ON FUNCTORS THAT DETECT $S_n$

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Abstract. Let $A$ be a Noetherian ring. For each $k$ where $0 \leq k \leq \dim A$ we construct left exact functors $D_k$ on $\text{Mod}(A)$. Let $D_k^i$ be the $i$th-right derived functor of $D_k$. Let $M$ be a finitely generated $A$-module. Under mild conditions on $A$ and $M$ we prove that vanishing of some finitely many $D_k^i(M)$ is equivalent to $M$ satisfying $S_n$.

1. INTRODUCTION

Let $A$ be a Noetherian ring and let $M$ be a finitely generated $A$-module. Let $n \geq 0$ be a non-negative integer. Recall that $M$ satisfies $S_n$ if

$$\text{depth } M_p \geq \min\{n, \dim M_p\}$$

for all primes $p$ in $A$.

Note that by convention the zero module has depth $+\infty$ and dimension $-1$. In this paper we construct functors which (under mild conditions) detect whether $M$ satisfies $S_n$.

Let $E$ be a not-necessarily finitely generated $A$-module. By $\dim E$ we mean dimension of the support of $E$ considered as a subspace of $\text{Spec } A$. Let $k \geq 0$ be an integer. Set

$$D_k(E) = \sum_{\substack{N \text{ submodule of } E \\dim N \leq k}} N$$

Clearly $D_k(E)$ is a submodule of $E$. Also if $\phi: E \to F$ is $A$-linear then it is easy to verify that $\phi(D_k(E)) \subseteq D_k(F)$. Set $D_k(\phi): D_k(E) \to D_k(F)$ to be the restriction of $\phi$ on $D_k(E)$. Clearly we have an additive functor $D_k$ on $\text{Mod}(A)$. It can be shown that $D_k$ is left exact; see section 2. Let $D_k^i$ be the $i$th-right derived functor of $D_k$.

To prove our results we need to assume that the ring $A$ satisfies certain conditions.

1.1. We assume that $A$ satisfies the following properties:

(1) $\dim A$ is finite.
(2) $A$ is catenary.
(3) $A$ is equi-dimensional, i.e., $\dim A/p = \dim A$ for all minimal primes $p$ of $A$.
(4) If $m$ is a maximal ideal in $A$ and $p$ is a minimal prime of $A$ then $\text{height}(m/p) = \dim A$.

We now give examples of rings which satisfy the hypotheses in 1.1.

(i) $A = R/I$ where $R = K[X_1, \ldots, X_n]$ and $I$ is an equi-dimensional ideal in $R$, i.e., $\text{height } p = \text{height } I$ for all minimal primes $p$ of $I$.  

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(ii) $A = R/I$ where $R = \mathcal{O}[X_1, \ldots, X_n]$; $\mathcal{O}$ is the ring of integers in a number field (i.e., a finite extension of $\mathbb{Q}$) and $I$ is an unmixed ideal of $R$.

(iii) $A = R/I$ where $R$ is a Cohen-Macaulay local ring and $I$ is an equi-dimensional ideal.

(iv) $A$ is a catenary local domain.

Recall a finitely generated $A$-module $M$ is said to be equi-dimensional if $\dim M$ is finite and $\dim A/p = \dim M$ for all minimal primes of $M$. Our main result is

**Theorem 1.2.** Let $A$ be a Noetherian ring satisfying the hypotheses in (1.1) and let $M$ be a finitely generated equi-dimensional $A$-module of dimension $\geq 1$. Let $n$ be an integer between $1$ and $\dim M$. Then the following conditions are equivalent:

(i) $M$ satisfies $S_n$.

(ii) $D_i^k(M) = 0$ for $i = 0, 1, \ldots, n-1$ and $0 \leq k < \dim M - i$.

Here is an overview of the contents of the paper. In section two we define our functors $D_k$ and prove a few basic properties. In section three we prove a crucial result regarding localization of our functors $D_k$. Finally in section four we prove Theorem 1.2.

**2. The functors $D_k$**

In this section we define the functors $D_k$ and prove some of its basic properties. Throughout $A$ is a Noetherian ring. The $A$-modules considered in this section need not be finitely generated.

2.1. Let $E$ be a $A$-module. Let $\text{Supp } E$ denote the support of $E$. Set $\dim E = \dim \text{Supp } E$. The following result is well-known

**Proposition 2.2.** Let $0 \to E_1 \to E_2 \to E_3 \to 0$ be an exact sequence of $A$-modules. Then

(a) $\text{Supp } E_2 = \text{Supp } E_1 \cup \text{Supp } E_3$.

(b) $\dim E_2 = \max\{\dim E_1, \dim E_3\}$.

2.3. We now define our functors $D_k$. Let $k \geq 0$ be an integer. Let $E$ be an $A$-module. Set

$$D_{k,A}(E) = \sum_{\text{N submodule of } E}^{\dim N \leq k} N.$$ 

We suppress $A$ in $D_{k,A}(E)$ if it is clear from the context. Clearly $D_k(E)$ is a submodule of $E$. The following Lemma is useful.

**Lemma 2.4.** Let $\xi \in D_k(E)$. Then there exists a finitely generated $A$-submodule $M$ of $E$ with $\xi \in M$ and $\dim M \leq k$.

**Proof.** There exists $A$-submodules $N_1, \cdots, N_s$ of $E$ with $\dim N_i \leq k$ and $\xi = n_1 + n_2 + \cdots + n_s$ where $n_i \in N_i$.

Set $N = N_1 + N_2 + \cdots + N_s$. There is a natural surjective map $\bigoplus_{i=1}^s N_i \to N$. By 2.2 it follows that $\dim N \leq k$. Also $\xi \in N$.

Set $M = A\xi \subseteq N$. By 2.2 it follows that $\dim M \leq k$. Also $\xi \in M$. \qed

**Proposition 2.5.** Let $\phi : E \to F$ be $A$-linear. Then $\phi(D_k(E)) \subseteq D_k(F)$. 

Proof. Let $\xi \in D_k(E)$. Then by Lemma 2.4 there exists a finitely generated $A$-submodule $N$ of $E$ with $\dim N \leq k$ and $\xi \in N$. Then $\phi(\xi) \in \phi(N)$. Clearly $\phi(N)$ is an $A$-submodule of $F$. Furthermore $\phi$ induces a surjective map $N \to \phi(N)$. By 2.2 we get that $\dim \phi(N) \leq k$. Thus $\phi(\xi) \in D_k(F)$. □

2.6. Set $D_k(\phi): D_k(E) \to D_k(F)$ to be the restriction of $\phi$ on $D_k(E)$. Clearly we have an additive functor $D_k$ on $\text{Mod}(A)$. We show

**Proposition 2.7.** $D_k$ is left exact.

Proof. Let $0 \to E \xrightarrow{\alpha} F \xrightarrow{\beta} G$ be an exact sequence. We want to prove that the sequence

$$0 \to D_k(E) \xrightarrow{D_k(\alpha)} D_k(F) \xrightarrow{D_k(\beta)} D_k(G),$$

is exact.

Clearly $D_k(\alpha)$ is injective. Also

$$D_k(\beta) \circ D_k(\alpha) = D_k(\beta \circ \alpha) = D_k(0) = 0,$$

as $D_k$ is an additive functor. Therefore image $D_k(\alpha) \subseteq \ker D_k(\beta)$.

Let $\xi \in \ker D_k(\beta)$. In particular $\xi \in \ker \beta$. So there exists $e \in E$ with $\alpha(e) = \xi$. As $\xi \in D_k(F)$, by Lemma 2.4 there exists a finitely generated $A$-submodule $N$ of $F$ with $\dim N \leq k$ and $\xi \in N$. Note that $\alpha$ induces an exact sequence

$$0 \to \alpha^{-1}(N) \to N.$$

By 2.2 we get that $\dim \alpha^{-1}(N) \leq k$. Also $e \in \alpha^{-1}(N)$. It follows that $e \in D_k(E)$. Thus $D_k$ is left exact. □

We need the following two properties of $D_k$.

**Proposition 2.8.** (a) Let $E$ be an $A$-module and let $L$ be an $A$-submodule of $E$. Then $D_k(L) = D_k(E) \cap L$.

(b) Let $E_\alpha$ be a family of $A$-modules with $\alpha \in \Gamma$. Then

$$D_k \left( \bigoplus_{\alpha \in \Gamma} E_\alpha \right) = \bigoplus_{\alpha \in \Gamma} D_k(E_\alpha).$$

Proof. (a) Clearly $D_k(L) \subseteq D_k(E) \cap L$. Let $\xi \in D_k(E) \cap L$. By 2.4 there exists a finitely generated $A$-submodule $N$ of $E$ with $\dim N \leq k$ and $\xi \in N$. So $\xi \in N \cap L$. By 2.2 $\dim N \cap L \leq \dim N \leq k$. So $\xi \in D_k(L)$.

(b) As $D_k$ is an additive functor the result holds if $\Gamma$ is a finite set.

It is clear that

$$\bigoplus_{\alpha \in \Gamma} D_k(E_\alpha) \subseteq D_k \left( \bigoplus_{\alpha \in \Gamma} E_\alpha \right).$$

Let $\xi \in D_k \left( \bigoplus_{\alpha \in \Gamma} E_\alpha \right)$. By 2.4 there exists a finitely generated $A$-submodule $N$ of $\bigoplus_{\alpha \in \Gamma} E_\alpha$ with $\dim N \leq k$ and $\xi \in N$. Say

$$\xi = \sum_{i=1}^{s} \xi_{\alpha_i} \text{ with } \xi_{\alpha_i} \in M_{\alpha_i}.$$

Then $\xi \in N'$ where $N' = N \cap (\bigoplus_{i=1}^{s} M_{\alpha_i})$. By 2.2 $\dim N' \leq k$. So

$$\xi \in D_k \left( \bigoplus_{i=1}^{s} M_{\alpha_i} \right) = \bigoplus_{i=1}^{s} D_k(M_{\alpha_i}) \subseteq \bigoplus_{\alpha \in \Gamma} D_k(E_\alpha).$$
We will also need the following computation.

**Lemma 2.9.** Assume dim $A$ is finite. Let $q$ be a prime ideal in $A$ and let $E(A/q)$ is the injective hull of $A/q$. Then

$$D_k(E(A/q)) = \begin{cases} E(A/q), & \text{if } \dim A/q \leq k \\ 0, & \text{otherwise.} \end{cases}$$

**Proof.** Let $N$ be a non-zero finitely generated $A$-submodule of $E(A/q)$. Let $p$ be a minimal prime of $N$ with $\dim A/p = \dim N$. Note $p \in \Ass N \subseteq \Ass E(A/q) = \{ q \}$. So $p = q$. It follows that $\dim N = \dim A/q$. As a consequence we have that $D_k(E(A/q)) = 0$ if $\dim A/q > k$.

Now assume $\dim A/q \leq k$. Let $\xi \in E(A/q)$ be non-zero. Set $N = A\xi$. Then $\dim N = \dim A/q \leq k$. So $\xi \in D_k(E(A/q))$. It follows that $D_k(E(A/q)) = E(A/q)$ if $\dim A/q \leq k$. \hfill $\Box$

### 3. Localization

In this section we assume that $A$ satisfies our assumptions \[1.1\]. The goal of this section is to prove the following:

**Theorem 3.1.** Assume $A$ satisfies \[1.1\]. Let $M$ be an $A$-module and let $p$ be a prime ideal in $A$. Set $r = \dim A/p$. Then for all $k \geq 0$ we have

$$D^i_{k+r,A}(M)_p \cong D^i_{k,A_p}(M_p) \quad \text{for all } i \geq 0.$$

To prove Theorem 3.1 we need several preparatory results. We first prove:

**Lemma 3.2.** Assume $A$ satisfies \[1.1\]. Let $p,q$ be prime ideals in $A$ with $q \subseteq p$. Then

$$\dim A/q = \height(p/q) + \dim A/p = \dim A_p/qA_p + \dim A/p.$$

**Proof.** It is easy to see that if $m$ is a maximal ideal of $A$ then $A_m$ satisfies the conditions of \[1.1\]. We also get

$$\dim A/p + \height p = \dim A.$$

We first note the following: if $p_0 \subseteq p_1 \subseteq \cdots \subseteq p_r = p$ is a saturated chain of prime ideals with $p_0$ a minimal prime then $r = \height p$. To see this extend it to a maximal chain $p_0 \subseteq p_1 \subseteq \cdots \subseteq p_s = m$ where $m$ is a maximal ideal in $A$. Then by assumption on $A$ we get $s = \dim A$. Localize at $m$. Then by \[1.1\] Lemma 2, p. 250 we get that $\height m = \height p_m + \height(m/p)$. Note $\height p_m = \height p \geq r$ and $\height(m/p) \geq s - r$. As $\height m = \dim A = s$ we get that $r = \height p$ and $s - r = \height(m/p)$. It is now elementary to see that $\height(p/q) = \height p - \height q$.

Note that by \(\dag\) we get $\dim A/q - \dim A/p = \height p - \height q$. The result follows. \hfill $\Box$

**Lemma 3.3.** Assume $A$ satisfies \[1.1\]. Let $p$ be a prime ideal in $A$. Set $r = \dim A/p$. Let $q$ be a prime ideal in $A$ with $q \subseteq p$. Let $k \geq 0$. Then

$$D_{k+r,A}(E_A(A/q)) \cong D_{k+r,A}(E(A/q))_p \cong D_{k,A_p}(E_A(A_p/qA_p)).$$
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Proof. As $A$ satisfies 1.1 by 3.2 we get

\[(*) \quad \dim A/q = \text{height}(p/q) + \dim A/p = \dim A_p/qA_p + r.\]

To prove our result we consider two cases.

Case 1: $\dim A/q \leq k + r$.

By $(*)$ this holds if and only if $\dim A_p/qA_p \leq k$. By Lemma 2.9 we have

$$D_{k+r,A}(E_A(A/q)) = E_A(A/q)$$

and $D_{k,A_p}(E_{A_p}(A_p/qA_p)) = E_{A_p}(A_p/qA_p)$.

The result follows since $E_A(A/q) \sim E_A(A/q)_p \sim E_{A_p}(A_p/qA_p)$.

Case 2: $\dim A/q > k + r$.

By $(*)$ this holds if and only if $\dim A_p/qA_p > k$. By Lemma 2.9 we have

$$D_{k+r,A}(E_A(A/q)) = 0 \text{ and } D_{k,A_p}(E_{A_p}(A_p/qA_p)) = 0.$$

The result follows.

We now show:

**Proposition 3.4.** Assume $A$ satisfies 1.1. Let $p$ be a prime ideal in $A$. Set $r = \dim A/p$. Let $M$ be an $A$-module. Let $k \geq 0$. Then

$$D_{k+r,A}(M)_p \cong D_{k,A_p}(M_p).$$

Proof. We consider two cases.

Case 1: $M$ is an injective $A$-module. By Matlis theory, cf. [1, 18.5]

$$M = \bigoplus_{q \in \text{Spec } A} E_A(A/q)^{\mu_q}.$$

Notice $\mu_q = \dim_{\kappa(q)} \text{Hom}_{A_q}(\kappa(q), M_q)$ (here $\kappa(q)$ is the residue field of $A_q$). By Proposition 2.8 we have

$$D_{k+r,A}(M) = \bigoplus_{q \in \text{Spec } A} D_{k+r,A}(E_A(A/q))^{\mu_q}.$$

Now note that

$$M_p = \bigoplus_{q \subseteq p} E_{A_p}(A_p/qA_p)^{\mu_q}.$$

Therefore by Proposition 2.8 we get that

$$D_{k,A_p}(M_p) = \bigoplus_{q \subseteq p} D_{k,A_p}(E_{A_p}(A_p/qA_p))^{\mu_q}.$$

The result now follows from Proposition 3.3.

Case 2: $M$ is an arbitrary $A$-module.

Embed $M$ into an injective $A$-module $I$. Then note that $M_p$ is a submodule of $I_p$.

By Proposition 2.8 we get $D_{k+r,A}(M) = D_{k+r,A}(I) \cap M$. So we get

$$D_{k+r,A}(M)_p = (D_{k+r,A}(I) \cap M)_p,$$

$$\cong D_{k+r,A}(I)_p \cap M_p; \quad \text{by [1] 7.4(i)},$$

$$\cong D_{k,A_p}(I_p) \cap M_p; \quad \text{by Case 1},$$

$$= D_{k,A_p}(M_p); \quad \text{by Proposition 2.8}.$$

We now give
Proof of Theorem 4.1. Let $I$ be a minimal injective resolution of $M$. Then note that $I_p$ is a minimal injective resolution of $M_p$. [1] Lemma 6, p. 149. Consider the complex $D = D_{k+r,A}(I)$. By [2,4] we get that $D_p = D_{k,A_p}(I_p)$. As $D$ is a complex of injectives, the map $D \rightarrow D_p$ is a surjective map of complexes. So we have an exact sequence of complexes $0 \rightarrow K \rightarrow D \rightarrow D_p \rightarrow 0$. Observe that by 2.4

$$K^i = \bigoplus_{q \notin p} E_A(A/q)^{i(M,A)}_{\dim A/q \leq r+k}$$

It follows that $K_p = 0$.

The short exact sequence of complexes $0 \rightarrow K \rightarrow D \rightarrow D_p \rightarrow 0$ yields a long exact sequence

$$\cdots \rightarrow H^i(K) \rightarrow H^i(D) \rightarrow H^i(D_p) \rightarrow H^{i+1}(K) \cdots$$

As $K_p = 0$ we get that $H^i(K)_p = 0$ for all $i$. Thus $H^i(D)_p \cong H^i(D_p)_p = H^i(D_p)$. The result follows. □

4. Proof of Theorem 1.2

In this section we prove Theorem 1.2 by induction on $n$. We prove the base case $n = 1$ separately.

Proposition 4.1. Assume $A$ satisfies 1.1. Let $M$ be a finitely generated equidi-

mensional $A$-module of dimension $\geq 1$. The following conditions are equivalent:

(i) $M$ satisfies $S_1$.
(ii) $D_k(M) = 0$ for all $k < \dim M$.

Proof. We first assume $M$ satisfies $S_1$. Then $\dim M_p \geq 1$ if and only if $p \notin \text{Ass } M$. Suppose $\xi \in D_k(M)$ is non-zero. Then by 2.3 there exists a finitely generated submodule $N$ of $M$ with $\dim N \leq k$ and $\xi \in N$. Let $p \in \text{Ass } N$ be such that $\dim A/p = \dim N$. Note $p \in \text{Ass } M$. It follows that $p \in \text{Min } M$. So $\dim N = \dim M$. It follows that $k \geq \dim M$. Thus $D_k(M) = 0$ for $k < \dim M$.

Conversely assume that $D_k(M) = 0$ for all $k < \dim M$. Suppose if possible $M$ does not satisfy $S_1$. Then there exists $p$ with $\dim M_p \geq 1$ and depth $M_p = 0$. Thus $p \in \text{Ass } M$. So we have an injection $A/p \rightarrow M$. Notice $c = \dim A/p < \dim M$. Thus $D_c(M) \neq 0$, a contradiction. □

We now give

Proof of Theorem 1.2. We prove the result by induction on $n$. We have proved the result for $n = 1$, see 1.1. We assume the result for $n - 1 \geq 1$ and prove it for $n$.

We first assume that $M$ satisfies $S_n$-property. As $M$ also satisfies $S_{n-1}$ we get by induction hypothesis that $D^j_{k}(M) = 0$ for $k < \dim M - j$ and $j = 0, 1, \cdots, n - 2$ Let $I$ be a minimal injective resolution for $M$. As $M$ satisfies $S_n$ we get that for $i \leq n - 1$,

$$I^i = \bigoplus_{\dim M_p \leq i} E(A/p)^{i(M)}_{\dim M_p}.$$  

Suppose $\xi \in D_k(I^{n-1})$ is non-zero. Then by 2.4 there exists a finitely generated $A$-submodule $N$ of $I^{n-1}$ with $\dim N \leq k$ and $\xi \in N$. Let $p$ be a minimal prime of
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$N$ with $\dim A/p = \dim N \leq k$. Then $p \in \text{Ass}^n$. So $\dim M_p \leq n - 1$. Let $q$ be a minimal prime of $M$ contained in $p$. Then by 3.2 we get

$\dim M = \dim A/q = \dim A_p/q_A_p + \dim A/p \leq \dim M_p + \dim N \leq n - 1 + \dim N$.

So $\dim N \geq \dim M - n + 1$. It follows that $D_{k}(\mathbb{Z}^{n-1}) = 0$ for $k < \dim M - n + 1$. Thus $D_{r}^{n-1}(M) = 0$ for $k < \dim M - n + 1$.

We now assume that $D_{i}^{n-1}(M) = 0$ for $i = 0, 1, \ldots, n - 1$ and $0 \leq k < \dim M - i$. By induction hypotheses it follows that $M$ satisfies $S_{n-1}$. Suppose if possible $M$ does not satisfy $S_n$. Then there exists a prime ideal $p$ with $\dim M_p \geq n$ and $\text{depth } M_p = n - 1$. We localize at $p$. We get that $D_{0, A_p}^{n-1}(M_p) \neq 0$. By Theorem 3.1 it follows that $D_{r}^{n-1}(M) \neq 0$ where $r = \dim A/p$.

Claim: $\dim M = \dim M_p + r$.

Assume the claim for the moment. Then $r = \dim M - \dim M_p \leq \dim M - n < \dim M - n + 1$. Also $D_{r}^{n-1}(M) \neq 0$. This contradicts our assumption.

Proof of claim. Let $q$ be a minimal prime of $M$ contained in $p$ and let $m$ be an arbitrary maximal ideal of $A$ containing $p$. By 3.2 we get that $\dim M = \dim A/q = \text{height}(m/q)$. As $A$ is catenary we get that $\text{height}(m/q) = \text{height}(m/p) + \text{height}(p/q)$. We take $q$ with $\text{height}(p/q) = \dim M_p$. Also note that again by 3.2 $\text{height}(m/p) = \dim A/p = r$. 

References

[1] H. Matsumura, *Commutative ring theory*, Translated from the Japanese by M. Reid. Second edition. Cambridge Studies in Advanced Mathematics, 8. Cambridge University Press, Cambridge, 1989.

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