EXISTENCE AND AXIAL SYMMETRY OF MINIMAL ACTION ODD SOLUTIONS FOR 2-D SCHRÖDINGER-NEWTON EQUATION

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Abstract. We consider the following 2-D Schrödinger-Newton equation
\[
\begin{aligned}
-\Delta u + u &= w|u|^{p-1}u \\
-\Delta w &= 2\pi|u|^p
\end{aligned}
\]
in $\mathbb{R}^2$

for $p \geq 2$. Using variational method with the Cerami compactness property, we prove the existence of minimal action odd solutions. Also by carefully applying the method of moving plane to a similar but more complex equation on the upper half space, we prove these solutions are in fact axially symmetric. Our results partially can be seen as the counterpart of results in paper [13] for the 2-D case, or the extension of the results [10] to the odd solution case.

Keywords: Logarithmic convolution potential, Cerami compactness, Method of moving plane, Schrödinger-Newton equation.

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1. Introduction

In this paper, we study the following Schrödinger-Newton equation in $\mathbb{R}^2$
\[
\begin{aligned}
-\Delta u + u &= w|u|^{p-1}u \\
-\Delta w &= 2\pi|u|^p
\end{aligned}
\]
in $\mathbb{R}^2$ for $p \geq 2$. It arises in many different physical models, see [24, 25]. Depending on these different models, it could be given different names: Choquard or Choquard-Pekar equation, Schrödinger-Newton equation, or stationary Hartree equation. Here we call it Schrödinger-Newton equation, based on the Penrose’s model for Newton gravitation coupled with the quantum physics. In 3-D case, this equation has been widely studied, where the fundamental solution for $-\Delta$ is $\frac{1}{|x|^2}$, a special Riesz potential of order 2. First Lieb [17] proved the the existence and uniqueness of radial positive ground state solutions for $p = 2$. Then Lions proved there are infinitely many radial solutions, see [19]. For general $p$ and other results, see [11, 2, 5, 13, 22, 26]. For the complete mathematical results, we strongly recommend the impressive survey [21] and the references therein, where the authors also listed many interesting open problems.

In 2-D case, the analysis for this equation is harder, because of the sign-changing property of the log function, which is the fundamental solution of $\Delta$ in $\mathbb{R}^2$. First Choquard, Stubby and Vuffray proved there is a unique radial ground state solution by an ODE method for $p = 2$, see [9]. Then Stubby established the variational framework and proved a stronger result using constraint minimization argument, see [28]. Based on this variational framework, Cingolani and Weth [10] discovered the energy functional or aciton functional ($p = 2$) satisfies the so-called Cerami compactness property, and used the minimax procedure to give the variational characterization of the ground state solution. They also show the symmetry of these solutions and other properties. Additionally they proved the existence of infinitely many solutions of which the energies go to infinity and have many different types of symmetry in terms of group $G$, see also [12]. Later Cao,
Dai and Zhang extended these results to the general $p \geq 2$ using the same method, see [6]. For the sharp decay and non-degeneracy, see [3].

In [13], the authors considered the existence of the minimal action odd solutions and minimal action nodal solutions in $\mathbb{R}^3$. From the results in [10, 6] for 2-D case, we know there indeed exist odd solutions and nodal solutions. So the odd solutions set and nodal solutions set are not empty. The natural questions for us are whether there is a minimal odd solution among all the odd solutions, and whether these minimal action odd solutions are axially symmetric. We will give these two questions a firmative answer. Our results can be seen as the counterpart of [13] for the 2-D case, or can be seen as the extension of [10, 6] to the odd solutions case.

We consider the energy functional or actional functional by

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^2} (|\nabla u|^2 + u^2) dx + \frac{1}{2p} \iint \log |x-y| |u|^p(x)|u|^p(y) dxdy$$

defined on the functions space

$$X = \left\{ u \in H^1(\mathbb{R}^2) : \int_{\mathbb{R}^2} \log (1 + |x|)|u|^p(x) dx < \infty \right\} := H^1 \cap L^p(d\mu),$$

where the Radon measure is $d\mu = \log (1 + |x|) dx$. Formally, the Schrödinger-Newton equation is the corresponding Euler-Lagrange equation for this energy functional. The properties of the actional functional and function space $X$ will be given below, see also [10, 6].

Now, we define the odd function space

$$X_{\text{odd}} := \left\{ u \in X \left| u(x_1, -x_2) = -u(x_1, x_2) \text{ for almost every } x = (x_1, x_2) \in \mathbb{R}^2 \right. \right\}$$

where $N_{\text{odd}}$ is not empty, since we can always choose $u$ with $\iint \log |x-y| |u|^p(x)|u|^p(y) dxdy < 0$, such that $\langle I'(tu), tu \rangle = 0$ for some $t > 0$. Also we define the odd ground state value by

$$c_{g,\text{odd}} := \inf_{u \neq 0} I(u)$$

We shall call $c_{g,\text{odd}}$ the minimal action value, and the corresponding solutions are the minimal action odd solutions, if they exist. The first minimax value is regularly defined on the function space $X_{\text{odd}}$ by

$$c_{\text{mm,odd}} := \inf_{u \neq 0} \sup_{t > 0} I(tu).$$

Also the mountain pass value is defined by

$$c_{\text{mp,odd}} := \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} I \circ \gamma(t),$$

where $\Gamma = \{ \gamma \in C([0,1], X_{\text{odd}}) : \gamma(0) = 0, I \circ \gamma(1) < 0 \}$. Our first result is the existence of minimal action odd solutions:
By the HLS inequality, we can bound $V$ and the corresponding functionals $p$ for the robust method, see [8, 11, 14, 16, 20, 29].

Assume Theorem 1.1.

Let introduce the bilinear form by convergence of Cerami sequence and the axial symmetry of minimal action odd solutions. We perpendicular to $\partial$.

Using these notations, we can rewrite the action functional in a compact form $\int \log(1 + 1|a+b|^p - |b|^p) \leq \epsilon|b|^p + C_\epsilon|a|^p$.

The proof can be seen in [18]. We will apply this simple $\epsilon$-inequality in proving the strong convergence of Cerami sequence and the axial symmetry of minimal action odd solutions. We introduce the bilinear form by

\[
B_1(f, g) = \iint \log(1 + |x - y|) f(x)g(y)dxdy,
\]

\[
B_2(f, g) = \iint \log(1 + \frac{1}{|x - y|}) f(x)g(y)dxdy,
\]

\[
B_0(f, g) = B_1(f, g) - B_2(f, g) = \iint \log(|x - y|) f(x)g(y)dxdy
\]

and the corresponding functionals

\[
V_1(u) = B_1(|u|^p, |u|^p) = \iint \log(1 + |x - y|)|u|^p(x)|u|^p(y)dxdy,
\]

\[
V_2(u) = B_2(|u|^p, |u|^p) = \iint \log(1 + \frac{1}{|x - y|})|u|^p(x)|u|^p(y)dxdy,
\]

\[
V_0(u) = B_0(|u|^p, |u|^p) = \iint \log(|x - y|)|u|^p(x)|u|^p(y)dxdy.
\]

By the HLS inequality, we can bound $V_2(u)$ by:

\[
|V_2(u)| \leq C||u|^p||_{L^{\frac{4}{3}}}^2 = C\|u\|^{2p}_{L^{\frac{4p}{4p - 1}}}.
\]

Using these notations, we can rewrite the action functional in a compact form

\[
I(u) = \frac{1}{2}\|u\|_{H^1}^2 + \frac{1}{2p}V_0(u)
\]

\[
= \frac{1}{2}\|u\|_{H^1}^2 + \frac{1}{2p} (V_1(u) - V_2(u))
\]
defined on the odd function space \( X_{\text{odd}} \) with the norm \( \|u\|_X = \|u\|_{H^1} + \|u\|_s \).

The next are the properties of the action functional and function space.

**Lemma 2.2.** (1) The function space \( X = H^1 \cap L^p(d\mu) \) is compactly embedding in \( L^s(\mathbb{R}^2) \) for all \( s \in [p, \infty) \);

(2) The functionals \( V_1, V_2, V_0 \) and \( I \) is \( C^1(X) \) : for each \( u, v \) in \( X \), \( i = 0, 1, 2 \), \( \langle V_i'(u), v \rangle = 2pB_i(|u|^p, |u|^{p-2}uv) \);

(3) \( V_1 \) is weakly lower semicontinuous on \( H^1(\mathbb{R}^2) \); \( I \) is weakly lower semicontinuous on \( X \) and is lower semicontinuous on \( H^1 \).

**Proof.** We only prove property (1). We have already known the embedding \( H^1(\mathbb{R}^2) \hookrightarrow L^r(\mathbb{R}^2)(dx) \) for \( 2 \leq r < \infty \) is locally compact by the Rellich-Kondrachov compactness theorem. Now by the Kolmogrov-M. Reiz- Frechet compactness criteria, see [4], we only need to check the uniformly integrability at infinity for \( (u_n) \subset X \) bounded. Notice for each \( \epsilon > 0 \), choose \( R \) large enough, then we have

\[
M \geq \int_{|x| \geq R} \log(1 + |x|)|u_n|^p dx \geq \int_{|x| \geq R} \log(1 + R)|u_n|^p dx,
\]

\[
\int_{|x| \geq R} |u_n|^p dx \leq \frac{M}{\log(1 + R)} \leq \epsilon, \quad \text{for all } R \text{ large enough},
\]

yielding the uniformly integrability. Hence the embedding \( X \hookrightarrow \hookrightarrow L^p \) is compact. By the Gagliardo-Nirenberg interpolation inequality, for all \( s \in [p, \infty) \), the embedding is also compact. \( \square \)

The properties of the solutions are listed in the following lemma.

**Lemma 2.3.** (1) If \( u \) is the critical point of the energy functional, then \( u \) is the weak solution of the following Euler-Lagrange equation:

\[-\Delta u + u + (\log |\cdot| \ast |u|^p)|u|^{p-2}u = 0.\]

(2) The potential function defined by \( w(x) := \int_{\mathbb{R}^2} \log |x - y||u|^p(y)dy \) is of class \( C^3 \), hence \(-\Delta w = 2\pi |u|^p \) classically. Moreover, we have \( w(x) - \log |x| \int_{\mathbb{R}^2} |u|^p \to 0 \), as \( x \to \infty \), and \(|\nabla w| \to 0 \) as \( x \to \infty \);

(3) \( u \) decay exponentially : for any \( \epsilon > 0 \), there is a \( C_\epsilon > 0 \), such that:

\[|u(x)| \leq C_\epsilon \exp^{-(1-\epsilon)|x|};\]

(4) \( u \) is \( W^{2,r}(\mathbb{R}^2) \), for every \( r \in (1, \infty) \), hence is the strong solution of the Euler-Lagrange equation, in fact \( u \in C^{2,\alpha}_{\text{loc}} \).

**Remark 2.4.** By Lemma 2.3 of property (2), if \( u \in H^1_0(\mathbb{R}^2_+) \cap L^p(d\mu) \subset X \), then we have

\[w(x) - \log |x| \int_{\mathbb{R}^2_+} |u|^p \to 0, \quad \text{as } x \to \infty.\]

Now for \( u \in X_{\text{odd}} = H^1_{\text{odd}} \cap L^p(d\mu) \), we have another asymptotics:

\[w(x) - \log |x| \int_{\mathbb{R}^2} |u|^p \to 0.\]
But by the odd symmetry, we have
\[
w(x) = \int_{\mathbb{R}^2} \log |x - y||u|^p(y)dy
\]
\[
= \frac{1}{2} \int_{\mathbb{R}^2} \left( \log \left[ (x_1 - y_1)^2 + (x_2 - y_2)^2 \right] \right) |u|^p(y)dy
\]
\[
= \frac{1}{2} \int_{\mathbb{R}^2_+} \left( \log \left( |x - y|^2 \right) \right) |u|^p(y)dy + \frac{1}{2} \int_{\mathbb{R}^2_+} \left( \log \left[ (x_1 - y_1)^2 + (x_2 + y_2)^2 \right] \right) |u|^p(y)dy.
\]
Combining these two asymptotics, we have
\[
\frac{1}{2} \int_{\mathbb{R}^2_+} \left( \log \left[ (x_1 - y_1)^2 + (x_2 + y_2)^2 \right] \right) |u|^p(y)dy - \log |x| \int_{\mathbb{R}^2_+} |u|^p \to 0, \quad \text{as } x \to \infty.
\]
Here, we view \( u \in X_{odd} \) defined on the upper halfspace. We will use this asymptotics in the proof of axial symmetry.

The following is the general Mountain Pass Lemma for Cerami sequence, see in [15].

**Lemma 2.5.** Assume \( X \) is Banach space, \( M \) is a metric space, \( M_0 \subset M \) is a closed subspace, \( \Gamma_0 \subset C(M_0; X) \). Define
\[
\Gamma := \{ \gamma \in C(M; X) : \gamma|_{M_0} \in \Gamma_0 \}.
\]
If \( I \in C^1(X; \mathbb{R}) \) satisfies
\[
\infty > c := \inf_{\gamma \in \Gamma} \sup_{t \in M} I(\gamma(t)) > a := \inf_{\gamma \in \Gamma_0} \sup_{t \in M_0} I(\gamma_0(t)),
\]
then for each \( \epsilon \in (0, \frac{c - a}{2}) \), \( \delta > 0, \gamma \in \Gamma \) with \( \sup_{t \in M} I(\gamma(t)) \leq c + \epsilon \), there exists a \( u \in X \) such that
\begin{enumerate}
  \item \( c - 2\epsilon \leq I(u) \leq c + 2\epsilon; \)
  \item \( \text{dist}(u, \gamma(M)) \leq \delta; \)
  \item \( \|I'(u)\|_{X'}(1 + \|u\|_X) \leq \frac{8\epsilon}{\delta}. \)
\end{enumerate}

3. **Proof of Theorem 1.1**

We use the idea of [10, 6]. First we verify the Cerami compactness property of the action functional on the closed subspace \( X_{odd} \). Then using the Mountain Pass Pass Lemma [2, 5], we can creat the Cerami sequence. Hence by the compactness, we get a critical point in \( X_{odd} \). According to the Palais’ principle of symmetric criticality, see [25], it’s a critical point in \( X \).

**Proposition 3.1.** Let \( (u_n) \) be a sequence in \( L^p(\mathbb{R}^2) \), s.t. \( u_n \xrightarrow{a.e.} u \in L^p(\mathbb{R}^2) \setminus \{0\} \). \( (v_n) \) be a bounded sequence in \( L^p \) s.t. \( \sup_{n} B_1(|u_n|^p, |v_n|^p) < \infty \). Then, there exists \( n_0 \in \mathbb{Z} \) and \( C > 0 \) s.t. \( \|v_n\|_* \leq C \) for \( n \geq n_0 \). Furthermore, if \( B_1(|u_n|^p, |v_n|^p) \to 0 \) and \( \|v_n\|_{L^p} \to 0 \), then \( \|v_n\|_* \to 0 \).

**Proposition 3.2.** Let \( (\tilde{u}_n) \) be a bounded sequence in \( X \) such that \( \tilde{u}_n \rightharpoonup u \) weakly and a.e. in \( X \). Then up to a subsequence \( B_1(|\tilde{u}_n|^p, |u|^p - 2u(\tilde{u}_n - u)) \to 0 \).

The two propositions can be seen in [10] for \( p = 2 \) and in [6] for \( p \geq 2 \). Based on this two propositions, we can verify the Cerami compactness property for the action functional.

**Lemma 3.3.** Let \( (u_n) \subset X_{odd} \) satisfied
\[
I(u_n) \to d > 0, \quad \|I'(u_n)\|_{X'_{odd}} (1 + \|u_n\|_X) \to 0, \quad \text{as } n \to \infty.
\]
Then up to a subsequence, there exist points \( (x_n) \subset \mathbb{Z}^2 \), such that
\[
u_n(\cdot - x_n) \to u \quad \text{strongly in } X_{odd}, \quad \text{as } n \to \infty,
\]
for some nonzero critical point \( u \in X_{odd} \) of \( I \).
Below we give a refined and rigorous proof for all $p \geq 2$ for this key lemma.

**Proof.** For clarity, We divide the proof into several steps.

**Step 1:** If $(u_n)$ satisfy condition (3.1), then $(u_n)$ is bounded in $H^1$. In fact, we have

$$o(1) = \langle I'(u_n), u_n \rangle = \|u_n\|_{H^1}^2 + V_0(u_n),$$

$$d \leftarrow I(u_n) = \frac{1}{2}\|u_n\|_{H^1}^2 + \frac{1}{2p}V_0(u),$$

then $d - \frac{1}{2p}o(1) = \left(\frac{1}{2} - \frac{1}{2p}\right)\|u_n\|_{H^1}^2$, so $(u_n)$ is bounded in $H^1$.

**Step 2:** We claim $(u_n)$ is non-vanishing:

$$\liminf_{n \to \infty} \sup_{x \in \mathbb{Z}^2} \int_{B_2(x)} u_n^2(y)dy > 0.$$  

If not, by the Lion’s vanishing lemma, see [27, 30], for each $s > 2$, we have $u_n \to 0$ in $L^s$. From

$$o(1) = \langle I'(u_n), u_n \rangle = \|u_n\|_{H^1}^2 + V_1(u_n) - V_2(u_n),$$

we get

$$\|u_n\|_{H^1}^2 + V_1(u_n) = o(1) + V_2(u_n).$$

Substituted into $I(u_n)$, yield

$$d \leftarrow I(u_n) = \frac{1}{2}\|u_n\|_{H^1}^2 + \frac{1}{2p}(V_1(u_n) - V_2(u_n)) \to 0,$$

absurd. Therefore, there exist points $(x_n) \in \mathbb{Z}^2$ such that $\inf_n \int_{B_2(x_n)} u_n^2(x - x_n)dx > 0$. Now we define the translation functions $\tilde{u}_n = u_n(x - x_n)$. Also $\tilde{u}_n \to u$ in $H^1$ for some $u \in H^1$. By the nonvanishing lemma and local compactness, $u \not\equiv 0$.

**Step 3:** $(\tilde{u}_n)$ is bounded in $L^p(\mu)$. In fact, from

$$V_1(\tilde{u}_n) = \langle I'(\tilde{u}_n), \tilde{u}_n \rangle - \|\tilde{u}_n\|_{H^1}^2 + V_2(\tilde{u}_n)$$

$$= \langle I'(u_n), u_n \rangle - \|u_n\|_{H^1}^2 + V_2(u_n)$$

$$= o(1) - \|u_n\|_{H^1}^2 + V_2(u_n),$$

we get $V_1(\tilde{u}_n)$ is bounded. By the Proposition 3.1 we get $\|\tilde{u}_n\|_{L^p}^p$ is bounded. Hence $(\tilde{u}_n)$ is bounded in $X$, which is compactly embedding in $L^s$ for all $s \geq p$.

**Step 4:** $\tilde{u}_n \to u$ strongly in $X_{odd}$. First, we claim that:

$$\langle I'(\tilde{u}_n), \tilde{u}_n - u \rangle \to 0 \text{ as } n \to \infty.$$ 

In fact, by the $\mathbb{Z}^2$-translation invariance, we have

$$\langle I'(\tilde{u}_n), \tilde{u}_n - u \rangle = \langle I'(u_n), u_n - u(x + x_n) \rangle \leq \|I'(u_n)\|_{X'_{odd}} \left(\|u_n\|_X + \|u(x + x_n)\|_X\right).$$

Now, we estimate the last two terms in the following way:

$$\|u_n\|_p^p - \log(1 + |x_n|)\|\tilde{u}_n\|_{L^p}^p = \int \log(1 + |x - x_n|)\tilde{u}_n^p dx - \int \log(1 + |x_n|)\tilde{u}_n^p dx$$

$$= \int_A \log \left(\frac{1 + |x - x_n|}{1 + |x_n|}\right)\tilde{u}_n^p + \int_B \log \left(\frac{1 + |x_n|}{1 + |x - x_n|}\right)\tilde{u}_n^p$$

$$= \int_A \log \left(\frac{1 + |x - x_n|}{1 + |x_n|}\right)\tilde{u}_n^p - \int_B \log \left(\frac{1 + |x|}{1 + |x - x_n|}\right)\tilde{u}_n^p,$$
where $A = \{ |x - x_n| \geq |x_n| \}$, $B = \{ |x - x_n| \leq |x_n| \}$. Then we choose $\delta \in (0, 1)$ fixed, set $D_1 = \{ |x - x_n| \leq \delta \} \cap B$, $D_2 = \{ \delta \leq |x - x_n| \leq |x_n| \}$, and each term is bounded by a constant independet of $n$:

$$
\int_{|x-x_n| \geq |x_n|} \log \left( \frac{1 + |x - x_n|}{1 + |x_n|} \right) \left| \tilde{u}_n \right|^p \leq \int_{A} \log \left( \frac{1 + |x|}{1 + |x_n|} \right) \left| \tilde{u}_n \right|^p
\leq \int_{A} (1 + |x|) \left| \tilde{u}_n \right|^p
\leq C,
$$

$$
\int_{\delta \leq |x-x_n| \leq |x_n|} \log \left( \frac{1 + |x_n|}{1 + |x - x_n|} \right) \left| \tilde{u}_n \right|^p \leq \int_{D_2} \log \left( 1 + \frac{|x|}{1 + \delta} \right) \left| \tilde{u}_n \right|^p
\leq \int_{D_2} (1 + |x|) \left| \tilde{u}_n \right|^p
\leq C,
$$

$$
\int_{\{ |x - x_n| \leq \delta \} \cap B} \log(1 + |x_n|) \left| \tilde{u}_n \right|^p \leq \int_{D_1} \log(1 + |x| + \delta) \left| \tilde{u}_n \right|^p
\leq \int_{D_1} (2(1 + |x|)^2) \left| \tilde{u}_n \right|^p
\leq 2 \int_{D_1} \left| \tilde{u}_n \right|^p + 2 \int_{D_1} \log(1 + |x|) \left| \tilde{u}_n \right|^p
\leq C,
$$

$$
\int_{\{ |x - x_n| \leq \delta \} \cap B} \log(1 + |x - x_n|) \left| \tilde{u}_n \right|^p \leq \left( \int_{D_1} \left( \log(1 + |x - x_n|) \right)^2 \right)^{\frac{1}{2}} \cdot \left( \int \left| \tilde{u}_n \right|^{2p} \right)^{\frac{1}{2}}
= \left( \int_{|y| \leq \delta} \left( \log(1 + |y|) \right)^2 \right)^{\frac{1}{2}} \cdot \left( \int \left| \tilde{u}_n \right|^{2p} \right)^{\frac{1}{2}}
\leq C_\delta.
$$

Hence the above estimates yield

$$
\left| \left\| u_n \right\|_p^p - \log(1 + |x_n|) \left\| \tilde{u}_n \right\|_{L^p}^p \right| \leq C,
$$

since $u \not\equiv 0$, we get

$$
\left| \left\| u_n \right\|_p^p - C_2 \log(1 + |x_n|) \right| \leq C_1.
$$
Also for $\|u(\cdot + x_n)\|_p^p$, we have

$$\|u(\cdot + x_n)\|_p^p = \int \log(1 + |x|)|u|^p(x + x_n)dx$$

$$= \int \log(1 + |x - x_n|)|u|^p$$

$$\leq \int \log(1 + |x|)|u|^p + \int \log(1 + |x_n|)|u|^p$$

$$\leq C_3 + C_4 \log(1 + |x_n|)$$

Combining the two estimates, we have

$$\left(\|u_n\|_X + \|u(\cdot + x_n)\|_X\right) \leq C\left(1 + \|u_n\|_X\right).$$

Then by the assumption (3.1) in Lemma 3.3, we have

$$\langle I'(u_n), \tilde{u}_n - u \rangle \to 0$$

as claimed. But on the other side, we get

$$o(1) = \langle I'(\tilde{u}_n), \tilde{u}_n - u \rangle$$

$$= \langle I'(\tilde{u}_n), \tilde{u}_n \rangle - \langle I'(\tilde{u}_n), u \rangle$$

$$= \|\tilde{u}_n\|_{H^1}^2 - \|u\|_{H^1}^2 + o(1) + \langle V_0'(\tilde{u}_n), \tilde{u}_n - u \rangle.$$

Estimating the each term yield

$$\langle V_0'(\tilde{u}_n), \tilde{u}_n - u \rangle \to 0$$

by the compact embedding of $X \hookrightarrow L^s$ and the Hardy-Littlewood-Sobolev Inequality. Now we estimate the $V_0'$:

$$\langle V_0'(\tilde{u}_n), \tilde{u}_n - u \rangle = B_1\left(\|\tilde{u}_n\|^p, \|\tilde{u}_n\|^{p-2}(\tilde{u}_n - u)\right)$$

$$= B_1\left(\|\tilde{u}_n\|^p, \|\tilde{u}_n\|^{p-2}\left((\tilde{u}_n - u)^2 + u(\tilde{u}_n - u)\right)\right)$$

$$= B_1\left(\|\tilde{u}_n\|^p, \|\tilde{u}_n\|^{p-2}\|\tilde{u}_n - u\|^2\right) + B_1\left(\|\tilde{u}_n\|^p, \|\tilde{u}_n\|^{p-2}u(\tilde{u}_n - u)\right).$$

But $B_1\left(\|\tilde{u}_n\|^p, \|\tilde{u}_n\|^{p-2}u(\tilde{u}_n - u)\right) \to 0$ by the Proposition 3.2. Let $v_n^p = \|\tilde{u}_n\|^{p-2}\|\tilde{u}_n - u\|^2$, then

$$B_1\left(\|\tilde{u}_n\|^p, \|\tilde{u}_n\|^{p-2}\|\tilde{u}_n - u\|^2\right) = B_1\left(\|\tilde{u}_n\|^p, v_n^p\right) \geq 0$$

and we get

$$o(1) = \langle I'(\tilde{u}_n), \tilde{u}_n - u \rangle$$

$$= o(1) + \|\tilde{u}_n\|_{H^1}^2 - \|u\|_{H^1}^2 + B_1\left(\|\tilde{u}_n\|^p, v_n^p\right)$$

$$\geq o(1) + \|\tilde{u}_n\|_{H^1}^2 - \|u\|_{H^1}^2,$$

which implies $\|\tilde{u}_n\|_{H^1}^2 \to \|u\|_{H^1}^2$ and $B_1\left(\|\tilde{u}_n\|^p, v_n^p\right) \to 0$. So we get the strong convergence in $H^1: \tilde{u}_n \to u$. Again, by the compact embedding of $X \hookrightarrow L^p$, we get $v_n^p = \|\tilde{u}_n\|^{p-2}\|\tilde{u}_n - u\|^2 \to 0$.
in $L^1$. By the Proposition 3.1, we get $\|v_n\|_s \to 0$.

Now, applying the Lemma 2.1, we have

\[
o(1) = \int \log(1 + |x|)|\tilde{u}_n|^p - |u|^2 = \int \left(\frac{|\tilde{u}_n - u|^p}{2} - |\tilde{u}_n - u|^2\right) d\mu \\
= \int |\tilde{u}_n - u|^p d\mu + \int \left(\frac{|\tilde{u}_n|^p}{2} - |\tilde{u}_n - u|^2\right) d\mu \\
\geq \int |\tilde{u}_n - u|^p d\mu - \epsilon \int |\tilde{u}_n|^p d\mu - C_\epsilon \int |u|^p - |\tilde{u}_n - u|^2 d\mu \\
\geq \int |\tilde{u}_n - u|^p d\mu - M\epsilon - C_\epsilon o(1),
\]

So we get $\|\tilde{u}_n - u\|_s \to 0$. Combining with the $H^1$ convergence, we get the strong convergence in $X$:

\[\|\tilde{u}_n - u\|_X \to 0.\]

**Step 5:** We prove $u$ is the critical point: $I'(u) = 0$. This is easily checked.

Let $v \in X_{odd}$, as we have already shown

\[\|v(\cdot + x_n)\|_s \leq C(1 + \log(1 + |x_n|)) \leq C(1 + \|u_n\|_s).\]

By this and $u \neq 0$, we have $\|v(\cdot + x_n)\|_X \leq C(1 + \|u_n\|_X)$. Notice that $\langle I'(u), v \rangle = \lim_{n \to \infty} \langle I'(\tilde{u}_n), v \rangle$. But we also have

\[\left|\langle I'(\tilde{u}_n), v \rangle\right| = \left|\langle I'(u_n), v(\cdot + x_n) \rangle\right| \leq \|I'(u_n)\|_{X'_{odd}} \|v(\cdot + x_n)\|_X \leq C\|I'(u_n)\|_{X'_{odd}} \left(1 + \|u_n\|_X\right) \to 0\]

by the assumption. Hence $\langle I'(u), v \rangle = 0$. And we finish the proof. \qed

**Proof of Theorem 1.1.** (1): $c_{mp,odd} \geq c_{mp} > 0$ obviously.

(2): First we use the Mountain Pass Lemma 2.5 to construct the Cerami sequence $(u_n) \subset X_{odd}$, then applying the Cerami compactness property of Lemma 3.3 for the action functional $I(u)$, we can extract a subsequence converges to a nonzero critical point $u$, and $I(u) = c_{mp,odd}$. By the Palais’ principle of symmetric criticality, $u$ is a critical point in $X$, and satisfies the corresponding properties of Lemma 2.3.

(3): First we notice $c_{g,odd} \geq c_{odd} = c_{mm,odd} \geq c_{mp,odd}$, but also $c_{g,odd} \leq c_{mp,odd}$, so $c_{g,odd} = c_{odd} = c_{mm,odd} = c_{mp,odd}$. The equality $c_{odd} = c_{mm,odd}$ is by the monotonicity of $I(tu)$ for $t$, see [?].

(4): This is obvious, since all the ground state solutions have constant sign, see [?]. And it can not be zero on the $\{x = \{x_1, x_2\} : x_2 = 0\}$. \qed

4. PROOF OF THEOREM 1.2

In this section, we prove all the minimal action odd solution are axially symmetric and $\frac{\partial u}{\partial x_1} < 0$ along the ray starting from the axis in $x_1$ direction. To prove this, we first reformulate the minimal action odd solution problem into a ground state problem in the upper half plane for some similar but more complex equation. Then we will carefully apply the method of moving plane to this equation to derive the axial symmetry.
Proposition 4.1. For every $v \in X_{odd}(\mathbb{R}^2)$, we have

$$I(v) = \tilde{I}(v|_{\mathbb{R}^2_+}),$$

where the new action functional $\tilde{I} : \tilde{X} := H_0^1(\mathbb{R}^2_+) \cap L^p(d\mu)|_{\mathbb{R}^2_+} \to \mathbb{R}$ is defined on the upper halfplane by

$$\tilde{I}(v) = \int_{\mathbb{R}^2_+} (|\nabla v|^2 + |v|^2)dx + \frac{1}{2p} \left( 2 \int_{\mathbb{R}^2_+} \int_{\mathbb{R}^2_+} \log |x-y||v|^p(x)|v|^p(y)dxdy + \int_{\mathbb{R}^2_+} \int_{\mathbb{R}^2_+} \log \left( (x_1-y_1)^2 + (x_2+y_2)^2 \right) |v|^p(x)|v|^p(y)dxdy \right).$$

In particular, if $u$ is the minimal action odd solution, then $u \in \tilde{X}$ with

$$\tilde{X} := \left\{ w \in \tilde{X} : w \neq 0, \langle \tilde{I}(w), w \rangle = 0 \right\},$$

and we have

$$\tilde{I}(u) = \inf_{\tilde{X}} \tilde{I}(w).$$

Proof. The proofs are direct computations and use the fact $u \in X_{odd}$ if and only if $u|_{\mathbb{R}^2_+} \in \tilde{X}$. □

From now on, we will freely view $u \in X_{odd}$ or $u \in \tilde{X}(\mathbb{R}^2_+)$. Note that the existence of minimum problem for $\tilde{I}(u) = \inf_{\tilde{X}} \tilde{I}(w)$ have already been proved by the Theorem 1. Also the minimal odd solution $u$ satisfies the new Euler-Lagrange equation:

$$-\Delta u + u + \frac{1}{2}H|u|^{p-2}u = 0 \quad \text{in } \mathbb{R}^2_+,$$

where $H = H_1 + H_2$ is defined by

$$H_1(x) := 2 \int_{\mathbb{R}^2_+} \left( \log |x-y||u|^p(y)dy, \right.$$

$$H_2(x) := \int_{\mathbb{R}^2_+} \left[ \log \left( (x_1-y_1)^2 + (x_2+y_2)^2 \right) |u|^p(y)dy \right].$$

Recall that we have shown in the Remark 2.4 that

$$\frac{1}{2}H_2(x) - \log |x| \int_{\mathbb{R}^2_+} |u|^p \to 0, \quad \text{as } x \to \infty.$$

From this, we can see $H_2$ is at most log growth, as $H_1$ does. But we need to give an explicit boundedness in terms of $\|u\|_{\tilde{X}}$ to show $H_2$ is well-defined on the $\tilde{X}$.

Proposition 4.2. $H_2(x) = F(x) - G(x)$, where $F, G$ are nonnegative functions bounded by

$$F(x) \leq C \left( \int_{\mathbb{R}^2_+} |u|^pdy + \int_{\mathbb{R}^2_+} |u|^p d\mu + \log(1 + |x|) \int_{\mathbb{R}^2_+} |u|^p d\mu \right),$$

$$G(x) \leq C \int_{\mathbb{R}^2_+} \frac{1}{|x-y|} |u|^p(y)dy.$$

Proof. This is a regular computation. First we define spherical cap over the upper half plane:

$$\Omega_1(x) := \left\{ y = (y_1, y_2) \in \mathbb{R}^2_+ : |x_1 - y_1|^2 + |x_2 + y_2|^2 \leq 1 \right\}, \quad \Omega_1^c(x) := \mathbb{R}^2_+ \backslash \Omega_1(x).$$
Then we have
\[ H_2(x) := \left( \int_{\Omega^+_1(x)} + \int_{\Omega^-_1(x)} \right) \left[ \log \left( (x_1 - y_1)^2 + (x_2 + y_2)^2 \right) \right] |u|^p(y) dy \]
\[ = F(x) - G(x). \]

We estimate \( F(x) \) and \( G(x) \) in the following way:
\[
F(x) = \int_{\Omega^+_1(x)} \left[ \log \left( (x_1 - y_1)^2 + (x_2 + y_2)^2 \right) \right] |u|^p(y) dy \\
\leq \int_{\Omega^+_1(x)} \left[ \log \left( (1 + (x_1 - y_1)^2)(1 + (x_2 + y_2)^2) \right) \right] |u|^p(y) dy \\
\leq \int_{\Omega^+_1(x)} \left[ \log \left( 1 + (x_1 - y_1)^2 \right) \right] |u|^p(y) dy + \int_{\Omega^+_1(x)} \left[ \log \left( 1 + (x_2 + y_2)^2 \right) \right] |u|^p(y) dy \\
\leq \int_{\Omega^+_1(x)} \left[ \log 2(1 + (x_1 - y_1)^2) \right] |u|^p(y) dy + \int_{\Omega^+_1(x)} \left[ \log 2(1 + (x_2 + y_2)^2) \right] |u|^p(y) dy \\
\leq C \int_{\Omega^+_1(x)} |u|^p(y) dy + C \int_{\Omega^+_1(x)} \left[ \log \left( (1 + |x|) \right) \right] |u|^p(y) dy \\
\quad + C \int_{\Omega^+_1(x)} \left[ \log \left( (1 + |x|) \right) \right] |u|^p(y) dy \\
\leq C \left( \int_{\mathbb{R}^2_+} |u|^p dy + \int_{\mathbb{R}^2_+} |u|^p d\mu + \log(1 + |x|) \int_{\mathbb{R}^2_+} |u|^p d\mu \right).
\]

For \( G(x) \) we have
\[
G(x) = \int_{\Omega^-_1(x)} \left( \frac{1}{(x_1 - y_1)^2 + (x_2 + y_2)^2} \right) |u|^p(y) dy \\
\leq \int_{\Omega^-_1(x)} \left( \frac{1}{(x_1 - y_1)^2 + (x_2 - y_2)^2} \right) |u|^p(y) dy \\
\leq C \int_{\mathbb{R}^2_+} \frac{1}{|x-y|} |u|^p(y) dy.
\]

This is done. \( \square \)

Now we can prove

**Proposition 4.3.** If \( u \) is the minimal odd solution, then \( u > 0 \) or \( u < 0 \) in \( \mathbb{R}^2_+ \).

**Proof.** By the characterization of \( \widetilde{I}(u) = \inf_{\mathcal{M}} \widetilde{I}(w) \) for \( u \neq 0 \), we see \(|u| \) is also the minimum for the new action functional \( \widetilde{I} \), and satisfies the new Euler-Lagrange equation (4.1). Applying the maximum principle of Serrin, \(|u| > 0 \). So \( u \) has constant sign in the upper halfplane \( \mathbb{R}^2_+ \). \( \square \)

Based on the semilinear elliptic equation (4.1) and the positivity of \( u \) in \( \mathbb{R}^2_+ \), we will use the method of moving plane carefully to deduce the symmetry property of the solutions. First we fix some solution \( u \) and all the constants below will depend on this solution \( u \), but independent of the moving plane \( T_\lambda \). To carry out the method of moving plane, we define \( T_\lambda := \{ x = (x_1, x_2) \in \mathbb{R}^2_+: x_1 = \lambda \} \), where we will move \( T_\lambda \) from \( \lambda = -\infty \) to some limiting position. Define \( \Sigma_\lambda \) is the left part of \( T_\lambda : \Sigma_\lambda := \{ x = (x_1, x_2) \in \mathbb{R}^2_+: x_1 < \lambda \} \). Let \( x^\lambda \) be the reflection point with respect to \( T_\lambda \) for the point \( x \in \Sigma_\lambda: x^\lambda = (2\lambda - x_1, x_2) = (x^\lambda_1, x^\lambda_2) \). We will compare the values of \( u \) at the points \( x^\lambda \) and \( x \). For this, we let \( u_\lambda(x) := u(x^\lambda) \), \( w_\lambda := u_\lambda - u \), and \( L_\lambda := H_1(x^\lambda) - H_1(x) \), \( M_\lambda(x) := H_2(x^\lambda) - H_2(x) \). We first need the integral representation of \( L_\lambda \) and \( M_\lambda \).
Proposition 4.4. For $x \in \Sigma_\lambda$, we have
\[
L_\lambda(x) = 2 \int_{\Sigma_\lambda} \left( \log \frac{|x - y|}{|x - y^\lambda|} \right) \left( |u_\lambda|^p - |u|^p \right) dy,
\]
\[
M_\lambda(x) = \int_{\Sigma_\lambda} \left( \log \frac{|x_1 - y_1|^2 + |x_2 + y_2|^2}{|x_1 - y_1^\lambda|^2 + |x_2 + y_2|^2} \right) \left( |u_\lambda|^p - |u|^p \right) dy.
\]

Proof. We check it directly.
\[
H_1(x) = 2 \int_{R^2_+} \left( \log |x - y| \right) |u|^p(y) dy
= 2 \int_{\Sigma_\lambda} \left( \log |x - y| \right) |u|^p(y) dy + 2 \int_{R^2_+ \setminus \Sigma_\lambda} \left( \log |x - y| \right) |u|^p(y) dy
= 2 \int_{\Sigma_\lambda} \left( \log |x - y| \right) |u|^p(y) dy + 2 \int_{\Sigma_\lambda} \left( \log |x - y^\lambda| \right) |u_\lambda|^p(y) dy
\]
\[
H_2(x) := \int_{R^2_+} \left[ \log ((x_1 - y_1)^2 + (x_2 + y_2)^2) \right] |u|^p(y) dy;
\]
\[
= \int_{\Sigma_\lambda} \left[ \log ((x_1 - y_1)^2 + (x_2 + y_2)^2) \right] |u|^p(y) dy + \int_{R^2_+ \setminus \Sigma_\lambda} \left[ \log ((x_1 - y_1)^2 + (x_2 + y_2)^2) \right] |u|^p(y) dy
= \int_{\Sigma_\lambda} \left[ \log ((x_1 - y_1)^2 + (x_2 + y_2)^2) \right] |u|^p(y) dy + \int_{\Sigma_\lambda} \left[ \log ((x_1 - y_1^\lambda)^2 + (x_2 + y_2)^2) \right] |u_\lambda|^p(y) dy.
\]
Substituted into $L_\lambda := H_1(x^\lambda) - H_1(x)$, $M_\lambda(x) := H_2(x^\lambda) - H_2(x)$, yield the integral representations.

Proof of Theorem 1.2. By the Euler-Lagrange equation (4.1), we know $w_\lambda = u_\lambda - u$ satisfies the equation
\[
-\Delta w_\lambda + w_\lambda + \frac{1}{2} L_\lambda |u_\lambda|^{p-2} u_\lambda + \frac{1}{2} M_\lambda |u_\lambda|^{p-2} u_\lambda + \frac{1}{2} (p - 1) H(x) |\psi_\lambda|^{p-2} w_\lambda = 0,
\]
where $\psi_\lambda$ between $u_\lambda$ and $u$. We define the negative part $\Sigma_\lambda^-$ of $w_\lambda$ in $\Sigma_\lambda$ by
\[
\Sigma_\lambda^- := \left\{ x \in \Sigma_\lambda : w_\lambda(x) = u_\lambda(x) - u(x) < 0 \right\}.
\]
Our aim is to show this set is empty $\Sigma_\lambda = \emptyset$ to give $w_\lambda = u_\lambda - u \geq 0$, until $T_\lambda$ arrive at some limiting position $\lambda_0$, which we will have $w_{\lambda_0}(x) = u_{\lambda_0}(x) - u(x) = 0$, the desired symmetry property. We divide this process of moving plane into two steps.

Step 1: Start moving the plane from $\lambda = -\infty$.
We multiply the equation (4.2) by $w_\lambda$, and integrate over $\Sigma_\lambda^-$. Notice that on the set $\Sigma_\lambda^-$, $0 < u_\lambda \leq \psi_\lambda \leq u$. We have
\[
\int_{\Sigma_\lambda^-} |\nabla w_\lambda|^2 + \int_{\Sigma_\lambda^-} |w_\lambda^-|^2 = -\frac{1}{2} \int_{\Sigma_\lambda^-} L_\lambda |u_\lambda|^{p-2} u_\lambda w_\lambda - \frac{1}{2} \int_{\Sigma_\lambda^-} M_\lambda |u_\lambda|^{p-2} u_\lambda w_\lambda
- \int_{\Sigma_\lambda} \frac{1}{2} (p - 1) H(x) |\psi_\lambda|^{p-2} (w_\lambda)^2.
\]
Since \( H(x) \to +\infty \), we choose \( \lambda \) negative enough, such that \( \int_{\Sigma_\lambda^-} H(x)|\psi_\lambda|^{p-2}(w_\lambda^-)^2 \geq 0 \). Then we get

\[
\int_{\Sigma_\lambda^-} |\nabla w_\lambda^-|^2 + \int_{\Sigma_\lambda^-} |w_\lambda^-|^2 \leq \frac{1}{2} \int_{\Sigma_\lambda^-} L_\lambda |u_\lambda|^{p-2}u_\lambda w_\lambda^- + \frac{1}{2} \int_{\Sigma_\lambda^-} M_\lambda |u_\lambda|^{p-2}u_\lambda w_\lambda^- \\
\leq \frac{1}{2} \int_{\Sigma_\lambda^-} L_\lambda^+ |u_\lambda|^{p-2}u_\lambda w_\lambda^- + \frac{1}{2} \int_{\Sigma_\lambda^-} M_\lambda^+ |u_\lambda|^{p-2}u_\lambda w_\lambda^-.
\]  

(4.3)

Now we estimate \( L_\lambda^+ \) and \( M_\lambda^+ \) separately using the integral representations of them. For \( x \in \Sigma_\lambda^- \), we have

\[
L_\lambda(x) = 2 \int_{\Sigma_\lambda^-} (\log \frac{|x-y|}{|x-y^\lambda|}) \left( |u_\lambda|^p - |u|^p \right) dy \\
= 2 \int_{\Sigma_\lambda^+} (\log \frac{|x-y|}{|x-y^\lambda|}) \left( |u_\lambda|^p - |u|^p \right) dy \\
= 2 \int_{\Sigma_\lambda^-} (\log \frac{|x-y^\lambda|}{|x-y|}) \left( |u|^p - |u_\lambda|^p \right) dy \\
\leq 2 \int_{\Sigma_\lambda^-} \left( \log (1 + \frac{|y^\lambda - y|}{|x-y|}) \right) \left( |u|^p - |u_\lambda|^p \right) dy \\
\leq C \int_{\Sigma_\lambda^-} \left( \log (1 + \frac{2|\lambda - y|}{|x-y|}) \right) |\phi_\lambda|^{p-2} \phi_\lambda w_\lambda^- dy \\
\leq C \int_{\Sigma_\lambda^-} \frac{1}{|x-y|} (\lambda - y_1) |\phi_\lambda|^{p-2} w_\lambda^- dy \\
\leq C \int_{\Sigma_\lambda^-} \frac{1}{|x-y|} (\lambda - y_1) |u|^p w_\lambda^- dy.
\]

So by the Hardy-Littlewood-Sobolev inequality, we have

\[
\|L_\lambda^+\|_{L^4(\Sigma_\lambda^-)} \leq C\|(\lambda - y_1)|u|^{p-1}w_\lambda^-\|_{L^4(\Sigma_\lambda^-)} \\
\leq C\|(\lambda - y_1)|u|^{p-1}\|_{L^4(\Sigma_\lambda^-)}\|w_\lambda^-\|_{L^2(\Sigma_\lambda^-)}.
\]

Hence

\[
\int_{\Sigma_\lambda^-} L_\lambda^- |u_\lambda|^{p-2}u_\lambda w_\lambda^- \leq C\|L_\lambda^+\|_{L^4(\Sigma_\lambda^-)}\|u_\lambda|^{p-1}\|_{L^4(\Sigma_\lambda^-)}\|w_\lambda^-\|_{L^2(\Sigma_\lambda^-)} \\
\leq C\|(\lambda - y_1)|u|^{p-1}\|_{L^4(\Sigma_\lambda^-)}\|u|^{p-1}\|_{L^4(\Sigma_\lambda^-)}\|w_\lambda^-\|_{L^2(\Sigma_\lambda^-)}^2.
\]

Since \( u \) is exponential decay, we have \( \|(\lambda - y_1)|u|^{p-1}\|_{L^4(\Sigma_\lambda^-)} \to 0 \) as \( \lambda \to -\infty \), also for \( \|u|^{p-1}\|_{L^4(\Sigma_\lambda^-)} \). Choose \( \lambda \) negative enough again, then we have

\[
\int_{\Sigma_\lambda^-} L_\lambda^- |u_\lambda|^{p-2}u_\lambda w_\lambda^- \leq \frac{1}{8}\|w_\lambda^-\|_{L^2(\Sigma_\lambda^-)}^2.
\]
We estimate the $M^+_\lambda$ in the following way.

$$M_\lambda = 2 \int_{\Sigma_\lambda} \left( \log \frac{|x_1 - y_1|^2 + |x_2 + y_2|^2}{|x_1 - y_1|^2 + |x_2 + y_2|^2} \right) \left( |u_\lambda|^p - |u|^p \right) dy$$

$$= 2 \int_{\Sigma_\lambda} \left( \log \frac{|x_1 - y_1|^2 + |x_2 + y_2|^2}{|x_1 - y_1|^2 + |x_2 + y_2|^2} \right) \left( |u_\lambda|^p - |u|^p \right) dy$$

$$+ 2 \int_{\Sigma_\lambda^+} \left( \log \frac{|x_1 - y_1|^2 + |x_2 + y_2|^2}{|x_1 - y_1|^2 + |x_2 + y_2|^2} \right) \left( |u_\lambda|^p - |u|^p \right) dy;$$

$$M^+_\lambda \leq 2 \int_{\Sigma_\lambda^+} \left( \log \frac{|x_1 - y_1|^2 + |x_2 + y_2|^2}{|x_1 - y_1|^2 + |x_2 + y_2|^2} \right) \left( |u_\lambda|^p - |u|^p \right) dy$$

$$= 2 \int_{\Sigma_\lambda^+} \left( \log \frac{|x_1 - y_1|^2 + |x_2 + y_2|^2}{|x_1 - y_1|^2 + |x_2 + y_2|^2} \right) \left( |u|^p - |u_\lambda|^p \right) dy$$

$$\leq C \int_{\Sigma_\lambda^+} \left( \log \frac{|x_1 - y_1|^2 + |x_2 + y_2|^2}{|x_1 - y_1|^2 + |x_2 + y_2|^2} \right) \left( |\eta_\lambda|^{p-2} \eta_\lambda w^-_\lambda \right) dy$$

$$\leq C \int_{\Sigma_\lambda^+} \left( \log (1 + \frac{|x_1 - y_1|^2 - |x_1 - y_1|^2}{|x_1 - y_1|^2 + |x_2 + y_2|^2}) \right) \left( |\eta_\lambda|^{p-2} \eta_\lambda w^-_\lambda \right) dy$$

$$\leq C \int_{\Sigma_\lambda^+} \left( \log (1 + \frac{|x_1 - y_1|^2 - |x_1 - y_1|^2}{|x - y|^2}) \right) \left( |\eta_\lambda|^{p-2} \eta_\lambda w^-_\lambda \right) dy$$

$$\leq C \int_{\Sigma_\lambda^+} \left( \log (1 + \frac{\sigma |x_1 - y_1|^2 + C_\sigma |y_1^\lambda - y_1|^2}{|x - y|^2}) \right) \left( |\eta_\lambda|^{p-2} \eta_\lambda w^-_\lambda \right) dy$$

$$\leq C \int_{\Sigma_\lambda^+} \left( \log (1 + \frac{\sigma |x_1 - y_1|^2 + C_\sigma |y_1^\lambda - y_1|^2}{|x - y|^2}) \right) \left( |\eta_\lambda|^{p-2} \eta_\lambda w^-_\lambda \right) dy$$

So we have the estimates for the second term in (4.3):

$$\int_{\Sigma_\lambda^+} M^+_\lambda |u_\lambda|^{p-2} u_\lambda w^-_\lambda \leq C \sigma^{\frac{1}{2}} \|u|^{p-1}\|_{L^2(\Sigma_\lambda^+)} \|w^-_\lambda\|_{L^2(\Sigma_\lambda)} \int_{\Sigma_\lambda^+} |u|^{p-1} w^-_\lambda dy$$

$$+ C_\sigma \int_{\Sigma_\lambda^-} N_\lambda |u|^{p-1} w^-_\lambda dy$$

$$\leq C \sigma^{\frac{1}{2}} \|u|^{p-1}\|_{L^2(\Sigma_\lambda^-)} \|w^-_\lambda\|_{L^2(\Sigma_\lambda)}$$

$$+ C_\sigma \|N_\lambda\|_{L^4(\Sigma_\lambda)} \|u|^{p-1}\|_{L^4(\Sigma_\lambda)} \|w^-_\lambda\|_{L^2(\Sigma_\lambda)}$$

$$\leq C \sigma^{\frac{1}{2}} \|u|^{p-1}\|_{L^2(\Sigma_\lambda^-)} \|w^-_\lambda\|_{L^2(\Sigma_\lambda)}$$

$$+ C_\sigma \|N_\lambda\|_{L^4(\Sigma_\lambda)} \|u|^{p-1}\|_{L^4(\Sigma_\lambda)} \|w^-_\lambda\|_{L^2(\Sigma_\lambda)}.$$
Now again taking $\sigma \to 0$, $\lambda \to -\infty$, we get

$$\int_{\Sigma_{\lambda}} M_{\lambda}^+ |u_{\lambda}|^{p-2} u_{\lambda} w_{\lambda}^- \leq \frac{1}{8} \|w_{\lambda}^-\|_{L^2(\Sigma_{\lambda})}^2.$$  

Combining these two estimates, we arrive at

$$\int_{\Sigma_{\lambda}} |\nabla w_{\lambda}^-|^2 + \int_{\Sigma_{\lambda}} |w_{\lambda}^-|^2 \leq \frac{1}{4} \|w_{\lambda}^-\|_{L^2(\Sigma_{\lambda})}^2.$$  

It follows $\int_{\Sigma_{\lambda}} |w_{\lambda}^-|^2 = 0$. Then the set $\Sigma_{\lambda}$ is of measure zero: $\mathcal{L}^n(\Sigma_{\lambda}) = 0$. Since if it has positive measure, say $\mathcal{L}^n(\Sigma_{\lambda}) \geq 2\delta$, then we can choose a compact subset $K \subset \Sigma_{\lambda}$ such that $\mathcal{L}^n(K) \geq \delta$. Notice the definition of the set $\Sigma_{\lambda}$. We get $w_{\lambda}^-$ is positive in $\Sigma_{\lambda}$, hence has a positive lower bound on $K$ by continuity, say $w_{\lambda} \geq \kappa > 0$. Then $\int_{\Sigma_{\lambda}} |w_{\lambda}^-|^2 \geq \kappa^2 \delta > 0$. Then by the continuity of $w_{\lambda}^-$, we get $\Sigma_{\lambda} = \emptyset$ from $\mathcal{L}^n(\Sigma_{\lambda}) = 0$. In fact, we can use $\int_{\Sigma_{\lambda}} |\nabla w_{\lambda}^-|^2 = 0$ to derive $w_{\lambda}^- = 0$. Anyway we have $\Sigma_{\lambda} = \emptyset$ and $w_{\lambda} = u_{\lambda} - u \geq 0$ for $\lambda$ negative enough.

**Step 2:** Move the plane to the limiting position.

Define $\lambda_0 := \sup \{\lambda | w_{\mu} \geq 0 \text{ for all } \mu \leq \lambda\}$. Then by the same argument as in Step 1 from the right direction $x_1 = +\infty$, we see $\lambda_0 < +\infty$. Now, we prove $w_{\lambda_0} = u_{\lambda_0} - u = 0$ to get the axial symmetry. We show this by contradiction. If not, we will prove there exists an $\epsilon > 0$ small enough, such that for all $\lambda \in (\lambda_0, \lambda_0 + \epsilon)$, we still have have $w_{\lambda} \geq 0$, which will be contradicted with definition of $\lambda_0$.

Suppose now $w_{\lambda_0} \neq 0$. Then $w_{\lambda_0} \geq 0$ and $w_{\lambda_0}(x_0) > 0$ for some $x_0 \in \Sigma_{\lambda_0}$. By the integral representation of $L_{\lambda_0}$ and $M_{\lambda_0}$, we see $L_{\lambda_0} < 0$, $M_{\lambda_0} < 0$ strictly. Then by the Euler-Lagrange equation of $w_{\lambda_0}$, we have

$$-\Delta w_{\lambda} + w_{\lambda} + \frac{1}{2}(p-1)H(x)|\psi_{\lambda}|^{p-2}w_{\lambda} = \frac{1}{2} L_{\lambda} |u_{\lambda}|^{p-2}u_{\lambda} - \frac{1}{2} M_{\lambda} |u_{\lambda}|^{p-2}u_{\lambda} > 0.$$  

From this, by the maximum principle, we get $w_{\lambda_0} > 0$ in $\Sigma_{\lambda_0}$, and $\frac{\partial w_{\lambda}}{\partial n_{\Sigma_{\lambda_0}}} > 0$ along the $-x_1$ direction of the ray. Now taking $R$ large enough such that $H(x)$ large enough in $\mathbb{R}_+^2 \setminus B_R(0)$. For $\Sigma_{\lambda_0} \cap B_R$, we have $w_{\lambda_0} > 0$. Then by the continuity of $w_{\lambda_0}(x) = w(\lambda, x)$, there is an $\epsilon > 0$ small enough, such that $w_{\lambda} |_{\Sigma_{\lambda} \cap B_R} > 0$ for all $\lambda \in (\lambda_0, \lambda_0 + \epsilon)$, which give us $w_{\lambda}^- = 0$. Then we estimate the integrals in (4.3) as in the Step 1, we have

$$\int_{\Sigma_{\lambda}} |\nabla w_{\lambda}^-|^2 + \int_{\Sigma_{\lambda}} |w_{\lambda}^-|^2 \leq \frac{1}{2} \int_{\Sigma_{\lambda} \cap B_R^c} L_{\lambda}^+ |u_{\lambda}|^{p-2}u_{\lambda}w_{\lambda}^- + \frac{1}{2} \int_{\Sigma_{\lambda} \cap B_R^c} M_{\lambda}^+ |u_{\lambda}|^{p-2}u_{\lambda}w_{\lambda}^-.$$  

All the estimates are exactly the same as in Step 1, except the integrals are over $\Sigma_{\lambda}^- \cap B_R^c$. Now taking $R \to \infty$ in place of $\lambda \to -\infty$ in Step 1, we again get

$$\int_{\Sigma_{\lambda}^-} |\nabla w_{\lambda}^-|^2 + \int_{\Sigma_{\lambda}^-} |w_{\lambda}^-|^2 \leq \frac{1}{4} \|w_{\lambda}^-\|_{L^2(\Sigma_{\lambda} \cap B_R^c)}^2.$$  

From this, we get $\Sigma_{\lambda}^- = \emptyset$ hence $w_{\lambda} = u_{\lambda} - u \geq 0$ for all $\lambda \in (\lambda_0, \lambda_0 + \epsilon)$, a contradiction with the definition of $\lambda_0$. So we have $w_{\lambda_0} \equiv 0$ and $\frac{\partial w_{\lambda}}{\partial x_1} < 0$ along the ray starting from the axis in $x_1$ direction.

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