Thermal and quantum noise in active systems

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We present a quantum network approach to the treatment of thermal and quantum fluctuations in measurement devices. The measurement is described as a scattering process of input fluctuations towards output ones. We present the results obtained with this method for the treatment of a cold damped capacitive accelerometer.

I. NON IDEAL QUANTUM MEASUREMENTS

Active systems are fundamental elements in high precision measurements. Amplifiers are used either for amplifying the signal to a macroscopic level or to make the system work around its optimal working point with the help of feedback loops. With techniques such as cold damping, it is possible to manipulate actively the fluctuations and to reduce the effective noise temperature of the devices well below the operating temperature. The analysis of sensitivity limits in these devices raises many questions related to fundamental processes as well as experimental constraints. How far is it possible to reduce the measurement temperature? How are these processes related to the fluctuation dissipation theorem? Are there quantum limits to this noise reduction associated with Heisenberg inequalities? How do the experimental constraints interplay with the fundamental limitations of the sensitivity?

The aim of the present paper is to address these questions with quantum network theory. This approach provides a rigorous thermodynamical framework able to withstand the constraints of a quantum analysis of the measurement. In the same time, it makes possible a realistic description of real measurement devices. Thermodynamic and quantum fluctuations are treated in the same footing. The measurement process is described as a scattering process allowing for a modular analysis of real quantum systems. Active systems such as the linear amplifier or the ideal operational amplifier are described in this framework. Here, the approach will be illustrated by analyzing the sensitivity of a cold damped capacitive accelerometer developed for fundamental physics applications in space.

We first present the analysis of passive electrical systems in term of quantum networks. Then, we use this approach to present the quantum analysis of an operational amplifier working in the ideal limit of infinite gain, infinite input impedance and null output impedance. In the last section, we illustrate the theoretical framework with the example of a cold damped accelerometer.

II. COUPLING WITH THE ENVIRONMENT

Relations between fluctuations and dissipation have first been discovered by Einstein who studied the viscous damping of mechanical systems. Another important application was the study of Johnson-Nyquist noise in resistive electrical elements. This classical result was extended to take into account the quantum statistical properties of fluctuations. A general approach of these relations was widely studied in the framework of linear response theory.

A. Dissipation and Fluctuations

A first insight into the physical effect of the coupling of an electrical circuit to the environment is provided by the analysis of an antenna in an electrical resonator. When a current flows through the antenna, electromagnetic radiation is emitted and the resonator energy decreases. As far as the electric circuit is concerned, the effect of the antenna is the same as a resistance. The antenna is also able to detect electromagnetic fields. An incoming wave puts into motion the electrons in the antenna and causes an electrical current to flow in the circuit. For thermal radiation, the detection radiation leads to a random current which brings the electrical oscillator to thermal equilibrium. In the high temperature limit, it leads to the usual thermodynamic $\frac{1}{2}k_B T$ per degree of freedom, with $k_B$ being Boltzmann constant and $T$ the radiation temperature. In the zero temperature limit, the detected field corresponds to the vacuum fluctuations of the electromagnetic field and the induced energy of the oscillator is the zero point energy $\frac{1}{2}\hbar \omega_0$, with $\omega_0$ the resonance frequency of the oscillator.
and that these two representations are related through the line of characteristic impedance Nyquist’s analysis. It consists in a semi infinite coaxial Figure 1 b corresponds to a model that originates from a system of interest through a line considered as the detection channel. The first equation corresponds to the relation (1) and

\[ I(x, t) = I_{\text{out}} \left( t + \frac{x}{c} \right) - I_{\text{in}} \left( t - \frac{x}{c} \right) \]

\[ U(x, t) = R \left( I_{\text{out}} \left( t + \frac{x}{c} \right) + I_{\text{in}} \left( t - \frac{x}{c} \right) \right) \]

At the end of the line, we deduce the following relations:

\[ U = RI + 2RI_{\text{in}} = RI + U_n \]

\[ I_{\text{out}} = I + I_{\text{in}} \]

The first equation corresponds to the relation (1) and leads to the identification of the noise as the input current \( I_{\text{in}} \). The second equation describes the output fields \( I_{\text{out}} \) emitted back to the line. This output field may be used either to feed other elements of the system or to perform a measurement by extracting information on the system of interest through a line considered as the detection channel.

**B. Treatment with quantum fields**

In an infinite line, current and voltage may be treated as quantum fields propagating in a two dimensional space-time. Throughout the paper, we will consider that a function \( f \) is defined in the time domain (notation \( f(t) \)) or in the frequency domain (Kubo’s notation \( f[\omega] \)) and that these two representations are related through the Fourier transform with the convention of quantum mechanics

\[ f(t) = \int \frac{d\omega}{2\pi} f[\omega] e^{-i\omega t} \]

The electronics convention may be recovered by substituting \( j \) to \(-i\).

Free field operators \( a^{\text{in}} \) and \( a^{\text{out}} \) can be defined as the Fourier components of \( I_{\text{in}} \) and \( I_{\text{out}} \)

\[ I(x, t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \sqrt{\frac{\hbar}{2\omega}} \left( a^{\text{out}}[\omega] \exp \left[ -i\omega \left( t + \frac{x}{c} \right) \right] - a^{\text{in}}[\omega] \exp \left[ -i\omega \left( t - \frac{x}{c} \right) \right] \right) \]

\[ U(x, t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \sqrt{\frac{\hbar}{2\omega}} R \left( a^{\text{out}}[\omega] \exp \left[ -i\omega \left( t + \frac{x}{c} \right) \right] + a^{\text{in}}[\omega] \exp \left[ -i\omega \left( t - \frac{x}{c} \right) \right] \right) \]

They are normalized so that they obey the standard commutation relations

\[ [a^{\text{in}}[\omega], a^{\text{in}}[\omega']] = [a^{\text{out}}[\omega], a^{\text{out}}[\omega']] = 2\pi \delta(\omega - \omega') \varepsilon(\omega) \]

where \( \varepsilon(\omega) \) denotes the sign of the frequency \( \omega \). This relation just means that the positive and negative frequency components correspond respectively to the annihilation \( a^{\omega} \) and creation \( a^{\omega}_\dagger \) operators of quantum field theory

\[ a^{\text{in}}[\omega] = a^{\omega}_\dagger \theta(\omega) + a^{\omega}_\dagger \theta(-\omega) \]

\( \theta(\omega) \) denotes the Heavyside function.

To characterize the fluctuations of these noncommuting operators, we use the correlation function defined as the average value of the symmetrized product. With stationary noise, the correlation function depends only on the time difference

\[ \langle a^{\text{in}}(t) \cdot a^{\text{in}}(t') \rangle = \sigma_{aa}^{\text{in}}(t - t') \]

\[ \langle a^{\text{in}}[\omega] \cdot a^{\text{in}}[\omega'] \rangle = 2\pi \delta(\omega + \omega') \sigma_{aa}^{\text{in}}[\omega] \]

The dot symbol denotes a symmetrized product for quantum operators.

In the case of a thermal bath, the noise spectrum is

\[ \sigma_{aa}^{\text{in}}[\omega] = \frac{1}{\exp \left( \frac{\hbar |\omega|}{k_B T_a} \right) - 1} + \frac{1}{2} \coth \left( \frac{\hbar |\omega|}{2k_B T_a} \right) \]

One recognizes the black body spectrum or the number of bosons per mode for a field at temperature \( T_a \) and a term \( \frac{1}{2} \) corresponding to the quantum fluctuations. The energy per mode will be denoted in the following as an effective temperature \( T_a \)

\[ k_B T_a = \hbar |\omega| \sigma_{aa}^{\text{in}}[\omega] = \frac{\hbar |\omega|}{2} \coth \left( \frac{\hbar |\omega|}{2k_B T_a} \right) \]

In the high temperature limit the classical energy for an harmonic field of \( k_B T_a \) per mode is recovered. In the low temperature limit, the energy \( \frac{\hbar |\omega|}{2} \) corresponding to the ground state of a quantum harmonic oscillator is obtained. Note that the term \( \frac{1}{2} \) corresponding to the zero point quantum fluctuations was added by Planck so that
the difference with the classical result $k_BT_a$ tends to zero in the high temperature limit [14].

These results are easily translated to obtain the expression of the Johnson Nyquist noise power

$$\sigma_{U_nU_n}[\omega] = 2Rh|\omega|\sigma_{aa}[\omega] = 2RkB\Theta_a$$

(11)

Our symmetric definition of the noise power spectrum leads to a factor 2 difference with the electronic convention where only positive frequencies are considered.

C. Quantum networks

The elementary systems described up to now as well as more complex devices to be studied later in this paper may be described by using a systematic approach which may be termed as “quantum network theory”. Initially designed as a quantum extension of the classical theory of electrical networks [14], this theory was mainly developed through applications to optical systems [12,13]. It has also been viewed as a generalized quantum extension of the linear response theory which is of interest for electrical systems as well [14]. It is fruitful for analyzing non-ideal quantum measurements containing active elements [15,16].

In this quantum network approach, the various fluctuations entering the system, either by dissipative or by active elements, are described as input fields in a number of lines as depicted on 1 b.

![Diagram of a quantum network](image)

FIG. 2. Representation of an electrical circuit as a quantum network. The central box is a reactive multipole which connects noise lines corresponding to the fluctuations entering the system, either by dissipative or by active elements. For example, the upper left port $n$ with voltage $U_n$ and current $I_n$ is connected to a line of impedance $R_n$ with inward and outward fields $a_{in}^n$ and $a_{out}^n$.

We first consider a passive linear network built with resistances and reactive elements like capacitances or inductances. Each resistance $R_n$ is modeled as a semi-infinite coaxial line $a_n$ with characteristic impedance $R_n$. The voltage $U_n$ and current $I_n$ associated with the resistance are the inward and outward fields $a_{in}^n$ and $a_{out}^n$ evaluated at the end of this line

$$I = R^{-\frac{1}{2}}\sqrt{\frac{h}{2}}|\omega|(a_{out}^n - a_{in}^n)$$

$$U = R^\frac{1}{2}\sqrt{\frac{h}{2}}|\omega|(a_{out}^n + a_{in}^n)$$

(12)

Here, $X = I$, $U$, $a_{in}^n$, $a_{out}^n$ denotes the column vector with components $X_n$ and $R$ is the diagonal matrix formed with the characteristic impedances $R_n$.

Input fields corresponding to different lines commute with each other. For simplicity, we also consider that the fields entering through the various ports are uncorrelated with each other. The interaction with the reactive elements is described by a reactive impedance matrix $Z$

$$U = -ZI$$

$$Z^\dagger = -Z$$

(13)

The whole network is then associated with a scattering $S$ matrix, also called repartition matrix [15], describing the transformation from the input fields to the output ones

$$a_{out}^n = S a_{in}^n$$

$$S = \frac{R^{-\frac{1}{2}}ZR^{-\frac{1}{2}} - 1}{R^{-\frac{1}{2}}ZR^{-\frac{1}{2}} + 1}$$

(14)

The output fields $a_{out}^n$ are also free fields which obey the same commutation relations as the input ones. In other words, $S$ matrix is unitary. In the case of the passive network, this property is an immediate consequence of the reactive nature of the impedance matrix $Z$

$$S^\dagger = \frac{R^{-\frac{1}{2}}ZR^{-\frac{1}{2}} - 1}{R^{-\frac{1}{2}}ZR^{-\frac{1}{2}} + 1} = \frac{-R^{-\frac{1}{2}}ZR^{-\frac{1}{2}} - 1}{-R^{-\frac{1}{2}}ZR^{-\frac{1}{2}} + 1} = S^{-1}$$

(15)

More generally, the unitarity of the $S$ matrix is required to ensure the quantum consistency of the description. In the following section, we will make use of this property to deduce general properties of amplifiers.

III. FLUCTUATIONS IN AMPLIFIERS

Quantum noise associated with linear amplifiers has been the subject of numerous works. In the line of thought initiated by early works on fluctuation-dissipation relations, active systems have been studied in the optical domain when maser and laser amplifiers were developed [16,20]. General thermodynamical constraints impose the existence of fluctuations for amplification as well as dissipation processes. The added noise determines the ultimate performance of linear amplifiers [21,22] and plays a key role in the question of optimal information transfer in optical communication systems [23,24].

We first consider the amplification of a field, for example in long distance telecommunication systems with repeaters.

The amplification of the field $a_{in}^n$ with a phase insensitive gain $G$ is given by the following equation

$$a_{out}^n = Ga_{in}^n + B_{in}^n$$

(16)
where $B^{in}$ is a noise added by the amplification. The gain $G$ may be frequency dependent. The commutator of the output field is then

$$[a^{out} [\omega], a^{out} [\omega']] = |G|^2 \left[ a^{in} [\omega], a^{in} [\omega'] \right] + [B^{in} [\omega], B^{in} [\omega']]$$  \hspace{1cm} (17)

The unitarity of the input output transformation and the preservation of the commutation implies a non zero commutator for $B^{in}$

$$[B^{in} [\omega], B^{in} [\omega']] = [a^{in} [\omega], a^{in} [\omega']] - [a^{out} [\omega], a^{out} [\omega']] = \left( 1 - |G|^2 \right) 2\pi \delta (\omega + \omega') \varepsilon (\omega)$$  \hspace{1cm} (18)

This result does not depend on the specific amplification process. For a gain larger than unity, the added noise can be represented by a free field $b^{in}$ with the usual commutation relation \[1\]

$$B^{in} [\omega] = \sqrt{|G|^2 - 1} b^{in} [-\omega] = \sqrt{|G|^2 - 1} (b^{in} [\omega])^\dagger$$  \hspace{1cm} (19)

The presence of the conjugation is characteristic of amplification processes and is encountered as soon as gain is present \[21,22\]. We may use this example to describe the noise analysis in a measurement process. Let us consider that the field $a^{in}$ carries a signal $A$ superimposed with fluctuations $c^{in}$

$$A = \langle a^{in} \rangle, \quad c^{in} = a^{in} - \langle a^{in} \rangle$$  \hspace{1cm} (20)

The input noise power $\Sigma^{in}_{AA}$ corresponds to the fluctuations $\sigma^{in}_{cc}$

$$\Sigma^{in}_{AA} = \sigma^{in}_{cc}$$  \hspace{1cm} (21)

The measurement corresponds to the output $a^{out}$ of the amplifier

$$a^{out} = GA + Gc^{in} + \sqrt{|G|^2 - 1} b^{in}$$  \hspace{1cm} (22)

To analyze the noise of this amplified signal, we define an estimator $\hat{A}$ by normalizing the output field $a^{out}$ of the amplifier so that it is the sum of $A$ and an extra noise

$$\hat{A} = \frac{1}{G} a^{out} = A + c^{in} + \sqrt{1 - \frac{1}{|G|^2}} b^{in}$$  \hspace{1cm} (23)

The added noise $\Sigma^{out}_{AA}$ is then described by a spectrum

$$\Sigma^{out}_{FF} = \sigma^{in}_{cc} + \left( 1 - \frac{1}{|G|^2} \right) \sigma^{in}_{bb}$$  \hspace{1cm} (24)

In the limit of large gain $G$ and for thermal fluctuations, it corresponds to

$$\hbar |\omega| \Sigma^{out}_{FF} = k_B (\Theta_a + \Theta_b)$$  \hspace{1cm} (25)

When the temperatures are equal, this corresponds to a loss of 3dB in the signal to noise ratio. This effect has been observed since the beginning of radiowave communications. It also sets a limit in the number of repeaters in optical fiber communications \[23,24\].

Most practical applications of amplifiers in measurements involve ideal operational amplifiers operating in the limits of infinite gain, infinite input impedance and null output impedance. In order to deal with the pathologies that could arise in such a system, we consider that it operates with a feedback loop which fixes its effective gain and effective impedances \[23\].

We first analyze the amplifier as depicted on figure 3a where the noise sources are represented as a current generator $I$ and a voltage generator $U$. By coupling two coaxial lines denoted $l$ and $r$ respectively on the left port and the right port of the amplifier, one realizes a measurement model. The left line comes from a monitored electrical system so that the inward field $l^{in}$ plays the role of the signal to be measured. Meanwhile, the right line goes to an electrical meter so that the outward field $r^{out}$ plays the role of the meter readout. In connection with the discussions of Quantum Non Demolition measurements \[24,27\], $l^{out}$ appears as the back-action field sent back to the monitored system and $r^{in}$ represents the fluctuations coming from the readout line. A reactive impedance $Z_f$ acts as feedback for the amplifier.

We now present the electrical equations associated with this measurement device. We first write the characteristic relations between the voltages and currents

$$U = U_l = U_r + Z_f I_f$$

$$I = I_l + I_f$$  \hspace{1cm} (26)

Here, $U_p$ and $I_p$ are the voltage and current at the port $p$, i.e. at the end of the line $p = l$ or $r$, while $U$ and $I$ are the voltage and current noise generators associated with the operational amplifier itself (see Fig.1). $Z_f$ is the impedance feedback. All equations are implicitly writ-
ten in the frequency representation and the impedances
are functions of frequency. Equations (26) take a simple
form because of the limits of infinite gain, infinite input
impedance and null output impedance assumed for the
ideal operational amplifier. We also suppose that the
fields incoming through the various ports are uncorre-
lated with each other as well as with amplifier noises.

As already emphasized, the output fields $p^{\text{out}}$ obey
the commutation relations (6) of free fields. To make
this property explicit, we use the characteristic equations
(24) associated with the amplifier and the lines to
rewrite the output fields $p^{\text{out}}$ and $r^{\text{out}}$ in terms of input
fields $l^{\text{in}}$, $r^{\text{in}}$ and of amplifier noise sources $U$ and $I$

$$r^{\text{out}} = -r^{\text{in}} - \frac{Z_f}{\sqrt{R_f R_t}} r^{\text{in}}$$
$$p^{\text{out}} = p^{\text{in}} + \frac{2}{\hbar |\omega| R_t} U$$

We then deduce from (27) that the voltage and current
fluctuations $U$ and $I$ obey the following commutation
relations

$$[U[\omega], U[\omega']] = [I[\omega'], I[\omega]] = 0$$
$$[U[\omega], I[\omega']] = 2\pi \hbar \omega \delta(\omega + \omega')$$

Hence, voltage and current fluctuations verify Heisenberg
inequalities which determine the ultimate performance
of the ideal operational amplifier used as a measurement
device [23].

To push this analysis further it is worth introducing
new quantities $a^{\text{in}}$ and $a'^{\text{in}}$ as linear combinations of the
noises $U$ and $I$ depending on a factor $R$ having the di-

$$U[\omega] = \sqrt{2\hbar |\omega| R} (a^{\text{in}}[\omega] - a'^{\text{in}}[-\omega])$$
$$I[\omega] = \sqrt{2\hbar |\omega| R} (a^{\text{in}}[\omega] + a'^{\text{in}}[-\omega])$$

For an arbitrary value of $R$, the quantities $a^{\text{in}}$ and $a'^{\text{in}}$
satisfy the free field commutation relations. In other
words, the voltage and current noises associated with the
amplifier may be replaced by the coupling to 2 further
lines $a$ and $a'$ and the presence of amplification requires
a conjugation of fluctuations coming in one of these two
lines. This representation of the amplifier as a quantum
network is depicted on figure 3b.

We may then fix the parameter $R$ to a value $R_a$ chosen
so that the fluctuations $a^{\text{in}}$ and $a'^{\text{in}}$ are uncorrelated.
This specific value is determined by the ratio between
voltage and current noise spectra

$$R_a = \sqrt{\frac{\sigma_{UU}}{\sigma_{II}}}$$

The 2 noise spectra $\sigma_{U}$ and $\sigma_{II}$ are defined as symme-
tric correlation functions. The fields $a^{\text{in}}$ and $a'^{\text{in}}$ are thus
described by temperatures $T_a$ and $T_a'$. We have assumed
that these fluctuations are the same for all field quadra-
tures, i.e. that the amplifier noises are phase-insensitive.
Although this assumption is not mandatory for the forth-
coming analysis, we also consider for simplicity that the
specific impedance $R_a$ is constant over the spectral do-
main of interest.

IV. THE COLD DAMPED ACCELEROMETER

We come to the discussion of the ultimate performance
of the cold damped capacitive accelerometer designed for
fundamental physics experiments in space [16].

FIG. 4. Scheme of the capacitive sensor. The proof mass
is placed between two electrodes formed by the inner walls
of the accelerometer cage. The position dependent capaci-
tances are polarized by an AC sinewave source which induces
a mean current at frequency $\omega_0$ in the symmetrical mode. The
mass displacement is read as the current induced in the an-
tisymmetric mode. An additional capacitance is inserted to
make the antisymmetric mode resonant with $\omega_0$. The signal
is detected after an ideal operational amplifier with capaci-
tive feedback followed by a synchronous demodulation. The
impedance of the detection line plays the role of a further
resistance $R_i$. The detected signal then feeds the servo loop
used to keep the mass centered with respect to the cage.

The central element of the capacitive accelerometer is a
parallelepipedic proof mass placed inside a box. The
walls of these box are electrodes distant from the mass
off a hundred micrometers. The proof mass is kept at the
center of the cage by an electrostatic suspension. Since a
dimension electrostatic suspension is instable, it is
necessary to use an active suspension.

In the cage reference frame, an acceleration is trans-
formed in an inertial force acting on the proof mass. The
force necessary to compensate this inertial force is mea-
sured. In fact, as in most ultrasensitive measurements,
the detected signal is the error signal used to compensate
the effect of the measured phenomenon.

The essential elements of the accelerometer are pre-
sent in figure 4. The proof mass and the cage form
two condensators. Any mass motion unbalances the dif-
ferential detection bridge and provides the error signal. In order to avoid low frequency electrical noise, the electrical circuit is polarized with an AC voltage with a frequency of a hundred kilohertz. After demodulation, this signal is used for detection and as an error signal for a servo control loop which allows to keep the mass centered in its cage.

Furthermore, the derivative of this signal provides a force proportional to the mass velocity and simulates a friction force. This active friction is called cold damping since it may be noiseless. More precisely, the effective friction force. This active friction is called cold damping force proportional to the mass velocity and simulates a servo control loop which allows to keep the mass centered in its cage.

The detection is performed with the output detection signal $r_1^\text{out}$. It is a linear combination of the external force $F_{\text{ext}}$ and of input fields in the various noise lines. We normalize this expression so that the coefficient of proportionality appearing in front of the external force $F_{\text{ext}}$ is reduced to unity. With this normalization, we obtain a force estimator $\hat{F}_{\text{ext}}$ which is just the sum of the true force $F_{\text{ext}}$ to be measured and of an equivalent input force noise. In the absence of feedback, the force estimator reads [16]:

$$\hat{F}_{\text{ext}} = F_{\text{ext}} + \sum_\alpha \mu_\alpha \alpha^{\text{in}}$$

(31)

where $\alpha^{\text{in}}$ denote the various input fields corresponding to the active and passive elements in the accelerometer.

When the feedback is active, the servo loop efficiently maintains the mass at its equilibrium position and the velocity is no longer affected by the external force $F_{\text{ext}}$. The residual motion is interpreted as the difference between the real velocity of the mass and the velocity measured by the sensor. This means that the servo loop efficiently corrects the motion of the mass except for the sensing error. However the sensitivity to external force is still present in the correction signal. Quite remarkably, in the limit of an infinite loop gain and with the same approximations as above, the expression of the force estimator $\hat{F}_{\text{ext}}$ is the same as in the free case [16].

The added noise spectrum $\Sigma_{FF}$ is obtained as

$$\Sigma_{FF} = \sum_\alpha |\mu_\alpha|^2 \sigma^{\text{in}}_{\alpha\alpha}$$

We have evaluated the whole noise spectrum $\Sigma_{FF}$ for the specific case of the instrument proposed for the $\mu$SCOPE space mission devoted to the test of the equivalence principle. Some of the main parameters of this system are listed below

$$M = 0.27 \text{ kg} \quad H_m = 1.3 \times 10^{-5} \text{ kg s}^{-1}$$

$$\frac{\Omega}{2\pi} \simeq 5 \times 10^{-4} \text{ Hz} \quad \omega_{\text{in}} \simeq 10^5 \text{ Hz}$$

$$R_a = 0.15 \times 10^6 \Omega \quad \Theta_a = 1.5 \text{ K}$$

(32)

$M$ is the mass of the proof mass, $H_m$ is the residual mechanical damping force, $\frac{\Omega}{2\pi}$ is the frequency of the measured mechanical motion, $\frac{\omega_{\text{in}}}{2\pi}$ is the operating frequency of the electrical detection circuit. $R_a$ and $\Theta_a$ are the characteristic impedance and temperature of the amplifier.

In these conditions, the added noise spectrum is dominated by the mechanical Langevin forces

$$\Sigma_{FF} = 2H_m k_B \Theta_m$$

$$= 1.1 \times 10^{-25} (\text{kg m s}^{-2})^2 /\text{Hz}$$

(33)

This corresponds to a sensitivity in acceleration

$$\frac{\sqrt{\Sigma_{FF}}}{M} = 1.2 \times 10^{-12} \text{ m s}^{-2}/\sqrt{\text{Hz}}$$

(34)

Taking into account the integration time of the experiment, this leads to the expected instrument performance corresponding to a test accuracy of $10^{-15}$.

In the present state-of-the-art instrument, the sensitivity is thus limited by the residual mechanical Langevin forces. The latter are due to the damping processes in the gold wire used to keep the proof mass at zero voltage [3]. With such a configuration, the detection noise is not a limiting factor. This is a remarkable result in a situation where the effective damping induced through the servo loop is much more efficient than the passive mechanical damping. This confirms the considerable interest of the cold damping technique for high sensitivity measurement devices.

Future fundamental physics missions in space will require even better sensitivities. To this aim, the wire will be removed and the charge of the test mass will be controlled by other means, for example UV photoemission. The mechanical Langevin noise will no longer be a limitation so that the analysis of the ultimate detection noise will become crucial for the optimization of the instrument performance. This also means that the electromechanical design configuration will have to be reoptimized taking into account the various noise sources associated with detection [16].

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