The natural quiver of an artinian algebra

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Abstract

The motivation of this paper is to study the natural quiver of an artinian algebra, a new kind of quivers, as a tool independing upon the associated basic algebra.

In [5], the notion of the natural quiver of an artinian algebra was introduced and then was used to generalize the Gabriel theorem for non-basic artinian algebras splitting over radicals and non-basic finite dimensional algebras with 2-nilpotent radicals via pseudo path algebras and generalized path algebras respectively.

In this paper, firstly we consider the relationship between the natural quiver and the ordinary quiver of a finite dimensional algebra. Secondly, the generalized Gabriel theorem is obtained for radical-graded artinian algebras. Moreover, Gabriel-type algebras are introduced to outline those artinian algebras satisfying the generalized Gabriel theorem here and in [5]. For such algebras, the uniqueness of the related generalized path algebra and quiver holds up to isomorphism in the case when the ideal is admissible. For an artinian algebra, there are two basic algebras, the first is that associated to the algebra itself; the second is that associated to the correspondent generalized path algebra. In the final part, it is shown that for a Gabriel-type artinian algebra, the first basic algebra is a quotient of the second basic algebra.

In the end, we give an example of a skew group algebra in which the relation between the natural quiver and the ordinary quiver is discussed.

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1 Natural quiver and the relation with ordinary quiver

Suppose that $A$ is a left artinian algebra over a field $k$, and $r = r(A)$ is the radical of $A$. In this paper left artinian algebras are written briefly as “artinian algebras”.

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Let \( \{S_1, S_2, \cdots, S_n\} \) be the complete set of non-isomorphic simple \( A \)-modules of \( A \). One can define a finite quiver \( \Gamma_A \), called the ordinary quiver of \( A \) as follows: \( \Gamma_0 = \{1, 2, \cdots, n\} \), and the number \( m_{ij} \) of arrows from \( i \) to \( j \) equals to the dimensional number \( \dim_k \text{Ext}_A(S_i, S_j) \). By [2], when \( A \) is a finite-dimensional basic algebra over an algebraically closed field \( k \) and \( 1_A = \varepsilon_1 + \cdots + \varepsilon_n \) a decomposition of \( 1_A \) into a sum of primitive orthogonal idempotents. Then, we can re-index \( \{S_1, S_2, \cdots, S_n\} \) such that \( S_i \cong A\varepsilon_i/r\varepsilon_i \), and moreover, \( \dim_k \text{Ext}_A(S_i, S_j) = \dim_k(\varepsilon_j r/r^2\varepsilon_i) \). Clearly, if \( Q \) is a finite quiver without oriented cycles, the ordinary quiver of the path algebra \( kQ \) is just \( Q \).

Now, we introduce the so-called natural quivers from artinian algebras.

Write \( A/r = \bigoplus_{i=1}^s \mathcal{A}_i \), where \( \mathcal{A}_i \) is a simple ideal of \( A/r \) for each \( i \). Then, the algebra \( r/r^2 \) is an \( A/r \)-bimodule by \( \bar{a} \cdot (r/r^2) \cdot \bar{b} = arb/r^2 \) for any \( \bar{a} = a + r, \bar{b} = b + r \in A/r \). Let \( _IM_j = \mathcal{A}_i \cdot r/r^2 \cdot \mathcal{A}_j \), then \( _IM_j \) is finitely generated as \( \mathcal{A}_i \mathcal{A}_j \)-bimodule for each pair \((i, j)\).

For two artinian algebras \( A \) and \( B \), the rank of a finitely generated \( A-B \)-bimodule \( M \) is defined as the least cardinal number of the sets of generators. Clearly, for any finitely generated \( A-B \)-bimodule, such rank always exists uniquely.

Now we can associate with \( A \) a quiver \( \Delta_A = (\Delta_0, \Delta_1) \), which is called the natural quiver of \( A \), in the following way. Let \( \Delta_0 = \{1, \cdots, s\} \) as the set of vertices. For \( i, j \in \Delta_0 \), let the number \( t_{ij} \) of arrows from \( i \) to \( j \) in \( \Delta_A \) be the rank of the finitely generated \( \mathcal{A}_j\mathcal{A}_i \)-bimodules \( jM_i \). Obviously, if \( jM_i = 0 \), there are no arrows from \( i \) to \( j \).

The notion of natural quiver was firstly introduced in [5], where the aim of the author is to use the generalized path algebra from the natural quiver of an artinian algebra \( A \) to characterize \( A \) through the generalized Gabriel theorem. In the further research, one is motivated to study the representation of an artinian algebra via the associated generalized path algebra or pseudo path algebra but not the basic algebra of the artinian algebra.

In order to clean the relation between the ordinary quiver and the natural quiver of an artinian algebra, it is necessary to note that the natural quiver defined here is indeed opposite to the quiver defined in [5]. Now, we consider the relation between the ordinary quiver and the natural quiver of an artinian \( k \)-algebra over an algebraically closed field \( k \).

Clearly the number of the vertices in two quivers are equal since \( A \) and \( A/r \) have the same simple modules, that is, we have \( n = s \) as above.

When \( A \) is a finite-dimensional basic algebra over an algebraically closed field \( k \), \( A/r \cong \prod k \) where the number of copies of \( k \) equals the number of primitive orthogonal idempotents. As mentioned above, \( \dim_k \text{Ext}_A^1(S_i, S_j) = \dim_k(\varepsilon_j r/r^2\varepsilon_i) \) where \( S_i, S_j \) are the simple modules of \( A \) corresponding to the primitive orthogonal idempotents \( \varepsilon_i, \varepsilon_j \) respectively, which means that the number of arrows from \( i \) to \( j \) in the ordinary quiver \( \Gamma_A \) of \( A \) is equal to that in the natural quiver \( \Delta_A \) of \( A \). Thus, we have:

Lemma 1.1. For a finite-dimensional basic algebra \( A \) over an algebraically closed field \( k \), the ordinary quiver and the natural quiver of \( A \) coincide. In particular, if \( Q \) is a
finite quiver without oriented cycles, the ordinary quiver and the natural quiver of the path algebra $kQ$ are both $Q$.

In order to discuss similarly for non-basic algebras, we introduce the following notion:

Let $Q$ be a quiver and $Q'$ a sub-quiver of $Q$. If $(Q')_0 = Q_0$ and for any vertices $i, j$, there exist arrows from $i$ to $j$ in $Q'$ if and only if there exist arrows from $i$ to $j$ in $Q$, then we call this $Q'$ a dense sub-quiver of $Q$.

When $A$ is over an algebraically closed field $k$, by Proposition 7.4.4 in [6], the relation:

$$t_{ij} \leq m_{ij} \leq n_{ij}$$

holds where $n_i$ and $n_j$ are integers such that $\overline{A_i} \cong M_{n_i}(k)$ and $\overline{A_j} \cong M_{n_j}(k)$. Trivially, if each $n_i = 1$, then $t_{ij} = m_{ij}$, thus the ordinary quiver $\Delta_A$ and $\Gamma_A$ the natural quiver of $A$ are coincided. But, when some $n_i \neq 1$, it is possible that $t_{ij} < m_{ij}$, and $t_{ij} \neq 0$ if and only if $m_{ij} \neq 0$, which means usually, $\Delta_A$ is a dense sub-quiver of $\Gamma_A$.

As well-known, for an artinian algebra $A$, there is the correspondent basic algebra $B$ and they are Morita-equivalent, i.e. the module categories $\text{Mod}A$ and $\text{Mod}B$ are equivalent, which follows that there is an equivalent functor $F$ such that $\text{Hom}_A(S_i, S_j) \cong F(\text{Hom}_B(F(S_i), F(S_j)))$ for any simple modules $S_i$ and $S_j$ in $\text{Mod}A$. Moreover, $\text{Ext}_A^1(S_i, S_j) \cong \text{Ext}_B^1(F(S_i), F(S_j))$. It means the ordinary quiver of $A$ is the same with that of $B$. If $A$ is of finite dimension, its basic algebra is also of finite dimension. In the summary, we have:

**Proposition 1.2.** Let $A$ be a finite dimensional algebra over a field $k$ with $r$ its radical and $A/r = \overline{A_1} \oplus \cdots \oplus \overline{A_n}$ the direct sum of simple ideals, and $B$ is the corresponding basic algebra of $A$. Let $\Delta_A$ and $\Delta_B$ be the ordinary quivers of $A$ and $B$ respectively, meanwhile $\Delta_A$ and $\Delta_B$ the natural quivers of $A$ and $B$ respectively. Then,

(i) $\Gamma_A = \Gamma_B$;

(ii) $\Gamma_B = \Delta_B$ if $k$ is algebraically closed;

(iii) $\Delta_A$ is a dense sub-quiver of $\Gamma_A$, also of $\Gamma_B$ and $\Delta_B$, if $k$ is algebraically closed.

2 Generalized Gabriel theorem in the radical-graded case

The concepts of generalized path algebras were introduced early in [3] in order to find a generalization of path algebras so as to obtain a generalized type of the Gabriel Theorem for arbitrary finite dimensional algebras which would admit this algebra to be isomorphic to a quotient algebra of such a generalized path algebra. It is natural to ask how we look for a generalized path algebra via the natural quiver to cover the artinian algebra. Unfortunately, in general, as shown by the counter-example in [5], an artinian algebra with lifted quotient may not be a homomorphic image of its correspondent $\mathcal{A}$-path-type tensor algebra. In this reason, the concepts of pseudo path algebras were introduced in [3] and it was shown that when the quotient algebra of an artinian algebra can be lifted,
the algebra is covered by a pseudo path algebra via the natural quiver under an algebra homomorphism.

However, there still exists some special class of artinian algebras which can be covered by their correspondent $\mathcal{A}$-path-type tensor algebras and equivalently by the generalized path algebras. This point can be seen in [5] from the generalized Gabriel theorem for a finite dimensional algebra with 2-nilpotent radical in the case it is splitting over its radical.

In this section, we will give another class of artinian algebras which can be covered by the generalized path algebra via the natural quiver, that is, the generalized Gabriel theorem for this class of artinian algebras is true, too. This class of artinian algebras are just the so-called radical-graded artinian algebra as follows.

For an artinian algebra $A$, let $r = \text{rad}A$ be the radical of $A$ and the Loewy length $rl(A) = s$. Define $\text{gr}A = A/r \oplus r/r^2 \oplus \cdots \oplus r^{s-2}/r^{s-1} \oplus r^{s-1}$ as a graded-algebra with multiplication $(x + r^{i+1})(y + r^{j+1}) = xy + r^{i+j+1}$ for $x \in r^i$, $y \in r^j$. Trivially, this graded algebra is strict.

An artinian algebra $A$ is said to be radical-graded if $A = \bigoplus_{i \geq 0} A_i$ is strictly graded with $A_0$ semisimple. In this case, there is a minimal positive integer $t$ such that $A_i = 0$ for all $i \geq t$ since $A$ is artinian. By this definition, it is easy to see that for any artinian algebra $A$, $\text{gr}A$ is always radical-graded. We have the following characterization:

**Proposition 2.1.** An artinian algebra $A$ is radical-graded if and only if $A \cong \text{gr}A$. In this situation, $A_0 \cong A/r$, $r \cong r/r^2 \oplus \cdots \oplus r^{s-2}/r^{s-1} \oplus r^{s-1}$ as algebras.

**Proof.** “$\Leftarrow$” is trivial since $\text{gr}A$ is radical-graded.

“$\Rightarrow$” Suppose that $A = \bigoplus_{i \geq 0} A_i$ is strictly graded with $A_0$ semisimple. Thus, there is a minimal positive integer $t$ such that $A_i = 0$ for $i \geq t$. Write $r' = \bigoplus_{i \geq 1} A_i$, clearly it is an ideal of $A$, $A/r' = A_0$ semisimple and $r'^j = \bigoplus_{i \geq j} A_i$ which is zero when $j \geq t$ and hence $r'$ is nilpotent. So $r' = r$ the radical of $A$, and clearly $r'^i/r'^{i+1} \cong A_i$ for $i \geq 1$ which is zero when $i \geq t$, and $r'^l = 0$ if and only if $A_i = 0$ for any $l$. Therefore $t = s = rl(A)$, $\text{gr}A = A/r \oplus r/r^2 \oplus \cdots \oplus r^{s-1} \cong A/r' \oplus r'/r'^2 \oplus \cdots \oplus r'^{s-1} = \bigoplus_{i \geq 0} A_i = A$. □

From the above proposition, we get

**Corollary 2.2.** For any artinian algebra $A$, $\text{gr}(\text{gr}A) \cong \text{gr}A$.

Now, we introduce briefly some notions about generalized path algebras.

Let $Q = (Q_0, Q_1)$ be a quiver and $\mathcal{A} = \{A_i : i \in Q_0\}$ a family of $k$-algebras $A_i$ with identity $e_i$, indexed by the vertices of $Q$. The elements $a_j \neq 0$ of $\bigcup_{i \in Q_0} A_i$ are called $\mathcal{A}$-paths of length zero, with starting vertex $s(a_j)$ and the ending vertex $e(a_j)$ are both $j$. For each $n \geq 1$, an $\mathcal{A}$-path $P$ of length $n$ is given by $a_1a_2a_3 \cdots a_n$, where $s(\beta_1)\beta_1 \cdots \beta_n e(\beta_n))$ is a path in $Q$ of length $n$, for each $i = 1, \ldots, n$, $0 \neq a_i \in A_{s(\beta_i)}$ and $0 \neq a_{n+1} \in A_{e(\beta_n)}$. $s(\beta_1)$ and $e(\beta_n)$ are also called respectively the starting vertex and the ending vertex of $P$. Write $s(P) = s(\alpha_1)$ and $e(P) = e(\alpha_n)$. Now, consider the quotient
2 GENERALIZED GABRIEL THEOREM IN THE RADICAL-GRADED CASE

$R$ of the $k$-linear space with basis the set of all $A$-paths by the subspace generated by all the elements of the form

$$a_1 \beta_1 \cdots \beta_{j-1}(a_j^1 + \cdots + a_j^m) \beta_j a_{j+1} \cdots \beta_n a_{n+1} - \sum_{i=1}^m a_1 \beta_1 \cdots \beta_{j-1} a_i^j \beta_j a_{j+1} \cdots \beta_n a_{n+1}$$

where \((s(\beta_i))[1 \cdots \beta_n]\) is a path in $Q$ of length $n$, for each $i = 1, ..., n$, $a_i \in A_i(\beta_i)$, $a_{n+1} \in A_{s(\beta_n)}$ and $a_j^l \in A_{s(\beta_j)}$ for $l = 1, ..., m$. In $R$, given two elements $[a_1 \beta_1 a_2 \beta_2 \cdots a_n \beta_n a_{n+1}]$ and $[b_1 \gamma_1 b_2 \gamma_2 \cdots b_n \gamma_n b_{n+1}]$, define the multiplication as follows:

$$[a_1 \beta_1 a_2 \beta_2 \cdots a_n \beta_n a_{n+1}] \cdot [b_1 \gamma_1 b_2 \gamma_2 \cdots b_n \gamma_n b_{n+1}]$$

$$= \begin{cases} [a_1 \beta_1 a_2 \beta_2 \cdots a_n \beta_n (a_{n+1} b_1) \gamma_1 b_2 \gamma_2 \cdots b_n \gamma_n b_{n+1}], & \text{if } a_{n+1}, b_1 \in A_i \text{ for the same } i \\ 0, & \text{otherwise} \end{cases}$$

It is easy to check that the above multiplication is well-defined and makes $R$ to become a $k$-algebra. This algebra $R$ defined above is called an $A$-path algebra of $Q$ respecting to $A$, or generally generalized path algebras. Denote it by $R = k(Q, A)$. Clearly, $R$ is an $A$-bimodule, where $A = \oplus_{i \in Q_0} A_i$.

A generalized path algebra $k(Q, A)$ is said to be normal if all algebras $A_i (i \in Q_0)$ are simple algebras for $A = \{A_i : i \in Q_0\}$.

Associated with the pair $(A, _A MA)$ for a $k$-algebra $A$ and an $A$-bimodule $M$, we write the $n$-fold $A$-tensor product $M \otimes_A M \otimes \cdots \otimes_A M$ as $M^n$. Writing $M^0 = A$, then $T(A, M) = A \oplus M \oplus M^2 \oplus \cdots \oplus M^n \oplus \cdots$ becomes a $k$-algebra with multiplication induced by the natural $A$-bilinear maps $M^i \times M^j \to M^{i+j}$ for $i \geq 0$ and $j \geq 0$. $T(A, M)$ is called the tensor algebra of $M$ over $A$.

Define a special class of tensor algebras so as to characterize generalized path algebras. An $A$-path-type tensor algebra is defined to be the tensor algebra $T(A, M)$ satisfying that (i) $A = \bigoplus_{i \in I} A_i$ for a family of $k$-algebras $A = \{A_i : i \in I\}$, (ii) $M = \bigoplus_{i,j \in I} M_{ij}$ where $iM_j$ are finitely generated $A_i$-$A_j$-bimodules for all $i$ and $j$ in $I$ and $A_k \cdot M_j = 0$ if $k \neq i$ and $iM_j \cdot A_k = 0$ if $k \neq j$. A free $A$-path-type tensor algebra is the $A$-path-type tensor algebra $T(A, M)$ whose each finitely generated $A_i$-$A_j$-bimodule $iM_j$ for $i$ and $j$ in $I$ is a free bimodule with a basis and the cardinality of this basis is equal to the rank of $iM_j$ as a finitely generated $A_i$-$A_j$-bimodule.

In an $A$-path algebra $k(Q, A)$, let $A = \bigoplus_{i \in Q_0} A_i$. For any $i, j$, let $iM_j^F$ be the free $A_i$-$A_j$-bimodule with basis given by the arrows from $j$ to $i$. Then the number of free generators in the basis is the rank of $iM_j^F$ as a finitely generated bimodule. Define $A_k \cdot iM_j^F = 0$ if $k \neq i$ and $iM_j^F \cdot A_k = 0$ if $k \neq j$. Then $M^F = \bigoplus_{j \rightarrow i} iM_j^F$ is an $A$-bimodule. We get the unique free $A$-path-type tensor algebras $T(A, M^F)$.

Conversely, given an $A$-path-type tensor algebra $T(A, M)$ with $A = \{A_i : i \in I\}$ and finitely generated $A_i$-$A_j$-bimodules $iM_j$ for $i, j \in I$ such that $A = \bigoplus_{i \in I} A_i$, $M = \bigoplus_{i,j \in I} iM_j$, $A_k \cdot iM_j = 0$ if $k \neq i$ and $iM_j \cdot A_k = 0$ if $k \neq j$. Trivially, $iM_j = A_i MA_j$. Let
$r_{ij}$ be the rank of $jM_i$. One can associate with $T(A,M)$ a quiver $Q = (Q_0, Q_1)$, called the quiver of $T(A,M)$, via $Q_0 = I$ as the set of vertices and for $i,j \in I$, $r_{ij}$ as the number of arrows from $i$ to $j$ in $Q$. Its $A$-path algebra $k(Q,A)$ is called the corresponding $A$-path algebra of $T(A,M)$. By definition, the quiver of $T(A/r,r/r^2)$ is just $\Delta_A$.

From the above discussion, every $A$-path-type tensor algebra $T(A,M)$ can be used to construct its corresponding $A$-path algebra $k(Q,A)$; but, from this $A$-path algebra $k(Q,A)$, we can get uniquely the free $A$-path-type tensor algebra $T(A,M^F)$. In summary, we have the following in [5]:

**Lemma 2.3.** (i) Every $A$-path-type tensor algebra $T(A,M)$ can be used to construct uniquely the free $A$-path-type tensor algebra $T(A,M^F)$. There is a surjective $k$-algebra morphism $\pi: T(A,M^F) \to T(A,M)$ such that $\pi(iM^F_i) = iM_j$ for any $i,j \in I$;

(ii) Let $T(A,M^F)$ be the free $A$-path-type tensor algebra built by a $A$-path algebra $k(Q,A)$. Then there is a $k$-algebra isomorphism $\tilde{\phi}: T(A,M^F) \to k(Q,A)$ such that for any $t \geq 1$, $\tilde{\phi}(\bigoplus_{j \geq t}(M^F_j)) = J^t$;

(iii) Let $T(A,M)$ be an $A$-path-type tensor algebra with the corresponding $A$-path algebra $k(Q,A)$. Then there is a surjective $k$-algebra homomorphism $\phi: k(Q,A) \to T(A,M)$ such that for any $t \geq 1$, $\phi(J^t) = \bigoplus_{j \geq t} M^t$.

Here $J$ denotes the ideal generated by all $A$-paths of length 1 in $k(Q,A)$.

In the sequel, we always denote by $J$ the ideal generated by all generalized paths of length one in the discussed generalized path algebras. When $Q$ is admissible, i.e. is acyclic, $J$ is just the radical of a normal generalized path algebra $k(Q,A)$ (see [3]).

A relation $\sigma$ on an $A$-path algebra $k(\Delta,A)$ is a $k$-linear combination of some $A$-paths $P_i$ with the same starting vertex and the same ending vertex, that is, $\sigma = k_1P_1 + \cdots + k_nP_n$ with $k_i \in k$ and $s(P_1) = \cdots = s(P_n)$ and $e(P_1) = \cdots = e(P_n)$. If $\rho = \{\sigma_i\}_{i \in T}$ is a set of relations on $k(\Delta,A)$, the pair $(k(\Delta,A),\rho)$ is called an $A$-path algebra with relations. Associated with $(k(\Delta,A),\rho)$ is the quotient $k$-algebra $k(\Delta,A,\rho) \overset{\text{def}}{=} k(\Delta,A)/\langle \rho \rangle$, where $\langle \rho \rangle$ denotes the ideal in $k(\Delta,A)$ generated by the set of relations $\rho$. When the length $l(P_i)$ of each $P_i$ is at least $j$, it holds $\langle \rho \rangle \subset J^j$.

Now, let $M = r/r^2$ as $A/r$-bimodule, $iM_j = \overline{A_i \cdot r/r^2 \cdot A_j}$, then $iM_j$ is finitely generated as $\overline{A_i A_j}$-bimodule for each pair $(i,j)$ and $M = \bigoplus_{i,j} iM_j$. Thus, we get the tensor algebra $T(A/r,r/r^2)$ and the corresponding generalized path algebra $k(\Delta_A,A)$ from the natural quiver $\Delta_A$ of $A$.

A set of some $A$-paths or their linear combinations in $k(Q,A)$ is said to be $A$-finite if all $A$-paths in this set are constructed from a finite number of paths in $Q$ with elements of $\bigcup_{i \in Q_0} A_i$. A quotient or an ideal of $k(Q,A)$ is said to be $A$-finitely generated if it is generated by an $A$-finite set.

The following is the main result in this section:
Theorem 2.4. (Generalized Gabriel Theorem in radical-graded case) Assume that $A$ is a radical-graded artinian $k$-algebra. Then, there is an $A$-finite set $\rho$ of relations of $k(\Delta_A, A)$ such that $A \cong k(\Delta_A, A)/\langle \rho \rangle$ with $J^s \subset \langle \rho \rangle \subset J$ for some positive integer $s$.

Proof: Let $r$ be the radical of $A$ with the Loewy length $rl(A) = s + 1$. Since $A$ is radical-graded, we have $A \cong A/r \oplus r/r^2 \oplus r^2/r^3 \oplus \cdots \oplus r^{s-1}/r^s \oplus r^s$. Thus, $r \cong r/r^2 \oplus r^2/r^3 \oplus \cdots \oplus r^{s-1}/r^s \oplus r^s$ and $A \cong A/r \oplus r$ as algebras.

Write $A/r = \bigoplus_{i=1}^s \mathbf{A}_i$ with simple ideals $\mathbf{A}_i$ for all $i$. Then, we have the $A$-path type tensor algebra $T(A/r, r/r^2)$ with $A = \{\mathbf{A}_i : i = 1 \cdots s\}$. Firstly, we can find a surjective morphism of algebras from $T(A/r, r/r^2)$ to $A$. In fact, for $r^m/r^{m+1}$, define $f_m : r^m/r^{m+1} \otimes_{A/r} r/r^2$ (with $m$ copies of $r/r^2$) $\rightarrow r^m/r^{m+1}$ satisfying that $f_m(\mathbf{A}_i \otimes \cdots \otimes \mathbf{A}_m) = \mathbf{A}_1 \cdots \mathbf{A}_m$ where $\mathbf{A}_i \subset r/r^2$ for $x_i \in r$ and $\mathbf{A}_i \cdot \mathbf{A}_m \subset r^m/r^{m+1}$. It is easy to see that $f_m$ is well-defined as a morphism of $A/r\cdot A/r$-bimodules and trivially, $f_m$ is surjective. Then, $f = id_{A/r} \oplus f_1 \oplus \cdots \oplus f_m \oplus \cdots$ is a surjective algebra morphism from $T(A/r, r/r^2)$ to $A$, where $f_m = 0$ when $m \geq s + 1$.

Moreover, by Lemma 2.3, there is a surjective $k$-algebra homomorphism $\varphi : k(\Delta_A, A) \rightarrow T(A/r, r/r^2)$ such that for any $t \geq 1$, $\varphi(J^t) = \bigoplus_{j \geq t}(r/r^2)^{\otimes j}$, where $(r/r^2)^{\otimes j}$ denotes $r/r^2 \otimes_{A/r} r/r^2 \otimes_{A/r} \cdots \otimes_{A/r} r/r^2$ with $j$ copies of $r/r^2$. Then, $f \varphi : k(\Delta_A, A) \rightarrow A$ is a surjective algebra morphism. Therefore, for the kernel $I = ker(\varphi)$, we obtain $k(\Delta_A, A)/I \cong A$.

Now, we prove that $\bigoplus_{j \geq rl(A)}(r/r^2)^{\otimes j} \subset Kerf \subset \bigoplus_{j \geq 2}(r/r^2)^{\otimes j}$. In fact, by the definition, $f_1 = id_{r/r^2}$, so $f|_{A/r \otimes r/r^2} = id_{A/r} \oplus f_1 : A/r \otimes r/r^2 \rightarrow A$ is a monomorphism with image intersecting $r^2$ trivially. It follows that $Kerf \subset \bigoplus_{j \geq 2}(r/r^2)^{\otimes j}$. On the other hand, $f((r/r^2)^{\otimes j}) = 0$ for $j \geq rl(A)$ since $r^j = 0$ in this case. Therefore we get $\bigoplus_{j \geq rl(A)}(r/r^2)^{\otimes j} \subset Kerf$.

But, by Lemma 2.3, for $t = rl(A)$ and $t = 2$ respectively, $\varphi(J^{rl(A)}) = \bigoplus_{j \geq rl(A)}(r/r^2)^{\otimes j}$ and $\varphi(J^2) = \bigoplus_{j \geq 2}(r/r^2)^{\otimes j}$. So, $\varphi(J^{rl(A)}) \subset Kerf \subset \varphi(J^2)$.

Then, we prove $J^t \subset \varphi^{-1}(\varphi(J^t)) \subset J^t + \tilde{\phi}(\bigoplus_{j \leq t-1}((r/r^2)^{\otimes j}) \cap \tilde{\phi}(Kerf)$ for $t \geq 1$, where $\tilde{\varphi}$ is that in Lemma 2.3. Trivially, $J^t \subset \varphi^{-1}(\varphi(J^t))$. On the other hand, $\tilde{\varphi} = \pi \tilde{\phi}^{-1}$ and then $\varphi^{-1} = \tilde{\phi}^{-1}$. By Lemma 2.3(iii), $\varphi(J^t) = \bigoplus_{j \geq t}(r/r^2)^{\otimes j}$. From the definition of $\pi$ in Lemma 2.3, it can be seen that $\pi^{-1}(\bigoplus_{j \leq t}((r/r^2)^{\otimes j}) \subset \bigoplus_{j \geq t}((r/r^2)^{\otimes j} + \bigoplus_{j \leq t-1}((r/r^2)^{\otimes j}) \cap Kerf$. Thus, by Lemma 2.3, we have

$$\varphi^{-1}(\varphi(J^t)) = \tilde{\phi}^{-1}(\pi^{-1}(\bigoplus_{j \geq t}(r/r^2)^{\otimes j})) \subset \tilde{\phi}(\bigoplus_{j \geq t}((r/r^2)^{\otimes j})) \cap \tilde{\phi}(Kerf) \cap \tilde{\phi}(Kerf) = J^t + \tilde{\phi}(\bigoplus_{j \leq t-1}((r/r^2)^{\otimes j}) \cap \tilde{\phi}(Kerf))$$

Hence,

$$J^{rl(A)} \subset \varphi^{-1}(\varphi(J^{rl(A)})) \subset \varphi^{-1}(Kerf) = I \subset \varphi^{-1}(\varphi(J^2)) \subset J^2 \oplus \tilde{\phi}(\bigoplus_{j \leq t-1}((r/r^2)^{\otimes j}) \cap \tilde{\phi}(Kerf) \subset J^2 + J \cap \tilde{\phi}(Kerf)$$

But, $\tilde{\phi}(Kerf) = \tilde{\phi}(\pi^{-1}(0)) = \tilde{\phi}^{-1}(0) = Ker\tilde{\phi}$. Then, $J^{rl(A)} \subset I \subset J^2 + J \cap Ker\tilde{\phi} \subset J$.

Lastly, we present $I$ through an $A$-finite set of relations on $k(\Delta_A, A)$. $J^{rl(A)}$ is the ideal $A$-finitely generated in $k(\Delta_A, A)$ by all $A$-paths of length $rl(A)$. $k(\Delta_A, A)/J^{rl(A)}$
3 Two basic algebras from an artinian algebra

For an artinian algebra \( A \), write \( A/r = \bigoplus_{i=1}^{s} \overline{A}_i \) with simple ideals \( \overline{A}_i \), we get \( k(\Delta_A,A) \) where \( \Delta_A \) is the natural quiver of \( A \) and \( A = \{ \overline{A}_i : i = 1, 2, \cdots, s \} \).

It is known that the associated basic algebra \( B \) which is Morita-equivalent to \( A \) is important for representations of \( A \). In order to realize our approach, it is valid to consider the associated basic algebra \( C \) of the generalized path algebra \( k(\Delta_A,A) \) of the natural quiver \( \Delta_A \) of \( A \) and moreover, the relationship between \( B \) and \( C \).

However, in general, the generalized path algebra \( k(\Delta_A,A) \) is not an artinian algebra, e.g. when the natural quiver \( \Delta_A \) contains an oriented cycle. So, \( k(\Delta_A,A) \) has not the so-called related basic algebra under the meaning of “artinian” such that they are Morita-equivalent each other. In this reason, \( C \) is different from that for artinian algebras.

A complete set of non-isomorphic primitive orthogonal idempotents of \( A \) is a set of primitive orthogonal idempotents \( \{ \varepsilon_i : i \in I \subset (\Delta_A)_0 \} \) such that \( A\varepsilon_i \not\cong A\varepsilon_j \) as left \( A \)-modules for any \( i \neq j \) in \( I \) and for each primitive idempotent \( \varepsilon_s \) the module \( A\varepsilon_s \) is isomorphic to one of the modules \( A\varepsilon_i \) (\( i \in I \)).

Every indecomposable projective module \( P \) is decided by a primitive idempotent \( \varepsilon_i \), that is, \( P \cong A\varepsilon_i \) for some \( i \). And, there exists a bijective correspondence between the iso-classes of indecomposable projective modules and the iso-classes of simple modules. The set of the latter is equal to the vertex set \( (\Gamma_A)_0 \) of the ordinary quiver \( \Gamma_A \) of \( A \), and then to the vertex set \( (\Delta_A)_0 \) of the natural quiver \( \Delta_A \) of \( A \). Hence, \( I = (\Delta_A)_0 \). Let each
$P_i$ be chosen as a representative from the iso-class of indecomposable projective module $A\varepsilon_i$ and let $i$ run over the vertex set $(\Delta_A)_0$. Then the basic algebra $B$ of $A$ is given by $B = \text{End}(\coprod_{i \in (\Delta_A)_0} P_i) \cong \bigoplus_{i,j \in (\Delta_A)_0} \text{Hom}_A(P_i, P_j) \cong \bigoplus_{i,j \in (\Delta_A)_0} \varepsilon_i A \varepsilon_j$.

**Lemma 3.1.** Let $A$ be an artinian algebra. Then the complete set of non-isomorphic primitive orthogonal idempotents of $A$, $A/r$ ($r$ is the radical of $A$) and $k(\Delta_A, A)$ are the same, whose cardinality is equal to that of the vertex set of the natural quiver of $A$.

**Proof:** Let $\overline{\varepsilon}_i$ be the image of $\varepsilon_i$ under the canonical homomorphism from $A$ to $A/r$. Since $A$ and $A/r$ have the same simple modules, $\{\overline{\varepsilon}_i : i \in (\Delta_A)_0\}$ is a complete set of non-isomorphic primitive orthogonal idempotents of $A/r$. But the idempotents of $k(\Delta_A, A)$ must have length zero, hence $\{\overline{\varepsilon}_i : i \in (\Delta_A)_0\}$ is also a complete set of non-isomorphic primitive orthogonal idempotents of $k(\Delta_A, A)$. □

As discussed before Lemma 3.1, the basic algebra $C$ satisfies

$$C = \text{End}(\coprod_{i \in (\Delta_A)_0} k(\Delta_A, A)\overline{\varepsilon}_i) \cong \bigoplus_{i,j \in (\Delta_A)_0} \overline{\varepsilon}_i k(\Delta_A, A)\overline{\varepsilon}_j.$$

Then, we get the following:

**Proposition 3.2.** For an artinian algebra $A$ over a field $k$ with the natural quiver $\Delta_A$, let $\{\varepsilon_i : i \in (\Delta_A)_0\}$ be the complete set of non-isomorphic primitive orthogonal idempotents of $A$. Denote by $\overline{\varepsilon}_i$ the image of $\varepsilon_i$ under the canonical morphism from $A$ to $A/r$. Then,

(i) the basic algebra $B$ of $A$ is isomorphic to $\bigoplus_{i \in (\Delta_A)_0} \varepsilon_i A \varepsilon_j$;

(ii) the basic algebra $C$ of the associated generalized path algebra $k(\Delta_A, A)$ of $A$ is isomorphic to $\bigoplus_{i \in (\Delta_A)_0} \overline{\varepsilon}_i k(\Delta_A, A)\overline{\varepsilon}_j$.

As we have said, $k(\Delta_A, A)$ is not artinian when $\Delta_A$ has an oriented cycle. Hence, we cannot affirm whether $C$ is Morita equivalent to $k(\Delta_A, A)$ in general. But, $C$ is still decided uniquely by $k(\Delta_A, A)$ and then by $A$.

For an arbitrary artinian algebra $A$, we still cannot obtain the explicit relation between two basic algebras $B$ and $C$ depending upon Proposition 3.2. However, for the following special case, that is, for the so-called Gabriel-type algebras, we will give an exact conclusion for the two basic algebras.

**Definition 3.1.** Let $A$ be an artinian algebra over a field $k$ and $k(\Delta_A, A)$ its associated normal generalized path algebra. If there exists an ideal $I$ of $k(\Delta_A, A)$ such that $A \cong k(\Delta_A, A)/I$, then we say $A$ to be of Gabriel-type.

Since in [5], we have the Generalized Gabriel Theorem for a finite dimensional algebra $A$ with 2-nilpotent radical $r = r(A)$ in the case $A$ is splitting over $r$, that is, $A \cong k(\Delta_A, A)/I(\rho)$ with $J^2 \subset (\rho) \subset J^2 + J \cap \text{Ker}\overline{\varphi}$ where $(\rho)$ is an ideal generated by the set of relations $\rho$ of $k(\Delta, A)$ and $\overline{\varphi}$ is that in Lemma 2.3. It means that any such finite dimensional algebra is always of Gabriel-Type.
Another example of Gabriel-type algebra is radical-graded artinian algebra as mentioned in Theorem 2.4.

For a Gabriel-type algebra, as Theorem 3.5 and 4.4 in [5], the uniqueness of the correspondent generalized path algebra and quiver holds up to isomorphism in the case the ideal is admissible, that is, if there exists another quiver and its related generalized path algebra such that the same isomorphism relation as in Definition 3.1 is satisfied for some admissible ideal \( I \), then this quiver and the related generalized path algebra are just respectively the natural quiver and the corresponding one of this algebra. Exactly, we have the following statement on the uniqueness:

**Theorem 3.3.** Assume \( A \) is an artinian algebra, \( r = r(A) \) is the radical of \( A \). Let \( A/r = \bigoplus_{i=1}^{p} \overline{A}_{i} \) with simple ideals \( \overline{A}_{i} \). If there is a quiver \( Q \) and a normal generalized path algebra \( k(Q, B) \) with a set of simple algebras \( B = \{ B_{1}, \ldots, B_{q} \} \) and an admissible ideal \( I \) of \( k(Q, B) \) (i.e. for some \( s, J^{s} \subset I \subset J^{2} \)) such that \( A \cong k(Q, B)/I \) where \( J \) the ideal of \( k(Q, B) \) generated by all \( B \)-paths of length one, then \( Q \) is just the natural quiver \( \Delta_{A} \) of \( A \) and \( p = q \) such that \( \overline{A}_{i} \cong B_{i} \) for \( i = 1, \ldots, p \) after reindexed. It follows that \( A \) is a Gabriel-type algebra.

**Proof:** Since \( (J/I)^{s} \subseteq J^{s}/I = 0 \) and \( k(Q, B)/I/J/I \cong k(Q, B)/J = B_{1} \oplus \cdots \oplus B_{q} \) semisimple, then \( \text{rad}(k(Q, B)/I) = J/I \). From the isomorphism \( A \cong k(Q, B)/I \), we have \( A/\text{rad}A \cong k(Q, B)/I/J/I \), i.e. \( \overline{A}_{1} \oplus \cdots \oplus \overline{A}_{p} \cong B_{1} \oplus \cdots \oplus B_{q} \). Thus \( p = q \) and \( \overline{A}_{i} \cong B_{i} \) for \( i = 1, \ldots, p \) after reindexed.

By the isomorphism, the two algebras \( A \) and \( k(Q, B)/I \) have the same natural quivers, i.e. \( \Delta_{A} = \Delta \). Then we only need to show that the natural quiver \( \Delta \) of \( k(Q, B)/I \) is just \( Q \). Firstly since \( p = q, \Delta_{0} = Q_{0} \). And the number of arrows from \( i \) to \( j \) in \( \Delta_{1} \) is \( \text{rank}(B_{j}(J/I^{2}/I)B_{i}) \), \( \text{rank}(B_{j}(J/J^{2})B_{i}) \), which is just the number of arrows from \( i \) to \( j \) in \( Q_{1} \). Therefore \( Q = \Delta = \Delta_{A} \). \( \square \)

**Lemma 3.4.** Let \( A \) be a Gabriel-type artinian algebra with \( A \cong k(\Delta_{A}, A)/I \) for an ideal \( I \) of \( k(\Delta_{A}, A) \) satisfying \( I \subset J \). Assume that \( \{ \varepsilon_{i} : i \in (\Delta_{A})_{0} \} \) is the complete set of non-isomorphic primitive orthogonal idempotents of \( A \). Then, there is a complete set of non-isomorphic primitive orthogonal idempotents \( \{ d_{i} : i \in (\Delta_{A})_{0} \} \) of \( A/r \) such that \( \pi(\varepsilon_{i}) = d_{i} + I \) for any \( i \in (\Delta_{A})_{0} \).

**Proof:** Let \( \pi(\varepsilon_{i}) = \tilde{\varepsilon}_{i} + I \), then \( \{ \tilde{\varepsilon}_{i} + I : i \in (\Delta_{A})_{0} \} \) is a complete set of non-isomorphic primitive orthogonal idempotents of \( k(\Delta_{A}, A)/I \) since \( \pi \) is an isomorphism.

Since \( (\tilde{\varepsilon}_{i} + I)^{2} = \tilde{\varepsilon}_{i} + I \), we get \( \tilde{\varepsilon}_{i}^{2} = \tilde{\varepsilon}_{i} \in I \). Note that \( I \) lies in the ideal of \( k(\Delta_{A}, A) \) generated by all \( A \)-paths of length one. Because the square of any non-cyclic path is zero, either \( \tilde{\varepsilon}_{i} + I = E_{i}c_{i} + I \) or \( \tilde{\varepsilon}_{i} + I = d_{i} + I \) where \( E_{i} \) are circles in \( \Delta_{A} \), \( c_{i} \) and \( d_{i} \) are primitive idempotents in \( k(\Delta_{A}, A) \), or equivalently in \( A/r \).
Let \( w = 1_{k(\Delta_A,A)/I} - \sum_{i \in (\Delta_A)_0} (\tilde{e}_i + I) \), then \( w \) is an idempotent and can be decomposed into a sum of some primitive orthogonal idempotents \( \tilde{f}_j + I \), write \( w = (\tilde{f}_1 + I) + \cdots + (\tilde{f}_t + I) \). Thus, \( 1_{k(\Delta_A,A)/I} = \sum_{i \in (\Delta_A)_0} (\tilde{e}_i + I) + \sum_{j=1}^t (\tilde{f}_j + I) \).

Let \( X + I \) and \( Y + I \) denote the sums of those idempotents in \( \{ \tilde{e}_i + I : i \in (\Delta_A)_0 \} \cup \{ \tilde{f}_j + I : j = 1 + \cdots + t \} \) respectively in the forms \( E_p c_p + I \) and \( d_q + I \), where \( c_p, d_q \in A/r \). Thus, \( 1_{k(\Delta_A,A)/I} = 1_{k(\Delta_A,A)/I} = (X + I) + (Y + I) \), it follows that \( X + I = (1_{k(\Delta_A,A)/I} - Y) + I \).

Suppose there are some \( i \) such that \( \tilde{e}_i + I = E_i c_i + I \neq 0 \). Then \( X + I \neq 0 \). Hence \( 1_{k(\Delta_A,A)/I} - Y \neq 0 \), then \( 1_{k(\Delta_A,A)/I} - Y \in X + I \subset J \), which is impossible due to \( 1_{k(\Delta_A,A)/I} - Y \in k((\Delta_A)_0,A) \).

The above contradiction means that each \( \tilde{e}_i + I = d_i + I \) where each \( d_i \) is primitive idempotent in \( A/r \).

Clearly \( \{ d_i : i \in (\Delta_A)_0 \} \) is a set of non-isomorphic primitive orthogonal idempotents of \( A/r \), by Lemma 3.1 it is a complete set of non-isomorphic primitive orthogonal idempotents of \( A/r \). □

**Theorem 3.5.** Let \( A \) be a Gabriel-type artinian algebra over a field \( k \) with \( A \cong k(\Delta_A,A)/I \) for an ideal \( I \) of \( k(\Delta_A,A) \) satisfying \( I \subset J \). Then for the basic algebra \( B \) of \( A \) and the basic algebra \( C \) of \( k(\Delta_A,A) \), it holds that \( B \cong (C + I)/I \).

**Proof:** Let \( \{ \varepsilon_i : i \in (\Delta_A)_0 \} \) be a complete set of non-isomorphic primitive orthogonal idempotents of \( A \). Then, by Lemma 3.4, there is a complete set of non-isomorphic primitive orthogonal idempotents \( \{ d_i : i \in (\Delta_A)_0 \} \) of \( A/r \) such that \( \pi(\varepsilon_i) = d_i + I \) for each \( i \in (\Delta_A)_0 \).

By Lemma 3.1, \( \{ d_i : i \in (\Delta_A)_0 \} \) is also a complete set of non-isomorphic primitive orthogonal idempotents of \( k(\Delta_A,A) \). Thus, by Proposition 3.2, we have \( C \cong \bigoplus_{i,j \in (\Delta_A)_0} d_i k(\Delta_A,A) d_j \). Moreover, under the isomorphism \( \pi \),

\[
B \cong \bigoplus_{i,j \in (\Delta_A)_0} \varepsilon_i A \varepsilon_j \\
\cong \bigoplus_{i,j \in (\Delta_A)_0} (d_i + I)(k(\Delta_A,A)/I)(d_j + I) \\
\cong \bigoplus_{i,j \in (\Delta_A)_0} (d_i k(\Delta_A,A) d_j + I)/I \\
\cong (C + I)/I. \quad \square
\]

This theorem mentions the relation between the two basic algebras \( B \) and \( C \) which are both decided by the same artinian algebra \( A \).

In general, for a Gabriel-type artinian \( A \) whose the ideal \( I \) is admissible (even only with \( I \subset J \)), the two natural quivers \( \Delta_B \) and \( \Delta_C \) of the associated basic algebras \( B \) of \( A \) and \( C \) of \( k(\Delta_A,A) \) are not equal. In fact, although \( B \cong (C + I)/I \), \( \text{rad}B \cong (\text{rad}C + I)/I \), in general \( B/\text{rad}B \not\cong C/\text{rad}C \) and \( (\text{rad}B)/(\text{rad}B)^2 \not\cong (\text{rad}C)/(\text{rad}C)^2 \).

**Proposition 3.6.** For a Gabriel-type artinian algebra \( A \) with \( A \cong k(\Delta_A,A)/I \), if \( I \) is an admissible ideal, the natural quivers of \( A \) and \( k(\Delta_A,A) \) are the same, i.e. \( \Delta_A = \Delta_{k(\Delta_A,A)} \).
3 TWO BASIC ALGEBRAS FROM AN ARTINIAN ALGEBRA

Proof: Since $I$ is admissible, there is a positive integer $s$ such that $J^s \subset I \subset J^2$. $\text{rad}A \sim J/I$ as proved in Theorem 3.3. And $A \cong k(\Delta_A, A)/I$, then $A/\text{rad}A \cong k(\Delta_A, A)/J$. Moreover, $\text{rad}A/(\text{rad}A)^2 \cong J/J^2$. Thus, by the definition, $\Delta_A = \Delta_{k(\Delta_A, A)}$. □

In the other case, for an artinian algebra $A$, when $\Delta_A$ is admissible (i.e. is acyclic), it is true that $\Delta_A = \Delta_{k(\Delta_A, A)}$, since $J$ is just the radical of $k(\Delta_A, A)$.

To sum up, for a finite dimensional algebras $A$ over algebraically closed field $k$, when either $\Delta_A$ is admissible or $A$ is of Gabriel-type satisfying $A \cong k(\Delta_A, A)/I$ with admissible $I$, we have the following diagram:

\[
\begin{array}{cccc}
\Gamma_A & \supset & \Delta_A & = \Delta_{k(\Delta_A, A)} & \subset & \Gamma_{k(\Delta_A, A)} \\
\Gamma_B & = & \Delta_B & \cap & \Delta_C & = \Gamma_C \\
\end{array}
\]

where $\Gamma_A$ is the ordinary quiver of $A$, etc.; $\subset$, $\supset$ and $\cap$ mean the embeddings of the dense sub-quivers.

We feel the relations in this diagram would still hold for any artinian algebras. This point of view will be discussed in the subsequent work.

As we say above, the ordinary quiver and the natural quiver of a finite dimensional basic algebra coincide each other. In the end of this section, we give an example which means the coincidence is also possible to happen for some non-basic algebras. Meanwhile, in this example, we show a method of computing the number of arrows of the natural quiver of an artinian algebra.

Example Let $k$ be an algebraically closed field of characteristic different from 2 and let $Q$ be the quiver:

\[
e_3 \bullet \xrightarrow{\beta} \bullet \xrightarrow{\alpha} \bullet \xrightarrow{\alpha'} \bullet \xrightarrow{\beta'} \bullet e_3'
\]

Denote the path algebra $kQ$ by $\Lambda$ and let $G = \langle \sigma \rangle$ be the group of order 2. For the elements $e_1, e_2, e_3, \alpha, \beta$ in $\Lambda$, let $\sigma e_1 = e_1, \sigma e_2 = e_2, \sigma e_3 = e_3, \sigma \alpha = \alpha', \sigma \beta = \beta'$. Then, there is only one way of extending $\sigma$ to a $k$-algebra automorphism of $Q$ and this is the way we will consider $G$ as a group of automorphisms of $Q$. Now, we consider the ordinary quiver and the natural quiver of the skew group algebra $\Lambda G$ (see [2]).

Let $\mathfrak{r}$ be the radical of $\Lambda$. By Proposition 4.11 in [2], $\mathfrak{r} \Lambda G = \text{rad}(\Lambda G)$. It is easy to see that $(\Lambda G)/(\mathfrak{r} \Lambda G) \cong (\Lambda/\mathfrak{r})G$. In the page 84 of [2], it was given that $(\Lambda/\mathfrak{r})G \cong A_1 \times A_2 \times A_3 \times A_4 = k \times k \times \left( \begin{array}{cc} k & k \\ k & k \end{array} \right) \times \left( \begin{array}{cc} k & k \\ k & k \end{array} \right)$ as algebras and the associated basic algebra $B$ is obtained in the reduced form from $\Lambda G$, which is Mortia-equivalent to $\Lambda G$, and moreover, it was proved in [2] that $B$ is isomorphic to the path algebra of the following quiver:
This quiver is just the ordinary quiver $\Gamma_{\Lambda G}$ of $\Lambda G$. Therefore, all $m_{ij} = 0$, or 1.

For $i = 1, 2, 3, 4$, $\dim_k A_i = n_i^2$ where $n_1 = n_2 = 1$, $n_3 = n_4 = 2$. By definition, for $i, j = 1, 2, 3, 4$, $t_{ij}$ is the rank of $j M_i = A_j(r\Lambda G)/(r\Lambda G)^2 A_i$ as $A_j A_i$-bimodule, equivalently, as a right $A_i \otimes A_j^{op}$-module. $A_i \otimes A_j^{op} \cong M_{n_i n_j}(k)$ is a simple algebra with dimension $n_i^2 n_j^2$. Thus, $j M_i$ is semisimple over this simple algebra. Let $j M_i = L_1 \oplus \cdots \oplus L_s$ where all $L_v$ are simple $A_i \otimes A_j^{op}$-modules for $v = 1, \cdots, s$. $L_v$ can be considered as a simple right ideal of $A_i \otimes A_j^{op}$, therefore, $L_v \cong (k k \cdots k)$ the whole set of all $1 \times n_i n_j$ matrices over $k$. For any $0 \neq x_v \in L_v$, $L_v = x_v(A_i \otimes A_j^{op})$. Then, $j M_i = \bigoplus_{v=1}^s x_v(A_i \otimes A_j^{op})$ as $(A_i \otimes A_j^{op})$-modules.

First, we prove $s = m_{ij}$ the number of the arrows from the vertex $i$ to the other vertex $j$ in the ordinary quiver $\Gamma_{\Lambda G}$ of $\Lambda G$.

Let $\{S_1, S_2, \cdots, S_n\}$ be the complete set of non-isomorphic simple $\Lambda G$-modules. Then $m_{ij} = \dim_k \text{Ext}_A(S_i, S_j)$. By [6][2], $\dim_k \text{Ext}_A(S_i, S_j) = \dim_k(k \varepsilon_j(r \Lambda G)/(r \Lambda G)^2 k \varepsilon_i)$ where $\{\varepsilon_1, \cdots, \varepsilon_n\}$ is the complete set of primitive orthogonal idempotents of $(\Lambda G)/\text{rad}(\Lambda G)$ with $\varepsilon_i \in A_i$. And, $k \varepsilon_j(r \Lambda G)/(r \Lambda G)^2 k \varepsilon_i = \varepsilon_j A_j(r \Lambda G)/(r \Lambda G)^2 A_i \varepsilon_i = \varepsilon_j M_i \varepsilon_i \cong j M_i(\varepsilon_i \otimes \varepsilon_j) = (L_1 + \cdots + L_s)(\varepsilon_i \otimes \varepsilon_j) = (x_1 + \cdots + x_s)(A_i \otimes A_j^{op})(\varepsilon_i \otimes \varepsilon_j) \cong (x_1 + \cdots + x_s)M_{n_i n_j}(k)E_{ij}$ where $\varepsilon_i \otimes \varepsilon_j$ is a primitive idempotent of $A_i \otimes A_j^{op}$ so let $E_{ij}$ be the correspondent element of $\varepsilon_i \otimes \varepsilon_j$ in $M_{n_i n_j}(k)$ under the isomorphism.

Obviously, $\dim_k x_v M_{n_i n_j}(k)E_{ij} = 1$ for all $v = 1, \cdots, s$. Then, $\dim_k(x_1 + \cdots + x_s)M_{n_i n_j}(k)E_{ij} = s$. It follows that $m_{ij} = s$.

For each pair $(i, j)$, when $m_{ij} = 0$, we have $k \varepsilon_j(r \Lambda G)/(r \Lambda G)^2 k \varepsilon_i = 0$. Then $j M_i = 0$. Thus, the rank $t_{ij}$ of $j M_i$ equals 0. When $m_{ij} = 1$, then $s = m_{ij} = 1$, that is, $j M_i = L_1$ is a simple $(A_i \otimes A_j^{op})$-module. Hence, the rank $t_{ij}$ of $j M_i$ is 1 in this case.

According to the above discussion, for each pair $(i, j)$, we have $t_{ij} = m_{ij} = 0$ or 1. Therefore, the natural quiver $\Delta_{\Lambda G}$ is equal to the ordinary quiver $\Gamma_{\Lambda G}$.

4 Interpretations

In [2][1], given a finite dimensional algebra $A$, the ordinary quiver $\Gamma_A$ can be constructed by the indecomposable projective modules and the irreducible morphisms between them. So the ordinary quiver of $A$ provides a convenient way to study its projective (or injective) modules and morphisms between them, even when $A$ is not a basic algebra. By the Gabriel theorem, the ordinary quiver of a finite-dimensional algebra $A$ is used as a tool to characterize the structure of its associated basic algebra but not of $A$. In this reason, the ordinary quiver is not effective enough to characterize a non-basic algebra. The generalized
Gabriel theorem in [5] shows the arrival of our goal via the natural quiver under some conditions.

Note that the AR-quiver of the sub-category of $\text{proj} A$ with irreducible morphisms is isomorphic to the opposite of the ordinary quiver of $A$.

Through [5] and here, we think the method of natural quiver may offset some shortage of ordinary quiver and AR-quiver. In certain sense, the natural quiver of an artinian algebra $A$ will also be available for the theory of representations of an artinian algebra.

Under certain condition, the representation category $\text{Rep} A$ of $A$ can be decided wholly by the ordinary quiver and the AR-quiver. The category of representations of $A$ may be partially induced from the category of representations of $\Gamma_B$ through the basic algebra $B$. For example, when $A$ is Gabriel-type, that is, $A$ is isomorphic to some quotient of the generalized path algebra of $\Delta_A = \Delta_B$, any representations of $A$ can be induced directly from some of representations of the generalized path algebra of $\Delta_A$. In the classical theory of representations of artin algebras (see [2][1][4] etc.), one wants to characterize $\text{Rep} A$ through representations of $\Delta_B$ with $B$. However, the difficulty is that in general, it is not easy to construct concretely the basic algebra $B$ from $A$. By comparison, the method of natural quivers is more straightforward through representations of the generalized path algebra of $\Delta_A$. Therefore, we hope to set up this new approach to representations of an artinian algebra via representations of the generalized path algebra of its natural quiver.

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