4.46),

\[
\mu^{(J)}(N_{G,n}) = 0, \quad \text{for all} \quad G \in \mathcal{G}, \ n \geq 1. \tag{4.49}
\]

By (4.46) and (4.49) we get (4.44) and the proof is complete. \( \Box \)
5 Concluding remarks

In this section we would like to discuss some possible improvements of the results obtained in this paper, which were done in [6].

First, though the typical examples of operators $J$ might be the ones given in Assumptions 3.1 and 3.4 from the pedagogical point of view, the class of operators $J$ (and hence the operators $K$) that have the properties in the main results of this paper is in fact much larger than the one we considered. It is possible to include, for example, the operators with non-continuous kernel functions and the ones that are not of translation-invariant. To extend the theory to that generality, we are, however, confronted with some subtlety of version-problems of the kernel function. This was thoroughly investigated in [6].

Second, as remarked in the paragraph after the proof of Lemma 4.3, the key idea to show the Gibbsianness is to get a continuity of the function $\xi \mapsto \gamma_\Lambda(f|\xi)$ for any continuous function $f$ with local support. The condition of finite range and small intensity was in fact introduced to guarantee such continuity. Some models, however, possess the continuity without the finiteness condition of the range. An example is the renewal process on the real line [4, 9], for which the function $k(\cdot)$ in (4.42) is given by

$$k(x) := \rho e^{-a|x|},$$

where $\rho, a > 0$ and $\rho < a/2$. For the details, we refer to [6, Example 3.11].

The most stringent condition is the boundedness of the operator $J \geq 0$, or the strictness in the ordering $K \leq I$ for the operator $K = J(I+J)^{-1}$. From this restriction, we have to exclude the most interesting models, for example, the Dyson’s model [20], where the defining operator $K$ has a sine kernel and 1 belongs to its spectrum. For those models, the operator $J := K(I - K)^{-1}$ is not even defined. So, one asks whether FRPF’s can still be Gibbs measures, and in this case, what the corresponding interactions are.
A Appendix

In this appendix we provide with some examples satisfying the conditions in Assumption 3.4. Let \( a : \mathbb{R} \to \mathbb{R} \) be a function defined by

\[
a(x) := \begin{cases} 
-\frac{1}{R}(x - R), & 0 \leq x < R \\
\frac{1}{R}(x + R), & -R < x < 0 \\
0, & |x| \geq R.
\end{cases}
\]  
(A.1)

Then its Fourier transform is

\[
\hat{a}(t) := \int e^{-ixt}a(x)dx = \frac{2}{Rt^2}(1 - \cos Rt) \geq 0.
\]  
(A.2)

For \( x = (x^1, \ldots, x^d) \in \mathbb{R}^d \), let

\[
u(x) := \prod_{l=1}^{d} a(x^l).
\]  
(A.3)

Then the Fourier transform is

\[
\hat{\nu}(t) = \prod_{l=1}^{d} \hat{a}(t^l), \quad t = (t^1, \cdots, t^d) \in \mathbb{R}^d.
\]  
(A.4)

Now let \( \hat{\varphi}(t) \geq 0, \ t \in \mathbb{R}^d \), be any function such that

\[
\int_{\mathbb{R}^d} \hat{\varphi}(t) \, dt < \infty.
\]  
(A.5)

and that its inverse Fourier transform \( \varphi(x) \) defines a kernel function of a bounded linear integral operator on \( L^2(\mathbb{R}^d) \). Define

\[
j(x) := \varphi(x)\nu(x).
\]  
(A.6)

Then

\[
\hat{j}(t) = \frac{1}{2\pi} \hat{\varphi} * \hat{\nu}(t) = \frac{1}{2\pi} \int \hat{\varphi}(t - s)\hat{\nu}(s)ds \geq 0
\]  
(A.7)

and

\[
\int \hat{j}(t)dt < \infty.
\]  
(A.8)

Therefore, \( \hat{j}(t)dt \) is a finite measure on \( \mathbb{R}^d \). By Bochner’s theorem, its inverse Fourier transform \( j(x) \) is a bounded and continuous function of positive type \([12]\). Obviously we have

\[
j(x) = 0 \text{ if } |x| \geq R.
\]  
(A.9)
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