Abstract

Small, illuminated aerosol particles embedded in a gas experience a photophoretic force. Most approximations assume the mean particle surface temperature to be effectively the gas temperature. This might not always be the case. If the particle temperature or the thermal radiation field strongly differs from the gas temperature (optically thin gases), given approximations for the free molecule regime overestimate the photophoretic force by an order of magnitude on average and for individual configurations up to three magnitudes. We apply the radiative equilibrium condition from the previous paper (Paper 1) — where photophoresis in the free molecular flow regime was treated — to the slip flow regime. The slip-flow model accounts for thermal creep, frictional and thermal stress gas slippage and temperature jump at the gas-particle interface. In the limiting case for vanishing Knudsen numbers — the continuum limit — our derived formula has a mean error of only 4% compared to numerical values. Eventually, we propose an equation for photophoretic forces for all Knudsen numbers following the basic idea from Rohatschek by interpolating between the free molecular flow and the continuum limit.

Keywords: photophoresis; rarefied gas; aerosols; transition regime; black body; thermal radiation

1. INTRODUCTION

Illuminated particles suspended in a gas experience photophoretic forces [Yalamov et al. (1976a,b); Rohatschek (1995); Loesche et al. (2013)]. For directed illumination like in Fig. 1 a simple description for high Knudsen numbers is based on a kinetic description of the momentum transfer between impinging gas molecules and the particles, which is stronger on one particular side of the particles. Often this is related to a temperature gradient across the particles’ surface which leads to a motion away from the radiation source.

Several experiments show photophoresis [Wurm & Krauss (2008); Loesche et al. (2014); van Eymeren & Wurm (2012)] and the theoretical treatment of photophoretic forces in different pressure regimes has also progressed (Malai et al. (2012); Beresnev et al. (1993); Yalamov et al. (1976a,b); Reed (1977)).

The findings in the first paper (Loesche et al. (2016)), (hereafter referred to as Paper 1) are based on work by Hidy & Brock (1967); Tong (1973); Yalamov et al. (1976a), which allow only low radiative fluxes \( I \) and small gas-particle temperature differences. It presented a new free molecular flow \( (fm) \) approximation, that now also supports the case of considerably higher radiative fluxes \( (I) \) and hotter/lower surface temperatures with respect to the surrounding gas \( (T_\infty) \), while assuming the particle to be in equilibrium with an external radiation field at \( T_{\text{rad}} \). It also performs very well for particles of low thermal conductivity \( k \) which so far only Yalamov et al. (1976a) does, too. Paper 1 showed that the optimized linearizations used have an excellent effect on the results, reducing the minimum and maximum relative error of the analytical equation (within the model) to \( \approx -50\% \) and \( 7\% \), respectively.
do not necessarily have to be the same, we propose another sf model in Section 5. From the equation for the sf regime we obtain the limiting case for vanishing Knudsen numbers (co).

In the sf regime we account for thermal creep, frictional and thermal stress gas slippage and temperature jump at the gas-particle interface. We will not include temperature dependent $k$, $k_t$ and $\eta$ but show how to account for that in Section 5. For smaller particles the boundary conditions in the sf regime can also be extended by some additional addends which are linear in the Knudsen number (Malai et al. 2012b), introducing several more parameters. However, as mentioned before, we interpolate between the co and fm approximations. Therefore we do not incorporate too many Knudsen-number dependent boundary conditions into this model which vanish in the limiting case $Kn \to 0$ (co). A discussion of the results and a comparison to other models is done in Section 5. 

All variables in this paper are also listed in Tab. A.4, including some basic relations. Section A provides some additional information for the interested reader in the supplementaries.

2. CLARIFICATION/KNUDSEN REGIMES

The Knudsen number $Kn$ is defined as the ratio of the mean free path of the gas molecules/atoms $\lambda$ and the characteristic length of the problem $r_0$ (here this is the particle radius)

$$Kn = \frac{\lambda}{r_0} \tag{1}$$

The fm and co regimes are the limits $Kn \to \infty$ and $Kn \to 0$, respectively. For fixed characteristic particle sizes $r_0$, both limits basically infer $p \to 0$ and $p \to \infty$, respectively. For high Knudsen numbers, the photophoretic force is linear in $p$ (Paper 1). Conversely, for low Knudsen numbers, the force goes with $p^{-1}$ (this paper). That means, for both limits it is $\lim_{Kn \to \infty} F_{phot}(p) = \lim_{Kn \to 0} F_{phot}(p) \neq 0$. This is obviously not useful.

Our considerations made in the fm and co regimes are hence for large and small enough Knudsen numbers, respectively.

Technically, $Kn \geq 10$ is associated with the fm regime, the transition regime is assumed for a Knudsen number range between $0.25 \lesssim Kn \lesssim 10$, but the lower bound varies with different transfer processes on particles (Hidy & Brock 1970). For low Knudsen numbers $Kn \ll 1$, the co regime is extended with a slip-flow boundary condition. This sub-regime is called the slip-flow regime. Here, no general bounds can be provided (Hidy & Brock 1970). A sketch of the different regimes is appended in Fig. A.6.

Therefore it is more exact to say, the considerations made in the co regime are actually made in the sf regime and only the limiting case for vanishing Knudsen numbers is associated with the co regime. On the other hand, as fm photophoresis is not meant for zero pressure ($Kn \neq 0$), one can also talk about co photophoresis ($Kn \neq \infty$).

3. PHOTOPHORESIS AT LOW KNUDSEN NUMBERS

For solid particles at low Knudsen numbers (e.g. large aerosols) the photophoretic force is a direct result of thermal creep along a surface $\partial V$ of the suspended particle (Reed 1977, Bakanov 2004), which occurs in case of a temperature gradient in the gas, which is tangential to $\partial V$.

For directed illumination of a homogeneous, spherical particle embedded in an effectively infinite gas as shown in Fig. 1 an equation for the ensuing longitudinal photophoretic force at low Knudsen numbers is proposed. The particle is supposed to be in a radiative equilibrium with an external radiation field at temperature $T_{rad}$. This radiation field can also be emitted by the gas itself, which has the temperature $T_{\infty}$ far away from the suspended particle. We present two means to describe photophoresis for directed illumination at a radiative flux of $I$. One is solely for the slip-flow regime with the limiting case of $Kn \to \infty$ (co) and the second one interpolates between all regimes, using the co limit and the fm limit from Paper 1.

The model consists of a hydrodynamic part and a heat transfer part. In this setting (Fig. 1), both problems are axisymmetric in $\xi$, i.e. they only depend on the coordinates $r$ and $\xi$. The $z$-axis is therefore set parallel to the direction of illumination and motion at speed $u$, and especially: $e_z = -e_r$. Gases and fluids with a small dynamic viscosity can be treated as ideal fluids. Additionally, if the fluid is incompressible and the flow is free of vortices, the flow can be treated like a potential flow. However, this statement is right for almost every point in the fluid except at the particle-fluid interface. Friction will definitely contribute here, large flow speed gradients occur, and friction forces will be comparable to inertial forces. Therefore, the boundary conditions in this model account for thermal creep as well as frictional and thermal stress gas slippage at the gas-particle interface.

Before setting up the hydrodynamic model, we give a short insight into thermal creep.

![Figure 1: Visualization of the situation considered. Illumination is directed along z-axis, thus for a homogeneous particle the surface temperature only depends on $\xi$ (spherical coordinate system $(r, \zeta, \xi)$). The sphere’s radius is $r_0$. The temperature of the gas is $T_{\infty}$ ($r \to \infty$), the temperature of the radiation field is $T_{rad}$.](image-url)
3.1. Thermal creep

Thermal creep causes a gas flow tangential to a surface (tangent \( \mathbf{t} \), normal \( \mathbf{n} \)) at a mass speed \( v \) which obeys the equation (Brenner, 2009)

\[
(\mathbf{I} - \mathbf{n} \otimes \mathbf{n}) \cdot (\mathbf{v} - \mathbf{u}) = \kappa_s \eta_{\text{kin}} (\mathbf{I} - \mathbf{n} \otimes \mathbf{n}) \cdot \nabla \log T_g \quad \text{on} \ \partial V .
\]

The mass velocity \( \mathbf{v} \) obeys the continuity equation

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{v} \rho = 0 \quad \text{on} \ \partial V ,
\]

\( \mathbf{u} \) is the velocity of the surface \( \partial V \) relative to the gas, \( \eta_{\text{kin}} \) denotes the kinematic viscosity of the gas, and \( \rho \) and \( T_g \) the gas mass density and gas temperature, respectively. (Brenner, 2005). \( \kappa_s \) is the thermal creep coefficient (also thermal slip coefficient) points out that, various experts on molecular dynamics agree on the correctness of this equation for gases, even though the underlying gas-kinetic molecular theory is not rigorous but only semi-quantitative.

The original value of the thermal creep coefficient \( \kappa_s = \frac{3}{4} \) goes back to Maxwell (1879). Bakanov (1992) lists a couple of different models which relate \( \kappa_s \) and the momentum accommodation coefficient \( \alpha_m \) by the equation

\[
\kappa_s (\alpha_m) = \frac{3}{4} (a_s + b_s \alpha_m),
\]

where \( a_s \) is close to 1 and \( b_s \) around 0.5, thus the thermal creep coefficient can be expected to obtain values between 0.75 \( \leq \kappa_s \leq 1.24 \). Rohatschek (1995) assumes a value of \( \kappa_s = 1.14 \) for \( \alpha_m = 0.9 \) and this value is also used by Loesche et al. (2014); Hesse (2011). Ivchenko et al. (1993) also suggested a model with more accurate values for \( \kappa_s \). One of the latest works is Ivchenko et al. (2007).

3.2. Hydrodynamic model

The momentum balance in the fluid is given by

\[
\rho \frac{d \mathbf{v}}{dt} \equiv \rho \left( \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right) = -\nabla p + \nabla \cdot \mathbf{\sigma} + \rho \mathbf{F}_{\text{ext}} ,
\]

where \( \mathbf{\sigma} \) denotes the stress tensor, that is related to the friction tensor \( \mathbf{F}_\mathbb{F} \)

\[
\sigma_{ik} = -p \delta_{ik} + R_{ik} \equiv \eta_{\text{dyn}} \left( \frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} \right) - \delta_{ik} \frac{2}{3} \eta_{\text{dyn}} \nabla \cdot \mathbf{v} .
\]

Introducing the fluid’s self-diffusion coefficient \( D \) and the fluid’s thermal expansion coefficient (at constant pressure) \( \gamma_{g\text{sp}} = \frac{1}{\rho} \left( \frac{\partial \rho}{\partial T} \right) \). In contrast to this equation, Eq. 3 is only valid for gases and no restrictions on the solids are imposed.

3.2.1. Ansatz

In orthogonal coordinates \( q_1, q_2, q_3 \) (with the accompanying scaling factors \( h_1, h_2, h_3 \)) a three-dimensional, stationary flow of an incompressible Newtonian fluid with symmetry in \( q_3 \) has a vector potential that only depends on two variables \( \mathbf{\Psi} = \Psi(q_1, q_2) \). Therefore it can be set \( \mathbf{\Psi} \sim \psi_3 \), and the velocity can subsequently be written as

\[
\mathbf{v} = \nabla \times \mathbf{\Psi} = -\nabla \times \left( \frac{\psi_3}{h_3} \right) \quad \text{(9a)}
\]

\[
= -\nabla \times \left( \frac{\psi_3}{h_3} \right) \quad \text{(9b)}
\]

\[
= \frac{\psi_3}{h_3} \times \nabla \psi(q_1, q_2) - \psi(q_1, q_2) \nabla \times \frac{\psi_3}{h_3} \quad \text{(9c)}
\]

\[
= \frac{\psi_3}{h_3} \times \nabla \psi(q_1, q_2) \quad \text{(9d)}
\]

\( \psi \) is called the Stokes stream function. Applying \( \frac{\psi_3}{h_3} \times \nabla \) on Eq. 7 (and using Eq. 9d) yields the equation that \( \psi \) satisfies, that is also the governing equation for the flow (Schubert, 2015)

\[
E^2 \psi = 0 \quad \text{(10a)}
\]

\[
E^2 = \frac{h_3}{h_1 h_2} \left[ \frac{\partial}{\partial q_1} \left( \frac{h_2}{h_1 h_3} \frac{\partial}{\partial q_1} \right) + \frac{\partial}{\partial q_2} \left( \frac{h_1}{h_2 h_3} \frac{\partial}{\partial q_2} \right) \right] \quad \text{(10b)}
\]

In spherical coordinates \( (r, \xi, \eta) \) the scaling factors \( (h_1, h_2, h_3) = (1, r, r \sin \xi) \), and hence it is \( E^2 = \partial_{r^2} + \frac{1}{r^2} \partial_r + \frac{1}{r^2 \sin \xi} \partial_{\sin \xi} \). The velocity subsequently reads

\[
\mathbf{v} = \left( \frac{v_r}{v_\xi} \right) = \frac{1}{r \sin \xi} \begin{pmatrix} \frac{1}{r} \partial_r \psi - \frac{1}{\sin \xi} \partial_\xi \psi \hfill 0 \end{pmatrix} .
\]

The ansatz for \( \psi \) is (Reed, 1977)

\[
\psi(r, \xi) = \alpha \psi_0(r) \psi_2(\xi) \quad \text{(12a)}
\]

\[
\psi_2(\xi) = \frac{1}{2} \sin^2 \xi .
\]
The radial part \( \psi_\rho(r) \) is determined by the governing equation
\[ E^3 \psi = 0, \]
which formulates an ordinary differential equation for \( \psi_\rho(r) \). Its solution is
\[ \psi_\rho(r) = \frac{a}{r} + br + cr^2 + dr^4. \]  
(13)

In the following, the gas temperature is expanded into a Legendre series
\[ T_g(r, \zeta) = T_\infty + \sum_{\nu=0}^{\infty} C_\nu \left( \frac{r}{R} \right)^{\nu+1} P_\nu(\cos \zeta). \]  
(14)

### 3.2.2. Boundary conditions

Like in [Reed (1977)], we use an inertial reference frame at rest with the fluid far away from the particle (Eq. 18c), where the \( z \)-axis is parallel to the direction of illumination (due to symmetry in \( \zeta \), see Fig. 1). The fluid does not penetrate the particles surface, therefore the fluid velocity has no additional normal component than \( u \cos \zeta \) (Eq. 18a).

The radial part \( \psi \) is determined by the governing equation
\[ \psi = \psi_m(r) + \sum_{\nu=1}^{\infty} \frac{a_{\nu}}{r^{\nu+1}} + \frac{b_{\nu}}{r^{\nu+1}} P_{\nu}(\cos \zeta) \]  
which is related to the momentum accommodation coefficient \( \alpha \) (1977; Chang & Keh 2012); the boundary conditions are given as (Reed 1977; Chang & Keh 2012; Reed 1977).

The values for the thermal stress slip coefficient \( \kappa \) vary between 1 and 3 (Chang & Keh 2012); \( \kappa_m \) is the gas-kinetic gas kinetic accommodation coefficient with values around 1.00 \( \leq \kappa_m \leq 1.35 \) and typically taking about 1.25 (Reed 1977).

The boundary conditions are given as
\[ \left\{ \begin{array}{ll}
\nu_r &= \cos \zeta \\
\nu_z &= -u \sin \zeta + \frac{\kappa_m K_n c_i}{\eta_{dyn}} (\nu_r + \kappa \nu_t) \\
\nu_r \rightarrow 0, & \nu_z \rightarrow 0
\end{array} \right. \]  
(18a)

\[ \nu_r = \cos \zeta, \quad \nu_z = -u \sin \zeta + \frac{\kappa_m K_n c_i}{\eta_{dyn}} (\nu_r + \kappa \nu_t) \quad \text{on} \partial V \]  
(18b)

\[ \nu_r \rightarrow 0, \quad \nu_z \rightarrow 0. \]  
(18c)

The values for the thermal stress slip coefficient \( \kappa \) vary between 1 and 3 (Chang & Keh 2012); \( \kappa_m \) is the gas-kinetic gas kinetic accommodation coefficient with values around 1.00 \( \leq \kappa_m \leq 1.35 \) and typically taking about 1.25 (Reed 1977).

### 3.2.3. Solution

The velocity \( \mathbf{v} = v_r \hat{e}_r + v_z \hat{e}_z \) is completely given by Eqs. 11-14 and the unknown parameters \( a, b, c \) and \( d \) are restricted by the boundary conditions. From Eq. 18c it can be concluded that
\[ c = d = 0. \]  
(19)

Because of Eq. 18a it is
\[ a + r^2 (b + c) = 0. \]  
(20)

Eq. 18b involves a derivation of the Legendre series of the gas temperature in \( \zeta \) (Eq. 14). As it is \( \partial_\nu P_\nu(\cos \zeta) = P_\nu(\cos \zeta) \), only the polynomials \( P_\nu \) occur in Eq. 18b. All terms with \( u \) and \( a, b \) are linear in \( \sin \zeta \) \( P_\nu \). As pairwise different \( P_\nu \) are orthogonal to each other (see Eq. A.1 in the appendix for details), a scalar product of this boundary condition with \( P_\nu \) will isolate the interesting addends containing \( a \) and \( b \). Together with Eq. 20 it is
\[ a = \frac{r_0^3}{1 + 3 \kappa_m K_n} \left( \frac{1}{2} + (\kappa_m + 3 \kappa_m K_n) \right) \frac{\eta_{dyn}}{\rho r_0 T_{\|} u} C_1 \]  
(22a)

\[ b = \frac{r_0}{1 + 3 \kappa_m K_n} \left( \frac{1}{2} + (\kappa_m + 3 \kappa_m K_n) \right) \frac{\eta_{dyn}}{\rho r_0 T_{\|} u} C_1 - r_0, \]  
(21b)

and subsequently
\[ v_r = \frac{\cos \zeta r_0^2}{2 C_1 \eta_{dyn}} \left( \frac{1}{2} + \zeta \right) (\kappa_m + 3 \kappa_m K_n), \]  
(22b)

\[ v_z = \frac{\sin \zeta r_0^2}{4 \eta_{dyn}} \left( \frac{1}{2} - \zeta \right) (\kappa_m + 3 \kappa_m K_n), \]  
(22c)

\[ v_r \rightarrow \frac{\cos \zeta r_0^2}{2 C_1 \eta_{dyn}} \left( \frac{1}{2} + \zeta \right) (\kappa_m + 3 \kappa_m K_n) \frac{\eta_{dyn}}{\rho r_0 T_{\|} u} C_1 - \frac{1}{2} + \frac{1}{3 \kappa_m K_n}, \]  
(22d)

Here, due to the symmetry of the setting, only \( F_z \) is not zero (Happel & Brenner 1983).

\[ F_z = -8 \pi \eta_{dyn} \lim_{r \rightarrow 0} \frac{r \psi(r, \zeta)}{r^2 \sin^2 \zeta}, \]  
(23)

\[ = -4 \pi \eta_{dyn} u b. \]  
(24)

Inserting \( v_r \) and \( v_z \) into Eq. 23 yields the force as
\[ F_z = -4 \pi \frac{\eta_{dyn}}{\rho T_{\|} u} \left( \kappa_m + 3 \kappa_m K_n \right) \frac{\kappa_m}{1 + 3 \kappa_m K_n} C_1 - 6 \pi \eta_{dyn} r_0 u \frac{1}{1 + 3 \kappa_m K_n}. \]

(25)

In the steady state, where the particle moves at constant speed \( u \), it is \( F_z = 0 \). That means, two forces are compensable each
other, that is the photophoretic force

\[ F_{\text{phot}} = -4\pi \frac{\eta_{\text{dyn}}^2}{\rho T_{\text{av}}} \frac{\kappa_t + 3\kappa_{tn}Kn}{1 + 3\kappa_{tn}Kn} C_1 e_z \]  
(26)

and the drag/resistance force

\[ F_{\text{drag}} = -6\pi \eta_{\text{dyn}} r_0 \frac{1 + 2\kappa_{tn}Kn}{1 + 3\kappa_{tn}Kn} u_{\text{phot}} . \]  
(27)

The ensuing steady state velocity is

\[ u_{\text{phot}} = -\frac{2}{3} \frac{\eta_{\text{dyn}}}{\rho T_{\text{av}} r_0} \frac{\kappa_t + 3\kappa_{tn}Kn}{1 + 2\kappa_{tn}Kn} C_1 e_z . \]  
(28)

Instead of this equation, the Millikan drag equation can be used here, which is more accurate for \( Kn \approx 1 \) (Mackowski, 1989).

In the following section, the unknown expansion coefficient \( C_1 \) of the gas temperature is determined by solving a heat transfer problem.

3.3. Heat transfer model

We follow the assumptions made in Paper 1, the particle is heated by directed illumination, which is described by the inhomogeneity \( I(q(r, \cos \xi)) \) in the heat transfer equation. The heat transfer model supports energy exchange with the gas, thermal radiation and a temperature jump at the gas-particle interface. The gas is supposed to be at temperature \( T_\infty \) far away from the suspended particle. The particle is required to be in radiative equilibrium with an external radiation field at temperature \( T_{\text{rad}} \). This can also be the gas itself as \( T_{\text{rad}} = T_\infty \).

3.3.1. Ansatz

The Péclet number \( Pe \) (Eq. 1.3) is required to be small, then thermal diffusive transport predominates advective transport. Therefore the governing equations are

\[ \begin{align*}
  k \Delta T & = -I q(r, \cos \xi) & \text{(29a)} \\
  k_T \Delta T_T & = 0 , & \text{(29b)}
\end{align*} \]

for the particle and gas, respectively. \( I = \varepsilon I_0 \) is the absorbed radiative flux, \( \varepsilon \) denotes the emissivity.

The ansatz for the particle temperature \( T \) is constructed insofar that on the surface it is given by the simple equation

\[ T(r_0, \xi) = \sum_{n=0}^{\infty} A_n P_n(\cos \xi) . \]  
(30)

For the general solution \( T(r, \xi) = T_1(r, \xi) + T_2(r, \xi) \), the homogeneous and particular ansatz functions are

\[ T_1(r, \xi) = \sum_{n=0}^{\infty} (A_n - B_n J_n(r_0)) \left( \frac{r}{r_0} \right)^\nu P_n(\cos \xi) \]  
(31a)

\[ T_2(r, \xi) = \sum_{n=0}^{\infty} B_n J_n(r_0) P_n(\cos \xi) . \]  
(31b)

Then, \( T_1 + T_2 \) yield Eq. 30 on the surface. The particular solution employs the asymmetry factor \( J_\nu \) for perfectly absorbing spheres, the asymmetry factors yield \( J_0 = 1/4 \) and \( J_1 = \pm 1/2 \) (positive for irradiation into direction \( e_\xi = -e_\xi \)).

3.3.2. Boundary conditions

To account for thermal radiation and a temperature jump at the surface, the following boundary conditions were chosen

\[ \begin{align*}
  k \frac{\partial T}{\partial n} & = k_T \frac{\partial T_T}{\partial n} - \sigma_{SB} \varepsilon \left( T^4 - T_{\text{rad}}^4 \right) & \text{at } \partial V \& \\
  T_\xi & = T - \kappa_t r_0 \kappa_t \frac{\partial T_T}{\partial n} & \text{at } \partial V \& \\
  T_\xi & \rightarrow \infty \rightarrow T_\infty . \&
\end{align*} \]  
(33a)

(33b)

(33c)

The last boundary condition is already met by the ansatz for the gas temperature in Eq. 14. To prevent nonlinear mixing of the expansion coefficients \( A_n \) and \( B_n \) at multiple orders in the first boundary condition, the term \( \sigma_{SB} \varepsilon (T^4 - T_{\text{rad}}^4) \) will be linearized at the mean temperature \( \bar{T} \)

\[ \sigma_{SB} \varepsilon (T^4 - T_{\text{rad}}^4) = \sigma_{SB} \varepsilon (4\bar{T}^3 - 3T^4) + . . . \]  
(34)

\[ \bar{T} = \frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi T(\zeta)^4 \sin \zeta \, d\zeta \, d\zeta \]  
(35)

which is given by integrating the boundary conditions. The second boundary condition is the temperature jump condition at the gas-particle surface. For \( Kn \rightarrow 0 \) (co regime) the sphere and the gas layer surrounding it are in thermal equilibrium. The thermal accommodation coefficient \( \alpha \) defines the temperature jump coefficient \( \kappa_\alpha \) as Reed (1977)

\[ \kappa_\alpha(\alpha) = \frac{15}{8} \left( 1 - \frac{\alpha}{\alpha} \right) . \]  
(36)
3.3. Solution

In a similar procedure as in Paper 1 the coefficients A and C can be obtained from the boundary conditions in Eqs. 33a and 33b by using the orthogonality relations of the Legendre polynomials \( P_r = P_r^\nu \) (Eq. A.1). B is obtained by the inhomogeneous heat transfer equation (Eq. 29a, upper index sf means slip flow)

\[
A_0^{\nu} = \frac{I J_\nu}{\nu^1 R_1 + \nu^3 R_{1+3} + 4\sigma S B E (\bar{T}^\nu)^3} \quad \nu \geq 1 \tag{37a}
\]

\[
A_0^{\nu} = \frac{I J_0 + \frac{\nu^1 R_1}{\nu^1 R_{1+3}} T_\infty + \sigma S B E (3 (\bar{T}^\nu)^4 + T_r^4)}{\nu^1 R_1 + 4\sigma S B E (\bar{T}^\nu)^3} \tag{37b}
\]

3.4. Result

Summarizing all previous findings, the photophoretic force in the slip flow regime with a gas temperature \( T_\infty \) and a radiation field temperature \( T_{\text{rad}} \) is given as

\[
F_{\text{phot}}^{\nu} = -4\pi \nu^2 \eta_0 \kappa + 3\kappa B_k T_\infty \quad \nu \geq 1 \tag{44a}
\]

\[
F_{\text{phot}}^{\nu} = -4\pi \nu^2 \eta_0 \kappa + 3\kappa B_k T_\infty \quad \nu \geq 1 \tag{44b}
\]

Apart from the additional radiative term \( 4\sigma S B E T_{\text{rad}}^3 \), the results are in agreement with Eq. 36 from Reed [1977] and Eq. 29 from Mackowski [1989] for \( k_0 = 0 \).

3.5. Continuum limit

The rotationally symmetric solution above (Eq. 44) is reused to describe the force in the continuum limit (\( k_0 \to 0 \)). From the boundary condition Eq. 33c it is \( T_{\text{bb}}^\nu = \bar{T} \) at the surface, and therefore the force reads

\[
F_{\text{phot}}^{c} = -4\pi \nu^2 \eta_0 \kappa + 3\kappa B_k T_\infty \quad \nu \geq 1 \tag{46a}
\]

\[
\tilde{T} = \frac{I J_1}{\nu^1 R_1 + 4\sigma S B E T_{\text{rad}}^3} \tag{46b}
\]

Then, the temperature \( \bar{T} \) (Eq. 35) meets the balance

\[
\pi r_0^2 \rho I_0 = 4\pi r_0^2 \sigma S B E (\bar{T}^4 - T_r^4) \tag{42}
\]
The photophoretic velocity is

\[ u_{\text{phot}} = \frac{2}{3} \frac{k_s}{\rho T r_0} \frac{\eta_{\text{dyn}}}{k_s} \left( \frac{I_1}{r_0} + 2 \frac{k_s}{r_0} + 4 \sigma_{SB} \varepsilon T_{bb}^3 \right) \varepsilon_c. \] (47)

4. INTERPOLATING BETWEEN FM- AND CO-PHOTOPHORESIS

An empirical method is used to describe the photophoretic force in the transition regime due to the complexity of transport processes in this regime. Rohatschek (1995) presents a phenomenological equation satisfying the linear proportionality of the force with the pressure in the fm regime and the inverse proportionality in the co regime by

\[ \frac{F_{\text{phot}}}{F} = \frac{2 + \delta}{2 + \delta + \frac{\delta_0}{p}}, \] (48)

with the free parameter \( \delta \) to be adjusted along the experimental values (Fig. 2). Because of the changing proportionality at an unknown pressure \( \hat{p} \), the force peaks at \( F = F(\hat{p}) \). Hettner (1928) already suggests the same equation with \( \delta = 0 \). Rohatschek (1995) also favors it, justifying it to be the best-fitting version of conducted experiments in the past, including work of other researchers such as Tong (1975), Arnold & Amani (1980). Nonetheless, the experiments of Rosen & Orr (1964) with large carbon agglomerates do not obey above’s law. Rohatschek (1985) gave evidence that for large agglomerates, theories of \( \Delta T \)-photophoresis cannot be applied because of the superposition of \( \Delta \rho \) and \( \Delta T \)-photophoresis. One of the experimental results mentioned by Rohatschek (1995) implied \( \delta = 0.8 \). The gas-kinetic calculations made by Chernyak & Beresnev (1993) suggested \( \delta \approx 1 \), and for slip-flow theories, e.g. in Reed (1977) it is even \( \delta \geq 2 \), both fitting about 67% and less than 50%, respectively, of the experimental findings Rohatschek (1995) discussed.

Hettner (1928) suggested an interpolation (Eq. 20 in the respective publication), which is

\[ \frac{1}{F_{\text{phot}}} = \frac{1}{F_{\text{phot}}^{\text{co}}} + \frac{1}{F_{\text{phot}}^{\text{fm}}} \] (49)

We therefore present a new interpolation along Rohatschek (1995) for \( \delta = 0 \), whose scope of validity includes not only \( T/T_{\infty} \approx 1 \) but also stronger temperature deviations and therefore higher intensities. This interpolation is based on the fm equation from Paper 1 and the co equation from this paper (Eq. 46).

4.1. Longitudinal photophoresis in the transition regime

To determine \( \hat{F} \) and \( \hat{p} \) for longitudinal photophoresis in the description of Eq. 48 for \( \delta = 0 \), a few more steps have to be made. Starting with Eq. 49, the force in the fm and the co regimes is \( A_{\text{f}}^{\text{co}} := \lim_{k_s \to 0} A_{\text{f}}^{\text{co}} \) etc., \( A_{\text{f}}^{\text{co}} \equiv C^\alpha_{\text{f}}, A_{\text{f}}^{\text{fm}} = \frac{1}{2} \frac{\eta_{\text{dyn}}}{\rho T r_0} \varepsilon_c \left( \frac{I_1}{r_0} + 2 \frac{k_s}{r_0} + 4 \sigma_{SB} \varepsilon T_{bb}^3 \right) \varepsilon_c. \) (47)

Here, the ideal gas equation \( \rho = \frac{\mu}{R_g T_g} \) was used to express the mean thermal gas speed as \( v_{th} = \sqrt{8p/(\pi \rho)} \). Eq. 51b is valid for mono-atomic gas, for di-atomic gas the factor \( 1/2 \) has
to be replaced with 3/4 \cite{Rohatschek1995}. The mean temperatures $\bar{T}^\text{fm}$ and $\bar{T}$ for $h > 0$ can be determined by solving Eq. \eqref{51a} iteratively with $\tau^\text{fm} = \bar{T}$ starting at $T_{bb}$. For a relatively small $h$, $\tau^\text{fm} = T_{bb}$ (Paper 1). The dimensionless scaling coefficients $\tau$ are subsequently

$$\tau^\text{fm} = \sqrt{A^\text{gf}_0} \frac{\alpha_{Rm}}{2} \frac{1}{k_{Rm} + h + 40\sigma_{SB} \epsilon (\bar{T}^\text{fm})^3}$$

and

$$\tau^\text{co} = \sqrt{A^\text{gf}_0} \frac{1}{\kappa_{Rm} + 2 \frac{\epsilon_{Rm}}{k_{Rm}^2} + 40\sigma_{SB} \epsilon \bar{T}_{bb}^3}.$$  

(52a)

(52b)

The interpolation equation Eq. \eqref{49} enables — together with the equations above — to derive

$$\hat{F} = \Xi \sqrt{\tau^\text{co} \tau^\text{fm} r_0^3 J_1 I}$$

(53a)

$$\hat{p} = \sqrt{\frac{\tau^\text{co}}{\tau^\text{fm} \rho^*} p^*}.$$  

(53b)

Compared to the work in \cite{Rohatschek1995}, the maximum force $\hat{F}$ is determined by the geometric mean of $\tau^\text{co}$ and $\tau^\text{fm}$ as additional factor. Similarly, the pressure $\hat{p}$ where the forces maximizes is given by extending the result in \cite{Rohatschek1995} by an additional factor, i.e. the square root of the ratio of the two $\tau$.

5. DISCUSSION

In this section, the underlying $fm$ and $co$ limit approximations are discussed. A brief comparison to the original model by \cite{Rohatschek1995}, \cite{Hettner1928} is given afterwards.

5.1. fm limit equation accuracy

In Paper 1 a new approximation for photophoretic forces in the $fm$ regime following

$$F^\text{fm}_{\text{phot}} \approx \frac{\pi}{3} \alpha \frac{\rho p}{\sqrt{A^\text{gf}_0 \bar{T}^3_{gg}} T_{gg}^2} \frac{I J_1}{k_{Rm} + h + 40\sigma_{SB} \epsilon \bar{T}_{bb}^3},$$

was introduced. As shown in that paper, this formula for photophoresis on spherical particles with surface temperatures strongly deviating from the gas temperature $T_{\infty}$ or high intensities $I$ significantly increases the accuracy of analytically determined photophoretic forces with respect to numerical values. Different classic approximation for the photophoretic force in the $fm$ regime which are not supporting these gas temperatures and intensity conditions were compared to the new approximation to emphasize the need for an additional equation in the $fm$ regime with an extended scope of validity. While still covering the classic scope of validity (for $\alpha_m = 1$, this equation can very well be approximated by the $fm$ equation from \cite{Berestnev1993} for $\alpha_k = 1$), Eq. \eqref{54} has an average relative error of about 1% for particles with a radius of up to 1.1 mm. With a maximum and minimum relative errors of only 7% and $\approx -50\%$, respectively, (for details see Paper 1), it is far more reliable under rather extreme conditions than the classic $fm$ approximations, which then overestimate the force up to orders of magnitude, as they were designed for basically low intensities.

5.2. co limit equation accuracy

Table 1: Intervals for the parameter sweep in COMSOL, where the heat transfer equation (Eq. \eqref{29a} with the boundary condition given by Eq. \eqref{33a} was solved ([a, b] denotes an interval between the numbers $a$ and $b$). All intervals are equally subdivided (log scale; the additional ‘1 m’ for $r_0$ means, there is a gap between 1 m and 0.11 m concerning this equal subdivision). Details on the subdivision can be found in \cite{Loesche2015}.

| parameter | parameter sweep intervals |
|-----------|---------------------------|
| $r_0$     | $[1.1 \times 10^{-4}, 1.1 \times 10^{-1}]$ m, and 1 m |
| $k$       | $[10^{-3}, 8]$ W m$^{-1}$ K$^{-1}$ |
| $I$       | $[0.5, 40]$ kW m$^{-2}$ |
| $T_{rad}$ | $[0, 350]$ K |

Figure 3: Parameter sweep histogram for 196,344 parameter combinations. The parameter sweep intervals are given in Tab. \ref{tab:1}. Like in Paper 1, the effectively exact numerical result was obtained with COMSOL by solving the heat transfer equation (Eq. \eqref{29a} with the boundary condition given by Eq. \eqref{33a} The bin size is 0.005 (0.5%). Color-coded arrows points towards the respective histogram’s peak. The histogram for $A^\text{co}_{II}$ (black) is very close to the exact value $A^\text{co}_{II}$, whereas the other expansion coefficients $A^\text{co}_{I}$ \cite{Reed1977} and $A^\text{co}_{I}$ \cite{Mackowski1989} are overestimated up to several orders of magnitude. The particle mean temperature $A^\text{co}_{II}$ (blue) is also very close to the exact value $A^\text{co}_{II}$, the ratio is within 1.00 $\leq A^\text{co}_{II}/A^\text{co}_{II}$ $\leq$ 1.63.

In this paper, we use the same radiation term in the boundary condition as in the previous paper (see Section 3.3.2). As the force depends on the first expansion coefficient of the gas temperature $C^\text{co}_{II} \propto A^\text{co}_{II}$ ($C^\text{co}_{II} = A^\text{co}_{II}$) very close to the surface, a more accurate expansion coefficient $A^\text{co}_{II}$ will obviously also improve the quality of the calculated force. This is especially true for high intensities $I$, where the radiation term $40\sigma_{SB} \epsilon \bar{T}_{bb}^3$ will strongly contribute to the solution. As the description of the entire pressure regimes photophoresis in this paper is based on the interpolation between the $fm$ and $co$ approximations, only the
Table 2: Statistical properties of the ratio of the particle temperature expansion coefficients (see Eq. 55). A parameter sweep of 196,344 parameter combinations was performed along the parameter intervals given in Tab. 1. Values in round brackets are for $r_0$ restricted to the interval [0.11, 11] mm.

| ratio of particle temperature expansion coefficients: | min   | max   | mean  | median | STD  |
|-----------------------------------------------------|-------|-------|-------|--------|------|
| analytic/numerical                                   | (1.00)| 38.368| 46.9  | 2.25   | 357  |
| $A_{1}^{co,0}/A_1$                                   | (1.00)| 20.802| 8.10  | 1.45   | 83.2 |
| $A_{1}^{co,rad}/A_1$                                 | (0.57)| 1.07  | 0.97  | 1.00   | 0.08 |
| $A_{1}^{co}/A_1$                                      | (0.66)| 1.07  | 0.99  | 1.00   | 0.05 |
| $A_{0}^{co}/A_0$                                      | (1.00)| 1.63  | 1.06  | 1.01   | 0.10 |
| $A_{1}^{co,bb}/A_0$                                  | (0.35)| 1.05  | 0.93  | 1.00   | 0.13 |
| $A_{1}^{co}/A_1$                                      | (0.46)| 1.05  | 0.96  | 1.00   | 0.10 |

Thermal radiation contributes as additional term in comparison to Rohatschek (1995), while those boundary conditions which are linear in the Knudsen-number disappear in the $co$ limit. We performed a parameter sweep along the values in Tab. 1 and visualize the strong influence of the black body radiation term in the histogram in Fig. 3, where the histograms of each ratio of one of the three expansion coefficients Eq. 55 and its true value are shown. This true value was obtained from temperature distribution across the spheres, calculated with COMSOL. The boundary conditions used in COMSOL are Eq. 33a. Tab. 2 shows minimum and maximum ratios. Beside that, it also shows other distribution information, which does not have a strict mathematical meaning but show a tendency, just as Fig. 3. For simplicity, we restrict ourselves to the $co$ limit in this discussion and neglect $k_g$ in the COMSOL calculations, as for gases like air, high intensities and not too small particles the radiation term dominates $2k_g/r_0 \ll 4\sigma_{SB}\epsilon T_{bb}^4$. In the other cases, the consideration of $k_g$ will not prevent any error here but only complicate our considerations, because the term $2k_g$ in the expansion coefficients did not arise from any linearizations or simplifications in the boundary condition. To investigate the influence of the black body radiation term in the first expansion coefficient of the particle surface temperature $A_1$, we therefore either set $\bar{T}$ to 0, $T_{in}$ and our result $T_{bb}$:

- no thermal radiation: $A_{1}^{co,0} = \frac{IJ_1}{k_0 + 2k_0}$ \hspace{1cm} (55a)
- simple thermal rad.: $A_{1}^{co,rad} = \frac{IJ_1}{k_0 + 2k_0 + 4\sigma_{SB}\epsilon T_{rad}^4}$ \hspace{1cm} (55b)
- black body rad.: $A_{1}^{co,bb} = \frac{IJ_1}{k_0 + 2k_0 + 4\sigma_{SB}\epsilon T_{bb}^4}$. \hspace{1cm} (55c)

The first equation $A_{1}^{co,0}$ was obtained by Reed (1977), Mackowski (1989) as they did not include thermal radiation. The second equation resembles the term used in the $fm$ approximation by Beresnev et al. (1993), and the last equation is our previously obtained result (see Eq. 37a). Fig. 5 clearly shows the good performance of $A_{1}^{co,bb}$, i.e. when the black body temperature is used. Surprisingly, the coefficient with no radiation $A_{1}^{co,0}$ and the one that assumes the particle to radiate with $T_{rad}$ both perform equally bad, although $A_{1}^{co,0}$ with $k_g = 0$ belongs to a boundary condition that does not allow a steady state solution of the heat transfer equation. As in the $co$ limit, the photophoretic force depends on $1/\bar{T}$, the histogram of the ratio of the mean temperature $\bar{T} = A_{1}^{co}$ and its numerically obtained value is also shown. It is mirrored at the ratio 1. In Paper 1 it was shown, that these two dimensionless variables

\[
\varphi_{rad} = \frac{\epsilon I_0 r_0}{k T_{rad}^4}, \quad \theta_{rad} = \frac{\sigma_{SB} T_{rad}^4}{I_0}.
\]

characterize different results of the heat transfer problem (scaled to unit sphere; here for omitted $k_g$). In Fig. 4 the ratio of $A_{1}^{co}$ and $A_{1}^{co}$ to their respective exact numerical values are plotted over $\varphi_{rad}$ and $\theta_{rad}$. From the plots one can conclude, that in the given parameter range (see Tab. 1), the relative error of $A_{1}^{co}$ and $A_{1}^{co}$ is less than 2\% for $\varphi_{rad} < 1$. Within the model, the results in Eq. 46 carry about the same error.
The dimensionless variables Eq. 56. The plot for Figure 4: Ratio of the particle temperature expansion coefficient $A_0^0$ based on two robust equations, that are the approximations for the interpolation model in Rohatschek (1995), the model is also assuming very low deviations from gas and mean surface temperature.

Additionally, for simplicity, Rohatschek (1995) omitted $k_g$ in $A_1^{co,0} = \frac{1}{2} \left( \frac{1}{\pi^2 \eta^2} \right)$ (Eq. 55a) in his model so that $A_1^{co,0}$ is equal to $A_1^{fm,0} = \frac{r_0}{T} (h = 0, \text{ too})$. For very low gas heat conductivities $k_g$ such as air this will not introduce a significant error, but for hydrogen-helium gases it will. Beside that, the introduced error will grow strongly as the discussed particles get larger (see Paper 1). In our model this is not the case anymore. But for low intensities $I$ and low gas heat conductivity $k_g$, the interpolation proposed in this paper is basically the same as in Rohatschek (1995), which performs well (see Rohatschek (1995) and Sec. 4) within its scope. We calculated the changes of $F$ and $\hat{p}$ with respect to the values obtained with the model from Rohatschek (1995) ($F_R$ and $\hat{p}_R$). For extreme values, the force ratio $\hat{F}/F_R$ can reach values between 2.7·10^{-5} and 2.7. The minimum pressure ratio $\hat{p}/p_R$ can be as low as 0.13, the maximum one 1.8.

Fig. 5 shows photophoretic forces for these extreme values as well as two more realistic cases in comparison to the predictions made in Rohatschek (1995). The corresponding parameters and values are listed in Tab. 3. We chose a laser illuminated mm-sized particle in a cooled experimental setup and a particle in an astrophysical context as example studies which results in force/pressure ratios of $\hat{F}/F_R$ = 0.34 with $\hat{p}/p_R$ = 0.72, and $\hat{F}/F_R$ = 0.17 with $\hat{p}/p_R$ = 0.15, respectively. Both values — especially in the last case — show significantly different predictions. However, experimental investigations on the interpolation for high intensities are subject to future work at this moment and beyond the scope of this paper. One should mention here that rotation of illuminated particles — which are observed especially in experimental studies e.g. by van Eymeren & Wurm (2012) — have high influence on the photophoretic force (Loesche et al. 2014).

As for high $I$ the temperature-dependence of $k_g$ and $\eta_{dyne}$ can be important, the mean temperature Eq. 45c can be iteratively calculated if $k_g = k_g(T)$. Then, the force in the $co$ limit $F_{co} = F_{co}(k(T), k_g(T), \eta_{dyne}(T))$ with $T = A_0^{co}$ can be obtained. In the $fm$ regime, the mean temperature Eq. 51a determines the heat conductivity $k = k(T = A_0^{fm})$, and therefore the force.

5.3. Changes for the entire range of pressures

As the $fm$ approximation shows even smaller relative errors for the same parameter sweep (Paper 1), the interpolation is based on two robust equations, that are the $fm$ and $co$ limit approximation of the photophoretic force.

In the following we discuss the predictions of this model for the entire range of pressures and compare them to those made in Rohatschek (1995):

\[ \hat{F}_R = \Xi_R \sqrt{\frac{\alpha}{2}} \frac{r_0}{s} I_1 J \quad (57a) \]
\[ \hat{p}_R = \sqrt{\frac{\alpha}{2}} p^* \quad (57b) \]
\[ \Xi_R = \frac{\pi}{2} \sqrt{\frac{\pi}{3}} \frac{v_{th} \eta_{dyne}}{T_{co}} \quad (57c) \]
\[ p^* = \frac{1}{2} \sqrt{3\pi} \frac{v_{th} \eta_{dyne}}{r_0} . \quad (57d) \]

In the model introduced in Paper 1 (Loesche et al. 2016) as well as here we incorporate possible temperature differences in the model.
between the illuminated object and the surrounding gas. This also includes the case of higher radiative fluxes $I$. The solutions for the free molecule regime (Paper 1) and the slip flow regime (Eq. (11)) can be calculated using the given formulae. The usage of the interpolation between the $fm$ and $co$ regimes is more complicated. The basic approximation in this paper follows (for simplicity, omitting $h$)

$$F_{\text{phot}} = \frac{2 \hat{F}}{p + \hat{p}}$$

$$\hat{F} = \frac{\pi}{2} \sqrt{\frac{2}{3}} \frac{v_{\text{th}} \eta_{\text{dyn}}}{\sqrt{\tau_{\text{co}} T_{\infty} r_{0}}} J_{1} I$$

$$\hat{p} = \sqrt{\frac{\tau_{\text{co}}}{2}} \frac{v_{\text{th}} \eta_{\text{dyn}}}{r_{0}}$$

with the mean thermal speed of the gas

$$v_{\text{th}} = \sqrt{\frac{8 \rho}{\pi \rho}}$$

and the mean temperatures

$$A_{0}^{\text{co}} = \frac{I J_{0} + \frac{h}{r_{0}} T_{\infty} + \sigma_{\text{SB}} \epsilon \left(3 T_{\text{bb}} + T_{\text{rad}}^{4}ight)}{\frac{k_{\text{B}}}{r_{0}} + 4 \sigma_{\text{SB}} \epsilon T_{\text{bb}}}$$

$$T_{g}^{\infty} = T_{\infty} + \alpha \left(\frac{I J_{0} + \sigma_{\text{SB}} \epsilon \left(3 T_{\text{bb}}^{4} + T_{\text{rad}}^{4}\right)}{4 \sigma_{\text{SB}} \epsilon T_{\text{bb}}} - T_{\infty}\right)$$

and the scaling factors

$$\tau_{\text{co}} = \frac{\sqrt{A_{0}^{\text{co}}}}{\sqrt{\frac{1}{r_{0}} + 4 \sigma_{\text{SB}} \epsilon T_{\text{bb}}}}$$

$$\tau_{\text{co}} = \frac{\frac{k_{\text{B}}}{r_{0}} + 2 \frac{h}{r_{0}} + 4 \sigma_{\text{SB}} \epsilon T_{\text{bb}}}{\sqrt{A_{0}^{\text{co}}}}$$

The importance of this model considering strong temperature deviations and high intensities for longitudinal photophoresis becomes apparent when calculating drift motion of dust particles in a (pre-)transitional protoplanetary disk, where the mean free path of the gas is often in the same order as the particles diameters. Especially near the central star the temperatures of the illuminated particles can get significantly higher than the temperature of the surrounding gas. Since photophoresis can dominate the force balance for small particles, the accuracy of the approximation used is highly important and therefore the model given in this paper has to be favored. Also, particles illuminated with lasers (Daun et al. 2008, Loesche et al. 2014) can lead to rather extreme conditions, previously not supported by approximations for the $fm$ and transition regimes.

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A. SUPPLEMENTARIES

A.1. One orthogonality relation for associated Legendre polynomials

$$\int_{-1}^{1} P_{\nu}^{\ell}(x) P_{\nu}^{\ell}(x) \, dx = \delta_{\nu \nu} \frac{2}{2 \nu + 1} (\nu + \mu)!$$

(A.1)

A.2. Average

The mean temperature of the scattered gas $T_{g}^{\infty}$ ($fm$, see Paper 1) is (with $\alpha$ denoting the thermal accommodation coefficient)

$$\overline{T_{g}^{\infty}} = T_{\infty} + \alpha \left(\overline{T} - T_{\infty}\right).$$

(A.2)

A.3. Transport numbers

$$P_{e} = Re Pr = \frac{\rho c_{p} u l}{k}$$

(A.3)

$$Re = \frac{\rho u l}{\eta_{\text{dyn}}}$$

(A.4)

$$Pr = \frac{c_{p}}{k} \eta_{\text{dyn}}$$

(A.5)

A.4. Force

The force exerted onto the suspended particle is given by

$$\mathbf{F} = \int \mathbf{\Pi} \cdot d\mathbf{A}$$

(A.6a)

$$\mathbf{\Pi} = -\rho \mathbf{v} \otimes \mathbf{v} + \mathbf{\sigma}.$$  

(A.6b)
Here, due to the symmetry of the problem, only \( F_z \) is not zero. As it is \( dA = n \, dA \) where \( n = e_r \), is the normal vector, the product \( \Pi \cdot n \) has to be determined

\[
\Pi \cdot n = \Pi \cdot e_r = \begin{bmatrix}
\Pi_{rr} & \Pi_{r\xi} & \Pi_{r\eta} \\
\Pi_{\xi r} & \Pi_{\xi\xi} & \Pi_{\xi\eta} \\
\Pi_{\eta r} & \Pi_{\eta\xi} & \Pi_{\eta\eta}
\end{bmatrix} \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix} = \begin{bmatrix}
\Pi_{rr} \\
\Pi_{\xi r} \\
\Pi_{\eta r}
\end{bmatrix} = \Pi_{rr} e_r + \Pi_{\xi r} e_\xi + \Pi_{\eta r} e_\eta.
\]

As \( \Pi \) is given by Eqs. A.6b and 6, the respective parts in spherical coordinates, and with incompressibility are

\[
R_{rr} = \frac{2 \eta_{\text{dyn}}}{r} \frac{\partial v_r}{\partial r},
\]

\[
R_{\xi r} = \frac{\eta_{\text{dyn}}}{r} \left( \frac{1}{r} \frac{\partial v_r}{\partial r} + \frac{\partial}{\partial r} \left( \frac{V_\xi}{r} \right) \right),
\]

\[
\Pi_{rr} = \begin{bmatrix}
\Pi_{rr} \\
\Pi_{\xi r} \\
\Pi_{\eta r}
\end{bmatrix} = \begin{bmatrix}
\Pi_{rr} \\
0 \\
0
\end{bmatrix},
\]

\[
\Pi_{\xi r} = 0.
\]

\( \Pi_{\xi r} = 0 \) as \( v_\xi = 0 \), and \( v \) is independent of \( \xi \). The \( z \)-component of the product in Eq. A.7 is

\[
(\Pi \cdot n)_z = \Pi_{rr} \cos \xi - \Pi_{\xi r} \sin \xi,
\]

and therefore the \( z \)-component of the force (Eq. A.6a) reads

\[
F_z = 2 \pi r_0^3 \int_0^1 d\xi \sin \xi \left( \Pi_{rr} \cos \xi - \Pi_{\xi r} \sin \xi \right).
\]
| Symbol | Description |
|--------|-------------|
| $J_v$  | asymmetry factor (dimensionless, Eq. 32) |
| $k$    | thermal conductivity of suspended particle in W m$^{-1}$ K$^{-1}$ |
| $k_g$  | thermal conductivity of the gas |
| $\eta_{\text{kin}}, \eta_{\text{dyn}}$ | kinematic and dynamic viscosity, $\eta_{\text{kin}} = \eta_{\text{dyn}}/\rho$, $\eta_{\text{dyn}}$ in Pa s |
| $Pe$   | Péclet number (Eq. A.3) |
| $Re$   | Reynolds number (Eq. A.4) |
| $Pr$   | Prandtl number (Eq. A.5) |
| $h$    | heat transfer coefficient (Eq. 51b) in W m$^{-2}$ K$^{-1}$ |
| $I$    | effective intensity $I = \varepsilon I_0$ in W m$^{-1}$ |
| $\varepsilon$ | (mean) emissivity |
| $\sigma_{\text{SB}}$ | Stefan-Boltzmann constant in W m$^{-2}$ K$^{-4}$ |
| $\lambda$ | mean free path of the gas in m |
| $Kn$   | Knudsen number (dimensionless, Eq. 1) |
| $q$    | normalized source function (Eq. 29) in m$^{-1}$ |
| $A_\nu, B_\nu, C_\nu, q_\nu$ | expansion coefficients ($\nu \geq 0$) |
Table 3: Changes of $\hat{F}$ and $\hat{p}$ with respect to the values obtained with the model from [Rohatschek 1995]. Extreme situations as well as two example studies are also sketched in Fig. [5].

|           | $I$  | $k$   | $r_0$ | $k_g$    | $T_{\infty}$ | $T_{rad}$ | $\hat{F}/\hat{F}_R$ | $\hat{p}/\hat{p}_R$ |
|-----------|------|-------|-------|----------|--------------|-----------|----------------------|----------------------|
| CASE I:   | $10^4$ | 1     | $10^{-3}$ | $2\cdot10^{-2}$ | 70           | 70        | 0.34                 | 0.72                 |
| CASE II:  | $10^3$ | $10^{-2}$ | $10^{-4}$ | 0.2      | 500          | 3         | 0.17                 | 0.14                 |
| MAX($\hat{F}/\hat{F}_R$) | 520   | 8     | $1.1\cdot10^{-3}$ | $10^{-3}$ | 1500         | 250       | 2.7                  | 1.2                  |
| MIN($\hat{F}/\hat{F}_R$) | $4\cdot10^4$ | $10^{-3}$ | $1.1\cdot10^{-3}$ | $2\cdot10^{-2}$ | 10          | 1500      | $2.7\cdot10^{-5}$ | 0.28                 |
| MAX($\hat{p}/\hat{p}_R$) | 6900  | 8     | $1.1\cdot10^{-3}$ | $10^{-3}$ | 1500         | 1         | 1.3                  | 2.4                  |
| MIN($\hat{p}/\hat{p}_R$) | 10    | $10^{-3}$ | $8.7\cdot10^{-6}$ | $2\cdot10^{-2}$ | 10          | 1500      | 0.22                 | $2\cdot10^{-3}$     |
