Abstract. We develop the theory of hyperelliptic Kleinian functions. As applications we consider construction of the explicit matrix realization of the hyperelliptic Kummer varieties, differential operators to have the hyperelliptic curve as spectral variety, solution of the KdV equations by Kleinian functions.

Contents

Introduction 2
1. Preliminaries 2
1.1. Hyperelliptic curves 2
1.2. Differentials 3
1.2.1. Holomorphic differentials 3
1.2.2. Meromorphic differentials 4
1.2.3. Differentials of the third kind 4
1.2.4. Fundamental 2–differential of the second kind 5
1.3. Riemann \( \theta \)-function 6
2. Kleinian functions 6
2.1. \( \sigma \)-function 6
2.1.1. \( \sigma \)-function as \( \theta \)-function 7
2.2. Functions \( \zeta \) and \( \wp \) 7
2.2.1. Realization of the fundamental 2–differential of the second kind by Kleinian functions 8
2.2.2. Solution of the Jacobi inversion problem 9
3. Basic relations 11
3.1. Fundamental cubic and quartic relations 13
3.2. Analysis of fundamental relations 14
3.2.1. Sylvester’s identity 14
3.2.2. Determinantal form 15
3.2.3. Extended cubic relation 17
4. Applications 18
4.1. Matrix realization of hyperelliptic Kummer varieties 18
4.2. Hyperelliptic \( \Phi \)-function 19
4.3. Solution of KdV equations by Kleinian functions 22
Concluding remarks 22
Acknowledgments 23
References 23

Date: October 24, 2018.
INTRODUCTION

In this paper we develop the Kleinian construction of hyperelliptic Abelian functions, which is a natural generalization of the Weierstrass approach in the elliptic functions theory to the case of a hyperelliptic curve of genus $g > 1$. Kleinian $\zeta$ and $\wp$–functions are defined as

$$
\zeta_i(u) = \frac{\partial}{\partial u_i} \ln \sigma(u), \quad \wp_{ij}(u) = -\frac{\partial^2}{\partial u_i \partial u_j} \ln \sigma(u), \quad i, j = 1, \ldots, g,
$$

where the vector $u$ belongs to Jacobian $\text{Jac}(V)$ of the hyperelliptic curve $V = \{(y, x) \in \mathbb{C}^2 : y^2 = \sum_{i=0}^{2g+2} \lambda_i x^i = 0\}$ and the $\sigma(u)$ is the Kleinian $\sigma$–function.

The systematical study of the $\sigma$–functions, which may be related to the paper of Klein [1], was an alternative to the developments of Weierstrass [2, 3] (the hyperelliptic generalization of the Jacobi elliptic functions $\text{sn}$, $\text{cn}$, $\text{dn}$) and the purely $\theta$–functional theory Göppel [4] and Rosenhain [5] for genus 2, generalized further by Riemann. The $\sigma$ approach was contributed by Burkhardt [6], Wiltheiss [7], Bolza [8], Baker [9] and others; the detailed bibliography may be found in [10]. We would like to cite separately H.F. Baker’s monographs [11, 12], worth special attention.

The paper is organized as follows. We recall the basic facts about hyperelliptic curves in the Section 1. In the Section 2 we construct the explicit expression for the fundamental 2–differential of the second kind and derive the solution of the Jacobi inversion problem in terms of the hyperelliptic $\wp$–functions. We give in the Section 3 the proof and the analysis of basic relations for $\wp$–functions. It is given an explicit description of the $\text{Jac}(V)$ in $\mathbb{C}^{2g+1}$ as the intersection of cubics. The Kummer variety $\text{Kum}(V) = \text{Jac}(V)/\pm$ appears to be the intersection of quartics in $\mathbb{C}^{2g+2}$ and is described in a whole by the condition $\text{rank} (\{h_{ij}\}_{i,j=1,\ldots,g+2}) < 4$. The Section 4 describes some natural applications of the Kleinian functions theory.

The paper is based on the recent results partially announced in [13, 14, 15, 16]. The given results are already used to describe a 2–dimensional Schrödinger equation [17].

1. Preliminaries

We recall some basic definition from the theory of the hyperelliptic curves and $\theta$–functions; see e.g. [11, 12, 13, 14, 20, 21] for the detailed exposition.

1.1. Hyperelliptic curves. The set of points $V(y, x)$ satisfying the

$$
y^2 = \sum_{i=0}^{2g+2} \lambda_i x^i = \lambda_2 x^{2g+2} \prod_{k=1}^{2g+2} (x - e_k) = f(x)
$$

is a model of a plane hyperelliptic curve of genus $g$, realized as a 2–sheeted covering over Riemann sphere with the branching points $e_1, \ldots, e_{2g+2}$. Any pair $(y, x)$ in $V(y, x)$ is called an analytic point; an analytic point, which is not a branching point is called a regular point. The hyperelliptic involution $\phi( )$ (the swap of the sheets of covering) acts as $(y, x) \mapsto (-y, x)$, leaving the branching points fixed.

To make $y$ the singlevalued function of $x$ it suffices to draw $g+1$ cuts, connecting pairs of branching points $e_i - e_i'$ for some partition of $\{1, \ldots, 2g+2\}$ into the set
of \( g + 1 \) disjoint pairs \( i, i' \). Those of \( e_j \), at which the cuts start we will denote \( a_i \),

ending points of the cuts we will denote \( b_i \), respectively; except for one of the cuts

which is denoted by starting point \( a \) and ending point \( b \). In the case \( \lambda_{2g+2} \mapsto 0 \)

this point \( a \mapsto \infty \). The equation of the curve, in case \( \lambda_{2g+2} = 0 \) and \( \lambda_{2g+1} = 4 \) can

be rewritten as

\[
(1.2) \quad y^2 = 4P(x)Q(x),
\]

\[
P(x) = \prod_{i=1}^{g}(x - a_i), \quad Q(x) = (x - b) \prod_{i=1}^{g}(x - b_i).
\]

The local parametrisation of the point \((y, x)\) in the vicinity of a point \((w, z)\):

\[
x = z + \begin{cases}
\xi, & \text{near regular point } (\pm w, z); \\
\xi^2, & \text{near branching point } (0, e_i); \\
\frac{1}{\xi}, & \text{near regular point } (\pm \infty, \infty); \\
\frac{1}{\xi}, & \text{near branching point } (\infty, \infty)
\end{cases}
\]

provides the structure of the hyperelliptic Riemann surface — a one-dimensional compact complex manifold. We will employ the same notation for the plane curve and the Riemann surface — \( V(y, x) \) or \( V \). All curves and Riemann surfaces through

the paper are assumed to be hyperelliptic, if the converse not stated.

A marking on \( V(y, x) \) is given by the base point \( x_0 \) and the canonical basis of cycles \( (A_1, \ldots, A_g; B_1, \ldots, B_g) \) — the basis in the group of one-dimensional homologies \( H_1(V(y, x), \mathbb{Z}) \) on the surface \( V(y, x) \) with the symplectic intersection

matrix \( I = \begin{pmatrix} 0 & -1_g \\ 1_g & 0 \end{pmatrix} \), where \( 1_g \) is the unit \( g \times g \)-matrix.

1.2. Differentials. Traditionally three kinds of differential 1–forms are distingui

sh ed on a Riemann surface.

1.2.1. Holomorphic differentials. or the differentials of the first kind, are the differential 1–forms \( du \), which can be locally given as \( du = (\sum_{i=0}^{\infty} \alpha_i \xi^i) d\xi \) in the vicinity of any point \((y, x)\) with some constants \( \alpha_i \in \mathbb{C} \). It can be checked directly, that forms satisfying such a condition are all of the form \( \sum_{i=0}^{g-1} \beta_i x^i \frac{dx}{y} \). Forms \( \{du_i\}_{i=1}^{g}, \)

\[
du_i = \frac{x^{i-1} dx}{y}, \quad i \in 1, \ldots, g
\]

are the set of canonical holomorphic differentials in \( H^1(V, \mathbb{C}) \). The \( g \times g \)-matrices of their \( A \) and \( B \)-periods,

\[
2\omega = \left( \oint_{A_k} du_l \right), \quad 2\omega' = \left( \oint_{B_k} du_l \right)
\]

are nondegenerate. Under the action of the transformation \( (2\omega)^{-1} \) the vector \( du = (du_1, \ldots, du_g)^T \) maps to the vector of normalized holomorphic differentials \( dv = (dv_1, \ldots, dv_g)^T \) — the vector in \( H^1(V, \mathbb{C}) \) to satisfy the conditions \( \oint_{A_k} dv_l = \delta_{kl}, k, l = 1, \ldots, g \). It is known, that \( g \times g \) matrix,

\[
\tau = \left( \oint_{B_k} dv_l \right) = \omega^{-1} \omega'
\]

belongs to the upper Siegel halfspace \( \mathcal{H}_g \) of degree \( g \), i.e. it is symmetric and has a

positively defined imaginary part.
Let us denote by $\text{Jac}(V)$ the Jacobian of the curve $V$, i.e. the factor $\mathbb{C}^g/\Gamma$, where $\Gamma = 2\omega \oplus 2\omega'$ is the lattice generated by the periods of canonical holomorphic differentials.

A divisor $D$ is a formal sum of subvarieties of codimension 1 with coefficients from $\mathbb{Z}$. Divisors on Riemann surfaces are given by formal sums of analytic points $D = \sum_i m_i(y_i, x_i)$, and $\deg D = \sum_i m_i$. The effective divisor is such that $m_i > 0 \forall i$.

Let $D$ be a divisor of degree 0, $D = X - Z$, with $X$ and $Z$ — the effective divisors $\deg X = \deg Z = n$ presented by $X = \{(y_1, x_1), \ldots, (y_n, x_n)\}$ and $Z = \{(w_1, z_1), \ldots, (w_n, z_n)\} \in (V)^n$, where $(V)^n$ is the $n$-th symmetric power of $V$.

The Abel map $\mathfrak{A} : (V)^n \to \text{Jac}(V)$ puts into correspondence the divisor $D$, with fixed $Z$, and the point $u = (u_1, \ldots, u_g)^T \in \text{Jac}(V)$ according to the theorem says that the points of the divisors $Z$ and $X$ are respectively the poles and zeros of a meromorphic function on $V(y, x)$ iff $\int_{X}^{Z} \frac{dx}{y} = 0 \mod \Gamma$.

The Jacobi inversion problem is formulated as the problem of inversion of the map $\mathfrak{A}$, when $n = g$ the $\mathfrak{A}$ is $1 \to 1$, except for so called special divisors. In our case special divisors of degree $g$ are such that at least for one pair $j$ and $k \in 1 \ldots g$ the point $(y_j, x_j)$ is the image of the hyperelliptic involution of the point $(y_k, x_k)$.

1.2.2. Meromorphic differentials. or the differentials of the second kind, are the differential 1-forms $dr$ which can be locally given as $dr = (\sum_{i=0}^{\infty} \alpha_i \xi^i) dx$ in the vicinity of any point $(y, x)$ with some constants $\alpha_i$, and $\alpha_{(-1)} = 0$. It can be also checked directly, that forms satisfying such a condition are all of the form $\sum_{i=0}^{g-1} \beta_i x^{i+g} \frac{dx}{y}$ (mod holomorphic differential). Let us introduce the following canonical Abelian differentials of the second kind

\[
\begin{align*}
\sum_{k=j}^{2g+1-j} (k+1-j) \lambda_{k+1+j} \frac{x^k dx}{4y}, & \quad j = 1, \ldots, g. 
\end{align*}
\]

We denote their matrices of $A$ and $B$-periods,

\[
2\eta = \left( -\oint_{A_k} dr_l \right), \quad 2\eta' = \left( -\oint_{B_k} dr_l \right).
\]

From Riemann bilinear identity, for the period matrices of the differentials of the first and second kind follows:

Lemma 1.1. $2g \times 2g$-matrix $\mathfrak{G} = \begin{pmatrix} \omega & \omega' \\ \eta & \eta' \end{pmatrix}$ belongs to $\text{PSp}_{2g}$:

\[
\mathfrak{G} \begin{pmatrix} 0 & -1_g \\ 1_g & 0 \end{pmatrix} \mathfrak{G}^T = -\frac{\pi i}{2} \begin{pmatrix} 0 & -1_g \\ 1_g & 0 \end{pmatrix}.
\]

1.2.3. Differentials of the third kind. are the differential 1-forms $d\Omega$ to have only poles of order 1 and 0 total residue, and so are locally given in the vicinity of any of the poles as $d\Omega = (\sum_{i=-1}^{\infty} \alpha_i \xi^i) dx$ with some constants $\alpha_i$, $\alpha_{-1}$ being nonzero. Such forms (mod holomorphic differential) may be presented as:

\[
\sum_{i=0}^{n} \beta_i \left( \frac{y + y_i^+}{x - x_i^+} - \frac{y + y_i^-}{x - x_i^-} \right) \frac{dx}{y},
\]
where \((y^+_i, x^+_i)\) are the analytic points of the poles of positive (respectively, negative) residue.

Let us introduce the canonical differential of the third kind

\[
(1.4) \quad d\Omega(x_1, x_2) = \left( \frac{y + y_1}{x - x_1} - \frac{y + y_2}{x - x_2} \right) \frac{dx}{2y},
\]

for this differential we have \(\int_{x_1}^{x_2} d\Omega(x_1, x_2) = \int_{x_1}^{x_2} d\Omega(x_3, x_4)\).

1.2.4. **Fundamental 2–differential of the second kind.** For \(\{(y_1, x_1), (y_2, x_2)\} \in (V)^2\) we introduce function \(F(x_1, x_2)\) defined by the conditions

\[
(1.5) \quad \begin{align*}
(i) & \quad F(x_1, x_2) = F(x_2, x_1), \\
(ii) & \quad F(x_1, x_1) = 2f(x_1), \\
(iii) & \quad \frac{\partial F(x_1, x_2)}{\partial x_2} |_{x_2=x_1} = \frac{df(x_1)}{dx_1}.
\end{align*}
\]

Such \(F(x_1, x_2)\) can be presented in the following equivalent forms

\[
(1.6) \quad F(x_1, x_2) = 2y_2^2 + 2(x_1 - x_2)y_2 \frac{dy_2}{dx_2} + (x_1 - x_2)^2 \sum_{j=1}^{g} x_1^{j-1} \sum_{k=j}^{2g+1-j} (k - j + 1)\lambda_{k+j+1} x_2^k,
\]

\[
(1.7) \quad F(x_1, x_2) = 2\lambda_2 + 2x_1^{g+1} x_2^{g+1} + \sum_{i=0}^{g} x_1^{i} x_2^{i} (2\lambda_2 + \lambda_{2i+1}(x_1 + x_2)).
\]

Properties \((1.5)\) of \(F(x_1, x_2)\) permit to construct the *global Abelian 2–differential of the second kind* with the unique pole of order 2 along \(x_1 = x_2\):

\[
(1.8) \quad \omega(x_1, x_2) = \frac{2y_1 y_2 + F(x_1, x_2) dx_1 dx_2}{4(x_1 - x_2)^2 y_1 y_2},
\]

which expands in the vicinity of the pole as

\[
\omega(x_1, x_2) = \left( \frac{1}{2(\xi - \zeta)^2} + O(1) \right) d\xi d\zeta,
\]

where \(\xi\) and \(\zeta\) are the local coordinates at the points \(x_1\) and \(x_2\) correspondingly.

Using the \((1.6)\), rewrite the \((1.8)\) in the form

\[
(1.9) \quad \omega(x_1, x_2) = \frac{\partial}{\partial x_2} \left( \frac{y_1 + y_2}{2y_1(x_1 - x_2)} \right) dx_1 dx_2 + du^T(x_1) dr(x_2),
\]

where the differentials \(du, dr\) are as above. So, the periods of this 2-form (the double integrals \(\oint A \oint B \omega(x_1, x_2)\)) are expressible in terms of \((2\omega, 2\omega')\) and \((-2\eta, -2\eta')\), e.g., we have for \(A\)-periods:

\[
\left\{ \oint_{A_i} \oint_{A_k} \omega(x_1, x_2) \right\}_{i,k=1,\ldots,g} = -4\omega^T \eta.
\]
1.3. Riemann $\theta$-function. The standard $\theta$-function $\theta(v|\tau)$ on $\mathbb{C}^g \times S_g$ is defined by its Fourier series,

$$\theta(v|\tau) = \sum_{m \in \mathbb{Z}^g} \exp \pi i \left\{ m^T \tau m + 2v^T m \right\}$$

The $\theta$-function possesses the periodicity properties $\forall k \in 1, \ldots, g$

$$\theta(v_1, \ldots, v_k + 1, \ldots, v_g|\tau) = \theta(v|\tau),$$
$$\theta(v_1 + \tau_1, \ldots, v_k + \tau_k, \ldots, v_g + \tau_g|\tau) = e^{i\pi \tau \tau_k - 2\pi i v \cdot \theta(v|\tau)}.$$

$\theta$-functions with characteristics $[\varepsilon] = \left[ \varepsilon' \ldots \varepsilon_g \right] \in \mathbb{C}^{2g}$

$$\theta[v](\varepsilon|\tau) = \sum_{m \in \mathbb{Z}^g} \exp \pi i \left\{ (m + \varepsilon')^T \tau (m + \varepsilon') + 2(v + \varepsilon)\tau(m + \varepsilon') \right\},$$

for which the periodicity properties are

$$\theta[v](v_1, \ldots, v_k + 1, \ldots, v_g|\tau) = e^{2\pi i v \cdot \theta(v|\tau)},$$
$$\theta[v](v_1 + \tau_1, \ldots, v_k + \tau_k, \ldots, v_g + \tau_g|\tau) = e^{i\pi \tau \tau_k - 2\pi i v \cdot \theta(v|\tau)}.$$

Further, consider half-integer characteristics $[\varepsilon]$; the $\theta$-function $\theta[v](\varepsilon|\tau)$ is even or odd whenever $4\varepsilon^T \varepsilon = 0$ or 1 modulo 2. There are $\frac{1}{2}(4^g - 2^g)$ even characteristics and $\frac{1}{2}(4^g - 2^g)$ odd.

Let $w^T = (w_1, \ldots, w_g) \in \text{Jac}(V)$ be some fixed vector, the function,

$$\mathcal{R}(x) = \theta \left( \int_{x_0}^x dv - w|\tau \right), \quad x \in V$$

is called Riemann $\theta$-function.

The Riemann $\theta$-function $\mathcal{R}(x)$ is either identically 0, or it has exactly $g$ zeros $x_1, \ldots, x_g \in V$, for which the Riemann vanishing theorem says that

$$\sum_{k=1}^g \int_{x_0}^{x_k} dv = w + K_{x_0},$$

where $K_{x_0}^T = (K_1, \ldots, K_g)$ is the vector of Riemann constants with respect to the base point $x_0$ and is defined by the formula

$$K_j = \frac{1 + \tau_{jj}}{2} - \sum_{l \neq j} \int_{A_j} dv_l(x) \int_{x_0}^x dv_j, \quad j = 1, \ldots, g.$$

2. KLEINIAN FUNCTIONS

Let $m, m' \in \mathbb{Z}^g$ be two arbitrary vectors; denote periods $\mathbf{E}(m, m') = 2\eta m + 2\eta' m'$, $\Omega(m, m') = 2\omega m + 2\omega' m'$.

2.1. $\sigma$-function. In [1, 2] it was shown, that the properties (2.1) and (2.2) define the function, which plays the central role in the theory of Kleinian functions.

**Definition 1.** An integral function $\sigma(u)$ is the Kleinian fundamental $\sigma$-function iff

1. for any vector $u \in \text{Jac}(V)$

$$(2.1) \quad \sigma(u + \Omega(m, m')) = \exp \left\{ \mathbf{E}^T(m, m')(u + \frac{1}{2}\Omega(m, m')) + \pi i m^T m' \right\} \sigma(u).$$
2. \( \sigma(u) \) has 0 of \( \left[ \frac{g+1}{2} \right] \) order at \( u = 0 \) and

\[
\lim_{u \to 0} \frac{\sigma(u)}{\delta(u)} = 1,
\]

where \( \delta(u) = \det \left( -\{u_{i+j-1}\}_{i,j=1}^{[g+1]/2} \right) \).

For small genera we have, \( \sigma = u_1 + \ldots \) for \( g = 1 \) and 2; \( \sigma = u_1u_3 - u_2^2 + \ldots \) for \( g = 3 \) and 4; \( \sigma = -u_3^3 + 2u_2u_3u_4 - u_1u_2^2 - u_2^2u_5 + u_1u_3u_5 + \ldots \) for \( g = 5 \) and 6 etc.

We introduce the \( \sigma \)-functions with characteristic \( \sigma_{r,r'} \) for vectors \( r^T, r' \in \frac{1}{2}\mathbb{Z}^g/\mathbb{Z}^g \) defined by the formula

\[
\sigma_{r,r'}(u) = e^{-\Theta^T(r,r')u} \frac{\sigma(u + \Omega(r,r'))}{\sigma(\Omega(r,r'))}.
\]

These functions are completely analogous to the Weierstrass’ \( \sigma \)-function appearing in the elliptic theory \([22]\).

2.1.1. \( \sigma \)-function as \( \theta \)-function. Fundamental hyperelliptic Kleinian \( \sigma \)-function belongs to the class of generalized \( \theta \)-functions. We give the explicit expression of the \( \sigma \) in terms of standard \( \theta \)-function as follows:

\[
\sigma(u) = Ce^{u^T\kappa u}(2\omega)^{-1}u - K_a|\tau),
\]

where \( \kappa = (2\omega)^{-1}\eta \), \( K_a \) is the vector of Riemann constants with the base point \( a \) and the constant

\[
C = \frac{\epsilon_4}{\theta(0|\tau)} \prod_{r=1}^g \frac{\sqrt{P'(a_r)}}{\sqrt{f'(a_r)}} \prod_{k<l} \sqrt{\epsilon_k - \epsilon_l}
\]

where \( (\epsilon_4)^4 = 1 \).

Direct calculation shows, that the function defined by \((2.3)\) satisfies \((2.1)\) and \((2.2)\), we only note that, in our case the vector of Riemann constants \((1.10)\) is, as follows from Riemann vanishing theorem,

\[
K_a = \sum_{k=1}^g \int_a^{a_k} dv.
\]

Putting \( g = 1 \) and fixing the elliptic curve \( y^2 = f(x) = 4x^3 - g_2x - g_3 \) in \((2.3)\), we see that the function

\[
\sigma(u) = \frac{1}{\vartheta_3(0|\tau)^{1/2}(\epsilon_1 - \epsilon_2)(\epsilon_2 - \epsilon_3)} e^{\frac{u^2}{2\omega}} \frac{\partial}{\partial u} \left( \frac{u}{2\omega} \right)
\]

is the standard Weierstrass \( \sigma \)-function, were we have used the standard notation for Jacobi \( \theta \)-functions (see e.g. \([22]\) ) .

2.2. Functions \( \zeta \) and \( \wp \). Kleinian \( \zeta \) and \( \wp \)-functions are defined as logarithmic derivatives of the fundamental \( \sigma \)

\[
\zeta_i(u) = \frac{\partial \ln \sigma(u)}{\partial u_i}, \quad i \in 1, \ldots, g;
\]

\[
\wp_{ij}(u) = -\frac{\partial^2 \ln \sigma(u)}{\partial u_i \partial u_j}, \quad \wp_{ijk}(u) = -\frac{\partial^3 \ln \sigma(u)}{\partial u_i \partial u_j \partial u_k}, \ldots, i, j, k, \ldots \in 1, \ldots, g.
\]
The functions $\zeta_i(u)$ and $\varphi_{ij}(u)$ have the following periodicity properties
\[
\zeta_i(u + \Omega(m, m')) = \zeta_i(u) + E_i(m, m'), \quad i \in 1, \ldots, g,
\]
\[
\varphi_{ij}(u + \Omega(m, m')) = \varphi_{ij}(u), \quad i, j \in 1, \ldots, g.
\]

2.2.1. **Realization of the fundamental 2–differential of the second kind by Kleinian functions.** The construction is based on the following

**Theorem 2.1.** Let $(y(a_0), a_0), (y, x)$ and $(\nu, \mu)$ be arbitrary distinct points on $V$ and let $\{(y_1, x_1), \ldots, (y_g, x_g)\}$ and $\{(\nu_1, \mu_1), \ldots, (\nu_g, \mu_g)\}$ be arbitrary sets of distinct points $\in (V)^g$. Then the following relation is valid

\[
\int_0^x \sum_{i=1}^g \int_{y_i}^{x_i} 2yy_i + F(x, x_i) \, dx \, dx_i \quad \frac{4(x - x_i)^2}{y \, y_i}
\]

is obtained using (1.8).

(2.5) $= \ln \left\{ \sigma \left( \int_{a_0}^x du - \sum_{i=1}^g \int_{y_i}^{x_i} du \right) \right\} - \ln \left\{ \sigma \left( \int_{a_0}^x du - \sum_{i=1}^g \int_{y_i}^{x_i} du \right) \right\},$

where the function $F(x, z)$ is given by [1.7].

**Proof.** Let us consider the sum

(2.6) $\sum_{i=1}^g \int_0^x \int_{\mu_i}^{x_i} \left[ \omega(x, x_i) + duT(x) \theta(du) (x_i) \right],$

with $\omega(\cdot, \cdot)$ given by (1.3). It is the normalized Abelian integral of the third kind with the logarithmic residues in the points $x_i$ and $\mu_i$. By Riemann vanishing theorem we can express (2.6) in terms of Riemann $\theta$–functions as

(2.7) $\ln \left\{ \frac{\theta \left( \int_{a_0}^x dv - \left( \sum_{i=1}^g \int_{y_i}^{x_i} dv - K_{a_0} \right) \right)}{\theta \left( \int_{a_0}^x dv - \left( \sum_{i=1}^g \int_{y_i}^{x_i} dv - K_{a_0} \right) \right)} \right\} - \ln \left\{ \frac{\theta \left( \int_{a_0}^x dv - \left( \sum_{i=1}^g \int_{y_i}^{x_i} dv - K_{a_0} \right) \right)}{\theta \left( \int_{a_0}^x dv - \left( \sum_{i=1}^g \int_{y_i}^{x_i} dv - K_{a_0} \right) \right)} \right\},$

and to obtain right hand side of (2.7) we have to combine the (2.3), expression of the vector $K_{a_0} \{2.4\}$, matrix $\theta = (2\omega)^{-1}\eta$ and Lemma 1.1. Left hand side of (2.5) is obtained using (2.8). \(\square\)

The fact, that right hand side of the (2.5) is independent on the arbitrary point $a_0$, to be employed further, has its origin in the properties of the vector of Riemann constants. Consider the difference $K_{a_0} - K_{a_0'}$ of vectors of Riemann constants with arbitrary base points $a_0$ and $a_0'$ by (1.10) we find

\[
K_{a_0} - K_{a_0'} = (g - 1) \int_{a_0'}^{a_0} \, dv,
\]

this property provides that

\[
\int_{a_0}^{x_0} \, dv - \left( \sum_{i=1}^g \int_{a_0}^{x_i} \, dv - K_{a_0} \right) = \int_{a_0'}^{x_0} \, dv - \left( \sum_{i=1}^g \int_{a_0'}^{x_i} \, dv - K_{a_0'} \right)
\]

for arbitrary $x_i$, with $i \in 0, \ldots, g$ on $V$, so the arguments of $\sigma$'s in (2.3) which are linear transformations by $2\omega$ of the arguments of $\theta$'s in (2.7), do not depend on $a_0.$
Corollary 2.1.1. From Theorem 2.1 for arbitrary distinct \((y(a_0), a_0)\) and \((y, x)\) on \(V\) and arbitrary set of distinct points \(\{(y_1, x_1), \ldots, (y_g, x_g)\} \in (V)^g\) follows:

\[
(2.8) \quad \sum_{i,j=1}^{g} \psi_{ij} \left( \int_{a_0}^{x} dx + \sum_{k=1}^{g} \int_{a_k}^{x_k} dx \right) x^{i-1} y^{j-1} = \frac{F(x, x_r) - 2yy_r}{4(x - x_r)^2}, \quad r = 1, \ldots, g.
\]

Proof. Taking the partial derivative \(\partial^2/\partial x \partial x\) from the both sides of (2.5) and using the hyperelliptic involution \(\psi(y, x) = (-y, x)\) and \(\psi(y(a_0), a_0) = (-y(a_0), a_0)\) we obtain (2.8).

In the case \(g = 1\) the formula (2.8) actually the addition theorem for the Weierstrass elliptic functions,

\[
\phi(u + v) = -\phi(u) - \phi(v) + \frac{1}{4} \left( \phi'(u) - \phi'(v) \right)^2
\]

on the elliptic curve \(y^2 = f(x) = 4x^3 - g_2x - g_3\).

Now we can give the expression for \(\omega(x, x_r)\) in terms of Kleinian functions. We send the base point \(a_0\) to the branch place \(\phi(x, x_r)\) and for \(r \in 1, \ldots, g\) the fundamental 2–differential of the second kind is given by

\[
\omega(x, x_r) = \sum_{i,j=1}^{g} \psi_{ij} \left( \int_{a_0}^{x} dx + \sum_{k=1}^{g} \int_{a_k}^{x_k} dx \right) \frac{x^{i-1} dx}{y} \frac{x^{j-1} dx}{y_r}.
\]

Corollary 2.1.2. \(\forall r \neq s \in 1, \ldots, g\)

\[
(2.9) \quad \sum_{i,j=1}^{g} \psi_{ij} \left( \sum_{k=1}^{g} \int_{a_k}^{x_k} dx \right) x^{i-1} x^{j-1} = \frac{F(x_s, x_r) - 2y_s y_r}{4(x_s - x_r)^2}.
\]

Proof. In (2.8) we have for \(s \neq r\)

\[
\int_{\phi(a_0)}^{x} dx + \sum_{k=1}^{g} \int_{a_k}^{x_k} dx = -2\omega \left( \int_{\phi(a_0)}^{x} dx \right) - \omega \left( \sum_{i=1}^{g} \int_{a_i}^{x_i} dx - K_{a_0} \right) =
\]

\[
-2\omega \left( \int_{x_s}^{x_r} dx \right) - \omega \left( \sum_{i=1}^{g} \int_{\phi(a_0)}^{x_i} dx - K_{x_s} \right) = \int_{x_s}^{x_r} dx + \sum_{i=1}^{g} \int_{a_i}^{x_i} dx
\]

and the change of notation \(x \to x_s\) gives (2.9).

2.2. Solution of the Jacobi inversion problem. The equations of Abel map in conditions of Jacobi inversion problem

\[
(2.10) \quad u_i = \sum_{k=1}^{g} \int_{a_k}^{x_k} \frac{x^{i-1} dx}{y},
\]

are invertible if the points \((y_k, x_k)\) are distinct and \(\forall j, k \in 1, \ldots, g\) \(\phi(y_k, x_k) \neq (y_j, x_j)\). Using (2.8) we find the solution of Jacobi inversion problem on the curves with \(a = \infty\) in a very effective form.

Theorem 2.2. The Abel preimage of the point \(u \in \Jac(V)\) is given by the set \(\{(y_1, x_1), \ldots, (y_g, x_g)\} \in (V)^g\), where \(x_1, \ldots, x_g\) are the zeros of the polynomial

\[
(2.11) \quad \mathcal{P}(x; u) = 0,
\]
where

\[ \mathcal{P}(x; u) = x^g - x^{g-1} \varphi_{g,g}(u) - x^{g-2} \varphi_{g,g-1}(u) - \ldots - \varphi_{g,1}(u), \]

and \( \{y_1, \ldots, y_g\} \) are given by

\[ y_k = -\frac{\partial \mathcal{P}(x; u)}{\partial u_g} \bigg|_{x=x_k}, \]

Proof. We tend in (2.8) \( a_0 \to a = \infty \). Then we take

\[ \lim_{x \to \infty} \frac{F(x, x_r)}{4x^g - (x - x_r)^2} = \sum_{i=1}^{g} \varphi_{gi}(u)x_r^{i-1}. \]

The limit in the left hand side of (2.14) is equal to \( x_0^g \), and we obtain (2.11).

We find from (2.10),

\[ \sum_{i=1}^{g} \frac{x_i^{k-1}}{y_i} \frac{\partial x_i}{\partial u_j} = \delta_{jk}, \quad \frac{\partial x_k}{\partial u_g} = \frac{y_k}{\prod_{i \neq k} (x_k - x_i)}. \]

On the other hand we have

\[ \frac{\partial \mathcal{P}}{\partial u_g} \bigg|_{x=x_k} = -\frac{\partial x_k}{\partial u_g} \prod_{i \neq k} (x_i - x_k), \]

and we obtain (2.13).\]

Let us denote by \( \varphi, \varphi' \) the \( g \)-dimensional vectors,

\[ \varphi = (\varphi_{g1}, \ldots, \varphi_{gg})^T, \quad \varphi' = \frac{\partial \varphi}{\partial u_g} \]

and the companion matrix \( \mathcal{B} \) of the polynomial \( \mathcal{P}(z; u) \), given by \( 2.12 \)

\[ \mathcal{B} = \mathcal{B} + \varphi e_g^T, \quad \text{where} \quad \mathcal{B} = \sum_{k=1}^{g} e_k e_k^T. \]

The companion matrix \( \mathcal{C} \) has the property

\[ x_k^n = X_k^T e_n - e_n^T e_k \varphi, \quad \forall n \in \mathbb{Z}, \]

with the vector \( X_k^T = (1, x_k, \ldots, x_k^{g-1}) \), where \( x_k \) is one of the roots of \( 2.11 \). From \( 2.9 \) we find \(-2y_{ix} = 4(x_r - x_s) \sum_{j=1}^{g} \varphi_{ij}(u)x_r^{i-1}x_s^{j-1} - F(x_r, x_s) \). Introducing matrices \( \Pi = (\varphi_{ij}), \Lambda_0 = \text{diag}(\lambda_{g2}, \ldots, \lambda_g) \) and \( \Lambda_1 = \text{diag}(\lambda_{g2-1}, \ldots, \lambda_1) \), we have, taking into account \( 2.13 \),

\[ 2X_k^T \varphi' \varphi'^T X_s = -4X_k^T (\mathcal{C}^2 \Pi - 2\mathcal{C} \Pi \mathcal{T} + \Pi \mathcal{T}^2) X_s \]

\[ + 4X_k^T (\mathcal{C} \varphi \varphi^T + \varphi \varphi^T \mathcal{C}) X_s + 2X_k^T \Lambda_0 X_s + X_k^T (\mathcal{C} \Lambda_1 + \Lambda_1 \mathcal{C}^T) X_s. \]

Whence, (see \( 10 \)):

Corollary 2.2.1. The relation

\[ 2\varphi' \varphi'^T = -4(\mathcal{C}^2 \Pi - 2\mathcal{C} \Pi \mathcal{T} + \Pi \mathcal{T}^2) + 4(\mathcal{C} \varphi \varphi^T + \varphi \varphi^T \mathcal{C}) + \mathcal{C} \Lambda_1 + \Lambda_1 \mathcal{C}^T + 2\Lambda_0. \]

connects odd functions \( \varphi_{gij} \) with poles order 3 and even functions \( \varphi_{jk} \) with poles of order 2 in the field of meromorphic functions on \( \text{Jac}(V) \).
Definition 2. The umbral derivative $D_s(p(z))$ of a polynomial $p(z) = \sum_{k=0}^{n} p_k z^k$ is given by

$$D_s p(z) = \left(\frac{p(z)}{z^s}\right) + \sum_{k=s}^{n} p_k x^{k-s},$$

where $(\cdot)_+$ means taking the purely polynomial part.

Considering polynomials $p = \prod_{k=1}^{n} (z - z_k)$ and $\tilde{p} = (z - z_0)p$, the elementary properties of $D_s$ are immediately deduced:

$$D_s(p) = zD_{s+1}(p) + p_s = zD_{s+1}(p) + S_{n-s}(z_1, \ldots, z_n),$$

(2.17) $D_s(\tilde{p}) = (z - z_0)D_s(p) + p_{s-1} = (z - z_0)D_s(p) + S_{n+1-s}(z_1, \ldots, z_n),$

where $S_l(\cdots)$ is the $l$-th order elementary symmetric function of its variables times $(-1)^l$ (we assume $S_0(\cdots) = 1$).

From (2.17) we see that $S_{n-s}(z_0, \ldots, \hat{z}_i, \ldots, z_n) = (D_{s+1}(\tilde{p})|_{z=z_i})$. This is particularly useful to write down the inversion of (2.13)

(2.18) $$\varphi_{ggk}(u) = \sum_{l=1}^{g} y_l \left. \left(\frac{D_k(P(z))}{\partial z} \right) \right|_{z=x_l},$$

where $P(z) = \prod_{k=1}^{g} (z - x_k)$.

It is of importance to describe the set of common zeros of the functions $\varphi_{ggk}(u)$.

Corollary 2.2.2. The vector function $\varphi'(u)$ vanishes iff $u$ is a halfperiod.

Proof: The equations $\varphi_{ggk}(u) = 0$, $k \in 1, \ldots, g$ yield to (2.18) the equalities $y_i = 0, \forall i \in 1, \ldots, g$. The latter is possible if and only if the points $x_1, \ldots, x_g$ coincide with any $g$ points $e_{i_1}, \ldots, e_{i_g}$ from the set branching points $e_1, \ldots, e_{2g+2}$. So the point

$$u = \sum_{l=1}^{g} \int_{a_i}^{e_{i_l}} du \in \text{Jac}(V)$$

is of the second order in Jacobian and hence is a halfperiod.

3. Basic relations

In this section we are going to derive the explicit algebraic relations between the generating functions in the field of meromorphic functions on $\text{Jac}(V)$. After some preparations just below, we will in the section 3.1 find the explicit cubic relations between $\varphi_{ggi}$ and $\varphi_{ij}$. These, in turn, lead to very special corollaries: the variety $\text{Kum}(V) = \text{Jac}(V)/\pm$ is mapped into the space of symmetric matrices of rank not greater than 3.

We start with, the conditions $\lambda_{2g+2} = 0$, $\lambda_{2g+1} = 4$ being imposed, the following Theorem 3.1 which is the starting point for derivation of the basic relations.

Theorem 3.1. Let $(y_0, x_0) \in V$ be an arbitrary point and $\{(y_1, x_1), \ldots, (y_g, x_g)\} \in (V)^g$ be the Abel preimage of the point $u \in \text{Jac}(V)$. Then

(3.1) $$-\zeta_j \left(\int_{a}^{x_0} du + u\right) = \int_{a}^{x_0} dr_j + \sum_{k=1}^{g} \int_{a_k}^{x_k} dr_j - \frac{1}{2} \sum_{k=0}^{g} y_k \left. \left(\frac{D_j(R'(z))}{R'(z)}\right) \right|_{z=x_k},$$
where \( R(z) = \prod_{j=0}^{q}(z - x_j) \) and \( R'(z) = \frac{d}{dz} R(z) \).

And

\[
(3.2) \quad -\zeta_j(u) = \sum_{k=1}^{g} \int_{a_k}^{x_k} dr_j - \frac{1}{2} \varphi_{gg,j+1}(u).
\]

**Proof.** Putting in (2.25) \( \mu_i = a_i \) we have

\[
(3.3) \quad \ln \left\{ \frac{\sigma(f_{a_0}^x du - u)}{\sigma(f_{a_0}^x du)} \right\} - \left\{ \frac{\sigma(f_{a_0}^x du - u)}{\sigma(f_{a_0}^x du)} \right\} = \int_{\mu}^x dr \ u + \sum_{k=1}^{g} \int_{a_k}^{x_k} d\Omega(x, \mu),
\]

where \( d\Omega \) is as in (3.4). Taking derivative over \( u_j \) from the both sides of the equality (3.3), after that letting \( a_0 \to \mu \) and applying \( \phi(y, x) = (-y, x) \) and \( \phi(\nu, \mu) = (-\nu, \mu) \), we have

\[
\zeta_j \left( \int_{\mu}^x du + u \right) + \int_{\mu}^x dr_j - \frac{1}{2} \sum_{k=1}^{g} \frac{1}{y_k} \frac{\partial x_k y_k - y_k}{\partial u_j x_k - x} = \zeta_j(u) - \frac{1}{2} \sum_{k=1}^{g} \frac{1}{y_k} \frac{\partial x_k y_k - \nu_k}{\partial u_j x_k - \mu_k}.
\]

Put \( x = x_0 \). Denoting \( P(z) = \prod_{i=1}^{q}(z - x_j) \) we find

\[
\sum_{k=1}^{g} \frac{1}{y_k} \frac{\partial x_k y_k - y_k}{\partial u_j x_k - x} = \sum_{k=1}^{g} \left( \frac{D_j(P(z))}{P'(z)} \right) \left( \frac{y_k - y_k}{x_k - x} \right) \]

\[
= \sum_{k=0}^{g} y_k \left( \frac{D_j(R(z))}{R'(z)} \right) \left( \frac{y_k - y_k}{x_k - x} \right) - \sum_{k=1}^{g} \left( \frac{D_{j+1}(P(z))}{P'(z)} \right) \left( \frac{y_k - y_k}{x_k - x} \right).
\]

Hence, using (2.18) and adding to both sides \( \sum_{k=1}^{g} \int_{a_k}^{x_k} dr_j \), we deduce

\[
\zeta_j \left( \int_{\mu}^{x_0} du + u \right) + \int_{\mu}^{x_0} dr_j + \sum_{k=1}^{g} \int_{a_k}^{x_k} dr_j - \frac{1}{2} \sum_{k=0}^{g} y_k \left( \frac{D_j(R(z))}{R'(z)} \right) \left( \frac{y_k - y_k}{x_k - x} \right)
\]

\[
(3.4) \quad = \zeta_j(u) + \sum_{k=1}^{g} \int_{a_k}^{x_k} dr_j - \frac{1}{2} \sum_{k=1}^{g} \frac{1}{y_k} \frac{\partial x_k y_k - \nu_k}{\partial u_j x_k - \mu_k} - \frac{1}{2} \varphi_{gg,j+1}.
\]

Now see, that the left hand side of the (3.4) is symmetrical in \( x_0, x_1, \ldots, x_g \), while the right hand side does not depend on \( x_0 \). So, it does not depend on any of \( x_i \). We conclude, that it is a constant depending only on \( \mu \). Tending \( \mu \to a \) and applying the hyperelliptic involution to the whole aggregate, we find this constant to be 0.

**Corollary 3.1.1.** For \((y, x) \in V\) and \( \alpha = \int_{a}^{x} du \):

\[
(3.5) \quad \zeta_j(u + \alpha) - \zeta_j(u) - \zeta_j(\alpha) = \frac{(-y D_j + \partial_j P(x; u))}{2P(x; u)},
\]

where \( \partial_j = \partial/\partial u_j \).

**Proof.** To find \( \zeta_j(\alpha) \) take the limit \( \{x_1, \ldots, x_g\} \to \{a_1, \ldots, a_g\} \) in (3.1). The right hand side of (3.3) is obtained by rearranging \( \frac{1}{2} \sum_{k=1}^{g} \frac{1}{y_k} \frac{\partial x_k y_k - y_k}{\partial u_j x_k - x_k} \).

**Corollary 3.1.2.** The functions \( \varphi_{ggk} \), for \( k = 1, \ldots, g \) are given by

\[
(3.6) \quad \varphi_{ggi} = (6 \varphi_{gg} + \lambda_{2g}) \varphi_{gj} + 6 \varphi_{g,i-1} - 2 \varphi_{g-1,i} + \frac{1}{2} \delta_{gi} \lambda_{2g-1}.
\]
Proof. Consider the relation (3.2). The differentials $d\zeta_i$, $i = 1, \ldots, g$ can be presented in the following forms

$$-d\zeta_i = \sum_{k=1}^g \varphi_{ik} du_k = \sum_{k=1}^g dr_i(x_k) - \frac{1}{2} \sum_{k=1}^g \varphi_{ggi,i+1,k} du_k.$$  

Put $i = g - 1$. We obtain for each of the $x_k$, $k = 1, \ldots, g$

$$\left(12x_k^{g+1} + 2\lambda_2g x_k^g + \lambda_2g - 1x_k^{g-1} - 4 \sum_{j=1}^g \varphi_{g-1,j} x_k^{j-1} \right) \frac{dx_k}{y_k} = 2 \sum_{j=1}^g \varphi_{gggj} x_k^{j-1} \frac{dx_k}{y_k}.$$

Applying the formula (2.12) to eliminate the powers of $x_k$ greater than $g - 1$, and taking into account, that the differentials $dx_k$ are independent, we come to

$$\sum_{i=1}^g \left[(6\varphi_{gg} + \lambda_2g)\varphi_{gi} + 6\varphi_{g,i-1} - 2\varphi_{g-1,i} + \frac{1}{2} \delta_{gi}\lambda_2g - 1 \right] x_k^{i-1} = \sum_{j=1}^g \varphi_{gggj} x_k^{j-1}.$$

Let us calculate the difference $\frac{\partial \varphi_{gggk}}{\partial u_i} - \frac{\partial \varphi_{gggi}}{\partial u_k}$ according to the (3.4). We obtain

**Corollary 3.1.3.**

(3.7) $\varphi_{ggk}\varphi_{gi} - \varphi_{ggi}\varphi_{gk} + \varphi_{g,i-1,k} - \varphi_{gi,k-1} = 0.$

This means that the 1–form $\sum_{i=1}^g (\varphi_{gg}\varphi_{gi} + \varphi_{g,i-1}) du_i$ is closed. We can rewrite this as $d\mathbf{u}^T \mathbf{C} \mathbf{u}$.

Differentiation of (3.7) by $u_g$ yields

(3.8) $\varphi_{gggk}\varphi_{gi} - \varphi_{ggsi}\varphi_{gk} + \varphi_{ggi,i-1,k} - \varphi_{gi,gk-1} = 0.$

And the corresponding closed 1–form is $d\mathbf{u}^T \mathbf{C} \mathbf{u}'.$

3.1. **Fundamental cubic and quartic relations.** We are going to find the relations connecting the odd functions $\varphi_{ggi}$ and even functions $\varphi_{ij}$. These relations take in hyperelliptic theory the place of the Weierstrass cubic relation

$$\psi^2 = 4\psi^3 - g_2\psi - g_3,$$

for elliptic functions, which establishes the meromorphic map between the elliptic Jacobian $\mathbb{C}^g/(2\omega; 2\omega')$ and the plane cubic.

The theorem below is based on the property of an Abelian function to be constant if any gradient of it is identically 0, or, if for Abelian functions $G(\mathbf{u})$ and $F(\mathbf{u})$ there exist such a nonzero vector $\alpha \in \mathbb{C}^g$, that $\sum_{i=1}^g \alpha_i \frac{\partial}{\partial u_i} (G(\mathbf{u}) - F(\mathbf{u}))$ vanishes, then $G(\mathbf{u}) - F(\mathbf{u})$ is a constant.

**Theorem 3.2.** The functions $\varphi_{ggi}$ and $\varphi_{ik}$ are related by

$$\varphi_{ggi}\varphi_{ggi} = 4\varphi_{gg}\varphi_{gi}\varphi_{gk} - 2(\varphi_{gj}\varphi_{gj-1,k} + \varphi_{g,k}\varphi_{gj-1,i})$$

$$+ 4(\varphi_{ggk}\varphi_{gj} + \varphi_{gggi}\varphi_{gj} + \varphi_{gj,gk,j-1}) + 4\varphi_{k,j-1,i-1} - 2(\varphi_{k,i-2} + \varphi_{i,k-2} - \varphi_{k,i-1})$$

$$+ \lambda_2^2g\varphi_{gj}\varphi_{gj} + \frac{\lambda_2^2g-1}{2} (\delta_{ij}\varphi_{kg} + \delta_{kg}\varphi_{ig}) + c_{i,k},$$

where

$$c_{i,k} = \lambda_{2i-2}\delta_{ik} + \frac{1}{2} (\lambda_{2i-1}\delta_{i,k+1} + \lambda_{2k-1}\delta_{i,k+1}).$$
Proof. We are looking for such a function \( G(u) \) that \( \frac{\partial}{\partial u_i}(\varphi g_{i} \varphi g_{k} - G) = 0 \). Direct check using (3.7) shows that

\[
\frac{\partial}{\partial u_i}(\varphi g_{i} \varphi g_{k} - (4\varphi g_{i}\varphi g_{i}\varphi g_{k} - 2(\varphi g_{i}\varphi g_{i} - k + \varphi g_{k}\varphi g_{k} - i))
+ 4(\varphi g_{k}\varphi g_{i} - i + \varphi g_{i}\varphi g_{k} - i) + 4\varphi g_{k} - i - 1 - 2(\varphi g_{i} - i - 2)
+ \lambda_{2g}\varphi g_{k}\varphi g_{i} + \frac{\lambda_{i}^{2g-1}}{2}(\delta_{gk}\delta_{kg} + \delta_{kg}\delta_{kg})) = 0.
\]

It remains to determine \( c_{ij} \). From (3.10), we conclude, that \( c_{i, k} \) for \( k = i \) is equal to \( \lambda_{2g-2} \), for \( k = i + 1 \) to \( \frac{1}{2}\lambda_{2i-1} \), otherwise 0. So \( c_{ij} \) is given by (3.10).

Consider \( C^{g+\frac{g(g+1)}{2}} \) with coordinates \( (z, p = \{p_{i,j}\}_{i,j=1}^{g}; \) with \( z^{T} = (z_{1}, \ldots, z_{g}) \) and \( p_{ij} = p_{ji} \), then we have

**Corollary 3.2.1.** The map

\[ \varphi : \text{Jac}(V) \setminus \sigma \rightarrow C^{g+\frac{g(g+1)}{2}}, \quad \varphi(u) = (\varphi(u), \Pi(u)), \]

where \( \Pi = \{\varphi_{ij}\}_{i,j=1}^{g} \), is meromorphic embedding.

The image \( \varphi(\text{Jac}(V) \setminus \sigma) \subset C^{g+\frac{g(g+1)}{2}} \) is the intersection of \( \frac{g(g+1)}{2} \) cubics, induced by (5).\(^{(6)}\)

\( \sigma \) denotes the divisor of 0’s of \( \sigma \).

Consider projection

\[ \pi : C^{g+\frac{g(g+1)}{2}} \rightarrow C^{g+\frac{g(g+1)}{2}}, \quad \pi(z, p) = p. \]

**Corollary 3.2.2.** The restriction \( \pi \circ \varphi \) is the meromorphic embedding of the Kummer variety \( \text{Kum}(V) = (\text{Jac}(V) \setminus \sigma)/\pm \rightarrow C^{g+\frac{g(g+1)}{2}} \). The image \( \pi(\varphi(\text{Jac}(V) \setminus \sigma)) \subset C^{g+\frac{g(g+1)}{2}} \) is the intersection of \( \frac{g(g+1)}{2} \) quartics, induced by (3.11)

\[
(\varphi g_{i} \varphi g_{i})(\varphi g_{k} \varphi g_{i}) - (\varphi g_{j} \varphi g_{k})(\varphi g_{i} \varphi g_{j}) = 0,
\]

where the parentheses mean, that substitutions by (3.9) are made before expanding.

The quartics (3.11) have no analogue in the elliptic theory. The first example is given by genus 2, where the celebrated Kummer surface \( \text{Kum} \) appears.

### 3.2. Analysis of fundamental relations

Let us take a second look at the fundamental cubics (3.9) and quartics (3.11).

**3.2.1. Sylvester’s identity.** For any matrix \( K \) of entries \( k_{ij} \) with \( i, j = 1, \ldots, N \) we introduce the symbol \( K[i,j]^{i_{1}, \ldots, i_{m}}_{j_{1}, \ldots, j_{n}} \) to denote the \( m \times n \) submatrix:

\[ K[i_{1}, \ldots, i_{m}]_{j_{1}, \ldots, j_{n}} = \{k_{i_{k}, j_{l}}\}_{k=1, \ldots, m; l=1, \ldots, n} \]

for subsets of rows \( i_{k} \) and columns \( j_{l} \).

We will need here the Sylvester’s identity (see, for instance [23]). Let us fix a subset of indices \( \alpha = \{i_{1}, \ldots, i_{k}\} \), and make up the \( N - k \times N - k \) matrix \( S(K, \alpha) \) assuming that

\[ S(K, \alpha)_{\mu, \nu} = \det K[i_{\mu}, j_{\nu}]^{i_{\alpha}}_{j_{\alpha}} \]

and \( \mu, \nu \) are not in \( \alpha \), then

\[ \det S(K, \alpha) = \det K[i_{\alpha}]^{(N-k-1)}_{j_{\alpha}} \det K. \]


3.2.2. Determinantal form. We introduce (cf. [13]) new functions \( h_{ik} \) defined by the formula

\[
h_{ik} = 4\varphi_{i-1,k-1} - 2\varphi_{i-1,k-2} - 2\varphi_{i,k-2} + \frac{1}{2} (\delta_{ik}(\lambda_{2i-2} + \lambda_{2k-2}) + \delta_{i+1,k-1}\lambda_{2i-1} + \delta_{i,k+1}\lambda_{2k-1}),
\]

(3.13)

where the indices \( i, k \in 1, \ldots, g+2 \). We assume that \( \varphi_{nm} = 0 \) if \( n \) or \( m \) is \( < 1 \) and \( \varphi_{nm} = 0 \) if \( n \) or \( m \) is \( > g \). It is evident that \( h_{ij} = h_{ji} \). We shall denote the matrix of \( h_{ik} \) by \( H \).

The map (3.13) from \( \varphi \)'s and \( \lambda \)'s to \( h \)'s respects the grading

\[
\deg h_{ij} = i + j, \quad \deg \varphi_{ij} = i + j + 2, \quad \deg \lambda = i + 2,
\]

and on a fixed level \( L \) (3.13) is linear and invertible. From the definition follows

\[
\sum_{i=1}^{L-1} h_{i,L-i} = \lambda_{L-2} \Rightarrow X^T H X = \sum_{i=0}^{2g+2} \lambda_i x^i
\]

for \( X^T = (1, x, \ldots, x^{g+1}) \) with arbitrary \( x \in \mathbb{C} \). Moreover, for any roots \( x_r \) and \( x_s \) of the equation \( \sum_{j=1}^{g+2} h_{g+j+2,j} x^j - 1 = 0 \) we have (cf. (2.9)) \( y_r y_s = X^T H X_s \).

From (3.13) we have

\[
-2\varphi_{g+i} = \frac{\partial}{\partial u}\varphi_{g+2,i} = \frac{\partial}{\partial u} h_{g+2,i} = -\frac{1}{2} \frac{\partial}{\partial u} h_{g+1,i+1},
\]

(3.14)

\[
2(\varphi_{g+i-1,1} - \varphi_{g,i-1,k}) = \frac{\partial}{\partial u} h_{g+2,i-1} - \frac{\partial}{\partial u} h_{g+2,k-1} = \frac{1}{2} \frac{\partial}{\partial u} h_{g+1,1} - \frac{1}{2} \frac{\partial}{\partial u} h_{g+2,i},
\]

\[
\ldots \text{ etc., and (see (3.6)) :}
\]

Using (3.13), we write (3.9) in more effective form:

\[
4\varphi_{g+i} \varphi_{g+k} = \frac{\partial}{\partial u} h_{g+2,i} \frac{\partial}{\partial u} h_{g+2,k} = -\det H_{[g+i,g+k+2]} - H_{[g+i,g+2]} - H_{[g+i,g+1,g+2]}.
\]

(3.15)

Consider, as an example, the case of genus 1. We define on the Jacobian of a curve

\[
y^2 = \lambda_4 x^4 + \lambda_3 x^3 + \lambda_2 x^2 + \lambda_1 x + \lambda_0
\]

the Kleinian functions: \( \sigma_k(u_1) \) with expansion \( u_1 + \ldots \), its second and third logarithmic derivatives \( -\varphi_{11} \) and \( -\varphi_{111} \). By (3.13) and following the definition (3.13)

\[
-4\varphi_{1111}^2 = \det H_{[1.2,3]} = \det \left( \begin{array}{ccc}
\lambda_0 & \frac{1}{2} \lambda_1 & -2 \varphi_{11} \\
\frac{1}{2} \lambda_1 & 4 \varphi_{11} + \lambda_2 & \frac{1}{2} \lambda_3 \\
-2 \varphi_{11} & \frac{1}{2} \lambda_3 & \lambda_4
\end{array} \right);
\]

the determinant expands as:

\[
\varphi_{1111}^2 = 4\varphi_{11}^2 + \frac{\lambda_1 \lambda_3 - 4 \lambda_4 \lambda_0}{4} + \frac{\lambda_0 \lambda_3^2 + \lambda_4 (\lambda_2^2 - 4 \lambda_2 \lambda_0)}{16},
\]

and the (3.14), in complete accordance, gives

\[
\varphi_{1111} = 6\varphi_{11}^2 + \frac{\lambda_1 \lambda_3 - 4 \lambda_4 \lambda_0}{8}
\]
These equations show that $\sigma_K$ differs by $\exp(-\frac{1}{6}\lambda_2 u_1^2)$ from standard Weierstrass $\sigma_W$ built by the invariants $g_2 = \lambda_1 \lambda_0 + \frac{1}{12} \lambda_2^2 - \frac{1}{4} \lambda_3 \lambda_1$ and $g_3 = \det \left( \begin{array}{ccc} \lambda_0 & \frac{1}{6} \lambda_2 & \frac{1}{2} \lambda_3 \\ \frac{1}{2} \lambda_1 & \frac{1}{2} \lambda_2 & \frac{1}{2} \lambda_3 \\ \frac{1}{3} \lambda_2 & \frac{1}{3} \lambda_2 & \frac{1}{3} \lambda_3 \end{array} \right)$ (see, e.g. [22, 26]).

Further, we find, that rank $H = 3$ in generic point of Jacobian, rank $H = 2$ in halfperiods. At $u_1 = 0$, where $\sigma_K$ has is of order 1, we have rank $\sigma_K^2 H = 3$.

Concerning the general case, on the ground of $(3.17)$, we prove the following:

**Theorem 3.3.** rank $H = 3$ in generic point $\in \text{Jac}(V)$ and rank $H = 2$ in the halfperiods. rank $\sigma(\mathbf{u})^2 H = 3$ in generic point $\in (\sigma)$ and rank $\sigma(\mathbf{u})^2 H$ in the points of $(\sigma)_{\text{sing}}$.

Here $(\sigma) \subset \text{Jac}(V)$ denotes the divisor of 0's of $\sigma(\mathbf{u})$. The $(\sigma)_{\text{sing}} \subset (\sigma)$ is the so-called singular set of $(\sigma)$. $(\sigma)_{\text{sing}}$ is the set of points where $\sigma$ vanishes and all its first partial derivatives vanish. $(\sigma)_{\text{sing}}$ is known (see [18] and references therein) to be a subset of dimension $g - 3$ in hyperelliptic Jacobians of $g > 3$, for genus 2 it is empty and consists of single point for $g = 3$. Generally, the points of $(\sigma)_{\text{sing}}$ are presented by $\{(y_1, x_1), \ldots, (y_{g-3}, x_{g-3})\} \in (V)^{g-3}$ such that for all $i \neq j \in 1, \ldots, g - 3$, $\phi(y_i, x_i) \neq (y_j, x_j)$.

**Proof.** Consider the Sylvester’s matrix $S = \begin{pmatrix} \varphi_{gg1} & \varphi_{gg2} & \cdots & \varphi_{ggg} \\ \varphi_{gg2} & \varphi_{gg3} & \cdots & \varphi_{ggg} \\ \cdots & \cdots & \cdots & \cdots \\ \varphi_{ggg} & \varphi_{ggg} & \cdots & \varphi_{ggg} \end{pmatrix}$ and $\det S = 0$. By $(3.12)$ we have $\det S = 0$, so by $(3.12)$ we see, that $\det H_{[i,j,g+1,g+2]}^{k,l,g+1,g+2}$ vanishes identically. As $\det H_{[g+1,g+2]}^{g+1,g+2} = \lambda_2 g + \lambda_2 g - \frac{1}{4} \lambda_2^2 g + 1$ is not an identical 0, we infer that

$$\det H = \begin{pmatrix} i,j,g+1,g+2 \\ k,l,g+1,g+2 \end{pmatrix} = 0.$$

Remark, that this equation is actually the $(3.11)$ rewritten in terms of $h$'s. Now from the $(3.16)$, putting $j = l = g$, we obtain for any $i, k$, except for such $\mathbf{u}$, that $H_{[g+1,g+2]}^{g+1,g+2}$ becomes degenerate, and those where the entries become singular i.e. $\mathbf{u} \in (\sigma)$,

$$(3.17) \quad h_{ik} = (h_{i,g}, h_{i,g+1}, h_{i,g+2}) \begin{pmatrix} H_{[g,g+1,g+2]}^{g,g+1,g+2} & -1 \end{pmatrix} \begin{pmatrix} h_{k,g} \\ h_{k,g+1} \\ h_{k,g+2} \end{pmatrix}.$$

This leads to the skeleton decomposition of the matrix $H$

$$(3.18) \quad H = H_{[1, \ldots, g+2]}^{1, \ldots, g+2} \begin{pmatrix} H_{[g,g+1,g+2]}^{g,g+1,g+2} & -1 \end{pmatrix} \begin{pmatrix} H_{[1, \ldots, g+2]}^{g+1,g+2} & \cdots & \cdots \end{pmatrix},$$

which shows, that in generic point of $\text{Jac}(V)$ rank of $H$ equals 3.

Consider the case $\det H_{[g,g+1,g+2]}^{g,g+1,g+2} = 0$. As by $(3.15)$ we have $\det H_{[g,g+1,g+2]}^{g,g+1,g+2} = 4 \varphi_{ggg}^2$ this may happen only if $\mathbf{u}$ is a halfperiod. And therefore we have instead of $(3.17)$ the equalities $H_{[1, \ldots, g+2]}^{1, \ldots, g+2} = 0$ and consequently in halfperiods matrix $H$ is decomposed as

$$H = H_{[1, \ldots, g+2]}^{1, \ldots, g+2} \begin{pmatrix} H_{[1, \ldots, g+2]}^{g+1,g+2} & \cdots \end{pmatrix} \begin{pmatrix} H_{[1, \ldots, g+2]}^{g+1,g+2} & \cdots \end{pmatrix},$$

having the rank 2.
Next, consider \( \sigma(u)^2 H \) at the \( u \in (\sigma) \). We have \( \sigma(u)^2 h_{i,k} = 4\sigma_{i-1}\sigma_{k-1}-2\sigma_{i-2}\sigma_{k-2} - 2\sigma_{i-2}\sigma_{k} \), where \( \sigma = \frac{1}{\sigma_{i-1}}\sigma(u) \), and consequently, the decomposition

\[
\sigma(u)^2 H|_{u \in (\sigma)} = 2(s_1, s_2, s_3) \begin{pmatrix}
0 & 0 & -1 \\
0 & 2 & 0 \\
-1 & 0 & 0
\end{pmatrix} \begin{pmatrix}
s_1^T \\
s_2^T \\
s_3^T
\end{pmatrix},
\]

where \( s_1 = (\sigma_1, \ldots, \sigma_g, 0, 0)^T \), \( s_2 = (0, \sigma_1, \ldots, \sigma_g, 0)^T \) and \( s_3 = (0, 0, \sigma_1, \ldots, \sigma_g)^T \). We infer, that \( \text{rank}(\sigma(u)^2 H) \) is 3 in generic point of \( (\sigma) \), and becomes 0 only when \( \sigma_1 = \ldots = \sigma_g = 0 \), is in the points \( \in (\sigma)_{\text{sing}} \), while no other values are possible.

**Conclusion.** The map

\[
h : u \mapsto \{4\sigma_{i-1}\sigma_{k-1}-2\sigma_{i-2}\sigma_{k-2} - 2\sigma_{i-2}\sigma_{k} - \sigma(4\sigma_{i-1,k-1}-2\sigma_{i,k-2} - 2\sigma_{i-2,k}) + \frac{1}{2}\sigma^2(\delta_{ik}(\lambda_{2i-2} + \lambda_{2k-2}) + \delta_{i,k+1}\lambda_{2i-1}) \} \in \mathbb{C}[\sigma_{i,k}^{+}, \sigma_{i,k}^{-}]_{i,k \in 1, \ldots, g+2},
\]

induced by (3.13) establish the meromorphic map of the \( (\text{Jac}(V) \setminus (\sigma)_{\text{sing}}) / \pm \) into the space \( Q_3 \) of complex symmetric \( (g+2) \times (g+2) \) matrices of rank not greater than 3.

We give the example of genus 2 with \( \lambda_6 = 0 \) and \( \lambda_5 = 4 \):

\[
(3.19) \quad H = \begin{pmatrix}
\lambda_0 & \frac{3}{2}\lambda_1 & -2\psi_{11} & -2\psi_{12} \\
\frac{1}{2}\lambda_1 & \lambda_2 + 4\psi_{11} & \frac{1}{2}\lambda_3 - 2\psi_{12} & -2\psi_{22} \\
-2\psi_{11} & \frac{1}{2}\lambda_3 - 2\psi_{12} & \lambda_4 + 4\psi_{22} & 2 \\
-2\psi_{12} & -2\psi_{22} & 2 & 0
\end{pmatrix}.
\]

In this case \( (\sigma)_{\text{sing}} = \{0\} \), so the Kummer surface in \( \mathbb{CP}^3 \) with coordinates

\[
(X_0, X_1, X_2, X_3) = (\sigma^2, \sigma^2\psi_{11}, \sigma^2\psi_{12}, \sigma^2\psi_{22})
\]

is defined by the equation \( \det \sigma^2 H = 0 \).

**3.2.3. Extended cubic relation.** The extension \( [13] \) of (3.15) is given by

**Theorem 3.4.**

\[
(3.20) \quad R^T \pi_{jl} \pi_{ik}^T S = \frac{1}{4} \det \begin{pmatrix}
H_{[kg+1g+2]_j}^{[kg+1g+2]_i} & S \\
R^T & 0
\end{pmatrix},
\]

where \( R, S \in \mathbb{C}^4 \) are arbitrary vectors and

\[
\pi_{ik} = \begin{pmatrix}
-\psi_{gk} \\
\psi_{gt} \\
\psi_{g-1,i,k-1} - \psi_{g-1,i,k-1} \\
\psi_{g-1,i,k,i-1} + \psi_{g,k,i-2} - \psi_{g,i,k-2}
\end{pmatrix}.
\]

**Proof.** Vectors \( \tilde{\pi} = \pi_{ik} \) and \( \pi = \pi_{jl} \) solve the equations

\[
H_{[kg+1g+2]_j}^{[kg+1g+2]_i} \pi = 0; \quad \tilde{\pi}^T H_{[kg+1g+2]_j}^{[kg+1g+2]_i} = 0.
\]

The theorem follows.

**Conclusion.** The case of genus 2, when \( \pi_{21} = (-\psi_{222}, \psi_{221}, -\psi_{211}, \psi_{111})^T \) exhausts all the possible \( \psi_{ijk} \)-functions, the relation (3.20) was thoroughly studied by Baker [12].
4. Applications

4.1. Matrix realization of hyperelliptic Kummer varieties. Here we present the explicit matrix realization (see [14]) of hyperelliptic Jacobians $\text{Jac}(V)$ and Kummer varieties $\text{Kum}(V)$ of the curves $V$ with the fixed branching point $e_{2g+2} = a = \infty$. Our approach is based on the results of Section 3.2.

Let us consider the space $H$ of complex symmetric $(g+2) \times (g+2)$–matrices $H = \{h_{k,s}\}$, with $h_{g+2,g+2} = 0$ and $h_{g+1,g+2} = 2$. Let us put in correspondence to $H \in H$ a symmetric $g \times g$–matrix $A(H)$, with entries $a_{k,s} = \det H_{[k,g+1],g+2}$.

From the Sylvester’s identity (3.12) follows that rank of the matrix $H \in H$ does not exceed 3 if and only if rank of the matrix $A(H)$ does not exceed 1.

Let us put $K_H = \{H \in H : \text{rank} H \leq 3\}$. For each complex symmetric $g \times g$–matrix $A = \{a_{k,s}\}$ of rank not greater 1, there exists, defined up to sign, a $g$–dimensional column vector $z = z(A)$, such that $A = -4z \cdot z^T$.

Let us introduce vectors $h_k = \{h_{k,s} ; s = 1, \ldots , g\} \in \mathbb{C}^g$.

Lemma 4.1. Map

$$\gamma : K_H \to (\mathbb{C}^g/\pm) \times \mathbb{C}^g \times \mathbb{C}^g \times \mathbb{C}^1$$

$$\gamma(H) = - (z(A(H)), h_{g+1}, h_{g+2}, h_{g+1,g+1})$$

is a homeomorphism.

Proof. follows from the relation:

$$4 \hat{H} = 4z \cdot z^T + 2 (h_{g+2} h_{g+1}^T + h_{g+1} h_{g+2}^T) - h_{g+1,g+1} h_{g+2} h_{g+2}^T$$

where $\hat{H}$ is the matrix composed of the column vectors $h_k$, $k = 1, \ldots , g$, and $z = (z(A(H)))$.

Let us introduce the 2–sheeted ramified covering $\pi : JH \to KH$, which the covering $C^g \to (\mathbb{C}^g/\pm)$ induces by the map $\gamma$.

Corollary 4.1.1. $\hat{\gamma} : JH \cong \mathbb{C}^{3g+1}$.

Now let us consider the universal space $W_g$ of $g$–th symmetric powers of hyperelliptic curves

$$V = \left\{(y,x) \in \mathbb{C}^2 : y^2 = 4x^{2g+1} + \sum_{k=0}^{2g} \lambda_{2g-k} x^{2g-k}\right\}$$

as an algebraic subvariety in $(\mathbb{C}^2)^g \times \mathbb{C}^{2g+1}$ with coordinates

$$\{(y_1,x_1), \ldots , (y_g,x_g), \lambda_{2g}, \ldots , \lambda_0\},$$

where $(\mathbb{C}^2)^g$ is $g$–th symmetric power of the space $\mathbb{C}^2$.

Let us define the map

$$\lambda : JH \cong \mathbb{C}^{3g+1} \to (\mathbb{C}^2)^g \times \mathbb{C}^{2g+1}$$

in the following way:

- for $G = (z, h_{g+1}, h_{g+2}, h_{g+1,g+1}) \in \mathbb{C}^{3g+1}$ construct by Lemma 4.1 the matrix $\pi(G) = H = \{h_{k,s}\} \in KH$
• put

\[ \lambda(G) = \{(y_k, x_k), \lambda_r; \ k = 1, \ldots , g, r = 0, \ldots , 2g. \} \]

where \{x_1, \ldots , x_g \} is the set of roots of the equation \(2x^g + h_{g+2}^T X = 0, \) and \( y_k = z^T x_k, \) and \( \lambda_r = \sum_{i+j=r+2} h_{i,j}. \)

Here \( X_k = (1, x_k, \ldots , x_k^{g-1})^T. \)

**Theorem 4.2.** Map \(\lambda\) induces map \(J\mathcal{H} \cong \mathbb{C}^{3g+1} \to W_g.\)

**Proof.** Direct check shows, that the identity is valid

\[ X_k^T A X_s + 4 \sum_{i,j=1}^{g+2} h_{i,j} x_k^{i-1} x_s^{j-1} = 0, \]

where \(A = A(H)\) and \(H = \pi(G).\) Putting \(k = s\) and using \(A = 4z \cdot z^T,\) we have \(y_k^2 = 4x_k^{g+1} + \sum_{s=0}^{g} \lambda_{2s} x_k^{2g+s}.\)

Now it is all ready to give the description of our realization of varieties \(T^g = \text{Jac}(V)\) and \(K^g = \text{Kum}(V)\) of the hyperelliptic curves.

For each nonsingular curve \(V = \{(y, x), y^2 = 4x^2 + \sum_{s=0}^{g-1} \lambda_{2s} x^{2g+s}\}\) define the map

\[ \gamma : T^g \to \mathcal{H} : \gamma(u) = H = \{h_{k,s}\}, \]

where \(h_{k,s} = 4\varphi_{k-1,s-1} - 2(\varphi_{s,k-2} + \varphi_{s-2,k}) + \frac{1}{4}\delta_{k,s} (\lambda_{2s+2} + \lambda_{2k-2}) + \delta_{k+1,s} \lambda_{2k-1} + \delta_{k,s+1} \lambda_{2s-1}.\)

**Theorem 4.3.** The map \(\gamma\) induces map \(T^g \to \mathcal{K}\), such that \(\varphi_{ggk} \varphi_{ggs} = \frac{1}{4}a_{ks}(\gamma(u)), i.e.\) \(\gamma\) is lifted to

\[ \tilde{\gamma} : T^g \to \mathcal{H} \cong \mathbb{C}^{3g+1} \text{ with } z = (\varphi_{gg1}, \ldots , \varphi_{ggg})^T. \]

Composition of maps \(\lambda \tilde{\gamma} : T^g \to W_g\) defines the inversion of the Abel map \(\mathfrak{A} : (V)^g \to T^g\) and, therefore, the map \(\tilde{\gamma}\) is an embedding.

So we have obtained the explicit realization of the Kummer variety \(T^g \cap \mathcal{H}\) of the hyperelliptic curve \(V\) of genus \(g\) as a subvariety in the variety of matrices \(\mathcal{K}\). As a consequence of the Theorem 1.3, particularly, follows the new proof of the theorem by B.A. Dubrovin and S.P. Novikov about rationality of the universal space of the Jacobians of hyperelliptic curves \(V\) of genus \(g\) with the fixed branching point \(e_{2g+2} = \infty\).

### 4.2. Hyperelliptic \(\Phi\)-function.

In this section we construct the linear differential operators, for which the hyperelliptic curve \(V(y, x)\) is the spectral variety.

**Definition 3.** \(\Phi\)-function of the curve \(V(y, x)\) with fixed point \(a\)

\[ \Phi : \mathbb{C} \times \text{Jac}(V) \times V \to \mathbb{C} \]

\[ \Phi(u_0, u; (y, x)) = \frac{\sigma(\alpha - u)}{\sigma(\alpha) \sigma(u)} \exp(-\frac{1}{2} y u_0 + \zeta^T(\alpha) u), \]

where \(\zeta^T(\alpha) = (\zeta_1(\alpha), \ldots , \zeta_g(\alpha))\) and \((y, x) \in V, u \) and \(\alpha = \int_0^x du \in \text{Jac}(V).\)

Particularly, \(\Phi(0, u; (y, x))\) is the Baker function (see [11, page 421] and [28]).
Theorem 4.4. The function $\Phi = \Phi(u_0, u; (y, x))$ solves the Hill’s equation

$$ (\partial_g^2 - 2\varphi_{gg})\Phi = (x + \frac{\lambda_{2g}}{4})\Phi, $$

with respect to $u_g$, for all $(y, x) \in V$.

Proof. From (3.5)

$$ \partial_g \Phi = y + \partial_g \mathcal{P}(x; u) \Phi, $$

where $\mathcal{P}(x; u)$ is given by (2.12), hence:

$$ \partial^2_g \Phi = y^2 - (\partial_g \mathcal{P}(x; u))^2 + 2\mathcal{P}(x; u)\partial^2_g \mathcal{P}(x; u) $$

and by (3.9) and (3.6) we obtain the theorem.

Let us introduce the vector $\Psi = (\Phi, \Phi_g)$, where $\Phi_g$ stands for $\partial_g \Phi$. Then equation (4.1) may be written as

$$ \partial_g \Psi = L_g \Psi, \quad \text{where} \quad L_g = \begin{pmatrix} 0 & 1 \\ \lambda_{2g} & 0 \end{pmatrix}. $$

In regard of (4.2) and (3.5), it is natural to introduce the family of $g+1$ operators, presented by $2 \times 2$ matrices,

$$ \{L_0, L_1, \ldots, L_g\}, \quad L_k = \begin{pmatrix} V_k & U_k \\ W_k & -V_k \end{pmatrix}, $$

and defined by the equalities

$$ L_k \Psi = \partial_k \Psi, \quad k \in 0, \ldots, g. $$

The theory developed in previous sections leads to the following description of this family of operators.

Proposition 4.5. Entries of the matrices $L_k$ are polynomials in $x$ and $2g$-periodic in $u$:

$$ L_k = D_k L_0 - \frac{1}{2} \begin{pmatrix} 0 & 0 \\ h_{g+2,k} & 0 \end{pmatrix}, $$

with

$$ U_0 = \frac{1}{2} \sum_{i=1}^{g+2} x^{i-1} h_{g+2,i}, \quad V_0 = -\frac{1}{4} \sum_{i=1}^{g+2} x^{i-1} \partial_g h_{g+2,i}, $$

and

$$ W_0 = \frac{1}{4} \sum_{i=1}^{g+2} x^{i-1} \det \begin{pmatrix} h_{g+1,i} & h_{g+2,i} \\ h_{g+2,i} & h_{g+2,g+1} \end{pmatrix}. $$

And the compatibility conditions

$$ [L_k, L_i] = \partial_k L_i - \partial_i L_k $$

are satisfied.

Here $D_k$ is umbral derivative (see page 1). Proof is straightforward due to (3.5), (3.13), (3.14) and (3.15).
Theorem 4.6. The function $\Phi = \Phi(u_0, u; (y, x))$ solves the system of equations

$$(\partial_k \partial_l - \gamma_{kl}(x, u) \partial_g + \beta_{kl}(x, u)) \Phi = \frac{1}{4} D_{k+l}(f(x)) \Phi$$

with polynomials in $x$

$$\gamma_{kl}(x, u) = \frac{1}{4} \left[ \partial_k D_l + \partial_l D_k \right] \sum_{i=1}^{g+2} x^{i-1} h_{g+2,i} \quad \text{and}$$

$$\beta_{kl}(x, u) = \frac{1}{8} \left[ (\partial_g \partial_k + h_{g+2,k}) D_l + (\partial_g \partial_l + h_{g+2,l}) D_k \right] \sum_{i=1}^{g+2} x^{i-1} h_{g+2,i}$$

for all $k, l \in 0, \ldots, g$ and arbitrary $(y, x) \in V$.

Here $f(x)$ is as given in (1.1) with $\lambda_{2g+2} = 0$ and $\lambda_{2g+1} = 4$.

Proof. Construction of operators $L_k$ yields

$$\Phi_{lk} = \frac{1}{2} (\partial_l U_k + \partial_k U_l) \Phi_g + (V_l V_k + \frac{1}{2} (\partial_l V_k + \partial_k V_l + U_l W_k + W_k U_l)) \Phi.$$

To prove the theorem we use (4.5), and it only remains to notice, that (cf. Lemma 4.1):

$$D_k(V_0) D_l(W_0) + \frac{1}{4} D_k(U_0) D_l(W_0) + \frac{1}{4} D_l(U_0) D_k(W_0) =$$

$$- \frac{1}{16} \left( \det H \right)^{g+1} \left[ \sum_{i=1}^{g+2} x^i \right] =$$

$$= \frac{1}{16} \left( \det H \right)^{g+1} \left[ \sum_{i=1}^{g+2} x^i \right] =$$

$$= \left( \sum_{i=1}^{g+2} x^i \right) = \Phi,$$

having in mind that $h_{g+2, g+2} = 0$ and $h_{g+2, g+1} = 2$, we obtain the theorem due to properties of matrix $H$.

Consider as an example the case of genus 2.

$$(\partial^2 - 2\varphi_{22}) \Phi = \frac{1}{4} (4x^2 + \lambda_4) \Phi,$$

$$(\partial_2 \partial_1 + \varphi_{222} \varphi_2 - \varphi_{22} x + \varphi_{22} + \frac{1}{4} \lambda_4) + 2 \varphi_{12}) \Phi = \frac{1}{4} (4x^2 + \lambda_4 x + \lambda_3) \Phi,$$

$$(\partial^2 + \varphi_{111} \varphi_2 - 2 \varphi_{12} x + \varphi_{22} + \frac{1}{4} \lambda_4) \Phi = \frac{1}{4} (4x^3 + \lambda_4 x^2 + \lambda_3 x + \lambda_2) \Phi.$$

And the $\Phi = \Phi(u_0, u_1, u_2; (y, x))$ of the curve $g^2 = 4x^5 + \lambda_4 x^4 + \lambda_3 x^3 + \lambda_2 x^2 + \lambda_1 x + \lambda_0$ solves these equations for all $x$.

The most remarkable of the equations of Theorem 4.6 is the balance of powers of the polynomials $\gamma_{kl}$, $\beta_{kl}$ and of the “spectral part” — the umbral derivative $D_{k+l}(f(x))$:

$$\deg_x \gamma_{kl}(x, u) \leq g - 1 - \min(k, l),$$

$$\deg_x \beta_{kl}(x, u) \leq 2g - (k + l),$$

$$\deg_x D_{k+l}(f(x)) = 2g + 1 - (k + l).$$
4.3. Solution of KdV equations by Kleinian functions. The KdV system is
the infinite hierarchy of differential equations
\[ u_{t_k} = \mathcal{X}_k[u], \]
the first two are
\[ u_{t_1} = u_x, \quad \text{and} \quad u_{t_2} = -\frac{1}{6}(u_{xxx} - 6uu_x), \]
and the higher ones are defined by the relation
\[ \mathcal{X}_{k+1}[u] = R \mathcal{X}_k[u], \]
where \( R = -\frac{1}{2} \partial_x^2 + 2u + uu_x \) is the Lenard’s recursion operator.
Identifying time variables \( (t_1 = x, t_2, \ldots, t_g) \to (u_g, u_{g-1}, \ldots, u_1) \) we have

**Proposition 4.7.** The function \( u = 2\wp_g(u) \) is a \( g \)-gap solution of the KdV system.

**Proof.** Really, we have
\[ u_x = \partial_x 2\wp_g \]
and by (3.6)
\[ u_{t_2} = \partial_{g-1} 2\wp_g = -\wp_{gggg} + 2\wp_{gg}\wp_{ggg}. \]
The action of \( R \)
\[ \partial_{g-i} 2\wp_g = \left[ -\partial_g^2 + 8\wp_{gg} \right] \wp_{gg,g-i} + 4\wp_{g,g-i}\wp_{ggg} \]
is verified by (3.6) and (3.7).

On the \( g \)-th step of recursion the “times” \( u_i \) are exhausted and the stationary equation
\[ \mathcal{X}_{g+1}[u] = 0 \]
appears. A periodic solution of \( g+1 \) higher stationary equation is a \( g \)-gap potential (see [29]).

**Concluding remarks**

The Kleinian theory of hyperelliptic Abelian functions as, the authors hope, this paper shows is an important approach alternative to the generally adopted formalism based directly on the multidimensional \( \theta \)-functions in various branches of mathematical physics. Still, a number of remarkable properties of the Kleinian functions were left beyond the scope of our paper. We give some instructive examples for the case of genus two \( u = \{u_1, u_2\} \).

- the addition theorem
  \[ \frac{\sigma(u + v)\sigma(u - v)}{\sigma^2(u)\sigma^2(v)} = \wp_{22}(u)\wp_{12}(v) - \wp_{12}(u)\wp_{22}(v) + \wp_{11}(v) - \wp_{11}(u), \]
- the equation, capable of being interpreted as the Hirota bilinear relation:
  \[
  \begin{cases}
  \frac{1}{3} \Delta \Delta^T + \Delta^T \begin{pmatrix} 0 & 0 & 1 \\ 0 & -\frac{1}{6} & 0 \\ 1 & 0 & 0 \end{pmatrix} \epsilon_{\eta,\eta} \epsilon_{\eta,\eta} \cdot \epsilon_{\eta,\epsilon,\eta,\eta} T - (\xi - \eta)^4 \epsilon_{\eta,\epsilon,\eta,\xi} T \\
  \end{cases}
  \]
  \[ \sigma(u)\sigma(u') \bigg|_{u' = u}, \]
  is identically 0, where \( \Delta^T = (\Delta_1^2, 2\Delta_1 \Delta_2, \Delta_2^2) \) with \( \Delta_i = \frac{\partial}{\partial u_i} - \frac{\partial}{\partial u'_i} \) and also \( \epsilon_{\xi,\eta} = (1, \eta + \xi, \eta \xi) \). After evaluation the powers of parameters \( \eta \) and \( \xi \) are replaced according to rules \( \eta^k, \xi^k \to \lambda_k \frac{(6-k)!}{6!} \) by the constants defining the curve.
• for the Kleinian $\sigma$–functions the operation is defined

$$\sigma(u_1, u_2) = \exp \left\{ \frac{u_2}{u_1} \sum_{k=1}^{6} k\lambda_k \frac{\partial}{\partial \lambda_{k-1}} \right\} \sigma(u_1, 0),$$

which resembles the function executed by vertex operators.

We give these formulas with reference to [12].

Another interesting problem is the reduction of hyperelliptic $\wp$–functions to lower genera. In the case of genus two it, happens according to the Weierstrass theorem when the period matrix $\tau$ can be transformed to the form (see e.g. [11, 24])

$$\tau = \begin{pmatrix} \tau_{11} & \frac{1}{N} \\ \frac{1}{N} & \tau_{22} \end{pmatrix},$$

where so called Picard number $N > 1$ is a positive integer. The associated Kummer surface turns in this case to Plücker surface. The reductions of the like were studied in [30] in order to single out elliptic potentials among the finite gap ones. The problems of this kind were treated in [31, 32, 33] by means of the spectral theory. We remark that the formalism of Kleinian functions extremely facilitates the related calculations and makes the solution more descriptive.

These and other problems of hyperelliptic abelian functions will be discussed in our forthcoming publications.

Concluding we emphasize, that the Kleinian construction of the hyperelliptic Abelian functions does not exclude the theta functional realization but complements it, and to the authors’ experience the combination of the both approaches makes the whole picture more complete and descriptive.

ACKNOWLEDGMENTS

The authors are grateful to S.P. Novikov for the attention and stimulating discussions; we are also grateful to I.M. Krichever, S.M. Natanson and A.P. Veselov for the valuable discussions. Special thanks to G. Thieme for the help in the collecting the classical German mathematical literature.

The research described in this publication was supported in part by grants no. M3Z000 (VMB) and no. U44000 (VZE) from the International Science Foundation and also the INTAS grant no. 93-1324 (VZE and DVL), and grant no. 94-01-01444 from Russian Foundation of Fundamental Researches.

REFERENCES

[1] F Klein. Über hyperelliptische Sigmanfunctionen. Math. Ann., 32, 1888, 351–380.
[2] K Weierstrass. Beitrag zur Theorie der Abel’schen Integrale. Jahreber. Königl. katolischen Gymnasium zu Braunsberg in dem Schuljahre 1848/49, pages 3–23, 1849.
[3] K Weierstrass. Zur Theorie der Abelschen Functionen. J. reine und angew. Math., 47, 1854, 289–306.
[4] G Göppel. Theoriae transcendentium Abelianarum primi ordinis adumbrato levis. J. reine und angew. Math., 35, 1847, 277.
[5] G Rosenhain. Abhandlung über die Funktionen zweier Variabler mit vier Perioden. Mem. pres. l’Acad. de Sci. de France. des savants, 9, 1851, 361–455.
[6] H Burkhardt. Beiträge zur Theorie der hyperelliptischen Sigmafunctionen. Math. Ann., 32, 1888, 381–402.
[7] E Wiltheiss. Über die Potenzreihen der hyperelliptischen Thetafunctionen. Math. Ann., 32, 1888, 410–423.
[8] O Bolza. On the first and second derivatives of hyperelliptic $\sigma$–functions. Amer. Journ. Math., 17:11, 1895.
[9] H F Baker. On the hyperelliptic sigma functions. *Amer. Journ. Math.*, 20, 1898, 301–384.
[10] A Krazer and W Wirtinger. *Abelsche Funktionen und allgemeine Thetafunctionen*, in: *Enzyklopädie der Mathematischen Wissenschaften II* (2), Heft 7, pages 603–882, Teibner, 1915.
[11] H F Baker. *Abel's theorem and the allied theory including the theory of Theta functions*. Cambridge University Press, Cambridge, 1897.
[12] H F Baker. *Multiply Periodic Functions*. Cambridge University Press, Cambridge, 1907.
[13] D V Leykin. On Weierstrass cubic for hyperelliptic functions. *Uspekhi Matem. Nauk*, 50 (6), 1995, 191–192.
[14] V M Buchstaber, V Z Enolskii, and D V Leykin. Matrix realization of hyperelliptic Kummer varieties. *Uspekhi Matem. Nauk*, 51 (2), 1996, to appear.
[15] V Z Enolskii and D V Leykin. *On the multiply periodic Schrödinger operators*, in: Proceedings of the conference “Coherent structures in Physics and Biology” July 10–14 1995, Edinburg.
[16] V M Buchstaber and V Z Enolskii. Explicit algebraic description of hyperelliptic Jacobian on the background of Kleinian $\sigma$–functions. *Funkt. Analiz. Pril.*, 30 (1), 1996, to appear.
[17] V M Buchstaber and V Z Enolskii. Abelian Bloch solutions of two dimensional Schrödinger equation. *Uspekhi Matem. Nauk*, 50 (1), 1995, 191–192.
[18] J D Fay. *Theta functions on Riemann surfaces*. Lecture Notes in Math., volume 352, Springer, Berlin, 1973.
[19] D Mumford. *Curves and their Jacobians*. University of Michigan Press, Ann Arbor, 1975.
[20] P Griffith and J Harris. *Principles of Algebraic Geometry*. Wiley, New York, 1978.
[21] H M Farkas and I Kra. *Riemann Surfaces*. Graduate texts in Math., volume 71, Springer, New York, 1980.
[22] H Bateman and A Erdelyi. *Higher Transcendental Functions*, volume 2. McGraw-Hill, New York, 1955.
[23] S M Roman. *The umbral calculus*. Academic Press, 1984.
[24] R W H T Hudson. *Kummer’s quartic surface*. Cambridge University Press, Cambridge, 1994. First published 1905.
[25] R A Horn and C R Johnson. *Matrix Analysis*. Cambridge University Press, Cambridge, 1986.
[26] E T Whittaker and G N Watson. *A course of modern analysis*, Cambridge University Press, Cambridge, 1973.
[27] B A Dubrovin and S P Novikov. Doklady Ac. Sc. SSSR, 219:3, 1974, 531–534.
[28] I M Krichever. The method of algebraic geometry in the theory of nonlinear equations. *Russian. Math. Survey*, 32, 1977, 180–208.
[29] B A Dubrovin, V B Matveev and S P Novikov. Nonlinear equations of KdV type, finite-gap linear operators and Abelian varieties. *Uspekhi Matem. Nauk*, 31 (1), 1976, 56–136.
[30] E D Belokolos, A I Bobenko, V Z Enolskii, A R Its, and V B Matveev. *Algebro Geometrical Approach to Nonlinear Integrable Equations*. Springer, Berlin, 1994.
[31] F Gesztesy and R Weikard. Treibich-Verdier potentials and the stationary (m)KdV hierarchy. *Math. Z.*, 219:451–476, 1995.
[32] F Gesztesy and R Weikard. On Picard potentials. *Diff. Int. Eqs.*, 8:1453–1476, 1995.
[33] F Gesztesy and R Weikard. A characterization of elliptic finite gap potentials. *C. R. Acad. Sci. Paris*, 321:837–841, 1995.

**National Scientific and Research Institute of Physico-Technical and Radio-Technical Measurements, VNIIFTRI, Mendeleeevo, Moscow Region, 141570, Russia**

**Theoretical Physics Division, NASU Institute of Magnetism, 36–b Vernadsky str., Kiev-680, 252142, Ukraine**