TRAFFIC CONGESTION PRICING VIA NETWORK CONGESTION GAME APPROACH

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ABSTRACT. This paper investigates the optimization of traffic congestion systems via network congestion game approach. Firstly, using the semi-tensor product (STP) of matrices, the matrix expression of network congestion game is obtained. Secondly, a necessary and sufficient condition is proposed to guarantee that the traffic systems can be transformed into network congestion game with given performance criterion as its weighted potential function. Then an algorithm is provided to design the traffic congestion price in the case that conversion can be established. Thirdly, by designing proper learning rule, the optimization of traffic systems can be achieved when individuals optimize their own utility function. Moreover, two special cases which make our results more accord with reality and rich. Finally, an example is exploited to demonstrate the effectiveness of our obtained results.

1. Introduction. At present, traffic congestion has become one of the main bottlenecks restricting the sustainable development of cities [13][9], so it is necessary to optimize the traffic systems and relieve the congestion by taking some measures. The implementation of congestion tax has proved to be a good way to solve congestion. In [8], Eliasson et al. showed the significant improvements in travel times when the local governments in some urban areas introduced congestion taxes. So

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how to design a reasonable congestion pricing mechanism is the key to implement this measure. The classic way to solve such problem is to formulate a bi-level optimization model with system optimum as the upper level objective and traffic assignment problem as the lower level. Congestion pricing is the decision variable of such bi-level model. The network design problem (NDP) in which the objective function is to achieve system optimum subject to user equilibrium has been thoroughly studied in the area of transport engineering [31]–[35].

To analyze this decentralized system, another nature framework is game theoretical control. Gopalakrishnan et al. described an hourglass architecture of game theoretic approach in [10]. By designing utility function, some systems can be translated into potential game, then using proper learning rule, the optimization can be finally achieved. The process can be explained in detail in [14][15]. Rosenthal [26] described the situation of road congestion and proposed the network equilibrium model in [27]. Rosenthal pointed out that this problem can be viewed as an $n$-person non-cooperative game and showed that the associated strategic game has pure Nash equilibria. Motivated by Rosenthal’s results, Monderer and Shapley [24] proved that the existence of a potential function and that each finite potential game can be derived from a congestion form. Le et al. extended Rosenthal’s model to a weighted congestion model with player-specific utility in [16] and proved the existence of weighted potential function. Therefore, considering the network congestion game can be regarded as a special potential game. Similar to the two steps in game-theoretical control, we want to convert the traffic system into a network congestion game. We propose the approach of network congestion game, which can be shown in Fig.1 [10]. It is the same as the potential game control theory. Firstly, by designing the travelers’ utility function, and then through the design of learning rules, the traffic congestion system can be transformed into the network congestion game.

Because the travelers’ utility function is determined by the road congestion price, to design a reasonable congestion charging mechanism is an important issue. As we all know, traffic flow can be regarded as the result of game between travelers and traffic managers [27], travelers act in a purely selfish manner and try to minimize their own objective function by choosing a suitable roads. The Nash equilibrium implies that they can not decrease their travel time by changing to another road. However, it is well known that an equilibrium point in pure strategies (if exist) does not always optimize the social welfare as individual incentives are not always compatible with system objectives. The most famous example is the Prisoner Dilemma [7]. So we will consider the problem about how to guarantee that the travelers and traffic systems achieve optimization at the same time.

In this paper, we investigate the traffic systems with a system performance criterion being given. Normally travelers do not follow the path allocated by managers, but choose the path according to the principle of maximize their own utility function. We want to study whether it is possible for the manager to design appropriate congestion price so that the traffic system becomes a network congestion game with the given system performance criterion as its potential function. Then by designing learning rule, when the travelers use congestion price to maximize their utility functions, the corresponding objective function of overall traffic system can be optimized.

The basic tool employed in this paper is the semi-tensor product (STP) of matrices, which is proposed by Cheng et.al [2]–[4]. It is a generalization of conventional
matrix product. This tool has been well applied to study Boolean networks [36]-[39], logical dynamics system [29], nonlinear feedback shift registers [21] and other problems. Recently, it has also been used to investigate the problems of games [3]-[11].

The main contributions of this paper are listed as follows:

1. A network congestion game method is proposed to optimize the traffic systems. A necessary and sufficient condition is obtained to guarantee that the traffic systems can be converted to network congestion game with the given performance criterion as its weighted potential function, which can be verified by checking whether the corresponding linear equations have solutions or not.

2. An algorithm for designing traffic congestion price is provided in the case of transforming can be established. The results show that when the travelers consider congestion price to maximize their own utilities, the whole traffic systems can also achieve the optimization.

The rest of this paper is arranged as follows. In Section 2, some prepare knowledges are given. The main results of the paper are presented and proved in Section 3, including the matrix expression of network congestion game, the conversion problem about traffic systems and network congestion game. In Section 4, an example is carried out to demonstrate the derived results. Finally, a brief conclusion is given in Section 5.

2. Preliminaries. In this section, some necessary preparations including the definition and properties about STP [2]–[4], and the knowledge of potential game theory [5] are presented.

2.1. Semi-tensor product of matrices. Firstly, some notations which will be used are given in the following.

- $R$ denotes the set of real numbers.
- $R^{m \times n}$ denotes the set of $m \times n$ real matrices.
- $\mathcal{D} = \{1, 0\}$.
- $\mathcal{D}_k = \{1, 2, \ldots, k\}, k \geq 2$.
- $\delta^i_k$ denotes the $i$-th column of the identify matrix $I_k$.
- $\Delta_k = \{\delta^i_k | i = 1, 2, \ldots, k\}$.
- The $i$-th column of matrix $L$ is denoted by $\text{col}_i(L)$. 

![Figure 1. The approach of network congestion game](image-url)
A matrix $L \in R^{m\times n}$ is called a logical matrix if the columns of $L$ belong to $\Delta_m$, then $L$ can be expressed as $L = [\delta_{m1}, \delta_{m2}, \ldots, \delta_{mn}]$, we simply denoted it as $L = \delta_m[i_1,i_2,\ldots,i_m]$.

$L_{m\times n}$ denotes the set of $m \times n$ logical matrices.

1) $\mathbf{1}_k = (1,1\ldots,1)^T$.

$\cdot$ denotes the cardinality.

**Definition 2.1.** [4] The semi-tensor product of matrix $A \in R^{m\times n}$ and matrix $B \in R^{p\times q}$ is defined as follows:

$$A \times B = (A \otimes I_{\alpha/n})(B \otimes I_{\alpha/p}),$$

(1)

where $\alpha$ is the least common multiple of $n$ and $p$, and $\otimes$ is Kronecker product.

It is noted that if $n = p$, then $A \times B = AB$, the semi-tensor product of matrices $A$ and $B$ becomes the conventional product of matrices $A$ and $B$. Hence, we omit the symbol “$\times$” throughout the paper. It is clearly indicated that the STP of matrices is a generalization of conventional matrix product. Not only the basic properties can be maintained, but also it produces some new properties.

**Definition 2.2.** Let $p \geq 2$ and $q \geq 2$ be two integers. The rear (front) deleting operator is defined as

$$D^{[p,q]}_r = I_p \otimes 1^T_q,$$

$$D^{[p,q]}_f = 1^T_p \otimes I_q.$$ 

**Proposition 1.** Let $X \in \Delta_p$ and $Y \in \Delta_q$ be two columns. Then

$$D^{[p,q]}_r XY = X,$$

$$D^{[p,q]}_f XY = Y.$$ 

**Definition 2.3.** Let $M \in R^{p\times m}$ and $N \in R^{q\times m}$. Then the Khatri-Rao product of $M$ and $N$ is defined as

$$M \times N = [\text{Col}_1(M) \times \text{Col}_1(N) \ldots \text{Col}_m(M) \times \text{Col}_m(N)] \in R^{pq\times m}.$$ 

(2)

**Lemma 2.4.** [2] Let $x_i \in D_k, i = 1,2,\ldots,n$, $f : D^m_k \rightarrow D_k$ be a $k$-valued logical function. Then there exists a unique logical matrix $M_f \in L_{k\times k^n}$, such that the vector form expressed as

$$f(x_1,x_2,\ldots,x_n) = M_f \otimes_{i=1}^n x_i,$$

(3)

where $M_f$ is called the structure matrix of $f$.

**Proposition 2.** Let $u : D^m_k \rightarrow D_k$, and $v : D^m_k \rightarrow D_k$ be expressed in algebraic form as $u = M_u \otimes_{i=1}^n x_i, v = M_v \otimes_{i=1}^n x_i$, where $M_u \in L_{k\times k^n}, M_v \in L_{k\times k^n}$. Then

$$uv = (M_u \ast M_v) \otimes_{i=1}^n x_i,$$

(4)

where $\ast$ is the Khatri-Rao product.

Aimed at an $k$-valued pseudo-logical function, it is straightforward to obtain the similar result written in the following.

**Corollary 1.** [2] Let $x_i \in D_k, i = 1,2,\ldots,n$. $c : D^m_k \rightarrow R$ be an $k$-valued pseudo-logical function. Then there exists a unique row vector $V^c \in R^{1 \times k^n}$, such that the vector form expressed as

$$c(x_1,x_2,\ldots,x_n) = V^c \otimes_{i=1}^n x_i,$$

(5)

where $V^c$ is called the structure vector of $c$. 
2.2. Potential game.

**Definition 2.5.** [24] Consider a finite game $G=(N,S,C)$, $G$ is called a potential game if there is a function $P : S \rightarrow \mathbb{R}$, called the potential function of $G$, such that for every $i \in N$,

$$c_i(x, s^{-i}) - c_i(y, s^{-i}) = P(x, s^{-i}) - P(y, s^{-i}), \forall x, y \in S, s^{-i} \in S^{-i},$$

(6)

where $S_i$ is the set of strategies of player $i$, $S^{-i} = \prod_{j \neq i} S_j$.

Now, if a game is given, we want to know whether the game is potential and the way to calculate potential function. Cheng et al. [5] presented the method and here we briefly describe it.

Define

$$\psi_i = I_{k^{i-1}} \otimes 1_k \otimes I_{k^{n-i}}, i = 1, 2, \ldots, n.$$  

(7)

and set

$$\xi_i \in \mathbb{R}^{k^{n-1}}, i = 1, 2, \ldots, n.$$  

$$b_i := (V_i^c - V_1^c)^T \in \mathbb{R}^{k^n}, i = 2, 3, \ldots, n.$$  

(8)

where $V_i^c$ is the structure vector of the payoff $c_i, i = 1, 2, \ldots, n$. Then we can construct a linear system, called the potential equation as

$$\Psi \xi = b,$$  

(9)

where

$$\Psi = \begin{bmatrix} -\psi_1 & \psi_2 & 0 & \ldots & 0 \\ -\psi_1 & 0 & \psi_3 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ -\psi_1 & 0 & 0 & \ldots & \psi_n \end{bmatrix}; \xi = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{bmatrix}; b = \begin{bmatrix} b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix}.$$  

We have the following lemma to judge whether a game is potential.

**Lemma 2.6.** [5] A finite game $G$ is potential, if and only if the potential equation (9) has a solution. Moreover, when the game is potential, the potential function can be calculated by

$$P(x) = V_p(x) + c_0,$$  

(10)

where $x = \kappa_{i=1}^n x_i, V_p = V_1^c - \xi_1^T D_f^{[k,k]}$, and $c_0$ is an arbitrary number.

3. Main results. Consider a traffic congestion system with a given performance criterion $T = \{N, (V, E), (A_i), P\}$, where $N$ is the set of travelers, $(V, E)$ is a directed network graph with the edges representing roads shared among all travelers, traveler $i$ must select an appropriate directed path from its pair origin-destination nodes $(o_i, d_i) \in V \times V$. $A_i$ indicates the set of alternative feasible paths for $i$. $P$ is a system objective function which needs to be optimized. Considering the fact that all travelers care only their own profits, but not the benefits of overall system, our goal is to translate this problem into a network congestion game such that its weighted potential function is $P$. So designing proper traffic congestion price is the key issue to achieve optimization.
3.1. Matrix expression of network congestion game. In this section, we will give the algebraic representation of network congestion game based on the tool of semi-tensor product of matrices.

**Definition 3.1.** A traffic congestion form is a tuple \( F = \{ N, (V, E), (A_i)_{i\in N}, (r_i)_{i\in N}, (c_l)_{l\in E}, \omega \} \), where
- \( N = \{ 1, 2, ... n \} \) is the set of travelers.
- \( (V, E) \) is a directed network graph with the edge \( e_i \) playing the role of road. \( (o_i, d_i) \in V \times V \) is the pair of origin-destination for traveler \( i \).
- \( A_i \) is the set of all paths from \( o_i \) to \( d_i \) for traveler \( i \).
- \( r_i \) represents the reward when the traveler \( i \) arrives the destination.
- \( c_l \) is the cost function depending on the number of travelers on road \( l \). In this paper, we interpret it as the traffic congestion price.
- \( \omega = (\omega_1, \omega_2, ... \omega_n) \) and \( \omega_i \) is the specific scaling of traveler \( i \)'s cost.

As we all know, each network congestion game \( G \) can be derived from a network congestion form \( F \) [24]. Therefore, in the following, the definition of network congestion game is given.

**Definition 3.2.** [23] A network congestion game is a tuple \( G = \{ N, (V, E), (A_i)_{i\in N}, (u_i)_{i\in N} \} \).
- \( N = \{ 1, 2, ... n \} \) is the set of players.
- \( (V, E) \) is a directed network graph with the edge playing the role of road. \( (o_i, d_i) \in V \times V \) is the pair of origin-destination for player \( i \).
- \( A_i \) is the set of all paths from \( o_i \) to \( d_i \) for player \( i \), we call it as the \( i \)’s strategy set. Denote the set of strategy profiles as \( A = \prod_{i=1}^{n} A_i \).
- \( u_i : A \mapsto R \) represents the utility when the traveler \( i \) arrives the destination.

For a profile \( a = (a_1, a_2, ... a_n) \in A, a_i \in A_i, c_l \in a_i \) suggests that player \( i \) choose road \( c_l \) under \( a_i \). Denote \( n_{c_l}(a) = |\{ i \in N | c_l \in a_i \}| \). \( c_{c_l}(n_{c_l}(a)) \) expresses the traffic congestion price of road \( c_l \). So the utility function of player \( i \) is

\[
    u_i(a) = r_i(a_i) - \omega_i \sum_{c_l \in a_i} c_{c_l}(n_{c_l}(a)). \tag{11}
\]

**Remark 1.** It is worth noting that the utility of traveler is related to his reward and cost. So to design a proper congestion price for every road is equivalent to designing the utility function for every traveler.

Next, we consider whether the network congestion game is potential, similar to the result in [22] for congestion game, we can obtain the following theorem.

**Theorem 3.3.** The network congestion game which deriving from the traffic congestion form is a weighted potential game. Moreover the weighted potential function can be given as

\[
    P(a) = \sum_{i \in N} \frac{r_i(a_i)}{\omega_i} - \sum_{c_l \in E} \sum_{k=1}^{n_{c_l}(a)} c_{c_l}(k). \tag{12}
\]

In the following, in order to better find the relationship between utility function and potential function, we want to obtain the matrix expression of (11) and (12).

1. Firstly, we consider the algebraic representation of the strategy profile \( a = (a_1, a_2, ... a_n) \in A, a_i \in A_i \). Before this, we express the each player’s action \( a_i \) into a vector form.
Let $a_i = (a_i^{e_1}, a_i^{e_2}, ..., a_i^{e_m}) \in \mathbb{R}^m$, where

\[
a_i^{e_l} = \begin{cases} 1, & \text{if } e_l \in a_i \\ 0, & \text{if } e_l \notin a_i \end{cases}
\]

(13)

We enlarge the strategy set $A_i$ into $\bar{A}_i$, where $\bar{A}_i = A_i \cup \bigcup_{2m} A_i$. Then we arrange the strategy of player $i$ in lexicographic order and denote $0 \sim \delta_1, 1 \sim \delta_2$, so $a_i \sim \delta_2^m$.

$t = a_i^{e_1} \cdot 2^{m-1} + a_i^{e_2} \cdot 2^{m-2} + ... + a_i^{e_{m-1}} \cdot 2 + a_i^{e_m} + 1 = \sum_{l=1}^{m} a_i^{e_l} \cdot 2^{m-l} + 1$.

We use the $a_{it}$ to represent the $t$-th strategy for player $i$, $t \in \{1, 2, ..., 2^m\}$. Similarly, the strategy profile $a \in \mathbb{R}^{2^m}$ can also be arranged in lexicographic order.

\[
a^\lambda = (a_{1t_1}, a_{2t_2}, ..., a_{nt_n})
\]

where $\lambda = \sum_{i=1}^{n} (t_i - 1) \cdot 2^{(n-i)m} + 1, \lambda \in \{1, 2, ..., 2^{mn}\}$.

(2) Next we consider the algebraic representation of the player’s reward function. The reward function $r_i(a_i)$ can be expressed as a vector as

\[
r_i = (r_i(a_{i1}), r_i(a_{i2}), ..., r_i(a_{i2^m}))
\]

Therefore the reward matrix $R$ can be given as

\[
R = (r_1, r_2, ..., r_n)^T \in \mathbb{R}^{n \times 2^m}.
\]

By using the algebraic representation of $a_i$ and reward function, based on the tool of semi-tensor product (STP) of matrices, we can derive the following fact.

**Proposition 3.** The reward function $r_i(a_i)$ in network congestion game can be expressed as a pseudo-logical function

\[
r_i(a_i) = r_i \times a_i = r_i \times \bigcup_{l=1}^{m} a_i^{e_l}.
\]

**Remark 2.** Because the traveler’s reward will be affected by many aspects, the requirement of reward in this paper is only limited to that the traveler can finally reach the destination. Therefore, we assume that no matter which strategy is chosen, as long as the destination is reached, the final reward is the same.

(3) Then we consider the algebraic representation of road cost function (traffic congestion price function), which is the function need to be designed.

We define $c_{e_l} : R \mapsto \mathbb{R}, c_{e_l}(k) = c_{e_l}^k$, $k$ is the number of users using road $e_l$, $l \{1, 2, ..., m\}; k \{1, 2, ..., n\}$. So the vector form of $c_{e_l}$ can be given as

\[
c_{e_l} = (c_{e_l}^1, c_{e_l}^2, ..., c_{e_l}^n)^T \in \mathbb{R}^{n \times 1}, l = 1, 2, ..., m.
\]

Therefore the traffic congestion price function $C$ can be derived as

\[
C = \left( \begin{array}{c} c_{e_1} \\ c_{e_2} \\ \vdots \\ c_{e_m} \end{array} \right) \in \mathbb{R}^{mn \times 1},
\]

we can also show the traffic congestion price function as another form $\hat{C} = (c_{e_1} \ c_{e_2} \ \cdots \ c_{e_m})^T \in \mathbb{R}^{m \times n}$.
For player $i$, under the strategy profile $a^\lambda \in A$, the total resource cost is $c^a_i = \omega_i \sum_{e \in a_i} \text{Row}_l \delta_n^{n_{e_i}}(a^\lambda)$, so the total cost function matrix can be expressed as $C_G \in \mathbb{R}^{n \times 2^{mn}}$, where

$$C_G(i, \lambda) = c^a_i, \lambda = 1, 2, \ldots, 2^{mn}. \tag{14}$$

(4) At last, we consider the algebraic representation of the number of players using road. For an arbitrary strategy profile $a$, we construct $\mathbf{n}(a) = (n_{e_1}(a), n_{e_2}(a), \ldots, n_{e_m}(a))^T \tag{14}$

Based on the above representations of reward function and cost function, we can obtain the following result consequently.

**Theorem 3.4.** The structure matrix expression of utility function (11) in network congestion game can be expressed as

$$M_{u_i} = r_i D_f [2^{(i-1)m}, 2^m] D_r [2^m, 2^{(n-i)m}] - \omega_i \text{Row}_i(C_G) \tag{15}$$

**Proof.** By utilizing the properties of semi-tensor product of matrices and the representation of utility function, we can derive

$$u_i(a) = M_{u_i} \times_{i=1}^{n} a_i$$

$$= r_i \times a_i - \omega_i \sum_{e \in a_i} \text{Row}_l \delta_n^{n_{e_i}}(a)$$

$$= r_i D_f [2^{(i-1)m}, 2^m] D_r [2^m, 2^{(n-i)m}] \times_{i=1}^{n} a_i - \omega_i \text{Row}_i(C_G) \times_{i=1}^{n} a_i$$

$$= (r_i D_f [2^{(i-1)m}, 2^m] D_r [2^m, 2^{(n-i)m}] - \omega_i \text{Row}_i(C_G)) \times_{i=1}^{n} a_i. \qed$$

Now we define a new vector as

$$b_l(a) = (1, \ldots, 1, 0, \ldots, 0) \in \mathbb{R}^{1 \times n}, l = 1, 2, \ldots, m \tag{16}$$

Let $B(a) = (b_1(a) \ b_2(a) \ \ldots \ b_m(a)) \in \mathbb{R}^{1 \times mn}$, so the weighted potential function can be obtained in the following.

**Theorem 3.5.** The matrix expression of weighted potential function (12) in network congestion game can be expressed as

$$P(a) = \frac{r(a)}{\omega^T} - B(a)C, \tag{17}$$

where $r(a) = (r_1(a_1), r_2(a_2), \ldots, r_n(a_n)), \omega = (\omega_1, \omega_2, \ldots, \omega_n)$. 

**Proof.** By the definition of weighted potential function in Theorem 1, it can be expressed as
3.2. The design of traffic congestion price. Now we consider a traffic congestion system $T = \{N, (V, E), (A_i, r_i, w, P)\}$, where $P$ is the given system performance criterion, which need to be optimized. We hope that players can choose the strategy profile $a^* \in A$, such that

$$P(a^*) = \max\{P(a) | a \in A\}. \quad (18)$$

So the key problem is how to guarantee players choose the profile $a^*$ in the absence of centralized management. The method in our paper is to design a suitable traffic congestion price such the traffic system can be transformed into a network congestion game with the given $P$ as its weighted potential function. Then we can achieve the optimization by using the properties of the potential game.

However, there is a natural question: how to design the traffic congestion price? Can it always be designed?

As we all know, the traffic congestion is caused by multiple people choosing the same road. So when we design the traffic congestion price, it must assumed to be nonnegative and non-decreasing as a function of the number of people using it. Now we give the following Assumption.

**Assumption 1.** The road congestion price function $c_{e_i}$ is nonnegative and non-decreasing of variable $k$, where $k$ represents the number of travelers on road $e_i$.

**Theorem 3.6.** Consider a given performance criteria traffic congestion system $T = \{N, (V, E), (A_i, r_i, w, P)\}$. The traffic congestion price can be designed so that the system can be converted into a network congestion game with the given $P$ as its weighted potential function, if and only if the equation (19) has at least one solution satisfying the Assumption 1.

$$\begin{pmatrix} B(a_1) \\ B(a_2) \\ \vdots \\ B(a^n) \end{pmatrix} - C = \begin{pmatrix} P(a_1) \\ P(a_2) \\ \vdots \\ P(a^n) \end{pmatrix}$$

(19)
where $\hat{P}(a) = \frac{r(a)}{\omega} - P(a)$.

Proof. We can simplify (17) as $\frac{r(a)}{\omega} - P(a) = B(a)C$. Let

$$\hat{P}(a) = \frac{r(a)}{\omega} - P(a),$$

we have

$$\hat{P}(a) = B(a)C, a \in A.$$  \hspace{1cm} (21)

(Necessity) From the matrix expression of potential function in network congestion game, we denote the strategy profile $a^1, a^2, ..., a^s \in A, |A| = s$. Because the traffic congestion form can be translated as the network congestion game, it means that the system of linear equations which is established by the formula (21) has at least one solution. Considering the characteristic of congestion price, it also satisfies Assumption 1.

(Sufficiency) If (19) has a solution which satisfying the Assumption 1, then the solution $C$ can be interpreted as the traffic congestion price in traffic system. By further computation, we can show that the weighted potential function is exactly given $P(a)$.

As a consequence of the above proposition and theorem, an algorithm can be obtained to design the traffic congestion price. For convenience, we let

$$B = \begin{pmatrix} B(a^1) \\ B(a^2) \\ \vdots \\ B(a^s) \end{pmatrix} \in \mathbb{R}^{s \times mn}, \quad \hat{P} = \begin{pmatrix} \hat{P}(a^1) \\ \hat{P}(a^2) \\ \vdots \\ \hat{P}(a^s) \end{pmatrix}.$$  \hspace{1cm} (20)

Algorithm 1

1.35 Step 1. Give a traffic congestion system $T = \{N, (V, E), (A_i), r_i, w, P\}$ with the pre-assigned overall function $P(a)$.

Step 2. Obtain the corresponding network congestion game $G = \{N, (V, E), (A_i)_{i \in N}, (u_i)_{i \in N}\}$ by Definition 3.2.

Step 3. Consider the vector form of equation (14) and (16) for all possible strategy profiles.

Step 4. Construct the matrix $B$ and $\hat{P}$ by equation (20).

Step 5. Solve the system of linear equations (19). Figure out the solution $C := \{BC = \hat{P}|C$ is nonnegative and non-decreasing of the number of player$\}$.

Step 6. Determine the traffic congestion price $C$ from the above step.

3.3. The design of learning rule. When the learning rule is assured, the player can use the informations to form a decision, then they can reach a desirable situation at last. Currently, the commonly used strategy updating rules in theoretical research are usually designed by experts. In this paper, a rule which is called the myopic best response adjustment (MBRA) rule [25] will be used. Some descriptions about MBRA are given as follows.

From the standpoint of player $i$, it considers other people’s strategies at $T$ moment, and then chooses the best strategy to deal with them. Define

$$O_i(t) = \text{argmax} \ u_i(s_i, s^{-i}(t)),$$  \hspace{1cm} (22)
where $u_i$ is the utility function of player $i$, $s_i \in S_i, s^{-i}(t) \in S^{-i}$, $S_i$ is the set of strategies of player $i, i = 1, 2, \ldots, n$. $S^{-i} := \prod_{j \neq i} S_j$ is the set of strategies of other players except $i$.

- if $x_i(t) \in O_i(t)$, then $x_i(t+1) = x_i(t)$.
- if $x_i(t) \notin O_i(t)$, choose the smallest $j$ such that $s_j \in O_i(t)$, then let $x_i(t+1) = s_j$.

Considering the property of potential game, if at each moment, we only allow one player to update the strategy, and the MBRA rule is used, the dynamics profile will converge to its Nash equilibrium. In the traffic system, travelers will optimize their utilities by using the information about congestion price and other players’ strategies. If the equilibrium is unique, it is also the maximum of $P$. However, we must point out that in a network congestion game, there may exist more than one Nash equilibrium. Therefore there is more than one optimal path for traveler, and for traffic system, there is more than one optimal strategy profile.

3.4. Two special cases. In this section, we will consider two special cases which may happen in the above discussion to make our results closer to reality.

(1) If the road capacity is limited, some strategy profiles maybe infeasible, then we can divide the strategic profile into two disjoint parts as $A = \Omega \cup \Omega^c$, where $\Omega$ is the set of feasible profiles and $\Omega^c$ is the set of infeasible profiles. For this situation, we can re-express the utility function and potential function as

$$
\hat{u}_i(a) = \begin{cases} 
u_i(a), & \text{if } a \in \Omega \\ \kappa, & \text{if } a \notin \Omega^c, \end{cases}
$$

$$
\hat{P}(a) = \begin{cases} \hat{P}(a), & \text{if } a \in \Omega \\ \kappa, & \text{if } a \notin \Omega^c, \end{cases}
$$

where $\kappa >> \max\{\nu_i(a), \hat{P}(a)|a \in \Omega\}$.

Let $\{a_1, a_2, \ldots, a_m\} \in \Omega$, so the system of linear equations become

$$
\begin{pmatrix} B(a_1) \\ B(a_2) \\ \vdots \\ B(a_m) \end{pmatrix} C = \begin{pmatrix} \hat{P}(a_1) \\ \hat{P}(a_2) \\ \vdots \\ \hat{P}(a_m) \end{pmatrix}
$$

(23)

**Corollary 2.** In the case of limited capacity of road, we also can derive the conclusion that if (23) has at least one solution, the road price of traffic congestion can be designed.

(2) If the system of linear equations (19) has no solution, how should we design the utility function and congestion work, and whether the near network congestion game plays an role in the process of analysis. This issue will be studied in the future work, the near potential game [1] can be applied to investigate the optimization and convergence.

4. An illustrative example. In the following, we give an illustrative example to show the effectiveness of our obtained results.

Consider a traffic congestion system $T = \{N, (V, E), (A_i), P\}$ with a given performance criterion, where $N = \{1, 2, 3\}, (V, E)$ is shown in Fig.2.

For player 1 and 2, their strategy set are $A_1 = A_2 = \{a_1^{1,2} = (e_1, e_3), a_2^{1,2} = (e_2, e_4)\}$, and for player 3, $A_3 = \{a_1^3 = (e_1, e_5), a_2^3 = (e_2, e_6)\}$. 
Assume that the system objective function is given in Table 1.

| a     | 111 | 112 | 121 | 122 | 211 | 212 | 221 | 222 |
|-------|-----|-----|-----|-----|-----|-----|-----|-----|
| P(a)  | 63  | 70  | 72  | 71  | 72  | 71  | 72  | 62  |

We want to determine whether appropriate traffic congestion prices can be designed to make the system and individual all gain the best.

Now we let \( r_1 = r_2 = 30 \) and \( r_3 = 50 \) which represent the rewards when they arrive the destination. The weight of their cost is \( \omega = (2, 2, 1) \). Then the corresponding network congestion game can be obtained. We let the given \( P(a) \) as the weighted potential function.

Firstly, we express each player’s action into a vector form.

\[
\begin{align*}
a_1^{1.2} & = (1 \ 0 \ 1 \ 0 \ 0 \ 0)^T, \\
a_2^{1.2} & = (0 \ 1 \ 0 \ 1 \ 0 \ 0)^T, \\
a_3^1 & = (1 \ 0 \ 0 \ 0 \ 1 \ 0)^T, \\
a_2^3 & = (0 \ 1 \ 0 \ 0 \ 0 \ 1)^T.
\end{align*}
\]

Then using (14), we construct

\[
\begin{align*}
n(a^1) & = (3 \ 0 \ 2 \ 0 \ 1 \ 0)^T, \\
n(a^2) & = (2 \ 1 \ 2 \ 0 \ 0 \ 1)^T, \\
\vdots & \\
n(a^7) & = (1 \ 2 \ 0 \ 2 \ 1 \ 0)^T, \\
n(a^8) & = (0 \ 3 \ 0 \ 2 \ 0 \ 1)^T.
\end{align*}
\]

By equation (16), we can derive \( b_l(a) \), for example, \( b_1(a^1) = (1 \ 1 \ 1 \ 1) \), \( b_3(a^1) = (1 \ 1 \ 0 \ 0) \). Therefore \( B(a^1) = (1 \ 1 \ 1 \ 0 \ 0 \ 0 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0) \). Using the same way, we can get the form of other \( B(a^s), s = 2, 3, ..., 8 \), so the matrix \( B \) can
be given as

\[
B = \begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}
\in \mathbb{R}^{8 \times 18}
\] (24)

Finally, by formula (20), the matrix of \( \tilde{P} \) can be given as following.

\[
\tilde{P} = (17 & 10 & 8 & 9 & 8 & 8 & 8 & 8)^T
\] (25)

We can check whether the system of linear equation (19) has solutions. We can give one of the solutions as

\[
C = [1, 4, 9, 1, 4, 9, 1, 2, 3, 1, 1, 0, 0, 0, 1, 2, 3]
\] (26)

Therefore according to Theorem 3.6, we can conclude that the traffic system can be translated into a network congestion game with the given system objective function as its weighted potential function, and the traffic congestion price of each road can be shown in (26).

Next, when the traffic congestion price is determined, by calculating the reward and cost of each player, using equation (11), we can obtain the utility matrix in Table 2.

| Table 2 Utility matrix of network congestion game |
|--------------------------------------------------|
| \( u \) \quad s \quad 111 | 112 | 121 | 122 | 211 | 212 | 221 | 222 |
| \hline
| \( u_1 \) | 8 | 18 | 20 | 26 | 20 | 20 | 8 |
| \( u_2 \) | 8 | 18 | 26 | 20 | 20 | 20 | 8 |
| \( u_3 \) | 41 | 48 | 46 | 45 | 46 | 45 | 49 | 40 |

We can verify that when the strategy updating rule is used, the dynamics profile will converge the Nash equilibria \( \{(1, 2, 1), (2, 1, 1), (2, 2, 1)\} \) starting from the given initial profile, and we can observe that the system objective function in Table 1 can achieve maximization corresponding.

5. **Conclusion.** This paper investigates how to design an appropriate congestion price of each road in a traffic congestion system. Based on the tool of semi-tensor product of matrices, the algebraic expression of network congestion game is obtained. A necessary and sufficient condition is given to verify whether the traffic system can be translated into a network congestion game. Next in the case that conversion can be established, an algorithm is proposed to design the congestion price. The convergence to Nash equilibrium can be guaranteed when the learning rule is determined. Lastly, two special cases are considered to make the results closer to reality. An example is provided to show the effectiveness of our obtained results.

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