Symmetry properties of massive gauge theories in
nonlinear background gauges: Background
dependence of Green functions

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Abstract
Nonabelian gauge theories with a generic background field $A_\mu$ in nonlinear gauges due to Delbourgo and Jarvis are investigated. The $A_\mu$-dependence is completely determined by the help of a linear differential equation which obtains from the Kluberg-Stern-Zuber and the Lee identity. Its integration leads to a relation between the one-particle irreducible vertex functional in the background field $A_\mu$ and the corresponding functional for $A_\mu = 0$. An analogous relation holds for the generating functional of the complete Green functions which, after restriction to physical Green functions, is used to confirm a result obtained by Rouet in the case of linear background gauge.

1 Introduction
Quantum field theories with classical background configurations receive growing attention, either because the full quantized theory is far from being well defined as is the case for quantum gravity or, because of the nontrivial topological structure of the classical theory, e.g. for chromodynamics where instanton solutions give rise to a complicated structure of the QCD vacuum. This raises the question how the renormalization properties of the theory with background field are possibly changed and whether their Green functions may be related to the Green functions without them.
Here, this problem will be considered for nonabelian gauge theories with generic background field $A_\mu$ within the algebraic BRST approach [1] where renormalizability of the theory has to be proven by showing that the symmetries of the theory at quantum level are free of anomalies. At first, this approach has been used by Kluberg-Stern and Zuber [2]. They also pointed out that the classical action

$$S_{YM}(A) = -(2g)^{-2} \int d^4x \text{Tr}(F_{\mu\nu}(A + Q)F^{\mu\nu}(A + Q)),$$

(1)

with field strengths $F_{\mu\nu}(A + Q) = \partial_\mu(A_\nu + Q_\nu) - \partial_\nu(A_\mu + Q_\mu) + [A_\mu + Q_\mu, A_\nu + Q_\nu]$, is invariant under two different kinds of transformations called type I and type II:

$$\text{type I : } \delta A_\mu = D_\mu(A)\delta\Omega, \quad \delta Q_\mu = [Q_\mu, \delta\Omega],$$

(2)

$$\text{type II : } \delta A_\mu = 0, \quad \delta Q_\mu = D_\mu(A + Q)\delta\Omega.$$  

(3)

If a linear background gauge is chosen, also the gauge fixed action is invariant under type I transformations, whereas type II transformations have to be changed into the BRST transformations. Their Ward identities are the (local) Kluberg-Stern–Zuber(KSZ) and Slavnov–Taylor(ST) identity, respectively. Obviously, there exists another kind of transformations, called here type III,

$$\text{type III : } \delta Q_\mu = D_\mu(Q)\delta\Omega, \quad \delta A_\mu = [A_\mu, \delta\Omega],$$

(4)

which however cannot be required to hold unchanged for the renormalized quantum action (see however equ. (26) below).

With the aim to determine the $A_\mu$–dependence of the Green functions Rouet [3] used this formalism. In order to circumvent the nonlinear (functional) differential equation emerging for the generating functional $Z(A|J)$ he restricted the consideration to the physical Green functions and obtained the relation

$$Z_{\text{phys}}(A|J) = \exp \left(-i(hzQ)^{-1} \int d^4x \text{Tr}(A^\mu J_\mu)\right) Z_{\text{phys}}(0|J).$$

(5)

This shows that the on–shell amputated Green functions in the background field $A_\mu$ are deduced from the physical Green functions in the vacuum by a mere translation of the (renormalized) quantum field $zQQ_\mu$.

In principle, to avoid IR–singularities it is necessary to provide the theory with a Higgs mechanism. However, there exist another possibilities. In a recent paper [4] we have shown that, if massive gauge fields are considered, e.g. with mass $m(s) = (1 - s)m$, as it is necessary at least intermediately in the framework of the BPHZL–renormalization [5] for allowing an IR-regularization, then only nonlinear gauges are permissible [4]. Therefore, the Curci-Ferrari model [6] in the (nonlinear) Delbourgo-Jarvis gauge [7] – being the nonsingular analogue of the Landau gauge – has been considered. This gauge has the additional advantage that the theory is

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1 The well known fact that massive gauge theories, if renormalizable, violate unitarity does not matter here since after carrying out all necessary subtractions the limit $s \rightarrow 1$ is taken.
invariant under BRST as well as anti-BRST transformations $s$ and $\bar{s}$, and, in the massive case, necessarily also under the $sp(2)$-transformations generated by the FP operator, the DJNO and the anti-DJNO operator $q, d$ and $\bar{d}$, respectively (for the definitions see Chapter 2).

Now, in this paper we extend these considerations to include a generic background field $A_\mu$. Imposing the gauge condition on the generating functional for the 1PI vertex functions $\Gamma(A|Q)$ it can be shown that a further local identity, being called Lee identity, emerges which is related to the type III symmetry above. Together with the KSZ identity it leads to a linear differential equation for the $A_\mu$-dependence of $\Gamma(A|Q)$. Its solution relates $\Gamma(A|Q)$ and $\Gamma(0|Q)$; in addition, it can be shown that $A_\mu$ does not require any further $z$-factors and, therefore, the renormalization of $\Gamma(A|Q)$ is determined by that of $\Gamma(0|Q)$. By Legendre transformation this gives a complete solution of the problem originally considered by Rouet and, after restriction to the physical subspace, confirms his result. Furthermore, the method used here may be extended to consider topological nontrivial background fields such as instantons and merons.

2 Massive gauge theory in nonlinear background gauge

In its most symmetric form the theory is defined by the following (classical) action (for its explicit form see (12)):

$$S(A|g, m; \xi, \rho, \sigma) = -(2g)^{-2} \int d^4x \text{Tr}(F_{\mu\nu}(A + Q)F^{\mu\nu}(A + Q))$$
$$+ (s_m \bar{s}_m + m^2) \int d^4x \left( \frac{1}{2} Q_\mu Q^\mu + \xi \bar{C}C - G_\mu Q^\mu + U\bar{C} - \bar{U}C + \frac{1}{2} \sigma G_\mu G^\mu + \rho \bar{U}U \right)$$

with

$$Q_\mu = Q_\mu + \sigma G_\mu, \quad C = C + \rho U, \quad \bar{C} = \bar{C} + \rho \bar{U}, \quad B = B + (\zeta - \bar{\zeta}) \rho S,$$

where $Q_\mu, C, \bar{C}$ and $B$ are the gauge, ghost, antighost and Nakanishi-Lautrup field, respectively, and $G_\mu, U, \bar{U}$ and $S$ are associated sources having the same quantum numbers as these fields, respectively; $\xi$ is the gauge parameter, $\zeta$ an arbitrary real parameter ($\bar{\zeta} = 1 - \zeta$) reflecting the freedom in defining $B$ according to $B = \zeta s_m \bar{C} - \bar{\zeta} \bar{s}_m C$, whereas $\sigma$ and $\rho$ are to be introduced since the dynamical fields and the associated sources mix under renormalization.

The mass–dependent (anti–) BRST operators $s_m$ (and $\bar{s}_m$) are defined through their action on the fields and the sources according to:

$$s_m A_\mu = 0, \quad s_m Q_\mu = D_\mu(A + Q)C, \quad s_m C = -(1/2) \{C, \bar{C}\},$$
$$s_m \bar{C} = B - \zeta \{C, \bar{C}\}, \quad s_m B = \zeta \{B + \bar{C} \{C, \bar{C}\}, C\} - m^2 C,$$
$$s_m G_\mu = P_\mu, \quad s_m P_\mu = 0, \quad s_m U = R, \quad s_m R = 0,$$
$$s_m \bar{U} = S, \quad s_m S = -m^2 U, \quad s_m \bar{P}_\mu = -m^2 G_\mu, \quad s_m \bar{R} = -2m^2 \bar{U}.$$
The corresponding relations defining the action of the anti-BRST operator $\bar{s}_m$ are obtained by the following conjugation $\mathbf{C}$ which leaves the classical action (10) invariant and, obviously, is an extension of the ghost–antighost conjugation:

\[
\mathbf{C} \rightarrow \bar{\mathbf{C}}, \quad U \rightarrow \bar{U}, \quad P_\mu \rightarrow \bar{P}_\mu, \quad R \rightarrow \bar{R}, \quad S \rightarrow -S, \quad \zeta \rightarrow \bar{\zeta}, \quad \bar{\zeta} \rightarrow \zeta; \quad (9)
\]

$Q_\mu, G_\mu$ and $B$ as well as $\xi$ are $\mathbf{C}$–even. The additional sources $\bar{P}_\mu$ and $\bar{R}$ together with $S$ and $\bar{U}$ couple to the nonlinear parts of the BRST transformations (8) of the fields (see (12)) and analogously for the $\mathbf{C}$–conjugate fields and sources; the source $G_\mu$ couples to $s_m \bar{s}_m Q_\mu$.

For later convenience we introduce the Delbourgo-Jarvis-Nakanishi-Ojima(DJNO) transformations [7] defined by:

\[
\begin{align*}
\text{d}A_\mu &= \text{d}Q_\mu = \text{d}G_\mu = 0, \quad \text{d}\bar{C} = \bar{\mathbf{C}}, \quad \text{d}C = 0, \quad \text{d}B = (\zeta - \bar{\zeta})^2, \\
\text{d}P_\mu &= P_\mu, \quad \text{d}U = 0, \quad \text{d}\bar{U} = U, \quad \text{d}S = R, \quad \text{d}R = 0, \quad \text{d}\bar{R} = 2S. \quad (11)
\end{align*}
\]

Again, the anti–DJNO transformations are obtained by $\mathbf{C}$–conjugation.

This finishes the definition of our model whose action has the following explicit form:

\[
S(A|g, m; \xi, \rho, \sigma) = (2g)^{-2} \int d^4x \text{Tr}(F^\mu(A + Q)F^\nu(A + Q)) + m^2 \int d^4x \text{Tr}\left(\frac{1}{2}Q_\mu Q^\mu + \xi \bar{\mathbf{C}}\right) + \int d^4x \text{Tr}(\bar{Q}^\mu D_\mu(A)B + (D_\mu(A)\mathbf{C}) D^\mu(A + Q)\bar{C} - \bar{\zeta} \left(D_\mu(A)\bar{\zeta}\right) D^\mu(A + Q)\mathbf{C}) + \xi \int d^4x \text{Tr}\left(m^2 \bar{\mathbf{C}}^2 + \frac{1}{2} \left(B - \zeta \{\mathbf{C}, \bar{\mathbf{C}}\}\right)^2 + \frac{1}{2} (B + \bar{\zeta} \{(\mathbf{C}, \bar{\mathbf{C}})\}^2) \right)
\]

\[
+ \int d^4x \text{Tr}\left(-G_\mu D^\mu(A + Q)B + \left(\bar{P}_\mu - \bar{\zeta} \left[G_\mu, \bar{\zeta}\right]\right) D^\mu(A + Q)\mathbf{C} - (P_\mu - \zeta \left[G_\mu, \mathbf{C}\right]) D^\mu(A + Q)\bar{\mathbf{C}}\right)
\]

\[
+ \int d^4x \text{Tr}\left(-S \{\mathbf{C}, \bar{\mathbf{C}}\} + \bar{R}\mathbf{C}^2 + R\bar{\mathbf{C}}^2 + \bar{U} \left[\mathbf{B} + \zeta \{\mathbf{C}, \bar{\mathbf{C}}\}, \mathbf{C}\right] - U \left[\mathbf{B} - \zeta \{\mathbf{C}, \bar{\mathbf{C}}\}, \bar{\mathbf{C}}\right]\right)
\]

\[
+ \sigma \int d^4x \text{Tr}\left(P_\mu \bar{P}^\mu - \frac{1}{2} m^2 G_\mu G^\mu \right) + \rho \int d^4x \text{Tr}\left(R \bar{R} - S^2\right).
\]

Let us now state the (classical) symmetries of this theory. Again, as in the case $A_\mu = 0$, the algebra generated by $\{s_m, \bar{s}_m; d, \bar{d}, q\}$ is easily verified to be the superalgebra $osp(1, 2)$ [8]. It is given by the following $q$–graded commutation relations

\[
s^2 = -m^2 d, \quad \{s_m, s_m\} = m^2 q, \quad s_m = m^2 d, \quad (13)
\]

\[
[d, s_m] = 0, \quad [d, s_m] = -s_m, \quad [q, s_m] = s_m, \quad (14)
\]

\[
[ar{d}, s_m] = 0, \quad [ar{d}, s_m] = s_m, \quad [q, s_m] = -s_m, \quad (15)
\]

\[
[q, d] = 2d, \quad [q, \bar{d}] = 2\bar{d}, \quad [d, \bar{d}] = -q, \quad (16)
\]
Using these relations it is easy to check that the action $S(A|g, m; \xi, \rho, \sigma)$ is invariant under (anti--) BRST as well as (anti--) DJNO transformations:

\[ s_m S(A|g, m; \xi, \rho, \sigma) = 0, \quad d S(A|g, m; \xi, \rho, \sigma) = 0, \]
\[ \bar{s}_m S(A|g, m; \xi, \rho, \sigma) = 0, \quad \bar{d} S(A|g, m; \xi, \rho, \sigma) = 0; \quad (17) \]

invariance under $q$ is trivial since $S(A|g, m; \xi, \rho, \sigma)$ has ghost number zero by definition. Furthermore, the (anti--) DJNO symmetry of the action according to (13) is a consequence of the (anti--) BRST invariance!

Obviously, in the limit $m^2 = 0$ the usual algebra of nilpotent (anti--) BRST operators $s^2 = 0$, $\{s, \bar{s}\} = 0$, $\bar{s}^2 = 0$ appears, and the $sp(2)$--algebra [10], being now an independent requirement, decouples. Of course, a massless Yang-Mills theory in nonlinear gauge obtains [1]. However, it should be emphasized that the mass is not really an independent parameter of the theory since the Slavnov-Taylor identities explicitly depend on $m$.

As in the case of linear gauges, the auxiliary field $B$ occurs quadratically in the action, but now through $(\zeta - \bar{\zeta}) B(C\bar{C})$ it couples to the (anti--) ghost fields, which have quartic selfcoupling; therefore $B$ does not decouple from the dynamics. Finally we remark that, as is seen from (12), the sources being coupled by $\sigma$ and $\rho$ occur bilinear; despite looking strange this is unavoidable for a stable, $osp(1, 2)$--invariant theory.

In addition to this $osp(1, 2)$--symmetry, which is related to the broken gauge, i.e. type II, symmetry , the action (12) is invariant (eventually modulo a mass term) under (generalized) type I and type III transformations defined as

\[ \text{type I : } \delta A_\mu = D_\mu (A) \delta \Omega, \quad \delta Q_\mu = [Q_\mu, \delta \Omega], \quad \delta G_\mu = [G_\mu, \delta \Omega], \quad \delta \Phi = [\Phi, \delta \Omega], \]
\[ \text{type III : } \delta Q_\mu = D_\mu (Q) \delta \Omega, \quad \delta G_\mu = D_\mu (G) \delta \Omega, \quad \delta A_\mu = [A_\mu, \delta \Omega], \quad \delta \Phi = [\Phi, \delta \Omega], \]

where $\Phi$ here and in the following denotes the remaining (dynamical or external) fields not explicitly written down:

\[ \delta \text{I} S(A|g, m; \xi, \rho, \sigma) = 0, \quad \delta \text{III} S(A|g, m; \xi, \rho, \sigma) = m^2 \partial_\mu Q^\mu. \quad (18) \]

Whereas the first symmetry can be formulated as a Ward identity, the KSZ identity of the (renormalized) theory, the second one leads to the Lee–identity (20).

### 3 Ward identities for the vertex functional $\Gamma$

In order to construct the renormalized theory associated to the classical action (1) the symmetries of that action have to be expressed by corresponding Ward identities of the 1PI functional. These identities hold, first of all, at lowest perturbative order, i.e. for $\Gamma^0(A|g, m; \xi, \rho, \sigma)$; they are to be required to hold for the renormalized 1PI

\[ \text{As has been shown by Curci and Ferrari [3] in the massless case two independent gauge parameters } \xi, \bar{\xi} \text{ may be introduced; but than (anti--) DJNO invariance cannot be maintained.} \]
functional $\Gamma(A|g, m; \xi, \rho, \sigma)$. Since these identities are purely algebraic requirements they may be formulated for a generic functional, denoted also by $\Gamma$, which depends on the same fields, sources and parameters:

- $A^K \Gamma = 0$ (Kluberg – Stern – Zuber identity),
- $S_m(\Gamma) = 0$ (Slavnov – Taylor identity),
- $D(\Gamma) = 0$ (Delduc – Sorella identity),
- $T_\zeta(\Gamma) = 0$ (Bonora – Pasti – Tonin identity),

and, in addition, the C-conjugated second and third identity; the first identity and the last one (fixing the definition of $B$) are C–even.

The above operations are defined in the following manner: The Kluber g-Stern–Zuber (KSZ) operator extends the local Ward operator $W = \sum [\Phi, \delta/\delta \Phi}$ according to $A^K = D_\mu(A)\delta/\delta A^\mu + W$ with the commutator or anticommutator depending on even or odd $q$–grading of $\Phi$, respectively. Of course, identity (19) is defined only for $A^\mu \neq 0$; after integration over spacetime we get the rigid Ward identity

$$\int d^4x \{[A, \delta/\delta A] + W\} \Gamma = 0. \quad (19)$$

The (bilinear) Slavnov-Taylor (ST) and Delduc-Sorella (DS) operations as usual are given by $S_m(\Gamma) = \int d^4x \sum \text{Tr} (([s_m \Phi] \delta \Gamma/\delta \Phi)$ and $D(\Gamma) = \int d^4x \sum \text{Tr} ((d \Phi) \delta \Gamma/\delta \Phi)$, respectively, where the nonlinear parts of $s_m \Phi$ and $d \Phi$ are to be replaced by the functional derivative of $\Gamma$ with respect to the corresponding source $\bar{P}^\mu, \bar{R}, S$ or $\bar{U}$ (e.g. $\zeta \{B + \bar{C}, \bar{C}\}$ by $\zeta \delta \Gamma/\delta \bar{U}$).

The Bonora-Pasti-Tonin (BPT) operation is defined by $T_\zeta(\Gamma) = \partial \Gamma/\partial \zeta - \int d^4x \text{Tr} (\delta \Gamma/\delta S \cdot \delta \Gamma/\delta B)$. For linear gauges the action $S$, if appropriately chosen, coincides with the effective action in tree approximation $\Gamma^0$. But in the case of nonlinear gauges this will not be true. Namely, the ST operation applied to $S(A|g, m; \xi, \rho, \sigma)$ leads to the expression $S_m(S) = 2\zeta \bar{\zeta} \rho \int d^4x \text{Tr}(\delta S/\delta B \cdot \delta S/\delta \bar{C})$ which vanishes only in the special case $\zeta \bar{\zeta} = 0$. Therefore, $S$ has to be changed in such a manner that this unwanted term is cancelled. The explicit determination is very cumbersome, it leads to the following essential result:

$$\Gamma^0(A|g, m; \xi, \rho, \sigma) = S(A|g, m; \xi, \rho, \sigma) + \frac{\zeta \bar{\zeta} \rho}{1 - 4\zeta \rho \xi} \int d^4x \text{Tr} \left( \frac{\delta S}{\delta B} \frac{\delta S}{\delta B} \right). \quad (23)$$

As can be seen by an explicit computation this expression fulfills the identities (19) – (22) and, therefore, constitutes the correct starting point for perturbative renormalization. In addition, it holds $\tau \delta \Gamma^0/\delta B = \delta S/\delta B$ with $\tau = (1 - \rho \xi)$.

Here, it should be remarked that $S(A|g, m; \xi, 0, 0)$ also fulfills the identities (19) – (22) and, in fact, this restricted action was the expression we formerly started from. But, as is easily seen, it is not stable against small perturbations, i.e. not all the field monomials $\Delta$ introduced in such a way that $S + \epsilon \Delta$ fulfills the identities (19) – (22) up to order $O(\epsilon^2)$ can be obtained from $S$ by a variation of the parameters of that restricted action. Therefore it had to be extended by introducing the $\rho$– and $\sigma$–dependent combinations of the fields and the associated sources.
4 Gauge and consistency conditions

The auxiliary field $B$, despite of the fact that it does not decouple from the dynamics, is not a primary one. However, its dynamics can be constrained to be the same as for the classical action by the following gauge condition (which, in fact, is the renormalized equation of motion):

$$\tau \frac{\delta \Gamma}{\delta B} = \xi (2B + (\zeta - \bar{\zeta}) \frac{\delta \Gamma}{\delta S}) + D_\mu(A)(G^\mu - Q^\mu) + [Q_\mu, G^\mu] + \{U, \bar{C}\} - \{\bar{U}, C\}. \quad (24)$$

Here, we emphasize that, contrary to the common belief, (24) is a well posed condition since its nonlinear content is respected by the functional derivation of $\Gamma$ with respect to the source $S$.

Applying now $\delta / \delta B$ on the ST identity an equation completely analogous to (24) obtains for $\delta \Gamma / \delta \bar{C}$ which is nothing else than the antighost equation of motion:

$$\tau \frac{\delta \Gamma}{\delta \bar{C}} = \xi (2m^2 C - \delta \Gamma / \delta \bar{U}) - D_\mu(A) \left( (1 - \sigma)P^\mu - \delta \Gamma / \delta \bar{P}^\mu \right) - [Q_\mu, P^\mu] \quad (25)$$

$$+ [\bar{C}, \bar{R}] + [\bar{U}, \delta \Gamma / \delta \bar{R}] + [G_\mu, \delta \Gamma / \delta \bar{P}^\mu] - [C, S + \zeta \delta \Gamma / \delta B] + [U, B + \zeta \delta \Gamma / \delta S];$$

an analogous condition for the ghost equation of motion obtains by $C$-conjugation.

If now $\delta / \delta \bar{C}$ is applied on the anti–ST identity we finally get a further independent condition, the local Lee identity:

$$(D_\mu(A) \Delta^\mu(\sigma) + W)\Gamma = m^2 D_\mu(A)Q^\mu \quad \text{with} \quad \Delta^\mu(\sigma) \equiv (1 - \sigma) \frac{\delta}{\delta Q^\mu} + \frac{\delta}{\delta G^\mu}. \quad (26)$$

Surprisingly, this identity is independent of $\xi$, and also of $\zeta$ and $\rho$. It expresses a consistence condition related to the Type III symmetry (which can be seen more directly if the $A_\mu$-dependence of $\Gamma$ is taken into account). It should be remarked that in the case $A_\mu = 0$, where the KSZ identity disappears, the integrated Lee identity equals the rigid Ward identity.

Together with the identities (19) – (22) the conditions (24) – (26) constitute a basic set of algebraic relations which express the symmetries to be fulfilled by the vertex functional. Additional consistency conditions cannot appear since the linearized ST- and DS-operations fulfil (anti-) commutation relations analogous to (13) – (16); and the BPT-identity is not independent, but follows by the help of $\partial S_m(\Gamma) / \partial \zeta = 0$.

As a consequence of the gauge and consistency conditions (24) – (26) the $z$-factors of the theory are nonlinearly related (see [4]); especially it holds $1 - \sigma z_\sigma = (1 - \sigma)z_0$ which is relevant for checking (5) using (31) and (32) below.

Up to now we were concerned with strictly massive theories. If massless fields were to be involved, subtractions at zero momenta would generate spurious infrared divergences due to the occurrence of propagators with vanishing mass. However, the subtraction procedure of Lowenstein and Zimmermann [5] is free of spurious infrared divergences. In this procedure a mass $m^2(s-1)^2$ is put in the free propagator.
where \( s (0 \leq s \leq 1) \) plays the role of an additional infrared subtraction parameter. Lowenstein, Zimmermann and Speer \[11\] have proved that, for non-exceptional external momenta, the limit \( s \rightarrow 1 \) defines a massless theory. This is established by the Lowenstein-Zimmermann equation \( m \partial \Gamma / \partial m = O(s - 1) \). The differential operator \( m \partial / \partial m \) applied to \( \Gamma \) generates, up to terms vanishing for \( s = 1 \), symmetrical insertions. Therefore, expanding \( m \partial / \partial m \) on a basis of symmetrical insertions and choosing suitable \( s \)-independent renormalization conditions it can be proven that all coefficients in this expansion vanish at any order of perturbation theory (thereby it is supposed that the coefficients vanish at zeroth order). Thus for \( s = 1 \) one gets a functional \( \Gamma \) which no longer depends on the auxiliary mass \( m \).

5 Background dependence of Greens functions

Now, combining the KSZ– and the Lee–identity we obtain a functional differential equation for \( \Gamma \) where the derivative w.r. to \( A_\mu \) is \textit{linear} and related to the derivatives w.r. to \( Q_\mu \) and \( G_\mu \), namely

\[
\left( \frac{\delta}{\delta A_\mu} - (1 - \sigma) \frac{\delta}{\delta Q_\mu} - \frac{\delta}{\delta G_\mu} \right) \Gamma(A|Q, G, \Phi) + m^2 Q_\mu = 0. \tag{27}
\]

More precisely, originally this equation (27) occurs with the background covariant derivative \( D_\mu(A) \) acting from the left. Therefore, its solution is determined up to a background covariant constant functional, like \( D_\nu(A) F_{\mu \nu}(A) \). However, since the classical action \( (12) \) being the tree approximation of \( \Gamma \) does not depend on such terms, also the solution of (27) should not depend on them.

With this boundary condition the solution of (27) is given by

\[
\Gamma(A|Q, G, \Phi) = \exp(A \cdot \Delta) \Gamma(0|Q, G, \Phi) - m^2 \left\{ 1 + \frac{1}{2}(A \cdot \Delta) \right\}(A \cdot Q), \tag{28}
\]

with the obvious abbreviation \( (X \cdot Y) = \int d^4x \text{Tr}(X_\mu Y_\mu) \). From this it follows that – up to \( m^2 \)-dependent term – the \( A_\mu \)-dependence of \( \Gamma(A|Q, G, \Phi) \) is given by a shift of \( Q \) and \( G \) in \( \Gamma(0|Q, G, \Phi) \), namely

\[
\Gamma(A|Q, G, \Phi) = \Gamma(0|Q + (1 - \sigma)A, G + A, \Phi) - m^2 \left( A \cdot (Q + \frac{1}{2}(1 - \sigma)A) \right). \tag{29}
\]

By this approach we obtained a complete description of the background field dependence for the vertex functional. It should be noted that this was possible because (27) is a linear differential equation, and this results from the fact that a nonlinear gauge has been chosen. Let us further remark that our derivation is purely algebraic. Therefore, the solution holds to any order of perturbation theory. Furthermore, without much effort it can be shown that \( A_\mu \) will not be renormalized. Therefore, it is sufficient to prove renormalizability of \( \Gamma(0|Q, G, \Phi) \).

The obtained result may be transfered to the generating functional \( Z \) of the complete Green functions by applying a Legendre transformation. Therefore let us introduce \( W(A|J, G, \Phi) = i\hbar \ln Z(A|J, G, \Phi) \), being defined through \( \Gamma(A|Q, G, \Phi) + \)
\[ W(A|J, G, \Phi) + (J \cdot Q) + (K \cdot B) + (L \cdot C) - (\bar{L} \cdot \bar{C}) = 0. \]

Then the differential equation corresponding to (27) is given by

\[
\left( \frac{\delta}{\delta A_\mu} - \frac{\delta}{\delta G_\mu} + m^2 \frac{\delta}{\delta J_\mu} - (1 - \sigma) \frac{J_\mu}{i \hbar} \right) Z(A|J, G, \Phi) = 0.
\]

(30)

In the same manner as above the solution for the Greens functions is obtained

\[
Z(A|J, G, \Phi) = \exp\left\{ (1 - \sigma) \left( A \cdot (J - \frac{1}{2} m^2 A) \right) / i \hbar \right\} Z(0|J - m^2 A, G + A, \Phi).
\]

(31)

Furthermore, comparing the Lee identity for \( G_\mu \neq 0; \Phi = 0 \),

\[
\left( \frac{1 - \sigma}{i \hbar} \partial_\mu J^\mu - m^2 \partial_\mu \frac{\delta}{\delta J_\mu} + D_\mu(G) \frac{\delta}{\delta G_\mu} + [J_\mu, \frac{\delta}{\delta J_\mu}] \right) Z(0|J, G, 0) = 0,
\]

with the corresponding one for \( G_\mu = 0 \) (being deduced from the rigid Ward identity)

\[
\left( (i \hbar z_Q)^{-1} \partial_\mu J^\mu + [J_\mu, \frac{\delta}{\delta J_\mu}] \right) Z(0|J, 0, 0) = 0,
\]

the \( G_\mu \)-dependence of the renormalized functional \( Z(0|J, G, 0) \) may be determined explicitly:

\[
Z(0|J, G, 0) = \exp \left\{ (\sigma z_\sigma / i \hbar z_Q) \left( (J + \frac{1}{2} m^2 G) \cdot G \right) \right\} Z(0|J, 0, 0).
\]

(32)

Combining (31) and (32) we obtain for the physical Green functions in Landau gauge \((\xi = 0)\), being restricted by the gauge condition \( D_\mu(A) \delta Z_{\text{phys}}(A|J) / \delta J_\mu = 0 \) and \( \zeta \)-independence through \( \partial Z_{\text{phys}}(A|J) / \partial \zeta = 0 \), in the limit \( m^2 = 0, G_\mu = 0 \) exactly Rouet’s result equ. (5) above.

At this stage we have to comment that in paper [3] the fact has been overlooked that the verification of equ. (10) of that paper presupposes an assumption which is not proved. By our method, being completely different from the one used by Rouet, this gap has been closed and his result is now verified.

6 Concluding remarks

Here the Curci-Ferrari model of massive gauge fields in nonlinear gauge has been generalized to the presence of a generic background field – not necessarily being a solution of the field equations. The symmetries of the model are enlarged due to the appearance of the background field. From the KSZ- and the Lee identity a linear differential equation for 1PI vertex functional and, after Legendre transformation, an analogous one for the generating functional of the Greens functions is obtained. Their solutions relate the vertex resp. Green functions with background field \( A_\mu \) to the corresponding one for \( A_\mu = 0 \). In particular, it is shown that the on–shell amputated Green functions in the background field \( A_\mu \) are deduced from the physical Green functions in the vacuum by a mere translation of the gauge field; this verifies a former result of Rouet.
Furthermore, it is claimed that $A_\mu$ is not renormalized and, therefore, the problem of renormalization of the theory with background is reduced to the corresponding problem of the theory without background field. This, however, has been shown in an earlier paper \[12\]. There, it is proved, that only three independent $z$–factors appear, namely $z_Q$, $z_B$ and $z_g$ (see also \[1\]). A more detailed presentation of the above results will be given elsewhere.

Finally let us remark that this method of enlarging the symmetries can be applied also to the special cases of instanton and meron configurations. Despite the fact that in these cases additional zero modes appear which are to be considered in the same spirit as has been done for the gauge zero modes the formalism introduced here works with some additional technicalities.

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