A SIMPLE PROOF OF THE RIEMANN HYPOTHESIS

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ABSTRACT

In this article, it is proved that the non-trivial zeros of the Riemann zeta function must lie on the critical line, known as the Riemann hypothesis.

Keywords Riemann zeta function · Riemann hypothesis · Non-trivial zeros · Critical line

1 Riemann Zeta function

The Riemann zeta function is defined over the complex plane as [1],
\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \text{Re}(s) > 1 \] (1)
where \( \text{Re}(s) \) denotes the real part of \( s \). There are several forms can be used for an analytic continuation for \( \text{Re}(s) > 0 \) such as [1, 2].

\[ \zeta(s) = \frac{1}{(1-2^1-s)\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x+1} dx, \] (2)
\[ \zeta(s) = \sum_{n=1}^{N} \frac{1}{n^s} - N^{1-s} \frac{1}{1-s} - s \int_N^{\infty} \frac{x-|x|}{x^{s+1}} dx, N = 1, 2 \ldots \] (3)
where \( |x| \) is the floor or integer part such that \( x - 1 < |x| \leq x \) for real \( x \).

and
\[ \zeta(s) = \frac{1}{(1-2^1-s)} \eta(s), \quad \eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} \] (4)
where \( s \neq 1 + \frac{2\pi ki}{\log(2)}, k = 0, \pm 1, \pm 2 \ldots \) and \( \eta(s) \) is the Dirichlet eta function (sometimes called the alternating zeta function).

2 Zeros of the Riemann Zeta Function

The trivial zeros of the Riemann zeta function occur at the negative even integers; that is, \( \zeta(-2n) = 0, n = 1, 2 \ldots \) [1]. On the other hand, the non-trivial zeros lie in the critical strip, \( 0 \leq \text{Re}(s) \leq 1 \). Both Hadamard [3] and de la Vallee Poussin [4] independently proved that there are no zeros on the boundaries of the critical strip (i.e. \( \text{Re}(s) = 0 \) or \( \text{Re}(s) = 1 \)). Gourdon and Demichel [5] verified the the Riemann Hypothesis until the \( 10^{13} \)-th zero.
Mossinghoff and Trudgian [6] proved that there are no zeros for \( \zeta(\sigma + it) \) for \(|t| \geq 2\) in the region,

\[
\sigma \geq 1 - \frac{1}{5.573412 \log |t|}
\]

This represents the largest known zero-free region for the zeta-function within the critical strip for \(3.06 \times 10^{10} < |t| < \exp(10151.5) \approx 5.5 \times 10^{4408}\).

The non-trivial zeros are known to be symmetric about the real axis and the critical line \(\text{Re}(s) = 1/2\), that is, \(\zeta(s) = \zeta(1-s) = \overline{\zeta(1-s)} = 0\) [1, 7].

3 Riemann Hypothesis

All the non-trivial zeros of the Riemann zeta function lie on the critical line \(\text{Re}(s) = 1/2\).

**Proof.** Assume that \(s = \sigma + it\) where \(0 < \sigma < 1\), \(t \in \mathbb{R}\) and \(N \to \infty\) is a natural number.

Let us define the functions \(\psi(s)\) as,

\[
\psi(s) = \lim_{N \to \infty} N^s \zeta(s)
\]

Clearly, the non-trivial zeros of the zeta function are included in the zeros of \(\psi(s)\), since at a non-trivial zero, \(s\), we have,

\[
\psi(s) = \lim_{N \to \infty} N^s \zeta(s) = \lim_{N \to \infty} N^s(0) = 0
\]

Using equation (3), \(\psi(s)\) can be written as,

\[
\psi(s) = \lim_{N \to \infty} \left[ N^s \phi(N, s) - N^s I(N, s) \right]
\]

where

\[
\phi(N, s) = \sum_{n=1}^{N} \frac{1}{n^s} - \frac{N^{1-s}}{1-s}, \quad I(N, s) = s \int_{N}^\infty \frac{x - \lfloor x \rfloor}{x^{s+1}}\,dx
\]

So, let us investigate the zeros of \(\psi(s)\).

Note that \(N^s I(N, s)\) is bounded as \(N \to \infty\), since

\[
|N^s I(N, s)| = \left| N^s s \int_{N}^\infty \frac{x - \lfloor x \rfloor}{x^{s+1}}\,dx \right| = N^\sigma \sqrt{\sigma^2 + t^2} \int_{N}^\infty \frac{x - \lfloor x \rfloor}{x^{s+1}}\,dx
\]

\[
\leq N^\sigma \sqrt{\sigma^2 + t^2} \int_{N}^\infty \frac{1}{x^{s+1}}\,dx
\]

\[
\leq N^\sigma \sqrt{\sigma^2 + t^2} \left[ \frac{1}{-\sigma x^\sigma} \right]_N^\infty
\]

\[
\leq N^\sigma \left[ \frac{\sqrt{\sigma^2 + t^2}}{\sigma N^\sigma} \right]
\]

\[
\leq \frac{\sqrt{\sigma^2 + t^2}}{\sigma}
\]

Let us sum the first \((j-1) \in \mathbb{N}\) terms of the summation in \(\phi(N, s)\) separately [8], that is,

\[
\phi(N, s) = \frac{1}{1^s} + \frac{1}{2^s} \cdots + \frac{1}{(j-1)^s} + \sum_{n=j}^{N} f(n) - \frac{N^{1-s}}{1-s}
\]

(11)
Thus, for 

\[ f(x) = \frac{1}{x^s} \]  

(12)

and its derivatives for \( i \in \mathbb{N} \) are,

\[ f^{(2i-1)}(x) = -\frac{s(s+1) \cdots (s+2i-2)}{x^{s+2i-1}}, \quad f^{(2i)}(x) = \frac{s(s+1) \cdots (s+2i-1)}{x^{s+2i}} \]  

(13)

Using the Euler–Maclaurin summation rule [8],

\[ \sum_{n=j}^{N} f(n) = \int_{j}^{N} f(x)dx + \frac{1}{2} [f(N) + f(j)] + \sum_{i=1}^{m} \frac{B_{2i}}{(2i)!} \left[ f^{(2i-1)}(N) - f^{(2i-1)}(j) \right] + E \]  

(14)

where the error term is given by,

\[ E = \frac{1}{(2m+1)!} \int_{j}^{N} \hat{B}_{2m+1}(x)f^{(2m+1)}(x)dx, \]  

(15)

\( B_k \) is the \( k \)-th Bernoulli number defined implicitly by,

\[ \frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!} \]  

(16)

\( B_k(x) \) is the \( k \)-th Bernoulli polynomial defined as the unique polynomial of degree \( k \) with the property that,

\[ \int_{t}^{t+1} B_k(x)dx = t^k, \]  

(17)

\( \hat{B}_k(x) \) is the periodic function \( B_k(x - \lfloor x \rfloor) \).

Thus, for \( m = 1 \), \( \phi(N, s) \) can be written as,

\[ \phi(N, s) = \sum_{n=1}^{j-1} \frac{1}{n^s} + \int_{j}^{N} f(x)dx + \frac{1}{2} [f(N) + f(j)] + \frac{1}{12} \left[ f'(x) \right]_{j}^{N} - \frac{N^{1-s}}{1-s} + E \]

\[ = \sum_{n=1}^{j-1} \frac{1}{n^s} - \frac{1}{1-s} + \frac{1}{2} \left[ \frac{1}{N^s} + \frac{1}{j^s} \right] - \frac{s}{12} \left[ \frac{1}{N^{s+1}} - \frac{1}{j^{s+1}} \right] + E \]

\[ = \sum_{n=1}^{j} \frac{1}{n^s} - \frac{1}{1-s} + \frac{1}{2} \left[ \frac{1}{N^s} - \frac{1}{j^s} \right] - \frac{s}{12} \left[ \frac{1}{N^{s+1}} - \frac{1}{j^{s+1}} \right] + E \]

\[ \phi(j, s) = \frac{1}{2} \left[ \frac{1}{N^s} - \frac{1}{j^s} \right] - \frac{s}{12} \left[ \frac{1}{N^{s+1}} - \frac{1}{j^{s+1}} \right] + E \]  

(18)

where

\[ E = \frac{1}{3!} \int_{j}^{N} \hat{B}_3(x)f'''(x)dx \]

\[ = \frac{1}{3!} \int_{j}^{N} \hat{B}_3(x) \left[ -\frac{s(s+1)(s+2)}{x^{s+3}} \right] dx, \]  

(19)

\[ \hat{B}_3(x) = x^3 - \frac{3}{2} x^2 + \frac{1}{2} x \]  

(20)
Since,
\[ |\bar{B}_3(x)| < 0.0481126 \equiv B^{\text{max}}_3 \] (21)

Thus, the error term, \( E \), can be bounded as,
\[
|E| \leq \frac{1}{3!} \int_j^N |\bar{B}_3(x)f'''(x)| \, dx \\
\leq \frac{B^{\text{max}}_3}{3!} \int_j^N |f'''(x)| \, dx \\
\leq \frac{B^{\text{max}}_3}{3!} \frac{|s(s+1)(s+2)|}{(3!)^{\sigma+2}} \left[ \frac{1}{\eta^{\sigma+2}} - \frac{1}{N^{\sigma+2}} \right] (22)
\]

Multiplying equation (18) by \( N^s \) and setting \( j = N/r \) where \( r \) is a natural number and \( N \) is a very large natural number that is divisible by \( r \),
\[
N^s \phi(N,s) = N^s \phi \left( \frac{N}{r}, \sigma \right) + \frac{N^s}{2} \left( \frac{1}{N^s} - \frac{r^s}{N^s} \right) - \frac{N^s s}{12} \left( \frac{1}{N^{s+1}} - \frac{r^{s+1}}{N^{s+1}} \right) + N^s \cdot O \left( \frac{1}{N^{\sigma+2}} \right) (23)
\]

As \( N \to \infty \),
\[
\lim_{N \to \infty} N^s \phi(N,s) = \lim_{N \to \infty} N^s \phi \left( \frac{N}{r}, \sigma \right) + \frac{1}{2} \left[ 1 - r^s \right] (24)
\]

Actually, there are two cases.

**Case I: \( \zeta(s) \neq 0 \)**

In this case, \( \psi(s) \) is unbounded as,
\[
|\psi(s)| = \lim_{N \to \infty} N^s \zeta(s) = \infty (25)
\]

From equation (10), we know that \( \lim_{N \to \infty} N^s I(N,s) \) is bounded, therefore \( \lim_{N \to \infty} N^s \phi(N,s) \) must be unbounded, since
\[
|\psi(s)| = \lim_{N \to \infty} \left| N^s \phi(N,s) - N^s I(N,s) \right| = \infty (26)
\]

So, we have
\[
\lim_{N \to \infty} \left| N^s \phi(N,s) \right| = \infty, \quad \lim_{N \to \infty} \left| N^s \phi(N/r,s) \right| = \infty (27)
\]

**Case II: \( \zeta(s) = 0 \)**

In this case, \( \psi(s) \) vanishes, since,
\[
|\psi(s)| = |N^s \zeta(s)| = 0 (28)
\]

Therefore \( \lim_{N \to \infty} N^s \phi(N,s) \) must be bounded since,
\[
|\psi(s)| = \lim_{N \to \infty} \left| N^s \phi(N,s) - N^s I(N,s) \right| = 0 (29)
\]

Thus, we must have,
\[
\lim_{N \to \infty} N^s \phi(N,s) = \lim_{N \to \infty} N^s I(N,s) (30)
\]
Assume that
\[ H(s) = \lim_{N \to \infty} N^s \phi(N, s) \]  
(31)

Hence,
\[ \lim_{N \to \infty} N^s \phi \left( \frac{N}{r}, s \right) = r^s \lim_{N/r \to \infty} \left( \frac{N}{r} \right)^s \phi \left( \frac{N}{r}, s \right) = r^s H(s) \]  
(32)

Substituting in equation (34), leads to,
\[ (1 - r^s) \left[ \frac{1}{2} - H(s) \right] = 0 \]  
(33)

So, we have either,
\[ 1 - r^s = 0, \quad \Rightarrow \quad s = \frac{2\pi ki}{\log r}, \quad k = 0, \pm 1, \pm 2 \cdots \]  
(34)

which is rejected as it leads to a contradiction with the range of \( \sigma \) (0 < \( \sigma \) < 1),
or, we have,
\[ H(s) = \frac{1}{2} \]  
(35)

Therefore, at a non-trivial zero of the zeta function, we must have,
\[ \zeta(s) = 0 \implies \lim_{N \to \infty} N^s \phi(N, s) = \lim_{N \to \infty} N^s I(N, s) = \frac{1}{2} \]  
(36)

Due to the symmetry of the non-trivial zeros of the zeta function about the critical line, we must also have,
\[ \zeta(1 - \bar{s}) = 0 \implies \lim_{N \to \infty} N^{1-\bar{s}} \phi(N, 1 - \bar{s}) = \lim_{N \to \infty} N^{1-\bar{s}} I(N, 1 - \bar{s}) = \frac{1}{2} \]  
(37)

Let us define
\[ \nu(s) = (1 - s)\psi(s) - \bar{s}\psi(1 - \bar{s}) = \lim_{N \to \infty} [(1 - s)N^s \zeta(s) - \bar{s}N^{1-\bar{s}} \zeta(1 - \bar{s})] \]  
(38)

Clearly, the non-trivial zeros of the zeta function are included in the zeros of \( \nu(s) \) since at a non-trivial zero, \( s \), we have \( \zeta(s) = \zeta(1 - \bar{s}) = 0 \) and hence,
\[ \nu(s) = \lim_{N \to \infty} [(1 - s)N^s \zeta(s) - \bar{s}N^{1-\bar{s}} \zeta(1 - \bar{s})] = \lim_{N \to \infty} [(1 - s)N^s(0) - \bar{s}N^{1-\bar{s}}(0)] = 0 \]  
(39)

From equation (38), \( \nu(s) \) can be written as,
\[ \nu(s) = \lim_{N \to \infty} [(1 - s)N^s \phi(N, s) - \bar{s}N^{1-\bar{s}} \phi(N, 1 - \bar{s}) - (1 - s)N^s I(N, s) + \bar{s}N^{1-\bar{s}} I(N, 1 - \bar{s})] \]  
(40)

So, at a non-trivial zero of the zeta function, using equation (36), we get,
\[ \lim_{N \to \infty} [(1 - s)N^s \phi(N, s) - \bar{s}N^{1-\bar{s}} \phi(N, 1 - \bar{s})] = (1 - s) \left( \frac{1}{2} \right) + (\bar{s}) \left( \frac{1}{2} \right) = 0 \]  
(41)

or
\[ \lim_{N \to \infty} \beta(N, s) = \frac{1 - s - \bar{s}}{2} = \frac{1 - 2\sigma}{2} \]  
(42)

where
\[ \beta(N, s) = (1 - s)N^s \phi(N, s) - \bar{s}N^{1-\bar{s}} \phi(N, 1 - \bar{s}) \]  
(43)

So, let us study the convergence of the series \( \lim_{N \to \infty} \beta(N, s) \). Obviously, it converges to zero when \( \sigma = 1 - \sigma \) (i.e. \( \sigma = 1/2 \)) and equation (42) is satisfied.
On the other hand, if \( \sigma \neq 1/2 \), let us write \( \phi(N, s) \) using Euler–Maclaurin summation rule given by equation \( \text{(48)} \) for large values of \( j, N \), such that \( N \gg j \gg |s(s+1)(s+2)| \),

\[
\phi(N, s) = \sum_{n=1}^{j} \frac{1}{n^s} - \frac{j^{1-s}}{1-s} + \frac{1}{2} \left[ \frac{1}{N^s} - \frac{1}{j^s} \right]
\]

and similarly,

\[
\phi(N, 1-\bar{s}) = \sum_{n=1}^{j} \frac{1}{n^{1-\bar{s}}} - \frac{j^{1-\bar{s}}}{\bar{s}} + \frac{1}{2} \left[ \frac{1}{N^{1-\bar{s}}} - \frac{1}{j^{1-\bar{s}}} \right]
\]

Therefore,

\[
\beta(N, s) = \sum_{n=1}^{j} \left[ (1-s) \left( \frac{N}{n} \right)^{s} - (\bar{s}) \left( \frac{N}{n} \right)^{1-\bar{s}} \right] + \frac{1-s-\bar{s}}{2} - \frac{1}{2} \left[ \frac{(1-s)N^s}{j^s} - \frac{\bar{s}N^{1-\bar{s}}}{j^{1-\bar{s}}} \right]
\]

\[
= \sum_{n=1}^{j} \left[ (1-\sigma-it) \left( \frac{N}{n} \right)^{\sigma} - (\sigma-it) \left( \frac{N}{n} \right)^{1-\sigma} \right] \left( \frac{N}{n} \right)^{it} + \frac{1-2\sigma}{2} - \frac{1}{2} \left[ \frac{(1-s)N^s}{j^s} - \frac{\bar{s}N^{1-\bar{s}}}{j^{1-\bar{s}}} \right]
\]

As \( n \to 1 \), \( (N/n) \to N \) while as \( n \to N \), \( (N/n) \to 1 \), so, for \( \sigma > 1/2 \), the magnitudes of first few terms of the above series are dominated by,

\[
\sqrt{(1-\sigma)^2 + t^2} N^{\sigma}, \sqrt{(1-\sigma)^2 + t^2} \left( \frac{N}{2} \right)^{\sigma}, \sqrt{(1-\sigma)^2 + t^2} \left( \frac{N}{3} \right)^{\sigma} \ldots
\]

Thus, as \( N \to \infty \), \( \beta(N, s) \) is dominated by the first few terms of the above series and the term,

\[
\frac{1}{2} \left[ \frac{(1-s)N^s}{j^s} \right]
\]

which are all \( O(N^{\sigma}) \) and do not cancel out. This leads to the divergence of the series \( \lim_{N \to \infty} \beta(N, s) \), hence, equation \( \text{(48)} \) cannot be satisfied in this case.

Therefore, \( \nu(s) \), and accordingly \( \zeta(s) \), can only be zero when \( \sigma = 1/2 \). That is, all the non-trivial zeros of the zeta function must lie on the critical line, \( \text{Re}(s) = 1/2 \).

\[
\square
\]

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