A critical assessment of some inhomogeneous pressure Stephani models

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I. INTRODUCTION

One of the ways to solve the dark energy problem\textsuperscript{1} is to consider non-uniform models of the Universe which could explain the acceleration only due to inhomogeneity\textsuperscript{2,3}. There is a suggestion that we live in a spherically symmetric void of density described by the Lemaître-Tolman-Bondi (LTB) concentric dust spheres model\textsuperscript{4}. The simplest inhomogeneous cosmological models are spherically symmetric of which category LTB models are complementary to Stephani models examples\textsuperscript{5,6}. The former have inhomogeneous density \(\rho(t,r)\) (variable density dust shells) while the latter have inhomogeneous pressure \(p(t,r)\) (variable pressure shells). Apparently, due to a conservative approach related to the matter content (dust) most of the cosmologists investigate the LTB models rather than Stephani models.

In view of large expansion of investigations related to LTB models as nearly the only example of an inhomogeneous cosmology, we emphasize that in these models comoving observers do not follow geodesics, in particular comoving perfect fluids have necessarily a radial dependent pressure. We consider a subclass of these models characterized by some inhomogeneity parameter \(\beta\). We show that also the velocity of sound, like the (effective) equation of state parameter, of comoving perfect fluids acquire a transition from standard dust, presumably ruling this model out as a realistic cosmology. The only way to accept these models is to keep all standard matter components of the universe including dark energy and take an inhomogeneity parameter \(\beta\) small enough.

We consider spherically symmetric inhomogeneous Stephani universes, the center of symmetry being our location. We emphasize that in these models comoving observers do not follow geodesics, in particular comoving perfect fluids have necessarily a radial dependent pressure. We consider a subclass of these models characterized by some inhomogeneity parameter \(\beta\). We show that also the velocity of sound, like the (effective) equation of state parameter, of comoving perfect fluids acquire a transition from standard dust, presumably ruling this model out as a realistic cosmology. The only way to accept these models is to keep all standard matter components of the universe including dark energy and take an inhomogeneity parameter \(\beta\) small enough.

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In this paper we would like to fill in the gap. One of the benefits of Stephani cosmology is that it possesses a totally spacetime inhomogeneous generalization\textsuperscript{5,6}. Our investigations can be considered as the first step towards developing more models of such a type - i.e. the universes which describe real inhomogeneity of space (for a review see e.g. Ref. \textsuperscript{7,8}) - not only those which possess a rather unrealistic center of the Universe which is against the Copernican principle. The Stephani universes have been the first inhomogeneous models ever compared with supernovae data\textsuperscript{9} - and proved that they could be fitted to it. LTB models were theoretically explored much earlier, but observationally tested later\textsuperscript{10}.

In general there is a lot of inhomogeneous models which are exact solutions of the Einstein field equations - not just the perturbations of the isotropic and homogeneous Friedmann models. Curiously, observations are practically made from just one point in the Universe (apart from the redshift drift\textsuperscript{11}) and extend only onto the unique past light cone of the observer on the Earth. Even the cosmic microwave background radiation is observed from one point so that its observations prove isotropy of the Universe, but not necessarily its homogeneity\textsuperscript{12}. A more general question is whether one should first start with model-independent observations of the past light cone and then make conclusions related to geometry of...
the Universe (cf. Ref. 13). In fact, it is difficult to really differentiate between have an inhomogeneous model of the Universe with the same number of parameters as a homogeneous dark energy model which both fit observations.

In Ref. 14 it has been shown for a Stephani model that the inhomogeneity could mimic the dark energy in the sense that they produce the same redshift-magnitude relation. The inhomogeneity has dominated the universe quite recently, so it influenced only slightly the Doppler peaks and did not influence big-bang nucleosynthesis at all. Other Stephani models have been studied in Ref. 15, where they were tested against cosmic microwave background data. In Ref. 16 the effect of redshift drift for the Stephani models has been calculated. It emerged that for large redshifts the drift for Stephani models and ΛCDM models exhibits the behavior which is like the redshift drift in LTB models. Besides, the drift becomes positive for small redshifts and approaches the behavior of the ΛCDM model, which allows negative values of the drift, for very high redshifts. For a very large inhomogeneity, the drift becomes positive for larger and larger redshift, and in the limit of inhomogeneity-dominated universe the relation is linear drift and the drift is always positive.

Our paper is organized as follows. In Sec. II we present the properties of inhomogeneous pressure Stephani models. In Sec. III we present in details the particular model we will study, emphasizing analogies and important differences with standard cosmological models. In Sec. IV we derive expressions for standard quantities used in order to constrain our model. Sec. V is devoted to the comparison of our particular Stephani universe with observational data. Finally in Sec. VI we summarize our results and give our conclusion.

II. INHOMOGENEOUS PRESSURE COSMOLOGY

The inhomogeneous pressure Stephani universe appears as the only spherically symmetric solution of Einstein equations for a perfect-fluid energy-momentum tensor $T^{ab} = (\rho c^2 + p) u^a u^b + p g^{ab}$ (ρ is the mass density, p is the pressure, $g^{ab}$ is the metric tensor, $u^a$ is the 4-velocity vector) which is conformally flat and can be embedded in a five-dimensional flat pseudoeuclidean space $\mathbb{E}_5[5]$. A general model has no spacetime symmetries at all, but its three-dimensional hyperspaces of constant time are maximally symmetric (admit 6 Killing vectors acting on space) like in the Friedmann universe. In this paper we consider only a spherically symmetric subcase of the Stephani model which reads as (one uses a Friedmann-like time coordinate $\bar{u}$)

$$ds^2 = -\frac{a^2}{\bar{u}^2} \left[ \frac{d\bar{u}}{\bar{u}} \right]^2 c^2 dt^2 + \frac{a^2}{V^2} (dr^2 + r^2 d\Omega^2), \quad (II.1)$$

where

$$V(t, r) = 1 + \frac{1}{4} k(t) r^2, \quad (II.2)$$

and $(\ldots)' \equiv \partial/\partial t$, $c$ is the velocity of light. The function $a(t)$ plays the role of a generalized scale factor, $k(t)$ has the meaning of a time-dependent “curvature index,” and $r$ is the radial coordinate. The models are characterized by the nonzero expansion $\Theta$ and the nonzero acceleration $\ddot{a}$.

The mass density and the pressure for a comoving perfect fluid are given by

$$\rho(t) = \frac{3}{8\pi G} \left[ \frac{\dot{a}^2(t)}{a^2(t)} + \frac{k(t) c^2}{a^2(t)} \right], \quad (II.3)$$

$$p(t, r) = \left[ -1 + \frac{1}{5} \frac{\dot{a}(t)}{a(t)} \left( \frac{V(t, r)}{a(t)} \right) \right] \rho(t) c^2$$

$$\equiv w_c(t, r) \rho(t) c^2, \quad (II.4)$$

where $G$ is the gravitational constant, and $w_c(t, r)$ is an effective spatially dependent barotropic index. Of course one can have more than one (comoving) perfect fluid as it is the case in a realistic cosmology. We will adress in great details in Section III a particular class of these models and show the analogy and fundamental differences with standard Friedmann cosmological universes. The metric (II.1) is written in terms of the isotropic coordinate which is related to a standard Friedmann coordinate $\tilde{r}$ via the transformation $\tilde{r} = r/(1 + (1/4)k r^2)$ [or its inverse $r = 2\tilde{r}/(1 + \sqrt{1 - k \tilde{r}^2})$]. This allows to make a good reference to a general Stephani model which is usually represented in Cartesian coordinates $\mathbb{E}_5[6]$. The local curvature $k(t)$ of the spacelike constant time hypersurfaces changes and the Universe may either “open up” to become negatively curved or “close down” to become positively curved. In a standard de Sitter or Friedmann case only a constant local topology spatial sections $k = 1$ ($S^3$ topology), $k = 0$ ($R^3$ topology), $k = -1$ ($H^3$ topology) are allowed. Stephani models have standard big-bang singularities ($a \to 0$, $\rho \to \infty$, $p \to \infty$) as well as finite density (FD) singularities of pressure which appear at some particular value of the radial coordinate $r$ $[16, 17]$, which resemble sudden future singularities (SFS) $[18]$ of Friedmann cosmology. The difference is that singularities occur as singularities of spatial coordinates rather than a time coordinate. SFS may appear in inhomogeneous Stephani universes, independently of the FD singularities $[19]$. In Stephani models there is also the spacelike II boundary $[17]$ which divides each negative curvature $k(t) < 0$ hypersurface onto the two sheets (the “far sheet” and the “near sheet” $[5]$), and appears when the function $V(t, r)$ in (II.2) is zero. On a II boundary the Universe behaves asymptotically like de Sitter. Eqs. (II.3) and
show that there is no global equation of state - it changes from shell to shell, where it is momentarily fixed. The nonvanishing components of the 4-velocity and the 4-acceleration vectors are

\[ u_t = -\frac{c}{V}, \quad u_r = -c \frac{V_r}{V}. \] (II.5)

The acceleration scalar is

\[ \dot{u} \equiv (u_a \dot{u}^a)^{\frac{1}{2}} = \frac{V_r}{a}, \] (II.6)

The components of the vector tangent to a null geodesic are

\[ k^t = \frac{V^2}{a}, \quad k^r = \pm \frac{V^2}{a^2} \sqrt{1 - \frac{h^2}{r^2}}, \quad k^\theta = 0, \quad k^\phi = \frac{h V^2}{a^2 r^2}, \] (II.7)

where \( h = \text{const.}, \) and the plus sign applies to a ray moving away from the center of symmetry, while the minus sign applies to a ray moving towards the center. The constant \( h \) and the angle \( \phi \) between the direction of observation and the direction defined by the observer and the center of symmetry are related by

\[ \cos \phi = \pm \sqrt{1 - \frac{h^2}{r^2}}. \] (II.8)

The angle \( \phi \) should be taken into account when one considers off-center observers \( [20, 21] \).

Out of the inhomogeneous models most attraction has recently been given to the Lemaitre-Tolman-Bondi universe which is the only spherically symmetric solution of Einstein equations for a pressureless matter energy-momentum tensor \( T^{ab} = \rho c^2 u^a u^b \) with a spatially dependent “curvature index” \( k(r) \). In LTB models there exists “shell-crossing” singularities \([22]\), which are of a weak type in the sense of Tipler and Królock \([23]\) and are similar to generalized sudden future singularities (GSFS) \([24]\) in the Friedmann universes and do not exhibit geodesic incompleteness \([25, 26]\). LTB models are characterized by the nonzero expansion \( \Theta \) and nonzero shear \( \sigma_{ab} \). In LTB models a big-bang singularity is not necessarily instantaneous.

In Refs. \([6, 20]\) two exact spherically symmetric Stephani models were found:

- model I which fulfills the condition \( (V/a)^{-} = 0 \)
- model II which fulfills the condition \( (k/a)^{-} = 0 \).

An example of model I \([6, 20]\) is given by:

\[ a(t) = \frac{1}{\gamma t + \delta}, \quad k(t) = \frac{\alpha t + \sigma}{\gamma t + \delta}, \] (III.9)

with the units of the constants given by: \( \alpha = \text{Mpc s}^{-1} \), \( \sigma = \text{Mpc} \), \( \gamma = \text{Mpc}^{-1} \) s\(^{-1} \), and \( \delta = \text{Mpc}^{-1} \). The metric (III.1) takes the form

\[ ds^2 = \frac{a^2}{V^2} \left[ -\left(\frac{a}{a}\right)^2 \left(\gamma + \frac{\alpha}{4} r^2\right) + c^2 dt^2 + dr^2 + r^2 d\Omega^2 \right]. \] (III.10)

Using (III.3) and (III.4) one has for this model

\[ \varrho(t) = \frac{3}{8\pi G} \left[ \gamma^2 c^2 + \left(\gamma t + \delta\right)^2 + c^2 (at + \sigma)(\gamma t + \delta) \right], \] (III.11)

\[ p(t, r) = \frac{3 c^2}{8\pi G} \left[ \frac{1}{3} \left(\gamma t + \delta\right)^2 + c^2 (at + \sigma) \right] \times \left\{ -\left(\gamma^2 + c^2 \alpha (\gamma t + \delta) + c^2 \gamma (at + \sigma) \right) \right\}. \]

The simplest subcase of (III.9) is when \( \sigma = \delta = 0 \), since we obtain a Friedmann universe with

\[ a(t) = \frac{1}{\gamma t^2}, \quad k(t) = \frac{\alpha}{\gamma} = \text{const.} = \alpha(a) t, \] (III.12)

and this is a phantom-dominated model with \( w = -5/3 \) \([27]\) (which has an interesting null geodesic completeness features \([28]\)). In the limit \( t \to 0 \) one has a big-rip singularity with \( a \to \infty, \rho \to \infty, \) and \( p \to \infty \), while in the limit \( t \to \infty \) one has \( a \to 0, \rho \to \infty, \) and \( p \to \infty \) (though it also depends on the radial coordinate \( r \)). If \( \sigma \neq 0 \) and \( \delta \neq 0 \), then we have the limits: a) \( t \to 0, a \to 1/\delta, \) \( k \to 1/\delta, \rho \to \) const., and

\[ p \to \frac{3 c^2}{8\pi G} \left\{ \gamma^2 + c^2 \delta \right\} \]

\[ + \left(\gamma^2 + c^2 \sigma \right) \left(\gamma^2 + \frac{c^2}{2} \right) \]

\[ \left(\gamma^2 + \frac{c^2}{2} \right) \].

b) \( t \to \infty, a \to 0, k \to a/\gamma, \rho \to \infty, p \to \infty \) and the singularities of pressure appear for \( | r | = 2\sqrt{\gamma/\alpha} \). The expansion of the curvature function \( k(t) \) for small \( t \to 0 \) gives

\[ k(t) \approx \frac{\sigma}{\delta} - \left(\frac{\alpha}{\delta} + \frac{\gamma \sigma}{\delta} \right) t + O(t^2). \] (III.14)

On the other hand, for large \( t \gg \delta/\gamma \) one has the expansion

\[ k(t) \approx \frac{\alpha}{\gamma} + \left(\frac{\sigma}{\gamma} - \frac{\alpha}{\gamma} \right) \frac{1}{t} + O(1/t^2). \] (III.15)

III. MODEL II

We will consider in detail a subclass of Model II and constrain its parameters with observations. For this model, the factor in front of \( dt^2 \) in the metric (III.1) reduces to \(-1/V^2\), hence the line element reads

\[ ds^2 = -\frac{1}{V^2} dt^2 + \frac{a^2}{V^2} (dr^2 + r^2 d\Omega^2). \] (III.16)

We see that this model is conformally related to a flat FLRW model. A subclass of model II with

\[ k(t) = \beta a(t) \] (III.17)
\( \beta = \text{constant with units } [\beta] = \text{Mpc}^{-1} \) was found in Ref. [29]. We will consider models (III.17) here in more details and constrain them with observations.

Let us recast first the basic equations (II.3)-(II.4) in a more familiar form putting here and below \( c = 1 \)

\[
H^2 = \frac{8\pi G}{3} \varrho - \frac{k(t)}{a^2} = \frac{8\pi G}{3} \varrho - \frac{\beta}{a} \quad \text{III.18}
\]

\[
\dot{a} V = -3H (\varrho + p) \ . \quad \text{III.19}
\]

These two equations define completely the background evolution of this cosmological universe. Of course, we can put an arbitrary number of comoving perfect fluids, each of them satisfying separately equation III.19.

In the last equality of III.18 as well as in III.19, we have used (III.17) and we have adopted the standard notation \( H \equiv \dot{a}/a \).

It is clear from (III.3) or (III.18) that \( \varrho \) depends only on time and has no spatial dependence. On the other hand, it is seen that the appearance of \( V(r,t) \) modifies the standard energy conservation equation. This forces the pressure \( p \) to depend on the coordinate \( r \). We will return to this crucial point below.

Another important point is that \( \frac{\partial}{\partial r} \) is not the case but we have \( V(r=0,t) = 1 \). This implies in particular that the evolution of \( \varrho(t) \) can be derived in \( r = 0 \) from III.19 using the standard conservation equation.

We consider here a model with only one comoving perfect fluid at low redshifts. We can have more of them even at low redshifts as they are certainly needed at higher redshifts in a realistic universe. In equations (III.23) one has in mind that the comoving perfect fluid should correspond in good approximation to dust-like matter in conventional Friedmann universes.

Let us assume as in Ref. [29] that at the center of symmetry a standard barotropic equation of state (EoS) \( p(t) = \varrho_0(t) \) holds with a time independent \( w \). This assumption gives

\[
\frac{8\pi G}{3} \varrho = \frac{A^2}{a^2(1+w)} \quad \text{III.20}
\]

where

\[
A^2 = \frac{8\pi G \varrho_0}{3} a^3 \quad \text{III.21}
\]

so that (III.18) becomes

\[
H^2 = \frac{A^2}{a^2(1+w)} - \frac{\beta}{a} \quad \text{III.22}
\]

\[
= \frac{8\pi G}{3} \left( \frac{\varrho_0}{a^3} + \frac{\varrho_{\beta,0}}{a} \right) . \quad \text{III.23}
\]

We adopt here the conventional notation for quantities defined today. Actually, this universe reduces completely to the standard FLRW universe at the center of spherical symmetry \( r = 0 \) if we identify the last term of (III.18) or (III.22) with a comoving perfect fluid with \( w_\beta = -2/3 \) (analogous to domain walls [30]) and a trivial redefinition of its energy density \( \varrho_\beta \)

\[
\frac{8\pi G}{3} \varrho_\beta = -\frac{\beta}{a} . \quad \text{III.24}
\]

We have in particular

\[
\beta = -\frac{8\pi G}{3} \varrho_{\beta,0} a_0 . \quad \text{III.25}
\]

The acceleration of the (generalized) scale factor \( a \) satisfies

\[
\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \left( 1 + 3w \right) \varrho - \varrho_\beta \quad \text{III.26}
\]

\[
= -\frac{4\pi G}{3} \left( 1 + 3w \right) \varrho - \frac{\beta}{2a} . \quad \text{III.27}
\]

which is trivially generalized when more comoving perfect fluids (e.g. radiation) are taken into account. It is obvious that \( \beta \) must be negative if we want (III.21) to make sense. Of course one can also consider universes with positive values of \( \beta \). But in that case, this model cannot serve as an alternative to conventional dark energy models though such models are studied [31]. Note that in this analogy, in sharp contrast to genuine comoving perfect fluids, the equation of state parameter \( w_\beta \) is the same everywhere i.e. for all \( r \). We stress further that the expression for the redshift \( z \) as a function of the generalized scale factor \( a \) differs from the conventional one as we will see in Section IV. Hence the constraints on equations of state coming from luminosity-distance \( d_L(z) \) gets more complicated than in standard cosmological universes as we will also see explicitly in next Section.

However for genuine perfect fluids, the effective equation of state parameter \( w_e(r,t) \) defined everywhere reads

\[
w_e(r,t) = \left[ w + \frac{\beta}{4} (1+w) a(t) r^2 \right] \quad \text{III.28}
\]

with

\[
p(r,t) = w_e(r,t) \varrho(t) . \quad \text{III.29}
\]

Hence \( w_e(r,t) \) is both time and space-dependent and we have in particular \( w_e(r=0) = w \).

Actually the radial dependence of \( w_e(r,t) \) is due to the radial dependence of the fluids pressure, while at the same time the fluids energy density is homogeneous with no spatial dependence at all. The physical reason behind this dependence is the following: a comoving observer does not follow a geodesic. Actually a geodesic observer will have a four-velocity with a non vanishing radial component, it will move in the radial direction in addition to its movement due to the expansion. In other words, in order to be comoving one needs some extra radial force acting on the observer. Analogously, in order for a perfect fluid to be comoving requires some extra radial force
which is provided here by the pressure gradient due to the radial dependence of the fluids pressure. This has of course implications which we will address below.

However, let us first return to the equation of state defined at \( r = 0 \). There is no reason why the equation of state parameter \( w \) should be constant so that we will relax this assumption and allow for any arbitrarily time evolving equation of state parameter \( w(t) \) or \( w(a) \). Of course we still have in full generality

\[
w(a) = w_c(r = 0, a) .
\]

(III.30)

Due to the fact that the time evolution of \( \varrho(t) \) can be found at \( r = 0 \), the standard result holds

\[
\varrho(a) = \varrho_0 \exp \left[ -3 \int_{a_0}^{a} \frac{da'}{a'} \left( 1 + \frac{w(a')}{a'} \right) \right] = \varrho_0 f(a) .
\]

(III.31)

We have in particular \( f(a_0) = 1 \). Similarly to the Friedmann models, one can define the critical density as \( \varrho_c = (3H^2)/(8\pi G) \) and the density parameter \( \Omega = \varrho/\varrho_c \). We have from (III.18)

\[
\frac{\varrho}{\varrho_c} - \frac{\beta}{aH^2} \equiv \Omega + \Omega_\beta = 1 ,
\]

(III.32)

valid at all times, and in particular today (at \( t = t_0 \)), \( \Omega_0 + \Omega_{\beta,0} = 1 \), with (putting here explicitly \( c \))

\[
\Omega_\beta \equiv - \frac{\beta c^2}{aH^2} \quad \text{and} \quad \beta = \frac{\varrho_0 H_0^2 c^{-2}}{} (\Omega_0 - 1) < 0 .
\]

(III.33)

(III.34)

In function of \( a \) we have

\[
H^2(a) = H_0^2 \left[ \Omega_0 f(a) + \Omega_{\beta,0} \left( \frac{\varrho_0}{a} \right) \right] .
\]

(III.35)

As we emphasized already several times, in a realistic cosmology one will have to introduce at least one more perfect fluid, namely radiation. In that case the last equations above are trivially generalized as follows

\[
\frac{\varrho}{\varrho_c} + \frac{\varrho_{\text{rad}}}{\varrho_c} - \frac{\beta c^2}{aH^2} \equiv \Omega + \Omega_{\text{rad}} + \Omega_\beta = 1 ,
\]

(III.36)

and in particular today \( \Omega_0 + \Omega_{\text{rad}} + \Omega_{\beta,0} = 1 \),

\[
H^2(a) = H_0^2 \left[ \Omega_0 f(a) + \Omega_{\text{rad},0} \left( \frac{\varrho_0}{a} \right)^4 + \Omega_{\beta,0} \frac{\varrho_0}{a} \right] ,
\]

(III.37)

and finally

\[
\beta = \frac{\varrho_0 H_0^2 c^{-2}}{} (\Omega_0 + \Omega_{\text{rad},0} - 1) < 0 .
\]

(III.38)

One may wonder why we do not append any suffix to the first term, like we do with the radiation term. This is because the first term will not behave like dust-like matter, not even at \( r = 0 \) while the radiation component does (by choice) at \( r = 0 \) with

\[
w_{\text{e,rad}}(r, t) = \left[ \frac{\varrho_{\text{rad}} + \frac{\beta}{4}(1 + w_{\text{rad}})ar^2}{a} \right]^{\frac{1}{3}} [1 + \beta ar^2] .
\]

(III.39)

(III.40)

and

\[
p_{\text{rad}}(r, t) = w_{\text{e,rad}}(r, t) \varrho_{\text{rad}}(t) .
\]

(III.41)

Like for any comoving perfect fluid, \( w_{\text{e,rad}}(r, t) \) is both time and space-dependent and we have in particular

\[
w_{\text{e,rad}}(r = 0) = \varrho_{\text{rad}} = \frac{1}{3} .
\]

(III.42)

The standard behaviour for radiation holds at \( r = 0 \).

As we have mentioned above, comoving observers do not follow geodesics, the four-velocity of geodesic observers will have a non-vanishing radial component. For this reason, the three-momentum \( \vec{p} \) of a free particle will not evolve like \( \propto V/a \). This can have important consequences for the thermal history of our universe, e.g. the distribution function of relics. Clearly all these effects should remain rather small for an acceptable cosmology.

There is another very interesting point. For perfect fluids with a barotropic equation of state of the type \( p(r, t) = w_c(r, t) \varrho(t) \), it is straightforward to compute the corresponding velocity of sound \( c_s \). Specializing to our model with \( k(t) = \beta a(t) \), the following result is obtained

\[
c^2_S(r, a) = w + \frac{\beta}{4}(1 + w)ar^2 - \frac{a}{3(1 + w)} \left[ \frac{dw}{da} + \frac{d}{da} \left( \frac{\beta}{4}(1 + w)ar^2 \right) \right] .
\]

(III.43)

where \( w \) is the equation of state parameter at \( r = 0 \). This expression simplifies when \( w \) has no time dependence, viz.

\[
c^2_S(r, a) = w + \left( w + \frac{2}{3} \right) \frac{\beta}{4}ar^2
\]

\[
= c^2_S(r = 0) + \left( c^2_S(r = 0) + \frac{2}{3} \right) \frac{\beta}{4}ar^2 .
\]

(III.44)

For a perfect fluid behaving like radiation today at \( r = 0 \), we obtain

\[
c^2_{S, \text{rad}}(r, a) = \frac{1}{3} + \frac{\beta}{4}ar^2 ,
\]

(III.45)

and for a perfect fluid behaving like dustlike matter today at \( r = 0 \), we get

\[
c^2_{S, \text{m}}(r, a) = \frac{\beta}{6}ar^2 .
\]

(III.46)

We see firstly that the velocity of sound, even for constant \( w \), develops both radial and time dependence. Secondly, for dust \( (w(r = 0) = 0) \), if the parameter \( \beta \) is negative, not only the pressure but also the velocity of sound squared will become negative for \( ar^2 \neq 0 \). Such a situation is encountered already in standard cosmology for a dark energy component with constant negative \( w \). But here we face this conceptual problem even for dust. The departure from the standard velocity of sound resulting
from (III.46) must be addressed when considering the formation of structure. Of course these problems could be addressed in the same way as for dark energy clustering.

The departure coming from (III.45) and (III.46) could further affect the CMB sound horizon and acoustic oscillations but we will show that this effect is extremely small. Clearly all these effects may be acceptable for $\beta$ sufficiently small.

Finally, it is interesting to note that very generally for any perfect fluid with constant $w$, the same standard velocity of sound is obtained both at $r = 0$ and at the time of the Big Bang $a = 0$.

It is seen from (III.44) that the departure $\Delta c_S^2$ from the standard sound velocity for a barotropic perfect fluid with constant $w$ is proportional to (putting explicitly $c$)$$\Delta c_S^2(r, a) \propto \beta a r^2 c^2 \propto -\Omega_{\beta, 0} \frac{a}{a_0} H_0^2 (a_0 r)^2 . \quad (III.47)$$As expected, the quantity $H_0 a_0 r$ has dimensions of velocity and can be conveniently estimated from$$H_0 a_0 r = 100 \beta a_0 r \text{ km/s} , \quad (III.48)$$with $h \equiv H_0/(100 \text{km/s/Mpc})$. A rough estimate of (III.47) using (III.48) indicates that (III.47) does not become too large in observational data.

Actually this quantity can be accurately computed on our past lightcone for given cosmological parameters. Indeed, extending the results of [6, 14, 29] when we have a component with a time dependent equation of state parameter $w(r = 0) = w(a)$ and taking further into account a radiation component, we have for a lightray reaching us ($r = 0$) today$$r(x) = \frac{c}{H_0 a_0} I(x) \quad (III.49)$$where$$x \equiv a/a_0 \ , \ I(x) \equiv \int_x^1 \frac{dx'}{\sqrt{\Omega_0 f(x') x'^4 + \Omega_{\text{rad}, 0} + \Omega_{\beta, 0} x'^2}} . \quad (III.50)$$The quantity $\frac{1}{4} \beta a r^2$ computed on our past lightcone is shown as a function of $x \equiv a/a_0$ on figure 1 and we see that it has a minimal value of about $9\%$ at $z \sim 4$. As expected it vanishes both at the Big Bang and today. It is even very small in the primordial era of the universe as well as at late times.

We should finally add a few words about the redshift in such universes, it will be considered in details in next Section. We will show that it is a modified function of $x$ (see (IV.1)), viz.$$1 + z = x^{-1} \left(1 + \frac{1}{4} \beta a r^2(x)\right) \quad (III.51)$$$$= x^{-1} - \frac{\Omega_{\beta, 0}}{4} I^2(x) . \quad (III.52)$$Hence we have$$\frac{1 + z(x) - \frac{\Omega_{\beta, 0}}{4}}{\frac{\Omega_{\beta, 0}}{4} x} = \frac{1}{4} \beta a r^2(x) \quad (III.53)$$$$= -\frac{\Omega_{\beta, 0}}{4} x I^2(x) \quad (III.54)$$So we have the elegant result that the quantity $\frac{1}{4} \beta a r^2$ (times $c^2$) gives the order of magnitude of the change on our past lightcone in the velocity of sound, in the equation of state parameter of commoving perfect fluids, in the modification of the metric (through the function $V$) compared to a flat Robertson-Walker metric and finally the relative change of the redshift $z(x)$ as a function of $x$. And we can conclude from the discussion above that the differences with the standard dependence of the redshift on $x$ remains rather small though not negligible (see figure 1).

It is quite clear from the results of this section that this inhomogeneous universe cannot serve as an alternative to dark energy models. Actually the crucial problem is that the $\beta$ dependent term behaves like a perfect fluid with an equation of state parameter equals to $-\frac{1}{2}$. In order to comply with the data, the “matter” component will be forced to behave very differently from standard dust already at the background level. It is nevertheless interesting to study how close to a viable universe this universe can come. In that case we must obviously have $\beta < 0$ and $\Omega_{\beta, 0} \sim 1$. The effects discussed in this section are small enough at very high redshifts so that CMB cosmological constraints can be “translated” in good approximation to our model in a self-consistent approach. Doing this analysis will give us the opportunity to derive
IV. OBSERVATIONAL CONSTRAINTS

We should first consider the redshift, a crucial theoretical and observational quantity. We proceed as in Friedmann cosmology, and consider an observer located at \( r = r_0 = 0 \) at coordinate time \( t = t_0 \). The observer receives a light ray emitted at \( r = r_e \) at coordinate time \( t = t_e \) by a comoving source and the redshift reads as \( [20, 32] \)
\[
1 + z = \frac{\left( u_0 k^3 \right)_e}{\left( u_0 k^3 \right)_0} = \frac{V(t_e, r_e)}{\frac{V(t_0, r_0)}{a(t_0)}} = \frac{a(t_0)}{a(t_e)} V(t_e, r_e) \equiv \frac{a_0}{a_e} V_e, \tag{IV.1}
\]
where we have used
\[
\frac{u_0 k^3}{a(t)} = -1 + \frac{1}{2} k(t) r^2 = -\frac{V}{a}. \tag{IV.2}
\]
obtained from \((1.5)\) and \((1.7)\). At this stage we emphasize the following very important point. When observational data are given in function of redshift, what is meant by redshift is the ratio between the observed wavelength \( \lambda_0 \) at time \( t_0 \) (at \( r = 0 \)) and the wavelength \( \lambda_e \) at emission time \( t_e \) of light emitted by a comoving source, namely \( \lambda_0/\lambda_e = 1 + z \).

If we want to use the observational data we have to make sure that the redshift defined in \((IV.1)\) retains this physical meaning. While in standard cosmology we have \( \lambda_0/\lambda_e = a_0/a_e \), in our model we have
\[
\frac{\lambda_0}{\lambda_e} = \frac{a_0}{a_e} V_e, \tag{IV.3}
\]
which indeed corresponds to the expression for \( 1 + z \) defined in \((IV.1)\). We stress once more that this is true for light emitted by comoving sources. And this brings us to the following interesting question. While the comoving fluid is comoving due to a radial dependent pressure, if matter clusters in the course of expansion it is not clear that clustered objects remain comoving. As we have mentioned in the previous Section, a test particle following a geodesic will not be comoving. This is clearly a very hard problem to solve and is beyond the scope of this work. We can only assume that the departure from a comoving movement is small enough that we can use compact objects like SNIa as comoving objects and check that this is self-consistent with the obtained best-fit models.

a) luminosity distance

Analogously, one can show that the luminosity distance reads \( [14, 29] \)
\[
dl = (1 + z)a_0 r, \tag{IV.4}
\]
and the distance modulus is
\[
\mu(z) = 5 \log_{10} d_L(z) + 25. \tag{IV.5}
\]
Interestingly, the angular diameter distance \( d_A \) and the luminosity distance \( d_L \) are related to each other in our universe exactly like in Friedmann universe, viz.
\[
d_A = \frac{a_e}{V_e} r_e = (1 + z)^{-2} d_L. \tag{IV.6}
\]
Hence the relation between both distances does not allow to discriminate between our model and the standard cosmological model.

Using the definition of redshift \((IV.1)\) and using \((III.49)\) one can write the redshift along null geodesics in function of \( x \) \([14]\)
\[
z(x) = \frac{1}{x} - 1 - \frac{\Omega_{\beta,0}}{4} \left[ \int_{x}^{1} \frac{dx'}{\sqrt{\Omega_0 f(x')r'^4 + \Omega_{\text{rad,0}} + \Omega_{\beta,0} x'^2}} \right]^2. \tag{IV.7}
\]
Inverting this function (numerically) gives us \( x(z) \). Hence the luminosity distance \((IV.4)\) reads
\[
d_L(x) = c \frac{2(1 + z(x))}{H_0} \sqrt{\frac{1/x - [1 + z(x)]}{\Omega_{\beta,0}}}. \tag{IV.8}
\]
Combining equations \((IV.7)\), \((IV.8)\) one can obtain numerically the function \( d_L(z) \) to be compared with observational data
\[
d_L(z) = c \frac{2(1 + z)}{H_0} \sqrt{\frac{x^{-1}(z) - (1 + z)}{\Omega_{\beta,0}}}. \tag{IV.9}
\]

b) redshift drift

The SNIa data have a large degeneracy in the \((w, \Omega_0)\) plane which can be broken using cluster data. In our case however, due to the non standard behaviour of “matter”, we prefer to use other data. In view of Figure \((1)\) which shows a maximal deviation around redshifts \( z \sim 4 \), it is interesting to use probes in this redshift range. Such probes do not exist at the present time though they are expected in the future (see e.g. \([34]\) ). Here we choose to use the redshift drift and the corresponding expected data. The idea of redshift drift test is to collect data from the two light cones separated by 10-20 years to look for the change in redshift of a source as a function of time and it was first noticed by Sandage and later explored by Loeb \([11]\).

Contemporary technique will allow to detect this tiny effect using planned telescopes such as the European Extremely Large Telescope (EELT) \([33, 36]\), the Thirty Meter Telescope (TMT), the Giant Magellan Telescope (GMT) or even gravitational wave interferometers DECIGO/BBO (DECi-hertz Interferometer Gravitational Wave Observatory/Big Bang Observer) \([37]\). Theoretically, the effect has already been investigated for...
the ΛCDM model, the Dvali-Gabadadze-Porrati (DGP) brane model, the matter-dominated model (CDM) [38], for LTB models [39], for backreaction timescape cosmology [40], for the axially symmetric Szekeres models [41], for the Stephani models [16] and for specific DE models (see e.g. [42]).

For our Stephani model II in which we are in the center of symmetry, the redshift drift is given by (see (A.10) from Appendix A with \( r_0 = 0 \))

\[
\delta z \equiv - H_0 \left( \frac{H}{H_0} - (1 + z) \right),
\]

where (III.37) should be used in order to express \( \frac{H}{H_0} \).

We emphasize again that, as it was also assumed when deriving the expression for luminosity distances, the emitting sources are assumed to be comoving.

c) baryon acoustic oscillations

Baryon acoustic oscillations (BAO) provide us with a standard ruler [13]. Baryon oscillations generated at the time baryons were tightly coupled to photons are found after decoupling in the matter power spectrum. This gives a constraint on the universe evolution. At the present time, BAO are measured on relatively small redshifts. The constraint can be optimised with the quantity present time, BAO are measured on relatively small redshifts.

The location of the Cosmic Microwave Background (CMB) acoustic peaks depends on the physics between us and the last scattering surface so it provides a probe of dark energy models. One quantity that can be used here for large degeneracies which is broken here using SNIa and redshift drift constraints.

### A. Numerical results

We used a Bayesian framework to confront our Stephani model with the cosmological observations discussed in the previous sections. For each cosmological probe we took the likelihood function to be Gaussian in form, i.e.

\[
p(\text{data}|\Theta) \propto \exp\left(-\frac{1}{2} \chi^2\right),
\]

where \( \Theta \) denotes the parameters of the Stephani model and "data" denotes generically the observed data for one of the three cosmological probes. For the SNIa data \( \chi^2 \) takes the form

\[
\chi^2_{\text{SN}} = \sum_{i,j=1}^{N} (C^{-1})_{ij} (\mu_{\text{obs}}(z_i) - \mu_{\text{pred}}(z_i)) \times (\mu_{\text{obs}}(z_j) - \mu_{\text{pred}}(z_j)),
\]

where \( C \) is the covariance matrix, while \( \mu_{\text{obs}}(z_i) \) and \( \mu_{\text{pred}}(z_i) \) are respectively the observed and the predicted distance modulus of the \( i \)-th Union2.1 SNIa [53]. For the CMB shift parameter \( \chi^2 \) takes the form

\[
\chi^2_{\text{CMB}} = \frac{(R - 1.725)^2}{0.018^2}.
\]

Using the data for BAO at \( z = 0.2 \) and 0.35 taken from [50], the \( \chi^2 \) is given by:

\[
\chi^2_{\text{BAO}} = (v_i - \mu_i^{\text{BAO}})(C^{-1})_{ij}^{\text{BAO}}(v_j - \mu_j^{\text{BAO}})
\]

where

\[
v = \left\{ \begin{array}{c} r_d(x(z_d); \Omega_m, \Omega_b; \Theta) \\ D_V(0.2, \Omega_m; \Theta) \\ D_V(0.35, \Omega_m; \Theta) \end{array} \right\}
\]

\( v^{\text{BAO}} = (0.1905, 0.1097) \) and

\[
C^{-1} = \begin{pmatrix} 30124 & -17227 \\ -17227 & 86977 \end{pmatrix},
\]

is the inverse of the covariance matrix. In the formula above we have also used the formula for the size of the
comoving sound horizon at the baryon dragging epoch \( r_s \) proposed in [53]:

\[
r_s(z_{\text{drag}}) = 153.5 \left( \frac{\Omega_b h^2}{0.0273} \right)^{-0.134} \left( \frac{\Omega_m h^2}{0.1326} \right)^{-0.255},
\]

with the parameters \( \Omega_b h^2 \) and \( \Omega_m h^2 \) being respectively the physical baryon and dark matter density of the \( \Lambda \)CDM model.

For the redshift drift we use the simulated data set presented in [51] (see the blue error bars in Fig. 2). This data set it is assumed to be centered on the \( \Lambda \)CDM redshift drift curve with normally distributed errors.

With this simulated data set \( \chi^2 \) takes the form:

\[
\chi^2_{\text{RD}} = \sum_{i=1}^{5} \frac{(\Delta z_{\text{obs}}(z_i) - \Delta z_{\text{theo}}(z_i))^2}{\sigma_i^2},
\]

where \( \Delta z_{\text{obs}}(z_i) \) and \( \Delta z_{\text{theo}}(z_i) \) are respectively the “observed” and the predicted value of the drift at redshift \( z_i \) and \( \sigma_i \) is the estimated error of the “observed” value of the redshift drift at \( z_i \).

In Fig. 2 we present confidence intervals (three contours, denoting roughly 68%, 95% and 99% confidence regions) for each observable.

It is evident from Fig. 3 that our Stephani model fits well the data for the SNIa, redshift drift and BAO since the related contours overlap with each other with their 1σ CL regions. However the departure of dust from a standard behaviour implied would render this model unviable. In addition it cannot comply at the same time to the CMB shift constraint. An interesting way of overcoming the latter problem is to replace the constant barotropic index (EoS parameter) \( w \) with a function \( w(a) \). Because we do not want to change the contours obtained for SNIa, BAO, and redshift drift, we take \( w(a) \) constant on the redshift interval from today up to \( z = 5 \). Further, at some value of the redshift between \( z = 5 \) and the decoupling \( z_{\text{dec}} \), we assume that the function \( w(a) \) suddenly changes its value and then remains constant up to \( z_{\text{dec}} \).

An example of such function \( w(a) \) fulfilling the above requirements is the following (see Fig.4):

\[
w(a) = w_1 + \frac{w_2}{2} [1 + \tanh(\lambda(a_{\text{tr}} - a))]. \quad \text{(IV.10)}
\]

where \( w_1, w_2, \lambda, \) and \( a_{\text{tr}} \) are constants.

As expected the Stephani model with this barotropic index \( w(a) \) [IV.10] with \( \lambda = 40, a_{\text{tr}} = 0.08 \) and \( w_1 = -0.08, w_2 = 0.4 \) agrees with the SNIa and BAO data and the shift parameter, while it recovers essentially the redshift drift in a \( \Lambda \)CDM model (see Figs. 24). Of course this represents a significant departure from a standard dust behaviour.

V. RESULTS AND CONCLUSIONS

In this paper we have discussed the Stephani models of pressure-gradient spherical shells which are complementary to the energy density varying spherical shells of the Lemaître-Tolman-Bondi models. In our Stephani model there is also a spherical symmetry and in the simplest version considered here we are assumed to be at the center of symmetry \( r = 0 \). However, a crucial difference between
FIG. 4. The time dependent equation of state parameter \( w(r = 0) = w(a) \) (see eq. (IV.10)) is plotted for the particular set of parameters \( \lambda = 40, a_{tr} = 0.08, w_2 = 0.4, w_1 = -0.08 \) and \( \Omega_{\beta, 0} = 0.68 \). Here, the transition occurs for \( a \sim 0.08 \) which corresponds to the redshift \( z \sim 10.49 \).

FIG. 6. The speeds of sound \( c_s^2(r = 0, a) \) (blue dashed curve) and \( c_s^2(r, a) \) (black curve) are shown corresponding to the barotropic index \( w(r = 0) = w(a) \), eq. (IV.10) shown on Figure 4. While \( c_s^2(r = 0, a) \) differs from \( w(a) \) in the region where \( w(a) \) changes rapidly, \( c_s^2(r, a) \) includes also the effect of the pressure gradient away from the origin.

Both inhomogeneous models is the dependence on the radial coordinate \( r \) of \( g_{00} \) in Stephani models. Comoving observers are no longer following geodesics, and this is basically why a comoving perfect fluid requires a radial dependent pressure in order to counteract the movement in the radial direction. As we have seen, this implies that the real (“effective”) physical pressure depends on both \( t \) or \( a \) as well as on \( r \). As another general property of these models, we have seen that even if the EqS parameter (barotropic index) \( w \) (defined at \( r = 0 \)) is constant, the (adiabatic) speed of sound will depend on both \( r \) and \( a \). In particular dust \( (w = 0) \) would acquire a negative speed of sound and a negative equation of state parameter \( w_e \) away from the origin in an accelerating universe.

The relative change of the redshift as a function of the (generalized) scale factor \( a \) will have a similar behaviour. While we have shown that all these effects remain relatively small, though non negligible at redshifts \( z \sim 4 \), on our past lightcone in a universe mimicking the cosmic history of our universe, it is nevertheless an interesting physical property.

We have also seen that our best fit model requires the barotropic index \( w \) to depend on the generalized scale factor \( a \) and to be in the interval \(-0.08 \lesssim w(a) \lesssim 0.3 \), presumably ruling out this model. Indeed, the integrated Sachs-Wolfe effect for example constraints severely deviations from a standard dust behaviour. We expect that the formation of structure would also be strongly affected.

Another interesting issue concerns compact objects formed through gravitational collapse. While the background perfect fluid is comoving due to its pressure gradient, it is an interesting question whether compact objects that form out of the perfect fluids perturbations will remain essentially comoving and for how long. A detailed study of all these problems including the growth of perturbations can probably not be addressed analytically and is beyond the scope of this work.

Of course, this model can always yield an acceptable universe if all standard components, including some dark energy component, are present and taking a parameter \( \beta \) which is small enough. In that case \( \beta \) can be negative as well as positive. This would be the most natural use of such models if observations would point to some slight inhomogeneity of our universe with a residual spherical symmetry around us.
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Appendix A: Redshift-drift formula for a general spherically symmetric Stephani model

We remind following Ref. [16] that the light emitted by the source at two different times \( t_o \) and \( t_o + \delta t_o \) will be observed at \( t_e \) and \( t_e + \delta t_e \) related by

\[
\int_{t_e}^{t_o} \frac{dt}{a(t)} = \int_{t_e + \delta t_e}^{t_o + \delta t_o} \frac{dt}{a(t)}. \tag{A.1}
\]

For small \( \delta t_e \) and \( \delta t_o \) we have

\[
\frac{\delta t_e}{a(t_e)} = \frac{\delta t_o}{a(t_o)}. \tag{A.2}
\]

Bearing in mind (IV.1), the redshift drift has a general definition [11]

\[
\delta z = \frac{(u_0 k^0)(t_e, t_o + \delta t_o) - (u_0 k^0)(t_o, t_o)}{(u_0 k^0)(t_0, t_0) + \partial [(u_0 k^0)(t_0, t_0)] / \partial t}, \tag{A.3}
\]

and this can be calculated to first order using the expansions (for higher order expansion see Ref. [54])

\[
(u_0 k^0)_o = (u_0 k^0)(t_0, t_0) + \frac{\partial [(u_0 k^0)(t_0, t_0)]}{\partial t} \delta t_0, \tag{A.4}
\]

\[
(u_0 k^0)_e = (u_0 k^0)(t_e, t_e) + \frac{\partial [(u_0 k^0)(t_e, t_e)]}{\partial t} \delta t_e. \tag{A.5}
\]

From (IV.2) we have

\[
\frac{\partial}{\partial t} (u_0 k^0) = - \left( \frac{1}{a} \right) - \frac{1}{4} \left( \frac{k}{a} \right) r^2. \tag{A.6}
\]

Applying (A.4), (A.5), and (A.2) we obtain

\[
\frac{\delta z}{\delta t_0} = \frac{\left[ \left( \frac{1}{a} \right) - \frac{1}{4} \left( \frac{k}{a} \right) r^2 \right]}{1 + \frac{4}{3} \frac{\alpha r_0^2}{\gamma t_0}} a(t_e) \tag{A.7}
\]

\[
\quad - \frac{\left[ \left( \frac{1}{a} \right) - \frac{1}{4} \left( \frac{k}{a} \right) r^2 \right]_o a(t_0)(1 + z)}{1 + \frac{4}{3} \frac{\alpha r_0^2}{\gamma t_0}}. \tag{A.8}
\]

From (A.7) for the model I we have

\[
\frac{\delta z}{\delta t_0} = \frac{a(t_e) (\gamma + \frac{1}{3} \alpha r_0^2) - a(t_0)(1 + z) (\gamma + \frac{1}{3} \alpha r_0^2)}{1 + \frac{4}{3} \frac{\alpha r_0^2}{\gamma t_0} + \frac{4}{5} \frac{\alpha r_0^2}{\gamma t_0}}, \tag{A.9}
\]

while for the model II we obtain

\[
\frac{\delta z}{\delta t_0} = - \frac{H_0}{1 + \frac{4}{3} \frac{kr_0^2}{Ho} + \frac{4}{5} \frac{\alpha r_0^2}{\gamma t_0}} \left[ H_e - (1 + z) \right], \tag{A.10}
\]

where \( H_e \equiv H(t_e) = \dot{a}(t_e)/a(t_e). \)

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