Quantized hydrodynamic model and the dynamic structure factor for a trapped Bose gas

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Abstract

We quantize the recent hydrodynamic analysis of Stringari for the low-energy collective modes of a trapped Bose gas at $T=0$. This is based on the time-dependent Gross-Pitaevskii equation, but omits the kinetic energy of the density fluctuations. We diagonalize the hydrodynamic Hamiltonian in terms of the normal modes associated with the amplitude and phase of the inhomogeneous Bose order parameter. These normal modes provide a convenient basis for calculating observable quantities. As applications, we calculate the depletion of the condensate at $T=0$ as well as the inelastic light-scattering cross section $S(q,\omega)$ from low-energy condensate fluctuations. The latter involves a sum over all normal modes, with a weight proportional to the square of the Fourier component of the density fluctuation associated with a given mode. Finally, we show how the Thomas-Fermi hydrodynamic description can be derived starting from the coupled Bogoliubov equations.

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I. INTRODUCTION

The recent achievement of Bose-Einstein condensation (BEC) in trapped atomic gases at ultra-cold temperatures [1] has stimulated interest in the collective oscillations of a non-uniform Bose condensate [2,3]. These are of direct experimental interest [4]. At temperatures well below the BEC transition temperature, all atoms are in the condensate, and one may then base the analysis on the Gross-Pitaevskii (GP) equation of motion [5] for the time-dependent condensate wave function \( \Phi(\mathbf{r},t) \) [6]. Linearizing around the static equilibrium value \( \Phi_0(\mathbf{r}) \), this leads to the well-known Bogoliubov equations of motion for the excitations of the condensate (see, for example, Refs. [7,8]). Recently Stringari [9] has noted that the GP equation can be rewritten as equations for the condensate density and phase variables and that within a Thomas-Fermi approximation (TFA) which neglects the kinetic energy associated with density fluctuations, these have the familiar structure of the hydrodynamic equations of a superfluid at \( T = 0 \) [6]. This allows one to give a simple theory of the low-frequency excitations of a trapped Bose gas, without the necessity of solving the full set of coupled Bogoliubov equations [2,3].

In the present paper, we develop the approach of Ref. [3] by quantizing this “hydrodynamic” description and diagonalizing the associated Hamiltonian in terms of the normal modes of the condensate. This is a natural generalization of the well-known discussion used in a uniform Bose-condensed fluid [10]. These modes form a natural basis for understanding the effect of low-energy condensate fluctuations on various physical quantities. We illustrate this in this paper by evaluating the local depletion of the condensate at \( T = 0 \) as well as the inelastic light scattering cross section from the low-energy condensate fluctuations. The latter is found to be quite different from the case of a uniform weakly-interacting Bose gas (as discussed, for example, in Refs. [11,12]).

The present “hydrodynamic” description is only an adequate description of the low-energy excitations of the condensate at \( T = 0 \). In particular, as discussed by Stringari [9], the neglect of the kinetic energy in the TF approximation is only correct in the strong
interaction or large-density limit \[13\]. In Section V, we discuss the precise relation between the present analysis and that based on the full coupled Bogoliubov equations for condensate excitations. In spite of its deficiencies, the hydrodynamic approximation has the advantage that it allows one to illustrate the qualitative effects of the low-frequency modes in a very explicit manner.

II. QUANTIZED HYDRODYNAMIC THEORY

In a dilute trapped Bose gas \[1\] near \( T = 0 \), most of the atoms are Bose-condensed and thus the whole system can be well described by the Gross-Pitaevskii Hamiltonian \[5,6\]

\[
H = \int d\mathbf{r} \Phi^*(\mathbf{r}) \left[ -\frac{\hbar^2 \nabla^2}{2m} - \mu + V_{\text{ex}}(\mathbf{r}) + \frac{1}{2} g_0 |\Phi(\mathbf{r})|^2 \right] \Phi(\mathbf{r}).
\] (1)

In (1), the static external potential \( V_{\text{ex}}(\mathbf{r}) \) is responsible for trapping of the atoms. We have assumed a short-range interaction \( g_0 \equiv 4\pi\hbar^2 a / m \) between the atoms, with \( a \) being the \( s \)-wave scattering length. The Bose-condensate order parameter \( \Phi(\mathbf{r}) \) can also be written as

\[
\Phi(\mathbf{r}) = |\Phi(\mathbf{r})| e^{i\phi(\mathbf{r})},
\] (2)

in terms of amplitude and phase components. Substitution of (2) into (1) yields

\[
H = \int d\mathbf{r} \left[ \sqrt{\rho_c} \left( -\frac{\hbar^2}{2m} \right) \nabla^2 \sqrt{\rho_c} + \frac{1}{2} m \rho_c \mathbf{v}^2 - \rho_c \mu + \rho_c V_{\text{ex}} + \frac{1}{2} g_0 \rho_c^2 \right],
\] (3)

where the local condensate density and superfluid velocity are defined by

\[
\rho_c(\mathbf{r}) \equiv |\Phi(\mathbf{r})|^2,
\] (4a)

\[
\mathbf{v}(\mathbf{r}) \equiv \frac{\hbar}{2mi|\Phi(\mathbf{r})|^2} [\Phi^*(\mathbf{r}) \nabla \Phi(\mathbf{r}) - \nabla \Phi^*(\mathbf{r}) \Phi(\mathbf{r})] = \frac{\hbar \nabla \phi(\mathbf{r})}{m}.
\] (4b)

Expressed in terms of these variables, the GP theory can be formulated in terms of hydrodynamic-type equations. We can also consider a time-dependent order parameter \( \Phi(\mathbf{r}, t) \) given by the time-dependent GP equation \[\]
\begin{align*}
  i\hbar \frac{\partial}{\partial t} \Phi(r, t) &= \left[ -\frac{\hbar^2 \nabla^2}{2m} - \mu + V_{\text{ex}}(r) + g_0 |\Phi(r, t)|^2 \right] \Phi(r, t), \quad (5)
\end{align*}

which can be used to discuss the excitations from the equilibrium condensate wavefunction \( \Phi_0(r) \).

To quantize a hydrodynamic-type description, we first replace the dynamical variables in \((4)\) by the corresponding quantum mechanical operators. The density and velocity operators satisfy the exact commutation rule \((6)\)
\begin{align*}
  [\hat{\rho}(r), \hat{\mathbf{v}}(r')] &= i\frac{\hbar}{m} \nabla \delta(r - r').
\end{align*}

This can be used in the analysis of the \( T = 0 \) GP expression \((4)\) because almost all the atoms are in the condensate and hence \( \hat{\rho}_c(r) \approx \hat{\rho}(r) \). (By the same token, it is difficult to quantize a hydrodynamic-type theory at \( T \neq 0 \) when there is significant depletion.) In terms of the phase operator defined by \((2)\), \((6)\) is equivalent to \((7)\)
\begin{align*}
  [\hat{\rho}(r), \hat{\phi}(r')] &= i\delta(r - r').
\end{align*}

Using the operator version of the Hamiltonian in \((3)\), it is straightforward to work out the Heisenberg equations of motion for both \( \hat{\rho} \) and \( \hat{\mathbf{v}} \) making use of \((6)\). One obtains \((8)\)
\begin{align*}
  \frac{\partial \hat{\rho}(r, t)}{\partial t} &= -\nabla \cdot [\hat{\rho}(r, t)\hat{\mathbf{v}}(r, t)] \quad (8)
\end{align*}

and
\begin{align*}
  m \frac{\partial \hat{\mathbf{v}}(r, t)}{\partial t} &= -\nabla \left[ V_{\text{ex}}(r) - \mu + g_0 \hat{\rho}(r, t) - \frac{\hbar^2 \nabla^2 \sqrt{\hat{\rho}(r, t)}}{2m\sqrt{\hat{\rho}(r, t)}} + \frac{1}{2} m \hat{\mathbf{v}}^2(r, t) \right]. \quad (9)
\end{align*}

Assuming the validity of the Gross-Pitaevskii \((3)\) Hamiltonian given by \((1)\), \((8)\) and \((9)\) are exact and can be used to discuss the dynamics of the condensate. They are equivalent to the GP equation in \((5)\).

It is useful to separate various operators into equilibrium and small fluctuation parts
\begin{align*}
  \hat{\Phi}(r, t) &\equiv \Phi_0(r) + \delta\hat{\Phi}(r, t) \\
  \hat{\rho}(r, t) &\equiv \rho_0(r) + \delta\hat{\rho}(r, t) \\
  \hat{\phi}(r, t) &\equiv \phi_0(r) + \delta\hat{\phi}(r, t) \\
  \hat{\mathbf{v}}(r, t) &\equiv \mathbf{v}_0(r) + \delta\hat{\mathbf{v}}(r, t), \quad (10)
\end{align*}
where the static condensate density profile is \( \rho_0(\mathbf{r}) \equiv |\Phi_0(\mathbf{r})|^2 \) and \( \mathbf{v}_0(\mathbf{r}) \equiv \hbar \nabla \phi_0(\mathbf{r})/m \).

Before proceeding, we recall that the equilibrium values are given by the solutions of the time-independent GP equation [see (5)],

\[
\left[ -\frac{\hbar^2 \nabla^2}{2m} - \mu + V_{\text{ex}}(\mathbf{r}) + g_0 \rho_0(\mathbf{r}) \right] \Phi_0(\mathbf{r}) = 0,
\]

(11)

where \( \Phi_0(\mathbf{r}) = |\Phi_0(\mathbf{r})|e^{i\phi_0(\mathbf{r})} \). The static solution of (8) must satisfy

\[
\nabla \cdot \left[ |\Phi_0(\mathbf{r})|^2 \mathbf{v}_0(\mathbf{r}) \right] = 0,
\]

(12)

which is equivalent to

\[
2 \nabla |\Phi_0(\mathbf{r})|^2 \cdot \nabla \phi_0(\mathbf{r}) + |\Phi_0(\mathbf{r})| \nabla^2 \phi_0(\mathbf{r}) = 0.
\]

(13)

Using this in (11), the latter equation can be reduced to an equation for the amplitude

\[
|\Phi_0(\mathbf{r})| = \sqrt{\rho_0(\mathbf{r})}
\]

\[
\left[ -\frac{\hbar^2 \nabla^2}{2m} - \mu + V_{\text{ex}}(\mathbf{r}) + g_0 \rho_0(\mathbf{r}) + \frac{1}{2} m \mathbf{v}_0^2(\mathbf{r}) \right] |\Phi_0(\mathbf{r})| = 0.
\]

(14)

Solutions of (13) and (14) with a spatially-varying phase \( \phi_0(\mathbf{r}) \) correspond to states with a non-zero superfluid velocity \( \mathbf{v}_0(\mathbf{r}) \) (such as vortices).

In this paper, we are interested in small fluctuations of the static condensate \( \Phi_0(\mathbf{r}) \). In terms of (14) and (10), we obtain

\[
\delta \hat{\rho}(\mathbf{r}, t) = \Phi_0^*(\mathbf{r}) \delta \hat{\Phi}(\mathbf{r}, t) + \Phi_0(\mathbf{r}) \delta \hat{\Phi}^\dagger(\mathbf{r}, t).
\]

(15)

Writing \( \hat{\Phi}(\mathbf{r}, t) \equiv \Phi_0(\mathbf{r})[1 + \delta \hat{A}(\mathbf{r}, t)]e^{i\delta \hat{\phi}(\mathbf{r}, t)} \) in terms of Hermitian amplitude and phase fluctuation operators, we find

\[
\delta \hat{\Phi}(\mathbf{r}, t) = \Phi_0(\mathbf{r})[\delta \hat{A}(\mathbf{r}, t) + i \delta \hat{\phi}(\mathbf{r}, t)]
\]

(16)

and hence the density fluctuation operator in (15) is related (as expected) to the amplitude fluctuation operator

\[
\delta \hat{\rho}(\mathbf{r}, t) = 2|\Phi_0(\mathbf{r})|^2 \delta \hat{A}(\mathbf{r}, t) = 2\rho_0(\mathbf{r}) \delta \hat{A}(\mathbf{r}, t).
\]

(17)
We have used the fact that $\hat{\delta A} = \delta \hat{A}^\dagger$ and $\hat{\delta \phi} = \delta \hat{\phi}^\dagger$. With (10), the commutator in (9) can be rewritten in terms of fluctuation operators

$$[\hat{\delta \hat{\rho}}(\mathbf{r}), \hat{\delta \hat{\phi}}(\mathbf{r}')] = i\delta(\mathbf{r} - \mathbf{r}').$$

(18)

The key simplification to be used in analyzing the Eqs. (8) and (9) is to neglect the kinetic energy term associated with density fluctuations $\hbar^2 \nabla^2 \sqrt{\rho}/2m\sqrt{\rho}$ in determining both the equilibrium and time-dependent solutions. This is referred to as the Thomas-Fermi approximation (TFA) [13, 9]. In the TFA, for $v_0(\mathbf{r}) = 0$, the static density profile $\rho_0(\mathbf{r})$ is related to the external trapping potential by [15]

$$\mu = g_0 \rho_0(\mathbf{r}) + V_{\text{ex}}(\mathbf{r}),$$

(19)

where the chemical potential $\mu$ is fixed by the total number of atoms $N = \int d\mathbf{r} \rho_0(\mathbf{r})$ and the condition $\rho_0(\mathbf{r}) \geq 0$. This should be a good description if the number of atoms is large enough. More specifically, it is valid for $Na/a_{\text{HO}} \gg 1$, where $a_{\text{HO}} \equiv (\hbar/m\omega_0)^{\frac{1}{2}}$ is the characteristic harmonic oscillator length for an external potential $V_{\text{ex}}(\mathbf{r}) = \frac{1}{2}m\omega_0^2 r^2$. One can show that this implies $\mu \gg h\omega_0$. The use of the TFA precludes the consideration of a vortex state, where $v_0(\mathbf{r}) = n\hbar/mr_\perp$ ($r_\perp$ is the distance from the axis of the vortex and $n$ is an integer). In the presence of vortices, the kinetic energy terms in (14) plays a crucial role in the determination of the spatial variation of $|\Phi_0(\mathbf{r})|$ [14].

In the same approximation, (8) and (3) give (to first order in the perturbed quantities)

$$\frac{\partial \delta \hat{\rho}(\mathbf{r},t)}{\partial t} = -\nabla \cdot [\rho_0(\mathbf{r})\delta \hat{\mathbf{v}}(\mathbf{r},t)]$$

(20)

and

$$m \frac{\partial \delta \hat{\mathbf{v}}(\mathbf{r},t)}{\partial t} = -\nabla [g_0 \delta \hat{\rho}(\mathbf{r},t)].$$

(21)

To be explicit, we have neglected the following terms in the square bracket in (11)

$$\frac{\hbar^2}{4m\sqrt{\rho_0(\mathbf{r})}} \left\{ -\nabla^2 \sqrt{\rho_0(\mathbf{r})} \frac{\delta \hat{\rho}(\mathbf{r},t)}{\rho_0(\mathbf{r})} + \nabla^2 \left[ \frac{\delta \hat{\rho}(\mathbf{r},t)}{\sqrt{\rho_0(\mathbf{r})}} \right] \right\}$$

(22)
relative to $g_0\delta\hat{\rho}(r, t)$. Combining (20) and (21) gives

$$
\frac{\partial^2 \delta\hat{\rho}(r, t)}{\partial t^2} = \nabla \cdot \left[ \frac{g_0}{m} \rho_0(r) \nabla \delta\hat{\rho}(r, t) \right]
$$

(23)

and the equivalent

$$
\frac{\partial^2 \delta\hat{\phi}(r, t)}{\partial t^2} = \nabla \cdot \left[ \frac{g_0}{m} \rho_0(r) \nabla \delta\hat{\phi}(r, t) \right].
$$

(24)

The solutions of (23) or (24) give the low-frequency condensate collective modes [9] of an inhomogeneous Bose gas with a local condensate density $\rho_0(r)$ given by (19).

In this approximate treatment, the GP Hamiltonian (3) reduces to

$$
\hat{H} = H_0 + \frac{1}{2} \int dr \left[ m\rho_0(r)\delta\hat{v}^2(r) + g_0\delta\hat{\rho}^2(r) \right],
$$

(25)

where the ground state is described by

$$
H_0 \equiv -\frac{1}{2} \int dr g_0\rho_0^2(r).
$$

(26)

An expression identical to (24) is derived in § 5.2 of Ref. [6] by a different procedure. Clearly the quadratic form (24) can be diagonalized by expanding the operators $\delta\hat{\rho}(r)$ and $\delta\hat{\phi}(r)$ [or $\delta\hat{v}(r)$]

$$
\delta\hat{\rho}(r) = \sum_j \left[ A_j \psi_j(r) \hat{\alpha}_j + A_j^* \psi_j^*(r) \hat{\alpha}_j^\dagger \right]
$$

$$
\delta\hat{\phi}(r) = \sum_j \left[ B_j \psi_j(r) \hat{\alpha}_j + B_j^* \psi_j^*(r) \hat{\alpha}_j^\dagger \right],
$$

(27)

in terms of creation and annihilation operators $\hat{\alpha}_j^\dagger$ and $\hat{\alpha}_j$ which create and destroy excitations of the condensate with energy $\hbar\omega_j$. These operators satisfy the usual Bose commutation relations: $[\hat{\alpha}_j, \hat{\alpha}_{j'}] = [\hat{\alpha}_j^\dagger, \hat{\alpha}_{j'}^\dagger] = 0$ and $[\hat{\alpha}_j, \hat{\alpha}_{j'}^\dagger] = \delta_{j,j'}$. The hermiticity of $\delta\hat{\rho}$ and $\delta\hat{\phi}$ is ensured by the form used in (27). The eigenfunctions $\psi_j(r)$ are assumed to satisfy both the normalization relation

$$
\int dr \ \psi_j^*(r) \psi_{j'}(r) = \delta_{j,j'},
$$

(28)

and the completeness relation

7
\[ \sum_j \psi_j^*(r) \psi_j(r') = \delta(r - r'). \] (29)

Using (27) with the help of (29), the commutator (18) can be shown to require that \( A_j^* B_j = -i/2 \).

Assuming that (25) has been diagonalized to give

\[ \hat{H} = H_0 + \sum_j \hbar \omega_j \left( \hat{\alpha}_j \hat{\alpha}_j^\dagger + \frac{1}{2} \right), \] (30)

the time-dependent operators in Heisenberg representation are then immediately given by

\[
\begin{align*}
\delta \hat{\rho}(r, t) &= \sum_j [A_j \psi_j(r) e^{-i \omega_j t} \hat{\alpha}_j + A_j^* \psi_j^*(r) e^{i \omega_j t} \hat{\alpha}_j^\dagger], \\
\delta \hat{\phi}(r, t) &= \sum_j [B_j \psi_j(r) e^{-i \omega_j t} \hat{\alpha}_j + B_j^* \psi_j^*(r) e^{i \omega_j t} \hat{\alpha}_j^\dagger].
\end{align*}
\] (31)

Using (31) in (21) immediately gives a second relation between the coefficients in (27), namely (for \( \omega_j > 0 \))

\[ i \hbar \omega_j B_j = g_0 A_j. \] (32)

Using (23) or (24) in connection with the expression (31), we see that the eigenfunctions \( \psi_j(r) \) and eigenvalues \( \omega_j \) are determined by

\[ -\nabla \cdot \left[ \frac{g_0}{m} \rho_0(r) \nabla \psi_j(r) \right] = \omega_j^2 \psi_j(r). \] (33)

Combining (32) with the relation \( A_j^* B_j = -i/2 \) leads to (for \( \omega_j > 0 \))

\[ A_j = i \sqrt{\frac{\hbar \omega_j}{2g_0}}; \quad B_j = \sqrt{\frac{g_0}{2 \hbar \omega_j}}. \] (34)

where, for convenience, \( A_j \) is chosen to be purely imaginary and \( B_j \) is chosen to be real.

Making use of the results (34) in (27), one can now verify after some algebra that the Hamiltonian (25) does indeed reduce to (30), as assumed. The harmonic Hamiltonian (30) describes the fluctuations of the condensate \( \Phi_0(r) \) in terms of a non-interacting gas of Bose excitations.

Eq. (33) has been solved by Stringari [9] for both isotropic and anisotropic parabolic traps. For an isotropic harmonic potential, the solutions are \( j \) represents the usual quantum numbers \( (n, \ell, m) \) for a spherical potential.)
\[ \psi_j(r) = c_j P^{(2n)}_\ell \left( \frac{r}{R} \right) r^\ell Y_{\ell m}(\theta, \phi) \Theta(R - r), \]  
(35)

where \( \Theta(x) \) is the step function. The associated energy eigenvalues are found to be

\[ \omega_j = \omega_0 \left( 2n^2 + 2n\ell + 3n + \ell \right)^{\frac{1}{2}}. \]  
(36)

In (35), \( Y_{\ell m}(\theta, \phi) \) are spherical harmonics, while \( P^{(2n)}_\ell(x) = \sum_{m=0}^{n} \alpha_{2m}(\ell) x^{2m} \) is a polynomial with coefficients satisfying the recurrence relation: \( \alpha_{2m+2} = -\alpha_{2m}(n - m)(2\ell + 2m + 3)/(m + 1)(2\ell + 2m + 3) \), with \( \alpha_0 = 1 \). In the Thomas-Fermi approximation (19), the chemical potential is given by \( \mu = \frac{1}{2} m \omega_0^2 R^2 \), where \( R \) is the radius at which \( \rho_0(r) \) vanishes. This implies that the integration in (28) is over a sphere of radius \( R \). One also finds directly that \( \mu = 15a\hbar^2 N/2mR^3 \), which can be used to give \( R \) in terms of \( N \) and \( a \). One can show that the TF hydrodynamic approximation should be valid for \( \omega_j \ll \mu \). The normalization factor \( c_j \) is determined via (28),

\[ c_j = c_{n\ell m} = \left\{ R^{2\ell+3} \int_0^1 dx x^{2\ell+2} \left[ P^{(2n)}_\ell(x) \right]^2 \right\}^{-\frac{1}{2}}. \]  
(37)

For \( n = 0 \), one has \( c_{0\ell m} = [(2\ell + 3)/R^{2\ell+3}]^{\frac{1}{2}} \).

For an isotropic parabolic trap, (35) gives the normal-mode basis functions which appear in \( \delta \hat{\rho}(r, t) \) and \( \delta \hat{\phi}(r, t) \) in (27). The normal-mode frequencies (36) are independent of the interaction strength because in (33), we have used the relation \( g_0\rho_0(r) = \mu - V_{ex}(r) \) given by (19). The corresponding results of (35) and (36) for a uniform Bose fluid \( \mu = g_0\rho_0/m \). We note that only finite-energy solutions of (33) are involved in the fluctuations \( \hat{\rho} \). A zero energy solution would correspond to the ground state value. The \( n = 0, \ell = 0 \) solution of (35), which corresponds to \( \psi_j(r) = \text{constant} \) and \( \omega_j = 0 \), is excluded in all summations such as (27). For further discussion of the zero energy solution, see the end of Section V.
III. GREEN’S FUNCTIONS AND DEPLETION OF CONDENSATE

Writing the quantum field operators (see Ch. 3 of Ref. [11]) in terms of condensate and non-condensate contributions \( \hat{\psi}(\mathbf{r}) \equiv \Phi_0(\mathbf{r}) + \tilde{\psi}(\mathbf{r}) \), the diagonal single-particle Green’s function \( G_{11} \) is given by [using (15)-(17)]

\[
G_{11}(\mathbf{r}, \mathbf{r}', t) \equiv -\langle \tilde{\psi}(\mathbf{r}, t) \tilde{\psi}(\mathbf{r}') \rangle \\
= -\langle \delta \hat{\Phi}(\mathbf{r}, t) \delta \hat{\Phi}(\mathbf{r}') \rangle \\
= -\Phi_0^*(\mathbf{r}) \Phi_0(\mathbf{r}') \left[ \langle \delta \hat{A}(\mathbf{r}, t) \delta \hat{A}(\mathbf{r}') \rangle + \langle \delta \hat{\phi}(\mathbf{r}, t) \delta \hat{\phi}(\mathbf{r}') \rangle \\
+ i \langle \delta \hat{A}(\mathbf{r}, t) \delta \hat{\phi}(\mathbf{r}') \rangle - i \langle \delta \hat{\phi}(\mathbf{r}, t) \delta \hat{A}(\mathbf{r}') \rangle \right].
\]

(38)

Using the normal-mode expansion (27), we find after some algebra

\[
G_{11}(\mathbf{r}, \mathbf{r}', t) = -\Phi_0^*(\mathbf{r}) \Phi_0(\mathbf{r}') \sum_j \left\{ \left[ \frac{\hbar \omega_j}{8g_0} \frac{1}{\rho_0(\mathbf{r}) \rho_0(\mathbf{r}')} + \frac{g_0}{2\hbar \omega_j} \right] C_{j+}(\mathbf{r}, \mathbf{r}', t) \\
+ \frac{1}{4} \left[ \frac{1}{\rho_0(\mathbf{r})} + \frac{1}{\rho_0(\mathbf{r}')} \right] C_{j-}(\mathbf{r}, \mathbf{r}', t) \right\},
\]

(39)

where we have defined

\[
C_{j\pm}(\mathbf{r}, \mathbf{r}', t) \equiv \psi_j^\dagger(\mathbf{r}) \psi_j(\mathbf{r}') N^0(\omega_j) e^{i\omega_j t} \pm \psi_j(\mathbf{r}) \psi_j^\dagger(\mathbf{r}') N^0(\omega_j) + 1 \right\} e^{-i\omega_j t}
\]

(40)

and \( N^0(\omega_j) = \left[ \exp (\beta \hbar \omega_j) - 1 \right]^{-1} \) is the Bose distribution function, with \( \beta = 1/k_B T \). In (39), the first term involves amplitude fluctuations, the second term involves phase fluctuations, while the third and fourth terms involve the coupling between the amplitude and phase fluctuations. Strictly speaking, this result is only valid at \( T \simeq 0 \), since we started with the Gross-Pitaevskii description [11] which assumes all the atoms are in the Bose-condensate. The time-ordered Green’s function equivalent of (39) is easily worked out.

In an analogous way, one can calculate \( G_{22} \) to be

\[
G_{22}(\mathbf{r}, \mathbf{r}', t) \equiv -\langle \delta \hat{\phi}(\mathbf{r}, t) \delta \hat{\phi}^\dagger(\mathbf{r}') \rangle \\
= -\Phi_0(\mathbf{r}) \Phi_0^*(\mathbf{r}') \sum_j \left\{ \left[ \frac{\hbar \omega_j}{8g_0} \frac{1}{\rho_0(\mathbf{r}) \rho_0(\mathbf{r}')} + \frac{g_0}{2\hbar \omega_j} \right] C_{j+}(\mathbf{r}, \mathbf{r}', t) \\
- \frac{1}{4} \left[ \frac{1}{\rho_0(\mathbf{r})} + \frac{1}{\rho_0(\mathbf{r}')} \right] C_{j-}(\mathbf{r}, \mathbf{r}', t) \right\},
\]

(41)
and the off-diagonal single-particle Beliaev Green’s functions are

\[
G_{12}(\mathbf{r}, \mathbf{r}', t) \equiv -\langle \hat{\delta} \hat{\Phi}^i(\mathbf{r}, t) \hat{\delta} \hat{\Phi}^i(\mathbf{r}') \rangle \\
= -\Phi^*_0(\mathbf{r}) \Phi_0^*(\mathbf{r}') \sum_j \left\{ \left[ \frac{\hbar \omega_j}{8 g_0 \rho_0(\mathbf{r}) \rho_0(\mathbf{r}')} - \frac{g_0}{2 \hbar \omega_j} \right] C_{j+}(\mathbf{r}, \mathbf{r}', t) \right\} \\
- \frac{1}{4} \left[ \frac{1}{\rho_0(\mathbf{r})} - \frac{1}{\rho_0(\mathbf{r}')} \right] C_{j-}(\mathbf{r}, \mathbf{r}', t) \right\}
\]

(42)

and

\[
G_{21}(\mathbf{r}, \mathbf{r}', t) \equiv -\langle \hat{\delta} \hat{\Phi}(\mathbf{r}, t) \hat{\delta} \hat{\Phi}(\mathbf{r}') \rangle \\
= -\Phi_0(\mathbf{r}) \Phi_0(\mathbf{r}') \sum_j \left\{ \left[ \frac{\hbar \omega_j}{8 g_0 \rho_0(\mathbf{r}) \rho_0(\mathbf{r}')} - \frac{g_0}{2 \hbar \omega_j} \right] C_{j+}(\mathbf{r}, \mathbf{r}', t) \right\} \\
+ \frac{1}{4} \left[ \frac{1}{\rho_0(\mathbf{r})} - \frac{1}{\rho_0(\mathbf{r}')} \right] C_{j-}(\mathbf{r}, \mathbf{r}', t) \right\}
\]

(43)

As an application of the above results, we calculate the depletion of the condensate density near \( T = 0 \). Using (39) and (40), one obtains

\[
\langle \tilde{\rho}(\mathbf{r}) \rangle \equiv -G_{11}(\mathbf{r}, \mathbf{r}, t = 0) \\
= \sum_j |\psi_j(\mathbf{r})|^2 \left\{ \frac{\hbar \omega_j}{4 g_0 \rho_0(\mathbf{r})} + \frac{g_0 \rho_0(\mathbf{r})}{\hbar \omega_j} - 1 \right\} (T \approx 0).
\]

(44)

For a spatially uniform Bose-condensed gas (\( \rho_0(\mathbf{r}) = \text{constant} \) and \( \omega_j \to \omega(\mathbf{k}) = c k \)), the phase fluctuations (second term) are the dominant contribution to the expression in (44). It then reduces to \[10\]

\[
\tilde{\rho} = \rho - \rho_0 = \frac{1}{2} \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{g_0 \rho_0}{\hbar \omega(\mathbf{k})}.
\]

(45)

In contrast, the phase fluctuations are not so dominant in a trapped Bose gas because their energy is finite. We note that (44) is similar to recent results based on a variational solution of the full Bogoliubov equations \[16\].

As discussed by Stringari \[9\], the collective mode energies in (36) in the \( n = 0 \) case reduce to the simple expression \( \omega_j \to \omega_0 \ell = \omega_0 \sqrt{\ell} \), for \( \ell \) finite. The \( \ell \)-channel contribution in (44) for these \( n = 0 \) “surface modes” \[3\] is given by
\[ \langle \tilde{\rho}(r) \rangle_{0,\ell} = \frac{1}{16\pi R^3}(2\ell + 1)(2\ell + 3) \bar{r}^{2\ell} \left( \frac{\sqrt{\ell} \bar{a}_{HO}^2}{1 - \bar{r}^2} + \frac{1 - \bar{r}^2}{\sqrt{\ell} \bar{a}_{HO}^2} - 2 \right), \]  

(46)

where \( \sum_m |Y_{\ell m}(\theta, \phi)|^2 = (2\ell + 1)/4\pi \) has been used and we have defined \( \bar{r} \equiv r/R, \bar{a}_{HO} \equiv a_{HO}/R \). The \( n = 1, \ell = 0 \) contribution to the depletion is 

\[ \langle \tilde{\rho}(r) \rangle_{1,0} = \frac{63}{64\pi R^3} \left( 1 - \frac{5}{3} \bar{r}^2 \right)^2 \left( \frac{\sqrt{5} \bar{a}_{HO}^2}{1 - \bar{r}^2} + \frac{1 - \bar{r}^2}{\sqrt{5} \bar{a}_{HO}^2} - 2 \right). \]  

(47)

For illustration, in Fig. 4 we show the depletion \( \langle \tilde{\rho}(r) \rangle \) as a function of \( r \) from the lowest-energy modes described by (46) and (47) [17]. We have assumed \( N = 10^5 \) trapped atoms, an \( s \)-wave scattering length \( a = 50\ \text{Å} \) and \( a_{HO} = 10^4\ \text{Å} \). These parameters give \( R = 6 \times 10^4\ \text{Å} \) [14]. The local condensate density is given by (19), namely

\[ \rho_0(r) = \frac{R^2}{8\pi a a_{HO}^4} \left( 1 - \bar{r}^2 \right) \Theta(1 - \bar{r}). \]  

(48)

For the parameters used in Fig. 4 we find \( \rho_0(r) = (1 - \bar{r}^2) 2.8 \times 10^{14}\ \text{cm}^{-3} \) and thus the \( T = 0 \) condensate depletion is very small, as expected. For these parameters, we note that

\[ \mu = \frac{1}{2} \left( \frac{R}{a_{HO}} \right)^2 \hbar \omega_0 = 18\hbar \omega_0 \]  

and hence these frequencies satisfy the condition \( \omega \lesssim \mu \) for which the TF approximation is valid [4].

**IV. DYNAMIC STRUCTURE FACTOR**

We next calculate the light scattering cross section which is proportional to the dynamic structure factor \( S(q, \omega) \). The latter is given by the Fourier transform of the density-density correlation function. We recall that the density can be split into contribution involving atoms being excited in/out of the condensate and contribution from non-condensate atoms. Thus we have (see Ch. 3 of [11])

\[ \hat{\rho}(r) \equiv \hat{\psi}^\dagger(r) \hat{\psi}(r) \]

\[ = |\Phi_0(r)|^2 + [\Phi_0^*(r) \hat{\psi}(r) + \Phi_0(r) \hat{\psi}^\dagger(r)] + \hat{\psi}^\dagger(r) \hat{\psi}(r) \]

\[ \equiv \rho_0(r) + \delta \hat{\rho}_c(r) + \tilde{\rho}(r), \]  

(49)
where \( \tilde{\rho}(r) \equiv \tilde{\psi}^*(r)\tilde{\psi}(r) \) is the non-condensate local density operator. In the present discussion of a dilute Bose gas, it will be assumed that one can neglect \( \tilde{\rho}(r) \) as being small at \( T = 0 \). The density-density correlation function due to the condensate density fluctuations is easily calculated to be [using (17), (38), and (39)]

\[
\langle \delta \hat{\rho}_c(r, t) \delta \hat{\rho}_c(r') \rangle = 4 \rho_0(r) \rho_0(r') \langle \delta \hat{A}(r, t) \delta \hat{A}(r') \rangle = \sum_j \frac{\bar{\hbar} \omega_j^2}{2g_0} C_{j+}(r, r', t),
\]

where the function \( C_{j+}(r, r', t) \) has been defined in (40). Thus for the problem being considered, the condensate contribution to \( S(q, \omega) \) is given by

\[
S_c(q, \omega) \propto \int d\mathbf{r} e^{-i\mathbf{q}\cdot\mathbf{r}} \int d\mathbf{r}' e^{i\mathbf{q}\cdot\mathbf{r}'} \int_{-\infty}^{\infty} dt e^{i\omega t} \langle \delta \hat{\rho}_c(r, t) \delta \hat{\rho}_c(r') \rangle = \sum_j \frac{\bar{\hbar} \omega_j^2}{2g_0} |\psi_j(q)|^2 \left\{ N^0(\omega_j) + 1 \right\} \delta(\omega - \omega_j) + N^0(\omega_j) \delta(\omega + \omega_j),
\]

where

\[
\psi_j(q) = \int d\mathbf{r} e^{-i\mathbf{q}\cdot\mathbf{r}} \psi_j(r).
\]

This is for a momentum transfer \( \hbar \mathbf{q} \) and energy transfer \( \hbar \omega \) given to the Bose condensate.

Since we are only dealing with condensate fluctuations (density fluctuations involving atoms entering/leaving the condensate), it is trivial that the poles of \( S_c(q, \omega) \) in (51) are identical to those of the single-particle Green’s functions in (38) and (42). However, this sharing of poles is a general feature of Bose-condensed fluids even when one includes non-condensate contributions (see Ch. 5 of Ref. [11]). While (51) is based on a calculation which is only valid at \( T \approx 0 \), we expect that more generally, it will approximately describe the contribution related to exciting excitations out of the condensate. In this context, it is useful to give the two limiting expressions

\[
S_c(q, \omega) \rightarrow \begin{cases} \sum_j \frac{\bar{\hbar} \omega_j^2}{2g_0} |\psi_j(q)|^2 \delta(\omega - \omega_j) & \text{if } k_B T \ll \hbar \omega_j \\ \sum_j \frac{k_B T}{2g_0} |\psi_j(q)|^2 [\delta(\omega - \omega_j) + \delta(\omega + \omega_j)] & \text{if } k_B T \gg \hbar \omega_j \end{cases}
\]

13
Of course, (51) and (53) do not include the scattering from thermally excited atoms present at \( T \neq 0 \) or the two-excitation (“multiphonon”) contributions \([11,12]\). These are briefly discussed at the end of this section.

In the analogous calculation of \( S_c(\mathbf{q}, \omega) \) for a weakly interacting homogeneous Bose-condensed gas \([11,12]\), one picks up a single sharp peak at the quasiparticle energy \( \omega(\mathbf{q}) \) corresponding to the momentum transfer involved. In a trapped Bose gas, in contrast, \( S_c(\mathbf{q}, \omega) \) in (51) is seen to be a weighted sum of all the normal modes describing the fluctuating inhomogeneous condensate, of frequency \( \omega_j \) and with a weight proportional to \( \omega_j |\psi_j(\mathbf{q})|^2 \).

Using the well-known expansion (for \( \mathbf{q} \) parallel to \( z \)-axis)

\[
e^{-i\mathbf{q}\cdot\mathbf{r}} = \sum_{\ell=0}^{\infty} \sqrt{\frac{4\pi(2\ell+1)}{2\pi}} (-i)^\ell j_\ell(qr)Y_0(\theta),
\]

(54)

where \( j_\ell \) is the \( \ell \)-order Bessel function, one can use (53) to directly obtain the Fourier transform of the eigenfunction \( \psi_j(\mathbf{r}) \), namely

\[
\psi_{n\ell m}(\mathbf{q}) = c_{n\ell m} (-i)^\ell \delta_{m,0} \sqrt{\frac{4\pi(2\ell+1)}{2\pi}} R^{\ell+3} \int_0^1 dx \ x^{\ell+2} P_\ell^{(2n)}(x) j_\ell(qRx),
\]

(55)

where the normalization factor \( c_{n\ell m} \) is given by (37). We note that as a result of (54), only the states \( \psi_j(\mathbf{r}) \) with \( m = 0 \) have a finite weight in \( S_c(\mathbf{q}, \omega) \) in (51).

In Fig. 2, we have plotted \( \omega_j |\psi_j(\mathbf{q})|^2 \) which appears in the formula (51) for \( S_c(\mathbf{q}, \omega) \) as a function of the dimensionless wavevector \( qR \) for excitations \((n, \ell) = (0, 1), (0, 2), (0, 3) \) and \((1, 0)\), with \( m = 0 \) in all cases. As one can see from Fig. 2, the strongest weights for these collective modes appear for \( qR \lesssim 6 \). As an example, for the parameters used in Fig. 4 to give \( R = 6 \times 10^{-4} \text{cm} \), this means that ideally, the momentum transfer \( q \) in a light-scattering experiment should be \( q \lesssim 10^4 \text{cm}^{-1} \) in order to pick up the strong spectral weight from the low-energy collective modes. For \( qR \gtrsim 1 \), the density fluctuations being probed have a wavelength smaller than the size of the condensate \( 2R \). The Thomas-Fermi approximation (19) we have used should be adequate. In contrast, the TF approximation is probably not very good for \( qR \ll 1 \) since the results could be sensitive to the abrupt vanishing of \( \rho_0(\mathbf{r}) \) for \( r > R \).
In Fig. 3, we plot the $T=0$ dynamic structure factor $S_c(q, \omega)$ given by (53). For clarity and as a description of finite energy resolution, we have broadened the delta function $\delta(\omega)$ using a Lorentzian, with a width of $\Gamma = 0.05\omega_0$. The momentum transfer corresponds to $qR = 2$. In Fig. 4, we give results for a higher momentum transfer $qR = 10$ using the same arbitrary units as in Fig. 3, showing a much weaker light-scattering intensity.

Besides the condensate contribution to $S(q, \omega)$ given by (51), we can use our hydrodynamic approximation to estimate the contribution of the non-condensate part. Using (49), the density-density correlation function arising from the non-condensate fluctuations can be expressed in terms of the single-particle Green’s function $G_{\alpha\beta}$ discussed in Section III,

$$
\langle \tilde{\rho}(r, t)\tilde{\rho}(r') \rangle = \langle \delta\Phi^\dagger(r, t)\delta\Phi(r, t)\delta\Phi^\dagger(r')\delta\Phi(r') \rangle \\
= \langle \delta\Phi^\dagger(r, t)\delta\Phi(r') \rangle \langle \delta\Phi(r, t)\delta\Phi^\dagger(r') \rangle + \langle \delta\Phi(r, t)\delta\Phi(r') \rangle \langle \delta\Phi^\dagger(r, t)\delta\Phi^\dagger(r') \rangle \\
= G_{11}(r, r', t)G_{22}(r, r', t) + G_{21}(r, r', t)G_{12}(r, r', t).
$$

(56)

After some algebra, using (39)-(43), we obtain the $T=0$ contribution due to creating two excitations,

$$
\tilde{S}(q, \omega) \propto \int d\mathbf{r} e^{-i\mathbf{q}\cdot\mathbf{r}} \int d\mathbf{r}' e^{i\mathbf{q}\cdot\mathbf{r}'} \int_{-\infty}^{\infty} dt e^{i\omega t} \langle \tilde{\rho}(r, t)\tilde{\rho}(r') \rangle \\
= \sum_{i,j} 2 |B_{ij}(q)|^2 \delta(\omega - \omega_i - \omega_j),
$$

(57)

where we have defined

$$
B_{ij}(q) \equiv \int d\mathbf{r} e^{-i\mathbf{q}\cdot\mathbf{r}} \left[ \frac{g_0\rho_0(r)}{2\hbar\sqrt{\omega_i\omega_j}} - \frac{\hbar\sqrt{\omega_i\omega_j}}{8g_0\rho_0(r)} \right] \psi_i(r)\psi_j(r).
$$

(58)

This result is the generalization of a similar calculation for a uniform Bose gas given in Ref. [11]. Of course, at $T = 0$, this contribution is much smaller than the condensate contribution $S_c(q, \omega)$ in (51) since most of the atoms are in the condensate, i.e., $\tilde{\rho} \ll \rho_c$. Using (36), one can see that $\tilde{S}(q, \omega)$ contributes only when $\omega \geq 2\omega_0$.

The general expressions (51) and (57) given in this section are also valid for anisotropic parabolic wells. Stringari [9] has discussed the solutions $\psi_j(r)$ of (33) for traps with axial symmetry about the $z$-axis. In this case, the azimuthal quantum number $m$ still characterizes the normal-mode eigenvectors $\psi_j(r)$.
V. RELATION TO BOGOLIUBOV COUPLED EQUATIONS

The standard approach to study the dynamics of a \( T = 0 \) Bose condensate (uniform or non-uniform) is based on the Bogoliubov approximation \([7,8]\). The non-condensate field operators are decomposed into positive- and negative-energy Bose fluctuations

\[
\tilde{\psi}(r,t) = e^{i\phi_0(r)} \sum_j [u_j(r)e^{-iE_j t/\hbar} \hat{\alpha}_j - v_j^*(r)e^{iE_j t/\hbar} \hat{\alpha}_j^\dagger]
\]

\[
\tilde{\psi}^\dagger(r,t) = e^{-i\phi_0(r)} \sum_j [u_j^*(r)e^{iE_j t/\hbar} \hat{\alpha}_j^\dagger - v_j(r)e^{-iE_j t/\hbar} \hat{\alpha}_j],
\]

(59)

where \( \hat{\alpha}_j, \hat{\alpha}_j^\dagger \) satisfy Bose commutation relations. The amplitude functions \( u_j(r) \) and \( v_j(r) \) in (59) are given by the solutions of the coupled eigenvalue equations \([7]\)

\[
\hat{L}u_j(r) - g_0|\Phi_0(r)|^2v_j(r) = E_ju_j(r)
\]

\[
\hat{L}^*v_j(r) - g_0|\Phi_0(r)|^2u_j(r) = -E_jv_j(r)
\]

(60)

with

\[
\hat{L} \equiv \frac{1}{2m} \left[ -i\hbar \nabla + mv_0(r) \right]^2 - \mu + V_{ex}(r) + 2g_0|\Phi_0(r)|^2.
\]

(61)

As noted by Fetter \([16]\), the solutions of (60) for \( u_j \) and \( v_j \) are basically linear combinations of \( \delta \rho \) and \( \delta \phi \) in the TF approximation studied in this paper. Within this approximation and setting \( v_0(r) = 0 \), (16) and (17) give

\[
\tilde{\psi}(r,t) = \delta \hat{\Phi}(r,t) = \Phi_0(r) \left[ \frac{\delta \hat{\rho}(r,t)}{2\rho_0(r)} + i\delta \hat{\phi}(r,t) \right].
\]

(62)

Comparing terms in (62) with (24), (22) immediately gives

\[
u_j(r) = i \left( \sqrt{\frac{g_0\rho_0(r)}{2\hbar \omega_j}} + \sqrt{\frac{\hbar \omega_j}{8g_0\rho_0(r)}} \right) \psi_j(r)
\]

\[
v_j(r) = i \left( \sqrt{\frac{g_0\rho_0(r)}{2\hbar \omega_j}} - \sqrt{\frac{\hbar \omega_j}{8g_0\rho_0(r)}} \right) \psi_j(r),
\]

(63)

with \( E_j = \hbar \omega_j \). Here we have used \( |\Phi_0(r)| = \sqrt{\rho_0(r)} \). One can easily verify using (28) and (29) that \( u_j(r) \) and \( v_j(r) \) given by (63) satisfy the usual orthogonality and completeness relations.
For completeness, it is useful to show explicitly how the Bogoliubov equations (60) can be used to derive the key equations (20) and (21) on which the TF hydrodynamic description is based. Using (19), the operator in (61) can be written as \( \hat{L} = \hat{L}_{GP} + g_0 |\Phi_0(\mathbf{r})|^2 \), where

\[
\hat{L}_{GP} \equiv -\frac{\hbar^2}{2m} \nabla^2 - \mu + V_{\text{ex}}(\mathbf{r}) + g_0 |\Phi_0(\mathbf{r})|^2.
\]

(64)

Subtracting and adding the two equations in (60) gives

\[
\hat{L}_{GP} [u_j(\mathbf{r}) - v_j(\mathbf{r})] + 2g_0 \rho_0(\mathbf{r}) [u_j(\mathbf{r}) - v_j(\mathbf{r})] = E_j [u_j(\mathbf{r}) + v_j(\mathbf{r})]
\]

(65)

\[
\hat{L}_{GP} [u_j(\mathbf{r}) + v_j(\mathbf{r})] = E_j [u_j(\mathbf{r}) - v_j(\mathbf{r})].
\]

(66)

Making use of (63), (27), and (34), one finds by direct comparison

\[
u_j(\mathbf{r}) - v_j(\mathbf{r}) = i \sqrt{\frac{\hbar \omega_j}{2g_0 \rho_0(\mathbf{r})}} \psi_j(\mathbf{r}) \equiv \frac{\delta \rho_j(\mathbf{r})}{|\Phi_0(\mathbf{r})|}\]

\[
u_j(\mathbf{r}) + v_j(\mathbf{r}) = 2i \sqrt{\frac{g_0 \rho_0(\mathbf{r})}{2 \hbar \omega_j}} \psi_j(\mathbf{r}) \equiv 2i |\Phi_0(\mathbf{r})| \delta \phi_j(\mathbf{r}),
\]

(67)

where, using (27) and (34), we have defined \( \delta \rho_j(\mathbf{r}) \equiv i \sqrt{\frac{\hbar \omega_j}{2g_0}} \psi_j(\mathbf{r}) \) and \( \delta \phi_j(\mathbf{r}) \equiv \sqrt{\frac{g_0}{2 \hbar \omega_j}} \psi_j(\mathbf{r}) \).

Taking into account (19), we see that the first term on the r.h.s. of (63) reduces to \(-\frac{\hbar^2}{2m} \nabla^2 [\delta \rho_j(\mathbf{r})/|\Phi_0(\mathbf{r})|] \). Since this is omitted in the TFA [see (22)], (63) reduces to

\[
i E_j \delta \phi_j(\mathbf{r}) = g_0 \delta \rho_j(\mathbf{r}).
\]

(68)

Using this in (31), this is equivalent to (21) or

\[
h \frac{\partial \delta \phi(\mathbf{r}, t)}{\partial t} = -g_0 \delta \rho(\mathbf{r}, t).
\]

(69)

Similarly, substituting (67) into (59) gives

\[
\frac{\hbar}{m} |\Phi_0(\mathbf{r})| \nabla^2 [|\Phi_0(\mathbf{r})| \delta \phi_j(\mathbf{r})] = \frac{i E_j}{\hbar} \delta \rho_j(\mathbf{r}).
\]

(70)

Within the TF approximation (i.e., neglecting terms involving \( \nabla^2 |\Phi_0(\mathbf{r})| \)), (70) is equivalent to

\[
\nabla \cdot [|\Phi_0(\mathbf{r})|^2 \delta v_j(\mathbf{r})] = i \frac{E_j}{\hbar} \delta \rho_j(\mathbf{r}),
\]

(71)
which, using (31), is equivalent to (20).

We note that while (65) and (66) describe fluctuations of the equilibrium condensate, formally they also exhibit a special solution (labelled by $s$) corresponding to $u_s(r) = v_s(r)$ with zero energy $E_s = 0$ \[7,8\]. If one identifies $u_s(r)$ with $|\Phi(r)|$, (66) is then equivalent to the static GP equation in (14). This zero-energy solution still appears in the TF hydrodynamic approximation described by (67), in which case we see that $\delta \rho_s(r) = 0$ and $u_s(r) = i|\Phi_0(r)|\delta \phi_s(r)$. This is a zero energy solution of (24) corresponding to a phase change $\delta \phi_s(r)$ which is independent of $r$. This zero frequency fluctuation of the order parameter has a very simple physical meaning, namely it involves uniform (rigid) phase change $\delta \phi_s$ at all points, but with no associated amplitude (or density) fluctuation ($\delta \rho_s(r) = 0$). This is the expected zero frequency mode in a system with a Bose broken symmetry.

VI. CONCLUDING REMARKS

The quantized expressions of density and phase fluctuations give a simple way to calculate the contribution of low-energy condensate fluctuations to various observable quantities. The results given in this paper are based on the “hydrodynamic description” in terms of the amplitude and phase of the condensate wavefunction at zero temperature, as formulated by Stringari \[9\] for trapped Bose fluids, in which the kinetic energy contribution due to density fluctuations is neglected. While this Thomas-Fermi hydrodynamic description is only adequate for the low-frequency modes ($\omega < \sim \mu$), it is much simpler than dealing with the full set of coupled Bogoliubov equations given by (60).

Calculations based on these normal modes allow one to exhibit the role of amplitude and phase fluctuations of the condensate in a very explicit fashion, complementing a more complete calculation based on numerical solutions of the Bogoliubov equations \[3\–4\]. We have illustrated this formalism by evaluating the single-particle diagonal and off-diagonal Green’s functions, the $T = 0$ condensate depletion, and the dynamic structure factor $S(q, \omega)$, taking into account the contribution of the lowest-frequency modes. These specific calculations are
carried out for an isotropic parabolic trap but can be generalized for an anisotropic trap using the results in Ref. 9.

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REFERENCES

[1] M. H. Anderson et al., Science 269, 198 (1995); K. B. Davis et al., Phys. Rev. Lett. 75, 3969 (1995).

[2] M. Edwards, P. A. Ruprecht, K. Burnett, and C. W. Clark, Phys. Rev. A, in press.

[3] A. L. Fetter, Phys. Rev. A, in press; cond-mat/9510037.

[4] M. Edwards, P. A. Ruprecht, K. Burnett, R. J. Dodd, and C. W. Clark, cond-mat/9605170 (unpublished).

[5] E. P. Gross, Nuovo Cimento 20, 451 (1961); L. P. Pitaevskii, Sov. Phys.-JETP 13, 451 (1961).

[6] P. Nozières and D. Pines, The Theory of Quantum Liquids: Volume II (Addison-Wesley, Red Wood City, California, 1990), Chap. 10.

[7] A. L. Fetter, Ann. Phys. (N.Y.) 70, 67 (1972).

[8] A. Griffin, Phys. Rev. B 53, 9341 (1996).

[9] S. Stringari, cond-mat/9603126 (unpublished).

[10] E. M. Lifshitz and L. P. Pitaevskii, Statistical Physics, part 2 (Pergamon Press, Oxford, 1980), esp. Sections 24 and 37.

[11] A. Griffin, Excitations in a Bose-Condensed Liquid (Cambridge, N.Y., 1993), p. 63ff.

[12] R. Graham and D. Walls, Phys. Rev. Lett. 76, 1774 (1996); J. Javanainen, Phys. Rev. Lett. 75, 1927 (1995).

[13] G. Baym and C. Pethick, Phys. Rev. Lett. 76, 6 (1996).

[14] F. Dalfovo, L. P. Pitaevskii, and S. Stringari, cond-mat/9604069 (unpublished).

[15] We should note that the words “Thomas-Fermi approximation” can have different meanings in the literature. In the present analysis of the $T = 0$ condensate and its fluctu-
ations, TF is used to refer to the neglect of the kinetic energy contribution associated with density fluctuations in (9) and (11). More generally, the TF approximation involves expressing both the interaction and kinetic energy of a non-uniform system as functionals of the local density using expressions for a uniform system. This “local density approximation” is often used in calculating the thermodynamic properties of a Bose gas in an harmonic well, under the assumption that the level spacing $\hbar \omega_0 \ll k_B T$ (See, for example, V. Bagnato et al., Phys. Rev. A 35, 4354 (1987); T. T. Chou et al., Phys. Rev. A, in press; cond-mat/9602153). This semiclassical TF approximation for doing sums over single-particle states is only valid at finite temperatures and is distinct from the TF hydrodynamic approximation for the order parameter and collective modes at $T = 0$ which is discussed here and in Ref. [9].

[16] A. L. Fetter, LT21 Conference, Prague, Aug., 1996, to appear in Czechoslovak Journ. Phys; cond-mat/9606016.

[17] For these modes, there is a spurious peak at the condensate boundary $r = R$ which is not shown in Fig. 1. This arises in the TFA from the abrupt vanishing of $\rho_0(r)$ for $r > R$, as given by (19). Inclusion of the kinetic term contribution [14] in (11) in the vicinity of the boundary leads to a smooth decrease of $\rho_0(r)$ and would remove this spurious divergence at the condensate boundary.
FIG. 1. Plot of the depletion given by (44) of the local condensate density due to the contributions of several low-energy excitations specified by \((n, \ell)\) [17]. The parameters used are: \(a_{HO} = 10^4 \text{Å}, a = 50 \text{Å}, \) and \(N = 10^5\), which give \(R = 6 \times 10^4 \text{Å}\).
FIG. 2. Plot of $\omega_j|\psi_j(q)|^2$ given in (53) as a function of dimensionless wavevector $qR$, where $R$ is the size of the condensate. These functions determine the weight of the light-scattering cross section in (53) of the corresponding collective modes of energy $\omega_{n\ell\theta}$ (see Figs. 3 and 4).
FIG. 3. Plot of the $T = 0$ dynamic structure factor $S(q, \omega)$ vs $\omega$, using a Lorentzian broadening of the delta function peaks ($\Gamma = 0.05\omega_0$) and a momentum transfer $qR = 2$. Only the contributions from the low-frequency modes $(n, \ell, 0)$ are included.
FIG. 4. Same plot as Fig. 3, for $qR = 10$. Note the much smaller intensity compared to the results for $qR = 2$ in Fig. 3.