Localizations for quiver Hecke algebras II

Masaki Kashiwara1,2 | Myungho Kim3 | Se-jin Oh4 | Euiyong Park5

1Kyoto University Institute for Advanced Study, Research Institute for Mathematical Sciences, Kyoto University, Kyoto, Japan
2Korea Institute for Advanced Study, Seoul, South Korea
3Department of Mathematics, Kyung Hee University, Seoul, South Korea
4Department of Mathematics, Sungkyunkwan University, Suwon, South Korea
5Department of Mathematics, University of Seoul, Seoul, South Korea

Correspondence
Myungho Kim, Department of Mathematics, Kyung Hee University, Seoul 02447, Korea.
Email: mkim@khu.ac.kr

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Abstract
We prove that the localization $\tilde{\mathcal{C}}_w$ of the monoidal category $\mathcal{C}_w$ is rigid, and the category $\mathcal{C}_{w,v}$ admits a localization via a real commuting family of central objects. For a quiver Hecke algebra $R$ and an element $w$ in the Weyl group, the subcategory $\mathcal{C}_w$ of the category $R\text{-}gmod$ of finite-dimensional graded $R$-modules categorifies the quantum unipotent coordinate ring $A_q(\mathfrak{n}(w))$. In the previous paper, we constructed a monoidal category $\tilde{\mathcal{C}}_w$ such that it contains $\mathcal{C}_w$ and the objects $\{M(\omega \lambda_i, \lambda_i) \mid i \in I\}$ corresponding to the frozen variables are invertible. In this paper, we show that there is a monoidal equivalence between the category $\tilde{\mathcal{C}}_w$ and $(\tilde{\mathcal{C}}_{w^{-1}})^{\text{rev}}$. Together with the already known left-rigidity of $\tilde{\mathcal{C}}_w$, it follows that the monoidal category $\tilde{\mathcal{C}}_w$ is rigid. If $v \preceq w$ in the Bruhat order, there is a subcategory $\mathcal{C}_{w,v}$ of $\mathcal{C}_w$ that categorifies the doubly-invariant algebra $N^w(q)C[N]^{N(v)}$. We prove that the family $(M(\omega \lambda_i, v \lambda_i))_{i \in I}$ of simple $R$-module forms a real commuting family of graded central objects in the category $\mathcal{C}_{w,v}$ so that there is a localization $\tilde{\mathcal{C}}_{w,v}$ of $\mathcal{C}_{w,v}$ in which $M(\omega \lambda_i, v \lambda_i)$ are invertible. Since the localization of the algebra $N^w(q)C[N]^{N(v)}$ by the family of the isomorphism classes of $M(\omega \lambda_i, v \lambda_i)$ is isomorphic to the coordinate ring $C[R_{w,v}]$ of the open Richardson variety associated with $w$ and $v$.
In the previous work [13], we developed a general procedure for localizations of monoidal categories and studied in detail the case that the categories consist of modules over quiver Hecke algebras. This paper is a continuation of [13]. Roughly speaking, the localization of a monoidal category in [13] is a procedure to find a larger monoidal category in which the prescribed objects are invertible. Let \( k \) be a commutative ring. For a \( k \)-linear monoidal category \( \mathcal{T} \), a pair \((C, R_C)\) of an object \( C \) and a natural transformation \( R_C : (C \otimes -) \to (- \otimes C) \) is called a braider in \( \mathcal{T} \) if \( R_C \) is compatible with the tensor product \( \otimes \). A family \( \{(C_i, R_{C_i})\} \) of braiders in \( \mathcal{T} \) is called a real commuting family if \( R_{C_i}(C_i) \in k^\times \text{id}_{C_i} \otimes C_i \) and \( R_{C_i}(C_j) \circ R_{C_j}(C_i) \in k^\times \text{id}_{C_j} \otimes C_i \). Then one can construct a \( k \)-linear monoidal category \((\widetilde{\mathcal{T}}, \otimes, 1)\) and a monoidal functor \( \Phi : \mathcal{T} \to \widetilde{\mathcal{T}} \) such that the object \( \Phi(C_i) \) are invertible and the morphisms \( \Phi(R_{C_i}(X)) : \Phi(C_i) \otimes \Phi(X) \to \Phi(X) \otimes \Phi(C_i) \) are isomorphisms for all \( i \) and all \( X \in \mathcal{T} \). Moreover, the pair \((\widetilde{\mathcal{T}}, \Phi)\) is universal with respect to these
properties. By the construction, any object in $\tilde{T}$ is of the form $X \otimes (\bigotimes_i C_i \otimes a_i)$ for some $X \in T$ and $a_i \in \mathbb{Z}$. There is also a graded version of localization. Assume that a graded monoidal category $T$ has a decomposition $T = \bigoplus_{\lambda \in A} T_{\lambda}$ for some abelian group $A$, which is compatible with $\otimes$, and the grading shift operator $q$. Then one can define the notion of graded braider $(C, R_C, \phi_C)$, where $\phi : A \to \mathbb{Z}$ is a group homomorphism and $R_C(X) \in \text{Hom}_T(C \otimes X, q^{\phi(C)} X \otimes C)$ for $X \in T_\lambda$. For a real commuting family of graded braiders in $T$, there exists a monoidal category $\tilde{T}$ and a functor $\phi : T \to \tilde{T}$ which have the same properties as in the ungraded cases.

One of the motivations to develop such a general procedure is to localize monoidal categories consisting of modules over quiver Hecke algebras. Let $\mathfrak{g}$ be a symmetrizable Kac–Moody algebra and $Q_+$ the root lattice of $\mathfrak{g}$. The quiver Hecke algebra associated with $\mathfrak{g}$ is a family $\{R(\beta)\}_{\beta \in Q_+}$ of $\mathbb{Z}$-graded associative algebras over $\mathbb{k}$ such that the Grothendieck ring $K(R\text{-}gmod)$ is isomorphic to the quantum unipotent coordinate ring $A_q(n)$ that is isomorphic to the dual $(U_q^+(\mathfrak{g}))^*$ of the half of the quantum group $U_q(\mathfrak{g})$ ([16, 24]). Here, $R\text{-}gmod$ denotes the direct sum of the categories of finite-dimensional graded $R(\beta)$-modules. For an $(R(\beta))$-module $M$ and an $R(\gamma)$-module $N$, the convolution product $M \circ N$ is the $R(\beta + \gamma)$-module induced from the $R(\beta) \otimes R(\gamma)$-module $M \otimes N$ through the (nonunital) algebra embedding $R(\beta) \otimes R(\gamma) \to R(\beta + \gamma)$. The category $R\text{-}gmod$ together with the convolution product is a monoidal category and the convolution product corresponds to the multiplication of the algebra $A_q(n)$. One of main advantages in the case $T = R\text{-}gmod$ is that for any simple module $C$ in $R\text{-}gmod$, there exists a unique nondegenerate graded braider $(C, R_C, \phi_C)$. Here, a braider is nondegenerate if $R_C(L(i))$ does not vanish for any $i$, where $L(i)$ denotes the unique graded simple module over $R(\alpha_i)$ and $\{\alpha_i \mid i \in I\}$ is the set of simple roots of $\mathfrak{g}$. Hence, one can consider localizations for various monoidal subcategories of $R\text{-}gmod$.

For an element $w$ in the Weyl group $W$ of $\mathfrak{g}$, there is a subalgebra $A_q(n(w))$ of $A_q(n)$ called the quantum unipotent coordinate ring associated with $w$ whose limit at $q = 1$ becomes the coordinate ring of the unipotent subgroup $N(w)$ associated with $w$. Note that the Lie algebra of $N(w)$ is $\mathfrak{n}(w) := \bigoplus_{\alpha \in \Delta^+ \cap w\Delta} g_\alpha$, where $\Delta^+$ is the set of positive/negative roots of $\mathfrak{g}$. The algebra $A_q(n(w))$ is interesting because it equips a quantum cluster algebra structure [1–3]. We denote the set of frozen variables by $\{D(w\Lambda_i, \Lambda_i)\}_{i \in I}$. Note that the element $D(w\Lambda_i, \Lambda_i)$ is not invertible in the algebra $A_q(n(w))$. If one localizes $A_q(n(w))$ at the set $\{D(w\Lambda_i, \Lambda_i)\}_{i \in I}$, then one gets a $q$-deformation $A_q(N^w)$ of the coordinate ring $\mathbb{C}[N^w]$, where $N^w$ denotes the unipotent cell associated with $w$ which can be identified with an open subset of $N(w)$ (De Concini–Procesi isomorphism, see [18, Theorem 4.13]). Now one can naturally associate the algebra $A_q(n(w))$ with a full subcategory $\mathcal{C}_w$ of $R\text{-}gmod$ whose Grothendieck ring $K(\mathcal{C}_w)$ is isomorphic to $A_q(n(w))$. It is interesting not only that the Grothendieck ring is isomorphic to $A_q(n(w))$, but also the category $\mathcal{C}_w$ reflects the quantum cluster algebra structure on $A_q(n(w))$. Indeed, every cluster monomial in $A_q(n(w))$ corresponds to a real simple module in $\mathcal{C}_w$, provided that $\mathfrak{g}$ is symmetric and $\mathbb{k}$ is a field of characteristic zero ([7]). Now the localization at the category level is exactly as one might imagine: each element $D(w\Lambda_i, \Lambda_i)$ corresponds to a simple module $M(w\Lambda_i, \Lambda_i)$ in $\mathcal{C}_w$, and the set $(M(w\Lambda_i, \Lambda_i), R_M(w\Lambda_i, \Lambda_i), \phi_{\Lambda_i})_{i \in I}$ forms a real commuting family of graded braiders in $\mathcal{C}_w$ ([13, Proposition 5.1]). Hence, the localization $\mathcal{C}_w$ of $\mathcal{C}_w$ with respect to the family categorifies the $q$-deformation $A_q(N^w)$ of the coordinate ring $\mathbb{C}[N^w]$ of the unipotent cell $N^w$ ([13, Corollary 5.4]). We emphasize that the set $(M(w\Lambda_i, \Lambda_i), R_M(w\Lambda_i, \Lambda_i), \phi_{\Lambda_i})_{i \in I}$ is also a real commuting family of graded braiders in the category $R\text{-}gmod$, and hence, one has the localization $(R\text{-}gmod)\widehat{[w]}$ of $R\text{-}gmod$ with respect to the family. Let us denote $Q_w : R\text{-}gmod \to (R\text{-}gmod)\widehat{[w]}$ the localization functor. Since the composition $\mathcal{C}_w \hookrightarrow R\text{-}gmod \xrightarrow{Q_w} (R\text{-}gmod)\widehat{[w]}$ factors through the localization $\mathcal{C}_w$, one obtains a functor $i_w : \mathcal{C}_w \to (R\text{-}gmod)\widehat{[w]}$, which turns out to be an equivalence.
There is a monoidal equivalence between the category \( \mathcal{A}_{\mathcal{Q}}(N_w) \) and \( \mathcal{C}_{\psi}(R, \psi) \) (Theorem 3.9). Here, \( \mathcal{T}^{\text{rev}} \) denotes the monoidal category \( (\mathcal{T}, \otimes^{\text{rev}}) \) where the reversed tensor product \( \otimes^{\text{rev}} \) is defined by \( X \otimes^{\text{rev}} Y := Y \otimes X \) and \( f \otimes^{\text{rev}} g := g \otimes f \) for any objects \( X, Y \) and morphisms \( f, g \). Then the left-rigidity of \( \mathcal{C}_{\psi}(R, \psi) \) implies the right-rigidity of \( \mathcal{A}_{\mathcal{Q}}(N_w) \). The strategy for constructing the equivalence is briefly as follows. There is an algebra automorphism \( \psi \) on \( R(\beta) \) (see (1.8)) that induces a monoidal equivalence \( \psi_{\ast} : R(\text{gmod}) \to (R(\text{gmod})^{\text{rev}}) \). Then the composition \( R(\text{gmod}) \xrightarrow{\psi_{\ast}} (R(\text{gmod})^{\text{rev}}) \xrightarrow{Q_{w^{-1}}} (R(\text{gmod})^{\text{rev}}) \) factors as \( R(\text{gmod}) \xrightarrow{F_{w^{-1}}} ((R(\text{gmod})^{\text{rev}})^{\text{rev}}) \), and the functor \( F_{w^{-1}} \) is the desired equivalence of categories. One of key conditions to get such a factorization is that \( (Q_{w^{-1}} \circ \psi_{\ast})(M(w\Lambda_i, \Lambda_i)) \) is invertible in \( R(\text{gmod})^{\text{rev}} \). Hence, we need to study the structure of modules \( \psi_{\ast}(M(w\Lambda_i, \Lambda_i)) \). Recall that the modules \( M(w\Lambda_i, \Lambda_i) \) are examples of the determinantial modules that have been studied in detail \cite{11, 13}. In general, for a dominant integral weight \( \Lambda \) and Weyl group elements \( v \leq u \) in the Bruhat order, there exists a distinguished element \( D(u\Lambda, v\Lambda) \) of \( A_q(n) \), called the unipotent quantum minor, and the determinantial module \( M(u\Lambda, v\Lambda) \) is the simple module corresponding to \( D(u\Lambda, v\Lambda) \) under the isomorphism \( K(R(\text{gmod})) \cong A_q(n) \). Even the module \( \psi_{\ast}(M(w\Lambda_i, \Lambda_i)) \) is no longer a determinantial module in general, it turns out that it shares many properties of determinantial modules. We characterize such a family of simple modules and call them the generalized determinantial modules (Theorem 2.18). It enables us to calculate the module \( \psi_{\ast}(M(w\Lambda_i, \Lambda_i)) \) explicitly and to see that \( (Q_{w^{-1}} \circ \psi_{\ast})(M(w\Lambda_i, \Lambda_i)) \) is invertible in \( \mathcal{A}_{\mathcal{Q}}(N_w) \).

The other main result of this paper is a localization of the category \( \mathcal{C}_{w,v} \) for a pair of Weyl group elements \( w, v \) such that \( v \leq w \) in the Bruhat order. The category \( \mathcal{C}_{w,v} \) can be characterized as the full subcategory consisting of modules \( M \in R(\beta) \)-mod such that \( \text{Res}_{\gamma, \beta - \gamma}(M) \neq 0 \) implies that \( \gamma \in Q_+ \cap wQ_- \). Here \( \text{Res}_{\alpha, \beta} \) denotes the restriction functor \( R(\alpha + \beta) \)-mod \( \to R(\alpha) \otimes R(\beta) \)-mod. Similarly, we define the category \( \mathcal{C}_{s,v} \) as the full subcategory \( R(\beta) \)-mod consisting of modules \( M \in R(\beta) \)-mod such that \( \text{Res}_{\beta - \gamma, \gamma}(M) \neq 0 \) implies that \( \gamma \in Q_+ \cap vQ_- \). We set \( \mathcal{C}_{w,v} := \mathcal{C}_{w} \cap \mathcal{C}_{s,v} \). Then the Grothendieck ring \( K(\mathcal{C}_{w,v}) \) can be understood as a \( q \)-deformation of the doubly-invariant algebra \( N'(w)C[N]N(w) \), where \( N \) is the unipotent radical, \( N_- \) is the opposite of \( N \), and \( N'(w) := N \cap (wNw^{-1}) \), \( N(v) := N \cap (vNv^{-1}) \) (see \cite[Remark 2.19]{11}). It is known that the localization \( N'(w)C[N]N(w) \otimes \bigoplus_{\alpha \in I} \{D(u\Lambda_i, v\Lambda_i)\} \) is isomorphic to the coordinate ring \( C[R_{w,v}] \) of the open Richardson variety \( R_{w,v} \) associated with \( w \) and \( v \) \cite{21, Theorem 2.12}. Hence, a localization of the category \( \mathcal{C}_{w,v} \) with respect to the set of determinantial modules \( \{M(w\Lambda_i, v\Lambda_i)\}_{i \in I} \) would give a categorification of the \( (q \)-deformation of) the coordinate ring \( C[R_{w,v}] \). We show that there exists a graded braider \( (M(w\Lambda_i, v\Lambda_i), R_M(w\Lambda_i, v\Lambda_i), \psi_{w,v}(\Lambda_i)) \) in the category \( \mathcal{C}_{w,v} \) (Proposition 4.2). The key idea for this is to take a restriction of the braider \( R_{M(w\Lambda_i, v\Lambda_i)} \) in \( R(\gamma) \)-mod. Indeed, for an object \( X \in \mathcal{C}_{s,v} \cap (R(\gamma) \)-mod), one can show that the restriction \( \text{Res}_{\gamma + \beta, \beta}(R_{M(w\Lambda_i, v\Lambda_i)}(X)) \) is equal to \( (M(w\Lambda_i, v\Lambda_i) \circ X) \otimes M(v\Lambda_i, \Lambda_i) \to (X \otimes M(w\Lambda_i, v\Lambda_i)) \otimes M(v\Lambda_i, \Lambda_i), \) where \( \beta = \Lambda_i - v\Lambda_i \), and \( \alpha = v\Lambda_i - w\Lambda_i \). We have such a nice form of restriction because the pairs \( (M(w\Lambda_i, v\Lambda_i), M(v\Lambda_i, \Lambda_i)) \) and \( (X, M(v\Lambda_i, \Lambda_i)) \) are distinguished pairs.
of simple modules called *unmixed pairs* (see Section 2.1). Now since \( \text{End}(M(w\Lambda_i, v\Lambda_i)) \cong k \), we obtain the desired homomorphism \( R_{M(w\Lambda_i, v\Lambda_i)}(X) \) from \( M(w\Lambda_i, v\Lambda_i) \circ X \) to \( X \circ M(w\Lambda_i, v\Lambda_i) \). It follows that the family \( \{(M(w\Lambda_i, v\Lambda_i), R_{M(w\Lambda_i, v\Lambda_i)}, \phi_{w,v,\Lambda_i})\}_{i \in I} \) is a real commuting family of graded braiders in the category \( \mathcal{C}_{s,u} \) due to the corresponding properties of the family in \( R\text{-gmod} \). Let \( \mathcal{C}_{s,u}[w] \) and \( \mathcal{C}_{w,u} \) be the localization of \( \mathcal{C}_{s,u} \) and \( \mathcal{C}_{w,u} \), respectively, via the family \( (M(w\Lambda_i, v\Lambda_i), R_{M(w\Lambda_i, v\Lambda_i)}, \phi_{w,v,\Lambda_i}) \). Then similarly to the case of \( (R\text{-gmod}) \[w\] \) and \( \mathcal{C}_{w,u} \), there is a monoidal equivalence between \( \mathcal{C}_{s,u}[w] \) and \( \mathcal{C}_{w,u} \) (Theorem 4.5). It is expected that the category \( \mathcal{C}_{w,u} \) gives a monoidal categorification of a \( q \)-deformation of the cluster algebra arising from the open Richardson variety \( R_{w,u} \) given in [21].

Let us explain some miscellaneous results in this paper that are not only used for the main theorems but also interesting by themselves. We characterize the simple modules that vanish under the localization functor \( Q_w : R\text{-gmod} \to (R\text{-gmod}) \[w\] \). Recall that the self-dual simple modules in \( R\text{-gmod} \) are in bijection with the crystal basis \( B(\infty) \) of \( A_q(\mathfrak{g}) \) ([20]). It turns out that a simple module \( M \) does not vanish under \( Q_w \) if and only if \( M \) matches an element \( b \) in \( B_w(\infty) \), where \( B_w(\infty) \) is a subset of \( B(\infty) \) introduced in [8], which is the limit of the Demazure crystals. Let \( I(w) \) be the subspace of \( A_q(n) \) spanned by the upper global basis elements corresponding to the elements crystal basis in \( B(\infty) \setminus B_w(\infty) \). Then \( I(w) \) is a two-sided ideal and the quotient \( A_q(n)/I(w) \) is called the *quantum closed unipotent cell* in [17]. The equivalence \( t_w : \mathcal{C}_{w,u} \to (R\text{-gmod}) \[w\] \) can be understood as a categorification of the isomorphism in [18, Theorem 4.13], provided that \( \mathfrak{g} \) is symmetric and \( k \) is a field of characteristic zero (see Remark 3.6).

For simple modules \( X \in R(\beta)\text{-gmod} \) and \( Y \in R(\gamma)\text{-gmod} \) such that one of them is *affreal* (see Definition 1.10), there is a distinguished homomorphism \( r_{X,Y} : X \circ Y \to Y \circ X \), called the *\( R \)-matrix*. As the \( R \)-matrix is a crucial feature of the category \( R\text{-gmod} \), the integers \( \Lambda(X,Y) \) and \( \Lambda(X,Y) \) (which is nonnegative) play important roles in the representation theory of the quiver Hecke algebras (see, e.g., [11]). Let \( L, M, \) and \( N \) be simple modules and assume that \( L \) is affreal. Then, we show that for any simple subquotient \( S \) of \( M \circ N \), we have \( \Lambda(L,M) \leq \Lambda(L,S) \) (Theorem 2.11). This theorem is strong in the sense that \( \Lambda(L,S) \) is bounded below by a number that does not involve with \( N \) at all. A corollary of this theorem (Corollary 2.13) is an analog of [12, Lemma 4.17], which holds in the category of finite-dimensional modules over quantum affine algebras.

This paper is organized as follows. In Section 1, we recall some preliminaries containing the localization of monoidal categories, quiver Hecke algebras, determinantial modules, and so forth. In Section 2, we develop some features in \( R\text{-gmod} \), including unmixed pairs, normal sequences, and generalized determinantial modules. In Section 3, we show that there is a monoidal equivalence between \( \mathcal{C}_{w,u} \) and \( (\mathcal{C}_{w,u})^{\text{rev}} \), which implies that \( \mathcal{C}_{w,u} \) is rigid. In Section 4, we construct a real commuting family of graded braiders \( \{(M(w\Lambda_i, v\Lambda_i), R_{M(w\Lambda_i, v\Lambda_i)}, \phi_{w,v,\Lambda_i})\}_{i \in I} \) in the category \( \mathcal{C}_{s,u} \). It turns out that the localization \( \mathcal{C}_{s,u}[w] \) of \( \mathcal{C}_{s,u} \) and the localization \( \mathcal{C}_{w,u} \) of \( \mathcal{C}_{w,u} \) are monoidally equivalent.
(a) a category $\mathcal{T}$,
(b) a bifunctor $\cdot \otimes \cdot : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$,
(c) an isomorphism $a(X,Y,Z) : (X \otimes Y) \otimes Z \sim X \otimes (Y \otimes Z)$ that is functorial in $X, Y, Z \in \mathcal{T}$,
(d) an object $1$, called an **unit object**, endowed with an isomorphism $\epsilon : 1 \otimes 1 \sim 1$

such that

1. the diagram below commutes for all $X, Y, Z, W \in \mathcal{T}$:

$$
\begin{array}{c}
((X \otimes Y) \otimes Z) \otimes W \\
\downarrow \quad \quad \quad \downarrow \\
(X \otimes (Y \otimes Z)) \otimes W \\
\downarrow \\
X \otimes ((Y \otimes Z) \otimes W)
\end{array}
\xrightarrow{a(X,Y,Z,W)}
\begin{array}{c}
((X \otimes Y) \otimes Z) \otimes W \\
\downarrow \quad \quad \quad \downarrow \\
(X \otimes (Y \otimes Z)) \otimes W \\
\downarrow \\
X \otimes ((Y \otimes Z) \otimes W)
\end{array}
$$

2. the functors $\mathcal{T} \ni X \mapsto 1 \otimes X \in \mathcal{T}$ and $\mathcal{T} \ni X \mapsto X \otimes 1 \in \mathcal{T}$ are fully faithful.

We have canonical isomorphisms $1 \otimes X \simeq X \otimes 1 \simeq X$ for any $X \in \mathcal{T}$. For $n \in \mathbb{Z}_{\geq 0}$ and $X \in \mathcal{T}$, we set $X^{\otimes n} = X \otimes \cdots \otimes X$, and $X^{\otimes 0} = 1$.

For monoidal categories $\mathcal{T}$ and $\mathcal{T}'$, a functor $F : \mathcal{T} \to \mathcal{T}'$ is called **monoidal** if it is endowed with an isomorphism $\xi_F : F(X \otimes Y) \sim F(X) \otimes F(Y)$ that is functorial in $X, Y \in \mathcal{T}$ such that the diagram

$$
\begin{array}{c}
F((X \otimes Y) \otimes Z) \\
\downarrow \quad \quad \quad \downarrow \\
F(X \otimes Y) \otimes F(Z) \\
\downarrow \\
(F(X) \otimes F(Y)) \otimes F(Z)
\end{array}
\xrightarrow{\xi_F(X,Y,Z)}
\begin{array}{c}
F((X \otimes Y) \otimes Z) \\
\downarrow \quad \quad \quad \downarrow \\
F(X \otimes Y) \otimes F(Z) \\
\downarrow \\
(F(X) \otimes F(Y)) \otimes F(Z)
\end{array}
$$

commutes for all $X, Y, Z \in \mathcal{T}$. We omit to write $\xi_F$ for simplicity. A monoidal functor $F$ is called **unital** if $(F(1), F(\epsilon))$ is a unit object. In this paper, we simply write a “monoidal functor” for a unital monoidal functor.

We say that a monoidal category $\mathcal{T}$ is an **additive** (resp. abelian) monoidal category if $\mathcal{T}$ is additive (resp. abelian) and the bifunctor $\cdot \otimes \cdot$ is biadditive. Similarly, for a commutative ring $k$, a monoidal category $\mathcal{T}$ is $k$-**linear** if $\mathcal{T}$ is $k$-linear and the bifunctor $\cdot \otimes \cdot$ is $k$-bilinear.

An object $X \in \mathcal{T}$ is **invertible** if the functors $\mathcal{T} \to \mathcal{T}$ given by $Z \mapsto Z \otimes X$ and $Z \mapsto X \otimes Z$ are equivalence of categories. If $X$ is invertible, then one can find an object $Y$ and isomorphisms
The triple $(Y, f, g)$ is unique up to a unique isomorphism. We write $Y = X^\otimes -1$.

For a monoidal category $\mathcal{T}$, we define a new monoidal category $\mathcal{T}^{\text{rev}}$ as the category $\mathcal{T}$ endowed with the new bifunctor $\otimes^{\text{rev}}$ defined by $X \otimes^{\text{rev}} Y := Y \otimes X$ and $f \otimes^{\text{rev}} g := g \otimes f$ for any objects $X, Y$ in $\mathcal{T}$ and for any morphisms $f, g$ in $\mathcal{T}$, respectively. The associativity constraints are given as $a^{\text{rev}}(X, Y, Z) := a(Z, Y, X)^{-1}$. The unit $(1, \varepsilon)$ of $\mathcal{T}$ serves as a unit of $\mathcal{T}^{\text{rev}}$, too. Let $F : C \to \mathcal{T}$ be a monoidal functor. Then $F : C^{\text{rev}} \to \mathcal{T}^{\text{rev}}$ is again a monoidal functor.

A pair of morphisms $\varepsilon : X \otimes Y \to 1$ and $\eta : 1 \to Y \otimes X$ in $\mathcal{T}$ is called an adjunction if the composition $X \cong X \otimes 1 \xrightarrow{\varepsilon \otimes 1} X \otimes Y \otimes X \xrightarrow{\varepsilon \otimes X} 1 \otimes X \cong X$ is the identity of $X$, and the composition $Y \cong 1 \otimes Y \xrightarrow{\eta \otimes 1} Y \otimes X \otimes Y \xrightarrow{Y \otimes \varepsilon} Y \otimes 1 \cong Y$ is the identity of $Y$. In the case when $(\varepsilon, \eta)$ is an adjunction, we say that $X$ is a left dual to $Y$ and $Y$ is a right dual to $X$ in $\mathcal{T}$. A monoidal category $\mathcal{T}$ is left (respectively, right) rigid if every object in $\mathcal{T}$ has a left (respectively, right) dual. We call that $\mathcal{T}$ is rigid, if it is left-rigid and right-rigid. For more generalities on monoidal categories, we refer the reader to [15, Chapter 4].

1.1.2 Real commuting family of graded braiders

In this subsection, we recall the notions of braiders and localization introduced in [13]. We refer the reader to loc. cit. for more details.

**Definition 1.1.** A left braider, simply a braider in the sequel, of a monoidal category $\mathcal{T}$ is a pair $(C, R_C)$ of an object $C$ and a morphism

$$R_C(X) : C \otimes X \longrightarrow X \otimes C$$

which is functorial in $X \in \mathcal{T}$ such that the following diagrams commute:

$$\begin{array}{ccc}
C \otimes X \otimes Y & \xrightarrow{R_C(X) \otimes Y} & X \otimes C \otimes Y \\
\downarrow R_C(X \otimes Y) & & \downarrow X \otimes R_C(Y) \\
X \otimes Y \otimes C, & \overset{=} \longrightarrow & C.
\end{array}$$

A braider $(C, R_C)$ is called a central object if $R_C(X)$ is an isomorphism for any $X \in \mathcal{T}$.

Let $k$ be a commutative ring and let $A$ be a $\mathbb{Z}$-module. A $k$-linear monoidal category $\mathcal{T}$ is $\Lambda$-graded if $\mathcal{T}$ has a decomposition $\mathcal{T} = \bigoplus_{\lambda \in \Lambda} T_\lambda$ such that $1 \in \mathcal{T}_0$ and $\otimes$ induces a bifunctor $T_\lambda \times T_\mu \to T_{\lambda + \mu}$ for any $\lambda, \mu \in \Lambda$. Let $q$ be an invertible central object in a $\Lambda$-graded category $\mathcal{T}$, which belongs to $\mathcal{T}_0$. We write $q^n$ ($n \in \mathbb{Z}$) for $q^{\otimes n}$ for the sake of simplicity.
**Definition 1.2.** A graded braider is a triple $(C, R_C, \phi)$ of an object $C$, a $\mathbb{Z}$-linear map $\phi : \Lambda \to \mathbb{Z}$, and a morphism

$$R_C(X) : C \otimes X \to q^{\phi(\lambda)} \otimes X \otimes C$$

such that the diagrams

$$\begin{array}{ccc}
C \otimes X \otimes Y & \xrightarrow{R_C(X) \otimes Y} & q^{\phi(\lambda)} \otimes X \otimes C \otimes Y \\
\uparrow R_C(X \otimes Y) & & \downarrow X \otimes R_C(Y) \\
q^{\phi(\lambda+\mu)} \otimes X \otimes Y \otimes C & \xrightarrow{\sim} & C
\end{array}$$

and

$$\begin{array}{ccc}
C \otimes 1 & \xrightarrow{R_C(1)} & 1 \otimes C \\
\uparrow & & \downarrow 1 \\
& \xrightarrow{q^{\phi(\lambda)}} & q^{\phi(\lambda)+\phi(\lambda')} \otimes C
\end{array}$$

commute for any $X \in T_\lambda$ and $Y \in T_\mu$.

We denote by $\mathcal{T}_{br}$ the category of graded braiders in $\mathcal{T}$. A morphism from $(C, R_C, \phi)$ to $(C', R_{C'}, \phi')$ in $\mathcal{T}_{br}$ is a morphism $f \in \text{Hom}_\mathcal{T}(C, C')$ such that $\phi = \phi'$ and the following diagram commutes for any $\lambda \in \Lambda$ and $X \in T_\lambda$:

$$\begin{array}{ccc}
C \otimes X & \xrightarrow{f \otimes X} & C' \otimes X \\
\downarrow R_C(X) & & \downarrow R_{C'}(X) \\
q^{\phi(\lambda)} \otimes X \otimes C & \xrightarrow{q^{\phi(\lambda')} \otimes X \otimes f} & q^{\phi(\lambda)} \otimes X \otimes C'.
\end{array}$$

For graded braiders $(C_1, R_{C_1}, \phi_1)$ and $(C_2, R_{C_2}, \phi_1)$ of $\mathcal{T}$, let $R_{C_1 \otimes C_2}(X)$ be the composition

$$C_1 \otimes C_2 \otimes X \xrightarrow{R_{C_1}(X)} q^{\phi_1(\lambda)} \otimes C_1 \otimes C_2 \otimes X \xrightarrow{R_{C_2}(X)} q^{\phi_1(\lambda)+\phi_2(\lambda)} \otimes X \otimes C_1 \otimes C_2$$

for $X \in T_\lambda$. Then $(C_1 \otimes C_2, R_{C_1 \otimes C_2}, \phi_1 + \phi_2)$ is also a graded braider of $\mathcal{T}$. Hence, the category $\mathcal{T}_{br}$ is a monoidal category with a canonical faithful monoidal functor $\mathcal{T}_{br} \to \mathcal{T}$.

Let $I$ be an index set and let $\{(C_i, R_{C_i}, \phi_i)\}_{i \in I}$ be a family of graded braiders. We say that $\{(C_i, R_{C_i}, \phi_i)\}_{i \in I}$ is a real commuting family of graded braiders in $\mathcal{T}$ if

(a) $C_i \in T_{\lambda_i}$ for some $\lambda_i \in \Lambda$, and $\phi_i(\lambda_i) = 0$, $\phi_i(\lambda_j) + \phi_j(\lambda_i) = 0$,
(b) $R_{C_i}(C_i) \in \mathbb{k} \times \text{id}_{C_i \otimes C_i}$ for $i \in I$,
(c) $R_{C_j}(C_i) \circ R_{C_i}(C_j) \in \mathbb{k} \times \text{id}_{C_i \otimes C_j}$ for $i, j \in I$.

Set

$$\Gamma := \mathbb{Z}^{\oplus I} \quad \text{and} \quad \Gamma_{\geq 0} := \mathbb{Z}^{\oplus I}_{\geq 0}.$$
We choose a \( \mathbb{Z} \)-bilinear map \( H : \Gamma \times \Gamma \to \mathbb{Z} \) such that \( \phi_i(\lambda_j) = H(e_i, e_j) - H(e_j, e_i) \) for any \( i, j \in I \). Then, we have

\[
\phi(\alpha, L(\beta)) = H(\alpha, \beta) - H(\beta, \alpha)
\]

for any \( \alpha, \beta \in \Gamma \). (1.2)

Let us denote by \( \phi_\alpha \) the \( \mathbb{Z} \)-linear map \( \phi(\alpha, -) : \Lambda \to \mathbb{Z} \) for each \( \alpha \in \Gamma \).

**Lemma 1.3** [13, Lemma 2.3, Lemma 1.16]. Let \( \{(C_i, R_{C_i}, \phi_i)\}_{i \in I} \) be a real commuting family of graded braiders in \( \mathcal{T} \).

(i) There exists a family \( \{\eta_{ij}\}_{i, j \in I} \) of elements in \( \mathbb{k}^\times \) such that

\[
R_{C_i}(C_i) = \eta_{ii} \text{id}_{C_i \otimes C_i},
R_{C_i}(C_i) \circ R_{C_j}(C_j) = \eta_{ij} \eta_{ji} \text{id}_{C_i \otimes C_j}
\]

for all \( i, j \in I \).

(ii) There exist a graded braider \( C^\alpha = (C^\alpha, R_{C^\alpha}, \phi_\alpha) \) for each \( \alpha \in \Gamma_{\geq 0} \), and an isomorphism

\[
\xi_{\alpha, \beta} : C^\alpha \otimes C^\beta \sim q^{H(\alpha, \beta)} \otimes C^{\alpha + \beta} \in \mathbb{T}_{br}
\]

in \( \mathcal{T}_{br} \) for \( \alpha, \beta \in \Gamma_{\geq 0} \) such that

(a) \( C^0 = 1 \) and \( C^{e_i} = C_i \) for \( i \in I \),

(b) the diagram in \( \mathcal{T}_{br} \)

\[
\begin{array}{ccc}
C^\alpha \otimes C^\beta \otimes C^\gamma & \xrightarrow{\xi_{\alpha, \beta} \otimes C^\gamma} & q^{H(\alpha, \beta)} \otimes C^{\alpha + \beta} \otimes C^\gamma \\
C^\alpha \otimes q^{H(\beta, \gamma)} \otimes C^\beta & \xleftarrow{\xi_{\beta, \gamma} \otimes C^\alpha} & q^{H(\alpha, \beta) + H(\gamma, \beta)} \otimes C^{\alpha + \beta + \gamma}
\end{array}
\]

(iii) for any \( \alpha, \beta, \gamma \in \Gamma_{\geq 0} \),

(c) the diagrams in \( \mathcal{T}_{br} \)

\[
\begin{array}{ccc}
C^0 \otimes C^0 & \xrightarrow{\xi_{0, 0}} & C^0 \\
1 \otimes 1 & \xrightarrow{\sim} & 1
\end{array}
\]

and

\[
\begin{array}{ccc}
C^\alpha \otimes C^\beta & \xrightarrow{R_{C^\alpha}(C^\beta)} & q^{H(\alpha, \beta)} \otimes C^\alpha \otimes C^\beta \\
q^{H(\alpha, \beta)} \otimes C^{\alpha + \beta} & \xrightarrow{\eta(\beta, \gamma) \text{id}_{C^{\alpha + \beta}}} & q^{H(\alpha, \beta)} \otimes C^{\alpha + \beta}
\end{array}
\]

for any \( i, j \in I \) and \( \alpha, \beta, \gamma \in \Gamma_{\geq 0} \), where

\[
\eta(\alpha, \beta) := \prod_{i, j \in I} \eta_{i, j}^{a_i b_j} \in \mathbb{k}^\times \text{ for } \alpha = \sum_{i \in I} a_i e_i \text{ and } \beta = \sum_{j \in I} b_j e_j \text{ in } \Gamma.
\]

Note that we have \( \eta(\alpha, 0) = \eta(0, \alpha) = 1 \), and \( \eta(\alpha, \beta + \gamma) = \eta(\alpha, \beta) \cdot \eta(\alpha, \gamma) \) and \( \eta(\alpha + \beta, \gamma) = \eta(\alpha, \gamma) \cdot \eta(\beta, \gamma) \) for \( \alpha, \beta, \gamma \in \Gamma \).

We define an order \( \leq \) on \( \Gamma \) by

\[
\alpha \leq \beta \text{ for } \alpha, \beta \in \Gamma \text{ with } \beta - \alpha \in \Gamma_{\geq 0},
\]
and set
\[ D_{\alpha_1, \ldots, \alpha_k} := \{ \delta \in \Gamma \mid \alpha_i + \delta \in \Gamma_{\geq 0} \text{ for any } i = 1, \ldots, k \} \]

for \( \alpha_1, \ldots, \alpha_k \in \Gamma \).

For \( X \in T_\lambda, Y \in T_\mu \) and \( \delta \in D_{\alpha, \beta} \), we set
\[ H^\text{gr}_\delta((X, \alpha), (Y, \beta)) := \text{Hom}_T(C^{\delta + \alpha} \otimes X, q^{H(\delta, \beta - \alpha) + \phi(\delta, \beta, \mu)} \otimes Y \otimes C^{\delta + \beta}). \]

For \( \delta, \delta' \in D_{\alpha, \beta} \) with \( \delta \leq \delta' \) and \( f \in H^\text{gr}_\delta((X, \alpha), (Y, \beta)) \), we define \( \zeta^\text{gr}_{\delta', \delta}(f) \in H^\text{gr}_{\delta'}((X, \alpha), (Y, \beta)) \) to be the morphism such that the following diagram commutes:

Then, \( \zeta^\text{gr}_{\delta', \delta} \) is a map from \( H^\text{gr}_\delta((X, \alpha), (Y, \beta)) \) to \( H^\text{gr}_{\delta'}((X, \alpha), (Y, \beta)) \) and \( \zeta^\text{gr}_{\delta'}, \delta' \circ \zeta^\text{gr}_{\delta', \delta} \circ \zeta^\text{gr}_{\delta', \delta} = \zeta^\text{gr}_{\delta', \delta} \), for \( \delta \leq \delta' \leq \delta'' \), so that \( \{ \zeta^\text{gr}_{\delta', \delta} \}_{\delta, \delta' \in D_{\alpha, \beta}} \) forms an inductive system indexed by \( D_{\alpha, \beta} \).

Hence, we can define a new category \( \tilde{T} \) as
\[ \text{Ob}(\tilde{T}) := \text{Ob}(T) \times \Gamma, \]
\[ \text{Hom}_{\tilde{T}}((X, \alpha), (Y, \beta)) := \lim_{\substack{\delta \in D_{\alpha, \beta}, \\ \lambda + L(\alpha) = \mu + L(\beta)}} H^\text{gr}_\delta((X, \alpha), (Y, \beta)), \]

where \( X \in T_\lambda \) and \( Y \in T_\mu \). For the composition of morphisms in \( \tilde{T} \) and its associativity, see [13, Section 2.2, Section 2.3].

By the construction, we have the decomposition
\[ \tilde{T} = \bigoplus_{\mu \in \Lambda} \tilde{T}_\mu, \quad \text{where } \tilde{T}_\mu := \{(X, \alpha) \mid X \in T_\lambda, \lambda + L(\alpha) = \mu \}. \]

The category \( \tilde{T} \) is a monoidal category with the following tensor product: For \( \alpha, \alpha', \beta, \beta' \in \Gamma \), \( X \in T_\lambda, X' \in T_{\lambda'}, Y \in T_\mu, \) and \( Y' \in T_{\mu'} \), we define
\[ (X, \alpha) \otimes (Y, \beta) := (q^{-\phi(\beta, \lambda) + H(\alpha, \beta)} \otimes X \otimes Y, \alpha + \beta), \]

and, for \( f \in H^\text{gr}_\delta((X, \alpha), (Y', \beta')) \) and \( g \in H^\text{gr}_{\epsilon}((Y, \beta), (Y', \beta')) \), we define
\[ T^\text{gr}_{\delta, \epsilon}(f, g) := \eta(\epsilon, \alpha - \alpha') T^\text{gr}_{\delta, \epsilon}(f, g), \]
where \( \bar{T}_{\delta,\varepsilon}^\text{gr}(f, g) \) is the morphism such that the following diagram commutes:

\[
\begin{array}{c}
\begin{array}{c}
C^{\delta+\alpha} \otimes X \otimes C^{\varepsilon+\beta} \otimes Y \\
R_{\alpha+\beta}(X)
\end{array}
\begin{array}{c}
\begin{array}{c}
\downarrow f \otimes g \\
q^b \otimes X' \otimes C^{\delta+\alpha'} \otimes Y' \otimes C^{\varepsilon+\beta'}
\end{array}
\begin{array}{c}
\downarrow R_{\delta+\varepsilon'}(Y')
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
q^a \otimes C^{\delta+\varepsilon+\alpha+\beta} \otimes X \otimes Y
\end{array}
\begin{array}{c}
\downarrow \bar{T}_{\delta,\varepsilon}^\text{gr}(f, g)
\end{array}
\begin{array}{c}
\downarrow \xi_{\delta+\varepsilon,\alpha+\beta'}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
q^d \otimes X' \otimes Y' \otimes C^{\delta+\varepsilon+\alpha'+\beta'}
\end{array}
\end{array}
\end{array}
\end{array}
\]

where

\[
\begin{align*}
a &= -\phi(\varepsilon + \beta, \lambda) + H(\delta + \alpha, \varepsilon + \beta), \\
b &= H(\delta, \alpha' - \alpha) + \phi(\delta + \alpha', \lambda') + H(\varepsilon, \beta' - \beta) + \phi(\varepsilon + \beta', \mu'), \\
c &= b + \phi(\delta + \alpha', \mu'), \\
d &= c + H(\delta + \alpha', \varepsilon + \beta').
\end{align*}
\]

Then, we have

\[
T_{\delta,\varepsilon}^\text{gr}(f, g) \in H_{\delta+\varepsilon}^\text{gr}((X, \alpha) \otimes (Y, \beta), (X', \alpha') \otimes (Y', \beta')).
\]

Then the map \( T_{\delta,\varepsilon}^\text{gr} \) is compatible with the maps \( \bar{T}_{\delta,\varepsilon}^\text{gr} \), in the inductive system, and moreover, it yields a bifunctor \( \otimes \) on \( \bar{T} \) ([13, Proposition 2.5])

\[
\text{Hom}_\bar{T}((X, \alpha), (X', \alpha')) \times \text{Hom}_\bar{T}((Y, \beta), (Y', \beta')) \longrightarrow \text{Hom}_\bar{T}((X, \alpha) \otimes (Y, \beta), (X', \alpha') \otimes (Y', \beta')).
\]

For \((X, \alpha) \in \bar{T}\), define \( R_{(q,0)}((X, \alpha)) \in \text{Hom}_\bar{T}((q \otimes X, X), (X \otimes q, \alpha))\) as the image of \( R_q(X) = H_{\alpha}^\text{gr}((q \otimes X, \alpha), (X \otimes q, \alpha))\). Then \((q,0), R_{(q,0)}\) is an invertible braider in \( \bar{T}\).

**Theorem 1.4.** Let \( \{C_i = (C_i, R_{C_i}, \phi_i)\}_{i \in I} \) be a real commuting family of graded braiders in \( \mathcal{T} \). Then the category \( \bar{T} \) defined above becomes a monoidal category. There exists a monoidal functor \( Y : \mathcal{T} \to \bar{T} \) and a real commuting family of graded braiders \( \{\bar{C}_i = (\bar{C}_i, R_{\bar{C}_i}, \phi_i)\}_{i \in I} \) in \( \bar{T} \) satisfying the following properties:

(i) for \( i \in I \), \( Y(C_i) \) is isomorphic to \( \bar{C}_i \) and it is invertible in \( \bar{T}_{\text{br}} \),

(ii) for \( i \in I \) and \( X \in T_{\delta,\varepsilon} \), the diagram

\[
\begin{array}{c}
\xymatrix{
Y(C_i \otimes X) \ar[r]^-{\sim} \ar[d]_{Y(R_{C_i}(X))} & \bar{C}_i \otimes Y(X) \ar[d]^{R_{\bar{C}_i}(Y(X))} \\
Y(q^{\phi_i}(\lambda) \otimes X \otimes C_i) \ar[r]^-{\sim} & q^{\phi_i}(\lambda) \otimes Y(X) \otimes \bar{C}_i
}
\end{array}
\]

commutes.
Moreover, the functor \( \Upsilon \) satisfies the following universal property:

(iii) If there are another \( \Lambda \)-graded monoidal category \( \mathcal{T}' \) with an invertible central object \( q \in \mathcal{T}'_0 \) with and a \( \Lambda \)-graded monoidal functor \( \Upsilon' : \mathcal{T} \to \mathcal{T}' \) such that

(a) \( \Upsilon' \) sends the central object \( q \in \mathcal{T}_0 \) to \( q \in \mathcal{T}'_0 \),

(b) \( \Upsilon'(C_i) \) is invertible in \( \mathcal{T}' \) for any \( i \in I \), and

(c) for any \( i \in I \) and \( X \in \mathcal{T} \), \( \Upsilon'(R_{C_i}(X)) : \Upsilon'(C_i \otimes X) \to \Upsilon'(q^\phi_i(\lambda) \otimes X \otimes C_i) \) is an isomorphism,

then there exists a monoidal functor \( \tilde{\Upsilon} \), which is unique up to a unique isomorphism, such that the diagram

\[
\begin{array}{ccc}
\mathcal{T} & \xrightarrow{\Upsilon} & \tilde{\mathcal{T}} \\
\downarrow{\Upsilon'} & & \downarrow{\tilde{\Upsilon}} \\
\mathcal{T}' & &
\end{array}
\]

commutes.

We denote by \( \mathcal{T}[C_i^{\otimes -1} \mid i \in I] \) the localization \( \tilde{\mathcal{T}} \) in Theorem 1.4. Note that

\[
(X, \alpha + \beta) \simeq q^{-H(\beta, \alpha)} \otimes (C^{\alpha} \otimes X, \beta), \quad (1, \beta) \otimes (1, -\beta) \simeq q^{-H(\beta, \beta)}(1, 0)
\]

for \( \alpha \in \Gamma_{\geq 0} \) and \( \beta \in \Gamma \).

**Proposition 1.5.** Let \((C_i, R_{C_i}, \phi_i)_{i \in I}\) be a real commuting family of graded braiders in a graded monoidal category \( \mathcal{T} \), and set \( \tilde{\mathcal{T}} := \mathcal{T}[C_i^{\otimes -1} \mid i \in I] \). Assume that

(a) \( \mathcal{T} \) is an abelian category,

(b) \( \otimes \) is exact.

Then \( \tilde{\mathcal{T}} \) is an abelian category with exact \( \otimes \), and the functor \( \Upsilon : \mathcal{T} \to \tilde{\mathcal{T}} \) is exact.

### 1.2 Quiver Hecke algebras

#### 1.2.1 Cartan data

Let \( I \) be an index set. A Cartan datum \((A, P, \Pi, \Pi^\vee, (\cdot, \cdot))\) consists of

(i) a free abelian group \( P \), called the weight lattice,

(ii) \( \Pi = \{\alpha_i \mid i \in I\} \subset P \), called the set of simple roots,

(iii) \( \Pi^\vee = \{h_i \mid i \in I\} \subset P^\vee := \text{Hom}(P, \mathbb{Z}) \), called the set of simple coroots,

(iv) a \( \mathbb{Q} \)-valued symmetric bilinear form \((\cdot, \cdot)\) on \( P \),

which satisfy

(a) \( (\alpha_i, \alpha_i) \in 2\mathbb{Z}_{>0} \) for \( i \in I \),

(b) \( \langle h_i, \lambda \rangle = \frac{2(\alpha_i, \lambda)}{(\alpha_i, \alpha_i)} \) for \( i \in I \) and \( \lambda \in P \),

which satisfy

(a) \( (\alpha_i, \alpha_i) \in 2\mathbb{Z}_{>0} \) for \( i \in I \),

(b) \( \langle h_i, \lambda \rangle = \frac{2(\alpha_i, \lambda)}{(\alpha_i, \alpha_i)} \) for \( i \in I \) and \( \lambda \in P \),
(c) A := \((h_i, \alpha_j)\)_{i,j \in I} is a generalized Cartan matrix, that is, \(\langle h_i, \alpha_i \rangle = 2\) for any \(i \in I\) and \(\langle h_j, \alpha_j \rangle \in \mathbb{Z}_{\leq 0}\) if \(i \neq j\).

(d) \(\Pi\) is a linearly independent set,

(e) for each \(i \in I\), there exists \(\Lambda_i \in P\) such that \(\langle h_j, \Lambda_i \rangle = \delta_{ij}\) for any \(j \in I\).

Let \(\Delta\) (resp. \(\Delta_+, \Delta_-\)) be the set of roots (resp. positive roots, negative roots). We set \(P_+ := \{\lambda \in P \mid \langle h_i, \lambda \rangle \geq 0\}\) for \(i \in I\), \(Q = \bigoplus_{i \in I} \mathbb{Z} \alpha_i\), and \(Q_+ = \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i\), and write \(ht(\beta) = \sum_{i \in I} k_i\) for \(\beta = \sum_{i \in I} k_i \alpha_i \in Q_+\). For \(i \in I\), we define

\[s_i(\lambda) = \lambda - \langle h_i, \lambda \rangle \alpha_i\]  

for \(\lambda \in P\), and \(W\) is the subgroup of \(\text{Aut}(P)\) generated by \(\{s_i\}_{i \in I}\).

For \(w, v \in W\), we write \(w \succeq v\) if there exists a reduced expression of \(v\) that appears in a subexpression of a reduced expression of \(w\) (the Bruhat order on \(W\)).

For \(w \in W\), we say that an element \(\lambda \in P\) is \(w\)-dominant if

\[\langle h_i, s_{i+1} \cdots s_r \lambda \rangle \geq 0\]  

for \(1 \leq k \leq r\),

where \(w = s_{i_1} \cdots s_{i_r}\) is a reduced expression of \(w\).

Note that any \(\lambda \in P_+\) is \(w\)-dominant for any \(w \in W\).

### 1.2.2 Quiver Hecke algebras

Let \(k\) be a field. For \(i, j \in I\), we choose polynomials \(Q_{i,j}(u, v) \in k[u, v]\) such that

(a) \(Q_{i,j}(u, v) = Q_{j,i}(v, u)\),

(b) it is of the form

\[Q_{i,j}(u, v) = \begin{cases} \sum_{p(\alpha_i, \alpha_i) + q(\alpha_j, \alpha_j) = -2(\alpha_i, \alpha_j)} t_{i,j;p,q} u^p v^q & \text{if } i \neq j, \\ 0 & \text{if } i = j, \end{cases}\]

where \(t_{i,j;-,0} \in k^\times\).

For \(\beta \in Q_+\) with \(ht(\beta) = n\), we set

\[I^\beta := \left\{ v = (v_1, \ldots, v_n) \in I^n \mid \sum_{k=1}^{n} \alpha_{s_k} = \beta \right\},\]

on which the symmetric group \(S_n = \langle s_k \mid k = 1, \ldots, n-1 \rangle\) acts by place permutations.
Definition 1.6. For $\beta \in \mathbb{Q}_+$, the quiver Hecke algebra $R(\beta)$ associated with $\mathbb{A}$ and $(Q_{i,j}(u,v))_{i,j \in \mathbb{I}}$ is the $k$-algebra generated by

$$\{e(\nu) \mid \nu \in I^{\beta}\}, \{x_k \mid 1 \leq k \leq n\}, \{\tau_l \mid 1 \leq l \leq n-1\},$$

satisfying the following defining relations:

$$e(\nu)e(\nu') = \delta_{\nu,\nu'}e(\nu), \sum_{\nu \in I^{\beta}} e(\nu) = 1, x_ke(\nu) = e(\nu)x_k, x_kx_l = x lx_k,$$

$$\tau_l e(\nu) = e(s_l(\nu))\tau_l, \tau_k\tau_l = \tau_l\tau_k \text{ if } |k-l| > 1,$$

$$\tau_k^2 e(\nu) = Q_{\nu_k,\nu_{k+1}}(x_k, x_{k+1})e(\nu),$$

$$(\tau_k x_l - x_{s_k(l)}\tau_k)e(\nu) = \begin{cases} -e(\nu) & \text{if } l = k \text{ and } \nu_k = \nu_{k+1}, \\ e(\nu) & \text{if } l = k + 1 \text{ and } \nu_k = \nu_{k+1}, \\ 0 & \text{otherwise}, \end{cases}$$

$$(\tau_{k+1}\tau_k\tau_{k+1} - \tau_k\tau_{k+1}\tau_k)e(\nu) = \begin{cases} Q_{\nu_k,\nu_{k+1}}(x_k, x_{k+1}, x_{k+2})e(\nu) & \text{if } \nu_k = \nu_{k+2}, \\ 0 & \text{otherwise}, \end{cases}$$

where

$$Q_{i,j}(u, v, w) := Q_{i,j}(u, v) - Q_{i,j}(w, v) \in k[u, v, w].$$

The algebra $R(\beta)$ has the $\mathbb{Z}$-grading defined by

$$\deg(e(\nu)) = 0, \quad \deg(x_k e(\nu)) = (\alpha_{\nu_k}, \alpha_{\nu_k}), \quad \deg(\tau_l e(\nu)) = -(\alpha_{\nu_l}, \alpha_{\nu_{l+1}}).$$

We denote by $R(\beta)$-Mod the category of graded $R(\beta)$-modules with degree-preserving homomorphisms. We write $R(\beta)$-gmod for the full subcategory of $R(\beta)$-Mod consisting of the graded modules that are finite-dimensional over $k$, and $R(\beta)$-proj for the full subcategory of $R(\beta)$-Mod consisting of finitely generated projective graded $R(\beta)$-modules. We set $R$-Mod := $\bigoplus_{\beta \in \mathbb{Q}_+} R(\beta)$-Mod, $R$-proj := $\bigoplus_{\beta \in \mathbb{Q}_+} R(\beta)$-proj, and $R$-gmod := $\bigoplus_{\beta \in \mathbb{Q}_+} R(\beta)$-gmod. The trivial $R(0)$-module of degree 0 is denoted by $1$. For simplicity, we write “a module” instead of “a graded module.” We define the grading shift functor $q$ by $(qM)_k = M_{k-1}$ for a graded module $M = \bigoplus_{k \in \mathbb{Z}} M_k$. For $M, N \in R(\beta)$-Mod, $\text{Hom}_{R(\beta)}(M, N)$ denotes the space of degree-preserving module homomorphisms. We define

$$\text{HOM}_{R(\beta)}(M, N) := \bigoplus_{k \in \mathbb{Z}} \text{Hom}_{R(\beta)}(q^k M, N),$$

and set $\deg(f) := k$ for $f \in \text{Hom}_{R(\beta)}(q^k M, N)$. When $M = N$, we write $\text{END}_{R(\beta)}(M) = \text{HOM}_{R(\beta)}(M, M)$. We sometimes write $R$ for $R(\beta)$ in $\text{HOM}_{R(\beta)}(M, N)$ for simplicity.
For $M \in R(\beta)$-mod, we set $M^* := \text{HOM}_k(M, k)$ with the $R(\beta)$-action given by

$$(r \cdot f)(u) := f(\rho(r)u), \quad \text{for } f \in M^*, r \in R(\beta) \text{ and } u \in M,$$

where $\rho$ is the antiautomorphism of $R(\beta)$ that fixes the generators $e(\nu), x_k, \tau_k$. We say that $M$ is self-dual if $M \cong M^*$ in $R$-gmod.

For $\beta, \beta' \in Q_+$, set $e(\beta, \beta') := \sum_{\nu \in I(\beta), \nu' \in I(\beta') e(\nu^* \nu')}$, where $\nu^* \nu'$ is the concatenation of $\nu$ and $\nu'$. Then there is an injective ring homomorphism

$$R(\beta) \otimes R(\beta') \to e(\beta, \beta') R(\beta + \beta')$$

given by $e(\nu) \otimes e(\nu') \mapsto e(\nu, \nu'), x_k e(\beta) \otimes 1 \mapsto x_k e(\beta, \beta'), 1 \otimes x_k e(\beta') \mapsto x_{k+|\beta|} e(\beta, \beta'), \tau_k e(\beta) \otimes 1 \mapsto \tau_k e(\beta, \beta'),$ and $1 \otimes \tau_k e(\beta') \mapsto \tau_{k+|\beta|} e(\beta, \beta')$. For $a \in R(\beta)$ and $a' \in R(\beta')$, the image of $a \otimes a'$ is sometimes denoted by $a \boxtimes a'$.

For $M \in R(\beta)$-Mod and $N \in R(\beta')$-Mod, we set

$$M \circ N := R(\beta + \beta') e(\beta, \beta') \otimes R(\beta) \otimes R(\beta') (M \otimes N).$$

For $u \in M$ and $v \in N$, the image of $u \otimes v$ by the map $M \otimes N \to M \circ N$ is sometimes denoted by $u \boxdot v$. We also write $M \boxdot N \subset M \circ N$ for the image of $M \otimes N$ in $M \circ N$.

For $\alpha, \beta \in Q_+$, let $X$ be an $R(\alpha + \beta)$-module. Then $e(\alpha, \beta)X$ is an $R(\alpha) \otimes R(\beta)$-module. We denote it by

$$\text{Res}_{\alpha, \beta}X.$$

We have

$$\text{Hom}_{R(\alpha) \otimes R(\beta)}(M \otimes N, \text{Res}_{\alpha, \beta}(X)) \cong \text{Hom}_{R(\alpha + \beta)}(M \circ N, X),$$

$$\text{Hom}_{R(\alpha) \otimes R(\beta)}(\text{Res}_{\alpha, \beta}(X), M \otimes N) \cong \text{Hom}_{R(\alpha + \beta)}(X, q(\alpha, \beta)N \circ M)$$

(1.7)

for any $R(\alpha)$-module $M$, any $R(\beta)$-module $N$, and any $R(\alpha + \beta)$-module $X$. See [16, Section 2.6], [20, Theorem 2.2] and [23, Theorem 6.2].

We denote by $M \vee N$ the head of $M \circ N$ and by $M \Delta N$ the socle of $M \circ N$. We say that simple $R$-modules $M$ and $N$ strongly commute if $M \circ N$ is simple. A simple $R$-module $L$ is real if $L$ strongly commutes with itself. Note that if $M$ and $N$ strongly commute, then $M$ and $N$ commute, that is, $M \circ N \cong N \circ M$ up to a grading shift.

For $i \in I$ and the functors $E_i$ and $F_i$ are defined by

$$E_i(M) = e(\alpha_i, \beta - \alpha_i)M \in R(\beta - \alpha_i)$-Mod

$$F_i(M) = R(\alpha_i) \circ M \in R(\beta + \alpha_i)$-Mod

for an $R(\beta)$-module $M$.

For $i \in I$ and $n \in \mathbb{Z}_{>0}$, let $L(i)$ be the simple $R(\alpha_i)$-module concentrated on degree 0 and $P(i^n)$ the indecomposable projective $R(n\alpha_i)$-module whose head is isomorphic to $L(i^n) := q_i^{-\frac{n(n-1)}{2}} L(i)^{\circ n}$,
where $q_i := q(\alpha_i, \alpha_i)/2$. Then, for $M \in R(\beta)$-Mod, we define

$$E_i^{(n)} M := \text{HOM}_{R(n\alpha_i)}(P(i^n), e(n\alpha_i, \beta - n\alpha_i)M) \in R(\beta - n\alpha_i)$-Mod,$$

$$F_i^{(n)} M := P(i^n) \circ M \in R(\beta + n\alpha_i)$-Mod.$$

Then, we have $F_i^{(1)} \simeq F_i$ and $E_i^{(1)} \simeq E_i$.

For $i \in I$ and a nonzero $M \in R(\beta)$-Mod, we define

$$\text{wt}(M) = -\beta, \ \varepsilon_i(M) = \max\{k \geq 0 \mid E_i^k M \neq 0\}, \ \varphi_i(M) = \varepsilon_i(M) + \langle h_i, \text{wt}(M) \rangle.$$

For a simple module $M$, we set

$$E_i^{\text{max}}(M) := E_i^{(\varepsilon_i(M))} M.$$

We can also define $E_i^+, F_i^+, \varepsilon_i^+$, and so forth, in the same manner as above if we replace $e(\alpha_i, \beta - \alpha_i)$, $R(\alpha_i) \circ -$ and so on, with $e(\beta - \alpha_i, \alpha_i), - \circ R(\alpha_i)$, and so on.

We denote by $K(R$-proj) and $K(R$-gmod) the Grothendieck groups of $R$-proj and $R$-gmod, respectively.

The following proposition will be used frequently.

**Proposition 1.7** (Shuffle lemma, [22, Proposition 2.7], [19, Theorem 4.3]). Let $\{\beta_j\}_{1 \leq j \leq r}$ and $\{\gamma_k\}_{1 \leq k \leq s}$ be two families of elements in $Q_+$ such that $\sum_{j=1}^r \beta_j = \sum_{k=1}^s \gamma_k$. Let $M_j$ be an $R(\beta_j)$-module for each $1 \leq j \leq r$. Then $e(\gamma_1, \ldots, \gamma_s)(M_1 \circ \cdots \circ M_r)$ has a filtration of $\bigotimes_{k=1}^s R(\gamma_k)$-modules whose graduations are isomorphic to the modules of the form

$$q^N \left( \bigotimes_{k=1}^s R(\gamma_k)e(\beta_{1,k}, \ldots, \beta_{r,k}) \right) \bigotimes_{j,k} R(\beta_{j,k}) \left( \bigotimes_{j=1}^r e(\beta_{j,1}, \ldots, \beta_{j,s})M_j \right).$$

Here,

- $\{\beta_{j,k}\}_{1 \leq j \leq r, 1 \leq k \leq s}$ is a family of elements in $Q_+$ such that $\beta_j = \sum_{k=1}^s \beta_{j,k}$ and $\gamma_k = \sum_{j=1}^r \beta_{j,k}$,
- the right action of $\bigotimes_{j,k} R(\beta_{j,k})$ on $\bigotimes_{k=1}^s R(\gamma_k)e(\beta_{1,k}, \ldots, \beta_{r,k})$ is induced by the action of $R(\beta_{j,k})$ on $R(\gamma_k)e(\beta_{1,k}, \ldots, \beta_{r,k})$,
- the left action of $\bigotimes_{j,k} R(\beta_{j,k})$ on $\bigotimes_{j=1}^r e(\beta_{j,1}, \ldots, \beta_{j,s})M_j$ is induced by the left action of $R(\beta_{j,k})$ on $e(\beta_{j,1}, \ldots, \beta_{j,s})M_j$,
- $N = \sum_{1 \leq j < j' \leq r, 1 \leq k, k' \leq s} (\beta_{j,k}, \beta_{j',k'})$.

Let us denote by $\mathfrak{g}$ the symmetrizable Kac–Moody algebra associated with the Cartan datum $(A, P, \Pi, \Pi^\vee, (-, ))$. Then, the quiver Hecke algebra $R$ associated with the same Cartan datum categorifies the negative half $U_q^-(\mathfrak{g})$ of the quantum group $U_q(\mathfrak{g})$ and its crystal/global basis ([16, 20, 24, 25, 27]). Let $B(\infty)$ be the crystal basis of $U_q^-(\mathfrak{g})$. Then, there is a bijection between $B(\infty)$ and the set of the isomorphism classes of self-dual simple $R$-modules. For $b \in B(\infty)$, let $S(b)$ be the self-dual simple $R$-module corresponding to $b$. Then, we have

$$S(\bar{f}_i(b)) \simeq L(i) \vee S(b) \quad \text{and} \quad S(\bar{e}_i(b)) \simeq \text{hd}(E_i(S(b)))$$

for $i \in I$ up to grading shifts, where $\bar{f}_i$ and $\bar{e}_i$ denote the Kashiwara operators on $B(\infty)$. 

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**LOCALIZATIONS FOR QUIVER HECKE ALGEBRAS II**

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Let $\psi : R(\beta) \overset{\sim}{\to} R(\beta)$ be the ring automorphism

$$
eq_1 \mapsto e(n_1, \ldots, n_n), \quad \psi : x_k \mapsto x_{n+1-k} \quad (1 \leq k \leq n), \quad \tau_k \mapsto -\tau_{n-k} \quad (1 \leq k < n),$$

where $n = |\beta|$. It induces a monoidal functor

$$\psi_* : R\text{-gmod} \overset{\sim}{\to} (R\text{-gmod})^{\text{rev}}.$$ 

Here, for a monoidal category $\mathcal{T}$, $\mathcal{T}^{\text{rev}}$ is the new monoidal category with the reversed tensor product $\otimes^{\text{rev}}$ (see § 1.1.1). Hence, there is a functorial isomorphism

$$\psi_*(M \circ N) \simeq \psi_*(N) \circ \psi_*(M)$$

for graded $R$-modules $M$ and $N$. Since $\psi$ is involutive, so is $\psi_*$. Note that $\psi_*(L(i^n)) \simeq L(i^n)$ for $i \in I$ and $n \geq 0$.

### 1.3 Affinizations and R-matrices

Let $R$ be a quiver Hecke algebra. We recall the notions of affinizations and R-matrices introduced in [14]. For $\beta \in Q_+$ and $i \in I$, let

$$\mathcal{P}_{i,\beta} := \sum_{\nu \in \beta} \left( \prod_{a \in \{1, \ldots, \text{ht}(\beta)\}, \nu_a = i} x_a \right) e(\nu) \in R(\beta).$$

Then $\mathcal{P}_{i,\beta}$ belongs to the center of $R(\beta)$.

**Definition 1.8.** Let $M$ be a simple $R(\beta)$-module. An *affinization* of $M$ with degree $d_M$ is an $R(\beta)$-module $\hat{M}$ with an endomorphism $z \hat{M}$ of $\hat{M}$ with degree $d_M \in \mathbb{Z}_{>0}$ and an isomorphism $\hat{M}/z\hat{M} \hat{M} \simeq M$ such that

1. $\hat{M}$ is a finitely generated free module over the polynomial ring $k[z\hat{M}]$.
2. $\mathcal{P}_{i,\beta} \hat{M} \neq 0$ for all $i \in I$.

Let $\beta \in Q_+$ with $m = \text{ht}(\beta)$. For $k = 1, \ldots, m-1$ and $\nu \in \beta$, the *intertwiner* $\varphi_k$ is defined by

$$\varphi_k e(\nu) = \begin{cases} (\tau_k x_k - x_k \tau_k) e(\nu) = (x_{k+1} \tau_k - \tau_k x_{k+1}) e(\nu) \\ (\tau_k (x_k - x_{k+1}) + 1) e(\nu) = (x_{k+1} - x_k) \tau_k - 1 \quad \text{if } \nu_k = \nu_{k+1}, \\ \tau_k e(\nu) \quad \text{otherwise}. \end{cases}$$

**Lemma 1.9** [5, Lemma 1.5].

1. $\varphi_k^2 e(\nu) = (Q_{\nu_k,\nu_{k+1}}(x_k, x_{k+1}) + \delta_{\nu_k,\nu_{k+1}}) e(\nu)$.
2. $\{\varphi_k\}_{k=1,\ldots,m-1}$ satisfies the braid relation.
(iii) For a reduced expression $w = s_{i_1} \cdots s_{i_t} \in S_m$, we set $\varphi_w := \varphi_{i_1} \cdots \varphi_{i_t}$. Then $\varphi_w$ does not depend on the choice of reduced expression of $w$.

(iv) For $w \in S_m$ and $1 \leq k \leq m$, we have $\varphi_w x_k = x_w(k) \varphi_w$.

(v) For $w \in S_m$ and $1 \leq k < m$, if $w(k+1) = w(k) + 1$, then $\varphi_w \tau_k = \tau_{w(k)} \varphi_w$.

For $m, n \in \mathbb{Z}_{\geq 0}$, we set $w[m, n]$ to be the element of $S_{m+n}$ such that

$$w[m, n](k) := \begin{cases} 
  k + n & \text{if } 1 \leq k \leq m, \\
  k - m & \text{if } m < k \leq m + n.
\end{cases}$$

Let $\beta, \gamma \in Q_+$ and set $m := \text{ht}(\beta)$ and $n := \text{ht}(\gamma)$. For $M \in R(\beta)$-Mod and $N \in R(\gamma)$-Mod, the $R(\beta) \otimes R(\gamma)$-linear map $M \otimes N \to N \circ M$ defined by

$$u \otimes v \mapsto \varphi_w[n, m](v \otimes u)$$

can be extended to an $R(\beta + \gamma)$-module homomorphism (up to a grading shift)

$$R_{M, N} : M \circ N \to N \circ M.$$ 

Let $\hat{M}$ be an affinization of a simple $R$-module $M$, and let $N$ be a nonzero $R$-module. We define a homomorphism (up to a grading shift)

$$R_{\hat{M}, N}^{\text{norm}} := z^{-s}_{\hat{M}} R_{\hat{M}, N} : \hat{M} \circ N \to N \circ \hat{M},$$

where $s$ is the largest integer such that $R_{\hat{M}, N}(\hat{M} \circ N) \subseteq z^{-s}_{\hat{M}} (N \circ \hat{M})$. We define

$$r_{M, N} : M \circ N \to N \circ M$$

to be the homomorphism (up to a grading shift) induced from $R_{\hat{M}, N}^{\text{norm}}$ by specializing at $z_{\hat{M}} = 0$. By the definition, $r_{M, N}$ never vanishes. We now define

$$\Lambda(M, N) := \deg(r_{M, N}),$$

$$\tilde{\Lambda}(M, N) := \frac{1}{2} (\Lambda(M, N) + (\text{wt}(M), \text{wt}(N))),$$

$$\mathcal{B}(M, N) := \frac{1}{2} (\Lambda(M, N) + \Lambda(N, M)).$$

Then, we have ([13, Lemma 3.11])

$$\mathcal{B}(M, N) \text{ and } \tilde{\Lambda}(M, N) \text{ are nonnegative integers.}$$

We also have (cf. [7, Lemma 3.15])

$$\Lambda(M, N_1 \circ N_2) = \Lambda(M, N_1) + \Lambda(M, N_2) \quad (1.10)$$

for nonzero modules $N_1, N_2$ and a simple module $M$ that admits an affinization.
Definition 1.10. We say that an $R$-module module $M$ is affreal if $M$ is real simple and $M$ admits an affinization.

Proposition 1.11 [14, Proposition 2.10]. Let $M$ and $N$ be simple $R$-modules. Assume that one of them is affreal. Then we have

$$\text{HOM}(M \circ N, N \circ M) = k_r_{M,N}.$$  

In the case that $N$ has an affinization $(\hat{N}, z_{\hat{N}})$, we can define $R^\text{norm}_{M,N}$ in a similar way as above. Then we have $\deg R^\text{norm}_{M,N} = \deg R^\text{norm}_{M,N}$, and $R^\text{norm}_{M,N} |_{z_{\hat{N}} = 0} = R^\text{norm}_{M,N} |_{z_{\hat{N}} = 0}$ up to a constant multiple if $M$ or $N$ is real. Hence, $\Lambda(M, N)$ and $r_{M,N}$ (up to a constant multiple) are well defined when either $M$ or $N$ is affreal. Moreover, they do not depend on the choice of affinizations.

1.4 Determinantial modules

Let $\Lambda \in \mathcal{P}_+$ and let $t_i$ be an indeterminate for each $i \in I$. Set

$$a_{\Lambda,i}(t_i) := t_i{(h_i, \Lambda)} \in k[t_i] \quad \text{for } i \in I.$$  

Let $\lambda \in \Lambda - Q_+$, and write $\beta := \Lambda - \lambda \in Q_+$ and $n := \text{ht}(\beta)$. The cyclotomic quiver Hecke algebra is the quotient of $R(\beta)$ given by

$$R^\Lambda(\lambda) := \frac{R(\beta)}{\sum_{i \in I} R(\beta) a_{\Lambda,i}(x^n e(\beta - \alpha_i, \alpha_i)) R(\beta)}.$$  

(1.11)

See [13] for more details.

Let $R^\Lambda(\lambda)$-Mod be the category of graded $R^\Lambda(\lambda)$-modules.

We define the functors

$$F^\Lambda_i : R^\Lambda(\lambda)$-Mod \to R^\Lambda(\lambda - \alpha_i)$-Mod,$$

$$E^\Lambda_i : R^\Lambda(\lambda)$-Mod \to R^\Lambda(\lambda + \alpha_i)$-Mod$$

by $F^\Lambda_i M = R^\Lambda(\lambda - \alpha_i) e(\alpha_i, \beta) \otimes_{R^\Lambda(\lambda)} M$ and $E^\Lambda_i M = e(\alpha_i, \beta - \alpha_i) M$ for $M \in R^\Lambda(\lambda)$-Mod. Similarly, for $m \in \mathbb{Z}_{\geq 0}$, we define

$$F^\Lambda_i^{(m)} M = R^\Lambda(\lambda - m \alpha_i) e(\alpha_i, \beta) \otimes_{R^\Lambda(\lambda - m \alpha_i)} (F^\Lambda_i)^{(m)} M \in R^\Lambda(\lambda - m \alpha_i)$-Mod,$$

$$E^\Lambda_i^{(m)} M = E^\Lambda_i^{(m)} M \in R^\Lambda(\lambda + m \alpha_i)$-Mod.$$

For $\lambda, \mu \in \mathcal{P}$, we write $\lambda \leq \mu$ if there exists a sequence of real positive roots $\beta_k (1 \leq k \leq \ell')$ such that $\lambda = s_{\beta_{\ell'}} \cdots s_{\beta_1} \mu$ and $(\beta_k, s_{\beta_{k-1}} \cdots s_{\beta_1} \mu) \geq 0$ for $1 \leq k \leq \ell'$. Here, $s_{\beta} (\lambda) = \lambda - (\beta^\vee, \lambda) \beta$ with $\beta^\vee = \frac{2}{(\beta, \beta)} \beta$.

Let $\lambda, \mu \in W\Lambda$ such that $\lambda \leq \mu$. The module $M(\lambda, \mu)$, called the determinantial module, is defined as follows. Choose $w, v \in W$ such that $\lambda = w\Lambda$ and $\mu = v\Lambda$, and then take their reduced
expressions \( w = s_{i_1} \cdots s_{i_l} \) and \( v = s_{j_1} \cdots s_{j_t} \), and set \( m_k = \langle h_{j_k}, s_{i_k+1} \cdots s_{i_l} \Lambda \rangle \) for \( k = 1, \ldots, l \), and \( n_k = \langle h_{j_k}, s_{j_k+1} \cdots s_{j_t} \Lambda \rangle \) for \( k = 1, \ldots, t \). We define
\[
M(\lambda, \Lambda) := F_{i_1}(m_1) \cdots F_{i_l}(m_l) k, \\
M(\lambda, \mu) := E^{*}((n_1) \cdots E^{*}(n_t) M(\lambda, \Lambda).
\]

The determinantal module \( M(\lambda, \mu) \) does not depend on the choice of \( w, v \) and their reduced expressions.

We summarize properties of determinantal modules.

**Proposition 1.12** [4, 11, Lemma 1.7, Proposition 4.2], and [13, Theorem 3.26]. Let \( \Lambda \in \mathcal{P}_+ \), and \( \lambda, \mu \in \mathcal{W}_\Lambda \) with \( \lambda \leq \mu \).

(i) \( M(\lambda, \mu) \) is a real simple \( R^\Lambda \)-module with an affinization.
(ii) If \( \langle h_1, \lambda \rangle \leq 0 \) and \( s_i \lambda \leq \mu \), then
\[
\varepsilon_i(M(\lambda, \mu)) = -\langle h_1, \lambda \rangle \quad \text{and} \quad E^{(-\langle h_1, \lambda \rangle)} M(\lambda, \mu) \simeq M(s_i \lambda, \mu).
\]
(iii) If \( \langle h_1, \mu \rangle \geq 0 \) and \( \lambda \leq s_i \mu \), then
\[
\varepsilon_i^*(M(\lambda, \mu)) = \langle h_1, \mu \rangle \quad \text{and} \quad E^{*(\langle h_1, \mu \rangle)} M(\lambda, \mu) \simeq M(\lambda, s_i \mu).
\]
(iv) For \( \Lambda, \Lambda' \in \mathcal{P}_+ \) and \( w, v \in \mathcal{W} \) such that \( v \leq w \), we have
\[
M(w\Lambda, v\Lambda) \circ M(w\Lambda', v\Lambda') \simeq q^{-(v \Lambda, v \Lambda'-w \Lambda')} M(w(\Lambda + \Lambda'), v(\Lambda + \Lambda')).
\]
(v) For \( \lambda, \mu, \zeta \in \mathcal{W}_\Lambda \) with \( \lambda \leq \mu \leq \zeta \), we have \( M(\lambda, \zeta) \simeq M(\lambda, \mu) \triangleright M(\mu, \zeta) \).

### 1.5 | The categories \( \mathcal{C}_w \) and \( \mathcal{C}_{*, v} \)

In this subsection, we recall the definition of categories \( \mathcal{C}_w \), \( \mathcal{C}_{*, v} \), and \( \mathcal{C}_{w, v} \) appeared in [11] (see also [26]).

For \( M \in \mathcal{R}(\beta)-\text{Mod} \), we define
\[
W(M) := \{ \gamma \in Q_+ \cap (\beta - Q_+) \mid \varepsilon(\gamma, \beta - \gamma)M \neq 0 \}, \\
W^*(M) := \{ \gamma \in Q_+ \cap (\beta - Q_+) \mid \varepsilon(\beta - \gamma, \gamma)M \neq 0 \}.
\]
Hence, if \( M = 0 \), then \( W(M) = \emptyset \), and if \( M \neq 0 \), then \( 0, \beta \in W(M) \).

**Proposition 1.13** [26, Proposition 3.7]. For any \( \mathcal{R} \)-module \( M \), we have
\[
W(M) \subseteq \text{span}_{\mathbb{R}_{\geq 0}} (W(M) \cap \Delta_+).
\]
Here, for a subset \( A \subseteq \mathbb{R} \otimes \mathbb{Z} Q \), we denote by \( \text{span}_{\mathbb{R}_{\geq 0}} (A) \) the subset of \( \mathbb{R} \otimes \mathbb{Z} Q \) consisting of linear combinations of elements in \( A \cup \{0\} \) with nonnegative coefficients.
For $w \in W$, we denote by $\mathcal{C}_w$ the full subcategory of $R$-gmod consisting of objects $M$ such that

$$W(M) \subset \text{span}_{\mathbb{R}_{\geq 0}}(\Delta_+ \cap w\Delta_-).$$

By Proposition 1.13, this condition is equivalent to

$$W(M) \cap \Delta_+ \subset w\Delta_-.$$

Similarly, for $v \in W$, we denote by $\mathcal{C}_{*,v}$ the full subcategory of $R$-gmod consisting of objects $M$ such that

$$W^*(M) \subset \text{span}_{\mathbb{R}_{\geq 0}}(\Delta_+ \cap v\Delta_+).$$

For $w, v \in W$, we define $\mathcal{C}_{w,v}$ to be the full subcategory of $R$-gmod whose objects are contained in both of the subcategories $\mathcal{C}_w$ and $\mathcal{C}_{*,v}$.

The categories $\mathcal{C}_w$, $\mathcal{C}_{*,v}$, and $\mathcal{C}_{w,v}$ are stable under convolution products, grading shifts, extensions, and taking subquotients.

1.6  Graded braiders in $R$-gmod associated with a Weyl group element

**Definition 1.14.** A graded braider $(M, R_M, \phi)$ in $R$-gmod is called nondegenerate if

$$R_M(L(i)) : M \circ L(i) \to q^{\phi(\alpha_i)} L(i) \circ M$$

is a nonzero homomorphism for each $i \in I$.

If $(M, R_M, \phi)$ is nondegenerate, then $\phi(\alpha_i) = -\Lambda(M, L(i))$ and $R_M(L(i)) = c r_{M,L(i)}$ for some $c \in k^\times$.

**Theorem 1.15** [13, Proposition 4.1, Lemma 4.3]. Let $R$ be a quiver Hecke algebra and let $M$ be a simple $R$-module. Then there exists a nondegenerate graded braider $(M, R_M, \phi)$ in $R$-gmod. If $(M, R'_M, \phi')$ is another nondegenerate graded braider, then $\phi = \phi'$ and there exists a group homomorphism $c : Q \to k^\times$ such that $R'_M(X) = c(\beta)R_M(X)$ for any $X \in R(\beta)$-gmod.

Fix an element $w$ in the Weyl group $W$. For $\Lambda \in P_+$, set

$$\mathcal{C}_\Lambda = \mathcal{C}_{w, \Lambda} := \mathcal{C}_w(\Lambda, \Lambda) \quad \text{and} \quad \mathcal{C}_i := \mathcal{C}_{\Lambda_i} \quad (i \in I).$$

For $i \in I$, we set

$$\lambda_i := \begin{cases} w\Lambda_i + \Lambda_i & \text{if } w\Lambda_i \neq \Lambda_i, \\ 0 & \text{if } w\Lambda_i = \Lambda_i. \end{cases}$$

Note that $\mathcal{C}_i \cong 1$ if and only if $w\Lambda_i = \Lambda_i$. We have (see [11, Corollary 3.8])

$$\Lambda(C_i, L(j)) = (\alpha_j, \alpha_j) \epsilon^*(C_i) + (\alpha_j, w\Lambda_i - \Lambda_i) = (\lambda_i, \alpha_j). \quad (1.12)$$

Applying Theorem 1.15, we have a nondegenerate graded braider $(C_i, R_{C_i}, \phi_i)$ for $i \in I$. 
Proposition 1.16 [13, Proposition 5.1]. The family \((C_i, R_{C_i}, \phi_i)_{i \in I}\) is a real commuting family of nondegenerate graded braiders in \(R\)-gmod and

\[ \phi_i(\beta) = -((\lambda_i, \beta)) \quad \text{for any } \beta \in Q. \]

Theorem 1.17 [13, Theorem 5.2]. For \(i \in I\) and any \(R(\beta)\)-module \(N\) in \(\mathcal{C}_w\), \(R_{C_i}(N)\) is an isomorphism.

We set \(\Gamma := \bigoplus_{i \in I} \mathbb{Z} \Lambda_i\) and define a \(\mathbb{Z}\)-bilinear map \(H : \Gamma \times \Gamma \to \mathbb{Z}\) defined by

\[ H(\Lambda_i, \Lambda_j) := -\check{\Lambda}(C_i, C_j) = (\Lambda_i, w\Lambda_j - \Lambda_j). \]

Then, for \(i, j \in I\), we have

\[ \phi_i(-\text{wt}(C_j)) = (w\Lambda_i + \Lambda_i, w\Lambda_j - \Lambda_j) = (\Lambda_i, w\Lambda_j) - (\Lambda_j, w\Lambda_i) = H(\Lambda_i, \Lambda_j) - H(\Lambda_j, \Lambda_i). \]

Thanks to Theorem 1.4 and Proposition 1.16, we have the localization of \(\mathcal{C}_w\) by the nondegenerate graded braiders \(\{C_i \mid i \in I\}\) which we denote by

\[ \widetilde{\mathcal{C}}_w := \mathcal{C}_w[C_i^{o-1} \mid i \in I]. \]

By the choice above of \(H\), for \(\Lambda = \sum_{i \in I} m_i \Lambda_i \in \Gamma_{\geq 0}\), we have

\[ C_{\Lambda} \simeq (1, \Lambda), \quad C_{\Lambda}^{o-1} \simeq q^{H(\Lambda, \Lambda)}(1, -\Lambda). \]

Thus, for \(\Lambda = \sum_{i \in I} a_i \Lambda_i \in \Gamma \subset P\), we simply write \(C_{\Lambda} := (1, \Lambda) \in \widetilde{\mathcal{C}}_w\). We have

\[ C_{\Lambda} \circ C_{\Lambda'} \simeq q^{H(\Lambda, \Lambda')}(C_{\Lambda+\Lambda'}). \]

The following is a summary of the results in [13] on the localization \(\widetilde{\mathcal{C}}_w\) of \(\mathcal{C}_w\) and the localization functor \(\Phi_w : \mathcal{C}_w \to \widetilde{\mathcal{C}}_w\):

(i) the objects \(\Phi_w(C_i)\) are invertible in \(\widetilde{\mathcal{C}}_w\),

(ii) \(\Phi_w(R_{C_{\Lambda}}(X))\) is an isomorphism for any \(\Lambda \in P_+\) and \(X \in \mathcal{C}_w\),

(iii) for any simple object \(S\) of \(\mathcal{C}_w\), the object \(\Phi_w(S)\) is simple in \(\widetilde{\mathcal{C}}_w\),

(iv) every simple object of \(\widetilde{\mathcal{C}}_w\) is isomorphic to \(C_{\Lambda} \circ \Phi_w(S)\) for some simple object \(S\) of \(\mathcal{C}_w\) and \(\Lambda \in P_+\),

(v) for two simple objects \(S\) and \(S'\) in \(\mathcal{C}_w\) and \(\Lambda, \Lambda' \in P\), \(C_{\Lambda} \circ \Phi_w(S) \simeq C_{\Lambda'} \circ \Phi_w(S')\) in \(\widetilde{\mathcal{C}}_w\) if and only if \(q^{H(\Lambda, \mu)}C_{\Lambda+\mu} \circ S \simeq q^{H(\Lambda', \mu)}C_{\Lambda'+\mu} \circ S'\) in \(\mathcal{C}_w\) for some \(\mu \in P\) such that \(\Lambda + \mu, \Lambda' + \mu \in P_+\),

(vi) the category \(\widetilde{\mathcal{C}}_w\) is abelian and every objects has finite length,

(vii) the grading shift functor \(q\) and the contravariant functor \(M \mapsto M^*\) on \(\mathcal{C}_w\) are extended to \(\widetilde{\mathcal{C}}_w\),

(viii) for any simple module \(M \in \widetilde{\mathcal{C}}_w\), there exists a unique \(n \in \mathbb{Z}\) such that \(q^n M\) is self-dual.
Theorem 1.18 [13, Theorem 5.7]. Every simple object in $\widetilde{\mathcal{C}}_w$ has a right dual.

Applying Theorem 1.4 and Proposition 1.16 again, we obtain the localization

$$(R\text{-gmod})^{\sim}[w] := R\text{-gmod}[C_i^{0,-1} | i \in I]$$

of the category $R\text{-gmod}$ by the real commuting family $\{C_i | i \in I\}$ of nondegenerated graded braiders. We denote the localization functor by

$$Q_w : R\text{-gmod} \to (R\text{-gmod})^{\sim}[w].$$

Since $\mathcal{C}_w$ is a full subcategory of $R\text{-gmod}$, there is a fully faithful monoidal functor

$$t_w : \mathcal{C}_w \to (R\text{-gmod})^{\sim}[w].$$

Set $I_w := \{i \in I | w\Lambda_i \neq \Lambda_i\}$. Note that $I_w = \{i_1, \ldots, i_l\}$ for any reduced expression $w = s_{i_1} \cdots s_{i_l}$ of $w$.

Theorem 1.19 [13, Theorem 5.8, Theorem 5.9]. Assume that $I = I_w$.

(i) The functor $t_w : \mathcal{C}_w \to (R\text{-gmod})^{\sim}[w]$ is an equivalence of categories.

(ii) The category $\mathcal{C}_w$ is left-rigid, that is, every object of $\mathcal{C}_w$ has a left dual.

We shall prove later in Theorem 3.9 that $\mathcal{C}_w$ is rigid.

Proposition 1.20. Assume that $I = I_w$. Let $X \in R\text{-gmod}$ be a simple module. Then we have

(i) $Q_w(X)$ is either a simple module or zero,

(ii) $Q_w(X) \simeq q^{H(\Lambda, \Lambda)} C_{-\Lambda} \circ Q_w(C_{\Lambda} \triangledown X)$ for any $\Lambda \in P_+$.

Proof.

(i) Follows from [13, Proposition 4.8 (i)].

(ii) Since there is an epimorphism $Q_w(C_{\Lambda}) \circ Q_w(X) \twoheadrightarrow Q_w(C_{\Lambda} \triangledown X)$, we may assume that $Q_w(X) \not\cong 0$.

Applying the exact monoidal functor $Q_w$,

$$C_{\Lambda} \circ X \Rightarrow C_{\Lambda} \triangledown X \Rightarrow X \circ C_{\Lambda},$$

we obtain

$$Q_w(C_{\Lambda}) \circ Q_w(X) \Rightarrow Q_w(C_{\Lambda} \triangledown X) \Rightarrow Q_w(X) \circ Q_w(C_{\Lambda}),$$

whose composition is an isomorphism. Hence, we obtain $Q_w(C_{\Lambda} \triangledown X) \cong Q_w(C_{\Lambda}) \circ Q_w(X)$. □
2 | NORMAL SEQUENCES AND GENERALIZED DETERMINATIONAL MODULES

2.1 | Unmixed pair

We say that an ordered pair \((M, N)\) of \(R\)-modules is unmixed if
\[
\mathcal{W}^n(M) \cap \mathcal{W}(N) \subset \{0\}.
\]

**Proposition 2.1** [26, Lemma 2.6], [11, Proposition 2.12]. Let \(\beta, \gamma \in \mathbb{Q}_+\) with \(|\beta| = m\) and \(|\gamma| = n\). Let \(M\) and \(N\) be an \(R(\beta)\)-module and an \(R(\gamma)\)-module, respectively. Assume that \((M, N)\) is an unmixed pair. Then, we have \(e(\beta, \gamma)(M \circ N) = M \boxtimes N\) and \(e(\beta, \gamma)(N \circ M) = \tau_{w[n,m]}(N \boxtimes M)\). There is an \(R(\beta) \otimes R(\gamma)\)-module isomorphism \(q^{(\beta, \gamma)} M \otimes N \rightarrow e(\beta, \gamma)(N \circ M)\) given by
\[
r(u \otimes v) = \tau_{w[n,m]}(v \otimes u) \quad \text{for any } u \in M \text{ and } v \in N.
\]
In particular, it induces a homomorphism \(r : M \circ N \rightarrow q^{(\beta, \gamma)} N \circ M\).

We denote by \(r_{M,N}\) the above morphism \(r\).

**Proposition 2.2** [26, Lemma 2.6]. Let \((M, N)\) be an unmixed pair of simple \(R\)-modules. Then we have
\[
\text{HOM}(M \circ N, N \circ M) = k r_{M,N}^*.
\]
Moreover, the image of \(r_{M,N} : M \circ N \rightarrow q^{(\beta, \gamma)} N \circ M\) is simple and isomorphic to \(M \nabla N\) and \(q^{(\beta, \gamma)} N \Delta M\).

**Proof.** The first assertion follows from (1.7). For any nonzero submodule \(S\) of \(N \circ M\), we have \(e(\beta, \gamma)S = \tau_{w[n,m]}(N \boxtimes M)\) by (1.7) and Proposition 2.1. It follows that \(N \circ M\) has a simple socle that is generated by \(\tau_{w[n,m]}(N \boxtimes M)\). Since the image of \(r_{M,N}\) is generated by \(\tau_{w[n,m]}(N \boxtimes M)\), we get the second assertion. \(\square\)

**Corollary 2.3.** Let \((M, N)\) be an unmixed pair of simple modules such that one of them is affreal. Then we have
\[
\Lambda(M, N) = 0.
\]

**Proof.** Let \(r\) be the morphism in the above proposition. Since \(r = r_{M,N}\) up to a constant multiple, we have \(\Lambda(M, N) = -(\beta, \gamma)\). It follows that \(\Lambda(M, N) = \frac{1}{2}(\Lambda(M, N) + (\beta, \gamma)) = 0\), as desired. \(\square\)

**Proposition 2.4.** Let \(\alpha, \beta, \gamma \in \mathbb{Q}_+\), and let \(L\) be an \(R(\alpha)\)-module, \(M\) an \(R(\beta)\)-module, and \(N\) an \(R(\gamma)\)-module. Assume that
\[
(W^\alpha(L) + W^\alpha(M)) \cap W(N) = \{0\}.
\]
Then, we have
\[ e(\alpha + \beta, \gamma)(L \circ M \circ N) \simeq (L \circ M) \otimes N, \] (2.1)
\[ e(\alpha + \beta, \gamma)(L \circ N \circ M) \simeq q^{-(\beta, \gamma)}(L \circ M) \otimes N, \] (2.2)
\[ e(\alpha + \beta, \gamma)(N \circ L \circ M) \simeq q^{-(\alpha + \beta, \gamma)}(L \circ M) \otimes N. \] (2.3)

Assume further that \( L, M, N \) are simple. Then we have
\[ e(\alpha + \beta, \gamma)(L \circ (M \nabla N)) \simeq (L \circ M) \otimes N, \] (2.4)
\[ e(\alpha + \beta, \gamma)((L \nabla N) \circ M) \simeq q^{-(\beta, \gamma)}(L \circ M) \otimes N. \] (2.5)

**Proof.** The isomorphisms (2.1) and (2.3) follow from Proposition 2.1 since \( W^*(L \circ M) \cap W(N) \subset \{0\} \).

Let us prove the second isomorphism (2.2). By the shuffle lemma (Proposition 1.7), the \( R(\alpha + \beta) \otimes R(\gamma) \)-module \( e(\alpha + \beta, \gamma)L \circ N \circ M \) has a filtration whose graduations are of the form
\[ G := (R(\alpha + \beta)e(\alpha_1, \beta_1, \gamma_1) \otimes R(\gamma)e(\alpha_2, \gamma_2, \beta_2)) \otimes_A (e(\alpha_1, \alpha_2)L \otimes e(\gamma_1, \gamma_2)N \otimes e(\beta_1, \beta_2)M) \]
up to grade shifts. Here,
• \( \alpha_k, \beta_k, \gamma_k \in \mathbb{Q}_+ \) (\( k = 1, 2 \)) such that \( \alpha = \alpha_1 + \alpha_2, \beta = \beta_1 + \beta_2, \gamma = \gamma_1 + \gamma_2, \alpha + \beta = \alpha_1 + \beta_1 + \gamma_1, \) and \( \gamma = \alpha_2 + \beta_2 + \gamma_2, \)
• \( A = R(\alpha_1) \otimes R(\alpha_2) \otimes R(\gamma_1) \otimes R(\gamma_2) \otimes R(\beta_1) \otimes R(\beta_2), \)
• \( A \rightarrow R(\alpha + \beta) \otimes R(\gamma) \) is given by \( a_1 \otimes a_2 \otimes c_1 \otimes c_2 \otimes b_1 \otimes b_2 \mapsto (a_1 c_1 b_1) \otimes (a_2 c_2 b_2) \).

If \( G \neq 0 \), then \( \alpha_2 \in W^*(L) \) and \( \beta_2 \in W^*(M), \gamma_1 \in W(N) \). Since \( \alpha_2 + \beta_2 = \gamma_1 \), we have \( \alpha_2 = \beta_2 = \gamma_1 = 0 \). Hence, only one \( G \) survives, and we have
\[ G = (R(\alpha + \beta)e(\alpha, \beta) \otimes R(\gamma)) \otimes_{R(\alpha) \otimes R(\gamma) \otimes R(\beta)} (L \otimes N \otimes M), \]
where \( R(\alpha) \otimes R(\gamma) \otimes R(\beta) \rightarrow R(\alpha + \beta) \otimes R(\gamma) \) is given by \( a \otimes c \otimes b \mapsto (a \cdot b) \otimes c \). Hence, we obtain \( G \simeq (L \circ M) \otimes N \). It implies the isomorphism (2.2). Note that
\[ q^{-(\beta, \gamma)}(L \circ M) \otimes N \rightarrow e(\alpha + \beta, \gamma)(L \circ N \circ M) \]
is induced by \( q^{-(\beta, \gamma)}L \otimes M \otimes N \rightarrow L \otimes (N \circ M) \).

Finally, let us show the isomorphisms (2.4) and (2.5).

We have commutative diagrams
\[ e(\alpha + \beta, \gamma)(L \circ N \circ M) \xrightarrow{\lambda} e(\alpha + \beta, \gamma)(L \circ (M \nabla N)) \xrightarrow{\lambda} e(\alpha + \beta, \gamma)q^{(\beta, \gamma)}(L \circ N \circ M) \]
\[ (L \circ M) \otimes N \xrightarrow{\sim} q^{-(\beta, \gamma)-(\beta, \gamma)}(L \circ M) \otimes N, \]
Here, the vertical arrows are isomorphisms by (2.1), (2.2), and (2.3). Hence, we obtain (2.4) and (2.5).

**Proposition 2.5.** Let $M$ be an affreal simple module, and $L$ an $R$-module. We assume that $\Lambda(M, S) = \Lambda(M, L)$ for any simple quotient $S$ of $L$. Then the head of $M \circ \text{hd}(L)$ is equal to the head of $M \circ L$.

**Proof.** Note that, for any simple quotient $S$, the following diagram commutes up to a constant multiple by [7, Proposition 3.2.8]:

\[
\begin{array}{ccc}
M \circ L & \xrightarrow{r_{M,L}} & L \circ M \\
\downarrow & & \downarrow \\
M \circ S & \xrightarrow{r_{M,S}} & S \circ M.
\end{array}
\]

In particular, we have

the composition $M \circ L \xrightarrow{r_{M,L}} L \circ M \rightarrow S \circ M$ does not vanish. (2.6)

Let $K$ be a maximal submodule of $M \circ L$. In order to see the proposition, it is enough to show that $M \circ T \subset K$ for some maximal module $T$ of $L$. Let us consider the following commutative diagram where $r$ is the $R$-matrix $r_{M,L} : M \circ L \rightarrow L \circ M$:

\[
\begin{array}{ccc}
M \circ K & \xrightarrow{r_{M,K}} & K \circ M \\
\downarrow & & \downarrow \\
M \circ M \circ L & \xrightarrow{r_{M,M \circ L}} & M \circ M \circ L & \xrightarrow{M \circ r} & M \circ L \circ M.
\end{array}
\]

Note that the composition of bottom morphisms is equal to $r_{M,M \circ L}$ up to a constant multiple by [7, Lemma 3.1.5]. Hence, we have $M \circ r(K) \subset K \circ M$. Hence, there exists a submodule $P \subset L$ such that $r(K) \subset P \circ M$ and $M \circ P \subset K$ by [6, Lemma 3.1]. In particular, we have $P \neq L$. Let us take a maximal submodule $T \subset L$ such that $P \subset T$. Since the composition $M \circ L \rightarrow L \circ M \rightarrow (L/T) \circ M$ does not vanish by (2.6), $K \subset r^{-1}(T \circ M) \neq M \circ L$. Hence, we obtain $r^{-1}(T \circ M) = K$. Hence $M \circ T \subset r^{-1}(T \circ M) = K$. □

**Corollary 2.6.** Let $N_j$ ($j = 1, \ldots, n$) be a simple module, and set $L := N_1 \circ \cdots \circ N_n$. Let $M$ be an affreal simple module. We assume that $M$ and $N_j$ commute. Then $M \circ \text{hd}(L)$ is semisimple and is equal to the head of $M \circ L$. 

Proof. By [7, Proposition 3.2.10], we have \( \Lambda(M, S) = \sum_{i=1}^{n} \Lambda(M, N_i) \) for any quotient \( S \) of \( L \). Note that every simple quotient of \( L \) commutes with \( M \). Thus \( M \circ \text{hd}(L) \) is semisimple. Then by the proposition above,
\[
M \circ \text{hd}(L) \simeq \text{hd}(M \circ \text{hd}(L)) = \text{hd}(M \circ L),
\]
as desired. \( \square \)

2.2 Normal sequences

Definition 2.7. Let \((M_1, \ldots, M_r)\) be a sequence of simple modules in \( R\)-gmod such that \( M_k \) is affreal except for possibly one \( k \). The sequence \((M_1, \ldots, M_r)\) is called a normal sequence if the composition of \( r \)-matrices
\[
 r_{M_1,\ldots,M_r} := (r_{M_{r-1},M_r}) \circ \cdots \circ (r_{M_2,M_r}) \circ \cdots \circ (r_{M_1,M_2})
\]
\[
: M_1 \circ \cdots \circ M_r \longrightarrow M_r \circ \cdots \circ M_1
\]
does not vanish.

Note that if \((M_1, \ldots, M_r)\) is a normal sequence, then \( \text{Im}(r_{M_1,\ldots,M_r}) \) is simple, and it is isomorphic to the head of \( M_1 \circ \cdots \circ M_r \) and to the socle of \( M_r \circ \cdots \circ M_1 \).

Lemma 2.8 [10, Lemma 2.7, Lemma 2.8]. Let \((L_1, \ldots, L_r)\) be a sequence of simple modules in \( R\)-gmod such that \( L_k \) are affreal except for possibly one \( k \). Then the following three conditions are equivalent.

(a) \((L_1, \ldots, L_r)\) is a normal sequence.

(b) \((L_2, \ldots, L_r)\) is a normal sequence and
\[
\Lambda(L_1, \text{hd}(L_2 \circ \cdots \circ L_r)) = \sum_{2 \leq j \leq r} \Lambda(L_1, L_j).
\]

(c) \((L_1, \ldots, L_{r-1})\) is a normal sequence and
\[
\Lambda(\text{hd}(L_1 \circ \cdots \circ L_{r-1}), L_r) = \sum_{1 \leq j \leq r-1} \Lambda(L_j, L_r).
\]

Proposition 2.9. Let \( M_j \) \((j = 1, \ldots, n)\) be an affreal module. We assume that
\[
\tilde{\Lambda}(M_j, M_k) = 0 \text{ if } 1 \leq j < k \leq n \text{ and } 3 \leq k.
\]
Then \((M_1, \ldots, M_n)\) is a normal sequence.

Remark that if \((M_1, \ldots, M_n)\) satisfies condition (2.7), then \((M_1^{m_1}, \ldots, M_n^{m_n})\) also satisfies (2.7) for any \( m_1, \ldots, m_n \in \mathbb{Z}_{\geq 0} \).

Proof. Let us show it by induction on \( n \). We may assume that \( n \geq 3 \). We have
\[
0 \leq \tilde{\Lambda}(\text{hd}(M_1 \circ \cdots \circ M_{n-1}), M_n) \leq \sum_{i=1}^{n-1} \tilde{\Lambda}(M_i, M_n) = 0,
\]
which implies

$$\Lambda(\text{hd}(M_1 \circ \cdots \circ M_{n-1}), M_n) = \sum_{i=1}^{n-1} \Lambda(M_i, M_n).$$

Now the conclusion follows from Lemma 2.8. □

For $m, n \in \mathbb{Z}_{\geq 1}$ and $v \in \mathfrak{S}_m$ and $w \in \mathfrak{S}_n$, let $v \star w$ be the element of $\mathfrak{S}_{m+n}$ defined by

$$(v \star w)(k) = \begin{cases} v(k) & \text{if } 1 \leq k \leq m, \\ w(k-m) + m & \text{if } m < k \leq m+n. \end{cases}$$

Let $\leq$ be the Bruhat order on $\mathfrak{S}_m$. For $1 \leq k < m$ and $w \in \mathfrak{S}_m$, we have

$$s_k w \leq w \text{ if and only if } w^{-1}(k) > w^{-1}(k+1).$$

Note also that

$$\text{if } u \leq w, s_k w \leq w \text{ and } u \leq s_k u, \text{ then } u \leq s_k w \text{ and } s_k u \leq w.$$  

We denote by $\mathfrak{S}_{\ell,m}$ the set of minimal length left coset representatives in $\mathfrak{S}_{\ell+m}$ with respect to the subgroup $\mathfrak{S}_\ell \times \mathfrak{S}_m$. Namely,

$$\mathfrak{S}_{\ell,m} = \{w \in \mathfrak{S}_{\ell+m} \mid w \text{ is increasing on } [1, \ell] \text{ and on } [\ell+1, \ell+m]\}.$$  

The following lemma will be used in the proof of Theorem 2.11, one of the main results of this section. In the lemma, $1_n$ denotes the unit of $\mathfrak{S}_n$.

**Lemma 2.10.** Let $\ell, m, n \in \mathbb{Z}_{>0}$, $w \in \mathfrak{S}_{m,\ell}$, $v \in \mathfrak{S}_{n,\ell}$. Set $v_0 = w[n, \ell]$. Then one has

(i) if $1 \leq k < m + \ell$ satisfies $s_k w \leq w$, then

(a) $w^{-1}(k+1) \leq m < w^{-1}(k) \leq m + \ell$,

(b) $s_k w \in \mathfrak{S}_{m,\ell}$,

(c) $(s_k w \star 1_n) \cdot (1_m \star v) \leq (w \star 1_n) \cdot (1_m \star v)$,

(ii) $\ell((w \star 1_n) \cdot (1_m \star v)) = \ell(v) + \ell(w),$

(iii) $1_m \star v \leq (w \star 1_n) \cdot (1_m \star v),$

(iv) $(w \star 1_n) \cdot (1_m \star v_0) \in \mathfrak{S}_{m+n,\ell},$

(v) If $1_m \star v_0 \leq (w \star 1_n) \cdot (1_m \star v)$, then $v = v_0$.

**Proof.**

(i) Let us set $a := w^{-1}(k+1) < b := w^{-1}(k)$. Then we have $w(b) < w(a)$. Hence, we have $a \leq m < b$. We have
\[ s_k w(i) = \begin{cases} 
  w(i) < k & \text{if } 1 \leq i < a, \\
  k & \text{if } i = a, \\
  w(i) > k + 1 & \text{if } a < i \leq m, \\
  w(i) < k & \text{if } m < i < b, \\
  k + 1 & \text{if } i = b, \\
  w(i) > k + 1 & \text{if } b < i \leq m + \ell. 
\] 

Hence, we have \( s_k w \in \mathfrak{S}_{m,\ell} \).

Now we have
\[
(1_m \star v)^{-1} \cdot (w \star 1_n)^{-1}(k) = (1_m \star v^{-1})(b) > m,
\]
\[
(1_m \star v)^{-1} \cdot (w \star 1_n)^{-1}(k + 1) = (1_m \star v^{-1})(a) = a \leq m.
\]

Hence, we have \( (s_k w \star 1_n) \cdot (1_m \star v) \leq (w \star 1_n) \cdot (1_m \star v) \).

(ii) Follows from (i) by induction on \( \ell(w) \).

(iii) Follows from (i).

(iv) For \( 1 \leq k \leq m + n + \ell \), we have
\[
(w \star 1_n) \cdot (1_m \star v_0)(k) = \begin{cases} 
  w(k) \leq m + \ell & \text{if } 1 \leq k \leq m, \\
  k + \ell > m + \ell & \text{if } m < k \leq m + n, \\
  w(k - n) & \text{if } m + n < k \leq m + n + \ell.
\end{cases}
\]

Hence, \( (w \star 1_n) \cdot (1_m \star v_0) \) is increasing on \([1, m + n]\) and \([m + n + 1, m + n + \ell]\).

(v) We shall prove it by induction on \( \ell(w) \). When \( w = 1_{m+\ell} \), it is obvious. Assume that \( \ell(w) > 0 \).

Take \( k \) such that \( 1 \leq k < m + \ell \) and \( s_k w \leq w \). Set \( x = (w \star 1_n) \cdot (1_m \star v) \). Then \( s_k x \leq x \) by (i).

On the other hand, we have, for any \( i \) such that \( 1 \leq i \leq m + \ell \),
\[
(1_m \star w[n, \ell])^{-1}(i) = (1_m \star w[\ell, n])(i)
\]
\[
= \begin{cases} 
  i & \text{if } 1 \leq i \leq m, \\
  i + n & \text{if } m < i \leq m + \ell.
\end{cases}
\]

Hence, \( (1_m \star v_0)^{-1} \) is increasing on \([1, m + \ell]\) and hence \( 1_m \star v_0 \leq s_k(1_m \star v_0) \). Since \( s_k x \leq x \) and \( 1_m \star v_0 \leq x \), we obtain
\[
1_m \star v_0 \leq s_k x = (s_k w \star 1_n) \cdot (1_m \star v).
\]

Thus, the induction hypothesis implies that \( v = v_0 \).

Theorem 2.11. Let \( L \) be an affreal simple module, and let \( M, N \) be simple modules.

(i) For any simple quotient \( S \) of \( M \circ N \), we have
\[
\overline{\Lambda}(L, M) \leq \overline{\Lambda}(L, S).
\]
(ii) For any simple quotient $S$ of $M \circ N$, we have
$$\tilde{\Lambda}(N, L) \leq \tilde{\Lambda}(S, L).$$

(iii) For any simple submodule $S$ of $M \circ N$, we have
$$\tilde{\Lambda}(L, N) \leq \tilde{\Lambda}(L, S).$$

(iv) For any simple submodule $S$ of $M \circ N$, we have
$$\tilde{\Lambda}(M, L) \leq \tilde{\Lambda}(S, L).$$

**Proof.** We shall prove only (i), since the other statements are obtained from (i) by applying
$$\psi_\alpha: (R\text{-gmod})^{\text{rev}} \xrightarrow{\sim} R\text{-gmod}$$
or the duality functor $\star$.

Let $(\tilde{L}, z)$ be an affinization of $L$. We may assume that $\deg z = 1$ (see [14, Example 2.4]).

Let $L \in R(\alpha)\text{-gmod}$, $M \in R(\beta)\text{-gmod}$, $N \in R(\gamma)\text{-gmod}$, and $\ell = |\alpha|$, $m = |\beta|$, $n = |\gamma|$. Set $w_0 = w[m, \ell] \in \mathfrak{S}_{m+\ell}$ and $v_0 = w[n, \ell] \in \mathfrak{S}_{n+\ell}$. Note that
$$R(\alpha + \beta)e(\beta, \alpha) = \sum_{\tau \in \mathfrak{S}_m} \tau \left( R(\beta) \boxtimes R(\alpha) \right).$$

(2.8)

Write
$$\varphi_{w[m, \ell]}e(\beta, \alpha) = \sum_{\tau \in \mathfrak{S}_m} \tau \left( a_{w}^{(\beta, \alpha)} \right),$$

where $a_{w}^{(\beta, \alpha)} \in R(\beta) \boxtimes R(\alpha) \subset e(\beta, \alpha)R(\beta + \alpha)e(\beta, \alpha)$. Similarly, we define $a_{v}^{(\gamma, \alpha)} \in R(\gamma) \boxtimes R(\alpha)$ for $v \in \mathfrak{S}_{n+\ell}$.

Then we have
$$R_{\tilde{L}, M}(u \boxtimes x) = \sum_{\tau \in \mathfrak{S}_m} \tau \left( a_{w}^{(\beta, \alpha)}(x \boxtimes u) \right) \quad \text{for } u \in \tilde{L} \text{ and } x \in M.$$

Note that $a_{w}^{(\beta, \alpha)}(x \boxtimes u) \in M \boxtimes \tilde{L}$.

By (2.8), we have
$$M \circ \tilde{L} = \bigoplus_{w \in \mathfrak{S}_{m+\ell}} \tau_w (M \boxtimes \tilde{L}).$$

Hence, in order to see (i), it is enough to show that
$$a_{w}^{(\beta, \alpha)}(M \boxtimes \tilde{L}) \subset z^{-2\tilde{\Lambda}(L, S) + \deg a_{w_0}^{(\beta, \alpha)}} (M \boxtimes \tilde{L})$$

(2.10)

for all $w \in \mathfrak{S}_{m+\ell}$. 
Let $\xi: R(\beta) \boxtimes R(\alpha) \rightarrow R(\alpha + \beta + \gamma)$ be the algebra homomorphism $b \boxtimes a \mapsto b \boxtimes e(\gamma) \boxtimes a$. Then, for $u \in \tilde{L}$, $x \in M$ and $y \in N$, we have

\[
R_{\tilde{L},M \circ N}(u \boxtimes x \boxtimes y) = (M \circ R_{\tilde{L},N}) \left( \sum_{w \in \mathfrak{S}_m, r} \tau_w(\beta, \alpha) (x \boxtimes y \boxtimes u) \right) \\
= \sum_{w \in \mathfrak{S}_m, r} \left( \tau_w \boxtimes e(\gamma) \right) \left( e(\beta) \boxtimes \varphi_{w[1,n,r]} \right) \xi(a_w(\beta, \alpha)) (x \boxtimes y \boxtimes u) \\
= \sum_{w \in \mathfrak{S}_m, r, v \in \mathfrak{S}_n} \left( \tau_w \boxtimes e(\gamma) \right) \left( e(\beta) \boxtimes \tau_v \right) \left( e(\beta) \boxtimes a_y(\gamma, \alpha) \right) \xi(a_w(\beta, \alpha)) (x \boxtimes y \boxtimes u).
\]

Let us denote by $h: M \otimes N \rightarrow S$ the composition $M \otimes N \rightarrow M \circ N \rightarrow S$. It is $R(\beta) \otimes R(\gamma)$-linear and injective, since $M \otimes N$ is a simple $R(\beta) \otimes R(\gamma)$-module.

Then, by Lemma 2.10 (ii), we have

\[
R_{\tilde{L},S}(u \boxtimes h(x \otimes y)) = \sum_{w \in \mathfrak{S}_m, r, v \in \mathfrak{S}_n} \tau_w(1_n)(1_m \star v) a_{w,v}(\beta, \gamma, \alpha) (h(x \otimes y) \boxtimes u) \\
= \sum_{w \in \mathfrak{S}_m, r} \tau_w(1_n)(1_m \star v_0) a_{w,v_0}(\beta, \gamma, \alpha) (h(x \otimes y) \boxtimes u) \\
+ \sum_{w \in \mathfrak{S}_m, r, \nu \in \mathfrak{S}_n, r \setminus \{v_0\}} \tau_w(1_n)(1_m \star v) a_{w,v}(\beta, \gamma, \alpha) (h(x \otimes y) \boxtimes u),
\]

where $a_{w,v}(\beta, \gamma, \alpha) = (e(\beta) \boxtimes a_y(\gamma, \alpha) \boxtimes \xi(a_w(\beta, \alpha))) \in R(\beta) \boxtimes R(\gamma) \boxtimes R(\alpha)$.

We write $R_{\tilde{L},S}^{\text{norm}} = z^{-s} R_{\tilde{L},S}$. Note that

\[
\overline{\Lambda}(L, S) = \left( \deg a_{w_0,v_0}(\beta, \gamma, \alpha) - s \right) / 2 = \left( \deg a_{w_0}(\beta, \alpha) + \deg a_{v_0}(\gamma, \alpha) - s \right) / 2.
\]

Thus, we obtain

\[
\sum_{w \in \mathfrak{S}_m, r} \tau_w(1_n)(1_m \star v_0) a_{w,v_0}(\beta, \gamma, \alpha) (h(x \otimes y) \boxtimes u) \\
+ \sum_{w \in \mathfrak{S}_m, r, \nu \in \mathfrak{S}_n, r \setminus \{v_0\}} \tau_w(1_n)(1_m \star v) a_{w,v}(\beta, \gamma, \alpha) (h(x \otimes y) \boxtimes u) \\
\in z^s(S \circ \tilde{L}).
\]

By Lemma 2.10, we have $(w \star 1_n)(1_m \star v) \neq 1_m \star v_0$ for $v \in \mathfrak{S}_n \setminus \{v_0\}$. Hence, we have

\[
\tau_w(1_n)(1_m \star v) a_{w,v}(\beta, \gamma, \alpha) (S \boxtimes \tilde{L}) \subset \tau_w(1_n)(1_m \star v) (S \boxtimes \tilde{L}) \\
\subset \sum_{g \in \mathfrak{S}_m+n, r} \tau_g(S \boxtimes \tilde{L}).
\]

On the other hand, we have

\[
S \circ \tilde{L} = \bigoplus_{g \in \mathfrak{S}_m+n, r} \tau_g(S \boxtimes \tilde{L}).
\]
By Lemma 2.10, we have \((w \star 1_n)(1_m \star v_0) \in \mathfrak{S}_{m+n+\varepsilon}^i\), and \((w \star 1_n)(1_m \star v_0) \succeq 1_m \star v_0\). Hence, we have
\[
\begin{align*}
a_{w, v_0}^{(y, \gamma, \alpha)}(h(x \otimes y) \boxtimes u) \in z^c(S \boxtimes L) \cap (h(M \otimes N) \boxtimes L) \\
= h(M \otimes N) \boxtimes z^c L \quad \text{for any } w \in \mathfrak{S}_{m, \varepsilon}.
\end{align*}
\]
Since we have \(a_{v_0}^{(y, \gamma)}|_{N \boxtimes L} \in k^{x} z^c \text{id}_{N \boxtimes L}\) with \(c = \deg a_{v_0}^{(y, \gamma)}\) by (2.9) and [14, Lemma 2.7], we have
\[
z^c (a_w^{(\beta, \alpha)}(x \boxtimes u)) \boxtimes y \in z^c (M \boxtimes L \boxtimes N).
\]
Finally, we obtain
\[
a_w^{(\beta, \alpha)}(x \boxtimes u) \in z^{c-\varepsilon} (M \boxtimes L) = z^{-2\Lambda(L, S) + \deg a_w^{(\beta, \alpha)}} (M \boxtimes L).
\]
It is nothing but (2.10).

\[\square\]

**Corollary 2.12.** Let \(L\) be an affreal simple module, and let \(M, N\) be simple modules. Let \(S\) be a simple quotient of \(M \circ N\).

If \(\Lambda(L, N) = 0\), then we have
\[
\widehat{\Lambda}(L, S) = \widehat{\Lambda}(L, M),
\]
and hence,
\[
\Lambda(L, S) = \Lambda(L, M) + \Lambda(L, N).
\]

**Proof.** We have
\[
\widehat{\Lambda}(L, M) \leq \widehat{\Lambda}(L, S) \leq \widehat{\Lambda}(L, M \circ N) = \widehat{\Lambda}(L, M) + \widehat{\Lambda}(L, N) = \widehat{\Lambda}(L, M),
\]
where the first inequality follows form Theorem 2.11 and the second follows from [7, Proposition 3.2.10]. Hence, we have \(\widehat{\Lambda}(L, S) = \widehat{\Lambda}(L, M)\) and \(\widehat{\Lambda}(L, S) = \widehat{\Lambda}(L, M) + \widehat{\Lambda}(L, N)\), which is equivalent to \(\Lambda(L, S) = \Lambda(L, M) + \Lambda(L, N)\), as desired.

\[\square\]

**Corollary 2.13.** Let \(L\) be an affreal simple module, and let \(M, N\) be simple modules. Assume that \(M\) or \(N\) is affreal.

(i) If \(\overline{\Lambda}(L, N) = 0\), then \((L, M, N)\) is a normal sequence.

(ii) If \(\overline{\Lambda}(N, L) = 0\), then \((N, M, L)\) is a normal sequence.

**Proof.**

(i) By Corollary 2.12, we have \(\Lambda(L, M \triangleright N) = \Lambda(L, M) + \Lambda(L, N)\). Hence, \((L, M, N)\) is a normal sequence by Lemma 2.8.

(ii) A similar proof to (i) works for (ii).

\[\square\]
2.3 | Head simplicity of convolutions with $L(i)$

**Definition 2.14.** For $i \in I, \beta \in \mathbb{Q}_+$ and a simple $R(\beta)$-module $M$, define

$$d_i(M) := \varepsilon_i(M) + \varepsilon^*_i(M) + \langle h_i, \text{wt}(M) \rangle.$$  

Recall the following lemma.

**Lemma 2.15** [11, Corollary 3.8]. For $i \in I, \beta \in \mathbb{Q}_+$ and a simple module $R(\beta)$-module $M$, we have

$$\tilde{\Lambda}(L(i), M) = \frac{\alpha_i}{2} \varepsilon_i(M),$$

$$\tilde{\Lambda}(M, L(i)) = \frac{\alpha_i}{2} \varepsilon^*_i(M),$$

$$b(L(i), M) = \frac{\alpha_i}{2} d_i(M).$$

For an $R(\beta)$-module $M$ and $i \in I$, set

$$\text{wt}_i(M) := \langle h_i, \text{wt}(M) \rangle = -\langle h_i, \beta \rangle.$$  

**Proposition 2.16** cf. [20]. Let $i \in I$ and let $M$ be a simple module.

(i) If $d_i(M) = 0$, then we have $L(i) \nabla M \simeq L(i) \circ M \simeq M \circ L(i) \simeq M \nabla L(i)$ (up to a grading shift) and $d_i(L(i) \circ M) = 0$.

If $d_i(M) > 0$, then we have

$$d_i(L(i) \nabla M) = d_i(M \nabla L(i)) = d_i(M) - 1,$$

$$\varepsilon_i(M \nabla L(i)) = \varepsilon_i(M), \quad \varepsilon^*_i(L(i) \nabla M) = \varepsilon^*_i(M).$$

(ii) We have

$$d_i(L(i^n) \nabla M) = \max(d_i(M) - n, 0),$$

$$d_i(M \nabla L(i^n)) = \max(d_i(M) - n, 0). \tag{2.12}$$

(iii) We have

$$\varepsilon_i(M \nabla L(i^n)) = \max(\varepsilon_i(M), \; n - \text{wt}_i(M) - \varepsilon^*_i(M)),$$

$$\varepsilon^*_i(L(i^n) \nabla M) = \max(\varepsilon^*_i(M), \; n - \text{wt}_i(M) - \varepsilon_i(M)).$$

**Proof.**

(i) The first statement follows from [7, Lemma 3.2.3] and [13, Corollary 3.18].

Assume that $d_i(M) > 0$. By [13, Corollary 3.18], we have $d_i(M \nabla L(i)) < d_i(M)$. On the other hand, we have

$$\varepsilon_i(M \nabla L(i)) \geq \varepsilon_i(M),$$

and

$$\varepsilon^*_i(M \nabla L(i)) + \text{wt}_i(M \nabla L(i)) = \varepsilon^*_i(M) + 1 + \text{wt}_i(M) - 2 = \varepsilon^*_i(M) + \text{wt}_i(M) - 1.$$
Hence, we obtain \( d_i(M \vee L(i)) \geq d_i(M) - 1 \). It follows that \( d_i(M \vee L(i)) = d_i(M) - 1 \) and \( \varepsilon_i(M \vee L(i)) = \varepsilon_i(M) \). For the statements for \( L(i) \vee M \) can be similarly proved.

(ii) Follows from (i).

(iii) By (ii), we have

\[
\max(d_i(M) - n, 0) = d_i(M \vee L(i^n)) = \varepsilon_i(M \vee L(i^n)) = \varepsilon_i^*(M) + n = wt_i(M) - 2n.
\]

Hence, we obtain

\[
\varepsilon_i(M \vee L(i^n)) = \max(\varepsilon_i(M) + \varepsilon_i^*(M) + wt_i(M) - n, 0) - \varepsilon_i^*(M) - wt_i(M) + n
\]

\[
= \max(\varepsilon_i(M), -\varepsilon_i^*(M) - wt_i(M) + n).
\]

The statement for \( \varepsilon_i^* \) is similarly proved.

\[\Box\]

**Theorem 2.17.** Let \( i \in I \) and let \( M \) be a simple module. Assume that \( a, b \in \mathbb{Z}_{\geq 0} \) satisfy

\[
b_i(M) \geq a + b.
\]

Then \( L(i^a) \circ M \circ L(b^i) \) has a simple head and a simple socle. In particular, we have

\[
\tilde{F}_i^a(\tilde{F}_i^*)^b M \simeq (\tilde{F}_i^*)^b \tilde{F}_i^a M.
\]

**Proof.** Assume that \( M \) is an \( R(\beta) \)-module with \( n = |\beta| \), and set \( L_1 = L(i)^{\circ a} \) and \( L_2 = L(i)^{\circ b} \)

(i) First assume that \( \varepsilon_i(M) = \varepsilon_i^*(M) = 0 \). We shall show that

the graded \((R(ax_i) \mathcal{D} R(\beta) \mathcal{D} R(b \alpha_i))\)-module \( L_1 \mathcal{D} M \mathcal{D} L_2 \) appears only once in \( e(i^a, \beta, i^b)(L_1 \circ M \circ L_2) \) as a composition factor (including the grading).

Note that we have

\[
e(i^a, \beta, i^b)(L_1 \circ M \circ L_2) = \bigoplus_{w \in \mathcal{G}_{a+n+b}^i} e(i^a, \beta, i^b) \tau_w e(\nu)(L(i)^{\mathcal{D} a} \mathcal{D} M \mathcal{D} L(i)^{\mathcal{D} b}).
\]

Here, \( \mathcal{G}_{a+n+b}^i \) is the set of \( w \in \mathcal{G}_{a+n+b} \) such that \( w \mid_{[a+1, a+n]} \) is increasing. Since \( \varepsilon_i(M) = 0 \) and \( \varepsilon_i^*(M) = 0 \), we may assume that \( \nu_{a+1} \neq i \) and \( \nu_{a+n} \neq i \). We may assume \( \nu_{a+1} \neq i \) for \( k \in [1, a] \cup [a + n + 1, a + n + b] \). Hence, we have

\[
a + 1 \leq w(a + 1) \quad \text{and} \quad w(a + n) \leq a + n.
\]

Hence, we have \( w \mid_{[a+1, a+n]} = \text{id}_{[a+1, a+n]} \). Thus, we obtain

\[
e(i^a, \beta, i^b)(L_1 \circ M \circ L_2) = \sum_{\nu \in \mathcal{G}_{a+n+b}^i} (R(ax_i) \mathcal{D} e(\beta) \mathcal{D} R(b \alpha_i)) \tau_{\nu} (L(i)^{\mathcal{D} a} \mathcal{D} M \mathcal{D} L(i)^{\mathcal{D} b}).
\]

Here \( \mathcal{G}_{a+n+b}^i \) is the set of \( \nu \in \mathcal{G}_{a+n+b} \) such that \( \nu^{-1}[1, a] \) and \( \nu^{-1}_{[a+n+1, a+n+b]} \) are increasing and \( \nu \mid_{[a+1, a+n]} = \text{id}_{[a+1, a+n]} \). The above gives an \( R(ax_i) \mathcal{D} R(\beta) \mathcal{D} R(b \alpha_i) \)-submodule filtration of \( e(i^a, \beta, i^b)(L_1 \circ M \circ L_2) \) that is compatible with \( \leq \) on \( \mathcal{G}_{a+n+b}^i \).
More precisely, we have the following equality in the Grothendieck group of \( (R(a_{\alpha_i}) \otimes R(\beta) \otimes R(b\alpha_i)) \)-gmod:

\[
[e(i^a, \beta, i^b)(L_1 \circ M \circ L_2)] = \sum_{v \in \mathcal{S}_{a+n+b}} q^{N(v)}[L_1 \otimes M \otimes L_2].
\]

Here, \( N(v) \) is the homogeneous degree of \( e(i^a, \beta, i^b)\tau_v \).

For \( v \in \mathcal{S}_{a+n+b}' \), let \( k \) be the number of \( s \in [1, a] \) such that \( v^{-1}(s) \in [a + n + 1, a + n + b] \). Then \( k \) is also equal to the number of \( t \in [a + n + 1, a + n + b] \) such that \( v^{-1}(t) \in [1, a] \). Then, \( 0 \leq k \leq \min(a, b) \) and we have

\[
[1, a] \cap v([1, a]) = [1, a - k],
\]

\[
[1, a] \cap v([a + n + 1, a + n + b]) = [a - k + 1, a],
\]

\[
[a + n + 1, a + n + b] \cap v([1, a]) = [a + n + 1, a + n + k],
\]

\[
[a + n + 1, a + n + b] \cap v([a + n + 1, a + n + b]) = [a + n + k + 1, a + n + b].
\]

Then, the homogeneous degree \( N(v) \) of \( e(i^a, \beta, i^b)\tau_v \) is equal to

\[
N(v) := -2k(\alpha_i, \beta) - (\alpha_i, \alpha_i)\#A(v).
\]

Here,

\[
A(v) := \{(s, t) \mid s \in [1, a], \ t \in [a + n + 1, a + n + b], \ v^{-1}(s) > v^{-1}(t)\}
\]

with

\[
A_1 = \{(s, t) \mid s \in [a - k + 1, a], \ t \in [a + n + k + 1, a + n + b], \ v^{-1}(s) > v^{-1}(t)\},
\]

\[
A_2 = \{(s, t) \mid s \in [1, a - k], \ t \in [a + n + 1, a + n + k], \ v^{-1}(s) > v^{-1}(t)\},
\]

\[
A_3 = \{(s, t) \mid s \in [a - k + 1, a], \ t \in [a + n + 1, a + n + k]\}.
\]

Hence, one has

\[
\#A(v) = \#A_1 + \#A_2 + \#A_3 \leq k(b - k) + k(a - k) + k^2 = k(a + b) - k^2.
\]

Since \(-\alpha_i, \beta) = \frac{(\alpha_i, \alpha_i)}{2} \delta_i(M)\), we obtain

\[
N(v) = (\alpha_i, \alpha_i)(k\delta_i(M) - \#A(v)) \geq (\alpha_i, \alpha_i)\left(k(a + b) - (k(a + b) - k^2)\right) = (\alpha_i, \alpha_i)k^2.
\]

Hence, \( N(v) = 0 \) implies \( k = 0 \) which is equivalent to \( v = \text{id} \). Thus, we obtain (2.13).
Any $R$-submodule $K$ of $L_1 \circ M \circ L_2$ is a proper submodule if and only if $L_1 \boxtimes M \boxtimes L_2$ does not appear in the $(R(a_\alpha) \boxtimes R(\beta) \boxtimes R(b_\alpha))$-module $e(i^a, \beta, i^b)K$ as a composition factor. Since the last property is stable by taking sums, a proper maximal submodule of $L_1 \circ M \circ L_2$ is unique, and hence, $L(i^a) \circ M \circ L(i^b)$ has a simple head. By duality, $L(i^a) \circ M \circ L(i^b)$ has a simple socle.

(ii) General case. We may assume that $a + b > 0$ and hence $\lambda_i(M) > 0$. Set $b' = \varepsilon^*(i^b M)$ and $a' = \varepsilon_i(E_1^a E_i^{b'}(M))$. Then, $\varepsilon_i(M_0) = \varepsilon^*_i(M_0) = 0$ and we have $M \simeq (L(i^a') \vee M_0) \vee L(i^b)$. Then, (2.12) implies that

$$d_i(L(i^a') \vee M_0) = \max(d_i(M_0) - a', 0),$$

$$d_i(M) = \max(d_i(L(i^a') \vee M_0) - b', 0)$$

$$= \max(d_i(M_0) - a' - b', 0).$$

Hence, we obtain $d_i(M_0) \geq a + a' + b + b'$. Therefore, (i) implies that the convolution $L(i^a) \circ M \circ L(i^b)$ has a simple head. Hence, $L(i^a) \circ M \circ L(i^b)$ has a simple head. □

2.4 | Generalization of determinantal modules

Recall that $\lambda \in \mathcal{P}$ is $w$-dominant if $(\beta, \lambda) \geq 0$ for any $\beta \in \Delta^+ \cap w^{-1}\Delta^-$. In this case, we have $\lambda - w\lambda \in \mathbb{Q}_+^+$.

**Theorem 2.18.** Let $w \in W$ and let $\lambda \in \mathcal{P}$. Assume that $\lambda$ is $w$-dominant. Then there exists a self-dual simple $R(\lambda - w\lambda)$-module $M_w(\lambda, \lambda)$, which satisfies the following conditions.

(a) If $i \in I$ satisfies $\langle h_i, w\lambda \rangle \geq 0$, then $\varepsilon_i(M_w(\lambda, \lambda)) = 0$.

(b) If $i \in I$ satisfies $\langle h_i, \lambda \rangle \leq 0$, then $\varepsilon^*_i(M_w(\lambda, \lambda)) = 0$.

(c) If $i \in I$ satisfies $s_iw < w$, then $M_{\lambda, w}(\lambda, \lambda) \simeq L(i^m) \vee M_{\lambda, w}(s_iw\lambda, \lambda)$ where $m = \langle h_i, s_iw\lambda \rangle = \varepsilon_i(M_{\lambda, w}(\lambda, \lambda)) \in \mathbb{Z}_{\geq 0}$.

(d) If $i \in I$ satisfies $ws_i < w$, then $M_{\lambda, w}(\lambda, \lambda) \simeq M_{\lambda, w}(w\lambda, s_i\lambda) \vee L(i^m)$ where $m = \langle h_i, \lambda \rangle = \varepsilon^*_i(M_{\lambda, w}(\lambda, \lambda)) \in \mathbb{Z}_{\geq 0}$.

(e) For any $\mu \in \mathcal{P}_+$ such that $\lambda + \mu \in \mathcal{P}_+$, we have

$$M(\lambda + \mu) \vee M_w(\lambda, \lambda) \simeq M(\lambda, \mu) \vee \lambda + \mu)$$

up to a grading shift.

Moreover, such an $M_w(\lambda, \lambda)$ is unique up to an isomorphism.

**Proof.** The uniqueness of $M_w(\lambda, \lambda)$ follows from (e) together with [6, Corollary 3.7] and [13, Theorem 3.26].

Let us show the existence of $M_w(\lambda, \lambda)$ satisfying (a)–(e) for a $w$-dominant $\lambda$ by induction on $\ell(w)$. Assume that $\ell(w) > 0$.

Take $i \in I$ such that $w' := s_iw < w$. Then there exists $M_{w'}(w', \lambda)$ satisfying (a)–(e) with $w'$ instead of $w$, since $\lambda$ is $w'$-dominant. By (a), we have $\varepsilon_i(M_{w'}(w', \lambda)) = 0$. Set $m = \langle h_i, w'\lambda \rangle \geq 0$ and $M_{w}(\lambda, \lambda) := L(i^m) \vee M_{w'}(w', \lambda)$. Then, $M_w(\lambda, \lambda)$ is self-dual by [7, Lemma 3.1.4], since $\lambda(L(i^m), M_{w'}(w', \lambda)) = 0$ by Corollary 2.3.
Let us take $\mu \in P_+$ such that $\eta := \lambda + \mu \in P_+$. Then, we have
\[ M(w', \mu) \vee M_w(w', \lambda, \lambda) \simeq M(w', \eta, \eta) \]
up to a grading shift by (e) for $w'$. Set $n = \langle h_i, w' \mu \rangle \geq 0$. Note that $n$ is nonnegative, since any dominant weight is $w$-dominant. Then $m + n = \langle h_i, w' \eta \rangle$. Since $\varepsilon_i(M_{w'}(w', \lambda, \lambda)) = 0$ by (a) for $w'$, the triple $(L(i^{m+n}), M(w', \mu, \mu), M_w(w', \lambda, \lambda))$ is a normal sequence by Corollaries 2.13 and 2.3. Hence, we conclude that the convolution $L(i^{m+n}) \circ M(w', \mu, \mu) \circ M_w(w', \lambda, \lambda)$ has a simple head.

We have epimorphisms
\[ L(i^{m+n}) \circ M(w', \mu, \mu) \circ M_w(w', \lambda, \lambda) \]
\[ \simeq L(i^m) \circ L(i^n) \circ M(w', \mu, \mu) \circ M_w(w', \lambda, \lambda) \]
\[ \rightarrow L(i^m) \circ M(w, \mu, \mu) \circ M_w(w', \lambda, \lambda) \]
\[ \simeq M(w, \mu, \mu) \circ L(i^n) \circ M_w(w', \lambda, \lambda) \rightarrow M(w, \mu, \mu) \circ M_w(w, \lambda, \lambda) \]
\[ \rightarrow M(w, \mu, \mu) \vee M_w(w, \lambda, \lambda), \]
where $\simeq$ follows from [11, Lemma 4.9] together with [13, Theorem 3.26].

On the other hand, by (e) for $w'$, we have
\[ L(i^{m+n}) \circ M(w', \mu, \mu) \circ M_w(w', \lambda, \lambda) \]
\[ \rightarrow L(i^{m+n}) \circ M(w', \eta, \eta) \rightarrow M(w, \eta, \eta). \]

Hence, we obtain $M(w, \mu, \mu) \vee M_w(w, \lambda, \lambda) \simeq M(w, \eta, \eta)$.

Thus, we obtain (e). Since $M_w(w, \lambda, \lambda)$ satisfying (e) is unique, $M_w(w, \lambda, \lambda)$ satisfies (c).

Let us show (b). Let us take $j \in I$ such that $w' := s_j w < w$. Set $m = \langle h_j, w' \lambda \rangle \geq 0$. Then, by (c), we have
\[ M_w(w, \lambda, \lambda) \simeq L(j^m) \vee M_w(w', \lambda, \lambda). \]

Hence, if $i \neq j$, then we have $\varepsilon^*_i(M_w(w, \lambda, \lambda)) = \varepsilon^*_i(M_w(w', \lambda, \lambda)) = 0$, where the last equality follows from (b) for $w'$.

Assume that $i = j$. Since $\varepsilon_i(M_w(w', \lambda, \lambda)) = 0$ by (a) for $w'$, Proposition 2.16 implies that
\[ \varepsilon^*_i(M_w(w, \lambda, \lambda)) = \varepsilon^*_i(L(i^m) \vee M_w(w', \lambda, \lambda)) \]
\[ = \max(\varepsilon^*_i(M_w(w', \lambda, \lambda)), m - \text{wt}_i(M_w(w', \lambda, \lambda))) \]
\[ = \max(0, m - \langle h_i, w' \lambda - \lambda \rangle) = \max(0, \langle h_i, \lambda \rangle) = 0. \]

Let us show (d). It is trivial for $\varepsilon(w) \leq 1$. Hence, we assume that $\varepsilon(w) > 1$. Let us take $j \in I$ such that $s_j w < ws_j$. Set $w' = s_j w$ and $n = \langle h_j, w' \lambda \rangle \geq 0$. Then, we have
\[ M_w(w, \lambda, \lambda) \simeq L(j^n) \vee M_w(w, \lambda, \lambda) \quad \text{and} \]
\[ M_{ws_j}(w, s_i \lambda) \simeq L(j^n) \vee M_{ws_j}(w, s_i \lambda), \]
where the second isomorphism follows from (c) for the pair $ws_j$ and $s_i \lambda$. 
By the induction hypothesis, we have

\[ M_w'(w', \lambda) \simeq M_{w's_j}(w', s_j \lambda) \cup L(i^m), \]

where \( m = \langle h_i, \lambda \rangle \). Let us show that \( L(j^n) \circ M_{w's_j}(w', s_j \lambda) \circ L(i^m) \) has a simple head.

If \( i = j \), Theorem 2.17 implies that \( L(j^n) \circ M_{w's_j}(w', s_j \lambda) \circ L(i^m) \) has a simple head, since we have

\[
e_i(M_{w's_j}(w' \lambda, s_j \lambda)) = e_i(M_{w's_j}(w' \lambda, s_j \lambda)) + e^*_{i}(M_{w's_j}(w' \lambda, s_j \lambda)) + w t_i(M_{w's_j}(w' \lambda, s_j \lambda))
\]

where the second equality follows from (a) and (b) for \( w's_j \).

If \( i \neq j \), \( L(j^n) \circ M_{w's_j}(w', s_j \lambda) \circ L(i^m) \) has a simple head by Corollary 2.13. Hence, we conclude that \( L(j^n) \circ M_{w's_j}(w', s_j \lambda) \circ L(i^m) \) has a simple head in any case. Hence, we have

\[
M_w(w \lambda, \lambda) \simeq L(j^n) \cup M_{w'}(w' \lambda, \lambda) \simeq \text{hd}(L(j^n) \circ M_{w's_j}(w' \lambda, s_j \lambda) \circ L(i^m))
\]

Thus, we obtain (d).

Finally, let us show (a). Let us take \( j \in I \) such that \( w' := s_j w < w \). Set \( m = \langle h_j, \lambda \rangle \). Then, by (d), we have

\[
M_w(w \lambda, \lambda) \simeq M_{w's_j}(w, s_j \lambda) \cup L(j^m).
\]

By the induction hypothesis, \( e_i(M_{w's_j}(w, s_j \lambda)) = 0 \). Hence, if \( i \neq j \), then \( e_i(M_{w}(w \lambda, \lambda)) = 0 \). If \( i = j \), then \( e^*_{i}(M_{w's_j}(w, s_j \lambda)) = 0 \) by (b), and hence, Proposition 2.16 implies

\[
e_i(M_{w}(w \lambda, \lambda)) = e_i(M_{w's_j}(w, s_j \lambda) \cup L(i^m))
\]

\[
= \max(e_i(M_{w's_j}(w, s_j \lambda)), m - \langle h_i, \text{wt}(M_{w's_j}(w, s_j \lambda)) \rangle)
\]

\[
= \max(0, m - \langle h_i, w \lambda - s_j \lambda \rangle) = \max(0, -\langle h_i, w \lambda \rangle) = 0,
\]

as desired. \( \square \)

**Lemma 2.19.** Let \( w \in W \) and let \( \lambda, \mu \in \mathcal{P} \) be \( w \)-dominant weights. We assume that \( M_w(w \lambda, \lambda) \) and \( M_w(w \mu, \mu) \) strongly commute. Then we have

\[ M_w(w \lambda + \mu, \lambda + \mu) \simeq M_w(w \lambda, \lambda) \circ M_w(w \mu, \mu) \]

up to a grading shift.

**Proof.** Let us argue by induction on \( \mathcal{C}(w) \). If \( \mathcal{C}(w) > 0 \), take \( i \in I \) such that \( w' := s_i w < w \). Set \( m = \langle h_i, w' \lambda \rangle \) and \( n = \langle h_i, w' \mu \rangle \). Then \( m = e_i(M_w(w \lambda, \lambda)) \) and \( n = M_w(w \mu, \mu) \) by Theorem 2.18 (c), and hence, \( M_{w'}(w' \lambda, \lambda) \) commutes with \( M_{w'}(w' \mu, \mu) \) by [11, Lemma 3.1]. Hence, we have

\[ E_i^{(m+n)}(M_w(w \lambda, \lambda) \circ M_w(w \mu, \mu)) \simeq M_{w'}(w' \lambda, \lambda) \circ M_{w'}(w' \mu, \mu) \]

\[ \simeq M_{w'}(w' \lambda + \mu, \lambda + \mu) \]
by the induction hypothesis, which implies that
\[
M_w(w\lambda, \lambda) \circ M_w(w\mu, \mu) \simeq L(i^{m+n}) \vee M_w'(w'(\lambda + \mu), \lambda + \mu)
\]
\[
\simeq M_w(w(\lambda + \mu), \lambda + \mu)
\]
by Theorem 2.18 (c).

\[\square\]

**Remark 2.20.** For \(w\)-dominant \(\lambda, \mu \in P\), \(M_w(w\lambda, \lambda)\) and \(M_w(w\mu, \mu)\) do not commute in general. For example, when \(g = A_2\), \(I = \{1, 2\}\), \(w = s_1s_2\), \(\lambda = \Lambda_1\), \(\mu = s_1\Lambda_1\), \(M_w(w\lambda, \lambda) \simeq L(1)\) and \(M_w(w\mu, \mu) \simeq L(2)\) do not commute.

Conjecturally, \(M_w(w\lambda, \lambda)\) and \(M_w(w\mu, \mu)\) commute if \(\lambda\) and \(\mu\) are in the same Weyl chamber (i.e., \((\beta, \lambda)(\beta, \mu) \geq 0\) for any real root \(\beta\)).

**Lemma 2.21.** Let \(w \in W\) and let \(\lambda \in P\) be a \(w\)-dominant weight. Then, for any \(\mu \in P_+\), we have
\[
M(w\mu, \mu) \vee M_w(w\lambda, \lambda) \simeq M_w(w(\lambda + \mu), \lambda + \mu)
\]
up to a grading shift.

**Proof.** Let us take \(\Lambda \in P_+\) such that \(\Lambda + \lambda \in P_+\). Since \(M(w\Lambda, \Lambda) \circ (M(w\mu, \mu) \circ M_w(w\lambda, \lambda)) \simeq M(w(\Lambda + \mu), \Lambda + \mu) \circ M_w(w\lambda, \lambda)\) has a simple head, it follows that
\[
M(w\Lambda, \Lambda) \vee (M(w\mu, \mu) \vee M_w(w\lambda, \lambda))
\]
\[
\simeq \text{hd}(M(w\Lambda, \Lambda) \circ M(w\mu, \mu) \circ M_w(w\lambda, \lambda))
\]
\[
\simeq M(w(\Lambda + \mu), \Lambda + \mu) \vee M_w(w\lambda, \lambda)
\]
\[
\simeq M(w(\Lambda + \mu + \lambda), \Lambda + \mu + \lambda)
\]
\[
\simeq M(w\Lambda, \Lambda) \vee M_w(w(\mu + \lambda), \mu + \lambda).
\]
Hence, we obtain
\[
M(w\mu, \mu) \vee M_w(w\lambda, \lambda) \simeq M_w(w(\mu + \lambda), \mu + \lambda),
\]
as desired. \[\square\]

**Remark 2.22.**

(i) If \(\lambda \in W\Lambda\) for some \(\Lambda \in P_+\), then \(M_w(w\lambda, \lambda)\) coincides with the determinantal module \(M(w\lambda, \lambda)\).

(ii) The simple module \(M_w(w\lambda, \lambda)\) may not be real. For example, for \(g = A_1^{(1)}\), \(I = \{0, 1\}\), \(\lambda = \Lambda_1 - \Lambda_0\), and \(w = s_0s_1\), the module \(M_w(w\lambda, \lambda) \simeq L(0) \vee L(1)\) is not real.

(iii) In general, the class \([M_w(w\lambda, \lambda)] \in U_q^{-}(g) \simeq K(R\text{-gmod})\) depends on the choice of \(\{Q_{i,j}(u, v)\}_{i,j \in I}\). For example, for \(g = A_1^{(1)}\), \(I = \{0, 1\}\), \(\lambda = 2(\Lambda_1 - \Lambda_0)\), and \(w = s_0s_1\), the class of the module \(M_w(w\lambda, \lambda) \simeq L(0^2) \vee L(1^2)\) depends on the choice of \(Q_{01}(u, v)\) (see [9, Example 3.3]).

(iv) Let \(\lambda, \mu \in P\). Let \(w \in W\) be an element such that \(\mu = w\lambda\) and \(\lambda\) is \(w\)-dominant. Then \(M_w(\mu, \lambda)\) does depend on the choice of \(w\) in general.
Let $\mathfrak{g} = A_2^{(1)}$, $I = \{0, 1, 2\}$, and $\lambda = \Lambda_1 + 2\Lambda_2 - 2\Lambda_0, w = s_2s_1s_0s_2s_1$, and $v = s_1s_2s_0s_1s_2$. Then $\mu = w\lambda = v\lambda = \lambda - 3(\alpha_1 + \alpha_2) - \alpha_0$. We have

$$M_w(\mu, \lambda) \simeq M_w(w\lambda_1, \lambda_1) \circ M_w(w\lambda_2, \lambda_2) \simeq \langle 2, 1, 0, 2, 1 \rangle \circ \langle 1, 2 \rangle,$$

$$M_v(\mu, \lambda) \simeq M_v(v\lambda_2, \lambda_2) \circ M_v(v\lambda_1, \lambda_1) \simeq \langle 1, 2, 0, 1, 2 \rangle \circ \langle 2, 1 \rangle,$$

where $\lambda_k = \Lambda_k - \Lambda_0$ ($k = 1, 2$). Note that for $(\nu_1, ..., \nu_n) \in I^n$ such that $(\alpha_{\nu_k}, \alpha_{\nu_{k+1}}) < 0$ and $\alpha_{\nu_k} \neq \alpha_{\nu_{k+2}}$, we denote by $(\nu_1, ..., \nu_n)$ the one-dimensional $R(\sum_{k=1}^{n} \alpha_{\nu_k})$-module such that $e(\nu_1, ..., \nu_n)(\nu_1, ..., \nu_n) = (\nu_1, ..., \nu_n)$.

(v) When $\lambda \in \mathcal{P}$ is $w$-dominant and $\mu, \lambda + \mu \in \mathcal{P}_+$, we have $M(w\mu, \mu) \ntimes M_w(\mu, \lambda) \simeq M(\mu, \lambda + \mu)$ as seen in Theorem 2.18. However, $M(w\mu, \mu)$ and $M_w(\mu, \lambda)$ may not commute in general. For example, take $\mathfrak{g} = A_1^{(1)}, w = s_0s_1, \lambda = \Lambda_1 - \Lambda_0$ as in (ii). If we take $\mu = \Lambda_0$, then, $M(w\mu, \mu) \ntimes M_w(\mu, \lambda) \simeq M(w\Lambda_1, \Lambda_1) \simeq L(0) \ntimes \langle 0, 1 \rangle \simeq L(0^2) \ntimes L(1)$ and $M_w(\mu, \lambda) \ntimes M(w\mu, \mu) \simeq \langle 0, 1, 0 \rangle$ is one-dimensional.

Recall the definition of $\psi$ (see (1.8)).

**Lemma 2.23.** Let $w \in W$ and let $\lambda \in \mathcal{P}$. Assume that $\lambda$ is $w$-dominant. Then we have an isomorphism of graded modules

$$\psi_*(M_w(w\lambda, \lambda)) \simeq M_{w^{-1}}(-\lambda, -w\lambda).$$

**Proof.** Let us argue by induction on $\ell(w)$. Take $i \in I$ such that $w' = s_iw < w$. Set $n = \langle h_i, w'\lambda \rangle \geq 0$. Then we have

$$\psi_*(M_w(w\lambda, \lambda)) \simeq \psi_*(L(i^n) \ntimes M_{w'}(w'\lambda, \lambda) \simeq \psi_*(M_{w'}(w'\lambda, \lambda)) \ntimes L(i^n)$$

$$\simeq M_{w'-1}(-\lambda, -w'\lambda) \ntimes L(i^n)$$

$$\simeq M_{w'-1}(-\lambda, -s_iw'\lambda) \simeq M_{w^{-1}}(-\lambda, -w\lambda),$$

where the first and fourth isomorphisms follow from Theorem 2.18 (c) and (d). □

**Lemma 2.24.** Let $w \in W$ and let $\lambda \in \mathcal{P}$. Assume that $\lambda$ is $w$-dominant. Then, for any $\Lambda \in \mathcal{P}_+$, we have

$$\Lambda(M(w\Lambda, \Lambda), M_w(w\lambda, \lambda)) = -(w\Lambda + \Lambda, \text{wt}(M_w(w\lambda, \lambda))),$$

$$\Lambda(M(w\Lambda, \Lambda), M_w(w\lambda, \lambda)) = - \Lambda(w\Lambda + \Lambda, \text{wt}(M_w(w\lambda, \lambda))).$$

(2.14)

**Proof.** Let us take $\mu \in \mathcal{P}_+$ such that $\eta := \lambda + \mu \in \mathcal{P}_+$. Set $M_\Lambda = M(w\Lambda, \Lambda), M_\lambda = M_w(w\lambda, \lambda), M_\mu = M(w\mu, \mu)$ and $M_\eta = M(w\eta, \eta)$. Then we have

$$M_\mu \ntimes M_\lambda \simeq M_\eta$$ up to a grading shift.

Hence, we have

$$-(w\Lambda + \Lambda, \text{wt}(M_\mu) + \text{wt}(M_\lambda)) = -(w\Lambda + \Lambda, \text{wt}(M_\eta)) = \Lambda(M_\Lambda, M_\eta),$$

$$= \Lambda(M_\Lambda, M_\mu \ntimes M_\lambda) = \Lambda(M_\Lambda, M_\mu) + \Lambda(M_\Lambda, M_\lambda),$$

$$= -(w\Lambda + \Lambda, \text{wt}(M_\mu)) + \Lambda(M_\Lambda, M_\lambda),$$

$$= -(w\Lambda + \Lambda, \text{wt}(M_\lambda)) + \Lambda(M_\Lambda, M_\lambda),$$
where the second and the last equalities follow from [13, Proposition 4.4], and the fourth equality follows from [11, Proposition 3.2.13].

The last equality in (2.14) follows from
\[ 2 \bar{\Lambda}(M(w\Lambda, \Lambda), M_w(w\lambda, \lambda)) = \Lambda(M(w\Lambda, \Lambda), M_w(w\lambda, \lambda)) + (\text{wt}(M_{\lambda}), \text{wt}(M_{\lambda})) \]
\[ = -(w\Lambda + \Lambda, \text{wt}(M_{\lambda})) + (w\Lambda - \Lambda, \text{wt}(M_{\lambda})) \]
\[ = -2(\Lambda, \text{wt}(M_{\lambda})). \]

□

3 RIGIDITY OF THE CATEGORY $\mathcal{C}_w$

3.1 Kernel of the localization functor

There exists a unique family of subsets $\{B_w(\infty)\}_{w \in W}$ of $B(\infty)$ satisfying the following properties (see [8]):

(1) $B_w(\infty) = \{u\infty\}$ if $w = 1$,
(2) if $s_i w < w$, then

\[ B_w(\infty) = \bigcup_{k \geq 0} j_k f_k i B_{s_i w}(\infty). \]

For $i \in I$ and a simple module $M$, set $\bar{E}_{i_1}^{\text{max}} M := E_{i_1}^{(n)} M$ where $n = \varepsilon_i(M)$.

Let $w \in W$ and $w = s_{i_1} s_{i_2} \cdots s_{i_l}$ a reduced expression of $w$.

For $b \in B(\infty)$, we denote by $S(b)$ the self-dual simple $R$-module corresponding to $b$.

**Proposition 3.1.** Let $M$ be a self-dual simple $R$-module. Then the following conditions are equivalent.

(a) $\bar{E}_{i_1}^{\text{max}} \cdots \bar{E}_{i_l}^{\text{max}} M \simeq 1$.

(b) There exists $(a_k)_{1 \leq k \leq l} \in (\mathbb{Z}_{\geq 0})^l$ such that

\[ \text{wt}(M) = -\sum_{k=1}^{l} a_k \alpha_{i_k} \quad \text{and} \quad e(i_1^{a_1}, \ldots, i_{l-1}^{a_{l-1}}, i_l^{a_l})M \neq 0. \]

(c) There exists $(a_k)_{1 \leq k \leq l} \in \mathbb{Z}_{\geq 0}^l$ such that $L(i_1^{a_1}) \circ \cdots \circ L(i_{l-1}^{a_{l-1}}) \circ L(i_l^{a_l}) \rightarrow M$.

(d) $M \simeq S(b)$ for some $b \in B_w(\infty)$.

**Proof.** (a) $\Rightarrow$ (b) $\Leftrightarrow$ (c) are trivial. (a) $\Leftrightarrow$ (d) follows from that

\[ B_w(\infty) = \left\{ b \in B(\infty) \mid \bar{E}_{i_1}^{\text{max}} \cdots \bar{E}_{i_l}^{\text{max}} b \simeq 1 \right\}, \]

which is [8, Proposition 3.2.5].

Assume (c). We will show (a) by induction on $l$. When $l = 0$, it is trivial. Assume that $l > 0$. Let $n = \varepsilon_i(M)$ and $M_0 = \bar{E}_{i_1}^{\text{max}} M \simeq E_{i_1}^{(n)} M$. Then there exists a nonzero homomorphism

\[ K := E_{i_1}^{(n)} \left( L(i_1^{a_1}) \circ \cdots \circ L(i_{l-1}^{a_{l-1}}) \circ L(i_l^{a_l}) \right) \xrightarrow{\phi} M_0. \]
Then, by the shuffle lemma, there exists a filtration $(F_s)_{0 \leq s \leq t}$ of $K$ such that

$$F_s/F_{s-1} \simeq L(i_1^{b_1}) \circ \cdots \circ L(i_l^{b_l})$$

for some $b_1, \ldots, b_l \geq 0$.

Take the smallest $s$ such that $\phi(F_s) \neq 0$. Then $\phi$ induces a nonzero homomorphism $F_s/F_{s-1} \to M_0$. Since $M_0$ is simple, we obtain a surjective homomorphism $L(i_1^{b_1}) \circ \cdots \circ L(i_l^{b_l}) \to M_0$ for some $b_1, \ldots, b_l \geq 0$. Since $\varepsilon_i(M_0) = 0$, we conclude that $b_1 = 0$. By the induction hypothesis, we have

$$\tilde{E}_{i_l} \tilde{E}_{i_{l-1}} \cdots \tilde{E}_{i_2} M_0 \simeq 1$$

and hence

$$\tilde{E}_{i_l} \tilde{E}_{i_{l-1}} \cdots \tilde{E}_{i_2} \tilde{E}_{i_1} M \simeq 1,$$

as desired.

**Proposition 3.2.** Let $M$ be a simple $R$-module. The followings are equivalent.

(a) $\Lambda(M(w, \lambda, M) < -(w, \lambda, \text{wt}(M))$ for some $\lambda \in P_+$.

(b) $\tilde{E}_{i_l} \cdots \tilde{E}_{i_1} M \not\simeq 1$.

Note that the first condition is equivalent to $\tilde{\Lambda}(M(w, \lambda, M) < -(\lambda, \text{wt}(M))$.

**Proof.** We will proceed by induction on $l = \ell(w)$.

If $l = 0$, then $M(w, \lambda, M) \cong 1$. Hence, $0 < -(2\lambda, \text{wt}(M))$ for some $\lambda \in P_+$ if and only if $\text{wt}(M) \neq 0$.

This is equivalent to $M \cong 1$.

Assume that $l > 0$. Set $i := i_1$, $w' := s_i w, s := (h_i, w'),$ and $M_0 := \tilde{E}^n M,$ where $n = \varepsilon_i(M)$. For $\lambda \in P_+$, we have

$$\tilde{\Lambda}(M(w, \lambda, M) = \tilde{\Lambda}(M(w, \lambda, L(i^n) \vee M_0)) = \tilde{\Lambda}(M(w, \lambda, L(i^n)) + \tilde{\Lambda}(M(w, \lambda, M_0))

= (\lambda, n\alpha_i) + \tilde{\Lambda}(L(i^n) \vee M(w', \lambda, M_0)) = (\lambda, n\alpha_i) + \tilde{\Lambda}(M(w', \lambda, M)),$$

where the second and the third equalities follow from that $b(M(w, \lambda, L(i)) = 0$ together with [7, Proposition 3.2.13], and the last follows from Corollary 2.12 together with $\tilde{\Lambda}(L(i^n), M_0) = 0$.

Hence, we obtain

$$\tilde{\Lambda}(M(w, \lambda, M) + (\lambda, \text{wt}(M)) = (\lambda, n\alpha_i) + \tilde{\Lambda}(M(w', \lambda, M_0) + (\lambda, \text{wt}(M))

= \tilde{\Lambda}(M(w', \lambda, M_0) + (\lambda, \text{wt}(M))$$

It follows that $\tilde{\Lambda}(M(w, \lambda, M) + (\lambda, \text{wt}(M)) < 0$ if and only if $\tilde{\Lambda}(M(w', \lambda, M_0) + (\lambda, \text{wt}(M)) < 0$. It is obvious that $\tilde{E}_{i_l} \cdots \tilde{E}_{i_1} M \not\simeq 1$ if and only if $\tilde{E}_{i_l} \cdots \tilde{E}_{i_1} M_0 \not\simeq 1$. Hence, the induction hypothesis implies that $\tilde{\Lambda}(M(w, \lambda, M) < -(\lambda, \text{wt}(M))$ if and only if $\tilde{E}_{i_l} \cdots \tilde{E}_{i_1} M \not\simeq 1$. □

Assume that $I = I_w := \{i_1, \ldots, i_l\}$. Recall that $Q_w : R\text{-gmod} \to (R\text{-gmod}) [w] \simeq \tilde{E}_w$ is the localization of $R\text{-gmod}$ via the real commuting family of graded braiders $\{C_i, R_{C_i}, \phi_i\}_{i \in I}$.

Then we have the following proposition.

**Proposition 3.3** (cf. [17]). Assume $I = I_w$. Let $X$ be a module in $R(\beta)\text{-gmod}$. Then the following conditions are equivalent:
(a) \( Q_w(X) \simeq 0 \),
(b) every simple subquotient \( S \) of \( X \) satisfies that \( \bar{E}_{i_1}^{\max} \cdots \bar{E}_{i_1}^{\max} S \not\simeq 1 \),
(c) every simple subquotient \( S \) of \( X \) is isomorphic to \( S(b) \) for some \( b \notin B_w(\infty) \),
(d) \( e(i_1^{a_1}, \ldots, i_l^{a_l})X = 0 \) for any \( (a_k)_{1 \leq k \leq l} \in \mathbb{Z}_{\geq 0}^l \) such that \( \beta = \sum_{k=1}^l a_k \alpha_{i_k} \).

**Proof.** For a simple module \( S \) in \( R\text{-}g\text{mod} \) and \( \lambda \in \mathbb{P}_+ \), \( R_{C_2}(S) = 0 \) if and only if \( \Lambda(C_2, S) < -(w, \lambda + \lambda, wt(S)) \) by [13, Proposition 4.4, Proposition 5.1]. Recall that by the definition of localization, the identity \( Q_w(S) \) is the limit of the morphisms \( R_{C_2}(S) \) for \( \lambda \in \mathbb{P}_+ \). Hence, \( Q_w(S) \simeq 0 \) if and only if \( R_{C_2}(S) = 0 \) for some \( \lambda \in \mathbb{P}_+ \). Thus, the desired result follows from Proposition 3.1 and Proposition 3.2. \( \square \)

**Corollary 3.4.** For any \( w \)-dominant \( \lambda \in \mathbb{P} \), we have

\[ Q_w(M_w(w\lambda, \lambda)) \not\simeq 0. \]

**Proof.** We have \( \bar{E}_{i_1}^{\max} \cdots \bar{E}_{i_1}^{\max} M_w(w\lambda, \lambda) \simeq 1 \). \( \square \)

**Remark 3.5.** Even if \( \lambda \) is \( w \)-dominant, \( M_w(w\lambda, \lambda) \) may not belong to \( \mathcal{C}_w \). For example, take \( \mathfrak{g} = A_2 \), \( w = s_1 s_2 \), and \( \lambda = s_1 \Lambda_1 \). Then, \( M_w(w\lambda, \lambda) \simeq L(2) \) does not belong to \( \mathcal{C}_w \), since \( \alpha_2 \notin \Delta_+ \cap w\Delta_- \).

**Remark 3.6.** Let us denote by \( \text{Ker} \, Q_w \) the full subcategory of \( R\text{-}g\text{mod} \) consisting of objects \( X \) satisfying \( Q_w(X) \simeq 0 \). The Grothendieck group \( K(\text{Ker} \, Q_w) \) is a two-sided ideal of \( K(R\text{-}g\text{mod}) \). Then we have the following commutative diagram.

\[
\begin{array}{ccc}
\mathcal{C}_w & \xrightarrow{\Phi_w} & \mathcal{E}_w \\
\downarrow{Q_w} & & \downarrow{i_w} \\
R\text{-}g\text{mod} & \longrightarrow & R\text{-}g\text{mod}/\text{Ker} \, Q_w \\
& & \longrightarrow (R\text{-}g\text{mod}) \sim \{w\}.
\end{array}
\]

Taking their Grothendieck groups, we have

\[
\begin{array}{ccc}
A_q(n(w)) & \longrightarrow & A_q(n(w))[D(w\Lambda_i, \Lambda_i)^{-1}; i \in I] \\
\downarrow{[I_w]} & & \downarrow{[I_w]} \\
A_q(n) & \longrightarrow & A_q(n)/I(w)[D(w\Lambda_i, \Lambda_i)^{-1}; i \in I],
\end{array}
\]

where \( I(w) \) is the ideal corresponding to \( \text{Ker} \, Q_w \), and \( D(w\Lambda, \Lambda) \) denotes the quantum unipotent minor corresponding to \( M(w\Lambda, \Lambda) \). Recall that if \( \mathfrak{g} \) is symmetric and \( k \) is of characteristic zero, then one can identify the isomorphism classes of self-dual simple modules with the elements of the upper global basis. In this case, the ideal \( I(w) \) coincides with the ideal \( (U^-_{w, q})^{\perp} \) in [18, Definition 3.37]. And the above diagram recovers [18, Theorem 4.13], which asserts that \([I_w]\) is an isomorphism.
3.2 Equivalence between $\mathcal{C}_w^*$ and $\mathcal{C}_{w^{-1}}$

For $w \in W$, let us denote by $\mathcal{C}_w^*$ the full subcategory of $R$-mod consisting of $R$-modules $M \in R$-mod such that

$$W^*(M) \subset \text{span}_{R \geq 0} (\Delta_+ \cap w \Delta_-).$$

Recall that, for a subset $S$ of $\mathbb{R} \otimes \mathbb{Z} Q$, we write $\text{span}_{\mathbb{R} \geq 0} S$ for the subset of linear combinations of elements in $S \cup \{0\}$ with nonnegative coefficients.

Set $C_\Lambda^* = \psi_*(M(w \Lambda, \Lambda)) \simeq M(-\Lambda, -w \Lambda)$ for $\Lambda \in P_+$ and $C_i^* = C_{\Lambda_i}^*$. Then $\{C_i^* \mid i \in I\}$ is a family of central objects of $\mathcal{C}_w^*$. We denote by $\mathcal{C}_w^*$ the localization $\mathcal{C}_w^* [C_i^* \otimes^{-1} \mid i \in I]$ of $\mathcal{C}_w^*$.

Then the antiautomorphism $\psi$ (see (1.8)) of $R$ induces equivalences of monoidal categories

$$\psi_* : (\mathcal{C}_w^*)^{\text{rev}} \simto \mathcal{C}_w^*,$$

$$\psi_* : (\mathcal{C}_w^*)^{\text{rev}} \simto \mathcal{C}_w^*.$$ 

Recall that for a monoidal category $\mathcal{T}$, $\mathcal{T}^{\text{rev}}$ is the category $\mathcal{T}$ endowed with the new tensor product $\otimes^{\text{rev}}$ defined by $X \otimes^{\text{rev}} Y : = Y \otimes X$.

When $I = I_w := \{i \in I \mid w \Lambda_i \neq \Lambda_i\}$, we denote by

$$Q_w^* : R\text{-mod} \rightarrow \mathcal{C}_w^*$$

the localization functor, which is induced by $Q_w : R\text{-mod} \rightarrow \mathcal{C}_w$ and $\psi_*$. That is, $Q_w^*$ is the composition $R\text{-mod} \rightarrow (R\text{-mod})^{\text{rev}} \rightarrow \mathcal{C}_w^{\text{rev}} \rightarrow \mathcal{C}_w^*$. Note that the composition

$$\mathcal{C}_w^* \rightarrow R\text{-mod} \rightarrow \mathcal{C}_w^*$$

coincides with the localization functor of $\mathcal{C}_w^*$ by its central objects $\{C_\Lambda_i^* \}_{\Lambda \in P_+}$.

The following theorem is one of the main results of this paper.

**Theorem 3.7.** There is an equivalence of monoidal categories $\mathcal{C}_w^*$ and $\mathcal{C}_{w^{-1}}$. More precisely, we have a quasi-commutative diagram (when $I_w = I$)

$$\begin{array}{ccc}
R\text{-mod} & \xrightarrow{Q_w} & \mathcal{C}_w^* \\
\downarrow{Q_{w^{-1}}} & \sim & \downarrow{Q_w} \\
\mathcal{C}_w^{-1} & \simto & \mathcal{C}_w^*.
\end{array}$$

**Proof.** We may assume that $I = I_w$ without loss of generality. Recall the localization functor $Q_w : R\text{-mod} \rightarrow (R\text{-mod})^{-}[w]$ from $R\text{-mod}$ to its localization via the real commuting family of graded braiders $(C_i, R_{C_i}, \phi_i)_{i \in I}$. Then there is a monoidal equivalence of categories (Theorem 1.19)

$$t_w : \mathcal{C}_w \simto (R\text{-mod})^{-}[w].$$
Now we consider the chain of the morphisms

\[ \mathcal{R}-\text{gmod} \xrightarrow{\psi_*} (\mathcal{R}-\text{gmod})^{\text{rev}} \xrightarrow{Q_w} \left( (\mathcal{R}-\text{gmod})\left[w\right] \right)^{\text{rev}}. \]

We claim that the composition \( Q_w \circ \psi_* \) factors as

\[ \mathcal{R}-\text{gmod} \xrightarrow{Q_{w^{-1}}} \left( \mathcal{R}-\text{gmod} \right)_{\sim \left[ w^{-1} \right]} \xrightarrow{f_w} \left( (\mathcal{R}-\text{gmod})\left[w\right] \right)^{\text{rev}}. \] (3.1)

By Theorem 1.4, it is enough to show that

(a) \( (Q_w \circ \psi_*)(C^-_{\lambda}) \) is invertible in \( \left( (\mathcal{R}-\text{gmod})\left[w\right] \right)^{\text{rev}} \) for any \( \lambda \in \mathbb{P}_+ \), where \( C^-_{\lambda} := M(w^{-1} \lambda, \lambda) \), and

(b) for any \( i \in I \) and \( X \in \mathcal{R}-\text{gmod} \), \( (Q_w \circ \psi_*)(R_{C_i^{-}}(X)) : (Q_w \circ \psi_*)(C^-_{\mu} \circ X) \to (Q_w \circ \psi_*)(X \circ C_i^{-}) \) is an isomorphism.

In the course of the proof, we forget grading shifts.

First, note that

For \( X \in \mathcal{R}-\text{gmod} \), \( Q_w(\psi_*(X)) \simeq 0 \) if and only if \( Q_{w^{-1}}(X) \simeq 0 \), \hspace{1cm} (3.2)

which follows from Proposition 3.3.

(a) By Lemma 2.23 and Theorem 2.18 (e), we have

\[ M(w \mu, \mu) \triangleright \psi_* \left( M(w^{-1} \lambda, \lambda) \right) \simeq M(w \mu, \mu) \triangleright M_w((-\lambda, -w^{-1} \lambda)) \simeq M(w \eta, \eta), \] \hspace{1cm} (3.3)

where \( \lambda \in \mathbb{P}_+ \) and \( \mu, \eta \in \mathbb{P}_+ \) such that \( \eta - \mu = -w^{-1} \lambda \). Hence, Proposition 1.20 implies that

\[ Q_w(\psi_*C^-_{\lambda}) \simeq C^-_{-w^{-1} \lambda}. \]

Hence (a) follows.

(b) It is enough to show that \( (Q_w \circ \psi_*)(R_{C_i^{-}}(X)) \neq 0 \) for any \( \Lambda \in \mathbb{P}_+ \) and any simple module \( X \) in \( \mathcal{R}-\text{gmod} \) with \( (Q_w \circ \psi_*)(X) \neq 0 \).

Indeed, when \( X \) is simple, since \( (Q_w \circ \psi_*)(C^-_{\lambda}) \) is invertible by (a), the objects \( (Q_w \circ \psi_*)(C^-_{\mu} \circ X) \) and \( (Q_w \circ \psi_*)(X \circ C^-_{\mu}) \) are simple in \( \left( (\mathcal{R}-\text{gmod})\left[w\right] \right)^{\text{rev}} \) so that \( (Q_w \circ \psi_*)(R_{C_i^{-}}(X)) \) is an isomorphism. For general \( X \in \mathcal{R}-\text{gmod} \), the morphism \( (Q_w \circ \psi_*)(R_{C_i^{-}}(X)) \) is an isomorphism by induction on the length of \( X \).

Note that for any simple \( X \in \mathcal{R}-\text{gmod} \) such that \( (Q_w \circ \psi_*)(X) \neq 0 \), there exists \( \mu \in \mathbb{P}_+ \) and a simple module \( Y \) in \( \mathcal{R}_{w^{-1}} \) such that there exists an epimorphism in \( \mathcal{R}-\text{gmod} \).

\[ f : C^-_{\mu} \circ X \twoheadrightarrow Y. \] \hspace{1cm} (3.4)

Indeed, if \( (Q_w \circ \psi_*)(X) \neq 0 \), then \( Q_{w^{-1}}(X) \neq 0 \) by (3.2). Therefore, \( Q_{w^{-1}}(X) \simeq C^-_{-\mu} \circ Y \) for some \( \mu \in \mathbb{P}_+ \) and a simple \( Y \in \mathcal{R}_{w^{-1}} \), equivalently \( Q_{w^{-1}}(C^-_{-\mu} \circ X) \simeq Q_{w^{-1}}(Y) \). Hence, there is a nonzero homomorphism \( C^-_{-\mu} \circ C^-_{-\mu} \circ X \to C^-_{-\mu} \circ Y \) in \( \mathcal{R}-\text{gmod} \), which is an epimorphism since \( C^-_{-\mu} \circ Y \simeq Y \circ C^-_{-\mu} \) is simple. Then, replacing \( C^-_{-\mu} \circ C^-_{-\mu} \) and \( C^-_{-\mu} \circ Y \) with \( C^-_{-\mu} \) and \( Y \), respectively, we obtain an epimorphism (3.4).
Then for any $\Lambda \in P_+$, the following diagram is commutative.

Assume that $(Q_w \circ \psi_*)(R_{C^-}(X)) = 0$. Applying $Q_w \circ \psi_*$ to the above diagram, we obtain $(Q_w \circ \psi_*)(R_{C^-}(Y)) = 0$. Because $R_{C^-}(Y)$ is an isomorphism, we have $(Q_w \circ \psi_*)(C^-_\Lambda \circ Y) \simeq 0$. Since $(Q_w \circ \psi_*)(C^-_\Lambda)$ is invertible by (a), we obtain $(Q_w \circ \psi_*)(Y) \simeq 0$, which implies $Q_{w^{-1}}(Y) \simeq 0$ by (3.2), which contradicts that $Y$ is a simple module in $\mathcal{C}_{w^{-1}}$. Thus, we obtain (b).

Thus, we obtain the diagram (3.1).

By changing the roles of $w$ and $w^{-1}$ in the above argument, we obtain the lower square in the following commutative diagram:

Since $\psi_*$ is involutive, the composition $P_{w^{-1}} \circ P_w$ is isomorphic to the identity functor on $(R\text{-}\text{gmod})^{-1}[w^{-1}]$ by Theorem 1.4 (iii). It follows that $P_w$ is an equivalence of categories, as desired.

**Corollary 3.8.** Assume that $I = I_w$. Let $X$ be a simple module in $R\text{-}\text{gmod}$ satisfying $Q_w(X) \not\simeq 0$. Then the triple

$$(M(w\mu, \mu), X, \psi_*(M(w^{-1}\Lambda, \Lambda)))$$

is a normal sequence for any $\mu, \Lambda \in P_+$.

**Proof.** In the course of the proof in the above theorem, we showed that $(Q_w \circ \psi_*)(R_{C^-}(\psi_*(X))) \neq 0$ for any simple module $X \in R\text{-}\text{gmod}$ such that $Q_w(X) \not\simeq 0$ and any $\Lambda \in P_+$. That is,

$$X \circ \psi_*(C^-_\Lambda) \rightarrow \psi_*(C^-_\Lambda) \circ X$$

does not vanish under the functor $Q_w$. Hence, the homomorphism

$$C^-_\mu \circ X \circ \psi_*(C^-_\Lambda) \rightarrow C^-_\mu \circ \psi_*(C^-_\Lambda) \circ X \rightarrow \psi_*(C^-_\Lambda) \circ X \circ C^-_\mu$$
is nonzero for any $\mu \in P_+$. Since $C_\mu, \psi_\alpha(C_\lambda)$ and $X$ are simple modules, the above homomorphism is equal to the composition of the $r$-matrices $r_{X, \psi_\alpha(C_\lambda)}, r_{C_\mu, \psi_\alpha(C_\lambda)}$, and $r_{C_\mu X}$ up to a constant multiple. Thus, $(C_\mu, X, \psi_\alpha(C_\lambda))$ is a normal sequence. □

Recall that the category $\tilde{\mathcal{C}}_{w^{-1}}$ is left-rigid, that is, every object of $\tilde{\mathcal{C}}_{w^{-1}}$ has a left dual in $\tilde{\mathcal{C}}_{w^{-1}}$ ([13, Corollary 5.11]). It follows that $(\tilde{\mathcal{C}}_w)^{rev}$ is left-rigid by the above theorem. Hence, we obtain the following theorem as its corollary.

**Theorem 3.9.** The category $\tilde{\mathcal{C}}_w$ is a rigid monoidal category.

### 4 LOCALIZATION OF THE CATEGORY $\mathcal{C}_{w,v}$

Through this section, we assume that $w, v \in W$ satisfies $v \leq w$ and $I_w = I$.

Let $N$ be a (not necessarily simple) module in $\mathcal{C}_{w,v}$ and $\lambda \in P_+$. Set $\alpha := \nu \lambda - w \lambda, \beta := \lambda - \nu \lambda,$ and $\gamma := -wt(N)$. Note that the $R$-matrix $r_{M(w \lambda, v \lambda), M(\nu \lambda, \lambda)}$ decomposes into

$$M(w \lambda, \nu \lambda) \circ M(\nu \lambda, \lambda) \xrightarrow{\pi} M(w \lambda, \lambda) \xrightarrow{\iota} q^{(\alpha, \beta)} M(\nu \lambda, \lambda) \circ M(w \lambda, \nu \lambda)$$

by [11, Proposition 4.6].

Let $\rho_{w,v,\lambda}(N)$ be the composition of the following chain of homomorphisms:

$$M(w \lambda, \nu \lambda) \circ N \circ M(\nu \lambda, \lambda) \xrightarrow{\pi \circ N} q^{(\beta, \gamma)} M(w \lambda, \nu \lambda) \circ M(\nu \lambda, \lambda) \circ N \xrightarrow{R_{M(w \lambda, \lambda)}(N)} q^{-(w \lambda + \nu \lambda, \gamma)} N \circ M(w \lambda, \lambda).$$

Here, $R_{M(w \lambda, \lambda)}$ is the nondegenerate braider associated with $M(w \lambda, \lambda)$, and $r_{N, M(\nu \lambda, \lambda)}$ is well defined because $(N, M(\nu \lambda, \lambda))$ is unmixed (Proposition 2.1). Note that we have

$$r_{N, M(\nu \lambda, \lambda)}(u \otimes v) = \tau_w[ht(\beta), ht(\gamma)](v \otimes u) \quad \text{for any } u \in N \text{ and } v \in M(\nu \lambda, \lambda),$$

and hence $\Lambda(N, M(\nu \lambda, \lambda)) = (\beta, \gamma)$. Since $\phi_{M(w \lambda, \lambda)}(\gamma) = -(w \lambda + \lambda, \gamma)$, we have

$$-\deg(\rho_{w,v,\lambda}(N)) = (\beta, \gamma) - (w \lambda + \lambda, \gamma) = -(w \lambda + \nu \lambda, \gamma).$$

**Lemma 4.1.** For any $R(\gamma)$-module $N \in \mathcal{C}_{w,v}$, We have

$$e(\alpha + \gamma, \beta)(M(w \lambda, \lambda) \circ N) \simeq q^{-(\beta, \gamma)}(M(w \lambda, \nu \lambda) \circ N) \otimes M(\nu \lambda, \lambda)$$

and

$$e(\alpha + \gamma, \beta)(N \circ M(w \lambda, \lambda)) \simeq (N \circ M(w \lambda, v \lambda)) \otimes M(\nu \lambda, \lambda).$$

**Proof.** To obtain the first isomorphism, it is enough to apply (2.5) in Proposition 2.4 by taking $(M(\nu \lambda, \nu \lambda), N, M(\nu \lambda, \lambda))$ as $(L, M, N)$, and for the second, we can apply (2.4) by taking $(N, M(\nu \lambda, \nu \lambda), M(\nu \lambda, \lambda))$ as $(L, M, N)$. □
Proposition 4.2. For any $R(\gamma)$-module $N \in \mathcal{C}_{s,u}$, there exists a unique homomorphism

$$\psi_{w,v,\lambda}(N) : M(w\lambda, v\lambda) \circ N \to q^{-(w\lambda + v\lambda, \gamma)}N \circ M(w\lambda, v\lambda)$$

such that the following diagrams are commutative.

\[\begin{array}{ccc}
(M(w\lambda, v\lambda) \circ N) \otimes M(v\lambda, \lambda) & \xrightarrow{\psi_{w,v,\lambda}(N)} & q^{-(w\lambda + v\lambda, \gamma)}(N \circ M(w\lambda, v\lambda)) \otimes M(v\lambda, \lambda) \\
\downarrow \psi_{w,v,\lambda}(N) & & \downarrow \psi_{w,v,\lambda}(N) \\
q^{(\beta, \gamma)} e(\alpha + \gamma, \beta)(M(w\lambda, \lambda) \circ N) & \xrightarrow{R_{M(w\lambda, \lambda)}(N)} & q^{-(w\lambda + v\lambda, \gamma)}e(\alpha + \gamma, \beta)(N \circ M(w\lambda, \lambda)),
\end{array}\]

\[\begin{array}{ccc}
M(w\lambda, v\lambda) \circ N \circ M(v\lambda, \lambda) & \xrightarrow{\psi_{w,v,\lambda}(N) \circ M(v\lambda, \lambda)} & q^{-(w\lambda + v\lambda, \gamma)}N \circ M(w\lambda, \lambda).
\end{array}\]

Proof. Applying $e(\alpha + \gamma, \beta)$ to

$$R_{M(w\lambda, \lambda)}(N) : M(w\lambda, \lambda) \circ N \to q^{-(w\lambda + \lambda, \gamma)}N \circ M(w\lambda, \lambda),$$

we obtain by Lemma 4.1

$$q^{-(\beta, \gamma)}(M(w\lambda, v\lambda) \circ N) \otimes M(v\lambda, \lambda) \to q^{-(w\lambda + \lambda, \gamma)}(N \circ M(w\lambda, v\lambda)) \otimes M(v\lambda, \lambda).$$

Since we have $\text{END}(M(v\lambda, \lambda)) \simeq k \text{id}$, we obtain

$$\psi_{w,v,\lambda}(N) : M(w\lambda, v\lambda) \circ N \to q^{-(w\lambda + v\lambda, \gamma)}N \circ M(w\lambda, v\lambda).$$

The commutativity of (4.2) is then obvious. \qed

Definition 4.3. For $N \in \mathcal{C}_{s,u}$ and $\lambda \in \mathbb{P}_+$, we define

$$R_{M(w\lambda, v\lambda)}(N) := \psi_{w,v,\lambda}(N) : M(w\lambda, v\lambda) \circ N \to q^{\phi_{w,v,\lambda}(-w(\lambda))}N \circ M(w\lambda, v\lambda),$$

where

$$\phi_{w,v,\lambda}(\gamma) = -(w\lambda + v\lambda, \gamma) \quad \text{for} \quad \gamma \in Q.$$

Theorem 4.4. The family $\{(M(w\Lambda_i, v\Lambda_i), R_{M(w\Lambda_i, v\Lambda_i), \phi_{w,v,\Lambda_i}})\}_{i \in I}$ is a real commuting family of graded braidings in the category $\mathcal{C}_{s,u}$, and also, it is a family of central objects in $\mathcal{C}_{w,u}$.

Proof. We know that $\{(M(w\Lambda_i, \Lambda_i), R_{M(w\Lambda_i, \Lambda_i), \phi_{w,\Lambda_i,\Lambda_i}})\}_{i \in I}$ is a real commuting family of graded braidings in the category $R\text{-gmod}$ (Proposition 1.16), and also, it is a family of central objects in $\mathcal{C}_w$ (Theorem 1.17). Hence, our assertion follows from Proposition 4.2. \qed

Set

$$\mathcal{C}_{s,u}[w] := \mathcal{C}_{s,u}[M(w\Lambda_i, v\Lambda_i)]^{\circ -1} \mid i \in I],$$
\[ \mathcal{C}_{w,v} := \mathcal{C}_{w,v}[M(w_\Lambda, v_\Lambda)^{o-1} \mid i \in I]. \]

Since \( \mathcal{C}_{w,v} \) is a full subcategory of \( \mathcal{C}_{*,v} \), the canonical embedding induces a fully faithful monoidal functor

\[ t_{w,v} : \mathcal{C}_{w,v} \rightarrow \mathcal{C}_{*,v}[w]. \]

**Theorem 4.5.** The functor \( t_{w,v} : \mathcal{C}_{w,v} \rightarrow \mathcal{C}_{*,v}[w] \) is an equivalence of categories.

**Proof.** Let us denote by \( Q_{w,v} : \mathcal{C}_{*,v} \rightarrow \mathcal{C}_{*,v}[w] \) the localization functor. It is enough to show that for every object \( X \in \mathcal{C}_{*,v}[w] \), there exists an object \( Z \in \mathcal{C}_{w,v} \) such that \( t_{w,v}(Z) \simeq X \). Since \( \mathcal{C}_{w,v} \) is closed under taking extension by \([13, \text{Proposition 2.10}]\), we may assume further that \( X \) is a simple object. Since every simple object in \( \mathcal{C}_{*,v}[w] \) is of the form \( Q_{w,v}(Y) \circ M(w_\lambda, v_\lambda)^{o-1} \) for some \( \lambda \in \mathbb{P}_+ \) and a simple object \( Y \in \mathcal{C}_{*,v} \) \([13, \text{Proposition 4.8}]\), we may assume that \( X \) is a simple module in \( \mathcal{C}_{*,v} \).

Let \( X \) be a simple module in \( \mathcal{C}_{*,v} \). We shall show that \( Q_{w,v}(X) \in \mathcal{C}_{w,v} \).

Recall that \( Q_w : \mathcal{R} \rightarrow (\mathcal{R})[w] \simeq \mathcal{C}_w \) is the localization functor.

(i) Assume first that \( Q_w(X) \neq 0 \). Then, there exists \( \Lambda \in \mathbb{P}_+ \) and a simple module \( Y \in \mathcal{C}_w \) such that

\[ Q_w(X) \simeq C_{\Lambda}^{o-1} \circ Y, \]

where \( C_\Lambda = M(w_\Lambda, \Lambda) \). Hence, we have an epimorphism in \( \mathcal{R} \) (by replacing \( \Lambda \) if necessary)

\[ C_\Lambda \circ X \rightarrow Y. \]

Note that by Lemma 4.1, we have

\[ \text{Res}_{*,\Lambda-v_\Lambda}(C_\Lambda \circ X) \simeq (M(w_\Lambda, v_\Lambda) \circ X) \otimes M(v_\Lambda, \Lambda), \text{ and} \]

\[ \text{Res}_{*,\Lambda-v_\Lambda}(X \circ C_\Lambda) \simeq X \circ M(w_\Lambda, v_\Lambda) \otimes M(v_\Lambda, \Lambda). \]

(4.3)

Set \( \beta = \Lambda - v_\Lambda = -\text{wt}((M(v_\Lambda, \Lambda))) \). Applying \( \text{Res}_{*,\beta} \) to the diagram,

we get a commutative diagram

\[ \begin{array}{ccc}
C_\Lambda \circ X & \xrightarrow{R_{C_\Lambda}(X)} & X \circ C_\Lambda \\
& \searrow & \downarrow \\
& \downarrow & \\
Y & \longleftarrow & \end{array} \]

\[ \begin{array}{ccc}
(M(w_\Lambda, v_\Lambda) \circ X) \otimes M(v_\Lambda, \Lambda) & \xrightarrow{\text{Res}_{*,\beta}(R_{C_\Lambda}(X))} & (X \circ M(w_\Lambda, v_\Lambda)) \otimes M(v_\Lambda, \Lambda). \\
& \text{Res}_{*,\beta}(Y) & \\
& & \end{array} \]
Let \( \psi : M(w\Lambda, v\Lambda) \circ X \to X \circ M(w\Lambda, v\Lambda) \) be the homomorphism such that

\[
\psi \otimes M(v\Lambda, \Lambda) = \text{Res}_{\ast, \beta}(R_{C_{\lambda}}(X)).
\]

Set \( Z := \text{Im}(\psi) \). Since \( \psi = cr_{M(w\Lambda, v\Lambda), X} \) for some \( c \in k^{\times}, \) \( Z \) is simple. Since \( X \) and \( M(w\Lambda, v\Lambda) \) belong to \( \mathcal{C}_{s, \nu} \), so does \( Z \). Note that \( Z \) belongs to \( \mathcal{C}_{w, \nu} \), because

\[
W(Z) \subset W(Y) \subset Q_+ \cap wQ_-.
\]

Since \( Z \) is the image of \( M(w\Lambda, v\Lambda) \circ X \), we have

\[
Q_{s, \nu}(X) \simeq M(w\Lambda, v\Lambda)^{e-1} \circ Q_{s, \nu}(Z) \simeq t_{w, \nu}(M(w\Lambda, v\Lambda)^{e-1} \circ Q_{w, \nu}(Z)).
\]

(ii) Assume that \( X \in \mathcal{C}_{s, \nu} \) satisfies \( Q_{w}(X) \simeq 0 \). Then, there exists \( \Lambda \in \mathcal{P}_+ \) such that \( R_{C_{\lambda}}(X) : C_{\lambda} \to X \circ C_{\lambda} \) vanishes. Applying \( \text{Res}_{s, \beta} \) (\( \beta = \Lambda - v\Lambda \)), we deduce from (4.3) that

\[
R_{M(w\Lambda, v\Lambda)}(X) \otimes M(v\Lambda, \Lambda) : (M(w\Lambda, v\Lambda) \circ X) \otimes M(v\Lambda, \Lambda) \longrightarrow (X \circ M(w\Lambda, v\Lambda)) \otimes M(v\Lambda, \Lambda)
\]

vanishes. Hence, \( R_{M(w\Lambda, v\Lambda)}(X) \) vanishes, which means that \( Q_{w, \nu}(X) \simeq 0 \).

\[\square\]

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