Moving Planes and Singular Points of Rational Parametric Surfaces

Falai Chen\textsuperscript{a,}\textsuperscript{*}, Xuhui Wang\textsuperscript{b,}\textsuperscript{a}

\textsuperscript{a}Department of Mathematics, University of Science and Technology of China
Hefei, Anhui, 230026, China
\textsuperscript{b}School of Mathematics, Hefei University of Technology
Hefei, Anhui, 230009, China

Abstract
In this paper we discuss the relationship between the moving planes of a rational parametric surface and the singular points on it. Firstly, the intersection multiplicity of several planar curves is introduced. Then we derive an equivalent definition for the order of a singular point on a rational parametric surface. Based on the new definition of singularity orders, we derive the relationship between the moving planes of a rational surface and the order of singular points. Especially, the relationship between the \( \mu \)-basis and the order of a singular point is also discussed.

Keywords: Rational parametric surface; Moving plane; \( \mu \)-basis; Singular point

1. Introduction

Give an algebraic surface \( f(x, y, z, w) = 0 \) in homogeneous form, the singular points of the surface are the points at which all the partial derivatives simultaneously vanish. Geometrically, a singular point on the surface is a point where the tangent plane is not uniquely defined, and it embodies geometric shape and topology information of the surface. Detecting and computing singular points has wide applications in Solid Modeling and Computer Aided Geometric Design (CAGD).

\textsuperscript{*}Corresponding author

Email addresses: chenfl@ustc.edu.cn (Falai Chen), wangxh05@mail.ustc.edu.cn (Xuhui Wang)
To find the singular points of a parametric surface $P(s, t)$, one can solve $P_s(s, t) \times P_t(s, t) = 0$. However, to find the orders of singularities, one has to resort to implicit form. Let $f(x, y, z, w)$ be the implicit equation of $P(s, t)$. A singular point $Q = (x_0, y_0, z_0, w_0)$ of $P(s, t)$ has order $r$ if all the partial derivatives of $f(x, y, z, w)$ with order up to $r - 1$ vanish at $Q$, and at least one of the $r$-th derivative of $f$ at $Q$ is nonzero. Unfortunately, converting the parametric form of a surface into implicit form is a difficult task, which is still a hot topic of research [1, 2, 6, 10, 13, 15]. Other methods such as generalized resultants [3, 17, 18] to find singular points don’t need prior implicitization. However, they are not designed for detecting and computing the singular points of higher order (order $\geq 3$). In this paper, we develop methods to treat singularities of rational parametric surfaces directly. Specifically, we use moving surfaces technique to study singularities of rational parametric surfaces and the relationship between the order of the singularities and moving surfaces.

The remainder of this paper is organized as follows. In Section 2, we recalls some preliminary results about the $\mu$-basis of a rational surface. The notion of intersection multiplicity of several planar curves is also introduced. In Section 3, a new definition for the order of a singular point directly from the parametric equation of a surface is presented, and the equivalence of the new definition with the classic definition is proved. In Section 4, we discuss the relationship between the moving planes and $\mu$-basis of a rational surface and the singular points of the rational surface. We conclude this paper with future research problems.

2. Preliminaries

Let $\mathbb{R}[s, t]$ be the ring of bivariate polynomials in $s$, $t$ over the set of real numbers $\mathbb{R}$. A rational parametric surface in homogeneous form is defined

$$P(s, t) = (a(s, t), b(s, t), c(s, t), d(s, t)), \quad (1)$$

where $a, b, c, d \in \mathbb{R}[s, t]$ are polynomials with $\gcd(a, b, c, d) = 1$. In order to apply the theory of algebraic geometry, sometimes we need to work with homogeneous polynomials,

$$P(s, t, u) = (a(s, t, u), b(s, t, u), c(s, t, u), d(s, t, u)). \quad (2)$$

A base point of a rational surface $P(s, t)$ is a parameter pair $(s_0, t_0)$ such that $P(s_0, t_0) = 0$. Note that even if the rational surface $P(s, t)$ is real, the base points could be complex numbers and possibly at infinity.
A moving plane is a family of planes with parametric pairs \((s, t)\)\(^{[15]}\)
\[
A(s, t)x + B(s, t)y + C(s, t)z + D(s, t) = 0
\]
where \(A(s, t), B(s, t), C(s, t), D(s, t) \in \mathbb{R}[s, t]\). A moving plane is said to follow the rational surface \((1)\) if
\[
A(s, t)a(s, t) + B(s, t)b(s, t) + C(s, t)c(s, t) + D(s, t)d(s, t) \equiv 0.
\]
The moving plane \((3)\) can be written as a vector form
\[
\mathbf{L}(s, t) = (A(s, t), B(s, t), C(s, t), D(s, t)).
\]
Let \(\mathbf{L}_{st}\) be the set of the moving planes which follow the rational surface \(\mathbf{P}(s, t)\), then \(\mathbf{L}_{st}\) is exactly the syzygy module \(\text{syz}(a, b, c, d)\) and is a free module of rank 3 \(^{[6]}\).

A \(\mu\)-basis of the rational surface \((1)\) consists of three moving planes \(\mathbf{p}, \mathbf{q}, \mathbf{r}\) following \((1)\) such that
\[
[\mathbf{p}, \mathbf{q}, \mathbf{r}] = \kappa \mathbf{P}(s, t),
\]
where \(\kappa\) is nonzero constant and \([\mathbf{p}, \mathbf{q}, \mathbf{r}]\) is the outer product of \(\mathbf{p} = (p_1, p_2, p_3, p_4)\), \(\mathbf{q} = (q_1, q_2, q_3, q_4)\), and \(\mathbf{r} = (r_1, r_2, r_3, r_4)\) defined by
\[
[\mathbf{p}, \mathbf{q}, \mathbf{r}] = \begin{pmatrix}
    p_2 & p_3 & p_4 \\
    q_2 & q_3 & q_4 \\
    r_2 & r_3 & r_4
\end{pmatrix} - \begin{pmatrix}
    p_1 & p_3 & p_4 \\
    q_1 & q_3 & q_4 \\
    r_1 & r_3 & r_4
\end{pmatrix} - \begin{pmatrix}
    p_1 & p_2 & p_4 \\
    q_1 & q_2 & q_4 \\
    r_1 & r_2 & r_4
\end{pmatrix}.
\]

A \(\mu\)-basis forms a basis for the syzygy module \(\mathbf{L}_{st}\) \(^{[6]}\).

**Definition 1.** Let \(f(x, y, z, w) = 0\) be the implicit equation of the parametric surface \(\mathbf{P}(s, t)\). Then \(\mathbf{X}_0 = (x_0, y_0, z_0, w_0)\) is a singular point of order \(r\) or an \(r\)-fold point if all the derivatives of \(f\) of order up to \(r - 1\) are zero at \(\mathbf{X}_0\) and at least one \(r\)-th derivative of \(f\) does not vanish at \(\mathbf{X}_0\). Specifically, \(\mathbf{X}_0\) is a double point if and only if
\[
f_x(\mathbf{X}_0) = f_y(\mathbf{X}_0) = f_z(\mathbf{X}_0) = f_w(\mathbf{X}_0) = 0,
\]
and at least one of the second order derivatives is non-zero.

To discuss the order of singular points on a rational parametric surface, we need to recall some preliminary knowledge about the intersection multiplicity of several curves in \(\mathbb{P}^2(\mathbb{C})\), the projective plane over the complex numbers.
Definition 2. [12] Let $R$ be a local ring with maximal ideal $m$ and $M$ be a finitely generated $R$-module. Assume $R$ contains $k = R/m$. For $l \gg 0$, the Hilbert polynomial implies that

$$\dim_k(M/m^{l+1}M) = \frac{e}{d!}l^d + \ldots,$$

where $d = \dim(R)$ and $e = e(M)$ is the multiplicity of $M$. The refined case is as follows. Let $I$ be an ideal with $m^sM \subset IM$ for some $s$, then $l \gg 0$ implies that

$$\dim_k(M/I^{l+1}M) = \frac{\tilde{e}}{d!}l^d + \ldots,$$

where $\tilde{e} = e(I, M)$ is the multiplicity of $I$ in $M$.

According to the above definition, we can define the intersection multiplicity of several planar curves in $\mathbb{P}^2(\mathbb{C})$. For planar curves $C_1, C_2, \ldots, C_v$ which are defined by homogeneous equations $f_1(s, t, u) = 0, \ldots, f_v(s, t, u) = 0$ respectively. Let $\tilde{f}_1, \ldots, \tilde{f}_v$ be the local equation of $C_1, C_2, \ldots, C_v$ near point $p$, then the intersection multiplicity of these curves at point $p$ is

$$m(p) = e(I_p, R_p)$$

for $I_p = \langle \tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_v \rangle$ and $R_p = \mathcal{O}_{\mathbb{P}^2, p}$, which is the ring of rational functions defined at $p$.

Assume $m^s \subset I \subset R$, then

- If $m^s \subset J \subset I \subset R$, then $e(J, R) \geq e(I, R)$.
- If $I^lJ = I^{l+1}$, then $e(J, R) = e(I, R)$, $J$ is a reduction ideal of $I$.
- If $I$ is generated by a regular sequence, then $e(I, R) = \dim_k R/I$, $I$ is a complete intersection.

Proposition 3. [12]

1. $I$ has a reduction ideal which is generated by a regular sequence.
2. The regular sequence can be chosen to the generic linear combinations of the generators of $I$. 

3. The order of singular points on rational parametric surfaces

Given a rational parametric surface, we first give a definition about the order of singular points directly from the parametric equation.

**Definition 4.** For a rational surface \((x, y, z, w)\), a point \(X_0 = (x_0, y_0, z_0, w_0)\) (wlog, assume \(w_0 \neq 0\)) is a \(r\)-fold singular point if

\[
\begin{align*}
w_0 a(s, t, u) - x_0 d(s, t, u) &= w_0 b(s, t, u) - y_0 d(s, t, u) \\
&= w_0 c(s, t, u) - z_0 d(s, t, u) = 0
\end{align*}
\]

has \(r + \lambda\) intersection points (counting multiplicity) in the \((s, t, u)\) plane, where the multiplicity is defined in (2), and \(\lambda\) is the number of base point of the surface.

We will show that the above definition is equivalent to the classic definition of order of singularities (Definition 1).

**Theorem 5.** Definition 1 and Definition 4 are equivalent.

**Proof.** \(X_0\) is an \(r\)-fold singular point on a surface if and only if, for a generic line passing through \(X_0\), the line intersects the surface at \(n-r\) distinct points besides \(X_0\), here \(n\) is the implicit degree of the surface. Without loss of generality, assume the singular point is at the origin \(X_0 = (x_0, y_0, z_0, w_0) = (0, 0, 0, 1)\). Let the generic line be defined by

\[
\begin{align*}
&l = L_1 \cap L_2,
\end{align*}
\]

where

\[
\begin{align*}
L_1 &: \alpha_1 x + \alpha_2 y + \alpha_3 z = 0, \\
L_2 &: \beta_1 x + \beta_2 y + \beta_3 z = 0.
\end{align*}
\]

Consider the two planar curves

\[
\begin{align*}
C &: g_1 = \alpha_1 a(s, t, u) + \ldots + \alpha_3 c(s, t, u) = 0, \\
D &: g_2 = \beta_1 a(s, t, u) + \ldots + \beta_3 c(s, t, u) = 0.
\end{align*}
\]

Let \(Z\) be the common zeros of \(a, b, c, S\) be the surface, \(l^-\) be the line segments by removing the origin from the line \(l\), denote \(\varphi : (s, t, u) \mapsto P(s, t, u)\), then

\[
C \cap D = \varphi^{-1}(S \cap l^-) \cup Z.
\]
By Bezout’s theorem, one has

\[ n^2 = \#(S \cap l^-) + \sum_{p \in Z} \dim \mathcal{O}_p / \langle g_1, g_2 \rangle_p. \]  

(6)

From (3), we can get that \(g_1_p\) and \(g_2_p\) is a reduction ideal of \(\langle a(s, t), b(s, t), c(s, t) \rangle_p\), where \(p\) is the intersection point of \(g_1 = 0, g_2 = 0\). Thus,

\[ e(\langle a, b, c \rangle_p, \mathcal{O}_p) = e(\langle g_1, g_2 \rangle_p, \mathcal{O}_p) = \dim_k \mathcal{O}_p / \langle g_1, g_2 \rangle_p \]

Therefore, (6) is equivalent to

\[ n^2 = \#(S \cap l^-) + \sum_{p \in Z} e(\langle a, b, c \rangle_p, \mathcal{O}_p). \]  

(7)

Let \(Z_1\) be the point set which satisfies \(a = b = c = 0\) and \(d \neq 0\), \(Z_2\) be the point set which satisfies \(a = b = c = d = 0\), then

\[ Z = Z_1 \cup Z_2, \quad \text{and} \quad Z_1 \cap Z_2 = \emptyset. \]

Therefore, (7) is equivalent to

\[ n^2 = \#(S \cap l^-) + r + \lambda. \]  

(8)

Since the implicit degree of the surface is \(n^2 - \lambda\), we immediately get that Definition 1 and Definition 4 are equivalent. \(\square\)

4. Relationship between moving planes and singular points

In this section, we study the relationship between the moving planes and the order of singular points on a rational parametric surface.

**Theorem 6.** Let \(P(s, t, u)\) be a parametric surface with no base points, and \(L(s, t, u)\) be a moving plane following \(P(s, t, u)\). If \(X_0 = (x_0, y_0, z_0, w_0)\) (assume \(w_0 \neq 0\)) is an \(r\)-fold singular point on the surface, then

\[ w_0 a(s, t, u) - x_0 d(s, t, u) = 0, \quad w_0 b(s, t, u) - y_0 d(s, t, u) = 0, \]

\[ w_0 c(s, t, u) - z_0 d(s, t, u) = 0, \quad L(s, t, u) \cdot X_0 = 0 \]

have \(r\) intersection points (counting multiplicity).
Proof. The rational parametric surface $P(s, t)$ has the following special three moving planes

$$L_1 := (-d(s, t), 0, 0, a(s, t)),$$

$$L_2 := (0, -d(s, t), 0, b(s, t)),$$

$$L_3 := (0, 0, -d(s, t), c(s, t)), $$

and they belong to $L_{s,t}$. Given a moving plane $L(s, t) = (A(s, t), B(s, t), C(s, t), D(s, t))$ follow the rational surface $P(s, t)$, assume $A(s, t), B(s, t), C(s, t), D(s, t)$ are relatively prime (If they are not relatively prime, we can deal with it similarly). As four dimensional vectors, $L_1, L_2, L_3$ are all perpendicular to $P(s, t)$, and $L$ is also perpendicular to $P(s, t)$. Thus, there exist $h, h_1, h_2, h_3 \in \mathbb{R}[s, t]$, and $\gcd(h_1, h_2, h_3) = 1$, such that

$$hL(s, t) = h_1L_1(s, t) + h_2L_2(s, t) + h_3L_3(s, t)$$

$$= (-h_1d, -h_2d, -h_3d, h_1a + h_2b + h_3c).$$

Since $\gcd(A, B, C, D) = 1$ and $\gcd(h_1, h_2, h_3) = 1$,

$$h = \gcd(-h_1d, -h_2d, -h_3d, h_1a + h_2b + h_3c) = \gcd(d, h_1a + h_2b + h_3c).$$

Thus $h|d$.

For an $r$-fold singular point $X_0$ on the surface, (5) have $r$ intersection points (counting multiplicity):

$$(p_0, \ldots, p_r) = ((s_0, t_0, u_0), \ldots, (s_v, t_v, u_v)),$$  \hspace{1em} (10)

and the multiplicity at $p_i$ is $m_i, i = 0, \ldots, v$, thus $r = m_0 + \ldots + m_v$. From (9), we have

$$h(s, t)l(s, t) = h_1(s, t)l_1(s, t) + h_2(s, t)l_2(s, t) + h_3(s, t)l_3(s, t),$$ \hspace{1em} (11)

where $l(s, t) = L \cdot X_0, l_i(s, t) = L_i \cdot X_0, i = 1, 2, 3$.

Now we discuss the two possible cases.

- If the intersection point $p_i$ is not at infinity, assume $p_i = (s_i, t_i, 1)$, and from (11), we have

$$h(s - s_i, t - t_i)l(s - s_i, t - t_i) = h_1(s - s_i, t - t_i)l_1(s - s_i, t - t_i)$$

$$+ h_2(s - s_i, t - t_i)l_2(s - s_i, t - t_i) + h_3(s - s_i, t - t_i)l_3(s - s_i, t - t_i).$$
Thus, the ideals generated by local equations of \( h(s, t, u)l(s, t, u), l_1(s, t, u), l_2(s, t, u), l_3(s, t, u) \) near \( p_i \) and local equations of \( l_1(s, t, u), l_2(s, t, u), l_3(s, t, u) \) near \( p_i \) are same.

- If intersection point \( p_i \) is at infinity, assume \( p_i = (1, t_i, 0) \). Since \( \mathbf{L}(s, t) \cdot \mathbf{P}(s, t) \equiv 0 \), its homogeneous form also satisfies:

\[
\mathbf{L}(s, t, u) \cdot \mathbf{P}(s, t, u) \equiv 0,
\]

and the dehomogenized form also has \( \mathbf{L}(1, t, u) \cdot \mathbf{P}(1, t, u) \equiv 0 \). Similar to the above analysis, their exist \( h', h'_1, h'_2, h'_3 \in \mathbb{R}[t, u] \), and \( \gcd(h'_1, h'_2, h'_3) = 1 \), such that

\[
h' l(1, t, u) = h'_1 l_1(1, t, u) + h'_2 l_2(1, t, u) + h'_3 l_3(1, t, u),
\]

where \( l(s, t, u), l_i(s, t, u) \) is the homogeneous form of \( l(s, t), l_i(s, t), i = 1, 2, 3 \), and \( h' \) \( d(1, t, u) \). Therefore, the ideals generated by local equations of \( h'l(1, t, u), l_1(1, t, u), l_2(1, t, u), l_3(1, t, u) \) near \( p_i \) and local equations of \( l_1(1, t, u), l_2(1, t, u), l_3(1, t, u) \) near \( p_i \) are also same.

Since \( h(p_i) \neq 0, h'(p_i) \neq 0 \) (otherwise, \( p_i \) is a base point of \( \mathbf{P}(s, t) \)), and based on the definition of the multiplicity, \( p_i, i = 1, \ldots, v \) are also the intersection points of

\[
w_0 a(s, t, u) - x_0 d(s, t, u) = 0, \quad w_0 b(s, t, u) - y_0 d(s, t, u) = 0, \\
w_0 c(s, t, u) - z_0 d(s, t, u) = 0, \quad \mathbf{L}(s, t, u) \cdot \mathbf{X}_0 = 0.
\]

and the intersection multiplicity at \( p_i \) are also same. \( \square \)

**Remark** 1. \( \mu \)-basis \( \mathbf{p}, \mathbf{q}, \mathbf{r} \) of \( \mathbf{P}(s, t) \) are also three special moving planes of the surface, and from Theorem 6,

\[
\mathbf{L}_1(s, t, u) \cdot \mathbf{X}_0 = 0, \quad \mathbf{L}_2(s, t, u) \cdot \mathbf{X}_0 = 0, \quad \mathbf{L}_3(s, t, u) \cdot \mathbf{X}_0 = 0, \\
\mathbf{p}(s, t, u) \cdot \mathbf{X}_0 = 0, \quad \mathbf{q}(s, t, u) \cdot \mathbf{X}_0 = 0, \quad \mathbf{r}(s, t, u) \cdot \mathbf{X}_0 = 0
\]

(12) also have the \( r \) intersection points.

Next we discuss the relationship of the \( \mu \)-basis and the order of singular points on a rational parametric surface.

For any moving plane \( \mathbf{l}(s, t) \in \mathbf{L}_{s,t} \), there exist polynomials \( h_i(s, t), i = 1, 2, 3 \), such that \( \mathbf{l}(s, t) = h_1 \mathbf{p} + h_2 \mathbf{q} + h_3 \mathbf{r} \) (Theorem 3.2 in [6]), thus

\[
\langle \mathbf{L}_1 \cdot \mathbf{X}_0, \mathbf{L}_2 \cdot \mathbf{X}_0, \mathbf{L}_3 \cdot \mathbf{X}_0, \mathbf{p} \cdot \mathbf{X}_0, \mathbf{q} \cdot \mathbf{X}_0, \mathbf{r} \cdot \mathbf{X}_0 \rangle = \langle \mathbf{p} \cdot \mathbf{X}_0, \mathbf{q} \cdot \mathbf{X}_0, \mathbf{r} \cdot \mathbf{X}_0 \rangle.
\]
Similar to the proof of Theorem 6, we can get that

\[ p(s, t, u) \cdot X_0 = q(s, t, u) \cdot X_0 = r(s, t, u) \cdot X_0 = 0 \quad \text{(13)} \]

also have the same multiplicity at each intersection point \( p_i, i = 0, \ldots, v \) as Equation (5).

An immediate consequence of the above analysis is:

**Corollary 7.** For a rational parametric surface \( P(s, t) \) with no base point and its \( \mu \)-basis \( p, q, r \), then \( X_0 = (x_0, y_0, z_0, w_0) \) is \( r \)-fold singular point on the surface if and only if the intersection points of

\[ p(s, t, u) \cdot X_0 = q(s, t, u) \cdot X_0 = r(s, t, u) \cdot X_0 = 0 \]

is \( r \) (counting multiplicity).

Now we consider the moving surface ideal:

\[ I' = \langle dx - a, dy - b, dz - c, dw - 1 \rangle \cap \mathbb{R}[x, y, z, s, t]. \]

Theorem 3.4, 3.5 of [6] shows that \( I' \) is a prime ideal and \( f(x, y, z, s, t) \in I' \) if and only if \( f(x, y, z, s, t) = 0 \) is a moving surface following the rational surface \( P(s, t) \). Moreover, if \( P(s, t) \) contains no base point then

\[ I' = \langle p, q, r \rangle \quad \text{(14)} \]

where \( p = p \cdot (x, y, z, 1) \), \( q = q \cdot (x, y, z, 1) \), \( r = r \cdot (x, y, z, 1) \). Therefore, for a rational surface \( P(s, t) \) with no base point, and any moving surface \( f = 0 \) following it, \( f \in \langle p, q, r \rangle \). Based on the above analysis, we can improve Theorem 6 as the following theorem

**Theorem 8.** For a parametric surface \( P(s, t, u) \) with no base point and any moving surface \( f(x, y, z, w, s, t, u) = 0 \) following it. If \( X_0 = (x_0, y_0, z_0, w_0) \) (assume \( w_0 \neq 0 \) ) is an \( r \)-fold singular point on the surface, then

\[
\begin{align*}
  w_0 a(s, t, u) - x_0 d(s, t, u) &= w_0 b(s, t, u) - y_0 d(s, t, u) \\
  &= w_0 c(s, t, u) - z_0 d(s, t, u) = f(x_0, y_0, z_0, w_0, s, t, u) = 0
\end{align*}
\]

have and only have \( r \) intersection points (counting multiplicity), where \( f(x, y, z, w, s, t, u) \) is the homogeneous form of \( f(x, y, z, w, s, t) \).
5. Conclusion

To make the $\mu$-basis more applicable in computing the singular points, we discuss the relations between moving planes (Specially, $\mu$-basis) and singular point of the rational surface.

In the future, we will discuss how to detect and compute singular points on an rational parametric surface based on moving planes (or $\mu$-basis) in an efficient way.

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