Simulating frustrated antiferromagnets with quadratically driven QED cavities

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We propose a class of quantum simulators for antiferromagnetic spin systems, based on coupled photonic cavities in presence of two-photon driving and dissipation. By modeling the coupling between the different cavities through a hopping term with negative amplitude, we solve numerically the quantum master equation governing the dynamics of the open system and determine its non-equilibrium steady state. Under suitable conditions, the steady state can be described in terms of the degenerate ground states of an antiferromagnetic Ising model. When the geometry of the cavity array is incommensurate with the antiferromagnetic coupling, the steady state presents properties which bear full analogy with those typical of the spin liquid phases arising in frustrated magnets.

Since many years, quantum simulation has proven very useful to address fundamental problems in different fields of research, from quantum chemistry to condensed-matter physics or cosmology \cite{11,12}. Following the pioneering idea of Feynman \cite{8}, several experimental platforms have been proposed to implement quantum simulators, neutral atoms in optical lattices \cite{9}, trapped ions \cite{10}, superconducting circuits \cite{11} and photonic systems \cite{12}, among others.

In particular, extended lattices of coupled nonlinear photonic cavities, both at optical and microwave frequencies, has been applied to the simulation of quantum collective phenomena \cite{13,14,15}. The effective photon-photon interaction arising from the nonlinearity of the medium where the electromagnetic field propagates, combined with losses of the cavities, make these systems the ideal platform to investigate the non-equilibrium dynamics of strongly correlated open quantum systems. This has motivated an intense research activity during the last years, which has shown the emergence of interesting phenomena, such as fractional quantum Hall effects \cite{16,21} or dissipative phase transitions \cite{22,24,25,26,27,28,29,30,31,32,33,34,35,36,37,38,39,40,41,42,43,44,45,46,47,48,49,50,51,52,53,54,55,56,57,58,59,60,61,62,63,64,65,66,67,68,69,70}.

A fundamental issue in many-body physics, that is still object of intense investigation, concerns the behavior of frustrated systems. Frustration refers to the presence of competing constraints in the Hamiltonian, which cannot be satisfied simultaneously. This phenomenon is particularly relevant in magnetic systems, where frustration usually has a geometric origin and leads to a macroscopic degeneracy of the ground state \cite{60,61}. Frustrated magnets can be therefore characterized by strong fluctuations even in the limit of zero temperature and display configurations called spin liquids, i.e. highly correlated phases with extensive entropy and without static order \cite{47}. Depending on the nature of the fluctuations, spin liquids can be either classical or quantum. In particular, the latter are prototypical examples of system with long-range entanglement and, although they are more elusive than their classical counterpart, they can show remarkable collective phenomena, such as emergent gauge fields and fractional particle excitations.

Quantum simulators, such as trapped ions \cite{48} and Rydberg atoms \cite{49,50}, have been applied to the study of frustrated magnets. However, although photonic lattices in presence of frustration have been investigated in the past \cite{19,20,51,52,53,54,55,56,57,58,59}, the possibility of simulating spin liquids by means of photonic systems is yet to be explored. An interesting experimental platform able to mimic the behavior of spin systems is represented by QED cavities subjected to two-photon (i.e. quadratic in the field) driving and losses \cite{59}. The non-equilibrium steady state of these resonators, indeed, is approximately restricted to the quantum manifold spanned by two coherent states with opposite phase, which can be associated to the opposite magnetic states of a quantum $s = 1/2$-spin \cite{60,61}.

This peculiar feature has motivated a deep research activity about these photonic systems, showing not only the feasibility of quantum computers and quantum annealers \cite{62,63,64,65,66,67,68,69}, but also the emergence of a second-order phase transition, analogous to that separating the paramagnetic and the ferromagnetic phases in quantum magnets \cite{20,26,44,47,61}.

In this work, we show how an array of coupled quadratically driven QED cavities can simulate the triangular antiferromagnetic Ising model \cite{68} – a well-known theoretical model supporting the emergence of a spin liquid phase. A necessary condition to recover this result is to engineer the coupling between the cavities such that the photon hopping strength is negative: this regime is experimentally feasible with photonic crystals \cite{69} and a possible realization with QED cavities has been discussed recently \cite{64,70}. By studying the first-order coherence correlation function, the entropy and the response to a single-photon driving field, we show that three coupled quadratically driven cavities, in the limit of strong two-photon pump, behave as three interacting spins with an Ising antiferromagnetic coupling (see Fig. 1).

We consider systems of $N = 2$ and $N = 3$ coupled photonic resonators, in presence of a Kerr nonlinearity with energy $U$ and two-photon driving with frequency $\omega_p$ and amplitude $G$. These can be modeled, in the reference frame rotating at half of the pump frequency, by
driven cavities, it is necessary to consider two different operators $\hat{\Gamma}_{j,k}$
where $L_j$ with opposite phase between neighboring cavities, but this coupling with $J < 0$. In the limit of strong driving field, the photons in each cavity form a coherent state with phase $\alpha$ or $-\alpha$. The coupling with $J < 0$ tends to favor the emergence of states with opposite phase between neighboring cavities, but this condition cannot be satisfied in the frustrated system made up of three mutually coupled cavities. This system bears strong analogies with three antiferromagnetically interacting Ising spins.

The Hamiltonian (we set $\hbar = 1$)

$$
\hat{H} = \sum_{j=1}^{N} -\Delta \hat{a}_j^{\dagger} \hat{a}_j + \frac{U}{2} \hat{a}_j^{\dagger 2} \hat{a}_j^2 + \frac{G}{2} \hat{a}_j^{\dagger 2} + \frac{G^*}{2} \hat{a}_j^2 \\
- \sum_{j \neq k} \frac{J}{2} (\hat{a}_j^{\dagger} \hat{a}_{j'}^\dagger + \hat{a}_j^\dagger \hat{a}_{j'}^\dagger) .
$$

(1)

where $\hat{a}_j$ is the photon destruction operator acting on the $j$-th site. The quantity $\Delta = \omega_p / 2 - \omega_c$ is the detuning between half of the two-photon driving field frequency $\omega_p$ and the resonant cavity frequency $\omega_c$. The photon hopping between different cavities, with strength $J$, is described by the last term in the equation.

Assuming Markovian dissipative processes for each cavity, the dynamics of the system is described by the density matrix $\hat{\rho}(t)$ which obeys to the quantum master equation in the Lindblad form:

$$
\frac{\partial \hat{\rho}}{\partial t} = \mathcal{L} \hat{\rho} = -i [\hat{H}, \hat{\rho}] + \sum_{j,k} \hat{\Gamma}_{j,k} \hat{\rho} \hat{\Gamma}_{j,k}^\dagger - \frac{1}{2} \left\{ \hat{\Gamma}_{j,k}^\dagger \hat{\Gamma}_{j,k}, \hat{\rho} \right\} ,
$$

(2)

where $\mathcal{L}$ is the Liouvillian superoperator and the jump operators $\hat{\Gamma}_{j,k}$ describe the transition induced by the environment on the system. In the case of quadratically driven cavities, it is necessary to consider two different kinds of dissipative processes. First, one-photon losses, modeled by the jump operators $\hat{\Gamma}_{j,1} = \sqrt{\gamma} \hat{a}_j$. Furthermore, since we assume an input channel injecting photons in pairs, then dissipative processes will likely arise through the same channel and therefore it is necessary to consider two-photon losses, which are described by the jump operators $\hat{\Gamma}_{j,2} = \sqrt{\eta \gamma} \hat{a}_j^2$.

The dynamics of the system evolves at large times towards a steady-state $\hat{\rho}_{ss}$, which satisfies the condition $\partial \hat{\rho}_{ss} / \partial t = 0$. We determine the steady-state density matrix by numerically solving the linear system $\mathcal{L} \hat{\rho}_{ss} = 0$, with the constraint $\text{Tr}(\hat{\rho}_{ss}) = 1$. The Hilbert space is truncated by setting a maximum value $N_m$ for the photon occupancy per cavity and a maximum value $N_{m,T}$ for the total photon occupancy in the system: the accuracy of the numerical results is checked by studying their convergence with $N_m$ and $N_{m,T}$.

In the search for the antiferromagnetic behavior, we need to pay attention to the choice of the physical parameters in Eq. (2). In Ref. [44], it has been shown that a system of coupled quadratically driven cavities is well approximated by a spin lattice when losses are small with respect to the Hamiltonian parameters and when the two-photon driving field is resonant with the mode of the single particle spectrum of the Bose-Hubbard Hamiltonian at minimum energy (i.e. for $\Delta = -J$). By choosing a positive value for the hopping strength $J$, the spin model presents a ferromagnetic coupling which leads, in the limit of large $G$, to a steady state that is a statistical mixture of two separable coherent states $|\Psi_{\pm J} \rangle = \prod_j | \alpha \rangle_j$, each of them with equal probability. This can be interpreted as a ferromagnetic phase if one associates the local coherent states with opposite $\alpha$ to the spin-up and spin-down state ($|\alpha \rangle \rightarrow | \uparrow \rangle$ and $| - \alpha \rangle \rightarrow | \downarrow \rangle$). It is therefore natural to expect that, by changing the sign of $J$, one can obtain an antiferromagnetic coupling in the approximate spin model. Hence, we assume here $\gamma = \eta \ll J$ and $\Delta = J < 0$ and we vary the value of the two-photon driving $G$ (which we assume real) in order to extrapolate the asymptotic behavior at large $G$.

To show the analogies between our quadratically driven photonic system and a frustrated antiferromagnet, we focus at first on the first-order coherence correlation function

$$
g^{(1)}_{1,2} = \frac{\text{Tr}(\hat{\rho}_{ss} \hat{a}_1^\dagger \hat{a}_2)}{\text{Tr}(\hat{\rho}_{ss} \hat{a}_1^\dagger \hat{a}_1)} ,
$$

(3)

and the von-Neumann entropy

$$
S = -\text{Tr}(\hat{\rho}_{ss} \log \hat{\rho}_{ss}) .
$$

(4)

The behavior of $g^{(1)}_{1,2}$ and of $S$ as a function of the driving field amplitude $G$ are shown respectively in Fig. 2 and 3 both for the system with $N = 2$ and $N = 3$ cavities.

The results for the system made up of $N = 2$ cavities bear a clear signature of an antiferromagnetic interaction. The correlation function $g^{(1)}_{1,2}$, which plays the role...
FIG. 2. The first-order correlation function $g^{(1)}_{1,2}$ as a function of the amplitude of the two-photon driving, for the systems of $N = 2$ and $N = 3$ cavities. Inset: behavior of $g^{(1)}_{1,2}$ vs. $G$ in the regime of large $G/\gamma$, plotted on a log-log scale. The other Hamiltonian parameters are $U/\gamma = 10$, $\Delta/\gamma = J/\gamma = -10$.

FIG. 3. The von-Neumann entropy $S$ as a function of the amplitude of the two-photon driving, for the systems of $N = 2$ and $N = 3$ cavities. Inset: behavior of $S$ vs. $G$ in the regime of large $G/\gamma$, plotted on a log-log scale. The other Hamiltonian parameters are $U/\gamma = 10$, $\Delta/\gamma = J/\gamma = -10$.

The first-order correlation function is negative and, for increasing $G$, approaches the asymptotic value $g^{(1)}_{1,2} = -1$. Moreover, the entropy as a function of $G$ increases from the value $S = 0$ in the limit of a vanishing driving amplitude (notice that for $G = 0$, the steady state is pure and corresponds to the bosonic vacuum) to the asymptotic value $S = \log(2)$ in the limit $G/\gamma \to \infty$, indicating that the steady state is dominated by two equiprobable quantum states.

The results for $g^{(1)}_{1,2}$ and for $S$ indicate that, in the limit of $G \gg \gamma$, the steady state is described by a statistical mixture of two separable states, obtained as product of two local coherent states with opposite phase. Its density matrix can therefore be written as

$$\hat{\rho}_2 = \frac{1}{2} |\alpha_0, -\alpha_0\rangle \langle \alpha_0, -\alpha_0 | + \frac{1}{2} | -\alpha_0, \alpha_0 \rangle \langle -\alpha_0, \alpha_0 | .$$  (5)

The value of the phase $\alpha_0$ of the local coherent states can be obtained as $\alpha_0 = \sqrt{\text{Tr}(\hat{\rho}_{ss} \hat{\rho}_2^2)}$, and its behavior as a function of $G$ is shown in Fig. 4. To test the assumption given in Eq. (5), we compute the fidelity $F(\hat{\rho}_2, \hat{\rho}_{ss}) = \left( \text{Tr} \left( \sqrt{\sqrt{\hat{\rho}_2} \hat{\rho}_{ss} \sqrt{\hat{\rho}_2}} \right) \right)^2$ between $\hat{\rho}_2$ and the steady-state density matrix $\hat{\rho}_{ss}$ obtained by the numerical solution of the master equation (2). It turns out that $1 - F(\hat{\rho}_2, \hat{\rho}_{ss}) < 10^{-4}$ for all the values $G/\gamma \geq 30$, thus indicating that Eq. (5) is a very good description of the steady-state density matrix in the limit of strong driving.

The possibility to investigate the effects of frustration is highlighted in the results of the system made up of $N = 3$ cavities. In this case, the coupling between different cavities is at odds with the geometric constraints of the system, thus leading to a behavior similar to frustrated antiferromagnets. The correlation function $g^{(1)}_{1,2}$ (Fig. 2) presents a non monotonous behavior as a function of $G$ and, for large values of $G/\gamma$, it converges with a power law behavior to the asymptotic value $-1/3$. This value is typical of the spin-spin correlation function between nearest neighbors in a triangular antiferromagnetic Ising model [71]. The behavior of the entropy $S$ (Fig. 3) also confirms the analogy with an antiferromagnetic system. In this case, the frustration in the spin alignment results in the appearance of six possible configuration minimizing the energy (i.e. all those with two antiparallel and one parallel pair of spin). The asymptotic value reached in the limit of large $G/\gamma$ is $S = \log(6)$, consis-
The dependence of $|\langle \hat{a}_1 \rangle |$ on $|F|$ is particularly different in the two systems with $N = 2$ and $N = 3$ cavities. For $N = 2$, we can distinguish two different regimes, according to the amplitude of the one-photon driving. For small $|F|$, instead, the antiferromagnetic order is broken and the steady state of the photonic system is a pure coherent one-photon drive. For $N = 3$, we notice that, although the system maintains a certain antiferromagnetic order, as it happens in the absence of the one-photon driving. For large $|F|$, we can construct an approximation for the steady-state density matrix for the system of $N = 3$ cavities in the limit where $G \gg \gamma$ as a statistical mixture of states obtained as tensor products of local coherent states with opposite phase. From the analogy with the ground state of the three antiferromagnetically coupled spins, the approximate steady state is

$$\hat{\rho}_3 = \frac{1}{6} \left( |\alpha_0, \alpha_0, -\alpha_0 \rangle \langle \alpha_0, \alpha_0, -\alpha_0 | + |\alpha_0, -\alpha_0, \alpha_0 \rangle \langle \alpha_0, -\alpha_0, \alpha_0 | + |\alpha_0, \alpha_0, \alpha_0 \rangle \langle \alpha_0, \alpha_0, \alpha_0 | + |\alpha_0, -\alpha_0, \alpha_0 \rangle \langle -\alpha_0, -\alpha_0, \alpha_0 | + |\alpha_0, \alpha_0, \alpha_0 \rangle \langle -\alpha_0, -\alpha_0, \alpha_0 | + |\alpha_0, -\alpha_0, \alpha_0 \rangle \langle \alpha_0, -\alpha_0, -\alpha_0 | \right). \quad (6)$$

At $G/\gamma = 60$, the fidelity between the steady-state density matrix $\hat{\rho}_3$ and the approximation given by Eq. (6) is $F(\hat{\rho}_3, \hat{\rho}_{ss}) = 0.956$.

A further evidence of the spin analogy can be found in the non-linear response of the quadratically driven photonic system to an one-photon pump. This latter can be modeled with an additional term in the Hamiltonian of Eq. (4)

$$\hat{H}_F = \hat{H} + \sum_j (F \hat{a}_j^\dagger F^* \hat{a}_j) \quad (7)$$

According to the spin approximation discussed in Ref. [14], this term can be associated to an external field in the magnetic analog of our system. The direction of the analog external field depends on $\alpha_0$ and, in the limit of large $\alpha_0$ (i.e. large $G/\gamma$), it becomes parallel to the direction of the Ising antiferromagnetic coupling. We have calculated the steady-state density matrix $\hat{\rho}_F$ of the system in presence of a strong two-photon driving $G/\gamma = 60$ and a variable one-photon driving $F$. We show in Fig. 5 the expectation value of the induced coherence $\langle \hat{a}_1 \rangle = \text{Tr}(\hat{\rho}_F \hat{a}_1)$. The quantity $\langle \hat{a}_1 \rangle$ strongly depends on the phase of the one-photon driving $F$, and the effect of the one-photon pump is more evident when the quantity $F^* \alpha_0$ is purely real. For this reason, in the results of Fig. 5, we have set the phase of $F$ according to this condition and vary the absolute value $|F|$. The dependence of $\langle \hat{a}_1 \rangle$ on $|F|$ is particularly different in the two systems with $N = 2$ and $N = 3$ cavities. For $N = 2$, we can distinguish two different regimes, according to the amplitude of the one-photon driving. For small $|F|$, instead, the antiferromagnetic order is broken and the steady state of the photonic system is a pure coherent state, with $\langle \hat{a}_1 \rangle = \alpha_0$. For the system with $N = 3$ cavities, the behavior of $\langle \hat{a}_1 \rangle$ is similar to the previous case only for large $|F|$, but is notably different in the opposite regime. For small $|F|$, we notice that, after a steep increase of $\langle \hat{a}_1 \rangle$ with $|F|$ at very small values of the pump, there is a a broad interval where the induced coherence depends very weakly on $|F|$ and stabilizes around a value close to $\alpha_0/3$. This latter behavior is reminiscent of the 1/3-magnetization plateau which emerges in the triangular antiferromagnetic Ising model in presence of an external magnetic field along the direction of the coupling [72]. It corresponds to the minimal energy configurations where two thirds of the spins in the lattice point in the direction of the external field and the remaining one third in the opposite one.

In conclusion, we have considered a system of coupled photonic cavities subject to a two-photon driving and showed the existence of regimes where these can simulate the properties of Ising antiferromagnets. The key feature, allowing the emergence of antiferromagnetic correlations among the photonic states, is a negative hopping rate between different cavities, a condition already realized experimentally [69]. By comparing the behavior of the systems of two and three cavities, whose geometry are respectively commensurate and incommensurate to the antiferromagnetic coupling, we highlighted the effects due to the frustration of the lattice, analogous to those arising in spin models. The von-Neumann entropy in particular signals the increased fluctuations in the frustrated system, which can be ascribed to a larger degeneracy of the states at minimum energy. The response of the photonic system to a coherent one-photon drive shows the emergence of a plateau in the induced coherence, which is reminiscent of the behavior of frustrated...
antiferromagnets under an external magnetic field.

Thanks to the possibility of realizing and manipulating systems of quadratically driven nonlinear photonic cavities within current experimental techniques, our results point to a novel class of quantum simulators for antiferromagnets, which could allow to investigate the properties of spin liquids by means of a fully controllable and versatile experimental platform. From the theoretical point of view, an important question which should be addressed in the future concerns the possibility to realize entangled spins liquids among the photons and the effects the geometric frustration can have on these. This possibility could be relevant to investigate the elusive quantum spin liquid phase.

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