Yukawa Textures and Anomalies

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ABSTRACT: We augment the Minimal Supersymmetric Standard Model with a gauged family-dependent $U(1)$ to reproduce Yukawa textures compatible with experiment. In the simplest model with one extra chiral electroweak singlet field, acceptable textures require this $U(1)$ to be anomalous. The cancellation of its anomalies by a generic Green-Schwarz mechanism requires $\sin^2 \theta_w = 3/8$ at the string scale, suggesting a superstring origin for the standard model.
1. Introduction

The extension of the standard model to $N = 1$ supersymmetry\cite{1} allows for its perturbative extrapolation to near Planckian scales, where the gauge couplings \cite{2} and some Yukawa couplings\cite{3} appear to converge. This raises the hope that the $N = 1$ standard model at short distances is much simpler than at experimental scales. However we do not have sufficient information to determine exactly the type of structure it describes, a GUT theory\cite{4}, or a direct descendant of superstrings.

In this letter, we attempt to answer this question by considering the structure of the Yukawa couplings. While not known in detail, their orders of magnitude are well determined by experiment. Their most striking aspect is the hierarchy of the masses of the three chiral families. The experimental values of the quark and lepton masses, extrapolated near the Planck scale, satisfy the orders of magnitude estimates\cite{5}

\begin{align}
\frac{m_u}{m_t} = \mathcal{O}(\lambda^8) ; \quad \frac{m_c}{m_t} = \mathcal{O}(\lambda^4) ;
\end{align}

\begin{align}
\frac{m_d}{m_b} = \mathcal{O}(\lambda^4) ; \quad \frac{ms}{mb} = \mathcal{O}(\lambda^2) ,
\end{align}

where, following Wolfenstein’s parametrization\cite{6}, we use the Cabibbo angle $\lambda$, as expansion parameter. The charged lepton masses also satisfy similar relations

\begin{align}
\frac{me}{m_\tau} = \mathcal{O}(\lambda^4) ; \quad \frac{m_\mu}{m_\tau} = \mathcal{O}(\lambda^2) .
\end{align}

The mass hierarchy appears to be geometrical in each sector. The equality

\begin{align}
m_b = m_\tau ,
\end{align}

known to be valid in the ultraviolet\cite{3}, yields the estimate

\begin{align}
\frac{mdmsmb}{memuxmt} = \mathcal{O}(1) .
\end{align}

A question of intense theoretical speculation is the mechanism which sets these orders of magnitude. In this letter we explore the possibility that it is a family-dependent gauged Abelian symmetry\cite{7}. While hardly new, this idea has been revisited in the recent literature
[8,9,10], but in rather specific models. We find in the simplest model of this kind that in order to reproduce the orders of magnitude of the quarks and lepton masses, the Abelian family symmetry must be anomalous. When its anomaly is compensated by a Green-Schwarz mechanism, the Weinberg angle is fixed [11]. Remarkably its value is $\sin^2 \theta_w = 3/8$, in perfect agreement with data when extrapolated to the infrared.

Our framework is the minimal extension of the Standard Model to $N = 1$ supersymmetry, including the so-called $\mu$ term, $P = \mu H_u H_d$. We reserve to a forthcoming paper [12] the case without an $ab$ initio $\mu$ term. In this paper, we aim to determine to what extent an Abelian charge symmetry can help in narrowing down possible Yukawa textures.

2. Yukawa Textures

Consider the most general Abelian charge that can be assigned to the particles of the Supersymmetric Standard Model,

$$X = X_0 + X_3 + \sqrt{3}X_8,$$

where $X_0$ is the family-independent part, $X_3$ is along $\lambda_3$, and $X_8$ is along $\lambda_8$, the two diagonal Gell-Mann matrices of the $SU(3)$ family space in each charge sector. In a basis where the entries correspond to the components in the family space of the fields $Q, \pi, \bar{d}, L$, and $\bar{e}$, we can write the different components in the form

$$X_i = (a_i, b_i, c_i, d_i, e_i),$$

for $i = 0, 3, 8$. The Higgs doublets $H_{u,d}$ have zero $X$-charge because of the $\mu$ term.

Let us assume that the tree-level Yukawa coupling involves only the third family (implicitly choosing the third direction in family space),

$$y_t Q_3 \bar{u}_3 H_u + y_b Q_3 \bar{d}_3 H_d + y_\tau L_3 \bar{e}_3 H_d,$$

where the $y_i$'s are the Yukawa couplings. This generates the relations

$$a_0 + b_0 = 2(a_8 + b_8),$$

$$a_0 + c_0 = 2(a_8 + c_8),$$

$$d_0 + e_0 = 2(d_8 + e_8).$$
The other elements of the Yukawa matrices are zero at tree-level. We assume that the reason is conservation of X-charge: these entries do not have the correct X-charge for renormalizable couplings. Let $x_{ij}$ be the excess X-charges at each of their entries; they are

$$
\begin{align*}
\text{charge } \frac{2}{3} : & \begin{pmatrix} 3(a_8 + b_8) + a_3 + b_3 & 3(a_8 + b_8) + a_3 - b_3 & 3a_8 + a_3 \\ 3(a_8 + b_8) - a_3 + b_3 & 3(a_8 + b_8) - a_3 - b_3 & 3a_8 - a_3 \\ 3b_8 + b_3 & 3b_8 - b_3 & 0 \end{pmatrix} \\
\text{charge } - \frac{1}{3} : & \begin{pmatrix} 3(a_8 + c_8) + a_3 + c_3 & 3(a_8 + c_8) + a_3 - c_3 & 3a_8 + a_3 \\ 3(a_8 + c_8) - a_3 + c_3 & 3(a_8 + c_8) - a_3 - c_3 & 3a_8 - a_3 \\ 3c_8 + c_3 & 3c_8 - c_3 & 0 \end{pmatrix} \\
\text{charge } - 1 : & \begin{pmatrix} 3(d_8 + e_8) + d_3 + e_3 & 3(d_8 + e_8) + d_3 - e_3 & 3d_8 + d_3 \\ 3(d_8 + e_8) - d_3 + e_3 & 3(d_8 + e_8) - d_3 - e_3 & 3d_8 - d_3 \\ 3e_8 + e_3 & 3e_8 - e_3 & 0 \end{pmatrix}
\end{align*}
$$

For the empty entries in the Yukawa matrices to be filled, the tree-level chiral symmetry of the first two families must be broken. A generic mechanism is to couple the fields of the standard model with new vector-like fermions, and then give these new fermions a mass that breaks chiral symmetry, and sets the scale of chiral symmetry breaking. This is similar to the see-saw mechanism[13], where chiral symmetry is replaced by lepton number, but its implementation is different, since we deal with Dirac, rather than Majorana matrices. It is a matter for the model builder to offer specific models which realize this scenario; we assume it can be done, and refer the reader to models in the literature[14,15].

The excess charge at each entry, $x_{ij}$, is assumed to be made up by an operator of higher dimensions with no hypercharge [14]. For example, X-charge conservation allows the non-renormalizable term

$$
Q_i \vec{u}_j H_u \left( \frac{\theta}{M} \right)^{n_{ij}},
$$

provided that the $n_{ij}$ are positive numbers which satisfy

$$
x_{ij} - x n_{ij} = 0 .
$$

We have introduced an electroweak singlet field $\theta$ with X-charge $-x$, and $M$ is some large scale. It is simplest to assume the existence of only one such field. In general the $n_{ij}$ are
expected to be integers, unless one is willing to envisage fractional powers of the field $\theta$, stemming from non-perturbative effects.

In the simplest model, there is only one electroweak singlet chiral superfield $\theta$, not chaperoned by its vectorlike partner. Invariance under supersymmetry then naturally[8] generates a true texture zero whenever a Yukawa matrix element has negative excess $X$-charge in units of ($-x$), and non-zero entries correspond only to positive excess $X$-charge. Henceforth we normalize $X$ so that $x = 1$.

In slightly more complicated models, $\theta$ is accompanied by its vector-like partner $\overline{\theta}$, with opposite value of $X$-charge. Any entry with negative excess charge can be filled by terms like

$$Q_i \overline{u}_j H_u \left( \frac{\theta}{M} \right)^{n_{ij}} ,$$

showing that the excess charges $x_{ij}$ need not be of the same sign. This also allows $\theta$ to have an $X$-conserving mass.

Let us assume that $\theta$ has a vacuum expectation value smaller than $M$, producing a small parameter, $\theta/M$. The $n_{ij}$ then determine the order of magnitude of the entries in the Yukawa matrices[14]. The masses $M$, and thus the expansion parameters are in principle different in the three charge sectors. However, since the down quark and lepton sectors share the same electroweak quantum numbers, we expect them to be the same at least for the charge -1 and -1/3 matrices.

This simple picture of the orders of magnitude of Yukawa matrices is quite restrictive. Consider the general case, where the normalized Yukawa matrix is

$$Y_{ij} = \mathcal{O}(\lambda^{n_{ij}}) \, ,$$

and

$$n_{ij} = |x_{ij}| \, ,$$

normalized to the heaviest mass in each charge sector. From the constraints satisfied among the $x_{ij}$

$$x_{11} = x_{13} + x_{31} , \quad x_{22} = x_{23} + x_{32} ,$$

$$x_{12} = x_{13} + x_{32} , \quad x_{21} = x_{23} + x_{31} ,$$

(2.10)
we obtain the inequalities
\[ n_{11} \leq n_{13} + n_{31} ; \quad n_{22} \leq n_{23} + n_{32} , \]
\[ n_{12} \leq n_{13} + n_{32} ; \quad n_{21} \leq n_{23} + n_{31} , \]
These enable us to derive general results on the hierarchy of eigenvalues of these matrices, in terms of
\[ p = \min (n_{11}, n_{22}, n_{12}, n_{21}) , \]
\[ q = \min (n_{11} + n_{22}, n_{12} + n_{21}) . \]

By considering the characteristic equation of the hermitean combinations \( Y^\dagger Y \) in each charge sector, we find the following eigenvalue patterns
\[ p \geq \frac{q}{2} \text{ eigenvalues : } O(1) , \pm O(\lambda^p) , \]
\[ p \leq \frac{q}{2} \text{ eigenvalues : } O(1) , O(\lambda^p) , O(\lambda^{q-p}) . \]

The first pattern is in contradiction with data, leaving the second as the only physically acceptable case. The determination of the order of magnitudes has been reduced to that for the underlying \( 2 \times 2 \) Yukawa matrix. Let us note for further use that the geometric hierarchies of the type (1.1), (1.2), (1.3) are obtained for \( q = 3p \).

It is possible to classify the orders of magnitude of the eigenvalues, according to the ranges taken by the X-charges in the different charge sectors. When \( a_3 \) and \( b_3 \) have the same sign, the results can be summarized as follows:

- \( 3|a_3 + b_3| \geq |a_3 + b_3| \)
  \[ q = 6|a_8 + b_8| ; \quad p = 3|a_8 + b_8| - |a_3 + b_3| , \]
  with the geometric hierarchy for \( |a_3 + b_3| = |a_8 + b_8| \).

- \( |a_3 - b_3| \leq 3|a_8 + b_8| \leq |a_3 + b_3| \)
  \[ q = 6|a_8 + b_8| ; \quad p = \begin{cases} 3|a_8 + b_8| - |a_3 - b_3| , \text{ when } 6|a_8 + b_8| \leq |a_3 - b_3| + |a_3 + b_3| , \\ |a_3 + b_3| - 3|a_8 + b_8| , \text{ when } 6|a_8 + b_8| \geq |a_3 - b_3| + |a_3 + b_3| . \end{cases} \]

Geometric hierarchies for these two cases are possible only if \( |a_3 + b_3| \geq 5|a_3 - b_3| \), with the respective assignments \( |a_3 - b_3| = |a_8 + b_8| \) and \( |a_3 + b_3| = 5|a_8 + b_8| \).
\[ 3|a_8 + b_8| \leq |a_3 - b_3| \]

\[ q = 2|a_3 - b_3| ; \quad p = |a_3 - b_3| - 3|a_8 + b_8| . \] (2.15)

In this case the geometric hierarchy is obtained for \( 9|a_8 + b_8| = |a_3 - b_3| \).

When \( a_3 \) and \( b_3 \) have opposite signs, we obtain the same equations with \( b_3 \) replaced by \(-b_3\). The other two charge sectors are described by the same equations, by changing \( b_i \) to \( c_i \) for the down quarks, and \( a_i, b_i \) by \( d_i, e_i \), respectively for the charged leptons sector.

This analysis simplifies if restricted to the case of symmetric textures. A texture is said to be symmetric if \( |x_{ij}| = |x_{ji}| \), in a charge sector, which does not necessarily mean that the Yukawa matrices are symmetric, only their orders of magnitude. There is no fundamental reason to require symmetry of the textures, although it was found[5] that several symmetric textures are compatible with experiment. Symmetric textures that reproduce hierarchical eigenvalues imply

\[
\begin{align*}
\text{Charge} \quad \frac{2}{3} \quad \text{sector :} & \quad a_3 = b_3 , \quad a_8 = b_8 , \\
\text{Charge} \quad -\frac{1}{3} \quad \text{sector :} & \quad a_3 = c_3 , \quad a_8 = c_8 . \\
\text{Charge} \quad -1 \quad \text{sector :} & \quad d_3 = e_3 , \quad d_8 = e_8 .
\end{align*}
\] (2.16)

The excess X-charge, shown here for a quark Yukawa matrix, is

\[
\begin{pmatrix}
2|3a_8 + a_3| & 6|a_8| & |3a_8 + a_3| \\
6|a_8| & 2|3a_8 - a_3| & |3a_8 - a_3| \\
|3a_8 + a_3| & |3a_8 - a_3| & 0
\end{pmatrix}.
\]

Our general analysis now reduces to just three cases. In all three, \( q = 12|a_8| \), and

\[
\begin{align*}
p = & 2(3|a_8| - |a_3|) , \quad \text{when} \quad 3|a_8| \geq |a_3| , \\
p = & 2(|a_3| - 3|a_8|) , \quad \text{when} \quad 2|a_3| \geq 6|a_8| \geq |a_3| , \\
p = & 6|a_8| , \quad \text{when} \quad |a_3| \geq 6|a_8| .
\end{align*}
\] (2.17)

In the first case, geometric hierarchy is achieved for \( |a_3| = |a_8| \), corresponding to the U or V-spin of the family \( SU(3) \). The second case yields \( |a_3| = 5|a_8| \), and the third case does not allow for a geometric hierarchy. In the more constrained case of totally symmetric textures[5], equation (1.4) leads to

\[
|a_8| = |b_8| = |c_8| = |d_8| = |e_8| ,
\] (2.18)
and geometric hierarchy in all sectors implies for all the textures
\[|a_3|, |b_3|, |c_3|, |d_3|, |e_3| = \begin{cases} 0 & |a_3|, \\ 5 |a_3| & \end{cases}. \] (2.19)

From now on, we restrict our analysis to the case where $\theta$ is chiral and all the excess charges have the same sign. Then we always have $q = 6(a_8 + b_8)$, so that
\[\det Y_u = \mathcal{O}(\lambda_u^{6(a_8 + b_8)}). \] (2.20)

Hence the X-charge of the determinant in each charge sector is independent of the texture coefficients that distinguish between the two lightest families. We set
\[\det Y_u \sim y_u^3 \mathcal{O}(\lambda_u^U), \quad \det Y_d \sim y_u^3 \mathcal{O}(\lambda_d^D), \quad \det Y_l \sim y_u^3 \mathcal{O}(\lambda_d^E), \] (2.21)
where
\[U \equiv 6(a_8 + b_8), \quad D \equiv 6(a_8 + c_8), \quad E \equiv 6(d_8 + e_8). \]

Since the down and lepton matrices have the same quantum numbers, and couple to the same Higgs, we may assume they have the same expansion parameter. In that case we can relate the products of the down quark masses to that of the leptons (assuming $y_b = y_\tau$)
\[\frac{m_d m_s m_b}{m_c m_\mu m_\tau} \sim \mathcal{O}(\lambda_d^{(D-E)}). \] (2.22)

It is more difficult to compare the up and down sectors in this way because of the unknown value of both $\tan \beta$, which sets the normalization between the two sectors, and of the relative magnitudes of the expansion parameters. In terms of their geometric mean and their ratio in the charge 2/3 and -1/3 sectors,
\[\lambda_0 = \sqrt{\lambda_u \lambda_d}, \quad \chi = \sqrt{\frac{\lambda_u}{\lambda_d}}, \] (2.23)
we find
\[\frac{m_u m_c m_t}{m_d m_s m_b} \sim \tan^3 \beta \left(\frac{y_t}{y_b}\right)^3 \mathcal{O}(\lambda_0^{(U-D)} \chi(U+D)). \] (2.24)

Experimentally, this ratio is known to be much larger than one. If the expansion parameters are the same for all three charge sectors, the data implies
\[U \approx 2D \approx 2E \approx 12, \] (2.25)
The less model dependent conclusion is that $D = E$, barring the perverse possibility that numerically, $\chi_d^{(E-D)}$ turns out to be of order one, which involves the overall normalization of $X$. We cannot infer the value of $\tan \beta$ without assuming an order of magnitude for $\chi$ and for $y_t/y_b$. From (2.13), (2.14), (2.15), we see that the hierarchy among the eigenvalues, however, does depend on the charges of the first two families.

Finally, we note that the relations (2.10) imply testable order of magnitude estimates among the Yukawa matrix elements. For example

$$Y_{11} \sim \frac{Y_{13}Y_{31}}{Y_{33}}, \quad Y_{22} \sim \frac{Y_{23}Y_{32}}{Y_{33}},$$

valid for each of the three charge sectors, and they are consistent with many of the allowed textures[5].

3. Anomalies

In general, the $X$ family symmetry is anomalous. If it is not gauged, this is not a cause for concern, although its spontaneous breakdown will generate a massless familon[16]. If gauged, its anomalies must be accounted for. The three chiral families contribute to the mixed gauge anomalies as follows

$$C_3 = 3(2a_0 + b_0 + c_0),$$
$$C_2 = 3(3a_0 + d_0),$$
$$C_1 = a_0 + 8b_0 + 2c_0 + 3d_0 + 6e_0.$$

The subscript denotes the gauge group of the Standard Model, i.e. $1 \sim U(1)$, $2 \sim SU(2)$, and $3 \sim SU(3)$. The X-charge also has a mixed gravitational anomaly, which is simply the trace of the X-charge,

$$C_g = 3(6a_0 + 3b_0 + 3c_0 + 2d_0 + e_0) + C'_g,$$

where $C'_g$ is the contribution from the particles that do not appear in the minimal $N = 1$ model. One must also account for the mixed $YXX$ anomaly, given by

$$C_{YXX} = 6(a_0^2 - 2b_0^2 + c_0^2 - d_0^2 + e_0^2) + 4A_T.$$
with the texture-dependent part given by

\[ A_T = (3a_8^2 + a_3^2) - 2(3b_8^2 + b_3^2) + (3c_8^2 + c_3^2) - (3d_8^2 + d_3^2) + (3e_8^2 + e_3^2) . \] (3.6)

The last anomaly coefficient is that of the X-charge itself, \( C_X \), the sum of the cubes of the X-charge.

Extra particles with chiral X-charge other than those in the minimal model, will contribute to both \( C_g' \) and \( C_X \), for instance, right-handed neutrino partners of the charged leptons (left-right symmetric theories). With only three chiral families (and not a fourth with a massive neutrino), new particles with electroweak quantum numbers must be electroweak vector-like pairs in order to have large \( \Delta I_W = 0 \) masses, but they need not be vector-like with respect to X-charge, in which case they will contribute to the \( C_i \) coefficients, a possibility we do not address in this letter.

From the tree-level Yukawa couplings to the third family expressed through (2.4), we can write combinations of anomaly coefficients in terms of the family-dependent charges

\[ C_1 + C_2 - \frac{8}{3} C_3 = 12(d_8 + e_8 - a_8 - c_8) = 2(E - D) , \]
\[ C_3 = 6(2a_8 + b_8 + c_8) = U + D . \] (3.7)

These allow us to relate the anomaly coefficients to the ratio of products of quark and lepton masses (2.22), (assuming \( y_b = y_{\tau} \)),

\[ \frac{m_d m_s m_b}{m_e m_{\mu} m_{\tau}} \sim \mathcal{O}(\lambda_d^{-1}(C_1 + C_2 - 8/3C_3)/2) . \] (3.8)

Compatibility with the extrapolated data requires the exponent to vanish

\[ C_1 + C_2 - \frac{8}{3} C_3 = 0 , \] (3.9)

which expressed in other variables, reads \( E = D \). Another expression relates the product of the six quark masses

\[ \Pi m_q \sim v^6(y_{t}y_{b} \sin \beta \cos \beta)^3 \mathcal{O}(\lambda_0^C_3 \chi^{(n-p)}) . \] (3.10)

If the expansion parameters are the same in both sectors (\( \chi \approx 1 \)), then

\[ C_3 \approx 18 . \] (3.11)
These equations apply only when all the $x_{ij}$ are all of the same sign.

For the X-charge to be gauged, its anomalies must be cancelled. We consider two ways to achieve this. One is to arrange the charges so that the anomalies cancel directly. The second is to appeal to a Green-Schwarz mechanism. One could also add new chiral fields to the minimal model to soak up the anomalies of the minimal model fields, but we do not consider this complicated alternative.

Let us assume first that X is anomaly-free. Then we must have

$$C_1 = C_2 = C_3 = 0, \quad C_g = 0.$$  \hspace{1cm} (3.12)

The last equation is not constraining as there are likely more fields in the theory with chiral X-charge. These are nicely consistent with (3.8), but the vanishing of $C_3$ contradicts our hypothesis that all excess charges have the same sign. Indeed, using the tree-level Yukawa relations (2.4), (3.1), we see that

$$0 = C_3 = 6(a_8 + b_8) + 6(a_8 + c_8),$$

which is not consistent with our assumption that all excess charges are positive. Hence we must rely on the Green-Schwarz mechanism.

4. Green-Schwarz Cancellation of X Anomaly

If indeed, X is anomalous, we can appeal to the Green-Schwarz mechanism to cancel some of its anomalies, and demand that the others vanish. String theories naturally contain an antisymmetric tensor Kalb-Ramond field. In four dimensions, it is the Nambu-Goldstone boson of an anomalous $U(1)$ which couples like an axion through a dimension five term to the divergence of the anomalous current. Its anomalies are cancelled by the Green-Schwarz mechanism[17]. Under a chiral transformation, this term is capable of soaking up certain anomalies, by shifting the axion field, provided that they appear in commensurate ratios

$$\frac{C_i}{k_i} = \frac{C_X}{k_X} = \frac{C_g}{k_g},$$  \hspace{1cm} (4.1)
where the $k_i$ are the Kac-Moody levels. They need to be integers only for the non-Abelian factors. We have assumed that the mixed gravitational anomaly is also cancelled à la Green-Schwarz. In other theories, it could be cancelled in the traditional way.

In superstring theories, this $U(1)$ is broken spontaneously slightly below the string scale. The scale is set by the charge content of the theory[18]. It follows that singlets with masses protected by X can still be very massive, and not appear in the effective low-energy theory.

This chiral U(1) charge may be useful for string phenomenology. Ibàñez[11] remarked that it can fix the value of the Weinberg angle, without the use of a grand unified group. More recently, Ibàñez and Ross[9] applied it to the determination of symmetric textures. Following their approach, we investigate the constraints this hypothesis puts on allowed textures at the string unification scale.

In superstring theories, the non-Abelian gauge groups have the same Kac-Moody levels. For Green-Schwarz cancellation, it means that

$$C_2 = C_3 \quad \text{or} \quad d_0 = b_0 + c_0 - a_0 . \quad (4.2)$$

After this very generic requirement, we see that equation (3.8) reduces to

$$\frac{m_DM_Sm_B}{m_e m_H m_T} \sim O(\lambda_d^{-1}(C_1 - 5/3C_2)/2) , \quad (4.3)$$

valid whenever $\theta$ is chiral. Since the right-hand side is of order one, it means that the exponent vanishes, so that in models with an $ab\ initial\ \mu$ term, we deduce that

$$C_1 = \frac{5}{3} C_2 . \quad (4.4)$$

However the gauge coupling constants at string unification scale with the anomaly coefficients, so that

$$\frac{C_1}{C_2} = \frac{g_1^2}{g_2^2} , \quad (4.5)$$

which fixes the Weinberg angle to the value

$$\sin^2 \theta_w = \frac{3}{8} ,$$
at the string scale, in perfect agreement with extrapolated phenomenology! Hence with chiral $\theta$, and the $\mu$ term, the Green-Schwarz cancellation leads to the correct value of the Weinberg angle. This may be viewed as a strong hint that the $N = 1$ model does indeed come from superstrings! Alternatively we could have imposed the canonical Weinberg angle value

$$5C_2 = 3C_1 \quad \text{or} \quad \epsilon_0 = 2a_0 - b_0,$$

which would have led us to agreement with experiment. From (3.7), we see that $E = D$.

Equations (4.2) and (4.6), together with the tree level restrictions (2.4), allow us to express the family-independent charges, $X_0$, in terms of the two observable combinations $U$, $E$, and the excess mixed gravitational anomaly

$$a_0 = \frac{1}{9}(5E + 4U - C_g + C'_g),$$

$$b_0 = \frac{1}{9}(-5E - U + C_g - C'_g),$$

$$c_0 = \frac{1}{9}(-2E - 4U + C_g - C'_g),$$

$$d_0 = -\frac{1}{3}(4E + 3U - C_g + C'_g),$$

$$e_0 = \frac{1}{3}(5E + 3U - C_g + C'_g),$$

(4.7)

We note that the gravitational anomaly is exactly along the anomaly-free combination of baryon minus lepton numbers, $B - L$. In fact the most general X-charge can contain an arbitrary mixture along $B - L$, but this is already taken into account by our general parametrization.

The vanishing of the mixed anomaly relates the excess mixed gravitational anomaly to a combination of the family-dependent charges

$$C_g - C'_g = \frac{5}{2}(U + 2E) + \frac{9A_T}{2(U + E)},$$

(4.8)

where $A_T$ is the previously defined family-dependent part. We can use this equation to
express the family-independent charges in terms of $U$, $E$, and $A_T$, with the simple results

\[
\begin{align*}
    a_0 &= \frac{U}{6} - \frac{A_T}{2(U+E)} , \\
    b_0 &= \frac{U}{6} + \frac{A_T}{2(U+E)} , \\
    c_0 &= \frac{E}{3} - \frac{U}{6} + \frac{A_T}{2(U+E)} , \\
    d_0 &= \frac{E}{3} - \frac{U}{6} + \frac{3A_T}{2(U+E)} , \\
    e_0 &= \frac{U}{6} - \frac{3A_T}{2(U+E)} ,
\end{align*}
\] (4.9)

In superstring models, the Green-Schwarz mechanism extends to the mixed gravitational anomaly so that

\[
\frac{C_g}{C_3} = \frac{k_g}{k_3} = \eta .
\]

where $\eta$ is a normalization parameter; in the simplest level-one models, it is equal to 12.

In general, however,

\[
C_g = \eta(U + E) .
\] (4.10)

The family independent X-charges are seen to depend only on two parameters, $E$, and $U$, assuming we know the normalization $\eta$.

5. Results

We have seen that when all the excess X-charges are of the same sign, the family symmetry must be anomalous in order to produce textures in agreement with experiment. When the anomalies are compensated by the Green-Schwarz mechanism, coming from superstring theory, we find that the near equality of the products of charged lepton and down quark masses in the ultraviolet (3.8) fixes the Weinberg angle to be $\sin^2 \theta_w = \frac{3}{8}$, in perfect agreement with experiment.

Consider first a simple model where the textures are symmetric in all three charge sectors. This means that

\[
a_{3,8} = b_{3,8} = c_{3,8} , \quad d_{3,8} = e_{3,8} ,
\]

so that $U = E$, and $A_T = 0$. The mixed gravitational anomaly is fixed to be

\[
C_g - C_g' = \frac{15}{2} U ,
\]
and from (4.9), the family-independent charges reduce to
\[ a_0 = b_0 = c_0 = d_0 = e_0 = \frac{U}{6}. \]

Geometric hierarchy in all three sectors can be achieved by choosing \( X \) to be the third component of the U-spin subgroup of the family \( SU(3) \)
\[ a_8 = a_3 = b_3 = c_3 = d_3. \]

However the expansion parameter cannot be the same in the charge 2/3 and -1/3 sectors. Since \( U = D \), comparison of (1.1), (1.2), and (2.21) implies that \( \lambda_u = \lambda_d^2 \), which means that \( \lambda_d = M_d/M_u \), where \( M_{u,d} \) are the mass parameters in the non renormalizable interactions (2.5) and (2.7). If the expansion parameters are all the same, the three Yukawas cannot have symmetric textures.

If we want to have the same expansion parameter in all three sectors, we have to relax this constraint by supposing that the textures are symmetric only for the down quark and charged lepton sectors. Then we can have the same expansion parameter in the two quark sectors, by requiring that \( U = 2E \), that is \( b_8 = 3a_8 \). A particularly simple assignment is to take
\[ a_8 = c_8 = d_8 = e_8 = a_3 = c_3 = d_3 = e_3 = \frac{b_8}{3} = \frac{b_3}{3}. \]

This leaves us with a geometric hierarchy in all three sectors, and the same expansion parameter. The order of eigenvalues are
\[ \mathcal{O}(1), \mathcal{O}(\lambda^{8a_8}), \mathcal{O}(\lambda^{16a_8}), \]
for the charge 2/3 sector, and
\[ \mathcal{O}(1), \mathcal{O}(\lambda^{4a_8}), \mathcal{O}(\lambda^{8a_8}), \]
for the charge -1/3 and -1 sectors. The family independent X-charges are different in all charge sectors
\[ X_0 = \left(\frac{2}{3}\right)^2 a_8 (11, 7, -2, -6, 15), \]
in the notation of (2.2). However, if we subtract an appropriate amount of hypercharge, we find a much simpler assignment,

\[ X'_0 = \left( \frac{2}{3} \right)^2 \frac{3a_8}{5} (17, 17, -6, -6, 16), \]
yielding the same X-charge for members of the same \( SU(5) \) multiplets. The family-dependent part is along the third component of family U-spin, to reproduce the geometrical hierarchies.

There are clearly many other possible charge assignments which reproduce the geometric hierarchies and yield the same expansion parameter in all three sectors. To choose among them requires making detailed assumptions about the origin of the non-renormalizable operators, and the interaction of the electroweak singlet. It was not the purpose of this letter to offer a detailed model, but to show that many of the generic features in the data can be understood simply in terms of a gauged family Abelian symmetry. In the process, we have been able to relate observable quantities to its anomalies, with the startling result that the value of the Weinberg angle is fixed by simply demanding that the ratio of down quark to lepton masses be of order one!

We leave to a future publication[12] the analysis of more general models, where the electroweak singlet field is accompanied by its vector-like partner, and where the \( \mu \) term restriction is lifted.

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References

1) For reviews, see H. P. Nilles, \textit{Phys. Rep.} \textbf{110} (1984) 1 and H. E. Haber and G. L. Kane, \textit{Phys. Rep.} \textbf{117} (1985) 75.

2) U. Amaldi, W. de Boer, and H. Furstenuau, \textit{Phys. Lett.} \textbf{B260} (1991) 447; J. Ellis, S. Kelley and D. Nanopoulos, \textit{Phys. Lett.} \textbf{260B} (1991) 131; P. Langacker and M. Luo, \textit{Phys. Rev.} \textbf{D44} (1991) 817.
3) H. Arason, D. J. Castaño, B. Keszthelyi, S. Mikaelian, E. J. Piard, P. Ramond, and B. D. Wright, Phys. Rev. Lett. 67 (1991) 2933; A. Giveon, L. J. Hall, and U. Sarid, Phys. Lett. 271B (1991) 138.

4) J. C. Pati and A. Salam, Phys. Rev. D10 (1974) 275; H. Georgi and S. Glashow, Phys. Rev. Lett. 32 (1974) 438; H. Georgi, in Particles and Fields-1974, edited by C.E.Carlson, AIP Conference Proceedings No. 23 (American Institute of Physics, New York, 1975) p.575; H. Fritzsch and P. Minkowski, Ann. Phys. NY 93 (1975) 193; F. Gürsey, P. Ramond, and P. Sikivie, Phys. Lett. 60B (1975) 177.

5) P. Ramond, R.G. Roberts and G. G. Ross, Nucl. Phys. B406 (1993) 19.

6) L. Wolfenstein, Phys. Rev. Lett. 51 (1983) 1945.

7) We reported our preliminary results at the Fermilab workshop on Yukawa Couplings, Nov., 1994, where we learned that a similar line of enquiry was being followed by V. Jain and R. Shrock.

8) M. Leurer, Y. Nir, and N. Seiberg, Nucl. Phys. B398 (1993) 319.

9) L. Ibáñez and G. G. Ross, Phys. Lett. B332 (1994) 100.

10) E. Papageorgiou, “Yukawa Textures from an extra $U(1)$ Symmetry”, Orsay Preprint, LPTHE Orsay 40/94.

11) L. Ibáñez, Phys. Lett. B303 (1993) 55.

12) P. Binétruy, S. Lavignac, and P. Ramond, in preparation.

13) M. Gell-Mann, P. Ramond, and R. Slansky in Sanibel Talk, CALT-68-709, Feb 1979, and in Supergravity (North Holland, Amsterdam 1979). T. Yanagida, in Proceedings of the Workshop on Unified Theory and Baryon Number of the Universe, KEK, Japan, 1979.

14) C. Froggatt and H. B. Nielsen Nucl. Phys. B147 (1979) 277.

15) S. Dimopoulos, L. Hall, S. Raby, and G. Starkman, Phys. Rev. D49 (1994) 3660.

16) D. B. Reiss, Phys. Lett. B155 (1982) 217; F. Wilczek, Phys. Rev. Lett. 49 (1982) 1549.

17) M. Dine, N. Seiberg, and E. Witten, Nucl. Phys. B289 (1987) 585; J. Atick, L. Dixon, and A. Sen, Nucl. Phys. B292 (1987) 109; M. Dine, I. Ichinoise, and N. Seiberg, Nucl. Phys. B293 (1987) 253.

18) A. Font, L.E. Ibáñez, H. P. Nilles, and F. Quevedo, Nucl. Phys. B307 (1988) 109; Phys. Lett. B210 (1988) 101; J. A. Casas, E. K. Katehou, and C. Muñoz, Nucl. Phys. B317 (1989) 171; J. A. Casas, and C. Muñoz, Phys. Lett. B209 (1988) 214; Phys. Lett. B214 (1988) 63; A. Font, L.E. Ibáñez, F. Quevedo, and A. Sierra, Nucl. Phys. B331 (1990) 421.