Degenerations of log Hodge de Rham spectral sequences, log Kodaira vanishing theorem in characteristic $p > 0$ and log weak Lefschetz conjecture for log crystalline cohomologies

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Abstract
In this article we prove that the log Hodge de Rham spectral sequences of certain proper log smooth schemes of Cartier type in characteristic $p > 0$ degenerate at $E_1$. We also prove that the log Kodaira vanishings for them hold when they are projective. We formulate the log weak Lefschetz conjecture for log crystalline cohomologies and prove that it is true in certain cases.

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1 Introduction
Let $\kappa$ be a perfect field of characteristic $p > 0$. Let $W$ (resp. $W_n$ ($n \in \mathbb{Z}_{>0}$)) be the Witt ring of $\kappa$ (resp. the Witt ring of $\kappa$ of length $n$).

In [Mu] Mumford has shown that the $E_1$-degeneration of the Hodge de Rham spectral sequence of a proper smooth scheme over $\kappa$ does not hold in general unlike

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the case of characteristic 0 in [D1]. In [Ray] Raynaud has shown that the Kodaira vanishing for a projective smooth scheme over \( \kappa \) does not hold in general unlike the case of characteristic 0 in [Ko]. However, in their famous article [D1], Deligne and Illusie have given a sufficient condition for the \( E_1 \)-degeneration and the Kodaira vanishing theorem: if a proper (resp. projective) smooth scheme \( X \) over \( \kappa \) has a smooth lift over \( \mathcal{W}_2 \), then the \( E_1 \)-degeneration (resp. the vanishing theorem) holds in characteristic \( p \) in a restricted sense. However there is no concretely calculable criterion for the existence of a smooth lift over \( \mathcal{W}_2 \) of a given \( X/\kappa \) in general. Consequently one does not know whether the \( E_1 \)-degeneration and the vanishing theorem for \( X/\kappa \) hold a priori. On the other hand, in [AZ] Achinger and Zdanowicz have constructed projective smooth schemes over \( \kappa \) which do not have smooth lifts over \( \mathcal{W}_2 \) and for which the \( E_1 \)-degenerations hold. In [Ek] Ekedahl has shown that the Hirokado variety in \( \mathcal{Y}_1 \) does not have a smooth lift over \( \mathcal{W}_2 \) when \( p = 3 \). However, in [Tak] Takayama has proved that (a part of) Kodaira vanishing theorem holds for it.

On the other hand, Deligne has proved the hard Lefschetz theorem for the \( l \)-adic étale cohomologies of \( X/\kappa \) in [D3] as in the case of characteristic 0. Using this result and Berthelot’s weak Lefschetz theorem for crystalline cohomologies of \( X/\kappa \) for any hypersurface sections of high degrees in [BT], Katz and Messing have proved the hard Lefschetz theorem and the weak Lefschetz theorem for isocrystalline cohomologies of \( X/\kappa \) ([KM]). However we would like to point out that there is a gap in the proof of Berthelot’s weak Lefschetz theorem in [BI] and we fill this gap in the text.

Let \( X \) be a proper (smooth) scheme over \( \kappa \) of pure dimension \( d \geq 1 \). Let \( q \) be a nonnegative integer. Let \( \Phi^q_{X/\kappa} \) be the Artin-Mazur functor of \( X/\kappa \) in degree \( q \): \( \Phi^q_{X/\kappa} \) is the functor defined by the following

\[
\Phi^q_{X/\kappa}(A) := \ker(H^q_{et}(X \otimes \kappa A, \mathbb{G}_m) \rightarrow H^q_{et}(X, \mathbb{G}_m)) \in \text{(Ab)}.
\]

Here \( A \) is an artinian local \( \kappa \)-algebra with residue field \( \kappa \). If \( \Phi^q_{X/\kappa} \) is formally smooth, then \( \Phi^q_{X/\kappa} \) is pro-represented by a commutative formal group over \( \kappa \) ([AM]). Let \( h^q(X/\kappa) \) be the height of \( \Phi^q_{X/\kappa} \) if it is pro-representable. We call \( h^q(X/\kappa) \) the \( q \)-th Artin-Mazur height of \( X/\kappa \).

Let \( X \) be a geometrically irreducible proper smooth scheme over \( \kappa \) of dimension \( d \geq 1 \). We say that \( X \) is a Calabi-Yau variety over \( \kappa \) of dimension \( d \) if \( H^q(X, \mathcal{O}_X) = 0 \) \((0 < q < d)\) and \( \Omega^2_X/\kappa \simeq \mathcal{O}_X \).

In [YI] the second named author of this article has recently proved the following:

**Theorem 1.1 (YI).** Let \( X \) be a Calabi-Yau variety over \( \kappa \) of dimension \( d \geq 1 \). If \( h^d(X/\kappa) < \infty \), then there exists a proper smooth scheme \( X \) over \( \mathcal{W}_2 \) such that \( X \otimes_{\mathcal{W}_2} \kappa = X \).

Using Deligne-Illusie’s theorem and ([14]), we see that the Hodge de Rham spectral sequence

\[(1.1.1) \quad E_1^{ij} = H^j(X, \Omega^i_X/\kappa) \Rightarrow H^{i+j}_{dR}(X/\kappa)\]

of \( X/\kappa \) degenerates at \( E_1 \) if \( d \leq p \). In particular, if \( p \neq 2 \) and \( h^3(X/\kappa) < \infty \), then ([11.1.1]) degenerates at \( E_1 \) for a 3-dimensional Calabi-Yau variety \( X/\kappa \). Using Joshi’s theorem ([J]), we easily see that the slope spectral sequence

\[(1.1.2) \quad E_1^{ij} = H^j(X, \mathcal{W}\Omega^i_{X/\kappa}) \Rightarrow H^{i+j}_{Crys}(X/\mathcal{W})\]

degenerates at \( E_1 \) for a 3-dimensional Calabi-Yau variety \( X/\kappa \) such that \( h^3(X/\kappa) < \infty \). Furthermore we see that it is of Hodge-Witt type by a fundamental theorem.
in $\mathbb{H}$: $H^i_{\text{crys}}(X/W) = \bigoplus_{i+j=q} H^j(X, \mathcal{W} \Omega^i_{X/\kappa})$ ($q \in \mathbb{N}$). (This is a 3-dimensional analogue of the Hodge-Witt decomposition of a $K3$-surface over $\kappa$ with finite second Artin-Mazur height ($I_1$).) Using Ekedahl’s remark in $\mathbb{H}$, we see that the following spectral sequence

$$(1.1.3) \quad E_1^{ij} = H^j(X, \mathcal{W}_n \Omega^i_{X/\kappa}) \Rightarrow H^{i+j}_{\text{crys}}(X/W_n) \quad (n \in \mathbb{N})$$

degenerates at $E_2$.

For an ample line bundle $L$ on $X$, in $Y_2$ the second named author has also proved that $H^j(X, \mathcal{L}) = 0$ ($j > 0$) without any assumption on $d$ and $p$.

To prove $Y_1$, he has introduced a new invariant $h_F(X)$ of $X$ as follows.

Let $Y$ be a (proper smooth) scheme over $\kappa$. Let $F_Y$ be the Frobenius endomorphism of $Y$. Set $F := W_n(F_Y^*) : W_n(\mathcal{O}_Y) \to F_Y*(W_n(\mathcal{O}_Y))$. This is a morphism of $\mathcal{W}_n(\mathcal{O}_Y)$-modules. In $Y_1$ he has introduced the notion of the quasi-Frobenius splitting height $h_F(Y)$ for any (proper smooth) scheme $Y$ over $\kappa$. (In [loc. cit.] he has denoted it by $h^{S}(Y)$.) It is the minimum of positive integers $n$’s such that there exists a morphism $\rho: F_Y*(W_n(\mathcal{O}_Y)) \to \mathcal{O}_Y$ of $\mathcal{W}_n(\mathcal{O}_Y)$-modules such that $\rho \circ F: W_n(\mathcal{O}_Y) \to \mathcal{O}_Y$ is the natural projection. (If there does not exist such $n$, then we set $h_F(Y) = \infty$.) (Because the “quasi-Frobenius splitting height” is too long, we call this the quasi-$F$-split height simply.) This is a nontrivial generalization of the notion of the Frobenius splitting by Mehta and Ramanathan in $[MR]$ because they have said that, for a scheme $Z$ of characteristic $p > 0$, $Z$ is a Frobenius splitting($=F$-split) scheme if $F: \mathcal{O}_Z \to F_{Z*}(\mathcal{O}_Z)$ has a section of $\mathcal{O}_Z$-modules. Mehta has already remarked that any proper smooth $F$-split scheme over $\kappa$ has a proper smooth lift over $\mathcal{W}_2$ ($I_1$) as a corollary of Nori and Srinivas’ beautiful deformation theory with absolute Frobenius endomorphisms in $[NS]$ and $[Sr]$. By using their theory, the second named author has proved that any proper smooth scheme over $\kappa$ has a proper smooth lift over $\mathcal{W}_2$ if $h_F(Y) < \infty$ ($Y_1$). Furthermore he has proved a fundamental equality $h_F(X) = h^d(X/\kappa)$ by using Serre’s exact sequence in $[Se]$, the calculation of the dimensions of the cohomologies of sheaves of closed differential forms of degree 1 due to Katsura and Van der Geer ($vGK$) and Serre’s duality ($Y_1$). As a result, he has obtained ($Y_1$). Recently Achinger has proved that, if $Z/\kappa$ is a (proper smooth) scheme over $\kappa$ with finite quasi-split height, then $Z/\kappa$ has a (proper smooth) lift over $\mathcal{W}_2$ by a method in the Appendix of $AZ$.

This article is a continuation of $Y_1$, in an expanded form. The results in this article are the log versions of $Y_1$, a part of $Y_2$, $[NS]$, $[Sr]$, $[Bi]$ and more.

The philosophy of log geometry of Fontaine-Illusie-Kato ($Kk1$, $Kk2$) tells us that one can give statements and prove them for certain non-smooth schemes by similar methods for smooth schemes if one can endow them with fine or fs(=fine and saturated) log structures and if one makes multiplicative calculations of local sections of log structures in addition to multiplicative and additive calculations of local sections of structure sheaves of schemes with the use of various cohomologies of various sheaves. Supported by this philosophy, we give the log versions of results in the articles in the previous paragraph. Though the proofs of a lot of results in this article are not psychologically extremely seriously difficult (after giving nontrivial formulations), the results themselves are nontrivial generalizations of the results in the articles above. (Of course there are often technically hard points in the proofs.) This is the typical merit of the log geometry of Fontaine-Illusie-Kato: it gives us appropriate languages.

Next let us recall Kawamata-Namikawa’s result briefly. This gives a not a little influence to this article.
Let \( \kappa \) be a field of any characteristic. Let \( s \) be an fs log scheme whose underlying scheme is \( \text{Spec}(\kappa) \) and whose log structure is associated to a morphism \( \mathbb{N} \ni 1 \mapsto a \in \kappa \) for some \( a \in \kappa \) (see \([\text{KK}1]\) and \([\text{KK}2]\) for fundamental terminologies of log schemes). If \( a = 0 \), then \( s \) is called the log point of \( \kappa \); if \( a \neq 0 \), then \( s = (\text{Spec}(\kappa), \kappa^*) \).

For a log scheme \( Z \), we denote by \( \bar{Z} \) the underlying scheme of \( Z \). For a relative log scheme \( Z/s \), we denote the log de Rham complex of \( Z/s \) by \( \Omega_{Z/s}^* \), and we set \( H^q_{\text{dR}}(Z/s) := H^q(Z, \Omega_{Z/s}^*) \) \((q \in \mathbb{N})\).

When \( \kappa = \mathbb{C} \), Kawamata and Namikawa have proved the following theorem in \([\text{KwN}]\).

**Theorem 1.2** ([\text{KwN} (4.2)]). Let \( s \) be the log point of \( \mathbb{C} \). Let \( X \) be a proper SNCL (=simple normal crossing log) scheme over \( s \) of pure dimension \( d \). Assume that \( d \geq 3 \). Let \( S \) be a small disk with canonical log structure. Let \( \bar{X}^{(0)} \) be the disjoint union of the irreducible components of \( \bar{X} \). Assume that the following three conditions hold:

(a) \( H^{d-1}(X, \mathcal{O}_X) = 0 \),
(b) \( H^{d-2}(\bar{X}^{(0)}, \mathcal{O}_{\bar{X}^{(0)}}) = 0 \),
(c) \( \Omega^d_{X/s} \cong \mathcal{O}_X \).

Then there exists an analytically strict semistable family \( X \) over \( S \) such that \( X \times_S s = X^\text{an} \), where \( X^\text{an} \) is the associated log analytic space to \( X/s \) (cf. \([\text{KLN}]\)).

Let us go back to the case where \( \kappa \) is a perfect field of characteristic \( p > 0 \). Let \( s \) be an fs log scheme before \((1.2)\). Let \( X \) be a proper log smooth log scheme over \( s \) of Cartier type. Let \( \mathcal{I}_{X/s} \) be Tsuji’s ideal sheaf of the log structure \( M_X \) of \( X \) defined in \([\text{IK}]\) and denoted by \( I_f \), where \( f: X \rightarrow s \) is the structural morphism. Here \( \mathcal{I}_{X/s} \) stems from the “horizontal” log structure on \( X \); in the text we shall recall the definition of \( \mathcal{I}_{X/s} \). We say that \( X/s \) is of vertical type if \( \mathcal{I}_{X/s} \mathcal{O}_X = \mathcal{O}_X \). If \( X/s \) is an SNCL scheme (\([\text{Nak}2]\), \([\text{Nak}7]\)), more generally, if \( X/s \) is locally a product of SNCL schemes, then \( X/s \) is of vertical type. One of the main results in this article is the following theorem:

**Theorem 1.3.** Let \( X \) be a proper log smooth log scheme over \( s \) of Cartier type. Let \( \mathcal{W}_2(s) \) be a log scheme whose underlying scheme is \( \text{Spec}(\mathcal{W}_2) \) and whose log structure is associated to a morphism \( \mathbb{N} \ni 1 \mapsto (a, 0) \in \mathcal{W}_2 \). Then the following hold:

1. If \( h_F(\bar{X}) < \infty \), then there exists a proper log smooth log scheme \( \tilde{X} \) over \( \mathcal{W}_2(s) \) such that \( \tilde{X} \times_{\mathcal{W}_2(s)} s = X \).
2. Furthermore, assume that \( \tilde{X} \) is of pure dimension \( d \) and that \( X/s \) is of vertical type and that the following three conditions hold:
   (a) \( H^{d-1}(X, \mathcal{O}_X) = 0 \) if \( d \geq 2 \),
   (b) \( H^{d-2}(X, \mathcal{O}_X) = 0 \) if \( d \geq 3 \),
   (c) \( \Omega^d_{X/s} \cong \mathcal{O}_X \).

Then \( h_F(\tilde{X}) = h^d(\tilde{X}/\kappa) \).

By using K. Kato’s theorem in \([\text{KK}1] (= \text{the log version of Deligne-Illusie’s theorem}) \) and \((1.3) (1)\), we obtain the following:

**Theorem 1.4.** Let \( Y \rightarrow s \) be a proper log smooth morphism of Cartier type of dimension \( d \). Assume that \( h_F(\bar{Y}) < \infty \). Then the log Hodge de Rham spectral sequence

\[
E_1^{ij} = H^j(Y, \Omega^i_{Y/s}) \Rightarrow H^{i+j}_{\text{dR}}(Y/s)
\]
degenerates at $E_1$ if $d < p$. If $F_{Y*}(\mathcal{O}_Y)$ is a locally free $\mathcal{O}_Y$-modules (of finite rank) and if $d \leq p$, then this spectral sequence degenerates at $E_1$. Here $F_Y: Y \to Y$ is the absolute Frobenius endomorphism of $Y$.

We also give another short proof of Kato’s theorem by using our log deformation theory with absolute Frobenius endomorphisms explained soon later. This is the log version of a generalization of Srinivas’ another short proof of Deligne-Illusie’s theorem (S1).

Let $Y/s$ be a log smooth log scheme of Cartier type.

One of the new key ingredient for the proof of (K3) is our log deformation theory with absolute Frobenius endomorphisms. This is the log version of Nori and Srinivas’ deformation theory with absolute Frobenius endomorphisms in [NoS] and [S1]. In this theory, the sheaf

$$B_1\Omega^1_{Y/s} := F_{Y*}(B\Omega^1_{Y/s}) := F_{Y*}(\text{Im}(d: \mathcal{O}_Y \to \Omega^1_{Y/s}))(\simeq F_{Y*}(\mathcal{O}_Y/\mathcal{O}_Y))$$

plays an important role as follows (In the trivial log case, $B_1\Omega^1_{Y/s}$ in this article is equal to $B\Omega^1_{Y/s}$ in [loc. cit.]):

**Theorem 1.5.** Let $F_{W_2(s)}: W_2(s) \to W_2(s)$ be the Frobenius endomorphism of $W_2(s)$. Let $\text{Lift}_{(Y,F_Y)/(W_2(s),F_{W_2(s)})}$ be the following sheaf

$$\text{Lift}_{(Y,F_Y)/(W_2(s),F_{W_2(s)})}(U) := \{\text{isomorphism classes of } (\tilde{U}, \tilde{F}) \mid \tilde{U} \text{ is a log smooth lift of } U \text{ over } W_2(s) \text{ and } \tilde{F}: \tilde{U} \to \tilde{U} \text{ is a lift of } F_U \text{ over } F_{W_2(s)}\}$$

for each log open subscheme $U$ of $Y$, where $F_U$ is the absolute Frobenius endomorphism of $U$. Then $\text{Lift}_{(Y,F_Y)/(W_2(s),F_{W_2(s)})}$ on $\tilde{Y}$ is a torsor under $\text{Hom}_{\mathcal{O}_Y}(\Omega^1_{Y/s}, B_1\Omega^1_{Y/s})$. In particular, the obstruction class of a log smooth lift of $(Y,F_Y)/s$ over $W_2(s)$ is a canonical element of $\text{Ext}^1((\Omega^1_{Y/s}, B_1\Omega^1_{Y/s}), \Omega^1_{Y/s})$ if $\tilde{Y}$ is separated. This obstruction class is the extension class of the following exact sequence of $\mathcal{O}_Y$-modules:

$$0 \to B_1\Omega^1_{Y/s} \to Z_1\Omega^1_{Y/s} \xrightarrow{C} \Omega^1_{Y/s} \to 0,$$

where $Z_1\Omega^1_{Y/s} := F_{Y*}(\text{Ker}(d: \Omega^1_{Y/s} \to \Omega^2_{Y/s}))$ and $C$ is the log Cartier operator:

$$C: Z_1\Omega^1_{Y/s} \xrightarrow{\text{proj}} Z_1\Omega^1_{Y/s}/B_1\Omega^1_{Y/s} \xrightarrow{C^{-1}} \Omega^1_{Y/s}.$$ Here $C^{-1}$ is the log Cartier isomorphism defined in [Kk1].

This is a special case of the main result in [3] below. Note that, because the log structure of $W_2(s)$ has a chart $\mathbb{N} \to W_2$, the structural morphism $U \to W_2(s)$ is automatically integral ([Kk1]). In the case where a base log scheme is more general, we have to consider log smooth integral lifts instead of log smooth lifts; the integrality is an essential condition in log deformation theory: deformation theory for log smooth schemes in [Kk1] (and [KkII]) has a serious defect to be corrected in general.

To construct the log deformation theory with absolute Frobenius endomorphisms itself is our aim in this article. To give the correct proof of (K3) is very involved. Indeed, even in the trivial logarithmic case in [NoS], we need a new additional quite extraordinary argument. More generally, we construct the log deformation theory with two kinds of relative Frobenius morphisms instead of absolute Frobenius endomorphisms in [loc. cit.] because relative Frobenius morphisms go well with (log) inverse Cartier isomorphisms when we consider log deformation theory with Frobenius morphisms over a more general fine log base scheme of characteristic $p > 0$. 


Our log deformation theory with Frobenius morphisms also has an application for the canonical lift of a log ordinary projective log smooth smooth scheme over $s$ with trivial log cotangent bundle over the canonical lift $\mathcal{W}(s)$ of $s$ over $\text{Spec}(\mathcal{W})$ (Nakk8). (This is the log version of theory of a canonical lift in [NoS].)

Other necessary new ingredient for the proof of [1.3] is the calculation of dimension of $H^q(X, B_n^1\Omega^n_{X/s})$ ($d-2 \leq q \leq d$) by following the method of Katsura and Van der Geer in [vGR].

As a corollary of (1.4), we also prove the log version of Raynaud’s vanishing theorem (=an analogue in characteristic $p$ of Kodaira-Akizuki-Nakano’s vanishing theorem in characteristic 0) as in [DI].

**Theorem 1.6 (Log Kodaira-Akizuki-Nakano-Raynaud Vanishing theorem).** Let the notations and the assumption be as in (1.4). Furthermore, assume that $Y$ is $f$, that the structural morphism $\hat{Y} \to \hat{s}$ of schemes is projective and that $\hat{Y}$ is of pure dimension $d$. Let $\mathcal{L}$ be an ample invertible $\mathcal{O}_Y$-module. Then $H^j(Y, i_*\mathcal{I}_{Y/s}\mathcal{O}_{Y/s} \otimes \mathcal{O}_Y \mathcal{L}) = 0$ for $i+j > \max\{d, 2d-p\}$.

In the most important case $i = d$ in (1.6), we prove a stronger theorem than this theorem (this stronger theorem is also one of the main results in this article):

**Theorem 1.7 (Log Kodaira Vanishing theorem I).** Let the notations and the assumptions be as in (1.6). Then $H^j(Y, i_*\mathcal{I}_{Y/s}\mathcal{O}_{Y/s} \otimes \mathcal{O}_Y \mathcal{L}) = 0$ for $j > 0$.

This theorem is the log version of a nontrivial generalization of Mehta and Ramanathan’s vanishing theorem in [MR]; the proof of this theorem is more nontrivial than that of their theorem. This theorem is important because we can obtain the new class of log schemes such that Kodaira vanishing theorem holds in characteristic $p > 0$. The theorem (1.7) has an interesting application for congruences of the cardinalities of rational points of log Fano varieties with finite quasi-$F$-split heights over the log point of a finite field (Nakk5). This is a generalization of Esnault’s theorem in [Es] (under the (mild) assumption “the finiteness of the quasi-$F$-split height”). We hope that (1.7) will have more important applications for algebraic geometry in characteristic $p$.

As a corollary of the vanishing theorem (1.0), we prove an analogous vanishing theorem in characteristic 0.

Lastly in this introduction, we formulate the log weak Lefschetz conjecture for log crystalline cohomologies and we give an affirmative result for this conjecture.

Let $Y$ be a projective SNCL scheme over the log point $s$. Let $E$ be a horizontal smooth divisor on $Y$ which will be defined in the text; roughly speaking, $E$ is locally defined by a local coordinate which has “no relation with a nontrivial local section of $M_Y/\mathcal{O}_Y$”.

Let $q$ be a nonnegative integer. For a proper log smooth scheme $Y/s$, let $\mathcal{H}_{\text{crys}}^q(Y/\mathcal{W}(s))$ be the log crystalline cohomology of $Y/\mathcal{W}(s)$ (Kk1). By the works in [K0] and [Nakk3] (cf. [Nakk7]), $\mathcal{H}_{\text{crys}}^q(Y/\mathcal{W}(s))$ and $\mathcal{H}_{\text{crys}}^q(E/\mathcal{W}(s))$ have the weight filtrations $P$’s. Set $K_0 := \text{Frac}(\mathcal{W})$. For a module $M$ over $\mathcal{W}$, set $M_{K_0} := M \otimes_{\mathcal{W}} K_0$. Let $i: E \hookrightarrow Y$ be the closed immersion. By a general theorem in [Nakk7], the pull-back of $i$

\[(1.7.1) \quad \iota^*_\text{crys}: H^q_{\text{crys}}(Y/\mathcal{W}(s))_{K_0} \to H^q_{\text{crys}}(E/\mathcal{W}(s))_{K_0} \quad (q \in \mathbb{Z})\]

is strictly compatible with $P$’s. In this article we conjecture the following:

**Conjecture 1.8 (Log weak Lefschetz conjecture for log isocrystalline cohomologies).** Assume that $\mathcal{O}_Y(E)$ is ample. Then the morphism (1.7.1) is a filtered isomorphism with respect to $P$’s if $q \leq d-2$ and strictly injective for $q = d-1$. 

In the text we give affirmative results for this conjecture. For example, we prove the following:

**Theorem 1.9.** Assume that $Y$ and $E$ have log smooth lifts over $W_2(s)$. Assume also that $\dim Y \leq p$. Then the following pull-back

\[(\ref{1.9.1}) \quad \iota^* \colon H^q_{\text{crys}}(Y/W(s)) \rightarrow H^q_{\text{crys}}(E/W(s)) \quad (q \in \mathbb{Z})\]

is an isomorphism if $q < d - 1$ and injective for $q = d - 1$ with torsion free cokernel. In particular, (1.8) is true under the assumptions above.

We prove this theorem by following but nontrivially correcting the method of Berthelot in [B1]. In the future we would like to prove that this conjecture is true in general. Note that because in [Nakk4] and [Nakk7] we have proved that the log hard Lefschetz conjecture is true in the strict semistable cases in mixed characteristics and equal characteristic $p > 0$, we can prove that the log weak Lefschetz conjecture is true in these important cases as a corollary.

In [Nakk4] we have proved that the log hard Lefschetz conjecture is true in characteristic 0 by using M. Saito’s result ([Sa]). As a corollary, we can prove that the log weak Lefschetz conjecture in characteristic 0 is true. In this article we prove this theorem by an algebraic method as Deligne and Illusie have proved the $E_1$-degeneration of the Hodge-de Rham spectral sequence of a proper smooth scheme in characteristic 0 in [DI] by an algebraic method.

The contents of this article are as follows.

Let $Z$ be a proper scheme over $\kappa$. Let $q$ be a nonnegative integer. Assume that $H^q(Z, \mathcal{O}_Z) \simeq \kappa$, that $H^{q+1}(Z, \mathcal{O}_Z) = 0$ and that the $q$-th Artin-Mazur functor $\Phi^q_{Z/\kappa} := \Phi^q_{Z/\kappa}(\mathbb{G}_m)$ is pro-representable. Assume also that the Bockstein operator $\beta: H^{q-1}(Z, \mathcal{O}_Z) \rightarrow H^q(Z, \mathcal{W}_{n-1}(\mathcal{O}_Z))$ arising from the following exact sequence

\[0 \rightarrow \mathcal{W}_{n-1}(\mathcal{O}_Z) \xrightarrow{\nu} \mathcal{W}_n(\mathcal{O}_Z) \rightarrow \mathcal{O}_Z \rightarrow 0\]

is zero for any $n \in \mathbb{Z}_{\geq 2}$. In §2 we prove that the $q$-th Artin-Mazur height $h^q(Z/\kappa)$ of $Z/\kappa$ is equal to the minimum of positive integers $n$’s of the non-vanishing of the Frobenius endomorphism $F: H^q(Z, \mathcal{W}_n(\mathcal{O}_Z)) \rightarrow H^q(Z, \mathcal{W}_n(\mathcal{O}_Z))$ by imitating the proof in [vGK] completely. (However we have needed a work to give this generalized statement.) Recently it has turned out that this characterization of $h^q(Z/\kappa)$ also has two applications for the congruences of the cardinalities of rational points of (log) Calabi-Yau varieties over the log point of a finite field (Nakk5) and for the fundamental inequality between Artin-Mazur heights and a quasi-$F$-split height (Nakk6).

In §3 we prove that there exists the log version of Serre’s exact sequence in [Se] in an elementary but elegant way and calculate $\dim_{\kappa} H^q(X, B_n \Omega^1_{X/s})$ $(d - 2 \leq q \leq d)$ for $X/s$ in §3.

In §4 following the methods in [NoS] and [Sr] but modifying and generalizing them, we construct log deformation theory with relative Frobenius morphisms. Our new theory is a geometric key part for the proof of (1.3). This is the most complicated part in this article. In addition, we give an additional result for the deformation theory for log smooth schemes in [Kk1] (and [Kk2]), which is an important correction of the theory in [loc. cit.], and we establish a relationship between these two deformation theories.
In §5, as applications of our deformation theory, we give another short proof of Kato’s theorem (cf. (1.4)). We also prove (1.6) by using Tsuji’s log Serre duality in [Ts1]. As in [DI] we prove the log versions of the weak Lefschetz theorems for log de Rham cohomologies in characteristics $p > 0$ and 0. The proof of the weak Lefschetz theorem in the case characteristics $p > 0$ includes an immediate correction of an elementary error in [DI]. Using this theorem in characteristics $p > 0$, we prove (1.9). We also prove the log version of Berthelot’s weak Lefschetz theorem and we fill a gap in the proof in [B1].

In §7 we give the notion of quasi-$F$-split schemes, which is the relative version of the notion of quasi-$F$-split varieties in [Y1]. In this section we prove two fundamental theorems for quasi-$F$-split log schemes as in [loc. cit.]: a lifting theorem and two vanishing theorems for them. The lifting theorem and one of the vanishing theorems are the relative and log versions of theorems in [Y1] and [Y2]. This vanishing theorem is a generalization of one of Mehta and Ramanathan’s vanishing theorems in [MR]. We also prove another vanishing theorem (1.7), which is a generalization of their another vanishing theorem in [loc. cit.].

In §8 we prove (1.3) by following the method in [Y1] and by using results in §3~§7.

In §9 we give a short proof of the weak Lefschetz theorem for crystalline cohomologies of proper smooth schemes over $\kappa$ due to Berthelot-Katz-Messing ([KM]) by using theory of rigid cohomologies of Berthelot ([B2], [B3], [B4]).

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Notations. (1) For a commutative ring $A$ with unit element and two $A$-modules $M$ ($M$ has two distinct $A$-module structures) and for $f \in \text{Hom}_A(M, M)$, $fM$ (resp. $M/f$) denotes $\ker(f : M \to M)$ (resp. $\text{coker}(f : M \to M)$). We use the same notation for an endomorphism of two $A$-modules on a topological space, where $A$ is a sheaf of commutative rings with unit elements on the topological space.

(2) For a log scheme $Z$ in the sense of Fontaine-Illusie-Kato ([Kk1], [Kk2]), we denote by $\tilde{Z}$ (resp. $\tilde{M}_Z := (M_Z, \alpha_Z)$) the underlying scheme (resp. the log structure) of $Z$. In this article we consider the log structure on the Zariski site on $\tilde{Z}$.

(3) For a morphism $f : Z \to T$ of log schemes, we denote by $\tilde{f} : \tilde{Z} \to \tilde{T}$ the underlying morphism of schemes of $f$.

(4) For a morphism $Z \to T$ of log schemes, we denote by $\Omega^{\bullet}_{Z/T}$ the log de Rham complex of $Z/T$ which was denoted by $\omega^{\bullet}_{Z/T}$ in [Kk1].

Convention. We omit the second “log” in the terminology a “log smooth (integral) log scheme”.

2 The heights of Artin-Mazur formal groups of certain schemes

Let $\kappa$ be a perfect field of characteristic $p > 0$. Let $\mathcal{W}$ (resp. $\mathcal{W}_n$) be the Witt ring of $\kappa$ (resp. the Witt ring of $\kappa$ of length $n > 0$). Let $Y$ be a proper scheme over $\kappa$. Let
$q$ be a nonnegative integer. Assume that $H^q(Y, \mathcal{O}_Y) \simeq \kappa$, $H^{q+1}(Y, \mathcal{O}_Y) = 0$ and that the Bockstein operator

$$\beta : H^{q-1}(Y, \mathcal{O}_Y) \to H^q(Y, \mathcal{O}_Y)$$

arising from the following exact sequence

$$0 \to \mathcal{W}_{n-1}(\mathcal{O}_Y) \xrightarrow{\nu} \mathcal{W}_n(\mathcal{O}_Y) \to \mathcal{O}_Y \to 0$$

is zero for any $n \in \mathbb{Z}_{\geq 2}$. In this section we characterize the height of the $q$-th Artin-Mazur formal group of $Y/\kappa$ (if it is pro-representable) by using the operator

$$F : H^q(Y, \mathcal{W}_n(\mathcal{O}_Y)) \to H^q(Y, \mathcal{W}_n(\mathcal{O}_Y)) \quad (n \in \mathbb{Z}_{\geq 1}).$$

This is a generalization of a result of Katsura and Van der Geer (HGK). Though they have proved this characterization for a Calabi-Yau variety over $\kappa$, it is not necessary to assume this strong condition nor to assume even that $Y$ is smooth over $\kappa$. Though the proof of our generalization is essentially the same as that of their result, we reprove our generalization because we would like to clarify how the assumptions above are necessary for the characterization.

The following is easy to prove.

**Proposition 2.1.** Let $g : Z \to S_0$ be a proper morphism of schemes of characteristic $p \geq 0$. Let $\mathcal{W}_n(\mathcal{O}_Z)$ ($n \in \mathbb{Z}_{\geq 1}$) be the sheaf of Witt rings of $\mathcal{O}_Z$ of length $n$. Let $V : \mathcal{W}_n(\mathcal{O}_Z) \to \mathcal{W}_{n+1}(\mathcal{O}_Z)$ be the Verschiebung and let $F : \mathcal{W}_n(\mathcal{O}_Z) \to \mathcal{W}_n(\mathcal{O}_Z)$ be the Frobenius operator. Let $R : \mathcal{W}_n(\mathcal{O}_Z) \to \mathcal{W}_{n-1}(\mathcal{O}_Z)$ be the projection. Let $q$ be a nonnegative integer. Assume that the Bockstein operator

$$\beta : R^{q-1}g_*(\mathcal{O}_Z) \to R^qg_*(\mathcal{W}_{n-1}(\mathcal{O}_Z))$$

arising from the following exact sequence

$$0 \to \mathcal{W}_{n-1}(\mathcal{O}_Z) \xrightarrow{\nu} \mathcal{W}_n(\mathcal{O}_Z) \xrightarrow{R^{q-1}} \mathcal{O}_Z \to 0$$

of abelian sheaves on $Z$ is zero for any $n \in \mathbb{Z}_{\geq 2}$. Assume that $R^{q+1}g_*(\mathcal{O}_Z) = 0$. Then the following hold:

1. The following sequence

$$0 \to R^qg_*(\mathcal{W}_{n-1}(\mathcal{O}_Z)) \xrightarrow{\nu} R^qg_*(\mathcal{W}_n(\mathcal{O}_Z)) \xrightarrow{R^{q-1}} R^qg_*(\mathcal{O}_Z) \to 0$$

of abelian sheaves on $S_0$ is exact. Consequently, if the projective system \( \{ R^qg_*(\mathcal{W}_n(\mathcal{O}_Z)) \}_{n=1}^{\infty} \) satisfies the Mittag-Leffler condition, then the following sequence

$$0 \to R^qg_*(\mathcal{W}(\mathcal{O}_Z)) \xrightarrow{\nu} R^qg_*(\mathcal{W}(\mathcal{O}_Z)) \to R^qg_*(\mathcal{O}_Z) \to 0$$

of abelian sheaves on $S_0$ is exact.

2. Assume that $Z$ is reduced and that $S_0$ is perfect. Assume also that $g_*(\mathcal{O}_Z) = \mathcal{O}_{S_0}^{\oplus c}$ for some positive integer $c$. Then $g_*(\mathcal{W}_n(\mathcal{O}_Z)/F)$ is a subsheaf of $R^1g_*(\mathcal{W}_n(\mathcal{O}_Z))$ of $\mathcal{W}_n(\mathcal{O}_{S_0})$-modules.

3. Let the notations be as in (2). Assume that $R^{q-1}g_*(\mathcal{O}_Z) = 0$ if $q \geq 2$ and that $R^{q-2}g_*(\mathcal{O}_Z) = 0$ if $q \geq 3$. If $q = 2$, assume also that $g_*(\mathcal{O}_Z) = \mathcal{O}_{S_0}^{\oplus c}$ for some positive integer $c$. Then $R^{q-2}g_*(\mathcal{W}_n(\mathcal{O}_Z)/F) = 0$.

4. Let the assumptions be as in (2) and (3). If $R^ig_*(\mathcal{O}_Z) = 0$ ($0 < i < q$), then

$$R^ig_*(\mathcal{W}_n(\mathcal{O}_Z)/F) = 0 \quad (0 \leq i \leq q - 2).$$
Proof. (1): Taking the long exact sequence of the exact sequence (2.1.2), we have the following exact sequence

\[
\cdots \to R^q g_*(W_{n-1}(O_Z)) \overset{V}{\to} R^q g_*(W_n(O_Z)) \overset{R^{n-1}}{\to} R^q g_*(O_Z) \\
\text{hence} \ R^{n+1} g_*(W_n(O_Z)) \to \cdots.
\]

Hence \(R^{q+1} g_*(W_n(O_Z)) = 0\). By the assumption, the morphism \(V : R^q g_*(W_{n-1}(O_Z)) \to R^q g_3(W_n(O_Z))\) is injective. Hence we obtain the exact sequence (2.1.3). Taking the projective limit of (2.1.3), we obtain the exact sequence (2.1.4).

(2): Because \(Z\) is reduced, the following sequence

\[
0 \to W_n(O_Z) \overset{F}{\to} W_n(O_Z) \to W_n(O_Z)/F \to 0
\]

is exact. Taking the long exact sequence of this exact sequence, we have the following exact sequence

\[
\cdots \to R^q g_*(W_n(O_Z)) \overset{F}{\to} R^q g_*(W_n(O_Z)) \to R^q g_*(W_n(O_Z))/F \\
\text{hence} \ R^{q+1} g_*(W_n(O_Z)) \to \cdots.
\]

We claim that the following natural morphism

\[
W_n(O_{S_0})^{\oplus c} = W_n(O_{S_0}^{\oplus c}) \to W_n(g_*(O_Z)) = g_*(W_n(O_Z))
\]

is an isomorphism. Indeed, assume that \(g_*(W_{n-1}(O_Z)) = W_{n-1}(O_{S_0})^{\oplus c}\). Then, by the following commutative diagram

\[
\begin{array}{ccc}
0 & \to & W_n(O_{S_0})^{\oplus c} \\
\vert & & \vert \\
0 & \to & g_*(W_{n-1}(O_Z))
\end{array}
\]

of exact sequences, we see that \(W_n(O_{S_0})^{\oplus c} = g_*(W_n(O_Z))\). Because \(F : W_n(O_{S_0}) \to W_n(O_{S_0})\) is bijective by the assumption, the morphism \(g_*(W_n(O_Z))/F \to R^1 g_*(W_n(O_Z))\) is injective.

(3): By (2.1.5) we easily see that \(R^{n-1} g_*(W_n(O_Z)) = 0\) (\(n \in \mathbb{Z}_{\geq 1}\)) if \(q \geq 2\). Hence we have the following exact sequence

\[
R^{q-2} g_*(W_n(O_Z)) \overset{F}{\to} R^{q-2} g_*(W_n(O_Z)) \to R^{q-2} g_*(W_n(O_Z))/F \to 0 \quad (q \geq 2).
\]

First assume that \(q \geq 3\). Then \(R^{q-2} g_*(W_n(O_Z)) = 0\). Hence \(R^{q-2} g_*(W_n(O_Z))/F = 0\).

Assume that \(q = 2\). Then \(R^1 g_*(W_n(O_Z)) = 0\). Hence \(g_*(W_n(O_Z))/F = 0\) by (2).

(4): Because \(R^i g_*(O_Z) = 0\) (\(0 < i < q\)), \(R^i g_*(W_n(O_Z)) = 0\) (\(0 < i < q\)) by (2.1.5). Hence \(R^q g_*(W_n(O_Z))/F = 0\) (\(0 < i < q - 1\)) by (2.1.6). By (2), \(g_*(W_n(O_Z))/F = 0\).

\[\square\]

**Corollary 2.2.** Let the assumptions be as in (2.1) (1). Furthermore, assume that \(R^q g_*(O_Z)\) is equal to a line bundle \(L\) on \(S_0\). Then \(R^q g_*(W(O_Z))/V = L\).

Proof. Obvious. \[\square\]
Let Art$\kappa$ be the category of artinian local $\kappa$-algebras with residue fields $\kappa$. Let us go back to the beginning of this section. Let $q$ be a nonnegative integer. Let $\Phi_{Y/\kappa}^q: \text{Art}_\kappa \rightarrow (\text{Ab})$ be the following functor: for $A \in \text{Art}_\kappa$, set
\[ \Phi_{Y/\kappa}^q(A) := \text{Ker}(H^0_\text{et}(Y \otimes_{\kappa} A, \mathbb{G}_m) \rightarrow H^0_\text{et}(Y, \mathbb{G}_m)) \in (\text{Ab}). \]

By [AM II (2.11)], $\Phi_{Y/\kappa}^q$ is pro-represented by a formal group over $\kappa$ if $\Phi_{Y/\kappa}^{q-1}$ is formally smooth. ($\Phi_{Y/\kappa}^0$ is pro-represented by a formal group over $\kappa$ [Schl (3.2)].) By [AM II (4.3)] the covariant Dieudonné module $D(\Phi_{Y/\kappa}^q)$ of $\Phi_{Y/\kappa}^q$ is equal to $H^q(Y, W(\mathcal{O}_Y))$. Let $h^q(Y/\kappa)$ be the height of $\Phi_{Y/\kappa}^q$. If $H^{q+1}(Y, \mathcal{O}_Y) = 0$, then $\Phi_{Y/\kappa}^q$ is formally smooth over $\kappa$. Moreover, if $H^q(Y, \mathcal{O}_Y) \simeq \kappa$, then $\Phi_{Y/\kappa}^q$ is a formal Lie group over $\kappa$ of dimension 1 and $D(\Phi_{Y/\kappa}^q)$ is a free $W$-module of rank $h^q(Y/\kappa)$ if $h^q(Y/\kappa) < \infty$ ([Ha V (28.3.10)]).

The following is a generalization of Katsura and Van der Geer’s theorem ([vGK (5.1), (5.2), (16.4)]).

**Theorem 2.3 (cf. [vGK (5.1), (5.2), (16.4)])**. Let $Y$ be a proper scheme over $\kappa$. (We do not assume that $Y$ is smooth over $\kappa$.) Let $q$ be a nonnegative integer. Assume that $H^q(Y, \mathcal{O}_Y) \simeq \kappa$, that $H^{q+1}(Y, \mathcal{O}_Y) = 0$ and that $\Phi_{Y/\kappa}^q$ is pro-representable. Assume also that the Bockstein operator
\[ (2.3.1) \quad \beta: H^{q-1}(Y, \mathcal{O}_Y) \rightarrow H^q(Y, W_{n-1}(\mathcal{O}_Y)) \]

arising from the following exact sequence
\[ 0 \rightarrow W_{n-1}(\mathcal{O}_Y) \xrightarrow{\mathcal{V}} W_n(\mathcal{O}_Y) \xrightarrow{R^{n-1}} \mathcal{O}_Y \rightarrow 0 \]
is zero for any $n \in \mathbb{Z}_{\geq 2}$. Let $n^q(Y)$ be the minimum of positive integers $n$’s such that
\[ F: H^q(Y, W_n(\mathcal{O}_Y)) \rightarrow H^q(Y, W_n(\mathcal{O}_Y)) \]
is not zero. (If $F = 0$ for all $n$, then set $n^q(Y) := \infty$.) Then $h^q(Y/\kappa) = n^q(Y)$.

**Proof.** (Though the proof is essentially the same as that of [vGK (5.1)] as stated in the beginning of this section, we reproduce the proof because the setting of (2.3) is considerably more general than that in [loc. cit.].) Set $h := h^q(Y/\kappa)$, $M := H^q(Y, W(\mathcal{O}_Y))$ and $M_n := H^q(Y, W_n(\mathcal{O}_Y))$. It suffices to prove that $h - 1 \geq n$ if and only if the morphism $F: M_n \rightarrow M_n$ is zero.

By (2.4.3) we see that $\text{length}_{W_n}(M_n) = n$. First we prove the implication “if”-part. If $h = n$, then the implication is obvious. Hence we may assume that $h < \infty$. Since $M = D(\Phi_{Y/\kappa}^q)$ is $p$-torsion free, the following sequence
\[ 0 \rightarrow M/F \xrightarrow{\mathcal{V}} M/p \rightarrow M/V \rightarrow 0 \]
of abelian groups is exact. Let $\sigma: \kappa \rightarrow \kappa$ be the $p$-th power map. Since $V(\sigma(a) \cdot x) = aV(x)$ ($a \in \kappa, x \in M/F$) and $\sigma \in \text{Aut}(\kappa)$, we have the following exact sequence
\[ 0 \rightarrow \sigma_*(M/F) \xrightarrow{\mathcal{V}} M/p \rightarrow M/V \rightarrow 0 \]
of $\kappa$-vector spaces. Hence
\[ \dim_{\kappa}(M/F) = \dim_{\kappa}(\sigma_*(M/F)) = \dim_{\kappa}(M/p) - \dim_{\kappa}(M/V) = h - 1 \]
The surjective morphism $H^q(Y, W_n(O_Y)) \to H^q(Y, W_n(O_Y))$ induces a surjective morphism $M/F \to M_n/F = M_n$. Because $\dim \kappa M_n = n$ by (2.1.3), we obtain the inequality $h - 1 \geq n$.

Next we prove the converse implication. (In [vGK] $\kappa$ is assumed to be algebraically closed; it is not necessary to assume this.)

Let $\kappa \to \kappa'$ be a morphism of perfect fields. Set $Y' := Y \otimes_{\kappa} \kappa'$. In the proof of [11 1 (1.9.2)], Illusie has proved that $W_n(O_{Y'}) = W_n(O_Y) \otimes_{\kappa} W_n(\kappa')$. Since the morphism $W_n \to W_n(\kappa')$ is flat, $H^1(Y, W_n(O_{Y'})) = H^1(Y, W_n(O_Y)) \otimes_{\kappa} W_n(\kappa')$. Let $\pi$ be an algebraic closure of $\kappa$. Since the morphism $W_n \to W_n(\pi)$ is faithfully flat, we may assume that $\kappa$ is algebraically closed. If $h = \infty$, then $F = 0$ on $M = D(\Phi_{Y/\kappa}) = D(\tilde{\varphi}_a)$. Hence $F = 0$ on $M_n$ for all $n$. We may assume that $h < \infty$.

Let $D(\kappa)$ be the Cartier-Dieudonné algebra over $\kappa$. As explained in [vGK] p. 266, $M = D(\Phi_{Y/\kappa}) \simeq D(\kappa)/D(\kappa)(F - V^{-1})$. (In [loc. cit.] $D(\kappa)$ has been denoted by $W[F, V]$; this is misleading.)

(The following argument is due to the referee.) It is easy to see that $V^{-1} M = FM$ as in [loc. cit.]. Consider the composite morphism $V^{-1} M = FM \simeq M \sim M_{h-1}$. Obviously this composite morphism is a zero morphism, while the image of this morphism is equal to $FM_{h-1}$ since the following diagram

$$
\begin{array}{ccc}
M & \xrightarrow{F} & M \\
\downarrow & & \downarrow \\
M_{h-1} & \xrightarrow{F} & M_{h-1}
\end{array}
$$

is commutative and since the morphism $M \to M_{h-1}$ is surjective. Hence $FM_{h-1} = 0$ and $F = 0$ on $M_n$ for all $n \leq h - 1$.

The following is a generalization of [vGK] (5.6):

**Corollary 2.4 (cf. [vGK] (5.6)).** Set $FH^q(Y, W_n(O_Y)) := \text{Ker}(F : H^q(Y, W_n(O_Y)) \to H^q(Y, W_n(O_Y)))$. Then

$$
(2.4.1) \quad \dim \kappa (FH^q(Y, W_n(O_Y))) = \min\{n, h^q(Y/\kappa) - 1\}.
$$

Consequently

$$
(2.4.2) \quad \dim \kappa (H^q(Y, W_n(O_Y))/F) = \min\{n, h^q(Y/\kappa) - 1\}.
$$

**Proof.** Let the notations be as in the proof of (2.3). As in the proof of (2.3), we may assume that $\kappa$ is algebraically closed. We may assume that $h < \infty$. If $n \leq h - 1$, then $F = 0$ on $M_n$. Hence $\text{Ker}(F : M_n \to M_n) = M_n$ and this is an $n$-dimensional vector space over $\kappa$.

Assume that $n \geq h$. Because $M = D(\Phi_{Y/\kappa}^q) \simeq D(\kappa)/D(\kappa)(F - V^{-1})$, there exists an element $\omega \in M$ such that $\{\omega, V(\omega), \ldots, V^{h-1}(\omega)\}$ is a basis of $M$ over $W$. Let $\overline{\omega}$ be the image of $\omega$ in $M_n$. Let $R : M_m \to M_{m-1}$ ($m \geq 2$) be the induced morphism by the projection $R : W_{m+1}(O_Y) \to W_{m+1}(O_Y)$. Then we claim that

$$
\{V^n R^{n-1}(\overline{\omega}), \ldots, V^{n-(h-1)} R^{n-(h-1)}(\overline{\omega})\}
$$

is a basis of $FM_m$. Indeed, this follows from the consideration in the case $n = h$ and induction on $n$ (by using the injectivity of the morphism $V : M_n \to M_{n+1}$) and the relation $FV = VF$. The claim implies (2.4.1).
The equality \( \text{(2.3)} \) follows from the following exact sequence
\[
0 \rightarrow \ker(F) \rightarrow M_n \stackrel{F}{\rightarrow} \sigma_n(M_n) \rightarrow \coker(F) \rightarrow 0.
\]
(Note that, since \( \kappa \) is perfect, \( \text{length}_{W_n}(M_n) = \text{length}_{W_n}(\sigma_n(M_n)) \).

**Remark 2.5.** We can generalize a part of \( \text{(2.3)} \) as follows.
Assume that \( H^d(Y, \mathcal{O}_Y) \simeq \kappa^m \) for a positive integer \( m \) instead of the assumption \( H^d(Y, \mathcal{O}_Y) \simeq \kappa \) and the operator \( F: H^d(Y, W_n(\mathcal{O}_Y)) \rightarrow H^d(Y, W_n(\mathcal{O}_Y)) \) is zero. Then \( n \leq m^{-1}h^q(Y/\kappa) - 1 \). The proof of this fact is the same as that of a part of the proof of \( \text{(2.3)} \).

3 The dimensions of cohomologies of closed differential forms

Let \( S_0 \) be a fine log scheme of characteristic \( p > 0 \). Let \( F_{S_0}: S_0 \rightarrow S_0 \) be the Frobenius endomorphism of \( S_0 \). Let \( Y \) be a log smooth scheme of Cartier type over \( S_0 \). Let \( g: Y \rightarrow S_0 \) be the structural morphism. Set \( Y' \) := \( Y \times_{S_0, F_{S_0}} S_0 \). Let \( W: Y' \rightarrow Y \) be the projection and let \( F: Y \rightarrow Y' \) be the relative Frobenius morphism over \( S_0 \). First recall the log inverse Cartier isomorphism due to Kato ([Kato1 (4.12) (1)])). It is the following isomorphism of sheaves of \( \mathcal{O}_{Y'} \)-modules:
\[
C^{-1}: \Omega_{Y'/S_0} \sim \rightarrow F_\bullet(\mathcal{H}^i(\Omega^\bullet_{Y/S_0})).
\]

Consider the case \( i = 0 \) in \( \text{(3.0.1)} \). Then \( C^{-1}: \mathcal{O}_{Y'} \sim \rightarrow F_\bullet(\mathcal{H}^0(\Omega^\bullet_{Y/S_0})) \) is the following isomorphism
\[
\mathcal{O}_{Y'} \ni a \mapsto F^a(a) \in F_\bullet(\mathcal{H}^0(\Omega^\bullet_{Y/S_0})).
\]

In particular, the following composite morphism
\[
\mathcal{O}_{Y'} \sim \rightarrow F_\bullet(\mathcal{H}^0(\Omega^\bullet_{Y/S_0})) \hookrightarrow F_\bullet(\mathcal{O}_Y)
\]
is injective.

**Remark 3.1.** Assume that \( \mathcal{O}_{S_0} \) is reduced. Then \( F_{S_0}^\circ \) induces an injective morphism \( F_{S_0}^\circ: \mathcal{O}_{S_0} \rightarrow F_{S_0}^\circ(\mathcal{O}_{S_0}) \). By ([Kato1 (4.5)]) the structural morphism \( \check{Y} \rightarrow \check{S}_0 \) is flat. Hence the natural morphism \( \mathcal{O}_Y \rightarrow W_*(\mathcal{O}_{Y'}) \) is injective. Because the composite morphism of this morphism and \( W_*(\mathcal{O}_{Y'}) \) is the \( p \)-th power endomorphism of \( \mathcal{O}_Y \), \( \check{Y} \) is reduced (cf. [Sh (2.3.2)]). Tsuji's result \( \text{(3.2)} \) below).

The following is Tsuji's result \( \text{(3.2)} \), which will be used in later sections.

**Proposition 3.2** ([Tsuji2 II (2.11) (1), (2.11) (2), (2.13) (1), (2.13) (2), (2.14), (4.2)]). The following hold:

1. The composite morphism of two saturated morphisms of integral log schemes is saturated.
2. The saturated morphisms of integral log schemes are stable under the base change of integral log schemes.
3. Let \( g: Y \rightarrow Z \) be an integral morphism of (fine) saturated log schemes. Then \( g \) is saturated if and only if the base change \( Y' \) of \( Y \) with respect to any morphism \( Z' \rightarrow Z \) from any (fine) saturated log scheme are saturated.
(4) Let $g : Y \to Z$ be a morphism of integral log schemes in characteristic $p > 0$. Then $g$ is $p$-saturated if and only if $g$ is of Cartier type.

(5) Let $g : Y \to Z$ be a log smooth integral morphism of fs log schemes. Then $g$ is saturated if and only if every fiber of $\tilde{g}$ is reduced.

**Proposition 3.3.** Set $BO_Y^{1 \cdot} : = \text{Im}(d : \mathcal{O}_Y \to \Omega_Y^1_{/S_0})$. Then the following sequence

\[(3.3.1) \quad 0 \to \mathcal{O}_Y \to F^*(\mathcal{O}_Y) \overset{F_*(d)}{\to} F_*(BO_Y^{1 \cdot}) \to 0\]

of $\mathcal{O}_Y$-modules is exact.

**Proof.** Except the surjectivity of $F_*(d)$, this is nothing but a reformulation of Proposition 3.0.2.

Since $\tilde{F}$ is a homeomorphism (SGA 5, XV Proposition 2 a)), $R^q F_*(\mathcal{E}) = 0 \ (q > 0)$ for an abelian sheaf $\mathcal{E}$ on $Y$. Hence $F_*(d)$ is surjective. $\square$

Let us recall well-known sheaves $B_n\Omega_Y^{1 \cdot}/S_0$ and $Z_n\Omega_Y^{1 \cdot}/S_0 \ (n \geq 1)$ of $g^{-1}(\mathcal{O}_{S_0})$-modules on $\tilde{Y}$ as in \cite[I 0 (2.2)]{I1} and \cite[(4.3)]{HK} defined by induction on $n$.

Because $\tilde{F}$ is a homeomorphism, we can identify an abelian sheaf on $\tilde{Y}$ with an abelian sheaf on $Y'$. Under this identification, we can express \textbf{3.0.1} as the equality

\[(3.3.2) \quad C^{-1} : \Omega_Y^{1 \cdot}/S_0 = \mathcal{H}^i(\Omega_Y^{1 \cdot})\]

of abelian sheaves. Set $B_0\Omega_Y^{1 \cdot}/S_0 : = 0$ and $Z_0\Omega_Y^{1 \cdot}/S_0 : = \Omega_Y^1/S_0$. We define $B_n\Omega_Y^{1 \cdot}/S_0$ and $Z_n\Omega_Y^{1 \cdot}/S_0$ by the following equalities ($n \geq 1$):

\[C^{-1} : B_{n-1}\Omega_Y^{1 \cdot}/S_0 = B_n\Omega_Y^{1 \cdot}/S_0/BO_Y^{1 \cdot}/S_0, \quad C^{-1} : Z_{n-1}\Omega_Y^{1 \cdot}/S_0 = Z_n\Omega_Y^{1 \cdot}/S_0/BO_Y^{1 \cdot}/S_0.\]

Then we have the following inclusions:

\[0 \subset B_1\Omega_Y^{1 \cdot}/S_0 \subset \cdots \subset B_n\Omega_Y^{1 \cdot}/S_0 \subset \cdots \subset Z_n\Omega_Y^{1 \cdot}/S_0 \subset \cdots \subset Z_1\Omega_Y^{1 \cdot}/S_0 \subset \Omega_Y^{1 \cdot}/S_0.\]

Set $Y^{(p)} : = Y'$ and $Y^{(n+1)} : = (Y^{(p+1)})'$. We consider $Z_n\Omega_Y^{1 \cdot}/S_0$ and $B_n\Omega_Y^{1 \cdot}/S_0$ as $\mathcal{O}_{Y^{(n+1)}}$-submodules of $F_n(\Omega_Y^{1 \cdot}/S_0)$. We recall the following result:

**Lemma 3.4 (\cite[I 0 (2.2.8), L 1.13])**. The sheaves $B_n\Omega_Y^{1 \cdot}/S_0$ and $Z_n\Omega_Y^{1 \cdot}/S_0 \ (n \in \mathbb{N}, i \in \mathbb{N})$ are locally free sheaves of $\mathcal{O}_{Y^{(n+1)}}$-modules of finite rank. They commute with the base changes of $S_0$.

If $\tilde{S}_0$ is perfect, then $\tilde{Y}' \sim \tilde{Y}$. Hence the equality \textbf{3.3.2} induces the following isomorphism:

\[(3.4.1) \quad C^{-1} : \Omega_Y^{1 \cdot}/S_0 \sim \mathcal{H}^i(\Omega_Y^{1 \cdot}).\]

Then we have the following Cartier morphisms $C$'s:

\[C : B_{n+1}\Omega_Y^{1 \cdot}/S_0 \overset{\text{proj}}{\longrightarrow} B_{n+1}\Omega_Y^{1 \cdot}/S_0/BO_Y^{1 \cdot}/S_0 \overset{C^{-1}}{\sim} B_n\Omega_Y^{1 \cdot}/S_0 \overset{\sim}{\longrightarrow} B_n\Omega_Y^{1 \cdot}/S_0,\]

and

\[C : Z_{n+1}\Omega_Y^{1 \cdot}/S_0 \overset{\text{proj}}{\longrightarrow} Z_{n+1}\Omega_Y^{1 \cdot}/S_0/BO_Y^{1 \cdot}/S_0 \overset{C^{-1}}{\sim} Z_n\Omega_Y^{1 \cdot}/S_0 \overset{\sim}{\longrightarrow} Z_n\Omega_Y^{1 \cdot}/S_0.\]
These morphisms are only morphisms of abelian sheaves on $\mathring{Y}$.

Until the end of this section except the remark \textit{\textbf{[5.5]}} below, assume that $\mathring{S}_0$ is perfect. Let $F: \mathring{Y} \to Y$ be the absolute Frobenius endomorphism of $Y$. In \cite{Serre} §7 (18) Serre has defined the following morphism of abelian sheaves

\[ d_n: F_*(\mathcal{W}_n(\mathcal{O}_Y)) \to F^n_*(\Omega^1_{Y/\mathring{S}_0}) \]

defined by the following formula:

\[ d_n((a_0, \ldots, a_{n-1})) = \sum_{i=0}^{n-1} a_i^{p^{n-i-1}} da_i \quad (a_i \in \mathcal{O}_Y). \]

(In \textit{loc. cit.} he has denoted $d_n$ by $D_n$ and he has considered $D_n$ only in the case $\mathring{S}_0 = \text{Spec}(\kappa)$.) He has remarked that the following formula holds:

\[ d_n((a_0, \ldots, a_{n-1})(b_0, \ldots, b_{n-1})) = b_0^{p^{n-1}} d_n((a_0, \ldots, a_{n-1})) + a_0^{p^{n-1}} d_n((b_0, \ldots, b_{n-1})). \]

By \textit{\textbf{[3.4.3]}}, it is easy to check that $d_n: F_*(\mathcal{W}_n(\mathcal{O}_Y)) \to F^n_*(\Omega^1_{Y/\mathring{S}_0})$ is a morphism of $\mathcal{W}_n(\mathcal{O}_Y)$-modules.

\textbf{Remark 3.5.} Let $F: \mathring{Y} \to Y'$ be the relative Frobenius morphism as in the beginning of this section. Then the morphism

\[ d_n: F_*(\mathcal{W}_n(\mathcal{O}_Y)) \to F^n_*(\Omega^1_{Y/\mathring{S}_0}) \]

cannot be a morphism of $\mathcal{W}_n(\mathcal{O}_{Y'})$-modules in general except the case $n = 1$.

The following \textbf{[3.6.1]} is the log version of a generalization of Serre’s result in \cite{Serre} §7 Lemme 2. Our proof of \textbf{[3.6.1]} is more elementary and more elegant than his proof.

\textbf{Proposition 3.6.} Assume that $\mathring{S}_0$ is perfect. Let $F: \mathring{Y} \to Y$ be the absolute Frobenius endomorphism of $Y$. Let $n$ be a positive integer. Denote the following composite morphism

\[ F_*(\mathcal{W}_n(\mathcal{O}_Y)) \xrightarrow{d_n} F^n_*(\Omega^1_{Y/\mathring{S}_0}) \]

by $d_n$ again. Then $d_n$ factors through $B_n\Omega^1_{Y/\mathring{S}_0}$ and the following sequence

\[ 0 \to \mathcal{W}_n(\mathcal{O}_Y) \xrightarrow{F} F_*(\mathcal{W}_n(\mathcal{O}_Y)) \xrightarrow{d_n} B_n\Omega^1_{Y/\mathring{S}_0} \to 0 \]

is exact. Here we denote the morphism $\mathcal{W}_n(F^*)$ (resp. $F_*(\mathcal{W}_n(\mathcal{O}_Y)) \to B_n\Omega^1_{Y/\mathring{S}_0}$) by $F$ (resp. $d_n$) again by abuse of notation. Consequently $d_n$ induces the following isomorphism of $\mathcal{W}_n(\mathcal{O}_Y)$-modules:

\[ F_*(\mathcal{W}_n(\mathcal{O}_Y))/F(\mathcal{W}_n(\mathcal{O}_Y)) \xrightarrow{\sim} B_n\Omega^1_{Y/\mathring{S}_0}. \]

\textit{Proof.} First consider the case $n = 1$. In this case, \textbf{[3.6.1]} is obtained by \textbf{[3.3]}.

We proceed by induction on $n$. Assume that \textbf{[3.6.1]; $n - 1$} is exact.
By the definition of $B_n\Omega^1_{Y/S_0}$, the isomorphism $C^{-1}$ in (3.4.1) induces the isomorphism $C^{-1}: B_{n-1}\Omega^1_{Y/S_0} \cong B_n\Omega^1_{Y/S_0}/B\Omega^1_{Y/S_0}$. Let $R: W_n(O_Y) \to W_{n-1}(O_Y)$ be the projection. By the inductive definition of $B_n\Omega^1_{Y/S_0}$, it is easy to check that $\text{Im}(d_n) \subset B_n\Omega^1_{Y/S_0}$. Consider the following diagram

(3.6.3)

\[
\begin{array}{c}
0 \quad 0 \quad 0 \\
0 \quad \mathcal{O}_Y \quad V_{n-1}^* \quad \mathcal{W}_n(O_Y) \quad R \quad \mathcal{W}_{n-1}(O_Y) \quad 0 \\
0 \quad F \quad F \quad F \\
0 \quad \mathcal{F}_n(O_Y) \quad \mathcal{F}_n(\mathcal{W}_n(O_Y)) \quad \mathcal{F}_n(\mathcal{W}_{n-1}(O_Y)) \quad 0 \\
0 \quad d_n \quad d_n \quad d_{n-1} \\
0 \quad B_1\Omega^1_{Y/S_0} \quad \mathcal{C} \quad B_n\Omega^1_{Y/S_0} \quad \mathcal{C} \quad B_{n-1}\Omega^1_{Y/S_0} \quad 0 \\
0 \quad 0 \quad 0 \quad 0.
\end{array}
\]

The three rows above are exact sequences of abelian sheaves on $Y$. By (3.0.3) the morphism $F: \mathcal{W}_n(O_Y) \to \mathcal{F}_n(\mathcal{W}_n(O_Y))$ is injective. It is clear that $\text{Im}(F) \subset \text{Ker}(d_n)$. It is also clear that the upper two diagrams are commutative. The commutativity of the left square of the lower diagram follows from the obvious relation

(3.6.4)

\[d_n V = d_{n-1}.\]

Since

\[C^{-1}(a_i^{p^n-2-1}da_i) = (a_i^{p^n-2-1}p^n da_i) = a_i^{p^n-1-1}da_i \quad (i \in \mathbb{N}, a_i \in \mathcal{O}_Y),\]

we see that the right square of the lower diagram is commutative. Induction on $n$ and the snake lemma show that the middle column is exact. $\blacksquare$

**Remark 3.7.** In [Se] Serre has considered (3.6.1) in the trivial logarithmic case with the assumption of the normality of $\hat{Y}$ only as an exact sequence of sheaves of abelian sheaves. In this article we have to consider (3.6.1) as an exact sequence of sheaves of $\mathcal{W}_n(O_Y)$-modules.

**Definition 3.8.** We call the exact sequence (3.6.1) of $\mathcal{W}_n(O_Y)$-modules the log Serre exact sequence of $Y/S_0$ in level $n$.

It is worth stating the following (this has been used in Nakkō in a key point):

**Corollary 3.9.** The following diagram

(3.9.1)

\[
\begin{array}{c}
\mathcal{F}_n(\mathcal{W}_n(O_Y)) \quad \mathcal{F}_n(\mathcal{W}_{n-1}(O_Y)) \\
\mathcal{F}_n(\mathcal{W}_n(O_Y)) \quad \mathcal{F}_n(\mathcal{W}_{n-1}(O_Y)) \\
B_n\Omega^1_{Y/S_0} \quad \mathcal{C} \quad B_{n-1}\Omega^1_{Y/S_0} \\
\end{array}
\]

is commutative.
Corollary 3.10. Consider the case $\tilde{S}_0 = \text{Spec}(\kappa)$ as in the beginning of the previous section. Denote $S_0$ by $s$ in this case. Let the notations and the assumptions be as in (2.3). Then the following hold:

1. $H^q(Y, W_n(\mathcal{O}_Y))/F = H^q(Y, B_n \Omega^1_Y/s)$. Consequently

\begin{equation}
\dim_n H^q(Y, B_n \Omega^1_Y/s) = \min\{n, h^q(\bar{\kappa}/\kappa) - 1\}.
\end{equation}

2. Assume that $H^{q-1}(Y, \mathcal{O}_Y) = 0$ if $q \geq 2$. Then $F H^q(Y, W_n(\mathcal{O}_Y)) = H^{q-1}(Y, B_n \Omega^1_Y/s)$. Consequently

\begin{equation}
\dim_n H^{q-1}(Y, B_n \Omega^1_Y/s) = \min\{n, h^q(\bar{\kappa}/\kappa) - 1\}.
\end{equation}

3. Assume that $H^{q-1}(Y, \mathcal{O}_Y) = 0$ if $q \geq 2$ and that $H^{q-2}(Y, \mathcal{O}_Y) = 0$ if $q \geq 3$. Then $H^{q-2}(Y, B_n \Omega^1_Y/s) = 0$.

Proof. (1): Taking the long exact sequence of (3.6.1), we have the following exact sequence of $W_n$-modules:

\begin{equation}
\begin{array}{ll}
H^{q-1}(Y, W_n(\mathcal{O}_Y)) & \xrightarrow{F} H^{q-1}(Y, F_\ast(\mathcal{W}_n(\mathcal{O}_Y))) \\
& \xrightarrow{F} H^{q-1}(Y, B_n \Omega^1_Y/s) \\
& \xrightarrow{F} H^q(Y, W_n(\mathcal{O}_Y)) \\
\end{array}
\end{equation}

Since $\tilde{F}$ is finite, $H^q(Y, F_\ast(\mathcal{W}_n(\mathcal{O}_Y))) = \sigma H^q(Y, W_n(\mathcal{O}_Y))$, where $\sigma$ is the Frobenius automorphism of $\mathcal{W}_n$. Hence we have the following exact sequence of $\mathcal{W}_n$-modules:

\begin{equation}
\begin{array}{ll}
H^{q-1}(Y, W_n(\mathcal{O}_Y)) & \xrightarrow{F} \sigma H^{q-1}(Y, W_n(\mathcal{O}_Y)) \\
& \xrightarrow{F} H^{q-1}(Y, B_n \Omega^1_Y/s) \\
& \xrightarrow{F} H^q(Y, W_n(\mathcal{O}_Y)) \\
\end{array}
\end{equation}

Hence $H^q(Y, W_n(\mathcal{O}_Y))/F = H^q(Y, B_n \Omega^1_Y/s)$. By (2.3) the dimension of this vector space over $\kappa$ is $\min\{n, h^q(\bar{\kappa}/\kappa) - 1\}$.

(2): If $q \geq 2$, it is easy to see that $H^{q-1}(Y, W_n(\mathcal{O}_Y)) = 0$. Hence we have the following exact sequence of $\mathcal{W}_n$-modules:

\begin{equation}
\begin{array}{ll}
0 & \rightarrow H^{q-1}(Y, B_n \Omega^1_Y/s) \\
& \xrightarrow{F} H^q(Y, W_n(\mathcal{O}_Y)) \\
\end{array}
\end{equation}

In the case $q = 1$, we see that (3.10.5) is also exact by the proof of (2.4) (2). This tells us that $H^{q-1}(Y, B_n \Omega^1_Y/s) = F H^q(Y, W_n(\mathcal{O}_Y))$. By (2.4) the dimensions of these vector spaces over $\kappa$ are $\min\{n, h^q(\bar{\kappa}/\kappa) - 1\}$.

(3): By (3.6.2) $F_\ast(\mathcal{W}_n(\mathcal{O}_Y))/F(\mathcal{W}_n(\mathcal{O}_Y)) = B_n \Omega^1_Y/s$. Hence

\begin{equation}
H^q(Y, B_n \Omega^1_Y/s) = H^q(Y, F_\ast(\mathcal{W}_n(\mathcal{O}_Y))/F(\mathcal{W}_n(\mathcal{O}_Y))) = H^q(Y, W_n(\mathcal{O}_Y)/F) \quad (q' \in \mathbb{N})
\end{equation}

since $\tilde{F}$: $\mathcal{W}_n(Y) \rightarrow \mathcal{W}_n(Y)$ is a homeomorphism. Because $\bar{Y}$ is reduced by (3.1), (3) is nothing but a special case of (2.4) (3).
4 Log deformation theory vs log deformation theory with abrelative and relative Frobenius morphisms

In this section we give the log versions of two relative versions of Nori and Srinivas’ deformation theory in [NoS] and [Sr]. In [loc. cit.] they have considered the deformation theory with the absolute Frobenius endomorphisms over the spectrum of the Witt ring of finite length of a perfect field of characteristic $p > 0$. In this section we construct the theory of log deformations with non well-known relative Frobenius morphisms instead of the absolute Frobenius endomorphisms over a more general base fine log scheme; we also remark that we can construct the theory of log deformations with well-known relative Frobenius morphisms.

In (4.7) below we give an important correction of K. Kato’s deformation theory for log smooth schemes in [Kk1] (and [Kf1]) and we establish a relationship between our log deformation theories and the correction of his theory. First let us recall the following proposition due to K. Kato.

**Proposition 4.1 ([Kk1 (3.9)])** Let

\[
\begin{array}{ccc}
T_0 & \xrightarrow{c} & T \\
\downarrow s & & \downarrow \\
Z & \longrightarrow & S
\end{array}
\]

be a commutative diagram of fine log schemes such that the upper horizontal morphism is an exact closed immersion defined by a square zero ideal sheaf $\mathcal{I}$ of $\mathcal{O}_T$. Let $P(s)$ be a Zariski sheaf on $T$ such that, for a log open subscheme $U$ of $T$, $P(s)(U)$ is the set of morphisms $s: U \rightarrow Z$’s making the resulting two triangles commutative in (4.1).

where we replace $T$, $T_0$ and $s$ by $U$, $U_0 := T_0 \cap U$ and $s|_{U_0}$, respectively. Then $P(s)$ is a torsor under $\text{Hom}_{\mathcal{O}_{U_0}}(s^*(\Omega^1_{Z/S}), \mathcal{I})$ on $T$. That is, for a morphism $g: U \rightarrow Z$ making the resulting two triangles commutative, there exists a bijection between the set of morphisms $h: U \rightarrow Z$’s making the resulting two triangles commutative in (4.1) and the set $H^0(U, \text{Hom}_{\mathcal{O}_{U_0}}(s^*(\Omega^1_{Z/S}), \mathcal{I})) = H^0(Z, \text{Hom}_{\mathcal{O}_{Z}}(\Omega^1_{Z/S}, (s|_{U_0})^*(\mathcal{I}|_{U})))$.

The bijection in (4.1) is obtained by the following two maps

\[ h \mapsto (da \mapsto h^*(a) - g^*(a) \in \mathcal{I}) \quad (a \in \mathcal{O}_Z) \]

and

\[ h \mapsto (d \log m \mapsto u_{h,g}(m) - 1 \in \mathcal{I}) \quad (m \in \mathcal{O}_Z) \]

where $u_{h,g}(m) \in \mathcal{O}_T^* \otimes_M$ is a unique local section such that $h^*(m) = g^*(m)u_{h,g}(m)$. We denote the corresponding element to $h$ in $\text{Hom}_{\mathcal{O}_Z}(\Omega^1_{Z/S}, (s|_{U_0})^*(\mathcal{I}|_{U}))$ by $h^* - g^*$.

**Proposition 4.2.** Let the notations be as in (4.1). Then the following hold:

(1) (cf. [SGA 1, III (5.6)], [22 (2.11)]) In

\[ H^1(T_0, \text{Hom}_{\mathcal{O}_{U_0}}(s^*(\Omega^1_{Z/S}), \mathcal{I})) \]

there exists a canonical obstruction class of the existence of a morphism $T \rightarrow Z$ making the diagrams of the two resulting triangles commutative in (4.1). If $s^*(\Omega^1_{Z/S})$ is a locally free $\mathcal{O}_{T_0}$-module, then this group is equal to $\text{Ext}^1_{\mathcal{O}_{T_0}}(s^*(\Omega^1_{Z/S}), \mathcal{I})$. 

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(2) In
\[ H^1(Z, \mathcal{H}om_{O_Z}(\Omega^1_{Z/S}, s_*(\mathcal{I}))) \]
there exists a canonical obstruction class of the existence of a morphism \( T \to Z \)
making the diagrams of the two resulting triangles commutative in \([1.1]\). If \( \Omega^1_{Z/S} \)
is a locally free \( O_Z \)-module, then this group is equal to \( \text{Ext}^1_Z(\Omega^1_{Z/S}, s_*(\mathcal{I})) \).

Proof. (1): By \([1.1]\) this is only a special case of a general well-known result (see [SGA 1, p. 70–71], \[G\]). We can also give the proof of (1) which is similar to the proof of (2) below.

(2): Let \( \mathcal{U} := \{ Z_i \} \) be a log affine open covering of \( Z \). Let \( \tilde{s}_i : T \to Z_i \) be a local lift of a restriction of \( s : T_0 \to Z \) obtained by shrinking \( T \). It is easy to see that \( \{ \tilde{s}_j^* - \tilde{s}_i^* \}_{ij} \) is an element of \( Z^1(\mathcal{U}, \mathcal{H}om_{O_Z}(\Omega^1_{Z/S}, s_*(\mathcal{I}))) \). Then we claim that the obstruction class stated in \([2.2]\) is the class
\[ \{ \tilde{s}_j^* - \tilde{s}_i^* \}_{ij} \in \lim_{\mathcal{U}} H^1(\mathcal{U}, \mathcal{H}om_{O_Z}(\Omega^1_{Z/S}, s_*(\mathcal{I}))) = H^1(Z, \mathcal{H}om_{O_Z}(\Omega^1_{Z/S}, s_*(\mathcal{I}))). \]

Indeed, if \( \tilde{s}_i \) is the restriction of a global lift \( \tilde{s} \) of \( s \), then \( \{ \tilde{s}_j^* - \tilde{s}_i^* \}_{ij} = 0 \). Conversely, if it is the coboundary, then there exists a class \( \{ t_i \} \) \( t_i \in \text{Hom}_{O_Z}(\Omega^1_{Z/S}, s_*(\mathcal{I})) \) such that \( \tilde{s}_j^* - \tilde{s}_i^* = t_j - t_i \). Hence \( \tilde{s}_j^* - t_j = \tilde{s}_i^* - t_i \) and \( \tilde{s}_i^* - t_i \)'s patch together. These sections define a global morphism \( T \to Z \) over \( S \). It is clear that this morphism is a lift of \( s : T_0 \to Z \) over \( S \) since \( \text{Im}(t_i) \subset s_*(\mathcal{I}) \). We have to prove that the class \( \{ \tilde{s}_j^* - \tilde{s}_i^* \}_{ij} \) in \( H^1(Z, \mathcal{H}om_{O_Z}(\Omega^1_{Z/S}, s_*(\mathcal{I}))) \) is independent of the choice of \( \mathcal{U} \). Assume that we are given another covering \( \mathcal{V} := \{ Z'_v \} \) and another local lift \( \tilde{s}'_v : T \to Z'_v \). Then, by considering the refinement \( \mathcal{U} \cap \mathcal{V} := \{ Z_i \cap Z'_v \} \) of \( \mathcal{U} \) and \( \mathcal{V} \),
\[ \{ \tilde{s}'_v - \tilde{s}_i \}_{i,v} \in Z^1(\mathcal{U} \cap \mathcal{V}, \mathcal{H}om_{O_Z}(\Omega^1_{Z/S}, s_*(\mathcal{I}))) \]
gives us a 1-coboundary. This implies the desired independence.

Assume that \( \Omega^1_{Z/S} \) is a locally free \( O_Z \)-module. Then we obtain the equality
\[ H^1(Z, \mathcal{H}om_{O_Z}(\Omega^1_{Z/S}, s_*(\mathcal{I}))) = \text{Ext}^1_Z(\Omega^1_{Z/S}, s_*(\mathcal{I})) \]
by the following spectral sequence:
\[ E_2^{ij} = H^i(Z, \mathcal{E}xt^j_{O_Z}(\mathcal{F}, \mathcal{G})) \Rightarrow \text{Ext}^{i+j}_Z(\mathcal{F}, \mathcal{G}) \quad (i, j \in \mathbb{N}) \]
for \( O_Z \)-modules \( \mathcal{F} \) and \( \mathcal{G} \).

Let \( S \) be a fine log scheme. Let \( S_0 \hookrightarrow S \) be an exact closed immersion of fine log schemes defined by a square zero ideal sheaf \( \mathcal{I} \) of \( O_S \). Let \( Y' \) be a log smooth scheme over \( S_0 \). Recall that \( 
abla / S \) is called a log smooth lift of \( Y / S_0 \) if \( Y \) is a log smooth scheme over \( S \) such that \( Y \times_S S_0 = Y' \).

Let \( Y' / S \) be a log smooth lift of \( Y / S_0 \). As an immediate corollary of \([4.1]\), we obtain the \( \delta \) in \([4.3]\) below as an element of \( \text{Hom}_{O_Y}(\Omega^1_{Y/S}, \mathcal{I}O_Y) \):

**Corollary 4.3.** Let \( Y' / S \) be a log smooth lift of \( Y / S_0 \). Then the following hold:

1. Let \( g \) be an automorphism of \( Y' / S \) such that \( g|_Y = \text{id}_Y \). Express \( g^*(a) = a + \delta(\overline{a}) \) \((a \in \mathcal{O}_Y)\) with \( \delta(\overline{a}) \in \mathcal{I}O_Y \). Here \( \overline{a} \) is the image of \( a \) in \( \mathcal{O}_Y \). Then \( \delta : \mathcal{O}_Y \to \mathcal{I}O_Y \) is a derivation over \( O_S \).

2. Express \( g^*(m) = m(1 + \delta(\overline{m})) \) \((m \in M_Y)\) with \( \delta(\overline{m}) \in \mathcal{I}O_Y \). Here \( \overline{m} \) is the image of \( m \) in \( M_Y \). Then \( \delta(\overline{mm'}) = \delta(\overline{m}) + \delta(\overline{m'}) \) \((m, m' \in M_Y)\).

3. Let \( \alpha : M_Y \to \mathcal{O}_Y \) be the structural morphism. Then \( \alpha(m)\delta(m) = \delta(\alpha(m)) \) \((m \in M_Y)\).
We also recall the following result due to K. Kato:

**Proposition 4.4** ([Kk1, (3.14) (2), (3)])

1. Let \( \bar{Y}/S \) be a log smooth lift of \( Y/S_0 \). Let \( \text{Aut}_S(\bar{Y}, Y) \) be the group of automorphisms \( g: \bar{Y} \to \bar{Y} \) over \( S \) such that \( g|_Y = \text{id}_Y \). Then the morphism

\[
\text{Aut}_S(\bar{Y}, Y) \ni g \mapsto \delta \in \text{Hom}_{O_Y}(\Omega^1_{\bar{Y}/S_0}, \mathcal{I}\mathcal{O}_Y).
\]

obtained by \( \text{(4.3)} \) gives the following isomorphism of groups:

\[
\text{Aut}_S(\bar{Y}, Y) \cong H^0(Y, \text{Hom}_{O_Y}(\Omega^1_{\bar{Y}/S_0}, \mathcal{I}\mathcal{O}_Y)) = \text{Hom}_{O_Y}(\Omega^1_{\bar{Y}/S_0}, \mathcal{I}\mathcal{O}_Y).
\]

2. Let \( \text{Lift}'_{Y/(S_0 \subset S)} \) be the following sheaf

\[
\text{Lift}'_{Y/(S_0 \subset S)}(U) := \{\text{isomorphism classes of log smooth lifts of } U/S_0 \text{ over } S\}
\]

for each log open subscheme \( U \) of \( Y \). If \( Y/S_0 \) has a lift \( \bar{Y}/S \), then there exists the following (natural) bijection of sets:

\[
\text{Lift}'_{Y/(S_0 \subset S)}(Y) \cong H^1(Y, \text{Hom}_{O_Y}(\Omega^1_{\bar{Y}/S_0}, \mathcal{I}\mathcal{O}_Y)) = \text{Ext}^1_Y(\Omega^1_{\bar{Y}/S_0}, \mathcal{I}\mathcal{O}_Y).
\]

Let us recall the map \( \text{(4.4.3)} \).

Let \( \bar{Z}/S \) be a log smooth lift of \( Y/S_0 \). Let \( \{\bar{U}_i\}_{i \in I} \) be a log affine open covering of \( \bar{Z} \) such that there exists a morphism \( g_i: \bar{U}_i \to \bar{Y} \) making the resulting two triangles

\[
\begin{array}{ccc}
U_i & \to & \bar{U}_i \\
\downarrow & & \downarrow \\
\bar{Y} & \to & S
\end{array}
\]

commutative. Here \( U_i := \bar{U}_i \cap Y \). Set \( U := \{U_i\}_{i \in I} \). Then we have a section

\[
g_{ij} := g_j^* - g_i^* \in \text{Hom}_{O_Y}(\Omega^1_{\bar{Y}/S_0}, \mathcal{I}\mathcal{O}_Y)(U_{ij}),
\]

where \( U_{ij} := U_i \cap U_j \). These sections define an element of \( H^1(U, \text{Hom}_{O_Y}(\Omega^1_{\bar{Y}/S_0}, \mathcal{I}\mathcal{O}_Y)) \). Consequently we have an element of \( H^1(Y, \text{Hom}_{O_Y}(\Omega^1_{\bar{Y}/S_0}, \mathcal{I}\mathcal{O}_Y)) \).

**Remark 4.5.** Let the notations be as in [Kk1 (3.14) (4)]. There is a mistake in [loc. cit.]. The statement [Kk1 (3.14) (4)] has no sense since a lift \( \bar{X} \) of \( (X, M, f) \) appears in the sufficient condition

\[
H^2(X, \text{Hom}_{O_X}(\omega^1_{\bar{X}/Y}, \mathcal{I}\mathcal{O}_{\bar{X}})) = 0
\]

for an existence of a lift \( \bar{X} \) of \( (X, M, f) \). (If one claims that [Kk1 (3.14) (4)] has a sense, one has to prove that the sheaf \( \mathcal{I}\mathcal{O}_{\bar{X}} \) on \( X_{\text{et}} \) is independent of the choice of \( \bar{X} \).)

To make [Kk1 (3.14) (3)] (= (4.4) (2)) better and correct [Kk1 (3.14) (4)], more generally to define an obstruction class of a lift of \( Y/S_0 \) over \( S \), we moreover assume that \( Y/S_0 \) is integral. In the following we always assume this. That is, \( Y \) is assumed to be a log smooth integral scheme over \( S_0 \). We say that \( \bar{Y}/S \) is a log smooth integral lift (or simply a lift) of \( Y/S_0 \) if \( \bar{Y} \) is a log smooth integral scheme over \( S \) such that \( \bar{Y} \times S S_0 = Y \).

---

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Remark 4.6. The obvious analogues of (4.4) (1) and (2) hold for a log smooth integral scheme $Y/S_0$ by the proof of [Kk1] (3.14)].

The following includes an important correction of [Kk1] (3.14) (4) and [Kf1] (8.6)]. This is a log version of [SGA 1, III (6.3)] and a generalization of [KwN] (2.2)]

Theorem 4.7. Let the notations be as above. For a log scheme $Z$ over $S_0$, set $\mathcal{T}_Z/S_0 := \mathcal{H}om_{\mathcal{O}_Z}(\Omega^1_{Z/S_0}, \mathcal{O}_Z)$. Then the following hold:

1. Let $U$ be a log open subscheme of $Y$ and let $\tilde{U}$ be a log smooth integral lift of $U$ over $S$. Then

$$\hom_{\mathcal{O}_U}(\Omega^1_{U/S_0}, \mathcal{T}_Z) = \mathcal{T}_U/S_0 \otimes_{\mathcal{O}_{S_0}} \mathcal{I}. \quad (4.7.1)$$

2. Assume that $Y$ is separated. Then, in

$$H^2(Y, \mathcal{T}_Y/S_0) \otimes_{\mathcal{O}_{S_0}} \mathcal{I}, \quad (4.7.2)$$

there exists a canonical obstruction class $\text{obs}_{Y/(S_0 \subset S)}$ of a lift of $Y/S_0$ over $S$. Let $\text{Lift}_{Y/(S_0 \subset S)}$ be the following sheaf

$\text{Lift}_{Y/(S_0 \subset S)}(U) := \{ \text{isomorphism classes of log smooth integral lifts of } U/S_0 \text{ over } S \}$

for each log open subscheme $U$ of $Y$. If the obstruction class vanishes, then there exists the following (natural) bijection of sets:

$$\text{Lift}_{Y/(S_0 \subset S)}(Y) \xrightarrow{\sim} H^1(Y, \mathcal{T}_Y/S_0) \otimes_{\mathcal{O}_{S_0}} \mathcal{I}. \quad (4.7.3)$$

3. Let the assumption and the notations be as in (2). If $\tilde{S}$ is an affine scheme, say, Spec$(A)$, if we set $I := \Gamma(\tilde{S}, \mathcal{I})$ and $A_0 := A/I$ and if $I$ is a flat $A_0$-module, then the cohomology $H^2(Y, \mathcal{T}_Y/S_0) \otimes_{A_0} I = \text{Ext}^2_{A}(\Omega^1_{Y/S_0}, \mathcal{O}_Y) \otimes_{A_0} I$.

Proof. (1): (Because we have to give a comment in (4.9) (2) below, we have to give the following very easy proof of (1).) Consider the following obvious exact sequence:

$$0 \to \mathcal{I} \to \mathcal{O}_S \to \mathcal{O}_{S_0} \to 0.$$ 

Taking the tensorization $\mathcal{O}_Y \otimes \mathcal{I}$ of this exact sequence and noting that $\tilde{U} \to \tilde{S}$ is flat ([Kk1] (4.5)], we see that $\mathcal{I}\mathcal{O}_Y = \mathcal{I} \otimes_{\mathcal{O}_S} \mathcal{O}_{\tilde{U}} = \mathcal{I} \otimes_{\mathcal{O}_{S_0}} \mathcal{O}_U$. Hence we obtain the following equalities:

$$\hom_{\mathcal{O}_U}(\Omega^1_{U/S_0}, \mathcal{I}\mathcal{O}_Y) = \hom_{\mathcal{O}_U}(\Omega^1_{U/S_0}, \mathcal{O}_U \otimes_{\mathcal{O}_{S_0}} \mathcal{I})$$

$$= \hom_{\mathcal{O}_U}(\Omega^1_{U/S_0}, \mathcal{O}_U) \otimes_{\mathcal{O}_{S_0}} \mathcal{I}$$

since $\Omega^1_{U/S_0}$ is a locally free $\mathcal{O}_U$-module ([Kk1] (3.10)]).

(2): We construct the obstruction class $\text{obs}_{Y/(S_0 \subset S)}$ as in [SGA 1, p. 79]. Though the statement [SGA 1, III (6.3)] is well-known, the proof of it is not well-understood at all. (See [L9] (1) below.) Because we cannot find a detailed proof of (2) using cocycles in references, e. g., [SGA 1, III (6.3)], [L2] (2.12), [KwN] (2.2) nor [Kf1] (8.6) unfortunately, we have to give the detailed proof of (4.7.3) as follows.

Let $\mathcal{U} := \{ U_i \}_{i \in I}$ be a log affine open covering of $Y$ such that $U_i$ has a log smooth integral lift $\tilde{U}_i$ over $S$. Set $\tilde{U}_{ij} := \tilde{U}_i|_{U_{ij}}$ (Note that we cannot use [Kk1] (3.14) (1)] for
any log affine open subscheme of \( Y \) because we cannot use \([Kk1] (3.14) (4)\). However, by the proof of \([Kk1] (3.14)\) and \([Kk1] (4.1) (ii)\), we have the \( U_i \) over \( S_0 \) and the \( \tilde{U}_i \) over \( S \). Because \( \tilde{V} \) is separated, \( \tilde{U}_{ij} := \tilde{U}_i \cap \tilde{U}_j \) is affine. Since \( \tilde{U}_{ji} \) (resp. \( \tilde{U}_{ij} \)) is log smooth over \( S \), there exists a morphism \( g_{ij} : \tilde{U}_{ij} \to \tilde{U}_{ji} \) (resp. \( h_{ij} : \tilde{U}_{ij} \to \tilde{U}_{ji} \)) over \( S \) which is an extension of \( \text{id}_{\tilde{U}_{ij}} : U_{ij} \to U_{ji} \) and \( g_{ij} \circ h_{ij} \in \text{End}_{S}(\tilde{U}_{ij}) \) is an extension of \( \text{id}_{U_{ij}} \) over \( S_0 \). In particular, \( g_{ij} \) is an isomorphism.

This time we use the vanishing of \( H^1(\tilde{U}_{ij}, \text{Hom}_{\mathcal{O}_{U_{ij}}}(\Omega^1_{U_{ij}/S_0}, \mathcal{O}_{U_{ij}}) \otimes_{\mathcal{O}_{S_0}} \mathcal{I}) \), though it has been used in the proof of \([SGA 1] \text{III (6.3)}\). If one wants, one can assume that \( g_{ij} = g_{ij}^{-1} \) as in \([NoS]\) because we can endow \( \text{Aut}_S(\tilde{U}_{ij} \cap \tilde{U}_{ik}) \) corresponding to the right hand side of the equality above is equal to

\[
\begin{align*}
(4.7.3) \quad \mathfrak{g}_{ijk} := g_{ijk}^* &- \text{id}^*_U \in \text{Hom}_{\mathcal{O}_{U_{ijk}}}(\Omega^1_{U_{ijk}/S_0}, \mathcal{I})
\end{align*}
\]

By \((4.7.1)\) we obtain the following equality:

\[
\text{Hom}_{\mathcal{O}_{U_{ijk}}}(\Omega^1_{U_{ijk}/S_0}, \mathcal{I}) = \text{Hom}_{\mathcal{O}_{U_{ij}}}(\Omega^1_{U_{ij}/S_0}, \mathcal{O}_{U_{ij}}) \otimes_{\mathcal{O}_{S_0}} \mathcal{I}.
\]

Consequently

\[
\mathfrak{g}_{ijk} \in \Gamma(U_{ijk}, \text{Hom}_{\mathcal{O}_Y}(\Omega^1_{U_{ij}/S_0}, \mathcal{O}_Y) \otimes_{\mathcal{O}_{S_0}} \mathcal{I}).
\]

In fact, we can check

\[
\mathfrak{g} := (\mathfrak{g}_{ijk}) \in Z^2(U, \text{Hom}_{\mathcal{O}_Y}(\Omega^1_{U/S_0}, \mathcal{O}_Y) \otimes_{\mathcal{O}_{S_0}} \mathcal{I})
\]

(cited \([SGA 1] \text{p. 79}\)). Indeed, because \((\partial(\mathfrak{g}))_{ijkl} = \mathfrak{g}_{kij}^* + \mathfrak{g}_{ijl}^* - \mathfrak{g}_{ikl}^* \in \mathfrak{g}_{ijk}^* \), it suffices to prove that \( \mathfrak{g}_{ijkl} = \mathfrak{g}_{kij}^* + \mathfrak{g}_{ijl}^* - \mathfrak{g}_{ikl}^* \). The element of \( \text{Aut}_S(\tilde{U}_{ijkl}|U_{ijk}) \) corresponding to the right hand side of the equality above is equal to

\[
\begin{align*}
(4.7.4) \quad (g_{ijkl}^{-1} g_{kij}^* g_{ijk})(g_{ijkl}^{-1} g_{jkl}^* g_{ijl})(g_{ijkl}^{-1} g_{ijl}^* g_{klj}) = (g_{ijkl}^{-1} g_{kij}^* g_{ijk})(g_{ijkl}^{-1} g_{jkl}^* g_{ijl})(g_{ijkl}^{-1} g_{ijl}^* g_{klj})^{-1}.
\end{align*}
\]

Hence

\[
\begin{align*}
(4.7.5) \quad \{(g_{ijkl}^{-1} g_{kij}^* g_{ijk})(g_{ijkl}^{-1} g_{jkl}^* g_{ijl})(g_{ijkl}^{-1} g_{ijl}^* g_{klj})\}^* - \text{id}^*_U \in \mathfrak{g}_{ijkl}^* \in \Gamma(U_{ijkl}, \mathcal{O}_Y) \otimes_{\mathcal{O}_{S_0}} \mathcal{I} = (g_{ijkl}^{-1} g_{kij}^* g_{ijk})(g_{ijkl}^{-1} g_{jkl}^* g_{ijl})(g_{ijkl}^{-1} g_{ijl}^* g_{klj}) = (g_{ijkl}^{-1} g_{kij}^* g_{ijk})(g_{ijkl}^{-1} g_{jkl}^* g_{ijl})(g_{ijkl}^{-1} g_{ijl}^* g_{klj})^{-1}.
\end{align*}
\]

Here, to obtain the second equality above, we have used the lemma \((4.8)\) below. Now we have the desired element

\[
\text{obs} \frac{\mathcal{O}_Y}{S_0 \subset S} := \text{the class of } \mathfrak{g}
\]

in

\[
(4.7.6) \quad H^2(U, \text{Hom}_{\mathcal{O}_Y}(\Omega^1_{U/S_0}, \mathcal{O}_Y) \otimes_{\mathcal{O}_{S_0}} \mathcal{I}) = H^2(Y, \text{Hom}_{\mathcal{O}_Y}(\Omega^1_{Y/S_0}, \mathcal{O}_Y) \otimes_{\mathcal{O}_{S_0}} \mathcal{I}).
\]

Here we have used the assumption on the separatedness of \( \tilde{Y} \) to obtain the equality above.
We claim that \( g \) is independent of the choice of \( g_{ij} \)'s. Let \( g'_{ij} : \tilde{U}_{ij} \xrightarrow{\sim} \tilde{U}_{ij} \) be another isomorphism which is a lift of \( \text{id}_{U_{ij}} \). Then \( g'_{ij}g^{-1}_{ij} \) is an element of \( \text{Aut}_S(\tilde{U}_{ij}, U_{ij}) \).

Let \( \delta_{ij} \) be the \( \delta \) corresponding to \( g'_{ij}g^{-1}_{ij} \in \text{Hom}_{\mathcal{O}_{U_{ij}}}(\mathcal{O}^1_{U_{ij}/S_0}, \mathcal{O}_{U_{ij}}) \otimes_{\mathcal{O}_{S_0}} \mathcal{I} \): \( g'_{ij}^* (a) = g'_{ij}^* (a + \delta_{ij}(\overline{m})) \) \((a \in \mathcal{O}_{\tilde{U}_{ij}}, g'_{ij}^* (m) = g'_{ij}^* (m(1 + \delta_{ij}(\overline{m}))) \) \((m \in \mathcal{M}_{\tilde{U}_{ij}}) \).

Since \( g_{ij}|_{U_{ij}} = \text{id}_{U_{ij}} \) and since \( g_{ij} \) is a morphism over \( S \), \( g_{ij}^* (a) = g_{ij}^* (a + \delta_{ij}(\overline{m})) \) and \( g_{ij}^* (m) = g_{ij}^* (m)(1 + \delta_{ij}(\overline{m})) \). Using these relations and \( (4.8) \) below and making simple calculations, we obtain an equality \( g' := (g'_{ijk}) = g + \partial((\delta_{ij})) \). Indeed we have the following equations:

\[
(4.7.7) \quad g'_{ij}g'_{jk}g'_{ik}^{-1}(a) = g'_{ij}g_{jk}(g_{ik}^{-1}(a) + \delta_{ij}(\overline{m})) = \cdots \\
= g'_{ij}g_{jk}(g_{ik}^{-1}(a)) + \delta_{ij}(\overline{m}) + \delta_{jk}(\overline{m}) - \delta_{ik}(\overline{m}) \quad (a \in \mathcal{O}_{\tilde{U}_{ijk}}).
\]

Similarly we have the following equation:

\[
(4.7.8) \quad g'_{ij}g'_{jk}g'_{ik}^{-1}(m) = g_{ij}g_{jk}(g_{ik}^{-1}(m))(1 + \delta_{ij}(\overline{m}) + \delta_{jk}(\overline{m}) - \delta_{ik}(\overline{m})) \quad (m \in \mathcal{M}_{\tilde{U}_{ijk}}).
\]

Hence \( g' = g + \partial((\delta_{ij})) \). This shows that our claim holds.

We have to show that \( g \) is independent of the choice of the lift \( \tilde{U}_i \) of \( U_i \) over \( S \). Set \( \tilde{V}_i := (U_i|_{\tilde{V}_i})g|_{\tilde{V}_i}(i|_{\tilde{V}_i}) : \tilde{V}_i \xrightarrow{\sim} \tilde{V}_i \). Then it is easy to check that

\[
(4.7.9) \quad g'_{ijk} = g_{ijk}g_{ij}^{-1}.
\]

Let \( g'_{ijk} \) be the analogue of \( g_{ijk} \) for \( g'_{ij} \). Because \( \mathcal{I}\mathcal{O}_{\tilde{U}_{ijk}} = \mathcal{I} \otimes_{\mathcal{O}_S} \mathcal{O}_{\tilde{U}_{ijk}} = \mathcal{I} \otimes_{\mathcal{O}_{S_0}} \mathcal{O}_{\tilde{U}_{ijk}} \), we have an equality \( g_{ijk} = g'_{ijk} \) by \( (4.7.8) \) below. This implies that \( g \) is independent of the choice of the lift \( \tilde{U}_i \).

Next we claim that the class \( g \) is independent of the choice of the open cover \( \mathcal{U} \). Since two log affine open coverings of \( \mathcal{U} \) has a log affine refinement, we consider a refinement \( \mathcal{V} := \{ V_i \} \) of \( \mathcal{U} \) with a morphism \( \tau : \{ i' \} \rightarrow \{ i \} \) such that \( V_i \) is affine and such that \( V_i \subset U_{\tau(i')} \) over \( S \). It suffices to prove that \( g \in \tilde{H}^2(\mathcal{U}, \mathcal{O}_{\mathcal{O}_S, \mathcal{O}_S \otimes \mathcal{O}_{\mathcal{S}_0}}) \) is mapped to \( g \) in \( \tilde{H}^2(\mathcal{V}, \mathcal{O}_{\mathcal{S}_0 \otimes \mathcal{O}_{\mathcal{S}_0}}) \).

This is clear since the isomorphism \( g_{\tau(i')}\tau(j') : \tilde{U}_{\tau(i')}\tau(j') \xrightarrow{\sim} \tilde{U}_{\tau(j')} \) induces an isomorphism \( g_{\tau(i')}\tau(j') : \tilde{V}_{\tau(i')} \xrightarrow{\sim} \tilde{V}_{\tau(j')} \).

If \( g \) is coboundary, then there exists an element \( h_{ij} \) of \( \text{Aut}_S(\tilde{U}_{ij}, U_{ij}) \) such that \( \{ g_{ij} \} = \{ g_{ij}h_{ij} \} \) satisfies the transitivity condition \( g_{ik} = g_{jk}g_{ij}^{-1} \) as in \( \text{[SGA 1]} \) p. 79 (2). Consequently we have a lift \( \tilde{Y}/S \) of \( Y/S_0 \).

The last statement in (2) follows from (1) and \( (4.4) \) (2).

(3): (3) immediately follows from \( (4.7.9) \) and the assumption of the flatness of \( I \) (cf. \( \text{[SGA 1]} \) p. 75).

\[ \square \]

**Lemma 4.8.** Let \( F \in \text{Hom}_{\mathcal{O}_{\mathcal{U}_{jk}}}(\mathcal{O}^1_{\mathcal{U}_{jk}/S_0}, \mathcal{O}_{\mathcal{U}_{jk}/S_0}) \otimes_{\mathcal{O}_{S_0}} \mathcal{I} \) be the element corresponding to an element \( g \in \text{Aut}_S(\tilde{U}_{jk}, U_{jk}) \). Let \( h : \tilde{U}_{jk} \xrightarrow{\sim} \tilde{U}_{jk} \) be an isomorphism over \( S \) such that \( h|_{U_{jk}} = \text{id}_{U_{jk}} \) Then \( (h^{-1}|_{\tilde{U}_{jk}})^* F(h|_{\tilde{U}_{jk}})^* = F \).

**Proof.** This is obvious since \( h|_{U_{ij}} = \text{id}_{U_{ij}} \) and \( h \) is a morphism over \( S \). \[ \square \]
Remark 4.9. (1) It is doubtful whether arithmetic or algebraic geometers can read the proof of the very well-known result [SGA 1] III (6.3] rigorously and can give the detailed proof of it because to give the precise proof of it is tiresome and hard as shown in the proof of [4.7] and because it needs quite unacceptable patience. (We have never seen (4.7), (4.7.5), (4.7.7), (4.7.8) and (4.8) in other references.) For this reason, there exist the mistakes pointed out in (4.8) and (2) below and no one except us has noticed the mistakes.

The statements [KoN] (2.2) (3)] and [Kf1] (8.6) 3 seems obscure because Kawamata-Namikawa and F. Kato have not constructed the obstruction class in their article (they have only claimed that the construction is the same as that of SGA 1 III (6.3)]) and because we cannot understand whether the obstruction classes in their article is canonical.

(2) Let the notations be as in [Kf1] p. 338]. We do not understand why there exists an isomorphism $I : \mathcal{O}_X \rightarrow I \otimes_A \mathcal{O}_X$, in [loc. cit.]. Indeed, there exist a lot of counter-examples for this isomorphism. For example, log blow ups by Fujiwara-Kato (FK). [Ni] give us counter-examples: the underlying morphisms of log blow ups are not necessarily flat. One of the simplest examples is as follows.

Let $K$ be a field of any characteristic. Set $A = K[x_1, x_2]$ and $B = K[x_1, x_2, t]/(x_2 - x_1) = K[x_1, t]$. Endow Spec($A$) (resp. Spec($B$)) with a log structure associated to a morphism $\mathbb{N}^{\geq 2} \ni e_i \mapsto x_i \in A$ (resp. $\mathbb{N}^{\geq 2} \ni e_1 \mapsto x_1, e_2 \mapsto t \in B$), where $e_i$ ($i = 1, 2$) is a canonical basis of $\mathbb{N}^{\geq 2}$. Let $T$ (resp. $Y$) be the resulting log scheme. Let $A \rightarrow B$ be a morphism defined by $x_1 \mapsto x_1$ and $x_2 \mapsto x_2$. Let $\mathbb{N}^{\geq 2} \rightarrow \mathbb{N}^{\geq 2}$ be a morphism defined by the following: $e_1 \mapsto e_1, e_2 \mapsto e_1 + e_2$. Then we have a morphism $Y \rightarrow T$. This morphism is log étale by the criterion of K. Kato ([KK1] (3.5)].) Let $S$ be an exact closed subscheme of $T$ defined by the ideal sheaf $(x_1^2, x_2)$. Set $\tilde{X} := Y \times_T S$. Then the projection $\tilde{X} \rightarrow S$ is log étale since log étale morphisms are stable under base changes. Let $S_0$ be an exact closed subscheme of $S$ defined by the ideal sheaf $I := (x_1)$. Then the global sections of $I \mathcal{O}_{\tilde{X}}$ are equal to $x_1(K[x_1, t]/(x_1^2, tx_1)) = Kx_1$. On the other hand, the global sections of $I \otimes_{\mathcal{O}_S} \mathcal{O}_{\tilde{X}}$ are equal to

$$(x_1 K[x_1]/(x_1^2)) \otimes_{K[x_1]/(x_1^2)} K[x_1, t]/(x_1^2, tx_1) = Kx_1 \otimes_{K[x_1]/(x_1^2)} (K[x_1]/(x_1^2))[t]/(tx_1) \cong K \otimes_{K[x_1]/(x_1^2)} (K[x_1]/(x_1^2))[t]/(tx_1) = K[t].$$

Hence $I \mathcal{O}_{\tilde{X}}$ cannot be isomorphic to $I \otimes_{\mathcal{O}_S} \mathcal{O}_{\tilde{X}}$. In particular, the structural morphism $\tilde{X} \rightarrow \tilde{S}$ is not flat. This is a counter example of the claim after [Kf2] (4.1]: “underlying morphisms of log smooth liftings are flat”.

By virtue of this remark, the title “Log smooth deformation theory” of [Kf1] had to be replaced by “Log smooth integral deformation theory.”

(3) Once one proves [4.1] (1) and [4.7] (1), [4.7] (2) is a formal consequence obtained by a general theory as in [Oj] VII (1.2.2)] without using the assumption of the separatedness in (4.7) (2). However, in this article, we shall use the explicit description of the obstruction class in the proof of (4.7) (2) (see the proof of (4.12) (4)).

(4) In [Oj] (5.6), (8.36]) Olsson has already obtained (4.7) by using his theory of log cotangent complexes.

As already stated, in the following we always assume that the log smooth morphism $Y \rightarrow S_0$ is integral.

For a log scheme $Z$, let $Z_{\text{red}}$ be the log exact closed subscheme of $Z$ whose underlying scheme is $Z_{\text{red}}$ and whose log structure is the inverse image of that of $Z$. Assume
that $\hat{Z}$ is of characteristic $p > 0$. Let $F_Z \colon Z \rightarrow Z$ be the $p$-th power Frobenius endomorphism. Let $e$ be a fixed positive integer. Set $q := p^e$. Set $Z[q] := Z \times_{\hat{Z}, F_{\hat{Z}}^e} \hat{Z}$.

This is different from $Z^{(q)} := Z \times_{Z, F_Z^q} Z = Z$, though $(Z^{(q)})^o = \hat{Z} = (Z[q])^o$. Then we have the following two natural morphisms

$$Z \rightarrow Z[q] \text{ and } Z[q] \rightarrow Z.$$  

We denote the first morphism by $F^{[e]}_{Z/\hat{Z}}$. Let $W$ be a fine log scheme over $Z$. Set $W := W \times_Z Z[q]$. Then we have the following commutative diagram

$$
\begin{array}{ccc}
W & \longrightarrow & W \\
\downarrow & & \downarrow \\
Z & \longrightarrow & Z[q] \longrightarrow Z.
\end{array}
$$

(4.9.1)

Here the upper (resp. lower) horizontal composite endomorphism is the $q$-th power Frobenius endomorphism of $W$ (resp. $Z$). We call $F^{[e]}_{Z/\hat{Z}}$ the $e$-times iterated abrelative Frobenius morphism of a base log scheme. (The adjective “abrelative” is a coined word which implies “absolute and relative” or “far from being relative”.) We call the morphism $W \rightarrow W$ the abrelative Frobenius morphism of $W$ over $Z \rightarrow Z[q]$. Set also $W' := W \times_{Z, F_Z^q} Z$. If the structural morphism $W \rightarrow Z$ is integral, then $W' \rightarrow W$ is the Frobenius morphism of $W$ over $Z[q]$.

**Remark 4.10.** In [Og 197] Ogus has already defined $Z[p]$ (he has denoted it by $Z^{(1)}$).

Now assume that $S_{0,\text{red}}$ is of characteristic $p > 0$. Set $S_{00} := S_{0,\text{red}}$. Assume that there exists a lift $F^{[e]}_{S} : S \rightarrow S$ of the $q$-th power Frobenius endomorphism $F^{[e]}_{S_{00}} : S_{00} \rightarrow S_{00}$. We fix $F^{[e]}_{S}$ and assume that $F^{[e]}_{S}$ induces a morphism $F^{[e]}_{S_{00}} : S_{00} \rightarrow S_{00}$. Set $F^{[e]}_{S_{00}} := (F^{[e]}_{S_{00}})^o$ and $F^{[e]}_{S_{00}} := (F^{[e]}_{S_{00}})^o$. Set also $S'[q] := S \times_{S, F^{[e]}_{S_{00}}} S_{00}$ and $S_0[q] := S_0 \times_{S_0, F^{[e]}_{S_{00}}} S_0$ by abuse of notation. (These log scheme may depend on the choice of $F^{[e]}_{S}$. ) Because the following diagram

$$
\begin{array}{ccc}
S & \longrightarrow & S \\
\downarrow & & \downarrow \\
\hat{S} & \longrightarrow & \hat{S}
\end{array}
$$

is commutative, we have a natural morphism $F^{[e]}_{S_{0}/S_0} : S_0 \rightarrow S'[0]$. Similarly we have a natural morphism $F^{[e]}_{S_{0}/S_0} : S_0 \rightarrow S'[0]$. We also have two projections $S'[0] \rightarrow S$ and
Let \( S_0^{[q]} \rightarrow S_0 \). Then we have the following commutative diagram:

\[
\begin{array}{ccc}
S_0 & \xrightarrow{F^e_{S_0/S_0}} & S_0^{[q]} \\
\downarrow & & \downarrow \\
S & \xrightarrow{F_{S_0/S_0}} & S^{[q]}
\end{array}
\]

If \( e = 1 \), then we denote \( F_S^{[e]} \) and \( F_S^{[e]}_{S/S} \) by \( F_S \) and \( F_{S/S} \), respectively.

Set \( Y_0 := Y_{\text{red}} \) for simplicity of notation. Set \( 'Y := Y \times S_0 S_0^{[q]} \) and \( 'Y_0 := Y \times S_0 S_0^{[q]} \).

By (4.9.1) we have the following commutative diagram

\[
\begin{array}{ccc}
Y_0 & \xrightarrow{F_0} & 'Y_0 \\
\downarrow & & \downarrow \\
S_0 & \xrightarrow{F_0^{[e]}_{S_0/S_0}} & S_0^{[q]}
\end{array}
\]

where \( F_0 : Y_0 \rightarrow 'Y_0 \) is the \( e \)-times iterated abrelative Frobenius morphism over \( S_0 \rightarrow S_0^{[q]} \). We assume that there exists a lift \( F : Y \rightarrow 'Y \) of \( F_0 : Y_0 \rightarrow 'Y_0 \) over \( S_0 \rightarrow S_0^{[q]} \). That is, we assume that there exists a morphism \( F : Y \rightarrow 'Y \) fitting into the following commutative diagram

\[
\begin{array}{ccc}
Y_0 & \xrightarrow{c} & Y \\
F_0 \downarrow & & \downarrow F \\
'Y_0 & \xrightarrow{c} & 'Y
\end{array}
\]

over the commutative diagram

\[
\begin{array}{ccc}
S_0 & \xrightarrow{c} & S_0 \\
F_0^{[e]}_{S_0/S_0} \downarrow & & \downarrow F_0^{[e]}_{S_0/S_0} \\
S_0^{[q]} & \xrightarrow{c} & S_0^{[q]}
\end{array}
\]
We say that $(\widetilde{Y}, \widetilde{F})/(S \rightarrow S'^{[q]})$ is a log smooth integral lift (or simply a lift) of $(Y, F)/(S_0 \rightarrow S'^{[q]})$, if $Y$ is a log smooth integral scheme over $S$ such that $Y \times_S S_0 = Y$ and $\widetilde{F}$ is a morphism $\widetilde{Y} \rightarrow \widetilde{Y} := \widetilde{Y} \times_{S', F'^{[q]}} S$ over $S \rightarrow S'^{[q]}$. Let the notations be as above. Let $\text{Lift}_{(Y,F)/(S_0 \subset S, F'^{[q]})}(U)$ be the following sheaf defined by the following equality:

\[
\text{Lift}_{(Y,F)/(S_0 \subset S, F'^{[q]})}(U) := \{\text{isomorphism classes of lifts of } (U, F|_U)/(S_0 \rightarrow S'^{[q]}) \text{ over } (S \rightarrow S'^{[q]})\}
\]

for each log open subscheme $U$ of $Y$. Here $\iota := U \times_{S_0, F'^{[q]}} S_0, F|_U : U \rightarrow \iota U$ is the restriction of $F$ to $U$, and the isomorphism classes of lifts of $(U, F|_U)/(S_0 \rightarrow S'^{[q]})$ over $(S \rightarrow S'^{[q]})$ are defined in an obvious way.

Let the notations be as above. Let $\iota : \iota Y \rightarrow \iota Y$ be the closed immersion. Let $\widetilde{G} : \iota Y \rightarrow \iota Y$ be another lift of $F$. Take $s$ in (4.11) as the composite morphism $Y \xrightarrow{\iota F} \iota Y \xrightarrow{\iota} \iota Y$ over the composite morphism $S_0 \rightarrow S \rightarrow S'^{[q]}$. Then $\widetilde{G}$ defines an element $\widetilde{G} = \widetilde{F}$ of}

\[
(4.10.3) \quad \text{Hom}_{\mathcal{O}_Y}(F^* \iota^*(\Omega^1_{\iota Y/S'^{[q]}}, \mathcal{I}\mathcal{O}_{\iota Y}) = \text{Hom}_{\mathcal{O}_Y}(F^*(\Omega^1_{Y/S'^{[q]}}, \mathcal{I}\mathcal{O}_Y)
\]

\[
= \text{Hom}_{\mathcal{O}_Y}(F^*(\Omega^1_{Y/S'^{[q]}}, \mathcal{O}_Y) \otimes_{\mathcal{O}_{S_0}} \mathcal{I}) = \text{Hom}_{\mathcal{O}_Y}(\Omega^1_{\iota Y/S'^{[q]}}, F^*(\mathcal{O}_Y)) \otimes_{\mathcal{O}_{S_0}} \mathcal{I}.
\]

This is the log version of a generalization of [NoS] p. 208, iii). The following is the log version of [NoS] p. 208, iv). This is a key lemma for (4.11) below.

**Lemma 4.11.** Assume that there exists a lift $(\widetilde{Y}, \widetilde{F})/S$ of $(Y, F)/S_0$. Assume that $\mathcal{I} = \pi^n \mathcal{O}_S$ for a global section $\pi$ of $\mathcal{O}_S$ and that $q\pi^n = 0$ in $\mathcal{O}_S$ for a positive integer $n$. Assume also that $S_{00} = S$ mod $\pi$ and that the morphism $\mathcal{O}_{S_{00}} \ni 1 \mapsto \pi^n \in \mathcal{I}$ is a well-defined isomorphism. Let the notations be as in (4.4) (1) and denote $\delta$ in (4.4) (1) by $\pi^n \delta$ in this lemma; the new $\delta$ is an element of $\text{Hom}_{\mathcal{O}_{S_0}}(\Omega^1_{Y_{0}/S_{00}}, \mathcal{O}_{Y_0})$ since

\[
\text{Hom}_{\mathcal{O}_Y}(\Omega^1_{Y/S_0}, \mathcal{I}\mathcal{O}_Y) = \text{Hom}_{\mathcal{O}_Y}(\Omega^1_{Y/S_0}, \mathcal{O}_Y) \otimes_{\mathcal{O}_{S_0}} \mathcal{I} \leftarrow \text{Hom}_{\mathcal{O}_{Y_0}}(\Omega^1_{Y_{0}/S_{00}}, \mathcal{O}_{Y_0}).
\]

Denote by $g \in \text{Aut}_{S_{0}}(\widetilde{Y}, \iota Y)$ the induced automorphism of $\iota Y$ by an element $g \in \text{Aut}_{S} (\widetilde{Y}, Y)$. Let $\delta$ be the element of $\text{Hom}_{\mathcal{O}_{Y_0}}(\Omega^1_{Y_{0}/S_{00}}, \mathcal{O}_{Y_0})$ obtained by $\iota g$. Then the following hold:
(1) \( \delta(a) = \sum_i \delta(a_i) \otimes b_i \) for \( a = \sum_i a_i \otimes b_i \in \mathcal{O}_{Y_0} = \mathcal{O}_{Y_0} \otimes_{\mathcal{O}_{S_{00}}} F_{10}^* \mathcal{O}_{S_{00}} \) \( (a_i \in \mathcal{O}_{Y_0}, b_i \in \mathcal{O}_{S_{00}}) \).

(2) \( g^{-1} \tilde{F}^* g^* (a) - \tilde{F}^* (a) = \pi^n F_{0}^* (\delta(a)) \) for \( a = \sum_i a_i \otimes b_i \in \mathcal{O}_{\tilde{Y}} \). Here we denote the image of \( a \in \mathcal{O}_{\tilde{Y}} \) in \( \mathcal{O}_{Y_0} \) by \( 'a \) by abuse of notation.

(3) \( 1 + \pi^n \delta(m) = [1 + \pi^n \delta(m), 1] \) for \( m = [m, u] \in M_{Y_0} = M_{Y_0} \otimes_{\mathcal{O}_{S_{00}}} F_{10}^* \mathcal{O}_{S_{00}} \) \( (m \in M_{Y_0}, u \in \mathcal{O}_{S_{00}}) \).

(4) \( (g^{-1} \tilde{F}^* g^*)(a)(\sum_i a_i \otimes b_i)^{-1} = \pi^n F_{0}^* (\delta(m)) \) for \( m = [m, u] \in M_{Y_0} = M_{Y_0} \otimes_{\mathcal{O}_{S_{00}}} F_{10}^* \mathcal{O}_{S_{00}} \).

(5) Let \( (\tilde{Y}_1, \tilde{F}_1) \) and \( (\tilde{Y}_2, \tilde{F}_2) \) be two lifts of \( (Y, F)/S_0 \). Then there exists at most one element \( h \) of \( \text{Hom}_{S}(\tilde{Y}_1, \tilde{Y}_2) \) such that \( h|_{Y} = \text{id}_Y \) and \( h \circ \tilde{F}_1 = \tilde{F}_2 \circ h \). Here the \( 'h \) on the left hand side of this equality is the induced isomorphism \( \tilde{Y}_1 \overset{\sim}{\rightarrow} \tilde{Y}_2 \) by \( h \).

(6) Let \( F, (\tilde{Y}_1, \tilde{F}_1) \) and \( (\tilde{Y}_2, \tilde{F}_2) \) be as in (5). Let \( h \) be an element of \( \text{Hom}_{S}(\tilde{Y}_1, \tilde{Y}_2) \) such that \( h|_{Y} = \text{id}_Y \). Then the image of

\[ (\pi^n + 1)^i \tilde{F}_1 h^{-1} \tilde{F}_2^* \in \text{Hom}_{\mathcal{O}_{\tilde{Y}}}(\Omega_{\tilde{Y}/S_0}^1, F_*(T_{\mathcal{O}_{\tilde{Y}}})) = \text{Hom}_{\mathcal{O}_{\tilde{Y}}}(\Omega_{\tilde{Y}/S_0}^1, F_*(\mathcal{O}_{\tilde{Y}})) \otimes_{\mathcal{O}_{S_0}} \mathcal{I} \]

is \( \pi^n \delta(F_*(a)) = \pi^n \delta(\sum_i (a_i^q b_i + \pi c_i)) = 0 \) because \( \delta \) is a derivation over \( \mathcal{O}_{\tilde{Y}} \) and because \( qa^n = 0 = \pi^n + 1 \) in \( \mathcal{O}_{\tilde{Y}} \). Using these vanishings, we have the following equalities:

\[
g^{-1} \tilde{F}^* g^* (a) = g^{-1} \tilde{F}^* (a + \pi^n \delta(a)) = g^{-1} (\tilde{F}^* (a) + \pi^n \sum_i \delta(a_i) b_i)
\]

\[
= \tilde{F}^* (a) - \pi^n \delta(\tilde{F}^* (a)) + \pi^n \sum_i \delta(a_i) b_i = F^* (a) + \pi^n \sum_i a_i b_i = \tilde{F}^* (a) + \pi^n F_0^* (\delta(a)).
\]

(3): Because

\[
'm(1 + \pi^n \delta(m)) = g^* (m) = [g^* (m), u] = [m(1 + \pi^n \delta(m)), u] = 'm[1 + \pi^n \delta(m), 1],
\]

we obtain (3).

(4): Consider a local section \( 'm := [m, u] \in M_{\tilde{Y}} \) in (3). Express \( \tilde{F}^* (m) = \)

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completed the proof of (5).

(4.11.2) be the morphism corresponding to 

Because (2):

equality holds:

\[ \delta = \pi^{n} \delta(\pi(\delta(m))) + (1 + \pi^{n}(\delta(m))) \]

Here we have used the formula in (4.11.3) (3) for the sixth and the seventh equalities; we have also used (3) for the second and the last equalities.

(5): Let \( g_{i} : \tilde{Y}_{1} \sim \tilde{Y}_{2} \) be an isomorphism such that \( g_{i}|_{Y} = \text{id}_{Y} \) and \( \gamma_{1} \circ \tilde{F}_{1} = \tilde{F}_{2} \circ g_{1} \) on \( \tilde{Y}_{1} \). Set \( g := g_{1} \circ g_{2}^{-1} \in \text{Aut}_{S}(\tilde{Y}_{2}, Y) \). Let \( \delta \in \text{Hom}_{\mathcal{O}_{Y_{0}}}(\Omega_{Y_{0}/S}, \mathcal{O}_{Y_{0}}) \) be the morphism corresponding to \( g \). Then we obtain the following equalities by using (2):

\[
0 = (g_{1}F_{1}g_{1}^{-1})(\gamma) - (g_{1}F_{1}g_{1}^{-1})(\gamma) = (g_{1}F_{1}g_{1}^{-1}(\gamma)) - (g_{1}F_{1}g_{1}^{-1})(\gamma)
\]

Because \( \mathcal{O}_{Y_{0}} \sim \pi^{n} \mathcal{O}_{Y} \) (since \( \tilde{Y} \) is flat over \( \tilde{S} \)), \( F_{0}^{n}(\delta(\gamma)) = 0 \). Because \( \tilde{Y}_{0} \) is reduced, the morphism \( F_{0}^{n} : \mathcal{O}_{Y_{0}} \rightarrow F_{0}(\mathcal{O}_{Y_{0}}) \) is injective and hence \( \delta(\gamma) = 0 \).

Because \( \tilde{Y}_{0} \) is reduced, the pull-back \( \text{pr}_{0}^{*} : \mathcal{O}_{Y_{0}} \rightarrow \text{pr}_{*}(\mathcal{O}_{Y_{0}}) \) of the projection \( \text{pr}_{0} : Y_{0} \rightarrow Y_{0} \) is also injective. Hence

\[ \delta(a) = 0 \quad (a \in \mathcal{O}_{Y_{0}}) \]

On the other hand, we obtain the following equalities by using (4):

\[
1 = (g_{1}F_{1}g_{1}^{-1})(\gamma) + (g_{1}F_{1}g_{1}^{-1})(\gamma) = (g_{1}F_{1}g_{1}^{-1}(\gamma)) + (g_{1}F_{1}g_{1}^{-1})(\gamma)
\]

By the same argument as that in the previous paragraph, we see that the following equality holds:

\[ \delta(m) = 0 \quad (m \in M_{Y}) \]

By (4.11.2) and (4.11.1), we see that \( g = \text{id}_{Y} \) and consequently \( g_{1} = g_{2} \). We have completed the proof of (5).

(6) Let \( g_{j} : \tilde{Y}_{1} \sim \tilde{Y}_{2} \) be an isomorphism such that \( g_{i}|_{Y} = \text{id}_{Y} \). Set \( g := g_{1} \circ g_{2}^{-1} \in \text{Aut}_{S}(\tilde{Y}_{2}, Y) \). Then we obtain the following equalities as in (5):

\[
(g_{1}F_{1}g_{1}^{-1})(\gamma) - F_{2}^{n}(\gamma) = (g_{1}F_{1}g_{1}^{-1}g_{2}F_{2}g_{2}^{-1}g_{2}g_{1}^{-1})(\gamma) - F_{2}^{n}(\gamma)
\]

\[
= (g_{1}F_{1}g_{1}^{-1}g_{2}^{-1})(\gamma) - F_{2}^{n}(\gamma)
\]

\[
= (g_{2}F_{2}g_{2}^{-1})(\gamma) + \pi^{n}(g_{2}F_{2}g_{2}^{-1})(\delta(\gamma)) - F_{2}^{n}(\gamma)
\]

\[
= (g_{2}F_{2}g_{2}^{-1})(\gamma) - F_{2}^{n}(\gamma) + \pi^{n}F_{0}^{n}(\delta(\gamma))
\]
since \( g_2|_Y = \text{id}_Y \). Analogously we obtain the following equalities as in (5):

\[
\left( g_1 F_1 g_1^{-1}\right)^* (m)(\tilde{F}_2^*(m))^{-1} = \left( g_1 g_2^{-1}, g_2 F_1 g_2^{-1} g_2 g_1^{-1}\right)^* (m)(\tilde{F}_2^*(m))^{-1} = \left( g_2 F_1 g_2^{-1}\right)^* (m)(\tilde{F}_2^*(m))^{-1} = \left( g_2 F_1 g_2^{-1}\right)^* (m)(\tilde{F}_2^*(m))^{-1} \]

since \( g_2|_Y = \text{id}_Y \). Hence we see that the image of \( (h F_1 h^{-1})^* - \tilde{F}_2^* \) in the quotient of the map \( F_0^* : \text{Hom}_{\mathcal{O}_Y} (\Omega^1_{Y_0/S_{00}}, \mathcal{O}_Y) \to \text{Hom}_{\mathcal{O}_Y} (\Omega^1_{Y_0/S_{00}}, F_{0*}(\mathcal{O}_Y)) \) is independent of \( h \). This proves (6).

\[\square\]

The following (2) and (3) are the log versions of [NoS] Proposition 1 in p. 205; the following (4) is the log version of [St1] p. 104 (ii); the following (5) is an additional result. Roughly speaking, we follow the argument in [NoS] for the proof of (3). However to give the precise proof of it is very involved (cf. (4.14)). The most important result. Roughly speaking, we follow the argument in [NoS] for the proof of (3). Once one knows (2), (3) is a formal consequence of (2); however to prove (2), we use the argument in the proof of (3); the proof of (4.14) is logically more complicated than those of (4.11) and (4.12).

**Theorem 4.12.** Let \( \mathcal{I} \), \( \pi \) and \( n \) be as in (4.11). Then the following hold:

1. Assume that \((Y, F)/S_0\) has a lift \((\tilde{Y}, \tilde{F})/S\). Set \( \text{Aut}_{S,F_S^{[\mathcal{I}]}} (\tilde{Y}, Y) := \{ g \in \text{Aut}_S (\tilde{Y}) \mid g|_Y = \text{id}_Y, \tilde{F} \circ g = g \circ \tilde{F} \} \). Then \( \text{Aut}_{S,F_S^{[\mathcal{I}]}} (\tilde{Y}, Y) = \{ \text{id}_Y \} \).

2. Let \( \text{pr}_0 : Y_0 \to Y_0 \) be the projection. The sheaf \( \text{Lift}_{(Y, F)/((S_0 \subset S), F_S^{[\mathcal{I}]})} \) on \( \tilde{Y} \) is a torsor under \( \text{pr}_0 (\text{Hom}_{\mathcal{O}_Y} (\Omega^1_{Y_0/S_{00}}, F_{0*}(\mathcal{O}_Y))/\mathcal{O}_{Y_0}) \).

3. Assume that \( \tilde{Y} \) is \( \mathcal{O} \)-separated. In

\[
\text{Ext}^1_{Y_0} (\Omega^1_{Y_0/S_{00}}, F_{0*}(\mathcal{O}_Y))/\mathcal{O}_{Y_0},
\]

there exists a canonical obstruction class \( \text{obs}_{(Y, F)/((S_0 \subset S), F_S^{[\mathcal{I}]})} \) of a lift of \((Y, F)/((S_0 \to S), F_S^{[\mathcal{I}]})\) over \( S \to S^{[\mathcal{I}]} \) over \( S \to S^{[\mathcal{I}]} \).

4. Assume that \( \tilde{Y} \) is \( \mathcal{O} \)-separated. Let

\[
\partial : \text{Ext}^1_{Y_0} (\Omega^1_{Y_0/S_{00}^{[\mathcal{I}]}, F_{0*}(\mathcal{O}_Y))/\mathcal{O}_{Y_0}) \to \text{Ext}^2_{Y_0, S_{00}^{[\mathcal{I}]}} (\Omega^1_{Y_0/S_{00}^{[\mathcal{I}]}, \mathcal{O}_{Y_0})
\]

be the boundary morphism obtained by the following exact sequence (4.13):

\[
0 \to \mathcal{O}_{Y_0} \to F_{0*}(\mathcal{O}_Y) \to F_{0*}(\mathcal{O}_Y)/\mathcal{O}_{Y_0} \to 0.
\]

Then \( \partial (\text{obs}_{(Y, F)/((S_0 \subset S), F_S^{[\mathcal{I}]})}) = \text{obs}_{Y/((S_0^{[\mathcal{I}]}) \subset S^{[\mathcal{I}]})} \).

5. Assume that \( \tilde{Y} \) is \( \mathcal{O} \)-separated. Assume that there exists a lift \( \tilde{\mathcal{Y}}/S \) of \( Y/S_0 \). Let \( \tilde{Y} \) be the base change of \( \tilde{Y} \) by the morphism \( S^{[\mathcal{I}]} \to S \). Then, in

\[
\text{Ext}^1_{Y_0} (\Omega^1_{Y_0/S_{00}^{[\mathcal{I}]}, F_{0*}(\mathcal{O}_Y))/\mathcal{O}_{Y_0}),
\]

there exists a canonical obstruction class \( \text{obs}_{Y/S} (F) \) of a lift \( \tilde{F} : \tilde{Y} \to \tilde{Y} \) of \( F : Y \to Y \) and this is mapped to \( \text{obs}_{(Y, F)/((S_0 \subset S), F_S^{[\mathcal{I}]})} \) by the following natural morphism

\[
\partial : \text{Ext}^1_{Y_0} (\Omega^1_{Y_0/S_{00}^{[\mathcal{I}]}, F_{0*}(\mathcal{O}_Y))/\mathcal{O}_{Y_0}) \to \text{Ext}^1_{Y_0} (\Omega^1_{Y_0/S_{00}^{[\mathcal{I}]}, F_{0*}(\mathcal{O}_Y))/\mathcal{O}_{Y_0}.
\]

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Proof. If there exists a lift $(\tilde{Y}, \tilde{F})/(\tilde{S} \rightarrow S^{[e]})$ of $(Y, F)/(S_0 \rightarrow S_0^{[e]})$, identify $\text{Hom}_{\mathcal{O}_Y} (\Omega^1_{Y,S_0^{[e]}}, F_*(\mathcal{I} \mathcal{O}_Y))$ with $\text{Hom}_{\mathcal{O}_{Y_0}} (\Omega^1_{Y_0/S_0^{[e]}}, F_0_*(\mathcal{O}_{Y_0}))$.

(1) (1) immediately follows from (4.11) (5).

(2): Assume that there exists a lift $(\tilde{Y}, \tilde{F})/S$ of $(Y, F)$ over $S$. Let $(\tilde{Z}, \tilde{G})/S$ be another lift of $(Y, F)$ over $S$. Let $\tilde{U}_i := \{\tilde{U}_i\}_{i \in I}$ and (resp. $\tilde{V}_i := \{\tilde{V}_i\}_{i \in I}$) be an open covering of $\tilde{Y}$ (resp. an open covering of $\tilde{Z}$) such that there exists an isomorphism $g_i: \tilde{U}_i \rightarrow \tilde{V}_i$ such that $g_{i|U_i} = \text{id}_{U_i}$. Here $U_i := \tilde{U}_i|_Y$ and $\tilde{V}_i$ is an open log subscheme of $\tilde{Z}$ such that $\tilde{V}_i|_Y = U_i$. Set $\tilde{F}_i := \tilde{F}|_{\tilde{U}_i}: \tilde{U}_i \rightarrow \tilde{V}_i$ and $\tilde{G}_i := \tilde{G}|_{\tilde{V}_i}: \tilde{V}_i \rightarrow \tilde{V}_i$. Then we have a section $(\tilde{g}_i^{-1}\tilde{G}_i g_i)^* \in \text{Hom}_{\mathcal{O}_{Y_0}} (\Omega^1_{Y_0/S_0^{[e]}}, F_0_*(\mathcal{O}_{Y_0}))((\tilde{U}_i)_{\tilde{U}_i})$. If we change $g_i$, then this section may change. However, the image of this section in $\text{Hom}_{\mathcal{O}_{Y_0}} (\Omega^1_{Y_0/S_0^{[e]}}, F_0_*(\mathcal{O}_{Y_0}))((\tilde{U}_i)_{\tilde{U}_i})$ does not change by (4.11) (6) and it is a well-defined section. Consequently, we have an element of $\text{Hom}_{\mathcal{O}_{Y_0}} (\Omega^1_{Y_0/S_0^{[e]}}, F_0_*(\mathcal{O}_{Y_0})/\mathcal{O}_{Y_0})$.

Conversely, assume that we are given a global section of $\text{Hom}_{\mathcal{O}_{Y_0}} (\Omega^1_{Y_0/S_0^{[e]}}, F_0_*(\mathcal{O}_{Y_0})/\mathcal{O}_{Y_0})$. Take a local lift in $\text{Hom}_{\mathcal{O}_{Y_0}} (\Omega^1_{Y_0/S_0^{[e]}}, F_0_*(\mathcal{O}_{Y_0}))$ of this global section. There exists a lift $\tilde{U}_i/S$ of $U_i/S_0$ if $U_i$ is a small log affine open subscheme of $Y$. Set $\tilde{U}_i := \tilde{U}_i \times_S S^{[e]}$. Let $\tilde{F}_i: \tilde{U}_i \rightarrow \tilde{U}_i$ be a lift of $F_0|_{U_i}: U_i \rightarrow \tilde{U}_i$. (This lift exists.) By (4.1) the local section of $\text{Hom}_{\mathcal{O}_{Y_0}} (\Omega^1_{Y_0/S_0^{[e]}}, F_0_*(\mathcal{O}_{Y_0}))$ corresponds to a local lift $(\tilde{U}_i, \tilde{F}_i)$ of $(U_i, F_0)$. Since this is obtained by the global section of $\text{Hom}_{\mathcal{O}_{Y_0}} (\Omega^1_{Y_0/S_0^{[e]}}, F_0_*(\mathcal{O}_{Y_0})/\mathcal{O}_{Y_0})$, they patch together by the proof of (3): we have only to change $g_i$: $\tilde{U}_i \rightarrow \tilde{U}_j$ in the proof of (4.1) by $g_{ij}: \tilde{U}_i \rightarrow \tilde{U}_j$, where $g_{ij}$ is the isomorphism in the proof of (3) below.

(3): Let the notations be as in the proof of (4.1). On $\tilde{U}_i$ there exists a lift $\tilde{F}_i: \tilde{U}_i \rightarrow \tilde{U}_i$ of $F_0|_{U_i}: U_i \rightarrow \tilde{U}_i$. This morphism defines a morphism $\tilde{F}_i|_{\tilde{U}_{ji}}: \tilde{U}_{ji} \rightarrow \tilde{U}_{ji}$. Then $\tilde{F}_i|_{\tilde{U}_{ji}}$ and $'g_{ij}(\tilde{F}_i|_{\tilde{U}_{ji}})g_{ij}^{-1}$ are two lifts of $F_0|_{U_{ij}}: U_{ij} = U_i \rightarrow U_{ij}$ and $'U_{ij} = U_{ij}$. Hence we have an element \[
\tilde{\omega}_{ij} := (\tilde{F}_i|_{\tilde{U}_{ji}})^* - (g_{ij}(\tilde{F}_i|_{\tilde{U}_{ji}})g_{ij}^{-1})^* \in \text{Hom}_{\mathcal{O}_{U_{ij}}} (\Omega^1_{U_{ij}/S_0^{[e]}}, F_*(\mathcal{I} \mathcal{O}_{U_{ij}}))
\]
\[
\tilde{\omega}_{ij} \sim \text{Hom}_{\mathcal{O}_{U_{ij},0}} (\Omega^1_{U_{ij,0}/S_0^{[e]}}, F_0_*(\mathcal{O}_{U_{ij,0}})).
\]
Let $\omega_{ij}$ be the image of this element in $\text{Hom}_{\mathcal{O}_{U_{ij,0}}} (\Omega^1_{U_{ij,0}/S_0^{[e]}}, F_0_*(\mathcal{O}_{U_{ij,0}})/\mathcal{O}_{U_{ij,0}})$. Then, by (4.11) (6), $\omega_{ij}$ is independent of the choice of $g_{ij}$. We claim that the following equality holds:
\[
\omega_{ik} = \omega_{ij} + \omega_{jk}.
\]
Indeed, by (4.9), \[
\tilde{\omega}_{ij} + \tilde{\omega}_{jk} = g_{jk}^{-1}(\tilde{F}_i|_{\tilde{U}_{ik}})^* - (g_{ij}(\tilde{F}_i|_{\tilde{U}_{ik}})g_{ij}^{-1})^* + (g_{jk}(\tilde{F}_j|_{\tilde{U}_{ij}})g_{jk}^{-1})^* - (g_{jk}(\tilde{F}_j|_{\tilde{U}_{ij}})g_{jk}^{-1})^* = (\tilde{F}_i|_{\tilde{U}_{ik}})^* - g_{ik}^{-1}(g_{ij}(\tilde{F}_i|_{\tilde{U}_{ij}})g_{ij}^{-1})^* + g_{jk}^{-1}(g_{jk}(\tilde{F}_j|_{\tilde{U}_{ij}})g_{jk}^{-1})^* - (g_{jk}(\tilde{F}_j|_{\tilde{U}_{ij}})g_{jk}^{-1})^*.
\]
By (4.11) (6) again, the image of the last term in $\text{Hom}_{\mathcal{O}_{U_{ij,0}}} (\Omega^1_{U_{ij,0}/S_0^{[e]}}, F_0_*(\mathcal{O}_{U_{ij,0}})/\mathcal{O}_{U_{ij,0}})$ is equal to $\omega_{ik}$. Set $U_0 := \{U_{i,0}\}_{i \in I}$ and $'U_0 := \{U_{i,0}\}_{i \in I}$. As a result, we obtain the class $\{\omega_{ij}\}$ in $H^1(U_0, \text{Hom}_{\mathcal{O}_{Y_0}} (\Omega^1_{Y_0/S_0^{[e]}}, F_0_*(\mathcal{O}_{Y_0})/\mathcal{O}_{Y_0}))$.
We claim that the class \( \{ \omega_{ij} \} \), more strongly the class \( \{ \tilde{\omega}_{ij} \} \) is independent of the choice of the lift \( \tilde{F}_i \). Indeed, let us take another lift \( \tilde{F}'_i \). Then \( \{ \tilde{F}'_i - \tilde{F}_i \} \) defines an element of
\[
\prod_i \text{Hom}_{\mathcal{O}_{U_{i,0}}} (\Omega^1_{U_{i,0}/S_0'}, F_{0*}(\mathcal{O}_{U_{i,0}})).
\]
Hence the class of \( \{ \tilde{\omega}_{ij} \}_{ij} \) is independent of the choice of the lift \( \tilde{F}_i \) by the following equalities obtained by (4.12.8):
\[
\begin{align*}
(\tilde{F}'_i)_{ij}^* - (g_{ij}(\tilde{F}'_i)_{ij})^* - (\tilde{F}_i)_{ij}^* - (g_{ij}(\tilde{F}_i)_{ij})^* & = 0 \\
(\tilde{F}'_i)_{ij}^* - (\tilde{F}_i)_{ij}^* & = 0
\end{align*}
\]

(The calculation above is missing in [NoS].)

Next we claim that the class of \( \{ \omega_{ij} \}_{ij} \) is independent of the choice of the open covering \( U_0 \). This is clear since any two open coverings have a refinement and \( \tilde{F}_i|_V \) is a lift of \( F_i|_V \) for any open subscheme \( V \) of \( U_i \).

As in [NoS], we claim that the class \( \{ \omega_{ij} \} \) is the obstruction class of a lift of \( (Y,F)/((S_0 \rightarrow S_0') \) over \( S \rightarrow S_0' \). Indeed, if there exists a lift \( (\tilde{Y}, \tilde{F})/((S \rightarrow S_0') \) of \( (Y,F)/((S_0 \rightarrow S_0') \), then we can take \( g_{ij} \) (resp. \( \tilde{F}_i \)) as the identity \( \tilde{F}_i \) (resp. \( \tilde{F}_i \)) and hence \( \omega_{ij} = 0 \).

Conversely assume that \( \{ \omega_{ij} \} = 0 \) in \( H^1(U_0, \text{Hom}_{\mathcal{O}_{Y_0}} (\Omega^1_{Y_0/S_0'}, F_{0*}(\mathcal{O}_{Y_0})/\mathcal{O}_{Y_0})) \). Then there exists a section \( \omega_i \in \text{Hom}_{\mathcal{O}_{Y_0}} (\Omega^1_{Y_0/S_0'}, F_{0*}(\mathcal{O}_{Y_0})/\mathcal{O}_{Y_0})(U_{i,0}) \) such that \( \omega_{ij} = \omega_j - \omega_i \). Assume that the image of \( \tilde{U}_{i,0} \) in \( S_0 \) is contained in an affine open subscheme of \( S_0 \). Since \( \tilde{U}_{i,0} \) is affine, so is \( \tilde{U}_{i,0} \). Hence the following sequence
\[
(4.12.2)
\]
\[
0 \rightarrow \Gamma(U_{i,0}, \mathcal{O}_{Y_0}) \xrightarrow{F_{ij}^*} \Gamma(U_{i,0}, F_{0*}(\mathcal{O}_{Y_0})) \rightarrow \Gamma(U_{i,0}, F_{0*}(\mathcal{O}_{Y_0})/\mathcal{O}_{Y_0}) \rightarrow 0
\]
is exact. Because \( \hat{Y} \) is separated, \( \hat{U}_{ij,0} \) is affine and then we see that \( \hat{U}_{ij,0} \) is affine.

Because \( \hat{U}_{ij,0} \) is affine, the following sequence
\[
(4.12.3)
\]
\[
0 \rightarrow \Gamma(U_{ij,0}, \mathcal{O}_{Y_0}) \xrightarrow{F_{ij}^*} \Gamma(U_{ij,0}, F_{0*}(\mathcal{O}_{Y_0})) \rightarrow \Gamma(U_{ij,0}, F_{0*}(\mathcal{O}_{Y_0})/\mathcal{O}_{Y_0}) \rightarrow 0
\]
is exact. By using this exact sequence, we see that
\[
(4.12.4)
\]
\[
\tilde{\omega}_{ij} = \tilde{\omega}_j - \tilde{\omega}_i + F_{ij}^* (\eta_{ij})
\]
in \( \text{Hom}_{\mathcal{O}_{Y_0}} (\Omega^1_{Y_0/S_0'}, F_{0*}(\mathcal{O}_{Y_0}))(U_{i,0}) \) for a section \( \eta_{ij} \in \text{Hom}_{\mathcal{O}_{Y_0}} (\Omega^1_{Y_0/S_0'}, \mathcal{O}_{Y_0})(U_{ij,0}) \). Here \( \tilde{\omega}_i \in \text{Hom}_{\mathcal{O}_{Y_0}} (\Omega^1_{Y_0/S_0'}, F_{0*}(\mathcal{O}_{Y_0}))(U_{i,0}) \) is a lift of \( \omega_i \). Change \( \tilde{F}_i \) by \( \tilde{F}'_i \) such that \( \tilde{F}'_i * - \tilde{F}_i * = -\tilde{\omega}_i \) and change \( g_{ij} \) by \( g_{ij}' \) such that \( g_{ij}' * - g_{ij} * = \eta_{ij} \). In the following we denote \( \tilde{F}_{ij} \) by \( \tilde{F}_i \) for simplicity of notation. Then the equality (4.12.4) is equivalent to the following equality:
\[
\tilde{F}_j * - (g_{ij} \tilde{F}_i \eta_{ij}) = -\tilde{F}'_j * + \tilde{F}_j * + \tilde{F}_i * - \tilde{F}_i * + F_{ij}^* (g_{ij}' * - g_{ij} *).
\]
Hence

\[(4.12.5)\]
\[
\tilde{F}_j^* - (\langle g'_{ij} \rangle \tilde{F}_i (g_{ij}^{-1}))^* = (g_{ij} \tilde{F}_i (g_{ij}^{-1}))^* - (\langle g'_{ij} \rangle \tilde{F}_i (g_{ij}^{-1}))^* + \tilde{F}_i^* - \tilde{F}_i + F_0^*(\langle g'_{ij} - g_{ij} \rangle).
\]

We claim that the right hand side of (4.12.5) vanishes. To prove this vanishing, we have to make quite strange calculations (at least at first glance) as follows. (These calculations are missing in \[\text{NoSide}\].)

Let \(a\) be a local section of \(\mathcal{O}_{\tilde{\mathcal{U}}_{ij}}\). Let \(b \in \mathcal{O}_{\tilde{\mathcal{U}}_{ij}}\) be a lift of the image of \(a\) in \(\mathcal{O}_{U_{ij}} = \mathcal{O}_{\mathcal{U}_{ij}}\). By (4.13) (1) below, we have the following equalities:

\[
\langle g'_{ij} \rangle \tilde{F}_i (g_{ij}^{-1})^*(a) - \langle g'_{ij} \rangle \tilde{F}_i (g_{ij}^{-1})^*(a)
\]

\[
= (g_{ij}^{-1})^*(\tilde{F}_i^*(b) - \tilde{F}_i^*(b)) + \pi^n(F_0^*(\delta_{g_{ij}}(a, b)) - F_0^*(\delta_{g_{ij}}'(a, b)))
\]

Hence

\[
\langle g'_{ij} \rangle \tilde{F}_i (g_{ij}^{-1})^*(a) - \langle g'_{ij} \rangle \tilde{F}_i (g_{ij}^{-1})^*(a) + \tilde{F}_i^*(b) - \tilde{F}_i^*(b) = \pi^n(F_0^*(\delta_{g_{ij}}(a, b)) - F_0^*(\delta_{g_{ij}}'(a, b))).
\]

The last term is equal to \(F_0^*(\delta_{g_{ij}}(a, b) - \delta_{g_{ij}}'(a, b))\) via the identification \(\pi^n\mathcal{O}_{\tilde{\mathcal{U}}_{ij}} \simeq \mathcal{O}_{Y_{0}}\). This is equal to \(F_0^*(\langle g'_{ij} - g_{ij} \rangle(a))\) by (4.13) (2) below. Consequently the value of the right hand side of (4.12.5) for any \(a \in \mathcal{O}_{\tilde{\mathcal{U}}_{ij}}\) and any lift \(b \in \mathcal{O}_{\tilde{\mathcal{U}}_{ij}}\) of the image of \(a\) in \(\mathcal{O}_{U_{ij}}\) is equal to 0. Similarly, by using (4.13) (3) and (4) below, the value of the right hand side of (4.12.5) for any \(m \in M_{\tilde{\mathcal{U}}_{ij}}\) and any lift \(l \in M_{\tilde{\mathcal{U}}_{ij}}\) of the image of \(m\) in \(M_{U_{ij}}\) is equal to 0. In conclusion, the right hand side of (4.12.5) is 0.

Consequently

\[(4.12.6)\]
\[
\tilde{F}_j = (\langle g'_{ij} \rangle \tilde{F}_i (g_{ij}^{-1}))^*.
\]

By (4.11) (5), \(g_{ik}^* = (g'_{jk} g_{ij})^*\). Obviously this implies that \(g_{ik} = g'_{jk} g_{ij}\). In this way, we see that \(\tilde{U}_i\) and \(\tilde{F}_i\) patch together.

(4): Because \(\omega_{ik} = \omega_{ij} + \omega_{jk}\) in \(\text{Hom}_{\mathcal{O}_{\mathcal{U}_{ijk}}}((\Omega^j_{\mathcal{U}_{ijk} / S_{00}^j} F_0(\mathcal{O}_{\mathcal{U}_{ijk}} / \mathcal{O}_{\mathcal{U}_{ij}}), \omega_{ijk})\) \(\mathcal{O}_{\mathcal{U}_{ijk}}\), there exists an element \(\omega_{ijk} \in \text{Hom}_{\mathcal{O}_{\mathcal{U}_{ijk}}}((\Omega^j_{\mathcal{U}_{ijk} / S_{00}^j} \mathcal{O}_{\mathcal{U}_{ijk}}), \omega_{ijk})\) such that \(\tilde{\omega}_{ij} + \tilde{\omega}_{ik} = F_0^*(\omega_{ijk})\). By the definition of \(\partial\), \(\{\omega_{ijk}\} \in H^2(\mathcal{U}_0, \text{Hom}_{\mathcal{O}_{Y_{0}}}((\Omega^j_{Y_{0} / S_{00}^j} \mathcal{O}_{Y_{0}})))\) is the element \(\partial(\langle (Y, \mathcal{F}) / (S_{00} \subset S_{00}^j) \rangle)\). On the other hand, by the definition of \(\tilde{\omega}_{ij}\),

\[
\tilde{F}_j^* - (g_{ij}^{-1})^*(\tilde{F}_i^*)^* g_{ij}^* = \tilde{F}_j^* - (g_{ij} (\tilde{F}_i) g_{ij}^{-1})^* = \tilde{\omega}_{ij}.
\]

By this equality, we also have the following equality by (4.8):

\[
g_{ij} \tilde{F}_j^* |_{\tilde{\mathcal{U}}_{ij}} (g_{ij}^{-1})^* - \tilde{F}_i^* |_{\tilde{\mathcal{U}}_{ij}} = \tilde{\omega}_{ij}.
\]

Hence

\[
((g_{ik}^{-1} g_{jk} g_{ij})^{-1} \tilde{F}_i (g_{ik}^{-1} g_{jk} g_{ij}))^* - \tilde{F}_i^* = g_{ij}^* g_{ik}^* (\tilde{F}_i^* g_{ij}^* (g_{ik}^{-1})^* (g_{ij}^{-1})^* - \tilde{F}_i^*) = g_{ij}^* g_{ik}^* (\tilde{F}_i^* g_{ik}^* (\tilde{F}_i^* g_{ij}^* (g_{ik}^{-1})^* (g_{ij}^{-1})^* - \tilde{F}_i^*) = g_{ij}^* g_{ik}^* (\tilde{F}_i^* (g_{ik}^{-1})^* - \tilde{\omega}_{ik} (g_{ij}^{-1})^* - \tilde{F}_i^*) = -\tilde{\omega}_{ik} + \tilde{\omega}_{jk} + \tilde{\omega}_{ij} = F_0^*(\omega_{ijk}).
\]
Let $\delta_{ijk}$ be the $\delta$ for $g_{ijk} = g_{ik}^{-1} g_{jk} g_{ij}$. Then

$$
((g_{ik}^{-1} g_{jk} g_{ij})^{-1} \tilde{F}_i (g_{ik}^{-1} g_{jk} g_{ij}))^* = (g_{ijk} \tilde{F}_k g_{ijk}^{-1})^* = F_0^* (\delta_{ijk})
$$

by (4.11) (2) and (4). Hence $F_0^*(\delta_{ijk}) = F_0^*(\omega_{ijk})$. Since $F_0$ is injective, $\delta_{ijk} = \omega_{ijk}$. This implies the desired equality $\text{obs}_{Y/(S_0^{[i]} \subset S_0)} = \partial(\text{obs}_{(Y,F)/(S_0^{[i]} \\ S_0)})$.

(5): (5) follows from (4.11) and the argument in the proof of (4). Indeed, we have only to set $g_{ij} = \text{id}_{\tilde{U}_{ij}}$, where $\tilde{U}_{ij}$ is an open log subscheme of $\tilde{Y}$ corresponding to $U_{ij}$ in $Y$. □

We have to give analogues of (4.11) (2) and (4) for the proof of (4.12) (3). This is a non-trivial lemma:

**Lemma 4.13.** Let the notations be before (4.11). Let $\tilde{Y}_i$ ($i = 1,2$) be a lift of $Y$ over $S$. Assume that there exists an isomorphism $g: \tilde{Y}_1 \sim_{\tilde{Y}_2}$ over $S$ such that $g|_Y = \text{id}_Y$. Let 'g': $\tilde{Y}_1 \sim \tilde{Y}_2$ be the induced isomorphism by $g$. Let $(\tilde{Y}_1, \tilde{F})$ be a lift of $(Y,F)$ over $S$. Then the following hold:

1. For a local section $a$ of $O_{\tilde{Y}_i}$ let $\overline{a}$ be the image of $a$ in $O_Y$ and let $b$ be a lift of $\overline{a}$ in $O_{\tilde{Y}_i}$. Let $\delta_g(a,b)$ be a unique local section of $O_{\tilde{Y}_0}$ such that $\pi^n \delta_g(a,b) = 'g*(a) - b$. Here we have considered $\pi^n \delta_g(a,b)$ as a local section of $O_{\tilde{Y}_1}$. Then

$$
(g^{-1})^* \tilde{F}^*(g)^*(a) = (g^{-1})^* \tilde{F}^*(b) + \pi^n F_0^*(\delta_g(a,b)).
$$

2. Let the notations be as in (1). Let $h$ be another isomorphism $h: \tilde{Y}_1 \sim \tilde{Y}_2$ over $S$ such that $h|_Y = \text{id}_Y$. Then $\pi^n(h_*(a) - \delta_g(a,b)) = 'h*(a) - g*(a)$. In particular this is independent of the choice of $b$.

3. For a local section $m$ of $M_{\tilde{Y}_i}$, let $\overline{m}$ be the image of $m$ in $M_Y$ and let $l$ be a lift of $\overline{m}$ in $M_{\tilde{Y}_i}$. Let $\delta_g(m,l)$ be a unique local section of $O_{\tilde{Y}_0}$ such that $'g*(m) = l(1 + \pi^n \delta_g(m,l))$. Then

$$
(g^{-1})^* \tilde{F}^*(g)^*(m) = (g^{-1})^* (\tilde{F}^*(l))(1 + \pi^n F_0^*(\delta_g(m,l))).
$$

4. Let the notations be as in (2) and (4). Then $\delta_h(m,l) - \delta_g(m,l) = 'h*(m)('g*(m))^{-1}$. In particular this is independent of the choice of $l$.

**Proof.** (1): We have the following equalities since $g|_Y = \text{id}_Y$:

$$
(g^{-1})^* \tilde{F}^*(g)^*(a) = (g^{-1})^* \tilde{F}^*(b + \pi^n \delta_g(a,b))
= (g^{-1})^* (\tilde{F}^*(b) + \tilde{F}^*(\pi^n \delta_g(a,b)))
= (g^{-1})^* \tilde{F}^*(b) + \pi^n F_0^*(\delta_g(a,b)).
$$

(2): Obvious.

(3): We have the following equalities since $g|_Y = \text{id}_Y$:

$$
(g^{-1})^* \tilde{F}^*(g)^*(m) = (g^{-1})^* \tilde{F}^*(l(1 + \pi^n \delta_g(m,l)))
= (g^{-1})^* (\tilde{F}^*(l)(1 + \tilde{F}^*(\pi^n \delta_g(m,l))))
= (g^{-1})^* (\tilde{F}^*(l))(1 + \pi^n F_0^*(\delta_g(m,l))).
$$

(4): Obvious. □
Remark 4.14. The lemma (4.13) in the trivial logarithmic case is missing in [NoS]. The strange calculation to prove the equality (4.12.6) in the trivial logarithmic case is also missing in [loc. cit.]. These are indispensable in [loc. cit.] for the proof of the equality (4.12.6); the complicatedness for the proof arises because we have to make calculations for local sections in $F_{0*}(\mathcal{O}_{Y'})$ not in $F_{0*}(\mathcal{O}_{Y})/\mathcal{O}_{Y_0}$.

Corollary 4.15. Assume that $e = 1$. Assume also that $Y_0/S_{00}$ is of Cartier type. Then the following hold:

1. The sheaf $\text{Lift}_{(Y,F)/(S_0 \subset S,F_0)}$ on $\tilde{Y}$ is a torsor under $\text{pr}_{0*}(\text{Hom}_{\mathcal{O}_{Y_0}}(\Omega^1_{Y_0/S_{00}}; F_{0*}(B\Omega^1_{Y_0/S_{00}})))$.

2. Assume that $\tilde{Y}$ is separated. In

$$\text{Ext}^1_{Y_0}(\Omega^1_{Y_0/S_{00}}; F_{0*}(B\Omega^1_{Y_0/S_{00}})), $$

there exists a canonical obstruction class $\text{obs}_{(Y,F)/(S_0 \subset S,F_0)}$ of a lift of $(Y_0,F)/(S_0 \to S_{00})$ over $S \to S_{00}$.

3. Assume that $\tilde{Y}$ is separated. Let

$$\partial: \text{Ext}^1_{Y_0}(\Omega^1_{Y_0/S_{00}}; F_{0*}(B\Omega^1_{Y_0/S_{00}})) \to \text{Ext}^2_{Y_0}(\Omega^1_{Y_0/S_{00}}; \mathcal{O}_{Y_0})$$

be the boundary morphism obtained by the following exact sequence (6.3):

$$0 \to \mathcal{O}_{Y_0} \to F_{0*}(\mathcal{O}_{Y_0}) \xrightarrow{\partial} F_{0*}(B\Omega^1_{Y_0/S_{00}}) \to 0.$$ 

Then $\partial(\text{obs}_{(Y,F)/(S_0 \subset S,F_0)}) = \text{obs}_{(Y_0,F)/(S_0 \subset S,F_0)}$.

Proof. Recall that $Y'_0 := Y_0 \times_{S_{00},F_{0*}} S_{00}$ and that $\tilde{Y}'_0 = \tilde{Y}_0$. By (3.3.1)

$$F_{0*}(\mathcal{O}_{Y}/S_{00}) = F_{0*}(B\Omega^1_{Y_0/S_{00}}).$$

Now (4.15) follows from (4.12) and (4.15.2). \qed

Next we develop a log deformation theory with (standard) relative Frobenius morphisms. Because the proof of the main result (4.16) below are very similar to that of (4.12), we omit it. The log deformation theory with abrelative Frobenius morphisms and the theory with relative Frobenius morphisms turns out equivalent theories by (4.17) below. To obtain this equivalence, we can also use W. Zheng’s proof in [SS]. See (4.19) below for this.

Let the notations be as before. Set $Y' := Y \times_{S_0,F_{0*}} S_0$ and $Y'_0 := Y \times_{S_{00},F_{0*}} S_{00}$.

Let $F_0: Y_0 \to Y'_0$ be the $e$-iterated relative Frobenius morphism of $Y_0/S_{00}$. Assume that there exists a lift $F: Y \to Y'$ of $F_0: Y_0 \to Y'_0$ over $S$. We say that $(\tilde{Y}, \tilde{F})/S$ is a log smooth integral lift (or simply a lift) of $(Y,F)/S_0$ if $\tilde{Y}$ is a log smooth integral scheme over $S$ such that $\tilde{Y}_0/S_0 = Y$ and $\tilde{F}$ is a morphism $\tilde{Y} \to \tilde{Y}' := \tilde{Y} \times_{S,F_{0*}} S$ over $S$ fitting into the following commutative diagram

$$
\begin{array}{ccc}
Y & \xrightarrow{c} & \tilde{Y} \\
F \downarrow & & \downarrow \tilde{F} \\
Y' & \xrightarrow{c} & \tilde{Y}'
\end{array}
$$

over the morphism $S_0 \xrightarrow{c} S$. 

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Let \( \text{Lift}_{(Y,F)/(S_0 \subset S, F_S^{[1]})} \) be the following sheaf

\[
\text{Lift}_{(Y,F)/(S_0 \subset S)}(U) := \{ \text{isomorphism classes of lifts of } (U, F|_U)/S_0 \text{ over } S \}
\]

for each log open subscheme \( U \) of \( Y \). The isomorphism class of a lift of \((U, F|_U)/S_0 \) over \( S \) is defined in an obvious way. Let \( \iota' : Y' \xrightarrow{\sim} \tilde{Y}' \) be the closed immersion. Let \( \tilde{G} : \tilde{Y} \rightarrow \tilde{Y}' \) be another lift of \( F \). Take \( s \) in (4.1) as the composite morphism \( Y \xrightarrow{F} Y' \xrightarrow{\iota} \tilde{Y}' \) over the composite morphism \( S_0 \subset S \). Then \( \tilde{G} \) defines an element \( \tilde{G}^* - F^* \) of

(4.15.3)
\[
\text{Hom}_{\mathcal{O}_Y}(F^* \iota'^* (\Omega^1_{Y'/S}), \mathcal{I} \mathcal{O}_Y) = \text{Hom}_{\mathcal{O}_Y}(F^* (\Omega^1_{Y'/S_0}), \mathcal{I} \mathcal{O}_Y) = \text{Hom}_{\mathcal{O}_Y}(\Omega^1_{Y'/S_0}, F_*(\mathcal{I} \mathcal{O}_Y)).
\]

**Theorem 4.16.** Let \( \mathcal{I}, \pi \) and \( n \) be as in (4.1). Assume that \( Y_0 \) is reduced. Then the following hold:

1. Assume that \((Y, F)/S_0 \) has a lift \((\tilde{Y}, \tilde{F})/S \). Set \( \text{Aut}_{S,F_S^{[1]}}(\tilde{Y}, Y) := \{ g \in \text{Aut}_S(\tilde{Y}) \mid g|_Y = \text{id}_Y, \tilde{F} \circ g = g' \circ \tilde{F} \} \). Then \( \text{Aut}_{S,F_S^{[1]}}(\tilde{Y}, Y) = \{ \text{id}_Y \} \).

2. Let \( \text{pr}_0 : Y' \rightarrow Y_0 \) be the projection. Then the sheaf \( \text{Lift}_{(Y,F)/(S_0 \subset S, F_S^{[1]})}^{\circ} \) on \( \tilde{Y} \) is a torsor under \( \text{pr}_0_* (\text{Hom}_{\mathcal{O}_{Y_0}}(\Omega^1_{Y'/S_0}, F_*(\mathcal{O}_{Y_0})/\mathcal{O}_{Y_0}^\circ)) \).

3. Assume that \( \tilde{Y} \) is separated. In

\[
\text{Ext}^1_{\mathcal{O}_Y}(\Omega^1_{Y'/S_0}, F_*(\mathcal{O}_{Y_0})/\mathcal{O}_{Y_0}^\circ),
\]

there exists a canonical obstruction class \( \text{obs}_{(Y,F)/(S_0 \subset S, F_S^{[1]})} \) of a lift of \((Y, F)/S_0 \) over \( S \).

4. Assume that \( \tilde{Y} \) is separated. Let

\[
\partial : \text{Ext}^1_{\mathcal{O}_Y}(\Omega^1_{Y'/S_0}, F_*(\mathcal{O}_{Y_0})/\mathcal{O}_{Y_0}^\circ) \rightarrow \text{Ext}^2_{\mathcal{O}_Y}(\Omega^1_{Y'/S_0}, \mathcal{O}_{Y_0}^\circ)
\]

be the boundary morphism obtained by the following exact sequence (4.16.1):

(4.16.1)
\[
0 \rightarrow \mathcal{O}_{Y_0} \rightarrow F_*(\mathcal{O}_{Y_0}) \rightarrow F_*(\mathcal{O}_{Y_0})/\mathcal{O}_{Y_0}^\circ \rightarrow 0.
\]

Then \( \partial (\text{obs}_{(Y,F)/(S_0 \subset S, F_S^{[1]})}) = \text{obs}_{(Y,F)/S} \).

5. Assume that \( \tilde{Y} \) is separated. Assume that there exists a lift \( \tilde{Y}/S \) of \( Y/S_0 \). Let \( \tilde{Y}' \) be the base change of \( \tilde{Y} \) by the morphism \( F_S^{[1]} : S \rightarrow S \). Then the obstruction class \( \text{obs}_{(Y,F)} \) of a lift \( \tilde{F} : \tilde{Y} \rightarrow \tilde{Y}' \) of \( F : Y \rightarrow Y' \) is an element of \( \text{Ext}^1_{\mathcal{O}_Y}(\Omega^1_{Y'/S_0}, F_*(\mathcal{O}_{Y_0})) \) and this is mapped to \( \text{obs}_{(Y,F)/(S_0 \subset S)} \) by the natural morphism

\[
\partial : \text{Ext}^1_{\mathcal{O}_Y}(\Omega^1_{Y'/S_0}, F_*(\mathcal{O}_{Y_0})) \rightarrow \text{Ext}^1_{\mathcal{O}_Y}(\Omega^1_{Y'/S_0}, F_*(\mathcal{O}_{Y_0})/\mathcal{O}_{Y_0}^\circ).
\]

Proof. We omit the proof because it is the same as that of 4.12.

**Corollary 4.17.** Let \( \beta_0 : Y_0' \rightarrow Y_0 \) and \( \beta : Y' \rightarrow Y \) be the natural morphisms. Set \( F := \beta \circ F \). Then the following hold:

1. Let

(4.17.1)
\[
\text{Lift}_{(Y,F)/(S_0 \subset S, F_S^{[1]})} \rightarrow \text{Lift}_{(Y,F)/(S_0 \subset S, F_S^{[1]})}
\]
be the natural morphism of sheaves in $Y_{zar}$ obtained by the base changes of the lifts of the open log subschemes of $Y$ by the morphism $F_{[e]} : S \to S^{[q]}$. Assume that $\text{Lift}_{(Y,F)/(S_0 \subset S,F_0^{[e]})}(Y)$ is not empty. Then the following diagram is commutative:

$$\text{Lift}_{(Y,F)/(S_0 \subset S,F_0^{[e]})}(Y) \xrightarrow{\sim} \text{Hom}_{\mathcal{O}_Y}(\Omega^1_{Y_0/S_0}, F_{0*}(\mathcal{O}_Y)/\mathcal{O}_{Y_0}) \xrightarrow{\sim} \beta_0^e$$

(4.17.2)

$$\text{Lift}_{(Y,F)/(S_0 \subset S,F_0^{[e]})}(Y) \xrightarrow{\sim} \text{Hom}_{\mathcal{O}_Y}(\Omega^1_{Y_0/S_0^{[p]}}, F_{0*}(\mathcal{O}_Y)/\mathcal{O}_{Y_0})$$

(2) Assume that $\hat{Y}$ is separated. Let

$$\beta_0^e : \text{Ext}^1_{Y_0}(\Omega^1_{Y_0/S_0^{[p]}}, F_{0*}(\mathcal{O}_Y)/\mathcal{O}_{Y_0}) \xrightarrow{\sim} \text{Ext}^1_{Y_0}(\Omega^1_{Y_0/S_0^{[p]}}, F_{0*}(\mathcal{O}_Y)/\mathcal{O}_{Y_0})$$

be the natural isomorphism. Then

$$\beta_0^e(\text{obs}_{(Y,F)/(S_0 \subset S,F_0^{[e]})}) = \text{obs}_{(Y,F)/(S_0 \subset S,F_0^{[e]})}.$$  

(3) The following diagram is commutative for $q \in \mathbb{Z}$:

$$\begin{array}{ccc}
\text{Ext}^q_{Y_0}(\Omega^1_{Y_0/S_0}, F_{0*}(\mathcal{O}_Y)/\mathcal{O}_{Y_0}) & \xrightarrow{\partial} & \text{Ext}^q_{Y_0}(\Omega^1_{Y_0/S_0}, \mathcal{O}_{Y_0}) \\
\beta_0^e & \circlearrowleft & \beta_0^e
\end{array}$$

(4.17.3)

$$\begin{array}{ccc}
\text{Ext}^q_{Y_0}(\Omega^1_{Y_0/S_0^{[p]}}, F_{0*}(\mathcal{O}_Y)/\mathcal{O}_{Y_0}) & \xrightarrow{\partial} & \text{Ext}^q_{Y_0}(\Omega^1_{Y_0/S_0^{[p]}}, \mathcal{O}_{Y_0}) \\
\beta_0^e & \circlearrowleft & \beta_0^e
\end{array}$$

Proof. (1), (2), (3): If $f : V \to W$ is a morphism of fine log schemes over $S^{[q]}$, then we obtain the base change $V \times_{S^{[q]}f_{[e]}} S \to W \times_{S,F_{[e]}} S$ of $f$ over $S$. This base change defines the left vertical morphism in (4.17.2). Recall that $Y'_0 := Y_0 \times_{S_0,F_0^{[e]}} S_0$. Because $Y_0/S_0$ is integral, $\hat{Y}' = \hat{Y}_0$. By the isomorphism before [Kk1 (1.8)], we also obtain the equality $\Omega^1_{Y_0/S_0^{[p]}} = \Omega^1_{Y_0/S_0^{[p]}}$; $\beta_0^e$ is nothing but the identity. Hence we obtain (1), (2) and (3).

\textbf{Corollary 4.18.} Assume that $e = 1$. Assume that $Y_0/S_0$ is of Cartier type. Then the following hold:

(1) Let $\text{pr}_0 : Y'_0 \to Y_0$ be the projection. Then the sheaf $\text{Lift}_{(Y,F)/(S_0 \subset S,F_0^{[e]})}$ on $\hat{Y}$ is a torsor under $\text{pr}_0(\text{Hom}_{\mathcal{O}_Y}(\Omega^1_{Y_0/S_0}, F_{0*}(B\Omega^1_{Y_0/S_0})))$.

(2) Assume that $\hat{Y}$ is separated. In

$$\text{Ext}^1_{Y_0}(\Omega^1_{Y_0/S_0}, F_{0*}(B\Omega^1_{Y_0/S_0})),$$

there exists a canonical obstruction class $\text{obs}_{(Y,F)/(S_0 \subset S,F_0^{[e]})}$ of a lift of $(Y,F)/S_0$ over $S$.

(3) Assume that $\hat{Y}$ is separated. Let

$$\partial : \text{Ext}^1_{Y_0}(\Omega^1_{Y_0/S_0}, F_{0*}(B\Omega^1_{Y_0/S_0})) \to \text{Ext}^2_{Y_0}(\Omega^1_{Y_0/S_0}, \mathcal{O}_{Y_0})$$

be the boundary morphism obtained by the following exact sequence (4.18.1):

$$0 \longrightarrow \mathcal{O}_{Y_0} \longrightarrow F_{0*}(\mathcal{O}_Y) \xrightarrow{F_{0*}(\partial)} F_{0*}(B\Omega^1_{Y_0/S_0}) \longrightarrow 0.$$  

Then $\partial(\text{obs}_{(Y,F)/(S_0 \subset S,F_0^{[e]})}) = \text{obs}_{Y'/S_0}$.  

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Proof. By (4.3.1)

\[(4.18.2) \quad F_0^\bullet(\cO_{Y_0})/\cO_{Y_0} = F_0^\bullet(B\Omega^1_{Y_0/S_{00}}).\]

Now (4.18) immediately follows from (4.17) and (4.18.2), \qed

**Remark 4.19.** Let the notations be as in (4.17). More directly, we also obtain the following isomorphism of sheaves on \(\bar{Y}\) by W. Zheng's proof in [SS] (2.5):

\[(4.19.1) \quad \text{Lift}_{(\bar{Y}, F)/(\bar{S}_0 \subset \bar{S}, F^e_{\bar{S}})} \cong \text{Lift}_{(Y, F)/(S_0 \subset S, F^e_S)};\]

Indeed, we have only to construct the inverse of the natural morphism (4.17.1). Assume that we are given a representable of an element \((\bar{U}, \bar{\cO}_U, \bar{\cO}_U)\) of \(\text{Lift}_{(Y, F)/(S_0 \subset S, F^e_S)}\).

Then, following [loc. cit.], consider the sum \(\bar{U}' \coprod_U \bar{U}\) of schemes, where \(\bar{U}' \to \bar{U}'\) is the natural exact closed immersion and the morphism \(\bar{U}' \to \bar{U}\) is the identity.

Hence this sum of the schemes is isomorphic to \(\bar{U}\). Endow this scheme with the log structure

\[M_{\bar{U}'} \times_{M_U} M_U = M_{\bar{U}} \times_{M_{\bar{U}' \times M_U}} M_{\bar{U}'} \times_{M_U} M_U\]

with natural composite structural morphism \(M_{\bar{U}'} \times_{M_U} M_U \subset M_{\bar{U}'} \to M_{\bar{U}'} \to \cO_{\bar{U}'}\). Let \(\bar{U}\) be the resulting log scheme. Then \(\bar{F}: \bar{U} \to U'\) induces a morphism \(\bar{F}: \bar{U} \to \bar{U}'\). The triple \((\bar{U}, \bar{U}', \bar{F})\) is the desired object of \(\text{Lift}_{(\bar{Y}, F)/(\bar{S}_0 \subset \bar{S}, F^e_{\bar{S}})}\) since \(M_{\bar{U}'} \times_{M_U} M_{\bar{U}'} \to M_{\bar{U}' \times M_U} \to M_U\) is flat over \(S_0\).

Assume that \(S_0\) is of characteristic \(p > 0\) and that \(Y/S_0\) is of Cartier type. Let \(\text{Sec}_{C}\) be the following sheaf

\[\text{Sec}_{C}(U) := \{f(U', O_{U'})\text{-linear sections of } C: F_* (Z\Omega^1_{U'/S_0}) \to \Omega^1_{U'/S_0}\}\]

each log open subscheme \(U\) of \(Y_0\). Here recall the following exact sequence

\[0 \to F_* (B\Omega^1_{U'/S_0}) \to F_* (Z\Omega^1_{U'/S_0}) \xrightarrow{C} \Omega^1_{U'/S_0} \to 0.\]

The following is the log version of a generalization of [YI] (2.2.1) (cf. [DI] Theorem 3.5).

**Theorem 4.20.** Let the assumptions be as in (4.16). Assume that \(e = 1\) and \(n = 1\) and that \(Y = Y_0\) and \(S_0 = S_{00}\). Assume also that \(\pi = p\) and that \(S\) is flat over \(\text{Spec}(\mathbb{Z}/p^2)\). Then there exists the following canonical isomorphism of sheaves on \(\bar{Y}\):

\[(4.20.1) \quad \text{Lift}_{(\bar{Y}, F)/(\bar{S}_0 \subset \bar{S}, F_{\bar{S}})} \cong \text{Sec}_{C}.\]

**Proof.** Let \((\bar{U}, \bar{F})\) be a representative of an element of \(\text{Lift}_{(\bar{Y}, F)/(\bar{S}_0 \subset \bar{S}, F_{\bar{S}})}(U)\). We have to construct a morphism \(\text{Lift}_{(\bar{Y}, F)/(\bar{S}_0 \subset \bar{S}, F_{\bar{S}})}(U) \to \text{Sec}_{C}(U)\) of sets. Let \(t: F \to G\) be a morphism of abelian sheaves on \(\bar{U}\). If \(G\) is a sheaf of flat \(\mathbb{Z}/p^2\)-modules in \(\bar{U}_{\text{zar}}\) and if \(\text{Im}(t) \subset pG\), then we can define a unique morphism \(p^{-1}t: F/p \to G/p\) fitting into the following commutative diagram:

\[
\begin{array}{ccc}
\mathcal{F} & \xrightarrow{t} & \mathcal{G} \\
\downarrow \text{proj} & & \downarrow \text{proj} \\
\mathcal{F}/p & \xrightarrow{p^{-1}t} & \mathcal{G}/p.
\end{array}
\]
Since $\tilde{F}$ is a lift of the relative Frobenius morphism of $X \to X'$ over $S$, the image of the pull-back morphism $F^*: \Omega^1_{U'/S} \to \tilde{F}_*(\Omega^1_{U'/S})$ is contained in $p\tilde{F}_*(\Omega^1_{U'/S})$. Similarly, because of the expression $F^*(m') = \prod_{i} m_i^n(i(1 + p\eta(m'))$ for $m' = \prod_{i} [m_i, n_i] \in M_{U'}$, we see that the image of the morphism $d\log F^*: M_{U'} \to \tilde{F}_*(\Omega^1_{U'/S})$ is contained in $p\tilde{F}_*(\Omega^1_{U'/S})$. The morphism $\tilde{F}^*: \Omega^1_{U'/S} \to \tilde{F}_*(\Omega^1_{U'/S})$ induces the following morphism: $\tilde{F}^*: \Omega^1_{U'/S} \to p\tilde{F}_*(\Omega^1_{U'/S})$. This morphism induces the following morphism

\[(4.20.2) \quad \tilde{F}_{\Omega'} := p^{-1}\tilde{F}^*: \Omega^1_{U'/S} \to \tilde{F}_*(\Omega^1_{U'/S}).\]

In fact, this morphism induces the following morphism

\[(4.20.3) \quad \tilde{F}_{\Omega'}: \Omega^1_{U'/S} \to F_*(\Omega^1_{U'/S})\]

(cf. the formulas \[(4.20.4)\) and \[(4.20.5)\) below). Express $\tilde{F}^*(a') = \sum_i a_i^0 b_i + \eta(a')$ for $a' \in \mathcal{O}_{U'}$ with $a' = \sum_i a_i \otimes b_i (a_i \in \mathcal{O}_{U'}, b_i \in \mathcal{O}_S)$ and $\eta(a') \in \mathcal{O}_{U'}$. We can easily check that the following equalities hold:

\[(4.20.4) \quad \tilde{F}_{\Omega'}(d(a \otimes b)) = b(a^{p-1} da) + d\eta(a \otimes b), \quad (a \in \mathcal{O}_{U'}, b \in \mathcal{O}_S)\]

and

\[(4.20.5) \quad \tilde{F}_{\Omega'}(d\log([m, n])) = d\log m + d\eta([m, n]) \quad (m \in M_{U'}, n \in M_S)\]

(cf. [St] p. 106). The morphism $\tilde{F}_{\Omega'}$ is compatible with the restrictions of log open subschemes of $Y$. Hence the morphism \[(4.20.3)\] is a section of $C: F_*(\Omega^1_{U'/S_0}) \to \Omega^1_{U'/S_0}$. Because $\text{Lift}_{(Y, F)}(S_0 \subset S_0)$ and $\text{Sec}_{C}$ is a torsor under $\text{pr}_{0, \mathcal{H}_{\mathcal{O}_{U'}}(\Omega^1_{Y'/S_0}, F_*(\Omega^1_{Y'/S_0}))}$ on $Y'$, these are isomorphic.

The following statement is the log version of [St] p. 103 (i)):

**Theorem 4.21.** Let the assumptions be as in \[(4.20)\]. Assume that $Y$ is separated. The obstruction class $\text{obs}_{(Y, F)}(S_0 \subset S_0)$ in $\text{Ext}^1_{Y'}(\Omega^1_{Y'/S_0}, F_*(\Omega^1_{Y'/S_0}))$ is equal to the extension class of the following exact sequence

\[(4.21.1) \quad 0 \to F_*(\Omega^1_{Y'/S_0}) \to F_*(\Omega^1_{Y'/S_0}) \to \Omega^1_{Y'/S_0} \to 0.\]

**Proof.** Let the notations be as in the proof of \[(1.12)\]. Let $\tilde{F}_i: \tilde{U}_i \to \tilde{U}'_i$ be a lift of $F_i: U_i \to U'_i$ over $S$. Let $\eta_i$ be the $\eta$ in the proof of \[(1.20)\] for $F_i$. Let $\tilde{m}_{ij} = [\tilde{m}_{ij}, \tilde{n}_{ij}] \in M_{U'_i}$ and $\tilde{a}_{ij} = \tilde{a}_{ij} \otimes \tilde{b}_{ij} \in \mathcal{O}_{U'_i}$ are lifts of local sections $m_{ij} = [m_{ij}, n_{ij}] \in M_{U_i}$ and $a_{ij} = a_{ij} \otimes b_{ij} \in \mathcal{O}_{U_i}$, respectively. Set $\tilde{m}_{ij}': = (g_{ij}')^*(\tilde{m}_{ij}) = [g_{ij}'(m_{ij}), n_{ij}] \in M_{U'_i}$ and $\tilde{a}_{ij}' = (g_{ij}')^*(\tilde{a}_{ij}) = g_{ij}'(\tilde{a}_{ij}) \otimes \tilde{b}_{ij} \in \mathcal{O}_{U'_i}$. Let $d\log m_{ij} + d(\eta_i[U_{ij}](m_{ij}')) + b_{ij}a_{ij}' - d(\eta_i[U_{ij}](a_{ij}'))$ be elements of $\Gamma(U_{ij}, Z\Omega^1_{Y'/S_0})$. Then we have an element

\[
\{d\log m_{ij} + d(\eta_i[U_{ij}](m_{ij}')) - \{d\log m_{ij} + d(\eta_i[U_{ij}](m_{ij}'))\}\} = d((\eta_i[U_{ij}](m_{ij}')) - (\eta_i[U_{ij}])'(m_{ij}'))
\]

of $\Gamma(U_{ij}', F_*(\Omega^1_{Y'/S_0}))$. We also have an element $d((\eta_i[U_{ij}])'(a_{ij}')) - (\eta_i[U_{ij}])'(a_{ij}')$ of $\Gamma(U_{ij}', F_*(\Omega^1_{Y'/S_0}))$. Hence we have an element of $\Gamma(U_{ij}', \mathcal{H}_{\mathcal{O}_{U'}}(\Omega^1_{Y'/S_0}, F_*(\Omega^1_{Y'/S_0})))$.
Via the identification \((F_*(\mathcal{O}_Y)/\mathcal{O}_{Y'})/(U'_i)) \overset{d \sim}{\longrightarrow} F_*(B\Omega^1_{Y/S_0})(U'_i)\), this is nothing but a 1-cocycle arising from

\[
(g_{ij}' \bar{F}_j|_{\tilde{E}_j} g_{ij}^{-1})^* (\tilde{m}'_{ij} \bar{m}'_{ij})^{-1} \equiv (g_{ij}' \bar{F}_j|_{\tilde{E}_j} g_{ij}^{-1})^* (\tilde{m}'_{ij} \bar{m}'_{ij})^{-1} \equiv (g_{ij}' \bar{F}_j|_{\tilde{E}_j} g_{ij}^{-1})^* (\tilde{m}'_{ij} \bar{m}'_{ij})^{-1} \equiv 1 + p((\eta_j|_{U_j})(m'_{ij}) - p(\eta_j|_{U_j})(m'_{ij}))
\]

and

\[
(g_{ij}' \bar{F}_j|_{\tilde{E}_j} g_{ij}^{-1})^* (\tilde{a}'_{ij} \bar{a}'_{ij}) \equiv (g_{ij}' \bar{F}_j|_{\tilde{E}_j} g_{ij}^{-1})^* (\tilde{a}'_{ij} \bar{a}'_{ij}) = (g_{ij}' \bar{F}_j|_{\tilde{E}_j} g_{ij}^{-1})^* (\tilde{a}'_{ij} \bar{a}'_{ij}) + pF^* (\delta F_{ij} (\tilde{a}'_{ij}; \bar{a}'_{ij})) - (g_{ij}' \bar{F}_j|_{\tilde{E}_j} g_{ij}^{-1})^* (\tilde{a}'_{ij})
\]

\[
= p((\eta_j|_{U_j})(a'_{ij}) - p(\eta_j|_{U_j})(a'_{ij})).
\]

Here \(\equiv\) means the equality in the quotient \((F_*(\mathcal{O}_Y)/\mathcal{O}_{Y'})/(U'_i)\) and we have used \(\textbf{[14.13]}\).

\[\textbf{Remark 4.22.} \text{In [57] there is no proof of the trivial version of [14.24]. In particular, [14.13] is missing in [loc. cit.].}\]

In the rest of this section, we consider the log deformation theory with absolute Frobenius endomorphism when \(\tilde{S}\) is perfect.

Let \(F_0: Y_0 \longrightarrow Y_0\) be the \(e\)-times iterated absolute Frobenius endomorphism over \(F_{S_0}^e: S_{00} \longrightarrow S_{00}\). We assume that there exists a lift \(F: Y \longrightarrow Y\) of \(F_0: Y_0 \longrightarrow Y_0\) over \(F_{S_0}^e\). We say that \((\tilde{Y}, \bar{F})/F_{S}^e\) is a log smooth integral lift (or simply a lift) of \((Y, F)/F_{S_0}^e\) if \(\tilde{Y}\) is a log smooth integral scheme over \(S\) such that \(\tilde{Y} \times_S S_0 = \tilde{Y}\) and \(\bar{F}\) is a morphism \(\tilde{Y} \longrightarrow \tilde{Y}\) over \(F_{S}^e\) fitting into the following commutative diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{c} & \tilde{Y} \\
F \downarrow & & \downarrow \bar{F} \\
Y & \xrightarrow{c} & \tilde{Y}
\end{array}
\]

over the commutative diagram

\[
\begin{array}{ccc}
S_0 & \xrightarrow{c} & S \\
F_{S_0}^e \downarrow & & \downarrow F_{S}^e \\
S_0 & \xrightarrow{c} & S.
\end{array}
\]

Let \(\text{Lift}_{(Y,F)/(S_0 \subset S,F_{S_0}^e)}\) be the following sheaf defined by the following equality:

\[
\text{Lift}_{(Y,F)/(S_0 \subset S,F_{S_0}^e)}(U) := \{\text{isomorphism classes of lifts of } (U,F|_U)/F_{S_0}^e \text{ over } F_{S_0}^e\}
\]

for each log open subscheme \(U\) of \(Y\). Here the isomorphism classes of lifts of \((U,F|_U)/F_{S_0}^e\) over \(F_{S_0}^e\) are defined in an obvious way.

Then the following hold by the same proof as that of \([14,12]\):
Theorem 4.23. Let \( I, \pi \) and \( n \) be as in (4.11). Then the following hold:

1. Assume that \((Y, F)/S_0\) has a lift \((Y, \tilde{F})/S\). Set \( \text{Aut}_{S,F_{\tilde{}}^c}^c(Y, Y) := \{ g \in \text{Aut}_S(\tilde{Y}) \mid g|_Y = \text{id}_Y, \tilde{F} \circ g = g \circ \tilde{F} \} \). Then \( \text{Aut}_{S,F_{\tilde{}}^c}^c(Y, Y) = \{ \text{id}_Y \} \).

2. The sheaf \( \text{Lift}_{(Y, F)/(S_0 \subset S, F_{\tilde{}}^c)}^c \) on \( \tilde{Y} \) is a torsor under \( \text{Hom}_{\mathcal{O}_Y}(\Omega^1_{Y_0/S_0}, F_0^*(\mathcal{O}_Y)/\mathcal{O}_Y) \).

3. Assume that \( \tilde{Y} \) is separated. In

\[
\text{Ext}^1_\mathcal{Y}(\Omega^1_{Y_0/S_0}, F_0^*(\mathcal{O}_Y)/\mathcal{O}_Y),
\]

there exists a canonical obstruction class \( \text{obs}_{(Y, F)/(S_0 \subset S, F_{\tilde{}}^c)}^c \) of a lift of \((Y, F)/F_{S_0}^c\) over \( F_S^c \).

4. Assume that \( \tilde{Y} \) is separated. Let

\[
\partial: \text{Ext}^1_\mathcal{Y}(\Omega^1_{Y_0/S_0}, F_0^*(\mathcal{O}_Y)/\mathcal{O}_Y) \to \text{Ext}^2_\mathcal{Y}(\Omega^1_{Y_0/S_0}, \mathcal{O}_Y)
\]

be the boundary morphism obtained by the following exact sequence (3.3):

\[
0 \to \mathcal{O}_Y \to F_0^*(\mathcal{O}_Y) \to F_0^*(\mathcal{O}_Y)/\mathcal{O}_Y \to 0.
\]

Then \( \partial(\text{obs}_{(Y, F)/(S_0 \subset S, F_{\tilde{}}^c)}) = \text{obs}_{(Y, F)/(S_0 \subset S)} \).

5. Assume that \( \tilde{Y} \) is separated. Assume that there exists a lift \( \tilde{Y}/S \) of \( Y/S_0 \).

Then, in

\[
\text{Ext}^1_\mathcal{Y}(\Omega^1_{Y_0/S_0}, F_0^*(\mathcal{O}_Y)),
\]

there exists a canonical obstruction class \( \text{obs}_{(Y, F)/(S_0 \subset S, F_{\tilde{}}^c)}^c \) of a lift \((Y, F)/F_{S_0}^c\) over \( F_S^c \).

Corollary 4.24. Assume that \( e = 1 \). Assume also that \( Y_0/S_0 \) is of Cartier type and that \( S_{00} \) is perfect. Then the following hold:

1. The sheaf \( \text{Lift}_{(Y, F)/(S_0 \subset S, F_0)} \) on \( \tilde{Y} \) is a torsor under \( \text{Hom}_{\mathcal{O}_Y}(\Omega^1_{Y_0/S_0}, F_0^*(B\Omega^1_{Y_0/S_0})) \).

2. Assume that \( \tilde{Y} \) is separated. In

\[
\text{Ext}^1_\mathcal{Y}(\Omega^1_{Y_0/S_0}, F_0^*(B\Omega^1_{Y_0/S_0})),
\]

there exists a canonical obstruction class \( \text{obs}_{(Y, F)/(S_0 \subset S, F_0)} \) of a lift \((Y, F)/F_{S_0}\) over \( F_S \).

3. Assume that \( \tilde{Y} \) is separated. Let

\[
\partial: \text{Ext}^1_\mathcal{Y}(\Omega^1_{Y_0/S_0}, F_0^*(B\Omega^1_{Y_0/S_0})) \to \text{Ext}^2_\mathcal{Y}(\Omega^1_{Y_0/S_0}, \mathcal{O}_Y)
\]

be the boundary morphism obtained by the following exact sequence (3.3):

\[
0 \to \mathcal{O}_Y \to F_0^*(\mathcal{O}_Y) \xrightarrow{F_0^*(d)} F_0^*(B\Omega^1_{Y_0/S_0}) \to 0.
\]

Then \( \partial(\text{obs}_{(Y, F)/(S_0 \subset S, F_0)}) = \text{obs}_{(Y, F)/(S_0 \subset S)} \).

Proof. Recall that \( Y_0' := Y_0 \times_{S_0} F_{S_0} \). Let \( F_{01}^c: Y_0' \to Y_0' \) be the relative Frobenius morphism. Because \( S_{00} \) is perfect, the projection \( \tilde{Y}_0' \to \tilde{Y}_0' \) is an isomorphism. By (4.15.2) we obtain the following composite isomorphism

\[
F_0^*(\mathcal{O}_Y)/\mathcal{O}_Y \xrightarrow{\sim} F_0^*(\mathcal{O}_Y)/\mathcal{O}_Y' = F_{01}^c(B\Omega^1_{Y_0/S_0}) \xrightarrow{\sim} F_0^*(B\Omega^1_{Y_0/S_0}).
\]

Now (4.24) follows from (4.23) and (4.24).
5 Applications of log deformation theory with relative Frobenius morphisms

In this section we give another short proof of Kato’s theorem on the $E_1$-degeneration of the log Hodge de Rham spectral sequence ([Kk1]) by following the method of Srinivas ([Sr]). We also give the log versions of vanishing theorems of Kodaira-Akizuki-Nakano in characteristic $p$ and 0 following the method of Raynaud ([DI]).

In [Sr, p. 104–105] Srinivas has given another short proof of the $E_1$-degeneration of the Hodge de Rham spectral sequence due to Deligne and Illusie ([DI]) by using the deformation theory in [NoS]. (Strictly speaking, he has proved this only in the case where the base scheme is the spectrum of a perfect field of characteristic $p > 0$.) By using the theory in §4 and his idea, we can also give another short proof of the degeneration at $E_1$ of the log Hodge de Rham spectral sequence due to Kato in [Kk1] (4.12 (3)) in the case where there exists a lift of the Frobenius endomorphism of the base log scheme:

**Theorem 5.1 (A special case of [Kk1] (4.12 (2))).** Let the notations and the assumptions be as in (4.12). Assume that $\hat{S}$ is flat over $\text{Spec}(\mathbb{Z}/p^{2})$. Set $S_0 := S \bmod p$. Let $Y$ be a log smooth separated scheme of Cartier type over $S_0$. Let $F : Y \to Y'$ be the relative Frobenius morphism over $S$. If $Y'$ has a log smooth integral lift $Z$ over a fine log scheme $S$, then there exists an isomorphism

$$
\bigoplus_{i < p} \Omega^{i}_{Y'/S_0}[-i] \sim \tau_{<p} F_{*}(\Omega^{*}_{Y/S_0})
$$

in the derived category $D^{+}(Y'_{\text{zar}})$ of bounded above complexes of $\mathcal{O}_{Y'}$-modules.

**Proof.** By (3.3.1) we have the following exact sequence

$$
\text{Ext}^{1}_{Y'}(\Omega^{1}_{Y'/S_0}, F_{*}(\mathcal{O}_{Y})) \to \text{Ext}^{1}_{Y'}(\Omega^{1}_{Y'/S_0}, F_{*}(B\Omega^{1}_{Y/S_0})) \to \text{Ext}^{2}_{Y'}(\Omega^{1}_{Y'/S_0}, \mathcal{O}_{Y'}).
$$

By (4.16) (4) and the assumption, there exists the extension class of the following exact sequence

$$
0 \to F_{*}(\mathcal{O}_{Y}) \to \mathcal{V} \to \Omega^{1}_{Y'/S_0} \to 0
$$

whose image in $\text{Ext}^{1}_{Y'}(\Omega^{1}_{Y'/S_0}, F_{*}(B\Omega^{1}_{Y/S_0}))$ is equal to $\text{obs}_{(Y,F)/(S_0,S,F_{0})}$. By (5.1.4) this exact sequence fits into the following commutative diagram:

$$
\begin{array}{cccccccc}
0 & \longrightarrow & F_{*}(\mathcal{O}_{Y}) & \longrightarrow & \mathcal{V} & \longrightarrow & \Omega^{1}_{Y'/S_0} & \longrightarrow & 0 \\
\bigg\downarrow F_{*}(d) & & \bigg\downarrow & & \bigg\downarrow & & \bigg\downarrow & & \\
0 & \longrightarrow & F_{*}(B\Omega^{1}_{Y/S_0}) & \longrightarrow & F_{*}(\Omega^{1}_{Y'/S_0}) & \longrightarrow & \Omega^{1}_{Y'/S_0} & \longrightarrow & 0.
\end{array}
$$

Set $C_{1} := (F_{*}(\mathcal{O}_{Y}) \to \mathcal{V})$. This is quasi-isomorphic to $\Omega^{1}_{Y'/S_0}[-1]$. The diagram (5.1.3) induces the following morphism

$$
\begin{array}{cccccccc}
0 & \longrightarrow & F_{*}(\mathcal{O}_{Y}) & \longrightarrow & \mathcal{V} & \longrightarrow & 0 & \longrightarrow & \ldots \\
\bigg\downarrow & & \bigg\downarrow & & \bigg\downarrow & & \bigg\downarrow & & \\
0 & \longrightarrow & F_{*}(\mathcal{O}_{Y}) & \longrightarrow & F_{*}(\Omega^{1}_{Y'/S_0}) & \longrightarrow & F_{*}(\Omega^{2}_{Y'/S_0}) & \longrightarrow & \ldots
\end{array}
$$

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of complexes since $F_*(Z\Omega^1_{Y/S_0}) \subset F_*(\Omega^1_{Y/S_0})$. We denote this morphism by $\varphi_1: C_1 \rightarrow F_*(\Omega^*_Y/S_0)$. Because the following diagram

$$
\begin{array}{ccc}
\mathcal{H}^1(C_1) & \xrightarrow{\mathcal{H}^1(\varphi_1)} & \mathcal{H}^1(F_*(\Omega^*_Y/S_0)) \\
\cong & \swarrow \cong & c^{-1} \\
\Omega^1_{Y'/S_0} & \xrightarrow{\sim} & \Omega^1_{Y'/S_0}
\end{array}
$$

(5.1.5)

is commutative, $\mathcal{H}^1(\varphi_1)$ is an isomorphism. Let $\varphi^1$ be the following morphism in the derived category $D(X')$:

$$
\varphi^1: \Omega^1_{Y'/S_0}[-1] \xleftarrow{\sim} C_1 \xrightarrow{\varphi_1} F_*(\Omega^*_Y/S_0).
$$

Then $\mathcal{H}^1(\varphi^1) = C^{-1}: \Omega^1_{Y'/S_0} \xrightarrow{\sim} \mathcal{H}^1(F_*(\Omega^*_Y/S_0))$.

Remark 5.2. Assume that there exists a lift $\tilde{Y}/S$ of $Y/S_0$. Then we have the following composite morphism

$$
\mathcal{O}_{Y'}, C^{-1} \rightarrow \mathcal{H}^0(F_*(\Omega^*_Y/S_0)) \xhookrightarrow{\varphi^0} F_*(\Omega^*_Y/S_0).
$$

Let $i$ be a positive integer less than $p$. Consider the following splitting

$$
\Omega^1_{Y'/S_0} \rightarrow (\Omega^1_{Y'/S_0})^\otimes_i
$$

of a natural surjection $(\Omega^1_{Y'/S_0})^\otimes_i \rightarrow \Omega^1_{Y'/S_0}$ defined by the morphism

$$
\omega_1 \wedge \cdots \wedge \omega_i \mapsto (i!)^{-1} \sum_{\sigma \in S_i} \text{sgn}(\sigma)\omega_{\sigma(1)} \otimes \cdots \otimes \omega_{\sigma(i)}
$$

as in [DI, p. 251]. Then we have the following composite morphism

$$
\varphi^i: \Omega^1_{Y'/S_0}[-i] \rightarrow (\Omega^1_{Y'/S_0})^\otimes_i[-i] \xrightarrow{(\varphi^0)^{\otimes_i}} (F_*(\Omega^*_Y/S_0))^\otimes_i \xrightarrow{\text{product}} F_*(\Omega^*_Y/S_0).
$$

By the multiplicative property of $C^{-1}$, $\mathcal{H}^i(\varphi^i)$ is equal to the Cartier isomorphism $C^{-1}: \Omega^1_{Y'/S_0} \xrightarrow{\sim} \mathcal{H}^i(F_*(\Omega^*_Y/S_0))$. Hence $\sum_{i=0}^{p-1} \varphi^i$ is the desired isomorphism [5.1.1].

Remark 5.2. Assume that there exists a lift $\tilde{Y}/S$ of $Y/S_0$. Then, by (4.16) (4), we can take the element $\text{obs}_{\tilde{Y}/S}(F)$ as an element in $\text{Ext}^1_{Y'}(\Omega^1_{Y'/S_0}, F_*(\mathcal{O}_{Y'}))$ in the proof of (5.1). (In [Sr] this has not been mentioned.)

Corollary 5.3 (A special case of [Kk1 (4.12) (1)])]. Let the notations and the assumptions be as in (5.1). Then the following hold:

1) Let $f: Y \rightarrow S_0$ and $f': Y' \rightarrow S_0$ be the structural morphisms. Then there exists the following decomposition

$$
R^q f_*(\Omega^*_Y/S_0) = \bigoplus_{i+j=q} R^j f'_*(\Omega^1_{Y'/S_0})
$$

for $q < p$. Moreover, if $Y/S_0$ is proper, then $E^{ij}_1 = E^{ij}_\infty$ for $i + j < p$, where $E^{ij}_\infty$ ($\ast = 1, \infty$) is the $E^{ij}_\ast$-term of the following spectral sequence

$$
E^{ij}_1 = R^j f_*(\Omega^1_{Y/S_0}) \Rightarrow R^{i+j} f_*(\Omega^*_Y/S_0).
$$

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Furthermore, in this case, $R^q f_*(\Omega^\bullet_{Y/S_0}^i) (0 \leq q < p)$ is locally free and commutes with any base change of fine log schemes.

(2) Assume that the structural morphism $\hat{Y} \to \hat{S}$ of schemes is flat, that $\dim(\hat{Y}/\hat{S}) \leq p$, that $F_*(\mathcal{O}_Y)$ is a locally free $\mathcal{O}_{Y'}$-modules (of finite rank) and that

$$H^{p+1}(Y', \mathcal{H}om_{\mathcal{O}_{Y'}}(\Omega^p_{Y'/S}, \mathcal{O}_{Y'})) = 0.$$

Then there exists a decomposition

$$\bigoplus_{i \leq p} \Omega_{Y'/S_0}^i \sim - \to F_*(\Omega_{Y/S_0}^\bullet)$$

(5.3.3)

in the derived category $D^+_{\text{Y zar}}$. Consequently there exists a decomposition

$$R^q f_*(\Omega_{Y/S_0}^\bullet) = \bigoplus_{i+j = q} R^j f'_*(\Omega_{Y'/S_0}^i) \quad (q \in \mathbb{Z})$$

(5.3.4)

and the following spectral sequence

$$E_1^{ij} = R^j f_*(\Omega_{Y/S_0}^i) \implies R^{i+j} f_*(\Omega_{Y/S_0}^\bullet)$$

(5.3.5)

degenerates at $E_1$. Furthermore, $R^q f_*(\Omega_{Y/S_0}^\bullet) (0 \leq q \leq p)$ is locally free and commutes with any base change of fine log schemes.

Proof. (1): This immediately follows from (5.1.1) and the log version of the argument of [DI, (4.1.2), (4.1.4)].

(2) (The proof is the same as that of [DI, (2.3)].) In the case $\dim(\hat{Y}/\hat{S}) < p$, (5.3.3) follows from (5.1.1). Consider the case $\dim(\hat{Y}/\hat{S}) = p$. We may assume that $\hat{Y}$ is connected. Then the wedge product

$$\Omega_{Y/S_0}^i \times \Omega_{Y/S_0}^{p-i} \to \Omega_{Y/S_0}^p$$

is a perfect pairing. Because $F_*(\mathcal{O}_Y)$ is a locally free $\mathcal{O}_{Y'}$-modules, we can check that

$$F_*(\Omega_{Y/S_0}^i) \times F_*(\Omega_{Y/S_0}^{p-i}) \to F_*(\Omega_{Y/S_0}^p) \overset{\text{proj}}{\to} \mathcal{H}^p(F_*(\Omega_{Y/S_0}^\bullet)) \cong \Omega_{Y'/S_0}^p$$

is also a perfect pairing of locally free $\mathcal{O}_{Y'}$-modules of finite rank. The rest of the proof of (5.3.3) is completely the same as that of [DI, (2.3), (3.7)].

Now (5.3.4) follows from the equality $R^q f'_*(F_*(\Omega_{Y/S_0}^\bullet)) = R^q f_*(\Omega_{Y/S_0}^\bullet)$ (since $F'$ is finite).

The following has not been stated in literatures:

**Corollary 5.4.** Let the notations be as in (5.3) (1). Assume that $S_0$ is the log point $s$ of a perfect field of characteristic $p > 0$ and that $Y/s$ is of vertical type. Assume that $\hat{Y}$ is of pure dimension $d$. Then $E_1^{ij} = E_\infty^{ij}$ for $i + j > 2d - p$.

Proof. The equality in the statement follows from (5.3) (1) and Tsuji’s duality for log de Rham cohomologies and his log Serre duality (see (5.6) below).
Remark 5.5. (1) In [Ts1] it is not necessary to assume that \( \Gamma(s, \mathcal{O}_s) \) is perfect. In fact, one has only to take the perfection of \( \Gamma(s, \mathcal{O}_s) \).

(2) Let \( K \) be a field of characteristic 0. Let \( T \) be an fs log scheme whose underlying scheme is \( \text{Spec}(K) \). Let \( g: Z \rightarrow T \) be a proper log smooth integral morphism of fs log schemes. Assume that \( g \) is saturated. Then, in [IKN p. 37], by using [K3] (4.12) (1), Illusie, Kato and Nakayama have proved that the following spectral sequence

\[
E_1^{ij} = R^ij_*\Omega^i_{Z/T} \Rightarrow R^{i+j}g_*\Omega^*_{Z/T}
\]
degenerates at \( E_1 \).

More strongly, in [IKN (7.2)], they have proved the \( E_1 \)-degeneration of (5.5.1) if \( g \) is proper log smooth and exact. They have also proved that \( E_1^{ij} \) is locally free if any stalk of \( M_Y/\mathcal{O}^*_Y \) is a free monoid. See also [Nakk4] (9.15) and [I4] for the log Hodge symmetry.

Next we give the log version of Raynaud’s result in [DI (2.8)]. To give it, we need to recall Tsuji’s ideal sheaf.

Let \( g: Y \rightarrow Z \) be a morphism of fs log schemes. Secondly let us recall Tsuji’s ideal sheaf \( \mathcal{I}_{Y/Z} \) of the log structure \( M_Y \) denoted by \( I_g \) in [Ts1] for the review of Tsuji’s log Serre duality.

For a commutative monoid \( P \) with unit element, an ideal is, by definition, a subset \( I \) of \( P \) such that \( PI \subset I \). An ideal \( p \) of \( P \) is called a prime ideal if \( P \setminus p \) is a submonoid of \( P \) ([KK2 (5.1)]). For a prime ideal \( p \) of \( P \), the height \( \text{ht}(p) \) is the maximal length of sequence’s \( p \supset p_1 \supset \cdots \supset p_r \) of prime ideals of \( P \). Let \( h: Q \rightarrow P \) be a morphism of monoids. A prime ideal \( p \) of \( P \) is said to be horizontal with respect to \( h \) if \( h(Q) \subset P \setminus p \) ([Ts1 (2.4)]).

Let \( Y \rightarrow Z \) be a morphism of fs log schemes. Let \( h: Q \rightarrow P \) be a local chart of \( g \) such that \( P \) and \( Q \) are saturated. Set

\[
I := \{a \in P \mid a \in p \text{ for any horizontal prime ideal of } P \text{ of height 1 with respect to } h\}.
\]

Let \( \mathcal{I}_{Y/Z} \) be the ideal sheaf of \( M_Y \) generated by \( \text{Im}(I \rightarrow M_Y) \). In [Ts1 (2.6)] Tsuji has proved that \( \mathcal{I}_{Y/Z} \) is independent of the choice of the local chart \( h \). Let \( \mathcal{I}_{Y/Z}\mathcal{O}_Y \) be the ideal sheaf of \( \mathcal{O}_Y \) generated by the image of \( \mathcal{I}_{Y/Z} \). For a quasi-coherent sheaf \( \mathcal{F} \) of \( \mathcal{O}_Y \)-modules, denote \( (\mathcal{I}_{Y/Z}\mathcal{O}_Y)\mathcal{F} \) by \( \mathcal{I}_{Y/Z}\mathcal{F} \).

Theorem 5.6 ([Ts1, (2.21)]). Let \( A \) be a discrete valuation ring with uniformizer \( \pi \). Let \( Z \) be an fs log scheme whose underlying scheme is \( \text{Spec}(A/\pi^m) \) for some \( m \geq 1 \) and whose log structure is associated to the morphism \( \mathbb{N} \ni 1 \mapsto a \in A/\pi^m \) for some \( a \in A/\pi^m \). Let \( g: Y \rightarrow Z \) be a saturated morphism of fs log schemes such that \( \hat{g} \) is of finite type. Assume that \( \Omega^d_{Y/Z} \) is a locally free \( \mathcal{O}_Y \)-modules of constant rank \( d \). Then \( g^!(\mathcal{O}_Z) = \mathcal{I}_{Y/Z}\Omega^d_{Y/Z}[d] \).

Definition 5.7. We say that \( Y/Z \) is of vertical type if \( \mathcal{I}_{Y/Z}\mathcal{O}_Y = \mathcal{O}_Y \).

Example 5.8. If \( X/s \) is an SNCL scheme ([Nakk2, Nakk7]), then \( X/s \) is of vertical type.

Corollary 5.9 (The log version of the vanishing theorem of Kodaira-Akizuki-Nakano in characteristic p). Let \( \kappa \) be a perfect field of characteristic \( p > 0 \). Let \( s \) be the log point of \( \kappa \) or \( (\text{Spec}(\kappa), \kappa^+) \). Let \( Y \rightarrow s \) be a projective log smooth morphism of Cartier type of fs log schemes which has a log smooth integral lift over \( W_2(s) \). Assume that \( \hat{Y} \) is of pure dimension \( d \). Let \( \mathcal{I}_{Y/s} \) be Tsuji’s ideal sheaf of \( M_Y \). Let \( \mathcal{L} \) be an ample invertible \( \mathcal{O}_Y \)-module. Then the following hold:
(1) $H^i(Y, \Omega^i_{Y/s} \otimes L^{-1}) = 0$ for $i + j < \min\{d, p\}$.

(2) $H^i(Y, I_{Y/s} \Omega^i_{Y/s} \otimes L) = 0$ for $i + j > \max\{d, 2d - p\}$.

Proof. (1): The proof is completely the same as that of [DI (2.8), (2.9)] by using Tsuji’s log Serre duality [5.6].

Indeed, set $F(m) := F \otimes_{\mathcal{O}_Y} L^m (m \in \mathbb{Z})$ for a coherent $\mathcal{O}_Y$-module $F$ and $\mathcal{M} := L^{-1}$ and $b := \min\{d, p\}$. If $m$ is large enough, then $H^i(Y, I_{Y/s} \Omega^i_{Y/s}(m) = 0$ for any $i$ and any $q > 0$ by Serre’s theorem [EGA III-1 (2.2.1)]. By Tsuji’s log Serre duality, $H^i(Y, \Omega^i_{Y/s}(-m)) = 0$ for any $i \in \mathbb{N}$ and $j < d$. In particular, $H^i(Y, \Omega^i_{Y/s}(-m)) = 0$ for any $i + j < d (i, j \in \mathbb{N})$ and hence $H^i(Y, \Omega^i_{Y/s}(m)) = 0$ for any $i + j < b (i, j \in \mathbb{N})$.

Assume that $H^i(Y, \Omega^i_{Y/s}(-p^n)) = 0$ for all $i + j < b$ and a positive integer $n$. Then we claim that $H^i(Y, \Omega^i_{Y/s}(-p^n)) = 0$. Indeed, let $W: Y' \to Y$ be the projection. Because the differential $d: F_*(\Omega^i_{Y/s}) \to F_*(\Omega^{i+1}_{Y/s})$ is $\mathcal{O}_{Y'}$-linear, we can consider the complex $W^*(\mathcal{M}^\otimes p^{-n}) \otimes_{\mathcal{O}_{Y'}} F_*(\Omega^i_{Y/s})$. Take the tensorization with $W^*(\mathcal{M}^\otimes p^{-n})$ for the isomorphism $\bigoplus_{i < b} \Omega^i_{Y/s}[i] \to F_*(\Omega^i_{Y/s})$ in $D^+(\mathcal{O}_{Y'})$:

$$\bigoplus_{i < b} W^*(\mathcal{M}^\otimes p^{-n}) \otimes_{\mathcal{O}_{Y'}} \Omega^i_{Y'/s}[i] \xrightarrow{\sim} W^*(\mathcal{M}^\otimes p^{-n}) \otimes_{\mathcal{O}_{Y'}} F_*(\Omega^i_{Y/s}).$$

(Note that $W^*(\mathcal{M}^\otimes p^{-n})$ is a flat $\mathcal{O}_{Y'}$-module.) We have the following spectral sequence:

$$E^1_{ij} = H^i(Y', W^*(\mathcal{M}^\otimes p^{-n}) \otimes_{\mathcal{O}_{Y'}} F_*(\Omega^i_{Y/s})) \implies H^{i+j}(Y', W^*(\mathcal{M}^\otimes p^{-n}) \otimes_{\mathcal{O}_{Y'}} F_*(\Omega^i_{Y/s})).$$

By the projection formula and the assumption, $E^1_{ij} = R^j f_* F_*(\mathcal{M}^\otimes p^{-n}) \otimes_{\mathcal{O}_Y} \Omega^i_{Y/s} = H^j(Y, \mathcal{M}^\otimes p^{-n} \otimes_{\mathcal{O}_Y} \Omega^i_{Y/s}) = H^j(Y, \Omega^i_{Y/s}(-p^n)) = 0$. Hence $H^{i+j}(Y', W^*(\mathcal{M}^\otimes p^{-n}) \otimes_{\mathcal{O}_{Y'}} F_*(\Omega^i_{Y/s})) = 0$ for $i + j < b$. By (5.9.1), $H^j(Y', W^*(\mathcal{M}^\otimes p^{-n}) \otimes_{\mathcal{O}_{Y'}} \Omega^i_{Y/s}) = 0$. Since $Y/s$ is integral and $\tilde{s}$ is perfect, $Y' \equiv Y \simeq \tilde{Y}$ and $\Omega^i_{Y'/s} = \Omega^i_{Y/s}[i] \simeq \Omega^i_{Y/s}$. Hence $H^j(Y, \Omega^i_{Y/s}(-p^n)) = H^j(Y', W^*(\mathcal{M}^\otimes p^{-n}) \otimes_{\mathcal{O}_{Y'}} \Omega^i_{Y/s}) = 0$.

(2): follows from (1) and Tsuji’s log Serre duality.

The following is a generalization of Norimatsu’s vanishing theorem ([No Theorem 1]). The following vanishing theorem is not a special case of Ambro-Fujino’s vanishing theorem ([A Theorem 3.2], [F4 Theorem 5.7]) and Fujino’s vanishing theorem ([F2 Theorem 1.1]).

**Corollary 5.10 (A log version of vanishing theorem of Kodaira-Akizuki-Nakano in characteristic 0).** Let $K$, $T$ and $Z$ be as in (5.5) (2). Assume that the log structure of $T$ is associated to a morphism $N \ni 1 \to a \in K$ for some $a \in K$.

Assume also that $\tilde{Z}$ is projective over $K$. Let $\mathcal{L}$ be an ample invertible $\mathcal{O}_Z$-module.

Then the following hold:

1. $H^i(Z, \Omega^i_{Z/T} \otimes \mathcal{L}^{-1}) = 0$ for $i + j < d$.

2. $H^i(Z, I_{Z/T, \Omega^i_{Z/T} \otimes \mathcal{L}}) = 0$ for $i + j > d$.

**Proof.** The proof is the same as that of [HKN (7.1.2)] by using Kato-Tsuji’s result [4.2].
6 Log weak Lefschetz conjecture

In this section we give the precise definition of the horizontal divisor appearing in the log weak Lefschetz conjecture ([LS]) and we prove the log weak Lefschetz conjecture in characteristic $0$ and we prove this conjecture in characteristic $p > 0$ in certain cases.

First we give the following definitions:

**Definition 6.1.** (1) Let $S_0$ be a family of log points ([Nakk7 (1.1)]) and let $X/S_0$ be an SNCL scheme (loc. cit., (1.1.16)). Let $A_{S_0}(a, d + e) (a \leq d)$ be a log scheme whose underlying scheme is $\text{Spec} \mathcal{O}_{S_0}[x_0, \ldots, x_d, y_1, \ldots, y_e]/(x_0 \cdots x_a)$ and whose log structure is associated to the morphism

$$N^{\geq a+1} \ni c_i \mapsto x_i - 1 \in O_{S_0}[x_0, \ldots, x_d, y_1, \ldots, y_e]/(x_0 \cdots x_a).$$

Let $D$ be an effective Cartier divisor on $X/S_0$. Endow $D$ with the inverse image of the log structure of $X$ and let $D$ be the resulting log scheme. We call $D$ a relative simple normal crossing divisor ($=:\text{relative SNCD}$) on $X/S_0$ if there exists a family

$$\Delta := \{D_\lambda\}_{\lambda \in \Lambda}$$

of non-zero effective Cartier divisors on $X/S_0$ of locally finite intersection which are SNC(=simple normal crossing) schemes over $S_0$ (Nakk7 (1.1.9)) such that

$$\tag{6.1.1} D = \sum_{\lambda \in \Lambda} D_\lambda \quad \text{in} \quad \text{Div}(X/S_0)_{\geq 0}$$

and, for any point $z$ of $D$, there exist a Zariski open neighborhood $V$ of $z$ in $X$ and the following cartesian diagram

$$\begin{array}{ccc}
D|_V & \longrightarrow & (y_1 \cdots y_b = 0) \\
\cap & \downarrow & \\
V & \xrightarrow{g} & A_{T_0}(a, d + e) \\
\downarrow & \downarrow & \\
T_0 & \overset{\cong}{\longrightarrow} & T_0
\end{array}$$

for some positive integers $a$, $b$, $d$, and $e$ such that $a \leq d$ and $b \leq e$. Here $T_0$ is an open log subscheme of $S_0$ whose log structure is associated to the morphism $N \ni 1 \mapsto 0 \in O_{T_0}$, $(y_1 \cdots y_b = 0)$ is an exact closed log subscheme of $A_{T_0}(a, d + e)$ defined by an ideal sheaf $(y_1 \cdots y_b)$, $g$ is strictly étale and $A_{T_0}(a, d + e) \longrightarrow T_0$ is obtained by the diagonal embedding $N \overset{\cong}{\longrightarrow} N^{\geq a+1}$. Endow $D_\lambda$ with the inverse image of the log structure of $X$ and let $D_\lambda$ be the resulting log scheme. We call $D_\lambda$ an SNCL component of $D$ and the equality (6.1.1) a decomposition of $D$ by SNCL components of $D$.

(2) Let the notations be as in (1). Let $E$ be another SNCD on $X/S_0$. Let $D \cup E$ be a log scheme whose underlying scheme is $\hat{D} \cup \hat{E}$ and whose log structure is the inverse image of the log structure of $X$. Then we say that $D \cup E$ is an SNCD on $X/S_0$ if, in the diagram (6.1.2) for any point $z \in \hat{D} \cup \hat{E}$, $(D \cup E)|_V = (y_1 \cdots y_e = 0)$ for some $b \leq c \leq e$. In this case, we denote $D \cup E$ by $D + E$.

The following construction of $M(D)$ is the log version of the construction in [NaS] p. 61.
Let $\text{Div}_{\mathcal{D}}(\hat{X}/\hat{S}_0)_{\geq 0}$ be a submonoid of $\text{Div}(\hat{X}/\hat{S}_0)_{\geq 0}$ consisting of effective Cartier divisors $E$’s on $\hat{X}/\hat{S}_0$ such that there exists an open covering $X = \bigcup_{i \in I} V_i$ (depending on $E$) of $X$ such that $E|_{V_i}$ is contained in the submonoid of $\text{Div}(\hat{V}_i/\hat{S}_0)_{\geq 0}$ generated by $\hat{D}_{\lambda_i}|_{\hat{V}_i}$ ($\lambda \in \Lambda$). By [NaS, A.0.1] the definition of $\text{Div}_{\mathcal{D}}(\hat{X}/\hat{S}_0)_{\geq 0}$ is independent of the choice of $\Delta$. (We have only to set $S := \text{Spec}_T (\mathcal{O}_T \cdot [x_0, \ldots, x_d]/(x_0 \cdots x_a))$ in [loc. cit.] and to consider the projection $X_\mathfrak{T}_0 \times_{\mathfrak{T}_0} S \rightarrow X_{\mathfrak{T}_0}$.)

The pair $(X, D)$ gives the following fs log structure $M(D)$ in the zariski topos $\mathfrak{X}_{\text{zar}}$ as in [NaS, p. 61].

Let $M(D)'$ be a presheaf of monoids in $\mathfrak{X}_{\text{zar}}$ defined as follows: for an open subscheme $\mathfrak{X}$ of $\hat{X}$,

$$\Gamma_{\mathfrak{X}}(\hat{V}, M(D)') := \{(E, a) \in \text{Div}_{\mathcal{D}}(\hat{V}/\hat{S}_0)_{\geq 0} \times \Gamma(\hat{V}, \mathcal{O}_{\mathfrak{X}}) | a \text{ is a generator of } \Gamma(\hat{V}, \mathcal{O}_{\mathfrak{X}}(-E))\}$$

with a monoid structure defined by an equation $(E, a) \cdot (E', a') := (E + E', aa')$.

The natural morphism $M(D)' \rightarrow \mathcal{O}_{\mathfrak{X}}$ defined by the second projection $(E, a) \mapsto a$ induces a morphism $M(D)' \rightarrow (\mathcal{O}_{\mathfrak{X}}, *)$ of presheaves of monoids in $\mathfrak{X}_{\text{zar}}$. The log structure $M(D)'$ is, by definition, the associated log structure to the sheafification of $M(D)'$. Because $\text{Div}_{\mathcal{D}}(\hat{V}/\hat{S}_0)_{\geq 0}$ is independent of the choice of the decomposition of $D|_{\hat{V}}$ by smooth components, $M(D)$ is independent of the choice of the decomposition of $D$ by SNCL components of $D$.

**Proposition 6.2.** Let the notations be as above. Let $z$ be a point of $D$ and let $V$ be an open neighborhood of $z$ in $X$ which admits the diagram (6.1.2). Assume that $z \in \bigcap_{i=1}^{b} \{ y_i = 0 \}$. If $V$ is small, then the log structure $M(D)|_{V} \rightarrow \mathcal{O}_{\mathfrak{V}}$ is isomorphic to $\mathcal{O}_{\mathfrak{V}} y_1^i \cdots y_b^i \rightarrow \mathcal{O}_{\mathfrak{V}}$. Consequently $M(D)|_{V}$ is associated to the homomorphism $\mathbb{N}^b_{\mathfrak{V}} \ni e_i \mapsto y_i \in M(D)|_{V} (1 \leq i \leq b)$ of sheaves of monoids on $\mathfrak{V}$, where $\{ e_i \}_{i=1}^{b}$ is the canonical basis of $\mathbb{N}^{b}$. In particular, $M(D)$ is fs.

**Proof.** We claim that, by shrinking $V$ in (6.1.2), for any $1 \leq i \leq b$, there exists a unique element $\lambda_i \in \Lambda$ satisfying

$$D_{\lambda_i}|_{\hat{V}} = \text{div}(y_i) \text{ in } \text{Div}(\hat{V}/\hat{S}_0)_{\geq 0}.$$

This follows from [NaS, Proposition A.0.1] by setting $S := (\mathfrak{h}_{\mathfrak{T}_0}(a, d))$, $X := \hat{V}$ and $D := \hat{D}$ in [loc. cit.]. The rest of the proof is the same as that of [NaS, (2.1.9)].

Set $X(D) := (X, M_X \oplus_{\mathcal{O}_X} M(D) \rightarrow \mathcal{O}_X)$. Then $X(D)/\mathfrak{S}_0$ is log smooth, integral and saturated by (3.2) (4).

**Remark 6.3.** As in the classical case (e. g., [12]), we can consider the log de Rham complex $\Omega^\bullet_{X/\mathfrak{S}_0}(\log D)$ with logarithmic poles along $D$. It is clear that the complex $\Omega^\bullet_{X/\mathfrak{S}_0}(\log D)$ is equal to the log de Rham complex $\Omega^\bullet_{X(D)/\mathfrak{S}_0}$. Set $\Omega^i_{X/\mathfrak{S}_0}(\log D)(-D) := \Omega^i_{X}(-\hat{D}) \otimes_{\mathcal{O}_X} \Omega^i_{X/\mathfrak{S}_0}(\log D) (i \in \mathbb{N})$. It is easy to check that the family $\{ \Omega^i_{X/\mathfrak{S}_0}(\log D)(-D) \}_{i \in \mathbb{N}}$ gives a complex $\Omega^\bullet_{X/\mathfrak{S}_0}(\log D)(-D) : d\Omega^i_{X/\mathfrak{S}_0}(\log D)(-D) \subset \Omega^{i+1}_{X/\mathfrak{S}_0}(\log D)(-D)$.
It suffices to prove that
\[ H \]

following spectral sequence
\[ p > \text{field of characteristic } \]

for \( i \) is exact. Hence we have the following exact sequence
\[ (6.6) \]

As in [DI, (4.2.2) (c)], the following sequence
\[ \text{Proof.} \]

Corollary 6.5. The following is the log version of a generalization of [DI, (2.12)].
Proof. The proof is the same as those of [NaS] (2.2.14), (2.2.15). 

The following is the log version of a generalization of [DI] (2.12)].

\textbf{Corollary 6.5.} Let \( X \) be a projective SNCL scheme over the log point \( s \) of a perfect field of characteristic \( p > 0 \). Let \( D \) be a (relative) SNCD on \( X/s \). Let \( E \) be a (relative) SNCD on \( X/s \) such that \( D + E \) is also a (relative) SNCD on \( X/s \). Assume that \( \mathcal{O}_{\hat{X}}(\hat{E}) \) is an ample invertible \( \mathcal{O}_{\hat{X}} \)-module. Assume that \( X(D)/s \) and \( E(D)/s \) lift to \( W_2(s) \). For simplicity of notation, denote \( E(D \cap E) \) and \( E^{(k)}(D \cap E^{(k)}) \) by \( E(D) \) and \( E^{(k)}(D) \), respectively. Let \( a: E^{(1)}(D) \to E^{(2)}(D) \) be the natural morphism. Set \( K(E(D))^\bullet := \ker(\Omega_{E^{(1)}(D)/s} \to a_*(\Omega_{E^{(2)}(D)/s}^\bullet)). \) Then the following hold:

1. The restriction morphism
\[ H^q_{\text{dR}}(X(D)/s) \to H^q(E^{(1)}, K(E(D))^\bullet) \]
is an isomorphism for \( q < \min\{d, p\} - 1 \) and injective for \( q = \min\{d, p\} - 1 \).

2. The restriction morphism
\[ H^j(X(D), \Omega_X^j(D)/s) \to H^j(E^{(1)}, K(E(D))^\bullet) \]
is an isomorphism for \( i + j < \min\{d, p\} - 1 \) and injective for \( i + j = \min\{d, p\} - 1 \).

Proof. (1): (The following proof includes a correction of the proof of [DI] (2.12)] (see [Di1] below). As in [DI] (4.2.2) (c)], the following sequence
\[ 0 \to \Omega_X^{i(j)}(D+E)/s(-E) \to \Omega_X^i(D)/s \to \Omega_X^{i(j)}(D)/s \to \Omega_X^{i(j)}(D)/s \to \cdots \]
is exact. Hence we have the following exact sequence
\[ 0 \to \Omega_X^{i(j)}(D+E)/s(-E) \to \Omega_X^i(D)/s \to K(E(D))^\bullet \to 0. \]

It suffices to prove that \( H^q(X, \Omega_X^i(D+E)/s(-E)) = 0 \) for \( q < \min\{d, p\} \). By the following spectral sequence
\[ (6.5.1) \]

it suffices to prove that \( H^j(X, \Omega_X^i(D+E)/s(-E)) = 0 \) for \( i + j < \min\{d, p\} \). This is a special case of (6.5.1).

(2) As in the proof of (1), it suffices to prove that \( H^j(X, \Omega_X^i(D+E)/s(-E)) = 0 \) for \( i + j < \min\{d, p\} \). We have already proved this in the proof of (1).
Remark 6.6. (cf. the proof of [No, Theorem 1]) (1) Let the notations be as in [DI (2.12)]. There is an elementary error in the proof of [loc. cit.] because there does not exist complexes $\Omega_X^\bullet(-D)$ and $\Omega_D^\bullet(-D)$ in [loc. cit.]. Consequently we do not have an exact sequence

$$(6.6.1) \quad 0 \rightarrow \Omega_X^\bullet(-D) \rightarrow \Omega_X^\bullet(\log D)(-D) \rightarrow \Omega_D^{\bullet-1}(-D) \rightarrow 0.$$ 

of complexes in [loc. cit.].

The correction of the proof is easy. We have only to use the following spectral sequence (6.6.2) and the following exact sequence (6.6.3) and the following vanishing (6.6.4) (which follows from [DI, (2.8)]):

$$E_1^{ij} = H^j(X, \Omega^i_{X/\kappa}(\log D)(-D)) = \Rightarrow H^{i+j}(X, \Omega^i_{X/\kappa}(\log D)(-D)).$$

$$(6.6.3) \quad 0 \rightarrow \Omega^i_{X/\kappa}(-D) \rightarrow \Omega^i_{X/\kappa}(\log D)(-D) \rightarrow \Omega^i_{D/\kappa}(-D|D) \rightarrow 0.$$ 

$$(6.6.4) \quad H^j(X, \Omega^i_{X/\kappa}(-D)) = 0 = H^j(D, \Omega^{i-1}_{D/\kappa}(-D|D)) = 0 \quad \text{for} \quad i + j < \min\{d, p\}.$$ 

(2) As in the proof of (6.5), to prove [DI (2.8)], one can also use the theory for log de Rham complex in [DI 4.2].

The following is a generalization of Norimatsu’s results [No, Theorem 2, Corollary].

Corollary 6.7. Let the notations be as in (6.5.10). Assume that $a$ in (5.10) is equal to 0 and denote $T$ by $s$. Let $Z/s$ be a projective SNCL scheme. Let $D$ and $E$ be SNCD’s on $Z/s$ such that $D + E$ is also an SNCD on $Z/s$. Assume that $\mathcal{O} \mathcal{Z}_E$ is an ample invertible $\mathcal{O}_Z$-module. Let $a: E^{(1)}(D) \rightarrow E^{(2)}(D)$ be the natural morphism. Let $K(E(D))^\bullet := \ker(\Omega^{(1)}_{E^{(1)}(D)/s} \rightarrow a_*(\Omega^{(2)}_{E^{(2)}(D)/s}))$ be a complex defined similarly as in (5.5). Then the following hold:

(1) The restriction morphism

$$H^q_{\text{dr}}(Z(D)/s) \rightarrow H^q(E^{(1)}, K(E(D))^\bullet)$$

is an isomorphism for $q < d - 1$ and injective for $q = d - 1$.

(2) The restriction morphism

$$H^i(Z(D), \Omega^i_{Z(D)/s}) \rightarrow H^i(E^{(1)}, K(E(D))^i)$$

is an isomorphism for $i + j < d - 1$ and injective for $i + j = d - 1$.

Proof. The proof is an analogue of the proof of [IKN (7.1.2)]. \hfill \square

Corollary 6.8 (Log weak Lefschetz theorem in characteristic 0). Let the notations be as in (6.7). Assume that $D = \emptyset$ and $E^{(2)} = \emptyset$. Then the following hold:

(1) The following pull-back morphism by the inclusion $i: E \rightarrow Z$

$$i^*: H^q_{\text{dr}}(Z/s) \rightarrow H^q_{\text{dr}}(E/s)$$

is an isomorphism for $q < d - 1$ and injective for $q = d - 1$. 

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(2) Assume furthermore that $K = \mathbb{C}$. Let $Z^\log$ be the Kato-Nakayama space of $Z$ with natural morphism $Z \mapsto S^1$ ([KaN (1.2)]). Let $\mathbb{R} \ni t \mapsto \exp(2\pi \sqrt{-1}t) \in S^1$ be the universal cover of $S^1$ and set $Z_\infty := Z^\log \times_{S^1} \mathbb{R}$ ([Us]). The following pull-back morphism by the inclusion $\iota: E \hookrightarrow Z$

$$\iota^*: H^q(Z_\infty, \mathbb{Q}) \to H^q(E_\infty, \mathbb{Q})$$

is an isomorphism of mixed Hodge structures for $q < d - 1$ and a strictly injective morphism of mixed Hodge structures for $q = d - 1$.

**Proof.** (1): (1) is a special case of [D2].

(2): By [FN] the morphism $\iota^*$ is a morphism of mixed Hodge structures. Hence (2) follows (1) and theory of mixed Hodge structures in [D2].

**Remark 6.9.** (1) In [Nakk4 (9.14)] we have proved the log hard Lefschetz theorem over $\mathbb{C}$. The result is as follows.

Assume that $\mathfrak{s} = \text{Spec}(\mathbb{C})$. Let $Z/\mathfrak{s}$ be a projective SNCL variety. Let $\lambda_\infty := c_{1,\infty}(\mathcal{L}) \in H^2(Z_\infty, \mathbb{Q})$ be the log cohomology class of an ample invertible $\mathcal{O}_Z$-module $\mathcal{L}$. Then the left cup product of $\lambda_\infty$ ($j \geq 0$)

$$\lambda^j_\infty: H^{d-j}(Z_\infty, \mathbb{Q}) \to H^{d+j}(Z_\infty, \mathbb{Q})(j)$$

(6.9.1) is an isomorphism of mixed Hodge structures. We have proved this theorem by using a result of M. Saito ([Sai (4.2.2)]). Let

$$\delta_E: H^0_{\text{dR}}(E/\mathfrak{s}) \to H^2_{\text{dR}}(X/\mathfrak{s})(1).$$

be the morphism defined in [Nakk4 (10.1.2)]. This morphism induces the following morphism

$$\iota_*: H^q_{\text{dR}}(E/\mathfrak{s}) \to H^q_{\text{dR}}(X/\mathfrak{s})(1).$$

By [Nakk4 (10.1.3)], the composite morphism

$$\iota_* \iota^*: H^q_{\text{dR}}(X/\mathfrak{s}) \to H^q_{\text{dR}}(E/\mathfrak{s}) \to H^{q+2}_{\text{dR}}(X/\mathfrak{s})(1).$$

(6.9.4) is the cup product with $\lambda_\infty \cup (?)$. Hence we obtain [6.8] (2) and (1) by the hard Lefschetz theorem above as in [KM II Corollary].

(2) We would like to lay emphasis on the algebraic nature of the proof of [6.8] as in [DI].

Let us go back to the case $\text{ch}(k) = p > 0$. Let $E$ be an SNCD on $X$ such that $E^{(2)} = 0$. Let $q$ be a nonnegative integer. For a proper log smooth scheme $Y/\mathcal{W}$, set $H^q_{\text{dR}}(Y/\mathcal{W})$ be the log crystalline cohomology of $Y/\mathcal{W}$ ([Kk1]). By the works in [Mo, Nakk3 and Nakk7], $H^q_{\text{crys}}(X/\mathcal{W}(s))$ and $H^q_{\text{crys}}(E/\mathcal{W}(s))$ have the weight filtrations $P$'s. Set $K_0 := \text{Frac}(\mathcal{W})$. For a module $M$ over $\mathcal{W}$, set $M_{K_0} := M \otimes_{\mathcal{W}} K_0$. Let $\iota: E \hookrightarrow X$ be the closed immersion. By a general theorem about the strict compatibility of the pull-back of a morphism of proper SNCL schemes in [Nakk7 (5.4.7)] (see [6.12] below for the statement), the pull-back of $\iota$

$$\iota^*_q: H^q_{\text{crys}}(X/\mathcal{W}(s))_{K_0} \to H^q_{\text{crys}}(E/\mathcal{W}(s))_{K_0} \quad (q \in \mathbb{Z})$$

(6.9.5) is a strict filtered morphism with respect to the $P$'s.

As to the log weak Lefschetz conjecture [13S], we prove the following stimulated by the work of Berthelot ([B1]) in this article:
Theorem 6.10 (Log weak Lefschetz theorem in log crystalline cohomologies). Let the notations be as in \((6.5)\). Assume that \(E^{(2)} = \emptyset\). Let \(\iota : E(D) \hookrightarrow X(D)\) be the closed immersion. Then the following hold:

(1) The pull-back

\[
\iota_{\text{crys}}^* : H^q_{\text{crys}}(X(D)/\mathcal{W}(s)) \rightarrow H^q_{\text{crys}}(E(D)/\mathcal{W}(s)) \quad (q \in \mathbb{Z})
\]

is an isomorphism if \(q < \min\{d, p\} - 1\) and injective for \(q = \min\{d, p\} - 1\) with torsion free cokernel.

(2) Assume that \(D = \emptyset\). Then the morphism \((6.10.1)\) modulo torsion is a filtered isomorphism for \(q < \min\{d, p\} - 1\) and strictly injective for \(q = \min\{d, p\} - 1\).

(3) Assume that \(D = \emptyset\). Then the morphism \((6.10.1)\) modulo torsion is a filtered isomorphism for \(q < d - 1\) and strictly injective for \(q = d - 1\).

(4) Let the assumption be as in (3). Assume that \(D = \emptyset\). Then the morphism \((6.10.1)\) modulo torsion is a filtered isomorphism for \(q < d - 1\) and strictly injective for \(q = d - 1\).

Proof. The proof of (3) is slightly simpler than that in \([B1]\); the proof of (3) corrects the proof in \([B1]\).

(1): Set \(m := \min\{d, p\} - 1\). Let \(K^*\) be the mapping cone of the following morphism

\[
\iota_{\text{crys}}^* : R\Gamma_{\text{crys}}(X(D)/\mathcal{W}(s)) \rightarrow R\Gamma_{\text{crys}}(E(D)/\mathcal{W}(s))).
\]

Then it suffices to prove that \(H^q(K^*) = 0\) for \(q < m\) and that \(H^m(K^*)\) has no nontrivial torsion. By the universal coefficient theorem

\[
0 \rightarrow H^q(K^*) \otimes_{\mathcal{W}} \kappa \rightarrow H^q(K^* \otimes_{\mathcal{W}} \kappa) \rightarrow \text{Tor}_1^W(H^{q+1}(K^*), \kappa) \rightarrow 0,
\]

it suffices to prove that \(H^q(K^* \otimes_{\mathcal{W}} \kappa)\) is the mapping cone of the following morphism

\[
\iota_{\text{crys}}^* : R\Gamma_{\text{dR}}(X(D)/\kappa) \rightarrow R\Gamma_{\text{dR}}(E(D)/\kappa).
\]

Hence (1) follows from the following exact sequence

\[
\cdots \rightarrow H^q_{\text{dR}}(X(D)/\kappa) \rightarrow H^q_{\text{dR}}(E(D)/\kappa) \rightarrow H^q_{\text{dR}}(K^* \otimes_{\mathcal{W}} \kappa) \rightarrow \cdots
\]

and \((6.5)\) (1).

(2): (2) follows from (1) and \((6.12)\) below.

(3): Set \(e := \deg \hat{E}\). Since \(\text{Ker}(\Omega^i_{X/s} \rightarrow \Omega^i_{E/s}) = \Omega^i_{X/s}(\log E)(-E)\), it suffices to prove that \(H^q(X, \Omega^i_{X/s}(\log E)(-E)) = 0\) for \(q < d\) by the proof of (1). Let \(i\) and \(j\) be nonnegative integers. By \((6.5.1)\), it suffices to prove that \(H^j(X, \Omega^i_{X/s}(\log E)(-E)) = 0\) for \(i + j = d\). By the following sequence

\[
0 \rightarrow \Omega^i_{X/s}(-E) \rightarrow \Omega^i_{X/s}(\log E)(-E) \rightarrow H^{i-1}_{\text{Res}}(\Omega^i_{E/s}(\log E)(-E)) \rightarrow 0,
\]
it suffices to prove that $H^j(X, \Omega^i_{X/s}(-E)) = 0$ for $i + j < d$ and $H^j(E, \Omega^i_{E/s}(-E)) = 0$ for $i + j < d - 1$. By Serre’s theorem [EGA III-1 (2.2.1)], $H^{d-j}(X, \Omega^{d-1-i}_{X/s}(E)) = 0$ for $i + j < d$ if $e$ is large enough. Hence $H^j(X, \Omega^i_{X/s}(-E)) = 0$ for $i + j < d$ by the log Serre duality of Tsuji. The rest is to prove that $H^j(E, \Omega^i_{E/s}(-E)) = 0$ for $i + j < d - 1$ if $e$ is large enough. Though this is the dual of $H^{d-1-j}(E, \Omega^{d-1-i}_{E/s}(E)) = 0$ for $i + j < d - 1$, the vanishing of this cohomology is nontrivial since $\Omega^{d-1-i}_{E/s}$ depends on $E/s$.

Set $\mathcal{J} : = \mathcal{O}_X(-E)$. Because $E/s$ is log smooth, the following second fundamental exact sequence in [Nas (2.1.3)]

$$\mathcal{J}/\mathcal{J}^2 \rightarrow \Omega^1_{X/s} \otimes_{\mathcal{O}_X} \mathcal{O}_E \rightarrow \Omega^1_{E/s} \rightarrow 0$$

becomes the following exact sequence

$$0 \rightarrow (\mathcal{J}/\mathcal{J}^2) \rightarrow \Omega^1_{X/s} \otimes_{\mathcal{O}_X} \mathcal{O}_E \rightarrow \Omega^1_{E/s} \rightarrow 0.$$ 

Because $\mathcal{J}/\mathcal{J}^2 = \mathcal{O}_E(-E)$, this sequence is equal to the following exact sequence

$$0 \rightarrow \mathcal{O}_E(-E) \rightarrow \Omega^1_{X/s} \otimes_{\mathcal{O}_X} \mathcal{O}_E \rightarrow \Omega^1_{E/s} \rightarrow 0.$$ 

Hence we have the following exact sequences

$$0 \rightarrow \Omega^{i-1}_{X/s} \otimes_{\mathcal{O}_X} \mathcal{O}_E(-2E) \rightarrow \Omega^i_{X/s} \otimes_{\mathcal{O}_X} \mathcal{O}_E(-E) \rightarrow \Omega^i_{E/s}(-E) \rightarrow 0.$$ 

and

$$0 \rightarrow \Omega^{i-1}_{X/s} \otimes_{\mathcal{O}_X} \mathcal{O}_E(-mE) \rightarrow \Omega^i_{X/s} \otimes_{\mathcal{O}_X} \mathcal{O}_E(-mE) \rightarrow \Omega^i_{E/s}(-mE) \rightarrow 0.$$ 

Hence we have the following exact sequence

$$\cdots \rightarrow H^j(E, \Omega^{i-1}_{X/s} \otimes_{\mathcal{O}_X} \mathcal{O}_E(-2E)) \rightarrow H^j(E, \Omega^i_{X/s} \otimes_{\mathcal{O}_X} \mathcal{O}_E(-E)) \rightarrow H^j(E, \Omega^i_{E/s}(-E)) \rightarrow \cdots.$$ 

Thus it suffices to prove that $H^j(E, \Omega^i_{X/s} \otimes_{\mathcal{O}_X} \mathcal{O}_E(-mE)) = 0$ for $i + j < d - 1$ for $m \in \mathbb{Z}_{\geq 1}$. By the following exact sequence

$$(6.10.2) \quad 0 \rightarrow \Omega^i_{X/s}(-(m+1)E) \rightarrow \Omega^i_{X/s}(-mE) \rightarrow \Omega^i_{X/s} \otimes_{\mathcal{O}_X} \mathcal{O}_E(-mE) \rightarrow 0,$$

it suffices to prove that

$$H^j(X, \Omega^i_{X/s}(-(mE))) = 0 = H^{j+1}(X, \Omega^i_{X/s}(-(m+1)E)).$$

Because $H^j(X, \Omega^i_{X/s}(-mE))$ and $H^{j+1}(X, \Omega^i_{X/s}(-(m+1)E))$ are the duals of $H^{d-j}(X, \Omega^{d-1-i}_{X/s}(mE))$ and $H^{d-j-1}(X, \Omega^{d-1-i}_{X/s}(mE))$ by the log Serre duality of Tsuji, the vanishing of the latter cohomologies follows from Serre’s theorem if $e$ is large enough.

(4): (4) follows from (3) and (6.12) below.

**Remark 6.11.** (1) Let the notations be as in [BD]. There is a gap in the proof of «théorème de Lefschetz faible» in [loc. cit.] because the sheaf $\Omega^i_{Y/k}$ in [loc. cit.] depends on $Y$: it is not clear that $H^q(Y, \Omega^i_{Y/k}(Y)) = 0$ for $q > 0$ even if the degree of $Y$ is large enough. Strictly speaking, the proof of the weak and the hard Lefschetz theorems in [KM] for the crystalline cohomology is also incomplete because it depends on Berthelot’s result.

(2) In the Appendix we give an easy proof of the weak Lefschetz theorem in [KM] by using a theory of rigid cohomology of Berthelot.
In this case we say that $Y$ of $n$ of positive integers $X/S$ be the pull-back of the relative Frobenius morphism for a quasi-$Y$ the second named author ([Y1]). We give two types of log Kodaira vanishing theorems in this section we give a generalization of the definition of quasi-$Y$-split projective log smooth scheme. These are generalizations of Mehta and Ramanathan’s vanishing theorems for $F$-split varieties in [MR]. One of our log Kodaira vanishing theorems for the log scheme is a generalization of the Kodaira vanishing theorem in [Y1]; the other of them is much stronger than the log Kodaira vanishing theorem for the log scheme. The proof of our log Kodaira vanishing theorems are harder than those of Mehta and Ramanathan’s vanishing theorems. In this section we also give a generalization of the lifting theorem of quasi-$F$-split varieties in [Y1].

The following definition (1) (resp. (2)) is a relative version of the definition due to the second named author of this article (resp. Mehta and Ramanathan).

**Definition 7.1.** Let $Y \to T_0$ be a morphism of schemes of characteristic $p > 0$. Let $F_{T_0}: T_0 \to T_0$ be the $p$-th power Frobenius endomorphism of $T_0$. Set $Y': = Y \times_{T_0, F_{T_0}} T_0$.

(1) (cf. [Y1, (4.1)]) Let $F := F^*_n: W_n(O_{Y'}) \to F_n(W_n(O_Y)) = F_n(W_n(O_{Y'}))$ be the pull-back of the relative Frobenius morphism $F_n: W_n(Y) \to W_n(Y')$ of $W_n(Y)/T$. (This is a morphism of $W_n(O_{Y'})$-modules.) Let $n_0$ be the minimum of positive integers $n$’s such that there exists a morphism $\rho: F_n(W_n(O_Y)) \to O_{Y'}$ of $W_n(O_{Y'})$-modules such that $\rho \circ F_n: W_n(O_{Y'}) \to O_{Y'}$ is the natural projection.

In this case we say that $Y$ is quasi-$F$-split. If $Y/T_0 = \hat{X}/\hat{S}_0$ for a relative log scheme $X/S_0$, then we say that $X$ is quasi-$F$-split by abuse of terminology. (If there does not exist $n$, then set $n_0 := \infty$.) Note that $Y' = \hat{X}$ (not necessarily equal to $\hat{X}$) in this case. We call $n_0$ the quasi-$F$-split height and denote it by $h_F(Y/T_0)$. If $Y/T_0 = \hat{X}/\hat{S}_0$ for a relative log scheme $X/S_0$ of characteristic $p > 0$, then we denote $h_F(Y/T_0)$ by $h_F(X/S_0)$ by abuse of notation.

(2) (cf. [MR, Definition 2]) If $n_0 = 1$ in (1), then we say that $Y/T_0$ is $F$-split. If $Y/T_0 = \hat{X}/\hat{S}_0$ for a relative log scheme $X/S_0$, we say that $X/S_0$ is $F$-split by abuse of terminology.

**Remark 7.2.** (1) Assume that $T_0$ is perfect. Let $F: W_n(O_Y) \to F_n(W_n(O_Y))$ be the pull-back of the absolute Frobenius endomorphism. Because $Y' \to Y$, $h_F(Y/T_0)$ is equal to the minimum of positive integers $n$’s such that there exists a morphism...
Let \( \rho: F_*(W_n(O_Y)) \rightarrow O_Y \) of \( W_n(O_Y) \)-modules such that \( \rho \circ F: W_n(O_Y) \rightarrow O_Y \) is the natural projection. This is the original definition in [Y1] (4.1) in the case \( T_0 = \text{Spec}(\kappa) \).

(2) Let the notations be as in [Y1]. If there exists a morphism \( \rho: F_{n*}(W_n(O_Y)) \rightarrow O_Y \), for \( n \in \mathbb{Z}_{\geq 1} \) such that \( \rho \circ F_{n*}: W_n(O_Y) \rightarrow O_Y \) is the natural projection, then, for all \( m \geq n \), there exists a morphism \( \rho': F_{m*}(W_m(O_Y)) \rightarrow O_Y \) such that \( \rho' \circ F_{m*}: W_m(O_Y) \rightarrow O_Y \) is the natural projection. Indeed, we have only to set \( \rho' := \rho \circ R^{n-m} \), where \( R: F_{n*}(W_n(O_Y)) \rightarrow F_{n-1*}(W_{n-1}(O_Y)) \) \((m+1 \leq l \leq n)\) is the projection.

The following easy lemma is necessary for the theorem (7.6) below.

**Lemma 7.3.** Let \( q \) be a nonnegative integer. Let \( X/S_0 \) be as in [Y1]. Assume that \( S_0 \) is perfect and that \( X \) is a log smooth scheme of Cartier type over \( S_0 \). Let \( M \) be an invertible \( O_X \)-module. Let \( i \) be a positive integer and let \( q \) be a nonnegative integer. Let \( g: X \rightarrow Y \) be a morphism of schemes over \( S_0 \). Assume that the Frobenius endomorphism \( F_Y: Y \rightarrow Y \) of \( Y \) is finite. If \( R^q g_*(B^n_i X/S_0 \otimes_{O_X} M^\otimes p^n) = 0 \) for \( \forall e \geq e_0 \), then \( R^q g_*(B^n_{n+1} X/S_0 \otimes_{O_X} M^\otimes p^n) = 0 \) for \( \forall e \geq e_0 \) and \( \forall n \geq 1 \).

**Proof.** Let \( F: X \rightarrow X \) be the \( p \)-th power Frobenius endomorphism. Consider the following exact sequence of \( O_X \)-modules:

\[
0 \rightarrow F_*(B_{n-1}^1 X/S_0) \rightarrow B_n^1 X/S_0 \rightarrow B_1^1 X/S_0 \rightarrow 0 \quad (n \geq 1).
\]

By the projection formula and noting that \( \tilde{F} \) and \( F_Y \) are finite morphisms, we have the following formula for a quasi-coherent \( O_X \)-module \( F \) and an invertible \( O_X \)-module \( N \):

\[
R^q g_*(F_*(F \otimes_{O_X} N^\otimes p)) = R^q g_*(F_*(F \otimes_{O_X} N^\otimes p)).
\]

Hence we have the following exact sequence

\[
F_Y R^q g_*(B_{n-1}^1 X/S_0 \otimes_{O_X} M^\otimes p^{n-1}) \rightarrow R^q g_*(B_n^1 X/S_0 \otimes_{O_X} M^\otimes p^n) \rightarrow R^q g_*(B_1^1 X/S_0 \otimes_{O_X} M^\otimes p^n).
\]

Induction on \( n \) tells us that \( R^q g_*(B_n^1 X/S_0 \otimes_{O_X} M^\otimes p^n) = 0 \) for \( \forall e \geq e_0 \) and \( \forall n \geq 1 \). \( \square \)

Next we construct key exact sequences as in the proof in [Y2] (3.1).

Let the notations be as in (7.3). Assume that \( F_Y \) is finite. (We do not assume that there exists a nonnegative integer \( e_0 \) such that \( R^q g_*(B^n_{1} X/S_0 \otimes_{O_X} M^\otimes p^n) = 0 \) for \( \forall e \geq e_0 \).) Push out the exact sequence (3.6.1) by the morphism \( R^{n-1}: W_n(O_X) \rightarrow O_X \). Then we have the following exact sequence

\[
0 \rightarrow O_X \rightarrow E_n \rightarrow B_n^1 X/S_0 \rightarrow 0,
\]
where $\mathcal{E}_n := \mathcal{O}_X \oplus \mathcal{W}_n((\mathcal{O}_X), F \mathcal{W}_n(\mathcal{O}_X))$. Consider the following diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & F_*(B_{n-1}1 \Omega^1_{X/S_0}) & \rightarrow & B_n \Omega^1_{X/S_0} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \text{C}^{n-1} & & \\
0 & \rightarrow & \mathcal{O}_X & \rightarrow & \mathcal{E}_n & \rightarrow & B_n \Omega^1_{X/S_0} \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \mathcal{O}_X & \rightarrow & \mathcal{E}_1 = F_*(\mathcal{O}_X) & \rightarrow & \mathcal{E}_n \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \mathcal{O}_X & \rightarrow & B_1 \Omega^1_{X/S_0} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0
\end{array}
\]

of $\mathcal{O}_X$-modules with exact rows and exact columns. The snake lemma tells us that $\text{Ker}(\mathcal{E}_n \rightarrow \mathcal{E}_1) = F_*(B_{n-1}1 \Omega^1_{X/S_0})$. Hence we have the following exact sequence

(7.3.5) \[0 \rightarrow F_*(B_{n-1}1 \Omega^1_{X/S_0}) \rightarrow \mathcal{E}_n \rightarrow F_*(\mathcal{O}_X) \rightarrow 0.\]

**Definition 7.4.** We call the exact sequence (7.3.3) (resp. (7.3.5)) of $\mathcal{O}_X$-modules the **fundamental exact sequence of Type I** (resp. **fundamental exact sequence of Type II**) of $X/S_0$. (It maybe better to call (7.3.3) the modified log Serre exact sequence.)

Let $\mathcal{M}$ be an invertible $\mathcal{O}_X$-module. Then we have the following exact sequences:

(7.4.1) \[0 \rightarrow \mathcal{M} \rightarrow \mathcal{E}_n \otimes_{\mathcal{O}_X} \mathcal{M} \rightarrow B_n \Omega^1_{X/S_0} \otimes_{\mathcal{O}_X} \mathcal{M} \rightarrow 0,\]

(7.4.2) \[0 \rightarrow F_*(B_{n-1}1 \Omega^1_{X/S_0}) \otimes_{\mathcal{O}_X} \mathcal{M} \rightarrow \mathcal{E}_n \otimes_{\mathcal{O}_X} \mathcal{M} \rightarrow F_*(\mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{M} \rightarrow 0\]

and

(7.4.3) \[0 \rightarrow \mathcal{M} \rightarrow F_*(\mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{M} \rightarrow B_1 \Omega^1_{X/S_0} \otimes_{\mathcal{O}_X} \mathcal{M} \rightarrow 0.\]

By (7.4.2), (7.3.3) and (7.4.3), we have the following exact sequences:

(7.4.4) \[
\cdots \rightarrow F_Y r^{t-1} g_* (\mathcal{M}^{\otimes p}) \rightarrow F_Y r^t g_* (B_{n-1} \Omega^1_{X/S_0} \otimes_{\mathcal{O}_X} \mathcal{M}^{\otimes p}) \rightarrow R^q g_* (\mathcal{E}_n \otimes_{\mathcal{O}_X} \mathcal{M}) \\
\rightarrow F_Y r^q g_* (\mathcal{M}^{\otimes p}) \rightarrow \cdots \quad (q \in \mathbb{N})
\]

and

(7.4.5) \[
\cdots \rightarrow R^q g_* (\mathcal{M}) \rightarrow F_Y r^q g_* (\mathcal{M}^{\otimes p}) \rightarrow R^q g_* (B_1 \Omega^1_{X/S_0} \otimes_{\mathcal{O}_X} \mathcal{M}) \\
\rightarrow R^{q+1} g_* (\mathcal{M}) \rightarrow \cdots \quad (q \in \mathbb{N}).
\]

Now set $h := h_F(X/S_0)$ and assume that $h < \infty$ Then we have the following decomposition by (7.2) (2) and (7.4.1):

(7.4.6) \[R^q g_* (\mathcal{E}_n \otimes_{\mathcal{O}_X} \mathcal{M}) = R^q g_* (\mathcal{M}) \oplus R^q g_* (B_n \Omega^1_{X/S_0} \otimes_{\mathcal{O}_X} \mathcal{M}) \quad (n \geq h, q \in \mathbb{N}).\]

The following lemma is a key one for (7.6) and (7.7) below: (1) (resp. (2)) in this lemma is necessary for the proof of (7.6) (resp. 7.7).
Lemma 7.5. Let the notations be as in (7.3). Assume also that \( h_F(X/S_0) < \infty \). Let \( \mathcal{L} \) be an invertible \( \mathcal{O}_X \)-module. Let \( e_0 \) be a fixed positive integer. Then the following hold:

1. Let \( q_0 \) be a fixed nonnegative integer. If \( R^q g_*(\mathcal{L}^{\otimes p^r}) = 0 \) and \( R^q g_*(B_1 \Omega^1_{X/S_0} \otimes \mathcal{O}_X \mathcal{L}^{\otimes p^r}) = 0 \) for all \( e \geq e_0 \) and all \( q \geq q_0 \), then \( R^q g_*(\mathcal{L}) = 0 \) and \( R^q g_*(B_n \Omega^1_{X/S_0} \otimes \mathcal{O}_X \mathcal{L}) = 0 \) for all \( n \geq 1 \) and all \( q \geq q_0 \).

2. Let \( q \) be a fixed nonnegative integer. If, for all \( e \geq e_0 \), \( R^q g_*(\mathcal{L}^{\otimes p^r}) = 0 \) and if, for all \( e \geq e_0 \), there exists an integer \( n(e) \geq h-1 \) such that \( R^q g_*(B_{n(e)} \Omega^1_{X/S_0} \otimes \mathcal{O}_X \mathcal{L}^{\otimes p^r}) = 0 \), then \( R^q g_*(\mathcal{L}) = 0 \) and \( R^q g_*(B_{n(e)+1} \Omega^1_{X/S_0} \otimes \mathcal{O}_X \mathcal{L}) = 0 \).

Proof. (1): (Though the statement of (1) is different from [Y2] (4.1), (4.2)) and the following proof of (1) is a simplification of of [loc. cit.], the following proof is essentially the same as that of [loc. cit.]: the simplification is to focus on the vanishing of \( R^q g_*(B_1 \Omega^1_{X/S_0} \otimes \mathcal{O}_X \mathcal{L}^{\otimes p^r}) \) and not to consider the vanishing of \( R^q g_*(B_{1} \Omega^1_{X/S_0} \otimes \mathcal{O}_X \mathcal{L}^{\otimes p^r}) = 0 \) for other \( l \)'s as an assumption; this focus is possible by (7.3).)

Set \( h := h_F(X/S_0) \). For a fixed positive integer \( e_1 \) and for all \( e \geq e_1 \), consider the following two conditions:

(Hyp\(_1\)(\(e_1\))) \[ R^q g_*(\mathcal{L}^{\otimes p^r}) = 0 \quad (\forall q \geq q_0) \]

and

(Hyp\(_2\)(\(e_1\))) \[ R^q g_*(B_1 \Omega^1_{X/S_0} \otimes \mathcal{O}_X \mathcal{L}^{\otimes p^r}) = 0 \quad (\forall q \geq q_0). \]

By the assumption Hyp\(_i\)(\(e_1\)) \( (i = 1, 2) \) is satisfied for the case \( e_1 = e_0 \).

Now assume that Hyp\(_i\)(\(e_1\)) \( (i = 1, 2) \) holds. By (7.3)

(7.5.1) \[ R^q g_*(B_1 \Omega^1_{X/S_0} \otimes \mathcal{O}_X \mathcal{L}^{\otimes p^r}) = 0 \]

for all \( e \geq e_1 \), all \( n \geq 1 \) and all \( q \geq q_0 \). In particular, \( R^q g_*(B_{h-1} \Omega^1_{X/S_0} \otimes \mathcal{O}_X \mathcal{L}^{\otimes p^r}) = 0 \) for all \( e \geq e_1 \) and all \( q \geq q_0 \). By (7.4.4) in the case \( \mathcal{M} = \mathcal{L}^{\otimes p^{e-1}} \), we see that

(7.5.2) \[ R^q g_*(\mathcal{E}_h \otimes \mathcal{O}_X \mathcal{L}^{\otimes p^{e-1}}) = 0 \quad (\forall e \geq e_1, \forall q \geq q_0). \]

Hence, by (7.4.6) in the case \( \mathcal{M} = \mathcal{L}^{\otimes p^{e-1}} \),

(7.5.3) \[ R^q g_*(B_h \Omega^1_{X/S_0} \otimes \mathcal{O}_X \mathcal{L}^{\otimes p^{e-1}}) = 0 = R^q g_*(\mathcal{L}^{\otimes p^{e-1}}) \quad (\forall e \geq e_1, \forall q \geq q_0). \]

By (7.4.5) for the case \( \mathcal{M} = \mathcal{L}^{\otimes p^{e-1}} \), we see that

(7.5.4) \[ R^q g_*(B_1 \Omega^1_{X/S_0} \otimes \mathcal{O}_X \mathcal{L}^{\otimes p^{e-1}}) = 0 \quad (\forall e \geq e_1, \forall q \geq q_0). \]

By (7.5.3) and (7.5.4), we have proved that Hyp\(_i\)(\(e_1-1\)) \( (i = 1, 2) \) holds. Descending induction on \( e_1 \) shows (7.5).

(2): By (7.4.4) in the case \( \mathcal{M} = \mathcal{L}^{\otimes p^{e-1}} \) and \( n = n(e) + 1 \), \( R^q g_*(\mathcal{E}_{n(e)+1} \otimes \mathcal{O}_X \mathcal{L}^{\otimes p^{e-1}}) = 0 \). Since \( n(e) + 1 \geq h, R^q g_*(\mathcal{L}^{\otimes p^{e-1}}) = 0 = R^q g_*(B_{n(e)+1} \Omega^1_{X/S_0} \otimes \mathcal{O}_X \mathcal{L}^{\otimes p^{e-1}}) = 0 \) by (7.4.6). Continuing this process, we see that \( R^q g_*(\mathcal{L}) = 0 = R^q g_*(B_{n(e)+1} \Omega^1_{X/S_0} \otimes \mathcal{O}_X \mathcal{L}). \)

The following is the relative log version of [Y2] Theorem 4.1, which is a nontrivial generalization of [MR] Proposition 1].
Theorem 7.6. Let the notations be as in (7.3). Assume that the structural morphism \( g: \tilde{X} \to Y \) is projective. Let \( L \) be a relatively ample line bundle on \( \tilde{X} \) with respect to \( g \). Assume also that \( h_F(X/S_0) < \infty \). Then \( R^qg_*(L) = 0 \) and \( R^qg_*(B_1\Omega^1_{X/S_0} \otimes_{O_X} L) = 0 \) for any \( q \geq 1 \) and any \( n \geq 1 \).

Proof. By Serre’s theorem ([EGA III-1 (2.2.1)]), there exists a positive integer \( m_0 \) such that \( R^qg_*(L^{\otimes m}) = 0 \) and \( R^qg_*(B_1\Omega^1_{X/S_0} \otimes_{O_X} L^{\otimes m}) = 0 \) for \( \forall m \geq m_0 \) and \( \forall q \geq 1 \). By considering the case where \( q_0 = 1 \) and \( e_0 \) in (7.5) is a large integer, we immediately obtain (7.6).

The following vanishing theorem is much stronger than (5.9) in the case \( i = 0 \) and \( i = d \) for a projective log smooth variety a quasi-\( F \)-split height in characteristic \( p > 0 \). The following (1) is also a nontrivial generalization of Kodaira vanishing theorem in [MR, Proposition 2].

Theorem 7.7. Let the notations be as in (7.3). Assume moreover that \( S_0 \) is equal to the log point \( s \) of a perfect field of characteristic \( p > 0 \). Assume that \( \tilde{X} \) is of pure dimension \( d \). Assume also that \( h_F(X/s) < \infty \). Let \( L \) be an ample invertible \( O_X \)-module. Set \( h := h_F(X/s) \). Then the following hold:

1. \( H^q(X, L^{\otimes(-1)}) = 0 \) for \( \forall q < d \).
2. \( H^q(X, \mathcal{I}_{X/s} \Omega^1_{X/s} \otimes_{O_X} L) = 0 \) for \( \forall q > 0 \).

Proof. By (5.4) and the log Serre duality of Tsuji, we have only to prove (1).

Let \( m \) be a positive integer. By (5.6),

\[
H^q(X, L^{\otimes(-m)}) \quad \text{and} \quad H^q(X, B_1\Omega^1_{X/s} \otimes_{O_X} L^{\otimes(-m)})
\]

are the duals of

\[
H^{d-q}(X, \mathcal{I}_{X/s} \Omega^1_{X/s} \otimes_{O_X} L^{\otimes m}) \quad \text{and} \quad H^{d-q}(X, \mathcal{H}om_{O_X}(B_1\Omega^1_{X/s}, \mathcal{I}_{X/s} \Omega^1_{X/s} \otimes_{O_X} L^{\otimes m})),
\]

respectively. By Serre’s theorem ([EGA III-1 (2.2.1)]), there exists a positive integer \( m_0 \) such that, for \( \forall m \geq m_0 \) and \( \forall q < d \) the latter cohomologies vanish. Hence there exists a positive integer \( e_0 \) such that, for \( \forall e \geq e_0 \) and \( \forall q < d \),

\[
H^q(X, L^{\otimes(-p^e)}) = 0
\]

and

\[
H^q(X, B_1\Omega^1_{X/s} \otimes_{O_X} L^{\otimes(-p^e)}) = 0.
\]

By (7.3)

\[
H^q(X, B_1\Omega^1_{X/s} \otimes_{O_X} L^{\otimes(-p^e)}) = 0 \quad (\forall l \geq 1).
\]

By (7.2) (2), \( H^q(X, L^{\otimes(-1)}) = 0 \) and \( H^q(X, B_1\Omega^1_{X/s} \otimes_{O_X} L^{\otimes(-1)}) = 0 \) for \( \forall l \geq h - 1 + e_0 \).}

The following problem seems very interesting (cf. [MS, Conjecture 1.1]):

Problem 7.8. Let \( Y \) be a projective log smooth integral scheme over a fine log scheme \( s \) whose underlying scheme is a field \( K \) of characteristic zero. Assume that \( \tilde{Y} \) is of pure dimension \( d \). Set \( p(Y, r) := \dim_K H^0(Y, r \otimes \Omega^d_Y) \) and \( \kappa(Y/s) := \lim_{r \to \infty} \frac{\log p(Y, r)}{\log r} \).
Assume that \( \kappa(Y/s) \leq 0 \). Let \( S \) be a fine log scheme whose underlying scheme is the spectrum of an algebra of finite type over \( \mathbb{Z} \) and let \( s \to S \) be a morphism of fine log schemes. Let \( Y \) be a projective log smooth integral scheme over \( S \) such that \( Y = \mathcal{Y} \times_S s \). Then does there exist a dense set of exact closed points \( T \) of \( S \) such that \( (\mathcal{Y}_t)^0 \) is quasi-\( F \)-split (or more strongly \( F \)-split) for every \( t \in T \)?

The following is the log version of a generalization of \([Y1, (4.4)]\).

**Theorem 7.9.** Let \( S_0 \) be a fine log scheme of characteristic \( p > 0 \). Assume that \( S_0 \) is perfect. Let \( S \) be a fine log scheme with exact closed immersion \( S_0 \to S \). Let \( \mathcal{I} \) be the ideal sheaf of this exact closed immersion. Assume that \( \mathcal{I} = \pi \mathcal{O}_S \) for a global section \( \pi \) of \( \mathcal{O}_S \) and that \( p\pi = 0 \) in \( \mathcal{O}_S \). Assume also that the morphism \( \mathcal{O}_{S_0} \to \mathcal{O}_S \) is a well-defined isomorphism. Assume that there exists a lift \( F_\mathcal{I} : S \to S \) of the Frobenius endomorphism \( F_{S_0} : S_0 \to S_0 \). Let \( Y \) be a (not necessarily proper) log smooth integral separated scheme over \( S_0 \). Assume that \( Y/S_0 \) is of Cartier type and that \( h_F(Y/S_0) < \infty \). Let \( F : Y \to Y \) be the absolute Frobenius endomorphism of \( Y \) over \( F_{S_0} \). Then there exists a log smooth integral scheme \( \mathcal{Y} \) over \( S \) such that \( \mathcal{Y} \times_S S_0 = Y \).

*Proof.* (The following proof is the log version of the proof of \([Y1, (4.4)]\).)

For simplicity of notation, we denote the \( p \)-th power Frobenius endomorphism \( W_n(Y) \to W_n(Y) \) by \( F_n \) for any \( n \geq 1 \). Push out the exact sequence \( (3.6.1;n) \) over \( Y/S_0 \), the invariant \( h_F(Y/S_0) \) is the minimum of positive integers \( n \)'s such that the exact sequence \( (7.9.1;n) \) is split. By \( (3.9) \) we have the following commutative diagram

\[
\begin{align*}
F_n(W_n(\mathcal{O}_Y)) &\xrightarrow{d_n} B_n\mathcal{O}_Y \\
\xrightarrow{R^{n-1}} &\xrightarrow{c^{n-1}} \\
F_n(\mathcal{O}_Y) &\xrightarrow{d} B_1\mathcal{O}_{Y_0/S_0}.
\end{align*}
\]

Because \( \mathcal{E}_n = F_n(W_n(\mathcal{O}_Y)) \oplus W_n(\mathcal{O}_Y) \mathcal{O}_Y \), we have the following commutative diagram of exact sequences by using \( (7.9.2) \):

\[
\begin{align*}
0 &\to \mathcal{O}_Y \to \mathcal{E}_n \xrightarrow{d_n} B_n\mathcal{O}_{Y/S_0} \\
&\quad \downarrow \quad \downarrow \quad \downarrow \\
0 &\to \mathcal{O}_Y \to F_n(\mathcal{O}_Y) \xrightarrow{d} B_1\mathcal{O}_{Y_0/S_0} \to 0.
\end{align*}
\]

Hence we have the following commutative diagram

\[
\begin{align*}
\operatorname{Ext}^1_Y(\mathcal{O}_{Y_0/S_0}, B_n\mathcal{O}_{Y/S_0}) &\xrightarrow{d_n} \operatorname{Ext}^2_Y(\mathcal{O}_{Y_0/S_0}, \mathcal{O}_Y) \\
\xrightarrow{c^{n-1}} &\xrightarrow{c^{n-1}} \\
\operatorname{Ext}^1_Y(\mathcal{O}_{Y_0/S_0}, B_1\mathcal{O}_{Y/S_0}) &\xrightarrow{d_1} \operatorname{Ext}^2_Y(\mathcal{O}_{Y_0/S_0}, \mathcal{O}_Y),
\end{align*}
\]

where \( d_n \) is the boundary morphism obtained by the exact sequence \( (7.9.1;n) \).
Now assume that \((7.9.1.1n)\) is split for a positive integer \(n\). (Since \(h_F(Y/S_0) < \infty\), the \(n\) exists.) Because the sequence \((7.9.1n)\) is split, \(\partial_n\) is the zero morphism. Because \(\text{obs}Y/(S_0 \subset S) = \partial_1(\text{obs}(Y,F)/(S_0 \subset S,F_0))\) by \((8.1.2)\) (3), it suffices to prove that \(\text{obs}(Y,F)/(S_0 \subset S,F_0) \in \text{Im}(C^{n-1})\). Because \(\text{obs}(Y,F)/(S_0 \subset S,F_0)\) is equal to the extension class of the following exact sequence

\[
\begin{align*}
0 & \rightarrow B_1 \Omega^1_{Y/S_0} \rightarrow Z_1 \Omega^1_{Y/S_0} \xrightarrow{C} \Omega^1_{Y/S_0} \rightarrow 0
\end{align*}
\]

(7.9.5)

by \((8.1.2)\) and because we have the following commutative diagram

\[
\begin{array}{ccc}
0 & \rightarrow & B_1 \Omega^1_{Y/S_0} \\
\downarrow C^{n-1} & & \downarrow C^{n-1} \\
0 & \rightarrow & B_1 \Omega^1_{Y/S_0} \rightarrow Z_1 \Omega^1_{Y/S_0} \rightarrow \Omega^1_{Y/S_0} \rightarrow 0
\end{array}
\]

(7.9.6)

we see that \(\text{obs}(Y,F)/(S_0 \subset S) \in \text{Im}(C^{n-1})\).

We complete the proof. \(\square\)

The following is one of results what we want to obtain:

**Corollary 7.10.** The conclusions of \((6.3)\) (1) and (2) hold for \(Y/S_0\).

\section{Lifts of certain log schemes over \(W_2\)}

In this section we give the log version of the main result in \([Y1]\).

The following is the log version of a generalization of \([Y1] (4.5)\).

**Theorem 8.1.** Let \(s\) be as in \((5.4)\). Let \(X\) be a proper log smooth, integral and saturated log scheme over \(s\) of pure dimension \(d\). Assume that \(X/s\) is of Cartier type and of vertical type. Assume also that the following three conditions hold:

\begin{itemize}
  \item[(a)] \(H^{d-1}(X,\mathcal{O}_X) = 0\) if \(d \geq 2\),
  \item[(b)] \(H^{d-2}(X,\mathcal{O}_X) = 0\) if \(d \geq 3\),
  \item[(c)] \(\Omega^d_{X/s} \simeq \mathcal{O}_X\).
\end{itemize}

Then \(h_F(X/\kappa) = h^d(X/\kappa)\).

**Proof.** (The following proof is the log version of the proof of \([Y1] (4.5)\).) Set \(h = h^d(X/\kappa)\). Let \(F: X \rightarrow X\) be the Frobenius endomorphism of \(X\). Consider the following exact sequence of \(\mathcal{O}_X\)-modules:

\[
\begin{align*}
0 & \rightarrow F_* (B_{n-1} \Omega^1_{X/s}) \rightarrow B_n \Omega^1_{X/s} \xrightarrow{C^{n-1}} B_1 \Omega^1_{X/s} \rightarrow 0.
\end{align*}
\]

(8.1.1)

Here note that the direct image \(F_*\) is necessary for \(B_{n-1} \Omega^1_{X/s}\) as in \([\text{loc. cit.}]\). Taking \(\text{Ext}^1_X(\ast,\mathcal{O}_X)\) of the exact sequence \((8.1.1)\), we have the following exact sequence

\[
\begin{align*}
\text{Ext}^1_X (B_1 \Omega^1_{X/s},\mathcal{O}_X) & \xrightarrow{C^{n-1}} \text{Ext}^1_X (B_{n-1} \Omega^1_{X/s},\mathcal{O}_X) \rightarrow \text{Ext}^1_X (F_* (B_{n-1} \Omega^1_{X/s}),\mathcal{O}_X) \\
& \rightarrow \text{Ext}^2_X (B_1 \Omega^1_{X/s},\mathcal{O}_X).
\end{align*}
\]

(8.1.2)

By Tsuji’s log Serre duality \((5.3)\), we have the following isomorphism

\[
\begin{align*}
\text{Ext}^q_X (B_n \Omega^1_{X/s},\mathcal{O}_X) & \simeq \text{Ext}^q_X (B_n \Omega^1_{X/s},\Omega^d_{X/s}) \simeq H^{d-q}(X, B_n \Omega^1_{X/s})^*.
\end{align*}
\]

(8.1.3)
where $\ast$ means the dual of a finite dimensional $\kappa$-vector space. Hence $\operatorname{Ext}_X^2(B_nX, o_X) = H^{d-2}(X, B_nX)^\ast = 0$ by (3.10.3) and we have the following exact sequence

\[(8.1.4)\]
\[
\operatorname{Ext}_X^1(B_1X, o_X) \xrightarrow{C^{n-1}_n} \operatorname{Ext}_X^1(B_nX, o_X) \rightarrow \operatorname{Ext}_X^1(F_n(B_{n-1}X, o_X), o_X) \rightarrow 0
\]
of $\kappa$-modules. By (8.10.2) and (8.13),

\[(8.1.5)\]
\[
\dim_n \operatorname{Ext}_X^1(B_1X, o_X) = 1
\]
and
\[
\dim_n \operatorname{Ext}_X^1(B_nX, o_X) = \min\{n, h-1\}.
\]

Since $F: \tilde{X} \rightarrow \tilde{X}$ is a finite morphism, we also have the following isomorphism

\[
\operatorname{Ext}_X^1(F_n(B_{n-1}X, o_X), o_X) = \operatorname{Ext}_X^1(F_n(B_{n-1}X, o_X), o_X) = H^{d-q}(X, F_n(B_{n-1}X, o_X))\ast
\]
\[
= H^{d-q}(X, B_nX, o_X)\ast.
\]
Hence

\[
\dim_n \operatorname{Ext}_X^1(F_n(B_{n-1}X, o_X), o_X) = \min\{n-1, h-1\}.
\]

First consider the case $e_1 = 0$. Then the following exact sequence

\[
0 \rightarrow o_X \xrightarrow{F} F_n(o_X) \rightarrow B_1X \rightarrow 0
\]
is split. Hence $H^q(X, F_n(o_X)) = H^q(X, o_X) \oplus H^q(X, B_1X) (q \in \mathbb{N})$. Since $\tilde{F}$ is finite, $H^q(X, F_n(o_X)) = H^q(X, o_X)$. Hence $H^q(X, B_1X) = 0$. (We can find this argument in [JR] (2.4.1) in the trivial log case.) In particular, $H^d-1(X, B_1X) = 0$. By (8.10.2) we see that $h = 1$.

Next consider the case $e_1 \neq 0$. Then $\operatorname{Ext}_X^1(B_1X, o_X) = \kappa e_1$ by (8.1.5). Because $C^{n-1}_n(e_1) = e_n$ by (6.13), we see that $e_n = 0$ if and only if the morphism $\operatorname{Ext}_X^1(B_nX, o_X) \rightarrow \operatorname{Ext}_X^1(B_{n-1}X, o_X)$ is an isomorphism by (8.1.4). Hence $e_n = 0$ if and only if $\min\{n-1, h-1\} = \min\{n, h-1\}$. This is equivalent to $h \leq n$. 

**Corollary 8.2.** (1.3) holds.

**Proof.** (8.2) immediately follows from (7.9) and (8.1).

By using the degeneration at $E_1$ of the log Hodge spectral sequence due to Kato ([Kk1] (4.12) (1)) or (5.1) and (5.3), we obtain the following:

**Corollary 8.3.** Let the assumptions be as in (8.1). Then (1.4) holds.

**Corollary 8.4.** (1) (1.7) holds.

(2) Let the notations and the assumptions be as in (1.6). Then $H^j(Y, L^{-1}) = 0$ for $j < d$.

**Proof.** This follows from (7.7) and (8.1).

The following is of independent interest:

**Corollary 8.5.** Let the notations and the assumptions be as in (1.3). Assume that $h^0(\tilde{X}/\kappa) \geq 2$. Then there does not exist a lift $\tilde{F}: \tilde{X} \rightarrow X$ over the Frobenius endomorphism $F_W(s): W_2(s) \rightarrow W_2(s)$ which is a lift of the Frobenius endomorphism of $X$.
Proof. If there exists the $\tilde{F}$ in (8.5), then [Nakk1, (3.2)] and [I3, (8.6), (8.8)] (cf. [CL, (4.3)]) tell us that $X/s$ is log ordinary, that is, $H^q(X, B\Omega^i_{X/s}) = 0$ for all $q$’s and $i$’s. By (8.10.1) in the case $n = 1$, $h^d(X/\kappa) = 1$. This contradicts the assumption $h^d(X/\kappa) \geq 2$.

Example 8.6. Let $X/s$ be a log $K3$ surface of type II ([Nakk2, §3]). By [RS, Theorem 1], we have the following spectral sequence

$E^{ij} = H^j(X^{(i)}), W(O_{X^{(i)}})) \Rightarrow H^{i+j}(X, W(O_X))$ (8.6.1)

obtained by the following exact sequence

$0 \rightarrow W(O_X) \rightarrow W(O_{X^{(0)}}) \rightarrow W(O_{X^{(1)}}) \rightarrow 0.$

By using this spectral sequence, it is easy to prove that $H^2(X, W(O_X)) \simeq H^1(E, W(O_E)).$

Assume that the double elliptic curve $E$ is supersingular. Then $h_F^d(\tilde{X}/\kappa) = h^2(\tilde{X}/\kappa) = 2$. Let $\mathcal{X}$ be a log smooth lift of $X$ over $W_2(s)$. Then there does not exist a lift $\tilde{F}: \mathcal{X} \rightarrow \mathcal{X}$ over $F_{W_2(s)}$ which is a lift of the Frobenius endomorphism of $X$.

Remark 8.7. Let $X/s$ be as in (8.1). Assume that dim $\tilde{X} \geq 2$. In the case where $h^d(\tilde{X}/\kappa) = 1$, we do not know whether there does not exist a lift $\tilde{F}: \mathcal{X} \rightarrow \mathcal{X}$ over $F_{W_2(s)}$ which is a lift of the Frobenius endomorphism of $X$ in general.

In the case where dim $\tilde{X} = 2$, there does not exist the lift $\tilde{F}$ above if the log structures of $\mathcal{X}$ and $s$ are trivial ([X, (3.3)]). (If $h^2(\tilde{X}/\kappa) \geq 2$, the proof of (8.3) gives us another proof of this fact.)

In the case dim $\tilde{X} = 1$ and the log structure of $X$ is nontrivial, one can prove that there exists a lift $\tilde{F}: \mathcal{X} \rightarrow \mathcal{X}$ over $F_{W_2(s)}$ which is a lift of the Frobenius endomorphism of $X$ ([Nakk8]).

By using (1.4), we obtain the following as in [LN].

Corollary 8.8. Assume that the log structures of $s$ and $X$ are trivial and that $\text{NS}(X)$ is $p$-torsion-free. Then $H^0(X, \Omega^1_{X/\kappa}) = 0$. 62
Appendix

Yukiyoshi Nakkajima

9 Weak Lefschetz theorem for isocrystalline cohomologies

In this section we prove the weak Lefschetz theorem in [KM] (cf. [B1]) for crystalline cohomologies of proper smooth schemes over $\kappa$ by using rigid cohomologies. To prove this, we prove the following:

Theorem 9.1. Let $K$ be the fraction field of a complete discrete valuation ring $V$ of mixed characteristic with residue field $\kappa$. Let $X$ be a projective scheme over $\kappa$ with a closed immersion $X \rightarrow \mathbb{P}^n_{\kappa}$. Set $d := \dim X$. Let $H$ be a hypersurface of $\mathbb{P}^n_{\kappa}$. Set $Y := X \cap H$ and $U := X \backslash Y$. Assume that $U$ is smooth over $\kappa$. Let $\iota : Y \hookrightarrow X$ be the inclusion morphism. Then the pull-back of $\iota^*$

$$
\iota^* : H^q_{\text{rig}}(X/K) \rightarrow H^q_{\text{rig}}(Y/K) \quad (9.1.1)
$$
is an isomorphism for $q < d - 1$ and injective for $q = d - 1$.

Proof. By [B2, (3.1) (iii)] we have the following exact sequence

$$
\cdots \rightarrow H^i_{\text{rig}, c}(U/K) \rightarrow H^i_{\text{rig}}(X/K) \rightarrow H^i_{\text{rig}}(Y/K) \rightarrow \cdots \quad (9.1.2)
$$

Hence it suffices to prove that

$$
H^i_{\text{rig}, c}(U/K) = 0 \quad (i < d) \quad (9.1.3)
$$

Let $H^i_{\text{MW}}(U/K)$ be the $i$-th Monsky-Washnitzer cohomology of $U/K$. We have the following equalities by Berthelot’s duality ([B4, (2.4)]), Berthelot’s comparison theorem ([B3, (1.10.1)]):

$$
H^i_{\text{rig}, c}(U/K) = \text{Hom}_K(H^{2d-i}_{\text{rig}}(U/K), K(-d)) = \text{Hom}_K(H^{2d-i}_{\text{MW}}(U/K), K(-d)) \quad (9.1.4)
$$

Hence it suffices to prove that

$$
H^i_{\text{MW}}(U/K) = 0 \quad (i > d) \quad (9.1.5)
$$

Since $U$ is affine, express $U = \text{Spec}(A_0)$. Let $\mathcal{U}$ be a lift of $U$ over $V$ ([E1 Théorème 6]). Express $\mathcal{U} = \text{Spec}(A)$. Then, by the definition of Monsky-Washnitzer cohomology, $H^i_{\text{MW}}(U/K) = H^i(K \otimes_Y A^\dagger \otimes_A \Omega^*_{A/\mathcal{U}})$. Now the vanishing (9.1.5) is obvious. \hfill \Box

Remark 9.2. In [C] Caro has proved the hard Lefschetz Theorem in $p$-adic cohomologies. In particular, he has reproved the hard Lefschetz theorem proved in [KM]. However it seems that (9.1) cannot be obtained by the hard Lefschetz theorem in [KM] and [C] because $X$ nor $Y$ is not necessarily smooth.

Corollary 9.3. Let the notations be as in (9.1). Let $W$ be a Cohen ring of $\kappa$. Let $K_0$ be the fraction field of $W$. Assume that $X$ and $Y$ are smooth over $\kappa$. Then

$$
\iota^* : H^q_{\text{crys}}(X/W)_{K_0} \rightarrow H^q_{\text{crys}}(Y/W)_{K_0}
$$
is an isomorphism for $q < d - 1$ and injective for $q = d - 1$. 

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Proof. This follows from the comparison theorem of Berthelot \([B3 (1.9)]\):

\[
H^q_{\text{crys}}(Z/W)_{K_0} = H^q_{\text{rig}}(Z/K_0)
\]

for a proper smooth scheme \(Z/\kappa\).

\[\square\]

**Remark 9.4.** Because of the development of theory of rigid cohomology by Berthelot, we have been able to give a very short proof of the weak Lefshetz theorem for crystalline cohomologies of proper smooth schemes over \(\kappa\) without using the Weil conjecture nor the hard Lefschetz theorem for crystalline cohomologies (as in the \(l\)-adic case).

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