On extension of Ginzburg-Jiang-Soudry correspondence to certain automorphic forms on $Sp_{4mn}(\mathbb{A})$ and $\tilde{Sp}_{4mn\pm 2n}(\mathbb{A})$

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Abstract. Let $F$ be a number field, and $\mathbb{A} = \mathbb{A}_F$. In this paper, first, we provide a family of global Arthur parameters confirming all parts of a general conjecture on the relation between the structure of Fourier coefficients and the structure of global Arthur parameters, given by Jiang in 2012. Then we extend a correspondence between certain automorphic forms on $Sp_{4n}(\mathbb{A})$ and $\tilde{Sp}_{2n}(\mathbb{A})$, given by Ginzburg, Jiang and Soudry in 2012, to certain automorphic forms on $Sp_{4mn}(\mathbb{A})$ and $\tilde{Sp}_{4mn\pm 2n}(\mathbb{A})$, using the same idea of considering compositions of automorphic descent maps.

1. Introduction

Let $F$ be a number field, and $\mathbb{A} = \mathbb{A}_F$. Fourier coefficients play important roles in the study automorphic forms. For example, a basic and fundamental result in the theory of automorphic forms for $GL_n(\mathbb{A})$ is that cuspidal automorphic forms are globally generic, that is, have non-vanishing Whittaker-Fourier coefficients, due to Shalika ([S74]) and Piatetski-Shapiro ([PS79]) independently. This result has been extended to the discrete spectrum of $GL_n(\mathbb{A})$ in [JL13a]. As another example, for classical groups, Ginzburg, Rallis and Soudry ([GRS11]) developed the theory of automorphic descent by studying certain Fourier coefficients of special type residual representations, producing the inverse of special cases of Langlands functorial transfers from classical groups to the general linear groups. The idea of automorphic descent has been generalized to explicit constructions of endoscopy transfers for classical groups which can be found in [G12] and [J12].

For general connected reductive groups, there is a framework of attaching Fourier coefficients to nilpotent orbits (for symplectic groups, see [GRS03], [JL13b]). For classical groups $G$, nilpotent orbits are parametrized by partitions and certain non-degenerate quadratic forms.
For any irreducible automorphic representation $\pi$ of $G(A)$, let $p^m(\pi)$ be the set of maximal partitions (under the natural ordering of partitions) providing non-vanishing Fourier coefficients for $\pi$ (for precise definition, see [J12] and [JL13b]).

On the other hand, given a general connected reductive group $G$, a main theme in the theory of automorphic forms is to study the discrete spectrum, which consists of cuspidal spectrum and residual spectrum. For $G = GL_n$, the residual spectrum was constructed explicitly by Moeglin and Waldspurger ([MW89]). For symplectic and special orthogonal groups, the discrete spectrum was classified by Arthur ([Ar13]) up to automorphic $L^2$-packets parametrized by global Arthur parameters.

Towards understanding the Fourier coefficients information of members in the automorphic $L^2$-packets, Jiang ([J12]) made a conjecture, which relates the structure of the global Arthur parameter of an irreducible automorphic representation $\pi$ which is in the discrete spectrum, to the structure of $p^m(\pi)$. We recall the symplectic case of this conjecture as follows. Let $G_n = Sp_{2n}$, with symplectic form

$\begin{pmatrix} 0 & v_n \\ -v_n & 0 \end{pmatrix},$

where $v_n$ is an $n \times n$ matrix with 1’s on the second diagonal and 0’s elsewhere. Fix a Borel subgroup $B = TU$, where the maximal torus $T$ consists of elements of the following form

$\text{diag}(t_1, \ldots, t_n; t_n^{-1}, \ldots, t_1^{-1})$

and the unipotent radical $U$ consists of all upper unipotent matrices in $G_n$.

The set of global Arthur parameters for the discrete spectrum of $G_n = Sp_{2n}$ is denoted by $\tilde{\Psi}_2(Sp_{2n})$, the elements of which are of the form

$\psi := \psi_1 \boxplus \psi_2 \boxplus \cdots \boxplus \psi_r,$

where $\psi_i$ are pairwise different simple global Arthur parameters of orthogonal type and have the form $\psi_i = (\tau_i, b_i)$ with $\tau_i \in A_{\text{cusp}}(a_i)$, $2n+1 = \sum_{i=1}^r a_i b_i$ (the dual group of $Sp_{2n}$ is $SO_{2n+1}(\mathbb{C})$), and $\prod_i \omega_{\tau_i}^{a_i} = 1$ (the condition on the central characters of the parameter), following [Ar13, Section 1.4]. More precisely, $\psi_i = (\tau_i, b_i)$ satisfies the following conditions: if $\tau_i$ is of symplectic type (i.e., $L(s, \tau_i, \Lambda^2)$ has a pole at $s = 1$), then $b_i$ is even; if $\tau_i$ is of orthogonal type (i.e., $L(s, \tau_i, \text{Sym}^2)$ has a pole at $s = 1$), then $b_i$ is odd.
Theorem 1.1 (Theorem 1.5.2, [Ar13]). For each global Arthur parameter $\psi \in \tilde{\Psi}_2(\text{Sp}_{2n})$ there defines a global Arthur packet $\tilde{\Pi}_\psi$. The discrete spectrum of $\text{Sp}_{2n}(\mathbb{A})$ has the following decomposition

$$L^2_{\text{disc}}(\text{Sp}_{2n}(F) \backslash \text{Sp}_{2n}(\mathbb{A})) \cong \bigoplus_{\psi \in \tilde{\Psi}_2(\text{Sp}_{2n})} \bigoplus_{\pi \in \tilde{\Pi}_\psi(\epsilon_\psi)} m_\psi \pi,$$

where $\tilde{\Pi}_\psi(\epsilon_\psi)$ denotes the subset of $\tilde{\Pi}_\psi$ consisting of members which occur in the discrete spectrum.

As in [J12], $\tilde{\Pi}_\psi(\epsilon_\psi)$ is called the automorphic $L^2$-packet attached to $\psi$. For $\psi$ of the form in (1.1), let $p(\psi) = [(b_1)^{(a_1)} \cdots (b_r)^{(a_r)}]$. For $\pi \in \tilde{\Pi}_\psi(\epsilon_\psi)$, the structure of the global Arthur parameter $\psi$ deduces constraints on the structure of $p^m(\pi)$, which is given by the following conjecture.

Conjecture 1.2 (Conjecture 4.2, [J12]). For any $\psi \in \tilde{\Psi}_2(\text{Sp}_{2n})$, let $\tilde{\Pi}_\psi(\epsilon_\psi)$ be the automorphic $L^2$-packet attached to $\psi$. Then the following hold.

1. Any symplectic partition $p$ of $2n$, if $p > \eta_{\psi', \psi}(p(\psi))$, does not belong to $p^m(\pi)$ for any $\pi \in \tilde{\Pi}_\psi(\epsilon_\psi)$.
2. For a $\pi \in \tilde{\Pi}_\psi(\epsilon_\psi)$, any partition $p \in p^m(\pi)$ has the property that $p \leq \eta_{\psi', \psi}(p(\psi))$.
3. There exists at least one member $\pi \in \tilde{\Pi}_\psi(\epsilon_\psi)$ having the property that $\eta_{\psi', \psi}(p(\psi)) \in p^m(\pi)$.

Here $\eta_{\psi', \psi}$ denotes the Barbasch-Vogan duality map (see [BV85], Definition A1 and [Ac03], Section 3.5) from the partitions for the dual group $G'$ to the partitions for $G$.

We refer to [J12], Section 4] for more discussion on this conjecture and related topics. Part (1) of Conjecture 1.2 is completely proved in [JL13c]. In the first part of this paper, we provide a family of global Arthur parameters confirming all parts of Conjecture 1.2.

Let $\tau$ be an irreducible unitary cuspidal automorphic representation of $GL_{2n}(\mathbb{A})$, with the properties that $L(s, \tau, \wedge^2)$ has a simple pole at $s = 1$, and $L(\frac{1}{2}, \tau) \neq 0$.

By Theorem 2.3 of [GJS12], there is an irreducible representation $\tilde{\tau}$ of $\tilde{\text{Sp}}_{2n}(\mathbb{A})$, which is $\psi^1$-generic, lifts weakly to $\tau$ with respect to $\psi$.

Let $\Delta(\tau, m)$ be a Speh representation in the discrete spectrum of $GL_{2mn}(\mathbb{A})$. For more information about Speh representations, we refer to [MW89], or the Section 1.2 of [JLZ13].

Let $P_r = M_r N_r$ be the maximal parabolic subgroup of $Sp_{2l}$ with Levi subgroup $M_r$ isomorphic to $GL_r \times Sp_{2l-2r}$. Using the normalization in
the group $X_{M_r}$ of all continues homomorphisms from $M_r(\mathbb{A})$ to $\mathbb{C}^*$, which is trivial on $M_r(\mathbb{A})$ (see [MW95]), will be identified with $\mathbb{C}$ by $s \mapsto \lambda_s$. Let $\widetilde{P_r}(\mathbb{A})$ be the pre-image of $P_r(\mathbb{A})$ in $\widetilde{Sp}_2(\mathbb{A})$.

For any $\phi \in A(\mathbb{N}_{2mn}(\mathbb{A})M_{2mn}(F)\backslash Sp_{4mn}(\mathbb{A}))_{\Delta(\tau,m)}$, following [L76] and [MW95], an residual Eisenstein series can be defined by

$$E(\phi,s)(g) = \sum_{\gamma \in P_{2mn}(F)\backslash Sp_{4mn}(F)} \lambda_s \phi(\gamma g).$$

It converges absolutely for real part of $s$ large and has meromorphic continuation to the whole complex plane $\mathbb{C}$. By [JLZ13], this Eisenstein series has a simple pole at $m$, which is the right-most one. Denote the representation generated by these residues at $s=m$ by $E_{\Delta(\tau,m)}$. This residual representation is square-integrable. By Section 6.2 of [JLZ13], the global Arthur parameter of $E_{\Delta(\tau,m)}$ is $\psi = (\tau,2m) \boxplus (1_{GL_1},1)$.

Our first main result can be stated as follows.

**Theorem 1.3.** Assume that $F$ is any number field.

$$p^m(E_{\Delta(\tau,m)}) = [(2n)^{2m}].$$

This theorem was discussed by Ginzburg in [G08] with a quite sketchy argument. A fully detailed proof will be given in Section 2.

By Theorem 1.3, Proposition 6.4 and Remark 6.5 of [JL13c], Parts (1) and (2) of Conjecture 1.2 hold for these global Arthur parameters $\psi = (\tau,2m) \boxplus (1_{GL_1},1)$. Note that in this case

$$\eta_{so_{2n+1}(\mathbb{C}),sp_{2n}(\mathbb{C})}(p(\psi)) = \eta_{so_{2n+1}(\mathbb{C}),sp_{2n}(\mathbb{C})}([(2m)^{2n}(1)]) = [(2n)^{2m}].$$

Combining Theorem 1.3, we can see that all parts of Conjecture 1.2 are confirmed for this family of global Arthur parameters.

In [GRS03], for any irreducible cuspidal automorphic representation $\pi$ of symplectic groups or their double covers, Ginzburg, Rallis and Soudry found a maximal partition which has only even parts, providing non-vanishing Fourier coefficients for $\pi$. We denote this partition by $P(\pi)$.

Next, we assume that $F$ is not totally imaginary, and consider $\mathcal{N}_{Sp_{4mn}}$, the set of irreducible cuspidal automorphic representations $\pi$ which are nearly equivalent to $E_{\Delta(\tau,m)}$ and

$$p(\pi) = [(2n)^{2m-1}(2n_1)^{s_1}(2n_2)^{s_2} \cdots (2n_k)^{s_k}],$$

with $2n \geq 2n_1 > 2n_2 > \cdots > 2n_k$, $k \geq 1$. Note that these partitions are less than or equal to $[(2n)^{2m}]$.

$\mathcal{N}_{Sp_{4mn}}$ can be naturally decomposed into a disjoint union of two sets $\mathcal{N}_{Sp_{4mn}} \cup \mathcal{N}_1$, where $\mathcal{N}_{Sp_{4mn}}$ consists of elements having a nonzero
Fourier coefficient $FJ_{\psi_1^{-1}}$ (for definition, see [GRST11, Section 3.2]), while $\mathcal{N}^\prime_{\text{Sp}_{4mn}}$ consists of elements having no nonzero Fourier coefficients $FJ_{\psi_1^{-1}}$.

For any $\tilde{\phi} \in A(\mathcal{N}_{2kn}(\mathbb{A})\widetilde{M}_{2kn}(F)\setminus \widetilde{S}_{\text{Sp}_{4kn+2n}}(\mathbb{A}))_{\mu_\psi \Delta(\tau,k)\otimes \bar{\pi}}$, following [L76] and [MW95], an residual Eisenstein series can be defined by

$$\tilde{E}(\tilde{\phi},s)(g) = \sum_{\gamma \in P_{2kn}(F)\backslash S_{\text{Sp}_{4kn+2n}}(F)} \lambda_s \tilde{\phi}(\gamma g).$$

It converges absolutely for real part of $s$ large and has meromorphic continuation to the whole complex plane $\mathbb{C}$. By similar argument as that in [JLZ13], this Eisenstein series has a simple pole at $\frac{k+1}{2}$, which is the right-most one. Denote the representation generated by these residues at $s = \frac{k+1}{2}$ by $\tilde{E}_{\Delta(\tau,k)\otimes \bar{\pi}}$. This residual representation is square-integrable.

Let $\mathcal{N}_{\text{Sp}_{4(m-1)n+2n}}^\prime(\tau,\psi)$ be the set of irreducible genuine cuspidal automorphic representations $\tilde{\sigma}_{4(m-1)n+2n}$ of $\widetilde{S}_{\text{Sp}_{4(m-1)n+2n}}(\mathbb{A})$, which are nearly equivalent to the residual representation $\tilde{E}_{\Delta(\tau,m-1)\otimes \bar{\pi}}$, have no nonzero Fourier coefficients $FJ_{\psi_1^{-1}}$, and

$$p(\tilde{\sigma}_{4(m-1)n+2n}) = [(2n)^{2(m-1)}(2n_1)^{s_1}(2n_2)^{s_2} \cdots (2n_k)^{s_k}],$$

with $2n \geq 2n_1 > 2n_2 > \cdots > 2n_k$, $k \geq 1$.

Let $\mathcal{N}_{\text{Sp}_{4mn+2n}}^\prime(\tau,\psi)$ be the set of irreducible genuine cuspidal automorphic representations $\tilde{\sigma}_{4mn+2n}$ of $\widetilde{S}_{\text{Sp}_{4mn+2n}}(\mathbb{A})$, which are nearly equivalent to the residual representation $\tilde{E}_{\Delta(\tau,m)\otimes \bar{\pi}}$, have a nonzero Fourier coefficient $FJ_{\psi_1^{-1}}$, and

$$p(\tilde{\sigma}_{4mn+2n}) = [(2n)^{2m}(2n_1)^{s_1}(2n_2)^{s_2} \cdots (2n_k)^{s_k}],$$

with $2n \geq 2n_1 > 2n_2 > \cdots > 2n_k$, $k \geq 1$.

For any $\tilde{\phi} \in A(\mathcal{N}_{2mn}(\mathbb{A})\widetilde{M}_{2mn}(F)\setminus \widetilde{S}_{\text{Sp}_{4mn+2n}}(\mathbb{A}))_{\mu_\psi \tau \otimes \tilde{\sigma}_{4(m-1)n+2n}}$, by similar calculation as in Pages 996-997 of [GJS12], it is easy to see that the corresponding Eisenstein series has a simple pole at $s = m$. Let $\tilde{E}_{\tau,\tilde{\sigma}_{4(m-1)n+2n}}$ be the residual representation of $\widetilde{S}_{\text{Sp}_{4mn+2n}}(\mathbb{A})$ generated by the corresponding residues. This residual representation is square-integrable.

For any $\sigma_{4mn} \in \mathcal{N}_{\text{Sp}_{4mn}}^\prime(\tau,\psi)$, for any

$$\phi \in A(\mathcal{N}_{2mn}(\mathbb{A})\widetilde{M}_{2mn}(F)\setminus \text{Sp}_{4mn+2n}(\mathbb{A}))_{\tau \otimes \sigma_{4mn}},$$

by similar calculation as in Pages 996-997 of [GJS12], it is easy to see that the corresponding Eisenstein series has a simple pole at $s = m$.
also by similar calculation as in Pages 996-997 of [GJS12], it is easy to see that the corresponding Eisenstein series has a simple pole at $s = \frac{2m+1}{2}$. Let $E_{\tau,\sigma 4mn}$ be the residual representation of $Sp_{4(m+1)n}(\mathbb{A})$ generated by the corresponding residues. This residual representation is square-integrable.

For any $\sigma_{4mn} \in N_{Sp_{4mn}}(\tau, \psi)$, let $D_{2n, \psi^{-1}}^{4mn}(\sigma_{4mn})$ be the $\psi^{-1}$-descent of $\sigma_{4mn}$ from $Sp_{4mn}(\mathbb{A})$ to $Sp_{4(m-1)n+2n}(\mathbb{A})$ (defined in Chapter 3 of [GRS11]). Note that by the tower property (see Theorem 7.10 of [GRS11]), $D_{2n, \psi^{-1}}^{4mn}(\sigma_{4mn})$ is cuspidal.

For any $\tilde{\sigma}_{4mn+2n} \in N_{\tilde{Sp}_{4mn+2n}}(\tau, \psi)$, let $D_{2n, \psi^{1}}^{4mn+2n}(\tilde{\sigma}_{4mn+2n})$ be the $\psi^{1}$-descent of $\tilde{\sigma}_{4mn+2n}$ from $\tilde{Sp}_{4mn+2n}(\mathbb{A})$ to $Sp_{4mn}(\mathbb{A})$ (defined in Chapter 3 of [GRS11]). Note that by the tower property (see Theorem 7.10 of [GRS11]), $D_{2n, \psi^{1}}^{4mn+2n}(\tilde{\sigma}_{4mn+2n})$ is also cuspidal.

Our second main result is that there are correspondences between $N_{Sp_{4mn}}(\tau, \psi)$ and $N'_{\tilde{Sp}_{4(m-1)n+2n}}(\tau, \psi)$, and between $N'_{\tilde{Sp}_{4mn+2n}}(\tau, \psi)$ and $N'_{Sp_{4n}}(\tau, \psi)$, as follows.

**Theorem 1.4.** Assume that $F$ is a number field which is not totally imaginary.

1. There is a surjective map
   \[
   \Psi : N'_{\tilde{Sp}_{4mn}}(\tau, \psi) \to N'_{Sp_{4(m-1)n+2n}}(\tau, \psi)
   \]
   \[
   \sigma_{4mn} \mapsto D_{2n, \psi^{-1}}^{4mn}(\sigma_{4mn}).
   \]

2. If for any $\tilde{\sigma}_{4(m-1)n+2n} \in N'_{\tilde{Sp}_{4(m-1)n+2n}}(\tau, \psi)$, $\tilde{E}_{\tau,\tilde{\sigma}_{4(m-1)n+2n}}$ is irreducible, then $\Psi$ is also injective.

Note that the case of $m = 1$ has already been proved by Ginzburg, Jiang and Soudry in [GJS12]. We use the same idea here to extend the correspondence to higher ranks. Among others, one key idea is to consider compositions of automorphic descent maps. Also note that they include $E_{\Delta(\tau,1)}$ in the domain of the map $\Psi$. For simplicity, we just let the domain of the map $\Psi$ consist of only irreducible cuspidal automorphic representations, and consider the descent of $E_{\Delta(\tau,m)}$ separately in Section 7.

**Theorem 1.5.** Assume that $F$ is a number field which is not totally imaginary.

1. There is a surjective map
   \[
   \Psi : N_{\tilde{Sp}_{4mn+2n}}(\tau, \psi) \to N'_{Sp_{4n}}(\tau, \psi)
   \]
   \[
   \tilde{\sigma}_{4mn+2n} \mapsto D_{2n, \psi^{1}}^{4mn+2n}(\tilde{\sigma}_{4mn+2n}).
   \]
(2) If for any $\sigma_{4mn} \in \mathcal{N}'_{Sp_{4mn}}(\tau, \psi)$, $E_{\tau, \sigma_{4mn}}$ is irreducible, then $\Psi$ is also injective.

Due to the similarity of the proofs of Theorem 1.4 and Theorem 1.5, we only give the proof for Theorem 1.4.

Theorem 1.4 and Theorem 1.5 together give us the following diagram about correspondences between various sets of irreducible cuspidal automorphic representations:

\[
\begin{array}{cccc}
\tilde{S}_{p_{4mn}}(\mathbb{A}) & \tilde{N}_{4mn} \cup \tilde{N}'_{4mn} & \downarrow D^{4mn+2n} \\
S_{p_{4mn}}(\mathbb{A}) & N_{4mn} \cup N'_{4mn} & \downarrow D^{4mn} \\
\tilde{S}_{p_{4mn-2n}}(\mathbb{A}) & \tilde{N}_{4mn-2n} \cup \tilde{N}'_{4mn-2n} & \downarrow D^{4mn-2n} \\
\vdots & \vdots & \vdots \\
\end{array}
\]

In the above diagram, for short, we write that $N_{4mn} := N_{Sp_{4mn}}$, $N'_{4mn} := N'_{Sp_{4mn}}$, $\tilde{N}_{4mn+2n} := \tilde{N}_{Sp_{4mn+2n}}$, $\tilde{N}'_{4mn+2n} := \tilde{N}'_{Sp_{4mn+2n}}$.

**Remark 1.6.** In Theorem 1.4 and Theorem 1.5, we assume that $F$ is a number field which is not totally imaginary, the reason is that when $F$ is a totally imaginary number field, then our construction will "stop at some point, and can not go to higher levels". The explicit explanation of this phenomenon will appear elsewhere.

From Theorem 1.3, for the residual representation $E_{\Delta(\tau,m)}$, we know that $p^m(E_{\Delta(\tau,m)}) = [(2n)^{2m}]$. From its proof, and by Lemma 2.6 of [GRS03] or Lemma 3.1 of [JL13b], we can see that it has a non-vanishing Fourier coefficient attached to the partition $[(2n)^{14mn-2n}]$ with respect to the character $\psi_{[(2n)^{14mn-2n}], -1}$. In Section 7, for any number field $F$, we show that both $E_{\Delta(\tau,m)}$ and $D^{4mn}_{2n,\psi^{-1}}(E_{\Delta(\tau,m)})$ are irreducible. The result can be stated as follows.

**Theorem 1.7.** Assume that $F$ is any number field.

1. $D^{4mn}_{2n,\psi^{-1}}(E_{\Delta(\tau,m)})$ is square-integrable and is in the discrete spectrum.
Both $\mathcal{E}_{\Delta(\tau,m)}$ and $\mathcal{D}_{2n,\psi,-1}^{4n}(\mathcal{E}_{\Delta(\tau,m)})$ are irreducible.

Note that in general, it is difficult to prove the irreducibility of certain descent representations. The case of $m = 1$ of Theorem 1.7 was proved in Theorem 4.1 of [GJS12], noting that by Theorem 2.5 of [GJS12], $\mathcal{E}_{\Delta(\tau,1)}$ is irreducible. Also note that, the irreducibility of $\mathcal{D}_{2n,\psi,-1}^{4n}(\mathcal{E}_{\Delta(\tau,1)})$ actually has already been proved by Jiang and Soudry in [JS03], using different methods.

At the end of this introduction, we discuss the contents by section. In Section 2, we will show Theorem 1.3, whose proof is reduced to that of Lemma 2.3, which will be given in Section 3. The Section 4, we will prove Part (1) of Theorem 1.4, whose proof is reduced to that of Theorem 4.6, which will be given in Section 5. In Section 6, we completes the proof of Theorem 1.4, by proving its Part (2). In Section 7, we prove Theorem 1.7. We assume that $F$ is not totally imaginary only in Sections 4–6.

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2. Proof of Theorem 1.3

In this section, we prove Theorem 1.3, which was discussed by Ginzburg in [G08] with a quite sketchy argument. To be complete, we give the full details here. $F$ is any number field in this section, Sections 3 and 7.

**Theorem 2.1** (Ginzburg, Theorem 1 [G08]).

$$p^m(\mathcal{E}_{\Delta(\tau,m)}) = [(2n)^{2m}].$$

**Proof.** By Theorem 1.3, Proposition 6.4 and Remark 6.5 of [JL13c], we only have to show that $\mathcal{E}_{\Delta(\tau,m)}$ has a nonzero Fourier coefficient attached to $[(2n)^{2m}]$.

We will prove this by induction on $m$. For $m = 1$, this is proved in the book [GRS11]. Note that when $m = 1$, $\mathcal{E}_{\Delta(\tau,1)}$ has a nonzero Fourier coefficient attached to the partition $[(2n)^{12n}]$, and the descent to $\widetilde{Sp}_{2n}$ is generic (see Theorem 3.1 of [GRS11]). Therefore, $\mathcal{E}_{\Delta(\tau,1)}$ has a nonzero
Fourier coefficient attached to the composite partition \( [(2n)1^{2n}] \circ [(2n)] \), which implies that \( \mathcal{E}_{\Delta(\tau,1)} \) has a nonzero Fourier coefficient attached to the partition \( [(2n)3^n] \) by Lemma 2.6 of [GRS03], or Lemma 3.1 of [JL13b]. For definition of composite partitions, we refer to Section 1 of [GRS03].

Now we assume that the theorem is true for the case of \( m - 1 \), and consider the case of \( m \geq 2 \).

Take any \( \varphi \in \mathcal{E}_{\Delta(\tau,m)} \), its Fourier coefficients attached to \( \mathfrak{p} = [(2n)^{2m}] \) are of the following forms

\[
\varphi_{\psi_{\mathfrak{p}}}^{\mathfrak{p}}(g) = \int_{[V_{\mathfrak{p},2}]} \varphi(vg)^{\psi_{\mathfrak{p}}^{-1}}(v)dv,
\]

where \( \mathfrak{p} = \{a_1, a_2, \ldots, a_{2m}\} \subset (F^*/(F^*)^2)^{2m} \). For definitions of the unipotent group \( V_{\mathfrak{p},2} \) and its character \( \psi_{\mathfrak{p}}^{\mathfrak{p}} \), see Section 2 of [JL13b].

Assume that \( T \) is the maximal split torus in \( S_{\mathfrak{p}_{4mn}} \), consists of elements

\[
\text{diag}(t_1, t_2, \ldots, t_{2mn}, t_{2mn}^{-1}, \ldots, t_2^{-1}, t_1^{-1}).
\]

Let \( \omega_1 \) be the Weyl element of \( S_{\mathfrak{p}_{4mn}} \), sending elements \( t \in T \) to the following torus elements:

\[
t'(t) = \text{diag}(t^{(0)}, t^{(1)}, t^{(2)}, \ldots, t^{(n)}, t^{*}, \ldots, t^{(2),*}, t^{(1),*}, t^{(0),*}),
\]

where

\[
t^{(0)} = \text{diag}(t_1, t_{n+1}, t_2, t_{n+2}, \ldots, t_i, t_{n+i}, \ldots, t_n, t_{2n})
\]

\[
t^{(j)} = \text{diag}(t_{2n+j}, t_{3n+j}, \ldots, t_{2n+j}, \ldots, t_{(2m-1)n+j}),
\]

for \( 1 \leq j \leq n \).

Identify \( S_{\mathfrak{p}_{4(m-1)n}} \) with its image in \( S_{\mathfrak{p}_{4mn}} \) under the embedding \( g \mapsto \text{diag}(I_{2n}, g, I_{2n}) \). Denote the restriction of \( \omega_1 \) to \( S_{\mathfrak{p}_{4(m-1)n}} \) by \( \omega_1' \).

Conjugating cross by \( \omega_1 \), the Fourier coefficient \( \varphi_{\psi_{\mathfrak{p}}}^{\mathfrak{p}} \) becomes:

\[
\int_{[V_{\mathfrak{p},2}]} \varphi(u\omega_1 g)^{\psi_{\mathfrak{p}}^{\mathfrak{p}}}(u)^{-1}du,
\]

where \( U_{p,2} = \omega_1 V_{p,2} \omega_1^{-1} \), and \( \psi_{\mathfrak{p}}^{\mathfrak{p}}(u) = \psi_{\mathfrak{p}}^{\mathfrak{p}}(\omega_1^{-1}u \omega_1) \).

Now, we describe the structure of elements in \( U_{p,2} \). Any element in \( U_{p,2} \) has the following form:

\[
u = \begin{pmatrix} z_{2n} & q_1 & q_2 \\ 0 & u' & q_1' \\ 0 & 0 & z_{2n}' \end{pmatrix} \begin{pmatrix} I_{2n} & 0 & 0 \\ p_1 & I_{(4m-4)n} & 0 \\ 0 & p_1^* & I_{2n} \end{pmatrix},
\]

where \( z_{2n} \in V_{2n} \), the unipotent radical of the parabolic \( Q_{2n} \) of \( GL_{2n} \) with Levi isomorphic to \( GL_{2n}^{\times n} \); \( u' \in U_{[(2m)^{2m-2}],2} = \omega_1' V_{[(2m)^{2m-2}],2} \omega_1'^{-1} \);
$q_1 \in M_{2n \times (4m-4)n}$, $p_1 \in M_{(4m-4)n \times 2n}$, satisfy certain conditions, which we do not specify at this moment; $q_2 \in M_{(2n) \times (2n)}$, such that $q_1^tv_2n - v_2nq_2 = 0$, where $v_2n$ is a matrix only with ones on the second diagonal. Note that

$$\psi_{p,a}(u) = \psi(uv_1)\psi_{p,a}(u)$$

where $a_1, a_2$ come from the $a = \{a_1, a_2, \ldots, a_{2m}\}$ occurred in the Fourier coefficient $\varphi_{\omega_1}^a$.

Since to show that $\mathcal{E}_{\Delta(\tau, m)}$ has a nonzero Fourier coefficient attached to the partition $p = [(2n)^2m]$, we only need to show that it has a nonzero Fourier coefficient $\varphi_{\omega_1}^a$ for some $a$, we consider the following special type of $a$:

$$a = \{1, -1, a_3, \ldots, a_{2m}\},$$

where $a_3, \ldots, a_{2m}$ are arbitrary elements in $F^*/(F^*)^2$.

Let $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$, and $\epsilon = \text{diag}(A, \ldots, A; I_{(4m-4)n}; A^*, \ldots, A^*)$, as in (2.31) of [GJS12]. Conjugating cross the integral in (2.3) by $\epsilon$, it becomes:

$$\int_{[U_{\omega, a}]} \varphi(u\epsilon_1g)\psi_{p,a}^{\omega_1}(u)^{-1}du,$$

where $U_{\omega, 2} = \epsilon U_{\omega, 2}\epsilon^{-1}$ whose elements have the same structure as $U_{\omega, 2}$ (see (2.4)), and $\psi_{p,a}^{\omega_1}(u) = \psi_{p,a}(\epsilon^{-1}u\epsilon)$.

Note that now

$$\psi_{p,a}^{\omega_1}(\epsilon) = \psi(a_1q_2(2n-1, 2) + a_2q_2(2n, 1)),$$

where $\epsilon$ in (2.18) and (2.35) of [GJS12] is $-1$ here.
Let $\nu$ be the following Weyl element of $\text{Sp}_{4n}$ which is defined on Page 14 of [GJS12], also in (4.9) of [GRS99]:

\[
\begin{align*}
\nu_{i,2i-1} &= 1, i = 1, \ldots, 2n, \\
\nu_{2n+i,2i} &= -1, i = 1, \ldots, n, \\
\nu_{2n+i,2i-1} &= 1, i = n + 1, \ldots, 2n, \\
\nu_{i,j} &= 0, \text{ otherwise.}
\end{align*}
\]

(2.6)

Write $\nu$ as $\left(\begin{smallmatrix} \nu_1 & \nu_2 \\ \nu_3 & \nu_4 \end{smallmatrix}\right)$, where $\nu_i$'s are of size $2n \times 2n$.

Let $\omega_2 = \left(\begin{smallmatrix} \nu_1 & \nu_2 \\ \nu_3 & \nu_4 \end{smallmatrix}\right) I_{(4m-4)n}$, a Weyl element of $\text{Sp}_{4mn}$. Conjugating cross the integral in (2.5) by $\omega_2$, it becomes:

\[
\int_{U_{\mathbb{F},2}^{r,\omega_2}} \varphi(u \omega_2 \epsilon \omega_1 g) \psi_{\mathbb{F},\omega_2}^{r,\epsilon,\omega_2}(u)^{-1} du,
\]

where $U_{\mathbb{F},2}^{r,\omega_2} = \omega_2 U_{\mathbb{F},2}^{r,\omega_2} \omega_2^{-1}$, and $\psi_{\mathbb{F},\omega_2}^{r,\epsilon,\omega_2}(u) = \psi_{\mathbb{F},\omega_2}^{r,\epsilon,\omega_2}(\omega_2^{-1} u \omega_2)$.

Now let’s describe the structure of elements in $U_{\mathbb{F},2}^{r,\omega_2}$. Any element in $U_{\mathbb{F},2}^{r,\omega_2}$ has the following form:

\[
u = \left(\begin{array}{cc}
\begin{array}{cc}
z_{2n} & q_1 \\
0 & u'
\end{array}

& \begin{array}{cc}
0 & q_2 \\
p_1 & I_{(4m-4)n}
\end{array}

\
\begin{array}{cc}
0 & z_{2n}^*
\end{array}

& \begin{array}{cc}
p_2 & I_{2n}
\end{array}

\end{array}\right)
\]

(2.8)

where $z_{2n} \in V_{2n}$, the standard maximal unipotent subgroup of $GL_{2n}$ as before; $u' \in U_{\mathbb{F},2}^{r,\omega_2}$, $q_1 \in M_{2n \times (4m-4)n}$, with $q_1(i,j) = 0$, for $j \pmod{2m-2}$; $q_2(p_1, p_2) \in M_{(2n) \times (2n)}$, with $q_1(i,j) = p_2(i,j) = 0$, for $i \leq j$; $p_1 \in M_{(4m-4) \times 2n}$, with $p_1(i,j) = 0$, for $i \geq 2m-2$.

Note that now

\[
\psi_{\mathbb{F},\omega_2}^{r,\epsilon,\omega_2}(\left(\begin{array}{cc}
z_{2n} & q_1 \\
0 & I_{(4m-4)n}
\end{array}\right) \left(\begin{array}{cc}
q_2 & q_1^* \\
p_1 & z_{2n}^*
\end{array}\right))
\]

(2.9)

\[
= \psi(z_{2n}(1,2) + \cdots + z_{2n}(n,n+1)) - z_{2n}(n+1,n+2) - \cdots - z_{2n}(2n-1,2n)).
\]

To proceed, we need to define some unipotent subgroups. Let $R_i^1 = \prod_{j=1}^1 R_{i,j}$, for $1 \leq i \leq n$, where $R_{i,j}^1 = \prod_{s=1}^{2m-2} X_{\alpha_{j,s}}$, with $\alpha_{j,s} = e_i - e_{2n+(2m-2)(i-j+1)-s+1}$. Let $R_i^1 = \prod_{j=1}^1 R_{i,j}^1$, for $n+1 \leq i \leq 2n-1$, where
\( R_{i,j}^1 = \prod_{s=1}^{2m-2} X_{\alpha_{j,s}^i} \), with \( \alpha_{j,s}^i = e_i - e_{2n+(2m-2)(i-j+1)-s+1} \) if \( j \geq i-n+1 \), and \( \alpha_{j,s}^i = e_i + e_{2mn-(2m-2)(i-n)+(2m-1)(j-1)+s} \) if \( j \leq i-n \).

Let \( C_{i}^1 = \prod_{j=1}^{i} C_{i,j}^1 \), for \( 1 \leq i \leq n \), where \( C_{i,j}^1 = \prod_{s=1}^{2m-2} X_{\beta_{j,s}^i} \), with \( \beta_{j,s}^i = e_{2n+(2m-2)(i-j+1)-s+1} - e_{i+1} \). Let \( C_{i}^1 = \prod_{j=1}^{i} C_{i,j}^1 \), for \( n+1 \leq i \leq 2n-1 \), where \( C_{i,j}^1 = \prod_{s=1}^{2m-2} X_{\beta_{j,s}^i} \), with \( \beta_{j,s}^i = e_{2n+(2m-2)(i-j+1)-s+1} - e_{i+1} \) if \( j \geq i-n+1 \), and \( \beta_{j,s}^i = -e_{2mn-(2m-2)(i-n)+(2m-1)(j-1)+s} - e_{i+1} \) if \( j \leq i-n \).

Let \( R_i^2 = \prod_{j=1}^{i} X_{\alpha_{j}^i} \), for \( 1 \leq i \leq n \), with \( \alpha_{j}^i = e_i + e_{2n-i+j} \), and \( R_i^2 = \prod_{j=1}^{2n-i} X_{\alpha_{j}^i} \), for \( n+1 \leq i \leq 2n-1 \), with \( \alpha_{j}^i = e_i + e_{i+j} \). Let \( C_i^2 = \prod_{j=1}^{i} X_{\beta_{j}^i} \), for \( 1 \leq i \leq n \), with \( \beta_{j}^i = -e_{2n-i+j} - e_{i+1} \), and \( C_i^2 = \prod_{j=1}^{2n-i} X_{\beta_{j}^i} \), for \( n+1 \leq i \leq 2n-1 \), with \( \beta_{j}^i = -e_{i+j} - e_{i+1} \).

Now, we are ready to apply Lemma 2.3 of [JL13b] repeatedly to the integration over \( \prod_{i=1}^{2n-1} C_i^2 C_i^1 \). We will consider the following pairs of groups:

\[
(R_2^2 R_1^1, C_2^2 C_1^1), \ldots, (R_n^2 R_{n-1}^1, C_n^2 C_{n-1}^1); \]
\[
(R_{n+1}^2, C_{n+1}^2); (R_{n+1}^1, C_{n+1}^1); \]
\[
\ldots; \]
\[
(R_{2n-1}^2, C_{2n-1}^2); (R_{2n-1}^1, C_{2n-1}^1). \]

Let \( \tilde{U}_{\frac{p_2}{2}}^{\epsilon, \omega_2} \) be the subgroup of \( U_{\frac{p_2}{2}}^{\epsilon, \omega_2} \) with \( p_1 \) and \( p_2 \)-parts zero. Then, \( U_{\frac{p_2}{2}}^{\epsilon, \omega_2} = \tilde{U}_{\frac{p_2}{2}}^{\epsilon, \omega_2} \prod_{i=1}^{2n-1} C_i^2 C_i^1 \). Let \( W = U_{\frac{p_2}{2}}^{\epsilon, \omega}, \tilde{W} = \tilde{U}_{\frac{p_2}{2}}^{\epsilon, \omega}, \) and \( \psi_W = \psi_{\tilde{W}}^{\frac{p_2}{2}, \frac{p_1}{2}} \).

First, we apply Lemma 2.3 of [JL13b] to \( (R_2^2 R_1^1, C_2^2 C_1^1) \). For this, we need to consider the quadruple \( (\tilde{W} \prod_{i=2}^{2n-1} C_i^2 C_i^1, \psi_W, X_{\alpha_{i}^1} R_{1,i}^1, X_{\beta_{i}^1} C_{1,i}^1) \).

It is easy to see that this quadruple satisfies all the conditions for this lemma. By this lemma, the integral in (2.7) is non-vanishing if and only if the following integral is non-vanishing:

\[
(2.10) \quad \int_{[R_2^2 R_1^1 \tilde{W} \prod_{i=2}^{2n-1} C_i^2 C_i^1]} \varphi(rwc \omega_2 \epsilon \omega_1 g) \psi_W(w)^{-1} d\epsilon dw dr.
\]

Note that \( R_i^2 = X_{\alpha_{i}^1} \), and \( R_i^1 = R_{1,i}^1 \).
Then we apply Lemma 2.3 of [JL13b] to \((R^2_2 R^1_2, C^2_2 C^1_2)\). For this, we need to consider the following sequence of quadruples:

\[
(R^2_1 R^1_1 \tilde{W} \prod_{i=3}^{2n-1} C^2_i C^1_i X_{\beta^2_2 R^1_2, \psi}, X_{\alpha^2_2 R^1_2, 1}, X_{\beta^1_2 C_2, 2}),
\]

\[
(R^2_1 R^1_1 X_{\alpha^2_1} R^1_2 \tilde{W} \prod_{i=3}^{2n-1} C^2_i C^1_i, \psi, X_{\alpha^2_2 R^1_2, 2}, X_{\beta^1_2 C_2, 2}).
\]

Applying this lemma twice, we get that the integral in \((2.10)\) is non-vanishing if and only if the following integral is non-vanishing:

\[
(2.11) \int_{[R^2_1 R^1_1 R^2_2 R^1_2 \tilde{W} \prod_{i=3}^{2n-1} C^2_i C^1_i]} \varphi(r w c \omega_2 \epsilon_1 g) \psi_W(w)^{-1} \, dc dw dr.
\]

Then we continue the above procedure, applying Lemma 2.3 of [JL13b] to pairs \((R^2_3 R^1_3, C^2_3 C^1_3), \ldots, (R^2_n R^1_n, C^2_n C^1_n)\). For the pair \((R^2_n R^1_n, C^2_n C^1_n)\), we need to consider the following sequence of quadruples:

\[
\left( \prod_{i=1}^{n-1} R^2_i R^1_i \tilde{W} \prod_{i=n+1}^{2n-1} C^2_i C^1_i \prod_{s=2}^{n} X_{\beta^2_s R^1_n, \psi}, X_{\alpha^2_n R^1_n, 1}, X_{\beta^1_s C_{n,1}}, \right),
\]

\[
\ldots,
\]

\[
\left( \prod_{s=1}^{n-1} X_{\alpha^2_s} R^1_n \tilde{W} \prod_{i=n+1}^{2n-1} C^2_i C^1_i, \psi, X_{\alpha^2_n} R^1_n, X_{\beta^1_s C_{n,n}}. \right)
\]

Applying Lemma 2.3 of [JL13b] repeatedly, we get that the integral in \((2.11)\) is non-vanishing if and only if the following integral is non-vanishing:

\[
(2.12) \int_{[\prod_{s=1}^{n} R^2_s R^1_s \tilde{W} \prod_{i=n+1}^{2n-1} C^2_i C^1_i]} \varphi(r w c \omega_2 \epsilon_1 g) \psi_W(w)^{-1} \, dc dw dr.
\]

Before applying Lemma 2.3 of [JL13b] to pairs \((R^2_n, C^2_s), (R^1_n, C^1_s)\), \(s = n+1, n+2, \ldots, 2n-1\), we need to take Fourier expansion along the one-dimensional root subgroup \(X_{e_s + e_s}\). And then we need to consider the pair \((R^2_s, C^2_s)\) first, then \((R^1_s, C^1_s)\).

For example, for \(s = n + 1\), we first take the Fourier expansion of the integral in \((2.12)\) along the one-dimensional root subgroup \(X_{e_s + e_s}\). Under the action of \(GL_1\), we get two kinds of Fourier coefficients corresponding to the two orbits of the dual of \([X_{e_s + e_s}]\): the trivial one and the non-trivial one. For the Fourier coefficients attached to the non-trivial orbit, we can see that there is an inner integral \(\varphi([2n + 2]_{14}, n_{-2n-2}, \{\omega\})\), which is identically zero by Proposition 6.4 and Remark 6.5 of [JL13c].
Therefore only the Fourier coefficient attached to the trivial orbit, which actually equals to the integral in (2.12), survives.

Then, we can apply Lemma 2.3 of [JL13b] to pairs 
\[(R_{n+1}^2, C_{n+1}^2), (R_{n+1}^1, C_{n+1}^1)\].
We need to consider the following sequence of quadruples:
\[
\prod_{i=1}^{n} R_{i}^2 R_{i}^1 \tilde{W} X_{\epsilon_{n+1}+\epsilon_{n+1}} \prod_{i=n+2}^{2n-1} C_i^2 \prod_{t=n+1}^{2n-1} C_t \prod_{s=2}^{n-1} X_{\beta_{\alpha}^s}, \psi_W, X_{\alpha_{n+1}^1}, X_{\beta_{n+1}^1}),
\]
\[
\ldots
\]
\[
\prod_{i=1}^{n} R_{i}^2 R_{i}^1 R_{i,n+1} R_{n+1,n+1} \tilde{W} X_{\epsilon_{n+1}+\epsilon_{n+1}} \prod_{i=n+2}^{2n-1} C_i^2 \prod_{t=n+1}^{2n-1} C_t \prod_{s=2}^{n+1} C_{t,n+1,s} \psi_W, R_{n+1,n+1}^1, C_{n+1,n+1}),
\]
\[
\prod_{i=1}^{n} R_{i}^2 R_{i}^1 R_{i,n+1,n+1} R_{n+1,n+1} \tilde{W} X_{\epsilon_{n+1}+\epsilon_{n+1}} \prod_{i=n+2}^{2n-1} C_i^2 \prod_{t=n+1}^{2n-1} C_t \prod_{s=2}^{n+1} C_{t,n+1,s} \psi_W, R_{n+1,n+1}^1, C_{n+1,n+1})
\]
Applying Lemma 2.3 of [JL13b] 2n times, we get that the integral in (2.12) is non-vanishing if and only if the following integral is non-vanishing:
\[
\int_{[\prod_{i=1}^{n+1} R_{i}^2 R_{i}^1 \tilde{W} X_{\epsilon_{n+1}+\epsilon_{n+1}} \prod_{i=n+2}^{2n-1} C_{i}^2 C_{i}^1]} \varphi(rw\omega_2\omega_1 g)\psi_W(w)^{-1} dcdwdwr.
\]
After repeating the above procedure to the pairs \((R_s^2, C_s^2), (R_s^1, C_s^1)\), \(s = n+1, n+2, \ldots, 2n-1\), we get that the integral in (2.13) is non-vanishing if and only if the following integral is non-vanishing:
\[
\int_{[\prod_{i=1}^{n-1} R_{i}^2 R_{i}^1 \tilde{W} \prod_{j=n+1}^{2n-1} X_{e_j+e_j}]} \varphi(rw\omega_2\omega_1 g)\psi_W(w)^{-1} dxdwdwr.
\]
Now, let’s see the structure of elements in \(\prod_{i=1}^{2n-1} R_{i}^2 R_{i}^1 \tilde{W} \prod_{j=n+1}^{2n-1} X_{e_j+e_j}\).
Any element in \(\prod_{i=1}^{2n-1} R_{i}^2 R_{i}^1 \tilde{W} \prod_{j=n+1}^{2n-1} X_{e_j+e_j}\) has the following form:
\[
w = \begin{pmatrix}
z_{2n} & q_1 & q_2 \\
0 & u' & q_1^* \\
0 & 0 & z_{2n}^*
\end{pmatrix}.
\]
where \( z_{2n} \in V_{2n} \), the standard standard maximal unipotent subgroup of \( GL_{2n} \), \( u' \in U_{[(2n)^{2n-2}]_2} := \omega_1 V_{[(2n)^{2n-2}]_2} \omega_1^{-1} \), \( q_1 \in M_{(2n) \times (4m-4)n} \), with the last row zero, \( q_2 \in M_{(2n) \times (2n)} \), with \( q_2(2n, 1) = 0 \). And by (2.9), the restriction of \( \psi_W \) to the \( z_{2n} \)-part gives a Whittaker character of \( GL_{2n} \).

Note that \( \psi_W \left( \begin{array}{ccc} z_{2n} & 0 & 0 \\ 0 & I_{4mn-4n} & 0 \\ 0 & 0 & z_{2n}^* \end{array} \right) = \psi(2n(1, 2) + \cdots + z_{2n}(n, n + 1) - z_{2n}(n + 1, n + 2) - \cdots - z_{2n}(2n - 1, 2n)). \)

It is clear that the integral in (2.14) is non-vanishing if and only if the following integral is non-vanishing:

\[
(2.15) \quad \int_{\prod_{j=1}^{2n-1} R^2 R^1 \tilde{W} \prod_{j=n+1}^{2n-1} X_{x_j + s_j}} \varphi(rwx_2 \omega_1 \gamma) \psi_W'(w)^{-1} dx d\omega dr,
\]

where \( \psi_W'(\begin{array}{ccc} z_{2n} & 0 & 0 \\ 0 & I_{4mn-4n} & 0 \\ 0 & 0 & z_{2n}^* \end{array}) = \psi(\sum_{i=1}^{2n-1} z_{2n}(i, i + 1)). \)

Then it is easy to see that the integral in (2.15) has an inner integral which is exactly \( \varphi^{v_{12n-1}} \), using notation in Lemma 2.5. On the other hand, we know that by Lemma 2.5, \( \varphi^{v_{12n-1}} = \varphi^{v_{12n}} \). Therefore, the integral in (2.14) becomes

\[
(2.16) \quad \int_{\tilde{U}} \varphi(u \omega_2 \omega_1 \gamma) \psi_{\tilde{U}}(u)^{-1} du,
\]

where any element in \( \tilde{U} \) has the following form:

\[
u = \nu(2n, u', q_1, q_2) = \begin{pmatrix} z_{2n} & q_1 & q_2 \\ 0 & u' & q_1^* \\ 0 & 0 & z_{2n}^* \end{pmatrix},
\]

where \( z_{2n} \in V_{2n} \), the standard maximal unipotent subgroup of \( GL_{2n} \), \( u' \in U_{[(2n)^{2n-2}]_2} := \omega_1 V_{[(2n)^{2n-2}]_2} \omega_1^{-1} \), \( q_1 \in M_{(2n) \times (4m-4)n} \), \( q_2 \in M_{(2n) \times (2n)} \), such that \( q_2^* v_{2n} - v_{2n} q_2 = 0 \), where \( v_{2n} \) is a matrix only with ones on the second diagonal.

Hence, the integral in (2.16) can be written as

\[
(2.17) \quad \int_{u(z_{2n}, u', 0, 0)} \varphi(u \omega_2 \omega_1 \gamma) p_{2n} \psi_{\tilde{U}}(u)^{-1} du,
\]

where \( \varphi_{p_{2n}} \) is the constant term of \( \varphi \) along the parabolic subgroup \( P_{2n} = M_{2n} U_{2n} \) of \( Sp_{4mn} \) with Levi isomorphic to \( GL_{2n} \times Sp_{(4m-4)n} \).

By Lemma 2.3, there is an automorphic function

\[
f \in A(N_{2n}(\mathbb{A}) M_{2n}(F) \backslash Sp_{4mn}(\mathbb{A})) \int_{|\tau| = \frac{2m-1}{2} \otimes e_{\Delta(r, m-1)}} f, \]

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such that
\[ \varphi(g)_{P_{2n}} = f(g), \forall g \in \text{Sp}_{4mn}(\mathbb{A}). \]

Therefore, \( \varphi(u\omega_2\epsilon_1g)_{P_{2n}} \) is an automorphic form in \( \tau|\cdot|^{-\frac{2m-1}{2}} \otimes \mathcal{E}_{\Delta(\tau,m-1)}. \) Note that the restriction of \( \psi_{\tilde{U}} \) to the \( z_{2n} \)-part gives us a Whittaker coefficient, and the restriction to the \( u' \)-part gives a Fourier coefficient of \( \mathcal{E}_{\Delta(\tau,m-1)} \) attached to the partition \( [(2n)^{2m-2}] \) up to a conjugation of the Weyl element \( \omega'_1 \). On the other hand, \( \tau \) is generic, and by induction assumption, \( \mathcal{E}_{\Delta(\tau,m-1)} \) has a nonzero Fourier coefficient attached to the partition \( [(2n)^{2m-2}] \). Therefore, we can make the conclusion that \( \mathcal{O}(\mathcal{E}_{\Delta(\tau,m)}) = [(2n)^{2m}]. \)

Remark 2.2. By similar argument, we can also prove that for the residual representation \( \tilde{\mathcal{E}}_{\Delta(\tau,k)} \otimes \tilde{\pi} \) of \( \tilde{\text{Sp}}_{4kn+2n}(\mathbb{A}), \)
\[ p^m(\tilde{\mathcal{E}}_{\Delta(\tau,k)} \otimes \tilde{\pi}) = [(2n)^{2k+1}]. \]

We will not give the details here.

The following lemma is an Sp-analogue of the Lemma 4.1 of [JL13a]. It gives formulas for certain constant terms of automorphic forms in \( \mathcal{E}_{\Delta(\tau,m)}. \) The proof of this lemma will be given in the next section.

Lemma 2.3. Let \( P_{2ni} = M_{2ni} N_{2ni}, 1 \leq i \leq m \) be the parabolic subgroup of \( \text{Sp}_{4mn} \) with Levi part \( M_{2ni} \cong \text{GL}_{2ni} \times \text{Sp}_{4n(m-i)}. \) Let \( \varphi \) be an arbitrary automorphic form in \( \mathcal{E}_{\Delta(\tau,m)}. \) Denote by \( \varphi(g)_{P_{2ni}} \) the constant term of \( \varphi \) along \( P_{2ni}. \) Then for \( 1 \leq i \leq m-1, \) there is an automorphic function
\[ f \in A(N_{2ni}(\mathbb{A}) M_{2ni}(F) \backslash \text{Sp}_{4mn}(\mathbb{A})) \mathcal{E}_{\Delta(\tau,i)|\cdot|^{-\frac{2m-i}{2}} \otimes \mathcal{E}_{\Delta(\tau,m-1)}}, \]
such that
\[ \varphi(g)_{P_{2ni}} = f(g), \forall g \in \text{Sp}_{4mn}(\mathbb{A}). \]
For \( i = m, \) there is an automorphic function
\[ f \in A(N_{2mn}(\mathbb{A}) M_{2mn}(F) \backslash \text{Sp}_{4mn}(\mathbb{A})) \mathcal{E}_{\Delta(\tau,m)|\cdot|^{-\frac{m}{2}}}, \]
such that
\[ \varphi(g)_{P_{2mn}} = f(g), \forall g \in \text{Sp}_{4mn}(\mathbb{A}). \]

Next, we prove two important lemmas, which can be viewed as an Sp-analogue of Lemmas 6.1 and 6.2 of [JL13a].
Lemma 2.4. Let $N_{12mn}$ be the standard maximal unipotent subgroup of $Sp_{4mn}$ consisting of upper triangular matrices. For $p \geq 2n$, define a character of $N_{12mn}$ as follows:

$$\psi_p^\phi(n) := \psi(n_{1,2} + \cdots + n_{p-1,p} + n_{p,p+1})$$

(2.18)

$$\psi(\epsilon_1 n_{p+1,p+2} + \cdots + \epsilon_{2mn-p} n_{2mn,2mn+1}),$$

where $n \in N_{12mn}$, $\epsilon_i \in \{0, 1\}$, for $1 \leq i \leq 2mn - p - 1$, $\epsilon_{2mn-p} \in F/(F^*)^2$, $\mathcal{E} = \{\epsilon_1, \ldots, \epsilon_{2mn-p}\}$. Then for any automorphic form $\varphi \in \mathcal{E}_{\Delta(r,m)}$, the following $\psi_p^\phi$-Fourier coefficient is identically zero:

$$\varphi_{\psi_p^\phi}(g) := \int_{[N_{12mn}]} \varphi(ng)\psi_p^\phi(n)^{-1}dn \equiv 0.$$

(2.19)

Proof. If $\epsilon_i \neq 0$, $\forall 1 \leq i \leq 2mn - p$, then $\psi_p^\phi$ is a generic character of $G_n$. Since $\mathcal{E}_{\Delta(r,m)}$ is not generic, it has no nonzero $\psi_p^\phi$-Fourier coefficients, i.e., $\varphi_{\psi_p^\phi} \equiv 0$, $\forall \varphi \in \mathcal{E}_{\Delta(r,m)}$.

Assume that $1 \leq i \leq 2mn - p$ is the first number such that $\epsilon_i = 0$. Then $\varphi_{\psi_p^\phi}$ has an inner integral $\varphi_{p+i}$, which is the constant term of $\varphi$ along $P_{p+i}$, the parabolic subgroup of $Sp_{4mn}$ with Levi isomorphic to $GL_{p+i} \times Sp_{4mn-2(p+i)}$. Note that $p + i > 2n$.

By the cuspidal support of $\mathcal{E}_{\Delta(r,m)}$, we can see that $\varphi_{p+i} = 0$ unless $p + i = 2ns$ with $2 \leq s \leq m$. By Lemma 2.3, there is an automorphic function

$$f \in A(N_{2ns}(\mathbb{A})M_{2ns}(F)\backslash Sp_{4mn}(\mathbb{A}))_{\Delta(r,s)\mid \frac{2m-s}{s} \in \mathcal{E}_{\Delta(r,b-s)}};$$

such that

$$\varphi(g)_{2ns} = f(g), \forall g \in Sp_{4mn}(\mathbb{A}),$$

Therefore, after taking the constant term, $\varphi(g)_{2ns}$ is an automorphic function over $GL_{2ns}(\mathbb{A}) \times Sp_{4mn-m-s}(\mathbb{A})$. Note that the character $\psi(n_{1,2} + \cdots + n_{2ns-1,2ns})$ gives a Whittaker character of $GL_{2ns}$. Since by Proposition 2.1 of [IL13, D](r, s) is not generic for $s > 1$, i.e., it has no nonzero Whittaker Fourier coefficients, we conclude that $\varphi_{\psi_p^\phi} \equiv 0$.

This completes the proof of the lemma. 

Lemma 2.5. Let $N_{1p}$ be the unipotent radical of the parabolic subgroup $P_{1p}$ of $Sp_{4mn}$ with Levi part isomorphic to $GL_{1}^{\times p} \times Sp_{4mn-2p}$. Let

$$\psi_{N_{1p}}(n) := \psi(n_{1,2} + \cdots + n_{p,p+1}),$$

and

$$\tilde{\psi}_{N_{1p}}(n) := \psi(n_{1,2} + \cdots + n_{p-1,p}),$$
be two characters of \( N_{1p} \). For any automorphic form \( \varphi \in \mathcal{E}_{\Delta(\tau,m)} \), define \( \psi_{N_{1p}} \) and \( \tilde{\psi}_{N_{1p}} \)-Fourier coefficients as follows:

\[
\varphi_{N_{1p}}(g) := \int_{[N_{1p}]} \varphi(ng)\psi_{N_{1p}}(n)^{-1}dn,
\]

\[
\varphi_{\tilde{N}_{1p}}(g) := \int_{[N_{1p}]} \varphi(ng)\tilde{\psi}_{N_{1p}}(n)^{-1}du.
\]

Then \( \varphi_{N_{1p}} \equiv 0, \forall p \geq n; \varphi_{N_{12n-1}} = \varphi_{N_{12n}} \).

Proof. First we assume that \( p \geq n \). Let \( X_{e_p+1+e_p+1} \) be the root subgroup corresponding to the root \( e_p+1 + e_p+1 \). Since the conjugating action of \( X_{e_p+1+e_p+1} \) normalizes \( N_{1p} \), and preserves the character \( \psi_{N_{1p}} \), we can take the Fourier expansion of \( \varphi_{N_{1p}} \) along \( X_{e_p+1+e_p+1} \).

Under the action of \( GL_1 \), we will get Fourier coefficients corresponding the two orbits of the dual of \( X_{e_p+1+e_p+1} \); the trivial one and the non-trivial one. Note that the non-trivial orbit gives us Fourier coefficients which are exactly the Fourier coefficients attached to \([2(p+1)]_{4mn-2q-2}\), with \( p+1 > n \). On the other hand, by Proposition 6.4 and Remark 6.5 of [JL13c], all the Fourier coefficients attached to the non trivial orbit vanish, and the Fourier coefficient attached to the trivial orbit, i.e., the Fourier coefficient with respect to the trivial character, survives. Hence, \( \varphi_{N_{1p}} \) becomes:

\[
\sum_{\gamma} \varphi_{N_{1p+1}}(\gamma g) + \varphi_{\tilde{N}_{1p}}(g),
\]

where \( \gamma \) is in some quotient space which we will not specify here.
Then we continue the above process of Fourier expansion for the two kinds of Fourier coefficients \( \varphi^{\psi_{N_{1p+1}}^{\mathbb{G}^{N_{1p+1}}}} \) and \( \tilde{\varphi}^{\psi_{N_{1p+1}}^{\mathbb{G}^{N_{1p+1}}}} \) along the pair of groups \( \langle X_{e_{p+2}+e_{p+2}}, R_{p+2} \rangle \). We will get four kinds of Fourier coefficients:

\[
\int_{[N_{1p+2}]} \varphi(ng)\psi_{N_{1p+2}}^{\xi}(n)^{-1}dn,
\]

where

\[
\psi_{N_{1p+2}}^{\xi}(n) := \psi(n_{1,2} + \cdots n_{p-1,p} + n_{p,p+1}) \\
\cdot \psi(\varepsilon_{1}n_{p+1,p+2} + \varepsilon_{2}n_{p+2,p+3}),
\]

and \( \xi = \{\varepsilon_{1}, \varepsilon_{2}\}, \varepsilon_{1}, \varepsilon_{2} \in \{0, 1\} \). Then we can continue the above process of Fourier expansion for each of these four kinds of Fourier coefficients along the pair of groups \( \langle X_{e_{p+3}+e_{p+3}}, R_{p+3} \rangle \). We will get six kinds of Fourier coefficients:

\[
\int_{[N_{1p+3}]} \varphi(ng)\psi_{N_{1p+3}}^{\xi}(n)^{-1}dn,
\]

where

\[
\psi_{N_{1p+3}}^{\xi}(n) := \psi(n_{1,2} + \cdots n_{p-1,p} + n_{p,p+1}) \\
\cdot \psi(\varepsilon_{1}n_{p+1,p+2} + \varepsilon_{2}n_{p+2,p+3} + \varepsilon_{2}n_{p+3,p+4}),
\]

and \( \xi = \{\varepsilon_{1}, \varepsilon_{2}, \varepsilon_{3}\}, \varepsilon_{1}, \varepsilon_{2}, \varepsilon_{3} \in \{0, 1\} \).

Keep repeating the above procedure, we will get Fourier coefficients of the following form: \( \varphi^{\psi_{N_{1p+2}}} \), with \( \xi = \{\varepsilon_{1}, \ldots, \varepsilon_{2mn-p}\}, \varepsilon_{i} \in \{0, 1\}, 1 \leq i \leq 2mn - p - 1 \), and \( \varepsilon_{2mn-p} \in F/(F^*)^2 \).

By Lemma 2.4, all Fourier coefficients of this kind are zero, if \( p \geq 2n \).

Therefore, \( \varphi^{\psi_{N_{1p}}} \equiv 0 \), if \( p \geq 2n \).

For \( p = 2n - 1 \), by Lemma 2.3, we can see that \( \varphi^{\psi_{N_{12n-1}}} = \tilde{\varphi}^{\psi_{N_{12n}}} \), since \( \varphi^{\psi_{N_{12n}}} \equiv 0 \) by above discussion.

This completes the proof of the lemma. \( \square \)

### 3. Proof of Lemma 2.3

In this section, we prove Lemma 2.3. Before we start, we recall the definition of the Eisenstein series in Section 1:

\[
E(\phi, s)(g) = \sum_{\gamma \in P_{2mn}(F) \backslash Sp_{4mn}(F)} \lambda_{s} \phi(\gamma g),
\]

where \( \phi \in A(N_{2mn}(\mathbb{A}) M_{2mn}(F) \backslash Sp_{4mn}(\mathbb{A}))_{\Delta_{(\tau, m)}} \). Assume that \( \varphi = \text{Res}_{s=\frac{m}{2}} E(\phi, s) \).

To compute \( \varphi^{\psi_{p_{2n}}} = (\text{Res}_{s=\frac{m}{2}} E(\phi, s))_{p_{2n}} \), we use the fact that the residue operator and the constant term operator are interchangeable.
So, first, we are going to calculate the constant term of $E(\phi, s)$ along $P_{2ni}$. We follow the calculation in Section 2 of [JLZ13].

\[
E_{P_{2ni}}(\phi, s)(g) = \int_{N_{2ni}(F)\backslash N_{2ni}(A)} E(\phi, s)(ug)du
= \sum_{\omega^{-1} \in P_{2mn} \cap M_{2ni}(F) \backslash M_{2ni}(F)} \sum_{\gamma \in M_{2ni}(F) \backslash M_{2ni}(F)} \int_{[N_{2ni}]} \int_{N_{2ni}(A)} \lambda_s \phi(\omega^{-1_\gamma u'u''}g) du'du'',
\]

where we define $M_{2ni}^\omega := \omega P_{2mn} \omega^{-1} \cap M_{2ni}$ and $N_{2ni}^\omega := \omega P_{2mn} \omega^{-1} \cap N_{2ni}$ and $[N_{2ni}^\omega] := N_{2ni}(F) \backslash N_{2ni}(A)$. Note that the unipotent radical $N_{2ni}$ can be decomposed as a product $N_{2ni,\omega} N_{2ni}^\omega$, where $N_{2ni,\omega}$ satisfies $N_{2ni,\omega} \cap N_{2ni}^\omega = \{1\}$ and $N_{2ni} = N_{2ni,\omega} N_{2ni}^\omega = N_{2ni,\omega} N_{2ni}^\omega N_{2ni,\omega}$.

Using the explicit calculation about the generalized Bruhat decomposition $P_{2mn} \backslash S_{P_{4mn}} / P_{2ni}$ (see Chapter 4 of [GRS11]), we can see that all the double cosets are killed by the cuspidal support of the Eisenstein series except two, which have the following representatives: $\omega_0 = Id$, and

\[
\omega_1 = \begin{pmatrix}
0 & 0 & I_{2ni} & 0 \\
I_{2n(m-i)} & 0 & 0 & 0 \\
0 & 0 & 0 & I_{2n(m-i)} \\
0 & -I_{2ni} & 0 & 0
\end{pmatrix}.
\]

Then

\[
E_{P_{2ni}}(\phi, s)(g) = E_{P_{2ni}}(\phi, s)_{\omega_0}(g) + E_{P_{2ni}}(\phi, s)_{\omega_1}(g),
\]

where

\[
E_{P_{2ni}}(\phi, s)_{\omega_j}(g)
= \sum_{\gamma \in M_{2ni}^\omega_j(F) \backslash M_{2ni}(F)} \int_{[N_{2ni}^\omega_j]} \int_{N_{2ni,\omega_j}(A)} \lambda_s \phi(\omega_j^{-1_\gamma u'u''}g) du'du'',
\]

$j = 0, 1$.

We will consider these two terms separately in the next two subsections.
3.1. \( \omega_0 \)-term. Write elements in \( N_{2n_i} \) as follows:

\[
u(X, Z, W) = \begin{pmatrix} I_{2n_i} & X & Z & W \\ I_{2n(m-i)} & Z' \\ I_{2n(m-i)} & X' \\ I_{2n_i} & \end{pmatrix}.
\]

Note that \( P_{2mn} \cap M_{2n_i} \backslash M_{2n_i} \cong P_{2n(m-i)} \backslash Sp_{4n(m-i)} \), where \( P_{2n(m-i)} \) is the parabolic subgroup of \( Sp_{4n(m-i)} \) with Levi isomorphic to \( GL_{2n(m-i)} \times 1_{Sp_0} \). Then the \( \omega_0 \)-term of the constant term can be written as

\[
(3.3) \quad E_{P_{2n_i}}(\phi, s)\omega_0(g) = \sum_{\gamma \in P_{2n(m-i)} \backslash Sp_{4n(m-i)}(F)} \int_{[N_{2n_i}]} \lambda_s \phi(\gamma u g) du.
\]

The integral can be calculated as follows:

\[
\int_{[N_{2n_i}]} \lambda_s \phi(\gamma u g) du = \int_{[N_{2n_i}]} \lambda_s \phi(u \gamma g) du = \int_{[M_{2n_i} \times 2n(m-i)]} \lambda_s \phi(u u(X) \gamma g) du' dX = \int_{[M_{2n_i} \times 2n(m-i)]} \lambda_s \phi(u(X) \gamma g) dX,
\]

where \( u(X) = u(X, 0, 0) \) with \( X \in M_{2n_i} \times 2n(m-i) \).

As in Section 2.2 of [JLZ13], the integral \( \int_{[M_{2n_i} \times 2n(m-i)]} \lambda_s \phi(u(X) \gamma g) dX \) can be viewed as the constant term of the automorphic function in \( \Delta(\tau, m) : x \mapsto \phi(\text{diag}(x, x^*) g) \), along the maximal parabolic subgroup \( Q_{2n_i, 2n(m-i)} \) of \( GL_{2mn} \) with Levi isomorphic to \( GL_{2n_i} \times GL_{2n(m-i)} \). We will denote it by \( \phi_{Q_{2n_i, 2n(m-i)}} \).

Let \( P_{2n_i, 2n(m-i)} = M_{2n_i, 2n(m-i)} N_{2n_i, 2n(m-i)} \) be a standard parabolic subgroup of \( Sp_{4mn} \), whose Levi \( M_{2n_i, 2n(m-i)} \cong GL_{2n_i} \times GL_{2n(m-i)} \times 1_{Sp_0} \).

The following lemma is parallel to Lemma 2.1 of [JLZ13].

**Lemma 3.1.** The constant term \( \lambda_s \phi_{Q_{2n_i, 2n(m-i)}} \) belongs to the following space

\[
A(N_{2n_i, 2n(m-i)}(A) M_{2n_i, 2n(m-i)}(F) \backslash Sp_{4mn}(A)) \Delta(\tau, m)|s|^{\frac{1}{2}} \otimes \Delta(\tau, m)|s|^{\frac{1}{2}} \otimes 1_{Sp_0}.
\]

**Proof.** The proof is omitted here, since it is almost the same as that of Lemma 2.1 of [JLZ13], word-by-word. \( \Box \)
Therefore, by (3.3) and Lemma 3.1, we can see that $E_{P_{2ni}}(\phi, s)_{\omega_0}$ belongs to the following space

$$A(N_{2ni}(A)M_{2ni}(F) \backslash \mathcal{S}p_{4mn}(A))_{\Delta(\tau, i)|\cdot|^\frac{m-n}{4} \otimes (\Delta(\tau, m-i)|\cdot|^\frac{m-n}{4} \otimes 1_{Sp_0})}.$$ 

Hence, the residue operator will kill $E_{P_{2ni}}(\phi, s)_{\omega_0}$, since $s = \frac{m}{2}$ is not a pole of the Eisenstein series on $\mathcal{S}p_{4n(m-i)}$ with inducing data $\Delta(\tau, m-i)|\cdot|^\frac{m-n}{2} \otimes 1_{Sp_0}$.

### 3.2. $\omega_1$-term

Note that $N_{2ni}^{\omega_1} = \{u(Z) = u(0, Z, 0) \mid Z \in M_{2ni \times 2n(m-i)}\}$, and $M_{2ni}^{\omega_1}(F) \backslash M_{2ni}(F)$ is isomorphic to $P_{2n(m-i)}(F) \backslash \mathcal{S}p_{4n(m-i)}(F)$.

Therefore, we have

$$E_{P_{2ni}}(\phi, s)_{\omega_1}(g)$$

\[= \sum_{\gamma \in P_{2n(m-i)}(F) \backslash \mathcal{S}p_{4n(m-i)}} \int_{N_{2ni, \omega_1}(A) \backslash [M_{2ni \times 2n(m-i)}]} \int_{M_{2n(m-i)x2n}} \lambda_s \phi(\omega_1^{-1}u(Z)ug) dZ du \]

\[= \sum_{\gamma \in P_{2n(m-i)}(F) \backslash \mathcal{S}p_{4n(m-i)}} \int_{N_{2ni, \omega_1}(A) \backslash [M_{2n(m-i)x2n}]} \lambda_s \phi(u'(X)) \omega_1^{-1}u g dX du, \]

where

$$u'(X) = \begin{pmatrix} I_{2(n-m)} & X \\ I_{2ni} & X' \\ I_{2ni} & I_{2(n-m)} \end{pmatrix} \text{ for } X \in M_{2n(m-i) \times 2ni}.$$

As in case of $\omega_0$, the integral $\int_{[M_{2n(m-i) \times 2n}]^2} \phi(u'(X)) dX$ can be viewed as the constant term of the automorphic function in $\Delta(\tau, m) \colon x \mapsto \phi(\text{diag}(x, x^*))g$, along the maximal parabolic subgroup $Q_{2n(m-i), 2ni}$ of $GL_{2mn}$ with Levi isomorphic to $GL_{2n(m-i)} \times GL_{2ni}$. We will denote it by $\phi_{Q_{2n(m-i), 2ni}}$.

Let $P_{2n(m-i), 2ni} = M_{2n(m-i), 2ni}U_{2n(m-i), 2ni}$ be a standard parabolic subgroup of $\mathcal{S}p_{4mn}$, whose Levi $M_{2n(m-i), 2ni} \cong GL_{2n(m-i)} \times GL_{2ni} \times 1_{Sp_0}$. Then, by Lemma 3.1, the constant term $\lambda_s \phi_{Q_{2n(m-i), 2ni}}$ belongs to the following space

$$A(N_{2n(m-i), 2ni}(A)M_{2n(m-i), 2ni}(F) \backslash \mathcal{S}p_{4mn}(A))_{\Delta(\tau, m-i)|\cdot|^\frac{m-n}{4} \otimes (\Delta(\tau, i)|\cdot|^\frac{m-n}{4} \otimes 1_{Sp_0})}.$$ 

Note that the outer integral in (3.4) is the intertwining operator corresponding to $\omega_1$, which maps

$$A(N_{2n(m-i), 2ni}(A)M_{2n(m-i), 2ni}(F) \backslash \mathcal{S}p_{4mn}(A))_{\Delta(\tau, m-i)|\cdot|^\frac{m-n}{4} \otimes (\Delta(\tau, i)|\cdot|^\frac{m-n}{4} \otimes 1_{Sp_0}}.$$
to
\[ A(N_{2ni,2n(m-i)}(\mathbb{A})M_{ni,a(b-i)}(F)\backslash Sp_{4mn}(\mathbb{A})) \Delta(\tau,i)|^{-(s+\frac{m-1}{2})\otimes\Delta(\tau,m-i)|^{s+\frac{1}{2}}} \otimes 1_{Sp_0}. \]

Note that \( \tau \) is self-dual.

Therefore, by (3.4), and the above discussion, we can see that \( E_{P_{2ni}}(\phi, s)_{\omega_1} \) belongs to the following space
\[ A(N_{2ni}(\mathbb{A})M_{2ni}(F)\backslash Sp_{4mn}(\mathbb{A})) \Delta(\tau,i)|^{-(s+\frac{m-1}{2})\otimes\Delta(\tau,m-i)|^{s+\frac{1}{2}}} \otimes 1_{Sp_0}. \]

And for \( 1 \leq i \leq m-1 \), after taking the residue operator, \( \text{Res}_{s=\frac{m}{2}} E_{P_{2ni}}(\phi, s)\omega_1 \) belongs to the following space
\[ A(N_{2ni}(\mathbb{A})M_{2ni}(F)\backslash Sp_{4mn}(\mathbb{A})) \Delta(\tau,i)|^{\frac{2m-i}{2}}\otimes\Delta(\tau,m-i)|^{s+\frac{1}{2}}} \otimes 1_{Sp_0}, \]

since \( s = \frac{m}{2} \) is the rightmost simple pole of the (normalized) Eisenstein series with inducing data \( \Delta(\tau, m-i)|^{s+\frac{1}{2}}} \otimes 1_{Sp_0}, \) and it is not a pole of the intertwining operator corresponding to \( \omega_1 \).

Therefore, for \( 1 \leq i \leq m-1 \),
\[
\varphi_{P_{2ni}} = (\text{Res}_{s=\frac{m}{2}} E(\phi, s))_{P_{2ni}}
= \text{Res}_{s=\frac{m}{2}} (E_{P_{2ni}}(\phi, s))
= \text{Res}_{s=\frac{m}{2}} (E_{P_{2ni}}(\phi, s)_{\omega_1}),
\]

belongs to the following space
\[ A(N_{2ni}(\mathbb{A})M_{2ni}(F)\backslash Sp_{4mn}(\mathbb{A})) \Delta(\tau,i)|^{\frac{2m-i}{2}}\otimes\Delta(\tau,m-i)|^{s+\frac{1}{2}}} \otimes 1_{Sp_0}. \]

For \( i = m \), \( \varphi_{P_{2mn}} = \text{Res}_{s=\frac{m}{2}} (E_{P_{2mn}}(\phi, s)_{\omega_1}) \) belongs to the following space
\[ A(N_{2mn}(\mathbb{A})M_{2mn}(F)\backslash Sp_{4mn}(\mathbb{A})) \Delta(\tau,m)|^{\frac{m}{2}}\otimes\Delta(\tau,m)|^{s+\frac{1}{2}}} \otimes 1_{Sp_0}, \]

since \( E_{P_{2mn}}(\phi, s)_{\omega_1} \) belongs to the following space
\[ A(N_{2mn}(\mathbb{A})M_{2mn}(F)\backslash Sp_{4mn}(\mathbb{A})) \Delta(\tau,m)|^{s+\frac{1}{2}}} \otimes 1_{Sp_0}, \]

and \( s = \frac{m}{2} \) is a simple pole of the intertwining operator corresponding to \( \omega_1 \).

This completes the proof of Lemma 2.3.

4. Proof of Part (1) of Theorem 1.4

In this section, we will prove that \( \Psi \) is surjective, which comes from the computation of composition of two descents
\[ D_{2n,\omega_1}^{4mn} \circ D_{2n,\omega_1}^{4mn+2n}(\bar{\mathcal{E}},\bar{\mathcal{E}}_{\mathfrak{g}(m-1)_{n+2n}}). \]
where \( \tilde{\sigma}_{4(m-1)n+2n} \in \mathcal{N}_{\tilde{S}_{4(m-1)n+2n}}^\tau(\tau, \psi) \), and
\[
\mathcal{D}_{2n, \psi}^{4mn+2n} (\tilde{\sigma}_{4(m-1)n+2n}) \subset \mathcal{N}_{\tilde{S}_{4mn}}^\tau(\tau, \psi).
\]

It turns out that there is a similar identity:
\[
\mathcal{D}_{2n, \psi}^{4mn} \circ \mathcal{D}_{2n, \psi}^{4mn+2n} (\tilde{\sigma}_{4(m-1)n+2n}) = \tilde{\sigma}_{4(m-1)n+2n},
\]
as in Proposition 5.2 of [GJS12]. We will prove in the next section that \( \Psi \) is well-defined, i.e., \( \mathcal{D}_{2n, \psi}^{4mn} \) is irreducible (see Theorem 4.6).

Note that, from this section to Section 6, we assume that \( F \) is a number field which is not totally imaginary, unless specified.

To start, we recall Proposition 4.1 of [JL13c] which generalizes Theorem 6.3 of [GRS11] and is true for any number field.

**Proposition 4.1.** Assume that \( F \) is any number field.

1. Let \( \mu_i, 1 \leq i \leq r \), be characters of \( F_v^* \), \( a \in F_v^* \), then
\[
\text{Ind}_{\mathcal{P}_{m_1}^{2n} \cdots \mathcal{P}_{m_k}^{2n}}^{\mathcal{P}_{m_1}^{2n-2k} \cdots \mathcal{P}_{m_k}^{2n-2k}} (\chi_1^{\mu_1} \chi_2^{\nu_{m_1}} \cdots \chi_k^{\nu_{m_k}} (\det(GL_{m_1})))
\approx \text{Ind}_{\mathcal{P}_{m_1}^{2n-2k} \cdots \mathcal{P}_{m_k}^{2n-2k}}^{\mathcal{P}_{m_1}^{2n-2k} \cdots \mathcal{P}_{m_k}^{2n-2k}} (\chi_1^{\mu_1} \chi_2^{\nu_{m_1}} \cdots \chi_k^{\nu_{m_k}} (\det(GL_{m_1}))),
\]

2. Let \( \mu_i, 1 \leq i \leq r \), be characters of \( F_v^* \), \( a, b \in F_v^* \), then
\[
\text{Ind}_{\mathcal{P}_{m_1}^{2n} \cdots \mathcal{P}_{m_k}^{2n}}^{\mathcal{P}_{m_1}^{2n-2k} \cdots \mathcal{P}_{m_k}^{2n-2k}} (\chi_1^{\mu_1} \chi_2^{\nu_{m_1}} \cdots \chi_k^{\nu_{m_k}} (\det(GL_{m_1})))
\approx \text{Ind}_{\mathcal{P}_{m_1}^{2n-2k} \cdots \mathcal{P}_{m_k}^{2n-2k}}^{\mathcal{P}_{m_1}^{2n-2k} \cdots \mathcal{P}_{m_k}^{2n-2k}} (\chi_1^{\mu_1} \chi_2^{\nu_{m_1}} \cdots \chi_k^{\nu_{m_k}} (\det(GL_{m_1}))),
\]

where \( \chi_\frac{a}{b} \) is a quadratic character of \( F_v^* \) defined by Hilbert symbol as follows: \( \chi_\frac{a}{b} (x) = \left( \frac{b}{a} \right) x \), and \( \nu = |\det(\cdot)| \).

Note that when \( a = b = 1 \), Proposition 4.1 is exactly Theorem 6.3 of [GRS11].

Next, we prove the equality mentioned at the beginning of this section, which will imply later that \( \Psi \) is surjective.

**Theorem 4.2.** (1)
\[
\mathcal{D}_{2n, \psi}^{4mn} \circ \mathcal{D}_{2n, \psi}^{4mn+2n} (\tilde{\sigma}_{4(m-1)n+2n}) \neq 0.
\]

(2)
\[
\mathcal{D}_{2n, \psi}^{4mn} \circ \mathcal{D}_{2n, \psi}^{4mn+2n} (\tilde{\sigma}_{4(m-1)n+2n}) = \tilde{\sigma}_{4(m-1)n+2n}.
\]

**Proof.** The proof of Part (1) is similar to that of Theorem 2.1 and to that of Theorem 2.1 of [GJS12]. The proof of Part (2) is similar to those of Theorem 5.1 and Proposition 5.2 of [GJS12].
Proof of Part (1). Note that descents $D_{4mn+2n}^{4\mu n+2n}$ and $D_{2n,\psi}^{4\mu n+2n}$ are defined in Section 3.2 of [GRS11].

By Corollary 2.4 of [JL13b] or Lemma 1.1 of [GRS03], and the discussion at the end of Section 1 of [GRS03], $D_{2n,\psi}^{4\mu n+2n}(\tilde{\tau},\tilde{\sigma}_{(m-1)n+2n}) \neq 0$ if and only if the following integral is non-vanishing:

$$
\int_{[W]} \tilde{\varphi}(w\psi_1 g) \psi_{1n}^{-1}(w) dw,
$$

where $\tilde{\varphi} \in \tilde{\mathcal{E}}_{\tau,\mathcal{E}_{(m-1)n+2n}}$, $g \in \tilde{Sp}_{4mn-2n}(A)$, embedded into $\tilde{Sp}_{4mn+2n}(A)$ via the map $g \mapsto \text{diag}(I_{2n}, g, I_{2n})$; $Y_1$, $Y_2$ are the groups defined in (2.5) of [JL13b] corresponding to the partitions $[(2n)^{14mn}]$ and $[(2n)^{14mn-2n}]$ respectively; $V_{[(2n)^{14mn-2n}]}$ and $V_{[(2n)^{14mn}]}$ are defined in Section 2 of [JL13b].

Explicitly, let $N_{1n}$ be the unipotent radical of the parabolic $P_{1n}$ of $Sp_{4mn+2n}$ with the Levi subgroup isomorphic to $GL_1^n \times Sp_{4mn}$, then $Y_1 V_{[(2n)^{14mn}]}$ is a subgroup of $N_{1n}$ consists of elements $v$ with $v_{n,j} = 0$, for $n+1 \leq j \leq 2mn+n$.

Identify $Sp_{4mn}$ with its embedding into $Sp_{4mn+2n}$ via the map $g \mapsto \text{diag}(I_n, g, I_n)$. Let $N_{1n}$ be the unipotent radical of the parabolic $P_{1n}$ of $Sp_{4mn}$ with the Levi subgroup isomorphic to $GL_1^n \times Sp_{4mn-2n}$, then $Y_1 V_{[(2n)^{14mn-2n}]}$ is a subgroup of $N_{1n}$ consists of elements $v$ with $v_{n,j} = 0$, for $n+1 \leq j \leq 2mn$.

Let $\omega$ be the Weyl element of $GL_{2n}^n$ defined in (4.31) of [GRS99].

$$
\tilde{\omega}_{2i,i} = 1, \quad i = 1, \ldots, n,
\tilde{\omega}_{2i-1,i+n} = 1, \quad i = 1, \ldots, n,
\tilde{\omega}_{i,j} = 0, \quad \text{otherwise}.
$$

Let

$$
\omega_1 = \begin{pmatrix} \tilde{\omega} & I_{4mn-2n} \\ \tilde{\omega}^* & \omega \end{pmatrix} \in Sp_{4mn+2n}(F).
$$

As in [GJS12], we identify $Sp_{4mn+2n}(F)$ with the subgroup $Sp_{4mn+2n}(F) \times 1$ of $Sp_{4mn+2n}(A)$.

Conjugating cross the integral in (4.3) by $\omega_1$, it becomes:

$$
\int_{[W]} \tilde{\varphi}(w\omega_1 g) \psi_{1n}^{-1}(w) dw,
$$
where $W_1 = \omega_1 Y_2 V_{[(2n)1^{4m-n-2}],2} Y_1 V_{[(2n)1^{4m-n}],2} \omega_1^{-1}$, if $\omega_1^{-1} w \omega_1 = v_1 v$, $v, v_1 \in Y_2 V_{[(2n)1^{4m-n-2}],2}$, $v \in Y_1 V_{[(2n)1^{4m-n}],2}$, then

$$\psi_{W_1}(w) = \psi_{[(2n)1^{4m-n}],1}(v) \psi_{[(2n)1^{4m-n-2}],1}(v_1).$$

Note that the metaplectic cover splits over $W_1(\mathbb{A})$, and $W_1(\mathbb{A}) \times 1$ is a subgroup of $\tilde{Sp}_{4mn+2}(\mathbb{A})$. We identify $W_1$ with $W_1 \times 1$.

Elements in $W_1$ are of the following form

$$(4.6) \quad w = \begin{pmatrix} Z & q_1 & q_2 \\ I_{4mn-2n} & q_2^* & Z^* \end{pmatrix},$$

where $q_1(i, j) = 0$, for $i = 2n - 1, 2n$, $1 \leq j \leq 2mn - n$, $Z \in GL_{2n}$ has the form (4.34) of [GRS99]. Write $Z$ as an $n \times n$ matrix of $2 \times 2$ block matrices $Z = ([Z]_{i,j})$, $1 \leq i, j \leq n$, then $[Z]_{n,1} = \cdots = [Z]_{n,n-1} = 0$, $[Z]_{n,n} = I_2$; $[Z]_{i,j}$ is lower unipotent, for $i < n$; $[Z]_{j,i}$ is lower triangular, for $i < j$; $[Z]_{i,j}$ is lower nilpotent, for $j < i < n$. And

$$(4.7) \quad \psi_{W_1}(w) = \psi\left(\sum_{i=1}^{n-1} tr([Z]_{i,i+1}) + (q_2(2n, 1) - q_2(2n - 1, 2))\right).$$

Let $R^1 = \prod_{i=1}^n R^1_{i,j}$, for $1 \leq i \leq n - 1$, with $R^1_{i,j} = X_{\alpha_{i,j}}$, the root subgroup corresponding to the root $\alpha_{i,j} = e_{2i} - e_{2(j-1)+1}$. Let $R^1 = \prod_{i=1}^{n-1} R^1_{i,j}$. Actually $R^1$ is the subgroup of $W_1$ consists of lower unipotent matrices. Write $W_1 = R^1 \tilde{W}_1$, with $R^1 \cap \tilde{W}_1 = \{1\}$.

Let $C^1_i = \prod_{j=1}^{n} C^1_{i,j}$, for $1 \leq i \leq n - 1$, with $C^1_{i,j} = X_{\beta_{i,j}}$, the root subgroup corresponding to the root $\beta_{i,j} = e_{2(j-1)+1} - e_{2(i+1)}$.

We consider the quadruple

$$(4.8) \quad (\tilde{W}_1 \prod_{i=1}^{n-2} R^1_i \prod_{j=2}^{n-1} R^1_{n-1,j}, \psi_{W_1}, R^1_{n-1,1}, C^1_{n-1,1}).$$

It is easy to see that this quadruple satisfies all the conditions for Lemma 2.3 of [JL13b]. Hence, by Lemma 2.3 of [JL13b], the integral in (4.5) is non-vanishing if and only if the following integral is non-vanishing:

$$(4.9) \quad \int_{[C^1_{n-1,1} \tilde{W}_1 \prod_{i=1}^{n-2} R^1_i \prod_{j=2}^{n-1} R^1_{n-1,j}]} \tilde{\varphi}(cwr \omega_1 g) \psi_{W_1}^{-1}(w) drdwdc.$$
We continue to consider the following sequence of quadruples:

\[
(C_{n-1,1}^1 \tilde{W}_1 R_{i_1}^{n-1} R_{j_1}^{n-1}, \psi_{W_1}, R_{n-1,2}^1, C_{n-1,2}^1),
\]

\[
\prod_{k=1}^{2} C_{n-1,k}^1 \tilde{W}_1 R_{i_1}^{n-2} R_{j_1}^{n-1}, \psi_{W_1}, R_{n-1,3}^1, C_{n-1,3}^1),
\]

\[
\cdots,
\]

\[
\prod_{k=1}^{n-2} C_{n-1,k}^1 \tilde{W}_1 R_{i_1}^{n-2}, \psi_{W_1}, R_{n-1,n-1}^1, C_{n-1,n-1}^1).
\]

Applying Lemma 2.3 of \cite{JL13b} \((n - 1)\) times, we get that the integral in \eqref{4.9} is non-vanishing if and only if the following integral is non-vanishing:

\[
\int_{[C_{n-1}^1 \tilde{W}_1 \prod_{i=1}^{n-1} R_i^1]} \tilde{\varphi}(cwr \omega_1 g) \psi_{W_1}^{-1}(w) dr dw dc.
\]

Then, we repeat the above procedure for the pairs

\[(R_{n-2}^1, C_{n-2}^1), \ldots, (R_1^1, C_1^1).\]

For example, after repeating the above procedure for the pairs

\[(R_{n-2}^1, C_{n-2}^1), \ldots, (R_{s+1}^1, C_{s+1}^1),\]

we need the following sequence of quadruples for the pair \((R_s^1, C_s^1)\):

\[
\prod_{i=s+1}^{n-1} C_{i}^1 \tilde{W}_1 R_{i_1}^{s-1} R_{j_1}^{s}, \psi_{W_1}, R_{s+1}^1, C_{s+1}^1),
\]

\[
\prod_{i=s+1}^{n-1} C_{i}^1 C_{s+1}^1 \tilde{W}_1 R_{i_1}^{s-1} R_{j_1}^{s}, \psi_{W_1}, R_{s+1}^1, C_{s+1}^1),
\]

\[
\cdots,
\]

\[
\prod_{i=s+1}^{n-1} C_{i}^1 \tilde{W}_1 R_{i_1}^{s-1}, \psi_{W_1}, R_{s+1}^1, C_{s+1}^1).
\]

After applying Lemma 2.3 of \cite{JL13b} \(s\) times, the integral in \eqref{4.11} is non-vanishing if and only if the following integral is non-vanishing:

\[
\int_{[\prod_{i=s+1}^{n-1} C_{i}^1 \tilde{W}_1 \prod_{i=1}^{s-1} R_i^1]} \tilde{\varphi}(cwr \omega_1 g) \psi_{W_1}^{-1}(w) dr dw dc.
\]

After repeating the above procedure for all the pairs

\[(R_{n-2}^1, C_{n-2}^1), \ldots, (R_1^1, C_1^1),\]
we will see that the integral in (4.11) is non-vanishing if and only if the following integral is non-vanishing:

\[(4.14) \quad \int_{\prod_{i=1}^{n-1} C_i^i \tilde{W}_1} \tilde{\varphi}(cw_1 g) \psi_{W_1}^{-1}(w) dw dc.\]

Note that \(\prod_{i=1}^{n-1} C_i^i \tilde{W}_1 = \omega_1 V_{[(2n)21^{4mn-2n}],2}\omega_1^{-1}\), and \(\psi_{W_1}(cw) = \psi_{[(2n)21^{4mn-2n}],(-1,1)}(\omega_1^{-1}cw).\)

Let \(A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\) and \(\epsilon = \text{diag}(A,\ldots,A; I_{(4m-4)n}; A^*,\ldots,A^*)\), as in (2.31) of [GJS12]. Conjugating cross the integral in (4.14) by \(\epsilon\), it becomes:

\[(4.15) \quad \int_{[W_2]} \tilde{\varphi}(w_1 g) \psi_{W_2}^{-1}(w) dw,\]

where \(W_2 = \epsilon \prod_{i=1}^{n-1} C_i^i \tilde{W}_1 \epsilon^{-1}\), \(\psi_{W_2}(w) = \psi_{\prod_{i=1}^{n-1} C_i^i \tilde{W}_1}(\epsilon^{-1}w \epsilon)\).

Elements in \(W_2\) are of the following form

\[(4.16) \quad w = \begin{pmatrix} Z & q_1 \\ I_{4mn-2n} & q_2^* \\ Z^* & q_3 \\ \end{pmatrix},\]

where \(q_1(i, j) = 0\), for \(i = 2n - 1, 2n, 1 \leq j \leq 2mn - n\), \(Z\) is in the unipotent radical of the parabolic subgroup of \(GL_{2n}\) with the Levi subgroup isomorphic to \(GL_n^*\) and \(GL_n^*\). And

\[(4.17) \quad \psi_{W_2}(w) = \psi(\sum_{i=1}^{2n-2} Z_{i,i+2} + q_2(2n - 1, 1)).\]

Let \(\nu = \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \\ \nu_4 \end{pmatrix}\) be the Weyl element in (2.6). Let

\[(4.18) \quad \omega_2 = \begin{pmatrix} \nu_1 \\ I_{4mn-2n} \\ \nu_3 \\ \nu_4 \end{pmatrix},\]

a Weyl element of \(\text{Sp}_{4mn+2n}\).

Conjugating cross the integral in (4.15) by \(\omega_2\), it becomes:

\[(4.19) \quad \int_{[W_3]} \tilde{\varphi}(w_1 \omega_2 g) \psi_{W_3}^{-1}(w) dw,\]

where \(W_3 = \omega_2 W_2 \omega_2^{-1}, \psi_{W_3}(w) = \psi_{W_2}(\omega_2^{-1}w \omega_2)\).
Elements in $W_3$ have the following form:

\begin{equation}
(4.20) \\
w = \begin{pmatrix} Z & q_1 & q_2 \\ 0 & I_{4mn-2n} & q_1^* \\ 0 & 0 & Z^* \end{pmatrix} \begin{pmatrix} I_{2n} & 0 & 0 \\ p_1 & I_{4mn-2n} & 0 \\ p_2 & p_1^* & I_{2n} \end{pmatrix},
\end{equation}

where $Z$ is in the standard maximal unipotent subgroup of $GL_{2n}$; $q_1(i,j) = 0$, for $n + 1 \leq i \leq 2n$, and for $i = n$, $1 \leq j \leq 2mn - n$; $q_2(i,j) = p_2(i,j) = 0$, for $1 \leq j \leq i \leq 2n$; $p_1(i,j) = 0$, for $1 \leq j \leq n$, and for $j = n + 1$, $n + 1 \leq i \leq 2n$.

Let $C$ be the unipotent subgroup consisting of elements of the following form:

\begin{equation}
\begin{pmatrix} I_{2n} & 0 & 0 \\ p_1 & I_{4mn-2n} & 0 \\ 0 & p_1^* & I_{2n} \end{pmatrix},
\end{equation}

where $p_1(i,j) = 0$ for any $i, j$, except $j = n + 1, 1 \leq i \leq 2mn - n$. Let $\tilde{R}$ be the unipotent subgroup consisting of elements of the following form:

\begin{equation}
\begin{pmatrix} I_{2n} & q_1 & 0 \\ 0 & I_{4mn-2n} & q_1^* \\ 0 & 0 & I_{2n} \end{pmatrix},
\end{equation}

where $q_1(i,j) = 0$ for any $i, j$, except $i = n, 1 \leq j \leq 2mn - n$.

Write $W_3 = \tilde{C}\tilde{W}_3$, with $\tilde{C} \cap \tilde{W}_3 = \{1\}$. Consider the quadruple $(\tilde{W}_3, \psi_{W_3}, \tilde{R}, \tilde{C})$. It is easy to see that this quadruple satisfies all the conditions of Lemma 2.3 of [JL13b]. Hence, by Lemma 2.3 of [JL13b], the integral (4.19) is non-vanishing if and only if the following integral is non-vanishing:

\begin{equation}
(4.21) \\
\int_{[W_4]} \tilde{\varphi}(w\omega_2\omega_1 g)\psi^{-1}_{W_4}(w)dw,
\end{equation}

where $W_4 = \tilde{R}\tilde{W}_3$, for $w = rw', r \in \tilde{R}, w \in \tilde{W}_3$, $\psi_{W_4}(rw') = \psi_{W_3}(w')$.

Let $R_1^2 = \prod_{j=1}^{2} X_{\alpha_j^i}$, for $1 \leq i \leq n$, with $\alpha_j^i = e_i + e_{2n-i+j}$, and $R_1^2 = \prod_{j=1}^{2n-i} X_{\alpha_j^i}$, for $n + 1 \leq i \leq 2n - 1$, with $\alpha_j^i = e_i + e_{i+j}$. Let $C_1^2 = \prod_{j=1}^{2n-i} X_{\beta_j^i}$, for $1 \leq i \leq n$, with $\beta_j^i = -e_{2n-i+j} - e_{i+1}$, and $C_1^2 = \prod_{j=1}^{2n-i} X_{\beta_j^i}$, for $n + 1 \leq i \leq 2n - 1$, with $\beta_j^i = -e_{i+j} - e_{i+1}$.

For $n + 1 \leq i \leq 2n - 1$, let $R_3^2$ be the unipotent subgroup consisting of elements of the following form:

\begin{equation}
\begin{pmatrix} I_{2n} & q_1 & 0 \\ 0 & I_{4mn-2n} & q_1^* \\ 0 & 0 & I_{2n} \end{pmatrix},
\end{equation}
where \( q_1(k, j) = 0 \) for any \( k, j \), except \( k = i, 1 \leq j \leq 4mn - 2n \). For \( n + 1 \leq i \leq 2n - 1 \), let \( C_i^3 \) be the unipotent subgroup consisting of elements of the following form:

\[
\begin{pmatrix}
I_{2n} & 0 & 0 \\
p_1 & I_{4mn-2n} & 0 \\
0 & p_1^t & I_{2n}
\end{pmatrix},
\]

where \( p_1(k, j) = 0 \) for any \( k, j \), except \( j = i + 1, 1 \leq k \leq 4mn - 2n \).

Write \( W_4 = \prod_{i=1}^{2n-1} C_i^2 \prod_{j=n+1}^{2n-1} C_j^3 \tilde{W}_4 \), with \( \prod_{i=1}^{2n-1} C_i^2 \prod_{j=n+1}^{2n-1} C_j^3 \cap \tilde{W}_4 = \{1\} \).

Then, we apply Lemma 2.3 of [JL13b] to the pairs

\( (R_1^2, C_1^2), (R_2^2, C_2^2), \ldots, (R_n^2, C_n^2) \).

For example, for \( (R_s^2, C_s^2), 1 \leq s \leq n \), we need to consider the following sequence of quadruples:

\[
\prod_{i=1}^{s-1} R_i^2 \prod_{j=n+1}^{2n-1} C_j^3 \tilde{W}_4 \prod_{t=s+1}^{2n-1} C_t^2 \prod_{l=2}^{s} X_{\beta_l^1}, \psi W_4, X_{\beta_l^1}, X_{\alpha_l^1},
\]

\[
\prod_{i=1}^{s-1} R_i^2 X_{\alpha_i^1} \prod_{j=n+1}^{2n-1} C_j^3 \tilde{W}_4 \prod_{t=s+1}^{2n-1} C_t^2 \prod_{l=3}^{s} X_{\beta_l^1}, \psi W_4, X_{\beta_2^1}, X_{\alpha_2^1},
\]

\[
\prod_{i=1}^{s-1} R_i^2 X_{\alpha_i^1} \prod_{k=1}^{s-1} X_{\alpha_k^1} \prod_{j=n+1}^{2n-1} C_j^3 \tilde{W}_4 \prod_{t=s+1}^{2n-1} C_t^2 \prod_{l=s+1}^{2n-1} X_{\beta_s^1}, \psi W_4, X_{\beta_s^1}, X_{\alpha_s^1}.
\]

After applying Lemma 2.3 of [JL13b] to all the pairs

\( (R_1^2, C_1^2), (R_2^2, C_2^2), \ldots, (R_n^2, C_n^2) \),

we get that the integral \( (4.21) \) is non-vanishing if and only if the following integral is non-vanishing:

\[
(4.22) \quad \int_{\prod_{i=1}^{n+1} R_i^2 \tilde{W}_4 \prod_{i=n+1}^{2n-1} C_i^2 C_i^3} \tilde{\varphi}(rw\omega r_2 \omega_1 g) \psi^{-1}(w) dc dw dr,
\]

Next, we apply Lemma 2.3 of [JL13b] to the pairs

\( (R_{n+1}^2, C_{n+1}^2), (R_{n+1}^3, C_{n+1}^3); \ldots; (R_{2n-1}^2, C_{2n-1}^2), (R_{2n-1}^3, C_{2n-1}^3) \).

Note that before applying Lemma 2.3 of [JL13b] to each pair \( (R_s^2, C_s^2), n + 1 \leq s \leq 2n - 1 \), we need to take the Fourier expansion along the one-dimensional root subgroup \( X_{e_s + e_s} \), as in the proof of Theorem 2.1.

For example, for \( s = n + 1 \), we first take the Fourier expansion of the integral in \( (4.22) \) along the one-dimensional root subgroup \( X_{e_s + e_s} \).
Under the action of $GL_1$, we get two kinds of Fourier coefficients corresponding to the two orbits of the dual of $[X_{e_s+e_t}]$: the trivial one and the non-trivial one. For any Fourier coefficient attached to the non-trivial orbit, we can see that there is an inner integral $\varphi([2n+2]_{4n+2}, \{z\}_1)$, which is identically zero by (1) in the proof of Theorem 4.4. Therefore, only the Fourier coefficient attached to the trivial orbit, which actually equals to the integral in (1.22), survives.

After applying Lemma 2.3 of [1133] to all the pairs

$$(R_1^{2n+1}, C_{n+1})_1, (R_2^{3n+1}, C_{n+1})_1; \ldots; (R_1^{2n-1}, C_{2n-1})_1, (R_2^{3n-1}, C_{2n-1})_1,$$

the integral (1.22) is non-vanishing if and only if the following integral is non-vanishing:

$$\int_{[W_{1n}]} \varphi (r x w \omega_2 \omega_1 g) \psi^{-1}_W (w) \, dw \, dx \, dr,$$

Note that elements in $\prod_{i=1}^{2n-1} R_i \prod_{t=n+1}^{2n-1} R_t^{3} X_{e_t+e_t} \tilde{W}_4$ have the following form:

$$w = \begin{pmatrix}
Z & q_1 & q_2 \\
0 & I_{4mn-2n} & q_1^* \\
0 & 0 & Z^*
\end{pmatrix},$$

where $Z$ is in the standard maximal unipotent subgroup of $GL_{2n}$; the last row of $q_1$ is zero. And $\psi_W (\begin{pmatrix}
Z & 0 & 0 \\
0 & I_{4mn-2n} & 0 \\
0 & 0 & Z^*
\end{pmatrix}) = \psi (\sum_{i=1}^{n-1} Z_{i,i+1} - \sum_{j=n}^{2n-1} Z_{i,i+1}).$

Clearly the integral (4.23) is non-vanishing if and only if the following integral is non-vanishing:

$$\int_{[W_{1n}]} \varphi (r x w \omega_2 \omega_1 g) \psi_{W_4}^{-1} (w) \, dw \, dx \, dr,$$

where $\psi_{W_4} (\begin{pmatrix}
Z & 0 & 0 \\
0 & I_{4mn-2n} & 0 \\
0 & 0 & Z^*
\end{pmatrix}) = \psi (\sum_{i=1}^{2n-1} Z_{i,i+1}).$ Note that the integral (4.25) is exactly $\varphi^{\psi_{N_12n-1}}$, using notation in Lemma 2.5. On the other hand, we know that by Lemma 2.5 $\varphi^{\psi_{N_12n-1}} = \varphi^{\psi_{N_12n}}$. Note that Lemma 2.5 also applies to metaplectic groups.

Therefore, the integral in (4.25) becomes

$$\int_{[U]} \varphi (w \omega_2 \omega_1 g) \psi_{U}^{-1} (w) \, du,$$
where any element in $\tilde{U}$ has the following form:
\[
u = \nu(Z, q_1, q_2) = \begin{pmatrix} Z & q_1 & q_2 \\ 0 & I_{4mn-2n} & q_1^* \\ 0 & 0 & Z^* \end{pmatrix},
\]
where $Z$ is in the standard maximal unipotent subgroup of $GL_{2n}$, $q_1 \in M(2n) \times (4m-4)n$, $q_2 \in M(2n) \times (2n)$, such that $q_2^t v_{2n} - v_{2n} q_2 = 0$, where $v_{2n}$ is a matrix only with ones on the second diagonal. $\psi_{\tilde{U}}(u) = \psi(\sum_{i=1}^{2n-1} u_{i,i+1})$.

Hence, the integral in (4.26) can be written as
\[
(4.27) \quad \int_{u(Z,0,0)} \tilde{\varphi}(u \omega_2 \omega_1 g) \tilde{P}_{2n} \psi_{\tilde{U}}(u)^{-1} du,
\]
where $\tilde{\varphi}_{P_{2n}}$ is the constant term of $\tilde{\varphi}$ along the pre-image of the parabolic subgroup $P_{2n} = M_{2n} U_{2n}$ of $Sp_{4mn+2n}$ with Levi isomorphic to $GL_{2n} \times Sp_{(4m-2)n}$.

By the similar calculation as in the proof of Lemma [2.3], or the calculation at the end of Theorem 2.1 of [GJS12], there is an automorphic function
\[
f \in A(N_{2n}(\mathbb{A}) \tilde{M}_{2n}(F) \backslash \tilde{S}_{p_{4mn+2n}}(\mathbb{A}))_{\tau}\ll m \otimes \tilde{\varphi}_{4(m-1)n+2n}^{-1};
\]
such that
\[
\tilde{\varphi}(g) \tilde{P}_{2n} = f(g), \forall g \in \tilde{S}_{p_{4mn+2n}}(\mathbb{A}).
\]

Therefore, the integral (4.27) is the Whittaker Fourier coefficient of an element in $\tau$, hence not identically zero. This completes the proof of Part (1).

**Proof of Part (2).** By definition of Fourier-Jacobi coefficients ((3.14) of [GRS11]), for $\phi_1 \in \mathcal{S}(A^{2mn-n})$, $\phi_2 = \phi_2 \otimes \phi_2$, $\phi_2 \in \mathcal{S}(A^n)$, $\phi_2 \in \mathcal{S}(A^{2m+2n-2n})$, we need to compute the composition of two Fourier-Jacobi coefficients $F J_{\psi_{n-1}}^{\phi_1}$ and $F J_{\psi_{n-1}}^{\phi_2}$:

\[
(4.28) \quad F J_{\psi_{n-1}}^{\phi_1} \circ F J_{\psi_{n-1}}^{\phi_2} (\tilde{\varepsilon})(g)
\]

\[
= \int_{[\tilde{V}([2n]_{14mn-2n}),1]} \int_{[\tilde{V}([2n]_{14mn}),1]} \tilde{\varphi}(uv g) \theta_{\psi_{n-1},\phi_1}^{2mn,\phi_2}(l_2(u) v g) \psi_{14mn-2n,-1}^{-1}(u) du
\]

\[
\theta_{\psi_{n-1},\phi_1}^{2mn-n,\phi_1}(l_1(v) g) \psi_{14mn-2n,-1}^{-1}(v) dv,
\]
where $\tilde{\varphi} \in \tilde{\mathcal{E}}_{\tau,\tilde{\varphi}_{4(m-1)n+2n}}$, $g \in \tilde{S}_{p_{4mn+2n}}(\mathbb{A})$, the theta series are defined in Section 1.2 [GRS11]. $V_{([2n]_{14mn-2n}),1}$ and $V_{([2n]_{14mn}),1}$ and $\psi_{([2n]_{14mn-2n}),-1}$ are defined in Section 2 of [LL13]. $V_{([2n]_{14mn-2n}),1}$ and $V_{([2n]_{14mn}),1}$ are as $N_n$ in (3.14) of [GRS11]. Explicitly, $V_{([2n]_{14mn}),1}$ is
the unipotent radical of the parabolic subgroup $P_{4mn+2n}$ of $Sp_{4mn+2n}$ with Levi subgroup isomorphic to $GL^n_1 \times Sp_{4mn}$. $V_{[(2n)1^{4mn-2n}],1}$ is the unipotent radical of the parabolic subgroup $P_{4mn}$ of $Sp_{4mn}$, with Levi subgroup isomorphic to $GL^n_1 \times Sp_{4mn-2n}$. Note that $Sp_{4mn}$ is embedded into $Sp_{4mn+2n}$ via the map $g \mapsto \text{diag}(I_n, g, I_n)$, and we identify it with the image. Then for $u \in V_{[(2n)1^{4mn}],1}$, $\psi_{[(2n)1^{4mn}],1}(u) = \psi(\sum_{i=1}^{n-1} u_{i,i+1})$, $l_2(u) = \prod_{i=1}^{2mn} X_i(u_{n,n+i})$, with $\alpha_i = e_n - e_{n+i}$. And for $v \in V_{[(2n)1^{4mn-2n}],1}$, $\psi_{[(2n)1^{4mn-2n}],1-1}(v) = \psi(\sum_{i=1}^{n-1} v_{n+i,n+i+1})$, $l_1(v) = \prod_{i=1}^{2mn-n} X_{\beta_i}(v_{2n,2n+i})$, with $\beta_i = e_{2n} - e_{2n+i}$.

First, we want to unfold the theta series $\theta_{\psi_{-1}}^{2mn,\phi_2}(l_2(u)vg)$. Write $l_2(u)$ as $l_2(u) = (q_1, q_2, q_3; z)$, where $q_1, q_3 \in A^n, q_2 \in A^{4mn-2n}$, $z \in A$. Then

\begin{align}
\theta_{\psi_{-1}}^{2mn,\phi_2}(l_2(u)vg) &= \sum_{\xi \in F^{2mn}} \omega_{\psi_{-1}}^{2mn}(l_2(u)vg)\phi_2(\xi) \\
&= \sum_{\xi_1 \in F^n, \xi_2 \in F^{2mn-n}} \omega_{\psi_{-1}}^{2mn}((q_1, q_2, q_3; z)vg)\phi_2(\xi_1, \xi_2) \\
&= \sum_{\xi_1 \in F^n, \xi_2 \in F^{2mn-n}} \omega_{\psi_{-1}}^{2mn}((\xi_1, 0, 0; 0)(q_1, q_2, q_3; z)vg)\phi_2(0, \xi_2) \\
&= \sum_{\xi_1 \in F^n, \xi_2 \in F^{2mn-n}} \omega_{\psi_{-1}}^{2mn}((q_1 + \xi_1, q_2, q_3; z + \xi_1)q_n q_3')vg)\phi_2(0, \xi_2) \\
&= \sum_{\xi_1 \in F^n, \xi_2 \in F^{2mn-n}} \omega_{\psi_{-1}}^{2mn}((0, q_2, q_3; z + \tilde{\xi}_1)(q_1 + \xi_1, 0, 0; 0)vg)\phi_2(0, \xi_2) \\
&= \sum_{\xi_1 \in F^n, \xi_2 \in F^{2mn-n}} \omega_{\psi_{-1}}^{2mn}((0, \tilde{q}_2, q_3; z + \tilde{\xi}_1)vg(q_1 + \xi_1, 0, 0; 0))\phi_2(0, \xi_2)
\end{align}

where $\nu_n$ is the matrix only with 1’s on the second diagonal, $\tilde{\nu}_n = 2\xi_1 \nu_n q_3' + q_3 \nu_n q_1', \tilde{\xi}_1 = 2\xi_1 \nu_n q_3' + q_3 \nu_n q_1'$, $\tilde{q}_2 = q_2 + (q_1 + \xi_1)(n)\nu, (q_1 + \xi_1)(n)$ is the $n$-th coordinate of the vector $q_1 + \xi_1$. Note that $(q_1 + \xi_1, 0, 0; 0)$ commutes with $g$.

Plugging (4.29) into the integral in (4.28), collapsing the summation over $\xi_1$ with the integration over $q_1$, changing variables for $q_2$ and $z$, we will have
\[ FJ_{\psi^{-1}}^\phi_1 \circ FJ_{\psi^{-1}}^\phi_2 \left( \bar{\xi}(g) \right) = \int_{[V_{(2n)}14mn-2n],1} \int_{[V_{(2n)}14mn],1} \int_{A^n} \tilde{\varphi}(uvg\hat{q}_1) \]

(4.30)

\[
\sum_{\xi_2 \in F^{2mn-n}} \omega_{\psi^{-1}}^{2mn} \left( (0, q_2, q_3; z) vg(q_1, 0, 0; 0) \right) \phi_2(0, \xi_2)
\]

(4.31)

where \( \hat{q}_1 = \prod_{i=1}^n X_{\alpha_i} (q_1(n, n+i)) \), with \( \alpha_i = e_n - e_{n+i} \); \( V_{\xi}^{(2n)}14mn,1 \) is a subgroup of \( V_{\xi}^{(2n)}14mn,1 \) consisting of elements \( u \) with \( u_{n,n+i} = 0 \), for \( 1 \leq i \leq n \); \( l_2(u) = (0, q_2, q_3; z) \). Note that, \( V_{\xi}^{(2n)}14mn,1 \) is actually \( YV_{\xi}^{(2n)}14mn,2 \), where \( Y \) is defined in (2.5) of [JL13b] corresponding to the partition \( [(2n)14mn] \), and \( V_{\xi}^{(2n)}14mn,2 \) is defined Section 2 of [JL13b].

For short, we will write \( q_1 \) for \( (q_1, 0, 0; 0) \).

By Formula (1.4) [GRSII],

\[
\sum_{\xi_2 \in F^{2mn-n}} \omega_{\psi^{-1}}^{2mn} \left( (0, q_2, q_3; z) vg(q_1, 0, 0; 0) \right) \phi_2(0, \xi_2)
\]

(4.32)

since we assumed that \( \phi_2 = \phi_{21} \circ \phi_{22} \). Let \( l_3(u) = (q_2, z) \), for \( u \in V_{\xi}^{(2n)}14mn,1 \). Then, the integral in (4.30) becomes

\[
\int_{[V_{(2n)}14mn-2n],1} \int_{[V_{(2n)}14mn],1} \int_{A^n} \tilde{\varphi}(uvg\hat{q}_1) \phi_{21}(q_1) dq_1
\]

where \( \tilde{\varphi}'(uvg) = \int_{A^n} \tilde{\varphi}(uvg\hat{q}_1) \phi_{21}(q_1) dq_1 \). Note that we have that \( \tilde{\varphi}' \in \tilde{E}_{r, \bar{m}(m-1)n+2n} \).
Let $\omega_1$ be the Weyl element of $Sp_{4mn+2n}$ in (4.4). Conjugating cross by $\omega_1$, the integral in (4.32) becomes

$$\int_{[W'_1]} \tilde{\varphi}'(w\omega_1 g)\theta_{\psi^{-1}}^{2mn-n,\phi_{22}}(l_4(w)g)\theta_{\psi_1}^{2mn-n,\phi_1}(l_5(w)g)\psi^{-1}_{W'_1}(w)dw,$$

where $W'_1 = \omega_1 V_{[(2n)1^{4mn-2n}]}^1 V_{[(2n)1^{4mm}]}^1 \omega_1^{-1}$, its elements have the form as in (4.6), except that there is no requirement that $q_1(i,j) = 0$, for $i = 2n - 1, 2n, 1 \leq j \leq 2mn - n$. For $w = \left( \begin{array}{ccc} Z & q_1 & q_2 \\ I_{4mn-2n} & q_1^* & Z^* \end{array} \right)$ as in (4.6), $l_4(w) = (q_1(2n-1), q_2(2n-1, 2)), l_5(w) = (q_1(2n), q_2(2n, 1)), q_1(2n-1), q_1(2n)$ are the $(2n-1)$-th, $(2n)$-th rows of $q_1$, respectively. And $\psi_{W'_1}(w) = \psi(\sum_{i=1}^{2n-2} w_{i,i+2})$, with notation as in (4.7).

Next, we repeat the steps from (4.8) to (4.14), and use Lemma 2.5 of [JL13b] whenever Lemma 2.3 of [JL13b] is used. We will get that the integral in (4.33) becomes

$$\int_{[W''_1]} \tilde{\varphi}''(w\omega_1 g)\theta_{\psi^{-1}}^{2mn-n,\phi_{22}}(l_4(w)g)\theta_{\psi_1}^{2mn-n,\phi_1}(l_5(w)g)\psi^{-1}_{W''_1}(w)dw,$$

where $W''_1$ is unipotent radical of the parabolic subgroup $P_{2n}^{4mn+2n}$ of $Sp_{4mn+2n}$ with Levi subgroup isomorphic to $GL_2^m \times Sp_{4mn-2n}$. $\tilde{\varphi}'' \in \tilde{\mathcal{S}}_{\tilde{\epsilon}(4(m-1)n+2n)}$. And for $w \in W''_1$, $\psi_{W''_1}(w) = \psi(\sum_{i=1}^{2n-2} w_{i,i+2})$.

Next, we want to unfold the two theta series as before. To do this, we need to use certain property of theta series as in (5.9) [GJS12]:

$$\theta_{\psi^{-1}}^{2mn-n,\phi_{22}}((x_1, y_1; z_1)g)\theta_{\psi_1}^{2mn-n,\phi_1}((x_2, y_2; z_1)g)$$

$$= \theta_{\psi_1}^{4mn-2n,\phi_{22} \otimes \phi_1}((x_1, x_2, y_2, -y_1; z_2 - z_1)\tilde{g}),$$

where for $w \in W'_1$, we write $l_4(w) = (x_1, y_1; z_1), l_5(w) = (x_2, y_2; z_2), x_1, y_1, x_2, y_2 \in A^{2mn-n},$ and

$$\tilde{g} = \left( \begin{array}{cccc} A & B & -C \\ A & B & C \\ -C & D \\ D \end{array} \right),$$

if we write $g = \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \in SL_{8mn-4n}$.
Then,

\[(4.36)\]
\[(x_1, x_2, y_2, -y_1)\gamma = (x_1 + x_2, y_1 + y_2, \frac{1}{2}(x_1 - x_2), -\frac{1}{2}(y_1 - y_2)), \]
\[\hat{g} := \gamma^{-1}\tilde{g}\gamma = \left(\begin{array}{c} g \\ g^* \\ 1 \end{array}\right).\]

Therefore, by (4.36), the right hand side of (4.35) becomes:

\[(4.37)\]
\[\theta_\psi^{4mn-2n,\phi_3}((x_1, x_2, y_2, -y_1; z_2 - z_1)\hat{g}) = \theta_\psi^{4mn-2n,\phi_3'}((x_1 + x_2, y_1 + y_2, \frac{1}{2}(x_1 - x_2), -\frac{1}{2}(y_1 - y_2); z_2 - z_1)\hat{g}),\]

where \(\phi_3' = \omega_\psi^{4mn-2n}(\gamma^{-1})\phi_3\).

Let \(A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\), and \(\epsilon = \text{diag}(A, \ldots, A; I_{(4m-4)n}; A^*, \ldots, A^*)\), as in (2.31) of [GJS12]. Conjugating cross the integral in (4.34) by \(\epsilon\), it becomes:

\[(4.38)\]
\[\int_{[W_2']} \tilde{\varphi}''(w\epsilon_\omega_1 g)\theta_\psi^{4mn-2n,\phi_3'}(l_0(w)\hat{g})\psi_{W_2}^{-1}(w)dw \]
\[= \int_{[W_2']} \tilde{\varphi}''(wg\epsilon_\omega_1)\theta_\psi^{4mn-2n,\phi_3'}(l_0(w)\hat{g})\psi_{W_2}^{-1}(w)dw \]
\[= \int_{[W_2']} \tilde{\varphi}''(wg)\theta_\psi^{4mn-2n,\phi_3'}(l_0(w)\hat{g})\psi_{W_2}^{-1}(w)dw,\]

where \(W_2' = \epsilon W_1''\epsilon^{-1}\), and elements in \(W_2'\) are as in (4.16), except that there is no requirement that \(q_1(i, j) = 0\), for \(i = 2n - 1, 2n, 1 \leq j \leq 2mn - n\), still \(\tilde{\varphi}'' \in \tilde{E}_{r,\tilde{\sigma}(4m-1)n+2n}\). Note that \(g\) commutes with \(\epsilon\omega_1\). And

\[\psi_{W_2'}(w) = \psi(\sum_{i=1}^{2n-2} w_{i,i+2} + w_{2n-1,4mn+1}).\]

For \(w = \begin{pmatrix} Z & q_1 \\ I_{4mn-2n} & q_2 \end{pmatrix}\)

as in (4.16), for \(i = 2n - 1, 2n\), write the \(i\)-th row of \(q_1\) as \(q_1(i) = (x_i, y_i)\), then

\[l_0(w) = (x_{2n}, y_{2n}, \frac{1}{2}x_{2n-1}, -\frac{1}{2}y_{2n-1}, -q_2(2n - 1, 1)).\]
Then, we unfold the theta series \( \theta_{\psi}^{4mn-2n,\phi_3}(l_6(w)\hat{g}) \) as in (4.29), the integral in (4.38) becomes:

\[
\int_{[W'_3]} \int_{A^{4mn-2n}} \tilde{\varphi}'''(wg\xi)\phi_3(\xi)d\xi \psi^{-1}_{W'_2}(w)dw
\]

(4.39)

\[
= \int_{[W'_3]} \tilde{\varphi}^{(4)}(wg)\psi^{-1}_{W'_3}(w)dw
\]

where \( \xi = \prod_{i=1}^{2mn-n} X_{\alpha_i}(\xi(i)) \prod_{j=1}^{2mn-n} X_{\beta_j}(\xi(2mn - n + j)) \), with \( \alpha_i = e_{2n} - e_{2n+i} \), \( \beta_j = e_{2n} + e_{2n+m+n-j+1} \), and \( \tilde{\varphi}'\varphi^{(4)}(wg) = \int_{A^{4mn-2n}} \tilde{\varphi}'''(wg\xi)\phi_3(\xi) \), still \( \tilde{\varphi}'\varphi^{(4)} \in \widetilde{E}_{\tau,\tilde{\sigma}}^{4(m-1)n+2n} \). And \( W'_3 \) is the subgroup of \( W'_2 \) consisting of elements \( w = \begin{pmatrix} Z & q_1 & q_2 \\ I_{4mn-2n} & q_1^* & Z^* \end{pmatrix} \), with \( q_1(2n) = 0 \).

Conjugate cross the integral in (4.39) by the Weyl elements \( \omega_2 \) in (4.18), it becomes:

\[
\int_{[W_4]} \tilde{\varphi}^{(4)}(w\omega_2g)\psi^{-1}_{W_4}(w)dw
\]

(4.40)

\[
= \int_{[W_4]} \tilde{\varphi}^{(4)}(wg\omega_2)\psi^{-1}_{W_4}(w)dw
\]

\[
= \int_{[W_4]} \tilde{\varphi}^{(5)}(wg)\psi^{-1}_{W_4}(w)dw,
\]

where \( W_4, \psi_{W_4} \) are exactly as in (4.21).

Now, we repeat the steps from (4.21) to (4.23), and use Lemma 2.5 of [JL13b] whenever Lemma 2.3 of [JL13b] is used. Then, we get that the integral in (4.40) becomes:

(4.41)

\[
\int_{[W_5]} \tilde{\varphi}^{(5)}(wg)\psi^{-1}_{W_5}(w)dw,
\]

where \( W_5 = \prod_{i=1}^{n-1} R_i^2 \prod_{i=n+1}^{2n-1} R_i^3 X_{\epsilon_i+\epsilon_i} \tilde{\varphi} \) as in (4.23). And given \( w = \begin{pmatrix} Z & q_1 & q_2 \\ I_{4mn-2n} & q_1^* & Z^* \end{pmatrix} \) as in (4.24), \( \psi_{W_5}(w) = \psi(\sum_{i=1}^{n-1} Z_{i,i+1} - \sum_{j=n}^{2n-1} Z_{i,i+1}) \).
Let \( t = \text{diag}(I_{n-1}, -I_{n+1}; I_{4mn-2n}; -I_{n+1}, I_{n-1}) \in Sp_{4mn+2n}(F) \). Since \( \tilde{\varphi}^{(5)} \) is automorphic, the integral in (4.41) becomes:

\[
\int_{[W_5]} \tilde{\varphi}^{(5)}(twg)\psi_{W_5}^{-1}(w)dw
\]

(4.42)

\[
= \int_{[W_5]} \tilde{\varphi}^{(5)}(wg)\psi_{W_5}'^{-1}(w)dw,
\]

after changing variable \( w \mapsto t^{-1}w \), and \( \psi_{W_5}'(w) = \psi(\sum_{i=1}^{2n-1} Z_{i,i+1}) \), for

\[
w = \begin{pmatrix} Z & q_1 & q_2 \\ 0 & I_{4mn-2n} & q_1^* \\ 0 & 0 & Z^* \end{pmatrix} \in W_5.
\]

Note that the integral on the right hand side of the identity in (4.42) is exactly \( \tilde{\varphi}^{\psi N_{12}n+1} \), using notation in Lemma 2.5. On the other hand, we know that by Lemma 2.5, \( \tilde{\varphi}^{\psi N_{12}n+1} = \tilde{\varphi}^{\psi N_{12}2n} \). Note that Lemma 2.5 also applies to metaplectic groups.

Therefore, the integral in (4.42) becomes

\[
\int_{[U]} \tilde{\varphi}^{(5)}(ug)\psi_{U}^{-1}(u)du,
\]

where \( \tilde{U} \) and \( \psi_{\tilde{U}} \) are exactly as in (4.26).

Now, it follows easily from the end of the proof of Part (1) that as a function of \( g \in \tilde{Sp}_{4mn-2n} \), the integral in (4.43) gives a section in \( \tilde{\sigma}_{4(m-1)n+2n} \). Since starting from the integral in (4.28), we always get equalities, \( F.J_{\psi_{n-1}}^{\phi_{1}} \circ F.J_{\psi_{n-1}}^{\phi_{2}} (\tilde{\xi}) \in \tilde{\sigma}_{4(m-1)n+2n} \). Therefore,

\[
D_{2n,\psi^{-1}}^{4mn} \circ D_{2n,\psi_{1}}^{4mn+2n} (\tilde{\xi}, \tilde{\sigma}_{4(m-1)n+2n}) \subset \tilde{\sigma}_{4(m-1)n+2n}.
\]

On the other hand, by Part (1),

\[
D_{2n,\psi^{-1}}^{4mn} \circ D_{2n,\psi_{1}}^{4mn+2n} (\tilde{\xi}, \tilde{\sigma}_{4(m-1)n+2n}) \neq 0.
\]

Since \( \tilde{\sigma}_{4(m-1)n+2n} \) is irreducible, we have that

\[
D_{2n,\psi^{-1}}^{4mn} \circ D_{2n,\psi_{1}}^{4mn+2n} (\tilde{\xi}, \tilde{\sigma}_{4(m-1)n+2n}) = \tilde{\sigma}_{4(m-1)n+2n}.
\]

This finishes the proof of Part (2), and completes the proof of the theorem.

\[\square\]

**Remark 4.3.** Note that in the proof of Part (2), we could easily get a similar identity as in Theorem 5.1 [GJS12], but for simplicity, we did not write it down explicitly.

Theorem 4.2 easily implies the following result.
Theorem 4.4. $\mathfrak{p} = [(2n)^{2m}(2n_1)^{s_1}(2n_2)^{s_2} \cdots (2n_k)^{s_k}]$ is a maximal partition providing non-vanishing Fourier coefficients for $\tilde{E}_{r,\tilde{\sigma}_{4(m-1)n+2n}}$.

**Proof.** By Theorem 4.2

$$[(2n)^{4mn}] \circ [(2n)^{14mn-2n}] \circ [(2n)^{2m-2}(2n_1)^{s_1}(2n_2)^{s_2} \cdots (2n_k)^{s_k}]$$

is a composite partition providing non-vanishing Fourier coefficients for $\tilde{E}_{r,\tilde{\sigma}_{4(m-1)n+2n}}$. By Lemma 2.6 of [GRS03] or Lemma 3.1 of [JL13b], $[(2n)^{2m}(2n_1)^{s_1}(2n_2)^{s_2} \cdots (2n_k)^{s_k}]$ is a partition providing non-vanishing Fourier coefficients for $\tilde{E}_{r,\tilde{\sigma}_{4(m-1)n+2n}}$.

Since by Theorem 4.2 $\mathcal{D}_{2n,v^1}^{4mn+2n}(\tilde{E}_{r,\tilde{\sigma}_{4(m-1)n+2n}}) = \tilde{\sigma}_{4(m-1)n+2n}$, and $p(\tilde{\sigma}_{4(m-1)n+2n}) = [(2n)^{2m}(2n_1)^{s_1}(2n_2)^{s_2} \cdots (2n_k)^{s_k}]$, to show the maximality of $\mathfrak{p}$, we just have to show that

1. (1) at the step of taking $\mathcal{D}_{2n,v^1}^{4mn+2n}$, $\tilde{E}_{r,\tilde{\sigma}_{4(m-1)n+2n}}$ has no nonzero Fourier coefficients attached to the symplectic partitions $[(2l)^{14mn+2n-2l}]$ for any $l \geq n + 1$, or $[(2l+1)^{2l+2}4mn+2n-l-2]$, for any $l \geq n$;

2. (2) at the step of taking $\mathcal{D}_{2n,v^1}^{4mn+2n}$, $\mathcal{D}_{2n,v^1}^{4mn+2n}(\tilde{E}_{r,\tilde{\sigma}_{4(m-1)n+2n}})$ has no nonzero Fourier coefficients attached to the symplectic partitions $[(2l)^{14mn-2l}]$ for any $l \geq n + 1$, or $[(2l+1)^{2l+2}4mn-2l]$, for any $l \geq n$.

We will show (1) and (2) using calculations of unramified local components. Let $v$ be a finite place such that $\tilde{E}_{r,\tilde{\sigma}_{4(m-1)n+2n},v}$ is unramified. This means that both $\tau_v$ and $\tilde{\sigma}_{4(m-1)n+2n,v}$ are also unramified. Assume that $\tau_v = \times_{i=1}^n \nu^{\alpha_i} \chi_i \times \times_{i=1}^n \nu^{\alpha_i} \chi_i^{-1}$, where $\nu^{\alpha_i} = \text{det}((\cdot)^{\alpha_i}), 0 \leq \alpha_i < \frac{1}{2}$, and $\chi_i$’s are unitary unramified characters of $F_{v}^*$, for $1 \leq i \leq n$. Since $\tilde{\tau}$ lifts weakly to $\tau$ with respect to $v$, $\tilde{\tau}_v = \mu_{\psi} \times_{i=1}^n \nu^{\alpha_i} \times \times_{i=1}^n \chi_i \times 1_{\tilde{S}_{p_0}}$.

Since by the definition of the set $\mathcal{N}_{\tilde{S}_{4(m-1)n+2n}}(\tau,\psi)$, $\tilde{\sigma}_{4(m-1)n+2n}$ is an irreducible cuspidal automorphic representation of $\tilde{S}_{4(m-1)n+2n}(\tilde{A})$, which is nearly equivalent to the residual representation $\tilde{\Delta}_{(r,m-1)\otimes \tilde{\tau}}$, similarly as Lemma 3.1 of [GRS03], it is easy to see that $\tilde{E}_{r,\tilde{\sigma}_{4(m-1)n+2n},v}$ is the unique unramified component of the following induced representation

$$\text{Ind}_{\tilde{S}_{4mn+2n}(F_v^*)}^{\tilde{S}_{4mn+2n}(F_v^*)} \mu_{\psi} \otimes_{i=1}^n \nu^{\alpha_i} \chi_i \text{det}_{GL_{2m+1}} \otimes 1_{\tilde{S}_{p_0}},$$

where $P_{(2m+1)n}$ is the parabolic subgroup of $Sp_{4mn+2n}$ with Levi isomorphic to $GL_{2m+1}^n \times 1_{Sp_0}$, and $\tilde{P}_{(2m+1)n}$ is its full pre-image in $\tilde{S}_{4mn+2n}$. 


By Lemma 3.2 of [JL13c], we can easily see that (1) holds. By (4.2) of Proposition 4.1, we will prove that

\[ FJ_{\psi_{n-1}}(\text{Ind}_{P_{4m+2n}^n}^{Sp_{4mn+2n}}(F_\tau^*) \mu_\psi \otimes_{i=1}^n |\alpha_i \chi_i(det GL_{2m+1}) \otimes 1_{Sp_0}) \]

(4.45)

\[ \cong \text{Ind}_{P_{4m+2n}^n}^{Sp_{4mn}}(F_\tau^*) \mu_\psi \otimes_{i=1}^n |\alpha_i \chi_i(det GL_{2m}) \otimes 1_{Sp_0}, \]

which is actually irreducible, by results in [Jan96], and is an unramified local component of \( D_{4mn+2n}^{4mn+2n}(\widehat{\mathcal{E}}_\tau,\mathcal{S}_{4(m-1)n+2n}) \). Again, by Lemma 3.1 of [JL13c], we can easily see that (2) also holds. Therefore, we have shown that \( p = [(2n)^{2m}](2n_1)^{s_1}(2n_2)^{s_2} \cdots (2n_k)^{s_k} \) is a maximal partition providing non-vanishing Fourier coefficients for \( \widehat{\mathcal{E}}_\tau,\mathcal{S}_{4(m-1)n+2n} \), which completes the proof of this theorem.

To continue, we prove that \( D_{2n,\psi_1}^{4mn+2n}(\widehat{\mathcal{E}}_\tau,\mathcal{S}_{4(m-1)n+2n}) \) is a cuspidal representation, every component of which is inside \( \mathcal{N}_{Sp_4}^{4mn}(\tau,\psi) \).

**Theorem 4.5.** \( D_{2n,\psi_1}^{4mn+2n}(\widehat{\mathcal{E}}_\tau,\mathcal{S}_{4(m-1)n+2n}) \subset \mathcal{N}_{Sp_4}^{4mn}(\tau,\psi) \).

**Proof.** We will prove that

(1) \( D_{2n,\psi_1}^{4mn+2n}(\widehat{\mathcal{E}}_\tau,\mathcal{S}_{4(m-1)n+2n}) \) is a cuspidal representation;

(2) every irreducible component of \( D_{2n,\psi_1}^{4mn+2n}(\widehat{\mathcal{E}}_\tau,\mathcal{S}_{4(m-1)n+2n}) \) is inside \( \mathcal{N}_{Sp_4}^{4mn}(\tau,\psi) \).

To prove (1), we will show that the constant terms of elements in \( D_{2n,\psi_1}^{4mn+2n}(\widehat{\mathcal{E}}_\tau,\mathcal{S}_{4(m-1)n+2n}) \) along all maximal parabolic subgroups of \( Sp_{4mn} \) are all zero.

Recall that \( P_{4mn}^{4mn} = M_{4mn}^r N_{4mn}^r \) (with \( 1 \leq r \leq 2mn \)) is the standard parabolic subgroup of \( Sp_{4mn} \) with Levi part \( M_{4mn}^r \) isomorphic to \( GL_r \times Sp_{4mn-2r} \), \( N_{4mn}^r \) is the unipotent radical. Take any \( \tilde{\xi} \in \widehat{\mathcal{E}}_\tau,\mathcal{S}_{4(m-1)n+2n} \), we will calculate the constant term of \( FJ_{\psi_{n-1}}^\phi(\tilde{\xi}) \) along \( P_{4mn}^r \), which is denoted by \( C_{N_{4mn}^r}(FJ_{\psi_{n-1}}^\phi)(\tilde{\xi}) \).

By Theorem 7.8 of [GRST11],

(4.46)

\[ C_{N_{4mn}^r}(FJ_{\psi_{n-1}}^\phi(\tilde{\xi})) = \sum_{k=0}^r \sum_{\gamma \in P_{r-k,1k}^1(F)} \int_{L(A)} \phi_1(i(\lambda)) FJ_{\psi_{n-1+k}}^\phi(\gamma \lambda \beta) d\lambda, \]

where \( N_{4mn+2n}^{4mn+2n} \) is the unipotent radical of the parabolic subgroup \( P_{r-k,1k}^1 \) of \( Sp_{4mn+2n} \) with Levi isomorphic to \( GL_{r-k} \times Sp_{4mn+2n-2r+2k} \), and it is identified with it’s image in \( Sp_{4mn+2n} \); \( P_{r-k,1k}^1 \) is a subgroup of
GL_r consisting of matrices of the form \( \begin{pmatrix} g & x \\ 0 & z \end{pmatrix} \), with \( z \in U_k \), the standard maximal unipotent subgroup of \( GL_k \); for \( a \in GL_j, j \leq 2mn + n \), \( \hat{a} = \text{diag}(a, I_{4mn + 2n - 2j}, a^*) \); \( L \) is a unipotent subgroup, consisting of matrices of the form \( \lambda = \begin{pmatrix} I_r & 0 \\ x & I_n \end{pmatrix} \), and \( i(\lambda) \) is the last row of \( x \);

\[
\beta = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}^\wedge; \phi = \phi_1 \otimes \phi_2, \text{ with } \phi_1 \in \mathcal{S}(\mathbb{A}^r), \phi_2 \in \mathcal{S}(\mathbb{A}^{2mn - r});
\]

\[
FJ_{\psi_{n-1+k}}^{\phi_2} (C_{N_{r-k}}^{4mn+2n} (\tilde{\xi})) (\hat{\gamma} \lambda \beta) := FJ_{\psi_{n-1+k}}^{\phi_2} (C_{N_{r-k}}^{4mn+2n} (\rho(\hat{\gamma} \lambda \beta) \tilde{\xi}))(I),
\]

with \( \rho(\hat{\gamma} \lambda \beta) \) denoting the right translation by \( \hat{\gamma} \lambda \beta; C_{N_{r-k}}^{4mn+2n} (\rho(\hat{\gamma} \lambda \beta) \tilde{\xi}) \) is restricted to \( \tilde{Sp}_{4mn+2n-2r+2k} (\mathbb{A}) \), then we apply the Fourier-Jacobi coefficient \( FJ_{\psi_{n-1+k}}^{\phi_2} \), taking automorphic forms on \( \tilde{Sp}_{4mn+2n-2r+2k} (\mathbb{A}) \) to \( Sp_{4mn-2r} (\mathbb{A}) \).

By the cuspidal support of \( \tilde{\xi}, C_{N_{r-k}}^{4mn+2n} (\tilde{\xi}) \) is identically zero, unless \( r = k \) or \( r - k = 2n \). When \( r = k \), the corresponding term is zero, because \( FJ_{\psi_{n-1+k}}^{\phi_2} (\tilde{\xi}) \) is zero, by Theorem 4.4. When \( r - k = 2n \), the restriction of \( C_{N_{r-k}}^{4mn+2n} (\tilde{\xi}) \) to \( \tilde{Sp}_{4mn-2n} (\mathbb{A}) \) is actually a vector in \( \tilde{\sigma}_{4(m-1)n+2n} \). Hence, \( FJ_{\psi_{n-1+k}}^{\phi_2} (C_{N_{r-k}}^{4mn+2n} (\tilde{\xi})) \) is identically zero, for \( 0 \leq k \leq r \), because \( \tilde{\sigma}_{4(m-1)n+2n} \) has no nonzero Fourier coefficients \( FJ_{\psi_{n-1}}^{\phi} \), and \( p(\tilde{\sigma}_{4(m-1)n+2n}) = [(2n)^{2m-1}(2n_1)^{s_1}(2n_2)^{s_2} \ldots (2n_k)^{s_k}] \). So, when \( r - k = 2n \), the corresponding term is also zero.

Therefore, we have shown that \( C_{N_{r-k}}^{4mn} (FJ_{\psi_{n-1}}^{\phi} (\tilde{\xi})) \) is identically zero for any \( 1 \leq r \leq 2mn \), and for any \( \tilde{\xi} \in \tilde{\mathcal{E}}_{\tau, \tilde{\sigma}_{4(m-1)n+2n}} \), which implies that \( D_{2n, \psi_{n-1}}^{4mn+2n} (\tilde{E}_{\tau, \tilde{\sigma}_{4(m-1)n+2n}}) \) is a cuspidal representation. This completes the proof of (1).

To prove (2), we need to show that for every irreducible component \( \pi \) of \( D_{2n, \psi_{n-1}}^{4mn+2n} (\tilde{E}_{\tau, \tilde{\sigma}_{4(m-1)n+2n}}) \):

(2-1) \( p(\pi) = [(2n)^{2m-1}(2n_1)^{s_1}(2n_2)^{s_2} \ldots (2n_k)^{s_k}] \);

(2-2) \( \pi \) is nearly equivalent to the residual representation \( \mathcal{E}_{\Delta(\tau, m)} \);

(2-3) \( \pi \) has a nonzero Fourier coefficient \( FJ_{\psi_{n-1}}^{\phi} \).

(2-1) is obvious by Theorem 4.4, and by Lemma 2.6 of [GRS05] or Lemma 3.1 of [JJ13b]. (2-2) follows easily from (4.45), because the induced representation on the right hand side of (4.45) is also an unramified component of \( \mathcal{E}_{\Delta(\tau, m)} \).
To show (2-3), as in the proof of Proposition 3.4 of \cite{GJS12}, we need to consider the following integral

\begin{equation}
\langle \varphi, FJ_{\psi,1}^{\sigma} (\tilde{\xi}) \rangle = \int_{[Sp_{4mn}]} \varphi(h) FJ_{\psi,1}^{\sigma} (\tilde{\xi})(h) dh,
\end{equation}

which is nonzero for some data $\varphi \in \pi$, $\tilde{\xi} \in \tilde{E}_{\tau,\tilde{\sigma}_{4(m-1)n+2n}}$, since $\pi$ is an irreducible component of $D_{2n,\psi}^{4mn+2n}(\tilde{E}_{\tau,\tilde{\sigma}_{4(m-1)n+2n}})$.

Assume that $\tilde{\xi} = \text{Res}_{s=m+1/2} \tilde{E}(\phi_s, \cdot)$, then from (4.47), we know that the following integral is also nonzero for some choice of data:

\begin{equation}
\langle \varphi, FJ_{\psi,1}^{\sigma} (\tilde{E}(\phi_s, \cdot)) \rangle = \int_{[Sp_{4mn}]} \varphi(h) FJ_{\psi,1}^{\sigma} (\tilde{E}(\phi_s, \cdot))(h) dh.
\end{equation}

Then, by the unfolding in Theorem 3.3 of \cite{GJRS11} (take $m = 2n$, $r = n$ there), the non-vanishing of the integral in (4.47) implies the non-vanishing of $FJ_{\psi,1}^{\sigma} (\pi)$.

This finishes the proof of (2), and completes the proof of the theorem. \hfill \Box

The next theorem implies that $\Psi$ is well-defined. We will prove it in the next section.

**Theorem 4.6.** $D_{2n,\psi}^{4mn}(\sigma_{4mn})$ is irreducible, and

\[ D_{2n,\psi}^{4mn}(\sigma_{4mn}) \in N_{Sp_{4(m-1)n+2n}}'(\tau, \psi), \]

for any $\sigma_{4mn} \in N_{Sp_{4mn}}(\tau, \psi)$.

Now by Theorems 4.2, 4.5, 4.6, we are able to conclude the Part (1) of Theorem 1.4

**Theorem 4.7** (Part (1) of Theorem 1.4). There is a surjective map

\[ \Psi : N_{Sp_{4mn}}(\tau, \psi) \rightarrow N_{Sp_{4(m-1)n+2n}}'(\tau, \psi) \]

\[ \sigma_{4mn} \mapsto D_{2n,\psi}^{4mn}(\sigma_{4mn}). \]

**Proof.** Theorem 4.6 implies that $\Psi$ is well-defined. Theorem 4.2 and Theorem 4.5 imply that for any $\tilde{\sigma}_{4(m-1)n+2n} \in N_{Sp_{4(m-1)n+2n}}'(\tau, \psi)$, take any irreducible component $\pi$ of $D_{2n,\psi}^{4mn+2n}(\tilde{E}_{\tau,\tilde{\sigma}_{4(m-1)n+2n}})$, which is inside
\[ N_{Sp_{4mn}}(\tau, \psi), \]
\[ \Psi(\pi) = D_{2n, \psi^{-1}}^{4mn}(\pi) \]
\[ \subseteq D_{2n, \psi^{-1}}^{4mn} \circ D_{2n, \psi^{1}}^{4mn+2n}(\tilde{E}_{\tau, \tilde{\sigma}_{4(m-1)n+2n}}) \]
\[ = \tilde{\sigma}_{4(m-1)n+2n}. \]

Theorem 4.5 also implies that \( D_{2n, \psi^{-1}}^{4mn}(\pi) \neq 0 \). Since \( \tilde{\sigma}_{4(m-1)n+2n} \) is irreducible, actually we have \( D_{2n, \psi^{-1}}^{4mn}(\pi) = \tilde{\sigma}_{4(m-1)n+2n} \).

Hence \( \Psi \) is surjective. 

\[ \square \]

5. Proof of Theorem 4.6

For any \( \sigma_{4mn} \in N_{Sp_{4mn}}(\tau, \psi) \), we know that the Eisenstein series corresponding to \[ \text{Ind}_{P_{2n}(\tilde{A})}^{Sp_{4(m+1)n}(\tilde{A})} \tau | s \otimes \sigma_{4mn} \]
has a simple pole at \( s = \frac{m+1}{2} \). Let \( E_{\tau, \sigma_{4mn}} \) be the residual representation of \( Sp_{4(m+1)n}(\tilde{A}) \) generated by the corresponding residues.

First, we have a similar result as that in Theorem 4.2.

Theorem 5.1. (1)
\[ D_{2n, \psi^{1}}^{4mn+2n} \circ D_{2n, \psi^{-1}}^{4(m+1)n}(E_{\tau, \sigma_{4mn}}) \neq 0. \]

(2)
\[ D_{2n, \psi^{1}}^{4mn+2n} \circ D_{2n, \psi^{-1}}^{4(m+1)n}(E_{\tau, \sigma_{4mn}}) = \sigma_{4mn}. \]

Proof. The proof is very similar to that of Theorem 4.2. We omit it here.

Next, we have a similar result as that in Theorem 5.8 of [GJS12].

Theorem 5.2. For any \( \sigma_{4mn} \in N_{Sp_{4mn}}(\tau, \psi) \), the automorphic representation \( D_{2n, \psi^{-1}}^{4(m+1)n}(E_{\tau, \sigma_{4mn}}) \) is square-integrable. Moreover, there is an irreducible representation \( \tilde{\sigma}_{4(m-1)+2n} \), which is a component of \[ D_{2n, \psi^{-1}}^{4mn}(\sigma_{4mn}) \subset N_{Sp_{4(m-1)n+2n}}(\tau, \psi), \]
such that the representation space of \( D_{2n, \psi^{-1}}^{4(m+1)n}(E_{\tau, \sigma_{4mn}}) \) has a non-trivial intersection with the representation space of \( \tilde{E}_{\tau, \tilde{\sigma}_{4(m-1)n+2n}} \).

Proof. We follow the constant term calculation in the proof of the Theorem 4.5.

Recall that \( P_{r}^{4mn+2n} = M_{r}^{4mn+2n} N_{r}^{4mn+2n} \) (with \( 1 \leq r \leq 2mn + n \)) is the standard parabolic subgroup of \( Sp_{4mn+2n} \) with Levi part \( M_{r}^{4mn+2n} \).
isomorphic to $GL_r \times Sp_{4mn+2n-2r}$, $N^{4mn+2n}_r$ is the unipotent radical, and $\tilde{P}^{4mn+2n}_r$ is the pre-image of $P^{4mn+2n}_r$ in $\tilde{Sp}_{4mn+2n}$. Take any $\xi \in \mathcal{E}_{\tau, \sigma_{4mn}}$, we will calculate the constant term of $FJ_{\psi_{n-1}^\phi}^\phi (\xi)$ along $\tilde{P}^{4mn+2n}_r$, which is denoted by $C_{N^{4mn+2n}_r} (FJ_{\psi_{n-1}^\phi}^\phi (\xi))$.

By Theorem 7.8 of [GRST1],

\begin{equation}
C_{N^{4mn+2n}_r} (FJ_{\psi_{n-1}^\phi}^\phi (\xi)) = \sum_{k=0}^r \sum_{\gamma \in P^1_{r-k,1k}(F) \setminus GL_r(F)} \int_{L(\mathbb{A})} \phi_1(i(\lambda))FJ_{\psi_{n-1}^\phi}^\phi (C_{N^{4mn+4n}_r} (\xi))(\gamma \lambda \beta) d\lambda,
\end{equation}

where $N^{4mn+4n}_{r-k}$ is the unipotent radical of the parabolic subgroup $P^{4mn+4n}_{r-k}$ of $Sp_{4mn+4n}$ with Levi isomorphic to $GL_{r-k} \times Sp_{4mn+4n-2r+2k}$; $P^1_{r-k,1k}$ is a subgroup of $GL_r$ consisting of matrices of the form \[
\begin{pmatrix}
g & x \\
0 & z
\end{pmatrix},
\]
with $z \in U_k$, the standard maximal unipotent subgroup of $GL_k$; for $a \in GL_j$, $j \leq 2mn + 2n$, $\hat{a} = \text{diag}(a, I_{4mn+4n-2j}, a^*)$; $L$ is a unipotent subgroup, consisting of matrices of the form $\lambda = \begin{pmatrix} I_r & 0 \\ x & I_n \end{pmatrix}$, and $i(\lambda)$ is the last row of $x$; $\beta = \begin{pmatrix} 0 & I_r \\ I_n & 0 \end{pmatrix}$; $\phi = \phi_1 \otimes \phi_2$, with $\phi_1 \in S(\mathbb{A}^r)$, $\phi_2 \in S(\mathbb{A}^{2mn+n-r})$;

\[FJ_{\psi_{n-1}^\phi}^\phi_{n-1+k} (C_{N^{4mn+4n}_r} (\xi))(\gamma \lambda \beta) := FJ_{\psi_{n-1}^\phi}^\phi_{n-1+k} (C_{N^{4mn+4n}_r} (\rho(\gamma \lambda \beta) \xi))(I),\]

with $\rho(\gamma \lambda \beta)$ denoting the right translation by $\gamma \lambda \beta$; $C_{N^{4mn+4n}_{r-k}} (\rho(\gamma \lambda \beta) \xi)$ is restricted to $Sp_{4mn+4n-2r+2k}(\mathbb{A})$, then we apply the Fourier-Jacobi coefficient $FJ_{\psi_{n-1}^\phi}^\phi_{n-1+k}$, taking automorphic forms on $Sp_{4mn+4n-2r+2k}(\mathbb{A})$ to $\tilde{Sp}_{4mn+2n-2r}(\mathbb{A})$.

By the cuspidal support of $\xi$, $C_{N^{4mn+4n}_{r-k}} (\xi)$ is identically zero, unless $r = k$ or $r - k = 2n$. When $r = k$, the corresponding term is zero, because $FJ_{\psi_{n-1}^\phi}^\phi_{n-1+k} (\xi)$ is zero, by Theorem 4.4. When $r - k = 2n$, the restriction of $C_{N^{4mn+4n}_{2n}} (\xi)$ to $Sp_{4mn}(\mathbb{A})$ is actually a vector inside $\sigma_{4mn}$. Hence, $FJ_{\psi_{n-1}^\phi}^\phi_{n-1+k} (C_{N^{4mn+4n}_{2n}} (\xi))$ is not zero for $k = 0$, and is identically zero for $1 \leq k \leq r$, because $\sigma_{4mn}$ has a nonzero Fourier coefficient $FJ_{\psi_{n-1}^\phi}^\phi_{n-1+k}$, and $\sum (\sigma_{4mn}) = [(2n)^{2m-1}(2n_1)^{s_1}(2n_2)^{s_2} \ldots (2n_k)^{s_k}]$. 


Therefore,

\[ C_{N_{2n}^{4m+2n}}(FJ_{\psi_{-1}}^{\sigma} (\xi)) \]

(5.2)

\[ = \int_{L(\mathbb{A})} \phi_1(i(\lambda))FJ_{\psi_{-1}}^{\sigma_2}(C_{N_{2n}^{4m+4n}}(\xi)) (\lambda \beta) \, d\lambda. \]

By similar calculation as in the proof of Lemma 2.3 when restricting to \( GL_{2n}(\mathbb{A}) \times Sp_{4mn}(\mathbb{A}) \),

\[ C_{N_{2n}^{4m+4n}}(\xi) \in \delta_{P_{2n}^{4m+4n}}^{1/2} |\det|^{-2n+1} \tau \otimes \sigma_{4mn}. \]

As in the proof of Theorem 2.5 [GJS12], we need to calculate the automorphic exponents attached to this non-trivial constant term (for definition see I.3.3 [MW95]). We consider the action of

\[ \hat{g} = \text{diag}(g, I_{4mn-2n}, g^*) \in GL_{2n}(\mathbb{A}) \times Sp_{4mn-2n}(\mathbb{A}). \]

Since \( r = 2n \), \( \beta = \left( \begin{array}{cc} 0 & I_{2n} \\ I_n & 0 \end{array} \right) \). \( \beta \text{diag}(I_n, \hat{g}, I_n) \beta^{-1} = \text{diag}(g, I_{4mn}, g^*) =: \tilde{g} \). Then changing variables in (5.2) \( \lambda \mapsto \tilde{g} \lambda \tilde{g}^{-1} \) will give a Jacobian \(|\det(g)|^{-n} \). On the other hand, by Formula (1.4) [GRS11], the action of \( \hat{g} \) on \( \phi_1 \) gives \( \gamma_\psi(|\det(g)|) |\det(g)|^{1/2} \). Therefore, the \( \hat{g} \) acts by \( \tau(g) \) with character

\[ \delta_{P_{2n}^{4m+4n}}^{1/2} |\det(g)|^{-2n+1} |\det(g)|^{-n} \gamma_\psi(|\det(g)|) |\det(g)|^{1/2} \]

\[ = \gamma_\psi(|\det(g)|) \delta_{P_{2n}^{4m+4n}}^{1/2} |\det(g)|^{-m}. \]

Hence, by Langlands square-integrability criterion (Lemma I.4.11 [MW95]), the automorphic representation \( D_{2n,\psi}^{4(m+1)n}(E_{\tau,\sigma_{4mn}}) \) is square-integrable.

And as a representation of \( GL_{2n}(\mathbb{A}) \times Sp_{4mn-2n}(\mathbb{A}) \),

(5.3)

\[ C_{N_{2n}^{4m+4n}}(D_{2n,\psi}^{4(m+1)n}(E_{\tau,\sigma_{4mn}})) = \gamma_\psi \delta_{P_{2n}^{4m+2n}}^{1/2} |\det|^{-m} \tau \otimes D_{2n,\psi}^{4m}\sigma_{4mn}(\sigma_{4mn}). \]

By (5.3), it is easy to see that any non-cuspidal irreducible subrepresentation of \( D_{2n,\psi}^{4(m+1)n}(E_{\tau,\sigma_{4mn}}) \) must be an irreducible subrepresentation of \( \tilde{E}_{\tau,\sigma_{4(m+1)n+2n}} \), for some irreducible subrepresentation \( \sigma_{4(m+1)+2n} \)

of \( D_{2n,\psi}^{4m}\sigma_{4mn}(\sigma_{4mn}) \).

To prove \( D_{2n,\psi}^{4m}(\sigma_{4mn}) \subset N_{Sp_{4(m+1)n+2n}}(\tau, \psi) \), we need to show that for every irreducible component \( \sigma \) of \( D_{2n,\psi}^{4m}(\sigma_{4mn}) \),

1. \( \sigma \) is cuspidal;
2. \( \overline{p}(\sigma) = [(2n)^{2n-2}(2n_1)^{s_1}(2n_2)^{s_2} \cdots (2n_k)^{s_k}]; \)
3. \( \sigma \) is nearly equivalent to the residual representation \( \tilde{E}_{\Delta(\tau,m-1)\otimes \mathbb{R}} \);
(4) $\sigma$ has no nonzero Fourier coefficient $FJ_{\psi_n^4}$.

(1) follows easily from the tower property (Theorem 7.10 [GRS11]).

(2) is implied by Lemma 2.6 [GRS03] or Lemma 3.1 [JL13b]. (3) can be read out easily from the right hand side of (4.45) by (4.1) of Proposition 4.1. Note that the right hand side of (4.45) is an unramified component of $E_{\Delta(\tau,m)}$, hence unramified component of $\sigma_{4mn}$. By Theorem 5.2 of [JL13b], as a cuspidal representation, $\sigma_{4mn}$ has no nonzero Fourier coefficient with respect to character $\psi|_{(2n)^{2m-1}(2n_1)^*1(2n_2)^*2(2n_3)^*k}|_a$, where $a = \{-1,1\} \cup a'$. Now (4) follows easily from Lemma 3.1 [JL13b].

Therefore, the representation space of $D_{2n,\psi^{-1}}^{1(m+1)n} (E_{\tau,\sigma_{4mn}})$ has a nontrivial intersection with the representation space of $\bar{E}_{\tau,\bar{\sigma}_{4(m-1)n+2n}}$; for some component $\bar{\sigma}_{4(m-1)n+2n}$ of

$$\mathcal{D}_{2n,\psi^{-1}}^{1(m+1)n} (\sigma_{4mn}) \subset N_{Sp_{4(m-1)n+2n}}^\psi (\tau, \psi).$$

This completes the proof of the theorem. □

**Proof of Theorem 4.10**

For any $\sigma_{4mn} \in N_{Sp_{4n+1}} (\tau, \psi)$, by Theorem 5.1

$$D_{2n,\psi^{-1}}^{1(m+1)n} (E_{\tau,\sigma_{4mn}}) = \sigma_{4mn}.$$  \hspace{1cm} (5.4)

By Theorem 5.2, there is an irreducible representation $\bar{\sigma}_{4(m-1)n+2n}$, which is a component of $D_{2n,\psi^{-1}}^{1(m+1)n} (\sigma_{4mn}) \subset N_{Sp_{4(m-1)n+2n}}^\psi (\tau, \psi)$, such that the representation space of $D_{2n,\psi^{-1}}^{1(m+1)n} (E_{\tau,\sigma_{4mn}})$ contains an irreducible subrepresentation $\pi$ of $\bar{E}_{\tau,\bar{\sigma}_{4(m-1)n+2n}}$. Since $\sigma_{4mn}$ is irreducible, by (5.4),

$$\sigma_{4mn} = D_{2n,\psi^{-1}}^{1(m+1)n} (\pi) \subset D_{2n,\psi^{-1}}^{1(m+1)n+2n} (\bar{E}_{\tau,\bar{\sigma}_{4(m-1)n+2n}}).$$  \hspace{1cm} (5.5)

Therefore,

$$D_{2n,\psi^{-1}}^{1(m+1)n} (\sigma_{4mn}) \subset D_{2n,\psi^{-1}}^{1(m+1)n+2n} (\bar{E}_{\tau,\bar{\sigma}_{4(m-1)n+2n}}) = \bar{\sigma}_{4(m-1)n+2n},$$

by Theorem 4.2. Hence, $D_{2n,\psi^{-1}}^{1(m+1)n} (\sigma_{4mn}) = \bar{\sigma}_{4(m-1)n+2n}$, irreducible as an element in $N_{Sp_{4(m-1)n+2n}}^\psi (\tau, \psi)$.

This completes the proof of Theorem 4.10 showing that $\Psi$ is well-defined. □

**6. Proof of Part (2) of Theorem 1.4**

In this section, we will prove that $\Psi$ is injective. For this, we need to assume that for any $\bar{\sigma}_{4(m-1)n+2n} \in N_{Sp_{4(m-1)n+2n}}^\psi (\tau, \psi)$, $\bar{E}_{\tau,\bar{\sigma}_{4(m-1)n+2n}}$ is irreducible.
For any \( \sigma_{4mn} \in \mathcal{N}_{Sp_{4mn}}(\tau, \psi) \), by Theorem 4.6 \( D_{2n, \psi}^{4mn} (\sigma_{4mn}) = \tilde{\sigma}_{4(m-1)n+2n} \in \mathcal{N}'_{\tilde{Sp}_{4(m-1)n+2n}}(\tau, \psi) \), which is irreducible. To show \( \Psi \) is injective, we only need to show that \( \sigma_{4mn} \) is an irreducible subrepresentation of \( \tilde{\sigma}_{4(m-1)n+2n} \).

By (5.5), \( \sigma_{4mn} = D_{2n, \psi}^{4mn+2n}(\pi) \subset D_{2n, \psi}^{4mn+2n}(\tilde{\sigma}_{\tau, \tilde{\sigma}_{4(m-1)n+2n}}) \), where \( \pi \) is an irreducible subrepresentation of \( \tilde{\sigma}_{\tau, \tilde{\sigma}_{4(m-1)n+2n}} \). Since we assume that \( \tilde{\sigma}_{\tau, \tilde{\sigma}_{4(m-1)n+2n}} \) is irreducible, we have that \( \pi = \tilde{\sigma}_{\tau, \tilde{\sigma}_{4(m-1)n+2n}} \). Hence \( \sigma_{4mn} = D_{2n, \psi}^{4mn+2n}(\tilde{\sigma}_{\tau, \tilde{\sigma}_{4(m-1)n+2n}}, \) which means that \( \sigma_{4mn} \) is uniquely determined by \( \tilde{\sigma}_{4(m-1)n+2n} \).

This completes the proof of Part (2) of Theorem 1.4.

7. Irreducibility of Certain Descent Representations

In Theorem 2.1 for the residual representation \( \mathcal{E}_{\Delta(\tau, m)} \), we have proved that \( p^m(\mathcal{E}_{\Delta(\tau, m)}) = [(2n)^{2m}] \). From the proof, and by Lemma 2.6 [GRS03] or Lemma 3.1 [JL13b], we can see that it has a nonzero Fourier coefficient attached to the partition \([2n]^{4mn-2n}\) with respect to the character \( \psi(\mathcal{E}_{\Delta(2n)^{4mn-2n}}) \). In this section, for any number field \( F \), we show that both \( \mathcal{E}_{\Delta(\tau, m)} \) and \( D_{2n, \psi}^{4mn} (\mathcal{E}_{\Delta(\tau, m)}) \) are irreducible. The result can be stated as follows.

**Theorem 7.1.** Assume that \( F \) is any number field.

1. \( D_{2n, \psi}^{4mn} (\mathcal{E}_{\Delta(\tau, m)}) \) is square-integrable and is in the discrete spectrum.

2. Both \( \mathcal{E}_{\Delta(\tau, m)} \) and \( D_{2n, \psi}^{4mn} (\mathcal{E}_{\Delta(\tau, m)}) \) are irreducible.

**Proof.** **Proof of Part (1).** As in Theorem 5.2, we follow the constant term calculation in the proof of the Theorem 4.5.

Recall that \( P_r^{4mn-2n} = M_r^{4mn-2n} N_r^{4mn-2n} \) (with \( 1 \leq r \leq 2mn - n \)) is the standard parabolic subgroup of \( Sp_{4mn-2n} \) with Levi part \( M_r^{4mn-2n} \) isomorphic to \( GL_r \times Sp_{4mn-2n-2} \), \( N_r^{4mn-2n} \) is the unipotent radical, and \( \tilde{P}_r^{4mn-2n} \) is the pre-image of \( P_r^{4mn-2n} \) in \( \tilde{Sp}_{4mn-2n} \). Take any \( \xi \in \mathcal{E}_{\Delta(\tau, m)} \), we will calculate the constant term of \( F J_{\psi}^{4mn-2n} (\xi) \) along \( \tilde{P}_r^{4mn-2n} \), which is denoted by \( \mathcal{C}_{N_r^{4mn-2n}} (F J_{\psi}^{4mn-2n} (\xi)) \).
By Theorem 7.8 of [GRS11],

\[(7.1)\]

\[C_{N_4^{4mn-2n}}(F_J^{\phi}_{\psi_{n-1}}(\xi))\]

\[= \sum_{k=0}^{r} \sum_{\gamma \in P^1_{r-k,1k}(F)\backslash GL_r(F)} \int_{L(\mathbb{A})} \phi_1(i(\lambda)) F_J^{\phi_2}_{\psi_{n-1+k}}(C_{N_4^{4mn}}(\xi))(\gamma \lambda \beta) d\lambda,\]

where \(N_4^{4mn}\) is the unipotent radical of the parabolic subgroup \(P_{r-k}^{4mn}\) of \(Sp_{4mn}\) with Levi isomorphic to \(GL_{r-k} \times Sp_{4mn-2r+2k}\); \(P_1^{1}_{r-k,1k}\) is a subgroup of \(GL_r\) consisting of matrices of the form \(\begin{pmatrix} g & x \\ 0 & z \end{pmatrix}\), with \(z \in U_k\), the standard maximal unipotent subgroup of \(GL_k\); for \(a \in GL_j, j \leq 2mn\), \(\hat{a} = \text{diag}(a, I_{4mn-2j}, a^*)\); \(L\) is a unipotent subgroup, consisting of matrices of the form \(\lambda = \begin{pmatrix} I_r & 0 \\ x & I_n \end{pmatrix}\), and \(i(\lambda)\) is the last row of \(x\);

\[\beta = \begin{pmatrix} 0 \\ I_n \\ 0 \end{pmatrix}; \phi = \phi_1 \otimes \phi_2, \text{ with } \phi_1 \in S(\mathbb{A}^r), \phi_2 \in S(\mathbb{A}^{2mn-n-r});\]

\[F_J^{\phi_2}_{\psi_{n-1+k}}(C_{N_4^{4mn}}(\xi))(\gamma \lambda \beta) := F_J^{\phi_2}_{\psi_{n-1+k}}(C_{N_4^{4mn}}(\rho(\gamma \lambda \beta)\xi))(I),\]

with \(\rho(\gamma \lambda \beta)\) denoting the right translation by \(\gamma \lambda \beta\); \(C_{N_4^{4mn}}(\rho(\gamma \lambda \beta)\xi)\) is restricted to \(Sp_{4mn-2r+2k}(\mathbb{A})\), then we apply the Fourier-Jacobi coefficient \(F_J^{\phi_2}_{\psi_{n-1+k}}\), taking automorphic forms on \(Sp_{4mn-2r+2k}(\mathbb{A})\) to \(\tilde{S}p_{4mn-2n-2r}(\mathbb{A})\).

By the cuspidal support of \(\xi\), \(C_{N_4^{4mn}}(\xi)\) is identically zero, unless \(r = k\) or \(r - k = 2ln, 1 \leq l \leq m - 1\). When \(r = k\), the corresponding term is zero, because \(F_J^{\phi_2}_{\psi_{n-1+k}}(\xi)\) is zero, by Theorem 4.4. When \(r - k = 2ln, 1 \leq l \leq m - 1\), \(F_J^{\phi_2}_{\psi_{n-1+k}}(C_{N_4^{4mn}}(\xi))\) is not zero for \(k = 0\), and is identically zero for \(1 \leq k \leq r\), because \(p^{m}(E_{\Delta(r,m)}) = [(2n)^{2m}]\).

Therefore, \(C_{N_4^{4mn-2n}}(F_J^{\phi}_{\psi_{n-1}}(\xi)) \neq 0\), only for \(r = 2ln, 1 \leq l \leq m - 1\). And for \(1 \leq l \leq m - 1\),

\[(7.2)\]

\[= \int_{L(\mathbb{A})} \phi_1(i(\lambda)) F_J^{\phi_2}_{\psi_{n-1}}(C_{N_4^{2mn}}(\xi))(\lambda \beta) d\lambda.\]

To prove square-integrability of \(D_{2n, \psi_{-1}}^{4mn}(E_{\Delta(r,m)})\), it turns out we only need to consider \(r = 2(m - 1)n\), which will be clear from the following discussion.
For \( r = 2(m - 1)n \),
\[
C_{N^{4mn-2n}_{2(m-1)n}}(FJ^\phi_{\psi^{-1}_{n-1}}(\xi))
\]
\[
= \int_{L(k)} \phi_1(i(\lambda)) FJ^\phi_{\psi^{-1}_{n-1}}(C_{N^{4mn}_{2(m-1)n}}(\xi))(\lambda) d\lambda.
\]
(7.3)

By Lemma \[\text{2.3}^\text{a}\] when restricted to \( GL_{2(m-1)n}(\mathbb{A}) \times Sp_{4n}(\mathbb{A}) \),
\[
C_{N^{4mn}_{2(m-1)n}}(\xi) \in \delta_{P^{4mn}_{2(m-1)n}} \left| \det \right|^{-\frac{m+1}{2}} \Delta(\tau, m - 1) \otimes E_{\Delta(\tau,1)}.
\]

As in the proof of Theorem 2.5 \[\text{GJS12}\], to calculate the automorphic exponent attached to this non-trivial constant term (for definition see \[\text{I.3.3}^\text{MW95}\]), we need to consider the action of \( \hat{g} = \text{diag}(g, I_{2n}, g^*) \in GL_{2(m-1)n}(\mathbb{A}) \times \widetilde{Sp}_{2n}(\mathbb{A}) \).

Since \( r = 2(m - 1)n \), \( \beta = \left( \begin{array}{cc} 0 & I_{2(m-1)n} \\ I_n & 0 \end{array} \right) \). \( \beta \text{diag}(I_n, \hat{g}, I_n)^{-1} = \text{diag}(g, I_{4n}, g^*) =: \tilde{g} \). Then changing variables in (7.3) \( \lambda \mapsto \tilde{g}\lambda^{-1} \) will give a Jacobian \( \left| \det(g) \right|^{-n} \). On the other hand, by Formula (1.4) \[\text{GRST1}\], the action of \( \hat{g} \) on \( \phi_1 \) gives \( \gamma_\psi(\det(g)) \left| \det(g) \right|^{\frac{1}{2}} \). Therefore, the \( \hat{g} \) acts by \( \tau(g) \) with character
\[
\gamma_\psi(\det(g)) \delta_{P^{4mn}_{2(m-1)n}} \left| \det \right|^{-\frac{m+1}{2}} \Delta(\tau, m - 1) \otimes D_{2n,\psi^{-1}}(\xi) \Delta(\tau,1).
\]

Therefore, as a function on \( GL_{2(m-1)n}(\mathbb{A}) \times \widetilde{Sp}_{2n}(\mathbb{A}) \),
\[
C_{N^{4mn-2n}_{2(m-1)n}}(FJ^\phi_{\psi^{-1}_{n-1}}(\xi))
\]
\[
\in \gamma_\psi \delta_{P^{4mn}_{2(m-1)n}} \left| \det \right|^{-\frac{m+1}{2}} \Delta(\tau, m - 1) \otimes D_{2n,\psi^{-1}}(\xi) \Delta(\tau,1).
\]
(7.4)

By Theorem 2.3 \[\text{GJS12}\], we know that \( D_{2n,\psi^{-1}}(\xi) \) is an irreducible, genuine, \( \psi \)-generic, cuspidal automorphic representation of \( \widetilde{Sp}_{2n}(\mathbb{A}) \), which lifts to \( \tau \) with respect to \( \psi \). Hence, as a representation of \( GL_{2(m-1)n}(\mathbb{A}) \times \widetilde{Sp}_{2n}(\mathbb{A}) \),
\[
C_{N^{4mn-2n}_{2(m-1)n}}(D_{2n,\psi^{-1}}(\xi) \Delta(\tau, m - 1))
\]
\[
= \gamma_\psi \delta_{P^{4mn}_{2(m-1)n}} \left| \det \right|^{-\frac{m+1}{2}} \Delta(\tau, m - 1) \otimes D_{2n,\psi^{-1}}(\xi) \Delta(\tau,1).
\]
(7.5)

Since, the cuspidal exponent of \( \Delta(\tau, m - 1) \) is \( \left\{ \left( \frac{2-m}{2}, \frac{4-m}{2}, \ldots, \frac{m-2}{2} \right) \right\} \),
the cuspidal exponent of \( C_{N^{4mn-2n}_{2(m-1)n}}(FJ^\phi_{\psi^{-1}_{n-1}}(\xi)) \) is \( \left\{ \left( \frac{2-2m}{2}, \frac{4-2m}{2}, \ldots, -1 \right) \right\} \).
Hence, by Langlands square-integrability criterion [MW95, Lemma I.4.11], the automorphic representation $D_{2n, \psi}^{4m}(\mathcal{E}_{\Delta(\tau, m)})$ is square-integrable and is in the discrete spectrum.

This completes the proof of Part (1).

Proof of Part (2).

The proof of irreducibility of $\mathcal{E}_{\Delta(\tau, m)}$ is similar to that of $\mathcal{E}_{\Delta(\tau, 1)}$ which is given on Page 982 of [GJS12] and in the proof of Theorem 2.1 of [GRST11]. To show the square-integrable residual representation $\mathcal{E}_{\Delta(\tau, m)}$ is irreducible, it suffices to show that at each local place $v$,

\begin{equation}
\text{Ind}_{P_{2mn}(F_v)}^{Sp_{2mn}(F_v)} \Delta(\tau_v, m)|\frac{m}{2}
\end{equation}

has a unique quotient. Since $\Delta(\tau_v, m)$ is the unique quotient of the following induced representation

\[
\text{Ind}_{Q_{(2n)^m}(F_v)}^{GL_{2mn}(F_v)} \tau_v|\frac{m-1}{2} \otimes \tau_v|\frac{m-3}{2} \otimes \cdots \otimes \tau_v|\frac{1}{2},
\]

where $Q_{(2n)^m}$ is the parabolic subgroup of $GL_{2mn}$ with Levi subgroup isomorphic to $GL_{2n}^m$. We just have to show that the following induced representation has a unique quotient

\begin{equation}
\text{Ind}_{P_{(2n)^m}(F_v)}^{Sp_{4mn}(F_v)} \tau_v|\frac{2m-1}{2} \otimes \tau_v|\frac{2m-3}{2} \otimes \cdots \otimes \tau_v|\frac{1}{2},
\end{equation}

where $P_{(2n)^m}$ is the parabolic subgroup of $Sp_{4mn}$ with Levi subgroup isomorphic to $GL_{2n}^m$.

Since $\tau_v$ is generic and unitary, by [T86] and [V86], $\tau_v$ is full parabolic induction from its Langlands data with exponents in the open interval $(-\frac{1}{2}, \frac{1}{2})$. Explicitly, we can assume that

\[\tau_v \cong \rho_1|\alpha_1 \times \rho_2|\alpha_2 \times \cdots \times \rho_r|\alpha_r,\]

where $\rho_i$’s are tempered representations, $\alpha_i \in \mathbb{R}$ and $\frac{1}{2} > \alpha_1 > \alpha_2 > \cdots > \alpha_r > -\frac{1}{2}$. Therefore, the induced representation in (7.7) can be written as follows:

\[
\rho_1|\frac{2m-1}{2} + \alpha_1 \times \rho_2|\frac{2m-1}{2} + \alpha_2 \times \cdots \times \rho_r|\frac{2m-1}{2} + \alpha_r \times \rho_1|\frac{2m-3}{2} + \alpha_1 \times \rho_2|\frac{2m-3}{2} + \alpha_2 \times \cdots \times \rho_r|\frac{2m-3}{2} + \alpha_r \times \cdots \times \rho_1|\frac{1}{2} + \alpha_1 \times \rho_2|\frac{1}{2} + \alpha_2 \times \cdots \times \rho_r|\frac{1}{2} + \alpha_r \times 1_{Sp_0}.
\]
Since \( \alpha_i \in \mathbb{R} \) and \( \frac{1}{2} > \alpha_1 > \alpha_2 > \cdots > \alpha_r > -\frac{1}{2} \), we can easily see that the exponents satisfy

\[
\frac{2m - 1}{2} + \alpha_1 > \frac{2m - 1}{2} + \alpha_2 > \cdots > \frac{2m - 1}{2} + \alpha_r > \frac{2m - 3}{2} + \alpha_1 > \frac{2m - 3}{2} + \alpha_2 > \cdots > \frac{2m - 3}{2} + \alpha_r > \frac{1}{2} + \alpha_1 > \frac{1}{2} + \alpha_2 > \cdots > \frac{1}{2} + \alpha_r > 0.
\]

By Langlands classification, it is easy to see that the induced representation in (7.7) has a unique quotient which is the Langlands quotient. This completes the proof of irreducibility of \( E_{\Delta(\tau, m)} \).

The proof of irreducibility of \( D_{4n}^{m+2n}(E_{\Delta(\tau, m)}) \) is similar to that in Theorem 4.6. We just sketch all the steps needed.

Recall that \( P_{2n}^{4mn+4n} = M_{2n}^{4mn+4n} N_{2n}^{4mn+4n} \) is the parabolic subgroup of \( Sp_{4mn+4n} \) with Levi subgroup \( M_{2n}^{4mn+4n} \) isomorphic to \( GL_{2n} \times Sp_{4mn} \).

For any \( \phi \in A(N_{2n}^{4mn+4n}(BA) M_{2n}^{4mn+4n}(F) \backslash Sp_{4mn+4n}(\mathbb{A})_{\tau} \otimes \mathcal{E}_{\Delta(\tau, m)}) \), the corresponding Eisenstein series defined as follows has a pole at \( s = \frac{m+1}{2} \):

\[
E(\phi, s)(g) = \sum_{\gamma \in P_{2n}^{4mn+4n}(F) \backslash Sp_{4mn+4n}(F)} \lambda_s \phi(\gamma g).
\]

The resulting residual representation generated by all the residues is actually \( \mathcal{E}_{\Delta(\tau, m+1)} \).

Then, by a similar argument as in the proof of Theorem 4.2 we get that

\[
(7.8) \quad D_{2n, \psi^1}^{4mn+2n} \circ D_{2n, \psi^{-1}}^{4(m+1)n}(E_{\Delta(\tau, m+1)}) \neq 0,
\]

\[
D_{2n, \psi^1}^{4mn+2n} \circ D_{2n, \psi^{-1}}^{4(m+1)n}(E_{\Delta(\tau, m+1)}) = E_{\Delta(\tau, m)}.
\]

Note that, as indicated at the end of the proof of Theorem 4.2 the irreducibility of \( E_{\Delta(\tau, m)} \) plays an essential role in proving the equality in (7.8).

From Part (1), we see that \( D_{2n, \psi^1}^{4mn}(E_{\Delta(\tau, m)}) \) is square-integrable and is in the discrete spectrum. For any irreducible component \( \pi \) of \( D_{2n, \psi^1}^{4mn}(E_{\Delta(\tau, m)}) \), for any \( \phi \in A(N_{2n}^{4mn+2n}(\mathbb{A}) \hat{M}_{2n}^{4mn+2n}(F) \backslash \hat{Sp}_{4mn+2n}(\mathbb{A}))_{\mu_\pi \tau \otimes \pi} \), the corresponding Eisenstein series defined as follows has a pole at \( s = m \). Denote the residual representation generated by all the residues by \( \tilde{\mathcal{E}}_{\tau, \pi} \).
Since \( \pi \) is irreducible, also by a similar argument as in the proof of Theorem 4.2 we get that
\[
\mathcal{D}^{4mn}_{2n,\psi^{-1}} \circ \mathcal{D}^{4mn+2n}_{2n,\psi^1} (\tilde{E}_{\tau,\pi}) \neq 0, \\
\mathcal{D}^{4mn}_{2n,\psi^{-1}} \circ \mathcal{D}^{4mn+2n}_{2n,\psi^1} (\tilde{E}_{\tau,\pi}) = \pi.
\]
(7.9)

Then, using a similar argument as in the proof of Theorem 5.2, we have that there is an irreducible component \( \pi \) of \( \mathcal{D}^{4mn}_{2n,\psi^1} (\mathcal{E}_{\Delta(\tau,m)}) \), such that the representation space of \( \mathcal{D}^{4mn+4n}_{2n,\psi^1} (\mathcal{E}_{\Delta(\tau,m+1)}) \) has a non-trivial intersection with the representation space of \( \tilde{E}_{\tau,\pi} \). Let \( \pi' \) be an irreducible subrepresentation of \( \tilde{E}_{\tau,\pi} \), which is in this intersection.

Since \( \mathcal{E}_{\Delta(\tau,m)} \) is irreducible, by the identity in (7.8) we have
\[
\mathcal{E}_{\Delta(\tau,m)} = \mathcal{D}^{4mn+2n}_{2n,\psi^1} (\pi') \subseteq \mathcal{D}^{4mn+2n}_{2n,\psi^1} (\tilde{E}_{\tau,\pi}).
\]

Therefore,
\[
\mathcal{D}^{4mn}_{2n,\psi^{-1}} (\mathcal{E}_{\Delta(\tau,m)}) \subseteq \mathcal{D}^{4mn}_{2n,\psi^1} \circ \mathcal{D}^{4mn+2n}_{2n,\psi^1} (\tilde{E}_{\tau,\pi}) = \pi,
\]
by (7.9). Hence, \( \mathcal{D}^{4mn}_{2n,\psi^{-1}} (\mathcal{E}_{\Delta(\tau,m)}) = \pi \), irreducible.

This completes the proof of the theorem. \( \square \)

Remark 7.2. Write \( \tilde{\pi} = \mathcal{D}^{4n}_{2n,\psi^{-1}} (\mathcal{E}_{\Delta(\tau,1)}) \). For
\[
\tilde{\phi} \in A(N^{4mn-2n}_{2(m-1)n}(\mathbb{A}) \backslash M^{4mn-2n}_{2(m-1)n}(\mathbb{A}) \backslash \text{Sp}_{4mn-2n}(\mathbb{A}))_{\mu_\psi \Delta(\tau,m-1) \otimes \tilde{\pi}},
\]
it is easy to see that the corresponding Eisenstein series has a simple pole at \( \frac{m}{2} \). Denote the residual representation by \( \tilde{E}_{\Delta(\tau,m-1),\tilde{\pi}} \).

From the proof of Part (1) of Theorem 7.1 (in particular, (7.3)), it is easy to see that if the residual representation \( \tilde{E}_{\Delta(\tau,m-1),\tilde{\pi}} \) is irreducible, then actually we have proved that \( \mathcal{D}^{4mn}_{2n,\psi^1} (\mathcal{E}_{\Delta(\tau,m)}) = \tilde{E}_{\Delta(\tau,m-1),\tilde{\pi}} \). And, with the assumption that \( \tilde{E}_{\Delta(\tau,m-1),\tilde{\pi}} \) is irreducible, using similar argument as that in Theorem 7.1, we can also prove that \( \mathcal{D}^{4mn-2n}_{2n,\psi^{-1}} (\tilde{E}_{\Delta(\tau,m-1),\tilde{\pi}}) \) is irreducible, square-integrable and is in the discrete spectrum. Furthermore, since \( \mathcal{E}_{\Delta(\tau,m-1)} \) is also irreducible by Theorem 7.1, we actually have
\[
\mathcal{D}^{4mn-2n}_{2n,\psi^{-1}} (\tilde{E}_{\Delta(\tau,m-1),\tilde{\pi}}) = \mathcal{E}_{\Delta(\tau,m-1)}.
\]

References

[Ac03] P. Achar, An order-reversing duality map for conjugacy classes in Lusztig’s canonical quotient. Transform. Groups 8 (2003), no. 2, 107–145.

[Ar13] J. Arthur, The endoscopic classification of representations: Orthogonal and Symplectic groups. Colloquium Publication Vol. 61, 2013, American Mathematical Society.

[BV85] D. Barbasch and D. Vogan, Unipotent representations of complex semisimple groups. Ann. of Math. (2) 121 (1985), no. 1, 41–110.
[G08] D. Ginzburg, *Endoscopic lifting in classical groups and poles of tensor L-functions*. Duke Math. J. **141** (2008), no. 3, 447–503.

[G12] D. Ginzburg, *Constructing automorphic representations in split classical groups*. Electron. Res. Announc. Math. Sci. **19** (2012), 18–32.

[GJRS11] D. Ginzburg, D. Jiang, S. Rallis and D. Soudry, *L-functions for symplectic groups using Fourier-Jacobi models*. Arithmetic geometry and automorphic forms, 183207, Adv. Lect. Math. (ALM), **19**, Int. Press, Somerville, MA, 2011.

[GJS12] D. Ginzburg, D. Jiang and D. Soudry, *On correspondences between certain automorphic forms on Sp_{4n} and \tilde{Sp}_{2n}*. Published online by Israel J. of Math. (2012). DOI: 10.1007/s11856-012-0058-4.

[GRS99] D. Ginzburg, S. Rallis and D. Soudry, *On a correspondence between cuspidal representations of GL_{2n} and \tilde{Sp}_{2n}*. J. Amer. Math. Soc. **12** (1999), no. 3, 849907.

[GRS03] D. Ginzburg, S. Rallis and D. Soudry, *On Fourier coefficients of automorphic forms of symplectic groups*. Manuscripta Math. **111** (2003), no. 1, 1–16.

[GRS05] D. Ginzburg, S. Rallis and D. Soudry, *Construction of CAP representations for Symplectic groups using the descent method*. Automorphic representations, L-functions and applications: progress and prospects, 193–224, Ohio State Univ. Math. Res. Inst. Publ., 11, de Gruyter, Berlin, 2005.

[GRS11] D. Ginzburg, S. Rallis and D. Soudry, *The descent map from automorphic representations of GL(n) to classical groups*. World Scientific, Singapore, 2011. v+339 pp.

[Jan96] C. Jantzen, *Reducibility of certain representations for symplectic and odd-orthogonal groups*. Compositio Math. **104** (1996), no. 1, 55–63.

[J12] D. Jiang, *Integral transforms and endoscopy correspondences for classical groups*. Preprint, 2012. arXiv 1212.6525. To appear in Proceedings of the Conference on Automorphic Forms and Related Geometry: Assessing the Legacy of I. I. Piatetski-Shapiro. Edited by: J. Cogdell, F. Shahidi, and D. Soudry. Contemporary Mathematics, AMS.

[JL13a] D. Jiang and B. Liu, *On Fourier coefficients of automorphic forms of GL(n)*. Int. Math. Res. Not. 2013 (17): 4029–4071. doi: 10.1093/imrn/rns153.

[JL13b] D. Jiang and B. Liu, *On special nilpotent orbits and Fourier coefficients for automorphic forms on symplectic groups*. Submitted. 2013.

[JL13c] D. Jiang and B. Liu, *Arthur parameters and Fourier coefficients for automorphic forms on symplectic groups*. Submitted. 2013.

[JLZ13] D. Jiang, B. Liu and L. Zhang, *Poles of certain residual Eisenstein series of classical groups*. Pacific J. of Math. Vol. **264** (2013), No. 1, 83–123

[JS03] D. Jiang and D. Soudry, *The local converse theorem for SO(2n+1) and applications*. Ann. of Math., **157** (2003), 743–806.

[L76] R. Langlands, *On the functional equations satisfied by Eisenstein series*. Springer Lecture Notes in Math. 544, 1976.

[MW89] C. Moeglin and J.-P. Waldspurger, *Le spectre residuel de GL(n)*. Ann. Sci. École Norm. Sup. (4) **22** (1989), no. 4, 605–674.
[MW95] C. Moeglin and J.-P. Waldspurger, *Spectral decomposition and Eisenstein series*. Cambridge Tracts in Mathematics, 113. Cambridge University Press, Cambridge, 1995.

[PS79] I. Piatetski-Shapiro, *Multiplicity one theorems*. Automorphic forms, representations and $L$-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1, pp. 209–212, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979.

[Sh10] F. Shahidi, *Eisenstein series and automorphic $L$-functions*, volume 58 of American Mathematical Society Colloquium Publications. American Mathematical Society, Providence, RI, 2010. ISBN 978-0-8218-4989-7.

[S74] J. Shalika, *The multiplicity one theorem for $GL_n$*. Ann. of Math. (2) **100** (1974), 171–193.

[T86] M. Tadić, *Classification of unitary representations in irreducible representations of general linear group (non-Archimedean case)*. Ann. Sci. École Norm. Sup. (4) **19** (1986), no. 3, 335–382.

[V86] D. Vogan, *The unitary dual of $GL(n)$ over an Archimedean field*. Invent. Math. **83** (1986), no. 3, 449–505.

[W01] J.-L. Waldspurger, *Intégrales orbitales nilpotentes et endoscopie pour les groupes classiques non ramifiés*. Astérisque 269, 2001.

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