From Gaudin Integrable Models to $d$-dimensional Multipoint Conformal Blocks

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In this work we initiate an integrability-based approach to multipoint conformal blocks for higher dimensional conformal field theories. Our main observation is that conformal blocks for $N$-point functions may be considered as eigenfunctions of integrable Gaudin Hamiltonians. This provides us with a complete set of differential equations that can be used to evaluate multipoint blocks.

1. INTRODUCTION

Conformal quantum field theories (CFTs) play an important role for our understanding of phase transitions, quantum field theory and even the quantum physics of gravity, through Maldacena’s celebrated holographic duality. Since they are often strongly coupled, however, they are very difficult to access with traditional perturbative methods. Polyakov’s famous conformal bootstrap program provides a powerful non-perturbative handle that allows to calculate critical exponents and other dynamical observables using only general features such as (conformal) symmetry, locality and unitarity/positivity [1]. The program has had an impressive early success in $d = 2$ dimensions [2] where it produced numerous exact solutions. During the last decade, the bootstrap has seen a remarkable revival in higher dimensional theories with new numerical as well as analytical incarnations. This has produced many stunning new insights, see e.g. [3] for a recent review and references, including record precision computations of critical exponents in the critical $d = 2$ Ising model [4, 5]. Despite these advances, it is evident that significant further developments are needed to make these techniques more widely applicable, beyond a few very special theories.

One promising avenue would be to study bootstrap consistency conditions for $N$-point correlators with $N > 4$ fields. Note that the enormous success in $d = 2$ is ultimately based on the ability to analyze correlation functions with any number of stress tensor insertions. But the extension of the bootstrap constraints in $d > 2$ beyond 4-point functions has been hampered by very significant technical problems, see [6–17] for recent publications on the subject. To overcome these challenges is the main goal of our work.

The central tool for CFTs in general and for the conformal bootstrap in particular are conformal partial wave expansions. These were introduced in [18] to separate correlation functions into kinematically determined conformal blocks (partial waves) [50] and expansion coefficients which contain all the dynamical information. For 4-point correlators, the relevant blocks are now well understood in any $d$, though only after some significant effort. Here we shall lay the foundations for a systematic extension to multipoint blocks. Our approach extends a remarkable observation in [19] about a relation between 4-point blocks and exactly solvable (integrable) Schrödinger problems.

To understand the key challenge in developing a theory of multipoint conformal blocks for $d > 2$, let us consider a 5-point function of scalar fields, the very simplest case of an $N$-point function with $N > 4$ field insertions. It is easy to see that one can build five independent conformally invariant cross ratios from $N = 5$ points as long as $d > 2$. On the other hand, we can reduce the evaluation of a 5-point function to a 3-point function by performing two operator product expansions (OPEs). The fields in the OPEs are characterized by a weight $\Delta$ and a spin $l$. So, from the two expansions we obtain a total of four quantum numbers, two weights and two spins of the fields that propagate in the intermediate channels, see Fig. 1. This is not sufficient to resolve the dependence of the 5-point function on the five cross ratios. In fact, we are missing one additional quantum number. Let us stress that $d = 3$ is the smallest dimension for which this happens. In $d = 2$ a 5-point function depends on four cross ratios and this matches the quantum numbers of intermediate fields. It is not difficult to see that the additional quantum numbers that are needed to characterize multipoint conformal blocks in $d > 2$ are associated with the choice of so-called tensor structures at the vertices of an OPE diagram, see Fig. 1. In the case of the 5-point function, the middle vertex in the OPE diagram gives rise to one additional quantum number. But what precisely is the nature of this quantum number and how can it be measured? Note that this question has not been addressed in any of the recent papers on multipoint blocks [6–17, 20, 21].

In order to describe our answer let us turn to the most basic description of conformal blocks, the so-called shadow formalism [22]. The latter provides integral formulas for conformal blocks that are reminiscent of Feynman integrals, in that they require a lot of technology to find analytical expressions in terms of special functions or even just efficient numerical tools to evaluate them. One crucial tool in the theory of Feynman integrals is to consider them as solutions of some differential equations. In their important work, Dolan and Osborn followed this same strategy and characterized shadow integrals as eigenfunctions of a set of Casimir differential operators [23]. The study of these differential equations
2. MULTIPOINT SHADOW INTEGRALS IN THE COMB CHANNEL

In order to state our results precisely, we shall briefly review some basics of the shadow integral formalism. The shadow formalism turns the graphical representation of a conformal block, such as that of Fig. 1, into an integral formula. Just as in the case of Feynman integrals, the ‘shadow integrand’ is built from relatively simple building blocks that are assigned to the links and 3-point vertices in the associated OPE diagram. For a scalar 5-point function, the most complicated vertex contains one scalar leg and two that are carrying symmetric traceless tensor (STT) representations. In order to write this vertex, we shall employ polarization spinors $z \in \mathbb{C}^d$ (see [28–31]) to convert spinning operators in STT representations into objects of the form

$$\mathcal{O}_{\Delta,t}(x; z) = \mathcal{O}^X_{\Delta,l}(x)z_{\nu_1} \cdots z_{\nu_l} = \mathcal{O}^X_{\Delta,l}(x)z_{\nu}.$$  \hfill (1)

The usual contraction of the STTs can be re-expressed as an integral over $\mathbb{C}^d$ as follows [32]

$$\mathcal{O}^X(x)\mathcal{O}'(x') = \int_{\mathbb{C}^d} d^dz \delta(z^2) \rho(z) \mathcal{O}(x; z)\mathcal{O}'(x'; z),$$  \hfill (2)

$$\rho(t) = \left(\frac{2}{\pi}\right)^{d-1} \frac{16t^{1-d/4}}{\Gamma(d/2-1)} K_{(d/2-1)}(2\sqrt{t}),$$  \hfill (3)

where $\mathcal{O}$ and $\mathcal{O}'$ are fields of equal spin and $K$ is the modified Bessel function of the second kind. In building shadow integrands, the function $\rho$ plays a role analogous to the propagator in Feynman integrals. Having converted field multiplets into functions, the 3-point vertex with one scalar leg in $d = 3$ takes the form

$$\Phi^{X}_{abc}(x; z) = (\mathcal{O}_{\Delta_a,l_a}(x_a; z_a)\mathcal{O}_{\Delta_c,l_c}(x_c; z_c)) \mathcal{O}_{\Delta_b,l_b}(x_b; z_b) = \frac{(X_{bc:a} \cdot z_b)^b}{(X_{ab:c} \cdot z_a)^c} \frac{(X_{ac:b} \cdot z_c)^c}{(X_{bc:a} \cdot z_b)^b} \frac{(X_{ab:c} \cdot z_a)^a}{(X_{ac:b} \cdot z_c)^c} \frac{(X_{bc:a} \cdot z_b)^b}{(X_{ac:b} \cdot z_c)^c} \frac{(X_{ac:b} \cdot z_c)^c}{(X_{bc:a} \cdot z_b)^b},$$  \hfill (4)

if $l_a - l_b \in 2\mathbb{Z}$ and vanishes otherwise. Here we have used the following standard notations

$$X^\mu_{ij,k} := \frac{x^\mu_{ik}}{x^\mu_{jk}} - \frac{x^\mu_{jk}}{x^\mu_{ik}} = -X^\mu_{kj,i}; \quad X^2_{ij,k} := \frac{x_{ij}}{x^\mu_{ik}x^\mu_{jk}},$$  \hfill (5)

with $x_{ij} = x_i - x_j$ and $X$ the unique independent cross-ratio that can be constructed from $(x_a, z_x, z_c; z_a, z_b)$,

$$X = \frac{1}{z_{ab}^2} \frac{z_{ac}^2}{(z_a \cdot X_{bc:a})(z_b \cdot X_{ca:b})}.$$  \hfill (6)

To a large extent, the function $t(X)$ that appears in the 3-point vertex is left undetermined by conformal symmetry. The only constraints come from the action of the stabilizer group $SO(d-1)$ of three points in $\mathbb{R}^d$. To describe these explicitly, let us restrict to $d = 3$ from now on. In this last setting, we are required to distinguish two cases, depending on the behavior of $t(X)$ under the action of the parity operator in $O(3)$:
where \( W_i^+ \): For parity-even functions, \( t(X) \) must be a polynomial of order at most \( \min(l_a, l_b) \).

\( W_i^- \): For parity-odd functions, \( t(X)/\sqrt{X(1-X)} \) must be a polynomial of order at most \( \min(l_a, l_b) - 1 \).

Taken together, these two linear spaces \( W_i^\pm \) of functions \( t(X) \) form a vector space of dimension

\[
n_{ab} = 2\min(l_a, l_b) + 1 = \min(2l_a + 1, 2l_b + 1).
\]  

(7)

The integer \( n_{ab} \) counts the number of 3-point tensor structures \([31]\). Note that \( n_{ab} = 1 \) if either \( l_a = 0 \) or \( l_b = 0 \) which means that \( t \) is a constant factor if there are two or three scalar legs. We shall therefore simply drop the corresponding vertex factors \( t \) when using formula (4) for vertices with two scalar legs.

Having described the vertex, we can now write down (shadow) integrals for any desired N-point function in the so-called comb channel, in which every OPE includes at least one of the external scalar fields. For \( N = 5 \) external scalar fields of weight \( \Delta_i, i = 1, \ldots, 5 \) the shadow integrals read

\[
\Psi^{(\Delta_1, \ldots, \Delta_5)}(\Delta_a, \Delta_b, l_a, l_b; t)(x_1, \ldots, x_5) = \prod_{s=a,b} \int_{\mathbb{R}^d} \frac{dz_s}{z_s^2} \frac{\rho(z_s \cdot z_s)}{\rho(z_s \cdot z_s)} \Phi_{1234}(x_1, x_2, x_3, x_5 \, \bar{z}_s) \times \Phi_{345}(x_1, x_2, x_3, x_5, z_s) \, \partial_X \, \partial_X \, \Phi_{45}(x_1, x_2, x_3, x_5, z_s). \]

(8)

Here the tilde on the indices of the first and third vertex means that we use eq. (4) for two scalar legs but with \( \Delta_a \) and \( \Delta_b \) replaced by \( d - \Delta_a \) and \( d - \Delta_b \), respectively.

After splitting off some factor \( \Omega \) that accounts for the nontrivial covariance law of the scalar fields under conformal transformations,

\[
\Psi^{(\Delta_1)}(\Delta_a, \Delta_b, l_a, l_b; t)(x_i) = \Omega^{(\Delta_1)}(x_i) \Psi^{(\Delta_1, \Delta_2, \Delta_3, \Delta_4)}(\Delta_a, \Delta_b, l_a, l_b; t)(u_1, \ldots, u_5),
\]

\[
\Omega^{(\Delta_1)}(x_i) := \left( X^2_{12345} \right)^{\frac{\Delta_1}{2}} \prod_{i=1}^{4} \left( X^2_{i+1,i+1,i-1;i} \right)^{\frac{\Delta_1}{2}} \left( X^2_{345} \right)^{\frac{\Delta_1}{2}},
\]

with \( \Delta_{ij} = \Delta_i - \Delta_j \) as usual, the shadow integral (8) gives rise to a finite conformal integral that defines the conformal block \( \psi \) as a function of five conformally invariant cross ratios \( u_i \). These integrals depend on the choice of \((\Delta_a, l_a), (\Delta_b, l_b)\) and the function \( t(X) \) at the middle vertex. Our goal is to compute this uninviting looking integral.

The strategy we have sketched in the introduction is to write down five differential equations for these blocks. Four of these are given by the eigenvalue equations for the second and fourth order Casimir operators for the intermediate channels,

\[
D_p^\alpha \Psi^{(\Delta_1, \Delta_2, \Delta_3, \Delta_4)}(\Delta_a, \Delta_b, l_a, l_b; t)(u) = C_p^\alpha \Psi^{(\Delta_1, \Delta_2, \Delta_3, \Delta_4)}(\Delta_a, \Delta_b, l_a, l_b; t),
\]

where \( p = 2, 4 \) and \( C_p^\alpha \) denotes the eigenvalue of the \( p \)-th order Casimir operator in the representation \((\Delta_s, l_s)\) for \( s = a, b \). The explicit form of the differential operators \( D_p^\alpha \) can be worked out and the resulting expressions resemble those found by Dolan and Osborn for \( N = 4 \).

But we are missing one more differential equation which we shall construct in the next section. It will turn out that shadow integrals are eigenfunctions of a fifth differential operator provided we prepare a very special basis \( t_n(X), n = 1, \ldots, n_{ab} \), in the space of 3-point tensor structures. We can characterize these functions \( t_n(X) \) as eigenfunctions of a particular fourth order differential operator

\[
H^{(\Delta, l_i)} = h_0(X) + \sum_{q=1}^{4} h_q(X)(X^q - 1)(X^q - 1) \partial_X^q, \quad (10)
\]

where \( h_q = h_q^{(\Delta, l_i)} \) are polynomials of order at most three, see Appendix A for concrete expressions. The operator \( H \) which has several remarkable properties, appears to be new. For our discussion it is most important to note that \( H \) leaves the two subspaces \( W_i^\pm \) invariant whenever both \( l_a \) and \( l_b \) are integer. Consequently, it specifies a special basis \( t_n \) of functions \( t(X) \) in the space of tensor structures,

\[
H^{(\Delta, l_i)} t_n(X) = \tau_n t_n(X), \quad n = 0, \ldots, n_{ab}. \quad (11)
\]

Explicit formulas for the eigenvalues \( \tau_n \) and the eigenfunctions \( t_n(X) \) can be worked out, and it is this basis of 3-point tensor structures that we will use to write down differential equations for the associated shadow integrals.

### 3. Multi-point blocks and Gaudin Hamiltonians

Our goal now is to characterize the shadow integrals through a complete set of five differential equations. These will take the form of eigenvalue equations for a set of commuting Gaudin Hamiltonians. In order to state precise formulas we need a bit of background on Gaudin models [25, 26]. Let us begin with a central object, the so-called Lax matrix,

\[
\mathcal{L}(w) = \sum_{i=1}^{N} \frac{T_{\alpha}(w)}{w - w_i} = \mathcal{L}_{\alpha}(w) T^\alpha. \quad (12)
\]

Here \( w_i \) are a set of complex numbers, \( T_{\alpha} \) denotes a basis of generators of the conformal Lie algebra and \( T^\alpha \) its dual basis with respect to an invariant bilinear form. The object \( T_{\alpha}^{(i)} \) is the standard first order differential operator that describes the behavior of a scalar primary field \( \mathcal{O}(x_i) \) of weight \( \Delta_i \) under the infinitesimal conformal transformation generated by \( T_{\alpha} \).

Given some conformally invariant symmetric tensor \( \kappa_p \) of degree \( p \) one can construct a family \( \mathcal{H}_p(w) \) of commuting operators as [33–35]

\[
\mathcal{H}_p(w) = \kappa_p^{\alpha_1 \cdots \alpha_p} \mathcal{L}_{\alpha_1}(w) \cdots \mathcal{L}_{\alpha_p}(w) + \ldots, \quad (13)
\]
where the dots represent correction terms expressible as lower degree combinations of the Lax matrix components \( L_\alpha(w) \) and their derivatives with respect to \( w \). For \( p = 2, 4 \) such correction terms are absent. The correction terms are necessary to ensure that the families commute,

\[
[ \mathcal{H}_p(w), \mathcal{H}_q(w') ] = 0 , \tag{14}
\]

for all \( p, q \) and all \( w, w' \in \mathbb{C} \). In the case where \( d = 3 \), the conformal algebra possesses two independent invariant tensors of second and fourth degree. Hence, we obtain two families of commuting differential operators that act on functions of the coordinates \( x_i \).

It is a well-known fact that these families commute with the diagonal action of the conformal algebra, i.e.

\[
[ T_\alpha, \mathcal{H}_p(w) ] = 0 \quad \text{where} \quad T_\alpha = \sum_{i=1}^N T^{(i)}_\alpha . \tag{15}
\]

Hence the commuting families \( \mathcal{H}_p(w) \) of operators descend to differential operators on functions \( \psi(u) \) of the conformally invariant cross ratios \( u \).

The functions \( \mathcal{H}_p(w) \) provide several continuous families of commuting operators. Only a finite set of these operators are independent. There are many ways of constructing such sets of independent operators, e.g. by taking residues of \( \mathcal{H}_p(w) \) at the singular points to give just one example. For the moment any such set still contains \( N \) parameters \( w_i, i = 1, \ldots, N \). Without loss of generality we can set these of three complex numbers to some specific value, e.g. \( w_1 = 0, w_{N-1} = 1, w_N = \infty \) so that we remain with \( N - 3 \) complex parameters our Gaudin Hamiltonians depend on.

Now we adapt the Gaudin model to the study of multipoint blocks. In the latter context we insist that the set of commuting operators we work with allows us to measure the weights \( \Delta \) and spins \( l \) of fields that are exchanged in intermediate channels, as do the multipoint Casimir operators. So, in order for the Gaudin Hamiltonians to be of any use to us, we must ensure that they include all such Casimir operators. For this to be the case, we are forced to make a very special choice of the remaining parameters \( w_r \) and to consider specific limits of these parameters (such limits have also been considered in \([36, 37]\) to study bending flow Hamiltonians and their generalisations \([38–41]\)). Let us explain this here for \( N = 5 \). Setting \( w_2 = \omega^2 \) and \( w_3 = \omega \) we can define

\[
\tilde{\mathcal{H}}_p(w) := \lim_{\omega \to 0} \omega^p \mathcal{H}_p(\omega w) , \quad p = 2, 4 . \tag{16}
\]

The new functions \( \tilde{\mathcal{H}}_p \) take values in the space of \( p \)th order differential operators on cross ratios. They possess singularities at three points only, namely at \( w = 0, 1, \infty \). Let us note that taking the limit \( \omega \to 0 \) does not spoil commutativity of these Hamiltonians.

After performing the special limit on the parameters \( w_r \), we can now extract the multipoint Casimir operators rather easily. In fact, it is not difficult to check that

\[
D^a_p = \lim_{w \to 0} w^p \tilde{\mathcal{H}}_p(w) , \quad D^b_p = \lim_{w \to \infty} w^p \tilde{\mathcal{H}}_p(w) . \tag{17}
\]

for \( p = 2, 4 \). Any additional independent operator we can obtain from \( \tilde{\mathcal{H}}_p(w) \) may be used to measure a fifth quantum number. One can show that the two second order Casimir operators \( D^2_2, s = a, b \) exhaust all the independent operators that can be obtained from \( \tilde{\mathcal{H}}_2(w) \).

The family \( \tilde{\mathcal{H}}_4(w) \), on the other hand, indeed supplies one independent operator in addition to the fourth order Casimir operators \( D^4_4, s = a, b \). We propose to use the operator \( \mathcal{V}_4 \) defined through

\[
\tilde{\mathcal{H}}_4(w = 1/2) = 16 \mathcal{V}_4 + \ldots , \tag{18}
\]

where the dots represent quadratic terms coming from the corrections in eq. (13). In the particular limit \( \omega \to 0 \) that we consider here, these corrections can be re-expressed in terms of the quadratic Casimirs \( D^2_2, s = a, b \), and can thus be discarded without spoiling commutativity of \( \mathcal{V}_4 \) with the Casimirs. An explicit computation then shows that \( \mathcal{V}_4 \) is expressed in terms of the conformal generators \( T^{(i)}_\alpha \) as

\[
\mathcal{V}_4 = \kappa_4^{\alpha_1 \cdots \alpha_4} S_{\alpha_1} \cdots S_{\alpha_4} , \quad S_{\alpha} = T^{(1)}_\alpha + T^{(2)}_\alpha - T^{(3)}_\alpha . \tag{19}
\]

The explicit form of \( \mathcal{V}_4 \) as a differential operator acting on functions \( \psi(u) \) of five cross ratios will be spelled out in our forthcoming publication \([27]\). Our central claim is that the 5-point shadow integrals \( \psi \) we discussed in the previous subsection are joint eigenfunctions of the four Casimir operators, see eq. (9), and of the vertex operator we defined through eq. (18),

\[
\mathcal{V}_4 \psi(\Delta_{12},\Delta_{13},\Delta_{45}) = \tau_n \psi(\Delta_{12},\Delta_{13},\Delta_{45}) \psi(t_n(X)) , \tag{20}
\]

where the eigenvalues \( \tau_n \) coincide with those that appeared in eq. (11) when describing the particular choice of a basis \( t_n(X) \) of tensor structures. These five differential equations characterize the shadow integral completely.

4. CONCLUSIONS AND OUTLOOK

In this work we initiated a systematic construction of multipoint conformal blocks in \( d \geq 3 \). Our advance relies on a characterization of multipoint conformal blocks as wave functions of Gaudin integrable models, which extends a similar relation between 4-point blocks and integrable Calogero-Sutherland models uncovered in \([19]\).

More specifically, we have explained that for a very special choice of tensor structures at the 3-vertices \( \Phi \) in the shadow integrand of eq. (8), the corresponding shadow integral becomes a joint eigenfunction of a complete set of commuting differential operators. The latter are Hamiltonians of special limits of the Gaudin model.

While we have explained the main ideas within the example of 5-point functions in \( d = 3 \), the strategy and in particular the relation with Gaudin models is completely general, i.e. it extends to \( N > 5 \), \( d > 3 \) and even spinning...
external operators, with appropriate changes. Starting with $N = 6$, there exist topologically distinct channels that can include vertices in which all three legs carry spin, such as the so-called snowflake channel for $N = 6$ [12]. Such vertices involve functions $t$ of more than one variable and hence the choice of basis in the space of tensor structures needs to be extended. Tensor structures for the generic vertex in $d = 3$, for example, are characterized as eigenfunctions of two commuting differential operators of fourth order in two variables $X, Y$ rather than a single such operator acting on $X$. As we go to higher dimension $d$, the vertices can become more and more involved and links can carry more than just STT representations. Treating such more general links only requires us to consider higher order Casimir operators. Through the relation to Calogero-Sutherland models [19], their solution theory is well known, see e.g. [42]. In this sense, links do not pose a significant new complication for the construction of multipoint blocks even if $d > 3$.

In forthcoming work [27] we will explain in detail how to construct the vertex differential operators, both for the shadow integrand and the shadow integral, and we shall spell out explicit formulas for all five differential operators that characterize the shadow integrals for 5-point functions in $d = 3$. This can then serve as a starting point to evaluate 5-point blocks explicitly, e.g. through series expansions or Zamolodchikov-like recursion formulas, similar to those used for 4-point blocks [24, 42–46].

Obviously, it would be very interesting to extend these constructions of differential operators to 6-point blocks, to develop an evaluation theory and to initiate a multipoint bootstrap for $d > 2$. As we have argued in the introduction, taking bootstrap constraints from multipoint correlation functions seems like a good strategy that is well aligned with the success of the 3-point tensor structures. Except for a constant term in $h_0$ which depends a bit on the precise choice of the fifth Gaudin Hamiltonian we extract, all coefficients are symmetric w.r.t. exchange of $a$ and $b$. Hence we will split them as

$$h_{(\Delta_a,l_a;\Delta_b,l_b)}(X) = \chi_{(\Delta_a,l_a;\Delta_b,l_b)}(X) + a \leftrightarrow b$$

and display the polynomials $\chi(X)$ instead of $h(X)$,

$$\chi_1 = 16X^2 (l_a - 1)(l_b - 1)(l_a + l_b - 2)$$

$$\chi_2 = 4X(4l_a^2 + 8l_a l_b + 2l_a(2\Delta_c - 21) + 2\Delta_a \Delta_b - 6\Delta_a - 7\Delta_c + 43) + 2 \left( (l_a + l_b)^2 + 2l_a(2\Delta_c - 10) - 2\Delta_a^2 + \Delta_c^2 + 2\Delta_a \Delta_b - 10\Delta_c + 22 \right)$$

$$\chi_3 = 32X (l_a - 2) - 4(4l_a + 2\Delta_c - 11)$$

$$\chi_4 = 8$$

$$\chi_5 = -8X^2 l_a (l_a - 1)l_b (l_b - 1) + 4Xl_a l_b (2l_a - 4l_a + 2\Delta_a \Delta_b - 6\Delta_a - \Delta_c + 8) + \text{const}.$$
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[50] In this letter we shall not distinguish between the two notions and simply use the term conformal block.