On Local Calabi-Yau Supermanifolds and Their Mirrors

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Abstract

We use local mirror symmetry to study a class of local Calabi-Yau supermanifolds with bosonic sub-variety $V_b$ having a vanishing first Chern class. Solving the usual super-CY condition, requiring the equality of the total $U(1)$ gauge charges of bosons $\Phi_b$ and the ghost like fields $\Psi_f$ one $\sum_b q_b = \sum_f Q_f$, as $\sum_b q_b = 0$ and $\sum_f Q_f = 0$, several examples are studied and explicit results are given for local $A_r$ super-geometries. A comment on purely fermionic super-CY manifolds corresponding to the special case where $q_b = 0$, $\forall b$ and $\sum_f Q_f = 0$ is also made.

Key words: Mirror Symmetry, Local CY super-manifolds, Topological string theory.

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1 Introduction

Mirror symmetry has played a crucial role in superstring dualities. It provides a map between Calabi-Yau (CY) manifolds used in the compactification of 10D superstring models and topological string theory [1,2]. In particular, it has been shown that the topological A- and B-models are connected by mirror symmetry [2]; see also discussion below. However, it has been realized, though, that rigid CY manifolds can have mirror manifolds which are not themselves CY geometries. An intriguing remedy is the introduction of CY super-manifolds in these considerations [3,4]. It has thus been suggested that mirror symmetry is between super-manifolds and manifolds alike, and not just between bosonic manifolds.

On the other hand, it has been found that there is a correspondence between the moduli space of holomorphic Chern-Simons theory on the CY super-manifold $\mathbb{CP}^{3|4}$ and, self-dual, four-dimensional $N = 4$ Yang-Mills theory [5,6]. This may also be related to the B-model of open topological string theory having $\mathbb{CP}^{3|4}$ as target space. Partly based on this work, CY super-manifolds and their mirrors have subsequently attracted a great deal of attention [7]-[18]. It has been found, for instance, that an A-model defined on the CY super-manifold $\mathbb{CP}^{3|4}$ is a mirror of a B-model on a quadric hypersurface in $\mathbb{CP}^{3|3} \times \mathbb{CP}^{3|3}$, provided the Kähler parameter of $\mathbb{CP}^{3|4}$ approaches infinity [7,8].

Following this observation, an effort has been devoted to go beyond these particular geometries. A special interest has been given to construct the mirror of Calabi-Yau super-manifolds whose bosonic parts are compact toric varieties [19]. One of the objectives of the present work is to extend the result of [19] by considering local Calabi-Yau manifolds which have been used in type II superstring compactifications in presence of D-branes. In particular, we discuss the mirror symmetry of the topological A-model on super-manifolds whose the bosonic part is a local CY variety. The corresponding theory is a supersymmetric $U(1)^{p}$ linear sigma model with $(n + p)$ chiral superfields with charge $q_{i}^{a}$ and $2p$ fermionic superfields with charge given by $Q_{\alpha}^{a}$ which is a $p \times 2p$ matrix. These charges satisfy the super local CY (SLCY) condition $\sum_{i=1}^{p+n} q_{i}^{a} - \sum_{\alpha=1}^{2p} Q_{\alpha}^{a} = 0$ requiring equality between the total charge of bosons and the ghost like fields.

In this paper we shall focus on the mirror super-geometry obtained by first choosing a special form of the full spectrum of $U(1)^{p}$ gauge charges and integrating out some fermionic fields in the topological B-model. In this way, the mirror B-models will still have some fermionic directions. Our interest will be on the mirror of $ADE$ super-geometries and mainly on the fermionic extension of the ordinary $A_{r}$ class. First, we study the case of $A_{1}$ super-geometry, which is found to be closely related to the equation of the bosonic case in agreement with analysis using LG models. Ordinary $A_{1}$ geometry is
recovered by canceling the fermionic directions. Then we work out the mirror of a class of $A_r$ local super CY manifold extending the $A_1$ super-geometry. Finally, we discuss the mirror symmetry of local higher dimensional super-CY geometries. In particular we specialize on the mirror symmetry of the topological A-model using a fermionic extension of line bundle over $\mathbb{CP}^n$.

The organization of this paper is as follows. In section 2, we review mirror symmetry of local super CY manifolds. In section 3, we study mirror of $ADE$ super-geometries by exhibiting the method on the ordinary $A_r$ series. In section 4, we consider mirror super-geometries beyond $ADE$ and in section 5, we give a conclusion.

## 2 Mirror symmetry of local super CY manifolds

In this section, we review mirror symmetry for local (bosonic) CY manifolds \cite{20,21}; then we give the extension to the super case.

### 2.1 Bosonic CY

To begin, let us consider a two-dimensional $\mathcal{N} = 2$ supersymmetric linear sigma model described in terms of $n+p$ chiral superfields $\Phi_i$ with charge $q_a^i$, $i = 1, \ldots, n+p$; $a = 1, \ldots, p$ under $U(1)^{\otimes p}$ gauge symmetry \cite{22}. The geometry of the topological A-model can be analyzed by solving the D-term potential ($D^a = 0$) of the $\mathcal{N} = 2$ linear sigma model; that is

$$
\sum_{i=1}^{n+p} q_a^i |\phi_i|^2 = r^a, \quad a = 1, \ldots, p,
$$

(2.1)

where the $r^a$'s are FI coupling parameters and where the $\phi_i$'s are the leading scalar fields of the chiral superfield $\Phi_i$. Dividing by $U(1)^{\otimes p}$ gauge symmetry, one gets a $n$-dimensional toric variety $^1$

$$
V^n = \frac{\mathbb{C}^{n+p} \setminus S}{\mathbb{C}^{\ast p}},
$$

(2.2)

where the $p$ copies of $\mathbb{C}^\ast$ actions indexed by $a = 1, \ldots, p$ are given by

$$
\mathbb{C}^{\ast p} : \phi_i \to \lambda^{q_a^i} \phi_i, \quad i = 1, \ldots, n + p,
$$

(2.3)

with $\lambda$ a non zero complex number. The requirement for $V^n$ to be a local CY manifold is to impose the condition

$$
\sum_{i=1}^{n+p} q_a^i = 0.
$$

(2.4)

$^1$Note that this geometry can be represented by a toric diagram $\Delta(V^n)$ spanned by $k = n + p$ vertices $v_i$ in a $\mathbb{Z}^n$ lattice satisfying $\sum_{i=1}^{n+p} q_a^i v_i = 0$, $a = 1, \ldots, p$. 

On the supersymmetric field theoretic level, this relation implies that the underlying linear sigma model flows in infrared to a conformal field theory.

Following [20]-[24], the mirror B-model is a Landau-Ginzburg (LG) model with periodic fields \( \{Y_i\} \) dual to \( \{\Phi_i\} \) and connected as,

\[
\text{Re}(Y_i) = |\Phi_i|^2, \quad i = 1, \ldots, n + p, \tag{2.5}
\]

where \( \text{Re}(Y_i) \) denotes the real part of \( Y_i \). Under mirror transformation, eq(2.1) is mapped to

\[
\sum_i q_i^a Y_i = t^a, \quad a = 1, \ldots, p, \tag{2.6}
\]

with \( r^a = \text{Re}(t^a) \). Moreover the LG superpotential of the topological B-model reads as

\[
W(Y_1, \ldots, Y_{n+p}) = \sum_{i=1}^{n+p} e^{-Y_i}. \tag{2.7}
\]

For convenience, it is useful to use the following field redefinitions

\[
\hat{y}_i = e^{-Y_i}, \quad i = 1, \ldots, n + p, \tag{2.8}
\]

Then the superpotential \( W(\hat{y}_1, \ldots, \hat{y}_{n+p}) \) reads as

\[
W = \sum_{i=1}^{n+p} \hat{y}_i, \tag{2.9}
\]

and so eq(2.6) translates into the following projective hypersurface

\[
\prod_{i=1}^{n+p} \hat{y}_i^{q_i^a} = e^{-t_a}, \quad a = 1, \ldots, p, \tag{2.10}
\]

with the manifest projective symmetry \( \hat{y}_i \to \lambda \hat{y}_i \) following from the CY condition (2.4). The solution of the constraint eqs(2.10) and projective symmetry defines a \((n + p) - p - 1 - 1 = n - 2\) dimensional toric manifold given by a holomorphic hypersurface in \( \mathbb{C}^{n-1} \)

\[
F(y_1, \ldots, y_{n-1}) = 0. \tag{2.11}
\]

To recover the right dimension of the original manifold; that is a complex dimension \( n \) local CY manifold, we generally use an ad-hoc trick which consist to add by hand two extra holomorphic variables \( u \) and \( v \) combined in a quadratic form \( uv \) and modify previous equation as

\[
F(y_1, \ldots, y_{n-1}) = uv. \tag{2.12}
\]

The main objective in what follows is to extend this analysis to a linear A-model with fermionic (ghosts) fields and study the resulting mirror B-model. Besides the generalization of above results to local super-CY manifolds, one of the results following from
this fermionic extension is the re-derivation of eq (2.12) without need of adding by hand of the term \( uv \) of right hand. As we will show later, the new manifold is given by a hypersurface of type,

\[
G(y_1, \ldots, y_{n-1}) = \chi \eta,
\]

(2.13)

where, instead of \( u \) and \( v \) variables, we have now the variables \( \chi \) and \( \eta \) which are ghost like fields. As we will see, this relation defines an even complex \( n \) dimension hypersurface of the complex superspace \( \mathbb{C}^{(n-1)|2} \). This geometry may be then viewed as alternative elevation of (2.11). The standard elevation eq (2.12) is given by the purely bosonic hypersurface in \( \mathbb{C}^{(n+1)|0} \).

2.2 Mirror of local super-CY

Here we want to study the mirror of the fermionic extension of the topological A-model on local toric CY manifolds discussed in the previous subsection. Actually, this may be viewed as an extension of paper [19] which has dealt with the case of compact bosonic toric manifolds. Important examples of that work have been projective spaces and products thereof.

2.2.1 Extended A-model

Roughly, the extension corresponds to adding, to the usual bosonic superfield \( \Phi_j \), a set of \( f \)-fermionic chiral superfields \( \Psi_\alpha \) with \( Q^j_a \) charge under \( U(1)^{\otimes p} \) gauge symmetry. We then have

\[
\Phi_j \rightarrow e^{i \sum_a \vartheta_a q^a_j} \Phi_j, \quad j = 1, \ldots, n + p,
\]

\[
\Psi_\alpha \rightarrow e^{i \sum_a \vartheta_a Q^a_\alpha} \Psi_\alpha, \quad \alpha = 1, \ldots, f,
\]

(2.14)

with same transformations for the leading component fields \( \phi_j \) and \( \psi_\alpha \) respectively and where the \( \vartheta_a \)'s are the gauge group parameters. The full spectrum of \( U(1)^{\otimes p} \) charge vectors \( q^a = (q^a|Q^a) \) thus takes the form

\[
(q^a|Q^a) = (q^a_1, \ldots, q^a_{p+n}|Q^a_1, \ldots, Q^a_f), \quad a = 1, \ldots, p.
\]

(2.15)

The extended D\( a \)-term equations resulting from the above generalized A-model is given by minimizing the Kähler potential of the 2D \( \mathcal{N} = 2 \) generalized superfield action

\[
S_{\mathcal{N}=2} = \int d^2 \sigma d^4 \theta K + \left( \int d^2 \sigma d^2 \theta W + cc \right),
\]

(2.16)

with respect to the gauge superfields \( V_a \). In above relation, \( K \) is the usual gauge invariant Kähler term and \( W \) is a chiral superpotential with superfield dependence as,

\[
K = K[\Phi_1, \ldots, \Phi^+_{n+p}; \Psi_1, \ldots, \Psi^+_f; V_1, \ldots, V_p],
\]

\[
W = W[\Phi_1, \ldots, \Phi^+_{n+p}; \Psi_1, \ldots, \Psi_f],
\]

(2.17)
as well as coupling constant moduli which have not been specified. Using the explicit expression of $K$ and putting back into

$$D^a = \frac{\partial K}{\partial V_a} |_{\theta = 0} = 0,$$  (2.18)

we get the following $D^a$-term equations

$$\sum_{i=1}^{p+n} q_i^a |\phi_i|^2 + \sum_{\alpha=1}^{f} Q_{\alpha}^a |\psi_{\alpha}|^2 = \text{Re}(t^a), \quad a = 1, \ldots, p,$$  (2.19)

where $\text{Re}(t^a)$ stands for FI coupling constant. Strictly speaking, this is an even hypersurface embedded in the complex supermanifold $\mathbb{C}^{p+n|f}$ with dimension $(p + n|f)$. Therefore, the space of vacua of above generalized supersymmetric action is a toric super-manifold $\mathcal{V}^n$ obtained by dividing $\mathbb{C}^{p+n|f}$ by $U(1)^g$ gauge symmetry group in same logic as in eq(2.2). Thus we have

$$\mathcal{V}^n = \frac{\mathbb{C}^{p+n|f}}{\mathbb{C}^* p} \setminus S.$$  (2.20)

With this relation at hand, one can go ahead and try to develop the toric super-geometry of these local super-manifolds by mimicking the standard toric geometry analysis of toric varieties. We shall not do this here; what we will do rather is to study some specific examples with direct link to type II superstring theory compactifications. The first class of these examples concerns specific fermionic extensions of $ADE$ geometries. The local super-CY (LSCY) condition reads as follows

$$\sum_{i=1}^{p+n} q_i^a = 0, \quad \sum_{\alpha=1}^{f} Q_{\alpha}^a = 0.$$  (2.21)

This constraint equation is required by the invariance of holomorphic measure $\Omega^{(p+n|f)}$ of the complex superspace $\mathbb{C}^{p+n|f}$,

$$\Omega^{(p+n|f)} = \left( \prod_{i=1}^{n+p} d\phi_i \right) \left( \prod_{\alpha=1}^{f} d\psi_{\alpha} \right),$$  (2.22)

under $\mathbb{C}^* p$ toric symmetry. In what follows, we shall fix our attention on those local CY super-manifolds $\mathcal{V}^n$ obeying the following special solution,

$$\sum_{i=1}^{p+n} q_i^a = 0, \quad \sum_{\alpha=1}^{f} Q_{\alpha}^a = 0, \quad a = 1, \ldots, p.$$  (2.23)

In this particular class of solutions of eq(2.21), we have taken the bosonic subvariety as a local CY manifold. This is the case for bosonic sub-varieties given by the fermionic extensions of $ADE$ geometries we are interested in here. It is also remarkable that fermionic directions obey as well a CY condition for bosonic manifold. In conclusion section, we will make a comment on this issue.
2.2.2 Extended B-model

Under T-duality, the bosonic superfield $\Phi_i$ of the linear super-toric sigma model is replaced by a dual superfield $Y_i$ as before, while the fermionic superfield $\Psi_\alpha$ is dualized by the triplet $(X_\alpha, \eta_\alpha, \chi_\alpha)$ \[8\]. The bosonic superfields $X_\alpha$ is related to $\Psi_\alpha$ as

$$\text{Re}(X_\alpha) = -|\Psi_\alpha|^2, \quad \alpha = 1, \ldots, f,$$

and the accompanying pair of chiral superfields $\{\eta_\alpha\}$ and $\{\chi_\alpha\}$ are fermionic superfields required by the preservation of the super-dimension and hence the total central charge. Under this dualization, the original complex superspace $\mathbb{C}^{p|n}$ gets mapped to

$$\mathbb{C}^{p+n+f|2f}. \quad \text{(2.25)}$$

The extended B-model, mirror to the above fermionic extended A-model with superfield action $S_{N=2}$, is given in terms of the following path integral, see also \[19\],

$$Z = \int \mathcal{D}F \left[ \prod_{a=1}^p \delta (F_a - t_a) \right] \exp \left[ \int W(Y, X, \eta, \chi) \right], \quad \text{(2.26)}$$

where we have set $\mathcal{D}F = (\prod_i dY_i) (\prod_\alpha dX_\alpha d\eta_\alpha d\chi_\alpha)$. In this relation the $F_a$’s are the D-terms of the extended A-model and $W = W(Y, X, \eta, \chi)$ is the extended LG superpotential of the topological B-model. They are as follows,

$$F_a = \sum_{i=1}^{n+p} q^a_i Y_i - \sum_{\alpha=1}^f Q^a_\alpha X_\alpha, \quad a = 1, \ldots, p,$$

$$W = \left( \sum_{i=1}^{n+p} e^{-Y_i} + \sum_{\alpha=1}^f e^{-X_\alpha} (1 + \eta_\alpha \chi_\alpha) \right). \quad \text{(2.27)}$$

To extract informations on the local super-geometry of the B-model, we need to integrate out the delta functions. Below, we shall focus our attention on the special case where $f = 2p$ and exemplify with models which have been used in type II superstring theory compactifications.

3 Mirror of $A_r$ super-geometries

Here we focus on the super-geometry extending the usual ordinary $A_r$ geometries. A quite similar analysis is a priori possible for the $DE$, affine and indefinite extensions.

3.1 Local super $A_1$ geometry

To illustrate the construction, we initially consider the example of the model $A_1$. This is a supersymmetric gauge theory with a $U(1)$ gauge symmetry and three chiral superfields
\( \Phi \) with charge \((1, -2, 1)\) together with a real gauge superfield \( V \). The D-term constraint (equation of motion of \( V \)) reads as
\[
|\Phi_1|^2 - 2|\Phi_2|^2 + |\Phi_3|^2 = \text{Re}(t). \tag{3.1}
\]
This geometry describes the Kahler deformation of the \( A_1 \) singularity of the ALE spaces
\[
uv = z^2, \tag{3.2}
\]
where \( u, v \) and \( z \) are the generators of gauge invariants. They are realized in terms of the scalar fields as follows
\[
u = \Phi_2^2 \Phi_2, \quad v = \Phi_2^2 \Phi_2, \quad z = \Phi_1 \Phi_2 \Phi_3. \tag{3.3}
\]
For generalizations to rank \( r \geq 2 \) ordinary \( ADE \) geometries as well as affine extensions and beyond see [25].

### 3.1.1 Extended model

Basically, there is an abundance of possible fermionic extensions of above model. It may be limited by imposing the LSCY condition [221]. Since we are interested in the case \( f = 2p = 2 \), the full spectrum of \( U(1) \) charge that one can have is given by the vector
\[
q' = (q|Q) = (1, -2, 1|1, -1). \tag{3.4}
\]
In this construction, \( A_1 \) model appears as a subsystem while, as noted before and as far as super-CY condition is concerned, there are several solution of eq(2.21). Using the extension (3.4), the D-term for the \( A_1 \) super-geometry becomes
\[
|\Phi_1|^2 - 2|\Phi_2|^2 + |\Phi_3|^2 + |\Psi_1|^2 - |\Psi_2|^2 = \text{Re}(t), \tag{3.5}
\]
where \( \text{Re}(t) \) is the unique Kahler parameter of the model.

### 3.1.2 Mirror of extended model

Applying mirror transformation to above extended A-model with \( A_1 \) super-geometry, the associated mirror B-model is obtained in the same way as presented in previous subsection. The corresponding extended LG path integral eqs(2.26-2.27) takes the following form
\[
Z = \int \mathcal{D}f \delta [Y_1 - 2Y_2 + Y_3 - X_1 + X_2 - t] \\
\times \exp \left( \sum_{i=1}^3 e^{-Y_i} + \sum_{\alpha=1}^2 e^{-X_\alpha} (1 + \eta_\alpha \chi_\alpha) \right). \tag{3.6}
\]
with \( \mathcal{D}F = (\prod_{i=1}^{3} dY_i) (\prod_{\alpha=1}^{2} dX_\alpha d\eta_\alpha d\chi_\alpha) \). As usually, to extract information on the mirror super-geometry of the B-model, we integrate out the fermionic fields \( \eta_1, \chi_1 \). Then solving the delta function constraint by integrating out \( X_1 \) yields,

\[
Z = \int \mathcal{D}\tilde{F} \left( e^{-Y_1 + 2Y_2 - Y_3} e^{-X_2} \right) \exp \left( \sum_{i=1}^{3} e^{-Y_i} + e^{-X_2} \left[ 1 + \eta_2 \chi_2 + e^t Y_1 + 2Y_2 - Y_3 e^{-X_2} \right] \right),
\] (3.7)

where \( \mathcal{D}\tilde{F} = (\prod_{i=1}^{3} dY_i) (dX_2 d\eta_2 d\chi_2) \). Now, introducing new complex variables \( x_i \) and \( y_i \) such that

\[
x = e^{-X_2}, \quad y_i = e^{-Y_i}, \quad i = 1, 2, 3,
\] (3.8)

the above partition function becomes

\[
Z = \int (dx d\eta_2 d\chi_2) \prod_{i=1}^{3} \left( \frac{dy_i}{y_i^3} \right) \exp \left( \sum_{i=1}^{3} y_i + x \left[ 1 + \eta_2 \chi_2 + e^t \frac{y_1 y_3}{y_2^2} \right] \right).
\] (3.9)

The rescaling \( \tilde{x} = (x/y_3^2) \) allows us to rewrite the above path integral as follows

\[
Z = \int dy_1 dy_2 dy_3 d\tilde{x} d\eta_2 d\chi_2 \exp \left( \sum_{i=1}^{3} y_i + \tilde{x} y_3^2 \left[ 1 + \eta_2 \chi_2 + e^t \frac{y_1 y_3}{y_2^2} \right] \right).
\] (3.10)

In order to get the mirror of local super-geometry \( A_1 \), we can see \( \tilde{x} \) as a Lagrange multiplier; integrating it out one gets the following equation of motion

\[
1 + \eta_2 \chi_2 + \frac{y_1 y_3}{y_2^2} e^t = 0.
\] (3.11)

The objective now is to interpret this equation as the mirror constraint equation of the topological A-model on \( A_1 \) super-geometry. In fact, we can solve (3.11) as

\[
\frac{y_1 y_3}{y_2^2} = -(1 + \eta_2 \chi_2)e^{-t}.
\] (3.12)

Replacing now \( t \) by \( t' = t + i\pi \), one absorbs the minus sign

\[
\frac{y_1 y_3}{y_2^2} = e^{-t'} + \eta_2 \chi_2 e^{-t'}.
\] (3.13)

Actually, this equation is quite similar to the bosonic one; except now we have the presence of the additional contribution \( \eta_2 \chi_2 e^{-t'} \) induced by the fermionic fields. It is easy to see that in the patch \( \eta_2 = \chi_2 = 0 \), we recover the bosonic mirror constraint equation of ALE space with \( A_1 \) singularity, namely

\[
\frac{y_1 y_3}{y_2^2} = e^{-t'}.
\] (3.14)

Return to equation (3.13); a straightforward computation reveals that this equation can be solved by taking the following parameterization

\[
y_1 = y, \quad y_3 = \frac{1}{y}, \quad y_2 = (1 + \eta_2 \chi_2)^{-\frac{1}{2}} e^\frac{t}{2}.
\] (3.15)
where we have set $t' = t$. We thus end with the following LG potential

$$y + \frac{1}{y} + (1 + \eta_2 \chi_2)^{-\frac{1}{2}} e^{\frac{t}{2}} = 0,$$

which is mirror to sigma model on $A_1$ super-geometry. This equation has the three following remarkable features:

1. For $\eta_2 = \chi_2$, we recover the usual bosonic LG superpotential mirror to the bosonic $A_1$ geometry

$$y + \frac{1}{y} + e^{\frac{t}{2}} = 0. \quad (3.17)$$

2. In the case $t \to 0$, one discovers the rule to define the super extension of the $A_1$ singularity with $U(1)$ charges as in eq (3.14). The mirror of super $A_1$ singularity can be then defined as follows

$$y_1 y_3 = y_2^2 (1 + \eta_2 \chi_2) \quad (3.18)$$

in agreement with indication from conformal Landau Ginzburg field models where adjunction of quadratic terms do not modify the total central charge. Moreover, by using fermionic statistics which forbids higher powers in $\eta_2$ and $\chi_2$, one may define extensions of above $A_1$ singularity.

3. In the limit where the condensate modulus $\eta_2 \chi_2$ is small, eq (3.16) reduces to

$$y + \frac{1}{y} + e^{\frac{t}{2}} - \frac{1}{2} \eta_2 \chi_2 e^{\frac{t}{2}} = 0. \quad (3.19)$$

By making the identification $\frac{1}{2} \eta_2 \chi_2 e^{-\frac{t}{2}}$ with the $uv$ of the relation $(2.11 \ 2.12)$, one discovers that the $uv$ term added by hand in the bosonic case to recover the right dimension of mirror manifold, is generated in a natural way in super-geometry. In the limit $t \to 0$, we have

$$y + \frac{1}{y} + 1 = \eta_2' \chi_2', \quad (3.20)$$

where we have set $\eta_2' \chi_2' = \frac{1}{2} \eta_2 \chi_2$. This is a complex two dimensions even hypersurface of $C^{1|2}$.

### 3.2 Super $A_p$

Now, we would like to push further the above results on super $A_1$ to the class of $A_p$ super-geometry series having usual $A_p$ geometry as local bosonic Calabi-Yau submanifolds. To start, recall that $A_p$ geometry has a description in terms of $U(1)^{\otimes p}$ sigma model
involving \((p+2)\) chiral fields with the bosonic charge \(p \times (p+2)\) matrix

\[
q_i^a = \begin{pmatrix}
1 & -2 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 1 & -2 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -2 & 1 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -2 & 1 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & -2 \\
\end{pmatrix}
\]  
(3.21)

Basically, there are several fermionic extensions of the above A-model. However as we mentioned before, we consider a model with \(2p\) fermionic fields. In this way, the SLCY condition may limit the choice of the charge matrix. For a reason to be specified later on, we propose the following \(U(1)^{\otimes p}\) charge spectrum for ghost like superfields

\[
Q_a^a = \begin{pmatrix}
1 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & -1 \\
\end{pmatrix}
\]  
(3.22)

This representation constitutes a simple and natural extension of eq(3.4) recovering \(A_1\) super-geometry as a leading example; other representations are obviously possible. This choice of \(U(1)^{\otimes p}\) charge matrix for ghost like fields allows us to handle each line as an individual \(A_1\) super-geometry. In this way, we can easily repeat the same lines that we have done for the super \(A_1\) case. Let us give some details below.

Roughly, LG mirror superpotential is given in terms of the following path integral

\[
Z = \int \mathcal{D}F \left[ \prod_{a=1}^{p} \delta (F_a - t_a) \right] \exp \left( \sum_i e^{-Y_i} + \sum_a e^{-X_a} (1 + \eta_a \chi_a) \right),
\]  
(3.23)

where now \(\mathcal{D}F = (\prod_{i=1}^{p+2} dY_i) (\prod_{a=1}^{2p} dX_a d\eta_a d\chi_a)\) and where we have set

\[
F_a = Y_a - 2Y_{a+1} + Y_{a+2} - X_{2a-1} + X_{2a}.
\]  
(3.24)

This partition function \(Z\) has \(p\) delta functions \(\delta (F_a - t_a)\). To get the mirror super-geometry, we first integrate out the fermionic field variables \((\eta_{1a}\chi_{1a})\) leaving only a dependence on \((\eta_{2a}\chi_{2a})\), \(a = 1, 2, \ldots, p\) and then we use delta functions to eliminate the field variables \(X_{2a-1}\). In doing so and following the same way as before, we get \(p\) equations of motion,

\[
1 + \eta_{2a}\chi_{2a} = \prod_i y_i^{q_i^a}, \quad a = 1, \ldots, p.
\]  
(3.25)
To see how to obtain these equations, let us consider the case of $A_2$ super-geometry. This is a $U(1)^2$ gauge theory with four chiral superfields ($\Phi_1, \Phi_2, \Phi_3, \Phi_4$) and four ghost like ones ($\Psi_1, \Psi_2, \Psi_3, \Psi_4$). The full spectrum of $U(1)^2$ gauge charges is given by

$$q'^1 = (1, -2, 1, 0|1, -1, 0, 0),$$
$$q'^2 = (0, 1, -2, 1|0, 0, 1, -1).$$ (3.26)

The above path integral reduces in present case to

$$Z = \int D\delta (F_1 - t_1) \delta (F_2 - t_2) \exp \left( \sum_{i=1}^{4} e^{-Y_i} + \sum_{\alpha=1}^{4} e^{-X_\alpha}(1 + \eta_\alpha \chi_\alpha) \right),$$ (3.27)

with field measure $D\delta = (\prod_{i=1}^{4} dY_i) (\prod_{\alpha=1}^{4} dX_\alpha d\eta_\alpha d\chi_\alpha)$ and D-terms as

$$F_1 = Y_1 - 2Y_2 + Y_3 - X_1 + X_2,$$
$$F_2 = Y_2 - 2Y_3 + Y_4 - X_3 + X_4.$$ (3.28)

Integrating in similar way as we have done for $A_1$ super-geometry and making the same variable changes, we get

$$Z = \int D\delta' \exp \left[ \sum_{i=1}^{4} y_i + \tilde{x}_1 y_2^2 \left( 1 + \eta_2 \chi_2 + \frac{e^{t'_1} y_1 y_3}{y_2^2} \right) \right]$$
$$\times \exp \left[ \tilde{x}_2 y_3^2 \left( 1 + \eta_4 \chi_4 + \frac{e^{t'_2} y_2 y_4}{y_3^2} \right) \right].$$ (3.29)

with $D\delta' = (\prod_{i=1}^{4} dy_i) (dx_1 dx_2 d\eta_2 d\eta_4 d\chi_4)$. In this case, we have two equations of motion which are given by

$$\frac{y_1 y_3}{y_2^2} = (1 + \eta_2 \chi_2) e^{-t'_1},$$
$$\frac{y_2 y_4}{y_3^2} = (1 + \eta_4 \chi_4) e^{-t'_2},$$ (3.30)

with $t'_a = t_a + i\pi$. After solving these two equations, we come up with the following mirror relation

$$\frac{1}{y} + \left( e^{\frac{t'_1}{2}} (1 + \eta_2 \chi_2) \right) + y + y^2 \left( e^{-t'_2} e^{\frac{t'_1}{2}} (1 + \eta_4 \chi_4)(1 + \eta_2 \chi_2)^{\frac{1}{2}} \right) = 0,$$ (3.31)

which should be compared with the usual mirror relation of ordinary $A_2$ geometry $\frac{1}{y} + 1 + y + y^2 = 0$ associated to the limit $t'_a = 0$ and $\eta_2 = \chi_2 = 0$.

4 More on mirror super-geometry

The method developed so far can be also used to build other local super CY manifolds. A simple extension of above $A_1$ super-geometry analysis is given by a sigma model
with target space involving a fermionic extension of line bundle over \( CP^p \) with \( p \geq 2 \). The case \( p = 1 \) corresponds exactly to the \( A_1 \) super-geometry studied previously. Let us analyze the case \( p = 2 \); that is the line bundle \( \mathcal{O}(-3) \) over \( CP^2 \). It admits a \( U(1) \) sigma model description in terms of four bosonic chiral fields with charge vector \((1, 1, 1, -3)\) and the corresponding D-term equation is given by

\[
|\Phi_1|^2 + |\Phi_2|^2 + |\Phi_3|^2 - 3|\Phi_4|^2 = \text{Re}(t)
\] (4.1)

Adding now 2 ghost like field variables \( \Psi_1 \) and \( \Psi_2 \) with vector charge \((1, -1)\), as required by SLCY condition, the D-term constraint equation of the extended A-model is given by

\[
|\Phi_1|^2 + |\Phi_2|^2 + |\Phi_3|^2 - 3|\Phi_4|^2 + |\Psi_1|^2 - |\Psi_2|^2 = \text{Re}(t).
\] (4.2)

The corresponding mirror super-geometry is given in terms of the following path integral

\[
Z = \int \mathcal{D}F \delta(F - t) \exp \left( \sum_{i=1}^{4} e^{-Y_i} + \sum_{\alpha=1}^{2} e^{-X_{\alpha}}(1 + \eta_\alpha \chi_\alpha) \right).
\] (4.3)

with \( \mathcal{D}F = (\prod_{i=1}^{4} dY_i)(\prod_{\alpha=1}^{2} dX_{\alpha} d\eta_\alpha d\chi_\alpha) \) and

\[
F = Y_1 + Y_2 + Y_3 - 3Y_4 - X_1 + X_2.
\] (4.4)

Now integrating out the fermionic fields \( \eta_1, \chi_1 \) and solving the delta function constraint by eliminating \( X_1 \), we get

\[
Z = \int \mathcal{D}F \left( e^{-Y_1 - Y_2 - Y_3 + 3Y_4} e^{-X_2} \right)
\times \exp \left( \sum_{i=1}^{4} e^{-Y_i} + e^{-X_2}(1 + \eta_2 \chi_2 + e^t e^{-Y_1 - Y_2 - Y_3 + 3Y_4}) \right),
\] (4.5)

with \( \mathcal{D}F = (\prod_{i=1}^{4} dY_i)(dX_2 d\eta_2 d\chi_2). \) Using the following field re-definition

\[
y_i = e^{-Y_i}, \quad x = e^{-X_2}
\] (4.6)

the above equation becomes

\[
Z = \int \left( \prod_{i=1}^{4} \frac{dy_i}{y_4^i} \right) (dxd\eta_2 d\chi_2) \exp \left[ \sum_{i=1}^{4} y_i + x \left( 1 + \eta_2 \chi_2 + \frac{e^t y_1 y_2 y_3}{y_4^i} \right) \right].
\] (4.7)

By help of the following rescaling \( \bar{x} = \frac{x}{y_4^i} \), the mirror geometry becomes

\[
Z = \int d\bar{x} d\eta_2 d\chi_2 \prod_{i=1}^{4} dy_i \exp \left[ \sum_{i=1}^{4} y_i + \bar{x} y_4^i \left( 1 + \eta_2 \chi_2 + \frac{e^t y_1 y_2 y_3}{y_4^i} \right) \right].
\] (4.8)

In this case, the equation of motion reads as

\[
\frac{y_1 y_2 y_3}{y_4^i} = -(1 + \eta_2 \chi_2)e^{-t}.
\] (4.9)
Absorbing the minus sign by replacing $t$ by $t + i\pi$, the above equation becomes

$$\frac{y_1y_2y_3}{y_4} = e^{-t'} + \eta_2\chi_2 e^{-t'}.$$  \hspace{1cm} (4.10)

This can be easily solved by the following parameterization

$$y_1 = x, \quad y_2 = y, \quad y_3 = \frac{1}{xy},$$

$$y_4 = (1 + \eta_2\chi_2)^{-\frac{1}{3}} e^{\frac{t'}{3}}.$$ \hspace{1cm} (4.11)

The superpotential describing the mirror of the super-geometry reads as

$$x + y + \frac{1}{xy} + (1 + \eta_2\chi_2)^{-\frac{1}{3}} e^{\frac{t'}{3}} = 0.$$ \hspace{1cm} (4.12)

For $\eta_2\chi_2 = 0$, we rediscover the usual bosonic relation.

5 Conclusion

In this paper, we have studied mirror symmetry of A-model on Calabi-Yau super-manifolds constructed as fermionic extensions of local toric CY satisfying the SLCY condition $\sum_{i=1}^{p+n} q_i^a = \sum_{\alpha=1}^{2p} Q_\alpha^a$. By solving this condition as $\sum_{i=1}^{p+n} q_i^a = 0$ and $\sum_{\alpha=1}^{2p} Q_\alpha^a = 0$ separately; we have considered two classes of mirror super-geometries. The first class deals with a special fermionic extension of ordinary geometries and the second class concerns a set of sigma models with target space involving a fermionic extension of line bundle over $\mathbb{C}P^n$ with $n \geq 2$. The representations studied here are not the general ones since the bosonic subvariety of super-manifold considered here is taken as a Calabi-Yau manifold. This condition is obviously not a necessary condition for building Calabi-Yau super-manifolds. This work may be viewed as a extension of [19] which has dealt with bosonic compact toric varieties. The mirror geometries studied in that paper have dealt only with bosonic variables. However, here the mirror B-models involve fermionic directions captured by the ghost like fields. In dealing with the mirror of $A_r$ super-geometries, we have shown that these local CY super-manifolds are described by algebraic geometry equations quite similar to the bosonic case. The later can be obtained by canceling fermionic directions. Moreover, we have found that in super-geometry, the right dimension of the bosonic CY subvariety is recovered in natural as shown on [19]. Finally we have shown that this approach applies as well to higher dimensional mirror super-geometries; mirror of A-model on super line bundle over $\mathbb{C}P^n$ studied in section 4 is an example amongst others.

In the end of this study, we would like to add that along with ordinary CY manifolds embedded in $\mathbb{C}^{n|0}$ and super-CY manifolds embedded in complex space $\mathbb{C}^{n|m}$, we
may also have super CY manifolds embedded in the purely fermionic space $\mathbb{C}^{0|m}$ without basic bosonic coordinates. These special super CY varieties are then hypersurface in $\mathbb{C}^{0|m}$ involving ghost fields like only.

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