INVOLUTORY HOPF GROUP-COALGEBRAS AND FLAT BUNDLES OVER 3-MANIFOLDS

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Abstract. Given a group $\pi$, we use involutary Hopf $\pi$-coalgebras to define a scalar invariant of flat $\pi$-bundles over 3-manifolds. When $\pi = 1$, this invariant equals to the one of 3-manifolds constructed by Kuperberg from involutary Hopf algebras. We give examples which show that this invariant is not trivial.

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Introduction

This paper is part of a program recently initiated by Turaev [8], called Homotopy quantum field theories (HQFT), whose purpose is the study of quantum invariants for maps. A $n$-dimensional HQFT with target a space $X$ consists in associating a vector space $V(g)$ to any map $g: N \rightarrow X$, where $N$ is an oriented closed $(n-1)$-manifold, and a linear map $L(f): V(f|\partial_{-}M) \rightarrow V(f|\partial_{+}M)$ to any map $f: M \rightarrow X$, where $M$ is an oriented $n$-cobordism with $\partial M = \partial_{+}M \cup (-\partial_{-}M)$. In particular, this assignment must only depend on the homotopy classes of the maps and must satisfy that to compose two such cobordisms amounts composing their associated linear maps. When $X$ is reduced to a single point, one recovers the notion of a topological quantum field theory, as described in [1].

Fix a group $\pi$. A HQFT with target an Eilenberg-Mac Lane space of type $K(\pi, 1)$ gives rise to invariants of flat $\pi$-bundles. The “quantum” approaches of 3-manifolds invariants can be generalized to this setting to get invariants of flat $\pi$-bundles over 3-manifolds (see [8] for the Reshetikhin-Turaev’s one and [11] for the Hennings-Kauffman-Radford’s one). The aim of this paper is to generalize the invariants of 3-manifolds constructed by Kuperberg [9] from involutary Hopf algebras to invariants of flat $\pi$-bundles over 3-manifolds.

The algebraic notion we use to replace that of a Hopf algebra is the notion of a Hopf $\pi$-coalgebra, introduced by Turaev in [8] and studied by the author in [12]. Briefly speaking, a Hopf $\pi$-coalgebra is a family $H = \{H_{\alpha}\}_{\alpha \in \pi}$ of algebras (over a field $\mathbb{k}$) endowed with a comultiplication $\Delta = \{\Delta_{\alpha,\beta}: H_{\alpha}\beta \rightarrow H_{\alpha} \otimes H_{\beta}\}_{\alpha,\beta \in \pi}$, a
morphism $\varepsilon: H_1 \to k$, and an antipode $S = \{ S_\alpha: H_\alpha \to H_{\alpha^{-1}} \}_{\alpha \in \pi}$ which verify some compatibility conditions. The case $\pi = \{1\}$ is the standard setting of Hopf algebras.

Fix a Hopf $\pi$-coalgebra $H = \{ H_\alpha \}_{\alpha \in \pi}$ which is involutory, that is, such that its antipode verifies $S_\alpha S_\alpha = \text{id}_{H_\alpha}$ for all $\alpha \in \pi$. Let $\varepsilon = (\varepsilon: p: M \to M)$ be a flat $\pi$-bundle over a 3-manifold $M$. Suppose that $\varepsilon$ is pointed and denote its base point by $\tilde{x} \in M$. Let $x = p(\tilde{x}) \in M$. Then there is a uniquely defined homomorphism $\pi_1(M, x) \to \pi$, called monodromy of $\varepsilon$ at $\tilde{x}$. We define a scalar invariant $K_H(\varepsilon, \tilde{x})$ of the pointed flat $\pi$-bundle $(\varepsilon, \tilde{x})$ as follows: we present the base space $M$ of $\varepsilon$ by a Heegaard diagram, we color this diagram with $\pi$ by using the monodromy of $\varepsilon$ at $\tilde{x}$, and we associate to this “$\pi$-colored” Heegaard diagram some structure constants of $H = \{ H_\alpha \}_{\alpha \in \pi}$. The proof of this result consists in showing that the “$\pi$-colored” Reidemeister-Singer moves report the equivalence of pointed flat $\pi$-bundles over 3-manifolds, and in verifying the invariance of $K_H$ under these moves by using the properties of involutory Hopf $\pi$-coalgebras.

The invariant $K_H$ is not trivial (we give examples of computation).

We study the dependence of $K_H(\varepsilon, \tilde{x})$ in the base point $\tilde{x}$. In particular, we have that $K_H(\varepsilon, \tilde{x})$ only depends on the path-connected component of $\tilde{x}$ and that $K_H(\varepsilon, \tilde{x})$ is independent of the choice of $\tilde{x}$ when $\pi$ is abelian.

If $\pi = \{1\}$ and $M$ is a 3-manifold, then $K_H(\text{id}_M: M \to M)$ coincides with the invariant of $M$ constructed by Kuperberg [4].

This paper is organized as follows. In Section 1, we review properties of Hopf $\pi$-coalgebras. In Section 2, we construct an invariant of $\pi$-colored Heegaard diagrams. In Section 3, we show that this invariant yields to an invariant of pointed flat $\pi$-bundles over 3-manifolds. Finally, we give examples in Section 4.

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Throughout this paper, we let $\pi$ be a group (with neutral element $1$) and $k$ be a field. All algebras are supposed to be over $k$, associative, and unitary. The tensor product $\otimes = \otimes_k$ is always assumed to be over $k$. If $U$ and $V$ are $k$-spaces, $\sigma_{U, V} : U \otimes V \to V \otimes U$ will denote the flip defined by $\sigma_{U, V}(u \otimes v) = v \otimes u$ for all $u \in U$ and $v \in V$.

## 1. Hopf Group-Coalgebras

In this section, we review definitions and properties concerning Hopf group-coalgebras. For a detailed treatment, we refer to [12].

### 1.1. Hopf $\pi$-coalgebras

Following [12], a Hopf $\pi$-coalgebra (over $k$) is a family $H = \{ H_\alpha \}_{\alpha \in \pi}$ of coalgebras endowed with a family $\Delta = \{ \Delta_{\alpha, \beta}: H_\alpha \otimes H_\beta \to H_\alpha \otimes H_\beta \}_{\alpha, \beta \in \pi}$ of algebra homomorphisms (the comultiplication), an algebra morphism $\varepsilon: H_1 \to k$ (the counit), and a family $S = \{ S_\alpha: H_\alpha \to H_{\alpha^{-1}} \}_{\alpha \in \pi}$ of $k$-linear maps (the antipode) such that, for all $\alpha, \beta, \gamma \in \pi$,

1. $$(\Delta_{\alpha, \beta} \otimes \text{id}_{H_\alpha}) \Delta_{\alpha, \beta, \gamma} = (\text{id}_{H_\alpha} \otimes \Delta_{\beta, \gamma}) \Delta_{\alpha, \beta, \gamma},$$

2. $$(\text{id}_{H_\alpha} \otimes \varepsilon) \Delta_{\alpha, 1} = \text{id}_{H_\alpha} = (\varepsilon \otimes \text{id}_{H_\alpha}) \Delta_{1, \alpha},$$
and
\[ m_\alpha(S_{\alpha^{-1}} \otimes \text{id}_{H_\alpha})\Delta_{\alpha^{-1}, \alpha} = \varepsilon 1_\alpha = m_\alpha(\text{id}_{H_\alpha} \otimes S_{\alpha^{-1}})\Delta_{\alpha, \alpha^{-1}}, \]
where \( m_\alpha : H_\alpha \otimes H_\alpha \to H_\alpha \) and \( 1_\alpha \in H_\alpha \) denote respectively the multiplication and unit element of \( H_\alpha \).

When \( \pi = 1 \), one recovers the usual notion of a Hopf algebra. In particular \((H_1, m_1, 1_1, \Delta_1, \varepsilon, S_1)\) is a Hopf algebra.

Remark that the notion of a Hopf \( \pi \)-coalgebra is not self-dual, and that if \( H = \{ H_\alpha \}_{\alpha \in \pi} \) is a Hopf \( \pi \)-coalgebra, then \( \{ \alpha \in \pi \mid H_\alpha \neq 0 \} \) is a subgroup of \( \pi \).

A Hopf \( \pi \)-coalgebra \( H = \{ H_\alpha \}_{\alpha \in \pi} \) is said to be of finite type if, for all \( \alpha \in \pi \), \( H_\alpha \) is finite-dimensional (over \( k \)). Note that it does not mean that \( \oplus_{\alpha \in \pi} H_\alpha \) is finite-dimensional (unless \( H_\alpha = 0 \) for all but a finite number of \( \alpha \in \pi \)).

The antipode of a Hopf \( \pi \)-coalgebra \( H = \{ H_\alpha \}_{\alpha \in \pi} \) is anti-multiplicative: each \( S_\alpha : H_\alpha \to H_{\alpha^{-1}} \) is an anti-homomorphism of algebras, and anti-comultiplicative: \( \varepsilon S_1 = \varepsilon \) and \( \Delta_{\beta^{-1}, \alpha^{-1}} S_{\alpha \beta} = \sigma \Delta_{\alpha^{-1}, \beta^{-1}} S_{\alpha \beta} \Delta_{\alpha, \beta} \) for any \( \alpha, \beta \in \pi \), see [12] Lemma 1.1.

The antipode \( S = \{ S_\alpha \}_{\alpha \in \pi} \) of \( H = \{ H_\alpha \}_{\alpha \in \pi} \) is said to be bijective if each \( S_\alpha \) is bijective. As for Hopf algebras, the antipode of a finite type Hopf \( \pi \)-coalgebra is always bijective (see [14] Corollary 3.7(a)).

We extend the Sweedler notation for a comultiplication to the setting of a Hopf \( \pi \)-coalgebra \( H = \{ H_\alpha \}_{\alpha \in \pi} \) in the following way: for any \( \alpha, \beta \in \pi \) and \( h \in H_{\alpha \beta} \), we write \( \Delta_{\alpha, \beta}(h) = \sum (h) h_{(1, \alpha)} \otimes h_{(2, \beta)} \in H_\alpha \otimes H_\beta \), or simply, if we leave the summation implicit, \( \Delta_{\alpha, \beta}(h) = h_{(1, \alpha)} \otimes h_{(2, \beta)} \). The coassociativity of \( \Delta \) gives that, for any \( \alpha, \beta, \gamma \in \pi \) and \( h \in H_{\alpha \beta \gamma} \),
\[ h_{(1, \alpha \beta)}(1, \alpha) \otimes h_{(1, \alpha \beta)}(2, \beta) \otimes h_{(2, \gamma)} = h_{(1, \alpha)} \otimes h_{(2, \beta \gamma)}(1, \beta) \otimes h_{(2, \beta \gamma)}(2, \gamma). \]
This element of \( H_\alpha \otimes H_\beta \otimes H_\gamma \) is written as \( h_{(1, \alpha)} \otimes h_{(2, \beta)} \otimes h_{(3, \gamma)} \). By iterating the procedure, we define inductively \( h_{(1, \alpha_1)} \otimes \cdots \otimes h_{(n, \alpha_n)} \) for any \( h \in H_{\alpha_1 \cdots \alpha_n} \).

If \( H = \{ H_\alpha \}_{\alpha \in \pi} \) is a Hopf \( \pi \)-coalgebra with bijective antipode and \( H_\alpha^{\text{cop}} \) denotes the opposite algebra to \( H_\alpha \), then \( H^{\text{cop}} = \{ H_\alpha^{\text{cop}} \}_{\alpha \in \pi} \), endowed with the comultiplication and counit of \( H \) and with the antipode \( S^{\text{cop}} = \{ S_\alpha^{\text{cop}} \}_{\alpha \in \pi} \), is a Hopf \( \pi \)-coalgebra, called opposite to \( H \).

Let \( H = \{ H_\alpha \}_{\alpha \in \pi} \) be a Hopf \( \pi \)-coalgebra with bijective antipode. Set \( H_\alpha^{\text{cop}} = H_{\alpha^{-1}} \) as an algebra, \( \Delta_{\alpha, \beta}^{\text{cop}} = \sigma \Delta_{\beta^{-1}, \alpha^{-1}}, \varepsilon^{\text{cop}} = \varepsilon \), and \( S_{\alpha}^{\text{cop}} = S_{\alpha^{-1}} \). Then \( H^{\text{cop}} = \{ H_\alpha^{\text{cop}} \}_{\alpha \in \pi} \) is a Hopf \( \pi \)-coalgebra, called coopposite to \( H \).

1.2. The case \( \pi \) finite. Let us suppose that \( \pi \) is a finite group. Recall that the Hopf algebra \( F(\pi) = k^\pi \) of functions on \( \pi \) has a basis \( \{ e_\alpha : \pi \to k \}_{\alpha \in \pi} \) defined by \( e_\alpha(\beta) = \delta_{\alpha, \beta} \) where \( \delta_{\alpha, \beta} = 1 \) and \( \delta_{\alpha, \beta} = 0 \) if \( \alpha \neq \beta \). The structure maps of \( F(\pi) \) are given by \( e_\alpha e_\beta = \delta_{\alpha, \beta} e_\alpha, 1_{F(\pi)} = \sum_{\alpha \in \pi} e_\alpha, \Delta(e_\alpha) = \sum_{\beta, \gamma} e_\beta \otimes e_\gamma, \varepsilon(e_\alpha) = \delta_{\alpha, 1}, \) and \( S(e_\alpha) = e_{\alpha^{-1}} \).

By a central prolongation of \( F(\pi) \) we shall mean a Hopf algebra \( A \) endowed with a morphism of Hopf algebras \( F(\pi) \to A \) which sends \( F(\pi) \) into the center of \( A \). The morphism \( F(\pi) \to A \) is called the central map of \( A \).

Since \( \pi \) is finite, any Hopf \( \pi \)-coalgebra \( H = \{ H_\alpha \}_{\alpha \in \pi} \) gives rise to a Hopf algebra \( \hat{H} = \oplus_{\alpha \in \pi} H_\alpha \) with structure maps given by \( \Delta|_{H_\alpha} = \sum_{\beta, \gamma} \delta_{\beta, \gamma} \Delta_{\beta, \gamma}, \varepsilon|_{H_\alpha} = \delta_{\alpha, 1}, \hat{m}|_{H_\alpha \otimes H_\beta} = \delta_{\alpha, \beta} m_\alpha, \hat{1} = \sum_{\alpha \in \pi} 1_\alpha, \) and \( \hat{S} = \sum_{\alpha \in \pi} S_\alpha \).
The $\mathbb{k}$-linear map $F(\pi) \to \tilde{H}$ defined by $e_\alpha \mapsto 1_\alpha$ clearly gives rise to a morphism of Hopf algebras which sends $F(\pi)$ into the center of $\tilde{H}$. Hence $\tilde{H}$ is a central prolongation of $F(\pi)$.

As noticed by Enriquez [3], the correspondence which assigns to every Hopf $\pi$-coalgebra $H = \{H_\alpha\}_{\alpha \in \pi}$ the central prolongation $\tilde{H}$ of $F(\pi)$ is one-to-one. Indeed, let $(A,m,\Delta,\varepsilon,S)$ be a central prolongation of $F(\pi)$. Denote by $1_\alpha \in A$ the image of $e_\alpha \in F(\pi)$ under the central map $F(\pi) \to A$ of $A$. Set $H_\alpha = A1_\alpha$ for any $\alpha \in \pi$. Since $F(\pi) \to A$ is a morphism of Hopf algebras and each $1_\alpha \in A$ is central, we have that the family $\{H_\alpha\}_{\alpha \in \pi}$ is a Hopf $\pi$-coalgebra with structure maps given by $m_\alpha = 1_\alpha \cdot m|_{H_\alpha \otimes H_\alpha}$, $\Delta_{\alpha,\beta} = (1_\alpha \otimes 1_\beta) \cdot \Delta|_{H_\alpha \otimes H_\beta}$, $\varepsilon = \varepsilon|_{H_1}$, and $S_\alpha = 1_{\alpha^{-1}} \cdot S|_{H_\alpha}$. Furthermore we have that $\tilde{H} = A$ as a central prolongation of $F(\pi)$, where $\tilde{H} = \oplus_{\alpha \in \pi} H_\alpha$ is the central prolongation of $F(\pi)$ associated to $\{H_\alpha\}_{\alpha \in \pi}$ as above.

1.3. $\pi$-integrals. Let us recall that a left (resp. right) integral for a Hopf algebra $(A,\Delta,\varepsilon,S)$ is an element $\lambda \in A^*$ such that $\lambda A = \varepsilon(x)\Lambda$ (resp. $\Lambda x = \varepsilon(x)\lambda$) for all $x \in A$. A left (resp. right) integral for the dual Hopf algebra $A^*$ is a $\mathbb{k}$-linear form $\lambda \in A^*$ verifying $(\text{id}_A \otimes \lambda)(\Delta(x)) = \lambda(x)1_A$ (resp. $(\lambda \otimes \text{id}_A)(\Delta(x)) = \lambda(x)1_A$) for all $x \in A$.

By a left (resp. right) $\pi$-integral for a Hopf $\pi$-coalgebra $H = \{H_\alpha\}_{\alpha \in \pi}$, we shall mean a family of $\mathbb{k}$-linear forms $\lambda = (\lambda_\alpha)_{\alpha \in \pi} \in \Pi_{\alpha \in \pi} H^*_\alpha$ such that

$$\text{(4) } (\text{id}_{H_\alpha} \otimes \lambda_\beta)(\Delta_{\alpha,\beta}(x)) = \lambda_{\alpha\beta}(x)1_\alpha \quad \text{(resp. } (\lambda_\alpha \otimes \text{id}_{H_\beta})(\Delta_{\alpha,\beta}(x)) = \lambda_{\alpha\beta}(x)1_\beta\text{)}$$

for all $\alpha, \beta \in \pi$ and $x \in H_{\alpha\beta}$.

Note that $\lambda_1$ is a usual left (resp. right) integral for the Hopf algebra $H_1^*$.

A $\pi$-integral $\lambda = (\lambda_\alpha)_{\alpha \in \pi}$ is said to be non-zero if $\lambda_\beta \neq 0$ for some $\beta \in \pi$. Note that a non-zero $\pi$-integral $\lambda = (\lambda_\alpha)_{\alpha \in \pi}$ verifies that $\lambda_\alpha \neq 0$ for all $\alpha \in \pi$ such that $H_\alpha \neq 0$ (and in particular $\lambda_1 \neq 0$).

It is known that the space of left (resp. right) integrals for a finite-dimensional Hopf algebra is one-dimensional. In the setting of Hopf $\pi$-coalgebras, we also have that the space of left (resp. right) $\pi$-integrals for a finite type Hopf $\pi$-coalgebra is one-dimensional (even when $\pi$ is infinite), see [2, Theorem 3.6].

1.4. Semisimplicity. A Hopf $\pi$-coalgebra $H = \{H_\alpha\}_{\alpha \in \pi}$ is said to be semisimple if each algebra $H_\alpha$ is semisimple. Note that a necessary condition for $H$ to be semisimple is that $H_1$ is finite-dimensional (since any infinite-dimensional Hopf algebra over a field is never semisimple, see [7, Corollary 2.7]). When $H$ is of finite type, then $H$ is semisimple if and only if $H_1$ is semisimple (see [4, Lemma 5.1]).

1.5. Cosemisimplicity. The notion of a comodule over a coalgebra may be extended to the setting of Hopf $\pi$-coalgebras. Namely, a right $\pi$-comodule over a Hopf $\pi$-coalgebra $H = \{H_\alpha\}_{\alpha \in \pi}$ is a family $M = \{M_\alpha\}_{\alpha \in \pi}$ of $\mathbb{k}$-spaces endowed with a family $\rho = \{\rho_{\alpha,\beta} : M_\alpha \to M_{\alpha\beta} \otimes H_\beta\}_{\alpha,\beta \in \pi}$ of $\mathbb{k}$-linear maps such that $(\rho_{\alpha,\beta} \otimes \text{id}_{H_\alpha})\rho_{\alpha,\beta,\gamma} = (\text{id}_{M_{\alpha\alpha}} \otimes \Delta_{\beta,\gamma})\rho_{\alpha,\beta,\gamma}$ and $(\text{id}_{M_{\alpha\alpha}} \otimes \varepsilon)\rho_{\alpha,1} = \text{id}_{M_{\alpha\alpha}}$ for all $\alpha, \beta, \gamma \in \pi$. A $\pi$-subcomodule of $M$ is a family $N = \{N_\alpha\}_{\alpha \in \pi}$, where $N_\alpha$ is a $\mathbb{k}$-subspace of $M_\alpha$ such that $\rho_{\alpha,\beta}(N_{\alpha\beta}) \subset N_\alpha \otimes H_\beta$ for all $\alpha, \beta \in \pi$. The notion of sum and direct sum of a family of $\pi$-submodules of a right $\pi$-comodule may be defined in the obvious way.

A right $\pi$-comodule $M = \{M_\alpha\}_{\alpha \in \pi}$ is said to be simple if it is non-zero (i.e., $M_\alpha \neq 0$ for some $\alpha \in \pi$) and if it has no $\pi$-subcomodules other than $0 = \{0\}_{\alpha \in \pi}$.
and itself. A right \( \pi \)-comodule which is a direct sum of a family of simple \( \pi \)-subcomodules is said to be cosemisimple. A Hopf \( \pi \)-coalgebra is called cosemisimple if it is cosemisimple as a right \( \pi \)-comodule over itself (with comultiplication as structure maps).

There exists (see \([4, \text{Theorem 5.4}]\)) a Hopf \( \pi \)-coalgebra version of the dual Maschke Theorem: a Hopf \( \pi \)-coalgebra \( H = \{H_\alpha\}_{\alpha \in \pi} \) is cosemisimple if and only if there exists a right \( \pi \)-integral \( \lambda = (\lambda_\alpha)_{\alpha \in \pi} \) for \( H \) such that \( \lambda_\alpha(1_\alpha) = 1 \) for all \( \alpha \in \pi \) with \( H_\alpha \neq 0 \). In particular, when \( H \) is of finite type, we have that \( H \) is cosemisimple if and only if the Hopf algebra \( H_1 \) is cosemisimple.

### 1.6. Crossed Hopf \( \pi \)-coalgebras

The notion of a crossing for a Hopf \( \pi \)-coalgebra is crucial to define the quasitriangularity of a Hopf \( \pi \)-coalgebra (see \([12, \text{Lemma 1.7}]\)). A Hopf \( \pi \)-coalgebra \( H = \{H_\alpha\}_{\alpha \in \pi} \) is said to be crossed if it is endowed with a family \( \varphi = \{\varphi_\beta : H_\alpha \to H_{(\alpha^\beta - 1)}\}_{\alpha, \beta \in \pi} \) of algebra isomorphisms (the crossing) such that each \( \varphi_\beta \) preserves the comultiplication and the counit, i.e., \( (\varphi_\beta \otimes \varphi_\beta)\Delta_\alpha = \Delta_{\alpha^\beta - 1, \alpha_{\beta^{-1}}} \varphi_\beta \) and \( \varepsilon \varphi_\beta = 1 \) for all \( \alpha, \beta, \gamma \in \pi \), and \( \varphi \) is multiplicative in the sense that \( \varphi_{\beta'} = \varphi_{\beta} \varphi_{\beta'} \) for all \( \beta, \beta' \in \pi \).

One easily verifies that a crossing preserves the antipode, that is, \( \varphi_\beta S_\alpha = S_{\beta \alpha^\beta - \varphi_\beta} \) for all \( \alpha, \beta \in \pi \).

A particular class of crossed Hopf \( \pi \)-coalgebras is that of Hopf \( \pi \)-coalgebras with \( \pi \) abelian: if \( \pi \) is an abelian group, then a Hopf \( \pi \)-coalgebra \( H = \{H_\alpha\}_{\alpha \in \pi} \) is always crossed (e.g., by taking \( \varphi_\beta[H_\alpha] = \text{id}_{H_\alpha} \)).

When \( \pi \) is a finite group, the notion of a crossing can be described by using the language of central prolongations of \( F(\pi) \) (see Section \([3, \text{Lemma 1.7}]\)). More precisely, a central prolongation \( A \) of \( F(\pi) \) is crossed if it is endowed with a group homomorphism \( \varphi : \pi \to \text{Aut}_{\text{Hopf}}(A) \) such that \( \varphi_\beta(1_\alpha) = 1_{\beta \alpha^\beta - 1} \) for all \( \alpha, \beta \in \pi \), where \( \text{Aut}_{\text{Hopf}}(A) \) is the group of Hopf automorphisms of the Hopf algebra \( A \) and \( 1_\alpha \) denotes the image of \( e_\alpha \in F(\pi) \) under the central map \( F(\pi) \to A \).

### 1.7. Involutory Hopf group-coalgebras

In this section we give some results concerning involutory Hopf \( \pi \)-coalgebras which are used for topological purposes in Sections \([2, \text{Lemma 1.7}]\) and \([4, \text{Lemma 1.7}]\).

A Hopf \( \pi \)-coalgebra \( H = \{H_\alpha\}_{\alpha \in \pi} \) is said to be involutory if its antipode \( S = \{S_\alpha\}_{\alpha \in \pi} \) is such that \( S_{\alpha^{-1}}S_\alpha = \text{id}_{H_\alpha} \) for all \( \alpha \in \pi \).

If \( \Lambda \) is an algebra and \( x \in \Lambda \), then \( r(x) \in \text{End}_k(\Lambda) \) will denote the right multiplication by \( x \) defined by \( r(x)(a) = xa \) for any \( a \in \Lambda \). Moreover, \( \text{Tr} \) will denote the usual trace of \( k \)-linear endomorphisms of a \( k \)-space.

**Lemma 1.** Let \( H = \{H_\alpha\}_{\alpha \in \pi} \) be a finite type Hopf \( \pi \)-coalgebra with antipode \( S = \{S_\alpha\}_{\alpha \in \pi} \). Let \( \lambda = (\lambda_\alpha)_{\alpha \in \pi} \) be a right \( \pi \)-integral for \( H \) and \( \Lambda \) be a left integral for \( H_1 \) such that \( \lambda_1(\Lambda) = 1 \). Let \( \alpha \in \pi \). Then

(a) \( \text{Tr}(f) = \lambda_\alpha(S_{\alpha^{-1}}(\Lambda_{(1)}))f(\Lambda_{(1,1)}) \) for all \( f \in \text{End}_k(H_\alpha) \);

(b) \( \text{Tr}(r(a) \circ S_{\alpha^{-1}}S_\alpha) = \epsilon(\Lambda)\lambda_\alpha(a) \) for all \( a \in H_\alpha \);

(c) If \( H_\alpha \neq 0 \), then \( \text{Tr}(S_{\alpha^{-1}}S_\alpha) \neq 0 \) if and only if \( H \) is semisimple and cosemisimple;

(d) If \( H_\alpha \neq 0 \), then \( \text{Tr}(S_{\alpha^{-1}}S_\alpha) = \text{Tr}(S_\alpha^2) \).

**Proof.** To show Part (a), identify \( H_\alpha^* \otimes H_\alpha \) and \( \text{End}_k(H_\alpha) \) by \( (p \otimes a)(x) = p(x)a \) for all \( p \in H_\alpha^* \) and \( a, x \in H_\alpha \). Under this identification, \( \text{Tr}(p \otimes a) = p(a) \). Let \( f \in \text{End}_k(H_\alpha) \). We may assume that \( f = p \otimes a \) for some \( p \in H_\alpha^* \) and \( a \in H_\alpha \). By
Applying Part (b) twice, we obtain that $\text{Tr}(\text{Tr}(S_{\alpha}^{-1}(\Lambda_{(2,\alpha^{-1})}))) = p(\Lambda(1,\alpha))S_{\alpha}^{-1}(\Lambda_{(2,\alpha^{-1})})$.

Therefore

$$
\text{Tr}(f) = p(a) = \lambda_a(ba) = \lambda_a(S_{\alpha}^{-1}(\Lambda_{(2,\alpha^{-1})}))p(\Lambda(1,\alpha)a) = \lambda_a(S_{\alpha}^{-1}(\Lambda_{(2,\alpha^{-1})}))f(\Lambda(1,\alpha)).
$$

Let us show Part (b). Let $a \in H_\alpha$. Then

$$
\text{Tr}(r(a) \circ S_{\alpha}^{-1}S_{\alpha}) = \lambda_a(S_{\alpha}^{-1}(\Lambda_{(2,\alpha^{-1})}))S_{\alpha}^{-1}S_{\alpha}(\Lambda(1,\alpha)a) \quad \text{by Part (a)}
$$

$$
= \lambda_a(S_{\alpha}^{-1}(\Lambda_{(2,\alpha^{-1})}))\alpha(1,\alpha))a
$$

$$
= \lambda_a(S_{\alpha}^{-1}(\epsilon(\Lambda)1_{\alpha^{-1}}))a \quad \text{by (3)}
$$

$$
= \epsilon(\Lambda)\lambda_a(a).
$$

To show Part (c), suppose $H_\alpha \neq 0$. Since $\text{Tr}(S_{\alpha}^{-1}S_{\alpha}) = \epsilon(\Lambda)\lambda_a(1_{\alpha})$ (by Part (b)), one easily concludes using the facts that $H$ is semisimple if and only if $\epsilon(\Lambda) \neq 0$ (by [14, Lemma 5.1] and [6, Theorem 5.1.8]) and $H$ is cosemisimple if and only if $\lambda_a(1_{\alpha}) \neq 0$ (by [12, Theorem 5.4] since $H_\alpha \neq 0$).

Let us show Part (d). By using (4), we have $\lambda_1(1_1)1_{\alpha} = (\lambda_1 \otimes \text{id}_{H_\alpha})\Delta_{1,\alpha}(1_{\alpha}) = \lambda_a(1_{\alpha})1_{\alpha}$, and so $\lambda_a(1_{\alpha}) = \lambda_1(1_1)$ since $1_{\alpha} \neq 0$ (because $H_\alpha \neq 0$). Therefore, by applying Part (b) twice, we obtain that $\text{Tr}(S_{\alpha}^{-1}S_{\alpha}) = \epsilon(\Lambda)\lambda_a(1_{\alpha}) = \epsilon(\Lambda)\lambda_1(1_1) = \text{Tr}(S_1^2)$.

**Lemma 2.** Let $H = \{H_\alpha\}_{\alpha \in \pi}$ be a finite type involutory Hopf $\pi$-coalgebra over a field of characteristic $p$. Let $\alpha \in \pi$ with $H_\alpha \neq 0$. If $p = 0$ or $p > \dim H_\alpha - \dim H_1$, then $\dim H_\alpha = \dim H_1$.

**Proof.** By Lemma 1(d), we have $\text{Tr}(S_{\alpha}^{-1}S_{\alpha}) = \text{Tr}(S_1^2)$, that is $(\dim H_\alpha)1_k = (\dim H_1)1_k$ (since $H$ is involutory). One easily concludes by using the hypothesis on the characteristic of the field $k$. 

**Lemma 3.** Let $H = \{H_\alpha\}_{\alpha \in \pi}$ be a finite type involutory Hopf $\pi$-coalgebra. Suppose that $\dim H_1 \neq 0$ in the ground field $k$ of $H$. Then $H$ is semisimple and cosemisimple.

**Proof.** This follows from Lemma 1(c), since $\text{Tr}(S_1^2) = \text{Tr}(\text{id}_{H_1}) = \dim H_1 \neq 0$. 

**1.8. Diagrammatic formalism of Hopf group-coalgebras.** The structure maps of a Hopf $\pi$-coalgebra $H = \{H_\alpha\}_{\alpha \in \pi}$ can be represented symbolically as in [3]. The multiplications $m_\alpha: H_\alpha \otimes H_\alpha \to H_\alpha$, the units elements $1_\alpha$, the comultiplication $\Delta_{\alpha,\beta}: H_{\alpha\beta} \to H_\alpha \otimes H_\beta$, the counit $\varepsilon: H_1 \to k$, and the antipode $S_\alpha: H_\alpha \to H_{\alpha^{-1}}$ are represented as in Figure 1. The inputs (incoming arrows) for the product symbols are read counterclockwise and the outputs arrows (outgoing arrows) for the comultiplication symbols are read clockwise.

In light of the associativity and coassociativity axioms (see Section 1.1), we adopt the abbreviations of Figure 1.

The combinatorics of the diagrams involving such symbolic representations of structure maps may be thought of as (sum of) products of structure constants. For example, if $(e_i)_i$ is a basis of $H_1$ and $\delta_i^{j,k} \in k$ are the structure constants of $\Delta_{1,1}$
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Figure 1. Symbolic representation of structure maps

```
m_α \rightarrow 1_α \rightarrow \Delta_{α,β} \rightarrow ε \rightarrow S_α
```

Figure 2. Diagrammatic abbreviations

defined by $Δ_{1,1}(e_i) = \sum_j δ_{i,j} e_j \otimes e_k$, then the element $C \in H_1$ represented in Figure 3(a) is given by $C = \sum_{i,k} δ_{i,k} e_k$.

```
C \rightarrow \Delta_{1,1} \rightarrow T_α = \begin{array}{c}
\Delta_{α,1}
\end{array}
```

(a) $C \in H_1$

(b) $T_α : H_α \rightarrow k$

Similarly, if $(e_i)_j$ is a basis of $H_α$ and $μ^k_{i,j} \in k$ are the structure constants of $m_α$ defined by $m_α(e_i \otimes e_j) = \sum_k μ^k_{i,j} e_k$, then the $k$-linear form $T_α : H_α \rightarrow k$ represented in Figure 3(b) is given by $T_α(e_i) = \sum_k μ^k_{i,j} e_k$. Note that $T_α(x) = Tr(r(x))$ for any $x \in H_α$, where $r(x) \in \text{End}_k(H_α)$ denotes the right multiplication by $x$ and $Tr$ is the usual trace of $k$-linear endomorphisms.

Until the end of this section, $H = \{H_α\}_{α ∈ π}$ will denote a finite type involutory Hopf $π$-coalgebra with $\dim H_1 ≠ 0$ in the ground field $k$ of $H$.

**Lemma 4.** $T = (T_α)_{α ∈ π}$ is a non-zero two-sided $π$-integral for $H$ and $C$ is a non-zero two-sided integral for $H_1$ which verify that $T_1(1_1) = ε(C) = T_1(C) = \dim H_1$. Moreover $S_1(C) = C$ and $T_{α-1} \circ S_α = T_α$ for all $α ∈ π$.

**Proof.** Recall that $H$ is semisimple and cosemisimple (by Lemma 3). Therefore, by [12, Theorem 5.4 and Corollary 5.7], there exists a two-sided $π$-integral $λ = (λ_α)_{α ∈ π}$ for $H$ such that $λ_α(1_α) = 1$ for all $α ∈ π$ with $H_α ≠ 0$. Let $Δ$ be a left integral for $H_1$ such that $λ_1(Δ) = 1$. By Lemma 3(b), we have that $T_α(x) = Tr(r(x)) = ε(Δ) λ_α(x)$ for any $x ∈ H_α$. Therefore $T = (T_α)_{α ∈ π}$ is a multiple of $λ = (λ_α)_{α ∈ π}$ and so is a two-sided $π$-integral for $H$, which is non-zero since $ε(Δ) ≠ 0$ (because $H_1$ is
semisimple, see [3, Theorem 5.1.8]). Likewise \( C = \lambda_1(1_1) \Lambda = \Lambda \) (by Lemma [3](b) applied to the Hopf 1-coalgebra \( H_1^1 \)) and so \( C \) is a non-zero left integral for \( H_1 \). Moreover \( C \) is a right integral for \( H_1 \) (since \( H_1 \) is semisimple and so its integrals are two-sided).

Since \( \lambda_1(1_1) = \lambda_1(\Lambda) = 1 \) and by Lemma [3](b), we have that \( T_1(C) = T_1(1_1) = \varepsilon(C) = \varepsilon(\Lambda) = \text{Tr}(\text{id}_{H_1}) = \dim H_1 \).

Since \( H \) is cosemisimple, [12, Theorem 4.2(c) and Corollary 5.7] give that \( T_{\alpha^{-1}} \circ S_{\alpha} = T_\alpha \) for all \( \alpha \in \pi \). Finally, \( S_1(C) \) is a left integral for \( H_1 \) and so there exists \( k \in \mathbb{k} \) such that \( S_1(C) = kC \). Now \( k = 1 \) since \( C = \Lambda \), \( \lambda_1(\Lambda) = 1 \), and \( \lambda_1 \circ S_1 = \lambda_1 \). Hence \( S_1(C) = C \). 

\[ \Delta_{\alpha_1,\ldots,\alpha_n} = C \xrightarrow{T_\alpha} \Delta_{\alpha_1,\ldots,\alpha_n} \quad \text{if } \alpha_1 \cdots \alpha_n = 1 \]

**Lemma 5.** The two tensors represented by the diagrams of Figure 4 are cyclically symmetric.

**Proof.** Let \( \alpha \in \pi \). Since \( (T_\alpha)_{\beta \in \pi} \) is a right \( \pi \)-integral for \( H \) (by Lemma [3]) and the Hopf algebra \( H_1 \) is semisimple and so unimodular, [12, Theorem 4.2(a)] gives that

\[ T_\alpha(xy) = T_\alpha(S^{-1}_\alpha S_\alpha(y)x) = T_\alpha(xy) \]

for all \( x, y \in H_\alpha \). Hence \( T_\alpha(x_1 x_2 \cdots x_n) = T_\alpha(x_2 \cdots x_n x_1) \) for all \( x_1, \ldots, x_n \in H_\alpha \).

Since \( H \) is cosemisimple and \( C \) is a left integral for \( H_1 \), by using [12, Corollaries 4.4 and 5.7] we have that, for all \( \alpha \in \pi \),

\[ C_{(1,\alpha)} \otimes C_{(2,\alpha^{-1})} = S^{-1}_{\alpha^{-1}} S_{\alpha}(C_{(2,\alpha)})C_{(1,\alpha^{-1})} = C_{(2,\alpha)} \otimes C_{(1,\alpha^{-1})}. \]

Therefore, for all \( \alpha_1, \ldots, \alpha_n \in \pi \) such that \( \alpha_1 \cdots \alpha_n = 1 \), we obtain

\[
C_{(1,\alpha_1)} \otimes \cdots \otimes C_{(n-1,\alpha_n^{-1})} \otimes C_{(n,\alpha_n)} \\
= \ (C_{(1,\alpha_n^{-1})} \otimes \cdots \otimes (C_{(1,\alpha_n^{-1})})_{(n-1,\alpha_n^{-1})} \otimes C_{(2,\alpha_n)} \\
= \ (C_{(2,\alpha_n^{-1})} \otimes \cdots \otimes (C_{(2,\alpha_n^{-1})})_{(n-1,\alpha_n^{-1})} \otimes C_{(1,\alpha_n)} \\
= \ C_{(2,\alpha_1)} \otimes \cdots \otimes C_{(n,\alpha_n)} \otimes C_{(1,\alpha_n)}.
\]

\[ \square \]

2. Invariants of colored Heegaard diagrams

In this section, we define \( \pi \)-colored Heegaard diagrams and their equivalence. Then, starting from an involutory Hopf \( \pi \)-coalgebra, we construct an equivalence invariant of \( \pi \)-colored Heegaard diagrams.
2.1. Colored Heegaard diagrams. By a Heegaard diagram, we shall mean a triple \( D = (S, u, l) \) where \( S \) is a closed, connected, and oriented surface of genus \( g \geq 1 \) and \( u = \{u_1, \ldots, u_g\} \) and \( l = \{l_1, \ldots, l_g\} \) are two systems of pairwise disjoint closed curves on \( S \) such that the complement to \( \cup_k u_k \) (resp. \( \cup_i l_i \)) is connected. Note that if a sphere with \( g \) handles is cut along \( g \) disjoint circles that do not split it, then a sphere from which \( 2g \) disks have been deleted is obtained.

The circles \( u_k \) (resp. \( l_i \)) are called the upper (resp. lower) circles of the diagram. By general position we can (and we always do) assume that \( u \) and \( l \) are transverse. Note that \( u \cap l \) is then a finite set. The Heegaard diagram \( D \) is said to be oriented if all its lower and upper circles are oriented.

Let \( D = (S, u, l) \) be an oriented Heegaard diagram. Denote by \( g \) the genus of \( S \). Fix an alphabet \( X = \{x_1, \ldots, x_g\} \) in \( g \) letters. For any \( 1 \leq i \leq g \), travelling along the lower circle \( l_i \) gives a word \( w_i(x_1, \ldots, x_g) \) as follows:

- Start with the empty word \( w_i = \emptyset \);
- Make a round trip along \( l_i \) following its orientation. Each time \( l_i \) encounters an upper circle \( u_k \) at some crossing \( c \in l_i \cap u_k \) (for some \( 1 \leq k \leq g \)), replace \( w_i \) by \( w_i x_k \) where:
  \[
  \nu = \begin{cases} 
  +1 & \text{if } (d_c l_i, d_c u_k) \text{ is an oriented basis for } T_c S, \\
  -1 & \text{otherwise};
  \end{cases}
  \]
- After a complete turn along \( l_i \), one gets \( w_j \).

Note that the word \( w_i \) is well-defined up to conjugacy by some word in the letters \( x_1, \ldots, x_g \) (this is due to the indeterminacy in the choice of the starting point on \( l_i \)).

We say that the oriented Heegaard diagram \( D \) is \( \pi \)-colored if each upper circle \( u_k \) is provided with an element \( \alpha_k \in \pi \) such that \( w_i(\alpha_1, \ldots, \alpha_g) = 1 \in \pi \) for all \( 1 \leq i \leq g \). The system \( \alpha = (\alpha_1, \ldots, \alpha_g) \) is called the color of \( D \).

Two \( \pi \)-colored Heegaard diagrams are said to be equivalent if one can be obtained from the other by a finite sequence of the following moves (or their inverse):

**Type I (homeomorphism of the surface):** By using an orientation-preserving homeomorphism of a (closed, connected, and oriented) surface \( S \) to a (closed, connected, and oriented) surface \( S' \), the oriented upper (resp. lower) circles on \( S \) are carried to the oriented upper (resp. lower) circles on \( S' \). The colors of the upper circles remain unchanged.

**Type II (orientation reversal):** The orientation of an upper or lower circle is changed to its opposite. For an upper circle \( u_i \), its color \( \alpha_i \) is changed to its inverse \( \alpha_i^{-1} \).

**Type III (isotopy of the diagram):** We isotop the lower circles of the diagram relative to the upper circles. If this isotopy is in general position, it reduces to a sequence of two-point moves shown in Figure 5. The colors of the upper circles remain unchanged.

![Figure 5. Two-point move](image-url)
Type IV (stabilization): We remove a disk from $S$ which is disjoint from all upper and lower circles and replace it by a punctured torus with one upper and one lower (oriented) circles. One of them corresponds to the standard meridian and the other to the standard longitude of the added torus, see Figure 6. The added upper circle is colored with $1 \in \pi$.

![Figure 6. Stabilization](image)

Type V (sliding a circle past another): Let $C_1$ and $C_2$ be two circles of a $\pi$-colored Heegaard diagram, both upper or both lower and let $b$ be a band on $S$ which connects $C_1$ to $C_2$ (that is, $b: I \times I \to S$ is an embedding of $[0,1] \times [0,1]$ for which $b(I \times I) \cap C_i = b(i \times I)$, $i = 1, 2$) but does not cross any other circle. The circle $C_1$ is replaced by

$$C_1' = C_1 \# b C_2 = C_1 \cup C_2 \cup b(I \times \partial I) \setminus b(\partial I \times I).$$

The circle $C_2$ is replaced by a copy $C_2'$ of itself which is slightly isotoped such that it has no point in common with $C_1'$. The new circle $C_1'$ (resp. $C_2'$) inherits of the orientation induced by $C_1$ (resp. $C_2$), see Figure 7.

![Figure 7. Circle slide](image)

If the two circles are both lower, then the colors of the upper circle remain unchanged. Suppose that the two circles are both upper, say $C_1 = u_i$ and $C_2 = u_j$ with colors $\alpha_i$ and $\alpha_j$ respectively. Set $p = b(0, \frac{1}{2}) \in u_i$ and $q = b(1, \frac{1}{2}) \in u_j$. Up to first applying a move of type II to $u_i$ and/or $u_j$, we can assume that $(d_q b(\cdot, \frac{1}{2}), d_p u_i)$ is a negatively-oriented basis for $T_p S$ and $(d_q b(\cdot, \frac{1}{2}), d_q u_j)$ is a positively-oriented basis for $T_q S$. Then the color of $u_i' = C_1'$ is $\alpha_i$ and the color of $u_j' = C_2'$ is $\alpha_j^{-1} \alpha_j$. The colors of the other upper circles remain unchanged.

One can remark that all these moves transform a $\pi$-colored Heegaard diagram into another $\pi$-colored Heegaard diagram. Indeed, for a move of type I, each word $w_i$ is replaced by a conjugate of itself. For a move of type II applied to an upper circle $u_k$, each word $w_i(x_1, \ldots, x_k, \ldots, x_g)$ is replaced by a conjugate of $w_i(x_1, \ldots, x_{k-1}, \ldots, x_g)$. For a move of type II applied to a lower circle $l_i$, the word $w_i$ is replaced by a conjugate of $w_i^{-1}$. For a move of type III between $u_k$ and $l_i$, the word $w_i$ is replaced by a conjugate of itself from which $x_k x_{k-1}^{-1}$ or $x_{k-1}^{-1} x_k$ has been inserted. For a move of type IV, the new word $w_{g+1}(x_1, \ldots, x_{g+1})$ is $x_{g+1}^{\pm 1}$. For a move of type V applied to two lower circles, say $l_i$ slides past $l_j$, the word $w_i$ is replaced by a conjugate of itself from which a conjugate of $w_j^{\pm 1}$ has been inserted.
and the other words remain unchanged (up to conjugation). For a move of type V applied to two upper circles, say \( u_i \) slides past \( u_j \), each word \( w_k(x_1, \cdots, x_j, \cdots, x_g) \) is replaced by a conjugate of \( w_k(x_1, \cdots, x_i x_j, \cdots, x_g) \) (see the assumptions on the orientation of the circles \( u_i \) and \( u_j \)). Therefore the conditions \( w_i(\alpha_1, \ldots, \alpha_g) = 1 \) are still verified when performing one of these moves.

2.2. Invariants of \( \pi \)-colored Heegaard diagrams. Fix a finite type involutory Hopf \( \pi \)-coalgebra \( H = \{ H_\alpha \}_{\alpha \in \pi} \) such that \( \dim H_1 \neq 0 \) in the ground field \( k \) of \( H \). Note that \( H \) is then semisimple and cosemisimple (by Lemma 3). Using this algebraic data, we give a method to define an invariant of \( \pi \)-colored Heegaard diagrams, which generalizes that of Kuperberg [4].

Let \( D = (S, u, l) \) be a \( \pi \)-colored Heegaard diagram with color \( \alpha = (\alpha_1, \ldots, \alpha_g) \).

To each upper circle \( u_k \), we associate the tensor of Figure 8(a), where \( c_1, \ldots, c_m \) are the crossings between \( u_k \) and \( l \) which appear in this order when making a round trip along \( u_k \) following its orientation. Since this tensor is cyclically symmetric (see Lemma 5), this assignment does not depend on the choice of the starting point on the circle \( u_k \) when making a round trip along it.

\[
\begin{array}{c}
c_1 \quad \cdots \quad c_m \\
\Delta \beta_1, \ldots, \beta_m \\
c_1 \quad \cdots \quad c_m
\end{array}
\]

(a) Tensor associated to \( u_k \)  
(b) Tensor associated to \( l_i \)

\textbf{Figure 8.}

To each lower circle \( l_i \), we associate the tensor of Figure 8(b), where \( c_1, \ldots, c_m \) are the crossings between \( l_i \) and \( u \) which appear in this order when making a round trip along \( l_i \) following its orientation, and the \( \beta_j \in \pi \) are defined as follows: if \( l_i \) intersects \( u_k \) at \( c_j \), then \( \beta_j = \alpha_k^\nu \) with \( \nu = +1 \) if \((d_{c_j} l_i, d_{c_j} u_k)\) is an oriented basis for \( T_{c_j} S \) and \( \nu = -1 \) otherwise. Note that \( \beta_1 \cdots \beta_m = w_i(\alpha_1, \ldots, \alpha_g) = 1 \) and so the tensor associated to \( l_i \) is well defined. Since this tensor is cyclically symmetric (see Lemma 5), this assignment does not depend on the choice of the starting point on the circle \( l_i \) when making a round trip along it.

Let \( c \) be a crossing point between an upper and a lower circle, say between \( u_k \) and \( l_i \). Let \( \nu \) be as above. If \( \nu = +1 \), we contract the tensors assigned to \( l_i \) and \( u_k \) as follows:

\[
\Delta \cdots \alpha_k \cdots \leftrightarrow c \quad \cdots \quad \leftrightarrow \quad \Delta \cdots \alpha_k \cdots \leftrightarrow \quad m_{\alpha_k}
\]

If \( \nu = -1 \), we contract the tensors assigned to \( l_i \) and \( u_k \) as follows:

\[
\Delta \cdots \alpha_k^{-1} \cdots \leftrightarrow c \quad \cdots \quad \leftrightarrow \quad \Delta \cdots \alpha_k^{-1} \cdots \leftrightarrow \quad S_{\alpha_k^{-1}} m_{\alpha_k}
\]

After all contractions, one gets \( Z(D) \in k \).
Finally, we set:

\[ K_H(D) = (\dim H_1)^{-g} Z(D) \in k. \]

**Theorem 1.** Let \( H = \{ H_{\alpha} \}_{\alpha \in \pi} \) be a finite type involutory Hopf \( \pi \)-coalgebra with \( \dim H_1 \neq 0 \) in the ground field \( k \) of \( H \). Then \( K_H \) is an equivalence invariant of \( \pi \)-colored Heegaard diagrams.

**Proof.** We have to verify that \( K_H \) is invariant under the moves of type I-V. The proof is similar to that of [4, Theorem 5.1], except that we have to take care of the colors of the Heegaard diagrams.

Clearly, \( K_H \) is invariant under a move of type I.

Consider a move of type II applied to an upper \( u_k \) circle with color \( \alpha_k \), that is, \( u_k \) is replaced by \( u'_k = u_k \) with the opposite orientation and with color \( \alpha_k^{-1} \). Let \( c_1, \ldots, c_n \) be the crossings between \( u_k \) and the lower circles which appear in this order following the orientation. Then the tensor associated to \( u_k \) (resp. \( u'_k \)) is:

\[ m_{\alpha_k} \quad \text{(resp. } m_{\alpha_k^{-1}} \text{)}. \]

Recall that the contraction rule applied to a crossing point \( c \in u_k \cap l_i \) is:

\[ \Delta \cdots \alpha_k \cdots \frac{\gamma}{c} \phi \quad \psi \frac{\gamma}{c} \]

where \( \nu = +1 \) and \( \phi = \text{id}_{H_{\alpha_k}} \) if \( (d_c, d_u, u_k) \) is a positively-oriented basis of \( T_cS \) and \( \nu = -1 \) and \( \phi = S_{\alpha_k^{-1}} \) otherwise. Then the contraction rule applied to the corresponding crossing point \( c' \in u'_k \cap l_i \) is:

\[ \Delta \cdots \alpha_k \cdots \frac{\gamma}{c} \psi \quad \phi \frac{\gamma}{c} \]

where \( \psi = \text{id}_{H_{\alpha_k^{-1}}} \) if \( (d_c, d_u', u'_k) \) is a positively-oriented basis of \( T_cS \) and \( \psi = S_{\alpha_k} \) otherwise. Now \( \psi = S_{\alpha_k} \circ \phi \) since the antipode is involutory. Therefore the invariance follows from the equality:

\[ S_{\alpha_k} \circ \phi \quad \psi \]

which comes from the anti-multiplicativity of the antipode (see [12, Lemma 1.1(a)]) and the fact that \( T_{\alpha^{-1}} \circ S_{\alpha} = T_{\alpha} \) for any \( \alpha \in \pi \) (by Lemma 4).

For a move of type II applied to a lower circle, the invariance follows from the equality:

\[ \Delta_{\beta_1, \ldots, \beta_m} \quad \Delta_{\beta_1, \ldots, \beta_m}^{-1} \]

which comes from the anti-multiplicativity of the antipode (see [12, Lemma 1.1(a)]) and the fact that \( T_{\alpha^{-1}} \circ S_{\alpha} = T_{\alpha} \) for any \( \alpha \in \pi \) (by Lemma 4).
which comes from the anti-comultiplicativity of the antipode (see [12, Lemma 1.1(c)]) and the fact that $S_1(C) = C$ (by Lemma [3]).

Consider now a two-point move between an upper circle with color $\alpha$ and a lower circle. Up to first applying a move of type II, we can consider that these two circles are oriented so that the invariance is a consequence of the following equality:

$$\Delta_{\ldots, \alpha, \alpha^{-1}, \ldots} \xrightarrow{S_1^{-1}} m_\alpha = \Delta_{\ldots, 1, \ldots} \xrightarrow{\varepsilon} 1_\alpha \xrightarrow{\cdot} m_\alpha = \Delta_{\ldots, \ldots}$$

which comes from [3].

A move of type IV contributes $C \to T_1 = \dim H_1$ (see Lemma [3]) to $Z(D)$, which is cancelled by normalization.

Consider a move of type V applied to two upper circles, say $u_i$ (with color $\alpha_i$) slides past $u_j$ (with color $\alpha_j$). We assume, as a representative case, that both circles have three crossings with the lower circles:

Using the anti-multiplicativity of the antipode (which allows us to consider only the positively-oriented case of the contraction rule), we have that the following factor of $Z(D)$:

$$a \xrightarrow{m_\alpha} b \xrightarrow{e} c \xrightarrow{f} d$$

is replaced by:

By using the multiplicativity of the comultiplication and the fact that $(T_\alpha)_{\alpha \in \pi}$ is a left $\pi$-integral for $H$ (see Lemma [3]), we obtain that these two factors are equal, as depicted in Figure [3].

Finally, suppose that a lower circle slides past another lower circle. We assume, as a representative case, that these two circles have both three crossings with the upper circles. Let $\alpha_1, \alpha_2, \alpha_3$ (resp. $\beta_1, \beta_2, \beta_3$) be the colors of the upper circles intersected (following the orientation) by the first (resp. second) lower circle considered. Then the invariance follows from the equality of Figure [4] which comes from the
multiplicativity of the comultiplication and the fact that \( C \) is a right integral for \( H_1 \) (see Lemma 4). This completes the proof of the theorem.

If \( D \) is a \( \pi \)-colored Heegaard diagram of genus \( g \) with color \( \alpha = (\alpha_1, \ldots, \alpha_g) \) and \( \beta \in \pi \), then \( \beta \alpha \beta^{-1} = (\beta \alpha_1 \beta^{-1}, \ldots, \beta \alpha_g \beta^{-1}) \) is clearly another color of the Heegaard diagram, said to be conjugate to the color \( \alpha \).

**Lemma 6.** Let \( H = \{ H_\alpha \}_{\alpha \in \pi} \) be a finite type involutory Hopf \( \pi \)-coalgebra with \( \dim H_1 \neq 0 \) in the ground field \( k \) of \( H \). Suppose that \( H = \{ H_\alpha \}_{\alpha \in \pi} \) is crossed (see Section 1.6). Then the invariant \( K_H(D) \) does not depend on the conjugacy class of the color of the \( \pi \)-colored Heegaard diagram \( D \).

**Proof.** Suppose that \( H = \{ H_\alpha \}_{\alpha \in \pi} \) admits a crossing \( \varphi = \{ \varphi_{\beta} : H_\alpha \to H_{\beta \alpha \beta^{-1}} \}_{\alpha, \beta \in \pi} \). Let \( D = (S,u,l) \) be a \( \pi \)-colored Heegaard diagram of genus \( g \) with color \( \alpha = (\alpha_1, \ldots, \alpha_g) \) and \( \beta \in \pi \). Then \( \beta \alpha \beta^{-1} = (\beta \alpha_1 \beta^{-1}, \ldots, \beta \alpha_g \beta^{-1}) \) is clearly another color of the Heegaard diagram, said to be conjugate to the color \( \alpha \).
Let us denote by $D^\beta$ the underlying oriented Heegaard diagram of $D$ endowed with the color $\beta \alpha$\textsuperscript{−1} = $(\beta \alpha_1 \beta^{-1}, \ldots, \beta \alpha_g \beta^{-1})$. We have to verify that $K_H(D^\beta) = K_H(D)$.

Let $1 \leq k, i \leq g$ and denote by $c_1, \ldots, c_n$ (resp. $c'_1, \ldots, c'_m$) the crossings between $u_k$ and $l$ (resp. $l_i$ and $u$) which appear in this order when making a round trip along $u_k$ (resp. $l_i$) following its orientation. Recall that, for $D$ (resp. $D^\beta$), the tensor of Figure 11(a) (resp. Figure 11(b)) is associated to the upper circle $u_k$ and the tensor of Figure 11(c) (resp. Figure 11(d)) is associated to the lower circle $l_i$ where, if $l_i$ intersects some $u_n$ at $d_j$, $\beta_j = \alpha_n^\nu$ with $\nu = 1$ if $(d_j, l_i, d_j, u_n)$ is an oriented basis for $T_{d_j} S$ and $\nu = -1$ otherwise.

Since $H$ is cosemisimple, \cite[Lemmas 6.3(a) and 7.2]{12} give that $\varphi_\beta(C) = C$ and \cite[Corollary 6.2 and Lemma 7.2]{12} give that $T_{\beta \alpha \beta^{-1}} \varphi_\beta = T_\alpha$ for all $\alpha \in \pi$. Therefore, using the multiplicativity and comultiplicativity of a crossing, we have the equalities of Figure 12.

3. Invariants of flat bundles over 3-manifolds

In this section, we use the invariant of colored Heegaard diagrams defined in Section 2 to construct an invariant of flat bundles over 3-manifolds.
3.1. Flat bundles over 3-manifolds. Fix a group $\pi$. By a flat $\pi$-bundle over a 3-manifold, we shall mean a principal $\pi$-bundle $\xi = (p: \tilde{M} \to M)$, where $M$ is a closed connected and oriented 3-manifold, which is flat, that is, such that its transition functions are locally constant. Such an object can be viewed as a regular covering $\tilde{M} \to M$ with group of automorphisms $\pi$. The space $\tilde{M}$ (resp. $M$) is called the total space (resp. base space) of $\xi$.

Two flat $\pi$-bundles over 3-manifolds $\xi$ and $\xi'$ are said to be equivalent if there exists an homeomorphism $\tilde{h}: \tilde{M} \to \tilde{M}'$ between their total spaces which preserves the action of $\pi$ and which induces an orientation-preserving homeomorphism $h: M \to M'$ between their base spaces.

A flat $\pi$-bundle $\xi = (p: \tilde{M} \to M)$ is said to be pointed when its total space $\tilde{M}$ is endowed with a base point $\tilde{x} \in \tilde{M}$. Two pointed flat $\pi$-bundles over 3-manifolds $(\xi, \tilde{x})$ and $(\xi', \tilde{x}')$ are said to be equivalent if there exists an equivalence $\tilde{h}: \tilde{M} \to \tilde{M}'$ between them such that $\tilde{h}(\tilde{x}) = \tilde{x}'$.

Let $(\xi, \tilde{x})$ be a pointed flat $\pi$-bundle over a 3-manifold. Set $x = p(\tilde{x}) \in M$, where $p: \tilde{M} \to M$ is the bundle map of $\xi$. We can associate to $(\xi, \tilde{x})$ a morphism $f: \pi_1(M, x) \to \pi$, called monodromy of $\xi$ at $\tilde{x}$, by the following procedure: any loop $\gamma$ in $(M, x)$ uniquely lifts to a path $\tilde{\gamma}$ in $\tilde{M}$ beginning at $\tilde{x}$. The path $\tilde{\gamma}$ ends at $\alpha \cdot \tilde{x}$ for a unique $\alpha \in \pi$. The monodromy is defined by $f([\gamma]) = \alpha$, where $[\gamma]$ denotes the homotopy class in $\pi_1(M, x)$ of the loop $\gamma$.

Any pointed flat $\pi$-bundle $(\xi, \tilde{x})$ over a 3-manifold $M$ leads to a triple $(M, x, f)$, where $x$ is the image of $\tilde{x}$ under the bundle map $\tilde{M} \to M$ of $\xi$ and $f: \pi_1(M, x) \to \pi$ is the monodromy of $\xi$ at $\tilde{x}$. Conversely, a triple $(M, x, f)$, where $M$ is a (closed, connected, and oriented) 3-manifold, $x \in M$, and $f: \pi_1(M, x) \to \pi$ is a group homomorphism, leads to a pointed flat $\pi$-bundle over $M$ uniquely determined up to equivalence (see Proposition 14.1]). Let us briefly recall the construction of the pointed flat $\pi$-bundle $\xi = (p: \tilde{M} \to M)$ associated to a triple $(M, x, f)$: the pointed manifold $(M, x)$ admits a universal covering $u: (Y, y) \to (M, x)$. The fundamental group $\pi_1(M, x)$ acts on the left on $Y$ as follows: given $[\sigma] \in \pi_1(M, x)$ and $z \in Y$, let $y'$ be the endpoint of the lift (in $Y$) of $\sigma$ to a path that starts on $y$. Choose a path $\gamma$ from $y$ to $z$ in $Y$. Then $[\sigma] \cdot z$ is defined to be the endpoint of the lift (in $Y$) of the path $u \cdot \gamma$ that starts at $y'$. Set $\tilde{M} = (Y \times \pi)/\pi_1(M, x)$, where $\pi_1(M, x)$ acts on the left on $Y \times \pi$ by the rule $[\sigma] \cdot (z, \alpha) = ([\sigma] \cdot z, \alpha f([\sigma])^{-1})$. Let $(z, \alpha) \in M$ denote the image of $(z, \alpha) \in Y \times \pi$ and define $p: \tilde{M} \to M$ by $p((z, \alpha)) = u(z)$.

Then $\xi = (p: \tilde{M} \to M)$ is a flat $\pi$-bundle over $M$ which is pointed with base point $\tilde{x} = (y, 1)$ and whose monodromy at $\tilde{x}$ is $f$.

It may be convenient to adopt this second point of view. In particular, under this point of view, two pointed flat $\pi$-bundles over 3-manifolds $(M, x, f)$ and $(M', x', f')$ are equivalent if there exists an orientation-preserving homeomorphism $h: M \to M'$ such that $h(x) = x'$ and $f' \circ h_* = f$, where $h_*: \pi_1(M, x) \to \pi_1(M', x')$ is the group isomorphism induced by $h$ in homotopy.

3.2. Heegaard diagrams of 3-manifolds. We first recall that a Heegaard splitting of genus $g$ of a closed, connected, and oriented 3-manifold $M$ is an ordered pair $(M_u, M_l)$ of submanifolds of $M$, both homeomorphic to a handlebody of genus $g$, such that $M = M_u \cup M_l$ and $M_u \cap M_l = \partial M_u = \partial M_l$. The handlebody $M_u$ (resp.
loops $\gamma_a$ Then the homotopy classes 
{loops in $D$}
3-balls have been deleted. 

$D_x$ for some Heegaard splitting $(M, \gamma)$ where the words $w_\gamma = D_\pi deleted, x \in g$ there exists $M_u \cap M_l$ by $S$. It is oriented as follows: for any point $p \in S$, a basis $(e_1, e_2)$ of $T_p S$ is positive if, when completing $(e_1, e_2)$ with a vector $e_3$ pointing from $M_l$ to $M_u$, we obtain a positively-oriented a basis $(e_1, e_2, e_3)$ of $T_p M$. Then $D = (S, u, l = \{u_1, \ldots, u_g\}, l = \{l_1, \ldots, l_g\})$ is a Heegaard diagram in the sense of Section 2.3. Such a Heegaard diagram is called a Heegaard diagram (of genus $g$) of $M$.

3.3. Kuperberg-type invariants of flat bundles over 3-manifolds. Fix a finite type involutory Hopf $\pi$-coalgebra $H = \{H_\alpha\}_{\alpha \in \pi}$ with $\dim H_1 \neq 0$ in the ground field $k$ of $H$.

Let $(\xi = (p: M \to M), \hat{x})$ be a pointed flat $\pi$-bundle over a 3-manifold $M$. Set $x = p(\hat{x}) \in M$ and let $f: \pi_1(M, x) \to \pi$ be the monodromy of $\xi$ at $\hat{x}$. Consider a Heegaard diagram $D = (S, u, l)$ of genus $g$ of $M$. Recall that $S = \partial M_u = \partial M_l \subset M$ for some Heegaard splitting $(M_u, M_l)$ of $M$. We arbitrarily orient the upper and lower circles so that $D$ is oriented. We can (and we always do) assume that $x \in S \setminus \{u_1\}$.

Since $S \setminus u$ is homeomorphic to a sphere from which $2g$ disks have been deleted, there exists $g$ pairwise disjoint (except in $x$) loops $\gamma_1, \ldots, \gamma_g$ on $(S, x)$ such that, for any $1 \leq i \leq g$,

- $\gamma_i$ intersects the upper circle $u_i$ in exactly one point $p_i$ in such a way that $(d_{p_i}, \gamma_i, d_{p_i} u_i)$ is a positively-oriented basis of $T_{p_i} S$;
- $\gamma_i$ does not intersect any other upper circle.

Then the homotopy classes $a_i = [\gamma_i] \in \pi_1(M, x)$ do not depend on the choice of the loops $\gamma_i$ verifying the above conditions (since each $\gamma_i$ is homotopic to a unique leaf of the $x$-based $g$-leafed rose formed by the core of the handelbody $M_u$).

**Lemma 7.** $\pi_1(M, x) = \langle a_1, \ldots, a_g \mid w_1(a_1, \ldots, a_g) = 1, \ldots, w_g(a_1, \ldots, a_g) = 1 \rangle$, where the words $w_i(x_1, \ldots, x_g)$ are defined as in Section 2.3.

**Proof.** Recall that there exists a finite collection $\{D_1, \ldots, D_g\}$ (resp. $\{D'_1, \ldots, D'_g\}$) of pairwise disjoint properly embedded 2-disks in $M_u$ (resp. $M_l$) which cut $M_u$ (resp. $M_l$) into a 3-ball and such that $u_i = \partial D_i$ and $l_i = \partial D'_i$ for $1 \leq i \leq g$. Since $\cup_{1 \leq i \leq g} D_i \cup S$ is a deformation retract of $M_u$ from which some 3-balls have been deleted, $\pi_1(\cup_{1 \leq i \leq g} D_i \cup S, x)$ is the free group generated by the homotopy classes of the loops $\gamma_1, \ldots, \gamma_g$. Now, by the Seifert-Van Kampen Theorem, gluing a disk $D'_i$ amounts adding the relation $w_I([\gamma_1], \ldots, [\gamma_g]) = 1$. Hence the lemma follows from the fact that $\cup_{1 \leq i \leq g} (D_i \cup D'_i) \cup S$ is a deformation retract of $M$ from which some 3-balls have been deleted. \[\square\]
For any $1 \leq i \leq g$, set $\alpha_i = f(a_i) \in \pi$. By Lemma 4, $\alpha = (\alpha_1, \cdots, \alpha_g)$ is a color of the oriented Heegaard diagram $D$. We say that the (oriented) Heegaard diagram $D$ of $M$ is colored by $f$.

Finally, we set:

$$K_H(\xi, \tilde{x}) = K_H(D) \in k,$$

where $K_H$ is the invariant of $\pi$-colored Heegaard diagrams of Theorem 1.

**Theorem 2.** Let $H = \{H_\alpha\}_{\alpha \in \pi}$ be a finite type involutory Hopf $\pi$-coalgebra with $\text{dim } H_1 \neq 0$ in the ground field $k$ of $H$.

(a) $K_H$ is an invariant of pointed flat $\pi$-bundles over 3-manifolds.

(b) Let $\xi = (p: M \to M)$ be a flat $\pi$-bundle over a 3-manifold $M$.

(i) The function $\tilde{x} \in \tilde{M} \mapsto K_H(\xi, \tilde{x}) \in k$ is constant on the path-connected components of $M$;

(ii) If $H$ is crossed or if $\pi$ is abelian or if the monodromy of $\xi$ is surjective, then $K_H(\xi, \tilde{x})$ does not depend on the choice of the base point $\tilde{x} \in \tilde{M}$. 

This theorem is proved in Section 3.3.

If $\pi = 1$ and $M$ is a 3-manifold, then $K_H(\text{id}_M: M \to M)$ coincides with the invariant of $M$ constructed by Kuperberg [4].

In Section 4, we give examples which show that the invariant $K_H$ is not trivial.

3.4. **Basic properties of $K_H$.** Let $H = \{H_\alpha\}_{\alpha \in \pi}$ be a finite type involutory Hopf $\pi$-coalgebra with $\text{dim } H_1 \neq 0$ in the ground field $k$ of $H$. Recall that $H^{\text{cop}}$ and $H^{\text{cop}}$ denote the opposite or coopposite Hopf $\pi$-coalgebra to $H$ (see Section 1.4).

Let $(\xi, \tilde{x})$ be a pointed flat $\pi$-bundle over a 3-manifold $M$. Denote by $-\xi$ the flat $\pi$-bundle $\xi$ whose base space $M$ is endowed with the opposite orientation. Then

$$K_H(-\xi, \tilde{x}) = K_{H^{\text{cop}}}(\xi, \tilde{x}) = K_{H^{\text{cop}}}(\xi, \tilde{x}).$$

Indeed, starting from an oriented Heegaard diagram $D = (S, u, l)$ for $M$, reversing the orientation of $M$ amounts reversing the orientation of the Heegaard surface $S$, and so the first equality of (3) can be easily obtained by reversing the orientation of the lower circles and the second one by reversing the orientation of the upper circles.

Let $(\xi_1, \tilde{x}_1)$ and $(\xi_2, \tilde{x}_2)$ be two pointed flat $\pi$-bundles over 3-manifolds. Denote by $x_1$ (resp. $x_2$) the image of $\tilde{x}_1$ (resp. $\tilde{x}_2$) under the bundle map of $\tilde{M}_1 \to M_1$ of $\xi_1$ (resp. $M_2 \to M_2$ of $\xi_2$), and by $f_1: \pi_1(M_1) \to \pi$ (resp. $f_2: \pi_1(M_2) \to \pi$) the monodromy of $\xi_1$ at $\tilde{x}_1$ (resp. of $\xi_2$ at $\tilde{x}_2$). Take closed 3-balls $B_1 \subset M_1$ and $B_2 \subset M_2$ such that $x_1 \in \partial B_1$ and $x_2 \in \partial B_2$. Glue $M_1 \setminus \text{Int}B_1$ and $M_2 \setminus \text{Int}B_2$ along a homeomorphism $h: \partial B_1 \to \partial B_2$ chosen so that $h(x_1) = x_2$ and that the orientations in $M_1 \setminus \text{Int}B_1$ and $M_2 \setminus \text{Int}B_2$ induced by those in $M_1$, $M_2$ are compatible. This gluing yields a closed, connected, and oriented 3-manifold $M_1 \# M_2$. For $i = 1$ or 2, consider the embeddings $j_i: M_i \setminus \text{Int}B_i \to M_i$ and $k_i: M_i \setminus \text{Int}B_i \to M_1 \# M_2$ and set $x = k_1(x_1) = k_2(x_2)$. By the Van Kampen theorem, since $\partial B_2 \cong h(\partial B_1)$ is simply-connected, there exists a unique group homomorphism $f: \pi_1(M_1 \# M_2, x) \to \pi$ such that $f \circ (k_i)_* = f_i \circ (j_i)_*$ ($i = 1, 2$). This leads to a triple $(M_1 \# M_2, x, f)$, and so to a pointed flat $\pi$-bundle over $M_1 \# M_2$, denoted by $(\xi_1 \# \xi_2, \tilde{x})$, whose monodromy is $f$ (see Section 3.4). Then

$$K_H(\xi_1 \# \xi_2, \tilde{x}) = K_H(\xi_1, \tilde{x}_1) K_H(\xi_2, \tilde{x}_2).$$
Indeed we can choose a Heegaard diagram for $M$ which is a connected sum of Heegaard diagrams for $M_1$ and $M_2$ and such that the colorations of these diagrams by the monodromies $f$, $f_1$, or $f_2$ are compatible with this connected sum.

3.5. **Proof of Theorem**. Let us prove Part (a) of Theorem 3. Adopting the second point of view of Section 3.1, let $(M, x, f)$ and $(M', x', f')$ be two equivalent pointed flat $\pi$-bundles over 3-manifolds. Let $D$ (resp. $D'$) be an oriented Heegaard diagrams of $M$ (resp. $M'$) colored by $f$ (resp. $f'$). By virtue of Theorem 3, it suffices to prove that $D$ and $D'$ are equivalent $\pi$-colored Heegaard diagrams, i.e., that $D$ can be obtained from $D'$ by a finite sequence of the moves of type I-V (or their inverses) described in Section 2.1.

Since $(M, x, f)$ and $(M', x', f')$ are equivalent, there exists an orientation-preserving homeomorphism $h : M \to M'$ with $f(x) = x'$ and $f = f' \circ h_\ast$, where $h_\ast : \pi_1(M, x) \to \pi_1(M', x')$ is the homomorphism induced by $h$ in homotopy. By the Reidemeister-Singer Theorem (see [3, Theorem 8] or [4, Theorem 4.1]), there exist:

- a finite sequence $M_0 = M, M_1, \ldots, M_{n-1}, M_n = M'$ of closed, connected, and oriented 3-manifolds;
- a Heegaard diagram $D_k = (S_k, u^k = \{u^k_1, \ldots, u^k_{g_k}\}, l^k = \{l^k_1, \ldots, l^k_{g_k}\})$ of genus $g_k$ of $M_k$ for each $0 \leq k \leq n$, with $D_0 = D$ and $D_n = D'$;
- a finite sequence $h_1 : M_0 \to M_1, \ldots, h_n : M_{n-1} \to M_n$ of orientation-preserving homeomorphisms;

such that $h = h_n \circ \cdots \circ h_1$ and, for any $1 \leq k \leq n$, the diagrams $D_{k-1}$ and $D_k$ are related by a move (or its inverse) of the following type:

**Type A:** $S_k = h_k(S_{k-1}), u^k_k = h_k(u^{k-1}),$ and $l^k_k = h_k(l^{k-1})$ relative to $u^k$;

**Type B:** $S_k = h_k(S_{k-1}), u^k_k = h_k(u^{k-1}),$ and $l^k = h_k(l^{k-1})$ relative to $u^k$;

**Type C:** $S_k = h_k(S_{k-1}) \# T^2, u^k = h_k(u^{k-1}) \cup \{C_1\}$, and $l^k = h_k(l^{k-1}) \cup \{C_2\}$, where $T^2$ is a torus and $\{C_1, C_2\}$ is the set formed by the standard meridian and longitude of $T^2$;

**Type D:** $S_k = h_k(S_{k-1}), u^k_k = h_k(u^{k-1}),$ and $l^k_k$ is obtained from $h_k(l^{k-1})$ by sliding one circle of $h_k(l^{k-1})$ past another circle of $h_k(l^{k-1})$, avoiding the other upper and lower circles of $h_k(S_{k-1})$;

**Type E:** $S_k = h_k(S_{k-1}), l^k = h_k(l^{k-1}),$ and $u^k$ is obtained from $h_k(u^{k-1})$ by sliding one circle of $h_k(u^{k-1})$ past another circle of $h_k(u^{k-1})$, avoiding the other upper and lower circles of $h_k(S_{k-1})$.

Set $x_0 = x \in M_0$ and define $x_k = h_k \circ \cdots \circ h_1(x) \in M_k$ for any $1 \leq k \leq n$. Note that $x_n = x'$ since $h(x) = x'$. Without loss of generality, we can assume that $x_k \in S_k \setminus \{u^k, l^k\}$. Set $f_0 = f : \pi_1(M_0, x_0) \to \pi$ and define $f_k = f \circ (h_k \circ \cdots \circ h_1)^{-1} : \pi_1(M_k, x_k) \to \pi$ for any $1 \leq k \leq n$. Since $f = h_\ast \circ f'$, we have that $f_n = f'$.

We arbitrarily orient the upper circles $u_k^k$ and the lower circles $l_k^k$ (so that each Heegaard diagram $D_k$ is oriented) and denote by $\alpha_k = (\alpha_k^1, \ldots, \alpha_k^{g_k})$ the coloration of the diagram $D_k$ by the homomorphism $f_k$.

Up to applying some moves of type II or to well-choosing the orientation of the added circles in a move of type C (or its inverse), we can assume that the orientation of the upper and lower circles are transported by the homeomorphisms $h_i$. Note that if we change the orientation of an upper circle $u_k^k$ to its inverse, then the color $\alpha_i^k = f([\gamma_i^k])$ is replaced by $f([\gamma_i^k)')^{-1}] = (\alpha_i^k)^{-1}$, where $\gamma_i^k$ is a loop on $(S_k, x_k)$.
which crosses (in a positively-oriented way) the upper circle \( u_i^k \) in exactly one point and does not intersect any other upper circle.

We have to verify that, for any \( 1 \leq k \leq n \), the colors of the diagrams \( D_{k-1} \) and \( D_k \) are related as described in the moves of type I-V of Section 2. Without loss of generality, we can assume that \( n = 1 \), that is that the diagrams \( D = (S, u, \ell) \) colored by \( f \) and \( D' = (S', u', \ell') \) colored by \( f' \) are related by moves of type I-V.

Denote the genus and color of \( u \) (resp. \( D' \)) by \( g \) (resp. \( g' \)) and \( \alpha = (\alpha_1, \ldots, \alpha_g) \) (resp. \( \alpha' = (\alpha'_1, \ldots, \alpha'_{g'}) \) respectively.

Suppose that \( D' \) is obtained from \( D \) by a move of type A. Let \( 1 \leq i \leq g' = g \) and \( \gamma_i \) be a loop on \((S, x)\) which crosses (in a positively-oriented way) the upper circle \( u_i \) in exactly one point and does not intersect any other upper circle of \( D \).

Then the orientations of \( u \) of type II to \( \gamma_i \) agree (by the same argument as above, since \( S' = h(S) \) and \( u' = h(u) \)). Therefore the \( \pi \)-colored Heegaard diagrams \( D \) and \( D' \) are related by a move of type I.

Suppose that \( D' \) is obtained from \( D \) by a move of type B. Then the colors of the upper circles \( u_i' \) and \( u_i \) agree (by the same argument as above, since \( S' = h(S) \) and \( u' = h(u) \)). Therefore the \( \pi \)-colored Heegaard diagrams \( D' \) is obtained from the \( \pi \)-colored Heegaard diagram \( D \) by a finite sequence of move of type I and III (by decomposing the isotopy into two-point moves, see Section 2).

Suppose that \( D' \) is obtained from \( D \) by a move of type C. Since \( u' = h(u) \cup \{C_1\} \), the color of the upper circle \( u_i' = h(u_i) \) agrees with that of the upper circle \( u_i \) for any \( 1 \leq i \leq g = g' - 1 \). Let \( \ell \) be a path connecting the point \( x' \) to the circle \( C_2 \) which does not intersect any upper circle of \( D' \). Then the loop \( \ell C_2 \ell^{-1} \) crosses \( C_1 \) in exactly one point and does not intersect any other upper circle of \( D' \). Set \( \nu = +1 \) if \( \ell C_2 \ell^{-1} \) crosses \( C_1 \) in a positively-oriented way and \( \nu = -1 \) otherwise. Therefore

\[
\alpha'_{g'} = f'(\ell C_2 \ell^{-1}) = f'(\ell C_2 \ell^{-1})^\nu.
\]

Now the circle \( C_2 \) is contractible in \( M' \). Thus \( \ell C_2 \ell^{-1} = 1 \in \pi_1(M', x') \) and so \( \alpha'_{g'} = 1 \in \pi \). Hence the \( \pi \)-colored Heegaard diagram \( D' \) is obtained from the \( \pi \)-colored Heegaard diagram \( D \) by a move of type I and then a move of type IV.

Suppose that \( D' \) is obtained from \( D \) by a move of type D. Since \( S' = h(S) \) and \( u' = h(u) \), the colors of the upper circles of \( D' \) and \( D \) agree. Then the \( \pi \)-colored Heegaard diagram \( D' \) is obtained from the \( \pi \)-colored Heegaard diagram \( D \) by a move of type I and then a move of type V.

Finally, suppose that \( D' \) is obtained from \( D \) by a move of type E, i.e., that \( u' \) is obtained from \( h(u) \) by sliding a circle \( h(u_i) \) past another circle \( h(u_j) \). Let \( b: I \times I \rightarrow S' \) be a band which connects \( h(u_i) \) to \( h(u_j) \) (that is, such that \( b(I \times I) \cap h(u_i) = b(0 \times I) \) and \( b(I \times I) \cap h(u_j) = b(1 \times I) \)) but does not intersect any other circle. We can also assume that \( x' \notin b(I \times I) \). Then

\[
u u_i' = h(u_i) \# h(u_j) = h(u_i) \cup h(u_j) \cup b(I \times \partial I) \setminus b(\partial I \times I)
\]

and \( u_i' \) is a copy \( h(u_j) \) which is slightly isotoped such that it has no point in common with \( u_i \). Set \( p = b(0, \frac{1}{2}) \in h(u_i) \) and \( q = b(1, \frac{1}{2}) \in h(u_j) \). Up to first applying a move of type II to \( u_i \) and/or \( u_j \), we can assume that the basis \( (d_p h(\cdot, \frac{1}{2}), d_q h(u_i)) \) for \( T_p S' \) is negatively-oriented and that the basis \( (d_p h(\cdot, \frac{1}{2}), d_q h(u_j)) \) for \( T_q S' \) is positively-oriented. Then the orientations of \( u_i' \) induced by \( h(u_i) \) and \( h(u_j) \) coincide and \( u_i' \) is
endowed with this orientation. Let $\gamma_i$ (resp. $\gamma_j$) be a loop on $(S,x)$ which crosses (in a positively-oriented way) the upper circle $u_i$ (resp. $u_j$) in exactly one point and does not intersect any other upper circle of $D$ neither the band $h^{-1}(b(I \times I))$. Let $\ell_1: I \to S'$ be a path with $\ell_1(0) = x'$ and $\ell_1(1) = p$ which does not intersect any upper circle of $D'$ and such that $(d_p\ell_1, d_p h(u_i))$ is a negatively-oriented basis for $T_p S'$. Let $\ell_2: I \to S'$ be a path with $\ell_2(0) = q$ and $\ell_2(1) = x'$ which does not intersect any upper circle of $D'$ and such that $(d_q \ell_2, d_q h(u_j))$ is a positively-oriented basis for $T_q S'$, see Figure 13.

Set $\gamma'_i = h(\gamma_i)$ (resp. $\gamma'_j = \ell_2 b(\cdot, \frac{1}{2}) \ell_1$). It is a loop on $(S', x')$ which crosses (in a positively-oriented way) the upper circle $u'_i$ (resp. $u'_j$) in exactly one point and does not intersect any other upper circle of $D'$. Therefore we have

$$\alpha'_i = f'([\gamma'_i]) = f'([h(\gamma_i)]) = f' \circ (h)_*([\gamma_i]) = f([\gamma_i]) = \alpha_i$$

and, since $\gamma'_j$ is homotopic (in $M'$) to the loop $h(\gamma_i)^{-1} h(\gamma_j)$,

$$\alpha'_j = f'([\gamma'_j]) = f'([h(\gamma_i)^{-1} h(\gamma_j)]) = f' \circ (h)_*([\gamma_i]^{-1} [\gamma_j]) = f([\gamma_i]^{-1} [\gamma_j]) = (\alpha_i)^{-1} [\gamma_j].$$

Hence the $\pi$-colored Heegaard diagram $D'$ is obtained from the $\pi$-colored Heegaard diagram $D$ by a move of type I and then a move of type V.

Let us prove Part (b.i) of Theorem 2. Let $\xi = (p: \tilde{M} \to M)$ be a flat $\pi$-bundle over a 3-manifold and $\hat{x}, \hat{x}'$ be two points in $\tilde{M}$ which belong to the same path-component. Consider a path $\hat{\gamma}$ in $\tilde{M}$ connecting $\hat{x}$ to $\hat{x}'$. Pushing $\hat{x}$ to $\hat{x}'$ along $\hat{\gamma}$ inside a tubular neighborhood of $\text{Im}(\hat{\gamma})$ in $\tilde{M}$ yields a self-homeomorphism of $\tilde{M}$ which is an equivalence between the pointed flat $\pi$-bundles $(\xi, \hat{x})$ and $(\xi, \hat{x}')$. Therefore $K_H(\xi, \hat{x}) = K_H(\xi, \hat{x}')$ by Part (a) of Theorem 2.
Let us prove Part (b.ii) of Theorem 2. Fix a $\pi$-bundle $\xi = (p: \tilde{M} \to M)$ over a 3-manifold $M$. Let $\tilde{x}, \tilde{x}'$ be two points in $\tilde{M}$. Set $x = p(\tilde{x})$ and $x' = p(\tilde{x}')$. Since $M$ is connected, there exists a path $\gamma$ in $M$ connecting $x$ to $x'$. Let $\tilde{z} \in M$ be the endpoint of the lift of $\gamma$ to a path that starts at $\tilde{x}$. Since $p(\tilde{z}) = x' = p(\tilde{x}')$, there exists $\beta \in \pi$ such that $\tilde{x}' = \beta \cdot \tilde{z}$.

Firstly, suppose that the monodromy $f$ of $\xi$ at $\tilde{z}$ is surjective. Then there exists a loop $\sigma$ based on $x'$ such that $f([\sigma]) = \beta$. Denote by $\tilde{\sigma}$ the lift of $\sigma$ to a path that starts at $\tilde{z}$. Since $\tilde{x}' = \beta \cdot \tilde{z} = f(\tilde{\sigma}) \cdot \tilde{z}$, the path $\tilde{\sigma}$ ends at $\tilde{x}'$. Finally, $\gamma \tilde{\sigma}$ is a path in $\tilde{M}$ which connects $\tilde{x}$ to $\tilde{x}'$. Hence $K_H(\xi, \tilde{x}) = K_H(\xi, \tilde{x}')$ by Part (b.i) of Theorem 2.

Secondly, suppose that the Hopf $\pi$-coalgebra $H$ is crossed (see Section 1.6). Since $\tilde{x}' = \beta \cdot \tilde{z}$, the monodromies $f_{\tilde{z}}, f_{\tilde{x}'}: \pi_1(M, x') \to \pi$ at $\tilde{z}$ and $\tilde{x}'$ are related by $f_{\tilde{x}'} = f_{\tilde{z}} \beta^{-1}$. Let $D = (S, u, l)$ be a Heegaard diagram of genus $g$ of $M$ whose upper and lower circles are arbitrarily oriented. Denote by $D_{\tilde{z}}$ and $D_{\tilde{x}'}$ the $\pi$-colored Heegaard diagrams obtained by coloring $D$ with $\tilde{z}$ and $\tilde{x}'$. Since $f_{\tilde{x}'} = f_{\tilde{z}} \beta^{-1}$, the colors of $D_{\tilde{z}}$ and $D_{\tilde{x}'}$ are conjugate. Therefore:

$$K_H(\xi, \tilde{x}') = K_H(D_{\tilde{x}'})$$
$$= K_H(D_{\tilde{z}}) \quad \text{by Lemma} 3$$
$$= K_H(\xi, \tilde{z})$$
$$= K_H(\xi, \tilde{x}) \quad \text{by Part (b.i) of Theorem 2}$$

Finally, suppose that $\pi$ is abelian. Then the Hopf $\pi$-coalgebra $H = \{H_\alpha\}_{\alpha \in \pi}$ is crossed (see Section 1.6) and so $K_H(\xi, \tilde{x}) = K_H(\xi, \tilde{x}')$ by the previous case. This completes the proof of Theorem 2.

4. Examples

In this section, we give some examples of computations of the scalar invariant of flat bundle over 3-manifolds constructed in Section 3.

4.1. Example. As remarked by Vainerman [10], the Kac-Paljutkin Hopf algebra $A = \mathbb{C}^4 \oplus \text{Mat}_2(\mathbb{C})$, viewed as a central prolongation of $F(\mathbb{Z}/2\mathbb{Z})$, leads to a finite type involutory Hopf $\mathbb{Z}/2\mathbb{Z}$-coalgebra $H = \{H_0, H_1\}$ over $\mathbb{C}$. Namely, set $H_0 = \mathbb{C}^4$ and $H_1 = \text{Mat}_2(\mathbb{C})$ as algebras. Let $\{e_1, e_2, e_3, e_4\}$ be the (standard) basis of $H_0$ and $\{e_{1,1}, e_{1,2}, e_{2,1}, e_{2,2}\}$ be the (standard) basis of $H_1$. The counit $\varepsilon: H_0 \to \mathbb{C}$ is given by $\varepsilon(e_1) = 1$ and $\varepsilon(e_2) = \varepsilon(e_3) = \varepsilon(e_4) = 0$. The comultiplication is given by

$$\Delta_{0,0}(e_1) = e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3 + e_4 \otimes e_4$$
$$\Delta_{0,0}(e_2) = e_1 \otimes e_2 + e_2 \otimes e_1 + e_3 \otimes e_4 + e_4 \otimes e_3$$
$$\Delta_{0,0}(e_3) = e_1 \otimes e_3 + e_3 \otimes e_1 + e_2 \otimes e_4 + e_4 \otimes e_2$$
$$\Delta_{0,0}(e_4) = e_1 \otimes e_4 + e_4 \otimes e_1 + e_2 \otimes e_3 + e_3 \otimes e_2$$

$$\Delta_{0,1}(e_{1,1}) = e_1 \otimes e_{1,1} + e_2 \otimes e_{2,2} + e_3 \otimes e_{1,1} + e_4 \otimes e_{2,2}$$
$$\Delta_{0,1}(e_{1,2}) = e_1 \otimes e_{1,2} - i e_3 \otimes e_{2,1} + e_4 \otimes e_{2,1}$$
$$\Delta_{0,1}(e_{2,1}) = e_1 \otimes e_{2,1} + i e_2 \otimes e_{1,2} - e_3 \otimes e_{2,1} - i e_4 \otimes e_{1,2}$$
$$\Delta_{0,1}(e_{2,2}) = e_1 \otimes e_{2,2} + e_2 \otimes e_{1,1} + e_3 \otimes e_{2,2} + e_4 \otimes e_{1,1}$$
The antipode is given by $S = e_{1,1} \otimes e_1 + e_{2,2} \otimes e_2 + e_{1,1} \otimes e_3 + e_{2,2} \otimes e_4$

The antipode is given by

$$
\Delta_{1,0}(e_1) = e_{1,1} \otimes e_1 + e_{2,2} \otimes e_2 + e_{1,1} \otimes e_3 + e_{2,2} \otimes e_4
$$

$$
\Delta_{1,0}(e_2) = e_{1,2} \otimes e_1 + i e_{2,1} \otimes e_2 - e_{1,1} \otimes e_3 - i e_{2,1} \otimes e_4
$$

$$
\Delta_{1,0}(e_3) = e_{2,1} \otimes e_1 - i e_{1,2} \otimes e_2 - e_{2,1} \otimes e_3 + i e_{1,2} \otimes e_4
$$

$$
\Delta_{1,0}(e_{2,2}) = e_{2,2} \otimes e_1 + e_{1,1} \otimes e_2 + e_{2,2} \otimes e_3 + e_{1,1} \otimes e_4
$$

The antipode is given by $S_0(e_k) = e_k$ for any $1 \leq k \leq 4$ and $S_1(e_{k,l}) = e_{1,k}$ for any $1 \leq k, l \leq 2$.

Let $n \geq 1$. There exists two flat $\mathbb{Z}/2\mathbb{Z}$-bundles $\xi_n^0$ and $\xi_n^1$ over the lens space $L(2n,1)$, whose monodromies $f_n^0, f_n^1: \pi_1(L(2n,1)) \cong \mathbb{Z}/2n\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ are respectively given by $f_n^0(1) = 0$ and $f_n^1(1) = 1$.

A Heegaard diagram $\{u_1, l_1\}$ of genus 1 of the lens space $L(2n,1)$ is given, on the torus $T = \mathbb{R}^2/\mathbb{Z}^2$, by $u_1 = \mathbb{R}(0,1) + \mathbb{Z}^2$ and $l_1 = \mathbb{R}(1, \frac{1}{2n}) + \mathbb{Z}^2$. Fix $k = 0, 1$ and set $\alpha = f_k^1(1) \in \mathbb{Z}/2\mathbb{Z}$. Denote by $D_\alpha$ the $\pi$-colored Heegaard diagram obtained from $(T, \{u_1, l_1\})$ by providing the circle $u_1$ with the color $\alpha$. Then

$$
K_H(\xi_n^k) = (\dim H_0)^{-1} K_H(D_\alpha) = \frac{1}{4} K_H(D_\alpha),
$$

where $K_H(D_\alpha) \in \mathbb{C}$ equals the tensor depicted in Figure 14(a).

\begin{figure}[h]
\centering
\begin{subfigure}{0.4 \textwidth}
\centering
\includegraphics[width=\textwidth]{khipq.png}
\caption{$K_H(D_\alpha)$}
\end{subfigure}
\hspace{2cm}
\begin{subfigure}{0.4 \textwidth}
\centering
\includegraphics[width=\textwidth]{fhipq.png}
\caption{$F_\alpha: H_0 \to H_\alpha$}
\end{subfigure}
\caption{}
\end{figure}

Let $F_\alpha: H_0 \to H_\alpha$ be the map defined in Figure 14(b). One easily checks by hand that $F_\alpha(x) = \varepsilon(x) 1_\alpha$ for all $x \in H_0$. Then, using the (co)associativity of the (co)multiplication, we get the equalities of Figure 15.

\begin{figure}[h]
\centering
\begin{subfigure}{0.4 \textwidth}
\centering
\includegraphics[width=\textwidth]{fig15.png}
\caption{}
\end{subfigure}
\hspace{2cm}
\begin{subfigure}{0.4 \textwidth}
\centering
\includegraphics[width=\textwidth]{fig15.png}
\caption{}
\end{subfigure}
\caption{}
\end{figure}
Therefore \( K_H(\xi_n^k) = K_H(\xi_n^k) \) for any \( n \geq 1 \). Hence, by computing by hand the values of \( K_H(\xi_n^k) \) for \( 1 \leq n \leq 4 \) and \( 0 \leq k \leq 1 \), we obtain that:

\[
K_H(\xi_n^0) = 4 \quad \text{and} \quad K_H(\xi_n^1) = \begin{cases} 
2 & \text{if } n \equiv 1 \pmod{2} \\
4 & \text{if } n \equiv 0 \pmod{4} \\
0 & \text{if } n \equiv 3 \pmod{4}
\end{cases}.
\]

4.2. Example. Let \( \pi \) and \( G \) be two finite groups, and \( \phi: G \to \pi \) be a group homomorphism. Then \( \phi \) induces a Hopf algebras morphism \( F(\pi) \to F(G) \), given by \( f \mapsto f \circ \phi \), whose image is central. Here \( F(G) = \mathbb{C}^G \) and \( F(\pi) = \mathbb{C}^\pi \) denote the Hopf algebras of complex-valued functions on \( G \) and \( \pi \) respectively. By Section 1.3, this data yields to a Hopf \( \pi \)-coalgebra \( H^\phi = \{ H^\phi_n \}_{n \in \pi} \).

Let us describe more precisely this Hopf \( \pi \)-coalgebra. Denote by \( (e_g)_{g \in G} \) the standard basis of \( F(G) \) given by \( e_g(h) = \delta_{g,h} \). Then, for any \( \alpha, \beta \in \pi \), we have that:

\[
H^\phi_n = \sum_{g \in \phi^{-1}(\alpha)} \mathbb{C} e_g, \quad m_\alpha(e_g \otimes e_h) = \delta_{g,h} e_g \quad \text{for any } g, h \in \phi^{-1}(\alpha),
\]

\[
1_\alpha = \sum_{g \in \phi^{-1}(\alpha)} e_g, \quad \varepsilon(e_g) = \delta_{g,1} \quad \text{for any } g \in \phi^{-1}(1),
\]

\[
\Delta_{\alpha,\beta}(e_g) = \sum_{h \in \phi^{-1}(\alpha), k \in \phi^{-1}(\beta)} e_h \otimes e_k \quad \text{for any } g \in \phi^{-1}(\alpha\beta),
\]

\[
S_\alpha(e_g) = e_{g^{-1}} \quad \text{for any } g \in \phi^{-1}(\alpha).
\]

Note that the Hopf \( \pi \)-coalgebra \( H^\phi = \{ H^\phi_n \}_{n \in \pi} \) is involutive and of finite type. Since \( \dim H^\phi_n = \#\phi^{-1}(1_G) = \# \ker \phi \) is non-zero in the field \( \mathbb{C} \), the invariant \( K_{H^\phi} \) of pointed flat \( \pi \)-bundles over 3-manifolds is well defined.

Lemma 8. Let \( (\xi, \hat{x}) \) be a pointed flat \( \pi \)-bundle over a 3-manifold \( M \). Then

\[ K_{H^\phi}(\xi, \hat{x}) = \# \{ g: \pi_1(M, x) \to G \mid \phi \circ g = f \}, \]

where \( x \) is the image of \( \hat{x} \) under the bundle map \( \hat{M} \to M \) of \( \xi \) and \( f: \pi_1(M, x) \to \pi \) is the monodromy of \( \xi \) at \( \hat{x} \).

Proof. Using the above explicit description of \( H^\phi \), one easily gets that, for any \( \alpha \in \pi \) and \( g_1, \ldots, g_m \in \phi^{-1}(\alpha) \),

\[
m_\alpha = \begin{cases} 
1 & \text{if } g_1 = g_2 = \cdots = g_m \\
0 & \text{otherwise}
\end{cases},
\]

and that, for any \( \alpha_1, \ldots, \alpha_n \in \pi \) such that \( \alpha_1 \cdots \alpha_n = 1 \) and \( g_1 \in \phi^{-1}(\alpha_1), \ldots, g_n \in \phi^{-1}(\alpha_n) \),

\[
\Delta_{\alpha_1, \ldots, \alpha_n}(e_{g_1} \otimes e_{g_n}) = \dim H^\phi (\delta_{g_1 \cdots g_n, 1}).
\]

Let \( (\xi, \hat{x}) \) be a pointed flat \( \pi \)-bundle over a 3-manifold \( M \). Let \( D = (S, u, l) \) be a Heegaard diagram of genus \( g \) of \( M \). Orient it and color it by the monodromy
f of $\xi$ at $\bar{x}$. Denote by $\alpha = (\alpha_1, \ldots, \alpha_g)$ the color of $D$. Let $w_1, \ldots, w_g$ be the words in the alphabet $\{X_1, \ldots, X_g\}$ as in Section 5.1. For any $1 \leq i \leq g$, write $w_i(X_1, \ldots, X_g) = X_{k_{i,1}}^{\varepsilon_{i,1}} \cdots X_{k_{i,n_i}}^{\varepsilon_{i,n_i}}$ where $\varepsilon_{i,j} = \pm 1$ and $1 \leq k_{i,j} \leq g$.

The non-zero factors of the tensor associated to an upper circle $u_i$ are of the form:

$$e_{g_i} \xrightarrow{m_{\alpha_i}} 1 \quad \text{where} \quad g_i \in \phi^{-1}(\alpha_i).$$

The non-zero factors of the tensor associated to a lower circle $l_i$ are of the form:

$$\Delta_{\alpha_{k_{i,1}},1, \ldots, \alpha_{k_{i,n_i}}}^{e_{h_{k_{i,j}}}} \xrightarrow{e_{h_{k_{i,j}}}} (\dim H_1^\phi) \delta_{h_{k_{i,j}}^{\varepsilon_{i,j}}}, \quad \text{where} \quad h_{k_{i,j}}^{\varepsilon_{i,j}} \in \phi^{-1}(\alpha_{k_{i,j}}^{\varepsilon_{i,j}}).$$

Now, at each crossing, that is, for $1 \leq i \leq g$ and $1 \leq j \leq n_i$, the contraction rule amounts to the equality of $h_{k_{i,j}}^{\varepsilon_{i,j}}$ with $g_{k_{i,j}}^{\varepsilon_{i,j}}$. Therefore we get that:

$$K_{H^\phi}(\xi, \bar{x}) = (\dim H_1^\phi)^{-g} \sum_{g_i \in \phi^{-1}(\alpha_i)} \left( \prod_{1 \leq i \leq g} (\dim H_1^\phi) \delta_{h_{k_{i,j}},1}^{\varepsilon_{i,j}} \right) \left( \prod_{1 \leq j \leq n_i} \delta_{h_{k_{i,j}}^{\varepsilon_{i,j}}, g_{k_{i,j}}^{\varepsilon_{i,j}}}, 1 \right)$$

$$= \sum_{g_i \in \phi^{-1}(\alpha_i)} \left( \prod_{1 \leq i \leq g} \delta_{w_i(g_1, \ldots, g_g), 1} \right)$$

$$= \#\{ (g_1, \ldots, g_g) \in G^g \mid w_i(g_1, \ldots, g_g) = 1 \text{ and } \phi(g_i) = \alpha_i \forall 1 \leq i \leq g \}$$

$$= \#\{ g : \pi_1(M, x) \to G \mid \phi \circ g = f \}. \quad \square$$

For example, suppose that $\pi = \mathbb{Z}/2\mathbb{Z}$, $G = S_3$, and $\phi : S_3 \to \mathbb{Z}/2\mathbb{Z}$ is the signature of 3-permutations. Let $M$ be a closed, connected, and oriented 3-manifold such that $\pi_1(M) = \mathbb{Z}/2\mathbb{Z}$. There is two flat $\mathbb{Z}/2\mathbb{Z}$-bundles $\xi_0$ and $\xi_1$ over $M$, whose monodromies are respectively trivial and $\text{id}_{\mathbb{Z}/2\mathbb{Z}}$. In this setting, we get that $K_{H^\phi}(\xi_0) = 1$ and $K_{H^\phi}(\xi_1) = 3$.

REFERENCES

1. M. Atiyah, Topological quantum field theories, Publ. Math. IHES 68 (1989), 175–186.
2. B. Enriquez, Private communication, 2001.
3. W. Fulton, Algebraic topology, Springer-Verlag, New York, 1995.
4. G. Kuperberg, Involutory Hopf algebras and 3-manifold invariants, Internat. J. Math. 2 (1991), no. 1, 41–66.
5. J. Singer, Three-dimensional manifolds and their Heegaard diagrams, Trans. Amer. Math. Soc. 35 (1933), no. 1, 88–111.
6. M. E. Sweedler, Hopf algebras, W.A. Benjamin, INC., New York, 1969.
7. , Integrals for Hopf algebras, Annals of Math. (2) 89 (1969), 323–335.
8. V. Turaev, Homotopy field theory in dimension 2 and group-algebras, preprint QA/9910010, 1999.
9. , Homotopy field theory in dimension 3 and group-categories, preprint GT/0005291, 2000.
10. L. Vainerman, Private communication, 2001.
11. A. Virelizier, *Algèbres de Hopf graduées et fibrés plats sur les 3-variétés*, Ph.D. thesis, 2001.
12. ______, *Hopf group-coalgebras*, J. Pure Appl. Algebra 171 (2002), no. 1, 75–122.

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