Research Article

Some Properties of Bifractional Bessel Processes Driven by Bifractional Brownian Motion

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Received 1 May 2020; Accepted 24 September 2020; Published 19 October 2020

Academic Editor: Wenguang Yu

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Let \( B = \{B_t^i : i = 1, \ldots, d\} \) be a \( d \)-dimensional bifractional Brownian motion and \( R_t = \sqrt{(B_0^1)^2 + \cdots + (B_0^d)^2} \) be the bifractional Bessel process with the index \( 2HK \geq 1 \). The Itô formula for the bifractional Brownian motion leads to the equation:

\[
R_t = \sum_{i=1}^{d} \int_0^t (B_s^i/R_s) dB_s^i + HK (d - 1) \int_0^t (s^{2HK-1}/R_s) ds.
\]

In the Brownian motion case \( K = 1 \) and \( (H = 1/2) \), \( X_t = \sum_{i=1}^{d} \int_0^t (B_s^i/R_s) dB_s^i \d s \geq 1 \) is a Brownian motion by Lévy’s characterization theorem. In this paper, we prove that process \( X_t \) is not a bifractional Brownian motion unless \( K = 1 \) and \( (H = 1/2) \). We also study some other properties and their application of this stochastic process.

1. Introduction

Given \( H \in (0, 1) \) and \( K \in [0, 1] \), the bifractional Brownian motion with the indices \( H \) and \( K \) is a mean zero Gaussian process \( B = \{B_t^{H,K}, t \geq 0\} \) such that \( B_0^{H,K} = 0 \) and

\[
E[B_t^{H,K}B_s^{H,K}] = R(t, s) = \frac{1}{2K} \left( (t^{2H} + s^{2H})^K - |t - s|^{2HK} \right),
\]

for all \( (s, t) \geq 0 \). This process was first introduced by Houdré and Villa [1]. More works for bifractional Brownian motion and their application can be found in [2–10] and the references therein. Clearly, the process is a fractional Brownian motion with Hurst the parameter \( H \) when \( K = 1 \). Particularly, the process is a Brownian motion when \( (K = 1) \) and \( (H = 1/2) \).

Since \( B_t^{H,K} \) is neither a Markov process nor a semimartingale unless \( (K = 1) \) and \( (H = 1/2) \), a lot of powerful techniques from classical stochastic analysis are not available to deal with it. As the generalization of the fractional Brownian motion, the bifractional Brownian motion also admits Hölder paths and self-similarity, but its increments are not stationary.

Let \( B = \{B_t^1, \ldots, B_t^d\} \) be a \( d \)-dimensional bifractional Brownian motion with the index \( \text{HK}(\geq 1/2) \). That is to say, each component of \( B \) is an independent one-dimensional bifractional Brownian motion with the index \( \text{HK}(\geq 1/2) \). Let \( R_t \) be the bifractional Bessel process defined by \( R_t = \sqrt{(B_0^1)^2 + \cdots + (B_0^d)^2} \).

There is an extensive literature on this process for the standard Brownian motion case \( (K = 1) \) and \( (H = 1/2) \) and the fractional Brownian motion case \( (K = 1) \) (see [11–15]). For \( (d \geq 2, HK > 1/2) \), by the Itô formula for the bifractional Brownian motion, we have (see Alós et al. [16] and Es-Sebaiy and Tudor [3])

\[
R_t = \sum_{j=1}^{d} \int_0^t \frac{B_j^2}{R_s} dB_s^j + HK (d - 1) \int_0^t \frac{s^{2HK-1}}{R_s} ds,
\]

for \( d = 1 \) and \( (HK \geq 1/2) \), one also has
\[ B^{H,K}_t = \int_0^t \text{sign}(B^{H,K}_s) \, dB^{H,K}_s + HK \int_0^t \delta(b^{H,K}_s) s^{2HK-1} \, ds, \]

where stochastic integrals are interpreted in the divergence sense and \( \delta \) denotes the Dirac delta function. When \( K = 1 \) and \( H = (1/2) \), the process

\[ X_t = \begin{cases} \int_0^t \text{sign}(B^{H,K}_s) \, dB^{H,K}_s, & d = 1, \\ \sum_{i=1}^d \int_0^t \frac{B_i^j}{R_i} \, dB^i_s, & d \geq 2 \end{cases} \]

is a standard Brownian motion by Lévy’s characterization theorem. Given \( K = 1 \), the fractional Brownian motion case was researched by Hu and Nualart [11]. So, it is natural and convenient as Lévy’s characterization theorem for general bifractional Brownian motion, to prove a stochastic process

\[ \text{is always negative}, \]

\[ \text{since it seems interesting to study bifractional Bessel processes driven by bifractional Brownian motions.} \]

The rest of the paper is organized as follows. In Section 2, we present some preliminaries for the bifractional Brownian motion. In Section 3, some properties to the process \( \int_0^t \) sign \( (B^{H,K}_s) \, dB^{H,K}_s \) are studied. In Section 4, we consider the process \( (\sum_{i=1}^d \int_0^t (B^i/R_i) \, dB^i_s) \) with \( d \geq 2 \). In Section 5, we consider the local time and Tanaka formula of the process \( \int_0^t \) sign \( (B^{H,K}_s) \, dB^{H,K}_s \).

2. Preliminaries

In this paper, we assume that \((1/2) < K < 1\) is arbitrary but fixed and let \( B = (B^{H,K}_t, 0 \leq t \leq T) \) be a bifractional Brownian motion with the index \( H \) and \( K \), which is defined on the complete probability space \((\Omega, \mathcal{F}, P)\). One can construct a stochastic calculus of variations with respect to the bifractional Brownian motion \( B^{H,K} \) by the Malliavin calculus method (see Alòs et al. [16] and Nualart [18]). We next recall the basic definitions and results for this calculus.

Bifractional Brownian motion \( B^{H,K} \) satisfies the estimates:

\[ 2^{-K} |t-s|^{2HK} \leq E \left[ (B_t - B_s)^2 \right] \leq 2^{1-K} |t-s|^{2HK}. \]  

One can write its covariance as follows:

\[ R(t, s) = R_1(t, s) + R_2(t, s), \]

where

\[ R_1(t, s) = \frac{1}{2^K} \left[ (s^{2H+t^{2HK}} + t^{2HK}) \right], \]

\[ R_2(t, s) = \frac{1}{2^K} \left[ t^{2HK} + s^{2HK} - |t-s|^{2HK} \right]. \]

Since \( R_1 \) is of the class \( C^2([0,T]^2) \) and \((\partial^2/\partial t \partial s)R_1(t, s)\) is always negative, \( R_1 \) is the distribution function and has \((\partial^2/\partial t \partial s)R_1(t, s)\) for density. \( R_2 \) is the distribution function with density \((\partial^2/\partial t \partial s)R_2(t, s) = 2(2HK-1)|HK-t-s|^{2HK-2} \) and belongs to \( L^1([0,T]^2) \). It follows that there exist two positive constants \( c_{H,K} \) and \( C_{H,K} \) which satisfy

\[ c_{H,K}|t-s|^{2HK-2} \leq \frac{3^2}{\partial t \partial s} R(t, s) \leq C_{H,K}|t-s|^{2HK-2}. \]

Denote

\[ \phi(t, s) = 2(2HK-1)HK|t-s|^{2HK-2}, \]  

for \( t, s \geq 0 \).

As a Gaussian process of \( B^{H,K} \), we can construct a stochastic calculus of variations with respect to this process. Suppose that \( \mathcal{H} \) is the completion of the space \( \mathcal{E} \) which is generated by \( \{1_{[0,T]}, t \in [0,T]\} \) with respect to the following inner product:

\[ \langle 1_{[0,T]} , 1_{[0,T]} \rangle_{\mathcal{H}} = R(s, t). \]  

Then, \( \phi \in \mathcal{E} \rightarrow B(\phi) \) is an isometry from \( \mathcal{E} \) to the Gaussian space generated by \( B \) which can be extended to \( \mathcal{H} \). We can write this Hilbert space \( \mathcal{H} \) as follows:

\[ \mathcal{H} = \{ \phi : [0, T] \rightarrow \mathbb{R} \| \phi \|_{\mathcal{H}} < \infty \}, \]

where \( \| \phi \|^2_{\mathcal{H}} := \int_0^T \phi(s)\phi(r)(\phi(s, r)dsdr). \] We can define the spaces of measurable functions as follows:

\[ \mathcal{M} = \{ \phi : [0, T] \rightarrow \mathbb{R} \| \phi \|^2_{\mathcal{M}} < \infty \}, \]

where

\[ \| \phi \|^2_{\mathcal{M}} := \int_0^T \int_0^T |\phi(s)||\phi(r)|\phi(s, r)dsdr < \infty. \]

It is easy to see that \( \mathcal{E} \) is dense in \( \mathcal{M} \) and \( \mathcal{M} \) is a Banach space. Suppose that \( \mathcal{E} \) is the set of smooth functional

\[ F = \int (B^{H,K}(\phi_1), B^{H,K}(\phi_2), \ldots, B^{H,K}(\phi_d)), \]

where \( F \in C_0^\infty(\mathbb{R}^n) \) and \( \phi_i \in \mathcal{E} \). The Malliavin derivative \( D \) of the above functional \( F \) is given as follows:
The derivative operator $D$ is a closable operator from space $L^2(\Omega)$ into space $L^2(\Omega; \mathcal{F})$. We denote $\mathcal{D}^{1,2}$, the closure of $\delta$, with respect to norm
\[ \|F\|_{1,2} = \sqrt{E|F|^2 + E\|DF\|^2_{\mathcal{F}}}. \]

The divergence integral $\delta$ is the adjoint operator of $D$. $\delta(u)$ can be defined by the duality relationship:
\[ E[F\delta(u)] = E\langle DF, u \rangle_{\mathcal{F}}, \]
for any $u \in \mathcal{D}^{1,2}$. For any $u \in \mathcal{D}^{1,2}$, one has $(\mathcal{D}^{1,2} \subset \text{Dom}(\delta))$ and
\[ E[\delta(u)^2] = E[u]^2_{\mathcal{F}} + E\int_{[0,T]^2} D_u D_{u\eta} \phi(\xi, s, \phi(\eta, r))d\xi d\eta 
\leq E[u]^2_{\mathcal{F}} + E\int_{[0,T]^2} D_u D_{u\eta} \phi(\xi, s, \phi(\eta, r))d\xi d\eta, \]
where
\[ \delta(u) = \int_0^T u_t dB_t^{HK}, \]
expressing the Skorokhod integral of a process $u$.

3. Case of One Dimension
We study the stochastic process $X = \{X_t, \ t \geq 0\}$ defined by
\[ X_t = \int_0^t \text{sign}(B_t^{HK}) dB_t^{HK}. \]

If $K = 1$ and $H = (1/2)$, $X_t$ is a standard Brownian motion from Levy’s characterization theorem. It is then natural to study any parameter $H$ and $K$. Next, we first prove $X$ is an HK-self-similar process for any $HK \geq (1/2)$.

**Proposition 1.** The stochastic process $X = \{X_t, \ t \geq 0\}$ is HK-self-similar.

**Proof.** Together with the HK-self-similarity property of the bifractional Brownian motion and Tanaka formula (4), for any $a > 0$, one can obtain
\[ X_{at} = \left[ B^{HK}_{at} - \text{HK} \int_0^t \delta(B_{s}^{HK}) B_{s}^{HK-1} ds \right] \]
\[ = \left[ B^{HK}_{at} - \text{HK} \int_0^t \delta(B_{su}^{HK}) (au)^{2HK-1} du \right] 
\leq \text{HK} \int_0^t \delta(B_{su}^{HK}) (au)^{2HK-1} du 
\leq \text{HK} \int_0^t \delta(B_{su}^{HK}) (au)^{2HK-1} du \]
\[ = \text{HK}^2 X_t, \]
where $\text{HK}$ denotes that both stochastic processes have the same distributions. This proof is completed.

For stochastic process sign $(B_t^{HK})$, we first obtain the Wiener chaos expansion. Let $I_n$ be the multiple Wiener integral of the stochastic process $B_t^{HK}$.

**Proposition 2.** For any $t \geq 0$, one can obtain
\[ \text{sign}(B_t^{HK}) = \sum_{m=0}^{\infty} b_{2m+1} I_{2m+1}(1), \]
where
\[ b_{2m+1} = \frac{2(-1)^m}{(2m+1)!2^m}. \]

**Proof.** For $\varepsilon > 0$, we denote
\[ p_\varepsilon(y) = \frac{1}{\sqrt{2\pi\varepsilon}} e^{-y^2/(2\varepsilon)}, \]
\[ f_\varepsilon(y) = 2 \int_{-\infty}^{y} p_\varepsilon(z) dz - 1, \quad y \in \mathbb{R}. \]
Then,
\[ p_{2m+1}(x) = \frac{1}{\sqrt{2\pi\varepsilon}} \exp\left( -\frac{y^2}{2\varepsilon} \right), \quad x \in \mathbb{R}, \]
which is a density function of the bifractional Brownian motion $B_t^{HK}$ and $f_\varepsilon(B_t^{HK}) \rightarrow \text{sign}(B_t^{HK})$ in $L^2(\Omega)$ as $\varepsilon \rightarrow 0$. By Stroock’s formula, one can obtain
\[ f_\varepsilon(B_t^{HK}) = \sum_{m=0}^{\infty} a_m^\varepsilon(t) \int_{0<s_1<\ldots<s_n<t} dB_{s_1}^{HK}, \ldots, dB_{s_n}^{HK}, \]
where
\[ a_m^\varepsilon(t) = E[D^m(f_\varepsilon(B_t^{HK}))] = 2E[p_{\varepsilon}^{(m-1)}(B_t^{HK})] \]
\[ = 2(-1)^{m-1} \frac{\partial^{m-1}}{\partial z^{m-1}} E[p_{\varepsilon}(B_t^{HK} - z)]_{z=0} \]
\[ = 2(-1)^{m-1} p_{\varepsilon^{(m-1)}}(0). \]
As $\varepsilon \to 0$, by taking the limit of (27) in the space $L^2(\Omega)$, one can obtain
\[
sign(B_t^{H,K}) = \sum_{m=0}^{\infty} a_n(t) \int_{0<s_1<\cdots<s_n<t} dB_t^{H,K}(s_1), \ldots, dB_t^{H,K}(s_n),
\] where $a_m(t) = \lim_{\varepsilon \to 0} a_m^{(\varepsilon)}(t) = 2(-1)^{m-1} P_{\varepsilon}^{(m-1)}(0)$, which implies
\[
a_m(t) = \begin{cases} 0, & n = 2k, \\ \frac{2(-1)^k(2k)!}{\sqrt{2\pi} t^{m+1} k!}, & n = 2k + 1. \end{cases}
\] (30)

The proof is completed.

In this paper, the notation $F \equiv G$ implies that there are two positive constants $c_1$ and $c_2$ such that
\[
c_1 G(x) \leq F(x) \leq c_2 G(x),
\] where $C$ denotes a generic positive constant and $F$ and $G$ have the common domain. □

**Proposition 3.** The random variable $\sign(B_t^{H,K})$ belongs to the Sobolev–Watanabe space $D^{a,2}$ for any $t \geq 0$ and $a < (1/2)$.

**Proof.** By Stirling's formula
\[
\lim_{k \to \infty} \frac{k!}{e^k} = \sqrt{2\pi},
\] we have
\[
E\left[I_{2m+1}(b_{2m+1})\right]^2 = (2m + 1)! (b_{2m+1}, b_{2k+1}) \rho^{2m+1} = \frac{(2m + 1)!4^{2m+1}}{(2m + 1)^2 2\pi (2m+1)H^2 K (m^2 m^2)^2} \approx \frac{C m^{-3/2}}{
\] where $c_m = (-1)^{m-1} (\sqrt{2\pi} (2m - 1)(m - 1)!2^{m-2})^{-1}$ and
\[
h_2m(B_1, \ldots, B_2m) = (B_1 V B_2 \cdots V B_2m)^{(2m-1)HK}. \quad (35)
\]

The above proposition is the chaos expansion of \( \int_0^t \sign(B_t^{H,K}) dB_t^{H,K} \) and implies the following result, which can be proved by the method similar to Proposition 3.

**Proposition 5.** For any $a < (1/2)$ and $t \geq 0$, the random variable $\int_0^t \sign(B_t^{H,K}) dB_t^{H,K}$ belongs to the Sobolev–Watanabe space $D^{a,2}$. Now, we consider the stochastic process $X$:
\[
\rho(n) = E[(X_{n+1} - X_n)(X_{n+1} - X_n)],
\] where $(0 < a \leq n)$.

**Definition 1.** We say a stochastic process $(X_t)_{t \geq 0}$ is long-range dependent (resp. short-range dependent) if for each $a > 0$,
\[
\sum_{n \geq a} |\rho(n)| = \infty, \quad \left( \text{resp. } \sum_{n \geq a} |\rho(n)| < \infty \right).
\] (37)

**Theorem 1.** The stochastic process $X$ of (21) is short-range dependent. Before proving this theorem, a lemma given by Yan et al. [9] is stated.

**Lemma 1.** Let $(0 \leq r < s)$ and $(0 < HK < 1)$, one defines
\[
\rho^2_{r,s} := s^{2HK} \rho^{2HK} - \mu^2,
\] where $\mu = E(B_r^{H,K} B_s^{H,K})$. Then, we have
\[
\rho^2_{r,s} = (s - r)^{2HK} \rho^{2HK}. \quad (39)
\]

**Remark 1.** The proof of estimate (39) uses the following two inequalities:
\[
(1 + x)^\alpha \leq 1 + (2\alpha - 1)x^\alpha, \quad 0 \leq x, \alpha \leq 1, \quad (40)
\]
\[
(u + v - 1)^K \leq u^K + v^K - 1 \leq (u + v - 1)^K + (uv)^K, \quad (0 \leq K \leq 1), \quad (41)
\]
where $0 \leq u, v \leq 1$ and $u + v \geq 1$. It is not difficult to prove inequality (40), which is stronger than the well-known inequality
\[
(1 + x)^\alpha \leq 1 + \alpha x^\alpha \leq 1 + x^\alpha,
\] because $(2\alpha - 1) \leq 1 - x$ for all $(0 \leq \alpha \leq 1)$.

**Proof of Theorem 1.** For $0 < a < k$, one can obtain
\[ \rho(k) = E\left[ \int_{a}^{a+1} \text{sign}(B_{t}^{H,K}) dB_{t}^{H,K} \right] \]
\[ = \int_{a}^{a+1} \int_{k}^{k+1} \phi(t,s)E(\text{sign}B_{t}^{H,K} \text{sign}B_{t}^{H,K}) \text{d}s \text{d}t + \int_{a}^{a+1} \int_{k}^{k+1} \phi(t,s)E(\text{sign}B_{t}^{H,K} \text{sign}B_{t}^{H,K}) \text{d}s \text{d}t \]
\[ = \int_{a}^{a+1} \int_{k}^{k+1} \phi(t,s)E(\text{sign}B_{t}^{H,K} \text{sign}B_{t}^{H,K}) \text{d}s + 4(HK)^{2} \int_{a}^{a+1} \int_{k}^{k+1} \psi_{1}(s,t)\psi_{2}(s,t) \text{d}s \text{d}t \]
\[ = \int_{a}^{a+1} \int_{k}^{k+1} \phi(t,s)E\left( \frac{B_{t}^{H,K}B_{t}^{H,K}}{B_{s}^{H,K}B_{s}^{H,K}} \right) \text{d}s \text{d}t + 4(HK)^{2} \int_{a}^{a+1} \int_{k}^{k+1} \psi_{1}(s,t)\psi_{2}(s,t) \text{d}s \text{d}t \]
\[ = \left[ a_{k} + b_{k} \right] \text{as } t \longrightarrow \infty \text{ and } s \in (0, 1), \text{ which implies} \]
\[ E\left( \frac{B_{t}^{H,K}B_{t}^{H,K}}{B_{s}^{H,K}B_{s}^{H,K}} \right) = E\left( \frac{B_{t}^{H,K}(\lambda_{s,t}B_{s}^{H,K} + \mathcal{N})}{B_{s}^{H,K}(\lambda_{s,t}B_{s}^{H,K} + \mathcal{N})} \right), \]
\[ = E\left( \frac{B_{1}^{H,K}(s \lambda_{s,t}B_{s}^{H,K} + \mathcal{N})}{B_{s}^{H,K}(s \lambda_{s,t}B_{s}^{H,K} + \mathcal{N})} \right) \]
\[ = O(t^{-HK}). \]

So, the term \( a_{k} \) behaves as \( k^{2HK-3}o(k^{-HK}) \). Now, we evaluate the second term \( b_{k} \). For \( s < t \), using Lemma 1, one can obtain
\[ E[\delta(B_{s}^{H,K}) \delta(B_{t}^{H,K})] = \int_{R^{2}} h(y, z) \delta(y) \delta(z) \text{d}y \text{d}z = h(0,0) \]
\[ = \frac{1}{2\pi \rho_{t,s}} s^{-HK} (t-s)^{-HK}, \]
where \( h(y, z) \) is the density function of \((B_{s}^{H,K}, B_{t}^{H,K})\). So,
\[ b_{k} = \frac{4(HK)^{2}}{2\pi} \int_{a}^{a+1} \int_{k}^{k+1} \left[ s^{2HK-1} + (t-s)^{2HK-1} \right] \frac{1}{[st]^{HK} - (1/4) \left( (1/2)^{K} \left( t^{2H} + s^{2H} \right)^{-1} \right) \left( t^{2H} - s^{2H} \right)^{-1}}]^{1/2} \text{d}s \text{d}t \]
\[ = \int_{a}^{a+1} \int_{k}^{k+1} \left[ (a+1)^{2HK-1} + (k+1)^{2HK-1} \right] \frac{1}{(a+1)^{2H} + (a+1)^{2H}} \left( (k+1)^{2H} - (a+1)^{2H} \right)^{2} \text{d}t \text{d}r \]
\[ = \lambda^{3HK-3}. \]

The proof is completed.
4. Case of Multidimension

We now consider the $d$-dimensional bifractional Brownian motion $B = (B_1^1, \ldots, B_d^d)_{t \geq 0}$ with the index $HK \geq (1/2)$, which implies the components $(B_i^i, \quad i = 1, \ldots, d)$ are independent bifractional Brownian motions with the same index $HK \geq (1/2)$. As in Section 2, we can define the derivative and divergence operators, $D^i$ and $\delta^i$, with respect to each component $B_i$. Suppose that $(D^i_+ P(\mathcal{F}))$ are the associated Sobolev spaces. Similarly, $(L^p_{HK})$ denotes the set of processes $u$ in $(D^i_+ P(\mathcal{F}))$ which satisfies

$$[dU]^p_{-HK} = E \left[ |U|^p_{L_{HK}([0,T])} \right] + E \left[ \| D^i U \|^p_{L_{HK}([0,T])} \right] < \infty.$$  

(52)

Let

$$R_i = |B_i| = \left( (B_i^1)^2 + \cdots + (B_i^d)^2 \right)^{1/2}$$  

(53)

be a bifractional Bessel process. In the following, we research the stochastic process:

$$X_t = \sum_{i=1}^{d} \int_{0}^{t} \frac{B_i^i}{R_i} dB_i.$$  

(54)

The next theorem can be proved similar to Es-Sebaiy and Tudor [3].

Theorem 2. Let $B = (B_1^1, B_2^2, \ldots, B_d^d)$ be a $d$-dimensional bifractional Brownian motion with $(2HK > 1)$ and $f$ be a function of class $C^2(\mathbb{R}^d)$. Then,

$$f(B_t) = f(0) + \sum_{i=1}^{d} \int_{0}^{t} \frac{\partial f}{\partial x_i}(B_s) dB_i^i + HK \sum_{i=1}^{d} \int_{0}^{t} \frac{\partial^2 f}{\partial x_i^2}(B_s) s^{2HK-1} ds.$$  

(55)

The following proposition gives an integral representation for bifractional Bessel processes and can be proved along the lines of the proof of Proposition 5.2 in Guerra and Nualart [19].

Proposition 6. Suppose that $R = \{R_i\}_{i \geq 0}$ is a bifractional Bessel process associated to the $d$-dimensional bifractional Brownian motion with index $HK > (1/2)$. For each $(i = 1, \ldots, d)$, one can obtain $(\{B_i^i/R_i\})_{i \geq 0} \in L^{(1/2)}_{HK, i}$ and

$$R_i = \sum_{j=1}^{d} \int_{0}^{t} \frac{B_j^j}{R_j} dB_j^i + HK(d-1) \int_{0}^{t} \frac{B_i^{2HK-1}}{R_i} ds.$$  

(56)
\[ g_i^2(R_i^2) = \frac{3}{8} \sqrt{\varepsilon} + \sum_{i=1}^{d} I(i, \varepsilon) + II(\varepsilon) + III(\varepsilon), \tag{63} \]

where

\[ I(i, \varepsilon) = \int_0^t \left[ \mathbb{1}_{[R_i^2 < \varepsilon]} \frac{1}{2 \sqrt{\varepsilon}} \left( 3 - \frac{R_i^2}{\varepsilon} \right) + \mathbb{1}_{[R_i^2 \geq \varepsilon]} \frac{1}{R_i} B_i^t dB_i^t, \right. \]

\[ II(\varepsilon) = HK(d-1) \int_0^t \mathbb{1}_{[R_i^2 < \varepsilon]} \frac{1}{R_i} B_i^t dB_i^t \]

\[ III(\varepsilon) = HK \int_0^t \mathbb{1}_{[R_i^2 < \varepsilon]} \frac{1}{R_i^2} \left[ 3d - (d+2) \frac{R_i^2}{\varepsilon} \right] B_i^t dB_i^t. \tag{64} \]

Together with \( \int_0^t (s^{2HK-1}/R_i) ds < \infty \) a.s. and the bounded convergence theorem, one can obtain

\[ \lim_{\varepsilon \to 0} II(\varepsilon) = (d-1)HK \int_0^t \frac{1}{R_i} s^{2HK-1} ds. \tag{65} \]

For the third term, by the substituting \( u = (\rho/s^{HK}) \) and Fubini’s theorem, we can obtain

\[ 0 \leq E(III(\varepsilon)) \leq \frac{3HKd}{2\sqrt{\varepsilon}} \int_0^t P\{R_i^2 < \varepsilon\} s^{2HK-1} ds \leq \frac{3HKd}{2\sqrt{\varepsilon}} \int_0^t \left[ \int_0^t \frac{1}{R_i} \left( \frac{1}{2} \int_0^\sqrt{\varepsilon} \frac{\rho}{2\pi s^{2HK}} e^{-\frac{\rho^2}{2s^{2HK}}} d\rho d\theta \right)^{2HK-1} ds \right. \]

\[ = \frac{3HKd}{2\sqrt{\varepsilon}} \int_0^\sqrt{\varepsilon} \left( \int_0^\infty \frac{1}{\nu^{(3/2)}} \right) \rho d\rho, \tag{66} \]

\[ \left. \right] \]

\[ B(\varepsilon) := E \left[ \int_0^T \int_0^T D_\rho \left( \left( \frac{B_i^t}{R_i} - \frac{B_i^t}{2\sqrt{\varepsilon}} \right) \left( 3 - \frac{R_i^2}{\varepsilon} \right) \right)^{1/1HK} ds dr \right] \]

\[ = \int_0^T r E \left[ \left( \frac{1}{R_i} - \frac{(B_i^t)^2}{R_i^2} \right) - \frac{3}{2\sqrt{\varepsilon}} - \frac{3R_i^2}{2\varepsilon} \right] 1_{[R_i^2 < \varepsilon]} dr \]

\[ \leq 4\pi \int_0^\sqrt{\varepsilon} E \left[ \frac{1}{R_i} \right]^{1/1HK} 1_{[R_i^2 < \varepsilon]} dr. \tag{70} \]

The distribution of \((B_1^t, \ldots, B_d^t)\) in spherical coordinates yields

\[ \int_0^T E \left[ |R|^{1/1HK} 1_{[R < \sqrt{\varepsilon}/\rho^{1HK}]} \right] dr \leq C_{d, HK} \int_0^\sqrt{\varepsilon} \int_0^T e^{-\frac{\rho^2}{2\varepsilon}} u^{d-1} \frac{1}{H^1HK} dudr, \tag{71} \]
where \( C_{d,H,K} > 0 \) is a constant which depends on \( H, K, \) and \( d. \)
For any \( r \in [0, T], \)
\[
\int_0^r e^{-\left(\frac{r}{\sqrt{t}}\right)^{1/HK}} e^{-\left(\frac{r}{\sqrt{t}}\right)^{1/(1/HK)}} \, dt \rightarrow 0, \quad \epsilon \rightarrow 0,
\]
\[
\int_0^\infty e^{-\left(\frac{r}{\sqrt{t}}\right)^{1/(1/HK)}} \, dt < \infty.
\]

We have
\[
\lim_{\epsilon \rightarrow 0} \int_0^r E\left|R_1\right|^{-(1/HK)}_{\{1, \leq (\sqrt{t})^{1/HK}\}} \, dr = 0,
\]
by the bounded convergence theorem, that is,
\[
\lim_{\epsilon \rightarrow 0} B(\epsilon) = 0.
\]
This proves the desired convergence (68), and the proposition follows. \( \square \)

**Proposition 7.** Stochastic process \( X \) which is given by (54) is HK-self-similar.

**Theorem 3.** Suppose \( f_j : \mathbb{R}^d \rightarrow \mathbb{R}, j = 1, 2, \ldots, d \) are with polynomial growth and smooth functions. Then, the stochastic process \( Z_t = \sum_{j=1}^d \int_0^t f_j(B_s)dB^i_s \) has the following stochastic chaos expansion:
\[
Z_t = \sum_{j=1}^d \sum_{k=1}^{\infty} \sum_{1 \leq j_1, \ldots, j_k \leq d} I_{i_1, \ldots, i_k}(g^{j_{i_1, \ldots, i_k}}(B_{i_1}, \ldots, B_{k+1})),
\]
where
\[
g^{j_{i_1, \ldots, i_k}}(B_{i_1}, \ldots, B_{k+1}) = \frac{(-1)^n (B_{i_1} \ldots \hat{V} \ldots B_{k+1})^{-1}_{HK}}{(2\pi)^{d/2}} g^{i_k} f_j(B_s) dy.
\]

**Proof.** For each \( (j = 1, 2, \ldots, d) \), using Stroock’s formula, we can obtain
\[
f_j(B_s) = \sum_{k=0}^{\infty} \sum_{1 \leq j_1, \ldots, j_k \leq d} \frac{1}{k!} I_{i_1, \ldots, i_k} \left(f^{i_{j_{i_1, \ldots, i_k}}}(s) ight)_{[0,d]},
\]
where
\[
I_{i_1, \ldots, i_k}(s) = E\left(D^{i_1}, \ldots, D^{i_k}(f_j(B_s))ight)
\]
\[
= E\left(\frac{\partial f_j}{\partial z_{i_1}, \ldots, \partial z_{i_k}}(B_s)\right)
\]
\[
= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \frac{\partial f_j}{\partial z_{i_1}, \ldots, \partial z_{i_k}} e^{-\left(\sum_{j=1}^d z^2\right)^{1/2}} \, dz
\]
\[
= \frac{\sqrt{\pi} e^{-\frac{1}{2}B_{i_1}^2}}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \frac{\partial f_j}{\partial z_{i_1}, \ldots, \partial z_{i_k}} e^{-\left(\sum_{j=1}^d z^2\right)^{1/2}} \, dy
\]
\[
= \frac{(-1)^n B_{i_1}^{-1}_{HK}}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \frac{\partial f_j}{\partial z_{i_1}, \ldots, \partial z_{i_k}} e^{-\left(\sum_{j=1}^d z^2\right)^{1/2}} \, dy.
\]
So,
\[
Z_t = \sum_{j=1}^d \int_0^t f_j(B_s)dB^i_s.
\]

This completes the proof.
Let \( f_j(x) = (x_j^2 + \cdots + x_{k+1}^2)^{1/2} \); then, \( f_j(tx) = f_j(x) \).

So, for such \( f_j \), one can obtain

\[
g_{i_1, \ldots, i_k}^j (B_1, \ldots, B_{k+1}) = \left( \frac{-1}{(2\pi)^{d/2}} \right)^{k+1} \int_{\mathbb{R}^d} \prod_{i,j} \frac{\partial^k}{\partial y_{i_1} \cdots \partial y_{i_k}} e^{-i(y_j^2/2)} f_j(y) \, dy. \tag{82}
\]

Then, \( (B_j^i/R_i) \) can be denoted by

\[
\frac{B_j^i}{R_i} = \sum_{k=0}^{\infty} \sum_{1 \leq i_1, \ldots, i_k \leq d} \left( \frac{-1}{(2\pi)^{d/2}} \right)^k \int_{\mathbb{R}^d} \prod_{i,j} \frac{\partial^k}{\partial y_{i_1} \cdots \partial y_{i_k}} e^{-i(y_j^2/2)} f_j(y) \, dy \int_{\{0 < s_1 < \cdots < s_k < t\}} \frac{dB_{i_1}^j}{\partial t} \cdots \frac{dB_{i_k}^j}{\partial t}. \tag{83}
\]

and the chaos expansion of \( \int_0^t (B_j^i/R_i) dB_k^j \) is

\[
\int_0^t \frac{B_j^i}{R_i} dB_k^j = \sum_{n=1}^{\infty} \sum_{1 \leq i_1, \ldots, i_n \leq d} \left( \frac{-1}{(2\pi)^{d/2}} \right)^n \int_{\mathbb{R}^d} \prod_{i,j} \frac{\partial^n}{\partial y_{i_1} \cdots \partial y_{i_n}} e^{-i(y_j^2/2)} f_j(y) \, dy \int_{\{0 < s_1 < \cdots < s_n < t\}} dB_{i_1}^j \cdots dB_{i_n}^j dB_{i_{n+1}}^j \cdots dB_{i_{k+1}}^j. \tag{84}
\]

The theorem is proved. \( \square \)

**Theorem 4.** The stochastic process \( X \) is short-range dependent.

**Proof.** Let

\[
\rho_k = \sum_{i,j=0}^d \int_0^t \int_0^{t-1} E \left( \frac{B_j^i}{B_k^i} \right) \phi(t,s) ds \, dt + \sum_{i,j=0}^d \int_0^t \int_0^{t-1} \int_0^t \int_0^t \int_0^t \phi(s,u) \phi(t,v) E \left( \frac{B_j^i}{B_k^i} \right) \frac{D_j^i}{D_k^i} \frac{B_j^i}{B_k^i} \, du \, dv \, ds \, dt
\]

\[
\equiv \rho_{k,1} + \rho_{k,2}.
\]

For \( \rho_{k,1} \), one can use the decomposition

\[
B_i^d = \frac{\partial H(s,t)}{\partial 
abla} B_i^d + \beta_{i,s} \mathcal{N}, \tag{87}
\]

where

\[
E \left( \langle B_i^d, B_j^d \rangle \right) = E \left( \frac{\langle B_i^d, B_j^d + \mathcal{N} \rangle}{\langle B_i^d, B_j^d + \mathcal{N} \rangle} \right) = E \left( \frac{\langle B_i^d, S^{H,K} \lambda_{i,s} B_1 + \mathcal{N} \rangle}{\langle B_i^d, S^{H,K} \lambda_{i,s} B_1 + \mathcal{N} \rangle} \right) = S^{H,K} \lambda_{i,s} E \left( \frac{|B_i^d|^2 - \langle B_1, \mathcal{N} \rangle^2}{|B_i^d, \mathcal{N} |} \right) + O(t^{-HK}). \tag{89}
\]

where \( \mathcal{N} \) is independent of \( B_i^d \) and is denoted as \( ad \)-dimensional standard normal random variable. By Lemma 13 in [11],
Thus,
\[
E\left( \frac{\langle B_2, B_1 \rangle}{|B_2||B_1|} \right)_{s \in H^1 \cap H^1} \geq (90)
\]
which implies that the term \( \rho_{k,1} \) behaves as \( k^{l-3} \).

For \( \rho_{k,2} \), one has
\[
\rho_{k,2} = (H^1)^2 \sum_{l=1}^{d} \sum_{k=1}^{n} \int_{k}^{k+1} E\left( \frac{\prod_{i,j=1}^{k} \delta_{ij} - B_i^T B_j}{|B_i|^2} \left( \prod_{i,j=1}^{k} \delta_{ij} - \frac{B_i^T B_j}{|B_i|^2} \right) \right) \psi_1(s,t) \psi_2(s,t) ds dt
\]

(91)

Since
\[
E\left( \frac{\langle B_2, B_1 \rangle^2}{|B_2||B_1|} \right) + \frac{d-2}{|B_2||B_1|} > 0
\]

behaves as \( Mt^{-H} \) as \( t \to \infty \), where
\[
M = E\left( \frac{\langle A^2 \rangle^2}{|A|^4} + \frac{d-2}{|A|^4} \right) > 0.
\]

We see that the term \( \rho_{k,2} \) also behaves as \( k^{H-3} \), and the theorem follows.

5. The Local Times of \( \int_0^t \text{sign}(B_{s}^{H,K})dB_{s}^{H,K} \)

Now, we consider the local times of the stochastic process
\[
X = \{ X_t \}_{t \geq 0}
\]
declared by
\[
X_t = \int_0^t \text{sign}(B_{s}^{H,K})dB_{s}^{H,K}.
\]

(94)

\[
E[\text{sign}(B_{s}^{H,K})\text{sign}(B_{u}^{H,K})] = \frac{4(2m)}{(2m+1)^2} \left( \frac{1}{2} \right)^{m+1} \left( \frac{2m+1}{2m+1} \right) (2m+1) \approx 2\pi k^2 (su)^{2m+1}
\]

(97)

\[
V_{t_0}(t) = \int_{t_0}^{t} \text{sign}(B_{s}^{H,K})D_sX_s ds, \quad 0 \leq t_0 \leq t,
\]

\[
\Phi(x) = 1_{x \in \Omega},
\]

(100)

where \( \Phi_0(x) = 0 \) for \( x > 0 \) and \( \Phi_0(x) = 1 \) for \( x < 0 \). Thus, one can obtain
which implies that
\[ \lim_{\varepsilon \to 0} E[\mathcal{I}_t]\] is nondecreasing. This completes the proof. \(\square\)

**Theorem 5.** Let the stochastic process \( X = \{X_t, t \geq 0\} \) be defined by
\[ X_t = \int_0^t \text{sign}(B_s^{H,K}) dB_s^{H,K}, \] and let \( \Phi : \mathbb{R}^+ \to \mathbb{R} \) be a convex function with polynomial growth. Then, there is a continuous increasing process \( A^\Phi \) which satisfies
\[ \Phi_t(x) = \Phi(0) + \int_0^t D^- (X_s) \text{sign}(B_s^{H,K}) dB_s^{H,K} + \frac{1}{2} A_t^\Phi, \] where \( D^- \Phi \) denotes the left-hand derivative of \( \Phi \).

**Proof.** If \( \Phi \in C^2 \), then this is the Itô formula, and
\[ A_t^\Phi = \int_0^t \Phi''(X_s) \text{sign}(B_s^{H,K}) D_s X_s ds, \] together with Lemma 2, implies that the stochastic process \( A^\Phi \) is increasing.

Now, let \( \Phi \not\in C^2 \). For \( x \in \mathbb{R} \) and \( \varepsilon > 0 \), one sets
\[ p_\varepsilon(x) = \frac{1}{\sqrt{2\pi\varepsilon}} e^{-\frac{(x)^2}{2\varepsilon}}, \] \[ \Phi_t(x) = \int_{\mathbb{R}} p_\varepsilon(x-y) \Phi(y) dy. \]

It is easy to see that \( \Phi_\varepsilon \in C^2 \) and has polynomial growth. So, for all \( \varepsilon > 0 \), there exists a continuous increasing process \( A^\Phi_\varepsilon \) such that
\[ A^\Phi_\varepsilon_t = \int_0^t \Phi''(X_s) \text{sign}(B_s^{H,K}) D_s X_s ds \]
\[ = \int R \Phi''(x) \left( \int_0^t \delta(X_s - x) \text{sign}(B_s^{H,K}) \right) D_s X_s ds dx. \]

Note that
\[ \lim_{\varepsilon \to 0} \Phi_\varepsilon(x) = \Phi(x) \] and one can obtain as \( \varepsilon \to 0 \)
\[ \int_0^t \Phi''(X_s) \text{sign}(B_s^{H,K}) dB_s^{H,K} \to \int_0^t D^- (X_s) \text{sign}(B_s^{H,K}) dB_s^{H,K}, \] in probability. So, \( A^\Phi \) converges to a stochastic process \( A^\Phi \) which, as a limit of increasing stochastic processes, is itself an increasing stochastic process and
\[ \Phi_t(x) = \Phi(0) + \int_0^t D^- (X_s) \text{sign}(B_s^{H,K}) dB_s^{H,K} + \frac{1}{2} A_t^\Phi, \] where \( A^\Phi \) can be chosen to be a.s. continuous. The proof is completed. \(\square\)

**Corollary 1.** For the process \( X_t = \int_0^t \text{sign}(B_s^{H,K}) dB_s^{H,K} \) and all \( x \in \mathbb{R} \), there exists a local time \( \mathcal{L}^x_t(X) \) such that
\[ |X_t - x| = |x| + \int_0^t \text{sign}(X_s - x) dX_s + \mathcal{L}^x_t(X). \]

**Proof.** Note that the left derivative of the function \( \Phi(y) = (y-x)^+ \) is equal to \( 1_{x \leq 0}(y) \). By Theorem 5, one can obtain
\[ (X_t - x)^+ = (-x)^+ \vee 0 + \int_0^t 1_{X_s > x} \text{sign}(B_s^{H,K}) dB_s^{H,K} + \frac{1}{2} A^+, \] where \( A^+ \) is a continuous increasing stochastic process. Similarly, there exists a continuous increasing stochastic process \( A^- \) which satisfies
\[ (X_t - x)^- = (x)^- \vee 0 + \int_0^t 1_{X_s > x} \text{sign}(B_s^{H,K}) dB_s^{H,K} + \frac{1}{2} A^- . \]

Therefore, one can obtain
\[ X_t = \int_0^t \text{sign}(B_s^{H,K}) dB_s^{H,K} + \frac{1}{2} (A^+ + A^-), \] which implies that \( A^+ = A^- \) a.s. and we set \( \mathcal{L}^x(X) = A_t^\gamma \). This completes the proof.

Combining this corollary with Es-Senbaï and Tudor [3], we can obtain the following results. \(\square\)
Corollary 2. Suppose that $\mathcal{L}_x^\infty(X)$ is the local time of the process $X$ and $\mathcal{L}_x^\infty(B_{H,K})$ is the weighted local time of the bifractional Brownian motion $B_{H,K}$ defined by

$$\mathcal{L}_x^\infty(B_{H,K}) = 2HK \int_0^t \delta(B_{H,K}^s - x) s^{2HK-1} ds. \quad (116)$$

Then, we have

$$|X_t - x| - |B_{H,K}^s - x| = \mathcal{L}_x^\infty(X) - \mathcal{L}_x^\infty(B_{H,K})$$

$$+ \int_0^t \text{sign}(X_s - x) \text{sign}(B_{H,K}^s - x) dB_{H,K}^s - \int_0^t \text{sign}(B_{H,K}^s - x) dB_{H,K}^s$$

$$= \mathcal{L}_x^\infty(X) - \mathcal{L}_x^\infty(B_{H,K}) + \int_0^t \text{sign}(X_s - x) - 1 \text{sign}(B_{H,K}^s - x) dB_{H,K}^s$$

$$= \mathcal{L}_x^\infty(X) - \mathcal{L}_x^\infty(B_{H,K}) - 2 \int_0^t \text{sign}(B_{H,K}^s - x) dB_{H,K}^s,$$

which implies that (117) holds.

Proof. By the Tanaka formula, one can obtain

$$\mathcal{L}_x^\infty(X) - \mathcal{L}_x^\infty(B_{H,K}^s) = |X_t - x| - |B_{H,K}^s - x|$$

$$+ 2 \int_0^t \text{sign}(B_{H,K}^s - x) dB_{H,K}^s. \quad (117)$$

Corollary 3. For any $t \geq 0$ and $x \in \mathbb{R}$, we have

$$\mathcal{L}_x^\infty(X) = \int_0^t \delta(X(s) - x) \text{sign}(B_{H,K}^s) D_s X_s ds. \quad (119)$$

Moreover, let $\Phi: \mathbb{R}^+ \to \mathbb{R}$ be a convex function with polynomial growth; one can obtain the following Itô–Tanaka formula:

$$\Phi(X(t)) = \Phi(0) + \int_0^t D^\cdot \Phi(X_s) \text{sign}(B_{H,K}^s) dB_{H,K}^s$$

$$+ \frac{1}{2} \int_0^t \mathcal{L}_x^\infty(X) \mu_d \, dx,$$

where $D^\cdot \Phi$ denotes the left derivative of $\Phi$ and signed measure $\mu_d$ which is defined by

$$\mu_d ([a,b]) = D^- \Phi(b) - D^- \Phi(a), \quad a < b, a, b \in \mathbb{R}. \quad (121)$$

Finally, one can prove that the local time of the process

$$\sum_{i=1}^d \int_0^t \frac{B_{H,K}^i}{R_t} dB_{H,K}^i, \quad d \geq 2$$

exists by the same method and can obtain the similar results.

6. Conclusions

This paper presents theorems and propositions associated with respect to the stochastic process $R_t = \sqrt{(B_t^1)^2 + \cdots + (B_t^d)^2}$, where $B = \{B_t^1, \ldots, B_t^d\}_{t \geq 0}$ is a $d$-dimensional bifractional Brownian motion and $(2HK \geq 1)$. Since there is no Lévy’s characterization theorem for a general bifractional Brownian motion, to prove whether a stochastic process

is a bifractional Brownian motion or not is difficult. Theorems 1 and 4 prove $X_t$ is short-range dependent in one-dimensional case and multidimensional case, respectively. Theorem 5 considers the local times of the stochastic process of $X_t$ in one-dimensional case. Theorem 2 gives the following chaos expansion of the stochastic process $X_t = \sum_{i=1}^d \int_0^t f_i(B_t^i) dB_{H,K}^i$. Moreover, significance results associated with the above theorem are given.

Data Availability

All the data generated during this study are included within this article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors’ Contributions

All the authors contributed equally and significantly in writing this paper. All the authors read and approved the final manuscript.

Acknowledgments

Xichao Sun acknowledges the National Natural Science Foundation of China (11971101 and 11426036), Natural Science Foundation of Anhui Province (1808085MA02), and Quality Engineering of Anhui Education (2018jyxm0497 and 2020zdtxsjg214). Ming Li acknowledges the National Natural Science Foundation of China under grant nos. 61672238, 61272402, and 61070214.
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