Persistent currents in n–fold twisted Moebius strips

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Abstract. – We investigate the influence of the topology on generic features of the persistent current in n–fold twisted Moebius strips formed of quasi one–dimensional mesoscopic rings, both for free electrons and in the weakly disordered regime. We find that there is no generic difference between the persistent current for untwisted rings and for Moebius strips with an arbitrary number of twists.

Introduction. – Persistent currents in mesoscopic rings threaded by a magnetic flux offer one of the prime examples of quantum coherence in mesoscopic physics [1]. The recent experimental realization of rings which have the shape of singly and doubly twisted Moebius strips and a diameter of about 50 µm [2] raises the intriguing question whether generic features of the persistent current depend upon the topology of the ring. An affirmative answer would imply a close connection between topology and quantum coherence. This is the problem we address in the present paper. We model the ring – both in its untwisted and its twisted form – in the standard way [3]: The electrons move either freely or diffusively through a two–dimensional conducting strip (see fig. 1). We work in the regime of zero or weak disorder and assume free motion in the transverse direction. Our model applies whenever the elastic mean free path is of the order of or larger than the transverse dimension of the ring.

Several theoretical papers have dealt with related problems. Predictions vary according to the dynamical regime under consideration [4,6,7,5]. In all these papers, however, the question whether there is a generic difference between the untwisted and the twisted case, seems not to have been addressed. We briefly describe the difference between our model and that of other recent work at the end of the paper.

We consider a metallic or semiconducting mesoscopic quasi one–dimensional ring at low temperature. A constant homogeneous magnetic field $\mathbf{B}$ threads the ring in a direction perpendicular to the plane of the ring. We take account only of the Aharonov–Bohm (AB) phase $\phi = 2\pi \Phi / \Phi_0$. Here $\Phi = AB$ is the magnetic flux through the ring as given by $B$ and by the area $A$ of the ring, and $\Phi_0$ is the elementary flux quantum. We disregard the effect of the magnetic field on the orbital motion of the electrons in the ring. We also disregard the spin of the electrons. The ring is twisted $n = 0, 1, \ldots$ times. The case $n = 0$ corresponds to the case of an ordinary plane ring, the case $n = 1$ to an ordinary Moebius strip, and higher values
of $n$ to multiply twisted Moebius strips. We ask: How does the number $n$ of twists affect the persistent current in the AB ring?

The case of free electrons and that of diffusive electron motion are considered in the two next sections. The last section contains a summary and a brief comparison with the work of refs. [5, 6, 7].

Free electrons. – It is useful to investigate first the case of free electrons (no disorder) at zero temperature, $T = 0$. For simplicity, we take the ring to be two–dimensional and assume that the transverse extension $d$ of the ring is very small compared to its circumference $L$. Then, the ring can be modelled as a rectangle with suitable boundary conditions at the surfaces. We introduce Cartesian coordinates $x, y$ with $x = [L/(2\pi)]\theta$ the coordinate in the longitudinal and $y$ the coordinate in the transverse direction. The values $\theta = 0, 2\pi$ and $y = \pm d/2$ define the surfaces of the rectangle.

The free Hamiltonian has the form
\[
\hat{H} = -\hbar^2 \frac{2}{\mu} \left[ \frac{(2\pi)^2}{L^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial y^2} \right].
\]
(1)

Here $\mu$ stands for the (effective) mass of the electron. The single–particle wave functions separate as $\Psi_{j,m}(\theta, y) = \chi_{j,m}(\theta) \psi_j(y)$, and the eigenvalues take the form
\[
E_{j,m} = E_j + E^m_j.
\]
(2)

The transverse modes $\psi_j(y)$ obey the boundary conditions
\[
\psi_j(-d/2) = 0 = \psi_j(+d/2).
\]
(3)
The associated eigenvalues $E_j$ are
\[
E_j = \frac{j^2 \pi^2 \hbar^2}{2\mu d^2}.
\]
(4)
The index $j$ defines the channels.

The boundary condition for the longitudinal modes depends on the AB phase $\phi$. We recall [3] that for an untwisted ring ($n = 0$), a single traversal of the ring by an electron multiplies the longitudinal wave function by the AB phase factor $\exp(\pm i\phi)$, the sign depending upon the direction in which the electron traverses the ring,
\[
\chi_{j,m}(\theta \pm 2\pi) = e^{\pm i\phi} \chi_{j,m}(\theta).
\]
(5)

In principle, condition (5) applies likewise to twisted rings ($n \neq 0$) but actually becomes modified because we also have to pay attention to the transverse modes. We demonstrate this for the case $n = 1$, the ordinary Moebius strip. If the electron starts out at point $(\theta, y)$, then a single traversal of the ring brings the electron to the point $(\theta \pm 2\pi, -y)$. The boundary condition now reads $\Psi_{j,m}(\theta \pm 2\pi, -y) = e^{\pm i\phi} \Psi_{j,m}(\theta, y)$. The effect of the traversal depends upon whether the transverse mode $\psi_j$ is even or odd with respect to a reflection of $y$ about the origin. Even modes (corresponding to odd values of $j$) remain unchanged while odd modes (even values of $j$) are multiplied by $(-1)$. Using this fact, we obtain the effective boundary condition
\[
\Psi_{j,m}(\theta \pm 2\pi, y) = (-1)^{j+1} e^{\pm i\phi} \Psi_{j,m}(\theta, y).
\]
(6)

It follows that for a general Moebius strip with $n$ twists, the form of the eigenfunctions in the longitudinal direction depends only on whether $n$ is even or odd and not on the actual value
of \( n \). For \( n \) even, the situation is the same as for the case \( n = 0 \) because an even number of twists leaves both even and odd transverse eigenfunctions unchanged. For \( n \) odd, the situation is the same as for the case \( n = 1 \) because an odd number of twists leaves the even transverse eigenfunctions unchanged and multiplies the odd ones with the factor \((-1)\).

Using the boundary condition (6), we obtain for the eigenvalues \( E_{\mu}^{m} \) of the longitudinal modes

\[
E_{\mu}^{m} = \frac{\hbar^2}{2\mu L^2} \left( \frac{2\pi}{L} \right)^2 \left[ m + \frac{\phi}{2\pi} + \delta_{j,n} \right]^2,
\]

with \( j \) defined above and \( m = 0, \pm 1, \pm 2, \ldots \). The phase \( \delta_{j,n} \) accounts for the symmetry of the transverse modes and the topology of the ring. It is given by

\[
\delta_{j,n} = \begin{cases} 
\frac{1}{2}, & \text{for } j \text{ even, } n \text{ odd}, \\
0, & \text{otherwise}.
\end{cases}
\]

We emphasize the difference between the cases where \( n \) is even or odd. In the first case, the energies in eq. (7), viewed as functions of \( \phi \), form a set of parabolas with minima at \( \phi = -2\pi m \) with \( m = 0, \pm 1, \pm 2, \ldots \). In the second case, we deal with two classes of parabolas. The energies in channels with odd values of \( j \) form the same kind of parabolas as for even values of \( n \). For channels with even values of \( j \), however, the minima of the parabolas lie at \( \phi = -2\pi m - \pi \), again with \( m = 0, \pm 1, \pm 2, \ldots \). This situation is illustrated in fig. 2. It has to be borne in mind, of course, that the energies \( E_{\mu}^{j} \) for the even and odd \( j \)–values are not the same, so that the minima of both types of parabolas do not have the same values. We conclude that on the level of the single–particle spectra and their dependence upon \( \phi \), a generic and qualitative difference does exist between Moebius strips with an even and those with an odd number of twists. The question is: Does this difference also manifest itself generically in the persistent current?

At \( T = 0 \), the persistent current \( I(\phi) \) is given by

\[
I(\phi) = -c \frac{\partial E_{\mu}}{\partial \Phi} = \frac{2\pi c}{\Phi_{0}} \frac{\partial E_{\mu}}{\partial \phi}.
\]

Here \( E_{\mu} \) is the ground–state energy of a system containing \( K \) electrons and is given by

\[
E_{\mu} = \sum_{m,j} p_{j,m} E_{j,m}.
\]

The occupation numbers \( p_{j,m} \) are equal to 0 or 1, sum up to \( K \), and must be chosen such that \( E_{\mu} \) is the lowest energy of the system. From eq. (7) we see that a shift of \( \phi \to \phi \pm 2\pi \) is equivalent to the shift \( m \to m \pm 1 \). This shows that \( E_{\mu} \) is periodic in \( \phi \) with period \( 2\pi \).

Before we look at the general case, we first address the situation of a single channel where the \( j \)–summation in eq. (10) is restricted to a fixed value of \( j \). We begin with the untwisted case, \( n = 0 \). This problem has been thoroughly discussed in ref. [3]. Let \( K_{j} \) be the number of electrons in channel \( j \). The form of \( E_{\mu} \) differs for \( K_{j} = 2K_{0} + 1 \) odd and \( K_{j} = 2K_{0} \) even. For \( K_{j} \) odd, the sum over \( m \) extends from \(-K_{j} \) to \(+K_{0} \). The persistent current is accordingly given by

\[
I(\phi) = -\frac{8\pi^2 c K_{j}}{\Phi_{0}^2} \frac{\hbar^2}{2\mu L^2} \frac{\phi}{2\pi}, \quad \text{for } -\pi \leq \phi \leq \pi \text{ and } K_{j} \text{ odd}.
\]
The persistent current is sawtooth–shaped, with discontinuous jumps of height \((8\pi^2cK_j/\Phi_0)(h^2/(2\mu L^2))\) at \(\phi = \pm (2m + 1)\pi\) for all integer \(m\). For \(K_j\) even, inspection of eq. (7) shows that the sum over \(m\) runs from \(-K_0\) to \(K_0 - 1\) for \(\phi > 0\) and from \(-K_0 + 1\) to \(K_0\) for \(\phi < 0\). The current is

\[
I(\phi) = -\frac{8\pi^2c}{\Phi_0} \frac{h^2}{2\mu L^2} [K_j \frac{\phi}{2\pi} + K_0], \text{ for } \phi < 0 \text{ and } K_j \text{ even,}
\]

while

\[
I(\phi) = -\frac{8\pi^2c}{\Phi_0} \frac{h^2}{2\mu L^2} [K_j \frac{\phi}{2\pi} - K_0], \text{ for } \phi > 0 \text{ and } K_j \text{ even.}
\]

The discontinuity of the sawtooth function is now located at \(\phi = 2\pi m\), with \(m = 0, \pm 1, \ldots\). In other words, in comparison with the case for odd \(K_j\), the sawtooth function for even \(K_j\) is shifted by \(\pi\). Thus, the current is characteristically different for even and for odd values of \(K_j\) but in both cases is periodic in \(\phi\) with period \(2\pi\).

We keep to the single–channel case and consider next the case of a singly twisted Moebius strip, \(n = 1\). In comparison with the untwisted case \((n = 0)\), there is no change for the “even” channels (odd \(j\)). However, the situation changes qualitatively for “odd” channels (even \(j\)) because of the changed shape of the spectrum, see fig. 2. For \(K_j = 2K_0\) even, the \(m\)–summation extends from \(-K_0\) to \(K_0 - 1\). The current is the same as in eq. (11). For \(K_j = 2K_0 + 1\) odd, the \(m\)–summation extends from \(-K_0\) to \(+K_0\) for \(\phi < 0\) and from \(-K_0 - 1\) to \(+K_0 - 1\) for \(\phi > 0\). The current is the same as in eqs. (12) and (13). Thus, for the “odd” channels, the persistent current changes in comparison with the untwisted case: The current for even \(K_j\) in the untwisted case equals the current for odd \(K_j\) in the twisted case, and vice versa. This modification directly reflects the change in the single–particle spectrum. The generalization to arbitrary values of \(n\) is obvious: Even (odd) values of \(n\) produce the same patterns as \(n = 0\) \((n = 1, \text{ respectively})\). If it were possible to prepare a Moebius strip with a single “odd” channel and a known number of electrons on it, then the sawtooth pattern of the persistent current would generically differ for Moebius strips with an even and an odd number of twists.

Unfortunately, this simple result becomes blurred as we consider the case of many channels. This is necessary because the number of electrons on a ring of realistic dimensions is at least several 1000 if not much larger, and the \(j\)–summation in eq. (11) can, therefore, not be restricted to a single value of \(j\). We first consider the case of even \(n\) and many channels. The number \(K_j\) of electrons in channel \(j\) will depend upon the relation between the energy \(E_j\) and the energies \(E_k\) of the channels \(k \neq j\). Therefore, the precise form of the current will depend upon both, the transverse width \(d\) of the ring, and the total number \(K\) of electrons on the ring. But the following general statements apply. Let us first assume that the number of electrons per channel does not change with \(\phi\). This assumption is not realistic but helps to clarify the situation. Then all channels with an odd number of electrons will jointly contribute a sawtooth function to the current which is discontinuous at \(\phi = 0\), while all channels with an even number of electrons will jointly contribute a sawtooth function which is discontinuous at \(\phi = \pm \pi\). The heights of the discontinuities are given in terms of the total number of electrons in either case. As a result, the current is periodic in \(\phi\) with period \(2\pi\) but within each period has double–sawtooth structure. It is only when the numbers of electrons on channels with even \(j\) and with odd \(j\) are equal that the period of the current becomes equal to \(\pi\). Our assumption that the number of electrons per channel is fixed, is unrealistic, however. This is because as we change \(\phi\), there will be crossings of levels pertaining to different channels. The values of \(\phi\) where the crossings occur depend on the \(j\)–values of the channels and (through the energies \(E_j\)) on the value of \(d\) and are, thus, not generic. Such crossings cause the numbers of electrons per
channel to change. As a result, the persistent current will acquire further sawtooth structures at values of \( \phi \) which differ from both 0 and \( \pm \pi \). However, the periodicity of the current with period \( 2\pi \) will not be affected. When we average over systems containing different numbers of electrons (as is done whenever we perform an ensemble–average over impurities), the period of the current is expected to be given by \( \pi \) as there is now equal weight given to all possible realizations.

We turn to the case of odd \( n \) and many channels. Again, we first assume that the number of electrons per channel does not change with \( \phi \). In the summation over \( j \) in eq. (10), the situation for channels with odd values of \( j \) is the same as described in the previous paragraph, while the roles of even and odd values of \( K_j \) are reversed for even values of \( j \). As long as these numbers are not known individually, the result for the persistent current is generically the same, however, as for even values of \( n \): The persistent current has sawtooth shape with discontinuities at \( \phi = 0 \) and at \( \phi = \pm \pi \). Further discontinuities arise because the numbers \( K_j \) change with \( \phi \). The resulting current is periodic with period \( 2\pi \) and not generically different from that for even values of \( n \).

The effect of finite temperature has been thoroughly discussed in ref. [3]: The occupation probabilities in eq. (10) no longer are equal to zero or one but are given in terms of Fermi functions. As a consequence, the sawtooth functions are smoothed, and the amplitude of the current decreases. But there are no qualitative changes in the picture that applies for \( T = 0 \).

**Diffusive motion.** Qualitatively, the influence of diffusion on the results obtained in the previous section can easily be guessed: Diffusion allows for elastic scattering between different channels and, thus, removes the level crossings displayed in fig. 2. As a result, the sawtooth pattern of the current (which is a consequence of such level crossings) is rounded off. The effect increases as the elastic mean free path decreases. This leads to an ever increasing suppression of the persistent current. The argument supports the naive expectation that gross properties of the persistent current like its periodicity should not depend upon the presence or absence of diffusion. For a quantitative description, it is necessary to resort to diagrammatic perturbation theory [8] or to the supersymmetry technique [9]. We have used the latter method and have convinced ourselves that the average current is identically the same for all values of \( n \), with \( n = 0, 1, \ldots \). The average current is periodic in \( \phi \) as expected. The proof is technically too complicated to be reproduced here. An extended paper containing the proof is in preparation [10].

**Conclusions.** We conclude that both for free electrons and for diffusive electron motion in the weakly disordered regime, there is no generic difference between the persistent current for untwisted rings and for Moebius strips with an arbitrary number \( n \) of twists. The persistent current generically has periodicity \( 2\pi \) and (at least) a double sawtooth structure. In detail, the structure depends upon the precise ordering of the energies of the transverse modes and on the number of electrons on the ring. For weak disorder, the ensemble–averaged current is independent of the number of twists.

The work of Yukubo et al. [5] uses a lattice model with hopping between sites. The authors include both longitudinal and transverse hopping amplitudes along the ring. We believe that their numerical work is done in the regime of strong disorder. The authors conclude that in this regime, the higher Fourier transforms of the persistent current might allow one to distinguish between a Moebius strip and an untwisted ring. Transverse hopping is also included in refs. [6, 7]. The authors conclude that topological features do affect the physical properties of the system. In both cases the distinction (if possible) seems to rest upon quantitative (rather than generic) features of the current. This would be consistent with our conclusions.
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Fig. 1 – Moebius strip of circumference $L$ and width $d$ threaded by magnetic flux $\Phi$.

Fig. 2 – Energy as function of the Aharonov-Bohm phase $\phi$ for two different transverse modes.