Error Compensated Loopless SVRG, Quartz, and
SDCA for Distributed Optimization

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September 21, 2021

Abstract
The communication of gradients is a key bottleneck in distributed training of large scale machine
learning models. In order to reduce the communication cost, gradient compression (e.g., sparsification
and quantization) and error compensation techniques are often used. In this paper, we propose and study
three new efficient methods in this space: error compensated loopless SVRG method (EC-LSVRG),
error compensated Quartz (EC-Quartz), and error compensated SDCA (EC-SDCA). Our method is
capable of working with any contraction compressor (e.g., TopK compressor), and we perform analysis
for convex optimization problems in the composite case and smooth case for EC-LSVRG. We prove linear
convergence rates for both cases and show that in the smooth case the rate has a better dependence on
the parameter associated with the contraction compressor. Further, we show that in the smooth case, and
under some certain conditions, error compensated loopless SVRG has the same convergence rate as the
vanilla loopless SVRG method. Then we show that the convergence rates of EC-Quartz and EC-SDCA
in the composite case are as good as EC-LSVRG in the smooth case. Finally, numerical experiments are
presented to illustrate the efficiency of our methods.

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1 Introduction

In this work we consider the composite finite-sum optimization problem

$$\min_{x \in \mathbb{R}^d} P(x) := \frac{1}{n} \sum_{\tau=1}^{n} f^{(\tau)}(x) + \psi(x), \quad (1)$$

where $f(x) := \frac{1}{n} \sum_{\tau} f^{(\tau)}(x)$ is an average of $n$ smooth convex functions $f^{(\tau)} : \mathbb{R}^d \to \mathbb{R}$ distributed over $n$ nodes (devices, computers), and $\psi : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ is a proper, closed and convex function representing a possibly nonsmooth regularizer. On each node, $f^{(\tau)}(x)$ is an average of $m$ smooth convex functions

$$f^{(\tau)}(x) := \frac{1}{m} \sum_{i=1}^{m} f^{(\tau)}_i(x),$$

representing the average loss over the training data stored on node $\tau$. We assume that problem (1) has at least one optimal solution $x^*$. For large scale machine learning problems, distributed training and parallel training are often used. While in such settings, communication is generally much slower than the computation, which makes the...
communication overhead become a key bottleneck. There are many ways to tackle this issue. For instance, there are large mini-batches strategy [Goyal et al., 2017, You et al., 2017], local SGD [Ma et al., 2017, Stich, 2020], asynchronous learning [Agarwal and Duchi, 2011, Lian et al, 2015, Recht et al., 2011], quantization and error compensation [Alistarh et al., 2017, Bernstein et al., 2018, Mishchenko et al., 2019, Seide et al., 2014, Wen et al., 2017].

For quantization, there are mainly two types, i.e., contraction compressor and unbiased compressor, which are defined as follows.

A possibly randomized map $Q: \mathbb{R}^d \to \mathbb{R}^d$ is called a contraction compressor if there is a $0 < \delta \leq 1$ such that
\[
\mathbb{E} [\|x - Q(x)\|^2] \leq (1 - \delta)\|x\|^2, \quad \forall x \in \mathbb{R}^d.
\]

$\tilde{Q}: \mathbb{R}^d \to \mathbb{R}^d$ is called an unbiased compressor if there is $\omega \geq 0$ such that
\[
\mathbb{E}[\tilde{Q}(x)] = x \quad \text{and} \quad \mathbb{E}[\|\tilde{Q}(x)\|^2] \leq (\omega + 1)\|x\|^2, \quad \forall x \in \mathbb{R}^d.
\]

Quantization can reduce the communicated bits to improve the communication efficiency, but it will slow down the convergence rate generally. Hence, error feedback (or called error compensation) scheme is often used to improve the performance of quantization algorithms [Seide et al., 2014]. For the unbiased compressor, if the accumulated quantization error is assumed to be bounded, the convergence rate of error compensated SGD is the same as vanilla SGD [Tang et al., 2018]. On the other hand, for the contraction compressor (for example TopK compressor [Alistarh et al., 2018]), the error compensated SGD actually has the same convergence rate as Vanilla SGD [Stich et al., 2018, Stich and Karimireddy, 2019, Tang et al., 2019]. The above results are all for the smooth case. If $f$ is non-smooth and $\psi \equiv 0$, error compensated SGD was studied by [Karimireddy et al., 2019] in the single node case, and the convergence rate is of order $O(1/\sqrt{\delta k})$ for $f$ being non-strongly convex.

For variance-reduced methods, there are QSVRG [Alistarh et al., 2017] for the smooth case where $\psi \equiv 0$ in problem (1), and VR-DIANA [Horváth et al., 2019a] for the composite or regularized case. However, the compressor of both algorithms need to be unbiased. Recently, an error compensated method called EC-LSVRG-DIANA which can achieve linear convergence for the strongly convex and smooth case was proposed by [Gorbunov et al., 2020], but besides the contraction compressor, the unbiased compressor is also needed in the algorithm. In this paper, we study the error compensated methods for loopless SVRG (L-SVRG) [Kovalev et al., 2019], Quartz [Qu et al., 2015], and SDCA [Shalev-Shwartz and Zhang, 2013], where only contraction compressors are needed.

1.1 Contributions

We now outline our main theoretical contributions.

1.1.1 Strongly convex case for EC-LSVRG

Denote the smoothness constants of functions $f$, $f(\tau)$, and $f_i(\tau)$ by $L_f$, $\bar{L}$, and $L$, respectively. Let $\mu$ be the strong convexity parameter, $p$ be the updating frequency of the reference point, and $\delta, \delta_1$ be the contraction compressor parameters. Let $p \leq O(1)\delta_1$.

**Composite case.** In the composite case, the iteration complexity of error compensated L-SVRG (EC-LSVRG) is
\[
O \left( \left( \frac{1}{\delta} + \frac{1}{p} + \frac{L_f}{\mu} + \frac{L}{\mu \delta} + \frac{(1 - \delta)\bar{L}}{\delta^2 \mu} + \frac{(1 - \delta)L}{\delta \mu} \right) \ln \frac{1}{\epsilon} \right).
\]
Under an additional assumption (Assumption 1.3) on the contraction compressor, the iteration complexity is improved to
\[
O \left( \left( \frac{1}{\delta} + \frac{1}{p} \frac{L_f}{\mu} + \frac{L}{n\mu} + \frac{(1-\delta)L_f}{\delta^2\mu} + \frac{(1-\delta)L}{n\delta \mu} \right) \ln \frac{1}{\epsilon} \right).
\]

**Smooth case.** In the smooth case, the iteration complexity of EC-LSVRG is
\[
O \left( \left( \frac{1}{\delta} + \frac{1}{p} \frac{L_f}{\mu} + \frac{L}{n\mu} + \sqrt{(1-\delta)L_fL} \frac{1}{\mu\delta} + \sqrt{(1-\delta)L_fL} \frac{1}{\mu\sqrt{\delta}} \right) \ln \frac{1}{\epsilon} \right).
\]

The iteration complexity of EC-LSVRG-DIANA [Gorbunov et al., 2020] is \(O((\omega + m + \frac{L_f}{\mu}) \ln \frac{1}{\epsilon})\). If the compressor \(Q_1\) in EC-LSVRG is obtained by scaling the unbiased compressor in EC-LSVRG-DIANA and we choose \(p = \min\{\frac{1}{m}, \frac{1}{\omega+1}\}\), then our iteration complexity becomes
\[
O \left( \left( \frac{1}{\delta} + \omega + m + \frac{L_f}{\mu} + \frac{L}{n\mu} + \sqrt{(1-\delta)L_fL} \frac{1}{\mu\delta} + \sqrt{(1-\delta)L_fL} \frac{1}{\mu\sqrt{\delta}} \right) \ln \frac{1}{\epsilon} \right),
\]
which is better than that of EC-LSVRG-DIANA since \(L_f \leq \tilde{L} \leq L\).

Under an additional assumption (Assumption 1.3) on the contraction compressor, the iteration complexity is improved to
\[
O \left( \left( \frac{1}{\delta} + \frac{1}{p} \frac{L_f}{\mu} + \frac{L}{n\mu} + \sqrt{(1-\delta)L_fL} \frac{1}{\mu\delta} \right) \ln \frac{1}{\epsilon} \right).
\]

In particular, if \(\frac{L_f}{\delta} \leq \frac{L}{n}\), then the above iteration complexity becomes
\[
O \left( \left( \frac{1}{p} + \frac{L_f}{\mu} + \frac{L}{n\mu} \right) \ln \frac{1}{\epsilon} \right),
\]
which is actually the iteration complexity of the uncompressed L-SVRG [Qian et al., 2019b]. Noticing that \(L_f \leq L \leq mnL_f\), this means that in the extreme case: \(L = mnL_f\), the error compensated L-SVRG has the same convergence rate as the uncompressed L-SVRG as long as \(\frac{1}{\delta} \leq m\) and \(p \leq O(\delta_1)\).

### 1.1.2 Non-strongly convex case for EC-LSVRG

Let \(p \leq O(\delta_1)\).

**Composite case.** In the composite case, the iteration complexity of EC-LSVRG is
\[
O \left( \left( \frac{1}{p} + L_f + \frac{L}{n} + \frac{(1-\delta)L}{\delta^2} + \frac{(1-\delta)L}{n\delta} \right) \frac{1}{\epsilon} \right).
\]

Under an additional assumption (Assumption 1.3) on the contraction compressor, the iteration complexity is improved to
\[
O \left( \left( \frac{1}{p} + L_f + \frac{L}{n} + \frac{(1-\delta)L_f}{\delta^2} + \frac{(1-\delta)L}{n\delta} \right) \frac{1}{\epsilon} \right).
\]

**Smooth case.** In the smooth case, the iteration complexity of EC-LSVRG is
\[
O \left( \left( \frac{1}{p} + L_f + \frac{L}{n} + \sqrt{(1-\delta)L_fL} \frac{1}{\delta} + \sqrt{(1-\delta)L_fL} \frac{1}{\sqrt{\delta}} \right) \frac{1}{\epsilon} \right).
\]

Under an additional assumption (Assumption 1.3) on the contraction compressor, the iteration complexity is improved to
\[
O \left( \left( \frac{1}{p} + L_f + \frac{L}{n} + \sqrt{(1-\delta)L_f} \frac{1}{\delta} \right) \frac{1}{\epsilon} \right).
\]
1.1.3 Strongly convex and composite case for EC-Quartz and EC-SDCA

We consider problem (8) for EC-Quartz and EC-SDCA in the strongly convex and composite case. The iteration complexities of EC-Quartz and EC-SDCA are

\[
O \left( \frac{1}{\delta} + m + \frac{R_m^2}{n\lambda\gamma} + \frac{R^2}{\lambda\gamma} + \frac{\sqrt{1 - \delta R^2}}{\delta\lambda\gamma} + \frac{\sqrt{1 - \delta RR_m}}{\lambda\gamma\sqrt{\delta}} \right) \ln \frac{1}{\epsilon}.
\]

Under an additional assumption (Assumption 1.3) on the contraction compressor, the iteration complexities are improved to

\[
O \left( \frac{1}{\delta} + m + \frac{R_m^2}{n\lambda\gamma} + \frac{R^2}{\lambda\gamma} + \frac{\sqrt{1 - \delta R^2}}{\delta\lambda\gamma} \right) \ln \frac{1}{\epsilon}.
\]

Noticing that for problem (8), we usually use \( R, \tilde{R}, \text{and } R_m \) (\( R, \tilde{R}, \text{and } R_m \) are defined in Algorithm 2) to estimate the smoothness constants of \( \sum_{i=1}^{m} \frac{1}{m} \phi_i(A_i^\top x) \) and \( \frac{1}{m} \sum_{i=1}^{m} \phi_i(A_i^\top x) \), respectively. In this sense, the iteration complexities of EC-Quartz and EC-SDCA in the composite case are as good as EC-LSVRG in the smooth case. Let \( \delta \) go to 1. Then the iteration complexity becomes

\[
O \left( \frac{mn + R_m^2}{n\lambda\gamma^2} \right) \ln \frac{1}{\epsilon},
\]

which has linear speed up with respect to the number of nodes \( n \) when \( n \leq \frac{R^2}{R_m^2} \). Noticing that there is no linear speed up with respect to \( n \) for Quartz with fully dense data, our result is actually better than Quartz in this case. This improvement for fully dense data benefits from the better estimation of expected mapable overapproximation (ESO). The ESO estimation for arbitrary sampling for Quartz can be found in the appendix.

1.2 Compression methods

We now give a few examples of contraction compressors:

**TopK compressor.** For a parameter \( 1 \leq K \leq d \), the TopK compressor [Stich et al., 2018] is defined as

\[
(\text{TopK}(x))_{\pi(i)} = \begin{cases} (x)_{\pi(i)} & \text{if } i \leq K, \\ 0 & \text{otherwise}, \end{cases}
\]

where \( \pi \) is a permutation of \( [d] := \{1, 2, \ldots, d\} \) such that \( (|x|)_{\pi(i)} \geq (|x|)_{\pi(i+1)} \) for \( i = 1, \ldots, d - 1 \).

**RandK compressor.** For a parameter \( 1 \leq K \leq d \), the RandK compressor is defined as

\[
(\text{RandK}(x))_i = \begin{cases} (x)_i & \text{if } i \in S, \\ 0 & \text{otherwise}, \end{cases}
\]

where \( S \) is chosen uniformly from the set of all \( K \) element subsets of \( [d] \).

It is known that after appropriate scaling, any unbiased compressor satisfying (3) becomes a contraction compressor [Beznosikov et al., 2020]. Indeed, it is easy to verify that for any \( \tilde{Q} \) satisfying (3), \( \frac{1}{\omega + 1} \tilde{Q} \) is a contraction compressor satisfying (2) with \( \delta = \frac{1}{(\omega + 1)} \) as follows.

\[
\mathbb{E} \left[ \left\| \frac{1}{\omega + 1} \tilde{Q}(x) - x \right\|^2 \right] = \frac{1}{(\omega + 1)^2} \mathbb{E} \left[ \left\| \tilde{Q}(x) \right\|^2 \right] + \left\| x \right\|^2 - \frac{2}{\omega + 1} \mathbb{E} \left[ \langle \tilde{Q}(x), x \rangle \right] \\
\leq \frac{1}{\omega + 1} \left\| x \right\|^2 + \left\| x \right\|^2 - \frac{2}{\omega + 1} \left\| x \right\|^2 = \left( 1 - \frac{1}{\omega + 1} \right) \left\| x \right\|^2.
\]

For more examples of contraction and unbiased compressors, we refer the reader to [Beznosikov et al., 2020]. For the TopK and RandK compressors, we have the following property.

**Lemma 1.1** (Lemma A.1 in [Stich et al., 2018]). For the TopK and RandK compressors with \( 1 \leq K \leq d \), we have

\[
\mathbb{E} \left[ \left\| \text{TopK}(x) - x \right\|^2 \right] \leq \left( 1 - \frac{K}{d} \right) \left\| x \right\|^2, \quad \mathbb{E} \left[ \left\| \text{RandK}(x) - x \right\|^2 \right] \leq \left( 1 - \frac{K}{d} \right) \left\| x \right\|^2.
\]
Now we propose some new contraction compressors.

**Composition of the unbiased compressor and contraction compressor.** Let $S \in [d]$, and denote $S_i \in S$ such that $S_1 < S_2 \cdots < S_{|S|}$, where $|S|$ represents the cardinality of $S$. For any $x \in \mathbb{R}^d$, define $x_S \in \mathbb{R}^{|S|}$ such that $(x_S)_i = x_i$ for $1 \leq i \leq |S|$. For any $y \in \mathbb{R}^{|S|}$, define $y_{S-1} \in \mathbb{R}^d$ such that $(y_{S-1})_{S_i} = y_i$ for $1 \leq i \leq |S|$ and $(y_{S-1})_j = 0$ for $j \notin S$. Then we have following result.

**Theorem 1.2.** For any unbiased compressor $\tilde{Q}$ with parameter $\omega$, and any contraction compressor $Q$ with parameter $\delta$, Define $\tilde{Q} \circ Q : x \to \left(\frac{1}{\omega+1}\tilde{Q}(Q(x)_S)\right)_{S-1}$, where $x \in \mathbb{R}^d$, and $S$ is any subset of $[d]$ such that $Q(x)_j = 0$ for $j \notin S$ ($S$ can depend on $Q(x)$). Then $\tilde{Q} \circ Q$ is a contraction compressor with parameter $\frac{\delta}{\omega+1}$.

**Proof.** For any $x \in \mathbb{R}^d$, we have

$$
\mathbb{E}\|x - Q \circ Q(x)\|_2^2 = \mathbb{E}\left\|x - \left(\frac{1}{\omega+1}\tilde{Q}(Q(x)_S)\right)_{S-1}\right\|_2^2
$$

$$
= \|x\|^2 + \mathbb{E}\left\|\left(\frac{1}{\omega+1}\tilde{Q}(Q(x)_S)\right)_{S-1}\right\|^2 - 2\mathbb{E}\left\langle x, \left(\frac{1}{\omega+1}\tilde{Q}(Q(x)_S)\right)_{S-1}\right\rangle
$$

$$
= \|x\|^2 + \mathbb{E}\left\|\frac{1}{\omega+1}\tilde{Q}(Q(x))\right\|^2 - 2\mathbb{E}\left\langle x, \frac{Q(x)}{\omega+1}\right\rangle
$$

$$
\leq \|x\|^2 + \frac{1}{\omega+1}\mathbb{E}\|Q(x)_S\|^2 - 2\mathbb{E}\left\langle x, \frac{Q(x)}{\omega+1}\right\rangle,
$$

where in the third equality we use $\|y_{S-1}\|^2 = \|y\|^2$ for any $y \in \mathbb{R}^{|S|}$,

$$
\mathbb{E}\left[\left(\frac{1}{\omega+1}\tilde{Q}(Q(x)_S)\right)_{S-1}\right] = \mathbb{E}\left[\left(\frac{1}{\omega+1}Q(x)_S\right)_{S-1}\right],
$$

and the fact that for any $x \in \mathbb{R}^d$, if $x_j = 0$ for $j \notin S$, then $(x_S)_{S^{-1}} = x$. Since $\|Q(x)_S\|^2 = \|Q(x)\|^2$, we further have

$$
\mathbb{E}\|x - \tilde{Q} \circ Q(x)\|^2 \leq \|x\|^2 + \frac{1}{\omega+1}\mathbb{E}\|Q(x)\|^2 - 2\mathbb{E}\left\langle x, \frac{Q(x)}{\omega+1}\right\rangle
$$

$$
= \left(1 - \frac{1}{\omega+1}\right)\|x\|^2 + \frac{1}{\omega+1}\mathbb{E}\left[\|Q(x)\|^2 - 2\left\langle x, Q(x)\right\rangle\right]
$$

$$
= \left(1 - \frac{1}{\omega+1}\right)\|x\|^2 + \frac{1}{\omega+1}\mathbb{E}\|x - Q(x)\|^2
$$

$$
\leq \left(1 - \frac{\delta}{\omega+1}\right)\|x\|^2.
$$

If we let $Q$ be the TopK compressor, $\tilde{Q}$ be the general exponential dithering operator [Beznosikov et al., 2020], and $S$ be $[d]$, then we recover the TopK combined with exponential dithering compressor in [Beznosikov et al., 2020].

We can construct some concrete contraction compressors based on Theorem 1.2. For example, let RTOPK be the composition of TopK and random dithering [Alistarh et al., 2017], and NTopK be the composition of TopK and natural compression [Horváth et al., 2019], where $S$ is composed of the $K$ indexes in $\text{TopK}(x)$, i.e., $S = \{\pi(1), ..., \pi(K)\}$.

We may use the following assumptions for the contraction compressor in some cases; and this will allow us to obtained improved results.

**Assumption 1.3.** $\mathbb{E}[Q(x)] = \delta x$. $\mathbb{E}[Q_1(x)] = \delta_1 x$.

It is easy to verify that RandK compressor satisfies Assumption 1.3 with $\delta = K/\kappa$, and $\tilde{Q}/(\omega+1)$, where $\tilde{Q}$ is any unbiased compressor, also satisfies Assumption 1.3 with $\delta = 1/\omega+1$.
2 Error Compensated L-SVRG

We now describe the error compensated L-SVRG algorithm (Algorithm 1). Roughly speaking, EC-LSVRG is a combination of L-SVRG, the error feedback technique, and the learning scheme method in VR-DIANA [Horváth et al., 2019a]. The search direction in L-SVRG is

\[ \frac{1}{n} \sum_{\tau=1}^{n} \nabla f_{i_{\tau}^{k}}^{(\tau)}(x^{k}) - \nabla f_{i_{\tau}^{k}}^{(\tau)}(w^{k}) + \nabla f^{(\tau)}(u^{k}), \]

where \( i_{\tau}^{k} \) is sampled uniformly and independently from \( \{1, 2, \ldots, m\} \) on \( \tau \)-th node for \( 1 \leq \tau \leq n \), \( x^{k} \) is the current iteration, and \( w^{k} \) is the reference point. Since when \( \psi \) is nonzero in problem (3), \( \nabla f(x^{\ast}) \) is nonzero in general, and so is \( \nabla f^{(\tau)}(x^{\ast}) \). Thus, compressing the direction

\[ \nabla f_{i_{\tau}^{k}}^{(\tau)}(x^{k}) - \nabla f_{i_{\tau}^{k}}^{(\tau)}(w^{k}) + \nabla f^{(\tau)}(u^{k}) \]

directly on each node would cause nonzero noise even when \( x^{k} \) and \( w^{k} \) goes to the optimal solution \( x^{\ast} \). We adopt a learning scheme method in VR-DIANA [Horváth et al., 2019a], but with the contraction compressor rather than the unbiased compressor. We maintain a vector \( h_{k}^{\ast} \in \mathbb{R}^{d} \) on each node, and use it to learn \( \nabla f^{(\tau)}(w^{k}) \) iteratively. The same copy of \( h^{k} \), which is the average of \( h_{k}^{\ast} \), is also maintained on each node. We subtract \( h_{k}^{\ast} \) from the search direction in L-SVRG on each node, and then add \( h^{k} \) back after the aggregation. The search direction of EC-LSVRG becomes

\[ g_{\tau}^{k} = \nabla f_{i_{\tau}^{k}}^{(\tau)}(x^{k}) - \nabla f_{i_{\tau}^{k}}^{(\tau)}(w^{k}) + \nabla f^{(\tau)}(u^{k}) - h_{\tau}^{k}, \]

that could be small if \( w^{k} \) is close to \( x^{k} \) and \( h_{\tau}^{k} \) close to \( \nabla f^{(\tau)}(w^{k}) \). Before compressing \( g_{\tau}^{k} \), we apply the error feedback technique. The accumulated error vector \( e_{\tau}^{k} \) is maintained on each node and will be added to \( \eta g_{\tau}^{k} \), where \( \eta \) represents the stepsize, before the compression. \( e_{\tau}^{k+1} \) is updated by the compression error at iteration \( k \) for each node. On each node, a scalar \( u_{\tau}^{k} \) is also maintained, and only \( u_{\tau}^{k} \) will be updated. The summation of \( u_{\tau}^{k} \) is \( u^{k} \), and we use \( u^{k} \) to control the updating frequency of the reference point \( w^{k} \). All nodes maintain the same copies of \( x^{k}, w^{k}, y^{k}, z^{k}, \) and \( u^{k} \). Each node sends their compressed vectors \( y_{\tau}^{k}, z_{\tau}^{k}, \) and \( u_{\tau}^{k+1} \) to the other nodes. At last, the proximal step is taken on each node, where we use the standard proximal operator:

\[ \text{prox}_{\eta \psi}(x) := \arg \min_{y} \left\{ \frac{1}{2} \| x - y \|^{2} + \eta \psi(y) \right\}. \]

The reference point \( w^{k} \) will be updated if \( u^{k+1} = 1 \). It is easy to see that \( w^{k} \) will be updated with probability \( p \) at each iteration.

In algorithm 1 let \( e^{k} = \frac{1}{n} \sum_{\tau=1}^{n} e_{\tau}^{k}, g^{k} = \frac{1}{n} \sum_{\tau=1}^{n} g_{\tau}^{k} \), and \( \tilde{x}^{k} = x^{k} - e^{k} \) for \( k \geq 0 \). Then we have

\[ e^{k+1} = \frac{1}{n} \sum_{\tau=1}^{n} \left( e_{\tau}^{k} + \eta g_{\tau}^{k} - y_{\tau}^{k} \right) = e^{k} + \eta g^{k} - y^{k}, \]

and

\[ \tilde{x}^{k+1} = x^{k+1} - e^{k+1} \]
\[ = x^{k+0.5} - \eta \partial \psi(x^{k+1}) - e^{k+1} \]
\[ = x^{k} - y^{k} - \eta h^{k} - \eta \partial \psi(x^{k+1}) - e^{k} - \eta g^{k} + y^{k} \]
\[ = \tilde{x}^{k} - \eta (g^{k} + h^{k} + \partial \psi(x^{k+1})). \]

2.1 Composite case \((\psi \neq 0)\) of EC-LSVRG

We need the following assumptions in this subsection.
Algorithm 1 Error compensated loopless SVRG (EC-LSVRG)

1: **Parameters:** stepsizes $\eta > 0$; probability $p \in (0, 1]$
2: **Initialization:** $x^0 = w^0 \in \mathbb{R}^d$; $e_0 = 0 \in \mathbb{R}^d$; $u^0 = 1 \in \mathbb{R}$; $h^0_\tau = 0 \in \mathbb{R}^d$; $h^0 = \frac{1}{n} \sum_{\tau = 1}^n h^0_\tau$
3: for $k = 0, 1, 2, \ldots$ do
   4:   for $\tau = 1, \ldots, n$ do
      5:      Sample $i^\tau_\tau$ uniformly and independently in $[m]$ on each node
      6:      $g^\tau_k = \nabla f_i^\tau(x^k) - \nabla f_i^\tau(w^k) + \nabla f(\tau)(w^k) - h^k_\tau$
      7:      $y^k_\tau = Q(\eta h^k_\tau + e^k_\tau)$
      8:      $e^k+1_\tau = e^k_\tau + \eta g^k_\tau - y^k_\tau$
      9:      $z^k_\tau = Q_1(\nabla f(\tau)(w^k) - h^k_\tau)$
     10:     $h^k+1_\tau = h^k_\tau + z^k_\tau$
     11:     $u^k+1_\tau = 0$ for $\tau = 2, \ldots, n$
     12:     $u^k_1 = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } 1 - p \end{cases}$
     13:     Send $y^k_\tau$, $z^k_\tau$, and $u^k+1_\tau$ to the other nodes
     14:     Receive $y^k_\tau$, $z^k_\tau$, and $u^k+1_\tau$ from the other nodes
     15:     $y^k = \frac{1}{n} \sum_{\tau = 1}^n y^k_\tau$
     16:     $z^k = \frac{1}{n} \sum_{\tau = 1}^n z^k_\tau$
     17:     $u^{k+1} = \sum_{\tau = 1}^n u^{k+1}_\tau$
     18:     $x^{k+0.5} = x^k - (y^k + \eta h^k)$
     19:     $x^{k+1} = \text{prox}_{\eta \psi}(x^{k+0.5})$
     20:     $w^{k+1} = \begin{cases} x^k & \text{if } u^{k+1} = 1 \\ w^k & \text{otherwise} \end{cases}$
     21:     $h^{k+1} = h^k + z^k$
   4: end for
3: end for

Assumption 2.1. The two compressors $Q$ and $Q_1$ in Algorithm 1 are contraction compressors with parameters $\delta$ and $\delta_1$, respectively.

Assumption 2.2. $f_i(\tau)$ is $L$-smooth, $f(\tau)$ is $L$-smooth, $f$ is $L_f$-smooth, and $\psi$ is $\mu$-strongly convex. $L_f \geq \mu \geq 0$.

The following are the main results. We use two Lyapunov functions for two cases: with or without Assumption 1.3 in the following two theorems.

Theorem 2.3. Let Assumption 2.1 and Assumption 2.2 hold. Define

$$
\Phi^k_1 \ := \left(1 + \frac{\eta h}{2}\right) \mathbb{E} \left[\left\|\hat{x}^k - x^*\right\|^2\right] + \frac{9}{\delta_n} \sum_{\tau = 1}^n \mathbb{E} \left[\left\|e^k_\tau\right\|^2\right] + \frac{164(1 - \delta)\eta^2}{\delta^2 \delta_1} \sum_{\tau = 1}^n \mathbb{E} \left[\left\|h^k_\tau - \nabla f(\tau)(w^k)\right\|^2\right]
+ \frac{4\eta^2}{3p} \left(\frac{41(1 - \delta)}{\delta} \left(\frac{4L}{\delta} + L + \frac{16Lp}{\delta \delta_1} \left(1 + \frac{2p}{\delta_1}\right) + \frac{16L}{n}\right)\right) \mathbb{E} [P(w^k) - P(x^*)].
$$

If $\eta \leq \frac{1}{4L_f}$, then we have

$$
\mathbb{E} \left[\Phi^{k+1}_1\right] \leq \left(1 - \min \left\{\frac{\eta h}{3}, \frac{\delta \delta_1}{4}, \frac{p}{4}\right\}\right) \mathbb{E} \left[\Phi^k_1\right] + 2\eta \mathbb{E} [P(x^*) - P(x^{k+1})] + \left(\frac{96(1 - \delta)}{\delta} \left(\frac{4L}{\delta} + L + \frac{16Lp}{\delta \delta_1} \left(1 + \frac{2p}{\delta_1}\right) + \frac{38L}{n}\right)\right) \eta^2 \mathbb{E} [P(x^k) - P(x^*)].
$$
Theorem 2.4. Let Assumption [2.1] and Assumption [2.2] hold. Define

\[
\Phi_k := \left(1 + \frac{\nu \mu}{2}\right) \mathbb{E}[\|\bar{x} - x_n\|^2 + \frac{9}{\delta} \mathbb{E}[\|e_n\|^2 + \frac{84(1 - \delta)}{\delta n^2} \sum_{\tau=1}^n \mathbb{E}[\|e^\tau\|^2]}
\]

\[\quad + \frac{164(1 - \delta)\eta^2}{\delta^2 \delta_1} \mathbb{E}[\|h^k - \nabla f(w^k)\|^2] + \frac{2000(1 - \delta)\eta^2}{\delta^2 \delta_1 n^2} \sum_{\tau=1}^n \mathbb{E}[\|h^k - \nabla f^\tau(w^k)\|^2]
\]

\[\quad + \frac{4\eta^2}{3p} \left(1 - \frac{\delta}{\delta_1}\right) \left(\frac{164L_f}{\delta} + \frac{1344L}{\delta n} + \frac{544L}{n} + \frac{(656L_f + 8000\delta)}{n \delta_1} \left(1 + \frac{2p}{\delta_1}\right) + \frac{16L}{n}\right) \mathbb{E}[P(w^k) - P(x^*])
\]

Under Assumption [1.3], if \( \eta \leq \frac{1}{4L_f} \), then we have

\[
\mathbb{E}[\Phi_k^{k+1}] \leq \left(1 - \min \left\{ \frac{\mu \delta}{3}, \frac{\delta_1}{4}, \frac{p}{4} \right\} \right) \mathbb{E}[\Phi_k^k] + 2\eta \mathbb{E}[P(x^*) - P(x^{k+1})]
\]

\[\quad + \left(1 - \frac{\delta}{\delta_1}\right) \left(\frac{383L_f}{\delta} + \frac{3136\delta_1}{\delta n} + \frac{1263L}{\delta n} + \frac{(1531L_f + 18667\delta)}{\delta_1 n} \left(1 + \frac{2p}{\delta_1}\right) + \frac{38L}{n}\right) \eta^2 \mathbb{E}[P(x^k) - P(x^*)].
\]

From the above two theorems, we can get the iteration complexity.

Theorem 2.5. Let Assumption [2.1] and Assumption [2.3] hold. Let \( w_k = (1 - \min \left\{ \frac{\nu \eta}{2}, \frac{\delta_1}{4}, \frac{p}{4} \right\})^{-k} \), \( W_k = \sum_{i=0}^k w_i \), and \( \bar{x}^k = \frac{1}{\nu \eta} \sum_{i=0}^k w_i x_i \).

(i) Let \( \eta \leq \frac{1}{4L_f} \left(\frac{4L_\bar{x}}{\delta_1} + L + \frac{16L\delta_1}{\delta n} \left(1 + \frac{2p}{\delta_1}\right) + 4L_f + \frac{42\delta}{n}\right) \). If \( \mu > 0 \), we have

\[
\mathbb{E}[P(\bar{x}^k) - P(x^*)] \leq \frac{\mu}{2} \|x^0 - x_n\|^2 + \frac{\nu}{2} \|P(x^0) - P(x^*)\| + \frac{1}{10L_\bar{x}} \sum_{\tau=1}^n \|\nabla f^\tau(x^0) - h^\tau_n\|^2 \left(1 - \min \left\{ \frac{\mu \delta}{3}, \frac{\delta_1}{4}, \frac{p}{4} \right\}\right)^{k+1}.
\]

If \( \mu = 0 \), we have

\[
\mathbb{E}[P(\bar{x}^k) - P(x^*)] \leq \frac{1}{k + 1} \left[\frac{9}{8\eta} \|x^0 - x_n\|^2 + \frac{1}{10L_\bar{x}} \sum_{\tau=1}^n \|\nabla f^\tau(x^0) - h^\tau_n\|^2 + \frac{8}{3\eta} \|P(x^0) - P(x^*)\|\right]
\]

(ii) Let \( \eta = \frac{1}{4L_f} \left(\frac{4L_\bar{x}}{\delta_1} + L + \frac{16L\delta_1}{\delta n} \left(1 + \frac{2p}{\delta_1}\right) + 4L_f + \frac{42\delta}{n}\right) \) and \( p \leq O(\delta_1) \). If \( \mu > 0 \) and \( \epsilon \leq \frac{\mu}{2} \|x^0 - x^*\|^2 + \frac{\nu}{2} \|P(x^0) - P(x^*)\| + \frac{1}{10L_\bar{x}} \sum_{\tau=1}^n \|\nabla f^\tau(x^0) - h^\tau_n\|^2 \), we have \( \mathbb{E}[P(\bar{x}^k) - P(x^*)] \leq \epsilon \) as long as

\[
k \geq O \left(\left(\frac{1}{\delta} + \frac{1}{p} + \frac{L_f}{\mu} + \frac{(1 - \delta)\bar{L}}{\mu \delta} + \frac{(1 - \delta)\bar{L}}{\delta} \right) \frac{1}{\epsilon}\right)
\]

If \( \mu = 0 \), we have \( \mathbb{E}[P(\bar{x}^k) - P(x^*)] \leq \epsilon \) as long as

\[
k \geq O \left(\left(\frac{1}{p} + \frac{L_f}{\mu} + \frac{(1 - \delta)\bar{L}}{\delta} \right) \frac{1}{\epsilon}\right)
\]

Theorem 2.6. Let Assumption [2.1] and Assumption [2.3] hold. Assume the compressors \( Q \) and \( Q_1 \) also satisfy Assumption [1.3]. Let \( w_k = (1 - \min \left\{ \frac{\nu \eta}{2}, \frac{\delta_1}{4}, \frac{p}{4} \right\})^{-k} \), \( W_k = \sum_{i=0}^k w_i \), and \( \bar{x}^k = \frac{1}{\nu \eta} \sum_{i=0}^k w_i x_i \).

(i) Let \( \eta \leq \frac{1}{4L_f} \left(\frac{418L_\bar{x}}{\delta_1} + \frac{4294L}{\delta n} + \frac{3349L}{n} + \frac{(1671L_f + 2930\bar{L})}{\delta_1 n} \left(1 + \frac{2p}{\delta_1}\right) + 4L_f + \frac{42\delta}{n}\right) \). If \( \mu > 0 \), we have

\[
\mathbb{E}[P(\bar{x}^k) - P(x^*)] \leq \frac{\mu}{2} \|x^0 - x_n\|^2 + \frac{\nu}{2} \|P(x^0) - P(x^*)\| + \frac{1}{10L_\bar{x}} \sum_{\tau=1}^n \|\nabla f^\tau(x^0) - h^\tau_n\|^2 \left(1 - \min \left\{ \frac{\mu \delta}{3}, \frac{\delta_1}{4}, \frac{p}{4} \right\}\right)^{k+1}.
\]
If $\mu = 0$, we have

\[
\mathbb{E}[P(x^k) - P(x^*)] \leq \frac{1}{k+1} \left( \frac{9}{8\eta} \|x^0 - x^*\|^2 + \frac{1}{10Lfp} \|\nabla f(x^0)\|^2 + \frac{6}{5L_fpn} \sum_{r=1}^{n} \|\nabla f(\tau)(x^0) - h^0_r\|^2 + \frac{8}{3p} (P(x^0) - P(x^*)) \right).
\]

(ii) Let $\eta = 1/\left( \frac{1-\delta}{\delta} \right) \left( \frac{418L_f}{\delta} + \frac{3422L}{\delta^3} n + 1349L \right) + \left( \frac{1671L_f + 2034L}{8\delta^4} \right) \left( 1 + \frac{2p}{\delta_1} \right) + 4L_f + \frac{42L}{n} \right) + p \leq O(\delta_1)$. If $\mu > 0$ and $\epsilon \leq \frac{\|x^0 - x^*\|^2}{2} + \frac{1}{2} (P(x^0) - P(x^*)) + \frac{1}{6L_f} (\|\nabla f(x^0)\|^2 + \frac{1}{10L_fn} \sum_{r=1}^{n} \|\nabla f(\tau)(x^0) - h^0_r\|^2)$, we have $\mathbb{E}[P(x^k) - P(x^*)] \leq \epsilon$ as long as

\[
k \geq O \left( \frac{\left( \frac{1}{\delta} + \frac{1}{p} + \frac{L_f}{\mu} + \frac{L}{n\mu} + \frac{(1-\delta)L_f}{\delta^2\mu} + \frac{(1-\delta)L}{n\delta\mu} \right) \ln \frac{1}{\epsilon}}{\frac{1}{L_f} \left( \frac{1}{(1-\delta)} + \frac{1}{L} + \frac{(1-\delta)L_f}{\delta^2} + \frac{(1-\delta)L}{n\delta} \right) \frac{1}{\epsilon}} \right).
\]

If $\mu = 0$, we have $\mathbb{E}[P(x^k) - P(x^*)] \leq \epsilon$ as long as

\[
k \geq O \left( \frac{\left( \frac{1}{\delta} + \frac{1}{p} + \frac{L}{n\mu} + \frac{(1-\delta)L}{\delta^2\mu} \right) \ln \frac{1}{\epsilon}}{\frac{1}{L_f} \left( \frac{1}{(1-\delta)} + \frac{1}{L} + \frac{(1-\delta)L_f}{\delta^2} + \frac{(1-\delta)L}{n\delta} \right) \frac{1}{\epsilon}} \right).
\]

**Remark.** The iteration complexity of EC-LSVRG in the case where $p \geq O(\delta_1)$ can be obtained easily, however, it is no better than the iteration complexity of EC-LSVRG in the case where $p \leq O(\delta_1)$. Thus we omit that case for simplicity.

Noticing that $L_f \leq \bar{L} \leq nL_f$ and $\bar{L} \leq L \leq m\bar{L}$, the iteration complexity in Theorem 2.6 could be better than that in Theorem 2.5. On the other hand, if $L_f = \bar{L} = L$, then both iteration complexities in Theorem 2.5 and Theorem 2.6 in the strongly convex case become

\[
O \left( \left( \frac{1}{\delta} + \frac{1}{p} + \frac{L}{\mu} + \frac{(1-\delta)L}{\delta^2\mu} \right) \ln \frac{1}{\epsilon} \right).
\]

### 2.2 Smooth case ($\psi = 0$) of EC-LSVRG

In this subsection, we study the Algorithm 1 for problem (1) with $\psi = 0$. We need the following assumption in this subsection.

**Assumption 2.7.** $f^{(\tau)}$ is $L$-smooth, $f^{(\tau)}$ is $\bar{L}$-smooth, $f$ is $L_f$-smooth and $f$ is $\mu$-strongly convex.

We also use two Lyapunov functions for two cases: with or without Assumption 1.3 in the following two theorems.

**Theorem 2.8.** Let Assumption 2.1 and Assumption 2.7 hold. Define

\[
\Phi_3^k := \mathbb{E}[\tilde{x}^k - x^*]^2 + \frac{12L_f}{n\delta} \sum_{r=1}^{n} \mathbb{E}[v^k_r]^2 + \frac{192(1-\delta)L_f}{\delta^2\delta_1} \sum_{r=1}^{n} \mathbb{E}[h^k_r - \nabla f^{(\tau)}(w^k)]^2 + \frac{4}{3p} \left( \frac{48(1-\delta)L_f}{\delta} \right) \left( \frac{4\bar{L}}{\delta} + L + \frac{16\bar{L}p}{\delta\delta_1} \left( \frac{1}{2p} + \frac{2p}{\delta_1} \right) \right) \mathbb{E}[f(w^k) - f(x^*)].
\]

If $\eta \leq \frac{1}{4L_f + 8L_f/n}$, then

\[
\mathbb{E}[\Phi_3^{k+1}] \leq \left( 1 - \min \left\{ \frac{\mu \eta^2}{4}, \frac{\delta_1}{4} \right\} \right) \mathbb{E}[\Phi_3^k] - \frac{\eta}{2} \left( 1 - \frac{224(1-\delta)L_f}{\delta} \right) \left( \frac{4\bar{L}}{\delta} + L + \frac{16\bar{L}p}{\delta\delta_1} \left( \frac{1}{2p} + \frac{2p}{\delta_1} \right) \right) \mathbb{E}[f(x^k) - f(x^*)].
\]
Theorem 2.9. Let Assumption 2.1 and Assumption 2.7 hold. Define
\[
\Phi_k := E\|\bar{x}^k - x^*\|^2 + \frac{12L_f \eta}{\delta} E\|e^k\|^2 + \frac{96(1 - \delta) L_f \eta}{n^2 \delta} \sum_{\tau=1}^{n} E\|e^\tau\|^2 \\
+ \frac{192(1 - \delta) L_f \eta^3}{\delta^2 \delta_1} E\|h^k - \nabla f(w^k)\|^2 + \frac{2304(1 - \delta) L_f \eta^3}{\delta^2 \delta_1 n^2} \sum_{\tau=1}^{n} E\|h^\tau - \nabla f(\bar{r})\|^2 \\
+ \frac{4}{3p} \left( \frac{48(1 - \delta) L_f \eta^3}{\delta} \left( \frac{4L_f}{\delta} + \frac{32L}{n\delta} + \frac{13L}{n} + \frac{16p(L_f + \frac{12L}{n})}{\delta \delta_1} \left( 1 + \frac{2p}{\delta_1} \right) \right) + \frac{4L \eta^2}{n} \right) E[f(w^k) - f(x^*)].
\]

Under Assumption 1.3 if \( \eta \leq \frac{1}{4L_f + 8L/n}, \) then
\[
E[\Phi_{k+1}^+] \leq \left( 1 - \min \left\{ \frac{\mu \eta}{2}, \frac{\delta}{4}, \frac{\delta_1}{4}, \frac{p}{4} \right\} \right) E[\Phi_k] \\
- \frac{\eta}{2} \left( 1 - \frac{224(1 - \delta) L_f \eta^2}{\delta} \left( \frac{4L_f}{\delta} + \frac{32L}{n\delta} + \frac{13L}{n} + \frac{16p(L_f + \frac{12L}{n})}{\delta \delta_1} \left( 1 + \frac{2p}{\delta_1} \right) \right) - 11L \eta \right) E[f(x^k) - f(x^*)].
\]

From the above two theorems, we can get the iteration complexity.

Theorem 2.10. Let Assumption 2.1 and Assumption 2.7 hold.
Let \( w_k = (1 - \min \{ \frac{\mu \eta}{2}, \frac{\delta}{4}, \frac{\delta_1}{4}, \frac{p}{4} \} ) \), \( W_k = \sum_{i=0}^{k} w_i \), and \( x^k = \frac{1}{W_k} \sum_{i=0}^{k} w_i x^i \).

(i) Let \( \eta \leq \min \left\{ \frac{1}{4L_f + 33L/n}, \frac{\delta}{60\sqrt{1 - \delta) L_f L}}, \frac{\sqrt{3}}{120 \sqrt{(1 - \delta) L_f L_p (1 + \frac{2p}{\delta_1})}} \right\} \). If \( \mu > 0 \), we have
\[
E[f(\bar{x}^k) - f(x^*)] \leq \frac{9\mu \|x^0 - x^*\|^2 + 2(f(x^0) - f(x^*)) + \frac{1}{15L \eta} \sum_{\tau=1}^{n} \|\nabla f(\bar{r})(x^0) - h_0^\tau\|^2}{1 - \left( 1 - \min \left\{ \frac{\mu \eta}{2}, \frac{\delta}{4}, \frac{\delta_1}{4}, \frac{p}{4} \right\} \right)^{k+1}} \left( 1 - \min \left\{ \frac{\mu \eta}{2}, \frac{\delta}{4}, \frac{\delta_1}{4}, \frac{p}{4} \right\} \right)^{k}.
\]

If \( \mu = 0 \), we have
\[
E[f(\bar{x}^k) - f(x^*)] \leq \frac{1}{k + 1} \left( \frac{18}{\eta} \|x^0 - x^*\|^2 + \frac{1}{15L \eta} \sum_{\tau=1}^{n} \|\nabla f(\bar{r})(x^0) - h_0^\tau\|^2 + \frac{6}{p} (f(x^0) - f(x^*)) \right).
\]

(ii) Let \( \eta = \min \left\{ \frac{1}{4L_f + 33L/n}, \frac{\delta}{60\sqrt{1 - \delta) L_f L}}, \frac{\sqrt{7}}{120 \sqrt{(1 - \delta) L_f L_p (1 + \frac{2p}{\delta_1})}} \right\} \) and \( p \leq O(\delta_1) \). If \( \mu > 0 \) and \( \epsilon \leq 9\mu \|x^0 - x^*\|^2 + 2(f(x^0) - f(x^*)) + \frac{1}{15L \eta} \sum_{\tau=1}^{n} \|\nabla f(\bar{r})(x^0) - h_0^\tau\|^2 \), we have \( E[f(\bar{x}^k) - f(x^*)] \leq \epsilon \) as long as
\[
k \geq O \left( \frac{1}{\delta + \frac{1}{p} + \frac{L_f}{\mu} + \frac{L \eta}{\mu}} + \frac{\sqrt{(1 - \delta)L_f L}}{\delta \mu} + \frac{\sqrt{(1 - \delta)L_f L}}{\sqrt{\delta \mu}} \right) \ln \frac{1}{\epsilon}.
\]

If \( \mu = 0 \), we have \( E[P(\bar{x}^k) - P(x^*)] \leq \epsilon \) as long as
\[
k \geq O \left( \frac{1}{p} + \frac{L_f}{n} + \frac{\sqrt{(1 - \delta)L_f L}}{\delta} + \frac{\sqrt{(1 - \delta)L_f L}}{\sqrt{\delta}} \right) \frac{1}{\epsilon}.
\]

Theorem 2.11. Let Assumption 2.1 and Assumption 2.7 hold. Assume the compressors \( Q \) and \( Q_1 \) also satisfy Assumption 1.3. Let \( w_k = (1 - \min \{ \frac{\mu \eta}{2}, \frac{\delta}{4}, \frac{\delta_1}{4}, \frac{p}{4} \} ) \), \( W_k = \sum_{i=0}^{k} w_i \), and \( \bar{x}^k = \frac{1}{W_k} \sum_{i=0}^{k} w_i x^i \).
(i) Let $\eta \leq \min \left\{ \frac{1}{4L_f + 33L/n}, \frac{\delta}{60\sqrt{1-\delta}L_f}, \frac{\sqrt{n}\delta}{229\sqrt{(1-\delta)L_f}L}, \frac{\sqrt{n}\delta}{360\sqrt{(1-\delta)L_f}L}, \frac{\delta\sqrt{\delta_1}}{120\sqrt{(1-\delta)pL_f \left( L_f + \frac{15L}{n}\right) \left( 1 + \frac{2\eta}{\alpha}\right)}} \right\} \right)$. If $\mu > 0$, we have

$$E[f(\bar{x}^k) - f(x^*)] \leq \frac{18\eta \|x^0 - x^*\|^2 + \frac{2}{L_f p n} \sum_{i=1}^n \|\nabla f(\tau_i)(x^0) - h_{\tau_i}^0\|^2 + 5(f(x^0) - f(x^*))}{1 - (1 - \min\left\{ \frac{\mu\eta\delta}{2}, \frac{\delta_1}{4}, \frac{\mu\eta\delta_1}{4}\right\} )^k + 1.}
$$

If $\mu = 0$, we have

$$E[f(\bar{x}^k) - f(x^*)] \leq \frac{\frac{36}{k+1} \eta \|x^0 - x^*\|^2 + \frac{6}{L_f p n} \sum_{i=1}^n \|\nabla f(\tau_i)(x^0) - h_{\tau_i}^0\|^2 + \frac{18}{p} (f(x^0) - f(x^*))}{k \left( 1 - \frac{\mu\eta\delta}{2}, \frac{\delta_1}{4}, \frac{\mu\eta\delta_1}{4}\right) \ln \frac{1}{\epsilon}}.$$

(ii) Let $\eta$ equal to the upper bound in (2) and $p \leq O(\delta_1)$. If $\mu > 0$ and $\epsilon \leq 9\mu \|x^0 - x^*\|^2 + \frac{1}{\sqrt{\tau_i n} \sum_{i=1}^n \|\nabla f(\tau_i)(x^0) - h_{\tau_i}^0\|^2 + 3(f(x^0) - f(x^*))$, we have $E[f(\bar{x}^k) - f(x^*)] \leq \epsilon$ as long as

$$k \geq O \left( \frac{\frac{1}{p} + \frac{L_f}{\mu} + \frac{L}{n\mu} + \sqrt{(1-\delta)\frac{L_f}{\mu\delta}}}{\ln \frac{1}{\epsilon}} \right).$$

If $\mu = 0$, we have $E[P(\bar{x}^k) - P(x^*)] \leq \epsilon$ as long as

$$k \geq O \left( \frac{\frac{1}{p} + \frac{L_f}{n} + \sqrt{(1-\delta)\frac{L_f}{\delta}}}{\ln \frac{1}{\epsilon}} \right).$$

Same as the composite case, the iteration complexity in Theorem 2.11 could be better than that in Theorem 2.10. On the other hand, if $L_f = L = L$, then both iteration complexities in Theorem 2.10 and Theorem 2.11 in the strongly convex case become

$$O \left( \frac{\frac{1}{p} + \frac{L}{\mu} + \sqrt{(1-\delta)\frac{L}{\delta\mu}}}{\ln \frac{1}{\epsilon}} \right).$$

3 Error Compensated Quartz and Error Compensated SDCA

In this section, we study the following problem:

$$\min_{x \in \mathbb{R}^d} P(x) = \frac{1}{N} \sum_{r=1}^n \sum_{i=1}^m \phi_{ir} (A_{ir}^T x) + \lambda g(x),$$

where $N = mn$ and $A_{ir} \in \mathbb{R}^{d \times t}$. The corresponding dual problem of problem (8) is

$$\max_{\alpha \in \mathbb{R}^N} D(\alpha) = -\tilde{f}(\alpha) - \tilde{\psi}(\alpha),$$
Algorithm 2 Error compensated Quartz (EC-Quartz) and error compensated SDCA (EC-SDCA)

1: Parameters: \( \theta > 0; R_m := \text{max}_{i,\tau} ||A_{i\tau}||; \)
\[ R^2 := \frac{1}{m} \lambda_{\text{max}}(\sum_{i=1}^{m} A_{i\tau}A_{i\tau}^T); \]
\[ v_{i\tau} = \frac{1}{m} R + nR^2 \in \mathbb{R} \text{ for } i \in [m] \text{ and } \tau \in [n]; \]
2: Initialization: \( \alpha^0 \in \mathbb{R}^t; x^0 \in \mathbb{R}^d; \)  \( u^0 = \frac{1}{\lambda N} \sum_{\tau=1}^{\tau} \sum_{i=1}^{m} A_{i\tau} \alpha^0_{i\tau} \in \mathbb{R}^d; e^0_\tau = 0 \in \mathbb{R}^d \text{ for } \tau \in [n] \)
3: for \( k = 0, 1, 2, \ldots \) do
4: for \( \tau = 1, \ldots, n \) do
5: EC-Quartz: \( x^{k+1} = (1 - \theta)x^k + \theta \nabla g^*(u^k) \)
6: EC-SDCA: \( x^{k+1} = \nabla g^*(u^k) \)
7: \( \alpha^{k+1}_{i\tau} = \alpha^{k}_{i\tau} \text{ for } i \in [m] \)
8: Sample \( i^\tau_k \) uniformly and independently in \([m]\) on each node
9: \( \Delta \alpha^{k+1}_{i\tau} = -\theta p^{-1}_{i\tau} \alpha^{k}_{i\tau} - \theta \nabla \phi_{i\tau}^\ast(A_{i\tau}^T x^{k+1}) \)
10: \( \alpha^{k+1}_{i\tau} = \alpha^{k}_{i\tau} + \Delta \alpha^{k+1}_{i\tau} \)
11: \( y^k_\tau = Q \left( \frac{1}{\lambda m} A_{i\tau}^T \Delta \alpha^{k+1}_{i\tau} + e^k \right) \)
12: \( e^{k+1}_\tau = e^k + \frac{1}{\lambda m} A_{i\tau}^T \Delta \alpha^{k+1}_{i\tau} - y^k_\tau \)
13: Send \( y^k_\tau \) to the other nodes
14: Receive \( y^k_\tau \) from the other nodes
15: \( u^{k+1}_\tau = u^k + \frac{1}{n} \sum_{\tau=1}^{\tau} y^k_\tau \)
16: end for
17: end for

where \( \hat{f}(\alpha) := \lambda g^\ast(\frac{1}{\lambda m} \sum_{i=1}^{m} A_{i\tau} \alpha i\tau), \hat{\psi}(\alpha) := \frac{1}{\lambda m} \sum_{i=1}^{m} \sum_{\tau=1}^{\tau} \phi^\ast_{i\tau}(-\alpha_{i\tau}), \alpha_{i\tau} \in \mathbb{R}^t, \)
\[ \alpha = (\alpha^T_{i1}, \ldots, \alpha^T_{im1}, \alpha^T_{i2}, \ldots, \alpha^T_{im2}, \ldots, \alpha^T_{in1}, \ldots, \alpha^T_{imn})^T \in \mathbb{R}^{tN}, \]
\( \phi^\ast_{i\tau} \) and \( g^\ast \) are the conjugate functions of \( \phi_{i\tau} \) and \( g \) respectively. Generally, for any vector \( h \in \mathbb{R}^tN, \) we use \( h_{i\tau} \in \mathbb{R}^t \) with \( i \in [m] \) and \( \tau \in [n] \) to denote the \( i+m(\tau-1) \)-th block vector of \( h \).

We need the following assumptions in this section.

Assumption 3.1. The compressor \( Q \) in Algorithm 2 is a contraction compressor with parameters \( \delta \).

Assumption 3.2. \( \phi_{i\tau} \) is \( \frac{1}{\gamma} \)-smooth. \( g \) is 1-strongly convex. \( \frac{R^2}{\gamma} \geq \lambda > 0 \).

The error compensated Quartz and error compensated SDCA are described in Algorithm 2. Quartz is a variance reduced primal-dual method, and also a minibatch version of SDCA. The updates of \( x^k \) in Quartz and SDCA are slightly different, and the rest steps are the same. In distributed Quartz, each node needs to communicate \( \frac{1}{\lambda m} A_{i\tau}^T \Delta \alpha^{k+1}_{i\tau} \) with each other at each step. The error feedback technique can be applied easily in this case. We maintain an accumulated vector \( e^k \) on each node, and add it to \( \frac{1}{\lambda m} A_{i\tau}^T \Delta \alpha^{k+1}_{i\tau} \) before compression. \( e^{k+1}_\tau \) is updated by the compression error at iteration \( k \) for each node. All nodes maintain the same copies of \( x^k \) and \( u^k \). The rest of Algorithm 2 is the same as Quartz and SDCA.

3.1 Convergence of EC-Quartz

Theorem 3.3. Let Assumption 3.1 and Assumption 3.2 hold. Assume \( \delta < 1 \). Define
\[ \Psi^k_\tau := P(x^k) - D(\alpha^k) + \frac{2(\rho + \theta \lambda)}{\delta n} \sum_{\tau=1}^{n} ||e^k_\tau||^2, \]
where \( \alpha^k := ((\alpha^k_{11})^T, \ldots, (\alpha^k_{m1})^T, (\alpha^k_{12})^T, \ldots, (\alpha^k_{m2})^T, \ldots, (\alpha^k_{n1})^T, \ldots, (\alpha^k_{mn})^T)^T \in \mathbb{R}^{tN} \), \( \rho = \frac{\delta R}{2\sqrt{n}} \) and \( a_1 = (1 - \delta)(2\bar{R}^2 + \delta R_m^2) \). Let

\[
\theta = \min \left\{ \frac{2\delta \lambda \gamma}{\delta \lambda \gamma m + \delta^2 \lambda^2 \gamma^2 m^2 + 48 \lambda \gamma a_1}, \frac{N \lambda \gamma \rho_{\tau}}{3 \nu_{\tau} + N \lambda \gamma}, \frac{\delta \lambda \gamma}{\delta \lambda \gamma m + 12 \sqrt{\lambda} \gamma} \right\}. \tag{9}
\]

Then \( \mathbb{E}[\Psi_0^k] \leq (1 - \min \{ \theta, \frac{\delta}{4} \})^k \psi_0^0 \), and we have \( \mathbb{E}[\Psi_0^k] \leq \epsilon \) as long as

\[ k \geq O \left( \left( \frac{1}{\delta} + m + \frac{R_m^2}{\lambda \gamma} + \frac{R^2}{\lambda \gamma} + \sqrt{\frac{1 - \delta}{\delta}} \bar{R} \right) \ln \frac{1}{\epsilon} \right). \tag{10} \]

**Theorem 3.4.** Let Assumption 3.3, Assumption 3.1, and Assumption 3.2 hold. Assume \( \delta < 1 \). Define

\[ \psi_2^k := P(x^k) - D(\alpha^k) + \frac{2(\rho + \theta \lambda)}{\delta} \|e^k\|^2 + \frac{2(a_2 - 1)}{\delta^2 R^2} \|e^k\|^2, \]

where \( e^k = \frac{1}{n} \sum_{\tau=1}^n e^k \), \( \rho = \frac{\delta R}{2\sqrt{n}} \) and \( a_2 = (1 - \delta)(2\bar{R}^2 + \frac{16\bar{R}^2}{n} + \frac{9\bar{R}^2}{n}) \). Let

\[
\theta = \min \left\{ \frac{2\delta \lambda \gamma}{\delta \lambda \gamma m + \delta^2 \lambda^2 \gamma^2 m^2 + 48 \lambda \gamma a_2}, \frac{N \lambda \gamma \rho_{\tau}}{3 \nu_{\tau} + N \lambda \gamma}, \frac{\delta \lambda \gamma}{\delta \lambda \gamma m + 12 \sqrt{\lambda} \gamma a_2} \right\}. \tag{11}
\]

Then \( \mathbb{E}[\Psi_2^k] \leq (1 - \min \{ \theta, \frac{\delta}{4} \})^k \psi_0^0 \), and we have \( \mathbb{E}[\Psi_2^k] \leq \epsilon \) as long as

\[ k \geq O \left( \left( \frac{1}{\delta} + m + \frac{R_m^2}{\lambda \gamma} + \frac{R^2}{\lambda \gamma} + \sqrt{\frac{1}{\delta}} \frac{\bar{R}^2}{\gamma} \right) \ln \frac{1}{\epsilon} \right). \tag{12} \]

Noticing that \( R^2 \leq \bar{R}^2 \leq nR^2 \) and \( \bar{R}^2 \leq R_m^2 \leq m\bar{R}^2 \), the iteration complexity in Theorem 3.4 could be better than that in Theorem 3.3. On the other hand, if \( R^2 = \bar{R}^2 = R_m^2 \), then both iteration complexities in Theorem 3.3 and Theorem 3.4 become

\[ O \left( \left( \frac{1}{\delta} + m + \frac{R^2}{\lambda \gamma} + \sqrt{\frac{1}{\delta}} \frac{\bar{R}^2}{\gamma} \right) \ln \frac{1}{\epsilon} \right). \tag{13} \]

### 3.2 Convergence of EC-SDCA

Define \( \psi_3^k := \psi_2^k + \frac{2(\rho + \theta \lambda)}{\delta} \|e^k\|^2 \), where \( \rho = \frac{\delta R}{2\sqrt{n}} \) and \( a_1 = (1 - \delta)(2\bar{R}^2 + \delta R_m^2) \). Choose \( \theta \) as in (2). Then \( \mathbb{E}[\Psi_3^k] \leq (1 - \min \{ \theta, \frac{\delta}{4} \})^k \psi_0^0 \), and we have \( \mathbb{E}[\Psi_3^k] \leq \epsilon \) as long as \( k \) satisfies (10). Let \( w_k = (1 - \min \{ \theta, \frac{\delta}{4} \})^{-k} \), \( W_k = \sum_{i=1}^k w_i \), and

\[ \bar{x}^k = \frac{1}{W_k} \sum_{i=1}^k w_i x^i. \]

Then we have

\[ \mathbb{E}[P(\bar{x}^k) - P(x^*)] \leq \frac{(1 - \min \{ \theta, \frac{\delta}{4} \})^k \epsilon_0^0}{1 - (1 - \min \{ \theta, \frac{\delta}{4} \})^k}, \tag{14} \]

and for \( \epsilon \leq \epsilon_0^0 \), \( \mathbb{E}[P(\bar{x}^k) - P(x^*)] \leq \epsilon \) as long as \( k \) satisfies (10).
Theorem 3.6. Let Assumption 1.3, Assumption 3.1, and Assumption 3.2 hold. Assume \( \delta < 1 \). Define
\[
\Psi_4^k := \frac{1}{T_D} + \frac{2(\rho + \theta \lambda)}{\delta} \|e^k\|^2 + \frac{16(1 - \delta)(\rho + \theta \lambda)}{\delta n^2} \sum_{\tau = 1}^{n} \|e^k\|_\tau^2,
\]
where \( e^k = \frac{1}{n} \sum_{i=1}^{n} e_i^k \), \( \rho = \frac{\delta_1 R}{T_D / 2\delta} \) and \( a_2 = (1 - \delta)(2R^2 + \frac{16\delta R^2}{n} + \frac{6\delta^2 R^2}{n}) \). Choose \( \theta \) as in (11). Then \( \mathbb{E}[\Psi_4^k] \leq (1 - \min\{\theta, \frac{1}{4}\})^k \Psi_4^1 \), and we have \( \mathbb{E}[\Psi_4^k] \leq \epsilon \) as long as \( k \) satisfies (12). Let \( w_k = (1 - \min\{\theta, \frac{1}{4}\})^{-k} \), \( W_k = \sum_{i=1}^{k} w_i \), and \( x_k = \frac{1}{W_k} \sum_{i=1}^{k} w_i x_i \). Then \( \mathbb{E}[P(x_k) - P(x^*)] \) satisfies (14), and for \( \epsilon \leq \epsilon_3^0 \), \( \mathbb{E}[P(x_k) - P(x^*)] \leq \epsilon \) as long as \( k \) satisfies (12).

![Figure 1: EC-LSVRG and EC-SDCA with different compressors](image)

### 4 Experiments

In this section, we run experiments with EC-LSVRG, EC-Quartz, and EC-SDCA to demonstrate the empirical effectiveness. In particular, we should highlight the linear convergence rate of our algorithm with the biased compressor and non-smooth objective function. Also, the communication complexity performance is competitive to other compressed algorithms.

**Setting.** We implement the logistic regression problem with \( L_1-L_2 \) regularization
\[
\min_{x \in \mathbb{R}^d} \left\{ \frac{1}{n} \sum_{i=1}^{n} \log(1 + \exp(-b_i a_i^T x)) + \lambda_1 \|x\|_1 \right\} + \frac{\lambda_2}{2} \|x\|_2^2,
\]
where \( \{a_i, b_i\}_{i \in [n]} \) are data samples. We use Python 3.7 to perform experiments on a server with 2 processors (Intel Xeon Gold 5120 @ 2.20GHz), 28 cores in total. Library include numpy, sklearn. We search the optimal step size from \( \{10^t, 3 \times 10^t\} \), where \( t \in \{-4, \cdots, 0, 1\} \) for all tested algorithms. We choose the same contraction compressors for \( Q \) and \( Q_1 \) for EC-LSVRG. Without more specification, we use \( L_1-L_2 \) regularization with \( \lambda_1 = \lambda_2 = 10^{-3} \) and \( \rho = \delta \) for EC-LSVRG. For EC-LSVRG-DIANA, we choose \( p = \frac{1}{m} \). For VR-DIANA, we use the optimal \( \alpha = \frac{1}{(w+1)} \). For smooth experiments, we use \( \lambda_1 = 0 \) and \( \lambda_2 = 10^{-3} \).

**Datasets.** We have four real data sets: \texttt{a5a}, \texttt{a9a}, \texttt{mushrooms}, and \texttt{w6a}, from the LIBSVM library [Chang and Lin, 2011].
Compressors. We use RandK, TopK, random dithering in [Alistarh et al., 2017], natural compression in [Horváth et al., 2019b], RTopK, and NTopK. It should be noticed that the random dithering and natural compression are both unbiased compressors. When we use them as contraction compressors, we mean the ones scaled by \( 1/(\omega+1) \). RandK is a contraction compressor. When we use it as an unbiased compressor, we mean the one scaled by \( d/K \). For RandK and TopK, \( \delta = K/d \), and the number of communicated bits for the compressed vector in \( \mathbb{R}^d \) is \( (64 + \lceil \log d \rceil)K \). For random dithering, we choose the level \( s = \sqrt{d} \), the number of communicated bits for the compressed vector is \( 2.8d + 64 \), and \( \omega = 1 \). For natural compression, the number of communicated bits for the compressed vector is \( 12K + K\lceil \log d \rceil \), and \( \delta = 8K/ad \).

4.1 TopK, random dithering, natural compression vs no compression

We firstly compare our algorithm with different compressors in Figure 1. It shows that, for the communication complexity, EC-LSVRG and EC-SDCA with contraction compressors are superior to the uncompressed ones, especially for Top1 compressor.

4.2 Comparison with ECSGD, ECGD, and EC-LSVRG-DIANA

We also compare our algorithms with the baseline EC-GD and state-of-the-art competitor EC-LSVRG-DIANA [Gorbunov et al., 2020], VR-DIANA [Horváth et al., 2019a] in Figures 2-9. For the subfigures where the title is “Top1/Rand1”, we use Top1 in EC-GD, EC-LSVRG-DIANA and our algorithms and Rand1 in VR-DIANA. For EC-LSVRG-DIANA, an unbiased compressor is also needed. Thus, for the Top1/Rand1 case, we use Rand1; for random dithering and natural compression cases, we use random dithering and natural compression, respectively, for the unbiased compressor in EC-LSVRG-DIANA.

Because of the compression error, EC-GD could not converge to the optimal solution. For EC-LSVRG-DIANA, it converges linearly to the optimal solution when the objective function is smooth, and the communication complexity performance of it is almost the same as that of EC-LSVRG. However, EC-LSVRG-DIANA does not support non-smooth objective function well, leading to a biased solution. These figures show that in the most cases, EC-LSVRG or EC-SDCA performs the best. VR-DIANA is not compatible with Top1 compressor, which is extremely efficient. While our methods, including EC-LSVRG and EC-SDCA, perform well on either smooth or non-smooth case with Top1 compressor.
Figure 4: Comparison with ECGD, EC-LSVRG-DIANA, VR-DIANA on a5a (non-smooth case)

Figure 5: Comparison with ECSGD, ECGD, and EC-LSVRG-DIANA on a5a (smooth case)

Figure 6: Comparison with ECGD, EC-LSVRG-DIANA, VR-DIANA on a5a (non-smooth case)

Figure 7: Comparison with ECSGD, ECGD, and EC-LSVRG-DIANA on a9a (smooth case)

Figure 8: Comparison with ECGD, EC-LSVRG-DIANA, VR-DIANA on w6a (non-smooth case)
4.3 TopK vs NTopK vs RTopK

Previous experiments have shown the efficiency of the contraction compressor. In this context, we consider using random dithering + TopK (RTopK) and natural compression + TopK (NTopK) to further improve the performance. It should be noted that NTopK is suitable for any $K$, while for RTopK, we usually require $K > 1$. By Figures 10–13, we can notice that either NTopK or RTopK reduces the communication costs than TopK only.

![Figure 10: Comparison among TopK, NTopK, and RTopK on mushrooms](image)

![Figure 11: Comparison among TopK, NTopK, and RTopK on a5a](image)

Figure 9: Comparison with ECSGD, ECGD, and EC-LSVRG-DIANA on w6a (smooth case)
4.4 Impact of the update frequency parameter $p$

Our default setting in EC-LSVRG is $p = \delta$. In this part, we investigate the impact of the update frequency $p$. From the theoretical results, it is easy to verify that an optimal choice of $p$ is $\Theta(\delta)$, and too large or small $p$ may lead to a slower convergence. By Figure 14 when $p = \delta/3$, the convergence is usually much slower (mushrooms, w6a). When $p = 1$, the performance is no better than $p = \delta$, generally. In particular, large $p$ makes convergence slower on a5a and a9a.
4.5 EC-SDCA vs EC-Quartz

Although in theory, EC-SDCA and EC-Quartz have the same iteration complexities, they perform differently in actual data. Figures 15 - 18 show that EC-SDCA is usually comparable to EC-Quartz or better than EC-Quartz, and sometimes much better, especially for Top1 compressor. Thus, we prefer EC-SDCA for more general scenarios.

Figure 15: EC-SDCA vs EC-Quartz on mushrooms

Figure 16: EC-SDCA vs EC-Quartz on a5a

Figure 17: EC-SDCA vs EC-Quartz on a9a

Figure 18: EC-SDCA vs EC-Quartz on w6a
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Appendix

A ESO Estimation for Arbitrary Sampling for Quartz

For simplicity, in this section we consider problem (8) with \( m = 1 \), and replace \( \phi_i \tau \) and \( A_i \tau \) with \( \phi_\tau \) and \( A_\tau \) respectively. We consider arbitrary proper set sampling, i.e., \( S \in [n] \) with \( p_i := \Pr[i \in S] > 0 \) for all \( i \in [n] := \{1, ..., n\} \).

**Assumption A.1.** There exist constants \( A_i \geq 0 \) for each \( i \in [n] \) and \( 0 \leq B \leq 1 \) such that for any matrix \( M \in \mathbb{R}^{t \times n} \) and sampling \( S \),

\[
E \left[ \left\| \sum_{i \in S} \frac{1}{p_i} M_i \right\|^2 \right] \leq \sum_{i=1}^{n} A_i \| M_i \|^2 + B \sum_{i=1}^{n} \| M_i \|_2^2,
\]

where \( M_i \) denotes the \( i \)th column vector of \( M \).

Assumption A.1 appeared in [Qian et al., 2019a] and [Qian et al., 2019b] for the convergence analysis of SAGA and L-SVRG. The estimations of \( A_i \) and \( B \) for arbitrary set sampling, \( \tau \)-nice sampling, and group sampling can be found in [Qian et al., 2019b].

**Lemma A.2.** Under Assumption A.1, the following inequality holds for all \( h_i \in \mathbb{R}^t \) with \( i \in [n] \),

\[
E \left[ \left\| \sum_{i \in S} A_i h_i \right\|^2 \right] \leq \sum_{i=1}^{n} p_i v_i \| h_i \|^2,
\]

where \( v_i = A_i p_i R^2_m + B p_i R^2 \).

**Proof.** By applying Assumption A.1 with \( M_i = p_i A_i h_i \), we have

\[
E \left[ \left\| \sum_{i \in S} A_i h_i \right\|^2 \right] = E \left[ \left\| \sum_{i \in S} \frac{1}{p_i} p_i A_i h_i \right\|^2 \right] \\
\leq \sum_{i=1}^{n} A_i \| p_i A_i h_i \|^2 + B \| \sum_{i=1}^{n} p_i A_i h_i \|_2^2 \\
\leq \sum_{i=1}^{n} A_i p_i^2 \| A_i \|^2 \| h_i \|^2 + B \sum_{i=1}^{n} \| p_i A_i h_i \|_2^2 \\
\leq \sum_{i=1}^{n} A_i p_i^2 R^2_m \| h_i \|^2 + B \sum_{i=1}^{n} \| p_i A_i h_i \|_2^2,
\]

where we use \( R_m = \max_i \{ \| A_i \| \} \) in the last inequality.
For $\|\sum_{i=1}^n p_i A_i h_i \|^2$, since $\sum_{i=1}^n p_i A_i h_i = [A_1, \ldots, A_n][p_1 h_1^\top, \ldots, p_n h_n^\top]^\top$, we have

$$\left\| \sum_{i=1}^n p_i A_i h_i \right\|^2 \leq \|[A_1, \ldots, A_n]\|^2 \cdot \|[p_1 h_1^\top, \ldots, p_n h_n^\top]^\top\|^2 = \lambda_{\text{max}}([A_1, \ldots, A_n][A_1, \ldots, A_n]^\top) \cdot \sum_{i=1}^n p_i^2 \|h_i\|^2$$

$$= \lambda_{\text{max}} \left( \sum_{i=1}^n A_i A_i^\top \right) \cdot \sum_{i=1}^n p_i^2 \|h_i\|^2 \leq \sum_{i=1}^n n R^2 p_i^2 \|h_i\|^2,$$

where we use $R^2 = \frac{1}{n} \lambda_{\text{max}} \left( \sum_{i=1}^n A_i A_i^\top \right)$ in the last inequality. Combining the above two inequalities, we arrive at

$$\mathbb{E} \left[ \left\| \sum_{i \in S} A_i h_i \right\|^2 \right] \leq \sum_{i=1}^n A_i p_i^2 R_m^2 \|h_i\|^2 + B \sum_{i=1}^n n R^2 p_i^2 \|h_i\|^2 \leq \sum_{i=1}^n p_i \left( A_i p_i R_m^2 + B n p_i R^2 \right) \|h_i\|^2.$$

\[\Box\]

## B Proofs for EC-LSVRG in the Composite Case

### B.1 Lemmas

Let $\mathbb{E}_k[\cdot]$ denote the expectation conditional on $x^k, w^k, h^k, u^k$, and $e^k_k$.

The following lemma shows the progress at iteration $k$ for the auxiliary points $\hat{x}^k$ and $\hat{x}^{k+1}$.

**Lemma B.1.** If $\eta \leq \frac{1}{4\mu}$, then

$$\left(1 + \frac{\eta \mu}{2}\right) \mathbb{E}_k \|\hat{x}^{k+1} - x^k\|^2 \leq \|\hat{x}^k - x^k\|^2 + 2\eta \mathbb{E}_k [P(x^k) - P(x^{k+1})] + \|h^k\|^2 + (1 + \frac{\eta \mu}{2}) \mathbb{E}_k \|e^{k+1}\|^2 + 4\eta^2 \mathbb{E}_k \|g^k + h^k - \nabla f(x^k)\|^2.$$

**Proof.** Since $\hat{x}^{k+1} = \hat{x}^k - \eta (g^k + h^k + \partial \psi(x^{k+1}))$, we have

$$\langle \eta (g^k + h^k), x^k - x^{k+1} \rangle$$

$$= \langle \hat{x}^k - \hat{x}^{k+1} - \eta \partial \psi(x^{k+1}), x^k - x^{k+1} \rangle$$

$$= \langle \hat{x}^k - x^{k+1}, x^k - x^{k+1} \rangle + \langle x^{k+1} - \hat{x}^{k+1}, x^k - x^{k+1} \rangle - \eta \langle \partial \psi(x^{k+1}), x^k - x^{k+1} \rangle$$

$$\geq \frac{1}{2} (\|\hat{x}^k - x^k\|^2 + \|\hat{x}^{k+1} - x^{k+1}\|^2 + \|x^{k+1} - x^k\|^2) + \frac{1}{2} (\|x^{k+1} - x^k\|^2 - \|x^{k+1} - x^{k+1}\|^2 - \|x^{k+1} - x^{k+1}\|^2 + \eta \left( \psi(x^{k+1}) - \psi(x^k) + \frac{\mu}{2} \|x^{k+1} - x^k\|^2 \right))$$

$$= \frac{1}{2} \|\hat{x}^{k+1} - x^k\|^2 - \frac{1}{2} \|\hat{x}^k - x^k\|^2 + \frac{1}{2} \|\hat{x}^{k+1} - x^{k+1}\|^2 - \frac{1}{2} \|x^{k+1} - x^{k+1}\|^2 + \eta \left( \psi(x^{k+1}) - \psi(x^k) + \frac{\mu}{2} \|x^{k+1} - x^k\|^2 \right).$$

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From $\|x^k - x^{k+1}\|^2 \geq \frac{1}{2}\|x^{k+1} - x^k\|^2 - \|x^k - x^k\|^2$, and $\|x^{k+1} - x^*\|^2 \geq \frac{1}{2}\|x^{k+1} - x^*\|^2 - \|x^k - x^k\|^2$, we arrive at

$$
\langle \eta (g^k + h^k), x^* - x^{k+1} \rangle 
\geq \frac{1 + \eta \mu/2}{2}\|x^{k+1} - x^*\|^2 - \frac{1}{2}\|x^k - x^*\|^2 + \frac{1}{4}\|x^{k+1} - x^*\|^2
- \frac{1 + \eta \mu}{2}\|x^{k+1} - x^+2\|^2 + \eta (\psi(x^{k+1}) - \psi(x^*)).
$$

(16)

Since $f$ is convex and $E_k[g^k + h^k] = \nabla f(x^k)$, we have

$$
f(x^*) \geq f(x^k) + \langle \nabla f(x^k), x^* - x^k \rangle 
= f(x^k) + E_k[(g^k + h^k, x^* - x^{k+1} + x^{k+1} - x^k)] 
= f(x^k) + E_k[(g^k + h^k, x^* - x^{k+1})] + E_k[(g^k + h^k - \nabla f(x^k), x^{k+1} - x^k)] 
+ E_k[\nabla f(x^k), x^{k+1} - x^k] 
\geq E_k[f(x^{k+1})] - \frac{L_f}{2}E_k[\|x^{k+1} - x^k\|^2] + E_k[(g^k + h^k, x^* - x^{k+1})]
+ E_k[(g^k + h^k - \nabla f(x^k), x^{k+1} - x^k)] 
\geq E_k[f(x^{k+1})] - \frac{L_f}{2}E_k[\|x^{k+1} - x^k\|^2] + E_k[(g^k + h^k, x^* - x^{k+1})]
- \frac{1}{2\beta}E_k[\|g^k + h^k - \nabla f(x^k)\|^2] - \frac{\beta}{2}E_k[\|x^{k+1} - x^k\|^2],
$$

where the second inequality comes from that $f$ is $L_f$-smooth and the last inequality comes from Young’s inequality with any $\beta > 0$.

By choosing $\beta = \frac{1}{4\eta}$, we have

$$
f(x^*) 
\geq E_k[f(x^{k+1})] - \left(\frac{L_f}{2} + \frac{1}{8\eta}\right)E_k[\|x^{k+1} - x^k\|^2] + E_k[(g^k + h^k, x^* - x^{k+1})]
- 2\eta E_k[\|g^k + h^k - \nabla f(x^k)\|^2] 
\geq E_k[f(x^{k+1})] + \left(\frac{1}{4\eta} - \frac{L_f}{2} - \frac{1}{8\eta}\right)E_k[\|x^{k+1} - x^k\|^2] + \frac{1 + \eta \mu/2}{2\eta}E_k[\|x^{k+1} - x^*\|^2]
- \frac{1}{2\eta}\|x^k - x^*\|^2 - \frac{1}{2\eta}\|x^k - x^*\|^2 - \frac{1 + \eta \mu}{2\eta}E_k[\|x^{k+1} - x^k\|^2]
+ E_k[\psi(x^{k+1})] - \psi(x^*) - 2\eta E_k[\|g^k + h^k - \nabla f(x^k)\|^2].
$$

Noticing that $\frac{1}{4\eta} - \frac{L_f}{2} - \frac{1}{8\eta} \geq 0$ if $\eta \leq \frac{1}{4L_f}$, we can get the result after rearrangement.

\[\square\]

**Lemma B.2.** We have

$$
\frac{1}{n} \sum_{\tau=1}^{n} E_k \left\| \nabla f^{(\tau)}(x^k) - \nabla f^{(\tau)}(w^k) \right\|^2 \leq 4L[P(x^k) - P(x^*)] + 4L[P(w^k) - P(x^*)],
$$

(17)

and

$$
\frac{1}{n} \sum_{\tau=1}^{n} \left\| \nabla f^{(\tau)}(x^k) - \nabla f^{(\tau)}(w^k) \right\|^2 \leq 4\bar{L}[P(x^k) - P(x^*)] + 4\bar{L}[P(w^k) - P(x^*)],
$$

(18)
\[
\mathbb{E}_k \left[ \frac{1}{n} \sum_{\tau=1}^{n} \left( \nabla f^{(\tau)}_i(x^k) - \nabla f^{(\tau)}_i(w^k) \right) \right]^2 \leq 4 \left( L_f + \frac{L}{n} \right) [P(x^k) - P(x^*)] + 4 \left( L_f + \frac{L}{n} \right) [P(w^k) - P(x^*)].
\]

and

\[
\mathbb{E}_k \left[ \frac{1}{n} \sum_{\tau=1}^{n} \left( \nabla f^{(\tau)}_i(x^k) - \nabla f^{(\tau)}_i(w^k) \right) + \nabla f(w^k) - \nabla f(x^k) \right]^2 \leq \frac{4L}{n} [P(x^k) - P(x^*) + P(w^k) - P(x^*)].
\]

Proof. Since \( f_i^{(\tau)} \) is \( L \)-smooth and \( f \) is \( L_f \)-smooth, we have ([Nesterov, 2004], Theorem 2.1.5)

\[
\|\nabla f_i^{(\tau)}(x) - \nabla f_i^{(\tau)}(y)\|^2 \leq 2L(f_i^{(\tau)}(x) - f_i^{(\tau)}(y) - \langle \nabla f_i^{(\tau)}(y), x - y \rangle),
\]

and

\[
\|\nabla f(x) - \nabla f(y)\|^2 \leq 2L_f (f(x) - f(y) - \langle \nabla f(y), x - y \rangle),
\]

for any \( x, y \in \mathbb{R}^d \). Therefore,

\[
\mathbb{E}_k \|\nabla f_i^{(\tau)}(x^k) - \nabla f_i^{(\tau)}(w^k)\|^2 \leq 2\mathbb{E}_k \|\nabla f_i^{(\tau)}(x^k) - \nabla f_i^{(\tau)}(x^*)\|^2 + 2\mathbb{E}_k \|\nabla f_i^{(\tau)}(w^k) - \nabla f_i^{(\tau)}(x^*)\|^2 \leq 4L[f_i^{(\tau)}(x^k) - f_i^{(\tau)}(x^*) - \langle \nabla f_i^{(\tau)}(x^*), x^k - x^* \rangle] + 4L[f_i^{(\tau)}(w^k) - f_i^{(\tau)}(x^*) - \langle \nabla f_i^{(\tau)}(x^*), w^k - x^* \rangle],
\]

and

\[
\mathbb{E}_k \left[ \frac{1}{n} \sum_{\tau=1}^{n} \left( \nabla f_i^{(\tau)}(x^k) - \nabla f_i^{(\tau)}(w^k) \right) \right]^2 = \mathbb{E}_k \left[ \frac{1}{n} \sum_{\tau=1}^{n} q_i^k \right]^2 = \frac{1}{n^2} \mathbb{E}_k \left( \sum_{\tau=1}^{n} q_i^k, \sum_{\tau=1}^{n} q_i^k \right) = \frac{1}{n^2} \sum_{\tau_1 \neq \tau_2} \mathbb{E}_k \left( q_i^{\tau_1}, q_i^{\tau_2} \right) = \frac{1}{n^2} \sum_{\tau_1 \neq \tau_2} \mathbb{E}_k \|q_i^{\tau_1} - q_i^{\tau_2}\|^2 + \frac{1}{n^2} \sum_{\tau_1 \neq \tau_2} \|\nabla f^{(\tau_1)}(x^k) - \nabla f^{(\tau_2)}(x^k) - \nabla f^{(\tau_2)}(w^k)\|^2 \leq \frac{1}{n^2} \sum_{\tau=1}^{n} \mathbb{E}_k \|q_i^{\tau}\|^2 + 2\|\nabla f(x^k) - \nabla f(w^k)\|^2 + 2\|\nabla f(w^k) - \nabla f(x^k)\|^2 \leq \left( \frac{4L}{n} + 4L_f \right) [f(x^k) - f(x^*) - \langle \nabla f(x^*), x^k - x^* \rangle] + \left( \frac{4L}{n} + 4L_f \right) [f(w^k) - f(x^*) - \langle \nabla f(x^*), w^k - x^* \rangle],
\]

26
where we denote $q_k^k = \nabla f^{(\tau)}_k(x^k) - \nabla f^{(\tau)}_k(w^k)$.

Since $x^*$ is an optimal solution, we have $-\nabla f(x^*) \in \partial \psi(x^*)$, which implies that

$$f(x^k) - f(x^*) - \langle \nabla f(x^*), x^k - x^* \rangle \leq P(x^k) - P(x^*). \quad (21)$$

Thus,

$$\frac{1}{n} \sum_{\tau=1}^{n} \mathbb{E}_k \left\| \nabla f^{(\tau)}_k(x^k) - \nabla f^{(\tau)}_k(w^k) \right\|^2 \leq 4L[P(x^k) - P(x^*)] + 4L[P(w^k) - P(x^*)],$$

and

$$\mathbb{E}_k \left\| \frac{1}{n} \sum_{\tau=1}^{n} \left( \nabla f^{(\tau)}_k(x^k) - \nabla f^{(\tau)}_k(w^k) \right) \right\|^2 \leq 4 \left( L_f + \frac{L}{n} \right) [P(x^k) - P(x^*)] + 4 \left( L_f + \frac{L}{n} \right) [P(w^k) - P(x^*)].$$

For $\mathbb{E}_k \left\| \frac{1}{n} \sum_{\tau=1}^{n} \left( \nabla f^{(\tau)}_k(x^k) - \nabla f^{(\tau)}_k(w^k) \right) + \nabla f(w^k) - \nabla f(x^k) \right\|^2$, we have

$$\mathbb{E}_k \left\| \frac{1}{n} \sum_{\tau=1}^{n} \left( \nabla f^{(\tau)}_k(x^k) - \nabla f^{(\tau)}_k(w^k) \right) + \nabla f(w^k) - \nabla f(x^k) \right\|^2 \leq \mathbb{E}_k \left\| \frac{1}{n} \sum_{\tau=1}^{n} \left( \nabla f^{(\tau)}_k(x^k) - \nabla f^{(\tau)}_k(w^k) \right) \right\|^2 - \| \nabla f(x^k) - \nabla f(w^k) \|^2 \leq \frac{1}{n^2} \sum_{\tau=1}^{n} \mathbb{E}_k \| q_k^k \|^2 - \frac{1}{n^2} \sum_{\tau=1}^{n} \| \nabla f^{(\tau)}(x^k) - \nabla f^{(\tau)}(w^k) \|^2 \leq \frac{4L}{n} [P(x^k) - P(x^*)] + 4\frac{L}{n} [P(w^k) - P(x^*)].$$

Since $f^{(\tau)}$ is $\bar{L}$-smooth, we have

$$\| \nabla f^{(\tau)}(x) - \nabla f^{(\tau)}(y) \|^2 \leq 2\bar{L} (f^{(\tau)}(x) - f^{(\tau)}(y) - \langle \nabla f^{(\tau)}(y), x - y \rangle).$$

Then similarly, we can get

$$\frac{1}{n} \sum_{\tau=1}^{n} \| \nabla f^{(\tau)}(x^k) - \nabla f^{(\tau)}(w^k) \|^2 \leq 4\bar{L} [P(x^k) - P(x^*)] + 4\bar{L} [P(w^k) - P(x^*)].$$

The following two lemmas show the evolution of $e_t^k$ and $e^k$, which will be used to construct the Lyapunov functions.

**Lemma B.3.** We have

$$\frac{1}{n} \sum_{\tau=1}^{n} \mathbb{E}_k \| e_t^{k+1} \|^2 \leq \left( 1 - \frac{\delta}{2} \right) \frac{1}{n} \sum_{\tau=1}^{n} \| e_t^k \|^2 + \frac{4(1 - \delta)\eta^2}{\delta n} \sum_{\tau=1}^{n} \| \nabla f^{(\tau)}(w^k) - h_t^k \|^2 + 4(1 - \delta)\eta^2 \left( \frac{4\bar{L}}{\delta} + \bar{L} \right) (P(x^k) - P(x^*) + P(w^k) - P(x^*)).$$
Proof. First, we have

\[ \mathbb{E}_k[\|e^{k+1}\|^2] \]

\[ \leq \ (1 - \delta) \mathbb{E}_k[\|e^k + \eta h^k\|^2] \]

\[ = \ (1 - \delta) \mathbb{E}_k[\|e^k + \eta(\nabla f(\tau)(x^k) - \nabla f(\tau)(w^k)) + \eta h^k - \eta(\nabla f(\tau)(x^k) - \nabla f(\tau)(w^k))\|^2] \]

\[ = \ (1 - \delta) \mathbb{E}_k[\|e^k + \eta\nabla f(\tau)(x^k) - \nabla f(\tau)(w^k) + \nabla f(\tau)(w^k) - h^k\|^2] \]

\[ + (1 - \delta) \eta^2 \mathbb{E}_k[\|\nabla f^{\tau}(x^k) - \nabla f^{\tau}(w^k) - (\nabla f^{\tau}(x^k) - \nabla f^{\tau}(w^k))\|^2] \]

\[ \leq \ (1 - \delta) \|e^k + \eta(\nabla f(\tau)(x^k) - h^k)\|^2 + (1 - \delta) \eta^2 \mathbb{E}_k[\|\nabla f^{\tau}(x^k) - \nabla f^{\tau}(w^k)\|^2] \]

where we use Young’s inequality in the third inequality and choose \( \beta = \frac{\delta}{2(1 - \delta)} \) when \( \delta < 1 \). When \( \delta = 1 \), it is easy to see that the above inequality also holds.

Then from Young’s inequality, we can get

\[ \frac{1}{n} \sum_{\tau=1}^n \mathbb{E}_k[\|e^{k+1}\|^2] \]

\[ \leq \ \left( 1 - \frac{\delta}{2} \right) \frac{1}{n} \sum_{\tau=1}^n \|e^k\|^2 + \frac{4(1 - \delta)\eta^2}{\delta n} \sum_{\tau=1}^n \|\nabla f(\tau)(x^k) - \nabla f(\tau)(w^k)\|^2 \]

\[ + \frac{4(1 - \delta)\eta^2}{\delta n} \sum_{\tau=1}^n \|\nabla f^{\tau}(w^k) - h^k\|^2 + (1 - \delta)\eta^2 \frac{1}{n} \sum_{\tau=1}^n \mathbb{E}_k[\|\nabla f^{\tau}(x^k) - \nabla f^{\tau}(w^k)\|^2] \]

\[ \leq \ \left( 1 - \frac{\delta}{2} \right) \frac{1}{n} \sum_{\tau=1}^n \|e^k\|^2 + \frac{4(1 - \delta)\eta^2}{\delta n} \sum_{\tau=1}^n \|\nabla f^{\tau}(w^k) - h^k\|^2 + (1 - \delta)\eta^2 \frac{1}{n} \sum_{\tau=1}^n \mathbb{E}_k[\|\nabla f^{\tau}(x^k) - \nabla f^{\tau}(w^k)\|^2] \]

\[ \leq \ \left( 1 - \frac{\delta}{2} \right) \frac{1}{n} \sum_{\tau=1}^n \|e^k\|^2 + \frac{4\eta^2}{\delta n} \sum_{\tau=1}^n \|\nabla f^{\tau}(w^k) - h^k\|^2 \]

\[ + 4(1 - \delta)\eta^2 \left( \frac{4L}{\delta} + L \right) (P(x^k) - P(x^*) + P(w^k) - P(x^*)) \]

\[ \leq \ \left( 1 - \frac{\delta}{2} \right) \frac{1}{n} \sum_{\tau=1}^n \|e^k\|^2 + \frac{2(1 - \delta)\delta}{n^2} \left( \left( \frac{\eta}{\delta} - \delta \right) \sum_{\tau=1}^n \|\nabla f^{\tau}(w^k) - h^k\|^2 \right) \]

\[ + 4(1 - \delta)\eta^2 \left( \frac{4L}{\delta} + \frac{5L}{n} \right) (P(x^k) - P(x^*) + P(w^k) - P(x^*)) + \frac{4(1 - \delta)\eta^2}{\delta} \|\nabla f(w^k) - h^k\|^2. \]
Proof. Under Assumption 1.3 we have $E[Q(x)] = \delta x$, and

\[
E_k \|e^{k+1}\|^2 = E_k \left\| \frac{1}{n} \sum_{\tau=1}^{n} e^{k+1}_\tau \right\|^2 \\
= \frac{1}{n^2} \sum_{i,j} E_k(e^{k+1}_i, e^{k+1}_j) \\
= \frac{1}{n^2} \sum_{\tau=1}^{n} E_k\|e^{k+1}\|^2 + \frac{1}{n^2} \sum_{i \neq j} E_k(e^{k+1}_i, e^{k+1}_j)
\]

\[
\leq (1 - \delta)E_k\|e^k + \eta g^k\|^2 + \frac{(1 - \delta)^2}{n^2} \sum_{\tau=1}^{n} E_k\|e^{k+1}_\tau\|^2 + \frac{(1 - \delta)^2}{n^2} \sum_{i \neq j} E_k\|e^k_i + \eta g^k_j\|^2 \\
\leq (1 - \delta)E_k\|e^k + \eta g^k\|^2 + \frac{(1 - \delta)^2}{n^2} \sum_{\tau=1}^{n} E_k\|e^{k+1}_\tau\|^2 + \frac{(1 - \delta)^2}{n^2} \sum_{i \neq j} E_k\|e^k_i + \eta g^k_j\|^2 + \frac{4(1 - \delta)^2}{n^2} \sum_{\tau=1}^{n} \|\nabla f^{(\tau)}(x^k) - \nabla f^{(\tau)}(w^k)\|^2
\]

\[
\leq (1 - \delta)E_k\|e^k + \eta g^k\|^2 + \frac{2(1 - \delta)^2}{n^2} \sum_{\tau=1}^{n} \|e^{k+1}_\tau\|^2 + \frac{4(1 - \delta)^2}{n^2} \sum_{\tau=1}^{n} \|\nabla f^{(\tau)}(x^k) - \nabla f^{(\tau)}(w^k)\|^2
\]

where we use the definitions of $e^k$ and $g^k$ in the last inequality. Then we can obtain

\[
E_k\|e^{k+1}\|^2 \\
\leq (1 - \delta)E_k\|e^k + \eta g^k\|^2 + \frac{(1 - \delta)^2}{n^2} \sum_{\tau=1}^{n} E_k\|e^{k+1}_\tau\|^2 + \frac{(1 - \delta)^2}{n^2} \sum_{i \neq j} E_k\|e^k_i + \eta g^k_j\|^2 \\
\leq (1 - \delta)E_k\|e^k + \eta g^k\|^2 + \frac{2(1 - \delta)^2}{n^2} \sum_{\tau=1}^{n} \|e^{k+1}_\tau\|^2 + \frac{4(1 - \delta)^2}{n^2} \sum_{\tau=1}^{n} \|\nabla f^{(\tau)}(x^k) - \nabla f^{(\tau)}(w^k)\|^2
\]

\[
+ \frac{4(1 - \delta)^2}{n^2} \sum_{\tau=1}^{n} \|\nabla f^{(\tau)}(w^k) - h^k\|^2 + \frac{16(1 - \delta)|\delta L_\eta|^2}{n} [P(x^k) - P(x^*) + P(w^k) - P(x^*)],
\]

where in the second and third inequalities we use the Young’s inequality.
For $(1 - \delta)E_k \|e^k + \eta g^k\|^2$, we have
\[
(1 - \delta)E_k \|e^k + \eta g^k\|^2
= \left(1 - \frac{\delta}{2}\right) \|e^k\|^2 + \frac{2(1 - \delta)\eta^2}{\delta} \|\nabla f(x^k) - h^k\|^2 + 4(1 - \delta)\eta^2 \|\nabla f(w^k)\|^2.
\]
Since $f$ is $L_f$-smooth, we have
\[
\|\nabla f(x^k) - h^k\|^2 \leq 2\|\nabla f(x^k) - \nabla f(x^*)\|^2 + 2\|\nabla f(w^k) - \nabla f(x^*)\|^2 \leq 4L_f \left[ f(x^k) - f(x^*) - \langle \nabla f(x^*), x^k - x^* \rangle \right] + 4L_f \left[ f(w^k) - f(x^*) - \langle \nabla f(x^*), w^k - x^* \rangle \right]
\]
which implies that
\[
\|\nabla f(x^k) - h^k\|^2 \leq 2\|\nabla f(x^k) - \nabla f(w^k)\|^2 + 2\|\nabla f(w^k) - h^k\|^2 \leq 2\|\nabla f(x^k) - \nabla f(w^k)\|^2 + 2\|\nabla f(w^k) - h^k\|^2 \leq 4L_f \left[ f(x^k) - f(x^*) - \langle \nabla f(x^*), x^k - x^* \rangle \right] + 4L_f \left[ f(w^k) - f(x^*) - \langle \nabla f(x^*), w^k - x^* \rangle \right]
\]
Hence, we arrive at
\[
(1 - \delta)E_k \|e^k + \eta g^k\|^2 \leq \left(1 - \frac{\delta}{2}\right) \|e^k\|^2 + \frac{2(1 - \delta)\eta^2}{\delta} \|\nabla f(w^k) - h^k\|^2 + 4(1 - \delta)\eta^2 \left[ f(x^k) - f(x^*) + f(w^k) - f(x^*) \right] + 4(1 - \delta)\eta^2 \|\nabla f(w^k) - h^k\|^2.
\]
Combining (22) and the above inequality, we can get
\[
E_k \|e^{k+1}\|^2 \leq \left(1 - \frac{\delta}{2}\right) \|e^k\|^2 + \frac{2(1 - \delta)\eta^2}{\delta} \sum_{\tau=1}^n \|e^\tau\|^2 + \frac{4(1 - \delta)\eta^2}{\delta} \sum_{\tau=1}^n \|\nabla f(\tau)(w^k) - h^k\|^2 + 4(1 - \delta)\eta^2 \left[ f(x^k) - f(x^*) + f(w^k) - f(x^*) \right] + 4(1 - \delta)\eta^2 \|\nabla f(w^k) - h^k\|^2
\]
\[
\leq \left(1 - \frac{\delta}{2}\right) \|e^k\|^2 + \frac{2(1 - \delta)\eta^2}{\delta} \sum_{\tau=1}^n \|e^\tau\|^2 + \frac{4(1 - \delta)\eta^2}{\delta} \sum_{\tau=1}^n \|\nabla f(\tau)(w^k) - h^k\|^2 + 4(1 - \delta)\eta^2 \left[ f(x^k) - f(x^*) + f(w^k) - f(x^*) \right] + 4(1 - \delta)\eta^2 \|\nabla f(w^k) - h^k\|^2.
\]

Lemma B.5. We have
\[
\frac{1}{n} \sum_{\tau=1}^{n} \mathbb{E}_k[||h^{k+1}_{\tau} - \nabla f(\tau)(w^{k+1})||^2] \leq \left( 1 - \frac{\delta_1}{2} \right) \frac{1}{n} \sum_{\tau=1}^{n} [||h^{k}_{\tau} - \nabla f(\tau)(w^{k})||^2] \\
+ 4Lp \left( 1 + \frac{2p}{\delta_1} \right) [P(x^k) - P(x^*) + P(w^k) - P(x^*)].
\]

Proof. First, from the update rule of \( w^k \), we have
\[
\mathbb{E}_k[||h^{k+1}_{\tau} - \nabla f(\tau)(w^{k+1})||^2] \\
= p\mathbb{E}_k[||h^{k+1}_{\tau} - \nabla f(\tau)(w^{k})||^2] + (1-p)\mathbb{E}_k[||h^{k+1}_{\tau} - \nabla f(\tau)(w^{k})||^2] \\
\leq p \left( 1 + \frac{2p}{\delta_1} \right) \mathbb{E}_k[||\nabla f(\tau)(x^k) - \nabla f(\tau)(w^{k})||^2] + \left( 1 + \frac{\delta_1}{2} \right) \mathbb{E}_k[||h^{k+1}_{\tau} - \nabla f(\tau)(w^{k})||^2] \\
+ (1-p)\mathbb{E}_k[||h^{k+1}_{\tau} - \nabla f(\tau)(w^{k})||^2] \\
= p \left( 1 + \frac{2p}{\delta_1} \right) [||\nabla f(\tau)(x^k) - \nabla f(\tau)(w^{k})||^2] + \left( 1 + \frac{\delta_1}{2} \right) [||h^{k}_{\tau} - \nabla f(\tau)(w^{k})||^2] \\
\leq p \left( 1 + \frac{2p}{\delta_1} \right) [||\nabla f(\tau)(x^k) - \nabla f(\tau)(w^{k})||^2] + \left( 1 - \frac{\delta_1}{2} \right) [||h^{k}_{\tau} - \nabla f(\tau)(w^{k})||^2],
\]
where the first inequality comes from the Young’s inequality and the last inequality comes from the contraction property of \( Q_1 \).

Then we can obtain
\[
\frac{1}{n} \sum_{\tau=1}^{n} \mathbb{E}_k[||h^{k+1}_{\tau} - \nabla f(\tau)(w^{k+1})||^2] \\
\leq p \left( 1 + \frac{2p}{\delta_1} \right) \frac{1}{n} \sum_{\tau=1}^{n} [||\nabla f(\tau)(x^k) - \nabla f(\tau)(w^{k})||^2] + \left( 1 - \frac{\delta_1}{2} \right) \frac{1}{n} \sum_{\tau=1}^{n} [||h^{k}_{\tau} - \nabla f(\tau)(w^{k})||^2] \\
\leq \left( 1 - \frac{\delta_1}{2} \right) \frac{1}{n} \sum_{\tau=1}^{n} [||h^{k}_{\tau} - \nabla f(\tau)(w^{k})||^2] + 4Lp \left( 1 + \frac{2p}{\delta_1} \right) [P(x^k) - P(x^*) + P(w^k) - P(x^*)].
\]

Lemma B.6. Under Assumption \( \mathbf{A3} \), we have
\[
\mathbb{E}_k||h^{k+1} - \nabla f(\tau)(w^{k+1})||^2 \leq \left( 1 - \frac{\delta_1}{2} \right) ||h^{k} - \nabla f(w^{k})||^2 + \frac{1 - 4p}{n^2} \sum_{\tau=1}^{n} [||h^{k}_{\tau} - \nabla f(\tau)(w^{k})||^2] \\
+ 4pLr \left( 1 + \frac{2p}{\delta_1} \right) [P(x^k) - P(x^*) + P(w^k) - P(x^*)].
\]

Proof. First, from the update rule of \( w^k \), we can obtain
\[
\mathbb{E}_k||h^{k+1} - \nabla f(w^{k+1})||^2 \\
= p\mathbb{E}_k[||h^{k+1} - \nabla f(w^{k})||^2] + (1-p)\mathbb{E}_k[||h^{k+1} - \nabla f(w^{k})||^2] \\
\leq p \left( 1 + \frac{2p}{\delta_1} \right) [||\nabla f(x^k) - \nabla f(w^{k})||^2] + p \left( 1 + \frac{\delta_1}{2p} \right) \mathbb{E}_k[||h^{k+1} - \nabla f(w^{k})||^2] + (1-p)\mathbb{E}_k[||h^{k+1} - \nabla f(w^{k})||^2] \\
= p \left( 1 + \frac{2p}{\delta_1} \right) [||\nabla f(x^k) - \nabla f(w^{k})||^2] + \left( 1 + \frac{\delta_1}{2} \right) \mathbb{E}_k[||h^{k+1} - \nabla f(w^{k})||^2].
\]

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For \( \mathbb{E}_k \| h^{k+1} - \nabla f(w_k) \|^2 \), under Assumption B.3 same as \( \mathbb{E}_k \| e^{k+1} \|^2 \), we have

\[
\mathbb{E}_k \| h^{k+1} - \nabla f(w_k) \|^2 \leq (1 - \delta_1) \| h^k - \nabla f(w_k) \|^2 + \frac{(1 - \delta_1)}{n^2} \sum_{r=1}^n \| h^r_k - \nabla f^{(r)}(w_k) \|^2.
\]

Hence, we arrive at

\[
\mathbb{E}_k \| h^{k+1} - \nabla f(w_k) \|^2 \leq p \left( 1 + \frac{2p}{\delta_1} \right) \| \nabla f(x^k) - \nabla f(w_k) \|^2 + \left( 1 - \frac{\delta_1}{2} \right) \| h^k - \nabla f(w_k) \|^2 + \frac{(1 - \delta_1)}{n^2} \sum_{r=1}^n \| h^r_k - \nabla f^{(r)}(w_k) \|^2 \leq \left( 1 - \frac{\delta_1}{2} \right) \| h^k - \nabla f(w_k) \|^2 + \frac{(1 - \delta_1)}{n^2} \sum_{r=1}^n \| h^r_k - \nabla f^{(r)}(w_k) \|^2 + 4pL_f \left( 1 + \frac{2p}{\delta_1} \right) [P(x^k) - P(x^*) + P(w^k) - P(x^*)].
\]

\[\square\]

### B.2 Proof of Theorem 2.3

Let \( \eta \leq \frac{1}{4L_f} \). From \( \| e^k \|^2 \leq \frac{1}{4} \sum_{r=1}^n \| e^k_r \|^2 \) and Lemma B.3, we have

\[
\left( 1 + \frac{\eta \mu}{2} \right) \mathbb{E}_k \| \tilde{z}^{k+1} - x^* \|^2 - \| \tilde{z}^k - x^* \|^2 - 2\eta \mathbb{E}_k (P(x^*) - P(x^{k+1})) \leq \| e^k \|^2 + (1 + \eta \mu) \mathbb{E}_k \| e^{k+1} \|^2 + 4\eta^2 \mathbb{E}_k \| g^k + h^k - \nabla f(x^k) \|^2 \leq \frac{1}{n} \sum_{r=1}^n \| e^k_r \|^2 + \frac{5}{4n} \sum_{r=1}^n \mathbb{E}_k \| e^{k+1}_r \|^2 + 4\eta^2 \mathbb{E}_k \| g^k + h^k - \nabla f(x^k) \|^2 \leq \frac{1}{n} \sum_{r=1}^n \| e^k_r \|^2 + \frac{5}{4n} \left( 1 - \frac{\delta}{2} \right) \frac{1}{n} \sum_{r=1}^n \| e^k_r \|^2 + \frac{5(1 - \delta)\eta^2}{\delta n} \sum_{r=1}^n \| \nabla f^{(r)}(w^k) - h^r_k \|^2 + 5(1 - \delta)\eta^2 \left( \frac{4L}{\delta} + L \right) [P(x^k) - P(x^*) + P(w^k) - P(x^*)] + 4\eta^2 \mathbb{E}_k \| g^k + h^k - \nabla f(x^k) \|^2 \leq \frac{9}{4} \cdot \frac{1}{n} \sum_{r=1}^n \| e^k_r \|^2 + \frac{5(1 - \delta)\eta^2}{\delta n} \sum_{r=1}^n \| \nabla f^{(r)}(w^k) - h^r_k \|^2 + \left( 5(1 - \delta) \left( \frac{4L}{\delta} + L \right) + \frac{16L}{n} \right) \eta^2 [P(x^k) - P(x^*) + P(w^k) - P(x^*)] \leq \mathbb{E}_k [P(w^{k+1}) - P(x^*)] = p[P(x^k) - P(x^*)] + (1 - p)[P(w^k) - P(x^*)].
\]

From the definition of \( w^{k+1} \), we have

\[
\mathbb{E}_k [P(w^{k+1}) - P(x^*)] = p[P(x^k) - P(x^*)] + (1 - p)[P(w^k) - P(x^*)].
\]

From Lemma B.3, we have
\[(1 + \frac{\eta \mu}{2}) \mathbb{E}_k \|\hat{x}^{k+1} - x^*\|^2 - \|\hat{x}^k - x^*\|^2 - 2\eta \mathbb{E}_k [P(x^*) - P(x^{k+1})] + \frac{9}{\delta n} \sum_{\tau=1}^{n} \mathbb{E}_k \|e_{\tau}^{k+1}\|^2 \leq \frac{9}{\delta n} \left(1 - \frac{\delta}{4}\right) \sum_{\tau=1}^{n} \|e_{\tau}^{k}\|^2 + \frac{41(1 - \delta)\eta^2}{\delta^2 n} \sum_{\tau=1}^{n} \|\nabla f^{(\tau)}(w^k) - h_{\tau}^k\|^2 + \left(\frac{41(1 - \delta)}{\delta} \left(\frac{4L}{\delta} + L + \frac{16L}{n}\right) \eta^2 [P(x^k) - P(x^*) + P(w^k) - P(x^*)]\right).\]

Then from Lemma [B.3] we have

\[(1 + \frac{\eta \mu}{2}) \mathbb{E}_k \|\hat{x}^{k+1} - x^*\|^2 - \|\hat{x}^k - x^*\|^2 - 2\eta \mathbb{E}_k [P(x^*) - P(x^{k+1})] + \frac{9}{\delta n} \sum_{\tau=1}^{n} \mathbb{E}_k \|e_{\tau}^{k+1}\|^2 \leq \frac{9}{\delta n} \left(1 - \frac{\delta}{4}\right) \sum_{\tau=1}^{n} \|e_{\tau}^{k}\|^2 + \frac{164(1 - \delta)\eta^2}{\delta^2 \delta_1 n} \sum_{\tau=1}^{n} \|h_{\tau}^{k+1} - \nabla f^{(\tau)}(w^{k+1})\|^2 \leq \frac{9}{\delta n} \left(1 - \frac{\delta}{4}\right) \sum_{\tau=1}^{n} \|e_{\tau}^{k}\|^2 + \frac{164(1 - \delta)\eta^2}{\delta^2 \delta_1 n} \left(1 - \frac{\delta}{4}\right) \sum_{\tau=1}^{n} \|h_{\tau}^{k} - \nabla f^{(\tau)}(w^k)\|^2 \leq \frac{4\eta^2}{3p} \left(\frac{41(1 - \delta)}{\delta} \left(\frac{4L}{\delta} + L + \frac{16Lp}{\delta \delta_1} \left(1 + \frac{2p}{\delta_1}\right) + \frac{16L}{n}\right) \mathbb{E}_k [P(w^{k+1}) - P(x^*)]\right) + 2\eta \mathbb{E}_k [P(x^*) - P(x^{k+1})] + \left(\frac{96(1 - \delta)}{\delta} \left(\frac{4L}{\delta} + L + \frac{16Lp}{\delta \delta_1} \left(1 + \frac{2p}{\delta_1}\right) + \frac{38L}{n}\right) \eta^2 [P(x^k) - P(x^*)]\right) \leq \left(1 - \min \left(\frac{\mu \eta}{3}, \frac{\delta}{4}, \frac{\delta_1}{4}, \frac{p}{4}\right)\right) \Phi_k + 2\eta \mathbb{E}_k [P(x^*) - P(x^{k+1})] + \left(\frac{96(1 - \delta)}{\delta} \left(\frac{4L}{\delta} + L + \frac{16Lp}{\delta \delta_1} \left(1 + \frac{2p}{\delta_1}\right) + \frac{38L}{n}\right) \eta^2 [P(x^k) - P(x^*)]\right),\]

where we use \((1 + \eta \mu)^{-1} \leq 1 - \frac{\eta \mu}{4}\) for \(\mu \eta < 1\). Taking expectation again and applying the tower property, we can get the result.

**B.3 Proof of Theorem [2.4]**

Let \(\eta \leq \frac{1}{4Lf}\). From Lemma [B.1] we have
Then from Lemmas B.3 and B.4, we have

\[
\begin{align*}
\left(1 + \frac{\eta \mu}{2}\right) \mathbb{E}_k \|x^{k+1} - x^*\|^2 + \frac{9}{\delta} \mathbb{E}_k \|e^{k+1}\|^2 + \frac{84(1 - \delta)}{\delta n^2} \sum_{\tau=1}^n \|e^{k+1}_\tau\|^2 \\ - \mathbb{E}[\|\tilde{x} - x^*\|^2 - 2\eta \mathbb{E}[P(x^*) - P(x^{k+1})]]
\end{align*}
\]

Lemma B.4

\[
\begin{align*}
\left(1 - \frac{\delta}{2} + \frac{\delta}{4}\right) \frac{9}{\delta} \|e^k\|^2 + \frac{41(1 - \delta)\eta^2}{\delta} \sum_{\tau=1}^n \|h^\tau - \nabla f(\tau)\|P(x^k) - P(x^*) + P(w^k) - P(x^*)]
\end{align*}
\]

Lemma B.3

\[
\begin{align*}
\left(1 - \frac{\delta}{4}\right) \frac{9}{\delta} \|e^k\|^2 + \left(1 - \frac{\delta}{4}\right) \frac{84(1 - \delta)}{\delta n^2} \sum_{\tau=1}^n \|e^{k+1}_\tau\|^2 + \frac{336(1 - \delta)\eta^2}{\delta^2 n^2} \sum_{\tau=1}^n \|h^\tau - \nabla f(\tau)\|P(x^k) - P(x^*) + P(w^k) - P(x^*)]
\end{align*}
\]
Then from Lemma B.5 and Lemma B.6 we can get

\[
\begin{align*}
&\left(1 + \frac{\eta\mu}{2}\right)E_k\|\hat{x}^{k+1} - x^*\|^2 + \frac{9}{\delta}E_k\|e^{k+1}\|^2 + \frac{84(1 - \delta)}{\delta n^2}\sum_{\tau=1}^{n}E_k\|e^{\tau k+1}\|^2 \\
&+ \frac{164(1 - \delta)\eta}{\delta^2 \delta_1}E_k\|h^{k+1} - \nabla f(w^{k+1})\|^2 + \frac{2000(1 - \delta)\eta^2}{\delta^2 \delta_1 n^2}\sum_{\tau=1}^{n}E_k\|h^{k+1} - \nabla f^{(\tau)}(w^{k+1})\|^2 \\
&- \|\hat{x}^k - x^*\|^2 - 2\eta\|E_k[P(x^*) - P(x^{k+1})]\|
\end{align*}
\]

**Lemma B.6**

\[
\begin{align*}
&\left(1 - \frac{\delta}{4}\right)\frac{9}{\delta}E_k\|e^{k}\|^2 + \left(1 - \frac{\delta}{4}\right)\frac{84(1 - \delta)}{\delta n^2}\sum_{\tau=1}^{n}E_k\|e^{\tau k}\|^2 + \left(1 - \frac{\delta}{4}\right)\frac{164(1 - \delta)\eta}{\delta^2 \delta_1}E_k\|h^{k} - \nabla f^{(k)}(w^{k})\|^2 \\
+ &\frac{500(1 - \delta)\eta^2}{\delta^2 n^2}\sum_{\tau=1}^{n}E_k\|h^{k} - \nabla f^{(\tau)}(w^{k})\|^2 + \frac{2000(1 - \delta)\eta^2}{\delta^2 \delta_1 n^2}\sum_{\tau=1}^{n}E_k\|h^{k+1} - \nabla f^{(\tau)}(w^{k+1})\|^2 \\
+ &\left(1 - \frac{\delta}{4}\right)\left(\frac{164L_f}{\delta} + \frac{1344L}{\delta n} + \frac{541L}{n}\right) + \frac{16L}{n} + \frac{656(1 - \delta)pL_f}{\delta^2 \delta_1}(1 + \frac{2p}{\delta_1})
\end{align*}
\]

**Lemma B.5**

\[
\begin{align*}
&\left(1 - \frac{\delta}{4}\right)\frac{9}{\delta}E_k\|e^{k}\|^2 + \left(1 - \frac{\delta}{4}\right)\frac{84(1 - \delta)}{\delta n^2}\sum_{\tau=1}^{n}E_k\|e^{\tau k}\|^2 + \left(1 - \frac{\delta}{4}\right)\frac{164(1 - \delta)\eta^2}{\delta^2 \delta_1}E_k\|h^{k} - \nabla f^{(k)}(w^{k})\|^2 \\
+ &\frac{2000(1 - \delta)\eta^2}{\delta^2 \delta_1 n^2}\sum_{\tau=1}^{n}E_k\|h^{k} - \nabla f^{(\tau)}(w^{k})\|^2 \\
+ &\left(1 - \frac{\delta}{4}\right)\left(\frac{164L_f}{\delta} + \frac{1344L}{\delta n} + \frac{541L}{n}\right) + \frac{16L}{n} + \frac{(656L_f + \frac{8000L}{n})(1 - \delta)p}{\delta^2 \delta_1}(1 + \frac{2p}{\delta_1})
\end{align*}
\]

Combining the above inequality and \ref{eq:lemma3}, we can obtain

\[
E_k[\Phi_2^{k+1}]
\]

\[
= \left(1 + \frac{\eta\mu}{2}\right)E_k\|\hat{x}^{k+1} - x^*\|^2 + \frac{9}{\delta}E_k\|e^{k+1}\|^2 + \frac{84(1 - \delta)}{\delta n^2}\sum_{\tau=1}^{n}E_k\|e^{\tau k+1}\|^2 \\
+ \frac{164(1 - \delta)\eta}{\delta^2 \delta_1}E_k\|h^{k+1} - \nabla f^{(k)}(w^{k+1})\|^2 + \frac{2000(1 - \delta)\eta^2}{\delta^2 \delta_1 n^2}\sum_{\tau=1}^{n}E_k\|h^{k+1} - \nabla f^{(\tau)}(w^{k+1})\|^2 \\
+ \frac{4\eta^2}{3p}\left(1 - \delta\right)\left(\frac{164L_f}{\delta} + \frac{1344L}{\delta n} + \frac{541L}{n}\right) + \frac{16L}{n}\right) + \frac{16L}{n}
\end{align*}
\]

\[
\cdot E_k[P(w^{k+1}) - P(x^*)]
\]

\[
\leq \left(1 - \min\left\{\frac{\mu\eta}{3}, \frac{\delta}{4}, \frac{p}{4}\right\}\right)\Phi_2^k + 2\eta E_k[P(x^*) - P(x^{k+1})] \\
+ \left(1 - \frac{\delta}{4}\right)\left(\frac{383L_f}{\delta n} + \frac{3136L}{\delta n} + \frac{1263L}{n} + \frac{1531L_f + \frac{18667L}{n}}{\delta_1}(1 + \frac{2p}{\delta_1})\right) + \frac{38L}{n}
\end{align*}
\]

where we use \(1 + \frac{\eta\mu}{2} \leq 1 - \frac{\eta\mu}{3}\) for \(\mu\eta < 1\). Taking expectation again and applying the tower property, we can have the result.
B.4 Proof of Theorem 2.5

Let \( \eta \leq \frac{1}{4L_f} \). From Theorem 2.3 we have

\[
\mathbb{E} [\Phi^k_1] \\
\leq \left( 1 - \min \left\{ \frac{\mu \eta}{3}, \frac{\delta}{3}, \frac{\delta_1}{4}, \frac{p}{4} \right\} \right) \mathbb{E} [\Phi^{k-1}_1] + 2\eta \mathbb{E} [P(x^*) - P(x^k)] \\
+ \left( \frac{96(1 - \delta)}{\delta} \left( \frac{4L}{\delta} + L + \frac{16L_p}{\delta_1} \left( 1 + \frac{2p}{\delta_1} \right) \right) + \frac{38L}{n} \right) \eta^2 \mathbb{E} [P(x^{k-1}) - P(x^*)] \\
\leq \left( 1 - \min \left\{ \frac{\mu \eta}{3}, \frac{\delta}{3}, \frac{\delta_1}{4}, \frac{p}{4} \right\} \right)^k \Phi_1^0 - 2\eta \sum_{i=1}^k \left( 1 - \min \left\{ \frac{\mu \eta}{3}, \frac{\delta}{3}, \frac{\delta_1}{4}, \frac{p}{4} \right\} \right)^{k-i} \mathbb{E} [P(x^i) - P(x^*)] \\
+ \left( \frac{96(1 - \delta)}{\delta} \left( \frac{4L}{\delta} + L + \frac{16L_p}{\delta_1} \left( 1 + \frac{2p}{\delta_1} \right) \right) + \frac{38L}{n} \right) \eta^2 \\
\cdot \sum_{i=0}^{k-1} \left( 1 - \min \left\{ \frac{\mu \eta}{3}, \frac{\delta}{3}, \frac{\delta_1}{4}, \frac{p}{4} \right\} \right)^{k-1-i} \mathbb{E} [P(x^i) - P(x^*)] \\
= \frac{1}{w_k} \Phi_1^0 - 2\eta \frac{\sum_{i=1}^k w_i \mathbb{E} [P(x^i) - P(x^*)]}{w_k} \\
+ \left( \frac{96(1 - \delta)}{\delta} \left( \frac{4L}{\delta} + L + \frac{16L_p}{\delta_1} \left( 1 + \frac{2p}{\delta_1} \right) \right) + \frac{38L}{n} \right) \eta^2 \sum_{i=0}^k \frac{w_i \mathbb{E} [P(x^i) - P(x^*)]}{w_k}
\]

where we use \( w_1 \leq \frac{12}{\eta} \) in the last inequality. Rearranging the above inequality, we can get

\[
\frac{2}{w_k} \sum_{i=0}^k w_i \mathbb{E} [P(x^i) - P(x^*)] \\
\leq \frac{1}{\eta w_k} \Phi_1^0 - \frac{1}{\eta} \mathbb{E} [\Phi^k_1] + 2\left( \frac{P(x^0) - P(x^*)}{w_k} \right) \\
+ \left( \frac{105(1 - \delta)}{\delta} \left( \frac{4L}{\delta} + L + \frac{16L_p}{\delta_1} \left( 1 + \frac{2p}{\delta_1} \right) \right) + \frac{42L}{n} \right) \eta \sum_{i=0}^k \frac{w_i \mathbb{E} [P(x^i) - P(x^*)]}{w_k}
\]

\[
\leq \frac{1}{\eta w_k} \Phi_1^0 + 2\left( \frac{P(x^0) - P(x^*)}{w_k} \right) \\
+ \left( \frac{105(1 - \delta)}{\delta} \left( \frac{4L}{\delta} + L + \frac{16L_p}{\delta_1} \left( 1 + \frac{2p}{\delta_1} \right) \right) + \frac{42L}{n} \right) \eta \sum_{i=0}^k \frac{w_i \mathbb{E} [P(x^i) - P(x^*)]}{w_k}.
\]

Hence, if

\[
\eta \leq \left( \frac{105(1 - \delta)}{\delta} \left( \frac{4L}{\delta} + L + \frac{16L_p}{\delta_1} \left( 1 + \frac{2p}{\delta_1} \right) \right) + 4L_f + \frac{42L}{n} \right)^{\frac{1}{n}}.
\]

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then
\[ \sum_{i=0}^{k} w_i E[P(x^i) - P(x^*)] \leq \frac{1}{\eta} \Phi_1^0 + 2(P(x^0) - P(x^*)) \] (25)

When \( \mu > 0 \), since
\[ W_k = \sum_{i=0}^{k} w_i = \frac{1 - \frac{1}{(1-\min\{\frac{\mu}{\eta}, \frac{\delta}{4}, \frac{\tau}{10}, \frac{\eta}{4}\})k+1}}{1 - \min\{\frac{\mu}{\eta}, \frac{\delta}{4}, \frac{\tau}{10}, \frac{\eta}{4}\} (1 - \min\{\frac{\mu}{\eta}, \frac{\delta}{4}, \frac{\tau}{10}, \frac{\eta}{4}\})k} \]
we can get
\[ \frac{1}{W_k} \sum_{i=0}^{k} w_i E[P(x^i) - P(x^*)] \]
\[ \leq \frac{1}{W_k} \left( \frac{1}{\eta} \Phi_1^0 + 2(P(x^0) - P(x^*)) \right) \]
\[ = \frac{\min\{\frac{\mu}{\eta}, \frac{\delta}{4}, \frac{\tau}{10}, \frac{\eta}{4}\}}{1 - \min\{\frac{\mu}{\eta}, \frac{\delta}{4}, \frac{\tau}{10}, \frac{\eta}{4}\} (1 - \min\{\frac{\mu}{\eta}, \frac{\delta}{4}, \frac{\tau}{10}, \frac{\eta}{4}\})k+1} \left( \frac{1}{\eta} \Phi_1^0 + 2(P(x^0) - P(x^*)) \right) \left( 1 - \min\left\{\frac{\mu}{\eta}, \frac{\delta}{4}, \frac{\tau}{10}, \frac{\eta}{4}\right\} \right)^k. \]

From the definition of \( \Phi_k^1 \) and \( \epsilon^1_r = 0 \), we have
\[ \min\{\frac{\mu}{3}, \frac{\delta}{4}, \frac{\tau}{10}, \frac{\eta}{4}\} \cdot \frac{1}{\eta} \Phi_1^0 \]
\[ = \min\{\frac{\mu}{3}, \frac{\delta}{4}, \frac{\tau}{10}, \frac{\eta}{4}\} \left( \frac{1}{\eta} \left( 1 + \frac{\mu}{2} \right) \|x^0 - x^*\|^2 + \frac{164(1 - \delta)\eta}{\delta^2 \delta n} \sum_{\tau=1}^{n} \|\nabla f^{(\tau)}(x^0) - h_0^0\|^2 + \frac{4\eta}{3p} \left( \frac{41(1 - \delta)}{\delta} \cdot \left( \frac{4L}{\delta} + L + \frac{16Lp}{\delta} \right) \left( 1 + \frac{2p}{\delta_1} \right) + \frac{16L}{\delta} \right) \right) \]
\[ \leq \frac{\mu}{3} \left( 1 + \frac{\mu}{2} \right) \|x^0 - x^*\|^2 + \min\{\frac{\mu}{3}, \frac{\delta}{4}, \frac{\tau}{10}, \frac{\eta}{4}\} \cdot \frac{1}{10Lpn} \sum_{\tau=1}^{n} \|\nabla f^{(\tau)}(x^0) - h_0^0\|^2 \]
\[ + \min\{\frac{\mu}{3}, \frac{\delta}{4}, \frac{\tau}{10}, \frac{\eta}{4}\} \cdot \frac{2}{3p} \left[ P(x^0) - P(x^*) \right] \]
\[ \leq \frac{\mu}{2} \|x^0 - x^*\|^2 + \frac{1}{3} \left[ P(x^0) - P(x^*) \right] + \frac{1}{40Lpn} \sum_{\tau=1}^{n} \|\nabla f^{(\tau)}(x^0) - h_0^0\|^2. \]

Therefore, we arrive at
\[ \frac{1}{W_k} \sum_{i=0}^{k} w_i E[P(x^i) - P(x^*)] \]
\[ \leq \frac{\mu/2 \|x^0 - x^*\|^2 + 1/2 \left(P(x^0) - P(x^*)\right) + 1/40Lpn \sum_{\tau=1}^{n} \|\nabla f^{(\tau)}(x^0) - h_0^0\|^2}{1 - \min\{\frac{\mu}{3}, \frac{\delta}{4}, \frac{\tau}{10}, \frac{\eta}{4}\} (1 - \min\{\frac{\mu}{3}, \frac{\delta}{4}, \frac{\tau}{10}, \frac{\eta}{4}\})k+1} \]
\[ \left( 1 - \min\left\{\frac{\mu}{3}, \frac{\delta}{4}, \frac{\tau}{10}, \frac{\eta}{4}\right\} \right)^k. \]

For \( \bar{x}^k = \frac{1}{W_k} \sum_{i=0}^{k} w_i x^i \), from the convexity of \( P \), we have
\[ E[P(\bar{x}^k) - P(x^*)] \leq \frac{\mu}{W_k} \sum_{i=0}^{k} w_i E[P(x^i) - P(x^*)] \]
\[ \leq \frac{\mu/2 \|x^0 - x^*\|^2 + 1/2 \left(P(x^0) - P(x^*)\right) + 1/40Lpn \sum_{\tau=1}^{n} \|\nabla f^{(\tau)}(x^0) - h_0^0\|^2}{1 - \min\{\frac{\mu}{3}, \frac{\delta}{4}, \frac{\tau}{10}, \frac{\eta}{4}\} (1 - \min\{\frac{\mu}{3}, \frac{\delta}{4}, \frac{\tau}{10}, \frac{\eta}{4}\})k+1} \]
\[ \left( 1 - \min\left\{\frac{\mu}{3}, \frac{\delta}{4}, \frac{\tau}{10}, \frac{\eta}{4}\right\} \right)^k. \]
If we choose \( \eta = \left( \frac{1}{16(1-\frac{1}{4})} \left( \frac{4L}{\delta} + L + \frac{16L}{\delta} \left( 1 + 2\eta \right) \right) + 4L_f + \frac{4K}{\delta} \right)^{-1} \), then in order to guarantee \( \mathbb{E}[P(\hat{x}^k) - P(x^*)] \leq \epsilon \), we first let
\[
\left( 1 - \min \left( \frac{\mu \eta \sigma}{3 \cdot 4}, \frac{\sigma_1 \eta}{4}, \frac{p \eta}{4} \right) \right)^{k+1} \leq \frac{1}{2},
\]
which implies that
\[
\mathbb{E}[P(\hat{x}^k) - P(x^*)] \leq \left( \mu \|x^0 - x^*\|^2 + P(x^0) - P(x^*) + \frac{1}{20L_n} \sum_{\tau=1}^{n} \|\nabla f(\tau)(x^0) - h_\tau^0\|^2 \right) \left( 1 - \min \left( \frac{\mu \eta \sigma}{3 \cdot 4}, \frac{\sigma_1 \eta}{4}, \frac{p \eta}{4} \right) \right)^k.
\]
Hence, when \( \epsilon \leq \frac{\delta}{2} \|x^0 - x^*\|^2 + \frac{1}{2}(P(x^0) - P(x^*)) + \frac{1}{20L_n} \sum_{\tau=1}^{n} \|\nabla f(\tau)(x^0) - h_\tau^0\|^2 \), \( \mathbb{E}[P(\hat{x}^k) - P(x^*)] \leq \epsilon \) as long as
\[
\left( 1 - \min \left( \frac{\mu \eta \sigma}{3 \cdot 4}, \frac{\sigma_1 \eta}{4}, \frac{p \eta}{4} \right) \right)^k \leq \frac{\epsilon}{\mu \|x^0 - x^*\|^2 + P(x^0) - P(x^*) + \frac{1}{20L_n} \sum_{\tau=1}^{n} \|\nabla f(\tau)(x^0) - h_\tau^0\|^2}.
\]
which is equivalent to
\[
k \geq \frac{1}{-\ln \left( 1 - \min \left( \frac{\mu \eta \sigma}{3 \cdot 4}, \frac{\sigma_1 \eta}{4}, \frac{p \eta}{4} \right) \right)} \ln \left( \frac{\mu \|x^0 - x^*\|^2 + P(x^0) - P(x^*) + \frac{1}{20L_n} \sum_{\tau=1}^{n} \|\nabla f(\tau)(x^0) - h_\tau^0\|^2}{\epsilon} \right).
\]
Since \(-\ln(1 - x) \geq x\) for \( x \in [0,1) \), if \( p \leq O(\delta_1) \), we have \( \mathbb{E}[P(\hat{x}^k) - P(x^*)] \leq \epsilon \) as long as
\[
k \geq O \left( \left( \frac{1}{\delta} + \frac{L_f}{\mu} + \frac{L}{n \mu} + \frac{(1-\delta)L}{\delta^2 \mu} + \frac{(1-\delta)L}{\delta \mu} \right) \ln \frac{1}{\epsilon} \right).
\]
When \( \mu = 0 \), from (25) and the convexity of \( P \), we have
\[
\mathbb{E}[P(\hat{x}^k) - P(x^*)] \leq \frac{1}{k+1} \sum_{i=0}^{k} \mathbb{E}[P(x^i) - P(x^*)] \leq \frac{1}{k+1} \left( \frac{1}{\eta} \Phi_i^0 + 2(P(x^0) - P(x^*)) \right).
\]
From the definition of \( \Phi_i^0 \) and \( e_i^0 = 0 \), we have
\[
\frac{1}{\eta} \Phi_i^0 = \frac{1}{\eta} \left( 1 + \frac{\mu \eta \sigma}{2} \right) \|x^0 - x^*\|^2 + \frac{16(1-\delta)\eta \eta}{\delta \sigma_1 n} \sum_{\tau=1}^{n} \|\nabla f(\tau)(x^0) - h_\tau^0\|^2
\]
\[
+ \frac{4\eta}{3p} \left( \frac{41(1-\delta)}{\delta} \left( \frac{4L}{\delta} + L + \frac{16Lp}{\delta} \left( 1 + \frac{2p}{\delta_1} \right) \right) + \frac{16L}{n} \right) [P(x^0) - P(x^*)].
\]
Hence, we arrive at
\[
\mathbb{E}[P(\hat{x}^k) - P(x^*)] \leq \frac{1}{k+1} \left( \frac{9}{8\eta} \|x^0 - x^*\|^2 + \frac{1}{10L_p m} \sum_{\tau=1}^{n} \|\nabla f(\tau)(x^0) - h_\tau^0\|^2 + \frac{8}{3p} (P(x^0) - P(x^*)) \right).
\]
In particular, if we choose \( \eta = \left( \frac{1}{16(1-\frac{1}{4})} \left( \frac{4L}{\delta} + L + \frac{16L}{\delta} \left( 1 + 2\eta \right) \right) + 4L_f + \frac{4K}{\delta} \right)^{-1} \) and \( p \leq O(\delta_1) \), we have \( \mathbb{E}[P(\hat{x}^k) - P(x^*)] \leq \epsilon \) as long as
\[
k \geq O \left( \left( \frac{1}{p} + L_f + \frac{L}{n} + \frac{(1-\delta)L}{\delta^2} + \frac{(1-\delta)L}{\delta} \right) \frac{1}{\epsilon} \right).
\]
B.5 Proof of Theorem 2.6

Same as the proof of Theorem 2.6 if

\[ \eta \leq \frac{1}{\left(\frac{1-\delta}{8}\right) \left(\frac{418L_L}{3} + \frac{3422L}{\delta n} + \frac{1349L}{n} + \frac{1671L_f + \frac{20364L}{n}}{\delta_1} \right) + 4L_f + \frac{42L}{n}}, \]

then

\[ \sum_{i=0}^{k} w_i \mathbb{E}[P(x^i) - P(x^*)] \leq \frac{1}{\eta} \Phi_0^2 + 2(\mathbb{P}(x^0) - P(x^*)). \tag{26} \]

When \( \mu > 0 \), notice that

\[
\begin{align*}
\min\{\frac{\mu \eta}{3}, \frac{\delta_1}{4}, \frac{p}{4}\} \cdot \frac{1}{\eta} \Phi_0^2 & = \\
\leq \frac{\mu}{3} \left(1 + \frac{\mu \eta}{2}\right) \|x^0 - x^*\|^2 + \min\{\frac{\mu \eta}{3}, \frac{\delta_1}{4}, \frac{p}{4}\} \cdot \frac{1}{10L_f \rho} \|\nabla f(x^0)\|^2 & + \min\{\frac{\mu \eta}{3}, \frac{\delta_1}{4}, \frac{p}{4}\} \cdot \frac{6}{5L_f \rho n} \sum_{\tau=1}^{n} \|\nabla f(\tau)(x^0) - h_\tau^0\|^2 & + \min\{\frac{\mu \eta}{3}, \frac{\delta_1}{4}, \frac{p}{4}\} \cdot \frac{2}{3p} [P(x^0) - P(x^*)] & \leq \frac{\mu}{2} \|x^0 - x^*\|^2 + \frac{1}{3} [P(x^0) - P(x^*)] + \frac{1}{40L_f} \|\nabla f(x^0)\|^2 & + \frac{3}{10L_f \rho n} \sum_{\tau=1}^{n} \|\nabla f(\tau)(x^0) - h_\tau^0\|^2.
\end{align*}
\]

Hence,

\[
\begin{align*}
\frac{1}{W_k} \sum_{i=0}^{k} w_i \mathbb{E}[P(x^i) - P(x^*)] & \leq \left(1 - \min\left\{\frac{\mu \eta}{3}, \frac{\delta_1}{4}, \frac{p}{4}\right\}\right)^k \cdot \frac{\frac{\mu}{2} \|x^0 - x^*\|^2 + \frac{1}{3} [P(x^0) - P(x^*)] + \frac{1}{40L_f} \|\nabla f(x^0)\|^2 + \frac{3}{10L_f \rho n} \sum_{\tau=1}^{n} \|\nabla f(\tau)(x^0) - h_\tau^0\|^2}{1 - \left(1 - \min\left\{\frac{\mu \eta}{3}, \frac{\delta_1}{4}, \frac{p}{4}\right\}\right)^k+1}.
\end{align*}
\]

When \( \mu = 0 \), notice that

\[
\begin{align*}
\frac{\Phi_0^2}{\eta} & = \frac{1}{\eta} \left(1 + \frac{\mu \eta}{2}\right) \|x^0 - x^*\|^2 & + \frac{164(1-\delta)\eta}{\delta^2 \delta_1} \|\nabla f(x^0)\|^2 & + \frac{2000(1-\delta)\eta}{\delta^2 \delta_1 n} \sum_{\tau=1}^{n} \|\nabla f(\tau)(x^0) - h_\tau^0\|^2 & + \frac{4\eta}{3p} \left(1 + \frac{\mu \eta}{2}\right) \left(\frac{164L_f}{\delta} + \frac{1344L}{\delta n} + \frac{541L}{n} + \frac{(656L_f + \frac{8000L}{n} p)}{\delta \delta_1} \right) \left(1 + \frac{2p}{\delta_1}\right) & + \frac{16L}{n} \right) \mathbb{E}[P(x^0) - P(x^*)] & \leq \frac{9}{8\eta} \|x^0 - x^*\|^2 + \frac{1}{10L_f \rho} \|\nabla f(x^0)\|^2 & + \frac{6}{5L_f \rho n} \sum_{\tau=1}^{n} \|\nabla f(\tau)(x^0) - h_\tau^0\|^2 + \frac{2}{3p} \left(1 - \min\left\{\frac{\mu \eta}{3}, \frac{\delta_1}{4}, \frac{p}{4}\right\}\right)^k+1.
\end{align*}
\]

From \( \frac{L}{n} \leq L_f \), the rest is the same as that of Theorem 2.5.
C Proofs for EC-LSVRG in the Smooth Case

C.1 A lemma

Thanks to the following lemma, we can get better results than the composite case. The main difference between Lemma [B.1] and Lemma [C.1] is that there is an additional stepsize $\eta$ before $E\|e^k\|^2$. The following lemma is similar to Lemma 7 in [Stich and Karimireddy, 2019]. However, for completeness, we give the proof.

Lemma C.1. If $\eta \leq \frac{1}{4L_f + 8L/n}$, then

$$E_k\|\hat{x}^{k+1} - x^*\|^2 \leq (1 - \frac{4\eta}{2})\|\hat{x}^k - x^*\|^2 - \frac{\eta}{2}[f(x^k) - f(x^*)] + 3L_f\eta\|e^k\|^2 + \frac{4L}{n}\eta^2[f(w^k) - f(x^*)].$$

Proof. Since $\psi = 0$, we have $\hat{x}^{k+1} = \hat{x}^k - \eta(g^k + h^k)$. Hence

$$E_k\|\hat{x}^{k+1} - x^*\|^2 = \|\hat{x}^k - x^*\|^2 - \eta\langle \hat{x}^k - x^*, \nabla f(x^k) \rangle + \eta^2E_k\|g^k + h^k\|^2$$

$$\leq \|\hat{x}^k - x^*\|^2 - 2\eta\langle x^k - x^*, \nabla f(x^k) \rangle + 2\eta\langle x^k - \hat{x}^k, \nabla f(x^k) \rangle + \eta^2E_k\|g^k + h^k\|^2$$

$$\leq \|x^k - x^*\|^2 - 2\eta(f(x^k) - f(x^*)) - \mu\eta\|x^k - x^*\|^2 + 2\eta\|e^k\|^2 + \eta^2E_k\|g^k + h^k\|^2,$$

where the last inequality comes from the $\mu$-strongly convexity of $f$.

For $\|x^k - x^*\|^2$, we have

$$\|x^k - x^*\|^2 \leq 2\|x^k - x^*\|^2 + 2\|e^k\|^2.$$

For $2\langle e^k, \nabla f(x^k) \rangle$, we have

$$2\langle e^k, \nabla f(x^k) \rangle \leq \frac{1}{2L_f}\|\nabla f(x^k)\|^2 + 2L_f\|e^k\|^2 \leq f(x^k) - f(x^*) + 2L_f\|e^k\|^2.$$

Thus, we arrive at

$$E_k\|\hat{x}^{k+1} - x^*\|^2 \leq \left(1 - \frac{\mu\eta}{2}\right)\|x^k - x^*\|^2 - \eta(f(x^k) - f(x^*)) + (2L_f + \mu)\eta\|e^k\|^2 + \eta^2E_k\|g^k + h^k\|^2$$

Finally, for $E_k\|g^k + h^k\|^2$, we have

$$E_k\|g^k + h^k\|^2 \leq E_k\left\| \frac{1}{n} \sum_{r=1}^{n} (\nabla f^{(r)}(w^k) - \nabla f^{(r)}(x^k)) + \nabla f(w^k) - \nabla f(x^k) - \nabla f(x^*) \right\|^2$$

$$\leq E_k\left\| \frac{1}{n} \sum_{r=1}^{n} (\nabla f^{(r)}(x^k) - \nabla f^{(r)}(w^k)) + \nabla f(w^k) - \nabla f(x^k) \right\|^2 + E\|\nabla f(x^k) - \nabla f(x^*)\|^2$$

$$\leq E_k\left\| \frac{1}{n} \sum_{r=1}^{n} (\nabla f^{(r)}(x^k) - \nabla f^{(r)}(w^k)) + \nabla f(w^k) - \nabla f(x^k) \right\|^2 + 2L_f(f(x^k) - f(x^*))$$

$$\leq \left(2L_f + \frac{4L}{n}\right)[f(x^k) - f(x^*)] + \frac{4L}{n}[f(w^k) - f(x^*)].$$
Therefore,

\[ E_k \|\tilde{x}^{k+1} - x^*\|^2 \leq \left(1 - \frac{\mu \eta}{2}\right) \|\tilde{x}^k - x^*\|^2 - \eta \left(1 - \frac{2L_f + 4L}{n} \right) \eta \left[ f(x^k) - f(x^*) \right] + 3L_f \eta \|e^k\|^2 + \frac{4L}{n} \eta^2 \|f(w^k) - f(x^*)\|. \]

By choosing \( \eta \leq \frac{1}{4L_f + 8L/n} \), we can get the result.

\[ \square \]

### C.2 Proof of Theorem 2.8

Let \( \eta \leq \frac{1}{4L_f + 8L/n} \). From Lemma [C.1](#LemmaC.1), Lemma [B.3](#LemmaB.3), and \( \|e^k\|^2 \leq \frac{1}{n} \sum_{\tau=1}^n \|e_{\tau}^k\|^2 \), we can obtain

\[ E_k \|\tilde{x}^{k+1} - x^*\|^2 + \frac{12L_f \eta}{n \delta} \sum_{\tau=1}^n \|e_{\tau}^k\|^2 + \left(1 - \frac{\mu \eta}{2}\right) \|\tilde{x}^k - x^*\|^2 - \frac{48(1 - \delta)L_f \eta^3}{\delta^2 n} \sum_{\tau=1}^n \|h_{\tau}^k - \nabla f^{(\tau)}(w^k)\|^2 \]

\[ + \frac{12L_f \eta}{n \delta} \left(1 - \frac{\delta}{2}\right) \sum_{\tau=1}^n \|e_{\tau}^k\|^2 + \frac{12L_f \eta}{n \delta} \left(1 - \frac{\delta}{4}\right) \sum_{\tau=1}^n \|h_{\tau}^k - \nabla f^{(\tau)}(w^k)\|^2 + \frac{4L}{n} \eta^2 \|f(w^k) - f(x^*)\| \]

Then from Lemma [B.5](#LemmaB.5), we can get

\[ E_k \|\tilde{x}^{k+1} - x^*\|^2 + \frac{12L_f \eta}{n \delta} \sum_{\tau=1}^n \|e_{\tau}^k\|^2 + \frac{12L_f \eta}{n \delta} \sum_{\tau=1}^n \|e_{\tau}^{k+1}\|^2 + \frac{12L_f \eta}{n \delta} \sum_{\tau=1}^n \|h_{\tau}^{k+1} - \nabla f^{(\tau)}(w^{k+1})\|^2 \]

\[ - \left(1 - \frac{\mu \eta}{2}\right) \|\tilde{x}^k - x^*\|^2 - \frac{4L}{n} \eta^2 \|f(w^k) - f(x^*)\| \]

\[ \leq \left(1 - \frac{\delta}{4}\right) \frac{12L_f \eta}{n \delta} \sum_{\tau=1}^n \|e_{\tau}^k\|^2 + \left(1 - \frac{\delta}{4}\right) \frac{12L_f \eta}{n \delta} \sum_{\tau=1}^n \|h_{\tau}^k - \nabla f^{(\tau)}(w^k)\|^2 \]

\[ + \frac{48(1 - \delta)L_f \eta^3}{\delta} \left(\frac{4L}{\delta} + 1 + \frac{16L}{\delta} \left(1 - \frac{2p}{\delta_1}\right) \right) \|f(x^k) - f(x^*) + f(w^k) - f(x^*)\|. \]
Combining (24) and the above inequality, we can obtain

\[ E_k[\Phi_{3,1}^k] \]

\[ = E_k\|\hat{x}^{k+1} - x^*\|^2 + \frac{12L_f n}{n^2 \delta} \sum_{\tau=1}^n \|e_{\tau}^{k+1}\|^2 + \frac{96(1 - \delta)L_f n}{n^2 \delta} \sum_{\tau=1}^n \|e_{\tau}^{k+1}\|^2 \]

\[ - \left( 1 - \frac{\mu n}{2} \right)\|\hat{x}^k - x^*\|^2 + \frac{n}{2} [f(x^k) - f(x^*)] - \frac{4L_f}{n} \eta^2 [f(w^k) - f(x^*)] \]

\[ \leq 3L_f n\|e^k\|^2 + \frac{12L_f n}{n^2 \delta} \sum_{\tau=1}^n \|e_{\tau}^{k+1}\|^2 + \frac{96(1 - \delta)L_f n}{n^2 \delta} \sum_{\tau=1}^n \|e_{\tau}^{k+1}\|^2 \]

\[ \leq 12L_f n \left( 1 - \frac{\delta}{4} \right)\|e^k\|^2 + \frac{24(1 - \delta)L_f n^2}{n^2 \delta} \sum_{\tau=1}^n \|e_{\tau}^{k+1}\|^2 + \frac{96(1 - \delta)L_f n}{n^2 \delta} \sum_{\tau=1}^n \|h_{\tau}^{k+1}\|^2 \]

\[ + \frac{48(1 - \delta)L_f n}{n^2 \delta} \left( \frac{4L_f}{\delta} + \frac{5L}{n} \right) [f(x^k) - f(x^*) + f(w^k) - f(x^*)] \]

\[ + \frac{48(1 - \delta)L_f n^2}{n^2 \delta} \sum_{\tau=1}^n \|h_{\tau}^{k} - \nabla f^{(r)}(w^k)\|^2 + \frac{48(1 - \delta)L_f n}{n^2 \delta} [f(x^k) - f(x^*) + f(w^k) - f(x^*)] \]

\[ \leq 3L_f n \left( 1 - \frac{\delta}{4} \right)\|e^k\|^2 + \frac{96(1 - \delta)L_f n}{n^2 \delta} \left( 1 - \frac{\delta}{4} \right) \sum_{\tau=1}^n \|e_{\tau}^{k+1}\|^2 \]

\[ + \frac{48(1 - \delta)L_f n^2}{n^2 \delta} \left( \frac{4L_f}{\delta} + \frac{32L}{n\delta} + \frac{13L}{n} \right) [f(x^k) - f(x^*) + f(w^k) - f(x^*)] \]

\[ + \frac{384(1 - \delta)L_f n^2}{n^2 \delta} \sum_{\tau=1}^n \|h_{\tau}^{k} - \nabla f^{(r)}(w^k)\|^2 + \frac{48(1 - \delta)L_f n}{n^2 \delta} [f(x^k) - f(x^*) + f(w^k) - f(x^*)]. \]
Then from Lemma B.5 and Lemma B.6, we can obtain

\[ E_k[\|\hat{x}^{k+1} - x^*\|^2] + \frac{12L_f \eta}{\delta} E_k[\|e^{k+1}\|^2] + \frac{96(1-\delta)L_f \eta}{n^2 \delta} \sum_{\tau=1}^{n} E_k[\|e^{\tau+1}\|^2] - \left(1 - \frac{\delta}{4}\right) \frac{12L_f \eta}{\delta} \|e^{k}\|^2 \\
+ \frac{192(1-\delta)L_f \eta^3}{\delta^2 \delta_1} E_k[\|h^{k+1} - \nabla f(w^{k+1})\|^2] + \frac{2304(1-\delta)L_f \eta^3}{\delta^2 \delta_1 n^2} \sum_{\tau=1}^{n} E_k[\|h^{\tau+1} - \nabla f(\tau)(w^{k+1})\|^2] \\
- \left(1 - \frac{\eta n}{2}\right) \|\hat{x}^{k} - x^*\|^2 + \frac{n}{2} \left[ f(x^{k}) - f(x^*) \right] - \frac{4L_n \eta}{n} \left[ f(w^{k}) - f(x^*) \right] \]
Hence, from Theorem 2.8 we have
\[
\mathbb{E}[\Phi_{k+1}] \leq \left(1 - \min\left\{\frac{\mu_\eta}{2}, \frac{\delta}{4}, \frac{\delta_1}{4}, \frac{p}{4}\right\}\right) \mathbb{E}[\Phi_k] - \frac{\eta}{18} \mathbb{E}[f(x^k) - f(x^*)]
\]
\[
\leq \left(1 - \min\left\{\frac{\mu_\eta}{2}, \frac{\delta}{4}, \frac{\delta_1}{4}, \frac{p}{4}\right\}\right)^{k+1} \Phi_k - \frac{\eta}{18} \sum_{i=0}^{k} \left(1 - \min\left\{\frac{\mu_\eta}{2}, \frac{\delta}{4}, \frac{\delta_1}{4}, \frac{p}{4}\right\}\right)^{k-i} \mathbb{E}[f(x^i) - f(x^*)]
\]
\[
\leq \left(1 - \min\left\{\frac{\mu_\eta}{2}, \frac{\delta}{4}, \frac{\delta_1}{4}, \frac{p}{4}\right\}\right)^k \Phi_0 - \frac{\eta}{18} \sum_{i=0}^{k} \left(1 - \min\left\{\frac{\mu_\eta}{2}, \frac{\delta}{4}, \frac{\delta_1}{4}, \frac{p}{4}\right\}\right)^{k-i} \mathbb{E}[f(x^i) - f(x^*)]
\]
\[
= \frac{1}{w_k} \Phi_0 - \frac{\eta}{18 w_k} \sum_{i=0}^{k} w_i [f(x^i) - f(x^*)],
\]
which implies that
\[
\frac{1}{W_k} \sum_{i=0}^{k} w_i [f(x^i) - f(x^*)] \leq \frac{18}{\eta W_k} \Phi_0 - \frac{18 w_k}{\eta W_k} \mathbb{E}[\Phi_{k+1}] \leq \frac{18}{\eta W_k} \Phi_0.
\]
(27)

When \(\mu > 0\), from (27) and
\[
W_k = \sum_{i=0}^{k} w_i = \frac{1}{1 - (1 - \min\left\{\frac{\mu_\eta}{2}, \frac{\delta}{4}, \frac{\delta_1}{4}, \frac{p}{4}\right\})^{k+1}} = \frac{1}{1 - (1 - \min\left\{\frac{\mu_\eta}{2}, \frac{\delta}{4}, \frac{\delta_1}{4}, \frac{p}{4}\right\})^{k+1} \min\left\{\frac{\mu_\eta}{2}, \frac{\delta}{4}, \frac{\delta_1}{4}, \frac{p}{4}\right\}(1 - \min\left\{\frac{\mu_\eta}{2}, \frac{\delta}{4}, \frac{\delta_1}{4}, \frac{p}{4}\right\})^k},
\]
we can obtain
\[
\frac{1}{W_k} \sum_{i=0}^{k} w_i [f(x^i) - f(x^*)] \leq \frac{\min\left\{\frac{\mu_\eta}{2}, \frac{\delta}{4}, \frac{\delta_1}{4}, \frac{p}{4}\right\}}{1 - (1 - \min\left\{\frac{\mu_\eta}{2}, \frac{\delta}{4}, \frac{\delta_1}{4}, \frac{p}{4}\right\})^{k+1}} \frac{18}{\eta} \Phi_0 \left(1 - \min\left\{\frac{\mu_\eta}{2}, \frac{\delta}{4}, \frac{\delta_1}{4}, \frac{p}{4}\right\}\right)^k.
\]

From the definition of \(\Phi_k\) and \(e_0^0 = 0\), we have
\[
\min\left\{\frac{\mu_\eta}{2}, \frac{\delta}{4}, \frac{\delta_1}{4}, \frac{p}{4}\right\} \cdot \frac{1}{\eta} \Phi_0\n\]
\[
= \min\left\{\frac{\mu_\eta}{2}, \frac{\delta}{4}, \frac{\delta_1}{4}, \frac{p}{4}\right\} \left(\frac{1}{\eta} \|x^0 - x^*\|^2 + \frac{192(1 - \delta)L f_\eta^2}{\delta^2 \delta_1 n} \sum_{\tau=1}^{n} \|\nabla f(\tau)(x^0) - h_\tau^0\|^2
\]
\[
+ \frac{4}{3p} \left(\frac{48(1 - \delta)L f_\eta^2}{\delta} \left(\frac{4L}{\delta} + L + \frac{16Lp}{\delta_1} \left(1 + \frac{2p}{\delta_1}\right) + \frac{4Lp}{\delta}\right)\right) \frac{f(x^0) - f(x^*)}{n}\right)\]
\[
\leq \frac{\mu}{2} \|x^0 - x^*\|^2 + \min\left\{\frac{\mu_\eta}{2}, \frac{\delta}{4}, \frac{\delta_1}{4}, \frac{p}{4}\right\} \frac{1}{70 L p n} \sum_{\tau=1}^{n} \|\nabla f(\tau)(x^0) - h_\tau^0\|^2
\]
\[
+ \min\left\{\frac{\mu_\eta}{2}, \frac{\delta}{4}, \frac{\delta_1}{4}, \frac{p}{4}\right\} \frac{4}{3p} \left(\frac{1}{18} + \frac{1}{84} + \frac{1}{18} + \frac{1}{8}\right) \frac{f(x^0) - f(x^*)}{n}\right)\]
\[
\leq \frac{\mu}{2} \|x^0 - x^*\|^2 + \frac{1}{280 L n} \sum_{\tau=1}^{n} \|\nabla f(\tau)(x^0) - h_\tau^0\|^2 + \frac{1}{5} [f(x^0) - f(x^*)].
\]

Therefore, we can get
\[
\frac{1}{W_k} \sum_{i=0}^{k} w_i [f(x^i) - f(x^*)] \leq \frac{9\mu \|x^0 - x^*\|^2 + 2(f(x^0) - f(x^*))}{1 - (1 - \min\left\{\frac{\mu_\eta}{2}, \frac{\delta}{4}, \frac{\delta_1}{4}, \frac{p}{4}\right\})^{k+1}} \left(1 - \min\left\{\frac{\mu_\eta}{2}, \frac{\delta}{4}, \frac{\delta_1}{4}, \frac{p}{4}\right\}\right)^k.
\]

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For $\bar{x}^k = \frac{1}{n_k} \sum_{i=0}^{k} u_i x^i$, from the convexity of $f$ and the above inequality, we have

$$
\mathbb{E}[f(\bar{x}^k) - f(x^*)] 
\leq \frac{9\mu \|x^0 - x^*\|^2 + 2(f(x^0) - f(x^*))}{1 - (1 - \min\left\{ \frac{\mu \eta}{2}, \frac{\delta}{4}, \frac{\delta_1}{4}, \frac{p}{4} \right\})^{k+1}} \left( 1 - \min\left\{ \frac{\mu \eta}{2}, \frac{\delta}{4}, \frac{\delta_1}{4}, \frac{p}{4} \right\} \right)^k.
$$

If we choose $\eta = \min \left\{ 1, \frac{\delta}{2L}, \frac{\delta}{60(1-\delta)L_f L}, \frac{1}{64(1-\delta)L_f L}, \frac{1}{120(1-\delta)L_f L} p \right\}$, then in order to guarantee $\mathbb{E}[f(\bar{x}^k) - f(x^*)] \leq \epsilon$, we first let

$$
\left( 1 - \min\left\{ \frac{\mu \eta}{2}, \frac{\delta}{4}, \frac{\delta_1}{4}, \frac{p}{4} \right\} \right)^k \leq \frac{1}{2},
$$

which implies that

$$
\mathbb{E}[f(\bar{x}^k) - f(x^*)] 
\leq \left( 18\mu \|x^0 - x^*\|^2 + 4(f(x^0) - f(x^*)) + \frac{2}{15Ln} \sum_{\tau=1}^{n} \|\nabla f^{(\tau)}(x^0) - h^0_\tau\|^2 \right) \left( 1 - \min\left\{ \frac{\mu \eta}{2}, \frac{\delta}{4}, \frac{\delta_1}{4}, \frac{p}{4} \right\} \right)^k.
$$

Hence, when $\epsilon \leq 9\mu \|x^0 - x^*\|^2 + 2(f(x^0) - f(x^*)) + \frac{2}{15Ln} \sum_{\tau=1}^{n} \|\nabla f^{(\tau)}(x^0) - h^0_\tau\|^2$, $\mathbb{E}[f(\bar{x}^k) - f(x^*)] \leq \epsilon$ as long as

$$
\left( 1 - \min\left\{ \frac{\mu \eta}{2}, \frac{\delta}{4}, \frac{\delta_1}{4}, \frac{p}{4} \right\} \right)^k \leq \frac{\epsilon}{18\mu \|x^0 - x^*\|^2 + 4(f(x^0) - f(x^*)) + \frac{2}{15Ln} \sum_{\tau=1}^{n} \|\nabla f^{(\tau)}(x^0) - h^0_\tau\|^2},
$$

which is equivalent to

$$
k \geq \frac{1}{-\ln(1 - x)} \ln \left( \frac{18\mu \|x^0 - x^*\|^2 + 4(f(x^0) - f(x^*)) + \frac{2}{15Ln} \sum_{\tau=1}^{n} \|\nabla f^{(\tau)}(x^0) - h^0_\tau\|^2}{\epsilon} \right),
$$

where $-\ln(1 - x) \geq x$ for $x \in [0, 1)$, if $p \leq O(\delta_1)$, we have $\mathbb{E}[f(\bar{x}^k) - f(x^*)] \leq \epsilon$ as long as

$$
k \geq O \left( \left( \frac{1}{\delta} + \frac{1}{p} + \frac{L_f}{\mu} + \frac{L}{\mu} + \frac{\sqrt{(1-\delta)L_f L}}{\delta \mu} + \frac{\sqrt{(1-\delta)L_f L}}{\sqrt{\delta \mu}} \right)^{-1} \ln \left( \frac{18\mu \|x^0 - x^*\|^2 + 4(f(x^0) - f(x^*)) + \frac{2}{15Ln} \sum_{\tau=1}^{n} \|\nabla f^{(\tau)}(x^0) - h^0_\tau\|^2}{\epsilon} \right) \right).
$$

When $\mu = 0$, from (27) and the convexity of $f$, we have

$$
\mathbb{E}[f(\bar{x}^k) - f(x^*)] \leq \frac{1}{k+1} \sum_{i=0}^{k} \mathbb{E}[f(x^i) - f(x^*)] \leq \frac{18}{\eta(k+1)} \Phi^0_3.
$$

From the definition of $\Phi^k_3$ and $e^0_\epsilon = 0$, we have
\[
\frac{1}{\eta} \Phi_0^c = \frac{1}{\eta} \|x^0 - x^*\|^2 + \frac{192(1 - \delta)L_f \eta^2}{\delta^2 \delta_1 n} \sum_{\tau = 1}^n \|\nabla f^{(\tau)}(x^0) - h_0^\tau\|^2 \\
+ \frac{4}{3p} \left( \frac{48(1 - \delta)L_f \eta^2}{\delta} \left( \frac{4L_f}{\delta} + L + \frac{16L_p}{\delta \delta_1} \left( 1 + \frac{2p}{\delta_1} \right) \right) + \frac{4L \eta}{n} \right) [f(x^0) - f(x^*)] \\
\leq \frac{1}{\eta} \|x^0 - x^*\|^2 + \frac{1}{70L_p n} \sum_{\tau = 1}^n \|\nabla f^{(\tau)}(x^0) - h_0^\tau\|^2 + \frac{4}{3p} \left( \frac{18}{18} + \frac{1}{18} + \frac{1}{18} + \frac{1}{8} \right) [f(x^0) - f(x^*)] \\
\leq \frac{1}{\eta} \|x^0 - x^*\|^2 + \frac{1}{70L_p n} \sum_{\tau = 1}^n \|\nabla f^{(\tau)}(x^0) - h_0^\tau\|^2 + \frac{1}{3p} [f(x^0) - f(x^*)].
\]

Hence, we arrive at

\[
E[f(\bar{x}^k) - f(x^*)] \leq \frac{1}{k + 1} \left( \frac{18}{\eta} \|x^0 - x^*\|^2 + \frac{1}{3p} \sum_{\tau = 1}^n \|\nabla f^{(\tau)}(x^0) - h_0^\tau\|^2 + \frac{6}{p} (f(x^0) - f(x^*)) \right).
\]

In particular, if we choose \( \eta = \min \left\{ \frac{1}{12L_f + 33L_f / n}, \frac{\sqrt{3} \delta}{60\sqrt{(1 - \delta)L_f L}}, \frac{\sqrt{7} \delta}{64\sqrt{(1 - \delta)L_f L}}, \frac{\delta \sqrt{\eta}}{120\sqrt{(1 - \delta)pL_f (L_f + \frac{12L_p}{n})}} \right\} \) and \( p \leq O(\delta_1) \), we have \( E[f(\bar{x}^k) - f(x^*)] \leq \epsilon \) as long as

\[
k \geq O \left( \left( \frac{1}{p} + L_f + \frac{\sqrt{(1 - \delta)L_f L}}{\delta} + \frac{\sqrt{(1 - \delta)L_f L}}{\sqrt{\delta}} \right) \frac{1}{\epsilon} \right).
\]

### C.5 Proof of Theorem 2.11

Let \( \eta \leq \min \left\{ \frac{1}{4L_f + 33L_f / n}, \frac{\sqrt{3} \delta}{60\sqrt{(1 - \delta)L_f L}}, \frac{\sqrt{7} \delta}{64\sqrt{(1 - \delta)L_f L}}, \frac{\delta \sqrt{\eta}}{120\sqrt{(1 - \delta)pL_f (L_f + \frac{12L_p}{n})}} \right\} \). Then we have

\[
\frac{896(1 - \delta)L_f^2 \eta^2}{\delta^2} \leq \frac{1}{4}, \quad \frac{7168(1 - \delta)L_f L \eta^2}{\delta} \leq \frac{1}{18}, \quad \frac{2912(1 - \delta)L_f L \eta^2}{\delta} \leq \frac{1}{18}, \quad \frac{11L \eta}{n} \leq \frac{1}{3}.
\]

and \( \frac{3584(1 - \delta)pL \eta L_f (L_f + \frac{12L_p}{n}) (1 + \frac{2p}{\delta})}{\delta^2} \leq \frac{1}{4} \).

Therefore, from Theorem 2.9 we have

\[
E[\Phi_4^{k+1}] \leq \left( 1 - \min \left\{ \frac{\mu \gamma}{2}, \frac{\delta}{4}, \frac{\delta_1}{4}, \frac{p}{4} \right\} \right) E[\Phi_4^k] - \frac{\eta}{36} E[f(x^k) - f(x^*)]
\]

\[
\leq \left( 1 - \min \left\{ \frac{\mu \gamma}{2}, \frac{\delta}{4}, \frac{\delta_1}{4}, \frac{p}{4} \right\} \right)^{k+1} \Phi_0^c - \frac{\eta}{36} \sum_{i = 0}^k \left( 1 - \min \left\{ \frac{\mu \gamma}{2}, \frac{\delta}{4}, \frac{\delta_1}{4}, \frac{p}{4} \right\} \right)^{k-i} E[f(x^i) - f(x^*)]
\]

\[
\leq \left( 1 - \min \left\{ \frac{\mu \gamma}{2}, \frac{\delta}{4}, \frac{\delta_1}{4}, \frac{p}{4} \right\} \right)^{k} \Phi_0^c - \frac{\eta}{36} \sum_{i = 0}^k \left( 1 - \min \left\{ \frac{\mu \gamma}{2}, \frac{\delta}{4}, \frac{\delta_1}{4}, \frac{p}{4} \right\} \right)^{k-i} E[f(x^i) - f(x^*)]
\]

\[
= \frac{1}{w_k} \Phi_0^c - \frac{\eta}{36 w_k} \sum_{i = 0}^k w_i E[f(x^i) - f(x^*)],
\]

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When $\mu > 0$, notice that

$$
\min \left\{ \frac{\mu \eta}{2}, p \right\} \frac{1}{\delta \Phi} \frac{1}{\eta} \Phi
= \min \left\{ \frac{\mu \eta}{2}, p \right\} \frac{1}{\delta \Phi} \left( \frac{1}{\eta} \left\| x_0 - x^* \right\|^2 + \frac{192(1-\delta)L_I \eta^2}{\delta \Phi} \left\| \nabla f(x_0) \right\|^2 + \frac{2304(1-\delta)L_I \eta^2}{\delta \Phi} \sum_{\tau=1}^{n} \left\| \nabla f^{(\tau)}(x_0) - h_0 \right\|^2
+ \frac{4}{3p} \left( \frac{48(1-\delta)\eta^2}{\delta} \right) \left( \frac{4L_I}{\delta} + \frac{32\bar{L}}{n\eta} + \frac{13L}{n} + \frac{16pL_I + 12\bar{L}}{\delta \Phi} \right) \left( 1 - \frac{2p}{\delta \Phi} \right) + \frac{4L \eta}{\delta \Phi} \right) \left[ f(x_0) - f(x^*) \right]
\right\}
\leq \frac{\mu}{2} \left\| x_0 - x^* \right\|^2 + \min \left\{ \frac{\mu \eta}{2}, p \right\} \frac{1}{\delta \Phi} \left( \frac{1}{\eta} \left\| \nabla f(x_0) \right\|^2 + \frac{1}{24L \eta \Phi} \sum_{\tau=1}^{n} \left\| \nabla f^{(\tau)}(x_0) - h_0 \right\|^2 + \frac{1}{9} \left[ f(x_0) - f(x^*) \right] \right\}
\leq \frac{\mu}{2} \left\| x_0 - x^* \right\|^2 + \frac{1}{24L \eta \Phi} \sum_{\tau=1}^{n} \left\| \nabla f^{(\tau)}(x_0) - h_0 \right\|^2 + \frac{1}{9} \left[ f(x_0) - f(x^*) \right].
$$

Then same as the proof of Theorem 2.10 we have

$$
\mathbb{E}[f(x^k) - f(x^*)] \leq \frac{18\mu \left\| x_0 - x^* \right\|^2 + \frac{2}{L_I n} \sum_{\tau=1}^{n} \left\| \nabla f^{(\tau)}(x_0) - h_0 \right\|^2 + 5 \left[ f(x_0) - f(x^*) \right]}{1 - \left( 1 - \min \left\{ \frac{\mu \eta}{2}, p \right\} \right)^k + 1}
\cdot \left( 1 - \min \left\{ \frac{\mu \eta}{2}, p \right\} \right)^k,
$$

and if we choose

$$
\eta = \min \left\{ \frac{1}{4L_I + 33L / n}, \frac{\delta}{\sqrt{\eta}}, \frac{1}{222(1-\delta)L_I L}, \frac{\sqrt{\eta}}{360(1-\delta)L_I L}, \frac{\sqrt{\eta}}{120(1-\delta)L_I L}, \frac{\delta}{\sqrt{\eta}} \right\},
$$

and $p \leq O(\delta_1)$, then $\mathbb{E}[f(x^k) - f(x^*)] \leq \epsilon$ with $\epsilon \leq 9\mu \left\| x_0 - x^* \right\|^2 + \frac{1}{L_I n} \sum_{\tau=1}^{n} \left\| \nabla f^{(\tau)}(x_0) - h_0 \right\|^2 + 3 \left[ f(x_0) - f(x^*) \right]$ as long as

$$
k \geq O \left( \left( \frac{1}{\delta} + \frac{1}{p} + \frac{L_I}{n \mu} + \sqrt{\frac{1-\delta}{\mu \delta}} \right) \ln \frac{1}{\epsilon} \right),
$$

which is equivalent to

$$
k \geq O \left( \left( \frac{1}{\delta} + \frac{1}{p} + \frac{L_I}{n \mu} + \sqrt{\frac{1-\delta}{\mu \delta}} \right) \ln \frac{1}{\epsilon} \right),
$$

since $\tilde{L} \leq L_I$, and

$$
2\sqrt{\frac{(1-\delta)L_I L}{\mu \delta}} \leq \sqrt{\frac{1-\delta}{\delta}} + \sqrt{\frac{1-\delta}{n \mu}} \leq \sqrt{\frac{1-\delta}{\delta}} + \frac{L_I}{n}.
$$

When $\mu = 0$, notice that
where $p$ and $p$ For brevity, let

$$h = \sum_{\tau=1}^{n} \|\nabla f(\tau)(x^0) - h^0\|^2$$

Lemma D.1. \[ESO\] The following inequality holds for all $h \in \mathbb{R}^{tN}$:

$$E[\|Ah|S\|^2] \leq \sum_{\tau=1}^{n} \sum_{\tilde{\tau}=1}^{m} p_{\tilde{\tau}} v_{\tilde{\tau}} \|h_{\tilde{\tau}}\|^2,$$

where $p_{\tilde{\tau}} = \frac{1}{m}$ and $v_{\tilde{\tau}} = R_{m}^2 + nR^2$.

D Proofs for EC-Quartz

D.1 Lemmas

For brevity, let

$$A = [A_{11}, ..., A_{m1}, A_{12}, ..., A_{m2}, ..., A_{n1}, ..., A_{mn}] \in \mathbb{R}^{d \times tN}.$$ Let $S = \{(i^\tau, \tau) | i^\tau \text{ is chosen from } [m] \text{ uniformly and independently for all } \tau \in [n]\}$ For any vector $h \in \mathbb{R}^{tN}$, let $h|S \in \mathbb{R}^{tN}$ be defined by

$$(h|S)_{i\tau} = \begin{cases} h_{i\tau} & \text{if } (i, \tau) \in S \\ 0 & \text{otherwise} \end{cases}.$$
Proof. We give two proofs.

First proof. Notice that $S$ can be regarded as a group sampling [Qian et al., 2019b] where the index set on each node is a group and $P[(i, \tau) \in S] = p_{i\tau} = \frac{1}{m}$ for all $i \in [m]$ and $\tau \in [n]$. Hence, from Lemma 6.6 in [Qian et al., 2019b], we have

$$m^2 \mathbb{E} \left[ \left\| \sum_{(i, \tau) \in S} A_{i\tau} h_{i\tau} \right\|^2 \right] = \mathbb{E} \left[ \left\| \sum_{(i, \tau) \in S} \frac{1}{p_{i\tau}} A_{i\tau} h_{i\tau} \right\|^2 \right]$$

$$\leq \sum_{\tau=1}^n \sum_{i=1}^n \frac{1}{p_{i\tau}} \|A_{i\tau} h_{i\tau}\|^2 + \left\| \sum_{\tau=1}^n \sum_{i=1}^n A_{i\tau} h_{i\tau} \right\|^2$$

$$= m \sum_{\tau=1}^n \sum_{i=1}^n \|A_{i\tau} h_{i\tau}\|^2 + \|Ah\|^2$$

$$\leq m \sum_{\tau=1}^n \sum_{i=1}^n \|A_{i\tau}\|^2 \|h_{i\tau}\|^2 + \lambda_{\max}(A^TA) \|h\|^2$$

$$= m \sum_{\tau=1}^n \sum_{i=1}^n \|A_{i\tau}\|^2 \|h_{i\tau}\|^2 + \lambda_{\max}(AA^T) \|h\|^2$$

$$\leq (mR_m^2 + NR^2) \sum_{\tau=1}^n \sum_{i=1}^n \|h_{i\tau}\|^2,$$

where in the last inequality, we use $\|A_{i\tau}\| \leq \max_{i, \tau} \|A_{i\tau}\| = R_m$ and

$$\frac{1}{N} \lambda_{\max}(AA^T) = \frac{1}{N} \lambda_{\max} \sum_{\tau=1}^n \sum_{i=1}^m A_{i\tau} A_{i\tau}^T = R^2.$$ 

Then we arrive at

$$\mathbb{E} \left[ \left\| Ah_{(S)} \right\|^2 \right] = \mathbb{E} \left[ \left\| \sum_{(i, \tau) \in S} A_{i\tau} h_{i\tau} \right\|^2 \right] \leq \frac{1}{m} (R_m^2 + nR^2) \sum_{\tau=1}^n \sum_{i=1}^n \|h_{i\tau}\|^2 \leq \sum_{\tau=1}^n \sum_{i=1}^n p_{i\tau} v_{i\tau} \|h_{i\tau}\|^2.$$
Second proof. From the definition of $S$, we have

$$
\mathbb{E}[\|Ah[S]\|^2] = \mathbb{E} \left[ \left\| \sum_{(i, \tau) \in S} A_{i\tau} h_{i\tau} \right\|^2 \right] = \mathbb{E} \left[ \sum_{\tau_1 \neq \tau_2} \left\langle A_{i_1 \tau_1}, h_{i_1 \tau_1}, A_{i_2 \tau_2} h_{i_2 \tau_2} \right\rangle \right] + \mathbb{E} \left[ \sum_{\tau_1} \left\| A_{i\tau} h_{i\tau} \right\|^2 \right]
$$

$$
= \frac{1}{m^2} \sum_{\tau_1 \neq \tau_2} \frac{1}{m} \sum_{i_1 = 1}^{m} \sum_{j_1 = 1}^{m} \left\| A_{i_1 \tau_1} h_{i_1 \tau_1}, A_{j_1 \tau_2} h_{j_1 \tau_2} \right\| + \frac{1}{m} \sum_{\tau_1} \sum_{i_1 = 1}^{m} \left\| A_{i\tau} h_{i\tau} \right\|^2
$$

$$
\leq \frac{1}{m^2} \|Ah\|^2 + \frac{1}{m} \sum_{\tau_1} \sum_{i_1 = 1}^{m} \left\| A_{i\tau} h_{i\tau} \right\|^2
$$

$$
\leq \frac{NR^2}{m^2} \sum_{\tau_1} \sum_{i_1 = 1}^{m} \left\| h_{i\tau} \right\|^2 + \frac{1}{m} \sum_{\tau_1} \sum_{i_1 = 1}^{m} \left\| h_{i\tau} \right\|^2
$$

$$
= \sum_{\tau_1} \sum_{i_1 = 1}^{m} p_{i\tau} v_{i\tau} \left\| h_{i\tau} \right\|^2,
$$

where in the fourth equality, we use the fact that $i^{\tau_1}$ is independent of $i^{\tau_2}$ for $\tau_1 \neq \tau_2$. 

\[ \square \]

Lemma D.2. [Lemma 18 in Qu et al. [2013]] Function $\hat{f} : \mathbb{R}^{tN} \to \mathbb{R}$ satisfies the following inequality:

$$
\hat{f}(\alpha + h) \leq \hat{f}(\alpha) + \langle \nabla \hat{f}(\alpha), h \rangle + \frac{1}{2\lambda N^2} h^\top A^\top A h, \quad \forall \alpha, h \in \mathbb{R}^{tN}.
$$

(30)

Lemma D.3. [Lemma 19 in Qu et al. [2013]] For all $\alpha, h \in \mathbb{R}^{tN}$, the following holds:

$$
\mathbb{E}[-D(\alpha + h[S])]
\leq \hat{f}(\alpha) + \sum_{\tau_1 = 1}^{n} \sum_{i_1 = 1}^{m} p_{i\tau} \left\langle \frac{1}{N} A_{i\tau} \nabla g^*(\frac{1}{\lambda N} A\alpha), h_{i\tau} \right\rangle + \frac{1}{2\lambda N^2} \sum_{\tau_1 = 1}^{n} \sum_{i_1 = 1}^{m} p_{i\tau} v_{i\tau} \left\| h_{i\tau} \right\|^2
$$

$$
+ \frac{1}{N} \sum_{\tau_1 = 1}^{n} \sum_{i_1 = 1}^{m} [(1 - p_{i\tau}) \phi_{i\tau}^*(\alpha_{i\tau}) - \alpha_{i\tau} - h_{i\tau}] - (1 - p_{i\tau}) \phi_{i\tau}^*(-\alpha_{i\tau} - h_{i\tau})]
$$

(31)
Lemma D.4. Fixing \( \alpha \in \mathbb{R}^{1N} \) and \( x \in \mathbb{R}^d \), let \( \hat{u} = \frac{1}{\lambda N} A \alpha \) and \( h \in \mathbb{R}^{1N} \) be defined by:

\[
h_{i\tau} = -\theta p_{i\tau}^{-1} (\alpha_{i\tau} + \nabla \phi_{i\tau}(A_{i\tau}^\top x)), \quad i \in [m], \; \tau \in [n],
\]

where \( \theta > 0 \). Then

\[
\begin{align*}
\hat{f}(\alpha) &+ \frac{1}{N} \sum_{\tau=1}^{n} \sum_{i=1}^{m} p_{i\tau} \left( \frac{1}{N} A_{i\tau}^\top \nabla g^* \left( \frac{1}{\lambda N} A \alpha \right), h_{i\tau} \right) + \frac{1}{2\lambda N^2} \sum_{\tau=1}^{n} \sum_{i=1}^{m} p_{i\tau} v_{i\tau} \| h_{i\tau} \|^2 \\
&+ \frac{1}{N} \sum_{\tau=1}^{n} \sum_{i=1}^{m} \left[ (1 - p_{i\tau}) \phi_{i\tau}^* (-\alpha_{i\tau}) + p_{i\tau} \phi_{i\tau}^* (-\alpha_{i\tau} - h_{i\tau}) \right] \\
\leq -(1 - \theta) D(\alpha) - \theta \lambda g(\nabla g^*(u)) - \frac{1}{N} \sum_{\tau=1}^{n} \sum_{i=1}^{m} (\theta \nabla g^*(u), A_{i\tau} \nabla \phi_{i\tau}(A_{i\tau}^\top x)) \\
&+ \frac{\theta}{N} \sum_{\tau=1}^{n} \sum_{i=1}^{m} \phi_{i\tau}^*(\nabla \phi_{i\tau}(A_{i\tau}^\top x)) + \frac{\rho + \theta \lambda}{2} \| \hat{u} - u \|^2 \\
&+ \frac{1}{2\lambda N^2} \sum_{\tau=1}^{n} \sum_{i=1}^{m} \left( p_{i\tau} v_{i\tau} + \frac{N \lambda p_{i\tau}^2 R^2}{\rho} - \frac{N \lambda \gamma p_{i\tau}^2 (1 - \theta p_{i\tau}^{-1})}{\gamma} \right) \| h_{i\tau} \|^2,
\end{align*}
\]

for any \( u \in \mathbb{R}^d \) and \( \rho > 0 \).

Proof. First, for any \( u \in \mathbb{R}^d \) and \( \rho > 0 \), we have

\[
\begin{align*}
\hat{f}(\alpha) &+ \frac{1}{N} \sum_{\tau=1}^{n} \sum_{i=1}^{m} p_{i\tau} \left( \frac{1}{N} A_{i\tau}^\top \nabla g^* \left( \frac{1}{\lambda N} A \alpha \right), h_{i\tau} \right) \\
&= \hat{f}(\alpha) + \frac{1}{N} \sum_{\tau=1}^{n} \sum_{i=1}^{m} p_{i\tau} \left( A_{i\tau}^\top \nabla g^*(\hat{u}), h_{i\tau} \right) \\
&= \hat{f}(\alpha) + \frac{1}{N} \sum_{\tau=1}^{n} \sum_{i=1}^{m} p_{i\tau} \langle A_{i\tau}^\top \nabla g^*(u), h_{i\tau} \rangle + \frac{1}{N} \sum_{\tau=1}^{n} \sum_{i=1}^{m} p_{i\tau} \langle (\nabla g^*(\hat{u}) - \nabla g^*(u)), A_{i\tau} h_{i\tau} \rangle \\
&= \hat{f}(\alpha) + \frac{1}{N} \sum_{\tau=1}^{n} \sum_{i=1}^{m} p_{i\tau} \langle A_{i\tau}^\top \nabla g^*(u), h_{i\tau} \rangle + \langle (\nabla g^*(\hat{u}) - \nabla g^*(u)), \frac{1}{N} A h^p \rangle \\
&\leq \hat{f}(\alpha) + \frac{1}{N} \sum_{\tau=1}^{n} \sum_{i=1}^{m} p_{i\tau} \langle A_{i\tau}^\top \nabla g^*(u), h_{i\tau} \rangle + \frac{\rho}{2} \| \nabla g^*(\hat{u}) - \nabla g^*(u) \|^2 + \frac{1}{2\rho N^2} \| A h^p \|^2 \\
&\leq \hat{f}(\alpha) + \frac{1}{N} \sum_{\tau=1}^{n} \sum_{i=1}^{m} p_{i\tau} \langle A_{i\tau}^\top \nabla g^*(u), h_{i\tau} \rangle + \frac{\rho}{2} \| \hat{u} - u \|^2 + \frac{1}{N} \sum_{\tau=1}^{n} \sum_{i=1}^{m} \frac{p_{i\tau}^2 R^2}{2\rho} \| h_{i\tau} \|^2,
\end{align*}
\]

where we denote \( h^p \in \mathbb{R}^m \) such that \( h^p_{i\tau} = p_{i\tau} h_{i\tau} \) in the third equality, in the first inequality we use the Young’s inequality and the last inequality comes from \( g^* \) is 1-smooth since \( g \) is 1-strongly convex. For the first two terms in the above inequality, we have
\[ \tilde{f}(\alpha) + \frac{1}{N} \sum_{\tau=1}^{n} \sum_{i=1}^{m} p_{i\tau} \left( A_{i\tau}^{\top} \nabla g^*(u), h_{i\tau} \right) \]

\[ = \lambda g^*(\tilde{u}) - \frac{\theta}{N} \sum_{\tau=1}^{n} \sum_{i=1}^{m} \langle \nabla g^*(u), A_{i\tau} \alpha_{i\tau} + A_{i\tau} \nabla \phi_{i\tau}(A_{i\tau}^{\top} x) \rangle \]

\[ = \lambda g^*(\tilde{u}) - \frac{\theta}{N} \sum_{\tau=1}^{n} \sum_{i=1}^{m} \langle \nabla g^*(u), A_{i\tau} \nabla \phi_{i\tau}(A_{i\tau}^{\top} x) \rangle \]

\[ = (1 - \theta) \lambda g^*(\tilde{u}) + \theta \lambda g^*(\tilde{u}) - \theta \lambda g^*(u) - \theta \lambda(\nabla g^*(u), \tilde{u} - u) + \theta \lambda g^*(u) - \theta \lambda(\nabla g^*(u), u) \]

\[ - \frac{1}{N} \sum_{\tau=1}^{n} \sum_{i=1}^{m} \langle \nabla g^*(u), A_{i\tau} \nabla \phi_{i\tau}(A_{i\tau}^{\top} x) \rangle \]

\[ \leq (1 - \theta) \lambda g^*(\tilde{u}) + \frac{\theta \lambda}{2} \| \tilde{u} - u \|^2 + \theta \lambda(g^*(u) - \langle \nabla g^*(u), u \rangle) \]

\[ - \frac{1}{N} \sum_{\tau=1}^{n} \sum_{i=1}^{m} \langle \nabla g^*(u), A_{i\tau} \nabla \phi_{i\tau}(A_{i\tau}^{\top} x) \rangle \]

\[ = (1 - \theta) \lambda g^*(\tilde{u}) + \frac{\theta \lambda}{2} \| \tilde{u} - u \|^2 - \theta \lambda g(\nabla g^*(u)) - \frac{1}{N} \sum_{\tau=1}^{n} \sum_{i=1}^{m} \langle \nabla g^*(u), A_{i\tau} \nabla \phi_{i\tau}(A_{i\tau}^{\top} x) \rangle, \]

where the first inequality comes from \( g^* \) is 1-smooth and the last equality comes from the definition of conjugate functions.

From (50) in [Qu et al., 2013], we also have

\[ \frac{1}{N} \sum_{\tau=1}^{n} \sum_{i=1}^{m} \left( (1 - p_{i\tau}) \phi_{i\tau}^*(-\alpha_{i\tau}) + p_{i\tau} \phi_{i\tau}^*(-\alpha_{i\tau} - h_{i\tau}) \right) \]

\[ \leq (1 - \theta) \psi(\alpha) + \frac{\theta}{N} \sum_{\tau=1}^{n} \sum_{i=1}^{m} \phi_{i\tau}^{\ast}((\nabla \phi_{i\tau}(A_{i\tau}^{\top} x)) - \frac{1}{2 \lambda N^2} \sum_{\tau=1}^{n} \sum_{i=1}^{m} \frac{N \lambda \gamma p_{i\tau}^2 (1 - \theta p_{i\tau}^{-1})}{\theta} \| h_{i\tau} \|^2. \]

Combining the above three inequalities, we arrive at

\[ \tilde{f}(\alpha) + \sum_{\tau=1}^{n} \sum_{i=1}^{m} p_{i\tau} \left( \frac{1}{N} A_{i\tau}^{\top} \nabla g^* \left( \frac{1}{N} A\alpha \right), h_{i\tau} \right) + \frac{1}{2 \lambda N^2} \sum_{\tau=1}^{n} \sum_{i=1}^{m} p_{i\tau} v_{i\tau} \| h_{i\tau} \|^2 \]

\[ + \frac{1}{N} \sum_{\tau=1}^{n} \sum_{i=1}^{m} \left( (1 - p_{i\tau}) \phi_{i\tau}^*(-\alpha_{i\tau}) + p_{i\tau} \phi_{i\tau}^*(-\alpha_{i\tau} - h_{i\tau}) \right) \]

\[ \leq -(1 - \theta) D(\alpha) - \theta \lambda g(\nabla g^*(u)) - \frac{1}{N} \sum_{\tau=1}^{n} \sum_{i=1}^{m} \langle \nabla g^*(u), A_{i\tau} \nabla \phi_{i\tau}(A_{i\tau}^{\top} x) \rangle \]

\[ + \frac{\theta}{N} \sum_{\tau=1}^{n} \sum_{i=1}^{m} \phi_{i\tau}^{\ast}((\nabla \phi_{i\tau}(A_{i\tau}^{\top} x)) + \frac{\rho + \theta \lambda}{2} \| \tilde{u} - u \|^2 \]

\[ + \frac{1}{2 \lambda N^2} \sum_{\tau=1}^{n} \sum_{i=1}^{m} \left( p_{i\tau} v_{i\tau} + \frac{N \lambda \gamma p_{i\tau}^2 R^2}{\rho} - \frac{N \lambda \gamma p_{i\tau}^2 (1 - \theta p_{i\tau}^{-1})}{\theta} \right) \| h_{i\tau} \|^2. \]

Let \( E_k[\cdot] \) denote the expectation conditional on \( x^k, \alpha^k, u^k, \) and \( e^k \). Define \( \Delta \alpha_{i\tau}^{k+1} := -\theta p_{i\tau}^{-1} \alpha_{i\tau}^k - \theta p_{i\tau}^{-1} \nabla \phi_{i\tau}(A_{i\tau} x^{k+1}) \) for \( k \geq 0 \). Notice that \( E_k[x^{k+1}] = x^{k+1} \). Hence, \( E_k[\Delta \alpha_{i\tau}^{k+1}] = \Delta \alpha_{i\tau}^{k+1} \).
Lemma D.5. We have

$$
\frac{1}{n} \sum_{\tau=1}^{n} E_k [||e_{\tau}^{k+1}||^2] \leq \left(1 - \frac{\delta}{2}\right) \frac{1}{n} \sum_{\tau=1}^{n} ||e_{\tau}^{k}||^2 + \frac{(1 - \delta)}{\lambda^2 n^2} \left(\frac{2 \bar{R}^2}{\delta} + R_m^2\right) \sum_{\tau=1}^{n} ||\Delta \alpha_{\tau}^{k+1}||^2.
$$

Proof. First, from the contraction property of $Q$, we have

$$
E_k[||e_{\tau}^{k+1}||^2] \leq (1 - \delta) E_k[||e_{\tau}^{k}||^2] + \frac{1}{\lambda m} A_i^{\tau} \Delta \alpha_{\tau}^{k+1} ||^2
$$

$$
= (1 - \delta) E_k[||e_{\tau}^{k}||^2] + \frac{1}{\lambda m^2} \sum_{i=1}^{m} A_i^{\tau} \Delta \alpha_{\tau}^{k+1} ||^2 + \frac{1}{\lambda m} A_i^{\tau} \Delta \alpha_{\tau}^{k+1} ||^2 - \frac{1}{\lambda m^2} \sum_{i=1}^{m} A_i^{\tau} \Delta \alpha_{\tau}^{k+1} ||^2
$$

$$
\leq (1 - \delta) E_k[||e_{\tau}^{k}||^2] + \frac{1}{\lambda m^2} \sum_{i=1}^{m} A_i^{\tau} \Delta \alpha_{\tau}^{k+1} ||^2 + (1 - \delta) E_k[||\frac{1}{\lambda m} A_i^{\tau} \Delta \alpha_{\tau}^{k+1} ||^2]
$$

$$
\leq (1 - \delta) E_k[||e_{\tau}^{k}||^2] + \frac{1}{\lambda m^2} \sum_{i=1}^{m} \sum_{i=1}^{N} \|\Delta \alpha_{\tau}^{k+1}\|^2 + (1 - \delta) \frac{R_m^2}{\lambda^2 m^3} \sum_{i=1}^{m} \|\Delta \alpha_{\tau}^{k+1}\|^2
$$

$$
\leq (1 - \delta)(1 + \beta)||e_{\tau}^{k}||^2 + (1 - \delta) \left(1 + \frac{1}{\beta}\right) \frac{1}{\lambda^2 m^4} \left(\sum_{i=1}^{m} A_i^{\tau} \Delta \alpha_{\tau}^{k+1} \right)^2 + \frac{(1 - \delta) R_m^2}{\lambda^2 m^3} \sum_{i=1}^{m} \|\Delta \alpha_{\tau}^{k+1}\|^2
$$

$$
= \left(1 - \frac{\delta}{2}\right) ||e_{\tau}^{k}||^2 + \frac{2(1 - \delta)}{\lambda^2 m^3} \sum_{i=1}^{m} \|\Delta \alpha_{\tau}^{k+1}\|^2 + \frac{(1 - \delta) R_m^2}{\lambda^2 m^3} \sum_{i=1}^{m} \|\Delta \alpha_{\tau}^{k+1}\|^2
$$

where we use Young’s inequality in the third inequality and choose $\beta = \frac{\delta}{2(1 - \delta)}$ when $\delta < 1$. When $\delta = 1$, it is easy to see that the above inequality also holds.

Taking the average of the above inequality from $\tau = 1$ to $n$, we can get the result.

$\square$

Lemma D.6. Let $e^{k} = \frac{1}{n} \sum_{\tau=1}^{n} e_{\tau}^{k}$ for $k \geq 0$. Under Assumption I.3, we have

$$
E_k[||e^{k+1}||^2] \leq \left(1 - \frac{\delta}{2}\right) ||e^{k}||^2 + \frac{2(1 - \delta)}{n^2} \sum_{\tau=1}^{n} ||e_{\tau}^{k}||^2 + \frac{(1 - \delta)}{\lambda^2 N m^2} \left(\frac{2 \delta + 1}{n} + \frac{2 R_m^2}{\delta}\right) \sum_{\tau=1}^{n} \sum_{i=1}^{m} \|\Delta \alpha_{\tau}^{k+1}\|^2.
$$
Proof. Under Assumption 1.3, we have $E[Q(x)] = \delta x$, and

$$
E_k\|e^{k+1}\|^2 = E_k\left\| \frac{1}{n} \sum_{\tau=1}^{n} e_{\tau}^{k+1} \right\|^2
= \frac{1}{n^2} \sum_{j,s} E_k((e_j^k, e_s^k) \cdot (e_j^{k+1}, e_s^{k+1}))
= \frac{1}{n^2} \sum_{\tau=1}^{n} E_k\|e_{\tau}^{k+1}\|^2 + \frac{1}{n^2} \sum_{j \neq s} E_k((e_j^k, e_s^k) \cdot (e_j^{k+1}, e_s^{k+1}))
\leq \frac{1}{n^2} \sum_{\tau=1}^{n} \left( e_{\tau}^{k} + \frac{1}{\lambda m} A_{i_{\tau}}^k \Delta \alpha_{i_{\tau}^{k+1}} \right) \| e_{\tau}^{k+1} \|^2 + \frac{1}{n^2} \sum_{j \neq s} E_k((e_j^k, e_s^k) \cdot (e_j^{k+1}, e_s^{k+1}))
= \frac{1}{n^2} \sum_{\tau=1}^{n} \left( e_{\tau}^{k} + \frac{1}{\lambda m} A_{i_{\tau}}^k \Delta \alpha_{i_{\tau}^{k+1}} \right) \| e_{\tau}^{k} \|^2 + \frac{1}{n^2} \sum_{j \neq s} E_k((e_j^k, e_s^k) \cdot (e_j^{k+1}, e_s^{k+1}))
\leq \left( 1 - \delta \right) \sum_{\tau=1}^{n} \left( e_{\tau}^{k} + \frac{1}{\lambda m} A_{i_{\tau}}^k \Delta \alpha_{i_{\tau}^{k+1}} \right) \| e_{\tau}^{k} \|^2 + \frac{1}{n^2} \sum_{j \neq s} E_k((e_j^k, e_s^k) \cdot (e_j^{k+1}, e_s^{k+1}))
$$

where we use the definitions of $e^k$ in the last inequality.

Let $S_k = \{(i_{\tau}^k, \tau) | i_{\tau}^k \}$ is chosen from $[m]$ uniformly and independently for all $\tau \in [n]$. Then $\sum_{\tau=1}^{n} A_{i_{\tau}}^k \Delta \alpha_{i_{\tau}^{k+1}} = \sum_{(i, \tau) \in S_k} A_{i} \Delta \alpha_{i_{\tau}^{k+1}}$, and we can obtain

$$
E_k\|e^{k+1}\|^2 - \left( 1 - \delta \right) E_k\left\| e_{\tau}^{k} + \frac{1}{\lambda N} \sum_{(i, \tau) \in S_k} A_{i} \Delta \alpha_{i_{\tau}^{k+1}} \right\|^2
\leq \frac{1}{n^2} \sum_{\tau=1}^{n} \left( e_{\tau}^{k} + \frac{1}{\lambda m} A_{i_{\tau}}^k \Delta \alpha_{i_{\tau}^{k+1}} \right) \| e_{\tau}^{k} \|^2
\leq \frac{2(1 - \delta)\beta}{n^2} \sum_{\tau=1}^{n} \| e_{\tau}^{k} \|^2 + \frac{2(1 - \delta)\beta}{\lambda^2 m^2 n^2} \sum_{\tau=1}^{n} E_k\| A_{i_{\tau}^k} \Delta \alpha_{i_{\tau}^{k+1}} \|^2
\leq \frac{2(1 - \delta)\beta}{n^2} \sum_{\tau=1}^{n} \| e_{\tau}^{k} \|^2 + \frac{2(1 - \delta)\beta R_{\alpha}^{\text{m}}}{\lambda^2 N^2 m} \sum_{\tau=1}^{n} \| \Delta \alpha_{i_{\tau}^{k+1}} \|^2,
$$

where in the second and third inequalities we use the Young’s inequality.
For \((1 - \delta)E_k \left\| e^k + \frac{1}{\lambda N} \sum_{(i, r) \in S_k} A_{ir} \Delta \alpha_{ir}^{k+1} \right\|^2\), we have

\[
(1 - \delta)E_k \left\| e^k + \frac{1}{\lambda N} \sum_{(i, r) \in S_k} A_{ir} \Delta \alpha_{ir}^{k+1} \right\|^2
\]

\[
= (1 - \delta)E_k \left\| e^k + \frac{1}{\lambda N m} \sum_{\tau=1}^{m} \sum_{i=1}^{n} A_{ir} \Delta \alpha_{ir}^{k+1} + \frac{1}{\lambda N} \sum_{(i, r) \in S_k} A_{ir} \Delta \alpha_{ir}^{k+1} - \frac{1}{\lambda N m} \sum_{\tau=1}^{m} \sum_{i=1}^{n} A_{ir} \Delta \alpha_{ir}^{k+1} \right\|^2
\]

\[
+ (1 - \delta)E_k \left\| \frac{1}{\lambda N m} \sum_{\tau=1}^{m} \sum_{i=1}^{n} A_{ir} \Delta \alpha_{ir}^{k+1} - \frac{1}{\lambda N m} \sum_{\tau=1}^{m} \sum_{i=1}^{n} A_{ir} \Delta \alpha_{ir}^{k+1} \right\|^2
\]

\[
\leq \left(1 - \frac{\delta}{2}\right) \left\| e^k \right\|^2 + \frac{2(1 - \delta)}{\delta^2 \lambda^2 N^2 m^2} \sum_{\tau=1}^{m} \sum_{i=1}^{n} \|A_{ir} \Delta \alpha_{ir}^{k+1}\|^2 + \frac{(1 - \delta)}{\lambda^2 N^2} \left\| e_k \right\|^2 \sum_{(i, r) \in S_k} A_{ir} \Delta \alpha_{ir}^{k+1}
\]

\[
\leq \left(1 - \frac{\delta}{2}\right) \left\| e^k \right\|^2 + \frac{(1 - \delta)R^2}{\lambda^2 N m^2} \left(\frac{2}{\delta} - 1\right) \sum_{\tau=1}^{m} \sum_{i=1}^{n} \|\Delta \alpha_{ir}^{k+1}\|^2 + \frac{(1 - \delta)}{\lambda^2 N^2} \sum_{\tau=1}^{m} \sum_{i=1}^{n} p_{ir} v_{ir} \|\Delta \alpha_{ir}^{k+1}\|^2.
\]

Recall that \(p_{ir} = \frac{1}{m}\) and \(v_{ir} = R_m^2 + nR^2\), we have

\[
(1 - \delta)E_k \left\| e^k + \frac{1}{\lambda N} \sum_{(i, r) \in S_k} A_{ir} \Delta \alpha_{ir}^{k+1} \right\|^2
\]

\[
\leq \left(1 - \frac{\delta}{2}\right) \left\| e^k \right\|^2 + \frac{(1 - \delta)}{\lambda^2 N m^2} \left(\frac{2R^2}{\delta} + \frac{R^2}{n}\right) \sum_{\tau=1}^{m} \sum_{i=1}^{n} \|\Delta \alpha_{ir}^{k+1}\|^2.
\]

Combining \(\text{[E1]}\) and the above inequality, we can get

\[
E_k \|e^{k+1}\|^2
\]

\[
\leq \left(1 - \frac{\delta}{2}\right) \left\| e^k \right\|^2 + \frac{2(1 - \delta)\delta}{n^2} \sum_{\tau=1}^{m} \|e^k\|^2 + \frac{(1 - \delta)}{\lambda^2 N m^2} \left(\frac{2\delta R^2}{n} + \frac{2R^2}{n}\right) \sum_{\tau=1}^{m} \|\Delta \alpha_{ir}^{k+1}\|^2
\]

\[
= \left(1 - \frac{\delta}{2}\right) \left\| e^k \right\|^2 + \frac{2(1 - \delta)\delta}{n^2} \sum_{\tau=1}^{m} \|e^k\|^2 + \frac{(1 - \delta)}{\lambda^2 N m^2} \left(\frac{(2\delta + 1)R^2_m}{n} + \frac{2R^2}{n}\right) \sum_{\tau=1}^{m} \|\Delta \alpha_{ir}^{k+1}\|^2.
\]
D.2 Proof of Theorem 3.3

Let $h^k \in \mathbb{R}^n$ be defined by:
\[ h^k_{i\tau} = \Delta \alpha^{k+1}_{i\tau} = -\theta p_{i\tau}^{-1}(\alpha^k_{i\tau} + \nabla \phi_{i\tau}(A_{i\tau}^T x^{k+1})), \quad i \in [m], \quad \tau \in [n], \]
for $k \geq 0$. Then we have $\alpha^{k+1} = \alpha^k + h^k_{[i\tau]}$. By Lemma D.3

\[
\mathbb{E}_k[-D(\alpha^{k+1})] 
\leq f(\alpha^k) + \sum_{\tau=1}^{n} \sum_{i=1}^{m} p_{i\tau} \left\langle \frac{1}{N} A_{i\tau}^T \nabla g^* \left( \frac{1}{\lambda N} A \alpha^k \right), h^k_{i\tau} \right\rangle + \frac{1}{2\lambda N^2} \sum_{\tau=1}^{n} \sum_{i=1}^{m} p_{i\tau} v_{i\tau} \|h^k_{i\tau}\|^2 
+ \frac{1}{N} \sum_{\tau=1}^{n} \sum_{i=1}^{m} \left( (1 - p_{i\tau}) \phi^*_{i\tau}(-\alpha^k_{i\tau}) + p_{i\tau} \phi^*_{i\tau}(-\alpha^k_{i\tau} - h^k_{i\tau}) \right). 
\]

Define $\tilde{u}^k = u^k + \epsilon^k$ for $k \geq 0$. Then we have
\[
\tilde{u}^{k+1} = u^{k+1} + \epsilon^{k+1} 
= u^k + \frac{1}{n} \sum_{\tau=1}^{n} y^k_{\tau} + \frac{1}{n} \sum_{\tau=1}^{n} e^{k+1}_{\tau} 
= u^k + \frac{1}{n} \sum_{\tau=1}^{n} y^k_{\tau} + \frac{1}{n} \sum_{\tau=1}^{n} \left( e^{k}_{\tau} + \frac{1}{\lambda m} A_{i\tau}^\top \Delta \alpha^{k+1}_{i\tau} - y^k_{\tau} \right) 
= u^k + \epsilon^k + \frac{1}{\lambda N} \sum_{\tau=1}^{n} A_{i\tau}^\top \Delta \alpha^{k+1}_{i\tau} 
= \tilde{u}^k + \frac{1}{\lambda N} \sum_{\tau=1}^{n} A_{i\tau}^\top \Delta \alpha^{k+1}_{i\tau}. 
\]

Moreover, since $\tilde{u}^0 = u^0 = \frac{1}{\lambda N} \sum_{\tau=1}^{n} \sum_{i=1}^{m} A_{i\tau} \alpha^0_{i\tau}$, we have
\[
\tilde{u}^k = \frac{1}{\lambda N} \sum_{\tau=1}^{n} \sum_{i=1}^{m} A_{i\tau} \alpha^k_{i\tau}, \tag{33}
\]
for $k \geq 0$.

Next we use Lemma D.4 to further bound $\mathbb{E}_k[-D(\alpha^{k+1})]$ by choosing $\alpha = \alpha^k$, $x = x^{k+1}$, and $u = u^k$. We have
\[
\mathbb{E}_k[-D(\alpha^{k+1})] 
\leq -(1 - \theta) D(\alpha^k) - \lambda g(\nabla g^*(u^k)) - \frac{1}{N} \sum_{\tau=1}^{n} \sum_{i=1}^{m} \left( \theta \nabla g^*(u^k), A_{i\tau} \nabla \phi_{i\tau}(A_{i\tau}^T x^{k+1}) \right) 
+ \frac{\theta}{N} \sum_{\tau=1}^{n} \sum_{i=1}^{m} \phi^*_{i\tau}(\nabla \phi_{i\tau}(A_{i\tau}^T x^{k+1})) + \frac{\theta + \theta \lambda}{2} \|\epsilon^k\|^2 
+ \frac{1}{2\lambda N^2} \sum_{\tau=1}^{n} \sum_{i=1}^{m} \left( p_{i\tau} v_{i\tau} + \frac{N\lambda p_{i\tau}^2 R^2}{\rho} - \frac{N\lambda p_{i\tau}^2 (1 - \theta p_{i\tau})}{\theta} \right) \|h^k_{i\tau}\|^2. 
\]
In order to guarantee the above coefficient to be nonpositive, we let

By convexity of $g$,

$$P(x^{k+1}) = \frac{1}{N} \sum_{\tau=1}^{n} \sum_{i=1}^{m} \phi_{\tau}(A_{\tau}^{T}x^{k+1}) + \lambda g((1-\theta)x^{k} + \theta \nabla g^{*}(u^{k}))$$

$$\leq \frac{1}{N} \sum_{\tau=1}^{n} \sum_{i=1}^{m} \phi_{\tau}(A_{\tau}^{T}x^{k+1}) + (1-\theta)\lambda g(x^{k}) + \theta \lambda g(\nabla g^{*}(u^{k})).$$

By combining the above two inequalities, we can get

$$\mathbb{E}_{k}[P(x^{k+1}) - D(\alpha^{k+1})]$$

$$\leq \frac{1}{N} \sum_{\tau=1}^{n} \sum_{i=1}^{m} \phi_{\tau}(A_{\tau}^{T}x^{k+1}) + (1-\theta)\lambda g(x^{k}) - (1-\theta)D(\alpha^{k})$$

$$- \frac{1}{N} \sum_{\tau=1}^{n} \sum_{i=1}^{m} (\theta \nabla g^{*}(u^{k}), A_{\tau}^{T} \phi_{\tau}(A_{\tau}^{T}x^{k+1})) + \frac{\theta}{N} \sum_{\tau=1}^{n} \sum_{i=1}^{m} \phi_{\tau}^{*}(\nabla \phi_{\tau}(A_{\tau}^{T}x^{k+1}))) + \frac{\rho + \theta \lambda}{2} \|e^{k}\|^{2}$$

$$+ \frac{1}{2LN^{2}} \sum_{\tau=1}^{n} \sum_{i=1}^{m} \left( p_{\tau} v_{\tau} + \frac{N\lambda p_{\tau}^{2}R^{2}}{\rho} - \frac{N\lambda \gamma p_{\tau}^{2}(1-\theta p_{\tau}^{-1})}{\theta} \right) \|h_{\tau}^{k}\|^{2}.$$

By convexity of $g$,

$$E_{k}[P(x^{k+1}) - D(\alpha^{k+1})]$$

$$\leq (1-\theta)(P(x^{k}) - D(\alpha^{k})) + \frac{\rho + \theta \lambda}{2} \|e^{k}\|^{2}$$

$$+ \frac{1}{2LN^{2}} \sum_{\tau=1}^{n} \sum_{i=1}^{m} \left( p_{\tau} v_{\tau} + \frac{N\lambda p_{\tau}^{2}R^{2}}{\rho} - \frac{N\lambda \gamma p_{\tau}^{2}(1-\theta p_{\tau}^{-1})}{\theta} \right) \|h_{\tau}^{k}\|^{2}.$$

Since $\nabla g^{*}(u^{k}) = x^{k+1} - (1-\theta)x^{k}$, same as the proof of Theorem 9 in [Qu et al. 2015], we can simply the above inequality to the following form

$$\mathbb{E}_{k}[P(x^{k+1}) - D(\alpha^{k+1})]$$

$$\leq (1-\theta)(P(x^{k}) - D(\alpha^{k})) + \frac{\rho + \theta \lambda}{2} \|e^{k}\|^{2}$$

$$+ \frac{1}{2LN^{2}} \sum_{\tau=1}^{n} \sum_{i=1}^{m} \left( p_{\tau} v_{\tau} + \frac{N\lambda p_{\tau}^{2}R^{2}}{\rho} - \frac{N\lambda \gamma p_{\tau}^{2}(1-\theta p_{\tau}^{-1})}{\theta} \right) \|h_{\tau}^{k}\|^{2}.$$

$$\text{(34)}$$

Since $\|e^{k}\|^{2} \leq \frac{1}{n} \sum_{\tau=1}^{n} \|e_{\tau}^{k}\|^{2}$ from the convexity of the Euclidean norm $\| \cdot \|^{2}$, from (34) we have

$$\mathbb{E}_{k}[\Psi_{1}^{k+1}] \leq (1-\theta)(P(x^{k}) - D(\alpha^{k})) + \frac{2(\rho + \theta \lambda)}{\delta n} \sum_{\tau=1}^{n} \mathbb{E}_{k}\|e_{\tau}^{k+1}\|^{2} + \frac{\rho + \theta \lambda}{2n} \sum_{\tau=1}^{n} \|e_{\tau}^{k}\|^{2}$$

$$+ \frac{1}{2LN^{2}} \sum_{\tau=1}^{n} \sum_{i=1}^{m} \left( 4(1-\delta)(\rho + \theta \lambda)m \frac{2R^{2}}{\delta} + R_{m}^{2} \right) + p_{\tau} v_{\tau}$$

$$+ \frac{N\lambda p_{\tau}^{2}R^{2}}{\rho} - \frac{N\lambda \gamma p_{\tau}^{2}(1-\theta p_{\tau}^{-1})}{\theta} \|h_{\tau}^{k}\|^{2}.$$

Recall that $p_{\tau} = \frac{1}{m}$, by choosing $\rho = \frac{\delta R^{2}}{\sqrt{m}}$, where $a_{1} = (1-\delta)(2R^{2} + \delta R_{m}^{2})$, the coefficient of $\|h_{\tau}^{k}\|^{2}$ becomes

$$\frac{4\rho n p_{\tau} R \sqrt{a_{1}}}{\delta} + \frac{4\rho a_{1}}{\delta^{2} m} + p_{\tau} v_{\tau} = \frac{N\lambda \gamma p_{\tau}^{2}(1-\theta p_{\tau}^{-1})}{\theta}.$$

In order to guarantee the above coefficient to be nonpositive, we let

$$\frac{4\rho a_{1}}{\delta^{2} m} \leq \frac{1}{3} \frac{N\lambda \gamma p_{\tau}^{2}(1-\theta p_{\tau}^{-1})}{\theta}, \quad p_{\tau} v_{\tau} \leq \frac{1}{3} \frac{N\lambda \gamma p_{\tau}^{2}(1-\theta p_{\tau}^{-1})}{\theta},$$

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\[
\frac{4np_i \tau R \sqrt{a_1}}{\delta} \leq \frac{1}{3} \cdot \frac{N \lambda \gamma p_i^2 (1 - \theta p_i^{-1})}{\theta},
\]
which is equivalent to
\[
\theta \leq \min \left\{ \frac{2\delta \lambda \gamma}{\delta \lambda \gamma m + \sqrt{\delta^2 \lambda \gamma^2 m^2 + 48 \lambda \gamma a_1}}, \frac{N \lambda \gamma p_i}{3n}, \frac{\delta \lambda \gamma}{\delta \lambda \gamma m + 12R \sqrt{a_1}} \right\}.
\]
By choosing the upper bound in the above inequality for \( \theta \), we arrive at
\[
E_k[\Psi_k^{k+1}] \leq (1 - \theta)(P(x^k) - D(\alpha^k)) + \left(1 - \frac{\delta}{2} + \frac{\delta}{4}\right) \frac{2(\rho + \theta \lambda)}{\delta n} \sum_{\tau=1}^{n} \|e_k^\tau\|^2
= \left(1 - \min \left\{ \theta, \frac{\delta}{4} \right\} \right) \Psi_1^k.
\]
By using the tower property, we can obtain
\[
E[\Psi_k^k] \leq \left(1 - \min \left\{ \theta, \frac{\delta}{4} \right\} \right) \Psi_1^k.
\]
Therefore, \( E[\Psi_k^k] \leq \epsilon \) as long as
\[
k \geq O \left( \left( \frac{1}{\theta} + \frac{1}{\delta} \right) \ln \frac{1}{\epsilon} \right)
= O \left( \frac{1}{\delta} + m + \frac{R_m^2}{n \lambda \gamma} + \frac{R^2}{\lambda \gamma} + \frac{1}{\delta} \sqrt{\frac{(1 - \delta)(R^2 + \delta R_m^2)}{\lambda \gamma}} \right) \ln \frac{1}{\epsilon}
= O \left( \frac{1}{\delta} + m + \frac{R_m^2}{n \lambda \gamma} + \frac{R^2}{\lambda \gamma} + \frac{R \sqrt{(1 - \delta)(R^2 + \delta R_m^2)} \ln 1}{\delta \lambda \gamma} \right)
= O \left( \frac{1}{\delta} + m + \frac{R_m^2}{n \lambda \gamma} + \frac{R^2}{\lambda \gamma} + \frac{\sqrt{1 - \delta RR \lambda \gamma}}{\delta \lambda \gamma} + \frac{\sqrt{1 - \delta RR \lambda \gamma}}{\lambda \gamma \delta} \right) \ln \frac{1}{\epsilon},
\]
where we use \( \frac{R^2}{\gamma} \geq \lambda \) in the second equality.

**D.3 Proof of Theorem 3.4**
First, from (34) and Lemma D.6 we have
\[
E_k[P(x^{k+1}) - D(\alpha^{k+1}) + \frac{2(\rho + \theta \lambda)}{\delta} ||e^{k+1}||^2]
\leq (1 - \theta)(P(x^k) - D(\alpha^k)) + \left(1 - \frac{\delta}{2} + \frac{\delta}{4}\right) \frac{2(\rho + \theta \lambda)}{\delta n} \sum_{\tau=1}^{n} ||e^\tau||^2
+ \frac{1}{2\lambda N^2} \sum_{\tau=1}^{n} \sum_{i=1}^{m} \left(4(\rho + \theta \lambda)(1 - \delta)n \left(\frac{(2\delta + 1)R_m^2}{\delta \lambda m} + \frac{2R^2}{\delta} \right) + p_{i\tau}v_{i\tau} + \frac{N \lambda \gamma p_i^2 R^2}{\rho} - \frac{N \lambda \gamma p_i^2 (1 - \theta p_i^{-1})}{\theta} \right) \|h_{i\tau}^k\|^2.
\]
Combining the above inequality and Lemma D.5 yields

\[ E_k[\Psi_{k+1}^2] \leq \left(1 - \min\left\{ \theta, \frac{\delta}{4}\right\}\right) \Psi_k^2 + \frac{1}{2\lambda N^2} \sum_{i=1}^{n} \sum_{m}^{m} \left(\frac{32(1-\delta)^2(\rho + \theta \lambda)}{\delta \lambda m} \left(\frac{2R^2}{\delta} + R_m^2\right) + \frac{4(\rho + \theta \lambda)(1-\delta)n}{\delta \lambda m} \left(\frac{(2\delta + 1)R_m^2}{n} + \frac{2R^2}{\delta}\right) + p_{i\tau} v_{i\tau} + \frac{N \lambda p_{i\tau}^2 R_m^2}{\rho} - \frac{N \lambda \gamma p_{i\tau}^2 (1 - \theta p_{i\tau}^{-1})}{\theta} \right) \|h_{i\tau}^k\|^2 \]

\[ \leq \left(1 - \min\left\{ \theta, \frac{\delta}{4}\right\}\right) \Psi_k^2 + \frac{1}{2\lambda N^2} \sum_{i=1}^{n} \sum_{m}^{m} \left(\frac{4(1-\delta)(\rho + \theta \lambda)n}{\delta \lambda m} \left(\frac{9R_m^2}{n} + \frac{16R^2}{\delta} + \frac{2R^2}{\delta}\right) + p_{i\tau} v_{i\tau} + \frac{N \lambda p_{i\tau}^2 R_m^2}{\rho} - \frac{N \lambda \gamma p_{i\tau}^2 (1 - \theta p_{i\tau}^{-1})}{\theta} \right) \|h_{i\tau}^k\|^2. \]

By choosing \( \rho = \frac{\delta R}{\lambda \gamma a_2} \), where \( a_2 = (1-\delta)(2R^2 + \frac{16R^2}{n} + \frac{9\delta R^2}{n}) \), the coefficient of \( \|h_{i\tau}^k\|^2 \) becomes

\[ \frac{4np_{i\tau} R \sqrt{a_2}}{\rho} + \frac{4\theta n a_2}{\delta^2 m} + p_{i\tau} v_{i\tau} - \frac{N \lambda \gamma p_{i\tau}^2 (1 - \theta p_{i\tau}^{-1})}{\theta}. \]

Same as the proof of Theorem D.3, we can choose

\[ \theta = \min\left\{ \frac{2\delta \lambda \gamma}{\delta \lambda \gamma + \lambda \gamma \sqrt{2\delta^2 R^2 + 48\delta R^2}} , \frac{N \lambda \gamma p_{i\tau}}{3\gamma m + \lambda \gamma \sqrt{2\delta^2 R_m^2 + 48\delta R_m^2}} , \frac{\delta \lambda \gamma}{5\lambda \gamma m + \delta \lambda \gamma \sqrt{2\delta^2 R_m^2 + 48\delta R_m^2}} \right\}, \]

and get \( E_k[\Psi_{k+1}^2] \leq \left(1 - \min\left\{ \theta, \frac{\delta}{4}\right\}\right) \Psi_k^2. \)

By using the tower property, we can obtain

\[ E[\Psi_k^2] \leq \left(1 - \min\left\{ \theta, \frac{\delta}{4}\right\}\right)^k \Psi_2^0. \]

Therefore, \( E[\Psi_k^2] \leq \epsilon \) as long as

\[ k \geq O\left(\left(\frac{1}{\theta} + \frac{1}{\delta}\right) \ln \frac{1}{\epsilon}\right) \]

\[ = O\left(\left(\frac{1}{\theta} + \frac{m + \frac{R_m^2}{\lambda \gamma}}{\frac{2n}{\lambda \gamma}} + \frac{1}{\delta} \sqrt{(1 - \delta)R^2} + \frac{\sqrt{(1 - \delta)R_m^2}}{\sqrt{\delta m \lambda \gamma}} + \frac{\sqrt{1 - \delta} \frac{R_m}{\lambda \gamma}}{\frac{\sqrt{\delta m}}{\lambda \gamma}} + \frac{\sqrt{1 - \delta} \frac{R}{\lambda \gamma}}{\frac{\sqrt{\delta m}}{\lambda \gamma}} \right) \ln \frac{1}{\epsilon}\right) \]

\[ = O\left(\left(\frac{1}{\theta} + \frac{m + \frac{R_m^2}{\lambda \gamma}}{\frac{2n}{\lambda \gamma}} + \frac{1 - \delta R_m}{\delta \lambda \gamma} + \frac{1 - \delta R}{\delta \lambda \gamma} \right) \ln \frac{1}{\epsilon}\right) \]

\[ = O\left(\left(\frac{1}{\theta} + \frac{m + \frac{R_m^2}{\lambda \gamma}}{\frac{2n}{\lambda \gamma}} + \frac{1 - \delta R^2}{\delta \lambda \gamma} \right) \ln \frac{1}{\epsilon}\right), \]

where in the first equality we use \( \frac{R^2}{n} \leq R^2 \), in the second equality we use

\[ \frac{2}{\delta} \sqrt{\frac{(1 - \delta)R^2}{\lambda \gamma}} \leq \frac{\sqrt{1 - \delta}}{\delta} + \frac{\sqrt{1 - \delta} R^2}{\delta \lambda \gamma}; \]

and

\[ \frac{2 \sqrt{(1 - \delta)R_m^2}}{\sqrt{\delta m \lambda \gamma}} \leq \frac{1 - \delta}{\delta} + \frac{R_m^2}{n \lambda \gamma}; \]

and in the last equality, we use

\[ \frac{2 \sqrt{1 - \delta R_m}}{\sqrt{\delta m \lambda \gamma}} \leq \frac{\sqrt{1 - \delta} R^2}{\delta \lambda \gamma} + \frac{\sqrt{1 - \delta} R_m^2}{n \lambda \gamma}. \]
E Proofs for EC-SDCA

E.1 A lemma

Lemma E.1. For error compensated SDCA, we have

\[
\mathbb{E}_k[e^k] \leq \mathbb{E}_D[e] - \mathbb{E}_D[e^k] + \rho \frac{(1 - \theta) e}{\rho} + \frac{\rho - \theta \lambda}{2} \|e\|^2 
- \frac{1}{2\lambda N^2} \sum_{i=1}^{m} \sum_{r=1}^{n} \left( \frac{\lambda N^2 p_{tr}^2 (1 - \theta p_{tr}^{-1})}{\theta} - p_{tr} \right) \|\Delta a_{tr}^k\|^2, \tag{35}
\]

for any \( \rho > 0 \).

Proof. Denote \( s^k_r = \theta p_{tr}^{-1} \) and \( z^k_r = -\nabla \phi_{tr}(A_{tr}^k x^k) \). Then from (33) we have

\[
N[D(x^k) - D(\alpha^k)] = \sum_{r=1}^{m} \left( -\phi_{tr}^*(\tilde{z}_{tr}) - \nabla g^*(\tilde{u}_r) \right) - \sum_{r=1}^{m} \phi_{tr}^*(\tilde{z}_{tr}) - \nabla g^*(\tilde{u}_r)
\]

\[
= \sum_{r=1}^{m} \left[ \phi_{tr}^*(\tilde{z}_{tr}) - \nabla g^*(\tilde{u}_r) \right] - \nabla g^*(\tilde{u}_r) - \nabla g^*(\tilde{u}_r) \left( \tilde{u}_r^1 - \tilde{u}_r \right)
\]

\[
\geq \sum_{r=1}^{m} \left[ s^k_r \phi_{tr}^*(-\alpha_{tr}^k - \nabla g^*(\tilde{u}_r)) - s^k_r \phi_{tr}^*(-\nabla g^*(\tilde{u}_r)) \right] + \frac{\gamma s^k_r (1 - s^k_r)}{2} \|\tilde{z}_{tr} - \alpha_{tr}^k\|^2
\]

\[
- \nabla g^*(\tilde{u}_r) - \nabla g^*(\tilde{u}_r) \left( \tilde{u}_r^1 - \tilde{u}_r \right)
\]

where in the first inequality, we use that \( \phi_{tr}^* \) is \( \gamma \)-strongly convex and \( g^* \) is 1-smooth. From (33) and the update of \( \alpha^k \), we know \( \tilde{u}_r^1 - \tilde{u}_r = \frac{1}{\lambda N} \sum_{r=1}^{m} A_{tr} \Delta a_{tr}^k \). Then we have

\[
N[D(x^k) - D(\alpha^k)]
\geq \sum_{r=1}^{m} \left[ s^k_r \phi_{tr}^*(-\alpha_{tr}^k - \nabla g^*(\tilde{u}_r)) - s^k_r \phi_{tr}^*(-\nabla g^*(\tilde{u}_r)) \right] + \frac{\gamma s^k_r (1 - s^k_r)}{2} \|\tilde{z}_{tr} - \alpha_{tr}^k\|^2
\]

\[
- \nabla g^*(\tilde{u}_r) - \nabla g^*(\tilde{u}_r) \left( \tilde{u}_r^1 - \tilde{u}_r \right)
\]

where in the last equality we use \( \phi_{tr}^*(-\tilde{u}_r) = -\phi_{tr}(A_{tr}^k x^k) \) which comes from \( z^k_r = -\nabla \phi_{tr}(A_{tr}^k x^k) \).
Since $x^{k+1} = \nabla g^*(u^k)$, we have $g(x^{k+1}) + g^*(u^k) = \langle x^{k+1}, u^k \rangle$. Therefore,

$$P(x^{k+1}) - D(\alpha^k) = \frac{1}{N} \sum_{\tau=1}^{N} \sum_{i=1}^{m} \phi_{i\tau}(A_{i\tau}^T x^{k+1}) + \lambda g(x^{k+1}) + \frac{1}{N} \sum_{\tau=1}^{N} \sum_{i=1}^{m} \phi_{i\tau}^*(\alpha_k^k) + \lambda g^*(\tilde{u}^k)$$

$$= \frac{1}{N} \sum_{\tau=1}^{N} \sum_{i=1}^{m} (\phi_{i\tau}(A_{i\tau}^T x^{k+1}) + \phi_{i\tau}^*(-\alpha_k^k) + \lambda g^*(\tilde{u}^k) - \lambda g^*(u^k) + \lambda \langle x^{k+1}, u^k \rangle,$$

which indicates that

$$\mathbb{E}_k \left[ \sum_{\tau=1}^{n} s_k \left( \phi_{i\tau}(A_{i\tau}^T x^{k+1}) + \phi_{i\tau}^*(-\alpha_k^k) + \langle A_{i\tau}^T x^{k+1}, \alpha_k^k \rangle \right) \right]$$

$$= \sum_{\tau=1}^{n} \sum_{i=1}^{m} \theta \left( \phi_{i\tau}(A_{i\tau}^T x^{k+1}) + \phi_{i\tau}^*(-\alpha_k^k) + \langle A_{i\tau}^T x^{k+1}, \alpha_k^k \rangle \right)$$

$$= \sum_{\tau=1}^{n} \sum_{i=1}^{m} \theta \left( \phi_{i\tau}(A_{i\tau}^T x^{k+1}) + \phi_{i\tau}^*(-\alpha_k^k) \right) + \theta \lambda N \langle x^{k+1}, \tilde{u}^k \rangle$$

$$\geq \theta N \left( P(x^{k+1}) - D(\alpha^k) \right) - \frac{\theta \lambda}{2} \| e^k \|^2 - \frac{1}{N} \sum_{\tau=1}^{n} \sum_{i=1}^{m} \gamma (1 - s_k^k) 2s_k^k \| \Delta \alpha_k^{k+1} \|^2$$

$$\geq \theta N \left( P(x^{k+1}) - D(\alpha^k) \right) - \frac{\theta \lambda}{2} \| e^k \|^2 - \frac{1}{2 \lambda N^2} \sum_{\tau=1}^{n} \sum_{i=1}^{m} A_{i\tau}^T \Delta \alpha_k^{k+1} \| ^2.$$
For $E_k \left( \nabla g^* (\tilde{u}^k) - x^{k+1}, \sum_{\tau=1}^n A_{i\tau}^T \Delta \alpha_{i\tau}^{k+1} \right)$, we have

\[
E_k \left( \nabla g^* (\tilde{u}^k) - x^{k+1}, \sum_{\tau=1}^n A_{i\tau}^T \Delta \alpha_{i\tau}^{k+1} \right) = \left\langle \nabla g^* (\tilde{u}^k) - \nabla g^* (u^k), \sum_{\tau=1}^n \sum_{i=1}^m A_{i\tau} p_{i\tau} \Delta \alpha_{i\tau}^{k+1} \right\rangle \\
\leq \frac{\rho N}{2} \| e^k \|^2 + \frac{1}{2\rho N} \left\| \sum_{\tau=1}^n \sum_{i=1}^m A_{i\tau} p_{i\tau} \Delta \alpha_{i\tau}^{k+1} \right\|^2 \\
\leq \frac{\rho N}{2} \| e^k \|^2 + \frac{\| A \|^2}{2\rho N} \sum_{\tau=1}^n \sum_{i=1}^m p_{i\tau}^2 \| \Delta \alpha_{i\tau}^{k+1} \|^2 \\
= \frac{\rho N}{2} \| e^k \|^2 + \frac{R^2}{2\rho} \sum_{\tau=1}^n \sum_{i=1}^m p_{i\tau}^2 \| \Delta \alpha_{i\tau}^{k+1} \|^2,
\]

where in the first inequality we use Young’s inequality for any $\rho > 0$ and that $g^*$ is 1-smooth, in the last equality we use the fact that $R^2 = \frac{1}{N} \lambda_{\max} \left( \sum_{\tau=1}^n \sum_{i=1}^m A_{i\tau}^T A_{i\tau} \right) = \frac{1}{N} \| A \|^2$.

Then we can obtain

\[
E_k [\epsilon_D^k - \epsilon_D^{k+1}] \geq \theta \left( P(x^{k+1}) - D(\alpha^k) \right) - \frac{\rho + \theta \lambda}{2} \| e^k \|^2 \\
+ \frac{1}{2\lambda N^2} \sum_{\tau=1}^n \sum_{i=1}^m \left( \frac{\lambda N \| p_{i\tau} \|^2 (1 - \theta p_{i\tau}^{-1})}{\rho} - p_{i\tau} v_{i\tau} - \frac{\lambda N \| p_{i\tau} \|^2 R^2}{\rho} \right) \| \Delta \alpha_{i\tau}^{k+1} \|^2 \\
= \theta \epsilon_P^{k+1} + \theta \epsilon_D^{k} - \frac{\rho + \theta \lambda}{2} \| e^k \|^2 \\
+ \frac{1}{2\lambda N^2} \sum_{\tau=1}^n \sum_{i=1}^m \left( \frac{\lambda N \| p_{i\tau} \|^2 (1 - \theta p_{i\tau}^{-1})}{\rho} - p_{i\tau} v_{i\tau} - \frac{\lambda N \| p_{i\tau} \|^2 R^2}{\rho} \right) \| \Delta \alpha_{i\tau}^{k+1} \|^2.
\]

After rearrangement, we can get the result.

E.2 Proof of Theorem 3.3

First, notice that (35) in Lemma E.1 is the same as (34) except that $\epsilon_D^{k+1}$ and $\epsilon_D^k$ are replaced by $P(x^{k+1}) - D(\alpha^{k+1})$ and $P(x^k) - D(\alpha^k)$ respectively, and there is an additional term $-\theta \epsilon_P^{k+1}$. Hence, same as the proof in Theorem 3.3, we can get

\[
E_k [\Psi_3^{k+1}] \leq \left( 1 - \min \left\{ \theta, \frac{\delta}{4} \right\} \right) \Psi_3^k - \theta \epsilon_P^{k+1},
\]

by when $\theta$ satisfies (9). Since $\epsilon_P^{k+1} \geq 0$, by using the tower property, we can obtain $E[\Psi_3^k] \leq (1 - \min \left\{ \theta, \frac{\delta}{4} \right\} )^k \Psi_3^0$.

From (38) and the tower property, we have

\[
E[\Psi_3^k] \leq \left( 1 - \min \left\{ \theta, \frac{\delta}{4} \right\} \right) E[\Psi_3^{k-1}] - \theta E[\epsilon_P^k],
\]

for $k \geq 1$. Let $w_k = (1 - \min \left\{ \theta, \frac{\delta}{4} \right\} )^{-k}$ and $W_k = \sum_{i=1}^k w_i$. By multiplying $w_k$ on the both sides of the above inequality, we can get

\[
E[w_k \Psi_3^k] \leq w_{k-1} \Psi_3^{k-1} - \theta E[w_k \epsilon_P^k] \\
\leq w_0 \Psi_3^0 - \theta \sum_{i=1}^k w_i E[\epsilon_P^i] \\
= \Psi_3^0 - \theta \sum_{i=1}^k w_i E[\epsilon_P^i],
\]

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which implies that

\[
\frac{1}{W_k} \sum_{i=1}^{k} w_i \mathbb{E}[\epsilon_p^i] \leq \frac{1}{\theta W_k} \Psi_3^0
\]

\[
= \min \left\{ \theta, \frac{\delta}{4} \right\} \frac{\Psi_3^0}{\theta \left( (1 - \min \left\{ \theta, \frac{\delta}{4} \right\})^{-k} - 1 \right) \Psi_3^0}
\]

\[
\leq \frac{(1 - \min \left\{ \theta, \frac{\delta}{4} \right\})^k \epsilon_D^0}{1 - (1 - \min \left\{ \theta, \frac{\delta}{4} \right\})^k},
\]

where we use \( \Psi_3^0 = \epsilon_D^0 \). Then from the convexity of \( P \), we have

\[
\mathbb{E}[P(\bar{x}^k) - P(x^*)] \leq \frac{1}{W_k} \sum_{i=1}^{k} w_i \mathbb{E}[\epsilon_p^i] \leq \frac{(1 - \min \left\{ \theta, \frac{\delta}{4} \right\})^k \epsilon_D^0}{1 - (1 - \min \left\{ \theta, \frac{\delta}{4} \right\})^k}.
\]

In order to guarantee \( \mathbb{E}[P(\bar{x}^k) - P(x^*)] \leq \epsilon \), we first let

\[
\left( 1 - \min \left\{ \theta, \frac{\delta}{4} \right\} \right)^k \leq \frac{1}{2},
\]

which indicates that

\[
\mathbb{E}[P(\bar{x}^k) - P(x^*)] \leq \left( 1 - \min \left\{ \theta, \frac{\delta}{4} \right\} \right)^k \cdot 2 \epsilon_D^0.
\]

Thus, when \( \epsilon \leq \epsilon_D^0 \), \( \mathbb{E}[P(\bar{x}^k) - P(x^*)] \leq \epsilon \) as long as

\[
\left( 1 - \min \left\{ \theta, \frac{\delta}{4} \right\} \right)^k \leq \frac{\epsilon}{2 \epsilon_D^0},
\]

which is equivalent to

\[
k \geq \frac{1}{- \ln \left( 1 - \min \left\{ \theta, \frac{\delta}{4} \right\} \right)} \ln \left( \frac{2 \epsilon_D^0}{\epsilon} \right).
\]

Finally, from \( - \ln(1 - x) \geq x \) for \( x \in [0, 1) \), we have \( \mathbb{E}[P(\bar{x}^k) - P(x^*)] \leq \epsilon \) as long as

\[
k \geq O \left( \left( \frac{1}{\delta} + \frac{1}{\theta} \right) \ln \left( \frac{2 \epsilon_D^0}{\epsilon} \right) \right)
\]

\[
= O \left( \left( \frac{1}{\delta} + m + \frac{R^2_m}{n \lambda \gamma} + \frac{R^2}{\lambda \gamma} + \frac{\sqrt{1 - \delta R \bar{R}}}{\delta \lambda \gamma} + \frac{\sqrt{1 - \delta R R_m}}{\lambda \gamma \sqrt{\delta}} \right) \ln \left( \frac{2 \epsilon_D^0}{\epsilon} \right) \right).
\]

### E.3 Proof of Theorem 3.6

The proof is the same as that of Theorem 3.5. Hence we omit it.