Global Attractor for Weakly Damped Forced KdV Equation in Low Regularity on $\mathbb{T}$

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Abstract In this paper we consider the long time behavior of the weakly damped, forced Korteweg-de Vries equation in the Sobolev spaces of the negative indices in the periodic case. We prove that the solutions are uniformly bounded in $\dot{H}^s(\mathbb{T})$ for $s > -\frac{1}{2}$. Moreover, we show that the solution-map possesses a global attractor in $\dot{H}^s(\mathbb{T})$ for $s > -\frac{1}{2}$, which is a compact set in $H^{s+3}(\mathbb{T})$.

Keywords: Korteweg-de Vries equation, attractor, asymptotic smoothing effect, Bourgain space, I-method.

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1 Introduction

In this paper we study the long time behavior of the weakly damped, forced Korteweg-de Vries (KdV) equations

$$\partial_t u + \partial_x^3 u + \frac{1}{2} \partial_x u^2 + \gamma u = f, \quad x \in \mathbb{T}, \; t \in \mathbb{R}^+, \quad (1.1)$$

$$u(x, 0) = u_0(x) \in \dot{H}^s(\mathbb{T}), \quad (1.2)$$

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where the unknown function $u$ is real-valued, the external force $f$ is time-independent and belongs to $\dot{H}^s(\mathbb{T})$, and the damping parameter $\gamma > 0$. Here $\mathbb{T} = \mathbb{R}/2\pi \mathbb{Z}$ and the space

$$
\dot{H}^s(\mathbb{T}) = \left\{ f \in H^s(\mathbb{T}) : f \text{ is zero mean, i.e. } \int_{\mathbb{T}} f(x) \, dx = 0 \right\}.
$$

When $\gamma = 0$, $f = 0$, it is known as KdV equation, which has been widely studied. The local well-posedness is known in $H^s(\mathbb{T})$ for $s \geq -1/2$ by Kenig, Ponce and Vega \[10\] by using an improved Bourgain argument. In \[4\], Colliander, Keel, Staffilani, Takaoka and Tao introduced the well-known I-method to extended the local result to a global one when $s \geq -1/2$. Further, there are examples given in \[3\] to show that the KdV equation is ill-posed for $s < -1/2$. On the other hand, the global attractor for the weakly damped, forced KdV equation was studied by many authors, see \[5\]–\[7\], \[8\], \[9\], \[11\], \[14\]–\[17\], etc.. Particularly, in \[7\], Goubet proved that the existence of the global attractor in $L^2(\mathbb{T})$, which is a compact set in $H^3(\mathbb{T})$. He showed these by the energy equations method (J. M. Ball’s argument) and Bourgain’s argument. In \[17\], Tsugawa consider (1.1) with $f \in L^2(\mathbb{T})$, and prove the existence of the global attractor in $\dot{H}^s(\mathbb{T})$ for $s > -\frac{3}{8}$, which is compact in $H^3(\mathbb{T})$. However, it seems that the method used in \[17\] depends strongly on the assumption $f \in L^2(\mathbb{T})$ to obtain the uniform boundness of the solution, moreover, it has a gap about the indix $s$ compared to the local well-posedness for $s \geq -\frac{1}{2}$. Meanwhile, the argument in \[17\] seems not suitably to the non-compact case.

In this paper, we consider (1.1) and prove that the global attractor exists in $\dot{H}^s(\mathbb{T})$ for $s > -\frac{1}{2}$, and is compact in $H^{s+3}(\mathbb{T})$, which was left open in \[7\] and \[17\]. Our main approach is combining the method in \[7\] and the idea of I-method. We first use the I-method to prove the uniform boundness of the solutions in Section 3. We remark that the argument used here should be somewhat different from \[4\], since what we need here is the uniform boundness of the solutions. However, it can be also succeeded in applying here, based on the brief that the weak damping shall prevent the unlimitedly increasing of the the energy in $H^s(\mathbb{T})$. Compared to \[17\], the strong condition $f \in \dot{L}^2(\mathbb{T})$ is relaxed to general: $f \in \dot{H}^s(\mathbb{T})$, and the index is lowered to be optimal by this argument. As a consequence, we obtain the existence of the bounded absorbing sets in
$H^s(\mathbb{T})$. Furthermore, we note that it lacks the conservation law in $H^s(\mathbb{T})$ when $s < 0$ for the KdV equation, so it is not sufficient to use the energy equation method to show the asymptotic compactness in $H^s(\mathbb{T})$. To overcome this problem, we pursue the asymptotic smoothing effect of the flow-map at first, and then prove the asymptotic compactness by using the asymptotic smoothing effect. More precisely, we split the solution into two parts, and prove that one part is decay to zero and the other is regular (bounded in $H^{s+3}(\mathbb{T})$), which is shown in Section 4. As we know, we need some infinitesimals (we denote them by $\epsilon(N)$ below, see Section 4 for the details), which indicate that the energies from nonlinearities can be controlled to ensure the decay of one part and uniformly bounded in a higher regular space of the other. However, it is not easy to give these $\epsilon(N)$ in this situation when the solutions are distributions. To cope with the difficulties, some decomposition tricks to treat the nonlinearities and special multilinear estimates included some commutators will be used throughout the article. Furthermore, some arguments in [13] and [18] are employed in this section. By employing this asymptotic smoothing effect, one may prove directly that the solution map is asymptotic compact in $H^l(\mathbb{T})$ for any $l < s + 3$. At last, we further show the compactness of the global attractor in $H^{s+3}(\mathbb{T})$.

We leave the existence of the global attractor in critical space $\dot{H}^{-\frac{1}{2}}(\mathbb{T})$ open in this paper.

**Notations.** We use $A \lesssim B$ or $B \gtrsim A$ to denote the statement that $A \leq CB$ for some large constant $C$ which may vary from line to line, and may depend on the data such as $\gamma, f, u_0$ and the index $s$ unless otherwise mentioned. When it is necessary, we will write the constants by $C_1, C_2, \cdots$, or $K_1, K_2, \cdots$ to see the dependency relationship. We use $A \ll B$ to denote the statement $A \leq C^{-1}B$, and use $A \sim B$ to mean $A \lesssim B \lesssim A$. The notation $a+$ denotes $a + \epsilon$ for any small $\epsilon$, and $a-$ for $a - \epsilon$. $\langle \cdot \rangle = (1 + |\cdot|^2)^{1/2}$, $J_x^\alpha = (1 - \partial_x^2)^{\alpha/2}$ and $D_x^\alpha = (-\partial_x^2)^{\alpha/2}$. We use $\|f\|_{L^p_t L^q_x}$ to denote the mixed norm $\left( \int \|f(x, \cdot)\|^p_{L^q} \, dx \right)^{\frac{1}{p}}$. Moreover, we denote $\mathcal{F}_x$ to be the Fourier transform corresponding to the variable $x$.

Further we define the Fourier spectral projector $P_N$, $Q_N$, $P_{\ll N}$ respectively as

$$P_N f(x) = \int_{|\xi| \leq N} e^{ix\xi} \hat{f}(\xi) \, d\xi, \quad Q_N f(x) = \int_{|\xi| \geq N} e^{ix\xi} \hat{f}(\xi) \, d\xi, \quad P_{\ll N} f(x) = \int_{|\xi| \ll N} e^{ix\xi} \hat{f}(\xi) \, d\xi.$$

We state our main result, as the end of this section.
Theorem 1.1 Let $-\frac{1}{2} < s < 0$ and $\gamma > 0$, $f \in \dot{H}^s(\mathbb{T})$. Then (1.1) (1.2) possess a global attractor in $\dot{H}^s(\mathbb{T})$, which is compact in $H^{s+3}(\mathbb{T})$.

2 Functional Spaces and Preliminary Estimates

We first introduce some notations and definitions. We define $(d\xi)_\mathbb{T}$ to be the normalized counting measure on $\mathbb{T}$ such that

$$\int a(\xi) (d\xi)_\mathbb{T} = \frac{1}{2\pi} \sum_{\xi \in \mathbb{Z}} a(\xi).$$

At the following, we always adopt the notation

$$\int_* a(\xi, \xi_1, \xi_2, \tau, \tau_1, \tau_2) = \int_{\xi_1 \neq 0, \xi_1 + \xi_2 = \xi} a(\xi, \xi_1, \xi_2, \tau, \tau_1, \tau_2) (d\xi_1)(d\xi_2)_\mathbb{T} d\tau_1 d\tau_2.$$

Define the Fourier transform of a function $f$ on $\mathbb{T}$ by

$$\hat{f}(\xi) = \int_{\mathbb{T}} e^{-2\pi i x \xi} f(x) dx,$$

and thus the Fourier inversion formula

$$f(x) = \int_{\mathbb{T}} e^{2\pi i x \xi} \hat{f}(\xi) (d\xi)_\mathbb{T}.$$

For $s, b \in \mathbb{R}$, we define the Bourgain space $X_{s,b}$ to be the completion of the Schwartz class under the norm

$$\|f\|_{X_{s,b}} \equiv \left( \int \int \langle \xi \rangle^{2s} \langle \tau - \xi^3 \rangle^{2b} |\hat{f}(\xi, \tau)|^2 (d\xi)_\mathbb{T} d\tau \right)^{\frac{1}{2}}. \quad (2.1)$$

For an interval $\Omega$, we define $X_{s,b}^\Omega$ to be the restriction of $X_{s,b}$ on $\mathbb{T} \times \Omega$ with the norm

$$\|f\|_{X_{s,b}^\Omega} = \inf \{ \|F\|_{X_{s,b}} : F|_{t \in \Omega} = f|_{t \in \Omega} \}. \quad (2.2)$$

When $\Omega = [-\delta, \delta]$, we write $X_{s,b}^\Omega$ as $X_{s,b}^{\delta}$. By the Functional analysis on Hilbert space we see that, for every $f \in X_{s,b}^\Omega$, there exists an extension $\tilde{f} \in X_{s,b}$ such that $\tilde{f} = f$ on $\Omega$ and

$$\|f\|_{X_{s,b}^\Omega} = \|\tilde{f}\|_{X_{s,b}}.$$
Further, we define the locally well-posed working spaces $Y^s$ via the norms

$$\| f \|_{Y^s} = \| f \|_{X_{s,\frac{1}{2}}} + \| \langle \xi \rangle^s \hat{f}(\xi, \tau) \|_{L^2((d\xi)\tau)L^1(d\tau)},$$

and the companion spaces $Z^s$:

$$\| f \|_{Z^s} = \| f \|_{X_{s,\frac{1}{2}}} + \| \langle \xi \rangle^s \hat{f}(\xi, \tau) \|_{L^2((d\xi)\tau)L^1(d\tau)}.$$

Then it is easy to see that $Y^s \subset C(\mathbb{R}; H^s(\mathbb{T})) \cap X_{s,\frac{1}{2}}$.

Let $s < 0$ and $N \gg 1$ be fixed, the Fourier multiplier operator $I_{N,s}$ is defined as

$$\hat{I}_{N,s}f(\xi) = m_{N,s}(\xi) \hat{f}(\xi), \quad (2.3)$$

where the multiplier $m_{N,s}(\xi)$ is a smooth, monotone function satisfying $0 < m_{N,s}(\xi) \leq 1$ and

$$m_{N,s}(\xi) = \begin{cases} 
1, & |\xi| \leq N, \\
N^{-s}|\xi|^s, & |\xi| > 2N. 
\end{cases} \quad (2.4)$$

Sometimes we denote $I_{N,s}$ and $m_{N,s}$ by $I$ and $m$ respectively for short if there is no confusion.

It is obvious that the operator $I_{N,s}$ maps $H^s(\mathbb{T})$ into $L^2(\mathbb{T})$ with equivalent norms for any $s < 0$. More precisely, there exists some positive constant $C$ such that

$$C^{-1}\| f \|_{H^s} \leq \| I_{N,s}f \|_{L^2} \leq CN^{-s}\| f \|_{H^s}. \quad (2.5)$$

Next, we give some estimates which will be used in the following sections.

**Lemma 2.1** [2]. Let $f \in X_{0,\frac{1}{4}}$, then

$$\| f \|_{L^4} \lesssim \| f \|_{X_{0,\frac{1}{4}}}. \quad (2.6)$$

**Lemma 2.2** Let $\varphi \in H^{\frac{1}{2}}(\mathbb{R})$, $f \in X_{s,\frac{1}{2}}$ for any $s \in \mathbb{R}$, then

$$\| \varphi f \|_{X_{s,\frac{1}{2}}} \leq C(\varphi)\| f \|_{X_{s,\frac{1}{2}}}. \quad (2.7)$$
Proof. Denote $U(t) \equiv e^{-\partial t}$, then by Hölder’s and Sobolev’s inequalities, we have

$$\|\varphi(t)U(-t)f(x,t)\|_{H^\frac{1}{2}} \lesssim \left\| J_t^\frac{1}{2}(\varphi) U(-t)f \right\|_{L_t^2} + \left\| \varphi J_t^\frac{1}{2}(U(-t)f) \right\|_{L_t^2}$$

$$\lesssim \left\| J_t^\frac{1}{2}(\varphi) \right\|_{L_t^2} \left\| U(-t)f \right\|_{L_t^{p'}} + \left\| \varphi \right\|_{L_t^{p'}} \left\| J_t^\frac{1}{2}U(-t)f \right\|_{L_t^p}$$

$$\lesssim \|\varphi\|_{H^\frac{1}{2}} \left\| J_t^\frac{1}{2}U(-t)f \right\|_{L_t^2},$$

where we set $p = 2^+$ such that $H_t^0(\mathbb{R}) \hookrightarrow L_t^p(\mathbb{R})$, and $\frac{1}{p} + \frac{1}{p'} = \frac{1}{2}$. Thus $H_t^\frac{1}{2}(\mathbb{R}) \hookrightarrow L_t^{p'}(\mathbb{R})$. Then the result follows by noting $\|\varphi f\|_{X_{s,b}} = \|\varphi(t)U(-t)f(x,t)\|_{H^\frac{1}{2}H^\frac{1}{2}}$ for any $s, b \in \mathbb{R}$. \hfill \Box

Remark. As a candidate, we keep in mind that $\varphi = \chi_{[0,\delta]}(t)e^{-\delta t}$ for some $\delta > 0$. A more precise version of Lemma 2.2 is, for any $0 \leq b < b' \leq \frac{1}{2}$

$$\|\varphi f\|_{X_{s,b}} \leq C(\varphi)\|f\|_{X_{s,b'}}.$$

We denote $\psi(t)$ to be an even smooth bump function of the interval $[-1,1]$.

Lemma 2.3 For any $u, v \in X^\delta_{-\frac{1}{2},\frac{1}{2}}$, such that $u, v$ are zero $x$–mean for all $t$, then

$$\|\psi(t/\delta) \partial_x (uv)\|_{Z^\delta_{-\frac{1}{2},\frac{1}{2}}} \lesssim \delta^{\frac{1}{2}} \|u\|_{X^\delta_{-\frac{1}{2},\frac{1}{2}}} \|v\|_{X^\delta_{-\frac{1}{2},\frac{1}{2}}}.$$ \hfill (2.8)

Proof. It is actually a more precise result from the proof of Proposition 3 in [4], where we shall combine the estimate

$$\|\psi(t/\delta)f\|_{X_{s,b'}} \lesssim \delta^{b-b'} \|f\|_{X_{s,b}},$$ \hfill (2.9)

for any $s \in \mathbb{R}, 0 \leq b' \leq b < \frac{1}{2}, f \in X_{s,b}$. \hfill \Box

Remark. Another version from the proof of Proposition 3 in [4] is

$$\|\varphi \psi(t/\delta) \partial_x (uv)\|_{X_{-\frac{1}{2},\frac{1}{2}}} \lesssim \delta^{\frac{1}{2}} \|u\|_{X^\delta_{-\frac{1}{2},\frac{1}{2}}} \|v\|_{X^\delta_{-\frac{1}{2},\frac{1}{2}}}.$$ \hfill (2.10)

for any $\varphi \in H^\frac{1}{2}(\mathbb{R})$, which follows by the estimate (2.7).

By (2.10) and the relation $|\xi| \leq |\xi_1| + |\xi_2|$ when $\xi = \xi_1 + \xi_2$, it leads to

$$\|\varphi \psi(t/\delta) \partial_x (uv)\|_{Z^\delta_{s,\frac{1}{2}}} \lesssim \|u\|_{X^\delta_{s,\frac{1}{2}}} \|v\|_{X^\delta_{s,\frac{1}{2}}} + \|u\|_{X^\delta_{s,\frac{1}{2}}} \|v\|_{X^\delta_{s,\frac{1}{2}}},$$ \hfill (2.11)
and thus
\[ \| \varphi \psi(t/\delta) \partial_x(uv) \|_{X_{s,-1/2}} \lesssim \|u\|_{X^{s,-1/2}} \|v\|_{X^{s,-1/2}} + \|u\|_{X^{s,-1/2}} \|v\|_{X^{s,-1/2}}, \]  
for any \( s \geq -\frac{1}{2} \), where we have dropped the factor \( \delta^{3/2} \) since it is smaller than 1.

Recall that the commutator \([A, B] = AB - BA\), we have the following result.

**Lemma 2.4** For any \( s \in \mathbb{R} \), if the functions \( u, v \) are zero-mean for all \( t \) and \( u = P_N u, v = Q_N v \) for some \( N > 0 \), then
\[ \| \partial_x [J^s_x, u] v \|_{X^{s,-1/2}} \lesssim \|u\|_{X^{s,1/2}} \|v\|_{X^{s,-1/2}}. \]  

**Proof.** By duality and Plancherel’s identity, it suffices to show
\[ \int \xi \langle \xi \rangle^{\frac{1}{2}} \langle \xi_1 \rangle^{-\frac{1}{2}} \langle \xi_2 \rangle^{-s+\frac{1}{2}} ((\xi)^s - (\xi_2)^s) \hat{f}(\xi, \tau) \hat{g}(\xi_1, \tau_1) \hat{h}(\xi_2, \tau_2) \lesssim \|f\|_{X_{0,1/2}} \|g\|_{X_{0,3/2}} \|h\|_{X_{0,3/2}} \]  
for any \( f, g, h \in X_{0,\frac{1}{2}} \) such that \( \text{supp} \xi \hat{g} \in [0, N] \) and \( \text{supp} \xi \hat{h} \in [N, +\infty) \) belongs to \([N, +\infty)\), thus \( |\xi_1| \leq |\xi_2| \). Note that \( |\xi| \sim |\xi_2| \) and \( \langle \xi \rangle^s - (\xi_2)^s \lesssim |\xi_1||\xi_2|^{s-1} \), the left-hand side is reduced to
\[ \int \xi \langle \xi \rangle^{\frac{1}{2}} \langle \xi_1 \rangle^{-\frac{1}{2}} \langle \xi_2 \rangle^{-\frac{1}{2}} \langle \xi \rangle^s \hat{f}(\xi, \tau) \hat{g}(\xi_1, \tau_1) \hat{h}(\xi_2, \tau_2). \]  

By the fact that
\[ \tau - \xi^3 - (\tau_1 - \xi_1^3) - (\tau_2 - \xi_2^3) = -3\xi_1 \xi_2, \]
we may assume that \( |\tau - \xi^3| \gtrsim |\xi_1 \xi_2| \) by symmetry. Therefore we reduce to control
\[ \int \langle \tau - \xi^3 \rangle^{\frac{1}{2}} \hat{f}(\xi, \tau) \hat{g}(\xi_1, \tau_1) \hat{h}(\xi_2, \tau_2). \]  

By Hölder’s inequality and Lemma 2.1, (2.15) is bounded by
\[ \|f\|_{X_{0,1/2}} \|g\|_{X_{0,3/2}} \|h\|_{X_{0,3/2}} \]  

This completes the proof of the lemma. \( \square \)

**Remark.** By (2.7) and the proof above (see (2.16)), we actually have
\[ \| \varphi \partial_x [J^s_x, u] v \|_{X^{s,-1/2}} \lesssim \|u\|_{X^{s,1/2}} \|v\|_{X^{s,-1/2}}. \]  

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for any \( \varphi \in H^{1/2}(\mathbb{R}) \) and \( u, v \) as Lemma 2.4.

We note that Lemma 2.4 gives a good estimate under the restrictions to \( u, v \) on the space-frequency space. A nature question is: what happens if we drop the restrictions? Now we give a general result about it.

**Lemma 2.5** For any \( s \in \mathbb{R} \), if the functions \( u, v \) are zero \( x \)-mean for all \( t \), then

\[
\| \partial_x [J_x^s, u] v \|_{X_{s_1}, -\frac{1}{2}} \lesssim \| u \|_{X_{s_2}, \frac{1}{2}} \| v \|_{X_{s_3}, \frac{1}{2}},
\]

where \( s_1, s_2, s_3 \) satisfy

\[
s_1 = -\frac{1}{2} + a - c; \quad s_2 = -\frac{1}{2} + s + a - b - c; \quad s_3 = -\frac{1}{2} + c - b,
\]

for any \( a, b, c \geq 0 \) and \( a + b - 2c \leq 1 - s \).

**Proof.** By duality and Plancherel’s identity, it suffices to show

\[
\int |\xi|^{s_1} (\xi_1)^{-s_2} (\xi_2)^{-s_3} (\langle \xi \rangle^s - \langle \xi_2 \rangle^s) \hat{f}(\xi, \tau) \hat{g}(\xi_1, \tau_1) \hat{h}(\xi_2, \tau_2) \lesssim \| f \|_{X_{0, \frac{1}{2}}} \| g \|_{X_{0, \frac{1}{2}}} \| h \|_{X_{0, \frac{1}{2}}},
\]

for any \( f, g, h \in X_{0, \frac{1}{2}} \).

First, if \( |\xi| \sim |\xi_2| \), then \( |\xi_1| \lesssim |\xi| \). The same proof as Lemma 2.4, the left-hand side is reduced to

\[
\int |\xi|^{s_1 - s_3 + s} |\xi_1|^{1-s_2} \hat{f}(\xi, \tau) \hat{g}(\xi_1, \tau_1) \hat{h}(\xi_2, \tau_2).
\]

It encounters (2.14) when

\[
s_1 - s_3 + s - 1 \leq 0, \quad \text{and} \quad s_1 - s_2 - s_3 + s - \frac{1}{2} \leq 0.
\]

Thus we have the claimed result.

Second, if \( |\xi| \ll |\xi_2| \sim |\xi_1| \), then the left-hand side is reduced to

\[
\int |\xi|^{1+s_1} |\xi_1|^{-s_2-s_3+s} \hat{f}(\xi, \tau) \hat{g}(\xi_1, \tau_1) \hat{h}(\xi_2, \tau_2).
\]

It encounters (2.14) again when

\[
-s_2 - s_3 + s - 1 \leq 0, \quad \text{and} \quad s_1 - s_2 - s_3 + s - \frac{1}{2} \leq 0.
\]
Third, if $|\xi| \sim |\xi_1| \gg |\xi_2|$, then the left-hand side is reduced to
$$\int_s |\xi|^{1+s_1-s_2+s}|\xi_2|^{-s_3} \hat{f}(\xi, \tau) \hat{g}(\xi_1, \tau_1) \hat{h}(\xi_2, \tau_2).$$

It also meets (2.14) when
$$s_1 - s_2 + s \leq 0, \quad \text{and} \quad s_1 - s_2 - s + \frac{1}{2} \leq 0. \quad (2.22)$$

We set that
$$s_2 + s_3 - s + 1 = a; \quad -s_1 + s_2 - s = b; \quad -s_1 + s_2 + s_3 - s + \frac{1}{2} = c,$$
then inserting them into (2.20)–(2.22), we give the condition (2.19) in the lemma. \hfill \square

As a consequence of Lemma 2.4, we have

**Corollary 2.6** For any $s \leq 0$, if the functions $u, v$ are zero $x$–mean for all $t$, then
$$\|\varphi \partial_x [J_x^s, u]v\|_{X^s_{\frac{1}{2} + s, -s}} \lesssim \|u\|_{X^s_{\frac{1}{2} + 2s, \frac{1}{2}}} \|v\|_{X^{-\frac{1}{2}, \frac{1}{2}}}, \quad (2.23)$$

for any $\varphi \in H^\frac{1}{2}(-\mathbb{R})$.

**Proof.** We choose $a = 1 + s, b = c = 0$ in (2.17), and use (2.7) to get the result. \hfill \square

At the end of this section, we give a special bilinear estimate as follows, which might be not sharp but enough to use in the article. A similar one has been presented in [18].

**Lemma 2.7** For any $s > -\frac{1}{4}$, $u \in X^s_{\frac{1}{2}}, a \in H^s(\mathbb{T})$ such that $u, a$ are zero $x$–mean, then
$$\|\psi(t/\delta) \partial_x (ua)\|_{Z^s} \lesssim \delta^{\frac{1}{2}} \|u\|_{X^s_{\frac{1}{2}}} \|a\|_{H^s}. \quad (2.24)$$

**Proof.** According to the two contributions to the $Z^s$-norm, we need to show
$$\|\psi(t/\delta) \partial_x (ua)\|_{X^s_{\frac{1}{2}}} \lesssim \delta^{\frac{1}{2}} \|u\|_{X^s_{\frac{1}{2}}} \|a\|_{H^s}, \quad (2.25)$$
$$\left\|\frac{\delta^{\frac{1}{2}} \|u\|_{X^s_{\frac{1}{2}}} \|a\|_{H^s}}{\langle \xi - \xi \rangle \langle \tau - \xi^3 \rangle} \right\|_{L^2_{\xi} L^1_{\tau}} \lesssim \delta^{\frac{1}{2}} \|u\|_{X^s_{\frac{1}{2}}} \|a\|_{H^s}. \quad (2.26)$$

For (2.25), we only need to show
$$\int_{\mathbb{T}} \frac{\delta^{\frac{1}{2}} \|u\|_{X^s_{\frac{1}{2}}} \|a\|_{H^s}}{\langle \xi - \xi \rangle \langle \tau - \xi^3 \rangle} \lesssim \delta^{\frac{1}{2}} \|u\|_{X^s_{\frac{1}{2}}} \|a\|_{H^s} \quad (2.27)$$
for any \( f, g \in X_{0, \frac{1}{2}}^\delta \), \( h \in L_{\delta x}^2 \), where we denote \( h_\delta(x, t) = \psi(t/\delta)h(x) \). Moreover, we may assume that \( f = \psi(t/\delta)f, g = \psi(t/\delta)g \).

Note that
\[
|\tau - \xi^3 - (\tau_1 - \xi_1^3) - \tau_2| = |\xi_2(\xi^2 + \xi_1 + \xi_1^2)| \geq |\xi_1 \xi_2|,
\]
we may split the integral (2.27) into three parts:

(a) \( |\tau - \xi^3| \geq |\xi||\xi_1||\xi_2|; \) (b) \( |\tau_1 - \xi_1^3| \geq |\xi||\xi_1||\xi_2|; \) (c) \( |\tau_2| \geq |\xi||\xi_1||\xi_2|.

(a) : \( |\tau - \xi^3| \geq |\xi||\xi_1||\xi_2| \), we split the integral again into two parts: Part 1: \( |\xi| \ll |\xi_1| \);
Part 2: \( |\xi_1| \ll |\xi| \ll |\xi_2| \).

Part 1: \( |\xi| \ll |\xi_1| \). The left-hand side of (2.27) restricted in this part is controlled by
\[
\int_s |\xi_2|^{-\frac{1}{2}} (\tau - \xi^3) \frac{1}{2} \hat{f}(\xi, \tau) \hat{g}(\xi_1, \tau_1) \hat{h}_\delta(\xi_2, \tau_2) \leq \|f\|_{X_{0, \frac{1}{4}}} \|g\|_{L_{\delta x}^1} \left\| J_x^{-\frac{1}{2}} h_\delta \right\|_{L_{\delta x}^2}.
\]

Since \( s \geq -\frac{1}{4} \), by Sobolev's inequality, we have,
\[
\left\| J_x^{-\frac{1}{2}} h_\delta \right\|_{L_{\delta x}^2} = \|\psi(t/\delta)\|_{L_t^2} \left\| J_x^{-\frac{1}{2}} h_\delta \right\|_{L_t^2} \lesssim \delta^{\frac{1}{2}} \|h\|_{L_t^2},
\]
where we note that for any \( 1 \leq p \leq \infty \),
\[
\|\psi(\cdot/\delta)\|_{L^p} = \delta^{\frac{1}{2}} \|\psi\|_{L^p}.
\]

Then (2.27) restricted in this part follows by (2.6) and (2.9).

Part 2: \( |\xi_1| \ll |\xi| \ll |\xi_2| \). The left-hand side of (2.27) restricted in this part is bounded by
\[
\int_s |\xi_1|^{-\frac{1}{2}} (\tau - \xi^3) \frac{1}{2} \hat{f}(\xi, \tau) \hat{g}(\xi_1, \tau_1) \hat{h}_\delta(\xi_2, \tau_2) \leq \|f\|_{X_{0, \frac{1}{4}}} \left\| J_x^{-\frac{1}{2}} g \right\|_{L_{\delta x}^1} \left\| h_\delta \right\|_{L_t^1 L_x^2}.
\]

Further, since \( s > -\frac{1}{4} \), by Sobolev's inequality, \( \left\| J_x^{-\frac{1}{2}} g \right\|_{L_{\delta x}^1} \lesssim \|g\|_{L_{\delta x}^1} \), and \( \left\| h_\delta \right\|_{L_t^1 L_x^2} \lesssim \delta^{\frac{1}{2}} \|h\|_{L^2} \). Then (2.27) restricted in this part follows by (2.6) and (2.9) again.
(b) : $|\tau_1 - \xi_1^3| \gtrsim |\xi||\xi_1||\xi_2|$, we also split the integral again into two parts: **Part 1:** $|\xi| \lesssim |\xi_1|$; **Part 2:** $|\xi_1| \ll |\xi| \sim |\xi_2|$. But **Part 1** is similar to Part (a)(1), so we only consider Part 2.

**Part 2:** $|\xi_1| \ll |\xi| \sim |\xi_2|$. The left-hand side of (2.27) restricted in this part is controlled by

$$
\int |\xi_1|^{-\frac{1}{2}-s} \langle \tau_1 - \xi_1^3 \rangle^{\frac{1}{2}} \hat{f}(\xi, \tau) \hat{g}(\xi_1, \tau_1) \hat{h}_\delta(\xi_2, \tau_2)
\lesssim \|f\|_{L^1_{\tau t}} \left\| J_x^{-\frac{1}{2}-s} \Gamma g \right\|_{L^2_{x,t}} \|h_\delta\|_{L^1_{\tau t}},
$$

where the operator $\Gamma$ defined by $\hat{\Gamma} g(\xi, \tau) = \langle \tau - \xi_1^3 \rangle^{\frac{1}{2}} \hat{g}(\xi, \tau)$. By Sobolev’s inequality,

$$
\left\| J_x^{-\frac{1}{2}-s} \Gamma g \right\|_{L^2_{x,t}} \lesssim \|\Gamma g\|_{L^2_t} = \|g\|_{X_{0,\frac{1}{2}}} \text{ since } s \geq -\frac{1}{4}.
$$

Then (2.27) follows by (2.6), (2.28) and (2.9).

(c) : $|\tau_2| \gtrsim |\xi||\xi_1||\xi_2|$, then the left-hand side of (2.27) is bounded by

$$
\int |\tau_2|^{\frac{1}{2}} \hat{f}(\xi, \tau) \hat{g}(\xi_1, \tau_1) \hat{h}_\delta(\xi_2, \tau_2) \lesssim \|f\|_{L^1_{\tau t}} \|g\|_{L^1_{\tau t}} \left\| D^\frac{1}{2}_x h\right\|_{L^2_t} \lesssim \delta^{\frac{1}{2}-} \|f\|_{X_{0,\frac{1}{2}}} \|g\|_{X_{0,\frac{1}{2}}} \|h\|_{L^2}\]
$$

by (2.6) and the fact $\left\| D^\frac{1}{2}_x \psi(\cdot/\delta) \right\|_{L^2} = \left\| D^\frac{1}{2}_x \psi \right\|_{L^2}$. Thus we prove the result (2.27).

For (2.26), if $|\tau_1 - \xi_1^3| \gtrsim |\xi||\xi_1||\xi_2|$ or $|\tau_2| \gtrsim |\xi||\xi_1||\xi_2|$, then by Hölder’s inequality, (2.26) is sufficient if

$$
\left\| \frac{\xi [\psi(\cdot/\delta) * \hat{u} * \hat{a}]}{\langle \tau - \xi_1^3 \rangle^{\frac{1}{2}} -} \right\|_{L^2_{\tau t}} \lesssim \delta^{\frac{1}{2}-} \|u\|_{X_{0,\frac{1}{2}}} \|a\|_{H^s}. (2.29)
$$

But it can be shown as (2.25) above. Therefore, we only consider

$$
|\tau - \xi_1^3| \gtrsim |\xi||\xi_1||\xi_2|.
$$

Moreover, in the event that

$$
|\tau_1 - \xi_1^3| \gtrsim |\xi_1| \xi_2|^{0+},
$$

(2.26) is reduced to (2.29) and turned further to

$$
\int |\xi|^{\frac{1}{2}+s}|\xi_1|^{-\frac{1}{2}-s}|\xi_2|^{-\frac{1}{2}-s} \langle \tau_1 - \xi_1^3 \rangle^{0+} \hat{f}(\xi, \tau) \hat{g}(\xi_1, \tau_1) \hat{h}_\delta(\xi_2, \tau_2)
\lesssim \delta^{\frac{1}{2}-} \|f\|_{L^2_{\tau t}} \|g\|_{X_{0,\frac{1}{2}}} \|h\|_{L^2},
$$

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which can be shown by the same argument used in Part (a) above. The event that
|\tau_2| \gtrsim |\xi_1\xi_2|^0 is similar. Therefore, we only need to consider the case that
\[ \tau_1 - \xi_1^3 = -\xi_2(\xi^2 + \xi_1 + \xi_2^1) + O(|\xi_1\xi_2|^0). \]

Let the set
\[ \Omega(\xi) = \{ \eta \in \mathbb{R} : \eta = -\xi_2(\xi^2 + \xi_1 + \xi_2^1) + O(|\xi_1\xi_2|^0) \} \]
then similar to Lemma 7.6 in \[4\], we have
\[ |\Omega(\xi) \cap \{ \eta : |\eta| \sim M \}| \lesssim M^{\frac{2}{3}}, \]
which leads to
\[ \left( \int \langle \tau - \xi^3 \rangle^{-1} \chi_{\Omega(\xi)}(\tau - \xi^3) \, d\tau \right)^{\frac{1}{2}} \lesssim 1. \] (2.30)

Indeed, it can be easily proved by the dyadic decomposition to the integration.

Using (2.30) and Hölder's inequality, the left-hand side of (2.26) is controlled by
\[ \left\| \xi[\psi(\cdot/\delta) \ast \hat{\eta} \ast \hat{a}] \right\| \left\langle \tau - \xi^3 \right\rangle^{\frac{1}{2}} \lesssim 1. \]
Again (2.26) follows from the same argument used in Part (a).

Remark. By (2.7) and the proof above, we actually have
\[ \|\varphi \psi(t/\delta) \eta_x(x)\|_{L^2} \lesssim \delta^{-\frac{3}{4}} \|u\|_{X^{s,\frac{1}{2}}} \|a\|_{H^s}, \]
for any \( \varphi \in H^\frac{3}{2}(-\mathbb{R}) \) and \( u, v \) as Lemma 2.7. As a consequence, we have
\[ \|\varphi \psi(t/\delta) \eta_x(x)\|_{X^{s+\frac{1}{4},\frac{3}{4}}} \lesssim \|u\|_{X^{s,\frac{1}{2}}} \|a\|_{H^s} + \|u\|_{X^{s,\frac{1}{2}}} \|a\|_{H^{s+l}}, \] (2.31)
for any \( s > -\frac{1}{4}, l \geq 0. \)

3 Well-posedness and Bounded Absorbing Sets

In this section, we first give the local well-posedness, then we apply the I-method and
the multilinear correction technique to prove the global well-posedness, finally we obtain
bounded absorbing sets.
3.1 The Local Well-posedness

Compared to the KdV equation, the equation (1.1) lacks the solution of scale invariance, so we have to dig some additional factors of $\delta$ from the estimates, which is of importance for us. The first one is a refined local result which is a contrast to Proposition 4 in [4].

By employing the bilinear estimates (2.8) (which replace (7.33) in [4]), we give the following local result (which instead of Proposition 4 in [4]).

**Proposition 3.1** Let $s \geq -\frac{1}{2}$, $I = I_{N,s}$, $f \in \dot{H}^{s}(\mathbb{T})$, then (1.1) (1.2) are locally well-posed for the initial data $u_0$ satisfies $Iu_0 \in \dot{L}^2(\mathbb{T})$, with the lifetime $\delta$ satisfying

$$\delta \sim (\|I_{N,s}u_0\|_{L^2} + \|I_{N,s}f\|_{L^2})^{-3-}. \quad (3.1)$$

Further, the solution satisfies the estimate

$$\|I_{N,s}u\|_{Y^0} \lesssim \|I_{N,s}u_0\|_{L^2} + \|I_{N,s}f\|_{L^2}. \quad (3.2)$$

**Remark.** The improvement in this local result is to give a refined estimates on lifetime $\delta$, which strongly effect on the global well-posedness of the weakly damped, forced KdV equation who lacks the solution of scale invariance (see the following subsection for more details).

By (3.2), we have further the control of the solution as

$$\sup_{t \in [t_0 - \delta, t_0 + \delta]} \|I_{N,s}u(t)\|_{L^2} \lesssim \|I_{N,s}u(t_0)\|_{L^2} + \|I_{N,s}f\|_{L^2}, \quad (3.3)$$

if we take “$t_0$” for the initial time.

3.2 The Global Well-posedness

Now we are further to consider the global well-posedness and the existence of bounded absorbing sets. The argument here is mainly the I-method in [4]. However, some estimates and the iteration process used in [4] should be rebuilt. We show that, due to the presence of the weak damping, the energy will not increase unlimitedly.
First we define the symmetrization of a $k-$multiplier $m : \mathbb{Z}^k \to \mathbb{R}$ by

$$[m]_{\text{sym}(\xi)} = \frac{1}{n!} \sum_{\sigma \in S_n} m(\sigma(\xi)),$$

where $S_n$ is the group of all permutations on $n$ objects. We say $m$ is symmetric if $m(\xi) = m(\sigma(\xi))$.

Define the $k-$multiplier

$$\Lambda_k(m; u_1, \cdots, u_k) = \int_{\sum \xi_j = 0} m(\xi_1, \cdots, \xi_k) \prod_{j=1}^k \mathcal{F}_x u_j(\xi_j, t) (d\xi_1)_T \cdots (d\xi_{k-1})_T.$$

We write $\Lambda_k(m) = \Lambda_k(m; u, \cdots, u)$ for short. Then by a direct computation, we have

$$\frac{d}{dt} \Lambda_k(m) = -k\gamma \Lambda_k(m) + \Lambda_k(\alpha_k m) + k\Lambda_k(m; u, \cdots, u, f)$$

$$- \frac{i}{2} k\Lambda_{k+1}(m(\xi_1, \cdots, \xi_{k-1}, \xi_k + \xi_{k+1}), \xi_k + \xi_{k+1})), \quad (3.4)$$

where the multiplier $m$ is symmetric and

$$\alpha_k \equiv i(\xi_1^3 + \cdots + \xi_k^3).$$

Note that the fourth term of $(3.4)$ may be symmetrized.

Now let $m(\xi_1, \xi_2) = m(\xi_1)m(\xi_2)$, denote the modified energy as

$$E_1^2(t) = \|Iu(t)\|^2_{L^2} = \Lambda_2(m),$$

then by $(3.4)$ and note that $\alpha_2(\xi_1, \xi_2) = 0$ when $\xi_1 + \xi_2 = 0$, we have

$$\frac{d}{dt} E_1^2(t) = -2\gamma E_1^2(t) + 2\Lambda_2(m; u, f) + \Lambda_3(M_3),$$

where

$$M_3(\xi_1, \xi_2, \xi_3) = -\frac{i}{3} (m^2(\xi_1)\xi_1 + m^2(\xi_2)\xi_2 + m^2(\xi_3)\xi_3).$$

Define a new modified energy $E_1^3(t)$ by

$$E_1^3(t) = \Lambda_3(\sigma_3) + E_1^2(t), \quad (3.5)$$

where

$$\sigma_3 = -\frac{M_3}{\alpha_3}.$$
Then one has
\[ \frac{d}{dt} E_3^4(t) = -\gamma E_1^4(t) - E_2^4(t) - 2\gamma \Lambda_3(\sigma_3) + 2\Lambda_2(m; u, f) + 3\Lambda_4(\sigma_3; u, u, f) + \Lambda_4(M_4), \]
where
\[ M_4 = -\frac{3}{2}i[\sigma_3(\xi_1, \xi_1, \xi_3 + \xi_4)(\xi_3 + \xi_4)]_{\text{sym}}. \]

Define another new modified energy \( E_4^4(t) \) again by
\[ E_4^4(t) = \Lambda_4(\sigma_4) + E_1^3(t), \tag{3.6} \]
with
\[ \sigma_4 = -\frac{M_4}{\alpha_4}. \]

Then one has
\[ \frac{d}{dt} E_4^4(t) = -\gamma E_1^4(t) - E_2^4(t) - 2\gamma \Lambda_3(\sigma_3) - 3\gamma \Lambda_4(\sigma_4) + 2\Lambda_2(m; u, f) + 3\Lambda_3(\sigma_3; u, u, f) + 4\Lambda_4(\sigma_4; u, u, u, f) + \Lambda_5(M_5), \]
where
\[ M_5 = -2i[\sigma_4(\xi_1, \xi_1, \xi_3, \xi_4 + \xi_5)(\xi_4 + \xi_5)]_{\text{sym}}. \]

Therefore,
\[ E_4^4(t + \delta) = e^{-\gamma \delta} E_4^4(t) + \int_0^\delta e^{-\gamma(\delta-t')}[F(t' + t) - \gamma E_1^2(t' + t)] dt', \tag{3.7} \]
where
\[ F = -2\gamma \Lambda_3(\sigma_3) - 3\gamma \Lambda_4(\sigma_4) + 2\Lambda_2(m; u, f) + 3\Lambda_3(\sigma_3; u, u, f) + 4\Lambda_4(\sigma_4; u, u, u, f) + \Lambda_5(M_5). \]

First, by (3.3), we have
\[ \gamma \int_0^\delta e^{-\gamma(\delta-t')} E_1^2(t' + t) dt' \geq c\gamma \delta \| Iu(t) \|_{L^2}^2 - C\gamma \delta \| If \|_{L^2}^2, \tag{3.8} \]
for some small \( c \), large \( C > 0 \).

Now we give some multilinear estimates on \( \Lambda_k \). The first one is an improvement of the results in Lemma 6.1 in [4], which is of importance in this situation.
Lemma 3.2 Let $s \geq -\frac{3}{4}$, $I = I_{N,s}$, then

$$|\Lambda_3(\sigma_3; u_1, u_2, u_3)| \lesssim N^{-\frac{3}{2}} \prod_{j=1}^{3} \|Iu_j(t)\|_{L^2};$$

(3.9)

$$|\Lambda_4(\sigma_4; u_1, u_2, u_3, u_4)| \lesssim N^{-3} \prod_{j=1}^{4} \|Iu_j(t)\|_{L^2}.$$  

(3.10)

Proof. For (3.9), since $\xi_1 + \xi_2 + \xi_3 = 0$, by symmetry we may assume $|\xi_1| \sim |\xi_2| \geq |\xi_3|$. Note that $\sigma_3$ vanishes when $|\xi_j| \leq N$ for $j = 1, 2, 3$, so we may assume further that $|\xi_1|, |\xi_2| \gtrsim N$.

Set

$$\Delta \equiv \frac{|\sigma_3|}{m(\xi_1)m(\xi_2)m(\xi_3)} = \frac{|M_3(\xi_1, \xi_2, \xi_3)|}{|\alpha_3(\xi_1, \xi_2, \xi_3)|m(\xi_1)m(\xi_2)m(\xi_3)},$$

then (3.9) follows if we show

$$|\Lambda_3(\Delta; u_1, u_2, u_3)| \lesssim N^{-\frac{3}{2}} \|u_1\|_{L^2}\|u_2\|_{L^2}\|u_3\|_{L^2}.$$  

Note that

$$|M_3(\xi_1, \xi_2, \xi_3)| \lesssim m^2(\xi_{\text{min}})|\xi_{\text{min}}|, \quad \alpha_3(\xi_1, \xi_2, \xi_3) = 3\xi_1\xi_2\xi_3,$$

thus, noting that $s \geq -\frac{3}{4}$, we have

$$\Delta \lesssim \frac{1}{|\xi_1\xi_2| m(\xi_1)m(\xi_2)} \sim N^{2s}|\xi_1|^{-1-s}|\xi_2|^{-1-s} \lesssim N^{-\frac{3}{2}}|\xi_1|^{-\frac{3}{4}}|\xi_2|^{-\frac{1}{4}}.$$

Therefore, by Hölder and Sobolev’s inequalities, we have

$$|\Lambda_3(\Delta; u_1, u_2, u_3)| \lesssim N^{-\frac{3}{2}}|\Lambda_3(|\xi_1|^{-\frac{1}{4}}|\xi_2|^{-\frac{1}{4}}; u_1, u_2, u_3)|$$

$$\lesssim N^{-\frac{3}{2}} \left\|D_x^{-\frac{1}{4}}u_1\right\|_{L^4} \left\|D_x^{-\frac{1}{4}}u_2\right\|_{L^4} \left\|u_3\right\|_{L^2}$$

$$\lesssim N^{-\frac{3}{4}} \left\|u_1\right\|_{L^2} \left\|u_2\right\|_{L^2} \left\|u_3\right\|_{L^2}.$$  

Now we turn to (3.10), and set

$$\tilde{\Delta} \equiv \frac{|\sigma_4|}{\prod_{j=1}^{4} m(\xi_j)},$$

then (3.10) suffices if we show

$$|\Lambda_4(\tilde{\Delta}; u_1, u_2, u_3, u_4)| \lesssim N^{-3} \prod_{j=1}^{4} \|u_j(t)\|_{L^2}.$$
By Lemma 4.4 in [4], we have
\[ \tilde{\Delta} \lesssim \frac{1}{\prod_{j=1}^{4} m(\xi_j) \max\{N, |\xi_j|\}}. \]

Since \( m(\xi) \sim N^{-s} \max\{N, |\xi|\}^{s} \), and noting that \( s \geq -\frac{3}{4} \), we have
\[ \tilde{\Delta} \lesssim N^{-3} \prod_{j=1}^{4} |\xi_j|^{-\frac{1}{4}}. \]

Therefore, we control \(|\Lambda_4(\tilde{\Delta}; u_1, u_2, u_3, u_4)|\) by
\[ N^{-3} \prod_{j=1}^{4} \left\| D^{-\frac{1}{2}} u_j(t) \right\|_{L^4_t} \lesssim N^{-3} \prod_{j=1}^{4} \left\| u_j(t) \right\|_{L^2}, \]
by Sobolev’s inequality.

**Lemma 3.3** Let \( s \geq -\frac{1}{2} \), \( I = I_{N,s}, \beta = \frac{5}{2} \), \( 0 < \delta \leq 1 \), then
\[ \left| \int_{0}^{\delta} \Lambda_5(M_5) \, dt \right| \lesssim N^{-\beta} \delta^{\frac{\beta}{2}} \| Iu \|_{Y_0}^{5}. \]  

**Proof.** We only give the modification of the proof in [4] here. By the modified bilinear estimate (2.8), we replace (8.1) in [4] by
\[ \left| \int_{0}^{\delta} \prod_{j=0}^{5} w_j(x, t) \, dx dt \right| \lesssim \delta^{\frac{1}{2}} \prod_{j=0}^{3} \left\| w_j \right\|_{Y_{\frac{1}{2}}} \left\| w_4 \right\|_{X_{-\frac{1}{2}, \frac{1}{2}}} \left\| w_5 \right\|_{X_{-\frac{1}{2}, \frac{1}{2}}}, \]
where \( w_j(x, t) \) are \( T \)-periodic, and zero \( x \)-mean for any \( t \). Based on that, we lead to the result claimed.

Now we consider a priori estimates of the solutions, and from now on we set \( \delta \sim N^{3s-} \) by (3.11) (the large number \( N \) will be chosen later). Back to (3.7), we apply (3.9)–(3.11) to yield a bound of \( \int_{0}^{\delta} F(t' + t) \, dt' \) as
\[ \delta \sup_{t' \in [t - \delta, t + \delta]} \left( N^{-\frac{3}{2}} \| Iu(t') \|_{L^4_t}^{2} + N^{-3} \| Iu(t') \|_{L^2_t}^{4} + \| Iu(t') \|_{L^2_t} \| Iu(t') \|_{L^1_t} \right) \]
\[ + N^{-\frac{3}{2}} \| Iu(t') \|_{L^2_t}^{2} \| Iu(t') \|_{L^2_t} + N^{-3} \| Iu(t') \|_{L^2_t} \| Iu(t') \|_{L^2_t} + N^{-\frac{1}{2}} \| Iu \|_{Y_0} \]. \]

By (2.5) and recalling that \( \delta \sim N^{3s-} \), we have
\[ \delta N^{-\frac{3}{2}} \| Iu \|_{L^4_t}^{3}, \delta N^{-3} \| Iu \|_{L^2_t}^{4}, \delta \| Iu \|_{L^2_t}^{2}, \delta \| Iu \|_{L^2_t}^{2} \leq K(f), \]  

(3.13)
for some positive constant $K$ only dependent on $\|f\|_{H^s}$. Therefore, by (3.2), (3.3) and (3.13), (3.12) is controlled by

$$
K(f) + \frac{c}{2} \gamma \delta \|Iu(t)\|_{L^2_x}^2 + C \delta \|Iu(t)\|_{L^2_x}^2 \left( N^{-\frac{3}{2}} \|Iu(t)\|_{L^2_x} + N^{-3} \|Iu(t)\|_{L^2_x}^2 \right)
+ N^{-\frac{2}{3}} \|If\|_{L^2_x} + N^{-3} \|Iu(t)\|_{L^2_x} \|If\|_{L^2_x} \right) + CN^{-\beta} \delta^{\frac{3}{4}} - \|Iu(t)\|_{L^2_x}^3
$$

(3.14)

for some constant $C$ independent on $N, u$ and $f$. Then by (2.5) again, we control (3.14) and finally get

$$
\int_0^\delta F(t'+t) \, dt' \lesssim K(f) + \frac{c}{2} \gamma \delta \|Iu(t)\|_{L^2_x}^2 + C \delta \|Iu(t)\|_{L^2_x}^2 \cdot
\left( N^{-\frac{3}{2}} \|Iu(t)\|_{L^2_x} + N^{-3} \|Iu(t)\|_{L^2_x}^2 + N^{-\frac{2}{3}} \|f\|_{H^s} \right)
+ N^{-3-s} \|Iu(t)\|_{L^2_x} \|f\|_{H^s} + N^{-\beta} \delta^{-\frac{3}{4}} \|Iu(t)\|_{L^2_x}^3.
$$

(3.15)

Combining with (3.7), (3.8), (3.13), (3.15), and choosing $\epsilon$ small enough, we have

$$
E_1^4(t + \delta) \leq e^{-\gamma \delta} E_1^4(t) + K(f) + \delta \|Iu(t)\|_{L^2_x}^2 \left( CN^{-\frac{3}{2}} \|Iu(t)\|_{L^2_x} + CN^{-3} \|Iu(t)\|_{L^2_x}^2 \right)
+ CN^{-\frac{2}{3-s}} \|f\|_{H^s} + CN^{-3-s} \|Iu(t)\|_{L^2_x} \|f\|_{H^s}
+ CN^{-\beta} \delta^{-\frac{3}{4}} \|Iu(t)\|_{L^2_x}^3 - \frac{c}{2} \gamma \delta).
$$

(3.16)

The last term of (3.16) is negative if

$$
N^{-\frac{3}{2}} \|Iu(t)\|_{L^2_x}, N^{-3} \|Iu(t)\|_{L^2_x}^2, N^{-\frac{2}{3-s}} \|f\|_{H^s},
N^{-3-s} \|Iu(t)\|_{L^2_x} \|f\|_{H^s}, N^{-\beta} \delta^{-\frac{3}{4}} \|Iu(t)\|_{L^2_x}^3 = o(N).
$$

(3.17)

Now we consider it by iteration. When $t = 0$, by (2.5) and noting that $\delta \sim N^{3s}$, (3.17) is satisfied when $s > -\frac{1}{2}$. We just check the last term of (3.17) here, which is sufficient to show

$$
N^{-\beta} \cdot N^{-2s+} \cdot N^{-3s} \|u_0\|_{H^s} \lesssim N^{0-},
$$

that is exactly, $s > -\frac{1}{2}$. Therefore, (3.16) yields

$$
E_1^4(t) \leq e^{-\gamma t} E_1^4(0) + K(f), \quad \text{for any } t \in [0, \delta].
$$

(3.18)

By using (3.5), (3.6), (3.18) and Lemma 3.2 again, we get

$$
E_2^2(t) \leq e^{-\gamma t} E_1^4(0) + K(f) + CN^{-\frac{3}{2}} \|Iu(t)\|_{L^2_x}^3 + CN^{-3} \|Iu(t)\|_{L^2_x}^4
$$

(3.19)
for any $t \in [0, \delta]$. Note that
\[
E^4_I(0) \leq C \|Iu_0\|_{L^2_T}^2 + CN^{-\frac{3}{2}} \|Iu_0\|_{L^2_T}^3 + CN^{-3} \|Iu_0\|_{L^2_T}^4,
\]
then by (3.19), it is easy to see that for $t \in [0, \delta]$,\[
E^2_I(t) \leq 2e^{-\gamma \delta} E^4_I(0) + 2K(f) \tag{3.20}
\]
by choosing $N$, dependent on $\|u_0\|_{H^s}, \|f\|_{H^s}$, large enough. By (3.20), (3.17) is true if we take $t = \delta$. Repeating the process above, iterating (3.18), we conclude that (3.20) holds for any $t > 0$. Therefore, by (2.5) again, we have
\[
\limsup_{t \to +\infty} \|u(t)\|_{H^s} \leq \sqrt{2K(f)}. \tag{3.21}
\]

We state our results in this subsection.

**Theorem 3.4** Let $\gamma > 0, f \in \dot{H}^s(\mathbb{T})$ and $s > -\frac{1}{2}$, then (1.1) (1.2) are globally well-posed in $\dot{H}^s(\mathbb{T})$. Moreover, the solution $u(t)$ satisfies (3.21).

By Theorem 3.4, (1.1) (1.2) define a continuous semigroup $S(t)$ on $H^s(\mathbb{T})$, such that $u(t) = S(t)u_0 \in C([0, +\infty); H^s(\mathbb{T}))$ is the solution with the initial function $u_0$. Moreover, we have

**Corollary 3.5** Let $s > -\frac{3}{4}$, then the solution map $S(t)$ associated with system (1.1) possesses a bounded absorbing ball in $H^s(\mathbb{T})$, with the radius given by (3.21).

### 4 Split of the Solution

For the existence of the global attractor, we need to show the asymptotic compactness of the solution map in $H^s(\mathbb{T})$. Since the KdV equation possesses no conservation law for in $H^s(\mathbb{T})$ for $s < 0$ and lacks sufficient regularity of the solutions, we apply the asymptotic smoothing effect via a suitable decomposition of the solution map to obtain it.

For this purpose, we fix a large number $N$ (which will be chosen later and may be different from the one in Section 3) and split the solution $u$ into two parts as
\[
u = v + w,
\]
where
\[
\begin{align*}
\partial_t v + \partial_x^3 v + \frac{1}{2} \partial_x v^2 + \gamma v &= -\frac{1}{2} P_N(\partial_x u^2 - \partial_x v^2) + f, \\
\partial_t w + \partial_x^3 w + \gamma w &= -\frac{1}{2} Q_N(\partial_x u^2 - \partial_x v^2), \\
v(x, 0) &= P_N u_0(x), \quad w(x, 0) = Q_N u_0(x).
\end{align*}
\]

(4.1)

(4.2)

(4.3)

The local well-posedness of the systems (4.2)–(4.3) can be proved by employing the bilinear estimates (2.8) and the standard process of the fixed point argument. Especially, taking the initial time “\(t_0\)” (under the assumption of existence), we have the estimate
\[
\|w\|_{Y^s} \lesssim \|w(t_0)\|_{H^s},
\]
with the lifetime \(\delta \in (0, 1)\) depending on \(\gamma\), \(\|w(t_0)\|_{H^s}\) and \(\|f\|_{H^s}\) but independent of \(N\).

Further, by (4.4), we have
\[
\sup_{t \in [t_0 - \delta, t_0 + \delta]} \|w(t)\|_{H^s} \lesssim \|w(t_0)\|_{H^s}.
\]

(4.5)

For \(v\), by (3.2) (for \(N = 1\)) and (4.4), we have
\[
\|v\|_{Y^s} \lesssim \|v(t_0)\|_{H^s} + \|u(t_0)\|_{H^s} + \|f\|_{H^s},
\]
\[
\sup_{t \in [t_0 - \delta, t_0 + \delta]} \|v(t)\|_{H^s} \lesssim \|v(t_0)\|_{H^s} + \|u(t_0)\|_{H^s} + \|f\|_{H^s}.
\]

(4.6)

(4.7)

4.1 Decay of \(w\) in \(H^s(\mathbb{T})\)

Noting that \(w = Q_N w\), we rewrite (4.2) into
\[
\partial_t w + \partial_x^3 w + \gamma w = Q_N [ww_x - (uw)_x],
\]
and drive the energy equation of (4.8) in \(H^s(\mathbb{R})\) to find
\[
\begin{align*}
\|J_x^s w(t + \delta)\|_{L^2_x}^2 &= e^{-\gamma \delta} \|J_x^s w(t)\|_{L^2_x}^2 + 2 \int_0^\delta e^{-\gamma (\delta - t')} \int_T J_x^s [ww_x - (uw)_x] \cdot J_x^s w \, dx \, dt' \\& - \gamma \int_0^\delta e^{-\gamma (\delta - t')} \|J_x^s w\|_{L^2_x}^2 \, dt',
\end{align*}
\]

(4.9)

where we have omitted the variable \(t' - \delta + t\) of the functions inside the time integral for short. Since for each \(f \in X_{s_T^1 \frac{1}{2}}\), there exists an \(\tilde{f} \in X_{1, \frac{1}{2}}\) such that \(f|_{t' \in [t, t+\delta]} = \tilde{f}|_{t' \in [t, t+\delta]}\)
and \( \|f\|_{X^{s,t+\delta}} \leq \|\tilde{f}\|_{X^{s,\frac{1}{2}}} \), we may replace \( u, w \) by \( \tilde{u}, \tilde{w} \) in the following procedure. But we remove the tilde \( \tilde{} \) again for simplicity.

First, by (4.5) we have
\[
\gamma \int_0^\delta e^{-\gamma(\delta-t')} \|J_x^s w\|_{L_x^2}^2 \, dt' \geq c(1 - e^{-\gamma \delta}) \|w(t)\|_{H^s}^2.
\] (4.10)

For the second term in (4.9), We write
\[
\int_0^\delta e^{-\gamma(\delta-t')} \int_T J_x^s [ww_x - (uw)_x] \cdot J_x^s w \, dx \, dt' = J_1 + J_2,
\]
where
\[
J_1 = \frac{1}{2} \int_0^\delta e^{-\gamma(\delta-t')} \int_T P_N u \cdot \partial_x (J_x^s w)^2 \, dx \, dt',
\]
\[
J_2 = - \int_0^\delta e^{-\gamma(\delta-t')} \int_T \partial_x [J_x^s, P_N u] w \cdot J_x^s w \, dx \, dt',
\]
We rewrite \( J_2 \) again by
\[
J_2 = J_{21} + J_{22} + J_{23}
\]
where
\[
J_{21} = \frac{1}{2} \int_0^\delta e^{-\gamma(\delta-t')} \int_T P_N u \cdot \partial_x (J_x^s w)^2 \, dx \, dt',
\]
\[
J_{22} = - \int_0^\delta e^{-\gamma(\delta-t')} \int_T \partial_x [J_x^s, P_N u] w \cdot J_x^s w \, dx \, dt',
\]
\[
J_{23} = - \int_0^\delta e^{-\gamma(\delta-t')} \int_T \partial_x J_x^s (Q_N u \cdot w) \cdot J_x^s w \, dx \, dt',
\]
where \( 0 < \epsilon \ll 1, N \gg 1 \), and the commutator \([A, B] = AB - BA\). First,

**Lemma 4.1** For any \( s \geq \frac{1}{2}, \) the functions \( z_0, z \) are zero \( x \)-mean for all \( t \), and \( z_0 \in X^\delta_{0,\frac{1}{2}}, z \in X^\delta_{s,\frac{1}{2}} \) with \( z = Q_N z \), then,
\[
\int_0^\delta e^{-\gamma(\delta-t')} \int_T z_0 \cdot \partial_x (J_x^s z)^2 \, dx \, dt' \lesssim N^{-\frac{1}{2}} \|z_0\|_{X^{\delta,\frac{1}{2}}_0} \|z\|_{X^{\delta,\frac{1}{2}}_s}^2.
\] (4.11)

**Proof.** By duality and (2.12), the left-hand side of (4.11) is controlled by
\[
\|z_0\|_{X^{\delta,\frac{1}{2}}_0} \|\varphi \partial_x (J_x^s z)^2\|_{X^{\delta,\frac{1}{2}}_{0,\frac{1}{2}}} \lesssim \|z_0\|_{X^{\delta,\frac{1}{2}}_0} \|J_x^s z\|_{X^{\delta,\frac{1}{2}}_0} \|J_x^s z\|_{X^{\delta,\frac{1}{2}}_s},
\]
where \( \varphi = \chi_{[0, \delta]} e^{-\gamma(\delta-t')} \). Then the result follows by noting \( z = Q_N z \). \(\square\)

By this lemma, (3.2) (for \( N = 1 \)) and (4.4), we have
\[
J_{21} \lesssim N^{-\frac{1}{2} - \sigma} \|u\|_{X^{\delta,\frac{1}{2}}_s} \|w\|_{X^{\delta,\frac{1}{2}}_s} \lesssim N^{-\frac{1}{2} - \sigma} (\|u(t)\|_{H^s} + \|f\|_{H^s}) \|w(t)\|_{H^s}^2.
\] (4.12)
Lemma 4.2 For any $s \in \mathbb{R}$, the functions $z_0, z$ are zero $x$–mean for all $t$, and $z_0 \in X^\delta_{s_1, \frac{1}{2}}$, and $z \in X^\delta_{s, \frac{1}{2}}$ with $z_0 = P_{< N} z_0, z = Q_N z$, then,

$$
\int_0^\delta e^{-\gamma(\delta-t')} \int_T \partial_x [J_x^s, z_0] z \cdot J_x^s z \, dx \, dt' \lesssim N^{-1} \| z_0 \|_{X^\delta_{s_1, \frac{1}{2}}} \| z \|_{X^\delta_{s, \frac{1}{2}}}^2.
$$

(4.13)

Proof. It follows easily by duality and (2.17), where we set $\varphi = \chi_{[0,\delta]} e^{-\gamma(\delta-t')}$. 

By this lemma, (3.2) and (4.4), we have

$$
J_{22} \lesssim N^{-1+(\frac{3}{4}-s)} \| u \|_{X^\delta_{s_1, \frac{1}{2}}} \| w \|_{X^\delta_{s_1, \frac{1}{2}}} \lesssim N^{-1+(\frac{3}{4}-s)} (\| u(0) \|_{H^s} + \| f \|_{H^s}) \| w(t) \|_{H^s}^2.
$$

(4.14)

On the other hand, by the duality, (2.12), (3.2) and (4.5), we have

$$
J_{23} \lesssim \| \varphi \partial_x J_x^s(Q_N u \cdot w) \|_{X^\delta_{s_1, \frac{1}{2}}} \| J_x^s w \|_{X^\delta_{s_1, \frac{1}{2}}} \lesssim \left( \| Q_N u \|_{X^\delta_{s_1, \frac{1}{2}}} \| w \|_{X^\delta_{s_1, \frac{1}{2}}} + \| Q_N u \|_{X^\delta_{s_1, \frac{1}{2}}} \| w \|_{X^\delta_{s_1, \frac{1}{2}}} \| w \|_{X^\delta_{s_1, \frac{1}{2}}} \right) \| w \|_{X^\delta_{s_1, \frac{1}{2}}} \lesssim N^{-s+\frac{1}{2}} \| Q_N u \|_{X^\delta_{s_1, \frac{1}{2}}} \| w(t) \|_{H^s}^2.
$$

(4.15)

Similar to $J_{23}$, we have

$$
J_1 \lesssim N^{-s+\frac{1}{2}} \| w(t) \|_{H^s}^3.
$$

(4.16)

Summing up (4.12), (4.14) - (4.16), we have

$$
\int_0^\delta e^{-\gamma(\delta-t')} \int_T J_x^s [w w_x - (u w)_x] \cdot J_x^s w \, dx \, dt' \lesssim \epsilon(N) (1 + \| w(t) \|_{H^s}) \| w(t) \|_{H^s}^2
$$

(4.17)

for some $\epsilon(N) = o(N)$, where we have used (3.21).

Inserting (4.10), (4.17) into (4.9), we have

$$
\| J_x^s w(t+\delta) \|_{L^2}^2 \leq e^{-\gamma \delta} \| J_x^s w(t) \|_{L^2}^2 + [\epsilon(N) (1 + \| w(t) \|_{H^s}) - c(1 - e^{-\gamma \delta})] \| w(t) \|_{H^s}^2.
$$

(4.18)

Note that $\| w(0) \|_{H^s} = \| Q_N u_0 \|_{H^s} \leq \| u_0 \|_{H^s}$, therefore, we observe that the last term in (4.18) is negative when $t = 0$ by choosing $N$, dependent only on $\gamma, \delta, \| u_0 \|_{H^s}$, large enough. Hence, for $t \in [0, \delta],$

$$
\| w(t) \|_{H^s} \leq e^{-\gamma t} \| Q_N u_0 \|_{H^s}^2.
$$

(4.19)

By iteration, we conclude that (4.19) holds for any $t > 0$. 

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Proposition 4.3 Let $\gamma > 0$, $u_0 \in \dot{H}^s(\mathbb{T})$, $u$ is the solution of \((1.1)\) \((1.2)\) given in Proposition 3.1, then \((4.2)\) \((4.3)\) are global well-posedness in $\dot{H}^s(\mathbb{T})$ and the solution satisfies the decay estimate \((4.19)\). Moreover, for any $T > 0$,

$$
\|w\|_{X^{s,\frac{1}{2}}} \leq C \|Q_N u_0\|_{H^s},
$$

for some constant $C$ independent of $T$.

Proof. We only need to show \((4.20)\), which follows from \((4.4)\), \((4.19)\) and the estimate

$$
\|f\|_{X^{\alpha}_{s,b}} \leq \|f\|_{X^{\alpha_0}_{s,b}} + \|f\|_{X^{\alpha/\alpha_0}_{s,b}}.
$$

for any $f \in X^{\alpha}_{s,b}$, $\Omega_0 \subset \Omega$ (see \[12\] for the proof in the real-line case). \(\square\)

4.2 Regularity of $v$

We obtain the global well-posedness of $v$, by the global well-posedness of $u$ and $w$. Moreover, by \((3.21)\) and \((4.19)\), we have the bound

$$
\limsup_{t \to +\infty} \|v(t)\|_{H^s} \leq 2K(f).
$$

Now we consider the regularity of $v$. For this purpose, we split $v$ again into two parts by writing

$$
v = y + Q_N v,
$$

where $y = P_N v = P_N u$. Then by \((3.2)\) and \((3.21)\), we have

$$
\|y\|_{L^\infty_t \dot{H}^m_x}, \|y\|_{X^{s,\frac{m}{m-s},\frac{1}{2}}} \lesssim N^{m-s}, \text{ for any } m \geq s.
$$

Therefore, we only focus on the regularity of $Q_N v$. Note that the method used in \[7\] is seemly not suitable for this situation, since we hardly give enough estimates on $y_t$ in the low regularity case. For this purpose, we employ the one used in \[13\] and \[18\]. First, we introduce the functions

$$
\hat{g}(\xi) = \frac{\hat{f}}{i\xi^3 + \gamma}, \quad g_N = Q_N g,
$$

(4.24)
then $g$ is the solution of
\[ \partial^3_x g + \gamma g = f. \]

Let $z = Q_N v - g_N$, then
\[ \partial_t z + \partial^3_x z + \gamma z = -\frac{1}{2} \partial_x Q_N (v^2), \quad (4.25) \]
\[ z(0) = -g_N. \quad (4.26) \]

Now we turn to prove that $z$ is uniformly bounded in $H^{s+3}(\mathbb{T})$ when $f \in H^s(\mathbb{T})$. For this, some lemmas are needed.

**Lemma 4.4** Let $g_N$ defined in (4.24) for $f \in H^s(\mathbb{T})$ and $l < \frac{3}{2} + s$, then
\[ \|g_N\|_{H^{s+3}} \leq \|Q_N f\|_{H^s}; \quad \|g_N\|_{X_\ell^{\frac{3}{2}}} \lesssim \|Q_N f\|_{H^s}. \quad (4.27) \]

**Proof.** The first term follows from the definition (4.24). On the other hand,
\[
\|\psi(\tau/\delta) g_N\|_{X_{\ell}^{\frac{3}{2}}} = \delta \left\| \langle \tau - \xi^3 \rangle^\frac{1}{2} \langle \xi \rangle^l \psi(\delta \tau) \hat{g}_N \right\|_{L_x^2} \\
\lesssim \delta \left\| \langle \tau \rangle^\frac{1}{2} \langle \xi \rangle^l \hat{\psi}(\delta \tau) \hat{g}_N \right\|_{L_x^2} + \delta \left\| \langle \xi \rangle^{l+\frac{3}{2}} \hat{\psi}(\delta \tau) \hat{g}_N \right\|_{L_x^2} \\
= \delta \left\| \langle \tau \rangle^\frac{1}{2} \hat{\psi}(\delta \tau) \right\|_{L_x^2} \|g_N\|_{H^l} + \delta \left\| \hat{\psi}(\delta \tau) \right\|_{L_x^{\frac{3}{2}}} \|g_N\|_{H^{l+\frac{3}{2}}} \\
\lesssim \|\psi\|_{H^{\frac{3}{2}}} \|g_N\|_{H^{s+3}}.
\]

Then the second term follows from the result of the first term. \qed

**Lemma 4.5** For $s \geq -\frac{1}{2}$, $f \in \dot{H}^s(\mathbb{T})$, $\gamma \in \mathbb{R}$, (4.25), (4.26) are local well-posedness in $\dot{H}^s(\mathbb{T})$ with some lifetime $\delta$ which depends on $\gamma, \|u_0\|_{H^s}, \|f\|_{H^s}$ but is independent of $N$. Moreover,
\[ \|z\|_{Y_s} \lesssim K_1, \quad (4.28) \]
for some constant $K_1$ independent of $N$.

**Proof.** It follows easily from (2.8), (4.6), (3.21), (4.22) and the standard fixed point argument. \qed

Next we prove the regularity of $z$. 24
Proposition 4.6 For $s > -\frac{1}{2}$, $f \in \dot{H}^s(\mathbb{T})$, $\gamma \in \mathbb{R}$, the solution $z$ obtained in Lemma 4.5 belongs to $Y^{s+3} \subset C([-\delta, \delta]; H^{s+3}(\mathbb{T}))$ with some lifetime $\delta$, which depends on $\gamma$, $\|u_0\|_{H^s}$, $\|f\|_{H^s}$ but is independent of $N$. Moreover,

$$\|z\|_{Y^{s+3}} \lesssim K_2(N) + \|z(0)\|_{H^{s+3}}. \quad (4.29)$$

To prove this proposition, we need the following results.

Lemma 4.7 For any $r \geq 0$, and $h \in H^r(\mathbb{T})$, the following bilinear estimate holds,

$$\|\psi(t/\delta)\partial_x(h^2)\|_{Z^r} \leq C\|h\|^2_{H^r}. \quad (4.27)$$

Proof. By replacing $Z^s$ by $X_{s, \frac{1}{2}+}$, then it is indeed a consequence of Lemma 2.5 in [18], although it is given in the real-line case. \hfill \Box

Proof of Proposition 4.6. Rewrite the nonlinearity as

$$v^2 = y^2 + z^2 + g_N^2 + 2yz + 2yg_N + 2zg_N, \quad (4.30)$$

then for some $\mu > 0$,

$$\|\partial_x(v^2)\|_{Z^{s+3}} \lesssim C\|g_N\|_{H^{s+3}} + \delta^\mu \left( \|y\|_{X_{s+3, \frac{1}{2}}} + \|z\|_{X_{s+3, \frac{1}{2}}} \right) \cdot \left( \|y\|_{X_{s, \frac{1}{2}}} + \|z\|_{X_{s, \frac{1}{2}}} + \|g_N\|_{H^{s+3}} \right) \lesssim C\|f\|_{H^s} + \delta^\mu \left( \|y\|_{X_{s+3, \frac{1}{2}}} + \|z\|_{X_{s+3, \frac{1}{2}}} \right), \quad (4.31)$$

where we used (2.11) to treat the first, second and fourth terms in (4.30), used Lemma 4.7 to treat the third term and used (2.24) to treat the fifth, sixth terms. Taking the $Y^{s+3}$-norm onto the two sides of the Duhamel’s integral equation of (4.25), then the results easily follow from (4.23), (4.27) and (4.28). \hfill \Box

We drive the energy equation of $z$ in $H^{s+3}(\mathbb{T})$ to find

$$\|z(t + \delta)\|_{H^{s+3}}^2 = e^{-\gamma \delta} \|z(t)\|_{H^{s+3}}^2 + \int_0^\delta e^{-\gamma(\delta - t')} \left\{ - \int_{\mathbb{T}} \partial_x^4 J^s_x(v^2) \cdot \partial_x^2 J^s_x z \, dx - \gamma \|z\|_{H^{s+3}}^2 \right\} \, dt'. \quad (4.32)$$
where we have omitted the variable $t' - \delta + t$ of the functions inside the time integral. In the following, we will prove that

$$\int_0^\delta e^{-\gamma(\delta-t')} \int_T \partial^4_x J^s_x(\nu^2) \cdot \partial^3_x J^s_x z \, dx dt' \lesssim K_3(N) + \epsilon(N) \|z(t)\|_{H^{s+3}}^2. \quad (4.33)$$

For this, we need the following lemma.

Lemma 4.8 For any $s > -\frac{1}{2}$, and $z_0 \in X^\delta_{s, \frac{3}{2}}$, $z \in X^\delta_{s+3, \frac{1}{2}}$ with $z = Q_N z$, we have

$$\int_0^\delta e^{-\gamma(\delta-t')} \int_T \partial^4_x J^s_x(z_0 \cdot z) \cdot \partial^3_x J^s_x z \, dx dt' \lesssim \epsilon(N) \|z_0\|_{X^\delta_{s, \frac{3}{2}}} \|z\|_{X^\delta_{s+3, \frac{1}{2}}},$$

(4.34)

where $\epsilon(N) = o(N)$. On the other hand, by the duality and \[2.12\],

$$\int_0^\delta e^{-\gamma(\delta-t')} \int_T \partial^4_x J^s_x(Q_N \cdot z_0 \cdot z) \cdot \partial^3_x J^s_x z \, dx dt'$$

$$\lesssim \|\partial^4_x J^s_x(Q_N \cdot z_0 \cdot z)\|_{X^\delta_{s, \frac{3}{2}}} \|\partial^3_x J^s_x z\|_{X^\delta_{s+3, \frac{1}{2}}}$$

$$\lesssim \left( \|Q_N \cdot z_0\|_{X^\delta_{s+3, \frac{1}{2}}} \|z\|_{X^\delta_{s+3, \frac{1}{2}}} + \|Q_N \cdot z_0\|_{X^\delta_{s, \frac{3}{2}}} \|z\|_{X^\delta_{s+3, \frac{1}{2}}} \right) \|z\|_{X^\delta_{s+3, \frac{1}{2}}}$$

$$\lesssim \epsilon(N) \left( \|z_0\|_{X^\delta_{s+3, \frac{1}{2}}} \|z\|_{X^\delta_{s+3, \frac{1}{2}}} + \|z_0\|_{X^\delta_{s, \frac{3}{2}}} \|z\|_{X^\delta_{s+3, \frac{1}{2}}} \right) \|z\|_{X^\delta_{s+3, \frac{1}{2}}}. \quad (4.34)$$

Summing up the two estimates above, we have (4.34).

Now we return to the proof of (4.33). By (4.30), we split $\int_0^\delta e^{-\gamma(\delta-t')} \int_T \partial^4_x J^s_x(\nu^2) \cdot \partial^3_x J^s_x z \, dx dt'$ into four parts by writing

$$I_1 = \int_0^\delta e^{-\gamma(\delta-t')} \int_T \partial^4_x J^s_x(y^2) \cdot \partial^3_x J^s_x z \, dx dt';$$

$$I_2 = \int_0^\delta e^{-\gamma(\delta-t')} \int_T \partial^4_x J^s_x(g_N^2) \cdot \partial^3_x J^s_x z \, dx dt';$$

$$I_3 = 2 \int_0^\delta e^{-\gamma(\delta-t')} \int_T \partial^4_x J^s_x[(y + z)g_N] \cdot \partial^3_x J^s_x z \, dx dt';$$

$$I_4 = \int_0^\delta e^{-\gamma(\delta-t')} \int_T \partial^4_x J^s_x[(2y + z)] \cdot \partial^3_x J^s_x z \, dx dt'. $$
Then we treat $I_1$ by duality, (2.12), (4.23), (4.29) and Young's inequality to find

$$I_1 \lesssim \|y\|_{X^\delta_{s+\frac{1}{2}}} \|y\|_{X^\delta_{s+\frac{3}{2}}} \|z\|_{X^\delta_{s+\frac{5}{2}}} \lesssim N^3 (K_2(N) + \|z(t)\|_{H^{s+3}}) \leq K(N) + \epsilon(N) \|z(t)\|^2_{H^{s+3}}.$$ 

We treat $I_2$ by duality, Lemma 4.7, (4.27), (4.29) and Young’s inequality to find

$$I_2 \lesssim \|g_N\|^2_{H^{s+3}} \|z\|_{X^\delta_{s+\frac{3}{2}}} \leq K(N) + \epsilon(N) \|z(t)\|^2_{H^{s+3}}.$$ 

We treat $I_3$ by duality, (2.31), (4.23), (4.29) and Young’s inequality to find

$$I_3 \lesssim \left(\|y + z\|_{X^\delta_{s+\frac{1}{2}}} \|g_N\|_{H^{s+3}} + \|y + z\|_{X^\delta_{s+\frac{3}{2}}} \|g_N\|_{H^{s+3}} + \|y\|_{X^\delta_{s+\frac{1}{2}}} \|z\|_{X^\delta_{s+\frac{5}{2}}} \right) \|z\|_{X^\delta_{s+\frac{5}{2}}} \lesssim K(N) + \epsilon(N) \|z(t)\|^2_{H^{s+3}}.$$ 

We treat $I_4$ by employing Lemma 4.8, (1.23), (4.28), (4.29) and Young’s inequality to find

$$I_4 \lesssim \epsilon(N) \left(\|y\|_{X^\delta_{s+\frac{1}{2}}} \|z\|_{X^\delta_{s+\frac{3}{2}}} + \|y\|_{X^\delta_{s+\frac{3}{2}}} \|z\|_{X^\delta_{s+\frac{5}{2}}} + \|z\|_{X^\delta_{s+\frac{1}{2}}} \|z\|_{X^\delta_{s+\frac{5}{2}}} \right) \|z\|_{X^\delta_{s+\frac{5}{2}}} \lesssim K(N) + \epsilon(N) \|z(t)\|^2_{H^{s+3}}.$$ 

Then (4.33) follows from the estimates above.

On the other hand, by (4.29) we have

$$\gamma \int_0^t e^{-\gamma(t-t')} \|\gamma\|^2_{H^{s+3}} \, dt' \geq c(1 - e^{-\gamma t}) \|z(t)\|^2_{H^{s+3}} - K_2(N).$$

Therefore, combining this with (4.32), (4.33), we have

$$\|z(t + \delta)\|^2_{H^{s+3}} \leq e^{-\gamma \delta} \|z(t)\|^2_{H^{s+3}} + K_3(N) + \left[ C \epsilon(N) - c(1 - e^{-\gamma \delta}) \right] \|z\|^2_{H^{s+3}}.$$ 

Note that the last term is always negative if we choose $N$, dependent on $s, \gamma, \delta$, large enough. So by iteration we have,

$$\|z(t)\|^2_{H^{s+3}} \leq e^{-\gamma t} \|g_N\|^2_{H^{s+3}} + K_3(N).$$
Since \( v = y + z + g_N \), combining it with (4.23), (4.27), we have

\[
\|v(t)\|_{H^{s+3}}^2 \leq e^{-\gamma t} \|Qf\|_{H^s}^2 + K_4(N). \tag{4.35}
\]

**Proposition 4.9** For \( s > -\frac{1}{2} \), \( f \in H^s(\mathbb{T}) \), \( \gamma > 0 \), (4.1) (4.3) are global well-posedness in \( H^{s+3}(\mathbb{T}) \). Moreover, the solution satisfies (4.35) and

\[
\|v\|_{X_{l,\frac{1}{2}}^T} \lesssim K_5(T), \tag{4.36}
\]

for any \( l < \frac{3}{2} + s, T > 0 \), where \( K_5 \) is dependent on \( s, \gamma, \|u_0\|_{H^s}, \|f\|_{H^s}, T \).

**Proof.** We only need to see (4.36), which follows from (4.23), (4.27), (4.29) and (4.21).

\( \square \)

## 5 Existence of Global Attractor and Asymptotic Smoothing Effect

### 5.1 Existence of Global Attractor in \( \dot{H}^s(\mathbb{T}) \)

Based on Corollary 3.5, we need to show the asymptotically compact of the solution map \( S(t) \) in \( H^s(\mathbb{T}) \) to prove the existence of global attractor. Let \( \{u_{0n}\}_n \) be a bounded sequence of initial data in \( \dot{H}^s(\mathbb{T}) \) and the time sequence \( \{t_n\}_n \) tending to infinity. Let \( u_n(t) = S(t)u_{0n} \) be the corresponding solution of (1.1) (1.2) and write \( u_n(t) = v_n(t) + w_n(t) \), where \( v_n(t), w_n(t) \) are the solutions of (4.1)–(4.3) corresponding to the initial condition \( u_{0n} \). The plan now is to show that \( u_n(t_n) \) is precompact in \( H^s(\mathbb{T}) \), with \( w_n(t_n) \) decay to zero and \( v_n(t_n) \) bounded in \( H^{s+3}(\mathbb{T}) \) and precompact in \( H^s(\mathbb{T}) \).

By Proposition 4.9, we first get that for any \( T > 0 \),

\[
\{v_n(t_n + \cdot)\}_n \quad \text{is bounded in} \quad C([0, T]; H^{s+3}(\mathbb{T})). \tag{5.1}
\]

We recall that, a sequence \( \{f_n(t)\}_n \), for \( t \in \Omega \), is uniformly equicontinuous in a Banach space \( X \), if for any \( \epsilon > 0 \), there exist an \( \eta > 0 \), such that, for any \( n \in \mathbb{N}, t, t' \in \Omega \),

\[
\|f_n(t) - f(t')\|_X \leq \epsilon, \quad \text{if} \quad |t - t'| \leq \eta.
\]
Lemma 5.1 For $t \in [0, T]$, $\{v_n(t)\}_n$ is uniformly equicontinuous in $H^s(\mathbb{T})$.

Proof. For any $t, \eta \in \mathbb{R}$, we have
\[
\|v_n(t + \eta) - v_n(t)\|_{H_x^s} \leq \left\| \langle \xi \rangle^s \int |\hat{v}_n(\xi, \tau)| \, d\tau \right\|_{L_x^2}.
\] (5.2)

Given $\epsilon > 0$, since $\|\langle \xi \rangle^s \hat{v}_n\|_{L_x^2 L_t^1}$ is uniformly bounded, there exists a large $\tau_0 = \tau_0(\epsilon) > 0$, such that
\[
\|\langle \xi \rangle^s \int_{|\tau| \geq \tau_0} |\hat{v}_n(\xi, \tau)| \, d\tau\|_{L_x^2} \leq \epsilon.
\]
On the other hand,
\[
\sup_{|\tau| \leq \tau_0} |e^{i\eta \tau} - 1| \lesssim |\eta \tau_0|,
\]
which leads to
\[
\|\langle \xi \rangle^s \int_{|\tau| \leq \tau_0} |\hat{v}_n(\xi, \tau)| \, d\tau\|_{L_x^2} \lesssim |\eta \tau_0||\langle \xi \rangle^s \hat{v}_n||_{L_x^2 L_t^1} \leq \epsilon
\]
by choosing $\tau_0 = \tau_0(\epsilon)$ small enough. Then the claimed result follows from (5.2).

Combining (5.1) and Lemma 5.1, we come to the conclusion, by Arzela-Ascoli’s theorem, that there exists a function $\bar{u}$ such that for any $l < s + 3,
\bar{u} \in C([0, T]; H^l(\mathbb{T})) \cap C_w([0, T]; H^{s+3}(\mathbb{T})) \cap L^\infty([0, T]; H^{s+3}(\mathbb{T}))$, (5.3)
and there exists a subsequence of $\{n\}$ (we also denote it by $\{n\}$) such that for any $t \in \mathbb{R}$,
\[
v_n(t_n + t) \rightharpoonup \bar{u}(t) \quad \text{weakly in } H^{s+3}(\mathbb{T}),
\]
\[
\to \bar{u}(t) \quad \text{strongly in } H^l(\mathbb{T}) \quad \text{for any } l < s + 3.
\] (5.4)
Moreover, by (1.19), we have for any $t \in \mathbb{R}, t_n \to +\infty$,
\[
w_n(t_n + t) \to 0 \quad \text{strongly in } H^s(\mathbb{T}).
\] (5.5)
Therefore,
\[
u_n(t_n + t) \to \bar{u}(t) \quad \text{strongly in } H^s(\mathbb{T}).
\]
Hence, we establish the following result.

Proposition 5.2 Let $\gamma > 0, f \in \dot{H}^s(\mathbb{T})$, the weakly damped, forced KdV equation (1.17) possesses a global attractor $\mathcal{A}$ in $\dot{H}^s(\mathbb{T})$, which is bounded in $H^{s+3}(\mathbb{T})$ and compact in $H^l(\mathbb{T})$ for any $l < s + 3$.
5.2 Compactness of the Global Attractor in $H^{s+3}(\mathbb{T})$

In this subsection, we prove that the attractor is in fact compact in $H^{s+3}(\mathbb{T})$. For this purpose, we just restrict the flow on the global attractor and assume that the sequence of the initial data $\{u_{0n}\}_n$ belongs to $\mathcal{A}$. Since $S(t)\mathcal{A} = \mathcal{A}$ for any $t \geq 0$, it is easy to see that the corresponding trajectories $u_n(t)$ are uniformly bounded in $H^{s+3}(\mathbb{T})$.

We consider $u'_n(t) = \frac{d}{dt}u_n(t)$ in $H^s(\mathbb{T})$. Note that by (1.1), $u'_n(t)$ are uniformly bounded in $H^s(\mathbb{T})$, and satisfy that

$$u'_n(t_n + \cdot) \rightarrow \tilde{u}'(\cdot), \quad \text{in } C([0, T]; H^s(\mathbb{T})) \quad \text{for any } s' < s;$$

$$\rightarrow \tilde{u}'(\cdot), \quad \text{weakly in } C_w([0, T]; H^s(\mathbb{T})). \quad (5.6)$$

Moreover, they satisfy the equation

$$\partial_t u' + \partial_x^2 u' + \gamma u' + \partial_x(uu') = 0. \quad (5.7)$$

By the fixed point argument process and the estimate (2.8) and (3.2), we see that (5.7) is locally well-posed in $Y^s$ for $s \geq -\frac{1}{2}$, and restricted on the time interval $[-\delta, \delta]$,

$$\|u'\|_{Y^s} \lesssim \|u'(0)\|_{H^s}. \quad (5.8)$$

Especially, we have by the continuity and (5.6) that for any $t \geq 0$,

$$\|u'_n(t_n + t + \cdot) - \tilde{u}'(t + \cdot)\|_{X^s_{1/2, 1/2}} \lesssim \|u'_n(t_n + t) - \tilde{u}'(t)\|_{H^s_{1/2}} \rightarrow 0, \quad (5.9)$$

when $t_n \rightarrow +\infty$.

Now we drive the energy equations of $u'_n(t)$ to obtain that

$$\|u'_n(t_n)\|_{H^s}^2 = e^{-2\gamma T}\|u'_n(t_n - T)\|_{H^s}^2 - 2 \int_0^T e^{-2\gamma(T-t')} \int_\mathbb{T} \partial_x J^s_x(u_n' u'_n) \cdot J^s_x u'dxdt'. \quad (5.10)$$

We plan to prove that

$$\int_0^T e^{-2\gamma(T-t')} \int_\mathbb{T} \partial_x J^s_x(u_n' u'_n) \cdot J^s_x u'dxdt' \rightarrow \int_0^T e^{-2\gamma(T-t')} \int_\mathbb{T} \partial_x J^s_x(\tilde{u} \tilde{u}') \cdot J^s_x \tilde{u}'dxdt'. \quad (5.11)$$

First, we obtain that when $t_n \rightarrow +\infty$,

$$\int_0^\delta e^{-2\gamma(T-t')} \int_\mathbb{T} \left| \partial_x J^s_x(u_n' u'_n) - \partial_x J^s_x(\tilde{u} \tilde{u}') \right| \cdot J^s_x u'dxdt' \lesssim \|u_n - \tilde{u}\|_{X^s_{1/2, 1/2}} \|u'_n\|_{X^s_{1/2, 1/2}}^2 \rightarrow 0, \quad (5.12)$$
by duality, \((2.12)\), \((5.4)\) and the continuity given in Proposition 3.1.

On the other hand, applying the argument in Section 4.1, we write

\[
\int_{0}^{\delta} e^{-2\gamma(T-t')} \int_{T} \partial_x J_x^s(\bar{u}'_n) \cdot J_x^s u'_n \, dx \, dt' = H_1(\bar{u}, u'_n) + H_2(\bar{u}, u'_n),
\]

where

\[
 H_1(u, v) = -\frac{1}{2} \int_{0}^{\delta} e^{-2\gamma(T-t')} \int_{T} u \cdot \partial_x (J_x^s v)^2 \, dx \, dt';
\]

\[
 H_2(u, v) = \int_{0}^{\delta} e^{-2\gamma(T-t')} \int_{T} \partial_x [J_x^s, u] v \cdot J_x^s v \, dx \, dt'.
\]

Then by \((2.12)\) and \((5.9)\), we have

\[
 H_1(\bar{u}, u'_n) - H_1(\bar{u}, \bar{u}') \\
 \lesssim \|\bar{u}\|_{X^s_{\frac{3}{2}+\frac{s}{4}, \frac{1}{2}}} \|\varphi \partial_x [J_x^s(u'_n - \bar{u}') \cdot J_x^s (u'_n + \bar{u}')]\|_{X^s_{\frac{3}{2}+\frac{s}{4}, \frac{1}{2}}} \\
 \lesssim \|\bar{u}\|_{X^s_{\frac{3}{2}+\frac{s}{4}, \frac{1}{2}}} \|u'_n - \bar{u}'\|_{X^s_{\frac{3}{2}+\frac{s}{4}, \frac{1}{2}}} \|u'_n + \bar{u}'\|_{X^s_{\frac{3}{2}+\frac{s}{4}, \frac{1}{2}}} \\
 \rightarrow 0,
\]

(5.13)

where we note that \(\|\bar{u}\|_{X^s_{\frac{3}{2}+\frac{s}{4}, \frac{1}{2}}}\) is bounded for any \(l < \frac{3}{2} + s\), the proof can be given by the similar argument in Section 4.2.

We write \(H_2(\bar{u}, u'_n) - H_2(\bar{u}, \bar{u}')\) by

\[
\int_{0}^{\delta} e^{-2\gamma(T-t')} \int_{T} \partial_x [J_x^s, \bar{u}] (u'_n - \bar{u}') \cdot J_x^s u'_n \, dx \, dt' \\
+ \int_{0}^{\delta} e^{-2\gamma(T-t')} \int_{T} \partial_x [J_x^s, \bar{u}] \bar{u}' \cdot J_x^s (u'_n - \bar{u}') \, dx \, dt'.
\]

Then by duality, \((2.23)\) and \((5.9)\), we have

\[
 H_2(\bar{u}, u'_n) - H_2(\bar{u}, \bar{u}') \\
 \lesssim \|\varphi \partial_x [J_x^s, \bar{u}] (u'_n - \bar{u}')\|_{X^s_{\frac{3}{2}+\frac{s}{4}, \frac{1}{2}}} \|J_x^s u'_n\|_{X^s_{\frac{3}{2}+\frac{s}{4}, \frac{1}{2}}} \\
+ \|\varphi \partial_x [J_x^s, \bar{u}] \bar{u}'\|_{X^s_{\frac{3}{2}+\frac{s}{4}, \frac{1}{2}}} \|J_x^s (u'_n - \bar{u}')\|_{X^s_{\frac{3}{2}+\frac{s}{4}, \frac{1}{2}}} \\
 \lesssim \|\bar{u}\|_{X^s_{\frac{3}{2}+\frac{s}{4}, \frac{1}{2}}} \left(\|ar{u}'\|_{X^s_{\frac{3}{2}+\frac{s}{4}, \frac{1}{2}}} + \|u'_n\|_{X^s_{\frac{3}{2}+\frac{s}{4}, \frac{1}{2}}}\right) \|u'_n - \bar{u}'\|_{X^s_{\frac{3}{2}+\frac{s}{4}, \frac{1}{2}}} \\
 \rightarrow 0.
\]

(5.14)

Note that we only restrict the integral domain with respect to time on \([0, \delta]\), but it can be easily extended to \([0, T]\) by dividing it into small intervals.
By (5.12)–(5.14), we have (5.11). By the energy equations of $\bar{u}'(t)$ in (5.10) and let $T \to +\infty$, we obtain that
\[
\limsup_{t \to +\infty} \|u_n'(t_n)\|_{H^s}^2 \leq \|\bar{u}'(0)\|_{H^s}^2.
\]
Therefore,
\[
u_n'(t_n) \to \bar{u}'(0), \quad \text{in} \quad H^s(\mathbb{T})
\]
by combining (5.6). This implies
\[
u_n(t_n) \to \bar{u}(0), \quad \text{in} \quad H^{s+3}(\mathbb{T}).
\]
So we give the claim result in this subsection and thus finish the proof of Theorem 1.1.

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