An approximate analytical solution of free convection problem for vertical isothermal plate via transverse coordinate Taylor expansion

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Abstract

The model under consideration is based on approximate analytical solution of two dimensional stationary Navier-Stokes and Fourier-Kirchhoff equations. Approximations are based on the typical for natural convection assumptions: the fluid noncompressibility and Boussinesq approximation. We also assume that ortogonal to the plate component \((x)\) of velocity is neglectible small. The solution of the boundary problem is represented as a Taylor Series in \(x\) coordinate for velocity and temperature which introduces functions of vertical coordinate \((y)\), as coefficients of the expansion. The correspondent boundary problem formulation depends on parameters specific for the problem: Grashoff number, the plate height \((L)\) and gravity constant. The main result of the paper is the set of equations for the coefficient functions for example choice of expansion terms number. The nonzero velocity at the starting point of a flow appears in such approach as a development of convextional boundary layer theory formulation.

1 Introduction

A conventional boundary layer theory of fluid flow used for free convective description assumes zero velocity at leading edge of a heated plate. More advanced theories of self-similarity also accept this same boundary condition [1], [2], [3]. However experimental visualization definitely shows that in the vicinity of edge the fluid motion exists [4], [8], [9]. It is obvious from the point of view of the mass conservation law. In the mentioned convection descriptions the continuity equation is not taken into account that diminishes the number of necessary variables. For example the pressure is excluded by cross differentiation of Navier-Stokes equation component.
The consequence of zero value of boundary layer thickness at the leading edge of the plate yields in infinite value of heat transfer coefficient which is in contradiction with the physical fact that the plate do not transfer a heat at the starting point of the phenomenon. The whole picture of the phenomenon is well known: the profiles of velocity and temperature in normal direction to a vertical plate is reproduced by theoretical concepts of Prandtl and self-similarity. While the evolution of profiles along tangent coordinate do not look as given by visualisation of isotherms (see e.g. [5]). It is obvious that isotherms dependance on vertical coordinate \( y \) significantly differs from power low dependance \( \delta \sim y^{1/4} \) of boundary layer theories.

In this article we develop the model of convective heat transfer taking into account nonzero fluid motion at the vicinity of the starting edge. Our model is based on explicit form of solution of the basic fundamental equations (Navier-Stokes and Fourier-Kirchhoff ) as a power series in dependant variables. The mass conservation law in integral form is used to formulate a boundary condition that links initial and final edges of the fluid flow.

We consider a two-dimensional free convective fluid flow in \( x, y \) plane generated by vertical isothermal plate of height \( L \) placed in undisturbed surrounding.

The algorithm of solution construction is following. First we expand the basic fields, velocity and temperature in power series of horizontal variable \( x \), it substitution into the basic system gives a system of ordinary differential equations in \( y \) variable. Such system is generally infinite therefore we should cut the expansion at some power. The form of such cutting defines a model. The minimal number of term in the modeling is determined by the physical conditions of velocity and temperature profiles. From the scale analysis of the equations we neglect the horizontal (normal to the surface of the plate) component velocity. The minimum number of therms is chosen as three: the parabolic part guarantee a maximum of velocity existence while the third therm account gives us change of sign of the velocity derivative. The temperature behavior in the same order of approximation is defined by the basic system of equations.

The first term in such expansion is linear in \( x \), that account boundary condition on the plate (isothermic one). The coefficient, noted as \( C(y) \) satisfy an ordinary differential equation of the fourth order. It means that we need four boundary condition in \( y \) variable. The differential links of other coefficients with \( C \) add two constants of integrations hence a necessity of two extra conditions. These conditions are derived from conservation laws in integral form.

The solution of the basic system, however, need one more constant choice. This constant characterize linear term of velocity expansion and evaluated by means of extra boundary condition.

In the second section we present basic system in dimensional and dimensionless forms. By means of cross-differentiation we eliminate the pressure term and next neglect the horizontal velocity that results in two partial differential equations for temperature and vertical component of velocity.

In the third section we expand both velocity and temperature fields into Taylor series in \( x \) and derive ordinary differential equations for the coefficients by direct substitution into basic system. The minimal (cubic) version is obtained...
disconnecting the infinite system of equations by the special constraint.

The fourth and fives sections are devoted to boundary condition formulations
and its explicit form in terms of the coefficient functions of basic fields. It is
important to stress that the set of boundary conditions and conservation laws
determine all necessary parameters including the Grasshof and Rayleigh numbers
in the stationary regime under consideration.

The last section contains the solution \( C(y) \) in explicit form and results of its
numerical analysis. The solution parameters values as the function of the plate
height \( L \) and parameters which enter the Grasshof number \( GR(l) \) estimation
are given in the table form, which allows to fix a narrow domain of the scale
parameter \( l \) being the characteristic linear dimension of the flow at the starting
level.

2 The basic equations

Let us consider a two dimensional stationary flow of incompressible fluid in the
gravity field. The flow is generated by a convective heat transfer from solid plate
to the fluid. The plate is isothermal and vertical. In the Cartesian coordinates
\( x \) (horizontal and orthogonal to the plate), \( y \) (vertical and tangent to the plate)
the Navier-Stokes (NS) system of equations have the form \[1\]:

\[
\rho \left( W_x \frac{\partial W_x}{\partial x} + W_y \frac{\partial W_y}{\partial y} \right) = g \rho_\infty b (T - T_\infty) - \frac{\partial p}{\partial y} + \rho \nu \left( \frac{\partial^2 W_y}{\partial y^2} + \frac{\partial^2 W_y}{\partial x^2} \right) \tag{1}
\]

\[
\rho \left( W_x \frac{\partial W_x}{\partial x} + W_y \frac{\partial W_y}{\partial y} \right) = - \frac{\partial p}{\partial x} + \rho \nu \left( \frac{\partial^2 W_x}{\partial y^2} + \frac{\partial^2 W_x}{\partial x^2} \right) \tag{2}
\]

In the above equations the pressure terms are divided in two parts \( \bar{p} = p_0 + p \).
The first of them is the hydrostatic one that is equal to mass force \( -g \rho_\infty \), where:

\[
\rho = \rho_\infty (1 - b (T - T_\infty)) \tag{3}
\]
is the density of a liquid at the nondisturbed area where the temperature is
\( T_\infty \). The second one is the extra pressure denoted by \( -\nabla p \). The part of gravity
force \( gb (T - T_\infty) \) arises from dependence of the extra density on temperature,
\( b \) is a coefficient of thermal expansion of the fluid. In the case of gases \( b =
-\frac{1}{\rho} \left( \frac{\partial \rho}{\partial T} \right)_p = \frac{1}{T_\infty} \). The last terms of the above equations represents the friction
forces with the kinematic coefficient of viscosity \( \nu \).

The mass continuity equation in the conditions of natural convection of
incompressible fluid in the steady state \[2\] has the form:

\[
\frac{\partial W_x}{\partial x} + \frac{\partial W_y}{\partial y} = 0. \tag{4}
\]

The temperature dynamics is described by the stationary Fourier-Kirchhoff
(FK) equation:

\[
W_x \frac{\partial T}{\partial x} + W_y \frac{\partial T}{\partial y} = a \left( \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial x^2} \right), \tag{5}
\]
where \( W_x \) and \( W_y \) are the components of the fluid velocity \( \overline{W} \), \( T \) - temperature and \( p \) - pressure disturbances correspondingly and \( a \) is the thermal diffusivity.

From the point of clarity of further transformations we use the same scale \( l \) along both variables \( x \) and \( y \). We will return to the eventual difference between characteristic scales in different directions while the solution analysis to be provided. After introducing variables:

\[
x' = x/l, \quad y' = y/l, \quad T' = (T - T_w)/(T_w - T_\infty),
\]

\[
p' = p/p_\infty, \quad W'_x = W_x/W_0, \quad W'_y = W_y/W_0
\]

we obtain in Boussinesq approximation (in all terms besides of buoyancy one we put \( \rho \approx \rho_\infty \)).

\[
W'_x \frac{\partial W'_y}{\partial x'} - W'_y \frac{\partial W'_x}{\partial y'} = \frac{gb(T_w - T_\infty)l}{W_0^2} (T' + 1) - \frac{p_\infty}{\rho_\infty W_0^2} \frac{\partial p'}{\partial y'} + \nu' \left( \frac{\partial^2 W'_y}{\partial y'^2} + \frac{\partial^2 W'_x}{\partial x'^2} \right)
\]

\[
W'_x \frac{\partial W'_x}{\partial x'} + W'_y \frac{\partial W'_y}{\partial y'} = - \frac{p_\infty}{\rho_\infty W_0^2} \frac{\partial p'}{\partial x'} + \nu' \left( \frac{\partial^2 W'_x}{\partial y'^2} + \frac{\partial^2 W'_y}{\partial x'^2} \right)
\]

and FK equation is written as

\[
W'_x \frac{\partial T'}{\partial x'} + W'_y \frac{\partial T'}{\partial y'} = a' \left( \frac{\partial^2 T'}{\partial y'^2} + \frac{\partial^2 T'}{\partial x'^2} \right),
\]

where \( \nu' = \frac{\nu}{W_0} \), \( \frac{\nu}{W_0} = a' \), \( l \) is a characteristic linear dimension and \( W_0 \) is characteristic velocity:

\[
W_0 = \frac{\nu}{l}.
\]

then \( a' = Pr \), \( \nu' = 1 \) and \( \frac{gb(T_w - T_\infty)l}{W_0^2} = Gr_r \), is the Grashof number, which after plugging (10) takes the form:

\[
Gr_r = \frac{gb(T_w - T_\infty)l^3}{\nu^2}.
\]

After cross differentiation of equations (7) and (8) we have:

\[
\frac{\partial}{\partial x'} \left[ W'_x \frac{\partial W'_y}{\partial x'} + W'_y \frac{\partial W'_x}{\partial y'} - Gr_r (T' + 1) - \left( \frac{\partial^2 W'_y}{\partial y'^2} + \frac{\partial^2 W'_x}{\partial x'^2} \right) \right] =
\]

\[
= \frac{\partial}{\partial y'} \left[ W'_x \frac{\partial W'_x}{\partial x'} + W'_y \frac{\partial W'_y}{\partial y'} - \left( \frac{\partial^2 W'_x}{\partial y'^2} + \frac{\partial^2 W'_x}{\partial x'^2} \right) \right]
\]

\[
\text{The FK equation rescales as}
\]

\[
Pr \left( W'_x \frac{\partial T'}{\partial x'} + W'_y \frac{\partial T'}{\partial y'} \right) = \left( \frac{\partial^2 T'}{\partial y'^2} + \frac{\partial^2 T'}{\partial x'^2} \right)
\]
\[ \rho = \rho_\infty (1 - b (T - T_\infty)) = \rho_\infty (1 - b\Phi (T' + 1)). \quad (14) \]

where \( \Phi = T_w - T_\infty \).

Next we would formulate the problem of free convection around the heated vertical isothermal plate \( x = 0, y \in [0, l] \), dropping the primes. In this case we assume the angle between the plate and a stream line is small that means a possibility to neglect the horizontal component of velocity of fluid, denoting the vertical component as \( W(y, x) \). In this paper we restrict ourselves by the assumption that \( W_x = 0 \) and \( W_y = W \), that yields

\[ \frac{\partial}{\partial x} \left[ W \frac{\partial W}{\partial y} - G_r (T + 1) - \left( \frac{\partial^2 W}{\partial y^2} + \frac{\partial^2 W}{\partial x^2} \right) \right] = 0, \quad (15) \]

\[ \Pr W \frac{\partial T}{\partial y} = \left( \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial x^2} \right). \quad (16) \]

### 3 Method of solution and approximations

The aim of this paper is the theory application to the standard example of a finite vertical plate. Having only two basic functions we consider the power series expansions of the velocity and temperature in Cartesian coordinates:

\[ W(x, y) = \gamma (y) x + \alpha (y) x^2 + \beta (y) x^3 + \kappa (y) x^4 + \ldots \quad (17) \]

\[ T(x, y) = C (y) x + A (y) x^2 + B (y) x^3 + F (y) x^4 + \ldots \quad (18) \]

According to standard boundary conditions on the plate we assume that the both functions tend to zero when \( x \to 0 \), so we choose for a calculation the variable that has the zero value for nondimensional temperature \( T' \). It means that the value of \( T(x, y) \) outside of the convective flow tends to \( -1 \). Substituting expressions \( (17) \) and \( (18) \) into the equations \( (15) \) \( (16) \), we take into account the linear independence of monomials \( x^n \) that gives a system of coupled nonlinear equations for the coefficients \( \gamma (y), \alpha (y), \ldots \) and \( C(y), A(y), \ldots \). Such system is infinite hence for a practical use we need to choose appropriate scheme of closed formulation for finite number of variables. The formulation should be based on physical assumptions for a concrete conditions.

We would like to restrict ourselves by the fourth order approximation for both variables that means we neglect higher order terms starting from fifth one. The area of the approximations validity is defined by the comparison of terms in expansions \( (17) \) and \( (18) \).

As it will be clear from further analysis we should consider the functions: \( \alpha (y), \beta (y), C(y), B(y) \) as variables of the first order, while \( \gamma (y) \) and \( F(y) \) to be the second one. From the relations that appear after substitution of \( (17) \) and \( (18) \) into \( (15) \) and \( (16) \) it follows that \( A(y) = 0 \) and \( F(y) = 0 \). Finally from
both equations (15), (16) we obtain the system of equations for the coefficients $B(y), C(y), \alpha(y)$ and $\beta(y)$:

$$6B(y) + \frac{\partial^2 C(y)}{\partial y \partial y} = 0, \quad (19)$$

$$\text{Pr} \alpha(y) \frac{\partial C(y)}{\partial y} - \frac{\partial^2 B(y)}{\partial y \partial y} = 0, \quad (20)$$

$$-6\beta(y) - Gr C(y) = 0, \quad (21)$$

$$\gamma(y) \frac{\partial \gamma(y)}{\partial y} - \frac{\partial^2 \alpha(y)}{\partial y \partial y} = 0. \quad (22)$$

The first two (19), (20) arise from FK equation and the rest of them are from the NS one. The system of equations is closed if $\gamma(y) = \text{const} = \gamma$. It means that the number of equations and the number of unknown functions is the same.

In the first approximation the velocity and temperature are expressed as:

$$W(y,x) = \gamma x + \alpha(y)x^2 + \beta(y)x^3, \quad T(y,x) = C(y)x + B(y)x^3. \quad (23)$$

From (22) one has

$$\alpha(y) = C_1 y + C_2. \quad (24)$$

From (19) it follows that

$$B(y) = -\frac{1}{6} \frac{\partial^2 C(y)}{\partial y \partial y}, \quad (25)$$

hence (20) goes to:

$$\frac{1}{6} \frac{\partial^2 C(y)}{\partial y \partial y \partial y \partial y} + \text{Pr} \left( yC_1 + C_2 \right) \frac{\partial C(y)}{\partial y} = 0. \quad (26)$$

The equation (21) reads:

$$\beta(y) = -\frac{Gr}{6} C(y). \quad (27a)$$

This results in

$$W(y,x) = \gamma x + (C_1 y + C_2)x^2 - \frac{Gr}{6} C(y)x^3, \quad T(y,x) = C(y)x - \frac{1}{6} \frac{\partial^2 C(y)}{\partial y \partial y}x^3. \quad (28)$$

The form of the equation (26) indicates that for unique solution one needs four boundary conditions for given parameters $C_1$ and $C_2$. Apart from such conditions we should also have values for $\gamma$ and $Gr$. So for explicit determination of $W(y,x)$ and $T(y,x)$ we need eight conditions.
4 The analysis of the problem formulation

Looking for the boundary conditions let us apply conservation laws of mass, momentum and energy, applying the laws to a control volume $V$ (see Fig.1).

The first one is the conservation of mass in two dimensions that in steady state looks as:

$$\int \rho \vec{W} \cdot \vec{n} dS = 0$$

(29)

where: $S$ is the sum of all lateral surfaces $S = \sum_{i=1}^{6} \Sigma_i$ (Fig.1). The mass conservation law in the integral form (29) is formulated by a division of the surface $\Sigma$ to two the lower $\Sigma_1$ and upper $\Sigma_2$ boundaries only.

According to our main assumption about two-dimensionality of the stream we neglect a dependence of variables on $z$ coordinate and $x \in [0,1]$. Hence the condition of total mass conservation looks as follows:

$$\int_{\Sigma_1} \rho \vec{W} \cdot \vec{n} dS = \int_{\Sigma_2} \rho \vec{W} \cdot \vec{n} dS$$

(30)

Where the flow from below $\Sigma_1$ is approximately the product of density at temperature $T = -1$ and velocity of the incoming flow in the interval $x \in [0,1]$. We follow the idea of the velocity field continuity at $y = 0$, hence $W(0,x) = \gamma x + \alpha (0) x^2 + \beta (0) x^3$.

For the left side in approximations mentioned above one has (28, 24, 27a):

$$\int_{\Sigma_1} \rho \vec{W} \cdot \vec{n} dS = \rho_\infty \int_0^{x_L} (\gamma x + \alpha (0) x^2 + \beta (0) x^3) dx = \rho_\infty \left( \frac{\gamma}{2} + \frac{\alpha}{3} - \frac{\beta}{24} C (0) \right)$$

and outcoming flow $\Sigma_2$ is expressed similarly:

$$\int_{\Sigma_2} \rho \vec{W} \cdot \vec{n} dS = \rho_\infty \int_0^{x_L} (\gamma x + (C_1 L + C_2) x^2 - \frac{\alpha}{6} C (L) x^3) dx = \frac{1}{2\pi} \rho_\infty x_L \left( 12\gamma + 8C_2 x_L - x_L^2 G r C (L) + 8LC_1 x_L \right) .$$

Figure 1: Fig 1. General view.
The mass conservation law yields

\[-12\gamma - 8C_2 + 8C_2x_L^3 + G_r C(0) + 12\gamma x_L^2 + 8LC_1x_L^3 - x_L^4 G_r C(L) = 0.\] (31)

The next condition is connected with the conservation of energy in a control volume \(V\) (area \(S\) with unit width see Fig.1) arises from FK equation (5) by integration over the volume.

\[
\Pr \int_V \left( W \frac{\partial T}{\partial y} \right) dV = \int_V \left( \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial x^2} \right) dV = \int_S \left( \nabla T \right) \cdot \vec{n} dS. \quad (32)
\]

The left side of the energy conservation equation (32) is transformed similar applying the identity \(\nabla \cdot (TW) = T \nabla \cdot \vec{W} + \vec{W} \cdot \nabla T\) and (4).

According to our assumptions we left only flows across \(\Sigma_1, \Sigma_2, \Sigma_3\) and basing on homogenity of the problem with respect to the coordinate \(z\) we have:

\[
\int_0^L \frac{\partial T}{\partial x} \bigg|_{x=0} dy + \Pr \left( - \int_0^1 T(x, 0) W(x, 0) dx + \int_0^{x_L} T(x, L) W(x, L) dx \right) = 0. \quad (33)
\]

To link the incoming fluid temperature \(T = -1\) (from the bottom edge flow) with the solution at \(y = 0\) and the outgoing fluid (see 28) we put \(T(x, 0) = -1\) that results in:

\[
\frac{C_2}{4} C(L)x_L^4 - \frac{C_2}{36} B x_L^6 - \frac{C(L)^2}{30} G_r x_L^5 - \frac{x_L^5}{30} B \gamma + \frac{\gamma}{3} C(L)x_L^3 - \frac{B L}{36} C_1 x_L^6 + \frac{L}{4} C_1 C(L)x_L^4 + \frac{BG_r}{252} C(L)x_L^7 + \frac{\gamma}{2} + \frac{C_2}{3} - \frac{G_r}{24} C(0) + \frac{1}{Pr} \int_0^L C(y) dy = 0, \quad (34)
\]

where \(B = B(0)\). The equation (29) is the ordinary differential equation of the fourth order, therefore its solution needs four constants of integration. These constants depend on two parameters \(C_1\) and \(C_2\), which enter the coefficients of the Eq.(26). The function \(C(y)\) defines the rest functions \(\beta(y)\) and \(B(y)\) via above relations. It means that we have six constants determining the solution of problem and we need also six corresponding boundary conditions.

5 Boundary conditions for temperature

The temperature values in the vicinity of the boundary edge point and taken as value -1 (temperature of incoming from the bottom flow). In dimensional form the interval of consideration has the characteristic length \(l\) which we identify with a parameter we used when dimensional variables were introduced (6). Let us remind that scale \(l\) is connected with special (local, horizontal) Grashof number \(G_r\) (11). The total height of the plate is denoted \(L\).

For a stationary process an edge conditions may be considered as initial one for a Cauchy problem. Having a power series approximation of such conditions
we choose the coefficients of the series using Weierstrass theorem. It means that we equalize the coefficients to scalar product of initial conditions and orthonormal polynomials on the interval \([0, l]\).

In our case the temperature profile \(T(0, x)\) represents this condition, while the function is constant \((-1)\) on the interval \([0, 1]\) in nondimensional variables. In the approximation of the third power orthogonal polynomials we have:

\[
T(0, x) = x^3 B(0) + C(0) x = \gamma_2 p_{1t} + \beta_2 p_{3t} \approx -1
\]

because nondimensional temperature of the fluid at the lower half plane, according to above, is \(-1\), where the polynomials are defined as: \(p_{1t} = tx, p_{3t} = q_t x + u_t x^3\).

The normalization for \(p_{1t}\) is

\[
\frac{1}{\sqrt{3}} \int_0^1 (tx)^2 dx = \frac{1}{3} t^2 = 1, \quad t = \sqrt{3}
\]

and orthogonality condition

\[
\int_0^1 p_{1t} p_{3t} dx = \int_0^1 \sqrt{3} (qx + u_t x^3) dx = 0
\]

gives the link between constants: \(u_t = -\frac{5}{3} q_t\), which plugging into

\[
\int_0^1 p_{3t}^2 dx = \int_0^1 \left( x q_t - \frac{5}{3} x^3 q_t \right)^2 dx
\]

results in \(q_t = \frac{3}{2} \sqrt{7}\), finally \(p_{3t} = -\frac{5}{2} \sqrt{7} (5x^2 - 3)\).

Substituting the result into \(T(0, x) = x^3 B(0) + C(0) x = \gamma_2 p_{1t} + \beta_2 p_{3t}\) gives two equations:

\[
B(0) + \frac{5}{2} \sqrt{7} \beta_2 = 0, \quad C(0) - \sqrt{3} \gamma_2 - \frac{3}{2} \sqrt{7} \beta_2 = 0
\]

which solving and projecting

\[
\beta_2 = \int_0^1 x \frac{3}{2} \sqrt{7} (5x^2 - 3) dx = -\frac{1}{8} \sqrt{7}, \quad \gamma_2 = -\int_0^1 x \sqrt{3} dx = -\frac{1}{2} \sqrt{3}
\]

yield boundary conditions for the coefficients for temperature expansion:

\[
B(0) = \frac{35}{16}, \quad C(0) = -\frac{45}{16}
\]

Ploting the temperature approximation at the level \(y = 0\), \(T(0, x) = x^3 B(0) + C(0) x = x^3 \frac{35}{16} - \frac{45}{16} x\) is given by the Fig.2.

Let us recall that \(B(0) = -\frac{1}{6} C''(0)\) (see eq. (25)), therefore \(C''(0) = -\frac{105}{\pi}\) we will consider as boundary condition for \(C(y)\).

The temperature gradient values \(dT/dx\) on the plate decrease when \(y\) grows. At the leading edge we pose the condition \(\partial T/\partial x|_{x=0} = 0\) because the plate lose
the contact with the fluid. It gives third boundary condition (28)

\[ C(L) = 0 \]  \hspace{1cm} (36)

6 Boundary conditions for velocity and temperature

The phenomenon of free convective heat transfer from isothermal vertical plate \((T = 0)\) imply that temperature gradient on the plate is negative \((C < 0)\) and decrease along \(y\) \((\partial C/\partial y < 0)\). It is also known that velocity profile has maximum at the distance \(x_m > 0\). The extrema for the curve is defined by derivative of \(W(y,x)\) as a function of \(x\). Hence the relation \(\frac{dW}{dx} = \gamma + 2\alpha x + 3\beta x^2 = 0\) indicates that for \(\alpha < 0\, , \beta > 0\) and \(\gamma > 0\) we have two extremal points

\[ x_m = -\frac{\alpha}{3\beta} - \frac{1}{3} \sqrt{\frac{\alpha^2}{9\beta^2} - \frac{\gamma}{3\beta}} \quad \text{and} \quad x_0(y) = -\frac{\alpha}{3\beta} + \frac{1}{3} \sqrt{\frac{\alpha^2}{9\beta^2} - \frac{\gamma}{3\beta}} \]  \hspace{1cm} (37)

if \(\frac{\alpha^2}{9\beta^2} - \frac{\gamma}{3\beta} > 0\). Notations are chosen to mark maximum position point as \(x_m\) while the minimum one is \(x_0(y) > x_m\).

In the exeptional case of \(\beta(y = L) = 0\), the expression simplifies

\[ x_{mL} = -\frac{\gamma}{2\alpha(L)} \]  \hspace{1cm} (38)

which is positive for \(\alpha < 0\). The second extremum do not exist now (see Fig.3). There is a possibility to choose the value \(W(y, x_0) = 0\) considering the \(x_0\) as a conditional boundary of the upward stream. We define hence \(x_L = 2x_{mL}\).
At the starting horizontal edge of the vertical plate the vertical velocity component of incoming flow (28) varies slow so we assume that
\[ C_1 = 0 \] (39)
hence
\[ W(y, x) = \gamma x + C_2 x^2 - \frac{G_r}{6} C(y) x^3. \] (40)
The extrema of the velocity profile (37) after account of (39) and (27a) is transformed as, for maximum: \[ x_m(y) = \frac{2C_2}{G_r C(y)} - \sqrt{\left(\frac{C_2^2}{G_r C(y)}\right)^2 + \frac{2\gamma}{G_r C(y)}}, \]
and minimum one: \[ x_0(y) = \frac{2C_2}{G_r C(y)} + \sqrt{\left(\frac{C_2^2}{G_r C(y)}\right)^2 + \frac{2\gamma}{G_r C(y)}}. \]
The following identity \[ x_0^2 = \frac{2}{G_r C(y)} (\gamma + 2x_0 C_2) \] holds for: \( \gamma + 2x_0 C_2 < 0. \)
Suppose there exists a level \( y = Y \) at which
\[ W(Y, x_0(Y)) = 0 \] (41)
where \( x_0(Y) \equiv x_{0Y} \) denotes the boundary layer thickness analog. The equation (41) is solved with respect to \( C(Y) \) that gives:
\[ C(Y) = -\frac{3}{2} \frac{C_2^2}{G_r} \] (42)
as function of the problem parameters. Then plugging (42) for the expression for the \( x_{0Y} \) yields
\[ x_{0Y} = -2 \frac{\gamma}{C_2} \] (43)
Let us return to the expression for the temperature (28) with neglecting the last term in temperature (the possibility of such assumption will be explained below) on the level \( Y \) and substitute (42) and (43) into it equalizing to the temperature of surrounding \( (T = -1) \).
\[ T(Y, x = x_{0Y}) = C(Y) x_{0Y} = -1, \] (44)
we have:
\[ C_2 = -\frac{G_r}{3}, \quad x_{0Y} = 6 \frac{\gamma}{G_r} = 6a, \quad C(Y) = -\frac{1}{6a} \] (45)
where:
\[ a = \frac{\gamma}{G_r}. \] (46)
From the equation (26) after plugging \( C_2 \) (45) and taking into account \( C_1 = 0 \) (39) we have
\[ \frac{1}{2} \frac{\partial^2 C(y)}{\partial y \partial y} - \Pr G_r \frac{\partial C(y)}{\partial y} = 0 \] (47)
The equation was studied recently [1] where the solution was given by:
\[ C(y) = A_0 + A_1 \exp[sy] + \exp[-\frac{sy}{2}](B_1 \cos[\frac{\sqrt{3}}{2} sy] + B_2 \sin[\frac{\sqrt{3}}{2} sy]), \] (48)
where

\[ s = \sqrt[3]{2 \Pr G_r} \]  

(49)

is expressed via \( Ra = G_r \Pr \). We have also boundary conditions:

\[
\begin{align*}
C(0) &= A_0 + A_1 + B_1 = -\frac{45}{16}, \\
C''(0) &= -6B(0) = \frac{1}{2} s^2 (2A_1 - B_1 - \sqrt{3}B_2) = -\frac{105}{8}, \\
C(L) &= A_0 + e^{-\frac{L}{2}L_0} (B_1 \cos \frac{1}{2} \sqrt{3}L_0 + B_2 \sin \frac{1}{2} \sqrt{3}L_0) + A_1 e^{L_0} = 0.
\end{align*}
\]

Solution of the system results in a rather big expression for \( A_1 \) as function of \( A_0 \) which we skip in the text, going to following approximtion. The explicit form of the equation (48) shows that the three last terms have exponential behaviour as function of \( sy \). It means that there are three different domains of the fluid flow structure. The first is the starting one where all terms are significant. The leading edge is characterized by two first terms and the medium domain is described by the only first one. We choose the parameter \( y = Y \) such that it belongs to that medium range. In such conditions

\[ A_0 = C(Y) = -\frac{1}{6a} \]  

(50)

where \( a = \frac{\gamma}{G_r} \).

Plugging \( A_0 \) in the form of (50) into the table of boundary conditions gives

\[
\begin{align*}
A_1 &= -\frac{45}{16} - B_1 + \frac{1}{6a}, \\
B_1 &= \frac{1}{6a} - \frac{1}{3} \sqrt{3}B_2 + \frac{45}{4} s^2 - \frac{15}{8}, \\
B_2 &= \left( e^{-\frac{L}{2}L_0} (-\frac{1}{18} + \frac{25}{18} \sqrt{3}a + \frac{16}{5} \cos \frac{1}{2} \sqrt{3}L_0) e^{-\frac{L}{2}L_0} (\frac{1}{18} + \frac{25}{18} \sqrt{3}a - \frac{45}{8}) \right) e^{-\frac{L}{2}L_0} (\sin \frac{1}{2} \sqrt{3}L_0 - \frac{1}{2} \sqrt{3} \cos \frac{1}{2} \sqrt{3}L_0) + \frac{1}{2} \sqrt{3} e^{L_0}.
\end{align*}
\]

Let us consider the natural approximation \( e^{-\frac{L}{2}L_0} \ll 1 \). After substitution of the expression for \( B_2 \) into \( B_1 \) and next \( B_1 \) into \( A_1 \) we have approximate formulas:

\[
\begin{align*}
A_0 &= -\frac{1}{6a}, \\
A_1 &= \frac{1}{6a}, \\
B_1 &= \frac{1}{6a} - \frac{45}{16}, \\
B_2 &\approx \frac{25 \sqrt{3} s^2}{4} - \frac{1}{18} \sqrt{3} a + \frac{15}{16} \sqrt{3} + \frac{1}{2 a \sqrt{3} e^{L_0}}.
\end{align*}
\]

It defines the expression for \( C(y) \) as the function of parameters \( s \) \((49)\), the plate height \( L \) and the new one \((46)\) \(a \). The velocity profile at the level \( Y \) is defined by (40) and the parameters values (45):

\[
W(Y, x) = \frac{1}{36} x G_r \frac{(6a - x)^2}{a}
\]  

(51)

7 Conservation laws application

The mass conservation equation (31) after substitution of \( C_1 = 0, 2x_L = 3a \) \((38)\), \( C(L) = 0, C_2 = -gG_r/3 \) and denoting \( \gamma = aG_r \) has the form:

\[
\frac{1}{48} G_r (12a - 7) (84a + 144a^2 + 1) = 0.
\]  

(52)
The only real solution of the equation (52) value that have physical sense is  
\( a = \frac{7}{12} \).

Now we can return to the energy conservation equation (34) plugging the boundary conditions for the domain restricted by the plate on interval \((0, Y)\). It simplifies the expression for the integral along the plate surface (heat transfer from the plate on this interval). Consequently we change \( x_L \) to \( x_0Y \) and neglect the integrant oscillations at vicinity of \( y = 0 \).

\[
\frac{1}{Pr} \int_0^Y C(y) dy - \frac{Gr}{36} C(Y)^2 x_0^5 + \frac{C_2}{4} C(Y) x_0^4 + \frac{\gamma}{3} C(Y) x_0^3 + \frac{\gamma}{2} + \frac{C_2}{24} C(0) = 0.
\]

(53)

We estimate the heat flux integral from the plate as

\[
\frac{1}{Pr} \int_0^Y C(y) dy \approx -\frac{1}{6} \frac{Y}{Pr a}
\]

(54)

and take into account the expressions for parameters \( C_2 = -\frac{Gr}{3}, x_0Y = 6a, C(Y) = -\frac{1}{6a} \) that yields:

\[
Ra = \frac{1}{6} a \left( \frac{L}{\frac{1}{2} a - \frac{5}{6} a^3 + \frac{7}{1152}} \right).
\]

As further considerations show, the value of \( Y \) may be chosen as close to the plate height \( L \).

\[
Ra = \frac{1}{6} a \left( \frac{L}{\frac{1}{2} a - \frac{5}{6} a^3 + \frac{7}{1152}} \right) = Gr \cdot Pr
\]

(55)

8 Numerics

Choosing \( L = 10 \) and plugging the values of the parameters \( a = \frac{7}{12}, Ra = 4.798 \cdot L \) and \( s = \sqrt{2Ra} = 4.5782 \) into the table of the function \( C(y) \) coefficients gives

\[
A_0 = -0.286, \\
A_1 = 3.679 \times 10^{-21}, \\
B_1 = -2.527, \\
B_2 \approx 2.18.
\]

Substitution of the table values into (48) we have the expression which allows to plot the function \( C(y) \). In the same approximation the typical velocity profile \( W(Y, x) \) at the the stability interval \((y \in [2, 9.5]) : W(Y, x) = \frac{1}{2\pi} xGr (x - \frac{7}{2})^2 \). Substitution of the Grashof number \( Gr = \frac{Ra}{Pr} = \frac{4.798 \cdot 10^5}{0.7} = 68.54 \) gives \( W(Y, x) = \frac{68.543}{2\pi} x (x - \frac{7}{2})^2 \) that is represented by the plot. In the same condition the temperature profile \( T(Y, x) \) is defined by the expression (28) and results in the plot. To understand the phenomenon it is useful to return to dimensional picture. As a main space scale it is chosen the parameter \( l \) which is connected with the Grashof number by (11) \( l = \sqrt{\frac{1}{k\Phi^2} Gr} \).
where $\Phi = T_w - T_\infty$. For the air example and the temperature $T_w = 40^\circ C$, $T_\infty = 20^\circ C$, $T_{av} = 30^\circ C$, $\Phi = 20K$, the viscosity coefficient $\nu = 16 \cdot 10^{-6} m^2/s$, the coefficient of thermal expansion $b = \frac{1}{1000}$ and for conditions of our model $(L = 10)$ $Gr = 68.54$ we estimate $l$ as: $l = \sqrt{\frac{1}{3} \frac{1}{100} (16 \cdot 10^{-6})^2 \cdot 68.54} = 2.985 \times 10^{-3} m \approx 3 mm$
9 Conclusions

First of all we would stress again that the model we present here have the engineering character of approximations, but include direct possibilities for a development by simple taking next terms of expansions into account. A modification of boundary conditions which would improve the transient regimes at both ends of the y-dependence is also possible.

Nevertheless in this simple modeling we observe some important characteristic features of real convection phenomenon as almost parallel streamlines and isotherms in the stability region (as, for example in visualizations of interferometric study from \[5\]). It follows from functional parameter \(C(y)\) behaviour inside the domain and small contribution of cubic therm in the expression for temperature \(28\).

Our explicit solution form and parameter values estimation allows to conclude that:

1. the streamlines and isotherms of the flow are almost parallel to the vertical heating plate surface in the domain of stability,
2. velocity values of the fluid flow at starting edge of the plate are nonzero,
3. the set of boundary conditions yields in the complete set of the solution parameter including the local Grashof number and hence, the characteristic linear dimension length \(l\) in normal to plate direction \(x\),
4. the results allow to described the natural heat transfer phenomenon for given fluid in thersms only the temperature difference \(\Phi\) and the plate height \(L\),
   which are novel in comparison with former theories.

10 References

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