Stability of a Time-varying Fishing Model with Delay

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Abstract We introduce a delay differential equation model which describes how fish are harvested

\[
\dot{N}(t) = \left[ \frac{a(t)}{1 + \left( \frac{N(t)}{K(t)} \right)^\gamma} - b(t) \right] N(t) \tag{A}
\]

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In our previous studies we investigated the persistence of equation (A) and existence of a periodic solution for this equation. Here we study the stability (local and global) of the periodic solutions of equation (A).

Keywords—Fishery, Periodic Environment, Delay Differential Equations, Global and Local Stability.

1 Introduction and Preliminaries

Consider the following differential equation which is widely used in Fisheries

\[
\dot{N} = [\beta(t, N) - M(t, N)]N, \tag{1}
\]

where \( N = N(t) \) is the population biomass, \( \beta(t, N) \) is the per-capita fecundity rate, and \( M(t, N) \) is the per-capita fishing mortality rate due to natural mortality causes and harvesting.

In equation (1) let \( \beta(t, N) \) be a Hill’s type function \([1, 2, 5]\)

\[
\beta(t, N) = \frac{a}{1 + \left(\frac{N}{K}\right)^\gamma}, \tag{2}
\]

where \( a \) and \( K \) are positive constants, and \( \gamma > 0 \) is a parameter.

Traditional Population Ecology is based on the concept that carrying capacity does not change over time even though it is known \([3]\) that the values of carrying capacity related to the habitat areas might vary, e.g., year-to-year changes in weather affect fish population.

We assume that in (2) \( a = a(t) \), \( K = K(t) \), and \( M(t, N) = b(t) \) are continuous positive functions.

Generally, Fishery models \([1, 2]\) recognize that for real organisms it takes time to develop from newborns to reproductively active adults.

Let in equation (2) \( N = N(\theta(t)) \), where \( \theta(t) \) is the maturation time delay \( 0 \leq \theta(t) \leq t \). If we take into account that delay, then we have the following time-lag model based on equation (1)

\[
\dot{N}(t) = \left[\frac{a(t)}{1 + \left(\frac{N(t)}{K(t)}\right)^\gamma} - b(t)\right] N(t) \tag{3}
\]
for $\gamma > 0$, with the initial function and the initial value

$$N(t) = \varphi(t), \ t < 0, \ N(0) = N_0$$

under the following conditions:

(a1) $a(t), b(t), K(t)$ are continuous on $[0, \infty)$ functions, $b(t) \geq b > 0, \ K \geq K(t) \geq k > 0$;

(a2) $\theta(t)$ is a continuous function, $\theta(t) \leq t$, $\limsup_{t \to \infty} \theta(t) = \infty$;

(a3) $\varphi : (-\infty, 0) \to R$ is a continuous bounded function, $\varphi(t) \geq 0, N_0 > 0$.

**Definition 1.1** A function $N : R \to R$ with continuous derivative is called a (global) solution of problem (3), (4), if it satisfies equation (3) for all $t \in [0, \infty)$ and equalities (4) for $t \leq 0$.

If $t_0$ is the first point, where the solution $N(t)$ of (3), (4) vanishes, i.e., $N(t_0) = 0$, then we consider the only positive solutions of the problem (3), (4) on the interval $[0, t_0]$.

Recently [5] we proved the following results:

**Lemma 1.1** Suppose $a(t) > b(t)$,

$$\sup_{t \geq 0} \int_{\theta(t)}^{t} \left( a(s) - b(s) \right) ds < \infty, \sup_{t \geq 0} \int_{\theta(t)}^{t} b(s) ds < \infty.$$

Then there exists the global positive solution of (3), (4) and this solution is persistence:

$$0 < \alpha_N \leq N(t) \leq \beta_N < \infty.$$

**Lemma 1.2** Let $a(t), b(t), K(t), \theta(t)$ be $T$-periodic functions, $a(t) \geq b(t)$. If at least one of the following conditions hold:

(b1)

$$\inf_{t \geq 0} \left( \frac{a(t)}{b(t)} - 1 \right) K^\gamma(t) > 1,$$

(b2)

$$\sup_{t \geq 0} \left( \frac{a(t)}{b(t)} - 1 \right) K^\gamma(t) < 1,$$

then equation (3) has at least one periodic positive solution $N_0(t)$. 

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In what follows, we use a classical result from the theory of differential equations with delay [4, 6].

**Lemma 1.3** Suppose that for linear delay differential equation

\[ \dot{x}(t) + r(t)x(h(t)) = 0 \]  

where \( 0 \leq t - h(t) \leq \sigma \), the following conditions hold:

\[ r(t) \geq r_0 > 0, \]  

\[ \limsup_{t \to \infty} \int_{h(t)}^{t} r(s)ds < \frac{3}{2} \]  

Then for every solution \( x \) of equation (5) we have \( \lim_{t \to \infty} x(t) = 0 \).

## 2 Main Results

Let us study global stability of the periodic solutions of equation (3).

**Theorem 2.1** Let \( a(t), b(t), K(t), \theta(t) \) be \( T \)-periodic functions, satisfying conditions of Lemma 1.1 and one of conditions b1) or b2) of Lemma 1.2. Suppose also that

\[ a(t) \geq a_0 > 0, \quad \gamma \int_{\theta(t)}^{t} a(s)ds < 6. \]  

Then there exists the unique positive periodic solution \( N_0(t) \) of (3) and for every positive solution \( N(t) \) of (3) we have

\[ \lim_{t \to \infty} (N(t) - N_0(t)) = 0, \]  

i.e., the positive periodic solution \( N_0(t) \) is a global attractor for all positive solutions of (3).

**Proof.** Lemma 1.2 implies that there exists a positive periodic solution \( N_0(t) \). If that solution is an attractor for all positive solutions then it is the unique positive periodic solution.

We set \( N(t) = \exp(x(t)) \) and rewrite equation (3) in the form

\[ \dot{x}(t) = \frac{a(t)}{1 + \left(\frac{e^{x(h(t))}}{K(t)}\right)^{\gamma}} - b(t). \]  

Suppose \( u(t) \) and \( v(t) \) are two different solutions of (9). Denote \( w(t) = u(t) - v(t) \). To prove the Theorem 2.1 it is sufficient to show that \( \lim_{t \to \infty} w(t) = 0 \).

It follows

\[
\dot{w}(t) = a(t) \left[ \frac{1}{1 + \left( \frac{e^{u(\theta(t))}}{K(t)} \right)^\gamma} - \frac{1}{1 + \left( \frac{e^{v(\theta(t))}}{K(t)} \right)^\gamma} \right]
\]  

(10)

Let

\[
f(y, t) = \frac{1}{1 + \left( \frac{e^y}{K(t)} \right)^\gamma}.
\]  

(11)

Using the mean value theorem, we have for every \( t \)

\[
f(y, t) - f(z, t) = f'(c)(y - z),
\]  

(12)

where \( \min\{y, z\} \leq c(t) \leq \max\{y, z\} \).

Clearly,

\[
f'(y, t) = -\frac{\gamma \left( \frac{e^y}{K(t)} \right)^\gamma}{\left( 1 + \left( \frac{e^y}{K(t)} \right)^\gamma \right)^2}
\]  

(13)

and \( |f'(y, t)| < \frac{1}{4} \gamma \).

Equalities (11)- (12) imply that equation (10) takes the form

\[
\dot{w}(t) = -M(t)w(\theta(t)),
\]  

(14)

where

\[
M(t) = \frac{\gamma a(t) \left( \frac{e^{v(\theta(t))}}{K(t)} \right)^\gamma}{\left( 1 + \left( \frac{e^{v(\theta(t))}}{K(t)} \right)^\gamma \right)^2},
\]

and

\[
\min\{u(\theta(t)), v(\theta(t))\} \leq c(t) \leq \max\{u(\theta(t)), v(\theta(t))\}.
\]

Now we want to check that for equation (14) all conditions of Lemma 1.3 hold.

From (13) we have \( M(t) < \frac{1}{4} \gamma a(t) \). Therefore inequality (7) holds. Let us check inequality (6). Set \( N_1(t) = e^{u(t)} \), \( N_2(t) = e^{v(t)} \), where \( N_1(t), N_2(t) \) are two solutions of equation (3), corresponding to the solutions \( u(t) \) and \( v(t) \) of equation (9). Lemma 1.1 implies that

\[
M(t) \geq \frac{\gamma a_0 \left( \frac{\min\{\alpha N_1, \alpha N_2\}}{K} \right)^\gamma}{\left( 1 + \left( \frac{\max\{\beta N_1, \beta N_2\}}{K} \right)^\gamma \right)^2} > 0,
\]

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where $\alpha_N$ and $\beta_N$ are defined by Lemma 1.1. Hence inequality (6) holds and therefore Theorem 2.1 is proven.

Consider now equation (3) with proportional coefficients:

$$\dot{N}(t) = \left[ \frac{ar(t)}{1 + \left( \frac{N(\theta(t))}{K} \right)^{\gamma}} - br(t) \right] N(t),$$

where $r(t) \geq r_0 > 0$. Clearly, if $a > b$ then equation (15) has the unique positive equilibrium

$$N^* = \left( \frac{a}{b} - 1 \right)^{\frac{1}{\gamma}} K.$$

Corollary 1. If $a > b, \left( \frac{a}{b} - 1 \right) K^\gamma \neq 1, r(t) \geq r_0 > 0$, and

$$\gamma a \limsup_{t \to \infty} \int_{\theta(t)}^{t} r(s) ds < 6,$$

then the equilibrium $N^*$ is a global attractor for all positive solutions of equation (15).

Let us now compare the global attractivity condition (17) with the local stability conditions.

**Theorem 2.2** Suppose $a > b, r(t) \geq r_0 > 0$ and

$$\frac{\gamma(a - b)b}{a} \limsup_{t \to \infty} \int_{\theta(t)}^{t} r(s) ds < \frac{3}{2}.$$  

Then the equilibrium $N^*$ of equation (15) is locally asymptotically stable.

**Proof.** Set $x = N - N^*$ and from equation (15) we have

$$\dot{x}(t) = \left[ \frac{ar(t)}{1 + \left( \frac{x(\theta(t)) + N^*}{K} \right)^{\gamma}} - br(t) \right] (x(t) + N^*).$$

Denote

$$F(u, v) = \left[ \frac{ar(t)}{1 + \left( \frac{u + N^*}{K} \right)^{\gamma}} - br(t) \right] (v + N^*).$$

Clearly,

$$F_u'(0, 0) = -\frac{\gamma(a - b)b}{a} r(t)$$
and $F'_v(0, 0) = 0$. Hence for equation (15) the linearized equation has a form
\[
\dot{x}(t) = -\frac{\gamma(a - b)b}{a} r(t) x(\theta(t)).
\] (20)

Lemma 1.3 and condition (18) imply that equation (20) is asymptotically stable, therefore the positive equilibrium $N^*$ of equation (15) is locally asymptotically stable.

Compare now Theorem 2.1 and Theorem 2.2. We have $\max \{b(a - b)\} = a/4$. Therefore, if
\[
a\gamma \limsup_{t \to \infty} \int_{\theta(t)}^{t} r(s)ds < 6,
\]
then equation (15) has locally asymptotically stable equilibrium $N^*$.

The last condition does not depend on $b$, and is identical to condition (17) that guarantees the existence of a global attractor. Therefore in Theorem 2.2 we obtained the best possible conditions for global attractivity for equation (3).

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