SEMICLASSICAL SPECTRAL ASYMPTOTICS
FOR A MAGNETIC SCHRODINGER OPERATOR
WITH NON-VANISHING MAGNETIC FIELD

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Abstract. We consider a magnetic Schrödinger operator $H^h$, depending on the semiclassical parameter $h > 0$, on a compact Riemannian manifold. We assume that there is no electric field. We suppose that the minimal value $b_0$ of the intensity of the magnetic field $b$ is strictly positive. We give a survey of the results on asymptotic behavior of the eigenvalues of the operator $H^h$ in the semiclassical limit.

1. Introduction

Let $M$ be a compact oriented manifold of dimension $n \geq 2$ (possibly with boundary). Let $g$ be a Riemannian metric and $B$ a real-valued closed 2-form on $M$. Assume that $B$ is exact and choose a real-valued 1-form $A$ on $M$ such that $dA = B$. Thus, one has a natural mapping

$$u \mapsto ih du + Au$$

from $C^\infty_c(M)$ to the space $\Omega^1_c(M)$ of smooth, compactly supported one-forms on $M$. The Riemannian metric allows to define scalar products in these spaces and consider the adjoint operator

$$(ih d + A)^* : \Omega^1_c(M) \to C^\infty_c(M).$$

A Schrödinger operator with magnetic potential $A$ is defined by the formula

(1.1) $H^h = (ih d + A)^*(ih d + A).$

Here $h > 0$ is a semiclassical parameter. If $M$ has non-empty boundary, we will assume that the operator $H^h$ satisfies the Dirichlet boundary conditions.

From the geometric point of view, the 1-form $A$ defines a Hermitian connection $\nabla_A = d - iA$ on the trivial complex line bundle $\mathcal{L}$ over $M$. The curvature of this connection is $-iB$. Then the operator $H^h$ is related with the associated covariant (or Bochner) Laplacian

$$H_A = \nabla_A^* \nabla_A$$

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by the formula
\[ H^h = h^2 (d - ih^{-1}A)^* (d - ih^{-1}A) = h^2 H_{h^{-1}A}. \]
This formula shows, in particular, that the semiclassical limit \( h \to 0 \) is clearly equivalent to the large magnetic field limit.

We choose local coordinates \( x = (x_1, \ldots, x_n) \) on \( M \). We write the 1-form \( A \) in the local coordinates as
\[ A = \sum_{j=1}^{n} A_j(x) \, dx_j, \]
the matrix of the Riemannian metric \( g \) as
\[ g(x) = (g_{j\ell}(x))_{1\leq j,\ell \leq n}, \]
and its inverse as
\[ g(x)^{-1} = (g^{j\ell}(x))_{1\leq j,\ell \leq n}. \]
We denote the determinant of \( g \) by:
\[ |g(x)| = \det(g(x)). \]
Then the magnetic field \( B \) is given by the following formula
\[ B = \sum_{j<k} B_{jk} \, dx_j \wedge dx_k, \quad B_{jk} = \frac{\partial A_k}{\partial x_j} - \frac{\partial A_j}{\partial x_k}. \]
Moreover, the operator \( H^h \) has in these coordinates the form
\[ H^h = \frac{1}{\sqrt{|g(x)|}} \sum_{1\leq j,\ell \leq n} \left( ih \frac{\partial}{\partial x_j} + A_j(x) \right) \left[ \sqrt{|g(x)|} g^{j\ell}(x) \left( ih \frac{\partial}{\partial x_\ell} + A_\ell(x) \right) \right]. \]
In the case when \( M = \mathbb{R}^n \) is the flat Euclidean space, the operator \( H^h \) takes the form
\[ H^h = \sum_{1\leq j \leq n} \left( h D_{x_j} - A_j(x) \right)^2, \]
where, as usual, \( D_{x_j} = \frac{1}{i} \frac{\partial}{\partial x_j}, \ j = 1, \ldots, n. \)

When \( n = 2 \), the magnetic two-form \( B \) is a volume form on \( M \) and therefore can be identified with the function \( b \in C^\infty(M) \) given by
\[ B = b \, dx_g, \]
where \( dx_g \) denotes the Riemannian volume form \( M \) associated with \( g \).

When \( n = 3 \), the magnetic two-form \( B \) can be identified with a magnetic vector field \( \vec{b} \) by the Hodge star-operator. If \( M \) is the Euclidean space \( \mathbb{R}^3 \), we have
\[ \vec{b} = (b_1, b_2, b_3) = \text{curl} \ A = (B_{23}, -B_{13}, B_{12}), \]
with the usual definition of curl.

We are interested in asymptotic behavior of the spectrum of the operator \( H^h \) in the semiclassical limit. This problem was studied in [7, 13, 19, 20, 23].
(32, 33, 34) (see [8, 14, 36] for surveys including the case of problems with boundary).

After the pioneering works by Kato [30] and his school, the starting reference for the spectral analysis of self-adjoint realizations of the magnetic Schrödinger operator is the paper by Avron-Herbst-Simon [1] where the role of the module of the magnetic field in the three-dimensional case appears for the first time. Further investigations were inspired by R. Montgomery [34], who was asking “Can we hear the locus of the magnetic field” (by analogy with the celebrated question by M. Kac). In [34], this question was studied for the two-dimensional magnetic Schrödinger operator. Motivated by the question of R. Montgomery, the first author and Mohamed in [19] investigated the asymptotic behavior of the low-lying eigenvalues of the Dirichlet realization of the magnetic Schrödinger operator in the case when the magnetic field vanishes. This study was continued more recently in [35, 12, 13, 5] (see also [14]). The case when the magnetic field never vanishes was analyzed in detail for the Dirichlet realization in the two-dimensional case in [20] and more recently in [15, 18, 37]. Moreover, there is a big literature devoted to the spectral analysis of the Neumann realization because of its connection with problems in superconductivity (see [8] and the references therein). Finally, we do not also give a complete description of the semiclassical results obtained in the case when an electric potential $V$ is creating the main localization and refer to [23] and [4] for a presentation and references therein.

The purpose of this paper is to give a survey of the results obtained in the case when the magnetic field never vanishes. First, we suppose that $M$ is two-dimensional. Let

$$b_0 = \min_{x \in M} |b(x)|.$$  

(1.4)

Remark that if $M$ is without boundary then we necessarily have $b_0 = 0$, since

$$\int_M b(x)d\mathbf{x}_g = \int_M d\mathbf{A} = 0.$$  

If we suppose that $M$ has a non-empty boundary and the operator $H^h$ satisfies the Dirichlet boundary conditions, it was observed by many authors [34, 31, 40, 41] (as the immediate consequence of the Weitzenböck-Bochner type identity and the positivity of the square of a suitable Dirac operator) that, if $U$ is a domain in $M$, then, for any $u \in C_c^\infty(U)$, the following estimate holds:

$$\|(ih \partial + \mathbf{A})u\|_U^2 \geq h \int_U |b|u|^2d\mathbf{x}_g.$$  

(1.5)

In particular, for any $h > 0$,

$$\lambda_0(H^h) \geq hb_0.$$  

(1.6)
In the case $M = \mathbb{R}^2$, this estimate follows from the formula

$$hb(x) = -i[hD_{x_1} - A_1, hD_{x_2} - A_2],$$

which implies (after an integration by parts) that

$$h \int b(x)|u(x)|^2 \, dx \leq \|(hD_{x_1} - A_1)u\|^2 + \|(hD_{x_2} - A_2)u\|^2.$$

Due to this estimate, the function $hb$ can be considered in many spectral problems as an effective electric potential, that is, as a magnetic analog of the electric potential $V$ in a Schrödinger operator $-h^2 \Delta + V$.

Any connected component of the minimum set

$$U = \{x \in M : b(x) = b_0\}$$

can be understood as a magnetic well (attached to the given energy $hb_0$). In particular, an asymptotic description of the spectrum near the bottom strongly depends on the geometry of the magnetic wells and the behavior of $b$ near them.

In higher dimensions, the role of magnetic potential is played by the function $x \mapsto h \cdot \text{Tr}^+(B(x))$, which can be defined in the following way. For any $x \in M$, denote by $B(x)$ the anti-symmetric linear operator on the tangent space $T_xM$ associated with the 2-form $B$:

$$g_x(B(x)u, v) = B_x(u, v), \quad u, v \in T_xM.$$  

Recall that the intensity of the magnetic field is defined as

$$\text{Tr}^+(B(x)) = \sum_{i\lambda_j(x) > 0} \lambda_j(x) = \frac{1}{2} \text{Tr}([B^*(x) \cdot B(x)]^{1/2}).$$

In the (3D) case the only positive eigenvalue is $|B(x)|$ and we get

$$\text{Tr}^+(B(x)) = |B(x)|.$$  

In the general case, we do not have the equivalent of (1.5) but only the weaker estimate [19):

$$\inf_x (\text{Tr}^+(B(x)) - C\hbar^2) \int |u(x)|^2 \, dx \leq \langle H^h u, u \rangle, \quad \forall u \in C_c^\infty(M).$$

When $U$ is not connected, the spectrum is essentially obtained by analyzing (the union of) the spectra of Dirichlet Laplacians attached to each component. This is true modulo exponentially small errors. This corresponds to the so called magnetic tunneling. We will not focus on this question (which is widely open) and will more emphasis on the presentation of the known semi-classical results in dimension 2 and 3 which are purely magnetic first at the bottom (Sections 2 and 3 in dimension 2 and Section 3 in dimension 3), secondly in Section 4 for excited states in dimension 2 where we present the newest contributions (Helffer-Kordyukov and Raymond-Vu Ngoc) but will give a few examples in the last section.
In this section, we will discuss the case of discrete wells. We assume that:

\[(2.1)\]

\[b_0 > 0,\]

and that there exist a unique point \(x_0\), which belongs to the interior of \(M\), \(k \in \mathbb{N}\) and \(C > 0\) such that for all \(x\) in some neighborhood of \(x_0\) the estimates hold:

\[(2.2)\]

\[C^{-1} d(x, x_0)^2 \leq b(x) - b_0 \leq C d(x, x_0)^2.\]

We introduce:

\[a = \text{Tr} \left( \frac{1}{2} \text{Hess} b(x_0) \right)^{1/2}, \quad d = \text{det} \left( \frac{1}{2} \text{Hess} b(x_0) \right)^{1/2},\]

and denote by \(\lambda_0(H^h) \leq \lambda_1(H^h) \leq \lambda_2(H^h) \leq \ldots\) the eigenvalues of the operator \(H^h\) in \(L^2(M)\).

**Theorem 2.1.** Under current assumptions, for any \(j \in \mathbb{N}\), there exists a sequence \((\alpha_{j,\ell})_{\ell \in \mathbb{N}}\) with

\[\alpha_{j,0} = b_0, \quad \alpha_{j,1} = 0, \quad \alpha_{j,2} = \frac{2d^{1/2}}{b_0} j + \frac{a^2}{2b_0},\]

such that

\[(2.3)\]

\[\lambda_j(H^h) \sim h \sum_{\ell=0}^{N} \alpha_{j,\ell} h^{\frac{\ell}{2}}.\]

In other words, for any \(N\), there exist \(C_{j,N} > 0\) and \(h_{j,N} > 0\) such that, for any \(h \in (0, h_{j,N})\),

\[|\lambda_j(H^h) - h \sum_{\ell=0}^{N} \alpha_{j,\ell} h^{\frac{\ell}{2}}| \leq C_{j,N} h^{\frac{N+3}{2}}.\]

In particular, we have for the groundstate energy \(\lambda_0(H^h)\) a two term asymptotics:

\[\lambda_0(H^h) = h b_0 + h^2 \frac{a^2}{2b_0} + O(h^{5/2}), \quad h \to 0,\]

and the asymptotics of the splitting between the groundstate energy and the first excited state:

\[\lambda_1(H^h) - \lambda_0(H^h) \sim h^2 \frac{2d^{1/2}}{b_0}.\]

This theorem is proved in [15]. A two-terms asymptotics for the ground state energy in the flat case was previously obtained in [20]. Recent improvements by Helffer-Kordyukov [18] and Raymond-Vu Ngoc [37] (see also Section [4]) show that no odd powers of \(h^{\frac{1}{2}}\) actually occur in the flat case. We believe that this fact also holds in the general case of Riemannian manifold.
The proof of the upper bound is based on a construction of approximate eigenfunctions for the operator $H^h$. More precisely, we prove in [15] the following accurate upper bound for the eigenvalues of the operator $H^h$.

**Theorem 2.2.** Under current assumptions, for any $j$ and $k$ in $\mathbb{N}$, there exists a sequence $(\mu_{j,k,\ell})_{\ell \in \mathbb{N}}$ with

$$\mu_{j,k,0} = (2k + 1)b_0, \quad \mu_{j,k,1} = 0,$$

and

$$\mu_{j,k,2} = (2j + 1)(2k + 1)\frac{d^{1/2}}{b_0} + (2k^2 + 2k + 1)\frac{t}{2b_0} + \frac{1}{2}(k^2 + k)R(x_0),$$

where $R$ is the scalar curvature, and

$$t = \text{Tr} \left( \frac{1}{2} \text{Hess} b(x_0) \right),$$

and for any $N$, there exist $\phi^h_{j,k,N} \in C^\infty(M)$, $C_{j,k,N} > 0$ and $h_{j,k,N} > 0$ such that

$$\phi^h_{j,k_1,N} \phi^h_{j,k_2,N} = \delta_{j_1,j_2}\delta_{k_1,k_2} + O_{j_1,j_2,k_1,k_2}(h),$$

and, for any $h \in (0, h_{j,k,N})$,

$$\|H^h\phi^h_{j,k,N} - \mu^h_{j,k,N}\phi^h_{j,k,N}\| \leq C_{j,k,N}h^{\frac{N+1}{2}}\|\phi^h_{j,k,N}\|,$$

where

$$\mu^h_{j,k,N} = h \sum_{\ell = 0}^N \mu_{j,k,\ell}h^{\frac{\ell}{2}}.$$

Since the operator $H^h$ is self-adjoint, using the Spectral Theorem, we immediately deduce the existence of eigenvalues near the values $\mu^h_{j,k,N}$.

**Corollary 2.3.** For any $j$, $k$ and $N$ in $\mathbb{N}$, there exist $C_{j,k,N} > 0$ and $h_{j,k,N} > 0$ such that, for any $h \in (0, h_{j,k,N})$,

$$\text{dist}(\mu^h_{j,k,N}, \text{Spec}(H^h)) \leq C_{j,k,N}h^{\frac{N+3}{2}}.$$

**Remark 2.4.** The low-lying eigenvalues of the operator $H^h$, as $h \to 0$, are obtained by taking $k = 0$ in Theorem 2.2. Therefore, as an immediate consequence of Theorem 2.2 we deduce that, for any $j$ and $N$ in $\mathbb{N}$, there exists $h_{j,N} > 0$ such that, for any $h \in (0, h_{j,N})$, we have

$$\lambda_j(H^h) \leq \mu^h_{j,0,N} + C_{j,0,N}h^{\frac{N+3}{2}}.$$

In particular, this implies the upper bound in Theorem 2.1.

**Remark 2.5.** Our interest in the case of arbitrary $k$ in Theorem 3.2 is motivated, in particular, by its importance for proving the existence of gaps in the spectrum of the operator $H^h$ in the semiclassical limit [11].
Remark 2.6. The term
\[(2k + 1)\hbar b_0 + \frac{1}{2} \hbar^2 (k^2 + k) R,\]
in the right-hand side of (2.5) (see also (3.5) below) has a natural interpretation as Landau levels. The interpretation depends on whether \(R\) is zero, positive or negative and, in all three cases, is given in terms of eigenvalues of the associated magnetic Laplacian with constant magnetic field (Landau operator) on the corresponding simply connected Riemann surface of constant curvature (see [15] for more details).

We also mention the paper [6] by Ferapontov and Veselov, who prove that these three model magnetic Laplacians are integrable in some sense. This observation enables them to give the complete description of the spectra of these operators in the same way as it was done by Schrödinger for the harmonic oscillator.

3. Degenerate wells in dimension 2

In this section, following [16], we will discuss the case when the minimum of the magnetic field is attained on a regular curve \(\gamma\). We assume that:

- \(b_0 > 0\);
- the set \(\{x \in M : |b(x)| = b_0\}\) is a smooth curve \(\gamma\), which is contained in the interior of \(M\);
- there is a constant \(C > 0\) such that for all \(x\) in some neighborhood of \(\gamma\) the estimates hold:

\[(3.1)\quad C^{-1}d(x,\gamma)^2 \leq |b(x)| - b_0 \leq Cd(x,\gamma)^2.\]

3.1. Asymptotics near the bottom. The main purpose is to give an asymptotics of the groundstate energy \(\lambda_0(H^h)\) of the operator \(H^h\). Denote by \(N\) the external unit normal vector to \(\gamma\). Let \(\tilde{N}\) denote the natural extension of \(N\) to a smooth normalized vector field on \(M\), whose integral curves starting from a point \(x\) in a tubular neighborhood of \(\gamma\) are the minimal geodesics to \(\gamma\). Consider the function \(\beta_2\) on \(\gamma\) given by

\[(3.2)\quad \beta_2(x) = \tilde{N}^2|b(x)|, \quad x \in \gamma.\]

By (3.1), it is easy to see that

\[\beta_2(x) > 0, \quad x \in \gamma.\]

Theorem 3.1. There exists \(h_0 > 0\), such that, for any \(h \in (0, h_0]\),

\[(3.3)\quad \lambda_0(H^h) = \hbar b_0 + \hbar^2 \frac{\mu_0}{4b_0} + \mathcal{O}(\hbar^{17/8}).\]

where

\[(3.4)\quad \mu_0 := \inf_{x \in \gamma} \beta_2(x).\]
The proof of the upper bound is based on a construction of approximate eigenfunctions for the operator $H^h$. We denote by $R$ the scalar curvature of the Riemannian manifold $(M, g)$.

**Theorem 3.2.** For any $x \in \gamma$ and for any integer $k \geq 0$, there exist $C$ and $h_0 > 0$, such that, for any $h \in (0, h_0]$, there exists $\Phi^h_k \in C^\infty_c(M), \Phi^h_k \neq 0$, such that

$$
\left\| H^h \Phi^h_k - \lambda^h(k,x) \Phi^h_k \right\| \leq C h^{17/8} \| \Phi^h_k \|,
$$

where

$$
\lambda^h(k,x) = (2k^2 + 2k + 1) h b_0 + h^2 \left[ (2k^2 + 2k + 1) \frac{\beta(x)}{4b_0} + \frac{1}{2} (k^2 + k) R(x) \right].
$$

When $k = 0$, we get:

**Corollary 3.3.** For any $x \in \gamma$, there exist $C$ and $h_0 > 0$, such that, for any $h \in (0, h_0]$, there exists $\Phi^h_0 \in C^\infty_c(M), \Phi^h_0 \neq 0$, such that

$$
\left\| H^h \Phi^h_0 - \lambda^h(x) \Phi^h_0 \right\| \leq C h^{17/8} \| \Phi^h_0 \|,
$$

where

$$
\lambda^h(x) = h b_0 + h^2 \frac{\beta(x)}{4b_0}.
$$

3.2. Miniwells. Like in the case of the Schrödinger operator with electric potential (see [22]), one can introduce an internal notion of magnetic well for a fixed closed curve $\gamma$ in the minimum set of the magnetic field $B$. Such magnetic wells can be naturally called magnetic miniwells. They are defined by means of the function $\beta_2$ on $\gamma$ given by (3.2).

**Theorem 3.4.** Assume that there exists a unique minimum point $x_0 \in \gamma$ of the function $\beta_2$ on $\gamma$, which is nondegenerate:

$$
\mu_2 := \beta_2''(x_0) > 0.
$$

For any $j \in \mathbb{N}$, there exist $C_j$ and $h_j > 0$, such that for any $h \in (0, h_j)$

$$
\lambda_j(H^h) \leq h b_0 + h^2 \frac{\mu_0}{4b_0} + h^{5/2} \frac{(\mu_0 \mu_2)^{1/2}}{4b_0^{3/2}} (2j + 1) + C_j h^{11/4}.
$$

Here and below the derivative means the derivative with respect to the natural parameter on $\gamma$.

**Remark 3.5.** We conjecture that

$$
\lambda_0(H^h) = h b_0 + h^2 \frac{\mu_0}{4b_0} + h^{5/2} \frac{(\mu_0 \mu_2)^{1/2}}{4b_0^{3/2}} + o(h^{5/2}).
$$

The proof is based on a construction of approximate eigenfunctions, which can be made near an arbitrary Landau level. For $k \in \mathbb{N}$, consider the function $V^h_k$ on $\gamma$ given by (cf. (3.5))

$$
V^h_k(x) := (2k^2 + 2k + 1) \frac{\beta(x)}{4b_0} + \frac{1}{2} (k^2 + k) R(x).
$$
Assume that there exists a unique minimum $x_0 \in \gamma$ of the function $V_k$ on $\gamma$, which is nondegenerate, that is satisfying, for all $x \in \gamma$ in some neighborhood of $x_0$,

$$Cd(x, x_0)^2 \leq V_k(x) - V_k(x_0) \leq C^{-1}d(x, x_0)^2.$$  

Under these assumptions, one can give the following, more precise construction of approximate eigenvalues of the operator $H^h$.

**Theorem 3.6.** Under current assumptions, for any $j, k \in \mathbb{N}$, there exist $u_{jk}^h \in C^\infty_c(M)$, $C_{jk} > 0$ and $h_{jk} > 0$ such that

$$(u_{jk}^h, u_{jk}^h) = \delta_{j_1j_2} + O_{j_1j_2,k}(h)$$

and, for any $h \in (0, h_{jk}]$,

$$\|H^h u_{jk}^h - \mu_{jk}^h u_{jk}^h\| \leq C_{jk} h^{11/4} \|u_{jk}^h\|,$$

where

$$\mu_{jk}^h = \mu_{j,k,0} h + \mu_{j,k,4} h^2 + \mu_{j,k,6} h^{5/2},$$

with

$$\mu_{j,k,0} = (2k + 1)b_0, \quad \mu_{j,k,4} = V_k(x_0),$$

and

$$\mu_{j,k,6} = \frac{1}{2b_0} V_k''(x_0)^{1/2} \beta_2(x_0)^{1/2}(2k + 1)^{1/2}(2j + 1).$$

### 4. Excited states for discrete wells

If Theorem 2.1 is satisfactory for the analysis of a finite numbers of eigenvalues at the bottom, it appears to be useful to get a extended description of the bottom of the spectrum including more excited states. Motivated by Karasev’s paper [29], it seems to be interesting to produce an effective Hamiltonian whose spectrum will also describes the excited states. In 2013, Helffer-Kordyukov [18] on one side, and Raymond-Vu Ngoc [37] on the other side reanalyze the problem in the case of discrete wells with two different points of view leading in the two cases to the existence of an effective $(1D)$-Hamiltonian whose spectrum describes the spectrum of our magnetic Schrödinger operator.

#### 4.1. Using a Grushin’s problem (after Helffer-Kordyukov [18])

The approach of [18] is based on Grushin’s method. This method was initiated in the context of hypoellipticity by V. Grushin [9] and then exploited by J. Sjöstrand alone or with collaborators in many contexts. We refer to [38] for a survey on this method and references or Appendix D in [42]. In spectral theory a variant of this method is known under the name of “Feschbach projection method” or “Schur complement formula” in analytic Fredholm theory.
Let us consider the magnetic Schrödinger operator $H^h$ in the flat Euclidean space $\mathbb{R}^2$:

$$H^h = h^2 D_x^2 + (hD_y + A(x,y))^2.$$ 

The magnetic field $\mathbf{B}$ is given by

$$\mathbf{B} = b \, dx \wedge dy \text{ with } b(x,y) = \frac{\partial A}{\partial x}(x,y).$$

Let

$$b_0 = \min_{(x,y) \in \mathbb{R}^2} |b(x,y)| > 0.$$ 

We assume that at $\infty$, we have

$$b_0 < \liminf_{|x|+|y| \to +\infty} |b(x,y)| := b_0 + \eta_0.$$ 

Then one can prove easily [19] that, for any $0 \leq \eta_1 < \eta_0$, there exists $h_1 > 0$ such that

$$\sigma(H^h) \cap [0, h(b_0 + \eta_1)) \subset \sigma_d(H^h), \quad \forall h \in (0, h_1].$$

Next, as above, we assume that:

- $b_0 > 0$;
- the set $\{(x,y) \in \mathbb{R}^2 : |b(x,y)| = b_0\}$ is a single point $(x_0, y_0)$;
- $(x_0, y_0)$ is a non-degenerate minimum:

$$\text{Hess } b(x_0, y_0) > 0.$$ 

We have a diffeomorphism $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\phi(x,y) = (A(x,y), y), \quad (x,y) \in \mathbb{R}^2.$$ 

We then associate with $b$ a function $\hat{b} \in C^\infty(\mathbb{R}^2)$ by

$$\hat{b} = b \circ \phi^{-1}.$$ 

**Theorem 4.1.** There exist $h_0 > 0, \varepsilon_0 > 0, \gamma_0 \in (0, \eta_0), \ h \mapsto \gamma_0(h)$ defined for $(0, h_0]$ such that $\gamma_0(h) \to \gamma_0$ as $h \to 0$, and a semiclassical symbol $p_{\text{eff}}(y, \eta, h, z)$, which is defined in a neighborhood $\Omega \subset \mathbb{R}^2$ of the set $\{(y, \eta) \in \mathbb{R}^2 : \hat{b}(y, \eta) \leq b_0 + \gamma_0\}$ for $h \in (0, h_0]$ and $z \in \mathbb{C}$ such that $|z| < \gamma_0 + \varepsilon_0$, of the form

$$p_{\text{eff}}(y, \eta, h, z) \sim \sum_{j \in \mathbb{N}} p^j_{\text{eff}}(y, \eta, z) h^j,$$

with

$$p^0_{\text{eff}}(y, \eta, z) = \hat{b}(y, \eta) - b_0 - z,$$

such that $\lambda_h \in \sigma(H^h) \cap [0, h(b_0 + \gamma_0(h)))$, if and only if the associated $h$-pseudodifferential operator

$$p_{\text{eff}}(y, hD_y, h, z(h))$$

has an approximate 0-eigenfunction $u^{qm}_{h} \in C^\infty(\mathbb{R})$, i.e.

$$p_{\text{eff}}(y, hD_y, h, z(h)) u^{qm}_{h} = \mathcal{O}(h^\infty),$$

We use the Weyl semi-classical quantization of the symbol (see for example [26]).
with
\[ z(h) = \frac{1}{h} (\lambda - h b_0) + O(h^\infty), \]
\[ |z(h)| < \gamma_0(h) \text{ for any } h \in (0, h_0], \text{ and such that the frequency set } u_{h_0}^{qm} \text{ is non-empty and contained in } \Omega. \]

Remark 4.2. Here (4.3) makes sense modulo $O(h^\infty)$ by extending first the symbol $p_{\text{eff}}(y, \eta, h, z)$ outside the neighborhood $\Omega$ to a semiclassical symbol in $\mathbb{R}^2$ and defining then the operators $p_{\text{eff}}(y, hD_y, h, z)$ by the Weyl calculus.

Using the localization of the frequency set of $u_{h_0}^{qm}$, the left hand side of (4.3) does not depend on the extension up to an error which is $O(h^\infty)$.

Remark 4.3. By Theorem 4.1, for any $E \in [b_0, b_0 + \gamma_0)$, the spectrum of the operator $H^h$ (divided by $h$) is determined near $E$ (say in an interval $(E - Ch^{\frac{1}{2}}, E + Ch^{\frac{1}{2}})$) and modulo $O(h^{\frac{3}{2}})$ by the spectrum of $\hat{b}(y, hD_y) + h b_1(y, hD_y, E)$, where one can use the Bohr-Sommerfeld rule (see [21] or [24] for a mathematical justification) for determining the energy levels.

Corollary 4.4. There exists $\gamma_0 \in (0, \eta_0)$, $h_0 > 0$ and $C > 0$ such that
\[ \lambda_{j+1}(H^h) - \lambda_j(H^h) \geq \frac{1}{C} h^2, \forall h \in (0, h_0], \]
for any $j$ such that $\lambda_{j+1}(H^h) < h(b_0 + \gamma_0)$.

4.2. Using a Birkhoff Normal form (after Raymond–Vu Ngoc [37]).

The proof of Raymond–Vu Ngoc is reminiscent of Ivrii’s approach (see his book (old version or new version in progress on his Home Page) [28] and the – more accessible but without proofs – introductory article [24]) and uses a Birkhoff normal form. This approach has the advantage to be semi-global and uses more general symplectomorphisms and their quantizations.

Consider the magnetic Schrödinger operator $H^h$ in $\mathbb{R}^2$ given by (1.2). Let $H$ be its $h$-symbol:
\[
H(x, y, \xi, \eta) = |\xi - A_1(x, y)|^2 + |\eta - A_2(x, y)|^2, \quad (x, y, \xi, \eta) \in T^*\mathbb{R}^2 = \mathbb{R}^2 \times \mathbb{R}^2.
\]

By definition the energy surface $\Sigma_E$ corresponding to energy $E$ is the set $H^{-1}(E)$. The first result shows the existence of a smooth symplectic diffeomorphism that transforms the initial Hamiltonian into a normal form, up to any order in the distance to the zero energy surface $\Sigma_0$. Assume that the magnetic field $b$ does not vanish in an open set $\Omega \subset \mathbb{R}^2$.

Theorem 4.5 ([37], Theorem 1.1). There exists a symplectic diffeomorphism $\Phi$, defined in an open set $\bar{\Omega} \subset \mathbb{C}_{z_1} \times \mathbb{R}^2_{z_2}$, with values in $T^*\mathbb{R}^2$, which sends the plane $\{z_1 = 0\}$ to $\Sigma_0$, and such that
\[
H \circ \Phi = |z_1|^2 f(z_2, |z_1|^2) + O(|z_1|^\infty),
\]

See [42] for a discussion of the frequency set and references therein.
where \( f : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R} \) is smooth. Moreover, the map
\[
\varphi : \Omega \ni (x,y) \mapsto \Phi^{-1}(x,y,A(x,y)) \in (\{0\} \times \mathbb{R}^2) \cap \tilde{\Omega}
\]
is a local diffeomorphism and
\[
f \circ (\varphi(x,y),0) = |b(x,y)|.
\]

The next result gives the quantum counterpart of this theorem. We keep the notation of the previous theorem.

**Theorem 4.6** ([37], Theorem 1.6). For \( h \) small enough there exists a (semi-classical) Fourier Integral Operator \( U_h \) such that
\[
U_h^*U_h = I + Z_h, \quad U_hU_h^* = I + Z_h',
\]
where \( Z_h, Z_h' \) are \( h \)-pseudo-differential operators that microlocally vanish in a neighborhood of \( \tilde{\Omega} \cap \Sigma_0 \), and
\[
U_h^*H^hU_h = \mathcal{I}_hF_h + R_h,
\]
where:

1. \( \mathcal{I}_h := -h^2 \frac{\partial^2}{\partial x_1^2} + x_1^2 \).
2. \( F_h \) is a classical \( h \)-pseudo-differential operator that commutes with \( \mathcal{I}_h \).
3. For any Hermite function \( h_n(x_1) \) such that \( \mathcal{I}_h h_n = h(2n-1)h_n \), the operator \( F^{(n)}_h \) acting on \( L^2(\mathbb{R}^2) \) by
   \[
   h_n \otimes F^{(n)}_h(u) = F_h(h_n \otimes u)
   \]
is a classical \( h \)-pseudo-differential operator with principal symbol
   \[
   F^{(n)}(x_2,\xi_2) = b(x,y),
   \]
   where \( (0,x_2 + i\xi_2) = \phi(x,y) \).
4. Given any \( h \)-pseudo-differential operator \( D_h \) with principal symbol \( d_0 \) such that \( d_0(z_1,z_2) = c(z_2)|z_1|^2 + \mathcal{O}(|z_1|^3) \), and any \( N \geq 1 \), there exist classical pseudo-differential operators \( S_{h,N} \) and \( K_N \) such that
   \[
   R_h = S_{h,N}(D_h)^N + K_N + \mathcal{O}(h^\infty),
   \]
   with \( K_N \) compactly supported away from a fixed neighborhood of \( |z_1| = 0 \).
5. \( \mathcal{I}_hF_h = N_h = \mathcal{H}^0_0 + Q_h \), where \( \mathcal{H}^0_0 \) is the \( h \)-pseudodifferential operator of symbol \( H^0(z_1,z_2) = b(\varphi^{-1}(z_2))|z_1|^2 \), and the operator \( Q_h \) is relatively bounded with respect to \( \mathcal{H}^0_0 \) with an arbitrarily small relative bound.

As a consequence, Raymond and Vu Ngoc obtain the following theorem.

---

\(^3\)See [26, 42] for a definition.
Theorem 4.7 ([37], Theorem 1.5). Assume that the magnetic field $B$ is non-vanishing on $\mathbb{R}^2$ and confining: there exist constants $\tilde{C}_1 > 0, M_0 > 0$ such that

$$b(q) \geq \tilde{C}_1 \text{ for } |q| \geq M_0.$$ 

Let $\mathcal{H}_h^0 = \text{Op}_w^h(H^0)$, where $H^0 = b(\varphi^{-1}(z_2)|z_2|^2$ where $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ is a diffeomorphism. Then there exists a bounded classical pseudo-differential operator $Q_h$ on $\mathbb{R}^2$, such that

- $Q_h$ commutes with $\text{Op}_w^h(|z_1|^2)$;
- $Q_h$ is relatively bounded with respect to $\mathcal{H}_h^0$ with an arbitrarily small relative bound;
- its Weyl symbol is $O_{zz}(h^2 + h|z_1|^2 + |z_1|^4)$,

so that the following holds. Let $0 < C_1 < \tilde{C}_1$. Then the spectra of $H_h$ and $\mathcal{N}_h := \mathcal{H}_h^0 + Q_h$ in $(-\infty, C_1 h]$ are discrete. We denote by $0 < \lambda_1(h) \leq \lambda_2(h) \leq \cdots$ the eigenvalues of $H_h$ and by $0 < \mu_1(h) \leq \mu_2(h) \leq \cdots$ the eigenvalues of $\mathcal{N}_h$. Then for any $j \in \mathbb{N}^*$ such that $\lambda_j(h) \leq C_1 h$ and $\mu_j(h) \leq C_1 h$, we have

$$|\lambda_j(h) - \mu_j(h)| = O(h^\infty).$$

Remark 4.8. Theorem 4.7 is stronger than in Theorem 4.1 because Theorem 4.1 gives a description of the spectrum of $H_h$ in the interval $[hb_0, h(b_0 + \gamma_0))$ for some $\gamma_0 \in (0, \eta_0)$, whereas in Theorem 4.7, $\gamma_0 \in (0, \eta_0)$ is arbitrary. On the other hand, the symbol of the effective Hamiltonian in Theorem 4.7 seems to be less explicit than in Theorem 4.1. The other point could be that Theorem 4.1 allows us to treat an additional term $h^2 V(x, y)$. This will complete the analysis of Helffer-Sjöstrand [25], in the case of the constant magnetic field. The case with an additional term $hV$ could also be interesting.

Remark 4.9. As communicated to us by F. Faure, there is some hope that the results of [37] can be generalized under a generic assumption to the case of arbitrary even dimension. Some results are also presented in [28, Chapter 13].

5. Discrete wells in dimension 3

In this section, we discuss the three-dimensional case.

5.1. Upper bounds [17]. Consider the magnetic Schrödinger operator $H^h$ in a domain $\Omega$ of the flat Euclidean space $\mathbb{R}^3$ (see [1.2]). As usual, we assume that $H^h$ satisfies the Dirichlet boundary condition. Let $\vec{b} = (b_1, b_2, b_3)$ be the corresponding vector magnetic field (see [1.3]).

We assume that there exists a constant $C > 0$ such that for $j = 1, 2, 3$ we have

$$|(\nabla b_j)(x)| \leq C(|\vec{b}(x)| + 1), \quad \forall x \in \Omega.$$ 

Put

$$b_0 = \min\{|\vec{b}(x)| : x \in \Omega\}. $$
We assume that there exist a (connected) bounded domain $\Omega_1 \subset \subset \Omega$ and a constant $\epsilon_0 > 0$ such that
\[(5.2) \quad |\vec{b}(x)| \geq b_0 + \epsilon_0, \quad x \notin \Omega_1.\]

As shown in [19], under conditions (5.1) and (5.2), for any $\epsilon_1$ with $0 < \epsilon_1 < \epsilon_0$, there exists $h_1 > 0$ such that, for $h \in (0, h_1]$,
\[
\sigma(H^h) \cap [0, h(b_0 + \epsilon_1)) \subset \sigma_d(H^h).
\]

Denote by
\[
\lambda_0(H^h) \leq \lambda_1(H^h) \leq \lambda_2(H^h) \leq \ldots \text{ the eigenvalues of the operator } H^h \text{ contained in } [0, h(b_0 + \epsilon_0)).
\]

Finally, we assume that:
\[
b_0 > 0,
\]
and that there exists a unique minimum $x_0 \in \Omega$ such that $|\vec{b}(x_0)| = b_0$, which is non-degenerate: in some neighborhood of $x_0$
\[
C^{-1}|x - x_0|^2 \leq |\vec{b}(x)| - b_0 \leq C|x - x_0|^2.
\]

We also introduce:
\[
d = \det \operatorname{Hess} |\vec{b}|(x_0), \quad a = \frac{1}{2b_0^2} \left( \operatorname{Hess} |\vec{b}| \cdot \vec{b} \right)(x_0).
\]

**Theorem 5.1.** Under current assumptions, for any $m \in \mathbb{N}$, there exist $C_m > 0$ and $h_m > 0$ such that, for any $h \in (0, h_m]$,
\[(5.3) \quad \lambda_m(H^h) \leq h b_0 + h^{3/2} a^{1/2} + h^2 \left[ \frac{1}{2b_0} \left( \frac{d}{2a} \right)^{1/2} (2m + 1) + \nu \right] + C_m h^{9/4},
\]
where $\nu$ is some explicit constant.

The proof of Theorem 5.1 is based on a construction of quasimodes.

**Theorem 5.2.** Under current assumptions, for any $j, k$ and $m \in \mathbb{N}$, there exist $\phi^h_{j,k,m} \in C_c^\infty(\Omega)$, $C_{j,k,m} > 0$ and $h_{j,k,m} > 0$ such that
\[
(\phi^h_{j_1,k_1,m_1}, \phi^h_{j_2,k_2,m_2}) = \delta_{j_1,j_2} \delta_{k_1,k_2} \delta_{m_1,m_2} + O_{j_1,k_1,m_1,j_2,k_2,m_2}(h),
\]
and, for any $h \in (0, h_{j,k,m}]$,
\[
\|H^h \phi^h_{j,k,m} - \mu^h_{j,k,m} \phi^h_{j,k,m}\| \leq C_{j,k,m} h^{1/2} \|\phi^h_{j,k,m}\|
\]
where
\[
\mu^h_{j,k,m} = \mu_{j,k,m,0} h + \mu_{j,k,m,2} h^{3/2} + \mu_{j,k,m,4} h^2.
\]
with
\[
\mu_{j,k,m,0} = (2k + 1)b_0, \quad \mu_{j,k,m,2} = (2j + 1)(2k + 1)^{1/2} a^{1/2},
\]
and
\[
\mu_{j,k,m,4} = \frac{1}{2b_0} \left( \frac{d}{2a} \right)^{1/2} (2m + 1)(2k + 1) + \nu(j, k),
\]
which means that it is given by a rather complicated explicit formula.
where \( \nu(j, k) \) has the form
\[
\nu(j, k) = \nu_{22}(2k + 1)^2 + \nu_{11}(2j + 1)^2 + \nu_0,
\]
with some explicit constants \( \nu_0, \nu_{11}, \nu_{22} \).

**Remark 5.3.** It is conjectured that
\[
\lambda_m(H^h) \geq h b_0 + h^{3/2} a^{1/2} + h^2 \left[ \frac{1}{2b_0} \left( \frac{d}{2a} \right)^{1/2} (2m + 1) + \nu \right] - C_m h^{9/4}.
\]
At the moment, we only know from [19]
\[
\lambda_m(H^h) \geq h b_0 - C h^2,
\]
which is an improvement of the general lower bound (1.8).

### 5.2. On some statements of V. Ivrii [27, 28]

Here we refer to some results announced in [27] and developed in Chapter 18 in [28]. These results correspond in the (3D)-case to what was discussed in the (2D)-case in the subsection 4.2. Under the assumption that the magnetic field does not vanish, the claim is that (up to conjugation by an \( h \)-Fourier integral operator), our Schrödinger operator can microlocally be written in the form:
\[
\omega_1(x_1, x_2, hD_{x_2}) + \sum_{2m+n+\ell \geq 3} h^\ell a_m n_l(x_1, x_2, hD_{x_2}) (h^2 D_{x_3} + x_3^2)^n (hD_{x_1})^n,
\]
where \( \psi \) is some local unspecified diffeomorphism which plays the role of \( \varphi^{-1} \) in Theorem 4.5.

Once precisely stated and proved, let us explain what we could expect after. Reducing to the lowest Landau level (the first eigenvalue of \( h^2 D_{x_3}^2 + x_3^2 \)), we obtain that the spectrum of our initial operator near the minimum of \( |\vec{b}(x)| \) should be deduced from the spectral analysis in \( (-\infty, hb_0 + Ch^2) \) (for some fixed \( C > 0 \)) in the semi-classical limit of the following “formal” pseudo-differential operator:
\[
h \omega_1(x_1, x_2, hD_{x_2}) + h^2 D_{x_1}^2 + \sum_{2m+n+\ell \geq 3} h^{m+\ell} a_m n_l(x_1, x_2, hD_{x_2})(hD_{x_1})^n.
\]
If we only look for the principal term (and divide by \( h \)), we get as first “effective” operator to analyze for the spectrum now in \( (-\infty, b_0 + Ch^2) \):
\[
\omega_1(x_1, x_2, hD_{x_2}) + (h^{1/2} D_{x_1})^2 + ha_{020}(x_1, x_2, hD_{x_2}),
\]

5We have tried to correct many typos of the statement (more specifically the remainder in Formula (25)) in [27]. Note in particular that the sum in the right hand side is undefined (see however [18] which meets the same problem) and could only be meaningful for some subspace of functions whose energy is for example less than \( hb_0 + Ch^2 \).
with the hope to get in this way an approximation modulo $O(h^{\frac{5}{4}})$. This suggests a semi-classical analysis near the bottom of a pseudodifferential operator of the type met in Born-Oppenheimer theory $p(x_1, x_2, h^{\frac{5}{4}}D_{x_1}, hD_{x_2})$ with two semi-classical parameters (see [36] and references therein for a recent discussion on this subject). This would be coherent with the expansion obtained in the right hand side of (5.3) at least modulo $O(h^{\frac{7}{4}})$.

6. Some remarks and open questions

6.1. Geometry of magnetic fields. Consider the magnetic Schrödinger operator in the flat Euclidean space $\mathbb{R}^n$ (see (1.2)). Its semiclassical symbol (as defined in (4.4)) is a smooth function $H \in C^\infty(\mathbb{R}^{2n})$ whose zero set of $H$ given by

$$\Sigma_0 := H^{-1}(0) = \{(x, \xi) \in \mathbb{R}^{2n} : \xi_j = A_j(x), j = 1, \ldots, n\}.$$ 

Since it is a graph, it is an embedded submanifold of $\mathbb{R}^{2n}$, parameterized by $x \in \mathbb{R}^n$. It is easy to check that if we denote by $J : \mathbb{R}^n \to \Sigma$ the embedding $J(x) = (x, A(x))$, then, for the canonical symplectic form $\omega = \sum_{j=1}^n d\xi_j \wedge dx_j$ on $\mathbb{R}^{2n}$ we have

$$J^* \omega|_\Sigma \cong B.$$ 

When $n = 2$ and the magnetic field $b$ does not vanish, $\Sigma_0$ is symplectic. When $n = 3$, $\Sigma_0$ cannot be symplectic. If the magnetic field $\vec{b}$ does not vanish, then $B$ has constant rank, and $\Sigma_0$ is a presymplectic manifold. When $n$ is arbitrary even, we can hope that, under generic assumptions, $\Sigma_0$ is symplectic. This kind of analysis was basic in the seventies for the analysis of the hypoellipticity of operators with multiple characteristics.

Recall that, in Remark 2.6, we give a geometric interpretation of some terms, entering into the asymptotic formula (2.5) for approximate eigenvalues of the operator $H_b$ in the two-dimensional case. One can naturally consider similar questions in the three-dimensional case. First, observe that three cases $R = 0$, $R > 0$ and $R < 0$ mentioned in Remark 2.6 correspond to three cases of two-dimensional model geometries: Euclidean, spherical and hyperbolical, respectively. In the three-dimensional case, the situation is more complicated. There are eight three-dimensional model geometries introduced by Thurston (see, for instance, [39]). The interesting open problem is to construct the magnetic Schrödinger operators with constant magnetic field on each three-dimensional geometric model and compute its spectra. It is also interesting to find examples of integrable magnetic Schrödinger operators on three-dimensional Riemannian manifolds.

6.2. The tunneling effect. Although, as a consequence of magnetic Agmon estimates [23, 19, 36], it is possible to give upper bounds on the tunneling effect (see Section 7.2 in [12] or [10] for an introduction) due to the presence of multiconnected magnetic wells, essentially no results are known for lower bounds of this effect analogous to what is proved for the celebrated
double well problem for the Schrödinger operator $-\hbar^2 \Delta + V$. The only exception is [23], which involves $\sum_j (\hbar D_{x_j} - t(h) A_j)^2 + V$ but this last result is not a “pure magnetic effect” and it is assumed that the magnetic field is small enough ($|t(h)| = O(h|\log h|)$).

There are however a few models where one can “observe” this effect in particular in domains with corners [2] (numerics with some theoretical interpretation, see also [8] for a presentation of results due to V. Bonnaillie-Noel), the role of the magnetic wells being played by the corners of smallest angle. We describe other toy models, which are closer to the analysis which is presented in this survey:

**Example 6.1.** We consider in $\mathbb{R}^2$ the operator:

$$\hbar^2 D_x^2 + (\hbar D_y - a(x))^2 + y^2.$$ 

This model is rather artificial (and not purely magnetic) but by Fourier transform, it is unitary equivalent to

$$\hbar^2 D_x^2 + (\eta - a(x))^2 + \hbar^2 D_\eta^2,$$

which can be analyzed because it enters in the category of the miniwells problem treated in Helffer-Sjöstrand [22]. We have indeed a well defined in $\mathbb{R}^2_{x,\eta}$ by $\eta = a(x)$ which is unbounded but if we assume a varying curvature $\beta(x) = a'(x)$ (with $\lim \inf_{|x| \to +\infty} |\beta'(x)| > \inf_{|x|} |\beta(x)|$) we will have a miniwell localization. A double well phenomenon can be created by assuming $\beta = a'$ even.

**Example 6.2.** If we add an electric potential $V(x)$ to the previous example, we get:

$$\hbar^2 D_x^2 + (\hbar D_y - a(x))^2 + y^2 + V(x).$$

For $a(x) = x$, this example was considered by J. Brüning, S. Yu. Dobrokhotov and R.V. Nekrasov in [3].

Here one can measure the explicit effect of the magnetic field by considering

$$\hbar^2 D_x^2 + \hbar^2 D_\eta^2 + (\eta - a(x))^2 + V(x).$$

If $V$ admits as minimum value 0, the wells are defined in $\mathbb{R}^2_{x,\eta}$ by $\eta = a(x), V(x) = 0$ and one can use under suitable assumptions the semi-classical treatment of the double well problem for the Schrödinger operator with electric potential $W(x, \eta) = (\eta - a(x))^2 + V(x)$ (see [10]).

**Example 6.3.** One can also imagine that in the case of Sections 2 and 4, we have a magnetic double well, and that a tunneling effect could be measured using the effective $(1D)$-hamiltonian introduced in Subsection 1.1 $\hat{b}(x, hD_x)$ (actually a perturbation of it), assuming that $b$ and $A$ are holomorphic with respect to one of the variables. Here we are extremely far for a proof but we could hope for candidates for a formula for the splitting.
Example 6.4. Similarly, one can hope to measure the tunneling in the case of miniwells, in the situation considered in Subsection 3.2, when $|b|$ admits its minimum along a curve and $\beta_2$ has two symmetric miniwells.

Example 6.5. Finally one can come back to the Montgomery example [34] which was analyzed in [19, 35, 12, 5] and corresponds to the two dimensional case when the magnetic field vanish to some order on a compact curve. According to a personal communication of V. Bonnaillie-Noël, F. Hérau and N. Raymond, it seems to be reasonable to hope (work in progress) that one could analyze the splitting between the two lowest eigenvalues for the following model in $\mathbb{R}^2$:

$$h^2 D_x^2 + \left(hD_y - \frac{\gamma(y)x^2}{2}\right)^2,$$

where $\gamma$ is a positive even $C^\infty$ function with two non degenerate minima and $\inf \gamma < \lim \inf \gamma$. By dilation, this problem is unitary equivalent to the analysis of the spectrum of

$$h^{\frac{4}{3}} \left(D_x^2 + \left(h^{\frac{1}{3}}D_y - \frac{\gamma(y)x^2}{2}\right)^2\right).$$

After division by $h^{\frac{4}{3}}$, the guess is then that we can understand the tunneling by analyzing the spectrum of the $h^{\frac{4}{3}}$-pseudodifferential operator on $L^2(\mathbb{R})$ whose Weyl symbol is $\gamma(x)^{\frac{4}{3}} E(\gamma(x)^{-\frac{4}{3}}\xi)$, where $E(\alpha)$ is the ground state energy of the Montgomery operator $D_t^2 + \left(\frac{t^2}{2} - \alpha\right)^2$. This would involve a Born-Oppenheimer analysis like in [36].

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