LINEAR FUNCTIONS AND DUALITY ON THE INFINITE POLYTorus

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Abstract. We consider the following question: Are there exponents $2 < p < q$ such that the Riesz projection is bounded from $L^q$ to $L^p$ on the infinite polytorus? We are unable to answer the question, but our counter-example improves a result of Marzo and Seip by demonstrating that the Riesz projection is unbounded from $L^\infty$ to $L^p$ if $p \geq 3.31138$. A similar result can be extracted for any $q > 2$. Our approach is based on duality arguments and a detailed study of linear functions. Some related results are also presented.

1. Introduction

Let $T^\infty = T \times T \times \cdots$ denote the countably infinite cartesian product of the torus $T = \{z \in \mathbb{C} : |z| = 1\}$. We equip the $T^\infty$ with its Haar measure $\mu_\infty$, which is equal to the infinite product of the normalized Lebesgue arc measure on $T$ in each variable. Let $1 \leq p \leq \infty$. Every $f$ in $L^p(T^\infty)$ has a Fourier series expansion

$$f(z) = \sum_{\alpha \in \mathbb{Z}_0^\infty} c_\alpha z^\alpha,$$

where the Fourier coefficients are defined in the standard way and $\alpha \in \mathbb{Z}_0^\infty$ means that the multi-index $\alpha$ contains only a finite number of non-zero components. The Riesz projection on $T^\infty$ is defined by

$$Pf(z) = \sum_{\alpha \in \mathbb{N}_0^\infty} c_\alpha z^\alpha.$$

The initial motivation for the present paper is the following.

Question. What is the largest $p = p_\infty$ such that the Riesz projection $P$ is bounded from $L^\infty(T^\infty)$ to $L^p(T^\infty)$?

The Riesz projection is certainly a contraction on the Hilbert space $L^2(T^\infty)$ and since $\|f\|_{L^2(T^\infty)} \leq \|f\|_{L^\infty(T^\infty)}$, we get that $p_\infty \geq 2$. This question has previously been investigated by Marzo and Seip [8] who demonstrated that $p_\infty \leq 3.67632$. We will obtain the following improvement.

Theorem 1. $p_\infty \leq p = 3.31138 \ldots$, where $p$ denotes the unique positive solution of the equation

$$\Gamma \left(1 + \frac{p}{2}\right)^\frac{1}{p} = \frac{2}{\sqrt{\pi}}.$$
For $2 \leq p \leq q \leq \infty$, let $\|P\|_{q,p}$ denote the norm of the Riesz projection from $L^q(\mathbb{T}^\infty)$ to $L^p(\mathbb{T}^\infty)$. In the case that the Riesz projection is unbounded, we use the convention $\|P\|_{q,p} = \infty$. As explained in [8], for each fixed $2 \leq q \leq \infty$ there is a number $2 \leq p_q < q$, called the critical exponent, with the property that

$$\|P\|_{p,q} = \begin{cases} 1 & \text{if } p \leq p_q, \\ \infty & \text{if } p > p_q. \end{cases}$$

The dichotomy (2) is a direct consequence of the fact that we are on the infinite polytorus. Let $f$ be a function in the unit ball of $L^q(\mathbb{T}^\infty)$ such that $\|Pf\|_{L^p(\mathbb{T}^\infty)} > 1$. Consider the function

$$f_2(z) = f(z_1, z_3, z_5, \ldots) \cdot f(z_2, z_4, z_6, \ldots)$$

which is also in the unit ball of $L^q(\mathbb{T}^\infty)$. The Riesz projection (1) acts independently on the variables, so we find that

$$Pf_2(z) = Pf(z_1, z_3, z_5, \ldots) \cdot Pf(z_2, z_4, z_6, \ldots)$$

which implies that $\|P f_2\|_{L^p(\mathbb{T}^\infty)} = \|P f\|_{L^p(\mathbb{T}^\infty)}^2 > \|P f\|_{L^p(\mathbb{T}^\infty)}$. This procedure can be repeated and so we obtain (2). The example from [8] producing $p_\infty \leq 3.67632$ is a function of only two variables.

The present paper is inspired by [3], where linear functions are used as building blocks in an similar way to what was just described to construct a counter-example related to Nehari’s theorem for Hankel forms on $\mathbb{T}^\infty$. The example from [3] improves on an earlier example from [2] by replacing a function of two variables by a linear function in an infinite number of variables.

Our approach differs from that of [8] (and [2]) in that we do not attempt to directly construct a counter-example, but instead use duality arguments to infer its existence. This approach leads us to consider the Hardy spaces $H^p(\mathbb{T}^\infty)$, which are the subspaces of $L^p(\mathbb{T}^\infty)$ consisting of elements such that $Pf = f$. A standard argument involving the Hahn–Banach theorem (see e.g. [5, Sec. 7.2]) yields that

$$\inf_{P\psi = \varphi} \|\psi\|_{L^r(\mathbb{T}^\infty)} = \|\varphi\|_{H^r(\mathbb{T}^\infty)} = \sup_{f \in H^r(\mathbb{T}^\infty)} \left| \frac{(f, \varphi)_{L^2(\mathbb{T}^\infty)}}{\|f\|_{H^r(\mathbb{T}^\infty)}} \right|$$

for $1 \leq r < \infty$ and $q^{-1} + r^{-1} = 1$. We will choose $\varphi$ and try to find the optimal $f$ in $H^r(\mathbb{T}^\infty)$ attaining the supremum. This will ensure the existence of $\psi$ in $L^q(\mathbb{T}^\infty)$ attaining the infimum, which be our counter-example through (2).

As we shall see in Section 3 it turns out that if we know the optimal $f$ in the supremum on the right hand side of (3), we can in certain cases construct the element $\psi$ in $L^q(\mathbb{T}^\infty)$ of minimal norm such that $P\psi = \varphi$, thereby attaining the infimum on the left hand side of (3).

As in [3] we will primarily be working with linear functions, which are of the form

$$f(z) = \sum_{j=1}^{\infty} c_j z_j.$$ 

Clearly, $\|f\|_{H^q(\mathbb{T}^\infty)} = \sum_{j \geq 1} |c_j|^2$ and we easily check that $\|f\|_{H^\infty(\mathbb{T}^\infty)} = \sum_{j \geq 1} |c_j|$. For $1 \leq p < \infty$, optimal norm estimates are given by Khintchine’s inequality.
Define
\[ a_p = \min \left( 1, \Gamma \left( 1 + \frac{p}{2} \right) \right) \quad \text{and} \quad b_p = \max \left( 1, \Gamma \left( 1 + \frac{p}{2} \right) \right). \]

If \( f \) is a linear function \((1)\) and \( 1 \leq p < \infty \), then we restate a result from \((7)\) as
\[ a_p \| f \|_{H^2(T^\infty)} \leq \| f \|_{H^p(T^\infty)} \leq b_p \| f \|_{H^2(T^\infty)} \]
and the constants in \((5)\) are optimal. We shall obtain the following companion inequality for dual norms, which might be of independent interest.

**Theorem 2.** Let \( 1 \leq p < \infty \). If \( f \) is a linear function \((1)\), then
\[ b_p^{-1} \| f \|_{H^2(T^\infty)} \leq \| f \|_{(H^p(T^\infty))'} \leq a_p^{-1} \| f \|_{H^2(T^\infty)}. \]

The constants are optimal.

**Remark.** In the case \( p = \infty \), it is easy to deduce by similar considerations (Lemma \((3)\)) that \( \| f \|_{(H^\infty(T^\infty))'} = \sup_{j \geq 1} |c_j| \) if \( f \) is a linear function \((1)\).

Optimality of the constants containing the Gamma function in \((5)\) and \((6)\) both arise from the function
\[ f(z) = \frac{z_1 + z_2 + \cdots + z_d}{\sqrt{d}} \]
as \( d \to \infty \) through the central limit theorem. In view of \((2)\) and \((3)\), we can therefore obtain the following general result. Note that Theorem \((1)\) corresponds to the particular case \( q = \infty \), since \( \Gamma(3/2) = \sqrt{\pi}/2 \).

**Theorem 3.** Let \( 2 \leq p \leq q \leq \infty \) and set \( q^{-1} + r^{-1} = 1 \). If
\[ \Gamma \left( 1 + \frac{p}{2} \right) \Gamma \left( 1 + \frac{r}{2} \right) > 1, \]
then the Riesz projection is unbounded from \( L^q(T^\infty) \) to \( L^p(T^\infty) \).

**Remark.** Theorem \((3)\) is an improvement on the same statement with requirement \( p/2 \cdot r/2 > 1 \), which can be deduced from a one-variable example found in \((2)\), Sec. 4]. Here is an alternative example to that of \((2)\) obtained by our approach using the Hahn–Banach theorem. For \( w \in \mathbb{D} \), the functional of point evaluation \( f \mapsto f(w) \) has norm \((1 - |w|^2)^{-1/r}\) on \( H^r(T) \) and the analytic symbol is \( \varphi_w(z) = (1 - wz)^{-1} \). Hence, if \( w = \varepsilon > 0 \) then \( \| \varphi_\varepsilon \|_{(H^r(T))'} = 1 + r^{-1} \varepsilon^2 + O(\varepsilon^4) \) as \( \varepsilon \to 0 \). Furthermore,
\[ \| \varphi_\varepsilon \|_{H^r(T)} = \| (1 - \varepsilon z)^{-r/2} \|_{H^2(T)}^{2/p} = 1 + \frac{p}{4} \varepsilon^2 + O(\varepsilon^4), \]
so we obtain the desired counter-example as soon as \( r^{-1} > p/4 \) in view of \((2)\). The optimal \( \psi_w \) in \( L^q(T) \) for this functional can be found in \((4)\) Thm. 6.1], and we note that it is similar (but not equal to) the counter-example constructed in \((2)\).

The present paper is organised into two additional sections. In Section \((2)\) we prove Theorem \((2)\) and Theorem \((3)\). Section \((3)\) is devoted to constructing the element \( \psi \) in \( L^q(T^\infty) \) for \( 1 < q <\infty \) of minimal norm such that \( P\psi(z) = z_1 + z_2 + \cdots + z_d \), thereby realising the infimum \((3)\) in this special case, which is of particular interest due to the crucial role it plays in the proof of Theorem \((2)\) and Theorem \((3)\).
2. Linear functions on $\mathbb{T}^\infty$

In preparation for the proof of Theorem 2 and Theorem 3, let us recall some basic facts about linear functions and projections on $\mathbb{T}^\infty$. The projection $A_d$ obtained by formally setting $z_j = 0$ for $j > d$ has the representation

$$ A_d f(z_1, z_2, \ldots) = \int_{\mathbb{T}^\infty} f(z_1, z_2, \ldots, z_d, z_{d+1}, z_{d+2}, \ldots) \, d\mu_\infty(z_{d+1}, z_{d+2}, \ldots). $$

Since $A_d f$ is a function the first $d$ variables, we take $L^p$ norm with respect to these variables and use the triangle inequality to obtain

$$ \|A_d f\|_{L^p(\mathbb{T}^\infty)} \leq \|f\|_{L^p(\mathbb{T}^\infty)}. $$

Let $k \in \mathbb{Z}$. We say that $f$ is $k$-homogeneous if

$$ f(e^{i\theta} z_1, e^{i\theta} z_2, e^{i\theta} z_3, \ldots) = e^{i k \theta} f(z_1, z_2, z_3, \ldots). $$

Clearly every $f$ in $L^p(\mathbb{T}^\infty)$ can be decomposed in $k$-homogeneous parts, say

$$ f(z) = \sum_{k \in \mathbb{Z}} f_k(z), $$

where $f_k$ is $k$-homogeneous. The following simple lemma is well-known, but we include a short proof for the readers convenience.

**Lemma 4.** Let $1 \leq p \leq \infty$ and suppose that $f$ in $L^p(\mathbb{T}^\infty)$ is decomposed as in (8). Then $\|f_k\|_{L^p(\mathbb{T}^\infty)} \leq \|f\|_{L^p(\mathbb{T}^\infty)}$ for every $k \in \mathbb{Z}$.

**Proof.** By the decomposition (8), we find that

$$ f_k(z) = \int_{-\pi}^{\pi} f(e^{i\theta} z_1, e^{i\theta} z_2, e^{i\theta} z_3, \ldots) \, e^{-i k \theta} \frac{d\theta}{2\pi}. $$

By the triangle inequality and interchanging the order of integration, we obtain

$$ \|f_k\|_{L^p(\mathbb{T}^\infty)} \leq \int_{-\pi}^{\pi} \int_{\mathbb{T}^\infty} |f(e^{i\theta} z_1, e^{i\theta} z_2, e^{i\theta} z_3, \ldots)|^p \, d\mu_\infty(z) \, \frac{d\theta}{2\pi} = \|f\|_{L^p(\mathbb{T}^\infty)}, $$

since for each $\theta$ the rotation $z_j \mapsto e^{i\theta} z_j$ does not change the $L^p(\mathbb{T}^\infty)$ norm of $f$. □

Let $\text{Lin}(\mathbb{T}^\infty)$ denote the space of linear functions (4). Lemma 4 states that the projection from $H^p(\mathbb{T}^\infty)$ to $\text{Lin}(\mathbb{T}^\infty) \cap H^p(\mathbb{T}^\infty)$ is contractive. This fact is crucial to the proof of Theorem 2 and Theorem 3 since it allows us to compute the $(H^p(\mathbb{T}^\infty))^*$ norm of a linear function $\varphi$ by testing only against functions $f$ from $\text{Lin}(\mathbb{T}^\infty) \cap H^p(\mathbb{T}^\infty)$.

In view of Khintchine’s inequality (5), the space $\text{Lin}(\mathbb{T}^\infty) \cap H^p(\mathbb{T}^\infty)$ consists of linear functions (4) with square summable coefficients for each $1 \leq p < \infty$, although the norms are generally different.

Armed with these preliminaries, we will now obtain the key new ingredient needed in the proofs of Theorem 2 and Theorem 3.

**Lemma 5.** Let $1 \leq p < \infty$ and set $\varphi_d(z) = (z_1 + \cdots + z_d)/\sqrt{d}$. Then

$$ \|\varphi_d\|_{(H^p(\mathbb{T}^\infty))^*} = \|\varphi_d\|_{H^p(\mathbb{T}^\infty)}^{-1}. $$

**Proof.** For the lower bound, we simply note that since $\varphi_d$ is in $H^p(\mathbb{T}^\infty)$ we obtain

$$ \|\varphi_d\|_{(H^p(\mathbb{T}^\infty))^*} = \sup_{f \in H^p(\mathbb{T}^\infty)} \frac{|\langle f, \varphi_d \rangle_{H^2(\mathbb{T}^\infty)}|}{\|f\|_{H^p(\mathbb{T}^\infty)} \|\varphi_d\|_{H^p(\mathbb{T}^\infty)}} \geq \frac{\|\varphi_d\|_{H^2(\mathbb{T}^\infty)}^2}{\|\varphi_d\|_{H^p(\mathbb{T}^\infty)} \|\varphi_d\|_{H^p(\mathbb{T}^\infty)}} = \|\varphi_d\|_{H^p(\mathbb{T}^\infty)}^{-1}. $$
For the upper bound, we first use (7) and Lemma [4] to the effect that
\[ \| \varphi \|_{H^p(T^\infty)}^* = \sup_{f \in H^p(T^\infty)} \frac{|\langle f, \varphi \rangle_{H^2(T^\infty)}|}{\| f \|_{H^p(T^\infty)}} = \sup_{f \in \text{Lin}(T^d)} \frac{|\langle f, \varphi \rangle_{H^2(T^\infty)}|}{\| f \|_{H^p(T^\infty)}}. \]

Any non-trivial element \( f \) in \( \text{Lin}(T^d) \) is of the form
\[ f(z) = \sum_{j=1}^{d} c_j z_j \]
with at least one non-zero coefficient. Define
\[ \lambda = \langle f, \varphi \rangle_{H^2(T^\infty)} = \frac{c_1 + \cdots + c_d}{\sqrt{d}}. \]

After rotating each of the variables if necessary, we may assume that \( c_j \geq 0 \) for \( 1 \leq j \leq d \) so that \( \lambda > 0 \) whenever \( f \) is a non-trivial element in \( \text{Lin}(T^d) \).

For \( 1 \leq k \leq d \), let \( f_k \) denote the polynomial obtained by replacing the coefficient sequence \( (c_1, \ldots, c_d) \) of \( f \) with the shifted sequence \( (c_k, c_{k+1}, \ldots, c_d, c_1, \ldots, c_{k-1}) \).

By symmetry, we find that \( \| f_k \|_{H^p(T^\infty)} = \| f \|_{H^p(T^\infty)} \). Note also that
\[ \frac{1}{d} \sum_{k=1}^{d} f_k(z) = \frac{c_1 + \cdots + c_d}{d} \sum_{j=1}^{d} z_j = \lambda \varphi(d(z)). \]

The triangle inequality therefore allows us to conclude that
\[ \lambda \| \varphi \|_{H^p(T^\infty)} \leq \frac{1}{d} \sum_{k=1}^{d} \| f_k \|_{H^p(T^\infty)} = \| f \|_{H^p(T^\infty)}. \]

Using (10) with (11) and (12), we obtain the upper bound
\[ \| \varphi \|_{(H^p(T^\infty))^*} = \sup_{f \in H^p(T^\infty)} \frac{|\langle f, \varphi \rangle_{H^2(T^\infty)}|}{\| f \|_{H^p(T^\infty)}} \leq \frac{\lambda}{\lambda \| \varphi \|_{H^p(T^\infty)}} = \| \varphi \|_{H^p(T^\infty)}^{-1} \]
which, when combined with the lower bound (9), completes the proof. □

Another viewpoint is to consider \( (z_j)_{j \geq 1} \) a sequence of independently distributed random variables on the torus and \( f(z) = \sum_{j \geq 1} c_j z_j \) as a weighted random walk in the plane. The norms \( \| f \|_{H^p(T^\infty)} \) can now be interpreted as moments of this random walk. A simple computation (see Section 3) gives that \( \| z_1 + z_2 \|_{H^1(T^\infty)} = 4/\pi \) and it is demonstrated in [11] that
\[ \| z_1 + z_2 + z_3 \|_{H^1(T^\infty)} = \frac{3}{16} \Gamma^6 \left( \frac{1}{3} \right) \frac{21^3}{4} \Gamma^6 \left( \frac{2}{3} \right) = 1.57459 \ldots \]
In general it is difficult to compute \( \| f \|_{H^p(T^\infty)} \) even for simple linear polynomials \( f \) (when \( p \) is not an even integer). However, the central limit theorem gives that
\[ \lim_{d \to \infty} \left\| \frac{z_1 + z_2 + \cdots + z_d}{\sqrt{d}} \right\|_{H^p(T^\infty)}^p = \int_C |Z|^p e^{-|Z|^2} \frac{dZ}{\pi} = \Gamma \left( 1 + \frac{p}{2} \right), \]
since \( (z_1 + z_2 + \cdots + z_d)/\sqrt{d} \) has a limiting complex normal distribution.

We are now ready to prove Theorem [2]. To conform with the notations of the present section and to make the proof cleaner, we consider now \( \varphi \) in \( (H^p(T^\infty))^* \) and \( f \) in \( H^p(T^\infty) \), so \( \varphi \) plays the role of \( f \) in the statement of the theorem.
Proof of Theorem 2. Let \( \varphi \) be a linear function in \((H^p(T^\infty))^*\). By Lemma 2, the Cauchy–Schwarz inequality and Khintchine’s inequality (5), we find that

\[
\| \varphi \|_{(H^p(T^\infty))^*} = \sup_{f \in \text{Lin}(T^\infty)} \frac{|\langle f, \varphi \rangle_{H^2(T^\infty)}|}{\|f\|_{H^p(T^\infty)}} \leq \sup_{f \in \text{Lin}(T^\infty)} \frac{\|f\|_{H^2(T^\infty)} \|\varphi\|_{H^2(T^\infty)}}{\|f\|_{H^p(T^\infty)}} \leq \frac{\|\varphi\|_{H^2(T^\infty)}}{a_p}.
\]

Conversely, Khintchine’s inequality (5) also gives that

\[
\| \varphi \|_{(H^p(T^\infty))^*} = \sup_{f \in H^p(T^\infty)} \frac{|\langle f, \varphi \rangle_{H^2(T^\infty)}|}{\|f\|_{H^p(T^\infty)}} \geq \frac{\|\varphi\|_{H^2(T^\infty)}}{b_p},
\]

since \( \varphi \) is in \( H^p(T^\infty) \). To prove optimality of the constants, we appeal to Lemma 5 and consider \( \varphi_d(z) = (z_1 + \cdots + z_d)/\sqrt{d} \) for \( d = 1 \) and as \( d \to \infty \).

Theorem 3 also follows easily from Lemma 5 and (13).

Proof of Theorem 3. Let \( 2 \leq p \leq q \leq \infty \) and set \( q^{-1} + r^{-1} = 1 \). Suppose that

\[
(14) \quad \Gamma \left(1 + \frac{p}{2}\right)^\frac{1}{2} \Gamma \left(1 + \frac{r}{2}\right)^\frac{1}{2} > 1.
\]

We want to prove that the Riesz projection is unbounded from \( L^q(T^\infty) \) to \( L^p(T^\infty) \). In view of (2), it is sufficient to find \( \psi \) in \( L^q(T^\infty) \) such that

\[
\frac{\|P\psi\|_{L^p(T^\infty)}}{\|\psi\|_{L^q(T^\infty)}} > 1.
\]

We pick \( \psi_d \) of minimal norm such that \( P\psi_d = \varphi_d \), where \( \varphi_d \) denotes the function from Lemma 5. By (3) and Lemma 5, we obtain

\[
\frac{\|P\psi_d\|_{L^p(T^\infty)}}{\|\psi_d\|_{L^q(T^\infty)}} = \|\varphi_d\|_{L^p(T^\infty)} \|\varphi_d\|_{L^q(T^\infty)}.
\]

By (13) and our assumption (14), the right hand side is strictly larger than 1 for some sufficiently large \( d \).

3. Minimal \( L^q(T^\infty) \) norm

We will now solve the following problem: For \( 1 < q \leq \infty \), find the element \( \psi \) in \( L^q(T^\infty) \) of minimal norm such that

\[
P\psi(z) = z_1 + z_2 + \cdots + z_d = \varphi(z).
\]

The strict convexity of \( L^q(T^\infty) \) when \( 1 < q < \infty \) means that the minimizer is unique. Uniqueness of the minimizer holds also for \( q = \infty \), but in this case it is a consequence of the continuity of \( \varphi \) on the polytorus (see e.g. [5, Sec. 8.2]).

In view of (3) and (the proof of) Lemma 5, we know that \( \psi \) satisfies

\[
\|\psi\|_{L^q(T^\infty)} = \frac{\langle \Phi, \psi \rangle_{L^2(T^\infty)}}{\|\Phi\|_{L^p(T^\infty)}}
\]

with \( p^{-1} + q^{-1} = 1 \). However, from Hölder’s inequality we know that

\[
\|\psi\|_{L^q(T^\infty)} = \frac{\langle \Phi, \psi \rangle_{L^2(T^\infty)}}{\|\Phi\|_{L^p(T^\infty)}}
\]
holds if and only if $|\psi| = C|\Phi|^{p/q}$ almost everywhere. Since $p/q = p - 1$ we conclude that $|\psi| = C|\varphi|^{p-1}$. We now bring into play Lemma 4 which reveals to us that the $\psi$ of minimal norm is 1-homogeneous. We then recall that $\varphi$ is 1-homogeneous, which gives that

$$\psi(z) = C|\varphi(z)|^{p-2}\varphi(z).$$

Since $\psi$ is now 1-homogeneous, we only need to check that its Fourier coefficients at $z_j$, for $1 \leq j \leq d$, are correct in order to ensure that $P\psi = \varphi$. Clearly $\psi$ inherits the symmetry of $\varphi$, so we find that

$$\int_{T^n} \psi(z) \bar{z}_j^d \mu_\infty(z) = \int_{\mathbb{T}^n} \psi(z) \bar{z}_1 + \bar{z}_2 + \cdots + \bar{z}_d \ d\mu_\infty(z) = \frac{C}{d} \|\varphi\|_{L^p(T^n)}^p$$

and therefore $C = d\|\varphi\|_{L^p(T^n)}^p$. Hence we conclude that the element $\psi$ in $L^q(T^n)$ for $1 < q \leq \infty$ of minimal norm such that $P\psi(z) = z_1 + z_2 + \cdots + z_d = \varphi(z)$ is

$$\psi(z) = \frac{d}{\|\varphi\|_{L^p(T^n)}} |\varphi(z)|^{p-2}\varphi(z)$$

where $p^{-1} + q^{-1} = 1$.

When $d = 2$, we can actually compute the Fourier series explicitly. We begin by using the trick $z_1 + z_2 = z_2(1 + z_1\bar{z}_2)$ to write $\psi(z) = z_2\Psi(z_1\bar{z}_2)$, where

$$\Psi(z) = \frac{2}{\|1 + z\|^p_{L^p(T)}} |1 + z|^{p-2}(1 + z).$$

Then we get that

$$\frac{\|1 + z\|^p_{L^p(T)}}{2} = \frac{1}{2} \int_{-\pi}^{\pi} |1 + e^{i\theta}|^p \frac{d\theta}{2\pi} = 2^{p-1} \int_{-\pi}^{\pi} \cos^p \left(\frac{\theta}{2}\right) \frac{d\theta}{2\pi} = 2^p \int_0^{\pi/2} \cos^p(\vartheta) \ d\vartheta.$$  

Similarly, we compute:

$$\int_{-\pi}^{\pi} |1 + e^{i\theta}|^{p-2} (1 + e^{i\theta}) e^{-ik\vartheta} \frac{d\theta}{2\pi} = 2^{p-1} \int_{-\pi}^{\pi} \cos^{p-1} \left(\frac{\theta}{2}\right) e^{-i(k-1/2)\theta} \frac{d\theta}{2\pi} = 2^{p-1} \int_{-\pi/2}^{\pi/2} \cos^{p-1}(\vartheta) e^{-i(2k-1)\vartheta} \frac{d\vartheta}{\pi} = 2^p \int_0^{\pi/2} \cos^{p-1}(\vartheta) \cos((1 - 2k)\vartheta) \ d\vartheta$$

The latter integral, which contains the former as the special case $k = 0, 1$ is known (see e.g. [9] p. 399]) and we obtain that

$$\int_{-\pi}^{\pi} |1 + e^{i\theta}|^{p-2} (1 + e^{i\theta}) e^{-ik\vartheta} \frac{d\theta}{2\pi} = \frac{1}{p \Beta\left(\frac{p+1-2k+1}{2}, \frac{p-1+2k+1}{2}\right)}$$

for $\Beta(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x + y)$. Combining everything, we find that

$$\psi(e^{i\theta_1}, e^{i\theta_2}) = \sum_{k \in \mathbb{Z}} \frac{\Gamma(1 + p/2)\Gamma(p/2)}{\Gamma(1 + p/2 - k)\Gamma(p/2 + k)} e^{ik\theta_1}e^{i(1-k)\theta_2}.$$  

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