Hypersurfaces with constant sectional curvature of $S^n \times \mathbb{R}$ and $H^n \times \mathbb{R}$.

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Abstract

We classify the hypersurfaces of $S^n \times \mathbb{R}$ and $H^n \times \mathbb{R}$ with constant sectional curvature and dimension $n \geq 3$.

1 Introduction

The submanifold geometry of the product spaces $S^n \times \mathbb{R}$ and $H^n \times \mathbb{R}$ has been extensively studied in the last years. Here $S^n$ and $H^n$ denote the sphere and hyperbolic space of dimension $n$, respectively. Emphasis has been given on minimal and constant mean curvature surfaces in $S^2 \times \mathbb{R}$ and $H^2 \times \mathbb{R}$, starting with the work in [1] and [15], among others. See [11] for an updated list of references on this topic.

Surfaces of constant Gaussian curvature of $S^2 \times \mathbb{R}$ and $H^2 \times \mathbb{R}$ were investigated in [2] and [3], with special attention to their global properties (see also [12] for a local study in $H^2 \times \mathbb{R}$). In particular, nonexistence of complete surfaces of constant Gaussian curvature $c$ in $S^2 \times \mathbb{R}$ (respectively, $H^2 \times \mathbb{R}$) was established for $c < -1$ and $0 < c < 1$ (respectively, $c < -1$). It was also shown that a complete surface of constant Gaussian curvature $c > 1$ in $S^2 \times \mathbb{R}$ (respectively, $c > 0$ in $H^2 \times \mathbb{R}$) must be a rotation surface. Moreover, the profile curves of such surfaces have been explicitly determined.

Our aim in this paper is to classify all hypersurfaces with constant sectional curvature and dimension $n \geq 3$ of $S^n \times \mathbb{R}$ and $H^n \times \mathbb{R}$. It turns out that for $n \geq 4$ a hypersurface of constant sectional curvature $c$ in $S^n \times \mathbb{R}$ (respectively, $H^n \times \mathbb{R}$) only exists, even locally, if $c \geq 1$ (respectively, $c \geq -1$), and for any such values of $c$ it must be an open subset of a complete rotation hypersurface. In the case $n = 3$, exactly one class of nonrotational hypersurfaces of $S^n \times \mathbb{R}$ and $H^n \times \mathbb{R}$ with constant sectional curvature arises. Each hypersurface in this class in $S^3 \times \mathbb{R}$ (respectively, $H^3 \times \mathbb{R}$) has constant sectional curvature $c \in (0, 1)$ (respectively, $c \in (-1, 0)$), and is constructed in an explicit way by means of a family of parallel flat surfaces in $S^3$ (respectively, $H^3$). An interesting property of such a hypersurface is that its unit normal vector field makes a constant angle with the unit vector field spanning the factor $\mathbb{R}$. All surfaces in $S^2 \times \mathbb{R}$ and $H^2 \times \mathbb{R}$
with this property were classified in [S] and [T], where they were called constant angle surfaces. Here we give a simple proof of a generalization of this result to constant angle hypersurfaces of arbitrary dimension of both $S^n \times \mathbb{R}$ and $H^n \times \mathbb{R}$.

2 Preliminaries

Let $Q^n_\epsilon$ denote either the sphere $S^n$ or hyperbolic space $H^n$, according as $\epsilon = 1$ or $\epsilon = -1$, respectively. In order to study hypersurfaces $f: M^n \to Q^n_\epsilon \times \mathbb{R}$, our approach is to regard $f$ as an isometric immersion into $E^{n+2}$, where $E^{n+2}$ denotes either Euclidean space or Lorentzian space of dimension $(n + 2)$, according as $\epsilon = 1$ or $\epsilon = -1$, respectively. More precisely, let $(x_1, \ldots, x_{n+2})$ be the standard coordinates on $E^{n+2}$ with respect to which the flat metric is written as

$$ds^2 = \epsilon dx_1^2 + dx_2^2 + \ldots + dx_{n+2}^2.$$ 

Regard $E^{n+1}$ as

$$E^{n+1} = \{(x_1, \ldots, x_{n+2}) \in E^{n+2} : x_{n+2} = 0\}$$

and

$$Q^n_\epsilon = \{(x_1, \ldots, x_{n+1}) \in E^{n+1} : \epsilon x_1^2 + x_2^2 + \ldots + x_{n+1}^2 = \epsilon\} \text{ (with } x_1 > 0 \text{ if } \epsilon = -1).$$

Then we consider the inclusion

$$i: Q^n_\epsilon \times \mathbb{R} \to E^{n+1} \times \mathbb{R} = E^{n+2}$$

and study the composition $i \circ f$, which we also denote by $f$.

Given a hypersurface $f: M^n \to Q^n_\epsilon \times \mathbb{R}$, let $N$ denote a unit normal vector field to $f$ and let $\frac{\partial}{\partial t}$ be a unit vector field tangent to the second factor. Then, a vector field $T$ and a smooth function $\nu$ on $M^n$ are defined by

$$\frac{\partial}{\partial t} = f_* T + \nu N.$$ 

Notice that $T$ is the gradient of the height function $h = \langle f, \frac{\partial}{\partial t} \rangle$.

Two trivial classes of hypersurfaces of $Q^n_\epsilon \times \mathbb{R}$ arise if either $\nu$ or $T$ vanishes identically:

Proposition 1 Let $f: M^n \to Q^n_\epsilon \times \mathbb{R}$ be a hypersurface.

(i) If $T$ vanishes identically, then $f(M^n)$ is an open subset of a slice $Q^n_\epsilon \times \{t\}$.

(ii) If $\nu$ vanishes identically, then $f(M^n)$ is an open subset of a Riemannian product $M^{n-1} \times \mathbb{R}$, where $M^{n-1}$ is a hypersurface of $Q^n_\epsilon$. 

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Let $\nabla$ and $R$ be the Levi-Civita connection and the curvature tensor of $M^n$, respectively, and let $A$ be the shape operator of $f$ with respect to $N$. Then the Gauss and Codazzi equations are

$$R(X, Y)Z = (AX \wedge AY)Z + \epsilon((X \wedge Y)Z - \langle Y, T \rangle(X \wedge T)Z + \langle X, T \rangle(Y \wedge T)Z),$$

and

$$\nabla_X AY - \nabla_Y AX - A[X, Y] = \epsilon\nu(X \wedge Y)T,$$

respectively, where $X, Y, Z \in TM$. Moreover, the fact that $\frac{\partial}{\partial t}$ is parallel in $Q^n \times \mathbb{R}$ yields for all $X \in TM$ that

$$\nabla_X T = \nu AX,$$

and

$$X(\nu) = -\langle AX, T \rangle.$$

\section{A basic lemma}

Our main goal in this section is to prove the following lemma.

**Lemma 2** Let $f : M^n_c \to Q^n_\epsilon \times \mathbb{R}$ be a hypersurface of dimension $n \geq 3$ and constant sectional curvature $c \neq 0$. Assume that $T \neq 0$ at $x \in M^n_c$. Then $T$ is a principal direction at $x$.

Lemma 2 will follow by putting together Lemma 3 and Proposition 4 below:

**Lemma 3** Let $f : M^n \to Q^n_\epsilon \times \mathbb{R}$ be a hypersurface. Suppose that $T \neq 0$ at $x \in M^n$. Then $f$ has flat normal bundle at $x$ as an isometric immersion into $\mathbb{E}^{n+2}$ if and only if $T$ is a principal direction at $x$.

**Proposition 4** Any isometric immersion $g : M^n_c \to \mathbb{E}^{n+2}$ of a Riemannian manifold with dimension $n \geq 3$ and constant sectional curvature $c \neq 0$ has flat normal bundle.

Lemma 3 was first proved in [7] for $n = 2$ and $\epsilon = 1$. A proof of the general case can be found in [13]. For the proof of Proposition 4 we make use of standard facts from [13] on the theory of flat bilinear forms. Recall that a symmetric bilinear form $\beta : V \times V \to W$, where $V$ and $W$ are finite-dimensional vector spaces, is said to be flat with respect to an inner product $\langle \cdot, \cdot \rangle : W \times W \to \mathbb{R}$ if

$$\langle \beta(X, Y), \beta(Z, T) \rangle - \langle \beta(X, T), \beta(Z, Y) \rangle = 0$$
for all $X,Y,Z,T \in V$. Clearly, the standard example of a flat bilinear form is the second fundamental form of an isometric immersion between space forms with the same constant sectional curvature.

Denote by $N(\beta) \subset V$ the \textit{nullity subspace} of $\beta$, given by

$$N(\beta) = \{X \in V : \beta(X,Y) = 0 : Y \in V \},$$

and by $S(\beta) \subset W$ its \textit{image subspace}

$$S(\beta) = \text{span}\{\beta(X,Y) : X,Y \in V \}.$$

The next result is a basic fact on flat bilinear forms (cf. Corollary 1 and Corollary 2 in [13]):

**Theorem 5** [13] \ Let $\beta : V \times V \to W$ be a flat bilinear form with respect to an inner product $\langle , \rangle$ on $W$. Assume that $\langle , \rangle$ is either positive-definite or Lorentzian and, in the latter case, suppose that $S(\beta)$ is a nongenerate subspace of $W$, i.e., $S(\beta) \cap S(\beta)^\perp = \{0\}$. Then

$$\dim N(\beta) \geq \dim V - \dim S(\beta).$$

Another fact we will need in order to handle the case $n = 3$ in Proposition [4] is the following consequence of Theorem 2 in [13]:

**Theorem 6** [13] \ Let $\beta : V \times V \to W$ be a flat bilinear form with respect to an inner product $\langle , \rangle$ on $W$. Assume that $\dim V = \dim W$, that $N(\beta) = \{0\}$ and that $\langle , \rangle$ is either positive-definite or Lorentzian. Moreover, in the latter case suppose that there exists a vector $e \in W$ such that $\langle \beta( , e), e \rangle$ is positive definite. Then there exists a diagonalizing basis $\{e_1, \ldots, e_n\}$ for $\beta$, i.e., $\beta(e_i, e_j) = 0$ for $1 \leq i \neq j \leq n$.

**Proof of Proposition [4]** \ First recall that $\mathbb{R}^{n+2}$ admits an umbilical inclusion $i$ into both hyperbolic space $\mathbb{H}_c^{n+3}$ and the Lorentzian sphere $\mathbb{S}_c^{n+2,1}$ of constant sectional curvature $c$, according as $c < 0$ or $c > 0$, respectively, i.e., its second fundamental form $\alpha$ is

$$\alpha(X,Y) = \sqrt{|c|} \langle X,Y \rangle \eta,$$

where $\eta$ is one of the two normal vectors such that $\langle \eta, \eta \rangle = -\text{sgn}(c)$, where $\text{sgn}(c) = c/|c|$. Similarly, Lorentzian space $\mathbb{L}^{n+2}$ admits umbilical inclusions into $\mathbb{H}_c^{n+2,1}$ or $\mathbb{S}_c^{n+1,2}$, according as $c < 0$ or $c > 0$, respectively.

Then, the second fundamental form $\alpha_\phi = g^* \alpha + i_* \alpha_g$ of $\phi = i \circ g$ at every $x \in M^n_c$ is a flat bilinear form with respect to the inner product $\langle , \rangle$ on its three-dimensional normal space. The inner product $\langle , \rangle$ is positive-definite if $c < 0$ and $\mathbb{E}^{n+2} = \mathbb{R}^{n+2}$, Lorentzian if either $c > 0$ and $\mathbb{E}^{n+2} = \mathbb{R}^{n+2}$ or if $c < 0$ and $\mathbb{E}^{n+2} = \mathbb{L}^{n+2}$, and has index
two if $c > 0$ and $\mathbb{E}^{n+2} = \mathbb{L}^{n+2}$. In the latter case, $\alpha_\phi$ is also flat with respect to the Lorentzian inner product $-\langle , \rangle$. Moreover, since

$$\langle \alpha_\phi(\cdot), i_*\eta \rangle = \langle \alpha(\cdot), \eta \rangle = -\text{sgn}(c) \sqrt{|c|} \langle \cdot, \cdot \rangle,$$

it follows that $N(\alpha_\phi) = \{0\}$. Let us consider the two possible cases:

(i) $S(\alpha_\phi)$ is nondegenerate: in this case Theorem 5 gives

$$\dim S(\alpha_\phi) \geq n - \dim N(\alpha_\phi) = n.$$

Since $\dim S(\alpha_\phi) \leq 3$, this implies that $n = 3 = \dim S(\alpha_\phi)$. Since $\langle \alpha_\phi(\cdot), -\text{sgn}(c) i_*\eta \rangle$ is positive definite, it follows from Theorem 6 that there exists a basis $\{e_1, \ldots, e_n\}$ of $T_xM^n_c$ such that $\alpha_\phi(e_i, e_j) = 0$ for $i \neq j$. In particular, we have

$$0 = \langle \alpha_\phi(e_i, e_j), i_*\eta \rangle = -\text{sgn}(c) \sqrt{|c|} \langle e_i, e_j \rangle \text{ for } i \neq j,$$

that is, $\{e_1, \ldots, e_n\}$ is an orthogonal basis. Since $\{e_1, \ldots, e_n\}$ also diagonalizes $\alpha_g$, we conclude that $g$ has flat normal bundle.

(ii) $S(\alpha_\phi)$ is degenerate: in this case, there exists a nonzero vector $\rho \in S(\alpha_\phi) \cap S(\alpha_\phi)^\perp$. Writing $\rho = \eta + i_*\zeta$, with $\zeta$ a unit normal vector to $g$, we obtain from $0 = \langle \alpha_\phi(X, Y), \rho \rangle$ for all $X, Y \in T_xM^n_c$ that

$$\langle \alpha_g(X, Y), \zeta \rangle = \text{sgn}(c) \sqrt{|c|} \langle X, Y \rangle,$$

for all $X, Y \in T_xM^n_c$, i.e., $g$ has an umbilical normal direction. Since $g$ has codimension two, the Ricci equation implies that its normal bundle is flat.

The flat case $c = 0$ can also be handled by means of Theorem 5.

**Lemma 7** Let $f : M^n_0 \to \mathbb{Q}_c^n \times \mathbb{R}$ be a flat hypersurface of dimension $n \geq 3$. Assume that $T \neq 0$ at $x \in M^n_0$.

(i) If $\varepsilon = 1$, then $\nu$ vanishes at $x$.

(ii) If $\varepsilon = -1$, then either $\nu$ vanishes at $x$ or $A_N = A_\xi$ for one of the two possible choices of a unit normal vector $N$ to $f$ at $x$.

In any case, $T$ is a principal direction of $f$ at $x$.

**Proof:** Regard $f$ as an isometric immersion into $\mathbb{E}^{n+2}$. Then, its second fundamental form $\alpha$ is a flat bilinear map by the Gauss equation. Let $\xi$ denote the outward pointing unit normal vector field to $\mathbb{Q}_c^n \times \mathbb{R}$. Then it is easily seen that the shape operator of $f$ with respect to $\xi$ is given by

$$A_\xi T = -\nu^2 T \text{ and } A_\xi X = -X \text{ for } X \in \{T\}^\perp. \quad (6)$$
Assume that $\nu \neq 0$ at $x \in M_0^n$. Then $A_\xi$, and hence $\alpha$, has trivial kernel by (6). If $\epsilon = 1$, it follows from Theorem 5 that

$$2 \geq \dim S(\alpha) \geq n,$$

a contradiction that proves (i). If $\epsilon = -1$, Theorem 5 in the Lorentzian case implies that $S(\alpha)$ is a degenerate subspace of the two-dimensional normal space of $f$ in $E^{n+2}$ at $x$. Hence $S(\alpha)$ is spanned by the light-like vector $i_*N + \xi$ for one of the two unit normal vectors $N$ to $f$ in $Q^n_\epsilon \times \mathbb{R}$ at $x$. But the fact that $i_*N + \xi \in S(\alpha)\perp$ just means that $A_N = A_\xi$.

For the last assertion, notice that a point where $\nu$ vanishes is a local minimum for $\nu$, hence $A_NT = 0$ at $x$ by (5). ■

4 Rotation hypersurfaces

Rotation hypersurfaces of $S^n \times \mathbb{R}$ and $H^n \times \mathbb{R}$ have been defined and their principal curvatures computed in [6], as an extension of the work in [4] on rotation hypersurfaces of space forms.

With notations as in Section 2, let $P^3$ be a three-dimensional subspace of $E^{n+2}$ containing the $\frac{\partial}{\partial x_1}$ and the $\frac{\partial}{\partial x_{n+2}}$ directions. Then $(Q^n_\epsilon \times \mathbb{R}) \cap P^3 = Q^1_\epsilon \times \mathbb{R}$. Denote by $I$ the group of isometries of $E^{n+2}$ that fix pointwise a two-dimensional subspace $P^2 \subset P^3$ also containing the $\frac{\partial}{\partial x_{n+2}}$-direction. Consider a curve $\alpha$ in $Q^1_\epsilon \times \mathbb{R} \subset P^3$ that lies in one of the two half-spaces of $P^3$ determined by $P^2$.

**Definition 8** A rotation hypersurface in $Q^n_\epsilon \times \mathbb{R}$ with profile curve $\alpha$ and axis $P^2$ is the orbit of $\alpha$ under the action of $I$.

We will always assume that $P^3$ is spanned by $\frac{\partial}{\partial x_1}$, $\frac{\partial}{\partial x_{n+1}}$ and $\frac{\partial}{\partial x_{n+2}}$. In the case $\epsilon = 1$, we also assume that $P^2$ is spanned by $\frac{\partial}{\partial x_1}$ and $\frac{\partial}{\partial x_{n+2}}$, and that the curve $\alpha$ is parametrized by arc length as

$$\alpha(s) = (\sin(k(s)), 0, \ldots, 0, \cos(k(s)), h(s)),$$

where $s$ runs over an interval $I$ where $\cos(k(s)) \geq 0$, so that $\alpha(I)$ is contained in a closed half-space determined by $P^2$. Here $k, h : I \to \mathbb{R}$ are smooth functions satisfying

$$k'(s)^2 + h'(s)^2 = 1 \text{ for all } s \in I. \quad (7)$$

In this case, the rotation hypersurface in $S^n \times \mathbb{R}$ with profile curve $\alpha$ and axis $P^2$ can be parametrized by

$$f(s, t) = (\sin(k(s)), \cos(k(s))\varphi_1(t), \ldots, \cos(k(s))\varphi_n(t), h(s)), \quad (8)$$
where \( t = (t_1, \ldots, t_{n-1}) \) and \( \varphi = (\varphi_1, \ldots, \varphi_n) \) parametrizes \( \mathbb{S}^{n-1} \subset \mathbb{R}^n \). The metric induced by \( f \) is
\[
d\sigma^2 = ds^2 + \cos^2(k(s))dt^2,
\]
where \( dt^2 \) is the standard metric of \( \mathbb{S}^{n-1} \).

For \( \epsilon = -1 \), one has three distinct possibilities, according as \( P^2 \) is Lorentzian, Riemannian or degenerate, respectively. We call \( f \), accordingly, a rotation hypersurface of spherical, hyperbolic or parabolic type, because the orbits of \( I \) are spheres, hyperbolic spaces or horospheres, respectively. In the first case, we can assume that \( P^2 \) is spanned by \( \frac{\partial}{\partial x_1} \) and \( \frac{\partial}{\partial x_{n+2}} \) and that the curve \( \alpha \) is parametrized by
\[
\alpha(s) = (\cosh(k(s)), 0, \ldots, 0, \sinh(k(s)), h(s)).
\]

Then \( f \) can be parametrized by
\[
f(s, t) = (\cosh(k(s)), \sinh(k(s))\varphi_1(t), \ldots, \sinh(k(s))\varphi_n(t), h(s)).
\]
The induced metric is
\[
d\sigma^2 = ds^2 + \sinh^2(k(s))dt^2,
\]
where \( dt^2 \) is the standard metric of \( \mathbb{S}^{n-1} \).

In the second case, assuming that \( P^2 \) is spanned by \( \frac{\partial}{\partial x_{n+1}} \) and \( \frac{\partial}{\partial x_{n+2}} \), the curve \( \alpha \) can also be parametrized as in (10), and a parametrization of \( f \) is
\[
f(s, t) = (\cosh(k(s))\varphi_1(t), \ldots, \cosh(k(s))\varphi_n(t), \sinh(k(s)), h(s)),
\]
where \( t = (t_1, \ldots, t_{n-1}) \) and \( \varphi = (\varphi_1, \ldots, \varphi_n) \) parametrizes \( \mathbb{H}^{n-1} \subset \mathbb{L}^n \). The induced metric is
\[
d\sigma^2 = ds^2 + \cosh^2(k(s))dt^2,
\]
where \( dt^2 \) is the standard metric of \( \mathbb{H}^{n-1} \).

Finally, when \( P^2 \) is degenerate, we choose a pseudo-orthonormal basis
\[
e_1 = \frac{1}{\sqrt{2}} \left( -\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_{n+1}} \right), \quad e_{n+1} = \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_{n+1}} \right), \quad e_j = \frac{\partial}{\partial x_j},
\]
for \( j \in \{2, \ldots, n, n+2\} \), and assume that \( P^2 \) is spanned by \( e_{n+1} \) and \( e_{n+2} \). Notice that \( \langle e_1, e_1 \rangle = 0 = \langle e_{n+1}, e_{n+1} \rangle \) and \( \langle e_1, e_{n+1} \rangle = 1 \). Then, we can parametrize \( \alpha \) by
\[
\alpha(s) = \left( k(s), 0, \ldots, 0, \frac{1}{2k(s)}, h(s) \right),
\]
with
\[
k(s) > 0 \quad \text{and} \quad (\ln k)^2(s) + h'(s)^2 = 1,
\]
and a parametrization of \( f \) is

\[
    f(s, t_2, \ldots, t_n) = \left( k(s), k(s)t_2, \ldots, k(s)t_n, -\frac{1}{2k(s)} - \frac{k(s)}{2} \sum_{i=2}^{n} t_i^2, h(s) \right),
\]

whose induced metric is

\[
    d\sigma^2 = ds^2 + k^2(s)dt^2,
\]

where \( dt^2 \) is the standard metric of \( \mathbb{R}^{n-1} \).

**Remark 9** Our definition of a rotation hypersurface in \( Q^n_\epsilon \times \mathbb{R} \) was taken from [6], and it naturally extends the one given in [4] for space forms. For \( \epsilon = -1 \), it differs from that used in [2], where only rotation surfaces of spherical type were considered.

We are now in a position to classify rotation hypersurfaces of \( Q^n_\epsilon \times \mathbb{R} \) with constant sectional curvature \( c \) and dimension \( n \geq 3 \). We state separately the cases \( \epsilon = 1 \) and \( \epsilon = -1 \):

**Theorem 10.** Let \( f: M^n_c \to S^n \times \mathbb{R} \) be a rotation hypersurface with constant sectional curvature \( c \) and dimension \( n \geq 3 \). Then \( c \geq 1 \). Moreover,

(i) if \( c = 1 \) then \( f(M^n_c) \) is an open subset of a slice \( S^n \times \{t\} \).

(ii) if \( c > 1 \) then \( f(M^n_c) \) is an open subset of a complete hypersurface that can be parametrized by (8), with

\[
    k(s) = \arccos \left( \frac{1}{\sqrt{c}} \sin(\sqrt{c} s) \right) \quad \text{(18)}
\]

and

\[
    h(s) = -\sqrt{\frac{c - 1}{c}} \ln \left( \frac{\cos(\sqrt{c} s) + \sqrt{c - \sin^2(\sqrt{c} s)}}{1 + \sqrt{c}} \right), \quad s \in [0, \pi/\sqrt{c}]. \quad \text{(19)}
\]

**Theorem 11.** Let \( f: M^n_c \to H^n \times \mathbb{R} \) be a rotation hypersurface with constant sectional curvature \( c \) and dimension \( n \geq 3 \). Then \( c \geq -1 \). Moreover,

(i) if \( c = -1 \) then \( f(M^n) \) is an open subset of a slice \( H^n \times \{t\} \).

(ii) if \( c \in (-1, 0) \) then one of the following possibilities holds:

(a) \( f(M^n) \) is an open subset of a complete hypersurface of spherical type that can be parametrized by (11), with

\[
    k(s) = \arcsinh \left( \frac{1}{\sqrt{1-c}} \sinh(\sqrt{1-c} s) \right) \quad \text{(20)}
\]
and

\[ h(s) = \sqrt{\frac{c+1}{-c}} \ln \left( \frac{\cosh(\sqrt{-c}s) + \sqrt{-c + \sinh^2(\sqrt{-c}s)}}{1 + \sqrt{-c}} \right). \]  \hfill (21)

(b) \( f(M^n) \) is an open subset of a complete hypersurface of hyperbolical type that can be parametrized by (13), with

\[ k(s) = \text{arccosh} \frac{1}{\sqrt{-c}} \cosh(\sqrt{-c}s) \]  \hfill (22)

and

\[ h(s) = \sqrt{\frac{c+1}{-c}} \ln \left( \sinh(\sqrt{-c}s) + \sqrt{c + \cosh^2(\sqrt{-c}s)} \right). \]  \hfill (23)

(c) \( f(M^n) \) is an open subset of a complete hypersurface of parabolical type that can be parametrized by (16), with

\[ k(s) = \exp \sqrt{-c}s \]  \hfill (24)

and

\[ h(s) = \sqrt{1 + cs}. \]  \hfill (25)

(iii) if \( c = 0 \), then one of the following possibilities holds:

(a) \( f(M^n) \) is an open subset of a complete hypersurface of spherical type that can be parametrized by (11), with

\[ k(s) = \text{arcsinh}(s) \]  \hfill (26)

and

\[ h(s) = -1 + \sqrt{1 + s^2}. \]  \hfill (27)

(b) \( f(M^n) \) is an open subset of a Riemannian product \( M^{n-1} \times \mathbb{R} \), where \( M^{n-1} \) is a horosphere of \( \mathbb{H}^n \).

(iv) if \( c > 0 \), then \( f(M^n) \) is an open subset of a complete hypersurface of spherical type that can be parametrized by (11), with

\[ k(s) = \text{arcsinh} \left( \frac{1}{\sqrt{c}} \sin(\sqrt{c}s) \right) \]  \hfill (28)

and

\[ h(s) = -\sqrt{\frac{c+1}{c}} \arctan \left( \frac{\cos(\sqrt{c}s)}{\sqrt{c + \sin^2(\sqrt{c}s)}} \right). \]  \hfill (29)
Remark 12 The hypersurfaces in Theorems [10] and [11] also occur in dimension $n = 2$. In particular, those in parts (ii) − b) and (ii) − c) of Theorem [11] provide examples of complete surfaces of constant Gaussian curvature $c \in (-1, 0)$ in $\mathbb{H}^2 \times \mathbb{R}$ that do not appear in [2].

For the proof of Theorems [10] and [11] we make use of the following fact:

**Proposition 13.** Assume that the warped product $I \times_\rho \mathbb{Q}_\delta^n$, $n \geq 2$, $\delta \in \{-1, 0, 1\}$, has constant sectional curvature $c$.

(i) If $c > 0$, then $\delta = 1$ and $\rho(s) = \frac{1}{\sqrt{c}} \sin(\sqrt{c} s + \theta_0)$, $\theta_0 \in \mathbb{R}$.

(ii) If $c = 0$, then one of the following possibilities holds:
   - (a) $\delta = 1$ and $\rho(s) = \pm s + s_0$, $s_0 \in \mathbb{R}$.
   - (b) $\delta = 0$ and $\rho(s) = A \in \mathbb{R}$.

(iii) If $c < 0$, then one of the following possibilities holds:
   - (a) $\delta = -1$ and $\rho(s) = \frac{1}{\sqrt{-c}} \cosh(\sqrt{-c} s + \theta_0)$, $\theta_0 \in \mathbb{R}$.
   - (b) $\delta = 0$ and $\rho(s) = \exp(\pm \sqrt{-c} s + s_0)$, $s_0 \in \mathbb{R}$.
   - (c) $\delta = 1$ and $\rho(s) = \frac{1}{\sqrt{-c}} \sinh(\sqrt{-c} s + \theta_0)$, $\theta_0 \in \mathbb{R}$.

**Proof:** In a warped product $I \times_\rho \mathbb{Q}_\delta^n$, $n \geq 2$, the sectional curvature along a plane tangent to $\mathbb{Q}_\delta^n$ is $(\delta - (\rho')^2)/\rho^2$, whereas the sectional curvature along a plane spanned by unit vectors $\partial/\partial s$ and $X$ tangent to $I$ and $\mathbb{Q}_\delta^n$, respectively, is $-\rho''/\rho$. Therefore, $I \times_\rho \mathbb{Q}_\delta^n$ has constant sectional curvature $c$ if and only if

$$ (\rho')^2 + c \rho^2 = \delta. \quad (30) $$

Notice that $-\rho''/\rho = c$, or equivalently,

$$ \rho'' + c \rho = 0, \quad (31) $$

follows by differentiating (30). If $c > 0$, we obtain from (30) that $\delta = 1$. Moreover, by (31) we have that

$$ \rho(s) = A \cos \sqrt{c} s + B \sin \sqrt{c} s $$

for some $A, B \in \mathbb{R}$, which gives $(\rho')^2 + c \rho^2 = c(A^2 + B^2)$. From (30) we get $c(A^2 + B^2) = 1$, hence we may write

$$ A = \frac{1}{\sqrt{c}} \sin \theta_0 \quad \text{and} \quad B = \frac{1}{\sqrt{c}} \cos \theta_0 $$

for some $\theta_0 \in \mathbb{R}$. It follows that

$$ \rho(s) = \frac{1}{\sqrt{c}} \sin(\sqrt{c} s + \theta_0). $$
The remaining cases are similar. ■

Proof of Theorems \text{10 and 11}: First we determine the possible values of \( c \) for a rotation hypersurface \( f: M^n_c \to \mathbb{Q}_c^n \times \mathbb{R} \) with constant sectional curvature \( c \) and dimension \( n \geq 3 \). If \( T \) vanishes on an open subset, then \( c = \epsilon \) by Proposition \text{1}. Otherwise, we can assume that \( T \) is nowhere vanishing. Then \( f \) has exactly two distinct principal curvatures \( \lambda \) and \( \mu \neq 0 \), the first one being simple with \( T \) as principal direction (cf. \text{[6]}). Let \( \{T, X_1, \ldots, X_{n-1}\} \) be an orthogonal basis of eigenvectors of \( A \) at \( x \), with

\[
AT = \lambda T \quad \text{and} \quad AX_i = \mu X_i, \quad 1 \leq i \leq n - 1.
\]

From the Gauss equation (2) of \( f \) for \( X = X_i \) and \( Y = Z = X_j, i \neq j \), we get

\[
c - \epsilon = \mu^2,
\]

and hence \( c > \epsilon \). This proves the first assertions in Theorems \text{10 and 11}.

Now assume that \( \epsilon = 1 \). Then \( f \) can be parametrized by (8), with \( k(s) \) and \( h(s) \) satisfying (7), and the metric induced by \( f \) is given by (9). Since \( c \geq 1 \), by Proposition \text{13} we must have

\[
\cos(k(s)) = \frac{1}{\sqrt{c}} \sinh(\sqrt{c} s + \theta_0)
\]

for some \( \theta_0 \in \mathbb{R} \). Replacing \( s \) by \( s - \theta_0/\sqrt{c} \), we can assume that \( \theta_0 = 0 \). If \( c = 1 \), then \( f \) just parametrizes an open subset of a slice \( \mathbb{S}^n \times \{t\} \). If \( c > 1 \), we obtain that \( k(s) \) and \( h(s) \) are given by (18) and (19), respectively. The corresponding profile curve is exactly that of the complete surface of constant sectional curvature \( c \) in \( \mathbb{S}^2 \times \mathbb{R} \) determined in [2], and their argument also applies to show the completeness of \( f \) in any dimension \( n \geq 3 \).

From now on we deal with the case \( \epsilon = -1 \). Assume first that \( f \) is of spherical type. Then \( f \) can be parametrized by (11), with \( k(s) \) and \( h(s) \) satisfying (7), and the metric induced by \( f \) is given by (12). By Proposition \text{13} the warping function \( \sinh(k(s)) \) must be equal to

\[
\frac{1}{\sqrt{c}} \sin(\sqrt{c} s + \theta_0), \quad \frac{1}{\sqrt{-c}} \sinh(\sqrt{-c} s + \theta_0), \quad \theta_0 \in \mathbb{R}, \quad \text{or} \quad \pm s + s_0, \quad s_0 \in \mathbb{R},
\]

according as \( c > 0 \), \( c < 0 \) or \( c = 0 \), respectively. After suitably replacing the parameter \( s \), we can assume that \( \theta_0 = 0 \) in the first two cases, and that \( \sinh(k(s)) = s \) in the last one. Each possibility gives rise to the expressions (20), (28) and (26) for \( k(s) \), and (21), (29) and (27) for \( h(s) \), respectively. The corresponding profile curves are exactly those of the complete rotation surfaces with constant sectional curvature of spherical type determined in [2], and the completeness of the corresponding hypersurfaces can be seen in the same way as in [2].
Now suppose that $f$ is of hyperbolical type. Then, it can be parametrized by (13), with $k(s)$ and $h(s)$ satisfying (17), and the induced metric is (14). Since $c \geq -1$, by Proposition 13 we must have $c \in [-1, 0)$ and
\[
\cosh(k(s)) = \frac{1}{\sqrt{-c}} \cosh(\sqrt{-c}s + \theta_0), \quad \theta_0 \in \mathbb{R}.
\]
As before, we can assume that $\theta_0 = 0$. If $c = -1$, then $f(M^n)$ is an open subset of a slice $\mathbb{H}^n \times \{t\}$. Otherwise, $k$ and $h$ are given by (22) and (23), respectively.

Finally, suppose that $f$ is of parabolical type. Then, it can be parametrized by (16), with $k(s)$ and $h(s)$ satisfying (15), and the induced metric is (17). By Proposition 13, we must have $c \leq 0$ and
\[
k(s) = A \in \mathbb{R} \text{ or } k(s) = \exp(\pm \sqrt{-c}s + s_0), \quad s_0 \in \mathbb{R},
\]
according as $c = 0$ or $c < 0$, respectively. In the first case, $f$ just parametrizes an open subset of a Riemannian product $M^{n-1} \times \mathbb{R}$, where $M^{n-1}$ is a horosphere of $\mathbb{H}^n$. In the second case, we can assume that $k(s) = \exp(\sqrt{-c}s)$ and then $h$ is given by (25).

Completeness of the hypersurfaces in this and the preceding case is straightforward.

## 5 Constant angle hypersurfaces

Let $g: M^{n-1} \rightarrow \mathbb{Q}_c^n$ be a hypersurface and let $g_s: M^{n-1} \rightarrow \mathbb{Q}_c^n$ be the family of parallel hypersurfaces to $g$, that is,
\[
g_s(x) = C_\epsilon(s)g(x) + S_\epsilon(s)N(x),
\]  
where $N$ is a unit normal vector field to $g$,
\[
S_\epsilon(s) = \begin{cases} 
\cos s, & \text{if } \epsilon = 1 \\
\cosh s, & \text{if } \epsilon = -1
\end{cases}
\text{ and } \quad S_\epsilon(s) = \begin{cases} 
\sin s, & \text{if } \epsilon = 1 \\
\sinh s, & \text{if } \epsilon = -1
\end{cases}.
\]
For $\epsilon = 1$, write the principal curvatures of $g$ as
\[
\lambda_i = \cot \theta_i, \quad 0 < \theta_i < \pi, \quad 1 \leq i \leq m,
\]
where the $\theta_i$ form an increasing sequence. For $X$ in the eigenspace of the shape operator $A_N$ of $g$ corresponding to the principal curvature $\lambda_i$, $1 \leq i \leq m$, we have
\[
g_sX = g_s(\cos sX - \sin sA_NX) = (\cos s - \sin s \cot \theta_i)X = \frac{\sin(\theta_i - s)}{\sin \theta_i}X,
\]
Thus, $g_s$ is an immersion at $x$ if and only if $s \neq \theta_i(x)(\mod \pi)$ for any $1 \leq i \leq m$. 

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For $\epsilon = -1$, write the principal curvatures of $g$ with absolute value greater than 1 as

$$\lambda_i = \coth \theta_i, \quad \theta_i \neq 0, \quad 1 \leq i \leq m.$$  

As in the preceding case, for $X$ in the eigenspace of the shape operator $A_N$ corresponding to the principal curvature $\lambda_i$, $1 \leq i \leq m$, we have

$$g_{ss}X = \frac{\sinh(\theta_i - s)}{\sinh \theta_i}X,$$

Thus, $g_s$ is an immersion at $x$ if and only if $s \neq \theta_i(x)$ for any $1 \leq i \leq m$.

In the case $\epsilon = 1$, set

$$U := \{(x,s) \in M^{n-1} \times \mathbb{R} : s \in (\theta_m(x) - \pi, \theta_1(x))\}. \quad (33)$$

For $\epsilon = -1$, let $\theta_+$ (respectively, $\theta_-$) be the least (respectively, greater) of the $\theta_i$ that is greater than 1 (respectively, less than $-1$), and set

$$U := \{(x,s) \in M^{n-1} \times \mathbb{R} : s \in (\theta_-(x), \theta_+(x))\}. \quad (34)$$

In both cases, if $V \subset M^{n-1}$ is an open subset and $I$ is an open interval containing 0 such that $V \times I \subset U$, then $g_s$ is an immersion on $V$ for every $s \in I$, with

$$N_s(x) = -\epsilon S_\epsilon(s)g(x) + C_\epsilon(s)N(x) \quad (35)$$

as a unit normal vector at $x$.

Now define

$$f: M^n := V \times I \to \mathbb{Q}_\epsilon^n \times \mathbb{R} \subset \mathbb{E}^{n+2}$$

by

$$f(x,s) = g_s(x) + Bs \frac{\partial}{\partial t}, \quad B > 0. \quad (36)$$

Then

$$f_*X = g_{ss}X, \quad \text{for any } X \in TM^{n-1},$$

and

$$f_* \frac{\partial}{\partial s} = N_s + B \frac{\partial}{\partial t},$$

where

$$N_s(x) = -\epsilon S_\epsilon(s)g(x) + C_\epsilon(s)N(x). \quad (37)$$

Since $g_s$ is an immersion on $V$ for every $s \in I$, it follows that $f$ is an immersion on $M^n$ with

$$\eta(x,s) = -\frac{B}{a} N_s(x) + \frac{1}{a} \frac{\partial}{\partial t}, \quad a = \sqrt{1 + B^2} \quad (38)$$
as a unit normal vector field. Thus, $f$ has the property that

$$\langle \eta, \frac{\partial}{\partial t} \rangle = \frac{1}{a}$$

is constant on $M^n$. Following [8], $f$ was called in [10] a constant angle hypersurface. Constant angle surfaces in $S^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ have been classified in [8] and [9], respectively. The next result was obtained in [10] as a consequence of a more general theorem. For the sake of completeness we provide here a simple and direct proof.

**Theorem 14.** Any constant angle hypersurface $f: M^n \to \mathbb{Q}_e^n \times \mathbb{R}$ is either an open subset of a slice $\mathbb{Q}_e^n \times \{t_0\}$ for some $t_0 \in \mathbb{R}$, an open subset of a product $M^{n-1} \times \mathbb{R}$, where $M^{n-1}$ is a hypersurface of $\mathbb{Q}_e^n$, or it is locally given by the preceding construction.

**Proof:** Let $\eta$ be a unit normal vector field to $f$. By assumption, $\nu = \langle \eta, \partial/\partial t \rangle$ is a constant on $M^n$, which we can assume to belong to $[0, 1]$. Since $\|T\|^2 + \nu^2 = 1$, the vector field $T$ has also constant length. By Proposition 1, the cases $\nu = 1$ and $\nu = 0$ correspond to the first two possibilities in the statement, respectively. From now on, we assume that $\nu \in (0, 1)$, hence $T$ is a vector field whose length is also a constant in $(0, 1)$. Since $T$ is a gradient vector field, its integral curves are (not unit-speed) geodesics in $M^n$. The fact that $T$ is a gradient also implies that the orthogonal distribution $\{T\}^\perp$ is integrable. Thus, there exists locally a diffeomorphism $\psi: M^{n-1} \times I \to M^n$, where $I$ is an open interval containing 0, such that $\psi(x, \cdot): I \to M^n$ are integral curves of $T$ and $\psi(\cdot, s): M^{n-1} \to M^n$ are integral manifolds of $\{T\}^\perp$. Set $F = f \circ \psi$, with $f$ being regarded as an isometric immersion into $\mathbb{E}^{n+2}$. Then

$$X\langle F, \frac{\partial}{\partial t} \rangle = \langle f_* \psi_* X, \frac{\partial}{\partial t} \rangle = \langle \psi_* X, T \rangle = 0$$

for any $X \in TM^{n-1}$. Thus $\langle F(x,s), \frac{\partial}{\partial t} \rangle = \rho(s)$ for some smooth function $\rho$ on $I$.

On the other hand, it follows from

$$0 = dv(X) = -\langle AX, T \rangle$$

that $AT = 0$, hence $F(x, \cdot): I \to \mathbb{Q}_e^n \times \mathbb{R}$ are geodesics in $\mathbb{Q}_e^n \times \mathbb{R}$, where $F = f \circ \psi$. Therefore, the projections $\Pi_1 \circ F(x, \cdot): I \to \mathbb{Q}_e^n$ and $\Pi_2 \circ F(x, \cdot): I \to \mathbb{R}$ are geodesics of $\mathbb{Q}_e^n$ and $\mathbb{R}$, respectively.

That $\Pi_2 \circ F(x, \cdot): I \to \mathbb{R}$ are geodesics in $\mathbb{R}$ just means that $\rho(s) = Bs$, for some constant $B > 0$, after possibly a translation in the parameter $s$ and changing $s$ by $-s$. Now define $g: M^{n-1} \to \mathbb{Q}_e^n$ by

$$g(x) = \Pi_1 \circ F(x, 0).$$

Rescaling the parameter $s$ so that the geodesics $\Pi_1 \circ F(x, \cdot): I \to \mathbb{Q}_e^n$ have unit speed, the fact that they are normal to $g$ at $g(x)$ for any $x \in M^{n-1}$ just says that

$$\Pi_1 \circ F(x, s) = g_s(x),$$

where $g_s$ denotes the parallel hypersurface to $g$ at a distance $s$. \[\square\]
Remark 15 The proof of Theorem 14 also applies to hypersurfaces of $\mathbb{R}^{n+1}$ whose unit normal vector field makes a constant angle with a fixed direction $\partial/\partial t$. Namely, writing $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$, with the second factor being spanned by $\partial/\partial t$, it shows that any such hypersurface is either an open subset of an affine subspace $\mathbb{R}^n \times \{t_0\}$ for some $t_0 \in \mathbb{R}$, an open subset of a product $M^{n-1} \times \mathbb{R}$, where $M^{n-1}$ is a hypersurface of $\mathbb{R}^n$, or it is locally given by (35), where $g_s$ is the family of parallel hypersurfaces to some hypersurface $g$ in the first factor $\mathbb{R}^n$, namely, $g_s(x) = g(x) + sN(x)$ for a unit vector field $N$ to $g$. A proof of this fact for surfaces in $\mathbb{R}^3$ was given in [14].

6 Nonrotational examples in dimension three

Here we use the construction of the previous section to produce a family of nonrotational hypersurfaces of $S^3 \times \mathbb{R}$ (respectively, $\mathbb{H}^3 \times \mathbb{R}$) with constant sectional curvature $c$ for any $c \in (0, 1)$ (respectively, $c \in (-1, 0)$).

Given a hypersurface $g: M^{n-1} \to Q^n_\epsilon$ and the family $g_s: M^{n-1} \to Q^n_\epsilon$ of parallel hypersurfaces to $g$, an easy computation shows that, whenever $\cot \epsilon_s := C_\epsilon(s)/S_\epsilon(s)$ is not a principal curvature of $g$ at any $x \in M^{n-1}$, the shape operator $A_s$ of $g_s$ with respect to the unit normal vector field $N_s$ given by (37) is

$$A_s = (\cot \epsilon_s I - A)^{-1}(\cot \epsilon_s A + \epsilon I).$$

Let $g: M^2 \to Q^3_\epsilon$ be a surface and let

$$f: M^3 := V \times I \subset M^2 \times \mathbb{R} \to Q^3_\epsilon \times \mathbb{R} \subset \mathbb{E}^5$$

be defined as in the previous section in terms of $g$. The normal space of $f$, as a submanifold of $\mathbb{E}^5$, is spanned by the unit normal vector field $\eta$ given by (38) and by the unit normal vector field $\xi(x,s) = g_s(x)$, which is normal to $Q^3_\epsilon \times \mathbb{R}$ at $f(x,s)$. We have

$$a\hat{\nabla}_X \eta = Bg_s A^s X = Bf_s A^s X$$

and

$$a\hat{\nabla}_X \eta = \epsilon Bg_s = \epsilon B\xi,$$

hence the principal curvatures of $A^f_\eta$ are

$$-\frac{B}{a}k^s_1, \quad -\frac{B}{a}k^s_2 \quad \text{and} \quad 0,$$

where $k^s_1$ and $k^s_2$ are the principal curvatures of $g_s$, the principal curvature 0 corresponding to the principal direction $\partial/\partial s$. On the other hand,

$$\hat{\nabla}_X \xi = g_s X = f_s X.$$
and
\[ \tilde{\nabla}_{\frac{\partial}{\partial s}} \xi = N_s = \frac{1}{a^2} f_* \frac{\partial}{\partial s} - \frac{B}{a} \eta. \]

Thus, the principal curvatures of \( A^f_s \) are \(-1/a^2\) and \(-1\), the first being simple with \( \partial/\partial s \) as principal direction, and the second having multiplicity two with \( TV \) as eigenbundle.

Now assume that \( M^2 = M^2_0 \) is flat. Then, the principal curvatures \( k_1 \) and \( k_2 \) of \( g \) satisfy \( k_1 k_2 = -\epsilon \) everywhere. By (39), the principal curvatures of \( g_s \) with respect to \( N_s \) are
\[ k^s_i = \frac{\cot \epsilon s k_i + \epsilon}{\cot \epsilon s - k_i}, \quad 1 \leq i \leq 2, \]
hence \( k_1^s k_2^s = -\epsilon \), that is, \( g_s \) is also a flat surface. It follows that the sectional curvature of \( M^3 \) along \( TV \) is
\[ (\frac{-B}{a} k_1^s)(\frac{-B}{a} k_2^s) + \epsilon = \frac{\epsilon}{a^2}, \]
which is also the sectional curvature of \( M^3 \) along any plane spanned by \( \partial/\partial s \) and a vector \( X \in TV \).

**Remark 16** It is easily seen that if the hypersurface \( f \) just constructed is regarded as a submanifold of \( \mathbb{R}^5 \) for \( \epsilon = 1 \), then it does not have any umbilical normal direction at any point. Hence it provides a new example of a constant curvature submanifold of \( \mathbb{R}^5 \) with codimension two that is free of weak-umbilic points in the sense of [13].

**Example 17** As an explicit example, consider the Clifford torus
\[ g: M^2_0 := S^1(\cos \theta_0) \times S^1(\sin \theta_0) \to \mathbb{S}^3 \]
parametrized by
\[ g(t_1, t_2) = (\cos \theta_0 \cos t_1, \cos \theta_0 \sin t_1, \sin \theta_0 \cos t_2, \sin \theta_0 \sin t_2), \]
which has
\[ N(t_1, t_2) = (-\sin \theta_0 \cos t_1, -\sin \theta_0 \sin t_1, \cos \theta_0 \cos t_2, \cos \theta_0 \sin t_2) \]
as a unit normal vector field in \( \mathbb{S}^3 \). Then,
\[ f: M^2_0 \times \mathbb{R} \to \mathbb{S}^3 \]
given by (36) can be reparametrized by
\[ f(t_1, t_2, s) = (\cos s \cos t_1, \cos s \sin t_1, \sin s \cos t_2, \sin s \sin t_2, Bs), \]
after replacing \( s + \theta_0 \) by \( s \) and a translation in the \( \partial/\partial t \)-direction. This hypersurface appears in [5] as an example of a weak-umbilic free doubly-rotation surface with constant sectional curvature having the helix \( s \mapsto (\cos s, \sin s, Bs) \) as profile, in the sense of [10].
A similar example can be constructed in $\mathbb{H}^3 \times \mathbb{R}$, starting with the flat surface

$$g: M_0^2 := \mathbb{H}^1(\cosh \theta_0) \times \mathbb{S}^1(\sinh \theta_0) \to \mathbb{H}^3$$

parametrized by

$$g(t_1, t_2) = (\cosh \theta_0 \cos t_1, \cosh \theta_0 \sin t_1, \sinh \theta_0 \cos t_2, \sinh \theta_0 \sin t_2).$$

In this case, the corresponding constant curvature hypersurface of $\mathbb{H}^3 \times \mathbb{R}$ is

$$f(t_1, t_2, s) = (\cosh s \cos t_1, \cosh s \sin t_1, \sinh s \cos t_2, \sinh s \sin t_2, Bs),$$

These examples can be characterized as the only constant curvature hypersurfaces of $Q^3 \times \mathbb{R}$ with 0 as principal curvature in the $T$-direction and whose two remaining principal curvatures are constant along $\{T\}^\perp$.

## 7 The main result

In this section we prove our main result, namely, we provide a complete classification of all hypersurfaces with constant sectional curvature of $Q^3 \times \mathbb{R}$, $n \geq 3$. We state separately the cases $\epsilon = 1$ and $\epsilon = -1$. For $\epsilon = 1$ we have:

**Theorem 18.** Let $f: M^c_1 \to \mathbb{S}^n \times \mathbb{R}$, $n \geq 3$, be an isometric immersion of a Riemannian manifold of constant sectional curvature $c$. Then $c \geq 0$. Moreover,

(i) if $c = 0$ then $n = 3$ and $f(M^0_1)$ is an open subset of a Riemannian product $M^2_0 \times \mathbb{R}$, where $M^2_0$ is a flat surface of $\mathbb{S}^3$.

(ii) if $c \in (0, 1)$ then $n = 3$ and $f$ is locally given by the construction described in Section 6.

(iii) if $c = 1$ then $f(M^1_1)$ is an open subset of a slice $\mathbb{S}^n \times \{t\}$.

(iv) if $c > 1$ then $f(M^c_1)$ is an open subset of a rotation hypersurface given by Theorem 11(ii).

The classification of constant curvature hypersurfaces of $\mathbb{H}^n \times \mathbb{R}$ with dimension $n \geq 3$ reads as follows:

**Theorem 19.** Let $f: M^c_1 \to \mathbb{H}^n \times \mathbb{R}$, $n \geq 3$, be an isometric immersion of a Riemannian manifold of constant sectional curvature $c$. Then $c \geq -1$. Moreover,

(i) if $c = -1$ then $f(M^{-1}_1)$ is an open subset of a slice $\mathbb{H}^n \times \{t\}$.
(ii) if $c \in (-1,0)$ then either $n = 3$ and $f$ is locally given by the construction described in Section 6, or $f(M^n_0)$ is an open subset of one of the rotation hypersurfaces given by Theorem 11 (ii).

(iii) if $c = 0$ then one of the following possibilities holds:

(a) $n = 3$ and $f(M^n_0)$ is an open subset of a Riemannian product $M^n_0 \times \mathbb{R}$, where $M^n_0$ is a flat surface of $\mathbb{H}^3$.

(b) $f(M^n_0)$ is an open subset of a Riemannian product $M^n_0 \times \mathbb{R}$, where $M^n_0$ is a horosphere of $\mathbb{H}^n$.

(c) $f(M^n_0)$ is an open subset of the spherical rotation hypersurface given by Theorem 11 (iii)-(a).

(iv) if $c > 0$ then $f(M^n_0)$ is an open subset of the spherical rotation hypersurface given by Theorem 11 (iv).

Proof of Theorems 18 and 19: Assume that the vector field $T$ does not vanish at $x \in M^n$. Then $T$ is a principal direction of $f$ by Lemma 2 and Lemma 7. Let \{ $T$, $X_1$, \ldots, $X_{n-1}$ \} be an orthogonal basis of eigenvectors of $A_N$ at $x$, with

$$A_N T = \lambda T \quad \text{and} \quad A_N X_i = \lambda_i X_i, \quad 1 \leq i \leq n-1.$$ 

From the Gauss equation (2) of $f$ for $X = X_i$ and $Y = Z = X_j$, $i \neq j$, we get

$$c - \epsilon = \lambda_i \lambda_j, \quad i \neq j. \quad (40)$$

On the other hand, for $X = T$ and $Y = Z = X_i$ the Gauss equation yields

$$c - \epsilon = \lambda \lambda_i - \epsilon \|T\|^2. \quad (41)$$

Assume first that $c = \epsilon$. By (40), we can assume that $\lambda_i = 0$ for all $2 \leq i \leq n-1$. Then, applying (41) for $i \geq 2$ yields a contradiction with $T \neq 0$. We conclude that for $c = \epsilon$ the vector field $T$ vanishes identically, and this gives part (iii) of Theorem 18 and part (i) of Theorem 19.

Now suppose that $c \neq \epsilon$. Then $T$ can not vanish on any open subset. Thus, we can assume without loss of generality that it is nowhere vanishing. If $n \geq 4$, we obtain from (40) that all $\lambda_i's$ coincide for $2 \leq i \leq n-1$. Denote all of them by $\mu$. Then, the Gauss equations now read

$$c - \epsilon = \mu^2 \quad (42)$$

and

$$c - \epsilon = \lambda \mu - \epsilon \|T\|^2, \quad (43)$$

which can also be written as

$$c = \lambda \mu + \epsilon \nu^2. \quad (44)$$
In particular, it follows from (42) that \( c > \epsilon \).

Now, since \( T \neq 0 \), it follows from (42) and (43) that \( \lambda \neq \mu \). Moreover, since \( T \) is a principal direction, we obtain from (5) that \( \nu \) is constant along the leaves of \( \{ T \}^\perp \), and hence the same holds for \( \lambda \) by (44) (since \( \mu \) has multiplicity greater than one, one can show using the Codazzi equation that it is constant along its eigenbundle; cf. the proof of Theorem 1 in [6]). Then, one can use the following result to conclude that \( f \) is a rotation hypersurface. It slightly generalizes Theorem 1 in [6], but actually follows from its proof.

**Proposition 20** Let \( f : M^n \to \mathbb{Q}^n \times \mathbb{R} \) be a hypersurface with \( n \geq 3 \) and \( T \neq 0 \). Assume that \( f \) has exactly two principal curvatures \( \lambda \) and \( \mu \) everywhere, the first one being simple with \( T \) as a principal direction. If \( \lambda \) is constant along the leaves of the eigenbundle \( \{ T \}^\perp \) of \( \mu \), then \( f(M^n) \) is an open subset of a rotation hypersurface.

Thus, the proofs of Theorems 18 and 19 for \( n \geq 4 \) are completed by Theorems 10 and 11. This also applies to the case \( n = 3 \) when we have \( \lambda_2 = \lambda_3 \) everywhere. By (40) and (41), this is not the case only if \( \lambda = 0 \). In this situation, equation (44) reduces to

\[
\epsilon \nu^2 = c. \tag{45}
\]

If \( c = 0 \), then \( \nu \) vanishes identically, and thus \( f(M^3_0) \) must be an open subset of a Riemannian product \( M^2_0 \times \mathbb{R} \), where \( M^2_0 \) is a flat surface in either \( S^3 \) or \( H^3 \), according as \( \epsilon = 1 \) or \( \epsilon = -1 \), respectively. If \( c \neq 0 \), it follows from (45) that \( f \) is a constant angle hypersurface. Therefore, by Theorem 14 it is locally given by (36) for some surface \( g : M^2 \to \mathbb{Q}^3_0 \). Moreover, if we write \( \nu = 1/a \), it was shown in Section 6 that the principal curvatures of \( f \) are

\[
-\frac{B}{a}k_1^s - \frac{B}{a}k_2^s \quad \text{and} \quad 0,
\]

where \( k_1^s \) and \( k_2^s \) are the principal curvatures of \( g_\ast \). By the Gauss equation (40), we have

\[
c - \epsilon = (-\frac{B}{a}k_1^s)(-\frac{B}{a}k_2^s).
\]

Replacing \( c = \epsilon/a^2 \) and using that \( B^2 + 1 = a^2 \), it follows that \( k_1^s k_2^s = -\epsilon \), hence \( g_\ast \) is a flat surface.

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