THE PRIME NUMBER RACE AND ZEROS OF DIRICHLET $L$-FUNCTIONS OFF THE CRITICAL LINE. III

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ABSTRACT. We show, for any $q \geq 3$ and distinct reduced residues $a, b \pmod{q}$, the existence of certain hypothetical sets of zeros of Dirichlet $L$-functions lying off the critical line implies that $\pi(x; q, a) < \pi(x; q, b)$ for a set of real $x$ of asymptotic density 1.

1 Introduction

For $(a, q) = 1$, let $\pi(x; q, a)$ denote the number of primes $p \leq x$ with $p \equiv a \pmod{q}$. The study of the relative magnitudes of the functions $\pi(x; q, a)$ for a fixed $q$ and varying $a$ is known colloquially as the “prime race problem” or “Shanks–Rényi prime race problem”. For a survey of problems and results on prime races, the reader may consult the papers [4] and [5]. One basic problem is the study of $P_{q:a_1,...,a_r}$, the set of real numbers $x \geq 2$ such that $\pi(x; q, a_1) > \cdots > \pi(x; q, a_r)$. It is generally believed that all sets $P_{q:a_1,...,a_r}$ are unbounded. Assuming the Generalized Riemann Hypothesis for Dirichlet $L$-functions modulo $q$ (GRH$_q$) and that the nonnegative imaginary parts of zeros of these $L$-functions are linearly independent over the rationals, Rubinstein and Sarnak [12] have shown for any $r$-tuple of reduced residue classes $a_1, \ldots, a_r$ modulo $q$, that $P_{q:a_1,...,a_r}$ has a positive logarithmic density (although the density may be quite small in some cases).

In [2] and [3], Ford and Konyagin investigated how possible violations of the Generalized Riemann Hypothesis (GRH) would affect prime number races. In [2], they proved that the existence of certain sets of zeros off the critical line would imply that some of the sets $P_{q:a_1,a_2,a_3}$ are bounded, giving a negative answer to the prime race problem with $r = 3$. Paper [3] was devoted to similar questions for $r$-way prime races with $r > 3$. One result from [3] states that for any $q$, $r \leq \phi(q)$ and set $\{a_1, \ldots, a_r\}$ of reduced residues modulo $q$, the existence of certain hypothetical sets of zeros of Dirichlet $L$-functions modulo $q$ implies that at most $r(r-1)$ of the sets $P_{q: \sigma(a_1), \ldots, \sigma(a_r)}$ are unbounded, $\sigma$ running over all permutations of $\{a_1, \ldots, a_r\}$.

In this paper, we investigate the effect of zeros of $L$-functions lying off the critical line for two way prime races. This case is harder, since it is unconditionally proved that for certain races $\{q; a, b\}$ the set $P_{q:a,b}$ is unbounded. For example, Littlewood [11] proved that $P_{4:3,1}$, $P_{4:1,3}$, $P_{3:1,2}$ and $P_{3:2,1}$ are unbounded. Later Knapowski and Turán ([9], [10]) proved for many $q, a, b$ that $\pi(x; q, b) - \pi(x; q, a)$ changes sign infinitely often and more recently Sneed [13] showed that $P_{q:a,b}$ is unbounded for every $q \leq 100$ and all possible pairs $(a, b)$.

Nevertheless, we prove that the existence of certain zeros off the critical line would imply that the set $P_{q:a,b}$ has asymptotic density zero, in contrast with a conditional result of Kaczorowski [7] on GRH, which asserts that $P_{q:1,b}$ and $P_{q:b,1}$ have positive lower densities for all $(b, q) = 1$.

2010 Mathematics Subject Classification. Primary 11N13, 11M26.

Key words and phrases. The Shanks–Rényi prime race problem, primes in arithmetic progressions, zeros of Dirichlet $L$-functions.
Let \( q \geq 3 \) be a positive integer and \( a, b \) be distinct reduced residues modulo \( q \). Moreover, for any set \( S \) of real numbers we define \( S(X) = S \cap [2, X] \).

**Theorem 1.1.** Let \( q \geq 3 \) and suppose that \( a \) and \( b \) are distinct reduced residues modulo \( q \). Let \( \chi \) be a nonprincipal Dirichlet character with \( \chi(a) \neq \chi(b) \), and put \( \xi = \arg(\chi(a) - \chi(b)) \in [0, 2\pi) \).

Suppose \( \frac{1}{2} < \sigma < 1, 0 < \delta < \sigma - \frac{1}{2} \), \( A > 0 \), and \( B = B(\xi, \sigma, \delta, A) \) is a multiset of complex numbers satisfying the conditions listed in Section 2. If \( L(\rho, \chi) = 0 \) for all \( \rho \in B \), \( L(s, \chi) \) has no other zeros in the region \( \{ s : \Re(s) \geq \sigma - \delta, \Im(s) \geq 0 \} \), and for all other nonprincipal characters \( \chi' \) modulo \( q \), \( L(s, \chi') \neq 0 \) in the region \( \{ s : \Re(s) \geq \sigma - \delta, \Im(s) \geq 0 \} \), then

\[
\lim_{X \to \infty} \frac{\meas(P_{q,a,b}(X))}{X} = 0.
\]

**Remarks.** Such \( \chi \) exists whenever \( a \) and \( b \) are distinct modulo \( q \). The sets \( B \) have the property that any \( \rho \in B \) has real part in \( [\sigma - \delta, \sigma] \), imaginary part greater than \( A \), and multiplicity \( O((\log \Im(\rho))^{3/4}) \) (that is, the multiplicities are much smaller than known bounds on the multiplicity of zeros of Dirichlet \( L \)-functions). The number of elements of \( B \) (counted with multiplicity) with imaginary part less than \( T \) is \( O((\log T)^{5/4}) \), and thus \( B \) is quite a “thin” set. Also, we note that if \( L(\beta + i\gamma, \chi) = 0 \) then \( L(\beta - i\gamma, \overline{\chi}) = 0 \), which is a consequence of the functional equation for Dirichlet \( L \)-functions (See e.g. Ch. 9 of [11]). The point of Theorem 1.1 is that proving

\[
\limsup_{X \to \infty} \frac{\meas(P_{q,a,b}(X))}{X} > 0
\]

requires showing that the multiset of zeros of \( L(s, \chi) \) cannot contain any of the multisets \( B \). This is beyond what is possible with existing technology (see e.g. [6] for the best known estimates for multiplicities of zeros).

Our method works as well for the difference \( \pi(x) - \text{li}(x) \), the error term in the prime number theorem. Littlewood [11] established that this quantity changes sign infinitely often. Let \( P_1 \) be the set of real numbers \( x \geq 2 \) such that \( \pi(x) > \text{li}(x) \). In [8] Kaczorowski proved, assuming the Riemann Hypothesis, that both \( P_1 \) and \( \overline{P}_1 \) have positive lower densities. Assuming the Riemann Hypothesis and that the nonnegative imaginary parts of the zeros of the Riemann zeta function \( \zeta(s) \) are linearly independent over the rationals, Rubinstein and Sarnak [12] have shown that \( P_1 \) has a positive logarithmic density \( \delta_1 \approx 0.00000026 \). In contrast with these results we prove that the existence of certain zeros of \( \zeta(s) \) off the critical line would imply that the set \( P_1 \) has asymptotic density zero (or asymptotic density 1).

**Theorem 1.2.** Suppose \( \frac{1}{2} < \sigma < 1, 0 < \delta < \sigma - \frac{1}{2} \) and \( A > 0 \). (i) If \( \xi = 0, B = B(\xi, \sigma, \delta, A) \) satisfies the conditions of Section 2, \( \zeta(\rho) \neq 0 \) for all \( \rho \in B \), and \( \zeta(s) \) has no other zeros in the region \( \{ s : \Re(s) \geq \sigma - \delta, \Im(s) \geq 0 \} \), then

\[
\lim_{X \to \infty} \frac{\meas(P_1(X))}{X} = 0.
\]

(ii) If \( \xi = \pi, B \) satisfies the conditions of Section 2, \( \zeta(\rho) \neq 0 \) for all \( \rho \in B \), and \( \zeta(s) \) has no other zeros in the region \( \{ s : \Re(s) \geq \sigma - \delta, \Im(s) \geq 0 \} \), then

\[
\lim_{X \to \infty} \frac{\meas(P_1(X))}{X} = 1.
\]

We omit the proof of Theorem 1.2 as it is nearly identical to the proof of Theorem 1.1 in the case \( q = 4 \).
2 The construction of $B$

For $j \geq 1$, suppose that

$$\exp(j^8) \leq \gamma_j \leq 2 \exp(j^8), \quad \left| \delta_j - \frac{1}{j^8} \right| \leq \frac{1}{j^9},$$

(2.1)

and

$$\left| \theta_j - \frac{\xi - \pi/2}{j^{16}} \right| \leq \frac{1}{j^{17}}.$$

We choose $j_0$ so large that for all $j \geq j_0$, $\gamma_j > A$ and $\sigma - \delta \leq \sigma - \delta_j$. Then we take $B$ to be the union, over $j \geq j_0$ and $1 \leq k \leq j^3$, of $m(k, j) = k(j^3 + 1 - k)$ copies of $\rho_{j,k}$, where

$$\rho_{j,k} = \sigma - \delta_j + i(k\gamma_j + \theta_j).$$

3 Preliminary Results

The following classical-type explicit formula was established in Lemma 1.1 of [2] when $x' = x$. The slightly more general result below, which is more convenient for us, is proved in exactly the same way.

Lemma 3.1. Let $\beta \geq 1/2$ and for each non-principal character $\chi \mod q$, let $B(\chi)$ be the sequence of zeros (duplicates allowed) of $L(s, \chi)$ with $\text{Re}(s) > \beta$ and $\text{Im}(s) > 0$. Suppose further that all $L(s, \chi)$ are zero-free on the real segment $\beta < s < 1$. If $(a, q) = (b, q) = 1$, $x$ is sufficiently large and $x' \geq x$, then

$$\phi(q)(\pi(x; q, a) - \pi(x; q, b)) = -2\text{Re} \left( \sum_{\chi \neq \chi_0 \mod q} \sum_{\rho \in B(\chi)} f(\rho) \right) + O \left( x^\beta \log^2 x \right),$$

where

$$f(\rho) := \frac{x^\rho}{\rho \log x} + \frac{1}{\rho} \int_2^x \frac{t^\rho}{t \log^2 t} dt = \frac{x^\rho}{\rho \log x} + O \left( \frac{x^{\text{Re}(\rho)}}{|\rho|^2 \log^2 x} \right).$$

Remark. For Theorem 1.2, we use a similar explicit formula for $\pi(x)$ in terms of the zeros $B(\zeta)$ of the Riemann zeta function which satisfy $\Re \rho > \beta$ and $\Im \rho > 0$:

$$\pi(x) = \text{li}(x) - 2\Re \sum_{\rho \in B(\zeta), |\Im \rho| \leq x'} f(\rho) + O(x^\beta \log^2 x).$$

Using properties of the Fejér kernel we prove the following key proposition.

Proposition 3.2. Let $\gamma \geq 1$, $L \geq 4$ and $X \geq 2$. Define

$$F_{\gamma,L}(x) = \sum_{k=1}^{L-1} (L - k) \cos(k\gamma \log x).$$

Then

$$\text{meas} \left\{ x \in [1, X] : F_{\gamma,L}(x) \geq - \frac{L}{4} \right\} \ll \frac{X}{\sqrt{L}}.$$
Proof. The Fejér kernel satisfies the following identity
\[
\frac{1}{L} \left( \frac{\sin \left( \frac{L\theta}{2} \right)}{\sin \left( \frac{\theta}{2} \right)} \right)^2 = 1 + 2 \sum_{k=1}^{L-1} \left( 1 - \frac{k}{L} \right) \cos(k\theta).
\]
This yields
\[
F_{\gamma,L}(x) = \sin^2 \left( \frac{L\gamma \log x}{2} \right) - \frac{L}{2}.
\]
Therefore, if \( F_{\gamma,L}(x) \geq -L/4 \) then
\[
\sin^2 \left( \frac{\gamma \log x}{2} \right) \leq \frac{2}{L} \sin^2 \left( \frac{L\gamma \log x}{2} \right) \leq \frac{2}{L}.
\]
Hence,
\[
\left\| \frac{\gamma \log x}{2\pi} \right\| \leq \varepsilon := \frac{1}{\sqrt{2L}},
\]
where \( \|t\| \) denotes the distance to the nearest integer. This implies
\[
\text{meas} \left\{ x \in [1, X] : F_{\gamma,L}(x) \geq -\frac{L}{4} \right\} \leq \text{meas} \left\{ x \in [1, X] : \left\| \frac{\gamma \log x}{2\pi} \right\| \leq \varepsilon \right\} \leq \sum_{\gamma_j \leq x} \sum_{k=1}^{\gamma_j} e^{2\pi(k+\varepsilon)/\gamma} - e^{2\pi(k-\varepsilon)/\gamma}
\]
\[
\leq \frac{\varepsilon}{\gamma} \sum_{0 \leq k \leq \frac{\gamma \log X}{2\pi} + \varepsilon} e^{2\pi(k+\varepsilon)/\gamma} \ll \varepsilon X. \quad \square
\]

4 Proof of Theorem 1.1

Suppose \( X \) is large and \( \sqrt{X} \leq x \leq X \). For brevity, let
\[
\Delta = \phi(q) \left( \pi(x; q, a) - \pi(x; q, b) \right).
\]
It follows from Lemma 3.1 with \( x' = \max(x, \max\{j^3\gamma_j : \gamma_j \leq x\}) \) that
\[
\Delta = -\frac{2}{\log x} \text{Re} \left( \frac{\chi(a) - \chi(b)}{2} \sum_{\gamma_j \leq x} \sum_{k=1}^{j^3} \frac{x^{\sigma-\delta_j+i(k\gamma_j+\theta_j)} m(k, j)}{\sigma - \delta_j + i(k\gamma_j + \theta_j)} \right)
\]
\[
+ O \left( \frac{x^\sigma}{\log^2 x} \sum_{\gamma_j \leq x} \frac{x^{-\delta_j}}{\gamma_j^2} \sum_{k=1}^{j^3} \frac{m(k, j)}{k^2} + x^{\sigma-\delta} \log^2 x \right)
\]
\[
= \frac{2x^\sigma}{\log x} \text{Re} \left( \frac{i(\chi(a) - \chi(b))}{2} \sum_{\gamma_j \leq x} \frac{x^{-\delta_j}}{\gamma_j} \sum_{k=1}^{j^3} x^{i(k\gamma_j+\theta_j)} (j^3 + 1 - k) \right)
\]
\[
+ O \left( \frac{x^\sigma}{\log x} \sum_{\gamma_j \leq x} \frac{j^4x^{-\delta_j}}{\gamma_j^2} + x^{\sigma-\delta} \log^2 x \right),
\]
(4.1)
Note that
\[ \frac{x^{-\delta_j}}{\gamma_j} = \exp \left( -\frac{\log x}{j^8} \left( 1 + O \left( \frac{1}{j} \right) \right) - j^8 + O(1) \right). \]

The maximum of this function over \( j \) occurs around \( J = J(x) := [(\log x)^{1/16}] \). In this case we have \( \log x = J^{16}(1 + O(1/J)) \) so that
\[ \frac{x^{-\delta_j}}{\gamma_j} = \exp \left( -2J^8 + O(J^7) \right) = \exp \left( -2(\log x)^{1/2} + O((\log x)^{7/16}) \right). \]

We will prove that most of the contribution to the main term on the right hand side of (4.1) comes for the \( j \)'s in the range \( J - J^{3/4} \leq j \leq J + J^{3/4} \). First, if \( j \geq 3J/2 \) or \( j \leq J/2 \) then
\[ \frac{x^{-\delta_j}}{\gamma_j} \ll \exp \left( -4J^8 \right) \ll \exp \left( -4(\log x)^{1/2} \right) \frac{x^{-\delta_j}}{\gamma_j}. \]

Now suppose that \( J/2 < j < J - J^{3/4} \) or \( J + J^{3/4} < j < 3J/2 \). Write \( j = J + r \) with \( J^{3/4} < |r| < J/2 \). For \( x > 0 \), \( x + 1/x = 2 + (x - 1)^2/x \), hence
\[ \left( 1 + \frac{r}{J} \right)^8 + \left( 1 + \frac{r}{J} \right)^{-8} \geq \left( 1 + \frac{r}{J} \right)^8 + \left( 1 + \frac{r}{J} \right)^{-8} \geq 2 \left( \frac{8J^2}{1 + 8JR/2} \right) \geq 2 + 12(r/J)^2. \]

We infer from (4.2) that
\[ \frac{x^{-\delta_j}}{\gamma_j} = \exp \left( -\frac{J^{16}}{j^8} \left( 1 + O \left( \frac{1}{J} \right) \right) - j^8 \right) = \exp \left( -J^8 \left( 1 + \frac{r}{J} \right)^8 + \left( 1 + \frac{r}{J} \right)^{-8} + O(J^7) \right) \leq \exp \left( -2J^8 \left( 1 + \frac{6}{\sqrt{J}} \right) + O(J^7) \right) \ll \exp \left( -2(\log x)^{1/3} \right) \frac{x^{-\delta_j}}{\gamma_j}. \]

Since \( \gamma_j \leq x \) implies that \( j \ll (\log x)^{1/8} \), the contribution of the terms \( 1 \leq j < J - J^{3/4} \) or \( J + J^{3/4} < j \) to the main term of (4.1) is
\[ \ll \exp \left( -2(\log x)^{1/3} \right) \frac{x^{\sigma - \delta_j}}{\gamma_j} \sum_{j \leq (\log x)^{1/4}} \sum_{k=1}^{j^3} (j^3 + 1 - k) \ll \exp \left( -2(\log x)^{1/3} \right) \frac{x^{\sigma - \delta_j}}{\gamma_j}. \]

Similarly, we have
\[ \frac{x^{-\delta_j}}{\gamma_j^2} = \exp \left( -\frac{\log x}{j^8} \left( 1 + O \left( \frac{1}{j} \right) \right) - 2j^8 + O(1) \right) \ll \exp \left( -2\sqrt{2}(\log x)^{1/2}(1 + o(1)) \right) \ll \exp \left( -2(\log x)^{1/3} \right) \frac{x^{-\delta_j}}{\gamma_j}, \]
which follows from (4.2) along with the fact that the maximum of \( f(t) = -\log x/t^8 - 2t^8 \) occurs at \( t = (\log x/2)^{1/6} \). Hence, using (4.2), the contribution of the error term of (4.1) is

\[
(4.4) \quad \ll \exp \left( -2(\log x)^{1/3} \right) \frac{x^{\sigma-\delta_j}}{\gamma_j} \sum_{j \leq (\log x)^{1/4}} j^4 + x^{\sigma-\delta} \log^2 x \ll \exp \left( - (\log x)^{1/3} \right) \frac{x^{\sigma-\delta_j}}{\gamma_j}.
\]

Therefore, inserting the bounds (4.3) and (4.4) in (4.1) we deduce that

\[
\Delta = \frac{2x^\sigma}{\log x} \Re \left( i(\overline{\chi}(a) - \chi(b)) \sum_{|j-J| \leq J^{3/4}} x^{-\delta_j} \sum_{k=1}^{J^3} \exp \left( i(k\gamma_j + \theta_j) \log x \right) (j^3 + 1 - k) \right) + O \left( \exp \left( - (\log x)^{1/3} \right) \frac{x^{\sigma-\delta_j}}{\gamma_j} \right).
\]

(4.5)

Let \( J - J^{3/4} \leq j \leq J + J^{3/4} \). Then \( j^{16} = J^{16} \left( 1 + O(J^{-1/4}) \right) \). Hence we get

\[
\theta_j \log x = \left( \arg(\chi(a) - \chi(b)) - \frac{\pi}{2} \right) \frac{\log x}{j^{16}} + O \left( \frac{\log x}{j^{11/4}} \right)
\]

\[
= \left( \arg(\chi(a) - \chi(b)) - \frac{\pi}{2} \right) + O \left( \frac{1}{j^{1/4}} \right).
\]

This implies

\[
i(\overline{\chi}(a) - \chi(b)) \exp \left( i\theta_j \log x \right) = |\chi(a) - \chi(b)| \left( 1 + O \left( \frac{1}{j^{1/4}} \right) \right),
\]

since \( e^{i\arg z} = z/|z| \). Inserting this estimate in (4.5) we obtain

\[
\Delta = \left( 1 + O \left( \frac{1}{\log y/64} \right) \right) 2|\chi(a) - \chi(b)| \sum_{|j-J| \leq J^{3/4}} \frac{x^{\sigma-\delta_j}}{\gamma_j \log x} F_{\gamma_j,j^3}(x)
\]

\[
(4.6) \quad + O \left( \exp \left( - (\log x)^{1/3} \right) \frac{x^{\sigma-\delta_j}}{\gamma_j} \right).
\]

For \( x \in [\sqrt{X}, X] \) we have \( \frac{1}{4}(\log X)^{1/16} \leq J - J^{3/4} \) and \( J + J^{3/4} \leq 4(\log X)^{1/16} \) if \( X \) is sufficiently large, since \( J = (\log x)^{1/16} + O(1) \). We define

\[
\Omega := \left\{ x \in [\sqrt{X}, X] : F_{\gamma_j,j^3}(x) \leq -\frac{j^3}{4} \text{ for all } \frac{1}{4}(\log X)^{1/16} \leq j \leq 4(\log X)^{1/16} \right\}.
\]

Then it follows from Proposition 3.2 that

\[
\text{meas} \Omega = X + O \left( \frac{X}{\frac{1}{4}(\log X)^{1/16} \leq j \leq 4(\log X)^{1/16}} \right) \frac{1}{j^{3/2}} + \sqrt{X}
\]

\[
(4.7) \quad = X \left( 1 + O \left( (\log X)^{-1/32} \right) \right).
\]
Furthermore, if \( x \in \Omega \) then we infer from (4.6) that
\[
\Delta \leq -\frac{1}{3} |\chi(a) - \chi(b)| \sum_{|j-J| \leq J^{3/4}} \frac{j^3 x^{\sigma - \delta_j}}{\gamma_j \log x} + O \left( \exp \left( -\left( \log x \right)^{1/3} \frac{x^{\sigma - \delta_j}}{\gamma_j} \right) \right).
\]
\[
\leq -\frac{1}{3} |\chi(a) - \chi(b)| \frac{J^3 x^{\sigma - \delta_j}}{\gamma_j \log x} (1 + o(1)) < 0
\]
if \( X \) is sufficiently large, which completes the proof.

5 Acknowledgement

The research of K. F. was partially supported by National Science Foundation grant DMS-0901339. The research of S. K. was partially supported by Russian Fund for Basic Research, Grant N. 11-01-00329. The research of Y. L. was supported by a Postdoctoral Fellowship from the Natural Sciences and Engineering Research Council of Canada.

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