Exact half-BPS Type IIB interface solutions II: Flux solutions and multi-Janus

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Abstract

Regularity and topology conditions are imposed on the exact Type IIB solutions on $AdS_4 \times S^2 \times S^2 \times \Sigma$ with 16 supersymmetries, which were derived in a companion paper \cite{1}. We construct an infinite class of regular solutions with varying dilaton, and non-zero 3-form fluxes. Our solutions may be viewed as the fully back-reacted geometries of $AdS_5 \times S^5$ (or more generally, Janus) doped with D5 and/or NS5 branes. The solutions are parametrized by the choice of an arbitrary genus $g$ hyper-elliptic Riemann surface $\Sigma$ with boundary, all of whose branch points are restricted to lie on a line. For genus 0, the Janus solution with 16 supersymmetries and 6 real parameters is recovered; its topology coincides with that of $AdS_5 \times S^5$. The genus $g \geq 1$ solutions are parametrized by a total of $4g + 6$ real numbers, $2g - 1$ of which are the real moduli of $\Sigma$. The solutions have $2g + 2$ asymptotic $AdS_5 \times S^5$ regions, $g$ three-spheres with RR 3-form charge, and another $g$ with NSNS 3-form charge. Collapse of consecutive branch points of $\Sigma$ yields singularities which correspond to D5 and NS5 branes in the probe limit. It is argued that the AdS/CFT dual gauge theory to each of our solutions consists of a 2+1-dimensional planar interface on which terminate $2g+2$ half-Minkowski 3+1-dimensional space-time $\mathcal{N} = 4$ super-Yang-Mills theories. Generally, the $\mathcal{N} = 4$ theory in each Minkowski half-space-time may have an independent value of the gauge coupling, and the interface may support various operators, whose interface couplings are further free parameters of the dual gauge theory.
1 Introduction

In a companion paper \[1\], the complete solution with 16 supersymmetries was obtained for Type IIB supergravity on $AdS_4 \times S^2 \times S^2 \times \Sigma$ with $SO(2,3) \times SO(3) \times SO(3)$ isometry. The solutions of \[1\] were found analytically in terms of two locally harmonic functions $h_1$ and $h_2$ on a Riemann surface $\Sigma$ with boundary. Generally, these solutions have varying dilaton and non-vanishing 3-form RR and NSNS fluxes. The goal of this paper is to present the construction of an infinite subclass of such solutions which have non-singular geometry.

One motivation for this investigation derives from the fact that the AdS/CFT duals to planar interface super-Yang-Mills theories in four dimensions are of this type. In particular, a planar interface theory with 16 supersymmetries was predicted to exist in \[3\], and its AdS/CFT dual was indeed found to be a regular supersymmetric Janus solution in \[1\]. (The related problem of AdS/CFT duals to defect super-Yang-Mills theory in the probe limit was studied in \[4\].) Another motivation stems from the similarity of the problem with the construction of half-BPS “bubbling geometries” in \[5\], and of AdS/CFT duals to half-BPS Wilson loop operators in $\mathcal{N} = 4$ super Yang-Mills in 4-dimensional space-time \[6, 7, 8\]. In particular, the AdS/CFT dual to a half-BPS Wilson loop corresponds to the geometry $AdS_2 \times S^2 \times S^4 \times \Sigma$ (see \[6\]); its general solution will be obtained in a further companion paper \[9\]. A final motivation is the construction of fully back-reacted Type IIB supergravity solutions in which the $AdS_5 \times S^5$ geometry is doped with D5 and/or NS5 branes.

In the present paper, we shall derive a set of regularity and topology conditions under which some of the general solutions of \[1\] are non-singular. We shall restrict attention to solutions for which one $S^2$, or the other $S^2$, shrinks to zero size on the boundary $\partial \Sigma$ of $\Sigma$. This will partition the boundary $\partial \Sigma$ into segments, each segment corresponding to the vanishing of either one $S^2$, or the other $S^2$, but not both. While the segments lie on the boundary $\partial \Sigma$, they correspond to regular interior points of the full 10-dimensional geometry. Two consecutive segments meet at a point on $\partial \Sigma$, which actually corresponds to an asymptotic $AdS_5 \times S^5$ throat region.

A general class of regular solutions allows for an arbitrary even number $2g + 2$ of asymptotic $AdS_5 \times S^5$ regions, connected by a smooth geometry. Specifically, these solutions will be parametrized by a genus $g \geq 0$ hyperelliptic Riemann surface $\Sigma$ with boundary $\partial \Sigma$, whose $2g + 2$ branch points lie on the real line. The harmonic functions $h_1$ and $h_2$ will obey alternating Neumann and Dirichlet boundary conditions on $\partial \Sigma$, corresponding to whether one

\footnote{The corresponding BPS equations for this geometry were obtained in \[2\], but no solutions, other than $AdS_5 \times S^5$, were constructed there.}
$S^2$ or the other $S^2$ shrinks to zero radius. The regularity and topology conditions, discussed in the preceding paragraph, will impose constraints on the zeros and poles of the Abelian differentials $\partial h_1$ and $\partial h_2$. In particular, $\partial h_1$ and $\partial h_2$ will have a double pole at each branch point, turning the point into an asymptotic $AdS_5 \times S^5$ region. Regularity requires a specific ordering pattern of the real zeros of the Abelian differentials $\partial h_1$ and $\partial h_2$ with respect to the branch points. These conditions will be solved in this paper. The resulting solutions will be referred to as “multi-Janus” solutions.

Our half-BPS solutions have some resemblance to the “bubbling AdS” solutions found in \[5\]. The “coloring” of [5] would correspond here to the alternating Neumann and Dirichlet boundary conditions obeyed by $h_1$ and $h_2$ on $\partial \Sigma$. From this perspective, the solutions obtained in this paper might be referred to as “bubbling multi-Janus” solutions.

At genus $g = 0$, the Janus solution with 16 supersymmetries with 6 free parameters, first obtained in \[1\], is recovered. It indeed exhibits 2 asymptotic $AdS_5 \times S^5$ regions, in agreement with the above counting, and has the topology of $AdS_5 \times S^5$.

At genus $g \geq 1$, the regular solutions are parametrized by the $2g - 1$ real moduli of $\Sigma$, and the $2g + 2$ real zeros of the Abelian differentials $\partial h_1$ and $\partial h_2$. Each solution has five further real parameters; one for the overall scale of the dilaton; one for the overall scale of the 10-dimensional metric, and 3 others allowing for global $SU(1,1)$ S-duality rotations to the general solution with a non-zero axion field. The total number of free parameters is thus $4g + 6$. The differentials $\partial h_1$ and $\partial h_2$ may also have common pairs of complex conjugate zeros, but their positions are fixed in terms of the moduli and the real zeros of $\partial h_1$ and $\partial h_2$ by certain elliptic or hyperelliptic period relations, which will be presented explicitly.

The genus $g \geq 1$ solutions have $2g + 2$ asymptotic $AdS_5 \times S^5$ regions. In each of these asymptotic regions, the dilaton tends to a constant, but the constants for different $AdS_5 \times S^5$ regions will generally be different from one another. The complete boundary of the full 10-dimensional geometry consists of $2g + 2$ four-dimensional conformal Minkowski space-times, each of which is obtained as the boundary in each separate $AdS_5 \times S^5$ region. The AdS/CFT correspondence implies that a conformal field theory will live in each of these $2g + 2$ four-dimensional Minkowski space-times. On the inside of the 10-dimensional geometry, we find $g$ homology 3-spheres which carry non-vanishing RR 3-form charge (and vanishing NSNS 3-form charge), and another $g$ homology spheres which carry non-vanishing NSNS 3-form charge (and vanishing RR 3-form charge).

The genus $g = 1$ case will be worked out in complete detail in terms of elliptic functions. The global geometry of the solution indeed exhibits 4 asymptotic $AdS_5 \times S^5$ regions, each with a distinct constant limiting value of the dilaton field, as represented in Figure 1. Ex-
istence of regular solutions will be shown analytically, for parameters valued in some open neighborhood.

Figure 1: The genus 1 solution has 4 distinct asymptotic $AdS_5 \times S^5$ regions, each with a different constant limit $\phi_1, \phi_2, \phi_3, \phi_4$ for the dilaton field. The radii $f_1$ and $f_2$ of the 2-spheres vanish on alternating segments of the boundary. The locations of the homology 3-spheres corresponding to RR and NSNS charges are also indicated.

For higher genus $g$, we present an argument for the existence of regular solutions by induction on the genus $g$. (The argument applies when the free parameters are valued in an open neighborhood containing the parameter space of the genus $g - 1$ regular solutions.) We have confirmed the existence of regular genus 2 solutions also by numerical analysis.

There is a sense in which the families of regular solutions for different genera $g$ are connected to one another. We shall show that one may pass from a regular genus $g$ solution to a regular genus $g - 1$ solution by allowing a branch cut between two consecutive branch points to shrink to zero size. The regularity and topology conditions require that the collapse of a branch cut on $\Sigma$ always be accompanied by the convergence to this branch cut of one of the complex zeros, and one real zero of either $\partial h_1$ or $\partial h_2$. A first option is to also let a second real zero converge to the collapsing branch cut; the regular solution at genus $g - 1$ is thereby recovered. A second option is to leave the remaining real zeros arbitrary. The collapsing branch cut then leaves two simple poles behind, which are argued to correspond
to a naked D5 or NS5 brane. The limit where the residue of this pole tends to 0 yields these D5 and NS5 branes in the probe limit. The solution with $g$ collapsed branch cuts is derived explicitly and shown to correspond to a Janus geometry doped with $g$ naked D5/NS5 branes.

The global description of the genus $g$ parameter space of regular solutions will be presented in the form of the definite ordering prescription of the branch points of $\Sigma$ and the real zeros of the differentials $\partial h_1$ and $\partial h_2$, as well as the vanishing of the Abelian integrals of $\partial h_1$ and $\partial h_2$ between certain consecutive branch points. The analysis of the global structure and topology of these parameter spaces poses an exciting and challenging mathematical problem, which is comparable to the problem of the studying the moduli space of instantons and magnetic monopoles.

The remainder of this paper is organized as follows. In section 2, we review the Ansatz for the Type IIB supergravity fields corresponding to the geometry $AdS_4 \times S^2 \times S^2 \times \Sigma$ with $SO(2,3) \times SO(3) \times SO(3)$ isometry, as well as the form of the complete exact solution in terms of two locally harmonic functions $h_1$ and $h_2$ on $\Sigma$, derived in [1]. In section 3, we derive the general regularity and topology conditions on $h_1$, $h_2$, and their differentials $\partial h_1$ and $\partial h_2$. In section 4, we introduce our general solution by presenting the genus $g$ hyperelliptic Ansatz, and subjecting it to all the regularity and topology conditions of section 3. These conditions are solved in terms of a unique relative ordering of the branch points of $\Sigma$ and the real zeros of the differentials $\partial h_1$ and $\partial h_2$. For $g \geq 1$, the homology 3-spheres are constructed and the corresponding charges of the RR and NSNS 3-forms are evaluated and shown to be generally non-vanishing. In section 5, the genus 1 case is worked out in complete detail. In section 6, the genus 2 case is solved in part analytically, and in part numerically, and a general argument is given for the existence of solutions to all genera. In section 7, the collapse of branch cuts is carried out, and the presence of probe D5/NS5 branes is demonstrated by evaluating the 3-form fluxes on these solutions. In section 8, a dual gauge theory is proposed as a generalization of the interface conformal field theory discussed in [3]. In Appendix A we add poles to the differentials $\partial h_1$ and $\partial h_2$ that are on the inside of $\Sigma$, and show that these do not lead to regular solutions.
2 Review of the Ansatz and the general local solution

In Type IIB supergravity \[10, 11\], the bosonic fields are the metric \(ds^2\), the axion/dilaton fields, represented by the 1-forms \(P\) and \(Q\), the complex 3-form \(G\) and the self-dual 5-form \(F_{(5)}\). Upon use of the Bianchi identities, the field \(P, Q\) may equivalently be represented by the more customary dilaton and axion fields \(\Phi, \chi\), while the 3-form \(G\) and the 5-form \(F_{(5)}\) derive from the complex 2-form \(B_{(2)}\) and real 4-form \(C_{(4)}\) potentials. The fermionic fields are the gravitino \(\psi\) and the dilatino \(\lambda\). The field equations, and BPS equations \(\delta \psi = \delta \lambda = 0\) for \(\psi_M = \lambda = 0\), as well as our conventions for Dirac matrices were presented in detail in [1].

In this section, we shall review the Ansatz for the fields on the \(AdS_4 \times S^2_1 \times S^2_2 \times \Sigma\) geometry with \(SO(2, 3) \times SO(3) \times SO(3)\) isometry, as well as the complete solution with 16 supersymmetries derived in [1].

2.1 The Ansatz

The Ansatz for the metric is

\[
d s^2 = f_4^2 d s^2_{AdS_4} + f_1^2 d s^2_{S^2_1} + f_2^2 d s^2_{S^2_2} + d s^2_{\Sigma} \tag{2.1}
\]

Here, \(d s^2_{AdS_4}\), \(d s^2_{S^2_1}\) and \(d s^2_{S^2_2}\) are the maximally symmetric metrics respectively on \(AdS_4\), \(S^2_1\) and \(S^2_2\) with unit radii; \(d s^2_{\Sigma}\) is a Riemannian metric on \(\Sigma\), and \(f_1, f_2, f_4\) are real functions on \(\Sigma\). Since \(\Sigma\) has a metric and an orientation it is a Riemann surface, and we may choose local conformal complex coordinates \(w, \bar{w}\), in which \(d s^2_{\Sigma} = 4 \rho^2 |dw|^2\), for a real \(\rho\) on \(\Sigma\). It will be useful to recast the frame \(e_a\), \(a = 8, 9\) on \(\Sigma\) in terms of local complex coordinates and

\[
e^z = (e^8 + i e^9)/2 = \rho dw
\]

\[
e^\bar{z} = (e^8 - i e^9)/2 = \rho d \bar{w} \tag{2.2}
\]

The dilaton/axion fields \(P, Q\) are 1-forms, and their Ansatz is given as follows,

\[
P = p_a e^a \\
Q = q_a e^a \tag{2.3}
\]

while the anti-symmetric tensor 3-form \(G\) and self-dual 5-form \(F_{(5)}\) are given by

\[
G = g_a e^{45a} + i h_a e^{67a} \\
F_{(5)} = f_a (-e^{0123a} + \varepsilon^{ab} e^{4567b}) \tag{2.4}
\]

Here, \(f_a, q_a\) are real, while \(g_a, h_a, p_a\) are complex frame vectors on \(\Sigma\).

\[\text{We use conventions where the frame indices are denoted by } z \text{ and } \bar{z}; \text{ the frame metric has non-vanishing components } \delta_{zz} = \delta_{\bar{z}\bar{z}} = 2; \text{ and the orientation on } \Sigma \text{ is given by } \varepsilon^{89} = +1.\]
2.2 The general solution

In [1], it was shown that the BPS equations imply certain reality conditions on the supersymmetry transformation spinors as well as on the components \( p_a, q_a, g_a \) and \( h_a \) of the Ansatz. These reality conditions allow every solution of the BPS equations to be mapped, under the \( SU(1,1) \) symmetry of Type IIB supergravity, into a solution with vanishing axion field, for which \( p_a, q_a \) and \( h_a \) are real, and \( q_a = 0 \). We shall solve the case where these conditions hold. Then applying arbitrary \( SU(1,1) \) transformations will produce all solutions with non-vanishing axion field as well, and accounts for 3 parameters for every solution.

The general solution, with 16 supersymmetries, to Type IIB supergravity on the manifold \( AdS_4 \times S_1^2 \times S_2^2 \times \Sigma \) with \( SO(2,3) \times SO(3) \times SO(3) \) isometry, is parametrized by two real harmonic functions \( h_1, h_2 \) on \( \Sigma \). The form \( W \) of weight \( (1,1) \) is ubiquitous in the solution, and is defined by

\[
W \equiv \partial_w h_1 \partial_{\bar{w}} h_2 + \partial_w h_2 \partial_{\bar{w}} h_1
\]

The form \( W \) is the inner product between the vector fields \( \partial_w h_1 \) and \( \partial_w h_2 \), so that \( W \) will vanish when these fields are orthogonal to one another.

The solution for the dilaton field \( \Phi = 2\phi \) has the following form:

\[
e^{4\phi} = \frac{2h_1 h_2 |\partial_w h_2|^2 - h_2^2 W}{2h_1 h_2 |\partial_w h_1|^2 - h_1^2 W}
\]

while the solution for the conformal factor \( \rho^2 \) in the \( \Sigma \)-metric is given by

\[
\rho^8 = \frac{W^2}{h_1^2 h_2^2} \left( 2h_1 |\partial_w h_2|^2 - h_2 W \right) \left( 2h_2 |\partial_w h_1|^2 - h_1 W \right)
\]

The functions \( f_1, f_2, \) and \( f_4 \) entering the ten-dimensional metric (2.1) are most succinctly expressed by leaving some of the \( \phi \)- and \( \rho \)-dependence manifest in the solution,

\[
\rho f_1 = -2\nu \text{Re} \left( e^{-2\phi} |\partial_w h_2|^2 - e^{2\phi} |\partial_w h_1|^2 - iW \right)^{\frac{1}{2}}
\]
\[
\rho f_2 = -2 \text{Im} \left( e^{-2\phi} |\partial_w h_2|^2 - e^{2\phi} |\partial_w h_1|^2 - iW \right)^{\frac{1}{2}}
\]
\[
\rho f_4 = |e^{-\phi} \partial_w h_2 - i e^\phi \partial_w h_1| + |e^{-\phi} \partial_w h_2 + i e^\phi \partial_w h_1|
\]

Formulas directly for the metric factors \( f_1, f_2 \) and \( f_4 \) were derived in Appendix E of [1]. Of interest to us in the present paper will be the following combinations, recorded for \( W \leq 0 \),

\[
f_1^2 f_4^2 = 4e^{+2\phi} h_1^2
\]
\[
f_2^2 f_4^2 = 4e^{-2\phi} h_2^2
\]

\(^3\)For notational convenience, we use \( \phi = \Phi/2 \) to represent the dilaton, in accord with [1].
The expression for $W > 0$ will not be needed in this paper.

Finally, the expressions for the 3-form components $g_a$, $h_a$ are most usefully presented in terms of the $B_2$ gauge potential, as this will allow us to directly compute 3-form fluxes and charges. The values for the field strength $F(3) = dB(2)$ may be found in [1], but will not be needed here. We have,

$$B(2) = b_1 \hat{e}^{45} + ib_2 \hat{e}^{67}$$  \hspace{1cm} (2.10)

Here, $\hat{e}^{45}$ and $\hat{e}^{67}$ are the volume forms for the spheres $S^2_1$ and $S^2_2$ with unit radii. The functions $b_1$ and $b_2$ are given by

$$b_1 = +2 \tilde{h}_2 + 2i h_1 h_2 \frac{\partial_w h_1 \partial_{\bar{w}} h_2 - \partial_w h_2 \partial_{\bar{w}} h_1}{2h_2|\partial_w h_1|^2 - h_1 W}$$

$$b_2 = -2 \tilde{h}_1 + 2i h_1 h_2 \frac{\partial_w h_1 \partial_{\bar{w}} h_2 - \partial_w h_2 \partial_{\bar{w}} h_1}{2h_1|\partial_w h_2|^2 - h_2 W}$$  \hspace{1cm} (2.11)

where $\tilde{h}_1$ and $\tilde{h}_2$ are the real harmonic functions conjugate to the harmonic functions $h_1$ and $h_2$ respectively. As such, they satisfy the conjugation relations,

$$\partial_w h_1 = -i \partial_w \tilde{h}_1$$

$$\partial_w h_2 = -i \partial_w \tilde{h}_2$$  \hspace{1cm} (2.12)

These solutions are local in the sense that the range of the local complex coordinate $w$, $\bar{w}$, and thus the surface $\Sigma$, remains to be specified globally.

### 2.3 The AdS$_5 \times S^5$ and Janus solutions

A 2-parameter family of solutions is obtained from [1],

$$h_1 = 2e^{-\phi_+} \text{Im} (e^w) - 2e^{-\phi_-} \text{Im} (e^{-w})$$

$$h_2 = 2e^{\phi_+} \text{Re} (e^w) + 2e^{\phi_-} \text{Re} (e^{-w})$$  \hspace{1cm} (2.13)

on the infinite strip,

$$\Sigma = \left\{ w \in \mathbb{C}; 0 \leq \text{Im}(w) \leq \frac{\pi}{2} \right\}$$  \hspace{1cm} (2.14)

The dilaton $\phi$ tends to the constants $\phi_{\pm}$ as $\text{Re}(w) \to \pm \infty$. When $\phi_+ = \phi_-$, this is just the AdS$_5 \times S^5$ solution, while for $\phi_+ \neq \phi_-$, it is the Janus solution with 16 supersymmetries, predicted in [3] on the basis of its AdS/CFT dual interface super Yang-Mills theory, and calculated explicitly, including the expressions of $\phi, \rho, f_1, f_2, f_4, g_a, h_a, f_a$, in [1].
3 Regularity and topology conditions

To extend the general local solution, given in terms of harmonic functions $h_1$ and $h_2$, to a global non-singular solution, we need to specify further global conformal data, namely the domain $\Sigma$ on which $h_1$ and $h_2$ are harmonic. Choosing generic $\Sigma$, $h_1$, and $h_2$ will generally lead to singularities in the dilaton $\phi$ and other functions which specify the Ansatz, and thus to singular solutions to Type IIB supergravity. Regularity will require interrelations between $\Sigma$, $h_1$ and $h_2$, which we shall now exhibit.

3.1 Basic regularity conditions

We shall adopt the following general regularity conditions on all the solutions considered in this paper. The dilaton $\phi$ and the metric functions $\rho^2$, $f_1$, $f_2$, and $f_4$ are

(R1) non-singular in the interior of $\Sigma$;

(R2) non-singular on the boundary $\partial \Sigma$, except possibly at isolated points.

The $AdS_5 \times S^5$ asymptotic region is one example of an isolated point in $\partial \Sigma$ where $f_4$ diverges. But the nature of this singularity on $\partial \Sigma$ is well-understood; it corresponds to a regular 10-dimensional geometry, and thus must be allowed. The regularity conditions (R1) and (R2) will also allow for the singularities of probe D5 and/or NS5 branes, for which $f_1$, $f_2$, and $f_4$ all diverge at isolated points on $\Sigma$. The fact that local solutions which have poles in the interior of $\Sigma$ lead to singular geometries is demonstrated for a generic class of singularities in Appendix A.

3.2 Topology conditions

In this paper, we shall restrict attention to solutions whose boundary is locally that of $AdS_5 \times S^5$. Poles in $\partial h_1$ and $\partial h_2$, located at isolated points on the boundary $\partial \Sigma$ will correspond to asymptotic throat regions, as in the $AdS_5 \times S^5$ and supersymmetric Janus solutions. The remaining part of the boundary consists of open segments. We shall require that these open segments of $\partial \Sigma$ correspond to regular interior points (and not to boundary points) of the full 10-dimensional Type IIB solution. This can be achieved by requiring that throughout each such segment, one $S^2$, or the other $S^2$, but not both, shrink to zero size. This topological condition already holds on the $AdS_5 \times S^5$ and Janus solutions, where it guarantees that the spheres $S^2_1$ and $S^2_2$ shrink to zero size in a manner precisely needed to recover the overall $S^5$ topology.
To investigate this topology, it is useful to consider the following combinations of the radii \( f_1 \) and \( f_2 \), readily derived from (2.7),

\[
\rho^4 f_1^2 f_2^2 = 4W^2
\]  

(3.1)

As a result of our earlier topology condition, the product \( f_1 f_2 \) must vanish along the entire boundary \( \partial \Sigma \). Using the regularity condition (R2) that \( \rho^2 \) is non-singular on \( \partial \Sigma \), except possibly at isolated points, as well as equations (3.1), it is clear that the vanishing of \( f_1 f_2 \) on \( \partial \Sigma \) implies,

\[
W(w, \bar{w}) = 0 \quad \text{for all} \quad w \in \partial \Sigma
\]  

(3.2)

For the Janus solution of (2.13), \( \Sigma \) of (2.14) consist of a strip \( w \in \mathbb{R} \oplus i[0, \pi/2] \), and \( f_1 \) vanishes at 0 while \( f_2 \) vanishes at \( \pi/2 \). Furthermore, we have \( W = -4 \chi(\phi_+ - \phi_-) \sin(2 \Im(w)) \), which indeed vanishes throughout the boundary \( \partial \Sigma \) of \( \Sigma \) in (2.14).

We shall also require that \( \partial \Sigma \) have only a single connected component. From this condition, it follows that \( f_1 f_2 \) and \( W \) cannot vanish in the interior of \( \Sigma \), except at isolated points. By continuity, the sign of \( W \) must remain constant throughout the interior of \( \Sigma \). If it were not, there would arise a curve on the inside of \( \Sigma \) where \( W = 0 \), which would contradict the requirement of a single boundary component. Since we seek families of solutions connected to \( \text{AdS}_5 \times S^5 \), for which \( W < 0 \) inside \( \Sigma \), we shall require \( W < 0 \) on the inside of \( \Sigma \), except at isolated points where \( W \) may vanish.

### 3.3 Interior zeros

Negativity of \( W \) greatly restricts the allowed zeros of the Abelian differentials \( \partial h_1 \) and \( \partial h_2 \). We shall now argue that all zeros of \( \partial h_1 \) and \( \partial h_2 \) in the interior of \( \Sigma \) must be common. For example, if \( \partial h_1 \) has a zero of order \( m \) at an interior point \( w_0 \in \Sigma \), then \( \partial h_2 \) must have a zero of order precisely \( m \) at \( w_0 \) as well.

To show this, choose local coordinates \( v, \bar{v} \) around \( w_0 \) so that \( v = 0 \) at \( w_0 \), and

\[
\begin{align*}
\partial h_1 &= \partial_v h_1 dv = v^{m_1} dv + O(v^{m_1+1} dv) \\
\partial h_2 &= \partial_v h_2 dv = c v^{m_2} dv + O(v^{m_2+1} dv)
\end{align*}
\]  

(3.3)

Parametrizing \( v \) and \( c \) by polar coordinates \( v = |v| e^{i\theta} \), and \( c = |c| e^{i\gamma} \), we readily evaluate

\[
W = 2|c||v|^{m_1+m_2} \cos \left( (m_1 - m_2)\theta - \gamma \right) + O(|v|^{m_1+m_2+1})
\]  

(3.4)

Since \( w_0 \) is an interior point, the range of \( \theta \) is over the full circle, \( 0 \leq \theta \leq 2\pi \). The only manner in which we can have \( W < 0 \) for sufficiently small \( |v| \) and all \( \theta \in [0, 2\pi] \) is if \( m_1 = m_2 \), which proves our earlier assertion. It then suffices to require \( \cos \gamma < 0 \).
3.4 Boundary conditions on $h_1$ and $h_2$

Using (2.9), sharp boundary conditions on $h_1$ and $h_2$ may be obtained. First, we argue that the AdS$_4$-radius $f_4$ cannot vanish. If follows from equation (6.26) of [1] that, if $f_4$ vanished, then we also must have $f_1 = f_2 = 0$, resulting in an unphysical singularity of the 10-dimensional Type IIB geometry. In particular, in the AdS$_5 \times S^5$ and Janus solutions, the function $f_4$ remains bounded away from 0. Combining the facts that $f_4 \neq 0$, and that $\phi$ is non-singular (except perhaps at isolated points) with equations (2.9), it is clear that $f_1$ vanishes if and only if $h_1 = 0$, while $f_2$ vanishes if and only if $h_2 = 0$.

Next, we shall show that, if $h_2 = 0$ in an open neighborhood $U_0 \subset \partial \Sigma$, then $h_1 \neq 0$ in $U_0$, except possibly at isolated points, and $h_1$ satisfies Neumann boundary conditions in $U_0$. The same statement, but with the roles of $h_1$ and $h_2$ interchanged, also holds.

To prove that $h_1$ satisfies Neumann boundary conditions throughout $U_0$, we use the fact that $h_2$ is harmonic in $\Sigma$, and vanishes on $U_0$, to choose conformal coordinates $w = x + iy$ (with $x, y$ real), such that $h_2 = y$ in an open neighborhood $S_0 \subset \Sigma$ which contains $U_0$. In terms of this coordinate $w$ on $S_0$, $W$ takes the following form,

$$W = -\Im(\partial_w h_1) = -\partial_y h_1$$

(3.5)

The vanishing of $W$ on the boundary $\partial \Sigma$, as required by (3.2), implies that $\partial_y h_1 = 0$ for all points in $U_0$, i.e. $h_1$ satisfies Neumann boundary conditions on $U_0$.

To prove that $h_1 \neq 0$ in $U_0$, except possibly at isolated points, we make use of the expression for the dilaton field (2.6) on our solutions. Each term in the numerator and denominator of (2.6) vanishes on $U_0$, either because $W = 0$, or because $h_2 = 0$, or both. Since with our choice of coordinates $w$ we have $\partial_w h_2 = -i/2$, which is non-vanishing, the first term in the numerator of (2.6) dominates the second. Neglecting the second term allows us to simplify by a factor of $h_1$ and we are left with,

$$e^{4\phi} \sim \frac{1}{4|\partial_w h_1|^2 + 2y^{-1}h_1 \partial_y h_1}$$

(3.6)

If $h_1$ were to vanish throughout an open set $U'_0 \subset U_0$, then $|\partial_w h_1|^2$ must also vanish there, since then we would have both $\partial_x h_1 = 0$ and $\partial_y h_1 = 0$. Furthermore, since $\partial_y h_1 = 0$ in $U'_0$, it follows that, if $h_1$ also vanishes there, we actually must have $y^{-1}h_1 \to 0$ as $y \to 0$. As a result, the denominator has to vanish as $y \to 0$, and the dilaton would be singular throughout the open set $U'_0$, which contradicts assumption (R2) that the dilaton must be non-singular on $\partial \Sigma$, except possibly at isolated points. Thus, we must have $h_1 \neq 0$ on $U_0$, except possibly at isolated points. This completes the proof of our above statement.
In summary, we have a remarkable conclusion. The boundary $\partial \Sigma$ is partitioned into two open sets $\partial \Sigma_+$ and $\partial \Sigma_-$. The closures of $\partial \Sigma_+$ and $\partial \Sigma_-$ intersect at isolated points, and their union is $\partial \Sigma$. The following boundary conditions on the functions $h_1$ and $h_2$ now hold,

\[
\begin{align*}
\text{on } \partial \Sigma_+ & \quad f_1 = h_1 = 0 \quad h_1 = \text{vanishing Dirichlet} \\
& \quad \partial_n h_2 = 0 \quad h_2 = \text{Neumann} \\
\text{on } \partial \Sigma_- & \quad f_2 = h_2 = 0 \quad h_2 = \text{vanishing Dirichlet} \\
& \quad \partial_n h_1 = 0 \quad h_1 = \text{Neumann}
\end{align*}
\]

where $\partial_n$ denotes the derivative normal to the boundary $\partial \Sigma$.

### 3.5 Summary of all regularity and topology conditions

Having analyzed the behavior of the harmonic functions $h_1$ and $h_2$ on the boundary $\partial \Sigma$, it remains to determine their allowed behavior on the inside of $\Sigma$. Given that $f_1$ cannot vanish in $\Sigma$, as argued in the first paragraph of subsection 3.4, it follows from (2.9) that $h_1$ and $h_2$ cannot vanish on the inside of $\Sigma$. Indeed a line or domain of zeros would force $f_1$ or $f_2$ to vanish in the interior of $\Sigma$, which is inconsistent with the assumption of subsection 3.2 that $\partial \Sigma$ consists of only a single connected component. For the Janus solution in (2.13), we have $h_1, h_2 > 0$ in the interior of $\Sigma$. Since our solutions will be connected to $AdS_5 \times S^5$, we require $h_1, h_2 > 0$ on the inside of $\Sigma$ throughout.

In summary, we have the following regularity conditions, in addition to (R1) and (R2),

(R3) On $\partial \Sigma_+$, $h_1$ and $h_2$ obey respectively Dirichlet and Neumann boundary conditions; 
On $\partial \Sigma_-$, $h_1$ and $h_2$ obey respectively Neumann and Dirichlet boundary conditions; 
Note that these two conditions together imply $W = 0$ throughout $\partial \Sigma$.

(R4) All zeros of $\partial h_1$ and $\partial h_2$ on the inside of $\Sigma$ must be common;

(R5) $W < 0$ on the inside of $\Sigma$, except possibly at isolated points, where $W = 0$;

(R6) $h_1 > 0$ and $h_2 > 0$ on the inside of $\Sigma$;

(R7) All Dirichlet boundary conditions must be vanishing, as given in (3.7).

Remarkably, this combination of regularity and topology assumptions, leads to boundary conditions akin to those of 2-dimensional electro-statics, with Dirichlet and Neumann components corresponding respectively to perfect conductor and perfect insulator.
4 The hyperelliptic Ansatz

We shall now present a construction for the harmonic functions \( h_1 \) and \( h_2 \), in terms of hyperelliptic Riemann surfaces of genus \( g \), which automatically solve the conditions (R1-R5) of section 3.5. Conditions (R6) and (R7) are more subtle, and will lead to certain relations on the parameters, which we shall derive explicitly. The genus 0 case reduces to the Janus and \( AdS_5 \times S^5 \) solutions, which are completely non-singular. For general genus \( g \), the Ansatz has \( 4g + 6 \) parameters (including one for the overall shift of the dilaton one for the overall scale of the 10-dimensional metric, and 3 for the \( SU(1, 1) \) rotation parameters to the general solution with non-vanishing axion.) In the next section, all regularity conditions will be solved analytically for the case of genus 1, and we shall prove that, in a certain range of these parameters, the full geometry of the solution is non-singular.

4.1 The Janus solution re-expressed on the lower half-plane

To begin construction of the hyperelliptic Ansatz, we map the Janus and \( AdS_5 \times S^5 \) solutions to the lower half-plane \( \Sigma \), on which the complex coordinates will be denoted by \( u, \bar{u} \) with \( \text{Im}(u) \leq 0 \). This may be done with the help of the exponential mapping,

\[-\sqrt{u} = e^{-w-(\phi_+-\phi_-)/2} \quad 0 \leq \text{Im}(w) \leq \frac{\pi}{2} \]  

(4.1)

where \( \text{Re}(w), \text{Re}(u) \) take values through \( \mathbb{R} \). Notice that \( \text{Re}(w) = -\infty \) and \( \text{Re}(w) = +\infty \) are respectively mapped to \( u = \infty \) and \( u = 0 \). The harmonic functions \( h_1, h_2 \) and their differentials \( \partial h_1, \partial h_2 \) are given by,

\[ h_1(u) = i \frac{r - u}{\sqrt{u}} + \text{c.c.} \quad \partial h_1 = -\frac{i}{2} \frac{u + r}{\sqrt{u}} du \]

\[ h_2(u) = -\frac{1 + u}{\sqrt{u}} + \text{c.c.} \quad \partial h_2 = -\frac{1}{2} \frac{u - 1}{\sqrt{u}} du \]  

(4.2)

Here, we use the shorthand \( r = e^{2\phi_- - 2\phi_+} \) and, to simplify the form of the differentials, we have omitted overall multiplicative constants in \( h_1 \) and \( h_2 \) whose effect is to shift the dilaton by an overall constant; these factors may be easily restored. From the differentials, we see that \( \partial h_1 \) is real along the negative real axis and imaginary along the positive real axis, and thus respectively obeys Neumann and Dirichlet boundary conditions along these segments.

For \( \partial h_2 \), the situation is reversed, as depicted in Figure 2. Thus, the solution (4.2) satisfies the conditions (3.7) with \( \partial \Sigma_+ = ]0, +\infty[ \) and \( \partial \Sigma_- = ]-\infty, 0[ \). It was shown in [1] that the solution is everywhere non-singular, so that all conditions (R1-R7) are in fact satisfied.
4.2 The hyperelliptic Ansatz

The Janus solution may be generalized by having, instead of just a single vanishing Dirichlet (D) segment $]0, +\infty[$, and a single Neumann (N) segment $]-\infty, 0[$ for $h_1$, and opposite boundary conditions for $h_2$, a larger number of such segments. This may be achieved by arranging the differentials $\partial h_1$ and $\partial h_2$ to have alternating real and purely imaginary values over several segments along $\partial \Sigma$. Since topology requires the boundary $\partial \Sigma$ to have only a single connected component, we may conformally map $\partial \Sigma$ to the entire real axis, and the interior of $\Sigma$ to the lower half-plane. The hyperelliptic Ansatz is obtained by having $2g + 2$ instead of $2$ separate segments of alternating D and N boundary conditions on the real axis $\partial \Sigma$.

Multiple alternating D and N boundary conditions may be obtained in terms of $2g + 2$ branch points on the real line, $e_1, e_2, \cdots, e_{2g+1}$, and $e_{2g+2} = \infty$, and the polynomial

$$s^2 = (u - e_1) \prod_{k=1}^{g}(u - e_{2k})(u - e_{2k+1}) \quad (4.3)$$

Here, we have used the $SL(2, \mathbb{R})$ symmetry on the lower complex half-plane to fix one branch point at $\infty$; two further branch points may be chosen at arbitrary points as well, using $SL(2, \mathbb{R})$. The algebraic equation (4.3) defines a hyperelliptic surface of genus $g$. Its $2g - 1$ real moduli for $g \geq 1$, may be parametrized by the $2g - 1$ remaining branch points. It will be convenient to prescribe the following definite ordering for the branch points,

$$e_{2g+1} < e_{2g} < e_{2g-1} < \cdots < e_3 < e_2 < e_1 \quad (4.4)$$
The collapse of any pair of consecutive branch points will produce a non-separating degeneration to a hyperelliptic surface of genus $g - 1$.

To produce a suitable generalization which satisfies the regularity conditions (R1) and (R2), we shall assume that the singularities of the differentials $\partial h_1$ and $\partial h_2$ at each branch point are no worse than those of the $AdS_5 \times S^5$ solution, i.e. of the form $du/(u - e_i)^{3/2}$. The differentials then have at most double poles at the branch points. A double pole at $u = \infty$ translates to the asymptotic behavior $du/\sqrt{u}$ as $u \to \infty$. Poles on the inside of $\Sigma$ would lead to singular solutions, as shown in Appendix A, and are thus excluded by (R1) and (R2).

Combining these requirements, we have the following form of the differentials,

$$\partial h_1 = -i \frac{P_1(u)}{s(u)^3} du$$
$$\partial h_2 = - \frac{P_2(u)}{s(u)^3} du$$

(4.5)

where $P_1(u)$ and $P_2(u)$ are polynomials of degree $3g + 1$ in $u$. The overall $-i$ sign for both $\partial h_1$ and $\partial h_2$ has been introduced later convenience.

To realize condition (R3) on the differentials, $\partial h_1$ and $\partial h_2$ need to alternate between taking real and purely imaginary values on the real axis. Along the real axis, the sign of $s^2$ behaves as follows,

$$s^2 > 0 \quad u \in \partial \Sigma_+ \equiv [e_1, \infty \cup \bigcup_{i=1}^g e_{2i+1}, e_{2i}]$$
$$s^2 < 0 \quad u \in \partial \Sigma_- \equiv ] - \infty, e_{2g+1} \cup \bigcup_{i=1}^g e_{2i}, e_{2i-1}[$$

(4.6)

so that the denominator $s(u)^3$ alternates between being real and purely imaginary on the real axis. Thus, to realize (R3), we must require that $P_1(u)$ and $P_2(u)$ either both be real, or both be purely imaginary on the real axis. Without loss of generality, we may choose both $P_1$ and $P_2$ to be real, i.e. to be polynomials with all real coefficients. The boundary conditions (R3) are then automatically satisfied, and we have

$$u \in \partial \Sigma_+ \quad \partial_u h_1 = \text{imaginary} \quad \text{Dirichlet}$$
$$\partial_u h_2 = \text{real} \quad \text{Neumann}$$

$$u \in \partial \Sigma_- \quad \partial_u h_1 = \text{real} \quad \text{Neumann}$$
$$\partial_u h_2 = \text{imaginary} \quad \text{Dirichlet}$$

(4.7)

4 Certain types of poles on the real axis will lead to mildly singular space-time solutions, which are of the type of D5 or NS5 branes in the probe limit. These solutions will in fact be recovered as limits of regular hyperelliptic solutions, as will be demonstrated in section 7.
Note that, given the alternating structure of the D and N boundary conditions, it would have been unnatural for our solutions to use either \( h_1 \) or \( h_2 \) as the real or imaginary part of a global system of conformal coordinates on \( \Sigma \).

### 4.3 Structure of the complex zeros

Condition (R4) of subsection 3.5 requires all the interior zeros, for which \( \text{Im}(u) < 0 \), of \( \partial h_1 \) and \( \partial h_2 \) to be common. The polynomials \( P_1 \) and \( P_2 \) have real coefficients, so that their zeros are either real or come in complex pairs. These complex zeros must be common between \( P_1 \) and \( P_2 \) in view of (R4), so that we have the decomposition, \( P_1(u) = P(u)Q_1(u) \) and \( P_2(u) = P(u)Q_1(u) \), where \( P(u) \) is a real polynomial whose roots are all complex and \( Q_1(u) \) and \( Q_2(u) \) are polynomials with only real roots. Thus, we obtain the more specific forms,

\[
\overline{\partial h_1} = -i \frac{P(u)Q_1(u)}{s(u)^3} du \\
\overline{\partial h_2} = -\frac{P(u)Q_2(u)}{s(u)^3} du
\]  

(4.8)

We shall assume that the roots of \( Q_1 \) are distinct; coincident roots may be attained as a limit later. We shall parametrize and order the roots of these polynomials as follows,

\[
P(u) = \prod_{a=1}^{p} (u - u_a)(u - \bar{u}_a) \quad \text{Im}(u_a) < 0
\]

\[
Q_1(u) = \prod_{b=1}^{q} (u - \alpha_b) \quad \alpha_q < \alpha_{q-1} < \cdots < \alpha_2 < \alpha_1
\]

\[
Q_2(u) = \prod_{b=1}^{q} (u - \beta_b) \quad \beta_q < \beta_{q-1} < \cdots < \beta_2 < \beta_1
\]  

(4.9)

Here, \( \alpha_b, \beta_b \in \mathbb{R} \) and \( 3g + 1 = 2p + q \) in view of the fact that \( P_1 \) and \( P_2 \) have degree \( 3g + 1 \).

### 4.4 Negativity of \( W \)

Condition (R5) of subsection 3.5 requires that \( W < 0 \) for all \( \text{Im}(u) < 0 \), except possibly at isolated points, where we may have \( W = 0 \). Clearly, at the common complex zeros \( u_a \) both differentials \( \partial h_1 \) and \( \partial h_2 \) and thus \( W \) all vanish. The points \( u_a \) are, of course, isolated points, and these zeros of \( W \) fit with the terms of condition (R5). To investigate the condition \( W < 0 \) elsewhere, we recast \( W \) in terms of the polynomials \( P, Q_1, Q_2 \),

\[
W = -\frac{|P(u)Q_1(u)|^2}{|s|^6} \left( i \frac{Q_2(u)}{Q_1(u)} - i \frac{Q_2(\bar{u})}{Q_1(\bar{u})} \right)
\]  

(4.10)
Since all the zeros of $Q_1$ are real and simple, and $Q_1$ and $Q_2$ are of the same degree $q$ and start with the same highest monomial $u^q$, the ratio $Q_2/Q_1$ may be decomposed as follows,

$$Q(u) \equiv \frac{Q_2(u)}{Q_1(u)} = 1 - \sum_{b=1}^{q} \frac{\gamma_b}{u - \alpha_b}$$  \hspace{1cm} (4.11)

where the residues $\gamma_b$ are real. As a result, $W$ takes the form,

$$W = -i(u - \bar{u}) \frac{|P(u)Q_1(u)|^2}{|s|^6} \sum_{b=1}^{q} \frac{\gamma_b}{|u - \alpha_b|^2}$$  \hspace{1cm} (4.12)

The requirement $W \leq 0$ in the lower half-plane is equivalent to

$$\gamma_b \geq 0 \hspace{1cm} b = 1, \cdots, q$$  \hspace{1cm} (4.13)

Notice that if $\gamma_b = 0$ then $\alpha_b$ is a zero common to $Q_1$ and $Q_2$. We begin by assuming that $\gamma_b > 0$ for all $b = 1, \cdots, q$, the alternative may be attained as a limiting case hereof. The condition $\gamma_b > 0$ is equivalent to an ordering condition between the zeros of $Q_1$ and $Q_2$, as shown in the Lemma below.

Lemma

(a) For $\gamma_b > 0$, $b = 1, \cdots, q$, the zeros $\beta_b$ of $Q_2$ and the zeros $\alpha_b$ of $Q_1$, alternate,

$$\alpha_q < \beta_q < \alpha_{q-1} < \beta_{q-1} < \cdots < \alpha_2 < \beta_2 < \alpha_1 < \beta_1$$  \hspace{1cm} (4.14)

(b) Vice-versa, if $Q_1(x)$ and $Q_2(x)$ have highest monomial $x^q$ and real zeros which alternate as in (4.14), then $Q(x) = Q_2(x)/Q_1(x)$ admits a pole decomposition (4.11) with $\gamma_b > 0$.

To prove part (a), we analyze the behavior of $Q(x)$ for real $x$. Since all $\gamma_b > 0$, none of the $\alpha_b$ is a zero of $Q_2$ or $Q$. On the other hand, we have,

$$Q'(x) = \sum_{b=1}^{q} \frac{\gamma_b}{(x - \alpha_b)^2} > 0$$  \hspace{1cm} (4.15)

so that $Q(x)$ is monotonically increasing between any two consecutive poles. Indeed, the behavior close to a pole $\alpha_b$ is

$$Q(x) \sim \frac{-\gamma_b}{x - \alpha_b}$$  \hspace{1cm} (4.16)

so that $Q(x)$ tends to $+\infty$ on the left and to $-\infty$ on the right of every pole. Thus, within an interval $[\alpha_{b+1}, \alpha_b]$ between any two consecutive poles, $Q(x)$ goes from $-\infty$ (at $\alpha_{b+1}$) to
Therefore, it must attain the value zero once and only once in that interval. Therefore, in the interval \([\alpha_q, \alpha_1]\) the function \(Q_2(x)\) has precisely \(q - 1\) zeros. The single remaining zero of \(Q_2(x)\) is on the right of this interval, which proves part (a) of the Lemma.

To prove part (b) of the Lemma, we simply notice that

\[
\gamma_b = -\frac{Q_2(\alpha_b)}{Q'_1(\alpha_b)} = -(\alpha_b - \beta_b) \prod_{a \neq b} \frac{\alpha_b - \beta_a}{(\alpha_b - \alpha_a)} \quad (4.17)
\]

By inspection of \((4.14)\), it is clear that, given \(b\), there are just as many \(\alpha_a > \alpha_b\) as there are \(\beta_a > \alpha_b\), so that the product of ratios in the above formula is always positive. Since we also have \(- (\alpha_b - \beta_b) > 0\), it follows that \(\gamma_b > 0\), and this completes the proof of the Lemma.

We conclude that the condition \(W < 0\) throughout the lower half-plane, for polynomials \(Q_1\) with zeros \(\alpha_1, \ldots, \alpha_q\), and \(Q_2\), with zeros \(\beta_1, \ldots, \beta_q\), is equivalent to the alternation of their roots in \((4.14)\), which thus gives a complete description and parametrization of \(W < 0\). The limit of coincident zeros \(\alpha_c = \alpha_{c+1}\) for some \(1 \leq c \leq q - 1\) forces \(\beta_c = \alpha_c\).

### 4.5 Obtaining the harmonic functions \(h_1, h_2\)

Thus far, our attention has focussed on the differentials \(\partial h_1\) and \(\partial h_2\). The full supergravity solution, of course, also involves the harmonic functions \(h_1\) and \(h_2\) themselves, as will the final two regularity conditions (R6) and (R7) of section subsection 3.5. From the differentials \(\partial h_1\) and \(\partial h_2\), the harmonic functions are obtained as hyperelliptic Abelian integrals with double poles at each branch point. It will be important to simplify these Abelian integrals and express them in terms of Abelian integrals with a pole only at infinity. This is carried out in this subsection.

The functions \(h_1\) and \(h_2\) will have simple poles at the branch points. It will be convenient to expose these poles before analyzing the boundary conditions on \(h_1\) and \(h_2\). A convenient basis for meromorphic functions with a single pole at branch point \(e_i\) is given by

\[
\frac{s(u)}{u - e_i} \quad (4.18)
\]

A useful formula is obtained by taking the differential of these basis functions. Recall that the differential \(du/s(u)\) is holomorphic, with a single zero at \(\infty\), which is of order \(2g - 2\). To take the differentials of the functions \(s(u)/(u - e_i)\), we introduce the following notation,

\[
\begin{align*}
    s(u)^2 &= (u - e_i)F_i(u) \\
    F_i(u) &= (u - e_i)\left(G_i(u) + F'_i(u)\right) + E_i \\
    E_i &\equiv F_i(e_i) \quad (4.19)
\end{align*}
\]
By construction, we have \( G_i(e_i) = 0 \). The polynomials \( s(u)^2, F_i(u), G_i(u) \) are of respective degrees \( 2g + 1, 2g, \) and \( 2g - 1 \), while \( E_i \) do not depend on \( u \). Using (4.19), we may recast the double poles in terms of total derivatives of simple poles, so we shall use this formula in the following manner,

\[
\frac{E_i \, du}{(u - e_i)s(u)} = -G_i(u) \, \frac{du}{s(u)} - 2d \left( \frac{s(u)}{u - e_i} \right) \tag{4.20}
\]

To apply this formula to the differentials \( \partial h_1, \partial h_2 \), we decompose the ratios \( P(u)Q_{1,2}(u)/s(u)^2 \) onto the double poles at the branch points, and recast the differentials as follows,

\[
\begin{align*}
\partial h_1 &= -i \left( R_1(u) - \sum_{i=1}^{2g+1} \frac{E_i A_i}{u - e_i} \right) du \frac{1}{s(u)} \\
\partial h_2 &= - \left( R_2(u) - \sum_{i=1}^{2g+1} \frac{E_i B_i}{u - e_i} \right) du \frac{1}{s(u)} \tag{4.21}
\end{align*}
\]

where \( R_1(u) \) and \( R_2(u) \) are polynomials in \( u \) of degree \( g \), with highest degree monomial \( u^g \). The polynomials \( R_1(u) \) and \( R_2(u) \) may be evaluated using an expansion for large \( u \). By inspection, the Abelian differentials \( R_{1,2}(u)du/s(u) \) are meromorphic with a double pole at \( \infty \), and \( 2g \) zeros. Identifying residues at \( u = e_i \) gives,

\[
\begin{align*}
A_i &= -P(e_i)Q_1(e_i)(E_i)^{-2} \\
B_i &= -P(e_i)Q_2(e_i)(E_i)^{-2} \tag{4.22}
\end{align*}
\]

Expressing the double poles in terms of differentials of simple poles, using (4.20), gives

\[
\begin{align*}
\partial h_1 &= -ip_1(u) \frac{du}{s(u)} - idq_1(u) \\
\partial h_2 &= -p_2(u) \frac{du}{s(u)} - dq_2(u) \tag{4.23}
\end{align*}
\]

where

\[
\begin{align*}
q_1(u) &= 2 \sum_{i=1}^{2g+1} \frac{s(u)A_i}{u - e_i} \\
p_1(u) &= R_1(u) + \sum_{i=1}^{2g+1} G_i(u)A_i \\
q_2(u) &= 2 \sum_{i=1}^{2g+1} \frac{s(u)B_i}{u - e_i} \\
p_2(u) &= R_2(u) + \sum_{i=1}^{2g+1} G_i(u)B_i \tag{4.24}
\end{align*}
\]
Here, $p_1$ and $p_2$ are real polynomials of $u$ of degree $2g - 1$, and $q_1$ and $q_2$ are real algebraic functions of $u$. As a result, we have explicit formulas for the functions $h_1$ and $h_2$ themselves,

\begin{align*}
  h_1(u) &= h_1^{(0)} + 2 \text{Im}\left( q_1(u) \right) + 2 \text{Im}\left( \int_{u_1}^{u} \frac{p_1(u)du}{s(u)} \right) \\
  h_2(u) &= h_2^{(0)} - 2 \text{Re}\left( q_2(u) \right) - 2 \text{Re}\left( \int_{u_2}^{u} \frac{p_2(u)du}{s(u)} \right)
\end{align*}

(4.25)

where $h_1^{(0)}$ and $h_2^{(0)}$ are real integration constants and $u_1, u_2$ are the origins of integration on the real axis. The remaining Abelian integrals are smooth functions everywhere, including at the branch points $e_i$, except possibly at the branch point at $\pm \infty$.

### 4.6 Positivity of $h_1, h_2$ near the branch points

Condition (R6) of 3.5 requires that $h_1 > 0$ and $h_2 > 0$ for all $\text{Im}(u) < 0$. We begin by ensuring their positivity at the branch points $e_i$, where the behavior of the simple pole dominates. Using (4.25) near a branch point $u \sim e_i$, we have

\begin{align*}
  h_1(u) &= +2A_i \text{Im}\left( \frac{s(u)}{u-e_i} \right) + \mathcal{O}(1) \\
  h_2(u) &= -2B_i \text{Re}\left( \frac{s(u)}{u-e_i} \right) + \mathcal{O}(1)
\end{align*}

(4.26)

Some care is needed in analyzing this condition. Starting from $u = +\infty$, move $u$ along the real axis to the left. On the branch $]e_1, +\infty[$, we define the function $s(u)$ to be negative. (Making the positive choice would flip the signs of both $h_1$ and $h_2$, which would be an immaterial change in our set-up.) This fixes the phase of $s(u)$ for all $u \in \mathbb{R}$, as follows,

\begin{align*}
  s(u)/|s(u)| &= -1 & u &\in ]e_1, +\infty[ \cup \bigcup_{j=1}^{n_1} ]e_{4j+1}, e_{4j}[ \\
  s(u)/|s(u)| &= +i & u &\in \bigcup_{j=0}^{n_2} ]e_{4j+2}, e_{4j+1}[ \\
  s(u)/|s(u)| &= +1 & u &\in \bigcup_{j=0}^{n_3} ]e_{4j+3}, e_{4j+2}[ \\
  s(u)/|s(u)| &= -i & u &\in \bigcup_{j=0}^{n_4} ]e_{4j+4}, e_{4j+3}[ 
\end{align*}

(4.27)
The upper limits of these unions are given by

\[
\begin{align*}
    n_1 &= \lfloor g/2 \rfloor \\
    n_2 &= \lfloor (2g - 1)/4 \rfloor + n_{-\infty} \\
    n_3 &= \lfloor (g - 1)/2 \rfloor \\
    n_4 &= \lfloor (2g - 3)/2 \rfloor + 1 - n_{-\infty}
\end{align*}
\]

where \( n_{-\infty} = 1 \) when \( g \) is even and \( n_{-\infty} = 0 \) when \( g \) is odd, and \( e_{2g+2} = -\infty \).

We now turn to evaluating the signs of the pole contributions to \( h_1 \) and \( h_2 \). Approaching the branch points \( e_{2j} \) from the left and the branch points \( e_{2j-1} \) from the right yields zero contribution to \( h_1(u) \), because the ratio \( s(u)/(u - e_i) \) is real there, while approaching the branch points \( e_{2j} \) from the right and the branch points \( e_{2j-1} \) from the left yields zero contribution to \( h_2(u) \), because the ratio \( s(u)/(u - e_i) \) is imaginary there.

Non-vanishing contributions are obtained for \( h_1 \) when \( e_{2j} \) is approached from the right and \( e_{2j-1} \) from the left. For \( h_1 \), approaching \( e_{4j+2} \) from the right and \( e_{4j+1} \) from the left gives \( s/|s| = +i \); while approaching \( e_{4j+4} \) from the right and \( e_{4j+3} \) from the left gives \( s/|s| = -i \). For \( h_2 \), approaching \( e_{4j+2} \) from the left and \( e_{4j+3} \) from the right gives \( s/|s| = +1 \); while approaching \( e_{4j+4} \) from the left and \( e_{4j+1} \) from the right gives \( s/|s| = -1 \). These values suffice to establish the signs required on \( A_i \) and \( B_i \) to make \( h_1, h_2 > 0 \) at the branch points, and we find,

\[
\begin{align*}
    A_{4j} &< 0 & B_{4j} &< 0 \\
    A_{4j+1} &< 0 & B_{4j+1} &> 0 \\
    A_{4j+2} &> 0 & B_{4j+2} &> 0 \\
    A_{4j+3} &> 0 & B_{4j+3} &< 0
\end{align*}
\]

Using the fact that we always have \( P(e_i) > 0 \), it is straightforward to translate these conditions into equivalent conditions of the polynomials \( Q_1 \) and \( Q_2 \), and we find,

\[
\begin{align*}
    Q_1(e_{4j}) &> 0 & Q_2(e_{4j}) &> 0 \\
    Q_1(e_{4j+1}) &> 0 & Q_2(e_{4j+1}) &< 0 \\
    Q_1(e_{4j+2}) &< 0 & Q_2(e_{4j+2}) &< 0 \\
    Q_1(e_{4j+3}) &< 0 & Q_2(e_{4j+3}) &> 0
\end{align*}
\]

Together with the ordering of the branch points \( e_i \) and the zeros of \( Q_1 \) and \( Q_2 \) these conditions give us the possible relative orderings of these points. For higher genus, there are clearly many combinatorial possibilities.
Of course, having proven that \( h_1, h_2 > 0 \) near the branch points for the above assignments does not prove that positivity holds throughout the lower half-plane. This result requires first solving the regularity condition (R7) of subsection 3.5 which we do in the next subsection.

### 4.7 Period relations

Condition (R7) of subsection 3.5 requires that wherever \( h_1 \) or \( h_2 \) satisfies Dirichlet conditions, it actually must vanish there: all Dirichlet boundary conditions must actually be vanishing. Thus, we must require that \( h_1 \) and \( h_2 \) satisfy vanishing Dirichlet boundary conditions on \( \partial \Sigma_+ \) and \( \partial \Sigma_- \) respectively.

We begin by requiring \( h_1 = 0 \) on the Dirichlet segment \([e_1, +\infty] \subset \partial \Sigma_+\), and \( h_2 = 0 \) on the Dirichlet segment \([-\infty, e_{2g+1}] \subset \partial \Sigma_-\). As a result, the integration constants in (4.25) get fixed, and we have:

\[
\begin{align*}
  h_1(u) &= 2 \text{Im} \left( q_1(u) \right) + 2 \text{Im} \left( \int_{+\infty}^u \frac{p_1(u) du}{s(u)} \right) \\
  h_2(u) &= -2 \text{Re} \left( q_2(u) \right) - 2 \text{Re} \left( \int_{-\infty}^u \frac{p_2(u) du}{s(u)} \right) \quad (4.31)
\end{align*}
\]

Here, we have also used the fact that \( q_1 \) is manifestly real on \([e_1, +\infty]\), and \( q_2 \) manifestly imaginary on \([-\infty, e_{2g+1}]\). To guarantee that \( h_1(u) = 0 \) also when \( u \in [e_{2j+1}, e_{2j}] \), for \( j = 1, \ldots, g \), and \( h_2(u) = 0 \) for \( u \in [e_{2j}, e_{2j-1}] \) for \( j = 1, \ldots, g \), it will suffice to have

\[
\begin{align*}
  h_1(e_{2j}) - h_1(e_{2j-1}) &= 0 \quad j = 1, \ldots, g \\
  h_2(e_{2j+1}) - h_2(e_{2j}) &= 0 \quad j = 1, \ldots, g \quad (4.32)
\end{align*}
\]

The contributions from \( q_1 \) and \( q_2 \) vanish provided the branch points are approached from a suitable direction (the path of integration of the differentials \( \partial h_1 \) and \( \partial h_2 \) has to be smooth). The remaining conditions (4.32) amount to the following period relations,

\[
\begin{align*}
  \text{Im} \left( \int_{e_{2j}}^{e_{2j-1}} \frac{p_1(u) du}{s(u)} \right) &= 0 \quad j = 1, \ldots, g \\
  \text{Re} \left( \int_{e_{2j}}^{e_{2j+1}} \frac{p_2(u) du}{s(u)} \right) &= 0 \quad j = 1, \ldots, g \quad (4.33)
\end{align*}
\]

giving a system of \( 2g \) real linear relations on the coefficients of the polynomials \( p_1 \) and \( p_2 \).

\(^5\)To properly define these integrals up to infinity, one should first introduce a cutoff, and then take the limit. No contributions to \( h_1 \) or \( h_2 \) arise from these segments, however, whether finite or infinite, because we take the real or imaginary parts.
4.8 Positivity of $h_1$ and $h_2$ throughout $\Sigma$

Combining the results of positivity near the branch points, obtained in (4.30), with the vanishing of $h_1$ and $h_2$ on their respective Dirichlet segments, we can now derive a further positivity condition for $h_1$ and $h_2$. It stems from the fact that if $h_1$ and $h_2$ vanish on their Dirichlet segments and are to be positive in the upper half-plane, then their normal derivatives on the real axis must be negative, namely

$$\operatorname{Im}(\partial_u h_1)\bigg|_{\operatorname{Im}(u)=0} < 0 \quad u \in \partial \Sigma_+$$

$$\operatorname{Im}(\partial_u h_2)\bigg|_{\operatorname{Im}(u)=0} < 0 \quad u \in \partial \Sigma_-$$  \hspace{1cm} (4.34)

Using the fact that $P(u) > 0$ for all $\operatorname{Im}(u) = 0$, and the phase values of $s$ along the real axis, given in (4.27), we obtain the following inequalities for $Q_1$ and $Q_2$,

$$Q_1(u) > 0 \quad u \in [e_1, +\infty) \cup \bigcup_{j=1}^{n_1} [e_{4j+1}, e_{4j}]$$

$$Q_2(u) < 0 \quad u \in \bigcup_{j=0}^{n_2} [e_{4j+2}, e_{4j+1}]$$

$$Q_1(u) < 0 \quad u \in \bigcup_{j=0}^{n_3} [e_{4j+3}, e_{4j+2}]$$

$$Q_2(u) > 0 \quad u \in \bigcup_{j=0}^{n_4} [e_{4j+4}, e_{4j+3}]$$  \hspace{1cm} (4.35)

with the same assignments for $n_1, \ldots, n_4$ as in (4.28). These conditions imply the inequalities at the branch points in (4.30), but they are actually stronger, as they imply that neither $Q_1$ nor $Q_2$ can have any zeros in their respective Dirichlet segments.

4.9 The general hyperelliptic solution

First, we shall show that the sign conditions on $Q_1(e_i)$ and $Q_2(e_i)$ put a lower bound on the number of real zeros. The existence of a lower bound follows from the fact, in (4.33), that the sign of $Q_1(e_k)$ and $Q_2(e_k)$ alternates in $k$ with periodicity 4. This alternation requires the degrees of the polynomials to satisfy $q = \deg(Q_1) = \deg(Q_2) \geq g$. Using the relation $3g + 1 = 2p + q$ of (4.9), where $p$ is the number of complex zeros in the lower half-plane, it is clear that the bound $q = g$ can never be attained, and we have a more stringent bound,

$$g + 1 \leq q = \deg(Q_1) = \deg(Q_2)$$  \hspace{1cm} (4.36)
which in turn implies that the number \( p \) of complex zeros \( u_a \) obeys \( p \leq g \).

Second, we shall use the fact, derived in (4.35), that \( Q_1 \) and \( Q_2 \) have no zeros on their respective Dirichlet segments, so that,

\[
\{ \alpha_1, \ldots, \alpha_q \} \subset U_-
\]
\[
\{ \beta_1, \ldots, \beta_q \} \subset U_+
\]

as well as the fact that the condition \( W < 0 \) requires the zeros of \( Q_1 \) and \( Q_2 \) to alternate as in (4.14). Since each elementary interval \([e_1, +\infty]\) and \([e_{i+1}, e_i]\) for \( i = 1, \ldots, 2g - 1 \) either contains no \( \alpha \) roots or no \( \beta \) roots, we conclude that it cannot contain more than one root (whether \( \alpha \) or \( \beta \)). Suppose, for example, that the interval \([e_{4j+2}, e_{4j+1}]\) contained two consecutive \( \alpha \) roots as (4.14), then it would also have to contain the \( \beta \) root which lies in between the two \( \alpha \) roots and this is not allowed by (4.35). Thus, each elementary interval contains at most one root. This gives an upper bound on the total number of real roots,

\[
2q \leq 2g + 2
\]

which we simply obtain from counting the total number of elementary intervals, \( 2g + 2 \). Combining the lower bound on \( q \) in (4.36) and the upper bound on \( q \) in (4.38), it is immediate that we must have

\[
p = g
\]
\[
q = g + 1
\]

which implies a unique relative ordering of the roots and branch points, given by

\[
\alpha_{g+1} < e_{2g+1} < \beta_{g+1} < e_{2g} < \cdots < \alpha_b < e_{2b-1} < \beta_b < \cdots < e_2 < \alpha_1 < e_1 < \beta_1
\]

for \( g \geq b \geq 2 \). Once this ordering is satisfied, conditions (R5) and (R6) will be obeyed.

### 4.10 Summary of the hyperelliptic solution

In summary, the genus \( g \) hyperelliptic Riemann surface \( \Sigma \) is represented by the lower half-plane, and its boundary \( \partial \Sigma \) is represented by the real line. The \( 2g + 2 \) branch points \( e_1, \ldots, e_{2g+1}, e_{2g+2} = \infty \) are real. The harmonic functions are defined by their differentials, \( \partial h_1 \) and \( \partial h_2 \), given by

\[
\partial h_1 = -i \frac{P(u)Q_1(u)}{s(u)^3} du
\]
\[
\partial h_2 = -\frac{P(u)Q_2(u)}{s(u)^3} du
\]
Here, $s$ was given in (4.3); $P$ is a polynomial of degree $2g$ in $u$, whose zeros come in complex pairs $(u_a, \bar{u}_a)$, with $\text{Im}(u_a) < 0$; and $Q_1$ and $Q_2$ are polynomials of degree $g+1$ in $u$ with real roots $\alpha_1, \ldots, \alpha_{g+1}$ and $\beta_1, \ldots, \beta_{g+1}$ respectively. The form of the differentials $\partial h_1$ and $\partial h_2$, and the relative ordering (4.40) between the real zeros and the branch points guarantees that conditions (R1-R7) are satisfied provided we impose also the period relations

\[
\text{Im} \left( \int_{e^{2j}}^{e^{2j-1}} \partial h_1 \right) = 0 \quad j = 1, \ldots, g
\]

\[
\text{Im} \left( \int_{e^{2j+1}}^{e^{2j}} \partial h_2 \right) = 0 \quad j = 1, \ldots, g
\]

(4.42)

Given a set of real zeros and branch points with ordering (4.40), the $2g$ period relations determine the $g$ complex zeros $u_a$.

Solutions of these period relations may not exist for given moduli of $\Sigma$, and all values of the real roots $\alpha$ and $\beta$ consistent with (4.40). In the section 5, the allowed parameter space will be explored analytically for the genus 1 case, and it will be shown that non-singular solutions do indeed exist for open sets of the full parameter space. In section 6, the existence of regular solutions will be explored also for higher genus, in part by analytical and in part by numerical methods, and it will be shown for genus 2 that regular solutions exist in an open set of the full parameter space.

### 4.11 Asymptotic behavior near the branch points

In this subsection, the asymptotic behavior of the above non-singular solutions near a branch point $u = e_i$ will be analyzed. Defining $u = e_i + z$, the harmonic functions $h_1, h_2$ behave as follows near $z = 0$,

\[
h_1 = 2i \left( \gamma_1 \frac{1}{\sqrt{z}} - \delta_1 \sqrt{z} \right) + o \left( z^{3/2} \right) + \text{c.c.}
\]

\[
h_2 = 2 \left( \gamma_2 \frac{1}{\sqrt{z}} - \delta_2 \sqrt{z} \right) + o \left( z^{3/2} \right) + \text{c.c.}
\]

(4.43)

where the constants are easily obtained from (4.8)

\[
\gamma_1 = \frac{P(e_i)Q_1(e_i)}{\prod_{j \neq i}(e_i - e_j)^{3/2}}
\]

\[
\gamma_2 = \frac{P(e_i)Q_2(e_i)}{\prod_{j \neq i}(e_i - e_j)^{3/2}}
\]

(4.44)
and

\[ \delta_1 = \gamma_1 \left( \sum_k^p \frac{1}{e_i - u_k} + \frac{1}{e_i - \bar{u}_k} \right) + \sum_k^q \frac{1}{q_i - \alpha_k} - \frac{1}{3} \sum_{k \neq i} \frac{1}{e_k - e_i} \]

\[ \delta_2 = \gamma_2 \left( \sum_k^p \frac{1}{e_i - u_k} + \frac{1}{e_i - \bar{u}_k} \right) + \sum_k^q \frac{1}{q_i - \beta_k} - \frac{1}{3} \sum_{k \neq i} \frac{1}{e_k - e_i} \]  

(4.45)

Comparison with the genus 0 Janus solution (4.2) reveals that near each of the branch points the supergravity geometry approaches \( \text{AdS}_5 \times S^5 \) asymptotically. Introducing a new coordinate \( z = e^{-2x-2iy} \) the asymptotic region near the branch point \( z = 0 \) is mapped to \( x \to \infty \) and the coordinate \( y \in [0, \pi/2] \).

### 4.11.1 The dilaton and the metric

The asymptotic values of the dilaton near each \( i \)-th branch point \( e_i \) is given by

\[ e^{2\phi} = \left| \frac{Q_2(e_i)}{Q_1(e_i)} \right| + o(e^{-4x}) \]  

(4.46)

which are constants which depend on \( e_i, \alpha_b \) and \( \beta_b \). The metric becomes

\[ ds^2 = 4\sqrt{2\Delta} \left( dx^2 + dy^2 + (\cos y)^2 ds^2_{S^1} + (\sin y)^2 ds^2_{S^2} + e^{2x} ds^2_{\text{AdS}_4} \right) + o(e^{-2x}) \]  

(4.47)

where

\[ \Delta = \delta_2 \gamma_1 - \delta_1 \gamma_2 = \gamma_1 \gamma_2 \left( \sum_{k=1}^q \frac{1}{e_i - \beta_k} - \frac{1}{e_i - \alpha_k} \right) \]  

(4.48)

In the limit \( x \to \infty \) the metric becomes \( \text{AdS}_5 \times S^5 \), and \( 4\sqrt{2\Delta} \) is the radius squared of the \( \text{AdS}_5 \times S^5 \) geometry in the neighborhood of \( e_i \). Since there are \( 2g + 2 \) branch points (including infinity), the genus \( g \) solution has \( 2g + 2 \) asymptotically \( \text{AdS}_5 \times S^5 \) regions, where the dilaton approaches (generically) different constant values. The holographic interpretation of this geometry will be presented in section 8.

### 4.11.2 The 2-form potential \( B_{(2)} \)

The functions \( b_1, b_2 \) (2.11) parametrize the NSNS and RR 2-form potentials of \( B_{(2)} \). To compute \( b_1 \) and \( b_2 \), we need to evaluate the harmonic duals \( \tilde{h}_1 \) and \( \tilde{h}_2 \). Recall equation (9.24) of [1] for the harmonic functions \( h_1 \) and \( h_2 \), and equation (9.43) for their harmonic duals \( \tilde{h}_1 \) and \( \tilde{h}_2 \), in terms of the holomorphic functions \( A \) and \( B \),

\[ h_1 = -i(A - \bar{A}) \quad \tilde{h}_1 = A + \bar{A} \]

\[ h_2 = B + \bar{B} \quad \tilde{h}_2 = i(B - \bar{B}) \]  

(4.49)
The holomorphic functions are found as follows,

\[ A = A_0 - 2\left(\gamma_1 \frac{1}{\sqrt{z}} - \delta_1 \sqrt{z}\right) + o\left(z^{3/2}\right) \]

\[ B = iB_0 + 2\left(\gamma_2 \frac{1}{\sqrt{z}} - \delta_2 \sqrt{z}\right) + o\left(z^{3/2}\right) \]  \hspace{1cm} (4.50)

Here, \( A_0 \) and \( B_0 \) are real \( z \)-independent parameters. They arise because the splitting of the harmonic functions \( h_1 \) and \( h_2 \) into holomorphic ones is unique only up to additive constants, which cannot be determined from the local properties of \( h_1 \) and \( h_2 \). (In the next subsection, the difference between the values of these constants at different branch points will be determined using Abelian integrals.) Putting all together, we have now the following asymptotic expressions for the fields \( b_1 \) and \( b_2 \),

\[ b_1 = -4B_0 + \frac{32\Delta (\delta_2 \gamma_1 + \delta_1 \gamma_2)}{\gamma_1^2 \gamma_2} (\sin y)^3 e^{-3x} + o(e^{-5x}) \]

\[ b_2 = -4A_0 + \frac{32\Delta (\delta_2 \gamma_1 + \delta_1 \gamma_2)}{\gamma_2^2 \gamma_1} (\cos y)^3 e^{-3x} + o(e^{-5x}) \]  \hspace{1cm} (4.51)

The associated 3-form fluxes vanish in the asymptotic \( AdS_5 \times S^5 \) regions.

### 4.12 Homology 3-spheres

The hyperelliptic solutions exhibit non-trivial 3-cycles on which the 3-form fields have non-zero charges. In this section, we determine these 3-cycles and evaluate the 3-form charges.

A non-trivial 3-cycle arises when a 1-parameter family of 2-spheres, either \( S^2_1 \) or \( S^2_2 \), starts at zero radius (respectively \( f_1 \) or \( f_2 \)) and returns to zero radius in a manner consistent with the topology of 3-spheres (respectively \( S^3_1 \) or \( S^3_2 \)). The relevant 1-parameter families correspond to intervals on the real line \( \partial \Sigma \), located between consecutive branch points. The precise correspondence is as follows:

\[ S^3_{1j} = \left\{ [e_{2j}, e_{2j-1}] \times_f S^2_1 \right\} \quad j = 1, \ldots, g \]

\[ S^3_{2j} = \left\{ [e_{2j+1}, e_{2j}] \times_f S^2_2 \right\} \quad j = 1, \ldots, g \]  \hspace{1cm} (4.52)

Here, it is understood that the product \( \times_f \) of the branch cut and a 2-sphere stands for a fibration of the 2-sphere over the interval, and not for a product of sets. Notice that for genus 0, which corresponds to the \( AdS_5 \times S^5 \) and Janus solutions, no non-trivial 3-cycles are found to exist.
4.13 Evaluation of the RR and NSNS 3-form charges

The real NSNS and RR 3-form field strengths $H_{(3)}$ and $C_{(3)}$, are differentials of the complex 2-form potential, given by the relation $H_{(3)} + iC_{(3)} = dB_{(2)}$. The explicit expression for $B_{(2)}$ on our solutions is given by (2.10) and (2.11). As a result, the charges $\mathcal{H}_j$ and $\mathcal{C}_j$, respectively of the fields $H_{(3)}$ and $C_{(3)}$, across the non-trivial 3-cycles $S^3_{1j}$ and $S^3_{2j}$ are given as follows,

$$\mathcal{H}_j \equiv \int_{S^3_{1j}} db_1 \wedge \hat{e}^{45} = +8\pi \int_{e_{2j}}^{e_{2j-1}} d\tilde{h}_2$$

$$\mathcal{C}_j \equiv \int_{S^3_{2j}} db_2 \wedge \hat{e}^{67} = -8\pi \int_{e_{2j+1}}^{e_{2j}} d\tilde{h}_1$$

(4.53)

In using the relation (2.11) between $b_{1,2}$ and $\tilde{h}_{1,2}$ in the above expressions, the second terms on the right hand sides of (2.11) cancel because $h_1 = 0$ on the intervals entering into the calculation of $\mathcal{H}_j$, while $h_2 = 0$ on the intervals entering the calculation of $\mathcal{C}_j$. An alternative way to see that the second terms on the right hand sides of (2.11) do not contribute is by observing that the above line integrals are effectively around closed curved on the full hyperelliptic Riemann surface (including the upper half plane and both Riemann sheets), and that those contributions to $b_1$ and $b_2$ are single-valued, and thus cancel out of the integrals.

To evaluate the line integrals over $d\tilde{h}_{1,2}$, we use (2.12) to express the differentials of $\tilde{h}_{1,2}$ in terms of those of $h_{1,2}$, and then use (4.31) to evaluate the line integrals. The functions $q_1(u)$ and $q_2(u)$ of (4.31), which enter in the expression (4.31), are single-valued scalars and do not contribute to the line integrals of (4.53). The remaining integrals give,

$$\mathcal{H}_j = -16i\pi \int_{e_{2j}}^{e_{2j-1}} \frac{p_2(u) du}{s(u)}$$

$$\mathcal{C}_j = -16\pi \int_{e_{2j+1}}^{e_{2j}} \frac{p_1(u) du}{s(u)}$$

(4.54)

Both integrals are real, given the phase of $s(u)$ in (4.27), and generically non-vanishing.
5 Genus 1 solutions

The elliptic case provides the simplest solution of the hyperelliptic Ansatz that goes beyond the $AdS_5 \times S^5$ and Janus solutions. Generically, it will have four distinct asymptotic $AdS_5 \times S^5$ regions, each with a different value of the dilaton. The Abelian integrals may be recast in terms of the familiar elliptic functions on the torus, and the domain of parameter space that leads to non-singular solutions may be constructed explicitly and analytically.

5.1 Formulation on the lower half-plane

First, the parametrization given in the preceding section for all genera simplifies considerably at genus 1, and reduces to the following,

\[
\begin{align*}
  s(u)^2 &= (u - e_1)(u - e_2)(u - e_3) \\
  P(u) &= (u - u_1)(u - \bar{u}_1) \\
  Q_1(u) &= (u - \alpha_1)(u - \alpha_2) \\
  Q_2(u) &= (u - \beta_1)(u - \beta_2)
\end{align*}
\]  

(5.1)

where we may choose $e_1 + e_2 + e_3 = 0$, without loss of generality. The branch points $e_1, e_2, e_3$ and the roots $\alpha_1, \alpha_2, \beta_1, \beta_2$ are real, and subject to the ordering relation (4.40) for $g = 1$,

\[
\alpha_2 < e_3 < \beta_2 < e_2 < \alpha_1 < e_1 < \beta_1
\]  

(5.2)

and $\text{Im}(u_1) < 0$. We use the calculations of subsection 4.5 to derive the period relations in the elliptic case. To this end, we compute $G_i(u) = -u + e_i$, as well as the following objects,

\[
\begin{align*}
  p_1(u) &= u - A_4 - \sum_{i=1}^{3} A_i(u - e_i) \\
  p_2(u) &= u - B_4 - \sum_{i=1}^{3} B_i(u - e_i)
\end{align*}
\]  

(5.3)

The constants $A_i, B_i$ are obtained as residues and are given by

\[
\begin{align*}
  A_i &= -P(e_i)Q_1(e_i)(E_i)^{-2} \\
  B_i &= -P(e_i)Q_2(e_i)(E_i)^{-2}
\end{align*}
\]  

(5.4)

\footnote{Useful general references on elliptic functions, conformal mapping, and explicit formulas may be found in the Bateman manuscript \cite{Bateman}, and in \cite{Hille}.}
while the constants $A_4$ and $B_4$ may be obtained from the next-to-leading order behavior at $u = \infty$ and are found to be,

$$A_4 = \alpha_1 + \alpha_2 + u_1 + \bar{u}_1$$
$$B_4 = \beta_1 + \beta_2 + u_1 + \bar{u}_1$$  \hspace{1cm} (5.5)$$

Here, $E_i$ is given solely in terms of the branch points by $E_i \equiv (e_i - e_j)(e_i - e_k)$ where $e_j$ and $e_k$ are two distinct branch points which are distinct also from $e_i$. The period relations of $\text{(4.33)}$ take the following form,

$$\text{Im} \left( \int_{e_2}^{e_1} \frac{p_1(u)du}{s(u)} \right) = 0$$
$$\text{Re} \left( \int_{e_3}^{e_2} \frac{p_2(u)du}{s(u)} \right) = 0$$  \hspace{1cm} (5.6)$$

They may be rendered more explicit by using the expressions for $p_1$ and $p_2$ of $\text{(5.3)}$, as well as the following basic elliptic integrals,

$$\int_{e_2}^{e_1} \frac{du}{s(u)} = \omega_3$$
$$\int_{e_2}^{e_1} \frac{udu}{s(u)} = -\zeta(\omega_3)$$
$$\int_{e_3}^{e_2} \frac{du}{s(u)} = \omega_1$$
$$\int_{e_3}^{e_2} \frac{udu}{s(u)} = -\zeta(\omega_1)$$  \hspace{1cm} (5.7)$$

where $\zeta(u)$ is the Weierstrass $\zeta$-function. In view of $\text{(4.27)}$, the periods $\omega_1$ and $\zeta(\omega_1)$ are real, while the periods $\omega_3$ and $\zeta(\omega_3)$ are purely imaginary. The period relations may then be recast as follows,

$$A_4 = -\zeta_3 + \sum_{i=1}^{3} A_i (e_i + \zeta_3) \hspace{1cm} \zeta_3 \equiv \frac{\zeta(\omega_3)}{\omega_3}$$
$$B_4 = -\zeta_1 + \sum_{i=1}^{3} B_i (e_i + \zeta_1) \hspace{1cm} \zeta_1 \equiv \frac{\zeta(\omega_1)}{\omega_1}$$  \hspace{1cm} (5.8)$$

Together with the defining relations for $A_i, B_i, A_4$ and $B_4$, the period relations give two real equations which may be viewed as equations for the complex zero $u_1$, as a function of the real modulus of the torus (parametrized by one of the branch points $e_1, e_2, e_3$) and the real zeros $\alpha_1, \alpha_2, \beta_1, \beta_2$. The relations implied on $u_1$ are quadratic and of the following form,

$$a_0 |u_1|^2 - a_1 (u_1 + \bar{u}_1) + a_2 = 0 \hspace{1cm} \text{Im}(u_1) < 0$$
$$b_0 |u_1|^2 - b_1 (u_1 + \bar{u}_1) + b_2 = 0$$  \hspace{1cm} (5.9)$$
where the coefficients of these quadrics are given by

\[ a_n = -\delta_{n,1} + (\alpha_1 + \alpha_2 + \zeta_3)\delta_{n,2} + \sum_{i=1}^{3} e_i^n (e_i + \zeta_3)Q_1(e_i)E_i^{-2} \]

\[ b_n = -\delta_{n,1} + (\beta_1 + \beta_2 + \zeta_1)\delta_{n,2} + \sum_{i=1}^{3} e_i^n (e_i + \zeta_1)Q_2(e_i)E_i^{-2} \]  \hspace{1cm} (5.10)

for \( n = 0, 1, 2 \) and \( \delta_{n,1} \) and \( \delta_{n,2} \) are the Kronecker \( \delta \). The quadrics of (5.9) are two half-circles whose centers lie on the real axis.

### 5.2 Parameter space via four real zeros

To investigate the existence of solutions to (5.9) and (5.10), it is convenient to recast (5.9) in terms of the centers and radii of the circles,

\[ \left| u_1 - \frac{a_1}{a_0} \right|^2 = R_a^2 \]
\[ \left| u_1 - \frac{b_1}{b_0} \right|^2 = R_b^2 \]

The necessary and sufficient conditions for the existence of a solution are that these circles have positive \( R_a^2 \) and \( R_b^2 \), and have non-trivial intersection. When they do, the intersection will produce a single point \( u_1 \), which is the unique common complex zero of the differentials \( \partial h_1 \) and \( \partial h_2 \) in the lower half-plane. In summary, these conditions are

\[ 0 \leq R_a^2, R_b^2 \]
\[ \left| \frac{a_1}{a_0} - \frac{b_1}{b_0} \right| \leq R_a + R_b \]  \hspace{1cm} (5.12)

In the next subsection, we shall show that the coefficients \( a_n, b_n \) may be expressed in terms of genus 1 modular forms.

### 5.3 Parameter space via the complex and two real zeros

A more practical approach is to use the complex zero itself as a parameter, together with the two real zeros \( \alpha_1, \beta_2 \) whose range is compact. We may then solve for \( \alpha_2 \) and \( \beta_1 \), using the relations (5.9) together with (5.10). One obtains,

\[ \alpha_2 = \frac{\alpha_1 + u_1 + \bar{u}_1 + \zeta_3 + \sum_{i=1}^{3} (e_i + \zeta_3)e_i(e_i - \alpha_1)|u_1 - e_i|^2E_i^{-2}}{-1 + \sum_{i=1}^{3} (e_i + \zeta_3)(e_i - \alpha_1)|u_1 - e_i|^2E_i^{-2}} \]

\[ \beta_1 = \frac{\beta_2 + u_1 + \bar{u}_1 + \zeta_1 + \sum_{i=1}^{3} (e_i + \zeta_1)e_i(e_i - \beta_2)|u_1 - e_i|^2E_i^{-2}}{-1 + \sum_{i=1}^{3} (e_i + \zeta_1)(e_i - \beta_2)|u_1 - e_i|^2E_i^{-2}} \]  \hspace{1cm} (5.13)
subject to the range (5.2). Considering the modulus as given, then for each set of values $\alpha_1$ and $\beta_2$ in the range (5.2), the above expression, together with the ranges for $\alpha_2$ and $\beta_1$, limit the domain of $u_1$ in the complex plane by two semi-circles. The allowed range of the parameters will be obtained analytically for the case of the square torus, where many simplifications occur.

Figure 3: Allowed parameter region for the complex zero $u_1 = x + iy$ for given values of $\alpha_1$ and $\beta_2$ (equal to $(e_1 + e_2)/2$ and $(e_2 + e_3)/2$ in this case), and varying $\tau = i$ (black arcs), $1.2i$ (blue arcs), and $1.6i$ (red arcs). The allowed regions are the trigons bounded by the real axis and arcs of the same color.

### 5.4 Formulation in terms of elliptic functions and explicit solution

An entire genus 1 Riemann surface, i.e. the double cover of the plane, is uniformized by the Weierstrass function which maps the torus\(^7\) with half-periods $\omega_1$ and $\omega_3$, and modulus $\tau = \omega_3/\omega_1$ into the double cover of the plane by the map

$$z \longrightarrow (\wp(z), \wp'(z)) \quad (5.14)$$

where $\wp'(z)$ is given in terms of $\wp(z)$ through the defining equation,

$$\wp'(z)^2 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3) \quad (5.15)$$

\(^7\)We adopt the conventions of [12] with $\omega = \omega_1$ real, and $\omega' = \omega_3$, and $\tau = \omega_3/\omega_1$ purely imaginary.
up to a sign. This sign distinguishes between the upper and lower Riemann sheets. The branch points are related to the half-periods by $e_i = \wp(\omega_i)$, for $i = 1, 2, 3$, and $\infty = \wp(0)$.

The Weierstrass function $\wp(z)$ is periodic with periods $2\omega_1$ and $2\omega_3$. It is real if and only if either $z = m\omega_1 + i\rho$, or $z = n\omega_3 + \rho$, both with $\rho$ real, and $m$ and $n$ arbitrary integers. This divides the fundamental region for the torus into 4 regions (see Figure 4). Under $z \to 2\omega_1 + 2\omega_3 - z$, the torus is mapped into itself, $\wp \to \wp$ but $\wp' \to -\wp'$, so that the two Riemann sheets are interchanged, and regions I and III, as well as II and IV, are interchanged with one another. Furthermore, regions I and III map to the lower half-plane, while regions II and IV map to the upper half-plane. The lower half-plane of a single sheet is the image of region I. Using the map $u = \wp(z)$, the inversion formulas \[12\],

$$\wp(z + \omega_i) = e_i + \frac{E_i}{\wp(z) - e_i} \quad i = 1, 2, 3 \quad (5.16)$$

and the sign choice $s(u) = \wp'(z)/2$, the differentials $\partial h_1$ and $\partial h_2$ may be recast in terms of the Weierstrass function,

$$\partial h_1 = i \sum_{\alpha=0}^{3} A_{\alpha} \left( \wp(z + \omega_{\alpha}) + \zeta_{3} \right) dz$$
$$\partial h_2 = \sum_{\beta=0}^{3} B_{\beta} \left( \wp(z + \omega_{\beta}) + \zeta_{1} \right) dz \quad (5.17)$$

Figure 4: The cut plane and fundamental domain of the elliptic solution.
where $A_0 = B_0 = -1$. With the help of the relation between the Weierstrass functions, $\zeta'(z) = -\wp(z)$, it is immediate to derive the expressions of the harmonic functions $h_1$ and $h_2$ themselves. The overall additive integration constant generated in the process is fixed by the requirement of vanishing Dirichlet boundary conditions, on the segments $[0, \omega_1]$ and $[\omega_3, \omega_2]$ for $h_1$ and on the segments $[0, \omega_3]$ and $[\omega_1, \omega_2]$ for $h_2$, and we find,

$$h_1(z) = 2i\zeta(\omega_3)(A_2 + A_3) - \sum_{\alpha=0}^{3} iA_{\alpha} \left[ \zeta(z + \omega_{\alpha}) - \zeta(\bar{z} + \bar{\omega}_{\alpha}) - (z - \bar{z})\zeta_3 \right]$$

$$h_2(z) = 2\zeta(\omega_1)(B_1 + B_2) - \sum_{\alpha=0}^{3} B_{\alpha} \left[ \zeta(z + \omega_{\alpha}) + \zeta(\bar{z} + \bar{\omega}_{\alpha}) - (z + \bar{z})\zeta_1 \right] \tag{5.18}$$

To show the Dirichlet vanishing on the segments $[\omega_3, \omega_2]$ and $[\omega_1, \omega_2]$, we have made use of the following addition formula for the $\zeta$-function [12],

$$\zeta(z + z') = \zeta(z) + \zeta(z') + \frac{1}{2} \frac{\wp'(z) - \wp'(z')}{\wp(z) - \wp(z')} \tag{5.19}$$

and the fact that $\wp'(\omega_i) = 0$ for $i = 1, 2, 3$.

![Figure 5: Field lines on the square of half-periods.](image)
5.5 Regularity

The general arguments of the preceding section guarantee that $W < 0$ as long as the ordering (5.2) holds, and that $h_1$ and $h_2$ are positive on their respective Neumann segments. Positivity of $h_1$ and $h_2$ was argued on general grounds as well, but in the elliptic case, it may actually be given a solid proof using explicit formulas. The starting point is two expansion formulas for the Weierstrass $\zeta$-function,

$$
\zeta(z) = z\zeta_1 + \frac{\pi}{2\omega_1} \sum_{m=-\infty}^{+\infty} \cotg \left( \frac{\pi z}{2\omega_1} + m\pi \frac{\omega_3}{\omega_1} \right)
$$

$$
\zeta(z) = z\zeta_3 + \frac{\pi}{2\omega_3} \sum_{m=-\infty}^{+\infty} \cotg \left( \frac{\pi z}{2\omega_3} + m\pi \frac{\omega_1}{\omega_3} \right)
$$

(5.20)

The first of these formulas is familiar [13], while the second may be obtained from the first by interchanging the half-periods $\omega_1$ and $\omega_3$. The first formula will be applied to $h_2$, the second to $h_1$. Remarkably, all dependence on $\zeta(\omega_1)$ and $\zeta(\omega_3)$ cancels, and $h_1$ and $h_2$ are expressed as simple infinite series expansions. Using the formula $\cotg u + \cotg \bar{u} = \sin(u + \bar{u})/|\sin u|^2$, complex conjugate pairs of terms may be combined, to obtain the following formulas,

$$
h_1(z) = -\frac{\pi i}{2\omega_3} \sum_{\alpha=0}^{3} A_\alpha \sum_{m=-\infty}^{+\infty} \frac{\sin \left( \frac{\pi}{2\omega_3} (z - \bar{z} + \omega_\alpha - \bar{\omega}_\alpha) \right)}{\sin \left( \frac{\pi}{2\omega_3} (z + \omega_\alpha + 2m\omega_1) \right)}
$$

$$
h_2(z) = -\frac{\pi}{2\omega_1} \sum_{\alpha=0}^{3} B_\alpha \sum_{m=-\infty}^{+\infty} \frac{\sin \left( \frac{\pi}{2\omega_1} (z + \bar{z} + \omega_\alpha + \bar{\omega}_\alpha) \right)}{\sin \left( \frac{\pi}{2\omega_1} (z + \omega_\alpha + 2m\omega_3) \right)}
$$

(5.21)
Notice that the sin factors in the numerators are actually independent of the summation index \( m \) and may be moved out of the sum over \( m \). Since these numerators depend only on either \( z - \bar{z} \) for \( h_1 \) and \( z + \bar{z} \) for \( h_2 \), it is immediate that positivity on their Neumann boundary segments actually implies positivity everywhere in the fundamental region bounded by the half-periods, which thus proves that \( h_1, h_2 > 0 \) everywhere.

### 5.6 Analytical solution and modular polynomials

The solution (5.18) is completely explicit in terms of the branch points \( e_i \), and the real zeros \( \alpha_1, \alpha_2, \beta_1, \beta_2 \), chosen subject to (5.2), as long as the period relations (5.8) are solved, by satisfying (5.9) with (5.10). The elliptic parametrization will allow us to render the period relations more explicit by expressing the parameters \( a_n \) and \( b_n \) of (5.10) in terms of modular forms. Defining the modular objects,

\[
M_1^{(m)} = \sum_{i=1}^{3} e_i^m (e_i + \zeta_3) E_i^{-2} \\
M_2^{(m)} = \sum_{i=1}^{3} e_i^m (e_i + \zeta_1) E_i^{-2}
\]  

(5.22)

the parameters \( a_n \) and \( b_n \) of (5.10) may be expressed as follows,

\[
a_n = \alpha_1 \alpha_2 M_1^{(n)} + (\alpha_1 + \alpha_2) \left( \delta_{n,2} - M_1^{(n+1)} \right) - \delta_{n,1} + \zeta_3 \delta_{n,2} + M_1^{(n+2)} \\
b_n = \beta_1 \beta_2 M_2^{(n)} + (\beta_1 + \beta_2) \left( \delta_{n,2} - M_2^{(n+1)} \right) - \delta_{n,1} + \zeta_1 \delta_{n,2} + M_2^{(n+2)}
\]  

(5.23)

To compute the modular objects \( M_1 \) and \( M_2 \), we express the branch points in terms of genus one \( \vartheta \)-functions, using the Thomae formulas [12],

\[
e_1 - e_2 = \frac{\pi^2}{4 \omega_1^2} \vartheta_4 = - \frac{i \pi}{\omega_1^2} \partial_\tau \ln \frac{\vartheta_2}{\vartheta_3} \\
e_1 - e_3 = \frac{\pi^2}{4 \omega_1^2} \vartheta_3 = - \frac{i \pi}{\omega_1^2} \partial_\tau \ln \frac{\vartheta_2}{\vartheta_4} \\
e_2 - e_3 = \frac{\pi^2}{4 \omega_1^2} \vartheta_2 = - \frac{i \pi}{\omega_1^2} \partial_\tau \ln \frac{\vartheta_3}{\vartheta_4}
\]  

(5.24)

The relation \( e_1 + e_2 + e_3 = 0 \) holds in view of the famous Jacobi identity \( \vartheta_2^4 + \vartheta_3^4 = \vartheta_4^4 \). The branch points are modular forms under the subgroup of the full modular group \( SL(2, \mathbb{Z}) \) which leaves the half-periods invariant. With these results, we readily compute the combinations \( E_i \),

\[
E_1^2(e_2 - e_3)^2 = E_2^2(e_3 - e_1)^2 = E_3^2(e_1 - e_2)^2 = 2^8 \pi^{12} \eta(\tau)^{24}
\]  

(5.25)
where $2\eta^3 = \psi_2 \psi_3 \psi_4$ is the Dedekind $\eta$-function. It remains to compute the combinations $\zeta_{1,3}$ defined in (5.8). The starting point is [13],

$$\zeta_1 = \zeta_3 + \frac{i \pi}{2\omega_1 \omega_3} = -\frac{i \pi}{\omega_1^2} \partial_\tau \ln \eta(\tau)$$

which shows that $\zeta_1$ and its $\tau \to -1/\tau$ transform $\zeta_3$ are modular connections. Using the relations between $\vartheta_4$ and $\partial_\tau \ln \vartheta$ in the second column of (5.24), we also obtain the following formulas for $i = 1, 2, 3$,

$$e_i + \zeta_1 = -\frac{i \pi}{\omega_1^2} \partial_\tau \ln \vartheta_{i+1}(0, \tau)$$

We conclude by noticing that the modular objects $M_{1,2}^{(m)}$ are not modular forms, but modular connections, as the parts proportional to $\zeta_1$ and $\zeta_3$ transform inhomogeneously under the modular group $SL(2, \mathbb{Z})$.

### 5.7 The special case of the square torus

For the square torus, $\tau = i$, the above quantities may be evaluated in elementary terms and the period relations may be solved explicitly. Without loss of generality, we choose the half-periods in the canonical normalization, $\omega_1 = 1/2$ and $\omega_3 = \tau/2$. Using the symmetry of the square torus, we have $\zeta_3 = -\zeta_1$. Using the first equation in (5.26), we find $\zeta_1 = \pi$ and $\zeta_3 = -\pi$. In terms of the parameters of the lower half-plane representation, we have

$$e_1 = -e_3 = k > 0 \quad e_2 = 0$$

$$E_1 = E_3 = 2k^2 \quad E_2 = -k^2$$

where $k \equiv \wp(1/2)$, whose numerical value is approximately $k = 6.875185816$. The ranges of the real zeros are as follows,

$$\alpha_2 < -k < \beta_2 < 0 < \alpha_1 < k < \beta_1$$

The modular objects $M_1^{(m)}$ and $M_2^{(m)}$, defined in (5.22) take the form,

$$2M_1^{(2m)} = -\pi k^{2m-4} (1 + 2\delta_{m,0})$$

$$2M_2^{(2m)} = +\pi k^{2m-4} (1 + 2\delta_{m,0})$$

and are used to calculate the parameters $a_n$ and $b_n$ in the period relations (5.9), and we find,

$$2k^4 a_0 = -3\pi \alpha_1 \alpha_2 - (\alpha_1 + \alpha_2) k^2 - \pi k^2$$
\[ 2k^4b_0 = +3\pi\beta_1\beta_2 - (\beta_1 + \beta_2)k^2 + \pi k^2 \]
\[ 2k^2a_1 = +\alpha_1\alpha_2 + \pi(\alpha_1 + \alpha_2) - k^2 \]
\[ 2k^2b_1 = +\beta_1\beta_2 - \pi(\beta_1 + \beta_2) - k^2 \]
\[ 2k^2a_2 = -\pi\alpha_1\alpha_2 + (\alpha_1 + \alpha_2)k^2 - 3\pi k^2 \]
\[ 2k^2b_2 = +\pi\beta_1\beta_2 + (\beta_1 + \beta_2)k^2 + 3\pi k^2 \]  
(5.31)

One could now proceed and impose the conditions (5.12) for the existence of solutions.

It turns out that a more explicit description of the allowed parameter space may be obtained by leaving the complex zero \( u_1 \) as a known parameter and solving instead for the real zeros \( \alpha_2 \) and \( \beta_1 \), as was done also in subsection 5.3. It is convenient to scale a factor of \( k \) out of \( u_1 \),

\[ u_1 = kv \]  
(5.32)

In terms of \( v \) in the lower half-plane, \( \text{Im}(v) < 0 \), the relations (5.9) are linear in \( \alpha_2 \) and \( \beta_1 \), and are solved as follows,

\[ \alpha_2 = -\frac{\pi k^2(|v|^2 + 3) + \alpha_1 k^2(|v|^2 - 1) + (\pi \alpha_1 - k^2)k(v + \bar{v})}{\pi \alpha_1(3|v|^2 + 1) + k^2(|v|^2 - 1) + (\alpha_1 + \pi)k(v + \bar{v})} \]
\[ \beta_1 = \frac{\pi k^2(|v|^2 + 3) - \beta_2 k^2(|v|^2 - 1) + (\pi \beta_2 + k^2)k(v + \bar{v})}{-\pi \beta_2(3|v|^2 + 1) + k^2(|v|^2 - 1) - (\pi - \beta_2)k(v + \bar{v})} \]  
(5.33)

These results must be supplemented with the inequalities

\[ \alpha_2 < -k \]
\[ \beta_1 > +k \]  
(5.34)

For the special point \( v = -i \), it is clear that both inequalities (5.34) hold for all allowed values \(-k < \beta_2 < 0 < \alpha_1 < k\). In the next subsection, we shall show that the corresponding solution may actually be mapped onto the Janus solution.

For non-special points \( v \neq -i \), the solution is distinct from Janus. Given the ranges \(-k < \beta_2 < 0 < \alpha_1 < k\) with strict inequalities, it is clear from (5.34) that there will exist an open set containing \( v = -i \) for which the inequalities will be satisfied, and thus regular solutions will exist.

### 5.8 Supersymmetric Janus as a limiting case

We now return to the general torus in the formulation of (5.17) and (5.18). There is one type of symmetric assignment of the \( A_\alpha \) and \( B_\alpha \) for which all inequalities and periods relations
are automatically satisfied, and which precisely reproduces the genus 0 Janus solution. The assignment is given as follows,

\[
\begin{align*}
A_0 &= -A_2 = -1 \\
B_0 &= -B_2 = -1 \\
A_1 &= -A_3 = -a \\
B_1 &= -B_3 = +b
\end{align*}
\] (5.35)

where \(a, b > 0\). In \(\partial h_1\) and \(\partial h_2\), the dependences on \(\zeta\) cancel, and one is left with

\[
\begin{align*}
\partial h_1(z) &= -i\phi(z) + i\phi(z + \omega_2) - ia\phi(z + \omega_1) + ia\phi(z + \omega_3) \\
\partial h_2(z) &= -\phi(z) + \phi(z + \omega_2) + b\phi(z + \omega_1) - b\phi(z + \omega_3)
\end{align*}
\] (5.36)

Since \(\partial h_1(\omega_2/2) = \partial h_2(\omega_2/2) = 0\), the point \(z = \omega_2/2\) is the common complex zero \(u_1 = \phi(\omega_2/2)\). (For the square torus, and using formula (15) of section 13.13 of [12], we find indeed that \(u_1 = -ik\).) To identify this solution with Janus, we compute the functions \(h_1\) and \(h_2\) for this assignment, with the help of (5.18). Using the formula (5.19), as well as the fact that \(\zeta(\omega_1)\) is real and \(\zeta(\omega_3)\) is imaginary, we recast the functions as follows,

\[
\begin{align*}
h_1(z) &= i\frac{1}{2}\phi'(z) \left( -\frac{1}{\phi(z) - e_2} + \frac{a}{\phi(z) - e_1} - \frac{a}{\phi(z) - e_3} \right) + \text{c.c.} \\
h_2(z) &= \frac{1}{2}\phi'(z) \left( -\frac{1}{\phi(z) - e_2} - \frac{b}{\phi(z) - e_1} + \frac{b}{\phi(z) - e_3} \right) + \text{c.c.}
\end{align*}
\] (5.37)

We now identify with the variables of the Janus solution \(u, r\), by identifying the functions \(h_1\) and \(h_2\), which requires the following map,

\[
\begin{align*}
\sqrt{u} &= -\frac{1}{2} \frac{\phi'}{\phi - e_2} \\
\frac{1}{\sqrt{u}} &= -\frac{b}{2} \frac{\phi'}{\phi - e_1} + \frac{b}{2} \frac{\phi'}{\phi - e_3} \\
-r \frac{1}{\sqrt{u}} &= +\frac{a}{2} \frac{\phi'}{\phi - e_1} - \frac{a}{2} \frac{\phi'}{\phi - e_3}
\end{align*}
\] (5.38)

Consistency of these three equations requires first the proportionality of the last two equations, or, \(r = a/b\), as well as the relation obtained by taking the sidewise product of the first two equations. After simplification of the \(\phi\)-dependence, using (5.15), we have \(1 = b(e_1 - e_3)\). This is always possible since, by our conventions, we have \(e_1 > e_3\).
5.9 Homology 3-spheres and 3-form charges

The homology 3-spheres in this geometry are $S^3_1 = [e_2, e_1] \times f S^2_1$ and $S^3_2 = [e_3, e_2] \times f S^2_2$. The corresponding charges are

$$
\mathcal{H} = \frac{8 \pi^2}{\omega_1} \left( 1 - \sum_{i=1}^{3} B_i \right)
$$

$$
\mathcal{C} = \frac{8 i \pi^2}{\omega_3} \left( 1 - \sum_{i=1}^{3} A_i \right)
$$

(5.39)

Notice that these vanish on the Janus solution, but are otherwise generally non-vanishing.
6 Higher genus solutions

At genus higher than 1, the explicit solution involves hyperelliptic integrals, whose explicit form is less familiar than that of elliptic functions. Nonetheless, partial analytic and numerical study of the solutions is possible. In subsection 6.1 the general set-up needed to obtain regular solutions at genus 2 is presented, while in subsection 6.4 a general continuity argument is given for the existence of completely regular solutions, which satisfy all the positivity conditions, at all genera. Finally, the solution may be formulated in terms of higher genus $\vartheta$-functions, which is done in subsection 6.3.

6.1 The genus 2 solutions

We follow closely the construction given in subsections 4.5 and 4.7 valid for any genus. We normalize the genus 2 curve so that

\[
\begin{align*}
   s(u)^2 & = (u - e_1)(u - e_2)(u - e_3)(u - e_4)(u - e_5) \\
   0 & = e_1 + e_2 + e_3 + e_4 + e_5 \\
   2g_2 & = e_1^2 + e_2^2 + e_3^2 + e_4^2 + e_5^2 
\end{align*}
\] (6.1)

It is straightforward to compute $G_i(u)$, and we find,

\[
    G_i(u) = (u - e_i) \left( -3u^2 - 4ue_i - 3e_i^2 + g_2 \right) 
\] (6.2)

The complex zeros $u_1, u_2, \bar{u}_1, \bar{u}_2$, with $\text{Im}(u_1), \text{Im}(u_2) < 0$, and the real zeros $\alpha_1, \alpha_2, \alpha_3$ and $\beta_1, \beta_2, \beta_3$, enter as follows,

\[
\begin{align*}
   P(u) & = (u - u_1)(u - \bar{u}_1)(u - u_2)(u - \bar{u}_2) \\
   Q_1(u) & = (u - \alpha_1)(u - \alpha_2)(u - \alpha_3) \\
   Q_2(u) & = (u - \beta_1)(u - \beta_2)(u - \beta_3) 
\end{align*}
\] (6.3)

Positivity of $h_1, h_2$ and negativity of $W$ require the following ordering,

\[
    \alpha_3 < e_5 < \beta_3 < e_4 < \alpha_2 < e_3 < \beta_2 < e_2 < \alpha_1 < e_1 < \beta_1 
\] (6.4)

The polynomials $R_1(u)$ and $R_2(u)$ are obtained from the expansions at $u = \infty$, and we find,

\[
\begin{align*}
   R_1(u) & = u^2 - A_7u + A_6 \\
   R_2(u) & = u^2 - B_7u + B_6 
\end{align*}
\] (6.5)

41
where the coefficients are given by

\[
\begin{align*}
A_7 &= 2 \text{Re}(u_1 + u_2) + \alpha_1 + \alpha_2 + \alpha_3 \\
B_7 &= 2 \text{Re}(u_1 + u_2) + \beta_1 + \beta_2 + \beta_3 \\
A_6 &= |u_1|^2 + |u_2|^2 + 4 \text{Re}(u_1) \text{Re}(u_2) + 2 \text{Re}(u_1 + u_2)(\alpha_1 + \alpha_2 + \alpha_3) \\
&\quad + g_2 + \alpha_1 \alpha_2 + \alpha_2 \alpha_3 + \alpha_3 \alpha_1 \\
B_6 &= |u_1|^2 + |u_2|^2 + 4 \text{Re}(u_1) \text{Re}(u_2) + 2 \text{Re}(u_1 + u_2)(\beta_1 + \beta_2 + \beta_3) \\
&\quad + g_2 + \beta_1 \beta_2 + \beta_2 \beta_3 + \beta_3 \beta_1
\end{align*}
\]  

(6.6)

Putting all together, the period relations are as follows,

\[
\begin{align*}
k = 1, 2 & \quad \int_{e_{2k-1}}^{e_{2k}} \frac{du}{s(u)} \left[ u^2 - A_7 u + A_6 + \sum_{i=1}^{5} A_i G_i(u) \right] = 0 \\
k = 1, 2 & \quad \int_{e_{2k+1}}^{e_{2k}} \frac{du}{s(u)} \left[ u^2 - B_7 u + B_6 + \sum_{i=1}^{5} B_i G_i(u) \right] = 0
\end{align*}
\]  

(6.7)

where

\[
\begin{align*}
A_i &= -P(e_i)Q_1(e_i)(E_i)^{-2} \\
B_i &= -P(e_i)Q_2(e_i)(E_i)^{-2}
\end{align*}
\]  

(6.8)

The four period integrals above take on either real or purely imaginary values, as a result of the phase values of (4.27) for \(s(u)\). Since the \(s(u)\) retains constant phase inside any interval between consecutive branch points, the denominator \(s(u)\) in the period integrals above may be replaced by \(|s(u)|\). Given the moduli of the surface, and the real zeros \(\alpha_1, \alpha_2, \alpha_3\) and \(\beta_1, \beta_2, \beta_3\), these 4 real equations then determine the two complex zeros \(u_1\) and \(u_2\).

### 6.2 Numerical analysis

We have explored the solutions of these equations numerically for genus 2 surfaces with various degrees of \(\mathbb{Z}_2\) symmetry, obtained via the following arrangement of the branch points,

\[
\begin{align*}
e_5 &= -t_2, \quad e_4 = \frac{1}{t_2}, \quad e_3 = 0, \quad e_2 = \frac{1}{t_1}, \quad e_1 = t_1
\end{align*}
\]  

(6.9)

for \(t_1, t_2 > 1\). The first \(\mathbb{Z}_2\) is the hyperelliptic involution, while a second \(\mathbb{Z}_2\) is the inversion \(u \to 1/u\). An extra \(\mathbb{Z}_2\), given by \(u \to -u\), is obtained by setting \(t_2 = t_1\), while a Riemann
Figure 7: Genus 2 solutions for complex zeros $u_1$ (red) and $u_2$ (blue) in the lower half $u$-plane, for modulus $t_1$ in the interval $1.03 < t_1 < 2$, fixed modulus $t_2 = t_0$, and $\alpha_1 = (e_1 + e_2)/2$, $\alpha_2 = (e_3 + e_4)/2$, $\alpha_3 = 2e_5$, $\beta_1 = 2e_1$, $\beta_2 = (e_2 + e_3)/2$, and $\beta_3 = (e_4 + e_5)/2$. 

A surface with symmetry $(\mathbb{Z}_2)^4$ is obtained when $t_1 = t_2 = t_0$, where $t_0 \equiv 1.27201965$. The period matrix for the surface with $(\mathbb{Z}_2)^4$ symmetry assumes the symmetrical form,

$$ \Omega_{IJ} = \begin{pmatrix} 0.89442719i & -0.44721359i \\ 0.44721359i & 0.89442719i \end{pmatrix} \quad I, J = 1, 2 \quad (6.10) $$

In Figure 7, we make a choice for the real zeros indicated on the figure’s caption, and plot the complex zeros $u_1$ and $u_2$, obtained as solutions to the period relations (6.7), as a function of the modulus $t_1$. The existence of this numerical solution means that global regular solutions exist for $h_1$ and $h_2$ for the above assigned values. In addition, we have numerically shown that regular solutions exist for less symmetrical assignments of the real zeros, and modulus $t_2 \neq t_0$. It is an interesting problem to describe the region of parameters $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3$ and moduli, but this issue lies beyond the scope of this paper.

Note that the complex zero $u_2$ tends to the real axis as $t_1 \to 1$. This limiting case corresponds to a degeneration of the genus 2 surface, a phenomenon that will be analyzed in generality in the subsequent section.
### 6.3 Parametrization via $\vartheta$-functions

The Thomae formulas [14] allow us to express the branch points in terms of genus two $\vartheta$-functions. We begin by recalling some key facts about genus two $\vartheta$-functions [14, 15].

A general spin structure is a half-integer characteristic, consisting of an array of two 2-component vectors,

$$\kappa = (\kappa' | \kappa'') \quad \kappa' = \left( \begin{array}{c} \kappa'_1 \\ \kappa'_2 \end{array} \right) \quad \kappa'' = \left( \begin{array}{c} \kappa''_1 \\ \kappa''_2 \end{array} \right)$$

(6.11)

and the entries $\kappa'_{1,2}$ and $\kappa''_{1,2}$ take on values 0 or 1/2 mod 1.

The $\vartheta$-function is defined in terms of the period matrix $\Omega$, and a general complex 2-component vector $\zeta$ by

$$\vartheta[\kappa](\zeta, \Omega) = \sum_{n \in \mathbb{Z}^2} \exp \left\{ \pi i (n + \kappa')^t \Omega (n + \kappa') + 2 \pi i (n + \kappa')^t (\zeta + \kappa'') \right\}$$

(6.12)

For any spin structure $\kappa$, the combination $4(\kappa')^t \kappa''$ is an integer: even (resp. odd) spin structures correspond to $4(\kappa')^t \kappa''$ even (resp. odd).

For genus 2, there exists a sharper version of the Thomae formulas, which was heavily used in [15]. The key is a one-to-one map between the six branch points $e_i$, $i = 1, \cdots, 6$ with $e_6 = \infty$, and the six odd spin structures $\nu_i$ at genus 2. The relation with $\vartheta$-functions is then simply given by

$$(e_i - e_j)(e_k - e_l) = \frac{M_{\nu_i \nu_j}M_{\nu_k \nu_l}}{M_{\nu_i \nu_k}M_{\nu_j \nu_l}}$$

(6.13)

where $M_{\nu_i \nu_j}$ is a modular form of weight 2, defined by

$$M_{\nu_i \nu_j} = \partial_1 \vartheta[\nu_i](0, \Omega)\partial_2 \vartheta[\nu_j](0, \Omega) - \partial_2 \vartheta[\nu_i](0, \Omega)\partial_1 \vartheta[\nu_j](0, \Omega)$$

(6.14)

An alternative formula for $M_{\nu_i \nu_j}$, with $i \neq j$, is in terms of even spin structures, but is given only up to an overall sign (familiar from the standard Thomae formulas of [14]),

$$M_{\nu_i \nu_j}^2 = \pi^4 \prod_{k \neq i,j} \vartheta[\nu_i + \nu_j + \nu_k](0, \Omega)^2$$

(6.15)

Upon choosing $e_4 = 0, e_5 = 1$, and $e_6 = \infty$, by $SL(2, \mathbb{R})$ invariance, and $i = 4, k = 5$ and $l = 6$, the cross ratio formula yields an explicit formula for the 3 remaining real moduli,

$$e_j = \frac{M_{\nu_i \nu_4}M_{\nu_5 \nu_6}}{M_{\nu_i \nu_5}M_{\nu_4 \nu_6}}$$

(6.16)

Note that this formula actually holds for all possible values of $j = 1, \cdots, 6$. 

44
The period integrals that enter into the period relations (6.7) are half $A$- and $B$-cycle integrals of Abelian differentials. We shall declare the following correspondence,

\[
\begin{align*}
[e_2, e_1] & \sim A_1 \\
[e_3, e_2] & \sim B_1 \\
[e_4, e_3] & \sim A_2 \\
[e_5, e_4] & \sim B_2
\end{align*}
\]

We define the following periods for $k = 0, 1, 2, 3$,

\[
\begin{align*}
\int_{e_2}^{e_1} \frac{u^k du}{s(u)} &= K_{1,k} \\
\int_{e_3}^{e_2} \frac{u^k du}{s(u)} &= L_{1,k} \\
\int_{e_4}^{e_3} \frac{u^k du}{s(u)} &= K_{2,k} \\
\int_{e_5}^{e_4} \frac{u^k du}{s(u)} &= L_{2,k}
\end{align*}
\]

The periods $L$ for $k = 0, 1$ are given in terms of the period matrix $\Omega$ and the periods $K$ by

\[
\Omega = \frac{1}{K_{1,1}K_{2,0} - K_{1,0}K_{2,1}} \begin{pmatrix}
K_{2,0}L_{1,1} - K_{2,1}L_{1,0} & K_{2,0}L_{2,1} - K_{2,1}L_{2,0} \\
K_{1,1}L_{1,1} - K_{1,0}L_{1,1} & K_{1,1}L_{2,0} - K_{1,0}L_{2,1}
\end{pmatrix}
\]

The $K$ periods cancel out of the set of four period relations (6.7), just as an overall factor of $\omega_1$ cancelled out of the genus 1 period relations.

The periods $K$ and $L$ for $k = 2, 3$ are analogous to the quantities $\zeta(\omega_1)$ and $\zeta(\omega_3)$ encountered at genus 1. They are the periods of Abelian differentials with poles of order 2 and 4 at the branch point $e_6 = \infty$. We are aware of no known explicit formulas for these periods in terms of $\vartheta$-functions, but suspect that such formulas could be derived with the help of the conversion formulas between the $\vartheta$-function and hyperelliptic representations obtained in [15].

The differentials $\partial h_1, \partial h_2$ as well harmonic functions may also be expressed in terms $\vartheta$-functions. The key ingredient is the identification of the Abelian differentials with double poles at the branch points with their form in terms of $\vartheta$-functions, a result available through the use of the prime form [16]. Since the definition of the prime form involves the Abelian integrals of the first kind anyway, it is unclear how much would be gained from this alternative expression, and we shall therefore suppress these formulas.

### 6.4 Continuity argument for existence of solutions at all genera

The solutions for different genera form a connected set and the lower genus solutions may be obtained as a limit of higher genus solutions, by collapsing branch cuts. For example, a
genus \( g \) solutions, with branch points \( e_1, \ldots, e_{2g+1} \) and \( e_{2g+2} = \infty \), real zeros \( \alpha_1, \ldots, \alpha_{g+1} \), and \( \beta_1, \ldots, \beta_{g+1} \), and complex zeros \( u_1, \ldots, u_g \), is smoothly connected to the genus \( g - 1 \) solution by letting

\[
\begin{align*}
e_{2b}, e_{2b-1} & \rightarrow x \\
\alpha_b, \beta_b, u_b, \bar{u}_b & \rightarrow x
\end{align*}
\]

for some \( 1 \leq b \leq g \). In this limit, the resulting differentials \( \partial h_1 \) and \( \partial h_2 \) become independent of the point \( x \), and collapse onto the genus \( g - 1 \) solution, obtained from the above genus \( g \) solution with the branch points \( e_{2b}, e_{2b-1} \) and the zeros \( \alpha_b, \beta_b, u_b, \bar{u}_b \) removed. This limit is completely smooth. Importantly, the limit is a local process, which is largely insensitive to the global properties of \( \Sigma \).

Now consider the reverse problem. If a genus \( g - 1 \) solution exists, the question is naturally raised as to whether a genus \( g \) solution exists in an open neighborhood of parameter space of the genus \( g - 1 \) solution. In other words, can an extra genus be “turned on” by inserting 2 extra branch points \( e_{2g+1}, e_{2g+2} \) and adding zeros \( \alpha_g, \beta_g, u_g, \bar{u}_g \)? We have investigated this question numerically for genus 1 and 2, and have found an affirmative answer to this question. We believe that it should be possible to prove analytically that a genus \( g \) solution will indeed exist in a sufficiently small open neighborhood of any genus \( g - 1 \) solution.

One key ingredient in the above conjecture is the necessary condition that, as a branch cut collapses to a point \( x \), exactly one complex zero converges to the real axis and collapses to \( x \) as well. This collapse phenomenon is studied in the next section, where it is shown to hold in general. It is also shown there, that, if one allows for mild singularities in the 10-dimensional geometry, such as those produced by D5 and NS5 branes in the probe limit, then only one of the real zeros, either \( \alpha \) or \( \beta \) is required to collapse to \( x \) as well, but not both. Thus, allowing for such probe limit singularities, a higher-dimensional parameter space of degenerations is allowed.
7 Collapse of branch cuts and D- and NS-branes

A direct study of the collapse of branch cuts from the explicit hyper-elliptic solution is complicated by the fact that the splitting off of pole terms in $h_1$ and $h_2$, between the functions $p_{1,2}$ and $q_{1,2}$ as in formula (4.31) introduces artificial divergences in $p_{1,2}$ and $q_{1,2}$ which however cancel in the functions $h_1$ and $h_2$. For this reason, we shall take first the limit of collapsing branch cuts on the differentials $\partial h_1$ and $\partial h_2$, where this limit exists consistently, and then re-examine the questions of regularity.

7.1 The case of genus 1

For genus 1, the branch points and zeros of $Q_1$ and $Q_2$ are subject to the following ordering, $\alpha_2 < e_3 < \beta_2 < e_2 < \alpha_1 < e_1 < \beta_1$. We shall take the limit where $e_1 - e_2 \to 0$, which clearly forces also $e_1 - \alpha_1 \to 0$. Without loss of generality, we make an overall translation by $e_3$, and set $e_3 = 0$, which is the position of the only remaining branch point, and we use the designation $e_1, e_2, \alpha_1 \to k^2 > 0$, with $k > 0$, and $\alpha = \alpha_2$. Thus, we have the ordering

$$\alpha < 0 < \beta_2 < k^2 < \beta_1$$  (7.1)

and the differentials are

$$\partial h_1 = -i \frac{(u - u_1)(u - \bar{u}_1)(u - \alpha)}{(u - k^2)^2} \frac{du}{\sqrt{u}}$$

$$\partial h_2 = - \frac{(u - u_1)(u - \bar{u}_1)(u - \beta_1)(u - \beta_2)}{(u - k^2)^3} \frac{du}{\sqrt{u}}$$  (7.2)

As a smooth limit of the elliptic case with the ordering of $\alpha, k^2, \beta_1, \beta_2$ prescribed above, the regularity condition $W \leq 0$ is automatic. To uniformize the square root, we introduce a new coordinate $u = w^2$, which, in view of $\text{Im}(u) < 0$, takes values in the second quadrant,

$$\Sigma \equiv \{w; \text{Re}(w) < 0, \text{Im}(w) > 0\}$$  (7.3)

To work out the vanishing Dirichlet conditions on $h_1$ and $h_2$, and insist on their positivity, we decompose the fractions into elementary poles, and we have,

$$\partial h_1 = -2i \left[1 + \frac{A_1}{w^2} + \frac{2kB_1}{w^2 - k^2} + \frac{2C_1(w^2 + k^2)}{(w^2 - k^2)^2}\right]dw$$

$$\partial h_2 = -2 \left[1 + \frac{A_2}{w^2} + \frac{2kB_2}{w^2 - k^2} + \frac{2C_2(w^2 + k^2)}{(w^2 - k^2)^2} + \frac{2D_2(3w^2k + k^3)}{(w^2 - k^2)^3}\right]dw$$  (7.4)
It is straightforward to integrate these (up to an additive constant, which must vanish by the vanishing Dirichlet conditions),

\[
\begin{align*}
    h_1 &= -2i(w - \bar{w}) \left[ 1 + \frac{A_1}{|w|^2} + \frac{2C_1(|w|^2 + k^2)}{|w^2 - k^2|^2} \right] - 2iB_1 \ln \frac{(w - k)(\bar{w} + k)}{(w + k)(\bar{w} - k)} \\
    h_2 &= -2(w + \bar{w}) \left[ 1 - \frac{A_2}{|w|^2} + 4C_2 \frac{|w|^2 - k^2}{|w^2 - k^2|^2} + 4kD_2 \frac{|w|^2(w^2 + \bar{w}^2 - w\bar{w} - 2k^2) + k^4}{|w^2 - k^2|^3} \right] \\
         & \quad - 2B_2 \ln \frac{|w - k|^2}{|w + k|^2}. 
\end{align*}
\]

For \( w \in \Sigma \), we have \(-2i(w - \bar{w}) > 0\) and \(-2(w + \bar{w}) > 0\). Therefore, positivity of \( h_1 \) and \( h_2 \) at the poles \( w = 0, -k \) (note that \( w = +k \notin \Sigma \)) requires that

\[
\begin{align*}
    A_1 > 0 & \quad B_1 = 0 & \quad C_1 \geq 0 \\
    A_2 < 0 & \quad kB_2 \leq 0 & \quad C_2 = D_2 = 0. 
\end{align*}
\]

The requirements on \( A_1, A_2, C_1 \) are obvious. The requirement \( B_1 = 0 \) stems from the fact that as \( w \) crosses the point \( w = -k \), the logarithms pick up an additive contribution of \( 2\pi i \), so that the left and the right Dirichlet boundary values of \( h_1 \) differ by \( 2\pi B_1 \). But regularity of the solution requires that all Dirichlet boundary values vanish, so that we must have \( B_1 = 0 \). Finally, it is easy to see that the sign of the numerator of the \( D_2 \) term depends on how the point \( w = -k \) is being approached. Setting \( \text{Re}(w) = -k \) and varying its imaginary part produces a negative numerator, while setting \( \text{Im}(w) = 0 \) and varying its real part produces a positive numerator. Since this is the leading singularity at \( w = -k \), we must have \( D_2 = 0 \). Given \( D_2 = 0 \), the term in \( C_2 \) remains as the leading singularity and the same argument applies to conclude that \( C_2 = 0 \), which concludes the proof of \( (7.6) \).

Obtaining the values of \( A_1, B_1, C_1, A_2, B_2, C_2, D_2 \) from those of \( k, \alpha, \beta_1, \beta_2 \), we have

\[
\begin{align*}
    A_1 &= -\alpha \frac{|u_1|^2}{k^4} & \quad A_1 + 2C_1 &= -u_1 - \bar{u}_1 - \alpha + 2k^2 \\
    A_2 &= -\beta_1 \beta_2 |u_1|^2 \quad A_2 + 2kB_2 &= -u_1 - \bar{u}_1 - \beta_1 - \beta_2 + 3k^2 \\
    C_1 &= \frac{|k^2 - u_1|^2(k^2 - \alpha)}{4k^4} \quad B_1 &= 0 
\end{align*}
\]

(7.7)

together with the requirement that \((u - u_1)(u - \bar{u}_1)(u - \beta_1)(u - \beta_2)\) have a double zero in \( u \) at \( u = k^2 \), which is equivalent to \( C_2 = D_2 = 0 \). Note that the conditions \( A_1 > 0, A_2 < 0, C_1 > 0 \) of \( (7.6) \) follow directly from the above identifications and the ordering \( (7.1) \).
Figure 8: Genus 1 collapsing branch cut, starting with a square torus at the top of each curve, to \( e_1 - e_2 \to 0 \) at the real axis, for \( \alpha_1 = (e_1 + e_2)/2, \alpha_2 = 2e_3, \beta_1 = 2e_1 \) and various values of \( \beta_2 = e_3 + x(e_2 - e_3) \), given by \( x = 0.99 \) (blue), \( x = 0.90 \) (top black), \( x = 0.50 \) (red), \( x = 0.10 \) (bottom black), and \( x = 0.01 \) (green).

To solve the remaining conditions, it is helpful to first solve the \( C_2 = D_2 = 0 \) condition. There are two cases. Either the double zero of \( (u - u_1)(u - \bar{u}_1)(u - \beta_1)(u - \beta_2) \) is simply \( u_1 = k^2 \), or it is not: \( u_1 \neq k^2 \).

7.1.1 The case \( u_1 \neq k^2 \)

In this case, the double zero requires \( \beta_1 = \beta_2 = k^2 \), and we have \( 2k^3B_2 = |k^2 - u_1|^2 > 0 \), and hence the condition \( kB_2 \leq 0 \) can never be satisfied. Thus, this case is simply ruled out.

7.1.2 The case \( u_1 = k^2 \)

This leads to drastic simplifications, and we have

\[
A_1 = -\alpha \\
A_2 = -\frac{\beta_1 \beta_2}{k^2} \\
B_1 = C_1 = C_2 = D_2 = 0 \\
B_2 = \frac{1}{2k^3}(k^2 - \beta_1)(k^2 - \beta_2)
\]

(7.8)
We see that the condition $kB_2 \leq 0$ is now automatic as well. The solution may be written down explicitly. The harmonic functions are given by

$$h_1 = -2i(w - \bar{w}) \left[ 1 - \frac{\alpha}{|w|^2} \right]$$

$$h_2 = -2(w + \bar{w}) \left[ 1 + \frac{\beta_1 \beta_2}{k^2|w|^2} \right] + \frac{(\beta_1 - k^2)(k^2 - \beta_2)}{k^3} \ln \frac{|w - k|^2}{|w + k|^2}$$

Since $w \in \Sigma$ of (7.3), and using the ordering (7.1), it is immediate that $h_1, h_2 > 0$ everywhere in $w \in \Sigma$. In particular, the $\ln$ is always positive since for $w \in \Sigma$, we have $|w - k|^2 > |w + k|^2$. Notice that, if we also have $\beta_1 = k^2$ or $\beta_2 = k^2$, then we recover the Janus solution.

### 7.2 General collapse of a branch cut

Having derived the behavior under collapse of a branch cut in the genus 1 case, we are now in a position to generalize the result to the collapse of a branch cut for general genus. The reason that such a result can be obtained is that the regularity and positivity conditions are essentially local conditions, largely insensitive to the global properties of the surface $\Sigma$.

Below, we shall show the following general result. The collapse of any branch cut is accompanied by the convergence of precisely one of the complex zeros $u_a$ to the real axis, and more specifically, to the location of the collapsing branch cut.

For simplicity, we shall assume that two consecutive branch points $e_{2b}$ and $e_{2b-1}$ collapse to a zero of type $\alpha$, and that the adjacent branch points $e_{2b+1}$ and $e_{2b-2}$ remain a finite distance away from $\alpha$. (If adjacent branch points are allowed to collapse onto $\alpha$ as well, we are dealing with a multiple degeneration of the Riemann surface. The same methods, to be explained below, can also be adapted to cover those cases.)

The corresponding branch cut $[e_{2b}, e_{2b-1}]$ contains a single zero $\alpha_b$ of $Q_1$. (The argument may readily be carried over to an interval $[e_{2b+1}, e_{2b}]$ which would contain a single zero $\beta_{b+1}$ of $Q_2$ instead.) Since $\alpha_b \in [e_{2b}, e_{2b-1}]$, this means that also $\alpha_b \to \alpha$. The resulting ordering of the collapsed interval is thus,

$$\alpha_{g+1} < e_{2g+1} < \cdots < e_{2b+1} < \beta_{b+1} < \alpha < \beta_b < e_{2b-2} < \cdots < e_1 < \beta_1$$

Having taken the limit $e_{2b}, e_{2b-1}, \alpha_b \to \alpha$, we shall now study the problem locally around $u \sim \alpha$. To do so, it is convenient to define the following quantities,

$$s(u)^2 = (u - \alpha)^2 \hat{s}(u)^2$$

$$Q_1(u) = (u - \alpha) \hat{Q}_1(u)$$
where
\[
\tilde{s}(u)^2 = (u - e_{2g+1}) \prod_{j=1, j \neq b}^{g} (u - e_{2j-1})(u - e_{2j})
\]
\[
\tilde{Q}_1(u) = \prod_{a=1, a \neq b}^{g} (u - \alpha_a)
\]

and recast the differentials \( \partial h_1 \) and \( \partial h_2 \) in terms of the following functions,
\[
\partial h_1 = -iM_1(u) \frac{du}{(u - \alpha)^2} \quad M_1(u) \equiv \frac{P(u)\tilde{Q}_1(u)}{\tilde{s}(u)^3}
\]
\[
\partial h_2 = -M_2(u) \frac{du}{(u - \alpha)^3} \quad M_2(u) \equiv \frac{P(u)Q_2(u)}{\tilde{s}(u)^3}
\] (7.13)

Note that, because of the extra zero in \( Q_1 \) at \( u = \alpha \), the differential \( \partial h_1 \) has a pole of order 2 while \( \partial h_2 \) has a pole of order 3. Also, it is readily verified that, since \( e_{2b+1} < \alpha < e_{2b-2} \), we have \( \tilde{s}(\alpha)^2 > 0 \), so that \( \tilde{s}(u) \), and thus \( M_1(u) \) and \( M_2(u) \) are real for \( u \) real and in the neighborhood of \( \alpha \). Expanding the differentials in powers of \( w \), with \( u = \alpha + w \), and integrating to obtain the harmonic functions \( h_1 \) and \( h_2 \), we find,
\[
h_1 = M_1(\alpha) \frac{2\text{Im}(w)}{|w|^2} - i M'_1(\alpha) \ln \frac{w}{\bar{w}} + \text{regular}
\]
\[
h_2 = M_2(\alpha) \frac{\text{Re}(w)^2 - \text{Im}(w)^2}{|w|^4} + M'_2(\alpha) \frac{2\text{Re}(w)}{|w|^2} - M''_2(\alpha) \ln |w| + \text{regular} \quad (7.14)
\]

Positivity of \( h_2 \) clearly requires that
\[
M_2(\alpha) = M'_2(\alpha) = 0 \quad M''_2(\alpha) \geq 0 \quad (7.15)
\]

Positivity at the leading singularity of \( h_1 \) requires that (recall that \( u \) and thus \( w \) are in the lower half-plane, with \( \text{Im}(w) < 0 \)),
\[
M_1(\alpha) \leq 0 \quad (7.16)
\]

There is also a constraint on the \( M'_1(\alpha) \) term which is more subtle. The function \( h_1 \) must obey vanishing Dirichlet boundary conditions on both sides of \( \alpha \). The term proportional to \( M'_1(\alpha) \) introduces a discontinuity in \( h_1 \) along the boundary, adding to \( h_1 \) the quantity \( 2\pi M_1(\alpha) \) as \( w \) moves across \( \alpha \) along the real axis. No such discontinuity is allowed with vanishing Dirichlet boundary conditions, and no such discontinuity can be compensated for.
by the regular terms, which have not been explicitly exhibited. Therefore, we have the final condition that

\[ M'_1(\alpha) = 0 \]  

(7.17)

Clearly, condition (7.15) requires that \( M_2(u) \) must have a double zero at \( u = \alpha \). There are only two different ways of achieving a double zero. (It is here that we use the extra assumption, stated above, that the adjacent branch points \( e_{2b+1} \) and \( e_{2b-2} \) remain a finite distance away from \( \alpha \).)

7.2.1 The case \( P(\alpha) \neq 0 \)

In this case, we must have \( Q_2(\alpha) = 0 \), which requires that \( \beta_b = \beta_{b+1} = \alpha \). We shall now calculate the sign of \( M_2''(\alpha) = P(\alpha)Q_2''(\alpha)/\tilde{s}(\alpha)^3 \). Analyzing the position of \( \alpha \) relative to the branch points and zeros of \( Q_2'' \), we readily find,

\[
\text{sign}(\tilde{s}(\alpha)) = (-1)^b \\
\text{sign}(Q_2''(\alpha)) = (-1)^{b-1}
\]  

(7.18)

It follows that \( M_2''(\alpha) < 0 \); the inequality is strict because we have \( P(\alpha) > 0 \), as well as \( Q_2''(\alpha), \tilde{s}(\alpha) \neq 0 \). But this result is contradictory to the third result in (7.15). Therefore, the case \( P(\alpha) \neq 0 \) is ruled out, and we must have instead,

7.2.2 The case \( P(\alpha) = 0 \)

Since the zeros of \( P(u) \) come in complex conjugate pairs, a real zero \( \alpha \) must be a double zero of \( P \). It then follows that the first and second conditions of (7.15), as well as (7.16) and (7.17) hold automatically. Since we now have \( \text{sign}(Q_2(\alpha)) = (-1)^b \) and \( P''(\alpha) > 0 \), it is easy to see that the third condition in (7.15) is also satisfied.

7.3 Complete collapse of the higher genus case

When branch cuts are being collapsed at higher genus, there are a priori two options for the remaining branch point: it can have even or odd index. If the index is even, \( e_{2r} \), for \( 1 \leq r \leq g \), then either the branch point \( e_1 \) or the branch point \( e_{2g+1} \) should be collapsed onto the branch point at \( \infty \) in order to obtain maximal collapse. If the index is odd, \( e_{2r-1} \), \( 1 \leq r \leq g + 1 \), then the point \( \infty \) remains a branch point. It is the latter case we shall discuss in detail here, the other case is analogous. As before, we shift and set \( e_{2r-1} = 0 \). It is clear that the branch cuts to the left of 0 collapse with a \( \beta \)-zero, while the ones to
the right collapse with an $\alpha$-zero. Following the arguments of subsection 7.2, each of the $g$ complex zeros must collapse to one of the $g$ collapsing branch cuts. We shall denote the negative collapsed branch cuts $-l_j^2$ and the positive collapsed branch cuts by $+k_i^2$, with $i = 1, \cdots, m = r - 1$ and $j = 1, \cdots, n = g - r + 1$. In view of the above considerations, the form of the differentials reduces to

$$\partial h_1 = -2i dw \left[ 1 + C_0 \frac{w^2}{|w|^2} + \sum_{j=1}^{n} \frac{C_j}{w^2 + l_j^2} \right]$$

$$\partial h_2 = -2 dw \left[ 1 + D_0 \frac{1}{w^2} + \sum_{i=1}^{m} \frac{D_i}{w^2 - k_i^2} \right]$$ (7.19)

for $k_i, l_j > 0$. It is straightforward to compute $W$, and check that the condition $W < 0$ is fulfilled as soon as we have $C_0 - D_0 > 0$, and

$$C_j > 0 \quad j = 1, \cdots, m$$
$$D_i < 0 \quad i = 1, \cdots, n$$ (7.20)

The functions are also readily evaluated,

$$h_1 = -2i(w - \bar{w}) \left[ 1 + \frac{C_0}{|w|^2} + \sum_{j=1}^{n} \frac{C_j}{l_j^2} \ln \frac{|w + il_j|^2}{|w - il_j|^2} \right]$$

$$h_2 = -2(w + \bar{w}) \left[ 1 - \frac{D_0}{|w|^2} - \sum_{i=1}^{m} \frac{D_i}{k_i^2} \ln \frac{|w - ki|^2}{|w + ki|^2} \right]$$ (7.21)

It is immediate that these functions satisfy vanishing Dirichlet boundary conditions, $h_1 = 0$ whenever $w$ is real, and $h_2 = 0$ whenever $w$ is purely imaginary. The conditions $h_1, h_2 > 0$ require the stronger inequalities $C_0 > 0$ and $D_0 < 0$. For $w \in \Sigma$, using the inequalities, $|w + il_j|^2 > |w - il_j|^2$ and $|w - ki|^2 > |w + ki|^2$, and the fact that $C_j > 0$ while $D_i < 0$, as already assumed in (7.20), it is immediate that $h_1, h_2 > 0$. This completes our proof of regularity of the solutions with poles on the boundary of the Riemann surface $\Sigma$.

### 7.4 The probe limit

In this subsection, the behavior of the solutions with collapsed branch points will be analyzed near the pole $w = -k$. Note that the two branch points collapse at a zero $\alpha_b$ of $\partial h_1$. The goal is to identify the singular solution in this limit with the metric of the probe NS5-brane. The limit when two branch points collapse near a zero $\beta_b$ of $\partial h_2$ corresponds to a probe D5-brane.
We define $w = -k + z$ and introduce a new coordinate $z = re^{i\psi}$. The expansion near the collapsed branch points is around $r = 0$. The harmonic functions (7.9) have the expansion

$$
\begin{align*}
    h_1 &= 4c_1 r \sin \psi \\
    h_2 &= 2d_0 - 2d_1 \ln r^2 
\end{align*}
$$

where $d_0, d_1, c_1 > 0$ and subleading terms in $r$ and $\ln r$ are dropped. In this limit the solution becomes

$$
\begin{align*}
    e^{i\psi} &= \frac{d_1^2 |\ln(r)|}{c_1^2} + \cdots \\
    \rho^2 &= 4\sqrt[c_1]{d_1} \frac{1}{r^{2|\ln(r)|^{\frac{1}{2}}}} + \cdots \\
    f_1^2 &= 4\sqrt[c_1]{d_1} (\sin \psi)^2 \frac{r^{\frac{1}{2}}}{|\ln(r)|^{\frac{1}{4}}} + \cdots \\
    f_2^2 &= 4\sqrt[c_1]{d_1} r^{\frac{1}{2}} |\ln(r)|^{\frac{1}{4}} + \cdots \\
    f_4^2 &= 4\sqrt[c_1]{d_1} r^{\frac{1}{2}} |\ln(r)|^{\frac{1}{4}} + \cdots 
\end{align*}
$$

Note that the range of the angular parameter is $\psi \in [0, \pi]$. The behavior of the metric factors $f_1$ and $f_2$ given in (7.23) implies that for fixed $r$ the compact part of the metric is $S^3 \times S^2$. The leading terms for the 2-form potential $B_{(2)}$ are given by

$$
\begin{align*}
    b_1 &= -8d_1 \psi + 2\sin 2\psi + o\left(\frac{1}{|\ln(r)|}\right) 
\end{align*}
$$
\[ b_2 = 16c_1 r \cos \psi + o \left( \frac{r}{|\ln(r)|} \right) \quad (7.24) \]

The function \( b_1 \) is associated with the NSNS two form potential and remains finite as \( r \to 0 \). This implies that there is a non-zero NSNS flux through the three sphere parameterized by \( \psi \) and \( S_3^2 \). The flux is given by

\[
\lim_{r \to 0} \int_{S^3} H_3 = \lim_{r \to 0} \int_{S^3} db_1 \wedge \epsilon^{45} \\
= \lim_{r \to 0} 4\pi \left( b_1 \bigg|_{\psi=\pi} - b_1 \bigg|_{\psi=0} \right) \\
= -32\pi^2 d_1 \\
\quad (7.25)
\]

Clearly, the limit of collapsing branch points is singular. There is, however, a natural interpretation of this singular behavior. The fact that the dilaton goes to infinity, that there is a non-vanishing NSNS 3-form flux through an \( S^3 \) and that metric factors for the \( S^2_2 \) and \( AdS_4 \) behave in the same way all indicate that the singularity can be attributed to the presence of a NS5-brane source with worldvolume \( AdS_4 \times S^2_2 \). At this point, only the flat space NS-brane metric is known. A construction of a solution of the supergravity equations in this background with an NS5-brane source and a detailed comparison of the solutions is beyond the scope of this paper.

A similar analysis can be carried out for the collapse of two branch points near a zero of \( \partial h_2 \). We will not give the details of this limit here since they are analogous to the ones presented above. The dilaton goes to zero, the coordinate \( \psi \) and \( S^2_2 \) produce a non-trivial \( S^3 \) and there is non-vanishing RR 3-form flux through this three sphere. Hence the interpretation of this singular solution is given by a probe D5-brane with worldvolume \( S^2_1 \times AdS_4 \).
8 The AdS/CFT dual interface gauge theory

The holographic interpretation \[17, 18, 19\] (for reviews, see \[20, 21\]) of the original Janus solution as well as the \( \mathcal{N} = 1 \) and \( \mathcal{N} = 4 \) supersymmetric generalizations are given by interface conformal field theories respectively in \[22, 23\], and \[1\]. The dual four-dimensional field theory lives on two four-dimensional half-spaces glued together at a three-dimensional interface. The boundary structure is found by determining where the various components of

![Diagram](image)

Figure 10: Dual interface theory for the genus 1 multi-Janus solution

the metric blow up. Using the Poincaré patch metric for the \( AdS_4 \) metric the ten-dimensional metric is given by

\[
d s^2 = 4\rho^2|dw|^2 + f_4^2 \left( \frac{-dt^2 + dx_1^2 + dx_2^2 + dz^2}{z^2} \right) + f_1^2 ds_1^2 + f_2^2 ds_2^2
\]

(8.1)

It was shown in section 4 that the metric is completely regular for appropriate choices of the parameters of the genus \( g \) hyperelliptic Ansatz. The only place where a metric factor blows up is at the branch points \( e_i \), for \( i = 1, 2, \ldots, 2g - 1 \) (and infinity), where \( f_4 \) diverges. There is an additional divergence of the \( AdS_4 \) metric when \( z \rightarrow 0 \). The boundary structure
becomes clearer if one rewrites the metric as follows

\[
ds^2 = \frac{f_4^2}{z^2} \left\{ -dt^2 + dx_1^2 + dx_2^2 + dz^2 + \frac{z^2}{f_4^2} \left( 4 \rho^2 |du|^2 + f_1^2 ds_{S^2_1} + f_2^2 ds_{S^2_2} \right) \right\}
\]

(8.2)

Near the \(2g + 2\) branch points, \(f_4\) diverges and there are \(2g + 2\) different 3 + 1-dimensional half-spaces \(H_{1,3}^{(i)}, \ i = 1, 2, \ldots, 2g + 2\) each spanned by \(t, x_1, x_2, z\). Each one of these spaces has a boundary at \(z = 0\). Since at \(z = 0\) the overall metric factor blows up, too, there is an additional boundary component given by a 2 + 1-dimensional space \(I_{1,2}\) spanned by \(t, x_1, x_2\).

The behavior of the fields near the boundary was worked out in subsection 4.11. The standard holographic relations [19] allow the identification of the dual conformal field theory. The detailed analysis is the same as the one presented in [1] and will not be repeated here.

The result that the holographic theory is an interface theory defined by \(2g + 2\) copies of \(\mathcal{N} = 4\) SYM living on (independent) 3 + 1-dimensional half-spaces \(H_{1,3}^{(i)}\). The boundaries of the half-spaces are all connected via a 2 + 1-dimensional interface \(I_{1,2}\).

The SYM theories on the \(i\)-th half-space (which is associated with the branch point \(e_i\)), has the coupling constant which can be determined from (4.46).

\[
g_Y^{(i)} = \frac{|Q_2(e_i)|}{|Q_1(e_i)|}
\]

(8.3)

In figure 10, we sketch the situation for the \(g = 1\) case which has four boundary components. Note that in figure 10, the four-dimensional spaces are embedded in a higher-dimensional space. This is mainly done for clarity, but is also natural from a holographic perspective. A different interpretation uses a “folding trick” employed in [24] where \(2g + 2\) copies of \(\mathcal{N} = 4\) SYM fields with coupling constants (8.3) exist independently on a single 3 + 1-dimensional half-space and only interact on the 2 + 1-dimensional interface.
9 Discussion

In this paper, we have found an infinite family of non-singular solutions to the half-BPS equations and Type IIB supergravity field equations on $AdS_4 \times S^2 \times S^2 \times \Sigma$ with $SO(2,3) \times SO(3) \times SO(3)$ symmetry. The general local solution was constructed in the companion paper \[1\], and is specified by two harmonic functions $h_1$ and $h_2$ on a Riemann surface $\Sigma$ with boundary. The class of non-singular solutions found in this paper is parameterized by a genus $g$ hyper-elliptic surface $\Sigma$ with boundary. The branch points of $\Sigma$ are restricted to lie on the real axis. Global regularity is obtained by enforcing the conditions (R1-R7) of subsection 3.5 on the harmonic functions $h_1$ and $h_2$. The ordering (4.40) of the $2g+2$ branch points and $g+1$ branch cuts on the real line is a one-dimensional analog of the coloring of the two-dimensional plane in the “bubbling AdS” solutions\[5\] of \[3\]. The conditions (R1-R7) in addition require that the real zeros of the differentials $\partial h_1$ and $\partial h_2$ obey a relative order amongst themselves and with the branch points.

There are several directions for further research concerning the half-BPS interface theories. Our regular solutions were found amongst the general local solution of \[1\] by imposing regularity and topology conditions. The topology conditions we enforced guarantee that there are asymptotic regions, where the geometry approaches $AdS_5 \times S^5$. It would be interesting to investigate whether these conditions are necessary for regularity. In particular, is it possible to find solutions on Riemann surfaces with topology different from the ones we have used? For example, can one find regular solutions with $h_1, h_2 > 0$ inside $\Sigma$, but with $W > 0$ as well? What is their topology?

It would also be interesting to analyze the proposed dual interface CFT of the multi-Janus solutions in more detail; in particular how does one picture the $2g+2$ half-spaces glued together over a three-dimensional interface, and what are the properties of the corresponding interface super Yang-Mills theory? For example, one feature of our construction is that the number of half-spaces always has to be even. It is an interesting question whether this is required for the preservation of supersymmetry on the interface along the lines of the arguments of \[3\], or is already required in order to have a consistent Dirac equation for the fermion fields in the problem.

A new feature of the multi-Janus solutions (which has genus $g \geq 1$, while the supersymmetric Janus solution had genus $g = 0$) is the presence of topologically non-contractible 3-cycles, and associated non-vanishing NSNS and RR 3-form charges. This fact, and the behavior of our solution in the limit where a branch cut collapses to a point, as was discussed\[58\].

\textsuperscript{8}Similar observations were made in \[2\].
in subsection 7.4, shows that the solutions found in this paper are the fully back-reacted versions corresponding to the insertion of probe D5 and NS5 branes in $\text{AdS}_5 \times S^5$ considered in [25, 4, 26]. Therefore, our half-BPS interface solutions provide a new example of “delocalization”. The defect conformal field theories associated with the probe 5-branes contain extra degrees of freedom coming from the strings localized at the intersection of the three and five branes. In the fully back-reacted geometry, the localized probe branes are replaced by a full non-singular geometry with flux, and the extra degrees of freedom, supported only on the defect, have disappeared. It would be interesting to investigate whether this phenomenon is connected to the observations of [27, 28].

A closely related class of supergravity solutions has $SO(2, 1) \times SO(3) \times SO(5)$ symmetry and is described by an $\text{AdS}_2 \times S^2 \times S^4 \times \Sigma$ Ansatz. The gravitational solution is interpreted as the fully back reacted geometry dual to half-BPS Wilson loops [6, 7, 8]. A detailed analysis of this solution applying the methods of this paper can be found in a further companion paper [9].

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A No Regular solutions with complex poles and cuts

While it would seem natural to consider solutions in which poles and branch cuts appear on the inside of the Riemann surface \( \Sigma \) (as opposed to the branch cuts and poles on the boundary, which were explored in the main body of this paper), we shall show in this appendix that no such regular solutions exist. We begin by working out the example with a single interior pole.

A.1 Janus plus one complex pole

We work in the lower half-plane with the variable \( u \) such that \( \text{Im}(u) < 0 \). Including a complex pole \( p^2 \), with \( \text{Im}(p^2) < 0 \) requires that we also include the complex pole \( \bar{p}^2 \). To keep the behavior at \( u = \infty \) unchanged, we need to also include a complex zero \( c \) and its complex conjugate \( \bar{c} \). Combining all these ingredients, we have the following complex pole generalization of the Janus solution,

\[
\begin{align*}
\partial h_1 &= -i \frac{(u - \alpha)(u - c)(u - \bar{c})}{(u - p^2)(u - \bar{p}^2)\sqrt{u}} du \\
\partial h_2 &= - \frac{(u - \beta)(u - c)(u - \bar{c})}{(u - p^2)(u - \bar{p}^2)\sqrt{u}} du
\end{align*}
\]  

(A.1)

Clearly, when \( c = p^2 \), the Janus solution is obtained as a limiting case.

To uniformize the square root, we change variables to \( w^2 = u \) with \( \text{Re}(w) < 0 \), and \( \text{Im}(w) > 0 \), as in (7.3), so that \( du/\sqrt{u^3} = 2dw/w^2 \), and decompose into elementary fractions,

\[
\begin{align*}
\frac{(w^2 - \alpha)(w^2 - c)(w^2 - \bar{c})}{w^2(w^2 - p^2)(w^2 - \bar{p}^2)} &= 1 + \frac{A_0}{w^2} + \frac{A_1}{w^2 - p^2} + \frac{\bar{A}_1}{w^2 - \bar{p}^2} \\
\frac{(w^2 - \beta)(w^2 - c)(w^2 - \bar{c})}{w^2(w^2 - p^2)(w^2 - \bar{p}^2)} &= 1 + \frac{B_0}{w^2} + \frac{B_1}{w^2 - p^2} + \frac{\bar{B}_1}{w^2 - \bar{p}^2}
\end{align*}
\]  

(A.2)

where

\[
\begin{align*}
A_0 &= -\alpha \frac{|c|^2}{|p|^4} > 0 \\
B_0 &= -\beta \frac{|c|^2}{|p|^4} < 0 \\
A_1 &= \frac{(p^2 - \alpha)(p^2 - c)(p^2 - \bar{c})}{p^2(p^2 - \bar{p}^2)} \\
B_1 &= \frac{(p^2 - \beta)(p^2 - c)(p^2 - \bar{c})}{p^2(p^2 - \bar{p}^2)}
\end{align*}
\]

(A.3)

\( ^9 \)The square \( p^2 \) is used here for later convenience; without loss of generality, we choose \( \text{Re}(p) < 0 < \text{Im}(p) \).
The functions $h_1$ and $h_2$ may then be obtained by elementary integrals. Decomposing the ratios $A_1/p$ and $B_1/p$ into their real and imaginary parts, we obtain,

\begin{align*}
  h_1 &= h_1^{(0)} - 2i(w - \bar{w}) + 2iA_0 \frac{\bar{w} - w}{|w|^2} + \text{Re} \left( -i \frac{A_1}{p} \right) \ln \left| \frac{(w - p)(w + \bar{p})}{(w + p)(w - \bar{p})} \right|^2 \\
  &\quad + i \text{Im} \left( -i \frac{A_1}{p} \right) \ln \frac{(w - p)(\bar{w} + p)(\bar{w} + \bar{p})(w - \bar{p})}{(\bar{w} - p)(w + p)(w + p)(\bar{w} - \bar{p})} \\
  h_2 &= h_2^{(0)} - 2(w + \bar{w}) + 2B_0 \frac{\bar{w} + w}{|w|^2} - \text{Re} \left( \frac{B_1}{p} \right) \ln \left| \frac{(w - p)(w + \bar{p})}{(w + p)(w + \bar{p})} \right|^2 \\
  &\quad - i \text{Im} \left( \frac{B_1}{p} \right) \ln \frac{(w - p)(\bar{w} - p)(\bar{w} + \bar{p})(w + \bar{p})}{(\bar{w} - \bar{p})(w + p)(w + p)(\bar{w} + \bar{p})} \\
  & \quad (A.4)
\end{align*}

The logarithms in the last term in each expression are multiple-valued in $w$ around the pole $p$. Single-valuedness of $h_1$ and $h_2$ requires that $\text{Re}(A_1/p) = \text{Im}(B_1/p) = 0$. These conditions are equivalent to requiring vanishing periods of both differentials $\partial h_1$ and $\partial h_2$ around the pole $p$. We parametrize the solutions to these equations explicitly by

\begin{align*}
  A_1 &= -ipa_1 & B_1 &= pb_1 & a_1, b_1 \in \mathbb{R} & (A.5)
\end{align*}

Requiring $h_1$ to vanish for real $w < 0$, and $h_2$ to vanish for imaginary $w$, we find that we must have $h_1^{(0)} = h_2^{(0)} = 0$. We then have the following simplified expressions,

\begin{align*}
  h_1 &= -2i(w - \bar{w}) + 2iA_0 \frac{\bar{w} - w}{|w|^2} - a_1 \ln \left| \frac{(w - p)(w + \bar{p})}{(w + p)(w - \bar{p})} \right|^2 \\
  h_2 &= -2(w + \bar{w}) + 2B_0 \frac{\bar{w} + w}{|w|^2} - b_1 \ln \left| \frac{(w - p)(w + \bar{p})}{(w + p)(w + \bar{p})} \right|^2 & (A.6)
\end{align*}

Since we have assumed that both $w$ and $p$ lie in the second quadrant,

\begin{align*}
  \text{Re}(w) &< 0 & \text{Re}(p) &< 0 \\
  \text{Im}(w) &> 0 & \text{Im}(p) &> 0 & (A.7)
\end{align*}

we have the inequalities,

\begin{align*}
  |w - p| &< |w - \bar{p}| & |w + \bar{p}| &< |w + p| \\
  |w - p| &< |w + \bar{p}| & |w - \bar{p}| &< |w + p| & (A.8)
\end{align*}

Thus, positivity of $h_1$ and $h_2$ will be assured for $w$ in the second quadrant provided,

\begin{align*}
  a_1, b_1 &\geq 0 & (A.9)
\end{align*}
It remains to solve (A.3) and the period relations (A.5). Combining both relations, we have,

\[
(p^2 - \alpha)(p^2 - c)(p^2 - \bar{c}) = -ia_1 p^3(p^2 - \bar{p}^2)
\]

\[
(p^2 - \beta)(p^2 - c)(p^2 - \bar{c}) = +b_1 p^3(p^2 - \bar{p}^2)
\]  

(A.10)

We consider these two complex equations for \(a_1\) and \(b_1\) given, and view \(p^2\) and the position of the zero \(c\) as determined by \(a_1\), \(b_1\), and the above period relations. Taking the ratio of the two equations, we obtain an equation for \(p^2\),

\[
p^2 = \frac{\alpha b_1 + i\beta a_1}{b_1 + i a_1}
\]  

(A.11)

The imaginary part of \(p^2\) is given by

\[
\text{Im}(p^2) = \frac{a_1 b_1 (\beta - \alpha)}{|a_1 - i b_1|^2}
\]  

(A.12)

Under the conditions \(a_1, b_1 > 0\), and using the inequality on the real zeros \(\beta - \alpha > 0\), the pole \(p^2\) actually lies in the upper half-plane, contrarily to our assumptions.

Note that if we had taken \(p^2\) in the upper half-plane instead, and \(p\) in the first quadrant, the inequalities in the first line of (A.8) would remain unchanged, but the inequalities in the last line would get reversed, with the effect of now requiring \(a_1 \geq 0\) but \(b_1 \leq 0\). From (A.12), we would then find that now \(p^2\) must be in the lower half-plane, which is again in contradiction with the assumptions made. Thus, the formal solution with one complex pole can never satisfy all the necessary conditions of regularity.

### A.2 The addition of general complex poles and branch cuts

The arguments presented in the preceding subsection are essentially local on the surface \(\Sigma\). Thus, by analogous reasoning, the presence of multiple complex poles is also ruled out. The arguments may also be generalized to complex branch cuts.
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