Perseus: A Simple and Optimal High-Order Method for Variational Inequalities

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February 23, 2024

Abstract

This paper settles an open and challenging question pertaining to the design of simple and optimal high-order methods for solving smooth and monotone variational inequalities (VIs). A VI involves finding \(x^* \in \mathcal{X}\) such that \(\langle F(x), x - x^* \rangle \geq 0\) for all \(x \in \mathcal{X}\). We consider the setting in which \(F : \mathbb{R}^d \rightarrow \mathbb{R}^d\) is smooth with up to \((p - 1)\)th-order derivatives. For \(p = 2\), the cubic regularization of Newton’s method has been extended to VIs with a global rate of \(O(\epsilon^{-1})\) [Nesterov, 2006]. An improved rate of \(O(\epsilon^{-2/3} \log \log(1/\epsilon))\) can be obtained via an alternative second-order method, but this method requires a nontrivial line-search procedure as an inner loop. Similarly, the existing high-order methods based on line-search procedures have been shown to achieve a rate of \(O(\epsilon^{-2/(p+1)} \log \log(1/\epsilon))\) [Bullins and Lai, 2022, Lin and Jordan, 2023, Jiang and Mokhtari, 2022]. As emphasized by Nesterov [2018], however, such procedures do not necessarily imply the practical applicability in large-scale applications, and it is desirable to complement these results with a simple high-order VI method that retains the optimality of the more complex methods. We propose a \(p\)th-order method that does not require any line search procedure and provably converges to a weak solution at a rate of \(O(\epsilon^{-2/(p+1)})\). We prove that our \(p\)th-order method is optimal in the monotone setting by establishing a lower bound of \(\Omega(\epsilon^{-2/(p+1)})\) under a generalized linear span assumption. A restarted version of our \(p\)th-order method attains a linear rate for smooth and \(p\)th-order uniformly monotone VIs and another restarted version of our \(p\)th-order method attains a local superlinear rate for smooth and strongly monotone VIs. Further, the similar \(p\)th-order method achieves a global rate of \(O(\epsilon^{-2/p})\) for solving smooth and nonmonotone VIs satisfying the Minty condition. Two restarted versions attain a global linear rate under additional \(p\)th-order uniform Minty condition and a local superlinear rate under additional strong Minty condition.

1 Introduction

Let \(\mathbb{R}^d\) be a finite-dimensional Euclidean space and let \(\mathcal{X} \subseteq \mathbb{R}^d\) be a closed, convex and bounded set with a diameter \(D > 0\). Given that \(F : \mathbb{R}^d \rightarrow \mathbb{R}^d\) is a continuous operator, a fundamental assumption in optimization theory, generalizing convexity, is that \(F\) is monotone:

\[
\langle F(x) - F(x'), x - x' \rangle \geq 0, \quad \text{for all } x, x' \in \mathbb{R}^d.
\]

Another useful assumption in this context is that \(F\) is \((p - 1)\)th-order \(L\)-smooth; in particular, that it has Lipschitz-continuous \((p - 1)\)th-order derivative \((p \geq 1)\) in the sense that there exists a constant
$L > 0$ such that

$$\|\nabla^{(p-1)} F(x) - \nabla^{(p-1)} F(x')\|_{op} \leq L\|x - x'\|, \quad \text{for all } x, x' \in \mathbb{R}^d. \quad (1)$$

With these assumptions, we can formulate the main problem of interest in this paper—the *Minty variational inequality* problem [Minty, 1962]. This consists in finding a point $x^* \in \mathcal{X}$ such that

$$\langle F(x), x - x^* \rangle \geq 0, \quad \text{for all } x \in \mathcal{X}. \quad (2)$$

The solution to Eq. (2) is often referred to as a *weak* solution to the variational inequality (VI) corresponding to $F$ and $\mathcal{X}$ [Facchinei and Pang, 2007]. By way of comparison, the *Stampacchia variational inequality* problem [Hartman and Stampacchia, 1966] consists in finding a point $x^* \in \mathcal{X}$ such that

$$\langle F(x^*), x - x^* \rangle \geq 0, \quad \text{for all } x \in \mathcal{X}, \quad (3)$$

and the solution to Eq. (3) is called a *strong* solution to the VI corresponding to $F$ and $\mathcal{X}$. In the setting where $F$ is continuous and monotone, the solution sets of Eq. (2) and Eq. (3) are equivalent. However, these two solution sets are different in general and a weak solution need not exist when a strong solution exists. In addition, computing an approximate strong solution involves a higher computational burden than finding an approximate weak solution [Monteiro and Svaiter, 2010, 2011, Chen et al., 2017].

Earlier works have focused on the asymptotic global convergence analysis of various VI methods under mild conditions [Lemke and Howson, 1964, Scarf, 1967, Todd, 2013, Hammond and Magnanti, 1987, Fukushima, 1992, Magnanti and Perakis, 1997b]. Two notable exceptions are the generalizations of the ellipsoid method [Magnanti and Perakis, 1995] and the interior-point method [Ralph and Wright, 1997], both of which have been the subject of nonasymptotic complexity analysis.

VIs capture a wide range of problems in optimization theory and beyond, including saddle-point problems and models of equilibria in game-theoretic settings [Cottle et al., 1980, Kinderlehrer and Stampacchia, 2000, Trémolières et al., 2011]. Moreover, the challenge of designing solution methods for VIs with provable worst-case bounds has driven significant research over several decades; see [e.g., Harker and Pang, 1990, Facchinei and Pang, 2007]. This research has provided a foundation for work in machine learning in recent years, where general saddle-point problems have emerged in many settings, including generative adversarial networks (GANs) [Goodfellow et al., 2014] and multi-agent learning in games [Cesa-Bianchi and Lugosi, 2006, Mertikopoulos and Zhou, 2019]. Some of these applications in ML induce a nonmonotone structure, with representative examples including the training of robust neural networks [Madry et al., 2018] or robust classifiers [Sinha et al., 2018].

Building on seminal work in the context of high-order optimization [Baes, 2009, Birgin et al., 2017], we tackle the challenge of developing $p$th-order methods for VIs via an inexact solution of regularized subproblems obtained from a $(p-1)$th-order Taylor expansion of $F$. Accordingly, we make the following assumptions throughout this paper.

**A1.** $F : \mathbb{R}^d \to \mathbb{R}^d$ is $(p-1)$th-order $L$-smooth.

**A2.** The subproblem based on a $(p-1)$th-order Taylor expansion of $F$ and a convex and bounded set $\mathcal{X}$ can be computed approximately in an efficient manner (see Section 3 for details).

For the first-order VI methods (i.e., $p = 1$), Nemirovski [2004] has proved that the extragradient (EG) method [Korpelevich, 1976, Antipin, 1978] converges to a weak solution with a global rate of $O(\epsilon^{-1})$ if $F$ is monotone and Eq. (1) holds. There are other methods with the same global rate guarantee,
including forward-backward splitting method [Tseng, 2000], optimistic gradient (OG) method [Popov, 1980, Mokhtari et al., 2020, Kotsalis et al., 2022] and dual extrapolation method [Nesterov, 2007]. All these methods match the lower bound of Ouyang and Xu [2021] and are thus optimal. In addition, a general adaptive line search framework has been proposed to unify and extend several convergence results from the VI literature [Magnanti and Perakis, 2004].

The investigation of second-order and high-order ($p \geq 2$) counterparts of these first-order methods is less advanced, as exploiting high-order derivative information is much more involved for VIs [Nesterov, 2006, Monteiro and Svaiter, 2012]. Aiming to fill this gap, some work has been recently devoted to studying high-order extensions of first-order VI methods [Bullins and Lai, 2022, Lin and Jordan, 2023, Jiang and Mokhtari, 2022]. These extensions attain a rate of $O(\epsilon^{-2/(p+1)} \log \log(1/\epsilon))$ but require a nontrivial line-search procedure at each iteration. Although the log log(1/\epsilon) factor is modest by itself, it reflects complexity in current design of high-order VI methods, a complexity which might hinder practical application. Notably, Nesterov [2018, page 305] emphasized the difficulty of removing the line search procedure without sacrificing the global rate of convergence and highlighted the goal of obtaining a simple and optimal high-order method as an open and challenging question. We summarize the challenge as follows:

**Can we design a simple and optimal $p^{th}$-order VI method without line search?**

In this paper, we present an affirmative answer to this query by identifying a $p^{th}$-order method that achieves a global rate of $O(\epsilon^{-2/(p+1)})$ while dispensing entirely with the line-search inner loop. The core idea of the proposed method is to incorporate a simple adaptive strategy into a high-order generalization of the dual extrapolation method.

There are two main reasons why we choose the dual extrapolation method as a base algorithm for our high-order methods. First, the dual extrapolation method has its own merits as summarized in Nesterov [2007], and the second-order VI method to attain a global convergence rate of $O(\epsilon^{-1})$ [Nesterov, 2006] was firstly developed based on a dual extrapolation step. Our method can be interpreted as an adaptive variant of this method (see Section 2.2). Second, the dual extrapolation step is an important ingredient for algorithm design in optimization, given the close relationship between extrapolation and acceleration in the context of several first-order methods for smooth convex optimization [Lan and Zhou, 2018a,b]. This is in contrast to the EG method, which is an approximate proximal point method [Mokhtari et al., 2020]. It would deepen our understanding of dual extrapolation if we could design a simple and optimal high-order VI method based on this scheme.

**Contributions.** The contribution of this paper consists in fully closing the gap between the upper and lower bounds in the monotone setting and improving the state-of-the-art upper bounds in the strongly monotone and/or structured non-monotone settings. In further detail:

1. We present a new $p^{th}$-order method for solving smooth and monotone VIs where $F$ has a Lipschitz continuous $(p-1)^{th}$-order derivative and $X$ is convex and bounded. We prove that the number of calls of subproblem solvers required by our method to find an $\epsilon$-weak solution is bounded by

$$O \left( \left( \frac{LD^{p+1}}{\epsilon} \right)^{\frac{2}{p+1}} \right).$$

We prove that our $p^{th}$-order method is indeed optimal by establishing a matching lower bound of $\Omega(\epsilon^{-2/(p+1)})$ under a generalized linear span assumption. We propose a restarted version of our $p^{th}$-
order method for solving smooth and $p^{th}$-order uniformly monotone VIs [Bauschke and Combettes, 2017]. In particular, these are problems in which there exists a constant $\mu > 0$ such that\footnote{We refer to Bauschke and Combettes [2017, Chapter 22] for a more general definition of uniformly monotone operators and relevant discussions. This class of operators are closely related to a direct generalization of uniformly convex functions [Nesterov, 2008, Section 2].}

$$\langle F(x) - F(x'), x - x' \rangle \geq \mu \|x - x'\|^{p+1}, \quad \text{for all } x, x' \in \mathbb{R}^d.$$ 

We show that the number of calls of subproblem solvers required to find $\hat{x} \in X$ satisfying $\|\hat{x} - x^*\| \leq \epsilon$ is bounded by

$$O \left( \frac{\kappa^{\frac{2}{p+1}} \log_2 \left( \frac{D}{\epsilon} \right)}{\epsilon} \right),$$

where $\kappa = L/\mu$ refers to the condition number of $F$. Focusing on smooth and strongly monotone VIs, where there exists a constant $\mu > 0$ such that

$$\langle F(x) - F(x'), x - x' \rangle \geq \mu \|x - x'\|^2, \quad \text{for all } x, x' \in \mathbb{R}^d,$$

we show that another restarted version of our $p^{th}$-order method can achieve a local superlinear rate for the case of $p \geq 2$.

2. We show how to modify our framework such that it can be used for solving smooth and non-monotone VIs satisfying the so-called Minty condition (see Definition 2.5). Again, we note that a line-search procedure is not required. We prove that the number of calls of subproblem solvers to find an $\epsilon$-strong solution is bounded by

$$O \left( \left( \frac{LD^{p+1}}{\epsilon} \right)^{\frac{2}{p}} \right).$$

Two restarted version of our $p^{th}$-order method attain a global linear rate under additional $p^{th}$-order uniform Minty condition and a local superlinear rate (for the case of $p \geq 2$) under additional strong Minty condition.

**Comparison to Adil et al. [2022].** Concurrently appearing on arXiv, Adil et al. [2022] has established the same upper bounds as ours for a high-order generalization of the EG method for solving smooth and monotone VIs. Their method was later extended by two subsequent works to solve strongly monotone VIs [Huang and Zhang, 2022] and nonmonotone VIs satisfying the Minty condition [Huang and Zhang, 2023]. It is also worth remarking that the methods from Huang and Zhang [2022, 2023] are based on similar restarting strategies but their refined analysis leads to a better convergence rate guarantee for solving strongly monotone VIs (up to a log factor) (see the discussion after Huang and Zhang [2022, Theorem 3.2]).

A lower bound has been established in Adil et al. [2022] for a class of $p^{th}$-order methods restricted to solving the primal problem. This is a rather strong limitation that excludes both our method and their method. We derive the same lower bound for a broader class of $p^{th}$-order methods that include both our method and their method thanks to the construction of a new hard instance. Although the hard instance function is different (and the lower bound does improve), we do wish to acknowledge that the proof techniques from Adil et al. [2022] inspired our analysis.
Comparison to Bullins and Lai [2022], Lin and Jordan [2023], Jiang and Mokhtari [2022]. Prior to our work and the concurrent work of Adil et al. [2022], all existing high-order VI methods have been designed based on line-search procedures and are shown to achieve a rate of $O(\epsilon^{-2/(p+1)} \log \log(1/\epsilon))$ for solving smooth and monotone VIs. In this context, Bullins and Lai [2022] was the first to prove improved rates for solving monotone VIs with third-order smoothness and beyond. Their method requires an oracle for finding a fixed point of a nonlinear equation using an implicit update. This necessitates a nontrivial line-search procedure per iteration and leads to a global rate of $O(\epsilon^{-2/(p+1)} \log(1/\epsilon))$. Subsequently, Lin and Jordan [2023] investigated the dynamics of high-order VI methods from a continuous-time viewpoint by proposing a novel closed-loop control system. Their analysis offers a simplification of existing analyses in Bullins and Lai [2022] as well as new results concerning high-order VI methods. However, the method from Lin and Jordan [2023] still requires line search and the obtained rate is the same. By distilling the idea of optimism, Jiang and Mokhtari [2022] proposed a generalized optimistic method with a novel adaptive line search procedure that provably only requires an $O(\log \log(1/\epsilon))$ calls to a subproblem solver per iteration on average. This leads to an improve rate of $O(\epsilon^{-2/(p+1)} \log \log(1/\epsilon))$ in monotone setting and $O((\kappa D^{p-1})^{\frac{p}{p+1}} + \log \log(1/\epsilon))$ in strongly monotone setting. While these line-search-based VI methods are indeed an achievement and can be amenable to implementation, it would be important to understand whether or not there exists a simple and optimal high-order VI method that has no need for a line-search procedure.

In comparison to Bullins and Lai [2022], Lin and Jordan [2023], Jiang and Mokhtari [2022], we believe that our method offers advantages in terms of simplicity. From an algorithmic design viewpoint, we incorporate a simple adaptive strategy into a high-order generalization of the dual extrapolation method, dispensing entirely with the line-search inner loop. Regarding technical parts, our convergence analysis only depends on a simple Lyapunov function and is easy to understand. However, it is worth remarking that our work does not eliminate the potential advantages of using line search. In fact, while the optimal first-order VI methods do not require line search, Magnanti and Perakis [2004] showed the benefits of adaptive line search by proposing a general framework to unify and extend several convergence results from the literature. The preliminary numerical results also confirmed that the adaptive line search procedure is fast in practice yet not very stable [Lin et al., 2022]. Nonetheless, we believe it is promising to study the line search procedure from Jiang and Mokhtari [2022] and see if modifications can speed up high-order VI methods in a universal manner.

Further related work. In addition to the aforementioned works, we review relevant research on high-order convex optimization. We focus on $p^{th}$-order methods for $p \geq 2$.

The systematic investigation of the global convergence rate of second-order methods originates in work on the cubic regularization of Newton’s method (CRN) [Nesterov and Polyak, 2006] and its accelerated counterpart (ACRN) [Nesterov, 2008]. The ACRN method was then extended with a $p^{th}$-order regularization model, yielding an improved global convergence rate of $O(\epsilon^{-1/(p+1)})$ [Baes, 2009], while an adaptive $p^{th}$-order method was proposed in Jiang et al. [2020] with the same global rate guarantee. This extension was recently revisited by other works [Nesterov, 2021b, Grapiglia and Nesterov, 2023] with a discussion on an efficient implementation of a third-order method. Meanwhile, within the accelerated Newton proximal extragradient (ANPE) framework [Monteiro and Svaiter, 2013], a $p^{th}$-order method was also proposed by Gasnikov et al. [2019] with a global convergence rate of $O(\epsilon^{-2/(3p+1)} \log(1/\epsilon))$ for minimizing a convex function whose the $p^{th}$-order derivative is Lipschitz continuous. An additional log factor remains between the best known upper bound and the lower bound of $O(\epsilon^{-2/(3p+1)})$ [Arjevani et al., 2019]. This gap has been closed by two independent works [Kovalev and Gasnikov, 2022, Carmon et al., 2023].
that offer a complementary viewpoint to Monteiro and Svaiter [2013], Gasnikov et al. [2019] on how to remove the line-search scheme. Subsequently, the $p^{th}$-order ANPE framework was extended to a strongly convex setting [Marques Alves, 2022] and was shown to achieve a global linear rate and a local superlinear rate while the lower bound on deterministic $p^{th}$-order methods for minimizing a smooth and strongly convex function was established in Kornowski and Shamir [2020]. Beyond the setting with Lipschitz continuous $p^{th}$-order derivatives, these $p^{th}$-order methods have been adapted to a setting with Hölder continuous $p^{th}$-order derivatives [Grapiglia and Nesterov, 2017, 2019, 2020, Song et al., 2021, Doikov and Nesterov, 2022]. Further settings include smooth nonconvex minimization [Cartis et al., 2010, 2011a,b, 2019, Birgin et al., 2016, 2017, Martínez, 2017] as well as structured nonsmooth minimization [Bullins, 2020]. There is also a complementary line of research that studies the favorable properties of lower-order methods in the setting of higher-order smoothness [Nesterov, 2021a,c,d].

We are aware of various high-order methods obtained via discretization of continuous-time dynamical systems [Wibisono et al., 2016, Lin and Jordan, 2022]. In particular, Wibisono et al. [2016] showed that the ACRN method and its $p^{th}$-order variants can be obtained from implicit discretization of an open-loop system without Hessian-driven damping. Lin and Jordan [2022] have provided a control-theoretic perspective on $p^{th}$-order ANPE methods by recovering them from implicit discretization of a closed-loop system with Hessian-driven damping. Both of these two works proved the convergence rate of $p^{th}$-order ACRN and ANPE methods using Lyapunov functions.

**Organization.** In Section 2, we present the setup for variational inequality (VI) problems and provide definitions for the class of operators and optimality criteria we consider in this paper. In addition, we review the dual extrapolation method. In Section 3, we present our new method, its restarted version, and our main results on the global and local convergence guarantee for monotone and nonmonotone VIs. We also establish a matching lower bound for a broad class of $p^{th}$-order methods in the monotone setting. In Section 4, we provide the proofs for our results. In Section 5, we conclude the paper with a discussion on future research directions.

**Notation.** We use lower-case letters such as $x$ to denote vectors and upper-case letters such as $X$ to denote tensors. Let $\mathbb{R}^d$ be a finite-dimensional Euclidean space (the dimension is $d \in \{1, 2, \ldots\}$), endowed with the scalar product $\langle \cdot, \cdot \rangle$. For $x \in \mathbb{R}^d$, we let $\|x\|$ denote its $\ell_2$-norm. For $X \in \mathbb{R}^{d_1 \times \ldots \times d_p}$, we define

$$X[z^1, \ldots, z^p] = \sum_{1 \leq i_1 \leq d_1, \ldots, 1 \leq i_p \leq d_p} (X_{i_1, \ldots, i_p}) z_{i_1}^1 \cdots z_{i_p}^p,$$

and $\|X\|_{\text{op}} = \max_{\|z\|=1, \|z^1\|=1, \ldots, \|z^p\|=1} X[z^1, \ldots, z^p]$ as well. Fixing $p \geq 0$ and letting $F : \mathbb{R}^d \to \mathbb{R}^d$ be a continuous and high-order differentiable operator, we define $\nabla^{(p)} F(x)$ as the $p^{th}$-order derivative at a point $x \in \mathbb{R}^d$ and write $\nabla^{(0)} F = F$. To be more precise, letting $z_1, \ldots, z_k \in \mathbb{R}^d$, we have

$$\nabla^{(k)} F(x)[z^1, \ldots, z^k] = \sum_{1 \leq i_1, \ldots, i_k \leq d} \left( \frac{\partial F_{i_1}}{\partial x_{i_2} \cdots \partial x_{i_k}}(x) \right) z_{i_1}^1 \cdots z_{i_k}^k.$$

For a closed and convex set $\mathcal{X} \subset \mathbb{R}^d$, we let $\mathcal{P}_\mathcal{X}$ be the orthogonal projection onto $\mathcal{X}$ and let $\text{dist}(x, \mathcal{X}) = \inf_{x' \in \mathcal{X}} \|x' - x\|$ denote the distance between $x$ and $\mathcal{X}$. Finally, $a = O(b(L, \mu, \epsilon))$ stands for an upper bound $a \leq C \cdot b(L, \mu, \epsilon)$, where $C > 0$ is independent of parameters $L, \mu$ and the tolerance $\epsilon \in (0, 1)$, and $a = \tilde{O}(b(L, \mu, \epsilon))$ indicates the same inequality where $C > 0$ depends on logarithmic factors of $1/\epsilon$. 

6
2 Preliminaries and Technical Background

In this section, we present the basic formulation of variational inequality (VI) problems and provide definitions for the class of operators and optimality criteria considered in this paper. We further give a brief overview of Nesterov’s dual extrapolation concept from which our new method originates.

2.1 Variational inequality problem

The regularity conditions that we consider for $F : \mathbb{R}^d \rightarrow \mathbb{R}^d$ are as follows.

**Definition 2.1** $F$ is $k^{th}$-order $L$-smooth if

$$\|\nabla^{(k)}F(x) - \nabla^{(k)}F(x')\|_{op} \leq L\|x - x'\|,$$

for all $x, x'$.

**Definition 2.2** We have the following characterizations:

- $F$ is monotone if $\langle F(x) - F(x'), x - x' \rangle \geq 0$ for all $x, x'$.
- $F$ is $k^{th}$-order $\mu$-uniformly monotone if $\langle F(x) - F(x'), x - x' \rangle \geq \mu\|x - x'\|^{k+1}$ for all $x, x'$.
- $F$ is $\mu$-strongly monotone if $\langle F(x) - F(x'), x - x' \rangle \geq \mu\|x - x'\|^2$ for all $x, x'$.

With the definitions in mind, we state the assumptions that impose in addition to $A1$ and $A2$ in order to define highly smooth VI problems.

**Assumption 2.3** We assume that (i) $F : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is $(p - 1)^{th}$-order $L$-smooth, and (ii) $\mathcal{X}$ is convex and bounded with a diameter $D = \max_{x, x' \in \mathcal{X}} \|x - x'\| > 0$.

The convergence of derivative-based optimization methods to a weak solution $x^* \in \mathcal{X}$ depends on properties of $F$ near this point, and in particular some form of smoothness condition is needed. As for the boundedness condition for $\mathcal{X}$, it is standard in the VI literature [Facchinei and Pang, 2007]. This condition not only guarantees the validity of the most natural optimality criterion in the monotone setting—the gap function [Nemirovski, 2004, Nesterov, 2007]—but additionally it is satisfied in real application problems [Facchinei and Pang, 2007]. On the other hand, there is another line of work focusing on relaxing the boundedness condition via appeal to other notions of approximate solution [Monteiro and Svaiter, 2010, 2011, 2012, Chen et al., 2017]. For simplicity, we retain the boundedness condition and leave the analysis for cases with unbounded constraint sets to future work.

**Monotone setting.** For some of our results we focus on operators $F$ that are monotone in addition to Assumption 2.3. Under monotonicity, it is well known that any $\epsilon$-strong solution is an $\epsilon$-weak solution but the reverse does not hold true in general. Accordingly, we formally define $\hat{x} \in \mathcal{X}$ as an $\epsilon$-weak solution or an $\epsilon$-strong solution as follows:

($\epsilon$-weak solution) \hspace{1cm} $\langle F(x), \hat{x} - x \rangle \leq \epsilon$, \hspace{1cm} for all $x \in \mathcal{X}$,

($\epsilon$-strong solution) \hspace{1cm} $\langle F(\hat{x}), \hat{x} - x \rangle \leq \epsilon$, \hspace{1cm} for all $x \in \mathcal{X}$.
These definitions motivate the use of a gap function, \( \text{GAP}(\cdot) : \mathcal{X} \to \mathbb{R}_+ \), defined by
\[
\text{GAP}(\hat{x}) = \sup_{x \in \mathcal{X}} \langle F(x), \hat{x} - x \rangle,
\]
(4)
to measure the optimality of a point \( \hat{x} \in \mathcal{X} \) that is output by various iterative solution methods; see [e.g., Tseng, 2000, Nemirovski, 2004, Nesterov, 2007, Mokhtari et al., 2020]. The boundedness of \( \mathcal{X} \) and the existence of a strong solution guarantee that the gap function is well defined. Formally, we have

**Definition 2.4** A point \( \hat{x} \in \mathcal{X} \) is an \( \epsilon \)-weak solution to the monotone VI that corresponds to \( F : \mathbb{R}^d \to \mathbb{R}^d \) and \( \mathcal{X} \subseteq \mathbb{R}^d \) if we have \( \text{GAP}(\hat{x}) \leq \epsilon \). If \( \epsilon = 0 \), then \( \hat{x} \in \mathcal{X} \) is a weak solution.

In the strongly monotone setting, we let \( \mu > 0 \) denote the modulus of strong monotonicity for \( F \). Under Assumption 2.3, we define \( \kappa := L/\mu \) as the condition number of \( F \). It is worth mentioning that the condition number quantifies the difficulty of solving the optimization problem [Nesterov, 2018] and appears in the iteration complexity bound of derivative-based methods for optimizing a smooth and strongly convex function. Accordingly, the VI that corresponds to \( F \) and \( \mathcal{X} \) is more computationally challenging as \( \kappa > 0 \) increases.

**Structured nonmonotone setting.** We study the case where \( F \) is nonmonotone but satisfies the Minty condition. Imposing such a condition is crucial since the smoothness of \( F \) is not sufficient to guarantee that the problem is computationally tractable. This was shown by Daskalakis et al. [2021] who established that deciding whether an approximate min-max solution exists is NP hard in smooth and nonconvex-nonconcave min-max optimization (which is a special instance of nonmonotone VIs).

A line of recent works has shown that the nonmonotone VI problem satisfying the Minty condition is computationally tractable [Solodov and Svaiter, 1999, Dang and Lan, 2015, Iusem et al., 2017, Kannan and Shanbhag, 2019, Song et al., 2020, Liu et al., 2021, Diakonikolas et al., 2021]. We thus make the following formal definition.

**Definition 2.5** The VI corresponding to \( F : \mathbb{R}^d \to \mathbb{R}^d \) and \( \mathcal{X} \subseteq \mathbb{R}^d \) satisfies the Minty condition if there exists a point \( x^* \in \mathcal{X} \) such that \( \langle F(x), x - x^* \rangle \geq 0 \) for all \( x \in \mathcal{X} \).

We make some comments on the Minty condition. First, this condition simply assumes the existence of at least one weak solution. Second, Harker and Pang [1990, Theorem 3.1] guarantees that there is at least one strong solution since \( F \) is continuous and \( \mathcal{X} \) is closed and bounded. However, the set of weak solutions is only a subset of the set of strong solutions if \( F \) is not necessarily monotone, and the weak solution might not exist. From this perspective, the Minty condition gives a favorable structure. Furthermore, the Minty condition is weaker than generalized monotone assumptions [Dang and Lan, 2015, Iusem et al., 2017, Kannan and Shanbhag, 2019] that imply that the computation of an \( \epsilon \)-strong solution of nonmonotone VIs is tractable for first-order methods. Finally, we say the VI satisfies the \( p \)th-order \( \mu \)-uniform Minty condition if there exists a point \( x^* \in \mathcal{X} \) such that \( \langle F(x), x - x^* \rangle \geq \mu \|x - x^*\|^{p+1} \) for all \( x \in \mathcal{X} \), and the VI satisfies the \( \mu \)-strong Minty condition [Song et al., 2020] if there exists a point \( x^* \in \mathcal{X} \) such that \( \langle F(x), x - x^* \rangle \geq \mu \|x - x^*\|^2 \) for all \( x \in \mathcal{X} \).

Accordingly, we define the residue function \( \text{RES}(\cdot) : \mathcal{X} \to \mathbb{R}_+ \) given by
\[
\text{RES}(\hat{x}) = \sup_{x \in \mathcal{X}} \langle F(\hat{x}), \hat{x} - x \rangle,
\]
(5)
which measures the optimality of a point \( \hat{x} \in \mathcal{X} \) achieved by iterative solution methods; see [e.g., Dang and Lan, 2015, Iusem et al., 2017, Kannan and Shanbhag, 2019, Song et al., 2020]. It is worth
noting that the boundedness of $\mathcal{X}$ and the continuity of $F$ guarantee that the residue function is well defined. Formally, we have

**Definition 2.6** A point $\hat{x} \in \mathcal{X}$ is an $\epsilon$-strong solution to the nonmonotone VI corresponding to $F : \mathbb{R}^d \to \mathbb{R}^d$ and $\mathcal{X} \subseteq \mathbb{R}^d$ if we have $\text{RES}(\hat{x}) \leq \epsilon$. If $\epsilon = 0$, then $\hat{x} \in \mathcal{X}$ is a strong solution.

There are many application problems that can be formulated as nonmonotone VIs satisfying the Minty condition, such as competitive exchange economies [Brighi and John, 2002] and product pricing [Choi et al., 1990, Gallego and Hu, 2014, Ewerhart, 2014]. In addition, the Minty condition restricted to nonconvex optimization was adopted for analyzing the convergence of stochastic gradient descent for deep learning [Li and Yuan, 2017] and it has found real-world applications [Kleinberg et al., 2018].

**Comments on weak versus strong solutions.** First, the monotonicity assumption is assumed such that the averaged iterates make sense and we have proved that the averaged iterates converge to an $\epsilon$-weak solution with a faster convergence rate of $O(\epsilon^{-2/(p+1)})$ in this setting (see Theorem 3.1). Such a bound is stronger than that for convergence rate of best iterates to an $\epsilon$-strong solution under only the Minty condition (see Theorem 3.7). Further, if we impose the monotonicity assumption, we conjecture that the rate of convergence to an $\epsilon$-strong solution can be improved from $O(\epsilon^{-2/p})$ to $O(\epsilon^{-2/(p+1)})$. Such a result has been achieved for the case of $p = 1$ [Diakonikolas, 2020]. However, it is worth mentioning that the first-order method in Diakonikolas [2020] is different from the first-order extragradient method and the first-order dual extrapolation method which are known to achieve an optimal convergence to an $\epsilon$-weak solution. It remains unclear how to design a high-order generalization of such new Halpern iteration methods. Finally, the complexity bound of $O(\epsilon^{-2/(p+1)})$ can not be extended beyond the monotone setting if only the Minty condition holds. Indeed, the key ingredient for proving the complexity bound of $O(\epsilon^{-2/(p+1)})$ is the use of averaged iterates in our new method. Such an averaging technique is known to be crucial for the monotone setting [Magnanti and Perakis, 1997a] but is not known to be valid when only the Minty condition holds. In addition, the fast convergence of Halpern iteration in Diakonikolas [2020] for achieving an $\epsilon$-strong solution heavily relies on the monotonicity assumption and does not extend to the setting when only the Minty condition holds. We would be very surprised if the optimal complexity bound for the monotone setting (note that we have established the matching lower bound) can be achieved for the setting when only the Minty condition holds. Even for the case of $p = 1$, we are not aware of any relevant supporting evidence. Further exploration of this topic is beyond the scope of our paper.

**Comments on Euclidean versus non-Euclidean settings.** The non-Euclidean generalization of the first-order dual extrapolation method has been shown to outperform the original method in various application problem (e.g., the case where $\mathcal{X}$ is a simplex) [Nesterov, 2007]. It remains a possibility that such a benefit also occurs for the case of $p \geq 2$ and thus it seems promising to study the high-order dual extrapolation method in non-Euclidean settings. In fact, we can follow the approach from Adil et al. [2022] and extend our methods to the non-Euclidean setting using Bregman divergence. However, we can not say much about the superiority of high-order dual extrapolation methods in the non-Euclidean setting since the solution of the subproblem will become much more involved. This is different from the first-order case where each subproblem has a closed-form solution even in non-Euclidean settings. This is also a intriguing topic but again beyond the scope of our paper.
2.2 Nesterov’s dual extrapolation method

Nesterov’s dual extrapolation method [Nesterov, 2007] was shown to be an optimal first-order method for computing the weak solution of the VI when $F$ is zeroth-order $L$-smooth and monotone [Ouyang and Xu, 2021]. We recall the basic formulation in our setting of a VI defined via an operator $F : \mathbb{R}^d \to \mathbb{R}^d$ and a closed, convex and bounded set $\mathcal{X} \subseteq \mathbb{R}^d$. Starting with the initial points $x_0 \in \mathcal{X}$ and $s_0 = 0 \in \mathbb{R}^d$, the $k$th iteration of the scheme is given by ($k \geq 1$):

Find $v_k \in \mathcal{X}$ s.t. $v_k = \arg\max_{v \in \mathcal{X}} \langle s_{k-1}, v - x_0 \rangle - \frac{\beta}{2} \|v - x_0\|^2$,

Find $x_k \in \mathcal{X}$ s.t. $\langle F(v_k) + \beta(x_k - v_k), x - x_k \rangle \geq 0$ for all $x \in \mathcal{X}$,

$s_k = s_{k-1} - \lambda F(x_k)$.

This method can be viewed as an instance of the celebrated extragradient method in the dual space (we refer to $s \in \mathbb{R}^d$ as the dual variable). Indeed, the rule which transforms a point $s_{k-1}$ into the next point $s_k$ at the $k$th iteration is called a dual extrapolation step. Nesterov [2007, Theorem 2] showed that the dual extrapolation method, with $\beta = L$ and $\lambda = 1$, generates a sequence $\{x_k\}_{k \geq 0} \subseteq \mathcal{X}$ satisfying the condition that the average iterate, $\bar{x}_k = \frac{1}{k+1} \sum_{i=0}^k x_i$, is an $\epsilon$-weak solution after at most $O(\epsilon^{-1})$ iterations. Here, $L > 0$ is the Lipschitz constant of $F$.

Nesterov also considered the setting where $F$ is monotone and first-order $L$-smooth and proposed a second-order dual extrapolation method for computing the weak solution of the VI [Nesterov, 2006]. Starting with the initial points $x_0 \in \mathcal{X}$ and $s_0 = 0 \in \mathbb{R}^d$, the $k$th iteration of the scheme is given by ($k \geq 1$):

Find $v_k \in \mathcal{X}$ s.t. $v_k = \arg\max_{v \in \mathcal{X}} \langle s_{k-1}, v - x_0 \rangle - \frac{\beta}{3} \|v - x_0\|^3$,

Find $x_k \in \mathcal{X}$ s.t. $\langle F^1_v(x_k) + \frac{M}{2} \|x_k - v_k\|(x_k - v_k), x - x_k \rangle \geq 0$ for all $x \in \mathcal{X}$,

$s_k = s_{k-1} - \lambda F(x_k)$,

where $F^1_v(\cdot) : \mathbb{R}^d \to \mathbb{R}^d$ is defined as a first-order Taylor expansion of $F$ at a point $v \in \mathcal{X}$:

$F^1_v(x) = F(v) + \nabla F(v)(x - v)$.

This scheme is based on the dual extrapolation step which combines a different regularization with a first-order Taylor expansion of $F$. This makes sense since we have zeroth-order and first-order derivative information available and hope to use both of them to accelerate convergence. Similar ideas have been studied for convex optimization [Nesterov and Polyak, 2006], leading to a simple second-order method with a faster global rate [Nesterov, 2008] than optimal first-order methods [Nesterov, 1983]. Unfortunately, the second-order dual extrapolation method with $\beta = 6L$, $M = 5L$ and $\lambda = 1$ is only guaranteed to achieve an iteration complexity of $O(\epsilon^{-1})$ [Nesterov, 2006, Theorem 4].

3 A Regularized High-Order Model and Algorithm

In this section, we present our algorithmic derivation of Perseus and provide a theoretical convergence guarantee for the method. We provide intuition into why Perseus and its restarted version yield the fast rates of convergence for VI problems. We present a complete treatment of the global and local convergence of Perseus and several restarted versions of our method for both the monotone setting and the nonmonotone setting under the Minty condition.
Algorithm 1 Perseus($p, x_0, L, T, \text{opt}$)

Input: order $p$, initial point $x_0 \in \mathcal{X}$, parameter $L$, iteration number $T$ and opt $\in \{0, 1, 2\}$.

Initialization: set $s_0 = 0d \in \mathbb{R}^d$.

for $k = 0, 1, 2, \ldots, T$ do

STEP 1: If $x_k \in \mathcal{X}$ is a solution of the VI, then stop.

STEP 2: Compute $v_{k+1} = \arg\max_{v \in \mathcal{X}} \{(s_k, v - x_0) - \frac{1}{2}\|v - x_0\|^2\}$.

STEP 3: Compute $x_{k+1} \in \mathcal{X}$ such that Eq. (7) holds true.

STEP 4: Compute $\lambda_{k+1} > 0$ such that

$$\frac{1}{20p-8} \leq \lambda_{k+1}L\||x_{k+1} - v_{k+1}|^{p-1}\| \leq \frac{1}{10p+2}.$$ 

STEP 5: Compute $s_{k+1} = s_k - \lambda_{k+1}F(x_{k+1})$.

end for

Output: $\hat{x} =$

$$\begin{cases} \bar{x}_T = \frac{1}{\sum_{k=1}^{T} \lambda_k} \sum_{k=1}^{T} \lambda_k x_k, & \text{if opt} = 0, \\
x_{kT} \text{ for } kT = \arg\min_{1 \leq k \leq T} \|x_k - v_k\|, & \text{else if opt} = 1, \\
x_{kT} & \text{else if opt} = 2. \end{cases}$$

3.1 Algorithmic scheme

We present our $p^{\text{th}}$-order method—Perseus($p, x_0, L, T, \text{opt}$)—in Algorithm 1. Here $p \in \{1, 2, \ldots\}$ is the order, $x_0 \in \mathcal{X}$ is an initial point, $L > 0$ is a Lipschitz constant for ($p-1$)$^{\text{th}}$-order smoothness, $T$ is the maximum iteration number and opt $\in \{0, 1, 2\}$ is the type of output. Our method is a generalization of the dual extrapolation method [Nesterov, 2007] from first order to general $p^{\text{th}}$ order.

The novelty of our method lies in an adaptive strategy used for updating $\lambda_{k+1}$ (see Step 4). This modification is simple yet important. It is the key for obtaining a global rate of $O(\epsilon^{-2/(p+1)})$ (monotone) and that of $O(\epsilon^{-2/p})$ (nonmonotone with the Minty condition) under Assumption 2.3. Focusing on the case of $p = 2$ and the monotone setting, our results improve on the best existing global convergence rates of $O(\epsilon^{-1})$ [Nesterov, 2006] and that of $O(\epsilon^{-2/3}\log\log(1/\epsilon))$ [Monteiro and Svaiter, 2012] under Assumption 2.3, while not sacrificing algorithmic simplicity. In addition, our methods allow the sub-problem to be solved inexactly, and we give options for choosing the type of outputs under different assumptions.

Comments on inexact subproblem solving. We remark that Step 3 involves computing an approximate strong solution to the VI where we define the operator $F_{v_{k+1}}(x)$ as the sum of a high-order polynomial and a regularization term. Indeed, we have

$$F_{v_{k+1}}(x) = F(v_{k+1}) + \langle \nabla F(v_{k+1}), x - v_{k+1} \rangle + \ldots + \frac{1}{(p-1)!} \nabla^{(p-1)} F(v_{k+1}) |x - v_{k+1}|^{p-1} + \frac{5L}{(p-1)!} ||x - v_{k+1}||^{p-1}(x - v_{k+1}),$$

where we write the VI of interest in the subproblem as follows:

$$\text{Find } x_{k+1} \in \mathcal{X} \text{ such that } \langle F_{v_{k+1}}(x_{k+1}), x - x_{k+1} \rangle \geq 0 \text{ for all } x \in \mathcal{X}. \quad (6)$$

Since $F_{v_{k+1}}$ is continuous and $\mathcal{X}$ is convex and bounded, Harker and Pang [1990, Theorem 3.1] guarantees that a strong solution to the VI in Eq. (6) exists and the problem of finding an approximate strong solution is well defined.

\footnote{For ease of presentation, we choose the factor of 5 here. It is worth noting that other large coefficients also suffice to achieve the same global convergence rate guarantee.}
In the monotone setting, we can prove that the $p^{th}$-order regularization subproblem in Eq. (6) is monotone (in fact, it is relatively strongly monotone) if the original VI is $p^{th}$-order $L$-smooth and monotone. Indeed, the VI with $F$ is monotone if and only if the symmetric part of the Jacobian matrix $\nabla F(x)$ is positive semidefinite for all $x \in \mathbb{R}^d$ [Rockafellar and Wets, 2009, Proposition 12.3]. That is to say,

$$\frac{1}{2}(\nabla F(x) + \nabla F(x)^\top) \succeq 0_{d \times d}, \quad \text{for all } x \in \mathbb{R}^d.$$ 

For the case of $p = 1$, we have $\nabla F_{v_{k+1}}(x) = 5L \cdot I_{d \times d} \succeq 0_{d \times d}$ for all $x \in \mathbb{R}^d$ where $I_{d \times d} \in \mathbb{R}^{d \times d}$ is an identity matrix. Thus, the VI in Eq. (6) is $5L$-strongly monotone. For the case of $p \geq 2$, we have

$$\begin{align*}
\nabla F_{v_{k+1}}(x) &= \nabla F(v_{k+1}) + \ldots + \frac{1}{(p-2)!} \nabla^{(p-1)} F(v_{k+1}) + \frac{5L}{(p-2)!} \|x - v_{k+1}\|^{p-2} I_{d \times d} + \frac{5L}{(p-2)!} \|x - v_{k+1}\|^{p-2} (x - v_{k+1})(x - v_{k+1})^\top.
\end{align*}$$

Since the original VI is $p^{th}$-order $L$-smooth, we obtain from Jiang and Mokhtari [2022, Eq. (7)] that

$$\|\nabla F(x) - (\nabla F(v_{k+1}) + \ldots + \frac{1}{(p-2)!} \nabla^{(p-1)} F(v_{k+1}) + \frac{5L}{(p-2)!} \|x - v_{k+1}\|^{p-2})\|_{op} \leq \frac{L}{(p-1)!} \|x - v_{k+1}\|^{p-1}.$$ 

This implies that

$$\begin{align*}
&\frac{1}{2}(\nabla F_{v_{k+1}}(x) + \nabla F_{v_{k+1}}(x)^\top) \succeq \frac{1}{2}(\nabla F(x) + \nabla F(x)^\top) \\
&\quad + \frac{4L}{(p-1)!} \|x - v_{k+1}\|^{p-1} I_{d \times d} + \frac{5L}{(p-2)!} \|x - v_{k+1}\|^{p-2} (x - v_{k+1})(x - v_{k+1})^\top \\
&\quad \succeq \frac{4L}{(p-1)!} \|x - v_{k+1}\|^{p-1} I_{d \times d} + \|x - v_{k+1}\|^{p-2} (x - v_{k+1})(x - v_{k+1})^\top,
\end{align*}$$

where the second inequality holds since the original VI is monotone. Thus, the VI in Eq. (6) is monotone and $4L$-relatively strongly monotone with respect to the reference function $h(x) = \frac{1}{2}\|x - v_{k+1}\|^p$ (see Nesterov [2021b] for the precise definition). Putting these pieces together yields the desired result.

From a computational viewpoint, we can use the generalized mirror-prox method in Titov et al. [2022] to compute $x_{k+1} \in X$ satisfying the following approximation condition:

$$\sup_{x \in X} \langle F_{v_{k+1}}(x_{k+1}), x_{k+1} - x \rangle \leq \frac{L}{p!} \|x_{k+1} - v_{k+1}\|^{p+1}.$$  

(7)

Thus, the solution of the subproblem in our framework is computationally tractable for the monotone setting. Other efficient solvers have been developed for the case of $p = 2$ and $X = \mathbb{R}^d$ in the context of optimization [Grapiglia and Nesterov, 2021] and minimax optimization [Huang et al., 2022, Adil et al., 2022, Lin et al., 2022] and were shown to be efficient in practice. However, it is worth mentioning that the subproblem solution method in our approach is different from that in existing line-search-based methods [Bullins and Lai, 2022, Lin and Jordan, 2023, Jiang and Mokhtari, 2022], and their per-iteration computational costs can not be directly compared. It remains an open challenge to develop a systematic benchmarking for these methods.

In the nonmonotone setting, the VI in Eq. (6) is not necessarily monotone and computing a solution $x_{k+1}$ satisfying Eq. (7) is intractable in general [Daskalakis et al., 2021]. However, $F_{v_{k+1}}$ is defined as the sum of a polynomial and a regularization term, and this special structure might lend itself to efficient numerical methods. For example, we consider the optimization setting where $F = \nabla f$ for a nonconvex function $f : \mathbb{R}^d \to \mathbb{R}$ with a Lipschitz second-order derivative, $X = \mathbb{R}^d$ and $p = 2$. Solving the VI in Eq. (6) is equivalent to solving cubic regularization subproblems in unconstrained optimization: finding a global solution of the regularized polynomial in the following form of

$$\langle \nabla f(v_{k+1}), x - v_{k+1} \rangle + \frac{1}{2} \langle x - v_{k+1}, \nabla^2 f(v_{k+1})(x - v_{k+1}) \rangle + \frac{L}{4} \|x - v_{k+1}\|^3.$$
The above optimization problem is nonconvex but can be solved approximately in a provably efficient manner. Examples of cubic regularization solvers include some generalized conjugate gradient methods with the Lanczos process [Gould et al., 1999, 2010] and a simple variant of gradient descent [Carmon and Duchi, 2019]. A recent textbook of Cartis et al. [2022] provides a detailed discussion of the existing techniques. The generalization of these techniques to handle the VI in Eq. (6) is challenging, however, and beyond the scope of this paper.

Comments on adaptive strategies. Our adaptive strategy for updating \( \lambda_{k+1} \) was inspired by an in-depth consideration of the reason a nontrivial binary search procedure is needed in existing \( p \)th-order methods. These methods compute a pair, \( \lambda_{k+1} > 0, x_{k+1} \in \mathcal{X} \), that (approximately) solve the \( x \)-subproblem that contains \( \lambda \) and the \( \lambda \)-subproblem that contains \( x \). In particular, the conditions can be written as follows:

\[
\alpha_- \leq \frac{\lambda_{k+1}L\|x_{k+1} - v_{k+1}\|^{p-1}}{p!} \leq \alpha_+ \quad \text{for proper choices of } \alpha_- \text{ and } \alpha_+,
\]

\[
\langle F_{v_{k+1}}(x_{k+1}) + \frac{1}{\lambda_{k+1}}(x_{k+1} - v_{k+1}), x - x_{k+1} \rangle \geq 0 \quad \text{for all } x \in \mathcal{X},
\]

where

\[
F_v(x) = F(v) + \langle \nabla F(v), x - v \rangle + \ldots + \frac{1}{(p-1)!}\nabla^{(p-1)}F(v)[x - v]^{p-1} + \frac{L}{(p-1)!}\|x - v\|^{p-1}(x - v). \tag{8}
\]

A key observation is that there can be some \( x \)-subproblems that do not need to refer to \( \lambda \); e.g., the one employed in Algorithm 1. Indeed, we compute \( x_{k+1} \in \mathcal{X} \) that approximately satisfies the following condition:

\[
\langle F_{v_{k+1}}(x_{k+1}), x - x_{k+1} \rangle \geq 0 \quad \text{for all } x \in \mathcal{X}.
\]

It suffices to return \( x_{k+1} \in \mathcal{X} \) with a sufficiently good quality to give us \( \lambda_{k+1} > 0 \) using a simple update rule. Intuitively, such an adaptive strategy makes sense since \( \lambda_{k+1} \) serves as the stepsize in the dual space and we need to be aggressive as the iterate \( x_{k+1} \) approaches the set of optimal solutions to the VI. Meanwhile, the quantity \( \|x_{k+1} - v_{k+1}\| \) can be used to measure the distance between \( x_{k+1} \) and an optimal solution, and the order \( p \in \{1, 2, 3, \ldots\} \) quantifies the relationship between the closeness and the exploitation of high-order derivative information. In summary, \( \lambda_{k+1} \) becomes larger for a better iterate \( x_{k+1} \in \mathcal{X} \) and such a choice leads to a faster global rate of convergence.

Restart version of Perseus. We summarize the restarted versions of our \( p \)th-order method in Algorithm 2. These methods, which we refer to as \( \text{Perseus-restart}(p, x_0, L, \sigma, D, T, \text{opt}) \), combine Algorithm 1 with two restart strategies; [c.f. Nemirovski and Nesterov, 1985, Nesterov, 2013, O’donoghue and Candes, 2015, Nesterov, 2018].
Restart schemes stop an algorithm when a criterion is satisfied and then restart the algorithm with a new input. Originally studied in the setting of momentum-based methods, restarting has been recognized as an important tool for designing linearly convergent algorithms when the objective function is strongly/uniformly convex [Nemirovski and Nesterov, 1985, Nesterov, 2013, Ghadimi and Lan, 2013] or has some other structures [Freund and Lu, 2018, Necora et al., 2019, Renegar and Grimmer, 2022]. Note that strong monotonicity is a generalization of such regularity conditions. As such, it is natural to consider a restarted version of our method, hoping to achieve linear convergence. Accordingly, at each iteration of Algorithm 2, we use \( x_{k+1} = \text{Perseus}(p, x_k, L, t, \text{opt}) \) as a subroutine. In other words, we simply restart \text{Perseus} every \( t \geq 1 \) iterations and take advantage of average iterates or best iterates to generate \( x_{k+1} \) from \( x_k \). In addition, it is worth mentioning that the choice of \( t \) can be specialized to different settings and/or different type of convergence guarantees. In particular, we set \( \text{opt} = 0 \) for the uniformly monotone setting and \( \text{opt} = 1 \) for the strongly monotone setting.

In the context of VI, the restarting strategies have been used to extend high-order extragradient methods [Bullins and Lai, 2022, Adil et al., 2022] from the monotone setting to the strongly monotone setting [Ostrovshov et al., 2020, Huang and Zhang, 2022]. Several papers also focus on the investigation of adaptive restart schemes that speed up the convergence of classical first-order methods [Giselsson and Boyd, 2014, O’donoghue and Candes, 2015] and provide theoretical guarantees in a general setting where the objective function is smooth and has Hölderian growth [Roulet and d’Aspremont, 2017, Fercoq and Qu, 2019]. A drawback of these schemes is that they rely on knowing appropriately accurate approximations of problem parameters. The same issue arises for our method, given that Algorithm 2 needs to choose \( T_{\text{inner}} \geq 1 \). In the optimization setting, recent work by Renegar and Grimmer [2022] shows how to alleviate this problem via a simple restart scheme that makes no attempt to learn parameter values and only requires the information that is readily available in practice. It is an interesting open question as to whether such a scheme can be found in the VI setting for \text{Perseus}.

### 3.2 Main results

We provide our main results on the convergence rate for Algorithm 1 and 2 in terms of the number of calls of the subproblem solvers. Note that Assumption 2.3 will be made throughout and we impose the Minty condition (see Definition 2.5) for the nonmonotone setting.

**Monotone setting.** The following theorems give us the global convergence rate of Algorithm 1 and 2 for smooth and (uniformly/strongly) monotone VIs.

**Theorem 3.1** Suppose that Assumption 2.3 holds and \( F : \mathbb{R}^d \rightarrow \mathbb{R}^d \) is monotone and let \( \epsilon \in (0, 1) \). The required number of iterations is

\[
T = O \left( \left( \frac{L D^{p+1} \epsilon}{\epsilon^{p+1}} \right)^{\frac{2}{p+1}} \right),
\]

where \( \hat{x} = \text{Perseus}(p, x_0, L, T, 0) \) satisfies \( \text{GAP}(\hat{x}) \leq \epsilon \) and the total number of calls of the subproblem solvers is equal to \( T \). Here, \( p \in \{1, 2, \ldots\} \) is an order, \( L > 0 \) is a Lipschitz constant for \((p-1)\)th-order smoothness of \( F \) and \( D > 0 \) is the diameter of \( \mathcal{X} \).
Theorem 3.2 Suppose that Assumption 2.3 holds and $F : \mathbb{R}^d \to \mathbb{R}^d$ is $p$th-order $\mu$-uniformly monotone and let $\epsilon \in (0, 1)$. The required number of iterations is

$$T = O \left( \log_2 \left( \frac{D}{\epsilon} \right) \right),$$

such that $\hat{x} = \text{Perseus-restart}(p, x_0, L, \mu, D, T, 0)$ satisfies $\| \hat{x} - x^* \| \leq \epsilon$ and the total number of calls of the subproblem solvers is bounded by

$$O \left( \kappa^{\frac{1}{p+1}} \log_2 \left( \frac{D}{\epsilon} \right) \right),$$

where $p \in \{1, 2, \ldots\}$ is an order, $\kappa = L/\mu > 0$ is the condition number of $F$, $D > 0$ is the diameter of $\mathcal{X}$ and $x^* \in \mathcal{X}$ is one weak solution.

Remark 3.3 For the first-order methods (i.e., the case of $p = 1$), the convergence guarantee in Theorem 3.1 recovers the global rate of $O(L/\epsilon)$ in Nesterov [2007, Theorem 2]. The same rate has been derived for other first-order methods [Nemirovski, 2004, Monteiro and Svaiter, 2010, Mokhtari et al., 2020, Kotsalis et al., 2022] and is known to match the established lower bound [Ouyang and Xu, 2021]. For the second-order and high-order methods (i.e., the case of $p \geq 2$), our results improve upon the state-of-the-art results [Monteiro and Svaiter, 2012, Bullins and Lai, 2022, Lin and Jordan, 2023, Jiang and Mokhtari, 2022] by shaving off an additional $\log \log(1/\epsilon)$ factor.

Remark 3.4 For the first-order methods, Theorem 3.2 recovers the global linear convergence rate achieved by the dual extrapolation method and matches the lower bound [Zhang et al., 2022]. For the second-order and high-order methods, our results are new and we believe that these bounds can not be further improved although we do not know of lower bounds.

Local convergence. We present the local convergence property of our methods for the strongly monotone VIs.

Theorem 3.5 Suppose that Assumption 2.3 holds and $F : \mathbb{R}^d \to \mathbb{R}^d$ is $\mu$-strongly monotone and let $\{x_k\}_{k=0}^{T+1}$ be generated by $\text{Perseus-restart}(p, x_0, L, \mu, D, T, 1)$. Then, the following statement holds true,

$$\| x_{k+1} - x^* \| \leq \sqrt{\frac{2p(5p-2)\kappa}{p} \| x_k - x^* \|^{\frac{p+1}{2}}},$$

where $\kappa = L/\mu > 0$ is the condition number of the VI, $D > 0$ is the diameter of $\mathcal{X}$ and $x^*$ is the unique weak solution of the VI. As a consequence, if $p \geq 2$ and the following condition holds true,

$$\| x_0 - x^* \| \leq \frac{1}{2} \left( \frac{pl}{(5p-2)\kappa} \right)^{\frac{1}{p-1}},$$

the iterates $\{x_k\}_{k=0}^{T+1}$ converge to $x^* \in \mathcal{X}$ in at least a superlinear rate.

Remark 3.6 The local convergence guarantee in Theorem 3.5 is derived for the second-order and high-order methods (i.e., the case of $p \geq 2$) and is posited as their advantage over first-order method if we hope to pursue high-accuracy solutions. In this context, Jiang and Mokhtari [2022] provided the same local convergence guarantee for the generalized optimistic gradient methods as our results in Theorem 3.5 but without counting the complexity bound of binary search procedure.
Nonmonotone setting. We consider smooth and nonmonotone VIs satisfying the Minty condition and present the global rate of Algorithm 1 and 2 in terms of the number of calls of the subproblem solvers.

Theorem 3.7 Suppose that Assumption 2.3 and the Minty condition hold true and let $\epsilon \in (0, 1)$. The required number of iterations is

$$T = O \left( \left( \frac{LD^{p+1}}{\epsilon} \right)^\frac{2}{p} \right),$$

such that $\hat{x} = \text{Perseus}(p, x_0, L, T, 2)$ satisfies $\text{RES}(\hat{x}) \leq \epsilon$ and the total number of calls of the subproblem solvers is equal to $T$. Here, $p \in \{1, 2, \ldots\}$ is an order, $L > 0$ is the Lipschitz constant for $(p-1)$th-order smoothness of $F$ and $D > 0$ is the diameter of $\mathcal{X}$.

Remark 3.8 The convergence guarantee in Theorem 3.7 was derived for other first-order methods [Dang and Lan, 2015, Song et al., 2020] for the case of $p = 1$. They are completely new for the case of $p \geq 2$ in the literature to our knowledge.

Remark 3.9 For smooth and nonmonotone VIs, we obtain the same convergence rate as Theorem 3.2 to a weak solution rather than a strong solution under $p$th-order $\mu$-uniform Minty condition. The same local superlinear rate as Theorem 3.5 to a weak solution can be obtained under $\mu$-strong Minty condition. The proof would be the same as that used for proving Theorem 3.2 and 3.5.

Lower bound. We provide the lower bound for the monotone setting under a linear span assumption. Our analysis and hard instance are largely inspired by the constructions and techniques from Nesterov [2021b] and Adil et al. [2022]. However, different from Adil et al. [2022], our lower bound is established for a wide class of $p$th-order dual extrapolation methods that include our method, rather than $p$th-order methods restricted to solve the primal problem in Adil et al. [2022, Eq. (11)].

For constructing the problems that are difficult for our $p$th-order methods, it is convenient to consider the saddle point problem, $\min_{z \in Z} \max_{y \in Y} f(z, y)$, which is a special monotone VI defined via an operator $F$ and a closed, convex and bounded set $\mathcal{X}$ as follows:

$$x = \begin{bmatrix} z \\ y \end{bmatrix}, \quad F(x) = \begin{bmatrix} \nabla_z f(z, y) \\ -\nabla_y f(z, y) \end{bmatrix}, \quad \mathcal{X} = Z \times Y.$$

Let us describe the abilities of $p$th-order methods of degree $p \geq 2$ in generating the new iterates. In particular, the output of oracle at a point $\bar{x} \in \mathcal{X}$ consists in the set of multi-linear forms given by $F(\bar{x}), \nabla F(\bar{x}), \ldots, \nabla^{(p-1)} F(\bar{x})$.

To that end, we assume that the $p$th-order method in our algorithm class is able to generate a sequence of iterates $\{x_k\}_{k \geq 0}$ satisfying the recursive condition:

$$s \in \text{Lin}(F(x_0), \ldots, F(x_k)), \quad \bar{x} = \text{argmax}_{x \in \mathcal{X}} \{\langle s, x - x_0 \rangle - \frac{1}{2} \|x - x_0\|^2\},$$

$x_{k+1} \in \mathcal{X}$ satisfies that $\langle \Phi_{a,\gamma,x}(x_{k+1} - x_k), x - x_{k+1} \rangle \geq 0$ for all $x \in \mathcal{X},$

where $\Phi_{a,\gamma,x}(h) = a_0 F(\bar{x}) + \sum_{i=1}^{p-1} a_i \nabla^{(i)} F(\bar{x}) |h|^i + \gamma \|h\|^{p-1} h$ with $a \in \mathbb{R}^p$ and $\gamma > 0$. Our assumption about the form of $p$th-order methods in our algorithm class is summarized as follows:
Assumption 3.10 The $p^{th}$-order method generates a sequence of iterates $\{x_k\}_{k \geq 0}$ satisfying the following recursive condition: for all $k \geq 0$, we have that $x_{k+1} \in \mathcal{X}$ satisfies that $\langle \Phi_{a,\gamma,x}(x_{k+1} - x_k), x - x_{k+1} \rangle \geq 0$ for all $x \in \mathcal{X}$, where

$$\bar{x} = \arg\max_{x \in \mathcal{X}} \{ \langle s, x - x_0 \rangle - \frac{1}{2} \|x - x_0\|^2 \} \text{ and } s \in \text{Lin}(F(x_0), \ldots, F(x_k)).$$

Assumption 3.10 is a generalization of a classical linear span assumption [Nesterov, 2021b] and is well satisfied by various dual extrapolation methods, including Algorithm 1. However, it might not hold true for other VI methods, such as extragradient methods and their variants [Monteiro and Svaiter, 2013, Bullins and Lai, 2022, Huang et al., 2022, Jiang and Mokhtari, 2022, Lin and Jordan, 2023]. Removing Assumption 3.10 using the rotation techniques [Arjevani et al., 2019, Carmon et al., 2020, Ouyang and Xu, 2021] is a challenging task; indeed, due to the nonlinear coupling terms of our hard instances (see Subsection 4.6), the previous analysis cannot be directly applied. We leave further exploration of this topic to future work.

Note that the same lower bound has been recently established in Adil et al. [2022] for a special class of $p^{th}$-order methods restricted to solve the primal problem under Assumption 3.10. Indeed, their construction is based on a saddle-point problem $\min_{z \in Z} \max_{y \in Y} f(z, y)$ and they assume that any method in their algorithm class not only satisfies Assumption 3.10 but has the access to $\nabla \phi(z), \ldots, \nabla^{(p)} \phi(z)$ where $\phi(z) = \max_{y \in Y} f(z, y)$ refers to the objective function of primal problem (see Adil et al. [2022, Lemma 4.3]). Proving the lower bound for general $p^{th}$-order methods under Assumption 3.10 requires a new hard instance, which is a nonlinear generalization of the instance used in Adil et al. [2022].

The following theorem summarizes our main result. The proof details can be found in Section 4.

Theorem 3.11 Fixing $p \geq 2$, $L > 0$ and $T > 0$ and letting $d \geq 4T + 1$ be the problem dimension. There exists two closed, convex and bounded sets $Z, Y \subseteq \mathbb{R}^d$ and a function $f(z, y) : Z \times Y \to \mathbb{R}$ that is convex-concave with an optimal saddle-point solution $(z_*, y_*) \in Z \times Y$ such that the iterates $\{(z_k, y_k)\}_{k \geq 0}$ generated by any $p^{th}$-order method under Assumption 3.10 must satisfy

$$\min_{0 \leq k \leq T} \left\{ \max_{y \in Y} f(z_k, y) - \min_{z \in Z} f(z, y_k) \right\} \geq \left( \frac{1}{4^{\frac{p}{2} + 1}} \right) LD_Z D_Y^p T^{-\frac{p + 1}{2}}.$$

Remark 3.12 The lower bound in Theorem 3.11 shows that any $p^{th}$-order method satisfying Assumption 3.10 requires at least $\Omega((LD_Z D_Y^p)^{\frac{1}{p+1}} \epsilon^{-\frac{2}{p+1}})$ iterations to reach an $\epsilon$-weak solution. Combined this result with Theorem 3.1 shows that Algorithm 1 is an optimal $p^{th}$-order method for solving smooth and monotone VIs. As mentioned before, we have improved the results in Adil et al. [2022] by constructing a new hard instance and deriving the same lower bound for a more broad class of $p^{th}$-order methods that include both Algorithm 1 and the high-order extragradient method in Adil et al. [2022].

Remark 3.13 For the lower bound for finding an $\epsilon$-strong solution in monotone setting, the case for first-order VI methods have been investigated in Diakonikolas [2020]. The key idea is to use the lower bound for finding an $\epsilon$-weak solution [Ouyang and Xu, 2021] and the algorithmic reductions to derive lower bounds. However, such a reduction is mostly based on the high-order generalization of Halpern iteration and is thus beyond the scope of the current manuscript. In particular, we have developed a simple and optimal $p^{th}$-order VI method for finding an $\epsilon$-weak solution in the monotone setting. However, the optimal algorithm for finding an $\epsilon$-strong solution in the monotone setting is likely to be different as evidenced by Diakonikolas [2020]. Computing an $\epsilon$-strong solution and/or an $\epsilon$-weak solution are
complementary, yet different, and they indeed deserve separate study in their own right. Moreover, the lower bound for finding an $\epsilon$-strong solution under the Minty condition is largely unexplored and missing even in the current literature for first-order VI methods.

4 Convergence Analysis

We present the convergence analysis for our $p^{th}$-order method (Algorithm 1) and its restarted version (Algorithm 2). In particular, we provide the global convergence guarantee (Theorems 3.1 and 3.2) and local convergence guarantee for the monotone setting (Theorems 3.5). We analyze the nonmonotone setting under the Minty condition (Theorems 3.7). Finally, we establish the lower bound for the monotone setting under a linear span assumption (Theorem 3.11).

4.1 Technical lemmas

We define the following Lyapunov function for the iterates $\{x_k\}_{k\geq 0}$ that are generated by Algorithm 1:

$$\mathcal{E}_k = \max_{v \in \mathcal{X}} \langle s_k, v - x_0 \rangle - \frac{1}{2}\|v - x_0\|^2.$$ (9)

This function is used to prove technical results that pertain to the dynamics of Algorithm 1.

**Lemma 4.1** Suppose that Assumption 2.3 holds true. For every integer $T \geq 1$, we have

$$\sum_{k=1}^{T} \lambda_k \langle F(x_k), x_k - x \rangle \leq \mathcal{E}_0 - \mathcal{E}_T + \langle s_T, x - x_0 \rangle - \frac{1}{TM} \left( \sum_{k=1}^{T} \|x_k - v_k\|^2 \right),$$ for all $x \in \mathcal{X}$.

**Proof.** By combining Eq. (9) and the definition of $v_{k+1}$, we have

$$\mathcal{E}_k = \langle s_k, v_{k+1} - x_0 \rangle - \frac{1}{2}\|v_{k+1} - x_0\|^2.$$ Then, we have

$$\mathcal{E}_{k+1} - \mathcal{E}_k = \langle s_{k+1}, v_{k+2} - x_0 \rangle - \langle s_k, v_{k+1} - x_0 \rangle - \frac{1}{2} (\|v_{k+2} - x_0\|^2 - \|v_{k+1} - x_0\|^2)$$

$$= \langle s_{k+1} - s_k, v_{x+1} - x_0 \rangle + \langle s_{k+1}, v_{k+2} - v_{k+1} \rangle - \frac{1}{2} (\|v_{k+2} - x_0\|^2 - \|v_{k+1} - x_0\|^2).$$ (10)

By using the update formula for $v_{k+1}$ again, we have

$$\langle x - v_{k+1}, s_k - v_{k+1} + x_0 \rangle \leq 0,$$ for all $x \in \mathcal{X}$.

Letting $x = v_{k+2}$ in this inequality and using $\langle a, b \rangle = \frac{1}{2}(\|a + b\|^2 - \|a\|^2 - \|b\|^2)$, we have

$$\langle s_k, v_{k+2} - v_{k+1} \rangle \leq \langle v_{k+1} - x_0, v_{k+2} - v_{k+1} \rangle = \frac{1}{2} (\|v_{k+2} - x_0\|^2 - \|v_{k+1} - x_0\|^2 - \|v_{k+2} - v_{k+1}\|^2).$$ (11)

Plugging Eq. (11) into Eq. (10) and using the update formula of $s_{k+1}$, we obtain:

$$\mathcal{E}_{k+1} - \mathcal{E}_k \leq \langle s_{k+1} - s_k, v_{k+1} - x_0 \rangle + \langle s_{k+1} - s_k, v_{k+2} - v_{k+1} \rangle - \frac{1}{2}\|v_{k+2} - v_{k+1}\|^2$$

$$= \langle s_{k+1} - s_k, v_{k+2} - x_0 \rangle - \frac{1}{2}\|v_{k+2} - v_{k+1}\|^2$$

$$\leq \lambda_{k+1} \langle F(x_{k+1}), x_0 - v_{k+2} \rangle - \frac{1}{2}\|v_{k+2} - v_{k+1}\|^2$$

$$= \lambda_{k+1} \langle F(x_{k+1}), x_0 - x \rangle + \lambda_{k+1} (F(x_{k+1}), x - x_{k+1}) + \lambda_{k+1} (F(x_{k+1}), x_{k+1} - v_{k+2}) - \frac{1}{2}\|v_{k+2} - v_{k+1}\|^2,$$
for any $x \in \mathcal{X}$. Summing up this inequality over $k = 0, 1, \ldots, T - 1$ and changing the counter $k + 1$ to $k$ yields that

$$
\sum_{k=1}^{T} \lambda_k \langle F(x_k), x_k - x \rangle \leq \mathcal{E}_0 - \mathcal{E}_T + \sum_{k=1}^{T} \lambda_k \langle F(x_k), x_0 - x \rangle + \sum_{k=1}^{T} \lambda_k \langle F(x_k), x_k - v_{k+1} \rangle - \frac{1}{2} \| v_k - v_{k+1} \|^2.
$$

Using the update formula for $s_{k+1}$ and letting $s_0 = 0_d \in \mathbb{R}^d$, we have

$$
\mathbf{I} = \sum_{k=1}^{T} \langle \lambda_k F(x_k), x_0 - x \rangle = \sum_{k=1}^{T} \langle s_{k-1} - s_k, x_0 - x \rangle = \langle s_0 - s_T, x_0 - x \rangle = \langle s_T, x - x_0 \rangle.
$$

Since $x_{k+1} \in \mathcal{X}$ satisfies Eq. (7), we have

$$
\langle F_{v_k}(x_k), x - x_k \rangle \geq -\frac{L}{p} \| x_k - v_k \|^{p+1}, \quad \text{for all } x \in \mathcal{X},
$$

where $F_v(x) : \mathbb{R}^d \to \mathbb{R}^d$ is defined for any fixed $v \in \mathcal{X}$ as follows:

$$
F_{v_k}(x) = F(v_k) + \langle \nabla F(v_k), x - v_k \rangle + \ldots + \frac{1}{(p-1)!} \langle \nabla^{(p-1)} F(v_k)[x - v_k]^{p-1} + \frac{5L}{(p-1)!} \| x - v_k \|^{p-1} (x - v_k).
$$

Under Assumption 2.3, we obtain from Bullins and Lai [2022, Fact 2.5] or Jiang and Mokhtari [2022, Eq. (6)] that

$$
\| F(x_k) - F_{v_k}(x_k) + \frac{5L}{(p-1)!} \| x_k - v_k \|^{p-1} (x_k - v_k) \| \leq \frac{L}{p!} \| x_k - v_k \|^p.
$$

Letting $x = v_{k+1}$ in Eq. (14), we have

$$
\langle F_{v_k}(x_k), x_k - v_{k+1} \rangle \leq \frac{L}{p!} \| x_k - v_k \|^{p+1}.
$$

Inspired by Eq. (15) and Eq. (16), we decompose $\langle F(x_k), x_k - v_{k+1} \rangle$ as follows:

$$
\langle F(x_k), x_k - v_{k+1} \rangle
\leq \langle F(x_k) - F_{v_k}(x_k) + \frac{5L}{(p-1)!} \| x_k - v_k \|^{p-1} (x_k - v_k), x_k - v_{k+1} \rangle
+ \langle F_{v_k}(x_k), x_k - v_{k+1} \rangle - \frac{5L}{(p-1)!} \| x_k - v_k \|^{p-1} \langle x_k - v_k, x_k - x_{k+1} \rangle
\leq \| F(x_k) - F_{v_k}(x_k) + \frac{5L}{(p-1)!} \| x_k - v_k \|^{p-1} (x_k - v_k) \| \cdot \| x_k - v_{k+1} \| + \langle F_{v_k}(x_k), x_k - v_{k+1} \rangle - \frac{5L}{(p-1)!} \| x_k - v_k \|^{p-1} \langle x_k - v_k, x_k - x_{k+1} \rangle
\leq \frac{L}{p!} \| x_k - v_k \|^{p+1} + \frac{L}{p!} \| x_k - v_k \|^p \| v_k - v_{k+1} \| - \frac{5L}{(p-1)!} \| x_k - v_k \|^{p-1} \langle x_k - v_k, x_k - x_{k+1} \rangle.
$$

Note that we have

$$
\langle x_k - v_k, x_k - v_{k+1} \rangle = \| x_k - v_k \|^2 + \langle x_k - v_k, v_k - v_{k+1} \rangle \geq \| x_k - v_k \|^2 - \| x_k - v_k \| \| v_k - v_{k+1} \|.
$$

Putting these pieces together yields that

$$
\langle F(x_k), x_k - v_{k+1} \rangle \leq \frac{(5p+1)L}{p!} \| x_k - v_k \|^p \| v_k - v_{k+1} \| - \frac{(5p-2)L}{p!} \| x_k - v_k \|^{p+1}.
$$
Since \( \frac{1}{20p-8} \leq \frac{\lambda_k L \|x_k - v_k\|^{p-1}}{p!} \leq \frac{1}{10p+2} \) for all \( k \geq 1 \), we have
\[
II \leq \sum_{k=1}^{T} \left( \frac{(5p+1)\lambda_k L \|x_k - v_k\|^p \|v_k - v_{k+1}\|}{p!} - \frac{1}{2}\|v_k - v_{k+1}\|^2 - \frac{(5p-2)\lambda_k L \|x_k - v_k\|^{p+1}}{p!} \right)
\leq \sum_{k=1}^{T} \left( \frac{1}{2}\|x_k - v_k\|^{p-1} - \frac{1}{2}\|v_k - v_{k+1}\|^2 - \frac{1}{4}\|x_k - v_k\|^2 \right)
\leq \sum_{k=1}^{T} \left( \max_{\eta \geq 0} \left\{ \frac{1}{2}\|x_k - v_k\|\eta - \frac{1}{2}\eta^2 \right\} - \frac{1}{4}\|x_k - v_k\|^2 \right)
= -\frac{1}{8} \left( \sum_{k=1}^{T} \|x_k - v_k\|^2 \right).
\tag{17}
\]
Plugging Eq. (13) and Eq. (17) into Eq. (12) yields that
\[
\sum_{k=1}^{T} \lambda_k \langle F(x_k), x_k - x \rangle \leq \mathcal{E}_0 - \mathcal{E}_T + \langle s_T, x - x_0 \rangle - \frac{1}{8} \left( \sum_{k=1}^{T} \|x_k - v_k\|^2 \right).
\]
This completes the proof.

**Lemma 4.2** Suppose that Assumption 2.3 and the Minty condition hold true and let \( x \in \mathcal{X} \). For every integer \( T \geq 1 \), we have
\[
\sum_{k=1}^{T} \lambda_k \langle F(x_k), x_k - x \rangle \leq \frac{1}{2}\|x - x_0\|^2, \quad \sum_{k=1}^{T} \|x_k - v_k\|^2 \leq 4\|x^* - x_0\|^2,
\]
where \( x^* \in \mathcal{X} \) denotes the weak solution to the VI.

**Proof.** For any \( x \in \mathcal{X} \), we have
\[
\mathcal{E}_0 - \mathcal{E}_T + \langle s_T, x - x_0 \rangle = \mathcal{E}_0 - \left( \max_{v \in \mathcal{X}} \langle s_T, v - x_0 \rangle - \frac{1}{2}\|v - x_0\|^2 \right) + \langle s_T, x - x_0 \rangle.
\]
Since \( s_0 = 0 \), we have \( \mathcal{E}_0 = 0 \) and
\[
\mathcal{E}_0 - \mathcal{E}_T + \langle s_T, x - x_0 \rangle \leq - \left( \langle s_T, x - x_0 \rangle - \frac{1}{2}\|x - x_0\|^2 \right) + \langle s_T, x - x_0 \rangle = \frac{1}{2}\|x - x_0\|^2.
\]
This together with Lemma 4.1 yields that
\[
\sum_{k=1}^{T} \lambda_k \langle F(x_k), x_k - x \rangle + \frac{1}{8} \left( \sum_{k=1}^{T} \|x_k - v_k\|^2 \right) \leq \frac{1}{2}\|x - x_0\|^2,
\]
for all \( x \in \mathcal{X} \),
which implies the first inequality. Since the VI satisfies the Minty condition (see Definition 2.5), there exists \( x^* \in \mathcal{X} \) such that \( \langle F(x_k), x_k - x^* \rangle \geq 0 \) for all \( k \geq 1 \). Letting \( x = x^* \) in the above inequality yields the second inequality.

We provide a technical lemma establishing a lower bound for \( \sum_{k=1}^{T} \lambda_k \).

\[\text{20}\]
Lemma 4.3 Suppose that Assumption 2.3 and the Minty condition hold true. For every integer \( k \geq 1 \), we have
\[
\sum_{k=1}^{T} \lambda_k \geq \frac{p}{(2p-8)L} \left( \frac{1}{\| x^* - x_0 \|^2} \right)^{\frac{p-1}{2}} T^{\frac{p+1}{2}},
\]
where \( x^* \in X \) denotes the weak solution to the VI.

Proof. Without loss of generality, we assume that \( x_0 \neq x^* \). For \( p = 1 \), we have \( \lambda_k = \frac{1}{12T} \) for all \( k \geq 1 \). For \( p \geq 2 \), we have
\[
\sum_{k=1}^{T} (\lambda_k)^{-\frac{2}{p-1}} \left( \frac{p}{(2p-8)L} \right)^{\frac{2}{p-1}} \leq \sum_{k=1}^{T} (\lambda_k)^{-\frac{2}{p-1}} (\lambda_k \| x_k - v_k \|^{p-1})^{\frac{2}{p-1}} = \sum_{k=1}^{T} \| x_k - v_k \|^2 \leq 4 \| x^* - x_0 \|^2.
\]
By the Hölder inequality, we have
\[
\sum_{k=1}^{T} 1 = \frac{T}{\sum_{k=1}^{T} (\lambda_k)^{-\frac{2}{p-1}}} \left( \sum_{k=1}^{T} (\lambda_k)^{-\frac{2}{p-1}} \right)^{\frac{2}{p-1}} \leq \left( \sum_{k=1}^{T} (\lambda_k)^{-\frac{2}{p-1}} \right)^{\frac{2}{p-1}} \left( \sum_{k=1}^{T} (\lambda_k)^{-\frac{2}{p-1}} \right)^{\frac{2}{p-1}} \left( \sum_{k=1}^{T} (\lambda_k)^{-\frac{2}{p-1}} \right)^{\frac{2}{p-1}}.
\]
Putting these pieces together yields
\[
T \leq (4 \| x^* - x_0 \|^2)^{\frac{p-1}{2}} \left( \frac{20p-8L}{p} \right)^{\frac{2}{p-1}} \left( \sum_{k=1}^{T} (\lambda_k)^{-\frac{2}{p-1}} \right)^{\frac{2}{p-1}},
\]
Plugging this into the above inequality yields that
\[
\sum_{k=1}^{T} \lambda_k \geq \frac{p}{(2p-8)L} \left( \frac{1}{\| x^* - x_0 \|^2} \right)^{\frac{p-1}{2}} T^{\frac{p+1}{2}}.
\]
This completes the proof. \( \square \)

4.2 Proof of Theorem 3.1

We see from Harker and Pang [1990, Theorem 3.1] that at least one strong solution to the VI exists since \( F \) is continuous and \( X \) is convex, closed and bounded. Since any strong solution is a weak solution if \( F \) is further assumed to be monotone, we obtain that the VI satisfies the Minty condition.

Letting \( x \in X \), we derive from the monotonicity of \( F \) and the definition of \( \tilde{x}_T \) (i.e., \( \text{opt} = 0 \)) that
\[
\langle F(x), \tilde{x}_T - x \rangle = \frac{1}{\sum_{k=1}^{T} \lambda_k} \left( \sum_{k=1}^{T} \lambda_k \langle F(x), x_k - x \rangle \right) \leq \frac{1}{\sum_{k=1}^{T} \lambda_k} \left( \sum_{k=1}^{T} \lambda_k \langle F(x), x_k - x \rangle \right).
\]
Combining this inequality with the first inequality in Lemma 4.2 yields that
\[
\langle F(x), \tilde{x}_T - x \rangle \leq \frac{\| x - x_0 \|^2}{2(\sum_{k=1}^{T} \lambda_k)}, \quad \text{for all } x \in X.
\]
Since \( x_0 \in X \), we have \( \| x - x_0 \| \leq D \) and hence
\[
\langle F(x), \tilde{x}_T - x \rangle \leq \frac{D^2}{2(\sum_{k=1}^{T} \lambda_k)}, \quad \text{for all } x \in X.
\]
Then, we combine Lemma 4.3 and the fact that \( \|x^* - x_0\| \leq D \) to obtain that

\[
\langle F(x), \tilde{x}_T - x \rangle \leq \frac{2^p(5p-2)}{pt} LD^{p+1} T^{-\frac{p+1}{2}}, \quad \text{for all } x \in \mathcal{X}.
\]

By the definition of a gap function (see Eq. (4)), we have

\[
\text{GAP}(\tilde{x}_T) = \sup_{x \in \mathcal{X}} \langle F(x), \tilde{x}_T - x \rangle \leq \frac{2^p(5p-2)}{pt} LD^{p+1} T^{-\frac{p+1}{2}}.
\]

Therefore, we conclude from Eq. (18) that we can set

\[
T = O \left( \left( \frac{LD^{p+1}}{\epsilon} \right)^{\frac{1}{4p+1}} \right),
\]

such that \( \hat{x} = \text{Perseus}(p, x_0, L, T, 0) \) satisfies \( \text{GAP}(\hat{x}) \leq \epsilon \). The total number of calls of the subproblem solvers is equal to \( T \) since our algorithm calls the subproblem solvers once at each iteration. This completes the proof.

### 4.3 Proof of Theorem 3.2

In the uniformly monotone setting with a convex, closed and bounded set, the solution \( x^* \in \mathcal{X} \) to the VI exists and is unique [Facchinei and Pang, 2007] and the VI satisfies the Minty condition.

We first consider the relationship between \( \|\hat{x} - x\| \) and \( \|x_0 - x^*\| \) where \( \hat{x} = \text{Perseus}(p, x_0, L, T_{\text{inner}}, 0) \). We derive from Jensen’s inequality and the definition of \( \tilde{x}_{\text{inner}} \) that

\[
\|\tilde{x}_{\text{inner}} - x^*\|^{p+1} = \left\| \frac{1}{\sum_{k=1}^{T_{\text{inner}}} \lambda_k} \left( \sum_{k=1}^{T_{\text{inner}}} \lambda_k x_k \right) - x^* \right\|^{p+1} \leq \frac{1}{\sum_{k=1}^{T_{\text{inner}}} \lambda_k} \left( \sum_{k=1}^{T_{\text{inner}}} \lambda_k \|x_k - x^*\|^{p+1} \right).
\]

Since \( F \) is \( p \)-th-order \( \mu \)-uniformly monotone, we have

\[
\|x_k - x^*\|^{p+1} \leq \frac{1}{\mu} \langle F(x_k) - F(x^*), x_k - x^* \rangle \leq \frac{1}{\mu} \langle F(x_k), x_k - x^* \rangle.
\]

Putting these pieces together yields that

\[
\|\tilde{x}_{\text{inner}} - x^*\|^{p+1} \leq \frac{1}{\mu(\sum_{k=1}^{T_{\text{inner}}} \lambda_k)} \left( \sum_{k=1}^{T_{\text{inner}}} \lambda_k \langle F(x_k), x_k - x^* \rangle \right). \tag{19}
\]

Combining the first inequality in Lemma 4.2 with Eq. (19) yields that

\[
\|\tilde{x}_{\text{inner}} - x^*\|^{p+1} \leq \frac{1}{2\mu(\sum_{k=1}^{T_{\text{inner}}} \lambda_k)} \|x_0 - x^*\|^2.
\]

This together with Lemma 4.3 and the fact that \( \hat{x} = \tilde{x}_{\text{inner}} \) yields that

\[
\|\hat{x} - x^*\|^{p+1} \leq \left( \frac{4\|x_0 - x^*\|^2}{\mu T^{p-4}} \right)^{\frac{p+1}{2}} \left( \frac{10p-4}{\mu T^{p-4}} \right) \|x_0 - x^*\|^2 \tag{20}
\]

\[= \left( \frac{2^p(5p-2)}{pt} \right)^{\frac{1}{2}} \|x_0 - x^*\|^{p+1}.
\]
Since $x_{k+1} = \text{Perseus}(p, x_k, L, T_{\text{inner}}, 0)$ in the scheme of Algorithm 2 and

$$T_{\text{inner}} = \left\lceil \left( \frac{2^{p+1}(5p-2)}{p!} \frac{L}{\mu} \right)^{\frac{2}{p+1}} \right\rceil,$$

we have

$$\|x_{k+1} - x^*\|^{p+1} \leq \frac{1}{2} \|x_k - x^*\|^{p+1}, \quad \text{for all } k = 0, 1, 2, \ldots, T.$$  \hspace{1cm} (22)

Therefore, we conclude from Eq. (21) and Eq. (22) that we can set

$$T = O \left( \log_2 \left( \frac{D}{\epsilon} \right) \right),$$

such that $\hat{x} = \text{Perseus-restart}(p, x_0, L, \mu, T, 0)$ satisfies $\|\hat{x} - x^*\| \leq \epsilon$. The total number of calls of the subproblem solvers is bounded by

$$O \left( \left( \frac{L}{\mu} \right)^{\frac{2}{p+1}} \log_2 \left( \frac{D}{\epsilon} \right) \right).$$

This completes the proof.

### 4.4 Proof of Theorem 3.5

In the strongly monotone setting, the solution $x^* \in X$ to the VI exists and is unique [Facchinei and Pang, 2007] and the VI satisfies the Minty condition.

We first consider the relationship between $\|\hat{x} - x^*\|$ and $\|x_0 - x^*\|$ where $\hat{x} = \text{Perseus}(p, x_0, L, T_{\text{inner}}, 1)$. Since $F$ is $\mu$-strongly monotone, we apply the same argument from the proof of Theorem 3.2 and obtain that

$$\|\hat{x}_{T_{\text{inner}}} - x^*\|^2 \leq \frac{1}{\mu \sum_{k=1}^{T_{\text{inner}}} \lambda_k} \left( \sum_{k=1}^{T_{\text{inner}}} \lambda_k \langle F(x_k), x_k - x^* \rangle \right) \leq \frac{1}{2\mu \sum_{k=1}^{T_{\text{inner}}} \lambda_k} \|x_0 - x^*\|^2,$$

Lemma 4.2

This together with Lemma 4.3 and the fact that $\hat{x} = \hat{x}_{T_{\text{inner}}}$ yields that

$$\|\hat{x} - x^*\|^2 \leq \left( \frac{2^{p}(5p-2)}{p!} \frac{L}{\mu} \right) \|x_0 - x^*\|^{p+1}.$$

Since $x_{k+1} = \text{Perseus}(p, x_k, L, T_{\text{inner}}, 1)$ in Algorithm 2 and $T_{\text{inner}} = 1$, we have

$$\|x_{k+1} - x^*\|^2 \leq \left( \frac{2^{p}(5p-2)}{p!} \right) \|x_k - x^*\|^{p+1},$$

which implies that

$$\|x_{k+1} - x^*\| \leq \sqrt{\frac{2^{p}(5p-2)}{p!}} \|x_k - x^*\|^{\frac{p+1}{2}}.$$

For the case of $p \geq 2$, we have $\frac{p+1}{2} \geq \frac{3}{2}$ and $p - 1 \geq 1$. If the following condition holds true,

$$\|x_0 - x^*\| \leq \frac{1}{2} \left( \frac{p!}{2^{p}(5p-2)\mu} \right)^{\frac{1}{p+1}},$$

23
we have
\[
\left( \frac{2^{p}(5p-2)\kappa}{p!} \right)^{\frac{1}{p-1}} \| x_{k+1} - x^* \| \leq \left( \frac{2^{p}(5p-2)\kappa}{p!} \right)^{\frac{p+1}{p}} \| x_k - x^* \|^{\frac{p+1}{2}}
\]
\[
= \left( \left( \frac{2^{p}(5p-2)\kappa}{p!} \right)^{\frac{1}{p-1}} \| x_k - x^* \| \right)^{\frac{p+1}{p}} \leq \left( \frac{2^{p}(5p-2)\kappa}{p!} \right)^{\frac{1}{p-1}} \| x_0 - x^* \|^{\frac{p+1}{2}} \leq \left( \frac{1}{2} \right)^{(\frac{p+1}{2})^{k+1}}.
\]
This completes the proof.

### 4.5 Proof of Theorem 3.7

We see from the second inequality in Lemma 4.2 that
\[
\min_{1 \leq k \leq T} \| x_k - v_k \|^2 \leq \frac{1}{T} \sum_{k=1}^{T} \| x_k - v_k \|^2 \leq \frac{4\| x^*-x_0 \|^2}{T}.
\]
By the definition of $x_{k_T}$ (i.e., $\text{opt} = 2$), we have
\[
\| x_{k_T} - v_{k_T} \|^2 \leq \frac{4\| x^*-x_0 \|^2}{T}.
\]
(23)

Recalling that $x_{k+1} \in \mathcal{X}$ satisfies Eq. (7), we have
\[
\langle F_{v_k}(x_k), x - x_k \rangle \geq -\frac{L}{2p} \| x_k - v_k \|^{p+1}, \text{ for all } x \in \mathcal{X},
\]
where $F_v(x) : \mathbb{R}^d \to \mathbb{R}^d$ is defined for any fixed $v \in \mathcal{X}$ as follows:
\[
F_{v_k}(x) = F(v_k) + \langle \nabla F(v_k), x - v_k \rangle + \ldots + \frac{1}{(p-1)!} \nabla^{(p-1)} F(v_k)[x - v_k]^{p-1} + \frac{5L}{(p-1)!} \| x - v_k \|^{p-1}(x - v_k).
\]

Under Assumption 2.3, we have Eq. (15) which further leads to
\[
\| F(x_k) - F_{v_k}(x_k) \| \leq \frac{(5p+1)L}{p!} \| x_k - v_k \|^p.
\]

Putting these pieces together yields that
\[
\langle F(x_k), x_k - x \rangle = \langle F(x_k) - F_{v_k}(x_k), x_k - x \rangle + \langle F_{v_k}(x_k), x_k - x \rangle
\]
\[
\leq \| F(x_k) - F_{v_k}(x_k) \| \| x_k - x \| + \frac{L}{p!} \| x_k - v_k \|^{p+1}
\]
\[
\leq \frac{L}{p!} \| x_k - v_k \|^p ( (5p+1) \| x_k - x \| + \| x_k - v_k \| ), \text{ for all } x \in \mathcal{X}.
\]

This implies that (for all $x \in \mathcal{X}$)
\[
\langle F(x_k), x_k - x \rangle \leq \frac{(5p+1)L}{p!} \| x_k - v_k \|^p \| x_k - x \| + \frac{L}{p!} \| x_k - v_k \|^{p+1}. \tag{24}
\]

Then, we derive from the fact that $\| x_k - x \| \leq D$ and $\| x_k - v_k \| \leq D$ that
\[
\langle F(x_k), x_k - x \rangle \leq \frac{(5p+2)L D}{p!} \| x_k - v_k \|^p, \text{ for all } x \in \mathcal{X}.
\]
By the definition of a residue function (see Eq. (5)), we have
\[
\text{RES}(x_{k_T}) = \sup_{x \in \mathcal{X}} (F(x_{k_T}), x_{k_T} - x) \leq \frac{(5p+2)LD}{p!} \|x_{k_T} - v_{k_T}\|^p
\]
(see Eq. (23))
\[
\leq \frac{(5p+2)LD}{p!} \left( \frac{4\|x^* - x_0\|^2}{\eta^2} \right)^{\frac{p}{2}}.
\]
Since \(x_0, x^* \in \mathcal{X}\), we have \(\|x^* - x_0\| \leq D\) and hence
\[
\text{RES}(x_{k_T}) \leq \frac{2^p(5p+2)}{p!} LD^{p+1} T^{-\frac{p}{2}}. \tag{25}
\]
Therefore, we conclude from Eq. (25) that we can set
\[
T = O \left( \left( \frac{LD^{p+1}}{\epsilon} \right)^{\frac{2}{p}} \right),
\]
such that \(\hat{x} = \text{Perseus}(p, x_0, L, T, 1)\) satisfies \(\text{RES}(\hat{x}) \leq \epsilon\). The total number of calls of the subproblem solvers is equal to \(T\) since our algorithm calls the subproblem solvers once at each iteration. This completes the proof.

### 4.6 Proof of Theorem 3.11

We first construct a hard function instance for any \(p\)th-order method that satisfies Assumption 3.10. The basic function that we will use is as follows:
\[
\eta(z, y) = \frac{1}{p} \sum_{i=1}^{d} (z^{(i)})^p \cdot y^{(i)}, \quad z \in \mathbb{R}^d, \ y \in \mathbb{R}^d.
\]
Fixing \((z, y) \in \mathbb{R}_+^d \times \mathbb{R}_+^d\), \((h_1, h_2) \in \mathbb{R}^d \times \mathbb{R}^d\) and \(1 \leq m + n \leq p\), we have
\[
\nabla^{(m,n)} \eta(z, y)[h_1]^m[h_2]^n = \frac{(p-1)!}{(p-m)!} \cdot \left\{ \begin{array}{ll}
\sum_{i=1}^{d} (z^{(i)})^{p-m} y^{(i)} (h_1^{(i)})^m, & \text{if } n = 0, \\
\sum_{i=1}^{d} (z^{(i)})^{p-m} (h_1^{(i)})^m h_2^{(i)}, & \text{if } n = 1, \\
0, & \text{otherwise.}
\end{array} \right. \tag{26}
\]
Note that \(T \geq 1\) is an integer-valued parameter and \(d \geq 4T + 1\). We now define the following \(4T \times 4T\) triangular matrix with two nonzero diagonals [Nesterov, 2021b]:
\[
U = \begin{bmatrix}
1 & -1 & 0 & \ldots & 0 \\
0 & 1 & -1 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & 1 & -1 \\
0 & 0 & \ldots & 0 & 1
\end{bmatrix}, \quad U^{-1} = \begin{bmatrix}
1 & 1 & 1 & \ldots & 1 \\
0 & 1 & 1 & \ldots & 1 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & 1 & 1 \\
0 & 0 & \ldots & 0 & 1
\end{bmatrix}, \quad U^T = \begin{bmatrix}
1 & 0 & \ldots & 0 & 0 \\
0 & 1 & \ldots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & 1 & 0 \\
0 & 0 & \ldots & -1 & 1
\end{bmatrix}.
\]
Now, we introduce \(d \times d\) upper triangular matrix \(A\) with the following structure:
\[
A = \begin{bmatrix}
U & 0 \\
0 & I_{d-4T}
\end{bmatrix}.
\]
We are now ready to characterize a novel hard function and the corresponding two constraint sets:

\[
    f(z, y) = \frac{f}{2^{p+1}p!} \left( \eta(Az, y) - \frac{1}{p(p+1)} \sum_{i=2}^{4T} (y^{(i)})^{p+1} - (z^{(1)})^{4T + \frac{1}{p}} \cdot y^{(1)} \right),
\]

\[
    \mathcal{Z} = \left\{ z \in \mathbb{R}^d : \begin{array}{l}
    0 \leq z^{(i)} \leq 4T - i + 1 \\
    \text{and } z^{(i)} = 0 \text{ for all } i > 4T
    \end{array} \right\},
\]

\[
    \mathcal{Y} = \{ y \in \mathbb{R}^d : 0 \leq y^{(i)} \leq 1 \text{ for all } 1 \leq i \leq 4T \text{ and } y^{(i)} = 0 \text{ for all } i > 4T \}.
\]

If \( m + n = p \), Eq. (26) implies that the only nonzero \((m, n)\)th-order derivatives of \( f \) are \( \nabla^{(0,p)} f(z, y) \), \( \nabla^{(p-1,1)} f(z, y) \) and \( \nabla^{(p,0)} f(z, y) \). It is also clear that the function \( f : \mathcal{Z} \times \mathcal{Y} \to \mathbb{R} \) is convex in \( z \) and concave in \( y \). This implies that the computation of an optimal saddle-point solution of \( f(z, y) \) is equivalent to solving a monotone VI with \( x = \begin{bmatrix} z \\ y \end{bmatrix} \) and

\[
    F(x) = \begin{bmatrix}
    \nabla^{(1,0)} f(z, y) \\
    -\nabla^{(0,1)} f(z, y)
    \end{bmatrix} = \frac{f}{2^{p+1}p!} \cdot \begin{bmatrix}
    \nabla^{(1,0)} \eta(Az, y) - y^{(1)} \cdot e^{(1)}_d \\
    -\nabla^{(0,1)} \eta(Az, y) + \frac{1}{p} \sum_{i=2}^{4T} (y^{(i)})^p \cdot e^{(i)}_d + (z^{(1)})^{4T + \frac{1}{p}} \cdot e^{(1)}_d
    \end{bmatrix}.
\]

**Step 1.** We show that \( F : \mathcal{Z} \times \mathcal{Y} \to \mathbb{R}^{2d} \) is \((p - 1)\)th-order smooth with a Lipschitz constant \( L > 0 \). Indeed, we have

\[
    \|\nabla^{(p-1)} F(x) - \nabla^{(p-1)} F(x')\|_{\text{op}} \leq \|\nabla^{(0,p)} f(z, y) - \nabla^{(0,p)} f(z', y')\|_{\text{op}} + p\|\nabla^{(p-1,1)} f(z, y) - \nabla^{(p-1,1)} f(z', y')\|_{\text{op}} + \|\nabla^{(p,0)} f(z, y) - \nabla^{(p,0)} f(z', y')\|_{\text{op}}.
\]

We let \( h = (h_1, h_2) \in \mathbb{R}^d \times \mathbb{R}^d \) and have

\[
    \nabla^{(0,p)} f(z, y)[h_2]^p = -\frac{f}{2^{p+1}p!} \cdot \sum_{i=2}^{4T} (y^{(i)})^i h_2^{(i)},
\]

and

\[
    \nabla^{(p-1,1)} f(z, y)[h_1]^{p-1} h_2 = \begin{cases} 
    \frac{f}{L} \cdot (\nabla^{(1,1)} \eta(Az, y)[Ah_1][h_2] - h_1^{(1)} h_2^{(1)}), & \text{if } p = 2, \\
    \frac{f}{L^{2p+1}p!} \cdot (\nabla^{(p-1,1)} \eta(Az, y)[Ah_1][p-1] h_2), & \text{otherwise},
    \end{cases}
\]

and

\[
    \nabla^{(p,0)} f(z, y)[h_1]^p = \frac{f}{2^{p+1}p!} \cdot \nabla^{(p,0)} \eta(Az, y)[Ah_1]^p.
\]

By the Cauchy-Schwartz inequality and \( \|A\| \leq 2 \) (see Nesterov [2021b, Eq. (4.2)]), we have

\[
    \|\nabla^{(0,p)} f(z, y) - \nabla^{(0,p)} f(z', y')\|_{\text{op}} \leq \sup_{\|h\| = 1} \left\{ \|\nabla^{(0,p)} f(z, y)[h_2]^p - \nabla^{(0,p)} f(z', y')[h_2]^p\| \right\}
\]

\[
    \leq \sup_{\|h\| = 1} \left\{ \frac{f}{2^{p+1}p!} \cdot \|y - y'\|\|h_2\|^p \right\} \leq \frac{f}{16} \cdot \|x - x'\|,
\]
Plugging the above equation into Eq. \((f)\) yields the desired result.

\[ Eq. \ (26) \] and \( \|A\| \leq 2 \]
\[ \sup_{\|h\|=1} \left\{ \frac{L}{2^{p+1}p!} \cdot \left( \|\nabla^{(p-1,1)} f(z, y)[h_1]^{p-1}[h_2]\| \right) \right\} \]
\[ \leq \frac{L}{2^{p+1}p} \cdot 2^p \cdot \|z - z'\| \| h_1 \|^{p-1} \| h_2 \| \leq \frac{L}{2^{p}} \cdot \|x - x'\|, \]
and
\[ Eq. \ (26) \] and \( \|A\| \leq 2 \]
\[ \sup_{\|h\|=1} \left\{ \frac{L}{2^{p+1}p} \cdot 2^p \cdot \|y - y'\| \| h_1 \|^{p} \right\} \]
\[ \leq \frac{L}{2^{p}} \cdot \|x - x'\| \leq \frac{L}{4} \cdot \|x - x'\|.
\]

Plugging the above equation into Eq. \((27)\) yields the desired result.

**Step 2.** We show that there exists an optimal solution \(x_\star = (z_\star, y_\star) \in \mathcal{Z} \times \mathcal{Y}\) such that \(F(x_\star) = 0_{2d}\) and compute the optimal value of \(f(z_\star, y_\star)\). By the definition, we have \(F(x_\star) = 0_{2d}\) is equivalent to the following statement:

\[
\begin{align*}
A^T \nabla^{(1,0)} \eta(Az_\star, y_\star) - y_\star^{(1)} \cdot e_d^{(1)} &= 0_d, \\
\nabla^{(0,1)} \eta(Az_\star, y_\star) - \frac{1}{p} \sum_{i=2}^{4T} (y_\star^{(i)})^p \cdot e_d^{(i)} - (z_\star - 4T + \frac{1}{p}) \cdot e_d^{(1)} &= 0_d.
\end{align*}
\]

\[(28)\]

Note that
\[
\begin{align*}
\nabla^{(1,0)} \eta(Az_\star, y_\star) &= \sum_{i=1}^{d} ((Az_\star)^{(i)})^{p-1} y_\star^{(i)} e_d^{(i)}, \\
\nabla^{(0,1)} \eta(Az_\star, y_\star) &= \frac{1}{p} \left( \sum_{i=1}^{d} ((Az_\star)^{(i)})^p e_d^{(i)} \right).
\end{align*}
\]

We claim that an optimal solution \(x_\star = (z_\star, y_\star)\) is given by

\[
\begin{align*}
z_\star^{(i)} &= \begin{cases} 4T - i + 1, & \text{if } 1 \leq i \leq 4T, \\ 0 & \text{otherwise.} \end{cases} \quad y_\star^{(i)} &= \begin{cases} 1, & \text{if } 1 \leq i \leq 4T, \\ 0 & \text{otherwise.} \end{cases}
\end{align*}
\]

\[(29)\]

Indeed, we can see from the definition of \(\mathcal{Z} \times \mathcal{Y}\) that \((z_\star, y_\star) \in \mathcal{Z} \times \mathcal{Y}\) and the definition of \(A\) that
\[
(Az_\star)^{(i)} = \begin{cases} 1, & \text{if } 1 \leq i \leq 4T, \\ 0 & \text{otherwise.} \end{cases}
\]

\[ 27 \]
This implies that
\[
\nabla^{(1,0)} \eta(Az_*, y_*) = \sum_{i=1}^{4T} e_d^{(i)}, \quad \nabla^{(0,1)} \eta(Az_*, y_*) = \frac{1}{p} \left( \sum_{i=1}^{4T} e_d^{(i)} \right).
\]

By the definition of \(A\), we have \(A^T \nabla^{(1,0)} \eta(Az_*, y_*) = e_d^{(1)}\). As such, we can verify that Eq. \(28\) holds true. As such, we conclude that the optimal solution \(z_* = (z_*, y_*)\) defined in Eq. \(29\) belongs to \(Z \times \mathcal{Y}\) and the optimal value is
\[
f(z_*, y_*) = \frac{L}{2^{p+1}p!} \left( \eta(Az_*, y_*) - \frac{1}{p(p+1)} \sum_{i=2}^{4T} (y_*)^{p+1} - (z_*)^{p+1} - 4T + \frac{1}{p} \right).
\]

This implies the desired result.

**Step 3.** We now proceed to investigate the dynamics of any \(p\)-th-order method under Assumption 3.10. For simplicity, we denote
\[
\mathbb{R}^d_k = \{ z \in \mathbb{R}^d : z^{(i)} = 0 \text{ for all } i = k+1, k+2, \ldots, d \}, \quad \text{for all } 1 \leq k \leq d-1.
\]

Without loss of generality, we assume that \(x_0 = 0_{2d}\) is the initial iterate. Then, we show that the iterates \(\{(z_k, y_k)\}_{k \geq 0}\) generated by any \(p\)-th-order method under Assumption 3.10 satisfy
\[
z_k \in \mathbb{R}^d_{2k} \cap Z, \quad \text{for all } 1 \leq k \leq T. \tag{30}
\]

It is clear that \(z_k \in Z\) for all \(1 \leq k \leq T\) since Assumption 3.10 guarantees that \(\{(z_k, y_k)\}_{k \geq 1} \subseteq X\). Thus, it suffices to show that \(z_k \in \mathbb{R}^d_{2k}\) for all \(1 \leq k \leq T\).

First of all, we prove that \(\vec{x} = (\vec{z}, \vec{y}) \in \mathbb{R}^d_k \times \mathbb{R}^d_k\) with \(1 \leq k \leq 2T - 1\) implies that \(\nabla^{(j)} F(\vec{x})[h] = \mathbb{R}^d_{k+1} \times \mathbb{R}^d_{k+1}\) for all \(0 \leq j \leq p - 1\) and any vector \(h \in \mathbb{R}^{2d}\). In particular, we have \(A\) is an upper triangular matrix and \(\vec{z} \in \mathbb{R}^d_k\). Thus, \(A \vec{z} \in \mathbb{R}^d_k\). In addition, we have \(\vec{y} \in \mathbb{R}^d_k\). Recall that we have derived in Eq. \(26\) that
\[
\nabla^{(m,n)} \eta(z, y)[h_1]^m[h_2]^n = \begin{cases} \sum_{i=1}^d (z^{(i)})^{p-m}(h_1^{(i)})^m h_2^{(i)}, & \text{if } n = 1, \\
0, & \text{otherwise.}
\end{cases}
\]

Putting these pieces together yields
\[
\frac{\partial}{\partial h_1} (\nabla^{(m,n)} \eta(A \vec{z}, \vec{y})[Ah_1]^m[h_2]^n) = \sum_{i=1}^k c_i^{(m,n)} A^T e_d^{(i)},
\]

\[
\frac{\partial}{\partial h_2} (\nabla^{(m,n)} \eta(A \vec{z}, \vec{y})[h_1]^m[h_2]^{n-1}) = \sum_{i=1}^k d_i^{(m,n)} e_d^{(i)}.
\]
where \( c_i^{(m,n)} \) and \( d_i^{(m,n)} \) are certain coefficients for \( 1 \leq m + n \leq p \) and \( 1 \leq i \leq k \) (these parameters are defined in an implicit form as in Nesterov [2021b]). Thus, we have

\[
\frac{\partial}{\partial t_1} \left( \nabla^{(m,n)} \eta(A \bar{z}, \bar{y}) [Ah_1]^{m} [h_2]^n \right) \in \mathbb{R}^{d_{k+1}}, \quad \frac{\partial}{\partial t_2} \left( \nabla^{(m,n)} \eta(A \bar{z}, \bar{y}) [h_1]^{m} [h_2]^{n-1} \right) \in \mathbb{R}^{d_k} \subseteq \mathbb{R}^{d_{k+1}}.
\]

Let us compute \( F(\bar{x}) \) and \( \nabla F(\bar{x})[h] \) explicitly. We have

\[
F(\bar{x}) = \frac{L}{2^{p+1} p!} \cdot \left[ \frac{\partial}{\partial t_1} \left( \nabla^{(1,0)} \eta(A \bar{z}, \bar{y}) [Ah_1]^{2} \right) - \bar{y}^{(1)} \bar{e}_d^{(1)} \right] - \frac{\partial}{\partial t_2} \left( \nabla^{(0,1)} \eta(A \bar{z}, \bar{y}) [h_2] \right) + \frac{1}{p} \sum_{i=2}^{2T} (\bar{y}^{(i)} p \cdot \bar{e}_d^{(i)}) + (\bar{z} - 2T + \frac{1}{p} \bar{e}_d^{(1)}).
\]

and

\[
\nabla F(\bar{x})[h] = \frac{L}{2^{p+1} p!} \left[ \frac{\partial}{\partial t_1} \left( \nabla^{(2,0)} \eta(A \bar{z}, \bar{y}) [Ah_1]^{2} \right) + \frac{\partial}{\partial t_2} \left( \nabla^{(1,1)} \eta(A \bar{z}, \bar{y}) [Ah_1] [h_2] \right) - \bar{e}_d^{(1)} \right] - \frac{\partial}{\partial t_1} \left( \nabla^{(1,1)} \eta(A \bar{z}, \bar{y}) [Ah_1] [h_2] \right) + \bar{h}_1^{(1)} \bar{e}_d^{(1)} - \frac{\partial}{\partial t_2} \left( \nabla^{(0,2)} \eta(A \bar{z}, \bar{y}) [h_2]^{2} \right) + \sum_{i=2}^{2T} (\bar{y}^{(i)} p \cdot \bar{e}_d^{(i)} - \bar{h}_2^{(1)} \bar{e}_d^{(i)}).
\]

This together with

\[
\frac{\partial}{\partial t_1} \left( \nabla^{(m,n)} \eta(A \bar{z}, \bar{y}) [Ah_1]^{m} [h_2]^n \right), \quad \frac{\partial}{\partial t_2} \left( \nabla^{(m,n)} \eta(A \bar{z}, \bar{y}) [h_1]^{m} [h_2]^{n-1} \right) \in \mathbb{R}^{d_{k+1}}
\]

yields \( F(\bar{x}), \nabla F(\bar{x})[h] \in \mathbb{R}^{d_{k+1}} \times \mathbb{R}^{d_{k+1}} \) for any \( h \in \mathbb{R}^{2d} \). Similarly, we have

\[
\nabla^{(j)} F(\bar{x})[h]^j \in \mathbb{R}^{d_{k+1}} \times \mathbb{R}^{d_{k+1}}, \quad \text{for all } 0 \leq j \leq p - 1 \text{ and any } h \in \mathbb{R}^{2d}.
\]

Second, we prove that \( x_k = (z_k, y_k) \in \mathbb{R}^{d_k} \times \mathbb{R}^{d_k} \) with \( 1 \leq k \leq 2T - 1 \) and \( \nabla^{(j)} F(\bar{x})[h]^j \in \mathbb{R}^{d_{k+1}} \times \mathbb{R}^{d_{k+1}} \) for all \( 0 \leq j \leq p - 1 \) and any vector \( h \in \mathbb{R}^{2d} \) implies that \( x_{k+1} = (z_{k+1}, y_{k+1}) \in \mathbb{R}^{d_{k+1}} \times \mathbb{R}^{d_{k+1}} \). Indeed, we recall from Assumption 3.10 that: for all \( g \geq 0 \), we have that \( x_{k+1} \in \mathcal{X} \) satisfies that \( \langle \Phi_{a,\gamma,\bar{x}}(x_{k+1} - x_k), x - x_{k+1} \rangle \geq 0 \) for all \( x \in \mathcal{X} \), where \( \Phi_{a,\gamma,\bar{x}}(\cdot) \) is defined by

\[
\Phi_{a,\gamma,\bar{x}}(h) := a_0 F(\bar{x}) + \sum_{i=1}^{p-1} a_i \nabla^{(i)} F(\bar{x})[h]^i + \gamma \| h \|^{p-1} h.
\]

Since \( \nabla^{(j)} F(\bar{x})[h]^j \in \mathbb{R}^{d_{k+1}} \times \mathbb{R}^{d_{k+1}} \) for all \( 0 \leq j \leq p - 1 \) and any vector \( h \in \mathbb{R}^{2d} \), we have \( g := a_0 F(\bar{x}) + \sum_{i=1}^{p-1} a_i \nabla^{(i)} F(\bar{x})[h]^i \in \mathbb{R}^{d_{k+1}} \times \mathbb{R}^{d_{k+1}} \). This implies that \( x_{k+1} \in \mathcal{X} \) satisfies that

\[
\langle g + \gamma \| x_{k+1} - x_k \|^{p-1} (x_{k+1} - x_k), x - x_{k+1} \rangle \geq 0 \text{ for all } x \in \mathcal{X}, \quad (31)
\]

where \( x_k \in \mathbb{R}^{d_k} \times \mathbb{R}^{d_k}, g \in \mathbb{R}^{d_{k+1}} \times \mathbb{R}^{d_{k+1}} \) and \( \gamma > 0 \). Here, \( \mathcal{X} = \mathcal{Z} \times \mathcal{Y} \) where

\[
\mathcal{Z} = \left\{ z \in \mathbb{R}^d : 0 \leq z^{(i)} \leq 4T - i + 1 \text{ and } z^{(i+1)} \leq z^{(i)} \text{ for all } 1 \leq i \leq 4T \text{ and } z^{(i)} = 0 \text{ for all } i > 4T \right\},
\]

\[
\mathcal{Y} = \left\{ y \in \mathbb{R}^d : 0 \leq y^{(i)} \leq 1 \text{ for all } 1 \leq i \leq 4T \text{ and } y^{(i)} = 0 \text{ for all } i > 4T \right\}.
\]

We claim that \( x_{k+1} = (z_{k+1}, y_{k+1}) \in \mathbb{R}^{d_{k+1}} \times \mathbb{R}^{d_{k+1}} \) and consider using a proof by contradiction. Indeed, by definition, we have \( x_{k+1} \in \mathcal{X} = \mathcal{Z} \times \mathcal{Y} \). Thus, if \( x_{k+1} = (z_{k+1}, y_{k+1}) \notin \mathbb{R}^{d_{k+1}} \times \mathbb{R}^{d_{k+1}} \), we have that either \( z_{k+1} \notin \mathbb{R}^{d_{k+1}} \) or \( y_{k+1} \notin \mathbb{R}^{d_{k+1}} \). We study these two cases separately as follows:
For the former case, we have $z_{k+1}^{(i+1)} \leq z_{k+1}^{(i)}$ for all $1 \leq i \leq 2T$. This together with $z_{k+1} \notin \mathbb{R}_k^{d}$ implies that there must exist $j \geq k + 2$ such that $z_{k+1}^{(j)} > 0$ and $z_{k+1}^{(j+1)} = 0$. Since $x_k \in \mathbb{R}_k^{d} \times \mathbb{R}_k^{d}$ and $g \in \mathbb{R}_{k+1}^{d} \times \mathbb{R}_{k+1}^{d}$, the $j$th element of $g + \gamma \|x_{k+1} - x_{k}\|^{p-1}(x_{k+1} - x_{k})$ is strictly positive. We can let $x = x_{k+1}$ except for the $j$th element being 0. Then, it is clear that $x \in \mathcal{X}$ and

$$
(g + \gamma \|x_{k+1} - x_{k}\|^{p-1}(x_{k+1} - x_{k}), x - x_{k+1}) < 0,
$$

which violates Eq. (31) and thus contradicts the definition of $x_{k+1}$. The similar argument is valid for the latter case where $y_{k+1} \notin \mathbb{R}_{k+1}^{d}$ is assumed and leads to the same contradiction. Putting these pieces yields the desired result.

Finally, we prove that $x_k \in \mathbb{R}_k^{d} \times \mathbb{R}_k^{d}$ for all $0 \leq k \leq T$ using an inductive argument. For the case of $k = 0$, we have $x_0 = \mathbf{0}_2d \in \mathbb{R}_0^{d} \times \mathbb{R}_0^{d}$. For the case of $k = 1$, we have $F(x_0) \in \mathbb{R}_1^{d} \times \mathbb{R}_1^{d}$ which further implies that $s \in \text{Lin}(F(x_0)) \subseteq \mathbb{R}_1^{d} \times \mathbb{R}_1^{d}$ and $\bar{x} = \mathcal{P}_\mathcal{X}(x_0 + s) \in \mathbb{R}_1^{d} \times \mathbb{R}_1^{d}$. Then, we have $\nabla^{(j)}F(\bar{x})|_{j} \in \mathbb{R}_1^{d} \times \mathbb{R}_1^{d}$ for all $0 \leq j \leq p - 1$ and any vector $h \in \mathbb{R}_1^{d}$. The same argument holds for the case of $k = 2$. Indeed, $F(x_0) \in \mathbb{R}_1^{d} \times \mathbb{R}_1^{d}$ and $F(x_1) \in \mathbb{R}_3^{d} \times \mathbb{R}_3^{d}$. Thus, we have $s \in \text{Lin}(F(x_0), F(x_1)) \subseteq \mathbb{R}_3^{d} \times \mathbb{R}_3^{d}$ and $\bar{x} \in \mathbb{R}_3^{d} \times \mathbb{R}_3^{d}$. This implies that $\nabla^{(j)}F(\bar{x})|_{j} \in \mathbb{R}_3^{d} \times \mathbb{R}_3^{d}$ for all $0 \leq j \leq p - 1$ and any vector $h \in \mathbb{R}_3^{d}$. Since $x_1 \in \mathbb{R}_3^{d} \times \mathbb{R}_3^{d}$, we have $x_2 \in \mathbb{R}_3^{d} \times \mathbb{R}_3^{d}$. Repeating these arguments yields that $x_k \in \mathbb{R}_k^{d} \times \mathbb{R}_k^{d}$ for all $1 \leq k \leq T$. Thus, $z_k \in \mathbb{R}_k^{d} \times \mathbb{R}_k^{d}$ for all $1 \leq k \leq T$.

**Step 4.** We now compute a lower bound on $\max_{y \in \mathcal{W}} f(z_k, y)$. Eq. (30) in **Step 3** implies that

$$
\max_{y \in \mathcal{W}} f(z_k, y) \geq \min_{z \in \mathbb{R}_k^{d} \times \mathbb{R}_k^{d}} f(z, y) \geq \min_{z \in \mathbb{R}_k^{d} \times \mathbb{R}_k^{d}} \max_{y \in \mathcal{W}} f(z, y), \quad \text{for all } 0 \leq k \leq T.
$$

We claim that $\min_{z \in \mathbb{R}_k^{d} \times \mathbb{R}_k^{d}} \max_{y \in \mathcal{W}} f(z_k, y) \geq \frac{L}{2^{p+1}p!} (2T + \frac{2T-1}{p+1})$. We let $\mathcal{W} = \{w \in \mathbb{R}_k^{d} : w(i) \geq 0 \text{ for all } 1 \leq i \leq 4T \text{ and } w(i) = 0 \text{ for all } i > 4T\}$ and derive that

$$
\max_{y \in \mathcal{W}} f(z_k, y) \geq \min_{w \in \mathbb{R}_k^{d} \cap \mathcal{W}} \max_{y \in \mathcal{W}} \frac{L}{2^{p+1}p!} \left( \eta(w, y) - \frac{1}{p(p+1)} \sum_{i=2}^{4T} (y(i))^{p+1} - \left( \sum_{i=1}^{d} w(i) - 4T + \frac{1}{p} \right) \cdot y(1) \right).
$$

We see from $w \in \mathbb{R}_k^{d} \cap \mathcal{W}$ that $w(i) \geq 0$ for all $1 \leq i \leq 2T$ and $w(i) = 0$ for all $2T + 1 \leq i \leq 4T$. Fixing $w \in \mathbb{R}_k^{d} \cap \mathcal{W}$, we have

$$
\max_{y \in \mathcal{W}} \left( \frac{1}{p} \sum_{i=1}^{d} (w(i))^p \cdot y(i) - \frac{1}{p(p+1)} \sum_{i=2}^{4T} (y(i))^{p+1} - \left( \sum_{i=1}^{d} w(i) - 4T + \frac{1}{p} \right) \cdot y(1) \right).
$$

We write

$$
= \max \left\{ \frac{1}{p} (w(1))^p - \left( \sum_{i=1}^{2T} w(i) - 4T + \frac{1}{p} \right), 0 \right\} + \frac{1}{p} \sum_{i=2}^{2T} (w(i))^p \cdot \min\{w(i), 1\} - \frac{1}{p(p+1)} \sum_{i=2}^{2T} (\min\{w(i), 1\})^{p+1}.
$$
The key observation is that the second and third terms are independent of \(w^{(1)}\) on the right-hand side. We also have
\[
\min_{w^{(1)} \geq 0} \max \left\{ \frac{1}{p} (w^{(1)})^p - \left( \sum_{i=1}^{2T} w^{(i)} - 4T + \frac{1}{p} \right), 0 \right\}
\]
\[
= \min_{w^{(1)} \geq 0} \max \left\{ \frac{1}{p} (w^{(1)})^p - w^{(1)} - \sum_{i=2}^{2T} w^{(i)} + 4T - \frac{1}{p}, 0 \right\} \geq \max \left\{ 4T - 1 - \sum_{i=2}^{2T} w^{(i)}, 0 \right\}.
\]
For simplicity, we define the function \(g(w)\) as follows,
\[
g(w) = \max \left\{ 4T - 1 - \sum_{i=2}^{2T} w^{(i)}, 0 \right\} + \frac{1}{p} \sum_{i=2}^{2T} (w^{(i)})^p \cdot \min\{w^{(i)}, 1\} - \frac{1}{p(p+1)} \sum_{i=2}^{2T} (\min\{w^{(i)}, 1\})^{p+1}.
\]
Since the function \(g(\cdot)\) is convex and symmetric, we have that \(\min_{w \in \mathbb{R}_{\geq 0}^2 \cap W} g(w)\) is achieved by the point with the same value of \(w^{(i)}\) for all \(2 \leq i \leq 2T\). Then, it suffices to solve the following one-dimensional optimization problem:
\[
\min_{\eta \geq 0} h(\eta) = \max\{4T - 1 - \eta(2T - 1), 0\} + \frac{2T-1}{p} \eta^{p+1} \geq 2T + \frac{2T-1}{p+1}.
\]
For the case of \(0 \leq \eta \leq 1\), we have
\[
h(\eta) = 4T - 1 - \eta(2T - 1) + \frac{2T-1}{p} \eta^{p+1} \geq 2T + \frac{2T-1}{p+1}.
\]
For the case of \(1 \leq \eta \leq \frac{4T-1}{2T-1}\), we have
\[
h(\eta) = 4T - 1 - \eta(2T - 1) + \frac{2T-1}{p} \eta^{p} - \frac{2T-1}{p(p+1)} \geq 2T + \frac{2T-1}{p+1}.
\]
For the case of \(\eta \geq \frac{4T-1}{2T-1}\), we have \(\eta^{p} \geq 1 + p(\eta - 1) \geq 1 + \frac{2Tp}{2T-1}\). Then, we have
\[
h(\eta) = \frac{2T-1}{p} \eta^{p} - \frac{2T-1}{p(p+1)} \geq \frac{2T-1}{2T-1} \left[ 1 + \frac{2Tp}{2T-1} \right] - \frac{2T-1}{p(p+1)} \geq 2T + \frac{2T-1}{p+1}.
\]
Putting these pieces together yields that
\[
\min_{z \in \mathbb{R}_{\geq 0}^2 \cap \mathbb{Z}} \max_{y \in \mathcal{Y}} f(z, y) \geq \frac{L}{2^{p+1}} \left[ 2T + \frac{2T-1}{p+1} \right],
\]
which implies the desired result.

**Final Step.** Since the point \((z_*, y_*) \in \mathcal{Z} \times \mathcal{Y}\) is an optimal saddle-point solution, we have
\[
\max_{y \in \mathcal{Y}} f(z_k, y) - \min_{z \in \mathcal{Z}} f(z, y_k) \geq \max_{y \in \mathcal{Y}} f(z_k, y) - f(z_*, y_*).
\]
Combining the results from **Step 2** and **Step 4**, we have
\[
\min_{0 \leq k \leq T} \left\{ \max_{y \in \mathcal{Y}} f(z_k, y) - \min_{z \in \mathcal{Z}} f(z, y_k) \right\} \geq \frac{L}{2^{p+1}} \left( 2T - \frac{2T}{p+1} \right)^{p \geq 2} \geq \frac{2TL}{2^{p(p+1)!}}.
\]
Note that we set \(D_\mathcal{Z} = 8T^{3/2}\) and \(D_\mathcal{Y} = 2\sqrt{T}\) (cf. the definition of \(\mathcal{Z}\) and \(\mathcal{Y}\)) and have \(D_\mathcal{Z} D_\mathcal{Y}^p = 2^{p+3} T^{(p+3)/2}\). Then, we have
\[
\min_{0 \leq k \leq T} \left\{ \max_{y \in \mathcal{Y}} f(z_k, y) - \min_{z \in \mathcal{Z}} f(z, y_k) \right\} \geq \left( \frac{1}{4^{p+1}} \right) LD_\mathcal{Z} D_\mathcal{Y}^p T^{-\frac{p+1}{2}}.
\]
This completes the proof.
5 Conclusions

We have proposed and analyzed a new $p$th-order method—Perseus—for finding a weak solution of smooth and monotone variational inequalities (VIs) when $F$ is $(p - 1)$th-order $L$-smooth. All of our theoretical results are based on the standard assumption that the subproblem arising from a $(p - 1)$th-order Taylor expansion of $F$ can be computed approximately in an efficient manner. For the case of $p \geq 2$, the best existing $p$th-order methods can achieve a global rate of $O(\epsilon^{-2/(p+1)} \log \log(1/\epsilon))$ [Bullins and Lai, 2022, Lin and Jordan, 2023, Jiang and Mokhtari, 2022] but require a nontrivial line-search procedure at each iteration. The open question has been whether it is possible to design a simple and optimal high-order method that achieves a global rate of $O(\epsilon^{-2/(p+1)})$ while dispensing with line search.

Our results settle this open problem. In particular, our $p$th-order method converges to a weak solution with a global rate of $O(\epsilon^{-2/(p+1)})$. A lower bound is proved in the monotone setting under a generalized linear span assumption, showing that our method is optimal in the monotone setting. The restarted versions attain a global linear rate for $p$th-order uniformly monotone VIs and a local superlinear rate for strongly monotone VIs. Moreover, we prove a global rate of $O(\epsilon^{-2/p})$ for solving smooth and nonmonotone VIs satisfying the Minty condition and extend these results under $p$th-order uniform Minty and strong Minty conditions. Future research include the investigation of lower bounds for the nonmonotone setting with the Minty condition and the comparative study of various lower-order methods in high-order smooth VI problems; see Nesterov [2021a,c,d] for recent examples of such comparisons in convex optimization.

Acknowledgments

This work was supported in part by the Mathematical Data Science program of the Office of Naval Research under grant number N00014-18-1-2764 and by the Vannevar Bush Faculty Fellowship program under grant number N00014-21-1-2941.

References

D. Adil, B. Bullins, A. Jambulapati, and S. Sachdeva. Optimal methods for higher-order smooth monotone variational inequalities. ArXiv Preprint: 2205.06167, 2022. (Cited on pages 4, 5, 9, 12, 14, 16, and 17.)

A. S. Antipin. Method of convex programming using a symmetric modification of Lagrange function. Matekon, 14(2):23–38, 1978. (Cited on page 2.)

Y. Arjevani, O. Shamir, and R. Shiff. Oracle complexity of second-order methods for smooth convex optimization. Mathematical Programming, 178(1):327–360, 2019. (Cited on pages 5 and 17.)

M. Baes. Estimate sequence methods: extensions and approximations. Institute for Operations Research, ETH, Zürich, Switzerland, 2009. (Cited on pages 2 and 5.)

H. H. Bauschke and P. L. Combettes. Convex Analysis and Monotone Operator Theory in Hilbert Spaces. Springer, 2017. (Cited on page 4.)
E. G. Birgin, J. L. Gardenghi, J. M. Martinez, S. A. Santos, and P. L. Toint. Evaluation complexity for nonlinear constrained optimization using unscaled KKT conditions and high-order models. *SIAM Journal on Optimization*, 26(2):951–967, 2016. (Cited on page 6.)

E. G. Birgin, J. L. Gardenghi, J. M. Martínez, S. A. Santos, and P. L. Toint. Worst-case evaluation complexity for unconstrained nonlinear optimization using high-order regularized models. *Mathematical Programming*, 163(1-2):359–368, 2017. (Cited on pages 2 and 6.)

L. Brighi and R. John. Characterizations of pseudomonotone maps and economic equilibrium. *Journal of Statistics and Management Systems*, 5(1-3):253–273, 2002. (Cited on page 9.)

B. Bullins. Highly smooth minimization of nonsmooth problems. In *COLT*, pages 988–1030. PMLR, 2020. (Cited on page 6.)

B. Bullins and K. A. Lai. Higher-order methods for convex-concave min-max optimization and monotone variational inequalities. *SIAM Journal on Optimization*, 32(3):2208–2229, 2022. (Cited on pages 1, 3, 5, 12, 14, 15, 17, 19, and 32.)

Y. Carmon and J. Duchi. Gradient descent finds the cubic-regularized nonconvex Newton step. *SIAM Journal on Optimization*, 29(3):2146–2178, 2019. (Cited on page 13.)

Y. Carmon, J. C. Duchi, O. Hinder, and A. Sidford. Lower bounds for finding stationary points I. *Mathematical Programming*, 184(1-2):71–120, 2020. (Cited on page 17.)

Y. Carmon, D. Hausler, A. Jambulapati, Y. Jin, and A. Sidford. Optimal and adaptive Monteiro-Svaiter acceleration. In *NeurIPS*, pages 20338–20350, 2022. (Cited on page 5.)

C. Cartis, N. I. M. Gould, and P. L. Toint. On the complexity of steepest descent, Newton’s and regularized Newton’s methods for nonconvex unconstrained optimization problems. *SIAM Journal on Optimization*, 20(6):2833–2852, 2010. (Cited on page 6.)

C. Cartis, N. I. M. Gould, and P. L. Toint. Adaptive cubic regularisation methods for unconstrained optimization. Part I: motivation, convergence and numerical results. *Mathematical Programming*, 127(2):245–295, 2011a. (Cited on page 6.)

C. Cartis, N. I. M. Gould, and P. L. Toint. Adaptive cubic regularisation methods for unconstrained optimization. Part II: worst-case function-and derivative-evaluation complexity. *Mathematical Programming*, 130(2):295–319, 2011b. (Cited on page 6.)

C. Cartis, N. I. Gould, and P. L. Toint. Universal regularization methods: Varying the power, the smoothness and the accuracy. *SIAM Journal on Optimization*, 29(1):595–615, 2019. (Cited on page 6.)

C. Cartis, N. I. M. Gould, and P. L. Toint. *Evaluation Complexity of Algorithms for Nonconvex Optimization: Theory, Computation and Perspectives*. SIAM, 2022. (Cited on page 13.)

N. Cesa-Bianchi and G. Lugosi. *Prediction, Learning, and Games*. Cambridge University Press, 2006. (Cited on page 2.)

Y. Chen, G. Lan, and Y. Ouyang. Accelerated schemes for a class of variational inequalities. *Mathematical Programming*, 165(1):113–149, 2017. (Cited on pages 2 and 7.)
S. C. Choi, W. S. DeSarbo, and P. T. Harker. Product positioning under price competition. *Management Science*, 36(2):175–199, 1990. (Cited on page 9.)

R. Cottle, F. Giannessi, and J-L. Lions. *Variational Inequalities and Complementarity Problems: Theory and Applications*. John Wiley & Sons, 1980. (Cited on page 2.)

C. D. Dang and G. Lan. On the convergence properties of non-Euclidean extragradient methods for variational inequalities with generalized monotone operators. *Computational Optimization and Applications*, 60(2):277–310, 2015. (Cited on pages 8 and 16.)

C. Daskalakis, S. Skoulakis, and M. Zampetakis. The complexity of constrained min-max optimization. In *STOC*, pages 1466–1478, 2021. (Cited on pages 8 and 12.)

J. Diakonikolas. Halpern iteration for near-optimal and parameter-free monotone inclusions and strong solutions to variational inequalities. In *COLT*, pages 1428–1451. PMLR, 2020. (Cited on pages 9 and 17.)

J. Diakonikolas, C. Daskalakis, and M. I. Jordan. Efficient methods for structured nonconvex-nonconcave min-max optimization. In *AISTATS*, pages 2746–2754. PMLR, 2021. (Cited on page 8.)

N. Doikov and Y. Nesterov. Local convergence of tensor methods. *Mathematical Programming*, 193(1):315–336, 2022. (Cited on page 6.)

C. Ewerhart. Cournot games with biconcave demand. *Games and Economic Behavior*, 85:37–47, 2014. (Cited on page 9.)

F. Facchinei and J-S. Pang. *Finite-Dimensional Variational Inequalities and Complementarity Problems*. Springer Science & Business Media, 2007. (Cited on pages 2, 7, 22, and 23.)

O. Fercoq and Z. Qu. Adaptive restart of accelerated gradient methods under local quadratic growth condition. *IMA Journal of Numerical Analysis*, 39(4):2069–2095, 2019. (Cited on page 14.)

R. M. Freund and H. Lu. New computational guarantees for solving convex optimization problems with first order methods, via a function growth condition measure. *Mathematical Programming*, 170(2):445–477, 2018. (Cited on page 14.)

M. Fukushima. Equivalent differentiable optimization problems and descent methods for asymmetric variational inequality problems. *Mathematical Programming*, 53:99–110, 1992. (Cited on page 2.)

G. Gallego and M. Hu. Dynamic pricing of perishable assets under competition. *Management Science*, 60(5):1241–1259, 2014. (Cited on page 9.)

A. Gasnikov, P. Dvurechensky, E. Gorbunov, E. Vorontsova, D. Selikhanovych, C. A. Uribe, B. Jiang, H. Wang, S. Zhang, S. Bubeck, Q. Jiang, Y. T. Lee, Y. Li, and A. Sidford. Near optimal methods for minimizing convex functions with Lipschitz $p$-th derivatives. In *COLT*, pages 1392–1393. PMLR, 2019. (Cited on pages 5 and 6.)

S. Ghadimi and G. Lan. Optimal stochastic approximation algorithms for strongly convex stochastic composite optimization, II: shrinking procedures and optimal algorithms. *SIAM Journal on Optimization*, 23(4):2061–2089, 2013. (Cited on page 14.)
P. Giselsson and S. Boyd. Monotonicity and restart in fast gradient methods. In CDC, pages 5058–5063. IEEE, 2014. (Cited on page 14.)

I. Goodfellow, J. Pouget-Abadie, M. Mirza, B. Xu, D. Warde-Farley, S. Ozair, A. Courville, and Y. Bengio. Generative adversarial nets. In NIPS, pages 2672–2680, 2014. (Cited on page 2.)

N. I. M Gould, S. Lucidi, M. Roma, and P. L. Toint. Solving the trust-region subproblem using the Lanczos method. SIAM Journal on Optimization, 9(2):504–525, 1999. (Cited on page 13.)

N. I. M. Gould, D. P. Robinson, and H. S. Thorne. On solving trust-region and other regularised subproblems in optimization. Mathematical Programming Computation, 2(1):21–57, 2010. (Cited on page 13.)

G. Grapiglia and Y. Nesterov. Regularized Newton methods for minimizing functions with Hölder continuous Hessians. SIAM Journal on Optimization, 27(1):478–506, 2017. (Cited on page 6.)

G. Grapiglia and Y. Nesterov. Accelerated regularized Newton methods for minimizing composite convex functions. SIAM Journal on Optimization, 29(1):77–99, 2019. (Cited on page 6.)

G. Grapiglia and Y. Nesterov. Tensor methods for minimizing convex functions with Hölder continuous higher-order derivatives. SIAM Journal on Optimization, 30(4):2750–2779, 2020. (Cited on page 6.)

G. Grapiglia and Y. Nesterov. On inexact solution of auxiliary problems in tensor methods for convex optimization. Optimization Methods and Software, 36(1):145–170, 2021. (Cited on page 12.)

G. Grapiglia and Y. Nesterov. Adaptive third-order methods for composite convex optimization. SIAM Journal on Optimization, 33(3):1855–1883, 2023. (Cited on page 5.)

J. H. Hammond and T. L. Magnanti. Generalized descent methods for asymmetric systems of equations. Mathematics of Operations Research, 12(4):678–699, 1987. (Cited on page 2.)

P. T. Harker and J-S. Pang. Finite-dimensional variational inequality and nonlinear complementarity problems: A survey of theory, algorithms and applications. Mathematical Programming, 48(1):161–220, 1990. (Cited on pages 2, 8, 11, and 21.)

P. Hartman and G. Stampacchia. On some non-linear elliptic differential-functional equations. Acta Mathematica, 115:271–310, 1966. (Cited on page 2.)

K. Huang and S. Zhang. An approximation-based regularized extra-gradient method for monotone variational inequalities. ArXiv Preprint: 2210.04440, 2022. (Cited on pages 4 and 14.)

K. Huang and S. Zhang. Beyond monotone variational inequalities: Solution methods and iteration complexities. ArXiv Preprint: 2304.04153, 2023. (Cited on page 4.)

K. Huang, J. Zhang, and S. Zhang. Cubic regularized Newton method for the saddle point models: A global and local convergence analysis. Journal of Scientific Computing, 91(2):1–31, 2022. (Cited on pages 12 and 17.)

A. N. Iusem, A. Jofré, R. I. Oliveira, and P. Thompson. Extragradient method with variance reduction for stochastic variational inequalities. SIAM Journal on Optimization, 27(2):686–724, 2017. (Cited on page 8.)
B. Jiang, T. Lin, and S. Zhang. A unified adaptive tensor approximation scheme to accelerate composite convex optimization. *SIAM Journal on Optimization*, 30(4):2897–2926, 2020. (Cited on page 5.)

R. Jiang and A. Mokhtari. Generalized optimistic methods for convex-concave saddle point problems. *ArXiv Preprint: 2202.09674*, 2022. (Cited on pages 1, 3, 5, 12, 15, 17, 19, and 32.)

A. Kannan and U. V. Shanbhag. Optimal stochastic extragradient schemes for pseudomonotone stochastic variational inequality problems and their variants. *Computational Optimization and Applications*, 74(3):779–820, 2019. (Cited on page 8.)

D. Kinderlehrer and G. Stampacchia. *An Introduction to Variational Inequalities and Their Applications*. SIAM, 2000. (Cited on page 2.)

B. Kleinberg, Y. Li, and Y. Yuan. An alternative view: When does SGD escape local minima? In *ICML*, pages 2698–2707. PMLR, 2018. (Cited on page 9.)

G. Kornowski and O. Shamir. High-order oracle complexity of smooth and strongly convex optimization. *ArXiv Preprint: 2010.06642*, 2020. (Cited on page 6.)

G. M. Korpelevich. The extragradient method for finding saddle points and other problems. *Matecon*, 12:747–756, 1976. (Cited on page 2.)

G. Kotsalis, G. Lan, and T. Li. Simple and optimal methods for stochastic variational inequalities, I: Operator extrapolation. *SIAM Journal on Optimization*, 32(3):2041–2073, 2022. (Cited on pages 3 and 15.)

D. Kovalev and A. Gasnikov. The first optimal acceleration of high-order methods in smooth convex optimization. In *NeurIPS*, pages 35339–35351, 2022. (Cited on page 5.)

G. Lan and Y. Zhou. An optimal randomized incremental gradient method. *Mathematical Programming*, 171(1):167–215, 2018a. (Cited on page 3.)

G. Lan and Y. Zhou. Random gradient extrapolation for distributed and stochastic optimization. *SIAM Journal on Optimization*, 28(4):2753–2782, 2018b. (Cited on page 3.)

C. E. Lemke and J. T. Howson. Equilibrium points of bimatrix games. *Journal of the Society for Industrial and Applied Mathematics*, 12(2):413–423, 1964. (Cited on page 2.)

Y. Li and Y. Yuan. Convergence analysis of two-layer neural networks with ReLU activation. In *NIPS*, pages 597–607, 2017. (Cited on page 9.)

T. Lin and M. I. Jordan. A control-theoretic perspective on optimal high-order optimization. *Mathematical Programming*, 195(1):929–975, 2022. (Cited on page 6.)

T. Lin and M. I. Jordan. Monotone inclusions, acceleration, and closed-loop control. *Mathematics of Operations Research*, 48(4):2353–2382, 2023. (Cited on pages 1, 3, 5, 12, 15, 17, and 32.)

T. Lin, P. Mertikopoulos, and M. I. Jordan. Explicit second-order min-max optimization methods with optimal convergence guarantee. *ArXiv Preprint: 2210.12860*, 2022. (Cited on pages 5 and 12.)
M. Liu, H. Rafique, Q. Lin, and T. Yang. First-order convergence theory for weakly-convex-weakly-concave min-max problems. *Journal of Machine Learning Research*, 22(169):1–34, 2021. (Cited on page 8.)

A. Madry, A. Makelov, L. Schmidt, D. Tsipras, and A. Vladu. Towards deep learning models resistant to adversarial attacks. In *ICLR*, 2018. URL https://openreview.net/forum?id=rJzIBfZAb. (Cited on page 2.)

T. L. Magnanti and G. Perakis. A unifying geometric solution framework and complexity analysis for variational inequalities. *Mathematical Programming*, 71(3):327–351, 1995. (Cited on page 2.)

T. L. Magnanti and G. Perakis. Averaging schemes for variational inequalities and systems of equations. *Mathematics of Operations Research*, 22(3):568–587, 1997a. (Cited on page 9.)

T. L. Magnanti and G. Perakis. The orthogonality theorem and the strong-f-monotonicity condition for variational inequality algorithms. *SIAM Journal on Optimization*, 7(1):248–273, 1997b. (Cited on page 2.)

T. L. Magnanti and G. Perakis. Solving variational inequality and fixed point problems by line searches and potential optimization. *Mathematical Programming*, 101(3):435–461, 2004. (Cited on pages 3 and 5.)

M. Marques Alves. Variants of the A-HPE and large-step A-HPE algorithms for strongly convex problems with applications to accelerated high-order tensor methods. *Optimization Methods and Software*, 37(6):2021–2051, 2022. (Cited on page 6.)

J. Martínez. On high-order model regularization for constrained optimization. *SIAM Journal on Optimization*, 27(4):2447–2458, 2017. (Cited on page 6.)

P. Mertikopoulos and Z. Zhou. Learning in games with continuous action sets and unknown payoff functions. *Mathematical Programming*, 173(1):465–507, 2019. (Cited on page 2.)

G. J. Minty. Monotone (nonlinear) operators in Hilbert space. *Duke Mathematical Journal*, 29(3):341–346, 1962. (Cited on page 2.)

A. Mokhtari, A. E. Ozdaglar, and S. Pattathil. Convergence rate of o(1/k) for optimistic gradient and extragradient methods in smooth convex-concave saddle point problems. *SIAM Journal on Optimization*, 30(4):3230–3251, 2020. (Cited on pages 3, 8, and 15.)

R. D. C. Monteiro and B. F. Svaiter. On the complexity of the hybrid proximal extragradient method for the iterates and the ergodic mean. *SIAM Journal on Optimization*, 20(6):2755–2787, 2010. (Cited on pages 2, 7, and 15.)

R. D. C. Monteiro and B. F. Svaiter. Complexity of variants of Tseng’s modified FB splitting and Korpelevich’s methods for hemivariational inequalities with applications to saddle-point and convex optimization problems. *SIAM Journal on Optimization*, 21(4):1688–1720, 2011. (Cited on pages 2 and 7.)

R. D. C. Monteiro and B. F. Svaiter. Iteration-complexity of a Newton proximal extragradient method for monotone variational inequalities and inclusion problems. *SIAM Journal on Optimization*, 22(3):914–935, 2012. (Cited on pages 3, 7, 11, and 15.)
R. D. C. Monteiro and B. F. Svaiter. An accelerated hybrid proximal extragradient method for convex optimization and its implications to second-order methods. *SIAM Journal on Optimization*, 23(2):1092–1125, 2013. (Cited on pages 5, 6, and 17.)

I. Necoara, Y. Nesterov, and F. Glineur. Linear convergence of first order methods for non-strongly convex optimization. *Mathematical Programming*, 175(1):69–107, 2019. (Cited on page 14.)

A. Nemirovski. Prox-method with rate of convergence o(1/t) for variational inequalities with Lipschitz continuous monotone operators and smooth convex-concave saddle point problems. *SIAM Journal on Optimization*, 15(1):229–251, 2004. (Cited on pages 2, 7, 8, and 15.)

A. Nemirovski and Y. Nesterov. Optimal methods of smooth convex minimization. *USSR Computational Mathematics and Mathematical Physics*, 25(2):21–30, 1985. (Cited on pages 13 and 14.)

Y. Nesterov. A method of solving a convex programming problem with convergence rate o(k^-2). In *Doklady Akademii Nauk*, volume 269, pages 543–547. Russian Academy of Sciences, 1983. (Cited on page 10.)

Y. Nesterov. Cubic regularization of Newton’s method for convex problems with constraints. Technical report, Université catholique de Louvain, Center for Operations Research and Econometrics (CORE), 2006. (Cited on pages 1, 3, 10, and 11.)

Y. Nesterov. Dual extrapolation and its applications to solving variational inequalities and related problems. *Mathematical Programming*, 109(2):319–344, 2007. (Cited on pages 3, 7, 8, 9, 10, 11, and 15.)

Y. Nesterov. Accelerating the cubic regularization of Newton’s method on convex problems. *Mathematical Programming*, 112(1):159–181, 2008. (Cited on pages 4, 5, and 10.)

Y. Nesterov. Gradient methods for minimizing composite functions. *Mathematical Programming*, 140(1):125–161, 2013. (Cited on pages 13 and 14.)

Y. Nesterov. *Lectures on Convex Optimization*, volume 137. Springer, 2018. (Cited on pages 1, 3, 8, and 13.)

Y. Nesterov. Inexact high-order proximal-point methods with auxiliary search procedure. *SIAM Journal on Optimization*, 31(4):2807–2828, 2021a. (Cited on pages 6 and 32.)

Y. Nesterov. Implementable tensor methods in unconstrained convex optimization. *Mathematical Programming*, 186(1):157–183, 2021b. (Cited on pages 5, 12, 16, 17, 25, 26, and 29.)

Y. Nesterov. Inexact accelerated high-order proximal-point methods. *Mathematical Programming*, pages 1–26, 2021c. (Cited on pages 6 and 32.)

Y. Nesterov. Superfast second-order methods for unconstrained convex optimization. *Journal of Optimization Theory and Applications*, 191(1):1–30, 2021d. (Cited on pages 6 and 32.)

Y. Nesterov and B. Polyak. Cubic regularization of Newton method and its global performance. *Mathematical Programming*, 108(1):177–205, 2006. (Cited on pages 5 and 10.)

P. Ostroukhov, R. Kamalov, P. Dvurechensky, and A. Gasnikov. Tensor methods for strongly convex strongly concave saddle point problems and strongly monotone variational inequalities. *ArXiv Preprint: 2012.15595*, 2020. (Cited on page 14.)
Y. Ouyang and Y. Xu. Lower complexity bounds of first-order methods for convex-concave bilinear saddle-point problems. *Mathematical Programming*, 185(1):1–35, 2021. (Cited on pages 3, 10, 15, and 17.)

B. O’donoghue and E. Candes. Adaptive restart for accelerated gradient schemes. *Foundations of Computational Mathematics*, 15(3):715–732, 2015. (Cited on pages 13 and 14.)

L. D. Popov. A modification of the Arrow-Hurwicz method for search of saddle points. *Mathematical notes of the Academy of Sciences of the USSR*, 28(5):845–848, 1980. (Cited on page 3.)

D. Ralph and S. J. Wright. Superlinear convergence of an interior-point method for monotone variational inequalities. *Complementarity and Variational Problems: State of the Art*, pages 345–385, 1997. (Cited on page 2.)

J. Renegar and B. Grimmer. A simple nearly optimal restart scheme for speeding up first-order methods. *Foundations of Computational Mathematics*, 22(1):211–256, 2022. (Cited on page 14.)

R. T. Rockafellar and R. J-B. Wets. *Variational Analysis*, volume 317. Springer Science & Business Media, 2009. (Cited on page 12.)

V. Roulet and A. d’Aspremont. Sharpness, restart and acceleration. In *NIPS*, pages 1119–1129, 2017. (Cited on page 14.)

H. Scarf. The approximation of fixed points of a continuous mapping. *SIAM Journal on Applied Mathematics*, 15(5):1328–1343, 1967. (Cited on page 2.)

A. Sinha, H. Namkoong, and J. Duchi. Certifiable distributional robustness with principled adversarial training. In *ICLR*, 2018. URL https://openreview.net/forum?id=Hk6kPgZA-. (Cited on page 2.)

M. V. Solodov and B. F. Svaiter. A new projection method for variational inequality problems. *SIAM Journal on Control and Optimization*, 37(3):765–776, 1999. (Cited on page 8.)

C. Song, Z. Zhou, Y. Zhou, Y. Jiang, and Y. Ma. Optimistic dual extrapolation for coherent non-monotone variational inequalities. In *NeurIPS*, pages 14303–14314, 2020. (Cited on pages 8 and 16.)

C. Song, Y. Jiang, and Y. Ma. Unified acceleration of high-order algorithms under general Hölder continuity. *SIAM Journal on Optimization*, 31(3):1797–1826, 2021. (Cited on page 6.)

A. A. Titov, S. S. Ablaev, M. S. Alkousa, F. S. Stonyakin, and A. V. Gasnikov. Some adaptive first-order methods for variational inequalities with relatively strongly monotone operators and generalized smoothness. In *ICOPTA*, pages 135–150. Springer, 2022. (Cited on page 12.)

M. J. Todd. *The Computation of Fixed Points and Applications*. Springer, 2013. (Cited on page 2.)

R. Trémolières, J-L. Lions, and R. Glowinski. *Numerical Analysis of Variational Inequalities*. Elsevier, 2011. (Cited on page 2.)

P. Tseng. A modified forward-backward splitting method for maximal monotone mappings. *SIAM Journal on Control and Optimization*, 38(2):431–446, 2000. (Cited on pages 3 and 8.)

A. Wibisono, A. C. Wilson, and M. I. Jordan. A variational perspective on accelerated methods in optimization. *Proceedings of the National Academy of Sciences*, 113(47):E7351–E7358, 2016. (Cited on page 6.)
J. Zhang, M. Hong, and S. Zhang. On lower iteration complexity bounds for the convex concave saddle point problems. *Mathematical Programming*, 194(1):901–935, 2022. (Cited on page 15.)