GLOBAL EXISTENCE IN A CHEMOTAXIS SYSTEM WITH SINGULAR SENSITIVITY AND SIGNAL PRODUCTION

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ABSTRACT. In this work we consider the chemotaxis system with singular sensitivity and signal production in a two dimensional bounded domain. We present the global existence of weak solutions under appropriate regularity assumptions on the initial data. Our results generalize some well-known results in the literature.

1. Introduction. Chemotaxis, the directed movement of cells or organisms in response to the gradients of concentration of the chemical stimuli, plays essential roles in various biological processes such as embryonic development, wound healing, disease progression, the location of food sources, avoidance of predators, attracting mates, slime mold aggregation, tumour angiogenesis, and primitive steak formation [16]. The pioneering works of the chemotaxis model was introduced by Keller and Segel in [26], describing the aggregation of cellular slime mold toward a higher concentration of a chemical signal, which reads

\[
\begin{align*}
  u_t &= \Delta u - \chi \nabla \cdot (u \nabla v), & x &\in \Omega, & t &> 0, \\
  v_t &= \Delta v - v + u, & x &\in \Omega, & t &> 0.
\end{align*}
\]  

(1.1)

The mathematical analysis of (1.1) and the variants thereof mainly concentrates on the boundedness and blow-up of the solutions [15, 45]. In addition to the original model, a variety of mathematical models to describe chemotaxis have been proposed, including the systems with the logistic terms [32], chemotaxis-haptotaxis...
models [29], multi-species chemotaxis systems [30, 35], attraction-repulsion chemotaxis system [31, 34], chemotaxis-fluid models [25, 33, 48] and so on. In the past few decades, classical Keller-Segel model and the variants has attracted extensive attentions. For a helpful overview of many models arising out of this fundamental description we refer to the survey [3, 12, 14].

This paper is concerned with the following chemotaxis system with singular sensitivity and signal production:

\[
\begin{aligned}
    u_t &= \nabla \cdot (u^{\beta} \nabla u) - \chi \nabla \cdot (\frac{v}{u} \nabla v), \quad x \in \Omega, \quad t > 0, \\
v_t &= \Delta v - v + u + g(x, t), \quad x \in \Omega, \quad t > 0, \\
\frac{\partial u}{\partial \nu} &= \frac{\partial v}{\partial \nu} = 0, \quad x \in \partial \Omega, \quad t > 0, \\
u(x, 0) &= u_0(x), \quad v(x, 0) = v_0(x), \quad x \in \Omega,
\end{aligned}
\]  

(1.2)

where \( \Omega \subset \mathbb{R}^2 \) is a bounded convex domain with smooth boundary \( \partial \Omega \) and \( \frac{\partial}{\partial \nu} \) denotes the derivative with respect to the outer normal of \( \partial \Omega \), \( u \) represents the cell density and \( v \) denotes the concentration of the chemical signal. \( \chi > 0, \theta > 1 \) is a given parameter, initial data \( u_0, v_0 \) are known functions satisfying

\[
\begin{aligned}
u_0 \in W^{1,\infty}(\Omega) \text{ is nonnegative and} \\
v_0 \in W^{1,\infty}(\Omega) \text{ is positive in } \Omega.
\end{aligned}
\]  

(1.3)

The additional external production of the signal chemical \( g \) is nonnegative function fulfilling

\[
g \in L^{\infty}(0, \infty); L^{1+\delta_0}(\Omega)) \cap C^{\alpha}(\Omega \times (0, \infty)) \text{ with some } \delta_0 \in (0, 1) \text{ and } \alpha > 0.
\]  

(1.4)

Being distinctive from the model (1.2), Keller and Segel [27] introduced a phenomenological model of wave-like solution behavior without any type of cell kinetics, the signal is consumed rather than produced by the cells, a prototypical version of which is given by:

\[
\begin{aligned}
u_t &= \Delta u - \chi \nabla \cdot (\frac{u}{v} \nabla v), \quad x \in \Omega, \quad t > 0, \\
v_t &= \Delta v - uv, \quad x \in \Omega, \quad t > 0.
\end{aligned}
\]  

(1.5)

where \( u \) represents the density of bacteria and \( v \) denotes the concentration of the nutrient. The second equation models consumption of the signal. In the first equation, the chemotactic sensitivity is determined according to the Weber-Fechner law, which says that the chemotactic sensitivity is proportional to the reciprocal of signal density. When system (1.5) without logistic source, Winkler [46] proved that if initial data satisfying appropriate regularity assumptions, the system (1.5) possesses at least one global generalized solution in two dimensional bounded domains. Moreover, he took into account asymptotic behavior of solutions to the system (1.5), and proved that \( v(\cdot, t) \xrightarrow{\mathcal{D}} 0 \) in \( L^{\infty}(\Omega) \) and \( v(\cdot, t) \to 0 \) in \( L^p(\Omega) \) as \( t \to \infty \) provided \( \int_{\Omega} u_0 \leq m, -\int_{\Omega} \ln \left( \frac{v_0}{\|v_0\|_{L^\infty(\Omega)}} \right) \leq M \), where \( m, M \) are positive constants. The same author [47] showed that the system (1.5) admits at least one global renormalized solution which is radially symmetric if initial data \( (u_0, v_0) \) are radially symmetric and \( \Omega := B_R(0) \subset \mathbb{R}^N \) with \( R > 0 \). When \( \frac{u}{v} \) and \( uv \) are replaced by \( \frac{u^\beta}{v} \) and \( f(u)v \), respectively, \( f \in C^1(\mathbb{R}) \) and \( 0 \leq f(u) \leq u^\beta, \beta \in (0, 1] (\beta \neq 0) \) and \( \gamma \in (0, 1) \), Viglialoro [38] proved that the system (1.5) admits a unique global classical solution in two-dimensional bounded domain if \( \|v_0\|_{L^{\infty}(\Omega)} < \chi^{-1/\beta} \). When \( uv \) is replaced by \( f(u)v, f \in C^1(\mathbb{R}) \) and \( 0 \leq f(u) \leq u^\beta, \beta \in (0, 1), \chi \in (0, 1) \) and
any sufficiently regular initial data, Lankeit and Viglialoro [23] showed that the system (1.5) has a global classical solution. Moreover, if additionally \( m = \|u_0\|_{L^1(\Omega)} \) is sufficiently small, then their boundedness is achieved. As \( \Delta u \) is replaced by \( \nabla \cdot (D(u)\nabla u) \) and \( D(u) \) satisfying \( D \in C^1([0, \infty)) \) and \( D(u) \geq \delta u^{m-1} \), Lankeit [22] showed the existence of locally bounded global solutions to the system (1.5) provided that \( m > 1 + \frac{N}{2} \). When \( \Delta u \) is replaced by \( \Delta u^m \), Yan and Li [51] proved that for all reasonably regular initial data, the corresponding Neumann initial-boundary value problem possesses a global generalized solution provided that \( m > 1 + \frac{N-2}{2N} \).

When the system (1.5) has a logistic source \( f(u) \), Lankeit and Lankeit [20] showed that the system (1.5) possesses a global generalized solution for any \( \chi \geq 0, r \geq 0 \) and \( \mu > 0 \) if \( f(u) = \kappa u - \mu u^2 \). As \( f(u) = ru - \mu u^2 \) and \( \frac{1}{\mu} \) is replaced by \( \phi(v) \), \( \phi(v) \in C^1(0, \infty) \) satisfying \( \phi(v) \to 0 \) as \( v \to \infty \). Zhao and Zheng [55] proved that the system (1.5) possesses a unique positive global classical solution provided \( k > 1 \) with \( N = 1 \) or \( k > 1 + \frac{N}{2} \) with \( N \geq 2 \). When \( f(u) = bu - \mu u^2 \) and \( \chi > 1 \), Wang [40] showed that the system (1.5) possesses a unique global classical solution in two-dimensional bounded domain if \( \mu > \mu_* \) with some constant \( \mu_* \). Moreover, the asymptotic behavior of solution is discussed. For more recent outcomes, one can see [7, 18, 19].

When system (1.2) without logistic source and \( g \equiv 0 \), \( \theta = 1 \), \( \frac{1}{\delta} \) is replaced by \( \chi(v) \), \( \chi(v) \leq \frac{\chi_0}{1 + \alpha v} \) with some \( \chi_0 > 0 \), \( \alpha > 0 \) and \( k > 1 \), Winkler [42] proved that for any choice of appropriate initial data, the system (1.2) possesses a unique global classical solution that is bounded in \( \Omega \times (0, \infty) \) for \( N \geq 1 \). Stinner and Winkler [37] showed that for any \( \lambda \in (0, \min\{1, \frac{1}{\sqrt{\chi_0}}\}) \), the system (1.2) admits at least one couple \((u, v)\) of nonnegative functions defined in \( \Omega \times (0, \infty) \) such that \((u, v)\) is a global weak power-\( \lambda \) solution of (1.2) for \( N \geq 2 \). Winkler [44] proved that \( 0 < \chi < \sqrt{\frac{2}{N}} \) then for any such data there exists a global-in-time classical solution. Moreover, global existence of weak solutions is established whenever \( 0 < \chi < \sqrt{\frac{N+2}{3N-4}} \). The boundedness of solution is leave an open problem. Fuije [8] solved the open problem of uniform-in-time boundedness of solutions for \( 0 < \chi < \sqrt{\frac{2}{N}} \), which was conjectured by Winkler [44]. Lankeit [21] showed that the system (1.2) possesses a global bounded classical solution if \( \chi \in (0, \chi_0) \) with some constant \( \chi_0 > 1 \). Recently, Winkler and Yokota [50] proved that the system (1.2) possesses a uniquely determined global classical solution if \( \chi \in (0, \chi_0] \) and \( \chi^2 \leq \delta \), where \( \chi \in (0, \sqrt{\frac{2}{N}}) \), \( \delta > 0 \) are constants. Furthermore, the solution of (1.2) converges to the homogeneous steady state \((\bar{u}_0, \bar{u}_0)\) at an exponential rate with respect to the norm in \((L^\infty(\Omega))^2\) as \( t \to \infty \), where \( \bar{u}_0 = \frac{1}{|\Omega|} \int_{\Omega} u_0 \). Lankeit and Winkler [24] introduced an apparently novel type of generalized solution, and proved that under the hypothesis that

\[
\chi < \begin{cases} 
\infty, & \text{if } t \in [0, T], \\
\sqrt{\delta}, & \text{if } t \in (T, T + \delta), \\
\frac{N}{N-2}, & \text{if } t \in [T + \delta, \infty);
\end{cases}
\]

for all initial data satisfying suitable assumptions on regularity and positivity, an associated no-flux initial-boundary value problem admits a globally defined generalized solution. This solution inter alia has the property that \( u \in L^{1+}(\Omega \times (0, \infty)) \). When the second equation degenerates into an elliptic equation, \( \nabla u \) is replaced by \( \nabla \chi(v) \), \( \chi \in C^{2+}((0, \infty)) \) with some \( \omega \in (0, 1) \), \( \chi' > 0 \), \( \chi(v) \to 0 \) as \( v \to \infty \).
Fujie and Senba [9] showed that the system (1.2) admits a global classical positive solution which is uniformly bounded. Ahn et al. [1] proved that the system (1.2) possesses at least one global nonnegative weak solution if \( \chi < \frac{(4 + \sqrt{N})^2}{4} \). Black [4] showed that the system (1.2) admits at least one global generalized solution if \( 0 < \chi < \frac{N}{N - 2} \). Zhigun [59] showed that system (1.2) admits a generalised supersolution with appropriate condition, and assumed that \((u, v)\) satisfies additional conditions and that the generalised supersolution is a classical solution. When the system (1.2) has a logistic source \( f(u) \) and \( g \equiv 0, \theta \equiv 1 \), as \( f(u) = ru - \mu u^2 \), Zhao and Zheng [54] proved that the system (1.2) possesses a global bounded classical solutions if \( r > \chi^2 \) for \( 0 < \chi \leq 2 \), or \( r > \chi - 1 \) for \( \chi > 2 \) in two-dimensional bounded domains. Based on existent results in [54], Zheng et al. [58] showed that the global bounded solution \((u, v)\) exponentially converges to the steady state \((r \mu, r \mu)\). Zhao and Zheng [56] generalized their own work [54] to the higher dimensional case. For more recent outcomes, one can see [5, 6, 10, 11, 36, 39, 41, 49, 53, 57].

Throughout the above analysis, compared with the system (1.2) with linear diffusion, it is so fragmentary that the system (1.2) with nonlinear diffusion. This paper will do further research in this direction, we show the system (1.2) possesses at least one global weak solutions.

Now the main result read as follows.

**Theorem 1.1.** Let \( \Omega \subset \mathbb{R}^2 \) be a bounded convex domain with smooth boundary, \( \chi > 0 \). Assume that \( g \) fulfills (1.4), and that

\[
\theta > 2.
\]

Then for any choice of initial data \((u_0, v_0)\) satisfy (1.3), the system (1.2) possesses at least one global weak solution in the sense of Definition 2.1 below which is locally bounded in the sense that

\[
\sup_{t \in [0, T)} \|u(\cdot, t)\|_{L^\infty(\Omega)} + \sup_{t \in [0, T)} \|v(\cdot, t)\|_{W^{1, q}(\Omega)} < \infty \text{ for all } T > 0 \text{ and } q > 2. \quad (1.6)
\]

**Remark 1.** Theorem 1.1 partly generalizes the result in [8, 21, 24, 37, 44, 50, 53].

In this paper, we use symbols \( C_i \) and \( c_i \) \((i = 1, 2, \cdots)\) as some generic positive constants which may vary in the context. For simplicity, \( u(x, t) \) is written as \( u \), the integral \( \int_{\Omega} u(x)dx \) is written as \( \int_{\Omega} u(x) \) and \( \int_0^t \int_{\Omega} u(x, s)dxds \) is written as \( \int_0^t \int_{\Omega} u(x, s) \).

The present paper has the following layout. In Section 2, we summarize useful lemmas in order to prove the main results. In Section 3, we give some fundamental estimates for the solution to the system (1.2) and prove Theorem 1.1.

2. **Preliminaries.** Under the assumptions of \( u \), the first equation of system (1.2) may be degenerate at \( u = 0 \). Therefore, the system (1.2) does not allow for classical solvability in general as the well-known porous medium equations. We introduce the following definition of weak solutions.

**Definition 2.1.** Assume that \( \theta \geq 1, \chi > 0 \), and that (1.3) and (1.4) hold. Then a pair \((u, v)\) of functions \( u \in L^\theta_{loc}(\Omega \times [0, \infty)) \) and \( v \in L^1_{loc}([0, \infty); W^{1, 1}(\Omega)) \) will be called a global weak solution of (1.2) if \( u \geq 0 \) and \( v > 0 \) a.e. in \( \Omega \times (0, \infty) \), if

\[
\frac{u}{v} \nabla v \text{ belongs to } L^1_{loc}(\Omega \times [0, \infty); \mathbb{R}^2),
\]
the integral identities
\[-\int_0^\infty \int_\Omega w \psi_t - \int_\Omega u_0 \psi(\cdot, 0) = \frac{1}{\delta} \int_0^\infty \int_\Omega u^\theta \Delta \psi + \chi \int_0^\infty \int_\Omega \frac{u}{\psi} \nabla \psi \cdot \nabla \psi \]  
(2.1)

and
\[-\int_0^\infty \int_\Omega v \psi_t - \int_\Omega v_0 \psi(\cdot, 0) = - \int_0^\infty \int_\Omega \nabla v \cdot \nabla \psi - \int_0^\infty \int_\Omega v \psi + \int_0^\infty \int_\Omega w_\psi + \int_0^\infty \int_\Omega g \psi \]  
(2.2)

hold for all \( \psi \in C^0_0(\overline{\Omega} \times [0, \infty)) \) fulfilling \( \frac{\partial \psi}{\partial \nu} = 0 \) on \( \partial \Omega \times (0, \infty) \).

In order to construct weak solutions by an approximation procedure, we introduce the following regularized problems:
\[
\begin{align*}
\left\{ \begin{array}{ll}
u_{\varepsilon t} &= \nabla \cdot ((u_\varepsilon + \varepsilon)^{\theta-1} \nabla u_\varepsilon) - \chi \nabla \cdot (\frac{u_\varepsilon}{\psi_\varepsilon} \nabla v_\varepsilon), & x \in \Omega, \ t > 0, \\
u_{\varepsilon t} &= \Delta u_\varepsilon - v_\varepsilon + \frac{u_\varepsilon}{1 + cu_\varepsilon} + g(x, t), & x \in \Omega, \ t > 0, \\
\frac{\partial u_\varepsilon}{\partial \nu} &= \frac{\partial v_\varepsilon}{\partial \nu} = 0, & x \in \partial \Omega, \ t > 0, \\
u_\varepsilon(x, 0) &= u_0(x), \ v_\varepsilon(x, 0) = v_0(x), & x \in \Omega,
\end{array} \right.
\]  
(2.3)

for \( \varepsilon \in (0, 1) \). All of these problems (2.3) are indeed globally solvable in the classical sense.

**Lemma 2.2.** Let \( \Omega \subset \mathbb{R}^2 \) be a bounded convex domain with smooth boundary, \( \chi > 0 \). Assume that \( g \) fulfills (1.3), and let \( \theta > 1 \) and \( \varepsilon \in (0, 1) \). Then there exist functions
\[
\begin{align*}
u_\varepsilon &\in C^0(\overline{\Omega} \times [0, \infty)) \cap C^2(\overline{\Omega} \times (0, \infty)), \\
v_\varepsilon &\in \bigcap_{p>2} C^0([0, \infty); W^{1,p}(\Omega)) \cap C^2(\overline{\Omega} \times (0, \infty)),
\end{align*}
\]
which solve (2.3) classically in \( \overline{\Omega} \times [0, \infty) \), and which are such that \( u_\varepsilon > 0 \) in \( \overline{\Omega} \times (0, \infty) \) and \( v_\varepsilon > 0 \) in \( \overline{\Omega} \times [0, \infty) \).

**Proof.** This can be seen by a straightforward adaptation of the reasoning in [49] on the basis of standard results on local existence and extensibility, as provided by the general theory in [2]. This completes the proof. \( \square \)

**Lemma 2.3.** Let \( \theta > 1 \). Then there exists \( \eta = \eta(T) > 0 \) such that whenever \( T > 0 \)
\[
\inf_{x \in \Omega} v_\varepsilon(x, t) \geq \eta > 0 \text{ for all } t \in (0, T) \text{ and } \varepsilon \in (0, 1).
\]  
(2.4)

**Proof.** Although the proof is similar to Lemma 2.1 in [8], we still intend to give a more detailed proof for the convenience of the reader. We use a known result for the Neumann heat semigroup \( e^{\Delta} \). In the same way as in the proof of Lemma 3.1 in [13], we can obtain the pointwise estimate from below
\[e^{\Delta}w(x) \geq \frac{1}{4\pi t} e^{-\frac{(\text{diam}\Omega)^2}{4t}} \cdot \int_\Omega w > 0 \text{ for all nonnegative } w \in C^0(\overline{\Omega}), \ (x, t) \in \Omega \times (0, \infty),\]

where \( \text{diam}\Omega := \max_{x, y \in \Omega} |x - y| \). First by the positivity of \( v_0 > 0 \) in \( \overline{\Omega} \) and the maximum principle we have
\[v_\varepsilon(t) \geq \min_{x \in \Omega} v_0(x) \cdot e^{-t} \text{ for all } t \geq 0.\]

Now fix \( \tau > 0 \). Then it follows that
\[v_\varepsilon(t) \geq \min_{x \in \Omega} v_0(x) \cdot e^{-\tau} =: \eta_\varepsilon > 0 \text{ for all } t \in [0, \tau].\]
Next, the representation formula of $v$, the maximal principle and (2.3) imply that
\[
v_v(t) = e^{(\Delta - 1)}v_0 + \int_0^t e^{(t-s)(\Delta - 1)} \frac{u_\varepsilon}{1 + \varepsilon u_\varepsilon}(\cdot, s) ds + \int_0^t e^{(t-s)(\Delta - 1)} g(\cdot, s) ds
\]
\[
\geq e^{(\Delta - 1)}v_0 + \int_0^t e^{(t-s)(\Delta - 1)} g(\cdot, s) ds
\]
\[
\geq \int_0^t \frac{1}{4\pi(t-s)} e^{-((t-s)+\frac{diam(\Omega)^2}{4\pi})} \cdot \left( \int_\Omega g(\cdot, s) dx \right) ds
\]
\[
= \left\{ \inf_{s > \tau} \int_\Omega g(\cdot, s) \right\} \cdot \int_0^t \frac{1}{4\pi r} e^{-r \frac{(diam(\Omega)^2)}{4\pi}} dr
\]
\[
\geq \left\{ \inf_{s > \tau} \int_\Omega g(\cdot, s) \right\} \cdot \int_\tau^t \frac{1}{4\pi r} e^{-r \frac{(diam(\Omega)^2)}{4\pi}} dr =: \eta_2 > 0 \text{ for all } t \in (r, \infty).
\]

Therefore we have $v_v(t) \geq \min\{\eta_1, \eta_2\}$ for all $t \geq 0$. This completes the proof. \qed

**Lemma 2.4.** Let $\theta > 1$. Then there exists $C_1 := C_1(T) > 0$ such that for all $T > 0$,
\[
\|v_v(\cdot, t)\|_{L^1(\Omega)} = \|u_0\|_{L^1(\Omega)}
\] (2.5)
and
\[
\|v_v(\cdot, t)\|_{L^1(\Omega)} = e^{-\frac{1}{\theta}} \|v_0\|_{L^1(\Omega)} + \|u_0\|_{L^1(\Omega)} (1 - e^{-\frac{1}{\theta}}) + \frac{1}{\theta} \int_0^t \|g(\cdot, s)\|_{L^1(\Omega)} e^{\frac{s-t}{\theta}} ds.
\] (2.6)

In particular,
\[
\|v_v(\cdot, t)\|_{L^1(\Omega)} \leq \|v_0\|_{L^1(\Omega)} + \|u_0\|_{L^1(\Omega)} + C_1 \|g(\cdot, s)\|_{L^{\infty}(0, \infty); L^{1+\theta}(\Omega))}
\] (2.7)
holds.

**Proof.** The results follows almost immediately from integrating the first and second equations of (2.3) and we omit giving details on this here. This completes the proof. \qed

**Lemma 2.5** (Gagliardo-Nirenberg interpolation inequality [34]). Let $0 < \theta \leq p \leq \frac{2N}{(N-2)^+}$. Then there exists positive constant $C_{GN}$ such that for all $u \in W^{1,2}(\Omega) \cap L^\theta(\Omega)$,
\[
\|u\|_{L^p(\Omega)} \leq C_{GN}(\|\nabla u\|_{L^2(\Omega)}^{\alpha} \|u\|_{L^\theta(\Omega)}^{1-\alpha} + \|u\|_{L^p(\Omega)})
\]
is valid with $\alpha = \frac{\theta - \frac{N}{2}}{1 - \frac{N}{2} + \frac{N}{\theta}} \in (0, 1)$, where $(N-2)_+$ denotes the positive part of $(N-2)$.

3. **Proof of Theorem 1.1.** In this section, we establish some priori estimates for solutions to the system (2.3), it is crucial ingredient for the proof of our main result.

**Lemma 3.1.** Assume that $\theta > 1$, and let $p \in (0, 1)$. Then there exists $C_2 := C_2(p, T) > 0$ such that for all $T > 0$,
\[
\int_t^{t+1} \int_\Omega v_{\varepsilon}^{p-2} |\nabla v_{\varepsilon}|^2 \leq C_2 \text{ for all } t \in (0, T) \text{ and } \varepsilon \in (0, 1).
\] (3.1)

**Proof.** From Lemma 2.4, there exists $c_1(T) > 0$ such that for all $T > 0$
\[
\int_\Omega v_v(\cdot, t) \leq c_1(T) \text{ for all } t \in (0, T + 1) \text{ and } \varepsilon \in (0, 1).
\]
Owing to \( p \in (0,1) \), using Young’s inequality, we have
\[
\int_{\Omega} v_p^\varepsilon (\cdot,t) \leq \int_{\Omega} (u^\varepsilon (\cdot,t) + 1) \leq c_1(T) + |\Omega| \text{ for all } t \in (0, T+1) \text{ and } \varepsilon \in (0,1).
\] (3.2)
Multiplying the second equation in (2.3) by \( v_p^{p-1} \), integrating by parts, we obtain
\[
\frac{1}{p} \frac{d}{dt} \int_{\Omega} v_p^\varepsilon = (1-p) \int_{\Omega} v_p^{p-2} |\nabla v_p^\varepsilon|^2 - \int_{\Omega} v_p^\varepsilon + \int_{\Omega} u^\varepsilon v_p^{p-1} + \int_{\Omega} g v_p^{p-1}
\]
\[
\geq (1-p) \int_{\Omega} v_p^{p-2} |\nabla v_p^\varepsilon|^2 - \int_{\Omega} v_p^\varepsilon \text{ for all } t > 0 \text{ and } \varepsilon \in (0,1),
\]
which yields
\[
(1-p) \int_t^{t+1} \int_{\Omega} v_p^{p-2} |\nabla v_p^\varepsilon|^2 \leq \frac{1}{p} \int_{\Omega} v_p^\varepsilon (\cdot,t+1) - \frac{1}{p} \int_{\Omega} v_p^\varepsilon (\cdot,t) + \int_t^{t+1} \int_{\Omega} v_p^\varepsilon
\]
\[
\leq \frac{1}{p} \int_{\Omega} v_p^\varepsilon (\cdot,t+1) + \int_t^{t+1} \int_{\Omega} v_p^\varepsilon \leq \frac{1}{p} (c_1(T) + |\Omega|) + c_1(T) + |\Omega|
\]
for all \( t > 0 \) and \( \varepsilon \in (0,1) \), by means of (3.2), we have
\[
\frac{1}{p} \int_{\Omega} v_p^\varepsilon (\cdot,t+1) + \int_{\Omega} v_p^{p-2} |\nabla v_p^\varepsilon|^2 \leq \frac{1}{p} (c_1(T) + |\Omega|) + c_1(T) + |\Omega|
\]
for all \( t \in (0, T) \) and \( \varepsilon \in (0,1) \). Letting \( C_2 := \frac{1+p}{p(1-p)} (c_1(T) + |\Omega|) \), we readily obtain (3.1). This completes the proof. 

**Lemma 3.2.** Assume that \( \theta > 1 \), and let \( q \in [1,2) \) close to 2. Then there exists \( C_3 := C_3(q,T) > 0 \) with the property that for all \( T > 0 \), any \( \varepsilon \in (0,1) \) and each \( t \in (0, T) \) satisfying \( t \geq 2 \), we can find \( t_* = t_*(\varepsilon) \in (t-2, t-1) \) such that
\[
\|v_\varepsilon (\cdot,t_*)\|_{W^{1,q}(\Omega)} \leq C_3.
\]
(3.3)

**Proof.** From Lemma 2.4 and Lemma 3.1, there exist \( c_1, c_2 > 0 \) such that for all \( T \geq 2 \),
\[
\int_{t-2}^{t-1} \int_{\Omega} v_\varepsilon^{-\frac{3}{2}} |\nabla v_\varepsilon|^2 \leq c_1 \text{ for all } t \in [2, T] \text{ and } \varepsilon \in (0,1).
\]
(3.4)
and
\[
\int_{\Omega} v_\varepsilon \leq c_2 \text{ for all } t \in (0, T) \text{ and } \varepsilon \in (0,1).
\]
(3.5)
Moreover, given \( q \in [1,2) \), we let \( p = p(q) := \frac{3q}{2q-2} > 1 \). Using the continuous embedding \( W^{1,2}(\Omega) \hookrightarrow L^p(\Omega) \), there exists \( c_3 := c_3(q) > 0 \) such that
\[
\|\phi\|_{L^p(\Omega)} \leq c_3 \|\nabla \phi\|_{L^2(\Omega)} + c_3 \|\phi\|_{L^q(\Omega)} \text{ for all } \phi \in W^{1,2}(\Omega).
\]
(3.6)
Letting \( T \geq 2 \) and \( t \in [2, T] \) be arbitrary, from (3.4) there exists \( t_* = t_*(\varepsilon) \in (t-2, t-1) \) such that
\[
\int_{\Omega} v_\varepsilon^{-\frac{3}{2}} (\cdot,t_*) |\nabla v_\varepsilon (\cdot,t_*)|^2 \leq c_1,
\]
(3.7)
comply with (3.5) and (3.6), we obtain
\[
\int_{\Omega} v_\varepsilon^\varepsilon (\cdot,t_*) = \|v_\varepsilon^\varepsilon (\cdot,t_*)\|_{L^p(\Omega)}^p
\]
\[
\leq c_3 \|\nabla v_\varepsilon^\varepsilon (\cdot,t_*)\|_{L^2(\Omega)}^p + c_3 \|v_\varepsilon^\varepsilon (\cdot,t_*)\|_{L^q(\Omega)}^p
\]
\[
\leq \frac{c_3}{4^p} + c_1 c_2^p := c_4.
\]
Applying (3.7) and Young’s inequality, we have
\[
\int_{\Omega} |\nabla v_\varepsilon(\cdot, t_*)|^q = \int_{\Omega} \left\{ v_\varepsilon^{-\frac{2}{q}}(\cdot, t_*) \right\}^q |\nabla v_\varepsilon(\cdot, t_*)|^2 v_\varepsilon^{\frac{3q}{2}}(\cdot, t_*) \\
\leq \int_{\Omega} v_\varepsilon^{-\frac{2}{q}}(\cdot, t_*) |\nabla v_\varepsilon(\cdot, t_*)|^2 + \int_{\Omega} v_\varepsilon^{\frac{3q}{2}}(\cdot, t_*) \\
\leq c_1 + c_4.
\]
This completes the proof. \(\square\)

**Lemma 3.3.** Let \(\theta > 2\). Then there exists \(C_4 := C_4(T) > 0\) such that if \(T > 0\) then
\[
\int_t^{t+1} \int_{\Omega} (u_\varepsilon + \varepsilon)^{2\theta-4} |\nabla u_\varepsilon|^2 \leq C_4 \quad \text{for all} \quad t \in (0, T) \quad \text{and} \quad \varepsilon \in (0, 1).
\]  

**Proof.** From Lemma 2.3 and Lemma 3.1, there exist \(c_1(T) > 0\) and \(c_2(T) > 0\) such that if \(T > 0\) then
\[
v_\varepsilon \geq \frac{1}{c_1(T)} \quad \text{in} \quad t \in (0, T) \quad \text{and for} \quad \varepsilon \in (0, 1).
\]  

and
\[
\int_t^{t+1} \int_{\Omega} v_\varepsilon^{-\frac{2}{q}} |\nabla v_\varepsilon|^2 \leq c_2(T) \quad \text{for all} \quad t \in (0, T) \quad \text{and} \quad \varepsilon \in (0, 1).
\]  

Letting \(\theta > 2\) and using the Young inequality, there exists \(c_3(T) > 0\) such that for all \(T > 0\),
\[
c_1(T) \zeta^{\theta-2} \leq \frac{c_1(T)}{2} \zeta^{\theta-1} + c_3(T).
\]

For such \(\theta\), multiplying the first equation in (2.3) by \((u_\varepsilon + \varepsilon)^{\theta-2}\), integrating by parts, we have
\[
\frac{1}{\theta - 1} \frac{d}{dt} \int_{\Omega} (u_\varepsilon + \varepsilon)^{\theta-1} + (\theta - 2) \int_{\Omega} (u_\varepsilon + \varepsilon)^{2\theta-4} |\nabla u_\varepsilon|^2 \\
= (\theta - 2) \chi \int_{\Omega} u_\varepsilon (u_\varepsilon + \varepsilon)^{\theta-3} \nabla u_\varepsilon \cdot \frac{\nabla v_\varepsilon}{v_\varepsilon} \\
\leq \frac{\theta - 2}{2} \int_{\Omega} (u_\varepsilon + \varepsilon)^{2\theta-4} |\nabla u_\varepsilon|^2 + \frac{(\theta - 2) \chi^2}{2} \int_{\Omega} \left( \frac{u_\varepsilon}{u_\varepsilon + \varepsilon} \right)^2 \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2} \\
\leq \frac{\theta - 2}{2} \int_{\Omega} (u_\varepsilon + \varepsilon)^{2\theta-4} |\nabla u_\varepsilon|^2 + \frac{(\theta - 2) \chi^2}{2} c_1^2(T) \int_{\Omega} v_\varepsilon^{-\frac{2}{q}} |\nabla v_\varepsilon|^2
\]
by the Gagliardo-Nirenberg inequality and Young’s inequality, we deduce
\[
\frac{1}{\theta - 1} \int_{\Omega} (u_\varepsilon + \varepsilon)^{\theta-1} = \frac{1}{\theta - 1} ||(u_\varepsilon + \varepsilon)^{\theta-1}||_{L^1(\Omega)} \\
\leq \frac{C_{GN}}{\theta - 1} ||\nabla (u_\varepsilon + \varepsilon)^{\theta-1}||_{L^2(\Omega)} \| (u_\varepsilon + \varepsilon)^{\theta-1} \|_{L^{\frac{2(\theta-1)}{\theta-2}}(\Omega)} \\
+ \frac{C_{GN}}{\theta - 1} \| (u_\varepsilon + \varepsilon) \|_{L^{\frac{1}{\theta-1}}(\Omega)} \\
\leq \frac{\theta - 2}{4} \int_{\Omega} (u_\varepsilon + \varepsilon)^{2\theta-4} |\nabla u_\varepsilon|^2 + c_3(T).
\]
Inserting (3.12) into (3.11), we get
\[
\frac{1}{\theta - 1} \frac{d}{dt} \int_\Omega (u_\varepsilon + \varepsilon)^{\theta - 1} + \frac{1}{\theta - 1} \int_\Omega (u_\varepsilon + \varepsilon)^{\theta - 1} + \frac{\theta - 2}{4} \int_\Omega (u_\varepsilon + \varepsilon)^{2\theta - 4} |\nabla u_\varepsilon|^2
\leq \frac{(\theta - 2)c_1^2}{2} \int_\Omega u_\varepsilon^{-2} |\nabla u_\varepsilon|^2 + c_3(T)
\]
for all \( t \in (0, T) \) and \( \varepsilon \in (0, 1) \). Integrating (3.13) over \((t, t + 1)\) and using (3.10), we readily obtain (3.8). This completes the proof.

**Lemma 3.4.** Let \( \theta > 2 \). Then there exists \( C_5 := C_5(T) > 0 \) such that for all \( T > 0 \),
\[
\int_t^{t+1} \int_\Omega u_\varepsilon^{2\theta - 1} \leq C_5 \text{ for all } t \in (0, T) \text{ and } \varepsilon \in (0, 1).
\]

**Proof.** Given \( \varrho \in C^0([0, \infty)) \) such that \( \varrho \equiv 0 \) on \([0, 1]\), \( \varrho(\zeta) = \zeta^{\theta-2} \) for all \( \zeta \geq 2 \) and \( 0 \leq \varrho(\zeta) \leq \zeta^{\theta-2} \) for all \( \zeta \geq 0 \). Letting \( f(\zeta) = \int_0^{\zeta} \varrho(\sigma) d\sigma \) for \( \zeta \geq 0 \). Then \( f \in C^1([0, \infty)) \) and fulfills \( f(\zeta) \leq \frac{\varrho_{\theta-1}}{\theta-1} \) as well as
\[
f(\zeta) \geq \int_2^{\zeta} \varrho(\sigma) d\sigma = \frac{\zeta^{\theta-1} - 2^{\theta-1}}{\theta - 1} \geq c_1 \zeta^{\theta-1} \text{ for all } \zeta \geq 3
\]
with \( c_1 := \frac{1 - (\frac{2}{\theta-1})^{\theta-1}}{\theta-1} > 0 \). Since \( f(\zeta) \leq \frac{\varrho_{\theta-1}}{\theta-1} \) and \( \varrho(\zeta) \leq \zeta^{\theta-2} \), we have
\[
\|f(u_\varepsilon)\|_{L^{\frac{1}{\theta-1}}(\Omega)} \leq (\theta - 1)^{-\frac{1}{\theta-1}} \int_\Omega u_\varepsilon = (\theta - 1)^{-\frac{1}{\theta-1}} m := c_2 \text{ for all } t > 0 \text{ and } \varepsilon \in (0, 1)
\]
and
\[
\|\nabla f(u_\varepsilon)\|_{L^2(\Omega)}^2 = \int_\Omega \varrho^2(u_\varepsilon)|\nabla u_\varepsilon|^2 \leq \int_\Omega u_\varepsilon^{2\theta - 4} |\nabla u_\varepsilon|^2 \text{ for all } t > 0 \text{ and } \varepsilon \in (0, 1).
\]
Due to \( \theta > 2 \), it is easy to know that
\[
\int_{\{u_\varepsilon \geq 1\}} u_\varepsilon^{2\theta - 4} |\nabla u_\varepsilon|^2 \leq \int_\Omega (u_\varepsilon + \varepsilon)^{2\theta - 4} |\nabla u_\varepsilon|^2 \text{ for all } t > 0 \text{ and } \varepsilon \in (0, 1).
\]
From Lemma 2.4 and Lemma 3.3, there exists \( c_3(T) > 0 \) such that
\[
\|f(u_\varepsilon)\|_{L^{\frac{1}{\theta-1}}(\Omega)} \leq c_3(T) \text{ for all } t \in (0, T + 1) \text{ and } \varepsilon \in (0, 1)
\]
and
\[
\int_t^{t+1} \|\nabla f(u_\varepsilon)\|_{L^2(\Omega)}^2 \leq c_3(T) \text{ for all } t \in (0, T) \text{ and } \varepsilon \in (0, 1).
\]
By the Gagliardo-Nirenberg inequality, there exists \( c_4 > 0 \) such that
\[
\int_t^{t+1} \int_\Omega f(u_\varepsilon) u_\varepsilon^{\theta - 1} \leq c_4 \int_t^{t+1} \|\nabla f(u_\varepsilon(\cdot, s))\|_{L^2(\Omega)}^2 \|f(u_\varepsilon(\cdot, s))\|_{L^{\frac{1}{\theta-1}}(\Omega)} ds
\]
\[
+ c_4 \int_t^{t+1} \|f(u_\varepsilon(\cdot, s))\|_{L^{\frac{2\theta-1}{\theta-1}}(\Omega)}^{\frac{2\theta-1}{\theta-1}} ds
\]
\[
\leq c_4 c_2^{\frac{1}{\theta-1}} c_3(T) + c_4 c_2^{\frac{2\theta-1}{\theta-1}} \text{ for all } t \in (0, T) \text{ and } \varepsilon \in (0, 1).
\]
This completes the proof.

**Lemma 3.5.** Let \( \theta > \frac{3}{2} \). Then we can find a constant \( C_6(T) > 0 \) such that if \( T > 0 \),
\[
\|v_\varepsilon(\cdot, t)\|_{W^{1, 2\theta-1}(\Omega)} \leq C_6(T) \text{ for all } t \in (0, T) \text{ and } \varepsilon \in (0, 1).
\]
Proof. Let $p := 2\theta - 1$, owing to $\theta > \frac{3}{4}$, it is easy to see that $p > 2$. By the continuous embedding $W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega) \hookrightarrow L^1(\Omega)$, there exists $c_1 > 0$ such that
\[
\|\phi\|_{L^\infty(\Omega)} \leq c_1 \|\phi\|_{W^{1,p}(\Omega)} \|\phi\|_{L^1(\Omega)}^{1-\alpha} \quad \text{for all } \phi \in W^{1,p}(\Omega) \tag{3.16}
\]
with $\alpha = \frac{2p}{3p-2} \in (0, 1)$, thanks to $2p^2 - 3p + 2 = 2(p-1)^2 + p > 0$, we have
\[
\frac{1}{p} + \frac{p-1}{p\alpha} = \frac{1}{p} + \frac{(p-1)(3p-2)}{2p^2} = \frac{3p^2 - 3p + 2}{2p^2} > \frac{1}{2},
\]
and thus, we can choose $q \in (1, 2)$ close to 2 such that
\[
\frac{1}{q} < \frac{1}{p} + \frac{p-1}{p\alpha}. \tag{3.17}
\]
From the well-known Neumann heat semigroup $(e^{t\Delta})_{t \geq 0}$ on $\Omega$ [43], there exist $k_1, k_2, k_3 > 0$ satisfying
\[
\|e^{t\Delta}\phi\|_{W^{1,p}(\Omega)} \leq k_1 \delta^{-\left(\frac{1}{2} - \frac{1}{2}\right)} \|\phi\|_{W^{1,p}(\Omega)} \quad \text{for all } \delta \in (0, 2) \text{ and } \phi \in C^1(\overline{\Omega}) \tag{3.18}
\]
and
\[
\|e^{t\Delta}\phi\|_{W^{1,\infty}(\Omega)} \leq k_2 \|\phi\|_{W^{1,\infty}(\Omega)} \quad \text{for all } \delta \in (0, 2) \text{ and } \phi \in C^1(\overline{\Omega}) \tag{3.19}
\]
as well as
\[
\|e^{t\Delta}\phi\|_{L^p(\Omega)} \leq k_3 \delta^{-\frac{1}{2}} \|\phi\|_{L^p(\Omega)} \quad \text{for all } \delta \in (0, 2) \text{ and } \phi \in C^0(\overline{\Omega}) \tag{3.20}
\]
From Lemma 2.4 and Lemma 3.4, there exist $c_1(T), c_2(T) > 0$ such that for all $T > 0$,
\[
\|v_e(\cdot, t)\|_{L^1(\Omega)} \leq c_1(T) \text{ for all } t \in (0, T) \text{ and } \phi \in C^0(\overline{\Omega}) \tag{3.21}
\]
and
\[
\int_{(t-2)^+}^t \int_{\Omega} u_{e}^p \leq c_2(T) \text{ for all } t \in (0, T) \text{ and } \phi \in C^0(\overline{\Omega}) \tag{3.22}
\]
and that for any such $T$, each $t_* \in (0, T)$ and arbitrary $\varepsilon \in (0, 1)$, we can find $\tilde{t} := \tilde{t}(t_*, \varepsilon) \geq 0$ such that
\[
\tilde{t} \in ((t_* - 2)^+, (t_* - 1)^+) \text{ and } \|v_e(\cdot, \tilde{t})\|_{W^{1,q}(\Omega)} \leq c_3(T) \text{ if } t_* \geq 2 \tag{3.23}
\]
and that
\[
\tilde{t} = 0 \text{ and } \|v_e(\cdot, \tilde{t})\|_{W^{1,\infty}(\Omega)} \leq c_4 := \|v_0\|_{W^{1,\infty}(\Omega)} \text{ if } t_* \in (0, 2). \tag{3.24}
\]
Applying the variation-of-constants formula of $v_e(\cdot, t)$ and (3.20), we have
\[
\|v_e(\cdot, t)\|_{W^{1,p}(\Omega)} = \left\| e^{(t-\tilde{t})(\Delta-1)}v_e(\cdot, \tilde{t}) + \int_{\tilde{t}}^t e^{(t-s)(\Delta-1)} \frac{u_{e}(\cdot,s)}{1 + \varepsilon u_{e}(\cdot,s)} ds \right\|_{W^{1,p}(\Omega)}
\]
\[
\leq e^{-(t-\tilde{t})} \|e^{(t-\tilde{t})}\Delta v_e(\cdot, \tilde{t})\|_{W^{1,p}(\Omega)} + k_3 \int_{\tilde{t}}^t (t-s)^{-\frac{1}{2}} e^{-(t-s)} \left\| \frac{u_{e}(\cdot,s)}{1 + \varepsilon u_{e}(\cdot,s)} \right\|_{L^p(\Omega)} ds
\]
\[
+ k_2 \int_{\tilde{t}}^t e^{-(t-s)} \|g(\cdot, s)\|_{L^p(\Omega)} ds \tag{3.25}
\]
for all \( t \in (\tilde{t}, t_\ast) \). If \( t_\ast \geq 2 \), by (3.18)
\[
e^{-(t-\tilde{t})\Delta}v_\varepsilon(\cdot, \tilde{t})\|_{W^{1, p}(\Omega)} \leq k_1(t - \tilde{t})^{-\left(\frac{1}{2} - \frac{1}{p}\right)}\|v_\varepsilon(\cdot, \tilde{t})\|_{W^{1, p}(\Omega)}
\leq k_1 c_3(T)(t - \tilde{t})^{-\left(\frac{1}{2} - \frac{1}{p}\right)}
\]
(3.26)
for all \( t \in (\tilde{t}, t_\ast) \). As \( 0 < t_\ast < 2 \), from (3.19) and (3.23), we obtain
\[
e^{-(t-\tilde{t})\Delta}v_\varepsilon(\cdot, \tilde{t})\|_{W^{1, p}(\Omega)} \leq k_2\|v_\varepsilon(\cdot, \tilde{t})\|_{W^{1, \infty}(\Omega)} \leq k_2 c_4,
\]
(3.27)
and (1.4) guarantees that
\[
k_2 \int_{\tilde{t}}^{t} e^{-(t-s)}\|g(\cdot, s)\|_{L^p(\Omega)} ds \leq k_2 c_5 \int_{\tilde{t}}^{t} e^{-(t-s)} ds
\leq k_2 c_5 (1 - e^{t-\tilde{t}}) \leq k_2 c_5
\]
(3.28)
for all \( t \in (\tilde{t}, t_\ast) \). For the second last summand on the right of (3.24), as \( t_\ast \geq 2 \), we get
\[
k_3 \int_{\tilde{t}}^{t} (t-s)^{-\frac{1}{2}} e^{-(t-s)}\|u_\varepsilon(\cdot, s)\|_{L^p(\Omega)} ds
\leq k_3 \int_{\tilde{t}}^{t} (t-s)^{-\frac{1}{2}} e^{-(t-s)}\|u_\varepsilon(\cdot, s)\|_{L^p(\Omega)} ds
\leq k_3 \left( \int_{\tilde{t}}^{t} \int_{\Omega} u_\varepsilon^p \frac{1}{2} \left( \int_{\tilde{t}}^{t} (t-s)^{-\frac{p}{p-1}} ds \right)^{\frac{p-1}{p}} \right)
\leq k_3 c_2(T)^{\frac{1}{p}} c_6 2^{1-\frac{p}{p-1}}
\]
(3.29)
where \( c_6 := \int_{0}^{1} (1 - \sigma)^{-\frac{p}{2(p-1)}} d\sigma \) is finite because of \( \frac{p}{2(p-1)} < 1 \). If \( 0 < t_\ast < 2 \), in a way similar to (3.28), we omit it. Collecting (3.25)-(3.29), we obtain (3.15). This completes the proof. \( \square \)

**Lemma 3.6.** Let \( \theta > \frac{3}{4} \) and \( p > \max\{2, \theta - 1 + \frac{3\theta - 3}{2\theta - 1}\} \). Then there exists \( C_7(T) > 0 \) such that for all \( T > 0 \),
\[
\int_{\Omega} u_\varepsilon^p \leq C_7(T) \quad \text{for all } t \in (0, T) \text{ and } \varepsilon \in (0, 1).
\]
(3.30)

**Proof.** Multiplying the first equation in (2.3) by \( u_\varepsilon^{p-1} \) and using Young’s inequality, we have
\[
\frac{1}{p} \frac{d}{dt} \int_{\Omega} u_\varepsilon^p + \frac{2(p-1)}{p + \theta - 1} \int_{\Omega} |\nabla u_\varepsilon|^{p+\theta-1} \leq \frac{p-1}{2} \int_{\Omega} u_\varepsilon^{p+\theta-3} |\nabla u_\varepsilon|^2 - (p-1) \int_{\Omega} u_\varepsilon^{p-2}(u_\varepsilon + \varepsilon)^{\theta-1} |\nabla u_\varepsilon|^2 + (p-1) \chi \int_{\Omega} u_\varepsilon^{p-1} \nabla u_\varepsilon \cdot \frac{\nabla v_\varepsilon}{v_\varepsilon} \leq \frac{p-1}{2} \int_{\Omega} u_\varepsilon^{p+\theta-3} |\nabla u_\varepsilon|^2 - (p-1) \int_{\Omega} u_\varepsilon^{p-2}(u_\varepsilon + \varepsilon)^{\theta-1} |\nabla u_\varepsilon|^2 + \frac{p-1}{2} \int_{\Omega} u_\varepsilon^{p+\theta-3} |\nabla u_\varepsilon|^2 + (p-1) \chi \int_{\Omega} u_\varepsilon^{p-\theta+1} \cdot \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2} \leq \frac{(p-1)\chi}{2} \int_{\Omega} u_\varepsilon^{p-\theta+1} \cdot |\nabla v_\varepsilon|^2 \, v_\varepsilon^2 \]
(3.31)
From Lemma 2.3 and Lemma 3.5, there exists a $c_1(T) > 0$ such that if $T > 0$ then

$$
\int_{\Omega} \frac{\|\nabla u_\varepsilon\|^{2\theta-1}}{v_\varepsilon^{\theta-1}} \leq c_1(T) \text{ for all } t \in (0, T) \text{ and } \varepsilon \in (0, 1).
$$

Applying the Hölder’s inequality, we get

$$
\frac{(p-1)\chi^2}{2} \int_{\Omega} u_\varepsilon^{p-\theta-1} \frac{\|\nabla u_\varepsilon\|^2}{v_\varepsilon^2}
\leq \frac{(p-1)\chi^2}{2} \left( \int_{\Omega} u_\varepsilon^{(2\theta-1)(p-\theta+1)} \right)^{\frac{2\theta-3}{p-\theta+1}} \left( \int_{\Omega} \frac{\|\nabla u_\varepsilon\|^{2\theta-1}}{v_\varepsilon^{\theta-1}} \right)^{\frac{2}{p-\theta+1}}
\leq c_2(T) \left( \int_{\Omega} u_\varepsilon^{(2\theta-1)(p-\theta+1)} \right)^{\frac{2\theta-3}{p-\theta+1}}
$$

(3.32)

for all $t \in (0, T)$ and $\varepsilon \in (0, 1)$ with $c_2(T) := \frac{(p-1)\chi^2}{2} c_1(T)^{\frac{2\theta-3}{p-\theta+1}}$. Since $p > \theta - 1 + \frac{2\theta-3}{p-\theta+1}$, it is easy to see that $\frac{2}{p+\theta-1} < \frac{2}{p+\theta+1}$. By the Gagliardo-Nirenberg inequality, there exists $c_3 := c_3(p) > 0$ such that

$$
\left( \int_{\Omega} u_\varepsilon^{(2\theta-1)(p-\theta+1)} \right)^{\frac{2\theta-3}{p-\theta+1}} = \left\| u_\varepsilon^{\frac{p-\theta+1}{2}} \right\|_{L^{\frac{2(p-\theta+1)}{2\theta-3}}(\Omega)}^{\frac{2(p-\theta+1)}{p+\theta+1}-1} \leq c_3 \left\| \nabla u_\varepsilon \right\|_{L^{\frac{2\theta-1}{p-\theta+1}}(\Omega)}^{a_1} \left\| u_\varepsilon \right\|_{L^{\frac{p+\theta-1}{p-\theta+1}}(\Omega)}^{a_2} + c_3 \left\| u_\varepsilon \right\|_{L^{\frac{p+\theta-1}{p-\theta+1}}(\Omega)}^{a_1}
$$

for all $t > 0$ and $\varepsilon \in (0, 1)$, where

$$
a_1 := \frac{(2\theta-1)(p-\theta+1)-2\theta+3}{(2\theta-1)(p-\theta+1)} \in (0, 1)
$$

and

$$
a_2 := \frac{2(p-\theta+1)(p-\theta+1) - 2\theta + 3}{(2\theta-1)(p-\theta+1)} = \frac{2(2\theta-1)(p-\theta+1) - 2\theta + 3}{(2\theta-1)(p-\theta+1)} - 2
$$

Using Young’s inequality, we derive

$$
c_2(T) \left( \int_{\Omega} u_\varepsilon^{(2\theta-1)(p-\theta+1)} \right)^{\frac{2\theta-3}{p-\theta+1}} \leq c_4(T) \left\| \nabla u_\varepsilon \right\|_{L^{\frac{2\theta-1}{p-\theta+1}}(\Omega)}^{a_1} + c_5(T)
$$

(3.33)

for all $t \in (0, T)$ and $\varepsilon \in (0, 1)$. Applying the Gagliardo-Nirenberg inequality once more, there exists $c_6 := c_6(p) > 0$ such that

$$
\frac{1}{p} \left\| u_\varepsilon \right\|_{L^p(\Omega)}^p \leq \frac{1}{p} \left\| u_\varepsilon \right\|_{L^{\frac{2\theta-1}{p-\theta+1}}(\Omega)}^{\frac{p-\theta-1}{2}} \left\| \nabla u_\varepsilon \right\|_{L^{\frac{2\theta-1}{p-\theta+1}}(\Omega)}^{\frac{p-\theta-1}{2}}
\leq c_6 \left\| \nabla u_\varepsilon \right\|_{L^{\frac{2\theta-1}{p-\theta+1}}(\Omega)}^{a_3} \left\| u_\varepsilon \right\|_{L^{\frac{p+\theta-1}{p-\theta+1}}(\Omega)}^{a_4} + c_6 \left\| u_\varepsilon \right\|_{L^{\frac{p+\theta-1}{p-\theta+1}}(\Omega)}^{a_3}
$$
with \( \alpha_2 := 1 - \frac{1}{p'} \). Since \( \theta > \frac{3}{2} \), it is easy to see that \( \frac{\alpha_2}{p+\theta-1} < 1 \), by the Young’s inequality again, there exists \( c_7(T) > 0 \) such that
\[
\frac{1}{p} \| u_\varepsilon \|_{L^p(\Omega)}^p \leq \frac{(p-1)}{(p+\theta-1)^2} \int_\Omega \| \nabla u_\varepsilon \|_{L^p(\Omega)}^{p+\theta-1} + c_7(T).
\] (3.34)
Collecting (3.30), (3.32) and (3.33), we have
\[
\frac{1}{p} \frac{d}{dt} \int_\Omega u_\varepsilon^p + \frac{1}{p} \int_\Omega u_\varepsilon^p \leq c_6(T) + c_7(T) \text{ for all } t \in (0, T) \text{ and } \varepsilon \in (0, 1),
\]
the desired result is directly from ODE comparison argument. This completes the proof.

**Lemma 3.7.** Let \( \theta > \frac{3}{2} \) and \( q > 2 \). Then there exists \( C_8(T) > 0 \) such that for all \( T > 0 \),
\[
\| v_\varepsilon(\cdot, t) \|_{W^{1,q}(\Omega)} \leq C_8(T) \text{ for all } t \in (0, T) \text{ and } \varepsilon \in (0, 1).
\] (3.35)
**Proof.** Since \( \theta > \frac{3}{2} \), we know that \( W^{1,2\theta-1}(\Omega) \hookrightarrow L^\infty(\Omega) \). Letting
\[
f_\varepsilon(x, t) = \frac{u_\varepsilon}{1 + \varepsilon u_\varepsilon}(x, t) + g(x, t), \ (x, t) \in \Omega \times (0, \infty),
\]
combining Lemma 3.6 with (1.4), there is \( c_1(T) > 0 \) such that for all \( T > 0 \),
\[
\| f_\varepsilon(x, t) \|_{L^p(\Omega)} \leq c_1(T) \text{ for all } t \in (0, T) \text{ and } \varepsilon \in (0, 1),
\] (3.34) can be obtained by the Neumann heat semigroup, the procedures similar to Lemma 3.5, to avoid repetition, we omit giving details on this here. This completes the proof.

**Lemma 3.8.** Let \( \theta > \frac{3}{2} \). Then there exists \( C_9(T) > 0 \) such that for all \( T > 0 \),
\[
\| u_\varepsilon(\cdot, t) \|_{L^\infty(\Omega)} \leq C_9(T) \text{ for all } t \in (0, T) \text{ and } \varepsilon \in (0, 1).
\] (3.36)
**Proof.** The proof similar to our recent work [31], to avoid repetition, we omit giving details on this here. This completes the proof.

**Lemma 3.9.** Let \( \theta > 2 \) and \( \gamma \geq \theta - 1 \). Then for all \( T > 0 \) there exists \( C_{10}(\gamma, T) > 0 \) such that
\[
\int_0^T \int_\Omega | \nabla(u_\varepsilon + \varepsilon)^\gamma |^2 \leq C_{10}(\gamma, T) \text{ for all } \varepsilon \in (0, 1).
\] (3.37)
**Proof.** From Lemma 3.8, given \( T > 0 \) we can find \( c_1(T) > 0 \) satisfying
\[
u_\varepsilon \leq c_1(T) \text{ in } \Omega \times (0, T) \text{ for all } \varepsilon \in (0, 1).
\] (3.38)
Thus, as \( \theta > 2 \), we have \( 2\gamma \geq 2\theta - 2 \), and thus,
\[
| \nabla(u_\varepsilon + \varepsilon)^\gamma |^2 = \{ (u_\varepsilon + \varepsilon)^{2\theta-4} | \nabla u_\varepsilon |^2 \} \cdot (u_\varepsilon + \varepsilon)^{2\gamma-2\theta+2} \leq \{ (u_\varepsilon + \varepsilon)^{2\theta-4} | \nabla u_\varepsilon |^2 \} \cdot (c_1(T) + 1)^{2\gamma-2\theta+2}
\] (3.39)
in \( \Omega \times (0, T) \) for all \( \varepsilon \in (0, 1) \). Integrating (3.39) over \( (0, T) \) and using Lemma 3.3, we readily obtain (3.37). This completes the proof.

**Lemma 3.10.** Let \( \theta > 2 \) and \( \gamma \geq \theta - 1 \). Then for all \( T > 0 \) there exists \( C_{11}(\gamma, T) > 0 \) such that
\[
\int_0^T \| \partial_t(u_\varepsilon + \varepsilon)^\gamma \|_{W^{2,2}(\Omega)} \cdot dt \leq C_{11}(\gamma, T) \text{ for all } \varepsilon \in (0, 1).
\] (3.40)
Proof. Fixed $t > 0$ and $\psi \in C^\infty(\Omega)$, by the straightforward calculation, we have

\[
\left| \int_{\Omega} \partial_t (u_\varepsilon + \varepsilon)^{\gamma} \psi \right| \\
= \left| \gamma \int_{\Omega} (u_\varepsilon + \varepsilon)^{\gamma-1} \psi \cdot \left\{ \nabla \cdot ((u_\varepsilon + \varepsilon)^{\theta-1} \nabla u_\varepsilon) - \chi \nabla \cdot \left( \frac{u_\varepsilon}{v_\varepsilon} \nabla u_\varepsilon \right) \right\} \right| \\
= -\gamma(\gamma - 1) \int_{\Omega} (u_\varepsilon + \varepsilon)^{\gamma+\theta-3} |\nabla u_\varepsilon|^2 \psi - \gamma \int_{\Omega} (u_\varepsilon + \varepsilon)^{\gamma+\theta-2} \nabla u_\varepsilon \cdot \nabla \psi \\
+ \gamma \int_{\Omega} u_\varepsilon (u_\varepsilon + \varepsilon)^{\gamma-2} (\nabla u_\varepsilon \cdot \frac{\nabla u_\varepsilon}{v_\varepsilon}) \psi \\
+ \gamma \int_{\Omega} u_\varepsilon (u_\varepsilon + \varepsilon)^{\gamma-1} (\nabla u_\varepsilon \cdot \nabla \psi) \right| \tag{3.41}
\]

for all $\varepsilon \in (0, 1)$. Given $T > 0$, from Lemma 3.8, Lemma 2.3, Lemma 3.5 and (1.4), there exist $c_i(T)$ ($i = 1, 2, 3, 4$) such that for all $\varepsilon \in (0, 1)$,

\[
u_\varepsilon \leq c_1(T), \quad c_2(T) \leq v_\varepsilon \leq c_3(T) \quad \text{and} \quad g \leq c_4(T) \quad \text{in} \quad \Omega \times (0, T). \tag{3.42}
\]

Using Young’s inequality, we have

\[
\left| \gamma \int_{\Omega} u_\varepsilon (u_\varepsilon + \varepsilon)^{\gamma-1} \nabla u_\varepsilon \cdot \nabla \psi \right| \\
\leq \int_{\Omega} v_\varepsilon^{-\frac{1}{2}} |\nabla u_\varepsilon|^2 + \frac{\gamma^2}{4} \int_{\Omega} \frac{u_\varepsilon^2 (u_\varepsilon + \varepsilon)^{2\gamma-2}}{v_\varepsilon} |\nabla \psi|^2 \\
\leq \int_{\Omega} v_\varepsilon^{-\frac{1}{2}} |\nabla u_\varepsilon|^2 + \frac{\gamma^2}{4} \frac{c_2(T)(c_1(T) + 1)^{2\gamma-2}}{c_4^2} \cdot \|\nabla \psi\|_{L^2(\Omega)}^2. \tag{3.43}
\]

As $\theta > 2$ and $\gamma \geq \theta - 1$, we have

\[
\left| -\gamma(\gamma - 1) \int_{\Omega} (u_\varepsilon + \varepsilon)^{\gamma+\theta-3} |\nabla u_\varepsilon|^2 \psi \right| \\
\leq \gamma(\gamma - 1)(c_1(T) + \varepsilon)^{-\theta+\gamma+1} \left\{ \int_{\Omega} (u_\varepsilon + \varepsilon)^{2\theta-4} |\nabla u_\varepsilon|^2 \right\} \|\psi\|_{L^\infty(\Omega)} \tag{3.44}
\]

and

\[
\left| \gamma \int_{\Omega} (u_\varepsilon + \varepsilon)^{\gamma-2} (\nabla u_\varepsilon \cdot \frac{\nabla u_\varepsilon}{v_\varepsilon}) \psi \right| \\
\leq \int_{\Omega} v_\varepsilon^{-\frac{1}{2}} |\nabla u_\varepsilon|^2 + \frac{\gamma^2(\gamma - 1)^2 \chi^2}{4} \frac{(u_\varepsilon + \varepsilon)^{-2\theta+2\gamma+2}}{c_4^2} \\
\times \left\{ \int_{\Omega} (u_\varepsilon + \varepsilon)^{2\theta-4} |\nabla u_\varepsilon|^2 \right\} \|\psi\|_{L^\infty(\Omega)} \tag{3.45}
\]

as well as

\[

\leq (c_1(T) + \varepsilon)^{2\gamma} \int_{\Omega} (u_\varepsilon + \varepsilon)^{2\theta-4} |\nabla u_\varepsilon|^2 + \frac{\gamma^2}{4} \|\nabla \psi\|_{L^2(\Omega)}^2 \tag{3.46}
\]

for all $t \in (0, T)$ and $\varepsilon \in (0, 1)$. Since $W^{2,2}(\Omega) \hookrightarrow L^\infty(\Omega)$, combining (3.40) with (3.42)-(3.45), then for each $T > 0$, there exists $c_5(T) > 0$ such that for all $t \in (0, T)$ and $\varepsilon \in (0, 1)$,

\[
\|\partial_t (u_\varepsilon + \varepsilon)\|_{W^{2,2}(\Omega)} \leq c_5(T) \left\{ \int_{\Omega} (u_\varepsilon + \varepsilon)^{2\theta-4} |\nabla u_\varepsilon|^2 + \int_{\Omega} v_\varepsilon^{-\frac{1}{2}} |\nabla v_\varepsilon|^2 + 1 \right\}. \tag{3.47}
\]
Integrating (3.46) over \((0,T)\) and using Lemma 3.1 and Lemma 3.3, we obtain (3.39). This completes the proof.

**Lemma 3.11.** Let \(\theta > 2\). The for all \(T > 0\), there exist \(\epsilon = \epsilon(T) \in (0,1)\) and \(C_{12}(T) > 0\) such that

\[
\|v_\epsilon\|_{C^\infty(\overline{\Omega} \times [0,T])} \leq C_{12}(T) \text{ for all } \epsilon \in (0,1).
\] (3.48)

**Proof.** Letting \(f_\epsilon := \frac{u_\epsilon}{1+\epsilon u_\epsilon} + g\) in \(\Omega \times (0,T)\) for \(\epsilon \in (0,1)\), from Lemma 3.8 and (1.4), we know that \((f_\epsilon)_{\epsilon \in (0,1)}\) is bounded in \(L^\infty_loc(\overline{\Omega} \times [0,\infty))\), (3.48) is directly from standard theory on Hölder regularity in parabolic equations [28]. This completes the proof.

**Lemma 3.12.** Let \(\theta > 2\). Then there exist \((\epsilon_j)_{j \in \mathbb{N}} \subset (0,1)\) as well as functions

\[
\begin{cases}
  u \in L^\infty_loc(\overline{\Omega} \times [0,\infty)), \\
  v \in C^0(\overline{\Omega} \times [0,\infty)) \cap \bigcap_{q>2} L^\infty_loc([0,\infty);W^{1,q}(\Omega))
\end{cases}
\] (3.49)

such that \(\epsilon_j \searrow 0\) as \(j \to \infty\), that \(u \geq 0\) a.e. in \(\Omega \times (0,\infty)\) and \(v > 0\) in \(\overline{\Omega} \times [0,\infty)\), that as \(\epsilon = \epsilon_j \searrow 0\) we have

\[
\begin{align*}
u_\epsilon &\to u \text{ in } \bigcap_{p \geq 1} L^p_loc(\overline{\Omega} \times [0,\infty)) \text{ and a.e. in } \Omega \times (0,\infty), \\
v_\epsilon &\to v \text{ in } C^0_loc(\overline{\Omega} \times [0,\infty)), \\
\nabla v_\epsilon &\rightharpoonup \nabla v \text{ in } \bigcap_{q>2} L^\infty_loc([0,\infty);L^q(\Omega)),
\end{align*}
\] (3.50) (3.51) (3.52)

and that \((u,v)\) is a global weak solution of (1.1) in the sense of Definition 2.1.

**Proof.** Taking any \(\gamma > 0\) such that \(\gamma \geq \theta -1\) if \(\theta > 2\), from Lemma 3.9 and Lemma 3.10, we have

\[
\{u_\epsilon + \epsilon \gamma\}_{\epsilon \in (0,1)} \text{ is bounded in } L^2((0,T);W^{1,2}(\Omega)) \text{ for all } T > 0
\]

and

\[
\{\partial_t(u_\epsilon + \epsilon \gamma)\}_{\epsilon \in (0,1)} \text{ is bounded in } L^1((0,T);(W^{2,2}(\Omega))^*) \text{ for all } T > 0.
\]

Thus, applying Aubin-Lions lemma implies \((\epsilon_j)_{j \in \mathbb{N}} \subset (0,1)\) and nonnegative function \(u\) on \(\Omega \times (0,\infty)\) such that \(\epsilon_j \searrow 0\) as \(j \to \infty\), and that as \(\epsilon = \epsilon_j \searrow 0\) we have \((u_\epsilon + \epsilon \gamma) \to u\gamma \in L^2_loc(\overline{\Omega} \times [0,\infty))\) and a.e. in \(\Omega \times (0,\infty)\), and thus, in particular, we have \(u_\epsilon \to u\) a.e. in \(\Omega \times (0,\infty)\). From Lemma 3.8 and Vitali convergence theorem, we know that (3.50) and the first result in (3.49) hold. From Lemma 3.7 and Lemma 3.11, Arzelà-Ascoli theorem and the Banach-Alaoglu theorem, we know that (3.51), (3.52) and the second result in (3.49) hold. Depending on (3.50)-(3.52) when taking \(\epsilon = \epsilon_j \searrow 0\) in the corresponding weak formulation associated with (2.3), we readily obtain (2.1) and (2.2). This completes the proof.

Finally, we prove the main theorem.

**The proof of Theorem 1.1**. All statements have actually been covered by Lemma 3.12 already. This completes the proof.
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