On Some New Proof of the Bogoliubov — Parasiuk Theorem
(Nonequilibrium Renormalization Theory II)

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Abstract

It is usually used a complicated combinatorics to prove the Bogoliubov — Parasiuk theorem. In the present paper we give a proof of the Bogoliubov — Parasiuk theorem which use a simple combinatorics. To give this proof we interpret Feynman amplitudes as distributions on the space of $\alpha$-parameters.

We will use this technique in the next paper to give a proof that the divergences in nonequilibrium diagram technique can be subtracted by means the counterterms of asymptotical state.

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1 Introduction

The mathematical theory of renormalization (R-operation) has been developed by N.N. Bogoliubov and O.S. Parasiuk [1,2]. K. Hepp has elaborated on their proofs [3].

In standard framework based on Bogoliubov — Parasiuk — Hepp sectors, renormalization theory needs in complicated combinatorics. In the present paper we give a new proof of the Bogoliubov — Parasiuk theorem which needs simple combinatorics. The main idea of this proof is an interpretation of the Feynman amplitudes as distributions on the space of $\alpha$ — parameters. The counterterms in our framework are local as in $\alpha$ — parameters space as in coordinate space. This fact makes possible to use decomposition of unite to solve the problem of overlapping divergences.

Similar technique in renormalization theory have been used in [4,5], but in coordinate and momenta representations.

We will use the technique, developed in the present paper to prove that the divergences in nonequilibrium diagram technique can be renormalized by means of counterterms of asymptotical state.

2 Definition of the Feynman Diagrams

**Definition.** A Feynman graph is a triple

$$\Phi = (V, R, F),$$

where $V$ is a finite set, called the set of vertices, $R$ is a finite set, called the set of lines, and $f$ is a map:

$$f : R \rightarrow (V \times V) \cup (V \times \{+, -\}).$$

The set $R_{\text{int}} = f^{-1}(V \times V)$ is called the set of internal lines. The set $R_{\text{in}} = f^{-1}(V \times \{+\})$ is called the set of lines coming into the graph. The set $R_{\text{out}} = f^{-1}(V \times \{-\})$ is called the set of lines coming from The Feynman graph. The set $R_{\text{ext}} = R_{\text{in}} \cup R_{\text{out}}$ is called the set of external lines.

Let $r$ be a line such that $f(r) = (v_1, v_2)$. We say that the line $r$ comes from the vertex $v_2$ and comes into the vertex $v_1$. We say also that the vertices $v_1$ and $v_2$ are connected by the line $r$. 

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Let $r$ be a line such that $f(r) = (v, +)$. We say that the line $r$ is an external line coming into the vertex $v$. Let $r$ be a line such that $f(r) = (v, -)$. In this case we say that the line $r$ is an external line coming from $v$.

Below we will consider only graphs such that for each vertex $v$ there exists a line $r$ such that $r$ comes into $v$ or $r$ comes from $v$.

By definition the path which connects the vertices $v'$ and $v''$ is a sequence of vertices

$$v' = v_0, v_1, ..., v_n = v''$$

such that for each $i$, $0 \leq i \leq n - 1$ the vertices $v_i, v_{i+1}$ are connected by some line.

The graph is called connected if and only if any two vertices are connected by some path.

A Feynman graph $Φ$ is called one particle irreducible (1PI) if it is connected and can not be done disconnected by removing a single line.

Now let us give some definitions necessary to give a definition of the Feynman graphs.

Let $Φ = (V_Φ, R_Φ, f_Φ)$ be a Feynman graph and $v \in V_Φ$ be some its vertex. Let

$$\mathcal{P}'_v = \bigotimes_{r \in R_\to v} \mathcal{P} \bigotimes_{r \in R_\from v} \mathcal{P}.$$  

Here $\mathcal{P}$ is a space of polynomials on $\mathbb{R}^4$, $R_\to v$ is a set of all lines coming into the vertex $v$ and $R_\from v$ is a set of all lines coming from the vertex $v$. We can see at $\mathcal{P}'_v$ as at polynomials $p(p_1, ..., p_n)$ of $n$ four-vectors arguments. $n = \sharp R_\to v + \sharp R_\from v$. Consider the subspace $\mathcal{N}_v$ of the space $\mathcal{P}'_v$ of all polynomials $p(p_1, ..., p_n)$ such that $p(p_1, ..., p_n) = 0$ if $p_1 + ... + p_n = 0$. The space of vertex operators by definition is

$$\mathcal{P}_v = \mathcal{P}'_v / \mathcal{N}_v.$$  

**Definition.** The Feynman diagram is a pair

$$\Gamma = (Φ_Γ, \varphi_Γ),$$

where $Φ_Γ$ is a Feynman graph and $\varphi_Γ$ is a map which assigns to each vertex $v$ an element $\varphi(v)$ of $\mathcal{P}_v$.

**Definition.** The Feynman diagram $\Gamma = (Φ_Γ, \varphi_Γ)$ is called connected, one particle irreducible, if the corresponding Feynman graph $Φ_Γ$ connected, one particle irreducible respectively.
Definition. Let $\Gamma = (\Phi, \varphi)$ be an one particle irreducible diagram. $\Phi = (V, R, F)$. Power of divergence of $\Gamma$ by definition is a number

$$\Omega_\Gamma = \sum_{v \in V} (\deg \varphi_\Gamma(v) - 4) + \sum_{r \in R_{\text{in}}} (4 - 2) + 4.$$ 

The Feynman Propagator. By definition Feynman propagator (Euclidean) is a map

$$\Delta : \mathbb{R}^4 \rightarrow \mathbb{R},$$

$$\Delta : p \mapsto \Delta(p) = \frac{1}{p^2 + m^2},$$

where $m > 0$.

Note, that

$$\Delta(p) = \int_0^{+\infty} e^{-\alpha(p^2 + m^2)} d\alpha.$$ 

This representation is called the Schwinger de Witt representation.

Define an analytically regularized propagator as follows

$$\Delta^z(p) = \int_0^{+\infty} \alpha^z e^{-\alpha(p^2 + m^2)} d\alpha.$$ 

It is easy to see that

$$\Delta^z(p) = \frac{\Gamma(1 + z)}{(p^2 + m^2)^{1+z}}.$$

The Feynman Amplitudes. Let $\Phi_\Gamma$ be a Feynman diagram. Let $n = \sharp R_{\text{ext}}$ be a number of external lines. Let $p_1,\ldots,p_n$ be momenta of particles coming into (or from) the diagram. The Feynman amplitude is a function of $p_1,\ldots,p_n$ defined as follows,

$$U^z_\Gamma(p_1,\ldots,p_n) \delta(p_1+\ldots+p_n)$$

$$= \int \prod_{r \in R_{\text{in}}} dp'_r \prod_{r \in R_{\text{in}}} \Delta^z(p'_r) \prod_{v \in V} \varphi_\Gamma(\pm p_{r_{1v}},\ldots,\pm p_{r_{kv}}) \delta(\sum_{r \rightarrow v} p'_r - \sum_{r \leftarrow v} p'_r). \quad (1)$$ 

Let us describe the basis elements of this formula.

a) The integration is over all internal momenta.
b) The symbol \( r \rightarrow v \) means that the line \( r \) comes into the vertex \( v \), the symbol \( r \leftarrow v \) means that the line \( r \) comes from the vertex \( v \).

c) Let \( r \in R_{\text{ext}} \). By definition \( p'_r = p_k \) ( \( k \) is a number of the line \( r \)) if \( r \) comes into \( v \), and \( p'_r = -p_k \) if \( r \) comes from \( v \).

d) \( k_v \) is a number of lines coming into the vertex \( v \) or coming from the vertex \( v \). The lines \( r^i_v \) are the lines coming into the vertex \( v \) or coming from the vertex \( v \). We take the sign + in arguments of \( \varphi \Gamma \) if the corresponding line comes into the vertex and the sign - if the corresponding line comes from the vertex.

We will prove below that the integral (1) converges and defines an enough many times differentiable function of \( p_1, \ldots, p_n \) if \( \text{Re } z \) is enough large. Let

\[
\Omega^c_\Gamma = \sum_{v \in V} (\deg \varphi \Gamma(v) - 4) + \sum_{r \in R_{\text{in}}} (4 - 2(1 + z)) + 4.
\]

3 The Bogoliubov — Parasiuk R-operation

Let \( \Gamma = (\Phi \Gamma, \varphi \Gamma) \), \( \Phi = (V, R, f) \).

Let \( V' \) be a subset of \( V \). Let \( \tilde{R}' \) be a subset of \( R \) satisfying the following condition. For each line \( r \) of \( \tilde{R}' \) there exist the vertices \( v, v' \in V' \) such that \( (v, v') = f(r) \).

Let \( R' = \tilde{R}' \cup \tilde{R}'' \), where \( \tilde{R}'' \subset (R \setminus \tilde{R}') \times \{+, -\} \) which consists of all pairs \( (r, +) \) such that \( r \) comes into \( V' \) and all pairs \( (r, -) \) such that \( r \) comes from \( V' \) (\( r \in R \setminus \tilde{R}' \)).

Consider the graph \( \Phi_\gamma = (V', R', f') \), where \( V' \) and \( R' \) are just defined, and

\[
\begin{align*}
  f'(r) &= f(r) \text{ if } r \in \tilde{R}' \\
  f'(r) &= (v, +) \text{ if } r \in \tilde{R}'', \ r = (r', +) \text{ for some } r' \in R \text{ and } r' \text{ comes into } v \\
  f'(r) &= (v, -) \text{ if } r \in \tilde{R}'', \ r = (r', -) \text{ for some } r' \in R \text{ and } r' \text{ comes from } v.
\end{align*}
\]

Let \( \gamma = (\Phi_\gamma, \varphi_\gamma) \) be a diagram, where \( \varphi_\gamma \) is a restriction of \( \varphi \Gamma \) on \( V' \). If \( \gamma \) is one particle irreducible diagram than \( \gamma \) is called an one particle irreducible subdiagram of \( \Gamma \).

Let \( \Gamma \) be one particle irreducible diagram and \( \gamma \) be its one particle irreducible subdiagram. Let \( C_\gamma(p_1, \ldots, p_l) \) be a polynomial of external momenta \( p_1, \ldots, p_l \) of \( \gamma \).
Define \( C_\gamma \ast U_\Gamma(p_1, ..., p_n) \) by the following way: consider a diagram \( \Gamma/\gamma := (\Phi_{\Gamma/\gamma}, \varphi_{\Gamma/\gamma}) \), where \( \Phi_{\Gamma/\gamma} \) is so called quotient graph obtained by replacing of \( \Phi_\gamma \) by a vertex \( v_0 \). \( \varphi_{\Gamma/\gamma} = \varphi_{\Gamma}(v) \) if \( v \neq v_0 \) and \( \varphi_{\Gamma/\gamma}(v_0)(p_1, ..., p_l) = C_\gamma(p_1, ..., p_l) \). Put by definition \( C_\gamma \ast U_\Gamma(p_1, ..., p_n) = U_{\Gamma/\gamma}(p_1, ..., p_l) \).

We can define \( C_{\gamma_1} \ast \ast C_{\gamma_n} \ast U_\Gamma \) by a similar way. Here \( \gamma_1, ..., \gamma_n \) are one particle irreducible subdiagrams of \( \Gamma \) such \( \forall (i, j) \ i \neq j \ i, j = 1, ..., n \) the diagrams \( \gamma_i, \gamma_j \) do not intersect.

Now let us define the R-operation. Let us suppose that our theory satisfy the following condition: at each order of perturbation theory there exists only a finite number of diagrams. Let us define the amplitudes \( R^z_{\Gamma}(p_1, ..., p_n), C^z_{\Gamma}(p_1, ..., p_n) \) by the following recurrent relation. Suppose that for each diagram \( G \) such that the number of its vertices less than \( n \) the amplitudes \( R^z_{\Gamma} \) and \( C^z_{\Gamma} \) are defined. To define the amplitudes \( R^z_{\Gamma} \) and \( C^z_{\Gamma} \) at \( n \)-th order, we must move all diagrams of order \( n \) starting from the diagrams with maximal numbers of external lines and going to the diagrams with minimal number of external lines and use the following formulas.

\[
C^z_{\Gamma} = -T(U^z_{\Gamma} + \sum_{\gamma_1 \ast \ast \ast \gamma_n \subset \Gamma} C^z_{\gamma_1} \ast \ast \ast C^z_{\gamma_n} \ast U^z_{\Gamma}),
\]

\[
R^z_{\Gamma} = (1 - T)(U^z_{\Gamma} + \sum_{\gamma_1 \ast \ast \ast \gamma_n \subset \Gamma} C^z_{\gamma_1} \ast \ast \ast C^z_{\gamma_n} \ast U^z_{\Gamma}).
\]

(2)

Here \( T \) is an operator which to each Laurant series with the center at zero \( \frac{a_{-n}}{z^n} + ... + \frac{a_{-1}}{z} + a_0 + a_1 z + \ldots \) assigns its pole part \( \frac{a_{-n}}{z^n} + ... + \frac{a_{-1}}{z} + a_0 \). \( \subset \) means the strong inclusion.

**Theorem (Bogoliubov — Parasiuk).** For each diagram \( \Gamma \) there exists a polynomial \( C^z_{\Gamma}(p_1, ..., p_n) \) of external momenta such that \( \deg C^z_{\Gamma}(p_1, ..., p_n) \leq \Omega_\Gamma \), its coefficient are the polynomials on the inverse powers of \( z \), and the following conditions are satisfied.

a) \( (U^z_{\Gamma} + \sum_{\gamma_1 \ast \ast \ast \gamma_n \subset \Gamma} C^z_{\gamma_1} \ast \ast \ast C^z_{\gamma_n} \ast U^z_{\Gamma}) \) has an unique analytical continuation into the point \( z = 0 \).

b) The recurrent relations (2) holds.
Our aim is to prove this theorem. Let us give some useful definitions. Let $n \in \mathbb{Z}$. Put by definition $\mathbb{R}_+ := \{ x \in \mathbb{R} \mid x \geq 0 \}$, and let $\alpha_1, ..., \alpha_n$ are the coordinates on $\mathbb{R}_+^n$, $\alpha_i \geq 0$.

The space of all test functions on $\mathbb{R}_+^n \mathcal{S}(\mathbb{R}_+^n)$ consists by definition of all smooth functions on $\mathbb{R}_+^n$ which decay faster than any inverse polynomial on $\alpha_1 + ... + \alpha_n$ with all its derivatives as $\alpha_1 + ... + \alpha_n$ tends to infinity.

This space is a Frechet space with respect to the following set of semi-norms:

$$\|f\|_n = \sup_{\{\vec{\alpha} \in \mathbb{R}_+^n, |\vec{m}| \leq n\}} |(1 + |\vec{\alpha}|)^n f^{(\vec{m})}|,$$

where $\vec{\alpha} = (\alpha_1, ..., \alpha_n)$,

$\vec{m} = (m_1, ..., m_n)$ $m_i \in \mathbb{Z}$,

$|\vec{m}| = m_1 + ... + m_n$,

$|\vec{\alpha}| = \alpha_1 + ... + \alpha_n$,

$f^{(\vec{m})} = \frac{\partial^{\vec{m}} f}{\partial^{m_1} \partial^{m_n}}$.

Denote by $\mathcal{S}'(\mathbb{R}_+^n)$ the topological dual of $\mathcal{S}(\mathbb{R}_+^n)$. The space $\mathcal{S}'(\mathbb{R}_+^n)$ is called the space of distributions on $\mathbb{R}_+^n$. If $f \in \mathcal{S}'(\mathbb{R}_+^n)$ then there exist such $m \in \mathbb{Z}$, $C \in \mathbb{R}_+$ as

$$|\langle f, g \rangle| \leq C \|g\|_m$$

$\forall g \in \mathcal{S}'(\mathbb{R}_+^n)$

(The Laurent Swartz theorem).

Let us denote by $\mathcal{S}'_m(\mathbb{R}_+^n)$ the space of all distributions which are continuous with respect to the norm $\|\|_m$. Let us introduce in $\mathcal{S}'_m(\mathbb{R}_+^n)$ the norm $\|\|'_m$ as follows:

$$\|f\|'_m = \inf \{ C \|\langle f, g \rangle\| \leq C \|g\|_m \forall g \in \mathcal{S}(\mathbb{R}_+^n) \}.$$

We have $\mathcal{S}'_0 \subset \mathcal{S}'_1 \subset ... \subset \mathcal{S}'_n$.

All injection are continuous and $\bigcup_{i=0}^{\infty} S'_i = \mathcal{S}'$.

Let $\mathcal{S}_m$ be a completion of $\mathcal{S}$ with respect to the norm $\|\|_m$. We can see at $\mathcal{S}'_m$ as at dual of the $\mathcal{S}_m$. We have $\mathcal{S}_0 \supset \mathcal{S}_1 \supset ... \supset \mathcal{S}_n$. All injection are continuous. Let us fix an one particle irreducible diagram $\Gamma$. Let us define the Feynman amplitude as a distribution on $(\mathbb{R}_+^n) := R_+^n$ $(n$ is a number of elements of $R_{in}$) by the formula

$$U_\Gamma^\pi(p_1, ..., p_n)(\vec{\alpha})\delta(p_1 +, ..., p_n)$$
\[
= \int \prod_{r \in R_{in}} \alpha_r^z \prod_{r \in R_{in}} dp'_r \prod_{v \in V} \varphi_T(\pm p_{r_1} \ldots \pm p_{r_v}) \delta(\sum_{r \to v} p'_r - \sum_{r \leftarrow v} p'_r) \prod_{r \in R_{in}} e^{-\alpha_r p'_r^2}. \tag{6}
\]

4 Estimates of the Feynman amplitudes

Now we will prove that if \( \text{Re } z \) is enough large the Feynman amplitude is an integrable function. Consider the Feynman graph \( \Phi \) corresponding to \( \Gamma \). Let \( \Phi'_{\Delta} \) be its some maximal tree, and \( R'_{in} \) is a set of all internal lines of \( \Phi \) which do not belong to \( \Phi'_{\Delta} \). The set of momenta \( p'_r, r \in R'_{in} \) (after removing of all \( \delta \)-functions) determines uniquely all others momenta. The momenta \( p'_r, r \in R'_{in} \) are called the loop momenta. We have

\[
U^z_\Gamma(p_1, \ldots, p_n)(\bar{\alpha})\delta(p_1 + \ldots, p_n) = \int \prod_{r \in R_{in}} \alpha_r^z \prod_{r \in R'_{in}} dp'_r \prod_{v \in V} \varphi_T(\pm p_{r_1} \ldots \pm p_{r_v}) \prod_{r \in R_{in}} e^{-\alpha_r p'_r^2}. \tag{7}
\]

The integrand has the form of polynomial multiplied by the Gauss function.

**Lemma.** Let \( P \) be a polynomial of \( x_1, \ldots, x_n \) \( x_i \in \mathbb{R}, i = 1, \ldots, n \) and \( Q \) be a positive definite quadratic form of \( x_1, \ldots, x_n \). Then

\[
| \int |P|e^{-Q}dx_1 \ldots dx_n | \leq C\left(1 + \frac{1}{\lambda_{\min}} + 1\right), \tag{8}
\]

where \( \lambda_{\min} \) is a minimal eigenvalue of \( Q \), and the constant \( C \) do not depend of \( P \).

**Proof.**

To prove the lemma it is enough to consider the integral

\[
\int |x|^p e^{-\lambda x^2} dx \tag{9}
\]

But for this case the lemma is evidence.

So we must to find the lower estimate for the quadratic form \( \sum_r \alpha_r p'_r^2 \).

We suppose that all momenta \( p'_r \) are defined by the loop momenta, and all external momenta are supposed to be equal to zero.

**Lemma.** There exists an absolute constant \( C \) (\( C \) do not depend of \( \alpha_r \)) such that:

\[
Q \geq C(\sum p'_r^2)\min\{\alpha_r\}. \tag{10}
\]
Proof.

Let us use the following electrotechnical analogy. The Feynman graph corresponds to the electrical scheme. Momenta $p_r$ corresponds to the currents. The parameters $\alpha_r$ corresponds to the resistances. The low of momenta conservation corresponds to the first Kirhhoff low. According to the Joule — Lenz low the heat generating by resister number $r$ is equal to $Q_r = \alpha_r p_r^2$. Let $Q$ be a total heat generating by the scheme. Suppose that $Q \leq 1$. The heat generating by resister number $r$ less or equal than $Q$. Therefor $p_r^2 \leq \frac{1}{\alpha_r}$. The lemma is proved.

Lemma. Let $p_1, ..., p_n$ be external momenta. Let $q_r$ be internal momenta corresponding to the minimum of the quadratic form $Q$. Variables $q_r$ satisfy to the following inequality (maximum principle):
\[
\|q_r\| \leq C \|p_r\| \tag{11}
\]
for some constant $C$ that does not depend of $\alpha_r$ and $p_r, p'_r$. It is possible to use an arbitrary norms in this inequality.

Proof.

Let $N$ be a maximal number of lines coming into (or from) a vertex, and $P := \max_{r \in R_{ext}} |p_r|$, $R = \sharp R_{in}$. Put $C = 2PN^R$. Let us prove that for all $r \in R_{in}$ $q_r \leq C$.

Note that $q_r$ satisfy the condition: the fall of voltage at each closed contour is equal to zero (The second Kirhhoff low). But according to the Ohm low the fall of voltage at the resister $r$ is equal to $\alpha_r q_r$. Suppose that there exists a line $r_1 \in R_{in}$ such that $|q_{r_1}| > C$. Let $v$ be a vertex such that a line $r_1$ coming into $v$. Let us consider other lines coming into the vertex $v$. It is evidence that there exists an internal line $r_2 r_1 \neq r_2$ such that $|q_{r_2}| > 2PN^{R-1}$. This fact follows from the first Kirhoff low. By the same way for the line $r_2$ we find the line $r_3$ such that: $|q_{r_3}| > 2PN^{R-2}$ e.t.c. After at most $R$ steps we find lines $r_1, ..., r_m$ such that for each $i = 1, ..., m - 1$ the lines $r_i$ and $r_{i+1}$ ends at a common vertex, the current flows in the same direction at each line and there exists a vertex $v$ and number $i = 1, 2, ... m$ such that the lines $r_i$ and $r_m$ ends at the same vertex. So we have a closed contour such that the fall of voltage at this contour is not equal to zero. This contradiction with the second Kirhoff low proves the lemma.

We find from two previous lemmas that for fixed $p_1, ..., p_n$ there exists a
constant $C$ such that

\[
|U_\Gamma(\vec{\alpha})(p_1, \ldots, p_n)| \leq C \left( \prod_{r \in R_{in}} \alpha_r^z \right) \left\{ \frac{1}{(\min\{\alpha_r\})^{2(\|R_{in}\| - \|V\| + 1) + 1/2\sum_{v \in V} \deg\varphi_v}} + 1 \right\}
\]

\[
= C \left( \prod_{r \in R_{in}} \alpha_r^z \right) \left\{ \frac{1}{(\min\{\alpha_r\})^{1/2\|R_{in}\| + \|R_{in}\| + 1}} + 1 \right\}.
\]  

We see that if $\Re z > 1/2\Omega_{\Gamma} + \|R\|$ then $U_\Gamma(\alpha)(p_1, \ldots, p_n)$ is a continuous function of variables $p_1, \ldots, p_n$. It is also clear that if $\Re z > 1/2\Omega_{\Gamma} + 1/2\|R_{in}\| + m$ than $U_\Gamma(\alpha)(p_1, \ldots, p_n)$ is $m$-times continuously differentiable function.

**Homogeneity of $U_\Gamma(\vec{\alpha})(p_1, \ldots, p_n)$.** Let us introduce an operation $\Lambda_\lambda$ which acts by the formula

\[
(\Lambda_\lambda U_\Gamma)(\vec{\alpha})(p_1, \ldots, p_n) := U_\Gamma(\lambda \vec{\alpha})(\frac{p_1}{\lambda^{1/2}}, \ldots, \frac{p_n}{\lambda^{1/2}}).
\]

It is clear that if all $\varphi_v$ are homogeneous then

\[
\Lambda_\lambda(U_\Gamma) = \frac{1}{\lambda^{1/2\|R_{in}\| + 1/2\|R_{in}\| + \|R_{in}\| + 1/2\|R_{in}\|}} U_\Gamma = \frac{1}{\lambda^{1/2\|R_{in}\| + 1/2\|R_{in}\| + \|R_{in}\|}} U_\Gamma.
\]

5 Local amplitudes, theorems II, III

**Definition.** Let $\Gamma$ be a Feynman diagram. Distribution $C_\Gamma(\vec{\alpha})(p_1, \ldots, p_n)$ is called local if it is a linear combination of $\delta$-function and its derivatives ($\delta(\vec{\alpha}) = \delta(\alpha_1) \ldots \delta(\alpha_n)$) with polynomial on external momenta coefficients, homogenous with respect $\Lambda_\lambda$ of degree $1/2\Omega_{\Gamma} + \|R_{in}\|$. In other words the function is local if it has the form

\[
C_\Gamma(\vec{\alpha})(p_1, \ldots, p_n) = \sum \delta^{(\vec{m})}(\vec{\alpha}) P_{\vec{m}}(p_1, \ldots, p_n),
\]

and

\[
\|\vec{m}\| + 1/2\deg P_{\vec{m}} = (1/2)\Omega_{\Gamma} + \|R_{in}\|.
\]

Let $U_\Gamma$ be a Feynman amplitude, $\gamma$ be a one particle irreducible diagram, and $C_\gamma$ be a local amplitude. Let us define $C_\gamma \ast U_\Gamma$.

Let $(\Gamma/\gamma)^\vec{m}$ be a Feynman diagram such that $\Phi_\Gamma/\Phi_\gamma$ be a corresponding Feynman graph, and the function $\varphi_{(\Gamma/\gamma)^\vec{m}}$ is defined as follows: if $v$ is not a
vertex, obtained by removing of $\Phi$ by a point, then $\varphi_{(\Gamma/\gamma)^n} = \varphi_{\Gamma}$, otherwise $\varphi_{(\Gamma/\gamma)^n} = P_m(p_1, ..., p_n)$. Let

$$(C_\gamma \star U_\Gamma)(\vec{\alpha})(p_1, ..., p_n) = \sum_{\vec{m}} U_{(\Gamma/\gamma)^n}(p_1, ..., p_n) \bigotimes_{\Gamma/\gamma} \delta^{(\vec{m})}(\vec{\alpha}). \quad (17)$$

Here the tensor product $\bigotimes$ has the following meaning. If $f(\vec{\alpha}) = \bigotimes_{r \in R_{\gamma}} f_r(\alpha_r)$ is a distribution on $R^\Gamma$, then $f \bigotimes_{\Gamma/\gamma} g(\vec{\gamma})$ is a distribution on $R^\Gamma$ equal to $\bigotimes_{r \in R_{\gamma}} l(\gamma_r)$, where $l(\gamma_r) = f(\gamma_r)$, if $r \in R_{\gamma}$; $l(\gamma_r) = g(\gamma_r)$, if $r \in R_{\gamma}^m$.

Let $\gamma_1, ..., \gamma_m$ be a set of one particle irreducible diagrams such that $\forall i, j = 1, ..., n$, $i \neq j$ the set of vertices of diagrams $\gamma_i, \gamma_j$ are not intersecting. Let $C_{\gamma_i}$, $i = 1, ..., m$ be some local amplitudes. As above we can define the distribution $C_{\gamma_1} \star ... \star C_{\gamma_m} \star U_\Gamma$. Now let us formulate our main theorem. The Bogoliubov — Parasiuk theorem follows from this theorem immediately.

**Theorem II.** For each one particle irreducible diagram $\Gamma$ there exists a local distribution $C_\Gamma(\vec{\alpha})(p) \in S'(R^\Gamma_+)$, that is at the same time is a polynomial on $\frac{1}{z}$, such that:

a) $$(U^z_\Gamma + \sum_{\gamma_1, ..., \gamma_m \subset \Gamma} C_{\gamma_1}^z \star ... \star C_{\gamma_m}^z \star U^z_\Gamma)$$

has an unique analytical extension in some punctured neighborhood of the point $z = 0$.

b) $$C^z_\Gamma = -T(U^z_\Gamma + \sum_{\gamma_1, ..., \gamma_m \subset \Gamma} C_{\gamma_1}^z \star ... \star C_{\gamma_m}^z \star U^z_\Gamma)$$

$T$ is an operator, which to each function holomorphic in some punctured neighborhood of zero assigns its pole part.
Before we begin to prove this theorem let us give some preliminary definitions.

Let $\Upsilon_{m,n}^\Gamma$ be a space of all functions $f$ belongs to $S_{m}^{\Gamma}$ such that their are smooth functions of $p_1, ..., p_f$ (i.e. $\forall \varphi \in S_{m}^{\Gamma} \langle f, \varphi \rangle$ — is a smooth function of external momenta) and,

$$\|(\Lambda_{\lambda} f)(\tilde{\alpha})(p)\|_m' \leq C_{\varepsilon}^f \lambda^{-\varepsilon - n}$$

(19)

if $|p_1|^2 + ... + |p_f|^2 \leq 1$, $\lambda \leq 1 \forall \varepsilon > 0$. $C_{\varepsilon}^f$ depend of $f$ and $\varepsilon$.

Now let us investigate the behavior of $\langle f(\tilde{\alpha})(p), g(\tilde{\alpha}) \rangle$ at large momenta. If $\|p\|^2 \leq 1$, then

$$\langle f(\tilde{\alpha})(p), g(\tilde{\alpha}) \rangle \leq C_{\varepsilon}^f \|g\|_m.$$  

(20)

Suppose that $\|p\| > 1$. Let $\frac{1}{\lambda^2} = \|p\|$ and let us introduce $p'$ such, that $p = \frac{p'}{\lambda^2}$.

$$|\langle f(\tilde{\alpha})(p), g(\tilde{\alpha}) \rangle| = \langle (\Lambda_{\lambda} f)(\tilde{\alpha})(\lambda^2 p), g(\tilde{\alpha}) \rangle \leq \langle (\Lambda_{\lambda} f)(\tilde{\beta})(p'), g(\lambda \tilde{\beta}) \rangle \lambda^{2 R_{\varepsilon}} \leq C_{\varepsilon}^f \lambda^{2 R_{\varepsilon}} \lambda^{-\varepsilon - n} \|g(\lambda \tilde{\beta})\|_m.$$  

$$\|g(\lambda \tilde{\beta})\|_m \leq \lambda^{-m} \|g(\tilde{\beta})\|_m.$$  

(21)

So we have the following

**Lemma.** If $f \in \Upsilon_{m,n}^\Gamma$, $g \in S_{m}^{\Gamma}$ then $|\langle f(\tilde{\alpha})(p), g(\tilde{\alpha}) \rangle| \leq C_{\varepsilon}^f (1 + |p|^{2(\varepsilon + m + n - 2R_{\varepsilon})}) \|g(\tilde{\alpha})\|_m.$

Let us make a remark. If $f \in \Upsilon_{m,n}^\Gamma$ and $\|(\Lambda_{\lambda} f)(\tilde{\alpha})(p)\|_m' \leq C_{\varepsilon}^f \lambda^{-\varepsilon - n}$, where $|p| \leq 1$ and $\lambda \leq 1$ then $\Lambda_{\mu}(f) \in \Upsilon_{m,n}^\Gamma$ ($\mu < 1$) and $\|(\Lambda_{\lambda}(\Lambda_{\mu} f))(\tilde{\alpha})(p)\|_m' \leq C_{\varepsilon}^{\mu f} \lambda^{-\varepsilon - n}$, where $C_{\varepsilon}^{\mu f} = C_{\varepsilon}^f \mu^{-\varepsilon - n}$.

We say, that the function $f_z \in \Upsilon_{m,n}^\Gamma$ is a holomorphic function of $z$ in the region $O$, if $\forall p \forall g \langle f_z(\tilde{\alpha})(p), g(\tilde{\alpha}) \rangle$ is a holomorphic function of $z$ in this region.

Now we formulate some theorem. Theorem II follows from this theorem.

**Theorem III** For each diagram $\Gamma$ we can construct an local amplitudes $C_{\varepsilon}^{\mu f}(\tilde{\alpha})(p)$, such that the following conditions are satisfied:

a) $C_{\varepsilon}^{\mu f}(\tilde{\alpha})(p)$ is a polynomial on $1/z$.  

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b) 

\[(U_\Gamma^z + \sum_{\gamma_1 \cdots \gamma_n \subset \Gamma} C_{\gamma_1}^z \ast \cdots \ast C_{\gamma_n}^z \ast U_\Gamma^z)\]

has an unique analytical continuation to the punctured neighborhood of the point \(z = 0\).

c) Amplitudes \(C_{\Gamma}^z\) satisfy to the following recurrent relation:

\[C_{\Gamma}^z = -T(U_\Gamma^z + \sum_{\gamma_1 \cdots \gamma_n \subset \Gamma} C_{\gamma_1}^z \ast \cdots \ast C_{\gamma_n}^z \ast U_\Gamma^z). \quad (22)\]

Denote by \((l)U_\Gamma\) some l-th derivatives of the Feynman amplitude \(U_\Gamma\). Let

\[R_\Gamma^z = (1 - T)(U_\Gamma^z + \sum_{\gamma_1 \cdots \gamma_n \subset \Gamma} C_{\gamma_1}^z \ast \cdots \ast C_{\gamma_n}^z \ast U_\Gamma^z). \quad (23)\]

Then

d) i) The function \(R_\Gamma^z\) is holomorphic for enough large \(\text{Re} \ z\) and holomorphic in the whole open complex plane except \(D \setminus \{0\}\), where \(D\) — some discrete set.

ii) \(R_\Gamma^z\) is a smooth function of external momenta and

\[\text{if } \text{Re} \ z > -\epsilon_\Gamma. \quad \text{Here } m \text{ depends of } \Gamma, \ l \text{ is some positive number, } \epsilon_\Gamma \text{ is a positive number, } (\text{Re} \ z)^\sim := \max(-\text{Re} \ z, 0). \text{ Constants } C_\varepsilon \text{ depends continuously on } z \text{ in the whole complex plane except } D \setminus \{0\}.\]

iii) \(R_\Gamma^z\) can be represented as a sum

\[R_\Gamma^z = \sum_{\delta} (R_\Gamma^z)^{(\delta)},\]

where \((R_\Gamma^z)^{(\delta)}\) are homogenous amplitudes with respect \(\Lambda_{\lambda};\)

\[\Lambda_{\lambda}(R_\Gamma^z)^{(\delta)} = (R_\Gamma^z)^{(\delta)} \lambda^{(-\frac{1}{2} + \frac{l}{2} + \frac{m}{2} + y_\delta^z)}, \quad (y_\delta^z > 0) \text{ and meromorphic on } z. \quad (R_\Gamma^z)^{(\delta)} \text{ has poles only at points of the set } D.\]

iv) \(U_\Gamma^z\) is a meromorphic function with poles belongs to \(D.\)
6 Beginning of the proof of theorem iii. Radial integration

We will prove the previous theorem. Our nearest goal is for all $\lambda > 0$ to define the distribution $C_\gamma \star U_\Gamma(\vec{\alpha})(p)\delta(|\vec{\alpha}| - \lambda)$.

Here $\Gamma$ is one particle irreducible diagram, and $\gamma$ is its one particle irreducible subdiagram. $C_\gamma$ is a local amplitude. Let $\delta_{\kappa}(x) = \frac{1}{\kappa} \chi(\frac{x}{\kappa})$, where $\chi$ is a smooth function such that, $\chi > 0$, $\int \chi(x)dx = 1$, supp$\chi \in [-\frac{1}{2}, \frac{1}{2}]$

**Lemma.** For enough large Re $z$ the following limit exists in the sense of distributions:

$$\lim_{\kappa \to +0} C_\gamma \star U_\Gamma(\vec{\alpha})(p)\delta_{\kappa}(|\vec{\alpha}| - \lambda - \kappa). \quad (25)$$

**Proof.** It is enough to prove that for all $\Gamma$ and $\vec{m}$ there exists a limit

$$\lim_{\kappa \to +0} U_\Gamma(\vec{\alpha})(p) \otimes \delta^{(\vec{m})}(\vec{\beta})\delta_{\kappa}(|\vec{\alpha}| + |\vec{\beta}| - \lambda - \kappa). \quad (26)$$

Let us find a value of this distribution at $\Phi(\vec{\alpha}, \vec{\beta})$. By definition of tensor product of distributions we find

$$I_\kappa = \langle U_\Gamma(\vec{\alpha})(p) \otimes \delta^{(\vec{m})}(\vec{\beta})\delta_{\kappa}(|\vec{\alpha}| + |\vec{\beta}| - \lambda - \kappa), \Phi(\vec{\alpha}, \vec{\beta}) \rangle$$

$$= \langle \delta^{(\vec{m})}(\vec{\beta}) U_\Gamma(\vec{\alpha})(p)\delta_{\kappa}(|\vec{\alpha}| + |\vec{\beta}| - \lambda - \kappa), \Phi(\vec{\alpha}, \vec{\beta}) \rangle. \quad (27)$$

Let $\eta_{\lambda}(\vec{\beta})$ be a function which is equal to 1 in some neighborhood of zero and equal to zero if $|\vec{\beta}| \geq \frac{1}{2}$. We have

$$I_\kappa = \langle \delta^{(\vec{m})}(\vec{\beta})\eta_{\lambda}(\vec{\beta}) U_\Gamma(\vec{\alpha})(p)\delta_{\kappa}(|\vec{\alpha}| + |\vec{\beta}| - \lambda - \kappa), \Phi(\vec{\alpha}, \vec{\beta}) \rangle. \quad (28)$$

The function $U_\Gamma(\vec{\alpha})$ is enough times differentiable if Re $z$ is enough large, and for each positive integer $n$ there exists a real number $A$ such that $U^{\vec{m}}(\vec{\alpha}) = 0$ if Re $z > A$, $|\vec{m}| \leq n$ and for some $i = 1, ..., 2R \alpha_i = 0$. The function

$$\eta_{\lambda}(\vec{\beta}) U_\Gamma(\vec{\alpha})(p)\delta_{\kappa}(|\vec{\alpha}| + |\vec{\beta}| - \lambda - \kappa), \Phi(\vec{\alpha}, \vec{\beta}) \rangle \quad (29)$$

tends to

$$\eta_{\lambda}(\vec{\beta}) \int d\vec{\alpha} U_\Gamma(\vec{\alpha})(p)\delta(|\vec{\alpha}| + |\vec{\beta}| - \lambda)\Phi(\vec{\alpha}, \vec{\beta}) =: f(\vec{\beta}) \quad (30)$$
with respect the norm \( \|\cdot\|_m \). So the limit \( \lim_{\kappa \to 0} I_\kappa \) exists and equal to \( \langle \delta^{(m)}(\beta), f(\beta) \rangle \). The lemma is proved.

The aim of the following lemma is to extract radial integration in the following expression

\[
\langle C_\gamma \ast U_\Gamma(\vec{\alpha})(p), \Phi(\vec{\alpha}) \rangle.
\]

**Lemma.** If \( \text{Re} \ z \) is enough large then

\[
\langle C_\gamma \ast U^z_\Gamma, (\vec{\alpha})(p), \Phi(\vec{\alpha}) \rangle
= \int_0^\infty d\lambda \langle C_\gamma \ast U^z_\Gamma(\vec{\alpha})(p), \delta(\lambda - |\vec{\alpha}|)\Phi(\vec{\alpha}) \rangle.
\]

**Proof.** It is enough to prove the lemma for the distribution

\[
U^z_\Gamma, (\vec{\alpha}) \otimes \delta^{\vec{m}}(\beta)
= (\alpha_1, \ldots, \alpha_k).
\]

Let \( \eta_\delta \) has the same meaning as in the previous lemma. If \( \text{Re} \ z \) is enough large \( U_\Gamma(\vec{\alpha})(1 - \eta_\delta(|\vec{\alpha}|)) \to U_\Gamma(\vec{\alpha}) \) in the sense of topology of \( S' \) as \( \delta \to 0 \).

The tensor product of distributions is separately continuous. So

\[
I = \langle U_\Gamma, (\vec{\alpha}) \otimes \delta^{(\vec{m})}(\beta), \Phi(\vec{\alpha}, \vec{\beta}) \rangle
= \lim_{\delta \to 0} \langle U_\Gamma(\vec{\alpha})(1 - \eta_\delta(|\vec{\alpha}|)) \otimes \delta^{(\vec{m})}(\beta), \Phi(\vec{\alpha}, \vec{\beta}) \rangle.
\]

Let \( \delta_\kappa \) has the same meaning as in the previous lemma. Note that

\[
\int d\lambda \delta_\kappa(\lambda - |\vec{\beta}| - |\vec{\alpha}| - \kappa) = 1.
\]

We have by using the standard argumentation based on an approximation of the integral by means Riemann sums:

\[
I = \lim_{\delta \to 0} \int_0^\infty d\lambda \langle U_\Gamma(\vec{\alpha})(1 - \eta_\delta(|\vec{\alpha}|)) \otimes \delta^{(\vec{m})}(\beta), \Phi(\vec{\alpha}, \vec{\beta})\delta_\kappa(\lambda - |\vec{\beta}| - |\vec{\alpha}| - \kappa) \rangle.
\]

We have by definition of the tensor product of distributions:

\[
I = \lim_{\delta \to 0} \int_0^\infty d\lambda \langle \delta^{(\vec{m})}(\beta) \langle U_\Gamma(\vec{\alpha})(1 - \eta_\delta(|\vec{\alpha}|)), \delta_\kappa(\lambda - |\vec{\beta}| - |\vec{\alpha}| - \kappa)\Phi(\vec{\alpha}, \vec{\beta}) \rangle.
\]

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Let us fix some positive $\lambda$. The following function of $\tilde{\beta}$
\[
\langle U_\Gamma(\bar{\alpha})(1 - \eta_\delta(|\alpha|)), \delta_\kappa(\lambda - |\tilde{\beta}| - |\alpha| - \kappa)\Phi(\bar{\alpha}, \tilde{\beta}) \rangle.
\]
(38)

has the following limit in $S^m$ as $\kappa \to 0$:
\[
\langle U_\Gamma(\bar{\alpha})(1 - \eta_\delta(|\alpha|))\delta(\lambda - |\tilde{\beta}| - |\alpha|), \Phi(\bar{\alpha}, \tilde{\beta}) \rangle,
\]
(39)

and integrand in (36) has an integrable majorant. Therefore
\[
I = \lim_{\delta \to 0} \int_0^\infty d\lambda \langle \delta^{(\bar{m})}(\tilde{\beta}) U_\Gamma(\bar{\alpha})(1 - \eta_\delta(|\alpha|))\delta(\lambda - |\tilde{\beta}| - |\alpha|), \Phi(\bar{\alpha}, \tilde{\beta}) \rangle.
\]
(40)

Note that the integrand in the right hand side of the last equality has the following limit as $\delta \to 0$
\[
\langle \delta^{(\bar{m})}(\tilde{\beta}) U_\Gamma(\bar{\alpha})\delta(\lambda - |\tilde{\beta}| - |\alpha|), \Phi(\bar{\alpha}, \tilde{\beta}) \rangle.
\]
(41)

If Re $z$ is enough large $U_\Gamma(\bar{\alpha})$ tends to zero enough fast with some number of its derivatives as $|\alpha| \to 0$ and the integrand in (40) has an integrable majorant.

So we can finish the proof of the lemma by using Lebegues theorem.

It is obvious that these two lemmas are hold for
\[
C_{\gamma_1} \ast \ldots \ast C_{\gamma_n} \ast U_\Gamma.
\]
(42)

We have proved this lemmas only for particular case in the purpose of simplicity.

Suppose that the theorem is proved for all diagrams such that their have less than $N$ vertices.

The basis of induction is evidence because all diagrams with one vertex are tree diagrams.

Let us define
\[
\tilde{R}_\Gamma^z = (U_\Gamma^z + \sum_{\gamma_1 \ast \ldots \ast \gamma_n \subset \Gamma} C_{\gamma_1}^z \ast \ldots \ast C_{\gamma_n}^z \ast U_\Gamma^z).
\]
(43)

Let $g \in S(R_\Gamma^z)$ and Re $z$ is enough large. It follows from two previous lemmas that:
\[
\langle \tilde{R}_\Gamma^z(\bar{\alpha})(p), g(\bar{\alpha}) \rangle = \int_0^\infty d\lambda \langle \tilde{R}_\Gamma^z(\bar{\alpha})(p)\delta(\lambda - |\bar{\alpha}|), g(\bar{\alpha}) \rangle.
\]
(44)
Let us use the following substitution in the right hand side of last equation $\vec{\alpha} = \lambda \vec{\beta}$. We find

$$\langle \tilde{R}_t^z(\vec{\alpha})(p), g(\vec{\alpha}) \rangle = \int_0^\infty d\lambda \lambda^{2R_t} \langle \Lambda_\lambda(\tilde{R}_t^z)(\sqrt{\lambda} p) \delta(1 - |\vec{\beta}|), g(\lambda \vec{\beta}) \rangle.$$ (45)

**7 Decomposition of unite. Factorization of R-operation**

Note that the ”integration” in brackets in the right hand side of (44) is over some simplex $T$. Baricentric coordinates on it are $\beta_1, \ldots, \beta_{2R_t}$. Now let us construct some decomposition of unit on $T$. Let us decompose $T$ into $2^n$ nonintersecting sets $O_A$, where $A$ is a subset of $\{1, 2, \ldots, n\}$, $n = \#R$.

$$O_A = \{ \vec{\alpha} \in T | a_i > \delta \text{ if } i \notin A, a_i < \delta \text{ if } i \in A \}. \quad (46)$$

$\delta$ is very small. Note that the set $O_{\{1,2,\ldots,n\}}$ is empty, because $|\vec{\alpha}| \equiv 1$ on $T$.

Let us now consider the following closed sets

$$\tilde{O}_A = \{ \vec{\alpha} \in T | a_i \geq \delta(1 - \gamma) \text{ if } i \notin A, a_i \leq \delta(1 + \gamma) \text{ if } i \in A \}; \quad (47)$$

$\gamma$ is very small. We have $O_A \subseteq \tilde{O}_A$, $\tilde{O}_{\{1,2,\ldots,n\}} = \emptyset$. Let us consider the following sets $X_A = \{ \vec{\alpha} | a_i = 0 \text{ if } i \in A \}$. The sets $X_A$ are closed. We have $X_A \cap \tilde{O}_{A'} = \emptyset$, if $A \notin A'$ and $X_A \cap O_{A'} = \emptyset$, if $A \subseteq A'$.

Let $\eta_A(\vec{\alpha})$ be a function which is equal to 1 on $O_A$ and which is equal to zero outside the region $\tilde{O}_A$. It is obvious that $\sum_A \eta_A > 0$. Consider the following functions $\tilde{\eta}_A(\vec{\alpha}) = \frac{\eta_A(\vec{\alpha})}{\sum_A \eta_A(\vec{\alpha})}$. It is obvious that the set of this functions is a decomposition of unit on $T$. One can think that $\tilde{\eta}(\vec{\alpha})$ can be extended to the decomposition of unit in some neighborhood of $T$ in $\mathbb{R}^n$. Note that $\tilde{\eta}_{A'}(\vec{\alpha})$ is equal to zero in some small neighborhood of $X_A$ if $A \notin A'$.

We can rewrite the right hand side of (45) as follows:

$$\langle \tilde{R}_t^z(\vec{\alpha})(p), g(\vec{\alpha}) \rangle = \sum_{A \subseteq \{1,2,\ldots,n\}} \int_0^\infty d\lambda \lambda^{2R_t} \langle \Lambda_\lambda(\tilde{R}_t^z)(\sqrt{\lambda} p) \delta(1 - |\vec{\beta}|), \tilde{\eta}_A(\vec{\beta}) g(\lambda \vec{\beta}) \rangle.$$ (48)
Let us introduce the following notations. For each $A \subset \{1, 2, \ldots, n\}$ assign the subdiagrams $\gamma^A_1, \ldots, \gamma^A_l$ as follows. To the set $A \subset \{1, 2, \ldots, n\}$ corresponds some set of lines. If we paint these lines in red color we obtain some subgraph in $\Phi_\Gamma$. Consider all one particle irreducible components of this subgraph. Let us complete these components by external lines. In result we obtain subdiagrams $\gamma^A_1, \ldots, \gamma^A_l$, where $l$ is a number of one particle irreducible components. We have

$$\langle \tilde{R}_\Gamma^z(\bar{\alpha})(p), g(\bar{\alpha}) \rangle$$

$$= \sum_{A \subset \{1, 2, \ldots, n\}} \int_0^\infty d\lambda \lambda^{2R_\Gamma - 1} \langle \Lambda_\lambda(\{1 + \sum_{\gamma_1 \cdots \gamma_n \subset \Gamma} C^z_{\bar{\gamma}_1} \ast \cdots \ast C^z_{\bar{\gamma}_n}\} \ast U_\Gamma^z)(\bar{\beta})(\sqrt{\lambda} p) \times \delta(1 - |\bar{\beta}|), \tilde{\eta}_A(\bar{\beta})g(\lambda \bar{\beta}) \rangle. \quad (49)$$

Note that there absents all terms in the internal sum, corresponding to sets $\gamma_1 \ast \cdots \ast \gamma_n \subset \Gamma$ such that for some $i$ the subdiagram $\gamma_i$ are not contained in $\gamma^A_k$ for all $k = 1, \ldots, l$. We can factorize the contribution of all other terms. We have, in evidence notations:

$$\langle \tilde{R}_\Gamma^z(\bar{\alpha})(p), g(\bar{\alpha}) \rangle = \sum_{A \subset \{1, 2, \ldots, n\}} \int_0^\infty d\lambda \lambda^{2R_\Gamma - 1}$$

$$\langle \Lambda_\lambda(\{\prod_{\gamma^A_i} \{1 + \sum_{\gamma_1 \cdots \gamma_n \subset \Gamma} C^z_{\bar{\gamma}_1} \ast \cdots \ast C^z_{\bar{\gamma}_n}\} \ast U_\Gamma^z)(\bar{\beta})(\sqrt{\lambda} p) \times \delta(1 - |\bar{\beta}|), \tilde{\eta}_A(\bar{\beta})g(\lambda \bar{\beta}) \rangle. \quad (50)$$

Before analyse this expression let us introduce some new definitions. Let $\Gamma$ be one particle irreducible diagram. Let us introduce the following amplitude $U_\Gamma^z(p)(\bar{\alpha})[q]$ which depends on loop momenta $q$ by the following formula:

$$U_\Gamma^z(p_1, \ldots, p_n)(\bar{\alpha})[q] = \prod_{r \in R_{in}} \alpha_r \prod_{r \in V} \varphi_r(\pm p_{r_1} \pm \cdots \pm p_{p_{r_k}}) \prod_{r \in R_{in}} e^{-\alpha_r p_r^2}. \quad (51)$$

It is supposed at last formula that momenta in the right hand side of this formula are expressed trough $q$. To point out the fact that some expression depends on $q$ we use the symbol $[q]$. Analogously we can define:

$$(C_\gamma \ast U_\Gamma)(\bar{\alpha})(p_1, \ldots, p_n)[q] = \sum_{\gamma} U_{(\gamma/\gamma)^\bar{\alpha}}(p_1, \ldots, p_n) \bigotimes_{\gamma/\gamma} \delta(\alpha)[q]. \quad (52)$$
This quantity depends on loop momenta of $\Gamma/\gamma$. Here we use denotations from the page (11). It is easy to see, (as in lemma 2), that $U^{\tilde{\gamma}}(p)(\tilde{\alpha})[q]$ can be multiplied by $\delta(\lambda - |\tilde{\alpha}|)$ and it is possible to extract integration on $\lambda$ (If Re $z$ is enough large).

Let $\Gamma$ be one particle irreducible diagram and $\gamma_1, \ldots, \gamma_k$ be a set of one particle irreducible subdiagrams such that for each $i, j = 1, \ldots, k$ $i \neq j$ the sets of vertices of diagrams $\gamma_i$ and $\gamma_j$ do not intersect. Let us define the quotient diagram

$$\Gamma/\gamma_1 \ast \ldots \ast \gamma_k.$$  

(53)

We replace each subdiagram $\gamma_i$ by a vertex $\tilde{v}_i$ and put

$$\varphi_{\Gamma/\gamma_1 \ast \ldots \ast \gamma_k}(v) = \varphi_{\Gamma}(v)$$  

(54)

if $v \neq \tilde{v}_i$ and $\forall i = 1, \ldots, k$

$$\varphi_{\Gamma/\gamma_1 \ast \ldots \ast \gamma_k}(\tilde{v}_i) = 1.$$  

(55)

Let $\gamma \subset \gamma' \subset \Gamma$ be one particle irreducible diagrams. Let $[q]_{\Gamma/\gamma'}$ be a set of loop momenta of $\Gamma/\gamma'$. Let us define $(C_{\gamma} \ast U_{\Gamma})(p_1, \ldots, p_n)[q]_{\Gamma/\gamma'}$ by the formula

$$(C_{\gamma} \ast U_{\Gamma})(p_1, \ldots, p_n)[q]_{\Gamma/\gamma'} = \sum_{\vec{m}} \int [dq]_{\gamma/\gamma'}(U_{\Gamma/\gamma'}^{\vec{m}})(p_1, \ldots, p_n)[q]_{\Gamma/\gamma'} \times_{\Gamma/\gamma} \delta(\vec{m})((\tilde{\alpha})).$$  

(56)

Now let us consider the following situation. We have two one particle irreducible diagrams $\gamma \subset \gamma' \subset \Gamma$ and a set $A$ of painted lines of $\Gamma$ which are satisfy the following condition: If we replace $\gamma'$ by a point the set $A$ becomes a tree. Let $\tilde{\eta}_A$ be a function which have been previously described. In this situation the following lemma holds.

**Lemma.** If Re $z$ is enough large than:

$$(C_{\gamma} \ast U_{\Gamma})(\tilde{\alpha})(p)\delta(|\tilde{\alpha}| - \lambda), \tilde{\eta}_A(\tilde{\alpha})g(\tilde{\alpha}))$$

$$= \sum_{\vec{m}} \int [dq]_{\Gamma/\gamma'}((U_{\Gamma/\gamma'}^{\vec{m}})(p)[q]_{\Gamma/\gamma'} \times_{\Gamma/\gamma} \delta(\vec{m}))(\tilde{\alpha})\delta(|\tilde{\alpha}| - \lambda), \tilde{\eta}_A(\tilde{\alpha})g(\tilde{\alpha})).$$  

(57)
Proof.
Let us divide variables \( \alpha \) into three groups:
- Variables \( \beta' \) correspond to the lines of \( \gamma \).
- Variables \( \beta'' \) corresponds to the lines of \( \gamma'/\gamma \).
- Variables \( \beta''' \) corresponds to the lines of \( \Gamma/\gamma' \). We have

\[
\langle (C_\gamma * U_\Gamma)(p) \delta(|\alpha| - \lambda), \tilde{\eta}_A(\tilde{\alpha})g(\tilde{\alpha}) \rangle = \sum_m \langle \delta(m) (\tilde{\beta}'(U^m_{(\Gamma/\gamma)}(p)(\tilde{\beta}'', \tilde{\beta}''')\delta(|\tilde{\beta}'| + |\tilde{\beta}''| + |\tilde{\beta}'''| - \lambda), \tilde{\eta}_A(\tilde{\beta}', \tilde{\beta}'', \tilde{\beta}''')g(\tilde{\beta}', \tilde{\beta}'', \tilde{\beta}''')) \rangle. \tag{58}
\]

We have

\[
\langle U^m_{(\Gamma/\gamma)}(p)(\tilde{\beta}'', \tilde{\beta}''')\delta(|\tilde{\beta}'| + |\tilde{\beta}''| + |\tilde{\beta}'''| - \lambda), \tilde{\eta}_A(\tilde{\beta}', \tilde{\beta}'', \tilde{\beta}''')g(\tilde{\beta}', \tilde{\beta}'', \tilde{\beta}''') \rangle = \int d\tilde{\beta}''d\tilde{\beta}'''\delta(|\tilde{\beta}'| + |\tilde{\beta}''| + |\tilde{\beta}'''| - \lambda),
\]

\[
= \{\tilde{\eta}_A(\tilde{\beta}', \tilde{\beta}'', \tilde{\beta}''')g(\tilde{\beta}', \tilde{\beta}'', \tilde{\beta}''')\} \int [dq]_{\Gamma/\gamma'} U^m_{(\Gamma/\gamma)}(p)[q]_{\Gamma/\gamma'}(\tilde{\beta}'', \tilde{\beta}'''). \tag{59}
\]

It is easy to see that all variables \( \tilde{\beta}''' \) except variables corresponding to some tree graph separated from zero. Variables \( \tilde{\beta}'' \) take values in some bounded region. The function \( U^m_{(\Gamma/\gamma)}(p)[q]_{\Gamma/\gamma'} \) is a function of fast decay at infinity (as \( e^{-A|q|^2}, A > 0 \)) on \( [q]_{\Gamma/\gamma'} \). Moreover, if \( [q]_{\Gamma/\gamma'} \) and \( \tilde{\beta}', \tilde{\beta}'' \) take a value in some bounded region, this function is uniformly continuous on \( [q]_{\Gamma/\gamma'} \) and \( \tilde{\beta}', \tilde{\beta}'' \) in this region. By using this remark we can approximate the integral over \( d[q]_{\Gamma/\gamma'} \) in (59) by Riemann sum. By other words:

\[
\sum_{[q]_{\Gamma/\gamma'}} [\Delta q]_{\Gamma/\gamma'} \int d\tilde{\beta}''d\tilde{\beta}'''\delta(|\tilde{\beta}'| + |\tilde{\beta}''| + |\tilde{\beta}'''| - \lambda),
\]

\[
\{\tilde{\eta}_A(\tilde{\beta}', \tilde{\beta}'', \tilde{\beta}''')g(\tilde{\beta}', \tilde{\beta}'', \tilde{\beta}''')U^m_{(\Gamma/\gamma)}(p)[q]_{\Gamma/\gamma'}(\tilde{\beta}'', \tilde{\beta}''') \rightarrow 59. \tag{60}
\]

uniformly on \( \beta' \) at each compact \( \tilde{\beta}' \).

Analogously, it is easy to prove, that the left hand side of (60) tends to the right hand side of (60) in the topology of \( C|\tilde{m}| \) at each compact if Re \( z \) is
enough large. Therefore the Riemann sums for the integral in the right hand side of (57) tend to the left hand side of (57). It is easy to prove that the integral in the right hand side exist also in the Lebegues sense. The lemma is proved.

The trivial generalization of the previous lemma is the following lemma.

**Lemma.** Let $A$ be a some subset of the set of lines of $\Gamma$, and $\gamma^A_1, ..., \gamma^A_f$ be corresponding subdiagrams. Let us divide the parameters $\{\alpha\}$ into $f + 1$ groups; $\vec{\beta}_i$ are parameters corresponding to $\gamma^A_i$ and $\vec{\beta}''$ are the other parameters. If $\text{Re } z$ is enough large, we have:

$$
\langle \tilde{R}^z_\Gamma(\vec{\alpha})(p)\delta(1 - |\vec{\alpha}|), \tilde{\eta}_A(\vec{\alpha})g(\vec{\alpha}) \rangle = \int [dq]_{\Gamma/\gamma^A_1 \ast \ldots \ast \gamma^A_f}^f \left( \bigotimes_{i=1}^f R^z_{\gamma_i}(\vec{\beta}_i)(p)[q]_{\Gamma/\gamma^A_1 \ast \ldots \ast \gamma^A_f} \{ \int d\vec{\beta}'' \delta(1 - \sum_{i=1}^f |\vec{\beta}_i| - |\vec{\beta}''|) \tilde{\eta}_A(\vec{\beta}_1, ..., \vec{\beta}_f, \vec{\beta}'') U_{\Gamma/\gamma^A_1 \ast \ldots \ast \gamma^A_f}(\vec{\alpha})[q]_{\Gamma/\gamma^A_1 \ast \ldots \ast \gamma^A_f}(p)g(\vec{\alpha}) \} \right). \quad (61)
$$

Here $R^z_{\Gamma}(\vec{\beta}_i)(p)[q]_{\Gamma/\gamma^A_1 \ast \ldots \ast \gamma^A_f}$ depends on the external momenta and the loop momenta $[q]_{\Gamma/\gamma^A_1 \ast \ldots \ast \gamma^A_f}$. We will prove below that the right hand side of (61) has an unique analytical on whole open complex plane except $D \setminus \{0\}$ ($D$ is some discrete set). Analogously:

$$
\langle \tilde{R}^z_\Gamma(\vec{\alpha})(p)\frac{\lambda}{\sqrt{\lambda}}\delta(\lambda - |\vec{\alpha}|), \tilde{\eta}_A(\vec{\alpha})g(\vec{\alpha}) \rangle = \int [dq]_{\Gamma/\gamma^A_1 \ast \ldots \ast \gamma^A_f}^f \left( \bigotimes_{i=1}^f (\Lambda_{\gamma_i} R^z_{\gamma_i})(\vec{\beta}_i)(p)[q]_{\Gamma/\gamma^A_1 \ast \ldots \ast \gamma^A_f} \{ \delta(1 - \sum_{i=1}^f |\vec{\beta}_i| - |\vec{\beta}''|) \tilde{\eta}_A(\vec{\beta}_1, ..., \vec{\beta}_f, \vec{\beta}'') U_{\Gamma/\gamma^A_1 \ast \ldots \ast \gamma^A_f}(\vec{\alpha})[q]_{\Gamma/\gamma^A_1 \ast \ldots \ast \gamma^A_f}(p)g(\lambda \vec{\alpha}) \} \right). \quad (62)
$$

Here $L_{\Gamma/\gamma^A_1 \ast \ldots \ast \gamma^A_f}$ is a number of loop of the diagram.
8 Estimates of the integrand in the integral over \(d\lambda\)

Note that the expression in the figured bracket in (62) is an infinitely differentiable function and its norm \(\|\cdot\|_m\) admit an estimate: there exists a constant \(A > 0\) such that for each \(N\) there exists a constant \(C > 0\) and a polynomial \(P(p, [q]_{\Gamma/\gamma_1^A \ldots \gamma_f^A})\) such that

\[
\|\cdot\|_m \leq C|P(p, [q]_{\Gamma/\gamma_1^A \ldots \gamma_f^A})| \frac{1}{(1 + \lambda)^N} e^{-A|q|^2_{\Gamma/\gamma_1^A \ldots \gamma_f^A}}, \tag{63}
\]

for some polynomial \(P\) and an arbitrary positive integer \(m\). The constant \(C\) depends on \(N\). We have by using the inductive assumption d) ii)

\[
|\langle \tilde{R}_T^{\bar{\gamma}}(\tilde{\alpha})(\frac{p}{\sqrt{\lambda}})\delta(\lambda - |\tilde{\alpha}|), \tilde{\eta}_A(\frac{\tilde{\alpha}}{\lambda})g(\tilde{\alpha}) \rangle| \leq C\frac{\lambda^{-\varepsilon}}{(1 + \lambda)^N} \prod \{\lambda^{-\frac{\theta_v}{2}}
\]

\[
(1 + z)^\varepsilon R_{\Gamma/\gamma_1^A \ldots \gamma_f^A} - \frac{1}{2} \{4L_{\Gamma/\gamma_1^A \ldots \gamma_f^A} + \sum_{v \in \Gamma/\gamma_1^A \ldots \gamma_f^A} \text{deg} \phi(v)\} \}
\]

\[
\leq C\frac{\lambda^{-\varepsilon}}{(1 + \lambda)^N} \lambda^{-\frac{\theta_v}{2} - 1} \lambda^{-x_T(\text{Re} z)^-}. \tag{64}
\]

Estimates on derivatives on external momenta. By using differentiability by external momenta and inductive assumption on \(R\) we have:

\[
|\langle (i) \tilde{R}_T^{\bar{\gamma}}(\tilde{\alpha})(p)(\frac{\sqrt{\lambda}}{\lambda})\delta(\lambda - |\tilde{\alpha}|), \tilde{\eta}_A(\frac{\tilde{\alpha}}{\lambda})g(\tilde{\alpha}) \rangle| \leq C\frac{\lambda^{-\varepsilon + \frac{1}{2}}}{(1 + \lambda)^N} \lambda^{-\frac{\theta_v}{2} - 1} \lambda^{-x_T(\text{Re} z)^-}. \tag{65}
\]

Holomorphic property. Let us prove

\[
\langle \tilde{R}_T^{\bar{\gamma}}(\tilde{\alpha})(\frac{p}{\sqrt{\lambda}})\delta(\lambda - |\tilde{\alpha}|), \tilde{\eta}_A(\frac{\tilde{\alpha}}{\lambda})g(\tilde{\alpha}) \rangle \tag{66}
\]

is a holomorphic function of \(z\) on the \(\{z \in \mathbb{C} | \text{Im} z > A\}\) for some positive \(A\). This quantity can be represented as an integral over \([dq]_{\Gamma/\gamma_1^A \ldots \gamma_f^A}\). It is
clear that the integrand holomorphic on \( z \). By using the Cauchy estimates, we find that the derivative on \( z \) of the integrand increases slowly than some polynomials \( \times e^{-Ap^2} \). From other hand the derivative of the integrand on the loop momenta increase analogously. Therefore the integrand is uniformly continuous on \( z \) and \( p \) if \( z \) belongs to some compact \( K \) such that \( K \subset \{ z | z \in \mathbb{C} : \text{Im} z > A \} \) for some positive \( A \). Therefore the integrand is measurable. Analogously one can prove that (65) is continuous on \( z \) on just described set. Let \( \pi \) be a small contour homotopic in \( \{ z \in \mathbb{C} | z > A \} \). Let us consider the integral of absolute value of integrand in (62) over the direct product of our contour and the space of external momenta. If we integrate at first (62) over \( \pi \) by \( z \) and at second by loop momenta we obtain a zero. By using Morera’s and Fubini’s theorem one has that (65) is holomorphic on \( z \) in our region.

9 End of the proof. Check of the inductive assumptions

The analytical extension of \( \tilde{R}^z_\Gamma(\tilde{\alpha})(p) \). Let \( M_{\Omega_\Gamma} \) be a Taylor projector which to each function of external momenta of diagram \( \Gamma \) assigns its Taylor polynomial with the center at zero of degree \( \Omega_\Gamma \). \( 1 - M_{\Omega_\Gamma} \) by Schlomilch theorem can be expressed through the integral of partial derivatives of degree \( \Omega_\Gamma + 1 \). By using the estimate (65) and an argumentation based on The Morera theorem we find, that \( (1 - M_{\Omega_\Gamma})\tilde{R}^z \) has an analytical continuation into some neighborhood of zero. If we use the substitution \( \tilde{\alpha} \mapsto \lambda\tilde{\alpha}, p \mapsto \frac{p}{\sqrt{\lambda}} \) and estimate (64) we find, that

\[
(l)(1 - M_{\Omega_\Gamma})\tilde{R}^z \in \Upsilon^m_{\Gamma}((1/2)\Omega_\Gamma + \frac{\Omega_\Gamma}{2} + x_1^\Gamma(\text{Re } z)^-).
\]

In other words for \( (1 - M_{\Omega_\Gamma})\tilde{R}^z \) the inductive assumption d) ii) holds.

Now let us investigate \( M_{\Omega_\Gamma} \tilde{R}^z(\tilde{\alpha})(p) \). Let \( M^n_\Gamma \) be a projector on homogeneous polynomials of degree \( n \). We have

\[
M^n_\Gamma \tilde{R}^z(\tilde{\alpha})(p)
= \int_0^\infty d\lambda \lambda^{\Omega_\Gamma - 1 + \frac{n}{2}} \langle A_\lambda (M^n_\Gamma \tilde{R}^z_\Gamma)(\tilde{\alpha})(p) \delta(1 - |\tilde{\alpha}|), g(\lambda\tilde{\alpha}) \rangle.
\]
For each diagram $\Gamma \tilde{R}^z$ can be decomposed (by inductive assumption) into the sum of homogenous functions with respect the operation $\Lambda_\lambda (R^z)^{\delta}$ of degree $\frac{\Omega^2}{2} + y^z_{\gamma} z$. Therefore (67) is the sum of terms of degree

$$M^n_{\Gamma} \tilde{R}^z (\vec{\alpha})(p)$$

$$= \sum_A \int_0^\infty d\lambda \lambda^{-1 - \frac{\Omega^2}{2} + y^z_{\gamma} z} (M^n_{\Gamma} (\tilde{R}^z)^{\delta}(\vec{\alpha})(p)\delta(1 - |\vec{\alpha}|), \tilde{\eta}_A g(\lambda \vec{\alpha}))$$

(69)

for some $y^z_{\gamma, A}$. By using the standard analytical property of the distribution $\theta(\lambda)\lambda^z$ we find, that (68) has an analytical continuation into some punctured neighborhood of zero. So we have prove the inductive assumption b).

**Locality of counterterms $C_{\Gamma}$ (beginning).** $C_{\Gamma}$ is a polynomial on the external momenta. This fact follows from the fact that $(1 - M^n_{\Gamma}) \tilde{R}^z$ has an analytical continuation into some neighborhood of zero. Let us prove that $C_{\Gamma}$ is a finite linear combination of $\delta$-functions on $\vec{\alpha}$. The pole part came from $M^n_{\Gamma} \tilde{R}^z$, more precisely from:

$$\int_0^1 d\lambda\lambda^{2R - 1 + \frac{\bar{z}}{2}} \langle \Lambda_\lambda (M^n_{\Gamma} \tilde{R}^z)(\vec{\alpha})(p)\delta(1 - |\vec{\alpha}|), g(\lambda \vec{\alpha}) \rangle.$$  

(70)

Let us write a representation

$$g(\lambda \vec{\alpha}) = \sum_{k=1}^\Phi \lambda^k \frac{d^k}{d\lambda^k} g(\lambda \vec{\alpha})|_{\lambda=0} + \lambda^{\Phi+1} \psi(\lambda)$$

(71)

$\psi(\lambda)$ is a smooth function. The contribution into the pole part comes from the terms of the form

$$\int_0^1 d\lambda\lambda^{2R - 1 + \frac{\bar{z}}{2}} \langle \Lambda_\lambda (\tilde{R}^z)(\vec{\alpha})(p)\delta(1 - |\vec{\alpha}|), \tilde{\eta}_A (\frac{d}{d\lambda})^k g(\lambda \vec{\alpha})|_{\lambda=0} \rangle.$$  

(72)

These terms has an analytical continuation into some punctured neighborhood of the point $z = 0$. We have to prove that these terms have the form of linear combination of $\delta$-functions and their derivatives. But

$$\frac{d}{d\lambda} g(\lambda \vec{\alpha}) = g(0),$$

$$\frac{d}{d\lambda} (\frac{\partial}{\partial \alpha_i} g(\lambda \vec{\alpha})|_{\vec{\alpha}=0}) = \sum_i \alpha_i (\frac{\partial}{\partial \alpha_i} g(\lambda \vec{\alpha})|_{\vec{\alpha}=0}),$$

(73)

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The proof of the point d) i) is analogues to the proof of the point b). We need only the Taylor projector $M_{Ω}$ replace by $M_{N}$ and chose the number $N$ enough large.

The proof of the points d) iii) and d) iv). We have proved before that the functions $\tilde{R}^{z}$ is a sum of meromorphic functions with respect to the operation $Λ_{λ}$. This fact and the inductive assumptions imply that the function $U_{ Γ}$ is meromorphic too. This fact and the formula for $R$-operation implies that the function $\tilde{R}^{z}$ can be represented as a sum of homogenous function with proper coefficients of homogeneity.

Locality of counterterms $C_{ Γ}$ (the end). Now let us prove that the constructed counterterms have a proper power of homogeneity with respect $Λ_{λ}$. This statement follows from the facts that counterterms are a) the linear combination of homogenous functions, b) the sum of the pole parts of holomorphic functions in some neighborhood of zero, which power of homogeneity is equal to $\frac{Ω_{ Γ}}{2} + 2^{R} - y_{Γ} δ_{Γ}$. So the inductive assumption c) is proved.

The inductive assumption a) follows from the form of subtract operator. Proof of the point d) ii). The fact that

$$
(1 - M_{Ω}) \tilde{R}^{z}_{ Γ} \in \gamma_{ Γ}\left((1/2)Ω_{ Γ} + 2R - \frac{1}{2}x_{Γ}^{(Re z)}\right)
$$

if $Re z \geq -\epsilon_{Γ}$ for some $\epsilon_{Γ}$ is proved. $M_{Ω}, \tilde{R}^{z}_{ Γ}$ can be represented as a linear combination of homogenous functions with proper power of homogeneity, so $\tilde{R}^{z}_{ Γ} \in \gamma_{ Γ}\left((1/2)Ω_{ Γ} + 2R - \frac{1}{2}x_{Γ}^{(Re z)}\right)$ if $Re z \geq -\epsilon_{Γ}$ except probably zero. $C^{z}_{ Γ}$ is homogenous and its homogenous power is equal to $Ω_{ Γ} + 2R$. $\tilde{R}^{z}_{ Γ}$ belongs to the needed class if $Re z \geq -\epsilon_{Γ}$ except probably zero. The fact that $\tilde{R}^{z}_{ Γ}$ belongs to the needed class if $Re z \geq -\epsilon_{Γ}$ can be proven by using the Cauchy theorem.

10 Conclusion.

In the present paper we gave some new proof of the Bogoliubov — Parasiuk theorem based on the theory of distributions. This technique will be used in the next paper of this series to prove that the divergences in nonequilibrium diagram technique can be renormalized by the counterterms of the asymptotical state.
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