TIME-DEPENDENT, COMPOSITIONALLY DRIVEN CONVECTION
IN THE OCEANS OF ACCRETING NEUTRON STARS

ZACH MEDIN$^1$ AND ANDREW CUMMING$^2$

$^1$ Los Alamos National Laboratory, Los Alamos, NM 87545, USA; zmedin@lanl.gov
$^2$ Department of Physics, McGill University, 3600 rue University, Montreal, QC H3A2T8, Canada; cumming@physics.mcgill.ca

Received 2014 August 28; accepted 2015 January 8; published 2015 March 18

ABSTRACT

We discuss the effect of convection driven by chemical separation at the ocean–crust boundary of accreting neutron stars. We extend the steady-state results of Medin & Cumming to transient accretors, by considering the time-dependent cases of heating during accretion outbursts and cooling during quiescence. During accretion outbursts, inward heat transport has only a small effect on the temperature profile in the outer layers until the ocean is strongly enriched in light elements, a process that takes hundreds of years to complete. During quiescence, however, inward heat transport rapidly cools the outer layers of the ocean while keeping the inner layers hot. We find that this leads to a sharp drop in surface emission at around a week followed by a gradual recovery as cooling becomes dominated by the crust. Such a dip should be observable in the light curves of these neutron star transients, if enough data is taken at a few days to a month after the end of accretion. If such a dip is definitively observed, it will provide strong constraints on the chemical composition of the ocean and outer crust.

Key words: dense matter – stars: neutron – X-rays: binaries

1. INTRODUCTION

The outermost $\approx 30$ m of an accreting neutron star is expected to form a fluid ocean that overlies the kilometer-thick solid crust of the star (Bildsten & Cutler 1995). This ocean is the site of non-radial oscillations (Bildsten & Cutler 1995; Piran & Bildsten 2005) and thermonuclear flashes of intermediate duration (in ’t Zand et al. 2005; Cumming et al. 2006) and long duration (Cumming & Bildsten 2001; Strohmayer & Brown 2002; Kuulkers et al. 2004). The thermal properties of the ocean determine the initial cooling of an accreting neutron star following the onset of quiescence (Brown & Cumming 2009, hereafter BC09), as observed for six sources (Wijnands et al. 2001, 2002, 2003, 2004; Cackett et al. 2006, 2008, 2013; Homan et al. 2007; Fridriksson et al. 2011; Degenaar & Wijnands 2011; Degenaar et al. 2011, 2013b). During both accretion and quiescence the matter in the ocean solidifies to form new crust (e.g., Medin & Cumming 2011, 2014); the manner in which this solidification occurs determines the physical properties of the crust (Haensel & Zdunik 1990; Brown & Bildsten 1998; Schatz et al. 1999).

The composition of the ocean is expected to consist of mostly heavy elements, formed by rapid proton capture (the rp-process) during nuclear burning of the accreted hydrogen and helium at low densities and subsequent electron captures at higher densities, although some carbon may also be present (Schatz et al. 2001; Gupta et al. 2007). At the ocean–crust boundary, as the matter transitions from liquid to solid it also undergoes chemical separation, with the lighter elements being preferentially left behind in the liquid (Horowitz et al. 2007). In Medin & Cumming (2011, hereafter Paper I) we showed that the buoyant liquid created by this chemical separation drives a continual mixing of the ocean, carrying light-element-enriched material outward toward the surface and heat inward to the ocean–crust boundary. In Medin & Cumming (2014, hereafter Paper II) we showed that during quiescence the inward heat transport is particularly strong, leading to rapid cooling of the outer ocean and a significant drop in the light curve compared with standard cooling models (e.g., BC09).

One motivation for studying the problem of “compositionally driven” convection in the neutron star ocean comes from superbursts, thermonuclear flashes of long duration and high energy, which are thought to occur in the deep ocean at carbon fractions of $\approx 20\%$ and ignition temperatures $\approx 6 \times 10^8$ K (Cumming & Bildsten 2001; Strohmayer & Brown 2002; Cumming et al. 2006). These conditions are difficult to achieve in standard models of hydrogen and helium burning (Schatz et al. 2003; Woosley et al. 2004; Stevens et al. 2014; but see in ’t Zand et al. 2003) and crust heating (e.g., Brown 2004; Cumming et al. 2006; Keek et al. 2008), respectively.

Observations of quiescent transiently accreting neutron stars also provide strong motivation for studying ocean convection. In addition to the large inward heat flux in the inner ocean/outer crust of the sources MXB 1659–29 and KS 1731–260 inferred by BC09, other anomalous behavior from transiently accreting neutron stars includes a rebrightening during a cooling episode in XTE J1701–462 (Fridriksson et al. 2011), and very rapid cooling a few days after accretion shut off in XTE J1709–267 (Degenaar et al. 2013b). Though both the rebrightening and the rapid cooling can be explained by a spurt of accretion during quiescence, as we show here these features may naturally arise from the cooling ocean when chemical separation occurs.

In this paper we generalize and expand on the results of Papers I and II. We place the steady-state calculations of Paper I in a larger context by adding the relevant physics into a full envelope–ocean–crust model (see Brown 2004; BC09; Paper II) and by considering the evolution toward that steady state; we examine the quiescence calculations of Paper II in greater detail and provide cooling curve fits for several additional sources. We begin in Section 2 by reviewing the picture of steady-state, compositionally driven convection as presented in Paper I, and discuss how the picture changes when time dependence is considered. In Section 3 we describe our calculation of the time-dependent temperature and composition structure of the ocean and crust. In Sections 4 and 5 we present results from our calculation during accretion and during cooling after accretion turns off, respectively; in Section 5.1 we additionally provide an analytic approximation to our cooling model. In Section 6 we...
compare the cooling light curves we generate to observations of transiently accreting neutron stars during quiescence. Finally, in Section 7 we discuss the implications of our results.

2. COMPOSITIONALLY DRIVEN CONVECTION IN THE OCEAN

2.1. Phase Diagrams and Chemical Separation

As in Paper I, to understand the effect of compositionally driven convection on the ocean we first consider the phase diagram for the ocean mixture. Though the ocean in an accreting neutron star is likely made up of a wide variety of elements (Schatz et al. 2001; Gupta et al. 2007), for computational tractability we only consider a two-component mixture of oxygen and selenium in this paper. Our O–Se mixture is based on the 17 component, rp-process ash mixture in the numerical models of Horowitz et al. (2007; see also Gupta et al. 2007); in that latter mixture selenium is the most abundant element and oxygen is the most abundant low-Z element. While the relative abundances and mass numbers of each element change with depth due to electron captures, we ignore any such effects and use $^{16}$O–$^{79}$Se throughout the ocean. It is unclear whether including two components is enough to accurately model the effects of chemical and phase separation in the ocean, and if so, what the charge values of those two components should be. Calculations of multicomponent phase diagrams using both extrapolation (see Medin & Cumming 2010) and molecular dynamics techniques (e.g., Hughto et al. 2012) are in progress to address these issues. Note that the equations in the body of the paper are specific to two-component mixtures, but that unless otherwise specified the equations in Appendices A–F are applicable generally to multicomponent mixtures.

The Coulomb coupling parameter is an important quantity for determining the phase diagram of two-component mixtures. The Coulomb coupling parameter for ion species $i$ is

$$\Gamma_i = \frac{Z_i^{5/3} e^2}{k_B T} \left( \frac{4\pi \rho Y_e}{3m_p} \right)^{1/3},$$  \hspace{1cm} (1)

while that for the mixture is

$$\Gamma = \left( \frac{Z^{5/3} e^2}{k_B T} \right) \left( \frac{4\pi \rho Y_e}{3m_p} \right)^{1/3} \left[ \left( \frac{T_8}{3} \right)^{1/3} \left( \frac{Z^{5/3}}{357} \right) \left( \frac{Y_e}{0.43} \right) \right]^{1/3}.$$  \hspace{1cm} (2)

Here $Z$ and $A$ are the ion charge and mass number, $Y_e = (Z)/(A)$ is the electron fraction, $\rho_0 = \rho/(10^9$ g cm$^{-3}$) the density, and $T_8 = T/(10^8$ K) the temperature; $(Q)$ signifies the number average of quantity $Q$ for the mixture, such that $(Q) = x_1 Q_1 + x_2 Q_2$, where $x_i$ is the number fraction of species $i$.

Figure 1 shows phase diagrams for a two-component mixture with charge ratio $Z_2/Z_1 = 34/8$ (top panel), and its approximation as used in our simulation (bottom panel). The Coulomb coupling constants $\Gamma_1$ and $\Gamma_2$ are given in terms of $\Gamma_{\text{crit}} \approx 175$, the value at which a single-species plasma crystallizes. The stable liquid region of each phase diagram is labeled “L.” The stable solid regions are labeled “S1” and “S2,” and the unstable region is shaded. A particle in the ocean/crust moves down the phase diagram during cooling or accretion driving, and up during rapid heating. In the bottom panel, the liquid composition marked by a filled triangle is in equilibrium with the solid composition marked by a filled pentagon; the liquid composition marked by a filled square is in equilibrium with the solid composition marked by a filled circle. (See Figure 1 of Paper I.)

A liquid with only one ion species solidifies when $\Gamma > \Gamma_{\text{crit}} \approx 175$ (e.g., Potehin & Chabrier 2000). A liquid of multicomponent mixture becomes unstable to phase separation at a $\Gamma$ value that varies with composition, known as the liquidus curve (in Figure 1, the upper boundary of the unstable region). For the $Z_2/Z_1 = 4.25$ charge mixture shown in Figure 1 the liquidus curve is almost linear in $\Gamma^{-1}$ versus $x_2$, with $\Gamma \approx \Gamma_{\text{crit}}$; we have therefore chosen $\Gamma = \Gamma_{\text{crit}}$ as the liquidus curve for our approximate phase diagram. Using Equation (2) and the equation of state of relativistic, degenerate electrons (applicable for $\rho \gtrsim 10^7$ g cm$^{-3}$)

$$y = \frac{P}{g} = \left( \frac{3\pi^2}{4} \right)^{1/3} \frac{\hbar c}{g} \left( \frac{\rho Y_e}{m_p} \right)^{4/3},$$  \hspace{1cm} (3)

where $y$ is the column depth, $P$ is the pressure, and $g$ is the surface gravity, we have that the liquidus in our phase diagram corresponds to the column depth

$$y_L = \left[ \frac{3}{16} \left( \frac{9}{4\pi^2} \right)^{1/3} \frac{\hbar c}{g} \left( \frac{\Gamma_{\text{crit}} k_B T}{(Z^{5/3})^4} \right)^4 \right]^{1/2}.$$  \hspace{1cm} (4)

where $T$ and $(Z^{5/3})$ are evaluated at the liquidus. Although a multicomponent liquid becomes unstable to phase separation at
the liquidus, in general it does not completely solidify until a much larger value of $\Gamma$. However, we found in Paper I that in the neutron star ocean any liquid–solid mixture formed during phase separation will differentiate spatially due to sedimentation of the solid particles at a rate much faster than any of the other mixing processes (including accretion driving). This means that all of the liquid in the ocean–crust region will lie above all of the solid there, such that the liquid effectively solidifies at the liquidus; the liquidus depth is also the depth of the ocean–crust boundary $y_b$. In other words, from Equation (4)

$$\dot{y}_b \equiv y_l$$

$$= 5.57 \times 10^{12} \left( \frac{T_{0.8}}{3} \right)^4 \left( \frac{Z_b}{357} \right)^{-4} \left( \frac{g_{14}}{2.45} \right)^{-1} \text{g cm}^{-2},$$

where $T_b$ and $Z_b$ are the temperature and ion charge at the base of the ocean.

2.2. Regimes of Chemical Separation

The fate of ocean–crust particles as they cross into the unstable region of the phase diagram and undergo phase/chemical separation is determined by the composition of the parcels before crossing and the direction they are moving on the diagram. The initial composition of the parcels depends on the accretion history of the neutron star. The direction each parcel moves on the phase diagram depends on whether accretion is ongoing or not, and if accretion is ongoing, whether the rate at which the ocean–crust boundary moves inward is greater than the rate at which which particles are driven inward, i.e., whether $\dot{y}_b > m$, where $\dot{y}_b$ is the rate of change in $y_b$ and $m$ is the local accretion rate per unit area. There are three regimes to consider: steady-state accretion, cooling after accretion shuts off, and rapid heating shortly after accretion turns on.

1. When the neutron star is accreting and the ocean–crust system is near or at its steady-state configuration, $\dot{y}_b < m$. In this regime accreted material is driven to higher pressure, such that material at the base of the ocean moves across the ocean–crust boundary and freezes (as shown by the arrow marked “driving” in Figure 1). According to the simplified phase diagram in Figure 1, if the ocean base has a composition $x_2 > 0.95$ or $x_2 = 0$, there will be no chemical separation upon freezing. If $0.95 > x_2 > 0.25$, some material will remain liquid and some will form a solid of composition S2. If $0.25 > x_2 > 0$, some material will remain liquid and some will form a solid of composition S1.

2. When accretion turns off and the neutron star is cooling, $\dot{y}_b < 0$ and $m = 0$. In this regime the temperature drops in the ocean, such that the ocean–crust boundary moves outward and the base of the ocean freezes (“cooling” in Figure 1). In this case the behavior of chemical separation with $x_2$ will be the same as that described above.

3. When accretion turns on again and the system moves toward its steady-state configuration, the heating is initially very strong. In this regime the crust melts faster than new material can be driven across the ocean–crust boundary, such that $\dot{y}_b > m$ (and material follows the “heating” arrow in Figure 1). According to our simplified phase diagram, if the crust has a composition $x_2 = 0$, $x_2 > 0.95$, or $x_2$ there is no chemical separation of the solid upon melting. If the crust is of composition S1, there will be chemical and phase separation into a light liquid and an S2 solid. However, assuming diffusion between solid–solid phases is slow (see Hughto et al. 2011), as heating continues the liquid and the solid will travel together up the phase diagram until the solid melts and recombines with the liquid; since the distance over which this occurs is relatively short compared to the size of the ocean, we assume for simplicity that in our calculations a solid of composition S1 will just melt to form a liquid of composition S1 ($x_2 = 0.44$). Therefore, in our calculations there is no chemical or phase separation when $\dot{y}_b > m$, regardless of crust composition.

2.3. Compositional Buoyancy and Convection

The thermal profile in the ocean is stable against convection (e.g., Medin & Cumming 2011), such that after phase separation and sedimentation of the solid the liquid parcels left behind will tend to remain in place. During steady-state accretion or cooling (regimes 1 and 2 of the previous subsection), however, the liquid parcels are lighter than the layers above them; these parcels can overcome the stabilizing thermal gradient and buoyantly rise if $\mathcal{A} > 0$, where $\mathcal{A}$ is the convective (Schwarzschild) discriminant (see, e.g., Cox 1980). In Appendix A we derive a general expression for $\mathcal{A}$ for a two-component mixture we can write (Equation (A7); see also Kippenhahn & Weigert 1994)

$$A_{HP} \chi_F = \chi_T (\nabla - \nabla_{ad}) + \chi_1 \nabla_{X_1},$$

Here,

$$\chi_1 = \chi_{X_1} = \chi_{X_2} = Y_i \left( \frac{X_i - X_2 Y_1}{Y_e} \right),$$

$$X_i = x_i A_i / (A)$$ is the mass fraction of species $i$, $Y_i = Z_i / A_i$ is the electron fraction of species $i$, $H_P = -dP/d\ln y = y/\rho$ is the scale height, and $\nabla = -H_P (d \ln T / dr) + \nabla_{X_1} = -H_P (d \ln X_1 / dr)$ are the temperature and composition gradients. The adiabatic temperature gradient is taken at constant (specific) entropy $s$ and composition $\{X_i\}$.

As steady-state accretion or cooling continues, light elements are continually deposited at the base of the ocean and transported outward by convection. For efficient convection $\mathcal{A}$ adjusts to be close to but slightly greater than zero. In Paper I we found that convection is extremely efficient throughout the ocean during steady-state accretion; in Appendix B of this paper we demonstrate that convection in the ocean is extremely efficient even during time-dependent heating or cooling, and even when effects due to rapid rotation ($\sim 10^2 \text{s}^{-1}$) and moderate magnetic fields ($\sim 10^{10} \text{G}$) are considered. We therefore assume in the main body of the paper that

$$\chi_1 \nabla_{X_1} = \chi_T (\nabla_{ad} - \nabla)$$
where convection is active (i.e., across the ocean convection zone).

### 2.4. Convection Equations

Here we assume Newtonian physics, plane-parallel geometry, mixing length theory, and efficient convection (Equation (9)). In mixing length theory the value of the mixing length is highly uncertain; but note below that under the efficient convection assumption this parameter does not appear in our equations. The continuity equation for the flow of species \( i \) is given by (e.g., Brown & Bildsten 1998)

\[
\dot{X}_i + m \frac{\partial X_i}{\partial y} = \frac{\partial F_{i,\text{fr}}}{\partial y} + \epsilon_i, \tag{10}
\]

where \( F_{i,\text{fr}} = F_{i,\text{fr}} \hat{r} \) is the composition flux for species \( i \) and \( \epsilon_i \) is the sum of all composition sources. The entropy balance equation is given by (e.g., Brown & Bildsten 1998)

\[
T \frac{\partial s}{\partial t} + T \dot{m} \frac{\partial s}{\partial y} = \frac{\partial F_{\text{cd}}}{\partial y} + \epsilon, \tag{11}
\]

where

\[
F = F_{\text{fr}} \hat{r} = F_{\text{cd}} + F_{\text{conv}} \tag{12}
\]

is the total flux,

\[
F_{\text{cd}} = F_{\text{cd}} \hat{r} = K T \nabla \hat{r} \tag{13}
\]

is the conductive heat flux, \( F_{\text{conv}} = F_{\text{conv}} \hat{r} \) is the convective heat flux, \( K \) is the thermal conductivity, and \( \epsilon \) is the sum of all heat sources. The terms on the left-hand side of Equation (11) can be written (Equations (A14) and (A13))

\[
T \frac{\partial s}{\partial t} = c_p \frac{\partial T}{\partial t} - b_1 \frac{X_1}{X_t} \frac{\partial X_1}{\partial t} \tag{14}
\]

and

\[
T \dot{m} \frac{\partial s}{\partial y} = c_p T \dot{m} \frac{\partial T}{\partial y} \left( \nabla - \nabla_{\text{ad}} - \frac{b_1 T}{c_p} \nabla X_1 \right), \tag{15}
\]

where \( c_p = T(\partial s/\partial T)_{P,X_i,Y_j} \) is the specific heat capacity,

\[
b_1 = b_{p,1} - b_{p,2} \frac{X_1}{X_2} + b_{p,3} \frac{(Y_1 - Y_2)X_1}{Y_e}, \tag{16}
\]

\( b_{p,1} \) \( b_{p,2} \) \( b_{p,3} \) \( Y_1 \) \( Y_2 \) \( X_1 \) \( X_2 \) \( Y_e \)

\( b_{p,1} \) \( b_{p,2} \) \( b_{p,3} \) \( Y_1 \) \( Y_2 \) \( X_1 \) \( X_2 \) \( Y_e \)

With the assumption of efficient convection, the convective heat flux in the ocean becomes (Equation (B8))

\[
F_{\text{conv}} = -\frac{c_p T \chi X_1}{X_t} \left( 1 + \frac{\chi b_1}{\chi c_p} \right) F_{r,\text{fr}}, \tag{17}
\]

the entropy balance equation in the ocean becomes (Equation (B10))

\[
\frac{c_p}{\partial t} + c_p \frac{\chi T X_1}{X_t} \frac{\partial X_1}{\partial t} = \frac{\partial F_{\text{cd}}}{\partial y} - F_{r,\text{fr}} \frac{\partial}{\partial y} \left[ c_p T X_1 \chi T \left( 1 + \frac{\chi b_1}{\chi c_p} \right) \right] + \epsilon, \tag{18}
\]

and the entropy advection term in the ocean becomes

\[
T \dot{m} \frac{\partial s}{\partial y} = -\frac{c_p T \dot{m} X_1}{\chi T} \left( 1 + \frac{\chi b_1}{\chi c_p} \right) \nabla X_1, \tag{19}
\]

where \( c_p, T, \chi, X_1, X_2, X_3 \) are found from the thermodynamic equations of Appendix D; \( K, c_p, \) and the other thermodynamic derivatives are found as in Paper I (see, e.g., Equation (39) of that paper).

The ocean is bounded from above by the hydrogen and helium burning layer, which ends at a column depth \( y_0 \). Rather than

The steady-state versions of the above equations are similar to the convection equations from Paper I. Using Equation (17) with the steady-state composition flux \( F_{r,\text{fr}} = \dot{m}(X_1 - X_{1,0}) \) (Paper I or Equation (34)), we have that the steady-state convective flux is given by

\[
F_{\text{conv}} = -\frac{c_p T \dot{m} X_1}{\chi T} \left( 1 + \frac{\chi b_1}{\chi c_p} \right) \left( 1 - \frac{X_{1,0}}{X_1} \right). \tag{20}
\]

This equation differs from Equation (43) of Paper I (which is in error) by a factor of \( 1 + (\chi T/\chi T) b_1/c_p \), which is less than 1.2 in any part of the ocean; the extra factor does not qualitatively change the results of our earlier paper. In the deep ocean, because \( c_p, T, \chi, X_1, \) and \( b_1 \) have only a weak dependence on \( y \) but \( \chi T \propto E_F \propto y^{-1/4} \), we also have that

\[
F_{\text{conv}} \propto y^{1/4} \tag{21}
\]

(see the result during cooling, \( F_{\text{conv}} \propto y^{5/4} \); see Paper II) and

\[
\frac{\partial F_{\text{conv}}}{\partial y} \simeq \frac{c_p T \dot{m} X_1}{4 y \chi T} \left( 1 - \frac{X_{1,0}}{X_1} \right) \tag{22}
\]

Since \( \nabla X_1 \ll 1 \), we have from Equations (19) and (22) that \( T \dot{m}(\partial s/\partial y) \ll \partial F_{\text{conv}}/\partial y \). Therefore, using Equation (11) we have that

\[
\frac{\partial F_{\text{fr}}}{\partial y} \equiv \frac{\partial F_{\text{cd}}}{\partial y} + \frac{\partial F_{\text{conv}}}{\partial y} \simeq -\epsilon \tag{23}
\]

in steady state, as we assumed in Paper I.

### 3. A MODEL OF THE ENVELOPE, CRUST, AND OCEAN IN THE TIME-DEPENDENT CASE

In our model we place the top of the envelope at \( y = 10^{-4} \) g cm\(^{-2} \) (i.e., at the surface) and the base of the crust at \( y = 3 \times 10^{18} \) g cm\(^{-2} \). The envelope structure is found as in Brown et al. (2002; see also Potekhin et al. 1997). We assume an \([X_H, X_He] \) \([0.7, 0.3] \) composition throughout the envelope. The crust structure and composition is found as in BC09, except that we leave the core temperature \( T_e \) as a free parameter rather than solving for it self-consistently, and use Newtonian physics with a surface gravity \( g \) constant across the envelope, ocean, and crust. General relativistic corrections are included only as an overall redshift of the time, \( t_{\text{eff}} = t(1 + z_{\text{surf}}) \), and effective temperature, \( T_{\text{eff,\,surf}} = T_{\text{eff}}/(1 + z_{\text{surf}}) \), from the local value to that seen by an observer at infinity; here the neutron star mass and radius are \( 1.62 M_\odot \) and 11.2 km, giving a redshift factor of \( 1 + z_{\text{surf}} = 1.32 \). Note that while \( g \) varies by about ten percent across the crust of a neutron star, it varies by less than a percent across the ocean, such that the assumption of constant \( g \) will modify the crust structure somewhat but will have very little effect on the ocean structure (for a given heat flux coming from the crust). As in BC09, we characterize the thermal conductivity in the inner crust with a single number, the impurity parameter \( Q_{\text{imp}} = \langle Z^2 \rangle - \langle Z \rangle^2 \). Our treatment of \( \epsilon \) in the crust, as well as our treatment of \( \epsilon \) across all layers, is described in Appendix C. In the ocean the components of \( b \) are found from the thermodynamic equations of Appendix D; \( K, c_p, \) and the other thermodynamic derivatives are found as in Paper I (see, e.g., Equation (39) of that paper).

The ocean is bounded from above by the hydrogen and helium burning layer, which ends at a column depth \( y_0 \). Rather than
tracking the physics of this layer, we leave $y_0$ as a free parameter; in Sections 4 and 5 we choose $y_0 = 10^8 \text{g cm}^{-2}$ (e.g., Bildsten & Brown 1997; see Figure 2). The mass fraction of species $i$ at the top of the ocean, $X_{i,0}$, is determined by the nuclear reactions within the burning layer (see Schatz et al. 2001). For the $^{16}$O-$^{79}$Se mixture described in Section 2 we nominally choose $\{X_{1,0}, X_{2,0}\} = \{0.02, 0.98\}$, as in Paper I (but see below). This is approximately the mixture the ocean would have if all of the light elements ($Z \leq 20$) were oxygen and all of the heavy elements ($Z > 20$) were selenium; further calculations, involving mixtures of more than two components, are required to determine whether this is a reasonable approximation.

The heavy-element ocean cannot penetrate into the light-element burning layer, such that the burning layer is stable to convection and the convective velocity drops to zero at the boundary (Equation (B2)). To include the stabilizing effect of this layer in our model we set

$$F_{r,X_i}(y_0) = 0$$

and

$$F_{r,\text{conv}}(y_0) = 0.$$  

The boundary conditions Equations (24) and (25) can occasionally be inconsistent with our assumption $X_{1,0} = 0.02$ above, in which case we allow $X_{1,0}$ to grow as necessary. Figure 2 shows an example of a case where $X_{1,0} > 0.02$ for our model. Note that in the more accurate three-component model of Paper I there is no inconsistency between $F_{r,X_i}(y_0)$ = 0 and $X_{1,0} = 0.02$, because the required rapid rise in $X_O$ with increasing column depth is stabilized by the rapid drop in $X_{He} + X_{He}$.

A similar situation occurs when convection is thermally driven ($V \geq V_{ad}$ and $\sum_i X_i \nabla X_i = 0$) at the top of the ocean.

The ocean is bounded from below by the crust, which begins at a column depth $y_b$. The mass fraction of species $i$ at the top of the crust, $X_{i,c}$, is determined by the mass fraction of species $i$ at the base of the ocean, $X_{i,b}$, according to the relevant phase diagram. For $^{16}$O-$^{79}$Se we use Figure 1 (see also Section 2.2); converting from number fraction to mass fraction gives

$$X_{1,c} = \begin{cases} X_{1,b}, & X_{1,b} \leq 0.01 \text{or } X_{1,b} = 1; \\ 0.01(\text{"S2"}), & 0.01 < X_{1,b} < 0.37; \\ 0.2(\text{"S1"}), & 0.37 < X_{1,b} < 1 \end{cases}$$

for $y_b < \dot{m}$, and

$$X_{1,c} = X_{1,b}$$

for $y_b > \dot{m}$.

Convection cannot occur for $y > y_b$, since the region is solid; therefore, at the ocean–crust boundary we set

$$F_{r,\text{conv}}(y_b^-) = 0,$$

where the superscript “−” signifies that the flux is evaluated on the deep (i.e., crust) side of the boundary. The composition flux at the ocean base is

$$F_{r,X_i}(y_b^-) = (\dot{m} - y_b) \Delta X_{i,b,c},$$

where $\Delta X_{i,b,c} = X_{i,b} - X_{i,c}$ and the superscript “−” signifies that the flux is evaluated on the shallow (i.e., ocean) side of the boundary (see the steady-state accretion expression $\dot{m}(X_{i,b} - X_{i,0}$) of Paper I). If $y_b \geq \dot{m}$, there will be no chemical separation at the boundary (Section 2.2) and therefore no compositionally driven convection in the ocean, such that

$$F_{r,X_i} = 0$$

and

$$F_{r,\text{conv}} = 0.$$  

throughout the ocean. Note that from Equations (26) and (29), $F_{r,X_i}(y_b^+) > 0$ for the O–Se system; with Equation (17) this means that $F_{r,\text{conv}}(y_b^+) < 0$ or that there is an inward heat flux at the base of the ocean due to compositionally driven convection (see Paper I).

We use a stationary grid for all column depths except $y_b$, which we track continuously. The rate at which the ocean–crust boundary moves is constrained by the heat flux continuity condition

$$F_{r,\text{cd}}(\dot{y}_b^-) + F_{r,\text{conv}}(\dot{y}_b^-) = F_{r,\text{cd}}(\dot{y}_b^+),$$

which using Equations (17) and (29) becomes

$$c_P T_b (\dot{m} - \dot{y}_b) \frac{\Delta T}{\Delta y} \left(1 + \frac{\chi r h_1}{\chi T} \right) \left(1 - \frac{X_{1,c}}{X_{1,b}} \right) = F_{r,\text{cd}}(\dot{y}_b^-) - F_{r,\text{cd}}(\dot{y}_b^+).$$

To estimate $F_{r,\text{cd}}(\dot{y}_b^-)$ and $F_{r,\text{cd}}(\dot{y}_b^+)$ we use the temperature gradient between $y_b$ and the nearest grid point on the low-$y$ side and on the high-$y$ side, respectively; for sufficiently small grid spacing this is a reasonable approximation. During rapid heating this approximation gives $F_{r,\text{cd}}(\dot{y}_b^-) < F_{r,\text{cd}}(\dot{y}_b^+)$ or $\dot{y}_b > \dot{m}$, such that to maintain self-consistency between Equations (31) and (32) we cannot use Equation (33) to find $\dot{y}_b$ but must find $\dot{y}_b$ from Equation (6).

In this paper we assume for simplicity that $\epsilon_{X_i} = 0$ in the ocean. In particular, this means that we ignore the effect of electron captures on the ocean composition. Therefore, using Equations (10) and (24), the composition flux at any point in the ocean satisfies

$$F_{r,X_i}(y) = \int_{y_0}^{y} \frac{\partial X_i(y')}{\partial t} dy' + \dot{m}(X_i - X_{i,0}).$$
From Equations (29) and (34) we have that the total change in ocean composition with time is given by

$$\int_{y_{b}}^{y_{0}} \frac{\partial X_{i}(y')}{\partial t} dy' = \dot{m} \Delta X_{i,0c} - \dot{y}_{b} \Delta X_{i,bc}$$  \tag{35}

with $\Delta X_{i,0c} = X_{i,0} - X_{i,c}$; this expression is used as a consistency check when we solve for $\partial X_{i}/\partial t$ below. The first term in Equation (35) represents the balance between the composition $\{X_{i,0}\}$ entering the ocean from the burning layer and the composition $\{X_{i,c}\}$ leaving the ocean to the crust, as driven by accretion; the second term represents the exchange of particles in the ocean to convert a solid block of composition $\{X_{i,c}\}$ into a liquid block of composition $\{X_{i,b}\}$, as the boundary moves inward (or vice versa as the boundary moves outward).

For the two-component ocean mixture considered in this paper, we solve for the evolution of the ocean composition and temperature structure as follows. In each time step we first guess a value for $\partial X_{i,b}/\partial t$. Our guess comes from the fact that composition changes slowly with depth near the base of the ocean, i.e., $\nabla_{X} = \chi_{T} (\nabla_{ad} - \nabla_{1}) \chi_{1} \ll 1$, which with Equation (35) gives the approximation

$$\frac{\partial X_{i,b}}{\partial t} \approx \frac{\dot{m} \Delta X_{i,0c} - \dot{y}_{b} \Delta X_{i,bc}}{\chi_{1}},$$  \tag{36}

where $\dot{y}_{b}$ is obtained from Equation (33). Once $\partial X_{i,b}/\partial t$ is chosen, the update value $\dot{X}_{i,b}$ is found from

$$\dot{X}_{i,b} = X_{i,b} + \Delta t \frac{\partial X_{i,b}}{\partial t},$$  \tag{37}

where $\Delta t$ is the current time step; $\dot{X}_{i,b}$ at every other depth in the ocean is found from $\dot{X}_{i,b}$ and Equation (9), and then Equation (37) is used to find $\partial X_{i}/\partial t$ at each depth. Finally, Equation (34) is used with $\partial X_{i}/\partial t$ to find $F_{r,X_{i}}$ for Equation (18). The value of $\partial X_{i,b}/\partial t$ is modified and the procedure repeated until Equation (35) holds true. The new value of $T_{b}$ is found from $X_{i,b}, \dot{y}_{b}$, and Equation (6). We find that our initial guess, Equation (36), often requires no extra iterations for reasonable accuracy.

From Equation (35), the steady-state $\partial X_{i}/\partial t = 0$ is reached when $\dot{y}_{b} = 0$ and $\Delta X_{i,0c} = 0$; that is, when the ocean–crust boundary stops moving and, if accretion is ongoing, when the composition at the top of the ocean is the same as that at the top of the crust (see Paper I). The latter condition happens in our $^{16}$O–$^{79}$Se simulations when $X_{1,b} \approx 0.37$ (see Equation (26)) through a simple feedback mechanism: if $X_{1,b} < 0.37$ at the ocean base, $\Delta X_{1,0c} > 0$ (the crust has composition S2), such that $\partial X_{1}/\partial t > 0$ and $X_{1,b}$ rises above 0.37; if $X_{1,b} > 0.37$, $\Delta X_{1,0c} < 0$ (the crust has composition S1), such that $\partial X_{1}/\partial t < 0$ and $X_{1,b}$ drops below 0.37; the composition of the ocean base hovers around $X_{1,b} = 0.37$. In reality the eutectic nature of the phase diagram at $X_{1,b} = 0.37$ will most likely cause the ocean base to solidify in vertically lamellar sheets of alternating S1, S2 composition (e.g., Woodruff 1973).

4. CONVECTION DURING ACCRETION

Here we evolve an example O–Se ocean from quiescence to steady state after accretion turns on, using the time-dependent equations of Sections 2 and 3 and assuming that compositionally driven convection is active. For this example $m = 10^{5}$ g cm$^{-2}$ s$^{-1}$ ($\sim 1.1 m_{\text{Edd}}$), $T_{c} = 10^{8}$ K, $Q_{\text{imp}} = 0$, and $y_{0} = 10^{6}$ g cm$^{-2}$. For our initial conditions at the start of accretion we choose $T_{\text{mill}} = T_{c}$ throughout the crust and ocean and $X_{\text{mill}} = 0.01$ (i.e., “S2” in Figure 1) throughout the ocean. The latter assumption is made because, even though $X_{1,b}$ can be large after a cooling episode, during the initial heating there is rapid inward movement of the ocean–crust boundary but no chemical separation, such that the bulk of the ocean has the same composition as the accreted crust (Section 2.2; but see Section 7).

Figure 3 shows the composition profile at various times during its evolution to steady state. The composition is shown only for the ocean; the top of the ocean is located at $y = 10^{6}$ g cm$^{-2}$, while the base of the ocean is located at the rightmost extent of each curve and moves inward as $X_{1,b}$ increases. The top of the convection zone can also be seen in the figure, as the depth where $X_{T}$ reaches the burning layer level of 0.02 and flattens out.

Figure 4 shows the temperature profile at various times during its evolution to steady state. As is discussed in Section 2.2, the ocean moves through two regimes to reach steady state. Initially
there is no compositionally driven convection, because of the strong accretion heating such that \( y_b > n_i \); the temperature profile reaches a quasi-steady state that matches the steady-state profile in the case without convection (the red dotted curve in Figure 4) in only a few years. Once that quasi-steady state is reached, \( y_b \ll n_i \) and the system slowly evolves over hundreds of years to the final steady state (the black solid curve with \( t_\infty = 490 \) yr).

While the ocean reaches the efficient convection state given by Equation (9) relatively quickly, in approximately one convective turnover time \( t_{\text{conv}} \sim 0.01 y_b / \dot{m} \) (months to a few years; see Paper I), it takes much longer to reach steady state, as can be seen in Figures 3 and 4. The time from the start of the accretion outburst to the start of steady state can be estimated from Equation (36): for a steady-state composition at the base of the ocean \( X_{1,b} = 0.37 \) (Section 3), and a difference between the composition at the top of the ocean and the top of the crust \( \Delta X_{1,0c} = 0.01 \) (Equation (26)), we have

\[
t_{ss} \sim \frac{X_{1,b}}{\Delta X_{1,0c}} \sim \frac{y_b}{\dot{m}} X_{1,b} \sim 1000 t_{\text{conv}}
\]

(i.e., hundreds to thousands of years). Using Equation (38) with \( y_b \sim 10^{24} \text{ g cm}^{-2} \), we find a steady-state time of \( t_{ss} \sim 10^3 \) yr (see Figure 3). Note that if we instead use \( \dot{m} \sim 10^4 \text{ g cm}^{-2} \text{s}^{-1} \), as is typical for low-mass X-ray binaries, \( t_{ss} \sim 10^5 \) yr. If \( y_b \sim 10^{22} \text{ g cm}^{-2} \), as in the model of Horowitz et al. (2007) for \( T_b \sim 3 \times 10^8 \) K, the time to reach steady state is still large: \( t_{ss} \sim 10^7 \) yr. If the mass fraction of light elements entering the ocean is ten times larger (\( X_{1,0} \sim 0.2 \)), however, as is the case for stable hydrogen and helium burning (e.g., Stevens et al. 2014), \( t_{ss} \sim y_b / \dot{m} \) is of order the accretion time.

Figure 5 shows the steady-state profiles for the total flux \( F_r = F_{r,\text{cd}} + F_{r,\text{conv}} \) and the conduction and convection contributions. The convection contribution has a \( y^{1/4} \) dependence, as in Equation (21). At the top of the figure we have marked the locations of the three accretion heat sources considered in this paper: hydrogen and helium burning, electron captures, and pycnonuclear fusion (see Appendix C). Since in steady state \( \partial F_r / \partial y \sim -\epsilon \) (Equation (23)), \( F_r \) is constant outside of the heat source regions and drops by \( \int_{y_{\text{low}}}^{y_{\text{high}}} \epsilon dy \) across each region. For example, electron captures are active from a column depth of \( y_{\text{low}} = 5 \times 10^{12} \text{ g cm}^{-2} \) to \( y_{\text{high}} = 5 \times 10^{15} \text{ g cm}^{-2} \) and release a total energy of \( Q_{\text{EC}} = (m_p / \dot{m}) \int_{y_{\text{low}}}^{y_{\text{high}}} \epsilon dy = 0.2 \text{ MeV nucleon}^{-1} \) (e.g., Haensel & Zdunik 2008), such that the total drop in \( F_r m_p / \dot{m} \) over that range is 0.2 MeV.

Note that the results shown in Figures 4 and 5 are qualitatively different from those in Figure 6 of Paper I, despite the similarity in the parameters used. This is due to a simplification made in our earlier paper: \( F_{\text{crust}} \), the outward radial heat flux coming from the crust, is not the same for a neutron star with compositionally driven convection as without. In reality, \( F_{\text{crust}} \) must be solved self-consistently with \( T_b \); because \( T_b \) is larger with convection, less heat flows into the ocean from the crust and therefore \( F_{\text{crust}} \) is smaller (or more negative, as is the case in Figure 5), which limits the growth of \( T_b \) (see Figure 1 and discussion of Brown 2004). For the example of Figure 4 the steady-state temperature at \( y = 10^{14} \text{ g cm}^{-2} \) (near the ocean–crust boundary) is only 4% larger with convection than without, whereas for the model of Paper I it is 20% larger. The inclusion of hydrogen and helium burning has a comparable impact on our model, increasing the temperature at \( y = 10^{14} \text{ g cm}^{-2} \) by 5% (compare the blue dashed curve and the red dotted curve in Figure 4).

### 5. CONVECTION AFTER ACCRETION TURNS OFF

We now consider the evolution of the ocean as it cools in quiescence. The evolution proceeds in four stages; these stages are discussed in detail in Paper II, but we outline them below for reference (see Figures 6 and 7):

In stage 1 there is no convection and the ocean cools as in BC09. In stage 2 the ocean base begins to cool, driving chemical separation and convection. Inward heat transport by
convection rapidly cools the envelope and ocean but maintains the ocean–crust boundary at a nearly constant temperature and depth. The temperature profile in the ocean steepens with time, and that the ocean–crust boundary from cooling. In stage 4, the crust is thermally relaxed to the point that only weak inward heating is necessary to maintain the boundary at a constant depth, and the temperature profile in the ocean flattens. 5.1. Analytic Approximation

Figure 7. Cooling light curve of a neutron star with $X_b^{init} = 0.37$ and compositionally driven convection, and the analytic approximation to this light curve (Equations (47) and (34)). The labels that appear above the graph denote the duration of the stages of cooling (see the text). For the analytic approximation, the transition from stage 1 to 2 and the transition from stage 2 to 3 are marked with an open square and an open circle, respectively. The parameters used are the same as in Figure 6, with the addition of $\gamma_b^{(3)} = 6.05 \times 10^{13} \text{ g cm}^{-2}$ and $T_b^{(3)}(1) = 2.73 \times 10^8 \text{ K}$. The light curve and analytic approximation for the case with $X_b^{init} = 0.1$ are also plotted for comparison. (See Figure 1 of Paper II.)

where the superscript “(s)” signifies that the quantity is evaluated at the beginning of stage $s$ of cooling. To solve for the evolution of $T_t$, we assume that the temperature profile through the ocean and crust has an initially constant gradient $\nabla T^{(1)}$ (see Figure 4; see also below). During cooling, there is a transition between the thermally relaxed outer layers with constant outward heat flux $\sigma B T_{t,eff}^4$, where $\sigma B$ is the Stefan–Boltzmann constant, and the inner layers still in steady state with outward heat flux $KT \nabla T^{(1)}/H_P$ (Equation (13) with $\nabla = \nabla T^{(1)}$). While the cooling wave is still in the ocean, this transition is defined by the thermal time

$$\tau \approx \frac{\rho c_P H_P}{2K} \propto y T^{-1}$$

(42)

(see Henyey & L’Ecuyer 1969); the factor of two in Equation (42) comes from integrating Equation (7) of BC09 assuming Equation (39) for $K$ and $H_P$ and that $c_P$ and $T$ are constant. We define the transition depth $y_t$ as the depth where the thermal time is equal to the cooling time $t$.

During stage 1, the transition depth is above the base of the ocean, i.e., $y_t < y_b^{(1)}$, and there is no compositionally driven convection. The transition from the steady-state heat flux for $y > y_t$ to the surface heat flux $\sigma B T_{t,eff}^4$ for $y \ll y_t$ is not sharp (see, e.g., the “20 days” curve of Figure 6). For lack of a better model, and to maintain continuum between stages 1 and 2, we use a modified version of Equation (50) for the heat flux: the (conductive) heat flux at any point $y \leq y_t$ is given by

$$\frac{K T}{H_P} \nabla = \left[ \frac{K T}{H_P} \right]_{y=y_t}^{\nabla = \nabla T^{(1)}} \left( \frac{y}{y_t} \right)^{5/4} + \sigma B T_{t,eff}^4.$$  (43)

Here $Q|_{y=y_t}$ signifies that the quantity $Q$ is evaluated at depth $y_t$. Equation (43) has the desired properties of being continuous and giving the correct heat flux values in the limiting cases $y \gg y_t$ and $y \ll y_t$. With Equation (39) we can solve Equation (43) for the temperature profile through the ocean,

$$T = T_t \left[ 1 - \frac{8}{5} \left( \nabla T^{(1)} - \nabla T_{eff,r} \right) \left( 1 - \left( \frac{y}{y_t} \right)^{5/4} \right) \right]$$

$$+ 2 \nabla T_{eff,r} \ln \left( \frac{y}{y_t} \right)^{1/2},$$  (44)

and the temperature at depth $y_t$ near the top of the ocean,

$$T_t \simeq T_t \left[ 1 - \frac{8}{5} \left( \nabla T^{(1)} - \nabla T_{eff,r} \right) + 2 \nabla T_{eff,r} \ln \left( \frac{y_t}{y_t} \right)^{1/2} \right].$$  (45)

Here we assume $y_t \ll y_t$ and have defined

$$\sigma T_{eff}^4 = \frac{K T}{H_P} \left|_{y=y_t} \nabla T_{eff,r}.$$  (46)

for convenience. From Equations (41) and (45) we obtain the scaling relation

$$T_{eff} = T_{eff}^{(2)} \left( \frac{T_t}{T_t} \right)^{1/2}$$

$$\times \left[ 1 - \frac{8}{5} \nabla T^{(1)} + 2 \nabla T_{eff,r} \ln \left( \frac{y_t}{y_t} \right)^{1/4} \right].$$  (47)

In deriving Equation (47) we grouped $T_{eff}$ terms and used the fact that $\nabla T_{eff,r} \propto T_{eff}^4/T_t^4$ (Equation (46)).
To determine $T_t$ and $y_t$ as a function of time during stage 1, we use Equation (42):

$$T_t = T^{(2)}_t \left( \frac{y_t}{y_b} \right)^{5/4} = T^{(2)}_t \left( \frac{t}{t_2} \right)^{5/4(1-\nu^{(1)})}$$  (48)

and therefore

$$y_t = y^{(2)}_b \left( \frac{t}{t_2} \right)^{1/(1-\nu^{(1)})}$$  (49)

where $t_2$ is the time at the beginning of stage 2 (see below). Along with $T^{(2)}_t = T^{(2)}_b$, $y^{(2)}_t = y^{(2)}_b$, and $\nu^{(2)}_t = \nu^{(2)}$ (where $\nu^{(2)}$ is $\nu^{(2)}_t$ taken at the base of the ocean; Equation (53)), Equations (48) and (49) can be inserted into Equation (47) to solve for $T_{eff}$ during stage 1 (see Equation (8) of BC09).

During stages 2 and 3, the ocean is thermally relaxed, such that the flux through the ocean satisfies $F_{conv} + F_{cd} = \sigma_b T^{4}_{eff}$. Since $F_{conv} \propto y^{3/4}$ (Paper I; see also Equation (B8) with Equations (29) and (36)) and $F_{cd} = KTV/H_P$, we have that the conductive flux in the ocean is given by

$$\frac{KT}{H_P} \nabla = \left[ \frac{KT}{H_P} \right]_{y=y_b} \nu_b - \sigma T^{4}_{eff} \left( \frac{y}{y_b} \right)^{5/4} + \sigma T^{4}_{eff}.$$  (50)

Similar to our method for stage 1 above, we use Equation (50) with Equation (39) to solve for the temperature profile through the ocean,

$$T = T^{(2)}_b \left[ 1 - \frac{8}{5} (\nabla_b - \nu^{(2)}) \right] \left\{ 1 - \left( \frac{y}{y_b} \right)^{5/4} \right\} + 2 \nu^{(2)} \ln \left( \frac{y}{y_b} \right)^{1/2},$$  (51)

and the temperature at depth $y_t$ near the top of the ocean,

$$T_t \simeq T^{(2)}_b \left[ 1 - \frac{8}{5} (\nabla_b - \nu^{(2)}) + 2 \nu^{(2)} \ln (y_t/y_b) \right]^{1/2}.$$  (52)

where

$$\sigma T^{4}_{eff} \equiv \left[ \frac{KT}{H_P} \right]_{y=y_b} \nu^{(2)}$$  (53)

and we assume that $y_t \ll y_b$. From Equations (41) and (52) we obtain the scaling relation

$$T^{(2)}_t = T^{(2)}_b \left( \frac{y_b}{T^{(2)}_b} \right)^{1/2} \times \left[ 1 - \frac{8}{5} \nabla_b + 2 \nu^{(2)} \ln (y_b/y_b^{(2)}) \right]^{1/4}.$$  (54)

In deriving Equation (54) we grouped $T^{(2)}_b$ terms and used the fact that $\nabla^{(2)} \propto T^{(2)}_b/\nabla^{(2)}_b$ (Equation (53)).

Stage 2 begins at a time $t_2 = t^{(2)}_b$, where $t^{(2)}_b$ is the thermal time evaluated at the base of the ocean. At the beginning of this stage the ocean base is still in steady state, $\nabla^{(2)}_b = \nabla^{(1)}$. We assume that the transition depth is stationary, i.e., that $y_t = y_b^{(2)}$ and $T_t = T^{(2)}_b$ are constant. To determine $\nabla_b$ as a function of time we look at the ocean energetics. The total energy stored in the ocean is

$$E = A \int_{y_t}^{y_b} c_p T dy,$$  (55)

where $A$ is the surface area; using Equation (51) and assuming that $c_p$ is constant in the ocean, that $y_t \ll y_b$, and that $\nabla_b$ and $\nabla^{(2)}_b$ are small ($\nabla^{(1)} \ll \nabla_b \ll \nabla_L$ in this stage and $\nabla^{(2)}_b \ll \nabla^{(1)}_b$ typically), we have

$$E \simeq A c_p T_b y_b \left( 1 - \frac{4}{9} \nabla_b - \frac{5}{9} \nu^{(2)} \right).$$  (56)

As $\nabla_b$ increases and the ocean cools, this energy is slowly depleted; using Equation (56) and the fact that $y_b$ and $T_b$ are constant during stage 2, we have that the ocean energy changes at a rate

$$\frac{dE}{dt} \simeq -A c_p T_b y_b \left( 4 \frac{d\nabla_b}{dt} + 5 \frac{d\nu^{(2)}}{dt} \right).$$  (57)

The depleted energy is released at the ocean base and must mask the cooling due to the difference between the flux entering the ocean from the crust and the flux leaving the ocean through the top; i.e.,

$$\frac{d\nabla^{(2)}}{dt} = -\frac{8\nabla^{(2)}_b}{5(1 - \frac{8}{5} \nu^{(2)})} \frac{d\nabla_b}{dt}.$$  (58)

From Equation (54) we have that

$$\frac{d\nu^{(2)}}{dt} = -\left( \frac{8\nabla^{(2)}_b}{5} \right) \frac{d\nabla_b}{dt},$$  (59)

combining Equations (57)–(59) with $\nu^{(2)} \ll 1$ gives

$$\nabla_b \simeq \frac{9}{8} \left( \nu^{(2)} - \nu^{(1)} \right) \left( \frac{t}{t_2} - 1 \right) + \nu^{(1)}.$$  (60)

Along with $T_b = T^{(2)}_b$ and $y_b = y_b^{(2)}$, Equation (60) can be inserted into Equation (54) to solve for $T_{eff}$ during stage 2.

Stage 3 begins when $\nabla_b = \nabla_L$, or at a time $t_3 = t^{(2)}_b \langle 8(\nabla_L - \nabla^{(1)})/9(\nabla^{(2)}_b - \nabla^{(1)}_b) + 1 \rangle$. We assume that $\nabla_b = \nabla_L$ is constant. To determine $T_b$ and $y_b$ as a function of time during stage 3, we use Equation (42) with $c_p$ and $K$ at their solid values such that $\tau \propto y^{3/4}$ (BC09). Because conduction is very efficient at transporting heat in the crust, we assume that the temperature gradient in the crust from the ocean–crust boundary to the transition depth is flat (see Paper I); i.e., the ocean–crust boundary cools at the same rate as the transition depth, $\partial \ln T_b/\partial \ln t = \partial \ln T_b/\partial \ln t = (\partial \ln T_b/\partial \ln y_t)(\partial \ln y_t/\partial \ln r) = 4\nu^{(1)}/3$, or equivalently,

$$T_b = T^{(3)}_b \left( \frac{t}{t_3} \right)^{4\nu^{(1)}/3}.$$  (61)

If enrichment is low, $\partial \ln y_b/\partial t \simeq 4(\partial \ln T_b/\partial t)$ (Equation (E10)), and we have

$$y_b \simeq y^{(3)}_b \left( \frac{t}{t_3} \right)^{16\nu^{(1)}/3};$$  (62)

but if enrichment is high (as is the case in Figure 6; see Figure 3 of Paper II), $\partial \ln y_b/\partial t \ll \partial \ln T_b/\partial t$, and we have

$$y_b \simeq y^{(3)}_b.$$  (63)

Along with $\nabla_b = \nabla_L$, $T^{(3)}_b = T^{(2)}_b$, and $y^{(3)}_b = y^{(2)}_b$, Equations (61)–(63) can be inserted into Equation (54) to solve for $T_{eff}$ during stage 3.
5.2. Results

Here we evolve the O–Se ocean from Section 4 as it cools after accretion turns off. For stages 1, 2, and 4 of cooling we use the equations from Sections 2 and 3, with $m = 0$ and $\epsilon = 0$ as is appropriate during cooling. We can also use these equations for stage 3, but the resulting light curves are noisy unless the simulation time step and spatial resolution are very small, due to the quasi-periodic activation/deactivation of convection that occurs during this stage (see above). Instead, we use the following method, which has the advantage of requiring a much coarser time and spatial grid for (empirically) comparably smooth and accurate light curves. We assume that once compositionally driven convection is strong enough for $\nabla b = \nabla_L$, it will remain at that critical level as cooling continues; i.e., we assume that when $\nabla b \geq \nabla_L$,

$$F_{r,\text{comp}}(y_b^*) = F_{r,\text{cd}}(y_b^*) - \frac{K T_b}{H_P} \nabla_L$$  \hspace{1cm} (64)

(see Equation (32)). Equation (33) can no longer be used to find $y_b^*$, instead we use the ocean–crust boundary equations of Appendix E. From Equation (E2) we have (see Equation (6))

$$\dot{y}_b = \frac{4 y_b}{T_b} \frac{\partial T_b}{\partial t} + \frac{\partial X_{1,b}}{\partial t},$$  \hspace{1cm} (65)

where $\dot{y}_b = \frac{\partial y_b}{\partial x}$ and $\frac{\partial X_{1,b}}{\partial x}$. We solve for $\frac{\partial T_b}{\partial t}$ using the entropy balance equation (Equation (E7); see Equation (11))

$$T_b \frac{\partial \dot{y}_b}{\partial t} - \dot{y}_b \left. \frac{\partial s_b}{\partial y} \right|_{y_b=y_b^*} = \frac{\partial F_r}{\partial y} \left. \frac{\partial y}{\partial y} \right|_{y=y_b^*}$$  \hspace{1cm} (66)

$$\approx \left[ \frac{K T (\nabla_L - \nabla)}{H_P \Delta y} \right]_{y=y_b^*},$$  \hspace{1cm} (67)

where $\Delta y$ is the grid spacing. Note that Equation (67) drives the temperature gradient $\nabla b$ to $\nabla_L$. We solve for $\frac{\partial X_{1,b}}{\partial t}$ using the iteration method described earlier, except that our initial guess is (Equation (E9))

$$\frac{\partial X_{1,b}}{\partial t} \approx - \frac{4 \Delta X_{1,bc}/T_b}{1 + y_b^* \Delta X_{1,bc}/y_b} \frac{\partial T_b}{\partial t}.$$  \hspace{1cm} (68)

We use the above method whenever $\nabla b > \nabla_L$ (i.e., during stage 3), for all of the calculations shown here and in Section 6. Note that even with this method, the light curves are slightly noisy in stage 3 (see, e.g., Figure 7).

In this section we choose initial conditions at the start of cooling $T^\text{init}(y = 10^{12} \text{ g cm}^{-2}) = 4 \times 10^9 \text{ K}$ near the base of the ocean, $T^\text{init}_c = 10^8 \text{ K}$ at the base of the crust, and a constant temperature gradient in between; and $X^\text{init}_{1,b} = 0.37$ at the base of the ocean with a composition profile given by Equation (9) throughout the ocean. These are approximately the steady-state conditions from Section 4 (see also BC09; Paper II). Note that our assumption of an initially constant temperature gradient in the ocean and crust means that the convective flux is zero at the start of cooling, which is not entirely consistent with the steady-state results of Section 4. Our intent here is to show the effects of compositionally driven convection on cooling only. In Section 6 we run our simulations over an entire accretion cycle from outburst to quiescence, such that the convective fluxes during cooling are calculated in a self-consistent way.

Figure 6 shows the temperature profiles at various times during cooling, along with the analytic approximation to these profiles. As can be seen in the figure, the ocean–crust boundary moves outward more quickly during cooling than during accretion: Equation (33) as it applies to the cooling case is given by

$$\dot{y}_b = -\frac{1}{X_{1,b} F_{r,\text{cd}}(y_b^*) - F_{r,\text{cd}}(y_b^*)} \frac{X_{1,b} F_{r,\text{cd}}(y_b^*) - F_{r,\text{cd}}(y_b^*) c_P T_b}{c_P T_b};$$  \hspace{1cm} (69)

using Equation (13) with a temperature gradient at the ocean base $\nabla y_b \approx 0.25$ (see below), we find $\dot{y}_b \approx -10^6 \text{ g cm}^{-2} \text{s}^{-1}$. This is markedly different from the situation in Section 4, where $|\dot{y}_b| \ll m$ over most of the evolution. The composition also evolves more quickly during cooling than during accretion. From Equation (36) we have $\frac{\partial X_{1,bc}}{\partial t} \approx -\dot{y}_b \Delta X_{1,bc}/y_b$, which is a factor of $\Delta X_{1,bc}/\Delta X_{1,oc} \geq 20$ times larger than the accretion value (see Equation (38)).

Figure 7 shows the cooling light curve along with the analytic approximation. As can be seen in the figure, changing $X^\text{init}_{1,b}$ has a strong effect on the light curve. This is for two reasons. First, for a larger light-element fraction in the ocean the thermal conductivity $K \propto (Z)^{-1}$ is also larger; a larger $K$ reduces the temperature gradient in the ocean (while self-consistently increasing the flux there), which keeps the outer layers hotter both during steady-state accretion and at the end of cooling when the crust and core are equilibrated (see BC09). Second, for a larger light-element fraction in the ocean the ocean–crust boundary is deeper, which delays the onset of strong ocean cooling due to compositionally driven convection. Note that in Figure 7 the analytic approximation deviates strongly from the model light curve for $t_\infty \gtrsim 100$ days. This is because our analytic expressions (Equations (47) and (54)) only account for cooling of the ocean and crust by heat conduction out through the envelope, not for late-time cooling by heat conduction into the core (see BC09).

6. COMPARISON TO OBSERVATIONS

In Paper II we presented fits to observations of XTE J1701–462 and IGR J17480–2446, using our model of compositionally driven convection; here we present fits to observations of several additional quiescent, transiently accreting neutron stars. Our goal in making these fits was to understand qualitatively how including convection in the ocean changes the fitting parameters for these sources. Therefore, we did not attempt to accurately fit our model to the observational data using rigorous parameter searches. Similar to BC09, each source was fit by running our simulations from the onset of accretion through the duration of the accretion outburst, then turning off accretion and tracking the cooling light curve out to the end of the observation. In our fits we take $m$ and the duration of the accretion outburst from observations and fit to the parameters $T_c$, $Q_{\text{imp}}$, $\gamma_0$, and $X^\text{init}_{1,b}$. Note that in Degenaar et al. (2014) we assumed shallow heating in our fit of EXO 0748–676 (see also BC09). In this paper we do not include shallow heating in our model directly. Instead, we vary $\gamma_0$ to provide the necessary shallow heating, with a larger $\gamma_0$ placing the burning layer closer to the bulk of the ocean and heating it more.

Figures 8 and 9 (see also Figure 4 of Paper II) show our fits to cooling light curves from several quiescent sources. Note that in most of our fits we use $\gamma_0 \sim 10$ times larger than the standard value of a few $\times 10^8 \text{ g cm}^{-2}$ (e.g., Bildsten & Brown 1997; Paper I); i.e., we must invoke a significant shallow heat source. Convection does not directly reduce the required shallow
heat source for each fit: for the same shallow heating (same values of $y_0$) the fits for our model with and without convection are generally equally valid (e.g., in Figure 9); in addition, the fitted ocean temperature at the start of cooling is similar in both our model and that of BC09, implying the use of a comparable shallow heating model. Instead, convection justifies the use of larger values of $X_{1,b}$ in our models due to light-element enrichment, which in turn increases the thermal conductivity in the ocean and makes it hotter without the need for shallow heating (see, e.g., the effect of different $X_{1,b}^\text{init}$ values in Figure 7).

In our fits here and in Paper II, several trends appear when comparing the light curve from the model with compositionally driven convection to that from the model without convection, for the same parameters (see also Figure 7). First, at $t_\infty \sim 1$–100 days post-outburst the light curve with convection drops below the light curve without convection then flattens out, as the cooling transitions from stage 1 to stage 2 to stage 3 (Section 5). This arises because the compositionally driven convection transports heat inward, rapidly cooling the ocean and temporarily slowing the cooling in the crustal layers where the phase separation occurs. Second, at late times the light curve with convection crosses above the light curve without convection, due to light-element enrichment during convection increasing the ocean thermal conductivity (stage 4); for several of our fits (IGR J17480–2446 from Paper II, EXO 0748–676, and MXB 1659–29) this happens within the observation. Third, our fits that have steep (shallow) light curves with convection will have correspondingly steep (shallow) light curves without convection; compare, e.g., our fits for MXB 1659–29 versus those for EXO 0748–676 in Figure 9. This is because, whether or not compositionally driven convection is in effect, the crust ultimately drives the cooling (see Section 5). We do not discuss the behavior of the crust cooling in this paper; a detailed discussion can be found in BC09.

Most of the light curves considered in this section are fit equally well by models with convection and without (see Figure 9). XTE J1709–267 ($\dot{m} = 2 \times 10^7 \text{ g cm}^{-2}\text{s}^{-1}$ with a 10 week outburst; Degenaar et al. 2013b) is the exception to the above generalization, since the model with convection fits better to the observed rapid decrease in the cooling light curve (Figure 8; see stage 2 of Figure 7). Convection also provides an explanation for the observed increase in the equilibrium flux level in IGR J17480–2446 from 2009 to 2014 (Degenaar et al. 2013a; see Paper II), because it allows the composition, and therefore the equilibrium temperature profile, to change across accretion episodes. For XTE J1701–462 (Friddriksson et al. 2011; see Paper II), models with convection can simultaneously fit the drop in the light curve at 100–200 days and the flattening at $\sim1000$ days post-outburst (Figure 10). Similarly for EXO 0748–676 ($\dot{m} = 2 \times 10^7 \text{ g cm}^{-2}\text{s}^{-1}$ with a 24 yr outburst; Degenaar et al. 2011, 2014), inclusion of convection leads to a plateau of slow cooling between $\sim150$–750 days, broadly consistent with the data (Figure 9). We note that the model with convection is not statistically preferred over the model without convection.

BC09 fit the light curve of MXB 1659–29 ($\dot{m} = 9 \times 10^7 \text{ g cm}^{-2}\text{s}^{-1}$ with a 2.5 yr outburst; Wijnands et al. 2003, 2004; Cackett et al. 2008, 2013) with a standard cooling model. Using similar parameters and including convection gives a model light curve with a “stage 2” drop at $\sim50$ days. Because of the gap in the data at $\sim40$–400 days, we have the freedom in our fits to choose where this drop occurs; we can instead move the drop to $\sim200$ days by increasing $X_{1,b}^\text{init}$ to an unphysical 0.8 (but see below). Note that in our model there is an abrupt jump in the light curve at late time ($t_\infty \sim 1000$ days in Figure 9), where the base of the ocean is saturated with light elements and compositionally driven convection halts (Section 5). This is a general feature of our fits to MXB 1659–29, as long as $X_{1,b}^\text{init} \gtrsim 0.3$, and it arises due to the steep drop in the light curve which causes rapid and prolonged outward motion of the ocean–crust boundary and strong chemical separation ($X_{1,b} \to 1$). The observations at late...
times neither support nor dispute the existence of this predicted bump (see Figure 9).

With the model of Sections 2 and 3 it is impossible to fit both “anomalous” data points in the light curve of XTE J1701–462 (the two points from \( t_{\text{obs}} \approx 200 \) to 300 days post-outburst in Figure 10; see also Figure 4 of Paper II). However, we can partially fit this data by considering ocean mixtures other than oxygen–selenium and/or large values of \( X_{\text{ini},b} \). Two such fits, one for a calcium-selenium ocean and one for an iron-selenium ocean (see Horowitz et al. 2007), are shown in Figure 10; here the light-element saturation in the ocean produces a bump in the light curve that matches the second anomalous data point and that has a peak occurring at the same time as the first anomalous data point. We believe these modifications to our basic model to be reasonable, considering our uncertainty regarding what two-component mixture to use for the ocean or whether a two-component mixture is an accurate representation of the ocean. The large changes produced in the light curves when different compositions are used (Figure 4 of Paper II versus Figure 10) emphasizes the need for models with three or more components. In addition, as we discuss in Section 7, it may be possible to reproduce the amplitude of the rebrightening in XTE J1701–462 by including heating due to electron captures self-consistently in our convection model.

7. DISCUSSION

In this paper we have continued the exploration begun in Paper I of the consequences of chemical separation and subsequent compositionally driven convection in the ocean of accreting neutron stars; while the model described in Paper I included only a steady-state ocean, here we use a full envelope–ocean–crust model and track its behavior from the onset of accretion to the end of cooling.

We have discovered a strong effect due to compositionally driven convection on the light curves of cooling, transiently accreting neutron stars. As the neutron star cools after an accretion outburst, the ocean–crust boundary moves outward. We find that this leads to chemical separation, and then convective mixing and inward heat transport, in a manner similar to that during accretion but at a much faster rate. The inward heat transport cools the outer layers of the ocean rapidly, but keeps the inner layers hot; the result is a sharp drop in surface emission at around a week (depending on parameters), followed by a gradual recovery as the ocean base moves outward. Such a dip should be observable in the light curves of these neutron star transients, if enough data is taken at a few days to a month after the end of accretion. If such a dip is definitively observed, it will provide strong constraints on the chemical composition of the ocean and outer crust.

We find that chemical separation can enrich the ocean to a light-element fraction \( X_1 \approx 0.1 \) within a few months of cooling after an accretion outburst (Section 5). This is the required fraction for unstable ignition of carbon (Schatz et al. 2003) and is reached well within the estimated superburst recurrence time of \( 1–3 \) yr (Kuulkers 2002; in ’t Zand et al. 2003), far more quickly than with either chemical separation during accretion heating or gravitational sedimentation during quiescence. The rapid enrichment during cooling can allow a superburst to occur right at the beginning of an accretion outburst, if the initial outburst is hot enough to ignite the carbon layer; this may help explain the puzzling superburst observed in EXO 1745–248 (Altamirano et al. 2012; see Paper II).

In order for chemical separation to occur, however, the composition at the base of the ocean and at the top of the crust must differ; this will not happen if accretion outbursts are too short to push accreted material to the base of the ocean. Ultimately the carbon excess is being supplied by the ashes of the hydrogen and helium burning layer, with carbon mass fraction \( X_{C,0} \approx 0.01 \) during unstable burning (Woosley et al. 2004). This excess is driven to the ignition depth \( y_{\text{ign}} \approx 10^{12} \text{ g cm}^{-2} \) within a few months to a year \( (y_{\text{ign}}/m) \); but it takes ten times longer to build the excess up to the required fraction 0.1 (see Figure 3). The total build-up time is at best a factor of three longer than the estimated superburst recurrence time of \( 1–3 \) yr (Kuulkers 2002; in ’t Zand et al. 2003). We suggest that while a recurrence rate of a few years cannot be sustained through compositionally driven convection, it is possible to have several bursts in a row at that rate if a small fraction of carbon can be “stored” in the deep ocean or crust (perhaps in lamellar sheets; see Section 3) after each burst. On the other hand, in ’t Zand et al. (2003) inferred observationally that stable burning is happening in superburst sources. Although the physical mechanism for the stable burning is not understood, it could produce much larger carbon fractions \( X_{C,0} \approx 0.2 \) (Stevens et al. 2014), which would reduce the timescale needed to enrich the ocean even during accretion (Section 4).

Note that while compositionally driven convection may help superburst models reach the levels of carbon enrichment required for carbon ignition, it does not help the models reach the required large ocean temperatures \( T_{\text{ign}} \approx 6 \times 10^9 \text{ K} \) (Cumming et al. 2006). In fact, we find (Sections 4 and 5) that temperatures in the bulk of the ocean are slightly lower with convection than without.

Two issues presented in Paper I have been resolved in the Appendix of this paper. In Appendix B we discuss what happens when \( \nabla > \nabla_{\text{ad}} \) in the ocean (see also Section 3). As we alluded to in Paper I, the small amount of hydrogen and helium in the transition region between the burning layer and the ocean stabilize the density gradient at the top of the ocean and allows for an unstable temperature gradient and heavy-element composition gradient simultaneously, such that there is no contradiction between a small convective velocity at the
top of the ocean and a smooth composition transition from the burning layer to the ocean. In Appendix F we discuss how rotation and magnetic fields affect our model. We find that the efficient convection assumption Equation (9) remains valid even in the presence of rapid rotation and moderate magnetic fields: the remaining temperature and composition evolution equations in the paper follow directly from it and are therefore also unaffected by rotation or magnetic field.

There are many possible directions for future work. In addition to the ideas suggested in Paper II, more work is needed to understand how electron captures affect our model; the heat released due to electron captures during mixing and sedimentation of the ocean could be observable in the light curves of neutron star transients. Consecutive accretion outburst-quiescence cycles should also be simulated to obtain self-consistent composition profiles in the ocean and outer crust.

We thank Chuck Horowitz, Nathalie Degenaar, and Chris Fontes for useful discussions. Z.M. was supported by a LANL Director’s Postdoctoral Fellowship. A.C. is supported by an NSERC Discovery grant, and is a member of the CIFAR Cosmology and Gravity program. This research was carried out in part under the auspices of the National Nuclear Security Administration of the U.S. Department of Energy at Los Alamos National Laboratory and supported by Contract No. DE-AC52-06NA25396.

APPENDIX A

CONVECTIVE STABILITY

Here we derive expressions for the convective discriminant \( A \) and other quantities related to entropy production and convective stability in the multicomponent oceans of neutron stars.

The usual stability requirement for a displaced fluid element (e.g., Kippenhahn & Weigert 1994)

\[
A < 0, \tag{A1}
\]

where

\[
A = \frac{d \ln \rho}{dr} - \left( \frac{d \ln \rho}{dr} \right)_{s,X_i,Y_i}, \tag{A2}
\]

with \( d/dr \) the gradient in the star and \( d/dr \) the gradient felt by an element displaced at constant entropy \( s \) and chemical composition \( \{X_i, Y_i\} \) (i.e., we assume the element is displaced with no radiated energy and no chemical diffusion). In the neutron star ocean, the sound speed is much larger than the convective velocity, such that a displaced element is always in pressure balance with its surroundings:

\[
\frac{d \ln P}{dr} = \left( \frac{d \ln P}{dr} \right)_{s,X_i,Y_i}. \tag{A3}
\]

Using Equation (A3) and

\[
d \ln P = \chi_T d \ln T + \chi_P d \ln \rho + \sum_{i=1}^{n} \chi_{Xi} d \ln X_i + \chi_{Ye} d \ln Y_e,
\]

we can rewrite \( A \) as

\[
A = -\frac{1}{\chi_P} \left[ \chi_T \frac{d \ln T}{dr} + \sum_{i=1}^{n} \chi_{Xi} \frac{d \ln X_i}{dr} + \chi_{Ye} \frac{d \ln Y_e}{dr} - \chi_T \left( \frac{d \ln T}{dr} \right)_{s,X_i,Y_i} \right]; \tag{A4}
\]

defining

\[
\chi_i = \chi_{Xi} - \chi_{X_n} \frac{X_i}{X_n} + \chi_{Ye} \frac{(Y_i - Y_n)X_i}{Y_e}; \tag{A5}
\]

and enforcing the constraints \( \sum_{i=1}^{n} X_i = 1 \) and \( Y_e = \sum_{i=1}^{n} Y_i X_i \), the convective discriminant becomes

\[
A = \frac{1}{HP \rho} \left[ \chi_T (\nabla - \nabla_{ad}) + \sum_{i=1}^{n-1} \chi_i \nabla X_i \right]. \tag{A7}
\]

Using

\[
\left( \frac{\partial s}{\partial T} \right)_{P,X_i,Y_i} = \frac{c_P}{T}, \tag{A8}
\]

\[
\left( \frac{\partial s}{\partial P} \right)_{T,X_i,Y_i} = - \left( \frac{\partial T}{\partial P} \right)_{s,X_i,Y_i} \left( \frac{\partial s}{\partial T} \right)_{P,X_i,Y_i} = - \frac{\nabla_{ad}}{P} c_P, \tag{A9}
\]

and Equations (D7) and (D8), we have

\[
ds = c_P d \ln T - c_P \nabla_{ad} d \ln P - \sum_{i=1}^{n} b_{P,i} d \ln X_i - b_{P,e} d \ln Y_e; \tag{A10}
\]

defining

\[
b_i = b_{P,i} - b_{P,n} \frac{X_i}{X_n} + b_{P,e} \frac{(Y_i - Y_n)X_i}{Y_e} \tag{A11}
\]

and again enforcing \( \sum_{i=1}^{n} X_i = 1 \) and \( Y_e = \sum_{i=1}^{n} Y_i X_i \), we can rewrite Equation (A10) as

\[
ds = c_P d \ln T - c_P \nabla_{ad} d \ln P - \sum_{i=1}^{n-1} b_i d \ln X_i \tag{A12}
\]

Therefore

\[
\frac{ds}{dr} = -\frac{1}{HP} \left[ c_P (\nabla - \nabla_{ad}) - \sum_{i=1}^{n-1} b_i \nabla X_i \right]. \tag{A13}
\]

Assuming that the pressure at a given depth does not change with time (see Appendix A of Brown & Bildsten 1998) we also have from Equation (A12) that

\[
\frac{\partial s}{\partial t} = \frac{c_P}{T} \frac{\partial T}{\partial t} - \sum_{i=1}^{n-1} \frac{b_i}{X_i} \frac{\partial X_i}{\partial t}. \tag{A14}
\]

APPENDIX B

MIXING LENGTH EQUATIONS AND EFFICIENT CONVECTION

Here we derive or define expressions for several quantities related to heat transfer and composition mixing, first using mixing length theory and then using the efficient convection assumption of Equation (9). We discuss the regimes in which either model is appropriate. Finally we discuss how to make these models consistent with an ocean that has \( \nabla > \nabla_{ad} \).

In mixing length theory (e.g., Kippenhahn & Weigert 1994), a displaced element feels an average force per unit mass of

\[
-\frac{gD\rho}{2\rho} = -\frac{g}{2\rho} \left( \frac{d\rho}{dr} \right)_{s,X_i,Y_i} = \frac{1}{2} g_\ell \Delta \mu \tag{B1}
\]
Figure 11. Composition profiles in the ocean of a neutron star with compositionally driven convection, in mixing length theory for various values of $\xi$ and in the efficient convection assumption. The left panel shows the steady-state composition profiles as $\xi$ increases from $5 \times 10^{-6}$ to $2 \times 10^{-5}$; the right panel shows the composition profile at various times for $\xi = 2 \times 10^{-5}$ and in the efficient convection assumption.

applied over an average distance of $l_m/2$; assuming that approximately half of this work goes into the kinetic energy of the particle, the convective velocity is given by

$$v_{\text{conv}}^2 = c_s^2 \frac{\xi^2}{8 \rho} \left( \chi_T (\nabla - \nabla_{\text{ad}}) + \sum_{i=1}^{n-1} \chi_i \nabla X_i \right),$$

(B2)

where $c_s = (g \rho H_p)^{1/2}$ is the sound speed and $\xi = l_m/\rho H_p$ is the ratio between the convection mixing length $l_m$ and the scale height (but see Appendix F). The composition flux for species $i$ is given by

$$D_{X_i} = -\frac{l_m}{2} \frac{dX_i}{dr} = \frac{\xi}{2} X_i \nabla X_i;$$

(B4)

the convective heat flux is given by

$$F_{r,\text{conv}} = \rho v_{\text{conv}} T \bar{D} X_i \hat{r}$$

(B5)

where

$$\bar{D} = -\frac{l_m}{2} \frac{d\bar{s}}{dr} = \frac{\xi}{2 c_p} \left( \nabla - \nabla_{\text{ad}} \right) - \frac{1}{c_p} \sum_{i=1}^{n-1} b_i \nabla X_i,$$

(B6)

using Equation (A13). To solve for the evolution of the ocean using mixing length theory, we assume a value for $\xi$ and use Equations (B2)–(B6) to find $F_{r,\text{X}}$ and $F_{r,\text{conv}}$ for Equations (10) and (11).

For efficient convection (Equation (9)), Equation (A13) becomes

$$\frac{T m}{y} \frac{\partial \bar{s}}{\partial y} = -\frac{c_p T m}{y} \sum_{i=1}^{n-1} \frac{\chi_i}{\chi_T} \frac{1 + \chi_T b_i}{\chi_i c_p} \nabla X_i;$$

(B7)

while Equation (B5) with Equation (B3) becomes

$$F_{r,\text{conv}} = -\frac{c_p T}{\chi_T} \sum_{i=1}^{n-1} \frac{\chi_i}{\chi_T} \left( 1 + \frac{\chi_T b_i}{\chi_i c_p} \right) F_{r,\text{X}} X_i.$$  

(B8)

Using Equations (10), (B7), and (B8) with $\epsilon_x = 0$, we have

$$\frac{\partial F_{r,\text{conv}}}{\partial y} = -\sum_{i=1}^{n-1} \left[ c_p T \chi_i \frac{1 + \chi_T b_i}{\chi_i c_p} \right] \frac{\partial X_i}{\partial y} + m \frac{\partial \bar{s}}{\partial y} \right] \right],$$

(B9)

such that with Equations (12), (A14), and (B7) the energy balance equation Equation (11) becomes

$$c_p \frac{\partial T}{\partial t} + \sum_{i=1}^{n-1} c_p T \chi_i \frac{\partial X_i}{\partial t} = -\sum_{i=1}^{n-1} F_{r,\text{X}} \frac{\partial}{\partial y} \left[ c_p T \chi_i \frac{1 + \chi_T b_i}{\chi_i c_p} \right] + \epsilon.$$  

(B10)

To solve for the evolution of the ocean using the efficient convection assumption we use the procedure described in Section 3. Note that Equations (B7)–(B10) are independent of $\xi$, such that we do not need to assume a value for this parameter.

Figure 11 shows the composition profile for the example from Section 4, using mixing length theory with various values of $\xi$, and using the efficient convection assumption. The value of $\xi$ at which efficient convection becomes a good approximation in the neutron star ocean can be estimated using Equation (29) as an upper bound for the composition flux: Equation (B3) gives

$$\frac{\xi}{2} \rho v_{\text{conv,max}} \simeq \bar{m} - \bar{y}_b \frac{\bar{m}}{\sum_{i=1}^{n-1} \chi_i \chi_T} \frac{\Delta X_{i,bc}}{X_i} \sim 10^7 \text{g cm}^{-2} \text{s}^{-1};$$

(B11)

using Equation (B2), $v_{\text{conv}} \leq v_{\text{conv,max}}$, and $\bar{m} - \bar{y}_b \sim 10^5 \text{g cm}^{-2} \text{s}^{-1}$ we have

$$\sum_{i=1}^{n-1} \frac{\chi_i}{\chi_T} \frac{\Delta X_{i,bc}}{X_i} \sim 32 \chi_T \left( \frac{\xi}{\rho v_{\text{conv,max}}} \right)^2 \frac{\Delta X_{i,bc}}{\rho c_s} \sim \left( 10^{-5} \xi \right)^4.$$  

(B12)

This means that for mixing length parameters $\xi \gg 10^{-5}$ (as we assumed in Paper I), convection is efficient; i.e., $\sum_{i=1}^{n-1} \chi_i \nabla X_i$ is extremely close to its maximum stable value,
\( \chi_T (\nabla v - \nabla) \). Note that when \( \xi \gg 10^{-5} \), small numerical errors in \( \chi_T (\nabla v - \nabla_{\text{ad}}) + \sum_{i=1}^{n-1} X_i \nabla X_i \) lead to very large errors in \( \nu_{\text{conv}} \). In this case we cannot use the full mixing length procedure but must assume efficient convection.

In Paper I we suggested that a time-dependent calculation could help resolve what happens when a stable composition profile cannot extend from the burning layer ash at the top of the ocean to the steady-state mixture at the ocean base (i.e., from \( X_{i,0} = 0.02 \) to \( X_{i,b} = 0.37 \) for the \( ^{16}\text{O} - ^{90}\text{Se} \) system); this could happen, e.g., when \( \nabla > \nabla_{\text{ad}} \) at the top of the ocean. In the current paper we remove the inconsistency by allowing the composition at the top of the ocean to be different from that provided by the burning layer (e.g., Figure 2; see also Figure 3 from Paper II), and use the stabilizing effect of the burning layer on convection as justification. If we instead fix \( \{ X_{i,0} \} \), we find that once the convection zone reaches the top of the ocean there is a flux at the outer boundary

\[
F_{r,X_i}(y_0) = \dot{m} \Delta X_{i,0c} \tag{B13}
\]

(Equations (10) and (29)), with \( X_{i,0} \neq X_{i,c} \) because the system is not yet in steady state. In the O–Se system, this means that a large quantity of oxygen is being ejected from the ocean into the envelope, a fact that we are ignoring because of our assumption of a fixed envelope composition. We conclude that the only way to make our models consistent with a fully ionized multicomponent plasma, such that formally we have \( \nabla > \nabla_{\text{ad}} \) ocean is to consider the envelope, by either allowing ocean material to mix into the envelope (through Equation (B13)), or by including hydrogen and helium burning in the envelope to prevent mixing (as in Figure 2).

**APPENDIX C**

**HEAT AND COMPOSITION SOURCES**

Here we derive expressions for the sources \( \epsilon_X \), and \( \epsilon \), as used in the continuity equation Equation (10) and entropy balance equation Equation (11), respectively.

There are sources of composition change \( \epsilon_X \), at three locations in our model.

1. At the ocean–crust boundary, composition changes abruptly due to chemical separation and rapid sedimentation of the solid at the phase transition. Formally, we write

\[
\epsilon_{X_i} = -(\dot{m} - \dot{y}_h) \Delta X_{i,0c} \delta(y - y_h), \tag{C1}
\]

where \( \delta \) is the Dirac delta function; but in practice we simply adjust \( X_{i,c} \) manually without reference to Equation (10).

2. At the hydrogen and helium burning layer, composition changes quickly due to the strong temperature dependence of the thermonuclear reactions, from \( \{ X_{i,e} \} \) in the envelope to \( \{ X_{i,0} \} \) at the top of the ocean. We assume that the burning layer is infinitely thin, such that formally we have

\[
\epsilon_{X_i} = \dot{m} \Delta X_{i,0} \delta(y - y_0). \tag{C2}
\]

Note that \( X_{i,0} \) is not necessarily the value given by the burning layer ashes (\( \{ X_O, X_{Se} \} = [0.02, 0.98] \) for the O–Se ocean model of this paper); we allow \( X_{i,0} \) to vary based on the composition profile required by efficient convection in the ocean (see Section 3).

3. In the crust, composition changes gradually due to electron captures and pycnonuclear fusion. For simplicity we set \( \langle A \rangle_c = 56 \) as the composition at the top of the crust, regardless of the value of \( X_{i,c} \), and follow the procedure of BC09 to obtain \( \langle Z \rangle \) and \( \langle A \rangle \) at greater depths. With this approximation, \( X_{i,c} \) only determines the physics of the liquid–solid phase transition (e.g., in Equation (36)) and has no effect on the crust properties (thermal conductivity, etc.).

During accretion, there are heat sources \( \epsilon \) at three locations in our model (see Figure 5).

1. At the hydrogen and helium burning layer, particles are driven to a critical depth (temperature) for thermonuclear reactions by accretion; for \( \{ X_{H}, X_{He} \} = [0.7, 0.3] \) these reactions release \( Q = 5 \text{ MeV nucleon}^{-1} \) (e.g., Brown & Bildsten 1998). Following BC09, we assume that the heat is released uniformly in the logarithm of column depth, over a region from \( y = y_{\text{low}} \) to \( y_{\text{high}} \), such that

\[
\epsilon = \begin{cases} \frac{Q m_i m_p}{h \hbar c^{\text{high}} / \text{low}} \ y_{\text{low}} < y < y_{\text{high}}; \\ 0, \text{ otherwise.} \end{cases} \tag{C3}
\]

Here we choose \( y_{\text{low}} = 0.2 y_0 \) and \( y_{\text{high}} = y_0 \).

2. In the outer crust, electron captures release \( Q = 0.2 \text{ MeV nucleon}^{-1} \) (e.g., Haensel & Zdunik 2008); we use \( y_{\text{low}} = 5 \times 10^{12} \text{ g cm}^{-2} \) and \( y_{\text{high}} = 5 \times 10^{15} \text{ g cm}^{-2} \).

3. In the inner crust, pycnonuclear fusion reactions release \( Q = 1.2 \text{ MeV nucleon}^{-1} \); we use \( y_{\text{low}} = 5 \times 10^{15} \text{ g cm}^{-2} \), and \( y_{\text{high}} = 3 \times 10^{18} \text{ g cm}^{-2} \).

**APPENDIX D**

**THERMODYNAMIC QUANTITIES**

Here we derive or define expressions for several thermodynamic quantities in multicomponent plasmas that are used in this paper.

The total differential for the Gibbs free energy is given by

\[
dG = -SdT + VdP + \sum_{i=1}^{n} \mu_i \dot{N}_i + \mu_e \dot{N}_e, \tag{D1}
\]

where \( G \) includes the energy of the ions and the electrons, \( V \) is the total volume, \( n \) is the total number of ion species in the plasma, \( \mu_i \) is the chemical potential of ion species \( i \), \( N_i \) is the number of ions of species \( i \), \( \mu_e \) is the electron chemical potential, and \( N_e \) is the number of electrons. Note that although \( N_e = \sum_{i=1}^{n} Z_i N_i \) for the fully ionized multicomponent plasma, such that \( N_e \) is not an independent thermodynamic variable, here we treat it as such in order to express the various relations derived in this section in terms of both ion and electron quantities. The ion and electron terms are combined in the rest of the paper (Equations (8) and (16)) to simplify the appearance of the equations. Using \( X_i = A_i N_i / \langle A \rangle N_e \), \( Y_e = \sum_{i=1}^{n} Y_i X_i \), and the Euler integral

\[
G = \sum_{i=1}^{n} \mu_i N_i + \mu_e N_e, \tag{D2}
\]

where \( G \) is the Gibbs free energy, we have

\[
dg = -SdT + \frac{1}{\rho} dP + \sum_{i=1}^{n} \frac{\mu_i}{A_i m_p} dX_i + \frac{\mu_e}{m_p} dY_e. \tag{D3}
\]

Here \( q \) is the “specific” version of the quantity \( Q \); i.e., \( q = Q / M \), where \( M = \langle A \rangle m_p N \) is the total mass of the system and
\[ N = \sum_{i=1}^{n} N_i \] is the total number of ions. Using Equation (D3) we can derive two useful Maxwell relations. Since

\[ \left( \frac{\partial^2 g}{\partial X_i \partial T} \right)_{P,X_j,X_k,Y} = \left( \frac{\partial^2 g}{\partial T \partial X_i} \right)_{P,X_j,Y} , \]  

we have

\[ -\left( \frac{\partial s}{\partial X_i} \right)_{T,P,X_j,Y} = \frac{1}{A_i m_p} \left( \frac{\partial \mu_i}{\partial T} \right)_{P,X,Y} . \]  

Similarly,

\[ -\left( \frac{\partial s}{\partial Y_e} \right)_{T,P,X_i} = \frac{1}{m_p} \left( \frac{\partial \mu_e}{\partial T} \right)_{P,X,Y} . \]  

We define

\[ b_{P,i} \equiv -X_i \left( \frac{\partial s}{\partial X_i} \right)_{T,P,X_j,Y} = \frac{X_i}{A_i m_p} \left( \frac{\partial \mu_i}{\partial T} \right)_{P,X,Y} \]  

and

\[ b_{P,e} \equiv -Y_e \left( \frac{\partial s}{\partial Y_e} \right)_{T,P,X_i} = \frac{Y_e}{m_p} \left( \frac{\partial \mu_e}{\partial T} \right)_{P,X,Y} . \]  

These terms are analogous to the ion and electron specific heat terms \( c_{P,i} = T(\partial s_i/\partial T)_{P,X,Y} \) and \( c_{P,e} = T(\partial s_e/\partial T)_{T,P,Y} \) for the composition. For degenerate electrons

\[ b_{P,e} \equiv -\frac{\pi^2 k_B T}{3 E_F} \frac{Y_e}{m_p} = -\frac{2}{3} c_{P,e} , \]  

where \( E_F = m_e c^2 (\sqrt{1 + x_F^2} - 1) \) is the Fermi energy with \( x_F = 10.0 \rho_{10}^{1/3} Y_e^{1/3} \); for the models we consider here the electrons are degenerate and Equation (D9) holds throughout the ocean, since \( k_B T / E_F \lesssim 0.2 \) for \( \rho \gtrsim 10^6 \) g cm\(^{-3}\) and \( T \gtrsim 3 \times 10^8 \) K. An accurate expression for \( b_{P,i} \) in the ocean can be obtained from \( \mu_i = (\partial F_i/\partial N_i)_{Y_e,Y,\mu} \) and the free energy of a multicomponent liquid

\[ F_i = k_B T \sum_{j=1}^{n} N_j \left[ f_{\text{OPR}}^{\text{CP}}(T_i) + \ln \left( \frac{N_i Z_i}{N_e} \right) \right] , \]  

where \( f_{\text{OPR}}^{\text{CP}} \) (including the ideal gas part) is defined in Equations (1) and (2) of Medin & Cumming (2010); we use this accurate expression in our numerical calculations. However, an approximation for \( b_{P,i} \) can be obtained by considering only the ideal gas term, the dominant temperature-dependent term in \( \mu_i \):

\[ b_{P,i} \approx \frac{X_i}{A_i m_p} k_B \ln \left( \frac{N_i h^2}{2 \pi A_i m_p k_B T} \right)^{3/2} , \]

\[ \sim 30 \frac{X_i k_B}{A_i m_p} \sim 10 c_{P,i} . \]  

### APPENDIX E

**TRACKING THE OCEAN–CRUST BOUNDARY**

Here we derive expressions for the motion of the ocean–crust boundary, as well as the changes in entropy and composition at the ocean–crust boundary.

Using Equation (6) and

\[ \langle Z_b^{5/3} \rangle = \sum_{i=1}^{n} x_{i,b} Z_i^{5/3} = \langle A \rangle \sum_{i=1}^{n} x_{i,b} Z_i^{5/3} / A_i , \]  

we have that the ocean–crust boundary moves at a rate

\[ \dot{y}_b = \dot{y}_{b,T} + \dot{y}_{b,X} , \]  

where

\[ \dot{y}_{b,T} = \frac{\partial y_b}{\partial T_{b}} \frac{T_{b}}{T_{b}} = \frac{4 y_b}{3 T_{b}} \]  

and

\[ \dot{y}_{b,X} = \sum_{i=1}^{n} \frac{\partial y_b}{\partial X_{i,b}} \frac{\partial X_{i,b}}{\partial t} = \sum_{i=1}^{n} y_{i,b} \frac{\partial X_{i,b}}{\partial t} \]  

with

\[ \frac{\partial y_b}{\partial X_{i,b}} = 4 y_b A_i \frac{Z_b^{5/3} - Z_i^{5/3}}{(Z_b^{5/3})} . \]  

For an \(^{16}\)O—\(^{79}\)Se mixture, \( y_{b,1} \approx 10 y_b \). At the ocean–crust boundary, the energy balance equation Equation (11) in the frame moving with the boundary is

\[ T_b \frac{\partial s_b}{\partial t} + T_b (\dot{m} - \dot{y}_b) \frac{\partial s_b}{\partial y} \bigg|_{y=y_b} = \frac{\partial F_b}{\partial y} \bigg|_{y=y_b} + \epsilon , \]  

where \( \epsilon \) indicates that the derivative is evaluated on the “downstream” side of the boundary: \( \dot{y}_b^{\text{down}} \) during heating \( (\dot{y}_b > 0) \), and \( \dot{y}_b^{\text{down}} \) during cooling \( (\dot{y}_b < 0) \). Note that \( \partial s / \partial y \big|_{y=y_b} = 0 \) and that \( \epsilon = 0 \) during cooling.

For a two component mixture, near the base of the ocean \( \nabla_X = \nabla_T (\nabla_{ad} - \nabla) / \chi_T \ll 1 \) (see Paper I), such that \( \partial X_1 / \partial t \) is almost constant there. Therefore, using Equation (35),

\[ \frac{\partial X_{1,b}}{\partial t} \approx \frac{\dot{m} \Delta X_{1,0c} - \dot{y}_b \Delta X_{1,bc}}{y_b} \]  

with Equations (E2)–(E4) we have

\[ \frac{\partial X_{1,b}}{\partial t} \approx 1 + y_{b,1} \Delta X_{1,bc} / y_b \left( \frac{\Delta X_{1,0c}}{y_b} - \frac{4 \Delta X_{1,bc}}{T_b} \right) \]  

During cooling we have from Equations (35) and (E9) that

\[ 4 \frac{\partial \ln T_b}{\partial t} \approx \frac{\Delta X_{1,bc}}{y_b} \]  

for \( \Delta X_{1,bc} \ll 1, 4 (\partial \ln T_b / \partial t) \approx \partial \ln y_b / \partial t \); while for \( \Delta X_{1,bc} \approx 1, \partial \ln T_b / \partial t \approx 3 (\partial \ln y_b / \partial t) \).

### APPENDIX F

**EFFECTS OF ROTATION AND MAGNETIC FIELD ON CONVECTION**

The effects of rotation and magnetic fields on convection have been examined in many places (e.g., Stevenson 1979, 2003;
Jones 2000; Christensen & Aubert 2006; see also Showman, Kaspi, & Flierl 2011). Here we use simple arguments to show that in the neutron star ocean, the efficient convection assumption Equation (9) is very good even in the presence of rapid rotation (∼10^2 s⁻¹) and moderate magnetic fields (∼10^{10} G).

We consider a two-component ocean mixture with a plane-parallel geometry and governed by Newtonian physics. We impose a gravitational field −g⃗r, rotation ⃗Ω, and magnetic field B₀, all uniform. We assume that during convective mixing, displaced fluid elements do not exchange heat or material with their surroundings until they have traveled a distance of order the mixing length lₘ, but by rapidly contracting or expanding they maintain pressure balance with their surroundings (see Appendix A). We therefore have

$$\left( \frac{dP_{tot}}{dr} \right)_{s,X,Y} = \frac{dP_{tot}}{dr},$$

where \( P_{tot} = P + P_{mag} \) and

$$P_{mag} = \frac{B^2}{8\pi},$$

(Note that the equations of Appendices D–B are unaffected by the inclusion of this magnetic “pressure” term because \( P_{mag} \ll P \). In the rotating frame, the equation of motion for a fluid element displaced from its equilibrium position \( r₀ \) is

$$\frac{\partial \mathbf{v}}{\partial t} = f_{grav} + f_{rot} + f_{mag}$$

where

$$\mathbf{v} = \frac{\partial \delta \mathbf{r}}{\partial t},$$

$$f_{grav} = -\frac{\delta \mathbf{r}}{\partial t}.$$

is the buoyancy force (per unit volume) felt by the element,

$$f_{rot} = -2\rho \Omega \times \mathbf{v}$$

is the Coriolis force, and

$$f_{mag} = \frac{1}{4\pi}(\nabla \times \mathbf{B}) \times \mathbf{B} + \nabla P_{mag} = \frac{1}{4\pi}(\mathbf{B} \cdot \nabla)\mathbf{B} \simeq \frac{1}{4\pi} (\mathbf{B}_0 \cdot \nabla)\delta \mathbf{B}$$

is the magnetic “tension” force. Here

$$\delta \mathbf{r} = \mathbf{r} - \mathbf{r}_0$$

is the displacement of the element from its equilibrium position and

$$\delta \mathbf{B} = \mathbf{B} - \mathbf{B}_0$$

is the perturbation to the magnetic field caused by this displacement. The magnetic tension term \( f_{mag} \) acts as a restoring force (see Kulsrud 2005): for a typical wavelength \( l_m \) and a perpendicular field displacement of \( \delta \mathbf{r} \) we have that

$$\delta \mathbf{B} \approx \frac{B_0}{l_m} \delta \mathbf{r}$$

such that

$$f_{mag} \approx \rho \omega_B^2 \delta \mathbf{r}.$$
Since \( A = \chi_T (\nabla - \nabla_{ad})/(\chi_\rho H_P) + \chi_X \nabla_X/(\chi_\rho H_P) \simeq \omega_B^2/g \sim 10^{-10} \text{cm}^{-1}\) while \( \chi_T (\nabla_{ad} - \nabla)/(\chi_\rho H_P) \sim 10^{-6} \text{cm}^{-1} \gg A \), we have

\[
\chi_X \nabla_X \simeq \chi_T (\nabla_{ad} - \nabla); \tag{F22}
\]

i.e., the efficient convection assumption Equation (9) is good even in the presence of rotation and magnetic fields.

Note that there is some ambiguity in the typical length scale for the problem. Putting Equation (F21) back into Equations (F14) and (F15) gives

\[
\delta x \sim \frac{2 \Omega \sigma}{\omega_B} \delta r; \tag{F23}
\]

since \( \sigma \ll \omega_B \), the typical perturbation in the horizontal direction is much smaller than in the vertical direction. This may require an average displacement smaller than \( l_m/2 \) to be used in Equation (F17) (see, e.g., Stevenson 1979), which will increase the oscillation frequency \( \sigma \) required to generate the composition flux \( F_X \). However, we still have \( \sigma^2 \ll g A \) such that \( g A \) will not change much (unless \( \omega_B \) also changes) and our conclusion remains the same. Note also that if the star is non-magnetic such that \( \omega_B = 0 \), Equation (F16) instead gives

\[
\sigma^2 \sim g A - 4 \Omega^2 \tag{F24}
\]

and we have

\[
A \simeq 4 \Omega^2/g \sim 10^{-10} \text{ cm}^{-1}; \tag{F25}
\]

we again find that efficient convection is a good assumption.

REFERENCES

Altamirano, D., Keek, L., Cumming, A., et al. 2012, MNRAS, 426, 927
Bildsten, L., & Brown, E. F. 1997, ApJ, 477, 897
Bildsten, L., & Cutler, C. 1995, ApJ, 449, 800
Brown, E. F. 2004, ApJL, 614, L57
Brown, E. F., & Bildsten, L. 1998, ApJ, 496, 915
Brown, E. F., & Chang, P. 2002, ApJ, 574, 920
Brown, E. F., & Cumming, A. 2009, ApJ, 698, 1020
Cackett, E. M., Brown, E. F., Cumming, A., et al. 2013, ApJ, 774, 131
Cackett, E. M., Wijnands, R., Linares, M., et al. 2006, MNRAS, 372, 479
Cackett, E. M., Wijnands, R., Miller, J. M., Brown, E. F., & Degenaar, N. 2008, ApJL, 687, L87
Christensen, U. R., & Aubert, J. 2006, GeoIL, 166, 97
Cox, J. P. 1980, Theory of Stellar Pulsation (Princeton, NJ: Princeton Univ. Press)
Cumming, A., & Bildsten, L. 2001, ApJL, 559, L127
Cumming, A., Macbeth, J., in ’t Zand, J. J. M., & Page, D. 2006, ApJ, 646, 429
Degenaar, N., Medin, Z., Cumming, A., et al. 2014, ApJ, 791, 47
Degenaar, N., & Wijnands, R. 2011, MNRAS, 412, 68
Degenaar, N., Wijnands, R., Brown, E. F., et al. 2013a, ApJ, 775, 48
Degenaar, N., Wijnands, R., & Miller, J. M. 2013b, ApJL, 767, L31
Degenaar, N., Wolff, M. T., Ray, P. S., et al. 2011, MNRAS, 412, 1409
Fridriksson, J. K., Homan, J., Wijnands, R., et al. 2011, ApJ, 736, 162
Gupta, S., Brown, E. F., Schatz, H., Möller, P., & Kratz, K.-L. 2007, ApJ, 662, 1188
Haensel, P., & Zdunik, J. L. 1990, A&A, 227, 431
Haensel, P., & Zdunik, J. L. 2008, A&A, 480, 459
Heney, L., & L’Ecuyer, J. L. 1969, ApJ, 156, 549
Homan, J., van den Klis, M., Wijnands, R., et al. 2007, ApJ, 656, 420
Horowitz, C. J., Berry, D. K., & Brown, E. F. 2007, PhRvE, 75, 066101
Hughto, H., Horowitz, C. J., Schneider, A. S., et al. 2012, PhRvE, 86, 066413
Hughto, J., Schneider, A. S., Horowitz, C. J., & Berry, D. K. 2011, ApJ, 730, 97
in ’t Zand, J. J. M., Cumming, A., van der Sluys, M. V., Verbunt, F., & Pols, O. R. 2005, A&A, 441, 675
in ’t Zand, J. J. M., Kuulkers, E., Verbunt, F., Heise, J., & Cornelisse, R. 2003, A&A, 411, L487
Jones, C. A. 2000, JSPTA, 358, 873
Keek, L., in ’t Zand, J. J. M., Kuulkers, E., et al. 2008, A&A, 479, 177
Kippenhahn, R., & Weigert, A. 1994, Stellar Structure and Evolution (Berlin: Springer)
Kulsrud, R. M. 2005, Plasma Physics for Astrophysics (Princeton, NJ: Princeton Univ. Press)
Kuulkers, E. 2002, A&A, 383, L5
Kuulkers, E., in ’t Zand, J. J. M., Homan, J., et al. 2004, in AIP Conf. Proc. 714, X-Ray Timing 2003; Rossi and Beyond, ed. P. Kaaret, F. K. Lamb, & J. H. Swank (Melville, NY: AIP), 257
Medin, Z., & Cumming, A. 2010, PhRvE, 81, 036107
Medin, Z., & Cumming, A. 2011, ApJ, 730, 97
Medin, Z., & Cumming, A. 2014, ApJL, 783, L3
Piro, A. L., & Bildsten, L. 2005, ApJ, 619, 1054
Potekhin, A. Y., & Chabrier, G. 2000, PhRvE, 62, 8554
Potekhin, A. Y., Chabrier, G., & Yakovlev, D. G. 1997, A&A, 323, 415
Schatz, H., Aprahamian, A., Barnard, V., et al. 2001, PhRvL, 86, 3471
Schatz, H., Bildsten, L., Cumming, A., & Ouellette, M. 2003,NuPhA, 718, 247
Schatz, H., Bildsten, L., Cumming, A., & Wiescher, M. 1999, ApJ, 524, 1014
Showman, A. P., Kaspi, Y., & Fierl, G. R. 2011, Icar, 211, 1258
Stevens, J., Brown, E. F., Cumming, A., Cyburt, R., & Schatz, H. 2014, ApJ, 791, 106
Stevenson, D. J. 1979, GApFD, 12, 139
Stevenson, D. J. 2003, IES, 208, 1
Strohmayer, T. E., & Brown, E. F. 2002, ApJ, 566, 1045
Wijnands, R., Guainazzi, M., van der Klis, M., & Méndez, M. 2002, ApJ, 573, L45
Wijnands, R., Homan, J., Miller, J. M., & Lewin, W. H. G. 2004, ApJL, 606, L61
Wijnands, R., Miller, J. M., Groot, P. J., et al. 2001, ApJL, 560, L159
Wijnands, R., Nowak, M., Miller, J. M., et al. 2003, ApJ, 594, 952
Woodruff, D. P. 1973, The Solid–Liquid Interface (London: Cambridge Univ. Press)
Woosley, S. E., Heger, A., Cumming, A., et al. 2004, ApJS, 151, 75