A HIGH-ORDER HDG METHOD FOR THE BIOT’S CONSOLIDATION MODEL

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ABSTRACT. We propose a novel high-order HDG method for the Biot’s consolidation model in poroelasticity. We present optimal error analysis for both the semi-discrete and full-discrete (combined with temporal backward differentiation formula) schemes. Numerical tests are provided to demonstrate the performance of the method.

1. Introduction

Biot’s seminar work [6–8] laid the foundation of the theory of poroelasticity, which models the interaction between the fluid flow and deformation in an fluid-saturated porous medium. The model is used in several industries such as petroleum and environmental engineering [46,55] and medical applications such as the modeling of the intestinal oedema [54].

In this paper, we consider the numerical solution of the following quasi-static Biot’s consolidation model

\[ c_s \dot{p} + \alpha \text{div}(\dot{u}) - \text{div}(\kappa \nabla p) = f \quad \text{in } \Omega, \]
\[ -\text{div}(2\mu \nabla s(u) - \lambda \text{div}(u) I) + \alpha \nabla p = g \quad \text{in } \Omega, \]

with homogeneous Dirichlet boundary conditions and proper initial data:

\[ u = 0, \quad p = 0 \quad \text{on } \partial \Omega, \]
\[ u(0, x) = u_0(x), \quad p(0, x) = p_0(x) \quad \text{in } \Omega, \]

where \( \Omega \subset \mathbb{R}^d \), \( d = 2, 3 \), is a bounded polygonal/polyhedral domain, \( p \) is the pressure and \( u \) is the deformation, \( c_s \geq 0 \) is the constrained specific storage coefficient which is close to zero in many applications, \( \alpha \) is the Biot-Willis constant which is close to one, \( \kappa \) is the permeability tensor, \( \lambda \) and \( \mu \) are the Lamé constants, and \( \nabla_s u = (\nabla u + \nabla^T u)/2 \) is the symmetric gradient operator. Here we consider homogeneous boundary condition for simplicity. More general boundary conditions, c.f. [32], can be handled with minor modification.

There are extensive literature on the study of spatial discretization for the Biot’s consolidation model with the finite element methods. The early work of Murad et. al. [30,32] studied the stability of the scheme using stable pair of Stokes finite elements for displacement and pressure. The monograph of Lewis and Schrefler [27], c.f. also references therein, discussed the finite element discretization using continuous Galerkin method for both the the displacement and pressure. Phillips and Wheeler proposed and analyzed an algorithm that combines the mixed methods for pressure and a continue/discontinuous Galerkin method for displacement [38].

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See also the discontinuous Galerkin methods [12, 28, 42], Galerkin least square method [21], the pressure-stabilized methods [4, 47], the mixed methods [2, 14, 23, 24, 36, 52], and the nonconforming methods [9, 19, 51].

In this paper, we consider the discretization to (1) using a displacement-pressure formulation with a high-order, superconvergent HDG method for the pressure Poisson operator [25, 35], and a high-order, divergence-conforming HDG method for the elasticity operator [15, 16, 25, 26]. The resulting differential algebraic system (DAE) is solved using backward differentiation formula (BDF) [17]. We present optimal a priori error estimates for the resulting semi-discrete and full-discrete schemes. The method is proven to be free from Poisson locking as \( \lambda \to \infty \), and is numerically shown to be also free from pressure oscillation in the case of low permeability with small time step size [41]. To reach a convergence rate of \( k + 1 \) for an energy norm, the fully discrete scheme has a set of globally coupled degrees of freedom (after static condensation) consists of polynomials of degree \( k + 1 \) for the normal displacement, polynomials of degree \( k \) for the tangential displacement, and polynomials of degree \( k - 1 \) for the pressure per facet (edge in 2D, face in 3D). We also discuss an improvement of this base scheme by slightly relaxing the \( H(\text{div}) \)-conformity of the displacement space so that only unknowns of polynomial degree \( k \) are involved for normal-continuity, c.f. [16, 22]. This modification results a globally coupled degrees of freedom consists of (vector) polynomials of degree \( k \) for the displacement, and degree \( k - 1 \) for the pressure per facet. It does not deteriorate the convergence rate, and allow for optimality of the method also in the sense of superconvergent HDG methods.

The rest of the paper is organized as follows. In Section 2, the semi-discrete scheme is introduced and analyzed. In Section 3, the fully-discrete scheme is introduced and analyzed. The numerical results supporting the theory is presented in Section 4. And a conclusion is drawn in Section 5.

2. Semi-discrete Scheme

2.1. Preliminaries. Let \( \mathcal{T}_h = \{ T \} \) be a conforming simplicial triangulation of \( \Omega \). Let \( \mathcal{F}_h = \{ F \} \) be the collection of facets (edges in 2D, faces in 3D) in \( \mathcal{T}_h \). For any element \( T \in \mathcal{T}_h \), we denote by \( h_T \) its diameter and we denote by \( h \) the maximum diameter over all mesh elements.

We distinguish functions with support only on facets indicated by a hat notation, e.g. \( \hat{\phi}, \hat{\xi} \), with functions with support also on the volume elements. Compositions of functions supported on volume elements (without hat notation) and those only on facets are used for the HDG discretization and indicated by underlining, e.g., \( \underline{\phi} = (\phi, \hat{\phi}), \underline{\xi} = (\xi, \hat{\xi}) \). To simplify notation, we denote the compound spaces

\[
W(h) := H^2_0(\Omega) \times H^1_0(\mathcal{F}_h), \quad \text{and} \quad U(h) := [H^2_0(\Omega)]^d \times [H^1_0(\mathcal{F}_h)]^d.
\]

We denote the tangential component of a vector \( v \) on a facet \( F \) by \( (v)^t = v - (v \cdot n)n \), where \( n \) is the normal direction on \( F \). Furthermore, for any function \( \phi \in H^2_0(\Omega) \), we denote \( \phi := (\phi, \phi|_{\mathcal{F}_h}) \in W(h) \), and for any function \( \xi \in [H^2_0(\Omega)]^d \), we denote \( \xi := (\xi, (\xi)^t|_{\mathcal{F}_h}) \in U(h) \).

For a domain \( D \in \mathbb{R}^d \), we denote \( (\cdot, \cdot)_D \) as the standard \( L^2 \)-inner product on \( D \). Whenever there is no confusion, we simply denote \( (\cdot, \cdot) \) as the inner product on the whole domain \( \Omega \).
Finally, to simplify the presentation of our analysis, we assume the permeability tensor $\kappa$ is a constant scalar throughout the domain $\Omega$. However, we note that the method is applicable to the more general case of a fully tensorial (possibly piecewise defined) permeability.

2.2. Finite elements. We consider an HDG method which approximates the pressure and displacement on the mesh $T_h$, and the pressure and tangential component of the displacement on the mesh skeleton $\mathcal{F}_h$:

$$W_h := \prod_{T \in T_h} \mathbb{P}^k(T),$$

$$\hat{W}_h := \{ \hat{w} \in \prod_{F \in \mathcal{F}_h} \mathbb{P}^{k-1}(F), \; \hat{w} = 0 \; \forall F \subset \partial \Omega \},$$

$$V_h := \{ v \in \prod_{T \in T_h} [\mathbb{P}^{k+1}(T)]^d, \; [v \cdot n]_F = 0 \; \forall F \in \mathcal{F}_h \} \subset H_0(\text{div}, \Omega),$$

$$\hat{V}_h := \{ \hat{v} \in \prod_{F \in \mathcal{F}_h} [\mathbb{P}^k(F)]^d, \; \hat{v} \cdot n = 0 \; \forall F \in \mathcal{F}_h, \; \hat{v} = 0 \; \forall F \subset \partial \Omega \},$$

where $[\cdot]$ is the usual jump operator, $\mathbb{P}^m$ the space of polynomials up to degree $m$. Note that functions in $\hat{W}_h$ and $\hat{V}_h$ are defined only on the mesh skeleton, and the normal component of functions in $\hat{V}_h$ is zero. Here the polynomial degree $k \geq 1$ is a positive integer.

To further simplify notation, we denote the composite spaces

$$\overline{W}_h := W_h \times \hat{W}_h, \text{ and } \overline{U}_h := V_h \times \hat{V}_h.$$

2.3. The semi-discrete numerical scheme. First, we introduce the following $L^2$ projections on the facets:

$$\Pi_{\overline{W}} : L^2(F) \rightarrow \mathbb{P}_{k-1}(F), \quad \int_F (\Pi_{\overline{W}} f) w \text{ ds} = \int_F f w \text{ ds} \quad \forall w \in \mathbb{P}_{k-1}(F),$$

$$\Pi_{\hat{V}} : [L^2(F)]^d \rightarrow [\mathbb{P}_k(F)]^d, \quad \int_F (\Pi_{\hat{V}} f) v \text{ ds} = \int_F f v \text{ ds} \quad \forall v \in [\mathbb{P}_k(F)]^d.$$  

Then, for all $\phi = (\psi, \hat{\psi}), \psi = (\psi, \hat{\psi}) \in \overline{W}_h + \overline{W}(h)$, and $\xi = (\xi, \hat{\xi}), \eta = (\eta, \hat{\eta}) \in \overline{U}_h + \overline{U}(h)$, we introduce the bilinear forms for the diffusion and elasticity operators, respectively:

$$a_h(\phi, \psi) := \sum_{T \in T_h} \int_T \kappa \nabla \phi \cdot \nabla \psi - \int_{\partial T} \kappa \nabla \phi \cdot n \| \psi \| \text{ ds}$$

$$(3a)$$

$$- \int_{\partial T} \kappa \nabla \psi \cdot n \| \phi \| \text{ ds} + \int_{\partial T} \kappa \frac{\tau}{h} \Pi_{\overline{W}} [\phi] \Pi_{\overline{W}} [\psi] \text{ ds},$$

$$b_h(\xi, \eta) := \sum_{T \in T_h} \int_T 2\mu \nabla_s(\xi) : \nabla_s(\eta) + \lambda \text{div}(\xi)\text{div}(\eta) \text{ dx}$$

$$(3b)$$

$$- \int_{\partial T} 2\mu \nabla_s(\xi) n \cdot [\eta] \text{ ds} - \int_{\partial T} 2\mu \nabla_s(\eta) n \cdot [\xi] \text{ ds}$$

$$+ \int_{\partial T} \frac{\tau}{h} \Pi_{\hat{V}} [\xi] \cdot \Pi_{\hat{V}} [\eta] \text{ ds},$$

$$+ \int_{\partial T} \frac{\tau}{h} \Pi_{\overline{W}} [\phi] \Pi_{\overline{W}} [\psi] \text{ ds}.$$
where \( \hat{\phi} = \phi - \bar{\phi} \) and \( \hat{\xi} = (\xi)^t - \bar{\xi} \) denote the jumps between interior and facet unknowns, and \( \tau = \tau_0k^2 \) with \( \tau_0 \) a sufficiently large positive constant.

We note that as long as \( \phi \) and \( \xi \) are finite element functions in \( W_h \) and \( V_h \), respectively, we have

\[
\int_{\partial T} \kappa \nabla \phi \cdot \mathbf{n} \, |\phi| \, ds = \int_{\partial T} \kappa \nabla \phi \cdot \Pi \mathbf{\hat{\psi}} \, |\phi| \, ds, \tag{4}
\]

\[
\int_{\partial T} 2\mu \nabla_s(\xi) \cdot \mathbf{n} \, ||\xi|| \, ds = \int_{\partial T} 2\mu \nabla_s(\xi) \cdot \Pi \mathbf{\hat{\psi}} \, ||\xi|| \, ds \tag{5}
\]
as \( \kappa \nabla \phi \cdot \mathbf{n} \) is a polynomial of degree \( k - 1 \), and \( 2\mu \nabla_s(\xi) \mathbf{n} \) is a polynomial of degree \( k \) on each facet.

The semi-discrete numerical scheme then reads: Find \( p_h = (p_h, \hat{p}_h) \in W_h \) and \( w_h = (u_h, \bar{u}_h) \in U_h \) such that

\[
(c_s \hat{p}_h + \alpha \text{div}(u_h), w_h) + a_h(p_h, w_h) = (f, w_h), \quad \forall w_h = (u_h, \bar{u}_h) \in W_h, \tag{6a}
\]

\[
b_h(u_h, v_h) - (p_h, \alpha \text{div}(v_h)) = (g, v_h), \quad \forall v_h = (u_h, \bar{u}_h) \in U_h. \tag{6b}
\]

2.4. Semi-discrete error estimates. We write

\[ A \leq B \]
to indicate that there exists a constant \( C \), independent of the mesh size \( h \), the parameters \( c_s, \alpha, \mu, \lambda, \kappa \) and the numerical solution, such that \( A \leq CB \).

We denote the following (semi)norms:

\[
\|w\|_{1,h} := \left( \sum_{T \in T_h} \|\nabla w\|^2_T + \frac{1}{h} \|\Pi \mathbf{\hat{\psi}}\|^2_{\partial T} \right)^{1/2}, \tag{7a}
\]

\[
\|v\|_{\mu,h} := \left( \sum_{T \in T_h} 2\mu \|\nabla_s v\|^2_T + \frac{2\mu}{h} \|\Pi \mathbf{\hat{\psi}}\|^2_{\partial T} \right)^{1/2}, \tag{7b}
\]

\[
\|v\|_{\mu,s,h} := \left( \|v\|^2_{\mu,h} + \sum_{T \in T_h} 2\mu h \|\nabla_s (v) \mathbf{n}\|^2_{\partial T} \right)^{1/2}, \tag{7c}
\]

\[
\|\{w, v\}\|_{h} := (c_s \|\bar{w}\|^2 + \|v\|^2_{\mu,h} + \lambda \|\text{div} u\|^2)^{1/2}. \tag{7d}
\]

We also denote the \( H^s \)-norm on \( \Omega \) as \( \| \cdot \|_s \), and when \( s = 0 \), we simply denote \( \| \cdot \| \) as the \( L^2 \)-norm on \( \Omega \).

Coercivity of the bilinear forms (3) follows directly from [16][35].

**Lemma 1.** Let the stabilization parameter \( \tau_0 \) be sufficiently large. Then, for any function \( \bar{w}_h \in W_h \), there holds

\[
\kappa \|\bar{w}_h\|^2_{1,h} \leq a_h(\bar{w}_h, \bar{w}_h), \tag{8a}
\]

and for any function \( \bar{v}_h \in U_h \), there holds

\[
\|\bar{v}_h\|^2_{\mu,h} + \lambda \|\text{div} \bar{v}_h\|^2 \leq b_h(\bar{v}_h, \bar{v}_h). \tag{8b}
\]

Consistency of the semi-discrete scheme (6) follows directly from integration by parts.
Lemma 2. Let \((p, u) \in H_0^1(\Omega) \times H_0^1(\Omega)\) be the solution to the equations (1). We have

\[
(c_\text{r} \hat{p} + \alpha \text{div}(\hat{u}), w) + ah(p, w) = (f, w), \quad \forall w \in W_h + W(h),
\]

\[
b_h(u, v) - (p, \alpha \text{div}(v)) = (g, v), \quad \forall v \in U_h + \tilde{U}(h).
\]

We use the technique of elliptic projectors \([49]\) to derive optimal convergent error estimates. Let \(\Pi p = (\Pi p, \hat{\Pi} p) \in W_h\) and \(\Pi u = (\Pi u, \hat{\Pi} u) \in U_h\) be the projectors defined as follows:

\[
a_h(\Pi p - p, w_h) = 0, \quad \forall w_h = (w_h, \hat{w}_h) \in W_h, \quad (9a)
\]

\[
b_h(\Pi u - u, v_h) - (\Pi p - p, \alpha \text{div}(v_h)) = 0, \quad \forall v_h = (v_h, \hat{v}_h) \in U_h. \quad (9b)
\]

Note that the above coupling is weak since the pressure projector is purely determined by the first set of equations \((9a)\).

The approximation properties of these elliptic projectors follows directly from the corresponding analysis for the elliptic problems \([16, 35]\).

We shall assuming the following full \(H^2\)-regularity

\[
\|\phi\|_2 \leq \|\theta\| \quad \text{(10)}
\]

for the dual problem \(-\nabla(\kappa \nabla \phi) = \theta\) with homogeneous Dirichlet boundary conditions for any source term \(\theta \in L^2(\Omega)\). The estimate \((10)\) holds on convex domains.

Lemma 3. Let the stabilization parameter \(\tau_0\) be sufficiently large. Let \(\Pi p \in W_h\) and \(\Pi u \in U_h\) be given by \([9]\). Assume the elliptic regularity result \([10]\) holds. Then, the following estimates holds:

\[
\|p - \Pi p\| \leq h^{k+1} \|p\|_{k+1}, \quad (11a)
\]

\[
\|u - \Pi u\|_{\mu, h} \leq h^{k+1} \left( \mu^{1/2} \|u\|_{k+2} + \frac{\alpha}{\lambda^{1/2}} \|p\|_{k+1} \right), \quad (11b)
\]

\[
\|\text{div}(u - \Pi u)\| \leq h^{k+1} \left( \mu^{1/2} \|u\|_{k+2} + \|\text{div} u\|_{k+1} + \frac{\alpha}{\lambda} \|p\|_{k+1} \right). \quad (11c)
\]

Proof. The pressure estimate follows from \([35]\). The displacement estimates follow from \([16]\). In particular, we introduce \(v_h := (\Pi_V u, \Pi_V \dot{u})\) where \(\Pi_V\) is the classical BDM interpolator, \([10]\) Proposition 2.3.2, and estimate the error by first applying a triangle inequality to split

\[
\|u - \Pi u\|_{\mu, h} = \|v_h - u\|_{\mu, h} + \|\Pi u - v_h\|_{\mu, h}.
\]

Using coercivity result in Lemma 1, we get

\[
\|\Pi u - v_h\|_{\mu, h}^2 + \lambda \|\text{div}(\Pi u - v_h)\|^2 \leq b_h(\Pi u - v_h, \Pi u - v_h)
\]

\[
= b_h(u - v_h, \Pi u - v_h) + (\Pi p - p, \alpha \text{div}(\Pi u - v_h))
\]

\[
\leq \|u - v_h\|_{\mu, h} \|\Pi u - v_h\|_{\mu, h} + \alpha \|\Pi p - p\| \|\text{div}(\Pi u - v_h)\|.
\]

Hence, applying the triangle inequality,

\[
\|u - \Pi u\|_{\mu, h} \leq \|u - v_h\|_{\mu, h} + \alpha \|\Pi p - p\|,
\]

\[
\|\text{div}(u - \Pi u)\| \leq \frac{1}{\lambda^{1/2}} \|u - v_h\|_{\mu, h} + \|\text{div}(u - \Pi_V u)\| + \frac{\alpha}{\lambda} \|p - \Pi p\|.
\]

The estimates \((11b), (11c)\) now follows from the standard approximation properties of the BDM interpolator \(\Pi_V\). \qed
To further simplify notation, we denote
\[ \varepsilon_u = u_h - \Pi u, \quad \varepsilon_p = p_h - \Pi p, \quad \delta_u = u - \Pi u, \quad \delta_p = p - \Pi p. \] (12)

Combining the numerical scheme with the consistency result in Lemma 2 adding and subtracting the above elliptic projectors, we arrive at the following error equations:

\[ \begin{align*}
(c_s \varepsilon_p + \alpha \text{div}(\varepsilon_u), w_h) + a_h(\varepsilon_p, w_h) &= (c_s \delta_p + \alpha \text{div}(\delta_u), w_h), \\
b_h(\varepsilon_u, w_h) - (\varepsilon_p, \alpha \text{div}(v_h)) &= 0,
\end{align*} \tag{13a,b} \]

for all \( w_h = (w_h, \tilde{w}_h) \in W_h \) and \( v_h = (v_h, \tilde{v}_h) \in U_h \).

By the inf-sup stability \([10]\) of the finite elements pair \( W_h \times V_h \subset L^2(\Omega) \times H_0(\text{div}, \Omega) \), we have the following pressure estimate.

**Lemma 4.** Let \( \varepsilon_p \) be the average of \( \varepsilon_p \) on \( \Omega \). Then, we have

\[ \alpha \|\varepsilon_p - \varepsilon_p\| \leq \mu^{1/2} \|\varepsilon_u\|_{\mu, h} + \lambda \|\text{div} \varepsilon_u\|. \]

**Proof.** By inf-sup stability \([10]\), there exists a function \( \hat{w}_h = (\hat{w}_h, \tilde{w}_h) \in U_h \) such that

\[ \text{div} \hat{w}_h = \varepsilon_p - \varepsilon_p, \quad \text{and} \quad \|\hat{w}_h\|_{\mu, h} \leq \mu^{1/2} \|\varepsilon_p - \varepsilon_p\|. \]

The estimate in Lemma 4 follows directly by taking \( v_h = \hat{w}_h \) in (13b), using the fact that \( (\text{div} \hat{w}_h, \varepsilon_p) = 0 \), and applying the Cauchy-Schwarz inequality. \qed

Now, we are ready to present our main results on the semi-discrete error estimates.

**Theorem 1.** Let the stabilization parameter \( \tau_0 \) be sufficiently large. Let \( (p_h, u_h) \in W_h \times U_h \) be the solution to (6) with initial data \( p_h(0) = \Pi p(0) \) and \( u_h(0) = \Pi u(0) \). Then, the following estimate holds for all \( T > 0 \):

\[ \|\{\varepsilon_p(T), \varepsilon_u(T)\}\|^2_h + \int_0^T a_h(\varepsilon_p, \varepsilon_p) \, dt \leq h^{2k+2} \Xi_1, \tag{14} \]

where

\[ \Xi_1 = T \int_0^T (c_s + \alpha^2(\lambda + \mu)/\lambda^2)\|\hat{p}\|^2_{k+1} + \mu(\lambda + \mu)/\lambda \|\hat{u}\|^2_{k+1} + (\lambda + \mu)\|\text{div} \hat{u}\|^2_{k+1} \, dt. \]

**Remark 1 (Robust displacement estimate).** The above estimate for the displacement is robust with respect to the incompressible limit \( c_s \to 0 \) and \( \lambda \to +\infty \), as long as the term \( \lambda \|\text{div} \hat{u}\|^2_{k+1} \) is bounded. It is also robust in the degenerate case as the permeability \( \kappa \to 0 \).

**Proof.** We use a standard energy argument. Taking \( \hat{w}_h = \varepsilon_p \) and \( \hat{v}_h = \varepsilon_u \) in the error equations (13a) and adding, we get

\[ (c_s \varepsilon_p, \varepsilon_p) + b_h(\varepsilon_u, \varepsilon_p) + a_h(\varepsilon_p, \varepsilon_p) = (c_s \delta_p + \alpha \text{div}(\delta_u), \varepsilon_p) = (c_s \delta_p, \varepsilon_p) + (\alpha \text{div}(\delta_u), \varepsilon_p - \varepsilon_p) \]

Applying the Cauchy-Schwarz inequality on the above right hand side and using the estimate in Lemma 4, we have

\[ (c_s \delta_p, \varepsilon_p) + (\alpha \text{div}(\delta_u), \varepsilon_p - \varepsilon_p) \leq c_s \|\delta_p\| \|\varepsilon_p\| + \|\text{div} \delta_u\| \|\delta_p\| (\mu^{1/2} \|\varepsilon_u\|_{\mu, h} + \lambda \|\text{div} \varepsilon_u\|) \]

\[ \leq \left( c_s \|\delta_p\|^2 + (\mu + \lambda)\|\text{div} \delta_u\|^2 \right)^{1/2} \left( c_s \|\varepsilon_p\|^2 + \|\varepsilon_u\|_{\mu, h}^2 + \lambda \|\text{div} \varepsilon_u\|^2 \right)^{1/2}. \]
Combing this estimate with the above identity, and invoking the coercivity result (8b), we get
\[ \frac{1}{2} \partial_t \left( c_s(\varepsilon_p, \varepsilon_p) + b_h(\varepsilon_u, \varepsilon_u) \right) + a_h(\varepsilon_p, \varepsilon_p) \preceq \Theta^{1/2} \left( c_s(\varepsilon_p, \varepsilon_p) + b_h(\varepsilon_u, \varepsilon_u) \right)^{1/2} \]
Recalling that $\varepsilon_p(0) = 0$ and $\varepsilon_u(0) = 0$, then an application of the Gronwall's inequality implies that
\[ c_s(\varepsilon_p(T), \varepsilon_p(T)) + b_h(\varepsilon_u(T), \varepsilon_u(T)) + \int_0^T a_h(\varepsilon_p, \varepsilon_p) \, dt \preceq T \int_0^T \Theta \, dt, \]
for all $T > 0$. Combining the above estimate with (8b) and (11), we get the desired inequality in Theorem 1.

**Corollary 1.** Let assumptions of Theorem 1 holds. Then, the following estimate holds for all $T > 0$:
\[ \left\| \{\varepsilon_p(T), \varepsilon_u(T)\} \right\|_h^2 \preceq h^{2k+2} \Xi_2, \]
where
\[ \Xi_2 = T \int_0^T \left( c_s + \frac{\alpha^2(\lambda + \mu)}{\lambda^2} \|\hat{p}\|_{k+1}^2 + \frac{\mu(\lambda + \mu)}{\lambda} \|\hat{\mathbf{u}}\|_{k+2}^2 + (\lambda + \mu) \|\text{div} \hat{\mathbf{u}}\|_{k+1} \right) \, dt. \]

**Proof.** Take one time derivative of the error equations (13). Then proceed as in the proof of Theorem 1.

Now, we give a robust pressure estimate, with respect to $c_s$, under the assumption that permeability $\kappa$ is away from zero.

**Theorem 2.** Let the assumptions of Theorem 1 hold. Then, for all $T > 0$, the following estimate holds
\[ \kappa \|\varepsilon_p(T)\|_1 \preceq \|\hat{p}(T)\|_k + \frac{\alpha}{\lambda^{1/2}} \Xi_2^{1/2} + \Xi_3, \]
where $\Xi_3$ is given in Corollary 1, and
\[ \Xi_3 = (c_s + \frac{\alpha^2}{\lambda} \|\hat{p}(T)\|_{k+1} + \alpha \left( \frac{\mu^{1/2}}{\lambda^{1/2}} \|\hat{\mathbf{u}}(T)\|_{k+2} + \|\text{div} \hat{\mathbf{u}}(T)\|_{k+1} \right). \]

**Proof.** Taking $\mathbf{w}_h = \varepsilon_p$ in (13a), reordering terms, and applying the Cauchy-Schwarz inequality, we have
\[ a_h(\varepsilon_p, \varepsilon_p) = \left( c_s(\delta_p - \varepsilon_p) + \alpha \text{div}(\delta_u - \varepsilon_u), \varepsilon_p \right) \preceq \left( c_s(\|\delta_p\| + \|\varepsilon_p\|) + \alpha(\|\text{div} \delta_u\| + \|\text{div} \varepsilon_u\|) \right) \|\varepsilon_p\| \]
Invoking the discrete Poincaré inequality [13], $\|\mathbf{w}_h\| \preceq \|\mathbf{w}_h\|_{1,h}$ for all $\mathbf{w}_h \in \mathbf{W}_h$, and using the coercivity result (8b), we get
\[ \kappa \|\varepsilon_p\|_{1,h} \preceq c_s(\|\delta_p\| + \|\varepsilon_p\|) + \alpha(\|\text{div} \delta_u\| + \|\text{div} \varepsilon_u\|). \]
Combining the above estimate with Lemma 3 and Corollary 1, we get the desired inequality in Theorem 2.

We conclude this section with a remark on (slightly) relaxing the $H(\text{div})$-conformity of the displacement space to reduce global coupling.
Remark 2 (Relaxed $H(\text{div})$-conformity). We noticed that to reach a convergence rate of $k+1$ for the “energy norm” $\|\cdot\|_{k,h}$, we need unknowns of polynomial degree $k+1$ on the facets. We follow the idea of [22] to relax the highest-order normal conformity of the displacement space:

$$V_h^{-} := \{ v \in \prod_{T \in T_h} [P^{k+1}(T)]^d, \; \Pi_F^k [v \cdot n]_F = 0 \forall F \in F_h \} \subset H(\text{div}, \Omega),$$

where $\Pi_F^k : L^2(F) \to P^k(F)$ is the $L^2$-projection. The resulting semi-discrete scheme still use the formulation (4), but with the space $U_h^{-} := V_h^{-} \times \hat{V}_h$ for displacement and $W_h$ for pressure. The globally coupled degrees of freedom (after static condensation) for this modification consists of polynomials of degree $k$ for the displacement and polynomials of degree $k-1$ for the pressure per facet; while the that for the original scheme consists of polynomials of degree $k+1$ for the normal-component of the displacement, polynomials of degree $k$ for the tangential-component of the displacement, and polynomials of degree $k-1$ for the pressure per facet.

We present numerical results in Section 4 to validate the optimality of such modification, and refer interested reader to [16, 22] for the analysis.

3. Fully-discrete Scheme

For the temporal discretization of the semi-discrete DAE (6), we consider the $m$-step BDF [17, Chapter V] method with step size $\Delta t > 0$: for $n \geq m$, find $(\hat{P}_h^n, \hat{U}_h^n) \in \hat{W}_h \times \hat{U}_h$ such that

$$\sum_{j=0}^{m} \frac{\delta_j}{\Delta t} (c_j p_h^{n-j} + \alpha \text{div}(u_h^{n-j}), w_h) + a_h(p_h^n, w_h) = (f(t^n), w_h), \quad (16a)$$

$$b_h(u_h^n, v_h) - (p_h^n, \alpha \text{div}(v_h)) = (g(t^n), v_h), \quad (16b)$$

for all $(w_h, v_h) \in W_h \times U_h$, with given starting values $(\hat{P}_h^i, \hat{U}_h^i)_{i=0}^{m-1}$, where $t^n = n\Delta t$. The method coefficients $\delta_j$ are determined from the relation

$$\delta(\zeta) = \sum_{j=0}^{m} \delta_j \zeta^j = \sum_{\ell=1}^{m} \frac{1}{\ell} (1 - \zeta)^\ell. \quad (17)$$

The BDF method is known to have order $m$ for $m \leq 6$, and is A-stable for $m = 1$ and $m = 2$, but not for $m \geq 3$.

Next, we provide error estimates for the fully discrete scheme (16) with $m = 2$ using an energy argument. We remark that the analysis for the cases with $3 \leq m \leq 5$ is similar but more technical as one needs to use the multiplier technique [1, 33].

To simplify notation, we denote the backward difference operator

$$d_t \phi^n := \frac{3\phi^n - 4\phi^{n-1} + \phi^{n-2}}{2\Delta t}. \quad (18)$$

Let $(\cdot, \cdot)$ be an inner product with associated norm $|\cdot|$. Then, a straightforward calculation yields

$$(d_t \phi^n, \phi^n) = \frac{1}{4\Delta t} \left( |\phi^n|^2 + |2\phi^n - \phi^{n-1}|^2 - |\phi^{n-1}|^2 - |2\phi^{n-1} - \phi^{n-2}|^2 \right)$$

$$+ |\phi^n - 2\phi^{n-1} + \phi^{n-2}|^2 \quad (19)$$
We continue to use the notation \( \{ \cdot, \cdot \} \). Denoting the norm
\[
\| \phi \|_{L^\infty (H^1)} := \sup_t \| \phi(t) \|_{H^1(\Omega)},
\]
we have the following result on the consistency of the scheme \(16\).

**Lemma 5.** Let \((p^n_h, u^n_h) \in W_h \times U_h\), \(n \geq 2\), be the solution to equations \(16\) with \(m = 2\) and starting values \((p^0_h, u^0_h)\) and \((p^1_h, u^1_h)\). Let \(p^n := p(t^n)\) and \(u^n := u(t^n)\) be the exact solution to equations \(1\) at time \(t^n\). Then, there holds, for \(n \geq 2\),
\[
(c_s d_\epsilon \epsilon^n, \omega_h) + a_h(\epsilon^n, \omega_h) = E^n_h(\omega_h)
\]
for all \(\omega_h = (w_h, \hat{\omega}_h) \in W_h\) and \(v_h = (v_h, \hat{v}_h) \in U_h\), where
\[
E^n_h(\omega_h) := c_s (d_\epsilon \delta^n_p - d_\epsilon p^n + \hat{p}^n, w_h) + \alpha (\text{div}(d_\epsilon \delta^n_u - d_\epsilon u^n + \hat{u}^n), w_h - \omega_h),
\]
and \(\omega_h\) is the average of \(w_h\) on \(\Omega\). Moreover, there holds
\[
\| E^n_h(\omega_h) \| \leq c_s O_{s,I} \| w_h \| + \alpha O_{s,II} \| w_h - \omega_h \|
\]
where, for integer \(s \geq 1\),
\[
O_{s,I} := h^{s+1} \left\| \frac{\partial p}{\partial t} \right\|_{L^\infty (H^{s+1})} + \Delta t^{s} \left\| \frac{\partial^s p}{\partial t^s} \right\|_{L^\infty (L^2)}
\]
\[
O_{s,II} := h^{s+1} \left( \frac{h^{1/2}}{\lambda^{1/2}} \left\| \frac{\partial u}{\partial t} \right\|_{L^\infty (H^{s+1})} + \| \text{div}(\frac{\partial u}{\partial t}) \|_{L^\infty (H^{s+1})} + \alpha \left\| \frac{\partial p}{\partial t} \right\|_{L^\infty (H^{s+1})} \right)
\]
\[
+ \Delta t^s \| \text{div} \left( \frac{\partial^s u}{\partial t^s} \right) \|_{L^\infty (L^2)}
\]

**Proof.** The error equations \(21\) follows from the scheme \(16\) and the consistency result in Lemma 2. The estimate \(22\) follows from the Cauchy-Schwarz inequality, the approximation properties in Lemma 3 of the elliptic projector, and Taylor expansion in time. \(\square\)

Our main result on the fully-discrete error estimates is given below.

**Theorem 3.** Let \((p^n_h, u^n_h) \in W_h \times U_h\), \(n \geq 2\), be the solution to equations \(16\) with \(m = 2\) and starting values \((p^0_h, u^0_h)\) and \((p^1_h, u^1_h)\). Let \(p^n := p(t^n)\) and \(u^n := u(t^n)\) be the exact solution to equations \(1\) at time \(t^n\). Then, there holds, for \(N \geq 2\),
\[
\| \{ \epsilon^n_p, \epsilon^n_u \} \|_h^2 + \Delta t \sum_{n=2}^{N} a_h(\epsilon^n_p, \epsilon^n_p) \leq \exp(N \Delta t) \left( \sum_{i=0}^{1} \| \{ \epsilon^i_p, \epsilon^i_u \} \|_h^2 + N \Delta t (c_s O_{s,I}^2 + (\lambda + \mu) O_{s,II}^2) \right)
\]

**Proof.** Taking \(w_h = \epsilon^n_p\) in equation \(21a\) and \(v_h = d_\epsilon \epsilon^n_u\) in equation \(21b\), and adding and summing the resulting expression for \(n = 2, \cdots, N\), we get
\[
\sum_{n=2}^{N} (c_s d_\epsilon \epsilon^n_p, \epsilon^n_p) + b_h(\epsilon^n_p, d_\epsilon \epsilon^n_u) + a_h(\epsilon^n_p, \epsilon^n_p) = \sum_{n=2}^{N} E^n_h(\epsilon^n_p)
\]
\[
\leq \sum_{n=2}^{N} (c_s O_{s,I}^2 + (\lambda + \mu) O_{s,II}^2) \| \{ \epsilon^n_p, \epsilon^n_u \} \|_h
\]
where we used a combination of Lemma 5 and Lemma 4 to derive the last inequality.
The identity (19) implies that
\[
\sum_{n=2}^{N} (c_s d \varepsilon^n_p, \varepsilon^n_p) + b_h(\varepsilon^n_u, d \varepsilon^n_u) \geq \frac{1}{4\Delta t} \left[ (c_s \| \varepsilon^N_p \|^2 + b_h(\varepsilon^N_u, \varepsilon^N_u) - c_s \| \varepsilon^1_p \|^2 \\
- c_s (2 \varepsilon^1_p - \varepsilon^0_p)^2) - b_h(\varepsilon^1_u, \varepsilon^1_u) - b_h(2 \varepsilon^1_u - \varepsilon^0_u, 2 \varepsilon^1_u - \varepsilon^0_u) \right].
\]

Hence,
\[
\frac{1}{\Delta t} \left( \| \{ \varepsilon^N_p, \varepsilon^N_u \} \|_h^2 - \sum_{i=0}^{1} \| \{ \varepsilon^i_p, \varepsilon^i_u \} \|_h^2 \right) \leq \sum_{n=2}^{N} (c_s d \varepsilon^n_p, \varepsilon^n_p) + b_h(\varepsilon^n_u, d \varepsilon^n_u)
\]

Combining this estimate with (24), we get
\[
\| \{ \varepsilon^N_p, \varepsilon^N_u \} \|^2_h + \Delta t \sum_{n=2}^{N} a_h(\varepsilon^n_p, \varepsilon^n_p) \leq \Delta t \sum_{n=2}^{N} (c_s \mathcal{O}^2_{2,1} + (\lambda + \mu) \mathcal{O}^2_{2,1})^{1/2} \| \{ \varepsilon^n_p, \varepsilon^n_u \} \|_h
\]
\[
+ \sum_{i=0}^{1} \| \{ \varepsilon^i_p, \varepsilon^i_u \} \|_h^2.
\]

Finally, the estimate (23) follows from a discrete Gronwall’s inequality, c.f. [18, Lemma 5.1].

**Remark 3** (Higher order BDF method). For m-step BDF methods with \( m = 1 \) or \( 3 \leq m \leq 5 \), we can still use a similar energy argument to derive the following estimate
\[
\| \{ \varepsilon^N_p, \varepsilon^N_u \} \|^2_h + \Delta t \sum_{n=2}^{N} a_h(\varepsilon^n_p, \varepsilon^n_p) \leq \exp(N\Delta t) \left( \sum_{i=0}^{m-1} \| \{ \varepsilon^i_p, \varepsilon^i_u \} \|_h^2 \\
+ N\Delta t(c_s \mathcal{O}^2_{m,1} + (\lambda + \mu) \mathcal{O}^2_{1,1}) \right).
\]

In the cases for \( 3 \leq m \leq 5 \), we need to apply the multiplier technique [33], and take in the energy argument the test function in the error equation (21a) to be \( \varepsilon^0_p := \varepsilon^0_p - \varepsilon^0_{p-1} \) with the multiplier \( \eta = 0.0836 \) for \( m = 3 \), \( \eta = 0.2878 \) for \( m = 4 \), and \( \eta = 0.8160 \) for \( m = 5 \). More details of the multiplier technique can be found in the recent publications [7, 29].

**Remark 4** (Starting values for BDF2 and BDF3). The m-step BDF method needs \( m - 1 \) starting values to begin with.

For BDF2, we can simply take \( (p^0, u^0) \) to be the Backward Euler solution to equations (16) with \( m = 1 \) and \( (p^0, u^0) = (\Pi p(0), \Pi u(0)) \). This implies
\[
\| \{ \varepsilon^0_p, \varepsilon^0_u \} \|_h = 0, \quad \| \{ \varepsilon^1_p, \varepsilon^1_u \} \|_h \leq \Delta t(c_s \mathcal{O}^2_{1,1} + (\lambda + \mu) \mathcal{O}^2_{1,1}).
\]

Combining these estimates with (23), we readily have \( \| \{ \varepsilon^N_p, \varepsilon^N_u \} \|_h \) converges \( k \)-th order in space, and \( k \)-th order in time.

For BDF3, we take \( (p^0, u^0) = (\Pi p(0), \Pi u(0)), (p^1, u^1), i = 1, 2, \) to be the solution with Crank-Nicolson time stepping. Similarly, the local error for \( \| \{ \varepsilon^i_p, \varepsilon^i_u \} \|_h \), \( i = 1, 2 \), are third-order in time. Then, the estimates in Remark 3 yields that \( \| \{ \varepsilon^N_p, \varepsilon^N_u \} \|_h \) converges \( k + 1 \)-th order in space, and third-order in time.
Remark 5 (diagonally implicit Runge-Kutta time stepping). Alternatively, we can apply the (one-step, multi-stage) diagonally implicit Runge-Kutta (DIRK) methods to solve the DAE \([k]\). We refer the interested reader to the references \([20, 34]\) for a setup. However, in our numerical experiments not documented here, we do observe the order reduction \([11, 44]\) in high-order \((\geq 3)\) DIRK schemes due to inappropriate boundary treatment in the intermediate stages. We only observe second order accuracy for third- and fourth-order DIRK schemes.

4. Numerical results

In this section, we present several numerical experiments to illustrate the performance of the proposed method. The numerical results are performed using the NGSolve software \([45]\).

4.1. Accuracy for a smooth solution with a large \(\lambda\). In order to confirm the optimal convergence rates in Section 2 and Section 3, we consider a manufactured smooth exact solution, similar to the one considered in \([53, \text{Section 7.1}]\). Specifically, we take the domain to be \(\Omega = (0, 1)^2\), with the exact displacement \(u = (u, v)\) and exact pressure \(p\) given by

\[
\begin{align*}
    u(x, t) &= -e^{-t} \cos(\pi x) \sin(\pi y) + \frac{1}{\mu + \lambda} e^{-t} \sin(\pi x) \sin(\pi y) \\
    v(x, t) &= e^{-t} \sin(\pi x) \cos(\pi y) + \frac{1}{\mu + \lambda} e^{-t} \sin(\pi x) \sin(\pi y) \\
    p(x, t) &= e^{-t} \sin(\pi x) \sin(\pi y).
\end{align*}
\]

Note that the solution is designed to satisfy

\[
\text{div } u = \pi e^{-t} \sin(\pi(x + y))/(\mu + \lambda) \to 0 \quad \text{as} \quad \lambda \to +\infty.
\]

We impose Dirichlet boundary conditions for both \(u\) and \(p\), and choose the following material parameters:

\[
c_0 = 0, \quad \alpha = 1, \quad \kappa = 1, \quad \lambda = 10^5, \quad \mu = 1.
\]

The final computational time is \(T = 0.5\).

Our computation is based on uniform triangular meshes; see Figure 1 for the coarsest mesh with mesh size \(h = 1/4\). We consider the fully discrete scheme \((16)\), with the (spatial) polynomial degree \(k\) in the finite element spaces \([2]\) varying from 1 to 3, and the (temporal) BDF3 method \((m = 3)\). We also present numerical results using the relaxed \(H(\text{div})\)-conformity approach, c.f. Remark 2. The stabilization parameter \(\tau\) in the bilinear forms \([3]\) is taking to be \(\tau = 10k^2\) for all the tests. We take the time step size to be \(\Delta t = h^\max((k+1)/3, 1)\), where \(h\) is the spatial mesh size. The error in the norm \(\|\cdot\|_{b,i}\), and the \(L^2\)-norms for displacement and pressure at the final time \(T = 0.5\) are recorded in Table 1 on a sequences of uniformly refined meshes for the original scheme \([6]\) and in Table 2 for the relaxed \(H(\text{div})\)-conforming scheme, c.f. Remark 2. In both tables, we observe the optimal convergence rates for the norm \(\|\cdot\|_{b,i}\), in full agreements with our main result in Theorem 3 and Remark 3, we also observe optimal convergence rates in the \(L^2\)-norm of the displacement \((k + 2)\), and the \(L^2\)-norm of the pressure \((k + 1)\).
Figure 1. The coarsest mesh with $h = 1/4$

Table 1. Convergence study at the final time $T = 0.5$: The original scheme.

| $k$ | $h$ | error | order | error | order | error | order |
|-----|-----|-------|-------|-------|-------|-------|-------|
| 1/4 | 7.214e-02 | - | 3.589e-03 | - | 2.190e-02 | - |
| 1/8 | 1.854e-02 | 1.96 | 4.76e-04 | 3.00 | 5.194e-03 | 2.08 |
| 1/16 | 4.692e-03 | 1.98 | 5.58e-05 | 3.01 | 1.304e-03 | 1.99 |
| 1/32 | 1.178e-03 | 1.99 | 6.91e-06 | 3.01 | 3.263e-04 | 2.00 |
| 1/64 | 2.951e-04 | 2.00 | 8.60e-07 | 3.00 | 8.159e-05 | 2.00 |
| 1/4 | 8.342e-03 | - | 3.306e-04 | - | 1.832e-03 | - |
| 1/8 | 1.034e-03 | 3.01 | 2.024e-05 | 4.03 | 2.421e-04 | 2.92 |
| 1/16 | 1.283e-04 | 3.01 | 1.239e-06 | 4.03 | 3.037e-05 | 3.00 |
| 1/32 | 1.598e-05 | 3.01 | 7.658e-08 | 4.02 | 3.799e-06 | 3.00 |
| 1/64 | 1.994e-06 | 3.00 | 4.759e-09 | 4.01 | 4.750e-07 | 3.00 |
| 1/4 | 7.739e-04 | - | 2.583e-05 | - | 1.851e-04 | - |
| 1/8 | 4.785e-05 | 4.02 | 7.659e-07 | 5.08 | 1.130e-05 | 4.03 |
| 1/16 | 3.019e-06 | 3.99 | 2.357e-08 | 5.02 | 7.095e-07 | 3.99 |
| 1/32 | 1.879e-07 | 4.01 | 7.244e-10 | 5.02 | 4.409e-08 | 4.01 |
| 1/64 | 1.178e-08 | 4.00 | 4.849e-11 | 3.90 | 2.761e-09 | 4.00 |

4.2. Barry and Mercer’s problem. We consider the Barry and Mercer’s problem [3], for which an exact solution is available in terms of infinite series (we refer the reader to the cited paper and also to [37, Section 4.2.1] for the expression). It models the behavior of a rectangular uniform porous material with a pulsating point source, drained on all sides, and on which zero tangential displacements are assumed on the whole boundary. The point-source corresponds to a sine wave on the rectangular domain $(0, a) \times (0, b)$ and is given as

$$ f(t) = 2\beta \delta_{x_0} \sin(\beta t), $$

where $\beta = (\lambda + 2\mu) \kappa_{ab}$ and $\delta_{x_0}$ is the Dirac delta at the point $x_0$. The computational domain together with the boundary conditions are depicted in Figure 2.
Table 2. Convergence study at the final time $T = 0.5$: The relaxed $H(\text{div})$-conforming scheme.

| $k$  | $h$  | error     | order | error     | order | error     | order |
|-----|-----|-----------|------|-----------|------|-----------|------|
| 1/4 |     | 6.201e-02 | -    | 3.441e-03 | -    | 2.190e-02 | -    |
| 1/8 |     | 1.641e-02 | 1.92 | 4.691e-04 | 2.87 | 5.194e-03 | 2.08 |
| 1/16|     | 4.262e-03 | 1.94 | 6.326e-05 | 2.89 | 1.304e-03 | 1.99 |
| 1/32|     | 1.085e-03 | 1.97 | 8.237e-06 | 2.94 | 3.263e-04 | 2.00 |
| 1/64|     | 2.732e-04 | 1.99 | 1.048e-06 | 2.97 | 8.159e-05 | 2.00 |

$2$ $1/4$ $8.170e-03$ - $3.253e-04$ - $1.832e-03$ -
| 1/8 |     | 1.036e-03 | 2.98 | 2.040e-05 | 4.00 | 2.421e-04 | 2.92 |
| 1/16|     | 1.295e-04 | 3.00 | 1.271e-06 | 4.01 | 3.037e-05 | 3.00 |
| 1/32|     | 1.618e-05 | 3.00 | 7.921e-08 | 4.00 | 3.799e-06 | 3.00 |
| 1/64|     | 2.021e-06 | 3.00 | 4.943e-09 | 4.00 | 4.750e-07 | 3.00 |

$3$ $1/4$ $7.633e-04$ - $2.562e-05$ - $1.851e-04$ -
| 1/8 |     | 4.741e-05 | 4.01 | 7.749e-07 | 5.05 | 1.130e-05 | 4.03 |
| 1/16|     | 3.924e-06 | 3.97 | 2.442e-08 | 4.99 | 7.095e-07 | 3.99 |
| 1/32|     | 1.894e-07 | 4.00 | 7.615e-10 | 5.00 | 4.409e-08 | 4.01 |
| 1/64|     | 1.191e-08 | 3.99 | 3.930e-11 | 4.28 | 2.761e-09 | 4.00 |

Figure 2. Computational domain and boundary conditions for the Barry and Mercer’s problem.

As in [38, 43], we consider the rectangular domain $(0, 1) \times (0, 1)$, and the following values of the material parameters:

$c_0 = 0, \ \alpha = 1, \ \ E = 10^5, \ \nu = 0.1, \ \kappa = 10^{-2},$
where $E$ and $\nu$ denotes Young’s modulus and the Poisson ratio, respectively, and

$$
\mu = \frac{E}{2(1+\nu)}, \quad \lambda = \frac{E\nu}{(1-2\nu)(1+\nu)}.
$$

The source is positioned at the point $(1/4,1/4)$.

We consider the fully discrete scheme (16), with the (spatial) polynomial degree $k = 1$ in the finite element spaces (2) and the (temporal) BDF2 method ($m = 2$). We use a relatively large time step of $\Delta t = \frac{\pi}{20\beta}$. The solution for the pressure on the deformed domain on a uniform triangular mesh with mesh size $h = 1/64$ is plotted in Figure 3 for two different “normalized time” $\hat{t} = \beta t$ of values $\hat{t} = \pi/2$ and $\hat{t} = 3\pi/2$. We observe that depending on the sign of the source term (positive for $\hat{t} = \pi/2$, negative for $\hat{t} = 3\pi/2$) the resultant displacements cause an expansion or a contraction of the medium. We also plot the pressure and $x$-component of the displacement profiles on three consecutive meshes, with mesh size $h = 1/32, 1/64, 1/128$, along the diagonal line $(0,0)–(1,1)$ of the domain, along with the exact solution in Figure 4. We observe form Figure 4 that the numerical solution resemble the exact solution very precisely.

Finally, to check the robustness of the method with respect to pressure oscillations for small permeability combined with small time steps, we show in Figure 5 the pressure profile after one step of backward Euler with $\kappa = 10^{-6}$ and $\Delta t = 10^{-4}$ on the uniform triangular mesh with $h = 1/64$. We do not observe significant oscillation.

5. Conclusion

In this paper we have analyzed the convergence property of a novel high-order HDG discretization of Biot’s consolidation model in poroelasticity combined with BDF time stepping. The method produce optimal convergence rates, and is free from Poisson locking when $\lambda \to \infty$.  

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3.png}
\caption{Numerical solution for the pressure on the deformed domain at different time}
\end{figure}
Figure 4. Numerical solution for pressure and x-component of displacement along the diagonal (0,0)-(1,1) of the domain for different time.

Figure 5. Numerical solution for pressure after one time step. Left: numerical pressure on Ω; Right: numerical pressure on the diagonal line (0,0)-(1,1).

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DIVERGENCE-CONFORMING HDG FOR THE BIOT PROBLEM

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