Hamiltonian cycles for the square of the augmentation graphs and Gray codes for restricted permutations and ascent sequences

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Abstract

In this paper, we construct a listing for the vertices of the augmentation graph of given size, and as a consequence, we obtain a Hamiltonian cycle for the square of the augmentation graph of given size. As applications, we have a Gray code for the 132-312 avoiding permutations of given length such that two successive permutations differ by at most 2 adjacent transpositions. Also we obtain Gray codes of strong distance 2 for the 001 avoiding ascent sequences and the 010 avoiding ascent sequences of given length.

1 Introduction

The combinatorial Gray codes is a mathematical model to generate the given combinatorial object such that each element is generated exactly once and any successive elements differ in some pre-specified, usually small, way. There are many examples of minimal change listings of combinatorial objects. Though there are many bijections among the combinatorial objects of same cardinality, in general, a bijection does not preserve the Gray code structures and hence we should construct Gray codes by ad hoc methods in many cases [15].

It is known that a Gray code problem can be formulated as a Hamiltonian path or a Hamiltonian cycle problem: the vertices of the graph are the objects and two vertices are joined by an edge if they differ from each other in the pre-specified way.

The augmentation graph of size $n$, see Section 3, can be defined on the binary words of length $n$. Douglas West posed the problem to determine whether there is a Hamiltonian path in the augmentation graph. When $\frac{n(n-1)}{2}$ is even, there are no Hamiltonian paths for the augmentation graph of size $n$ and to the best of our knowledge, the problem is open for $n \geq 7$ with $\frac{n(n-1)}{2}$ being odd [15]. In Section 3 we construct a Hamiltonian cycle for the square of the augmentation graph of size $n$.

The theory of pattern avoidance is a very active research area and many papers are written in this theme, see Kitaev’s survey [11]. Basic notations about pattern avoiding permutations are given in
Section 2.2 The most popular results on the pattern avoidance are the enumerations of permutations avoiding a pattern of length three, which is enumerated by the Catalan numbers [12] [13] [17]. For further information on pattern avoidance, see [5]. One research direction is to construct Gray codes for the permutations of given length avoiding a set of patterns and many Gray codes are found by several authors. We denote the set of the permutations of length $n$ avoiding a pattern $p$ by $S_n(p)$ and that avoiding any pattern in $A$ by $S_n(A)$, where $A$ is a set of permutations. Juarna and Vajnovszki constructed Gray codes for $S_n(123,132)$ and $S_n(123,132, p(p−1)· · ·1(p+1))$ [10]. In [8], Dukes et al. gave Gray codes for large families of pattern avoiding permutations including many fundamental classes. Baril improved their results [1]. Their proofs are based on ECO method [2] [4].

The most fundamental cases are Gray codes for the permutations of length $n$ avoiding a single pattern of length three. In particular, Baril constructed Gray codes for $S_n(p)$ for $p ∈ S_3$, where two consecutive permutations differ by at most three positions and his results are optimal for odd $n$ [1]. Next, we should consider the permutations of length $n$ avoiding two patterns of length three. By the reversal and complementation and their compositions, there are five symmetry classes avoiding two pattern of length three, they are $S_n(123,321)$, $S_n(123,231)$, $S_n(231,312)$, $S_n(123,312)$ and $S_n(132,312)$ [17].

The set $S_n(123,321)$ is empty set for $n ≥ 5$. Also it is easy to see that $S_n(123,231)$ and $S_n(231,312)$ have no Gray codes such that successive permutations differ by at most $k$ transpositions, where the constant $k$ does not depend on $n$. For instance, consider $\lfloor \frac{n}{2} \rfloor (\lfloor \frac{n}{2} \rfloor − 1) · · · 21n(n−1) · · · (\lfloor \frac{n}{2} \rfloor + 2)(\lfloor \frac{n}{2} \rfloor + 1)$, where $\lfloor · \rfloor$ is the floor function that gives the greatest integer less than or equal to the number. Juarna and Vajnovszki gave a Gray code for $S_n(123,132)$ in [10]. The remaining case is $S_n(132,312)$, which is not treated in the previous works. In section 4, we give a Gray code for $S_n(132,312)$ as an application of our Hamiltonian cycle for $G(n – 1)$ in Section 3.

The ascent sequences, which we define in Section 2.3 play an important role in the study of $(2 + 2)$-free posets and several researchers have found the connections between the ascent sequences and other combinatorial objects enumerated by Fishburn number, see [6] and section 3.2.2 of [11].

One can define the patterns in the ascent sequences analogous to the patterns in permutations, see Section 2.3. Pattern avoidance in the ascent sequences was studied by Baxter, Duncan, Pudwell and Steingrimsson and enumerative results and further properties about the pattern avoiding ascent sequences are found at [3], [9] and their references.

In [13], Sabri and Vajnovszki constructed a Gray code for the ascent sequences of given length. Sabri gave Gray codes for $A_n(p)$ for $p ∈ \{011,101,021,201,012\}$, where $A_n(p)$ denotes the set of ascent sequences of length $n$ avoiding a pattern $p$ [10]. His Gray codes are based on the Hamming distance, the number of positions at which two sequences differ. In this paper, we use more restrictive distance, we call it the strong distance, see Section 2.3 and construct Gray codes for $A_n(001)$ and $A_n(010)$ such that the strong distance of two successive sequences is at most 2.

Throughout the paper, we set $F_2 := \{0,1\}$ and $F^n_2 := \{ε_1ε_2· · ·ε_n | ε_i ∈ F_2, 1 ≤ i ≤ n\}$, the set of binary words of length $n$.

2 Preliminaries

In this section, we give several definitions and notations which we use in this paper.
2.1 The Hamiltonian cycles and the Gray codes

In this section, we introduce several notations from graph theory, see Diestel’s text \[7\]. A graph is a pair $G = (V, E)$ of sets such that $E \subset V \times V$. Sometimes we denote $G$ for short. A element of $V$ is called a vertex and that of $E$ is called an edge.

A path is a tuple of vertices such that consecutive vertices are adjacent in the graph and the vertices are all distinct. The length of a path is the number of the vertices minus one. For example, an edge is a path with two vertices and its length is 1. A cycle is a path such that the starting vertex and the ending vertex are adjacent. A graph is called connected if there is a path connecting each pair of vertices. For a connected graph $G$ and two vertices $u, v \in G$, the distance between $u$ and $v$ is the length of a shortest path from $u$ to $v$ and we denote it by $d_G(u, v)$. A Hamiltonian path (resp. Hamiltonian cycle) is a path (resp. cycle) that visits each vertex exactly once. For a graph $G = (V, E)$, define $G^2$ to be the graph on $V$ such that two vertices are adjacent if and only if their distance in $G$ is at most 2. The graph $G^2$ is called the square of $G$.

The combinatorial Gray codes is a mathematical model to generate the given combinatorial object such that each element is generated exactly once and any successive elements differ in some pre-specified, usually small, way. It is known that the Gray code problem can be formulated as a Hamiltonian path or a Hamiltonian cycle problem: the vertices of the graph are the objects and two vertices are joined by an edge if they differ from each other in the pre-specified way.

A listing of a combinatorial object is a sequence such that each element appears exactly once. Let $X$ and $Y$ be combinatorial objects with $X \cap Y = \phi$ and let $L_X = (x_1, x_2, \ldots, x_m)$ and $L_Y = (y_1, y_2, \ldots, y_n)$ be listings of $X$ and $Y$ respectively. Write $L_X \circ L_Y := (x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n)$, i.e., the concatenation of $X$ and $Y$, which is a listing of $X \cup Y$.

2.2 Permutations and pattern avoiding permutations

We use one-line notation, i.e., we denote a permutation $\omega \in S_n$ by the sequence $\omega(1)\omega(2)\cdots\omega(n)$, where $S_n$ is the set of all permutations on $\{1, 2, \ldots, n\}$.

For two permutations $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ and $\tau = \tau_1\tau_2\cdots\tau_n$, we call they differ by an adjacent transposition if and only if $\sigma_k = \tau_k$ for $k \neq i, i + 1, \sigma_i = \tau_{i+1}$ and $\sigma_{i+1} = \tau_i$ for some $1 \leq i \leq n - 1$. In other words, $\sigma$ can be obtained from $\tau$ by changing adjacent positions in the one-line notation of $\tau$. For example, 352641 and 325614 differ by an adjacent transposition, changing the second and the third positions.

Similarly, we call two permutations of same length $\sigma$ and $\tau$ differ by two adjacent transpositions if and only if $\sigma \neq \tau$ and there exists a permutation $\theta$ such that $\sigma$ and $\theta$ differ by an adjacent transposition and also $\theta$ and $\tau$ do. For example, 352641 and 325614 differ by two adjacent transpositions. We say that $\sigma$ and $\tau$ differ by at most two adjacent transpositions if $\sigma$ and $\tau$ differ by an adjacent transposition or two adjacent transpositions. Similarly, one can also define when two permutations differ by at most $k$ adjacent transpositions.

For $\omega = \omega_1\omega_2\cdots\omega_n \in S_n$ and $\pi = \pi_1\pi_2\cdots\pi_k \in S_k$ with $k \leq n$, we say that a permutation $\omega$ has a $\pi$-pattern if $st(\omega_{i_1}\omega_{i_2}\cdots\omega_{i_k}) = \pi_1\pi_2\cdots\pi_k$ for some $1 \leq i_1 < i_2 < \cdots < i_k \leq n$, where $st(\omega_{i_1}\omega_{i_2}\cdots\omega_{i_k})$ is a permutation in $S_k$ defined by the following process: the smallest value of $\omega_{i_1}\omega_{i_2}\cdots\omega_{i_k}$ is replaced with 1, the second smallest value is replaced with 2, and so on. We
call \( st(\omega_1, \omega_2, \ldots, \omega_k) \) the standardization of \( \omega_1, \omega_2, \ldots, \omega_k \). If \( st(\omega_1, \omega_2, \ldots, \omega_k) \neq \pi_1 \pi_2 \cdots \pi_k \) for any \( 1 \leq i_1 < i_2 < \cdots < i_k \leq n \), we say that \( \omega \) is a \( \pi \)-avoiding permutation. We denote the set of \( \pi \)-avoiding permutations of length \( n \) by \( S_n(\pi) \) and also for a set of permutations \( P \), let \( S_n(\pi) \) denote the set of permutations of length \( n \) avoiding each pattern in \( P \).

In this paper, we define Gray codes for a set of permutations as follows.

**Definition 2.1.** For \( n \in \mathbb{N} \) and a subset \( \mathcal{X} \) of \( S_n \), a Gray code for \( \mathcal{X} \) is a listing of \( \mathcal{X} \) such that each successive permutations differ by at most \( d \) adjacent transpositions, where \( d \in \mathbb{N} \) is independent from \( n \).

### 2.3 Ascent sequences and restricted ascent sequences

An **ascent sequence** is a sequence \( x_1 x_2 \cdots x_n \) of nonnegative integers such that \( x_1 = 0 \) and

\[
x_i \leq \text{asc}(x_1 x_2 \cdots x_{i-1}) + 1
\]

for \( 2 \leq i \leq n \), where \( \text{asc}(x_1 x_2 \cdots x_{i-1}) \) is the number of ascents in the sequence \( x_1 x_2 \cdots x_{i-1} \), that is, the number of \( 1 \leq j \leq i-2 \) such that \( x_j < x_{j+1} \). For example, 01201014216 is an ascent sequence of length 11 and 012010635 is not an ascent sequence, because the 7th position is greater than \( \text{asc}(012010) + 1 = 4 \). The number of ascent sequences of length \( n \) is enumerated by Fishburn number, see A022493 in OEIS. We denote the set of ascent sequences of length \( n \) by \( \mathcal{A}_n \).

For a nonnegative integer sequence \( x = x_1 x_2 \cdots x_n \), the **reduction** of \( x \) is the sequence obtained by replacing the \( i \)-th smallest digits of \( x \) with \( i - 1 \) and we denote it by \( \text{red}(x) \). For example, \( \text{red}(200424) = 100212 \). One can define the patterns in the ascent sequences analogous to the patterns in permutations.

**Definition 2.2.** \(^{[3]} \) Let \( p_1 p_2 \ldots p_k \) be a sequence of nonnegative integers with \( \text{red}(p_1 p_2 \ldots p_k) = p_1 p_2 \ldots p_k \). An ascent sequence \( a_1 a_2 \ldots a_n \) is called a \( p_1 p_2 \ldots p_k \) pattern avoiding ascent sequence if and only if for any \( 1 \leq i_1 < i_2 < \cdots < i_k \leq n \), \( \text{red}(a_{i_1} a_{i_2} \ldots a_{i_k}) \neq p_1 p_2 \ldots p_k \). We denote the set of \( p_1 p_2 \ldots p_k \) avoiding ascent sequences of length \( n \) by \( \mathcal{A}_n(p_1 p_2 \ldots p_k) \).

For a set of integer sequences \( \mathcal{I} \) of given length \( n \), a Gray code for \( \mathcal{I} \) is sometimes defined as a listing of \( \mathcal{I} \) such that the **Hamming distance** between any two successive elements (the number of positions at which two sequences differ) is bounded by a given constant which is independent from \( n \). Sabri gave Gray codes of Hamming distance 3, that is a listing such that the Hamming distance of any consecutive elements is at most 3, for \( \mathcal{A}_n(p) \), where \( p \in \{011, 101, 021, 201\} \) and that of Hamming distance 1 for \( \mathcal{A}_n(012) \) \(^{[10]} \).

We define a Gray code by using another distance. For \( a_1 a_2 \ldots a_n, b_1 b_2 \ldots b_n \in \mathcal{A}_n \), define

\[
d_{\text{str}}(a_1 a_2 \ldots a_n, b_1 b_2 \ldots b_n) := \sum_{k=1}^{n} |a_k - b_k|,
\]

where \( | \cdot | \) is the absolute value of given number.

We call \( d_{\text{str}} \) **strong distance**. The strong distance is essentially equivalent to the **adjacent interchange property** which is referred in \(^{[15]} \). In analogy with Sabri’s definition in \(^{[16]} \), we can state the following definition.
Definition 2.3. For a subset $X$ of $A_n$, a Gray code of strong distance $d$ for $X$ is a listing of $X$ such that each strong distance of two successive elements is at most $d$ and there exists two successive elements of strong distance $d$, where $d \in \mathbb{N}$ is independent from $n$.

The strong distance of given two ascent sequences is larger than or equal to the Hamming distance of them and the strong distance is a more restrictive distance than the Hamming distance.

3 A Hamiltonian cycle for the square of the augmentation graph

Douglas West defined the augmentation graph of size $n$ on the subsets of $\{1,2,\ldots,n\}$ [15]. We realize the augmentation graph on $\mathbb{F}_2^n$ and we denote it by $G(n)$. No confusion should results when we call our graph $G(n)$ the augmentation graph of size $n$.

Set $V(G(n)) := \mathbb{F}_2^n$ and draw an edge between $\epsilon_1\epsilon_2\cdots\epsilon_n$ and $\epsilon'_1\epsilon'_2\cdots\epsilon'_n \in \mathbb{F}_2^n$ if

1. $\epsilon_1 \neq \epsilon'_1$ and $\epsilon_i = \epsilon'_i$ for $2 \leq i \leq n$ or
2. for some $1 \leq i \leq n-1$, $\epsilon_i\epsilon_{i+1} = 01$ and $\epsilon'_i\epsilon'_{i+1} = 10$ or $\epsilon_i\epsilon_{i+1} = 10$ and $\epsilon'_i\epsilon'_{i+1} = 01$ and $\epsilon_j = \epsilon'_j$ for $j \neq i, i+1$.

In other words, two vertices are adjacent if and only if they only differ in the first positions or differ by an interchange of a 0 and a 1 in adjacent positions.

Remark 3.1. In $G(n)$, the degree of $00\cdots0$ and $11\cdots1$ is 1. Hence $G(n)$ has no Hamiltonian cycles for $n \geq 2$. To the best of our knowledge, there are no Hamiltonian paths for $G(n)$ with $\frac{n(n-1)}{2}$ being even and the existence of Hamiltonian paths is not known for $n \geq 7$ with $\frac{n(n-1)}{2}$ being odd [15].

Figure 1 shows $G(3)$, left hand side, and $G^2(3)$, right hand side.

![Figure 1: The graphs $G(3)$ and $G^2(3)$.](image-url)
Remark 3.2. A listing of $V(G(n))$ is a $2^n$-tuple of $V(G(n))$ such that each vertex appears exactly once. For a listing $(\alpha_1, \alpha_2, \cdots, \alpha_{2^n})$ such that every distance of successive vertices is at most 2, we say that distance 2 jumps do not appear consecutively in the listing if there is no $2 \leq i \leq 2^n - 1$ such that $d_{G(n)}(\alpha_i, \alpha_{i+1}) = 2$.

Definition 3.1. For $\epsilon = \epsilon_1\epsilon_2\cdots\epsilon_m \in \mathbb{F}_2^m$ and $\eta, \mu \in \mathbb{F}_2$, define $\epsilon\eta := \epsilon_1\epsilon_2\cdots\epsilon_m\eta$ (resp. $\epsilon\eta\mu := \epsilon_1\epsilon_2\cdots\epsilon_m\eta\mu$) which is the concatenation of $\epsilon$ and $\eta$, (resp. $\epsilon$, $\eta$ and $\mu$) and it is an element of $\mathbb{F}_2^{m+1}$ (resp. $\mathbb{F}_2^{m+2}$).

The following is the main result of this section.

Theorem 3.1. For $n \geq 2$, there is a listing

$$(\epsilon_1, \epsilon_2, \cdots, \epsilon_{2^n})$$

of $V(G(n))$, which satisfies:

(L1) $\epsilon_1 = 00 \cdots 0$, all entries are 0, and $\epsilon_{2^n} = 100 \cdots 0$, the first entry is 1 and the rest of the entries are all 0;

(L2) $d_{G(n)}(\epsilon_i, \epsilon_{i+1}) \leq 2$ for $1 \leq i \leq 2^n - 1$;

(L3) if $d_{G(n)}(\epsilon_i, \epsilon_{i+1}) = 2$, then $d_{G(n)}(\epsilon_{i-1}, \epsilon_i) = d_{G(n)}(\epsilon_{i+1}, \epsilon_{i+2}) = 1$ for $2 \leq i \leq 2^n - 2$.

In other words, there is a listing which starts from $00 \cdots 0$ and ends at $100 \cdots 0$ such that every distance of successive elements is at most 2 and distance 2 jumps do not appear consecutively.

In the above listing $(\epsilon_1, \epsilon_2, \cdots, \epsilon_{2^n})$, the distance of $\epsilon_1$ and $\epsilon_{2^n}$ is 1. Hence we obtain the following Corollary.

Corollary 3.1. For $n \in \mathbb{N}_{\geq 2}$, the square of $G(n)$ has a Hamiltonian cycle.

To prove Theorem 3.1 we show Proposition 3.1 and Proposition 3.2.

Proposition 3.1. For $k \geq 2$, suppose that there is a listing

$$(\epsilon_1, \epsilon_2, \cdots, \epsilon_{2^{2k-1}})$$

of $V(G(2k-1))$, which satisfies:

(A1) $\epsilon_1 = 00 \cdots 00$, all entries are 0, and $\epsilon_{2^{2k-1}} = 10 \cdots 00$, the first entry is 1 and the rest of the entries are all 0;

(A2) $d_{G(2k-1)}(\epsilon_i, \epsilon_{i+1}) \leq 2$ for $1 \leq i \leq 2^{2k-1} - 1$ and distance 2 jumps do not appear consecutively;

(A3) the element $00 \cdots 01$, the last entry is 1 and the rest of the entries are all 0, appears next to $10 \cdots 01$, the first and the last entries are 1 and the rest of the entries are all 0;

(A4) $\epsilon_{2^{2k-1} - 1}$, the second from the last entry of the listing, is $001 \cdots 00$, the third entry is 1 and the rest of entries are all 0.

Then there is a listing

$$(\eta_1, \eta_2, \cdots, \eta_{2^{2k}})$$

(3)
of \( V(G(2k)) \), which satisfies:

(B1) \( \eta_1 = 00 \cdots 00 \), all entries are 0, and \( \eta_{2k} = 10 \cdots 00 \), the first entry is 1 and the rest of the entries are all 0;

(B2) \( d_{G(2k)}(\eta_i, \eta_{i+1}) \leq 2 \) for \( 1 \leq i \leq 2^k - 1 \) and distance 2 jumps do not appear consecutively;

(B3) 001 \cdots 01, the third and the last entries are 1 and the rest of the entries are all 0, appears next to 100 \cdots 01, the first and the last entries are 1 and the rest of the entries are all 0;

(B4) \( \eta_{2k-1} \), the second from the last entry in the listing, is 001 \cdots 0, the third entry is 1 and the rest of the entries are all 0.

Proof. Let \( L_{2k-1} = (\epsilon_1, \epsilon_2, \cdots, \epsilon_{2k-1}) \) be a listing of \( V(G(2k-1)) \) which satisfies the conditions from (A1) to (A4). From (A3), we have \( \epsilon_i = 10 \cdots 01 \), the first and the last entries are 1 and the rest of the entries are all 0, and \( \epsilon_{i+1} = 00 \cdots 01 \), the last entry is 1 and the rest of entries are all 0, for some \( 2 \leq i \leq 2^{k-1} - 2 \). Remark that \( \epsilon_{2^{k-1}-1} = 001 \cdots 00 \), the third entry is 1 and the rest of entries are all 0.

Set

\[
X_1 := (\epsilon_1 0, \epsilon_2 0, \cdots, \epsilon_i 0),
\]

\[
X_2 := (\epsilon_{2^{k-1} 1}, \epsilon_{2^{k-1} 1} 0, \cdots, \epsilon_2 1, \epsilon_1 1),
\]

\[
X_3 := (\epsilon_{i+1} 0, \epsilon_{i+2} 0, \cdots, \epsilon_{2^{k-1} 0}, \epsilon_{2^{k-1} 0}).
\]

The concatenation \( X_1 \circ X_2 \circ X_3 \) is a listing of \( G(2k) \) and put \( X_1 \circ X_2 \circ X_3 = (\eta_1, \eta_2, \cdots, \eta_{2k}) \). Remark that \( |X_3| \geq 2 \).

From the construction, we have \( \eta_1 = 00 \cdots 00 \) and \( \eta_{2k} = 10 \cdots 00 \). We see (B1).

For each \( X_1, X_2 \) and \( X_3 \), every distance of successive elements is at most 2 and distance 2 jumps do not appear consecutively. Also, the distance between the last element of \( X_1 \) and the first element of \( X_2 \) is 1 and that of between the last element of \( X_2 \) and the first element of \( X_3 \) is 1. Hence \( X_1 \circ X_2 \circ X_3 \) satisfies (B2).

The first entry of \( X_2 \) is 100 \cdots 01, the first and the last entries are 1 and the rest of entries are all 0. Because \( \epsilon_{2^{k-1}-1} = 001 \cdots 00 \), the third entry is 1 and the rest of entries are all 0, the second entry of \( X_2 \) is 001 \cdots 01, the third and the last entries are 1 and the rest of entries are all 0. Hence we see (B3).

The second from the last entry of \( X_1 \circ X_2 \circ X_3 \) is in \( X_3 \), because \( |X_3| \geq 2 \), and that is \( \epsilon_{2^{k-1}-1} 0 = 001 \cdots 00 \), the third entry is 1 and the rest of the entries are all 0. Hence we see (B4). This completes the proof.

\[ \square \]

Example 3.1. For \( G(3) \), put

\[
\epsilon_1 = 000, \quad \epsilon_2 = 010, \quad \epsilon_3 = 110, \quad \epsilon_4 = 011, \quad \epsilon_5 = 111, \quad \epsilon_6 = 101, \quad \epsilon_7 = 001, \quad \epsilon_8 = 100
\]
and then, the listing \((e_1, e_2, \cdots, e_8)\) satisfies from (A1) to (A4). Set
\[
X_1 := (e_1 0, e_2 0, e_3 0, e_4 0, e_5 0, e_6 0),
\]
\[
X_2 := (e_8 1, e_7 1, e_6 1, e_5 1, e_4 1, e_3 1, e_2 1, e_1 1)
\]
and
\[
X_3 := (e_7 0, e_8 0).
\]
Then the listing \(X_1 \circ X_2 \circ X_3\) is
\[
(0000, 0100, 1100, 0110, 1010, 1001, 0011, 1011, 1111, 0111, 1101, 0101, 0001, 0010, 1000)
\]
and that satisfies from (B1) to (B4).

**Proposition 3.2.** For \(k \geq 2\), suppose that there is a listing
\[
(e_1, e_2, \cdots, e_{2^k})
\]
of \(V(G(2k))\), which satisfies:

(B1) \(e_1 = 00 \cdots 00\), all entries are 0, and \(e_{2^k} = 10 \cdots 00\), the first entry is 1 and the rest of the entries are all 0;

(B2) \(d_{G(2k)}(e_i, e_{i+1}) \leq 2\) for \(1 \leq i \leq 2^k - 1\) and distance 2 jumps do not appear consecutively;

(B3) \(001 \cdots 01\), the third and the last entries are 1 and the rest of the entries are all 0, appears next to \(100 \cdots 01\), the first and the last entries are 1 and the rest of the entries are all 0;

(B4) \(e_{2^k-1}\), the second from the last entry in the listing, is \(001 \cdots 00\), the third entry is 1 and the rest of the entries are all 0.

Then there is a listing
\[
(\eta_1, \eta_2, \cdots, \eta_{2^{k+1}})
\]
of \(V(G(2k + 1))\), which satisfies:

(A1) \(\eta_1 = 00 \cdots 00\), all entries are 0, and \(\eta_{2^{k+1}} = 10 \cdots 00\), the first entry is 1 and the rest of the entries are all 0;

(A2) \(d_{G(2k+1)}(\eta_i, \eta_{i+1}) \leq 2\) for \(1 \leq i \leq 2^{k+1} - 1\) and distance 2 jumps do not appear consecutively;

(A3) the element \(00 \cdots 01\), the last entry is 1 and the rest of the entries are all 0, appears next to \(10 \cdots 01\), the first and the last entries are 1 and the rest of the entries are all 0;

(A4) \(\eta_{2^{k+1}-1}\), the second from the last entry in the listing, is \(001 \cdots 00\), the third entry is 1 and the rest of entries are all 0.

**Proof.** Let \(L_{2k} = (e_1, e_2, \cdots, e_{2^k})\) be a listing of \(V(G(2k))\) which satisfies the conditions from (B1) to (B4). From (B3), we have \(e_i = 10 \cdots 01\), the first and the last entries are 1 and the rest of entries are all 0, and \(e_{i+1} = 001 \cdots 01\), the third and the last entries are 1 and the rest of entries are all 0,
for some \(2 \leq i \leq 2^{2k} - 2\). Remark that \(\mathbf{e}_{2^{2k-1}} = 001 \cdots 00\), the third entry is 1 and the rest of the entries are all 0.

Set

\[ \mathbb{Y}_1 := (\mathbf{e}_1 0, \mathbf{e}_2 0, \cdots, \mathbf{e}_i 0), \]  

\[ \mathbb{Y}_2 := (\mathbf{e}_{2^{2k}1}, \mathbf{e}_1 1, \mathbf{e}_2 1, \cdots, \mathbf{e}_{2^{2k}-2}, \mathbf{e}_{2^{2k}1}), \]  

\[ \mathbb{Y}_3 := (\mathbf{e}_{i+1} 0, \mathbf{e}_{i+2} 0, \cdots, \mathbf{e}_{2^{2k}1}). \]  

The concatenation \(\mathbb{Y}_1 \circ \mathbb{Y}_2 \circ \mathbb{Y}_3\) is a listing of \(G(2k+1)\) and put \(\mathbb{Y}_1 \circ \mathbb{Y}_2 \circ \mathbb{Y}_3 = (\mathbf{\eta}_1, \mathbf{\eta}_2, \cdots, \mathbf{\eta}_{2^{2k+1}})\). Remark that \(|\mathbb{Y}_3| \geq 2\).

From the construction, we have \(\mathbf{\eta}_1 = 00 \cdots 00\), all entries are 0 and \(\mathbf{\eta}_{2^{2k+1}} = 10 \cdots 00\), first entry is 1 and the rest of the entries are all 0. Hence we see (A1).

In \(\mathbb{Y}_2\), the distance between the first element and the second element is 1. Hence, for each \(\mathbb{Y}_1, \mathbb{Y}_2\) and \(\mathbb{Y}_3\), every distance of successive elements is at most 2 and distance 2 jumps do not appear consecutively. Also the distance between the last element of \(\mathbb{Y}_1\) and the first element of \(\mathbb{Y}_2\) is 1 and that of between the last element of \(\mathbb{Y}_2\) and the first element of \(\mathbb{Y}_3\) is 1. Hence \(\mathbb{Y}_1 \circ \mathbb{Y}_2 \circ \mathbb{Y}_3\) satisfies (A2).

The first element of \(\mathbb{Y}_2\) is 10 \cdots 01, the first and the last entries are 1 and the rest of entries are all 0, and the second element of \(\mathbb{Y}_2\) is 00 \cdots 01, the last entry is 1 and the rest of entries are all 0. Hence we obtain (A3). The second from the last entry of \(\mathbb{Y}_1 \circ \mathbb{Y}_2 \circ \mathbb{Y}_3\) is in \(\mathbb{Y}_3\), because \(|\mathbb{Y}_3| \geq 2\), and that is \(\mathbf{e}_{2^{2k}-1} 0 = 001 \cdots 00\). Hence we have (A4). This completes the proof.

\[\square\]

**Example 3.2.** From example 3.1 put

\[ \mathbf{e}_1 = 0000, \quad \mathbf{e}_2 = 0100, \quad \mathbf{e}_3 = 1100, \quad \mathbf{e}_4 = 0110, \quad \mathbf{e}_5 = 1110, \quad \mathbf{e}_6 = 1010, \quad \mathbf{e}_7 = 1001, \]
\[ \mathbf{e}_8 = 0011, \quad \mathbf{e}_9 = 1011, \quad \mathbf{e}_{10} = 1111, \quad \mathbf{e}_{11} = 0111, \quad \mathbf{e}_{12} = 1101, \quad \mathbf{e}_{13} = 0101, \quad \mathbf{e}_{14} = 0001, \]
\[ \mathbf{e}_{15} = 0010, \quad \mathbf{e}_{16} = 1000. \]

The listing \((\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_{16})\) of \(V(G(4))\) satisfies from (B1) to (B4).

Set

\[ \mathbb{Y}_1 := (\mathbf{e}_1 0, \mathbf{e}_2 0, \mathbf{e}_3 0, \mathbf{e}_4 0, \mathbf{e}_5 0, \mathbf{e}_6 0, \mathbf{e}_7 0), \]  

\[ \mathbb{Y}_2 := (\mathbf{e}_{16} 1, \mathbf{e}_1 1, \mathbf{e}_2 1, \mathbf{e}_3 1, \mathbf{e}_4 1, \mathbf{e}_5 1, \mathbf{e}_6 1, \mathbf{e}_7 1, \mathbf{e}_8 1, \mathbf{e}_9 1, \mathbf{e}_{10} 1, \mathbf{e}_{11} 1, \mathbf{e}_{12} 1, \mathbf{e}_{13} 1, \mathbf{e}_{14} 1, \mathbf{e}_{15} 1) \]  

and

\[ \mathbb{Y}_3 := (\mathbf{e}_{8} 0, \mathbf{e}_9 \mathbf{e}_{10} 0, \mathbf{e}_{11} 0, \mathbf{e}_{12} 0, \mathbf{e}_{13} 0, \mathbf{e}_{14} 0, \mathbf{e}_{15} 0 \mathbf{e}_{16} 0). \]

Then the listing \(\mathbb{Y}_1 \circ \mathbb{Y}_2 \circ \mathbb{Y}_3 = (\mathbf{\eta}_1, \mathbf{\eta}_2, \cdots, \mathbf{\eta}_{32})\) of \(V(G(5))\) is
In this section, we construct a Gray code for $S_n(132, 312)$.

The 132-312 avoiding permutations is called Gilbreath permutations. A permutation $\sigma = \sigma_1\sigma_2\ldots\sigma_n \in S_n$ with $\sigma_1 = k$, where $1 \leq k \leq n$, is a 132-312 avoiding permutation if and only if $\sigma$ is a shuffle of the sequences $k(k-1)\ldots21$ and $(k+1)(k+2)\ldots(n-1)n$ with $\sigma_1 = k$, see Proposition 12 in [17]. For example, 567438291 $\in S_9$ is a shuffle of 54321 and 6789 and it is a 132-312 avoiding permutation. In this section, we construct a Gray code for $S_n(132, 312)$ for $n \geq 2$.

First we construct a natural bijection from $V(G(n-1))$ to $S_n(132, 312)$ which preserves minimal changes of these objects. We note that our map is essentially the same bijection presented at the proof of Proposition 12 in [17].

**Definition 4.1.** For $n \in \mathbb{N}_{\geq 2}$ and $\epsilon = \epsilon_1\epsilon_2\ldots\epsilon_{n-1} \in V(G(n-1))$, define $\Psi_n(\epsilon)$ to be a positive integer sequence of length $n$, say $a_1a_2\ldots a_n$, such that

1. $a_1$ is the cardinality of 0 in $\epsilon$ plus 1,
2. if $\epsilon_{i-1} = 0$ with $i \geq 2$, then $a_i$ is the cardinality of $j$ such that $j \geq i - 1$ and $\epsilon_j = 0$,
3. if $\epsilon_{i-1} = 1$ with $i \geq 2$, then $a_i$ is $a_1$ plus the cardinality of $j$ such that $j \leq i - 1$ and $\epsilon_j = 1$.

For example, $\Psi(1001011) = 45326178$ and $\Psi(0001011) = 54326178$. It is easy to see the following Lemma.

**Lemma 4.1.** Notation is as above, $a_1a_2\ldots a_n$ is a permutation of length $n$. Furthermore, it is a shuffle of $a_1(a_1 - 1)\ldots21$ and $(a_1 + 1)(a_1 + 2)\ldots(n-1)n$ and hence it is a 132-312 avoiding permutation. Moreover, $\Psi_n$ is a bijection from $G(n-1)$ to $S_n(132, 312)$. 

\[
\begin{align*}
\eta_1 &= 00000, & \eta_2 &= 01000, & \eta_3 &= 11000, & \eta_4 &= 01100, & \eta_5 &= 11100, & \eta_6 &= 10100, & \eta_7 &= 10010, \\
\eta_8 &= 10001, & \eta_9 &= 00001, & \eta_{10} &= 01001, & \eta_{11} &= 11001, & \eta_{12} &= 01101, & \eta_{13} &= 11101, & \eta_{14} &= 10101, \\
\eta_{15} &= 10011, & \eta_{16} &= 00111, & \eta_{17} &= 10111, & \eta_{18} &= 11111, & \eta_{19} &= 01111, & \eta_{20} &= 11011, & \eta_{21} &= 01011, \\
\eta_{22} &= 00011, & \eta_{23} &= 00101, & \eta_{24} &= 00110, & \eta_{25} &= 10110, & \eta_{26} &= 11110, & \eta_{27} &= 01110, & \eta_{28} &= 11010, \\
\eta_{29} &= 01010, & \eta_{30} &= 00010, & \eta_{31} &= 00100, & \eta_{32} &= 10000,
\end{align*}
\]

and it satisfies from (A1) to (A4).
**Proposition 4.1.** If \( \epsilon = \epsilon_1 \epsilon_2 \cdots \epsilon_{n-1} \) and \( \eta = \eta_1 \eta_2 \cdots \eta_{n-1} \) are adjacent in \( G(n-1) \), then the corresponding permutations \( \Psi_n(\epsilon) \) and \( \Psi_n(\eta) \) differ by an adjacent transposition.

**Proof.** By assumption, \( \eta \) can be obtained from \( \epsilon \) by interchanging adjacent 0 and 1 or by changing the first position of \( \epsilon \).

**Case 1.** We discuss the case when \( \eta \) is obtained by interchanging adjacent 0 and 1 in \( \epsilon \). Without loss of generality, we can assume that \( \eta_x = \epsilon_x \) for \( x \neq i-1, i \), \( \epsilon_{i-1} = \eta_i = 1 \) and \( \epsilon_i = \eta_{i-1} = 0 \) for some \( 1 \leq i \leq n-1 \), i.e.,

\[
\epsilon = \epsilon_1 \epsilon_2 \cdots \epsilon_{i-1} 1 0 \epsilon_{i+1} \cdots \epsilon_{n-1},
\eta = \epsilon_1 \epsilon_2 \cdots \epsilon_{i-1} 0 1 \epsilon_{i+1} \cdots \epsilon_{n-1}.
\]

Set \( \Psi_n(\epsilon) := a_1 a_2 \ldots a_n \). If \( \Psi_n(\eta) = b_1 b_2 \ldots b_n \), then by Definition \[4.1\], we see \( a_x = b_x \) for \( x \neq i, i+1 \), \( a_i = b_{i+1} \) and \( a_{i+1} = b_i \). Hence \( \Psi_n(\epsilon) \) and \( \Psi_n(\eta) \) differ by an adjacent transposition.

**Case 2.** We discuss the case when \( \eta \) is obtained by changing the first position of \( \epsilon \). Without loss of generality, we can assume that \( \eta_i = \epsilon_i \) for \( 2 \leq i \leq n-1 \) and \( \epsilon_1 = 0 \) and \( \eta_1 = 1 \), i.e.,

\[
\epsilon = 0 \epsilon_2 \epsilon_3 \cdots \epsilon_{n-1},
\eta = 1 \epsilon_2 \epsilon_3 \cdots \epsilon_{n-1}.
\]

Set \( \Psi_n(\epsilon) := a_1 a_2 \ldots a_n \) and suppose that \( \Psi_n(\eta) = b_1 b_2 \ldots b_n \). If \( \epsilon_{i-1} = 0 \) with \( i \geq 3 \), then we see \( a_i = b_i \) by Definition \[4.1\]. If \( \epsilon_{i-1} = 1 \) with \( i \geq 3 \), then \( a_i \) (resp. \( b_i \)) is \( n \) minus the cardinality of \( j \) with \( j \geq i \) and \( \epsilon_j = 1 \) (resp. \( \eta_j = 1 \)) and hence we see \( a_i = b_i \). Therefore we see \( a_p = b_p \) for \( p \geq 3 \).

Let \( k \) be the cardinality of 0 in \( \epsilon \), then \( a_1 = (k+1), a_2 = k, b_1 = k \) and \( b_2 = (k+1) \). Hence \( \Psi_n(\epsilon) \) and \( \Psi_n(\eta) \) differ by an adjacent transposition.

\[ \square \]

From Theorem \[3.1\] and the definition of \( \Psi_n \), we obtain the following result.

**Theorem 4.1.** For \( n \geq 3 \), there exists a Gray code

\[
(a_1, a_2, \ldots, a_{2^n-1}) \tag{19}
\]

for \( S_n(213, 312) \) which satisfies:

(P1) \( a_1 = n(n-1) \ldots 321 \), the reversal of the identity permutation, and \( a_{2^n-1} = (n-1)n(n-2)(n-3) \ldots 312 \), the first (resp. second) entry is \( n-1 \) (resp. \( n \)) and the \( i \)-th entry is \( n-i+1 \) for \( i \geq 3 \);)

(P2) two consecutive elements differ by at most 2 adjacent transpositions;

(P3) if \( a_i \) and \( a_{i+1} \) differ by 2 adjacent transpositions, then \( a_{i-1} \) and \( a_i \) differ by an adjacent transposition and also \( a_{i+1} \) and \( a_{i+2} \) do for \( 1 \leq i \leq 2^n-1 \).

**Remark 4.1.** For \( p_1 p_2 \ldots p_n \in S_n \), its reversal is defined by \( p_n p_{n-1} \ldots p_1 \in S_n \) \[17\]. By applying the reversal for every element in the Gray code in Theorem \[4.1\], we obtain a Gray code which starts from \( 123 \cdots (n-1)n \) and ends at \( 123 \cdots (n-2)n(n-1) \) which satisfies P(2) and (P3) in Theorem \[4.1\].
Corollary 4.1. For \( n \geq 3 \), there is a Gray code for \( S_n(213, 312) \) such that any successive permutations differ by at most two adjacent transpositions.

Remark 4.2. On our Gray code in Theorem 4.1, the first permutation and the second permutation differ by two adjacent transpositions. If one obtains a Hamiltonian cycle for \( G(n - 1) \), then one gets a Gray code for \( S_n(213, 312) \) such that two successive permutations differ by only one adjacent transposition.

Example 4.1. From Example 3.2 and \( \Psi_6 \), we obtain a following listing \((a_1, a_2, \ldots, a_{32})\) for \( S_6(132, 312) \).

\[
\begin{align*}
a_1 &= 654321, \quad a_2 = 546321, \quad a_3 = 456321, \quad a_4 = 435621, \quad a_5 = 345621, \quad a_6 = 453621, \\
a_7 &= 453261, \quad a_8 = 534261, \quad a_9 = 543261, \quad a_{10} = 435261, \quad a_{11} = 435261, \quad a_{12} = 324516, \\
a_{13} &= 234516, \quad a_{14} = 342516, \quad a_{15} = 342156, \quad a_{16} = 321456, \quad a_{17} = 231456, \quad a_{18} = 123456, \\
a_{19} &= 213456, \quad a_{20} = 234156, \quad a_{21} = 324156, \quad a_{22} = 432156, \quad a_{23} = 432516, \quad a_{24} = 432561, \\
a_{25} &= 342561, \quad a_{26} = 234561, \quad a_{27} = 324561, \quad a_{28} = 345261, \quad a_{29} = 435261, \quad a_{30} = 543621, \\
a_{31} &= 543621, \quad a_{32} = 564321. 
\end{align*}
\]

5 A Gray code for \( A_n(001) \)

In this section, we construct a Gray code of strong distance 2 for \( A_n(001) \) for \( n \in \mathbb{N} \). An ascent sequence avoids 001 pattern if and only if it starts with a strictly increasing ascent sequence, we denote it by 012...k, followed by an arbitrary weakly decreasing sequence whose letters are smaller than or equal to \( k \). First, we define a map \( \Phi_{F_{n-1} \to A_n(001)} \) from \( F_{n-1} \) to \( A_n(001) \).

For a binary word \( \alpha = \alpha_1 \alpha_2 \ldots \alpha_{n-1} \), we say that the \( x \)-th entry of 0 is in the \( i \)-th position when \( \alpha_i = 0 \) and the cardinality of \( j \) with \( j \leq i \) and \( \alpha_j = 0 \) is \( x \). For example, The 6-th entry of 0 in 00010101011001101 is in the 9-th position, we denote it in bold style.

For \( \epsilon \in F_{n-1} \), we can write

\[
\epsilon = \underbrace{11\ldots1}_{p \text{ times}} \underbrace{01\ldots1}_{(k_1-k_2) \text{ times}} \underbrace{01\ldots1}_{(k_2-k_3) \text{ times}} \underbrace{01\ldots1}_{(k_3-k_4) \text{ times}} \underbrace{01\ldots1}_{(k_{q-1}-k_q) \text{ times}} \underbrace{01\ldots1}_{k_q \text{ times}},
\]

for some \( n-1 \geq k_1 \geq k_2 \geq \ldots \geq k_q \geq 0 \) and \( p, q \geq 0 \) such that \( k_1 + p + q = n - 1 \), where \( k_i \) is the number of 1 on the right hand side of the \( i \)-th entry of 0. We have \( k_i = k_{i+1} \) when the \( i \)-th entry of 0 is adjacent to the \((i+1)\)-th entry of 0.

We set \( \phi_1^{(1)}(\epsilon) := 012\ldots(k_1+p) \) and \( \phi_2^{(1)}(\epsilon) := k_1k_2\ldots k_q \). Remark that the cardinality of 1 is \((k_1+p)\) and that of 0 is \( q \). Define \( \Phi_{F_{n-1} \to A_n(001)}(\epsilon) \) to be the concatenation of \( \phi_1^{(1)}(\epsilon) \) and \( \phi_2^{(1)}(\epsilon) \), that is

\[
\Phi_{F_{n-1} \to A_n(001)}(\epsilon) := 012\ldots(k_1+p)k_1k_2\ldots k_q
\]

and it is a 001 avoiding ascent sequence.

For example, for \( \epsilon = 010111010011 \in F_{12} \), we see \( k_1 = 7, k_2 = 6, k_3 = 3, k_4 = k_5 = 2, p = 0 \) and \( q = 5 \) and we have \( \phi_1^{(13)}(\epsilon) = 01234567 \) and \( \phi_2^{(13)}(\epsilon) = 76322 \). Hence we obtain \( \Phi_{F_{12} \to A_{13}(001)}(010111010011) = 0123456776322 \). We state the above observation as Lemma.
Lemma 5.1. For a binary word \( \epsilon = \epsilon_1 \epsilon_2 \ldots \epsilon_{n-1} \), suppose that \( \phi_2^{(n)}(\epsilon) = k_1 k_2 \ldots k_q \).

1. The number of 0 in \( \epsilon \) is \( q \).
2. For \( 1 \leq i \leq q \), the cardinality of 1 in \( \epsilon \) on the right hand side of the \( i \)-th entry of 0 equals \( k_i \).
3. \( \phi_1^{(n)}(\epsilon) = 012 \ldots (n - 1 - q) \), where \((n - 1 - q)\) is the number of 1 in \( \epsilon \).

Next, we define a map \( \Phi_{A_n(001)}^{n-1} : \mathbb{A}_n(001) \to \mathbb{F}_2^{n-1} \). A 001 avoiding ascent sequence of length \( n \) can be written as \( 012 \ldots r x_1 x_2 \ldots x_s \), where \( 012 \ldots r \) is a strictly increasing ascent sequence and \( x_1 x_2 \ldots x_s \) is a weakly decreasing sequence whose letters are smaller than or equal to \( r \) with \( r + s = n - 1 \).

Let \( \Phi_{A_n(001)}^{n-1}((012 \ldots r x_1 x_2 \ldots x_s)) \) be the binary words of length \((n - 1)\) such that:

1. the cardinality of 0 (resp. 1) is \( s \) (resp. \( r \));
2. the cardinality of 1 on the right hand side of the \( i \)-th entry of 0 equals \( x_i \) for \( 1 \leq i \leq s \), i.e.,

\[
\frac{11 \ldots 1}{(r-x_1) \text{ times}} \frac{0}{(x_1-x_2) \text{ times}} \frac{11 \ldots 1}{(2-x_2) \text{ times}} \frac{0}{(x_2-x_3) \text{ times}} \frac{11 \ldots 1}{(3-x_3) \text{ times}} \cdots \frac{0}{(x_{s-1}-x_s) \text{ times}} \frac{11 \ldots 1}{s \text{ times}}
\]

For \( 0123422100 \in \mathbb{A}_{10}(001) \), the strictly increasing sequence part is 01234 and the weakly decreasing part is 22100. From this, we see \( r = 4 \), \( s = 5 \), \( x_1 = x_2 = 2 \), \( x_3 = 1 \) and \( x_4 = x_5 = 0 \). We obtain

\[
\frac{11}{r-x_1=2 \text{ times}} \frac{0}{x_1-x_2=0 \text{ times}} \frac{\phi}{x_2-x_3=1 \text{ times}} \frac{0}{x_3-x_4=1 \text{ times}} \frac{1}{x_4-x_5=0 \text{ times}} \frac{\phi}{x_5=0 \text{ times}}
\]

and \( \Phi_{A_{10}(001)}^{9}((0123422100)) = 110010100 \). By the construction, the following Lemma is straightforward to check.

Lemma 5.2. \( \Phi_{F_2^{n-1} \to A_n(001)}^{n-1} \circ \Phi_{A_n(001) \to F_2^{n-1}} \) (resp. \( \Phi_{A_n(001) \to F_2^{n-1}} \circ \Phi_{F_2^{n-1} \to A_n(001)}^{n-1} \)) is an identity map on \( A_n(001) \) (resp. \( \mathbb{F}_2^{n-1} \)). Hence \( \Phi_{F_2^{n-1} \to A_n(001)}^{n-1} \) and \( \Phi_{A_n(001) \to F_2^{n-1}} \) are bijections.

Proposition 5.1. If \( \epsilon = \epsilon_1 \epsilon_2 \ldots \epsilon_{n-1} \) and \( \eta = \eta_1 \eta_2 \ldots \eta_{n-1} \) are adjacent in \( G(n-1) \), then the strong distance of \( \Phi_{F_2^{n-1} \to A_n(001)}^{n-1}(\epsilon) \) and \( \Phi_{F_2^{n-1} \to A_n(001)}^{n-1}(\eta) \) is 1.

Proof. By the assumption, \( \eta \) can be obtained from \( \epsilon \) by interchanging adjacent 0 and 1 in \( \epsilon \) or by changing the first position of \( \epsilon \).

Case 1. We discuss the case when \( \eta \) is obtained by interchanging adjacent 0 and 1 in \( \epsilon \). Without loss of generality, we can assume that \( \eta_i = \epsilon_x \) for \( x \neq i, i+1, \epsilon_i = \eta_{i+1} = 1 \) and \( \epsilon_{i+1} = \eta_i = 0 \).

Suppose that the \( x \)-th entry of 0 in \( \epsilon \) is in the \((i+1)\)-th position. The \( x \)-th entry of 0 in \( \eta \) is in the \( i \)-th position, i.e.,

\[
\epsilon = \epsilon_1 \epsilon_2 \ldots \epsilon_{i-1} \begin{array}{c} 1 \end{array} 0 \begin{array}{c} x \text{-th entry of 0} \end{array} \epsilon_{i+2} \ldots \epsilon_{n-1},
\]

\[
\eta = \epsilon_1 \epsilon_2 \ldots \epsilon_{i-1} \begin{array}{c} 0 \end{array} 1 \begin{array}{c} x \text{-th entry of 0} \end{array} \epsilon_{i+2} \ldots \epsilon_{n-1}.
\]
From (3) of Lemma 5.1, we have \( \phi_1^{(n)}(\epsilon) = \phi_1^{(n)}(\eta) \). Set \( \phi_2^{(n)}(\epsilon) = \theta_1 \theta_2 \ldots \theta_s \) and \( \phi_2^{(n)}(\eta) = \tau_1 \tau_2 \ldots \tau_s \). From (1) of Lemma 5.1, we see \( r = s \). We have \( \theta_u = \tau_u \) for \( 1 \leq u \neq x \leq r \) and we get \( \tau_x = \theta_x + 1 \) from (2) of Lemma 5.1. Hence the strong distance of \( \Phi_{F_2^{n-1} \to A_n(001)}(\epsilon) \) and \( \Phi_{F_2^{n-1} \to A_n(001)}(\eta) \) is 1.

Case 2. We discuss the case when \( \eta \) is obtained by changing the first position of \( \epsilon \). Without loss of generality, we can assume that \( \eta_i = \epsilon_i \) for \( 2 \leq i \leq n - 1 \), \( \epsilon_1 = 1 \) and \( \eta_1 = 0 \), i.e.,

\[
\begin{align*}
\epsilon &= 1 \epsilon_2 \epsilon_3 \ldots \epsilon_{n-1}, \\
\eta &= 0 \epsilon_2 \epsilon_3 \ldots \epsilon_{n-1}.
\end{align*}
\]

Suppose that the number of 1 (resp. 0) in \( \epsilon \) is \( t \) (resp. \( n - 1 - t \)). Then the number of 1 (resp. 0) in \( \eta \) is \( t - 1 \) (resp. \( n - t \)).

From (3) of Lemma 5.1, we see \( \phi_1^{(n)}(\epsilon) = 012 \ldots t \) and \( \phi_1^{(n)}(\eta) = 012 \ldots (t - 1) \). Also \( \phi_2^{(n)}(\epsilon) \) has \( n - 1 - t \) letters and set \( \phi_2^{(n)}(\epsilon) = b_1 b_2 \ldots b_{n-1-t} \). Then \( \phi_2^{(n)}(\eta) = (t-1)b_1 b_2 \ldots b_{n-1-t} \), because the first entry of 0 in \( \eta \) is at the first position of \( \eta \) and \( \eta_i = \epsilon_i \) for \( 2 \leq i \leq n - 1 \). Hence we have

\[
\begin{align*}
\Phi_{F_2^{n-1} \to A_n(001)}(\epsilon) &= 012 \ldots (t - 1) t \ b_1 b_2 \ldots b_{n-1-t}, \\
\Phi_{F_2^{n-1} \to A_n(001)}(\eta) &= 012 \ldots (t - 1) (t - 1) b_1 b_2 \ldots b_{n-1-t}
\end{align*}
\]

and the strong distance of \( \Phi_{F_2^{n-1} \to A_n(001)}(\epsilon) \) and \( \Phi_{F_2^{n-1} \to A_n(001)}(\eta) \) is 1.

From Theorem 3.1, Remark 3.3 and Proposition 5.1, we obtain the following result.

**Theorem 5.1.** For \( n \geq 3 \), there exists a Gray code \( (a_1, a_2, \ldots, a_{2^{n-1}}) \) of strong distance 2 for \( A_n(001) \) which satisfies:

1. \( a_1 = 00 \ldots 0 \), all entries are 0, and \( a_{2^{n-1}} = 0100 \ldots 00 \), the second entry is 1 and the rest of the entries are all 0;
2. \( d_{str}(a_1, a_2) = 2 \);
3. if \( d_{str}(a_i, a_{i+1}) = 2 \), then \( d_{str}(a_{i-1}, a_i) = d_{str}(a_{i+1}, a_{i+2}) = 1 \) for \( 1 \leq i \leq 2^{n-1} - 1 \).

**Corollary 5.1.** For \( n \geq 3 \), \( A_n(001) \) has a Gray code of strong distance 2 which starts from 00\ldots0 and ends at 0100\ldots0.

**Remark 5.1.** For \( n \geq 3 \), if one obtain a Hamiltonian cycle for \( G(n-1) \), then one gets a Gray code of strong distance 1 for \( A_n(001) \).

**Example 5.1.** By applying \( \Phi_{F_2 \to A_6(001)} \) to the listing in Example 3.2, we have a Gray code \( (a_1, a_2, \ldots, a_{32}) \) for \( A_6(001) \), where
6 A Gray code for $A_n(010)$

In this section, we construct a Gray code of strong distance 2 for $A_n(010)$ for $n \in \mathbb{N}$. An ascent sequence avoids 010 pattern if and only if it is a weakly increasing ascent sequence [9].

For a binary word $\epsilon = \epsilon_1\epsilon_2 \ldots \epsilon_n$ and $\eta = \eta_1\eta_2 \ldots \eta_{n-1}$ be the sequence such that $a_1 = 0$ and $a_i$ is the number of 1 in $\epsilon_n-a_{i+1} \epsilon_n-a_{i+2} \ldots \epsilon_{n-1}$ for $2 \leq i \leq n$. The sequence is weakly increasing and hence it is a 010 avoiding ascent sequence. Define $\Phi_{F^2_{n-1} \rightarrow A_n(010)}(\epsilon) = a_1 a_2 \ldots a_n$. Obviously, it is a bijection from $F^n_{n-1}$ to $A_n(010)$. For example, $\Phi_{F^2_{n-1} \rightarrow A_n(010)}(100110100) = 0001123334$.

**Proposition 6.1.** If $\epsilon = \epsilon_1\epsilon_2 \ldots \epsilon_n$ and $\eta = \eta_1\eta_2 \ldots \eta_{n-1}$ are adjacent in $G(n-1)$, then the strong distance of $\Psi_{F^2_{n-1} \rightarrow A_n(010)}(\epsilon)$ and $\Psi_{F^2_{n-1} \rightarrow A_n(010)}(\eta)$ is 1.

**Proof.** By the assumption, $\eta$ can be obtained from $\epsilon$ by interchanging adjacent 0 and 1 in $\epsilon$ or by changing the first position of $\epsilon$.

**Case 1.** We consider the case where $\eta$ is obtained by interchanging adjacent 0 and 1 in $\epsilon$. Without loss of generality, we can assume that $\eta_x = \epsilon_x$ for $x \neq i, i + 1$, $\epsilon_i = \eta_{i+1} = 1$ and $\epsilon_{i+1} = \eta_i = 0$ for some $1 \leq i \leq (n-1)$, i.e.,

$$\epsilon = \epsilon_1\epsilon_2 \ldots \epsilon_{i-1}10\epsilon_{i+2} \ldots \epsilon_{n-1},$$
$$\eta = \epsilon_1\epsilon_2 \ldots \epsilon_{i-1}01\epsilon_{i+2} \ldots \epsilon_{n-1}.$$

Put $\Phi(\epsilon) = a_1 a_2 \ldots a_n$. Then $a_{n-i} = a_{n-i+1} + 1$ and $b_{n-i} = b_{n-i+1} + 1$. If $\Phi(\epsilon) = b_1 b_2 \ldots b_n$, then by the construction, we see $b_k = a_k$ for $k \neq (n-i), (n-i+1)$. Also we have $b_{n-i} = b_{n-i+1} + 1$ and $b_{n-i+1} = b_{n-i}$ and hence we obtain $a_{n-i+1} = b_{n-i+1}$ and $a_{n-i} = b_{n-i} - 1$. Therefore the strong distance of $\Phi_{F^2_{n-1} \rightarrow A_n(010)}(\epsilon)$ and $\Phi_{F^2_{n-1} \rightarrow A_n(010)}(\eta)$ is 1.

**Case 2.** We discuss the case where $\eta$ is obtained by changing the first position of $\epsilon$. Without loss of generality, we can assume that $\eta_i = \epsilon_i$ for $2 \leq i \leq (n-1)$, $\epsilon_1 = 0$ and $\eta_1 = 1$, i.e.,

$$\epsilon = 0 1 \epsilon_2 \ldots \epsilon_{n-1},$$
$$\eta = 1 \epsilon_2 \epsilon_3 \ldots \epsilon_{n-1}.$$

If $\Phi_{F^2_{n-1} \rightarrow A_n(010)}(\epsilon) = a_1 a_2 \ldots a_{n-1} a_n$, then we see $\Phi_{F^2_{n-1} \rightarrow A_n(010)}(\eta) = a_1 a_2 \ldots a_{n-1} (a_n + 1)$. Therefore the strong distance of $\Phi_{F^2_{n-1} \rightarrow A_n(001)}(\epsilon)$ and $\Phi_{F^2_{n-1} \rightarrow A_n(001)}(\eta)$ is 1.

$\blacksquare$
From Theorem 3.1, Remark 3.3 and Proposition 6.1 we obtain the following result.

**Theorem 6.1.** For \( n \geq 3 \), there is a Gray code \((\alpha_1, \alpha_2, \ldots, \alpha_{2^n-1})\) of strong distance 2 for \( A_n(010) \) which satisfies:

1. \( \alpha_1 = 00\ldots0 \), all entries are 0, and \( \alpha_{2^n-1} = 00\ldots01 \), the last entry is 1 and the remaining entries are all 0;
2. \( d_{str}(a_1, a_2) = 2 \);
3. if \( d_{str}(\alpha_i, \alpha_{i+1}) = 2 \), then \( d_{str}(\alpha_{i-1}, \alpha_i) = d_{str}(\alpha_{i+1}, \alpha_{i+2}) = 1 \) for \( 1 \leq i \leq 2^n - 1 - 1 \).

**Corollary 6.1.** For \( n \geq 3 \), \( A_n(010) \) has a Gray code of strong distance 2 which starts from 00\ldots0 and ends at 000\ldots01.

**Remark 6.1.** For \( n \geq 3 \), if one obtain a Hamiltonian cycle for \( G(n-1) \), then one gets a Gray code of strong distance 1 for \( A_n(010) \).

**Example 6.1.** By applying \( \Phi_{E_2^5} \rightarrow A_6(010) \) to the listing in Example 3.2, we have a Gray code \((a_1, a_2, \ldots, a_{32})\) for \( A_6(010) \), where

\[
\begin{align*}
    a_1 &= 000000, & a_2 &= 000011, & a_3 &= 000012, & a_4 &= 000122, & a_5 &= 000123, & a_6 &= 000112, \\
    a_7 &= 001112, & a_8 &= 011111, & a_9 &= 011112, & a_{10} &= 011122, & a_{11} &= 011123, & a_{12} &= 011233, \\
    a_{13} &= 011234, & a_{14} &= 011223, & a_{15} &= 012223, & a_{16} &= 012333, & a_{17} &= 012334, & a_{18} &= 012345, \\
    a_{19} &= 012344, & a_{20} &= 012234, & a_{21} &= 012233, & a_{22} &= 012222, & a_{23} &= 011222, & a_{24} &= 001222, \\
    a_{25} &= 001223, & a_{26} &= 001234, & a_{27} &= 001233, & a_{28} &= 001123, & a_{29} &= 001122, & a_{30} &= 001111, \\
    a_{31} &= 000111, & a_{32} &= 000001.
\end{align*}
\]

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