ON ARTIN’S CONJECTURE: PAIRS OF ADDITIVE FORMS

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Abstract. It is established that for every pair of additive forms \( f = \sum_{i=1}^{s} a_i x_i^k, g = \sum_{i=1}^{s} b_i x_i^k \) of degree \( k \) in \( s > 2k^2 \) variables the equations \( f = g = 0 \) have a non-trivial \( p \)-adic solution for all odd primes.

1. Introduction

Let \( k \geq 1 \) be a natural number and \( a_i \) and \( b_i \) integer coefficient for \( 1 \leq i \leq s \). A special case of Artin’s conjecture \([1]\) states that the pair of additive equations

\[
\sum_{i=1}^{s} a_i x_i^k = \sum_{i=1}^{s} b_i x_i^k = 0
\]

have a non-trivial \( p \)-adic solution for all primes \( p \) provided that \( s > 2k^2 \).

Davenport and Lewis \([4]\) started to answer the question whether this statement is true by proving that \( s > 2k^2 \) variables are sufficient if \( k \) is odd, whereas for even \( k \) they only obtained the bound \( s \geq 7k^3 \). Brüdern and Godinho \([2]\) proved that the expected bound \( s > 2k^2 \) holds for even \( k \) which are not of the shape \( k = \tau \cdot 3^r \) or \( k = p^r (p-1) \) for \( p \) prime and \( \tau \geq 1 \) as well. For each of these excluded shapes they proved for all but one prime that a non-trivial \( p \)-adic solution exists if \( s > 2k^2 \). The missing primes are \( p = 2 \) in the case \( k = 3 \cdot 2^r \) and \( p \) if \( k = p^r (p-1) \). Here, they gave the bounds \( s \geq \frac{8}{3} k^2 \) for \( p = 2 \) and \( k = 3 \cdot 2^r \), \( s \geq 8k^2 \) for \( p = 2 \) and \( k = 2^r \), and \( s \geq 4k^2 \) for \( p \geq 3 \) and \( k = p^r (p-1) \). All in all, the bound \( s \geq 8k^2 \) holds for all \( p \) and all \( k \).

There was some further progress for \( p = 2 \) and \( k = 2^r \) for \( \tau = 1, \tau = 2 \) and \( \tau \geq 16 \). For \( k = 2 \) the expected bound \( s > 8 \) follows from the general result by Dem’yanov \([5]\) that for two quadratic forms \( f_1, f_2 \) in at least nine variables the equations \( f_1 = f_2 = 0 \) have a non-trivial \( p \)-adic solution for all primes \( p \). Poehler \([11]\) proved for \( k = 4 \), that \( 49 = 3k^2 + 1 \) variables suffice and Kränzelin \([10]\) showed for \( k = 2^r \) with \( \tau \geq 16 \) that the expected \( 2k^2 + 1 \) variables are sufficient.

For \( p \geq 3 \) and \( k = p^r (p-1) \) on the other hand, the bound was further sharpened by Godinho and de Souza Neto \([6,7]\) who proved that \( s \geq 2\frac{k^2}{p+1} - 2k \) suffices for \( p \in \{3,5\} \) and if \( \tau \geq \frac{p-1}{2} \) for \( p \geq 7 \) as well. For \( k = 6 = 3 \cdot 2 \), the bound \( s > 2k^2 \) was reached by Godinho, Knapp and Rodrigues \([8]\) while later Godinho and Ventura \([9]\) showed that this bound suffices for \( k = 3^r \cdot 2 \) with \( \tau \geq 2 \) as well. Therefore, all pairs of diagonal forms of equal degree \( k \) in more than \( 2k^2 \) variables have a non-trivial \( 3 \)-adic solution. The aim of this paper is to prove the following theorem, which shows that this statement does not only hold for \( p = 3 \) but for all \( p \geq 3 \), by taking care of the degrees \( k = p^r (p-1) \) for \( p \geq 5 \) and \( \tau \geq 1 \).

Theorem. Let \( p \geq 5 \) be a prime, \( \tau \geq 1 \) and \( k = p^r (p-1) \). Then for \( a_i, b_i \in \mathbb{Z} \) with \( 1 \leq i \leq s \), the equations

\[
\sum_{i=1}^{s} a_i x_i^k = \sum_{i=1}^{s} b_i x_i^k = 0
\]

have a non-trivial \( p \)-adic solution for all \( s > 2k^2 \).

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This completes the proof of Artin’s conjecture for two diagonal forms of the same degree for all primes \( p \neq 2 \). For \( p = 2 \) there are only the questions whether there is a non-trivial 2-adic solution for \( k = 3 \cdot 2^\tau \) for \( \tau \geq 2 \) and \( k = 2^\tau \) for \( 2 \leq \tau \leq 15 \) provided that \( s > 2k^2 \). The argument by Kränzlein [10] can be easily applied for the case \( k = 3 \cdot 2^\tau \) as well if \( \tau \geq 16 \). Thus, only finitely many \( k \) remain for which the bound \( s > 2k^2 \) is not reached.

The proof of the theorem follows a pattern by Davenport and Lewis [4] while making use of some improvements by Brüdern and Godinho [2]. Section 2 defines an equivalence relation on the set of all systems (\( 1.1 \)), introduced by Davenport and Lewis [4]. This equivalence relation is defined in a way that solubility of (\( 1.1 \)) in \( \mathbb{Q}_p \setminus \{0\} \) is preserved, which allows to pick representatives with useful properties from each class and prove the existence of a non-trivial \( p \)-adic solution only for them. Due to a version of Hensel’s lemma, one can show that a system (\( 1.1 \)) has a non-trivial \( p \)-adic solution by proving that the congruences

\[
\sum_{i=1}^{s} a_i x_i^k \equiv \sum_{i=1}^{s} b_i x_i^k \equiv 0 \mod p^{\tau+1}
\]

have a solution for which the matrix

\[
\begin{pmatrix}
 a_1 x_1 & \ldots & a_s x_s \\
 b_1 x_1 & \ldots & b_s x_s
\end{pmatrix}
\]

has rank 2 modulo \( p \). Section 3 recalls the notions of coloured variables, introduced by Brüdern and Godinho [2], and contractions which were established by Davenport and Lewis [4]. Together, they are the foundation of the proof. Coloured variables and a refinement of them provide a way to take care of the rank of the matrix (2.3), whereas Section 4 which issues more restrictions on the pairs of equations one has to find a solution for. Section 5 is a collection of combinatorial results which are frequently used, directly and indirectly, in the remaining sections. A description on how the notion of coloured variables is used in combination with contractions to obtain a solution of (\( 1.2 \)) such that the matrix (\( 1.3 \)) has rank 2 is contained in Section 6 whereas Section 7 consists of a collection of lemmata which describe situations in which one can lift some solutions modulo \( p^j \) to solutions of a higher modulus. The remaining two sections contain the actual proof which is divided into Section 8 for the case \( k = p(p-1) \) and Section 9, where the remaining cases with \( k = p^\tau (p-1) \) and \( \tau \geq 2 \) are handled. This division is due to the different modulus in (\( 1.2 \)). For big \( \tau \), one has more variables whose coefficients are not both congruent to 0 modulo \( p^{\tau+1} \), which is balanced in the case \( \tau = 1 \) by a permutation argument.

2. \( p \)-NORMALISATION

This section will recall an equivalence relation on the set of systems (\( 1.1 \)) which was introduced by Davenport and Lewis [4] in order to choose representatives with specific characteristics.

Define for any pair of additive forms

\[
f = \sum_{i=1}^{s} a_i x_i^k, \quad g = \sum_{i=1}^{s} b_i x_i^k
\]

with rational coefficients \( a_i \) and \( b_i \) (\( 1 \leq i \leq s \)) a rational number

\[
\vartheta(f,g) := \prod_{\substack{1 \leq i,j \leq s \\mid i \neq j}} (a_i b_j - a_j b_i).
\]

For integers \( i_1 \) (\( 1 \leq i \leq s \)) consider the pair

\[
f' = f (p^{\nu_1} x_1, \ldots, p^{\nu_s} x_s), \quad g' = g (p^{\nu_1} x_1, \ldots, p^{\nu_s} x_s)
\]

and for rational numbers \( \lambda_1, \lambda_2, \mu_1 \) and \( \mu_2 \) with \( \lambda_1 \mu_2 - \lambda_2 \mu_1 \neq 0 \) the pair

\[
f'' = \lambda_1 f + \lambda_2 g, \quad g'' = \mu_1 f + \mu_2 g.
\]
If another pair \( \tilde{f}, \tilde{g} \) with rational coefficients can be obtained by a finite succession of the operations (2.2) and (2.3) on the pair \( f, g \), then they are called \( p \)-equivalent. If \( (x'_1, \ldots, x'_s) \) is a non-trivial solution of \( f' = g' = 0 \) then \( (p^{r_1}x'_1, \ldots, p^{r_s}x'_s) \) is a non-trivial solution of \( f = g = 0 \), whereas if \( (x_1, \ldots, x_s) \) is a non-trivial solution for \( f = g = 0 \), then one has a non-trivial solution for \( f' = g' = 0 \) as well, given via \( (p^{r_1}x_1, \ldots, p^{r_s}x_s) \). Therefore, solubility is preserved under the operation (2.2).

The same holds for the operation (2.3). Here, one direction is obvious, and the other holds, because the transformation is invertible. Consequently, the existence of a non-trivial solution for \( f = g = 0 \) in \( \mathbb{Q}_p \) implies that there is one for all pairs \( \tilde{f}, \tilde{g} \) which are \( p \)-equivalent to \( f, g \). It can also be easily deduced from the definition of \( \vartheta(f, g) \), that if \( \vartheta(f, g) = 0 \), the same holds for \( \vartheta(f', g') \) and \( \vartheta(f''', g''') \) and therefore, for the whole \( p \)-equivalent class.

**Definition 1.** A pair \( f, g \) given by (2.1) with integers coefficients and \( \vartheta(f, g) \neq 0 \) is called \( p \)-normalised, if the power of \( p \) dividing \( \vartheta(f, g) \) is as small as possible amongst all pairs of forms (2.1) with integer coefficients in the same \( p \)-equivalent class.

As each \( p \)-equivalent class contains pairs, for which all coefficients \( a_i, b_i \) are integers, it follows that the existence of a non-trivial solution for all \( p \)-normalised pairs induces a non-trivial solution for all pairs of forms with rational coefficients \( a_j, b_j \) and \( \vartheta(f, g) \neq 0 \). Using a compactness argument, Davenport and Lewis [4] showed that it induces the existence of a solution for all pairs of forms \( f, g \) with \( \vartheta(f, g) = 0 \) as well.

**Lemma 1.** Suppose for an fixed \( s \) that the equations \( f = g = 0 \) have a non-trivial solution in \( \mathbb{Q}_p \) for all \( p \)-normalised pairs \( f, g \). Then, for any rational coefficients \( a_i, b_i \), the equations (1.1) have a non-trivial solution in \( \mathbb{Q}_p \).

**Proof.** See [4, Section 5]. \( \square \)

Consequently, it suffices to focus on finding non-trivial \( p \)-adic solutions for \( p \)-normalised pairs \( f, g \) in more than \( 2k^2 \) variables. The following lemma gives information about the properties of them.

**Lemma 2.** A \( p \)-normalised pair of additive forms \( f, g \) of degree \( k \) in \( s \) variables can be written as

\[
\begin{align*}
f &= f_0 + pf_1 + \cdots + p^{k-1}f_{k-1}, \\
g &= g_0 + pg_1 + \cdots + p^{k-1}g_{k-1},
\end{align*}
\]

where \( f_i, g_i \) are forms in \( m_i \) variables, and these sets of variables are disjoint for \( i = 0, 1, \ldots, k-1 \). Moreover, each of the \( m_i \) variables occurs in at least one of \( f_i, g_i \) with a coefficient not divisible by \( p \). One has

\[
m_0 + \cdots + m_j \geq \frac{(j+1)s}{k} \quad \text{for} \quad j = 0, 1, \ldots, k-1.
\]

Moreover, if \( q_i \) denotes the minimum number of variables appearing in any form \( \lambda f_i + \mu g_i \) (\( \lambda \) and \( \mu \) not both divisible by \( p \)) with coefficients not divisible by \( p \), then

\[
m_0 + \cdots + m_{j-1} + q_j \geq \frac{(j + \frac{1}{2})s}{k} \quad \text{for} \quad j = 0, 1, \ldots, k-1.
\]

**Proof.** See [4, Lemma 9]. \( \square \)

At least one integer coefficient \( a_i \) or \( b_i \) of a variable \( x_i \) of a \( p \)-normalised pair \( f, g \) is non-zero, because else one would have \( \vartheta(f, g) = 0 \). Consequently, there is a maximal power \( l \) of \( p \), which divides both \( a_i \) and \( b_i \). Due to the previous lemma, one can deduce, that \( 0 \leq l \leq k-1 \) for all variables \( x_i \) of a \( p \)-normalised pair.

**Definition 2.** A variable \( x_i \) of a pair \( f, g \) with integer coefficients is said to be at level \( l \) if its coefficients \( a_i \) and \( b_i \) are both divisible by \( p^l \) but not both divisible by \( p^{l+1} \).

By Lemma 2, a \( p \)-normalised pair has exactly \( m_i \) variables at level \( l \) for \( 0 \leq l \leq k-1 \). The integers \( \tilde{a}_i, \tilde{b}_i \) are defined for a variable \( x_i \) at level \( l \) with integer coefficients \( a_i, b_i \) via \( \tilde{a}_i = p^{-l}a_i \),
and \( \tilde{b}_i = p^{-1}b_i \). These integers \( \tilde{a}_i, \tilde{b}_i \) are the coefficients of the forms \( f_i, g_i \) as defined in Lemma \ref{lemma2} and the vector \( \left( \frac{\tilde{a}_i}{\tilde{b}_i} \right) \) is called the \textit{level coefficient vector} of a variable \( x_i \).

One can restrict the question of the existence of a non-trivial \( p \)-adic solution to one of congruences. To this end, it is useful to adopt the notation 
\[
\gamma := \begin{cases} 
1, & \text{if } \tau = 0 \\
\tau + 1, & \text{if } \tau > 0 \text{ and } p > 2 \\
\tau + 2, & \text{if } \tau > 0 \text{ and } p = 2,
\end{cases}
\]
(2.5)

by Davenport and Lewis \cite{davenport1960} which is used in the following lemma.

**Lemma 3.** If the congruences
\[
\sum_{i=1}^{s} a_i x_i^k \equiv 0 \pmod{p^\gamma}, \quad \sum_{i=1}^{s} b_i x_i^k \equiv 0 \pmod{p^\gamma}
\]
(2.6)

have a solution in the integers for which the matrix
\[
\begin{pmatrix}
    a_1 x_1 & \cdots & a_s x_s \\
    b_1 x_1 & \cdots & b_s x_s
\end{pmatrix}
\]

has rank 2 modulo \( p \), then the equations (2.6) have a non-trivial \( p \)-adic solution.

**Proof.** See \cite{davenport1960} Lemma 7. \hfill \Box

Such a solution is called a \textit{non-singular solution}. The remainder of the proof of the theorem will focus on finding non-singular solutions for \( p \)-normalised pairs \( f, g \).

The next section will introduce the methods used to find non-singular solutions.

3. Coloured Variables and Contractions

This section will recall the concept of coloured variables, first used by Br"udern and Godinho \cite{bruedern1999}, and refine it in a way such that it meets the requirements of the special case \( k = p^\tau (p - 1) \). It will also describe the method of contractions which was introduced by Davenport and Lewis \cite{davenport1960}. Together, both concepts form the foundation of this proof.

To have more control over the non-singularity of a solution of (2.6), Br"udern and Godinho \cite{bruedern1999} divided the set of variables at level \( l \) into \( p + 1 \) sets, depending on their level coefficient vector. For that, they defined the vectors \( e_0 = \left( \begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right) \) and \( e_\nu = \left( \begin{smallmatrix} 1 \\ \nu \end{smallmatrix} \right) \) for \( \nu \in \{1, \ldots, p\} \). Viewed as vectors in \((\mathbb{Z}/p\mathbb{Z})^2\) the vectors define the sets
\[
\mathcal{L}_\nu := \{ ce_\nu \mid c \in (\mathbb{Z}/p\mathbb{Z})^* \}
\]
for \( 0 \leq \nu \leq p \). Modulo \( p \), each level coefficient vector \( (\tilde{a}_i, \tilde{b}_i) \) lies in exactly one of the disjoint sets \( \mathcal{L}_\nu \).

**Definition 3.** A variable \( x_i \) at level \( l \) is said to be of colour \( \nu \), if the level coefficient vector \( (\tilde{a}_i, \tilde{b}_i) \) interpreted as a vector in \( \mathbb{F}_p^2 \) lies in \( \mathcal{L}_\nu \). The parameter \( I_\nu^l \) of a pair \( f, g \) is the number of variables \( x_i \) at level \( l \) of colour \( \nu \).

The parameter \( q_\nu \) introduced in Lemma \ref{lemma2} denotes the minimum number of variables appearing with a coefficient not divisible by \( p \) in any form \( \lambda f_i + \mu g_i \) with \( (\lambda, \mu) \notin \{(0,0)\} \) modulo \( p \). This is closely related to the concept of coloured variables. By setting \( \lambda \equiv 0 \) modulo \( p \) for \( \nu = 0 \) or \( \mu \equiv -\lambda \nu \) for \( \nu \in \{1, \ldots, p\} \) the variables which appear in \( \lambda f_i + \mu g_i \) with a coefficient divisible by \( p \) are exactly those of colour \( \nu \). Consequently, if \( I_\nu^l \geq I_\mu^l \) for all \( 0 \leq \mu \leq \nu \) it follows that \( I_\nu^l = m_l - q_\nu \). Define \( I_{\max}^l = m_l - q_\nu \). This notation can be generalised as follows.

**Definition 4.** For a set \( \mathcal{K} \) of indices \( i \) of variables \( x_i \) at level \( l \) define \( I_\nu (\mathcal{K}) \) as the number of \( i \in \mathcal{K} \) with \( x_i \) of colour \( \nu \), \( I_{\max} (\mathcal{K}) = \max_{0 \leq \nu \leq p} I_\nu (\mathcal{K}) \) and \( q (\mathcal{K}) = |\mathcal{K}| - I_{\max} (\mathcal{K}) \).
Note, that if $\mathcal{K}$ is the set of all indices of variables at level $l$, then $|\mathcal{K}| = m_l$, $I_\nu(\mathcal{K}) = I_\nu$, $I_{\text{max}}(\mathcal{K}) = I_{\text{max}}^l$, and $q(\mathcal{K}) = q$.

From the definition of a non-singular solution it follows, that whether a solution of (2.6) is non-singular depends exclusively on the variables at level 0. If a solution of (2.6) has variables at level 0 of at least two different colours set to a value which is not congruent to 0 modulo $p$, the corresponding matrix has rank 2 modulo $p$ making it a non-singular solution. To use variables at different levels one can take sets of variables at one level and combine them in a way that they can be seen as a variable of a higher level. This method was introduced by Davenport and Lewis [3] and applied in combination with the notion of coloured variables by Brüdern and Godinho [2].

Definition 5. Let $\mathcal{K}$ be a set of indices $j$ with $x_j$ at level $l$. Let $h \in \mathbb{N}$ with $h > l$ and suppose that there are integers $y_j$ with $p \nmid y_j$ such that

$$\sum_{j \in \mathcal{K}} a_j y_j^k \equiv \sum_{j \in \mathcal{K}} b_j y_j^k \equiv 0 \mod p^h.$$ 

Then $\mathcal{K}$ is called a contraction from level $l$ to level at least $h$. If either $\sum_{j \in \mathcal{K}} a_j y_j^k$ or $\sum_{j \in \mathcal{K}} b_j y_j^k$ is not congruent to 0 modulo $p^{h+1}$, then $\mathcal{K}$ is called a contraction from level $l$ to level $h$.

Recall for variables at level $l$ that $\tilde{a}_j = p^{-l}a_j$ and $\tilde{b}_j = p^{-l}b_j$. Hence, a set $\mathcal{K}$ of variables at level $l$ is a contraction to a variable at level at least $l + n$ if there are $y_j$ not divisible by $p$ such that

$$\sum_{j \in \mathcal{K}} \tilde{a}_j y_j^k \equiv \sum_{j \in \mathcal{K}} \tilde{b}_j y_j^k \equiv 0 \mod p^n.$$ 

If $\mathcal{K}$ is a contraction from level $l$ to some level $h$, one can set $x_j = y_j X_0$ for all $j$ in the contraction $\mathcal{X}$. Through this, one obtains a variable $X_0$ at level $h$. One says that the variable $X_0$ can be traced back to the variables $x_j$ with $j \in \mathcal{K}$. Assume that there are other variables $X_i$ at level $h$ with $i \in \{1, \ldots, n\}$, where each of the variables $X_i$ is a variable at level $h$ which either occurred in the pair $f, g$ or is the result of a contraction. If the set of indices $\{0, 1, \ldots, n\}$ of the variables $X_0, X_1, \ldots, X_n$ is a contraction to a variable $Y$ at a level at least $h + 1$, then one says that the variable $Y$ can be traced back not only to the variables $X_i$ for $i \in \{0, 1, \ldots, n\}$ but also to all the variables that those variables can be traced back to. For example, $Y$ can be traced back to all $x_j$ with $j \in \mathcal{K}$.

Definition 6. A variable is called a primary variable if it can be traced back to two variables at level 0 of different colours.

If one can contract a primary variable at level at least $\gamma$, then by setting this contracted variable and everything else zero, one obtains a non-singular solution of (2.6) and therefore a non-trivial $p$-adic solution.

In some cases the knowledge of the exact level and colour of a variable that was contracted will give quite an advantage. To gain control about this, the concept of coloured variables is not strong enough because it can only give the information whether a certain set of variables at level $l$ is a contraction to a variable at level at least $l + 1$, but one does not know the behaviour of the variables modulo $p^{l+2}$. Therefore, one cannot use it to extract information about the exact level and colour of the contracted variable. To gain this information, one can divide the set of variables of one colour into smaller sets, which consider the level coefficient vectors $(\frac{b_j}{h_i})$ not only modulo $p$ but modulo $p^2$.

For that, view the vectors $e_0 = (l_0)$ and $e_\nu = (l_1)$ as vectors in $\left(\mathbb{Z}/p^2\mathbb{Z}\right)^2$ and define the vectors $e^0 = (0_p)$ and $e^\nu = (0_p)$ for $\nu \in \{1, \ldots, p - 1\}$. This enables one to define sets similar to the sets $\mathcal{L}_\nu$ via

$$\mathcal{L}_\nu = \left\{ e(\nu + \mu e^\nu) \mid e \in \left(\mathbb{Z}/p^2\mathbb{Z}\right)^* \right\}$$

for $0 \leq \nu \leq p$ and $0 \leq \mu \leq p - 1$. Here again, a level coefficient vector $(\tilde{a}_j)$ lies modulo $p^2$ in exactly one of the disjoint sets $\mathcal{L}_\nu$. 


Definition 7. A variable $x_i$ is said to be of colour nuance $(\nu, \mu)$ if the level coefficient vector $(\bar{a}_i, \bar{b}_i)$ interpreted as a vector in $\left(\mathbb{Z}/p^2\mathbb{Z}\right)^2$ lies in $\mathcal{L}_{\nu\mu}$. The parameter $l_i^{\nu\mu}$ of a pair $f, g$ is the number of variables $x_i$ at level $l$ of colour nuance $(\nu, \mu)$.

For all variables $x_i$ of colour nuance $(\nu, \mu)$ there is a unique integer $c_i \in \{1, 2, \ldots, p^2\}\backslash p\mathbb{Z}$ for which $\left(\bar{a}_i / c_i\right) \equiv c_i (e_\nu + \mu e_\gamma) \mod p^2$. The integer $c_i$ is said to be the corresponding integer to $x_i$.

Lemmas 4 and 5 show that it suffices to find a non-singular solution for all $p$-normalised pairs in order to prove that for any rational coefficients $a_j, b_j$ the equations (1.1) have a non-trivial solution in $\mathbb{Q}_p$. Due to Lemma 2 one already has some information about the number of variables at certain levels and the distribution of these variables in the different colours of $p$-normalised forms $f, g$. One can further exploit that every $p$-equivalence class contains more than just one $p$-normalised pair. The next lemma shows further properties that are fulfilled by at least one $p$-normalised pair in each $p$-equivalence class for which $\vartheta(f, g) \neq 0$ holds.

Lemma 4. Each pair of additive forms $(f, g)$, with rational coefficients and $\vartheta \neq 0$, is $p$-equivalent to a $p$-normalised pair $f, g$ possessing the following properties:

(i) $g_0$ contains exactly $q_0$ variables with coefficients not divisible by $p$.

(ii) One of $f_1, g_1$ contains exactly $q_1$ variables with coefficients not divisible by $p$.

(iii) $g_0$ has the form

$$g_0 = p^2 \sum_{i=1}^{n_0} a_i x_i^k + p \sum_{i=i_0^{n_0+1}}^{n_0} \beta_i x_i^k + \sum_{I_i^{n_0}+1}^{m_0} \gamma_i x_i^k,$$

where $\beta_{i_0^{n_0+1}}, \ldots, \beta_{I_1^{n_0}}, \gamma_{I_1^{n_0}+1}, \ldots, \gamma_{m_0}$ are not divisible by $p$, and

$$m_0 + m_1 - I_0^1 \geq \frac{m_0 - q_0}{p}.$$

Furthermore, $I_0^0 \geq I_0^0$ for all $0 \leq \mu \leq p - 1$.

Proof. See [4, Lemma 10].

It follows from the first property, that $n_0^{\max} = I_0^1 = m_0 - q_0$. The second property shows, that either $I_1^0 = m_1 - q_1$ or $I_0^1 = m_1 - q_1$ and therefore, either the colour $0$ or the colour $p$ has the most variables at level $1$. Note, that it follows from the third property, that

$$I_0^0 + q_0 + m_1 - I_0^1 \geq \frac{m_0 - q_0}{p},$$

and thus, that

$$I_0^0 - I_0^0 \geq \frac{n_0 - q_0 - (m_1 - I_0^1)}{p}. \tag{3.2}$$

As every $p$-normalised pair is $p$-equivalent to a $p$-normalised pair possessing the properties of the previous lemma, it suffices to prove the existence of a non-singular solutions for $p$-normalised pairs with these properties.

By using only the variables at level $0$ it was proved by Brüdern and Godinho [2, Section 4] that a pair $f, g$ for which $q_0$ is large has a non-singular solution as displayed in the following.

They said that a colour $\nu$ is zero-representing if there is a subset $\mathcal{K}$ of variables at level $0$ of colour $\nu$ for some $0 \leq \nu \leq p$, which is a contraction to a variable at level at least $\gamma$. The following Lemma is an immediate result from this definition.

Lemma 5. If a pair $f, g$ has two colours that are zero-representing, then there exists a non-singular solution of (2.3).

Proof. See [2, Lemma 4.1].

Using a theorem of Olson [12], they then provided a lower bound of the amount of variables at level $0$ of colour $\nu$ which are required in order to ensure that $\nu$ is zero-representing.

Lemma 6. If $I_0^0 \geq p^\gamma + p^{\gamma-1} - 1$, then the colour $\nu$ is zero-representing.
Proof. See [2] Lemma 4.2.

Using these two lemmata and the theorem of Olson [12] again, they concluded the following statement.

**Lemma 7.** If a pair $f,g$ has $q_0 \geq 2p^\gamma - 1$, then there exists a non-singular solution of \((2.6)\).

**Proof.** See [2] Lemma 4.4.

Therefore, it suffices to focus on $p$-normalised forms $f,g$ that fulfil the properties of Lemma 3 and have $q_0 \leq 2p^\gamma - 2$.

4. **Combinatorial Results**

This section contains a collection of lemmata with combinatorial results on congruences modulo $p$ and $p^2$ for primes $p$, which will later be convenient for finding contraction in certain sets.

**Lemma 8.** Let $n > \text{ggT} (k,p-1)$ and $c_1, \ldots, c_n$ be any integers coprime to $p$. Then, the congruence
\[ c_1 x_1^k + \cdots + c_n x_n^k \equiv 0 \mod p\]
has a solution with $x_1 \not\equiv 0 \mod p$.

**Proof.** See [3] Lemma 1.

**Lemma 9.** Let $\alpha_{ij} \in \mathbb{Z}^n$ for $1 \leq i \leq n$ and $1 \leq j \leq s$ with $s \geq np - n + 1$. Then the equation
\[ \sum_{j=1}^{s} \varepsilon_j \binom{\alpha_{ij}}{\alpha_{nj}} \equiv 0 \mod p\]
has a solution with $\varepsilon_j \in \{0,1\}$ for $1 \leq j \leq s$ and some $\varepsilon_j \neq 0$.

**Proof.** This is the special case $G = (\mathbb{Z}/p\mathbb{Z})^n$ of the theorem of Olson [12].

**Lemma 10.** Let $s \geq 3p - 2$ and $a_j, b_j \in \mathbb{Z}$ for $1 \leq j \leq s$. Then there exists a non-empty subset $J \subset \{1,2,\ldots,s\}$ with $|J| \leq p$ and $\sum_{j \in J} a_j \equiv \sum_{j \in J} b_j \equiv 0 \mod p$.

**Proof.** See [13] Lemma 1.1.

**Lemma 11.** Let $d_j \in \mathbb{Z}/p\mathbb{Z}$ for $1 \leq j \leq 3p - 2$. Then there exists a non-empty subset $J \subset \{1,\ldots,3p-2\}$ with $|J| \leq p$,
\[ \sum_{j \in J} d_j \equiv 0 \mod p \quad \text{and} \quad \sum_{j \in J} d_j \not\equiv 0 \mod p^2.\]

**Proof.** See [1] Lemma 3.7.

**Lemma 12.** Let $d_j \in \mathbb{Z}/5\mathbb{Z}$ for $1 \leq j \leq 9$. Then there exists a non-empty subset $J \subset \{1,\ldots,9\}$ with $|J| \leq 5$,
\[ \sum_{j \in J} d_j \equiv 0 \mod 5 \quad \text{and} \quad \sum_{j \in J} d_j \not\equiv 0 \mod 25.\]

**Proof.** See [6] Proposition 3.1

5. **Strategy**

This section contains a general description of the remainder of the proof, for which further notation is introduced. Assume for the remainder of this paper that $\tau \geq 1$ is an integer, $p \geq 5$ a prime and $k = p^\tau (p-1)$. This will not be repeated in the following but nonetheless assumed in all following lemmata.

**Definition 8.** A $p$-normalised pair of additive forms $f,g$ as in (2.1) is called a proper $p$-normalised pair if $s \geq 2k^2 + 1$, $q_0 \leq 2p^{\tau+1} - 2$ and it satisfies the properties of Lemma 3.
The restrictions on $k$, $p$ and $\tau$ show that $\gamma = \tau + 1$. Therefore, it follows from Lemmata 1 and 4 and 7 that it suffices to prove for every proper $p$-normalised pair $f, g$ that the equations $f = g = 0$ have a non-trivial $p$-adic solution.

The bound $s \geq 2k^2+1$ and Lemma 2 show, that a proper $p$-normalised pair has the lower bounds

$$m_0 + \cdots + m_j \geq (2j + 2)p^\tau + 1,$$

$$m_0 + \cdots + m_{j-1} + q_j \geq (2j + 1)p^\tau + 1$$

for $j \in \{0, \ldots, k-1\}$ and Lemma 4 provides furthermore

$$l_0^0 - l_0^0 \geq 2p^\tau + 1 - 2p^\tau - q_0 - (m_1 - l_0^1).$$

To find a non-trivial $p$-adic solution for a proper $p$-normalised pair, it suffices, due to Lemma 3, to show that a non-singular solution exists. Using contractions as described in Section 3, this can be done by showing, that one can construct a primary variable at level $\tau + 1$.

In the following there will be two different strategies to construct a primary variable at level at least $\tau + 1$. For the first, one contracts the variables at level 0 to primary variables at level at least 1. Using contractions recursively, one can obtain primary variables at higher levels, until one eventually reaches at least level $\tau + 1$.

The second strategy will be used if $I_0^0 \geq p^{\tau+1} + p^\tau - 1$. By Lemma 6 with $\gamma = \tau + 1$, it follows that the colour 0 is zero-representing. In this case it suffices to have a contraction to a variable at level at least $\tau + 1$, which can be traced back to at least one variable at level 0 of a different colour than 0. If such a variable can also be traced back to a variable at level 0 of colour 0, the variable is already primary. Else, there is a contraction to another variable at level at least $\tau + 1$, using only the variables at level 0 of colour 0. Setting both of these variables 1 and everything else zero proves, that there is a non-singular solution of $f = g = 0$.

**Definition 9.** A variable which is either a variable at level 0 of a different colour than 0 or can be traced back to one is called *colourful*.

Thus, if $l_0^0 \geq p^{\tau+1} + p^\tau - 1$, the goal is to create a colourful variable at level at least $\tau + 1$.

The gain of this second strategy are the variables at level 0 of colour 0. To contract primary variables at level at least 1, one usually uses the variables at level 0. If the goal is only to contract colourful variables at level at least 1, it will suffice to use the $q_0$ variables at level 0 which are colourful. Then, the variables at level 0 of colour 0 can be used to create variables at a higher level, to help contracting the colourful variables to colourful variables at an even higher level, until one eventually contracts them to a colourful variable at level at least $\tau + 1$. This works, because then, one encounters one of the following two scenarios. Either the colourful variable at level at least $\tau + 1$ can be traced back to a variable at level 0 of colour 0. Then one has used one of those variables, which were created using the variables at level 0 of colour 0, some way along the way, and the colourful variable at level at least $\tau + 1$ is also primary. If on the other hand, the colourful variable at level at least $\tau + 1$ cannot be traced back to a variable at level 0 of colour 0, those helpful variables were not needed, to create a colourful variable at level at least $\tau + 1$. Hence, one can create a colourful variable at level at least $\tau + 1$, without using any of the variables at level 0 of colour 0, which still enables one to create a variable at level at least $\tau + 1$, using only those.

The process of creating a colourful or primary variable at level at least $\tau + 1$ will follow the same pattern. If one has a colourful or primary variable at level at least $l$, either this variable is already at level at least $l + 1$, or one tries to find a contraction to a variable at level at least $l + 1$, which contains the colourful or primary variable and thus ensures, that the resulting variable at level $l + 1$ is colourful or primary, as well. To find such a contraction, one needs to guarantee, that there are other variables at the same level with certain properties. Thus, one differs between the colourful and primary variables, for which one only needs to know a lower bound of their level, and the remaining variables, which will be useful, to contract colourful or primary variables to colourful and primary variables at a higher level. For them it is important to know the precise level they are at. This will be considered by the following notation.

A primary variable at level at least $l$ of colour nuance $\nu, \mu$ will be denoted by $P^l_{\nu^\mu}$, whereas a colourful variable which otherwise has the same properties will be denoted by $C^l_{\nu^\mu}$. The notation
$E_{\nu\mu}$ will be used to describe a variable at the exact level $l$ of colour nuance ($\nu, \mu$). Note that for $S \in \{C, P\}$ a variable of type $S_{\nu\mu}^l$ can either be of type $S_{\nu\mu}^{l+1}$ or of type $E_{\nu\mu}^l$, but not both. It will be said throughout the proof that a set of variables contracts to a variable with certain properties, if one the following cases occur. Either one of the variables in the set is already a variable with the desired properties, or the set of indices of these variables contains a contraction to a variable with these properties. This will help to minimize the amount of cases in which one has to distinguish between an $S_{\nu\mu}^l$ variables being of type $S_{\nu\mu}^{l+1}$ or $E_{\nu\mu}^l$ for $S \in \{C, P\}$. Sometimes one only wants to establish the level and the colour of one variable. Then, this is denoted by $P_{\nu}^l$, $C_{\nu}^l$, or $E_{\nu}^l$. If even the colour is of no importance, such a variable is said to be of type $P^l$, $C^l$ or $E^l$. In some cases, one has to denote, that a variable of type $E^l$ is not of colour $\nu$, or that a variable of type $E_{\nu}^l$ is not of colour nuance ($\nu, \mu$). This is denoted by $E_{\nu}^l$ and $E_{\nu,\mu}^l$, respectively.

It will turn out, that the number of $C^1$ and $P^1$ variables one can contract the $E^0$ variables to, is at least partly dependent on the parameter $q_0$. Therefore, it will be useful to define a further parameter $r = r(f, g)$ for a pair $f, g$ which restricts the area for $q_0$ to

\[(5.2)\]

\[p^{r+1} + rp^r \leq q_0 \leq p^{r+1} + (r + 1)p^r - 1.\]

For a proper $p$-normalised pair $f, g$ it follows that $r = r(f, g) \in \{-1, 0, 1, \ldots, p - 1\}$ due to $p^{r+1} - p^r + 1 \leq q_0 \leq 2p^{r+1} - 2$.

6. Contraction Related Auxiliaries

This section is a compilation of settings in which sets of variables contract to variables at a higher level.

6.1. Contracting One Specific Variable. The lemmata in this subsection describe situations in which one contracts sets of variables to one variable with specific properties.

**Lemma 13.** Let $\mathcal{K}$ be a set of indices of $E^l$ variables. If $|\mathcal{K}| \geq 2p - 1$ and $q(\mathcal{K}) \geq p$, then $\mathcal{K}$ contains a contraction $J$ to a variable at level at least $l + 1$, such that $J$ contains variables of at least two different colours.

**Proof.** This is a restatement of [4] Lemma 3].

**Lemma 14.** Let $S \in \{C, P\}$. A set of $2p - 1$ variables of type $S^l$ contracts to an $S^{l+1}$ variable.

**Proof.** Either one of the $S^l$ variables is already a variable of type $S^{l+1}$ or Lemma [3] can be used with $n = 2$ to show that the set of indices of the $2p - 1$ variables of type $S^l$ contains a contraction to a variable at level at least $l + 1$ which can be traced back to at least one of the $S^l$ variables. Therefore, it is an $S^{l+1}$ variable.

**Lemma 15.** Let $S \in \{C, P\}$ and let there be $3p - 2$ variables of type $S^l$. Then one can contract them to a variable of type $S^{l+1}$, using at most $p$ of them.

**Proof.** Either one of the $S^l$ variables is already a variable of type $S^{l+1}$ or one can contract the $S^l$ variables to a variable at level at least $l + 1$ using at most $p$ of them due to Lemma [10]. This variable can be traced back to at least one of the $S^l$ variables, thus it is an $S^{l+1}$ variable.

**Lemma 16.** Let there be $3p - 2$ variables of type $E_{\nu}^l$ for $p \geq 5$ and $2p - 1$ variables of type $E_{\nu}^l$ for $p = 5$. Then one can contract at most $p$ of these variables to a variable of type $E_{\nu}^{l+1}$.

**Proof.** For $p \geq 5$ see [7] Lemma 3.10 and for $p = 5$ see [6] Lemma 3.8].

**Lemma 17.** Let there be $3p - 2$ variables of type $E_{\nu,\mu}^l$ for $p \geq 5$ or $2p - 1$ variables for $p = 5$. Then one can contract at most $p$ variables to a variable of type $E_{\nu,\mu}^{l+1}$.
Proof. Let \( \mathcal{X} \) be the set of indices of these variables. Let \( c_i \) be the corresponding integer of the variable \( x_i \). Due to Lemma 11 for \( p \geq 5 \) and Lemma 12 for \( p = 5 \), there is a non-empty subset \( J \subseteq \mathcal{X} \) with \( |J| \leq p \), such that \( \sum_{j \in J} c_j = 0 \mod p \) while \( \sum_{j \in J} c_j \neq 0 \mod p^2 \) and it follows that

\[
\sum_{j \in J} \left( \frac{a_j}{b_j} \right) \equiv \sum_{j \in J} c_j (e_\nu + \mu e^{\nu'}) \equiv (e_\nu + \mu e^{\nu'}) \sum_{j \in J} c_j \not\equiv 0 \mod p^2,
\]

while \( \sum_{j \in J} c_j \equiv 0 \mod p \). As \( p \mid e^{\nu'} \), this leaves

\[
\sum_{j \in J} \left( \frac{a_j}{b_j} \right) \equiv e_\nu \sum_{j \in J} c_j \equiv p e_\nu \mod p^2
\]

for some \( c \) not congruent to 0 modulo \( p \). Hence, by setting \( x_i = 1 \) for all \( i \in J \), one can see that \( J \) is a contraction of at most \( p \) variables to a variable of type \( E^{i+1}_\nu \).

\( \square \)

Lemma 18. Let there be \( p-1 \) variables of type \( E^l_{\nu \mu_1} \) and one of type \( E^l_{\nu \mu_2} \) with \( \mu_1 \neq \mu_2 \). Then one can contract them to an \( E^{i+1}_\nu \) variable.

Proof. Define \( x^{-1} \) for an integer \( x \in \mathbb{Z} \setminus p \mathbb{Z} \) as the element in \( \{1, \ldots, p-1\} \) which solves \( x \cdot x^{-1} \equiv 1 \mod p \).

Let \( \mathcal{X} \) be the set of indices of those \( p \) variables and \( c_i \) be the corresponding integer for \( i \in \mathcal{X} \). Let \( x_{ia} \) be the \( E^l_{\nu \mu_2} \) variable. Due to Lemma 8 there is a solution of

\[
\sum_{i \in \mathcal{X}} c_i y_i^k \equiv t p \mod p^2
\]

for some \( t \in \{1, \ldots, p\} \) with \( y_{ia} \not\equiv 0 \mod p \). Consequently, one has \( y_{ia} \equiv 1 \mod p \) because \( p-1 \mid k \) and it follows that

\[
\sum_{i \in \mathcal{X}} \left( \frac{a_i}{b_i} \right) y_i^k \equiv \sum_{i \in \mathcal{X} \setminus \{ia\}} c_i (e_\nu + \mu_1 e^{\nu'}) y_i^k + c_{ia} (e_\nu + \mu_2 e^{\nu'}) y_{ia} \equiv t p e_\nu + c_{ia} e^{\nu'} (\mu_2 - \mu_1) \mod p^2
\]

which is divisible by \( p \) because \( e^{\nu'} \) is. For \( \nu = 0 \) one has

\[
t p e_\nu + c_{ia} e^{\nu'} (\mu_2 - \mu_1) \equiv p \left( t \left( \begin{array}{c} 1 \\ 0 \end{array} \right) + c_{ia} \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \right) (\mu_2 - \mu_1) \equiv p \left( c_{ia} (\mu_2 - \mu_1) \left( t e_{\nu_0}^{-1} (\mu_2 - \mu_1)^{-1} \right) \right) \mod p^2
\]

because \( p \) divides neither \( c_{ia} \) nor \( \mu_2 - \mu_1 \). It follows that the resulting variable lies at level \( l+1 \) and is of colour \( \nu' \neq 0 \) with \( \nu' \equiv t e_{\nu_0}^{-1} (\mu_2 - \mu_1)^{-1} \mod p \). For \( \nu \neq 0 \) one gets

\[
t p e_\nu + c_{ia} e^{\nu'} (\mu_2 - \mu_1) \equiv p \left( \left( \begin{array}{c} \nu' \\ 1 \end{array} \right) + c_{ia} \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \right) (\mu_2 - \mu_1) \ mod p^2
\]

which is for \( t \equiv 0 \mod p \) congruent to

\[
p \left( c_{ia} (\mu_2 - \mu_1) \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \right)
\]

and else congruent to

\[
p \left( t \left( \nu + t e_{\nu_0}^{-1} (\mu_2 - \mu_1) \right) \right).
\]

Hence, again because \( p \) divides neither \( c_{ia} \) nor \( \mu_2 - \mu_1 \) one obtains a variable at level \( l+1 \), which is for \( t \equiv 0 \mod p \) of colour 0 and for \( t \equiv 0 \mod p \) of colour \( \nu' \) for \( \nu' \equiv \nu + t e_{\nu_0}^{-1} (\mu_2 - \mu_1) \mod p \) with \( \nu' \neq \nu \).

\( \square \)

Lemma 19. Let \( S \in \{C, P\} \) and \( 0 \leq m \leq p-1 \). Let there be \( p-m-1 \) variables of type \( E^l_{\nu} \) and \( m+1 \) of type \( S^l_{\nu} \). Then they contract to a variable of type \( S^{l+1}_{\nu} \).
Proof. Either one of the $S_l^i$ variables is already a $S_l^{i+1}$ variable, or one can assume, that they are all of type $E_l^i$ as well. The cases $l > 0$ can be reduced to the case $l = 0$ by working with the level coefficient vector $(\bar{a}_l/b_l)$ instead of the coefficient vector $(a_l/b_l)$. See [10, Lemma 3.7] for the case $l = 0$. \qed

Lemma 20. Let $\mathcal{H}$ be a set of indices of variables of type $E_l^i$ with $|\mathcal{H}| \geq 4p - 3$ and either for all $i \in \mathcal{H}$ the corresponding integer $c_i$ is congruent to an element in the set $\{1, 2, \ldots, \frac{p-1}{2}\}$ modulo $p$ or all $c_i$ are congruent to elements in the set $\{\frac{p+1}{2}, \ldots, p-1\}$. Then $\mathcal{H}$ contains a contraction $\mathcal{K}$ to a variable of type $E_l^{i+1}$, with $|\mathcal{K}| \leq 2p - 2$.

Proof. For all $i \in \mathcal{H}$, let $(c_i, \mu_i)$ be the colour nuance of the variable $x_i$ and let $d_i \in \{1, 2, \ldots, p - 1\}$ and $f_i \in \{0, 1, \ldots, p - 1\}$ be such that as $c_i = d_i + pf_i$.

For the proof one can assume that $|\mathcal{H}| = 4p - 3$. If this is not the case, one can take a subset of $\mathcal{H}$ to obtain the desired result. The first part proves the weaker claim that $\mathcal{H}$ contains a subset $\mathcal{K}$ containing at most $2p$ variables such that

$$\sum_{i \in J} \left( \frac{d_i}{d_i \mu_i} \right) \equiv 0 \mod p \quad \text{and} \quad \sum_{i \in J} \left( \frac{\bar{a}_i}{b_i} \right) \equiv dp \epsilon \mod p^2,$$

for some $d \not\equiv 0 \mod p$. By Lemma 9, the set $\mathcal{H}$ contains a non-empty subset $J$ such that

$$(6.1) \quad \sum_{i \in J \cup J} \left( \frac{d_i}{d_i \mu_i} \right) \equiv 0 \mod p.$$ \hfill (6.1)

This leads to

$$\sum_{i \in J} \left( \frac{\bar{a}_i}{b_i} \right) \equiv \sum_{i \in J} c_i (\epsilon \mu_i \epsilon') \equiv \sum_{i \in J} (d_i + f_i p) (\epsilon \mu_i \epsilon')$$

$$\equiv \sum_{i \in J} d_i \epsilon \mu_i + \sum_{i \in J} d_i \mu_i \epsilon' + \sum_{i \in J} f_i \epsilon \mu_i \epsilon' + \sum_{i \in J} f_i \epsilon \mu_i \epsilon'$$

$$\equiv \epsilon \mu_i \sum_{i \in J} d_i \mod p^2,$$

where the last equivalence holds due to $p \mid \epsilon'$ and the second and third entry in (6.1). The first entry shows that this is congruent to 0 modulo $p$. As $J$ is a non-empty subset of $\mathcal{H}$, it follows from the fourth entry, that $|J| \in \{p, 2p, 3p\}$. If $|J| = 3p$, take a subset $J \subseteq J$ containing $3p - 2$ elements. By Lemma 9 with $n = 3$, there is a subset $J \subseteq J$ with

$$\sum_{i \in J \cup J} \left( \frac{d_i}{d_i \mu_i} \right) \equiv 0 \mod p,$$

and hence,

$$\sum_{i \in J} \left( \frac{\bar{a}_i}{b_i} \right) \equiv \epsilon \mu_i \sum_{i \in J} d_i \mod p^2,$$

as before, which again is congruent to 0 modulo $p$. As $J = J \cup (J \setminus J)$, it follows that

$$\sum_{i \in J \setminus J} \left( \frac{d_i}{d_i \mu_i} \right) \equiv 0 \mod p,$$

and therefore,

$$\sum_{i \in J \setminus J} \left( \frac{\bar{a}_i}{b_i} \right) \equiv \epsilon \mu_i \sum_{i \in J \setminus J} d_i \mod p^2,$$
which is congruent to 0 modulo $p$ as well. Furthermore, both sets $\mathcal{J}$ and $\mathcal{J} \setminus \mathcal{J}$ are non-empty, and the smallest of them has at most $\frac{2p}{2} \leq 2p$ elements. It follows, that in every case there is a non-empty set $\mathcal{K} \subset \mathcal{H}$ containing at most $2p$ elements, such that

$$\sum_{i \in \mathcal{J}} \frac{\bar{a}_i}{\bar{b}_i} \equiv e_\nu \sum_{i \in \mathcal{J}} d_i \mod p^2,$$

and

$$\sum_{i \in \mathcal{J}} \left( d_i, \mu_i \right) \equiv 0 \mod p.$$

Assume now for such a set $\mathcal{K}$ that all corresponding integers $c_i$ are congruent to elements in the set $\{1, 2, \ldots, \frac{p-1}{2}\}$ modulo $p$. It follows, that $d_i$ lies in the same set for all $i \in \mathcal{K}$. Hence, it can be deduced from

$$1 \leq \sum_{i \in \mathcal{X}} d_i \leq \sum_{i \in \mathcal{X}} \frac{p-1}{2} \leq p(p-1),$$

that $\sum_{i \in \mathcal{X}} d_i \not\equiv 0 \mod p^2$ and therefore,

$$\sum_{i \in \mathcal{X}} \left( \frac{\bar{a}_i}{\bar{b}_i} \right) \equiv dp e_\nu \mod p^2$$

for some $d \not\equiv 0 \mod p$. This proves the weaker claim if all $c_i$ are modulo $p$ congruent to an element in the set $\{1, 2, \ldots, \frac{p-1}{2}\}$. Now let all $c_i$ be congruent to elements in the set $\left\{ \frac{p+1}{2}, \ldots, p-1 \right\}$. It follows that

$$\left( -\frac{\bar{a}_i}{\bar{b}_i} \right) \equiv (p^2 - c_i) (e_\nu + \mu_i e_\nu') \equiv (p - d_i + p(p - f_i - 1)) (e_\nu + \mu_i e_\nu') \mod p^2,$$

and, the corresponding integers $p - d_i + p(p - f_i - 1)$ lie modulo $p$ in $\{1, 2, \ldots, \frac{p-1}{2}\}$, again. Using the obtained results, there is a subset $\mathcal{K} \subset \mathcal{H}$ with $|\mathcal{K}| \leq 2p$ and

$$\sum_{j \in \mathcal{J}} \left( -\frac{\bar{a}_j}{\bar{b}_j} \right) \equiv dp e_\nu \mod p^2$$

for some $d \not\equiv 0 \mod p$ and, as $\left( \frac{\bar{a}_i}{\bar{b}_i} \right)$ lies in the same set $\mathcal{L}_\nu$ as $\left( -\frac{\bar{a}_i}{\bar{b}_i} \right)$, one further has

$$\sum_{j \in \mathcal{J}} \left( \frac{p - d_j}{(p - d_j, \mu_j) \mu_j} \right) \equiv 0 \mod p.$$

It follows that

$$\sum_{j \in \mathcal{J}} \left( \frac{\bar{a}_j}{\bar{b}_j} \right) = -\sum_{j \in \mathcal{J}} \left( -\frac{\bar{a}_j}{\bar{b}_j} \right) \equiv -dp e_\nu \mod p^2$$

for some $d \not\equiv 0 \mod p$ and it further holds that

$$\sum_{j \in \mathcal{J}} \left( \frac{d_j}{d_j, \mu_j} \right) \equiv 0 \mod p.$$

This completes the proof for the weaker claim. Now let $\mathcal{K} \subset \mathcal{H}$ be a subset with $|\mathcal{K}| \leq 2p$,

$$\sum_{i \in \mathcal{X}} d_i \equiv 0 \mod p \quad \text{and} \quad \sum_{i \in \mathcal{X}} \left( \frac{\bar{a}_i}{\bar{b}_i} \right) \equiv pde_\nu \mod p^2$$

for some $d \not\equiv 0 \mod p$. Assuming that $|\mathcal{K}| \geq 2p - 1$, there is, according to Lemma 9 with $n = 2$, a subset $\tilde{\mathcal{K}} \subset \mathcal{K}$ with $|\tilde{\mathcal{K}}| \leq 2p - 1$ and

$$\sum_{i \in \mathcal{K}} \left( \frac{d_i}{d_i, \mu_i} \right) \equiv 0 \mod p.$$

It follows, that

$$\sum_{i \in \mathcal{K}} \left( \frac{\bar{a}_i}{\bar{b}_i} \right) \equiv e_\nu \sum_{i \in \mathcal{X}} d_i + pe_\nu \sum_{i \in \mathcal{X}} f_i \mod p^2.$$
which is congruent to 0 modulo $p$, but not necessarily incongruent to 0 modulo $p^2$. As

\[ \sum_{i \in \mathcal{X} \setminus \mathcal{X}} \left( \frac{d_i}{d_i \mu_i} \right) \equiv 0 \mod p \]

holds as well, one can deduce, that

\[ \sum_{i \in \mathcal{X} \setminus \mathcal{X}} \left( \frac{\bar{a}_i}{b_i} \right) \equiv e_{\nu} \sum_{i \in \mathcal{X} \setminus \mathcal{X}} d_i + pe_{\nu} \sum_{i \in \mathcal{X} \setminus \mathcal{X}} f_i \mod p^2, \]

which is again congruent to 0 modulo $p$. For at least one of those sets, either $\mathcal{X}$ or $\mathcal{X} \setminus \mathcal{X}$, the sum is not congruent to 0 modulo $p^2$ as the sum over all $i \in \mathcal{X}$ is not, and therefore, it is impossible for both subsums to be congruent to 0 modulo $p^2$. The set for which this sum is incongruent to 0 modulo $p^2$ is a contraction to a variable of type $E_i^{s+1}$.

Both subsets are non-empty and hence, as all $d_i$ are incongruent to 0 modulo $p$, they contain at least 2 elements. Thus, each one has a most $2p - 2$ elements, which proves the claim.

**Lemma 21.** Let $S \in \{C, P\}$ and $0 \leq m \leq p - 1$. Let there be $p + m$ variables of type $S^l$ and further $p - m - 1$ variables of type $E_i^l$. Then one can contract them to an $S^{l+1}$ variable.

**Proof.** If one of the $S^l$ variables is already an $S^{l+1}$ variable, the claim is fulfilled. Thus, one can assume, that the $S^l$ variables are $E^l$ variables as well. If there are $p$ variables of the same colour $\mu$, then at least one of them is an $S^l$ variable, because there are at most $p - 1$ variables which are not. Hence, Lemma 19 shows that one can contract them to an $S^{l+1}$ variable.

Else, there are at most $p - 1$ variables of the same colour. Let $\mathcal{X}$ be the set of indices of all $2p - 1$ variables. Then, one has $I_{\max}(\mathcal{X}) \leq p - 1$, and thus, $q(\mathcal{X}) \geq p$. By Lemma 13, the set $\mathcal{X}$ contains a contraction to a variable at level at least $l + 1$, using at least two different colours. One can trace that variable back to at least one of the $S^l$ variables, because the variables which are not of type $S^l$ are all of the same colour, which proves the claim.

**Lemma 22.** Let $S \in \{C, P\}$ and $0 \leq m \leq p - 1$. Let there be $p - 1$ variables of type $E_i^l$, $p - m - 1$ variables of type $E_i^l$ and $m + 1$ variables of type $S^l$. Then one can contract them to an $S^{l+1}$ variable.

**Proof.** If one of the variable of type $S^l$ is already an $S^{l+1}$ variable, the claim is fulfilled, thus one can assume that these variables are of type $E^l$ as well. Furthermore, one can assume, that none of the $S^l$ variables is of type $E_i^l$, because else, Lemma 13 can be use to contract the $p - 1$ variables of type $E_i^l$ together with the $S^l$ variable to an $S^{l+1}$ variable.

Therefore, one can assume that one has $p - 1$ variables of type $E_i^l$ and $p$ variables of type $E_i^l$ from which at least one is an $S^l$ variable. For convenience name the $E_i^l$ variables $x_1, \ldots, x_{p-1}$ and the $E_i^l$ variables $x_p, \ldots, x_{2p-1}$, where $x_{2p-1}$ is an $S^l$ variable. Furthermore, let $c_i$ be the corresponding integer of $x_i$ for $1 \leq i \leq 2p - 1$ and $\nu_i \neq \nu$ the colour of the variables $x_i$ for $p \leq i \leq 2p - 1$. These $2p - 1$ variables contract to an $S^{l+1}$ variable if there is a solution of

\[ \sum_{i=1}^{p-1} c_i e_{\nu_i} x_i^k + \sum_{i=p}^{2p-1} c_i e_{\nu_i} x_i^k \equiv 0 \mod p, \]

with $x_{2p - 1} \neq 0 \mod p$. The existence of such a solution follows from the proof of Theorem 2 by Olson and Mann [11], but not from the statement of the theorem, from which one can only conclude the existence of a solution, but not that one has one with $x_{2p - 1} \neq 0 \mod p$. Thus, for the convenience of the reader, the following contains a proof that such a solution exists. In essence the proof uses the same methods as the proof by Olson and Mann, but is tailored for this exact case.

By applying the linear transformation induced by

\[ \begin{pmatrix} 1 & 0 \\ 1 & -\nu \end{pmatrix} \]


if \( \nu \neq 0 \), one can transform the case \( \nu \neq 0 \) to the case \( \nu = 0 \), because

\[
\begin{pmatrix}
1 & 0 \\
0 & \nu
\end{pmatrix}

\begin{pmatrix}
e_{\nu}
\end{pmatrix}

= \nu
\begin{pmatrix}
e_0
\end{pmatrix}

and

\[
\begin{pmatrix}
1 & 0 \\
1 & \nu
\end{pmatrix}

\begin{pmatrix}
e_{\nu}
\end{pmatrix}

\in \mathcal{L}_0
\]

for some \( \tilde{\nu} \neq \nu \). All that remains is to solve a system of the kind

\[
\sum_{i=1}^{p-1} (\alpha_i x_i + \sum_{i=p}^{2p-1} \beta_i x_i) \equiv 0 \mod p
\]

with \( y_{2p-1} \equiv 0 \mod p \). This reduces the system \([6.2]\) by setting \( x_i = y_i \) for \( p \leq i \leq 2p - 1 \) to

\[
\sum_{i=p}^{2p-1} \gamma_i y_i^k \equiv 0 \mod p
\]

for \( C = \sum_{i=p}^{2p-1} \beta_i y_i^k \). Now consider an additional variable \( y_0 \). If \( p \mid y_0 \) then \( y_0^k \equiv 1 \mod p \), hence, applying Lemma 5 again, this time to the system

\[
\sum_{i=1}^{p-1} \gamma_i y_i^k \equiv 0 \mod p
\]

provides a solution \( y_i \) with \( p \mid y_0 \). It follows that \( x_i = y_i \) for \( 1 \leq i \leq p - 1 \) is also a solution for \([6.3]\), and therefore, one has a solution of \([6.2]\) given by \( x_i = y_i \) with \( 1 \leq i \leq 2p - 1 \) with \( p \mid x_{2p-1} \).

\( \square \)

6.2. Contracting Several Variables. The lemmata in this section show how to contract a set of variables at level at least \( l \) to another set of variables at level at least \( l + 1 \).

Lemma 23. Let \( \mathcal{H} \subset \{1, \ldots, m_0\} \) be a subset of indices of variables at level 0. Then \( \mathcal{H} \) contains at least

\[
\min\left(\left\lfloor \frac{|\mathcal{H}|}{2p-1} \right\rfloor, \frac{q(\mathcal{H})}{p} \right)
\]

pairwise disjoint contractions to variables of type \( P^1 \).

Proof. This is the special case \( \delta = \gcd(k, p-1) = p-1 \) of a result from Lemmata 1 and 3 of \([4]\) which is proved in the second paragraph of Section 6 of that paper.

\( \square \)

Lemma 24. Let \( S \in \{C, P\} \) and let there be \( x \) variables of type \( S^l \). They contract to \( \left\lfloor \frac{x+3}{p} \right\rfloor - 3 \) variables of type \( S^{l+1} \), where each contraction contains at most \( p \) variables, leaving at least \( \min\{2p-2, x\} \) variables of type \( S^l \) unused.

Proof. For \( x \leq 3p-3 \) the statement is trivial. Therefore, let \( x \geq 3p-2 \). Assume first, that all \( x \) variables are also of type \( E^l \). Then there is a contraction of at most \( p \) variables to an \( S^{l+1} \) variable due to Lemma 15. Hence, after doing this \( \left\lfloor \frac{x+3}{p} \right\rfloor - 4 \) times, there are still at least

\[
x - \left( \left\lfloor \frac{x+3}{p} \right\rfloor - 4 \right) p 
\geq x - (x+3+p-1-4p) = 3p-2
\]

unused \( S^l \) variables. Hence, one can apply Lemma 15 once more to obtain \( \left\lfloor \frac{x+3}{p} \right\rfloor - 3 \) contractions, leaving at least \( 2p-2 \) variables unused. Thus, in this case, the claim holds.

Now assume that of the \( x \) variables of type \( S^l \) there are \( y \) variables already of type \( S^{l+1} \) while the remaining \( x-y \) variables are of type \( E^l \). One has

\[
y \geq \left\lfloor \frac{x+3}{p} \right\rfloor - 3 + 2p - 2 - (x-y)
\]
because of $x \geq 3p - 2$. If $x - y \leq 2p - 2$, one can divide the $y$ variables of type $S^{l+1}$ in one set containing $\left\lceil \frac{x + 3}{p} \right\rceil - 3$ and one set containing $2p - (x - y)$ of them. The variables in the second set together with the remaining $x - y$ variables of type $S^l$ are at least $2p - 2$ variables of type $S^l$, while the first set contains the $\left\lceil \frac{x + 3}{p} \right\rceil - 3$ variables of type $S^{l+1}$. Thus one can assume, that $x - y \geq 2p - 1$ and use the first part of this proof. The set of the $x - y$ variables of type $E^l$ contains at least

$$\left\lceil \frac{x - y + 3}{p} \right\rceil - 3$$

contractions to variables of type $S^{l+1}$, leaving at least $2p - 2$ variables of type $S^l$ unused. Together with the $y$ variables of type $S^{l+1}$ this gives at least

$$\left\lceil \frac{x - y + 3}{p} \right\rceil - 3 + y = \left\lceil \frac{x - y + 3}{p} \right\rceil - 3 + \left\lceil \frac{x + y(p - 1) + 3}{p} \right\rceil - 3 \geq \left\lceil \frac{x + 3}{p} \right\rceil - 3$$

to variable of type $S^{l+1}$.

**Lemma 25.** Let there be $x$ variables of type $E^l_\nu$. They contract to $\left\lceil \frac{x}{2p - 2} \right\rceil - 4$ variables of type $E^{l+1}_\nu$, leaving at least $\min\{6p - 9, x\}$ variables of type $E^l_\nu$ unused.

**Proof.** For $x < 8p - 7$ the statement is trivial. If $x \geq 8p - 7$, one can divide the $x$ variables in two sets. Those for which the corresponding integer $c_i$ is congruent to one element in \{1, \ldots, \frac{p - 1}{2}\} modulo $p$, and the remaining variables. As long as there are at least $8p - 7$ variables left, at least one of these sets contains at least $4p - 3$ variables, which indicates that one can contract at most $2p - 2$ of them to a variable of type $E^{l+1}_\nu$ due to Lemma 20. Doing this $\left\lceil \frac{x}{2p - 2} \right\rceil - 5$ times leaves at least

$$x - (2p - 2) \left(\left\lceil \frac{x}{2p - 2} \right\rceil - 5\right) \geq x - x - 2p + 3 + 10p - 10 = 8p - 7$$

unused variables, hence, there is another contraction, leaving at least $6p - 9$ variables unused. □

**Lemma 26.** A set of $x \geq 3p^2 - 3p + 1$ variables of type $E^l_\nu$ contracts to $\left\lceil \frac{x}{p} \right\rceil - 2p + \frac{p - 1}{2}$ variables of type $E^{l+1}_\nu$ for $p \geq 5$. A set of $x \geq 2p^2 - 2p + 1$ variables of type $E^l_\nu$ contracts to $\left\lceil \frac{x}{p} \right\rceil - 2p + 3$ variables of type $E^{l+1}_\nu$ for $p = 5$. In both cases, this leaves at least $6p - 9$ of the $E^l_\nu$ variables unused.

**Proof.** A set of at least $(3p - 3)p + 1$ variables of type $E^l_\nu$ contains at least $3p - 2$ variables which are of the same colour nuance. By Lemma 17, one can contract at most $p$ variables of them to a variable of type $E^{l+1}_\nu$. Repeating this as often as possible provides $\left\lceil \frac{x}{p} \right\rceil - 3p + 3$ variables of type $E^{l+1}_\nu$ and leaves at least

$$x - p \left(\left\lceil \frac{x}{p} \right\rceil - 3p + 3\right) \geq x - (x + p - 1 - 3p^2 + 3p) = 3p^2 - 4p + 1$$

unused $E^l_\nu$ variables. For $p = 5$ this can be done as long as there are at least $(2p - 2)p + 1$ variables left. Therefore, one can do it $\left\lceil \frac{x}{p} \right\rceil - 2p + 2$ times, leaving at least

$$x - p \left(\left\lceil \frac{x}{p} \right\rceil - 2p + 2\right) \geq x - (x + p - 1 - 2p^2 + 2p) = 2p^2 - 3p + 1$$

unused variables. Using Lemma 20 provides another $p + \frac{p - 1}{2} - 4$ variables of type $E^{l+1}_\nu$ for $p \geq 5$ and one for $p = 5$, while leaving at least $6p - 9$ unused variables. All in all, one obtains

$$\left\lceil \frac{x}{p} \right\rceil - 3p + 3 + p + \frac{p - 1}{2} - 4 = \left\lceil \frac{x}{p} \right\rceil - 2p + p - \frac{3}{2}$$

variables of type $E^{l+1}_\nu$ for $p \geq 5$ and

$$\left\lceil \frac{x}{p} \right\rceil - 2p + 2 + 1 = \left\lceil \frac{x}{p} \right\rceil - 2p + 3$$

for $p = 5$. □
Lemma 27. Let \( S \in \{C, P\} \) and \( x, y, z \) be non-negative integers with \( y + z \geq (2 - m) p - 2 \) for some \( m \in \{0, 1, 2\} \) and \( x - m \geq 0 \). Let there be \( (p - 1) y \) variables of type \( E^l_{\nu} \), \( (p - 1) y \) variables of type \( E^r_{\nu} \) and \( px + y + z \) variables of type \( S^l \). Then one can contract them to \( x + y - m \) variables of type \( S^{l+1} \) without using \( z + mp \) of the variables of type \( S^l \).

Proof. Using Lemma 15 to contract \( p \) of the variables of type \( S^l \) to an \( S^{l+1} \) variable can be done \( x - m \) times. This leaves \( y + z + mp \geq 2p - 2 \) variables of type \( S^l \). Then, one can construct \( y \) sets, each consisting of one \( S^l \) variable, \( p - 1 \) variables of type \( E^l_{\nu} \) and \( p - 1 \) variables of type \( E^r_{\nu} \). By Lemma 22 each of these sets contains a contraction to an \( S^{l+1} \) variable, giving a total of \( x + y - m \) variables of type \( S^{l+1} \) as claimed, without using \( z + mp \) variables of type \( S^l \). \( \square \)

Lemma 28. Let \( S \in \{C, P\} \) and \( x \) be a non-negative integer. Let \( \mathcal{X} \) be a set of \( E^l \) variables with \( |\mathcal{X}| \geq (2p - 2) x + p^2 - 3p + 1 \) and \( q(\mathcal{X}) \geq (p - 1) x \) and let there be further \( x \) variables of type \( S^l \). Then one can contract them to \( x \) variables of type \( S^{l+1} \).

Proof. The first part of the proof will show via induction on \( x \) that the set \( \mathcal{X} \) contains \( x \) distinct sets \( S_i \) with \( |S_i| = 2p - 2 \) and \( q(S_i) = p - 1 \) for all \( 1 \leq i \leq x \).

For \( x = 0 \) the statement is true. It suffices to show for \( x \geq 1 \) that \( \mathcal{X} \) contains a set \( \mathcal{X}' \) with \( |\mathcal{X}'| = (2p - 2) x + p^2 - 3p + 1 + \alpha \) and \( q(\mathcal{X}') = x(\alpha) + \beta \) with \( \alpha, \beta \in \mathbb{N}_0 \). As \( x \geq 1 \) it follows that \( q(\mathcal{X}') \geq p - 1 \) and \( |\mathcal{X}'| \geq p^2 - p - 1 = (p + 1)(p - 2) + 1 \), hence, \( I_{\max}(\mathcal{X}) = I_{\nu}(\mathcal{X}) \geq p - 1 \) for some \( 0 \leq \nu \leq p \). Thus, one can take \( \mathcal{X}' \) as a set containing \( p - 1 \) variables of type \( E^l_{\nu} \) and \( p - 1 \) variables of type \( E^r_{\nu} \) from which it follows that \( |\mathcal{X}'| = 2p - 2, q(\mathcal{X}') = p - 1 \) and

\[
|\mathcal{X}'| = |\mathcal{X}| - 2p + 2 \geq (x - 1)(2p - 2) + p^2 - 3p + 1.
\]

For \( \beta \geq p - 1 \) one has the trivial bound

\[
q(\mathcal{X}') \geq q(\mathcal{X}) - 2(x - 1)(p - 1) + \beta - (p - 1) \geq (x - 1)(p - 1),
\]

whereas for \( \beta \leq p - 2 \) it follows that

\[
I_{\max}(\mathcal{X}) = q(\mathcal{X}) - (p - 1) \geq (x - 1)(p - 1).
\]

and thus

\[
q(\mathcal{X}') = q(\mathcal{X}) - (p - 1) \geq (x - 1)(p - 1).
\]

It follows, that the set \( \mathcal{X} \) contains \( x \) distinct sets \( S_i \) with \( |S_i| = 2p - 2 \) and \( q(S_i) = p - 1 \).

For each set \( S_i \) there is a \( \nu_i \) such that \( I_{\max}(S_i) = I_{\nu_i}(S_i) = p - 1 \). For \( i \in \{1, \ldots, x\} \) take the set \( S_i \) and one variable of type \( S^l \), which gives \( p - 1 \) variables of type \( E^l_{\nu_i} \), \( p - 1 \) variables of type \( E^r_{\nu_i} \) and one \( S^l \) variable. Such a set contains a contraction to an \( S^{l+1} \) variable due to Lemma 22. Thus, one obtains \( x \) variables of type \( S^{l+1} \). \( \square \)

Lemma 29. Let \( S \in \{C, P\} \) and \( x, y, z \) be non-negative integers with \( y + z \geq (2 - m) p - 2 \) for some \( m \in \{0, 1, 2\} \) and \( x - m \geq 0 \). Let there be \( (2p - 2) y + p^2 - 3p + 1 \) variables of type \( E^l_{\nu} \) from which at least \( (p - 1) y \) variables are of type \( E^l_{\nu} \) for any \( 0 \leq \nu \leq p \). Furthermore, let there be \( px + y + z \) variables of type \( S^l \). Then one can contract them to \( x + y - m \) variables of type \( S^{l+1} \) without using \( z + mp \) of the variables of type \( S^l \).

Proof. Using Lemma 15 to contract \( p \) of the variables of type \( S^l \) to an \( S^{l+1} \) variable can be done \( x - m \) times. This leaves \( y + z + mp \geq 2p - 2 \) variables of type \( S^l \). One can contract \( y \) of them together with the variables of type \( E^l_{\nu} \) to \( y \) variables of type \( S^{l+1} \) due to Lemma 28. This gives a total of \( x + y - m \) variables of type \( S^{l+1} \) as claimed, without using \( z + mp \) variables of type \( S^l \). \( \square \)

Lemma 30. Let \( x \) be a non-negative integer. Let there be at least \( px + p^2 - 3p + 3 \) variables of type \( E^l_{\nu} \) from which at least \( x \) are of type \( E^l_{\nu} \) for some \( \mu \) and at least \( x \) are of type \( E^r_{\nu} \). Then one can contract \( px \) of them to \( x \) variables of type \( E^l_{\nu} \).
Proof. Divide the $E_{\nu}^1$ variables in three sets. One contains $x$ variables of type $E_{\nu}^1$, the next one contains $y$ variables of type $E_{\nu}^1$ and the last one contains the remaining variables.

The statement is trivial for $x = 0$, thus one can assume that $x \geq 1$. Assume now, that the last set contains $z \geq (p-2)p+1 = p^2-2p+1$ variables, and the first two both contain $y \geq 1$ variables. Then there is an $\eta$ such that the last set contains at least $p-1$ variables of type $E_{\nu}^1$ and one can choose one variable in one of the first two sets, which is of type $E_{\nu}^1$. These $p$ variables contract to an $E_{\nu}^{l+1}$ variable due to Lemma [18]. Then, one can take one variable in the untouched set and put it in the last set, such that the first two sets both contain $y-1$ variables and the last one contains $z - p + 2$ variables.

Starting with $z \geq (p-2)x + p^2-3p+3$ and $y = x$, after following this process $x-1$ times, one still has at least $p^2-2p+1$ variables in the last set left, while the other two each contain one variable.

It follows, that one can contract one more variable of type $E_{\nu}^{l+1}$ as described above, giving a total of $x$ variables of type $E_{\nu}^{l+1}$. \hfill \qed

### 6.3. Inductive Contractions

This subsection uses induction to contract sets of variables at some level to variables more than one level higher.

**Lemma 31.** Let $S \in \{C, P\}$ and $i, j \in \mathbb{N}_0$ with $i \leq j \leq \tau$ as well as $m \in \mathbb{Z}$ with $m \geq 1$. Let there be $p^{\tau-i+1} + mp^{\tau-i-1} - 2$ variables of type $S^l$. Then one can contract them to $p^{\tau-j+1} + mp^{\tau-j-1} - 2$ variables of type $S^l$ and at least $2p - 2$ variables of type $S^l$ for all $l \in \{i, \ldots, j-1\}$.

**Proof.** For $i = j$ the statement is trivial, thus, the cases $i < j \leq \tau$ remain. Assume for an $l \in \{i, \ldots, j-1\}$ that there are $p^{\tau-i+1} + mp^{\tau-i-1} - 2$ variables of type $S^l$ and $2p-2$ variables of type $S^l$ for all $n \in \{i, \ldots, l-1\}$. Lemma [24] shows that these variables can be contracted to

$$\left\lfloor \frac{p^{\tau-i+1} + mp^{\tau-i-1} + 1}{p} \right\rfloor - 3 = p^{\tau-i} + mp^{\tau-i-1} - 2$$

variables of type $S^{l+1}$. This leaves at least $2p - 2$ variables of type $S^l$ unused. The claim follows via induction. \hfill \qed

**Lemma 32.** Let $S \in \{C, P\}$ and $i, j \in \mathbb{N}_0$ with $i \leq j \leq \tau$ as well as $m \in \mathbb{Z}$ with $m \geq -1$. Let there be $p^{\tau-i+1} + mp^{\tau-i}$ variables of type $S^l$ and for all $l \in \{i, \ldots, j-1\}$ let there be an $n_l$ and $2p-2$ variables of type $S^l$. Then one can contract them to $p^{\tau-j+1} + mp^{\tau-j}$ variables of type $S^l$.

**Proof.** For $i = j$ the statement is trivial, thus, the cases $i < j \leq \tau$ remain. Assume for an $l \in \{i, \ldots, j-1\}$ there are $p^{\tau-i+1} + mp^{\tau-i}$ variables of type $S^l$ and $2p-2$ variables of type $E_{\nu}^1$. Lemma [24] shows that there exist

$$\left\lfloor \frac{p^{\tau-i+1} + mp^{\tau-i} + 3}{p} \right\rfloor - 3 = p^{\tau-i} + mp^{\tau-i-1} - 2$$

contractions to variables $S^{l+1}$, each of them containing at most $p$ variables. Therefore, there are even $2p$ variables of type $S^l$ remaining. Together with the $2p-2$ variables of type $E_{\nu}^1$, they can be contracted to another two $S^{l+1}$ variables, using Lemma [24] twice. This gives a total of $p^{\tau-l} + mp^{\tau-l}$ variables of type $S^{l+1}$. The claim follows via induction. \hfill \qed

**Lemma 33.** Let $m \leq p - 1$ be an integer and let there be a $j \in \{0, 1, \ldots, \tau - 1\}$ such that there are

$$2p^{\tau-j} + (4 - 2m)p^{\tau-j} = \frac{p-1}{2} \sum_{i=1}^{\tau-j-1} p^i + (2m-1) p^{\tau-j} + 3 \sum_{i=0}^{\tau-j-2} p^i - 2p - 2,$$

variables of type $E_{\nu}^1$. Then one can contract them to $p - m - 1$ variables of type $E_{\nu}^1$ and $2p - 2$ variables of type $E_{\nu}^1$ for all $l \in \{j, j+1, \ldots, \tau - 1\}$.

**Proof.** If $j \leq \tau - 2$, assume that for some $l \in \{j, j+1, \ldots, \tau - 2\}$ one can contract the variables to $2p^{\tau-l} + (4 - 2m)p^{\tau-l} - \frac{p-1}{2} \sum_{i=1}^{\tau-l-1} p^i + (2m-1) p^{\tau-l} + 3 \sum_{i=0}^{\tau-l-2} p^i - 2p - 2$ variables of type $E_{\nu}^1$.
and $2p - 2$ variables of type $E^l_{i'}$ for all $i \in \{j, j + 1, \ldots, l - 1\}$. Using Lemma 25, the variables of type $E^l_{i'}$ can be contracted to
\[
\left[2p^{\tau_{i}+1} + (4 - 2m)p^{\tau_{i}} - \frac{p - 1}{2} \sum_{i=1}^{\tau_{i}-1} p^{i} + (2m - 1)p^{\tau_{i}-1} + 3 \sum_{i=0}^{\tau_{i}-2} p^{i} - 2p + 1\right] - 2p + p - \frac{3}{2}
\]
\[= 2p^{\tau_{i}} + (4 - 2m)p^{\tau_{i}-1} - \frac{p - 1}{2} \sum_{i=0}^{\tau_{i}-2} p^{i} + (2m - 1)p^{\tau_{i}-1} + 3 \sum_{i=0}^{\tau_{i}-3} p^{i} - 1 - 2p + \frac{p - 3}{2}
\]
\[= 2p^{\tau_{i}-(t+1)+1} + (4 - 2m)p^{\tau_{i}-(t+1)} - \frac{p - 1}{2} \sum_{i=1}^{\tau_{i}-(t+1)-1} p^{i} + (2m - 1)p^{\tau_{i}-(t+1)-1} + 3 \sum_{i=0}^{\tau_{i}-(t+1)-2} p^{i} - 2p - 2
\]
variables of type $E^l_{i'}$, while leaving at least $6p - 9 \geq 2p - 2$ variables of type $E^l_{i'}$ unused. Hence, by induction, one can contract the $E^l$ variables to
\[
2p^{\tau_{i}-(\tau - 1)+1} + (4 - 2m)p^{\tau_{i}-(\tau - 1)} - \frac{p - 1}{2} \sum_{i=1}^{\tau_{i}-(\tau - 1)-1} p^{i} + (2m - 1)p^{\tau_{i}-(\tau - 1)-1} + 3 \sum_{i=0}^{\tau_{i}-(\tau - 2)-2} p^{i} - 2p - 2
\]
variables of type $E^l_{i'}$ and $2p - 2$ variables of type $E^l_{i}$ for all $i \in \{j, j + 1, \ldots, \tau - 2\}$. This reduced the cases $j \leq \tau - 2$ to the case $j = \tau - 1$. For $j = \tau - 1$, one can contract the variables of type $E^l_{i'}$ to
\[
\left[\frac{2p^{2} + (2 - 2m)p + 2m - 3}{2p - 2}\right] - 4 = p - m - 1
\]
variables of type $E^l_{i'}$ with Lemma 25 while leaving at least $6p - 9 \geq 2p - 2$ variables of type $E^l_{i'}$. This proves the claim.

**Lemma 34.** Let $p = 5$ and $m \leq p - 1$ be an integer. Let there be
\[
3p^{\tau_{i}+1} - mp^{\tau_{i}} - 3p^{\tau_{i}} - \sum_{i=0}^{\tau_{i}-1} p^{i} - 2p + 2,
\]
variables of type $E^l_{i'}$ for some $j \in \{0, 1, \ldots, \tau\}$. Then one can contract them to $p - m - 1$ variables of type $E^l_{i'}$ and $2p - 2$ variables of type $E^l_{i}$ for all $i \in \{j, j + 1, \ldots, \tau - 1\}$.

**Proof.** For $j = \tau$ the claim is trivial, thus, one can assume that $j \in \{0, 1, \ldots, \tau - 1\}$.

For $j \leq \tau - 2$, assume that for some $l \in \{j, j + 1, \ldots, \tau - 2\}$ one can contract the variables to $3p^{\tau_{i}+1} - mp^{\tau_{i}} - 3p^{\tau_{i}} - \sum_{i=0}^{\tau_{i}-1} p^{i} - 2p + 2$ variables of type $E^l_{i'}$ and $2p - 2$ variables of type $E^l_{i}$ for all $i \in \{j, j + 1, \ldots, l - 1\}$. Using Lemma 26 for $p = 5$, the variables of type $E^l_{i'}$ can be contracted to
\[
\left[3p^{\tau_{i}+1} - mp^{\tau_{i}} - 3p^{\tau_{i}} - \sum_{i=0}^{\tau_{i}-1} p^{i} - 2p + 2\right] - 2p + 3
\]
\[= 3p^{\tau_{i}} - mp^{\tau_{i}-1} - 3p^{\tau_{i}-1} - \sum_{i=0}^{\tau_{i}-2} p^{i} - 1 - 2p + 3
\]
\[= 3p^{\tau_{i}-(t+1)+1} - mp^{\tau_{i}-(t+1)} - 3p^{\tau_{i}-(t+1)} - \sum_{i=0}^{\tau_{i}-(t+1)-2} p^{i} - 2p + 2
\]
variables of type $E^l_{i'}$, while leaving at least $6p - 9 \geq 2p - 2$ variables of type $E^l_{i'}$ unused. By induction, it follows that one can contract
\[
3p^{\tau_{i}-(\tau - 1)+1} - mp^{\tau_{i}-(\tau - 1)} - 3p^{\tau_{i}-(\tau - 1)} - \sum_{i=0}^{\tau_{i}-(\tau - 2)-2} p^{i} - 2p + 2 = 3p^{2} - mp - 5p + 1
\]
variables of type $E^l_{i'}$ and $2p - 2$ variables of type $E^l_{i}$ for all $i \in \{j, j + 1, \ldots, \tau - 2\}$. This reduced the cases $j \leq \tau - 2$ to the case $j = \tau - 1$. 

\[\square\]
For $j = \tau - 1$ one has $3p^2 - mp - 5p + 1$ variables of type $E_\nu^j$. This is at least as big as $2p^2 - 2p + 1$ for $m \leq 2$. Thus, one can use Lemma 24 for $p = 5$ to contract them to

$$\left[\frac{3p^2 - mp - 5p + 1}{p}\right] - 2p + 3 = 3p - m - 5 + 1 - 2p + 3 = p - m - 1$$

variables of type $E_\nu^\tau$ while leaving at least $2p - 2$ variables of type $E_\nu^{\tau - 1}$ unused. For $m = 4$ the claim follows because $p - 4 - 1 = 0$, which leaves $3p^2 - 4p - 5p + 1 = 6p + 1 \geq 2p - 2$ variables of type $E_\nu^{\tau - 1}$. In the remaining case $m = 3$, one obtains

$$\left[\frac{3p^2 - 3p - 5p + 1}{2p - 2}\right] - 4 = 1 = p - 3 - 1 = p - m - 1$$

variables of type $E_\nu^\tau$ with Lemma 25 while leaving at least $6p - 9 \geq 2p - 2$ variables of type $E_\nu^{\tau - 1}$ unused.

Lemma 35. Let there be $4p^{\tau - j} - \frac{p - 1}{2} \sum_{i=0}^{\tau - j - 1} p^i + 3 \sum_{i=0}^{\tau - j - 2} p^i - 2p - 2$ variables of type $E_\nu^j$ for some $j \in \{0, 1, \ldots, \tau - 1\}$. Then one can contract $2p - 2$ variables of type $E_\nu^i$ for all $i \in \{j, \ldots, \tau - 1\}$, simultaneously.

Proof. For $j = \tau - 1$ the statement is trivial, thus, the cases $j \in \{0, 1, \ldots, \tau - 2\}$ remain. Assume that for some $l \in \{j, \ldots, \tau - 2\}$ one can contract the variables of type $E_\nu^l$ to $2p - 2$ variables in $E_\nu^j$ for all $i \in \{j, \ldots, l - 1\}$ and $4p^{\tau - l} - \frac{p - 1}{2} \sum_{i=0}^{\tau - l - 1} p^i + 3 \sum_{i=0}^{\tau - l - 2} p^i - 2p - 2$ variables of type $E_\nu^i$. Then they can be contracted with Lemma 26 to

$$\left[\frac{4p^{\tau - l} - \frac{p - 1}{2} \sum_{i=0}^{\tau - l - 1} p^i + 3 \sum_{i=0}^{\tau - l - 2} p^i - 2p - 2}{p}\right] - 2p + \frac{p - 3}{2}$$

$$= 4p^{\tau - l - 1} - \frac{p - 1}{2} \sum_{i=0}^{\tau - l - 2} p^i + 3 \sum_{i=0}^{\tau - l - 3} p^i - 2p - 2 + 1 - 2p + \frac{p - 3}{2}$$

$$= 4p^{\tau - (l + 1)} - \frac{p - 1}{2} \sum_{i=1}^{\tau - (l + 1) - 1} p^i + 3 \sum_{i=0}^{\tau - (l + 1) - 2} p^i - 2p - 2$$

variables of type $E_\nu^{l+1}$, while leaving at least $6p - 9 \geq 2p - 2$ variables of type $E_\nu^i$. Via induction one can deduce that on can contract $2p - 2$ variables of type $E_\nu^i$ for all $i \in \{j, \ldots, \tau - 2\}$ and $4p^{\tau - (l + 1)} - \frac{p - 1}{2} \sum_{i=1}^{\tau - (l + 1) - 1} p^i + 3 \sum_{i=0}^{\tau - (l + 1) - 2} p^i - 2p - 2 = 2p - 2$ variables of type $E_\nu^{l+1}$.

7. Pairs of Forms with $\tau = 1$

This section contains the proof that for all proper $p$-normalised pairs $f, g$ with $\tau = 1$ the equations $f = g = 0$ have a non-trivial $p$-adic solution. This is primarily done by contracting a $C^\tau + 1 = C^2$ variable if $I_0^n \geq p^2 + p - 1$, which indicates that the colour 0 is zero-representing, and else by contracting a $P^{\tau + 1} = P^2$ variable.

The following lemma will exploit $p$-equivalence classes by transforming some pairs $f, g$ into $p$-equivalent pairs $f, g$, for which one can contract a $P^2$ variable.

Lemma 36. Let $1 \leq m \leq p$ be a natural number and $j \in \{0, \ldots, k - 1\}$. Let $f, g$ be a pair given by (24) with integer coefficients, $\tau = 1$, $q_j \geq pm$, $m_1 \geq m(2p - 1)$, $q_{j+1} \geq p - m$ and $I_{\text{max}}^n \geq p - 1$. Then there exists a non-trivial $p$-adic solution of $f = g = 0$.

Proof. Apply $x \mapsto px$ for all variables at level $l$ for all $l \in \{0, \ldots, j - 1\}$, and then multiply both equations with $p^{-j}$. This transforms the pair $f, g$ into a $p$-equivalent pair with integer coefficients, $q_0 \geq pm$, $m_0 \geq m(2p - 1)$, $q_1 \geq p - m$ and $I_1^n \geq p - 1$ for some $\nu$. Using Lemma 24 one can contract the $E^0$ variables to $m$ variables of type $P^1$. The $p - 1$ variables of type $E_\nu^1$ and the $p - m$ variables of type $E_\nu^0$ can be contracted together with the $P^1$ variables to a $P^2$ variable due to Lemma 24. Thus, the transformed pair has a non-trivial $p$-adic solution, from which it follows that the $p$-equivalent pair $f, g$ has one as well.
Due to this lemma, one can assume that if \( q_j \geq p \) and \( m_j \geq m(2p - 1) \) for some \( j \in \{0, \ldots, k-1\} \) that either \( q_{j+1} \leq p - m - 1 \) or \( I_{\text{max}}^1 \leq p - 2 \). For a \( p \)-normalised pair, one has \( m_0 \geq 2p^2 - 2p + 1 \geq (p - 1)(2p - 1) \) and \( q_0 \geq p^2 - p + 1 \geq (p - 1)p \). Therefore, one can assume that one has either \( q_1 = 0 \) or \( I_{\text{max}}^1 \leq p - 2 \). The following two lemmata will divide the case \( \tau = 1 \) into \( I_{\text{max}}^1 \geq p - 1 \) and thus \( q_1 = 0 \) and \( I_{\text{max}}^1 \leq p - 2 \).

**Lemma 37.** Let \( f, g \) be a proper \( p \)-normalised pair with \( \tau = 1 \) and \( I_{\nu}^1 = I_{\text{max}}^1 \geq p - 1 \). Then the equations \( f = 0 \) have a non-trivial \( p \)-adic solution.

**Proof.** As described above, one can assume that \( q_1 = 0 \) and thus \( I_{\nu}^1 = I_{\text{max}}^1 = m_1 \). It follows that

\[
m_0 \geq 3p^2 - 3p + 1 - q_1 = 3p^2 - 3p + 1 \geq 2p^2 - p.
\]

Assume first, that \( r(\nu, \nu) = r \geq 0 \). Then one can use Lemma 24 to contract the \( E^0 \) variables to \( p \) variables of type \( P^1 \) and Lemma 21 to contract the \( P^1 \) variables together with the \( E_{\nu}^1 \) variables to a \( P^2 \) variable. Consequently, one can assume, that \( r = -1 \) which leads to

\[
I_0^0 \geq 3p^2 - 3p + 1 - q_0 - q_1 \geq 2p^2 - 3p + 2 \geq 2p^2 - p - 1.
\]

Hence, the colour 0 is zero-representing and it suffices to show that one can contract a \( C^2 \) variable.

By Lemma 3 one knows that \( \nu \in \{0, p\} \). If \( \nu = p \), one can contract \( 2p - 2 \) of the variables of type \( E_{\nu}^0 \) to an \( E_0^1 \) variable, using Lemma 20 once, because \( 2p^2 - 3p + 2 \geq 8p - 7 = 2(4p - 4) + 1 \) for all \( p \geq 5 \). If on the other hand \( \nu = 0 \), one has

\[
I_0^0 \geq \frac{r_0^0}{p} \geq 2p - 3 \geq p - 1,
\]

due to Lemma 3 and

\[
I_0^0 - I_0^0 \geq 2p^2 - 2p - q_0 - (m_1 - I_1^0) \geq p^2 - 2p + 1 \geq 1,
\]

by Lemma 3. Hence, one can contract \( p - 1 \) variables of type \( E_{0}^1 \) and one \( E_{0}^0 \) variable to an \( E_0^1 \) variable due to Lemma 18.

In both cases, there are still at least \( 2p^2 - 3p + 2 - (2p - 2) = 2p^2 - 5p + 4 \geq 2p^2 - 2 \) variables of type \( E_{0}^0 \) remaining. Those contract with \( p^2 - p \) of the \( C^0 \) variables to \( p - 1 \) variables of type \( C^1 \) due to Lemma 22. All in all, one has \( p - 1 \) variables of type \( E_{\nu}^1 \), one \( E_0^1 \) variable and \( p - 1 \) variables of type \( C^1 \). Due to Lemma 22 these can be contracted to a \( C^2 \) variable, which completes the proof. \( \square \)

**Lemma 38.** Let \( f, g \) be a proper \( p \)-normalised pair with \( \tau = 1 \) and \( I_{\nu}^1 = I_{\text{max}}^1 \leq p - 2 \). Then the equations \( f = g = 0 \) have a non-trivial \( p \)-adic solution.

**Proof.** By \( I_{\mu}^1 \leq I_{\text{max}}^1 \leq p - 2 \) for all \( 0 \leq \mu \leq p \), it follows that

\[
m_1 \leq (p - 2)(p + 1) = p^2 - p - 2.
\]

If one has \( q_1 \geq p \) and \( m_1 \geq 2p - 1 \) one can assume, due to Lemma 36 that either \( q_2 \leq p - 2 \) or \( I_{\text{max}}^2 \leq p - 2 \). For \( q_2 \leq p - 2 \) it follows that

\[
m_0 \geq 5p^2 - 5p + 1 - p^2 + p + 2 - p + 2 = 4p^2 - 5p + 5 \geq 4p^2 - 6p + 3,
\]

while for \( I_{\text{max}}^2 \leq p - 2 \) it follows that \( m_2 \leq p^2 - p - 2 \) and thus

\[
m_0 \geq 6p^2 - 6p + 1 - p^2 + p + 2 = 4p^2 - 4p + 5 \geq 4p^2 - 6p + 3.
\]

Else, one has either \( q_1 \leq p - 1 \) or \( m_1 \leq 2p - 2 \). If \( q_1 \leq p - 1 \) it follows that \( m_1 \leq 2p - 2 \) as well, because \( m_1 = I_{\text{max}}^1 + q_1 \). Then one obtains

\[
m_0 \geq 4p^2 - 4p + 1 - 2p + 2 = 4p^2 - 6p + 3.
\]

One of these three bounds holds in any case, thus, one can assume that

\[
m_0 \geq 4p^2 - 6p + 3.
\]

This lower bound for \( m_0 \) leads to

\[
I_0^0 = m_0 - q_0 \geq 4p^2 - 6p + 3 - p^2 - (r + 1)p + 1 = 3p^2 - rp - 7p + 4.
\]
For \( r \leq p - 2 \) this is at least as big as \( p^2 + p - 1 \) for \( p \geq 5 \), hence, it suffices to contract a \( C^2 \) variable, whereas one has to contract a \( P^2 \) variable for \( r = p - 1 \). The remaining proof will be divided into three cases, based on the value of \( r = r(f, g) \).

**Case** \( r = p - 1 \). If \( m_0 \geq (2p - 1)(2p - 1) = 4p^2 - 4p + 1 \), one can use Lemma 23 to contract the \( E^0 \) variables to \( 2p - 1 \) variables of type \( P^1 \). By Lemma 14, it follows that one can contract those \( P^1 \) variables to a \( P^2 \) variable. Hence, one can assume that \( m_0 \leq 4p^2 - 4p \) and thus \( m_1 \geq 1 \). Due to (7.2) one has \( m_0 \geq 4p^2 - 6p + 2 = (2p - 1)(2p - 2) \). Therefore, Lemma 23 shows, that one can contract the \( E^0 \) variables to \( 2p - 2 \) variables of type \( P^1 \). Lemma 9 with \( n = 2 \) shows that one can contract them together with one of the \( E^1 \) variables to a variable of a level at least 2. This contraction cannot contain only the \( E^2 \) variable, thus the resulting variable has to be a \( P^2 \) variable.

**Case** \( 0 \leq r \leq p - 2 \). One can assume that \( I^1_m = I^1_{max} \leq p - r - 2 \), because else, Lemma 32 can be used to contract \( p^2 + rp \) of the \( C^0 \) variables together with \( 2p - 2 \) variables of type \( E^0 \) to \( p + r \) variables of type \( C^1 \). Then one can contract them together with the \( E^1_{n} \) variables to a \( C^2 \) variable due to Lemma 21. It follows that (7.4)

\[
m_1 \leq p^2 - (r + 1)p - r - 2.
\]

If \( q_2 \geq p - 1 \) and \( I^2_{max} \geq p - 1 \), one can use Lemma 14 to contract \( p(p - r - 1) \) of the variables of type \( E^0 \) to \( p - r - 1 \) variables of type \( E^1 \). This is possible, because afterwards, there are still at least

\[
3p^2 - rp - 7p + 4 + p(p - r - 1) = 2p^2 - 6p + 4 \geq 3p - 2
\]

of the \( E^0 \) variables unused. Lemma 32 can be used to contract \( p^2 + rp \) of the \( C^0 \) and \( 2p - 2 \) of the remaining \( E^0 \) variables to \( p + r \) variables of type \( C^1 \). One can assume, that the \( C^1 \) variables are of type \( E^1 \), because else one already has a \( C^2 \) variable. Take the set \( \mathcal{K} \) of the \( 2p - 1 \) variables of type \( E^1 \) that were contracted, from which \( p + r \) are of type \( C^1 \). If there is a \( \mu \) with \( I^1_{\mu}(\mathcal{K}) \geq p \) there are at least \( p \) variables of type \( E^1_{\mu} \) in \( \mathcal{K} \). As \( p + r \) of the variables in \( \mathcal{K} \) are type \( C^1 \), it follows that there is at least one \( C^1_{\mu} \) variable in \( \mathcal{K} \). Thus, one can contract the \( E^1_{\mu} \) variables in \( \mathcal{K} \) with Lemma 19 to a \( C^2 \) variable. Else, one has \( q(\mathcal{K}) \geq p \) and thus, one has transformed the pair \( f, g \) into another one with \( m_1 \geq 2p - 1 \) and \( q_1 \geq p \). This new pair has the same values for \( q_2 \) and \( I^2_{max} \), thus it follows from Lemma 19 that it has a non-trivial \( p \)-adic solution. Consequently the pair \( f, g \) has one as well. Thus, one can assume, that either \( q_2 \leq p - 2 \) or \( I^2_{max} \leq p - 2 \).

By (7.4), it follows for \( q_2 \leq p - 2 \) that

\[
m_0 \geq 5p^2 - 5p + 1 - p^2 + (r + 1)p + r + 2p + 2 = 4p^2 - 5p + rp + 4 + r
\]

and for \( I^2_{max} \leq p - 2 \) that \( m_2 \leq p^2 - p - 2 \) and therefore

\[
m_0 \geq 5p^2 - 5p + 1 - p^2 + (r + 1)p + r + 2p + 2 - p^2 + p + 2 = 4p^2 - 4p + rp + 5 + r.
\]

In both cases, one obtains the lower bound

\[
m_0 \geq 4p^2 - 5p + rp + 5 + r,
\]

which leads to

\[
I^0_0 = m_0 - q_0 \geq 3p^2 - 6p + 6 + r \geq 2p^2 - 2rp + 2r - 1.
\]

Now one can distinguish between the cases \( m_1 \geq 1 \) and \( m_1 = 0 \).

**Case** \( m_1 \geq 1 \). One can use Lemma 25 to contract the \( E^0 \) variables to

\[
\left\lfloor \frac{2p^2 - 2rp + 2r - 1}{2p - 2} \right\rfloor - 4 = p - r - 2
\]

variables of type \( E^1 \). This leaves at least \( 6p - 9 \geq 2p - 2 \) variables of type \( E^0 \). Hence, one can use Lemma 32 to contract them with \( p^2 + rp \) of the \( C^0 \) variables to \( p + r \) variables of type \( C^1 \). The set \( \mathcal{H} \), containing the \( p - r - 2 \) variables of type \( E^1 \), the \( p + r \) variables of type \( C^1 \) and one further \( E^1 \) variables, which exists due to \( m_1 \geq 1 \), contains a contraction to a \( C^2 \) variable. If none of the \( C^1 \) variables is already of type \( C^2 \), there is either a \( \mu \) such that \( I^1_{\mu}(\mathcal{K}) \geq p \) or \( q(\mathcal{K}) \geq p \). If \( I^1_{\mu}(\mathcal{K}) \geq p \), then at least one of the \( E^1_{\mu} \) variables in \( \mathcal{K} \) is a \( C^1 \) variable and thus \( \mathcal{K} \) contains a contraction to a \( C^2 \) variable due to Lemma 19. If on the other hand \( q(\mathcal{K}) \geq p \), then \( \mathcal{K} \) contains
Therefore, one can take \( p \) which can be contracted to \( p \). Then there are at least 3 variables of type \( E^1 \). Hence, one can assume that the second entry \( b_i \) of the level coefficient vector is not congruent to 0 modulo \( p \). But if a subset \( \mathcal{A} \) of \( \mathcal{H} \) contains this variable and additionally only variables of type \( E^0 \), then it cannot be a contraction to a variable at level at least 2, because then one has exactly one \( i \in \mathcal{A} \) for which

\[
\sum_{j \in \mathcal{A}} b_j y_j^i \equiv 0 \mod p \quad \text{with all } y_j \neq 0 \mod p.
\]

Consequently, this cannot occur, and the resulting variable is a \( C^2 \) variable.

**Case** \( m_1 = 0 \). This leads to the even better bound

\[
m_0 \geq 4p^2 - 4p + 1
\]

and thus

\[
l_0^0 \geq 4p^2 - 4p + 1 - p^2 - (r + 1)p + 1 = 3p^2 - (5 + r)p + 2.
\]

For \( p \geq 7 \), this is at least as big as \( 2p^2 + 2p - 2p^2 + 2r - 3 \), thus, one can use Lemma 13 to contract the \( E^0_0 \) variables to \( r - 1 \) variables of type \( E^1 \), while leaving at least \( 2p - 2 \) variables of type \( E^0 \) unused. For \( p = 5 \), this is at least as big as \( 3p^2 - rp - 5p + 1 \), thus Lemma 14 shows that one can contract the \( E^0_0 \) variables to \( r - 1 \) variables of type \( E^1_0 \), as well, while leaving at least \( 2p - 2 \) variables of type \( E^0_0 \) unused. In both cases, one can use Lemma 22 to contract the \( 2p - 2 \) variables of type \( E^0_0 \) with \( p^2 + rp \) of the \( C^0 \) variables to \( p + r \) variables of type \( C^1 \). Then one can contract them together with the \( p - r - 1 \) variables of type \( E^1_0 \) to a \( C^2 \) variable due to Lemma 21.

**Case** \( r = -1 \). Note first, that one has \( m_1 - I_0^1 - I_0^0 \leq p^2 - 2p = (p - 2)p \) due to \( I_{\text{max}} \leq p - 2 \), and thus

\[
l_0^0 - I_0^0 \geq 2p^2 - 2p - q_0 - (m_1 - I_0^1) \geq 1,
\]

by (5.1). If \( m_0 \geq 4p^2 - 4p \), one obtains the lower bound

\[
l_0^0 \geq 3p^2 - 4p + 1,
\]

and consequently

\[
l_0^0 \geq \frac{l_0^0}{p} \geq 3p - 4 \geq p - 1.
\]

Therefore, one can take \( p - 1 \) variables of type \( E^0_0 \) and one of type \( E^0_{00} \), which can be contracted to a \( E^1_0 \) variable by Lemma 13. There are at least \( 3p^2 - 5p + 1 \) variables of type \( E^0_0 \) remaining, which can be contracted to \( p - 1 \) variables of type \( E^1_0 \) using Lemma 25 for \( p \geq 7 \) and Lemma 26 for \( p = 5 \). This leaves at least \( 6p - 9 \geq 2p - 2 \) variables of type \( E^0_0 \), which can be contracted with \( p^2 - p \) of the \( C^0 \) variables to \( p - 1 \) variables of type \( C^1 \) using Lemma 32. Then one can use Lemma 22 to contract the \( p - 1 \) variables of type \( E^1_0 \), the \( p - 1 \) variables of type \( C^1 \) and the \( E^0_0 \) variable to a \( C^2 \) variable. Hence, one can assume that

\[
m_0 \leq 4p^2 - 4p - 1.
\]

It follows that \( m_1 \geq 2 \). Note, that one has

\[
l_0^0 \geq 3p^2 - 6p + 4 \geq 2p^2 - 1 = (2p - 2)(p + 1) + 1 \quad \text{and} \quad l_0^0 \geq 3p - 6 \geq p - 1
\]

due to (7.3).

**Case** \( m_1 - I_0^1 = 0 \). Due to \( m_1 \geq 2 \), one has \( I_0^1 \geq 2 \). Take a set, which contains \( p - 1 \) variables of type \( E^0_0 \) and one \( E^0_{00} \) variable. This set contains a contraction to an \( E^1_0 \) variable due to Lemma 13. Then there are at least \( 3p^2 - 7p + 4 \geq 2p^2 - 2p + 1 \) variables of type \( E^0_0 \) left. Therefore, one can use Lemma 25 to contract them to \( p - 3 \) variables of type \( E^1_0 \), giving a total of \( p - 1 \), while leaving at least \( 6p - 9 \geq 2p - 2 \) variables of type \( E^0_0 \) unused. Lemma 32 can be used to contract \( 2p - 2 \) of the remaining \( E^0_0 \) variables together with \( p^2 - p \) of the \( C^0 \) variables to \( p - 1 \) variables of type \( C^1 \). One can contract the \( p - 1 \) variables of type \( E^0_0 \), the \( E^1_0 \) variable and the \( p - 1 \) variables of type \( C^1 \) to a \( C^2 \) variable, due to Lemma 22.
Case $m_1 - I_0^1 \geq 1$. Use Lemma 25 to contract the $E_0^0$ variable to $p - 2$ variables of type $E_0^0$ while leaving at least $6p - 9 \geq 2p - 2$ unused. Then one can take Lemma 22 to contract $p^2 - p$ of the $C_0^0$ variables together with $2p - 2$ of the remaining $E_0^0$ variables to $p - 1$ variables of type $C_1^1$. If $I_0^1 \geq 1$, then one can use Lemma 23 to contract the $p - 1$ variables of type $E_0^1$, the $p - 1$ variables of type $C_1^1$ and one of the $E_0^1$ variables to a $C^2$ variable. Thus, one can assume, that $I_0^1 = 0$, $m_1 - I_0^1 \geq 2$ and

$$m_1 \leq p^2 - 2p,$$

because $I_{\text{max}}^p \leq p - 2$. If none of the $C_1^1$ variables is already of type $C^2$, they are all $E^1$ variables. Take a set $\mathcal{X}$ containing the $C_1^1$ variables, two of the $E_0^1$ variables which exist due to $m_1 - I_0^1 \geq 2$ and the $p - 2$ variables of type $E_0^1$. If there is a $\mu$ such that $I_\mu(\mathcal{X}) \geq 2$, then there is at least one $C_\mu^1$ variable in $\mathcal{X}$. Due to Lemma 19, one can contract the variables in $\mathcal{X}$ of colour $\mu$ to an $C^2$ variable. Else, one has $q(\mathcal{X}) \geq p$, because $|\mathcal{X}| = 2p - 1$. It follows, that one has transformed the pair $f, g$ into a pair with $m_1 \geq 2p - 1$ and $q_1 \geq p$. The new pair either has a non-trivial $p$-adic solution due to Lemma 50 from which it would follow that $f, g$ has one as well, or it has $q_2 \leq p - 2$ or $I_{\text{max}}^p \leq p - 2$. As the new pair has the same parameter $q_2$ and $I_{\text{max}}^p$ as the pair $f, g$, one can assume, that $q_2 \leq p - 2$ or $I_{\text{max}}^p \leq p - 2$ holds for $f, g$ as well. This contradicts the $p$-normalisation, because then one of the inequalities

$$m_0 + m_1 + q_2 \leq 4p^2 - 4p - 1 + p^2 - 2p + p - 2 = 5p^2 - 5p - 3 < 5p^2 - 5p + 1,$$

and

$$m_0 + m_1 + m_2 \leq 4p^2 - 4p - 1 + p^2 - 2p + p^2 - p - 2 = 6p^2 - 7p - 3 < 6p^2 - 6p + 1,$$

holds, hence, it follows that this case cannot occur.

This concludes the case $r = -1$ and with that the claim follows. $\square$

This shows that for every proper $p$-normalised pair $f, g$ the equations $f = g = 0$ have a non-trivial $p$-adic solution provided that $\tau = 1$.

8. Pairs of Forms with $\tau \geq 2$

This section will prove the theorem for $\tau \geq 2$, which completes the proof. In general, the proof relies on the same techniques independent on the actual value of $\tau$, but sometimes one has to separate the cases $\tau = 2$ and $\tau = 3$, because the proof is easier for bigger $\tau$ and hence, the cases $\tau \in \{2, 3\}$ require some extra effort.

In order to avoid a repetition of the same argument, the following lemma will point out a situation in which one can contract a $C_{\tau + 1}$ or a $P_{\tau + 1}$ variable, which will appear constantly in the proof for $\tau \geq 2$.

Lemma 39. Let $S \in \{C, P\}$ and $0 \leq m \leq p - 1$. Let there be $p^\tau j + 1 + mp^\tau j$ variables of type $S_j$ for some $j \in \{0, \ldots, \tau - 1\}$ and $p - m - 1$ variables of type $E_{\nu_i}^j$ for some $\nu$. Furthermore for $i \in \{\tau, j + 1, \ldots, \tau - 1\}$ let there be $2p - 2$ variables of type $E_{\nu_i}^j$ for some colours $\nu_i$. Then one can contract them to a variable of type $S_{\tau + 1}$.

Proof. One can contract the variables of type $S_j$ and type $E_{\nu_i}^j$ for $i \in \{\tau, j + 1, \ldots, \tau - 1\}$ to $p + m$ variables at level of type $S_{\tau}$ due to Lemma 32. Those and the $p - m - 1$ variables of type $E_{\nu_i}^j$ can be contracted to a variable of type $S_{\tau + 1}$ using Lemma 24. $\square$

The following lemma focuses on cases, where the number of variables at level 0 is small.

Lemma 40. Let $f, g$ be a proper $p$-normalised pair with $\tau \geq 2$ and $m_0 \leq 3p^\tau + 1 - 4p - 2p^\tau + p + 3$. Then the equations $f = g = 0$ have a non-trivial $p$-adic solution.

Proof. By the $p$-normalisation of $f, g$, one has $y_0 \geq p^\tau + 1$ and $m_0 \geq 2p^\tau + 1$, from which it follows that one can contract the variables at level 0 to $p^\tau - p^\tau - 1$ variables of type $P^1$ due
to Lemma 23. The upper bound of $m_0$ provides the bounds
\[
m_1 \geq 4p^{r+1} - 4p^r + 1 - 3p^{r+1} + 4p^r + 2p^{r-1} - p - 3 = p^{r+1} + 2p^{r-1} - p - 2 \geq 2p^r + 4p^{r-1} + p^2 - p - 7 = \left(p^{r-1} + 3 \sum_{i=0}^{\tau-2} p^i - 1\right)(2p - 2) + p^2 - 5p - 3 \]
and
\[
q_1 \geq 3p^{r+1} - 3p^r + 1 - 3p^{r+1} + 4p^r + 2p^{r-1} - p - 3 = p^r + 2p^{r-1} - p - 2 = \left(p^{r-1} + 3 \sum_{i=0}^{r-2} p^i - 1\right)(p - 1).
\]
Therefore, there are at least \(p^{r-1} + 3 \sum_{i=0}^{r-2} p^i - 1\) variables of type \(E^1\) from which at least \(p^{r-1} + 3 \sum_{i=0}^{r-2} p^i - 1\) are of type \(E^1\) for all \(0 \leq \nu \leq p\). Those variables can be contracted together with the \(p^1\) variables to \(2p^{r-1} + p^{r-2} - 2\) variables of type \(P^2\) by using Lemma 29 with \(x = p^{r-1} - 2p^{r-2} - 3 \sum_{i=0}^{r-3} p^i - 1\), \(y = p^{r-1} + 3 \sum_{i=0}^{r-3} p^i - 1\) and \(z = p - 2\). Then Lemma 31 can be used to contract the \(P^2\) variables to \(2p - 1\) variables of type \(P^3\), which contract to a \(P^{r+1}\) variable due to Lemma 14.

For bigger \(m_0\) it will be helpful to divide the cases depending on the value of \(r(f,g)\). The following three lemmata will complete the proof that \(a\) for a proper \(p\)-normalised pair \(f, g\) with \(r \geq 2\) and \(r = r(f, g) \geq 0\) the equations \(f = g = 0\) have a non-trivial \(p\)-adic solution.

This will be done by using different strategies depending on the size of \(m_0\). The area of the value of \(m_0\) in which one has to use a certain strategy differs between \(p \geq 7\) and \(p = 5\). This is due to some inequalities, which do not hold if \(p\) is too small. To counter this, the lemmata that are stronger in the case \(p = 5\) will be used, which results in the different areas.

**Lemma 41.** Let \(f, g\) be a proper \(p\)-normalised pair with \(r \geq 2\), \(r = r(f, g) \geq 0\) and \(m_0 \geq 3p^{r+1} + 8p^r\) for \(p \geq 7\) and \(m_0 \geq 3p^{r+1} + 3p^r\) for \(p = 5\). Then the equations \(f = g = 0\) have a non-trivial \(p\)-adic solution.

**Proof.** As \(I_0 = m_0 - q_0\), one can estimate \(I_0\) via
\[
I_0 = m_0 - q_0 \geq 3p^{r+1} + 8p^r - p^{r+1} - (r + 1) p^r + 1 = 2p^{r+1} + (7 - r) p^r + 1,
\]
for all primes \(p \geq 7\), and via
\[
I_0 = m_0 - q_0 \geq 3p^{r+1} + 3p^r - p^{r+1} - (r + 1) p^r + 1 = 2p^{r+1} + (2 - r) p^r + 1,
\]
for \(p = 5\). Both are at least as big as \(p^{r+1} + p^r - 1\), because \(r \leq p - 1\), from which it follows that the colour 0 is zero-representing, and hence, it suffices to contract a \(C^{r+1}\) variable. Furthermore, the lower bound for \(I_0\) implies that
\[
I_0 \geq 2p^{r+1} + (4 - 2r) p^r + (2r - 1) p^{r-1} + 3 \sum_{i=0}^{r-2} p^i - 2p - 2
\]
for \(p \geq 7\) and
\[
I_0 \geq 3p^{r+1} - rp^r - 3p^{r-1} - \sum_{i=0}^{r-1} p^i - 2p + 2
\]
for \(p = 5\). Thus, one can contract the \(E^0_i\) variables to \(p - r - 1\) variables of type \(E^0_r\), using Lemma 26 for \(p \geq 7\) and Lemma 27 for \(p = 5\), while leaving at least \(2p - 2\) variables of type \(E^0_i\) for all \(i \in \{0, 1, \ldots, r - 1\}\). Then one can contract \(p^{r+1} + rp^r\) variables of type \(C^0\) together with the \(2p - 2\) variables of type \(E^0_i\) for all \(i \in \{0, 1, \ldots, r - 1\}\) and the \(E^r_i\) variables to a \(C^{r+1}\) variable due to Lemma 28.

**Lemma 42.** Let \(f, g\) be a proper \(p\)-normalised pair \(f, g\) with \(r \geq 2\), \(r = r(f, g) \geq 0\), and \(m_0 \geq 3p^{r+1} + p^r - 3\) which has \(m_0 \leq 3p^{r+1} + 8p^r - 1\) for \(p \geq 7\) and \(m_0 \leq 3p^{r+1} + 3p^r - 1\) for \(p = 5\). Then the equations \(f = g = 0\) have a non-trivial \(p\)-adic solution.
Proof. By \( q_0 \leq 2p^{r+1} - 2 \), one obtains
\[
I_0^p = m_0 - q_0 \geq 3p^{r+1} + p^r - 3 - 2p^{r+1} + 2 = p^{r+1} + p^r - 1,
\]
from which it follows that the colour 0 is zero-representing. Therefore, it suffices to contract a \( C^{r+1} \) variable. The variables of type \( E_0^p \) can be contracted with Lemma 55 to \( 2p \) variables of type \( E_0^0 \) for all \( i \in \{0, 1, \ldots, \tau - 1\} \) as
\[
I_0^p \geq p^{r+1} + p^r - 1 \geq 4p^r - \frac{p - 1}{2} \sum_{i=1}^{r-1} p^i + 3 \sum_{i=0}^{r-2} p^i - 2p - 2.
\]
If \( I_0^p \geq p - r - 1 \) for some \( \nu \), then one can contract the \( p^{r+1} + rp^r \) variables of type \( C^0 \) together with the \( p - r - 1 \) variables of type \( E_0^p \) and the \( 2p - 2 \) variables of type \( E_0^0 \) for all \( i \in \{0, 1, \ldots, \tau - 1\} \) to one variable of type \( C^{r+1} \) with Lemma 55. Thus one can assume that
\[
(8.1) \quad m_\nu \leq (p - r - 2)(p + 1) = p^2 - (r + 1)p - r - 2 \leq p^2.
\]
Likewise, if \( I_0^p \geq 2p^{r+j+1} + (4 - 2r)p^{r-j} - \frac{p^2}{2} \sum_{i=1}^{r-j-1} p^i + (2r - 1)p^{r-j+1} + 3 \sum_{i=0}^{r-j-2} p^i - 2p - 2 \) for some \( j \in \{1, \ldots, \tau - 1\} \) and some \( \nu \), one can contract the variables of type \( E_0^p \) to \( p - r - 1 \) variables of type \( E_0^0 \) due to Lemma 55. Then one can contract the \( p^{r+1} + rp^r \) variables of type \( C^0 \) together with the \( p - r - 1 \) variables of type \( E_0^p \) and the \( 2p - 2 \) variables of type \( E_0^0 \) for all \( i \in \{0, 1, \ldots, \tau - 1\} \) to a \( C^{r+1} \) variable with Lemma 55. Hence, one can assume that this is not the case, giving the upper bound
\[
(8.2) \quad I_{\text{max}} \leq 2p^{r+j+1} + (4 - 2r)p^{r-j} - \frac{p^2}{2} \sum_{i=1}^{r-j-1} p^i + (2r - 1)p^{r-j+1} + 3 \sum_{i=0}^{r-j-2} p^i - 2p - 3.
\]
If \( m_1 \geq 2p^{r+j+1} - (2r + 2)p^{r-j} + p^2 - 3p + 2r + 1 \) and \( q_1 \geq p^{r+j+1} - (r + 1)p^{r-j} + r \) for some \( j \in \{1, \ldots, \tau - 1\} \), one can contract \( p^{r+1} + rp^r \) of the \( C^0 \) variables together with the \( 2p - 2 \) variables of type \( E_0^0 \) for \( i \in \{0, 1, \ldots, j - 1\} \) to \( p^{r+j+1} + rp^{r-j} \) variables of type \( C^j \), using Lemma 21. It follows from the lower bounds for \( m_1 \) and \( q_1 \), that one can contract the variables of type \( E_0^p \) together with the \( p^{r+j+1} + rp^{r-j} \) variables of type \( C^j \) to \( 2p^{r-j} - p^{r-j+1} - 1 \) variables of type \( C^{r+1} \), using Lemma 21 with \( x = p^{r-j} - p^{r-j+1} + r \sum_{i=0}^{r-j-1} p^i, y = p^{r-j} - r \sum_{i=0}^{r-j-1} p^i \) and \( z = r \). This leaves at least \( p + r \) of the \( C^j \) variables unused. Furthermore, the \( 2p - 2 \) variables of type \( E_0^0 \) which were contracted at the beginning of the proof are unused as well. Hence, Lemma 21 can be used to contract \( p - 1 \) of them and \( p \) of the remaining \( C^j \) variables to another \( C^{r+1} \) variable. All in all, one has \( 2p^{r-1} \) variables of type \( C^{r+1} \) and \( 2p - 2 \) variables of type \( E_0^0 \) for all \( i \in \{j + 1, \ldots, \tau - 1\} \). By Lemma 55 these variables contract to a \( C^{r+1} \) variable. One can therefore assume, that either \( m_j \leq 2p^{r+j+1} - (2r + 2)p^{r-j} + p^2 - 3p + 2r \) or \( q_j \leq p^{r+j+1} - (r + 1)p^{r-j} + r - 1 \) for \( j \in \{1, \ldots, \tau - 1\} \). It follows that either one has \( m_j \leq 2p^{r+j+1} - (2r + 2)p^{r-j} + p^2 - 3p + 2r \) for \( p \leq 3p \) or \( q_j \leq p^{r+j+1} - (r + 1)p^{r-j} + r - 1 \) otherwise, one obtains, due to (8.2), the upper bound
\[
m_j \leq 3p^{r+j+1} + (3 - 3r)p^{r-j} - \frac{p^2}{2} \sum_{i=1}^{r-j-1} p^i + (2r - 1)p^{r-j+1} + 3 \sum_{i=0}^{r-j-2} p^i - 2p + r - 4.
\]
Both upper bounds are smaller than \( 4p^{r+j+1} \), thus one can assume, that \( m_j \leq 4p^{r+j+1} \) for \( j \in \{1, \ldots, \tau - 1\} \). It follows that one has \( m_\tau \leq 4p^\tau \) for all \( \tau \geq 2 \) and \( m_2 \leq 4p^2 \) for \( \tau = 3 \). Furthermore, one has \( m_2 \leq p^\tau \) for \( \tau = 2 \) due to (8.1). It follows that
\[
m_0 + m_1 + m_2 \leq 3p^{r+1} + 13p^\tau - 1 \leq 6p^{r+1} - 6p^\tau
\]
for all \( p \geq 7 \), whereas one obtains
\[
m_0 + m_1 + m_2 \leq 3p^{r+1} + 8p^\tau - 1 \leq 6p^{r+1} - 6p^\tau
\]
for \( p = 5 \). This contradicts the \( p \)-normalisation of \( f, g \), from which the claim follows. □

Lemma 43. Let \( f, g \) be a proper \( p \)-normalised pair with \( \tau \geq 2 \), \( r = r(f, g) \geq 0 \) and \( 3p^{r+1} - 4p^r - 2p^{r-1} + p + 4 \leq m_0 \leq 3p^{r+1} + p^r - 4 \). Then the equations \( f = g = 0 \) have a non-trivial \( p \)-adic solution.
Proof. By Lemma \[25\] \( r \geq 0 \) and \( m_0 \geq 2p^{r+1} - p^r \), one can contract the \( E^0 \) variables to \( p^r \) variables of type \( P^1 \).

If there is a \( \nu \) such that \( I^1_\nu \geq 2p^r + 4p^{r-1} - \frac{1}{2} \sum_{i=1}^{r-2} p^i - p^{r-2} - 3 \sum_{i=0}^{r-3} p^i - 2p - 2 \), one can contract the variables of type \( E^1_\nu \) with Lemma \[33\] and the resulting variables together with the variables of type \( P^1 \) to a variable of type \( P^{r+1} \) with Lemma \[39\]. From now on, one can assume that

\[
I^1_\nu \leq 2p^r + 4p^{r-1} - \frac{p-1}{2} \sum_{i=1}^{r-2} p^i - p^{r-2} - 3 \sum_{i=0}^{r-3} p^i - 2p - 3
\]

for all \( \nu \).

If \( m_1 \geq 3p^r + 5p^{r-1} - \frac{1}{2} \sum_{i=1}^{r-2} p^i - p^{r-2} - 3 \sum_{i=0}^{r-3} p^i - 3p - 4 \), it follows therefore, that

\[
q_1 = m_1 - I^1_{\max} \geq p^r + p^{r-1} - p - 1 = \left( p^{r-1} + 2 \sum_{i=0}^{r-2} p^i - 1 \right)(p-1)
\]

and

\[
m_1 \geq 2p^r + 2p^{r-1} + p^2 - 5p - 1 = \left( p^{r-1} + 2 \sum_{i=0}^{r-2} p^i - 1 \right)(2p - 2) + p^2 - 3p + 1.
\]

Hence, one can use Lemma \[29\] with \( x = p^{r-1} - p^{r-2} - 2 \sum_{i=0}^{r-3} p^i - 1, y = p^{r-1} + 2 \sum_{i=0}^{r-2} p^i - 1 \) and \( z = p - 1 \) to contract the \( E^1 \) variables together with the \( P^1 \) variables to obtain \( 2p^{r-1} + p^{r-2} - 2 \) variables of type \( P^2 \). Then one can contract them to \( 2p - 1 \) variables of type \( P^r \) with Lemma \[31\] and these to one \( P^{r+1} \) variable with Lemma \[14\]. Thus, one can assume, that

\[
m_1 \leq 3p^r + 5p^{r-1} - \frac{p-1}{2} \sum_{i=1}^{r-2} p^i - p^{r-2} - 3 \sum_{i=0}^{r-3} p^i - 3p - 5.
\]

If one has the even stronger upper bound \( m_1 \leq 2p^2 - p - 3 \), the \( p \)-normalisation of \( f, g \) can be used to obtain the lower bounds

\[
m_2 \geq 6p^{r-1} - 6p^r + 1 - 3p^{r+1} - p^r + 4 - 2p^2 + p + 3
\]

\[
= 3p^{r+1} - 7p^r - 2p^2 + p + 8 \geq 2p^{r-1} + 2p^{r-2} + p^2 - 3p - 3
\]

and

\[
q_2 \geq 5p^{r+1} - 5p^r + 1 - 3p^{r+1} - p^r + 4 - 2p^2 + p + 3
\]

\[
= 2p^{r+1} - 6p^r - 2p^2 + p + 8 \geq p^{r-1} + p^{r-2} - 2
\]

\[
= \left( p^{r-2} + 2 \sum_{i=0}^{r-3} p^i \right)(p-1).
\]

One can contract the \( P^1 \) variables to \( p^{r-1} - 2 \) variables of type \( P^2 \) using Lemma \[31\]. For \( r = 2 \) one can use Lemma \[25\] to contract one of the \( P^2 \) variables together with the \( E^2 \) variables to a \( P^3 = P^{r+1} \) variable, because \( p^2 - 2 + 2 \sum_{i=0}^{r-3} p^i = 1 \). For \( r \geq 3 \) on the other hand, one can use Lemma \[29\] with \( x = p^{r-2} - p^{r-3} - 2 \sum_{i=0}^{r-4} p^i - 1, y = p^{r-2} + 2 \sum_{i=0}^{r-3} p^i \) and \( z = p - 4 \) to contract the \( P^2 \) variables to \( 2p^{r-2} + p^{r-3} - 1 \) variables of type \( P^3 \). Then one can use Lemma \[31\] to contract them to \( 2p - 1 \) variables of type \( P^r \) and Lemma \[14\] to obtain a \( P^{r+1} \) variable. One can therefore assume, that \( m_1 \geq 2p^2 - p - 2 = (2p - 3)(p+1) + 1 \), from which it follows that there is a \( \nu \) such that

\[
I^1_\nu \geq 2p - 2.
\]

One can contract the \( p^r \) variables of type \( P^1 \) together with the \( 2p - 2 \) variables of type \( E^1_\nu \) to \( p^{r-1} \) variables of type \( P^2 \) with Lemma \[32\]. The \( p \)-normalisation of \( f, g \) can be used to obtain the lower bound

\[
m_2 \geq 6p^3 - 6p^2 + 1 - 3p^3 - p^2 + 4 - 3p^2 - 2p + 6 = 3p^3 - 10p^2 - 2p + 11 \geq p^2 - p - 1
\]
for \( \tau = 2 \) and

\[
m_2 \geq 6p^{\tau+1} - 6p^\tau + 1 - 3p^{\tau+1} - p^\tau + 4 - 3p^\tau - 5p^{\tau-1} + \frac{p-1}{2} \sum_{i=1}^{\tau-2} p^i + p^{\tau-2} - 3 \sum_{i=0}^{\tau-3} p^i + 3p + 5
\]

\[
= 3p^{\tau+1} - 10p^\tau - 5p^{\tau-1} + \frac{p-1}{2} \sum_{i=1}^{\tau-2} p^i + p^{\tau-2} - 3 \sum_{i=0}^{\tau-3} p^i + 3p + 10
\]

\[
\geq 2p^\tau + 6p^{\tau-1} + 3p^{\tau-2} + 2p^{\tau-3} + 6 \sum_{i=0}^{\tau-4} p^i \geq (p+1) \left( 2p^{\tau-1} + 4p^{\tau-2} - p^{\tau-3} + 3 \sum_{i=0}^{\tau-4} p^i \right).
\]

for \( \tau \geq 3 \). Thus, there is an \( \mu \) with \( I_\mu^2 \geq p-1 \) for \( \tau = 2 \) and a \( \mu \) with

\[
I_\mu^2 \geq 2p^{\tau-1} + 4p^{\tau-2} - \frac{p-1}{2} \sum_{i=1}^{\tau-3} p^i - p^{\tau-3} + 3 \sum_{i=0}^{\tau-4} p^i - 2p - 2,
\]

for \( \tau \geq 3 \). For \( \tau = 2 \), one can contract the \( p-1 \) variables of type \( E_\mu^2 \) together with the \( p \) variables of type \( P^2 \) to a \( P^3 = P^{\tau+1} \) variable with Lemma 21. If \( \tau \geq 3 \), one can obtain a \( P^{\tau+1} \) by contracting the \( E_\mu^2 \) variables with Lemma 23 and the resulting ones together with the \( P^2 \) variables with Lemma 39.

This completes the case \( r(f, g) \geq 0 \). The following three lemmata will complete the case \( \tau \geq 2 \) by showing that for every proper \( p \)-normalised pair \( f, g \) with \( \tau \geq 2 \) and \( r(f, g) = -1 \) the equations \( f = g = 0 \) have a non-trivial \( p \)-adic solution. Here, it is useful to choose strategies depending on the value of \( I_0 \). As for \( r(f, g) \geq 0 \), some of the bounds will differ for \( p = 5 \) in order to balance that some inequalities only hold for \( p \geq 7 \).

**Lemma 44.** Let \( f, g \) be a proper \( p \)-normalised pair with \( \tau \geq 2 \), \( r = r(f, g) = -1 \), and \( I_0^0 \geq 2p^{\tau+1} + \frac{1}{2} p^\tau - p^{\tau-1} + 3 \sum_{i=0}^{\tau-2} p^i + 2p^2 - \frac{1}{2} p - 2 \). Then the equations \( f = g = 0 \) have a non-trivial \( p \)-adic solution. For \( p = 5 \) even \( I_0^0 \geq 3p^{\tau+1} - \sum_{i=0}^{\tau-2} p^i + 2p^2 - 6p + 2 \) is sufficient.

**Proof.** It is sufficient to contract a \( C^{\tau+1} \) variable because \( I_0^0 \geq p^{\tau+1} + p^{\tau-1} \) is given, which implies that the colour 0 is zero-representing. By Lemma 43 it follows that

\[
I_0^0 \geq \frac{I_0^0}{p} \geq 2p^{\tau} + 11 \sum_{i=0}^{\tau-2} p^i - p^{\tau-2} + 3 \sum_{i=0}^{\tau-3} p^i + 2p - \frac{11}{2}
\]

for \( p \geq 5 \) and

\[
I_0^0 \geq 3p^{\tau} - \sum_{i=0}^{\tau-1} p^i + 2p - 6,
\]

for \( p = 5 \), which is both bigger than \( (p-1) \left( p^{\tau-1} + p - 2 \right) = p^\tau - p^{\tau-1} + p^2 - 3p + 2 \). Furthermore, by (5.1), one obtains

\[
I_0^0 - I_0^0 \geq 2p^{\tau+1} - 2p^\tau - q_0 - (m_1 - I_0^0) \geq p^{\tau+1} - 2p^\tau + 1 - (m_1 - I_0^0),
\]

as \( q_0 \leq p^{\tau+1} - 1 \) due to \( r = -1 \). This is bigger than \( p^{\tau+1} + p - 2 - (m_1 - I_0^0) \), therefore, one can take \( p^{\tau+1} + p - 2 - (m_1 - I_0^0) \) sets containing one variable of type \( E_0^0 \) and \( p-1 \) variables of type \( E_0^0 \). By Lemma 15, each of this set contains a contraction to a \( E_0^1 \) variable. For \( p \geq 5 \) there are at least

\[
2p^{\tau+1} + 5p^\tau - \frac{p-1}{2} \sum_{i=1}^{\tau-1} p^i - p^{\tau-1} + 3 \sum_{i=0}^{\tau-2} p^i + p^2 - 4p - 2
\]

and for \( p = 5 \) at least

\[
3p^{\tau+1} - 2p^\tau = \sum_{i=0}^{\tau-1} p^i + p^2 - 4p + 2
\]

variables of type \( E_0^0 \) left, which is both at least as big as

\[
p^\tau + p^2 - 6p + 1 = p \left( p^{\tau-1} + p - 2 \right) + 3p^2 - 4p + 1.
\]
As long as there are at least \( p (3p - 3) + 1 = 3p^2 - 3p + 1 \) variables of type \( E_0^0 \) left, one has at least \( 3p - 2 \) variables of type \( E_0^0 \) for some \( \mu \). Therefore, one can use Lemma 17 to contract \( p^\tau + p^\rho - 2p \) of the \( E_0^0 \) variables to \( p^\tau - p - 2 \) variables of type \( E_0^1 \), using each time \( p \) variables of the same colour nuance. Now, one has \( p^\tau - p - 2 \) variables of type \( E_0^1 \) and \( p^\tau - p - 2 \) variables of type \( E_0^1 \). This leaves at least
\[
2p^\tau + 4p^\rho - \frac{p - 1}{2} \sum_{i=1}^{\tau-1} p^i - p^\tau - 1 + 3 \sum_{i=0}^{\tau-2} p^i - 2p - 2
\]
variables of type \( E_0^1 \) for \( p \geq 5 \) and
\[
3p^\tau - 3p^\rho - \frac{\tau - 1}{2} \sum_{i=0}^{\tau-1} p^i - 2p + 2
\]
for \( p = 5 \) remaining. Use Lemma 33 for \( p \geq 5 \) and Lemma 34 for \( p = 5 \) to contract the \( E_0^1 \) variables to \( p - 1 \) variables of type \( E_0^2 \) and \( 2p - 2 \) variables of type \( E_0^2 \) for all \( i \in \{0, 1, \ldots, \tau - 1\} \). With Lemma 32 one can contract \( p^\tau + p^\rho - 1 \) of the variables of type \( E_0^2 \) and the \( 2p - 2 \) variables of type \( E_0^2 \) to \( p^\tau - p^\rho - 1 \) variables of type \( C^1 \). Use Lemma 27 with \( x = \frac{\tau - 1}{2} \sum_{i=0}^{\tau-2} p^i - 1 \), \( y = \sum_{i=0}^{\tau-2} p^i + 1 \) and \( z = p - 2 \) to contract \( p^\tau + p - 2 \) variables of type \( E_0^2 \) and \( p^\tau + p - 2 \) variables of type \( E_0^2 \) together with the \( C^1 \) variables to \( p^\tau - 1 \) variables of type \( C^2 \), without using \( 2p - 2 \geq p \) of the \( C^1 \) variables. The \( 2p - 2 \geq p - 1 \) variables of type \( E_0^3 \) which were contracted while the \( p - 1 \) variables of type \( E_0^2 \) were contracted are also unused. One can contract \( p - 1 \) of them together with \( p \) of the remaining \( C^1 \) variables to an additional \( C^2 \) variable using Lemma 21. This gives a total of \( p^\tau - 1 \) variables of type \( C^2 \). Then one can contract the \( C^2 \) variables for \( E_0^3 \) variables for \( i \in \{2, \ldots, \tau - 1\} \) and the \( E_0^3 \) variables to a \( C^{\tau + 1} \) variable due to Lemma 39.

**Lemma 45.** Let \( f, g \) be a proper \( p \)-normalised pair with \( \tau \geq 2 \), \( r = r (f, g) = -1 \) and \( p^\tau + p^\rho - 1 \leq I_p^0 \leq 2p^\tau + 1 \frac{4p^\tau}{2} p^\tau - p - 1 + 3 \sum_{i=0}^{\tau-2} p^i + 2p^2 - \frac{4p^\tau}{2} p - 3 \) for \( p \geq 7 \) and \( p^\tau + p^\rho - 1 \leq I_p^0 \leq 3p^\tau - 3p^\rho + 2p^2 - 6p + 1 \) for \( p = 5 \). Then the equations \( f = g = 0 \) have a non-trivial \( p \)-adic solution.

**Proof.** It follows from \( r = -1 \) and the restrictions on \( I_p^0 \) that
\[
(8.3) \quad m_0 \leq 3p^\tau + \frac{11}{2} p^\rho - p^\tau - 1 + 3 \sum_{i=0}^{\tau-2} p^i + 2p^2 - \frac{11}{2} p - 4,
\]
for \( p \geq 7 \), whereas one can obtain for \( p = 5 \) the even better bound
\[
(8.4) \quad m_0 \leq 4p^\tau - \frac{\tau - 1}{2} \sum_{i=0}^{p^\tau - 1} p^i + 2p^2 - 6p.
\]
As \( I_p^0 \geq p^\tau + p - 1 \) the colour \( 0 \) is zero-representing, hence, it suffices to show that one can contract a \( C^{\tau + 1} \) variable. Due to the \( p \)-normalisation of \( f, g \) and \( r = -1 \), one has the lower bound
\[
I_p^0 \geq 3p^\tau - 3p^\rho + 1 - q_0 - q_1 \geq 2p^\tau + 3p^\rho - 2 - q_1
\]
as well.

Assume first, that \( I_p^1 = I_p^{1 \text{ max}} \geq p^\tau + p^\rho - 1 - \frac{p - 1}{2} \sum_{i=1}^{\tau - 2} p^i + 3 \sum_{i=0}^{\tau - 3} p^i - p - 4 \). Then one can make sure, that additionally, one has \( p^\tau + p - 2 \) variables of type \( E_0^3 \) by contracting the \( E_0^3 \) variables to at least \( p^\tau + p - 2 - q_1 \) variables of type \( E_0^3 \) as described in the following paragraph.

One can assume that \( q_1 \leq p^\tau + p - 3 \), because else, there is nothing to be done. If \( \nu \neq 0 \), one can contract the variables of type \( E_0^3 \) to
\[
\left\lfloor \frac{2p^\tau + 3p^\rho - 2 - q_1}{p} \right\rfloor - 2p + \frac{p - 3}{2} \geq 2p^\tau - 3p^\rho - 1 + \frac{2 - q_1}{p} - 2p + \frac{p - 3}{2}
\]
variables of type \( E_0^4 \) with Lemma 20 for \( p \geq 7 \), which is at least as big as \( p^\tau + p - 2 - q_1 \) for \( p \geq 7 \) and to contract
\[
\left\lfloor \frac{2p^\tau + 3p^\rho - 2 - q_1}{p} \right\rfloor - 2p + 3 \geq 2p^\tau - 3p^\rho - 1 + \frac{2 - q_1}{p} - 2p + 3 \geq p^\tau + p - 2 - q_1
\]
variables of type $E_0^1$ with Lemma 20 for $p = 5$. This leaves $6p - 9 \geq 2p - 2$ variables of type $E_0^0$ unused in both cases. If on the other hand, one has $\nu = 0$, it follows that

$$I_0^0 \geq \frac{I_0}{p} \geq 2p^\tau - 3p^{\tau - 1} + \frac{2 - q_1}{p} \geq p^\tau + p - 2 - q_1$$

and by $m_1 - I_0^1 = q_1$ and (5.1) that

$$I_0^0 - I_{00}^0 \geq 2p^{\tau + 1} - 2p^\tau - q_0 - q_1 \geq p^{\tau + 1} - 2p^\tau + 1 - q_1 \geq p^\tau + p - 2 - q_1.$$ 

Furthermore, one has

$$I_0^0 \geq 2p^{\tau + 1} - 3p^\tau + 2 - 2q_1 \geq p^{\tau + 1} + 2p^2 - 5p - q_1p + 3 \geq p(p^\tau + p - 2 - q_1) + p^2 - 3p + 3.$$ 

Thus one can contract $p^{\tau + 1} + p^2 - 2p - q_1p$ of the $E_0^0$ variables to $p^\tau + p - 2 - q_1$ variables of type $E_0^1$ due to Lemma 50 leaving at least $p^{\tau + 1} - 3p^\tau - p^2 + 2p + (p - 1)q_1 \geq 2p - 2$ variables of type $E_0^0$ unused.

In both cases, one has contracted enough $E_0^1$ variables to have at least $p^\tau + p - 2$ variables of type $E_0^2$, while there are $2p - 2$ variables of type $E_0^0$ remaining. The $E_0^0$ variables can be contracted together with $p^{\tau + 1} - p^\tau$ of the $C_1^0$ variables to $p^\tau - p^{\tau - 1}$ variables of type $C_1$, using Lemma 32.

Then, one can contract $4p^{\tau + 1} - p^2 \sum_{i=0}^{\tau - 2} p_i^3 + 3 \sum_{i=0}^{\tau - 2} p_i^2 - 2p - 2$ of the variables of type $E_0^1$ with Lemma 33 to $2p - 2$ variables of type $E_0^2$ for all $j \in \{1, ..., \tau - 1\}$. The remaining $p^\tau + p - 2$ variables of type $E_0^1$ together with the $p^\tau + p - 2$ variables of type $E_0^2$ and the $C_1^0$ variables can be contracted, using Lemma 27 with $x = p^{\tau - 1} - 2p^{\tau - 2} - \sum_{i=0}^{\tau - 2} p_i + 1$, $y = \sum_{i=0}^{\tau - 1} p_i + 1$ and $z = p - 2$, to $2p^{\tau - 2} - p^{\tau - 2}$ variables of type $C_2$. With Lemma 39 those and the $2p - 2$ variables in $E_0^2$ for $j \in \{2, ..., \tau - 1\}$ can be contracted to a $C^{\tau + 1}$ variable. Thus, from now on, one can assume that

$$I_1^{\text{max}} \leq p^\tau + 4p^{\tau - 1} - \frac{p - 2}{2} \sum_{i=1}^{\tau - 2} p_i^2 + 3 \sum_{i=0}^{\tau - 3} p_i^2 - p - 5.$$ 

If $q_1 \geq p^\tau + p - 2 = \left(\sum_{i=0}^{\tau - 1} p_i^3 + 1\right)(p - 1)$ and $m_1 \geq 2p^\tau + p^2 - p - 3 = \left(\sum_{i=0}^{\tau - 1} p_i^2 + 1\right)(2p - 2) + p^2 - 3p + 1$, one can use Lemma 33 to contract the $E_0^0$ variables to $2p - 2$ variables of type $E_0^2$ for all $i \in \{0, ..., \tau - 1\}$ because $I_0^0 \geq p^{\tau + 1} + p^\tau - 1 \geq 4p^\tau - \frac{p^2}{2} \sum_{i=0}^{\tau - 4} p_i^3 + 3 \sum_{i=0}^{\tau - 2} p_i^2 - 2p - 2$. By Lemma 32, the $p^{\tau + 1} - p^\tau$ variables of type $C_1^0$ can be contracted together with the $2p - 2$ variables of type $E_0^2$ to $p^\tau - p^{\tau - 1}$ variables of type $C_1$. Using Lemma 29 with $x = p^{\tau - 1} - 2p^{\tau - 2} - \sum_{i=0}^{\tau - 2} p_i^2 + 1$, $y = \sum_{i=0}^{\tau - 1} p_i + 1$ and $z = p - 2$, one can contract the $E_1^0$ variables together with the $C_1^0$ variables to $2p^{\tau - 2} - p^{\tau - 2}$ variables of type $C_2^1$, which contract together with the $2p - 2$ variables of type $E_0^2$ for $i \in \{2, ..., \tau - 1\}$ to a $C^{\tau + 1}$ variable due to Lemma 39. Therefore, one can assume that either $m_1 \leq 2p^\tau + p^2 - p - 4$ or $q_1 \leq p^\tau + p - 3$. The latter case leads to $m_1 = q_1 + I_1^{\text{max}} \leq 2p^\tau + 4p^{\tau - 1} - \frac{p - 2}{2} \sum_{i=1}^{\tau - 2} p_i^2 + 3 \sum_{i=0}^{\tau - 3} p_i^2 - 8$ due to (5.5). Hence, from now on, one can assume that

$$m_1 \leq 2p^\tau + 4p^{\tau - 1} - \frac{p - 2}{2} \sum_{i=1}^{\tau - 2} p_i^2 + 3 \sum_{i=0}^{\tau - 3} p_i^2 + p^2 - 8,$$

because this is an upper bound for the upper bound for $m_1$ in both cases.

By the $p$-normalisation of $f, g$, it follows that

$$I_0^0 \geq 4p^\tau + 1 - q_0 - m_1 \geq 3p^{\tau + 1} - 6p^\tau - 4p^{\tau - 1} - 3 \sum_{i=0}^{\tau - 3} p_i^2 - p - 10.$$ 

Therefore, one has $I_0^0 \geq 3p^\tau - 6p^{\tau + 1} - 4p^{\tau - 2} - 3 \sum_{i=0}^{\tau - 2} p_i^2 - p \geq p^{\tau + 1} - 2p - 3$ and, due to (5.1), it follows that

$$I_0^0 - I_{00}^0 \geq 2p^{\tau + 1} - 2p^\tau - q_0 - \left(m_1 - I_0^1\right) \geq p^{\tau + 1} - 2p^\tau + 1 - \left(m_1 - I_0^1\right) \geq p^{\tau + 1} + p - 3 - \left(m_1 - I_0^1\right).$$

It follows from (5.7) that $I_0^0 \geq p^\tau + 2p^2 - 3p - p \left(m_1 - I_0^1\right) + p^2 - 3p + 3$, thus, if $m_1 - I_0^1 \leq p^{\tau + 1} + 2p - 3$, one can contract $p^\tau + 2p^2 - 3p - p \left(m_1 - I_0^1\right)$ of the $E_0^0$ variables to $p^{\tau + 1} + 2p - 3 - \left(m_1 - I_0^1\right)$ variables.
of type $E^1_0$ with Lemma \[3\] There are at least $3p^{\tau+1} - 7p^\tau - 4p^{\tau-1} - 3 \sum_{i=0}^{\tau-3} p^i - 3p^2 + 3p + 10$ variables of type $E^1_0$ remaining, which contract to
\[
\left[\frac{3p^{\tau+1} - 7p^\tau - 4p^{\tau-1} - 3 \sum_{i=0}^{\tau-3} p^i - 3p^2 + 3p + 10}{p}\right] - 2p + \frac{p - 3}{2} \geq 3p^\tau - 7p^{\tau-1} - 4p^{\tau-2} - 3 \sum_{i=0}^{\tau-4} p^i - 5p + \frac{p - 3}{2} + 4
\]
variables of type $E^1_0$ with Lemma \[28\] while leaving at least $6p - 9 \geq 2p - 2$ variables of type $E^0_0$ unused. This is at least as big as $p^{\tau-1} + 2p - 3$. Thus, one has at least $p^{\tau-1} + 2p - 3$ variables of type $E^1_0$, as well as a total of $p^{\tau-1} + 2p - 3$ variables of type $E^1_0$.

By Lemma \[34\] one can contract $p^{\tau+1} - p^\tau$ of the $C^0$ variables with the remaining $2p - 2$ variables of type $E^0_0$ to $p^\tau - p^{\tau-1}$ variables of type $C^1$ and then use Lemma \[27\] with $x = p^{\tau-1} = \sum_{i=0}^{\tau-2} p^i - 1$, $y = \sum_{i=0}^{\tau-3} p^i + 2$ and $z = p - 3$ to contract them together with the $E^1$ variables to $p^{\tau-1}$ variables of type $C^2$.

For $\tau = 2$ it follows for $p \geq 7$ due to \[8.3\] and \[8.6\], that
\[
m_2 \geq 3p^3 - \frac{33}{2}p^2 + \frac{5}{2}p + 10 \geq p^2 - p - 1 = (p - 2)(p + 1) + 1,
\]
and for $p = 5$ due to \[8.3\] and \[8.6\], that
\[
m_2 \geq 2p^3 - 10p^2 + 3p + 10 \geq p^2 - p - 1 = (p - 2)(p + 1) + 1.
\]
Therefore, one has a $\mu$ with $I_\mu^2 \geq p - 1$, from which it follows that one can contract the $p$ variables of type $C^2$ and the $p - 1$ variables of type $E^0_n$ to a $C^3 = C^{\tau+1}$ variable due to Lemma \[21\] Thus, from now on one can assume, that $\tau \geq 3$.

If $I_\mu^2 = I_{max}^2 \geq 2p^{\tau+1} + 4p^{\tau-2} - \frac{p - 1}{2} \sum_{i=1}^{\tau-3} p^i - p^{\tau-3} + 3 \sum_{i=0}^{\tau-4} p^i - 2p - 2$, one can use Lemma \[33\] to contract the $E^2_n$ variables to $p - 1$ variables of type $E^3_n$ and $2p - 2$ variables of type $E^p_n$ for all $i \in \{2, \ldots, \tau - 1\}$. It follows that one can contract them together with the $C^2$ variables to a $C^{\tau+1}$ variable due to Lemma \[29\]. From now on, one can assume, that
\[
I_{max}^2 \leq 2p^{\tau-1} + 4p^{\tau-2} - \frac{p - 1}{2} \sum_{i=1}^{\tau-3} p^i - p^{\tau-3} + 3 \sum_{i=0}^{\tau-4} p^i - 2p - 3,
\]
and therefore
\[
(8.8) \quad m_2 \leq 2p^\tau + 6p^{\tau-1} + 3p^{\tau-2} + 2p^{\tau-3} + 6 \sum_{i=0}^{\tau-4} p^i - 2p^2 - 5p - 3.
\]
Then, one can contract the $p^{\tau-1}$ variables of type $C^2$ to $p^{\tau-2} - 2$ variables of type $C^3$ using Lemma \[31\] Due to \[8.3\], \[8.6\] and \[8.8\], it follows that
\[
m_0 + m_1 + m_2 \leq 3p^{\tau+1} + \frac{19}{2}p^{\tau} - \frac{17}{2}p^{\tau-1} + 6p^{\tau-2} + 8p^{\tau-3} + 12 \sum_{i=0}^{\tau-4} p^i + 2p^2 - 10p - 15,
\]
which does not only hold for $p \geq 7$ but also for $p = 5$ because the upper bound \[8.3\] is in the case $p = 5$ bigger than the upper bound \[8.3\]. This leads to
\[
q_3 \geq 4p^{\tau+1} - \frac{33}{2}p^\tau - \frac{17}{2}p^{\tau-1} - 6p^{\tau-2} - 8p^{\tau-3} - 12 \sum_{i=0}^{\tau-4} p^i - p^2 + 10p + 16 \geq p^{\tau-2} + p^{\tau-3} - 2
\]
and
\[
m_3 \geq 5p^{\tau+1} - \frac{35}{2}p^\tau - \frac{17}{2}p^{\tau-1} - 6p^{\tau-2} - 8p^{\tau-3} - 12 \sum_{i=0}^{\tau-4} p^i - p^2 + 10p + 16 \geq 2p^{\tau-2} + 2p^{\tau-3} + p^2 - 3p - 3.
\]
For $\tau = 3$ one can contract one of the $C^3$ variables together with the $E^3$ variables to a $C^4$ variable using Lemma \[23\] with $x = 1$. For $\tau \geq 4$ the $C^3$ variables can be contracted with the $E^3$ variables, using Lemma \[20\] with $x = p^{\tau-3} - p^{\tau-4} - 2 \sum_{i=0}^{\tau-6} p^i - 1$, $y = p^{\tau-3} + 2 \sum_{i=0}^{\tau-4} p^i$ and $z = p - 4$, to $2p^{\tau-3} + p^{\tau-4} - 1$
variables of type $C^4$. Then one can use Lemma 31 to contract them to $2p - 1$ variables of type $C^7$ and then Lemma 14 to contract them to a $C^{r+1}$ variable.

**Lemma 46.** Let $f, g$ be a proper $p$-normalised pair with $r \geq 2$, $r = r(f, g) = -1$ and $I_0^f \leq p^{r+1} + p^r - 2$. Then the equations $f = g = 0$ have a non-trivial $p$-adic solution.

**Proof.** Due to the upper bound for $I_0^f$ and $r = -1$ it follows that

(8.9) $\quad m_0 \leq p^{r+1} + p^r - 2 + p^{r+1} - 1 = 2p^{r+1} + p^r - 3$

and hence,

(8.10) $\quad q_1 \geq 3p^{r+1} - 3p^r + 1 - 2p^{r+1} + p^r + 3 = p^{r+1} - 4p^r + 4 \geq p^r + p - 2$

and

(8.11) $\quad m_1 \geq 4p^{r+1} - 4p^r + 1 - 2p^{r+1} - p^r + 3 = 2p^{r+1} - 5p^r + 4 \geq 2p^r + p^2 - p - 3$.

Use Lemma 23 to contract the $E_0^0$ variables to $p^r - p^{r-1}$ variables of type $P^1$. If $I_0^f = I_{\max}^f \geq p^r + 4p^{r-1} - \frac{p}{2} \sum_{i=1}^{2} p^j + 3 \sum_{i=0}^{3} p^j - p - 4$, one can contract $4p^{r-1} - \frac{p}{2} \sum_{i=1}^{3} p^j + 3 \sum_{i=0}^{3} p^j - 2p - 2$ of the variables of type $E_0^f$ to $2p - 2$ variables of type $E_0^f$ for all $j \in \{1, \ldots, \tau - 1\}$ using Lemma 33, which leaves $p^r + p - 2$ variables of type $E_0^f$ unused. Then, Lemma 27 can be used with $x = p^{r-1} - 2p^{r-2} - \sum_{i=0}^{3} p^j - 1$, $y = \sum_{i=0}^{4} p^i + 1$ and $z = p - 2$ to contract the remaining $E_0^f$ variables together with the $p^r + p - 2$ variables of type $E_0^f$ and the $P^1$ variables to $2p^{r-1} - p^{r-2}$ variables of type $P^2$. Those and the $2p - 2$ variables in $E_0^f$ for $j \in \{2, \ldots, \tau - 1\}$ can be contracted to a $P^{r+1}$ variable, using Lemma 39.

Thus, one can furthermore assume that one has $I_0^f = I_{\max}^f \leq p^r + 4p^{r-1} - \frac{p}{2} \sum_{i=1}^{2} p^j + 3 \sum_{i=0}^{3} p^j - p - 5$. It follows, that

(8.12) $\quad m_1 \leq p^{r+1} + 5p^r + 4p^{r-1} + 3p^{r-2} + 6 \sum_{i=0}^{\tau - 1} p^i$.

Due to (8.10) and (8.11), one can use Lemma 29 with $x = p^{r-1} - 2p^{r-2} - \sum_{i=0}^{3} p^j - 1$, $y = \sum_{i=0}^{4} p^i + 1$ and $z = p - 2$ to contract the $p^r - p^{r-1}$ variables of type $P^1$ and the $E_1^0$ variables to $2p^{r-1} - p^{r-2}$ variables of type $P^2$. For $\tau = 2$, one can use Lemma 14 to contract the $2p - 1$ variables of type $P^2$ to a $P^3 = P^{r+1}$ variable. Hence, one can assume that $\tau \geq 3$. As a consequence of (8.9) and (8.12), it follows that

$m_2 \geq 6p^{r+1} - 6p^r + 1 - 2p^{r+1} - p^r + 3 - p^{r+1} - 5p^r - 4p^{r-1} - 3p^{r-2} - 6 \sum_{i=0}^{3} p^i$

$= 3p^{r+1} - 12p^r - 4p^{r-1} - 3p^{r-2} - 6 \sum_{i=0}^{\tau - 1} p^j + 4,$

which is bigger than $(p + 1) \left(4p^{r-2} - \frac{p}{2} \sum_{i=1}^{3} p^j + 3 \sum_{i=0}^{3} p^j - 2p - 2\right)$. Hence, there is a $\mu$ such that $I_0^f \geq 4p^{r-2} - \frac{p}{2} \sum_{i=1}^{3} p^j + 3 \sum_{i=0}^{3} p^j - 2p - 2$; thus, one can contract the $E_0^0$ variables using Lemma 35 and then the resulting variables together with the $P^2$ variables to a $P^{r+1}$ variable, using Lemma 39.

It follows that for a proper $p$-normalised pair $f, g$ with $\tau \geq 2$ the equations $f = g = 0$ have a non-trivial $p$-adic solution, which in combination with Section 7 proves the claim of the theorem.

**References**

[1] E. Artin. *The collected papers of Emil Artin.* Edited by Serge Lang and John T. Tate. Addison-Wesley Publishing Co., Inc., Reading, Mass.-London, 1965.

[2] J. Brüdern and H. Godinho. On Artin’s conjecture. II. Pairs of additive forms. *Proc. London Math. Soc.* (3), 84(3):513–538, 2002.

[3] H. Davenport and D. J. Lewis. Homogeneous additive equations. *Proc. Roy. Soc. London Ser. A*, 274:443–460, 1963.

[4] H. Davenport and D. J. Lewis. Two additive equations. In *Number Theory (Proc. Sympos. Pure Math., Vol. XII, Houston, Tex., 1967)*, pages 74–98. Amer. Math. Soc., Providence, R.I., 1969.
[5] V. B. Dem’yanov. Pairs of quadratic forms over a complete field with discrete norm with a finite field of residue classes. Izv. Akad. Nauk SSSR. Ser. Mat., 20:307–324, 1956.

[6] H. Godinho and T. C. de Souza Neto. Pairs of additive forms of degrees $2.3^\tau$ and $4.5^\tau$. J. Comb. Number Theory, 3(2):87–102, 2011.

[7] H. Godinho and T. C. de Souza Neto. Pairs of additive forms of degree $p^\tau(p–1)$. Funct. Approx. Comment. Math., 48(part 2):197–211, 2013.

[8] H. Godinho, M. P. Knapp, and P. H. A. Rodrigues. Pairs of additive sextic forms. J. Number Theory, 133(1):176–194, 2013.

[9] H. Godinho and L. Ventura. Pairs of diagonal forms of degree $3^\tau.2$ and Artin’s conjecture. J. Number Theory, 177:211–247, 2017.

[10] C. Kränzelin. Pairs of additive forms of degree $2n$. PhD thesis, Universität Stuttgart, 2009.

[11] H. B. Mann and J. E. Olson. Sums of sets in the elementary Abelian group of type $(p, p)$. J. Combinatorial Theory, 2:275–284, 1967.

[12] J. E. Olson. A combinatorial problem on finite Abelian groups. I. J. Number Theory, 1:8–10, 1969.

[13] J. E. Olson. A combinatorial problem on finite Abelian groups. II. J. Number Theory, 1:195–199, 1969.

[14] S. Poehler. Two additive quartic forms. PhD thesis, Universität Stuttgart, 2007.