Generalized Bloch analysis and propagators on Riemannian manifolds with a discrete symmetry

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Abstract

We consider an invariant quantum Hamiltonian $H = -\Delta_{LB} + V$ in the $L^2$ space based on a Riemannian manifold $\tilde{M}$ with a countable discrete symmetry group $\Gamma$. Typically, $\tilde{M}$ is the universal covering space of a multiply connected Riemannian manifold $M$ and $\Gamma$ is the fundamental group of $M$. On the one hand, following the basic step of the Bloch analysis, one decomposes the $L^2$ space over $\tilde{M}$ into a direct integral of Hilbert spaces formed by equivariant functions on $\tilde{M}$. The Hamiltonian $H$ decomposes correspondingly, with each component $H_\Lambda$ being defined by a quasi-periodic boundary condition. The quasi-periodic boundary conditions are in turn determined by irreducible unitary representations $\Lambda$ of $\Gamma$. On the other hand, fixing a quasi-periodic boundary condition (i.e., a unitary representation $\Lambda$ of $\Gamma$) one can express the corresponding propagator in terms of the propagator associated to the Hamiltonian $H$. We discuss these procedures in detail and show that in a sense they are mutually inverse.
I. Introduction

It has been demonstrated in Ref. 4 that in the formalism of quantum mechanics vector potentials are more significant than in classical mechanics. Geometrically this observation consists of the fact that over a connected but multiply connected manifold $M$ there exist non-equivalent vector bundles with connections (covariant derivatives) whose curvatures are equal. Here we are going to focus exclusively on the case when the curvature vanishes. It is not difficult to see that all flat Hermitian connections on $M$, up to equivalence, are in one-to-one correspondence with unitary representations of the fundamental group $\Gamma = \pi_1(M)$. Given a flat Hermitian connection the corresponding unitary representation of $\Gamma$ is defined by the parallel transport along closed paths with a fixed base point. In the physical terminology, the parallel transport along a closed path produces nothing but the nonintegrable phase factor.\(^{39,15}\) Conversely, denote by $\tilde{M}$ the universal covering space of $M$. Then $\tilde{M} \to M$ is a principal fiber bundle with the structure group $\Gamma$, and since $\dim \tilde{M} = \dim M$ there exists exactly one connection in this bundle which is necessarily flat. Given a unitary representation $\Lambda$ of $\Gamma$ one can associate to this principal fiber bundle a Hermitian vector bundle with a connection which is again flat.\(^{18}\)

On the physical level this means that there may exist, depending on $\pi_1(M)$, non-equivalent quantum Hamiltonians describing a particle moving on $M$ in a vanishing gauge field. In the physical literature there was described a construction of the propagator associated to such a Hamiltonian. The construction is based on the notion of the Feynman path integral and assumes the knowledge of the free propagator on the universal covering space $\tilde{M}$. The propagator on $M$ connecting points $x$ and $y$ is then constructed as a weighted sum running over all homotopy classes of paths from $x$ to $y$, the summands are expressed in terms of the free propagator on $\tilde{M}$ and the weights depend on a given representation $\Lambda$ of $\Gamma$. In Ref. 26 one treats the case of a circle, and in Ref. 27 the example of the Aharonov-Bohm effect is worked out. All this material can be also found in Ref. 28. Generalizations to non-Abelian gauge groups are discussed in Refs. 36, 25. Let us also note that in the mathematical literature an analogous formula is known to be valid, under certain assumptions, for heat kernels.\(^{6,35}\)

Though the formula for the propagator is derived in Ref. 27 rather formally it turns out to be quite effective even when considering more complicated topologies. As a distinguished example of this kind one may point out two-dimensional quantum systems describing the Aharonov-Bohm effect with two solenoids, possibly with an additional scalar potential.\(^{24}\) The propagator for such a system is expressed in an
explicit manner as an infinite series in Ref. 30, and more details on the method and computations are provided in Ref. 19. The same example with a spin is discussed in Ref. 9. Moreover, this formula makes it possible to treat the scattering problem in this model as well.\textsuperscript{31,32,33} The scattering amplitude has been also derived in Ref. 16 by completely different technical tools, see also Refs. 17, 37. As one can verify by a straightforward computation, this is an encouraging observation that both methods finally lead to the same result (compare formula (12) in Ref. 32 to Theorem 1.1 in Ref. 16). Let us note that some other two-dimensional systems with similar features have been discussed in the literature recently.\textsuperscript{13,10}

The formula for the propagator on multiply connected spaces, as derived in Ref. 27 in the framework of the Feynman path integral, is the central topic of the current paper. We shall call it loosely the Schulman’s ansatz. Our goal is to find a mathematically rigorous interpretation of this formula. In a more general setting, we shall consider the situation when $\tilde{M}$ is a manifold with a discrete symmetry group $\Gamma$ and $M = \tilde{M}/\Gamma$. Thus $\tilde{M}$ is a covering space of $M$ though not necessarily universal. Treating this problem we realized quickly that one has to consider the construction also from the opposite side. By the Schulman’s ansatz one relates to the free propagator on $\tilde{M}$ and to any unitary representation $\Lambda$ of $\Gamma$ a propagator corresponding to a Hamiltonian on $M$. Conversely, one may ask how to reconstruct the free propagator on $\tilde{M}$ from the knowledge of the family of propagators on $M$ when $\Lambda$ runs over all unitary irreducible representations (up to equivalence) of $\Gamma$. The inverse procedure leads to a generalization of the Bloch decomposition which we are going to discuss as well.

A generalization of the Bloch analysis has been proposed by Sunada.\textsuperscript{34} This method has been used systematically to reveal the band structure of spectrum for a $\Gamma$-periodic elliptic operator on a non-compact Riemannian manifold $\tilde{M}$ with a discrete symmetry group $\Gamma$ under the assumption that the quotient $\tilde{M}/\Gamma$ is compact. For recent progress in this direction see also Refs. 22, 23. Further, in Ref. 5 one considers magnetic Hamiltonians on the two-dimensional torus in the case when the magnetic field is determined by an integral two-form (or, in other words, when the magnetic flux through 2-cycles is quantized in accordance with the famous Dirac’s rule for magnetic monopoles). It is shown there that the Bochner Laplacian over the universal covering of the torus (the plane) decomposes into a direct integral whose components are all equivalence classes of Bochner Laplacians over the torus with the prescribed magnetic field. This result is then extended in Ref. 11 to more general compact Riemannian manifolds $M$ whose fundamental group $\Gamma$ is Abelian. Again one constructs a direct integral over the Pontryagin dual $\hat{\Gamma}$. A more algebraic approach in the spirit of the
Gelfand’s representation theorem is developed in Ref. 12. Given a discrete symmetry group $\Gamma$ one does not work directly with the dual space $\hat{\Gamma}$ but instead with a $C^*$-algebra of continuous functions on that space. Thus the basic topic in Ref. 12 is the spectral analysis of elliptic operators on Hilbert $C^*$-modules over noncommutative $C^*$-algebras. The symmetry group $\Gamma$ can be noncommutative, its action is assumed to be co-compact.

In the current paper we recall the Bloch decomposition in a form which is rather close to that due to Sunada.\textsuperscript{34} For a more detailed discussion see Remark \textsuperscript{33} below in the text. Let us stress however that our motivation is quite different than are those in the above cited papers and thus we do not aim at all at the spectral analysis of elliptic operators in question. For our purposes we need only to construct a decomposition of a Hamiltonian on the covering space $\tilde{M}$ into a direct integral over the dual space $\hat{\Gamma}$ whose components are Hamiltonians based on the manifold $M = \tilde{M}/\Gamma$.

Let us point out some other features of our approach. Firstly, here we are interested only in the case when the gauge field vanishes, and thus, in a suitable formalism, we deal just with Laplace-Beltrami operators $\Delta_{LB}$. Furthermore, rather than working with $L^2$ spaces of sections in vector bundles over $M$ we prefer to work with spaces of equivariant functions on the covering space $\tilde{M}$ since this way one avoids the explicit use of vector potentials. Secondly, we do not require the manifold $M$ to be compact. For this more general setting we have to distinguish the Friedrichs extension as the preferred self-adjoint extension of a semibounded symmetric operator $-\Delta_{LB} + V$ defined on smooth functions with compact supports. Finally, we restrict ourselves to type I discrete groups $\Gamma$ since for these groups the generalized harmonic analysis is well established.\textsuperscript{29} Unfortunately, with this restriction, some interesting physical models are not covered. Nevertheless one may hope that further generalizations of our approach are possible since the generalized Fourier analysis has been developed also for some other groups which are not included among the type I groups.

The paper is organized as follows. In Section 2 we introduce some basic notation and notions, and we specify more precisely the goals of the paper. Several facts concerning the generalized Fourier analysis that are important for our approach are recalled in Section 3. In Section 4 we discuss the generalized Bloch decomposition. Further we concentrate on the propagator which is regarded as the generalized kernel of the unitary evolution operator. Thus, to deal with the propagator, we apply the Schwartz kernel theorem. In addition, we have to adjust this theorem to our purposes. This is done in Section 5. Finally, Section 6 is devoted to the desired interpretation of the Schulman’s ansatz.
II. Formulation of the problem

We consider a smooth connected Riemannian manifold $\tilde{M}$ (which is supposed to be Hausdorff and second countable). Denote by $\tilde{\mu}$ the measure on $\tilde{M}$ induced by the Riemannian metric. All $L^p$ spaces based on $\tilde{M}$ will be understood with this measure. Furthermore, there is given an at most countable discrete group $\Gamma$ acting on $\tilde{M}$ as a symmetry group, i.e., the Riemannian metric is $\Gamma$-invariant. The action is assumed to be smooth, free and proper (or, by another frequently used terminology, properly discontinuous). Let us recall that under these assumptions any element $s \in \Gamma$ different from the unity has no fixed points on $\tilde{M}$, and for any compact set $K \subset \tilde{M}$, the intersection $K \cap s \cdot K$ is nonempty only for finitely many elements $s \in \Gamma$. This also implies that any point $y \in \tilde{M}$ has a neighborhood $U$ such that the sets $s \cdot U$, $s \in \Gamma$, are mutually disjoint (Ref. 21, Corollary 12.10).

The quotient $M = \tilde{M}/\Gamma$ is again a connected Riemannian manifold (Ref. 1, Proposition 4.1.23). Let $\mu$ denote the induced measure on $M$. Again, all $L^p$ spaces based on $M$ are understood with this measure. Thus we get a principal fiber bundle $\pi : \tilde{M} \rightarrow M$ with the structure group $\Gamma$.

In some applications the following example is of interest. One starts from a connected Riemannian manifold $M$. Let $\tilde{M}$ be the universal covering space of $M$ and $\Gamma = \pi_1(M)$ be the fundamental group of $M$. Then $\Gamma$ is at most countable (Ref. 21, Theorem 8.11), $\Gamma$ acts on $\tilde{M}$ smoothly, freely and properly (Ref. 20, Chapter 21), and one can naturally identify $M$ with $\tilde{M}/\Gamma$ (see also Proposition 5.9 in Ref. 18).

Let us denote by $L_s$ the left action of $s \in \Gamma$ on $\tilde{M}$, i.e., $L_s(y) = s \cdot y$ for $y \in \tilde{M}$. Given a unitary representation $\Lambda$ of $\Gamma$ on a separable Hilbert space $\mathcal{L}_\Lambda$ one constructs the Hilbert space $\mathcal{H}_\Lambda$ formed by $\Lambda$-equivariant vector-valued functions on $\tilde{M}$ (more precisely, by their equivalence classes modulo measure zero). In more detail, any function $\psi \in \mathcal{H}_\Lambda$ is measurable with values in $\mathcal{L}_\Lambda$ and

$$\forall s \in \Gamma, L_s^*\psi = \Lambda(s)\psi.$$ 

Furthermore, the norm of $\psi$ induced by the scalar product, as introduced below, must be finite. Here, as usual, $f^* : \text{Fun}(N) \rightarrow \text{Fun}(M)$ is the pull-back mapping associated to $f : M \rightarrow N$ and some appropriate function spaces $\text{Fun}(M)$, $\text{Fun}(N)$ based on the sets $M$ and $N$, respectively. In order to keep the notation simple the same symbol, $f^*$, will be used independently of the concrete nature of functional spaces in question (they may be formed, for example, by smooth functions or square integrable functions). The
scalar product of $\psi_1, \psi_2 \in \mathcal{H}_\Lambda$ is defined by

$$\langle \psi_1, \psi_2 \rangle = \int_M \langle \psi_1(y), \psi_2(y) \rangle \, d\mu(x). \quad (1)$$

**Remark 1.** In (1) and everywhere in what follows we use the following convention. If $f$ is a measurable function on $\tilde{M}$ such that $f$ is constant on the fibers of $\pi$ (equivalently, $f$ is $\Gamma$-invariant) then $f = \pi^* g$ for some, essentially unique, measurable function $g$ on $M$. If $g \in L^1(M)$ then by the integral

$$\int_M f(y) \, d\mu(x)$$

we mean $\int_M g \, d\mu$. Notice that, with this convention, if $f \in L^1(\tilde{M})$ is arbitrary then

$$\int_{\tilde{M}} f \, d\tilde{\mu} = \int_M \sum_{s \in \Gamma} L_s^* f(y) \, d\mu(x). \quad (2)$$

One can easily see that the sum $\sum_{s \in \Gamma} L_s^* f(y)$ converges absolutely almost everywhere on $\tilde{M}$.

Let $\Delta_{LB}$ be the Laplace-Beltrami operator on $\tilde{M}$. As a differential operator on $C^\infty_0(\tilde{M})$, $\Delta_{LB}$ is unambiguously determined by the equality

$$\forall \varphi_1, \varphi_2 \in C^\infty_0(\tilde{M}), \quad \langle \varphi_1, -\Delta_{LB} \varphi_2 \rangle = \int_M \tilde{g}(d\varphi_1, d\varphi_2) \, d\tilde{\mu}$$

where the scalar product on the LHS is understood in $L^2(\tilde{M})$ and $\tilde{g}$ is the Riemannian metric defined on the cotangent spaces on $\tilde{M}$. Suppose further that there is given a measurable $\Gamma$-invariant bounded real function $V(y)$ on $\tilde{M}$. Then one can introduce with the aid of the Friedrichs extension the Hamiltonian $H = -\Delta_{LB} + V$ as a selfadjoint operator on $L^2(\tilde{M})$. Actually, the differential operator $-\Delta_{LB} + V$ is symmetric and bounded below on the domain $C^\infty_0(\tilde{M})$. Moreover, the invariance of the Riemannian metric and the invariance of $V$ imply that $H$ is a $\Gamma$-invariant operator (i.e., $H$ commutes with all $L_s^*$, $s \in \Gamma$).

To the same differential operator, $-\Delta_{LB} + V$, one can relate a selfadjoint operator $H_\Lambda$ on the space $\mathcal{H}_\Lambda$ for any unitary representation $\Lambda$ of $\Gamma$. First one constructs a linear subspace in $\mathcal{H}_\Lambda$ formed by smooth vector-valued functions. Let us define

$$\Phi_\Lambda := \sum_{s \in \Gamma} L_s^* \otimes \Lambda(s^{-1}) : C^\infty_0(\tilde{M}) \otimes \mathcal{L}_\Lambda \to \mathcal{H}_\Lambda \quad (3)$$
If $\varphi \in C_0^\infty(\tilde{M})$, $v \in \mathcal{L}_\Lambda$, then
\[
(\Phi_\Lambda \varphi \otimes v)(y) = \sum_{s \in \Gamma} \varphi(s \cdot y) \Lambda(s^{-1}) v,
\]
and on any compact set $K \subset \tilde{M}$, only a finite number of summands on the RHS of (4) do not vanish (for the action of $\Gamma$ is proper). Consequently, the vector-valued function $\Phi_\Lambda \varphi \otimes v$ is smooth.

From definition (3) one immediately finds that
\[
\forall s \in \Gamma, \quad L_s^* \circ \Phi_\Lambda = \Lambda(s) \Phi_\Lambda,
\]
\[
\forall s \in \Gamma, \quad \Phi_\Lambda \circ (L_s^* \otimes 1) = \Phi_\Lambda \circ (1 \otimes \Lambda(s)).
\]
Property (5) implies that $\Phi_\Lambda \varphi \otimes v$ is actually $\Lambda$-equivariant. It is also not difficult to see from (4) that the norm of $\Phi_\Lambda \varphi \otimes v$ in the Hilbert space $\mathcal{H}_\Lambda$ is finite.

Furthermore, using (2) and some simple manipulations one finds that for any $\varphi \in C_0^\infty(\tilde{M})$ and $v \in \mathcal{L}_\Lambda$,
\[
\forall \psi \in \mathcal{H}_\Lambda, \quad \langle \psi, \Phi_\Lambda \varphi \otimes v \rangle = \int_{\tilde{M}} \varphi(y) \langle \psi(y), v \rangle \, d\tilde{\mu}(y).
\]
From this one deduces that the range of $\Phi_\Lambda$ is dense in $\mathcal{H}_\Lambda$. As a particular case of (7) we have
\[
\langle \Phi_\Lambda \varphi_1 \otimes v_1, \Phi_\Lambda \varphi_2 \otimes v_2 \rangle = \sum_{s \in \Gamma} \langle \Lambda(s^{-1}) v_1, v_2 \rangle \langle L_s^* \varphi_1, \varphi_2 \rangle.
\]
Here the scalar product $\langle L_s^* \varphi_1, \varphi_2 \rangle$ is understood in $L^2(\tilde{M})$ and, once again, it is nonzero only for finite number of elements $s \in \Gamma$ for the action of $\Gamma$ is proper.

The Laplace-Beltrami operator is well defined on $\text{Ran}(\Phi_\Lambda)$. Since $\Delta_{LB}$ commutes with $L_s^*$, $s \in \Gamma$, one has
\[
\Delta_{LB} \Phi_\Lambda[\varphi \otimes v] = \Phi_\Lambda[\Delta_{LB} \varphi \otimes v].
\]

**Lemma 2.** The differential operator $-\Delta_{LB}$ is positive on the domain $\text{Ran}(\Phi_\Lambda) \subset \mathcal{H}_\Lambda$.

**Proof.** Denote by $\mathfrak{g}$ the Riemannian metric defined on the cotangent spaces on $M$. Recall that $\tilde{\mathfrak{g}}$ has a similar meaning on the Riemannian manifold $\tilde{M}$. Using (8) and
equality (2) in Remark 1 one derives that

$$\langle \Phi_A \varphi_1 \otimes v_1, -\Delta_{LB} \Phi_A \varphi_2 \otimes v_2 \rangle = \sum_{s \in \Gamma} \langle v_1, \Lambda(s^{-1})v_2 \rangle \int_M \tilde{g}(d\varphi_1, L_s^*d\varphi_2) \, d\tilde{\mu}$$

$$= \int_M \sum_{t \in \Gamma} \sum_{s \in \Gamma} \langle \Lambda(t^{-1})v_1, \Lambda(s^{-1})v_2 \rangle \tilde{g}(L_t^*d\varphi_1(y), L_s^*d\varphi_2(y)) \, d\mu(x).$$

(10)

Choose $\psi \in \text{Ran} \Phi_A$. Then $K = \pi(\text{supp} \, \psi)$ is a compact subset of $M$. There exists a finite open covering of $K$, $K \subset \bigcup_{i=1}^n U_i$, such that the fiber bundle $\pi : \tilde{M} \to M$ is trivial over each $U_i$, i.e., there exist smooth sections $\sigma_i : U_i \to \tilde{M}$, $\pi \circ \sigma_i = \text{id}_{U_i}$. Furthermore, for this covering there exists a partition of unity $\{\eta_i \}_{i=1}^n$ where $\eta_i \in C_0^\infty(U_i)$, $0 \leq \eta_i \leq 1$, and $\sum \eta_i \equiv 1$ on a neighborhood of $K$. Choose also auxiliary functions $\xi_i \in C_0^\infty(U_i)$ so that $0 \leq \xi_i \leq 1$ and $\xi_i \equiv 1$ on a neighborhood of $\text{supp} \, \eta_i$. Set

$$\tilde{\xi}_i = \pi^* \xi_i \in C^\infty(\pi^{-1}(U_i)).$$

The only reason of introducing the functions $\tilde{\xi}_i$ is to cope with the fact, in the formulas below, that the sections $\sigma_i$ are defined only locally. One simply uses the natural inclusion $C_0^\infty(U_i) \subset C_0^\infty(M)$. If $\varphi_1, \varphi_2 \in C_0^\infty(\tilde{M})$ satisfy $\text{supp} \, \varphi_1, \text{supp} \, \varphi_2 \subset \pi^{-1}(K)$ then expression (10) equals

$$\sum_{i=1}^n \sum_{t \in \Gamma} \sum_{s \in \Gamma} \langle \Lambda(t^{-1})v_1, \Lambda(s^{-1})v_2 \rangle \int_M \eta_i \, g(\sigma_i^* d(\tilde{\xi}_i L_t^* \varphi_1), \sigma_i^* d(\tilde{\xi}_i L_s^* \varphi_2)) \, d\mu.$$

Let $s_i$ be a positive sesquilinear form on $C_0^\infty(M) \otimes \mathcal{L}_A$ defined by

$$s_i(\zeta \otimes v_1, \zeta_2 \otimes v_2) = \langle v_1, v_2 \rangle \int_M \eta_i \, g(d\zeta_1, d\zeta_2) \, d\mu.$$

With this notation we have

$$\langle \Phi_A \varphi_1 \otimes v_1, -\Delta_{LB} \Phi_A \varphi_2 \otimes v_2 \rangle = \sum_{i=1}^n s_i(\sigma_i^* (\tilde{\xi}_i \Phi_A \varphi_1 \otimes v_1), \sigma_i^* (\tilde{\xi}_i \Phi_A \varphi_2 \otimes v_2)).$$

Thus we arrive at the following conclusion. Set $\phi_i = \sigma_i^* (\tilde{\xi}_i \psi) \in C_0^\infty(M) \otimes \mathcal{L}_A$. Then

$$\langle \psi, -\Delta_{LB} \psi \rangle = \sum_{i=1}^n s_i(\phi_i, \phi_i) \geq 0.$$
This completes the verification.

Clearly, since the function $V(y)$ is $\Gamma$-invariant the multiplication operator by $V$ is well defined in the Hilbert space $\mathcal{H}_\Lambda$. Now the definition of the Hamiltonian $H_\Lambda$ is straightforward. This is the Friedrichs extension of the differential operator $-\Delta_{LB} + V$ considered on the domain $\text{Ran} \Phi_\Lambda$.

Let us denote by $U(t) = \exp(-itH)$, $t \in \mathbb{R}$, the evolution operator in $L^2(\tilde{M})$. Similarly, $U_\Lambda(t) = \exp(-itH_\Lambda)$, $t \in \mathbb{R}$, is the evolution operator in $\mathcal{H}_\Lambda$ where $\Lambda$ is a unitary representation of $\Gamma$. Denote by $\hat{\Gamma}$ the dual space to $\Gamma$ (the quotient space of the space of irreducible unitary representations of $\Gamma$). In the current paper we wish to address the following two mutually complementary problems. First, to express $U(t)$ in terms of $U_\Lambda(t)$, $\Lambda \in \hat{\Gamma}$. Second, to express $U_\Lambda(t)$ in terms of $U(t)$ for a fixed unitary representation $\Lambda$ of $\Gamma$. It turns out that answers to both problems do exist. A solution to the former one is provided by the generalized Bloch decomposition. A solution to the latter problem is given by a formula known from the theoretical physics (the Schulman’s ansatz).\textsuperscript{27, 28}

III. The generalized Fourier analysis

The generalized harmonic analysis is well established for locally compact groups of type I.\textsuperscript{29} This is why we restrict ourselves to the case when $\Gamma$ is a type I group. Countable discrete groups of type I are well characterized (Ref. 38, Satz 6).

**Theorem 3** (Thoma). A countable discrete group is type I if and only if it has an Abelian normal subgroup of finite index.

Unfortunately, there are multiply connected configuration spaces of interest whose fundamental group is not of type I. For example, the configuration space for the two-dimensional model describing a charged quantum particle moving in the magnetic field of $r$ Aharonov-Bohm fluxes is a plane with $r$ excluded points. It is well known that $\pi_1(\mathbb{R}^2 \setminus \{p_1, \ldots, p_r\})$ is the free group with $r$ generators. However, a freely generated group with two and more generators is not of type I. In this case, though, the situation is not completely lost since a harmonic analysis has been proved to exist on free groups as well.\textsuperscript{8} But we do not cover this example in the current paper.

The discrete group $\Gamma$ is understood to be equipped with the counting measure. Let $d\hat{\mu}$ be the Plancherel measure on $\hat{\Gamma}$. Denote by $\mathcal{I}_2(\mathcal{L}_\Lambda) \equiv \mathcal{L}_\Lambda^* \otimes \mathcal{L}_\Lambda$ Hilbert space formed by Hilbert-Schmidt operators on $\mathcal{L}_\Lambda$ (here $\mathcal{L}_\Lambda^*$ is the dual space to $\mathcal{L}_\Lambda$). The
Fourier transformation is constructed as a unitary mapping

\[ \mathcal{F} : L^2(\Gamma) \rightarrow \int_{\hat{\Gamma}} \mathcal{I}_2(\mathcal{L}_\Lambda) \, d\hat{m}(\Lambda). \]  

(11)

Let us list its basic properties.\(^{29}\) If \( f_i \in L^2(\Gamma) \), \( \hat{f}_i = \mathcal{F}[f_i] \), \( i = 1, 2 \), then

\[ \sum_{s \in \Gamma} f_1(s)f_2(s) = \int_{\Gamma} \text{Tr}[\hat{f}_1(\Lambda)^*\hat{f}_2(\Lambda)] \, d\hat{m}(\Lambda). \]

Furthermore, for \( f \in L^1(\Gamma) \subset L^2(\Gamma) \), one has

\[ \mathcal{F}[f](\Lambda) = \sum_{s \in \Gamma} f(s)\Lambda(s). \]

There exists an inversion formula: if \( f \) is of the form \( f = g * h \) (the convolution) where \( g, h \in L^1(\Gamma) \), and \( \hat{f} = \mathcal{F}[f] \) then

\[ f(s) = \int_{\Gamma} \text{Tr}[\Lambda(s)^*\hat{f}(\Lambda)] \, d\hat{m}(\Lambda). \]

Under our restrictions, one does not encounter any problems when interpreting the above formulas. This is guaranteed in an obvious manner by the following theorem (Ref. 38, Korollar I).

**Theorem 4** (Thoma). If \( \Gamma \) is a countable discrete group of type I then \( \dim \mathcal{L}_\Lambda \) is a bounded function of \( \Lambda \) on the dual space \( \hat{\Gamma} \).

Consequently, \( \mathcal{I}_2(\mathcal{L}_\Lambda) \) coincides with the space of all linear operators on \( \mathcal{L}_\Lambda \), and the trace is well defined in the usual sense. For example, let \( \delta_g \in L^2(\Gamma) \), \( g \in \Gamma \), be defined by \( \delta_g(s) = \delta_{g,s}, \forall s \in \Gamma \). Then \( \mathcal{F}[\delta_g](\Lambda) = \Lambda(g) \) and

\[ \|\delta_g\|^2 = \|\mathcal{F}[\delta_g]\|^2 = \int_{\Gamma} \text{Tr}[\Lambda(g)^*\Lambda(g)] \, d\hat{m}(\Lambda) = \int_{\Gamma} \dim \mathcal{L}_\Lambda \, d\hat{m}(\Lambda). \]

Hence

\[ \int_{\Gamma} \dim \mathcal{L}_\Lambda \, d\hat{m}(\Lambda) = 1 \]

and

\[ \hat{m}(\hat{\Gamma}) \leq 1. \]  

(12)

Finally, let us note that the Fourier transformation decomposes the regular representation \( \mathcal{R} \) of \( \Gamma \) into a direct integral of irreducible representations. The regular
representation acts on $L^2(\Gamma)$ as $R_s = L_{s^{-1}}^*$, $\forall s \in \Gamma$, and one has

$$\forall s \in \Gamma, \ F R_s F^{-1} = \int_\Gamma 1 \otimes \Lambda(s) \, d\tilde{m}(\Lambda)$$

(with the identification $\mathcal{F}(\mathcal{L}_\Lambda) \equiv \mathcal{L}_\Lambda^* \otimes \mathcal{L}_\Lambda$). This relation means nothing but

$$\forall s \in \Gamma, \forall f \in L^2(\Gamma), \ F[L_s f](\Lambda) = \Lambda(s^{-1})F[f](\Lambda).$$

(13)

In this context, of course, $L_s$ stands for the left action of $\Gamma$ on itself.

**IV. The generalized Bloch decomposition**

An application of the harmonic analysis on $\Gamma$ makes it possible to carry out the first step in the Bloch analysis. This means a decomposition of the Hilbert space $L^2(\tilde{M})$ into a direct integral jointly with a corresponding decomposition of the Hamiltonian $H$. Let us describe the procedure in detail. In the notation below, the variable $y$ usually runs over $\tilde{M}$ while $x$ runs over $M$. Recall also Remark 11 used repeatedly throughout this section and, in particular, the meaning of the symbol $\int_M f(y) \, d\mu(x)$ for a $\Gamma$-invariant function $f$ on $\tilde{M}$.

For $f \in L^2(\tilde{M})$ and $y \in \tilde{M}$ set

$$\forall s \in \Gamma, \ f_y(s) = f(s^{-1} \cdot y).$$

Obviously,

$$\forall s \in \Gamma, \ f_{s \cdot y} = L_{s^{-1}}^* f_y$$

(14)

(here again, $L_s$ stands for the left action of $\Gamma$ on itself). Thus the norm $\|f_y\|$ taken in $L^2(\Gamma)$ is a $\Gamma$-invariant function of $y \in \tilde{M}$ and one easily finds that

$$\|f\|^2 = \int_M \|f_y\|^2 \, d\mu(x).$$

Hence for almost all $x \in M$ and all $y \in \pi^{-1}(\{x\})$ one has $f_y \in L^2(\Gamma)$. Observe that the tensor product $\mathcal{L}_\Lambda^* \otimes \mathcal{H}_\Lambda$ can be naturally identified with the Hilbert space of $1 \otimes \Lambda$-equivariant operator-valued functions on $\tilde{M}$ with values in $\mathcal{L}_\Lambda^* \otimes \mathcal{L}_\Lambda \equiv \mathcal{F}(\mathcal{L}_\Lambda)$. 

11
Definition 5. The mapping

\[ \Phi : L^2(\tilde{M}) \rightarrow \int_{\hat{\Gamma}} L^*_{\Lambda} \otimes H_{\Lambda} \, d\hat{m}(\Lambda) \]

is defined so that for \( f \in L^2(\tilde{M}) \) and \( \Lambda \in \hat{\Gamma} \), the component \( \Phi[f](\Lambda) \) is a measurable operator-valued function on \( \tilde{M} \),

\[ \Phi[f](\Lambda) (y) := \mathcal{F}[f_y](\Lambda) \in \mathcal{S}_2(L_{\Lambda}). \] (15)

In particular, if \( f \in L^1(\tilde{M}) \cap L^2(\tilde{M}) \) then

\[ \Phi[f](\Lambda)(y) = \sum_{s \in \Gamma} f(s^{-1} \cdot y)\Lambda(s). \]

From here one can also deduce a simple relation between \( \Phi \) and the mappings \( \Phi_{\Lambda}, \Lambda \in \hat{\Gamma} \), as introduced in (3). For \( \varphi \in C^\infty_0(\tilde{M}) \), \( v \in L_{\Lambda} \) and \( y \in \tilde{M} \),

\[ \Phi[\varphi](\Lambda)(y)v = (\Phi_{\Lambda}\varphi \otimes v)(y). \] (16)

Proposition 6. \( \Phi \) is a well defined unitary mapping.

Proof. (i) According to the above discussion, if \( f \in L^2(\tilde{M}) \) then for a.a. \( y \in \tilde{M} \), \( \mathcal{F}[f_y](\Lambda) \) is well defined for a.a. \( \Lambda \in \hat{\Gamma} \). By the Fubini theorem, for a.a. \( \Lambda \in \hat{\Gamma} \), the vector-valued function \( \Phi[f](\Lambda) \) is defined almost everywhere on \( \tilde{M} \). Moreover, it immediately follows from (15), (14) and (13) that \( \Phi[f](\Lambda) \) is \( 1 \otimes \Lambda \)-equivariant.

(ii) \( \Phi \) is an isometry. Indeed (see the defining relation (1), Remark 1 and (2)),

\[ \|\Phi[f]\|^2 = \int_{\hat{\Gamma}} \left( \int_M \|\Phi[f](\Lambda)(y)\|^2 \, d\mu(x) \right) \, d\hat{m}(\Lambda) \]

\[ = \int_M \left( \int_{\hat{\Gamma}} \|\mathcal{F}[f_y](\Lambda)\|^2 \, d\hat{m}(\Lambda) \right) \, d\mu(x) \]

\[ = \int_M \|f_y\|^2 \, d\mu(x) = \|f\|^2. \]

(iii) \( \Phi \) is surjective. Let \( \psi \in \int_{\hat{\Gamma}} L^*_{\Lambda} \otimes H_{\Lambda} \, d\hat{m}(\Lambda) \). This implies that \( \psi(\Lambda) \) is well defined for a.a. \( \Lambda \in \hat{\Gamma} \), and for such \( \Lambda \) and for a.a. \( y \in \tilde{M} \), \( \psi(\Lambda)(y) \in L^*_{\Lambda} \otimes L_{\Lambda} \equiv \mathcal{S}_2(L_{\Lambda}) \). By the Fubini theorem,

\[ \|\psi\|^2 = \int_M \left( \int_{\hat{\Gamma}} \|\psi(\Lambda)(y)\|^2 \, d\hat{m}(\Lambda) \right) \, d\mu(x). \]
Hence for a.a. \( x \in M \) and all \( y \in \pi^{-1}(\{x\}) \), \( \psi(\cdot)(y) \in L^2(\hat{\Gamma}) \). Since the Fourier transform is surjective there exists an essentially unique measurable function \( \check{\psi}(s, y) \) on \( \Gamma \times \hat{M} \) such that for a.a. \( y \in \hat{M} \), \( \check{\psi}(\cdot, y) \in L^2(\Gamma) \) and \( \mathcal{F}[\check{\psi}(\cdot, y)](\Lambda) = \psi(\Lambda)(y) \). The \( \mathcal{A}(\mathcal{L}_\Lambda) \)-valued function \( \psi(\Lambda) \) is equivariant, i.e., \( L^*_s \psi(\Lambda) = \Lambda(s) \psi(\Lambda) \), \( \forall s \in \Gamma \).

Recalling \( \mathcal{L}_\Lambda \) we have

\[
\mathcal{F}[\check{\psi}(\cdot, s \cdot y)](\Lambda) = \Lambda(s) \psi(\Lambda)(y) = \Lambda(s) \mathcal{F}[\check{\psi}(\cdot, y)](\Lambda) = \mathcal{F}[L^*_s \check{\psi}(\cdot, y)](\Lambda).
\]

From the injectivity of the Fourier transform it follows that

\[
\forall s, r \in \Gamma, \text{ for a.a. } y \in \hat{M}, \quad \check{\psi}(r, s \cdot y) = \check{\psi}(s^{-1} r, y).
\]

In particular, letting \( r = 1 \), we have

\[
\forall s \in \Gamma, \text{ for a.a. } y \in \hat{M}, \quad \check{\psi}(s, y) = \check{\psi}(1, s^{-1} \cdot y).
\]

Set \( f(y) = \check{\psi}(1, y) \). Then one has \( f_y(s) = \check{\psi}(s, y) \) and so \( f_y \in L^2(\Gamma) \) for a.a. \( y \in \hat{M} \). Moreover, \( \mathcal{F}[f_y](\Lambda) = \psi(\Lambda)(y) \). From here one easily concludes that \( f \in L^2(\hat{M}) \) and \( \Phi[f] = \psi \). \( \square \)

**Proposition 7.** The decomposition

\[
\Phi H \Phi^{-1} = \int_{\Gamma} 1 \otimes H_\Lambda \, d\hat{m}(\Lambda) \quad (17)
\]

holds true and, consequently,

\[
\Phi U(t) \Phi^{-1} = \int_{\Gamma} 1 \otimes U_\Lambda(t) \, d\hat{m}(\Lambda). \quad (18)
\]

**Proof.** Relation \( \mathcal{L}_\Lambda(\Phi) \) between \( \Phi \) and \( \Phi_\Lambda \) and equality \( (9) \) imply that

\[
\forall \varphi \in C^\infty_0(\hat{M}), \quad \Delta_{LB} \Phi[\varphi] = \Phi[\Delta_{LB} \varphi]. \quad (19)
\]

Denote by \( H^0 \) the Friedrichs extension of the differential operator \( -\Delta_{LB} \) with the domain \( C^\infty_0(\hat{M}) \) in the Hilbert space \( L^2(\hat{M}) \). Similarly, let \( H^0_\Lambda \) be the Friedrichs extension of the differential operator \( -\Delta_{LB} \) with the domain \( \text{Ran} \Phi_\Lambda \) in the Hilbert space \( \mathcal{H}_\Lambda \). Equality \( (19) \) implies that for all \( \varphi_1, \varphi_2 \in C^\infty_0(\hat{M}) \),

\[
\langle \varphi_1, -\Delta_{LB} \varphi_2 \rangle = \langle \Phi[\varphi_1], \Phi[-\Delta_{LB} \varphi_2] \rangle = \int_{\Gamma} \langle \Phi[\varphi_1](\Lambda), 1 \otimes (-\Delta_{LB}) \Phi[\varphi_2](\Lambda) \rangle \, d\hat{m}(\Lambda).
\]
Closing the quadratic forms one finds that

$$\Phi H^0 \Phi^{-1} = \int_{\Gamma} ^{\oplus} 1 \otimes H^0_{\Lambda} \, \text{d}\tilde{m}(\Lambda). \tag{20}$$

With some abuse of notation we denote by $V$ the multiplication operator by the function $V(y)$ in the Hilbert space $L^2(\tilde{M})$. According to our assumptions, the function $V(y)$ is bounded and so $V$ is a bounded operator. Since the function $V(y)$ is $\Gamma$-invariant the corresponding multiplication operator can be introduced also in the Hilbert space $\mathcal{H}_\Lambda$. In this case the operator will be denoted by the symbol $V_\Lambda$. The $\Gamma$-invariance of $V(y)$ implies that for any $f \in L^2(\tilde{M})$, $(Vf)_y = V(y)f_y$, whence

$$\Phi [Vf](\Lambda)(y) = V(y)\Phi [f](\Lambda)(y).$$

This equality means nothing but

$$\Phi V \Phi^{-1} = \int_{\Gamma} ^{\oplus} 1 \otimes V_\Lambda \, \text{d}\tilde{m}(\Lambda). \tag{21}$$

Since $H = H^0 + V$ and $H_\Lambda = H^0_{\Lambda} + V_\Lambda$ relation (17) follows from (20) and (21). $\square$

Remark 8. Various forms of decomposition (17) can be found in the literature. Let us make a short comparison to some previous works. The generalized Bloch theory has been proposed and used by Sunada and collaborators\textsuperscript{34,2,3} to investigate the character of spectra of $\Gamma$-periodic elliptic operators on $\tilde{M}$ under the assumption that the quotient $\tilde{M}/\Gamma$ is compact. In particular, in this case one can demonstrate the band structure of spectrum, see also Ref. 7 for an asymptotic estimate of the number of bands. These ideas have been also applied in Refs. 22, 23 to construct coverings $\tilde{M} \to M = \tilde{M}/\Gamma$ such that $M$ is a compact Riemannian manifold, $\tilde{M}$ is non-compact and the Laplace-Beltrami operator $\Delta_{LB}$ on $\tilde{M}$ has at least a prescribed finite number of spectral gaps. The Bloch decomposition used in Refs. 34, 23 basically coincides with that given in the current paper, only some details of presentation somewhat differ (for example, we avoid fixing a fundamental domain for the action of $\Gamma$ on $\tilde{M}$). In fact, the situation treated in these references is more abstract in the following sense. Instead of the Fourier transform (11) one can consider a unitary mapping

$$\mathcal{F} : L^2(\Gamma) \to \int_{Z} ^{\oplus} \mathcal{H}(z) \, \text{d}z$$

such that the regular representation $\mathcal{R}_s = L^s_{\Lambda-1}$ of $\Gamma$ in $L^2(\Gamma)$ decomposes correspond-
\[ \forall s \in \Gamma, \ q(R_s q^{-1} = \int_Z^{\oplus} R_s(z) \, dz. \quad (22) \]

Here \((Z, dz)\) is supposed to be a separable Hausdorff space with a regular Borel measure, and \(R_s(z), z \in Z\), is a unitary operator in the Hilbert space \(\mathcal{H}(z)\). The unitary mapping \(\Phi : L^2(\tilde{M}) \rightarrow \int_Z^{\oplus} \mathcal{H}(z) \, dz\) is again defined so that for \(f \in L^2(\tilde{M})\) and \(z \in Z\), the component \(\Phi[f](z)\) is a measurable \(R(z)\)-equivariant vector-valued function on \(\tilde{M}\),

\[ \Phi[f](z)(y) := \mathcal{F}[f_y](z) \in \mathcal{H}(z). \]

Observe that for \(h \in L^2(\Gamma)\), \(h = \sum_{s \in \Gamma} h(s) R_s \delta_e \) (\(e \in \Gamma\) is the unit element). Furthermore, (22) means that \(\forall s \in \Gamma, \forall h \in L^2(\Gamma)\),

\[ \mathcal{F}[R_s h](z) = R_s(z) \mathcal{F}[h](z) \text{ a.e. on } Z. \]

It follows that for \(\varphi \in C_0^\infty(\tilde{M})\),

\[ \Phi[\varphi](z)(y) = \sum_{s \in \Gamma} \varphi(s^{-1} \cdot y) R_s(z) \mathcal{F}[\delta_e](z). \quad (23) \]

This is the form of \(\Phi\) used in Refs. 34, 23. From (23) one deduces that \(\Phi\) decomposes the Friedrichs extension of the Laplace-Beltrami operator \(\Delta_{LB}\) on \(\tilde{M}\).

In Ref. 11 the group \(\Gamma\) is supposed to be Abelian. On the other hand, one considers therein magnetic Hamiltonians with in general non-vanishing magnetic fields. In more detail, let \((L, h, \nabla)\) be a Hermitian line bundle with connection over \(M\) and \((\tilde{L}, \tilde{h}, \tilde{\nabla}) = \pi^*(L, h, \nabla)\) where \(\pi : \tilde{M} \rightarrow M\) is the projection. Denote by \(C^\infty(\tilde{L})\) the vector space of smooth sections in \(\tilde{L}\). If \(\sigma \in C^\infty(\tilde{L})\) then \(\tilde{\nabla}\sigma\) belongs to \(C^\infty(\pi^* M \otimes \tilde{L})\). The Bochner Laplacian \(\Delta_B\) as a differential operator acting on \(C_0^\infty(\tilde{L})\) is unambiguously determined by the equality

\[ \forall \sigma_1, \sigma_2 \in C_0^\infty(\tilde{L}), \quad \int_{\tilde{M}} h(\sigma_1, -\Delta_B \sigma_2) \, d\tilde{\mu} = \int_{\tilde{M}} \tilde{g} \otimes \tilde{h}(\tilde{\nabla}\sigma_1, \tilde{\nabla}\sigma_2) \, d\tilde{\mu}. \]

The magnetic Schrödinger operator \(H^\nabla\) in the Hilbert space \(L^2(\tilde{L})\) (the Hilbert space of square integrable sections in \(\tilde{L}\)) is the Friedrichs extension of \(-\Delta_B\) with the domain
\[ C^\infty_0(\tilde{L}) \] The action of \( \Gamma \) on \( \tilde{M} \) lifts in a canonical way to an isometric linear action \( \gamma \) of \( \Gamma \) on \( \tilde{L} \). For \( \chi \in \tilde{\Gamma} \) let \( \mathcal{H}_{\tilde{L}, \chi} \) be the Hilbert space of measurable sections in \( \tilde{L} \) which satisfy
\[ \forall s \in \Gamma, \quad \sigma(s \cdot y) = \chi(s) \gamma_s(\sigma(y)) \quad \text{a.e. on } \tilde{M} \]
and have a finite norm (see Remark 1)
\[ \int_{\tilde{M}} \tilde{h}(\sigma(y), \sigma(y)) \, d\mu(x) < \infty. \]

In Ref. 11 one constructs in a way very similar to that of the current paper a unitary mapping
\[ \Phi : L^2(\tilde{L}) \to \int_{\tilde{\Gamma}} \mathcal{H}_{\tilde{L}, \chi} \, d\tilde{m}(\chi) \]
which decomposes \( H^\nabla \).

V. The Schwartz kernel theorem

Equality (18) represents a way how to express the evolution operator \( U(t) \) in terms of \( U_\Lambda(t) \), \( \Lambda \in \tilde{\Gamma} \). Our next task is to invert this relationship. The final formula will concern kernels of operators rather than directly the operators. Let us recall the fundamental kernel theorem due to Schwartz (see, for example, Theorem 5.2.1 in Ref. 14).

**Theorem 9** (Schwartz). Let \( X_i \subset \mathbb{R}^n_i, \ i = 1, 2, \) open, \( K \in \mathcal{D}'(X_1 \times X_2) \). Then by the equation
\[ \forall \varphi_1 \in C^\infty_0(X_1), \varphi_2 \in C^\infty_0(X_2), \quad (K \varphi_1)(\varphi_2) = K(\varphi_1 \otimes \varphi_2) \quad (24) \]
there is defined a continuous linear map \( K : C^\infty_0(X_1) \to \mathcal{D}'(X_2) \). Conversely, to every such continuous linear map \( K \) there is one and only one distribution \( \mathcal{K} \) such that (24) is valid. One calls \( \mathcal{K} \) the kernel of \( K \).

**Corollary 10.** To every \( B \in \mathcal{B}(L^2(\tilde{M})) \) there exists one and only one \( \beta \in \mathcal{D}'(\tilde{M} \times \tilde{M}) \) such that
\[ \forall \varphi_1, \varphi_2 \in C^\infty_0(\tilde{M}), \quad \beta(\varphi_1 \otimes \varphi_2) = \langle \varphi_1, B \varphi_2 \rangle. \]
Moreover, the map \( B \mapsto \beta \) is injective.

We call \( \beta \) the kernel of \( B \).
This corollary of the kernel theorem can be also extended to Hilbert spaces formed by equivariant vector-valued functions. Let us stress that kernels in this case are operator-valued distributions.

**Theorem 11.** Let $\Lambda$ be an irreducible unitary representation of $\Gamma$ in a Hilbert space $L_\Lambda$. To every $B \in \mathcal{B}(L_\Lambda)$ there exists one and only one $\beta \in \mathcal{D}'(\tilde{M} \times \tilde{M}) \otimes \mathcal{B}(L_\Lambda)$ such that

$$\forall \varphi_1, \varphi_2 \in C_0^\infty(\tilde{M}), \forall v_1, v_2 \in L_\Lambda,$$

$$\langle v_1, \beta(\varphi_1 \otimes \varphi_2) v_2 \rangle = \langle \Phi_\Lambda \varphi_1 \otimes v_1, B \Phi_\Lambda \varphi_2 \otimes v_2 \rangle.$$  

The distribution $\beta$ is $\Lambda$-equivariant in the following sense

$$\forall s \in \Gamma, \quad \beta \circ (L_s \otimes 1) = \Lambda(s) \beta, \quad \beta \circ (1 \otimes L_s) = \beta \Lambda(s^{-1})$$  

(25)

(here $L_s \otimes 1$ and $1 \otimes L_s$ are regarded as diffeomorphisms on $\tilde{M} \times \tilde{M}$). Moreover, the map $B \mapsto \beta$ is injective.

**Proof.** From (8) one can see that

$$\|\Phi_\Lambda \varphi \otimes v\|^2 \leq \|v\|^2 \sum_{s \in \Gamma} |\langle \varphi, L^*_s \varphi \rangle|.$$  

Let $K \subset \tilde{M}$ be a compact set. Since the action of $\Gamma$ on $\tilde{M}$ is proper there exists a number $n_K \in \mathbb{N}$ depending only on $K$ such that

$$\forall \varphi \in C_0^\infty(\tilde{M}) \text{ s.t. supp} \varphi \subset K, \forall v \in L_\Lambda, \quad \|\Phi_\Lambda \varphi \otimes v\| \leq n_K \|\varphi\| \|v\|$$  

(26)

(here $\|\varphi\|$ is the norm of $\varphi$ in $L^2(\tilde{M})$). This implies that the linear map

$\Phi_\Lambda : C_0^\infty(\tilde{M}) \otimes L_\Lambda \to \mathcal{H}_\Lambda$ is continuous.

Fix $v_1, v_2 \in L_\Lambda$ and consider the linear map

$$C_0^\infty(\tilde{M}) \to \mathcal{D}'(\tilde{M}) : \varphi_1 \mapsto f_1$$

defined by

$$\forall \varphi_2 \in C_0^\infty(\tilde{M}), \quad f_1(\varphi_2) = \langle \Phi_\Lambda \varphi_1 \otimes v_1, B \Phi_\Lambda \varphi_2 \otimes v_2 \rangle.$$  

Estimate (26) implies that this linear map is continuous. By the Schwartz kernel theorem, there exists $\beta_{v_1,v_2} \in \mathcal{D}'(\tilde{M} \times \tilde{M})$ such that $\beta_{v_1,v_2}(\varphi_1 \otimes \varphi_2) = f_1(\varphi_2).$ For
\( \varphi_1, \varphi_2 \in C^\infty(\tilde{M}) \), the expression \( \beta_{v_1,v_2}(\varphi_1 \otimes \varphi_2) \) is linear in \( v_2 \), anti-linear in \( v_1 \), and one has

\[
|\beta_{v_1,v_2}(\varphi_1 \otimes \varphi_2)| \leq C(\varphi_1, \varphi_2) \|v_1\| \|v_2\|
\]

where \( C(\varphi_1, \varphi_2) \) depends only on \( \varphi_1, \varphi_2 \). Hence there exists a unique bounded operator \( \beta(\varphi_1 \otimes \varphi_2) \in \mathcal{B}(L_\Lambda) \) such that

\[
\forall v_1, v_2 \in L_\Lambda, \quad \langle v_1, \beta(\varphi_1 \otimes \varphi_2) v_2 \rangle = \beta_{v_1,v_2}(\varphi_1 \otimes \varphi_2).
\]

Moreover, \( \beta(\varphi_1 \otimes \varphi_2) \) depends on \( \varphi_1, \varphi_2 \in C^\infty(\tilde{M}) \) continuously. This defines an operator-valued distribution \( \beta \in \mathcal{D}'(\tilde{M} \times \tilde{M}) \otimes \mathcal{B}(L_\Lambda) \).

The diffeomorphism \( L_s, s \in \Gamma \), acting on \( \tilde{M} \) preserves the measure \( \tilde{\mu} \). By the definition of composition of distributions with diffeomorphisms we have, for any \( f \in \mathcal{D}'(\tilde{M}) \) and \( L_s \),

\[
\forall \varphi \in C^\infty(\tilde{M}), \quad f \circ L_s(\varphi) = f(L_{s^{-1}}^s \varphi).
\]

Recalling (6) we get

\[
\langle v_1, \beta \circ (L_s \otimes 1)(\varphi_1 \otimes \varphi_2) v_2 \rangle = \langle v_1, \beta(L_{s^{-1}}^s \varphi_1 \otimes \varphi_2) v_2 \rangle = \langle \Phi_{\Lambda} \circ (L_{s^{-1}}^s \otimes 1) \varphi_1 \otimes v_1, B \Phi_{\Lambda} \varphi_2 \otimes v_2 \rangle = \langle \Phi_{\Lambda} \varphi_1 \otimes \Lambda(s^{-1}) v_1, B \Phi_{\Lambda} \varphi_2 \otimes v_2 \rangle = \langle v_1, \Lambda(s) \beta(\varphi_1 \otimes \varphi_2) v_2 \rangle.
\]

This verifies the first relation in (25). The second relation in (25) can be verified very similarly.

The injectivity immediately follows from the fact that \( \text{Ran} \Phi_{\Lambda} \) is dense in \( \mathcal{H}_\Lambda \). \( \square \)

**VI. The Schulman’s ansatz**

Everywhere in this section \( t \) is a real parameter. We deal with propagators as distributions introduced as kernels of the corresponding evolution operators. Let \( \mathcal{K}_t \in \mathcal{D}'(\tilde{M} \times \tilde{M}) \) be the kernel of \( U(t) \in \mathcal{B}(L^2(\tilde{M})) \), and let \( \mathcal{K}_{\lambda}^A \in \mathcal{D}'(\tilde{M} \times \tilde{M}) \otimes \mathcal{B}(L_\Lambda) \) be the kernel of \( U_{\Lambda}(t) \in \mathcal{B}(L_\Lambda) \). Recall that \( \mathcal{K}_{\lambda}^A \) is \( \Lambda \)-equivariant which means that

\[
\forall s \in \Gamma, \quad \mathcal{K}_{\lambda}^A(s \cdot y_1, y_2) = \Lambda(s) \mathcal{K}_{\lambda}^A(y_1, y_2), \quad \mathcal{K}_{\lambda}^A(y_1, s \cdot y_2) = \mathcal{K}_{\lambda}^A(y_1, y_2) \Lambda(s^{-1}). \quad (27)
\]

For each \( \Lambda \in \hat{\Gamma} \) choose an orthonormal basis \( \{u^\Lambda_n\} \) in \( L_\Lambda \), and let \( \{\phi^\Lambda_n\} \) be the dual
basis in $\mathcal{L}_A^*$. First we wish to rewrite the Bloch decomposition of the propagator (18) in terms of kernels. Note that if $A \in \mathcal{I}_2(\mathcal{L}_A^* \otimes \mathcal{L}_A)$ then

$$A = \sum_n \phi_n^A \otimes A u_n^A.$$ 

In particular, if $\varphi \in C_0^\infty(\tilde{M})$ then

$$\Phi[\varphi](\Lambda) = \sum_n \phi_n^A \otimes (\Phi_\Lambda \varphi \otimes u_n^A) \in \mathcal{L}_A^* \otimes \mathcal{H}_A.$$ 

With the aid of this relation one derives that

$$\langle \Phi[\varphi_1](\Lambda), (1 \otimes U_\Lambda(t)) \Phi[\varphi_2](\Lambda) \rangle = \sum_m \sum_n \langle \phi_m^A, \phi_n^A \rangle \langle \Phi_\Lambda \varphi_1 \otimes u_m^A, U_\Lambda(t) \Phi_\Lambda \varphi_2 \otimes u_n^A \rangle \quad \text{on } \hat{\Gamma}.$$ 

Lemma 12. For all $\varphi_1, \varphi_2 \in C_0^\infty(\tilde{M})$, the function $\Lambda \mapsto \text{Tr}[\mathcal{K}_t^A(\varphi_1 \otimes \varphi_2)]$ is integrable on $\hat{\Gamma}$.

Proof. From (28) one derives the estimate

$$\text{Tr}[\mathcal{K}_t^A(\varphi_1 \otimes \varphi_2)] \leq \| \Phi[\varphi_1](\Lambda) \| \| \Phi[\varphi_2](\Lambda) \|$$

whence

$$\int_{\hat{\Gamma}} \| \text{Tr}[\mathcal{K}_t^A(\varphi_1 \otimes \varphi_2)] \| d\hat{m}(\Lambda) \leq \left( \int_{\hat{\Gamma}} \| \Phi[\varphi_1](\Lambda) \|^2 d\hat{m}(\Lambda) \right)^{1/2} \left( \int_{\hat{\Gamma}} \| \Phi[\varphi_2](\Lambda) \|^2 d\hat{m}(\Lambda) \right)^{1/2} = \| \varphi_1 \| \| \varphi_2 \| < \infty.$$ 

This concludes the verification.

Proposition 13. The kernel $\mathcal{K}_t$, $t \in \mathbb{R}$, decomposes into a direct integral in the following sense:

$$\forall \varphi_1, \varphi_2 \in C_0^\infty(\tilde{M}), \quad \mathcal{K}_t(\varphi_1 \otimes \varphi_2) = \int_{\hat{\Gamma}} \text{Tr}[\mathcal{K}_t^A(\varphi_1 \otimes \varphi_2)] d\hat{m}(\Lambda).$$

Proof. Applying successively the defining relation for $\mathcal{K}_t$, equalities (18) and (28) one finds that

$$\mathcal{K}_t(\varphi_1 \otimes \varphi_2) = \langle \Phi[\varphi_1], \Phi U(t) \Phi^{-1} \Phi[\varphi_2] \rangle$$

19
The proof is complete. □

Next we wish to invert relation (29). An inverse relation, which we call here the Schulman’s ansatz, was derived in the theoretical physics in the framework of path integration\textsuperscript{27, 28} and reads

\[
K_{t}^{\Lambda}(x, y) = \sum_{s \in \Gamma} \Lambda(s) K_{t}(s^{-1} \cdot x, y).
\]

Our main goal in the current section is a mathematically rigorous derivation and interpretation of this formula.

Suppose that $\varphi_{1}, \varphi_{2} \in C_{0}^{\infty}(\tilde{M})$ are fixed but otherwise arbitrary. Set

\[
F_{t}(s) = K_{t} \circ (L_{s^{-1}} \otimes 1)(\varphi_{1} \otimes \varphi_{2}) \quad \text{for } s \in \Gamma,
\]

and

\[
G_{t}(\Lambda) = K_{t}^{\Lambda}(\varphi_{1} \otimes \varphi_{2}) \in \mathcal{B}(\mathcal{L}_{\Lambda}) \quad \text{for } \Lambda \in \hat{\Gamma}.
\]

Lemma 14. $F_{t} \in L^{2}(\Gamma)$, $G_{t}$ is bounded in the Hilbert-Schmidt norm on $\hat{\Gamma}$.

Proof. (i) Since the action of $\Gamma$ is proper one can write any test function $\varphi \in C_{0}^{\infty}(\tilde{M})$ as a finite sum, $\varphi = \sum_{j=1}^{n} \eta_{j}$, with $\eta_{j} \in C_{0}^{\infty}(\tilde{M})$, so that

\[
\forall j = 1, \ldots, n, \forall s \in \Gamma \setminus \{1\}, \quad \text{supp} \eta_{j} \cap \text{supp} L_{s}^{*} \eta_{j} = \emptyset.
\]

This is why one can assume, without loss of generality, that the test function $\varphi_{1}$ fulfills

\[
\forall s \in \Gamma \setminus \{1\}, \quad \text{supp} \varphi_{1} \cap \text{supp} L_{s}^{*} \varphi_{1} = \emptyset.
\]

In that case, \(\{||\varphi_{1}||^{-1} L_{s}^{*} \varphi_{1}\}_{s \in \Gamma}\) is an orthonormal system in $L^{2}(\tilde{M})$. One has

\[
F_{t}(s) = K_{t}(L_{s}^{*} \varphi_{1} \otimes \varphi_{2}) = \langle L_{s}^{*} \varphi_{1}, U(t) \varphi_{2} \rangle,
\]

and so, by the Bessel inequality,

\[
\sum_{s \in \Gamma} |F_{t}(s)|^{2} = \|\varphi_{1}\|^{2} \sum_{s \in \Gamma} \left| \frac{1}{\|\varphi_{1}\|} L_{s}^{*} \varphi_{1}, U(t) \varphi_{2} \right|^{2} \leq \|\varphi_{1}\|^{2} \|\varphi_{2}\|^{2}.
\]
(ii) Using the defining relation for $K_t^\Lambda$ one can estimate the Hilbert-Schmidt norm of $G_t(\Lambda)$ as follows
\[
\| G_t(\Lambda) \|_2^2 = \sum_m \sum_n |\langle u_m^\Lambda, K_t^\Lambda(\varphi_1 \otimes \varphi_2) u_n^\Lambda \rangle |^2 \leq \sum_m \| \Phi_\Lambda \varphi_1 \otimes u_m^\Lambda \|_2^2 \sum_n \| \Phi_\Lambda \varphi_2 \otimes u_n^\Lambda \|_2^2.
\]
Again, without loss of generality, one can assume that $\varphi_1$ fulfills condition (32). Then in the expression
\[
(\Phi_\Lambda \varphi_1 \otimes u_m^\Lambda)(y) = \sum_{s \in \Gamma} \varphi_1(s^{-1} \cdot y) \Lambda(s) u_m^\Lambda
\]
only at most one summand on the RHS does not vanish. It follows that
\[
\| (\Phi_\Lambda \varphi_1 \otimes u_m^\Lambda)(y) \|_2^2 = \sum_{s \in \Gamma} |\varphi_1(s^{-1} \cdot y)|^2 \| u_m^\Lambda \|_2^2
\]
and (recalling (2))
\[
\| \Phi_\Lambda \varphi_1 \otimes u_m^\Lambda \|_2^2 = \int_M \| (\Phi_\Lambda \varphi_1 \otimes u_m^\Lambda)(y) \|_2^2 \, d\mu(x) = \int_M \sum_{s \in \Gamma} |\varphi_1(s^{-1} \cdot y)|^2 \, d\mu(x) = \| \varphi_1 \|_2^2.
\]
Hence, if condition (32) is true then
\[
\sum_m \| \Phi_\Lambda \varphi_1 \otimes u_m^\Lambda \|_2^2 = \dim(L_\Lambda) \| \varphi_1 \|_2^2.
\]
In virtue of Theorem 4 this sum is uniformly bounded in $\Lambda$. The other sum in (33), $\sum_n \| \Phi_\Lambda \varphi_2 \otimes u_n^\Lambda \|_2^2$, can be analogously shown to have the same property.

Remark 15. Since we know that the total measure $\hat{m}(\hat{\Gamma})$ is finite (see (12)) Lemma 14 implies that $\| G_t(\cdot) \| \in L^1(\hat{\Gamma}) \cap L^2(\hat{\Gamma})$.

**Proposition 16.** For all $\varphi_1, \varphi_2 \in C_0^\infty(\tilde{M})$, the functions $F_t(s)$ and $G_t(\Lambda)$ defined respectively by (30) and (31) satisfy
\[
F_t = \mathcal{F}^{-1}[G_t].
\]

**Proof.** Replacing the test function $\varphi_1$ in (29) by $L_s^* \varphi_1$, $s \in \Gamma$, and using (27) one
arrives at the equality
\[
\mathcal{K}_t \circ (L_{s-1} \otimes 1)(\varphi_1 \otimes \varphi_2) = \int_T \text{Tr}[\Lambda(s)^* \mathcal{K}_t^A(\varphi_1 \otimes \varphi_2)] \, d\hat{m}(\Lambda).
\]
In virtue of Lemma 14, this means exactly that \( F_t(s) = \mathcal{F}^{-1}[G_t](s) \). \qed

Corollary 17. Conversely,
\[
G_t = \mathcal{F}[F_t]. \tag{34}
\]

Rewriting (34) formally gives
\[
\mathcal{K}_t^A(\varphi_1 \otimes \varphi_2) = \sum_{s \in T} \Lambda(s) \mathcal{K}_t \circ (L_{s-1} \otimes 1)(\varphi_1 \otimes \varphi_2)
\]
which is nothing but the Schulman’s ansatz.

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