DIMENSIONAL ANALYSIS OF FRACTAL INTERPOLATION FUNCTIONS

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Abstract. We provide a rigorous study on dimensions of fractal interpolation function defined on a closed and bounded interval of \( \mathbb{R} \) which is associated to a continuous function with respect to a base function, scaling functions and a partition of the interval. In particular, we provide an exact estimation of the box dimension of \( \alpha \)-fractal functions.

1. INTRODUCTION

The idea of fractal interpolation functions was introduced by Barnsley [3]. Many authors attempted to calculate the box and Hausdorff dimensions of the graph of fractal interpolation functions corresponding to a data set, see for instance \([3, 14, 18, 23, 27]\). Furthermore, some authors \([19, 29]\) have studied the properties of fractal interpolation functions corresponding to a data set.

Here we first start with iterated function system, for more details see \([4]\).

1.1. Iterated Function System. Let \((X, d)\) be a complete metric space, and let \(H(X)\) be the family of all nonempty compact subsets of \(X\). We define the Hausdorff metric

\[
h(A, B) := \max \left\{ \max_{a \in A} \min_{b \in B} d(a, b), \max_{b \in B} \min_{a \in A} d(a, b) \right\}.
\]

It is well known that \((H(X), h)\) is a complete metric space. Let \(k\) be a positive integer, and let, for \(i = 1, 2, \ldots, k\), \(w_i\) be contraction selfmap of \(X\), i.e., there exist real numbers \(R_i \in [0, 1)\) such that

\[d(w_i(x), w_i(y)) \leq R_id(x, y) \quad \forall \ x, y \in X.
\]

Definition 1.1. The system \(\{(X, d); w_1, w_2, \ldots, w_k\}\) is called an iterated function system, IFS for short.

The IFS generates the mapping \(W\) from \(H(X)\) into \(H(X)\) given by

\[W(A) = \bigcup_{i=1}^{k} w_i(A).
\]

The Hutchinson-Barnsley map \(W\) defined above is then a contraction mapping, with respect to the Hausdorff metric \(h\), the Lipschitz constant \(R_\ast := \max\{R_1, R_2, \ldots, R_k\}\).

Thus, by the Banach contraction principle, there exists a unique nonempty compact subset \(A\) such that \(A = \bigcup_{i=1}^{k} w_i(A)\). Such a set \(A\) is termed the attractor of the IFS.

The reader is referred to \([3, 4, 20]\) for the upcoming technical introduction. The method of constructing fractal interpolation functions (FIFs) is as follows:

Key words and phrases. Iterated function systems, Fractal interpolation functions, Hausdorff dimension, Box dimension, Open set condition.
1.2. Fractal Interpolation Functions. Consider a set of interpolation points \( \{(x_n, y_n) : n = 1, 2, \ldots, N\} \) with strictly increasing abscissa. Set \( J = \{1, 2, \ldots, N - 1\} \), \( I = [x_1, x_N] \) and for \( j \in J \), let \( I_j = [x_j, x_{j+1}] \). For \( j \in J \), let \( L_j : I \to I_j \) be a contraction homeomorphism such that
\[
L_j(x_1) = x_j, L_j(x_N) = x_{j+1}, j \in J.
\]
For \( j \in J \), let \( F_j : I \times \mathbb{R} \to \mathbb{R} \) be a mapping satisfying
\[
|F_j(x, y) - F_j(x, y_*)| \leq \kappa_j |y - y_*|,
\]
\[
F_j(x_1, y_1) = y_j, F_j(x_N, y_N) = y_{j+1}, j \in J,
\]
where \((x, y), (x, y_*) \in K\) and \(0 \leq \kappa_j < 1\) for all \( j \in J \). We shall take
\[
L_j(x) = a_j x + b_j, \quad F_j(x, y) = \alpha_j y + q_j(x).
\]
In the above expressions \( a_j \) and \( b_j \) are determined so that the conditions \( L_j(x_1) = x_j, L_j(x_N) = x_{j+1} \) are satisfied. The multipliers \( \alpha_j \), called scaling factors, are such that \(-1 < \alpha_j < 1\) and \( q_j : I \to \mathbb{R} \), \( j \in J \) are suitable continuous functions satisfying the “join-up conditions” imposed for the bivariate maps \( F_j \). That is, \( q_j(x_1) = y_j - \alpha_j y_1 \) and \( q_j(x_N) = y_{j+1} - \alpha_j y_N \) for all \( j \in J \). Now define functions \( W_j : I \times \mathbb{R} \to I \times \mathbb{R} \) for \( j \in J \) by
\[
W_j(x, y) = (L_j(x), F_j(x, y)).
\]
Theorem 1 in \[3\] says that the IFS \( \mathcal{I} := \{I \times \mathbb{R}; W_1, W_2, \ldots, W_{N-1}\} \) defined above has a unique attractor which is the graph of a function \( g \) which satisfies the following functional equation reflecting self-referentiality:
\[
g(x) = \alpha_j g(L_j^{-1}(x)) + q_j(L_j^{-1}(x)), x \in I_j, j \in J.
\]
In \[3\], Barnsley gave an estimate for the Hausdorff dimension of an affine FIF. Falconer \[1\] also estimated the Hausdorff dimension of an affine FIF. Further, in \[4, 5, 14\], Barnsley and his collaborators calculated the box dimensions for classes of affine FIFs. In \[6\], Barnsley and Massopust computed the box dimensions of FIFs generated by bilinear maps. In \[15\], Hardin and Massopust produced a formula for the box dimension of vector-valued multivariate FIFs. In this article, we focus on dimensions of a special type of FIFs known as \( \alpha \)-fractal function.

1.3. \( \alpha \)-Fractal Functions: a Fractal Perturbation Process. The idea in the construction of a FIF can be adapted to obtain a class of fractal functions associated with a prescribed continuous function on a compact interval in \( \mathbb{R} \). To this end, as is customary, let us denote by \( C(I) \), the space of all continuous real-valued functions defined on a compact interval \( I = [x_1, x_N] \) in \( \mathbb{R} \). We shall endow \( C(I) \) with the uniform norm. Let \( f \in C(I) \) be prescribed, referred to as the germ function. Let us consider the following elements to construct the IFS.

1. A partition \( \Delta := \{x_1, x_2, \ldots, x_N : x_1 < x_2 < \cdots < x_N\} \) of \( I = [x_1, x_N] \).
2. For each \( j \in J \), let \( \alpha_j : I \to \mathbb{R} \) be continuous functions with \( \|\alpha_j\|_\infty = \max_j ||\alpha_j||_\infty < 1 \). These functions are referred to as scaling functions. Consider \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{N-1}) \in (C(I))^{N-1} \), referred to as scaling vector.
3. A continuous function \( b : I \to \mathbb{R} \) such that \( b(x_1) = f(x_1), b(x_1) = f(x_N) \) and \( b \neq f \), termed base function.
Let us define maps

\[ L_j(x) = a_j x + b_j, \]
\[ F_j(x, y) = \alpha_j(x)y + f(L_j(x)) - \alpha_j(x)b(x). \]  

(1.1)

By Theorem 1 in [3] one can see that the corresponding IFS \( \mathcal{I} := \{ I \times \mathbb{R}; W_1, W_2, \ldots, W_{N-1} \} \), where

\[ W_j(x, y) = \left( L_j(x), F_j(x, y) \right), \]

has a unique attractor, which is the graph of a continuous function, denoted by \( f_{\Delta, b}^{\alpha} \) and its graph has a non-integer Hausdorff-Besicovitch dimension. One can treat \( f_{\Delta, b}^{\alpha} \) as a “fractal perturbation” of \( f \).

**Definition 1.2.** The function \( f_{\Delta, b}^{\alpha} \) is called the \( \alpha \)-fractal function corresponding to \( f \) with respect to \( \Delta \) and \( b \).

There has been a great interest to study the properties of \( \alpha \)-fractal function \( f^{\alpha} \), the reader is referred to [20] [21]. Akhtar et-al [1] calculated the box dimension of graph of \( f^{\alpha} \). Recently the authors of [25] estimated the box dimension of the graph of \( f^{\alpha} \). We refer the reader to [26] for the bivariate \( \alpha \)-fractal functions and further developments. In this article, we study the box and Hausdorff dimensions of graph of \( f^{\alpha} \) deeply. Our results are generalizing certain existing results in an exciting manner.

We skip definitions of the box and Hausdorff dimensions and refer the reader to [10] for their definitions.

## 2. Main Theorems

**Note 2.1.** We define a metric \( d \) on \( I \times \mathbb{R} \) as follows

\[ d((x, y), (z, w)) = c_1|x - z| + c_2|(y - f^{\alpha}(x)) - (w - f^{\alpha}(z))| \quad \forall (x, y), (z, w) \in I \times \mathbb{R}. \]

Then \( d \) is a metric on \( I \times \mathbb{R} \). Furthermore, \( (I \times \mathbb{R}; d) \) is a complete metric space.

**Remark 2.2.** We recall the functional equation

\[ f^{\alpha}(x) = f(x) + \alpha_j(L_j^{-1}(x)).(f^{\alpha} - b)(L_j^{-1}(x)) \quad \forall x \in I_j, j \in J. \]

Now for every \( x \in I_j \) and \( j \in J \), we have

\[
|f^{\alpha}(x) - f(x)| = |\alpha_j(L_j^{-1}(x)).(f^{\alpha} - b)(L_j^{-1}(x))|
= |\alpha_j(L_j^{-1}(x))|| (f^{\alpha} - b)(L_j^{-1}(x))|
\leq ||\alpha_j||_{\infty}|| f^{\alpha} - b||_{\infty}
\leq ||\alpha||_{\infty}|| f^{\alpha} - b||_{\infty}.
\]
The above implies that \( \| f^\alpha - f \|_\infty \leq \| \alpha \|_\infty \| f^\alpha - b \|_\infty \). Using triangle inequality we obtain \( \| f^\alpha - f \|_\infty \leq \| \alpha \|_\infty \| f^\alpha - f \|_\infty + \| \alpha \|_\infty \| f - b \|_\infty \). Finally we have \( \| f^\alpha - f \|_\infty \leq \frac{\| \alpha \|_\infty}{1 - \| \alpha \|_\infty} \| f - b \|_\infty \). Further, we have
\[
\| f^\alpha \|_\infty - \| f \|_\infty \leq \| f^\alpha - f \|_\infty \leq \frac{\| \alpha \|_\infty}{1 - \| \alpha \|_\infty} \| f - b \|_\infty .
\]
Therefore, we get \( \| f^\alpha \|_\infty \leq \| f \|_\infty + \frac{\| \alpha \|_\infty}{1 - \| \alpha \|_\infty} \| f - b \|_\infty := M \).

We should mention that the techniques involved in the proof of the next proposition is same as [4, Theorem 4].

**Proposition 2.3.** The map \( W_j : I \times [-M, M] \rightarrow I \times [-M, M] \) is a contraction map with respect to the above metric provided
\[
\max \left\{ a_j + \frac{2c_2 Mk_{\alpha_j}}{c_1}, \| \alpha_j \|_\infty \right\} < 1
\]
and \( \alpha_j : I \rightarrow \mathbb{R} \) satisfies \( |\alpha_j(x) - \alpha_j(y)| \leq k_{\alpha_j}|x - y| \).

**Proof.** Let \( (x, y), (z, w) \in I \times [-M, M] \). We have
\[
d(W_j(x, y), W_j(z, w)) = c_1|L_j(x) - L_j(z)|
\]
\[
= c_2 \left| \alpha_j(x)y + f(L_j(x)) - \alpha_j(x)b(x) - f^\alpha(L_j(x)) \right|
\]
\[
= c_2 \left| \alpha_j(z)w + f(L_j(z)) - \alpha_j(z)b(z) - f^\alpha(L_j(z)) \right|
\]
\[
\leq c_2 |\alpha_j(x) - \alpha_j(z)| (w - f^\alpha(z))
\]
Further, we get
\[
d(W_j(x, y), W_j(z, w)) \leq c_1|L_j(x) - L_j(z)|
\]
\[
+ c_2 |\alpha_j|_\infty |(y - f^\alpha(x)) - (w - f^\alpha(z))| + 2M c_2 k_{\alpha_j} |x - z|
\]
\[
= c_1 |\alpha_j|_\infty |x - z|
\]
\[
+ c_2 |\alpha_j|_\infty |(y - f^\alpha(x)) - (w - f^\alpha(z))| + 2M c_2 k_{\alpha_j} |x - z|
\]
\[
= \left( a_j + \frac{2c_2 Mk_{\alpha_j}}{c_1} \right) c_1 |x - z|
\]
\[
+ c_2 |\alpha_j|_\infty |(y - f^\alpha(x)) - (w - f^\alpha(z))|
\]
\[
\leq \max \left\{ a_j + \frac{2c_2 Mk_{\alpha_j}}{c_1}, |\alpha_j|_\infty \right\} \left( c_1 |x - z| \right.
\]
\[
+ c_2 (y - f^\alpha(x)) - (w - f^\alpha(z)) \right) \right)
\]
\[
= \max \left\{ a_j + \frac{2c_2 Mk_{\alpha_j}}{c_1}, |\alpha_j|_\infty \right\} d((x, y), (z, w)) .
\]
This completes the proof. \( \square \)
We say that an IFS \( \{ X; w_1, w_2, \ldots, w_k \} \) satisfies the open set condition if there exists a non-empty open set \( U \) with
\[
\bigcup_{i=1}^{k} w_i(U) \subset U
\]
and terms present in the above union are disjoint. Further, if the above \( U \) satisfies \( U \cap A \neq \emptyset \) then we call that the IFS satisfies the strong open set condition. Now we are ready to prove the following.

**Theorem 2.4.** Let \( \mathcal{I} := \{ I \times \mathbb{R}; W_1, W_2, \ldots, W_{N-1} \} \) be the IFS as defined earlier such that
\[
|r_i||(x, y) − (w, z)||_2 ≤ ||W_j(x, y) − W_j(w, z)||_2 ≤ R_i||(x, y) − (w, z)||_2,
\]
for every \((x, y), (w, z)\) \( \in I \times \mathbb{R} \), where \( 0 < r_i ≤ R_i < 1 \) \( \forall i \in \{1, 2, \ldots, N−1\} \). Then \( s_* ≤ \dim_H(\text{Graph}(f^\alpha)) ≤ s^* \), where \( s_* \) and \( s^* \) are determined by \( \sum r_i^n = 1 \) and
\[
\sum_{i=1}^{N} R_i^n = 1 \text{ respectively.}
\]

**Proof.** Following Proposition 9.6 in [10] we have the required upper bound. For the lower bound of Hausdorff dimension of \( \text{Graph}(f^\alpha) \) we proceed as follows.

Define \( U = (x_1, x_N) \times \mathbb{R} \). It is plain to see that
\[
L_i((x_1, x_N)) \cap L_j((x_1, x_N)) = \emptyset,
\]
for every \( i, j \in J := \{1, 2, \ldots, N−1\} \) with \( i \neq j \). This immediately yields
\[
W_i(U) \cap W_j(U) = \emptyset,
\]
for every \( i, j \in J \) satisfying \( i \neq j \). Using \( U \cap \text{Graph}(f^\alpha) \neq \emptyset \), one deduces that the IFS satisfies the SOSC. Since \( U \cap \text{Graph}(f^\alpha) \neq \emptyset \), we have an index \( i \in J^* \) with \( \text{Graph}(f^\alpha)_i \subset U \), where \( J^* := \cup_{m \in \mathbb{N}} \{1, 2, \ldots, N−1\}^m \), that is, the set of all finite sequences which are made up of the elements of \( J \) and \( \text{Graph}(f^\alpha)_i := W_i(\text{Graph}(f^\alpha)) := W_i \circ W_{i_2} \circ \cdots \circ W_{i_m}(\text{Graph}(f^\alpha)) \) for \( i \in J^m \) (\( m \)-times Cartesian product of \( J \) with itself) and for \( m \in \mathbb{N} \). Now, it is obvious that for any \( n \) and \( j \in J^n \), the sets \( \text{Graph}(f^\alpha)_{ij} \) are disjoint. Furthermore, the IFS \( \{ W_{i_j} : j \in J^n \} \) satisfies the hypotheses of Proposition 9.7 in [10]. Therefore, with the notation \( r_j = r_{j_1}r_{j_2} \cdots r_{j_n} \) we have \( s_n ≤ \dim_H(G^*) \), where \( G^* \) is an attractor of the above-said IFS and \( \sum_{j \in J^n} r_j^n = 1 \). Since \( G^* \subset \text{Graph}(f^\alpha) \), we get \( s_n ≤ \dim_H(G^*) \leq \dim_H(\text{Graph}(f^\alpha)) \). Suppose that \( \dim_H(\text{Graph}(f^\alpha)) < s_*, \) where \( \sum_{i=1}^{N−1} r_i^n = 1 \). This gives \( s_n < s_* \). Using the previous estimates, we have
\[
(2.1) \quad r_i^{-r_i^n} \geq \sum_{j \in J^n} r_j^n \dim_H(\text{Graph}(f^\alpha)) \geq \sum_{j \in J^n} r_j^n \dim_H(\text{Graph}(f^\alpha)) \cdot r_j^n \geq \sum_{j \in J^n} r_j^n \max_{\text{max}} \dim_H(\text{Graph}(f^\alpha)) \cdot r_j^n = r_n(\dim_H(\text{Graph}(f^\alpha)) \cdot s_*),
\]
where \( r_{\text{max}} = \max\{r_1, r_2, \ldots, r_{N-1}\} \). Since \( r_{\text{max}} < 1 \) and the term on left side in the above expression is bounded, we have a contradiction as \( n \) tends to infinity. Thus our claim is wrong. This implies that \( \dim_H(\text{Graph}(f^\alpha)) \geq s_* \), which is the required result. \( \square \)

**Remark 2.5.** Under the assumptions of Proposition 2.3 we may find a upper bound for the Hausdorff dimension of graph of \( \alpha \)-fractal function \( f^\alpha \) using the above theorem.

**Remark 2.6.** Note that the open set \( (0, 1) \times \mathbb{R} \) will serve for our purpose to satisfy the strong open set condition for \( \{I \times \mathbb{R}; W_j : j = 1, 2, \ldots, N-1\} \). With the aid of the above theorem we are able to improve Theorem 4 in [3]. In particular, with the notation in [3] we can omit the following condition from that theorem:

\[
t_1, t_N \leq (\text{Min}\{a_1, a_N\})(\sum_{n=1}^{N-1} t_n^{l})^{1/l}.
\]

**Remark 2.7.** Schief [24] followed the technique of Bandt [2] and proved that open set and strong open set conditions are equivalent for similitudes. Further Peres, Rams, Simon, and Solomyak [22] showed Schief’s theorem for self-conformal maps. The same result with a different approach can be seen in [12, 16, 30]. We do not know whether or not the open set and strong open set conditions are equivalent for \( \alpha \)-fractal functions.

**Remark 2.8.** It is known that for a pure self-similar set or self-conformal set \( \mathcal{A} \), \( \dim_H(\mathcal{A}) = \dim_B(\mathcal{A}) = \dim_{\overline{B}}(\mathcal{A}) \), for more details see [11]. Note that the nature of \( \alpha \)-fractal functions depend on the IFS parameter. In particular, one can obtain pure self-similar or partial self similar \( \alpha \)-fractal functions by choosing suitable scaling functions and thus for \( \alpha \)-fractal functions we may or may not get the equal dimensions.

**Remark 2.9.** Here we talk about continuity of the Hausdorff dimension. Note that, in general Hausdorff dimension is not a continuous function. For example, \( A_n := [-\frac{1}{p^n}, \frac{1}{p^n}] \to A := \{0\} \) in Hausdorff metric but \( \dim_H(A_{\mathcal{A}}) = 1 \) does not converge to \( \dim_H(A) = 0 \). In [25], the continuous dependence of \( \alpha \)-fractal function on the parameters is studied. One may pose a question of continuity of the Hausdorff dimension of \( \alpha \)-fractal function with respect to the parameters involved. However, it seems that the result may not hold in general.

### 3. Oscillation Spaces

We refer the reader to [7, 9] for oscillation spaces. Let \( Q \subset [0, 1] \) \( p \)-adic subinterval so that \( Q = \left[ \frac{i}{p^m}, \frac{i+1}{p^m} \right] \) for some integers \( m \geq 0 \) and \( 0 \leq i < \frac{1}{p^m} \). For a continuous function \( f : [0, 1] \to \mathbb{R} \), we define oscillation of \( f \) over \( Q \)

\[
R_f(Q) = \sup_{x, y \in Q} |f(x) - f(y)| = \sup_{x \in Q} f(x) - \inf_{y \in Q} f(y),
\]

and total oscillation of order \( m \),

\[
\text{Osc}(m, f) = \sum_{|Q|=p^{-m}} R_f(Q),
\]
where the sum ranges over all $p$-adic intervals $Q \subset [0, 1]$ of length $|Q| = \frac{1}{p^m}$.

Let $\beta \in \mathbb{R}$. We define the oscillation space $\mathcal{V}^\beta(I)$ by

$$
\mathcal{V}^\beta(I) = \left\{ f \in \mathcal{C}(I) : \sup_{m \in \mathbb{N}} \frac{\text{Osc}(m, f)}{p^m(1-\beta)} < \infty \right\}.
$$

One can define

$$
\mathcal{V}^{\beta-}(I) = \{ f \in \mathcal{C}(I) : f \in \mathcal{V}^{\beta-}(I) \forall \epsilon > 0 \},
$$

and

$$
\mathcal{V}^{\beta+}(I) = \{ f \in \mathcal{C}(I) : f \notin \mathcal{V}^{\beta+}(I) \forall \epsilon > 0 \}.
$$

Now, let us write the next two theorems as a prelude.

**Theorem 3.1 ([7], Theorem 4.1).** Let $f$ be a real-valued continuous function defined on $I$, we have

$$
\dim_B(\text{Graph}(f)) \leq 2 - \gamma \iff f \in \mathcal{V}^{\gamma-}(I) \text{ if } 0 < \gamma \leq 1
$$

and

$$
\dim_B(\text{Graph}(f)) \geq 2 - \gamma \iff f \in \mathcal{V}^{\gamma+}(I) \text{ if } 0 \leq \gamma < 1.
$$

**Theorem 3.2 ([9], Theorem 3.1).** Let $f : I \to \mathbb{R}$ be a continuous function and let $0 < \gamma < 1$. Then we have

$$
\dim_B(\text{Graph}(f)) = 2 - \gamma \iff f \in \cap_{\beta<\gamma} \mathcal{V}^\beta(I) \cup \cup_{\beta>\gamma} \mathcal{V}^\beta(I).
$$

**Lemma 3.3.** Let $f, g \in \mathcal{C}(I)$ and $\lambda \in \mathbb{R}$. Then, for $m \in \mathbb{N}$ we have the following

1. $\text{Osc}(m, \lambda f) = |\lambda| \text{Osc}(m, f)$
2. $\text{Osc}(m, f + g) \leq \text{Osc}(m, f) + \text{Osc}(m, g)$
3. $\text{Osc}(m, fg) \leq \|g\|_\infty \text{Osc}(m, f) + \|f\|_\infty \text{Osc}(m, g)$.

**Proof.**

(1) For $m \in \mathbb{N}$ and $f, g \in \mathcal{C}(I)$, one proceeds as follows

$$
\text{Osc}(m, \lambda f) = \sum_{|Q|=p^{-m}} R_{\lambda f}(Q)
$$

$$
= \sum_{|Q|=p^{-m}} \sup_{x,y \in Q} |(\lambda f)(x) - (\lambda f)(y)|
$$

$$
= \sum_{|Q|=p^{-m}} |\lambda| \sup_{x,y \in Q} |f(x) - f(y)|
$$

$$
= |\lambda| \sum_{|Q|=p^{-m}} |f(x) - f(y)|
$$

$$
= |\lambda| \text{Osc}(m, f).
$$
(2) Turning to second item we have

\[ \text{Osc}(m, f + g) = \sum_{Q = p^{-m}} R_{f+g}(Q) \]

\[ = \sum_{Q = p^{-m}} \sup_{x, y \in Q} |(f + g)(x) - (f + g)(y)| \]

\[ \leq \sum_{Q = p^{-m}} \left( \sup_{x, y \in Q} |f(x) - f(y)| + \sup_{x, y \in Q} |g(x) - g(y)| \right) \]

\[ = \sum_{Q = p^{-m}} R_f(Q) + \sum_{Q = p^{-m}} R_g(Q) \]

\[ = \text{Osc}(m, f) + \text{Osc}(m, g). \]

(3) The third item follows through the following lines.

\[ \text{Osc}(m, fg) = \sum_{Q = p^{-m}} R_{fg}(Q) \]

\[ = \sum_{Q = p^{-m}} \sup_{x, y \in Q} |(fg)(x) - (fg)(y)| \]

\[ = \sum_{Q = p^{-m}} \sup_{x, y \in Q} |f(x)g(x) - f(y)g(x) + f(y)g(x) - f(y)g(y)| \]

\[ \leq \sum_{Q = p^{-m}} \left( \sup_{x, y \in Q} |g(x)||f(x) - f(y)| + \sup_{x, y \in Q} |f(y)||g(x) - g(y)| \right) \]

\[ \leq \sum_{Q = p^{-m}} \left( |g|_{\infty} \sup_{x, y \in Q} |f(x) - f(y)| + |f|_{\infty} \sup_{x, y \in Q} |g(x) - g(y)| \right) \]

\[ = |g|_{\infty} \sum_{Q = p^{-m}} R_f(Q) + |f|_{\infty} \sum_{Q = p^{-m}} R_g(Q) \]

\[ = |g|_{\infty} \text{Osc}(m, f) + |f|_{\infty} \text{Osc}(m, g). \]

Thus, the proof of the lemma is complete. \( \square \)

**Proposition 3.4.** Let \( f \in V^\beta(I) \), we define \( \|f\|_{V^\beta} := \|f\|_{\infty} + \sup_{m \in \mathbb{N}} \frac{\text{Osc}(m, f)}{p^m(1-\beta)}. \)

Then \( \|\cdot\|_{V^\beta} \) forms a norm on \( V^\beta(I) \).

**Proof.** Through simple and straightforward calculations, we have

\[ \|f\|_{V^\beta} = 0 \]

\[ \iff \|f\|_{\infty} = 0 \text{ and } \sup_{m \in \mathbb{N}} \frac{\text{Osc}(m, f)}{p^m(1-\beta)} = 0 \]

\[ \iff f = 0, \]

(2)

\[ \|\lambda f\|_{V^\beta} = \|\lambda f\|_{\infty} + \sup_{m \in \mathbb{N}} \frac{\text{Osc}(m, \lambda f)}{p^m(1-\beta)} \]

\[ = |\lambda| \|f\|_{\infty} + |\lambda| \sup_{m \in \mathbb{N}} \frac{\text{Osc}(m, f)}{p^m(1-\beta)} \]

\[ = |\lambda| \|f\|_{V^\beta}, \]

and
\begin{align*}
\|f + g\|_{V^\beta} &= \|f + g\|_\infty + \sup_{m \in \mathbb{N}} \frac{Osc(m, f + g)}{p^m(1-\beta)} \\
&\leq \|f\|_\infty + \|g\|_\infty + \sup_{m \in \mathbb{N}} \frac{Osc(m, f + g)}{p^m(1-\beta)} \\
&\leq \|f\|_\infty + \|g\|_\infty + \sup_{m \in \mathbb{N}} \frac{Osc(m, f)}{p^m(1-\beta)} + \sup_{m \in \mathbb{N}} \frac{Osc(m, g)}{p^m(1-\beta)} \\
&= \|f\|_{V^\beta} + \|g\|_{V^\beta},
\end{align*}

hence the proof. \hfill \Box

**Lemma 3.5.** Let \((f_n)\) be a sequence of continuous functions that converges uniformly to some \(f : I \to \mathbb{R}\) and \(m \in \mathbb{N}\), then we have
\[
Osc(m, f_n) \to Osc(m, f).
\]
Furthermore, let \((f_n)\) be a sequence in \(V^\beta(I)\) that converges uniformly to some \(f : I \to \mathbb{R}\), then we have
\[
\sup_{m \in \mathbb{N}} \frac{Osc(m, f)}{p^m(1-\beta)} \leq \liminf_{n \to \infty} \sup_{m \in \mathbb{N}} \frac{Osc(m, f_n)}{p^m(1-\beta)}.
\]

**Proof.** Let \(m \in \mathbb{N}\), we have
\[
\begin{align*}
\lim_{n \to \infty} Osc(m, f_n) &= \lim_{n \to \infty} \sum_{|Q|=p^{-m}} R_{f_n}(Q) \\
&= \lim_{n \to \infty} \sum_{|Q|=p^{-m}} \sup_{x,y \in Q} |f_n(x) - f_n(y)| \\
&= \sum_{|Q|=p^{-m}} \lim_{n \to \infty} \sup_{x,y \in Q} |f_n(x) - f_n(y)| \\
&= \sum_{|Q|=p^{-m}} \sup_{x,y \in Q} |f(x) - f(y)| \\
&= Osc(m, f).
\end{align*}
\]

Now for \(m \in \mathbb{N}\), we get
\[
\begin{align*}
\frac{Osc(m, f)}{p^m(1-\beta)} &= \sum_{|Q|=p^{-m}} \sup_{x,y \in Q} |f(x) - f(y)| \\
&= \sum_{|Q|=p^{-m}} \frac{\sup_{x,y \in Q} |f_n(x) - f_n(y)|}{p^m(1-\beta)} \\
&= \lim_{n \to \infty} \frac{\sum_{|Q|=p^{-m}} \sup_{x,y \in Q} |f_n(x) - f_n(y)|}{p^m(1-\beta)} \\
&= \lim_{n \to \infty} \frac{Osc(m, f_n)}{p^m(1-\beta)} \\
&\leq \liminf_{n \to \infty} \left( \sup_{m \in \mathbb{N}} \frac{Osc(m, f_n)}{p^m(1-\beta)} \right).
\end{align*}
\]

Thus, the proof of the lemma is complete. \hfill \Box
We claim that
\[ \|f_n - f_k\|_{V^\beta} < \epsilon \quad \forall \ n, k \geq n_0. \]
By definition of \( \|\cdot\|_{V^\beta} \) one gets \( \|f_n - f_k\|_{V^\beta} < \epsilon \quad \forall n, k \geq n_0. \) Since \( (C(I), \|\cdot\|) \) is a Banach space, we have a continuous function \( f \) with \( \|f_n - f\|_{\infty} \to 0 \) as \( n \to \infty. \) We claim that \( f \in V^\beta(I) \) and \( \|f_n - f\|_{V^\beta} \to 0 \) as \( n \to \infty. \) Let \( m \in \mathbb{N} \) and \( n \geq n_0. \)
In view of Lemma 3.3 we have
\[
\|f_n - f\|_{\infty} + \frac{\text{Osc}(m, f_n - f)}{p^m(1-\beta)} = \lim_{k \to \infty} \left( \|f_n - f_k\|_{\infty} + \frac{\text{Osc}(m, f_n - f_k)}{p^m(1-\beta)} \right)
\leq \lim_{k \to \infty} \left( \|f_n - f_k\|_{\infty} + \sup_{m' \in \mathbb{N}} \frac{\text{Osc}(m', f_n - f_k)}{p^{m'}(1-\beta)} \right)
\leq \sup_{k \geq n_0} \left( \|f_n - f_k\|_{\infty} + \sup_{m' \in \mathbb{N}} \frac{\text{Osc}(m', f_n - f_k)}{p^{m'}(1-\beta)} \right)
= \sup_{k \geq n_0} \|f_n - f_k\|_{V^\beta}
\leq \epsilon.
\]
The above is true for every \( m \in \mathbb{N}. \) Therefore, we obtain \( f - f_{n_0} \in V^\beta(I). \) Using Lemma 3.3 we have \( f = f - f_{n_0} + f_{n_0} \in V^\beta(I), \) and \( \|f_n - f\|_{V^\beta} \leq \epsilon \quad \forall n \geq n_0. \)

**Remark 3.7.** If \( |L_j(I)| = \frac{1}{p^j} \) for some \( k_j \in \mathbb{N} \) with \( \sum_{j \in J} \frac{1}{p^j} = 1 \), then for \( m \geq \max_{j \in J} \{k_j\} \) we have
\[
\text{Osc}(m, f) = \sum_{j \in J} \text{Osc}(m, f, L_j(I)),
\]
where \( \text{Osc}(m, f, L_j(I)) = \sum_{\|Q\|=p^{-m}, Q \subseteq L_j(I)} R_f(Q). \)

**Proof.** Since \( I = \bigcup_{j \in J} L_j(I) \) and \( \sum_{j \in J} \frac{1}{p^j} = 1 \) we have
\[
\text{Osc}(m, f) = \sum_{\|Q\|=p^{-m}} R_f(Q)
= \sum_{j \in J} \sum_{\|Q\|=p^{-m}, Q \subseteq L_j(I)} R_f(Q)
= \sum_{j \in J} \text{Osc}(m, f, L_j(I)).
\]

\[
(3.1)
\]

**Theorem 3.8.** Let \( f, b, \alpha_j \ (j \in J) \in V^\beta(I) \) be such that \( b(x_1) = f(x_1) \) and \( b(x_N) = f(x_N) \). Further we assume that \( |L_j(I)| = \frac{1}{p^j} \) for some \( k_j \in \mathbb{N} \) with \( \sum_{j \in J} \frac{1}{p^j} = 1. \)

For \( \max \left\{ \|\alpha\|_{\infty} + \sup_{m \in \mathbb{N}} \frac{\text{Osc}(m, \alpha_j)}{p^m(1-\beta)}, \sum_{j \in J} \|\alpha_j\|_{\infty} \right\} < 1, \) we have \( f^\alpha \in V^\beta(I). \)
Proof. Let \(\mathcal{V}_f^\beta(I) = \{g \in \mathcal{V}^\beta(I) : g(x_1) = f(x_1), g(x_N) = f(x_N)\}\). We observe that the space \(\mathcal{V}_f^\beta(I)\) is a closed subset of \(\mathcal{V}^\beta(I)\). It follows that \(\mathcal{V}_f^\beta(I)\) is a complete metric space with respect to the metric induced by norm \(\|\cdot\|_{\mathcal{V}^\beta}\). We define a map \(T : \mathcal{V}_f^\beta(I) \to \mathcal{V}_f^\beta(I)\) by
\[
(Tg)(x) = f(x) + \alpha_j(L_j^{-1}(x)) \ (g - b)(L_j^{-1}(x))
\]
for all \(x \in I_j\), where \(j \in J\). First we observe that the mapping \(T\) is well-defined. Using Remark \([x\overline{4}1]\) for \(g, h \in \mathcal{V}_f^\beta(I)\) we have
\[
\|Tg - Th\|_{\mathcal{V}^\beta} = \|Tg - Th\|_\infty + \frac{Osc(m, Tg - Th)}{p^{\alpha(1-\beta)}}
\]
\[
\leq \|\alpha\|_\infty \|g - h\|_\infty + \sum_{j \in J} \|\alpha_j\|_\infty \|g - h\|_\infty\sup_{m \in \mathbb{N}} \frac{Osc(m, g - h)}{p^{\alpha(1-\beta)}}
\]
\[
+ \sum_{j \in J} \|g - h\|_\infty \sup_{m \in \mathbb{N}} \frac{Osc(m, \alpha_j)}{p^{\alpha(1-\beta)}}
\]
\[
\leq \left( \|\alpha\|_\infty + \sum_{j \in J} \|\alpha_j\|_\infty \sup_{m \in \mathbb{N}} \frac{Osc(m, \alpha_j)}{p^{\alpha(1-\beta)}} \right) \|g - h\|_\infty
\]
\[
+ \left( \sum_{j \in J} \|\alpha_j\|_\infty \sup_{m \in \mathbb{N}} \frac{Osc(m, g - h)}{p^{\alpha(1-\beta)}} \right) \|g - h\|_{\mathcal{V}^\beta}
\]
\[
\leq \max \left\{ \|\alpha\|_\infty + \sum_{j \in J} \|\alpha_j\|_\infty \sup_{m \in \mathbb{N}} \frac{Osc(m, \alpha_j)}{p^{\alpha(1-\beta)}}, \sum_{j \in J} ||\alpha_j||_\infty \right\} \|g - h\|_{\mathcal{V}^\beta}.
\]
From the hypothesis, it follows that \(T\) is a contraction map on \(\mathcal{V}_f^\beta(I)\). Using the Banach contraction principle, we get a unique fixed point of \(T\), namely \(f^\alpha \in \mathcal{V}_f^\beta(I)\). Furthermore, since \(T(f^\alpha) = f^\alpha\), we write \(f^\alpha\) as a part of the functional equation:
\[
f^\alpha(L_j(x)) = f(L_j(x)) + \alpha_j(x) (f^\alpha - b)(x)
\]
for every \(x \in I\) and \(j \in J\). Now with \(J := \{1, 2, 3, \ldots, N - 1\}\) we define functions \(W_j : I \times \mathbb{R} \to I \times \mathbb{R}\) for \(j \in J\) by
\[
W_j(x, y) = \left( L_j(x), \alpha_j(x)y + f(L_j(x)) - \alpha_j(x)b(x) \right).
\]
We show in the last part of the proof that graph of the associated fractal function \(f^\alpha\) is an attractor of the IFS \(\{I \times \mathbb{R}; W_j, j \in J\}\). Following the proof of Theorem 1 appeared in \([3]\) we may prove that attractor of the above IFS is graph of a function. It remains to show that it is actually the graph of fractal perturbation \(f^\alpha\). To see that we use the functional equation and \(I = \cup_{j \in J} L_j(I)\) and get
\[
\cup_{j \in J} W_j(Graph(f^\alpha)) = \cup_{j \in J} \{W_j(x, f^\alpha(x)) : x \in I\}
\]
\[
= \cup_{j \in J} \left\{ (L_j(x), \alpha_j(x)f^\alpha(x) + f(L_j(x)) - \alpha_j(x)b(x)) : x \in I \right\}
\]
\[
= \cup_{j \in J} \left\{ (L_j(x), f^\alpha(L_j(x))) : x \in I \right\}
\]
\[
= \cup_{j \in J} \left\{ (x, f^\alpha(x)) : x \in L_j(I) \right\}
\]
\[
= Graph(f^\alpha),
\]
Remark 3.9. Let $0 < \gamma \leq 1$ and $f, b, \alpha_j$ be suitable functions satisfying the hypothesis of Theorem 3.12. Then, Theorem 3.1 yields that $\dim_B(\text{Graph}(f^\alpha)) \leq 2 - \gamma$.

Remark 3.10. Having Theorem 3.12 in mind, we may ask the assumptions on the parameters for which $f^\alpha \in \bigcap_{\beta < 1} \mathcal{V}^\beta(I) \setminus \bigcup_{\beta > \gamma} \mathcal{V}^\beta(I)$. This question remains open.

Before stating the upcoming remark, we define the Hölder space as follows:

$${\mathcal{H}}^s(I) := \{g : I \to \mathbb{R} : g \text{ is Hölder continuous with exponent } s\}.$$ 

If we equip the space $\mathcal{H}^s(I)$ with norm $\|g\|_s := \|g\|_\infty + |g|_s$, where $|g|_s = \sup_{x \neq y} \frac{|g(x) - g(y)|}{|x - y|^s}$, then it forms a Banach space.

Remark 3.11. Let us start with the following example: the function $f : [0, 1] \to \mathbb{R}$ defined by $f(x) = |x - \frac{1}{2}|^\beta$, where $0 < \beta < 1$, is a simple example of a function in $\bigcap_{\beta < 1} \mathcal{V}^\beta(I)$ which is only in the Hölder space $\mathcal{H}^\beta(I)$. Therefore, the dimension of the graph is 1 while the classical result only provides us with the upper bound $2 - \beta$. Note that [9] the spaces $\mathcal{V}^\beta(I)$ are refined version of Hölder spaces. Hence, our result obtained here generalizes many previous results, see, for instance, [1][25].

The next theorem has been proved in [25] using the series expansion. We here give a different proof which we feel, is more general and direct.

**Theorem 3.12.** Let $f, b$ and $\alpha$ be Hölder continuous with exponent $s$ such that $b(x_1) = f(x_1)$ and $b(x_N) = f(x_N)$. Then with the notation $a := \min\{\alpha_j : j \in J\}$ we have $f^\alpha$ is Hölder continuous with exponent $s$ provided $\frac{\|\alpha\|_\infty}{a^s} < 1$.

**Proof.** Let $\mathcal{H}_s^\alpha(I) := \{g \in \mathcal{H}^s(I) : g(x_1) = f(x_1), g(x_N) = f(x_N)\}$. Applying the definition of closed set, we see that the set $\mathcal{H}_s^\alpha(I)$ is a closed subset of $\mathcal{H}^s(I)$. Because $\mathcal{H}^\alpha(I)$ is a Banach space as mentioned, it follows that $\mathcal{H}_s^\alpha(I)$ is a complete metric space with respect to the metric induced by aforementioned norm $\|\cdot\|_s$ for $\mathcal{H}^s(I)$. We define a map $T : \mathcal{H}_s^\alpha(I) \to \mathcal{H}_s^\alpha(I)$ by

$$(Tg)(x) = f(x) + \alpha_j(L_j^{-1}(x))(g-b)(L_j^{-1}(x))$$

for all $x \in I_j$ where $j \in J$. First we shall show that $T$ is well-defined. For this let us note that

$${[Tg]}_s = \max_{j \in J} \frac{|Tg(x) - Tg(y)|}{|x - y|^s} \\ \leq \max_{j \in J} \left[ \sup_{x \neq y, x, y \in I_j} \frac{|f(x) - f(y)|}{|x - y|^s} + \sup_{x \neq y, x, y \in I_j} \frac{\alpha_j(L_j^{-1}(x)) (|g-b|L_j^{-1}(x)) - (g-b)(L_j^{-1}(y))}{|x - y|^s} + \sup_{x \neq y, x, y \in I_j} \frac{(|g-b)(L_j^{-1}(y))| \alpha_j(L_j^{-1}(x)) - \alpha_j(L_j^{-1}(y))}{|x - y|^s} \right]$$

$$\leq [f]_s + \frac{\|\alpha\|_\infty}{a^s} ([g]_s + [b]_s) + \frac{\|g-b\|_\infty}{a^s} [\alpha]_s,$$
where \( [\alpha]_s = \max_{j \in J} \sup_{x \neq y, x, y \in I} \frac{|\alpha_j(x) - \alpha_j(y)|}{|x - y|^s} \). For \( g, h \in \mathcal{H}_f^s(I) \), we have

\[
\|Tg - Th\|_\mathcal{H} = \|Tg - Th\|_\infty + [Tg - Th]
\leq \|\alpha\|_\infty \|g - h\|_\infty + \|\alpha\|_a^\infty [g - h]_s + \|g - h\|_a^\infty [\alpha]_s
\leq \|\alpha\|_a^\infty \|g - h\|_\mathcal{H}.
\]

Since \( \|\alpha\|_a^\infty < 1 \), it follows that \( T \) is a contraction self map on \( \mathcal{H}_f^s(I) \). Thanks to Banach contraction principle, a unique fixed point of \( f \) exists. This proves the result.

**Remark 3.13.** By dint of a more spirited effort (see [25]), we can observe with the help of the above theorem that the Hölder constant of the map \( f^{\alpha} \) depends only on the germ function \( f \), the partition \( \Delta \) and the parameter maps \( b, \alpha \).

Note that for equidistant nodes we have \( a = a_j = \frac{1}{N-1} \).

**Theorem 3.14.** Let \( f \) be a germ function, and \( b, \alpha_j \) be suitable continuous functions such that

\[
|f(x) - f(y)| \leq k_f|x - y|^{s},
\]

\[
|b(x) - b(y)| \leq k_b|x - y|^{s},
\]

\[
|\alpha_j(x) - \alpha_j(y)| \leq k_\alpha|x - y|^{s}
\]

for every \( x, y \in I, j \in J \), and for some \( k_f, k_b, k_\alpha > 0, s \in (0, 1] \). Further, assume that there are constants \( K_f, \delta_\alpha > 0 \) such that for each \( x \in I \) and \( \delta < \delta_0 \) there exists \( y \in I \) with \( |x - y| \leq \delta \) and \( |f(x) - f(y)| \geq K_f|x - y|^{s} \). We have \( \dim_B(G_{f^{\alpha}}) = 2 - s \) provided that \( \|\alpha\|_a < a^s \).

**Proof.** In the light of Theorem 3.12 and \( \|\alpha\|_a < a^s \), we have \( f^{\alpha} \) is Hölder continuous with the same exponent \( s \). That is, we may consider

\[
|f^{\alpha}(x) - f^{\alpha}(y)| \leq k_{f^{\alpha}}|x - y|^{s}
\]

for some \( k_{f^{\alpha}} > 0 \). We obtain a bound for upper box dimension of the graph of the fractal function \( f^{\alpha} \) as follows: For \( 0 < \delta < 1 \), let \( N_\delta(G_{f^{\alpha}}) \) be the number of \( \delta \)-boxes that cover graph of \( f^{\alpha} \), with \( [\cdot] \) the ceiling function, we have

\[
N_\delta(G_{f^{\alpha}}) \leq \sum_{i=1}^{\lceil \frac{1}{\delta} \rceil} \left( 1 + \left[ \frac{R_{f^{\alpha}}((i - 1)\delta, i\delta)}{\delta} \right] \right)
\leq \sum_{i=1}^{\lceil \frac{1}{\delta} \rceil} \left( 2 + \frac{R_{f^{\alpha}}((i - 1)\delta, i\delta)}{\delta} \right)
= 2 \left[ \frac{1}{\delta} \right] + \sum_{i=1}^{\lceil \frac{1}{\delta} \rceil} R_{f^{\alpha}}((i - 1)\delta, i\delta)
\leq 2 \left[ \frac{1}{\delta} \right] + \sum_{i=1}^{\lceil \frac{1}{\delta} \rceil} k_{f^{\alpha}} \delta^{s-1}.
\]
Consequently, we deduce

\[
\dim_B(\text{Graph}(f^\alpha)) = \lim_{\delta \to 0} \frac{\log N_\delta(\text{Graph}(f^\alpha))}{-\log \delta} \leq 2 - s.
\]

It is sufficient to prove the following bound for lower box dimension:

\[
\dim_B(\text{Graph}(f^\alpha)) \geq 2 - s.
\]

We recall the self-referential equation

\[
f^\alpha(x) = f(x) + \alpha_j(L_j^{-1}(x)) \left[ f^\alpha(L_j^{-1}(x)) - b(L_j^{-1}(x)) \right],
\]

for every \( x \in I_j \) and \( j \in J \). For \( x, y \in I_j \) such that \( |x - y| \leq \delta \), we obtain

\[
|f^\alpha(x) - f^\alpha(y)| = |f(x) - f(y) + \alpha_j(L_j^{-1}(x)) f^\alpha(L_j^{-1}(x)) - \alpha_j(L_j^{-1}(y)) f^\alpha(L_j^{-1}(y)) - \alpha_j(L_j^{-1}(x)) b(L_j^{-1}(x)) + \alpha_j(L_j^{-1}(y)) b(L_j^{-1}(y))| \\
\geq |f(x) - f(y)| - \|\alpha\|_\infty \left| f^\alpha(L_j^{-1}(x)) - f^\alpha(L_j^{-1}(y)) \right| \\
- \|\alpha\|_\infty \left| b(L_j^{-1}(x)) - b(L_j^{-1}(y)) \right| \\
- (\|b\|_\infty + \|f^\alpha\|_\infty) |\alpha_j(L_j^{-1}(x)) - \alpha_j(L_j^{-1}(y))|.
\]

With the help of Equation (3.2), we obtain

\[
|f^\alpha(x) - f^\alpha(y)| \geq K_f |x - y|^s - \|\alpha\|_\infty k_f a^s |L_j^{-1}(x) - L_j^{-1}(y)|^s \\
- \|\alpha\|_\infty k_b |L_j^{-1}(x) - L_j^{-1}(y)|^s \\
- (\|b\|_\infty + M) k_a |x - y|^s \\
\geq K_f |x - y|^s - \|\alpha\|_\infty k_f a^s |x - y|^s \\
- \|\alpha\|_\infty k_a a^{-s} |x - y|^s \\
- (\|b\|_\infty + M) a^{-s} k_a |x - y|^s \\
= (K_f - (k_f a + k_b) |\alpha|_\infty a^{-s} - (\|b\|_\infty + M) a^{-s} k_a) |x - y|^s.
\]

Let \( K := K_f - (k_f a + k_b) |\alpha|_\infty a^{-s} - (\|b\|_\infty + M) a^{-s} k_a \). For \( \delta = a^m \), we estimate

\[
N_\delta(\text{Graph}(f^\alpha)) \geq \sum_{i=1}^{a^{-m}} \max \left\{ 1, \left[ a^{-m} R_{f^\alpha} [(i-1)\delta, i\delta] \right] \right\} \\
\geq \sum_{i=1}^{a^{-m}} \left[ a^{-m} R_{f^\alpha} [(i-1)\delta, i\delta] \right] \\
\geq \sum_{i=1}^{a^{-m}} \left[ K a^{-m} a^{ms} \right] \\
\geq a^{-m} a^{-m} K a^{ms} \\
= K a^{m(s-2)}.
\]
Using the above bound for $N_\delta(\text{Graph}(f^\alpha))$, we obtain

$$
\lim_{\delta \to 0} \frac{\log \left( N_\delta(\text{Graph}(f^\alpha)) \right)}{-\log(\delta)} \geq \lim_{m \to \infty} \frac{\log \left( Ka^m(s-2i) \right)}{-m \log a} = 2 - s,
$$

establishing the result.

**Corollary 3.15.** If we consider the Bernstein polynomial as the base function, then for a Lipschitz $f$, we obtain a sequence of Bernstein $\alpha$-fractal functions (see [8, 28] for details). For each $n \in \mathbb{N}$, let $G$ be the graph of the Bernstein $\alpha$-fractal function. Then under the hypothesis of Theorem 3.14, we obtain $\dim_H(G) \leq 1$.

**Remark 3.16.** In [1], Nasim et al. computed the box dimension of $\alpha$-fractal function under certain condition. But for the Hölder exponent $s \in (0, 1)$ the author has calculated the obvious upper bound as $2 - s$. However, in this article, in Theorem 3.14, we have calculated the exact estimation of the box dimension of $\alpha$-fractal function under suitable condition.

**Theorem 3.17.** Let $f, \alpha_j (j \in J)$ and $b$ be Hölder continuous with exponent $s$ such that $b(x_1) = f(x_1)$ and $b(x_N) = f(x_N)$. If $\|a\|_H < a^s$ with $a = \min\{a_j : j \in J\}$ then

$$
1 \leq \dim_H(\text{Graph}(f^\alpha)) \leq 2 - s.
$$

**Proof.** We will proceed by defining a map $\Phi : \text{Graph}(f^\alpha) \to I$ by $\Phi((x, f(x))) = x$. Then

$$
|\Phi((x, f(x))) - \Phi((y, f(y)))| = |x - y| \leq \|(x, f(x)) - (y, f(y))\|_2.
$$

That is, $\Phi$ is a Lipschitz map. Using a properties of Hausdorff dimension (see [10]), we have $\dim_H(\text{Graph}(f^\alpha)) \leq \dim_H(\text{Graph}(f^\alpha))$. It is easy to check that the map $\Phi$ is onto. Hence we have $\dim_H(\text{Graph}(f^\alpha)) \geq \dim_H(I) = 1$. We recall a well-known result, see [10], which relates the Hausdorff dimension and box dimension in the following sense:

$$
\dim_H(C) \leq \dim_B(C) \leq \overline{\dim}_B(C)
$$

for any bounded set $C \subset \mathbb{R}^n$. Theorem 3.12 and the first part of Theorem 3.14 yield the required upper bound for the Hausdorff dimension of the graph of fractal function $f^\alpha$. □

**Definition 3.18.** Let $f : I \to \mathbb{R}$ be a function. For each partition $P : t_0 < t_1 < t_2 < \cdots < t_n$ of the interval $I$, we define

$$
V(f, I) = \sup_P \sum_{i=1}^n |f(t_i) - f(t_{i-1})|,
$$

where the supremum is taken over all partitions $P$ of the interval $I$. If $V(f, I) < \infty$, we say that $f$ is of bounded variation. The set of all functions of bounded variation on $I$ will be denoted by $BV(I)$. We define a norm on $BV(I)$ by $\|f\|_{BV} := \|f(t_0)\| + V(f, I)$. Moreover, the space $BV(I)$ is a Banach space with respect to this norm.

Liang [17] proved that
Theorem 3.19. If $f \in C(I) \cap BV(I)$, then $\dim_{H}(\text{Graph}(f)) = \dim_{B}(\text{Graph}(f)) = 1$.

The next remark is straightforward but useful for the upcoming theorem.

Remark 3.20. Let $f$ be real-valued function on $I = [0, 1]$. For $c, d \in \mathbb{R}$, we define a function $g(x) = f(cx + d)$ on a suitable domain. If $f$ is of bounded variation on $I$ then $g$ is also of bounded variation on its domain.

We present the following remark for the sake of independent interest.

Remark 3.21. We know that $f^{\alpha}$ satisfies the self-referential equation

$$f^{\alpha}(x) = f(x) + \alpha_{j}(L_{j}^{-1}(x)) \cdot (f^{\alpha} - b)(L_{j}^{-1}(x)) \quad \forall \ x \in I_{j}, \ j \in J.$$  

The self-referential equation may also be written in the following manner

$$\alpha_{j}(L_{j}^{-1}(x)) \cdot b(L_{j}^{-1}(x)) = f(x) - f^{\alpha}(x) + \alpha_{j}(L_{j}^{-1}(x)) \cdot f^{\alpha}(L_{j}^{-1}(x)) \quad \forall \ x \in I_{j}, \ j \in J.$$  

Further, we assume $f, f^{\alpha}$ and $\alpha_{j}$ ($j \in J$) be of bounded variation with $\alpha_{j} > 0$ or $< 0$ on $I$. Using algebra of bounded variation functions (see [13]), one concludes that $b$ is of bounded variation.

The following theorem is a generalization of [25, Theorem 4.8]. However, we present the proof for reader’s convenience.

Theorem 3.22. Let $f \in BV(I)$. Suppose that $\triangle = \{x_{1}, x_{2}, \ldots, x_{N} : x_{1} < x_{2} < \cdots < x_{N}\}$ is a partition of $I, b \in BV(I)$ satisfying $b(x_{1}) = f(x_{1}), b(x_{N}) = f(x_{N})$, and $\alpha_{j}$ ($j \in J$) are functions in $BV(I)$ with $\|\alpha\|_{BV} < \frac{1}{2(N-1)}$. Then, the fractal perturbation $f^{\alpha}$ corresponding to $f$ is of bounded variation on $I$.

Proof. Let $BV_{\alpha}(I) = \{g \in BV(I) : g(x_{1}) = f(x_{1}), \ g(x_{N}) = f(x_{N})\}$. We may see (using the definition of closed set) that $BV_{\alpha}(I)$ is a closed subset of $BV(I)$. Being a close subset of Banach space $BV(I)$, the space $BV_{\alpha}(I)$ is a complete metric space when endowed with metric induced by norm $\|g\|_{BV} := |g(x_{1})| + V(g, I)$. Define the RB operator $T : BV_{\alpha}(I) \rightarrow BV_{\alpha}(I)$ by

$$(Tg)(x) = f(x) + \alpha_{j}(L_{j}^{-1}(x)) \left[g(L_{j}^{-1}(x)) - b(L_{j}^{-1}(x))\right],$$

for every $x \in I_{j}$ and $j \in J$. As done in previous theorems we note that $T$ is well-defined. Let $P : t_{0} < t_{1} < t_{2} < \cdots < t_{m}$ be a partition of the interval $I_{j}$, where $m \in \mathbb{N}$. Consider

$$
\| (Tg - Th)(t_{i}) - (Tg - Th)(t_{i-1}) \| = \big| \alpha_{j}(L_{j}^{-1}(t_{i})) (g - h)(L_{j}^{-1}(t_{i})) \big| \\
- \big| \alpha_{j}(L_{j}^{-1}(t_{i-1})) (g - h)(L_{j}^{-1}(t_{i-1})) \big| \\
\leq \big| \alpha_{j}(L_{j}^{-1}(t_{i})) \big| \big| (g - h)(L_{j}^{-1}(t_{i})) \big| \\
- \big| (g - h)(L_{j}^{-1}(t_{i-1})) \big| + \big| (g - h)(L_{j}^{-1}(t_{i-1})) \big| \\
\cdot \big| \alpha_{j}(L_{j}^{-1}(t_{i})) - \alpha_{j}(L_{j}^{-1}(t_{i-1})) \big| \\
\leq \| \alpha \|_{\infty} \cdot \big| (g - h)(L_{j}^{-1}(t_{i})) - (g - h)(L_{j}^{-1}(t_{i-1})) \big| \\
+ \| g - h \|_{\infty} \big| \alpha_{j}(L_{j}^{-1}(t_{i})) - \alpha_{j}(L_{j}^{-1}(t_{i-1})) \big|.
$$
Summing over $i = 1$ to $m$, we have

$$
\sum_{i=1}^{m} \left| (Tg - Th)(t_i) - (Tg - Th)(t_{i-1}) \right|
\leq \|\alpha\|_\infty \sum_{i=1}^{m} \left| (g - h)(L_j^{-1}(t_i)) - (g - h)(L_j^{-1}(t_{i-1})) \right|
+ \|g - h\|_\infty \sum_{i=1}^{m} \left| \alpha_j(L_j^{-1}(t_i)) - \alpha_j(L_j^{-1}(t_{i-1})) \right|
\leq \|\alpha\|_\infty \|g - h\|_{BV} + \|g - h\|_\infty \|\alpha\|_{BV}
\leq \|\alpha\|_{BV} \left( \|g - h\|_{BV} + \|g - h\|_\infty \right)
\leq 2\|\alpha\|_{BV} \|g - h\|_{BV}.
$$

The above inequality holds for any partition of $I_j$. Therefore, one gets

$$
\|Tg - Th\|_{BV} \leq 2(N - 1)\|\alpha\|_{BV} \|g - h\|_{BV}.
$$

Since $\|\alpha\|_{BV} < \frac{1}{2(N - 1)}$, $T$ is a contraction on the complete metric space $BV_*(I)$. Applying the Banach fixed point theorem we have a unique fixed point $f^\alpha$ of $T$. Moreover, the fixed point $f^\alpha$ of $T$ satisfies the self-referential equation, that is,

$$
f^\alpha(x) = f(x) + \alpha_j(L_j^{-1}(x)) \left[ f^\alpha(L_j^{-1}(x)) - b(L_j^{-1}(x)) \right],
$$

for every $x \in I_j$ and $j \in J$.

\[ \square \]

**Theorem 3.23.** Let the germ function $f$ and the parameter $b$ be continuous functions of bounded variation. Suppose $\alpha_j (j \in J)$ are functions of bounded variation with $\|\alpha\|_{BV} < \frac{1}{2(N - 1)}$. Then $\dim_H(\text{Graph}(f^\alpha)) = \dim_B(\text{Graph}(f^\alpha)) = 1$.

**Proof.** Theorem 3.19 and Theorem 3.22 produce the result. \[ \square \]

We shall denote by $AC(I)$ the Banach space of all absolutely continuous functions on $I$ with its usual norm (denoted by $\|\cdot\|_{AC}$).

**Theorem 3.24.** Let $f \in AC(I)$. Suppose that $\Delta = \{x_1, x_2, \ldots, x_N : x_1 < x_2 < \cdots < x_N \}$ is a partition of $I$, $b \in AC(I)$ satisfying $b(x_1) = f(x_1)$, $b(x_N) = f(x_N)$, and $\alpha_j (j \in J)$ are functions in $AC(I)$ with $\|\alpha\|_{AC} < \frac{1}{2(N - 1)}$, where $a = \min\{a_j : j \in J\}$. Then, the fractal perturbation $f^\alpha$ corresponding to $f$ is absolutely continuous on $I$.

**Proof.** Let $AC_*(I) = \{g \in AC(I) : g(x_1) = f(x_1), \ g(x_N) = f(x_N)\}$. We observe (using the sequential definition of a closed set) that $AC_*(I)$ is a closed subset of $AC(I)$. Since $AC(I)$ endowed with $\|g\|_{AC} := |g(x_1)| + \int_{x_1}^{x_N} |g'(x)|dx$ is a complete normed linear space, the set $AC_*(I)$ is a complete metric space when equipped with metric induced by aforesaid norm. Define the RB operator $T : AC_*(I) \to AC_*(I)$ by

$$
(Tg)(x) = f(x) + \alpha_j(L_j^{-1}(x)) \left[ g(L_j^{-1}(x)) - b(L_j^{-1}(x)) \right],
$$
for every \( x \in I_j \) and \( j \in J \). We note that the conditions on \( f \) and \( b \) dictate the function \( T \) to be well-defined. Consider

\[
\int_{L_j(x_1)}^{L_j(x_N)} |(Tg - Th)'(x)| dx \leq \frac{1}{a_j} \int_{L_j(x_1)}^{L_j(x_N)} |\alpha_j'(L_j^{-1}(x))(g - h)(L_j^{-1}(x))| dx \\
+ \frac{1}{a_j} \int_{L_j(x_1)}^{L_j(x_N)} |\alpha_j(L_j^{-1}(x))(g - h)'(L_j^{-1}(x))| dx \\
= \frac{1}{a_j} \int_{x_1}^{x_N} |\alpha_j'(y)(g - h)(y)| dy \\
+ \frac{1}{a_j} \int_{x_1}^{x_N} |\alpha_j(y)(g - h)'(y)| dy \\
\leq \frac{\|g - h\|_\infty}{a_j} \int_{x_1}^{x_N} |\alpha_j'(y)| dy \\
+ \frac{\|\alpha_j\|_\infty}{a_j} \int_{x_1}^{x_N} |(g - h)'(y)| dy.
\]

Summing over \( j = 1 \) to \( N - 1 \), we have

\[
\sum_{j=1}^{N-1} \int_{L_j(x_1)}^{L_j(x_N)} |(Tg - Th)'(x)| dx \leq \frac{2(N - 1)\|\alpha\|_\text{AC}}{a} \|g - h\|_\text{AC}.
\]

Therefore, one gets

\[
\|Tg - Th\|_\text{AC} \leq \frac{2(N - 1)\|\alpha\|_\text{AC}}{a} \|g - h\|_\text{AC}.
\]

Since \( \|\alpha\|_\text{AC} < \frac{a}{2(N - 1)} \), we deduce that \( T \) is a contraction on the complete metric space \( \mathcal{AC} (I) \). Moreover, the fixed point \( f^\alpha \) of \( T \) satisfies the self-referential equation, that is,

\[
f^\alpha(x) = f(x) + \alpha_j(L_j^{-1}(x)) \left[ f^\alpha(L_j^{-1}(x)) - b(L_j^{-1}(x)) \right],
\]

for every \( x \in I_j \) and \( j \in J \).

\[ \square \]

Combining Theorem 3.22 and Theorem 3.22, one can immediately deduce the following.

**Theorem 3.25.** Let the germ function \( f \) and the parameter \( b \) be absolutely continuous functions. Suppose \( \alpha_j \ (j \in J) \) are absolutely continuous functions with \( \|\alpha\|_\text{AC} < \frac{a}{2(N - 1)} \). Then \( \dim_H(\text{Graph}(f^\alpha)) = \dim_B(\text{Graph}(f^\alpha)) = 1 \).

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