Wavy-roughness effect to thermal explosion of materials with power-law thermal absorptivity in microannuli

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Abstract

We obtain the approximate solutions for the steady temperature profiles of materials with a temperature-dependent thermal absorptivity inside a microannulus with wavy-rough surfaces considering a quasilinear partial differential equation by the boundary perturbation approach. We found the critical Frank-Kamenetskii parameter will depend on the small amplitude wavy-roughness.

Keywords : Boundary perturbation; Frank-Kamenetskii parameter; Corrugations

1 Introduction

Thermal instability such as the catastrophic phenomenon of thermal runaway in which a slight change of external heating (e.g., microwave) power causes the temperature to increase rapidly to the melting point of the material (e.g., ceramic) have been studied for simple geometry cases [1-7]. Up to now, the basic understanding of the (microwave) heating process still remains somewhat empirical and speculative due to its highly nonlinear character. The mathematical description of this process is accordingly fraught with nonlinearities (not to mention the external heating) : The heat equation is highly temperature dependent at the temperatures required for sintering. Moreover, the thermal boundary conditions must take into account both convective and radiation heat loss. The result is a highly nonlinear initial-boundary value problem. Even for simple geometry problems, because of the highly nonlinearity characteristic, numerical approaches were often adopted to solve the corresponding equations and boundary conditions [1,6-7]. Relevant approaches have also a wide field of applications in combustion theory or for safety of storage of materials capable of exothermic chemical reaction [8]. In this short paper, we shall consider the above mentioned problems with simple geometry (say, an annulus) in microdomain. However, real surfaces are rough at the micro- or even at the meso-scale [9-11]. We presume the roughness to be wavy-like in transverse direction along the outer and inner surfaces of the annulus which thus make the corresponding boundary conditions highly nonlinear. With this, we adopt the boundary perturbation approach [9-11] to handle the mathematically complicated boundary value problems.
2 Mathematical formulations

Considering the forced heat equation

\[ \rho c_p(T) \frac{\partial T}{\partial t} = \nabla \cdot \left( \nu(T) \nabla T \right) + \gamma(T)\phi^2 \]

where \( T \) is the temperature, \( \nu \) is the thermal conductivity, \( \gamma \) is the thermal absorptivity, \( \rho \) is the density that is usually assumed constant, \( c_p \) is the specific heat and \( \phi \) is the heating source. Here, the thermal absorptivity depends on the square of the external source (amplitude). Assuming constant specific heat and that the heating source (amplitude) is temperature independent, the appropriate form of above equation becomes

\[ \frac{\partial T}{\partial t} = \nabla \cdot \left( \kappa(T) \nabla T \right) + \gamma(T)\phi^2. \]  

(1)

where \( \phi \) might depend on the spatial coordinates.

2.1 Smooth Cylinder Geometry

Without loss of the generality, we consider a class of solutions for the nonlinear reaction-diffusion equations

\[ \frac{\partial T}{\partial t} = \frac{\partial}{\partial x} \left( xT^m \frac{\partial T}{\partial x} \right) + \delta K(x)T^m \]  

(2)

with the geometry of an infinite cylinder \((x \equiv r \text{ with } r \text{ being the radius})\) and \(m = -1\). Here, \( \delta \) is the Frank-Kamenetskii parameter [12] which incorporates the chemical properties of the combustible material, the temperature of assembly as well as its geometrical dimensions. \( K(x) = x^\mu \) and \( \mu < 0, \mu \text{ is a constant.} \) The critical value \( \delta_{cr} \) defines criticality and provides, for example, a criterion for safety of storage of materials capable of exothermic chemical reaction [8,12-13]. Thus, \( \delta_{cr} \) is important in characterizing the thermal stability properties of the material under consideration.

For the smooth geometry case, we consider the steady state thermal explosions between two infinite cylindrical material undergoing exothermic reaction by using the change of variable given by \( x = \exp(y), \ T^n = \exp(U) \) and \( \Theta = U + (2 + \mu)y \) and we can obtain that closed-form solutions which are available for all real values of \( \mu \) [12]. In fact, equation (2) becomes

\[ \frac{d^2 \Theta}{dy^2} + \delta n e^\Theta = 0 \]  

(3)

and the solutions are

\[ e^\Theta = \frac{C_0}{\cosh^2 \left( \sqrt{\frac{n\delta C_0}{2}} y - D_0 \right)} \quad \text{or} \quad T^n = \frac{C_0 x^{-2-\mu}}{\cosh^2 \left( \sqrt{\frac{n\delta C_0}{2}} \ln x - D_0 \right)} \]  

(4)
where $C_0$, $D_0$ are integration constants.

Note that for the pure cylindrical case ($x \equiv r = 0$), once $\mu \geq -1$, we must make sure the solution to be well defined considering the limit of $x \to 0$:

$$T^n = \frac{(\mu + 2)^2}{2n\delta x^{2+\mu} \cosh^2 \left( \frac{2+\mu}{2} \ln x - D_0 \right)}$$

with

$$C_0 = \frac{(2 + \mu)^2}{2n\delta}.$$ 

The boundary conditions for a smooth annulus could be

$$T^n(1) = 1 \quad \text{and} \quad T^n(s) = 1,$$

where $s = r_1/r_2$ (cf. Fig. 1) and $x = r/r_2$. These could help us to fix $C_0$ and $D_0$. Under this situation, we have

$$\sqrt{\frac{n\delta}{2}} \ln s \cosh(-D_0) = D_0 + \cosh^{-1}\left[ s^{-\frac{\mu-2}{2}} \cosh(-D_0) \right],$$

where $\mu < -1$.

### 2.2 Wavy-Rough Annulus Geometry

We start to consider a steady state heat flow in a wavy-rough microannulus of $r_2$ (in mean-averaged outer radius) with the outer wall being a fixed wavy-rough surface: $r = r_2 + \epsilon \sin(k\theta)$ and $r_1$ (in mean-averaged inner radius) with the inner wall being a fixed wavy-rough surface: $r = r_1 + \epsilon \sin(k\theta + \beta)$, where $\epsilon$ is the amplitude of the (wavy) roughness, and the wave number: $k = 2\pi/L$ ($L$ is the wave length), $\beta$ is the phase shift. The schematic is illustrated in Fig. 1.

With the boundary perturbation approach from Refs. [9-11], we shall derive the temperature field along the wavy-rough microannuli considering dimensionless values. We firstly select $L_a \equiv r_2$ to be the characteristic length scale and set

$$r' = r/L_a, \quad R_o = r_2/L_a, \quad R_i = r_1/L_a, \quad \epsilon' = \epsilon/L_a.$$ 

After this, for simplicity, we drop all the primes. It means, now, $r$, $R_o$, $R_i$, and $\epsilon$ become dimensionless. The walls are prescribed as $r = R_o + \epsilon \sin(k\theta)$, $r = R_i + \epsilon \sin(k\theta + \beta)$ and the presumed steady state heat flow is along the $z$-direction (microannulus-axis direction).

Let $T$ be expanded in $\epsilon$:

$$T = T_0 + \epsilon T_1 + \epsilon^2 T_2 + \cdots,$$

and on the boundary, we expand $T(r_0 + \epsilon dr, \theta(= \theta_0))$ into

$$T(r, \theta)|_{(r_0 + \epsilon dr, \theta_0)} = T(r_0, \theta) + \epsilon [d\theta T_r(r_0, \theta)] + \epsilon^2 \left[ \frac{dr^2}{2} T_{rr}(r_0, \theta) \right] + \cdots$$
where the subscript means the partial differentiation (say, $T_r \equiv \partial T / \partial r$).

The corresponding equation, for simplicity (as the wavy roughness being imposed) as we select firstly $n = m + 1$, becomes

$$
\frac{\partial}{r \partial r} \left( r^{m} \frac{\partial T}{\partial r} \right) + \delta r^\mu T^{m+1} = 0
$$

or

$$
\frac{\partial}{r \partial r} \left( r \frac{\partial \Theta}{\partial r} \right) + (m+1)\delta r^\mu \Theta = 0
$$

with $\Theta = T^{m+1}$.

Now, we can set $\eta = r^{(2+\mu)/2}$ and then obtain

$$
\frac{\partial^2 \Theta}{\partial \eta^2} + \frac{\partial \Theta}{\eta \partial \eta} + \frac{4(m+1)\delta}{(2+\mu)^2} \Theta = 0, \quad \mu \neq -2.
$$

Above equation has the zeroth order solution in terms of Bessel functions

$$
\Theta_0 = A_0 J_0(K \eta) + B_0 Y_0(K \eta),
$$

once we consider $\Theta = \Theta_0 + \epsilon \Theta_1 + \cdots$. Here, $A_0, B_0$ are integration constants. $K = 2\sigma/(2+\mu)$ and $\sigma = [(m+1)\delta]^{1/2}$. Thus, we obtain, for $\mu \neq -2$,

$$
\Theta_0 = T_0^n = T_0^{n+1} = A_0 J_0 \left( \frac{2[(m+1)\delta]^{1/2}}{2+\mu} r^{(2+\mu)/2} \right) + B_0 Y_0 \left( \frac{2[(m+1)\delta]^{1/2}}{2+\mu} r^{(2+\mu)/2} \right)
$$

and for $\mu = -2$,

$$
\Theta_0 = T_0^n = T_0^{n+1} = A^* \sin \left( [(m+1)\delta]^{1/2} \ln r \right) + B^* \cos \left( [(m+1)\delta]^{1/2} \ln r \right).
$$

Here, $A_0,B_0$ or $A^*,B^*$ could be uniquely determined after imposing the boundary conditions. This leads to $T_0 = \Theta_0^{1/(m+1)}$.

As for the first order correction or solution, with the wavy-roughness effect, we can only have (considering $T^n \equiv (T_0 + \epsilon T_1 + \cdots)^n = T_0^n + n \epsilon T_1 T_0^{n-1} + \cdots$ with $T_0$ known), for $m = 0$,

$$
\frac{\partial^2 T_1}{\partial r^2} + \frac{\partial T_1}{r \partial r} + \frac{1}{r^2} \frac{\partial^2 T_1}{\partial \theta^2} + \delta r^\mu T_1 = 0,
$$

due to the complicated orientational characteristics ($\theta$-dependent). Using the boundary conditions from the equation (6), we have

$$
T_0 \bigg|_{r=s} = 1, \quad T_0 |_{r=1} = 1,
$$

$$
\frac{\partial T_0}{\partial r} \bigg|_{r=1} \sin(k \theta) + T_1 |_{r=1} = 0,
$$

and

$$
\frac{\partial T_0}{\partial r} |_{r=s} \sin(k \theta + \beta) + T_1 |_{r=s} = 0,
$$
Equation (15) can be transformed to
\[ \frac{\partial^2 R}{\partial \xi^2} + \frac{\partial R}{\partial \xi \partial \xi} + \left( \frac{2}{2 + \mu} \right)^2 (\delta r^{\mu+2} - K^2) R = 0, \quad T_1 = i e^{iK\theta} R(\xi), \]
and solved after setting
\[ \xi^2 = r^{\mu+2}, \quad \nu = \frac{2}{2 + \mu} \delta^{1/2}, \quad \bar{K} = \frac{2}{2 + \mu} K, \quad \mu \neq -2, \]
where
\[ R(\xi) = A_1 J_{\bar{K}}(\nu \xi) + B_1 J_{-\bar{K}}(\nu \xi). \]

We thus have
\[ T_1 = i e^{iK\theta} \left[ A_1 J_{\frac{2}{2 + \mu} K} \left( \frac{2}{2 + \mu} \delta^{1/2} r^{(2+\mu)/2} \right) + B_1 J_{-\frac{2}{2 + \mu} K} \left( \frac{2}{2 + \mu} \delta^{1/2} r^{(2+\mu)/2} \right) \right], \]
with \( \mu \neq -2 \) or
\[ T_1 = i e^{iK\theta} \left[ C_1 \cos \left( (\delta - K^2)^{1/2} \ln r \right) + D_1 \sin \left( (\delta - K^2)^{1/2} \ln r \right) \right], \quad \mu = -2. \]

Here, \( A_1, B_1, C_1, D_1 \) could be fixed by boundary conditions or equations (16-18). In fact, from the boundary condition (17) or the wavy roughness along the annulus, we can identify \( K = k \). With the first order perturbed solution due to wavy roughness: \( T_1 \), we can thus obtain the approximate solution for especially \( m = 0 \)
\[ T = T_0 + \epsilon T_1 + \cdots, \]
where
\[ T_0 = A_0 J_0 \left( \frac{2 \delta^{1/2}}{2 + \mu} r^{(2+\mu)/2} \right) + B_0 Y_0 \left( \frac{2 \delta^{1/2}}{2 + \mu} r^{(2+\mu)/2} \right), \quad \mu \neq -2 \]
or
\[ T_0 = A^* \sin \left( \delta^{1/2} \ln r \right) + B^* \cos \left( \delta^{1/2} \ln r \right), \quad \mu = -2, \]
with \( T_1 \) given by equations (22) and (23). \( T_0 \) and \( T_1 \) are illustrated for \( s = 0.4 \) in Figs. 1 and 2.

### 3 Conclusion

We adopt the boundary perturbation approach to obtain the approximate solutions for the steady temperature profiles of materials with a temperature-dependent thermal absorptivity inside a microannulus with wavy-rough surfaces considering a quasilinear partial differential equation. Acknowledgement. The first author stayed at the Chern Shiing-Shen Institute of Mathematics, Nankai University for one month. Thus the first author should thank their hospitality for the first stage Visiting-Scholar Program.
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Fig. 1. Approximate zeroth order temperature profiles: $T_0(r)$ for different $\mu (=-4, -1, 1)$ with $\delta = 2$. The boundary conditions are satisfied at both sides ($T_0(s = 0.4) = T_0(1) = 1$).
Fig. 2. Approximate $T_1(r)$ temperature profiles for different $\mu (= -4, -1, 1)$ with $\delta = 2$. Here, the wave number of roughness $k$ is equal to 8, $\beta = \pi/4$ and the inner boundary is $s = 0.4$. We can observe that the small-amplitude wavy-roughness changes the near-inner-wall temperature distribution significantly once $\mu = -4$. 