Čech complexes for covers of small categories

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Abstract

We present a combinatorial analogue of the nerve theorem for covers of small categories, using the Grothendieck construction. We apply our result to prove the inclusion-exclusion principle for the Euler characteristic of a finite category.

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1 Introduction

For an open cover $U = \{U_a\}_{a \in A}$ of a space $X$, the Čech nerve $\check{C}(U)$ of $U$ is a simplicial space defined by

$$\check{C}(U)_n = \coprod_{a_0, \ldots, a_n \in A} U_{a_0 \ldots a_n},$$

where $U_{a_0 \ldots a_n}$ denotes the intersection $U_{a_0} \cap \ldots \cap U_{a_n}$. Here, the face maps omit indices, and the degeneracy maps insert copies of indices. Dugger and Isaksen have proved that the natural map hocolim $\check{C}(U) \to X$ from the homotopy colimit of the Čech nerve to the original space is a weak homotopy equivalence [DI04].

In this paper, a combinatorial analogue of their result is provided for small categories. Let $C$ be a small category, and let $D = \{D_a\}_{a \in A}$ be a collection of subcategories of $C$ satisfying $C = \bigcup_{a \in A} D_a$. We can define the Čech nerve $\check{C}(D)$ of $D$ as a simplicial object in the category of small categories, similarly to the above. In the context of small categories, the Grothendieck construction is known as a model of the homotopy colimit of a diagram in small categories [Tho79]. The Grothendieck construction $\text{Gr}(\check{C}(D))$ of the Čech nerve is also equipped with the natural functor $\text{Gr}(\check{C}(D)) \to C$. A natural question to ask is when this induces a homotopy equivalence on classifying spaces.

Unfortunately, it does not hold for every cover. For example, let $C$ be a small category, formed as follows:

$$x \overrightarrow{y} \overleftarrow{z},$$
with the terminal object $z$. If $D_1$ is the full subcategory with $\text{ob}(D_1) = \{x, y\}$, and $D_2$ is the full subcategory with $\text{ob}(D_2) = \{y, z\}$, then $C = D_1 \cup D_2$. The classifying space of $C$ is contractible; however, the classifying space of the Grothendieck construction for the Čech nerve of the cover $\{D_1, D_2\}$ is homotopy equivalent to a circle.

In order to solve the problem above, we introduce two classes of covers of small categories consisting of ideals and filters, respectively. These are generalized notions of ideals and filters of posets [Sta12, Zap98]. An ideal $D$ of a small category $C$ is a full subcategory such that an object $y$ of $C$ belongs to $D$, whenever $C(y, x) \neq \emptyset$ for some object $x$ of $D$. A cover $\mathcal{D} = \{D_a\}_{a \in A}$ of a small category $C$ is called an ideal cover, if $D_a$ is an ideal of $D$ for any $a \in A$. We can also define the dual notions; filters and filter covers.

**Theorem 1.1.** Let $\mathcal{D}$ be an ideal cover or a filter cover of a small category $C$. The natural functor $\text{Gr}(\check{C}(\mathcal{D})) \rightarrow C$ induces a homotopy equivalence between their classifying spaces.

As an application of this result, we will show that we can obtain the inclusion-exclusion principle of the Euler characteristics for finite categories with ideal covers or filter covers. The Euler characteristic $\chi(C)$ of a finite category $C$ is introduced by Leinster in [Lei08]. He calculates the Euler characteristic of the Grothendieck construction for a diagram in finite categories. By applying this to the (reduced) Čech nerve of a cover for a finite category, we obtain the following formula.

**Theorem 1.2** (Inclusion-exclusion principle). Let $\mathcal{D} = \{D_a\}_{a \in A}$ be an ideal cover or a filter cover of a finite category $C$, indexed by a totally ordered finite set $A$. If each intersection $D_{a_0...a_i}$ and $C$ have Euler characteristics, then we have 

$$\chi(C) = \sum_{i=0}^{\infty} \sum_{a_0 < ... < a_i} (-1)^i \chi(D_{a_0...a_i}).$$

Indeed, Leinster has demonstrated the inclusion-exclusion principle of the Euler characteristic for finite sets, as Example 3.4 (d) in [Lei08]. The theorem above is a generalization of that result.

The rest of this paper is organized as follows. Section 2 describes the homotopy theory of Čech complexes, for covers of small categories. Subsequently, we focus on the inclusion-exclusion principle for Euler characteristics of finite categories in section 3.

## 2 The Čech nerve for covers of small categories

Let $C$ be a small category, and let $\mathcal{D} = \{D_a\}_{a \in A}$ be a collection of subcategories of $C$ indexed by a set $A$. The intersection $\bigcap_{a \in A} D_a$ is a subcategory of $C$, whose set of objects is $\bigcap_{a \in A} \text{ob}(D_a)$ and set of morphisms is $\bigcap_{a \in A} D_a(x, y)$, for each pair of objects $x$ and $y$. 

On the other hand, the union \( \bigcup_{a \in A} D_a \) is a subcategory of \( C \), whose set of objects is \( \bigcup_{a \in A} \text{ob}(D_a) \) and the set of morphisms is generated by compositions of morphisms of \( D_a \). That is:

\[
\left( \bigcup_{a \in A} D_a \right)(x, y) = \{ f_n \cdots f_0 \mid f_i \in D_{a_i}(z_i, z_{i+1}), z_0 = x, z_{n+1} = y \},
\]

for each pair of objects \( x \) and \( y \).

**Definition 2.1.** Let \( C \) be a small category. A collection of subcategories \( \mathcal{D} = \{ D_a \}_{a \in A} \) of \( C \) is called a cover if \( C = \bigcup_{a \in A} D_a \).

**Definition 2.2 (Čech nerve).** Let \( \mathcal{D} = \{ D_a \}_{a \in A} \) be a cover of a small category \( C \). The Čech nerve \( \hat{\mathcal{C}}(\mathcal{D}) \) is a simplicial object in the category \( \text{Cat} \) of small categories, defined as

\[
\hat{\mathcal{C}}(\mathcal{D})_n = \coprod_{a_0, \ldots, a_n \in A} D_{a_0 \cdots a_n},
\]

where \( D_{a_0 \cdots a_n} \) denotes the intersection \( D_{a_0} \cap \cdots \cap D_{a_n} \). Let us denote an object (a morphism) \( x \) of \( \hat{\mathcal{C}}(\mathcal{D})_n \) belonging to \( D_{a_0 \cdots a_n} \) by \( x_{a_0 \cdots a_n} \). The face map \( d_i : \hat{\mathcal{C}}(\mathcal{D})_n \to \hat{\mathcal{C}}(\mathcal{D})_{n-1} \) is the functor omitting the index \( a_i \). That is:

\[
d_i(x_{a_0 \cdots a_n}) = x_{a_0 \cdots \hat{a}_i \cdots a_n},
\]

on objects and morphisms. Similarly, the degeneracy map \( s_j : \hat{\mathcal{C}}(\mathcal{D})_n \to \hat{\mathcal{C}}(\mathcal{D})_{n+1} \) is the functor inserting the index \( a_j \) between \( a_j \) and \( a_{j+1} \). That is:

\[
s_j(x_{a_0 \cdots a_n}) = x_{a_0 \cdots a_j a_j+1 \cdots a_n},
\]

on objects and morphisms.

In other words, \( \hat{\mathcal{C}}(\mathcal{D}) \) is a functor from \( \Delta^{\text{op}} \) to the category \( \text{Cat} \), where \( \Delta \) consists of totally ordered finite sets \( [n] = \{0, \cdots, n\} \) and order preserving maps. It takes \( [n] \) to \( \hat{\mathcal{C}}(\mathcal{D})_n \), and an order preserving map \( \varphi : [m] \to [n] \) in \( \Delta \) to the functor \( \varphi_* : \hat{\mathcal{C}}(\mathcal{D})_n \to \hat{\mathcal{C}}(\mathcal{D})_m \). Here, \( \varphi_* \) is given by \( \varphi_* (x_{a_0 \cdots a_n}) = x_{a_0 \cdots a_m} \) on objects and morphisms, satisfying \( b_j = a_{\varphi(j)} \) for each \( j \in \{0, \ldots, m\} \).

A (pseudo) functor from a small category to \( \text{Cat} \) provides a new small category, called the Grothendieck construction.

**Definition 2.3 (Grothendieck construction).** Let \( C \) be a small category, and let \( F : C \to \text{Cat} \) be a functor. The **Grothendieck construction** \( \text{Gr}(F) \) is a small category defined by the following:

- The set of objects consists of pairs \( (c, x) \) of an object \( c \) of \( C \) and an object \( x \) of \( F(c) \).
- For each pair of objects \( (c, x) \) and \( (d, y) \) of \( \text{Gr}(F) \), the set of morphisms \( \text{Gr}(F)((c, x), (d, y)) \) consists of pairs of morphisms \( (f, g) \), where \( f \in C(c, d) \) and \( g \in F(d)(F(f)(x), y) \).
• For each composable pair of morphisms \((f_1, g_1) \in \text{Gr}(F)((c, x), (d, y))\) and \((f_2, g_2) \in \text{Gr}((d, y), (e, z))\), the composition is given by \( (f_2, g_2)(f_1, g_1) = (f_2f_1, g_2(F(f_2)(g_1))) \).

We can take the Grothendieck construction of the Čech nerve \(\check{C}(D) : \Delta^{op} \to \text{Cat} \) for a cover \(D\) of a small category \(C\). Let us see the details of this category. The set of objects of \(\text{Gr}(\check{C}(D))\) consists of objects \(x_{a_0...a_n}\), indexed by \(a_0, \ldots, a_n \in A\) and \(n \geq 0\). The set of morphisms \(\text{Gr}(\check{C}(D))(x_{a_0...a_n}, y_{b_0...b_m})\) consists of pairs \((\varphi, f_{b_0...b_m})\) of an order preserving map \(\varphi : [n] \to [m]\) such that \(b_j = a_{\varphi(j)}\) for each \(j\), and a morphism \(f_{b_0...b_m} : x_{b_0...b_m} \to y_{b_0...b_m}\) of \(D_{b_0...b_m}\). It is equipped with the canonical functor \(\rho : \text{Gr}(\check{C}(D)) \to C\) that eliminates indices. That is, \(\rho(x_{a_0...a_n}) = x\) on objects, and \(\rho(\varphi, f_{a_0...a_n}) = f\) on morphisms. A natural question is to ask when the functor \(\rho\) induces a homotopy equivalence on classifying spaces. The classifying space \(BC\) of a small category \(C\) is constructed as the geometric realization of the nerve of \(C\) (see \text{Seg68}, \text{Qui73}, \text{Hir03}, \text{for homotopical properties of classifying spaces}). For a cover of a small category, the classifying space of the Grothendieck construction for the Čech nerve is called the Čech complex.

**Definition 2.4.** Let \(C\) be a small category. An **ideal** \(D\) of \(C\) is a full subcategory such that an object \(y\) of \(C\) belongs to \(D\), whenever \(C(y, x) \neq \emptyset\) for some object \(x\) of \(D\). Dually, an **filter** \(D\) of \(C\) is a full subcategory such that an object \(y\) of \(C\) belongs to \(D\), whenever \(C(x, y) \neq \emptyset\) for some object \(x\) of \(D\). A cover \(D = \{D_a\}_{a \in A}\) of \(C\) is called an **ideal** (a **filter**) **cover** if \(D_a\) is an ideal (a filter) for any \(a \in A\).

The notions above are generalizations of ideals and filters of posets \text{Sta12, Zap98}. Let \(D\) be a full subcategory of a small category \(C\). The category of complement \(C\setminus D\) is defined as the full subcategory whose set of objects is \(\text{ob}(C) \setminus \text{ob}(D)\). If \(D\) is an ideal (a filter), then \(C\setminus D\) is a filter (an ideal). In other words, the ideal \(D\) yields a functor \(C \to \mathcal{P}\), where \(\mathcal{P}\) is the poset formed of \(0 < 1\). The functor sends \(\text{ob}(D)\) to \(\{0\}\) and \(\text{ob}(C\setminus D)\) to \(\{1\}\). Conversely, for a functor \(F : C \to \mathcal{P}\), the category of fiber \(F^{-1}(0)\) is an ideal, and \(F^{-1}(1)\) is a filter of \(C\).

**Theorem 2.5.** Let \(D = \{D_a\}_{a \in A}\) be an ideal cover or a filter cover of a small category \(C\). Then, the natural functor \(\rho : \text{Gr}(\check{C}(D)) \to C\) induces a homotopy equivalence between their classifying spaces.

**Proof.** We first consider the case in which \(D\) is an ideal cover. We use Quillen’s theorem A \text{[Qui73]} for \(\rho\). In order to apply this, we examine the left homotopy fiber \(\rho \downarrow x\) of \(\rho\), over an object \(x\) of \(C\). The set of objects consists of pairs \((y_{b_0...b_n}, f)\) of an object \(y_{b_0...b_n}\) of \(\text{Gr}(\check{C}(D))\), and a morphism \(f : y \to x\) of \(C\). The set of morphisms \((\rho \downarrow x)((y_{b_0...b_n}, f), (z_{c_0...c_m}, g))\) consists of morphisms \((\varphi, h_{c_0...c_m}) : y_{b_0...b_n} \to z_{c_0...c_m}\) of \(\text{Gr}(\check{C}(D))\), such that \(g \circ h = f\).

On the other hand, we choose and fix an index \(a \in A\) such that \(x \in \text{ob}(D_a)\). The set of objects of the over category \(\text{Gr}(\check{C}(D)) \downarrow x_a\) consists of morphisms...
(v_j, f_a) : y_{a_0...a_n} \to x_a of \Gr(\hat{C}(D)). Here, v_j is the map \([0] \to [n]\) choosing \(j \in \{0, \ldots, n\}\), and \(f_a\) is a morphism from \(y_a = y_{a_0} \to x_a of D_a\).

The set of morphisms \((\Gr(\hat{C}(D)))\downarrow x_a)\) \(\{(v_j, f_a), (v_k, g_a)\}\) consists of morphisms \((\varphi, h)\) of \(\Gr(\hat{C}(D))\), such that \((v_k, g_a) \circ (\varphi, h) = (v_j, f_a)\).

A functor

\[
F : \rho^{-1}x \to \Gr(\hat{C}(D))\downarrow x_a
\]

is defined by \(F(y_{a_0...a_n}, f) = (v_{n+1}, f_a) : y_{a_0...a_n} \to x_a\) on objects. Note that the object \(\varphi\) belongs to \(D_a\), since \(D_a\) is an ideal. A morphism \((\varphi, h_{b_0...b_m}) : (y_{a_0...a_n}, f) \to (z_{b_0...b_m}, g)\) of \(\rho^{-1}x\) is sent to a morphism \((\tilde{\varphi}, h_{b_0...b_m}) : (v_{n+1}, f_a) \to (v_{m+1}, g_a)\) of \(\Gr(C(D))\downarrow x_a\), where \(\tilde{\varphi} : [m] \to [n+1]\) is the extension of \(\varphi : [m] \to [n]\) with \(\varphi(m+1) = n+1\). Conversely, a functor

\[
G : \Gr(\hat{C}(D))\downarrow x_a \to \rho^{-1}x
\]

is defined by \(G(v_j, g_a) = (y_{a_0...a_n}, g)\), for an object \((v_j, g_a) : y_{a_0...a_n} \to x_a\) of \(\Gr(C(D))\downarrow x_a\). There exists a natural transformation \(t : GF \Rightarrow \text{id}\) defined by

\[
t(y_{a_0...a_n}, f) = (d_{n+1}, \text{id}_{a_0...a_n}) : (y_{a_0...a_n}, f) \to (y_{a_0...a_n}, f).
\]

Note that the classifying space of the over category \(\Gr(\hat{C}(D))\downarrow x_a\) is contractible, since it has a terminal object. We conclude that the classifying spaces of \(\Gr(C(D))\downarrow x_a\) and \(\rho^{-1}x\) are homotopy equivalent and contractible. Quillen’s theorem A completes the proof for the case of ideal covers. If \(\mathcal{D}\) is a filter cover of \(\mathcal{C}\), the opposite cover \(\mathcal{D}^{\text{op}} = \{D_a^{\text{op}}\}_{a \in A}\) is an ideal cover of \(\mathcal{C}^{\text{op}}\). Hence, the classifying space of the left homotopy fiber \(\rho'\downarrow x\) is contractible for the natural functor \(\rho' : \Gr(C(D)^{\text{op}}) \to \mathcal{C}^{\text{op}}\), and any object \(x\) of \(\mathcal{C}^{\text{op}}\). The left homotopy fiber \(\rho'\downarrow x\) of \(\rho'\) coincides with the right homotopy fiber \(x_\downarrow \rho\) of \(\rho\) for any object \(x \in \text{ob}(\mathcal{C}) = \text{ob}(\mathcal{C}^{\text{op}})\). Again, Quillen’s theorem A implies the result.

Thomason’s homotopy colimit theorem \[Th97\] tells us that the Čech complex for a cover \(\mathcal{D}\) of a small category is homotopy equivalent to the homotopy colimit \(\lim_{\to} (BC(\mathcal{D}) : \Delta^{\text{op}} \to \text{Top})\), in the category \text{Top} of spaces. The diagram \(BC(\mathcal{D})\) is a Reedy cofibrant simplicial space \[Hir03\], with respect to the Strom model structure on \text{Top}, since all the degeneracy maps are closed cofibrations. The Bousfield-Kan map \(hocolim BC(\mathcal{D}) \to |BC(\mathcal{D})|\) is a homotopy equivalence by Theorem 18.7.4 of \[Hir03\]. Consequently, we have a homotopy equivalence

\[
|BC(\mathcal{D})| \simeq BC,
\]

for an ideal cover or a filter cover \(\mathcal{D}\) of a small category \(\mathcal{C}\). Note that the collection \(\{BD_a\}_{a \in A}\) is a cover consisting of subcomplexes of \(BC\). In \[Bjo03\], Björner shows the above homotopy equivalence for a simplicial complex with a cover consisting of subcomplexes, where the connectivity of intersections is assumed. We do not require such connectivity of intersections for the proof of Theorem 2.3.

If the index set \(A\) is equipped with a total order, then we can consider the ordered Čech nerve.

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**Definition 2.6** (Ordered Čech nerve). Let \( \mathcal{D} = \{D_a\}_{a \in A} \) be a cover of a small category \( C \), indexed by a totally ordered set \( A \). The ordered Čech nerve \( \check{C}^o(\mathcal{D}) \) is a simplicial object in \( \textbf{Cat} \), defined as

\[
\check{C}^o(\mathcal{D})_n = \prod_{a_0 \leq \cdots \leq a_n} D_{a_0 a_1 \cdots a_n},
\]

The face and degeneracy maps are defined similarly to the ordinary Čech nerve.

The ordered Čech nerve is smaller, and often easier to treat, than the ordinary Čech nerve. There exists an inclusion \( \check{C}^o(\mathcal{D}) \to \check{C}(\mathcal{D}) \) of simplicial objects in \( \textbf{Cat} \). This induces a functor \( I : \text{Gr}(\check{C}^o(\mathcal{D})) \to \text{Gr}(\check{C}(\mathcal{D})) \) between their Grothendieck constructions.

**Proposition 2.7.** Let \( \mathcal{D} = \{D_a\}_{a \in A} \) be a cover of a small category \( C \), indexed by a totally ordered set \( A \). The inclusion functor \( I : \text{Gr}(\check{C}(\mathcal{D})) \to \text{Gr}(\check{C}(\mathcal{D})) \) induces a homotopy equivalence between their classifying spaces.

**Proof.** For an object \( x_{a_0...a_n} \) of \( \text{Gr}(\check{C}(\mathcal{D})) \), there exists a permutation \( \sigma \in \Sigma_n \), such that \( a_{\sigma(0)} \leq \cdots \leq a_{\sigma(n)} \) in \( A \). In the case that \( a_i = a_j \) for some \( i < j \), we always choose \( \sigma(i) < \sigma(j) \). This yields an inverse functor \( J : \text{Gr}(\check{C}(\mathcal{D})) \to \text{Gr}(\check{C}(\mathcal{D})) \), sending \( x_{a_0...a_n} \) to \( x_{a_{\sigma(0)}...a_{\sigma(n)}} \), on objects. For a morphism \( \langle \varphi, f \rangle : x_{a_0...a_n} \to y_{b_0...b_m} \) of \( \text{Gr}(\check{C}(\mathcal{D})) \), let \( \sigma \in \Sigma_n \) and \( \tau \in \Sigma_m \) be permutations for the canonical reordering of \( a_0...a_n \) and \( b_0...b_m \), respectively. For simplicity, denote \( \alpha_i = a_{\sigma(i)} \) and \( \beta_j = b_{\tau(j)} \), for each \( i \) and \( j \). For every \( j \in \{0, \cdots, m\} \), we have

\[
\beta_j = b_{\tau(j)} = a_{\sigma \tau(j)} = a_{\sigma^{-1} \varphi \tau(j)}.
\]

Consider the composition of maps \( J(\varphi) = \sigma^{-1} \varphi \tau : [m] \to [n] \). We can describe

\[
\varphi^{-1}(0) \quad \varphi^{-1}(1) \quad \varphi^{-1}(n)
\]

\[
\sigma^{-1}(0) \quad \sigma^{-1}(1) \quad \sigma^{-1}(n)
\]

\[
b_0...b_m = a_0...a_0 a_1...a_1 ... a_n...a_n.
\]

The permutation \( \tau \) acts on \( b_0...b_m \) as \( \sigma \):

\[
b_{\tau(0)}...b_{\tau(m)} = a_{\sigma(0)}...a_{\sigma(0)} a_{\sigma(1)}...a_{\sigma(1)} ... a_{\sigma(n)}...a_{\sigma(n)}.
\]

It follows that \( J(\varphi) \) preserves order. The functor \( J \) sends \( \langle \varphi, f \rangle \) to \( (J(\varphi), f) \) on morphisms.

Trivially, we have \( JJ = \text{id} \) on \( \text{Gr}(\check{C}(\mathcal{D})) \). Let us examine the composition \( JJ \). A functor \( K : \text{Gr}(\check{C}(\mathcal{D})) \to \text{Gr}(\check{C}(\mathcal{D})) \) is defined by \( K(x_{a_0...a_n}) = x_{a_0...a_n a_{\sigma(0)}...a_{\sigma(n)}} \) on objects. For a morphism \( \langle \varphi, f \rangle \) of \( \text{Gr}(\check{C}(\mathcal{D})) \), define \( K(\varphi, f) = (\varphi * J(\varphi), f) \), where \( \varphi * J(\varphi) : [2m + 1] \to [2n + 1] \) is given as

\[
\left\{
\begin{array}{ll}
\varphi * J(\varphi)(i) = \varphi(i) & \text{if } 0 \leq i \leq m, \\
\varphi * J(\varphi)(i) = J(\varphi)(i - m - 1) & \text{if } m + 1 \leq i \leq 2m + 1.
\end{array}
\right.
\]
There exist natural transformations $t : K \Rightarrow IJ$ with $t(x_{a_0 \ldots a_n}) = (d^n_0, \text{id}_x)$, and $s : K \Rightarrow \text{id}$ with $s(x_{a_0 \ldots a_n}) = (d_{n+1} \ldots d_{2n+1}, \text{id}_x)$, for an object $x_{a_0 \ldots a_n}$ of Gr($\mathcal{C}(\mathcal{D})$). Therefore, the induced maps from $I$ and $J$ on classifying spaces are homotopy inverses to each other. 

For a cover $\mathcal{D} = \{D_a\}_{a \in A}$ of a small category $C$, indexed by a totally ordered set $A$, the natural functor $\rho^o : \text{Gr}(\mathcal{C}^o(\mathcal{D})) \to C$ is given as the composition $\rho^o = \rho I$. Theorem 2.5 and Proposition 2.7 imply the following.

**Corollary 2.8.** Let $\mathcal{D} = \{D_a\}_{a \in A}$ be an ideal cover or a filter of a small category $C$, indexed by a totally ordered set $A$. The natural functor $\rho^o : \text{Gr}(\mathcal{C}^o(\mathcal{D})) \to C$ induces a homotopy equivalence between their classifying spaces.

Note that the Grothendieck construction of the ordered Čech nerve is independent of the order on the index set.

**Proposition 2.9.** Let $\mathcal{D} = \{D_a\}_{a \in A}$ be a cover of a small category $C$, and let the index set $A$ be equipped with two total order $\leq_1$ and $\leq_2$. Denote the ordered Čech nerve induced from $\leq_i$ by $\mathcal{C}^o(\mathcal{D})_i$, for $i = 1, 2$, respectively. Then, the Grothendieck constructions $\text{Gr}(\mathcal{C}^o(\mathcal{D})_1)$ and $\text{Gr}(\mathcal{C}^o(\mathcal{D})_2)$ are isomorphic to each other as categories.

**Proof.** Similarly to the proof of Proposition 2.7, we obtain a permutation $\sigma$ for an object $x_{a_0 \ldots a_n}$ of $\text{Gr}(\mathcal{C}^o(\mathcal{D})_1)$, such that $x_{a_{\sigma(0)} \ldots a_{\sigma(n)}}$ belongs to $\text{Gr}(\mathcal{C}^o(\mathcal{D})_2)$. This yields a functor $F : \text{Gr}(\mathcal{C}^o(\mathcal{D})_1) \to \text{Gr}(\mathcal{C}^o(\mathcal{D})_2)$. We also obtain an inverse functor $G : \text{Gr}(\mathcal{C}^o(\mathcal{D})_2) \to \text{Gr}(\mathcal{C}^o(\mathcal{D})_1)$ by reordering the indexes. A straightforward calculation shows that $FG = \text{id}$ and $GF = \text{id}$. 

Next, we define the reduced Čech nerve, smaller than the ordered Čech nerve.

**Definition 2.10 (Reduced Čech nerve).** Let $\Delta_{\text{inj}}$ denote the subcategory of $\Delta$, consisting of the same objects as $\Delta$ and injective order preserving maps. Let $\mathcal{D} = \{D_a\}_{a \in A}$ be a cover of a small category $C$, indexed by a totally ordered set $A$. The reduced Čech nerve of $\mathcal{D}$ is a functor $\mathcal{C}^o(\mathcal{D}) : \Delta_{\text{inj}} \to \text{Cat}$, defined by

$$\mathcal{C}^o(\mathcal{D})_n = \coprod_{a_0 < \ldots < a_n} D_{a_0a_1 \ldots a_n}.$$

This is the diagram in $\text{Cat}$ where the degenerate parts of the ordered Čech nerve have been removed. It has only face maps omitting indices. Note that if $A$ is finite, then $\mathcal{C}^o(\mathcal{D})_n$ is the empty category for $n$ greater than the cardinality $A^2$ of $A$.

The inclusion $\Delta_{\text{inj}} \to \Delta$ induces a functor $L : \text{Gr}(\mathcal{C}^o(\mathcal{D})) \to \text{Gr}(\mathcal{C}^o(\mathcal{D}))$ on Grothendieck constructions.

**Lemma 2.11.** Let $\mathcal{D} = \{D_a\}_{a \in A}$ be a cover of a small category. For two morphisms $(\varphi, f), (\chi, g) : x_{a_0 \ldots a_n} \to y_{b_0 \ldots b_m}$ of $\text{Gr}(\mathcal{C}(\mathcal{D}))$, if the indices $a_0, \ldots, a_n$ are distinguished, then $\varphi = \chi$. 

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Proof. For any \( j \in \{0, \ldots, m\} \), the equality \( a_{\varphi(j)} = b_j = a_{\chi(j)} \) holds. Since the indices \( a_0, \ldots, a_n \) are distinguished, \( \varphi(j) \) and \( \chi(j) \) must be equal to each other.

**Proposition 2.12.** Let \( \mathcal{D} = \{ D_a \}_{a \in A} \) be a cover of a small category \( C \), indexed by a totally ordered set \( A \). The inclusion functor \( L : \text{Gr}(\hat{C}^o(\hat{D})) \to \text{Gr}(\hat{C}^o(D)) \) has a right adjoint functor.

Proof. For an arbitrary object \( x_{a_0 \ldots a_n} \) of \( \text{Gr}(\hat{C}^o(\hat{D})) \), we obtain an object \( x_{\alpha_0 \ldots \alpha_{n'}} \) of \( \text{Gr}(\hat{C}^o(\tilde{D})) \) and a surjective order preserving map \( \psi : [n] \to [n'] \), such that \( \alpha_j = a_{\psi(j)} \), by removing duplicate indices. This determines the reduction functor \( R : \text{Gr}(\hat{C}^o(\hat{D})) \to \text{Gr}(\hat{C}^o(\tilde{D})) \), sending \( x_{a_0 \ldots a_n} \) to \( x_{\alpha_0 \ldots \alpha_{n'}} \) on objects. For a morphism \( \varphi : x_{a_0 \ldots a_n} \to y_{b_0 \ldots b_m} \) of \( \text{Gr}(\hat{C}^o(\hat{D})) \), the order preserving map \( \varphi' : [m] \to [n] \) induces \( \varphi' : [m'] \to [n'] \), making the following diagram commutative:

\[
\begin{array}{ccc}
[m] & \xrightarrow{\varphi} & [n] \\
\downarrow{\psi} & & \downarrow{\psi'} \\
[m'] & \xrightarrow{\varphi'} & [n']
\end{array}
\]

The reduction functor \( R \) sends \( (\varphi, f) \) to \( (\varphi', f) \) on morphisms. We take objects \( x_{a_0 \ldots a_n} \) of \( \text{Gr}(\hat{C}^o(\hat{D})) \) and \( y_{b_0 \ldots b_m} \) of \( \text{Gr}(\hat{C}^o(\hat{D})) \), respectively. Since \( a_0, \ldots, a_n \) are distinguished, we have that

\[
\text{Gr}(\hat{C}^o(\hat{D}))(x_{a_0 \ldots a_n}, R(y_{b_0 \ldots b_m})) = \text{Gr}(\hat{C}^o(\hat{D}))(x_{a_0 \ldots a_n}, y_{\beta_0 \ldots \beta_{m'}})
\]

\[
\cong D_{\beta_0 \ldots \beta_{m'}}(x, y)
\]

\[
= D_{b_0 \ldots b_m}(x, y)
\]

\[
\cong \text{Gr}(\hat{C}^o(\hat{D}))(L(x_{a_0 \ldots a_n}), y_{b_0 \ldots b_m}),
\]

by Lemma 2.11. Therefore, we conclude that the reduction functor \( R \) is right adjoint to \( L \).

The reduced Čech nerve is also equipped with the natural functor \( \check{\rho} = \rho \circ L : \hat{C}^o(\hat{D}) \to C \), the same as the ordered Čech nerve and the ordinary Čech nerve. Form Proposition 2.12 and Corollary 2.8 we can deduce the following.

**Corollary 2.13.** Let \( \mathcal{D} = \{ D_a \}_{a \in A} \) be an ideal cover or a filter cover of a small category \( C \), indexed by a totally ordered set \( A \). The natural functor \( \check{\rho} : \hat{C}^o(\hat{D}) \to C \) induces a homotopy equivalence between their classifying spaces.

**Definition 2.14.** A cover \( \mathcal{D} = \{ D_a \}_{a \in A} \) of a small category \( C \) is called **locally finite** when

\[
\{ a \in A \mid x \in \text{ob}(D_a) \}^2 < \infty,
\]

for any object \( x \) of \( C \).
Proposition 2.15. Let $D = \{D_a\}_{a \in A}$ be a locally finite ideal cover of a small category $C$, indexed by a totally ordered set $A$. The natural functor $\tilde{\rho} : \text{Gr}(\tilde{C}^{\text{op}}(D)) \to C$ has a left adjoint functor.

Proof. A functor $\pi : C \to \text{Gr}(\tilde{C}^{\text{op}}(D))$ is defined by $\pi(x) = x_{a_0...a_n}$ on objects, where $D_{a_0}, \ldots, D_{a_n}$ are whole distinguished ideals in $D$ that contain $x$. For a morphism $f : x \to y$ of $C$, the object $x$ belongs to an ideal $D$ if $y$ does. When we describe $\pi(x) = x_{a_0...a_n}$ and $\pi(y) = y_{b_0...b_m}$, the inclusion relation $\{b_0, \ldots, b_m\} \subset \{a_0, \ldots, a_n\}$ holds. This yields an injection $\varphi : [n] \to [m]$, such that $b_j = a_{\varphi(j)}$. We define $\pi(f) = (\varphi, f) : \pi(x) \to \pi(y)$ on morphisms.

Take objects $x$ of $C$ and $y_{b_0...b_m}$ of $\text{Gr}(\tilde{C}^{\text{op}}(D))$, respectively. Then, by Lemma 2.11

$C(x, \tilde{\rho}(y_{b_0...b_m})) = C(x, y) = D_{b_0...b_m}(x, y) \cong \text{Gr}(\tilde{C}^{\text{op}}(D))(\pi(x), y_{b_0...b_m})$.

Therefore, we conclude that the functor $\pi$ is left adjoint to $\tilde{\rho}$. □

When $D = \{D_a\}_{a \in A}$ is a filter cover of a small category $C$, the opposite cover $D^{\text{op}} = \{D_a^{\text{op}}\}_{a \in A}$ is an ideal cover of $C^{\text{op}}$. The following corollary then follows immediately.

Corollary 2.16. Let $D = \{D_a\}_{a \in A}$ be a locally finite filter cover of a small category $C$, indexed by a totally ordered set $A$. The natural functor $\text{Gr}(\tilde{C}^{\text{op}}(D^{\text{op}})) \to C^{\text{op}}$ has a left adjoint functor.

3 The inclusion-exclusion principle for the Euler characteristics of finite categories

In this section, we focus on the inclusion-exclusion principle for the Euler characteristic of a finite category. The Euler characteristic of a finite category was introduced by Leinster [Lei08]. This is a generalization of Möbius inversion for posets [Rot64]. Let us briefly review the definition. In this paper, we use the rational numbers $\mathbb{Q}$ as the value of Euler characteristics of finite categories.

Definition 3.1. Suppose that $C$ is a finite category that has finitely many objects and morphisms.

1. The similarity matrix of $C$ is the function $\zeta : \text{ob}(C) \times \text{ob}(C) \to \mathbb{Q}$, given by the cardinality of the set of morphisms $\zeta(a, b) = C(a, b)^2$.

2. Let $u : \text{ob}(C) \to \mathbb{Q}$ denote the column vector with $u(a) = 1$, for any object $a$ of $C$. A weighting on $C$ is a column vector $w : \text{ob}(C) \to \mathbb{Q}$ such that $\zeta w = u$. Dually, a coweighting on $C$ is a row vector $v : \text{ob}(C) \to \mathbb{Q}$, such that $v^* w = u^*$, where $u^*$ is the transposed matrix of $u$.

Note that we have

$$\sum_{i \in \text{ob}(C)} w(i) = u^* w = v^* w = vu = \sum_{j \in \text{ob}(C)} v(j),$$
if both a weighting and a coweighting exist. Moreover,
\[ \sum_{i \in \text{ob}(C)} w(i) = u^* w = v_\zeta w = v_\zeta w' = \sum_{i \in \text{ob}(C)} w'(i), \]
for two (co)weightings \( w \) and \( w' \) on \( C \). This guarantees the following definition.

**Definition 3.2.** Let \( C \) be a finite category. We say that \( C \) has Euler characteristic if it has both a weighting \( w \), and a coweighting \( v \), on \( C \). Then, the Euler characteristic of \( C \) is defined by

\[ \chi(C) = \sum_{i \in \text{ob}(C)} w(i) = \sum_{j \in \text{ob}(C)} v(j). \]

It is well-known that the topological Euler characteristic \( \chi_T \) has the following formula:

\[ \chi_T(A \cup B) = \chi_T(A) + \chi_T(B) - \chi_T(A \cap B), \]
for subcomplexes \( A \) and \( B \) of a finite CW-complex \( X \). Recall the example given in our introduction. The finite category \( C \) is

\[ x \xrightarrow{1} y \xrightarrow{1} z, \]
with the terminal object \( z \), \( D_1 \) is the full subcategory with \( \text{ob}(D_1) = \{x, y\} \), and \( D_2 \) is the full subcategory with \( \text{ob}(D_2) = \{y, z\} \). The Euler characteristic \( \chi(C) = \chi(D_1 \cup D_2) = 1 \), since \( C \) has a terminal object. However,

\[ \chi(D_1) + \chi(D_2) - \chi(D_1 \cap D_2) = 0 + 1 - 1 = 0. \]

The inclusion-exclusion principle does not hold in this case. In [FLST11], Fiore, Luck, and Sauer used the homotopy colimit of diagram in \( \text{Cat} \), instead of the genuine colimit or union.

Now, we consider the following situation. Let \( C \) be a finite category, and let \( \mathcal{D} = \{D_a\}_{a \in A} \) be a cover of \( C \), indexed by a totally ordered finite set \( A \). We denote the full subcategory of \( \Delta_{n+1} \), consisting of \( [n] \) for \( 0 \leq n \leq A^2 - 1 \) by \( \Delta_A \). The reduced Čech nerve \( \text{Gr}(\breve{C}o(\breve{D})) \) can be regarded as a functor \( \Delta_A \to \text{Cat} \). The category \( \Delta_A \) is a finite acyclic category, which never has circuit of morphisms. Moreover, each \( \breve{C}o(\breve{D}) \) is a finite coproduct of finite categories. Leinster provided the product formula for the Euler characteristic of finite categories, with respect to Grothendieck construction in [Lei08]. By applying this to \( \text{Gr}(\breve{C}o(\breve{D})) \), we obtain the inclusion-exclusion principle for ideal (filter) covers of a finite category.

**Theorem 3.3 (Inclusion-exclusion principle).** Let \( \mathcal{D} = \{D_a\}_{a \in A} \) be an ideal cover or a filter cover of a finite category \( C \), indexed by a finite set \( A \). If each \( D_{a_0...a_i} \) and \( C \) have Euler characteristics, then we have

\[ \chi(C) = \sum_{i=0}^{A^2 - 1} \sum_{a_0 < \ldots < a_i} (-1)^i \chi(D_{a_0...a_i}). \]
Proof. Let $\mathcal{D}$ be an ideal cover of $C$. The category $\Delta_{\mathcal{A}}$ has a unique coweighting $v[n] = (-1)^n$ (see Example 3.4 (d) of [Lei08]). By applying Proposition 2.8 of [Lei08] to $\text{Gr}(\tilde{\mathcal{C}}^{\mathcal{O}}(\mathcal{D}))$, we have

$$
\chi(\text{Gr}(\tilde{\mathcal{C}}^{\mathcal{O}}(\mathcal{D}))) = \sum_{i=0}^{A^I-1} (-1)^i \chi(\tilde{\mathcal{C}}^{\mathcal{O}}(\mathcal{D})_i) = \sum_{i=0}^{A^I-1} \sum_{a_0 < \ldots < a_i} (-1)^i \chi(D_{a_0 \ldots a_i}).
$$

Then, Proposition 2.4 of [Lei08] and our Proposition 2.15 imply the result. If $\mathcal{D}$ is a filter cover of $C$, then $\mathcal{D}^{\text{op}}$ is an ideal cover of $C^{\text{op}}$. Since the equality $\chi(X) = \chi(X^{\text{op}})$ holds for any finite category $X$ having Euler characteristic, we obtain the desired formula.

Note that the order on the index set is not essential, since we can choose a total order on any finite set, and Proposition 2.15 holds in the case of reduced Čech nerves too.

Corollary 3.4. Let both $A$ and $B$ be filters or ideals of a finite category $C$. If each of $A$, $B$, $A \cap B$, and $A \cup B$ has Euler characteristic, then we have

$$
\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B).
$$

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