Turbulence in non-integer dimensions by fractal Fourier decimation

Uriel Frisch,1 Anna Pomyalov,2 Itamar Procaccia,2 and Samriddhi Sankar Ray1
1UNS, CNRS, OCA, Laboratoire Cassiopée, B.P. 4229, 06304 Nice Cedex 4, France
2Department of Chemical Physics, The Weizmann Institute of Science, Rehovot 76100, Israel
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Fractal decimation reduces the effective dimensionality of a flow by keeping only a (randomly chosen) set of Fourier modes whose number in a ball of radius \( k \) is proportional to \( k^D \) for large \( k \). At the critical dimension \( D = 4/3 \) there is an equilibrium Gibbs state with a \( k^{-5/3} \) spectrum, as in [V. L’vov et al., Phys. Rev. Lett. 89, 064501 (2002)]. Spectral simulations of fractally decimated two-dimensional turbulence show that the inverse cascade persists below \( D = 2 \) with a rapidly rising Kolmogorov constant, likely to diverge as \( (D - 4/3)^{-2/3} \).

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In theoretical physics a number of interesting results have been obtained by extending the dimension \( d \) of space from directly relevant values such as 1, 2, 3 to non-integer values. Dimensional regularization in field theory \[5\] and of fractal decimation in Fourier space, appropriate for different way of switching to non-integer dimensions, shall come back – should happen in dimension 4/3. A new way of fractal decimation in Fourier space, appropriate for hydrodynamics. Since, here, we are primarily interested in dimensions less than two, we shall do our decimation starting from the standard dimension \( d = 2 \) case.

The forced incompressible Navier-Stokes equations for the velocity field can be written in abstract notation as

\[
\partial_t \textbf{u} = \textbf{B}(\textbf{u}, \textbf{u}) + \textbf{f} + \Lambda \textbf{u} , \quad (1)
\]

\[
\textbf{B}(\textbf{u}, \textbf{u}) = -\textbf{u} \cdot \nabla \textbf{u} + \nabla p, \quad \Lambda = \nu \nabla^2 , \quad (2)
\]

where \( \textbf{u} \) stands for the velocity field \( \textbf{u}(x_1, x_2, t) \), \( \textbf{f} \) for \( \textbf{f}(x_1, x_2, t) \), \( p \) is the pressure and \( \nu \) the viscosity. The velocity \( \textbf{u} \) is taken in the space of divergence-less velocity fields which are \( 2\pi \) periodic in \( x_1 \) and \( x_2 \), such that \( \textbf{u}(t = 0) = \textbf{u}_0 \). Now, we define a Fourier decimation operator \( P_D \) on this space of velocity fields:

\[
\textbf{u} = \sum_{k \in \mathbb{Z}^2} e^{ik \cdot x} \hat{\textbf{u}}_k , \quad \text{then} \quad P_D \textbf{u} = \sum_{k \in \mathbb{Z}^2} e^{ik \cdot x} \theta_k \hat{\textbf{u}}_k . \quad (3)
\]

Here \( \theta_k \) are random numbers such that

\[
\theta_k = \begin{cases} 1 & \text{with probability } h_k \\ 0 & \text{with probability } 1 - h_k , \end{cases} \quad k = |k| . \quad (4)
\]

To obtain \( D \)-dimensional dynamics we choose

\[
h_k = C(k/k_0)^{D-2} , \quad 0 < D < 2, \quad 0 < C \leq 1 , \quad (5)
\]

where \( k_0 \) is a reference wavenumber; here \( C = k_0 = 1 \). All the \( \theta_k \) are chosen independently, except that \( \theta_k = \theta_{-k} \) to preserve Hermitian symmetry. Our fractal decimation procedure removes at random — but in a time-frozen (quenched) way — many modes from the \( k \) lattice, leaving on average N\( (k) \propto k^D \) active modes in a disk of radius \( k \). The randomness in the choice of the decimation will be called the disorder \[6\].

Observe that \( P_D \) is a projector, that it commutes with the viscous diffusion operator \( \Lambda \) and that it is self-adjoint for the energy (\( L^2 \)) norm, defined as usual as \( |\textbf{u}|^2 = (1/(2\pi)^2) \int |\textbf{u}(x)|^2 d^2 x \), where the integral is over a \( 2\pi \times 2\pi \) periodicity square. The conservation of energy (by the nonlinear term) for sufficiently smooth solutions of the Navier–Stokes equation can be expressed as \( (\textbf{u}, B(\textbf{u}, \textbf{u})) = 0 \) where \( \textbf{u} \) and \( \textbf{w} \equiv (1/(2\pi)^2) \int |\textbf{u}(x)|^2 d^2 x \) is the \( L^2 \) scalar product.

The decimated Navier–Stokes equation, written for an incompressible field \( \textbf{v} \), takes the following form:

\[
\partial_t \textbf{v} = P_D B(\textbf{v}, \textbf{v}) + P_D \textbf{f} + P_D \Lambda \textbf{v} . \quad (6)
\]

\[2\] A more drastic and non-random decimation is the reduced wave-vector set approximation (REWA; see Ref. \[6\] and references therein), in which the number of active mode grows as \( \ln k \), so that from our point of view it has dimension \( D = 0 \).
The initial condition is $\mathbf{v}_0 \equiv \mathbf{v}(t = 0) = P_D \mathbf{u}_0$. Thus at any later time $P_D \mathbf{v} = \mathbf{v}$. Energy is again conserved; indeed $(\mathbf{v}, P_D B(\mathbf{v}, \mathbf{v})) = 0$, as is seen by moving the self-adjoint operator $P_D$ to the left hand side of the scalar product and using $P_D \mathbf{v} = \mathbf{v}$. Enstrophy conservation for the decimated problem is proved in a similar way by working with the vorticity.

If, in addition to decimation, we apply a Galerkin truncation which kills all the modes having wavenumbers beyond a threshold $K_G$, the surviving modes constitute a dynamical system having a finite number of degrees of freedom. Such truncated systems with no forcing and no viscosity have been studied by Lee, Kraichnan and others [8]. Using suitable variables related to the real and imaginary parts of the active modes, the dynamical equations may be written as

$$y_\alpha = \sum \beta \gamma A_{\alpha \beta \gamma} y_\beta y_\gamma.$$  

For the purely Galerkin-truncated (not decimated) case it is well known that the above dynamical system satisfies a Liouville theorem $\sum \alpha \partial y_\alpha / \partial y_\alpha = 0$ and thus conserves volume in phase space. This in turn implies the existence of (statistically) invariant Gibbs states for which the probability is a Gaussian, proportional to $e^{-\beta E}$ where $E = \sum_k |u_k|^2$ is the energy and $\Omega = \sum_k k^2 |u_k|^2$ is the enstrophy. Such Gibbs states, called by Kraichnan absolute equilibria, play an important role in his theory of the two-dimensional (2D) inverse energy cascade [9]. If we now combine inviscid, unforced Galerkin truncation and decimation, it is easily checked that the Liouville theorem still holds, provided the decimation preserves Hermitean symmetry. For such Gibbs states, and any active mode $\theta_k = 1$, one easily checks that the mean square energy $\langle |u_k|^2 \rangle = C'/(\alpha + \beta k^2)$, where $C' > 0$ does not depend on $k$. The corresponding energy spectrum is the mean energy $E(k)$ of modes having a wavenumber between $k$ and $k + 1$. Up to fluctuations of the disorder, the number of active modes in such a shell is $O(k^{D-1})$. Thus,

$$E(k) = \frac{k^{D-1}}{\alpha + \beta k^2}; \quad \beta > 0, \quad \alpha > -\beta,$$

where various positive constants have been absorbed into a new definition of $\alpha$ and $\beta$. An instance is enstrophy equipartition: $\alpha = 0$ (all the modes have the same enstrophy), for which the energy spectrum is $E(k) \propto k^{D-3}$. Following LPP, if we now set $D = 4/3$, we obtain a $k^{-5/3}$ spectrum, the spectrum predicted by Kolmogorov’s 1941 scaling theory and extended by Kraichnan to the inverse energy cascade of 2D turbulence. Note that such Gibbs states are only conditionally Gaussian, for a given disorder. Otherwise, they are highly intermittent, since a given high-$k$ mode will be active only in a small fraction of the disorder realizations. We also note that similar phenomena have been observed in shell models [10].

The form (7) of the D-dimensional absolute equilibria also allows for the kind of Bose condensation in the gravest modes (here, those with unit wavenumber) found by Kraichnan for 2D turbulence. For this the “inverse temperature” $\alpha$ must be taken negative, close to its minimum realizable value $-\beta$. The arguments used by Kraichnan to predict an inverse Kolmogorov $k^{-5/3}$ energy cascade for high-Reynolds number 2D turbulence with forcing near an intermediate wavenumber $k_{\text{inj}}$ carry over to the decimated case with $D < 2$. In particular the conservation of enstrophy blocks energy transfer to high wavenumbers. This in itself does not imply that the energy will cascade to wavenumbers smaller than $k_{\text{inj}}$, producing a $k$-independent energy flux: it might also linger around and accumulate near $k_{\text{inj}}$.

It is now our purpose to show that for $4/3 < D \leq 2$, when the energy spectrum is prescribed to be $E(k) = k^{-5/3}$ over the inertial range, there is a negative energy flux $\Pi_E$, vanishing linearly with $D - 4/3$ near the critical dimension $D = 4/3$. For this we shall assume that a key feature of the two-dimensional energy cascade carries over to lower dimensions, namely the existence of scaling solutions with local (in Fourier space) dynamics, so that the energy transfer is dominated by triads of wavenumbers with comparable magnitudes. Let us now decompose the energy inertial range into bands of fixed relative width, say one octave, delimited by the wavenumbers $k^0, k^1, k^2$, etc. Because of locality there is much intraband dynamics but, of course, interband interactions are needed to obtain an energy flux. Pure intraband dynamics (with no forcing and dissipation) would lead to thermalization. For dimensional reasons, thermalization and interband transfer have the same time scale, namely the eddy turnover time $k^{-3/2}E^{-1/2}(k)$.

To get a handle on the combined intraband and interband dynamics we perform a thermodynamic thought experiment in which we artificially separate them in time. In the first phase, starting from a $k^{-5/3}$ spectrum we prevent the various bands from interacting by introducing (impenetrable) interband barriers at their edges. In each band, the modes will then thermalize and achieve a Gibbs state with a spectrum (8) in which $\alpha$ and $\beta$ are determined by the constraints that the total band energy and enstrophy be the same as for the $-5/3$ spectrum. For example, in the first band this gives the constraints $(n = 0$ for the energy and $n = 2$ for the enstrophy)

$$\int_1^2 dk k^n \left[ k^{D-1} / (\alpha + \beta k^2) - k^{-5/3} \right] = 0.$$  

In general this constitutes a system of transcendental equations for the parameters $\alpha$ and $\beta$ that can only be solved numerically. This is illustrated in Fig. 11 for the 2D case. We observe that in 2D the absolute equilibrium spectrum and the $-5/3$ spectrum are very close to each other. Specifically, in 2D the absolute equilibrium spectrum exceeds the $-5/3$ spectrum by about 10% at any lower band edge and by about 5% at any upper band edge. Of course, as we approach the critical dimension $D = 4/3$ the discrepancy goes to zero and can easily be
calculated perturbatively in $D - 4/3$. In the second phase of our thought experiment, we remove just one of the barriers between two adjacent bands, say, the barrier at $2^1$. A new thermalization leads then to an absolute equilibrium in the band $[2^0, 2^2]$, which again, can be easily calculated. In 2D, before the removal, the energy between $2^0$ and $2^2$ was $0.555$. After the new thermalization, this energy is found to have increased by 0.00551. Thus energy has been transferred from the upper band $[2^1, 2^2]$ to the lower band $[2^0, 2^2]$. Close to $D = 4/3$, we can again apply elementary perturbation techniques and obtain for the upper-to-lower-band energy transfer 0.009($D - 4/3$) to leading order. Our thermodynamic thought experiment thus suggests that the energy flux vanishes linearly with $D - 4/3$, being negative above the critical dimension, which implies an inverse cascade. In the K41 inertial range, the energy spectrum and the energy flux $\Pi$ are mostly insensitive to $k_0$ for different values of $D$. Energy injection at the rate $\varepsilon$ is done in a band of width three around $k_{\text{max}} = 319$ by adding to the time-rate-of-change of the Fourier amplitude of the vorticity a term proportional to the inverse of its complex conjugate $\hat{v}_{\text{conj}}$. This allows a $k$-independent and time-independent energy injection. As $D$ is decreased the amplitude of this forcing is increased to keep the total energy injection on active modes fixed at $\varepsilon = 0.01$. The damping parameters are $\nu = 10^{-11}$ and $\mu = 0.1$. Runs are done concurrently for different values of $D$ on a high-performance cluster at the Weizmann Institute and take typically a few thousand hours of CPU per run to achieve a statistical steady state.

Energy spectra are obtained by angular averages over Fourier-space shells of unit width

$$E(K) = \frac{1}{2} \sum_{K \leq k < K+1} |\hat{v}(k)|^2,$$

where the $\hat{v}(k)$ are the Fourier coefficients of the solution of the decimated Navier–Stokes equation (10). We also need the energy flux $\Pi_E(K)$ through wavenumber $K$ due to nonlinear transfer, defined as

$$\Pi_E(K) \equiv \sum_{k \leq K} \hat{v}^* (k) \cdot \hat{N}L(k),$$

where $\hat{N}L(k)$ denotes the set of Fourier coefficients of the nonlinear term $P_D B(v, v)$ in the decimated Navier–Stokes equation (6) and the asterisk denotes complex conjugation. $E(k)$ and $\Pi_E(K)$ are mostly insensitive to the disorder realization.

Figs. 2 and 3 (inset) show the steady-state compensated energy spectra $k^{5/3}E(k)$ and the energy fluxes $\Pi_E(k)$, for various values of $D$ between 2 and 1.5, respectively. Both are quite flat, over a significant range of $k$ values, evidence that $D$-dimensional forced turbulence, Fourier decimated down from the two-dimensional case, preserves the key feature of two-dimensional turbulence of having an inverse cascade that follows the $-5/3$
law. Note that the inertial range (the flat region of the compensated energy spectrum) shrinks as the dimension $D$ decreases. The absolute value of the energy flux is about 80% of the energy injection $\varepsilon$ for $D = 2$, but drops to less than 50% for $D = 1.5$. Indeed, as the dimension $D$ is lowered, there are fewer and fewer pairs of active modes in the forcing band, capable through their beating interaction of draining the energy into the infrared direction; thus the energy injection will be more and more balanced by direct dissipation near injection. Preventing this would require a substantial lowering of the dissipation which in turn requires a substantial increase in the resolution at the high-$k$ end. Anyway, the fact that the flux $|\Pi_E|$ becomes substantially lower than injection does not prevent us from calculating the Kolmogorov constant, given (in terms of plateau values) by $C_{\text{Kol}} = k^{5/3}E(k)/(|\Pi_E(k)|^{2/3})$. Figure 3 shows the variation of the Kolmogorov constant with dimension. When lowering the dimension from 2 to 1.5, a combined effect of a rise in the compensated spectrum and a drop in flux yields a monotonic growth of about a factor ten in the Kolmogorov constant and a substantial growth of errors due to fluctuations within the averaging interval. Probing the conjectured divergence by moving closer to the critical point $D = 4/3$ would require much higher resolution. A state-of-the-art 16,384$^2$ simulation of sufficient length might shed light.

We finally observe that the fractal Fourier decimation procedure can be started from any integer dimension and can be applied to a large class of problems in hydrodynamics and beyond. It could be interesting, for example, to investigate how it affects the dissipative anomaly of shock-dominated compressible flow.

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