A SINGULAR DEMAILLY-PĂUN THEOREM

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Abstract. We give a numerical characterization of the Kähler cone of a possibly singular compact analytic variety which is embedded in a smooth ambient space.

1. Introduction

The classical Nakai-Moishezon ampleness criterion (see e.g. [8] and references therein) characterizes ample line bundles on a projective variety as those which have positive intersection against all subvarieties. This was later extended to $\mathbb{R}$-divisors by Campana-Peternell [1]. In a groundbreaking paper, Demailly and Păun [7] proved a vast generalization of this result, which holds for all real $(1,1)$ classes on a compact Kähler manifold. More precisely, they proved that the Kähler cone of a compact Kähler manifold is one of the connected components of the positive cone, consisting of classes which have positive intersection against all analytic subvarieties. Very recently, a new proof of this theorem was obtained by combining the main result of our previous work [3] with a result of Chiose [2].

In this note, we prove an extension of the Demailly-Păun theorem [7] to singular varieties which are embedded in a smooth ambient space. A $(1,1)$ class on the variety is just taken to be the restriction of a $(1,1)$ class from the ambient space, and such a class is Kähler if it is so in a neighborhood of the variety inside the ambient space. This is in fact equivalent to the more intrinsic definition of a Kähler class on a compact analytic space as given for example in [13], as shown by Păun [9], and this allows us to avoid discussing these more technical notions. With these observations in mind, our main theorem is the following:

Theorem 1.1. Let $(M, \omega)$ be a smooth (but possibly noncompact and incomplete) Kähler manifold, and $E \subset M$ be a compact analytic subvariety. Let $\alpha$ be a closed smooth real $(1,1)$ form on $M$ such that

$$\int_V \alpha^k \wedge \omega^{\dim V-k} > 0,$$

for all positive-dimensional irreducible analytic subvarieties $V \subset E$, and for all $1 \leq k \leq \dim V$. Then there exist an open neighborhood $U$ of $E$ in $M$ and a smooth function $\varphi : U \to \mathbb{R}$ such that $\alpha + \sqrt{-1} \partial \bar{\partial} \varphi$ is a Kähler metric on $U$. If $M$ is an open subset of the regular locus of some projective variety,
then the inequalities
\[
\int_V \alpha^\dim V > 0,
\]
for all \( V \) as above suffice to reach the same conclusion.

This theorem answers a question that was posed to us by R.J. Conlon and H.-J. Hein, in relation to their paper \[5\] (see also \[4, 1.3.5\]). Applications of this result to the study of the Kähler cone of asymptotically conical Calabi-Yau manifolds will appear in a forthcoming revision of \[5\].

The main tools we use are the Demailly-Păun theorem itself, for smooth compact Kähler manifolds, and our recent theorem \[3\] which shows that the non-Kähler locus of a nef and big class on a compact complex manifold equals the null locus of the class. The idea is to work by induction on the dimension on \( E \) (as in \[7\]), and to prove the result by working on a resolution of singularities (as in \[3\]). This way we avoid any technical discussion of currents on singular analytic spaces.

In future work, we hope to address the extension of the Demailly-Păun theorem \[7\] as well as the main result of our previous work \[3\] to general compact Kähler (reduced and irreducible) analytic spaces.

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2. Proof of Theorem 1.1

This section contains the proof of Theorem 1.1.

Clearly we may assume that no component of \( E \) is zero-dimensional, since for those the result is trivial.

Let us first assume that \( E \) is irreducible and 1-dimensional. Let \( \nu : \tilde{M} \to M \) be an embedded resolution of singularities of \( E \subset M \), so that \( \tilde{M} \) is smooth, connected and Kähler, and the proper transform \( \tilde{E} \) of \( E \) is a smooth compact Riemann surface. We will also write \( \nu : \tilde{E} \to E \) for the induced map, so that \( \nu^* \alpha \) is a smooth closed real \((1,1)\) form with \( \int_{\tilde{E}} \nu^* \alpha > 0 \).

Therefore the class \( [\nu^* \alpha] \) on \( \tilde{E} \) is Kähler, and we can find a smooth function \( \psi \) on \( \tilde{E} \) such that \( \nu^* \alpha + \sqrt{-1} \partial \bar{\partial} \psi > 0 \) on \( \tilde{E} \). It is elementary to find an open neighborhood \( \tilde{U} \) of \( \tilde{E} \) in \( \tilde{M} \) and a smooth extension of \( \psi \) to \( \tilde{U} \) (still denoted by \( \psi \)) such that \( \nu^* \alpha + \sqrt{-1} \partial \bar{\partial} \psi > 0 \) on \( \tilde{U} \) (see e.g. \[9\] Lemme 1, p.416). Note that \( U = \nu(\tilde{U}) \setminus E_{\text{sing}} \) is an open neighborhood of \( E_{\text{reg}} \) inside \( M \), but in general it is not the case that \( \nu(\tilde{U}) \) is an open neighborhood of \( E \).
inside $M$, because it may “pinch off” near $E_{\text{sing}}$. Furthermore, even if $\nu(\tilde{U})$ happens to be an open neighborhood of $E$, the pushforward function $\nu_\ast \psi$ is not well-defined wherever different branches of $\tilde{E}$ come together under the map $\nu$. Therefore, we need to work a bit harder to achieve our goal.

Let $\{p_1, \ldots, p_N\} \subset \tilde{E}$ be the exceptional locus of $\nu$ intersected with $\tilde{E}$, so that $\{\nu(p_1), \ldots, \nu(p_N)\}$ equals the singular set of $E$. For each point $p_j$ we add to $\psi$ a function of the form $\varepsilon \theta(z) \log |z - p_j|$, where $\varepsilon > 0$ is small enough, where $z = (z_1, \ldots, z_N)$ are local coordinates for $\tilde{M}$ near $p_j$, and $\theta$ is a smooth cutoff function supported in a small neighborhood of $p_j$ in $\tilde{M}$, so that we obtain a new function $\tilde{\psi}$, which is smooth away from the $p_j$’s and goes to $-\infty$ there, and such that $\nu_\ast \alpha + \sqrt{-1} \partial \bar{\partial} \tilde{\psi}$ is a Kähler current on $\tilde{U}$.

Then the smooth function $\tilde{\psi} = \nu_\ast \psi$ on $U$ satisfies $\alpha + \sqrt{-1} \partial \bar{\partial} \tilde{\psi} > 0$, but we are not done yet because $U$ does not contain the singular points of $E$. Let $\{\nu(p_1), \ldots, \nu(p_k)\}$ be all the singular points of $E$ (so $k \leq N$), and fix charts $U_j$ for $M$ centered at $\nu(p_j)$ for $1 \leq j \leq k$, with coordinates so that each $U_j$ is the Euclidean ball of radius 2. Call $U_j'$ the Euclidean ball of radius 1 in these coordinates, and let $A$ be the minimum of $\psi$ on the compact set $\bigcup_{j=1}^k (\partial U_j') \cap U$, which is a finite number because $\tilde{\psi}$ is smooth there. Choose a large constant $B > 0$ such that on each $U_j$ we have $\alpha + B \sqrt{-1} \partial \bar{\partial} |z|^2 > 0$. On $U \cap U_j$ then we have that $\tilde{\psi}$ and $B|z|^2 + A - B - 1$ are both strictly $\alpha$-plurisubharmonic, with $\tilde{\psi}$ approaching $-\infty$ at the center of the ball $U_j$, and with $\tilde{\psi} > B|z|^2 + A - B - 1$ on a neighborhood of $(\partial U_j') \cap U$. If $\max$ denotes a regularized maximum function (see, e.g. [6] I.5.18)), then

$$\psi_g = \max(\tilde{\psi}, B|z|^2 + A - B - 1)$$

is smooth and strictly $\alpha$-plurisubharmonic on $U_j \cap U$, it equals $\tilde{\psi}$ in a neighborhood of $(\partial U_j') \cap U$, and it equals $B|z|^2 + A - B - 1$ as we approach the origin. Therefore the function $\psi_g$ trivially glues to $\tilde{\psi}$ outside $U_j'$, and we can extend it to be equal to $B|z|^2 + A - B - 1$ in a small neighborhood of the origin in $U_j$. Repeating this construction for all $j$, and gluing each of them to $\tilde{\psi}$, we finally obtain an open neighborhood $\overline{U}$ of $E$ in $M$ and a smooth function $\varphi$ on $\overline{U}$ such that $\alpha + \sqrt{-1} \partial \bar{\partial} \varphi$ is a Kähler metric on $\overline{U}$, as required.

Next, we assume that $E$ has pure dimension 1, but need not be irreducible anymore. Then, writing $E = \cup_j E_j$ with $E_j$ irreducible, we can apply the result to each $E_j$ and obtain $U_j, \varphi_j$ as above, and “glue” them all together using [9] Lemme, p.419, and obtain the desired Kähler potential $\varphi$ on some neighborhood $\overline{U}$ of $E$.

We now deal with the general case, by induction on $\dim E$ (which is by definition the max of the dimensions of the irreducible components of $E$). The base of the induction is what we have just proved. For the induction
step, let \( \dim E = n \) and assume the result holds in all dimensions \(< n \). As we just did, it is enough to prove the theorem in the case when \( E \) is irreducible, since if there are several components then we work on each one separately, and in the end glue the resulting metrics as before. So we will assume that \( E \) is irreducible. Take \( \nu : \tilde{M} \to M \) to be an embedded resolution of singularities of \( E \subset M \), obtained as a composition of blowups with smooth centers, so that \( \tilde{M} \) is smooth and Kähler, and the proper transform \( \tilde{E} \) of \( E \) is smooth.

Then \( \nu^*\alpha \) is a smooth closed real \((1,1)\) form on \( \tilde{E} \), and we claim that its class \([\nu^*\alpha]\) on \( \tilde{E} \) is nef. If assume that \( M \) is an open subset of the regular locus of some projective variety, then this holds because we have \( \int_V (\nu^*\alpha)^{\dim V} \geq 0 \) for all positive-dimensional irreducible subvarieties \( V \) in \( \tilde{E} \) (using \( [11,1] \)), and so \([7, \text{Theorem 4.5(ii)}] \) gives that the class \([\nu^*\alpha]\) on \( \tilde{E} \) is nef. However, in our general setup (where there may be no projective compactification), to use \([7, \text{Theorem 4.3(ii)}] \) we would have to check instead that

\[
\int_V \nu^*\alpha^k \wedge \tilde{\omega}^{\dim E - k} \geq 0,
\]

for all positive-dimensional irreducible subvarieties \( V \subset \tilde{E} \), for some Kähler form \( \tilde{\omega} \) on \( \tilde{E} \) and for all \( 1 \leq k \leq \dim V \), and it does not seem easy to check this directly. Instead, we argue as follows. We have

\[
\int_{\tilde{E}} \nu^*(\alpha^k \wedge \tilde{\omega}^{\dim E - k}) > 0,
\]

for \( 1 \leq k \leq \dim E \), because \( \nu : \tilde{E} \to E \) is a modification, and using \( [11] \). Since the class \([\nu^*\omega]\) is nef on \( \tilde{E} \), we can find Kähler classes on \( \tilde{E} \) arbitrarily close to it, and therefore there exists a Kähler metric \( \tilde{\omega} \) on \( \tilde{E} \) such that

\[
\int_{\tilde{E}} \nu^*\alpha^k \wedge \tilde{\omega}^{\dim E - k} > 0,
\]

for \( 1 \leq k \leq \dim E \). Now for \( t \geq 0 \) sufficiently large, the class \([\nu^*\alpha + t\tilde{\omega}]\) is Kähler on \( \tilde{E} \). Let \( t_0 \) be the minimum value of \( t \) such that the class \([\nu^*\alpha + t\tilde{\omega}]\) is nef on \( \tilde{E} \), and suppose for a contradiction that \( t_0 > 0 \). By definition the class \([\nu^*\alpha + t_0\tilde{\omega}]\) is not Kähler on \( \tilde{E} \). Thanks to \([7, \text{Theorem 0.1]} \), there exists a positive-dimensional irreducible analytic subvariety \( V \subset \tilde{E} \), such that

\[
\int_V (\nu^*\alpha + t_0\tilde{\omega})^{\dim V} = 0,
\]

since if we had strict positivity for all such \( V \) then the class \([\nu^*\alpha + t_0\tilde{\omega}]\) would be Kähler. Also \( V \) must be properly contained in \( \tilde{E} \), because we have

\[
\int_{\tilde{E}} (\nu^*\alpha + t_0\tilde{\omega})^{\dim E} > 0,
\]

by \([21] \). Then \( \nu(V) \) is an irreducible analytic subvariety of \( E \) (possibly a point), of dimension strictly less than \( \dim E \), and with the same positivity property \([11]\), so by induction we can find an open neighborhood \( W \) of
\( \nu(V) \) in \( M \) and a smooth function \( \eta \) on \( W \) such that \( \alpha + \sqrt{-1} \partial \bar{\partial} \eta > 0 \). Therefore, in the open neighborhood \( \nu^{-1}(W) \) of \( V \) the smooth function \( \nu^* \eta \) satisfies \( \nu^* \alpha + \sqrt{-1} \partial \bar{\partial} (\nu^* \eta) \geq 0 \). Since \( \tilde{\omega} \) is Kähler on \( \tilde{E} \) and \( t_0 > 0 \), this implies that
\[
\int_V (\nu^* \alpha + t_0 \tilde{\omega})^{\dim V} > 0,
\]
contradicting (2.2). Therefore we must have \( t_0 \leq 0 \), and so the class \( [\nu^* \alpha] \) is indeed nef on \( \tilde{E} \).

This proves our claim that the class \( [\nu^* \alpha] \) is nef on \( \tilde{E} \), and since
\[
\int_{\tilde{E}} (\nu^* \alpha)^{\dim E} = \int_E \alpha^{\dim E} > 0,
\]
by (11), we can apply [7, Theorem 2.12] and see that this class is also big, i.e. it contains a Kähler current \( \nu^* \alpha + \sqrt{-1} \partial \bar{\partial} \psi \), which we may assume has analytic singularities thanks to Demailly’s regularization theorem (see [7, Theorem 3.2]). Also, if \( V \not\subseteq \text{Exc}(\nu) \cap \tilde{E} \) then \( \nu(V) \) is an irreducible subvariety of \( E \) of the same dimension as \( V \), and \( \nu : V \to \nu(V) \) is bimeromorphic and so we have \( \int_V (\nu^* \alpha)^{\dim V} = \int_{\nu(V)} \alpha^{\dim V} > 0 \), thanks to assumption (11). This means that the null locus of the class \( [\nu^* \alpha] \) on \( \tilde{E} \) is contained in \( \text{Exc}(\nu) \), and so using [3, Theorem 1.1], we may choose \( \psi \) to be smooth on \( \tilde{E} \setminus \text{Exc}(\nu) \).

We use [7, Lemma 2.1] to obtain a quasi-plurisubharmonic function with nontrivial analytic singularities along \( \text{Exc}(\nu) \), and add a small multiple of it to \( \psi \), to obtain a function \( \hat{\psi} \) which is smooth on \( \tilde{E} \setminus \text{Exc}(\nu) \) and goes to \(-\infty\) along \( \text{Exc}(\nu) \), and such that \( \nu^* \alpha + \sqrt{-1} \partial \bar{\partial} \hat{\psi} \) is a Kähler current on \( \tilde{E} \) with analytic singularities along \( \text{Exc}(\nu) \).

As in the first part of the proof of [3, Theorem 3.2], up to modifying \( \hat{\psi} \) slightly (maintaining its same properties) we can find an extension \( \hat{\psi}' \) to an open neighborhood \( \tilde{U} \) of \( \tilde{E} \setminus \text{Exc}(\nu) \) in \( \tilde{M} \).

Here are some details for this construction (see [3] for full details). By a resolution of singularities argument, we construct a modification \( \mu : \tilde{M} \to \tilde{M} \), which is a composition of blowups with smooth centers, such that \( \mu(\text{Exc}(\mu)) \) is equal to \( \text{Exc}(\nu) \), such that the proper transform \( \tilde{E} \) of \( E \) is smooth, and the pullback under \( \mu \) of the ideal sheaf on \( \tilde{E} \) which defines the singularities of the Kähler current \( \nu^* \alpha + \sqrt{-1} \partial \bar{\partial} \hat{\psi} \) is a principal ideal, supported along a simple normal crossings divisor, which is the restriction to \( \tilde{E} \) of a simple normal crossings divisor on \( \tilde{M} \) (which is equal to \( \text{Exc}(\mu) \)), which has normal crossings with \( \tilde{E} \). We then cover \( \tilde{E} \) by finitely many coordinate charts \( \{W_j\} \) for \( \tilde{M} \). To the pullback \( \mu^* \hat{\psi} \) we add a small multiple of \( \sqrt{-1} \partial \bar{\partial} \log |s|^2_h \), where \( s \) defines \( \text{Exc}(\mu) \) (and \( h \) is chosen suitably), to obtain a strictly \( \mu^* \nu^* \alpha \)-plurisubharmonic function \( \Psi \) on \( \tilde{E} \), with analytic singularities as before (in particular, smooth away from \( \text{Exc}(\mu) \)). For each \( j \), we then extend \( \Psi|_{W_j \cap \tilde{E}} \) to a function \( \psi_j \) on \( W_j \) in an elementary fashion, still
preserving strict $\mu^*\nu^*\alpha$-plurisubharmonicity. Then we use a gluing procedure inspired by a classical method of Richberg [11] (see e.g. [12] Lemma 3.3), but with the extra difficulty that now the functions $\psi_j$ have poles. Nevertheless, arguing exactly as in [3] Proof of Theorem 3.2, we can obtain an open neighborhood $U_1$ of $\hat{E}$ in $\hat{M}$ and a strictly $\mu^*\nu^*\alpha$-plurisubharmonic function $\hat{\Psi}$ on $U_1$, which restricts to $\Psi$ on $\hat{E}$, and is smooth on $\hat{E}\setminus \text{Exc}(\mu)$.

Here we highlight that since $\hat{E}$ is a complex submanifold of a complex manifold, constructing this extension $\hat{\Psi}$ on an open neighborhood $U_1$ of $\hat{E}$ in $\hat{M}$ would be standard by Richberg [11] if $\Psi$ was smooth (or even just continuous) on $\hat{E}$. On the other hand, if the singularities of $\Psi$ were completely arbitrary, then such an extension would not be possible in general. The key property that saves us here is that the singular locus of $\Psi$ is the intersection with $\hat{E}$ of a simple normal crossings divisor, $\text{Exc}(\mu)$.

Then we take $\tilde{U} = \mu(U_1)$, and $\tilde{\psi} = \mu_*\hat{\psi}$, which are as required. In particular, $\tilde{U}$ is an open neighborhood of $\hat{E}\setminus \text{Exc}(\nu)$ in $\hat{M}$, and $\tilde{\psi}$ is strictly $\nu^*\alpha$-plurisubharmonic, and it is smooth on $\tilde{U}\setminus \text{Exc}(\nu)$.

On the open set $U = \nu(\tilde{U})\setminus E_{\text{sing}}$ (which is a neighborhood of $E_{\text{reg}}$ in $M$) we have the smooth function $\hat{\psi} = \nu_*\tilde{\psi}'$ with $\alpha + \sqrt{-1}\partial \bar{\partial}(\nu_*\tilde{\psi}')$ a smooth Kähler metric there, and with $\nu_*\tilde{\psi}'$ approaching $-\infty$ along $E_{\text{sing}}$. Now $E_{\text{sing}}$ is a subvariety of $M$ of dimension strictly less than $n$, with the same positivity property [11], so by induction we can find an open neighborhood $W$ of $E_{\text{sing}}$ in $M$ and a smooth function $\tilde{\varphi}$ on $W$ with $\alpha + \sqrt{-1}\partial \bar{\partial}\tilde{\varphi} > 0$ on $W$. We may also assume that $\tilde{\varphi}$ is defined on a slightly larger open set, so that it is smooth up to $\partial W$.

If we let $A$ be the minimum of $\hat{\psi}$ on the compact set $(\partial W) \cap \tilde{U}$ and $B$ be the maximum of $\tilde{\varphi}$ on the same set, then $\hat{\psi} > \tilde{\varphi} + A - B - 1$ holds on a neighborhood of $(\partial W) \cap U$. Then

$$\psi_g = \max(\hat{\psi}, \tilde{\varphi} + A - B - 1)$$

is smooth and strictly $\alpha$-plurisubharmonic on $U \cap W$, equal to $\hat{\psi}$ near $(\partial W) \cap U$, and equal to $\tilde{\varphi} + A - B - 1$ as we approach $E_{\text{sing}}$. Therefore $\psi_g$ trivially glues to $\hat{\psi}$ outside $W$, and we can extend it to be equal to $\tilde{\varphi} + A - B - 1$ in a neighborhood of $E_{\text{sing}}$. In this way we obtain an open neighborhood $U$ of $E$ in $M$ and a smooth function $\varphi$ on $U$ such that $\alpha + \sqrt{-1}\partial \bar{\partial}\varphi$ is a Kähler metric on $U$, as required.

Lastly, the statement in the projective case follows from the Kähler one exactly as in [2], by choosing $\omega$ to be the curvature form of a very ample line bundle $L$ on the projective variety which contains $M$ as an open subset, and observing that

$$\int_V \alpha^k \wedge \omega^\dim V - k = \int_{V \cap H_1 \cap \cdots \cap H_{\dim V - k}} \alpha^k,$$
for generic members $H_1, \ldots, H_{\dim V - k}$ of the linear system $|L|$, so that $V \cap H_1 \cap \cdots \cap H_{\dim V - k}$ is an irreducible subvariety of dimension $k$.

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