A Note on Higher Order and Variable Order Logic over Finite Models

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Abstract

We show that descriptive complexity’s result extends in High Order Logic to capture the expressivity of Turing Machine which have a finite number of alternation and whose time or space is bounded by a finite tower of exponential. Hence we have a logical characterisation of ELEMENTARY. We also consider the expressivity of some fixed point operators and of monadic high order logic.

Finally, we show that Variable Order logic over finite structures, a notion introduced by [8] contain the Analytical Hierarchy.

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1 Introduction

Descriptive complexity is a field of computational complexity. It studies the relation between logical formalisms and complexity classes. For a given complexity class, what logic do we need to express languages in this class; for a formula in a given logic, what is the complexity of checking the truth value of this formula, over finite structures, as a function of the cardinality of the structure.

The relation between complexity classes and descriptive classes is strong, since a lot of well known complexity classes, such as $\mathsf{AC^0}$, $\mathsf{L}$, $\mathsf{NL}$, $\mathsf{P}$, $\mathsf{NP}$, $\mathsf{PSPACE}$, $\mathsf{EXPTIME}$ and $\mathsf{EXPSPACE}$, are exactly equal to some descriptive classes using first and second order relations, with either syntactic restrictions (Monadic relations, Horn and Krom formulae) or with operators like “fixed point” and “transitive closure” (see [10, 14]).

These issues in terms of capturing complexity classes are well understood in agreed upon notation for first and second-order logic, but beyond that there were open questions and a need for clarity and standardization of notation. The extension of those results to higher order logic began with [12], and was followed more recently by [8, 5, 11]. It is also called “Complex object” in database theory [1].

The article [8] also introduced the so-called “Variable-order logic”, extending the high-order logic where the order of a quantified variable is not fixed in the formulae. They stated that it is at least Turing-hard but did not give an upper bound for the expressivity of this language.

The main contributions of this paper are the following:
We give a definition of High-Order logic which is less restrictive than the usual one,

- we prove a normal-form theorem which respects the expressivity of the logic,
- we prove the equality between some subclasses of the High-Order logic and some complexity classes below ELEMENTARY,
- we prove that any formula in the analytical hierarchy can be written as a formula in Variable Order logic.

2 Definition

2.1 The core of the language

Let \( r \) be an integer. We will begin by defining the syntax of the \( r \)th order logic (\( \text{HO}^r \)) and its semantics over finite structures. First Order (\( \text{FO} \)) and Second Order (\( \text{SO} \)) are the special cases \( \text{HO}^1 \) and \( \text{HO}^2 \).

**Definition 2.1** (Universe). A universe \( A \) is the set \([0, n - 1]\).

**Definition 2.2** (Type). A type of order 1 is just the element \( \iota \), and a type of order \( r > 1 \) is a tuple of types of order at most \( r - 1 \).

**Example 2.3.** For example, \((\iota, ((\iota), \iota), \iota)\) is the type of a ternary relation of order 4 whose first and last elements are elements of the universe, and whose second one is a binary relation of order 3 whose first element is a monadic relation of order 2 and the second an element of the universe.

**Definition 2.4** (Relation). A relation of type \( \iota \) is an element of the universe, and a relation of type \( (t_1, \ldots, t_n) \), where the \( t_i \)'s are types, is a subset of the Cartesian product of the relations of type \( t_i \).

\( R^t_A \) is the set of all relations of type \( t \) over the universe \( A \).

**Notation 2.5.** In this article \( X^t \), where \( X \) is any symbol, will always be a variable of type \( t \). Hence \( X^1 \) is a first-order variable or a constant.

For \( t \neq \iota \), \( \top^t \) (resp. \( \bot^t \)) is the special case of the relation of type \( t \) that is always true (resp. false).

By extension, if \( \overline{X} = X_1^{t_1}, \ldots, X_n^{t_n} \) is a tuple of variables, then we say that \( t = (t_1, \ldots, t_n) \) is the type of \( \overline{X} \).
Definition 2.6 (Vocabulary). A vocabulary \( \sigma = \{ R_{t_1}^1, \ldots, R_{s}^t \} \) is a set of relation symbols. It is a vocabulary of order \( r \) if the type of every relation is of order at most \( r \).

We denote by \( R_{t_i}^i \) a relation symbol of type \( t_i \). A relation symbol of type \( t \) is called a constant. We sometimes omit the type-superscripts of variables and relations when this information is redundant.

For example, in \( \forall R(t, x), R(x, y) \lor \exists z, x + z = y \), it is clear that the second occurrence of \( R \) is also of type \((t, t)\), that \( x, y \) and \( z \) are of type \( t \) and that \( + \) is a predicate of type \((t, t, t)\).

Our definition is not standard in that a vocabulary may include relations whose type has order greater than 2.

In [8] the types are restricted to what we will call arity normal form in subsection 2.4.1. We use the more general definition of types from [5], and we will show in subsection 2.4.1 that our choice is equivalent to their choice (at least for the complexity classes that we study).

Definition 2.7 (Structure). For any type \( \sigma = \{ R_{t_1}^1 \ldots \} \), a \( \sigma \text{-structure} \) \( \mathfrak{A} = (A, \mathfrak{A}(R_1), \ldots) \) is a tuple such that \( A \) is a nonempty universe and each \( \mathfrak{A}(R_i) \in R_i^{A} \).

When \( \mathcal{X}^t \) and \( \mathcal{E}^t \) are a variable and a relation of the same type, then we write \( \mathfrak{A} = \mathfrak{A}[\mathcal{X}/\mathcal{R}] \) to speak of the \( \sigma \cup \{X\} \)-structure such that \( \mathfrak{A}'(\mathcal{X}) = \mathcal{R} \) and \( \mathfrak{A}'(\mathcal{Y}) = \mathfrak{A}(\mathcal{Y}) \) if \( \mathcal{Y} \neq \mathcal{X} \).

By extension, if \( \mathcal{X}^t = \mathcal{X}_1^{t_1}, \ldots, \mathcal{X}_n^{t_n} \) and \( \mathcal{R}^t = R_1^{t_1}, \ldots, R_{s}^{t_s} \) are tuples of variables and of relations of the same type then \( \mathfrak{A}[\mathcal{X}/\mathcal{R}] \) is syntactic sugar for \( \mathfrak{A}[\mathcal{X}_1/R_1] \ldots [\mathcal{X}_n/R_n] \).

Definition 2.8 (Formula). A high-order formula \( \varphi \) is defined recursively as usual, such that if \( \psi \) and \( \psi' \) are formulae then \( \psi \land \psi', \psi \lor \psi', \neg \psi, \forall \mathcal{X}^t \psi \) and \( \exists \mathcal{X}^t \psi \) are also formulae. Here \( t \) is the type of \( \mathcal{X} \).

Finally, for types \( t = (t_1, \ldots, t_a) \), \( \mathcal{X}^t(\mathcal{Y}_1^{t_1}, \ldots, \mathcal{Y}_a^{t_a}) \) and \( \mathcal{Y}^t = \mathcal{X}^t \) are the two kinds of atomic formulae.

Definition 2.9 (HO, \( \text{HO}^r \), \( \Sigma_j^r \) and \( \Sigma_j^{r,f} \)). The set \( \text{HO} \) contains every formulae with high order quantifiers, then \( \text{HO}^r \) is the subset of \( \text{HO} \) formulae whose quantified variables are of order at most \( r \). Hence \( \text{HO}^0 \) is the set of quantifier-free formulae.

The set \( \Sigma_j^r \) (resp. \( \Pi_j^r \)) for \( j > 0 \) is the class of formulae containing \( \Pi_{j−1}^r \) (resp. \( \Sigma_{j−1}^r \)) and closed by conjunction, disjunction and existential (resp. universal) quantification of variables of order at most \( r \). We have \( \Sigma_0^r = \text{HO}^r−1 \).

The normal form of \( \Sigma_j^r \) (resp. \( \Pi_j^r \)), where \( j \geq 0 \), is the set of formulae as in equation\( \Pi \) with \( \psi \in \text{HO}^r−1 \) in normal form and the types \( t_{i,j} \) are of order at most \( r \) (resp. the same kind of formulae, exchanging \( \lor \) and \( \land \)).
\[ \exists \lambda_{1,1}^{t,1} \ldots \exists \lambda_{1,i_1}^{t,i_1} \forall \lambda_{2,1}^{t,2} \ldots \forall \lambda_{2,i_2}^{t,i_2} \ldots Q \lambda_{1,1}^{t,1} \ldots Q \lambda_{1,j}^{t,j} \psi \]  

(1)

We will prove in Subsubsection 2.4.2 that any formula in \( \Sigma_f \) is equivalent to a formula in normal form.

Finally \( \text{HO}^r \) (resp. \( \Sigma_f^r \)) is the subset of \( \text{HO}^r \) (resp. \( \Sigma_f^r \)) without any free variables of order more than \( f \). This definition is different from the one of \([5]\) where \( f \) denotes the maximum arity of high order relations.

The classes of formulae for \( 0 \leq r \leq 2 \) and \( f = 2 \) are well studied since \( \text{HO}^{0,2} \) is the set of quantifier-free formulae, \( \text{HO}^{1,2} = \text{FO} \) and \( \text{HO}^{2,2} = \text{SO} \).

**Definition 2.10** (Semantics). For \( r \geq 1 \), \( \mathfrak{A} \models \mathcal{X}^{t,r} \mathcal{Y}^{a,r} \) if and only if

\[ (\mathfrak{A}(\mathcal{X}^{t,r})), \ldots, \mathfrak{A}(\mathcal{Y}^{a,r})) \in \mathfrak{A}(\mathcal{X}^{t,r+1}). \]

\( \mathfrak{A} \models \forall \mathcal{X}^{t} \mathcal{Y} \) if \( \mathfrak{A}(\mathcal{X}^{t}) = \mathfrak{A}(\mathcal{Y}) \), when the last equality is an equality of sets. It is decidable since the sets are well-founded.

Satisfaction for \( \psi \wedge \psi', \psi \lor \psi' \) and \( \neg \psi \) are defined in the usual way.

\( \mathfrak{A} \models \forall \mathcal{X}^{t} \psi \) is true if and only if for all \( R^t \in R_{\mathfrak{A}}^{t} \), \( \mathfrak{A}[\mathcal{X}^{t}/R^t] \models \psi \).

\( \mathfrak{A} \models \exists \mathcal{X}^{t} \psi \) is true if and only if there exists some \( R^t \in R_{\mathfrak{A}}^{t} \), \( \mathfrak{A}[\mathcal{X}^{t}/R^t] \models \psi \).

**2.2 Operators**

In this section, if \( L \) is a logic class and \( P \) is an operator, then \( L(P) \) is the set that contains the formulae of \( L \), closed by the operator \( P \).

**2.2.1 Transitive closure**

**Definition 2.11** (Transitive closure). Let \( \mathcal{X}^{t} = \mathcal{X}^{t}_1, \ldots, \mathcal{X}^{t}_n \) be an \( n \)-tuple and let \( \mathcal{Y}, \mathcal{Z} \) and \( \mathcal{T} \) be three other \( n \)-tuples of the same type and let \( \varphi \) be a \( (\sigma \cup \{ \mathcal{X}_1, \ldots, \mathcal{X}_n, \mathcal{Y}_1, \ldots, \mathcal{Y}_n \}) \)-formula in \( L \).

Then \( (\text{TC}(\mathcal{X}^{t}\varphi))(\mathcal{Y},\mathcal{T}) \) is a \( \sigma \)-formula in \( L(\text{TC}) \). The operator \( \text{TCXe} \) is called the “Transitive Closure” operator.

**Definition 2.12** (Semantics of \( \text{TC} \)). \( \mathfrak{A} \models (\text{TC}(\mathcal{X}^{t}\varphi))(\mathcal{Y},\mathcal{T}) \) is true if and only if \( \mathcal{T} = \mathcal{Z} \) or if there exists an \( n \)-tuple \( \mathcal{M} \) of type \( t \) such that \( \mathfrak{A}[\mathcal{X}/\mathfrak{A}(\mathcal{Z})][\mathcal{Y}/\mathfrak{A}(\mathcal{M})] \models \varphi \) and \( \mathfrak{A} \models (\text{TC}(\mathcal{X}^{t}\varphi))(\mathcal{M},\mathcal{T}) \).

**Example 2.13**. Let the universe be a directed graph and let the vocabulary contain only \( E \) such that \( E(x,y) \) is true if there is an edge from \( x \) to \( y \). Then \( \varphi = (\text{TC}_{x,y}E(x,y)) \) is a relation of type \( (t,t) \) such that \( \varphi(z,t) \) is true if and only if there is a path in the directed graph from \( z \) to \( t \).
2.2.2 Fixed Point

**Definition 2.14** (Fixed Point). Let $\vec{X}^t = X_1^{t_1}, \ldots, X_n^{t_n}$ be a tuple of type $t$ and $\vec{Y}$ be another tuple of the same type $t$, let $P$ be a variable of type $t = (t_1, \ldots, t_n)$, and let $\varphi$ and $\psi$ be some $\sigma \cup \{P, X_1^{t_1}, \ldots, X_n^{t_n}\}$-formulae. Then $(\text{IFP}_{\vec{X}, P, \varphi})(\vec{Y})$, $(\text{NIFP}_{\vec{X}, P, \varphi, \psi})(\vec{Y})$, $(\text{APFP}_{\vec{X}, P, \varphi, \psi})(\vec{Y})$, $(\text{AIFP}_{\vec{X}, P, \varphi, \psi})(\vec{Y})$ are $\sigma \cup \{\vec{Y}\}$-formulae in $L(\text{FP})$, $L(\text{IFP})$, $L(\text{NPFP})$, $L(\text{NIFP})$, $L(\text{APFP})$ and $L(\text{AIFP})$ respectively. The letters “N” and “A” stand for “nondeterministic” and “Alternating” respectively, “I” and “P” for “Inflationary” and “Partial”, and “FP” stands for “Fixed Point”.

We restrict the formulae of NIFP and NPFP such that there are no negation applied outside of a non-deterministic fixed-point operator.

**Definition 2.15** (Semantics of PFP). Let $(\text{PFP}_{\vec{X}, P, \varphi})(\vec{Y})$ be a formula. Then we can define the relations $(P_i)_{i \in N}$ by recursion on $i$.

For each $\vec{X} \in R_{\vec{X}}$, $P_0(\vec{X})$ is false and $P_1(\vec{X})$ is true if and only if $\varphi$. Hence the property $P_i$ is true on the input $\vec{X}$ if $\varphi$ is true on input $\vec{R}$ when the variable $P$ is replaced by the relation $P_{i-1}$.

Then, either this process leads to a fixed point, i.e. there exists $i$ such that $P_i = P_{i+1}$ and then $\models P_0(\vec{X})$ or the set of relation of $(P_i)_{i \in N}$ has a cycle of size strictly greater than 1 and then $\not \models P_0(\vec{X})$.

**Definition 2.16** (Semantics of IFP). Using the notation of the last definition, let $\varphi'(\vec{X}, P) = P(\vec{X}) \lor \varphi(\vec{X}, P)$. Then we can define $\text{IFP}(\varphi_{P, \vec{X}})(\vec{Y})$ as $\text{PFP}(\varphi'_{P, \vec{X}})(\vec{Y})$.

Another equivalent way to define it is to define $P_0$ as the predicate that is always false, and $P_1(\vec{X}) = P_{i-1}(\vec{X}) \lor \varphi(P_{i-1}, \vec{X})$.

We should note that to decide if the desired fixed point for $\varphi$ exists we must run the definition step by step and check whether $P_{i+1} = P_i$ for $i < |R_{\vec{X}}|$.

The definition of IFP makes the operator monotonically increasing so a fixed point will always be reached within $\log |R_{\vec{X}}|$ steps.

The nondeterministic fixed points and alternating fixed points are introduced in [2]. We choose not to use their notation “$FP(A, n)(\varphi_1, \varphi_2, S)(\vec{t})$”, but instead to use $(\text{APFP}_{\vec{X}, \varphi_1, \varphi_2})(\vec{t})$ to be coherent with the notation for PFP as defined in [10].

**Definition 2.17** (Semantics of NPFP and NIFP). Let $\vec{X}', \vec{Y}$ be two vectors of the same type $t$, $P'$ be a variable of type $t$, and $\varphi_0$ and $\varphi_1$ be $\sigma \cup \{P', \vec{X}'\}$-formulae.

We can define the relations $(P_i)_{i \in \{0,1\}^\ast}$, where $l$ is a list of bit. For each $\vec{X} \in R_{\vec{X}}$, $P_0(\vec{X})$ is false, and by induction for $i \in \{0, 1\}$, $P_i$ is true if $A[\vec{X}/\vec{R}][P/P_i] \models \varphi_i$. 

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Then $A[Y/R] |= (\text{NPFP}_{\mathcal{X},P,\varphi_0,\varphi_1})(Y)$ is true if and only if there exists an $l \in \{0, 1\}^*$ such that $A[Y/R] |= P_l(Y)$ and $P_0l = P_1l = P_l$.

This means that $(\text{NPFP}_{\mathcal{X},P,\varphi_0,\varphi_1})$ is the union of the relations which are fixed points $P$ for both $\varphi_i$ and such that $P$ is accessible by applying the $\varphi_i$ a finite number of time to $\bot$.

We could also consider this as a directed graph $G_{\mathcal{X},P,\varphi_0,\varphi_1}$ without self-loop, with a node from every relation $P$ to $\varphi_i(P)$. Then $(\text{NPFP}_{\mathcal{X},P,\varphi_0,\varphi_1},\psi)$ is defined as $\bigcup s \{P_s | P_0s = P_1s = P_s\}$ is the union of the leaves reachable from the relation $\bot$.

The semantic of NIFP is to NPFP what IFP is to PFP. This means that for every relations $P$, $\varphi(P) \subseteq P$ and $\psi(P) \subseteq P$. It is also possible to define $(\text{NIFP}_{\mathcal{X},P,\varphi_0,\varphi_1},\psi)$ as syntactic sugar for $(\text{NPFP}_{\mathcal{X},P^t,\varphi_0,\varphi_1},\psi(P^t,\mathcal{X})) \cup \varphi(P,\mathcal{X}), P(\mathcal{X}) \cup \psi(P,\mathcal{X}))$. The graph defined above is then acyclic.

**Definition 2.18 (Semantics of APFP and AIFP).** We use the notations of definition 2.17.

Let $s$ be a function from strings of bits to bits. Let $\sigma$ be a vocabulary, $A$ be a $\sigma$-structure and $\varphi_0$ and $\varphi_1$ be $\sigma \cup \{P^t, \overline{\mathcal{X}}^t\}$-formulae. Then we define the tree $T_{\varphi_0,\varphi_1,s,\overline{\mathcal{X}}}$ whose nodes are labelled by relations of type $t$. The root is the relation $\bot$, and for $n$, a list of bits that indicates a path from the root in the tree, we define the label of $n$ as $P_n$, as in definition 2.17. If $P_n = P_{0n} = P_{1n}$ then $n$ is a leaf, else if the depth of $n$ is even then its children are the nodes with labels $P^t_0n$ and $P^t_1n$ that are not equal to $P^t_n$, else its only child is $P^t_{s(n)n}$. We assume that $P^t_{s(n)n} \neq P_n$ else we consider that the tree $T_{\varphi_0,\varphi_1,s,\overline{\mathcal{X}}}$ does not exist.

A local alternating fixed point $A_{\varphi_0,\varphi_1,s,\overline{\mathcal{X}}}$ is a relation such that $\mathcal{A} |= A_{\varphi_0,\varphi_1,s,\overline{\mathcal{X}}}(\overline{\mathcal{X}})$ if and only if for every label $l$ that are leaves of an existing tree $T_{\varphi_0,\varphi_1,s,\overline{\mathcal{X}}}$ we have $\mathcal{A} |= P_l(\overline{\mathcal{X}})$. This means that a tuple is accepted by the tree if and only if it is accepted by every relations of its leaves.
The alternating fixed point, \( A_{\varphi_0, \varphi_1, \mathcal{A}} \) is a relation such that
\[ \mathcal{A} \models A_{\varphi_0, \varphi_1, \mathcal{A}}(\mathcal{Y}) \]
if and only if there exist an \( s \) such that
\[ \mathcal{A} \models A_{\varphi_0, \varphi_1, s, \mathcal{A}}(\mathcal{Y}) \].

Then \( \mathcal{A} \models (\text{APFP}_X, \varphi_0, \varphi_1)(\mathcal{Y}) \) is true if and only if \( \mathcal{A} \models A_{\varphi_0, \varphi_1, \mathcal{A}}(\mathcal{Y}) \).

AIFP is to APFP what NIFP is to NPFP.

This is almost the definition of [2], except that \( T_{\varphi_0, \varphi_1, \mathcal{A}} \), \( A_{\varphi_0, \varphi_1, \mathcal{A}} \) and \( A_{\varphi_0, \varphi_1, s, \mathcal{A}} \) are not named and \( s \) is not considered, but having a name for those values will help the proof of 2.20. On page 8 they speak of the "length of the longest branch", and it seems that they assume that the tree is of finite size. They do not seem to explain why this assumption can be true without loss of generality and without considering that the tree is instead a graph; it is easy to imagine a branch which repeats itself an infinite number of times when \( G_{\varphi_0, \varphi_1, \mathcal{A}} \) is cyclic. Hence we think it is interesting to give another definition of alternating fixed point where we can always give an answer in a finite time.

**Definition 2.19** \((T_{\varphi_0, \varphi_1, \mathcal{A}})\). We will write \(|\bigcup|\) for \(|\bigcap|\) and \(|\bigcap|\) for \(|\bigcup|\).

Let \( T_{\varphi_0, \varphi_1, \mathcal{A}} \) be a tree where each node’s label is a pair with either \(|\bigcup|\) or \(|\bigcap|\) as first element and a relation as second element, and where the root is \((|\bigcup|, \bot)\). The children of \((c, P)\) are \((\overline{c}, \varphi_0(P))\) and \((\overline{c}, \varphi_1(P))\) except if \(P = \varphi_1(P) = \varphi_0(P)\) in which case this node is a leaf. If in a branch we find two nodes with the same label \((c, P)\), we remove the second occurrence and its descendants.

We recursively define the output of the tree as the relation of the label if the tree is a leaf, else as \(c\) applied to the output of its children. By extension we write \( T_{\varphi_0, \varphi_1} \) instead of its output relation. It will be clear by the context if we mean the tree or its output.

**Proposition 2.20.** \( T_{\varphi_0, \varphi_1, \mathcal{A}} = A_{\varphi_0, \varphi_1, \mathcal{A}} \)

**Proof.** Let \( \mathcal{X} \) be an tuple. We are going to prove that \( \mathcal{X} \in T_{\varphi_0, \varphi_1, \mathcal{A}} \iff \mathcal{X} \in A_{\varphi_0, \varphi_1, \mathcal{A}} \).

\[ \Rightarrow \] Let us assume that \( \mathcal{X} \in A_{\varphi_0, \varphi_1, \mathcal{A}} \). Then there exists some function \( s \) such that \( \mathcal{X} \in A_{\varphi_0, \varphi_1, s, \mathcal{A}} \). It then suffices to see that on every node \( n \) of \( T_{\varphi_0, \varphi_1, \mathcal{A}} \) with label \(|\bigcup|\) we can keep only the child whose number is \( s(n) \), and we obtain a tree that is a subset of \( T_{\varphi_0, \varphi_1, s, \mathcal{A}} \). Since there is no negation in the tree, if we remove an element of an union we can not add any elements in the output of the tree, hence there is no loss of generality in doing that. We now have a tree \( T' \) whose only gates’ label are \(|\bigcap|\). It is trivial to see that any element \( \overline{\mathcal{Y}} \) is in the output of \( T' \) if and only if it is in every leaf. Since in the construction of \( T_{\varphi_0, \varphi_1, \mathcal{A}} \) we only removed nodes that are copies of nodes higher in the tree, then \( \overline{\mathcal{Y}} \) is also in any leaf of \( A_{\varphi_0, \varphi_1, s, \mathcal{A}} \), hence \( \mathcal{X} \) is in the output of \( T_{\varphi_0, \varphi_1} \).
Let us assume that $\overline{X} \in T_{\varphi_0, \varphi_1, A}$. We will define a function $s$ such that $\overline{X} \in A_{\varphi_0, \varphi_1, s, A}$. Note that $s(n) = 0$ if $n$ is a string of odd size, since then the value of $s$ does not matter in $A_{\varphi_0, \varphi_1, s, A}$. Let $n$ be the shortest string such that $s(n)$ is not defined. Then $n$ is of even length, hence by hypothesis over $T_{\varphi_0, \varphi_1, A}$, $\overline{X}$ is in the node $n$. Either $n$ is a union node, in which case there is a child $b$ such that $\overline{X}$ is in $bn$, and we define $s(n)$ as $b$ and for every finite string $m$, $s(mn)$ as 0 since those values do not matter. Else $n$ is a leaf. If it is because $P_n = P_{0n} = P_{1n}$ then $s(n) = 0$ since this value does not matter. Else it is because its children were already seen in this branch, in which case let $m$ be the other occurrence of a node with the same relation, $n = pm$ and for every $q$ we define $s(qn)$ as $s(qm)$, by hypothesis over $n$ it is well defined, since $n$ is the shortest non defined string, and $q$’s length is strictly positive.

When we cut in $T_{\varphi_0, \varphi_1, A}$, it was because the child was an infinite repetition of itself, and we can define the function $s$ in $A_{\varphi_0, \varphi_1, s, A}$ with the same repetition. It is then trivial to see that $\overline{X}$ is indeed in every leaf of $T_{\varphi_0, \varphi_1, s, A}$, hence in $A_{\varphi_0, \varphi_1, A}$.

Claim 2.21. In fact, the same proof would work for a tree bigger than $T_{\varphi_0, \varphi_1}$, choosing to cut later in the branches would not remove anything since there are no $\neg$ gates, and would not add anything in the output since the later $\cap$ gates would remove the eventual new elements of the set.

This will be useful since it means we will not have to remember the set of relations seen on a branch, and we only have to count until we have seen more nodes than the number of relations.

### 2.2.3 Operator normal form

It was proved in [10] that the transitive closure and deterministic fixed points can be in normal form without loss of generality. In fact algorithms were given to obtain equivalent formulae in normal form. Furthermore, [2] states that this normal form extends to alternating fixed points, and to nondeterministic fixed points that are not under negation (which is impossible by definition).

Those normal forms are $(\text{TC}_{\overline{X} \varphi})(\bot, \top)$ and $(F_{P, \overline{X} \varphi})(\bot)$ where $F$ is a fixed point operator and $\varphi$ a formulae in FO or SO, it is trivial that this result extends to high order.

### 2.3 Mathematics definitions and notations

#### 2.3.1 Mathematics functions

**Definition 2.22** (Iterated exponential). Using the standard notation for the tetration operator, we define: $\exp^n_i(x) = ^i\exp^{n-1}(x)$ and $\exp^0_i(x) = x$. That is
\[ \exp^n_i(x) = i^{x \cdot \cdots \cdot x} \] with \( n \) exponentiations of \( i \). We will also write \( \text{texp}^n_a(x, r) = a^{r \times \exp^{n-1}(x, r)} \) and \( \text{texp}^0_a(x, r) = x \).

**Definition 2.23** (Elementary function). Let \( f \) be a function from \( \mathbb{N} \) to \( \mathbb{N} \), it is an elementary function if \( f = \exp^{O(1)}(n) \), that is, there is a constant \( c \) such that \( f = O(\exp^c(n)) \). We denote by ELEMENTARY the set of languages decidable in elementary time.

Finally, we introduce some complexity classes that we will use.

**Definition 2.24.** Let \( f \) be a function from \( \mathbb{N} \) to \( \mathbb{N} \) and \( i \in \mathbb{N} \).

Let \( \text{ATIME}(f) \) be the set of languages accepted by an alternating Turing machine halting in \( O(f(n)) \) steps on input of size \( n \). The restriction with at most \( i - 1 \) alternation between universal and existential states, beginning by existential, \( \Sigma_i \text{TIME}(f) \), in particular \( \Sigma_1 \text{TIME}(f) \) is denoted by \( \text{NTIME}(f) \). The definition of \( \text{TIME}(f) \) is similar, but every steps are deterministic, and so on for \( \text{ASPACE}(f) \) and \( \text{SPACE}(f) \), where the limit is not on the number of step but on the number of cells used by the machine for the computation.

We could also define ELEMENTARY using space or bounded alternation since we have \( \text{TIME}(f(n)) \subseteq \Sigma_{O(1)} \text{TIME}(f(n)) \subseteq \text{SPACE}(f(n)) \subseteq \text{TIME}(2^{f(n)}) \).

### 2.3.2 Syntactic sugar in logic

**Notation 2.25.** Let \( Q \) be a quantifier, we will define “\( \oplus_Q \)”. We write “\( \oplus_a \)” for “\( \land \)” and “\( \oplus_y \)” for “\( \Rightarrow \)”.

When we define a language \( L' \) from a language \( L \), we will always assume that “1” is a letter that is not in the alphabet of \( L \).

Some formulae will be used often in this article, hence we are going to define some syntactic sugar in this subsubsection.

\[
\begin{align*}
\text{card}_{\leq a}(T^{(t_1, \ldots, t_n)}) &= \text{def} \ \forall 0 \leq i \leq a, 1 \leq j \leq a U_{i,j}^{t_i} [\land 0 \leq i \leq a U_{i,j}^{t_i} \Rightarrow \lor 0 \leq i < j \leq a \overline{U}_i = \overline{U}_j], \\
\text{card}_{\geq a}(T^{(t_1, \ldots, t_n)}) &= \text{def} \ \exists 1 \leq i \leq a, 1 \leq j \leq a U_{i,j}^{t_i} [\land 1 \leq i \leq a T(U_{i,1}, \ldots, U_{i,n}) \land 1 \leq i < j \leq a \overline{U}_i \neq \overline{U}_j], \\
\text{card}_{a}(T^p) &= \text{def} \ \text{card}_{\geq a}(T^p) \land \text{card}_{\leq a}(T^p).
\end{align*}
\]

On ordered set we define \( 0(x) = \text{def} \ \neg \exists y (y < x) \), \( \max(x) = \text{def} \ \neg \exists y (y > x) \), and \( 1(x) = \text{def} \ \text{card}(y < x) \) where \( y \) is the free variable of the formula “\( \text{card} \)” to means that \( x \) is 0, 1 or max. We will assume that we can use those constants without having to explicitly quantify them in the formulae.

Finally, \( (Q.x.\varphi)\psi \) is syntactic sugar for \( Qx(\varphi \oplus_Q \psi) \).
2.4 Normal form

In this subsection, we are going to discuss two ways to normalize the language and see that the definition we choose does not change the expressivity of the language. Hence we will be able to choose the more restrictive one to prove theoretical results, and the more expressive one to express queries. These results are on the syntax of the formula, hence they also extend as results for general logic, with finite or infinite models.

2.4.1 Types of fixed arity

We are going to restrain type to a special form and prove that it does not change the expressivity of the language.

Definition 2.26 (Arity relation). For each $a, r \geq 1$, we define $A(a, r)$ to be the type $(A(a, r - 1), \ldots, A(a, r - 1))$ if $r > 1$, with a copy of $A(a, r - 1)$ and we define $A(a, 1)$ to be $\iota$. We write $F^a_r$ for $R^{A(a, r)}$, the set of relations of type $A(a, r - 1)$.

We say that a formula is in arity normal form (ANF) if all of its types respect the arity definition. Let us define $\Sigma'$ to be the set of formulae in ANF.

Proposition 2.27. The class of queries of $\Sigma''_j$ is exactly the class of queries of $\Sigma'_j$. Formally for every formula $\varphi \in \Sigma'_j$ we can find an equivalent formula $\varphi' \in \Sigma''_j$.

Proof. The side $\subseteq$ is trivial, since the definition of $\Sigma'$ is a restriction of the definition of $\Sigma''_j$. Indeed “$a, r$” as a type is defined as “$\iota$” if $r = 1$ and as “$(a, r - 1), \ldots, (a, r - 1)$” where $(a, r - 1)$ is considered as a type.

To show $\supseteq$, let $\varphi \in \Sigma'_j$ and define $a$ to be the size of the bigger tuple, defined this way: $\text{size}(1) = \text{def} 1$ and $\text{size}(t_1, \ldots, t_a) = \text{def} \max(a, \max_{1 \leq i \leq a} \text{size}(t_i))$. There are two problems that we need to correct. First we need to change the type of every relation such that a type of order $r$ contains only type of order $r - 1$, and such that all relations of those types have the same arity.

Step normal form Let us define step normal form (SNF) to be the formulae that respect the first of these properties, that a type of order $r$ contains only types of order $r - 1$. We are going to show that each formula is equivalent to a formula in SNF. To do this, the encoding in order $j$ of a relation $\mathcal{R}^i$ of order $i$, when $i < j$, will be a relation of order $j$ whose type contains only one elements of arity $j - 1$, whose type contains only one elements, and so on until the one element of order $i$ which is of course $\mathcal{R}^i$. 

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Let us define a formula \( \equiv(S_i, i, S_j, j) \) with \( j - i \) pair of parenthesis, that is true if and only if \( S_j \) is interpreted as explained in the last paragraph.

\[
equiv(S_i, i, S_j, j) = \exists i < k < j S_k \bigwedge_{i < k \leq j} \big( \forall T S_k(T) \Rightarrow T = S_{k-1} \big) \tag{2}
\]

It now suffices to replace every instance of an atomic proposition like \( \lambda^r(Y_1^{r_1}, \ldots, Y_n^{r_n}) \) by \( \exists_1 \leq k \leq n S_k^{r-1} \bigwedge_{1 \leq k \leq n} \equiv \big( Y_k^{r_1}, r_i, S_k^{r-1}, r-1 \big) \). An easy induction shows us that we obtain an equivalent formula, and it is clear that it is in SNF.

**From SNF to arity normal form:** From now on we will assume that every type in the vocabulary respects ANF.

We will make sure that every quantified relation is of arity \( a \), and we will do it so that in every relation of arity \( b < a \), the last element will be copied \( a - b + 1 \) times. We need to check that, when relations are quantified, they respect this property.

\[
\text{encode}_{a, b}(\lambda^a, r) = \forall 1 \leq i \leq a \lambda_i^{a-r-1} \big( \lambda(\lambda_1, \ldots, \lambda_a) \Rightarrow \bigwedge_{b < j \leq a} \lambda_j = \lambda_j \big)
\]

We will assume that \( \varphi \) is in SNF. Syntactically we will replace every occurrence of \( Q\lambda^t, r \varphi \) by

\[
Q\lambda^a, r \left( \text{encode}_{a, b}(\lambda^a, r) \oplus Q \varphi \right) \tag{3}
\]

To be more precise, we don’t really need the quantification of the \( \lambda_i^{a-r-1} \) to be just after the quantification of \( \lambda^a, r \). Since the \( \lambda_i \) only interacts with \( \lambda^a \), we can put the quantifiers anywhere after the quantification of \( \lambda^a \). Hence if \( \varphi \) is in decreasing normal order as defined in the next section (the orders of the quantified variables decrease) we can postpone the quantifications in \( \lambda \) to put them at the right place of the list. Then we can, without loss of generality, extract the quantifiers of \( \varphi \) to put them after the quantification of the \( \lambda_i \). This way if \( \varphi \) was in decreasing normal form, it will remain in normal form.

We will replace every atomic formula \( \lambda^{t, r}(Y_1^{t_1, r-1}, \ldots, Y_n^{t_n, r-1}) \) by \( \lambda^{a, r}(Y_1^{a, r-1}, \ldots, Y_n^{a, r-1}) \) where \( Y_i^{a, r-1} \) means that the last element is repeated \( a - b + 1 \) times.

It is clear that those formulae are equivalent and in arity normal form.

**Increasing arity of input structure** Even if we can not accept an input structure with relations using “type”, we must at least accept the relation respecting the “arity” constraint, which creates a problem when we change the syntax of our
formula. Let say that \( R \) is an input structure of order \( r \) and arity \( b \), and we want to have a copy \( S \) of it of arity \( a \). Then we can state that \( S \) is a good copy with

\[
\text{copy}_r(R^{b,r}, S^{a,r}) = \text{def} \ \text{encode}_{a,b}(S) \land \forall_{1 \leq i \leq b} \text{copy}_{r-1}(Y_1, \ldots, Y_b) \Rightarrow \\
\forall_{1 \leq i \leq b} Z_i \land (\bigwedge_{1 \leq i \leq b} \text{copy}_{r-1}(Y_i, Z_i)) \Rightarrow S(Z_1, \ldots, Z_b) \tag{4}
\]

\[
\text{copy}_1(R^{b,r}, S^{a,r}) = \text{def} R = S \tag{5}
\]

Let \( \sigma = \{ R_1^{b_1,r_1}, \ldots, R_n^{b_n,r_n} \} \) and let \( \varphi \) be a \( \sigma \)-formula. When we apply the rules of the last paragraph to extend \( \varphi \) into an equivalent formula \( \varphi' \) of arity \( a \) we must in fact transform it into:

\[
\varphi'' = \text{def} \ \forall_{1 \leq i \leq n} S_i (\bigwedge_{1 \leq i \leq n} \text{copy}_{r_i}(R_i, S_i)) \Rightarrow \varphi'[R_i/S_i] \tag{6}
\]

**Respecting order of quantifiers**  In the next subsection we will take care of the order of the quantifiers, and we will need an algorithm for this normal form, hence here we must emphasize a few details. Both in the proof of step normal form and of arity normal form, we did not create any new quantification of order \( r \) or higher, so we respected the global form of the formula when we consider only \( r \)th order quantifiers. Now, let us suppose that we respect the decreasing normal form as in Definition 2.31. Then we can see that the new quantifiers of our formula do not have to be exactly where we put them – we can postpone them to be at the good place in the sequence, and postpone the new quantifier-free part to be with the quantifier-free part of the formula. Hence if the input is formula in decreasing normal form, the output is also a formula in decreasing normal form.

Then, as we stated, the normal form equivalent formula is indeed in \( \Sigma' \).

In this article, we will only use formulae in arity normal form, and this will simplify our proofs, and \( X^{a,r} \) will be a syntactic sugar for \( X^{A(a,r)} \).

### 2.4.2 Order of the quantifiers

As stated earlier, there is a normal form for \( \Sigma_j' \) and we will give an algorithm to obtain that normal form. The difference between this algorithm and the “folklore” one as given in [8] [13] (the latter is about high order in general and not in finite structures) is that the folklore algorithm sends HO’ to HO’ but does not respect the number of alternations. On the other hand, our algorithm sends \( \Sigma_j' \) to \( \Sigma_j' \). We should note that our algorithm does not respects the maximal arity.

**Prefix normal form:**
Definition 2.28 (Prefix normal form). A formula is in prefix normal form (PNF) if it begins with a sequence of quantifications and ends with a quantifier-free formula.

Lemma 2.29. Every formula in $\Sigma^r_j$ is equivalent to a formula in prefix normal form.

We will assume that there are not two variables of the same name. Thanks to $\alpha$-conversion this creates no loss of generality.

Notation 2.30. $Q$ will be a meta variable for quantifiers, $A$ for atomic formula, $\otimes$ for disjunction or conjunction and $\oplus$ for a polarity symbol (+ or -), where applying - to a symbol will give its dual while + will not change it. Hence $-\neg =_{\text{def}} \neg$, $-\epsilon =_{\text{def}} \neg \epsilon$, $-\exists =_{\text{def}} \forall$, $-\forall =_{\text{def}} \exists$ and we can even apply a polarity symbol to a polarity symbol, $++ =_{\text{def}} -- =_{\text{def}} +$, and $-\neg =_{\text{def}} \neg$. On the other hand $+\neg =_{\text{def}} \neg$, $+Q =_{\text{def}} Q$ and $+\otimes =_{\text{def}} \otimes$.

Proof. (of Lemma 2.29) We will do a constructive proof, by giving an algorithm to transform the formula. We will use three auxiliary recursive functions.

PrefixNormalForm($r, \varphi$):=Aux($r, \varphi, \exists, +$)

The result of the Aux function is such that negations are only on atomic predicates, so it must remember the parity of the number of $\neg$ it met. This is the information of the last argument. It will give an output in prefix normal form with as little alternation as possible and that is why it must know what was the last quantifier of order $r$. That is what its third argument is for. Since we want a normal form for $\Sigma^r_j$, we assume that the formula begins with an existential quantification of order $r$, and hence we can give an $\exists$ quantifier as argument.

If Aux meets a quantifier, it will write the very same quantifier and work inductively on the formula. If it meets a negation it will switch its polarity and continue inductively. Finally if it finds a conjunction or disjunction, it will act inductively on both parts to put them in prefix normal form, and then will combine them with Aux'.

Aux ($r, \varphi, Q, \oplus$):=match $\varphi$ with
| $Q'\mathcal{X}^{a,p}, \psi$ $\rightarrow$ let $\psi' =$(if $p = r$
then Aux ($r, \psi, (\oplus Q'), \oplus$)
else Aux ($r, \psi, Q, \oplus$)) in $A \otimes \psi'$
| $\neg \psi$ $\rightarrow$ Aux ($r, \psi, Q, (\oplus Q')$) in $A \otimes \psi'$
| $A \rightarrow \oplus A$ if $A$ is atomic where $-A =_{\text{def}} \neg A$.
| $\varphi \otimes \psi$ $\rightarrow$ let $\varphi' =_{\text{def}} \text{Aux} (r, \varphi, Q, \oplus)$ and $\psi' =_{\text{def}} \text{Aux} (r, \psi, Q, \oplus)$ in $\text{Aux'} (r, \varphi', \psi', Q, \oplus)$
Of course when we have $Q \mathcal{X}$ on the left of the arrow and $Q \mathcal{X}$ on the right of the arrow, we assume that both $\mathcal{X}$ are of the same type. This assumption will be true until the end of this proof.

Aux’ will take two inputs in prefix normal form, and a parameter to know if we must consider its conjunction or its disjunction. Then it will extract from them as many quantifiers of order $r$ of the last seen polarity as possible. When $\varphi$ and $\psi$ begin with quantifiers of the other polarity of order $r$, we will switch the polarity we want to extract. Finally when one formula has no more quantifiers (by the prefix normal form we have by induction, we know it is then a quantifier-free formula) we will extract all predicates of the other formula using $\text{Aux}''$. Finally we will link the two quantifier-free parts of the formula with the $\otimes$ relation.

$\text{Aux}'(r, \varphi, \psi, Q, \otimes) := \text{match } \varphi \text{ with}
\begin{cases}
Q'X^{p-p} \varphi' \rightarrow & \text{if } p < r \text{ or } Q = Q' \text{ then } Q'X \text{Aux}'(r, \varphi', \psi, Q, \otimes) \\
Q'Y \psi' \rightarrow & \text{if } q < r \text{ or } Q = Q'' \text{ then } Q''Y \text{Aux}'(r, \varphi, \psi', Q, \otimes) \\
\_ \rightarrow & \text{Aux}''(\varphi, \psi, \otimes)
\end{cases}$

$\text{Aux}''(\varphi, \psi, \otimes) := \text{match } \varphi \text{ with}
\begin{cases}
Q'X \varphi' \rightarrow & Q'X \text{Aux}''(\varphi', \psi, \otimes) \\
\_ \rightarrow & \varphi \otimes \psi
\end{cases}$

An easy induction over $\text{Aux}''$ and $\text{Aux}'$ shows that the number of alternations in the output is the larger number of alternations of the two elements of the input. Then an induction over $\text{Aux}$ shows that its output respects the same property. It is trivial to see that if the input was a formula of order $r$, so is the output.

This algorithm gives a normal form for $\Sigma^r_j$ only, as it does not promise in general to give the smallest number of alternations. For example:

$\text{PrefixNormalForm}((\forall X \exists Y \varphi) \land (\exists Z \psi)) = \exists Z \forall X \exists Z (\varphi \land \psi)$

The formula with the smaller number of alternations is $\forall X \exists Y Z (\varphi \land \psi)$. But since this formula begins with a $\forall$, it is still an $\Sigma^r_3$ formula.

We can easily change the algorithm to obtain a normal form for $\Pi^r_j$, as it is given by $\text{Aux}(r, \varphi, \forall, +)$. Finally, if we want an algorithm to obtain the smaller number of alternation, it suffices to run both algorithm and choose the formula with the smallest number of alternation.

**Decreasing normal form:**

**Definition 2.31** (Decreasing normal form). An $\Sigma^r_j$ formula, for $r \geq 1$, is in decreasing normal form (DNF) if it is in the form $\exists X_1^r \forall X_2^r \ldots Q X_j^r \psi$ where each $Q$ is a quantifier and $\psi$ is an $\text{HO}^{r-1}$ formula in decreasing normal form.
Definition 2.32 (Normal form). An $\Sigma_j^r$ formula is in normal form (NF) if it is in both arity normal form and decreasing normal form, and hence also in prefix normal form and step normal form. A formula in $\text{HO}^r(P)$, where $P$ is an operator, is in normal form if it is in operator’s normal form and its subformula in $\text{HO}^r$ is also in normal form.

Theorem 2.33. Every formula $\varphi \in \Sigma_j^r$ is equivalent to a formula $\varphi' \in \Sigma_j^r$ in normal form.

And in each group of quantifiers of order $r$, the number of quantifiers in $\varphi'$ is not greater than the number of quantifiers of that order in $\varphi$.

Proof. Let $\varphi$ be a formula. Thanks to property 2.29, we can assume it to be in prefix normal form and it will be straightforward that, while we transform it, it will remain in prefix normal form.

The proof will be by induction over the order $r$. It is trivial if $r = 0$ or $r = 1$ because a quantifier free formula and a first order formula in prefix normal form are in normal form. Hence we will assume that $r > 1$ and that the property is true for all $p < r$. Now we will prove the property by induction over the number $n$ of relations of order $r$. If $n = 0$ then it is a formula of order $r - 1$, and hence the property is true by induction. So we will suppose that $n > 0$ and that the property is true for every $m < n$. We will prove this property by induction over the number $q$ of quantifications. It is true if $q = 0$ because it is then a quantifier-free formula, which is in normal form; we will assume $q > 1$ and that the property is true for any formula with $r < q$ quantifications.

Then $\varphi = Q\chi^{t,i}.\psi$, and by induction over the number of quantifiers if $i < r$, or over the number of quantifiers of order $r$ if $i = r$, there exists a formula $\psi'$ in normal form equivalent to $\psi$. If $i = r$, then $\varphi' = Q\chi^r\psi'$ is in normal form and equivalent to $\varphi$, and hence the property is true.

We will now assume that $i < r$. If $\psi'$ contains fewer quantifiers of order $r$, then $\varphi' = Q\chi^t.i.\psi'$ is a formula, equivalent to $\varphi$, with fewer quantifiers of order $r$. Hence by the induction property over this number we can find an equivalent formula in normal form.

We will then assume that there are at least the same number of quantifiers of order $r$ in $\psi'$ as in $\psi$, and since the induction hypothesis tells us that there is not more quantification, we will assume that the number of quantifications is the same. Since $\psi'$ is in normal form and contains a quantifiers of order $r$, then $\psi' = Q\chi^{t',r}.\xi$ and we can now write $\varphi'$ as $Q\chi^{t':r}.Q\chi^{t'.r}.\xi[\chi(\Gamma_1,\ldots,\Gamma_a)/\chi(\chi',\Gamma_1,\ldots,\Gamma_a)]$ where $t'::t'$ is the tuple whose first element is $t$ and whose other elements are the elements of $t'$. Let us assume for now that this formula is equivalent to $\varphi$. $\psi'' = Q\chi^{t'.r}.\xi[\chi(\Gamma_1,\ldots,\Gamma_a)/\chi(\chi',\Gamma_1,\ldots,\Gamma_a)]$ is a formula with fewer quantifiers of order $r$ than in $\varphi$ and hence it has got a normal form $\psi''$ equivalent to $\psi''$, and then $\varphi'' = Q\chi^{t':r}.\psi''$ is a formula in normal form equivalent to $\varphi$. 17
Here with $\mathcal{Y}^{t,i}$ we have lost the normal form of last section, so let $\varphi'''$ be equivalent to $\varphi''$ and in arity normal form. We proved that it is possible, we just need to consider the free variables of $\varphi$ that are quantified in the entire formula as elements of the vocabulary of $\varphi$, which is coherent with our definition. And, as we explained at the end of last subsubsection, since $\varphi''$ is in decreasing normal form it will remain in this normal form.

Now, it remains to prove that $\varphi$ is equivalent to $\varphi'$ in the last case, which means that $\varphi = QA^{t,i}Q'Y^{t,i}Y \xi$ is equivalent to $\varphi' = Q'\psi''$ with $\psi'' = QA^{t,i}Y(\mathcal{Z}_1, \ldots, \mathcal{Z}_a)/Y(X, \mathcal{Z}_1, \ldots, \mathcal{Z}_a)$. There are four different cases, for the four possible values of the couple $(Q, Q')$. We are going to make a proof for $Q = \forall, Q' = \exists$; the three other cases use the same idea.

Let $\varphi = \forall X^{t,i} \exists Y^{t,i} \xi$ and $\varphi' = \exists Y^{t,i} \forall X^{t,i} \xi[Y(\mathcal{Z}_1, \ldots, \mathcal{Z}_a)/Y(X, \mathcal{Z}_1, \ldots, \mathcal{Z}_a)]$. We are going to prove their equivalence, first by proving that the truth of the first formula implies the truth of the second one. Let $A$ be a structure, and suppose that $A \models \varphi$, then for any relation $X^{t,i}$ there exists a relation $Y^{t,i}$ such that $A[X/X][Y/Y] \models \xi$, so let $\mathcal{Y}^{t,i} = \{X^{t,i} \mapsto \mathcal{T} | \mathcal{T} \in \mathcal{Y}_X \}$. Then, for any value of $X^{t,i}$, $\mathcal{Y}_X(\mathcal{T}) \iff Y''(X, \mathcal{T})$, and by induction over $\xi$, we have $A[X/X][Y/Y] \models \xi \iff A[X/X][Y/Y] \models \xi[\mathcal{Y}(\mathcal{Z}_1, \ldots, \mathcal{Z}_a)/Y(X, \mathcal{Z}_1, \ldots, \mathcal{Z}_a)]$.

Now we will show that the truth of the second statement implies the truth of the first. Suppose that $A \models \varphi'$. Then there exists an $\mathcal{Y}^{t,i}$ such that for all value of $X^{t,i}$ we have $A[X/X][Y/Y] \models \xi[\mathcal{Y}(\mathcal{Z}_1, \ldots, \mathcal{Z}_a)/Y(X, \mathcal{Z}_1, \ldots, \mathcal{Z}_a)]$. Let $X''^{t,i}$ be an arbitrary relation, then let $Y''_X = \{\mathcal{T} | \mathcal{Y}(\mathcal{X}', \mathcal{T})\}$, then $\mathcal{Y}_X(\mathcal{T}) \iff Y''(X, \mathcal{T})$, and by induction over $\xi$ we have $A[X/X][Y/Y] \models \xi \iff A[X/X][Y/Y] \models \xi[\mathcal{Y}(\mathcal{Z}_1, \ldots, \mathcal{Z}_a)/Y(X, \mathcal{Z}_1, \ldots, \mathcal{Z}_a)]$.

\[\square\]

**Infinite structures** As stated in the beginning of this subsection, every proof only used information about the formulae and there is not any use of the “structure”. Hence this normal form also applies to formulae in high-order over infinite structures.

### 3 High-order queries

#### 3.1 Number of relations

**Definition 3.1.** Let $r, a > 0$ be positive integers. We define $C(r, a)$ to be the maximum cardinality of a relation of $F^{r,a}$, $N(r, a)$ to be the number of relations in it, $T(r, a)$ is the number of $a$-tuples of relations and $B(r, a)$ is the number of
bits necessary to describe such a relation. These relations are also defined without
the “a”, for example \( C(r) = C(r, O(1)) \).

**Lemma 3.2.** We have the following equalities:

- \( C(r, a) = \exp_{2}^{r-2}(n^a, a) = \exp_{2}^{r-2}(n^{O(1)}) \)
- \( T(r, a) = \exp_{2}^{r-1}(n^a, a) = \exp_{2}^{r-1}(n^{O(1)}) \)
- \( N(r, a) = 2^{\exp_{2}^{r-2}(n^a, a)} = \exp_{2}^{r-1}(n^{O(1)}) \)
- \( B(r, a) = \exp_{2}^{r-2}(n^a, a) = \exp_{2}^{r-2}(n^{O(1)}) \)

This lemma is similar to the one stated in [8] but corrects a minor error there.

We need the “big O” to be inside of the exponent and not around it.

**Proof.** Indeed, \( T(1, a) \) is the size of the Cartesian product of \( a \) sets of \( n \) elements
each, so \( T(1, a) = n^a = \exp_{2}^{0}(n^a, a) = \exp_{2}^{0}(n^{O(1)}) \).

By induction, supposing the properties are true up to order \( r - 1 \geq 1 \):

- An \( a \)-ary relation of order \( r \) is a subset of the tuples of \( a \)-ary relations of
  order \( r - 1 \) so \( C(r, a) = T(r - 1, a) = \exp_{2}^{r-2}(n^a, a) \).

- Hence the number of \( a \)-ary relations of order \( r \) is the number of subsets
  of the \( a \)-tuples of \( a \)-ary relations of order \( r - 1 \), so \( N(r, a) = 2^{T(r - 1, a)} =
  2^{\exp_{2}^{r-2}(n^a, a)} = \exp_{2}^{r-1}(n^{O(1)}) \).

- The number of \( a \)-tuples of \( a \)-ary relations of order \( r \) is the size of Carte-
  sian product of \( a \) copies of the set of \( a \)-ary relations of order \( r \), so \( T(r, a) =
  N(r, a)^a = (2^{\exp_{2}^{r-2}(n^a, a)})^a = 2^{\exp_{2}^{r-2}(n^a, a) \times a} = \exp_{2}^{r-1}(n^a, a) = \exp_{2}^{r-1}(n^{O(1)}) \).

The proof for \( B(r, a) \) will be the subject of the next subsection. \( \square \)

### 3.2 Encoding relations

In this subsection we will explain how high order relations can be used and
checked in a space-efficient way such that queries of these relations are also ef-
ficient.

Since there are \( N(r, a) = 2^{\exp_{2}^{r-2}(n^a)} \) relations of order \( r \) and arity \( a \), we need
at least \( \log_{2}(N(r, a)) = \exp_{2}^{r-2}(n^a) = T(r - 1, a) \) bits to encode a relation \( R^{n,r} \)
as a string of bits. The last equality is not a surprise, because all the information
one needs to know the relation \( R^{n,r} \) explicitly is the set of \( a \)-tuples of relations of
order \( r - 1 \) in \( R^{n,r} \); except for the special case \( r = 1 \), where relations are on the
elements of the universe, but in this case it is well known that one needs \( \lceil \log(n) \rceil \)
bits. Since our code will use exactly this number of bits, it is impossible to find a more space efficient general encoding.

We will show that this is an exact bound when \( r \geq 2 \) by creating a one-to-one encoding function \( e \) from a relation of order \( r \) and arity \( a \) onto a string of bits of length \( \exp^r_2(n^a, a) \). Let \( b \) be a bit position of \( e(R^r) \). As a binary number \( b \) is a string of length \( \log_2(\exp^r_2(n^a, a)) = a \exp^r_2(n^a, a) \), so inductively, it can be considered as an \( a \)-tuple of codes of \( a \)-ary relations of order \( r - 1 \). The description will then be that the \( b \)th bit will be one if and only if this \( a \)-tuple encodes an element of \( R^r \).

It is clear that this is a one-to-one relation and that this encoding contains all the relevant information, and thus that the equality of relations is just an equality of strings of bits. It also gives us a canonical order over relations, which is the order over the binary code of the relation.

### 3.3 Encoding input

In subsection 3.2 we explained how to encode high order relations in a space efficient way. But it is efficient in the worst case; in graph theory it would be equivalent to the matrix encoding. But, as in graph theory, it can also be interesting to consider other codes for the input, especially for non-dense relations.

An example of a possible code would be a circuit such that the leaves are elements of the universe, the nodes of height \( 2n \) are a relations of order \( n \) and the nodes of height \( 2n + 1 \) are \( a \)-tuples of relations of order \( n \). There is an edge from an \( a \)-tuple into a relation if this tuple is an element of the relation, and the \( a \)-tuples are of in-degree \( a \), where there is an order on the edges, the \( a \) predecessors being of course the \( a \) elements of the tuple.

Since many different encodings could be imagined, depending on the assumptions about the problem one wants to solve, we are going to speak of a more general property.

**Definition 3.3** (acceptable code). An encoding of a \( \sigma \)-structure \( \mathfrak{A} \) is said to be acceptable if for every relation \( R^{n,r} \) and \( a \)-tuple \( S^{a,r-1} \) the property \( R(\mathfrak{S}) \) is decidable in time polynomial in the size of the description of \( R \) and \( \mathfrak{S} \).

**Definition 3.4** (reasonable input). A set of input is reasonable for a given code if the size of the code of the structures of this classes is bounded by a polynomial in the size of the structure.

By cardinality, it is clear that the class of every structure of order at least 3 can not be “reasonable”.

**Claim 3.5.** The circuit encoding and the encoding of section 3.2 are acceptable, and they are reasonable for inputs of order 2.
3.4 Reducing the order of the input

Since we mostly want to study the formulae in \( \text{HO}^{r,2} \) we are going to show how to reduce the order of the structures. That is, for a formula in \( \text{HO}^{r+1,f+1} \) for \( f \geq 2 \), how to obtain an equivalent formula in \( \text{HO}^{r,f} \) over an equivalent vocabulary of order \( f \), for a precise definition of “equivalent”.

We will define the function \( F : \Sigma_{j=0}^{r+1,f+1} \rightarrow \Sigma_j^{r,f} \), and the function \( V \) from vocabularies \( \sigma \) of order \( f+1 \geq 3 \) and \( \sigma \)-formulae into vocabularies of order \( f \), such that if \( \varphi \) is an \( \sigma \)-formula then \( F(\varphi) \) is a \( V(\sigma) \)-formula and the function \( S \) is from \( \sigma \)-structures into \( V(\sigma) \)-structures such that \( \mathcal{A} \models \varphi \iff \mathcal{V}(\mathcal{A}) \models F(\varphi) \).

The encoding will be such that \( |V(\mathcal{A})| = O(2^{3^mO(1)}) \). We consider that this size is acceptable since an \( f \)th order relation is encoded with \( \exp_2^{f-2}(n^{O(1)}) \) bits, and after we apply those functions we will have a structure with relations of order up to \( f-1 \). Hence the new size of the encoding of the input will be \( \exp_2^{f-3}(2^{n^{O(1)}}) = \exp_2^{f-2}(n^{O(1)}) \) bits.

Let \( R^{a,r} \) be a symbol of order \( r \) and arity \( a \). We define \( V(R) \) as a symbol of order \( \max\{1, r - 1\} \) and arity \( a \).

Let \( \sigma = \{<^2, R_1^{a_1}, \ldots, R_n^{a_n}\} \), \( \varphi \) a \( \sigma \)-formula, let \( a' \) be the highest arity of a quantified variable of \( \varphi \), let \( a = \max(a_1, \ldots, a_m, a', 1) \), then \( V(\sigma, \varphi) = \{<, n, T_1^{a_1}, \ldots, T_n^{a_n}, V(R_1), \ldots, V(R_n)\} \). If \( \mathcal{A} \) is a \( \sigma \)-structure of cardinality \( n \) where \( < \) is interpreted as a total order over the universe, then \( S(\mathcal{A}) \) contains exactly \( 2^a \) elements where the first \( n \) elements represent the \( n \) elements of \( \mathcal{A} \), and the first \( 2^a \) elements, with \( b \leq a \), represent the second-order \( b \)-ary relations, the exact representation being the same as in subsection 3.2. The \( c_i \) will be the same constants, \( n \) will be the \( n \)th element and represents the size of the input of the former universe, \( T_i^{a+1}(x_0, x_1, \ldots, x_i) \) will be true if \( x_0 \) represents a second-order \( i \)-ary relation \( R_i^{2,x_0} \) and the \( (x_j)_{1 \leq j \leq i} \) represent elements of the universe, and if \( R_{x_0}(x_1, \ldots, x_i) \).

This means that the same elements of the structures may represent both a first-order element and second order \( b \)-ary elements for any \( b \). The exact meaning is known only when the variable is queried. The former \( r_i \)th order \( a_i \)-ary relations now become \( r_i - 1 \)th order \( a_i \)-ary relations, the only other change is that when \( r_i - 1 = 2 \), we assume that the relation does not accept any first-order element which is not the representation of a former second order \( a \)-ary relation.

We must define what it means for a relation \( R^{e,r} \) to be a correct encoding of a \( b \)-ary relation of order \( q \). We will do it with \( \text{acc}(e,q,b) \) which means that it contains

---

1. \( a \) depends on \( a' \) which explains why \( V \) takes \( \varphi \) as an input.
2. There is no loss of generality since in high-order we can always create a linear order.
Table 1: F

| \( \varphi \)               | \( F(\varphi) \)               |
|-------------------------------|-------------------------------|
| \( R = S \)                   | \( R = S \)                   |
| \( R^{2,b}(x_1, \ldots, x_b) \) | \( T_b(R, x_1, \ldots, x_b) \) |
| \( \mathcal{R}^{p,b}(x_1, \ldots, x_b) \) | \( \mathcal{R}^{p-1,b}(x_1, \ldots, x_b) \) |
| \( \varphi \lor \psi \)       | \( F(\psi) \lor F(\psi) \)   |
| \( \neg \varphi \)            | \( \neg F(\varphi) \)        |
| \( \exists x. \psi \)         | \( \exists x.(x < n \land \psi) \) |
| \( \exists X^{2,b} \psi \)    | \( \exists X^1.(x < 2^{nb} \land \psi) \) |
| \( \exists \mathcal{X}^{p,b} \psi \) | \( \exists \mathcal{X}^{p-1,b} \psi \) |
| \( (TC_{\mathcal{X}^{\psi}})(\mathcal{Z}T) \) | \( (TC_{\mathcal{X}^{\psi}})(acc(\mathcal{V}(\mathcal{X})) \land F(\psi)))(\mathcal{Z}T) \) |
| \( (PFP_{\mathcal{X}^{\psi}})(\mathcal{Y}) \) | \( (PFP_{\mathcal{X}^{\psi}})(acc(\mathcal{V}(\mathcal{X})) \land F(\psi)))(\mathcal{Y}) \) |
| \( (IFP_{\mathcal{X}^{\psi}})(\mathcal{Y}) \) | \( (IFP_{\mathcal{X}^{\psi}})(acc(\mathcal{V}(\mathcal{X})) \land F(\psi)))(\mathcal{Y}) \) |

no first order elements that are not encodings of second order elements.

\[
\text{acc}(\mathcal{X}^{q,b}) = \text{def} \forall_{1 \leq i \leq b} \mathcal{Y}_i^{q-1,b}(\mathcal{X}(\mathcal{Y}_1, \ldots, \mathcal{Y}_n)) \Rightarrow \bigwedge_{1 \leq i \leq b} \text{acc}(\mathcal{Y}_i) \quad (7)
\]

\[
\text{acc}(\mathcal{X}^{1,b}) = \text{def} \mathcal{X} < 2^{nb} \quad (8)
\]

Here \( n \) is a constant of the new vocabularies which represents the size of the former universe, we could either add \( 2^{nb} \) to the input, or define \( x < 2^{nb} \) as \( \forall x_0, \ldots, x_b(T_{b+1}(x_0, x_0, \ldots, x_b) \Rightarrow x_0 = 0) \) if \( b < a \), else as \( \top = \text{def} \forall x (x = x) \) and of course \( x = 0 \) as \( \neg \exists y.(y < x) \).

**Lemma 3.6.** For any high-order relation \( \mathcal{R}^{r,v} \) we have \( \text{acc}(\mathcal{V}(\mathcal{R})) \). If \( \text{acc}(\mathcal{R}) \) is true then there is some \( \mathcal{S} \) such that \( \mathcal{V}(\mathcal{S}) = \mathcal{R} \).

**Proof.** The first part is by construction of \( V \), and the second one is a trivial induction over the order. \( \Box \)

Now we need to define \( F \), and we will do it recursively. We assume without loss of generality that there is no \( \forall \) or \( \land \). The algorithm is in table 3.4. In this algorithm, when \( \mathcal{Y} \) is a tuple of variables we denote \( \mathcal{X} = \mathcal{X}_1^{r_1,a_1}, \ldots, \mathcal{X}_r^{r_r,a_r} \), \( \mathcal{V}(\mathcal{X}) = \mathcal{V}(\mathcal{X}_1), \ldots, \mathcal{V}(\mathcal{X}_r) \) and if \( P \) is a variable whose type is equivalent to \( \mathcal{X} \) then \( \mathcal{V}(P) \)'s type is equivalent to \( \mathcal{V}(\mathcal{X}) \).

**Theorem 3.7.** If \( f \geq 2, r \geq 2 \) (resp. \( r = 1 \)) and \( \varphi \in \Sigma^{r+1,f+1}_j \) then \( F(\varphi) \in \Sigma^{r,f}_j \) (resp \( F(\varphi) \in \text{HO}^{r,f}_j \)). For any vocabulary \( \sigma \), \( \sigma \)-structure \( \mathfrak{A} \) and \( \sigma \)-formula \( \varphi \), \( \mathfrak{A} \models \varphi \Leftrightarrow \mathcal{V}(\mathfrak{A}, \varphi) \models F(\varphi) \).
Proof. For the first statement, as we can see, the only new quantifiers are of order lower than \( r - 1 \) (resp. of order 1), hence the number of alternations of the \( r \)th order quantification in \( F(\varphi) \) is the same as the number of \( r + 1 \)th order quantifiers in \( \varphi \).

For the second statement, we do the proof by induction over \( \varphi \). For the atomic formulae it is by construction, and for the negation, conjunction and disjunction it is trivial.

So assume that \( \varphi = \exists x. \psi \), and let us prove \( \Rightarrow \). If \( \mathfrak{A} \vDash \varphi \) then there is some \( i < n \) such that \( \mathfrak{A}[x/i] \models \psi \), by induction \( V(\mathfrak{A}[x/i]) = F(\psi) \) and since \( i < n \), then \( F(\varphi) \) is true.

For \( \Leftarrow \), if \( V(\mathfrak{A}) = F(\varphi) \) then there is some \( i \) such that \( V(\mathfrak{A}[x/i]) = x < c \land F(\psi) \), of course then \( i < n \), hence \( V(\mathfrak{A}[x/i]) = V(\mathfrak{A}[x/i]) \) and by induction \( V(\mathfrak{A}[x/i]) = F(\psi) \Leftarrow \mathfrak{A}[x/i] = \psi \), hence \( \mathfrak{A} \models \varphi \Leftarrow V(\mathfrak{A}) = F(\varphi) \).

The case \( \varphi = (TC_{\mathfrak{A}/\mathfrak{Z}}(\psi))(\mathfrak{Z}/\mathfrak{M}) \) Let us prove \( \Rightarrow \) by induction over the number \( s \) of steps of the transitive closure. If \( s = 0 \) it is trivial, let us suppose that \( s > 1 \) and it is true for \( s - 1 \). Then there exists \( \mathfrak{M} \) equivalent to \( \mathfrak{Z} \) such that \( \mathfrak{A}[\mathfrak{A}/\mathfrak{Z}][\mathfrak{Z}/\mathfrak{M}] \models \psi \) and \( \mathfrak{A} \models (TC_{\mathfrak{A}/\mathfrak{Z}}(\mathfrak{Z}/\mathfrak{M}))) \) and then by the induction hypothesis over \( \varphi \), we have \( V(\mathfrak{A}[\mathfrak{A}/\mathfrak{Z}][\mathfrak{Z}/\mathfrak{M}]) = F(\psi) \) hence \( V(\mathfrak{A}[\mathfrak{A}/\mathfrak{Z}][\mathfrak{Z}/\mathfrak{M}]) = F(\psi) \) and by lemma 3.6 \( \text{acc}(V(\mathfrak{M})) \) and by the induction hypothesis over \( s \), \( V(\mathfrak{A}) = (TC_{\mathfrak{A}/\mathfrak{Z}}(\psi))(\mathfrak{Z}/\mathfrak{M}) \).

Now, let us prove \( \Leftarrow \), it is also an induction over the number of steps \( s \) that close \( V(\mathfrak{A}) = (TC_{V(\mathfrak{A})}(\text{acc}(V(\mathfrak{A}))))(\mathfrak{Z}/\mathfrak{M}) \). If \( s = 0 \) then it is trivial, else there exists some \( \mathfrak{M} \) equivalent to \( \mathfrak{A} \) such that \( V(\mathfrak{A}[\mathfrak{A}/\mathfrak{Z}][\mathfrak{Z}/\mathfrak{M}]) = \text{acc}(\mathfrak{A}) \land F(\psi) \) hence by lemma 3.6 there is some \( \mathfrak{M} \) such that \( V(\mathfrak{M}) = \mathfrak{M} \) and by the induction hypothesis over \( \varphi \) we have \( V(\mathfrak{A}[\mathfrak{A}/\mathfrak{Z}][\mathfrak{Z}/\mathfrak{M}]) = \psi \), which ends the proof.

The proofs for the fixed points are similar, with induction on the size of the fixed point. \( \square \)

Claim 3.8. In this article we will give results for formulae over structures of order 2. In general, if the input structure is of order \( p - 1 \) and hence contains at least one relation of order \( p \) and no relation of higher order, the time and space bound will decrease, by \( p - 2 \) applications of the logarithm over the bound. In particular, a corollary will be that queries in \( \text{HO}^{r,r} \) are computable in polynomial time, as proven in \([7]\).
4 Arithmetic predicates

4.1 Predicates over relations

In finite model theory, the arithmetic predicates are important, especially in first order, where even partial fixed points can not express the parity of the size of the universe without an order relation. In next sections we will often use either bit predicates or addition over high order relations, so in this section we will first explain how to obtain those relations.

As it is already known, a linear order can be specified by a second order binary relation, hence, contrary to what happens in the first-order case, we will not make any statement about the existence or the absence of an order relation in the vocabulary.

We intend to show that the usual predicates that we may ask over first order, bit, plus, times, $<$, are redundant in high-order; all of these predicates can be defined thanks to a first-order total order.

We will speak of some arithmetic operations both over predicates and over tuples of predicates, as both will be useful in this article. To distinguish them, we adopt the convention that “predicate $a,r$” refers to a predicate over relations and “predicate $a,r$” refers to a predicate over tuples of relations.

Notation 4.1. In this section, “$\mathcal{P}^{a,r}$” will always be an $a$-tuple of relations of arity $a$ and order $r$, $\mathcal{P}_1^{a,r}, \ldots, \mathcal{P}_n^{a,r}$.

Claim 4.2 (arity of predicate). As we will see, to define a predicate over relations of arity $a$, quantification is over variables of arity $a$, and hence there is no increase of arity of the formula because of the arithmetic predicate. In particular, those predicates over monadic relations are monadic formulae.

4.1.1 Equality predicate

If there is a binary first-order equality predicate, then every other equality predicate can be defined in the logic. Define $=_{a,r}$ to be the equality predicate over relation of order $r$ and arity $a$, and then we can define it recursively as: $\mathcal{X}^{a,r} =_{a,r} \mathcal{Y}^{a,r} =_{a,r} \forall \mathcal{P}^{a,r} - 1 (\mathcal{X}(\mathcal{P}) \leftrightarrow \mathcal{Y}(\mathcal{P}))$. And of course $\mathcal{P}^{a,r} =_{a,r} \mathcal{Q}^{a,r} =_{a,r} \bigwedge_{0 \leq i < a} P_i =_{a,r} Q_i$.

4.1.2 Order relation

Suppose that we have an order relation on first-order variables, $x < y$. Then we can recursively encode a formula $\mathcal{X}^{a,r} <_{a,r} \mathcal{Y}^{a,r}$ over relations of arity $a$ and order $r$ considered as binary numbers.
\( X^{a,r} <_{a,r} Y^{a,r} = \text{def } \exists \exists \mathcal{T}^{a,r-1} \cdot (Y(\mathcal{T}) \land \neg X(\mathcal{T}) \land \forall \mathcal{Q}^{a,r-1} \cdot (\mathcal{T} <_{a,r-1} \mathcal{Q} \Rightarrow (Y(\mathcal{Q}) \leftrightarrow X(\mathcal{Q}))) \). \)

Here \( <_{a,r} \) is a relation over \( a \)-tuples of relations of order \( r \) defined as: \( X^{a,r} <_{a,r} Y^{a,r} = \text{def } \bigvee_{1 \leq i \leq a} (X_i <_{a,r} Y_i \land \bigwedge_{1 \leq j < i} (X_i =_{a,r} Y_i)) \).

4.1.3 Bit predicate

It is usual in descriptive complexity to use a “bit” relation, taking two first order variables \( x \) and \( y \), such that \( \text{bit}(x, y) \) is true if and only if the \( y \)th bit of the binary expression of \( x \) is 1.

For high order it is easier; since a relation \( R^{a,r} \) is equivalent to a string of \( T^{(r-1),a} \) bits, we can write the \( y \) as \( a \) relations of order \( i \), and then \( \text{bit}(R^{a,r}, S^{a,r-1}_1, \ldots, S^{a,r-1}_a) = \text{def } R^{a,r}(S^{a,r-1}_1, \ldots, S^{a,r-1}_a) \).

4.1.4 Addition

The addition of relations is defined as addition over the corresponding strings of bits.

\[
\varphi_{\text{carry}}(X^{a,r}, Y^{a,r}, Z^{a,r-1}) = \text{def } \exists \exists \mathcal{T}^{a,r-1} \cdot (\mathcal{T} <_{a,r-1} \mathcal{T} \land X(\mathcal{T}) \land Y(\mathcal{T}) \land \forall \mathcal{U}^{a,r-1} \cdot ((\mathcal{T} <_{a,r-1} \mathcal{U} <_{a,r-1} \mathcal{T}) \Rightarrow (X(\mathcal{U}) \lor Y(\mathcal{U})))) \quad (9)
\]

\[
\text{plus}^{a,r}(X^{a,r}, Y^{a,r}, Z^{a,r}) = \text{def } \forall \mathcal{T}^{a,r-1} \cdot (Z(\mathcal{T}) \leftrightarrow X(\mathcal{T}) \oplus Y(\mathcal{T}) \oplus \varphi_{\text{carry}}(X, Y, Z)) \quad (10)
\]

Here \( A \oplus B \) is syntactic sugar for \( A \iff \neg B \), and \( \varphi_{\text{carry}}(X^{a,r}, Y^{a,r}, Z^{a,r-1}) \) is true if there is a carry propagated in position \( Z \) in the addition of \( X \) and \( Y \).

4.1.5 Addition + Multiplication

In first-order, it is well-known that addition + multiplication \( \equiv \text{bit} \), and the proof does not specify that the predicate must be over a first-order object, so the very same proof works for higher order logic. Hence, addition + multiplication over first-order elements is equivalent to the bit predicate over first-order elements, which extends over higher-order relations as seen in subsubsection 4.1.3 which is then equivalent to addition + multiplication over higher order relations.

4.2 Addition over tuples

We will also need to add tuples of elements, and in this subsection we will show how to do it. Let us define \( p = T(r, a) = \text{texp}_{r-2}^{r}(n^a, a) \).
Overflow: We will define \( \text{plus}_{a,r} \) over \( a \)-tuples of relations of arity \( a \) and order \( r \). First, let \( C^{a,r}(\mathcal{X}^{a,r}, \mathcal{Y}^{a,r}) \) be a predicate indicating that the addition of \( \mathcal{X} \) and \( \mathcal{Y} \) overflows (\( \mathcal{X} + \mathcal{Y} \geq p \)).

\[
C^{a,r}(\mathcal{X}^{a,r}, \mathcal{Y}^{a,r}) = \text{def } \neg \exists Z^{a,r}. \text{plus}^{a,r}(\mathcal{X}, \mathcal{Y}, Z)
\]

This just means that there is no value \( Z \) such that \( \mathcal{X} + \mathcal{Y} = Z \), so \( Z \geq \text{exp}_2^{r-2}(n^a, a) \).

Addition modulo \( p \) Now we also need to speak of addition modulo \( p \), but using only numbers strictly smaller than \( p \). If the addition does not overflow, it suffices to test the addition. If it overflows, we can existentially quantify \( d, e, f, g \) and \( h \) such that:

\[
\begin{align*}
d + \mathcal{X} &= p - 1 & d &= p - 1 - \mathcal{X} \\
d + 1 &= e & e &= p - \mathcal{X} \\
f + \mathcal{Y} &= p - 1 & f &= p - 1 - \mathcal{Y} \\
e + g &= h & h &= 2p - \mathcal{X} - \mathcal{Y} - 1 \\
i + h &= p - 1 & i &= p - 1 - (2p - \mathcal{X} - \mathcal{Y} - 1) = \mathcal{X} + \mathcal{Y} - p
\end{align*}
\]

We can then see that if \( \mathcal{X} + \mathcal{Y} \geq p \) then each variable has exactly one possible value which is less than \( p \). It is trivial for \( d \) and \( f \), and for \( e \) it is enough to see that, since \( \mathcal{X} + \mathcal{Y} \geq p \) and \( \mathcal{X}, \mathcal{Y} < p \) then \( \mathcal{X}, \mathcal{Y} > 0 \), so \( p - \{\mathcal{X}, \mathcal{Y}\} < p \); \( g = 2p - \mathcal{X} - \mathcal{Y} - 1 \leq 2p - p - 1 = p - 1 \) since \( \mathcal{X} + \mathcal{Y} \geq p \), and a fortiori \( h = g + 1 < p - 1 + 1 = p \).

The exact equation of plus modulo (\( \text{plus}_m \)) is then:

\[
\text{plus}_m^{a,r}(\mathcal{X}^{a,r}, \mathcal{Y}^{a,r}, Z^{a,r}) = \text{def } \mathcal{X} + \mathcal{Y} = Z \lor \\
\exists d, e, f, g. d + \mathcal{X} = (p - 1) \land d + 1 = e \land f + \mathcal{Y} = (p - 1) \land e + f = g \land Z + g = (p - 1)
\]

Addition of tuples: We can consider an \( a \)-tuple of numbers as a number of length \( a \) in base \( p \), so addition extends naturally on it. Let us write \( \text{plus}_{a,r} \) for the addition of \( a \)-tuples of \( a \)-ary relations of order \( r \). The idea is the same as the addition of string of bits, with the difference that propagating overflows can be done in different ways. The creation of an overflow at position \( j \) happens only if \( C^{a,r}(\mathcal{X}_j, \mathcal{Y}_j) \) overflows, and then it propagates at position \( k \) if \( X_k + Y_k \geq p - 1 \). But since, if \( X_k + Y_k > p - 1 \) then we have \( C^{a,r}(\mathcal{X}_k, \mathcal{Y}_k) \), we can consider that the overflow was created at position \( k \). Hence we consider that the only way for a overflowing bit to propagate itself is when \( X_k + Y_k = p - 1 \):

\[
\text{plus}_{a,r}(\mathcal{X}, \mathcal{Y}, Z) = \text{def } \bigwedge_{1 \leq i \leq a} \big( \bigvee_{1 \leq j < i} C(\mathcal{X}_j, \mathcal{Y}_j) \big) \bigwedge_{j < k < i} \text{plus}_m^{a,r}(\mathcal{X}_j, \mathcal{Y}_j, p - 1)), \text{then } \exists T^{a,r}. (\text{plus}_m^{a,r}(\mathcal{X}_i, \mathcal{Y}_i, T) \land \text{plus}_m^{a,r}(T, 1, Z_i)) \text{ else(plus}_m^{a,r}(\mathcal{X}_i, \mathcal{Y}_i, Z_i)\)(11)
\]
5 Relations between High-Order queries and complexity classes

As stated in Section 2, we have decided to accept high-order vocabularies. For the logic of order \( r \) we accept formulae with quantifiers of order up to \( r \), but the vocabularies can contain relations of any order. We may usually assume that the order of the vocabulary is at most \( r + 1 \), which is coherent with FO which contains second order relations as its input. This is because, a relation of order \( r + 2 \), can only be used with relation of order \( r + 1 \) which could not be quantified, hence those relations are in the structure, and those relations could be replaced by their truth value without loss of generality.

5.1 High Order and Bounded Alternating Time

Theorem 5.1. For \( j > 0 \), \( \Sigma_j^{r,c} = \Sigma_j \text{TIME}(\exp_2^{r-2}(n^{O(1)})) \) for \( c \leq r + 1 \) with a reasonable input, as defined in Section 3.3.

This theorem is true for \( c = 2 \) since [8, 11] proved that \( \Sigma_j^{r,2} = \text{NTIME}(\exp_2^{j-2} \Sigma_{j-1}^{p}) \). They did not write the “2” since in their definitions every formulae are over structures of order 2.

We will then prove the theorem directly for queries over high-order structures.

Proof. Since \( \Sigma_j^{r,2} \subset \Sigma_j^{r,f} \), then \( \Sigma_j^{r,2} \) is at least as expressive as the definition of [8], so we have this side for free: \( \Sigma_j^{r,f} \supseteq \Sigma_j \text{TIME}(\exp_2^{r-2}(n^{O(1)})) \).

We now want to prove \( \subset \); let \( \varphi \) be a query in \( \Sigma_j^r \), so then \( \varphi = \exists \overline{x}_1 \forall \overline{x}_2 \ldots Q \overline{x}_j \psi \) where \( \psi \in \text{HO}^{r-1} \). We can begin by existentially guessing \( \overline{x}_1 \), which asks us to write \( O(\log(N(r))) = \exp_2^{r-2}(n^{O(1)}) \) bits for each variable of \( \overline{x}_1 \). Then we can universally choose a value for \( \overline{x}_2 \), and so on. This takes time and space \( O(\exp_2^{r-2}(n^{O(1)})) \) and \( j - 1 \) alternations.

Now everything we will do will use deterministic time. There are a finite number of variables, let us say \( v \) variables, of order up to \( r - 1 \). Hence each variables can take at most \( N(r - 1) \) values, and there are then \( N(r - 1)^k = \exp_2^{r-2}(n^{O(1)})^k = 2^{\exp_2^{r-3}(n^{O(1)}) \times k} = \exp_2^{r-2}(n^{O(1)})^k \) possible values for the \( k \) variables. Writing one of the possible values of those \( v \) variables on the tape will take \( kB(r - 1) = \exp_2^{r-3}(n^{O(1)}) \), so writing all of the possible tuples will take \( \exp_2^{r-3}(n^{O(1)})^k, \exp_2^{r-2}(n^{O(1)})^k = \exp_2^{r-2}(n^{O(1)})^k \) deterministic time and space.

Finally we want to check the quantifier-free part of the formula, and it is clear that every relation, either quantified relations or relations of the structure of order up to \( r \), can be checked in time at most \( \exp_2^{r-2}(n^{O(1)})^k \) thanks to the “acceptable encoding” assumption. We will check those formulæ at most \( \exp_2^{r-2}(n^{O(1)})^k \) times, so we will spend at most \( \exp_2^{r-2}(n^{O(1)})^{k}, \exp_2^{r-2}(n^{O(1)})^{k} = \exp_2^{r-2}(n^{O(1)})^{k} \)
times checking the quantifier-free part. If we use relations of order \( r + 1 \), to check \( R^{a,r+1}(S^{a,r}) \) we need to use random-access, to check if the \( S \) bit of \( R \) is 1 or not.

When we consider the total time, we see that it is indeed in \( \exp^2(n^{O(1)})^k \), and we used \( j - 1 \) alternations, so the theorem is true.

Taking the union of every classes considered in Theorem 5.1 we have the following corollary:

**Corollary 5.2.** Over any structure, we have \( \text{ELEMENTARY} = \text{HO} \).

### 5.2 Operators on HO

In this section, we will prove that the properties we obtain while adding operators to first and second order logic, relating those logics to space complexity and deterministic time complexity, extend naturally over HO.

In the paper [2], where the nondeterministic and alternating fixed points are introduced, a characterization of the expressivity of first order logic with operators was given in term of “relational machines”. The reason is the Turing machine model implies an order over the input, which is avoided by the relational machines, so that they are better simulations of general first order formulae. Since in second order we can quantify an order over the universe, and this order then extends over high order relations there is no loss of generality in working with Turing machines.

As explained in Subsubsection 2.2.3, there is a normal form for the formulae with operators. Every formula can be assumed to be either like \((\text{TC}_{\pi,Y,\varphi}(0,\max))\) or like \((F_{\varphi})(0)\), where \( F \) is a fixed point operator and \( \varphi \) a formula in HO. So in this subsection we are always going to assume that the formulae are in this form.

The table 2 summarizes the maximum number of steps an operator can make without looping, and the number of bits of information accessible at each state. There is no information about non deterministic and alternating computation since it does not change those numbers.

### 5.2.1 Inflationary fixed point and alternating partial fixed point

It is already known that \( P = \text{FO}(\text{IFP}) \) over ordered structures, and similarly \( \text{EXP} = \text{SO}(\text{IFP}) \). In [2] it was proved that \( \text{FO}(\text{NIFP}) \) is NP over first order with

| \( \text{HO}^*(P) \) | Maximal number of step \( P \) | Number of bits |
|-----------------|-----------------------------|----------------|
| \( P = \text{TC} \) | \( T(r) = \exp^{-1}(n^{O(1)}) \) | \( B(r) = \exp^{-2}(n^{O(1)}) \) |
| IFP             | \( C(r + 1) = \exp^{-1}(n^{O(1)}) \) | \( B(r + 1) = \exp^{-1}(n^{O(1)}) \) |
| PFP             | \( N(r + 1) = \exp^{r}(n^{O(1)}) \) | \( B(r + 1) = \exp^{-1}(n^{O(1)}) \) |
an order relation. They are special cases of the theorem:

**Theorem 5.3.** Over reasonable input we have $\text{ASPACE}(\exp r^{-1}(n^{O(1)})) = \text{HO}^{r,j}(\text{APFP}) = \text{HO}^{r+1}(\text{IFP}) = \text{DTIME}(\exp 2^{(n^{O(1)})})$.

The article [6] proved $\text{HO}^{r+1}(\text{IFP}) = \text{DTIME}(\exp 2^{(n^{O(1)})})$ assuming an order over the structure and a vocabulary of order 2. Our proof is similar, but we begin by constructing order and arithmetic relations thanks to second-order relation.

**Proof.** It has been proven in [4] that when $f$ is a function greater than the logarithm, $\text{ASPACE}(f) = \text{DTIME}(2^{O(f)})$ hence $\text{ASPACE}(\exp 2^{-1}(n^{O(1)})) = \text{DTIME}(\exp 2^{(n^{O(1)})})$.

**Proof of** $\text{HO}^{r+1}(\text{IFP}) \subseteq \text{DTIME}(\exp 2^{(n^{O(1)})})$. Let $\varphi \in \text{HO}^{r+1}(\text{IFP})$, such that $\varphi = (\text{IFP}_{\pi, P, \psi})(\pi)$. Suppose that $\pi = x_1, \ldots, x_n$, then there are $T(r + 1) = \exp 2^{r+1-1}(n^{O(1)})$ sets of tuples of relations equivalent to $\pi$, hence we find the fixed point after at most $\exp 2^{r+1-1}(n^{O(1)})$ steps. Since $\psi \in \text{HO}^{r} + 1$ we know that $\psi \in \Sigma_j \text{TIME}(\exp 2^{j-2}(n^{O(1)}))^P$ for some $j$ where “$P$” is an oracle in $P$. This class is a subset of $\text{DTIME}(\exp 2^{r+1-1}(n^{O(1)}))^P$, and since there are at most $\exp 2^{r+1-1}(n^{O(1)})$ elements in $P$ it can still be coded with a string of bits, and then checked in time $\exp 2^{r+1-1}(n^{O(1)})$. Since in time $\exp 2^{r+1-1}(n^{O(1)})$ there are at most $\exp 2^{r+1-1}(n^{O(1)})$ queries to the oracle, then checking $\psi$ takes time $\exp 2^{r+1-1}(n^{O(1)})^2 = \exp 2^{r+1-1}(n^{O(1)})$.

During the $i$th step we will check for every tuple of relation $\pi$ if $z \in P_i$, applying $\psi$ with input $P_{i-1}$. Since there are up to $N(r + 1) = \exp 2^{r+1-1}(n^{O(1)})$ possible relations, each step will take time $\exp 2^{r+1-1}(n^{O(1)}) \times \exp 2^{r+1-1}(n^{O(1)}) = \exp 2^{r+1-1}(n^{O(1)})$. Finally, since there are at most $\exp 2^{r+1-1}(n^{O(1)})$ steps, the entire computation will take time $\exp 2^{r+1-1}(n^{O(1)}) \times \exp 2^{r+1-1}(n^{O(1)}) = \exp 2^{r+1-1}(n^{O(1)})$, which ends this side of the proof.

**Proof of** $\text{HO}^{r,j}(\text{APFP}) \subseteq \text{HO}^{r+1,j}(\text{IFP})$. Let $\xi \in \text{HO}^{r}(\text{APFP}), \xi = (\text{APFP}_{\pi, P, \varphi, \psi})(\pi)$. We are going to use an inflationary fixed point to create the tree $T = T_{\varphi, \psi}$. We will associate the label of every node to its path in $T$.

Since there is at most $B(r + 1) = \exp 2^{r+1}(n^{O(1)})$ values that $P$ can take then there is at most $2\exp 2^{-1}(n^{O(1)}) = \exp 2^{2}(n^{O(1)})$ paths of such length. But it is correct since $Q$ can also take $C(r + 2) = \exp 2^{2}(n^{O(1)})$ values.

Since by Claim 2.21 the tree $T$ can be cut once we met twice the same relation in a branch, and that there is at most $\exp 2^{r+1}(n^{O(1)})$ relations, we can cut the tree at depth $\exp 2^{r+1}(n^{O(1)})$, hence using a simple fixed point is not a problem.

Then with a second fixed point, we recursively calculate the output of the circuit. We consider the leaves that are not a fixed point to be the relation $\perp$, the leaves which are fixed points we consider the relation in their label. Then we do union and intersection of the gates when we know their children’s value.
Proof of $\text{ASPACE}(\exp^{r-1}_2(n^{O(1)})) \subseteq \text{HO}^r(\text{APFP})$ The proof for $r = 1$ was given in [2]. The same proof works for $r > 1$, except that we can construct an arbitrary order as explained above.

Once again, accepting that the input contains high order relations does not change the expressivity, if we consider acceptable input, and that the input size is the size of the structure and not the size of the description. And since we have time $\exp^{r-1}_2(n^{O(1)})$ and not $\exp^{r-2}_2(n^{O(1)})$, we can even check element of relation of order $r + 1$.

5.2.2 (Non)deterministic partial fixed point, Transitive closure, Alternating inflationary fixed point and Space complexity

It is already known that $\text{FO}(\text{AIFP}) = \text{FO}(\text{NPFP}) = \text{FO}(\text{PFP}) = \text{SO}(\text{TC}) = \text{PSPACE}$ over ordered structures. These equality are special cases of Theorem 5.4:

**Theorem 5.4.** Over reasonable input we have $\text{HO}^r(\text{AIFP}) = \text{HO}^r(\text{NPFP}) = \text{HO}^r(\text{PFP}) = \text{HO}^{r+1}(\text{TC}) = \text{SPACE}(\exp^{r-1}_2(n^{O(1)}))$.

We are going to transform formulae from one formalism to another one without going through machines, giving a pattern of algorithms for the transformation. There will be an exception for $\text{HO}^r(\text{NPFP})$ that we only know how to transform into a space bounded TM, the equality using Savitch’s theorem [16].

The result for AIFP is not a surprise if we consider that IFP is time and A is alternations, so that this theorem is similar to $\text{ATIME}(f) = \text{SPACE}(f)$.

**Proof.** Proof of $\text{HO}^r(\text{AIFP}) \subseteq \text{HO}^r(\text{PFP})$: Let $\xi \in \text{HO}^r(\text{AIFP}), \xi \in (\text{AIFP}_{p_r,\xi} \phi,\psi)(\bar{\mathcal{V}})$. Then the fixed point can be obtained with at most $C(r) = \exp^{r-2}_2(n^{O(1)})$ iterations since it is inflationary, and there is at most $2^{C(r)} = \exp^{r-1}_2(n^{O(1)})$ paths.

We are going to transform $\xi$ in an $\text{HO}^r(\text{PFP})$ formula. In PFP we can do $T(r) = \exp^{r-1}_2(n^{O(1)})$ steps, which is enough to test every path. We will make a relation $Q$ which has 3 arguments. The first one is a path $p$ in the tree $T_{\varphi,\psi}$, i.e. a string of bits such that the $i^{th}$ bit is 0 if the $i^{th}$ step in AIFP was $\varphi$ else 1. When the second argument is 0 then the third argument is the relation $P_p$, else if the second argument is 1 then the third argument is 0 to mean that the relation $P_p$ was defined.

For first step, we let $Q(0, 0, 0) = 0$ and $Q(0, 1, 0) = 0$. During the next step if $Q(C/2, 1, 0)$ is true then we set $Q(C, 1, 0)$ and $Q(C, 0, \varphi(P_{C/2}))$ to be true. Finally we end the computation when for every $C$, $Q(C, 1, 0)$ is true, then we have the $\exp^{r-2}_2(n^{O(1)})$ level of the tree $T_{\varphi,\psi}$, and every relation $P_p$ can be checked in
$Q(p, 0, .)$. We check if there is one of those relations that is a fixed point, and that contains $\mathcal{Y}$. If yes, we accept $Q(2, 0, 0)$, else $Q(2, 0, 1)$. We can not miss a fixed point; since it is inflationary, we see it at or before step $\exp^{r-2}_2(nO(1))$, and if we discovered the fixed point sooner, it is not a problem if we continue to apply $\varphi$ or $\psi$. (We still have got the fixed point, by the very definition of fixed points.) If there is $Q(2, 0, b)$ with $b \in \{0, 1\}$ which is true, then we accept only $Q(2, 0, b)$ so we indeed have got a fixed point, and we accept only if $(2, 0, 0)$ is in the output of this PFP. This ends the proof.

**Proof of $\text{HO}^r(\text{PFP}) \subseteq \text{HO}^{r+1}(\text{TC})$:** Let $\varphi \in \text{HO}^{r+1}(\text{PFP})$. By the normal form property we can assume that $\varphi = (\text{PFP}_{pr+1, \overrightarrow{X}}' \psi(P, \overrightarrow{X}))(\overrightarrow{y})$. We also assume that $\overrightarrow{X}$ contains only $r$th order variable and $\psi$ is in $\text{HO}^{r+1,r+1}$. Then

$$\varphi' = \text{def } (\text{TC}_{x,pr+1,y,\overrightarrow{Q}r+1}\psi')(0, \overrightarrow{0}, 1, \max)$$

(12)

$$\psi' = \text{def } x = 0 \wedge (\text{if } P = \psi(P, .) \text{ then } (\text{if } P(0) \text{ then } y = 1 \wedge Q = \max))$$

else $\bot$ else $(y = 0 \wedge Q = \psi(P, .))$) (13)

Here $P = \psi(P, .)$ is syntactic sugar for $\forall \overrightarrow{Z} \varphi(P, \overrightarrow{Z})$.

**Proof of $\text{HO}^{r+1}(\text{TC}) \subseteq \text{HO}^r(\text{AIFP})$** Let $\xi \in \text{HO}^{r+1}(\text{TC})$. We suppose that $\xi$ is in normal form, hence $\xi = (\text{TC}_{\overrightarrow{A},pr+1}\psi)(0, \max)$ with $\psi \in \text{HO}^{r+1}$. Let us say that $P_0 = 0$ and $P'_0 = \max$, so that the transitive path from $P_0$ to $P'_0$ can take up to $T(r+1) = \exp^{r}_2(nO(1))$ steps. We are of course going to do a divide and conquer method, existentially guessing the middle $P''_0$ of the path, and universally checking both sides, that there is both a path from $P_1 = P_0$ to $P'_1 = P''_0$, and from $P_1 = P''_0$ to $P'_1 = P'_0$, and so on. Hence we need to make at most $\log(T(r+1)) = \exp^{r-1}_2(nO(1))$ guesses. For each choice there are $T(r+1) = \exp^{r}_2(nO(1))$ possible choices. In AIFP we can only choose one element of two ($\varphi$ or $\psi$) so we will need to guess the relation in the middle of the path bit by bit, so it will take $\log(T(r+1)) = \exp^{r-1}_2(nO(1))$ guesses of bit; we use a counter to find when we have guessed every bit, while there are bits to guess the universal choice does not do anything. In total this makes $\log^{2}(T(r+1)) = \exp^{r-1}_2(nO(1))$ existential guesses and $\log(T(r+1)) = \exp^{r-1}_2(nO(1))$ universal ones. This is possible in $\text{HO}^{r}(\text{AIFP})$.

Finally we existentially guess when the path is one step long, then we just check that indeed $\psi(P, P')$ is true.

**Proof of $\text{HO}^r(\text{PFP}) \subseteq \text{HO}^r(\text{NPFP})$:** This is trivial, it suffices to transform a formula of $\text{HO}^r(\text{PFP})$ into normal form, so that no negation are applied to the operator, and then transform $\text{PFP}_\varphi$ to $\text{NPFP}_\varphi$. 31
Theorem 5.5. \[ \text{This is a special case of Theorem 5.5} \]

Proof of \( \text{HO}^r(\text{NPFP}) \subseteq \text{SPACE}(\exp_2^{r-1}(n^{O(1)})) \): Let \( \varphi \in \text{HO}^r(\text{NPFP}) \), such that \( \varphi = (\text{NPFP}_P \psi, r \psi, \xi)(\overline{x}) \). We are going to give an algorithm in \( \text{NSPACE}(\exp_2^{r-1}(n^{O(1)})) \), which can be simulated in \( \text{SPACE}(\exp_2^{r-1}(n^{O(1)})) \) by Savitch Theorem.

Suppose that \( \overline{x} = x_1, \ldots, x_n \). Then there are \( T(r) = \exp_2^{r-1}(n^{O(1)}) \) sets of tuples of type \( t \), and hence writing a value of \( P_i \) takes \( \exp_2^{r-1}(n^{O(1)}) \) bits. We begin by writing \( P_0 = \top \), and we loop so that when we know \( P_i \) we guess if it is a fixed point, then we look if \( \psi(P_i) = \xi(P_i) = P_i \) and if \( P_i(\overline{j}) \); if yes we accept, else we reject. Else we guess if we need to apply \( \varphi \) or \( \psi \) to obtain \( P_{i+1} \), where the \( j \)th bit is 1 if the \( j \)th relation equivalent to \( \overline{x} \) is true. We can then loop over every possible relation of type \( t \) to see if it is in \( P_{i+1} \), enumerating these relations take space \( T(r) = \exp_2^{r-2}(n^{O(1)}) \), and it is already known that testing \( \psi \in \text{HO}^r \) is in \( \text{ATIME}(\exp_2^{r-2}(n^{O(1)})) \subseteq \text{PSPACE}(\exp_2^{r-1}(n^{O(1)})) \). Once \( P_{i+1} \) is known we can forget \( P_i \), so there is no need of more space.

Proof of \( \text{SPACE}(\exp_2^{r-1}(n^{O(1)})) \subseteq \text{HO}^r(\text{PFP}) \): As we already know, we can encode a configuration of a TM in \( \text{DTIME}(\exp_2^{r-1}(n^{O(1)})) \) and using space \( \exp_2^{r-1}(n^{O(1)}) \) using relations of order \( r+1 \); of course our relation will be \( P \). We now only need to be able to decide if one configuration is the successor of another one.

If we encode a configuration as a string of bits, 00 for 0, 01 for 1, and 1x for the head of the Turing machine in state \( x \), then to decide the value of the bit \( i \) at time \( t+1 \), we only need to look at up to \( \log |x| + 5 \) bits on the left and on the right of a bit at time \( t \). Since we have a “succ” relations over high-order relation we can easily do it in \( \text{HO}^r \) (because \( \log |x| \) is a constant for a given TM). This let us speak of the next step of the Turing machine.

We can assume without loss of generality that there is only one accepting configuration, with empty tape, and that the Turing machine loops on this configuration. Then the formula will check if the description of this configuration is accessible.

Once again, accepting that the input contains high order relations does not change the expressivity, if we consider only acceptable input, and that the input size is the size of the structure and not the size of the description. And since we have space \( \exp_2^{r-1}(n^{O(1)}) \) and not \( \exp_2^{r-2}(n^{O(1)}) \), we can even check elements of relations of order \( r+1 \).

5.2.3 Nondeterministic inflationary fixed point

In [2] it was proved that \( \text{FO}(\text{NIFP}) \) is NP over first order with an order relation. This is a special case of Theorem 5.5.

Theorem 5.5. Over reasonable input we have \( \text{HO}^{r,j}(\text{NIFP}) = \Sigma_1^{r+1,j} \).
Proof. \(\subseteq\): Let \(\xi = (\text{NIFP}_{p+1}X \varphi, \psi)(\overline{Y})\) with \(\varphi, \psi \in \text{HO}'\). Then we will existentially guess a relation \(Q\) whose type is a pair of \(\overline{X}\)'s type. The first half of the arguments is a time-stamp, such that \(Q(\overline{C}, .)\) is the relation \(PC\) where \(\overline{C}\) is considered as a number.

Since \(\xi\) is an inflationary point, it can take at most \(C(r + 1, a) = \exp^{r-1}_2(n^{O(1)})\) iterations; since the counter, which consists of variables of order \(r\), can count up to \(T(r) = \exp^{r-1}_2(n^{O(1)})\) we can indeed encode every steps in one relation.

We then just need to check if \(\overline{Y}\) is in \(Q(\overline{C}, .)\) for some \(\overline{C}\) such that \(Q(\overline{C}, .)\) is a fixed point for both \(\varphi\) and \(\psi\).

\[
\xi' = \exists Q^{1L}r.\{(\neg \exists \overline{X}.Q(\overline{X}), \overline{\overline{X}})) \land (\exists \overline{C}.Q(\overline{C}, \overline{X}) \land \forall \overline{X} \varphi(Q(\overline{C}, \overline{X})) \iff Q(\overline{C}, \overline{X}) \iff \psi(Q(\overline{C}, \overline{X}))) \land \\
\forall \overline{X}.Q(\overline{C}, \overline{X}) \iff (Q(\overline{T − 1}, \overline{X}) \lor \varphi[P/Q(\overline{T − 1}, .)][\overline{X}] \lor \psi[P/Q(\overline{T − 1}, .)][\overline{X}])\} \tag{14}
\]

Proof of \(\supseteq\): Let \(\varphi = \exists \overline{X}^{r+1}.\psi\) with \(\psi \in \text{HO}'\). We will nondeterministically guess every bit of \(Q\); there are \(C(r + 1) = \exp^{r-1}_2(n^{O(1)})\) such bits and we can do \(T(r) = \exp^{r+1}_2(n^{O(1)})\) steps in an inflationary fixed point.

We will create a relation \(P\) that takes three arguments. The second one is a time-stamp \(\overline{C}\). If the first argument is 0 then the last argument is the string of bits that we are constructing. Else if the first argument is 1 then the third argument is 0; this means that the string of bits at time \(\overline{C}\) was already defined.

When \(\overline{C} = 0\) we must have \(\overline{X} = 0\), and when \(\overline{C} > 0\), if \(\overline{C} − 1\) is defined and \(\overline{C}\) is not, then the values of \(\overline{X}\) is either multiplied by 2, in \(\psi'\), or by 2 and incremented by \(\xi'\). Finally, when the string of bits is completed, we check if \(\psi\) is true when \(Q(\overline{X})\) is replaced by \(P(0, \max, \overline{X})\).

If it is true, we accept the arguments \((2, 0, 0)\), else nothing. Since nothing else changes, this is a fixed point, and \(\varphi'\) will be true if and only if \(\psi\) is verified by this string of bits.

\[
\varphi' =_{\text{def}} \text{NIFP}_{p, b, c, X}^{1L} (b', \overline{X}, \overline{\overline{X}}) \tag{15}
\]

\[
\psi' =_{\text{def}} b = 1 \land \overline{C} = \overline{X} = 0 \lor \text{if } \exists \overline{X}'.P(0, \max, X')
\]

\[
\text{then(if } \psi[P/P(0, \max, .)] \text{ then}(b = 2 \land \overline{X} = \overline{0} \land C = \overline{0}) \text{ else } \bot)
\]

\[
\text{else}(P(1, \overline{C} − 1, 0) \land \neg P(1, \overline{C}, 0) \land)
\]

\[
((b = 0 \land P(0, C, .) = 2P(0, C − 1, .)) \lor (b = 1 \land \overline{X} = 0)) \tag{16}
\]

\[
\xi' =_{\text{def}} P(1, \overline{C} − 1, 0) \land \neg P(1, \overline{C}, 0) \land
\]

\[
((b = 0 \land P(0, C, .) = 2P(0, C − 1, .) + 1) \lor (b = 1 \land \overline{X} = 0)) \tag{17}
\]

We think that this is an equality (at least for \(r = 1\) it is one), but the other side of the relation seems harder to prove.
5.3 Horn and Krom formulae

Another important result in descriptive complexity theory is that $P = SO(\text{HORN})$ and $NL = SO(\text{KROM})$. We will discuss the problem of extending these results to higher-order.

Definition 5.6 (Horn and Krom formula). A literal is an atomic predicate or its negation, the first one is called a positive literal and the last one a negative literal. A disjunction of literals is a clause, and a conjunction of clauses is a quantifier free formula in conjunctive normal form (CNF). A CNF formula is then a formula $\varphi = Q_1 x_1 \lor \ldots \lor Q_n x_n \forall \psi$, where the $Q$ are quantifiers and $\psi$ is a quantifier-free CNF formula.

A Horn formula is a CNF formula such that in each clause there is exactly one positive quantified literal. A Krom formula is a CNF formula such that in each clause there are at most two literals.

Over second order, the proof of the equality begins by proving that those classes have a normal form where every second order quantifier in existential. Over higher order, it is not easy to see what this normal form would be. For example in $HO^3$ we can not require the second order quantifiers to both be universal and all be existential. And if we accept the first order to be also existential then problems like “clique”, which are known to be NP-complete, can be coded in $SO(\text{HORN})$, so finding the good restriction over quantifiers is mandatory to have an interesting result.

5.4 Monadic High-Order Logic (MHO)

Monadic Second Order MSO is a well-studied logic, we intend to study the monadic restriction of logic of order at least 3, as we will see the theory is really different.

Definition 5.7. The set of monadic relations of order $r \geq 1$ is the $(r - 1)$th power set of the universe, $P^{r-1}(A)$; where we define $P^0(E) = E$ and $P^r(A) = P(P^{r-1}(E))$ and $P$ is the usual power set operation.

The Monadic High-Order Logic of order $r$ (MHO$^r$) is defined as the subset of queries of HO$^r$ where all quantified relations are monadic. The definitions of MHO$^{r,j}$, M$\Sigma^r_j$ and M$\Sigma^r_{j,f}$ are straightforward extensions of the HO and $\Sigma$ definitions.

The definition only restricts the arity of quantified relation, and so the vocabulary of a formula may contain many-ary relations.

It is well known that one of the main problem with MSO is that one can not create an order over the structure. But in Monadic Third Order one can quantify a set of the form $\{[0, i]|0 \leq i < n\}$ and use this as a linear order over the structure.
This let us create addition with the set \( \{\{a, b, c\}|a+b = c\} \) and multiplication with the set \( \{\{a, b, c\}|a \times b = c\} \), hence we can define a “bit” predicate and simulate Turing Machine.

It is important to realize that Theorem 5.1 assumed that we can increase the arity to obtain more space and time. Since we can not do it anymore we see that the big \( O \) is not anymore in the top of the tower of exponential, but in the second floor. Hence we obtain similarly Theorem 5.8

**Theorem 5.8.** \( M^{\Sigma_r^c} = \Sigma_j^c \text{TIME}(2^{O(\exp^r_2(n^{(j)}))}) \) for \( c \leq r + 1 \) with a reasonable input.

### 6 Conditional relations among the classes

In this section, we will discuss theorems of the form “If \( A = B \) then \( C = D \)” where \( A, B, C \) and \( D \) are complexity classes or theories over finite models. Most results use a padding argument or are corollaries of theorems known on lower complexity classes. What will be more interesting is to study the results that seems intuitive but that we do not know how to prove.

There are conjectures in high complexity classes which seem to be copy of theorem over polynomial classes, we will explain why the known proof for polynomial classes fails on higher classes.

We are going to work mostly with Turing Machine, and we will also translate the results are descriptive complexity's theorem or question.

We also should emphasize the fact that when we do not explicitly state any assumptions over the function classes, then they could contains only one function, hence we also obtain theorem over complexity time bounded by a function.

#### 6.1 The \( r \)th exponential hierarchy

It is known that \( \text{SO} = \text{PH} \), the polynomial time hierarchy, and \( \text{SO}_j = \Sigma_j^2 = \Sigma_j^p \) is the \( j \)th level of the polynomial hierarchy. We are going to extend this hierarchy to higher order.

**Definition 6.1** (\( r \)th exponential hierarchy). Let \( \text{HO}_{r+2} \) be the \( r \)th exponential hierarchy, and \( \Sigma_j^{r+2} \) be the \( j \)th level of the \( r \)th exponential hierarchy.

We choose the name such that the (alternating) time of \( r \)th exponential hierarchy has \( r \) exponential under the \( n \). We have the polynomial hierarchy as the 0th exponential hierarchy. Our definition is different from the “Exponential hierarchy” of [15] in that his hierarchy is \( \bigcup_{i \in \mathbb{N}} \text{TIME}(\exp_2^i(n^{(1)})) \), and in each of our levels we also consider alternations.
Definition 6.2 (Collapsing). For a class of function \( C \) we say that \( C \) collapses to the \( j \)th level if \( \forall k \geq j, \Sigma_j \text{TIME}(C) = \Sigma_k \text{TIME}(C) \). By extension we say that \( \text{HO}^r \) (resp. \( \text{HO}^{r,j} \)) collapses to the \( j \)th level if for all \( k \geq j \) \( \Sigma_j = \Sigma_k \) (resp. \( k \geq j \) \( \Sigma_j^{r,f} = \Sigma_k^{r,f} \)).

6.2 General classes of functions

Lemma 6.3. Let \( C \) be a class of function, and \( j \geq 0 \), if \( \Sigma_j \text{TIME}(C) = \Sigma_{j+1} \text{TIME}(C) \) then \( \Sigma_j \text{TIME}(C) = \Pi_j \text{TIME}(C) = \Sigma_{j+1} \text{TIME}(C) = \Pi_{j+1} \text{TIME}(C) \).

Proof. The proof is almost identical to the one of the polynomial hierarchy, which is the special case \( C = n^{O(1)} \). If \( \Sigma_j \text{TIME}(C) = \Sigma_{j+1} \text{TIME}(C) \) then their complement are also equals, so we have \( \Pi_j \text{TIME}(C) = \Pi_{j+1} \text{TIME}(C) \) hence \( \Pi_j \text{TIME}(C) \subseteq \Sigma_{j+1} \text{TIME}(C) = \Sigma_j \text{TIME}(C) \subseteq \Pi_{j+1} \text{TIME}(C) = \Pi_j \text{TIME}(C) \).

Theorem 6.4. Let \( F \) and \( G \) be classes of functions such that for all \( f \in F \) there exists a function \( h_f \) computable in time \( f \) (resp. space \( f \), resp. space \( \log f \)) and \( g_f \in G \) such that \( f(n) = O(g_f(h_f(n) + n)) \) and for all \( g' \in G \) there exists \( f' \in F \) such that \( g'(h_f(n) + n) = O(f') \). Let \( 0 \leq j < k \) and assume that \( \Sigma_j \text{TIME}(G) = \Sigma_k \text{TIME}(G) \) then \( \Sigma_j \text{TIME}(F) = \Sigma_k \text{TIME}(F) \) (resp. assume that \( \Sigma_j \text{TIME}(G) = \text{SPACE}(G,k) \) then \( \Sigma_j \text{TIME}(F) = \text{SPACE}(F) \), resp. assume that \( \text{SPACE}(\log(G)) = \text{TIME}(G) \) then \( \text{SPACE}(\log(F)) = \text{TIME}(F) \)).

Proof. Let \( f \in F \) and \( L \) a language decided by a TM \( M \in \Sigma_k \text{TIME}(f) \) (resp. \( \text{SPACE}(f) \), resp \( \text{TIME}(f) \)) and let \( L' = \{ x1^{h_f(|x|)} | x \in L \} \). It can be decided by a TM \( M' \in \Sigma_k \text{TIME}(g_f(n)) \) (resp. \( \text{SPACE}(g_f) \), resp \( \text{TIME}(g_f) \)) which tests whether there is a correct number of 1 and then simulates \( M \) (it is possible in our bound since \( f \subseteq O(g_f(h_f(n) + n)) \) and \( h_f \) is constructible in \( \text{TIME}(f) \)), hence by our assumption there is \( g' \in G \) such that \( L' \) can be decided by a TM \( M'' \in \Sigma_j \text{TIME}(g'(n)) \) (resp. id., resp. \( \text{SPACE}(\log g) \)). Then \( L \) can be decided by a TM \( M''' \) which, on input \( x \), writes down \( X = x1^{h_f(|x|)} \), which takes time \( O(f) \) (resp. space \( O(f) \), resp time \( \log f \)), and then simulates \( M'' \) on \( X \), which takes \( g'(h_f(n) + n) \), and by hypothesis there exists \( f' \in F \) such that \( f + g'(h_f(n) + n) = O(f') \) (resp. id., resp. \( \log f + \log(g'(h_f(n) + n) = O(\log f') \)), hence we indeed have \( \Sigma_k \text{TIME}(F) \subseteq \Sigma_j \text{TIME}(F) \) (resp. \( \Sigma_k \text{TIME}(F) \subseteq \text{SPACE}(F) \), resp. \( \text{SPACE}(F) \subseteq \text{TIME}(F) \)). The proof of \( \supseteq \) is trivial since \( j < k \).

Corollary 6.5. Let \( f, g \) be integer functions such that there exists a function \( h \), computable in time \( O(f) \), such that \( f = \Theta(h(n) + n) \). Then for all \( 0 \leq j < k \) \( \Sigma_j \text{TIME}(g) = \Sigma_k \text{TIME}(g) \) implies \( \Sigma_j \text{TIME}(f) = \Sigma_k \text{TIME}(f) \), \( \Sigma_j \text{TIME}(g) = \text{ASPACE}(g) \) implies \( \Sigma_j \text{TIME}(f) = \text{ASPACE}(f) \) and \( \text{TIME}(g) = \text{SPACE}(\log(g)) \) implies \( \Sigma_j \text{TIME}(f) = \text{ASPACE}(f) \).
It is surprising that we do not know how to prove that if $\Sigma_j \text{TIME}(C) = \Sigma_{j+1} \text{TIME}(C)$ then $C$ collapses to level $j$. But we think that it must be true, or at least that it would be really hard to prove it to be false. First because if it was false it would imply $P \not\subseteq NP$, and also because it would be surprising that, for some complexity classes, having $j$ or $j+1$ alternations is as expressive, but having $j+2$ alternations is strictly more expressive.

**Lemma 6.6.** Let $2 \leq r < p$ and $0 < j < k$. Then $\Sigma_j^{r,2} = \Sigma_k^{r,2}$ implies that $\Sigma_j^{p,2} = \Sigma_k^{p,2}$, $\Sigma_j^{r,2} = \text{HO}^{r,2}(\text{TC})$ implies $\Sigma_j^{p,2} = \text{HO}^{p,2}(\text{TC})$ and $\text{HO}^{r,2}(\text{TFP}) = \text{HO}^{p,2}(\text{TC})$.

**Proof.** Let $F = \exp_2^{r-2}(n^{O(1)})$ and $G = \exp_2^{p-2}(n^{O(1)})$, the condition of Theorem 6.4 are respected since, for all $f \in F$, $g_f = \exp_2^{p-2}(n)$, $h_f = \exp_2^{p-r}(n)$, we have $f(n) = O(g(h(n) + n))$ and for all $g' \in G$ let $f' = g'(h_f(n) + n)$ it is easy to see that $f' \in F$ hence $g'(h_f(n) + n) = O(f'(n))$.

By Theorem 6.1 $\Sigma_j^{r,2}$ is equal to $\Sigma_j \text{TIME}(\exp_2^{r-2}(n^{O(1)}))$ and by Theorem 5.4 $\text{HO}^{r,2}(\text{TC})$ is equal to $\text{SPACE}(\exp_2^{r-2}(n^{O(1)}))$. Then the corollary is just a translation of Theorem 5.4 in a descriptive complexity setting. \hfill \blacksquare

### 6.3 Polynomial hierarchy and exponential hierarchies

First we are going to prove that hypothesis on the polynomial hierarchy and polynomial space imply results on the exponential hierarchy. Hence we may prove some interesting result on polynomial classes by proving them in exponential hierarchy.

**Theorem 6.7.** Let $D$ be a class of functions which contains at least every linear function and let $C$ be a class of time-constructible functions closed under addition and such that $\forall g \in D, f \in C(f \circ f \in C)$. If $\Sigma_j \text{TIME}(D) = \Sigma_{j+1} \text{TIME}(D)$ or $\Sigma_j \text{TIME}(D) = \Pi_j \text{TIME}(D)$ then $\forall k \geq j$, $\Sigma_k \text{TIME}(C) = \Sigma_j \text{TIME}(C)$ and if $\Sigma_j \text{TIME}(D) = \text{SPACE}(D)$ then $\Sigma_j \text{TIME}(C) = \text{SPACE}(C)$.

Here we use a definition of TM with one reading tape and one working tape, this way the linear time function can at least verify their bounds.

**Proof.** The first assumption implies the second one by Lemma 6.3 hence we are only going to suppose that $\Sigma_j \text{TIME}(D) = \Pi_j \text{TIME}(D)$ without loss of generality. We will do the proof by induction over $k$, for $k = j$, we want to prove that $\Sigma_j \text{TIME}(C) = \Pi_j \text{TIME}(C)$. We will only prove $\subseteq$ because $\supseteq$ will be true by symmetry. It is only a padding argument, let $f \in C$ and $L$ decided by a TM $M \in \Sigma_j \text{TIME}(f)$, then $L' = \{x1^{\ell(x)}|x \in L\}$. $L'$ can be decided by a TM $M'$ in $\Sigma_j \text{TIME}(O(n))$ hence in $\Pi_j \text{TIME}(g)$ for some $g \in D$, then $L$ can be decided.
by a TM $M'' \in \Pi_j \text{TIME}(f + g \circ f)$ which writes $f(n)$ “1” on his working tape and simulates $M'$. By our assumption on $C$ we then have that $M'' \in \Pi_j \text{TIME}(C)$, hence $\Sigma_j \text{TIME}(C) \subseteq \Pi_j \text{TIME}(C)$.

Now, let $k > j$ and suppose that the property is true for $k - 1$, that is that $\Sigma_{k-1} \text{TIME}(C) = \Pi_{k-1} \text{TIME}(C) = \Sigma_j \text{TIME}(C)$ and let $L$ be a language accepted by a TM $M \in \Sigma_k \text{TIME}(f)$ with $f \in C$. Then on input $x$ of size $n$, we may assume without loss of generality that $M$ makes $f(n)$ existential steps writing $O(f(n))$ symbols on the tape, and then make $k - j - 1$ alternations. Let us say that this first part is done by a TM $M_1$. Then $M$ make a second part in $\Theta_j \text{TIME}(f(n))$ where $\Theta$ is $\Pi$ or $\Sigma$ depending on the parity of $k - j$, let us call $M_2$ the TM that ends the computation of $M$, since it’s input tape is of size $O(f(n))$. $M_2 \in \Theta_j \text{TIME}(O(n)) \subseteq \Theta_j \text{TIME}(D)$. There is some $g \in D$ such that there is a TM $M'_2 \in \Theta_j \text{TIME}(g)$ where $\Pi = \Sigma$ and $\Sigma = \Pi$ equivalent to $M_2$, now we create a TM $M'$ which begin by simulating $M_1$ and then $M'_2$; we indeed have only $k - 1$ alternations, and the time of the computation is $f + (g \circ f)$ which is in $C$ by our assumptions, hence $L$ is also accepted by $M' \in \Sigma_{k-1} \text{TIME}(C) = \Sigma_j \text{TIME}(C)$ where the last equality is by the induction hypothesis. We obtain the result $\Pi_k \text{TIME}(C) = \Pi_j \text{TIME}(C)$ by symmetry.

The result about space is a corollary of theorem 6.4 when we take $G = D$ and $F = C$. We always take $g_f(n) = n$, $h_f = f$ and for any $g' \in G$ $g'(h(n) + n) = g'(f(n) + n) \in C$ by the closure assumption.

**Corollary 6.8.** If the polynomial (resp. linear) hierarchy collapses to the $j$th level then every exponential hierarchy collapses to the $j$th level. If $\text{PSPACE} \subseteq \Sigma_j^p$ then $\text{SPACE}(\exp_2^r(n^{O(1)})) \subseteq \Sigma_j \text{TIME}(\exp_2^r(n^{O(1)}))$.

**Proof.** We apply theorem 6.7 with $D = n^{O(1)}$ (resp. $D = O(n)$) and $C = \exp_2^r(n^{O(1)})$ for $r \geq 0$. It is easy to see that $C$ is closed under $D$ and under addition.

**6.4 Classes of formulae**

Now we will give results for the formula formalism, there may not be corollary of the results over general classes of formulae because of the order of the vocabularies of the formulae, something which does not have any exact translation in the TM setting.

**Lemma 6.9.** For $j \geq 0$:

1. if $\Sigma_j^{r,i} = \Sigma_j^{r,i+1}$ with $0 \leq i \leq r + 1$ then $\Sigma_j^{r,i} = \Pi_j^{r,i} = \Sigma_j^{r,i+1} = \Pi_j^{r,i+1}$.
2. If $\Sigma^r_i = \Pi^r_i$ with $i = r$ (resp. $i = r + 1$) then $\Sigma^r_k = \Sigma^r_{k+1} = \Pi^r_k = \Pi^r_{k+1}$ for all $k < r$ (resp. $k = r + 1$).

3. If $\Sigma^r_i = \Sigma^r_{i+1}$ or $\Sigma^r_i = \Pi^r_i$ for $i = r$ (resp. $i = r + 1$) then $\text{HO}^{r,k}$ collapses to the $j + 1$th level for $k \leq r$ (resp. $k = r + 1$).

The proofs are almost identical to the one for the polynomial hierarchy which is the special case $r = i = 2$.

**Proof.** (of the lemmas) For the first point, if $\Sigma^r_i = \Sigma^r_{i+1}$ then their complements are also equals, so we have $\Pi^r_i = \Pi^r_{i+1}$ hence $\Pi^r_i \subseteq \Sigma^r_{i+1} = \Sigma^r_i \subseteq \Pi^r_{i+1} = \Pi^r_i$.

For the second point, let $\varphi \in \Sigma^r_j$ with $i \leq r$ (resp. $i = r + 1$), then $\varphi = \exists X^r_0 \psi$ where $\psi \in \Pi^r_j$ (resp. $\Pi^r_{j+1}$), then there exists $\psi' \in \Sigma^r_j$ (resp. $\Sigma^r_{j+1}$) equivalent to $\psi$, then $\varphi' = \exists X^r_0 \psi'$ is equivalent to $\varphi$ and is in $\Sigma^r_j$, hence $\Sigma^r_j \subseteq \Sigma^r_i$. By symmetry we also have $\Pi^r_{j+1} \subseteq \Pi^r_i$. The other side, $\supseteq$, is trivial, and by transitivity $\Sigma^r_j = \Sigma^r_i = \Pi^r_i = \Pi^r_{j+1}$.

For the third point, by the first point of the lemma the first condition implies the second one, hence we are only going to use this condition, that $\Sigma^r_i = \Pi^r_i$ for $i = r$ (resp. $i = r + 1$). By induction over $l \geq j$, we will prove that $\Sigma^r_i = \Sigma^r_l = \Pi^r_i$ for $i \leq r$ (resp. $i = r + 1$). For $l = j$ this is the second point of the lemma, so assume that $l > j$ and that the property is true for $l - 1$, by the second part of the lemma we have $\Sigma^r_i = \Pi^r_i = \Sigma^r_{l-1}$ and the last equality is true by induction.

What is surprising is that it seems that we do not have a proof that if $\Sigma^r_2 = \Sigma^r_{j+1}$ then $\text{HO}^{r,2}$ collapses to level $j$. This is because, if $\varphi \in \Sigma^r_{j+1}$, then $\varphi = \exists X^{\omega r} \psi$ with $\varphi \in \Sigma^r_j$ and not in $\Sigma^r_2$; and we have no hypothesis about this class. Lemma 6.9 is almost what we would have wanted, but in the lemma we must bootstrap the property with an assumption over formulae with a free variable of order $r - 1$, and in the theorem with a formula whose highest free-variable is of degree 2. This is the descriptive complexity translation of the question raised in [9]: if two levels of the $r$th exponential hierarchy are equal, does the $r$th exponential hierarchy collapse? The proofs used for the polynomial hierarchy do not work because exponentials are not closed under composition.

### 7 Variable order

Variable order (VO) is an extension of high-order where the orders of the relations are not fixed any more but are variable. It was defined in [8], and it was proved
there that it is “complete”; and in fact more expressive than Turing machines, because it can decide the halting problem, and hence also its complement.

One problem with VO is that two \( \alpha \)-equivalent formula are not always equivalent.

\( \forall i \forall X_i \exists Y_i (X_i = Y_i) \) is false while \( \forall i \forall \alpha \forall i \exists Y_i (X_i = Y_i) \) is true.

In this section we first give a new definition of “Variable order” logic, equivalent to that of [8], but that we consider easier to use, at least because it respects the equivalence of \( \alpha \)-equivalent formulae. Then we prove that VO contains the analytical hierarchy.

### 7.1 A new definition

**Definition 7.1** (Sequence of relations). A sequence of relations (of arity \( a \)) is such that the relation number \( r \) of the sequence is of arity \( a \) and order \( r \).

We will write \( \mathcal{X}^a = (\mathcal{X}^a_r)_{r \in \mathbb{N}^+} \) to mean “\( \mathcal{X} \) is a sequence of arity \( a \)”.

**Definition 7.2** (Variable-order (VO)). Now the vocabularies will be over two sorts, the positive integers and the sequence of relations. The quantifiers of our logic will be over one of those two sorts.

A variable-order formula \( \varphi \) is defined recursively as usual, such that if \( \psi \) and \( \psi' \) are formulae then \( \psi \land \psi' \), \( \psi \lor \psi' \), \( \neg \psi \), \( \forall X_a,r \). \( \psi \), \( \exists X_a,r \). \( \psi \), \( \forall r. \psi \) and \( \exists r. \psi \) are also formula; where \( \mathcal{X}^a \) are sequences of relations and \( r \) is an order variable taking values in \( \mathbb{N}^+ \).

Finally \( \mathcal{Y}^r (Y_1, \ldots, Y_a) \), \( Y =_r \mathcal{X} \), \( r = p \) and \( r < p \) are the atomic formulae where \( r \) and \( p \) are variable orders and \( \mathcal{X} \) and the \( Y_i \) are untyped relation variables.

The closed formulae are defined as usual.

**Definition 7.3** (Semantics of VO). We will write \( \mathcal{X}^r \) to speak of the element of order \( r \) of the sequence \( \mathcal{X}^a = (\mathcal{X}^r_r)_{r \in \mathbb{N}} \). \( \land \), \( \lor \) and \( \neg \) have their usual meaning.

- \( \mathfrak{A} \models r = p \) if and only if \( \mathfrak{A}[r] = \mathfrak{A}[p] \)
- \( \mathfrak{A} \models r < p \) if and only if \( \mathfrak{A}[r] < \mathfrak{A}[p] \)
- \( \mathfrak{A} \models \mathcal{X}^r (X_1, \ldots, X_a) \) if and only if \( \mathfrak{A}[r] > 1 \) and \( (\mathfrak{A}[X_1]^a[r] - 1, \ldots, \mathfrak{A}[X_a]^a[r] - 1) \in \mathfrak{A}[\mathcal{X}]^a[r] \)
- \( \mathfrak{A} \models \mathcal{X} =_r \mathcal{X} \) if and only if \( \mathfrak{A}[\mathcal{X}]^a[r] = \mathfrak{A}[\mathcal{Y}]^a[r] \)
- \( \mathfrak{A} \models \forall \mathcal{X}^a. \varphi \) (resp. \( \mathfrak{A} \models \exists \mathcal{X}^a. \varphi \)) if and only if for all sequences (resp. if and only if there exists one sequence) \( \mathcal{R}^a = (\mathcal{R}^a_r)_{r \in \mathbb{N}^+} \) of \( a \)-ary relation of every positive order: \( \mathfrak{A}[\mathcal{X}/\mathcal{R}] \models \varphi \)
• $\mathfrak{A} \models \forall r. \varphi$ (resp. $\mathfrak{A} \models \exists r. \varphi$) if and only if for all (resp. if and only if there exist one) $i \in \mathbb{N}^+ : \mathfrak{A}[r/i] \models \varphi$

We are now going to define $\text{VO}'$, which is the “variable order” as defined in [8] and prove that our definition is equivalent to theirs.

**Definition 7.4.** We have an infinite number of order variables $r_1, \ldots, r_n, \ldots$ of first order variables $x_1, \ldots, x_n, \ldots$, and of untyped relation variables $X_1, \ldots, X_n, \ldots$. As in $\text{VO}$, there are quantification over order variables, but there is also quantification over first-order variables, and the quantification over relations “associates” with it a non-free order variable. The atomic formula are then $X^{r_i}_j(x_1, \ldots, x_a)$ where the exponent is associated with the relation variable, and the value of the exponent variable is the value of this variable in the scope of this formula.

We emphasize that the value of an order variable associated with an untyped relation variable can change between the association and the atomic formula if the variable is quantified again.

**Theorem 7.5.** $\text{VO}$ is equivalent to $\text{VO}'$.

*Proof.* Every formula in $\text{VO}'$ is also a formula in $\text{VO}$ and its semantics is the same, so $\text{VO}$ is at least as expressive that $\text{VO}'$.

Let $\varphi$ be an $\text{VO}$ formula over the vocabulary $\sigma$, let $X_1, \ldots, X_n$ be the variables of $\varphi$ and let $\sigma' = \{i_1, \ldots, i_n\}$ be $n$ new order variable. We will create an $\text{VO}'$ formula $\varphi'$ such that $\forall i_1, \ldots, i_n. \varphi'$ is equivalent to $\varphi$.

$\varphi'$ is $\varphi$ where the $QX_j$ are replaced by $QX_j^{i_j}$ and the atomic formulae containing $X_j^{p}$ will be replaced this way:

• $X_j^{r} = X_k^{r}$ is replaced by “$\exists i_j, i_k. (r = i_j \land r = i_k \land X_j^{i_j} = X_k^{i_k})$”
• $X_j^{r}(X_{k_1}, \ldots, X_{k_a})$ is replaced by “$\exists i_j, i_{k_1}, \ldots, i_{k_a}. (r = i_j \land \bigwedge_{1 \leq b \leq a} (r - 1) = i_{k_b} \land X_j^{i_j}(X_{k_1}^{i_{k_1}}, \ldots, X_{k_a}^{i_{k_a}}))$” where “$r - 1 = x$” is a syntactic sugar for “$x < r \land \forall o'. (\neg (x < o' \land o' < r))$”.
• $X_j^{p} \in X_j^{p}$ is replaced by “$\exists i_j, i_k. (i_j = r \land i_k = p \land X_j^{i_j} = X_k^{i_k})$”.

**Lemma 7.6.** Let $\sigma$ be a vocabulary, $\varphi$ a formula over $\sigma$ such that there are $n$ relation variables, $\sigma'$ a set distinct of $\sigma$ of cardinality $n$, $\mathfrak{A}$ a $\sigma$-structure and $\mathfrak{A}'$ an extension of $\mathfrak{A}$ over vocabulary $\sigma \cup \sigma'$. Then $\mathfrak{A} \models \varphi \Leftrightarrow \mathfrak{A}' \models \varphi'$.

This lemma implies that $\psi = \forall i_1, \ldots, i_n. \varphi'$ will be such that $\mathfrak{A} \models \varphi \Leftrightarrow \mathfrak{A}' \models \psi$.

*Proof.* Of the lemma

The proof for $\land, \lor$ and $\neg$ is an easy induction.
• If \( \varphi \) is \( \forall X_j \psi \), then \( \varphi' = \forall X_j \psi' \). Then \( \mathcal{A} \models \varphi \iff \mathcal{A}' \models \varphi' \) if and only if for all sequences of relations \( R \), \( \mathcal{A}[\mathcal{X}/R] \models \psi \iff \mathcal{A}'[\mathcal{X}/R] \models \psi' \), and since \( A'[\mathcal{X}/R] \) is a \( \sigma \cup \{X\} \cup \sigma' \)-structure which is an extension of the \( \sigma \cup \{X\} \)-structure \( A[\mathcal{X}/R] \) by induction we indeed have \( \mathcal{A}[\mathcal{X}/R] \models \psi \iff \mathcal{A}'[\mathcal{X}/R] \models \psi' \).

• If \( \varphi \) is \( \exists X_j \psi \) the proof by induction is the same.

• If \( \varphi \) is \( \forall j, \psi \), then \( \varphi' = \forall j, \psi' \). Then \( \mathcal{A} \models \varphi \iff \mathcal{A}' \models \varphi' \) if and only if for all positive integer \( r \), \( \mathcal{A}[i/r] \models \psi \iff \mathcal{A}'[i/r] \models \psi' \), and since \( A'[i/r] \) is a \( \sigma \cup \{i\} \cup \sigma' \)-structure which is an extension of the \( \sigma \cup \{i\} \)-structure \( A[i/r] \), then by induction \( \mathcal{A} \) we indeed have \( \mathcal{A}[i/r] \models \psi \iff \mathcal{A}'[i/r] \models \psi' \).

• If \( \varphi \) is \( \exists j, \psi \) the proof by induction is the same.

• If \( \varphi \) is \( X = Y \) then \( \varphi' = \exists i_j, i_k . (r = i_j \land r = i_k \land X_{j/k}^{ij} = X_{k/k}^{ik}) \). We will show \( \mathcal{A} \models \varphi \iff \mathcal{A}' \models \varphi' \) by two implication.

\( \Rightarrow \) by definition \( \mathcal{A} \models \varphi \) means that \( \mathcal{A}[\mathcal{Y}[r]] = \mathcal{A}[\mathcal{Y}[r]] \), so \( r \) is a correct value for both \( i_j \) and \( i_k \) such that \( r = i_j \land r = i_k \land X_{j/k}^{ij} = X_{k/k}^{ik} \), hence \( \mathcal{A} \models i_j, i_k . (r = i_j \land r = i_k \land X_{j/k}^{ij} = X_{k/k}^{ik}) \) is true.

\( \Leftarrow \) it is clear that if \( \mathcal{A} \models \exists i_j, i_k . (r = i_j \land r = i_k \land X_{j/k}^{ij} = X_{k/k}^{ik}) \) is true, then \( \mathcal{A} \models X_{j/k}^{ij} = X_{k/k}^{ik} \) must be true when \( i_j = i_k = r \), so \( \mathcal{A} \models X = Y \).

The important point in this case is that the value of \( i_k \) in \( \mathcal{A}' \) has no importance.

• If \( \varphi = X_{j/1}^{i_1}(Y_{k_1}, \ldots, Y_{k_a}) \) or \( \varphi = X_{k/1}^{i_1} \in X_{k/1}^{i_1} \) then \( \varphi' = \exists i_j, i_{k_1}, \ldots, i_{k_a} . (r = i_j \land r = i_{k_1} \land \ldots \land r = i_{k_a} \land X_{j/k}^{i_{k_1}}(Y_{k_1}^{i_{k_1}}, \ldots, Y_{k_a}^{i_{k_a}})) \) and a similar proof can be done, showing that the equality in \( \varphi' \) will make that the value in \( \mathcal{A}' \) has no importance, and will end the proof.

There is in fact one last difficulty not treated in this proof, VO accepts that the variable order can be free and that its value can be given in the vocabulary, which is forbidden in VO'. For inductive proofs it is easier to just consider that we can have order variables in the vocabulary. And even if we reject the free order variable in the formulae, we will see in section 7.2.1 how to encode them with relational variables in VO.
7.2 Arithmetic on order variables

Let \( r \) and \( p \) be order variable, we will show that we can define both \( r+p \) and \( r \times p \).
In this definition we will assume that there is at least 2 elements in the universe.

**Notation 7.7.** In this section \( \mathcal{X}_a \) will means that the variable \( \mathcal{X} \) is of arity \( a \).

We cannot write \( a \) as an exponent since exponent are used for order variables.
But since in the proofs we will not use list of variable there will be non confusion.

Also in this section “\( A \) contains \( B \)” means that \( B \in A \).
We will use many straightforward syntactic sugar:

\[
"i + c = j" =_{\text{def}} \text{if } c = 0 \text{ then } i = j \text{ else } \\
\exists k > i ((k + (c - 1) = j) \land \neg \exists l (i < l < k)) \tag{18}
\]

\[
"\mathcal{X} \cup \mathcal{Y} = \mathcal{Z}" =_{\text{def}} \forall r, p, A ((A^r \in \mathcal{X}^p \land A^r \in \mathcal{Y}^p) \iff (A^r \in \mathcal{Z}^p)) \tag{19}
\]

\[
"A(B(C))" =_{\text{def}} A(B) \land B(C) \tag{20}
\]

In equation (18) \( c \) is a constant. In (7.2) the relation can also be \( <, \in, \land, \lor \) or \( = \).

7.2.1 Variable order as input

We will first need to be able to take number as input, and create a formula \( \varphi_i \) such that the number of variable satisfying a monadic second order predicate \( P_i \) is equal to the order variable \( r_i \). Formally we want that \( r_i = |\{y \in A | P_i(y)\}| \) is the only value such that \( \varphi(P_i, r_i) \) is true. We will not use a binary encoding but this unary one for clarity; since we intend to prove calculability results and not complexity one, there is no difference.

The idea we will use is to create a class of binary high-order relation; let us call this class “unique”.

**Definition 7.8.** The binary relation \( \mathcal{X} \) is unique up to level \( r + 1 \) if every element of the sequence \( \mathcal{X} \) of order at most \( r \) contains only one relation, which is the precedent element of the sequence repeated twice, and the elements of order greater than \( r \) are empty. This imply that \( \mathcal{X}^r \) contains exactly \( r \) elements.

\[
\text{unique}(\mathcal{X}_2, r) =_{\text{def}} \forall i (1 \leq i \leq r \Rightarrow (\mathcal{X}^{i+1}(\mathcal{X}^i, \mathcal{X}^i) \land \\
\forall \mathcal{Y}_2, \mathcal{Z}_2 \mathcal{X}^{i+1}(\mathcal{Y}^i, \mathcal{Z}^i) \Rightarrow \mathcal{X} = i \mathcal{Y} = i \mathcal{Z})) \tag{21}
\]

We will then state that there is a bijection between the elements of \( \mathcal{X} \) and the variable \( y \) that respect some property \( P_i(y) \), this will create the wanted relation between the order (of \( \mathcal{X} \)) and the elements satisfying \( P_i \).
A bijection will be a set \( \mathcal{T}' \) of couple of elements \( \mathcal{U}' \) (with \( u = t - 1 \)), one of the element of the couple will be an element of \( \mathcal{X}' \) and the other one will be an \( y \) such that \( P_i(y) \). By definition of \( \mathcal{X}' \), if \( \mathcal{X}^r \in \mathcal{U}' \) then for all \( p < r \), \( \mathcal{X}^p \in \mathcal{U}' \), hence we will use a more precise definition; we will say that \( \mathcal{X}^r \) is an element of \( \mathcal{U}' \) if \( r \) is the biggest order \( p \) such that \( \mathcal{X}^p \in \mathcal{U}' \).

\[
element(\mathcal{X}_2, \mathcal{U}_2, r, u) = \text{def } \mathcal{X}^r \in \mathcal{U}' \land \mathcal{X}^{r+1} \notin \mathcal{U}'
\]

It is easy to obtain such an element, we define a list \( \mathcal{E}' \) this way; \( \mathcal{E}' = \mathcal{U}' \), \( \mathcal{E}' = \mathcal{X}' \), and for every \( r < p < u \) (\( \mathcal{E}'^{p-1}, \emptyset^{p-1} \)) is the only relation of \( \mathcal{E}' \) where \( \emptyset \) is the “false” relation. It is then clear that \( \mathcal{X}^r \in \mathcal{U}' \) and that \( \mathcal{X}^{r+1} \notin \mathcal{U}' \).

Of course, every element \( \mathcal{U}' \) of the set \( \mathcal{T}' \) will contain at most two elements, one element \( \mathcal{X} \) and a \( y \) verifying \( P_i \). \( \mathcal{U} \) can contains also one element if \( y = \mathcal{X} \). The fact that there are exactly one elements satisfying \( \varphi \) in \( \mathcal{U}' \) can be called “surjection”.

\[
surjection(\mathcal{U}, u, \varphi) = \text{def } \exists i \in \mathcal{U}(\varphi(i) \land \forall j \in \mathcal{U}(\varphi(j) \Rightarrow i = j))
\]

And we must also check that every element of \( \mathcal{X} \) and every \( y \) such that \( P_i(y) \) is an element of \( \mathcal{U}' \) is contained in an \( \mathcal{U}' \) of \( \mathcal{T}' \). It is here that it is important that \( \mathcal{X} \) contains at most one element at each level, this way we are sure that there is exactly one element of first order in \( \mathcal{X} \), if this element is an element of \( P_i \) then we will assume it is in bijection with itself; and there is no other element of \( \mathcal{X} \) that could imply that a first order element \( z \), which verify \( P_i \) is in \( \mathcal{U}' \). The fact that every element has got an image in \( \mathcal{T}' \) can be called the “injection”.

\[
injection(\mathcal{T}, t, \varphi) = \text{def } \forall y, \mathcal{X}'(\varphi'(y) \Rightarrow \exists \mathcal{U}(\mathcal{T}'(\mathcal{U}) \land y \in \mathcal{U}' \Rightarrow \mathcal{U} = \emptyset)) \land \forall \mathcal{V}(\mathcal{T}'(\mathcal{V}) \land y \in \mathcal{V}' \Rightarrow \mathcal{U} = \mathcal{V}))
\] (22)

Defining the bijection is just the conjunctions of injection and surjection.

\[
bijection(\mathcal{T}, t, \varphi) = \forall \mathcal{U}(\mathcal{T}'(\mathcal{U}, \emptyset) \Rightarrow \surjection(\mathcal{U}, t - 1, \varphi)) \land \injection(\mathcal{T}, t, \varphi).
\]

Assuming that there is at least one \( y \) verifying \( P_i \) we can tell that there are \( i \) elements \( y \) verifying \( P_i \), with this formula.

\[
equal^+(i, P_i) = \text{def } \exists \mathcal{X}', \mathcal{T}, t(\text{unique}(\mathcal{X}', i) \land \text{bijection}(\mathcal{T}, t, P_i) \land \text{bijection}(\mathcal{T}, t, \lambda i.\text{elements}(\mathcal{X}, \mathcal{T}, i, t - 1)))
\] (23)

Here \( \lambda i.\varphi(i) \) means that \( i \) is going to be the free variable of the property used in the formula of “bijection”.

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The problem here was that there is no relation of order 0, we are then going to encode them. We will do it this way: \((1,1)\) means 0, \((n,2)\) means \(n\) and \((n,m)\) for \(m > 2\) or \((m = 1\) and \(n > 1\)) means nothing.

\[
equal(i, i', \varphi) = \text{def} \ 0 \ \text{if} \ ¬\exists X \varphi(X) \ \text{then} \ i = i' = 1 \ \text{else equal}^+(i, \varphi) \ \land \ i' = 2
\]

**Theorem 7.9.** \(\text{VO is not more expressive if the formula can have free degree variable .} \)

*Proof.* Let \(n\) be an integer, \(\sigma' = \{v_1, \ldots, v_n, v'_1, \ldots, v'_n\}\) and \(\sigma'' = \{P_1, \ldots, P_n\}\) be sets of \(n\) order variables and monadic second order relations, let \(A\) be a finite universe, let \(\sigma\) be a vocabulary distinct from \(\sigma'\) and \(\sigma''\), let \(A\) be a \(\sigma \cup \sigma'\)-structure and let \(A'\) the \(\sigma \cup \sigma''\)-structure such that for \(P \in \sigma A[P] = A'[P]\), and for \(v_i \in \sigma'\) we have \((A[v_i], A[v'_i]) = |\{y \in A | u \in A'[P]\}|\), let \(\varphi\) be a formula over vocabulary \(\sigma \cup \sigma'.\) Then \(A |\varphi \Leftrightarrow A' |\forall 1 \leq i \leq n \ equal(v_i, v'_i, P_i)).\) \(\square\)

### 7.2.2 Addition

We now want to be able to add order variables. The idea will be the same, \(r + p = q\) if there is a bijection between a relation of order \(q\) and the union of a relation of order \(r\) and a relation of order \(p\). We will do it by having \(Y\) in \(Z^q\), and quantify a bijection between elements of \(X^r\) and the elements of \(Z^q\) of order higher than \(p\).

\[
\text{plus}^+(r, p, q) = \text{def} \ \exists T, X, Y, Z, t, (\text{unique}(X, r) \ \land \ \text{unique}(Y, p) \ \land \ \text{unique}(Z, q) \ \land \ Y^r \in Z^q \ \land \ \text{bijection}(T, t, \lambda i. \ \text{elements}(X, U, i, t - 1)) \ \land \ \text{bijection}(T, t, \lambda i. \ \text{elements}(Z, U, i, t - 1) \ \land \ i > p) \ \land \ \text{different}(X, Z))(24)
\]

We need to make sure that the bijection between elements of \(Z\) and the one of \(X\) is correct by checking that there is no element that are both in \(X\) and \(Z\), this is the point of \(\text{different}(X, Z)\).

\[
\text{different}(X, Z) = \text{def} \ \forall i \ X \neq i \ Z \quad (25)
\]

Finally, using the code for 0 and positive integers of the last subsection, we can define the addition of \(\mathbb{N}\).

\[
\text{plus}(r, o', p, p', q, q') = \text{def} \ \text{if} \ o' = 1 \ \text{then} \ (p = q \ \land \ p' = q') \ \text{else} \ (\text{if} \ p' = 1 \ \text{then} \ (r = q \ \land \ o' = q') \ \text{else} \ (\text{plus}^+(r, p, q) \ \land \ q' = 2)) \quad (26)
\]

### 7.2.3 Multiplication

Finally we want to code the multiplication of order, once again the formula \(r \times p = q\) will choose relations \(X, Y\) and \(Z\), unique up to order \(r, p\) and \(q\) respectively, such
that there is a bijection between the elements of \(Z\) and the Cartesian product of

the elements of \(X\) and of the elements of \(Y\).

\[
times^+(r, p, q) \overset{\text{def}}{=} \exists T, \mathcal{X}, \mathcal{Y}, Z, t, (\text{unique}(\mathcal{X}, r) \land
\text{bijection}(T, t, \lambda i, j. \text{elements}(\mathcal{X}, \mathcal{U}, i, t - 1) \land \text{elements}(\mathcal{Y}, \mathcal{U}, j, t - 1))
\land \text{unique}(\mathcal{Y}, p) \land \text{unique}(Z, q))(27)
\]

Of course we now can extend the multiplication over every nonnegative integers.

\[
times(r, o', p, p', q, q') \overset{\text{def}}{=} \text{if}(o' = 1 \lor p' = 1) \text{ then}(q' = q = 1)
\text{ else}(\times(r, p, q) \land q' = 2)(28)
\]

7.2.4 Set of natural numbers

We can define any set \(S \subseteq \mathbb{N}\) in \(\text{VO}\) as a sequence of relation \(X^i\) such that if

\(i - 1 \in S\) then \(X^i = \top\) else \(X^i = \bot\). We can of course assert that \(X\) is a correct
code with

\[
\text{correct-set}(X^1) = \forall i(X =^i \top \lor X =^i \bot)
\]

and that \(n \in X\) with

\[
\text{in}(n, X) = \text{def } X =^{n+1} \top.
\]

7.3 \(\text{VO}\) contains the analytical hierarchy

**Definition 7.10** (Analytical hierarchy (AnH)). Let \(\sigma = \{+, \times, =, c_1, \ldots, c_n, S_1, \ldots, S_m\}\) where the \(c_i\) are constant natural numbers and the \(S_i\) are constant sets of natural
numbers. Let \(\mathcal{N}\) be a \(\sigma\)-structure over the universe \(\mathbb{N}\) such that every arithmetical
operation has its usual meaning.

Then let \(\Sigma^0_0 = \Pi^0_0 = \Delta^0_0\) be the set of formula with quantification only on first
order variables. The formula \(\varphi\) is in \(\Sigma^1_i\) if it is in the form \(\varphi = \exists X \psi\) where \(\psi\)
is in \(\Pi_i\), \(\exists X\) is a quantification over the subset of \(\mathbb{N}\). A formula is in \(\Pi_i\) if it is
the negation of a formula in \(\Sigma^1_i\). Let \(\Delta^1_i = \Sigma^1_i \cap \Pi^1_i\), \(\Delta^1_i\) is the \(i\)th level of the
analytical hierarchy.

The analytical hierarchy (AnH) is equal to the union of the \(\Delta^1_i\); \(\text{AH} = \bigcup_{i \in \mathbb{N}} \Delta^1_i\).

**Theorem 7.11.** We have AnH \(\subseteq\) VO

**Proof.** This section explained how to transform input into order variable, and how
to add and multiply order variable; it also explained how to quantify sets of natural
numbers, and express that a number is inside of the set. Then every formula of
AnH can be easily encoded into VO.

\[\square\]
8 Open problems

Direct equality between classes  When many classes are equal, it may be interesting to find a way to directly transform the formulae without needing to encode a Turing machine. So we may want to find a direct translation from $\text{HO}^r(\text{NPFP})$ to $\text{HO}^r(\text{PFP})$, $\text{HO}^{r+1}(\text{TC})$ or $\text{HO}^r(\text{AIFP})$. We also would like to prove that $\text{HO}^{r+1}(\text{IFP}) \subseteq \text{HO}^r(\text{APFP})$.

$r$th exponential hierarchy  Is $\Sigma^r_j$ a strict subset of $\Sigma^r_{j+1}$? For $r = 2$ this question is: “Does the polynomial hierarchy collapse to the $j$th level?”. And as we saw in theorem 4.8, if we can prove that there is at least one $r$ such that the $r$th exponential hierarchy does not collapse to the $j$th level, then the same result is true for all $p < r$. This may eventually be a way to prove that the polynomial hierarchy does not collapse to some level, hence that $P \neq \text{NP}$.

We also wonder if $\text{HO}^r(\text{IFP})$ is strictly contained in $\Sigma^{r+1}$, for $r = 1$ it is the question $P \neq \text{NP}$.

More surprising, we leave as open the question: If $(\Sigma^r_{j,f} = \Pi^r_{j,f}$ or $\Sigma^r_{j,f} = \Sigma^r_{j+1})$, for $r > 2$, does $\text{HO}^r$ collapse to the the $j$th level? In general, for a class of function $C$ what is the condition over $C$ such that $\Sigma_j \text{TIME}(C) = \Pi_j \text{TIME}(C)$ or $\Sigma_j \text{TIME}(C) = \Sigma_{j+1} \text{TIME}(C)$ implies that the class $\Sigma_j \text{TIME}(C)$ collapse to the $j(j + 1)$ level. We gave sufficient condition but can not prove that they are necessary. We think that those implication must be true, because for them to be false we must have that, for some $j$, $j$ or $j + 1$ alternation does not change the expressivity, but for some $k > j$, $k$ alternations is more expressive; this seems to make no sens.

Relational machines  Relational machines where introduced in [3], and extended in [2]; they are an extension of the Turing machines with relation register. The input are given in the register and not on the tape, which remove the implicit order that Turing machines usually has on the input. The machine can, as usual, write on the tapes, read the tapes, but can also apply boolean operations to the registers and check if a register is empty. The input is then measured as the number of different types of elements in the input; because the size of the input can not be known by relational machines.

It was proven that relational-\text{P}, relational-\text{NP}, relational-\text{PSPACE} and relational-\text{EXP} are equivalent to $\text{FO(IIFP)}$, $\text{FO(NIFP)}$, $\text{FO(PFP)}$ and $\text{FO(AIFP)}$, and that two relational classes are equivalent if and only if the usual classes are equivalent.

We think that it may be interesting to find a correct extension to those relational machines to simulate high-order formulae. In particular it may give let us transform the “reasonable input” assumption into something more formal over
those relational machines.

**Fixed arity high-order** We discussed Monadic High Order, which is the special case of “maximal-arity” being 1 as defined in [5]. It may be interesting to give a better characterisation of expressivity of logics in function of maximal-arity, basic-arity [11] or other restriction of arity.

**Restrictions** Is there a good way to define Horn and Krom formulae in high-order? As stated in section 5.3, finding a correct definition with good properties seems to be not trivial. Finally, over high-order, is there some other syntactic restriction which give interesting properties?

**Games** In first and second order logic, games, like the Ehrenfeucht-Fraïssé (see chapter 4 of [14]) ones, are tools to prove that some queries are not expressible in a given logic. It would be interesting to extend these games over the high order classes. We might even define a game for every class, which would let us prove that some queries are not elementary.

Those games would be very hard to win for the duplicator, so it would then be interesting to try to find easier games.

**Other extensions** What would be the effect of adding counting quantifiers, or unary quantifiers, over high order logic? How would the different infinitary logics be more expressive with high order? (The definition of those logic can be found in chapter 8 of [14].)

**Variable order** What is the exact upper bound on the expressivity of variable order? We give the analytical hierarchy as a lower bound, $\text{AnH} \subseteq \text{VO}$, and we conjecture this to be an equality, but coding a variable order formula into the analytical hierarchies seems to be a nontrivial technical task.

What would be the expressivity of $\text{VO}'$ if the order variables could not be quantified many times? Since the variable should be quantified before the formula it is associated to is quantified, it could be a severe restriction to the expressivity of the language. The author thinks that this would express exactly the class of functions computable in elementary time. (This class is at least a lower bound, since this version of $\text{VO}'$ would be a superset of $\text{HO}'$ for any value of $r$).

The idea behind this assumption is that with a finite number of order of variable it is impossible to find difference between two relations of order sufficiently high, if we can decide what is the exact bound for a given number of order variable, let us say $b$, then we can replace every $Qi \cdot \varphi$ by $(Qi < b)\varphi$, hence the language
is decidable and it seems that this kind of formulae can be written as formulae in \( \text{HO} \) (with \( b \) different values of order from 1 to \( b \) for every relation variables).

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