Two variable logic with ultimately periodic counting

Michael Benedikt
University of Oxford
michael.benedikt@cs.ox.ac.uk

Egor V. Kostylev
University of Oxford
egor.kostylev@cs.ox.ac.uk

Tony Tan
National Taiwan University
tonytan@csie.ntu.edu.tw

Abstract

We consider the extension of $\text{FO}^2$ with quantifiers that state that the number of elements where a formula holds should belong to a given ultimately periodic set. We show that both satisfiability and finite satisfiability of the logic are decidable. We also show that the spectrum of any sentence is definable in Presburger arithmetic. In the process we present several refinements to the “biregular graph method”. In this method, decidability issues concerning two-variable logics are reduced to questions about Presburger definability of integer vectors associated with partitioned graphs, where nodes in a partition satisfy certain constraints on their in- and out-degrees.

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1 Introduction

In the search for expressive logics with decidable satisfiability problem, two-variable logic, denoted here as \( \text{FO}^2 \), is one yardstick. This logic is expressive enough to subsume basic modal logic and many description logics, while satisfiability and finite satisfiability coincide, and both are decidable \cite{22, 14, 8}. However, \( \text{FO}^2 \) lacks the ability to count. Two-variable logic with counting, \( \text{C}^2 \), is a decidable extension of \( \text{FO}^2 \) that adds counting quantifiers. In \( \text{C}^2 \) one can express, for example, \( \exists^5 x P(x) \) and \( \forall x \exists^5 y E(x, y) \) which, respectively, mean that there are exactly 5 elements in unary relation \( P \), and that every element in a graph has at least 5 adjacent edges. Satisfiability and finite satisfiability do not coincide for \( \text{C}^2 \), but both are decidable \cite{9, 15}. In \cite{15} the problems were shown to be NEXPTIME-complete under a unary encoding of numbers, and this was extended to binary encoding in \cite{17}. However, the numerical capabilities of \( \text{C}^2 \) are quite limited. For example, one can not express that the number of outgoing edges of each element in the graph is even.

A natural extension is to combine \( \text{FO}^2 \) with Presburger arithmetic where one is allowed to define collections of tuples of integers from addition and equality using boolean operators and quantifiers. The collections of \( k \)-tuples that one can define in this way are the semi-linear sets, and the collections of integers (when \( k = 1 \)) definable are the ultimately periodic sets. Prior work has considered the addition of Presburger quantification to fragments of two-variable logic. For every definable set \( \phi(x,y) \) and every ultimately periodic set \( S \), one has a formula \( \exists^S y \phi(x,y) \) that holds at \( x \) when the number of \( y \) such that \( \phi(x,y) \) is in \( S \). We let \( \text{FO}^2_{\text{Pres}} \) denote the logic that adds this construct to \( \text{FO}^2 \).

On the one hand, the corresponding quantification over general \( k \)-tuples (allowing semi-linear rather than ultimately periodic sets) easily leads to undecidability \cite{10, 3}. On the other hand, adding this quantification to modal logic has been shown to preserve decidability \cite{11, 8}. Related one-variable fragments in which we have only a unary relational vocabulary and the main quantification is \( \exists^5 x \phi(x) \) are known to be decidable (see, e.g. \cite{2}), and their decidability is the basis for a number of software tools focusing on integration of relational languages with Presburger arithmetic \cite{13}. The decidability of full \( \text{FO}^2_{\text{Pres}} \) is, to the best of our knowledge, open. There are a number of other extensions of \( \text{C}^2 \) that have been shown decidable; for example it has been shown that one can allow a distinguished equivalence relation \cite{21} or a forest-structured relation \cite{5, 4}. \( \text{FO}^2_{\text{Pres}} \) is easily seen to be orthogonal to these other extensions.

In this paper we show that both satisfiability and finite satisfiability of \( \text{FO}^2_{\text{Pres}} \) are decidable. Our result makes use of the biregular graph method introduced for analyzing \( \text{C}^2 \) in \cite{12}. The method focuses on the problem of existence of graphs equipped with a partition of vertices based on constraints on the out- and in-degree. Such a partitioned graph can be characterized by the cardinalities of each partition component, and the key step in showing these decidability results is to prove that the set of tuples of integers representing valid sizes of partition components is definable by a formula in Presburger arithmetic. From this “graph constraint Presburger definability” result one can reduce satisfiability in the logic to satisfiability of a Presburger formula, and from there infer decidability using known results on Presburger arithmetic.

The approach is closely-related to the machinery developed by Pratt-Hartmann (the “star types” of \cite{20}) for analyzing the decidability and complexity of \( \text{C}^2 \), its fragments \cite{18}, and its extensions \cite{21, 4}. An advantage of the biregular graph approach is that it is transparent how to extract more information about the shape of witness structures. In particular we can infer that the spectrum of any formula is Presburger definable, where the spectrum of a
formula \( \phi \) is the set of cardinalities of finite models of \( \phi \). It is also interesting to note that a more restricted version of our biregular graph method is used to prove the decidability of \( \text{FO}^2 \) extended with two equivalence relations [11].

Characterising the spectrum for general first order formulas is quite a difficult problem, with ties to major open questions in complexity theory [7]. This work can be seen as a demonstration of the power of the biregular graph method to get new decidability results. We make heavy use of both techniques and results in [12], adapting them to the richer logic. We also require additional inductive arguments to handle the interaction of ordinary counting quantifiers and modulo counting quantification.

**Organization.** Section 2 provides background on two variable logic and Presburger arithmetic. Section 3 gives our main result and provides a high-level idea of the proof, while Section 4 gives some of the details behind the core lemmas concerning Presburger definability of solutions to regular graph problems that underlie the proof. Section 5 gives more restricted versions of our techniques, while conclusions are given in Section 6. Many proofs are deferred to the appendix.

## 2 Preliminaries

Let \( N = \{0, 1, 2, \ldots \} \) and let \( N_\infty = N \cup \{\infty\} \).

**Linear and ultimately periodic sets.** A set of the form \( \{a + ip \mid i \in \mathbb{N}\} \), for some \( a, p \in \mathbb{N} \) is a linear set. We will denote such a set by \( a^p \), where \( a \) and \( p \) are called the offset and period of the set, respectively. Note that, by definition, \( a^0 = \{a\} \), which is a linear set.

For convenience, we define \( \emptyset \) and \( \{\infty\} \) (which may be written as \( \infty^p \)) to also be linear sets.

An *ultimately periodic set* (u.p.s.) \( S \) is a finite union of linear sets. Usually we write a u.p.s. \( \{c_1\} \cup \cdots \cup \{c_m\} \cup a_1^{p_1} \cup \cdots \cup a_n^{p_n} \) as just \( \{c_1, \ldots, c_m, a_1^{p_1}, \ldots, a_n^{p_n}\} \), and abusing notation, we write \( a^p \in S \) for a u.p.s. \( S \) if \( a + ip \in S \) for every \( i \in \mathbb{N} \).

**Two-variable logic with ultimately periodic counting quantifiers.** An *atomic formula* is either an atom \( R(\vec{u}) \), where \( R \) is a predicate, and \( \vec{u} \) is a tuple of variables of appropriate size, or an equality \( u = u' \), with \( u \) and \( u' \) variables, or one of the formulas \( \top \) and \( \bot \) denoting the True and False values. The logic \( \text{FO}^2_{\text{Pres}} \) is a class of first-order formulas using only variables \( x \) and \( y \), built up from atomic formulas and equalities using the usual boolean connectives and also *ultimately periodic counting quantification*, which is of the form \( \exists^S x \phi \) where \( S \) is a u.p.s. One special case is where \( S \) is a singleton \( \{a\} \), with \( a \in \mathbb{N}_\infty \), which we write \( \exists^a x \phi \); in case of \( a = 0 \), these are *counting quantifiers*. The semantics of \( \text{FO}^2_{\text{Pres}} \) is defined as usual except that, for every \( a \in \mathbb{N} \), \( \exists^a x \phi \) holds when there are exactly \( a \) number of \( x \)'s such that \( \phi \) holds, \( \exists^\infty x \phi \) holds when there are infinitely many \( x \)'s such that \( \phi \) holds, and \( \exists^S x \phi \) holds when there is some \( a \in S \) such that \( \exists^a x \phi \) holds.

Note that when \( S = \{\infty\} \cup 0^{+1} = \mathbb{N}_\infty \), \( \exists^S x \phi \) is equivalent to \( \top \). When \( S = 0^{+1} \), \( \exists^S x \phi \) semantically means that there are finitely many \( x \) such that \( \phi \) holds. We define \( \exists^a x \phi \) to be \( \bot \) for any formula \( \phi \). We also note that \( \exists^0 x \phi \) is equivalent to \( \forall x \neg \phi \), and \( \exists^0 x \phi \) is equivalent to \( \exists^\infty \neg x \phi \).

For example, we can state in \( \text{FO}^2_{\text{Pres}} \) that every node in a graph has even degree (i.e., the graph is Eulerian). Clearly \( \text{FO}^2_{\text{Pres}} \) extends \( C^2 \), the fragment of the logic where only counting quantifiers are used, and \( \text{FO}^2 \), the fragment where only the classical quantifier \( \exists \) is allowed.

**Presburger arithmetic.** An *existential Presburger formula* is a formula of the form \( \exists x_1 \ldots x_k \phi \), where \( \phi \) is a quantifier-free formula over the signature including constants 0, 1, a binary function symbol +, and a binary relation ≤. Such a formula is a *sentence* if it has no free variables. The notion of a sentence holding in a structure interpreting the function, relations,
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and constants is defined in the usual way. The structure $\mathcal{N} = (\mathbb{N}, +, \leq, 0, 1)$, is defined by interpreting $+, \leq, 0, 1$ in the standard way, while the structure $\mathcal{N}_\infty = (\mathbb{N}_\infty, +, \leq, 0, 1)$ is the same except that $a + \omega = \omega$ and $a \leq \omega$ for each $a \in \mathbb{N}_\infty$.

It is known that the satisfiability of existential Presburger sentences over $\mathcal{N}$ is decidable and belongs to NP [10]. Further, the satisfiability problem for $\mathcal{N}_\infty$ can easily be reduced to that for $\mathcal{N}$. Indeed, we can first guess which variables are mapped to $\omega$ and then which atoms should be true, then check whether each guessed atomic truth value is consistent with other guesses and determine additional variables which must be infinite based on this choice, and finally restrict to atoms that do not involve variables guessed to be infinite, and check that the conjunction is satisfiable by standard integers.

Theorem 1. The satisfiability problem for existential Presburger sentences over $\mathcal{N}$ and $\mathcal{N}_\infty$ are both in NP.

Main result

In this section we prove decidability of $\text{FO}^\text{res}_\text{Pres}$ satisfiability (relying on key lemmas proved later on). Our decidability procedure is based on the key notion of regular graphs. Note that whenever we talk about graphs or digraphs (i.e., directed graphs), by default we allow both finite and infinite sets of vertices and edges.

3.1 Regular graphs

In the following we fix an integer $p \geq 0$. Let $\mathbb{N}_\infty, +_p$ denote the set whose elements are either $a$ or $a +_p$, where $a \in \mathbb{N}_\infty$. For integers $t, m \geq 1$, let $\mathbb{N}^t \times \mathbb{N}_\infty, +_p$ denote the set of matrices with $t$ rows and $m$ columns where each entry is an element from $\mathbb{N}_\infty, +_p$.

A $t$-color bipartite (undirected) graph is $G = (U, V, E_1, \ldots, E_t)$, where $U$ and $V$ are sets of vertices and $E_1, \ldots, E_t$ are pairwise disjoint sets of edges between $U$ and $V$. Edges in $E_i$ are called $E_i$-edges. We will write an edge in a bipartite graph as $(u, v) \in U \times V$. For a vertex $u \in U \cup V$, the $E_i$-degree of $u$ is the number of $E_i$-edges adjacent to $u$. The degree of $u$ is the sum of the $E_i$-degrees for $i = 1 \ldots t$. We say that $G$ is complete, if $U \times V = \bigcup_{i=1}^t E_i$.

For two matrices $A \in \mathbb{N}^{t \times m}_\infty, +_p$ and $B \in \mathbb{N}^{n \times n}_\infty, +_p$, the graph $G$ is a $A|B$-biregular graph, if there exist partitions $U_1, \ldots, U_m$ of $U$ and $V_1, \ldots, V_n$ of $V$ such that for every $1 \leq i \leq t$, for every $1 \leq k \leq m$, for every $1 \leq l \leq n$, the $E_i$-degree of every vertex in $U_k$ is $A_{i,k}$ and the $E_i$ degree of every vertex in $V_l$ is $B_{i,l}$. For each such partition, we say that $G$ has size $M|N$, where $M = (|U_1|, \ldots, |U_m|)$ and $N = (|V_1|, \ldots, |V_n|)$. The partition $U_1, \ldots, U_m$ and $V_1, \ldots, V_n$ is called a witness partition. We should remark that some $U_i$ and $V_i$ are allowed to be empty.

The above definition can be easily adapted for the case of directed graphs that are not necessarily bipartite. A $t$-color directed graph (or digraph) is $G = (V, E_1, \ldots, E_t)$, where $E_1, \ldots, E_t$ are pairwise disjoint set of directed edges on a set of vertices $V$. As before, edges in $E_i$ are called $E_i$-edges. The $E_i$-indegree and -outdegree of a vertex $u$, is defined as the number of incoming and outgoing $E_i$-edges incident to $u$.

In a $t$-color digraph $G$ we will assume that (i) there are no self-loops—that is, $(v, v)$ is not an $E_i$-edge, for every vertex $v \in V$ and every $E_i$, and (ii) if $(u, v)$ is an $E_i$-edge, then its inverse $(v, u)$ is not an $E_j$-edge for any $E_j$. This will suffice for the digraphs that arise in

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1 By abuse of notation, when we say that an integer $z$ equals $a +_p$, we mean that $z \in a +_p$. Thus, when writing $A_{i,k} = a +_p$, we mean that the degree of the vertex is an element in $a +_p$. 

our decision procedure. We say that a digraph $G$ is complete, if for every $u,v \in V$ and $u \neq v$, either $(u,v)$ or $(v,u)$ is an $E_i$-edge, for some $E_i$.

We say that $G$ is a $A|B$-regular digraph, where $A, B \in \mathbb{N}_{\infty,+p}^m$, if there exists a partition $V_1, \ldots, V_m$ of $V$ such that for every $1 \leq i \leq t$, for every $1 \leq k \leq m$, the $E_i$-indegree and -outdegree of every vertex in $V_k$ is $A_{i,k}$ and $B_{i,k}$, respectively. We say that $G$ has size $(|V_1|, \ldots, |V_m|)$, and call $V_1, \ldots, V_m$ a witness partition.

Lemma 2 below will be the main technical tool for our decidability result. Let $\bar{x}$ and $\bar{y}$ be vectors of variables of length $m$ and $n$, respectively.

**Lemma 2.** For every $A \in \mathbb{N}_{\infty,+p}^m$ and $B \in \mathbb{N}_{\infty,+p}^m$, there exists (effectively computable) existential Presburger formula $c\text{-bireg}_{A|B}(\bar{x}, \bar{y})$ such that for every $(M, N) \in \mathbb{N}_\infty^m \times \mathbb{N}_\infty^n$, the following holds: there is complete $A|B$-biregular graph with size $M|N$ if and only if $c\text{-bireg}_{A|B}(M, N)$ holds in $\mathcal{N}_\infty$.

**Lemma 3.** For every $A \in \mathbb{N}_{\infty,+p}^m$ and $B \in \mathbb{N}_{\infty,+p}^m$, there exists (effectively computable) existential Presburger formula $c\text{-reg}_{A|B}(\bar{x})$ such that for every $M \in \mathbb{N}_\infty^m$ the following holds. There is complete $A|B$-regular digraph with size $M$ if and only if $c\text{-reg}_{A|B}(M)$ holds in $\mathcal{N}_\infty$.

Lemmas 2 and 3 can be easily readjusted when we are interested only in finite sizes, i.e., $M \in \mathbb{N}^m$ and $N \in \mathbb{N}^n$, by requiring the formulas to hold in $\mathcal{N}$, instead of $\mathcal{N}_\infty$. Alternatively, we can also state inside the formulas that none of the variables in $\bar{x}$ and $\bar{y}$ are equal to $\infty$.

The proofs of these two lemmas are discussed in Section 4.

### 3.2 Decision procedure

Theorem 4 below is the main result in this paper.

**Theorem 4.** For every $\text{FO}_\text{Pres}^2$ sentence $\phi$, there is an (effectively computable) existential Presburger formula $PRES_\phi$ such that (i) $\phi$ has a model iff $PRES_\phi$ holds in $\mathcal{N}_\infty$ and (ii) $\phi$ has a finite model iff $PRES_\phi$ holds in $\mathcal{N}$.

From the decision procedure for existential Presburger formulas (Theorem 1 mentioned in Section 2) we immediately will obtain the following corollary.

**Corollary 5.** Both satisfiability and finite satisfiability for $\text{FO}_\text{Pres}^2$ are decidable.

We will sketch how Theorem 4 is proven, making use of Lemmas 2 and 3. We start by observing that satisfiability (and spectrum analysis) for an $\text{FO}_\text{Pres}^2$ sentence can be converted effectively into the same questions for a sentence in a variant of Scott normal form:

$$\phi := \forall x \forall y \alpha(x, y) \land \bigwedge_{i=1}^k \forall x \exists y S_i y \beta_i(x, y) \land x \neq y, \quad (1)$$

where $\alpha(x, y)$ is a quantifier free formula, each $\beta_i(x, y)$ is an atomic formula and each $S_i$ is an u.p.s. The proof, which is fairly standard, can be found in the appendix. By taking the least common multiple, we may assume that all the (non-zero) periods in all $S_i$ are the same.

We recall some standard terminology. A 1-type is a maximally consistent set of atomic and negated atomic unary formulas using only variable $x$. A 1-type can be identified with the quantifier-free formula that is the conjunction of its constituent formulas. Thus, we say that an element $a$ in a structure $A$ has 1-type $\pi$, if $\pi$ holds on the element $a$. We denote
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by $A_\pi$ the set of elements in $A$ with 1-type $\pi$. Clearly the domain $A$ of a structure $A$ is partitioned into the sets $A_\pi$. Similarly, a 2-type is a maximally consistent set of atomic and negated atomic binary formulas using only variables $x, y$, containing the predicate $x \neq y$.

The notion of a pair of elements $(a, b)$ in a structure $A$ having 2-type $E$ is defined as with 1-types. We denote by $\Pi = \{\pi_1, \pi_2, \ldots, \pi_n\}$ and $\mathcal{E} = \{E_1, \ldots, E_1, \overline{E}_1, \ldots, \overline{E}_t\}$ the sets of all 1-types and 2-types, respectively, where $\overline{E}_i(x, y) = E_i(y, x)$ for each $1 \leq i \leq t$—that is, each $\overline{E}_i$ is the reversal of $E_i$.

Let $g : \mathcal{E} \times \Pi \rightarrow \mathbb{N}_{\infty, +p}$ be a function. We will use such a function $g$ to describe the “behavior” of the elements in the following sense. Let $A$ be a structure. We say that an element $a \in A$ behaves according to $g$, if for every $E \in \mathcal{E}$ and for every $\pi \in \Pi$, the number of elements $b \in A_\pi$ such that the 2-type of $(a, b)$ is $E$ belongs to $g(E, \pi)$. We denote by $A_{\pi, g}$ the set of all elements in $A_\pi$ that behave according to $g$. The restriction of $g$ on 1-type $\pi$ is the function $g_\pi : \mathcal{E} \rightarrow \mathbb{N}_{\infty, +p}$, where $g_\pi(E) = g(E, \pi)$. We call the function $g_\pi$ the behavior (function) towards 1-type $\pi$.

We are, of course, only interested in functions $g$ that are consistent with the sentence $\phi$ in $[\Pi]$, and we formalize this as follows:

- A 1-type $\pi \in \Pi$ and a function $g : \mathcal{E} \times \Pi \rightarrow \mathbb{N}_{\infty, +p}$ are incompatible (w.r.t. $\forall x \forall y \alpha(x, y)$), if there is $E \in \mathcal{E}$ and $\pi' \in \Pi$ such that $\pi(x) \land E(x, y) \land \pi'(y) \models \neg \alpha(x, y)$ and $g(E, \pi') \neq 0$.

- A function $g : \mathcal{E} \times \Pi \rightarrow \mathbb{N}_{\infty, +p}$ is a good function (w.r.t. $\bigwedge_{i=1}^k \forall x \exists y \beta_i(x, y) \land x \neq y$), if for every $\pi \in \Pi$ and for every $i$ the following holds\footnote{Here the operation $+ \circ \mathbb{N}_{\infty, +p}$ is defined to be commutative operation where $a + \infty = a + p + \infty = \infty$ and $a + p + b = a + p + b + p = (a + b) + p$. On integers from $\mathbb{N}$, it is the standard addition operation.}:

$$\sum_{E=\beta_i(x, y)} \sum_{\pi \in \Pi} g(E, \pi) = a$$

for some $a \in S_i$.

If $A \models \phi$ then $A_{(\pi, g)} = \emptyset$, whenever $\pi$ and $g$ are incompatible, and in addition every element in $A$ behaves only according to some good function.

The main idea is to construct the sentence $\text{PRES}_\phi$ that “counts” the cardinality $|A_{(\pi, g)}|$ in every structure $A \models \phi$, for every $\pi$ and $g$. Toward this end, let $\mathcal{G} = \{g_1, g_2, \ldots, g_m\}$ enumerate all good functions. Note that $\mathcal{G}$ can be computed effectively from the sentence $\phi$, since it suffices to consider functions $g : \mathcal{E} \times \Pi \rightarrow \mathbb{N}_{\infty, +p}$ with codomain $\{0, \ldots, a, 0^+p, \ldots, a^+p, \infty\}$, where $a$ is the maximal offset of the (non-$\infty$) elements in $\bigcup_{i=1}^k S_i$.

The sentence $\text{PRES}_\phi$ will be of the form

$$\text{PRES}_\phi := \exists \overline{X} \text{ consistent}_1(\overline{X}) \land \text{ consistent}_2(\overline{X}),$$

where $\overline{X}$ is a vector of variables $(X_{(\pi_1, g_1)}, X_{(\pi_2, g_2)}, \ldots, X_{(\pi_m, g_m)})$. Intuitively, each $X_{(\pi, g)}$ represents $|A_{(\pi, g)}|$. By the formulas $\text{consistent}_1(\overline{X})$ and $\text{consistent}_2(\overline{X})$, we capture the consistency of the integers $\overline{X}$ with the formulas $\forall x \forall y \alpha(x, y)$ and $\bigwedge_{i=1}^k \forall x \exists y \beta_i(x, y) \land x \neq y$, respectively.

We start by defining the formula $\text{consistent}_1(\overline{X})$. Letting $H$ be the set of all pairs $(\pi, g)$ where $\pi$ and $g$ are incompatible, the formula $\text{consistent}_1(\overline{X})$ can be defined as follows:

$$\text{consistent}_1(\overline{X}) := \bigwedge_{(\pi, g) \in H} X_{(\pi, g)} = 0.$$ 

(3)

\begin{align*}
\text{consistent}_1(\overline{X}) := \bigwedge_{(\pi, g) \in H} X_{(\pi, g)} = 0. & \\
\text{consistent}_2(\overline{X}) := \bigwedge_{(\pi, g) \in H} X_{(\pi, g)} = 0. & \\
\end{align*}
Towards defining the formula $\text{consistent}_2(X)$, we introduce some notations. For $\pi \in \Pi$, define the matrices $M_\pi, \overrightarrow{M}_\pi \in \mathbb{N}_{\infty}^{t \times m}$ as follows:

$$
M_\pi := \begin{pmatrix}
g_1(E_1, \pi) & \cdots & g_m(E_1, \pi) \\
\vdots & \ddots & \vdots \\
g_1(E_t, \pi) & \cdots & g_m(E_t, \pi)
\end{pmatrix}
$$

and

$$
\overrightarrow{M}_\pi := \begin{pmatrix}
g_1(E_1, \pi) & \cdots & g_m(E_1, \pi) \\
\vdots & \ddots & \vdots \\
g_1(E_t, \pi) & \cdots & g_m(E_t, \pi)
\end{pmatrix}.
$$

The idea is that $M_\pi$ captures all possible behavior towards 1-type $\pi$, where each column $j$ represents the behavior of $g_j$ towards $\pi$. Note that for a structure $A$ and 1-type $\pi$, the restriction of $A$ on the set $A_\pi$ can be viewed as a $t$-color digraph $G = (V, E_1, \ldots, E_t)$. It is sufficient to consider only the 2-types $E_1, \ldots, E_t$, because each $E_i$ determines its reversal $E_i$. Moreover, an element $a$ has an incoming $E_i$-edge if and only if it has an outgoing $E_i$-edge. Thus, if $A \models \phi$, the graph $G$ is a complete $M_\pi|\overrightarrow{M}_\pi$-regular digraph.

Now, we explain how to capture the behavior between elements with distinct 1-types. Define matrices $L_\pi, \overrightarrow{L}_\pi \in \mathbb{N}_{\infty}^{t \times m}$ as follows:

$$
L_\pi := \begin{pmatrix}
M_\pi \\
\overrightarrow{M}_\pi
\end{pmatrix}
$$

and

$$
\overrightarrow{L}_\pi := \begin{pmatrix}
\overrightarrow{M}_\pi \\
M_\pi
\end{pmatrix}.
$$

That is, in $L_\pi$ the first $t$ rows come from $M_\pi$ with the next $t$ rows from $\overrightarrow{M}_\pi$. On the other hand, in $\overrightarrow{L}_\pi$ the first $t$ rows come from $\overrightarrow{M}_\pi$, followed by the $t$ rows from $M_\pi$.

The idea is that for a structure $A$, the 2-types that are realized between $A_\pi$ and $A_{\pi'}$ can be viewed as a $2t$-color bipartite graph $G = (A_\pi, A_{\pi'}, E_1, \ldots, E_t, \overrightarrow{E_1}, \ldots, \overrightarrow{E_t})$, where the direction of the edges are ignored. Moreover, a pair $(a,b)$ has 2-type $E$ if and only if $(b,a)$ has 2-type $\overrightarrow{E}$. Thus, if $A \models \phi$, the graph $G$ is a complete $L_\pi|\overrightarrow{L}_\pi$-biregular graph.

Now we are ready to define the formula $\text{consistent}_2(X)$. We enumerate all the 1-types $\pi_1, \ldots, \pi_n$ and define $\text{consistent}_2$ as follows:

$$
\text{consistent}_2(X) := \bigwedge_{1 \leq i \leq n} \text{c-reg}_{M_\pi_i|\overrightarrow{M}_\pi_i}(\overrightarrow{X}_{\pi_i}) \land \bigwedge_{1 \leq i < j \leq n} \text{c-bireg}_{E_{\pi_i}}(\overrightarrow{X}_{\pi_i}, \overrightarrow{X}_{\pi_j}). \quad (4)
$$

The formula $\text{consistent}_1(\overrightarrow{X})$ is Presburger definable by inspection, while $\text{consistent}_2(\overrightarrow{X})$ is Presburger definable using Lemmas 2 and 3. The correctness comes directly from the following lemma.

**Lemma 6.** For every structure $A \models \phi$, $\text{consistent}_1(\overrightarrow{N}) \land \text{consistent}_2(\overrightarrow{N})$ holds, where $\overrightarrow{N} = ([A_{\pi_1,g_1}, \ldots, A_{\pi_n,g_m}])$. Conversely, for every $\overrightarrow{N}$ such that $\text{consistent}_1(\overrightarrow{N}) \land \text{consistent}_2(\overrightarrow{N})$ holds, there is $A \models \phi$ such that $\overrightarrow{N} = ([A_{\pi_1,g_1}, \ldots, A_{\pi_n,g_m}])$.

**Proof.** Let $\phi$ be in Scott normal form as in [1]. As before, $\Pi = \{\pi_1, \pi_2, \ldots, \pi_n\}$ denote the set of all 1-types and $E = \{E_1, \ldots, E_t\}$ the set of all 2-types, where $\overrightarrow{E_i}(x,y) = E_i(y,x)$ for each $1 \leq i \leq t$. Recall that each 2-type $E$ contains the predicate $x \neq y$ and that $G = \{g_1, \ldots, g_m\}$ is the set of all good functions.

Note that for $\pi, \pi' \in \Pi$ and $E \in E$, the conjunction $\pi(x) \land E(x,y) \land \pi'(y)$ corresponds to a boolean assignment of the atomic predicates in $\alpha(x,y)$. Thus, either $\pi(x) \land E(x,y) \land \pi'(y) \models \alpha(x,y)$ or $\pi(x) \land E(x,y) \land \pi'(y) \models \neg \alpha(x,y)$. Similarly, $\pi(x) \land x = y \models \alpha(x,y)$ or $\pi(x) \land x = y \models \neg \alpha(x,y)$.

We first prove the first statement in the lemma. Let $A \models \phi$. Partition $A$ into $A_{\pi,g}$’s. We will show that $\text{consistent}_1(\overrightarrow{X}) \land \text{consistent}_2(\overrightarrow{X})$ holds when each $X_{\pi,g}$ is assigned with the value $|A_{\pi,g}|$. 

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Since \( A \models \forall x \forall y \alpha(x, y) \), by definition \( A_{\pi, g} = \emptyset \), whenever \( \pi \) and \( g \) are incompatible. Thus, consistent\( _1(\bar{X}) \) holds.

Next, we will show that consistent\( _2(\bar{X}) \) holds. Let \( \pi \in \Pi \). By definition of \( A_{\pi, g} \), \( A_{\pi, g} \) is a complete \( M_{\pi} \)-regular digraph \( G = (V, E_1, \ldots, E_t) \), with size \(( |A_{\pi, g_1}|, \ldots, |A_{\pi, g_m}|) \). Thus, by Lemma 3, \( c\text{-reg}_{M_{\pi}}(\bar{X}_\pi) \) holds.

For \( \pi_i, \pi_j \in \Pi \), where \( i < j \), the structure \( A \) restricted to \( A_{\pi_i} \) and \( A_{\pi_j} \) can be viewed as a complete \( L_{\pi_i, \pi_j} \)-biregular graph \( G = (U, V, E_1, \ldots, E_t, \bar{E}_1, \ldots, \bar{E}_t) \), where \( U = A_{\pi_i} \) and \( V = A_{\pi_j} \) and for each \( 1 \leq i \leq t \), we have the interpretation denoted (by a slight abuse of notation) as \( E_i \) consist of all pairs \((a, b) \in A_{\pi_i} \times A_{\pi_j} \) whose 2-type is \( E_i \), and similarly for \( \bar{E}_i \). By Lemma 2, \( c\text{-bireg}_{G_{\pi_i}}(\bar{X}_{\pi_i}, \bar{X}_{\pi_j}) \) holds.

Now we prove the second statement. Suppose \( \text{PRES}_\phi \) holds. By definition, there exists an assignment to the variables in \( \bar{X} \) such that \( \text{consistent}_1(\bar{X}) \land \text{consistent}_2(\bar{X}) \) holds. Abusing notation as we often do in this work, we denote the value assigned to each \( X_{\pi, g} \) by the variable \( X_{\pi, g} \) itself.

For each \((\pi, g)\), we have a set \( V_{\pi, g} \) with cardinality \( X_{\pi, g} \). We denote by \( V_{\pi} = \bigcup_g V_{\pi, g} \).

We construct a structure \( A \) that satisfies \( \phi \) as follows.

- The domain is \( A = \bigcup_{\pi, g} V_{\pi, g} \).
- For each \( \pi \in \Pi \), for each \( a \in V_{\pi} \), the unary atomic formulas on \( a \) are defined such that the 1-type of \( a \) becomes \( \pi \).
- For each \( \pi \in \Pi \), the binary predicates on \((u, v) \in V_{\pi} \times V_{\pi} \) are defined as follows. Since \( c\text{-reg}_{M_{\pi}}(\bar{X}_\pi) \) holds, there is a complete \( M_{\pi} \)-regular digraph \( G = (V_{\pi}, E_1, \ldots, E_t) \) with size \( \bar{X}_\pi \). The edges \( E_1, \ldots, E_t \) define precisely the 2-types among elements in \( V_{\pi} \).
- For each \( \pi_i, \pi_j \), where \( i < j \), the binary predicates on \((u, v) \in V_{\pi_i} \times V_{\pi_j} \) are defined as follows. Since \( c\text{-bireg}_{G_{\pi_i}}(\bar{X}_{\pi_i}, \bar{X}_{\pi_j}) \) holds, there is a \( L_{\pi_i, \pi_j} \)-biregular graph \( G = (V_{\pi_i}, V_{\pi_j}, E_1, \ldots, E_t, \bar{E}_1, \ldots, \bar{E}_t) \) with size \( \bar{X}_{\pi_i}, \bar{X}_{\pi_j} \). The edges \( E_1, \ldots, E_t, \bar{E}_1, \ldots, \bar{E}_t \) define precisely the 2-types on \((u, v) \in V_{\pi_i} \times V_{\pi_j} \).

We first show that \( A \models \forall x \forall y \alpha(x, y) \). Indeed, suppose there exist \( u, v \in A \) such that \( \pi(u) \land E_i(u, v) \lor \pi'(v) \not\equiv \alpha(u, v) \). By definition, there is \( g \) such that \( u \in V_{\pi, g} \) and \( g(E_{\pi}, \pi') \neq 0 \). Thus, \( V_{\pi, g} \neq \emptyset \). This also means that \( \pi \) is incompatible with \( g \), which implies that \( X_{\pi, g} = 0 \) by consistent\( _1(\bar{X}) \), thus, contradicts the assumption that \( V_{\pi, g} \neq \emptyset \).

Next, we show that \( A \models \bigwedge_{i=1}^t \forall x \exists y \beta_i(x, y) \land x \not\equiv y \). Note that \( G = \{g_1, \ldots, g_m\} \) consists of only good functions. Thus, for every \( g \in G \), for every \( \beta_i \), the sum \( \sum_{(x, y) \in E} g(E, \pi) \) is an element in \( S_i \).

## 4 Proof ideas for Lemmas 2 and 3

We now discuss the proof of the main biregular graph lemmas. For now, we deal only with the 1-color case, which gives the flavour of the arguments. The general case, which is much more involved, is deferred to the appendix.

This section is organized as follows. In Subsection 4.1 we will focus on a relaxation of Lemma 2 where the requirement being complete is dropped. This will then be used to prove the complete case in Subsection 4.2. Finally, in Subsection 4.3 we present a brief explanation on how to modify the proof for the biregular graphs to the one for regular digraphs.

### 4.1 The case of incomplete 1-color biregular graphs

This subsection is devoted to the proof of the following lemma.
Lemma 7. For every $A \in \mathbb{N}^{1 \times m}_{\infty,+p}$ and $B \in \mathbb{N}^{1 \times n}_{\infty,+p}$, there exists (effectively computable) existential Presburger formula $\text{bireg}_{A|B}(\bar{x}, \bar{y})$ such that for every $(\bar{M}, \bar{N}) \in \mathbb{N}^{m \times n}_{\infty} \times \mathbb{N}^{n \times \infty}_{\infty}$ the following holds: there is an $A|B$-biregular graph with size $\bar{M}|\bar{N}$ if and only if $\text{bireg}_{A|B}(\bar{M}, \bar{N})$ holds in $\mathbb{N}_{\infty}$.

The desired formula $c\text{-bireg}_{A|B}(\bar{x}, \bar{y})$ for complete biregular graphs will be defined using the formula $\text{bireg}_{A|B}(\bar{x}, \bar{y})$.

We will use the following notations. The term vectors always refers to row vectors, and we usually use $\bar{a}, \bar{b}, \ldots$ (possibly indexed) to denote them. We write $(\bar{a}, \bar{b})$ to denote the vector $\bar{a}$ concatenated with $\bar{b}$. Obviously 1-row matrices can be viewed as row vectors. For $\bar{a} = (a_1, \ldots, a_k) \in \mathbb{N}^k_{\infty}$, we write $\bar{a}^{\text{p}}$ to denote the vector $(a_1^{\text{p}}, \ldots, a_k^{\text{p}})$.

Matrix entries of the form $a^{\text{p}}$ are called periodic entries. Otherwise, they are called fixed entries. By grouping the entries according to whether they are fixed/periodic, we write a 1-row matrix $\bar{M}$ as $(\bar{a}, \bar{b}^{\text{p}})$, where $\bar{a}$ and $\bar{b}^{\text{p}}$ correspond to the fixed and periodic entries in $\bar{M}$. Matrices that contain only fixed (or, periodic) entries are written as $\bar{a}$ (or, $\bar{a}^{\text{p}}$).

To specify $A|B$-biregular graphs, we write $(\bar{a}, \bar{b}^{\text{p}})|(\bar{c}, \bar{d}^{\text{p}})$-biregular graphs, where $A = (\bar{a}, \bar{b}^{\text{p}})$ and $B = (\bar{c}, \bar{d}^{\text{p}})$. Similarly, when, say, $A$ contains only fixed entries, it is written as $\bar{a}^{\text{p}}|(\bar{c}, \bar{d}^{\text{p}})$-biregular. The size of $(\bar{a}, \bar{b}^{\text{p}})|(\bar{c}, \bar{d}^{\text{p}})$-biregular graph is written as $(\bar{M}_0, \bar{M}_1)|(ar{N}_0, \bar{N}_1)$, where the lengths of $\bar{M}_0, \bar{M}_1, \bar{N}_0, \bar{N}_1$ are the same as $\bar{a}, \bar{b}, \bar{c}, \bar{d}$, respectively. The other cases, when some of $\bar{a}, \bar{b}^{\text{p}}, \bar{c}, \bar{d}^{\text{p}}$ are omitted, are treated in similar manner.

As before, we will write $\bar{x}, \bar{y}$ (possibly indexed) to denote a vector of variables. We write $\bar{1}$ to denote the vector with all components being 1. We use $\cdot$ to denote the standard dot product between two vectors. To avoid being repetitive, when dot products are performed, it is implicit that the vector lengths are the same. In particular, $\bar{1} \cdot \bar{x}$ is the sum of all the components in $\bar{x}$.

We now outline the proof of Lemma 7, focusing only on the case where there is no $\infty$ degree in the matrices. The case where such a degree exists is similar but simpler. Without loss of generality, we can also assume that none of the fixed entries are zero. For vectors $\bar{M}_0, \bar{M}_1, \bar{N}_0, \bar{N}_1$ with the same length as $\bar{a}, \bar{b}, \bar{c}, \bar{d}$, respectively, we say that $(\bar{M}_0, \bar{M}_1)|(ar{N}_0, \bar{N}_1)$ is big enough for $(\bar{a}, \bar{b}^{\text{p}})|(\bar{c}, \bar{d}^{\text{p}})$, if the following holds:

(a) $\bar{M}_0 \cdot \bar{1} + \bar{M}_1 \cdot \bar{1} + \bar{N}_0 \cdot \bar{1} + \bar{N}_1 \cdot \bar{1} \geq 2\delta_{\text{max}}^2 + 3$,

(b) $\bar{M}_1 \cdot \bar{1} \geq \delta_{\text{max}}^2 + 1$,

(c) $\bar{N}_1 \cdot \bar{1} \geq \delta_{\text{max}}^2 + 1$.

Here $\delta_{\text{max}} = \max(p, \bar{a}, \bar{b}, \bar{c}, \bar{d})$—that is, the maximal element among $p$ and the components in $\bar{a}, \bar{b}, \bar{c}, \bar{d}$. When $\bar{b}^{\text{p}}$ or $\bar{d}^{\text{p}}$ are missing, the same notion can be defined by dropping condition (b) or (c), respectively. For example, we say that $\bar{M}|\bar{N}$ is big enough for $\bar{a}|\bar{b}$, if $\bar{M} \cdot \bar{1} + \bar{N} \cdot \bar{1} \geq 2\delta_{\text{max}}^2 + 3$, where $\delta_{\text{max}} = \max(\bar{a}, \bar{b})$. Similarly, $(\bar{M}_0, \bar{M}_1)|\bar{N}$ is big enough for $(\bar{a}, \bar{b}^{\text{p}})|\bar{c}$, if $\bar{M}_0 \cdot \bar{1} + \bar{M}_1 \cdot \bar{1} + \bar{N} \cdot \bar{1} \geq 2\delta_{\text{max}}^2 + 3$, and $\bar{M}_1 \cdot \bar{1} \geq \delta_{\text{max}}^2 + 1$, where $\delta_{\text{max}} = \max(p, \bar{a}, \bar{b}, \bar{c})$.

The proof idea is as follows. We first construct a formula that deals with big enough sizes. Then, we construct a formula for each of the cases when one of the conditions (a), (b) or (c) is violated. The interesting case will be when condition (b) is violated. This means that the number of vertices with degrees from $\bar{b}^{\text{p}}$ is fixed, and they can be “encoded” inside the Presburger formula.

We start with the big enough case. When there are only fixed entries, we will use the following lemma.

Lemma 8. For $\bar{M}|\bar{N}$ big enough for $\bar{a}|\bar{b}$, there is a $\bar{a}|\bar{b}$-biregular graph with size $\bar{M}|\bar{N}$ if and only if $\bar{M} \cdot \bar{a} = \bar{N} \cdot \bar{b}$.

Proof. Note that if we have a biregular graph with the desired outdegrees on the left, then
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the total number of edges must be $M \cdot \bar{a}$, and similarly the total number of edges considering the requirement for vertices on the right, we see that the total number of edges must be $\bar{N} \cdot \bar{b}$. Thus this condition is always a necessary one, regardless of whether $M|\bar{N}$ is big enough.

When both $M$ and $\bar{N}$ do not contain $\infty$, \cite[Lemma 7.2]{2} shows that when $M|\bar{N}$ is big enough for $\bar{a}|\bar{b}$, the converse holds: $M \cdot \bar{a} = \bar{N} \cdot \bar{b}$ implies that there is a $\bar{a}|\bar{b}$-biregular graph with size $M|\bar{N}$. We briefly mention the proof idea there, which we will also see later (e.g., in the proof of Lemma \cite{3}). There is a preliminary construction that handles the requirement on vertices on one side in isolation, leaving the vertices on the right with outdegree 1. A follow-up construction merges vertices on the right in order to ensure the necessary number of incoming edges on the right. In doing so we exploit the “big enough” property in order to avoid merging two nodes on the right with a common adjacent edge on the left.

We will now prove that the condition is also sufficient when either $M$ or $\bar{N}$ contains $\infty$. So assume $M \cdot \bar{a} = \bar{N} \cdot \bar{b}$, and thus both $M, \bar{N}$ contain $\infty$.

We construct an $\bar{a}|\bar{b}$-biregular graph $G = \langle U, V, E \rangle$ with size $M|\bar{N}$ as follows. Let $\bar{a} = (a_1, \ldots, a_m)$ and $\bar{b} = (b_1, \ldots, b_n)$. Let $M = (M_1, \ldots, M_m)$ and $\bar{N} = (N_1, \ldots, N_n)$. We pick pairwise disjoint sets $U_1, \ldots, U_m$, where each $|U_i| = M_i$ and $V_1, \ldots, V_n$, where $|V_i| = N_i$.

We set $U = \bigcup_i U_i$ and $V = \bigcup_i V_i$.

The edges are constructed as follows. For each $i \leq i \leq m$, when $|U_i|$ is finite, we make each vertex $u \in U_i$ have degree $a_i$, as follows. For each $1 \leq j \leq t$, we pick $a_i$ “new” vertices from some infinite set $V_i$—that is, vertices that are not adjacent to any edge, and connect them to $u$. Likewise, for each vertex $v \in V_i$ when $|V_i|$ is finite. After performing this, every vertex in finite $U_i$ and $V_i$ has degree $a_i$ and $b_i$, respectively, and every vertex in infinite sets $U_i$ and $V_i$ has degree at most 1.

Finally, we iterate the following process. For every infinite $U_i$, if $u \in U_i$ has degree other than $a_i$, we change the degree to $a_i$ by picking “new” vertices from some infinite set $V_i$, and connect them to $u$ by an appropriate number of edges. Likewise, we can make each vertex $v$ in infinite $V_i$ to have degree $b_i$. Note that in any iteration, for every infinite set $U_i$, the degree of a vertex $u \in U_i$ is either $a_i$, 1, or 0. Likewise, in any iteration, for every infinite set $V_i$, the degree of a vertex $v \in V_i$ is either $b_i$, 1, or 0. Since there is an infinite supply of vertices, there are always new vertices that can be picked in any iteration.

Now we move to the case where the entries are still big enough, but some of the entries are periodic on one side. Then we consider the following formula $\Psi_{\bar{a}, \bar{b}+p}\bar{c}(\bar{x}_0, \bar{x}_1, \bar{y})$:

$$\exists z \ (z \neq \infty) \land (\bar{a} \cdot \bar{x}_0 + \bar{b} \cdot \bar{x}_1 + pz = \bar{c} \cdot \bar{y}).$$

(5)

Note that if $G = \langle U, V, E \rangle$ is a $(\bar{a}, \bar{b}+p)\bar{c}$-biregular graph with size $(\bar{M}_0, \bar{M}_1)|\bar{N}$, then the number of edges $|E|$ should equal the sum of the degrees of the vertices in $U$, which is $\bar{a} \cdot \bar{M}_0 + \bar{b} \cdot \bar{M}_1 + pz$, for some integer $z \geq 0$. Since this quantity must equal the sum of the degrees of the vertices in $V$, which is $\bar{c} \cdot \bar{N}$, we again conclude that this formula is a necessary condition—regardless of whether the entries are big enough. We again show the converse.

\textbf{Lemma 9.} For $(\bar{M}_0, \bar{M}_1)|\bar{N}$ big enough for $(\bar{a}, \bar{b}+p)\bar{c}$ the following holds. There is a $(\bar{a}, \bar{b}+p)\bar{c}$-biregular graph with size $(\bar{M}_0, \bar{M}_1)|\bar{N}$ if and only if $\Psi_{\bar{a}, \bar{b}+p}\bar{c}(\bar{M}_0, \bar{M}_1, \bar{N})$ holds.

\textbf{Proof.} Assume that $\Psi_{\bar{a}, \bar{b}+p}\bar{c}(\bar{M}_0, \bar{M}_1, \bar{N})$ holds. As before, abusing notation, we denote the value assigned to variable $z$ by $z$ itself. Suppose $\bar{a} \cdot \bar{M}_0 + \bar{b} \cdot \bar{M}_1 + pz = \bar{N} \cdot \bar{c}$. Since $(\bar{M}_0, \bar{M}_1)|\bar{N}$ is big enough for $(\bar{a}, \bar{b}+p)\bar{c}$, it follows immediately that $(\bar{M}_0, \bar{M}_1, z)|\bar{N}$ is big enough for $(\bar{a}, \bar{b}, p)\bar{c}$. Applying Lemma \cite{3} there is a $(\bar{a}, \bar{b}, p)\bar{c}$-biregular graph with size $(\bar{M}_0, \bar{M}_1, z)|\bar{N}$. That is, we have a graph that satisfies our requirements, but there is an
There exists a partition class $Z$ on the left of size $z$ where the number of adjacent vertices is $p$, rather than being $\hat{b} \cdot p$ as we require. Let $G = (U, V, E)$ be such a graph, and let $U = U_0 \cup U_1 \cup Z$, where $U_0$, $U_1$, and $Z$ are the sets of vertices whose degrees are from $\bar{a}$, $\bar{b}$, and from $p$. Note that $|U_0| = \bar{M}_0 \cdot \bar{1}$, $|U_1| = \bar{M}_1 \cdot \bar{1}$ and $|Z| = z$.

We will construct a $(\bar{a}, \bar{b}, p)(\bar{c})$-biregular graph with size $(\bar{M}_0, \bar{M}_1)$ on $\hat{N}$. The idea is to merge the vertices in $Z$ with vertices in $U_1$. Let $z_0 \in Z$. The number of vertices in $U_1$ reachable from $z_0$ in distance 2 is at most $\hat{d}_{\text{max}}^2$. Since $(\bar{M}_0, \bar{M}_1)$ on $\hat{N}$ is big enough for $(\bar{a}, \bar{b}, p)(\bar{c})$, we have $|U_1| = \bar{M}_1 \cdot 1 \geq \hat{d}_{\text{max}}^2 + 1$. Thus, there is a vertex $u \in U_1$ not reachable in distance 2. We merge $z_0$ and $u$ into one vertex. Since the degree of $z_0$ is $p$, such merging increases the degree of $u$ by $p$, which does not break our requirement. We perform such merging for every vertex in $Z$.

Finally, we turn to the big enough case where there are periodic entries on both sides. There we will deal with the following formula $\Psi_{(\bar{a}, \bar{b}, p)(\bar{c}, \bar{d})}(\bar{x}_0, \bar{x}_1, \bar{y}_0, \bar{y}_1)$:

$$\exists \bar{z}_1 \exists \bar{z}_2 \quad (z_1 \neq \infty) \land (z_2 \neq \infty) \land (\bar{a} \cdot \bar{x}_0 + \bar{b} \cdot \bar{x}_1 + p z_1 = \bar{c} \cdot \bar{y}_0 + \bar{d} \cdot \bar{y}_1 + p z_2).$$

\begin{lemma}
For $(\bar{M}_0, \bar{M}_1)$ on $(\hat{N}_0, \hat{N}_1)$ big enough for $(\bar{a}, \bar{b}, p)(\bar{c}, \bar{d})$ the following holds: there exists a $(\bar{a}, \bar{b}, p)(\bar{c}, \bar{d})$-biregular graph with size $(\bar{M}_0, \bar{M}_1)$ if and only if $\Psi_{(\bar{a}, \bar{b}, p)(\bar{c}, \bar{d})}(\bar{M}_0, \bar{M}_1, \hat{N}_0, \hat{N}_1)$ holds.
\end{lemma}

\begin{proof}
As before, the “only if” direction is straightforward, so we focus on the “if” direction.

Suppose $\Psi_{(\bar{a}, \bar{b}, p)(\bar{c}, \bar{d})}(\bar{M}_0, \bar{M}_1, \hat{N}_0, \hat{N}_1)$ holds. Thus, $\bar{a} \cdot \bar{M}_0 + \bar{b} \cdot \bar{M}_1 + p z_1 = \bar{c} \cdot \bar{N}_0 + \bar{d} \cdot \bar{N}_1 + p z_2$.

If $z_1 \geq z_2$, then the equation can be rewritten as $\bar{a} \cdot \bar{M}_0 + \bar{b} \cdot \bar{M}_1 + p (z_1 - z_2) = \bar{c} \cdot \bar{N}_0 + \bar{d} \cdot \bar{N}_1$.

By Lemma 9 there is a $(\bar{a}, \bar{b}, p)(\bar{c}, \bar{d})$-biregular graph with size $(\bar{M}_0, \bar{M}_1)$, which of course, is also $(\bar{a}, \bar{b}, p)(\bar{c}, \bar{d})$-biregular. The case when $z_2 \geq z_1$ is symmetric.

The previous lemmas give formulas that capture the existence of 1-color biregular graphs for big enough sizes. We now turn to sizes that are not big enough—that is, when one of the conditions (a), (b) or (c) is violated. When condition (a) is violated, we have restricted the total size of the graph, and thus we can write a formula that simply enumerate all possible valid sizes. We will consider the case when condition (b) is violated, with the case where condition (c) is violated being symmetric.

If (b) is violated we can fix the value of $\bar{M}_1 \cdot 1$ as some $r$, and it suffices to find a formula that works for this $r$. The idea is that a fixed number of vertices in a graph can be "encoded" as formulas. For $\bar{a} = (a_1, \ldots, a_n)$, $\bar{b} = (b_1, \ldots, b_1)$, $\bar{c} = (c_1, \ldots, c_m)$ and $\bar{d} = (d_1, \ldots, d_n)$, and for integer $r \geq 0$, define the formula $\Phi_{(\bar{a}, \bar{b}, p)(\bar{c}, \bar{d})}(\bar{x}_0, \bar{x}_1, \bar{y}_0, \bar{y}_1)$ as follows:

\begin{enumerate}
    \item when $r = 0$, let
    $$\Phi_{(\bar{a}, \bar{b}, p)(\bar{c}, \bar{d})}(\bar{x}_0, \bar{x}_1, \bar{y}_0, \bar{y}_1) := \bar{x}_1 \cdot \bar{1} = 0 \land \Psi_{(\bar{a}, \bar{b}, p)(\bar{c}, \bar{d})}(\bar{x}_0, \bar{y}_0, \bar{y}_1),$$

    \item when $r \geq 1$, let $\bar{x}_1 = (x_{1,1}, \ldots, x_{1,1})$ and
    \begin{align*}
        &\Phi_{(\bar{a}, \bar{b}, p)(\bar{c}, \bar{d})}(\bar{x}_0, \bar{x}_1, \bar{y}_0, \bar{y}_1) := \\
        &\exists \bar{z}_0 \exists \bar{z}_1 \exists \bar{z}_2 \exists \bar{z}_3 \sqrt{t} \left( (x_{1,1} \neq 0) \land \left( b_1 + p s = \bar{z}_1 \cdot \bar{1} + \bar{z}_3 \cdot \bar{1} \right) \land (s \neq \infty) \right) \land \\
        &\land \left( \begin{array}{c}
            \bar{z}_0 + \bar{z}_1 = \bar{y}_0 \\
            \bar{z}_2 + \bar{z}_3 = \bar{y}_1 \\
        \end{array} \right) \land \Phi_{(\bar{a}, \bar{b}, p)(\bar{c}, \bar{d})}(\bar{x}_0, \bar{x}_1 - e_i, \bar{y}_0, \bar{z}_1, \bar{z}_2, \bar{z}_3),
    \end{align*}
\end{enumerate}

where $e_i$ denotes the unit vector (with length $k$) where the $i$-th component is 1, and the lengths of $\bar{z}_0$ and $\bar{z}_1$ are the same as $\bar{y}_0$, and the lengths of $\bar{z}_2$ and $\bar{z}_3$ are the same as $\bar{y}_1$. 

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\end{flushright}
The motivation for these formulas will be explained in the proof of the following lemma.

**Lemma 11.** For every $\bar{a}, \bar{b}, \bar{c}, \bar{d}$, every integer $r \geq 0$ and every $\bar{M}_0, \bar{M}_1, \bar{N}_0, \bar{N}_1$ such that

- $\bar{M}_0 \cdot \bar{1} + \bar{N}_0 \cdot \bar{1} + \bar{N}_1 \cdot \bar{1} \geq 2\delta_{\text{max}}^2 + 3$,
- $\bar{N}_1 \cdot \bar{1} \geq \delta_{\text{max}}^2 + 1$,
- $\bar{M}_1 \cdot \bar{1} = r$,

where $\delta_{\text{max}} = \max(p, \bar{a}, \bar{c}, \bar{d})$, the following holds: there is a $(\bar{a}, \bar{b}^+ p) (\bar{c}, \bar{d}^+ p)$-biregular graph with size $(\bar{M}_0, \bar{M}_1)(\bar{N}_0, \bar{N}_1)$ if and only if $\Phi^{r}(\bar{a}, \bar{b}^+ p) (\bar{c}, \bar{d}^+ p) (\bar{M}_0, \bar{M}_1, \bar{N}_0, \bar{N}_1)$ holds.

**Proof.** The proof is by induction on $r$. The base case $r = 0$ follows from Lemma 9 so we focus on the induction step.

We begin with the “only if” direction, which provides the intuition for these formulas. Suppose $G = (U, V, E)$ is a $(\bar{a}, \bar{b}^+ p) (\bar{c}, \bar{d}^+ p)$-biregular with size $(\bar{M}_0, \bar{M}_1)(\bar{N}_0, \bar{N}_1)$. We let $U = U_{0,1} \cup \cdots \cup U_{0,k} \cup U_{1,1} \cup \cdots \cup U_{1,l}$, where $M_0 = (\{U_{0,1}, \ldots, U_{0,k}\})$ and $M_1 = (\{U_{1,1}, \ldots, U_{1,l}\})$. Likewise, we let $V = V_{0,1} \cup \cdots \cup V_{0,m} \cup V_{1,1} \cup \cdots \cup V_{1,n}$, where $N_0 = (\{V_{0,1}, \ldots, V_{0,m}\})$ and $N_1 = (\{V_{1,1}, \ldots, V_{1,n}\})$.

Since we are not in the base case, we can assume $\bar{M}_1 \cdot \bar{1} = \sum_{i=1}^{t} |U_{1,i}| = r \neq 0$. Thus we can fix some $i$ with $1 \leq i \leq t$ such that $U_{1,i} \neq \emptyset$, and fix also some $u \in U_{1,i}$. Based on this $u$, we define, for each $1 \leq j \leq m$, $Z_{0,j}$ to be the set of vertices in $V_{0,j}$ adjacent to $u$. For each $1 \leq j \leq n$ we let $Z_{1,j}$ be the set of vertices in $V_{1,j}$ adjacent to $u$. Figure 1 illustrates the situation.

If we omit the vertex $u$ and all its adjacent edges, we have the following:

- for every $1 \leq j \leq m$, every vertex in $Z_{0,j}$ has degree $c_j - 1$,
- for every $1 \leq j \leq n$, every vertex in $Z_{1,j}$ has degree $(d_j - 1)^+ p$.

Thus, we have a $(\bar{a}, \bar{b}^+ p) (\bar{c}, \bar{c} - 1, \bar{d}^+ p, (\bar{d} - \bar{1})^+ p)$-biregular graph with size $(\bar{M}_0, \bar{M}_1 - e_i)(\bar{K}_0, \bar{K}_1, \bar{K}_2, \bar{K}_3)$, where

\[
\begin{align*}
K_0 &= |V_{0,1}| - |Z_{0,1}|, \ldots, |V_{0,m}| - |Z_{0,m}|, & K_1 &= |Z_{0,1}|, \ldots, |Z_{0,m}|, \\
K_2 &= |V_{1,1}| - |Z_{1,1}|, \ldots, |V_{1,n}| - |Z_{1,n}|, & K_3 &= |Z_{1,1}|, \ldots, |Z_{1,n}|.
\end{align*}
\]
We can check that the sizes allow us to apply the induction hypothesis to this graph, keeping in mind that the sizes on the left have now decreased by one. We conclude that Φ_{(a,b,r+p)}((\bar{c},d+\rho, (d-1)+\rho)}(\bar{M}_0, \bar{M}_1 - e_i)|\langle \bar{K}_{0,0}, \bar{K}_{0,1}, \bar{K}_{1,0}, \bar{K}_{1,1} \rangle| holds. Moreover, since \( u \in U_{1,i} \), and hence the degree of \( u \) is \( b_+^p \), we have \( \bar{K}_1 \cdot \bar{1} + \bar{K}_3 \cdot \bar{1} = b_1 + ps \), for some integer \( s \geq 0 \).

Note also that \( \bar{K}_0 + \bar{K}_1 = \bar{N}_0 \) and \( \bar{K}_2 + \bar{K}_3 = \bar{N}_1 \). Thus, \( \Phi_{(a,b,r+p)}((\bar{c},d+\rho)}(\bar{M}_0, \bar{M}_1)|\langle \bar{N}_0, \bar{N}_1 \rangle| \)
holds where the variables \( \bar{z}_0, \bar{z}_1, \bar{z}_2, \bar{z}_3 \) are assigned with \( \bar{K}_0, \bar{K}_1, \bar{K}_2, \bar{K}_3 \), respectively.

For the “if” direction, suppose \( \Phi_{(a,b,r+p)}((\bar{c},d+\rho)}(\bar{M}_0, \bar{M}_1, \bar{N}_0, \bar{N}_1) \) holds. Then we can fix some \( s, \bar{z}_0, \bar{z}_1, \bar{z}_2, \bar{z}_3 \), and \( i \) such that (a) \( x_{1,i} \neq 0 \), (b) \( b_1 + ps = \bar{z}_1 \cdot \bar{1} + \bar{z}_3 \cdot \bar{1} \), (c) \( \bar{z}_0 + \bar{z}_1 = \bar{N}_0 \), (d) \( \bar{z}_2 + \bar{z}_3 = \bar{N}_1 \), and (e) \( \Phi_{(a,b,r+p)}((\bar{c},d+\rho, (d-1)+\rho)}(\bar{M}_0, \bar{M}_1 - e_i, \bar{z}_0, \bar{z}_1, \bar{z}_2, \bar{z}_3) \) holds.

We prove from this that a biregular graph of the appropriate size exists. Note that the hypothesis requires that \( \bar{M}_0 \cdot \bar{1} + \bar{N}_0 \cdot \bar{1} + \bar{N}_1 \cdot \bar{1} \geq 2\delta_{\text{max}}^2 + 3 \), where \( \delta_{\text{max}} \) as defined in the statement of the lemma. Since \( \max(\rho, a, b, c, \bar{c} - 1, d, d - 1) = \delta_{\text{max}} \), the equalities in (c) and (d) imply that \( \bar{M}_0 \cdot \bar{1} + \bar{N}_0 \cdot \bar{1} + \bar{z}_1 + \bar{z}_3 \cdot \bar{1} + \bar{z}_2 \cdot \bar{1} \) is bigger than \( 2\delta_{\text{max}}^2 + 3 \).

Note that \( (\bar{M}_1 - e_j) \cdot \bar{1} = r - 1 \). Thus we can apply the induction hypothesis and obtain a \( (a, b, r+p)|\langle \bar{c}, \bar{c} - 1, d, d+\rho, (d-1)+\rho) \)-biregular graph \( G = (U, V, E) \) with size \((\bar{M}_0, \bar{M}_1 - e_i)|\langle \bar{z}_0, \bar{z}_1, \bar{z}_2, \bar{z}_3 \rangle \).

Let \( V = V_0 \cup V_1 \cup V_2 \cup V_3 \) be the partition of \( V \), where

\[
\begin{align*}
V_0 &= V_{0,1} \cup \cdots \cup V_{0,m}, \\
V_1 &= V_{1,1} \cup \cdots \cup V_{1,m}, \\
V_2 &= V_{2,1} \cup \cdots \cup V_{2,n}, \\
V_3 &= V_{3,1} \cup \cdots \cup V_{3,n},
\end{align*}
\]

and such that:

- for every \( 1 \leq i \leq m \), the degree of vertices in \( V_{0,j} \) and \( V_{1,j} \) are \( c_j \) and \( c_j - 1 \), respectively;
- for every \( 1 \leq i \leq n \), the degree of vertices in \( V_{2,j} \) and \( V_{3,j} \) are \( d_j + p \) and \( (d_j - 1) + p \), respectively.

Note also that \( \bar{z}_0 = (|V_0,1|, \ldots, |V_{0,m}|) \), \( \bar{z}_1 = (|V_{1,1}|, \ldots, |V_{1,m}|) \), \( \bar{z}_2 = (|V_{2,1}|, \ldots, |V_{2,n}|) \), and \( \bar{z}_3 = (|V_{3,1}|, \ldots, |V_{3,n}|) \).

Let \( u \) be a fresh vertex. We can construct a \( (a, b, r+p)|\langle \bar{c}, \bar{c} - 1, d, d+\rho, (d-1)+\rho) \)-biregular graph \( G' = (U \cup \{u\}, V, E') \), by connecting the vertex \( u \) with every vertex in \( V_1 \cup V_3 \). Note that the formula states that \( \bar{z}_1 \cdot \bar{1} + \bar{z}_3 \cdot \bar{1} = b_1 + ps \), which equals to \( |V_1| + |V_3| \), thus the degree of \( u \) is \( b_1 + ps \), which satisfies our requirement for a vertex to be in \( U_1 \). Since prior to the connection, the degrees of \( V_{1,j} \) and \( V_{3,j} \) are \( c_j - 1 \) and \( (d_j - 1) + p \), after connecting \( u \) with each vertex in \( V_1 \cup V_3 \), their degrees become \( c_j \) and \( d_j + p \). That is, the right side vertices now have the desired degrees, i.e., \( G' \) is \( (a, b, r+p)|\langle \bar{c}, \bar{c} - 1, d, d+\rho, (d-1)+\rho) \)-biregular. Moreover, \( \bar{z}_0 + \bar{z}_1 = \bar{N}_0 \) and \( \bar{z}_2 + \bar{z}_3 = \bar{N}_1 \). Thus, the resulting graph \( G' \) has size \((\bar{M}_0, \bar{M}_1)|\langle \bar{N}_0, \bar{N}_1 \rangle \).

The formula \( \text{bireg}_{(a,b+r+p)}((\bar{c},d+\rho)}(x_0, x_1, y_0, y_1) \) characterizing the sizes of \( (a, b, r+p)|\langle \bar{c}, \bar{c} - 1, d, d+\rho, (d-1)+\rho) \)-biregular graphs can be defined by combining all the cases described above.

### 4.2 Proof of Lemma 2 for 1-color graphs (the complete case)

We now turn to bootstrapping the biregular case to add the completeness imposed in Lemma 2. Let \( a = (a_1, \ldots, a_k) \), \( b = (b_1, \ldots, b_l) \), \( c = (c_1, \ldots, c_m) \) and \( d = (d_1, \ldots, d_n) \). Let \( x_0 = (x_{0,1}, \ldots, x_{0,k}) \), \( \bar{x}_1 = (x_{1,1}, \ldots, x_{1,l}) \), \( y_0 = (y_{0,1}, \ldots, y_{0,m}) \), and \( y_1 = (y_{1,1}, \ldots, y_{1,n}) \).

The formula \( \text{c-bireg}_{(a,b+r+p)}((\bar{c},d+\rho)}(x_0, x_1, y_0, y_1) \) for the sizes of complete \( (a, b, r+p)|\langle \bar{c}, \bar{c} - 1, d, d+\rho, (d-1)+\rho) \)-biregular graphs is the conjunction of \( \text{bireg}_{(a,b+r+p)}((\bar{c},d+\rho)}(x_0, x_1, y_0, y_1) \) such that:

- for every \( 1 \leq i \leq k \), if \( x_{0,i} \neq 0 \), then \( y_0 \cdot 1 + y_1 \cdot 1 = a_i \);
- for every \( 1 \leq i \leq l \), if \( x_{1,i} \neq 0 \), then \( y_0 \cdot 1 + y_1 \cdot 1 = b_i + p z_i \), for some \( z_i \);
- for every \( 1 \leq i \leq m \), if \( y_{0,i} \neq 0 \), then \( x_0 \cdot 1 + x_1 \cdot 1 = c_i \);
- for every \( 1 \leq i \leq n \), if \( y_{1,i} \neq 0 \), then \( x_0 \cdot 1 + x_1 \cdot 1 = d_i + p z_i \), for some \( z_i \).
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To understand these additional conditions, consider a complete biregular graph meeting the cardinality specification. The completeness criterion for 1-color graphs implies that each element on the left is connected to every element on the right. Thus if the size of a partition required to have fixed outdegree $a_i$ is non-empty, we must have that $a_i$ is exactly the cardinality of the number of elements on the right. This is what is captured in the first item. If we have non-empty size for a partition whose outdegree is constrained to be $b_i$ plus a multiple of $p$, then the total number of elements on the right must be $b_i$ plus a multiple of $p$. This is what the second item specifies. Considering elements on the left motivates the third and fourth item. Thus we see that these conditions are necessary.

Suppose $c$-biregular $(\bar{a}, \bar{b}, \bar{n})$-biregular graph $G = (U, V, E)$ with size $(\bar{M}_0, \bar{M}_1)$, which are not necessarily complete. Note that $N_0 \cdot \bar{I} + N_1 \cdot \bar{I}$ is precisely the number of vertices in $V$. The first item states that the existence of a vertex $u$ with degree $a_i$ implies $u$ is adjacent to every vertex in $V$. Now, suppose there is a vertex $u \in U$ with degree $b_i$. If $u$ is not adjacent to every vertex in $V$, then we can add additional edges so that $u$ is adjacent to every vertex in $V$. The second item states that $|V| = b_i$. Thus, adding such edges is legal, since the degree of $u$ stays $b_i$.

We can make vertices in $V$ adjacent to every vertex in $U$ using the same argument.

4.3 The proof for regular digraphs

Recall that in the prior argument we consider only digraphs without any self-loop. Thus, a digraph can be viewed as a bipartite graph by splitting every vertex $u$ into two vertices, where one is adjacent to all the incoming edges, and the other to all the outgoing edges. Thus, $A|B$-regular digraphs with size $\bar{M}$ can be characterized as $A|B$-biregular digraphs with size $\bar{M}/M$. For more details, see [12] Section 8.

Some remarks on the general cases. We stress that although the 1-color case contains many of the key ideas, the multi-color case requires a finer analysis to deal with the “big enough” case, and also a reduction that allows one to restrict to matrices of a very special form (“simple matrices”). Both of these techniques are deferred to the appendix.

5 Extensions and applications

A type/behavior profile for a model $M$ is the vector of cardinalities of the sets $A_{\pi,g}$ computed in $M$, where $\pi$ ranges of 1-types and $g$ over behavior functions (for a fixed $\phi$). Recall that in the proof Theorem 4 we actually showed, in Lemma 6, that we can obtain existential Presburger formulas which define exactly the vectors of integers that arise as the type/behavior profiles of models of $\phi$. The domain of the model can be broken up as a disjoint union of sets $A_{\pi,g}$, and thus its cardinality is a sum of numbers in this vector. We can thus add one additional integer variable $x_{total}$ in $PRES_\phi$, which will be free, with an additional equation stating that $x_{total}$ is the sum of all $X_{\pi,g}$'s. This allows us to conclude definability of the spectrum.

Theorem 12. From an FO$^2_{PRes}$ sentence $\phi$, we can effectively construct a Presburger formula $\psi(n)$ such that $N \models \psi(n)$ exactly when $n$ is the size of a finite structure that satisfies $\phi$, and similarly a formulas $\psi_\infty(n)$ such that $N_\infty \models \psi_\infty(n)$ exactly when $n$ is the size of a finite or countably infinite model of $\phi$.

We say that $\phi$ has NP data complexity of (finite) satisfiability if there is a non-deterministic algorithm that takes as input a set of ground atoms $A$ and determines whether $\phi$ is satisfiable, running in time polynomial in the size of $A$. Pratt-Hartmann [19] showed that $C^2$ formulas have NP data complexity of both satisfiability and finite satisfiability.
Following the general approach to data complexity from [19], while plugging in our Presburger characterization of $\text{FO}^2_{\text{Pres}}$, we can show that the same data complexity bound holds for $\text{FO}^2_{\text{Pres}}$.

**Theorem 13.** $\text{FO}^2_{\text{Pres}}$ formulas have NP data complexity of satisfiability and finite satisfiability.

**Proof.** We give only the proof for finite satisfiability. We will follow closely the approach used for $\mathbf{C}^2$ in Section 4 of [19], and the terminology we use below comes from that work.

Given a set of facts $D$, our algorithm guesses a set of facts (including equalities) on elements of $D$, giving us a finite set of facts $D^+$ extending $D$, but with the same domain as $D$. We check that our guess is consistent with the universal part $\alpha$ and such that equality satisfies the usual transitivity and congruence rules.

Now consider 1-types and 2-types with an additional predicate Observable. Based on this extended language, we consider good functions as before, and define the formulas consistent$_1$ and consistent$_2$ based on them. 1-types with that contain the predicate Observable will be referred to as observable 1-types. The restriction of a behavior function to observable 1-types will be called an observable behavior. Given a structure $M$, an observable one-type $\pi$, and an observable behavior function $g_0$, we let $M_{\pi,g_0}$ be the elements of $M$ having 1-type $\pi$ and observable behavior $g_0$, and we analogously let $D_{\pi,g_0}$ be the elements of $D$ whose 1-type and behavior in $D^+$ match $\pi$ and $g_0$.

We declare that all elements in $A$ are in the predicate Observable. Add to the formulas consistent$_1$ and consistent$_2$ additional conjuncts stating that for each observable 1-type $\pi$ and for each observable behavior function $g_0$, the total sum of the number of elements with 1-type $\pi$ and a behavior function $g$ extending $g_0$ (i.e. the cardinality of $M_{\pi,g_0}$) is the same as $|D_{\pi,g_0}|$, with the cardinality being counted modulo equalities of $D^+$.

At this point our algorithm returns true exactly when the sentence obtained by existentially-quantifying this extended set of conjuncts is satisfiable in the integers. The solving procedure is certainly in NP. In fact, since the number of variables is fixed, with only the constants varying, it is in PTIME [13].

We argue for correctness, focusing on the proof that when the algorithm returns true we have the desired model. Assuming the constraints above are satisfied, we get a graph, and from the graph we get a model $M$. $M$ will clearly satisfy $\phi$, but its domain does not contain the domain of $D$. Letting $O$ be the elements of $M$ satisfying Observable, we know, from the additional constraints imposed, that the cardinality of $O$ matches the cardinality of the domain of $D$ modulo the equalities in $D^+$, and for each observable 1-type $\pi$, and observable behavior $g_0$, $|M_{\pi,g_0}| = |D_{\pi,g_0}|$.

Fix an isomorphism $\lambda$ taking each $M_{\pi,g_0}$ to (equality classes of) $D_{\pi,g_0}$. Create $M'$ by redefining $M$ on $O$ by connecting pairs $(o_1, o_2)$ via $E$ exactly when $\lambda(o_1), \lambda(o_2)$ are connected via $E$ in $D^+$. We can thus identify $O$ with $D^+$ modulo equalities in $M'$.

Clearly $M'$ now satisfies $D$. To see that $M'$ satisfies $\phi$, we simply note that since all of the observable behaviors are unchanged in moving from an element $e$ in $M$ to the corresponding element $\lambda(e)$ in $M'$, and every such $e$ modified has an observable type, it follows that the behavior of every element in $M$ is unchanged in moving from $M$ to $M'$. Since the 1-types are also unchanged, $M'$ satisfies $\phi$.

Note that the data complexity result here is best possible, since even for $\text{FO}^2$ the data complexity can be NP-hard [19].
6 Conclusion

We have shown that we can extend the powerful language two-variable logic with counting to include ultimately periodic counting quantifiers without sacrificing decidability, and without losing the effective definability of the spectrum of formulas within Presburger arithmetic. We believe that by refining our proof we can obtain a \(2\text{NEXPTIME}\) bound on complexity. However the only lower bound we know of is \text{NEXPTIME}, inherited from \(\text{FO}^2\). We leave the analysis of the exact complexity for future work.

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A Scott normal form

In this appendix we prove that every $\mathcal{FO}^2_{\mathcal{G}_{res}}$ formula can be converted into the normal form used in the body of the paper.

We will first give a couple of lemmas.

Lemma 14. Let $S \subseteq \mathbb{N}_\omega$ where $0 \notin S$ and $q$ be a unary predicate. The following sentence $\Psi_1$:

$$\Psi_1 := \forall x \ (q(x) \rightarrow \exists^S y \ \phi(x,y))$$

is equivalent to $\Psi_2$:

$$\Psi_2 := \forall x \ \exists^{S\cup\{0\}} y \ (q(x) \land \phi(x,y)) \land \forall x \ \exists^{N\cup\{0\}} y \ (q(x) \rightarrow \phi(x,y)).$$

Proof. It is worth noting that $q(x) \land \phi(x,y)$ is equivalent to $(q(x) \rightarrow \phi(x,y)) \land (\neg q(x) \rightarrow \bot)$.

Let $\mathcal{A}$ be a structure. For an element $a \in A$, define $W_{a,\phi(x,y)}$ as follows:

$$W_{a,\phi(x,y)} = \{b \in A \mid (\mathcal{A}, x \mapsto a, y \mapsto b) \models \phi(x,y)\}.$$ 

That is, $W_{a,\phi(x,y)}$ is the set of elements that can be assigned to $y$ so that $\phi(x,y)$ holds, when $x$ is assigned with element $a$.

Suppose $\mathcal{A} \models \Psi_1$. So, for every $a \in q^A$, $|W_{a,\phi}| \in S$, hence, the following holds:

$$\mathcal{A}, a \models \exists^S y \ q(x) \rightarrow \phi(x,y) \quad \text{and} \quad \mathcal{A}, a \models \exists^S y \ q(x) \land \phi(x,y).$$

(7)

For every $a \notin q^A$, the following holds:

$$\mathcal{A}, a \models \exists^{|A|} y \ q(x) \rightarrow \phi(x,y) \quad \text{and} \quad \mathcal{A}, a \models \exists^S y \ q(x) \land \phi(x,y).$$

(8)

Combining (7) and (8), we have $\mathcal{A} \models \Psi_2$.

For the other direction, suppose $\mathcal{A} \models \Psi_2$. Since $\mathcal{A} \models \forall x \ \exists^{S\cup\{0\}} y \ (q(x) \land \phi(x,y))$, either $|W_{a,\phi(x,y)}| = 0$ or $|W_{a,\phi(x,y)}| \in S$, for every $a \in A$. Since $\mathcal{A} \models \forall x \ \exists^{N\cup\{0\}} y \ (q(x) \rightarrow \phi(x,y))$, for every $a \in q^A$, $|W_{a,\phi(x,y)}| \neq 0$. Thus, for every $a \in q^A$, $|W_{a,\phi(x,y)}| \in S$. Therefore, $\mathcal{A} \models \Psi_1$. ◀

The following lemma is proven in a similar manner.

Lemma 15. Let $S \subseteq \mathbb{N}_\omega$ where $0 \in S$ and $q$ be a unary predicate. The sentence $\Psi_1$ below:

$$\Psi_1 := \forall x \ (q(x) \rightarrow \exists^S y \ \phi(x,y))$$

is equivalent to the following sentence $\Psi_2$:

$$\Psi_2 := \forall x \ \exists^S y \ (q(x) \land \phi(x,y)).$$

Obviously, Lemma 14 and 15 can be modified trivially when $q(x)$ is any quantifier-free formula with free variable $x$.  

\[ \]
Conversion into “almost” Scott normal form. We will first show how to convert an \( \text{FO}^2_{\text{Pres}} \) sentence into an equisatisfiable sentence in “almost” Scott normal form:

\[
\forall x \forall y \alpha(x, y) \land \bigwedge_{i=1}^{k} \forall x \exists^{S_i} y \beta_i(x, y).
\]

(9)

That is, the requirement \( x \neq y \) is dropped for \( \beta_i(x, y) \) to hold. In fact, we get more than equisatisfiability: each model of our sentence can be expanded to a model of the normal form. This will be important for our result about the spectrum. In the remainder of this section we omit similar statements for brevity.

The conversion is a rather standard renaming technique from two-variable logic. Let \( \Psi \) be an \( \text{FO}^2_{\text{Pres}} \) sentence. We first assume that \( \Psi \) does not contain any subformula of the form \( \forall x \phi \), by rewriting them into the form \( \exists^0 x \neg \phi \).

Whenever there is a subformula \( \psi(x) \) in \( \Psi \) of the form \( \exists S y \phi(x, y) \), where \( \phi(x, y) \) is quantifier free and \( S \) is a u.p.s., we perform a transformation. Let \( q \) be a fresh unary predicate, and replace the subformula \( \psi(x) \) in \( \Psi \) with atomic \( q(x) \), and add a sentence which states that \( q(x) \) is equivalent to \( \psi(x) \), i.e.,

\[
\forall x \left( q(x) \leftrightarrow \psi(x) \right),
\]

which is equivalent to

\[
\forall x \left( q(x) \rightarrow \exists^S y \phi(x, y) \right) \land \forall x \left( \neg q(x) \rightarrow \exists^{N_0 - S} y \phi(x, y) \right),
\]

which, in turn, by Lemma \[14\] and \[15\], can be converted into sentences of the form (9). We iterate this procedure until \( \Psi \) is in the “almost” Scott normal form described above.

Conversion into Scott normal form in (1). Now, we show the conversion from “almost” Scott normal form into Scott normal form. Note that

\[
\forall x \exists^S y \beta(x, y)
\]

is equivalent to

\[
\forall x \left( \neg \beta(x, x) \rightarrow \exists^S y \beta(x, y) \land x \neq y \right) \land \forall x \left( \beta(x, x) \rightarrow \exists^{S-1} y \beta(x, y) \land x \neq y \right),
\]

where \( S - 1 \) denotes the set \( \{ i - 1 \mid i \in S \} \).

Applying Lemma \[14\] and \[15\], a sentence of the form (9) can be converted into an equisatisfiable sentence of the form:

\[
\forall x \forall y \alpha(x, y) \land \bigwedge_{i=1}^{k} \forall x \exists^{S_i} y \beta_i(x, y) \land x \neq y,
\]

where each \( \beta_i(x, y) \) is quantifier free. To make it into Scott normal form, we introduce a new predicate \( \gamma_i(x, y) \), for each \( 1 \leq i \leq k \), and rewrite the sentence as:

\[
\forall x \forall y \left( \alpha(x, y) \land \bigwedge_{i=1}^{k} \left( \gamma_i(x, y) \leftrightarrow \beta_i(x, y) \right) \right) \land \bigwedge_{i=1}^{k} \forall x \exists^{S_i} y \gamma_i(x, y) \land x \neq y.
\]

The conversion described above takes \( O(Cn) \) time where \( n \) is the length of the original \( \text{FO}^2_{\text{Pres}} \) sentence and the factor \( C \) is the complexity of computing the complement \( N_0 - S \) of a u.p.s. \( S \), which of course, depends on the representation of a u.p.s. However, we should note that the number of new atomic predicates introduced is linear in \( n \).
Two variable logic with ultimately periodic counting

B Proof of Lemmas 2 and 3: Presburger definability for regular graph and digraph problems

Recall the statement of Lemma 2:

For every $A \in \mathbb{N}_{\infty, t + p}^{m \times n}$ and $B \in \mathbb{N}_{\infty, t + p}^{n \times m}$ there exists an (effectively computable) existential Presburger formula $c$-$bireg_{A|B}(\bar{x}, \bar{y})$ such that for every $(\bar{M}, \bar{N}) \in \mathbb{N}_{\infty}^{m \times n}$, the following holds. There is complete $|A|B$-biregular graph with size $\bar{M}|\bar{N}$ if and only if $c$-$bireg_{A|B}(\bar{M}, \bar{N})$ holds in $\mathcal{N}_{\infty}$.

The statement of Lemma 3 was the analog for directed graphs:

For every $A \in \mathbb{N}_{\infty, t + p}^{m \times n}$ and $B \in \mathbb{N}_{\infty, t + p}^{n \times m}$ there exists an (effectively computable) existential Presburger formula $c$-$reg_{A|B}(\bar{x})$ such that for every $\bar{M} \in \mathbb{N}_{\infty}^{m}$, the following holds. There is complete $A|B$-regular digraph with size $\bar{M}$ if and only if $c$-$reg_{A|B}(\bar{M})$ holds in $\mathcal{N}_{\infty}$.

The outline of the proof given in this appendix is as follows.

- We introduce some notation in Subsection B.1.
- We will first consider the construction of the Presburger formula for the case when the matrices $A$ and $B$ are what we call simple matrices in Subsection B.2.
- Then, we show how the general case of $A|B$-biregular graphs can be decomposed into a collection of $A'|B'$-biregular graphs where both $A', B'$ are simple matrices. This is presented in Subsection B.3.
- The case of regular digraphs is presented in Subsection B.4.

B.1 Notation and terminology

As before, the term “vectors” means row vectors. We use $\bar{x}, \bar{y}, \bar{z}$ (possibly indexed) to denote vectors of numbers from $\mathbb{N}_{\infty}$. Since we are now transitioning to general multiple color graphs, we will use matrix notation. We use $\cdot$ to denote matrix multiplication. When we perform matrix multiplication, we always assume that the sizes of the operands are appropriate. We write $I_t$ to denote the identity matrix with size $t \times t$.

Let $A \in \mathbb{N}_{\infty, t + p}^{m \times n}$. The transpose of a matrix $A$ is denoted by $A^T$. The entry in row $i$ and column $j$ is $A_{i,j}$. We write $A_{i,*}$ and $A_{*,j}$ to denote the $i^{th}$ row and $j^{th}$ column of $A$, respectively. We call an entry $A_{i,j}$ a fixed entry, if it is in $\mathbb{N}_{\infty}$. Otherwise, it is called a periodic entry. The characteristic matrix of $A$, denoted by $\chi(A)$, is the matrix taking values in $\{0, 1\}$ obtained by replacing its non-zero entries with $1$, while the zero entries remain zero.

Definition 16. A matrix is called a simple matrix, if every row consists of either only periodic entries or only fixed entries.

Here we insist that $\infty$ is regarded as periodic entry, since $\infty$ is regarded as $\infty + p$. Intuitively, the reason is that when a vertex has degree $\infty$, adding $p$ (or any arbitrary number) of additional new edges adjacent to it still make its degree $\infty$.

We write $\text{offset}(a + p)$ to denote the offset value $a$. For convenience, $\text{offset}(a)$ is $a$ itself and $\text{offset}(\infty) = \infty$. The offset of $A$, denoted by $\text{offset}(A)$, is the matrix obtained by replacing every entry $A_{i,j}$ with $\text{offset}(A_{i,j})$. Of course, if $A$ does not contain any periodic entry, then $\text{offset}(A)$ is $A$ itself. For example, if $A = \begin{pmatrix} 0^{+p} & 2^{+} \\ 0 & 3^{+p} \end{pmatrix}$, then $\chi(A) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\text{offset}(A) = \begin{pmatrix} 0 & 2 \\ 0 & 3 \end{pmatrix}$. 
The norm of a matrix $A$ is defined as $\|A\| = \max_{1 \leq j \leq l} \sum_{i=1}^{k} \text{offset}(A_{i,j})$. It is actually the standard 1-norm of its offset matrix. Of course, a vector $\vec{a} = (a_1, \ldots, a_n)$ can be viewed as a 1 row matrix. Thus, $\|\vec{a}\| = \max(\text{offset}(a_1), \ldots, \text{offset}(a_n))$ and $\|\vec{a}\| = \sum_{i=1}^{n} \text{offset}(a_i)$.

If $A$ and $B$ are matrices with the same number of columns, $(A \ B)$ denotes the matrix where the first sequence of rows are $A$ and the next sequence of rows are $B$. Likewise, if $A$ and $B$ have the same number of rows, $(A \ B)$ denotes the matrix where the first sequence of columns are $A$ and the next sequence of columns are $B$.

Next, we generalize the notion of “big enough” in Section 4.

The distinction between “big enough” and “not big enough” size vectors used for the 1-color case in Section 4 will need to be refined into two distinct notions.

**Definition 17.** Let $A \in \mathbb{N}_{\infty, p}^{l \times m}$ and $B \in \mathbb{N}_{\infty, p}^{l \times n}$. Let $\vec{M}$ and $\vec{N}$ be row vectors whose lengths are the same as the number of columns of $A$ and $B$, respectively. We say that $\vec{M} \ | \vec{N}$ is slightly big enough for $A|B$, if for every $1 \leq i \leq t$, $\chi(A_{i,*}) \cdot \vec{M}^\top + \chi(B_{i,*}) \cdot \vec{N}^\top \geq 2\delta_{\max}^2 + 3$, where $\delta_{\max} = \max(\|\text{offset}(A)\|, \|\text{offset}(B)\|)$.

**Definition 18.** Let $A \in \mathbb{N}_{\infty, p}^{l \times m}$ and $B \in \mathbb{N}_{\infty, p}^{l \times n}$. Let $\vec{M}$ and $\vec{N}$ be row vectors whose lengths are the same as the number of columns of $A$ and $B$, respectively. We say that $\vec{M} \ | \vec{N}$ is big enough for $A|B$, if the following holds.

(a) For every $1 \leq i \leq t$, $\chi(A_{i,*}) \cdot \vec{M}^\top + \chi(B_{i,*}) \cdot \vec{N}^\top \geq 2\delta_{\max}^2 + 3$.
(b) For every $1 \leq i \leq t$, $\chi(A_{i,*}) \cdot \vec{M}^\top \geq \delta_{\max}^2 + 1$.
(c) For every $1 \leq i \leq t$, $\chi(B_{i,*}) \cdot \vec{N}^\top \geq \delta_{\max}^2 + 1$.

Here, $\delta_{\max} = \max(\|\text{offset}(A)\|, \|\text{offset}(B)\|, p)$.

**Remark 19.** Some basic observations:

- The notion of “slightly big enough” is defined only on matrices $A|B$ which contain only fixed entries.
- In the notion of “big enough,” condition (a) requires that $\chi(A_{i,*}) \cdot \vec{M}^\top + \chi(B_{i,*}) \cdot \vec{N}^\top$ is at least $2\delta_{\max}^2 + 3$, which is quartic in $\delta_{\max}$, a jump from quadratic for the 1-color case.
- The reason is purely technical, because in multiple color graphs, in some cases periodic entries can be reduced to fixed entries but with quadratic blow-up on the matrix entries.
- Of course, big enough is stronger than slightly big enough.

Informally, “slightly big enough” entries are those that will allow the analogous results to Lemma 8 from the 1-color case, which concerned fixed-degree constraints, to go through. “Big enough” will have some additional margin over “slightly big enough”, which will allow us to handle the case of matrices with periodic entries by reduction to the fixed-entry case.

### B.2 The case when the matrices are simple

We start by proving the Presburger characterization assuming the matrices are simple. We will later give a way to reduce to the simple case.

Assuming the matrices are simple, the overall strategy is the same as in Section 4. We will first prove the following lemma for biregular graphs (with simple matrices) where the requirement being complete is dropped.

**Lemma 20.** For each simple matrix $A \in \mathbb{N}_{\infty, p}^{l \times m}$ and $B \in \mathbb{N}_{\infty, p}^{l \times n}$, there exists an (effectively computable) existential Presburger formula $\text{bireg}_{A|B}(\vec{x}, \vec{y})$ such that for every $(\vec{M}, \vec{N}) \in \mathbb{N}_\infty^r \times \mathbb{N}_\infty^r$, the following holds. There is an $A|B$-biregular graph with size $\vec{M} \ | \vec{N}$ if and only if $\text{bireg}_{A|B}(\vec{M}, \vec{N})$ holds in $\mathbb{N}_\infty$. 

The proof of Lemma \ref{20} is in Subsubsection \ref{B.2.1}. We will then show how to modify it for the case of complete biregular graphs.

\subsection*{B.2.1 Proof of Lemma \ref{20}: characterizing sizes of biregular graphs when the matrices are simple}

We will introduce a bit more terminology. Let $A \in \mathbb{N}^{t \times 1}$, i.e., $A$ is a column vector. In a $t$-color bipartite graph $G$, we say that the degree of a vertex $u$ is $A$ if its $E_i$-degree is $A_i, 1$, for each $1 \leq i \leq t$.

We start with the following lemma that deals with the case when both matrices contain only fixed entries. The proof is a generalization of \cite[Theorem 7.4]{12} to the case when the sizes may be infinite. To an extent, it is also a generalization of Lemma \ref{8} to the case of multiple color graphs.

\begin{lemma}
Let $A \in \mathbb{N}^{t \times m}$ and $B \in \mathbb{N}^{t \times n}$, i.e., the entries in both $A$ and $B$ are all fixed entries. For $M|N$ slightly big enough for $A|B$, the following holds. There is an $A|B$-biregular graph with size $M|N$ if and only if $A \cdot M^T = B \cdot N^T$.
\end{lemma}

\begin{proof}
The “only if” direction is as follows. Let $G = (U, V, E_1, \ldots, E_t)$ be a $A|B$-biregular graph with size $M|N$. The equality, as in the analogous 1-color case, comes from the fact that both $AM^T$ and $BN^T$ simply “count” the number of edges in each color, i.e., $AM^T = \left(|E_1|, \ldots, |E_t|\right)^T = BN^T$.

We now show the “if” direction. Suppose $A \cdot M^T = B \cdot N^T$. We will show that there is an $A|B$-biregular graph with size $M|N$.

The proof is by induction on $t$. The base case $t = 1$ has been shown in Lemma \ref{8}. For the induction hypothesis, we assume the lemma holds when the number of colors is $\leq t - 1$.

Let $A'$ and $B'$ be the matrices obtained by omitting the last row in $A$ and $B$, respectively. Since $M|N$ is slightly big enough for $A|B$, we infer that $M|N$ is slightly big enough for $A'|B'$. Applying the induction hypothesis, there is an $A'|B'$-biregular graph $G' = (U', V', E_1, \ldots, E_{t-1})$ with size $M|N$.

Arguing as above we see that $M|N$ is slightly big enough for $A_{t+1, B_{t+1}}$, i.e., the last rows of $A$ and $B$. Applying the induction hypothesis, there is an $A_{t+1, B_{t+1}}$-biregular graph $G'' = (U'', V'', E_t)$ with size $M|N$. Since $G'$ and $G''$ have the same size, we can assume that $U'' = U'$ and $V'' = V'$.

To obtain the desired $A|B$-biregular graph, we first merge the two graphs, obtaining a single graph $G = (U, V, E_1, \ldots, E_t)$. Such a graph $G$ is “almost” a $A|B$-biregular, except that it is possible we have an edge $(u, v)$ which is in $E_1 \cup \cdots \cup E_t$, as well as in $E_t$. We will again make use of the merging technique adapted from \cite[Theorem 7.4]{12}.

Note that the number of vertices reachable from $u$ and $v$ in distance 2 (with any of $E_1, \ldots, E_t$-edges) is at most $2\delta_{max}^2 + 2$ (the total includes $u$ and $v$). Thus, there is $(w, w') \in E_t$ such that $(u, w') \notin E_1 \cup \cdots \cup E_t$ and $(w, v) \notin E_1 \cup \cdots \cup E_t$. We can perform edge swapping where we omit the edges $(u, v), (w, w')$ from $E_t$, but add $(u, w'), (w, v)$ into $E_t$. See the illustration below. This edge swapping does not effect the degree of any of the vertices $u, v, w, w'$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{edge_swap.png}
\caption{Illustration of edge swapping.}
\end{figure}
The "big enough case" for simple matrices with both periodic and fixed entries, but no \( \infty \) entries. For simple matrices \( A \in \mathbb{N}^{t \times m} \) and \( B \in \mathbb{N}^{t \times n} \) define the following formula.

\[
\Phi_{A|B}(\bar{x}, \bar{y}) := \exists z_{1,1} \ldots \exists z_{1,t} \exists z_{2,1} \ldots \exists z_{2,t} \\
\text{offset}(A) \cdot \bar{x}^T + \begin{pmatrix} \alpha_1 p_{z_{1,1}} \\ \vdots \\ \alpha_t p_{z_{1,t}} \end{pmatrix} = \text{offset}(B) \cdot \bar{y}^T + \begin{pmatrix} \beta_1 p_{z_{2,1}} \\ \vdots \\ \beta_t p_{z_{2,t}} \end{pmatrix} \\
\land \bigwedge_{i=1}^{t} z_{1,i} \neq \infty \land z_{2,i} \neq \infty,
\]

where \( \alpha_i = 1 \) if row \( i \) in \( A \) consists of periodic entries and is 0 otherwise, and similarly \( \beta_i = 1 \) if row \( i \) in \( B \) consists of periodic entries and is 0 otherwise.

We are now ready to handle the “big enough” case when there are periods on both sides. Here we will see some additional complications not present in the 1-color case. Recall that in the analogous result for the 1-color case, Lemma [10] one of the ingredients was simply moving the smaller periodic factor to the other side, at which point we could reduce to the case where one side had only fixed degrees. In the general case there may not be one side that has the larger periodic factor for every color. This will require us to do a finer case analysis, where our additional distinction between “big enough” and “slightly big enough” will come into play.

**Lemma 22.** Let \( A \in \mathbb{N}^{t \times m} \) and \( B \in \mathbb{N}^{t \times n} \) be simple matrices which do not have columns that have only zero entries and do not have an \( \infty \) entry. For \( M|N \) big enough for \( A|B \), there is a \( A|B \)-biregular graph with size \( M|N \) if and only if \( \Phi_{A|B}(M, N) \) holds in \( N^\infty \).

**Proof.** The “only if” direction is as in the 1-color case. Suppose \( G = (U, V, E_1, \ldots, E_t) \) is \( A|B \)-biregular graph with size \( M|N \). For each \( 1 \leq i \leq t \), the number of \( E_i \)-edges is the sum of \( E_i \)-degrees of vertices in \( U \) which is \( \text{offset}(A) \cdot \bar{M} + \alpha_i p_{z_{1,i}} \), for some integer \( z_{1,i} \geq 0 \). This, of course, must equal to the sum of \( E_i \)-degrees of vertices in \( V \) which is \( \text{offset}(B) \cdot \bar{N} + \beta_i p_{z_{2,i}} \), for some integer \( z_{2,i} \geq 0 \).

We now prove the “if” direction. Suppose \( \Phi_{A|B}(M, N) \) holds. Abusing notation as before, we denote the values assigned to the variables \( z_{i,j} \)'s by the variables \( z_{i,j} \)'s themselves. We can rewrite \( \Phi_{A|B}(M, N) \) as follows:

\[
(\text{offset}(A), pI) \cdot \begin{pmatrix} \bar{M}^T \\ \alpha_{1z_{1,1}} \\ \vdots \\ \alpha_{tz_{1,t}} \end{pmatrix} = (\text{offset}(B), pI) \cdot \begin{pmatrix} \bar{N}^T \\ \beta_{1z_{2,1}} \\ \vdots \\ \beta_{tz_{2,t}} \end{pmatrix},
\]
Recall that $I_t$ is the identity matrix with size $t \times t$. Since $\tilde{M}|\tilde{N}$ is big enough for $A|B$, it is immediate that $M|N$ is also big enough for $(\text{offset}(A), p|I_t)|\text{offset}(B), p|I_t)$.  

We are going to construct $A|B$-biregular graph with size $M|N$. There are two cases, with the first case being analogous to the 1-color case, and the other case being where new complications arise.

**Case 1** $\alpha_i z_{1,i} \geq \beta_i z_{2,i}$, for every $1 \leq i \leq t$. We can rewrite Equation (11) into 

$$
(\text{offset}(A), p|I_t) \cdot \left( \begin{array}{c}
\alpha_i z_{1,i} - \beta_i z_{2,i} \\
\vdots \\
\alpha_i z_{1,t} - \beta_i z_{2,t}
\end{array} \right) = \text{offset}(B) \cdot \tilde{N}^r.
$$

Let $\tilde{K} = (\alpha_i z_{1,1} - \beta_1 z_{2,1}, \ldots, \alpha_i z_{1,t} - \beta_1 z_{2,t})$.

Since $\tilde{M}|\tilde{N}$ is big enough for $A|B$, it is immediate that $(\tilde{M}, \tilde{K})|\tilde{N}$ is also big enough for $(\text{offset}(A), p|I_t)|\text{offset}(B)$. By Lemma 21, there is a $(\text{offset}(A), p|I_t)|\text{offset}(B)$-biregular graph $G = (U, V, E_1, \ldots, E_t)$ with size $(\tilde{M}, \tilde{K})|\tilde{N}$. Let $U = U_1 \cup \cdots \cup U_m \cup W_1 \cup \cdots \cup W_t$, where for each $1 \leq i \leq t$:

- for each $1 \leq j \leq m$, the $E_i$-degree of every vertex in $U_j$ is offset($A_{j,i}$), and $|U_j| = M_i$;
- the $E_i$-degree of every vertex in $W_i$ is $p$, and $|W_i| = \alpha_i z_{1,i} - \beta_i z_{2,i}$, and for every $i' \neq i$, the $E_i$-degree of every vertex in $W_i$ is 0.

Observe that if $W_i \neq 0$, i.e., $\alpha_i z_{1,i} - \beta_i z_{2,i} \neq 0$, then $\alpha_i \neq 0$, which by definition, row $i$ in matrix $A$ consists of only periodic entries. For such an $i$ we are going to merge vertices in $W_i$ with vertices in $U$.

Let $w$ be a vertex in $W_i$, where $W_j \neq 0$. The number of vertices in $G$ reachable by $w$ in distance 2 (with any edges) is at most $\delta_2^2 w$, where $\delta_\text{max} = \max(||\text{offset}(A)||, ||\text{offset}(B)||, p)$. Since $\chi(A_{i,*}) \cdot \tilde{M}^r \geq \delta_2^2 + 1$, there is a vertex $u \in U$ which is not reachable from $w$ in distance 2. We can merge $w$ with $u$. We perform such merging for every vertex in $W_i$. Since the $E_i$-degree of every vertex in $W_i$ is $p$, and the $E_{i'}$-degree of vertices in $W_{i'}$ is 0, for every $i' \neq i$, such merging only increases the $E_i$-degree of a vertex in $U$ by $p$. We continue in this way for every $i$ where $W_i \neq 0$ resulting in a $A|B$-biregular graph with size $\tilde{M}|\tilde{N}$.

The case where $\beta_i z_{2,i} \geq \alpha_i z_{1,i}$, for every $1 \leq i \leq t$, can be handled in a symmetrical manner.

**Case 2** There is $1 \leq i, i' \leq t$ such that $\alpha_i z_{1,i} < \beta_i z_{2,i}$ and $\alpha_{i'} z_{1,i'} < \beta_{i'} z_{2,i'}$.

Let $\Gamma$ be the set of $i$ such that $\alpha_i z_{1,i} < \beta_i z_{2,i}$. This means that the following holds.

- For every $i \in \Gamma$, $\alpha_i \neq 0$, i.e., row $i$ in $A$ consists of only periodic entries,
- For every $i \notin \Gamma$, $\alpha_i \neq 0$, i.e., row $i$ in $B$ consists of only periodic entries.

For simplicity, we first assume that for $i \in \Gamma$, offset($A_{i,*}$) does not contain a 0 entry, and for $i \notin \Gamma$, offset($B_{i,*}$) does not contain a 0 entry.

Now, for every $i \in \Gamma$,

$$
0 \leq (\alpha_i z_{1,i} - \beta_i z_{2,i}) p = \text{offset}(B_{i,*}) \cdot \tilde{N}^r - \text{offset}(A_{i,*}) \cdot \tilde{M}^r \leq \delta_\text{max} ||\tilde{N}^r|| - ||\tilde{M}^r||
$$

and, for every $i \notin \Gamma$,

$$
0 < (\beta_i z_{2,i} - \alpha_i z_{1,i}) p = \text{offset}(A_{i,*}) \cdot \tilde{M}^r - \text{offset}(B_{i,*}) \cdot \tilde{N}^r \leq \delta_\text{max} ||\tilde{M}^r|| - ||\tilde{N}^r||.
$$

Combining these two inequalities, we obtain that for every $i \in \Gamma$

$$
(\alpha_i z_{1,i} - \beta_i z_{2,i}) p \leq \delta_\text{max} ||\tilde{M}^r|| - ||\tilde{M}^r|| \leq \delta_\text{max} ||\tilde{N}^r||
$$
We define a matrix
\[ \begin{pmatrix} \beta_{i}z_{1,i} - \alpha_{i}z_{1,i} \end{pmatrix} \]
with size \(M \times N\). These two inequalities state that the "extra" edges due to being periodic are linear in the number of vertices \(\|M\|\) and \(\|\bar{N}\|\). Thus, we distribute them uniformly among the vertices, each will get at most \(\delta_{\max}^{2}\) extra degree, which is constant.

We now formalize this intuition. We define a matrix \(C\).

- For each \(i \in \Gamma\)
  \[ C_{i,*} := \text{offset}(A_{i,*}) + (\delta_{1}, \ldots, \delta_{m})p, \]
  where \(\delta_{1}, \ldots, \delta_{m} \leq \delta_{\max}^{2}\) and \(C_{i,*} \cdot M^{\top} = \text{offset}(A_{i,*}) \cdot M^{\top} + (\alpha_{i}z_{1,i} - \beta_{i}z_{2,i})p\). Such \(\delta_{1}, \ldots, \delta_{m}\) exist due to the fact that \((\alpha_{i}z_{1,i} - \beta_{i}z_{2,i})p \leq \delta_{\max}^{2}\|M^{\top}\|\).
- For each \(i \notin \Gamma\), \(C_{i,*}\) is \(\text{offset}(A_{i,*})\).

We define a matrix \(D\) similarly from \(B\).

- For each \(i \in \Gamma\)
  \[ D_{i,*} := \text{offset}(B_{i,*}) + (\delta_{1}, \ldots, \delta_{m})p, \]
  where \(\delta_{1}, \ldots, \delta_{m} \leq \delta_{\max}^{2}\) and \(D_{i,*} \cdot \bar{N}^{\top} = \text{offset}(B_{i,*}) \cdot \bar{N}^{\top} + (\beta_{i}z_{2,i} - \alpha_{i}z_{1,i})p\). Such \(\delta_{1}, \ldots, \delta_{m}\) exist due to the fact that \((\beta_{i}z_{2,i} - \alpha_{i}z_{1,i})p \leq \delta_{\max}^{2}\|\bar{N}\|\).
- For each \(i \notin \Gamma\), \(D_{i,*}\) is \(\text{offset}(B_{i,*})\).

We see that the entries in \(C\) and \(D\) increase by \(\delta_{\max}^{2}\). Since \(\bar{M}|\bar{N}\) is big enough for \(A|B\), \(\bar{M}|\bar{N}\) is slightly big enough for \(C|D\). Applying Lemma 21, there is a \(C|D\)-biregular graph with size \((\bar{M}, \bar{N})\). Since for every \(i \in \Gamma\), \(A_{i,*}\) consists of only periodic entries whereas for every \(i \notin \Gamma\), \(B_{i,*}\) consists of only periodic entries, a \(C|D\)-biregular graph is also \(A|B\)-biregular.

Now we consider the case when some of the entries in \(\text{offset}(A_{i,*})\) are zero. For \(i \in \Gamma\), observe the equation
\[ \text{offset}(A_{i,*}) \cdot \bar{M}^{\top} + (\alpha_{i}z_{1,i} - \beta_{i}z_{2,i})p = \text{offset}(B_{i,*}) \cdot \bar{N}^{\top}. \]

Now, if there is \(j\) such that \(\text{offset}(A_{i,j}) = 0\), we rewrite the equation as
\[ \bar{a} \cdot \bar{M}^{\top} + z'p = \text{offset}(B_{i,*}) \cdot \bar{N}^{\top}, \]
where \(\bar{a} = (a_{1}, \ldots, a_{m})\) is such that \(a_{j} = 0\), if \(\text{offset}(A_{i,j}) = 0\) and \(a_{j} = \text{offset}(A_{i,j})\), otherwise; and \(z'p = \bar{a} \cdot \bar{M}^{\top} - \text{offset}(A_{i,*}) \cdot \bar{M}^{\top}\). Thus the quantity \(z'p\) represents the extra edges that we need to distributed among the \(\|\bar{M}^{\top}\|\) vertices, which by our proof above is at most \(\delta_{\max}^{2}\|\bar{M}^{\top}\|\).

Of course, if \(z'p < 0\), then there is nothing to prove because the original number of extra edges \((\alpha_{i}z_{1,i} - \beta_{i}z_{2,i})p\) is already small enough that when distributed among \(\|\bar{M}^{\top}\|\) vertices only increase the degree by \(p\). A symmetrical argument can be made when \(i \notin \Gamma\). This completes our proof of Lemma 22.

\begin{remark}
- It is only in the proof of Lemma 22 that we require the quantity \(\chi(\text{offset}(A_{i,*})) \cdot \bar{M}^{\top} + \chi(\text{offset}(B_{i,*})) \cdot \bar{N}^{\top}\) to be at least quartic on \(\delta_{\max}\), and not quadratic as in Section 4. This is due to the fact that the entries in the matrices \(C\) and \(D\) in case 2 above increase by \(\delta_{\max}^{2}\).
\end{remark}
Note also that the requirement that \( A \) and \( B \) are simple matrices allows us to “distribute” the extra edges \((\alpha_{i1,1} - \beta_{i2,1})p\) uniformly among \(|M^r|\) vertices.

The previous lemmas give formulas that capture the existence of biregular graphs for big enough sizes. We now turn to sizes that are not big enough. As in the 1-color case, this means that one of the conditions (a), (b) or (c) is violated, and when condition (a) is violated we can work via brute force enumeration of cardinalities. Thus we focus on the case when condition (b) is violated, with the case of condition (c) being violated argued symmetrically.

**Encoding of not “big enough” components in a Presburger formula.** Note that when (b) is violated, there is \( 1 \leq i \leq t \), \( \chi(A_{i,*}) \cdot \bar{M}^r \leq \delta_{\text{max}}^2 \). This means that the sum of the components in \( \bar{M} \) that corresponds to the non-zero components in \( A_{i,*} \) is at most \( \delta_{\text{max}}^2 \). We can fix this sum to be \( r \leq \delta_{\text{max}}^2 \), and these \( r \) vertices can then be encoded inside Presburger formula, as in Lemma 11.

Let \( A_0 \in \mathbb{N}^{t \times k}, B \in \mathbb{N}_n^{t \times l} \), and \( C \in \mathbb{N}_m^{t \times m} \), where all of them do not contain any \( \infty \) entry. We also assume that none of them have columns with only zero entries. We will construct a Presburger formula that gives us the sizes \((\bar{M}_0, \bar{M}_1)|\bar{N}\) of \((A,B)|C\)-biregular graphs when \( \bar{M}_1 \cdot \bar{1} \) is a fixed integer \( r \geq 0 \) and \( \bar{M}_0|\bar{N} \) is big enough for \( A|C \).

Let \( r \geq 0 \) be an integer and \( \bar{x}_0, \bar{x}_1, \bar{y} \) be vectors of variables with lengths \( k, l, m \), respectively. Define formula \( \Phi^r_{(A,B)|C}(\bar{x}_0, \bar{x}_1, \bar{y}) \) inductively on \( r \) as follows.

- When \( r = 0 \),
  \[
  \Phi^0_{(A,B)|C}(\bar{x}_0, \bar{x}_1, \bar{y}) := \bar{x}_1 \cdot \bar{1} = 0 \wedge \Phi_{A|C}(\bar{x}_0, \bar{y}).
  \]

  where \( \Phi_{A|C}(\bar{x}_0, \bar{y}) \) is as defined in Eq. 10.

- When \( r \geq 1 \),
  \[
  \Phi^r_{(A,B)|C}(\bar{x}_0, \bar{x}_1, \bar{y}) := \exists s_1 \ldots \exists s_t \exists z_1 \ldots \exists z_t \bigg( \bigg( \bigwedge_{j=1}^t (s_j = 0) \wedge \bar{y} = \sum_{i=0}^t z_i \wedge \bigwedge_{j=1}^t s_j \neq \infty \wedge \bar{x}_i \cdot \bar{1} = \text{offset}(B_{i,j}) + \alpha_i ps_i \bigg) \wedge \Phi^r_{(A,B)|(|C,C-J_1,\ldots,C-J_t)}(\bar{x}_0, \bar{x}_1 - e_i, \bar{z}_1, \bar{z}_1, \ldots, \bar{z}_t) \bigg).
  \]

  Here each \( \alpha_i \) is a constant in \( \{0,1\} \) with \( \alpha_i = 1 \) if and only if \( B_{i,j} \) is a periodic entry.

  Also each \( J_i \) is a matrix with size \( (t \times m) \) where row \( i \) consists of all 1 entries and all the other rows have only 0 entries.

We will show that the formula \( \Phi^r_{(A,B)|C}(\bar{x}_0, \bar{x}_1, \bar{y}) \) captures the sizes of \((A,B)|C\)-biregular graphs when \( \bar{x}_1 \cdot \bar{1} = r \), as stated below.

**Lemma 24.** For every \( A,B,C \), for every integer \( r \geq 0 \) and every \( \bar{M}_0, \bar{M}_1, \bar{N} \) such that \( \bar{M}_0|\bar{N} \) is big enough for \( A|C \) and \( ||M^r|| = r \), the following holds.

\[
\Phi^r_{(A,B)|C}(\bar{M}_0, \bar{M}_1, \bar{N}) \text{ holds if and only if there is a } (A,B)|C\text{-biregular graph with size } (\bar{M}_0, \bar{M}_1)|\bar{N} \text{ and } ||M^r|| = r.
\]

The proof of Lemma 24 is a routine adaptation of the proof of Lemma 11 hence, omitted. To conclude this part, by combining all the cases described above, we can define a Presburger formula \( \text{bireg}_{A,B}(\bar{x}, \bar{y}) \) that characterizes the sizes of \( A|B \)-biregular graphs where \( A \) and \( B \) are simple matrices that do not contain an \( \infty \) entry.
When the simple matrices do contain an $\infty$ entry. Now we will consider the case when the matrices contain an $\infty$ entry. We fix two matrices $A \in \mathbb{N}_{\infty}^{m \times n}$ and $B \in \mathbb{N}_{\infty}^{m \times n}$. The main idea will be to reduce to the case without an $\infty$ entry that we have already handled. We will do this naively by deleting all rows involving $\infty$. Then we will post-process the resulting graphs to deal with the rows we had deleted. This post-processing is conceptually simple, although it requires a lengthy case analysis; the intuition is that the presence of $\infty$ in a row heavily constrains the behavior.

We now formalize this idea. Let $\tilde{A}$ and $\tilde{B}$ be matrices obtained by deleting row $i$ in both $A$ and $B$ if either $A$ or $B$ contains an $\infty$ entry in row $i$. Note that $\tilde{A}$ and $\tilde{B}$ have the same number of rows and that both of them do not have any $\infty$ entry.

We have to refine the notion of big enough once more.

Definition 25. Let $\tilde{M} = (M_1, \ldots, M_m) \in \mathbb{N}_{\infty}^m$ and $\tilde{N} = (N_1, \ldots, N_n) \in \mathbb{N}_{\infty}^n$. We say that $\tilde{M}|\tilde{N}$ is big enough for $A|B$, if the following holds:

- $M|N$ is big enough for $\tilde{A}|\tilde{B}$;
- if $A_{i,j} = \infty$ and $M_j \neq 0$, then $M_j \geq \delta_0$;
- if $B_{i,j} = \infty$ and $N_j \neq 0$, then $N_j \geq \delta_0$.

Here $\delta_0$ is the sum of all the offsets of entries in $A$ and $B$ that are not $\infty$.

The motivation for the second item is that it allows us to construct a graph with the property that: if $|E_i| = \infty$, every vertex with finite $E_i$-degree is only $E_i$-adjacent to vertices with infinite $E_i$-degree. The choice of the sum of all offsets of non-infinite entries is arbitrary, but it is sufficient for our purpose. The motivation of the third item is analogous.

Lemma 26. Suppose that $\tilde{M}|\tilde{N}$ is big enough for $A|B$, and that in addition:

- $\Phi_{A|B}(\tilde{M}, \tilde{N})$ holds, where the formula $\Phi_{A|B}$ is as defined in Eq. \[40\].
- If $A_{i,j} = \infty$ and $M_j \neq 0$, then $\chi(B_{i,j})\tilde{N}^T = \infty$.
- Likewise, if $B_{i,j} = \infty$ and $N_j \neq 0$, then $\chi(A_{i,j})\tilde{M}^T = \infty$.

Then, there is a $A|B$-biregular graph with size $\tilde{M}|\tilde{N}$.

Proof. The intuitive meaning of the hypothesis is as follows. The formula $\Phi_{A|B}(\tilde{M}, \tilde{N})$ simply identifies two ways of counting the number of edges. The second item means that if there is a vertex with $\infty E_i$-degree in $U$, then there must be infinitely many vertices in $V$ that are incident to $E_i$-edge. The meaning of the third item is similar for $V$.

Let $\tilde{M} = (M_1, \ldots, M_m)$ and $\tilde{N} = (N_1, \ldots, N_n)$. Suppose $\tilde{M}|\tilde{N}$ be as in the hypothesis. By definition, for some finite $z_{1,1}, \ldots, z_{1,t}, z_{2,1}, \ldots, z_{2,t}$, we have

$$\text{offset}(A) \cdot \vec{x}^T + \begin{pmatrix} \alpha_1 p_{z_{1,1}} \\ \vdots \\ \alpha_t p_{z_{1,t}} \end{pmatrix} = \text{offset}(B) \cdot \vec{y}^T + \begin{pmatrix} \beta_1 p_{z_{2,1}} \\ \vdots \\ \beta_t p_{z_{2,t}} \end{pmatrix}.$$ 

Recall that $\alpha_i = 1$, if row $i$ in $A$ consists of non-fixed entries and is 0 otherwise; and similarly $\beta_i = 1$ if row $i$ in $B$ consists of non-fixed entries and is 0 otherwise.

For $i$ such that $\text{offset}(A_{i,*})\tilde{M}^T = \infty$, we can assume that $z_{1,i} = z_{2,i} = 0$, since the terms $\alpha_i p_{z_{1,i}}$ and $\beta_i p_{z_{2,i}}$, which are finite, become redundant. Similarly for $i$ such that $\text{offset}(B_{i,*})\tilde{N}^T = \infty$.

We are going to construct a $A|B$-biregular graph $G = (U, V, E_1, \ldots, E_t)$ with size $\tilde{M}|\tilde{N}$.

First, assume that $\tilde{A}$ and $\tilde{B}$ have $\ell_0 - 1$ rows. We can also assume that the first $\ell_0 - 1$ rows of $A$ and $B$ are $\tilde{A}$ and $\tilde{B}$, respectively.
Note that since $\Phi_{\hat{A}\hat{B}}(\hat{M}, \hat{N})$ holds, $\Phi_{\hat{A}\tilde{B}}(\hat{M}, \tilde{N})$ also holds. Thus, by Lemma 22, there is an $\tilde{A}\hat{B}$-biregular graph $G = (U, V, E_1, \ldots, E_{\ell_0-1})$ with size $\hat{M}|\tilde{N}$. Let $U = U_1 \cup \cdots \cup U_m$ and $V = V_1 \cup \cdots \cup V_n$ be the witness partition. We now define the edges $E_{\ell_0}, \ldots, E_t$ in $G$.

(For vertices in $U_j$, where $|U_j|$ is finite) For every $\ell_0 \leq i \leq t$, we first assign $E_i$-edges that are adjacent to some vertex $u \in U_j$, where $|U_j|$ is finite to make the $E_i$-degree of $u$ become $A_{i,j}$, whenever $A_{i,j} \neq \infty$.

Let $H := \{\ell_0 \leq i \leq t : A_{i,j} \neq \infty\}$, i.e., the set of indices between $\ell_0$ and $t$ where $A_{i,j} \neq \infty$.

We further partition $H$ into $H_0 \cup H_1$, where $H_0$ contains all the indices $i$ where $A_{i,j}$ contains an $\infty$ entry and $H_1 = H - H_0$.

For the colors in $H_0$, we do the following.

Let $H_0 = \{i_1, \ldots, i_k\}$, and let $j_1, \ldots, j_k$ be the indexes such that $B_{i_1,j_1} = \infty$.

Let $S_{i_1}, \ldots, S_{i_k}$ be pairwise disjoint sets of vertices from $V_{j_1}, \ldots, V_{j_k}$, respectively such that the cardinality $|S_{i_k}| = A_{i_k,j_k}$, and none of the vertices in $S_{i_1}, \ldots, S_{i_k}$ are adjacent to $u$ via any $E_{1}, \ldots, E_{\ell_0-1}$-edges. Recall that $j$ is such that $u \in U_j$.

It is possible to pick those sets. Indeed, because of the big enough assumption, we have $|V_{j_i}| = N_{j_i} \geq \delta_0$, which is the sum of all finite entries in $A$ and $B$, thus, bigger than the sum $\sum_{i \in H} \text{offset}(A_{i,j})$.

We connect $u$ to every vertex in $S_{i_k}$ via $E_{i_k}$-edges.

For the colors not in $H_0$, we do the following.

Let $H_1 = \{i_1, \ldots, i_k\}$, and let $j_1, \ldots, j_k$ be the indexes such that $|V_{j_i}| = \infty$.

Note that for each $i \geq \ell_0$, the terms offset$(A_{i,j}) M^j$ and offset$(B_{i,j}) N^j$ are both infinite.

Thus, for $i_k \in H_1$, there should some $j_k$ such that $V_{j_k}$ is infinite set.

Let $S_{i_1}, \ldots, S_{i_k}$ be pairwise disjoint sets of vertices from $V_{j_1}, \ldots, V_{j_k}$, respectively such that the cardinality $|S_{i_k}| = A_{i_k,j_k}$, and none of the vertices in $S_{i_1}, \ldots, S_{i_k}$ are adjacent to $u$ via any of the edges defined so far, i.e, $E_{1}, \ldots, E_{\ell_0-1}$-edges and $E_t$-edges where $i \in H_0$.

It is also possible to pick sets with the required property above, since each $V_{j_i}$ contains infinitely many elements and the sum $\sum_{i \in H} \text{offset}(A_{i,j})$ is finite.

We connect $u$ to every vertex in $S_{i_k}$ via $E_{i_k}$-edges.

After performing this step, for every $\ell_0 \leq i \leq t$, when $A_{i,j} \neq \infty$, the $E_i$-degree of $u$ is $A_{i,j}$, for every $u \in U_j$.

Now, we show how to assign $E_i$-edges adjacent to $u \in U_j$ when $A_{i,j} = \infty$.

Let $H := \{i : A_{i,j} = \infty\}$.

By the hypothesis of our lemma, for every $i \in H$, $\chi(B_{i,j}) N^i = \infty$, hence, there is $t$ such that $N_t = \infty$.

For ease of reference, let $H = \{i_1, \ldots, i_k\}$ and let $j_1, \ldots, j_k$ be such that $V_{j_1}, \ldots, V_{j_k}$ are all infinite finite and $\chi(B_{i_1,j_1}), \ldots, \chi(B_{i_k,j_k})$ are all 1.

Let $S_{i_1}, \ldots, S_{i_k}$ be pairwise disjoint sets of vertices from $V_{j_1}, \ldots, V_{j_k}$, respectively such that the cardinality $|S_{i_k}| = \infty$, and none of the vertices in $S_{i_1}, \ldots, S_{i_k}$ are adjacent to $u$ via any $E_t$-edges that are already defined.

Such sets exist because each $V_{j_i}$ contains infinitely many elements and the number of already-defined $E_t$-edges adjacent to $u$ is finite.

We connect $u$ to every vertex in $S_{i_k}$ via $E_{i_k}$-edges.

Note that after this step, every vertex $u$ in a finite $U_j$ has degree precisely $A_{i,j}$.

(For vertices in $V_j$, where $|V_j|$ is finite) This case can be treated symmetrically to the previous one.
(For vertices in $U_j$, where $|U_j|$ is infinite) After the two steps above, we note that vertices in the sets $U_j$ or $V_j$ that are infinite satisfy the following conditions:

- For every $\ell_0 \leq i \leq t$, every vertex $u \in U_j$, where $U_j$ is infinite, is adjacent to at most one $E_i$-edge, for some $\ell_0 \leq i \leq t$, i.e., $\sum_{i=\ell_0}^t \deg_{E_i}(u) \leq 1$.
  Similarly for every vertex in $V_j$, where $V_j$ is infinite.

- For every vertex $u \in U_j$, where $U_j$ is infinite, if $\sum_{i=\ell_0}^t \deg_{E_i}(u) = 1$, for some $\ell_0 \leq i \leq t$, then $A_{1,j} \neq 0$.
  Similarly for every vertex in $V_j$, where $V_j$ is infinite.

Recall that $\deg_{E_i}(u)$ denotes the $E_i$-degree of vertex $u$.

We will now assign $E_i$-edges adjacent to vertices in the infinite sets $U_j$ and $V_j$. The assignment is done as follows. We pick a vertex $u \in U_j$ whose degree is not $A_{*,j}$. We assign $E_{\ell_0}, \ldots, E_{t_i}$-edges adjacent to $u$ as follows.

1. Let $H = \{ \ell_0 \leq i \leq t : \text{there is } j \text{ s.t. } |V_j| = \infty \text{ and } B_{i,j} \neq 0 \}$. 
2. We will first assign $E_i$-edges when $i \in H$. 
- Let $H = \{ i_1, \ldots, i_k \}$, and let $j_1, \ldots, j_k$ be such that $V_{j_1}, \ldots, V_{j_k}$ are all infinite and $B_{i_1,j_1}, \ldots, B_{i_k,j_k}$ are all non-zero.
- Let $S_{i_1}, \ldots, S_{i_k}$ be pairwise disjoint sets of vertices from $V_{j_1}, \ldots, V_{j_k}$, respectively such that the cardinality $|S_{i_k}| = A_{i_1,j_1}$ and none of the vertices in $S_{i_1}, \ldots, S_{i_k}$ are adjacent to $u$ via any $E_i$-edges that are already defined.
  As above, it is possible to choose such sets, since each $V_{j_i}$ contains infinitely many elements and the number of already-defined $E_i$-edges adjacent to $u$ is finite. In this step we also insist that $V_{j_1} - (S_{i_1} \cup \cdots \cup S_{i_k})$ is still infinite, for every $j_i \in \{ j_1, \ldots, j_k \}$, which is still possible, since $V_{j_i}$ is infinite.
- We connect $u$ to every vertex in $S_{i_k}$ via $E_{i_k}$-edges.

3. Next, we assign $E_i$-edges when $i \notin H$.
- Note that for every $i \notin H$, since $|U_j| = \infty$, there is $j$ such that $B_{i,j} = \infty$ and $N_{i,j} \neq 0$ is finite. Moreover, if $i \notin H$, $A_{i,j} \neq \infty$. Otherwise, if $A_{i,j} = \infty$, by the hypothesis of our lemma, $\chi(B_{i,\ast}) \chi^T = \infty$, thus, there must be some $j'$ such that $V_{j'} = \infty$ and $N_{i,j'} \neq 0$.
- Let $i_1, \ldots, i_k$ be the indices not in $H$, and let $j_1, \ldots, j_k$ be the indexes such that $B_{i_1,j_1}, \ldots, B_{i_k,j_k}$ are all $\infty$ and $N_{i_1,j_1}, \ldots, N_{i_k,j_k} \neq 0$.
- Let $S_{i_1}, \ldots, S_{i_k}$ be pairwise disjoint sets of vertices from $V_{j_1}, \ldots, V_{j_k}$, respectively such that the cardinality $|S_{i_k}| = \text{offset}(A_{i_1,j_1})$, and none of the vertices in $S_{i_1}, \ldots, S_{i_k}$ are adjacent to $u$ via any $E_i$-edges that are already defined.
  We can reason as before that such sets exist. Because of the big enough assumption we know $|V_{j_1}| = N_{i,j} \geq \delta_0$, which is the sum of all finite entries in $A$ and $B$ and is therefore bigger than the sum $\sum_{i \in H \cup \{ 1, \ldots, t_0 \}} \text{offset}(A_{i,j})$.
- We connect $u$ to every vertex in $S_{i_k}$ via $E_{i_k}$-edges.

We can define symmetrical steps for every vertex $v \in V_j$ where $V_j$ is infinite. We iterate the steps (1)–(3) above infinitely often. Note that on each iteration, the following invariant holds. For every $\ell_0 \leq i \leq t$, every vertex $u \in U_j$, where $U_j$ is infinite, either:

- the degree of $u$ is precisely $A_{*,j}$, or
- $u$ is adjacent to at most one $E_i$-edge, for some $\ell_0 \leq i \leq t$, i.e., $\sum_{i=\ell_0}^t \deg_{E_i}(u) \leq 1$.
  Moreover, in this case, if $\deg_{E_i}(u) = 1$, $A_{1,j} \neq 0$.

The same invariant holds for every vertex $v \in V_j$, where $V_j$ is infinite. This completes our proof of Lemma 26.
Two variable logic with ultimately periodic counting

Remark 27. The graph \( G = (U, V, E_1, \ldots, E_t) \) constructed in the proof of Lemma 26 above satisfies the following condition.

- If \( |V| = \infty \), then for every vertex \( u \in U \), there are infinitely many vertices in \( V \) that are not adjacent to \( u \) via any edge.
- Likewise, if \( |U| = \infty \), then for every vertex \( v \in V \), there are infinitely many vertices in \( U \) that are not adjacent to \( v \) via any edge.

- Since the construction of \( G \) is via iteration on each vertex in \( U \cup V \), we can deduce that if \( |V| = \infty \), then for every finite subset \( U' \subseteq U \), there are infinitely many vertices in \( V \) that are not adjacent to any vertex in \( U' \) via any edge.
- Likewise, if \( |U| = \infty \), then for every finite subset \( V' \subseteq V \), there are infinitely many vertices in \( U \) that are not adjacent to any vertex in \( V' \) via any edge.

Obviously, the hypothesis in Lemma 26 is a necessary condition for the existence of an \( A|B \)-biregular graph with size \( M|N \), regardless of the whether \( M|N \) is big enough: see a brief explanation in the first paragraph of proof of Lemma 26.

For the sizes that are not big enough, we can use a similar encoding technique as in Lemma 23. Thus, this concludes the proof of Lemma 26.

B.2.2 Enforcing a completeness restriction on simple biregular graphs

We will now consider the formula defining possible partition sizes for complete biregular graphs. As in the 1-color case, this will be done by reduction to the case where the completeness restriction has not been enforced.

We introduce a further restriction on the matrices that will be useful.

Definition 28. For a pair of simple matrices \( A|B \) (with the same number of rows), we say that \( A|B \) is a good pair, if there is \( i \) such that row \( i \) in both \( A \) and \( B \) are periodic rows.

Here we should remark that \( A \) and \( B \) may contain \( \infty \) entry, and it is useful to recall that \( \infty \) entry is regarded as periodic entry.

Note that if \( A|B \) is not a good pair, then complete \( A|B \)-biregular graphs can only have vertices up to \( 2\delta_{\max} \), where \( \delta_{\max} \) is the maximal non-infinite entry in \( A \) and \( B \). Indeed, suppose \( G = (U, V, E_1, \ldots, E_t) \) is a complete \( A|B \)-biregular graph. Since \( A|B \) is not a good pair, for each \( 1 \leq i \leq t \), \( |E_i| \) is at most \( \delta_{\max}|U| \) or \( \delta_{\max}|V| \). Thus, \( \sum_{i=1}^{t} |E_i| \) is at most \( \delta_{\max}(|U| + |V|) \). On the other hand, the fact that \( G \) is complete implies that \( \sum_{i=1}^{t} |E_i| = |U||V| \) which is strictly bigger than \( \delta_{\max}(|U| + |V|) \), when \( |U| + |V| > 2\delta_{\max} \).

So we now are ready to define the formula \( c\text{-bireg}_{A|B}(\bar{x}, \bar{y}) \) for a good pair \( A|B \). The formula is similar to that in the 1-color case presented in the body of the paper. Let \( \bar{x} = (x_1, \ldots, x_m) \) and \( \bar{y} = (y_1, \ldots, y_n) \). Let \( A \in \mathbb{N}^{t \times m}_{\infty, +p} \) and \( B \in \mathbb{N}^{t \times n}_{\infty, +p} \) be simple matrices.

We define \( c\text{-bireg}_{A|B}(\bar{x}, \bar{y}) \) as the conjunction of \( \text{bireg}_{A|B}(\bar{x}, \bar{y}) \) with the following constraints.

\( (C1) \) For every \( 1 \leq j \leq m \), if \( x_j \neq 0 \), then \( \|\bar{y}\| = \|\text{offset}(A_{*, j})\| + \alpha p z \), for some finite \( z \).

Here \( \alpha \) is a constant in \( \{0, 1\} \) where \( \alpha = 1 \) if and only if \( A_{*, j} \) contains a periodic entry.

\( (C2) \) For every \( 1 \leq j \leq n \), if \( y_j \neq 0 \), then \( \|\bar{x}\| = \|\text{offset}(B_{*, j})\| + \alpha p z \), for some finite \( z \). Again, here \( \alpha \) is now a constant in \( \{0, 1\} \) with \( \alpha = 1 \) if and only if \( B_{*, j} \) contains a periodic entry.

Constraint \( (C1) \) states that the number of vertices on the right side must equal the degree of the vertices on the left side. Note that if \( A \) contains \( \infty \), i.e., some vertex on the left side has \( \infty \) degree, then the number of vertices on the right side must be infinite. Constraint \( (C2) \) states symmetrical meaning.
Lemma 29. For good simple matrices $A$ and $B$, $c$-bireg$_{A|B}(M, N)$ holds exactly when there is a complete biregular graph of size $M|N$.

Proof. That $c$-bireg$_{A|B}(M, N)$ holds is a necessary condition for the existence of $A|B$-biregular graph is pretty straightforward. This follows from the fact that if $G = (U, V, E_1, \ldots, E_t)$ is a complete $A|B$-biregular graph then the sum of all $E_i$-degrees of every vertex in $U$ must equal to $|V|$, and likewise, the sum of all $E_i$-degree of every vertex in $V$ must equal to $|U|$.

Now we show that it is also a sufficient condition. Suppose $c$-bireg$_{A|B}(M, N)$ holds. Thus, bireg$_{A|B}(M, N)$ holds, which implies there is a (not necessarily complete) $A|B$-biregular graph $G = (U, V, E_1, \ldots, E_t)$ with size $M|N$. Let $U_1, \ldots, U_m$ and $V_1, \ldots, V_n$ be the witness partition. Since $A|B$ is a good pair, there is $i$ such that row $i$ in both $A$ and $B$ are periodic rows.

We will now show how to make $G$ complete. There are a few cases to consider.

(Case 1) Both $U$ and $V$ are finite.

In this case, for every $(u, v) \notin E_1 \cup \cdots \cup E_t$, we define $(u, v)$ to be in $E_t$. To show that $G$ is still $A|B$-biregular, assume that $u \in U_j$ Note that constraint (C1) forces $|V|$ to be $[\text{offset}(A_{u,j})]^{+p}$. Since the degree of $u$ in the original $G$ is already $[\text{offset}(A_{u,j})]^{+p}$, we are simply adding a multiple of $p$ extra $E_i$-degrees to $u$. Likewise, constraint (C2) forces every vertex $v \in V_j$ to be $[\text{offset}(B_{v,j})]^{+p}$.

(Case 2) When $U$ is infinite and $V$ is finite.

Suppose $(u, v) \notin E_1 \cup \cdots \cup E_t$.

We pick $v_1, \ldots, v_{p-1}$ vertices from $V$ that are not adjacent to $u$ via any $E_i$-edges. Note that constraint (C1) states that $|V| = [\text{offset}(A_{u,j})]^{+p}$. Since degree of $u$ is already $[\text{offset}(A_{u,j})]^{+p}$, and $G$ is not complete, the number of vertices in $V$ that are not adjacent to $u$ must be a multiple of $p$. Since $u$ and $v$ are not adjacent, there must be at least $(p - 1)$ other vertices in $V$ that are not adjacent to $u$.

By Remark 27, there are vertices $u_1, \ldots, u_{p-1}$ from $U$ that are not adjacent to any of $v, v_1, \ldots, v_{p-1}$ via any $E_i$-edges. We set all the pairs in $\{u, u_1, \ldots, u_{p-1}\} \times \{v, v_1, \ldots, v_{p-1}\}$ to be in $E_t$. We iterate the process above for every $(u, v) \notin E_1 \cup \cdots \cup E_t$.

The case when $U$ is finite and $V$ is infinite is symmetric.

(Case 3) When both $U$ and $V$ are infinite.

Suppose $(u, v) \notin E_1 \cup \cdots \cup E_t$. If $E_i$-degree of $u$ and $v$ are $\infty$, then we simply set $(u, v)$ to be an $E_i$-edge.

Otherwise, we can apply a similar technique as in (case 2), where we pick vertices $u_1, \ldots, u_{p-1}$ and $v_1, \ldots, v_{p-1}$ and declare all pairs in $\{u, u_1, \ldots, u_{p-1}\} \times \{v, v_1, \ldots, v_{p-1}\}$ to be in $E_t$.

This completes our proof of Lemma 29.

To conclude this section, we can state the following lemma, the complete biregular analog of Lemma 20.

Lemma 30. For each simple matrix $A \in \mathbb{N}^{t \times m}_{\infty +p}$ and $B \in \mathbb{N}^{t \times m}_{\infty +p}$, there exists an (effectively computable) existential Presburger formula $c$-bireg$_{A|B}(\bar{x}, \bar{y})$ such that for every $(M, N) \in \mathbb{N}^{m}_{\infty} \times \mathbb{N}^{m}_{\infty}$, the following holds. There is a complete $A|B$-biregular graph with size $M|N$ if and only if $c$-bireg$_{A|B}(M, N)$ holds in $\mathbb{N}^{m}_{\infty}$.

B.3 The case of non-simple matrices

In this subsection we will show to decompose an arbitrary $A|B$-biregular graph into a collection of simple biregular graphs, i.e., where the matrices are all simple. We will first
present the intuition.

Let $G = (U, V, E_1, \ldots, E_t)$ be a $A|B|$-biregular graph, where $A \in \mathbb{N}^{t \times p}_{\infty}$ and $B \in \mathbb{N}^{t \times n}_{\infty}$. Let $U = U_1 \cup \cdots \cup U_m$ and $V = V_1 \cup \cdots \cup V_n$ be the witness partition. For each $1 \leq j \leq m$ and $1 \leq j' \leq n$, we define $G_{j, j'}$ as the subgraph of $G$ restricted to the vertices in $U_j \cup V_{j'}$. We will argue that for each $g$ graph, and let $A$, and for each $E_{j'}$ function of $C$ choices" is also fixed. the guesses on outdegrees into the different case where the constraint given by restrictions on the guesses we can make for different choices of consider constraints on adjacent edges for elements of $V$ be periodic. Similarly, if the constraint is of the classes, with each associated to a different constraint on outgoing edges in $G_{j, j'}$. But all of these constraints will be of fixed degree. Now conversely suppose the constraint is of the form $a^+p$ for some $a$. Again we can partition the $U_j$ according to whether the outdegree to elements of $V_j$ is $b^+p$ for $b \leq a$. All of the constraints for different partition classes will be periodic. Similarly, if the constraint is $\infty^p$, we partition $U_j$ according to whether the outdegree to elements of $V_j$ is $\infty^+p$, or $0^+p, \ldots, p - 1^+p$. The situation is the same if we consider constraints on adjacent edges for elements of $V_j$. Of course, there are additional restrictions on the guesses we can make for different choices of $j$ and $j'$. For example in the case where the constraint given by $A$ for $U_j$ is a fixed element $r$, we need to be sure that the guesses on outdegrees into the different $V_j$ add to one. But the number of these “global choices” is also fixed.

We now present the formalization of this idea. For a column vector $C \in \mathbb{N}^{t \times n}_{\infty \times +p}$, a behavior function of $C$ towards $n$ partitions is a function $g : [t] \times [n] \rightarrow \mathbb{N}_{\infty, +p}$ such that:

$$
\begin{pmatrix}
\sum_{j=1}^{n} g(1,j) \\
\vdots \\
\sum_{j=1}^{n} g(t,j)
\end{pmatrix} = C
$$

and for each $1 \leq i \leq t$ the following holds.

- If $C_{i,1}$ is periodic, then for each $1 \leq j \leq n$, $g(i,j)$ is periodic.
- If $C_{i,1}$ is fixed, then for each $1 \leq j \leq n$, $g(i,j)$ is fixed.

We define by $G[C, n]$ the set of all behavior function of $C$ towards $n$ partitions. For each $C$, for each $1 \leq j \leq n$, we define the following matrix $M[C, n, j]$:

$$
M(C, n, j) := \begin{pmatrix}
g_1(1,j) & g_2(1,j) & \cdots & g_k(1,j) \\
g_1(2,j) & g_2(2,j) & \cdots & g_k(2,j) \\
\vdots & \vdots & \ddots & \vdots \\
g_1(t,j) & g_2(t,j) & \cdots & g_k(t,j)
\end{pmatrix},
$$

where $g_1, \ldots, g_k$ are all the behavior functions of $C$ towards $n$ partitions.

It is not difficult now to show the following. Let $G = (U, V, E_1, \ldots, E_t)$ be $A|B$-biregular graph, and let $U = U_1 \cup \cdots \cup U_m$ and $V = V_1 \cup \cdots \cup V_n$ be any partitions.
If $G$ is $A|B$-biregular with witness partition $U = U_1 \cup \cdots \cup U_m$ and $V = V_1 \cup \cdots \cup V_n$, then for every $1 \leq i \leq m$ and $1 \leq j \leq n$, the subgraph $G_{i,j}$ is $M(A_{i,i}, n, j)|M(B_{j,j}, m, i)$-biregular.

If for every $1 \leq i \leq m$ and $1 \leq j \leq n$, the graph $G_{i,j}$ is $M(A_{i,i}, n, j)|M(B_{j,j}, m, i)$-biregular, then $G$ is $A|B$-biregular graph with witness partition $U = U_1 \cup \cdots \cup U_m$ and $V = V_1 \cup \cdots \cup V_n$.

From here, for $A \in \mathbb{N}_{\infty,+}^{t \times m}$ and $B \in \mathbb{N}_{\infty,+}^{t \times n}$, the Presburger formula for an arbitrary $A|B$-biregular graph can be defined by taking the conjunction of formulas for the simple case. Formally we proceed as follows.

For each $1 \leq i \leq m$, for each behavior function $g$ of $A_{i,i}$ towards $n$ partitions, we have a variable $X_{i,g}$.

Let $\bar{X}_i = (X_{i,g_1}, \ldots, X_{i,g_k})$, where $g_1, \ldots, g_k$ are all the behavior functions of $A_{i,i}$ towards $n$ partitions.

Similarly, for each $1 \leq i \leq n$, for each behavior function $g$ of $B_{i,i}$ towards $m$ partitions, we have a variable $Y_{i,g}$, and we can let $\bar{Y}_i = (Y_{i,g_1}, \ldots, Y_{i,g_k})$, where $g_1, \ldots, g_k$ are all the behavior functions of $B_{i,i}$ towards $m$ partitions.

The formula $\text{bireg}_{A|B}(\bar{x}, \bar{y})$ is defined as follows.:

\[
\text{bireg}_{A|B}(\bar{x}, \bar{y}) := \exists \bar{X}_1 \cdots \exists \bar{X}_m \exists \bar{Y}_1 \cdots \exists \bar{Y}_n \bigwedge_{i=1}^{n} \bigwedge_{j=1}^{n} \text{bireg}_{M(A_{i,i}, n, j)|M(B_{j,j}, m, i)}(\bar{X}_i, \bar{Y}_j) \wedge \bar{x} = (\bar{X}_1 \cdot \bar{1}, \ldots, \bar{X}_m \cdot \bar{1}) \wedge \bar{y} = (\bar{Y}_1 \cdot \bar{1}, \ldots, \bar{Y}_n \cdot \bar{1}).
\]

Intuitively, $X_{i,g}$ represents the number of vertices in $U_i$ with behavior function $g$.

This formula can then be modified for the complete case as follows:

\[
\text{c-bireg}_{A|B}(\bar{x}, \bar{y}) := \exists \bar{X}_1 \cdots \exists \bar{X}_m \exists \bar{Y}_1 \cdots \exists \bar{Y}_n \bigwedge_{i=1}^{n} \bigwedge_{j=1}^{n} \text{c-bireg}_{M(A_{i,i}, n, j)|M(B_{j,j}, m, i)}(\bar{X}_i, \bar{Y}_j) \wedge \bar{x} = (\bar{X}_1 \cdot \bar{1}, \ldots, \bar{X}_m \cdot \bar{1}) \wedge \bar{y} = (\bar{Y}_1 \cdot \bar{1}, \ldots, \bar{Y}_n \cdot \bar{1}).
\]

### B.4 Construction of the Presburger formula for complete regular digraphs

We will now present the proof for regular digraphs. We will first consider the formula for (non-complete) regular digraph. Like the formula for biregular graphs, the formula for the complete digraph can be obtained by adding the constraint that the number of vertices on one side must equal the degree of the vertices on the other side.

As mentioned in the body, a digraph $G$ can be viewed as a bipartite graph $G'$ by splitting every vertex $u$ in $G$ into two vertices $u_1$ and $u_2$ adjacent to all the incoming and outgoing edges, respectively. See Figure 2. Thus, it is straightforward that if $G$ is $A|B$-regular digraph with size $\bar{M}$, such vertex splitting will result in $A|B$-biregular graph with size $\bar{M}|\bar{M}$.

Now we will show the converse, i.e., when $\bar{M}|\bar{M}$ is big enough for $A|B$, if $\text{bireg}_{A|B}(\bar{M}, \bar{M})$ holds, then there is $A|B$-regular digraph with size $\bar{M}$. As shown in Subsection B.3, it suffices to consider simple matrices $A$ and $B$.

The idea is to extend the argument in [12 Section 8], where the same result was proven when both matrices $A$ and $B$ contain only fixed entries. We will explain briefly the idea. Let $A$ and $B$ contain only fixed entries. Suppose $\text{bireg}_{A|B}(\bar{M}, \bar{M})$ holds. Thus, there is
Two variable logic with ultimately periodic counting

\[ \begin{align*}
& \text{Figure 2 } \text{Splitting a vertex } u \text{ in a digraph } G \text{ into two vertices } u_1 \text{ and } u_2. \text{ One is adjacent to all the incoming edges and the other to all the outgoing edges.}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2}
\caption{Splitting a vertex $u$ in a digraph $G$ into two vertices $u_1$ and $u_2$. One is adjacent to all the incoming edges and the other to all the outgoing edges.}
\end{figure}

& \text{Figure 3 } \text{Swapping between the edges } (u_i, v_i) \text{ and } (w, w'), \text{ resulting in the edges } (u_i, w') \text{ and } (v_i, w').

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3}
\caption{Swapping between the edges $(u_i, v_i)$ and $(w, w')$, resulting in the edges $(u_i, w')$ and $(v_i, w')$.}
\end{figure}

A|B-biregular graph $G = (U, V, E_1, \ldots, E_t)$ with size $\bar{M}|\bar{M}$. We let $U = \{u_1, \ldots, u_k\}$ and $V = \{v_1, \ldots, v_k\}$, where $k = \|M^+\|$.

We orient all the edges with direction going from vertices in $U$ to vertices in $V$, and merge $u_i$ with $v_i$ for each $i = 1, \ldots, k$. This will result in an $A|B$-regular digraph with size $M|M$. When we do the merging, however, we have to make sure that there is no edge between vertex $u_i$ and $v_i$, since otherwise we will create a self-loop, which is forbidden in our definition of $A|B$-regular digraph. To avoid this, we argue that we can always “swap” edges. Suppose $(u_i, v_i)$ is an edge in $G$. Since $\bar{M}|\bar{M}$ is big enough for $A|B$, there is another edge $(w, w')$ such that both $w$ and $w'$ are not reachable from $u_i$ or $v_i$. We can then swap edges by deleting the edges $(u_i, v_i)$ and $(w, w')$ and replace them with $(u_i, w')$ and $(w, v_i)$. See Figure 3. This swapping preserves the degree of all vertices, while at the same time omitting the edge $(u_i, v_i)$.

The idea that we have just explained makes use of the fact that the degree of each vertex is bounded by a fixed small constant, i.e., $\max \|A\|, \|B\|$. In the case when $A$ and $B$ contains periodic entries, there is no such bound. In the following paragraphs, we will show how to construct an $A|B$-biregular graph $G$ as above where there is no edge between $u_i$ and $v_i$ for every $i = 1, \ldots, k$.

We proceed along the lines of the proof given for Lemma 22. We recall the formula that characterizes the sizes of $A|B$-biregular graph that are big enough, where $A$ and $B$ are simple
matrices:

\[
\Phi_{A|B}(\vec{x}, \vec{y}) := \exists z_{1,1} \ldots \exists z_{1,t} \exists z_{2,1} \ldots \exists z_{2,t}
\]

\[
\text{offset}(A) \cdot \vec{x}^T + \begin{pmatrix} \alpha_1 z_{1,1} \\ \vdots \\ \alpha_t z_{1,t} \end{pmatrix} = \text{offset}(B) \cdot \vec{y}^T + \begin{pmatrix} \beta_1 z_{2,1} \\ \vdots \\ \beta_t z_{2,t} \end{pmatrix}
\]

\[
\land \bigwedge_{i=1}^t z_{1,i} \neq \infty \land z_{2,i} \neq \infty,
\]

where \( \alpha_i = 1 \), if row \( i \) in \( A \) consists of periodic entries, and 0, otherwise; and similarly \( \beta_i = 1 \), if row \( i \) in \( B \) consists of periodic entries, and 0, otherwise.

(Case 1) \( \alpha_i z_{1,i} \geq \beta_i z_{2,i} \), for every \( 1 \leq i \leq t \).

Since the formula holds when \( \vec{x}, \vec{y} \) are assigned with \( \bar{M}, \bar{M} \), we rewrite the equation into

\[
(\text{offset}(A), pI_t) \cdot \begin{pmatrix} \bar{M}^T \\ \alpha_1 z_{1,1} - \beta_1 z_{2,1} \\ \vdots \\ \alpha_t z_{1,t} - \beta_t z_{2,t} \end{pmatrix} = \text{offset}(B) \cdot \bar{M}^T.
\]

Let \( K = (\alpha_1 z_{1,1} - \beta_1 z_{2,1}, \ldots, \alpha_t z_{1,t} - \beta_t z_{2,t}) \).

By Lemma \[21\] there is a \((\text{offset}(A), pI_t))\text{offset}(B)\)-biregular graph \( G = (U, V, E_1, \ldots, E_t) \) with size \((M, K)\). Let \( U = U_1 \cup \cdots \cup U_m \cup W_1 \cup \cdots \cup W_t \), where for each \( 1 \leq i \leq t \):

- for each \( 1 \leq j \leq m \), the \( E_i \)-degree of every vertex in \( U_j \) is \( \text{offset}(A_{i,j}) \), and \( |U_j| = M_j \);
- the \( E_i \)-degree of every vertex in \( W_i \) is \( p \) and \( |W_i| = \alpha_i z_{1,i} - \beta_i z_{2,i} \), and for every \( i' \neq i \), the \( E_{i'} \)-degree of every vertex in \( W_i \) is 0.

We let \( U = \{u_1, \ldots, u_k\} \) and \( V = \{v_1, \ldots, v_k\} \). We will call vertex \( v_i \) the mirror image of vertex \( u_i \), for each \( i = 1, \ldots, k \). By applying the edge swapping method, we can assume that there is no edge between \( u_i \) and \( v_i \) for every \( i = 1, \ldots, k \). Our aim is to merge the vertices in \( W_1 \cup \cdots \cup W_t \) with vertices in \( U \) while preserving this property. This is done as follows.

- We pick pairwise disjoint sets \( Z_1, \ldots, Z_t \subseteq U \) such that each \( |Z_i| = \delta^2_{\max} + 1 \). Such \( Z_i \) exist due to the fact that \( M \bar{M} \) is big enough for \( A|B \).
- Let \( Z = Z_1 \cup \cdots \cup Z_t \).
- Let \( Z'_1, \ldots, Z'_t \subseteq V \) be the set of mirror images of \( Z_1, \ldots, Z_t \), respectively.
- Let \( Z' = Z'_1 \cup \cdots \cup Z'_t \).
- We make sure that none of the vertices in \( W \) are adjacent (by any edge) with the vertices in \( Z' \).

We can ensure this because the number of edges adjacent to vertices in \( Z' \) and \( Z \) is bounded by a fixed constant, i.e., \( 2(\delta^2_{\max} + 1)t \). So, if \( M \bar{M} \) is big enough, and there is an \( E_i \)-edge between a vertex \( w_1 \in W \) and a vertex in some \( w_2 \in Z' \), there is another \( E_i \)-edge (\( w_3, w_4 \)) such that \( w_3 \notin Z \) and \( w_4 \notin Z' \).

We can apply the edge swapping method by omitting the edges \( (w_1, w_2) \) and \( (w_3, w_4) \), and replacing them with \( (w_1, w_4) \) and \( (w_3, w_2) \). Note that \( w_2 \) cannot be the mirror image of \( w_3 \).

Finally, we can merge the vertices in \( W \) with vertices in \( U \) without introducing any edge between a vertex and its mirror image. For each \( 1 \leq i \leq t \), for each vertex \( w \in W_i \), there is a vertex \( u \in Z_i \) such that \( u \) is not reachable from \( w \) in distance 2. Such a vertex \( u \) exists, since \( |Z_i| = \delta^2_{\max} + 1 \), and we can merge \( u \) with \( w \).
Note that since \( w \) is not adjacent to any vertex in \( Z' \), and the mirror image of \( u \) must be in \( Z'_i \subseteq Z' \), this merging does not yield any edge between \( u \) and its mirror image.

(Case 2) There is \( 1 \leq i, i' \leq t \) such that \( \alpha_i z_{1,i} > \beta_i z_{2,i} \) and \( \alpha_{i'} z_{1,i'} < \beta_{i'} z_{2,i'} \).

In this case, we have shown that the degree of the constructed \( A|B \)-biregular graph are also bounded above by some constant, i.e., \( \delta_{\max} + \delta_{2,\max} \). Thus, we can perform the same edge swapping to prevent the existence of the edges \((u_i, v_i)\).