ON “SMALL GEODESICS” AND FREE LOOP SPACES

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Abstract. A topological group is constructed which is homotopy equivalent to the pointed loop space of a path-connected Riemannian manifold $M$ and which is given in terms of “composable small geodesics” on $M$. This model is analogous to J. Milnor’s free group construction \cite{12} which provides a model for the pointed loop space of a connected simplicial complex. Related function spaces are constructed from “composable small geodesics” which provide models for the free loop space of $M$ as well as the space of continuous maps from a surface to $M$.

1. Introduction

The main purpose of this article is to describe a “combinatorial model” which is a topological group that is homotopy equivalent to the pointed loop space of a Riemannian manifold $M$. A second purpose is to give a model for the free loop space $\Lambda M$, the space of all continuous maps from the circle to $M$. As a consequence, a model for the space of continuous maps of a closed orientable surface to $M$ is also given.

The pointed loop space $\Omega M$ is useful here. That is the subspace of $\Lambda M$ given by functions which preserve a point, namely $f(*) = *_M$ for fixed points $*$ in the circle, and $*_M$ in $M$. The method used here is implicit in a method due to J. Milnor \cite{12}. Thus this article is partially an exposition of those methods, but in a somewhat different as well as extended context. The main point is the construction of a topological group arising from “small geodesics” which has the homotopy type of $\Omega M$.

Combinatorial models are obtained by patching together “composable small geodesics” on $M$ as follows. The model is given by the space of all ordered $n$-tuples of points where successive points are required (1) to differ and (2) to satisfy the additional property that there is an unique minimal geodesic between successive points. With mild hypotheses, the space of all such $n$-tuples for $n \geq 0$ are assembled into a topological group which is weakly homotopy equivalent to the pointed loop space $\Omega M$. Forming the homotopy orbit space for this group acting on itself by the adjoint representation then gives a model for the free loop space $\Lambda M$.

One further application is a combinatorial model for the space of pointed maps as well as free maps of a closed, orientable Riemann surface to $M$ which is given in terms of “composable small geodesics”.

These models arise by introducing a natural monoid structure on “composable minimal geodesics” as described explicitly below as well as exploiting the structure of the radius of convexity for a Riemannian manifold. One feature of one combinatorial model here is that it is naturally a topological group while the multiplication in the loop space $\Omega M$ is only associative up to homotopy.

Combinatorial models have been useful for 50 years. Some examples were first given in work of I. M. James \cite{9}, and J. Milnor \cite{12}. Later models for $\Omega^n \Sigma^n(X)$ were given by J. Milgram and J. P. May \cite{10,11}. Combinatorial models for the free loop space of a suspension are given in \cite{2,4} while models for the space of continuous maps of the $n$-sphere to an $(n + 1)$-fold suspension are given in \cite{3}.

1991 Mathematics Subject Classification. Primary: 55R99, Secondary: 58D99.
Throughout this article $M$ is assumed to be a Riemannian manifold. Some definitions as well as proofs are patterned quite closely on an early result of Milnor \[12\] which gave combinatorial models for loop spaces of connected simplicial complexes for which modifications are made here for “small geodesics”. Related models for $\Lambda M$ given by “small geodesics” are also given in \[1\]. This article assembles some of that structure in a language familiar to topologists with the main modification required here given by Lemma \[3.1\].

The authors thank Ryan Budney, and Mike Gage for explaining the convexity radius of a Riemannian manifold. The first author is grateful for the hospitality from the Institute for Advanced Study during some of the work on this article. Much of this article dates back to conversations of the authors in 2002.

2. Constructions, and statement of results

The purpose of this section is to assemble a space built out of “small geodesics” as well as recording basic properties of these constructions. Before giving these constructions, recall that a Riemannian manifold $M$ admits a cover by geodesic balls (for example page 1123 section 360C of [6]). This feature is used in the first definition as follows.

(1) Let

$$Z(M, k)$$

denote the subspace of $M^{k+1}$ given by $(k + 1)$-tuples $(x_k, x_{k-1}, \ldots, x_0)$ which satisfy the property that there is an unique minimal geodesic from $x_i$ to $x_{i+1}$ for all $0 \leq i < k$.

(2) Fix a point $v_0$ in $M$. Let $Z(M, k, v_0)$ denote the subspace of $Z(M, k)$ with $x_0 = v_0$.

(3) Let $X(M, k)$ denote the subspace of $Z(M, k)$ with $x_0 = x_k$.

(4) Let $G(M, k)$ is the subspace of $X(M, k)$ with $x_0 = x_k = v_0$.

Consider the disjoint union $\coprod_{k \geq 0} Z(M, k)$. Let

$$Z(M, \infty)$$

denote the identification space obtained from the equivalence relation on $\coprod_{k \geq 0} Z(M, k)$ generated by

$$(x_k, x_{k-1}, \ldots, x_i, \ldots, x_0) \sim (x_k, x_{k-1}, \ldots, \hat{x}_i, \ldots, x_0)$$

whenever $x_i = x_{i+1}$ or $x_{i+1} = x_{i-1}$. Notice that this equivalence relation is exactly that given in [12], page 274, where a topological group is constructed which is homotopy equivalent to the loop space of a connected simplicial complex.

Let

$$[x_k, x_{k-1}, \ldots, x_0]$$

denote the associated equivalence class of $(x_k, x_{k-1}, \ldots, x_0)$ in $Z(M, \infty)$. Similarly, let

(1) $X(M, \infty)$ be the subspace of $Z(M, \infty)$ given by the image of $\coprod_{k \geq 0} X(M, k)$ in $Z(M, \infty)$,

(2) $Z(M, \infty, v_0)$ be the subspace of $Z(M, \infty)$ given by the image of $\coprod_{k \geq 0} Z(M, k, v_0)$ in $Z(M, \infty)$, and

(3) $G(M, \infty)$ be the subspace of $Z(M, \infty)$ given by the image of $\coprod_{k \geq 0} G(M, k)$ in $Z(M, \infty)$.

Next notice that the first coordinate projection map $\pi_k : Z(M, k) \to M$ given by $\pi_k((x_k, x_{k-1}, \ldots, x_i, \ldots, x_0)) = x_k$ is continuous. Since the projection maps $\pi_k$ preserve the equivalence relation for $\coprod_{k \geq 0} Z(M, k)$, there is an induced continuous map

$$\pi : Z(M, \infty) \to M.$$
Two useful, technical lemmas are stated next. Proofs are analogous to that of Lemma 4.1 in section 4 and are omitted.

**Lemma 2.1.** If $M$ is a path-connected Riemannian manifold, then

- $Z(M, \infty, v_0)$,  
- $X(M, \infty)$, and  
- $G(M, \infty)$

are path-connected. Furthermore, $\pi$ is a surjection.

In addition, there is a “partial product” motivated by the fundamental groupoid $\mu$:

- $Z(M, j, v_0)$
- $G(M, k)$

given by the formula

$\mu((x_j, \cdots, x_0), (y_k, \cdots, y_0)) = (x_j, \cdots, x_0, y_k, \cdots, y_0)$

as $x_0 = y_k = y_0 = v_0$. The map $\mu$ is continuous as it is induced by the inclusion of a subspace (which has the subspace topology).

**Lemma 2.2.** There are continuous maps $\mu : Z(M, \infty, v_0) \times G(M, \infty) \to Z(M, \infty, v_0)$ together with the following commutative diagram.

$$
\begin{array}{ccc}
G(M, \infty) \times G(M, \infty) & \xrightarrow{\mu} & G(M, \infty) \\
\downarrow & & \downarrow \\
Z(M, \infty, v_0) \times G(M, \infty) & \xrightarrow{\mu} & Z(M, \infty, v_0)
\end{array}
$$

In addition $G(M, \infty)$ is a topological group with identity given by the equivalence class $[v_0]$, and $G(M, \infty)$ acts on $Z(M, \infty, v_0)$ via the map $\mu$.

This action has further properties as proven below.

**Lemma 2.3.** The natural orbit space obtained from the right-action of $\mu : Z(M, \infty, v_0) \times G(M, \infty) \to Z(M, \infty, v_0)$ gives the projection

$p : Z(M, \infty, v_0) \to Z(M, \infty, v_0)/G(M, \infty)$

which is the projection in a fibre bundle.

Let $P(M)$ denote the pointed path-space of $M$, that is the based continuous functions $f : [0,1] \to M$ such that $f(0) = v_0$.

**Theorem 2.4.** Assume that $M$ is a path-connected Riemannian manifold.

1. The space $Z(M, \infty, v_0)$ is contractible.
2. The projection $\pi : Z(M, \infty, v_0) \to M$ is the projection in a principle $G(M, \infty)$-bundle.
3. There is a map $\Theta : X(M, \infty) \to \Lambda(M)$ together with morphisms of fibrations

$$
\begin{array}{ccc}
G(M, \infty) & \xrightarrow{\Theta} & \Omega(M) \\
\downarrow & & \downarrow \\
Z(M, \infty, v_0) & \xrightarrow{\Theta} & P(M) \\
\downarrow \pi & & \downarrow \\
M & \xrightarrow{1} & M
\end{array}
$$

and
The maps

\[ \Theta : G(M, \infty) \to \Omega(M), \]

and

\[ \Theta : X(M, \infty) \to \Lambda(M) \]

are weak homotopy equivalences.

The proof of this theorem is given in sections 3, 4 and 5.

The analogous case for certain topological groups is given next where the free loop space of \( G/\Gamma \) is considered with \( G \) a simply-connected topological group and with \( \Gamma \) a closed discrete subgroup of \( G \). The structure of \( \Lambda(G/\Gamma) \) for some of these are then tied in below with the model above using “small geodesics”.

First, let \( \Gamma \) denote a discrete group which acts freely and properly discontinuously on a manifold \( M \) (via a left action). Let \( p : M \to M/\Gamma \) denote the natural associated covering space projection. The structure of \( \Lambda(M/\Gamma) \) is standard, follows from properties of covering spaces and is described next for purposes of exposition.

A point in the orbit space \( M/\Gamma \) is the orbit

\[ [m] = m \cdot \Gamma \]

for \( m \) in \( M \). Let \( g \) denote an element of \( \Gamma \) and define

\[ P_g(M) \]

to be the paths in \( M \) which start at a fixed point \( *_M \) which end at \( g(*_M) \). Define

\[ \Theta : \Pi_{g\Gamma} P_g(M) \to \Omega(M) \]

by the formula \( \Theta(f)(t) = p(f(t)) \).

Next, assume that \( M = G \) is a simply-connected topological group with \( \Gamma \) a discrete subgroup of \( G \). In this case, there is a natural left \( G \)-action on \( \Pi_{g\Gamma} P_g(G) \) given by conjugation specified by the formula

\[ \alpha(f)(t) = \alpha \cdot (f(t)) \cdot \alpha^{-1} \]

for \( f \) in \( \Pi_{g\Gamma} P_g(G) \) \( \alpha \) in \( \Pi \). Furthermore, let \( 1_G \) denotes the identity element in the topological group \( G \) which is assumed to be the base-point. Evidently, conjugation by any element in \( G \) preserves \( 1_G \). Using the action of \( G \) on itself via right multiplication by the inverse of an element \( \alpha \) in \( G \), there is an induced diagonal action of \( G \) on

\[ G \times \Pi_{g\Gamma} P_g(G) \]

together with an induced map

\[ \Phi : G \times \Pi_{g\Gamma} P_g(G) \to \Lambda(G/\Gamma) \]

given by

\[ \Phi(m, f)(t) = \pi[m \cdot f(t)]. \]

Notice that there is a commutative diagram
The next proposition is standard.

**Proposition 2.5.**

1. If \( M \) is a path-connected space, then \( \Theta : \Pi_g \Pi P_g(M) \to \Omega(M) \) is a homeomorphism.

2. If \( G \) is a simply-connected Lie group and \( \Pi \) is a discrete subgroup of \( G \), then \( \Phi : G \times \Pi \Pi_g \Pi P_g(G) \to \Lambda(G/\Pi) \) is a homeomorphism.

A combinatorial model for the spaces of maps of a surface \( S_g \) to \( M \) is given next. Fix a topological group \( G \) and consider the commutator map \( \chi_g : G^{2g} \to G \) given by

\[
\chi_g(x_1, x_2, \ldots, x_{2g}) = [x_1, x_2] \cdot [x_3, x_4] \cdots [x_{2g-1}, x_{2g}]
\]

for which \([a, b] = aba^{-1}b^{-1}\). Let

\[\xi(\chi_g)\]

denote the homotopy theoretic fibre of the map \( \chi_g \). Then it is classical that there is a map

\[\alpha : \xi(\chi_g) \to \text{map}_* (S_g, BG)\]

which is a homotopy equivalence. In case \( G = G(M, \infty) \), then \( BG \) is homotopy equivalent to \( M \) by Theorem 2.4.

A specific concrete model for this homotopy theoretic fibre is described next. Let \( PG \) denote the path-space for \( G \), that is the space of continuous functions \( f : [0, 1] \to G \) such that \( f(0) = 1_G \) (Any choice of base-point suffices here.)

Thus there is a projection

\[p : PG \to G\]

given by

\[p(f) = f(1)\].

Define \( \xi_g(G) \) as the pull-back in the following diagram:

\[
\begin{array}{ccc}
\xi_g(G) & \longrightarrow & PG \\
\downarrow & & \downarrow p \\
G^{2g}(Y) & \xrightarrow{\chi_g} & G \\
\end{array}
\]

One example is given by

\[G = G(M, \infty)\]

the topological group given above which is homotopy equivalent \( \Omega(M) \). This group will be used next to give a model for the space of pointed continuous maps \( \text{map}_*(S_g, M) \).
The final example is the space of maps $\text{map}(S^1, BG)$ where it is again classical that this space is homotopy equivalent to the homotopy orbit space (sometimes called the Borel construction)

$$EG \times_G G^{ad}$$

where $G^{ad}$ denotes the left $G$-space where the action is conjugation. Again, let

$$G = G(M, \infty)$$

and

$$EG = Z(M, \infty, v_0).$$

**Corollary 2.6.** Assume that $M$ is a simply-connected Riemannian manifold.

1. If $G = G(M, \infty)$, then there is a weak homotopy equivalence
   $$\xi_g(G) \rightarrow \text{map}(S_g, M).$$
2. If $G = G(M, \infty)$ and $EG = Z(M, \infty, v_0)$, then there is a weak homotopy equivalence
   $$EG \times_G \xi_g(G) \rightarrow \text{map}(S_g, M).$$
3. If $G = G(M, \infty)$ and $EG = Z(M, \infty, v_0)$, then there is a weak homotopy equivalence
   $$EG \times_G G^{ad} \rightarrow \Lambda M.$$
Proof. Fix a point $p$ in $M$. By 2.1 it may be assumed that there exists a neighborhood $U_p$ of $p$ for which there is a unique minimal geodesic between any two points $x$ and $y$ in $U_p$. Thus the pair $(x, p)$ is in $Z(M, 2)$ for $x$ in $U_p$.

Notice that by 2.1, $\pi^{-1}(p)$ is non-empty. Let $e_p$ denote a fixed choice of point in $\pi^{-1}(p)$ regarded as a subspace of $Z(M, \infty, v_0)$. Hence

$$e_p = [p, x_{k-1}, \cdots, x_1, v_0].$$

Write

$$e_p^{-1} = [v_0, x_1, x_2, \cdots, x_{k-1}, p],$$

a point in $Z(M, \infty)$, but not in $Z(M, \infty, v_0)$.

Let $g$ denote a point in $G(M, \infty)$, then $g = [v_0, y_{n-1}, \cdots, y_1, v_0]$. Thus by the definition of the map $\mu$, the element $[x, p] \cdot e_p \cdot g$ is an element of $Z(M, \infty, v_0)$. Define $p : U_p \times G(M, \infty) \to Z(M, \infty, v_0)$ by the formula

$$p(x, g) = [x, p] \cdot e_p \cdot g.$$ 

By 2.2, $p$ is continuous. Furthermore,

$$p(\mu(x, g)) = x$$

by definition. Thus $\mu$ takes values in $\pi^{-1}(U_p)$ and is a continuous function

$$\mu : U_p \times G(M, \infty) \to \pi^{-1}(U_p).$$

Let $e$ be any point in $\pi^{-1}(U_p)$. Since $e = [x_k, \cdots, x_1, v_0] \pi(e) = x_k$ in $U_p$, there is an unique minimal geodesic from $p$ to $\pi(e)$ $[p, x_k] = [p, \pi(e)]$. Notice that the product

$$e_p^{-1} \cdot [p, \pi(e)] \cdot e$$

is equal to

$$[v_0, x_1, \cdots, x_{k-1}, p] \cdot [p, x_k] \cdot [x_k, x_{k-1}\cdots, v_0].$$

Hence this product is equivalent to

$$[v_0, x_1, \cdots, x_{k-1}, p, x_k, x_{k-1}\cdots, v_0],$$

and is an element of $G(M, \infty)$.

Define

$$\theta_p : \pi^{-1}(U_p) \to G(M, \infty)$$

by the formula $\theta_p(e) = e_p^{-1} \cdot [p, \pi(e)] \cdot e$. By the previous paragraph, this function is continuous. There is an associated continuous function

$$\Theta : \pi^{-1}(U_p) \to U_p \times G(M, \infty)$$

given by

$$\Theta(e) = (\pi(e), \theta_p(e)).$$

The following formulas are satisfied which imply directly that both $\Theta$ and $\phi_p$ are homeomorphisms so the projection map is locally trivial.

(1) $(\phi_p)(\Theta(e)) = (\phi_p)(\pi(e), \theta_p(e))$

(2) $(\phi_p)(\Theta(e)) = [\pi(e), p] \cdot e_p \cdot \theta_p(e)$

(3) $(\phi_p)(\Theta(e)) = [\pi(e), p] \cdot e_p \cdot [p, \pi(e)] \cdot e = e,$

(1) $(\Theta)((\phi_p)(x, g)) = (\Theta)((x, p) \cdot e_p \cdot g)$

(2) $(\Theta)((\phi_p)(x, g)) = (\pi([x, p] \cdot e_p \cdot g), \theta_p([x, p] \cdot e_p \cdot g))$

(3) $(\Theta)((\phi_p)(x, g)) = (x \cdot g).$
To finish, it suffices to see that the locally trivial projection \( \pi : Z(M, \infty, v_0) \to M \) is that of a principal bundle. This follows by an inspection of the transition functions as follows. Let \( q \) denote a point in \( M \) together with an open set \( U_q \) in \( M \) such that \( U_p \cap U_q \) is non-empty. Let \( x \) be an element of \( U_p \cap U_q \). Thus the element \([v_p, x, v_q]\) is defined. Define the function

\[
g_{p,q} : U_p \cap U_q \to G(M, \infty)
\]

by the formula

\[
g_{p,q}(x) = c_p^{-1}[v_p, x, v_q] \cdot e_q.
\]

These formulas satisfy the following.

1. \( \pi \circ \phi_p(x, g) = x \),
2. \( \phi_p(\pi(e), \theta_p(g)) \pi \circ \phi_p(x, g) = x \),
3. \( \theta_p \circ \phi_p(x, g) = g \),
4. \( \theta_p \circ \phi_q(x, g) = g_{p,q}(x) \cdot g \).

The lemma follows. \( \square \)

Dold gave a sufficient condition which insures that certain functions are fibrations.

**Theorem 3.3.** If \( B \) is a paracompact Hausdorff space, a continuous map \( \pi : E \to B \) is a fibration if and only if the map \( \pi \) is locally a fibration.

The next technical lemma is as follows.

**Lemma 3.4.** The projection \( \pi : X(M, \infty, v_0) \to M \) is locally trivial and a surjection. Thus the projection \( \pi \) is a fibration (since \( M \) is paracompact and Hausdorff).

**Proof.** Fix a point \( v \) in \( M \). Since \( M \) is a Riemannian manifold, the exists a convex neighborhood \( U \) of \( v \) by [3.1] The main step in the proof of this lemma is to show that \( \pi^{-1}(U) \) is homeomorphic to \( U \times G(M, \infty) \) via a homeomorphism which preserves the projections to \( U \).

Since \( M \) is path-connected, there is a path \( \alpha \) from \( v \) to \( v_0 \) (where \( v_0 \) is a fixed point in \( M \) as given above). Cover the image of \( \alpha \) by convex neighborhoods choose a finite subcover (as the image of the interval is compact). Thus there is a finite collection of open convex sets \( U = U_0, U_1, \cdots, U_q \) such that

1. the image of \( \alpha \) is contained in the union \( \cup_{0 \leq i \leq n} U_i \),
2. \( v \) is in \( U_q \),
3. \( v_0 \) is in \( U = U_0 \),
4. \( U_i \cap U_{i+1} \) is non-empty for all \( i \).

Choose points \( w_i \) in \( U_i \cap U_{i+1} \) with the condition that \( w_0 = v_0 \ w_q = v \). Since \( w_i, w_{i+1} \) are in \( U_{i+1} \) for all \( i \), the ordered \( q+1 \)-tuple \((v, v_{q-1}, v_{q-2}, \cdots, v_0)\) is an element of \( Z(M, q+1, v_0) \) thus projects to the equivalence class \([v, v_{q-1}, v_{q-2}, \cdots, v_0] \) in \( Z(M, \infty, v_0) \).

Next, consider any element

\[ [v_0, x_s, x_{s-1}, \cdots, x_1, v_0] = [v_0, \bar{x}, v_0] \]

in \( G(M, \infty) \). Notice that if \( u \) is any point in \( U \), there is an unique minimal geodesic from \( u \) to \( v \). Thus \([u, v, v_{q-1}, v_{q-2}, \cdots, v_0]\) is a point in \( Z(M, \infty, v_0) \) such that \( \pi([u, v, v_{q-1}, v_{q-2}, \cdots, v_0]) = u \). Hence the map \( \lambda : G(M, \infty) \times U \to X(M, \infty) \) given by the formula

\[
\lambda([v_0, \bar{x}, v_0], u) = [u, v] \cdot [v, v_{q-1}, v_{q-2}, \cdots, v_0] \cdot [v_0, \bar{x}, v_0]
\]

is continuous takes values in \( \pi^{-1}(U) \). It follows that \( \lambda \) gives a map

\[ \lambda : G(M, \infty) \times U \to \pi^{-1}(U). \]
Consider any point \( \vec{z} = [y_t, y_{t-1}, \cdots, y_0] \) in \( \pi^{-1}(U) \). Thus \( y_t \) is a point in \( U \), there is an unique minimal geodesic from \( y_t \) to \( u \) \([u, y_t, y_{t-1}, \cdots, y_0]\) is a point of \( Z(M, \infty, v_0) \).

Define \( \gamma : \pi^{-1}(U) \to G(M, \infty) \times U \) by the formula
\[
\gamma(\vec{z}) = ([v, v_{q-1}, v_{q-2}, \cdots, v_0]^{-1} \cdot [v, \pi(\vec{z})], \pi(\vec{z})).
\]

Notice that \( \gamma \) is continuous is an inverse for \( \lambda \) hence locally triviality follows.

To show that the map \( \pi \) is a surjection, choose a path \( \alpha \) from \( v_0 \) to any given point \( v \) in \( M \). The above cover for the image of \( \alpha \) gives that there is a point \([v, v_n, v_{n-1}, \cdots, v_0]\) in \( X(M, \infty, v_0) \) which projects to the point \( v \).

\[ \square \]

4. Contractibility of \( Z(M, \infty, v_0) \)

The proof of Lemma 4.1 is given first. The statements to be proven are as follows where \( p \) denotes a point of \( M \).

1. There exists an \( \epsilon > 0 \) such that there is a convex ball of radius \( \epsilon \) containing \( p \).
2. Furthermore the function \( \Phi : G(M) \to map([0, 1], M) \) is continuous.

Proof. Let \( p \) be a point in \( M \), with \( T_p \) the tangent space at the point \( p \). The image of the exponential map applied to \( T_p \) contains a convex neighborhood of \( p \). Thus, there exists an unique geodesic arc joining any two points in this neighborhood there is a smooth geodesic arc \( f : [0, 1] \to M \) of minimal arc length between these points. Part (1) of the lemma follows.

Next, it must be shown that \( \Phi \) is continuous. Let \( U(K, V) \) denote the open set in the function space \( Map([0, 1], M) \) given by the functions which carry the compact set \( [0, 1] \) into the open set \( V \) in \( M \). Notice that since \( K \) is a compact subset of \([0, 1], \)

1. \( K \) is a union of the path components given by \( N \) closed intervals \([x_j, y_j]\) for a fixed integer \( N \) with
2. \( U(K, V) = U(\Pi_{1 \leq j \leq N}[x_j, y_j], V) \)
3. \( U(K, V) = \cap_{1 \leq j \leq N} U([x_j, y_j], V). \)

Assume that \( U(K, V) \) contains the point \( f_{(a,b)}(-) \) Thus the image \( f(K) \) is contained in \( V \) is compact. Cover the image \( f(K) \) by convex open balls contained in \( V \) ( by part (1) of the lemma as \( M \) is Riemannian ) choose a finite subcover \( O_1, O_2, \cdots, O_i \).

Notice that \( U(K, O_i) \) is an open subset of \( U(K, V) \). Thus the finite intersection
\[
\cap_{1 \leq i \leq N} U(K, O_i)
\]
is an open subset of \( U(K, V) \). Since the set \( K \) is a disjoint union of \( N \) closed intervals \([x_j, y_j] \), there is an equality \( U(K, V) = U(\Pi_{1 \leq j \leq N}[x_j, y_j], V) = \cap_{1 \leq j \leq N} U([x_j, y_j], V) \).

Thus \( \cap_{1 \leq N} \cap_{1 \leq i \leq N} U([x_j, y_j], O_i) \) is an open set in \( U(K, V) \) which contains \( f \).

To finish, notice that there is a homeomorphism
\[
\alpha : \Phi^{-1}(U([x_j, y_j], O_i)) \to O_i \times O_i
\]
which sends a function \( h \) to the pair \((h(x_i), h(x_j))\). Thus \( \Phi^{-1}(U([x_j, y_j], O_i)) \) is an open set which contains \( f_{(a,b)}(-) \) the set \( \cap_{1 \leq j \leq N} \cap_{1 \leq i \leq N} \Phi^{-1}(U([x_j, y_j], O_i)) \) satisfies the following properties:

1. the set is open,
2. the set contains \( f \)
3. \( f_{(a,b)}(-) \) is in \( \Phi^{-1}(U(K, V)) \).

The lemma follows. \[ \square \]

The previous lemma is used to prove

Lemma 4.1. If \( M \) is a path-connected Riemannian manifold, then \( Z(M, \infty, v_0) \) is contractible.
Proof. Consider the subspace of $Z(M, k, v_0)$ given by

$$D(M, k, v_0) = \{(x_k, x_{k-1}, \cdots, x_1, v_0)|x_i \neq x_{i+1}, x_{i-1} \neq x_i, i < k\}.$$  

Next, consider the continuous function $\Phi : G(M) \to map([0, 1], M)$ of $\Theta$ defined by $\Phi((a, b)) = f_{(a, b)}(-)$, the unique geodesic from $a$ to $b$ parameterized by arc length.

Assume that $k \geq 1$ let $\bar{x} = (x_k, x_{k-1}, \cdots, x_1, v_0)$ denote a point in $D(M, k, v_0)$. Define

$$h_k : [0, 1] \times D(M, k, v_0) \to Z(M, k, v_0)$$

as follows.

$$h_k(t, \bar{x}) = (\Phi(x_k, x_{k-1})(t), x_{k-1}, x_{k-2}, \cdots, x_1, v_0).$$

Since $\Phi$ is continuous, it follows that $h_k$ is continuous. Furthermore, $h_k(0, \bar{x}) = \bar{x}$ $h_k(1, \bar{x}) = (x_k-1, x_{k-1}, x_{k-2}, \cdots, x_1, v_0)$. In case $k = 0$, define $h_0$ to be the identity. Thus there is a homotopy

$$H : [0, 1] \times \Pi_{0 \leq k}D(M, k, v_0) \to \Pi_{0 \leq k}Z(M, k, v_0)$$

given by $\Pi_{0 \leq k}h_k$.

Observe that $H$ passes to quotient spaces to give an induced continuous function

$$\tilde{H} : [0, 1] \times Z(M, \infty, v_0) \to Z(M, \infty, v_0)$$

which is continuous when restricted to $\Pi_{0 \leq k}D(M, k, v_0)$. The homotopy $\tilde{H}$ gives the property that $Z(M, \infty, v_0)$ is contractible. The lemma follows. $\square$

5. Maps to free loop spaces

The purpose of this section is to exhibit the map

$$\Theta : X(M, \infty) \to \Lambda(M)$$

by patching together “small geodesics”.

Recall that $v_0$ is a fixed point of $M$ that $Z(M, k)$ is the subspace of $M^{k+1}$ given by $(k + 1)$-tuples $(x_k, x_{k-1}, \cdots, x_0)$ which satisfy the property that there is a unique minimal geodesic from $x_i$ to $x_{i+1}$ for all $1 \leq i < k$ while $X(M, k)$ denotes the subspace of $Z(M, k)$ with $x_0 = x_k$.

Consider the minimal geodesic from $x_i$ to $x_{i+1}$ which is specified by the path $\sigma : [0, 1] \to M$ given by the smooth function $f_{(x_i, x_{i+1})}(-)$. Thus given a point $\bar{z} = (x_k, x_{k-1}, \cdots, x_0)$ in $X(M, k)$, there are paths

$$\sigma_j : [0, 1] \to M$$

with $1 \leq j \leq k$ specified by $\sigma_j(t) = f_{(x_i, x_{i+1})}(t)$. Next, consider the sum of the arc-lengths of these paths

$$\lambda_1 + \lambda_2 + \cdots + \lambda_k = L.$$

The paths $\sigma_j$ are glued together to give a piecewise smooth closed curve as a function from $[0, 1]$ to $M$ which is parameterized by arc-length as follows.

Let $\delta_j = (\lambda_1 + \lambda_2 + \cdots + \lambda_j)/L$. Define

$$\sigma_{j, L} : [\delta_{j-1}, \delta_j] \to M$$

by the formula

$$\sigma_{j, L}(t) = \sigma_j(x)$$

for

$$x = (tL - \delta_{j-1})/(||tL - \delta_{j-1}||)$$

and $t$ in $[\delta_{j-1}, \delta_j]$. These functions “glue” together to give

$$\sigma_{\bar{z}} : [0, 1] \to M$$
which is given by $\sigma_{j,L}$ when restricted to $[\delta_{j-1}, \delta_j]$. There is an induced function

$$\Theta : X(M, k) \to \Lambda(M)$$

defined by

$$B(\vec{z}) = \sigma_{\vec{z}}.$$

**Theorem 5.1.** There is a morphism of fibrations

$$
\begin{array}{ccc}
G(M, \infty) & \xrightarrow{\Theta} & \Omega(M) \\
\downarrow & & \downarrow \\
X(M, \infty) & \xrightarrow{\Theta} & \Lambda(M) \\
\downarrow & & \downarrow \\
M & \xrightarrow{1} & M
\end{array}
$$

**Proof.** Notice that the adjoint of $\Theta : X(M, \infty) \to \Lambda M$, the map $\widetilde{\Theta} : S^1 \times X(M, \infty) \to M$ is continuous. Since all spaces here are Hausdorff and locally compact, the map $\Theta$ is continuous. That the diagram commutes is a direct inspection of the definitions. □

6. Two problems

It is natural to ask about the existence of natural representations of the group $G(M, \infty)$ in either $O(m)$ or $GL(m, \mathbb{R})$. Give a natural procedure for constructing such representations.

For example, give a homomorphism $G(M, \infty) \to O(m)$ which corresponds to either the stable normal bundle or the stable tangent bundle of $M$. That is, there is a map

$$\nu : M \to BO(m)$$

which represents the normal bundle of $M$. Then looping this map gives

$$\Omega(\nu) : \Omega(M) \to \Omega(BO(m))$$

where $\Omega(BO(m))$ is homotopy equivalent to $O(m)$. Describe a natural representation $\rho : G(M, \infty) \to GL(m, \mathbb{R})$ which is homotopic to $\Omega(\nu)$.

7. Acknowledgements

The second author was partially supported by National Science Foundation and the Institute for Advanced Study (in 2006).

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