DETERMINANT METHOD AND THE PSEUDO-EFFECTIVE
THRESHOLD

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Abstract. — In this paper, we will give an upper bound of the number of auxiliary
hypersurfaces in the determinant method, which reformulates an unpublished work
of Salberger by Arakelov geometry. One of the key constants will be determined by
the pseudo-effective threshold of certain line bundles.

Résumé (La méthode de déterminant et le seuil de pseudo-effectivité)
Dans cet article, on donnera une majoration du nombre de hypersurfaces auxi-
liaires dans la méthode déterminant, qui reformule un travail non publié de Sal-
berger par la géométrie d’Arakelov. Une des constantes clés sera déterminée par le
seuil de pseudo-effectivité de certains fibrés en droites.

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1. Introduction

Let $K$ be a number field, and $X \hookrightarrow \mathbb{P}^n_K$ be a projective variety. Let $\xi \in X(K)$,
and $H_K(\xi)$ be a height of $\xi$ with respect to the above closed immersion, for example,
the classic Weil height (cf. [21, §B.2, Definition]). A height function $H_K(\cdot)$ on the
set of rational points is able to be used to measure their arithmetic complexities. Let
$B \in \mathbb{R}$, and

$S(X; B) = \{\xi \in X(K) \mid H_K(\xi) \leq B\}$

be the set of rational points of bounded heights with respect to the above closed
immersion. Usually, a good height function has the so-called Northcott’s property,
which means that the cardinality $N(X; B) = \#S(X; B)$ is finite when $B$ is fixed. In this case, the map $N(X; \cdot) : \mathbb{R} \to \mathbb{N}$ is a function which gives a description of the density of rational points in $X$.

It is a central subject to understand different kinds of properties of the function $N(X; B)$ with the variable $B \in \mathbb{R}$ for different kinds of $X$. For this target, lots of methods have been developed. In this article, we will focus on the uniform upper bound of $N(X; B)$. The word "uniform" means that we want to obtain a good upper bound of $N(X; B)$ for a family of projective varieties satisfying certain common conditions, for example, with the same degree and dimension.

1.1. Determinant method. — In this article, we will focus on the so-called determinant method proposed in [20] to study the density of rational points in arithmetic varieties.

1.1.1. Basic ideas and the developments. — Traditionally, we consider a projective variety $X \hookrightarrow \mathbb{P}_\mathbb{Q}^n$ over $\mathbb{Q}$ for simplicity, since the operations over arbitrary number fields sometimes bring us extra technical troubles. In [2] (see also [29]), Bombieri and Pila proposed a method of determinant argument to study plane affine curves. In [20], Heath-Brown developed the so-called the $p$-adic determinant method, which generalized the method of [2] to the higher dimensional case. His idea is to focus on a subset of $S(X; B)$ whose reductions modulo a prime number are a same regular point, and he proved that this subset can be covered by a bounded degree hypersurface which do not contain the generic point of $X$. By Siegel's Lemma, we can assure the existence of such hypersurfaces in $\mathbb{P}_\mathbb{Q}^n$ with bounded degree. Then he counted the number of regular points over finite fields, and control the regular reductions. By this method, he proved that $N(X; B) \ll_{d, \delta, \epsilon} B^{d+\epsilon}$ for all $\epsilon > 0$, where $\delta = \deg(X)$. In [6], Broberg generalized it to the case over an arbitrary number field.

In [20], Heath-Brown also proposed a so-called the dimension growth conjecture. Let $\dim(X) = d$. It is said that for all $d \geq 2$ and $\delta \geq 2$, we have $N(X; B) \ll_{d, \delta, \epsilon} B^{d+\epsilon}$ for all $\epsilon > 0$. He proved this conjecture for some special cases. Later, Browning, Heath-Brown and Salberger had some contributions on this subject, see [7, 8, 9, 32, 33] for the refinements of the determinant method and the proofs under certain conditions.

In [34], Salberger considered the case of cubic hypersurfaces, where we have a better estimate on a key invariant than that was obtained in [9, 33]. Actually, this work essentially applied the refinement of the invariant mentioned above by the pseudo-effective thresholds of certain line bundles.

1.1.2. Reformulation by Arakelov geometry. — In [12, 13], H. Chen reformulated the works of Salberger [32] by the slope method in Arakelov geometry. By this formulation, we replace the matrix of monomials by the evaluation map which sends a global section of a particular line bundle to a family of rational points. By the slope inequalities, we can control the height of the evaluation map in the slope method, which replaces the role of Siegel's lemma in controlling heights.

There are two advantages by the approach of Arakelov geometry. First, Arakelov geometry gives a natural conceptual framework for the determinant method over an
arbitrary number field. Second, it is easier to obtain explicit estimates, since usually
the constants obtained by the slope method are given explicitly.

But in this article, because of certain obstructions in the study of the positivity
of line bundles, we are not able to give effective estimates for all invariants. We will
explain the exact reason later.

1.2. Application of the pseudo-effective threshold. — In a mini-course of the
summer school "Arakelov Geometry and Diophantine applications" at Institut Fourier
in 2017, and a mini-course of the thematic activity "Reinventing rational points" at
Institut Henri Poincaré in 2019, Salberger gave lectures on the application of the
pseudo-effective thresholds of certain line bundles on projective varieties to estimate
the number of auxiliary hypersurfaces in the determinant method. In [34], he has
applied this idea to study the density of rational points in the complement of the
union of all lines of cubic surfaces in $\mathbb{P}^3$.

In this article, we will reformulate the above works of Salberger by Arakelov
geometry following the strategy of [12, 13], where we will consider the case of general
projective varieties. Some ideas of this work has been applied in [34].

1.2.1. Role of pseudo-effective threshold. — Let $X \hookrightarrow \mathbb{P}_K^n$ be a projective variety
over the number field $K$ of degree $\delta$ and dimension $d$, $\pi : \tilde{X} \rightarrow X$ be the blowing up
at the non-singular rational point $\eta$, $E$ is the exceptional divisor of this blowing up,
$H$ be a Cartier divisor on $X$ given by a hyperplane section on $\mathbb{P}_K^n$, and $D, m \in \mathbb{N}$.
We consider the sum

$$R(\eta,D) = \sum_{m=1}^{\infty} \dim_K H^0\left(\tilde{X}, D\pi^*H - mE\right),$$

which plays a significant role in Salberger’s refinement of $p$-adic determinant men-
tioned above. Next, we denote

$$I_X(H,\eta) = \int_0^\infty \operatorname{vol}(\pi^*H - \lambda E)d\lambda,$$

where $\operatorname{vol}(\cdot)$ is the usual volume function of $\mathbb{R}$-divisors. In Theorem 4.6, we will give
a proof of the estimate

$$R(\eta,D) = \frac{I_X(H,\eta)}{d!}D^{d+1} + O_{d,\delta}(D^d).$$

By this fact, we can refine some former results on the determinant method.

1.2.2. An improved upper bound of the number of auxiliary hypersurface.
— Let $\mathcal{X} \hookrightarrow \mathbb{P}_K^n$ be the Zariski closure of $X \hookrightarrow \mathbb{P}_K^n$, and $p$ be a maximal ideal of $\mathcal{O}_K$ whose
residue field is $\mathbb{F}_p$. Let $\xi \in \mathcal{X}(\mathbb{F}_p)$, and we denote by $S(X;B,\xi)$ the subset of $S(X;B)$
the reduction modulo $p$ of whose Zariski closures in $\mathcal{X}$ is $\xi$. We can prove that the
invariant $I_X(H,\eta)$ only depends on its reduction class if its reduction is regular. By
Lemma 5.1, if for the family of maximal ideals $p_1, \ldots, p_r$ of $\mathcal{O}_K$, the point $\xi_j$ is regular
in $\mathcal{X}$ for all $j = 1, \ldots, r$ and $\bigcap_{j=1}^r S(X;B,\xi_j) \neq \emptyset$, then all $I_X(H,\xi_j)$ are equal, noted
by $I_X(H, \xi_j)$ for simplicity. Then we have the result below from (3), where Salberger has proved the case of $K = \mathbb{Q}$.

**Theorem 1.1 (Theorem 5.2).** — We keep all the above notations. Let $p_1, \ldots, p_r$ be a family of maximal ideals of $O_K$, $N(p_j) = \# (O_K/p_j)$, and $\epsilon > 0$. Suppose that the point $\xi_j \in \mathcal{X} (\mathbb{F}_{p_j})$ is regular in $\mathcal{X}$ for all $j = 1, \ldots, r$. If the inequality

$$\sum_{j=1}^r \log N(p_j) \geq K, n, \epsilon \frac{\delta}{I_X(H, \xi_j)} \log B$$

is verified, then there exists a hypersurface of degree $O_{d, \delta, \epsilon}(1)$, which covers $\bigcap_{j=1}^r S(X; B, \xi_j)$ but do not contain the generic point of $X$.

By this result, let $\epsilon > 0$ and

$$I_X(H) = \inf_{\eta \in S(X; B)} I_X(H, \eta).$$

Then we have the following estimate of the number of auxiliary hypersurfaces from Theorem 1.1, where Salberger has proved the case of $K = \mathbb{Q}$, too.

**Theorem 1.2 (Theorem 5.7).** — With all the notations above. There exists a constant $C_4(\epsilon, \delta, n, d, K)$ such that $S(X; B)$ is covered by no more than

$$C_4(\epsilon, \delta, n, d, K) B^{(\frac{1}{2} + \frac{d+1}{d})}$$

hypersurfaces of degree $O_{n, \delta, \epsilon}(1)$ which do not contain the generic point of $X$.

By an unpublished result of Salberger (see also [25, Corollary 4.2]), for every regular closed point $\eta$ in $X$, we have $I_X(H, \eta) \geq d\delta^{1+\frac{1}{d}} / (d + 1)$. In this sense, the upper bound of the number of auxiliary hypersurfaces given in (4) can be considered as an improvement of some former results ([20, 32, 13], for example). If we focus on some special varieties $X$ with clearer information on $I_X(H, \eta)$ defined at (2), we may obtain a better estimate on the number of auxiliary hypersurfaces, see [34] for such an example, where the case of cubic hypersurfaces in $\mathbb{P}^3$ is considered.

**1.2.3. Ineffective estimates.** — In the above argument, we have

$$\dim_K H^0 \left( \tilde{X}, D\pi^* H - mE \right) = \frac{D^d}{d!} \vol \left( \pi^* H - \frac{m}{D} E \right) + O_{d, \delta}(D^{d-1}).$$

However, up to the author’s knowledge, we are not able to obtain an effective version in the above estimate. Thus we are only able to make sure that the maximal degree of auxiliary hypersurfaces can depend only on $n$, $\delta$ and $\epsilon$, but we cannot get an explicit bound until now.
1.3. Organization of the article. — This article is organized as follows. In §2, we will recall some useful preliminaries and propose the basic setting, where we follow the approach of [12, 13]. In §3, we will give a bound relating to the invariant \( R(\eta, D) \) defined in (1) and both geometric and arithmetic Hilbert-Samuel functions of arithmetic varieties, which is a generalization of [34, Lemma 16.9]. In §4, we will prove the finiteness of the sum (1) and the asymptotic estimate (3), which reformulates some former results of Salberger. In §5, we will prove Theorem 1.1, and give the upper bound (4) in Theorem 1.2 by applying it.

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2. Preliminaries and the basic setting

In this section, we will provide some preliminaries that will be used to interpret the determinant method in terms of Arakelov theory, where we follow the strategy of H. Chen in [12, 13].

2.1. Classic height function of rational points. — Let \( K \) be a number field, and \( \mathcal{O}_K \) be its ring of integers. We denote by \( M_{K,f} \) the set of finite places of \( K \), and by \( M_{K,\infty} \) the set of infinite places of \( K \). In addition, we denote by \( M_K = M_{K,f} \cup M_{K,\infty} \) the set of places of \( K \). For every \( v \in M_{K,f} \), if \( Q_v \) is the \( p \)-adic field, we define the absolute value \( |x|_v = |N_{K_v/Q_v}(x)|_p^{1/[K_v:Q_v]} \), where \(|.|_p \) is the usual \( p \)-adic absolute value. For every \( v \in M_{K,\infty} \), we define \( |x|_v = |N_{K_v/Q_v}(x)|^{1/[K_v:Q_v]} \), where \(|.| \) is the usual absolute values over \( \mathbb{R} \) or \( \mathbb{C} \).

For every \( a \in K^\times \), we have the product formula (cf. [28, Chap. III, Proposition 1.3])

\[
(5) \quad \prod_{v \in M_K} |a|_v^{[K_v:Q_v]} = 1.
\]

Let \( \xi = [\xi_0 : \cdots : \xi_n] \in \mathbb{P}^n_K(K) \). We define the absolute height of \( \xi \) in \( \mathbb{P}^n_K \) as

\[
(6) \quad H_K(\xi) = \prod_{v \in M_K} \max_{0 \leq i \leq n} \{|\xi_i|_v|^{[K_v:Q_v]} \}.
\]

Next, we define the logarithmic height of \( \xi \) as

\[
(7) \quad h(\xi) = \frac{1}{[K:Q]} \log H_K(\xi),
\]

which is independent of the choice of \( K \) (cf. [21, Lemma B.2.1]).
Suppose \( X \) is a closed integral subscheme of \( \mathbb{P}^n_K \) of degree \( \delta \) and dimension \( d \), and \( \phi : X \hookrightarrow \mathbb{P}^n_K \) is the projective embedding. For \( \xi \in X(K) \), we define \( H_K(\xi) = H_K(\phi(\xi)) \) for simplicity, and usually we omit the closed immersion \( \phi \). Next, we define

\[
S(X; B) = \{ \xi \in X(K) | H_K(\xi) \leq B \}, \quad \text{and} \quad N(X; B) = \#S(X; B).
\]

By the Northcott’s property (cf. [21, Theorem B.2.3]), the cardinality \( N(X; B) \) is finite for a fixed real number \( B \geq 1 \).

The objective of counting rational points of bounded height is to understand the function \( N(X; B) \) with some particular projective varieties \( X \) and real numbers \( B \geq 1 \).

### 2.2. Multiplicity of points in a scheme.

In this part, we will define the multiplicity of closed points in schemes induced by the local Hilbert-Samuel function. This notion will be useful in the determinant method.

Let \( X \) be a Noetherian scheme of pure dimension \( d \), which means all its irreducible components have the same dimension. Let \( \xi \) be a closed point of \( X \), \( \mathfrak{m}_{X,\xi} \) be the maximal ideal of the local ring \( \mathcal{O}_{X,\xi} \), and \( \kappa(\xi) \) be its residue field. We define

\[
H_{\xi}(s) = \dim_{\kappa(\xi)} \left( \frac{\mathfrak{m}_{X,\xi}^s}{\mathfrak{m}_{X,\xi}^{s+1}} \right)
\]

as the local Hilbert-Samuel function of \( X \) at the closed point \( \xi \) with the variable \( s \in \mathbb{N} \), where we define \( \mathfrak{m}_{X,\xi}^0 = \mathcal{O}_{X,\xi} \) for simplicity. For this function, when \( d \geq 2 \), we have the polynomial asymptotic extension

\[
H_{\xi}(s) = \frac{\mu_{\xi}(X)}{(d-1)!} s^{d-1} + O(s^{d-2}),
\]

where we define the positive integer \( \mu_{\xi}(X) \) as the multiplicity of point \( \xi \) in \( X \). If \( d = 1 \), then \( \mathcal{O}_{X,\xi} \) is a local Artinian ring. The multiplicity \( \mu_{\xi}(X) \) is then defined as the length of the local ring \( \mathcal{O}_{X,\xi} \) as a \( \mathcal{O}_{X,\xi} \)-module.

If \( \mathcal{O}_{X,\xi} \) is a regular local ring, we say that \( \xi \) is regular in \( X \). In this case we have \( \mu_{\xi}(X) = 1 \). Otherwise we say that \( \xi \) is singular in \( X \). If \( X \) is pure dimensional and has no embedded component, then from the fact that \( \xi \) is singular in \( X \) by the above definition, we deduce \( \mu_{\xi}(X) \geq 2 \) (cf. [27, (40.6)]).

We denote by \( X^{\text{reg}} \) the regular locus of \( X \), and by \( X^{\text{sing}} \) the singular locus of \( X \). By the semi-continuity of the multiplicity function, the singular locus \( X^{\text{sing}} \) is a closed subset of \( X \). If \( X \) is reduced and pure dimensional, the set \( X^{\text{reg}} \) is open dense in \( X \) (cf. [19, Corollary 8.16, Chap. II]).

### 2.3. Normed vector bundles.

The normed vector bundle is one of the main research objects in Arakelov geometry. Let \( K \) be a number field and \( \mathcal{O}_K \) be its ring of integers. A normed vector bundle over \( \text{Spec} \mathcal{O}_K \) is a pair

\[
\mathcal{E} = \left( E, (\| \cdot \|_v)_{v \in \mathcal{M}_K, \infty} \right),
\]

where:

- \( E \) is a projective \( \mathcal{O}_K \)-module of finite rank;
- \( (\| \cdot \|_v)_{v \in \mathcal{M}_K, \infty} \) is a family of norms, where \( \| \cdot \|_v \) is a norm over \( E \otimes_{\mathcal{O}_K, v} \mathbb{C} \) which is invariant under the action of \( \text{Gal}(\mathbb{C}/K_v) \).
If all the norms $(\|\cdot\|_v)_{v \in M_{K,\infty}}$ are Hermitian, we say that $E$ is a Hermitian vector bundle over $\text{Spec} \mathcal{O}_K$. In particular, if $\text{rk}_K(E) = 1$, we say that $E$ is a Hermitian line bundle over $\text{Spec} \mathcal{O}_K$.

Suppose that $F$ is a sub-$\mathcal{O}_K$-module of $E$. We say that $F$ is a saturated sub-$\mathcal{O}_K$-module of $E$ if $E/F$ is a torsion-free $\mathcal{O}_K$-module.

Let $\mathcal{E} = \left( E, (\|\cdot\|_{E,v})_{v \in M_{K,\infty}} \right)$ and $\mathcal{F} = \left( F, (\|\cdot\|_{F,v})_{v \in M_{K,\infty}} \right)$ be two Hermitian vector bundles over $\text{Spec} \mathcal{O}_K$. If $F$ is a saturated sub-$\mathcal{O}_K$-module of $E$ and $\|\cdot\|_{F,v}$ is the restriction of $\|\cdot\|_{E,v}$ over $F \otimes \mathcal{O}_K,v \mathcal{O}_K$ for every $v \in M_{K,\infty}$, we say that $\mathcal{F}$ is a sub-Hermitian vector bundle of $\mathcal{E}$ over $\text{Spec} \mathcal{O}_K$.

We say that $G = \left( G, (\|\cdot\|_{G,v})_{v \in M_{K,\infty}} \right)$ is a quotient Hermitian vector bundle of $\mathcal{E}$ over $\text{Spec} \mathcal{O}_K$, if for every $v \in M_{K,\infty}$, the module $G$ is a projective quotient $\mathcal{O}_K$-module of $E$ and $\|\cdot\|_{G,v}$ is the induced quotient space norm of $\|\cdot\|_{E,v}$.

For simplicity, we denote by $E_K = E \otimes \mathcal{O}_K K$ in the remainder part of this article.

### 2.4. Arakelov invariants

We will introduce some useful invariants in Arakelov geometry in this part.

#### 2.4.1. Arakelov degree

Let $\mathcal{E}$ be a Hermitian vector bundle over $\text{Spec} \mathcal{O}_K$, and $(s_1, \ldots, s_r)$ be a $K$-basis of the vector space $E_K$. The Arakelov degree of $\mathcal{E}$ is defined as

$$\deg(\mathcal{E}) = - \sum_{v \in M_K} [K_v : Q_v] \log \|s_1 \wedge \cdots \wedge s_r\|_v$$

$$= \log \left( \# (E/\mathcal{O}_K s_1 + \cdots + \mathcal{O}_K s_r) \right) - \frac{1}{2} \sum_{v \in M_{K,\infty}} \log \det \left( \langle s_i, s_j \rangle_{v,1 \leq i,j \leq r} \right),$$

where $\|s_1 \wedge \cdots \wedge s_r\|_v$ follows the definition in [11, 2.1.9] for all $v \in M_{K,\infty}$, and $\langle s_i, s_j \rangle_{v,1 \leq i,j \leq r}$ is the Gram matrix of the basis $\{s_1, \ldots, s_r\}$ with respect to $v \in M_{K,\infty}$.

For those $v \in M_{K,f}$, we take the norms given by models.

We refer the readers to [18, 2.4.1] for a proof of the equivalence of the above two definitions. The Arakelov degree is independent of the choice of the basis $\{s_1, \ldots, s_r\}$ by the product formula (5). In addition, we define

$$\deg_n(\mathcal{E}) = \frac{1}{[K : Q]} \deg(\mathcal{E})$$

as the normalized Arakelov degree of $\mathcal{E}$, which is independent of the choice of the base field $K$.

#### 2.4.2. Slope

Let $\mathcal{E}$ be a non-zero Hermitian vector bundle over $\text{Spec} \mathcal{O}_K$, and $\text{rk}(E)$ be the rank of $E$. The slope of $\mathcal{E}$ is defined as

$$\mu(\mathcal{E}) := \frac{1}{\text{rk}(E)} \deg_n(\mathcal{E}).$$

In addition, we denote by $\mu_{\max}(\mathcal{E})$ the maximal slope of all its non-zero Hermitian sub-bundles, and by $\mu_{\min}(\mathcal{E})$ the minimal slope of all its non-zero Hermitian quotients bundles of $\mathcal{E}$.
2.4.3. Height of linear maps. — Let $\mathcal{E}$ and $\mathcal{F}$ be two non-zero Hermitian vector bundles over $\text{Spec} \mathcal{O}_K$, and $\phi : E_K \to F_K$ be a non-zero homomorphism of $K$-vector spaces. The height of $\phi$ is defined as

$$h(\phi) = \frac{1}{[K : \mathbb{Q}]} \sum_{v \in M_K} \log \|\phi\|_v,$$

where $\|\phi\|_v$ is the operator norm of $K_v$-linear map $\phi_v : E \otimes_K K_v \to F \otimes_K K_v$ induced by the above linear homomorphism with respect to every $v \in M_K$.

We refer the readers to [3, Appendix A] for some equalities and inequalities on Arakelov degrees and the heights of corresponding homomorphisms.

2.5. Arithmetic Hilbert-Samuel function. — Let $\mathcal{E}$ be a Hermitian vector bundle of rank $n + 1$ over $\text{Spec} \mathcal{O}_K$, and $\mathcal{P}(\mathcal{E})$ be the projective space which represents the functor from the category of commutative $\mathcal{O}_K$-algebras to the category of sets mapping all $\mathcal{O}_K$-algebra $A$ to the set of projective quotient $A$-module of $\mathcal{E} \otimes \mathcal{O}_A$ $A$ of rank $1$.

Let $\mathcal{O}_{\mathcal{P}(\mathcal{E})}(1)$ (or by $\mathcal{O}(1)$ if there is no confusion) be the universal bundle, and $\mathcal{O}_{\mathcal{P}(\mathcal{E})}(D)$ (or by $\mathcal{O}(D)$) be the line bundle $\mathcal{O}_{\mathcal{P}(\mathcal{E})}(1)^{\otimes D}$ for simplicity. The Hermitian metrics on $\mathcal{E}$ induce by quotient of Hermitian metrics (i.e. Fubini-Study metrics) on $\mathcal{O}_{\mathcal{P}(\mathcal{E})}(1)$ which define a Hermitian line bundle $\mathcal{O}_{\mathcal{P}(\mathcal{E})}(1)$ on $\mathcal{P}(\mathcal{E})$.

For every $D \in \mathbb{N}^+$, let

$$E_D = H^0(\mathcal{P}(\mathcal{E}), \mathcal{O}_{\mathcal{P}(\mathcal{E})}(D)),$$

and $r(n, D)$ be its rank over $\mathcal{O}_K$. In fact, we have

$$r(n, D) = \binom{n + D}{D}. \tag{10}$$

For each $v \in M_{K, \infty}$, we denote by $\|\cdot\|_{v, \text{sup}}$ the norm over $E_{D, v} = E_D \otimes_{\mathcal{O}_K, v} \mathbb{C}$ such that

$$\forall s \in E_{D, v}, \|s\|_{v, \text{sup}} = \sup_{x \in \mathcal{P}(\mathcal{E})_K, v, (\mathbb{C})} \|s(x)\|_{v, \text{FS}}, \tag{11}$$

where $\|\cdot\|_{v, \text{FS}}$ is the corresponding Fubini-Study norm.

Next, we will introduce the metric of John, see [37] for a systematic introduction to this notion. In general, for a given symmetric convex body $C$, there exists the unique ellipsoid $J(C)$, called ellipsoid of John, contained in $C$ whose volume is maximal.

For the $\mathcal{O}_K$-module $E_D$ and any place $v \in M_{K, \infty}$, we take the ellipsoid of John of its unit closed ball defined via the norm $\|\cdot\|_{v, \text{sup}}$, and this ellipsoid induces a Hermitian norm, noted by $\|\cdot\|_{v, \text{John}}$. For every section $s \in E_D$, the inequality

$$\|s\|_{v, \text{sup}} \leq \|s\|_{v, \text{John}} \leq \sqrt{r(n, D)} \|s\|_{v, \text{sup}} \tag{12}$$

is verified by [37, Theorem 3.3.6]. In fact, these constants do not depend on the choice of the symmetric convex body.

Let $A$ be a ring, and $E$ be an $A$-module. We denote by $\text{Sym}^D_A(E)$ the symmetric product of degree $D$ of the $A$-module $E$, or by $\text{Sym}^D(E)$ if there is no confusion on the base ring.
If we consider the above $E_D$ defined in (9) as an $O_K$-module, we have the isomorphism of $O_K$-modules $E_D \cong \text{Sym}^D(\mathcal{E})$. Then for every place $v \in M_{K,\infty}$, the Hermitian norm $\|\cdot\|_v$ over $\mathcal{E}_{v,\mathbb{C}}$ induces a Hermitian norm $\|\cdot\|_{v,\text{sym}}$ over $E_D$ by the symmetric product. More precisely, this norm is the quotient norm induced by the quotient morphism

$$\mathcal{E} \otimes D \to \text{Sym}^D(\mathcal{E}),$$

where the vector bundle $\mathcal{E} \otimes D$ is equipped with the norms induced by the tensor product of $\mathcal{E}$ over $\text{Spec} O_K$ (see [17, Definition 2.10] for the definition). We say that this norm is the symmetric norm over $\text{Sym}^D(\mathcal{E})$. For any place $v \in M_{K,\infty}$, the norms $\|\cdot\|_{v,\text{John}}$ and $\|\cdot\|_{v,\text{sym}}$ are invariant under the action of the unitary group $U(\mathcal{E}_{v,\mathbb{C}}, \|\cdot\|_v)$ of order $n + 1$. Then they are proportional and the ratio is independent of the choice of $v \in M_{K,\infty}$ (see [4, Lemma 4.3.6] for a proof). We denote by $R_0(n, D)$ the constant such that, for every section $0 \neq s \in E_{D,v}$, the equality

$$\log \|s\|_{v,\text{John}} = \log \|s\|_{v,\text{sym}} + R_0(n, D).$$

is verified.

**Definition 2.1.** — Let $E_D$ be the $O_K$-module defined in (9). For every place $v \in M_{K,\infty}$, we denote by $\overline{E}_D$ the Hermitian vector bundle over $\text{Spec} O_K$, where for every $v \in M_{K,\infty}$, $E_D$ is equipped with the norm of John $\|\cdot\|_{v,\text{John}}$ induced by the norm $\|\cdot\|_{v,\text{sup}}$ defined in (11). Similarly, we denote by $\overline{E}_{D,\text{sym}}$ the Hermitian vector bundle over $\text{Spec} O_K$ where $E_D$ is equipped with the norms $\|\cdot\|_{v,\text{sym}}$ introduced above.

With all the notations in Definition 2.1, we have the following result.

**Proposition 2.2 ([12], Proposition 2.7).** — With all the notations in Definition 2.1, we have

$$\hat{\mu}_\text{min}(\overline{E}_D) = \hat{\mu}_\text{min}(\overline{E}_{D,\text{sym}}) - R_0(n, D).$$

In the above equality, the constant $R_0(n, D)$ defined in the equality (13) satisfies the inequality

$$0 \leq R_0(n, D) \leq \log \sqrt{r(n, D)},$$

where the constant $r(n, D) = \text{rk}(E_D)$ follows the definition in the equality (10).

Let $X$ be a pure dimensional closed subscheme of $\mathbb{P}(\mathcal{E}_K)$, and $\mathcal{X}$ be the Zariski closure of $X$ in $\mathbb{P}(\mathcal{E})$. We denote by

$$\eta_{X,D} : E_{D,K} = H^0(\mathbb{P}(\mathcal{E}_K), \mathcal{O}_{\mathbb{P}(\mathcal{E}_K)}(D)) \to H^0(X, \mathcal{O}_{\mathbb{P}(\mathcal{E}_K)}(1)|_X^\otimes D)$$

the evaluation map over $X$ induced by the closed immersion from $X$ to $\mathbb{P}(\mathcal{E}_K)$. In addition, we denote by $F_D$ the largest saturated sub-$O_K$-module of $H^0(\mathcal{X}, \mathcal{O}_{\mathbb{P}(\mathcal{E}_K)}(1)|_X^\otimes D)$ such that $F_{D,K} = \text{Im}(\eta_{X,D})$. When the integer $D$ is large enough, the homomorphism $\eta_{X,D}$ is surjective, which means $F_D = H^0(\mathcal{X}, \mathcal{O}_{\mathbb{P}(\mathcal{E}_K)}(1)|_X^\otimes D)$ (cf. [19, Chap. III, Theorem 5.2 (b)]).

The $O_K$-module $F_D$ is equipped with the quotient metrics (from $\overline{E}_D$) such that $F_D$ is a Hermitian vector bundle over $\text{Spec} O_K$, noted by $\overline{F}_D$ this Hermitian vector bundle. Moreover, in the remainder part of this article, we denote by $r_1(D)$ the rank of the $O_K$-module $F_D$. 


Definition 2.3. — We denote by $\mathcal{F}_D$ the Hermitian vector bundle over $\text{Spec } \mathcal{O}_K$ defined above from (14). We define that the function which maps the positive integer $h$ to $\hat{\mu}(\mathcal{F}_D)$ is the arithmetic Hilbert-Samuel function of $X$ with respect to the Hermitian line bundle $\mathcal{O}_{\mathcal{F}(\mathcal{E})}(1)$.

Remark 2.4. — With all the notations in Definition 2.3. Let

$$h_{\mathcal{O}_{\mathcal{F}(\mathcal{E})}(1)}(X) = \deg_n \left( \hat{c}_1 \left( \mathcal{O}_{\mathcal{F}(\mathcal{E})}(1) \right)^{d+1} \cdot [\mathcal{X}] \right).$$

In fact, the Arakelov degree (15) defines a height of $X$ by the arithmetic intersection theory (cf. [14, Definition 2.5]). By [31, Théorème A], we have

$$h_{\mathcal{O}_{\mathcal{F}(\mathcal{E})}(1)}(X) = \lim_{D \to +\infty} \frac{\deg_n(F_D)}{D^{d+1/(d+1)}}.$$  

By [12, Corollary 2.9], we have the trivial lower bound of $\hat{\mu}(\mathcal{F}_D)$

$$\hat{\mu}(\mathcal{F}_D) \geq -\frac{1}{2}D \log(n+1).$$

2.6. Height of rational points given by Arakelov theory. — We will give a definition of the height of rational points by Arakelov theory in this part. Let $\mathcal{F}$ be a Hermitian vector bundle of rank $n+1$ over $\text{Spec } \mathcal{O}_K$, $P \in \mathbb{P}(\mathcal{E}_K)(K)$, and $\mathcal{P} \in \mathbb{P}(\mathcal{E})(\mathcal{O}_K)$ be its Zariski closure in $\mathbb{P}(\mathcal{E})$. Let $\mathcal{O}_{\mathcal{F}(\mathcal{E})}(1)$ be the universal bundle equipped with the corresponding Fubini-Study metric at each $v \in M_{K,\infty}$, then $\mathcal{P}^*\mathcal{O}_{\mathcal{F}(\mathcal{E})}(1)$ is a Hermitian line bundle over $\text{Spec } \mathcal{O}_K$. We define the height of the rational point $P$ as

$$h_{\mathcal{O}_{\mathcal{F}(\mathcal{E})}(1)}(P) = \deg_n \left( \mathcal{P}^* \mathcal{O}_{\mathcal{F}(\mathcal{E})}(1) \right).$$

In fact, (17) is the same as the definition (15) when we choose $X$ to be a rational point in $\mathbb{P}(\mathcal{E}_K)$ considered as one of its closed integral subschemes.

Remark 2.5. — We keep all the above notations in this part. Now we choose $\mathcal{F} = \left( \mathcal{O}_{K}^{(n+1)} ; (\| \cdot \|_v)_{v \in M_{K,\infty}} \right)$, where for every $v \in M_{K,\infty}$, $\| \cdot \|_v$ is the $\ell^2$-norm mapping $(t_0, \ldots, t_n)$ to $\sqrt{|v(t_0)|^2 + \cdots + |v(t_n)|^2}$. We suppose that $P$ has the $K$-rational projective coordinate $[x_0 : \cdots : x_n]$, then we have (cf. [26, Proposition 9.10])

$$h_{\mathcal{O}_{\mathcal{F}(\mathcal{E})}(1)}(P) = \sum_{v \in M_{K,\infty}} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \log \left( \max_{1 \leq i \leq n} |x_i|_v \right)$$

$$+ \frac{1}{2} \sum_{v \in M_{K,\infty}} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \log \left( \sum_{j=0}^{n} |v(x_j)|^2 \right).$$

In addition, let the $h(\cdot)$ be the height defined in (7). Then by some elementary calculation, the inequality

$$\left| h(P) - h_{\mathcal{O}_{\mathcal{F}(\mathcal{E})}(1)}(P) \right| \leq \frac{1}{2} \log(n+1).$$
is verified uniformly for all \( P \in \mathbb{P}(E_K) \) when we choose the above \( \mathcal{F} \).

**2.7. Further notations on counting rational points problem.** — Let \( \psi : X \hookrightarrow \mathbb{P}(E_K) \) be a closed immersion from \( X \) to \( \mathbb{P}(E_K) \), and \( P \in X(K) \). We denote the height of \( P \) by \( h_{\mathcal{O}_{\mathbb{P}(E)}(1)}(\psi(P)) \) at (17). We will use the notations \( h_{\mathcal{O}_{\mathbb{P}(E)}(1)}(P) \), \( h_{\mathcal{O}(1)}(P) \) or \( h(P) \) if there is no confusion of the morphism \( \psi \) and the Hermitian line bundle \( \mathcal{O}_{\mathbb{P}(E)}(1) \). This height also satisfies the Northcott’s property for arbitrary Hermitian vector bundle \( E \) (cf. [39, Theorem 5.3]), so it can be used in the counting rational points problem. Actually, the line bundle \( \mathcal{O}_{\mathbb{P}(E)}(1) \) can be replaced by arbitrary ample line bundle for the correctness of the Northcott’s property.

In the rest part of this article, unless specially mentioned, we will use the height function defined at (17), and we will use the notation \( h \) to denote this height function. The classic height defined at (6) and (7) will not be essentially used any longer.

### 3. An improved estimate of the determinant

In this section, we will improve an estimate in the determinant method. Parts of the construction are from [34].

**3.1. Estimates of norms.** — In this part, we will estimate the norms of some local homomorphisms, which can be viewed as a generalization of parts of [13, §3]. The same idea has been applied in [34, §16.2]. This estimate is finer than that in [32, Lemma 2.4] and [13, Proposition 3.4], but will be more implicit because of some technical obstructions.

First, we refer a useful auxiliary result in [13], which will be useful in the approach of Arakelov geometry. Before introducing it, we recall an useful notion. Let \((k, |\cdot|)\) be a non-Archimedean field, and \((V, \|\cdot\|)\) be normed vector space over \((k, |\cdot|)\). We say that \((V, \|\cdot\|)\) is ultranormed if for all \( x, y \in U \), we have \( \|x + y\| \leq \max\{\|x\|, \|y\|\} \).

**Lemma 3.1** ([13, Lemma 3.3]). — Let \( k \) be a field equipped with a non-Archimedean absolute value \(|\cdot|\), \( U \) and \( V \) be two \( k \)-linear ultranormed spaces of finite rank and \( \phi : U \to V \) be a \( k \)-linear homomorphism. Let \( m = \dim_k(U) \). For any integer \( 1 \leq i \leq m \), let

\[
\lambda_i = \inf_{\text{codim}_U(W) = i - 1} \|\phi|_W\|.
\]

If \( i > m \), let \( \lambda_i = 0 \). Then for any integer \( r > 0 \), we have

\[
\|\wedge^r \phi\| \leq \prod_{i=1}^{r} \lambda_i.
\]

In the remained part of this section, unless specially mentioned, we denote by \( K \) a number field, and by \( \mathcal{O}_K \) its ring of integers. We fix a Hermitian vector bundle \( \mathcal{E} \) of rank \( n + 1 \) over \( \text{Spec} \mathcal{O}_K \), a closed integral subscheme \( X \) of \( \mathbb{P}(E_K) \), and the Zariski closure \( \mathcal{X} \) of \( X \) in \( \mathbb{P}(E) \). We refer the readers to [34, Lemma 16.9] for the original ideas of the construction.
Let $\mathfrak{p}$ be a maximal ideal of $O_K$, $\mathbb{F}_p$ be the residue field of $O_K$ at $\mathfrak{p}$. Let $\xi$ be an $\mathbb{F}_p$-point of $\mathcal{X}$, and $k \in \mathbb{N}^+$. We suppose that $\{f_i\}_{1 \leq i \leq k}$ is a family of local homomorphisms of $O_{K,\mathfrak{p}}$-algebras from $O_{\mathcal{X},\xi}$ to $O_{K,\mathfrak{p}}$. Let $\mathfrak{a}$ be the kernel of $f_i$, then we have $O_{\mathcal{X},\xi}/\mathfrak{a} \cong O_{K,\mathfrak{p}}$, which shows that $\mathfrak{a}$ is a prime ideal. Furthermore, since $O_{\mathcal{X},\xi}$ is a local ring with the maximal ideal $m_\xi$, we have $m_\xi \supseteq \mathfrak{a}$. Moreover, for $f_1$ is a local homomorphism, we have $\mathfrak{a} + \mathfrak{p}O_{\mathcal{X},\xi} = m_\xi$.

In addition, we suppose that the point $\xi$ is regular in $\mathcal{X}$, which means $O_{\mathcal{X},\xi}$ is a regular local ring. In this case, the ideal $\mathfrak{a}$ is generated by $\dim (O_{\mathcal{X},\xi}) - 1$ regular parameters (cf. [1, Proposition 4.10]). Since these elements form a regular sequence on $O_{\mathcal{X},\xi}$ (cf. [36, Chap. III, Proposition 6]), we have $\text{Sym}^m(\mathfrak{a}/\mathfrak{a}^2) \cong \mathfrak{a}^m/\mathfrak{a}^{m+1}$ as free $O_{K,\mathfrak{p}}$-modules for all $m \geq 0$ by [16, Chap. IV, §2, Corollary 2.4], where we define $\mathfrak{a}^0 = O_{\mathcal{X},\xi}$ for convenience.

Let $S = O_{\mathcal{X},\xi} \setminus \mathfrak{a}$, and we denote by

$$R_{\mathcal{X},\xi} = S^{-1}(O_{\mathcal{X},\xi})$$

the localization of $O_{\mathcal{X},\xi}$ at the prime ideal $\mathfrak{a}$. We denote by $m_\xi$ the maximal ideal of the ring $R_{\mathcal{X},\xi}$, and then we have $m_\xi = \mathfrak{a}R_{\mathcal{X},\xi}$ by the definition of this localization.

Let $u \in S$ and $r \in \mathfrak{a}^m$ for every $m \geq 0$. If $ur \in \mathfrak{a}^{m+1}$ is verified, since we have $(u + \mathfrak{a})(r + \mathfrak{a}^{m+1}) = \mathfrak{a}^{m+1}$, then we obtain $r \in \mathfrak{a}^{m+1}$. Therefore, we deduce

$$m_\xi^{m+1} \cap \mathfrak{a}^m = (\mathfrak{a}^{m+1} \cap R_{\mathcal{X},\xi}) \cap \mathfrak{a}^m = \mathfrak{a}^{m+1}$$

for all $m \geq 0$.

Let $E$ be a free sub-$O_{K,\mathfrak{p}}$-module of finite type of $O_{\mathcal{X},\xi}$ and let

$$f = (f_i|_E)_{1 \leq i \leq k} : E \rightarrow O_{K,\mathfrak{p}}^k$$

be an $O_{K,\mathfrak{p}}$-linear homomorphism. As $f_1$ is a homomorphism of $O_{K,\mathfrak{p}}$-algebras, it is surjective.

We consider $(E \cap \mathfrak{a}^j)/(E \cap \mathfrak{a}^{j+1})$ and $(E \cap m_\xi^j)/(E \cap m_\xi^{j+1})$ as two free $O_{K,\mathfrak{p}}$-modules, where we consider $E$ as a sub-$O_{K,\mathfrak{p}}$-module of $R_{\mathcal{X},\xi}$ if it is necessary. Then we have the isomorphism of $O_{K,\mathfrak{p}}$-modules

$$E \cap (\mathfrak{a}^i)/(E \cap \mathfrak{a}^{i+1}) \cong (E \cap \mathfrak{a}^i)/( (E \cap \mathfrak{a}^i) \cap (E \cap m_\xi^{i+1}) )$$

$$\cong ( (E \cap \mathfrak{a}^i) + (E \cap m_\xi^{i+1}) )/(E \cap m_\xi^{i+1}) \cong (E \cap m_\xi^i)/(E \cap m_\xi^{i+1})$$

by (20), where we use the fact $\mathfrak{a}^iR_{\mathcal{X},\xi} + m_\xi^{i+1} = m_\xi^i$ in $R_{\mathcal{X},\xi}$.

Now we suppose that the reductions of all the above local homomorphisms $f_1, \ldots, f_k$ modulo $\mathfrak{p}$ are same, which means all the composed homomorphisms $O_{\mathcal{X},\xi} \xrightarrow{f_i} O_{K,\mathfrak{p}} \rightarrow \mathbb{F}_p$ are same for every $i = 1, \ldots, k$, where the last arrow is the canonical reduction morphism modulo $\mathfrak{p}$. Let $N(\mathfrak{p}) = \#\mathbb{F}_p$. In this case, the norm of the restriction of $f$ on $E \cap \mathfrak{a}^j$ is smaller than $N(\mathfrak{p})^{-1}$. In fact, for any $1 \leq i \leq k$, we have $f_i(\mathfrak{a}^j) \subseteq \mathfrak{p}O_{K,\mathfrak{p}}$, and hence we have $f_i(\mathfrak{a}^j) \subseteq \mathfrak{p}/O_{K,\mathfrak{p}}$.

From the above construction, we have the following result, which is a reformulation of the estimate in [34, Lemma 16.9].
Proposition 3.2. — Let \( \mathfrak{p} \) be a maximal ideal of \( \mathcal{O}_K \), and \( \xi \in \mathcal{X}(\mathfrak{p}) \) be a non-singular point. Suppose that \( \{f_i\}_{1 \leq i \leq k} \) is a family of local \( \mathcal{O}_{K, \mathfrak{p}} \)-linear homomorphisms from \( \mathcal{O}_{X, \xi} \) to \( \mathcal{O}_{K, \mathfrak{p}} \) whose reductions module \( \mathfrak{p} \) are same. Let \( E \) be a free \( \mathcal{O}_{K, \mathfrak{p}} \)-module of finite type of \( \mathcal{O}_{X, \xi} \), and let \( E = (f_i|_E)_{1 \leq i \leq k} \) be that defined in (21), and \( N(\mathfrak{p}) = \#(\mathcal{O}_K/\mathfrak{p}) \). We consider \( E \) as a sub-\( \mathcal{O}_{K, \mathfrak{p}} \)-module of \( \mathcal{R}_{X, \xi} \), and let

\[
R_{\xi}(E) = \sum_{k=1}^{\infty} \dim_K \left( E \cap m_{\xi}^k \right)_K.
\]

Then if \( r = \dim_K(E_K) \), we have

\[
\log \| \wedge^r f_K \| \leq -R_{\xi}(E) \log N(\mathfrak{p}).
\]

Proof. — By the above notations and argument, we have the filtration

\[
F: E \supset E \cap a \supset \cdots \supset E \cap a^j \supset E \cap a^{j+1} \supset \cdots
\]

of \( E \), whose \( j \)-th subquotient \( (E \cap a^j)/(E \cap a^{j+1}) \) is a free \( \mathcal{O}_{K, \mathfrak{p}} \)-module. The restriction of \( f \) on \( E \cap a^j \) has norm smaller than \( N(\mathfrak{p})^{-j} \). Meanwhile, let \( \{q_{\xi}(m)\}_{m=1}^{\infty} \) be the series of non-negative integers where the integer \( m \) appears exactly \( \dim_K \left( E \cap m_{\xi}^m \right)_K - \dim_K \left( E \cap m_{\xi}^{m+1} \right)_K \) times. Then by the isomorphism (22), the free \( \mathcal{O}_{K, \mathfrak{p}} \)-modules \( (E \cap a^j)/(E \cap a^{j+1}) \) and \( (E \cap m_{\xi}^j)/(E \cap m_{\xi}^{j+1}) \) have the same rank for all \( j \geq 0 \). Thus we have

\[
\inf_{W \subset E_K} \inf_{\text{codim}_{E_K}(W) = j-1} \|f_K|_W\| \leq N(\mathfrak{p})^{-q_{\xi}(j)}.
\]

Since the above filtration \( F \) is of finite length, then by an elementary calculation, we obtain the equality

\[
\sum_{m=1}^{\infty} q_{\xi}(m) = \sum_{m=1}^{\infty} \dim_K \left( E \cap m_{\xi}^m \right)_K.
\]

Finally by applying Lemma 3.1 to (24), we obtain the result. \( \square \)

3.2. Existence of auxiliary hypersurfaces. — In this part, we will reformulate the determinant method by the slope method. Different from [32, Theorem 3.2] and [34, Theorem 16.12], our estimate will depend on the term \( R_{\xi}(E) \) defined in (23) for a special choice of \( E \). In §4, we will reformulate the estimate of \( R_{\xi}(E) \) for our application such that we are able to control the number of auxiliary hypersurfaces by this result. The strategy is similar to that of [13, Theorem 3.1].

The following slope equality is useful in this reformulation, which is obtained by the slope equalities and inequalities.

Proposition 3.3 ([12], Proposition 2.2). — Let \( \mathbf{T} \) be a Hermitian vector bundle of rank \( r > 0 \) over \( \text{Spec} \mathcal{O}_K \), and \( \{T_i\}_{i \in I} \) be a family of Hermitian line bundles over
Spec O_K. If \( \phi : E_K \to \bigoplus_{i \in I} L_{i,K} \) is an injective homomorphism of \( K \)-vector spaces, then there exists a subset \( I_0 \) of \( I \) whose cardinality is \( r \) such that the equality

\[
\tilde{\mu}(E) = \frac{1}{r} \left( \sum_{i \in I_0} \tilde{\mu}(L_i) + h \left( \wedge^r (pr_{I_0} \circ \phi) \right) \right)
\]

is verified, where \( pr_{I_0} : \bigoplus_{i \in I} L_{i,K} \to \bigoplus_{i \in I_0} L_{i,K} \) is the canonical projection.

The following result is a refined determinant method, which follows the strategy of [13, Theorem 3.1] by bringing the term \( R_\xi(E) \) defined in (23) into the estimate.

Before providing the statement, we will introduce the operation below. Let \( \mathcal{E} \) be a Hermitian vector bundle of rank \( n+1 \) over Spec \( O_K \), \( X \) be a closed integral subscheme of \( \mathbb{P}(E_K) \), and \( \mathcal{X} \) be the Zariski closure of \( X \) in \( \mathbb{P}(E) \). We choose a \( P \in X(K) \), and let \( \mathcal{P} \in \mathcal{X}(O_K) \) be the Zariski closure of \( P \) in \( \mathcal{X} \). If we say that the reduction of \( P \) modulo \( p \) is an ideal \( \xi \) of \( O_K \), we mean that we consider the reduction of \( \mathcal{P} \) modulo \( p \), whose image is \( \xi \). We will use this representation multiple times in this article below.

**Theorem 3.4.** — We keep all the above notations. Let \( \{ p_j \}_{j \in J} \) be a finite family of maximal ideals of \( O_K \), and \( \{ P_i \}_{i \in I} \) be a family of rational points of \( X \) such that, for any \( i \in I \) and any \( j \in J \), the reduction of \( P_i \) modulo \( p_j \) coincides with the same non-singular point \( \xi_j \in \mathcal{X}(\mathbb{F}_{p_j}) \). Let \( \overline{F}_D \) be that defined in Definition 2.3, \( R_{\xi_j}(F_D) \) be that defined in (23), \( r_1(D) = \text{rk}(F_D) \), \( N(p_j) = \#(O_K/p_j) \), and the height function \( h(\cdot) \) of rational points defined in (17) by Arakelov theory. If the inequality

\[
\sup_{i \in I} h(P_i) < \frac{\tilde{\mu}(F D)}{D} - \frac{\log r_1(D)}{2D} + \frac{1}{[K:\mathbb{Q}]} \sum_{j \in J} \frac{R_{\xi_j}(F_D)}{D r_1(D)} \log N(p_j)
\]

is verified for a positive integer \( D \), then there exists a section \( s \in E_{D,K} \) (see (9) for its definition), which contains \( \{ P_i \}_{i \in I} \) but does not contain the generic point of \( X \). In other words, \( \{ P_i \}_{i \in I} \) can be covered by a hypersurfaces of \( \mathbb{P}(E_K) \) of degree \( D \) which does not contain the generic point of \( X \).

**Proof.** — We suppose the section predicted by this theorem does not exist. Then the evaluation map

\[
f : F_{D,K} \to \bigoplus_{i \in I} P_i^* O_{\mathcal{X}(K)}(1)|_{\mathcal{X}}^D
\]

is injective. We can replace \( I \) by one of its subsets such that the above homomorphism \( f \) is an isomorphism.

For every \( v \in M_{K,\infty} \), we have

\[
\frac{1}{r_1(D)} \log \| \wedge^{r_1(D)} f \|_v \leq \log \| f \|_v \leq \log \sqrt{r_1(D)},
\]

where the first inequality comes from Hadamard’s inequality, and the second one is due to the definition of metrics of John introduced at §2.5.
For every \( v \in M_{K,f} \), let \( p \) be the maximal ideal of \( \mathcal{O}_K \) corresponding to the place \( v \). By definition, the isomorphism \( f \) is induced by a homomorphism \( \mathcal{O}_K \)-modules

\[
F_D \rightarrow \bigoplus_{i \in I} \mathcal{P}_i^* \mathcal{O}_{\mathcal{Y}(E)}(1)^{\otimes D},
\]

where \( \mathcal{P}_i \) is the \( \mathcal{O}_K \)-point of \( \mathcal{X} \) extending \( P_i \). Hence for any maximal ideal \( p \), we have \( \log \| \wedge^r(D) f \|_p \leq 0 \).

We fix a \( j \in J \). For each \( i \in I \), the \( \mathcal{O}_K \)-point \( \mathcal{P}_i \) defines a local homomorphism from \( \mathcal{O}_{\mathcal{X},\xi_j} \) to \( \mathcal{O}_{K,p_j} \), which is \( \mathcal{O}_{K,p_j} \)-linear. By taking a local trivialization of \( \mathcal{O}_{\mathcal{Y}(E)}(1)^{\otimes D} \) at \( \xi_j \), we identify \( F_D \) as a sub-\( \mathcal{O}_{K,p_j} \)-module of \( \mathcal{O}_{\mathcal{X},\xi_j} \). Then by Proposition 3.2, we have

\[
\log \| \wedge^r(D) f \|_{p_j} \leq - \mathcal{R}_{\xi_j}(F_D) \log N(p_j).
\]

From the above two upper bounds of the operator norms, combined with Proposition 3.3, we obtain

\[
\frac{\hat{h}(\mathcal{F},D)}{D} \leq \sup_{i \in I} h(P_i) + \frac{1}{2D} \log r_1(D) - \frac{1}{|K : \mathbb{Q}|} \sum_{j \in J} \frac{\mathcal{R}_{\xi_j}(F_D)}{Dr_1(D)} \log N(p_j),
\]

which leads to a contradiction.

4. Estimates of \( \mathcal{R}_{\xi_j}(F_D) \)

In order to apply Theorem 3.4, more information about the term \( \mathcal{R}_{\xi_j}(F_D) \) need to be gathered. The aim of this section is to give an asymptotic estimate of \( \mathcal{R}_{\xi_j}(F_D) \), which reformulate a result of Salberger by a more implicit approach.

4.1. Finiteness of \( \mathcal{R}_{\xi_j}(F_D) \). — Formally, the sum in \( \mathcal{R}_{\xi_j}(F_D) \) defined in (23) is infinite. But since the filtration \( \mathcal{F} \) introduced in the proof of Proposition 3.2 is finite, then \( \mathcal{R}_{\xi_j}(F_D) \) is essentially a finite sum. Then when the positive integer \( m \) is large enough in \( F_D \cap m_{\xi_j}^{\mathcal{P}} \), it will be a zero module, so essentially it is a finite sum.

The following result is a reformulation of [34, Lemma 16.10], at which the case of cubic hypersurfaces in \( \mathbb{P}^3 \) was considered only.

**Proposition 4.1.** — We keep all notations and conditions in Theorem 3.4. Let \( \eta_j \in X(K) \) be a rational point which specializes to \( \xi_j \) with respect to the operation in Theorem 3.4, \( m_{\xi_j} \) be the maximal ideal of \( \mathcal{R}_{\mathcal{X},\xi_j} \) defined in (19), and \( n_{\eta_j} \) be the maximal ideal of \( \mathcal{O}_X \) at the point \( \eta_j \). Then for every \( m \in \mathbb{N}^+ \) and \( j \in J \) in Theorem 3.4, we have

\[
\dim_K \left( F_D \cap m_{\xi_j}^m \right) = \dim_K \ker \left( F_{D,K} \rightarrow H^0 \left( X, \mathcal{O}_{\mathcal{Y}(E)}(1)^{\otimes D} \otimes \mathcal{O}_X / n_{\eta_j}^m \right) \right) \geq \max \left\{ 0, r_1(D) - \left( \frac{d + m - 1}{m - 1} \right) \right\},
\]

where we identify \( F_D \) as a sub-\( \mathcal{O}_{K,p_j} \)-module of \( \mathcal{O}_{\mathcal{X},\xi_j} \) for the above \( j \in J \).
Proof. — Let $s_1, \ldots, s_{r_1(D)} \in F_D$ which generate $F_D$. Let $T_0, \ldots, T_n$ be the homogeneous coordinate of $X \hookrightarrow \mathbb{P}(E)$. Without loss of generality, we suppose that $T_0(\xi_j) \neq 0$ with respect to the canonical morphism. Let $r_i = s_i/T_0^D$ for all $i = 1, \ldots, r_1(D)$. and $W_D \subset R_{X, \xi_j}$ be the vector space over $K$ generated by the images of $r_1, \ldots, r_{r_1(D)}$ in $R_{X, \xi_j}$, which is also of dimension $r_1(D)$. Thus for each $s \in F_D$, its image in $H^0 \left( X, \mathcal{O}_{\mathbb{P}^r(\xi_j)}(1) \otimes \mathcal{O}_X/m_{\eta_j}^m \right)$ is zero if and only if $s/T_0^D \in \ker \left( W_D \rightarrow W_D/m_{\xi_j}^m \right)$ considered as an element in $R_{X, \xi_j}$, which means it is verified if and only if $s/T_0^D \in W_D \cap m_{\xi_j}^m$. Thus there exists an isomorphism of $K$-vector spaces from $F_{D,K}$ to $W_D$, which maps $\ker \left( F_{D,K} \rightarrow H^0 \left( X, \mathcal{O}_{\mathbb{P}^r(\xi_j)}(1) \otimes \mathcal{O}_X/m_{\eta_j}^m \right) \right)$ onto $W_D \cap m_{\xi_j}^m$ and then we obtain the first equality in the assertion.

By the fact that the point $\xi_j$ is regular in $X$ and $\dim(X) = d$, then the point $\eta_j$ is also regular in $X$, and the ring $R_{X, \xi_j}$ is a regular local ring of Krull dimension $d$. By these facts, we have $\dim_K \left( R_{X, \xi_j}/m_{\xi_j}^m \right) = (d+m-1)/m!$ for all $m \in \mathbb{N}^+$. Furthermore, we have $\dim_K \left( W_D/ \left( W_D \cap m_{\xi_j}^m \right) \right) \leq \dim_K \left( R_{X, \eta_j}/m_{\eta_j}^m \right)$. Hence we have

$$
\dim_K \left( F_D \cap m_{\xi_j}^m \right)_K = \dim_K \left( W_D/ \left( W_D \cap m_{\xi_j}^m \right) \right) \geq r_1(D) - \left( d + m - 1 \right)/m - 1,
$$

which completes the proof.

Connection with Seshadri constant. — In this part, we will give a lower bound of the positive integer $m$ such that

$$
\dim_K \left( F_D \cap m_{\xi_j}^m \right)_K = \dim_K \ker \left( F_{D,K} \rightarrow H^0 \left( X, \mathcal{O}_{\mathbb{P}^r(\xi_j)}(1) \otimes \mathcal{O}_X/m_{\eta_j}^m \right) \right)
$$

are both zero, where all the above notations are same as those in Proposition 4.1. For this target, we will introduce some notions on the geometric positivity of line bundles. We refer the readers to [24, §5.1] for a systemic introduction to it.

Let $X$ be an closed integral projective scheme over a field, $L$ be a line bundle on $X$, and $\xi \in X$ be a regular point with the maximal ideal $\mathfrak{n}_\xi \subset \mathcal{O}_X$. We consider the natural map

$$
H^0 \left( X, L \right) \rightarrow H^0 \left( X, L \otimes \mathcal{O}_X/\mathfrak{n}_\xi^{s+1} \right)
$$

taking the global sections of $L$ to their $s$-jets at $\xi$. By definition, the kernel of the map (26) is $H^0 \left( X, L \otimes \mathfrak{n}_\xi^{s+1} \right)$.

In addition, let $L$ be a nef line bundle on $X$. We fix a closed point $\xi \in X$, and let $\pi : \tilde{X} \rightarrow X$ be the blowing up at $\xi$, and $E = \pi^{-1}(\xi)$ be the exceptional divisor. We define the Seshadri constant of $L$ at $\xi$ as

$$
\epsilon(X, L; \xi) = \epsilon(L, \xi) = \sup \{ \epsilon > 0 | \pi^* L - \epsilon E \text{ is nef} \}.
$$
By [24, Proposition 5.1.5], we have
\begin{equation}
\epsilon(L; \xi) = \inf_{\xi \in C \subseteq X} \left\{ \frac{(L \cdot C)}{\mu_{\xi}(C)} \right\},
\end{equation}
where \( C \) takes over all integral curves \( C \subseteq X \) passing through \( \xi \), and \( \mu_{\xi}(C) \) is the multiplicity of \( \xi \) in \( C \), see §2.2 for the definition.

Some properties of the Seshadri constant will be useful in the proof of the proposition below.

**Proposition 4.2.** — With all the notations and conditions in Proposition 4.1, when \( m \geq \sqrt[\delta D]{L} \), we have
\[
\ker \left( F_{D,K} \to H^0 \left( X, \mathcal{O}_{\mathcal{P}(\xi)}(1)_{X}^{\otimes D} \otimes \mathcal{O}_X/n_{\eta_j}^m \right) \right) = 0,
\]
where \([s]\) denotes the largest integer smaller than \( s \).

**Proof.** — By the definition of \( F_{D,K} \) induced in (14), the \( K \)-vector space \( F_{D,K} \) is a sub-\( K \)-vector space of \( H^0 \left( X, \mathcal{O}_{\mathcal{P}(\xi)}(1)_{X}^{\otimes D} \right) \), so it is enough to prove the bound for the \( K \)-linear map
\[
H^0 \left( X, \mathcal{O}_{\mathcal{P}(\xi)}(1)_{X}^{\otimes D} \right) \to H^0 \left( X, \mathcal{O}_{\mathcal{P}(\xi)}(1)_{X}^{\otimes D} \otimes \mathcal{O}_X/n_{\eta_j}^m \right).
\]
In other words, we need a bound of \( m \in \mathbb{N} \) such that \( H^0 \left( X, \mathcal{O}_{\mathcal{P}(\xi)}(1)_{X}^{\otimes D} \otimes n_{\eta_j}^m \right) \) is zero.

By definition, the space \( H^0 \left( X, \mathcal{O}_{\mathcal{P}(\xi)}(1)_{X}^{\otimes D} \right) \) is zero when \( m \) is strictly larger than the possibly maximal multiplicity of the point \( \eta_j \) in the divisors which are linearly equivalent to \( \mathcal{O}_{\mathcal{P}(\xi)}(1)_{X}^{\otimes D} \). We denote by \( \mu_{\eta_j} \left( \mathcal{O}_{\mathcal{P}(\xi)}(1)_{X}^{\otimes D} \right) \) the above maximal multiplicity. By [15, Corollary 12.4] and (28), we have
\begin{equation}
\mu_{\eta_j} \left( \mathcal{O}_{\mathcal{P}(\xi)}(1)_{X}^{\otimes D} \right) \leq \epsilon \left( \mathcal{O}_{\mathcal{P}(\xi)}(1)_{X}^{\otimes D}; \eta_j \right),
\end{equation}
where we consider the intersection in the regular locus of \( X \), and the multiplicity of a point in pure-dimensional schemes is considered at [15, Corollary 12.4]. In addition, the multiplicity satisfies the additivity of cycles by [5, Chap. VIII, §7, n° 1, Prop. 3].

By [24, Example 5.1.4], we have
\begin{equation}
\epsilon \left( \mathcal{O}_{\mathcal{P}(\xi)}(1)_{X}^{\otimes D}; \eta_j \right) = D \epsilon \left( \mathcal{O}_{\mathcal{P}(\xi)}(1)|_X; \eta_j \right).
\end{equation}
By [24, Proposition 5.1.9], we have
\begin{equation}
\epsilon \left( \mathcal{O}_{\mathcal{P}(\xi)}(1)|_X; \eta_j \right) \leq \sqrt[D]{\mathcal{O}_{\mathcal{P}(\xi)}(1)|_X} = \sqrt[\delta D]{L},
\end{equation}
for \( \eta_j \) is regular in \( X \) and \( \deg(X) = \delta \) with respect to \( \mathcal{O}(1) \).

By (29), (30) and (31), when \( m \geq \left[ \sqrt[\delta D]{L} \right] + 1 \), we have \( m > \mu_{\eta_j} \left( \mathcal{O}_{\mathcal{P}(\xi)}(1)|_X^{\otimes D} \right) \), and we will have the trivial kernel in this case. \( \square \)
4.2. Invariants induced by blowing up. — Let $\mathcal{E}$ be a Hermitian vector bundle of rank $n + 1$ over $\text{Spec} \mathcal{O}_K$, $X$ be a closed integral subscheme of $\mathbb{P}(\mathcal{E}_K)$ of dimension $d$ and degree $\delta$, and $\mathcal{X}$ be the Zariski closure of $X$ in $\mathbb{P}(\mathcal{E})$. If the positive integer $D$ is large enough, then we have $F_D = H^0(\mathcal{X}, \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)|_{\mathcal{X}}^{\otimes D})$ and $F_{D,K} = H^0(X, \mathcal{O}_{\mathbb{P}(\mathcal{E}_K)}(1)|_{X}^{\otimes D})$, where $F_D$ and $F_{D,K}$ are defined in Definition 2.3. By this fact, we will give an alternative description of the term $\mathcal{R}_{\xi_j}(F_D)$ in Theorem 3.4.

Let $\eta \in X(K)$ be non-singular, $\mathfrak{n}_\eta$ be the maximal ideal of $\mathcal{O}_X$ at the point $\eta$, and

$$\pi : \tilde{X} \to X$$

be the blowing up of $X$ at $\eta$. Let $E = \pi^{-1}(\eta)$ be the exceptional divisor of the above blowing up morphism $\pi$, and $I_E \subset \mathcal{O}_{\tilde{X}}$ be the ideal sheaf of $E \subset \tilde{X}$. By the projection formula (cf. [19, Chap. III, Exercise 8.3]) applied at (32), we have $R^i \pi_* (\pi^* (\mathcal{O}_{\mathbb{P}(\mathcal{E}_K)}(1)|_{\mathcal{X}}^{\otimes D})) = 0$ for all $i \geq 1$, and it deduces $\pi_* (\pi^* (\mathcal{O}_{\mathbb{P}(\mathcal{E}_K)}(1)|_{\mathcal{X}}^{\otimes D})) = \mathcal{O}_{\mathbb{P}(\mathcal{E}_K)}(1)|_{\mathcal{X}}^{\otimes D}$. So we obtain

$$H^0(X, \mathcal{O}_{\mathbb{P}(\mathcal{E}_K)}(1)|_{\mathcal{X}}^{\otimes D}) \cong H^0(\tilde{X}, \pi^* (\mathcal{O}_{\mathbb{P}(\mathcal{E}_K)}(1)|_{\mathcal{X}}^{\otimes D})).$$

From the above isomorphism, we have the commutative diagram

$$
\begin{array}{ccc}
H^0(X, \mathcal{O}_{\mathbb{P}(\mathcal{E}_K)}(1)|_{\mathcal{X}}^{\otimes D}) & \longrightarrow & H^0(X, \mathcal{O}_{\mathbb{P}(\mathcal{E}_K)}(1)|_{\mathcal{X}}^{\otimes D} \otimes \mathcal{O}_X/\mathfrak{n}_\eta^m) \\
\downarrow & & \downarrow \\
H^0(\tilde{X}, \pi^* (\mathcal{O}_{\mathbb{P}(\mathcal{E}_K)}(1)|_{\mathcal{X}}^{\otimes D})) & \longrightarrow & H^0(\tilde{X}, \pi^* (\mathcal{O}_{\mathbb{P}(\mathcal{E}_K)}(1)|_{\mathcal{X}}^{\otimes D} \otimes \mathcal{O}_{\tilde{X}}/I_E^m),
\end{array}
$$

where the kernel of the bottom map is isomorphic to $H^0(\tilde{X}, \pi^* (\mathcal{O}_{\mathbb{P}(\mathcal{E}_K)}(1)|_{\mathcal{X}}^{\otimes D}) \otimes I_E^m)$ for $m \geq 1$. By the above argument, we have the following result.

**Proposition 4.3.** — With all the above notations, we have

$$\dim_K \left( H^0(\tilde{X}, \pi^* (\mathcal{O}_{\mathbb{P}(\mathcal{E}_K)}(1)|_{\mathcal{X}}^{\otimes D}) \otimes I_E^m) \right) = \dim_K \ker(H^0(X, \mathcal{O}_{\mathbb{P}(\mathcal{E}_K)}(1)|_{\mathcal{X}}^{\otimes D}) \to H^0(X, \mathcal{O}_{\mathbb{P}(\mathcal{E}_K)}(1)|_{\mathcal{X}}^{\otimes D} \otimes \mathcal{O}_X/\mathfrak{n}_\eta^m))$$

for all $m \geq 1$.

4.3. The volume of certain line bundles. — In this part, we will give a connection between the above invariant $\mathcal{R}_{\xi_j}(F_D)$ in Theorem 3.4 and the volume of certain line bundles.

4.3.1. Definition of volume function. — In the first step, we will recall the definition of the volume of line bundles on projective varieties at [24, Definition 2.2.31]. For more details about this notion, see [24, §2.2.C].

Let $X$ be a projective integral scheme of dimension $d$ over a field, and $L$ be a line bundle on $X$. We denote by $h^0(X, L) = \dim H^0(X, L)$ for simplicity. Then the *volume* of the line bundle $L$ is defined to be the non-negative number

$$\text{vol}(L) = \text{vol}_X(L) = \lim_{D \to \infty} \frac{h^0(X, L^{\otimes D})}{D^d/d!}$$

(33)
Meanwhile, if $E$ is a Cartier divisor on $X$, we denote the volume by $\text{vol}(E)$ or $\text{vol}_X(E)$ for simplicity, or by passing $O_X(E)$.

Let $NS(X)$ be the Néron-Severi group of $X$ (see [24, Definition 1.1.15] for its definition). By [24, Proposition 2.2.41], the volume of a line bundle only depends on its class in Néron-Severi group. Let $NS(X)_\mathbb{R} = NS(X) \otimes \mathbb{R}$. By [24, Corollary 2.2.45], the volume function defined in (33) can be extended uniquely to a continuous function

$$\text{vol} : NS(X)_\mathbb{R} \to \mathbb{R},$$

where Cartier $\mathbb{R}$-divisors (see [24, §1.3.B] for its definition) are considered above.

4.3.2. Dependence on the reduction. — We keep all the notations as above. Let $H$ be a Cartier divisor on $X$ given by a hyperplane section in $\mathbb{P}(\mathcal{E}_K)$. Let $\eta_1, \eta_2 \in X(K)$ be non-singular, and $\pi_1 : \tilde{X}_1 \to X$ and $\pi_2 : \tilde{X}_2 \to X$ be the blowing ups of $X$ at $\eta_1$ and $\eta_2$ respectively, with respect to the exceptional divisors $E_1 \subset \tilde{X}_1$ and $E_2 \subset \tilde{X}_2$. By Proposition 4.1 and Proposition 4.3, if two rational points of $\eta_1, \eta_2 \in X(K)$ have the same non-singular specialization modulo a maximal ideal of $O_K$ in the sense of Theorem 3.4, then we have

$$h^0 \left( \tilde{X}_1, D\pi_1^*(H) - mE_1 \right) = h^0 \left( \tilde{X}_2, D\pi_2^*(H) - mE_2 \right)$$

for every $D, m \in \mathbb{N}$, which means it only depends on its specialization by the operation of Theorem 3.4.

4.3.3. Pseudo-effective thresholds. — By the fact stated in §4.3.2 above, we will introduce the following invariant.

**Definition 4.4.** — Let $X$ be a closed integral subscheme of $\mathbb{P}(\mathcal{E}_K)$ over the number field $K$, $\eta \in X(K)$ whose specialization modulo $p \in \text{Spm} O_K$ is the non-singular point $\xi$ in the sense of Theorem 3.4, $\pi : \tilde{X} \to X$ be the blowing up at $\eta$, and $E \subset \tilde{X}$ be its exceptional divisor. Let $H$ be a Cartier divisor on $X$ given by a hyperplane section in $\mathbb{P}(\mathcal{E}_K)$. We define

$$I_X(H, \xi) = \int_0^\infty \text{vol}(\pi^*H - \lambda E) \, d\lambda,$$

where the above volume function $\text{vol}(\cdot)$ follows the extended definition introduced in (34) over $\tilde{X}$.

**Remark 4.5 (History of $I_X(H, \xi)$).** — To the author’s knowledge, the invariant $I_X(H, \xi)$ given in Definition 4.4 is first introduced by Per Salberger in 2006 at a talk in Mathematical Science Research Institute (MSRI), Berkeley, USA. In [25, §4], D. McKinnon and M. Roth also introduced this invariant for the research of Diophantine approximations over higher dimensional projective varieties, which is a generalization of Roth’s theorem. In [25], they use the notation $\beta_x(L) = I_X(L, x)/\text{vol}_X(L)$ for a closed point $x$ and an ample line bundle $L$. 
4.4. The dominant term of $\mathcal{R}_{\xi_j}(F_D)$. — We keep all the above notations and conditions. We will give an asymptotic estimate of $\mathcal{R}_{\xi_j}(F_D)$ defined in (23) by the invariant $I_X(H, \xi_j)$, where $j \in J$ is given in Theorem 3.4.

**Theorem 4.6.** — Let $X$ be a closed integral subscheme of $\mathbb{P}(\mathcal{E}_K)$ of dimension $d$ and degree of the ring $\mathcal{E}$ in the sense of Theorem 3.4, $\pi$ be defined in (23), where $j \in J$ and $\xi_j \in \mathcal{E}(\mathbb{F}_p)$ are same as those in Theorem 3.4, and $H$ be a Cartier divisor on $X$ given by a hyperplane section in $\mathbb{P}(\mathcal{E}_K)$. Then we have

$$
\mathcal{R}_{\xi_j}(F_D) = I_X(H, \xi_j) \frac{D^{d+1}}{d!} + O_{d, \delta}(D^d),
$$

where $I_X(H, \xi_j)$ is defined in Definition 4.4.

**Proof.** — Let $\eta \in X(K)$ be a rational point whose reduction modulo $p$ is $\xi$ in the sense of Theorem 3.4, $\pi : \tilde{X} \to X$ be the blowing up of $X$ at $\eta$, $E = \pi^{-1}(\eta)$ be the exceptional divisor of $\pi$. If denote $B = H^0(X, O_X)$ and let $\dim B$ be the Krull dimension of the ring $B$, then by [30, Lemma 2.1], when $D \geq \dim B - 1$, we have $F_D = H^0(X, O_{\mathcal{E}(1)}(\mathcal{E}))$ and $F_{D, K} = H^0(X, O_{\mathcal{E}(K)}(1))\otimes_{\mathcal{E}} D$. Then by Proposition 4.1 and Proposition 4.3, we have

$$
\mathcal{R}_{\xi_j}(F_D) \sim \sum_{m=1}^{\infty} h^0 \left( \tilde{X}, D \pi^* H - mE \right)
$$

when $D$ tends into infinite.

Since $\text{vol}(\pi^* H) = \text{vol}(H) = \delta$, we have

$$
h^0 (X, DH) = h^0 \left( \tilde{X}, D \pi^* H \right) = \frac{\delta}{d!} D^d + O_{d, \delta}(D^{d-1}).
$$

Meanwhile, if $m \geq 1$, we have $0 \leq h^0 \left( \tilde{X}, D \pi^* H - mE \right) \leq h^0 \left( \tilde{X}, D \pi^* H \right)$ and $\text{vol}(\pi^* H - mE) \leq \text{vol}(\pi^* H) = \delta$ for every $m \in \mathbb{N}$. Then by the definition of volume at (33), when $m = 1, \ldots, \left[ \sqrt[d]{\delta} D \right] + 1$, we have

$$
h^0 \left( \tilde{X}, D \pi^* H - mE \right) = \frac{D^d}{d!} \text{vol} \left( \pi^* H - \frac{m}{D} E \right) + O_{d, \delta}(D^{d-1}),
$$

where $[s]$ denotes the largest integer smaller than $s \in \mathbb{R}$.

By Proposition 4.2, we have

$$
\sum_{m=1}^{\infty} h^0 \left( \tilde{X}, D \pi^* H - mE \right) = \left[ \sqrt[d]{\delta} D \right] + 1 \sum_{m=1}^{\left[ \sqrt[d]{\delta} D \right]} h^0 \left( \tilde{X}, D \pi^* H - mE \right).
$$

By the estimate of remainder term in (35) and Definition 4.4, we have

$$
\sum_{m=1}^{\left[ \sqrt[d]{\delta} D \right] + 1} h^0 \left( \tilde{X}, D \pi^* H - mE \right) = \frac{D^d}{d!} \sum_{m=1}^{\infty} \text{vol} \left( \pi^* H - \frac{m}{D} E \right) + O_{d, \delta}(D^d)
$$

$$
= \frac{I_X(H, \xi_j)}{d!} D^{d+1} + O_{d, \delta}(D^d),
$$

where $I_X(H, \xi_j)$ is defined in Definition 4.4.
and we obtain the result.

**Remark 4.7.** — By a result of Salberger announced in the MSRI lecture mentioned in Remark 4.5 (see also [25, Corollary 4.2]), when $X → P(E_K)$ is of degree $δ$ with respect to $O_{P(E_K)}(1)$, we have the following lower bound of $I_X(H, ξ)$ introduced in Definition 4.4, which is

$$I_X(H, ξ) \geq \frac{d \text{vol}(H)}{d + 1} \sqrt{\frac{\text{vol}(H)}{\mu_η(X)}} \geq \frac{d}{d + 1} ε_η(H) \text{vol}(H),$$

where the reduction of $η ∈ X(K)$ modulo $p ∈ \text{Spm} O_K$ is $ξ$ in the sense of Theorem 3.4, $μ_η(X)$ is the multiplicity of $η$ in $X$, and $ε_η(H)$ is the Seshadri constant of $H$ at $η$. For the application in this case, we have

$$I_X(H, ξ) \geq \frac{d \text{vol}(H)}{d + 1} \sqrt{\frac{\text{vol}(H)}{\mu_η(X)}} = \frac{dδ^{1 + \frac{1}{d}}}{(d + 1)},$$

since the point $η$ is regular in $X$, and $\text{vol}(H) = H^d = δ$ by definition. Then by Theorem 4.6, we have

$$R_ξ(F_D) \geq \frac{dδ^{1 + \frac{1}{d}}}{(d + 1)!} D^{d+1} + O_{d, δ}(D^d),$$

which is the same as that obtained in Proposition ?? and some other former results, for example, in [32, Main Lemma 2.5].

5. The number of auxiliary hypersurfaces

In this section, for a closed integral subscheme $X$ of $P(E_K)$, we will give an upper bound of the number of hypersurfaces which cover $S(X; B) = \{ξ ∈ X(K) | H_K(ξ) ≤ B\}$ but do not contain the generic point of $X$. The height function $H_K(ξ) = \exp([K : \mathbb{Q}] h(ξ))$, and $h(ξ)$ follows the definition (17) by Arakelov theory with respect to the Hermitian vector bundle $E$ over $\text{Spec} O_K$.

5.1. Application of the asymptotic estimate of $R_ξ(F_D)$. — Let $E$ be a Hermitian vector bundle of rank $n + 1$ over $\text{Spec} O_K$, $X$ be a closed integral subscheme of $P(E_K)$, and $X$ be the Zariski closure of $X$ in $P(E)$. Let $p ∈ \text{Spm} O_K$, and $ξ ∈ \mathcal{X}(F_p)$. We denote by $S(X; B, ξ)$ the subset of $S(X; B)$ whose reduction modulo $p$ is $ξ$ in the sense of Theorem 3.4.

**Lemma 5.1.** — We keep all the notations and conditions in Theorem 3.4. If $\bigcap_{j ∈ J} S(X; B, ξ_j)$ is not empty, then for every $j ∈ J$, all $\{I_X(H, ξ_j)\}_{j ∈ J}$ are equal, where $I_X(H, ξ_j)$ is defined in Definition 4.4.

**Proof.** — By Proposition 4.1, the invariant $I_X(H, ξ_j)$ only depends on its specialization. Then we obtain the assertion from Proposition 4.3 directly.  ■
We keep all the notations and conditions in Lemma 5.1, and we define
\begin{equation}
I_X(H,\xi_J) = I_X(H,\xi_j)
\end{equation}
for all \(j \in J\). Then by the asymptotic estimate of \(R_\xi(F_D)\), we have the result below deduced from Theorem 3.4.

**Theorem 5.2.** — We keep all the notations in Theorem 3.4. Let \(\{p_j\}_{j \in J}\) be a family of maximal ideals of \(O_K\) and \(B,\epsilon > 0\). For every \(j \in J\), let \(\xi_j \in \mathcal{X}(\mathbb{F}_{p_j})\) be a regular point. Let \(I_X(H,\xi_J)\) be defined in (37) (by Lemma 5.1 it is well defined). If the inequality
\begin{equation}
\sum_{j \in J} \log N(p_j) \geq (1 + \epsilon) \left( \log B + [K : \mathbb{Q}] \log \left( \frac{(n + 1)(d + 1)}{2} \right) \right) \frac{\delta}{I_X(H,\xi_J)}
\end{equation}
is verified, then there exists a hypersurface of degree \(O_d,\delta,\epsilon(1)\) in \(\mathbb{P}(E_K)\), which contains the set \(\bigcap_{j \in J} S(X;B,\xi_j)\) but do not contain the generic point of \(X\).

**Proof.** — We only need to prove the assertion for the case when \(\bigcap_{j \in J} S(X;B,\xi_j) \neq \emptyset\). Let \(D \in \mathbb{N}^+\), and we suppose that such there does not exist such a hypersurface of degree \(D\). By Theorem 3.4, we have
\begin{equation}
\frac{\log B}{[K : \mathbb{Q}]} \geq \frac{\hat{\mu}(F_D)}{D} - \frac{\log r_1(D)}{2D} + \sum_{j \in J} \frac{R_{\xi_j}(F_D) \log N(p_j)}{Dr_1(D)[K : \mathbb{Q}]}.
\end{equation}

For every \(j \in J, \xi_j\) is regular in \(\mathcal{X}\), and we have
\[r_1(D) = \frac{\delta}{d!} D^d + O_d,\delta(D^{d-1}).\]

Then we apply Theorem 4.6 by combining the above two facts, and we obtain that there exists a constant \(C(d,\delta)\) depending on \(d\) and \(\delta\), such that
\[\frac{R_{\xi_j}(F_D)}{Dr_1(D)} \geq \frac{I_X(H,\xi_j)}{\delta} + \frac{C(d,\delta)}{D}\]
is verified for each \(D \geq 1\) and \(j \in J\). By [10, §1.2], we have
\[r_1(D) \leq \delta \left( \frac{D + d}{D} \right) \leq \delta(d + 1)^D.\]

We combine the above arguments and the trivial lower bound of \(\hat{\mu}(F_D)\) introduced at (16). From the inequality (39), we have
\begin{equation}
\frac{\log B}{[K : \mathbb{Q}]} \geq -\frac{1}{2} \log(n + 1) - \frac{\log \delta}{2D} - \frac{1}{2} \log(d + 1) + \left( \frac{I_X(H,\xi_j)}{\delta} + \frac{C(d,\delta)}{D} \right) \sum_{j \in J} \frac{\log N(p_j)}{[K : \mathbb{Q}]},
\end{equation}
and we obtain

$$\left( \frac{I_X(H, \xi_J)}{\delta} \sum_{j \in J} \frac{\log N(p_j)}{[K : \mathbb{Q}]} \log B \right) - \frac{\log(2(n + 1) - \frac{1}{2} \log(d + 1)}{D} \leq \left( -\frac{\log \delta}{2} + C(d, \delta) \right) \sum_{j \in J} \frac{\log N(p_j)}{[K : \mathbb{Q}]}.$$ 

By the hypothesis (38), the left hand side of the above inequality is larger than or equal to

$$\epsilon \cdot I_X(H, \xi_J) \sum_{j \in J} \frac{\log N(p_j)}{[K : \mathbb{Q}]} D,$$

which implies

$$D \leq (\epsilon^{-1} + 1) \frac{\delta}{I_X(H, \xi_J)} \left( -\frac{\log \delta}{2} + C(d, \delta) \right).$$

By (36) and the fact that all $\xi_j$ is regular in $\mathcal{P}_{\mathbb{P}^1}$, for each $j \in J$, there exists a lower bound of $I_X(H, \xi_J)$ which only depends on the $d$ and $\delta$. Then we obtain a contradiction, which terminates the proof.

The following result can be considered as a generalization of [34, Main Lemma 16.3.1].

**Corollary 5.3.** — We keep all the notations and conditions in Theorem 5.2. Let

$$I_X(H) = \inf_{\eta \in S(X^{\mathrm{reg}}; B)} \{I_X(H, \eta)\}.$$ 

If the inequality

$$\sum_{j \in J} \log N(p_j) \geq (1 + \epsilon) \left( \log B + \frac{1}{2} |K : \mathbb{Q}| \log ((n + 1)(d + 1)) \right) \frac{\delta}{I_X(H)}$$

is verified, then there exists a hypersurface of degree $O_{n, \delta, \epsilon}(1)$ in $\mathbb{P}(E_K)$, which contains $\bigcap_{j \in J} S(X; B, \xi_j)$ but does not contain the generic point of $X$.

**Proof.** — By definition (40), we have

$$\frac{\delta}{I_X(H)} \geq \frac{\delta}{I_X(H, \xi_J)},$$

where $I_X(H, \xi_J)$ is defined in the assertion of Theorem 5.2. Then we obtain this result from (38) in Theorem 5.2 directly.

**5.2. Bertrand’s postulate of number fields.** — In order to apply Theorem 5.2 and Corollary 5.3, we need some estimate about the distribution of prime ideals of rings of algebraic integers. In fact, we need an analogue of Bertrand’s postulate for the case of number fields, which follows.
Lemma 5.4. — Let $K$ be a number field, and $\mathcal{O}_K$ be the ring of integers of $K$. There exists a constant $\alpha(K) \geq 2$ depending on $K$, such that for all number $N_0 \geq 1$, there exists at least one maximal ideal $p$ of $\mathcal{O}_K$, such that $N_0 < N(p) \leq \alpha(K)N_0$.

We refer to [23] or [35, Théorème 2] for a proof by admitting the Generalized Riemann Hypothesis, and to [38, Théorème 1.7] without admitting it.

5.3. Complexity of the singular locus. — Let $\mathcal{E}$ be a Hermitian vector bundle of rank $n+1$ over $\text{Spec } \mathcal{O}_K$, $X$ be a closed integral subscheme of $\mathbb{P}(\mathcal{E}_K)$ of degree $\delta$ and dimension $d$. In order to give an upper bound of the number of auxiliary hypersurfaces which cover $S(X; B)$ but do not contain the generic point of $X$, we divide $S(X; B)$ into two part: the part of regular points and the part of singular points. In this part, we will deal with the singular part $S(X^{\text{sing}}; B)$.

By [12, Theorem 3.10] (see also [13, §2.6]), we have the following control to the complexity of the singular locus.

Proposition 5.5. — Let $\mathcal{E}$ be a Hermitian vector bundle of rank $n+1$ over $\text{Spec } \mathcal{O}_K$, and $X$ be a closed integral subscheme of $\mathbb{P}(\mathcal{E}_K)$, which is of degree $\delta$ and of dimension $d$. Then there exists a hypersurface of degree $(\delta - 1)(n - d)$ in $\mathbb{P}(\mathcal{E}_K)$ which covers $S(X^{\text{sing}}; B)$ but do not contain the generic point of $X$.

5.4. Control of regular reductions. — Let $p \in \text{Spm } \mathcal{O}_K$, $S(X^{\text{reg}}; B)$ be the subset of $S(X; B)$ consisting of regular points, and $(S(X; B, \xi))$ be the subset of $S(X; B)$ whose reduction modulo $p$ is $\xi$, where the operation modulo $p$ follows the sense of Theorem 3.4. We denote

$$S(X^{\text{reg}}; B, p) = \bigcup_{\substack{\xi \in \mathcal{E}(\mathcal{O}_p) \\ \mu_\xi = 0}} S(X; B, \xi).$$

In other words, $S(X^{\text{reg}}; B, p)$ is the subset of $S(X^{\text{reg}}; B)$ with regular reduction modulo $p$.

Next, we will refer a result that $S(X^{\text{reg}}; B)$ can be covered by some $S(X^{\text{reg}}; B, p)$ for some particular $p \in \text{Spm } \mathcal{O}_K$. For this aim, we introduce the following constants original from [13, Notation 19]. Let

$$C_1 = (d + 2)\mu_{\text{max}} \left( \text{Sym}^\delta \left( \mathcal{E}' \right) \right) + \frac{1}{2}(d + 2) \log \text{rk} \left( \text{Sym}^\delta \mathcal{E} \right) + \delta \log \left( (d + 2)(n - d) \right) + \frac{\delta}{2}(d + 1) \log (n + 1),$$

$$C_2 = \frac{r}{2} \log \text{rk} \left( \text{Sym}^\delta \mathcal{E} \right) + \frac{1}{2} \log \text{rk} \left( \bigwedge^{n-d} \mathcal{E} \right) + \log \sqrt{(n - d)!} + (n - d) \log \delta,$$

and

$$C_3 = (n - d)C_1 + C_2.$$

The above constant $C_1$ is original from [12, (21)], and $C_2$ is from [12, Remark 3.9]. The constant $C_3$ firstly appeared at [12, Theorem 3.10], and we have

$$C_3 \ll_{n, d} \delta.$$
By the above notations, we state the following result.

**Lemma 5.6 ([13], Lemma 4.1).** — Let $N_0 > 0$ be a real number and $r$ be the integral part of

$$
\dfrac{(n - d)(\delta - 1) \log B + \left( (n - d)h_{\mathcal{O}_X(1)}(X) + C_3 \right) [K : \mathbb{Q}]}{\log N_0} + 1,
$$

where the constant $C_3$ is defined in (42), and the height $h_{\mathcal{O}_X(1)}(X)$ is defined in (15). If $p_1, \ldots, p_r$ are distinct maximal ideals of $\mathcal{O}_K$ such that $N(p_i) > N_0$ is verified for every $i = 1, \ldots, r$, then

$$S(X_{\text{reg}}; B) = \bigcup_{i=1}^r S(X_{\text{reg}}; B, p_i),$$

where every $S(X_{\text{reg}}; B, p_i)$ is defined in (41).

5.5. An upper bound of the number of auxiliary hypersurfaces. — In this part, we will estimate the number of auxiliary hypersurfaces which cover $S(X; B)$ but do not contain the generic point of $X$. In fact, by Proposition 5.5, we only need to consider the regular part $S(X_{\text{reg}}; B)$.

By [12, Theorem 4.8] and [12, Proposition 2.12], the rational points with small height in $X$ can be covered by one hypersurface of degree $O(n(\delta))$ not containing the generic point of $X$, where the "small" height means that the bound $B$ is small compared with the height of $X$. We will use the above argument to deal with the points with small height and the method of Theorem 3.4 and Proposition 5.3 to deal with the regular points with large height, and combine it with Lemma 5.6.

**Theorem 5.7.** — Let $K$ be a number field and $\mathcal{O}_K$ be its ring of integers. Let $E$ be a Hermitian vector bundle of rank $n+1$ over $\text{Spec} \mathcal{O}_K$, $X$ be a closed integral subscheme of $\mathbb{P}(E_K)$ of dimension $d$ and degree $\delta$, and $\epsilon > 0$ be an arbitrary real number. Then there exists an explicit constant $C_4(\epsilon, \delta, n, d, K)$, such that for every $B \geq e^\epsilon$, the set $S(X; B)$ can be covered by no more than $C_4(\epsilon, \delta, n, d, K)B^\frac{1 + d}{2d+1+d}$ hypersurfaces with degree of $O_{n,\delta,\epsilon}(1)$ which do not contain the generic point of $X$, where $I_X(H)$ is defined in (40).

**Proof.** — We divide this proof into two parts, the case of varieties with large height and the case of varieties with small height.

**Part 1. Case of large height varieties.** - Suppose that the inequality

$$h_{\mathcal{O}_X(1)}(X) > \left( \frac{(2d + 2)^{d+1}}{d!} \right)(\log B \left[ K : \mathbb{Q} \right] + \frac{3}{2} \log(n + 1) + 2d)$$

is verified, where $h_{\mathcal{O}_X(1)}(X)$ is defined in (15). Then by [12, Theorem 4.8] and [12, Proposition 2.12] (see also §2.1 and §2.3 of [13]), there exists a hypersurface in $\mathbb{P}(E_K)$ of degree $2(n - d)(\delta - 1) + d + 2$ which covers $S(X; B)$ but does not contain the generic point of $X$. 

Part 2. Case of small height varieties. - Now we suppose that the inequality
\[ h_{\mathcal{O}(K^{(1)})}(X) \leq \frac{(2d + 2)^{d+1}}{d!} \delta \left( \frac{\log B}{[K : \mathbb{Q}]} + \frac{3}{2} \log(n + 1) + 2^d \right) \]
is verified. Let
\[ \log N_0 = (1 + \epsilon) \left( \log B + \frac{1}{2}[K : \mathbb{Q}] (\log(n + 1) + \log(d + 1)) \right) \frac{\delta}{I_X(H)}, \]
and \( r \) be the positive integer defined in (44) of Lemma 5.6. In this case, we have
\[ r \leq A_1 \log B + A_2 \log N_0 + 1, \]
where the constants are
\[ A_1 = (n - d)(\delta - 1) + \frac{(2d + 2)^{d+1}}{d!} (n - d)\delta, \]
and
\[ A_2 = [K : \mathbb{Q}] \left( C_3 + \frac{(2d + 2)^{d+1}}{d!} \delta \left( \log(n + 1) + \frac{1}{2} \log(d + 1) + 2^d \right) \right) \]
with the constant \( C_3 \) is defined in (42). By the assumption \( \log B \geq \epsilon \), we obtain \( r \leq A_3 \), where
\[ A_3 = \frac{I_X(H)}{\delta} (A_1 + \epsilon^{-1} A_2) + 1. \]

By Bertrand’s postulate (cf. Lemma 5.4), there exists a family of maximal ideals \( p_1, \ldots, p_r \) of \( \mathcal{O}_K \), such that
\[ \alpha(K)^{i-1} N_0 \leq N(p_i) \leq \alpha(K)^i N_0 \]
for every \( i = 1, \ldots, r \), where the constant \( \alpha(K) \geq 2 \) depends only on the number field \( K \).

For each \( p_i \), we have
\[ \# \mathcal{X}'(F_{p_i}) \leq \delta \left( N(p_i)^d + \cdots + 1 \right) \leq \delta(d + 1)N(p_i)^d \leq \delta(d + 1)\alpha(K)^d N_0^d, \]
and then we obtain the following upper bound of the number of auxiliary hypersurfaces which cover \( S_1(X; B) \) but do not cover the generic point of \( X \). The upper bound mentioned above is
\[ \sum_{i=1}^{r} \# \mathcal{X}'(F_{p_i}) \leq \delta(d + 1)N_0^d \sum_{i=1}^{r} \alpha(K)^d \]
\[ = \delta(d + 1)N_0^d \frac{\alpha(K)^d(\alpha(K)^d - 1)}{\alpha(K)^d - 1} \leq C_4' B^{\frac{1 + t + d}{2X(M)'}}. \]
where the constant
\[
C'_4 = \delta(d+1) \frac{\alpha(K)^d(\alpha(K)^{A_3d} - 1)}{\alpha(K)^d - 1} \frac{\frac{(1+\epsilon)[K:H]d^d}{\delta}}{(d+1)(n+1)^{1+\frac{1}{d}}} \leq \delta(d+1) \frac{\alpha(K)^d(\alpha(K)^{A_3d} - 1)}{\alpha(K)^d - 1} \frac{\frac{(1+\epsilon)[K:H]d^d}{\delta}}{(d+1)(n+1)^{1+\frac{1}{d}}},
\]
(45) := C''_4(\epsilon, \delta, n, d, K).

In the above inequality, the second line is from the lower bound of \( I_X(H) \) provided at [25, Corollary 4.2] (see (36) for this lower bound in our application) and the definition of \( I_X(H) \) at (40). Then we obtain the assertion by Corollary 5.3.

**Conclusion.** - By the above argument, we obtain the result after combining it with Proposition 5.5, where we choose the constant \( C_4(\epsilon, \delta, n, d, K) = C'_4(\epsilon, \delta, n, d, K) + 1 \) introduced at (45).

**Remark 5.8.** — In the proof of Theorem 5.7, by the fact that \( A_1 \ll_{n,d} \delta \) and \( A_2 \ll_{n,d} \delta \), we have \( A_3 \ll_{n,d,\epsilon} \delta^{1+\frac{1}{d}} \), we obtain
\[
\log C_4(\epsilon, \delta, n, d, K) \ll_{n,K,\epsilon} \delta^{1+\frac{1}{d}},
\]
since we have \( 1 \leq d \leq n - 1 \). But the above estimate of \( C_4(\epsilon, \delta, n, d, K) \) is valueless unless we are able to obtain an explicit estimate of the degree of auxiliary hypersurfaces.

**Remark 5.9.** — In Theorem 5.7, we do not give an explicit upper bound of the degree of auxiliary hypersurfaces. The main obstruction is that in Theorem 4.6, when we estimate \( R_{\xi}(F_D) \), until the author’s knowledge, we cannot find an explicit lower bound of \( \dim H^0(X, L^\otimes m) \) for arbitrary line bundle \( L \). If \( L \) is ample, see [22, Page 92] for such an explicit lower bound. So by the strategy of this article, we are not able to control the dependence of \( S(X; H) \) on the degree of \( X \) due to the limit of the author’s ability.

**References**

[1] A. Altman & S. Kleiman – *Introduction to Grothendieck duality theory*, Lecture Notes in Mathematics, Vol. 146, Springer-Verlag, Berlin-New York, 1970.

[2] E. Bombieri & J. Pila – “The number of integral points on arcs and ovals”, *Duke Mathematical Journal* 59 (1989), no. 2, p. 337–357.

[3] J.-B. Bost – “Périodes et isogénies des variétés abéliennes sur les corps de nombres (d’après D. Masser et G. Wüstholz)”, *Astérisque* (1996), no. 237, p. Exp. No. 795, 4, 115–161, Séminaire Bourbaki, Vol. 1994/1995.

[4] J.-B. Bost, H. Gillet & C. Soulé – “Heights of projective varieties”, *Journal of the American Mathematical Society* 7 (1994), no. 4, p. 903–1027.

[5] N. Bourbaki – *Éléments de mathématique*, Masson, Paris, 1983, Algèbre commutative. Chapitre 8. Dimension. Chapitre 9. Anneaux locaux noethériens complets. [Commutative algebra. Chapter 8. Dimension. Chapter 9. Complete Noetherian local rings].
[6] N. Broberg – “A note on a paper by R. Heath-Brown: “The density of rational points on curves and surfaces” [Ann. of Math. (2) 155 (2002), no. 2, 553–595]”, Journal für die Reine und Angewandte Mathematik 571 (2004), p. 159–178.

[7] T. D. Browning & D. R. Heath-Brown – “The density of rational points on non-singular hypersurfaces. I”, The Bulletin of the London Mathematical Society 38 (2006), no. 3, p. 401–410.

[8] ———, “The density of rational points on non-singular hypersurfaces. II”, Proceedings of the London Mathematical Society. Third Series 93 (2006), no. 2, p. 273–303, With an appendix by J. M. Starr.

[9] T. D. Browning, D. R. Heath-Brown & P. Salberger – “Counting rational points on algebraic varieties”, Duke Mathematical Journal 132 (2006), no. 3, p. 545–578.

[10] M. Chardin – “Une majoration de la fonction de Hilbert et ses conséquences pour l’interpolation algébrique”, Bulletin de la Société Mathématique de France 117 (1989), no. 3, p. 305–318.

[11] H. Chen – “Convergence des polygones de Harder-Narasimhan”, Mémoires de la Société Mathématique de France 120 (2010), p. 1–120.

[12] ———, “Explicit uniform estimation of rational points I. Estimation of heights”, Journal für die Reine und Angewandte Mathematik 668 (2012), p. 59–88.

[13] ———, “Explicit uniform estimation of rational points II. Hypersurface coverings”, Journal für die Reine und Angewandte Mathematik 668 (2012), p. 89–108.

[14] G. Faltings – “Diophantine approximation on abelian varieties”, Annals of Mathematics. Second Series 133 (1991), no. 3, p. 549–576.

[15] W. Fulton – Intersection theory, second ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 2, Springer-Verlag, Berlin, 1998.

[16] W. Fulton & S. Lang – Riemann-Roch algebra, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 277, Springer-Verlag, New York, 1985.

[17] É. Gaudron – “Pentes de fibrés vectoriels adéliques sur un corps global”, Rendiconti del Seminario Matematico della Università di Padova 119 (2008), p. 21–95.

[18] H. Gillet & C. Soulé – “On the number of lattice points in convex symmetric bodies and their duals”, Israel Journal of Mathematics 74 (1991), no. 2-3, p. 347–357.

[19] R. Hartshorne – Algebraic geometry, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, No. 52.

[20] D. R. Heath-Brown – “The density of rational points on curves and surfaces”, Annals of Mathematics. Second Series 155 (2002), no. 2, p. 553–595.

[21] M. Hindry & J. H. Silverman – Diophantine geometry, Graduate Texts in Mathematics, vol. 201, Springer-Verlag, New York, 2000, An introduction.

[22] J. Kollár – Lectures on resolution of singularities, Annals of Mathematics Studies, vol. 166, Princeton University Press, Princeton, NJ, 2007.

[23] J. C. Lagarias & A. M. Odlyzko – “Effective versions of the Chebotarev density theorem”, in Algebraic number fields: L-functions and Galois properties (Proc. Sympos., Univ. Durham, Durham, 1975), Academic Press, London, 1977, p. 409–464.
[24] R. Lazarsfeld – *Positivity in algebraic geometry. I*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, vol. 48, Springer-Verlag, Berlin, 2004, Classical setting: line bundles and linear series.

[25] D. McKinnon & M. Roth – “Seshadri constants, diophantine approximation, and Roth’s theorem for arbitrary varieties”, *Inventiones Mathematicae* 200 (2015), no. 2, p. 513–583.

[26] A. Moriwaki – *Arakelov geometry*, Translations of Mathematical Monographs, vol. 244, American Mathematical Society, Providence, RI, 2014, Translated from the 2008 Japanese original.

[27] M. Nagata – *Local rings*, Interscience Tracts in Pure and Applied Mathematics, No. 13, Interscience Publishers a division of John Wiley & Sons New York-London, 1962.

[28] J. Neukirch – *Algebraic number theory*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 322, Springer-Verlag, Berlin, 1999, Translated from the 1992 German original and with a note by Norbert Schappacher, With a foreword by G. Harder.

[29] J. Pila – “Density of integral and rational points on varieties”, *Astérisque* (1995), no. 228, p. 4, 183–187, Columbia University Number Theory Seminar (New York, 1992).

[30] B. Poonen – “Bertini theorems over finite fields”, *Annals of Mathematics. Second Series* 160 (2004), no. 3, p. 1099–1127.

[31] H. Randriambololona – “Métriques de sous-quotient et théorème de Hilbert-Samuel arithmétique pour les faisceaux cohérents”, *Journal für die Reine und Angewandte Mathematik* 590 (2006), p. 67–88.

[32] P. Salberger – “On the density of rational and integral points on algebraic varieties”, *Journal für die Reine und Angewandte Mathematik* 606 (2007), p. 123–147.

[33] , “Counting rational points on projective varieties”, (2013), preprint.

[34] , “Uniform bounds for rational points on cubic hypersurfaces”, in *Arithmetic and geometry*, London Math. Soc. Lecture Note Ser., vol. 420, Cambridge Univ. Press, Cambridge, 2015, p. 401–421.

[35] J.-P. Serre – “Quelques applications du théorème de densité de Chebotarev”, *Institut des Hautes Etudes Scientifiques. Publications Mathématiques* (1981), no. 54, p. 323–401.

[36] , *Local algebra*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2000, Translated from the French by CheeWhye Chin and revised by the author.

[37] A. C. Thompson – *Minkowski geometry*, Encyclopedia of Mathematics and its Applications, vol. 63, Cambridge University Press, Cambridge, 1996.

[38] B. Winckler – “Intersection arithmétique et problème de lehmer elliptique”, Ph.D. Thesis, Université de Bordeaux, Talence, Novembre 2015.

[39] X. Yuan – “Algebraic dynamics, canonical heights and Arakelov geometry”, in *Fifth International Congress of Chinese Mathematicians. Part 1, 2*, AMS/IP Stud. Adv. Math., 51, pt. 1, vol. 2, Amer. Math. Soc., Providence, RI, 2012, p. 893–929.

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