A lower bound for the uniform Schoenberg operator

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Abstract

We present an estimate for the lower bound for the Schoenberg operator with equidistant knots in terms of the second order modulus of smoothness. We investigate the behaviour of iterates of the Schoenberg operator and in addition, we show an upper bound of the second order derivative of these iterates. Finally, we prove the equivalence between the approximation error and the second order modulus of smoothness.

Keywords: spline approximation, Schoenberg operator, iterates, inverse theorem

1. Introduction

L. Beutel et al. stated in [1] an interesting conjecture about the equivalence of the approximation error of the Schoenberg operator on \([0,1]\) and the second order modulus of smoothness. We prove that this conjecture holds true for the uniform Schoenberg operator if the degree of the splines is fixed and the mesh gauge tends to zero. To this end, we characterize the behaviour of the iterates of the Schoenberg operator. Related to our result is the work of Zapryanova et al. [2], who proved an inverse theorem for the uniform Schoenberg operator using the Ditzian-Totik modulus of smoothness. In contrast to their result, we give a direct lower bound.

More specifically, we show that for \(f \in C([0,1])\) we have the uniform estimate

\[
\omega_2(f, \delta) \leq 5 \cdot \|f - S_{n,k} f\|_{\infty},
\]

where \(\omega_2(f, \delta)\) is the classical modulus of smoothness.

1.1. The Schoenberg operator

For integers \(n, k > 0\), we consider the equidistant knots \(\{x_j = \frac{j}{n}\}_{j=0}^n\) as a partition of \([0,1]\). We extend this knot sequence by setting

\[
x_{-k} = \cdots = x_0 = 0 < x_1 < \ldots < x_n = \cdots = x_{n+k} = 1.
\]

For \(f \in C([0,1])\), the variation-diminishing spline operator of degree \(k\) with respect to the knots \(\{x_j\}_{j=-k}^{n+k}\) is then defined by

\[
S_{n,k} f(x) = \sum_{j=-k}^{n-1} f(\xi_{j,k}) N_{j,k}(x), \quad 0 \leq x < 1,
\]

\[
S_{n,k} f(1) = \lim_{y \uparrow 1} S_{n,k} f(y)
\]

with the nodes

\[
\xi_{j,k} := \frac{x_{j+1} + \cdots + x_{j+k}}{k}, \quad -k \leq j \leq n - 1,
\]
and the normalized B-splines
\[
N_{j,k}(x) := (x_{j+k+1} - x_j)[x_j, \ldots, x_{j+k+1}](\cdot - x)_+^k.
\]

This operator was introduced by Schoenberg in 1959 as a generalization of the Bernstein operator see, e.g., [3, 4]. The normalized B-splines form a partition of the unity
\[
\sum_{j=-k}^{n-1} N_{j,k}(x) = 1, \tag{1}
\]
and the Schoenberg operator can reproduce linear functions, i.e.,
\[
\sum_{j=-k}^{n-1} \xi_j N_{j,k}(x) = x, \tag{2}
\]
due to the chosen Greville nodes. A comprehensive overview of direct inequalities for this operator can be found in [1].

1.2. Notation

Throughout this paper, we will consider the Banach space \(C([0,1])\), i.e., the space of real-valued continuous functions on the interval \([0,1]\) endowed with the supremum norm \(\|\cdot\|_\infty\),
\[
\|f\|_\infty = \sup \{ |f(x)| : x \in [0,1] \}, \quad f \in C([0,1]).
\]
The space of bounded linear operators on \(C([0,1])\) will be denoted by \(\mathcal{B}(C([0,1]))\) equipped with the usual operator norm \(\|\cdot\|_{op}\). As a \(n+k\)-dimensional subspace of \(C([0,1])\), we denote by \(S(n,k)\) the spline space of degree \(k\) with respect to the knot sequence \(\{x_j\}_{j=-k}^{n+k}\),
\[
S(n,k) = \left\{ \sum_{j=-k}^{n-1} c_j N_{j,k} : c_j \in \mathbb{R}, \; j \in \{-k, \ldots, n-1\} \right\} \subset C^{k-1}([0,1]).
\]
Since \(S(n,k)\) is finite-dimensional, \(S(n,k)\) is a Banach space with the inherited norm \(\|\cdot\|_\infty\). For more information on spline spaces see, e.g., [5]. For \(f \in C([0,1])\) and points \(x_0, \ldots, x_k \in [0,1]\), the divided difference \([x_{j_0}, \ldots, x_{j_k}]f\) is defined to be the coefficient of \(x^k\) in the unique polynomial of degree \(k\) or less that interpolates \(f(x)\) at the points \(x_0, \ldots, x_k\).

2. The iterates of the Schoenberg operator

In the following, we discuss some basic properties of the iterates of the Schoenberg operator. For \(m \in \mathbb{N}\), we define
\[
(S_{n,k}^m f)(x) = (S_{n,k}^{m-1} (S_{n,k} f))(x) \quad \text{for all } x \in [0,1].
\]

Lemma 1. We can write the \(m\)-th iterate of the Schoenberg operator as
\[
S_{n,k}^m f(x) = S_{n,k}^{m-1} \left( \sum_{j=-k}^{n-1} f(\xi_{j,k}) N_{j,k}(x) \right)
= \sum_{j_1, \ldots, j_m=-k}^{n-1} f(\xi_{j_1,k}) N_{j_1,k}(\xi_{j_2,k}) \cdots N_{j_{m-1},k}(\xi_{j_m,k}) N_{j_m,k}(x).
\]

Proof. Induction over \(m\). □

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2.1. The first and second derivative of the iterates

In this section, we consider the derivatives and give explicit representations. For that, we define a discrete backward difference operator $\Delta_t$ by

$$\Delta_t f(\xi_j, k) := \frac{f(\xi_{j+1}, k) - f(\xi_{j-1}, k)}{\xi_{j+1} - \xi_{j-1}}.$$

With this, we can state:

**Lemma 2.** The following properties hold for the derivatives of the Schoenberg operator:

$$D_S n, k f = S_+^n k, -1 \Delta_k f$$

and

$$D^2 S n, k f = S_{++,}^n k, -1 \Delta_k \Delta_k f,$$

where $S_+^n k, f$ and $S_{++,}^n k, f$ are Schoenberg operators with shifted knots defined by

$$S_+^n k, f = \sum_{j=1-k}^{n-1} f(\xi_{j+1}, k) N_{j, k} \quad \text{and} \quad S_{++,}^n k, f = \sum_{j=1-k}^{n-1} f(\xi_{j+2}, k) N_{j, k}.$$

**Proof.** This lemma follows directly by the representation of the derivative, [4]:

$$D_S n, k f(x) = \sum_{j=1-k}^{n-1} \frac{f(\xi_{j+1}, k) - f(\xi_{j-1}, k)}{\xi_{j+1} - \xi_{j-1}} N_{j, k-1}(x),$$

and

$$D^2 S n, k f(x) = \sum_{j=2-k}^{n-1} \frac{f(\xi_{j+1}, k) - f(\xi_{j-1}, k)}{\xi_{j+1} - \xi_{j-1}} \frac{f(\xi_{j+2}, k) - f(\xi_{j-2}, k)}{\xi_{j+2} - \xi_{j-2}} N_{j, k-1}(x).$$

Applying the definition of the discrete backward difference operator $\Delta_t$ gives the required representation. □

Now we give an analogous representation for the iterates of the Schoenberg operator.

**Theorem 1.** The first and second derivative of the iterates of the Schoenberg operator have the following representation:

$$D_S^m n, k f(x) = \sum_{j_1=1-k}^{n-1} \sum_{j_1=1-k}^{n-1} f(\xi_{j_1}, k) N_{j_1, k}(\xi_{j_2}, k) \cdots N_{j_m-2, k}(\xi_{j_m-1, k}) \cdots$$

$$\cdots \frac{N_{j_m-1, k}(\xi_{j_m}, k) - N_{j_m-1, k}(\xi_{j_m-1, k})}{\xi_{j_m, k} - \xi_{j_m-1, k}} N_{j_m, k-1}(x)$$

$$= \sum_{j_1=1-k}^{n-1} \sum_{j_1=1-k}^{n-1} f(\xi_{j_1}, k) N_{j_1, k}(\xi_{j_2}, k) \cdots N_{j_m-2, k}(\xi_{j_m-1, k}) \Delta_k N_{j_m-1, k}(\xi_{j_m}) N_{j_m, k-1}(x).$$

and

$$D^2 S^m n, k f(x) = \sum_{j_2=2-k}^{n-1} \sum_{j_2=2-k}^{n-1} f(\xi_{j_1}, k) N_{j_1, k}(\xi_{j_2}, k) \cdots N_{j_m-2, k}(\xi_{j_m-1, k}) \cdots$$

$$\cdots \frac{N_{j_m-1, k}(\xi_{j_m}, k) - N_{j_m-1, k}(\xi_{j_m-1, k})}{\xi_{j_m, k} - \xi_{j_m-1, k}} \frac{N_{j_m-1, k}(\xi_{j_m+1, k}) - N_{j_m-1, k}(\xi_{j_m+2, k})}{\xi_{j_m+1, k} - \xi_{j_m+2, k}} N_{j_m, k-1}(x)$$

$$= \sum_{j_1=1-k}^{n-1} \sum_{j_1=1-k}^{n-1} f(\xi_{j_1}, k) N_{j_1, k}(\xi_{j_2}, k) \cdots N_{j_m-2, k}(\xi_{j_m-1, k}) \Delta_k N_{j_m-1, k}(\xi_{j_m}) N_{j_m, k-1}(x).$$

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Proof. Applying Lemma 1 and 2 to $S_n^{-1} f$ yields the result.

2.1.1. An upper bound for the second derivative of the iterates

Our idea is now to work with the shift invariant basis functions $N_{j,k}$, $j \in \{0, \ldots, n-k-1\}$, to stay away from the boundary of the interval $[0, 1]$. Then we can represent the Schoenberg operator as a convolution operator and apply known techniques for this kind of operators.

Therefore, let $x \in [x_{2k+2}, x_{n-2k-2}]$. Then we have

\[ x \not\in \bigcup_{j=-k}^{k+1} \text{supp } N_{j,k} \text{ and } x \not\in \bigcup_{j=-2k-2}^{n-1} \text{supp } N_{j,k}, \]

because $\text{supp } N_{j,k} \subset [x_j, x_{j+k+1}]$. Besides, we can simplify the notation of the iterates of the Schoenberg operator for $x \in [x_{2k+2}, x_{n-2k-2}]$ to

\[ S_n^m f(x) = \sum_{j_1, \ldots, j_m = 0}^{n-k-1} f(\xi_{j_1, k}) N_{j_1, k}(\xi_{j_2, k}) \cdots N_{j_{m-1}, k}(\xi_{j_m, k}) N_{j, k}(x). \]

Now, we show that the basis functions $\{N_{j,k}\}_{j=0}^{n-k-1}$ are shift invariant.

**Theorem 2.** The $N_{j,k}$ with $j \in \{0, \ldots, n-k-1\}$ are translates of each other, i.e.,

\[ N_{j+1,k}(\xi_i) = N_{j,k}(\xi_{i-1}), \]

and $\text{supp span } N_{j,k} \subset [0, 1]$.

Proof. As $\text{supp } N_{j,k} \subset [x_j, x_{j+k+1}]$ all corresponding knots $x_i$, $i \in \{j, \ldots, j+k+1\}$ are distinct from each other. Explicitly, we have $x_i = \frac{i}{n}$. Now let $h = 1/n$. Then, we get

\[
\begin{align*}
N_{j+1,k}(\xi_i) &= (x_{j+k+2} - x_{j+1}) [x_{j+1}, \ldots, x_{j+k+2}] (\cdot - \xi_i)^k \\
&= (x_{j+k+1} - x_j) \frac{1}{h^{k+1} \cdot k!} \sum_{l=j+1}^{j+k+2} \binom{k+1}{l-j-1} (-1)^{j+k+2-l} (x_l - \xi_i)^k \\
&= (x_{j+k+1} - x_j) \frac{1}{h^{k+1} \cdot k!} \sum_{l=j}^{j+k+1} \binom{k+1}{l-j} (-1)^{j+k+1-l} (x_{l+1} - \xi_i)^k \\
&= (x_{j+k+1} - x_j) \frac{1}{h^{k+1} \cdot k!} \sum_{l=j}^{j+k+1} \binom{k+1}{l-j} (-1)^{j+k+1-l} (x_1 - \xi_{i-1})^k \\
&= N_{j,k}(\xi_{i-1}).
\end{align*}
\]

The last line holds, because

\[
x_{i+1} - \xi_i = \frac{k \cdot (l+1) - \sum_{j=1}^{k}(i+j)}{nk} = \frac{k \cdot l - \sum_{j=1}^{k}(i+j-1)}{nk} = x_l - \xi_{i-1}.
\]

With Theorem 2, we get the following corollary:
Corollary 1. For $m \in \mathbb{N}$ and $x \in [x_{k+1}, x_{n-2k-2}]$ we get:
\[
DS_n f(x) = n^{-k-1} \sum_{j_1, \ldots, j_m=0}^n f(\xi_{j_1}) N_{j_1,k}(\xi_{j_2}) \cdots N_{j_{m-2},k}(\xi_{j_{m-1}}) \Delta_k N_{j_{m-1},k}(\xi_{j_m}) N_{j_m,k}(x)
\]
\[
= n^{-k-1} \sum_{j_1, \ldots, j_m=0}^n f(\xi_{j_1}) N_{j_1,k}(\xi_{j_2}) \cdots \Delta_k N_{j_{m-2},k}(\xi_{j_{m-1}}) N_{j_{m-1},k}(\xi_{j_m}) N_{j_m,k}(x)
\]
\[
\vdots
\]
\[
= n^{-k-1} \sum_{j_1, \ldots, j_m=0}^n f(\xi_{j_1}) \Delta_k N_{j_1,k}(\xi_{j_2}) N_{j_2,k}(\xi_{j_3}) \cdots N_{j_{m-1},k}(\xi_{j_m}) N_{j_m,k}(x),
\]
i.e., the backward difference operator can be applied to $N_{j,k}$ for every index $j$. Thus, we have $m-1$ possibilities to represent the first derivative of the iterated Schoenberg operator.

Analog for $D^2 S_n^m f$, where we have
\[
D^2 S_n^m f(x) = n^{-k-1} \sum_{j_1, \ldots, j_m=0}^n f(\xi_{j_1}) N_{j_1,k}(\xi_{j_2}) \cdots N_{j_{m-2},k}(\xi_{j_{m-1}}) \Delta_k \Delta_{k-1} N_{j_{m-1},k}(\xi_{j_m}) N_{j_m,k-2}(x)
\]
\[
= n^{-k-1} \sum_{j_1, \ldots, j_m=0}^n f(\xi_{j_1}) N_{j_1,k}(\xi_{j_2}) \cdots \Delta_k \Delta_{k-1} N_{j_{m-2},k}(\xi_{j_{m-1}}) N_{j_{m-1},k}(\xi_{j_m}) N_{j_m,k-2}(x)
\]
\[
\vdots
\]
\[
= n^{-k-1} \sum_{j_1, \ldots, j_m=0}^n f(\xi_{j_1}) \Delta_k \Delta_{k-1} N_{j_1,k}(\xi_{j_2}) N_{j_2,k}(\xi_{j_3}) \cdots N_{j_{m-1},k}(\xi_{j_m}) N_{j_m,k-2}(x).
\]

Similar to $DS_n^m f$, we have $m(m-1)$ possibilities to represent the second derivative of the $m$-th iterate of the Schoenberg operator.

We will abbreviate the last term by
\[
D^2 S_n^m f(x) = n^{-k-1} \sum_{j_1, \ldots, j_m=0}^n f(\xi_{j_1}) \cdot P(j_1, \ldots, j_m; x) \cdot I_{l_1,l_2}(j_1, \ldots, j_{m-1}; x),
\]
where
\[
P(j_1, \ldots, j_m; x) := \prod_{l=1}^{m-1} N_{j_l,k}(\xi_{j_{l+1}}) N_{j_m,k-2}(x),
\]
and for $l_1, l_2 \in \{1, \ldots, m-1\}$, $l_1 \leq l_2$,
\[
I_{l_1,l_2}(j_1, \ldots, j_{m-1}; x) = \begin{cases} \Delta_{k-1} N_{l_1,k}(x) \cdot \Delta_k N_{l_2,k}(x), & \text{for } l_1 \neq l_2, \\ \Delta_{k-1} \Delta_k N_{l_1,k}(x), & \text{for } l_1 = l_2. \end{cases}
\]

Now we are able to give an upper bound for the second order derivative of the iterated Schoenberg operator:

Theorem 3. For the integer $m \geq 2$, $h = 1/n$ and $x \in [x_{2k+2}, x_{n-2k-2}]$ we have the upper bound
\[
|D^2 S_n^m f(x)| \leq \frac{2\varepsilon_{n,k}}{(m-1)^{3/2}h^2} \cdot \|f\|_{\infty}.
\]
Proof. As we have \( \frac{m(m-1)}{2} \) possibilities to express \( D^2 S_{n,k}^m f(x) \), we write (3) as the following mean:

\[
D^2 S_{n,k}^m f(x) = \frac{2}{m(m-1)} \sum_{l_1 \leq l_2=1}^{m-1} D^2 S_{n,k}^m f(x)
\]

\[
= \frac{2}{m(m-1)} \sum_{j_1, \ldots, j_m=0}^{n-k-1} \left( f(\xi_{j_1,k}) \cdot P(j_1, \ldots, j_m; x) \cdot \sum_{l_1 \leq l_2=1}^{m-1} I_{l_1,l_2}(j_1, \ldots, j_m-1; x) \right).
\]

Since \( P \) is positive, we can split \( P = P^{1/2}P^{1/2} \), where \( P^{1/2} \) is the positive root. Then we apply the Cauchy-Schwarz inequality and get in abbreviated notation the following pointwise inequality for \( x \in [x_{2k+2}, x_{n-2k-2}] \):

\[
|D^2 S_{n,k}^m f| \leq \frac{2}{m(m-1)} \left\{ \sum_{j_1, \ldots, j_m=0}^{n-k-1} |f|^2 P \right\}^{1/2} \left\{ \sum_{j_1, \ldots, j_m=0}^{n-k-1} P \left( \sum_{l_1 \leq l_2=1}^{m-1} I_{l_1,l_2} \right) \right\}^{1/2}
\]

\[
\leq \frac{2}{m(m-1)} \left( \|f\|_{\infty} \cdot 1 \right) \left( \sum_{j_1, \ldots, j_m=0}^{n-k-1} P \left( \sum_{l_1 \leq l_2=1}^{m-1} I_{l_1,l_2} \right) \right)^{1/2}.
\]

(4)

Here, we used the partition of unity property of the B-splines, namely that \( \sum_{j=-k}^{n-1} N_{j,k}(x) = 1 \) holds for all \( x \in [0,1] \). Summation by parts, beginning with \( j_1, j_2, \ldots \), leads to

\[
\sum_{j_1, \ldots, j_m=-k}^{n-1} P(j_1, \ldots, j_m; x) = \sum_{j_m=0}^{n-k-1} N_{j_m,k-2}(x) \sum_{j_{m-1}=0}^{n-k-1} N_{j_{m-1},k}(\xi_{j_{m},k}) \cdots \sum_{j_1=0}^{n-k-1} N_{j_1,k}(\xi_{j_2,k}) = 1.
\]

Finally, we take the supremum norm of \( f \) and obtain the inequality used for the first term.

Next, we discuss the second product in (4). For the term \( \left( \sum_{l_1 \leq l_2=1}^{m-1} I_{l_1,l_2} \right)^2 \) we get formally

\[
\left( \sum_{l_1 \leq l_2=1}^{m-1} I_{l_1,l_2} \right)^2 = \sum_{l_1 \leq l_2=1}^{m-1} I_{l_1,l_2}^2 + \sum_{l_1 \neq l_2} I_{l_1,l_2} I_{s_1,s_2}.
\]

Note that the last sum vanishes, since for any indices \( i, j \in \{0, \ldots, n - k - 1\} \) we have

\[
\sum_{j=-k}^{n-1} \Delta_k N_{j,k}(\xi_i) = 0,
\]

and

\[
\sum_{j=-k}^{n-1} \Delta_{k-1} \Delta_k N_{j,k}(\xi_i) = 0,
\]

because of the partition of unity (1). That means, if the difference operator \( \Delta_k \) or \( \Delta_{k-1} \Delta_k \) is applied to \( N_{j,k} \) without being squared, the whole sum vanishes. Therefore, we get

\[
\sum_{j_1, \ldots, j_m=0}^{n-k-1} P(j_1, \ldots, j_m; x) \left( \sum_{l_1 \leq l_2=1}^{m-1} I_{l_1,l_2} \right)^2 = \sum_{j_1, \ldots, j_m=0}^{n-k-1} P(j_1, \ldots, j_m; x) \sum_{l_1=1}^{m-1} I_{l_1,l_1}^2.
\]
With this we obtain from (1) the final inequality
\[
|D^2 S_{n,k}^m f(x)| \leq \frac{2}{(m-1)^2} \|f\|_{\infty} \left( (m-1) \frac{\varepsilon_{n,k}}{h^2} \right)^{\frac{1}{2}} \\
\leq \frac{2 \varepsilon_{n,k}}{(m-1)^{3/2}h^2} \cdot \|f\|_{\infty},
\]
where
\[
\varepsilon_{n,k}^2 := \sup_{j=k}^{n-1} \frac{(N_{j,k}(\xi_{i,k}) - 2N_{j,k}(\xi_{i-1,k}) + N_{j,k}(\xi_{i-2,k}))^2}{N_{j,k}(\xi_{i,k})} = \sup_{j=k}^{n-1} \frac{(\Delta_{k-1} \Delta_k N_{j,k}(\xi_{i,k}))^2}{N_{j,k}(\xi_{i,k})}
\]
with
\[
N_{j,k}(\xi_{i,k}) := \begin{cases} N_{j,k}(\xi_{i,k}), & \text{if } N_{j,k}(\xi_{i,k}) = 0, \\ 1, & \text{if } N_{j,k}(\xi_{i,k}) \neq 0. \end{cases}
\]
The terms \(N_{j,k}(\xi_{i,k})\) are formally needed to avoid zero divisions in the term for \(\varepsilon_{n,k}^2\).

**Corollary 2.** Due to the uniform convergence, we get for \(k > 0\) fixed, \(m > 1\) and \(n \to \infty\) the uniform upper bound
\[
\|D^2 S_{n,k}^m f\|_{\infty} \leq \frac{2 \varepsilon_{n,k}}{h^2 \cdot (m-1)^{3/2}} \cdot \|f\|_{\infty}.
\]

3. The lower bound of the Schoenberg operator

In this section, we show that for \(0 < t < \frac{1}{k}\) and \(k \geq 3\), there exists a constant \(M > 0\), such that
\[
M \cdot \omega_2(f,t) \leq \|f - S_{n,k} f\|_{\infty},
\]
where the second order modulus of smoothness \(\omega_2 : C([0,1]) \times (0, \frac{1}{k}] \to [0, \infty)\) is defined by
\[
\omega_2(f,t) := \sup_{0 < h < t} \sup_{x \in [0.1 - 2h]} |f(x) - 2f(x + h) + f(x + 2h)|.
\]

As the modulus of smoothness is equivalent to the \(K\)-functional \(\|K\| \leq \|f\|_{\infty}^2 + t^2 \|D^2 S_{n,k} f\|_{\infty}^2\), we can derive the inequality
\[
\|f\|_{\infty}^2 \leq \|f - S_{n,k} f\|_{\infty} + t^2 \|D^2 S_{n,k} f\|_{\infty}^2.
\]
(5)

To prove our main result, we need to estimate the second term by the approximation error \(\|f - S_{n,k} f\|_{\infty}\). In a first step, we show that the second order differential operator \(D^2\) is bounded on the spline space.

**Lemma 3.** For \(k \geq 3\), the differential operator \(D^2 : S(n,k) \to S(n,k - 2)\) is bounded with
\[
\|D^2\|_{op} \leq \frac{4d_k}{h^2},
\]
where \(d_k > 0\) is a constant depending only on \(k\).

**Proof.** Let \(s \in S(n,k), s(x) = \sum_{j=k}^{n-1} c_j N_{j,k}(x), \) with \(\|s\|_{\infty} = 1\). According to M. Marsden [4], Lemma 2 on page 35, we can calculate the second order derivative by
\[
D^2 s(x) = \sum_{j=2-k}^{n-1} \frac{c_{j-1,k} - c_{j-1,k-1}}{\xi_{j,k-1} - \xi_{j-1,k-1}} - \frac{c_{j-1,k} - c_{j-2,k}}{\xi_{j-1,k} - \xi_{j-2,k}} \xi_{j,k-1} - \xi_{j-1,k-1} N_{j,k-2}(x).
\]
Then we obtain with the triangle inequality
\[
\|D^2 s\|_{\infty} = \left\| \sum_{j=2-k}^{n-1} \frac{\xi_{j,k-1} - \xi_{j-1,k-1}}{\xi_{j,k-1} - \xi_{j-1,k-1}} N_{j,k-2}(x) \right\|_{\infty}
\]
\[
\leq \|c\|_{\infty} + 2 \|c\|_{\infty} + \|c\|_{\infty} \cdot \sum_{j=1-k}^{n-1} \|N_{j,k-1}\|_{\infty},
\]
where
\[
\|c\|_{\infty} = \max \{|c_j| : j \in \{-k, \ldots, n-1\}\}.
\]
According to [8], there exists \(d_k > 0\), such that
\[
d_k^{-1} \|c\|_{\infty} \leq \sum_{j=-k}^{n-1} c_j N_{j,k} \leq \|c\|_{\infty}.
\]
Rewriting the first inequality yields \(\|c\|_{\infty} \leq D_k\), because \(\|s\|_{\infty} = 1\). Now we use the partition of unity to derive the estimate
\[
\|D^2 s\|_{\infty} \leq \frac{4}{h^2} d_k.
\]
Taking the supremum of all \(s \in \mathcal{S}(n,k)\) with \(\|s\|_{\infty} = 1\) yields the result.

Now we are able to prove our main result:

**Theorem 4.** For \(0 < t \leq \frac{1}{2}\) and \(k \geq 3\), there exists a constant \(M > 0\) only depending on \(n\) and \(k\), independent of \(f\), such that
\[
M \cdot \omega_2(f, t) \leq \|f - S_{n,k} f\|_{\infty}.
\]

**Proof.** We extend \(\|D^2 S_{n,k} f\|_{\infty}\) into a telescopic series:
\[
\|D^2 S_{n,k} f\|_{\infty} = \|D^2 S_{n,k} f - D^2 S_{n,k}^2 f + D^2 S_{n,k}^2 f - D^2 S_{n,k}^3 f + \ldots\|_{\infty}
\]
\[
\leq \sum_{m=1}^{\infty} \|D^2 S_{n,k}^m (f - S_{n,k} f)\|_{\infty}
\]
\[
= \|D^2 S_{n,k} (f - S_{n,k} f)\|_{\infty} + \sum_{m=2}^{\infty} \|D^2 S_{n,k}^m (f - S_{n,k} f)\|_{\infty}.
\]
Then we apply Corollary 2 and Lemma 3 and obtain
\[
\|D^2 S_{n,k} f\|_{\infty} \leq \frac{4d_k \|f - S_{n,k} f\|_{\infty}}{h^2} + \sum_{m=1}^{\infty} \frac{2\varepsilon_{n,k}}{h^2 \cdot m^{3/2}} \|f - S_{n,k} f\|_{\infty}
\]
\[
\leq \frac{4d_k + 2\varepsilon_{n,k} \cdot \zeta(\frac{3}{2})}{h^2} \|f - S_{n,k} f\|_{\infty}.
\]
Finally, applying the above result to (5) yields the estimate
\[
\omega_2(f, t) \leq \left( 4 + \frac{t^2(4d_k + 2\varepsilon_{n,k} \cdot \zeta(\frac{3}{2}))}{h^2} \right) \|f - S_{n,k} f\|_{\infty}
\]
Corollary 3. For $k \geq 3$, $n \to \infty$ and $f \in C([0,1])$ the following uniform estimate holds:
\[
\omega_2(f, \delta) \leq 5 \cdot \|f - S_{n,k} f\|_{\infty}.
\]
Proof. With
\[
\delta = \frac{h}{\sqrt{(4d_k + 2\varepsilon_{n,k} \cdot \zeta(\frac{1}{2}))}},
\]
the corollary follows, because for $n \to \infty$ we have that $h \to 0$ and hence, $\delta \to 0$.

Corollary 4. For $0 < t \leq \frac{1}{2}$ and $k \geq 3$, we have the equivalence
\[
\omega_2(f, t) \sim \|f - S_{n,k} f\|_{\infty}
\]
in the sense that there exist constants $M_1, M_2 > 0$ independent of $f$ and only depending on $n$ and $k$ such that
\[
M_1 \cdot \omega_2(f, t) \leq \|f - S_{n,k} f\|_{\infty} \leq M_2 \cdot \omega_2(f, t).
\]
Proof. We apply Theorem 4 to get the lower inequality and we use the inequality
\[
\|f - S_{n,k} f\|_{\infty} \leq \left(1 + \frac{1}{2t^2} \cdot \min \left\{\frac{1}{2k}, \frac{(k+1)H^2}{12}\right\}\right) \cdot \omega_2(f, t),
\]
from [1] to obtain the upper bound, where
\[
H := \max \{\{x_{j+1} - x_j\} : j \in \{-k, \ldots, n-1\}\}.
\]
Consequently, we have proved that the conjecture stated in [1] holds true under the conditions of Theorem 4. Additionally, we note that in Corollary 3 we have the relation $d_k \sim 2^k$. Therefore, $\delta$ tends to zero also for $k \to \infty$. With this note, we finally conclude with the following related conjecture:

Conjecture 1. For $n > 0$ fixed and $k \to \infty$, there exists $M > 0$ independent on $n$ and $k$ such that
\[
M \cdot \omega_2(f, \delta) \leq \|f - S_{n,k} f\|_{\infty}.
\]

References
[1] L. Beutel, H. Gonska, D. Kacso, G. Tachev, On variation-diminishing Schoenberg operators: new quantitative statements, Monografias de la Academia de Ciencias de Zaragoza 20 (2002) 9–58.
[2] T. Zapryanova, G. Tachev, Generalized Inverse Theorem for Schoenberg Operator, Journal of Modern Mathematics Frontier 1 (2).
[3] H. B. Curry, I. J. Schoenberg, On Pólya frequency functions. IV. The fundamental spline functions and their limits, Journal d'Analyse Mathématique 17 (1966) 71–107.
[4] M. Marsden, An identity for spline functions with applications to variation-diminishing spline approximation, Journal of Approximation Theory 3 (1970) 7–49.
[5] C. de Boor, A Practical Guide to Splines, Applied Mathematical Sciences, Springer, 1978.
[6] P. L. Butzer, H. Berens, Semi-groups of operators and approximation, Springer, 1967.
[7] H. Johnen, K. Scherer, On the Equivalence of the K-functional and Moduli of Continuity and Some Applications (1976).
[8] C. de Boor, The quasi-interpolant as a tool in elementary polynomial spline theory, Approximation Theory (GG Lorentz et al., eds), Academic Press (New York) (1973) 269–276.