Supersymmetric quantum mechanics with reflections

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Abstract

We consider a realization of supersymmetric quantum mechanics where supercharges are differential-difference operators with reflections. A supersymmetric system with an extended Scarf I potential is presented and analyzed. Its eigenfunctions are given in terms of little $-1$ Jacobi polynomials which obey an eigenvalue equation of Dunkl type and arise as a $q\to\mathbb{R}1$ limit of the little $q$-Jacobi polynomials. Intertwining operators connecting the wavefunctions of extended Scarf I potentials with different parameters are presented.

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1. Introduction

Since its introduction by Witten [1] 30 years ago, supersymmetric quantum mechanics has been widely developed and has found numerous applications, both physical and mathematical. Among the many reviews and books published on this topic, the reader might consult [2] for background relevant to this paper.

We consider here a realization of supersymmetric quantum mechanics that relies on the use of reflection operators [3, 4]. Unlike the most standard approach it does not (necessarily) involve the presence of ‘spin-like’ degrees of freedom and associated finite dimensional vector spaces, it implies however the presence of reflection operators in the Hamiltonians. We shall consider the simplest context of one-dimensional quantum mechanics.

Hamiltonians with reflection operators have most notably arisen in the context of quantum many-body integrable systems of Calogero–Sutherland type [5, 6] and their generalizations with internal degrees of freedom [7]. In these models, the constants of motion (and hence the Hamiltonians) are best expressed, and shown to be in involution, using exchange operators [8–12] known in the mathematical literature as Dunkl operators [13]. These are differential-difference operators that involve reflections. Such operators are needed to describe parabosonic
oscillators [14–16]. Associated are deformed Heisenberg algebras that were used in [3, 4] to
design a supersymmetry without fermions of the kind that will be of interest here. The
exchange formalism has also proved instrumental in demonstrating [17] the superintegrability
of certain models in the plane [18]. Recently, symmetry algebras with reflection operators
as elements have been examined, in the framework of a finite oscillator model [19] or as
the \( q \to -1 \) limit of the quantum algebra \( sl_q(2) \) [20]. Also, in the design of spin chains for
quantum information transport, the property of mirror symmetry [21] that is required for perfect
transmission brings in equations involving reflection operators. While these studies provide
many reasons to examine Hamiltonians with reflection operators, there is also intrinsic merit
in the identification of exactly solvable quantum mechanical problems where supersymmetry
manifests itself.

In mathematics, the Dunkl operators are central to the theory of multivariate orthogonal
polynomials [22] and there is currently much activity in the area of Dunkl harmonic analysis
[23]. Recently, two of us have authored and co-authored a series of papers [24–27] showing
that the set of classical orthogonal polynomials in one variable can be significantly enlarged
by studying polynomial eigenfunctions of first-order differential operators of Dunkl type. The
simplest of these heretofore ‘missing’ classical orthogonal polynomials are called little \(-1\)
Jacobi polynomials and will intervene below. As their name indicates, they can be obtained
[24] as a \( q \to -1 \) limit of the little \( q \)-Jacobi polynomials [28].

The outline of this paper is as follows. In section 2 we shall indicate in general terms
how supersymmetric Hamiltonians can be derived from Hermitian supercharges involving the
reflection operator. The difference with the standard approach will be pointed out. We shall
examine in section 3 the very simple case of a supersymmetric oscillator Hamiltonian with
reflection. A more elaborate example will be provided in section 4, where an extension of the
Scarf I potential [29] will be introduced and studied. The eigenfunctions associated to this
extended potential will be given in terms of little \(-1\) Jacobi polynomials. The normalization is
determined in appendix A. Furthermore, intertwining operators connecting the wavefunctions
of the supersymmetric Scarf I potentials with different parameters will be presented. A brief
conclusion will follow. In appendix B, we provide examples of one-dimensional quantum
Hamiltonians with reflection operator. They are not supersymmetric but their wavefunctions
involve the generalized Gegenbauer polynomials [30, 31] that share with the little \(-1\) Jacobi
polynomials the Dunkl-classical property [24, 32].

2. Supersymmetric quantum mechanics with Dunkl supercharges

To facilitate the comparison between the usual supersymmetric quantum mechanics and the one
with reflections, let us first recall the basics of the standard approach. Let \( H \) be a Hamiltonian;
it is said to be supersymmetric if there are supercharges \( Q, \tilde{Q} \) such that the superalgebra
relations

\[ H = \{ Q, \tilde{Q} \}, \quad [Q, H] = 0, \quad [\tilde{Q}, H] = 0 \]  \(2.1\)

are realized. As usual, \( \{ A, B \} = AB + BA, [A, B] = AB - BA \). In the most simple setting of
one-dimensional quantum mechanics, this is achieved by taking

\[ Q = \frac{1}{\sqrt{2}} (p - i W) b, \]  \(2.2\)

where \( p = -i d/dx, W = W(x) \) is the superpotential and \( b, b^\dagger \) are fermionic annihilation
and creation operators satisfying

\[ b^2 = (b^\dagger)^2 = 0, \quad \{ b, b^\dagger \} = 1, \]  \(2.3\)
and represented by the $2 \times 2$ matrices:

$$ b = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad b^\dagger = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}. \quad (2.4) $$

Upon calculating $\{Q, Q^\dagger\}$ with $Q$ given by (2.2), we readily find

$$ H = \{Q, Q^\dagger\} = \frac{1}{2} (p^2 + W^2) + \frac{1}{2} \frac{dW}{dx} \sigma_3, \quad (2.5) $$

where

$$ \sigma_3 = [b, b^\dagger] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \quad (2.6) $$

Equation (2.5) gives the form of a supersymmetric Hamiltonian in one dimension that has two supercharges $Q$ and $Q^\dagger$. One speaks of $N = 1$ supersymmetry. Note that there are systems with only one Hermitian supercharge $Q = Q^\dagger$ such that $H = Q^2$. The Pauli Hamiltonian in the presence of a magnetic monopole is one such system [33]. One speaks of $N = \frac{1}{2}$ supersymmetry. In the following, we shall consider mostly such $N = \frac{1}{2}$ (or chiral) supersymmetric problems.

For reference, let us record the specific form of $H$ when

$$ W = -\frac{\beta}{2 \cos x}, \quad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}, \quad (2.7) $$

$$ H = -\frac{1}{2} \frac{d^2}{dx^2} + \frac{(\frac{\xi}{2})^2}{2 \cos^2 x} - \frac{(\frac{\xi}{2}) \sin x}{2 \cos^2 x} \sigma_3 
\begin{bmatrix} H_\beta & 0 \\ 0 & H_{-\beta} \end{bmatrix} \quad (2.8) $$

with

$$ H_\beta = -\frac{1}{2} \frac{d^2}{dx^2} + \frac{\beta (\frac{\xi}{2} - \sin x)}{4 \cos^2 x}. \quad (2.9) $$

This is a supersymmetrization of the Scarf I potential. Note that a more general 2-parameter form can be obtained [2] by using $W = A \tan(\alpha x) - B \sec(\alpha x)$. This supersymmetric system has recently been further generalized in [34] using exceptional polynomials.

Let us now indicate how the relation $H = Q^2$ can be realized by introducing reflections instead of ‘spin’ degrees of freedom. Let $R$ denote the reflection operator:

$$ Rf(x) = f(-x). \quad (2.10) $$

A realization of supersymmetric quantum mechanics is obtained by taking as the supercharge the following differential-difference operator of Dunkl type:

$$ Q = \frac{1}{\sqrt{2}} \left( \frac{d}{dx} + U(x) \right) R + \frac{1}{\sqrt{2}} V(x), \quad (2.11) $$

where $U(x)$ is an even function and $V(x)$ an odd function,

$$ U(-x) = U(x), \quad V(-x) = -V(x). \quad (2.12) $$

The operator $R$, which can be identified with the one-dimensional parity operator, is readily seen to be self-adjoint, $R^\dagger = R$, with respect to the standard inner product of functions over symmetric domains; it follows that $Q$ shares that property $Q^\dagger = Q$. It is again a simple calculation to evaluate $Q^2$ and find the following form for a supersymmetric Hamiltonian $H$:

$$ H = Q^2 = -\frac{1}{2} \frac{d^2}{dx^2} + \frac{1}{2} (U^2 + V^2) + \frac{1}{2} \frac{dU}{dx} - \frac{1}{2} \frac{dV}{dx} R. \quad (2.13) $$
In this realization, unless $V$ is a constant, the operator $R$ appears in the Hamiltonian. It is of course possible to write (2.13) in a $2 \times 2$ matrix form. Consider to that end the Schrödinger equation $H\Psi = E\Psi$, split $\Psi$ into its even ($\Psi_{\text{even}}$) and odd ($\Psi_{\text{odd}}$) parts and write $\Psi$ as the 2-vector
\[\Psi = \begin{bmatrix} \Psi_{\text{even}} \\ \Psi_{\text{odd}} \end{bmatrix}.\] (2.14)

Obviously,
\[R\Psi = \sigma_3 \Psi\] (2.15)
in this notation. Moreover, when viewed as acting on wavefunctions written as in (2.14) the Hamiltonian (2.13) takes the form
\[H = -\frac{1}{2} \frac{d^2}{dx^2} + \frac{1}{2} (U^2 + V^2) + \frac{1}{2} \frac{dU}{dx}\sigma_1 - \frac{1}{2} \frac{dV}{dx}\sigma_3,\] (2.16)
where
\[\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.\] (2.17)

In this formalism, the supersymmetric Hamiltonian associated to the supercharge (2.11) with $R$, looks very similar to the one given in (2.5) especially if $U = 0$. It should be stressed however that the standard construction of supersymmetric Hamiltonians, reviewed at the beginning of this section, has nothing to do with the parity properties or parity decomposition of the wavefunctions. Hence the two supersymmetric realizations (the standard one and the one with reflections) are genuinely different even if they can be, in certain cases, presented in superficially similar forms.

### 3. A supersymmetric oscillator with reflections

Consider as a first example the system which is obtained from (2.11) and (2.13) by setting
\[U = 0, \quad V = x.\] (3.1)

This yields
\[Q = \frac{1}{\sqrt{2}} \left( \frac{d}{dx} R + x \right)\] (3.2)
and
\[H = -\frac{1}{2} \frac{d^2}{dx^2} + \frac{1}{2} x^2 - \frac{1}{2} R.\] (3.3)

This is simply the standard harmonic oscillator to which $(1/2)R$ has been added to render it supersymmetric. This system has also been analyzed in [35]. It is presented here as illustrative background to the novel supersymmetrization of the Scarf potential that is discussed in the next section. The associated Schrödinger equation is readily solved using the familiar orthonormal number states $|n\rangle$, with $n = 0, 1, \ldots$ and $\langle m|n\rangle = \delta_{m,n}$, of the quantum oscillator. Recall that the annihilation and creation operators $a$, $a^\dagger$, obeying $[a, a^\dagger] = 1$ and realized in the coordinate representation by
\[a = \frac{1}{\sqrt{2}} \left( \frac{d}{dx} + x \right), \quad a^\dagger = \frac{1}{\sqrt{2}} \left( -\frac{d}{dx} + x \right)\] (3.4)
act as follows on the state $|n\rangle$:
\[a|n\rangle = \sqrt{n}|n-1\rangle, \quad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle.\] (3.5)
The spectrum of $H$ is easily obtained by observing that

$$H = a^\dagger a + \frac{1}{2}(1 - R).$$  

(3.6)

Since

$$R[n] = (-1)^n |n\rangle,$$

(3.7)
in view of the fact that $[R, a] = [R, a^\dagger] = 0$ and that we shall take $R[0] = |0\rangle$, it follows that

$$E_n = n + \frac{1}{2}(1 - (-1)^n), \quad n = 0, 1, 2, \ldots.$$  

(3.8)

The spectrum will hence consist only of the even numbers starting with zero. Each level is degenerate except for the ground state which is unique.

It is instructive to diagonalize $Q$. First observe from (3.4) and (3.7) that

$$Q[n] = \begin{cases} \sqrt{n} |n - 1\rangle & \text{if } n \text{ is even}, \\ \sqrt{n + 1} |n + 1\rangle & \text{if } n \text{ is odd}. \end{cases}$$  

(3.9)

In view of (3.9), it is readily seen that the states

$$|n, \epsilon\rangle = \frac{1}{2} (|2n + 1 + \epsilon|2n + 2\rangle), \quad \epsilon = \pm 1,$$

(3.10)
obev

$$Q[n, \epsilon] = \epsilon \sqrt{2n + 2} |n, \epsilon\rangle, \quad Q[0] = 0, \quad n = 0, 1, \ldots.$$  

(3.11)

It thus immediately follows that

$$H[n, \epsilon] = (2n + 2) |n, \epsilon\rangle, \quad H[0] = 0, \quad n = 0, 1, \ldots,$$

(3.12)

which is tantamount by linearity to

$$H[2n + 1] = (2n + 2) |2n + 1\rangle; \quad H[2n + 2] = (2n + 2) |2n + 2\rangle.$$  

(3.13)

As is well known, in the coordinate representation, the wavefunctions $\langle x|n\rangle$ are given in terms of Hermite polynomials $H_n(x)$ by

$$\langle x|n\rangle = \frac{1}{\pi^{1/4}2^{n/2}\sqrt{n}} e^{-x^2/2} H_n(x).$$  

(3.14)

Using the relation between the Laguerre polynomials $L_n^\alpha$ and the Hermite polynomials [36], it is straightforward to find that

$$\langle x|n, \epsilon\rangle = \frac{(-1)^n}{\pi^{1/4}} \left[ \frac{n!}{(n + 1)_{n+1}} \right]^{1/2} e^{-x^2/2} \left( x L_n^{1/2}(x^2) + \epsilon (n + 1) L_{n+1}^{-1/2}(x^2) \right),$$  

(3.15)

where $(a)_n = a(a + 1) \cdots (a + n - 1)$ is the Pochammer symbol.

It is readily seen in this example that $R$ maps the degenerate eigenstates into one-another:

$$R[n, \epsilon] = -|n, -\epsilon\rangle.$$  

(3.16)

This follows from the fact that in this specific case

$$[Q, R] = 0, \quad [H, R] = 0.$$  

(3.17)

Hence, $R$ which was diagonalized simultaneously with $H$, transforms an eigenstate of $Q$ with eigenvalue $\epsilon \sqrt{2n + 1}$ into another eigenstate of $Q$, degenerate in energy, with eigenvalue $-\epsilon \sqrt{2n + 1}$. This explains why the levels of the system exhibit a twofold degeneracy at the exclusion of the ground state.
4. A novel supersymmetrization of the Scarf I potential

The example of the last section was of course very simple. We shall now present a more elaborate case by providing the supersymmetrization with reflections of the Scarf I Hamiltonian given in (2.9). The associated Schrödinger equation will be found to be exactly solvable in terms of the recently identified little $-1$ Jacobi polynomials.

In the formulation of section 2, let us take

$$U(x) = -\frac{\beta}{2 \cos x}, \quad V(x) = -\frac{\alpha}{2 \sin x}. \quad (4.1)$$

This choice of functions respect condition (2.12), that is, that $U$ be even and $V$ odd. With these $U$ and $V$, supercharge (2.11) and Hamiltonian (2.13) read

$$Q_{\alpha,\beta} = \frac{1}{\sqrt{2}} \left( \frac{d}{dx} - \frac{\beta}{2 \cos x} \right) R - \frac{\alpha}{2 \sqrt{2} \sin x} \quad (4.2)$$

$$H_{\alpha,\beta} = Q_{\alpha,\beta}^2 \quad (4.3)$$

This obviously offers an alternative to the standard supersymmetrization (2.8) of the Scarf I Hamiltonian $H_\beta = H_{0,\beta}$ given in (2.9). Note the presence of the reflection operator in $H_{\alpha,\beta}$ when $\alpha \neq 0$. Two free parameters are present. Observe that $H_\beta$ itself is supersymmetric:

$$H_{\beta} = Q_{0,\beta}^2. \quad \text{We shall now show that the Schrödinger equation } H_{\alpha,\beta} \Psi = E \Psi \text{ is exactly solvable and, to that end, we shall look for the eigenfunctions of } Q_{\alpha,\beta}. \quad (4.4)$$

Let us first remark that in $N = \frac{1}{2}$ supersymmetry, the wavefunctions are eigenstates of the single supercharge $Q$ and supersymmetry is broken whenever the Hamiltonian does not admit normalizable eigenfunctions with zero energies [1]. This is what happens here as it is readily found that the ground-state wavefunction $\Psi_{0,\alpha,\beta}$ is given by

$$\Psi_{0,\alpha,\beta} = N_0 |\sin x|^{\alpha/2} \cos^{\beta/2} x (1 + \sin x)^{1/2} \quad (4.5)$$

and satisfies

$$Q_{\alpha,\beta} \Psi_{0,\alpha,\beta} = -\frac{1}{2 \sqrt{2}} (\alpha + \beta + 1) \Psi_{0,\alpha,\beta}. \quad (4.6)$$

The normalization constant $N_0$ is such that

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx |\Psi_{0,\alpha,\beta}|^2 = 1. \quad (4.7)$$

It is found to be (see appendix A)

$$N_0 = \left[ \frac{\Gamma\left(\frac{\alpha}{2} + \frac{\beta}{2} + 1\right)}{\Gamma\left(\frac{\alpha}{2} + 1\right) \Gamma\left(\frac{\beta}{2} + 1\right)} \right]^{1/2} \quad (4.8)$$

with the help of the beta integral. As usual, $\Gamma(x)$ denotes the standard gamma function. Let us now carry out the ‘gauge’ transformation of $Q_{\alpha,\beta}$ with the ground state $\Psi_{0,\alpha,\beta}$. Let

$$\tilde{Q}_{\alpha,\beta} = \Psi_{0,\alpha,\beta}^{-1} Q_{\alpha,\beta} \Psi_{0,\alpha,\beta}. \quad (4.9)$$

It is straightforward to see that

$$\tilde{Q}_{\alpha,\beta} = \sec x - \tan x \frac{d}{dx} - \frac{\alpha \csc x}{2 \sqrt{2}} (1 - R) - \frac{1}{2 \sqrt{2}} (\alpha + \beta + 1) R. \quad (4.10)$$
Perform now the change of variables
\[ y = \sin x \] (4.11)
to find that
\[ \tilde{Q}_{\alpha,\beta} = \frac{1}{\sqrt{2}} (1-y) \frac{d}{dy} R - \frac{\alpha}{2\sqrt{2}y} (1-R) - \frac{1}{2\sqrt{2}} (\alpha + \beta + 1) R. \] (4.12)
We thus identify \( \tilde{Q}_{\alpha,\beta} \) as the Dunkl-type operator of which the little \(-1\) Jacobi polynomials are the eigenfunctions.

It has been shown in [24] that the little \(-1\) Jacobi polynomials \( P_{n}^{(\alpha,\beta)}(y) \) satisfy the following Dunkl-type eigenvalue equation:
\[ \left[ 2(1-y) \frac{d}{dy} R + \left( \alpha + \beta + 1 - \frac{\alpha}{y} \right) (1-R) \right] P_{n}^{(\alpha,\beta)}(y) = \lambda_{n,a,\beta} P_{n}^{(\alpha,\beta)}(y), \] (4.13)
where
\[ \lambda_{n,a,\beta} = \begin{cases} -2n & \text{for } n \text{ even}, \\
\frac{2(n+\alpha+\beta+1)}{2n+1} & \text{for } n \text{ odd}. \end{cases} \] (4.14)

These polynomials can be obtained as an appropriate \( q \rightarrow -1 \) limit of the little \( q \)-Jacobi polynomials from the Askey scheme [28] and have the following expressions in terms of the hypergeometric (terminating) series:
\[ P_{n}^{(\alpha,\beta)}(y) = \kappa_{n} \left[ \begin{align*}
\frac{2}{2F1} & \left( \frac{n+\alpha+\beta+2}{2} \alpha+1 \frac{n}{2} ; y^2 \right) \\
\frac{1}{2F1} & \left( 1-\frac{1}{2} \frac{n+\alpha+\beta+2}{2} \alpha+3 \frac{1}{2} ; y^2 \right) \end{align*} \right] \] (4.15)
for \( n \) even, and
\[ P_{n}^{(\alpha,\beta)}(y) = \kappa_{n} \left[ \frac{1-n}{2F1} \left( \frac{1-\frac{1}{2} \frac{n+\alpha+\beta+3}{2}}{\alpha+1} \frac{1}{2} \frac{1}{2} ; y^2 \right) \\
- \frac{(\alpha+\beta+1)y}{2F1} \left( \frac{1-\frac{1}{2} \frac{n+\alpha+\beta+3}{2}}{\alpha+1} \frac{1}{2} \frac{1}{2} ; y^2 \right) \right] \] (4.16)
for \( n \) odd. (For a definition of the \( 2F1 \) symbol, see (A.11).) The coefficients \( \kappa_{n} \) are chosen so as to make the polynomials \( P_{n}^{(\alpha,\beta)}(y) \) monic, i.e. \( P_{n}^{(\alpha,\beta)}(y) = y^{n} + \mathcal{O}(n-1) \). Through the identification of the factor of the leading term in (4.15) and (4.16), they are found to be
\[ \kappa_{n} = \begin{cases} (-1)^{\frac{n}{2}} \frac{\Gamma\left(\frac{n+\alpha+\beta+1}{2}\right)}{\Gamma\left(\frac{n+\alpha+\beta+1}{2}\right)} & \text{for } n \text{ even}, \\
(-1)^{\frac{n-1}{2}} \frac{\Gamma\left(\frac{n+\alpha+\beta+1}{2}\right)}{\Gamma\left(\frac{n+\alpha+\beta+1}{2}\right)} & \text{for } n \text{ odd}. \end{cases} \] (4.17)
For \( \alpha > -1, \beta > -1 \), they are orthogonal with respect to the weight function
\[ \omega(y) = |y|^\alpha (1-y^2)^{\beta/2} (1+y). \] (4.18)
It thus follows, comparing (4.12) and (4.13), that the wavefunctions \( \Psi_{\alpha,\beta} \) defined by

\[
\Psi_{n,\alpha,\beta}(x) = \frac{N_n}{N_0} \Psi_{0,\alpha,\beta} P_n^{(\alpha,\beta)}(\sin x)
\]

will satisfy the eigenvalue equation

\[
Q_{\alpha,\beta} \Psi_{n,\alpha,\beta}(x) = q_{n,\alpha,\beta} \Psi_{n,\alpha,\beta}(x)
\]

with

\[
q_{n,\alpha,\beta} = \frac{1}{2\sqrt{2}} \begin{cases} -(2n + \alpha + \beta + 1) & \text{for } n \text{ even}, \\ (2n + \alpha + \beta + 1) & \text{for } n \text{ odd}. \end{cases}
\]

Since \( H_{\alpha,\beta} = Q_{\alpha,\beta}^2 \), the spectrum \( E_{n,\alpha,\beta} \) of the Hamiltonian is given by

\[
E_{n,\alpha,\beta} = \frac{1}{2} (2n + \alpha + \beta + 1)^2,
\]

for \( n = 0, 1, 2, \ldots \), and its eigenfunctions are those of \( Q_{\alpha,\beta} \), that is, the functions \( \Psi_{n,\alpha,\beta}(x) \) given in (4.19). Note that the energy values depend only on the sum \( \alpha + \beta \).

The normalization constants \( N_n \) are chosen so that

\[
\int_{-\frac{1}{2}}^{\frac{1}{2}} dx |\Psi_{n,\alpha,\beta}(x)|^2 = 1.
\]

Their calculation, which is described in appendix A, makes use of the moments of the weight function (4.18) and relies on certain hypergeometric summations. They are given by

\[
N_n = \begin{cases} \frac{N_0 \binom{\frac{1}{2} + \frac{\alpha + \beta + 1}{2}}{n} \binom{\frac{1}{2} + \frac{\alpha + \beta}{2}}{n}}{\sqrt{(\frac{\alpha + \beta + 1}{2})!} (\frac{\alpha + \beta + 1}{2})_{\frac{1}{2}} (\frac{\alpha + \beta}{2})_{\frac{1}{2}} (\frac{\alpha + \beta}{2} + 1)_{\frac{1}{2}}}, & \text{for } n \text{ even}, \\ \frac{N_0 \binom{\frac{1}{2} + \frac{\alpha + \beta + 1}{2}}{n} \binom{\frac{1}{2} + \frac{\alpha + \beta}{2}}{n}}{\sqrt{(\frac{\alpha + \beta + 1}{2})!} (\frac{\alpha + \beta + 1}{2})_{\frac{1}{2}} (\frac{\alpha + \beta}{2})_{\frac{1}{2}} (\frac{\alpha + \beta}{2} + 1)_{\frac{1}{2}}}, & \text{for } n \text{ odd}. \end{cases}
\]

Our experience with the oscillator leads us to examine the action of the reflection operator \( R \) on the eigenstates of \( Q_{\alpha,\beta} \). The wavefunctions \( R \Psi_{n,\alpha,\beta} \) will obviously satisfy

\[
(RQ_{\alpha,\beta}R)R\Psi_{n,\alpha,\beta} = q_{n,\alpha,\beta} R\Psi_{n,\alpha,\beta},
\]

(4.25)

\[
(RH_{\alpha,\beta}R)R\Psi_{n,\alpha,\beta} = q_{n,\alpha,\beta}^2 R\Psi_{n,\alpha,\beta}.
\]

(4.26)

We may thus couple \( H_{\alpha,\beta} \) and \( RH_{\alpha,\beta}R \) in a 2 \times 2 matrix as follows:

\[
\mathcal{H} = \begin{bmatrix} H_{\alpha,\beta} & 0 \\ 0 & RH_{\alpha,\beta}R \end{bmatrix}
\]

(4.27)

to create a system with twofold degeneracy. As a result, the states

\[
\begin{bmatrix} \Psi_{n,\alpha,\beta} \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ R\Psi_{n,\alpha,\beta} \end{bmatrix},
\]

(4.28)

that are interchanged by the operator

\[
\begin{bmatrix} 0 & R \\ R & 0 \end{bmatrix},
\]

(4.29)

are degenerate eigenstates of the combined system with \( q_{n,\alpha,\beta}^2 \) as a common energy.

Now, it is easy to see that

\[
RQ_{\alpha,\beta}R = -Q_{n,-\beta}
\]

(4.30)

\[
RH_{\alpha,\beta}R = H_{\alpha,-\beta}.
\]

(4.31)
Hence
\[ H = \begin{bmatrix} H_{\alpha, \beta} & 0 \\ 0 & H_{\alpha, -\beta} \end{bmatrix}. \] (4.32)

When \( \alpha = 0 \), we return to the matrix Hamiltonian (2.8) that was obtained in the standard way. When \( \beta = 0 \), it is manifest that \( H_{\alpha, 0} \) is reflection invariant, \([H_{\alpha, 0}, R] = 0\), and has a degenerate spectrum. Indeed, all levels, the ground state included, exhibit a twofold degeneracy with \( \Psi_{n, \alpha, 0} \) and \( R \Psi_{n, \alpha, 0} \) satisfying
\[ Q_{\alpha, 0} \Psi_{n, \alpha, 0} = \lambda_{n, \alpha, \beta} \Psi_{n, \alpha, 0} \] (4.33)
\[ Q_{\alpha, 0} R \Psi_{n, \alpha, 0} = -\lambda_{n, \alpha, \beta} R \Psi_{n, \alpha, 0} \] (4.34)
and having the same energy. Therefore when \( \beta = 0 \), we find a situation similar to the one observed for the oscillator except that, here, the reflection symmetry is spontaneously broken.

Notwithstanding the properties of the ground state, it is not difficult to convince oneself that such degeneracies will occur whenever \( U(x) = 0 \), that is whenever \([H, R] = 0\).

Using the raising and lowering operators of the little \(-1\) Jacobi polynomials, we can obtain intertwining operators that map the eigenfunctions of \( Q_{\alpha, \beta} \) into those of \( Q_{\alpha, \beta + \pm 2} \).

Let
\[ X_{\alpha, \beta} = \frac{d}{dx} + \frac{1}{2} \beta \tan x - \frac{1}{2} \sec x - \frac{\alpha}{2} (1 + \csc x) R \] (4.35)
and
\[ Y_{\alpha, \beta} = -\frac{d}{dx} + \frac{1}{2} \beta \tan x - \frac{1}{2} \sec x - \frac{\alpha}{2} (1 - \csc x) R; \] (4.36)
then we have
\[ X_{\alpha, \beta} \Psi_{n, \alpha, \beta} = [n]_\alpha N_n \Psi_{n-1, \alpha, \beta + 2} \] (4.37)
\[ Y_{\alpha, \beta} \Psi_{n, \alpha, \beta} = (\beta - 1 + [n]_\alpha) N_n \Psi_{n+1, \alpha, \beta - 2}, \] (4.38)
where
\[ [n]_\alpha = n + \frac{\alpha}{2} (1 - (-1)^n). \] (4.39)
The product of \( X_{\alpha, \beta} \) and \( Y_{\alpha, \beta} \) is expressible in terms of \( H_{\alpha, \beta} \) and \( Q_{\alpha, \beta} \) as follows:
\[ Y_{\alpha, \beta + 1} X_{\alpha, \beta + 1} = 2H_{\alpha, \beta} + \sqrt{2\alpha}Q_{\alpha, \beta} + \frac{1}{4}(\alpha + \beta + 1)(\alpha - \beta - 1). \] (4.40)

Finally, the operators \( X_{\alpha, \beta} \) and \( Y_{\alpha, \beta} \) are seen to obey the intertwining relations
\[ Q_{\alpha, \beta + 2} X_{\alpha, \beta} = -X_{\alpha, \beta} Q_{\alpha, \beta} \] (4.41)
\[ Q_{\alpha, \beta - 2} Y_{\alpha, \beta} = -Y_{\alpha, \beta} Q_{\alpha, \beta}. \] (4.42)

5. Conclusion

Let us summarize our results and offer an outlook to conclude. We considered supersymmetric quantum Hamiltonians that have Dunkl-type operators as supercharges. This approach to supersymmetrization leads to systems that have reflection operators in their Hamiltonians. We introduced in this fashion a supersymmetric extension with two parameters of the Scarf I Hamiltonian in one dimension. We showed this system to be exactly solvable and found that its wavefunctions are expressed in terms of the little \(-1\) Jacobi polynomials.
One avenue of future research is to further explore models whose supersymmetric extension with reflections would prove exactly solvable. One might wish to adapt the concept of shape invariance to systems with reflections and to perform a classification analogous to the one that has been carried out for additive shape invariant systems in the standard supersymmetry realization [37]. It would also be worth examining how this approach of using reflection operators to create supersymmetric Hamiltonians applies to systems with more degrees of freedom.

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Appendix A. The normalization of the wavefunctions $\Psi_{n,\alpha,\beta}(x)$

The wavefunctions of Hamiltonian (4.4) are given by

$$\Psi_{n,\alpha,\beta}(x) = \frac{N_n}{N_0} \Psi_{0,\alpha,\beta}(x) P_n^{(\alpha,\beta)}(\sin x), \quad (A.1)$$

where

$$\Psi_{0,\alpha,\beta}(x) = N_0 |\sin x|^{\alpha/2} \cos^{\beta/2} x (1 + \sin x)^{1/2}. \quad (A.2)$$

We shall determine the constants $N_n$ so that

$$\int_{-\pi/2}^{\pi/2} |\Psi_{n,\alpha,\beta}(x)|^2 \mathrm{d}x = 1. \quad (A.3)$$

Let $y = \sin x$,

$$\int_{-\pi/2}^{\pi/2} |\Psi_{n,\alpha,\beta}(x)|^2 \mathrm{d}x = N_n^2 \int_{-1}^{1} \frac{d(y)}{\cos x} |\sin x|^{\alpha} |\cos x|^{\beta} (1 + \sin x) (P_n^{(\alpha,\beta)}(\sin x))^2$$

$$= N_n^2 \int_{-1}^{1} |y|^{\alpha} (1 - y^2)^{\frac{\beta-1}{2}} (1 + y) (P_n^{(\alpha,\beta)}(y))^2$$

$$= N_n^2 \int_{-1}^{1} \omega(y) (P_n^{(\alpha,\beta)}(y))^2, \quad (A.4)$$

where $\omega(y)$ is the measure (4.18) for which the polynomials $P_n^{(\alpha,\beta)}(y)$ are orthogonal [24].

The constant $N_0$ is chosen so that

$$N_0^2 \int_{-1}^{1} \omega(y) = 1. \quad (A.5)$$

It is straightforward to see that

$$\int_{-1}^{1} \omega(y) = \int_{0}^{1} \mathrm{d} t \ t^{\frac{\beta}{2}} (1 - t)^{\frac{\alpha-1}{2}} = B \left( \frac{\alpha + 1}{2}, \frac{\beta + 1}{2} \right) \quad (A.6)$$

and hence that

$$N_0 = \left[ \frac{\Gamma \left( \frac{\alpha}{2} + \frac{\beta}{2} + \frac{1}{2} \right) \Gamma \left( \frac{\alpha}{2} + \frac{1}{2} \right) \Gamma \left( \frac{\beta}{2} + \frac{1}{2} \right)}{\Gamma \left( \frac{\alpha + 1}{2} \right) \Gamma \left( \frac{\beta + 1}{2} \right)} \right]^{1/2}, \quad (A.7)$$
where $\Gamma(x)$ and $B(x, y)$ are the standard gamma and beta functions. The moments $c_n$, defined by

$$c_n = N_0^2 \int_{-1}^{1} dy \omega(y)y^n,$$

are similarly calculated and given by [24]

$$c_{2n} = c_{2n-1} = \left(\frac{\alpha}{2} + \frac{1}{2}\right)_n (\frac{\alpha}{2} + \frac{1}{2} + 1)_n$$

(A.8)

Now, from (A.3) we have

$$\frac{N_n}{N_0} = \frac{N_0^2}{N_0} \int_{-1}^{1} dy \omega(y) \left[P_n^{(a, \beta)}(y)\right]^2,$$

$$= N_0 \int_{-1}^{1} dy \omega(y) P_n^{(a, \beta)}(y)y^n$$

(A.9)

since the polynomials are monic and obey

$$\int_{-1}^{1} dy \omega(y) P_n^{(a, \beta)}(y)y^m = 0,$$

(A.10)

for $m \leq n - 1$. Recall that the (generalized) hypergeometric series with $r$ numerator parameters $a_1, \ldots, a_r$ and $s$ denominator parameters $b_1, \ldots, b_s$ are defined by [28, 38]

$$\_rF_s \left( \begin{array}{c} a_1, a_2, \ldots, a_r \cr b_1, \ldots, b_s \end{array} ; z \right) = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_r)_n z^n}{(b_1)_n \cdots (b_s)_n}.$$  

(A.11)

In order to evaluate integral (A.9), one uses the explicit expressions (4.15) and (4.16) of the polynomials $P_n^{(a, \beta)}(y)$ in terms of $_2F_1$ hypergeometric series and the values of the moments. For $n$ even, $n = 2k$, we have

$$\frac{N_0^2}{N_{2k}} = \kappa_{2k} (A_k + B_k).$$

(A.12)

where

$$A_k = \sum_{m=0}^{k} \frac{(-k)_m (k + 1 + \frac{\alpha}{2} + \frac{\beta}{2})_m (\frac{\alpha}{2} + \frac{1}{2})_{m+k}}{m! (\frac{\alpha}{2} + \frac{1}{2})_m (\frac{\alpha}{2} + \frac{1}{2} + 1)_{m+k}}$$

(A.13)

$$B_k = \frac{2k}{\alpha + 1} \sum_{m=0}^{k-1} \frac{(1-k)_m (k + 1 + \frac{\alpha}{2} + \frac{\beta}{2})_m (\frac{\alpha}{2} + \frac{1}{2})_{m+k+1}}{m! (\frac{\alpha}{2} + \frac{1}{2})_m (\frac{\alpha}{2} + \frac{1}{2} + 1)_{m+k+1}},$$

(A.14)

and $\kappa_{2k}$ are the coefficients (4.17) ensuring that $P_n^{(a, \beta)}$ is monic.

Using the identity

$$(a)_m = (a)_k (a + k)_m,$$

(A.15)

$A_k$ is reduced to

$$A_k = \frac{(\frac{\alpha}{2} + \frac{1}{2})_k}{(\frac{\alpha}{2} + \frac{1}{2} + 1)_k} \sum_{m=0}^{k} \frac{(-k)_m (k + \frac{\alpha}{2} + \frac{1}{2})_m}{m! (\frac{\alpha}{2} + \frac{1}{2})_m}$$

(A.16)

and, with the help of the Chu–Vandermonde summation formula [38]

$$\_2F_1 \left( \begin{array}{c} -n \cr c \end{array} ; b \right) = \frac{(c - b)_n}{(c)_n},$$

(A.17)
we find

\[ A_k = \frac{(-1)^k k!}{\left(\frac{a}{2} + \frac{b}{2} + 1\right)_k}. \]  

(A.18)

With the help of (A.15) again, \( B_k \) can be written as

\[ B_k = \frac{2k}{\alpha + 1} \left(\frac{a}{2} + \frac{b}{2} + 1\right)_{k+1} \sum_{m=0}^{k-1} \frac{(1-k)_m(k+1+\frac{a}{2}+\frac{b}{2})_m}{m!(\frac{a}{2} + \frac{b}{2})_m(\frac{a}{2} + \frac{b}{2} + k + 2)_m} (\alpha + 1)_k. \]  

(A.19)

At this point, let us use a summation formula for generalized hypergeometric equations (see for example [39])

\[ _3F_2 \left( 1-k, b, c+k - 1; b+1, c \right) = \frac{(k-1)!}{(b+1)_k} \sum_{\ell=0}^{k-1} \frac{p_\ell}{\ell!} F_1 \left( \ell, c+k-1; 1 \right) \]  

(A.20)

which is valid for \( k \geq 1 \). Using the Chu–Vandermonde summation formula (A.17) twice, we obtain

\[ _3F_2 \left( 1-k, b, c+k - 1; b+1, c \right) = \frac{(k-1)!}{(b+1)_k} \sum_{\ell=0}^{k-1} \frac{(-k)_\ell (1-b-\ell)_\ell}{\ell! c_\ell} \]  

\[ = \frac{(k-1)!}{(b+1)_k} \sum_{\ell=0}^{k-1} \frac{b_\ell (-k)_\ell}{\ell! c_\ell} \]  

\[ = \frac{(k-1)!}{(b+1)_k} \left( _3F_1 \left( -k, \frac{b}{c} - 1; 1 \right) - \frac{(-1)^k b_k}{c_k} \right) \]  

(A.21)

to see that \( B_k \), for \( k \geq 1 \), simplifies to

\[ B_k = \frac{(-1)^k k!}{\left(\frac{a}{2} + \frac{b}{2} + 1\right)_k} - \frac{(-1)^k k!}{\left(\frac{a}{2} + \frac{b}{2} + 1\right)} \]  

(A.22)

\[ = \frac{(-1)^k k!}{\left(\frac{a}{2} + \frac{b}{2} + 1\right)_k} - A_k. \]  

(A.23)

Note that for \( k = 0 \), \( B_k = 0 \) and so \( N_0^2/N_0^2 = A_0 = 1 \) as expected. Therefore, for \( n \) even, \( n = 2k, k = 0, 1, \ldots \),

\[ \frac{N_0^2}{N_{2k}^2} = \frac{k!(\frac{a}{2} + \frac{1}{2})_k (\frac{a}{2} + \frac{1}{2})_k (\frac{a}{2} + \frac{b}{2} + 1)_k}{(\frac{a}{2} + \frac{b}{2} + 1)_k}. \]  

(A.24)

For \( n \) odd, \( n = 2k-1, k = 1, 2, \ldots \), one proceeds similarly to find that

\[ \frac{N_{2k}^2}{N_{2k-1}^2} = \frac{(k-1)!(\frac{a}{2} + \frac{1}{2})_k (\frac{a}{2} + \frac{1}{2})_k (\frac{a}{2} + \frac{b}{2} + 1)_{k-1}}{(\frac{a}{2} + \frac{b}{2} + 1)_k}. \]  

(A.25)

This therefore provides the normalization factors \( N_n \) as they are given in (4.24).
Appendix B. Some (other) examples of Hamiltonians with reflections

Apart from the little and big $-1$ Jacobi polynomials, there are other systems of orthogonal polynomials which satisfy eigenvalue equations involving Dunkl-type operators. It was shown in [32] that the generalized Hermite and generalized Gegenbauer polynomials are the only symmetric orthogonal polynomials that obey such an equation, in these cases of second order with respect to the classical Dunkl operator. We indicate here that the equation for the generalized Gegenbauer polynomials can be presented in Schrödinger form with an additional ‘reflection’ term.

The generalized Gegenbauer polynomials [30, 31] $P_n(y)$ are symmetric polynomials (i.e. $P_n(-y) = (-1)^n P_n(y)$) which are orthogonal on the interval $[-1, 1]$ with respect to the weight function

$$w(x) = |x|^{2\mu} (1 - x^2)^\alpha.$$  \hfill (B.1)

The polynomials $P_n(y)$ satisfy the eigenvalue equation [32]

$$LP_n(y) = \lambda_n P_n(y),$$  \hfill (B.2)

where

$$L = (1 - y^2)T_\mu^2 - 2(\alpha + 1)yT_\mu$$  \hfill (B.3)

and $T_\mu$ is the classical Dunkl operator

$$T_\mu = \partial_y + \mu y^{\alpha+1} (I - R).$$  \hfill (B.4)

The eigenvalues are

$$\lambda_n = \begin{cases} -n(n+1+2\alpha+2\mu) & \text{for } n \text{ even}, \\ -(2\mu+n)(2\alpha+n+1) & \text{for } n \text{ odd}. \end{cases}$$  \hfill (B.5)

Change the independent variable $y = \sin x$ and consider the operator

$$H = -F_0(x)LF_0^{-1}(x),$$  \hfill (B.6)

where

$$F_0(x) = \sqrt{w(y) \cos x} = |\sin x|^{\mu} \cos^{\alpha+1/2} x.$$  \hfill (B.7)

It is assumed that $-\pi/2 < x < \pi/2$. It can be checked that the operator $H$ has the form

$$H = -\partial_x^2 + U_0(x) + U_1(x)R,$$  \hfill (B.8)

where

$$U_0(x) = \alpha^2 \cos^4 x + (\mu^2 - 2\alpha^2 + 1/4) \cos^2 x + \alpha^2 - 1/4 \cos^2 x \sin^2 x,$$  \hfill (B.9)

and

$$U_1(x) = (2\alpha + 1)\mu - \frac{\mu}{\sin^2 x}.$$  \hfill (B.10)

This is another example of an exactly solvable Schrödinger Hamiltonian which includes the reflection operator $R$.

Note that the function $F_0(x)$ defined in (B.7) is the ground-state wavefunction of the Hamiltonian $H$ corresponding to the lowest eigenvalue $\lambda_0 = 0$:

$$HF_0(x) = 0.$$  \hfill (B.11)

The bound state wavefunctions $\psi_n(x)$ that satisfy the Schrödinger equation

$$H\psi_n(x) = \lambda_n \psi_n(x)$$  \hfill (B.12)
have the form
\[ \psi_n(x) = F_0(x)P_n(\sin x), \quad (B.13) \]
where \( P_n(y) \) are generalized Gegenbauer polynomials.

There are two special cases of Hamiltonian (B.8) worth mentioning. If \( \mu = 0 \), then the term \( U_1(x) R \) with the reflection operator disappears and the Hamiltonian \( H \) becomes the usual trigonometric Pöschl–Teller trigonometric potential:
\[ H = -\partial_x^2 + \frac{\alpha^2 - 1/4}{\cos^2 x} - \frac{(2\alpha + 1)^2}{4}. \quad (B.14) \]
It is well known that its eigenfunctions are expressed in terms of ordinary Gegenbauer (ultraspherical) polynomials.

Another interesting special case occurs for \( \alpha = -1/2 \). We have then
\[ H = -\partial_x^2 + \frac{\mu^2}{\sin^2 x} - \mu^2 - \frac{\mu}{\sin^2 x} R, \quad (B.15) \]
which can be related to the two-particle Calogero–Sutherland–Moser (CSM) model with an exchange term. The \( N \)-body CSM Hamiltonian (with the exchange operators) is [11]
\[ H = -\sum_{j=1}^{N} \partial_x^2 + \beta \gamma x_j^2 \sum_{j>k=1}^{N} \frac{\beta/2 - S_{jk}}{\sin^2[\gamma(x_j - x_k)]}, \quad (B.16) \]
where \( \beta, \gamma \) are arbitrary real parameters and \( S_{jk} \) is the operator which exchanges the coordinates.

Let \( N = 2 \), put \( \gamma = 2^{-1/2} \), \( \beta = 2\mu \) and choose the coordinates
\[ x = \frac{x_1 - x_2}{\sqrt{2}}, \quad u = \frac{x_1 + x_2}{\sqrt{2}}. \quad (B.17) \]
We can then rewrite Hamiltonian (B.16) as
\[ H = -\partial_x^2 - \partial_u^2 + \frac{\mu^2}{\sin^2 x} - \frac{\mu}{\sin^2 x} R. \quad (B.18) \]
The term \( -\partial_u^2 \) corresponds to the conserved energy of the center-of-mass and can be separated out. Comparing (B.18) with (B.15) we see that these two Hamiltonians coincide up to an inessential constant term.

Another one-dimensional quantum Hamiltonian with a reflection term can similarly be obtained from the two-particle rational CSM model; the wavefunctions in this case are expressed in terms of generalized Hermite polynomials.

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