RECOVERING THE ORTHOGONAL POLYNOMIALS FROM THEIR SPECIFIC SPECTRAL TRANSFORMATIONS

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ABSTRACT. In this work, the concept of quasi-type Kernel polynomials with respect to a moment functional is introduced. Difference equation satisfied by these polynomials along with the criterion for orthogonality conditions are discussed. Recovering the source orthogonal polynomial involving the linear spectral transformation is provided. This process involves an expression of ratio of kernel polynomials. Special cases of such ratios in terms of certain continued fractions are exhibited.

1. INTRODUCTION

Let \( \mu \) be a non-trivial positive Borel measure with support containing infinitely many points. The support of \( \mu \) having only finitely many points leads to the linear dependence of monomials in \( L_2(d\mu) \)-known as trivial measure. Thus we deal with the measure \( \mu \) having infinitely many points in the support. The monomials \( \{x^j\}_{j=0}^{\infty} \) then become linearly independent in \( L_2(d\mu) \). Applying Gram-Schmidt process on \( \{x^j\}_{j=0}^{\infty} \) one obtains certain polynomials \( \{P_n\}_{n\geq 0} \) satisfying

\[
\mathcal{L}(P_n(x)P_m(x)) = \int P_n(x)P_m(x)d\mu = \delta_{nm}.
\]  

(1.1)

It can be noted that, by considering two sequences of complex constants \( \{\lambda_n\} \) and \( \{c_n\} \), where \( \lambda_n \)'s and \( c_n \)'s are related to moment functional \( \mathcal{L} \), the following three term recurrence relation (TTRR) \[11\]

\[
xP_n(x) = P_{n+1}(x) + c_{n+1}P_n(x) + \lambda_{n+1}P_{n-1}(x),
\]  

(1.2)

with \( P_{-1}(x) = 0, \ P_0(x) = 1 \), can also be used to obtain recursively the sequence of orthogonal polynomials \( \{P_n\}_{n\geq 0} \). Favard’s theorem \[15\] guarantees that there exists a unique linear moment functional \( \mathcal{L} \) such that the orthogonality condition (1.1) is satisfied with respect to \( \mathcal{L} \). Moreover, the definiteness property of the moment functional depends on the parameters \( \lambda_n \) and \( c_n \).

The concept of linear combination of two consecutive members of a sequence of orthogonal polynomials was first studied by Riesz \[34\] in 1923 in his solution to the Hamburger moment problem. Later, in 1937, Fejér \[23\] studied the linear combination of three consecutive member of sequence of orthogonal polynomials. Finally, Shohat \[36\] generalized the concept to finite linear combination of orthogonal polynomials in the study of mechanical quadrature. The concept of quasi-orthogonal polynomial on the unit circle has been studied by Alfaro and Moral \[6\]. However, the quasi-orthogonality on the unit circle with respect to linear functional defined on the space of Laurent polynomials is not so strong in comparison to the real line case. For further study of quasi-orthogonal polynomials,
we encourage the readers to see [1,2,3,13,15,18–20].

In [27], Grinshpun studied the necessary and sufficient conditions for the orthogonality of the linear combinations of polynomials which he called a special linear combination of orthogonal polynomials with respect to a weight function. The support of this weight function lies in an interval. These type of orthogonal families of polynomials appears in the solution to the problem of Peebles-Korous [33], approximate solution to the Cauchy problem for the ordinary differential equations [21] and Gelfond’s problem of the polynomials [24]. Grinshpun also proved that the Bernstein-Szego orthogonal polynomials of any kind can be written as a special linear combination of the Chebyshev polynomials of the same kind. The special feature of this representation is that the coefficients are independent of $n$. Orthogonality of the linear combination of orthogonal polynomials with constant coefficients is also discussed in [3–5]. Furthermore, the TTRR type relation satisfied by a quasi-orthogonal polynomial of order one along with the orthogonality of quasi-orthogonal polynomials is discussed in [29]. The second-order differential equations for quasi-orthogonal polynomials of order one is also addressed in [29].

When we deal with the measure $d\tilde{\mu}$ of the form $d\tilde{\mu} = (x - k)d\mu$, where $k$ does not belong to the support of measure $d\mu$, we obtain a sequence of orthogonal polynomials, which we call kernel polynomials. We refer to [8,14,15,31] and references therein for further details in this direction. In this article, we define the linear combination of two consecutive terms of a sequence of kernel polynomials, which we call quasi-type kernel polynomials of order one. Further, we study the properties of quasi-type kernel polynomials and their connection with orthogonal polynomials corresponding to spectral transformations. In particular, we recover the orthogonal polynomials $P_n(z)$ from the linear combination of quasi-type kernel polynomials and polynomials generated by Darboux transformations. Additionally, we express the ratio of consecutive degree of polynomials generated by Christoffel transformation in terms of infinite continued fractions.

1.1. Organization. In Section 2, we discuss the necessary and sufficient condition for the quasi-type kernel polynomial of order one. In addition, we discuss the criterion for the orthogonality of quasi-type kernel polynomials. Section 3 describes the recovery of orthogonal polynomials from the quasi-type kernel polynomial of order one and linear spectral transformations. Moreover, the recovery of orthogonal polynomials from the quasi-type kernel polynomial of order two and the iterated kernel polynomials is addressed. In Section 4, we calculate the limiting case of the ratio of kernel polynomials. Furthermore, the ratio of certain kernel polynomials (namely Laguerre and Jacobi) in terms of continued fraction is also discussed.

2. QUASI-TYPE KERNEL POLYNOMIAL AND ORTHOGONALITY

In this section, we discuss the known results about a linear combination of orthogonal polynomials known as quasi-orthogonal polynomial and the polynomials generated by Christoffel transformation known as kernel polynomials. Motivated by this, we define quasi-type kernel polynomial of order one. We also give an example that satisfies the condition of the quasi-type kernel polynomials. We conclude this section with the discussion of orthogonality of quasi-type kernel polynomials.
Definition 2.1. [1,15] A non-zero polynomial $p$ is called quasi-orthogonal polynomial of order one if it is of degree at most $n+1$ and
\[ \mathcal{L}(x^m p(x)) = \int x^m p(x) d\mu = 0 \text{ for } m = 0, 1, 2, \ldots, n-1. \]

Remark 2.1. Note that
\begin{enumerate}
  \item $\mathcal{L}(x^m p_{n+1}(x)) = 0$ for $m = 0, 1, 2, \ldots, n-1$, and
  \item $\mathcal{L}(x^m p_n(x)) = 0$ for $m = 0, 1, 2, \ldots, n-1$.
\end{enumerate}

This gives that $p_{n+1}(x)$ and $p_n(x)$ are both quasi-orthogonal polynomials of order one. Thus, one can think of $p(x)$ as a linear combination of $p_{n+1}(x)$ and $p_n(x)$.

Now, we will state the result which justifies the above statement.

Theorem 2.1. [1,15] Let $Q_{n+1}(x)$ be a quasi-orthogonal polynomial of order one, if, and only if, there are constants $a$ and $b$, not both zero simultaneously, such that
\[ Q_{n+1}(x) = a p_{n+1}(x) + b p_n(x). \]

For given $k \in \mathbb{C}$, we can define the new linear functional $\mathcal{L}^*$ for a polynomial $p(x)$ as
\[ \mathcal{L}^*(p(x)) = \mathcal{L}((x-k)p(x)). \]

This new linear functional is called the Christoffel transformation of $\mathcal{L}$ at $k$. We can define the corresponding kernel polynomials by the formula [15]
\[ p^*_n(k;x) = (x-k)^{-1} \left[ p_{n+1}(x) - \frac{p_{n+1}(k)}{p_n(k)} p_n(x) \right] \text{ for } n \geq 0 \text{ and } p_n(k) \neq 0. \]

\[ \{p^*_n(k;x)\}_{n=0}^\infty \] is a monic orthogonal polynomial sequence with respect to $\mathcal{L}^*$ [15], and hence by Favard’s theorem, it satisfies the TTRR:
\[ x p^*_n(k;x) = p^*_n+1(k;x) + c^*_n p_n(k;x) + \lambda^*_n p^*_n(k;x), \]
where
\[ \lambda^*_n = \lambda_n \frac{p_n(k)p_{n-2}(k)}{p_{n-1}(k)}, \quad c^*_n = c_{n+1} - \frac{p^2_n(k) - p_{n-1}(k)p_{n+1}(k)}{p_{n-1}(k)p_n(k)}. \]

The study of the kernel polynomials with respect to the non-trivial probability measure on the unit circle is also an active part of research. When $k = 1$, the sequence of kernel polynomials satisfies the three-term recurrence relation with their recurrence coefficients related to the positive chain sequences. The kernel polynomials on the unit circle are closely related to the para-orthogonal polynomials. For more information concerning kernel polynomials (known as Christoffel-Darboux kernel), we refer to [10,16] and references therein.

The Christoffel-Darboux identity [15, eq. 4.9] is given by
\[ \lambda_1\lambda_2\ldots\lambda_{n+1} \sum_{j=0}^n \frac{p_j(x)p_j(x')}{{\lambda_1\lambda_2\ldots\lambda_{j+1}}} = \frac{p_{n+1}(x)p_n(x') - p_{n+1}(x')p_n(x)}{x-x'}, \]
where $\{p_n(x)\}_{n \geq 0}$ is a orthogonal polynomial sequence with respect to $d\mu$.

With the use of (2.4), we can express the kernel polynomials as
\[ p_n^*(k;x) = \lambda_1\lambda_2\ldots\lambda_{n+1}(p_n(k))^{-1} K_n(x,k), \]
where
\[ K_n(x, k) = \sum_{j=0}^{n} p_j(x)p_j(k), \quad p_n(x) = (\lambda_1\lambda_2...\lambda_{n+1})^{-1/2} P_n(x). \] (2.6)

For a fixed \( k \), we can easily deduce from the Christoffel-Darboux identity that \((x - k)K_n(x, k)\) is a quasi-orthonormal polynomial of order one. On the other hand if we replace fixed number \( k \) by variable \( u \) then we can discuss the orthogonality of the polynomials \{(x - u)K_n(x, u)\}_{n \geq 0}.

**Theorem 2.2.** Let \( d\mu \) be a positive Borel measure on \( \mathbb{R} \) with finite moments. Then the sequence \( \{(x - u)K_n(x, u)\}_{n \geq 0} \) forms an orthogonal polynomial sequence with respect to the product measure \( d\mu(\mathbb{R} \times \mathbb{R}) \) on \( L^2(\mathbb{R}^2, d\mu(\mathbb{R} \times \mathbb{R})) \).

**Proof.**
\[ \iint |x-u|^2 K_n(x,u)K_m(x,u) d\mu(u) d\mu(x) = \begin{cases} 0 & \text{for } m \neq n, \\ 2\lambda_{n+1}^2 & \text{for } m = n. \end{cases} \] (2.7)

For \( m \leq n - 2 \), we have
\[ \iint (x-u)^2 K_n(x,u)K_m(x,u) d\mu(u) d\mu(x) = 0. \]

For \( m = n - 1 \), using the orthonormal property of \( p_n \). We have
\[ \iint (x-u)^2 K_n(x,u)K_m(x,u) d\mu(u) d\mu(x) = -\lambda_{n+1}^2 \int p_{n+1}(x)p_{n-1}(x) d\mu(x) = 0. \]

For \( m = n \), we have
\[ \iint (x-u)^2 K_n(x,u)K_m(x,u) d\mu(u) d\mu(x) = \lambda_{n+1}^2 \left( \int p_{n+1}(x) d\mu(x) + \int p_n^2(x) d\mu(x) \right) = 2\lambda_{n+1}^2. \]

This completes the proof. \( \square \)

Now, we give the following definition:

**Definition 2.2.** Let \( \{P_n^*(k;x)\}_{n=0}^{\infty} \) be the sequence of kernel polynomials which exists for some \( k \in \mathbb{C} \), and forms an orthogonal polynomial sequence with respect to \( \mathcal{L}^* \). A non-zero polynomial \( Q_{n+1}^*(k;\cdot) \) is called a quasi-type kernel polynomial of order one if it is of degree at most \( n+1 \) and \( \mathcal{L}^*(x^m Q_{n+1}^*(k;x)) = 0 \) for \( m = 0, 1, ..., n-1 \).

**Remark 2.2.** Note that
1. \( \mathcal{L}^*(x^m P_{n+1}^*(k;x)) = 0 \) for \( m = 0, 1, 2, ..., n-1 \), and
2. \( \mathcal{L}^*(x^m P_n^*(k;x)) = 0 \) for \( m = 0, 1, 2, ..., n-1 \),
so that both \( P_{n+1}^*(k;x) \) and \( P_n^*(k;x) \) are quasi-type kernel polynomials of order 1.
Remark 2.3. In general, we can say a polynomial $Q^*_{n+1}(k; \cdot) \neq 0$ is called quasi-type kernel polynomial of order $l \geq 1$ if, and only if, it is of degree at most $n + 1$, $n \geq l + 1$ and $\mathcal{L}^*(x^m Q^*_{n+1}(k; x)) = 0$ for $m = 0, 1, \ldots, n - l$.

Theorem 2.3. $Q^*_{n+1}(k; x)$ is a quasi-type kernel polynomial of order 1, if, and only if, there are constants $a$ and $b$, not zero simultaneously, such that

$$Q^*_{n+1}(k; x) = aP^*_{n+1}(k; x) + bP^*_n(k; x).$$

Proof. If $Q^*_{n+1}(k; x)$ is a quasi-type kernel polynomial of order 1, then for some constant $c_0, c_1, \ldots, c_{n+1}$, we can write

$$Q^*_{n+1}(k; x) = \sum_{m=0}^{n+1} c_m P^*_m(k; x)$$

with $c_m = \frac{\mathcal{L}^*[Q^*_{n+1}(k; x)P^*_m(k; x)]}{\mathcal{L}^*[P^*_m(k; x)]}$ and hence, $c_m = 0$ for $m \in \{0, 1, \ldots, n - 1\}$. Thus, we get

$$Q^*_{n+1}(k; x) = aP^*_{n+1}(k; x) + bP^*_n(k; x).$$

Conversely, if $a$ and $b$ are not simultaneously zero, then

$$\mathcal{L}^*(x^m Q^*_{n+1}(k; x)) = a\mathcal{L}^*(x^m P^*_{n+1}(k; x)) + b\mathcal{L}^*(x^m P^*_n(k; x)) = 0$$

for $m = 0, 1, \ldots, n - 1$.

This completes the proof. \square

Remark 2.4. In general, we can extend the above theorem for order $l$ and say $Q^*_n(k; x)$ is a monic quasi-type kernel polynomial of order $l$ if

$$Q^*_n(k; x) = P^*_n(k; x) + \sum_{m=1}^{l} \alpha_m P^*_{n-m+1}(k; x) \text{ for } n \geq l + 1.$$  

Next, we consider an example which supports Theorem 2.3. For this, first we easily show that polynomial $P^*_{n+1}(k; x)$ is a quasi-type kernel polynomial of order one with respect to $\mathcal{L}^*$. Indeed, $P^*_{n+1}(k; x)$ can be written as a linear combination of $P^*_n(k; x)$ and $P^*_m(k; x)$ with constant coefficients $[31$, eq. 2.5]:

$$P^*_{n+1}(k; x) = P^*_n(k; x) - \frac{P^*_{n}(k)}{P^*_{n+1}(k)} \lambda_{n+1} - P^*_m(k; x),$$  

where $\lambda_{n+1}$ is a strictly positive constant in TTRR (1.2).

Example 2.1. Let $\{C_n(x)\}_{n=0}^{\infty}$ be a sequence of polynomials which forms an orthogonal polynomial sequence with respect to the Chebyshev measure $\,d\mu = (1 - x^2)^{-1/2} \,dx$ with compact support $[-1, 1]$. This is referred to as Chebyshev polynomial of first kind. The corresponding monic Chebyshev polynomial can be written as

$$\hat{C}_0(x) = C_0(x),$$

$$\hat{C}_n(x) = 2^{-n}C_{n+1}(x), \; n \geq 0.$$  

The monic polynomial $\hat{C}_n(x)$ satisfies the following TTRR

$$\hat{C}_{n+1}(x) = x\hat{C}_n(x) - \frac{1}{4}\hat{C}_{n-1}(x), \; n \geq 2,$$

$$\hat{C}_2(x) = x\hat{C}_1(x) - \frac{1}{2}\hat{C}_0(x).$$
By using (2.2), we can write (2.9) as

\[
\hat{C}_{n+1}(k; x) = \frac{1}{x - k} \left[ \hat{C}_{n+2}(x) - \frac{\hat{C}_{n+2}(k)}{\hat{C}_{n+1}(k)} \hat{C}_{n+1}(x) \right].
\]

\{\hat{C}_n(k; x)\}_{n=0}^{\infty} is a monic orthogonal polynomial sequence with respect to the quasi definite linear functional \(L^*\). Then, by [28], we say that \(\hat{C}_{n+1}(x)\) is a quasi-type kernel polynomial of order one. Indeed,

\[
\hat{C}_{n+1}(x) = \hat{C}_{n+1}(k; x) - \frac{1}{4} \frac{\hat{C}_n(k)}{\hat{C}_{n+1}(k)} \hat{C}_n(k; x).
\]

2.1. Recurrence relation and orthogonality. It is known that we do not have the orthogonality of quasi-orthogonal polynomials with respect to \(L\), although it is interesting to obtain the difference equation similar to TTRR of quasi-orthogonal polynomials. In [29], Ismail and Wang discussed the TTRR type relation for quasi-orthogonal polynomials. In the next result, we generalize their result to obtain the difference equation with variable coefficients of quasi-type kernel polynomials.

**Theorem 2.4.** Let \(Q^*_n(k; x)\) be a monic quasi-type kernel polynomial of order 1. Then \(Q^*_n(k; x)\) satisfy the difference equation

\[
\mathcal{J}_n(x)Q^*_{n+2}(k; x) = \left[ D_{n+1}(x)\mathcal{J}_n(x) - b\mathcal{J}_{n+1}(x) \right] Q^*_n(k; x) - \lambda^*_n + 1 \mathcal{J}_{n+1}(x) Q^*_n(k; x),
\]

where

\[
D_{n+1}(x) = x - c^*_{n+2} + b, \quad \mathcal{J}_{n+1}(x) = bD_n(x) + \lambda^*_n.
\]

**Proof.** By the definition of \(Q^*_n(k; x)\), we have

\[
Q^*_n(k; x) = \mathcal{P}^*_n(k; x) + b\mathcal{P}^*_{n-1}(k; x).
\]

By using (2.2), we can write (2.9) as

\[
Q^*_n(k; x) = \mathcal{P}^*_n(k; x) + b\mathcal{P}^*_{n-1}(k; x) = -\frac{b}{\lambda^*_n} \mathcal{P}^*_n(k; x) + \left( (x - c^*_n) \frac{b}{\lambda^*_n} + 1 \right) \mathcal{P}^*_{n-1}(k; x).
\]

(2.10)

We can write the equations (2.9) and (2.10), in the matrix form as follows

\[
\begin{pmatrix}
Q^*_{n+1}(k; x) \\
Q^*_n(k; x)
\end{pmatrix} =
\begin{pmatrix}
1 & b \\
b \lambda^*_n & (x - c^*_n) \frac{b}{\lambda^*_n} + 1
\end{pmatrix}
\begin{pmatrix}
\mathcal{P}^*_{n+1}(k; x) \\
\mathcal{P}^*_n(k; x)
\end{pmatrix}.
\]

Since the right side of the matrix is invertible, we have

\[
\begin{pmatrix}
\mathcal{P}^*_{n+1}(k; x) \\
\mathcal{P}^*_n(k; x)
\end{pmatrix} = \lambda^*_n \begin{pmatrix}
x - c^*_n \frac{b}{\lambda^*_n} + 1 & -b \\
b \lambda^*_n & 1
\end{pmatrix}\begin{pmatrix}
Q^*_{n+1}(k; x) \\
Q^*_n(k; x)
\end{pmatrix}
\]

(2.11)

Further, using (2.2), we write

\[
Q^*_{n+2}(k; x) = (x - c^*_{n+2} + b)\mathcal{P}^*_{n+1}(k; x) - \lambda^*_n \mathcal{P}^*_n(k; x).
\]

Again, we can use (2.11) to obtain the expression of \(Q^*_{n+2}(k; x)\) in terms of \(Q^*_n(k; x)\) and \(Q^*_n(k; x)\) as

\[
Q^*_{n+2}(k; x) = \frac{\lambda^*_n}{b^2 + \lambda^*_n x - c^*_n + b}\begin{pmatrix}
(x - c^*_n) \frac{b}{\lambda^*_n} + 1 & -b \\
b \lambda^*_n & 1
\end{pmatrix}\begin{pmatrix}
Q^*_{n+1}(k; x) \\
Q^*_n(k; x)
\end{pmatrix}.
\]
\[ Q_n^*(k; x) = \frac{\lambda_{n+1}^*}{b^2 + \lambda_{n+1}^* + (x - c_{n+1}^*)b} \left( (x - c_{n+2}^* + b) \left[ \left( (x - c_{n+1}^*) \frac{b}{\lambda_{n+1}^*} + 1 \right) Q_{n+1}^*(k; x) - bQ_n^*(k; x) \right] - \lambda_{n+2}^* \left( \frac{b}{\lambda_{n+1}^*} Q_{n+1}^*(k; x) + Q_n^*(k; x) \right) \right). \]

After simplifying the above equation, we obtain the desired result
\[ J_n(x)Q_{n+2}^*(k; x) = [D_{n+1}(x)J_n(x) - bJ_{n+1}(x)] Q_{n+1}^*(k; x) - \lambda_{n+1}^*J_{n+1}(x)Q_n^*(k; x), \]
where
\[ D_{n+1}(x) = x - c_{n+2}^* + b, \quad J_{n+1}(x) = bD_n(x) + \lambda_{n+1}^*. \]
This completes the proof. \( \Box \)

Next, we discuss the necessary and sufficient conditions for orthogonality of quasi-type kernel polynomial of order \( l \). One may prove Theorem 2.5 in the same line as \( [4, \text{Theorem } 1] \), and hence we omit the proof.

**Theorem 2.5.** Suppose \( \{P_n(x)\}_{n=0}^\infty \) be a sequence of monic orthogonal polynomials with respect to a quasi definite linear functional \( \mathcal{L} \) and suppose \( \{P_n^*(k; x)\}_{n=0}^\infty \) be a sequence of kernel polynomials generated by Christoffel transformation \( \mathcal{L}^* \) at \( k \), which satisfy TTRR \([2,2]\) with recurrence parameters \( c_{n+1}^*, \lambda_{n+1}^* \) given by \([2,3]\). Further, let \( \{Q_n^*(k; x)\}_{n=0}^\infty \) be a sequence of quasi-type kernel polynomials
\[ Q_n^*(k; x) = P_n^*(k; x) + \sum_{m=1}^{l} \alpha_m P_{n-m}^*(k; x) \quad \text{for } n \geq l + 1, \]
where \( \{\alpha_m\}_{m=1}^\infty \) are scalars with nonzero value of \( \alpha_l \). Then \( \{Q_n^*(k; x)\}_{n=0}^\infty \) is monic orthogonal with respect to a linear functional, if, and only if, the following conditions hold:

(i) The polynomials \( Q_m^*(k; x) \) satisfy a TTRR given by
\[ Q_{m+1}^*(k; x) = (x - c_{m+1}^*)Q_m^*(k; x) + \hat{\lambda}_{m+1}^*Q_{m-1}^*(k; x) = 0, \]
with \( \hat{\lambda}_{m+1}^* \neq 0 \) for \( m \in \{0, 1, 2, \ldots, l\} \).

(ii) For \( n > l + 1 \),
\[ \lambda_{n+1}^* - \lambda_{n-l+1}^* = \alpha_1(c_{n+1}^* - c_n^*) \neq 0, \]
\[ \alpha_m(c_{n-m+1}^* - c_{n+1}^*) + \alpha_{m-1}[\lambda_{n-m+2}^* - \lambda_{n+1}^* - (\alpha_1(c_{n+1}^* - c_{n+1}^*))] = 0, \quad m \in \{1, 2, \ldots, l\}. \]

(iii) For \( m \in \{1, \ldots, l - 1\} \),
\[ \lambda_{l+2}^* \neq \alpha_1(c_{l+2}^* - c_{l+1}^*), \]
\[ \alpha_{m+1}(c_{l-m+1}^* - c_{l+1}^*) + \alpha_m\lambda_{l-j+2}^* = \alpha_m[\lambda_{l+2}^* - \alpha_1(c_{l+1}^* - c_{l+2}^*)], \]
\[ \lambda_{l+2}^* \neq \alpha_1\lambda_{l+2}^* + \alpha_1(\alpha_{l+1}^*)(c_{l+1}^* - c_{l+2}^*) = \alpha_l\lambda_{l+2}^*, \]
where \( \alpha_m \), \( m \in \{1, 2, \ldots, l\} \), represents the constant coefficients of \( \mathcal{P}_{l-m}^*(k; \cdot) \) in the Fourier representation of \( Q_l^*(k; \cdot) \).

Moreover, for \( n \geq l + 1 \), we have
\[ c_{n+1}^* = c_{n+1}, \quad \hat{\lambda}_{n+1}^* = \lambda_{n+1}^* + \alpha_1(c_n^* - c_{n+1}^*), \]
where \( \bar{c}_{n+1}^* \) and \( \hat{\lambda}_{n+1}^* \) are the recurrence coefficients in the TTRR expansion of \( Q_n^*(k; \cdot) \).
3. Recovery of Orthogonal Polynomials

In this section, our primary goal is to recover the original orthogonal polynomials from the linear combination of polynomials generated by Darboux transformations and quasi-type kernel polynomials of orders 1 and 2 via suitable coefficients. In this process, we identify the unique sequences of constants that are necessary to recover the orthogonal polynomials.

3.1. Christoffel transformation. The relations among the quasi-orthogonal polynomials, monic orthogonal polynomial sequence and kernel polynomials are discussed in [8]. In Theorem 3.1, we recover the polynomials \( P_n(x) \) from the linear combination of polynomial generated by Christoffel transformation and quasi-type kernel polynomials of order one with rational coefficients. We identify two sequences of parameters that are responsible for obtaining \( P_n(x) \). We work with the monic quasi-type kernel polynomials of order one for some \( k \in \mathbb{C} \), which can be defined as \( T_n^*(k, x) = P_n^*(k; x) + B_n P_{n-1}^*(k; x) \).

**Theorem 3.1.** Let \( \{P_n(x)\}_{n=0}^{\infty} \) be a monic orthogonal polynomial sequence with respect to the positive definite linear functional \( L \). Let \( T_n^*(k_1, x) \) be a monic quasi-type kernel polynomial of order one for some \( k_1 \in \mathbb{C} \). Suppose also that the sequence \( \{P_n^*(k_2; x)\}_{n=0}^{\infty} \) of kernel polynomials generated by Christoffel transformation exists for some \( k_2 \in \mathbb{C} \). Then there exist unique sequences of constants \( \{\gamma_n\} \) and \( \{\eta_n\} \) with an explicit expression such that the relation

\[
P_n(x) = \frac{x - k_1}{x - \gamma_n} T_n^*(k_1; x) + \eta_n \frac{x - k_2}{x - \gamma_n} P_{n-1}^*(k_2; x) \tag{3.1}
\]

holds. In particular, if \( k = k_1 = k_2 \in \mathbb{C} \) then

\[
\tilde{T}_n^*(k; x) = P_n^*(k; x) + \tilde{B}_n P_{n-1}^*(k; x) = \frac{x - \gamma_n}{x - k} P_n(x),
\]

and if \( \text{supp}(d\mu) \subset \mathbb{R} \) is compact then

\[
\tilde{T}_n^*(k; x) \in L^1(d\mu).
\]

**Proof.** Using the definitions of kernel polynomials and quasi-type kernel polynomial of order one, we have

\[
(x - k_1) T_{n+1}^*(k_1; x) - (x - \gamma_n) P_{n+1}(x) + \eta_n (x - k_2) P_n^*(k_2; x)
= (x - k_1) (P_{n+1}^*(k_1; x) + B_{n+1} P_n^*(k_1; x)) - (x - \gamma_n) P_{n+1}(x) + \eta_n (x - k_2) P_n^*(k_2; x)
= P_{n+2}(x) - \frac{P_{n+2}(k_1)}{P_{n+1}(k_1)} P_{n+1}(x) + B_{n+1} P_{n+1}(x) - B_{n+1} \frac{P_{n+1}(k_1)}{P_n(k_1)} P_n(x) - (x - \gamma_n) P_{n+1}(x)
+ \eta_n P_{n+1}(x) - \eta_n \frac{P_{n+1}(k_2)}{P_n(k_2)} P_n(x)
\]

Combining the coefficients of \( P_{n+2}(x) \), \( P_{n+1}(x) \) and \( P_n(x) \), we can write the above expression as

\[
(x - k_1) T_{n+1}^*(k_1; x) - (x - \gamma_n) P_{n+1}(x) + \eta_n (x - k_2) P_n^*(k_2; x) = P_{n+2}(x)
- \left( x - \gamma_n + \frac{P_{n+2}(k_1)}{P_{n+1}(k_1)} - B_{n+1} - \eta_n \right) P_{n+1}(x) - \left( \eta_n \frac{P_{n+1}(k_2)}{P_n(k_2)} + B_{n+1} \frac{P_{n+1}(k_1)}{P_n(k_1)} \right) P_n(x).
\tag{3.2}
\]
Consider
\[ \eta_n = - \left( \lambda_{n+2} + B_{n+1} \frac{P_{n+1}(k_1)}{P_n(k_1)} \right) \frac{P_n(k_2)}{P_{n+1}(k_2)}, \]
and
\[ \gamma_n = c_{n+2} + \frac{P_{n+2}(k_1)}{P_{n+1}(k_1)} - B_{n+1} + \left( \lambda_{n+2} + B_{n+1} \frac{P_{n+1}(k_1)}{P_n(k_1)} \right) \frac{P_n(k_2)}{P_{n+1}(k_2)}. \]

Then, we can write (3.2) as
\[ (x - k_1)T_{n+1}(k_1; x) - (x - \gamma_n)P_{n+1}(x) + \eta_n(x - k_2)P_n^*(k_2; x) = P_{n+2}(x) - (x - c_{n+2})P_n(x) + \lambda_{n+2}P_n(x), \]
where
\[ c_{n+2} = \gamma_n - \frac{P_{n+2}(k_1)}{P_{n+1}(k_1)} + B_{n+1} + \eta_n, \quad \] 
\[ \lambda_{n+2} = - \eta_n \frac{P_{n+1}(k_2)}{P_n(k_2)} - B_{n+1} \frac{P_{n+1}(k_1)}{P_n(k_1)}. \]

The above expression (3.3) must be equal to zero because \( P_n(x) \) is a monic orthogonal polynomial sequence with respect to measure \( d\mu \). Hence, by Favard’s theorem it satisfies the TTRR which implies (3.1). If both quasi-type kernel polynomial of order one and kernel polynomials exist for some \( k = k_1 = k_2 \in \mathbb{C} \), then (3.1) can be written as
\[ (x - k) \left( P_n^*(k; x) + \tilde{B}_{n+1} P_n^*(k; x) \right) - (x - \gamma_n)P_{n+1}(x) = 0, \]
where \( \tilde{B}_{n+1} = B_{n+1} + \eta_n \). This implies \( (x - k)\tilde{T}_{n+1}(k; x) - (x - \gamma_n)P_{n+1}(x) = 0 \), which further gives
\[ \tilde{T}_{n+1}(k; x) = \frac{x - \gamma_n}{x - k}P_{n+1}(x). \]

If the support of a measure \( \mu \) is a compact subset of real line and \( k \notin \text{supp}(d\mu) \) then
\[
\|\tilde{T}_{n+1}(k; x)\|_{L^1(d\mu)} = \int \frac{x - \gamma_n}{x - k}P_{n+1}(x) \, d\mu \\
\leq \int \frac{x}{x - k}P_{n+1}(x) \, d\mu + \gamma_n \int \frac{P_{n+1}(x)}{x - k} \, d\mu \\
\leq \left( \int \frac{1}{|x - k|^2} \, d\mu \right)^{1/2} \left[ \left( \int |xP_{n+1}|^2 \, d\mu \right)^{1/2} + \gamma_n \left( \int |P_{n+1}|^2 \, d\mu \right)^{1/2} \right] \\
< \infty.
\]

In the above, we used the triangle inequality and Hölder’s inequality to obtain the first and second inequalities, respectively. Moreover, finiteness follows directly from the fact that multiplication by \( x \) is in \( L^2(d\mu) \) and \( k \notin \text{supp}(d\mu) \). \( \square \)

3.2. Geronimus transformation. Let \( \mathcal{L} \) be a linear functional. For given \( k \in \mathbb{C} \), define
\[ \tilde{\mathcal{L}}((x - k)p(x)) = \mathcal{L}(p(x)) \]
for any polynomial \( p(x) \). This transformation \( \tilde{\mathcal{L}} \) is known as Geronimus transformation at \( k \in \mathbb{C} \). Geronimus transformation can be regarded as the inverse of Christoffel transformation at \( k \) [12].

For any polynomial \( p(x) \), we can write
\[ \tilde{L}(p(x)) = \tilde{L} \left( \frac{p(x) - p(k)}{x - k} \right) (x - k) + p(k) \]
\[ = \mathcal{L} \left( \frac{p(x) - p(k)}{x - k} \right) + p(k) \tilde{L}(1), \]

where \( \tilde{L}(1) \) is not uniquely determined, and hence an arbitrary constant. However, \( \tilde{L}(1) \neq 0 \), because there does not exist any orthogonal polynomial sequence with such property \([8]\).

Next we state the result in which a sequence of quasi-orthogonal polynomials of order one with suitable choice of \( A_n \) is taken, which forms an orthogonal polynomial sequence with respect to the Geronimus transformation at \( k \).

**Theorem 3.2.** \([8,12,26]\) Let \( \{\mathcal{P}_n(x)\}_{n=0}^{\infty} \) be the sequence of orthogonal polynomials with respect to the positive definite linear functional \( \mathcal{L} \). If \( k \in \mathbb{C} \setminus \text{supp}\mu \) then the sequence of monic polynomials
\[ \tilde{\mathcal{P}}_n(k; x) = \mathcal{P}_n(x) + A_n \mathcal{P}_{n-1}(x), \quad (3.5) \]
where
\[ A_n = -\frac{\int_{k}^{\infty} \mathcal{P}_n(x) d\mu(x)}{\int_{k}^{\infty} \mathcal{P}_{n-1}(x) d\mu(x)} \]
is an orthogonal polynomial sequence for the corresponding Geronimus transformation \( \tilde{\mathcal{L}} \) at \( k \).

We can also express the polynomial \( \mathcal{P}_n(x) \) as the linear combination of \( \tilde{\mathcal{P}}_n(k; x) \) and \( \tilde{\mathcal{P}}_{n+1}(k; x) \) as follows \([26]\):
\[ \mathcal{P}_n(x) = \frac{1}{x - k} \tilde{\mathcal{P}}_{n+1}(k; x) - \frac{1}{x - k} \frac{\tilde{\mathcal{P}}_{n+1}(k; k)}{\mathcal{P}_n(k; k)} \tilde{\mathcal{P}}_n(k; x). \]

In the next theorem, we recover the polynomial \( \mathcal{P}_n(x) \) from the linear combination of polynomials generated by Geronimus transformation and quasi-type kernel polynomials of order one with rational coefficients. We obtain three sequences of parameters that recover the polynomial \( \mathcal{P}_n(x) \).

**Theorem 3.3.** Let \( \{\mathcal{P}_n(x)\}_{n=0}^{\infty} \) be a monic orthogonal polynomial sequence with respect to the positive definite linear functional \( \mathcal{L} \). Let \( \mathcal{T}_n(k_2, x) \) be a quasi-type kernel polynomial of order one for some \( k_2 \in \mathbb{C} \). Further, suppose that the sequence \( \{\mathcal{P}_n(k_1; x)\}_{n=0}^{\infty} \) of the polynomials corresponding to Geronimus transformation exist for some \( k_1 \in \mathbb{C} \). Then there exist unique sequences of constants \( \{\alpha_n\}, \{\gamma_n\} \) and \( \{\eta_n\} \) such that
\[ \mathcal{P}_n(x) = \frac{1}{\alpha_n x - \gamma_n} \tilde{\mathcal{P}}_{n+1}(k_1; x) + \eta_n \frac{x - k_2}{\alpha_n x - \gamma_n} \mathcal{T}_n^*(k_2; x). \quad (3.6) \]

**Proof.** Considering \([8,5]\) together with the definition of kernel polynomials and quasi-type kernel polynomial of order one gives
\[ \tilde{\mathcal{P}}_{n+1}(k_1; x) - (\alpha_n x - \gamma_n) \mathcal{P}_n(x) + \eta_n (x - k_2) \mathcal{T}_n^*(k_2; x) \]
\[ = \mathcal{P}_{n+1}(x) + A_{n+1} \mathcal{P}_n(x) - (\alpha_n x - \gamma_n) \mathcal{P}_n(x) + \eta_n (x - k_2) (\mathcal{P}_n^*(k_2; x) + \tilde{B}_n \mathcal{P}_{n-1}^*(k_2; x)) \]
\begin{align}
= \mathcal{P}_{n+1}(x) + A_{n+1}\mathcal{P}_n(x) - (\alpha_n x - \gamma_n)\mathcal{P}_n(x) + \eta_n\mathcal{P}_{n+1}(x) - \eta_n \frac{\mathcal{P}_{n+1}(k_2)}{\mathcal{P}_n(k_2)} \mathcal{P}_n(x)
&+ \eta_n \tilde{B}_n \mathcal{P}_n(x) - \eta_n \tilde{B}_n \frac{\mathcal{P}_n(k_2)}{\mathcal{P}_{n-1}(k_2)} \mathcal{P}_{n-1}(x).
\end{align}

Combining the coefficients of \(\mathcal{P}_{n+1}(x), \mathcal{P}_n(x)\) and \(\mathcal{P}_{n-1}(x)\), we get
\begin{align}
\tilde{\mathcal{P}}_{n+1}(k_1; x) - (\alpha_n x - \gamma_n)\mathcal{P}_n(x) + \eta_n(x - k_2)\Gamma_n^*(k_2; x)
&= (1 + \eta_n) \left[ \mathcal{P}_{n+1}(x) - \left( \frac{\alpha_n x}{1 + \eta_n} - \frac{\gamma_n + A_{n+1} - \eta_n \frac{\mathcal{P}_{n+1}(k_2)}{\mathcal{P}_n(k_2)} + \eta_n \tilde{B}_n}{1 + \eta_n} \right) \mathcal{P}_n(x)
\right.
\left. - \eta_n \tilde{B}_n \mathcal{P}_n(k_2) \right] \frac{\mathcal{P}_{n-1}(k_2)}{1 + \eta_n} \mathcal{P}_{n-1}(x). \tag{3.7}
\end{align}

Since \(\mathcal{P}_n(k_2) \neq 0, \mathcal{P}_{n-1}(k_2) \neq 0\), by substituting
\begin{align}
\eta_n &= -\frac{\lambda_{n+1}}{\lambda_{n+1} + \tilde{B}_n \mathcal{P}_n(k_2)}, \quad \alpha_n = 1 - \frac{\lambda_{n+1}}{\lambda_{n+1} + \tilde{B}_n \mathcal{P}_n(k_2)},
\end{align}
and
\begin{align}
\gamma_n &= c_{n+1}(1 + \eta_n) - A_{n+1} + \eta_n \frac{\mathcal{P}_{n+1}(k_2)}{\mathcal{P}_n(k_2)} - \eta_n \tilde{B}_n,
\end{align}
we can write the right side of the expression (3.7) as
\begin{align}
(1 + \eta_n) \left[ \mathcal{P}_{n+1}(x) - (x - c_{n+1}) \mathcal{P}_n(x) + \lambda_{n+1} \mathcal{P}_{n-1}(x) \right], \tag{3.8}
\end{align}
where
\begin{align}
c_{n+1} &= \frac{\gamma_n + A_{n+1} - \eta_n \frac{\mathcal{P}_{n+1}(k_2)}{\mathcal{P}_n(k_2)} + \eta_n \tilde{B}_n}{1 + \eta_n}, \quad \lambda_{n+1} = -\frac{\eta_n \tilde{B}_n \mathcal{P}_n(k_2)}{(1 + \eta_n) \mathcal{P}_{n-1}(k_2)}.
\end{align}

The above expression (3.8) must be equal to zero. Since \(\mathcal{P}_n(x)\) is a monic orthogonal polynomial sequence, by Favard’s theorem it satisfies three term recurrence relation which implies (3.6). \(\square\)

3.3. Uvarov Transformation. Linear spectral transformations play a significant role in the study of perturbation of orthogonal polynomials. We can obtain one of the main transformations by adding point mass to the original measure. In other words, if \(\mathcal{L}\) is a quasi-definite linear functional, then we can define \(\hat{\mathcal{L}}\) by
\begin{align}
\hat{\mathcal{L}} = \mathcal{L} + R_o \delta(x - k),
\end{align}
where \(\delta(\cdot)\) is a mass point at \(k\) and \(R_o\) is a non zero constant. The new linear functional \(\hat{\mathcal{L}}\) is known as canonical Uvarov transformation \(\text{[37]}\) of \(\mathcal{L}\).

To study the structure of polynomials corresponding to Uvarov transformation, it is essential that the Uvarov transformation has at least the property of quasi definiteness. In this regard, the necessary and sufficient conditions for preserving the quasi definite property of the linear functional are given in \(\text{[32]}\). In addition, the condition for preserving the positive definite property of Uvarov transformation from the original positive definite linear functional is given in \(\text{[28]}\).
Theorem 3.4 (cf. [25, page 256]). Let \( \{ \mathcal{P}_n(x) \}_{n=0}^{\infty} \) be a monic orthogonal polynomial sequence corresponding to the quasi definite linear functional \( \hat{\mathcal{L}} \). Suppose that the sequence \( \{ \mathcal{P}_n^*(k; x) \}_{n=0}^{\infty} \) of kernel polynomials generated by Christoffel transformation exists for some \( k \in \mathbb{C} \). Then we have

\[
\hat{\mathcal{P}}_n(x) = \mathcal{P}_n(x) - T_n \mathcal{P}_{n-1}^*(k; x),
\]

where

\[
T_n = \frac{R_o \mathcal{P}_n^2(k)}{\lambda_1 \ldots \lambda_{n+1}} \left( 1 + \frac{R_o \mathcal{P}_{n-1}^2(k; k) \mathcal{P}_n(k)}{\lambda_1 \ldots \lambda_{n+1}} \right).
\]

The following result shows that one can recover the original sequence of orthogonal polynomials from the linear combination of quasi-type kernel polynomials of order one and polynomials generated by Uvarov transformation with rational coefficients and by suitably identifying three sequences of constants.

Theorem 3.5. Let \( \mathcal{P}_n^*(k_2; x) \) be a quasi-type kernel polynomial of order one for some \( k_2 \in \mathbb{C} \). Further, suppose that sequence \( \{ \hat{\mathcal{P}}_n(x) \}_{n=0}^{\infty} \) of the polynomials corresponding to Uvarov transformation. Then there exist unique sequences of constants \( \{ \alpha_n \}, \{ \gamma_n \} \) and \( \{ \eta_n \} \) such that

\[
\mathcal{P}_n(x) = \frac{x - k_1}{\alpha_n x - \gamma_n} \hat{\mathcal{P}}_n(x) + \eta_n \frac{x - k_2}{\alpha_n x - \gamma_n} \mathcal{T}_n^*(k_2; x).
\]

Proof. We have

\[
(x - k_1) \hat{\mathcal{P}}_n(x) - (\alpha_n x - \gamma_n) \mathcal{P}_n(x) + \eta_n (x - k_2) \mathcal{T}_n^*(k_2; x)
\]

\[
= (x - k_1) \mathcal{P}_n(x) - T_n \mathcal{P}_n(x) + T_n \frac{\mathcal{P}_n(k_1)}{\mathcal{P}_{n-1}(k_1)} \mathcal{P}_{n-1}(x) - \alpha_n x \mathcal{P}_n(x) + \beta_n \mathcal{P}_n(x)
\]

\[
+ \eta_n (x - k_2) \mathcal{P}^*_n(k_2; x) + \hat{\mathcal{B}} n \eta_n (x - k_2) \mathcal{P}^*_{n-1}(k_2; x)
\]

\[
= (1 - \alpha_n) x \mathcal{P}_n(x) + (\beta_n - k_1 - T_n) \mathcal{P}_n(x) + T_n \frac{\mathcal{P}_n(k_1)}{\mathcal{P}_{n-1}(k_1)} \mathcal{P}_{n-1}(x) + \eta_n \mathcal{P}_{n+1}(x)
\]

\[
- \eta_n \frac{\mathcal{P}_{n+1}(k_2)}{\mathcal{P}_{n+1}(k_2)} \mathcal{P}_n(x) + \eta_n \hat{\mathcal{B}} n \mathcal{P}_n(x) - \eta_n \hat{\mathcal{B}} n \mathcal{P}_n(k_2) \mathcal{P}_{n-1}(x)
\]

\[
= (1 - \alpha_n) [\mathcal{P}_{n+1}(x) + c_{n+1} \mathcal{P}_n(x) + \lambda_{n+1} \mathcal{P}_{n-1}(x)] + (\beta_n - k_1 - T_n) \mathcal{P}_n(x)
\]

\[
+ T_n \frac{\mathcal{P}_n(k_1)}{\mathcal{P}_{n-1}(k_1)} \mathcal{P}_{n-1}(x) + \eta_n \mathcal{P}_{n+1}(x) - \eta_n \frac{\mathcal{P}_{n+1}(k_2)}{\mathcal{P}_{n+1}(k_2)} \mathcal{P}_n(x)
\]

\[
+ \eta_n \hat{\mathcal{B}} n \mathcal{P}_n(x) - \eta_n \hat{\mathcal{B}} n \frac{\mathcal{P}_n(k_2)}{\mathcal{P}_{n-1}(k_2)} \mathcal{P}_{n-1}(x)
\]

\[
= (1 - \alpha_n + \eta_n) \mathcal{P}_{n+1}(x) + \left( \beta_n - k_1 - T_n - \eta_n \frac{\mathcal{P}_{n+1}(k_2)}{\mathcal{P}_{n+1}(k_2)} \right) \mathcal{P}_n(x)
\]

\[
+ \left( T_n \frac{\mathcal{P}_n(k_1)}{\mathcal{P}_{n-1}(k_1)} - \eta_n \hat{\mathcal{B}} n \right) \frac{\mathcal{P}_n(k_2)}{\mathcal{P}_{n-1}(k_2)} + \lambda_{n+1} - \alpha_n c_{n+1}, \mathcal{P}_{n-1}(x).
\]

Here to obtain (3.11), we have used the expression for multiplication by \( x \) with \( \mathcal{P}_n(x) \) and combining the coefficients of \( \mathcal{P}_{n+1}(x), \mathcal{P}_n(x) \) and \( \mathcal{P}_{n-1}(x) \) to obtain equation (3.11). Next, setting the right side of (3.11) equal to zero and by using the fact that \( \mathcal{P}_{n+1}(x), \mathcal{P}_n(x) \) and \( \mathcal{P}_{n-1}(x) \) are linearly independent, we get that the coefficients must be equal to zero.
orthogonal polynomials of order two. Define

\[ S_n = \frac{T_n P_n(k_1)}{B_n P_n(k_2)} + \lambda_{n+1} \]

Further, suppose that the sequence \( \{P_n(x)\} \) gives the expression (3.10) and this completes the proof.

\[ \square \]

3.4. Quasi-type kernel polynomials of order two. Now we define the monic quasi-orthogonal polynomials of order two. Define \( S_n(x) \) as follows:

\[ S_n(x) = P_n(x) + L_n P_{n-1}(x) + M_n P_{n-2}(x), \quad (3.12) \]

where \( \{P_n(x)\}_{n=0}^{\infty} \) is a monic orthogonal polynomial sequence with respect to the linear functional \( \mathcal{L} \) for any choice of \( L_n, M_n \in \mathbb{C} \).

If \( \mathcal{L}(x^m S_n(x)) = 0 \) for \( m = 0, 1, 2, ..., n-3 \), for any choice of \( L_n, M_n \in \mathbb{C} \), then \( S_n(x) \) is called quasi-orthogonal polynomial of order two.

Similarly, we can extend this definition to the quasi-type kernel polynomial of order two with respect to \( \mathcal{L}^* \). Define \( S_n^*(k; x) \) as follows:

\[ S_n^*(k; x) = P_n^*(k; x) + \tilde{L}_n P_{n-1}^*(k; x) + \tilde{M}_n P_{n-2}^*(k; x), \]

where \( \{P_n^*(k; x)\}_{n=0}^{\infty} \) is a sequence of kernel polynomials which exist for some \( k \in \mathbb{C} \) and form a monic orthogonal polynomial system with respect to the quasi-definite linear functional \( \mathcal{L}^* \).

If \( \mathcal{L}^*(x^m S_n^*(k; x)) = 0 \) for \( m = 0, 1, 2, ..., n-3 \) and for any choice of \( \tilde{L}_n, \tilde{M}_n \in \mathbb{C} \), then \( S_n^*(k; x) \) is called quasi-type kernel polynomial of order two.

In the next theorem, we recover the polynomials \( P_n(x) \) from the linear combination of iterated kernel polynomials \( \{P_n(x)\} \) with two parameters and quasi-type kernel polynomials of order two with rational coefficients. We obtain two sequences of constants responsible for obtaining \( P_n(x) \).

**Theorem 3.6.** Let \( \{P_n(x)\}_{n=0}^{\infty} \) be a monic orthogonal polynomial sequence with respect to the positive definite linear functional \( \mathcal{L} \). Let \( S_{n+1}^*(k_1; x) \) be a quasi-type kernel polynomial of order two for some \( k_1 \in \mathbb{C} \) with suitable choice of \( \tilde{L}_n, \tilde{M}_n \) which satisfy

\[ \tilde{L}_n + \tilde{M}_n \frac{P_{n+1}(k_1)}{\lambda_{n+1} P_{n-1}(k_1)} = \frac{P_{n+2}(k_1)}{P_{n+1}(k_1)} - \frac{P_{n+2}(k_2)}{P_{n+1}(k_2)} - \frac{P_{n+1}(k_2, k_3)}{P_{n+2}(k_2, k_3)}. \quad (3.13) \]

Further, suppose that the sequence \( \{P_n^*(k_2, k_3; x)\}_{n=0}^{\infty} \) of iterated kernel polynomials exists for some \( k_2 \in \mathbb{C}_+, k_3 \in \mathbb{C}_\pm \). Then, there exist unique sequences of constants \( \{\alpha_n\}, \{\beta_n\} \) such that the relation

\[ P_n(x) = \frac{x - k_1}{\alpha_n x - \beta_n} S_{n+1}^*(k_1; x) - \frac{(x - k_2)(x - k_3)}{\alpha_n x - \beta_n} P_n^*(k_2, k_3; x) \quad (3.14) \]
holds.

**Remark 3.1.** The iterated kernel polynomials sequence for some $k_2 \in \mathbb{C}_+, k_3 \in \mathbb{C}_\pm$ are given in [8, eq. 3.5], where $\mathbb{C}_\pm = \{z : \text{Im}z \leq 0\}$.

**Proof.** Using the definition of quasi-orthogonal polynomials of order two and the expression of kernel and iterated kernel polynomials, we have

$$(x - k_1)S_{n+1}^k(k_1; x) - (\alpha_n x - \beta_n)P_n(x) - (x - k_2)(x - k_3)P_n^*(k_2, k_3; x)$$

$$= P_{n+2}(x) - \frac{P_{n+2}(k_1)}{P_{n+1}(k_1)}P_{n+1}(x) + \tilde{L}_n P_{n+1}(x) - \frac{P_{n+1}(k_1)}{P_{n}(k_1)} P_n(x) + \tilde{M}_n P_n(x)$$

$$- \tilde{M}_n \frac{P_n(k_1)}{P_{n-1}(k_1)} P_{n-1}(x) - \alpha_n x P_n(x) + \beta_n P_n(x) - P_{n+2}(x) + \frac{P_{n+2}(k_2)}{P_{n+1}(k_2)} P_{n+1}(x)$$

$$+ \frac{P_{n+1}(k_2)}{P_{n+1}(k_2)} P_{n+1}(x) - \frac{P_n^*(k_2, k_3)}{P_n(k_2)} P_n(x).$$

Combining the coefficients of $P_{n+1}(x)$, $P_n(x)$ and $P_{n-1}(x)$, we get

$$(x - k_1)S_{n+1}^k(k_1; x) - (\alpha_n x - \beta_n)P_n(x) - (x - k_2)(x - k_3)P_n^*(k_2, k_3; x)$$

$$= \left( -\frac{P_{n+2}(k_1)}{P_{n+1}(k_1)} + \frac{P_{n+2}(k_2)}{P_{n+1}(k_2)} + \tilde{L}_n - \alpha_n + \frac{P_n^*(k_2, k_3)}{P_n(k_2)} \right) P_{n+1}(x)$$

$$+ \left( -\frac{P_n(k_1)}{P_{n-1}(k_1)} - \alpha_n \lambda_{n+1} \right) P_{n-1}(x) + \left( -\tilde{L}_n \frac{P_{n+1}(k_1)}{P_n(k_1)} + \tilde{M}_n + \beta_n - \alpha_n c_{n+1} \right) P_n(x).$$

Setting the left side of the above equation equal to zero and by using the fact that $P_{n+1}(x)$, $P_n(x)$ and $P_{n-1}(x)$ are linearly independent we get that the coefficients must be equal to zero. This gives

$$\alpha_n = \frac{1}{\lambda_{n+1}} \tilde{M}_n \frac{P_n(k_1)}{P_{n-1}(k_1)}.$$

$$\beta_n = \tilde{L}_n \frac{P_{n+1}(k_1)}{P_n(k_1)} - \tilde{M}_n + \lambda_{n+2} \left( \frac{\sum_{j=0}^{n+1} \frac{P_j(k_1)P_j(k_2)}{\lambda_{1,2,...,\lambda_{j+1}}} \right) - \frac{1}{\lambda_{n+1}} \frac{P_n(k_1)}{P_{n-1}(k_1)} c_{n+1}.$$

We used the Christoffel-Darboux formula [15] and the fact that zeros [33] of $P_n(x)$ lie on the real line for $n = 1, 2, \ldots$, for $k_2, k_3 \in \mathbb{C}_\pm$ to write the expression for $\beta_n$. $\square$
In the hypothesis of Theorem 4.1, we required the coefficients $\bar{L}_n$ and $\bar{M}_n$ of quasi-type kernel polynomial of order two to satisfy the expression (3.13). As a result, we obtained two unique sequences of constants $\bar{C}_n$ and $\bar{C}_n^*$ that are useful in recovering the polynomial $\mathcal{P}_n(x)$. It would be interesting to remove the hypothesis of this particular choice of the coefficients $\bar{L}_n$ and $\bar{M}_n$. More specifically, we end this section with the following question.

Problem 1. Is it possible to obtain three sequences of constants so that relaxation of the hypothesis (3.13) is permissible?

4. RATIO OF KERNEL POLYNOMIALS AND CONTINUED FRACTIONS

In this section, we deal with the ratio of kernel polynomials as $x$ tends to $k$. For this, we require the confluent form of Christoffel-Darboux formula which we recall in Theorem 4.1.

Then we discuss the ratio of kernel polynomials in terms of infinite continued fractions.

Theorem 4.1. Let $\{\mathcal{P}_n(x)\}_{n=1}^{\infty}$ be a sequence of monic orthogonal polynomials and $\lambda_n \neq 0$. Then

$$\sum_{j=0}^{n} \frac{\mathcal{P}_j^2(x)}{\lambda_1 \lambda_2 \ldots \lambda_{j+1}} = \frac{\mathcal{P}_{n+1}(x)\mathcal{P}_n(x) - \mathcal{P}_{n+1}(x)\mathcal{P}_n'(x)}{\lambda_1 \lambda_2 \ldots \lambda_{n+1}}.$$ 

Now we will compute the ratio of kernel polynomials as $x$ approaches $k$.

Theorem 4.2. Let $\{\mathcal{P}_n(k; x)\}_{n=0}^{\infty}$ be a sequence of kernel polynomials that exists for some $k \in \mathbb{C}$. Then

$$\lim_{x \to k} \frac{\mathcal{P}_{n+1}(k; x)}{\mathcal{P}_n(k; x)} = \frac{\mathcal{P}_n(k)}{\mathcal{P}_{n+1}(k)} \lambda_{n+2} \left( 1 + \frac{1}{\lambda_1 \lambda_2 \ldots \lambda_{n+2}} \sum_{j=0}^{n} \frac{\mathcal{P}_{n+1}(k)}{\lambda_1 \lambda_2 \ldots \lambda_{j+1}} \right).$$

and

$$\lim_{x \to k} \frac{\mathcal{P}_n(k; x)}{\mathcal{P}_{n+1}(k; x)} = \frac{\mathcal{P}_n(k)}{\mathcal{P}_{n+1}(k)} \frac{1}{\lambda_{n+2}} \left( 1 - \frac{\mathcal{P}_{n+1}(k)}{\lambda_1 \lambda_2 \ldots \lambda_{n+2}} \sum_{j=0}^{n+1} \frac{\mathcal{P}_j^2(k)}{\lambda_1 \lambda_2 \ldots \lambda_{j+1}} \right).$$

Proof. Using the Definition 2.1 and Theorem 4.1, we have

$$\lim_{x \to k} \frac{\mathcal{P}_{n+1}(k; x)}{\mathcal{P}_n(k; x)} = \frac{\mathcal{P}_n(k)}{\mathcal{P}_{n+1}(k)} \lim_{x \to k} \left( \frac{\mathcal{P}_{n+2}(x)\mathcal{P}_{n+1}(k) - \mathcal{P}_{n+2}(k)\mathcal{P}_{n+1}(x)}{\mathcal{P}_{n+1}(x)\mathcal{P}_n(k) - \mathcal{P}_{n+1}(k)\mathcal{P}_n(x)} \right)$$

$$= \frac{\mathcal{P}_n(k)}{\mathcal{P}_{n+1}(k)} \left( \frac{\mathcal{P}_{n+2}(k)\mathcal{P}_{n+1}(k) - \mathcal{P}_{n+1}(k)\mathcal{P}_{n+1}(k)}{\mathcal{P}_{n+1}(k)\mathcal{P}_n(k) - \mathcal{P}_{n+1}(k)\mathcal{P}_n(k)} \right)$$

$$= \frac{\mathcal{P}_n(k)}{\mathcal{P}_{n+1}(k)} \lambda_{n+2} \left( \frac{\sum_{j=0}^{n+1} \frac{\mathcal{P}_j^2(k)}{\lambda_1 \lambda_2 \ldots \lambda_{j+1}}}{\sum_{j=0}^{n} \frac{\mathcal{P}_j^2(k)}{\lambda_1 \lambda_2 \ldots \lambda_{j+1}}} \right).$$
\[ \frac{P_n(k)}{P_{n+1}(k)} \lambda_{n+2} \left( 1 + \frac{P_{n+1}^2(k)}{\lambda_1 \lambda_2 ... \lambda_{n+2}} \sum_{j=0}^{n} \frac{P_j^2(k)}{\lambda_1 \lambda_2 ... \lambda_{j+1}} \right). \]

In similar lines, we can obtain (1.2). This completes the proof. □

**Corollary 4.1.** Let \( \{\hat{C}_n(x)\}_{n=0}^{\infty} \) be a sequence of monic Chebyshev polynomials of first kind. Then

\[ \lim_{n \to \infty} \lim_{x \to 1} \frac{\hat{C}_{n+1}^*(1; x)}{\hat{C}_n^*(1; x)} = \frac{1}{2}. \]

**Proof.** By using Theorem 4.2 and Example 2.1, we have

\[ \lim_{x \to 1} \frac{\hat{C}_{n+1}^*(1; x)}{\hat{C}_n^*(1; x)} = \frac{1}{2} \left( 1 + \frac{12}{2^{2n+1} + 1} \right). \]

Allowing \( n \to \infty \), we get the desired result. □

Next, we will discuss the link between the ratio of kernel polynomials and infinite continued fractions. For this, we first need the definition of the hypergeometric functions. The Gauss hypergeometric function \[39\] \( {}_2F_1(p, q; r; z) \) is given by

\[ {}_2F_1(p, q; r; z) = \sum_{n=0}^{\infty} \frac{(p)_n (q)_n}{(r)_n n!} z^n \quad \text{for} \quad r \notin \{0, -1, -2, \ldots\}, \quad (4.3) \]

where the symbol \((\cdot)_n\) is known as Pochhammer symbol and is defined as

\[ (p)_n = p(p+1)(p+2)...(p+n-1) = \frac{\Gamma(p+n)}{\Gamma p}, \quad \text{with} \quad (p)_0 = 1. \]

The above series converges absolutely in \( \{z \in \mathbb{C} : |z| < 1\} \). Further we can analytically continue the series as a single valued function everywhere except any path joining the branch points 1 and infinity \[7\].

Note that if we take either \( p \) or \( q \) to be a negative integer, the terms of the series will vanish after some stage and we will be left with a finite linear combination of monomials. If this happens, the convergence of the hypergeometric series is not an issue.

If we replace \( z \) by \( z/q \) and allow \( q \to \infty \), then by using

\[ \lim_{n \to \infty} \frac{(q)_n}{q^n} = 1, \]

we obtain the Kummer or confluent hypergeometric function

\[ \phi(p; r; z) := {}_1F_1(p; r; z) = \lim_{q \to \infty} F(p, q; r; z/q) = \sum_{n=0}^{\infty} \frac{(p)_n}{(r)_n n!} z^n \quad \text{for} \quad r \notin \{0, -1, -2, \ldots\}. \]

We shall use the following contiguous relation satisfied by the Gaussian hypergeometric function to obtain the continued fraction of ratio of hypergeometric functions.

\[ F(p+1, q; r; z) = F(p, q; r; z) - \frac{q}{r} z F(p+1, q+1; r+1; z), \quad (4.4) \]

\[ F(p, q; r; z) = F(p, q+1; r+1; z) - \frac{p(r-q)}{r(r+1)} z F(p+1, q+1; r+2; z), \quad (4.5) \]
\[ F(p, q + 1; r + 1; z) = F(p + 1, q + 1; r + 2; z) \]
\[ - \frac{(q + 1)(r - p + 1)}{(r + 1)(r + 2)} z F(p + 1, q + 2; r + 3; z). \] (4.6)

We can use (4.4)-(4.6) to get the ratio of Gauss hypergeometric functions [38, p. 337] (see also [17, 30]).

\[ \frac{F(p + 1, q; r; z)}{F(p, q; r; z)} = \frac{1}{1 - \frac{(1 - g_0)g_1z}{1} - \frac{(1 - g_1)g_2z}{1} - \ldots} \] (4.7)

with
\[ g_j = g_j(p, q, r) := \begin{cases} 0 & \text{for } j = 0, \\ \frac{p+k}{r+2k-1} & \text{for } j = 2k \geq 2, k \geq 1, \\ \frac{q+k-1}{r+2k-2} & \text{for } j = 2k - 1 \geq 1, k \geq 1. \end{cases} \]

Hence, we can write the ratio of Kummer hypergeometric functions as a limit of the ratio of Gauss hypergeometric functions by

\[ \frac{\phi(p + 1; r; z)}{\phi(p; r; z)} = \lim_{q \to \infty} \frac{F(p + 1, q; r; z/q)}{F(p, q; r; z/q)} = \frac{1}{1 - \frac{d_1z}{1} - \frac{d_2z}{1} - \ldots} \] (4.8)

with \( d_j = \lim_{q \to \infty} \frac{(1 - g_{j-1})g_j}{q} \) for all \( j \geq 1 \). So

\[ d_j = d_j(p, r) := \begin{cases} \frac{1}{r} & \text{for } j = 1, \\ \frac{-(p+k)}{(r+2k-1)(r+2k-2)} & \text{for } j = 2k, k \geq 1, \\ \frac{q+k-1}{(r+2k-1)(r+2k-2)} & \text{for } j = 2k - 1, k \geq 2. \end{cases} \]

If we put \( p = -n, r = \gamma + 2 \) and \( z = -x \) in (4.8), we obtain

\[ \frac{\phi(-n + 1; \gamma + 2; -x)}{\phi(-n; \gamma + 2; -x)} = \frac{1}{1 + \frac{\tilde{d}_1x}{1} + \frac{\tilde{d}_2x}{1} + \ldots} \] (4.9)

with

\[ \tilde{d}_j = \tilde{d}_j(-n, \gamma + 2) := \begin{cases} \frac{1}{\gamma + 2} & \text{for } j = 1, \\ \frac{(n-k)(\gamma + 2k)}{(\gamma + 2k)(\gamma + 2k + 1)} & \text{for } j = 2k, k \geq 1, \\ \frac{\gamma + n + k + 1}{(\gamma + 2k)(\gamma + 2k + 1)} & \text{for } j = 2k - 1, k \geq 2. \end{cases} \] (4.10)

4.1. **Kernel of Laguerre polynomials.** We know that Laguerre polynomials with parameter \( \gamma \) can be written in the form of Kummer hypergeometric functions [22].

\[ L_n^\gamma(x) = \binom{n + \gamma}{n} F_1(-n; \gamma + 1; x), \quad n = 0, 1, 2, \ldots \]

\( \{L_n^\gamma(x)\}_{n=0}^\infty \) forms an orthogonal system on \([0, +\infty)\) with respect to the weight function

\( w(x) = x^\gamma e^{-x}, \gamma > -1. \)

We can normalize the Laguerre polynomials and define

\[ \mathbb{L}_n(x) = \mathbb{L}_n(\gamma ; x) := \frac{1}{\sqrt{\Gamma(\gamma + 1)}} \left( \begin{array}{c} n + \gamma \\ n \end{array} \right) L_n^\gamma(-x), \quad n = 0, 1, 2, \ldots \]
\[ \{L_n(x)\}_{n=0}^{\infty} \text{ forms an orthonormal system on } (-\infty, 0] \text{ with respect to the weight function } (-x)^{\gamma}e^x. \]

Considering (2.5) for the Laguerre polynomials with the particular value \( k = 0 \), we get

\[ L_n^\gamma(0; x) = \lambda_1 \lambda_2 \ldots \lambda_{n+1} (L_n^\gamma(0))^{-1} L_n(\gamma + 1; x), \]

where \( \lambda_{n+1} = n(n+\gamma) \) [15, p. 154].

So, the ratio of the kernel of Laguerre polynomials for \( k = 0 \) is given by

\[ \frac{L_{n-1}^\gamma(0; x)}{L_n^\gamma(0; x)} = \frac{1}{\lambda_{n+1}} \frac{L_n^\gamma(0) L_{n-1}(\gamma + 1; x)}{L_{n-1}^\gamma(0) L_n(\gamma + 1; x)}. \]

We can write the above expression as

\[ \frac{L_{n-1}^\gamma(0; x)}{L_n^\gamma(0; x)} = \frac{1}{n^2} \sqrt{\frac{\mathbb{B}(n, \gamma + 2)}{n \mathbb{B}(n, \gamma + 1)}} \phi(-n + 1; \gamma + 2; -x) \phi(n; \gamma; -x), \]

where \( \mathbb{B}(\cdot, \cdot) \) denotes the well-known Beta function.

Hence, we can use (4.9) to obtain the ratio of kernel of Laguerre polynomials for \( k = 0 \) in terms of the continued fractions as

\[ \frac{L_{n-1}^\gamma(0; x)}{L_n^\gamma(0; x)} = \frac{1}{n^2} \sqrt{\frac{\mathbb{B}(n, \gamma + 2)}{n \mathbb{B}(n, \gamma + 1)}} \left( \frac{\tilde{d}_1 x}{1} + \frac{\tilde{d}_2 x}{1} + \frac{\tilde{d}_3 x}{1} + \cdots \right), \]

where \( \tilde{d}_j \)'s are given by (4.10).

Similarly we can express the ratio of the kernels of Laguerre polynomials with different parametric value of \( \gamma > 0 \) for \( k = 0 \) as

\[ \frac{L_{n-1}^\gamma(0; x)}{L_n^\gamma(0; x)} = \frac{\gamma^2}{n^{3/2}(n+\gamma)(\gamma+1)(\gamma+1)} \frac{\phi(-n + 1; \gamma + 2; -x)}{\phi(-n; \gamma + 1; -x)}. \]

The above ratio can be simplified as

\[ \frac{L_{n-1}^\gamma(0; x)}{L_n^\gamma(0; x)} = \frac{\gamma^2}{n^{3/2}(n+\gamma)(\gamma+1)(\gamma+1)} \left( \frac{\tilde{d}_1 x}{1} + \frac{\tilde{d}_2 x}{1} + \frac{\tilde{d}_3 x}{1} + \cdots \right), \]

Now, we can use [38] eq. 91.1 to obtain

\[ \frac{L_{n-1}^{(-1)}(0; x)}{L_n^{(-1)}(0; x)} = \frac{\gamma^2}{n^{3/2}(n+\gamma)(\gamma+1)(\gamma+1)} \left( \frac{\tilde{d}_1 x}{1} + \frac{\tilde{d}_2 x}{1} + \frac{\tilde{d}_3 x}{1} + \cdots \right), \]

where

\[ \tilde{d}_j = d_j(n, r) := \begin{cases} \frac{n+k+\gamma+1}{(\gamma+2k+1)(\gamma+2k+2)} & \text{for } j = 2k + 1, k \geq 0, \\ \frac{1-n-k}{(\gamma+2k+1)(\gamma+2k+2)} & \text{for } j = 2k + 2, k \geq 0. \end{cases} \]

4.2. Kernel of Jacobi polynomials. We know that the Jacobi polynomials with parameter \((\gamma, \delta)\) can be written in the form of Gauss hypergeometric functions [22]

\[ P_n^{(\gamma, \delta)}(x) = \binom{n + \gamma}{n} F \left( -n, n + \gamma + \delta + 1; \gamma + 1; \frac{1-x}{2} \right), \quad n \in \mathbb{Z}_+. \]
Note that \( \{P_n^{(\gamma,\delta)}(x)\}_{n=0}^{\infty} \) forms an orthogonal system on \([-1, 1]\) with respect to the weight function \( w(x) = (1 - x)^\gamma(1 + x)^\delta, \gamma, \delta > -1 \).

The normalisation

\[
\tilde{P}_n^{(\gamma,\delta)}(x) = \sqrt{\frac{(2n + \gamma + \delta + 1)\Gamma(n + 1)\Gamma(n + \gamma + \delta + 1)}{2^{n+\delta+1}\Gamma(n + \gamma + 1)\Gamma(n + \delta + 1)}} P_n^{(\gamma,\delta)}(x) \tag{4.13}
\]

provides the sequence \( \{\tilde{P}_n^{(\gamma,\delta)}(x)\}_{n=0}^{\infty} \) as an orthonormal system on \([-1, 1]\) with respect to the weight function \( w(x) = (1 - x)^\gamma(1 + x)^\delta, \gamma, \delta > -1 \).

Considering (2.5) for the Jacobi polynomials with the particular value \( P_n(x) = P_n^{(\gamma,\delta)}(x) \) for \( \gamma > -1, \delta > 0 \) for \( k = 1 \) is given by

\[
P_n^{(\gamma+1,\delta)}(1; x) = \lambda_1\lambda_2...\lambda_n (P_n^{(\gamma,\delta)}(1))^{-1} \tilde{P}_n^{(\gamma+1,\delta)}(x),
\]

where [15, p. 153]

\[
\lambda_{n+1} = \frac{4n(n + \gamma)(n + \delta)(n + \gamma + \delta)}{(2n + \gamma + \delta)^2(2n + \gamma + \delta + 1)(2n + \gamma + \delta - 1)}.
\]

Hence, the ratio of the kernel of Jacobi polynomials with parameters \( \gamma > -1, \delta > 0 \) for \( k = 1 \) is given by

\[
\frac{P_n^{(\gamma+1,\delta)^*}(1; x)}{P_n^{(\gamma+1,\delta-1)^*}(1; x)} = \frac{\lambda_1\lambda_2...\lambda_n P_n^{(\gamma+1,\delta-1)}(1) \tilde{P}_n^{(\gamma+1,\delta)}(x)}{\lambda_1\lambda_2...\lambda_{n+1} P_{n-1}^{(\gamma+1,\delta)}(1) \tilde{P}_{n-1}^{(\gamma+1,\delta-1)}(x)}.
\]

The above ratio can be simplified as

\[
\frac{P_n^{(\gamma+1,\delta)^*}(1; x)}{P_n^{(\gamma+1,\delta-1)^*}(1; x)} = C(n, \gamma, \delta) \frac{F(-n + 1, n + \gamma + \delta + 1; \gamma + 2; \frac{1-x}{2})}{F(-n, n + \gamma + \delta + 1; \gamma + 2; \frac{1-x}{2})}, \tag{4.14}
\]

where

\[
C(n, \gamma, \delta) = \sqrt{\frac{(\gamma + \delta + 2)^2(2n + \gamma + \delta + 1)(2n + \gamma + \delta)^3}{32n^3(n + \gamma + 1)(\gamma + 1)^2\delta^2}}.
\]

Hence, we can use (4.7) to write the ratio of kernel of Jacobi polynomials in terms of continued fractions as

\[
\frac{P_n^{(\gamma+1,\delta)^*}(1; x)}{P_n^{(\gamma+1,\delta-1)^*}(1; x)} = C(n, \gamma, \delta) \left( \frac{1 - (1 - e_0) e_1 \frac{1-x}{2}}{1 - (1 - e_1) e_2 \frac{1-x}{2}} \right) \left( \frac{1 - (1 - e_2) e_3 \frac{1-x}{2}}{1} \right) \cdots, \tag{4.15}
\]

with

\[
e_j = e_j(n, \gamma, \delta) := \begin{cases} 0 & \text{for } j = 0, \\ \frac{-n + k}{\gamma + 2k + 1} & \text{for } j = 2k, k \geq 1, \\ \frac{n - \gamma + 3k}{\gamma + 2k} & \text{for } j = 2k - 1, k \geq 1. \end{cases}
\]

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