Moduli stacks and invariants of semistable objects on K3 surfaces

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Abstract

For a K3 surface $X$ and its bounded derived category of coherent sheaves $D(X)$, we have the notion of stability conditions on $D(X)$ in the sense of T. Bridgeland. In this paper, we show that the moduli stack of semistable objects in $D(X)$ with a fixed numerical class and a phase is represented by an Artin stack of finite type over $\mathbb{C}$. Then following D. Joyce’s work, we introduce the invariants counting semistable objects in $D(X)$, and show that the invariants are independent of a choice of a stability condition.

1 Introduction

The work of this paper is motivated by D. Joyce’s recent works [19], [20], [16], [17], [21], especially [17, Conjecture 6.25] on the counting invariants of semistable objects on K3 surfaces or abelian surfaces. Such invariants are expected to produce automorphic functions on the space of stability conditions in the sense of T. Bridgeland [7].

1.1 Stability conditions

Let $X$ be a smooth projective variety over $\mathbb{C}$, $\text{Coh}(X)$ the abelian category of coherent sheaves on $X$, and $D(X)$ the bounded derived category of $\text{Coh}(X)$. For an ample divisor $\omega$ on $X$, there is a notion of $\omega$-Gieseker stability on $\text{Coh}(X)$, and the moduli spaces of semistable sheaves have been studied in detail up to now [12]. The notion of stability conditions on a triangulated category $T$ (especially including the case of $T = D(X)$) was introduced by T. Bridgeland [7] motivated by M. Douglas’s $\Pi$-stability [10], [11]. Roughly it consists of data $\sigma = (Z, P)$,

$$Z: K(T) \rightarrow \mathbb{C}, \quad P(\phi) \subset T,$$

where $Z$ is a group homomorphism and $P(\phi)$ is a full subcategory for each $\phi \in \mathbb{R}$, and these data satisfy some axiom. (See Definition 2.1 below.) Then Bridgeland [7] showed that the set of good stability conditions has a structure of a complex manifold. When $T = D(X)$, the space of stability conditions $\text{Stab}(X)$ carries a map,

$$Z: \text{Stab}(X) \rightarrow \mathcal{N}(X)^{\mathbb{C}}_\mathbb{C},$$

where $\mathcal{N}(X) = K(X)/\equiv$ is a numerical Grothendieck group. (See Definition 2.1.) The precise descriptions of the space $\text{Stab}(X)$ have been studied in the articles [6], [5], [19], [27], [26], [24], [15], [2], [29], [28]. In particular when $X$ is a K3 surface or an abelian surface, Bridgeland [6] described $\text{Stab}^*(X)$, one of the connected components of $\text{Stab}(X)$, as a covering space over a certain open subset $\mathcal{P}_0^+(X) \subset \mathcal{N}(X)^{\mathbb{C}}_\mathbb{C}$, and related its Galois group to the group of autoequivalences of $D(X)$. 

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In general when $X$ is a Calabi-Yau manifold, it is expected that the space $\text{Stab}(X)$ describes the so-called “stringy Kähler moduli space”. More precisely Bridgeland conjectures in [4] that the double quotient space,

$$\text{Auteq } D(X) \setminus \text{Stab}(X) / \mathbb{C},$$

contains the stringy Kähler moduli space $\mathcal{M}_K(X)$, which is in a mirror side $\hat{X}$, isomorphic to the moduli space of the complex structures $\mathcal{M}_C(\hat{X})$. When $X$ is an elliptic curve, $\mathcal{M}_C(\hat{X})$ is nothing but the modular curve, and we have the following complete picture [7],

$$\text{Auteq } D(X) \setminus \text{Stab}(X) / \mathbb{C} \cong \mathcal{M}_C(\hat{X}) = \mathcal{H} / \text{SL}(2, \mathbb{Z}),$$

where $\mathcal{H} \subset \mathbb{C}$ is the upper half plane. On the space $\mathcal{H}$, several automorphic functions (Eisenstein series, $j$-invariant) have been studied. Thus it is interesting to construct automorphic functions on the space $\text{Stab}(X)$, purely from the categorical data of $D(X)$, and compare the classical theory in the mirror side.

### 1.2 Counting invariants of semistable sheaves

D. Joyce’s recent works [19], [20], [16], [17], [21] are attempts to introduce some structures on the space $\text{Stab}(X)$, such as Frobenius structures or automorphic functions. However for several technical reasons, his arguments work only on the space of stability conditions on an abelian category. What we are interested in this paper is the work [17], where D. Joyce studies certain counting invariants of semistable sheaves on a K3 surface $X$. We denote $C(X) \subset N(X)$ the image of $\text{Coh}(X) \to N(X)$, and let $\alpha \in C(X)$ be a numerical class and $\Lambda$ a $\mathbb{Q}$-algebra. We consider a motivic invariant,

$$\Upsilon: (\text{quasi-projective varieties}) \longrightarrow \Lambda.$$

As an example, one can take $\Lambda = \mathbb{Q}(z)$ and $\Upsilon(Y)$ to be the virtual Poincare polynomial of $Y$. Using $\Upsilon$, D. Joyce [17] constructs an invariant $\hat{I}^\alpha(\omega) \in \Lambda$ which counts $\omega$-Gieseker semistable sheaves of numerical type $\alpha$, and its weighted counting

$$\hat{J}^\alpha(\omega) = \sum_{\alpha_1 + \cdots + \alpha_n = \alpha} l^{-\sum_j (\alpha_j, \alpha)} (-1)^{n-1} (l-1)^{-n} \prod_{i=1}^n \hat{I}^\alpha_i(\omega) \in \Lambda.$$

Here $\alpha_i \in C(X)$ has the same reduced Hilbert polynomial with $\alpha$ and $l = \Upsilon(\mathbb{A}^1) \in \Lambda$. Then Joyce [17] showed that $\hat{J}^\alpha(\omega)$ does not depend on a choice of $\omega$, so one can denote it by $\hat{J}^\alpha \in \Lambda$.

The purpose of this paper is to translate the above work into the context of Bridgeland’s stability conditions. As we see below, it is related to the automorphic functions on the space of stability conditions. Based on the results in [17], D. Joyce proposes the following conjecture.

**Conjecture 1.1.** [17] **Conjecture 6.25** Let $X$ be a K3 surface or an abelian surface. For $\sigma \in \text{Stab}^*(X)$ and $\alpha \in N(X)$, there is $J^\alpha(\sigma) \in \Lambda$, a certain weighted counting of $\sigma$-semistable objects of numerical type $\alpha$, such that

(i) $J^\alpha(\sigma)$ does not depend on a choice of $\sigma$. Hence we can write it $J^\alpha \in \Lambda$.

(ii) If $\alpha \in C(X)$, then $J^\alpha = \hat{J}^\alpha$.

Suppose for instance Conjecture [1.1] is true. Let $\text{Auteq}^* D(X)$ be the group of autoequivalences on $D(X)$ which preserve the component $\text{Stab}^*(X)$. Then the property (i) of Conjecture [1.1] implies that $J^\alpha = J^{\Phi_\alpha}$ for $\Phi \in \text{Auteq}^* D(X)$. (See Corollary 5.26 below.) Based on
this observation, Joyce [17] suggests that the map (ignoring convergence)

\[ \text{Stab}^*(X) \ni \sigma = (Z, \mathcal{P}) \mapsto \sum_{\alpha \in N(X) \setminus \{0\}} \sum_{k \in \mathbb{Z}} \frac{J_{\alpha} Z(\alpha)^k}{Z(\alpha)^k} \in \Lambda \otimes_{\mathbb{Q}} \mathbb{C}, \tag{4} \]

for \( k \in \mathbb{Z} \) would give a holomorphic function on \( \text{Stab}^*(X) \) which is invariant under the action of \( \text{Aut}_{\mathbb{Q}}^* D(X) \), i.e. automorphic function on \( \text{Stab}^*(X) \). Our goal is the following.

**Theorem 1.2.** Conjecture 1.1 is true.

As stated in [8], [17], it is interesting to compare the formula (4) with the work of Borcherds [3] on the product expansions of the automorphic forms.

### 1.3 Moduli problems

The first issue in attacking Conjecture 1.1 is to develop the moduli theory of semistable objects in the sense of Bridgeland. The moduli theory of objects in \( D(X) \) is studied in some articles [13], [23], [1]. In the recent work of Inaba [13], he constructs some nice moduli spaces of complexes, using the notion of ample sequences. However the relationship between Bridgeland’s stability conditions [7] and Inaba’s stability conditions using ample sequences [13] is not clear. On the other hand, for our purpose we do not require the moduli spaces to have good properties, (projective, fine, etc). In fact we only need it to be an Artin stack of finite type. Thus in Section 3 we work over \( D(X) \) for an arbitrary smooth projective variety \( X \) and establish the general arguments to guarantee the moduli stacks to be Artin stacks of finite type.

In Section 3, the work of Lieblich [23] would help us. Let \( \mathcal{M} \) be the moduli stack of objects \( E \in D(X) \) which satisfies \( \text{Ext}^{<0}(E, E) = 0 \). Then he showed that \( \mathcal{M} \) is an Artin stack of locally finite type over \( \mathbb{C} \). For \( \alpha \in N(X), \phi \in \mathbb{R} \) and \( \sigma = (Z, \mathcal{P}) \in \text{Stab}(X) \), we study the substack,

\[ \mathcal{M}^{(\alpha, \phi)}(\sigma) \subset \mathcal{M}, \]

which is the moduli stack of \( E \in \mathcal{P}(\phi) \) and of numerical type \( \alpha \). At least we have to resolve the following two problems, addressed by [1].

- **Generic flatness problem**
  Let \( \mathcal{A} = \mathcal{P}((0,1]) \subset D(X) \) and \( \mathcal{E} \in D(X \times S) \) a family of objects in \( D(X) \). Then is the locus \( s \in S \) on which \( \mathcal{E}_s \in \mathcal{A} \) an open subset of \( S \)?

- **Boundedness problem**
  Does the set of objects \( E \in \mathcal{P}(\phi) \) of numerical type \( \alpha \) form a bounded family?

Let \( \text{Stab}^*(X) \) be one of the connected components of the space \( \text{Stab}(X) \), and suppose it satisfies the Assumption 3.1 below. Especially we require a certain subset \( \mathcal{V} \subset \text{Stab}^*(X) \), which in several examples can be taken to be the so called *neighborhoods of the large volume limits*. The main theorem in Section 3 is the following.

**Theorem 1.3.** [Theorem 3.20] Assume the generic flatness problem and the boundedness problem are true for any \( \sigma \in \mathcal{V} \). Then for any \( \sigma \in \text{Stab}^*(X), \alpha \in N(X) \) and \( \phi \in \mathbb{R} \), the stack \( \mathcal{M}^{(\alpha, \phi)}(\sigma) \) is an Artin stack of finite type over \( \mathbb{C} \).

In Section 4 we check that the assumption in Theorem 1.3 is satisfied when \( X \) is a K3 surface or an abelian surface, and \( \text{Stab}^*(X) \) is the connected component described in [5]. Thus we obtain the following.
Theorem 1.4. [Theorem 4.12] Let $X$ be a K3 surface or an abelian surface. Then for any $\sigma \in \text{Stab}^*(X)$, $\alpha \in \mathcal{N}(X)$ and $\phi \in \mathbb{R}$, the stack $\mathcal{M}^{(\alpha,\phi)}(\sigma)$ is an Artin stack of finite type over $\mathbb{C}$.

1.4 Counting invariants of semistable objects

The next step is to study the invariant determined by the moduli stack $\mathcal{M}^{(\alpha,\phi)}(\sigma)$. Given data $(\alpha,\phi)$, we introduce $J^\alpha(\sigma) \in \Lambda$ for $\alpha \in \mathcal{N}(X)$ and $\sigma \in \text{Stab}^*(X)$ in a completely similar way of $\hat{J}^\alpha(\omega)$. Then translating the arguments in [17] to the context of Bridgeland’s stability conditions, we show the following in Section 5.

Theorem 1.5. [Theorem 5.24] The invariant $J^\alpha(\sigma) \in \Lambda$ does not depend on a choice of $\sigma \in \text{Stab}^*(X)$.

Thus we may write $J^\alpha(\sigma) = J^\alpha$. Finally in Section 6 we compare $J^\alpha$ and $\hat{J}^\alpha$.

Theorem 1.6. [Theorem 6.6] For $\alpha \in C(X)$, we have $J^\alpha = \hat{J}^\alpha$.

By Theorem 1.5 and Theorem 1.6 the invariant $J^\alpha(\sigma)$ satisfies the required property of Conjecture 1.1.

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Convention

Throughout this paper we work over $\mathbb{C}$. For a variety $X$, we denote by $D(X)$ the bounded derived category of coherent sheaves on $X$. For a triangulated category $\mathcal{T}$, its Grothendieck group is denoted by $K(\mathcal{T})$. When $\mathcal{T} = D(X)$, we simply write it $K(X)$.

2 Generalities on stability conditions

The notion of stability conditions on triangulated categories was introduced in [7] to give the mathematical framework for the Douglas’s work on $\Pi$-stability [10], [11]. Here we collect some basic definitions and results in [7], [6].

2.1 Stability conditions on triangulated categories

Definition 2.1. A stability condition on a triangulated category $\mathcal{T}$ consists of data $\sigma = (Z, \mathcal{P})$, where $Z : K(\mathcal{T}) \to \mathbb{C}$ is a linear map, and $\mathcal{P}(\phi) \subset \mathcal{T}$ is a full additive subcategory for each $\phi \in \mathbb{R}$, which satisfy the following:

- $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$.
- If $\phi_1 > \phi_2$ and $A_1 \in \mathcal{P}(\phi_1)$, then $\text{Hom}(A_1, A_2) = 0$.
- If $E \in \mathcal{P}(\phi)$ is non-zero, then $Z(E) = m(E) \exp(i\pi \phi)$ for some $m(E) \in \mathbb{R}_{>0}$. 


For a non-zero object \( E \in \mathcal{T} \), we have the following collection of triangles:

\[
0 = E_0 \xrightarrow{} E_1 \xrightarrow{} E_2 \xrightarrow{} \cdots \xrightarrow{} E_n = E
\]

such that \( A_j \in \mathcal{P}(\phi_j) \) with \( \phi_1 > \phi_2 > \cdots > \phi_n \).

We denote \( \phi^+_{\sigma}(E) = \phi_1 \) and \( \phi^-_{\sigma}(E) = \phi_n \). The non-zero objects of \( \mathcal{P}(\phi) \) are called semistable of phase \( \phi \), and the objects \( A_j \) are called semistable factors of \( E \) with respect to \( \sigma \). For an object \( E \in \mathcal{T} \) the mass \( m_{\sigma}(E) \in \mathbb{R}_{>0} \) is defined by

\[
m_{\sigma}(E) = \sum_{i=1}^{n} |Z(A_i)|.
\]

The following proposition is useful in constructing stability conditions.

**Proposition 2.2.** [7, Proposition 4.2] Giving a stability condition on \( \mathcal{T} \) is equivalent to giving a heart of a bounded t-structure \( \mathcal{A} \subset \mathcal{T} \), and a group homomorphism \( Z : K(\mathcal{T}) \to \mathbb{C} \) called a stability function, such that for a non-zero object \( E \in \mathcal{A} \) one has

\[
Z(E) \in \{ r \exp(i\pi \phi) \mid r > 0, 0 < \phi < 1 \},
\]

and the pair \((Z, \mathcal{A})\) satisfies the Harder-Narasimhan property.

For the Harder-Narasimhan property, we refer [7, Definition 2.3]. For a non-zero object \( E \in \mathcal{A} \), one can find \( \phi(E) \in (0, 1) \) which satisfies \( Z(E) \in \mathbb{R}_{>0} e^{i\pi \phi(E)} \). We also call \( \phi(E) \) the phase of \( E \). The correspondence of Proposition 2.2 is given by

\[
(Z, \mathcal{P}) \mapsto (Z, \mathcal{P}((0, 1])).
\]

Here for an interval \( I \subset \mathbb{R} \), the subcategory \( \mathcal{P}(I) \subset \mathcal{T} \) is defined to be the smallest extension closed subcategory which contains \( \mathcal{P}(\phi) \) for \( \phi \in I \). In particular \( \mathcal{P}((0, 1]) \) is a heart of a t-structure on \( \mathcal{T} \), and similarly

\[
\mathcal{A}_\phi = \mathcal{P}((\phi - 1, \phi]),
\]

is also a heart of a t-structure for any \( \phi \in \mathbb{R} \). (See [7, Section 3].) On the other hand for \( \phi_1, \phi_2 \in \mathbb{R} \) with \( 0 < \phi_2 - \phi_1 < 1 \), the category \( \mathcal{P}((\phi_1, \phi_2]) \) is only a quasi-abelian category. We say a morphism \( E_1 \to E_2 \) in \( \mathcal{P}((\phi_1, \phi_2])) \) strict epimorphism if it fits into the triangle \( E_3 \to E_1 \to E_2 \) with \( E_3 \in \mathcal{P}((\phi_1, \phi_2)) \). For the detail, one can consult [7, Section 4], especially [7, Lemma 4.3].

### 2.2 The space of stability conditions

The set of stability conditions which satisfy the technical condition local finiteness [7, Definition 5.7] is denoted by \( \text{Stab}(\mathcal{T}) \). It is shown in [7, Section 6] that \( \text{Stab}(\mathcal{T}) \) has a natural topology. In fact for \( \sigma \in \text{Stab}(\mathcal{T}) \) and \( \varepsilon > 0 \), there is a subset

\[
B_{\varepsilon}(\sigma) \subset \text{Stab}(\mathcal{T}), \quad (5)
\]

and \( \{ B_{\varepsilon}(\sigma) \}_{\varepsilon, \sigma} \) gives an open basis of \( \text{Stab}(\mathcal{T}) \). We refer [7, Section 6] for the construction of \( B_{\varepsilon}(\sigma) \). Here we only note that for \( \tau = (W, Q) \in B_{\varepsilon}(\sigma) \), one has

\[
Q(\phi) \subset \mathcal{P}((\phi - \varepsilon, \phi + \varepsilon)),
\]

for any \( \phi \in \mathbb{R} \). (See [7, Lemma 6.1].) Forgetting the information of \( \mathcal{P} \), we have the map

\[
Z : \text{Stab}(\mathcal{T}) \to \text{Hom}_\mathbb{Z}(K(\mathcal{T}), \mathbb{C}).
\]
Theorem 2.3. [7, Theorem 1.2] For each connected component \( \Sigma \subset \text{Stab}(T) \), there exists a linear subspace \( V(\Sigma) \subset \text{Hom}_\mathbb{Z}(K(T), \mathbb{C}) \) with a norm such that \( Z \) restricts to a local homeomorphism, \( Z : \Sigma \rightarrow V(\Sigma) \).

Let \( \tilde{\text{GL}}^+(2, \mathbb{R}) \) be the universal cover of \( \text{GL}^+(2, \mathbb{R}) \). There is the right action of \( \tilde{\text{GL}}^+(2, \mathbb{R}) \), and the left action of the group \( \text{Auteq}(T) \) on \( \text{Stab}(T) \) [7, Lemma 8.2]. By the description in loc.cite., the action of \( \text{GL}^+(2, \mathbb{R}) \) does not change the set of semistable objects. The subgroup \( \mathbb{C} \subset \tilde{\text{GL}}^+(2, \mathbb{R}) \) acts on \( \text{Stab}(T) \) faithfully. Explicitly for \( \lambda \in \mathbb{C} \) and \( \sigma = (Z, \mathcal{P}), \lambda(\sigma) = (Z', \mathcal{P}') \) with

\[
Z'(\ast) = e^{-i\pi \lambda}Z(\ast), \quad \mathcal{P}'(\phi) = \mathcal{P}(\phi + \text{Re} \lambda).
\]

2.3 Numerical stability conditions

In general \( \text{Stab}(T) \) is infinite dimensional. So usually we consider the space of numerical stability conditions. (See [6] Section 4.) Let \( X \) be a smooth projective variety. Recall that we have the pairing,

\[
\chi : D(X) \times D(X) \ni (E, F) \mapsto \chi(E, F) = \sum_{i \in \mathbb{Z}} (-1)^i \dim \text{Hom}(E, F[i]) \in \mathbb{Z},
\]

and it descends to the paring on \( K(X) \).

Definition 2.4. We define the numerical Grothendieck group \( \mathcal{N}(X) \) to be the quotient group,

\[
\mathcal{N}(X) = K(X)/\equiv,
\]

where \( E_1 \equiv E_2 \) if and only if \( \chi(E_1, F) = \chi(E_2, F) \) for any \( F \in K(X) \). A stability condition \( \sigma = (Z, \mathcal{P}) \) on \( D(X) \) is numerical if \( Z : K(X) \rightarrow \mathbb{C} \) factors through

\[
Z : K(X) \rightarrow \mathcal{N}(X) \rightarrow \mathbb{C}.
\]

The set of locally finite numerical stability conditions is denoted by \( \text{Stab}(X) \). There exists a map [6, Theorem 4.1],

\[
Z : \text{Stab}(X) \rightarrow \mathcal{N}(X)_\mathbb{C}^*,
\]

and since the dimension of \( \mathcal{N}(X)_\mathbb{C} \) is finite, any connected component of \( \text{Stab}(X) \) is a complex manifold. In this paper, we introduce the notion of algebraic stability conditions, whose definition is not seen in the literatures.

Definition 2.5. We call a stability condition \( \sigma = (Z, \mathcal{P}) \in \text{Stab}(X) \) algebraic if the image of \( Z : \mathcal{N}(X) \rightarrow \mathbb{C} \) is contained in \( \mathbb{Q} \oplus \mathbb{Q}i \).

If \( \sigma = (Z, \mathcal{P}) \) is algebraic, then the image of \( Z \) is discrete and the abelian category \( \mathcal{P}((0, 1]) \) is noetherian [1, Proposition 5.0.1]. Here we put a notation and an easy remark.

Definition 2.6. Define \( \mathcal{I} \subset \mathbb{R} \) to be

\[
\mathcal{I} = \{ \phi \in \mathbb{R} \mid \text{there exists a rational point in } \mathbb{R}_{>0}e^{i\pi \phi} \}.
\]

Note that \( \mathcal{I} \) is a dense countable subset in \( \mathbb{R} \).

Remark 2.7. Let us take an algebraic stability condition \( \sigma = (Z, \mathcal{P}) \) and \( \phi \in \mathcal{I} \). By [6], we can find \( g \in \mathbb{C} \) such that \( g(\sigma) = (Z', \mathcal{P}') \) is also algebraic and \( \mathcal{P}'((0, 1]) = \mathcal{P}((\phi - 1, \phi]) \). Hence \( \mathcal{A}_\phi = \mathcal{P}((\phi - 1, \phi]) \) is also noetherian for \( \phi \in \mathcal{I} \).
2.4 Wall and chamber structures

Let $\text{Stab}^*(X)$ be one of the connected components of $\text{Stab}(X)$. We use the wall and chamber structure on the space $\text{Stab}^*(X)$. For the detail one can consult [6, Section 9]. For a fixed $\sigma \in \text{Stab}^*(X)$, we say a subset $S \subset D(X)$ has bounded mass if there exists $m > 0$ such that $m_\sigma(E) \leq m$ for any $E \in S$. Note that this notion does not depend on a choice of $\sigma \in \text{Stab}^*(X)$. (See [6, Definition 9.1].) The following is a slight generalization of [6, Proposition 9.3].

**Proposition 2.8.** Assume that for any bounded mass subset $S \subset D(X)$, the numerical classes
\[
\{ [E] \in \mathcal{N}(X) \mid E \in S \},
\]
is a finite set. Then for any compact subset $\mathcal{B} \subset \text{Stab}^*(X)$, there exists a finite number of real codimension one submanifolds $\{ W_\gamma \mid \gamma \in \Gamma \}$ on $\text{Stab}^*(X)$ such that if $\Gamma' \subset \Gamma$ is a subset of $\Gamma$ and $C$ is one of the connected components,
\[
C \subset \bigcap_{\gamma \in \Gamma'} (\mathcal{B} \cap W_\gamma) \setminus \bigcup_{\gamma \notin \Gamma'} W_\gamma,
\]
then if $E \in S$ is semistable in some $\sigma \in C$, then it is semistable for all $\sigma \in C$.

**Proof.** The statement is not seen in the literatures. However the proof is a straightforward adaptation of the proof of [6, Proposition 9.3] and we leave the readers to check the detail. Here we only recall the construction of the walls $\{ W_\gamma \}_{\gamma \in \Gamma}$, since it will be needed later. For a bounded mass subset $S \subset D(X)$, Bridgeland [6, Proposition 9.3] considered another bounded mass subset $S' \subset S' \subset D(X)$,
\[
S' = \{ A \in D(X) \mid \text{there is some } \sigma \in \mathcal{B} \text{ and } E \in S \text{ such that } m_\sigma(A) \leq m_\sigma(E) \},
\]
and let $v_1, \ldots, v_n \in \mathcal{N}(X)$ be the numerical classes of $S'$. Let $\Gamma$ be the set of pairs $(v_i, v_j)$ such that $v_i$ and $v_j$ are not proportional in $\mathcal{N}(X)$. Then for $\gamma = (v_i, v_j) \in \Gamma$, $W_\gamma$ is defined to be
\[
W_\gamma = \{ \sigma = (Z, P) \in \text{Stab}^*(X) \mid Z(v_1)/Z(v_2) \in \mathbb{R}_{>0} \}.
\]

It is proved in [6, Lemma 9.2] that the assumption of Proposition 2.8 is satisfied when $X$ is a K3 surface or an abelian surface.

We say a connected component $\text{Stab}^*(X)$ is full if the image of the map $\text{Stab}^*(X) \to \mathcal{N}(X)_C^*$ is an open subset of $\mathcal{N}(X)_C^*$. Note that if $\text{Stab}^*(X)$ is full, then the subset of algebraic stability conditions is dense in $\text{Stab}^*(X)$. Here we give the following easy lemma.

**Lemma 2.9.** Assume that $\text{Stab}^*(X)$ is full. Let $\mathcal{B}^o$ be an open subset of $\text{Stab}^*(X)$ and its closure $\mathcal{B}$ is compact. Then for a connected component $C$ of $\mathcal{B}$, the set of points $\sigma \in C$ which are algebraic is dense in $C$.

**Proof.** There is no proof in the literatures, however by the description of the walls [10], it is easy to check that any intersection $\cap_{\gamma \in \Gamma} W_\gamma$ contains a dense subset of algebraic stability conditions. 

7
3 Moduli stacks of semistable objects

The purpose of this section is to establish the general arguments to study the moduli stacks of
semistable objects. Throughout this section, $X$ is a smooth projective variety over $\mathbb{C}$, and $S$ is
a $\mathbb{C}$-scheme. We always assume $S$ is connected. For an object $E \in D(X \times S)$ and a $S$-scheme
$T \to S$, we denote by $E_T$ the derived pull-back of $E$ to $X \times T$. We denote

$$p: X \times S \to X, \quad q_S: X \times S \to S,$$

the projections respectively. For a set of objects $S \subset D(X)$, we say it is bounded if there is a
$\mathbb{C}$-scheme $Q$ of finite type and $F \in D(X \times Q)$ such that any object $E \in S$ is isomorphic to $F_q$
for some $q \in Q$. Also we say a map

$$\nu: S \to \mathbb{R},$$

is bounded (resp bounded above, bounded below) if there is $c \in \mathbb{R}$ such that $|\nu(E)| \leq c$
(resp $\nu(E) \leq c$, $\nu(E) \geq c$.) For the generalities of Artin stacks, one can consult [22]. In this
section, we work over a connected component $\text{Stab}^\ast(X) \subset \text{Stab}(X)$, which satisfies the following
assumption.

Assumption 3.1.

- For any bounded mass subset $S \subset D(X)$, the set of numerical classes is finite.
- There is a subset $\mathcal{V} \subset \text{Stab}^\ast(X)$ which consists of algebraic stability conditions and satisfies
  the following: for any algebraic $\sigma \in \text{Stab}^\ast(X)$, there exist $\Phi \in \text{Auteq} D(X)$ and
  $g \in \tilde{\text{GL}}^+(2, \mathbb{R})$ such that $g \circ \Phi(\sigma)$ is also algebraic and contained in $\mathcal{V}$.

The above assumption is known to hold in several examples. For instance, if $X$ is an elliptic
curve, one can take $\mathcal{V}$ to be just one point of an algebraic stability condition [7]. When $X$ is a
K3 surface or an abelian surface, we will see in the next section that Assumption 3.1 is satisfied.

3.1 Openness of stability conditions

Let $\mathcal{M}$ be the 2-functor

$$\mathcal{M}: (\text{Sch} / \mathbb{C}) \to \text{groupoid},$$

which sends a $\mathbb{C}$-scheme $S$ to the groupoid $\mathcal{M}(S)$ whose objects consist of $E \in D(X \times S)$ which
is relatively perfect [23] Definition 2.1.1] and satisfies

$$\text{Ext}^i(E_s, E_s) = 0, \text{ for all } i < 0 \text{ and } s \in S. \tag{11}$$

For the detail we refer [23]. Lieblich showed the following.

Theorem 3.2. [23] The 2-functor $\mathcal{M}$ is an Artin stack of locally finite type over $\mathbb{C}$.

Let us fix $\sigma = (Z, P) \in \text{Stab}^\ast(X)$, $\phi \in \mathbb{R}$ and $\alpha \in \mathcal{N}(X)$. Note that any object $E \in \mathcal{P}(\phi)$
satisfies (11). Thus it is possible to define the following.

Definition 3.3. We define $M^{(\alpha, \phi)}(\sigma)$ to be the set of $\sigma$-semistable objects of phase $\phi$ and
numerical type $\alpha$, and

$$\mathcal{M}^{(\alpha, \phi)}(\sigma) \subset \mathcal{M},$$

the substack of objects in $M^{(\alpha, \phi)}(\sigma)$. 

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Lemma 3.4. Assume $M^{(\alpha, \phi)}(\sigma)$ is bounded and $M^{(\alpha, \phi)}(\sigma)$ is an open substack of $M$. Then $M^{(\alpha, \phi)}(\sigma)$ is an Artin stack of finite type over $\mathbb{C}$.

Proof. Let $M \to \mathcal{M}$ be an atlas of $M$. The openness of $M^{(\alpha, \phi)}(\sigma)$ implies there is an open subset $M^0 \subset M$ which gives a surjective smooth morphism $M^0 \to M^{(\alpha, \phi)}(\sigma)$. Furthermore the boundedness of $M^{(\alpha, \phi)}(\sigma)$ implies there is a surjection $M' \to M^0$ from a finite type $\mathbb{C}$-scheme $M'$. This implies $M^0$ is also of finite type, and it gives an atlas of $M^{(\alpha, \phi)}(\sigma)$. □

Our purpose here is to give the sufficient condition for $M^{(\alpha, \phi)}(\sigma)$ to be an open substack of $M$. We consider the following claim.

Claim 3.5. For a smooth quasi-projective variety $S$ and $\mathcal{E} \in \mathcal{M}(S)$, assume that the locus
\[ S^0 = \{ s \in S \mid \mathcal{E}_s \text{ is of numerical type } \alpha \text{ and } \mathcal{E}_s \in \mathcal{P}(\phi) \}, \quad (12) \]
is not empty. Then there is an open subset $U \subset S$ which is contained in $S^0$.

By the following lemma, it is enough to consider Claim 3.5.

Lemma 3.6. Assume Claim 3.5 is true. Then $M^{(\alpha, \phi)}(\sigma)$ is an open substack of $M$.

Proof. By Theorem 3.2, it suffices to show that for an arbitrary affine $\mathbb{C}$-scheme $S$ of finite type and $\mathcal{E} \in \mathcal{M}(S)$, the locus (12) is open in $S$. Assume Claim 3.5 is true and take an affine $\mathbb{C}$-scheme $S$ of finite type and $\mathcal{E} \in \mathcal{M}(S)$. Assume that the locus (12) is not empty. Let $g: S' \to S$ be a resolution of singularities. Note that the locus $S'^0 \subset S'$ determined by $\mathcal{E}_{S'} \in \mathcal{M}(S')$ and (12) is not empty because $g$ is surjective. Applying Claim 3.5 to $\mathcal{E}_{S'}$, there is an open subset $U'_1 \subset S'$ such that $U'_1 \subset S'^0$. Restricting to the locus where $g$ is an isomorphism, we obtain an open subset $U_1 \subset S$ such that $U_1 \subset S^0$. Let $Z_1 = S \setminus U_1$. If $Z_1 \cap S^0$ is empty, we have $S^0 = U_1$. Otherwise take the pull-back $\mathcal{E}_{Z_1} \in \mathcal{M}(Z_1)$ and apply the same argument. Then we obtain an open subset $U_2 \subset Z_1 \cap S^0$ in $Z_1$ and a closed subset $Z_2 = Z_1 \setminus U_2$, which is also closed in $S$. Repeating this argument, we get a sequence of closed subsets in $S$,
\[ \cdots \subset Z_n \subset Z_{n-1} \subset \cdots \subset Z_1, \]
which must be terminate because $S$ is noetherian. Then $Z = \cap_i Z_i$ is a closed subset of $S$ and we have $S^0 = S \setminus Z$. Therefore $S^0$ is open. □

3.2 Sheaf of t-structures

Here we introduce the sheaf of t-structures studied by D.Abramovich and A.Polishchuk [1]. Let $A \subset D(X)$ be a heart of a bounded t-structure and assume that $A$ is noetherian. Take a smooth projective variety $S$ and an ample line bundle $L \in \text{Pic}(S)$.

Theorem 3.7. [1, Theorem 2.6.1] The subcategory
\[ A_S = \{ F \in D(X \times S) \mid \text{R}p_1(F \otimes L^n) \in A \text{ for all } n \gg 0 \}, \]
is a heart of a bounded t-structure on $D(X \times S)$, independent of a choice of $L$. Furthermore it is a noetherian abelian category.
The subcategory $\mathcal{A}_S \subset D(X \times S)$ extends to a sheaf of bounded t-structures \cite[Theorem 2.7.2]{1}, i.e. for an open subset $j: U \subset S$, there exists a heart of a bounded t-structure $\mathcal{A}_U \subset D(X \times U)$ such that

$$(\text{id} \times j)^*: D(X \times S) \longrightarrow D(X \times U),$$

takes $\mathcal{A}_S$ to $\mathcal{A}_U$. Moreover it is shown in \cite[Lemma 2.5.3]{1} that $\mathcal{A}_U$ does not depend on a projective compactification $U \subset S$. Thus one can define $\mathcal{A}_S$ for a smooth quasi-projective variety $S$. One of the necessary fact for our purpose is the following open heart property.

**Theorem 3.8.** \cite[Theorem 3.3.2]{1} For a smooth quasi-projective variety $S$ and $E \in D(X \times S)$, assume there exists $s \in S$ such that $E_s \in \mathcal{A}$. Then there exists an open subset $s \in U \subset S$ such that $E_U \in \mathcal{A}_U$.

We say $E \in \mathcal{A}_S$ is t-flat if for any $s \in S$ one has $E_s \in \mathcal{A}$. Since $U \mapsto \mathcal{A}_U$ is a sheaf of t-structures, if $E \in \mathcal{M}(S)$ satisfies $E_s \in \mathcal{A}$ for all $s \in S$, then Theorem 3.8 and \cite[Lemma 2.1.1]{1} show that $E \in \mathcal{A}_S$ and it is t-flat. For a closed point $s \in S$ and the inclusion $i_s: X \times \{s\} \hookrightarrow X \times S$, it is shown in \cite[Lemma 2.5.3]{1} that

$$Li_s^*: D(X \times S) \rightarrow D(X),$$

is right t-exact with respect to the t-structures with hearts $\mathcal{A}_S$, $\mathcal{A}$ respectively. Thus one has the following lemma.

**Lemma 3.9.** Let $0 \rightarrow \mathcal{H} \rightarrow E \rightarrow \mathcal{F} \rightarrow 0$ be an exact sequence in $\mathcal{A}_S$ and assume that $E, \mathcal{F}$ are t-flat. Then $\mathcal{H}$ is also t-flat.

For our purpose, we have to consider the following problem called generic flatness problem.

**Problem 3.10.** \cite[Problem 3.5.1]{1} For $E \in \mathcal{A}_S$, is there an open subset $U \subset S$ such that for each $s \in U$, we have $E_s \in \mathcal{A}$?

**Remark 3.11.** If Problem 3.10 is true, the same argument of Lemma 3.9 shows the following: for an arbitrary $\mathcal{C}$-scheme $S$ of finite type, the points $s \in S$ on which $E_s \in \mathcal{A}$ is in fact open.

In \cite{1}, there is a partial result for Problem 3.10.

**Proposition 3.12.** \cite[Proposition 3.5.3]{1} For $E \in \mathcal{A}_S$, there is a dense subset $U \subset S$ such that for each $s \in U$, we have $E_s \in \mathcal{A}$.

The generic flatness problem requires $U$ to be open in Zariski topology. Let us take an algebraic stability condition $\sigma = (Z, \mathcal{P}) \in \text{Stab}^s(X)$. The purpose here is to reduce Claim 3.5 to Problem 3.10.

**Lemma 3.13.** Let $S$ be a smooth quasi-projective variety, $\phi \in \mathbb{R}$ and $\alpha \in N(X)$.

(i) For $E \in \mathcal{M}(S)$, assume the locus $S^\circ$ defined by (12) is non-empty. Then $S^\circ$ is dense in $S$.

(ii) In the same situation of (i), assume Problem 3.10 is true for $\mathcal{A}_\phi = \mathcal{P}((\phi - 1, \phi])$. Then $S^\circ$ contains an open subset of $S$. (Thus Claim 3.5 is true for this $\phi \in \mathbb{R}$.)

**Proof.** (i) Because $S^\circ$ is non-empty and $\sigma$ is algebraic, we have $\phi \in I$. Hence $\mathcal{A}_\phi = \mathcal{P}((\phi - 1, \phi])$ is noetherian by Remark 2.7. Let us take $s \in S^\circ$. Note that $E_s \in \mathcal{P}(\phi) \subset \mathcal{A}_\phi$. Thus by Lemma 2.7 and Theorem 3.8 there exists an open subset $s \in U \subset S$ such that $E_U \in \mathcal{A}_\phi$. For our purpose, we have to consider the following problem called generic flatness problem. For our purpose, we have to consider the following problem called generic flatness problem. For our purpose, we have to consider the following problem called generic flatness problem.
Therefore by Proposition 3.12 there exists dense subset $U' \subset U$ such that for $s' \in U'$, we have $\mathcal{E}_{s'} \in A_\phi$. Since $\mathcal{E}_{s'}$ is numerically equivalent to $\mathcal{E}_s$, we have $Z(\mathcal{E}_{s'}) \in \mathbb{R}_{>0} e^{i\pi \phi}$. This implies $\mathcal{E}_{s'} \in \mathcal{P}(\phi)$, hence $U' \subset S^\circ$.

(ii) If we assume the generic flatness for $A_\phi$, then we can take $U'$ in the proof of (i) to be open. \qed

3.3 Boundedness of semistable objects

Here we discuss the boundedness of semistable objects and certain quotient objects. We fix an algebraic stability condition $\sigma = (Z, P) \in \text{Stab}^* (X)$, and consider the following problem.

**Problem 3.14.** Is the set of objects $M^{(\alpha, \phi)}(\sigma)$ bounded, for any $\alpha \in \mathcal{N}(X)$ and $\phi \in \mathbb{R}$?

Let $A = \mathcal{P}((0, 1])$. We show the following.

**Lemma 3.15.** Assume Problem 3.14 is true for a fixed $\sigma$. Then for any $\phi \in (0, 1)$ and $G \in A$, the following set of objects,

$$Q(G, \phi) = \{ E \in A \mid \text{there exists a surjection } G \to E \text{ in } A \text{ and } \phi(E) \leq \phi \},$$

is bounded.

**Proof.** For $E \in Q(G, \phi)$, let $F_1, F_2, \cdots, F_{n(E)}$ be the semistable factors of $E$ in $\sigma$ such that $F_i \in \mathcal{P}(\phi_i)$ and $\phi_1 > \phi_2 > \cdots > \phi_{n(E)}$. We have

$$\sum_{i=1}^{n(E)} \text{Im} Z(F_i) = \text{Im} Z(E) \leq \text{Im} Z(G). \quad (13)$$

Note that $\text{Im} Z(F_i) > 0$ except $i = 1$. Because $\sigma$ is algebraic, the image $\mathcal{N}(X) \xrightarrow{Z} \mathbb{C} \xrightarrow{\text{Im}} \mathbb{R}$ is discrete. Thus (13) implies that the map $E \mapsto n(E)$ on $Q(G, \phi)$ is bounded, and the following set

$$\{ \text{Im} Z(F_i) \in \mathbb{Q} \mid 1 \leq i \leq n(E), E \in Q(G, \phi) \}, \quad (14)$$

is a finite set.

Next there exist surjections, $G \to E \to F_{n(E)}$ in $A$, so we have $\phi_{\sigma}^{-}(G) \leq \phi_{n(E)} \leq \phi_i$ for $1 \leq i \leq n(E)$. (See [7, Lemma 3.4].) Thus the map on $Q(G, \phi)$,

$$E \mapsto \max \{ \text{Re} Z(F_i) \mid 1 \leq i \leq n(E) \} \in \mathbb{Q}, \quad (15)$$

is bounded above. On the other hand since $\phi(E) \leq \phi < 1$ and $\text{Im} Z(E) \leq \text{Im} Z(G)$, the following map on $Q(G, \phi)$,

$$E \mapsto \text{Re} Z(E) = \sum_{i=1}^{n(E)} \text{Re} Z(F_i), \quad (16)$$

is bounded below. Combined with the fact that (15) is bounded above, the following set

$$\{ \text{Re} Z(F_i) \in \mathbb{Q} \mid 1 \leq i \leq n(E), E \in Q(G, \phi) \}, \quad (17)$$
is a finite set. Then the finiteness of (14), (17) and Assumption 3.1 imply that the following set,
\[ \{ F_i \in N(X) \mid 1 \leq i \leq n(E), E \in Q(G, \phi) \}, \]  
(18)
is a finite set. Since we assume that Problem 3.14 is true, the finiteness of (18) implies that the set of objects
\[ \{ F_i \mid 1 \leq i \leq n(E), E \in Q(G, \phi) \}, \]
is bounded. Thus \( Q(G, \phi) \) is also bounded by Lemma 3.16 below.

Here we have used the following easy lemma.

Lemma 3.16. Let \( S_i \subset D(X) \) be the sets of objects for \( 1 \leq i \leq 3 \) and \( S_1, S_2 \) are bounded. Assume that for any object \( E_3 \in S_3 \), there is an \( E_i \in S_i \) for \( i = 1, 2 \) and a triangle,
\[ E_1 \rightarrow E_3 \rightarrow E_2. \]
Then \( S_3 \) is also bounded.

The proof is easy and leave it to the reader. In fact it is enough to notice that \( \text{Ext}^1(E_2, E_1) \) is finite dimensional.

Assuming Problem 3.10 and Problem 3.14 we can construct certain schemes which parameterize quotient objects. Let \( \mathcal{E} \in \mathcal{A}_S \) be a \( t \)-flat family and take \( \phi \in (0,1) \). We consider the following functors,
\[ \text{Quot}(\mathcal{E}, \phi), \text{Sub}(\mathcal{E}, \phi) : (\text{Sch}/S) \rightarrow (\text{Set}), \]
such that \( \text{Quot}(\mathcal{E}, \phi) \) (resp \( \text{Sub}(\mathcal{E}, \phi) \)) takes a \( S \)-scheme \( T \) to the isomorphism classes of objects \( \mathcal{F} \in \mathcal{M}(T) \) together with a morphism \( \mathcal{E}_T \rightarrow \mathcal{F} \), (resp \( \mathcal{F} \rightarrow \mathcal{E}_T \)) such that
- For each closed point \( t \in T \), \( \mathcal{F}_t \) is contained in \( \mathcal{A} \) and \( \phi(\mathcal{F}_t) \leq \phi \). (resp \( \phi(\mathcal{F}_t) > \phi \).)
- For each closed point \( t \in T \), the induced morphism \( \mathcal{E}_t \rightarrow \mathcal{F}_t \) is a surjection in \( \mathcal{A} \). (resp \( \mathcal{F}_t \rightarrow \mathcal{E}_t \) is an injection in \( \mathcal{A} \).)
We show the following.

Proposition 3.17. For a fixed \( \sigma \), assume Problem 3.10 for \( \mathcal{A} = \mathcal{P}((0,1]) \) and Problem 3.14 are true. Then for any \( \phi \in (0,1) \) there exist \( S \)-schemes \( Q(\mathcal{E}, \phi), \mathcal{S}(\mathcal{E}, \phi) \) which are of finite type over \( S \), and morphisms over \( S \),
\[ Q(\mathcal{E}, \phi) \rightarrow \text{Quot}(\mathcal{E}, \phi), \]
\[ \mathcal{S}(\mathcal{E}, \phi) \rightarrow \text{Sub}(\mathcal{E}, \phi), \]
which are surjective on \( \mathbb{C} \)-valued points of \( \text{Quot}(\mathcal{E}, \phi) \) and \( \text{Sub}(\mathcal{E}, \phi) \).

Proof. First let us construct \( Q(\mathcal{E}, \phi) \). By [II Lemma 2.6.2], there exists an object \( G \in \mathcal{A}, n \in \mathbb{Z} \) and a surjection \( G_S \otimes \mathcal{L}^{-n} \rightarrow \mathcal{E} \) in \( \mathcal{A}_S \). Note that the induced morphism \( \mathcal{G} \rightarrow \mathcal{E}_s \) is a surjection by Lemma 3.9. By the assumption and Lemma 3.15 there is a \( \mathbb{C} \)-scheme \( Q_1 \) of finite type over \( \mathbb{C} \) and an object \( \mathcal{F} \in \mathcal{M}(Q_1) \) such that any object in \( Q(G, \phi) \) is isomorphic to \( \mathcal{F}_q \) for some \( q \in Q_1 \). Let \( Q^*_1 \) be
\[ Q^*_1 = \{ q \in Q_1 \mid \mathcal{F}_q \in \mathcal{A} \}. \]
Since we assume the generic flatness for \(A\), the locus \(Q_1^f\) is open in \(Q_1\). Set \(Q_2 = Q_1^f \times S\) and we regard it as a \(S\)-scheme via the projection \(Q_2 \to S\). By \cite{23} Proposition 2.2.3, there exists an affine open subset \(U \subset Q_2\) such that the functor \(\text{Coh}(U) \to \text{Coh}(U)\) sending \(M\) to

\[
M \mapsto \mathcal{H}^0(\mathcal{R}q_{U}, \mathcal{R}\text{Hom}(\mathcal{E}_U, \mathcal{F}_U \otimes q_{U}^{*}M)), \quad (19)
\]

has the form \(\text{Hom}(\tilde{\mathcal{E}}_U, M)\) for some locally free sheaf \(\tilde{\mathcal{E}}_U\) on \(U\). Here \(\mathcal{F}_U\) is the pull-back of \(\mathcal{F}\) via

\[
U \subset Q_2 \to Q_1^f \subset Q_1.
\]

Set \(Q_2' = (Q_2 \setminus U) \bigsqcup U\) and apply the same procedure to \(\mathcal{E}_{Q_2'}\) and \(\mathcal{F}_{Q_2'}\) repeatedly. Then we obtain an affine scheme of finite type \(Q_3\) with a morphism \(Q_3 \to Q_2\), which is bijective on closed points, and a locally free sheaf \(\tilde{\mathcal{E}}\) on \(Q_3\) such that the functor \(\text{Coh}(Q_3) \to \text{Coh}(Q_3)\) given in the same way as \((19)\) has the form \(\text{Hom}(\tilde{\mathcal{E}}, \ast)\). Furthermore the functor

\[
(T \to Q_3) \mapsto \mathcal{H}^0(\mathcal{R}q_{T}, \mathcal{R}\text{Hom}(\mathcal{E}_T, \mathcal{F}_T)) \in \text{Coh}(T),
\]

is represented by \(\mathcal{V}(\tilde{\mathcal{E}})\) by \cite{23} Proposition 2.2.3. Thus there exists a universal morphism \(\mathcal{E}_{\mathcal{V}(\tilde{\mathcal{E}})} \to \mathcal{F}_{\mathcal{V}(\tilde{\mathcal{E}})}\). Let \(\mathcal{H}\) be its cone, i.e. \(\mathcal{H}\) fits into the distinguished triangle in \(D(X \times \mathcal{V}(\tilde{\mathcal{E}}))\),

\[
\mathcal{H} \longrightarrow \mathcal{E}_{\mathcal{V}(\tilde{\mathcal{E}})} \longrightarrow \mathcal{F}_{\mathcal{V}(\tilde{\mathcal{E}})}.
\]

For \(q \in \mathcal{V}(\tilde{\mathcal{E}})\), note that \(\mathcal{F}_q\) is contained in \(A\). Thus the induced morphism \(\mathcal{E}_q \to \mathcal{F}_q\) is surjective in \(A\) if and only if \(\mathcal{H}_q \in A\). Then define \(Q(\mathcal{E}, \phi)\) to be the locus,

\[
Q(\mathcal{E}, \phi) := \{q \in \mathcal{V}(\tilde{\mathcal{E}}) \mid \mathcal{H}_q \in A\}.
\]

Again \(Q(\mathcal{E}, \phi)\) is an open subscheme of \(\mathcal{V}(\tilde{\mathcal{E}})\), in particular it is of finite type over \(S\). The restriction of \(\mathcal{E}_{\mathcal{V}(\tilde{\mathcal{E}})} \to \mathcal{F}_{\mathcal{V}(\tilde{\mathcal{E}})}\) to \(Q(\mathcal{E}, \phi)\) induces a morphism,

\[
Q(\mathcal{E}, \phi) \longrightarrow \mathcal{U} \mathcal{O}t(\mathcal{E}, \phi),
\]

which is surjective on \(\mathbb{C}\)-valued points by the construction.

Next we construct \(\mathcal{S}(\mathcal{E}, \phi)\). Since \(\phi(\mathcal{E}_s)\) does not depend on \(s \in S\), we can easily see the following: there exists \(\phi' \in (0, 1)\) such that for any \(s \in S\) and a subobject \(\mathcal{H} \subset \mathcal{E}_s\) in \(\mathcal{A}\) with \(\phi(\mathcal{H}) > \phi\), we have \(\phi(\mathcal{E}_s/\mathcal{H}) \leq \phi'\). Let us consider \(Q(\mathcal{E}, \phi')\) and the universal quotient \(\mathcal{E}_{Q(\mathcal{E}, \phi')} \to \mathcal{F}\) on \(X \times Q(\mathcal{E}, \phi')\). We consider the distinguished triangle,

\[
\mathcal{H} \longrightarrow \mathcal{E}_{Q(\mathcal{E}, \phi')} \longrightarrow \mathcal{F}.
\]

Note that \(\mathcal{H}_q \in A\), thus one can define its phase \(\phi(\mathcal{H}_q) \in (0, 1]\). Then we construct \(\mathcal{S}(\mathcal{E}, \phi)\) as follows,

\[
\mathcal{S}(\mathcal{E}, \phi) = \{q \in Q(\mathcal{E}, \phi') \mid \phi(\mathcal{H}_q) > \phi\}.
\]

Since \(q \mapsto \phi(\mathcal{H}_q)\) is locally constant on \(Q(\mathcal{E}, \phi')\), the locus \(\mathcal{S}(\mathcal{E}, \phi)\) is a union of the connected components of \(Q(\mathcal{E}, \phi')\), in particular of finite type over \(S\). The induced morphism \(\mathcal{H}_{\mathcal{S}(\mathcal{E}, \phi)} \to \mathcal{E}_{\mathcal{S}(\mathcal{E}, \phi)}\) gives a morphism \(\mathcal{S}(\mathcal{E}, \phi) \to \mathcal{U} \mathcal{O}b(\mathcal{E}, \phi)\), which is surjective on \(\mathbb{C}\)-valued points. \(\blacksquare\)
3.4 Generic flatness for $A_\phi = \mathcal{P}((\phi - 1, \phi])$

Again we fix an algebraic stability condition $\sigma = (Z, \mathcal{P}) \in \text{Stab}^*(X)$. Here we study the generic flatness for $\mathcal{P}((\phi - 1, \phi])$ under several assumptions. The purpose here is the following.

**Proposition 3.18.** Under the same assumption as in Proposition 3.17, let us take $\phi \in \mathcal{I}$. Then Problem 3.10 is true for $A_\phi = \mathcal{P}((\phi - 1, \phi])$.

**Proof.** For $E \in A_{\phi,S}$, let us find an open subset $U \subset S$ on which $E_s \in A_\phi$. We may assume $0 < \phi \leq 1$. By [1, Lemma 3.2.1] we may also assume $S$ is projective, and let $L \in \text{Pic}(S)$ be an ample line bundle. Then $E \in A_{\phi,S}$ implies,

$$R^p_s(E \otimes L^n) \in \mathcal{P}((\phi - 1, \phi]),$$

(20)

for $n \gg 0$. For $A = \mathcal{P}((0, 1])$, we denote by $H^i_A(*)$, $H^i_{A_S}(*)$ the $i$-th cohomology functors on $D(X)$, $D(X \times S)$ with respect to the t-structures with hearts $A$, $A_S$ respectively. Then (20) implies

$$H^i_A(R^p_s(E \otimes L^n)) = 0 \text{ unless } i = 0, 1.$$ 

(21)

On the other hand, we have

$$R^p_s(H^i_{A_S}(E \otimes L^n)) = R^p_s(H^i_{A_S}(E) \otimes L^n) \in A,$$

(22)

for $n \gg 0$. The first equality comes from [1, Proposition 2.1.3]. Thus (21) and (22) imply

$$R^p_s(H^i_{A_S}(E) \otimes L^n) = 0 \text{ unless } i = 0, 1,$$

(23)

for $n \gg 0$. It is easy to deduce from (23) that $H^i_{A_S}(E) = 0$ unless $i = 0, 1$, by using the standard t-structure on $D(X \times S)$. For $i = 0, 1$, denote $E^i = H^i_{A_S}(E) \in A_S$. Since we assume the generic flatness for $A$, there exists an open set $S'_1 \subset S$ such that for $s \in S'_1$ one has $E^i_s \in A$. Since we have the distinguished triangle $E^0 \rightarrow E \rightarrow E^1[-1]$, we have the distinguished triangle in $D(X)$,

$$E^0_s \rightarrow E_s \rightarrow E^1_s[-1],$$

for $s \in S'_1$. Hence for $s \in S'_1$, $E_s \in A_\phi$ is equivalent to the following,

$$E^0_s \in \mathcal{P}((0, \phi]), \quad E^1_s \in \mathcal{P}((\phi, 1]).$$

(24)

Thus it is enough to find an open set $U \subset S'_1$ where (24) holds. Note that by Proposition 3.12 the set of points $s \in S'_1$ on which (24) hold is dense in $S$. First let us consider the locus where $E^1_s \in \mathcal{P}((\phi, 1])$ holds. Let $\pi_\phi$ be the composition of the morphisms,

$$\pi_\phi: Q(E^1, \phi) \rightarrow \mathcal{D}(E^1, \phi) \rightarrow S,$$

constructed in Proposition 3.17. Note that $E^1_s \in \mathcal{P}((\phi, 1])$ if and only if there is no surjection $E^1_s \rightarrow F$ in $A$ with $\phi(F) \leq \phi$. Thus $E^1_s \in \mathcal{P}((\phi, 1])$ if and only if $s \notin \text{im} \pi_\phi$, and such points are dense in $S'_1$. This implies $\pi_\phi$ is not dominant. Because $Q(E^1, \phi)$ is of finite type, there is an open subset $U \subset S'_1 \setminus \text{im} \pi_\phi$ and (24) holds on $U$.

We can argue in a similar way (using $S(E, \phi)$ instead of $Q(E, \phi)$) to find an open subset $U \subset S'_1$ where $E^0_s \in \mathcal{P}((0, \phi])$ holds. We leave the detail to the reader. \hfill $\square$
Lemma 3.19. Under the same assumption as in Proposition 3.17, take $Q$ by associating a $P$.

Now we can follow the same construction as in Proposition 3.17 and obtain $E$ is a morphism $q: P \rightarrow Q$ such that for each $t \in T$, the induced morphism $E_t \rightarrow F_t$ is a strict epimorphism in $P((\phi_0, \phi_1))$. We need the following.

Lemma 3.19. Under the same assumption as in Proposition 3.17 take $\phi_0, \phi_1 \in I$ as above. Then there exists a $S$-scheme $Q(E, \phi_0, \phi_1)$ of finite type over $S$ and a morphism

$$Q(E, \phi_0, \phi_1) \rightarrow \text{Quot}(E, \phi_0, \phi_1),$$

which is surjective on $C$-valued points.

Proof. Let us take $\phi_2 \in I$ which satisfies $\phi_0, \phi_1 \in (\phi_2 - 1, \phi_2)$, and $G \in A_{\phi_2} = P((\phi_2 - 1, \phi_2))$.

Then there exists a surjection $Q \rightarrow E$ in $A_{\phi_2}$, is bounded. As in Proposition 3.17 there exists a surjection $G \otimes L^{-n} \rightarrow E$ in $A_{\phi_2,S}$ for some $n \in Z$ and $L \in \text{Pic}(S)$ is an ample line bundle. By the boundedness of $Q(G, \phi_0, \phi_1)$, there exists a $C$-scheme $Q_1$ of finite type and $F_1 \in D(X \times Q_1)$ such that any object in $Q(G, \phi_0, \phi_1)$ is isomorphic to $F_1 q$ for some $q \in Q_1$. By the assumption and Proposition 3.17 the generic flatness holds for $P((\phi - 1, \phi]) \times \phi \in I$. Thus the locus

$$Q_1^c = \{q \in Q_1 \mid F_q \in P((\phi_0, \phi_1))\},$$

is open because we have,

$$P((\phi_0, \phi_1]) = P((\phi_0, \phi_0 + 1]) \cap P((\phi_1 - 1, \phi_1]).$$

Now we can follow the same construction as in Proposition 3.17 and obtain $Q_2 = Q_1^c \times S$, $Q_3 \rightarrow Q_2$, $E \in \text{Coh}(Q_3)$, and $Q(E, \phi_0, \phi_1) \subset \forall(\tilde{E})$ as desired.

The following is the main theorem in this section.

Theorem 3.20. Under the assumption 3.1 assume that for any $\sigma = (Z, P) \in V$, Problem 3.10 for $A = P((0, 1])$ and Problem 3.12 are true. Then for any $\sigma \in \text{Stab}^*(X)$, $\alpha \in N(X)$ and $\phi \in R$, the stack $\mathcal{M}^{(\alpha, \phi)}(\sigma)$ is an Artin stack of finite type over $C$.

Note that by Lemma 3.1 and Lemma 3.6 it suffices to check Claim 3.5 and Problem 3.14. Also note that by Proposition 3.18 and Lemma 3.13 (ii), the result holds for any $\sigma \in V$. We divide the proof into some steps.

Step 1. The result holds for an algebraic stability condition $\sigma = (Z, P) \in V$. 

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Proof. First we show Claim 3.15 holds. For a smooth quasi-projective variety $S$ and $E \in \mathcal{M}(S)$, assume the locus $S^\circ$ defined by $12$ is non-empty. Note that $S^\circ$ is dense in $S$ by Lemma 3.13 (i). Since $\sigma \in \overline{V}$, there exists $\sigma' = (Z', \mathcal{P}') \in \mathcal{V}$ and $\phi_i \in (\phi - 1/2, \phi + 1/2) \cap I$ for $0 \leq i \leq 5$ such that

$$\mathcal{P}(\phi) \subset \mathcal{P}'((\phi_0, \phi_1]) \subset \mathcal{P}'((\phi_2, \phi_3]) \subset \mathcal{P}'((\phi_4, \phi_5]) \subset \mathcal{P}\left(\left(\frac{1}{2}, \phi + \frac{1}{2}\right)\right).$$

For $E \in \mathcal{P}'((\phi - 1/2, \phi + 1/2])$, we denote by $\phi(E) \in (\phi - 1/2, \phi + 1/2]$ the phase with respect to the stability function $Z$. By the assumption for $\sigma' \in \mathcal{V}$ and Proposition 3.18 there is an open subset $S_1 \subset S$ on which $E_s \in \mathcal{P}'((\phi_0, \phi_1])$. Now we have

$$\{s \in S \mid E_s \text{ is not semistable in } \sigma\}$$

$$= \{s \in S \mid \text{there is a strict epimorphism } E_s \rightarrow F \text{ in } \mathcal{P}'((\phi_2, \phi_3]) \text{ with } \phi(F) < \phi(E_s)\}$$

$$\subset \{s \in S \mid \text{there is a strict epimorphism } E_s \rightarrow F \text{ in } \mathcal{P}'((\phi_4, \phi_5]) \text{ with } \phi(F) < \phi(E_s)\}.$$

On the other hand, assume there is a strict epimorphism $E_s \rightarrow F$ in $\mathcal{P}'((\phi_4, \phi_5])$ with $\phi(F) < \phi(E_s)$. Then it is a surjection in $\mathcal{P}'((\phi - 1/2, \phi + 1/2])$, and $\phi(F) < \phi(E_s)$ implies $E_s$ is not semistable in $\sigma$. Thus we obtain,

$$\{s \in S \mid E_s \text{ is not semistable in } \sigma\}$$

$$= \{s \in S \mid \text{there is a strict epimorphism } E_s \rightarrow F \text{ in } \mathcal{P}'((\phi_4, \phi_5]) \text{ with } \phi(F) < \phi(E_s)\}. $$

Let

$$\pi_{\phi_4, \phi_5} : \mathcal{Q}(\mathcal{E}, \phi_4, \phi_5) \rightarrow S$$

be the $S$-scheme constructed in Lemma 3.19 applied for $\sigma'$. Let $\mathcal{E}_{\mathcal{Q}(\mathcal{E}, \phi_4, \phi_5)} \rightarrow \mathcal{F}$ be the universal epimorphism on $X \times \mathcal{Q}(\mathcal{E}, \phi_4, \phi_5)$ and define $\mathcal{Q}^o(\mathcal{E}, \phi_4, \phi_5)$ to be the locus

$$\mathcal{Q}^o(\mathcal{E}, \phi_4, \phi_5) := \{q \in \mathcal{Q}(\mathcal{E}, \phi_4, \phi_5) \mid \phi(F_q) < \phi(E_q)\}.$$

Since $q \mapsto \phi(F_q)$ is locally constant on $\mathcal{Q}(\mathcal{E}, \phi_4, \phi_5)$, $\mathcal{Q}^o(\mathcal{E}, \phi_4, \phi_5)$ is a union of connected components of $\mathcal{Q}(\mathcal{E}, \phi_4, \phi_5)$, in particular it is of finite type over $S$. Let $\pi^o_{\phi_4, \phi_5}$ be the restriction of $\pi_{\phi_4, \phi_5}$ to $\mathcal{Q}^o(\mathcal{E}, \phi_4, \phi_5)$. Then for a point $s \in S$, $s \in S^\circ$ if and only if $s \notin \text{im } \pi^o_{\phi_4, \phi_5}$. This implies $S \setminus \text{im } \pi^o_{\phi_4, \phi_5}$ is dense, thus there exists an open subset $U \subset S \setminus \text{im } \pi^o_{\phi_4, \phi_5}$.

Next we check that $M^{(\alpha, \phi)}(\sigma)$ is bounded. Take $E \in M^{(\alpha, \phi)}(\sigma)$ and let $F_i \in \mathcal{P}'((\phi_0, \phi_1])$ for $1 \leq i \leq n(E)$ be the semistable factors of $E$ in $\sigma'$. Because $\sigma'$ is algebraic and $\phi_1 - \phi_0 < 1$, the map $E \mapsto n(E)$ is bounded on $M^{(\alpha, \phi)}(\sigma)$ and

$$\{Z'(F_i) \in \mathbb{C} \mid 1 \leq i \leq n(E), E \in M^{(\alpha, \phi)}(\sigma)\},$$

is a finite set. Since we assume that Problem 3.14 is true for $\sigma'$, the set of objects

$$\{F_i \mid 1 \leq i \leq n(E), E \in M^{(\alpha, \phi)}(\sigma)\},$$

is bounded. Thus $M^{(\alpha, \phi)}(\sigma)$ is also bounded by Lemma 3.16.

$\square$

**Step 2.** The result holds for any algebraic stability condition $\sigma \in \text{Stab}^\ast(X)$. 

Proof. Note that $\Phi \in \text{Auteq} D(X)$ induces a 1-isomorphism,

$$\mathcal{M} \ni E \mapsto \Phi(E) \in \mathcal{M}.$$ 

Also note that an action of $g \in \mathrm{GL}^+(2, \mathbb{R})$ does not change the set of semistable objects. Thus we have

$$\mathcal{M}^{(\alpha, \phi)}(\sigma) = \mathcal{M}^{(\alpha, \phi)}(g(\sigma)),$$

for some $\phi' \in \mathbb{R}$. Hence if the result holds for $\sigma \in \text{Stab}^*(X)$, then it also holds for $g \circ \Phi(\sigma)$ for any $\Phi \in \text{Auteq} D(X)$ and $g \in \mathrm{GL}^+(2, \mathbb{R})$. Thus the result holds for any algebraic stability condition $\sigma \in \text{Stab}^*(X)$ by Assumption 3.1 and Step 1. \[\Box\]

**Step 3.** The result holds for any $\sigma = (Z, P) \in \text{Stab}^*(X)$.

Proof. Let $\sigma \in \mathcal{B} \subset \text{Stab}^*(X)$ be an open neighborhood of $\sigma$ such that its closure $\mathcal{B}$ is compact. Let $S \subset D(X)$ be

$$S := \{E \in D(X) \mid E \text{ is of numerical type } \alpha \text{ and semistable in some } \sigma' \in \mathcal{B}\}.$$ 

Then $S$ has bounded mass, hence by Assumption 3.1 and Proposition 2.8, there exists a finite number of codimension one walls $\{W_\gamma\}_{\gamma \in \Gamma}$ which gives a wall and chamber structure on $\mathcal{B}$. Let $\Gamma' \subset \Gamma$ be the subset which satisfies,

$$\sigma \in \bigcap_{\gamma \in \Gamma'} W_\gamma \setminus \bigcup_{\gamma \notin \Gamma'} W_\gamma.$$  \hspace{1cm} (25)

Let $C$ be the connected component of the right hand side of (25) which contains $\sigma$. Then if $E$ is of numerical type $\alpha$ and semistable in $\sigma$, then it is semistable for any $\sigma' \in C$. We can take $\sigma' = (Z', P')$ to be algebraic by Lemma 2.9. Thus $\mathcal{M}^{(\alpha, \phi)}(\sigma) = \mathcal{M}^{(\alpha, \phi')}(\sigma')$ for some $\phi'$, and the result follows from Step 2. \[\Box\]

Remark 3.21. Note that Assumption 3.1 and Proposition 3.18 also imply the following: the set of $\sigma = (Z, P) \in \text{Stab}^*(X)$ such that $P((\phi - 1, \phi))$ satisfies the generic flatness for any $\phi \in \mathcal{I}$ is dense in $\text{Stab}^*(X)$.

4 Semistable objects on K3 surfaces

In this section we assume $X$ is a K3 surface or an abelian surface. The aim of this section is to show that the assumption in Theorem 3.20 is satisfied in this case.

4.1 Mukai lattices and Mukai vectors

Let $\text{NS}^*(X)$ be the Mukai lattice,

$$\text{NS}^*(X) = \mathbb{Z} \oplus \text{NS}(X) \oplus \mathbb{Z}.$$ 

For $v_i = (r_i, l_i, s_i)$ with $i = 1, 2$, its bilinear pairing is given by

$$(v_1, v_2) = l_1 \cdot l_2 - r_1 s_2 - r_2 s_1.$$  \hspace{1cm} (26)
For an object $E \in D(X)$ its Mukai vector is defined as follows.

$$v(E) = \text{ch}(E)\sqrt{\text{td}X} = (r(E), c_1(E), ch_2(E) + \epsilon \cdot r(E)).$$

Here $\epsilon = 1$ if $X$ is a K3 surface and $\epsilon = 0$ if $X$ is an abelian surface. Sending an object to its Mukai vector gives an isomorphism,

$$v : \mathcal{N}(X) \xrightarrow{\cong} \text{NS}^*(X). \quad (27)$$

Under the identification (27), the bilinear pairing $-\chi(E_1, E_2)$ on the left hand side goes to the pairing (26).

4.2 Twisted Gieseker-stability and $\mu$-stability

We recall the notion of twisted Gieseker-stability and $\mu$-stability on the category of coherent sheaves $\text{Coh}(X)$. For the detail, one can consult [12], [25]. Take $\mathcal{L}, \mathcal{M} \in \text{Pic}(X)$, and suppose $\mathcal{L}$ is ample. For $E \in \text{Coh}(X)$ one can write the twisted Hilbert polynomial as follows,

$$\chi(E \otimes \mathcal{M}^{-1} \otimes \mathcal{L}^n) = \sum_{i=0}^{d} a_i n^i,$$

for $a_i \in \mathbb{Q}$ and $a_d \neq 0$. For $\omega = c_1(\mathcal{L})$ and $\beta = c_1(\mathcal{M})$, define the twisted reduced Hilbert polynomial $P(E, \beta, \omega, n)$ to be

$$P(E, \beta, \omega, n) = \frac{\chi(E \otimes \mathcal{M}^{-1} \otimes \mathcal{L}^n)}{a_d} \quad (28)$$

When $\beta = 0$, we simply write it $P(E, \omega, n)$. Note that (28) is calculated by chern characters of $E$, $\mathcal{M}$ and $\mathcal{L}$. Thus by formally replacing the chern characters by their fractional, we can define $P(E, \beta, \omega, n)$ for $\mathbb{Q}$-divisors $\beta$ and $\omega$, and $E \in \mathcal{N}(X)$. Explicitly when $v(E) = (r, l, s)$ with $r > 0$, we have

$$P = n^2 + \frac{2(l - r\beta) \cdot r \cdot \omega}{r \omega^2} \cdot n - \frac{(l^2 - 2rs - (l - r\beta)^2)}{r^2 \omega^2} + \frac{2\epsilon}{\omega^2}, \quad (29)$$

and $P = n + (s - \beta \cdot l)/\omega \cdot l$ when $r = 0, l \neq 0$, and $P = s$ when $r = l = 0$. Also for a torsion free sheaf $E$, define $\mu_{\omega}(E) \in \mathbb{Q}$ to be

$$\mu_{\omega}(E) = \frac{l \cdot \omega}{r}.$$

**Definition 4.1.** For a pure sheaf $E \in \text{Coh}(X)$, we say $E$ is $(\beta, \omega)$-twisted (semi)stable if for any subsheaf $F \subsetneq E$ one has

$$P(F, \beta, \omega, n) < P(E, \beta, \omega, n), \; \text{(resp} \leq),$$

for $n \gg 0$. If $\beta = 0$, we say simply $\omega$-Gieseker (semi)stable. Also a torsion free sheaf $E$ is $\mu_{\omega}$-(semi)stable if for any subsheaf $F \subsetneq E$ one has

$$\mu_{\omega}(F) < \mu_{\omega}(E), \; \text{(resp} \leq).$$

There are notions of Harder-Narasimhan filtrations in both stability conditions [25].
4.3 Stability conditions on K3 surfaces

Here we recall the constructions of stability conditions on a K3 surface or an abelian surface $X$ studied in [6]. Let $\beta, \omega$ be $\mathbb{Q}$-divisors on $X$ with $\omega$ ample. For a torsion free sheaf $E \in \text{Coh}(X)$, one has the Harder-Narasimhan filtration

$0 = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = E,$

such that $F_i = E_i/E_{i+1}$ is $\mu_\omega$-semistable and $\mu_\omega(F_i) > \mu_\omega(F_{i+1})$. Then define $\mathcal{T}_{(\beta, \omega)} \subset \text{Coh}(X)$ to be the subcategory consists of sheaves whose torsion free parts have $\mu_\omega$-semistable Harder-Narasimhan factors of slope $\mu_\omega(F_i) > \beta \cdot \omega$. Also define $\mathcal{F}_{(\beta, \omega)} \subset \text{Coh}(X)$ to be the subcategory consists of torsion free sheaves whose $\mu_\omega$-semistable factors have slope $\mu_\omega(F_i) \leq \beta \cdot \omega$.

**Definition 4.2.** We define $\mathcal{A}_{(\beta, \omega)}$ to be

$$\mathcal{A}_{(\beta, \omega)} = \{ E \in D(X) \mid H^{-1}(E) \in \mathcal{F}_{(\beta, \omega)}, H^0(E) \in \mathcal{T}_{(\beta, \omega)} \}. $$

**Remark 4.3.** Note that different choices of $\beta, \omega$ may define the same category $\mathcal{A}_{(\beta, \omega)}$. For instance, we have $\mathcal{A}_{(\beta, k\omega)} = \mathcal{A}_{(\beta, \omega)}$ for $k \in \mathbb{Q}_{\geq 1}$.

We define $\mathcal{Z}_{(\beta, \omega)} : \mathcal{N}(X) \to \mathbb{C}$ by the formula.

$$\mathcal{Z}_{(\beta, \omega)}(E) = (\exp(\beta + i\omega), v(E)).$$

(30)

Explicitly if $v(E) = (r, l, s)$ and $r \neq 0$, then (30) is written as

$$\mathcal{Z}_{(\beta, \omega)}(E) = \frac{1}{2r} \left( (l^2 - 2rs) + r^2 \omega^2 - (l - r\beta)^2 \right) + i(l - r\beta) \cdot \omega. $$

(31)

If $r = 0$, (30) is written as

$$\mathcal{Z}(E) = (-s + l \cdot \beta) + i(l \cdot \omega).$$

We define $\sigma_{(\beta, \omega)}$ to be the pair $(\mathcal{Z}_{(\beta, \omega)}, \mathcal{A}_{(\beta, \omega)})$.

**Proposition 4.4.** [6, Lemma 6.2, Proposition 7.1] The subcategory $\mathcal{A}_{(\beta, \omega)} \subset D(X)$ is a heart of a bounded t-structure, and the pair $\sigma_{(\beta, \omega)}$ gives a stability condition on $D(X)$ if and only if for any spherical sheaf $E$ on $X$, one has $\mathcal{Z}_{(\beta, \omega)}(E) \notin \mathbb{R}_{\leq 0}$. This holds whenever $\omega^2 > 2$.

Let $\text{Stab}^*(X)$ be the connected component of $\text{Stab}(X)$ which contains $\sigma_{(\beta, \omega)}$, and define $\mathcal{V} \subset \text{Stab}^*(X)$ to be

$$\mathcal{V} = \{ \sigma_{(\beta, \omega)} \in \text{Stab}^*(X) \mid \sigma_{(\beta, \omega)} \text{ satisfies the assumption in Proposition 4.4} \}.$$ 

The following is stated in [6] Section 13.

**Theorem 4.5.** The connected component $\text{Stab}^*(X)$ and the subset $\mathcal{V} \subset \text{Stab}^*(X)$ satisfy Assumption 4.6.

Finally we give the following useful lemma.

**Lemma 4.6.** (i) If $E \in D(X)$ satisfies $\text{Hom}(E, E) = \mathbb{C}$, then $v(E)^2 \geq -2$.

(ii) For $\mathbb{Q}$-divisors $\beta, \omega$ with $\omega$ ample and $m \in \mathbb{R}_{\geq 0}$, the set of Mukai vectors

$$\{ v \in \text{NS}^*(X) \mid v^2 \geq -2, |(\exp(\beta + i\omega), v)| \leq m \},$$

is finite.

(iii) For $E \in \mathcal{N}(X)$, assume $v(E) = (0, l, s) \in \text{NS}^*(X)$ with $l \neq 0$. Then

$$P(E, \beta, \omega, n) = n - \frac{\text{Re} \mathcal{Z}_{(\beta, \omega)}(E)}{\text{Im} \mathcal{Z}_{(\beta, \omega)}(E)} \in \mathbb{Q}[n].$$
(iv) For \( E, E' \in \mathcal{N}(X) \), \( P(E, \beta, \omega, n) = P(E', \beta, \omega, n) \) if and only if
\[
\text{Im} \frac{Z_{(\beta, k\omega)}(E')}{Z_{(\beta, k\omega)}(E)} = 0,
\]
for infinitely many \( k \in \mathbb{Q} \).

Proof. (i) is proved in [6, Lemma 5.1] and (ii) is proved in [6, Lemma 8.2]. (iii) and (iv) follow easily from (29) and (31).

### 4.4 Generic flatness for \( A(\beta, \omega) \)

Here we show the generic flatness in a special case.

**Lemma 4.7.** Problem 3.10 is true for \( A = A(\beta, \omega) \).

**Proof.** Let \( S \) be a smooth projective variety over \( \mathbb{C} \), \( L \in \text{Pic}(S) \) be an ample line bundle. Let us take \( E \in A_S \). By the definition of \( A_S \), we have
\[
R^p_s(\mathcal{E} \otimes L^n) \in A(\beta, \omega),
\]
for \( n \gg 0 \). In particular \( R^p_s(\mathcal{E} \otimes L^n) \) is concentrated in degree \([-1, 0]\]. Note that the following spectral sequence
\[
E_2^{i,j} = R^p_s(H^i(\mathcal{E}) \otimes L^n) \Rightarrow R^{i+j}_s(\mathcal{E} \otimes L^n),
\]
degenerates for \( n \gg 0 \). Therefore \( H^i(\mathcal{E}) = 0 \) unless \( j = -1 \) or \( 0 \). By [12, Theorem 2.3.2], there is an open subset \( U \subset S \) and filtrations of coherent sheaves,
\[
H^{-1}(\mathcal{E})_U = F^0 \supset F^1 \supset \cdots \supset F^k, \quad H^0(\mathcal{E})_U = T^0 \supset T^1 \supset \cdots \supset T^l,
\]
such that
- Each \( F^i \) and \( T^i \) are flat sheaves on \( U \).
- For \( s \in U \), the filtrations
\[
H^{-1}(\mathcal{E})_s = F^0_s \supset F^1_s \supset \cdots \supset F^k_s, \quad H^0(\mathcal{E})_s = T^0_s \supset T^1_s \supset \cdots \supset T^l_s,
\]
are Harder-Narasimhan filtrations in \( \omega \)-Gieseker stability.

Note that \( \mathcal{E}_s \in A(\beta, \omega) \) is equivalent to
\[
\mu_\omega(F^k_s) \leq \beta \cdot \omega, \quad \mu_\omega(T^0_s/T^1_s) > \beta \cdot \omega \text{ or } H^0(\mathcal{E})_s \text{ is torsion }, \tag{33}
\]
and such points are dense in \( S \) by Proposition 3.12. For each \( i, s, s' \in U \), the coherent sheaves \( F^i_s, T^i_s \) are numerically equivalent to \( F^i_{s'}, T^i_{s'} \) respectively. Therefore (33) holds for any \( s \in U \). This implies \( \mathcal{E}_s \in A(\beta, \omega) \) for any \( s \in U \). \( \Box \)
4.5 Boundedness of semistable objects

Next we check the boundedness of \( M^\alpha(\sigma_{(\beta, \omega)}) \), where \( \sigma_{(\beta, \omega)} \in V \). Let us prepare some notation and lemmas. For \( E \in A_{(\beta, \omega)} \), let

\[
H^0(E)_{\text{tor}} \subset H^0(E),
\]

be the maximal torsion subsheaf of \( H^0(E) \), and set

\[
H^0(E)_{\text{fr}} = H^0(E)/H^0(E)_{\text{tor}}.
\]

Let

\[
T_1, \cdots, T_{a(E)} \in \text{Coh}(X),
\]

\[
F_1, \cdots, F_{d(E)}, F_{d(E)+1}, \cdots, F_{e(E)} \in \text{Coh}(X),
\]

be \( \mu_\omega \)-stable factors of \( H^0(E)_{\text{fr}}, H^{-1}(E) \) respectively. Also let

\[
T_{a(E)+1}, \cdots, T_{b(E)}, T_{b(E)+1}, \cdots, T_{c(E)} \in \text{Coh}(X)
\]

be \((\beta, \omega)\)-twisted stable factors of \( H^0(E)_{\text{tor}} \). For the numbering, we set as follows.

\[
\dim T_i = 2 \quad (1 \leq i \leq a(E)),
\]

\[
\dim T_i = 1 \quad (a(E) < i \leq b(E)),
\]

\[
\dim T_i = 0 \quad (b(E) < i \leq c(E)),
\]

\[
\text{Im } Z_{(\beta, \omega)}(T_i) > 0 \quad (1 \leq i \leq d(E)),
\]

\[
\text{Im } Z_{(\beta, \omega)}(T_i) = 0 \quad (d(E) < i \leq e(E)).
\]

Also for \( \alpha \in \mathcal{N}(X) \), define the set of objects \( M^\alpha(\beta, \omega) \) to be

\[
M^\alpha(\beta, \omega) = \{ E \in A_{(\beta, \omega)} \mid \text{Im } Z_{(\beta, \omega)}(E) \leq \text{Im } Z_{(\beta, \omega)}(\alpha) \}.
\]

We prepare the following lemma.

**Lemma 4.8.** The maps on \( M^\alpha(\beta, \omega) \),

\[
E \mapsto b(E), \quad E \mapsto d(E),
\]

are bounded. Furthermore the sets

\[
\{ \text{Im } Z_{(\beta, \omega)}(T_i) \in \mathbb{Q} \mid 1 \leq i \leq c(E), E \in M^\alpha(\beta, \omega) \}, \quad (34)
\]

\[
\{ \text{Im } Z_{(\beta, \omega)}(F_i[1]) \in \mathbb{Q} \mid 1 \leq i \leq e(E), E \in M^\alpha(\beta, \omega) \}, \quad (35)
\]

are finite sets.

**Proof.** For \( E \in M^\alpha(\beta, \omega) \), we have the inequality,

\[
\text{Im } Z_{(\beta, \omega)}(\alpha) \geq \text{Im } Z_{(\beta, \omega)}(E) = \sum_{i=1}^{b(E)} \text{Im } Z_{(\beta, \omega)}(T_i) + \sum_{i=1}^{d(E)} \text{Im } Z_{(\beta, \omega)}(F_i[1]).
\]

Note that each term of the above sum is positive. Noting that \( \beta \) and \( \omega \) are rational, we can conclude the result. \( \square \)
The next step is to bound the real parts of $Z_{(β,ω)}(T_i)$ and $Z_{(β,ω)}(F_i[1])$. For the later use, we also give the bound of real part of $Z_{(β,ω)}(\ast)$ for $k \geq Q \geq 1$.

**Lemma 4.9.** There exist constants $C, C', N$, which depend only on $α, β$ and $ω$ such that
\[
\frac{1}{k} \text{Re} Z_{(β,ω)}(T_i) \geq \text{Re} Z_{(β,ω)}(T_i) \geq C \quad (1 \leq i \leq a(E)),
\]
\[
\frac{1}{k} \text{Re} Z_{(β,ω)}(F_i[1]) \leq \text{Re} Z_{(β,ω)}(F_i[1]) \leq C' \quad (1 \leq i \leq e(E)),
\]
for any $E \in M^α(β, ω)$.

**Proof.** We give the proof of (36). The proof of (37) is similar. Denote
\[
v(T_i) = (r_i, l_i, s_i) \in \mathbb{Z} \oplus \text{NS}(X) \oplus \mathbb{Z}.
\]
Note that $r_i > 0$ for $1 \leq i \leq a(E)$, and
\[
\text{Im} Z_{(β,ω)}(T_i) = (l_i - r_i β) \cdot ω,
\]
which is bounded by Lemma 4.8. Thus the Hodge index theorem implies that there exists a constant $C'' > 0$ which depends only on $α, β$ and $ω$ such that
\[
(l_i - r_i β)^2 \leq C''.
\]
By Lemma 4.6 (i), we have
\[
v(T_i)^2 = l_i^2 - 2r_is_i \geq -2.
\]
Hence for $1 \leq i \leq a(E)$, we have
\[
\text{Re} Z_{(β,ω)}(T_i) = \frac{1}{2r_i} \left( (l_i^2 - 2r_is_i) + r_i^2ω^2 - (l_i - r_i β)^2 \right)
\]
\[
\geq \frac{1}{2r_i} - \frac{2 + C''}{2r_i}
\]
\[
> -\frac{2 + C''}{2}.
\]
Similarly we have
\[
\frac{1}{k} \text{Re} Z_{(β,kω)}(T_i) - \text{Re} Z_{(β,ω)}(T_i) \geq \frac{1}{2} (k - 1)ω^2 + \left( \frac{1}{k} - 1 \right) \frac{2 + C''}{2},
\]
Thus one can find a desired $N > 0$.

Finally we give the following preparation.

**Lemma 4.10.** Let $S$ be a subset of $M^α(β, ω)$.

(i) Assume
\[
E \mapsto \text{Re} Z_{(β,ω)} H^0(E)\text{tr},
\]
is bounded above on $S$. Then the following set,
\[
\{v(T_i) \in \text{NS}^*(X) \mid 1 \leq i \leq a(E), E \in S\},
\]

is a finite set.

(ii) Assume
\[ E \mapsto \text{Re} Z_{(\beta, \omega)} H^{-1}(E), \]
is bounded below on \( S \). Then the following set,
\[ \{ v(F_i) \in NS^+(X) \mid 1 \leq i \leq e(E), E \in S \}, \]
is a finite set.

**Proof.** We show (ii). The proof of (i) is similar and leave it to the reader. For \( E \in S \), we have
\[ \text{Re} Z_{(\beta, \omega)}(H^{-1}(E)[1]) = \sum_{i=1}^{d(E)} \text{Re} Z_{(\beta, \omega)}(F_i[1]) + \sum_{i=d(E)+1}^{e(E)} \text{Re} Z_{(\beta, \omega)}(F_i[1]). \] (40)

Note that \( Z_{(\beta, \omega)}(F_i[1]) \in \mathbb{R}_{<0} \) for \( d(E) < i \leq e(E) \), and \( \text{Re} Z_{(\beta, \omega)}(F_i[1]) \) is bounded above for \( 1 \leq i \leq d(E) \) by Lemma 4.19. Furthermore \( E \mapsto d(E) \) is bounded by Lemma 4.8. Therefore the map \( E \mapsto e(E) \) is bounded and the following set is a finite set:
\[ \{ \text{Re} Z_{(\beta, \omega)}(F_i[1]) \in \mathbb{Q} \mid 1 \leq i \leq e(E), E \in S \}. \]

Then combined with Lemma 4.8 the following set is a finite set:
\[ \{ Z_{(\beta, \omega)}(F_i[1]) \in \mathbb{C} \mid 1 \leq i \leq e(E), E \in S \}. \] (41)

By the finiteness of (41) and Lemma 4.6 (i), (ii), the set (39) is also finite.

Now we can show the following.

**Proposition 4.11.** Problem 3.14 is true for any \( \sigma_{(\beta, \omega)} \in \mathcal{V} \).

It is enough to show the boundedness of
\[ M^\alpha(\sigma_{(\beta, \omega)}) = \{ E \in \mathcal{A}_{(\beta, \omega)} \mid E \text{ is of numerical type } \alpha \text{ and semistable in } \sigma_{(\beta, \omega)} \}. \]

Note that we have \( M^\alpha(\sigma_{(\beta, \omega)}) \subset M^\alpha(\beta, \omega) \). Let \( T, T' \) and \( F \) be the sets of objects,
\[ T = \{ H^0(E)_{fr} \in \text{Coh}(X) \mid E \in M^\alpha(\sigma_{(\beta, \omega)}) \}, \]
\[ T' = \{ H^0(E)_{tor} \in \text{Coh}(X) \mid E \in M^\alpha(\sigma_{(\beta, \omega)}) \}, \]
\[ F = \{ H^{-1}(E) \in \text{Coh}(X) \mid E \in M^\alpha(\sigma_{(\beta, \omega)}) \}. \]

By Lemma 3.16, it suffices to show that each \( T \), \( T' \) and \( F \) are bounded. We divide the proof into two steps.

**Step 1.** The sets of objects \( T, F \) are bounded.

**Proof.** Take \( E \in M^\alpha(\sigma_{(\beta, \omega)}) \). Note that we have the exact sequence in \( \mathcal{A}_{(\beta, \omega)} \),
\[ 0 \rightarrow H^{-1}(E)[1] \rightarrow E \rightarrow H^0(E) \rightarrow 0, \]
and a surjection \( H^0(E) \rightarrow H^0(E)_{fr} \) in \( \mathcal{A}_{(\beta, \omega)} \). Thus we have
\[ \text{Im} Z_{(\beta, \omega)}(H^{-1}(E)[1]) \leq \text{Im} Z_{(\beta, \omega)}(\alpha), \quad \text{Im} Z_{(\beta, \omega)}(H^0(E)_{fr}) \leq \text{Im} Z_{(\beta, \omega)}(\alpha), \]

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and the semistability of $E$ implies
\[ \phi(H^{-1}(E)[1]) \leq \phi(E) \leq \phi(H^0(E)_{\text{fr}}). \]

Therefore if we consider the maps on $M^\alpha(\sigma_{(\beta, \omega)})$,
\[
\begin{align*}
E &\mapsto \text{Re } Z_{(\beta, \omega)}(H^0(E)_{\text{fr}}) \in \mathbb{Q}, \\
E &\mapsto \text{Re } Z_{(\beta, \omega)}(H^{-1}(E)[1]) \in \mathbb{Q},
\end{align*}
\]
then (42) is bounded above and (43) is bounded below. Thus one can apply Lemma 4.10 and conclude that the sets
\[
\{v(T_i) \in \text{NS}^*(X) \mid 1 \leq i \leq a(E), E \in M^\alpha(\sigma_{(\beta, \omega)}) \},
\]
\[
\{v(F_i[1]) \in \text{NS}^*(X) \mid 1 \leq i \leq e(E), E \in M^\alpha(\sigma_{(\beta, \omega)}) \},
\]
are finite sets. Since the set of Gieseker-stable sheaves with a fixed Mukai vector is bounded, (see [12]) the sets of sheaves
\[
\{T_i \in \text{Coh}(X) \mid 1 \leq i \leq a(E), E \in M^\alpha(\sigma_{(\beta, \omega)}) \},
\]
\[
\{F_i \in \text{Coh}(X) \mid 1 \leq i \leq e(E), E \in M^\alpha(\sigma_{(\beta, \omega)}) \},
\]
are bounded. Thus $T$ and $F$ are also bounded by Lemma 3.16 \hfill \Box

**Step 2.** The set of sheaves $T'$ is bounded.

*Proof.* For $a(E) < i \leq b(E)$ we may assume $P(T_i, \beta, \omega, n) > P(T_{i+1}, \beta, \omega, n)$. Hence by Lemma 4.6 (iii) we have
\[ \phi(T_{a(E)+1}) > \cdots > \phi(T_{b(E)}). \] (44)

Note that there is an exact sequence
\[ 0 \rightarrow T' \rightarrow H^0(E)_{\text{tor}} \rightarrow T_{b(E)} \rightarrow 0, \]
both in Coh($X$) and $A_{(\beta, \omega)}$. Let $H^0(E)/T' \in \text{Coh}(X)$ be the cokernel of the inclusion,
\[ T' \hookrightarrow H^0(E)_{\text{tor}} \rightarrow H^0(E), \]
in Coh($X$). Then the following composition,
\[ E \rightarrow H^0(E) \rightarrow H^0(E)/T', \]
is a surjection in $A_{(\beta, \omega)}$. Thus we have
\[ \text{Im } Z_{(\beta, \omega)}(H^0(E)/T') \leq \text{Im } Z_{(\beta, \omega)}(E), \]
and the semistability of $E$ implies $\phi(E) \leq \phi(H^0(E)/T')$. Hence the map
\[
\begin{align*}
E &\mapsto \text{Re } Z_{(\beta, \omega)}(H^0(E)/T') \\
&= \text{Re } Z_{(\beta, \omega)}(T_{b(E)}) + \text{Re } Z_{(\beta, \omega)}(H^0(E)_{\text{fr}}),
\end{align*}
\]
is bounded above. Since $T$ is bounded, it follows that $\text{Re} Z_{(\beta, \omega)}(T_{b(E)})$ is also bounded above. Hence by Lemma 4.8 and (44), there is a constant $C'''$ (which depends only on $\alpha$, $\beta$ and $\omega$) such that

$$\text{Re} Z_{(\beta, \omega)}(T_{i}) \leq C'''$$

(45)

On the other hand we have

$$\text{Re} Z_{(\beta, \omega)}(H_{0}^{0}(E)_{\text{tor}}) = \sum_{i=a(E)}^{b(E)} \text{Re} Z_{(\beta, \omega)}(T_{i}) + \sum_{i=b(E)+1}^{c(E)} \text{Re} Z_{(\beta, \omega)}(T_{i}).$$

(46)

Note that $E \mapsto \text{Re} Z_{(\beta, \omega)}(H_{0}^{0}(E)_{\text{tor}})$ is bounded on $M_{\alpha}(\sigma(\beta, \omega))$ because $\mathcal{T}$ and $\mathcal{F}$ are bounded. Thus the boundedness of (46) together with (45) and $\text{Re} Z_{(\beta, \omega)}(T_{i}) \in \mathbb{R}_{<0}$ for $b(E) < i \leq c(E)$ show that the set,

$$\{\text{Re} Z_{(\beta, \omega)}(T_{i}) \in \mathbb{Q} | a(E) < i \leq c(E), E \in M^{a}(\sigma(\beta, \omega))\},$$

(47)

is a finite set, and $E \mapsto c(E)$ is bounded. Hence by Lemma 4.8 the finiteness of (47), and using Lemma 4.6(ii), we conclude that the set

$$\{v(T_{i}) \in NS^{*}(X) | a(E) < i \leq c(E), E \in M^{a}(\sigma(\beta, \omega))\},$$

is a finite set. Again the set of sheaves,

$$\{T_{i} \in \text{Coh}(X) | a(E) < i \leq c(E), E \in M^{a}(\sigma(\beta, \omega))\},$$

is bounded, thus $\mathcal{T}'$ is also bounded by Lemma 3.16.

Combined with the result in the previous section, we obtain the following.

**Theorem 4.12.** Let $X$ be a K3 surface or an abelian surface. Then for any $\sigma \in \text{Stab}^{*}(X)$, $\alpha \in \mathcal{N}(X)$, $\phi \in \mathbb{R}$, the stack $\mathcal{M}^{(\alpha, \phi)}(\sigma)$ is an Artin stack of finite type over $\mathbb{C}$.

**Proof.** This follows from Theorem 3.20, Theorem 4.5, Lemma 4.7, and Proposition 4.11.

5 Invariants counting semistable objects

In this section, $X$ is a K3 surface or an abelian surface, $\mathcal{M}$ is the moduli stack of objects $E \in D(X)$ with $\text{Ext}^{<0}(E, E) = 0$ as in the previous section. The aim in this section is to introduce and study the the invariants, as an analogue of the work [17].

5.1 Stack functions

Let $\mathcal{D}$ be an Artin stack over $\mathbb{C}$. Following D. Joyce’s work [18], we introduce the notion of stack functions on $\mathcal{D}$. For the detail, one can consult [18, Section 3]. Let us consider pairs $(\mathcal{R}, \rho)$, where $\mathcal{R}$ is an Artin $\mathbb{C}$-stack of finite type over $\mathbb{C}$ with affine geometric stabilizers and $\rho: \mathcal{R} \to \mathcal{D}$ is a 1-morphism. We say two pairs $(\mathcal{R}, \rho)$, $(\mathcal{R}', \rho')$ equivalent if there exists a 1-isomorphism $\tau: \mathcal{R} \to \mathcal{R}'$ such that $\rho' \circ \tau$ is 2-isomorphic to $\rho$. 

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**Definition 5.1.** Define the $\mathbb{Q}$-vector space $SF(D)$ to be

$$SF(D) := \bigoplus_{(\mathcal{R}, \rho)} \mathbb{Q}[(\mathcal{R}, \rho)]/\sim.$$ 

Here $[(\mathcal{R}, \rho)]$ is an equivalence class of $(\mathcal{R}, \rho)$ and the relation $\sim$ is generated by

$$[(\mathcal{R}, \rho)] = [(\mathcal{R}^!, \rho|_{\mathcal{R}^!})] + [(\mathcal{R} \setminus \mathcal{R}^!, \rho|_{\mathcal{R} \setminus \mathcal{R}^!})],$$

where $\mathcal{R}^!$ is a closed substack of $\mathcal{R}$.

For $\rho: \mathcal{R} \to D$ and $\rho': \mathcal{R}' \to D$, there is a notion of fiber product [19, Definition 2.10],

$$\mathcal{R} \times_{\rho, D, \rho'} \mathcal{R}' \to \mathcal{R}, \quad \pi_{\mathcal{R}'}, \quad \rho' \to D.$$

As in [18, Definition 3.1], we can define a $\mathbb{Q}$-bilinear product $SF(D) \times SF(D) \to SF(D)$ by the formula,

$$[(\mathcal{R}_i, \rho_i)] \cdot [(\mathcal{R}_i', \rho_i') = [(\mathcal{R}_i \times_{\rho_i, D, \rho_i'} \mathcal{R}_i', \rho_i \circ \pi_{\mathcal{R}_i'})].$$

If $\Pi: C \to \mathcal{C}$ is a 1-morphism of Artin $\mathbb{C}$-stacks, then define the push-forward $\Pi^*: SF(D) \to SF(C)$ by

$$\Pi^*: \sum_{i=1}^{m} c_i[(\mathcal{R}_i, \rho_i)] \mapsto \sum_{i=1}^{m} c_i[(\mathcal{R}_i, \Pi \circ \rho_i)].$$

If $\Pi$ is of finite type, one can define the pull-back $\Pi_*: SF(C) \to SF(D)$,

$$\Pi_*: \sum_{i=1}^{m} c_i[(\mathcal{R}_i, \rho_i)] \mapsto \sum_{i=1}^{m} c_i[(\mathcal{R}_i \times_{\rho_i, C, \phi} D, \pi_D)].$$

The tensor product $\otimes: SF(D) \times SF(C) \to SF(D \times C)$ is

$$\left( \sum_{i=1}^{m} c_i[(\mathcal{R}_i, \rho_i)] \right) \otimes \left( \sum_{i=1}^{m'} c'_i[(\mathcal{R}'_i, \rho'_i)] \right) = \sum_{i,j} c_i d_j [(\mathcal{R}_i \times \mathcal{R}'_i, \rho_i \times \rho'_j)].$$

One can consult [18, Definition 3.1] for the detail of these definitions. For a substack $i: D^i \hookrightarrow D$, we write $[(D, i)]$ as $[D^i \hookrightarrow D]$. If $X$ is a K3 surface or an abelian surface, we have shown in Theorem 4.12 that the stack $\mathcal{M}^{(\alpha, \phi)}(\sigma)$ is an open substack of $\mathcal{M}$ and it is of finite type.

**Definition 5.2.** For $\sigma \in \text{Stab}^*(X)$, $\alpha \in \mathcal{N}(X)$ and $\phi \in \mathbb{R}$, we define $\delta^{(\alpha, \phi)}(\sigma)$ to be

$$\delta^{(\alpha, \phi)}(\sigma) = [\mathcal{M}^{(\alpha, \phi)}(\sigma) \hookrightarrow \mathcal{M}] \in SF(\mathcal{M}).$$

### 5.2 Ringel-Hall algebras

Take an algebraic stability condition $\sigma = (Z, \mathcal{P}) \in \text{Stab}(X)$ and let $\mathcal{A}_{\phi} = \mathcal{P}((\phi - 1, \phi))$ for $\phi \in I$. Assume the generic flatness holds for $\mathcal{A}_{\phi}$, then the stack of objects in $\mathcal{A}_{\phi}$ is an open substack of $\mathcal{M}$, thus in particular it is an Artin stack over $\mathbb{C}$. We denote it by $\mathcal{Obj}_{\mathcal{A}_{\phi}} \subset \mathcal{M}$.  

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Following [20, Definition 5.1], we introduce the associative multiplication \( \ast \) on \( \text{SF}(\text{Obj}_A) \) based on Ringel-Hall algebras. Let \( \text{Obj}(A_\phi, n) \) be the moduli stack of filtrations,

\[
0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_n,
\]

with \( E \in A_\phi \). It is shown in [19, Theorem 8.2] that \( \text{Obj}(A_\phi, n) \) is an Artin stack of locally finite type over \( \mathbb{C} \). We have the following 1-morphisms,

\[
\prod_{i=1}^n \text{Obj}_A \overset{\prod_{i=1}^n p_i}{\longrightarrow} \text{Obj}_A \rightarrow \text{Obj}_A.
\]

Here \( p_i : \text{Obj}(A_\phi, n) \to \text{Obj}_A \) is defined to be

\[
(0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_n) \mapsto F_i := E_i/E_{i-1},
\]

and \( \Pi_n : \text{Obj}(A_\phi, n) \to \text{Obj}_A \) is defined to be

\[
(0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_n) \mapsto E_n.
\]

It is shown in [19, Theorem 8.4] that \( \prod_{i=1}^n p_i \) is of finite type, thus one can define its pull-back. One has the following diagram,

\[
\begin{array}{ccc}
\text{SF}(\text{Obj}_A) \times \text{SF}(\text{Obj}_A) & \overset{\otimes}{\longrightarrow} & \text{SF}(\text{Obj}_A) \\
\downarrow & & \downarrow \\
\text{SF}(\text{Obj}_A) \times \text{Obj}_A(p_1 \times p_2)^* & \overset{\Pi_2^*}{\longrightarrow} & \text{SF}(\text{Obj}(A_\phi, 2)) \overset{\Pi_2^*}{\longrightarrow} \text{SF}(\text{Obj}_A),
\end{array}
\]

**Definition 5.3.** We define a bilinear operation \( \ast : \text{SF}(\text{Obj}_A) \times \text{SF}(\text{Obj}_A) \to \text{SF}(\text{Obj}_A) \) to be

\[
f \ast g = \Pi_2^*((p_1 \times p_2)^*(f \otimes g)).
\]

It is shown in [20, Theorem 5.2] that \( \ast \) is associative and \( \text{SF}(\text{Obj}_A) \) is a \( \mathbb{Q} \)-algebra with identity \( \delta_{[0]} : [(0 \to \text{Obj}_A)] \). In fact we have

\[
(f \ast g) \ast h = f \ast (g \ast h) = \Pi_3^*((p_1 \times p_2 \times p_3)^*(f \otimes g \otimes h)),
\]

\[
f_1 \ast \cdots \ast f_n = \Pi_{n\ast}(\prod_{i=1}^n p_i)^*(f_1 \otimes \cdots \otimes f_n).
\]

One can consult [20, Section 5] for the detail of the algebra (\( \text{SF}(\text{Obj}_A), \ast \)). For an interval \( I \subset \mathbb{R} \), set \( C^\sigma(I) \subset \mathcal{N}(X) \) to be

\[
C^\sigma(I) = \text{im}([P(I) \to \mathcal{N}(X)]) \setminus \{0\} \subset \mathcal{N}(X).
\]

**Definition 5.4.** For \( \alpha_1, \cdots, \alpha_n \in C^\sigma(\langle \phi - 1, \phi \rangle) \), we define the substack

\[
\mathcal{M}(\{\alpha_i\}_{1 \leq i \leq n}, A_\phi, \sigma) \subset \text{Obj}(A_\phi, n)
\]

to be the stack of filtrations \([18]\) such that \( F_i = E_i/E_{i-1} \) is semistable in \( \sigma \) and of numerical type \( \alpha_i \).

Note that for \( \alpha \in C^\sigma(\langle \phi - 1, \phi \rangle) \), there is a unique phase \( \phi(\alpha) \in (\phi - 1, \phi] \) with respect to the stability function \( Z \). Also note that the element \( \delta^{(\alpha, \phi(\alpha))}(\sigma) \in \text{SF}(\mathcal{M}) \) is regarded as the element of \( \text{SF}(\text{Obj}_A) \).
Lemma 5.5. For \( \alpha_1, \ldots, \alpha_n \in C^\sigma((\phi - 1, \phi)) \), we have the following equality in \( \text{SF}(\mathfrak{Obj}, \mathcal{A}_\phi) \),
\[
\Pi_{n^+}[\mathcal{M}(\{\alpha_i\}_{1 \leq i \leq n}, \mathcal{A}_\phi, \sigma) \hookrightarrow \mathfrak{Obj}(\mathcal{A}_\phi, n)] = \delta(\alpha_1, \phi(\alpha_1))(\sigma) \ast \cdots \ast \delta(\alpha_n, \phi(\alpha_n))(\sigma). \tag{51}
\]

Proof. By the definition we have
\[
[\mathcal{M}(\{\alpha_i\}_{1 \leq i \leq n}, \mathcal{A}_\phi, \sigma) \hookrightarrow \mathfrak{Obj}(\mathcal{A}_\phi, n)] = (\prod_{i=1}^{n} \pi_i)^* \prod \mathcal{M}(\alpha_i, \phi(\alpha_i))(\sigma) \hookrightarrow \prod \mathfrak{Obj}(\mathcal{A}_\phi), \tag{52}
\]
in \( \text{SF}(\mathfrak{Obj}(\mathcal{A}_\phi, n)) \). Thus it is enough to apply \( \Pi_{n^+} \) to (52) and use (50). \( \square \)

5.3 Motivic invariants of Artin stacks

Let \( K(\text{Var}) \) be the Grothendieck ring of quasi-projective varieties. This is a \( \mathbb{Z} \)-module generated by the isomorphism classes of quasi-projective varieties \( [X] \), and relations \( [X] = [Y] + [X \setminus Y] \) for closed subschemes \( Y \subset X \). The formula \( [X] \cdot [X'] = [X \times X'] \) extends to a ring structure on \( K(\text{Var}) \). Suppose \( \Lambda \) is a commutative \( \mathbb{Q} \)-algebra and \( \Upsilon \) is a ring homomorphism,
\[
\Upsilon: K(\text{Var}) \longrightarrow \Lambda. \tag{53}
\]

Write \( l = \Upsilon(k^1) \in \Lambda \). We assume \( l \) and \( t^k - 1 \) are invertible in \( \Lambda \) for \( k \geq 1 \). This assumption is required for the value
\[
\Upsilon(\text{GL}(m, \mathbb{C})) = l^{m(m-1)/2} \prod_{k=1}^{m} (t^k - 1),
\]
to be invertible in \( \Lambda \).

Example 5.6. We can take \( \Lambda = \mathbb{Q}(z) \) and \( \Upsilon([X]) = P(X; z) \) the virtual Poincare polynomial of \( X \). When \( X \) is smooth and projective, \( P(X; z) \) is the usual Poincare polynomial \( \sum_{k=0}^{\dim X} b^k(X)z^k \).

An algebraic \( \mathbb{C} \)-group \( G \) is called special if every principal \( G \)-bundle is locally trivial. It is shown in [18, Lemma 4.6] that if \( G \) is special then \( \Upsilon([G]) \) is invertible in \( \Lambda \).

Theorem 5.7. [18, Theorem 4.9] Under the above situation, there exists a unique morphism of \( \mathbb{Q} \)-algebras,
\[
\Upsilon': \text{SF}(\text{Spec} \mathbb{C}) \longrightarrow \Lambda,
\]
such that if \( G \) is a special algebraic \( \mathbb{C} \)-group which acts on a quasi-projective variety \( X \), then \( \Upsilon'([X/G]) = \Upsilon([X])/\Upsilon([G]) \).

Let \( \Pi: \mathcal{M} \rightarrow \text{Spec} \mathbb{C} \) be the structure morphism. Given a motivic invariant \( \Upsilon: K(\text{Var}) \rightarrow \Lambda \) as in (53), we have the following maps,
\[
\Upsilon' \circ \Pi_{+}: \text{SF}(\mathcal{M}) \longrightarrow \text{SF}(\text{Spec} \mathbb{C}) \longrightarrow \Lambda. \tag{54}
\]

Definition 5.8. Take \( \sigma = (Z, \mathcal{P}) \in \text{Stab}^b(X) \) and \( \alpha \in \mathcal{N}^b(X) \). We define \( I^\alpha(\sigma) \in \Lambda \) as follows. If \( Z(\alpha) = 0 \), we set \( I^\alpha(\sigma) = 0 \). Otherwise take \( \phi \in \mathbb{R} \) which satisfies \( Z(\alpha) \in \mathbb{R}_{>0}e^{i\pi}\phi \), and define \( I^\alpha(\sigma) \) to be
\[
I^\alpha(\sigma) = \Upsilon' \circ \Pi_{+} \delta^{(\alpha, \phi)}(\sigma) \in \Lambda.
\]

The definition of \( I^\alpha(\sigma) \) is an analogue of [17, Definition 6.1]. It is clear that the definition of \( I^\alpha(\sigma) \) does not depend on a choice of \( \phi \). Then as an analogue of [17, Definition 6.22], we introduce the invariant \( J^\alpha(\sigma) \in \Lambda \).

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Definition 5.9. We define $J^\alpha(\sigma) \in \Lambda$ as follows. If $Z(\alpha) = 0$, we set $J^\alpha(\sigma) = 0$. Otherwise choose $\phi \in \mathbb{R}$ which satisfies $Z(\alpha) \in \mathbb{R}_{>0}e^{i\phi}$, and define $J^\alpha(\sigma)$ to be

$$J^\alpha(\sigma) := \sum_{\alpha_1 + \cdots + \alpha_n = \alpha} l^{-\sum_{j>1} \chi(\alpha_j, \alpha_1)} \left(\frac{(-1)^{n-1}}{n} \prod_{i=1}^n I^\alpha(\sigma) \in \Lambda, \right.$$}

(55)

where $\alpha_i \in C^\alpha(\phi)$ for all $1 \leq i \leq n$.

Again the definition of $J^\alpha(\sigma)$ does not depend on a choice of $\phi$.

Remark 5.10. Suppose that any $E \in M^{(\alpha, \phi)}(\sigma)$ is stable. This occurs whenever $\alpha \in N(X)$ is primitive and $\sigma$ is not contained in a wall in Proposition 2.8. In this case, $J^\alpha(\sigma)$ coincides with $(l - 1)I^\alpha(\sigma)$. Furthermore any $E \in M^{(\alpha, \phi)}(\sigma)$ satisfies Hom$(E, E) = \mathbb{C}$. Hence by Inaba’s work [14] and the openness of stability proved in Section 4, there is an algebraic space $\tilde{M}^{(\alpha, \phi)}(\sigma)$ which parameterizes the objects in $M^{(\alpha, \phi)}(\sigma)$. Hence in this case we have $J^\alpha(\sigma) = \Upsilon(\tilde{M}^{(\alpha, \phi)}(\sigma)) \in \Lambda$.

The factor $l - 1 = \Upsilon(\mathbb{C}^\ast)$ is required to cancel out the contributions of the stabilizers Aut$(E) \cong \mathbb{C}^\ast$.

Remark 5.11. Suppose $\Lambda = \mathbb{Q}(z)$ and $\Upsilon$ be as in Example 5.6. Under the assumption in Remark 5.10, we can define the invariant $J^\alpha(\sigma)|_{z = -1} \in \mathbb{Q}$, as a virtual euler number of the moduli space. However in general, we don’t know whether the denominator of $J^\alpha(\sigma) \in \mathbb{Q}(z)$ is divided by $z + 1$ or not. So at this time, we do not define the invariant in $\mathbb{Q}$ in this way. Also see Remark 5.12 below.

We have to check the following.

Lemma 5.12. The sum (55) is a finite sum.

Proof. Note that for any $\varepsilon > 0$ there is an algebraic stability condition $\sigma' = (Z', \mathcal{P}') \in \text{Stab}^\ast(X)$ such that $\mathcal{P}(\phi) \subset \mathcal{P}'((\phi - \varepsilon, \phi + \varepsilon))$. Thus the possibilities of $n$ in the sum (55) is finite. Let us assume $\prod I^\alpha(\sigma) \neq 0$ in the sum (55). Then there are objects $E_i \in \mathcal{P}(\phi)$ of numerical type $\alpha_i$. By taking stable factors of $E_i$ and using Lemma 4.6 one can check that the possibilities for $\alpha_i$ is also finite. 

5.4 The algebra $A(A, \Lambda, \chi)$

Let $\sigma = (Z, \mathcal{P}) \in \text{Stab}^\ast(X)$ and $A_{\phi} = \mathcal{P}((\phi - 1, \phi))$ be as in (5.2). We introduce the $\Lambda$-algebra $A(A_{\phi}, \Lambda, \chi)$. For the detail, see [20] Section 6.

Definition 5.13. [20, Definition 6.3] We define the $\Lambda$-algebra $A(A_{\phi}, \Lambda, \chi)$ to be

$$A(A_{\phi}, \Lambda, \chi) = \bigoplus_{\alpha \in C^\ast((\phi - 1, \phi))} \Lambda c_\alpha,$$

such that the multiplication is given by $c_\alpha \cdot c_\beta = l^{-\chi(\beta, \alpha)}c_{\alpha + \beta}$.

Note that since we assume $X$ is K3 surface or an abelian surface, the algebra $A(A_{\phi}, \Lambda, \chi)$ is a commutative algebra. Let $i_{\alpha} : \text{Db}^{\alpha}A_{\phi} \subset \text{Db}_{i}A_{\phi}$ be the substack which parameterizes $E \in A_{\phi}$ of numerical type $\alpha$. We denote by $\Pi_{\alpha} : \text{Db}^{\alpha}A_{\phi} \rightarrow \text{Spec} \mathbb{C}$ the structure morphism. Given a motivic invariant $\Upsilon$ as in (53), we construct the map $\Theta : \text{SF}(\text{Db}^{\alpha}A_{\phi}) \rightarrow A(A_{\phi}, \Lambda, \chi)$ to be

$$\Theta : f \mapsto \sum_{\alpha \in C^\ast((\phi - 1, \phi))} \Upsilon'(\Pi_{\alpha}i_{\alpha}^*f) \cdot c_\alpha.$$
Definition 5.14. [17, Definition 3.18] For $\alpha \in C^\sigma((\phi - 1, \phi])$ define $\overline{\delta}(\sigma) \in A(A_\phi, \Lambda, \chi)$ to be

$$\overline{\delta}(\sigma) := \Theta(\delta(\alpha,\phi(\alpha)))(\sigma) = I^\alpha(\sigma)c_\alpha \in A(A_\phi, \Lambda, \chi),$$

and $\overline{\tau}(\sigma) \in A(A_\phi, \Lambda, \chi)$ to be

$$\overline{\tau}(\sigma) = \sum_{\alpha_1 + \cdots + \alpha_n = \alpha} \frac{(-1)^{n-1}}{n} \overline{\delta}(\sigma) \ast \cdots \ast \overline{\delta}(\sigma)$$

$$= \frac{J^\alpha(\sigma)}{I - 1}c_\alpha \in A(A_\phi, \Lambda, \chi),$$

where $\alpha_i \in C^\sigma(\phi(\alpha))$.

Remark 5.15. The definition of $J^\alpha(\sigma)$ is motivated by the weighted sum in the Ringel-Hall algebra. In fact Joyce [16, Theorem 8.7] showed that the following weighted sum in $\text{SF}(\mathcal{O}b_\mathcal{A}_\phi)$,

$$\epsilon(\alpha,\phi(\alpha))(\sigma) = \sum_{\alpha_1 + \cdots + \alpha_n = \alpha} \frac{(-1)^{n-1}}{n} \delta(\alpha_1,\phi(\alpha_1))(\sigma) \ast \cdots \ast \delta(\alpha_n,\phi(\alpha_n))(\sigma),$$

with $\alpha_i \in C^\sigma(\phi(\alpha))$ is contained in a certain Lie subalgebra $\text{SF}^\text{ind}(\mathcal{O}b_\mathcal{A}_\phi)$. Roughly it means that the stack function $\epsilon(\alpha,\phi(\alpha))(\sigma)$ is supported on indecomposable objects. Hence if $\Theta$ is a ring homomorphism, we have $\Theta(\epsilon(\alpha,\phi(\alpha))(\sigma)) = \overline{\epsilon}(\sigma)$ and in particular one can define $J^\alpha(\sigma)|_{z=1} \in \mathbb{Q}$ in Example [5.6]. However $\Theta$ is not ring homomorphism in our case, so instead $J^\alpha(\sigma)$ is defined as the weighted sum in the algebra $A(A_\phi, \Lambda, \chi)$ rather than the Ringel-Hall algebra. This is the motivation of the invariants explained in [17].

Although the map $\Theta$ is not a ring homomorphism, we have the following proposition.

Proposition 5.16. For $\alpha_1, \cdots, \alpha_n \in C^\sigma((\phi - 1, \phi])$, suppose $\phi(\alpha_1) > \phi(\alpha_2) > \cdots > \phi(\alpha_n)$ where $\phi(\alpha_i) \in (\phi - 1, \phi]$ the phase with respect to the stability function $Z$. Then we have the following equality in $A(A_\phi, \Lambda, \chi)$,

$$\Theta(\delta(\alpha_1,\phi(\alpha_1))(\sigma) \ast \cdots \ast \delta(\alpha_n,\phi(\alpha_n))(\sigma)) = \overline{\delta}(\sigma) \ast \cdots \ast \overline{\delta}(\sigma).$$

Proof. This is obtained by applying [17] Proposition 6.20] for the abelian category $\mathcal{P}((\phi - 1, \phi])$.

5.5 Behavior of invariants in a chamber

Let us investigate how the invariant vary under a change of stability conditions. From here until the end of section, we fix $\alpha \in N(X)$. Let $\mathcal{B}^\circ \subset \text{Stab}(X)$ be an open subset and its closure $\mathcal{B} = \overline{\mathcal{B}}$ is compact. Let $\mathcal{S} \subset D(X)$ be the set of objects,

$$\mathcal{S} := \{E \in D(X) \mid E \text{ is semistable in some } \sigma' = (Z', \mathcal{P}') \in \mathcal{B} \text{ with } |Z'(E)| \leq |Z'(\alpha)|\}.$$  

Then $\mathcal{S}$ has a bounded mass, thus there exists a finite number of codimension one submanifolds $\{\mathcal{W}_\gamma\}_{\gamma \in \Gamma}$ which gives a wall and chamber structure on $\mathcal{B}$. (See Proposition 2.8). Let $\mathcal{C}$ be one of the connected component,

$$\mathcal{C} \subset \mathcal{B} \setminus \bigcup_{\gamma} \mathcal{W}_\gamma.$$  

We show the following,
Proposition 5.17. Take \( \sigma_i = (Z_i, \mathcal{P}_i) \in \mathcal{C} \) for \( i = 0, 1 \). Then we have \( J^\alpha(\sigma_0) = J^\alpha(\sigma_1) \).

Proof. First assume \( Z_0(\alpha) \in \mathbb{R}_{>0}e^{i\pi \phi} \) for some \( \phi \in \mathbb{R} \) and \( Z_1(\alpha) = 0 \). Then there is no object \( F \in \mathcal{S} \) which is semistable in \( \sigma_1 \) and of numerical type \( \alpha \). Because \( \sigma_0 \) and \( \sigma_1 \) are contained in the same chamber, there is no object \( F \in \mathcal{S} \) which is semistable in \( \sigma_0 \) and of numerical type \( \alpha \). Note that if the sum \((55)\) for \( \sigma_0 \) is non-zero, there exist

\[
\alpha_1, \ldots, \alpha_n \in C^{\sigma_0}(\phi),
\]

such that \( \prod_{i=1}^n I^\alpha(\sigma_0) \) is non-zero, and \( \alpha_1 + \cdots + \alpha_n = \alpha \). By the definition of \( I^\alpha(\sigma_0) \), there must be an object \( E_i \in \mathcal{P}_0(\phi) \) of numerical type \( \alpha_i \) for each \( i \). Then \( \oplus_{i=1}^n E_i \) is semistable in \( \sigma_0 \) and of numerical type \( \alpha \), which is a contradiction. Hence in this case, one has

\[
J^\alpha(\sigma_0) = J^\alpha(\sigma_1) = 0.
\]

Thus we may assume \( Z_i(\alpha) \neq 0 \) for \( i = 0, 1 \). Again choose \( \phi \in \mathbb{R}, \alpha_1, \ldots, \alpha_n \in C^{\sigma_0}(\phi) \) in the sum \((55)\) for \( \sigma_0 \). If \( \prod_{i=1}^n I^\alpha(\sigma_0) \neq 0 \), then there exist \( E_i \in \mathcal{P}_0(\phi) \) of numerical type \( \alpha_i \). We have

\[
|Z_0(E_i)| = |Z_0(\alpha_i)| \leq |Z_0(\alpha)|.
\]

Thus we have \( E_i \in \mathcal{S} \). Note that for each \( i \) and \( j \), the values \( Z_0(\alpha_i) \) and \( Z_0(\alpha_j) \) are proportional. Furthermore we have \( \sigma_0 \notin \mathcal{W}_\gamma \) for any \( \gamma \). By the construction of \( \mathcal{W}_\gamma \) in Proposition 2.8 this implies \( \alpha_i \) and \( \alpha_j \) must be proportional in \( \mathcal{N}(X) \), hence \( \alpha_i \) is proportional to \( \alpha \). Choose a path

\[
\lambda: [0, 1] \to \mathcal{C},
\]

such that \( \lambda(0) = \sigma_0 \) and \( \lambda(1) = \sigma_1 \). We denote \( \lambda(t) = \sigma_t = (Z_t, \mathcal{P}_t) \). For an arbitrary \( E_i \in \mathcal{P}_0(\phi) \) of numerical type \( \alpha_i \), we have \( E_i \in \mathcal{S} \), thus \( E_i \) is also semistable in \( \sigma_t \). Hence \( Z_t(\alpha_i) \neq 0 \) for \( t \in [0, 1] \), and the phase of \( E_i \) in \( \sigma_t \) is uniquely determined independent of a choice of \( E_i \in \mathcal{P}_0(\phi) \). Thus there is \( \phi' \in \mathbb{R} \) which satisfies \( Z_1(\alpha_i) \in \mathbb{R}_{>0}e^{i\pi \phi'} \) such that

\[
\mathcal{M}^{(\alpha_i, \phi)}(\sigma_0) \subset \mathcal{M}^{(\alpha_i, \phi')}(\sigma_1).
\]

By the converse argument, we obtain \( \mathcal{M}^{(\alpha_i, \phi)}(\sigma_0) = \mathcal{M}^{(\alpha_i, \phi')}(\sigma_1) \), thus \( I^\alpha_i(\sigma_0) = I^\alpha_i(\sigma_1) \). Because \( \alpha_i \) is proportional to \( \alpha \), we have \( Z_1(\alpha) \in \mathbb{R}_{>0}e^{i\pi \phi'} \) and \( \alpha_i \in C^{\sigma_1}(\phi') \). Hence the sum \((55)\) for \( J^\alpha(\sigma_0) \) and \( J^\alpha(\sigma_1) \) are identified.

\[\square\]

5.6 Behavior of invariants near a wall

Next we investigate the behavior of the invariants near a wall \( \mathcal{W}_\lambda \). Here we use the same notation as in (5.3). Take \( 0 < \varepsilon < 1/6 \) and \( \sigma_i = (Z_i, \mathcal{P}_i) \in \text{Stab}^\ast(X) \) for \( i = 0, 1 \). We assume the following.

- \( \sigma_0 \) is algebraic, contained in \( \mathcal{C} \), and \( \mathcal{P}((\psi - 1, \psi]) \) satisfies the generic flatness for any \( \psi \in \mathcal{I} \). (See Remark 3.21)

- \( \sigma_1 \in \mathcal{W}_\gamma \cap \overline{\mathcal{C}} \) for some \( \gamma \) and \( \sigma_0 \in B_\varepsilon(\sigma_1) \). (See (5).)

First we give the following lemma.

Lemma 5.18. Assume \( Z_1(\alpha) = 0 \). Then we have

\[
J^\alpha(\sigma_0) = J^\alpha(\sigma_1) = 0.
\]
Lemma 5.19. We have the following equality in semistable in \( \sigma \) with respect to the stability functions \( Z \).

Proof. We show the following decomposition,

\[
\mathcal{M}^{(\alpha, \phi)}(\sigma) = \bigotimes_{\alpha_1 + \cdots + \alpha_n = \alpha} \mathcal{M}(\{\alpha_i\}_{i \leq n}, \mathcal{A}_\psi, \sigma_0),
\]

where \( \alpha_i \in C^{\sigma_0}((\phi - \varepsilon, \phi + \varepsilon)) \) and they satisfy (61). First note that any object \( E \in \mathcal{P}_1(\phi) \) of numerical type \( \alpha \), a \( C \)-valued point of the LHS of (62), has the unique filtration

\[
0 = E_0 \subset E_1 \subset \cdots \subset E_n = E,
\]

in \( \mathcal{A}_\psi \) such that \( F_i = E_i/E_{i-1} \) is semistable in \( \sigma_0 \), of numerical type \( \alpha_i \in C^{\sigma_0}((\phi - \varepsilon, \phi + \varepsilon)) \), and they satisfy \( \phi_0(\alpha_1) > \cdots > \phi_0(\alpha_n) \). Since \( 0 < \varepsilon < 1/6 \), we have

\[
|Z_0(F_i)| \leq |Z_0(E)| = |Z_0(\alpha)|,
\]

thus \( F_i \in \mathcal{S} \). Therefore \( F_i \) is semistable for any \( \sigma' \in C \), hence it is also semistable in \( \sigma_1 \). The condition \( \sigma_1 \in C \) implies that \( \phi_1(\alpha_1) \geq \cdots \geq \phi_1(\alpha_n) \). Because \( E \) is semistable in \( \sigma_1 \), we must have \( \phi_1(\alpha_1) = \cdots = \phi_1(\alpha_n) = \phi \). This means \( E \) is a \( C \)-valued point of the RHS of (62).

Conversely take \( \alpha_1, \cdots, \alpha_n \in C^{\sigma_0}((\phi - \varepsilon, \phi + \varepsilon)) \) which satisfy \( \alpha_1 + \cdots + \alpha_n = \alpha \) and (61). Suppose for \( E \in \mathcal{A}_\psi \), there is a filtration

\[
0 = E_0 \subset E_1 \subset \cdots \subset E_n = E,
\]

in \( \mathcal{A}_\psi \) such that \( F_i = E_i/E_{i-1} \) is semistable in \( \sigma_0 \) and of numerical type \( \alpha_i \), i.e. \( E \) is a \( C \)-valued point of the RHS of (62). Again we have \( |Z_0(F_i)| \leq |Z_0(\alpha)| \) thus \( F_i \in \mathcal{S} \). Hence \( F_i \) is also semistable in \( \sigma_1 \), and (61) implies \( E \) is also semistable in \( \sigma_1 \). This implies \( E \) is a \( C \)-valued point of the RHS of (62).
Now we have shown (62) at the level of $\mathbb{C}$-valued points. Finally we have the isomorphism of the stabilizers,

$$\text{Aut}(E_1 \subset \cdots \subset E_n) \cong \text{Aut}(E_n).$$

Hence we have the decomposition (62). Using (61), the formula (60) follows. (Also see the proof of [17, Theorem 5.11].)

Next we compare $\tau^i(\sigma_i) \in A(A_\psi, \Lambda, \chi)$ near a wall. Following [17, Definition 4.2], we introduce the following combinatorial values.

**Definition 5.20.** For $\alpha_1, \ldots, \alpha_n \in C^{\sigma_0}((\phi - \varepsilon, \phi + \varepsilon))$, consider the following two conditions.

(a) $\phi_0(\alpha_i) \leq \phi_0(\alpha_{i+1})$, and $\phi_1(\alpha_1 + \cdots + \alpha_i) > \phi_1(\alpha_{i+1} + \cdots \alpha_n)$.
(b) $\phi_0(\alpha_i) > \phi_0(\alpha_{i+1})$, and $\phi_1(\alpha_1 + \cdots + \alpha_i) \leq \phi_1(\alpha_{i+1} + \cdots \alpha_n)$.

If for all $i = 1, \ldots, n - 1$ one of the above two conditions is satisfied, then define

$$S(\{\alpha_i\}_{1 \leq i \leq n}, \sigma_0, \sigma_1) = (-1)^r,$$

where $r$ is the number of $i = 1, \ldots, n - 1$ satisfying (a). Otherwise define $S(\{\alpha_i\}_{1 \leq i \leq n}, \sigma_0, \sigma_1) = 0$.

The values $S(\{\alpha_i\}_{1 \leq i \leq n}, \sigma_0, \sigma_1)$ give the transformation coefficients of the invariants.

**Lemma 5.21.** We have the following equality in $A(A_\phi, \chi, \Lambda)$,

$$\delta^i(\sigma_1) = \sum_{\alpha_1 + \cdots + \alpha_n = \alpha} S(\{\alpha_i\}_{1 \leq i \leq n}, \sigma_0, \sigma_1) \delta^{\alpha_1}(\sigma_0) \cdots \delta^{\alpha_n}(\sigma_0),$$

where $\alpha_i \in C^{\sigma_0}((\phi - \varepsilon, \phi + \varepsilon))$.

**Proof.** Applying $\Theta$ to (60) and using Proposition 5.16 we have

$$\delta^i(\sigma_1) = \sum_{\alpha_1 + \cdots + \alpha_n = \alpha} \delta^{\alpha_1}(\sigma_0) \cdots \delta^{\alpha_n}(\sigma_0),$$

where $\alpha_1, \ldots, \alpha_n \in C^{\sigma_0}((\phi - \varepsilon, \phi + \varepsilon))$ satisfy (61). Therefore it is enough to check that the right hand sides of (64) and (63) are equal. Suppose that $\alpha_1, \ldots, \alpha_n$ in (63) satisfy

$$S(\{\alpha_i\}_{1 \leq i \leq n}, \sigma_0, \sigma_1) \delta^{\alpha_1}(\sigma_0) \cdots \delta^{\alpha_n}(\sigma_0) \neq 0. \quad (65)$$

Then for each $i$, there exists $E_i \in P_0(\phi_0(\alpha_i))$ which is of numerical type $\alpha_i$. Since $0 < \varepsilon < 1/6$, we have $|Z_0(E_i)| \leq |Z_0(\alpha)|$. Thus $E_i \in S$, and by the construction of walls $W_\chi$ in Proposition 2.9 we have the following:

For $i, j$, if $\phi_0(\alpha_i) \geq \phi_0(\alpha_j)$ then $\phi_1(\alpha_i) \geq \phi_1(\alpha_j)$. \hfill (66)

Thus the coefficient $S(\{\alpha_i\}_{1 \leq i \leq n}, \sigma_0, \sigma_1)$ for which $\alpha_1, \ldots, \alpha_n$ satisfy (65) is calculated by the property (66) in a purely combinatorial way. It is computed in [17, 5.2] and the result is

$$S(\{\alpha_i\}_{1 \leq i \leq n}, \sigma_0, \sigma_1) = \begin{cases} 1 & \text{if } \alpha_1, \ldots, \alpha_n \text{ satisfy } (61), \\ 0 & \text{otherwise}. \end{cases}$$

**Remark 5.22.** Take $\alpha' \in C^{\sigma_1}(\phi)$ with $|Z_1(\alpha')| \leq |Z_1(\alpha)|$. Then the above proof shows that the same formula (63) replaced $\alpha$ by $\alpha'$ also holds.
Now we can compare $J^\alpha(\sigma_0)$ and $J^\alpha(\sigma_1)$.

**Proposition 5.23.** We have $F^i(\sigma_0) = F^i(\sigma_1)$ in $A(\mathcal{A}_\psi, \chi, \Lambda)$. Thus we have $J^\alpha(\sigma_0) = J^\alpha(\sigma_1)$ in $\Lambda$.

The proof relies on the combinatorial result in [17]. So before beginning the proof, we show the simplest case for the reader’s convenience. Suppose the following decomposition is unique.

$$\alpha = \alpha_1 + \alpha_2, \quad \alpha_j \in C^{\sigma_0}(\phi - \varepsilon, \phi + \varepsilon).$$

Furthermore for such $\alpha_j$, we assume $\phi_1(\alpha_1) = \phi_1(\alpha_2)$, $\phi_0(\alpha_1) > \phi_0(\alpha_2)$. In this case we have

$$\begin{align*}
F^i(\sigma_1) &= F^i(\sigma_1) - \frac{1}{2} F^{\sigma_1}(\sigma_1) * F^{\sigma_2}(\sigma_1) - \frac{1}{2} F^{\sigma_2}(\sigma_1) * F^{\sigma_1}(\sigma_1), \\
F^i(\sigma_1) &= F^i(\sigma_0) + F^{\sigma_1}(\sigma_0) * F^{\sigma_2}(\sigma_0).
\end{align*}$$

Also we have $F^i(\sigma_0) = F^i(\sigma_0)$, and $F^{\sigma_1}(\sigma_1) = F^{\sigma_1}(\sigma_1)$. Thus we have

$$\begin{align*}
F^i(\sigma_1) &= F^i(\sigma_0) + \frac{1}{2} [F^{\sigma_1}(\sigma_0), F^{\sigma_2}(\sigma_0)], \\
&= F^i(\sigma_0).
\end{align*}$$

Here we used the fact that $A(\mathcal{A}_\psi, \chi, \Lambda)$ is commutative in our case. Now we give the proof of Proposition 5.24.

**Proof.** Using the proof of [16] Theorem 7.7 in the algebra $A(\mathcal{A}_\psi, \Lambda, \chi)$ (also see [17] Definition 6.22), we may write $F^i(\sigma_0)$ as follows,

$$F^i(\sigma_0) = \sum_{\alpha_1 + \cdots + \alpha_n = \alpha} \frac{1}{n!} F^{\alpha_1}(\sigma_0) * \cdots * F^{\alpha_n}(\sigma_0),$$

where $\alpha_i \in C^{\sigma_0}(\phi_0(\alpha))$. We can rewrite (67) as the same formula of [67] and $\alpha_i \in C^{\sigma_0}(\phi - \varepsilon, \phi + \varepsilon)$ with $\phi_0(\alpha_i) = \phi_0(\alpha_i)$. Then substituting (63), (67) to the definition of $F^i(\sigma_1)$ in (56), (also noting Remark 5.22) we can write $F^i(\sigma_1)$ in the following formula,

$$F^i(\sigma_1) = \sum_{\alpha_1 + \cdots + \alpha_n = \alpha} U(\{\alpha_i\}_{1 \leq i \leq n}, \sigma_0, \sigma_1) F^{\alpha_1}(\sigma_0) * \cdots * F^{\alpha_n}(\sigma_0),$$

where $\alpha_i \in C^{\sigma_0}(\phi - \varepsilon, \phi + \varepsilon)$. Here one can consult the explicit description of

$$U(\{\alpha_i\}_{1 \leq i \leq n}, \sigma_0, \sigma_1) \in Q,$$

in [17] Definition 4.4, after replacing $C(\mathcal{A})$ in loc.cite. by $C^{\sigma_0}(\phi - \varepsilon, \phi + \varepsilon)$. In [17] Theorem 5.2, it is proved that using [17] Theorem 5.4 the formula (68) is written as

$$F^i(\sigma_1) = F^i(\sigma_0) + \{\text{multiple commutators of } F^{\alpha_i}(\sigma_0)\}.$$ 

In our case $A(\mathcal{A}_\psi, \Lambda, \chi)$ is commutative since we assume $X$ is a K3 surface or an abelian surface. Hence we have $F^i(\sigma_1) = F^i(\sigma_0)$.

Now we can show the following.

**Theorem 5.24.** For $\sigma = (Z, \mathcal{P}) \in \text{Stab}^*(X)$ and $\alpha \in \mathcal{N}(X)$, the invariant $J^\alpha(\sigma) \in \Lambda$ does not depend on a choice of $\sigma$. 

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Proof. Take $\sigma = (Z, P) \in \text{Stab}^*(X)$ and $\tau = (W, Q) \in \text{Stab}^*(X)$. Let $\lambda$ be a path

$$\lambda: [0, 1] \rightarrow \text{Stab}^*(X),$$

such that $\lambda(0) = \sigma$ and $\lambda(1) = \tau$. We take a connected open set $\mathcal{B}^o \subset \text{Stab}^*(X)$ which contains $\lambda([0, 1])$ and its closure $\mathcal{B} = \overline{\mathcal{B}^o}$ is compact. We consider the set of objects $S$ as in (59) and the associated walls $\{W_\gamma\}_{\gamma \in \Gamma}$. We denote $\lambda(t) = \sigma_t = (Z_t, P_t)$. We may assume that the set of points $K \subset [0, 1]$ on which $\sigma_t$ is algebraic and $P_t((\psi - 1, \psi))$ satisfies the generic flatness for any $\psi \in \mathcal{I}$ is dense in $[0, 1]$. Take $s_0, s_1, s_2, \cdots, s_N, s_{N+1} \in [0, 1]$ and $t^\pm_i \in (s_i, s_{i+1}) \cap K$ such that

- For $1 \leq i \leq N$, $s_i \in W_\gamma$ for some $W_\gamma$, and $s_0 = 0$, $s_{N+1} = 1$.
- For any $t \in (s_i, s_{i+1})$, we have $\lambda(t) \notin W_\gamma$ for any $\gamma$.
- $\sigma_{t^+_i} \in B_\varepsilon(\sigma_{s_{i+1}})$, $\sigma_{t^-_i} \in B_\varepsilon(\sigma_{s_i})$ with $0 < \varepsilon < 1/6$.

By Proposition 5.17 and Proposition 5.23, we have

$$J^\alpha(\sigma_{s_i}) = J^\alpha(\sigma_{t^+_i}) = J^\alpha(\sigma_{t^-_i}) = J^\alpha(\sigma_{s_{i+1}}),$$

for each $i$. Thus $J^\alpha(\sigma) = J^\alpha(\tau)$ follows.

By Theorem 5.24, the following definition is well defined.

**Definition 5.25.** For $\alpha \in N(X)$, we define $J^\alpha \in \Lambda$ to be $J^\alpha(\sigma)$ for some $\sigma \in \text{Stab}^*(X)$.

Let $\text{Auteq}^* D(X)$ be the subgroup of $\Phi \in \text{Auteq} D(X)$ which preserves the connected component $\text{Stab}^*(X)$. Also for $\Phi \in \text{Auteq}^* D(X)$, we denote

$$\Phi_*: N(X) \rightarrow N(X),$$

the induced automorphism. We have the following corollary of Theorem 5.24.

**Corollary 5.26.** For $\Phi \in \text{Auteq}^* D(X)$, one has $J^\alpha = J^{\Phi_* \alpha}$.

**Proof.** We have

$$J^\alpha = J^\alpha(\sigma) = J^{\Phi_* \alpha}(\Phi(\sigma)) = J^{\Phi_* \alpha}.$$

\[\square\]

6 Comparison of invariants which count semistable objects and semistable sheaves

In this section we compare $J^\alpha$ and $\hat{J}^\alpha$, where $\hat{J}^\alpha$ is a counting invariant of semistable sheaves introduced in [17].
6.1 Counting invariants of semistable sheaves

Let \( \Lambda \) be a \( \mathbb{Q} \)-algebra and \( \Upsilon: K(\text{Var}) \to \Lambda \) be a motivic invariant as in [17]. We denote by \( C(X) = \text{im}(\text{Coh}(X) \to \mathcal{N}(X)) \).

For \( \alpha \in C(X) \), we recall the definition of \( \hat{J}^\alpha \in \Lambda \) introduced in [17]. Let \( \omega \) be an ample divisor on \( X \). We consider the moduli stack, \( \hat{\mathcal{M}}(\omega) \subset \mathcal{M} \), which is the stack of \( \omega \)-Gieseker semistable sheaves of numerical type \( \alpha \). Let \( \hat{\delta}^\alpha(\omega) \in \text{SF}(\mathcal{M}) \) be the associated stack function. We consider the map \( \Upsilon' \circ \Pi^*: \text{SF}(\mathcal{M}) \to \Lambda \) as in (54).

**Definition 6.1.** [17, Definition 6.1, Definition 6.22] We define \( \hat{I}^\alpha(\omega) \in \Lambda \) to be

\[
\hat{I}^\alpha(\omega) = \Upsilon' \circ \Pi^* \hat{\delta}^\alpha(\omega) \in \Lambda,
\]

and \( \hat{J}^\alpha(\omega) \in \Lambda \) to be

\[
\hat{J}^\alpha(\omega) = \sum_{\alpha_1 + \cdots + \alpha_n = \alpha} (-1)^{n-1} \frac{(l-1)}{n} \prod_{i=1}^n \hat{I}^\alpha_i(\omega) \in \Lambda,
\]

where \( \alpha_i \in C(X) \) satisfies \( P(\alpha_i, \omega, n) = P(\alpha, \omega, n) \).

Joyce [17] showed the following.

**Theorem 6.2.** [17, Theorem 6.24] The invariant \( \hat{J}^\alpha(\omega) \in \Lambda \) does not depend on a choice of an ample divisor \( \omega \).

For \( \alpha \in C(X) \), we define \( \hat{J}^\alpha(\omega) \) to be \( \hat{J}^\alpha(\omega) \) for some ample divisor \( \omega \), which is well defined by Theorem 6.2.

6.2 Comparison of \( J^\alpha \) and \( \hat{J}^\alpha \)

Here we compare \( J^\alpha \) and \( \hat{J}^\alpha \) for \( \alpha \in C(X) \). Let us take an ample divisor \( \omega \) and \( k \in \mathbb{Q}_{\geq 1} \). We use the following notation,

\[
Z_{k\omega} = Z_{(0,k\omega)}, \quad \mathcal{A}_\omega = \mathcal{A}_{(0,\omega)} = \mathcal{A}_{(0,k\omega)}, \quad \sigma_k = (Z_{k\omega}, \mathcal{A}_\omega).
\]

The idea is to compare the following two values,

\[
J^\alpha(\sigma_k) \in \Lambda, \quad \hat{J}^\alpha(\omega) \in \Lambda,
\]

in the limit \( k \to \infty \). In [6, Proposition 14.2], Bridgeland proved that (putting a certain assumption on a numerical class), an object \( E \in D(X) \) is semistable in \( \sigma_k \) for all \( k \gg 0 \) if and only if \( E \) is \( \omega \)-Gieseker semistable. This is what string theory predicts that BPS branes in the limit \( k \to \infty \) are in fact Gieseker stable sheaves. What we actually have to prove is that we can choose \( k > 0 \) uniformly so that it works for any semistable objects. First we give the following lemma.

**Lemma 6.3.** For an ample line bundle \( \mathcal{L} \in \text{Pic}(X) \), one has

\[
J^\alpha = J^\alpha \otimes \mathcal{L}, \quad \hat{J}^\alpha = \hat{J}^\alpha \otimes \mathcal{L}.
\]
Proof. Note that tensoring $\mathcal{L}$ gives an autoequivalence $\otimes \mathcal{L} \in \text{Auteq}^* D(X)$. Thus $J^\alpha = J^{\alpha \otimes \mathcal{L}}$ follows from Corollary \[5.26\]. Next let $\omega = c_1(\mathcal{L})$. Then by Theorem \[6.2\] we have $\hat{J}^\alpha = J^\alpha(\omega)$. The equality $\hat{J}^\alpha(\omega) = J^{\alpha \otimes \mathcal{L}}(\omega)$ follows easily from the fact that for any $\omega$-Gieseker semistable sheaf $E$ of numerical type $\alpha$, $E \otimes \mathcal{L}$ is also $\omega$-Gieseker semistable and it is of numerical type $\alpha \otimes \mathcal{L}$.

For $\alpha \in C(X)$ we denote $v(\alpha) = (r, l, s)$. We show the following proposition.

**Proposition 6.4.** Suppose $\omega \cdot l > 0$ or $r = l = 0$, and choose $0 < \phi_k \leq 1$ which satisfies $Z_{\omega}(\alpha) \in \mathbb{R}_{>0} e^{i\pi \phi_k}$. Then there exists $N > 0$ such that for all $k \geq N$ and $\alpha'$ which satisfies

$$\alpha' \in C^{\alpha_k}(\phi_k) \text{ with } |\text{Im} Z_\omega(\alpha')| \leq |\text{Im} Z_\omega(\alpha)|,$$

any $E \in M(\alpha', \phi_k)(\nu_k)$ is a $\omega$-Gieseker semistable sheaf.

**Proof.**

**Step 1.**

First by Lemma \[4.6\] (i), (ii), the set of $\alpha' \in N(X)$ which satisfies \[71\] is a finite set for a fixed $\alpha$. When $r = l = 0$, any object $E \in A_{\omega}$ of numerical type $\alpha$ is a zero dimensional sheaf, so the result is obvious. Thus we may assume $\omega \cdot l > 0$. In this case $\phi_k$ goes to zero for $k \to \infty$ when $r > 0$ and goes to $1/2$ when $r = 0$. Thus there is $N > 0$ so that $\phi_k \leq 3/4$ for all $k \geq N$. Take $E \in M(\alpha', \phi_k)(\nu_k)$, and $\alpha'$ satisfies \[71\]. Then we have

$$\phi_k(H^{-1}(E)[1]) \leq \phi_k \leq \frac{3}{4}.$$

Thus the map on $\bigcup_{k \geq N, \alpha'} M(\alpha', \phi_k)(\nu_k)$,

$$E \mapsto \frac{\text{Re} Z_{\omega}(H^{-1}(E)[1])}{\text{Im} Z_{\omega}(H^{-1}(E)[1])} = \frac{1}{k} \frac{\text{Re} Z_{\omega}(H^{-1}(E)[1])}{\text{Im} Z_{\omega}(H^{-1}(E)[1])},$$

is bounded below. Note that $E$ is contained in $M^\alpha(0, \omega)$, and the map $E \mapsto \text{Im} Z_\omega(E)$ on $M^\alpha(0, \omega)$ is bounded by Lemma \[4.8\]. Therefore the map on $\bigcup_{k \geq N, \alpha'} M(\alpha', \phi_k)(\nu_k)$,

$$E \mapsto \frac{1}{k} \text{Re} Z_{\omega}(H^{-1}(E)[1]),$$

is bounded below. Thus using Lemma \[4.9\] we may assume that

$$E \mapsto \text{Re} Z_{\omega}(H^{-1}(E)[1]),$$

is bounded below on $\bigcup_{k \geq N, \alpha'} M(\alpha', \phi_k)(\nu_k)$. Then one can apply Lemma \[4.10\] and the set

$$\{v(H^{-1}(E)[1]) \in \text{NS}^*(X) \mid E \in \bigcup_{k \geq N, \alpha'} M(\alpha', \phi_k)(\nu_k)\},$$

is a finite set. Let us denote the above set $v_1, \cdots, v_n$. Then $\lim_{k \to \infty} \phi_k(v_i) = 1$, so by replacing $N$ if necessary, we have $\phi_k(v_i) > 3/4$ for any $k \geq N$ and $1 \leq i \leq n$. This implies that for $k \geq N$ any object $E \in M(\alpha', \phi_k)(\nu_k)$ satisfies $H^{-1}(E) = 0$, so $E$ is a sheaf.
Step 2.

Next we show that any $E \in \cup_{k \geq N, \alpha'} M^{(\alpha', \phi_k)}(\sigma_k)$ is a $\omega$-Gieseker semistable sheaf, by replacing $N$ if necessary. Assume that $E$ is not $\omega$-Gieseker semistable, and let $T$ be the $\omega$-Gieseker semistable factor of $E$ of the smallest reduced Hilbert polynomial. We denote

$$v(E) = (r', l', s'), \quad v(T) = (r'', l'', s'').$$

Note that $r' = 0$ is equivalent to $r'' = 0$, and in this case $E$ must be $\omega$-semistable by Lemma 4.6 (iii). Thus we may assume $r' > 0$, $r'' > 0$. Note that in this case $\phi_k$ goes to 0 for $k \to \infty$. Since the map $E \to T$ is a surjection in $A_\omega$, and $E$ is $\sigma_k$-semistable for some $k \geq N$, one has $\phi_k(E) \leq \phi_k(T)$. Thus we have

$$\frac{\text{Re} Z_{k\omega}(E)}{\text{Im} Z_{k\omega}(E)} = \frac{\text{Re} Z_{k\omega}(T)}{\text{Im} Z_{k\omega}(T)}. \tag{72}$$

Explicitly (72) is equivalent to

$$\frac{\omega \cdot l'' - l'}{\omega \cdot l'} \left( -s' + \frac{1}{2}k^2 \omega^2 r' \right) \geq -s'' + \frac{1}{2}k^2 \omega^2 r''. \tag{73}$$

Also we note that

$$0 < r'' \leq r', \quad 0 < \omega \cdot l'' \leq \omega \cdot l', \quad l''^2 - 2r'' s'' \geq -2. \tag{74}$$

Here the third equality comes from Lemma 4.6 (i). Then (73) and (74) imply that the set

$$\{ v(T) \in \text{NS}^*(X) \mid E \in \cup_{k \geq N, \alpha'} M^{(\alpha', \phi_k)}(\sigma_k) \}, \tag{75}$$

is a finite set. Applying the same argument for other torsion free Gieseker-semistable factors, we deduce that the set

$$\{ v(E_{\text{tot}}) \in \text{NS}^*(X) \mid E \in \cup_{k \geq N, \alpha'} M^{(\alpha', \phi_k)}(\sigma_k) \},$$

is a finite set. It follows that the set

$$\{ v(E_{\text{tor}}) \in \text{NS}^*(X) \mid E \in \cup_{k \geq N, \alpha'} M^{(\alpha', \phi_k)}(\sigma_k) \},$$

is also a finite set, say $v'_1, \ldots, v'_m$. Since $\phi_k(v'_i)$ goes to 1/2 for each $i$, we have $\phi_k(v'_i) > \phi_k$ for all $1 \leq i \leq m$ and $k \geq N$, after replacing $N$ if necessary. Thus for such $N$ and $k \geq N$, if we take $E \in M^{(\alpha', \phi_k)}(\sigma_k)$, then $E$ must be a torsion free sheaf. By the definition of $T$, one has

$$\mu_\omega(E) > \mu_\omega(T) \quad \text{or} \quad \mu_\omega(E) = \mu_\omega(T), \quad \frac{s'}{r'} > \frac{s''}{r''}.$$

We have

$$\frac{Z_{k\omega}(E)}{r'} - \frac{Z_{k\omega}(T)}{r''} = -\left( \frac{s'}{r'} - \frac{s''}{r''} \right) + ik(\mu_\omega(E) - \mu_\omega(T)).$$

So after replacing $N$ we have $\phi_k(E) > \phi_k(T)$ for $k \geq N$. Such $N$ is determined by only a numerical class of $T$. Thus the finiteness of (75) implies that one can take $N$ uniformly so that $\phi_k(E) > \phi_k(T)$ for all $E \in M^{(\alpha', \phi_k)}(\sigma_k)$ and $k \geq N$. This contradicts that $E$ is $\sigma_k$-semistable, so $E$ must be $\omega$-Gieseker semistable.

Next we check the following.
Lemma 6.5. Suppose \( \omega \cdot l > 0 \) or \( r = l = 0 \). Then there is \( N > 0 \) so that for \( k \geq N \) and \( \alpha' \in C(X) \) which satisfies
\[
P(\alpha', \omega, n) = P(\alpha, \omega, n), \quad \text{Im} \, Z_\omega(\alpha') \leq \text{Im} \, Z_\omega(\alpha),
\]
any \( \omega \)-Gieseker semistable sheaf \( E \) of numerical type \( \alpha' \) is \( \sigma_k \)-semistable.

\textbf{Proof.} First using Lemma 4.6 (i), (ii), the set of \( \alpha' \in C(X) \) which satisfies (76) is finite for a fixed \( \alpha \). Thus we may assume \( \alpha' = \alpha \). Note that the case of \( r = l = 0 \) is obvious. The case of \( r > 0, \omega \cdot l > 0 \) is proved in [6, Proposition 14.2]. One can also check that in the proof of \textit{loc.cite.}, the desired \( N > 0 \) is taken to be uniformly for any \( \omega \)-Gieseker semistable sheaf \( E \) of numerical type \( \alpha \). (We leave the readers to check the detail. It is enough to notice in [6, Lemma 14.3] that the set of \( \omega \)-Gieseker semistable sheaves of numerical type \( \alpha \) is bounded.) Thus it is enough to check the case of \( r = 0 \) and \( l \neq 0 \). Let \( E \) be a \( \omega \)-Gieseker semistable sheaf with \( v(E) = (0, l, s) \). Since \( \phi_k \) goes to 1/2 for \( k \to \infty \), we may assume \( 1/4 < \phi_k < 3/4 \). For each \( k \), let \( E_k \in A_\omega \) be the \( \sigma_k \)-semistable factor of \( E \) whose phase is the largest. If \( E \) is not semistable in \( \sigma_k \), we have
\[
\phi_k(E_k) > \phi_k > \frac{1}{4}.
\]
We have the exact sequence in \( A_\omega \),
\[
0 \to E_k \to E \to E'_k \to 0
\]
Then the associated long exact sequence of (77) with respect to the standard t-structure implies that \( E_k \) is a sheaf. We have the sequence,
\[
0 \to (E_k)_{\text{tor}} \to E_k \to (E_k)_{\text{fr}} \to 0,
\]
which is exact in both \( A_\omega \) and \( \text{Coh}(X) \). Combining sequences (78) and (79), we obtain the exact sequence in \( A_\omega \),
\[
0 \to (E_k)_{\text{tor}} \to E \to F \to 0,
\]
Again the long exact sequence associated to (80) implies that (80) is also exact in \( \text{Coh}(X) \). Because \( E \) is \( \omega \)-Gieseker semistable, we have \( P((E_k)_{\text{tor}}, \omega, n) \leq P(E, \omega, n) \). Thus we have
\[
\phi_k((E_k)_{\text{tor}}) \leq \phi_k \leq 3/4,
\]
by Lemma 4.6 (iii). Then the sequence (79) and (77) imply the map
\[
k \mapsto \frac{\text{Re} \, Z_{k\omega}((E_k)_{\text{fr}})}{\text{Im} \, Z_{k\omega}((E_k)_{\text{fr}})},
\]
is bounded above. Then applying Lemma 4.8 and Lemma 4.9 there is \( N > 0 \) such that the map \( k \mapsto \text{Re} \, Z_{\omega}((E_k)_{\text{fr}}) \) is bounded above for \( k \geq N \). Hence by Lemma 4.10, the set
\[
\{v((E_k)_{\text{fr}}) \in \text{NS}^+(X) \mid k \in \mathbb{Q}_\geq N\},
\]
is a finite set. Thus we have
\[
\phi_k((E_k)_{\text{fr}}) \to 0,
\]
for \( k \to \infty \). However since we have (79) and (81), (82) implies that \( \phi_k(E_k) < 1/4 \) for \( k \geq N \) by replacing \( N \) if necessary. This contradicts to (77), thus for such \( N \) and \( k \geq N \), \( E \) must be \( \sigma_k \)-semistable. The above proof also shows that one can take \( N \) uniformly for all \( \omega \)-Gieseker semistable sheaf \( E \) of numerical type \( \alpha \).
Finally we show the following.

**Theorem 6.6.** For \( \alpha \in C(X) \), we have \( J^\alpha = \tilde{J}^\alpha \).

**Proof.** Since \( v(\alpha \otimes \mathcal{L}) = v(\alpha) \cdot \text{ch}(\mathcal{L}) \) for \( \mathcal{L} \in \text{Pic}(X) \), Lemma 6.3 implies that we may assume \( v(\alpha) = (r, l, s) \) with \( \omega \cdot l > 0 \) or \( r = l = 0 \). It is enough to compare \( J^\alpha(\sigma_k) \) and \( J^\alpha(\omega) \) for \( k \geq N \), where \( N \) is chosen as in Proposition 6.4 and Lemma 6.5. Take \( \alpha_1, \cdots, \alpha_n \in C^\sigma_k(\phi_k) \) such that \( \alpha_1 + \cdots + \alpha_n = \alpha \) and \( \prod_{i=1}^n I^{\alpha_i}(\sigma_k) \neq 0 \). Then first applying Proposition 6.4 we have

\[
\alpha_i \in C(X), \quad \mathcal{M}^{(\alpha_i, \phi_k)}(\sigma_k) \subset \tilde{\mathcal{M}}^{\alpha_i}(\omega).
\]

For a fixed \( k \geq N \), let \( \sigma_k \in \mathfrak{B}^\circ \) be an open neighborhood of \( \sigma_k \) such that its closure \( \mathfrak{B} \) is compact. Then there is a wall and chamber structure \( \{W_\gamma\}_{\gamma \in \Gamma} \) on \( \mathfrak{B} \) with respect to \( (\ref{eq:wall-chamber}) \). There is a subset \( \Gamma' \subset \Gamma \) and a connected component \( C \) as in \( (\ref{eq:wall-chamber}) \) such that infinitely many \( \sigma_k \) for \( k' \geq \mathbb{Q} \geq N \) are contained in \( C \). We may assume \( \sigma_k \in C \). Then if \( \alpha_i \) and \( \alpha_j \) are not proportional in \( \mathcal{N}(X) \), we have

\[
\text{Im} \frac{Z_{k'}^{\omega}(\alpha_j)}{Z_{k'}^{\omega}(\alpha_i)} = 0,
\]

for infinitely many \( k' \in \mathbb{Q} \geq N \). By Lemma 4.0 (iv), this implies

\[
P(\alpha_i, \omega, n) = P(\alpha_j, \omega, n) = P(\alpha, \omega, n),
\]

for any \( i, j \). Then one can apply Lemma 6.5 and conclude

\[
\mathcal{M}^{(\alpha_i, \phi_k)}(\sigma_k) = \tilde{\mathcal{M}}^{\alpha_i}(\omega).
\]

Hence we have \( \prod_{i=1}^n I^{\alpha_i}(\sigma_k) = \prod_{i=1}^n \tilde{I}^{\alpha_i}(\omega) \).

Conversely take \( \alpha_1, \cdots, \alpha_n \in C(X) \) such that \( \prod_{i=1}^n I^{\alpha_i}(\omega) \neq 0 \) and \( \alpha_1 + \cdots + \alpha_n = \alpha \), \( P(\alpha_i, \omega, n) = P(\alpha, \omega, n) \). Again \( (\ref{eq:sum}) \) holds for \( k \geq N \) by Proposition 6.4 and Lemma 6.5 so \( \prod_{i=1}^n I^{\alpha_i}(\sigma_k) = \prod_{i=1}^n \tilde{I}^{\alpha_i}(\omega) \) holds. Also \( P(\alpha_i, \omega, n) = P(\alpha, \omega, n) \) implies \( \alpha_i \in C^\sigma_k(\phi_k) \). Thus the sum \( (\ref{eq:sum}) \) and \( (\ref{eq:sum}) \) are equal.

\[\square\]

**Remark 6.7.** In this paper, we do not give the explicit computation of the invariant \( J^\alpha(\sigma) \). However if \( \alpha \in C(X) \) is primitive, then by the work of Yoshioka \( (\ref{ref-yoshioka}) \), \( J^\alpha(\sigma) \) can be computed by the invariant of the Hilbert scheme of points on \( X \). As commented in \( (\ref{ref-yoshioka}) \), it might be possible to compute the invariant for other \( \alpha \in \mathcal{N}(X) \) using this remark and Theorem 1.2.

**References**

[1] D. Abramovich and A. Polishchuk. Sheaves of t-structures and valuative criteria for stable complexes. *J.reine.angew.Math*, Vol. 590, pp. 89–130, 2006.

[2] A. Bergman. Stability conditions and Branes at Singularities. *preprint*. math.AG/0702092.

[3] R. Borcherds. Automorphic forms on \( O_{s+2, s}(\mathbb{R}) \) and infinite products. *Invent.math*, Vol. 120, pp. 161–213, 1995.

[4] T. Bridgendal. Spaces of stability conditions. *preprint*. math.AG/0611510.

[5] T. Bridgendal. Stability conditions and Kleinian singularities. *preprint*. math.AG/0508257.
[6] T. Bridgeland. Stability conditions on K3 surfaces. preprint. math.AG/0307164.

[7] T. Bridgeland. Stability conditions on triangulated categories. Ann of Math (to appear). math.AG/0212237.

[8] T. Bridgeland. Derived categories of coherent sheaves. Proceedings of the 2006 ICM, 2006. math.AG/0602129.

[9] T. Bridgeland. Stability conditions on a non-compact Calabi-Yau threefold. Comm. Math. Phys, Vol. 266, pp. 715–733, 2006.

[10] M. Douglas. D-branes, categories and N = 1 supersymmetry. J.Math.Phys, Vol. 42, pp. 2818–2843, 2001.

[11] M. Douglas. Dirichlet branes, homological mirror symmetry, and stability. Proceedings of the 1998 ICM, pp. 395–408, 2002. math.AG/0207021.

[12] D. Huybrechts and M.Lehn. Geometry of moduli spaces of sheaves, Vol. E31 of Aspects in Mathematics. Vieweg, 1997.

[13] M. Inaba. Moduli of stable objects in a triangulated category. preprint. math.AG/0612078.

[14] M. Inaba. Toward a definition of moduli of complexes of coherent sheaves on a projective scheme. J.Math.Kyoto Univ., Vol. 42-2, pp. 317–329, 2002.

[15] A. Ishii, K.Ueda, and H.Uehara. Stability Conditions on $A_n$-Singularities. math.AG/0609551.

[16] D. Joyce. Configurations in abelian categories III. Stability conditions and identities. preprint. math.AG/0410267.

[17] D. Joyce. Configurations in abelian categories IV. Invariants and changing stability conditions. preprint. math.AG/0410268.

[18] D. Joyce. Motivic invariants of Artin stacks and ‘stack functions’. preprint. math.AG/0509722.

[19] D. Joyce. Configurations in abelian categories I. Basic properties and moduli stack. Advances in Math, Vol. 203, pp. 194–255, 2006.

[20] D. Joyce. Configurations in abelian categories II. Ringel-Hall algebras. Advances in Math, Vol. 210, pp. 635–706, 2007.

[21] D. Joyce. Holomorphic generating functions for invariants counting coherent sheaves on Calabi-Yau 3-folds. Geometry and Topology, Vol. 11, pp. 667–725, 2007.

[22] G. Laumon and L. Moret-Bailly. Champs algébriques, Vol. 39 of Ergebnisse der Mathematik und ihrer Grenzgebiete. Springer Verlag, Berlin, 2000.

[23] M. Lieblich. Moduli of complexes on a proper morphism. J.Algebraic Geom, Vol. 15, pp. 175–206, 2006.

[24] E. Macri. Some examples of moduli spaces of stability conditions on derived categories. preprint. math.AG/0411613.
[25] K. Matsuki and R. Wentworth. Mumford-Thaddeus principle on the moduli space of vector bundles on an algebraic surface. *Internat. J. Math*, Vol. 8, pp. 97–148, 1997.

[26] S. Okada. Stability manifold of $\mathbb{P}^1$. *J. Algebraic Geom*, Vol. 15, pp. 487–505, 2006.

[27] R. Thomas. Stability conditions and the braid groups. *Comm. Anal. Geom*, Vol. 14, pp. 135–161, 2006.

[28] Y. Toda. Stability conditions and Calabi-Yau fibrations. *preprint*. math.AG/0068495.

[29] Y. Toda. Stability conditions and crepant small resolutions. *Trans. Amer. Math. Soc (to appear)*. math.AG/0512648.

[30] K. Yoshioka. Moduli spaces of stable sheaves on abelian surfaces. *Math. Ann*, Vol. 321, pp. 817–884, 2001.

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