Spectral saturation: inverting the spectral Turán theorem

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Abstract

Let $\mu(G)$ be the largest eigenvalue of a graph $G$ and $T_r(n)$ be the $r$-partite Turán graph of order $n$.

We prove that if $G$ is a graph of order $n$ with $\mu(G) > \mu(T_r(n))$, then $G$ contains various large supergraphs of the complete graph of order $r+1$, e.g., the complete $r$-partite graph with all parts of size $\log n$ with an edge added to the first part.

We also give corresponding stability results.

Keywords: complete $r$-partite graph; stability, spectral Turán's theorem; largest eigenvalue of a graph.

1 Introduction

This note is part of an ongoing project aiming to build extremal graph theory on spectral grounds, see, e.g., [3], [13, 20].

Let $\mu(G)$ be the largest adjacency eigenvalue of a graph $G$ and $T_r(n)$ be the $r$-partite Turán graph of order $n$. The spectral Turán theorem [16] implies that if $G$ is a graph of order $n$ with $\mu(G) > \mu(T_r(n))$, then $G$ contains a $K_{r+1}$, the complete graph of order $r+1$.

On the other hand, it is known (e.g., [2], [4], [9], [12]) that if $e(G) > e(T_r(n))$, then $G$ contains large supergraphs of $K_{r+1}$.

It turns out that essentially the same results also follow from $\mu(G) > \mu(T_r(n))$.

Recall first a family of graphs, studied initially by Erdős [7] and recently in [2]: an $r$-joint of size $t$ is the union of $t$ distinct $r$-cliques sharing an edge. Write $js_r(G)$ for the maximum size of an $r$-joint in a graph $G$. Erdős [7], Theorem 3’, showed that:

If $G$ is a graph of sufficiently large order $n$ satisfies $e(G) > e(T_r(n))$, then $js_{r+1}(G) > n^{r-1}/(10(r+1))^{6(r+1)}$.

Here is a explicit spectral analogue of this result.

Theorem 1 Let $r \geq 2$, $n > r^{15}$, and $G$ be a graph of order $n$. If $\mu(G) > \mu(T_r(n))$, then $js_{r+1}(G) > n^{r-1}/r^{2r+4}$.
Erdős [4] introduced yet another graph related to Turán’s theorem: let \( K_r^* (s_1, \ldots, s_r) \) be the complete \( r \)-partite graph with parts of size \( s_1 \geq 2, s_2, \ldots, s_r \), with an edge added to the first part. The extremal results about this graph given in [4] and [9] were recently extended in [12] to:

Let \( r \geq 2 \), \( 2/\ln n \leq c \leq r^{-(r+7)(r+1)} \), and \( G \) be a graph of order \( n \). If \( G \) has \( t_r(n) + 1 \) edges, then \( G \) contains a \( K_r^* ([c \ln n], \ldots, [c \ln n], [n^{1-\sqrt{c}}]) \).

Here we give a similar spectral extremal result.

**Theorem 2** Let \( r \geq 2 \), \( 2/\ln n \leq c \leq r^{-(2r+9)(r+1)} \), and \( G \) be a graph of order \( n \). If \( \mu(G) > \mu(T_r(n)) \), then \( G \) contains a \( K_r^* ([c \ln n], \ldots, [c \ln n], [n^{1-\sqrt{c}}]) \).

As an easy consequence of Theorem 2 we obtain

**Theorem 3** Let \( r \geq 2 \), \( c = r^{-(2r+9)(r+1)} \), \( n \geq e^{2/c} \), and \( G \) be a graph of order \( n \). If \( \mu(G) > \mu(T_r(n)) \), then \( G \) contains a \( K_r^* ([c \ln n], \ldots, [c \ln n]) \).

Theorems 1, 2, and 3 have corresponding stability results.

**Theorem 4** Let \( r \geq 2 \), \( 0 < b < 2^{-10r^{-6}}, n \geq r^{20} \), and \( G \) be a graph of order \( n \). If \( \mu(G) > (1 - 1/r - b)n \), then \( G \) satisfies one of the conditions:

(a) \( js_{r+1}(G) > n^{r-1}/r^{2r+5} \);

(b) \( G \) contains an induced \( r \)-partite subgraph \( G_0 \) of order at least \((1 - 4b^{1/3})n \) with minimum degree \( \delta(G_0) > (1 - 1/r - 7b^{1/3})n \).

**Theorem 5** Let \( r \geq 2 \), \( 2/\ln n \leq c \leq r^{-(2r+9)(r+1)/2}, 0 < b < 2^{-10r^{-6}} \), and \( G \) be a graph of order \( n \). If \( \mu(G) > (1 - 1/r - b)n \), then \( G \) satisfies one of the conditions:

(a) \( G \) contains a \( K_r^* ([c \ln n], \ldots, [c \ln n], [n^{1-2\sqrt{c}}]) \);

(b) \( G \) contains an induced \( r \)-partite subgraph \( G_0 \) of order at least \((1 - 4b^{1/3})n \) with minimum degree \( \delta(G_0) > (1 - 1/r - 7b^{1/3})n \).

**Theorem 6** Let \( r \geq 2 \), \( c = r^{-(2r+9)(r+1)/2}, 0 < b < 2^{-10r^{-6}}, n \geq e^{2/c} \), and \( G \) be a graph of order \( n \). If \( \mu(G) > (1 - 1/r - b)n \), then one of the following conditions holds:

(a) \( G \) contains a \( K_r^* ([c \ln n], \ldots, [c \ln n]) \);

(b) \( G \) contains an induced \( r \)-partite subgraph \( G_0 \) of order at least \((1 - 4b^{1/3})n \) with minimum degree \( \delta(G_0) > (1 - 1/r - 7b^{1/3})n \).

**Remarks**

- Obviously Theorems 1, 2, and 3 are tight since \( T_r(n) \) contains no \((r+1)\)-cliques.

- Theorems 2, 4, 5, and 6 are essentially best possible since for every \( \varepsilon > 0 \), choosing randomly a graph \( G \) of order \( n \) with \( e(G) = \lceil (1 - \varepsilon)n^2/2 \rceil \) edges we see that \( \mu(G) > (1 - \varepsilon)n \), but \( G \) contains no \( K_2(c \ln n, c \ln n) \) for some \( c > 0 \), independent of \( n \).
- Theorem 1 implies in turn spectral versions of other known results, like Theorem 3.8 in [8]:

Every graph $G$ of order $n$ with $\mu(G) > \mu(T_r(n))$ contains $cn$ distinct $(r+1)$-cliques sharing an $r$-clique, where $c > 0$ is independent of $n$.

- The relations between $c$ and $n$ in Theorems 2 and 5 need explanation. First, for fixed $c$, they show how large must be $n$ to get valid conclusions. But, in fact, the relations are subtler, for $c$ itself may depend on $n$, e.g., letting $c = 1/\ln \ln n$, the conclusions are meaningful for sufficiently large $n$.

- Note that, in Theorems 2 and 5, if the conclusion holds for some $c$, it holds also for $0 < c' < c$, provided $n$ is sufficiently large;

- The stability conditions (b) in Theorems 4, 5, and 6 are stronger than the conditions in the stability theorems of [6], [21] and [11]. Indeed, in all these theorems, condition (ii) implies that $G_0$ is an induced, almost balanced, and almost complete $r$-partite graph containing almost all the vertices of $G$;

- The exponents $1 - \sqrt{c}$ and $1 - 2\sqrt{c}$ in Theorems 2 and 5 are far from the best ones, but are simple.

The next section contains notation and results needed to prove the theorems. The proofs are presented in Section 3.

2 Preliminary results

Our notation follows [1]. Given a graph $G$, we write:

- $V(G)$ for the vertex set of $G$ and $|G|$ for $|V(G)|$;
- $E(G)$ for the edge set of $G$ and $e(G)$ for $|E(G)|$;
- $d(u)$ for the degree of a vertex $u$;
- $\delta(G)$ for the minimum degree of $G$;
- $k_r(G)$ for the number of $r$-cliques of $G$;
- $K_r(s_1, \ldots, s_r)$ for the complete $r$-partite graph with parts of size $s_1, \ldots, s_r$.

The following facts play crucial roles in our proofs.

**Fact 7** ([16], Theorem 1) Every graph $G$ of order $n$ with $\mu(G) > \mu(T_r(n))$ contains a $K_{r+1}$. □

**Fact 8** ([15], Theorem 5) Let $0 < \alpha \leq 1/4$, $0 < \beta \leq 1/2$, $1/2 - \alpha/4 \leq \gamma < 1$, $K \geq 0$, $n \geq (42K + 4)/\alpha^2 \beta$, and $G$ be a graph of order $n$. If

$$\mu(G) > \gamma n - K/n \quad \text{and} \quad \delta(G) \leq (\gamma - \alpha) n,$$

then $G$ contains an induced subgraph $H$ satisfying $|H| \geq (1 - \beta) n$ and one of the conditions:

(a) $\mu(H) > \gamma (1 + \beta \alpha/2) |H|$;

(b) $\mu(H) > \gamma |H|$ and $\delta(H) > (\gamma - \alpha) |H|$.

□
Fact 9 ([2], Lemma 6) Let \( r \geq 2 \) and \( G \) be a graph of order \( n \). If \( G \) contains a \( K_{r+1} \) and \( \delta (G) > (1 - 1/r - 1/r^4) n \), then \( j_{s_{r+1}} (G) > n^{r-1}/r^{r+3} \). \( \square \)

Fact 10 ([3], Theorem 2) If \( r \geq 2 \) and \( G \) is a graph of order \( n \), then
\[
k_r(G) \geq \left( \frac{\mu (G)}{n} - 1 + \frac{1}{r} \right) \frac{r (r - 1)}{r + 1} \left( \frac{n}{r} \right)^{r+1}.
\]
\( \square \)

Fact 11 ([3], Theorem 4) Let \( r \geq 2, 0 \leq b \leq 2^{-10} r^{-6} \), and \( G \) be a graph of order \( n \). If \( G \) contains no \( K_{r+1} \) and \( \mu (G) \geq (1 - 1/r - b) n \), then \( G \) contains an induced \( r \)-partite graph \( G_0 \) satisfying \( |G_0| \geq (1 - 3e^{1/3}) n \) and \( \delta (G_0) > (1 - 1/r - 6e^{1/3}) n \). \( \square \)

Fact 12 ([12], Theorem 6) Let \( r \geq 2, 2/\ln n \leq c \leq r^{-(r+8)} \), and \( g \) is a graph of order \( n \). If \( G \) contains a \( K_{r+1} \) and \( \delta (G) > (1 - 1/r - 1/r^4) n \), then \( G \) contains a \( K_r^+ \left( \lfloor c \ln n \rfloor, \ldots, \lfloor c \ln n \rfloor, \left\lceil n^{1-r^2} \right\rceil \right) \). \( \square \)

Fact 13 ([10], Theorem 1) Let \( r \geq 2, c^r \ln n \geq 1 \), and \( G \) be a graph of order \( n \). If \( k_r(G) \geq cn^r \), then \( G \) contains a \( K_r (s, \ldots, s, t) \) with \( s = \lfloor c^r \ln n \rfloor \) and \( t > n^{1-c^{-1}} \). \( \square \)

Fact 14 The number of edges of \( T_r(n) \) satisfies \( 2e(T_r(n)) \geq (1 - 1/r) n^2 - r/4 \). \( \square \)

3 Proofs

Below we prove Theorems [1], [2], [4] and [5]. We omit the proofs of Theorems [3] and [6] since they are easy consequences of Theorems [2] and [5].

All proofs have similar simple structure and follow from the facts listed above.

Proof of Theorem [1]

Let \( G \) be a graph of order \( n \) with \( \mu (G) > \mu (T_r(n)) \); thus, by Fact [7] \( G \) contains a \( K_{r+1} \). If
\[
\delta (G) > (1 - r^{-1} - r^{-4}) n,
\]
then, by Fact [9] \( j_{s_{r+1}} (G) > n^{r-1}/r^{r+3} \), completing the proof.

Thus, we shall assume that [11] fails. Then, letting
\[
\alpha = 1/r^4, \quad \beta = 1/2, \quad \gamma = 1 - 1/r, \quad K = r/4,
\]
we see that
\[
\delta (G) \leq (\gamma - \alpha) n \quad (3)
\]
and also, in view of Fact [14]
\[
\mu (G) > \mu (T_r(n)) \geq 2e(T_r(n))/n \geq (1 - 1/r) n - r/4n = \gamma n - K/n. \quad (4)
\]
Given (2), (3) and (4), Theorem 8 implies that, for \( n \geq r^{15} \), \( G \) contains an induced subgraph \( H \) satisfying \(|H| \geq n/2\) and one of the conditions:

(i) \( \mu(H) > (1 - 1/r + 1/(4r^4))|H| \);
(ii) \( \mu(H) > (1 - 1/r)|H| \) and \( \delta(H) > (1 - 1/r - 1/r^4)|H| \).

If condition (i) holds, Fact 10 gives

\[
js_{r+1}(G) \geq js_{r+1}(H) \geq \left( \frac{r + 1}{2} \right) \frac{k_{r+1}(H)}{e(H)} > r(r + 1) \frac{k_{r+1}(H)}{H^2}
\]

and so,

\[
js_{r+1}(G) > js_{r+1}(H) > \frac{|H|^{r-1}}{r^{r+3}} > \frac{1}{2^{r+1}r^{r+3}}n^{r-1} > \frac{1}{r^{2r+4}}n^{r-1},
\]

completing the proof.

If condition (ii) holds, then \( H \) contains a \( K_{r+1} \); thus, by Fact 9, \( js_{r+1}(H) > |H|^{r-1}/r^{r+3} \). To complete the proof, notice that

\[
js_{r+1}(G) > js_{r+1}(H) > \frac{|H|^{r-1}}{r^{r+3}} \geq \frac{1}{2^{r-1}r^{r+3}}n^{r-1} > \frac{1}{r^{2r+4}}n^{r-1}.
\]

\[\square\]

**Proof of Theorem 2**

Let \( G \) be a graph of order \( n \) with \( \mu(G) > \mu(T_r(n)) \); thus, by Fact 7, \( G \) contains a \( K_{r+1} \). If

\[
\delta(G) > (1 - 1/r - 1/r^4)n, \tag{5}
\]

then, by Fact 12, \( G \) contains a \( K_{r+1}^{+} \left( \lfloor c \ln n \rfloor, \ldots, \lfloor c \ln n \rfloor, \lceil n^{1-cr^3} \rceil \right) \), completing the proof, in view of \( cr^3 < \sqrt{e} \).

Thus, we shall assume that (5) fails. Then, letting

\[
\alpha = 1/r^4, \quad \beta = 1/2, \quad \gamma = 1 - 1/r, \quad K = r/4, \tag{6}
\]

we see that

\[
\delta(G) \leq (\gamma - \alpha)n \tag{7}
\]

and also, in view of Fact 14,

\[
\mu(G) > \mu(T_r(n)) \geq 2e(T_r(n))/n \geq (1 - 1/r)n - r/4n = \gamma n - K/n. \tag{8}
\]

Given (6), (7) and (8), Theorem 8 implies that, for \( n > r^{15} \), \( G \) contains an induced subgraph \( H \) satisfying \(|H| \geq n/2\) and one of the conditions:
(i) $\mu(H) > (1 - 1/r + 1/(4r^4)) |H|$;
(ii) $\mu(H) > (1 - 1/r) |H|$ and $\delta(H) > (1 - 1/r - 1/r^4) |H|$.

If condition (i) holds, Fact 10 gives

$$k_{r+1}(H) > \left( \frac{\mu(H)}{|H|} - 1 - \frac{1}{r^4} \right) \frac{r(r-1)}{n+1} \left( \frac{|H|}{r} \right)^{r+1} > \frac{r(r-1)}{4r^4} \left( \frac{|H|}{r} \right)^{r+1}$$

$$> \frac{1}{2^{r+3}r^{r+4}(r+1)} n^{r+1} > \frac{1}{r^{2r+9}} n^{r+1} \geq c^{1/(r+1)} n^{r+1}.$$

Thus, by Fact 13, $G$ contains a $K_{r+1}(s, \ldots, s, t)$ with $s = \lceil c \ln n \rceil$ and $t > n^{1-c/(r+1)} > n^{1/\sqrt{r}}$. Then, obviously, $G$ contains a $K_r^+\left(\lfloor c \ln n \rfloor, \ldots, \lfloor c \ln n \rfloor, \left\lceil \left| H \right|^{1-2c \alpha^3} \right\rceil\right)$.

To complete the proof, note that $2c \ln |H| \geq 2c \ln \frac{n}{2} > c \ln n$ and

$$|H|^{1-2c \alpha^3} \geq \left( \frac{n}{2} \right)^{1-2c \alpha^3} \geq \frac{1}{2} n^{1-2c \alpha^3} > n^{1-\sqrt{r}}.$$

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**Proof of Theorem 4** Let $G$ be a graph of order $n$ with $\mu(G) > (1 - 1/r - b) n$. If $G$ contains no $K_{r+1}$, then condition (b) follows from Fact 11; thus we assume that $G$ contains a $K_{r+1}$. If

$$\delta(G) > (1 - 1/r - 1/r^4) n,$$

then Fact 9 implies condition (a).

Thus, we shall assume that (9) fails. Then, letting

$$\alpha = 1/r^4 - b, \quad \beta = 4b/\alpha, \quad \gamma = 1 - 1/r - b, \quad K = 0,$$

we easily see that

$$\beta = \frac{4b}{1/r^4 - b} \leq \frac{1}{2}, \quad \delta(G) \leq (\gamma - \alpha) n,$$

and

$$\mu(G) > (1 - 1/r - b) n = \gamma n.$$  \hspace{1cm} (12)

Given (10), (11) and (12), Theorem 8 implies that, for $n \geq r^{20}$, $G$ contains an induced subgraph $H$ satisfying $|H| \geq (1 - \beta) n$ and one of the conditions:

(i) $\mu(H) > (1 - 1/r) |H|$;
(ii) $\mu(H) > (1 - 1/r - b) |H|$ and $\delta(H) > (1 - 1/r - 1/r^4) |H|$. 

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If condition (i) holds, by Theorem [1] we have

\[ j_{s_{r+1}}(G) \geq j_{s_{r+1}}(H) = \frac{|H|^{r-1}}{r^{2r+4}} \geq (1 - \beta)^{r-1} \frac{n^{r-1}}{r^{2r+4}} = \left(1 - \frac{4b}{1/r^4 - b}\right)^{r-1} \frac{n^{r-1}}{r^{2r+4}} \]

implying condition (a) and completing the proof.

Suppose now that condition (ii) holds. If \( H \) contains a \( K_{r+1} \), by Fact [3] we see that

\[ j_{s_{r+1}}(G) \geq j_{s_{r+1}}(H) \geq \frac{|H|^{r-1}}{r^{r+3}} \geq (1 - \beta)^{r-1} \frac{n^{r-1}}{r^{r+3}} > \frac{n^{r-1}}{2r-1/r^3+3} > \frac{n^{r-1}}{2r+5}, \]

implying condition (a).

If \( H \) contains no \( K_{r+1} \), by Fact [11], \( H \) contains an induced \( r \)-partite subgraph \( H_0 \) satisfying \(|H_0| > (1 - 3b^{1/3}) |H|\) and \( \delta(H_0) > (1 - 6b^{1/3}) |H|\). Now from

\[ \beta = \frac{4b}{1/r^4 - b} \leq \frac{4b}{1/r^4 - 1/(2^{10}r^6)} \leq 8r^4b < b^{1/3}, \]

we deduce that

\[ |H_0| \geq (1 - 3b^{1/3}) |H| \geq (1 - 3b^{1/3})(1 - \beta) n > (1 - 4b^{1/3}) n \]

and

\[ \delta(H_0) > (1 - 6b^{1/3}) |H| \geq (1 - 7b^{1/3})(1 - \beta) n > (1 - 7b^{1/3}) n. \]

Thus condition (b) holds, completing the proof.

\[ \square \]

**Proof of Theorem 5** Let \( G \) be a graph of order \( n \) with \( \mu(G) > (1 - 1/r - b) n \). If \( G \) contains no \( K_{r+1} \), then condition (b) follows from Fact [11] thus we assume that \( G \) contains a \( K_{r+1} \). If

\[ \delta(G) > (1 - 1/r - 1/r^4) n, \]

then Fact [12] implies condition (a).

Thus, we shall assume that [13] fails. Then, letting

\[ \alpha = 1/r^4 - b, \quad \beta = 4b/\alpha, \quad \gamma = 1 - 1/r - b, \quad K = 0, \]

we easily see that

\[ \beta = \frac{4b}{1/r^4 - b} \leq \frac{1}{2}; \quad \delta(G) \leq (\gamma - \alpha) n, \]

and

\[ \mu(G) > (1 - 1/r - b) n = \gamma n. \]

Given [14], [15] and [16], Theorem 5 implies that, for \( n \geq r^{20} \), \( G \) contains an induced subgraph \( H \) satisfying \(|H| \geq (1 - \beta) n\) and one of the conditions:
(i) \(\mu (H) > (1 - 1/r) |H|\);
(ii) \(\mu (H) > (1 - 1/r - b) |H|\) and \(\delta (H) > (1 - 1/r - 1/r^4) |H|\).

If condition (i) holds, Theorem 2 implies that \(H\) contains a

\[ K_r^+ \left( \lfloor 2c \ln |H| \rfloor, \ldots, \lfloor 2c \ln |H| \rfloor, \lceil |H|^{1-2c'r^3} \rceil \right). \]

Now condition (a) follows in view of \(2c \ln |H| \geq \frac{c}{2} > c \ln n\) and

\[ |H|^{1-2c'r^3} \geq \left( \frac{n}{2} \right)^{1-2c'r^3} \geq \frac{1}{2} n^{1-2c'r^3} > n^{1-\sqrt{c}}, \]

completing the proof.

Suppose now that condition (ii) holds. If \(H\) contains a \(K_{r+1}^+\), by Fact 12 \(H\) contains a

\[ K_r^+ \left( \lfloor 2c \ln |H| \rfloor, \ldots, \lfloor 2c \ln |H| \rfloor, \lceil |H|^{1-2c'r^3} \rceil \right). \]

This implies condition (a) in view of \(2c \ln |H| \geq \frac{c}{2} > c \ln n\) and

\[ |H|^{1-2c'r^3} \geq \left( \frac{n}{2} \right)^{1-2c'r^3} \geq \frac{1}{2} n^{1-2c'r^3} > n^{1-\sqrt{c}}. \]

If \(H\) contains no \(K_{r+1}^+\), the proof is completed as the proof of Theorem 4.

Concluding remarks

It is not difficult to show that if \(G\) is a graph of order \(n\), then the inequality \(e (G) > e (T_r (n))\) implies the inequality \(\mu (G) > \mu (T_r (n))\). Therefore, Theorems 1-6 imply the corresponding nonspectral extremal results with narrower ranges of the parameters.

Finally, a word about the project mentioned in the introduction: in this project we aim to give wide-range results that can be used further, adding more integrity to spectral extremal graph theory.

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