Asymptotic behaviour of non-radiative solution to the wave equations

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Abstract

In this work we consider weakly non-radiative solutions to both linear and non-linear wave equations. We first characterize all weakly non-radiative free waves, without the radial assumption. Then in dimension 3 we show that the initial data of non-radiative solutions to a wide range of nonlinear wave equations are similar to those of non-radiative free waves in term of asymptotic behaviour.

1 Introduction

1.1 Background and main topics

Channel of energy The channel of energy method plays an important role in the study of asymptotic behaviour of solutions to non-linear wave equations in the past decade. This method mainly discusses the distribution of energy as time tends to infinity. More precisely, if $u$ is a solution to either linear or non-linear wave equation defined for all time, then the following limits are considered for a given constant $R$.

$$\lim_{t \to \pm \infty} \int_{|x| > R + |t|} |\nabla_{t,x} u(x, t)|^2 dx.$$  

Here for convenience we use the notation $\nabla_{t,x} u = (u_t, \nabla u)$. This theory was first established for solutions to homogeneous linear wave equation, i.e. free waves, then applied to the study of non-linear wave equations. Please see, for instance, Côte-Kenig-Schlag [2], Duyckaerts-Kenig-Merle [3, 7] and Kenig-Lawrie-Schlag [14] for linear theory; and Duyckaerts-Kenig-Merle [5, 9] for the applications of the channel of energy on soliton resolution of focusing wave equation.

Non-radiative solutions A crucial part of the channel of energy theory is to discuss the property of non-radiative solutions. Let $u$ be a solution to the wave equation with a finite energy. We call it a non-radiative solution if and only if

$$\lim_{t \to \pm \infty} \int_{|x| > |t|} |\nabla_{t,x} u(x, t)|^2 dx = 0.$$  

We may also consider a more general case. We call a solution $u$ to be $R$-weakly non-radiative if and only if

$$\lim_{t \to \pm \infty} \int_{|x| > R + |t|} |\nabla_{t,x} u(x, t)|^2 dx = 0.$$  

Let us first consider (weakly) non-radiative solutions to the homogeneous linear wave equation in $\mathbb{R}^d$. It has been proved that any non-radiative free wave must be zero, see Duyckaerts-Kenig-Merle [4,7]. All radial weakly non-radiative free waves have also been well understood. The following result was first proved for odd dimensions $d \geq 3$ by Kenig et al. [15] then generalized to the even dimensions $d \geq 2$ in Li-Shen-Wei [16].

**Proposition 1.1** (Radial weakly non-radiative solutions). Let $d \geq 2$ be an integer and $R > 0$ be a constant. If initial data $(u_0, u_1) \in \dot{H}^1 \times L^2$ are radial, then the corresponding solution to the homogeneous linear wave equation $u$ is $R$-weakly non-radiative, i.e.,

$$\lim_{t \to \pm \infty} \int_{|x| > |t| + R} |\nabla_{t,x} u(x,t)|^2 \, dx = 0,$$

if and only if the restriction of $(u_0, u_1)$ in the region $\{ x \in \mathbb{R}^d : |x| > R \}$ is contained in

$$\text{Span} \left\{ (r^{2k_1-d},0),(0,r^{2k_2-d}) : 1 \leq k_1 \leq \left\lfloor \frac{d+1}{4} \right\rfloor, 1 \leq k_2 \leq \left\lfloor \frac{d-1}{4} \right\rfloor \right\}.$$

Here the notation $\lfloor q \rfloor$ is the integer part of $q$. In particular, all radial $R$-weakly non-radiative solution in dimension 2 are supported in $\{ (x,t) : |x| \leq |t| + R \}$.

**Goals of this work** The aim of this paper is two-fold. The first goal of this paper is to characterize all (possibly non-radial) initial data so that the corresponding solutions to free wave equation are $R$-weakly non-radiative. For convenience we define

$$P(R) \doteq \left\{ (u_0, u_1) \in \dot{H}^1 \times L^2(\mathbb{R}^d) : \lim_{t \to \pm \infty} \int_{|x| > |t| + R} |\nabla_{t,x} S_L(u_0,u_1)|^2 \, dx = 0 \right\}.$$

Here $S_L(u_0,u_1)$ is the corresponding solution of the free wave equation with given initial data $(u_0, u_1)$. We will give a decomposition of every element $(u_0, u_1) \in P(R)$ in term of spherical harmonic functions, whose details are given in Section 3. The second goal is to show that in the 3-dimensional case any weakly non-radiative solution to a wide range of non-linear wave equations share the same asymptotic behaviour as weakly non-radiative free waves, as given in Sarapu [2]. Our argument depends on a suitable decay estimate of weakly non-radiative free waves in the exterior region $\{ (x,t) : |x| > |t| + R \}$. The decay estimates of this kind are clearly true for radial non-radiative solutions, as given in Proposition 1.1. Although we expect that a similar estimate holds for non-radial non-radiative solutions in all dimensions $d \geq 2$ as well, this has been proved only in dimension 3, as far as the author knows. This is why we have to restrict our discussion to dimension 3 in this work.

**2 The characteristics of $P(R)$**

In this section we give an explicit expression of the element in the space $P(R)$. We use spherical harmonics and follow a similar argument as given in Duyckaerts-Kenig-Merle [8]. Let us first give a brief review on some basic properties of spherical harmonics. We recall that the eigenfunctions of the Laplace-Beltrami operator on $S^{d-1}$ are exactly the homogeneous harmonic polynomials of the variables $x_1, x_2, \ldots, x_d$. Such a polynomial $\Phi$ of degree $\nu$ satisfies

$$-\Delta_{S^{d-1}} \Phi = \nu (\nu + d - 2) \Phi.$$

We choose a Hilbert basis $\{ \Phi_k(\theta) \}_{k \geq 0}$ of the operator $-\Delta_{S^{d-1}}$ on the sphere $S^{d-1}$. Here we assume that the harmonic polynomial $\Phi_k$ is of degree $\nu_k$. In particular we assume $\nu_0 = 0$ and $\nu_k > 0$ if $k \geq 1$. Next we give the statement of our first main result. We start by the odd dimensional case and then deal with the even dimensional case. Please note that a similar result for odd dimensions has been proved in Côte-Laurent [1] by the Radon transform. The novelty of our result includes

2
• We give an $L^2$ decay estimate of $\partial_r u_0$ near infinity in addition;
• The argument works for even dimensions as well, with minor modifications.

2.1 Odd dimensions

Proposition 2.1. Assume that $d \geq 3$ is an odd integer and $\mu = (d - 1)/2$. Then $(u_0, u_1) \in P(R)$ is equivalent to saying that there exist two sequences of polynomials $\{P_k(z)\}_{k \geq 0}$ and $\{Q_k(z)\}_{k \geq 0}$ of the following form $(A_{k,k_1}, B_{k,k_2}$ are constants) $P_k(z) = \sum_{1 \leq k_1 \leq \lceil \frac{k+1}{2}\rceil} A_{k,k_1} z^{\mu + 1 + \nu_k - 2k_1};$ $Q_k(z) = \sum_{1 \leq k_2 \leq \lceil k/2\rceil} B_{k,k_2} z^{\mu + \nu_k - 2k_2};$

with $\sum_{k=0}^{\infty} \int_{0}^{1/R} (\nu_k(d - 2 + \nu_k)|P_k(z)|^2 + |zP_k'(z)|^2) \, dz < +\infty; \sum_{k=0}^{\infty} \int_{0}^{1/R} |Q_k(z)|^2 \, dz < +\infty;$

so that $u_0(r, \theta) = \sum_{k=0}^{\infty} r^{-\mu} P_k(1/r) \Phi_k(\theta), \quad r > R; \quad u_1(r, \theta) = \sum_{k=0}^{\infty} r^{-\mu - 1} Q_k(1/r) \Phi_k(\theta), \quad r > R. \tag{1}$

Here the first identity holds for every fixed $r > R$ in the sense of $L^2(S^{d-1})$ convergence. The second one holds in the sense of $L^2(\{x: |x| > R\})$ convergence. In addition, we have

(i) The derivative of $u_0$ can be given by $\nabla_x u_0 = \sum_{k=0}^{\infty} r^{-\mu - 1} \left\{ P_k(1/r) \nabla \Phi_k(\theta) - [\mu P_k(1/r) + (1/r)P_k'(1/r)] \Phi_k(\theta) \right\}.$

This identities holds in the sense of $L^2(\{x: |x| > R\})$ convergence. Here we naturally embed $\nabla \Phi_k(\theta)$ into $\mathbb{R}^d$ by the identity $\nabla \Phi_k(\theta) = \nabla_x \Phi_k(\theta)$. In the right hand side of this identity we understand $\Phi_k$ as a function defined in $\mathbb{R}^d \setminus \{0\}$ by polar coordinates.

(ii) The norms of $(u_0, u_1)$ can be determined by $P_k(z)$ and $Q_k(z)$'s:

$$\|\nabla u_0\|^2_{L^2(\{x: |x| > R\})} = \sum_{k=1}^{\infty} \nu_k(d - 2 + \nu_k) \int_{0}^{1/R} |P_k(z)|^2 \, dz;$$

$$\|u_1\|^2_{L^2(\{x: |x| > R\})} = \sum_{k=0}^{\infty} \int_{0}^{1/R} |Q_k(z)|^2 \, dz;$$

$$\|\partial_r u_0\|^2_{L^2(\{x: |x| > R\})} = \sum_{k=0}^{\infty} \int_{0}^{1/R} |zP''_k(z) + \mu P_k(z)|^2 \, dz < +\infty.$$

(iii) The derivative $\partial_r u_0$ satisfies the following decay estimates ($R_1 \geq 2R$)$\int_{|x| > R_1} |\partial_r u_0(x)|^2 \, dx \lesssim (R/R_1) \int_{|x| > R} |\nabla u_0(x)|^2 \, dx;$$

$$\int_{|x| > R_1} |\partial_r u_0(x)|^2 \, dx \lesssim (R/R_1) \int_{|x| > R} |\nabla u_0(x)|^2 \, dx.$$

Here $u_0^\circ$ is the non-radial part of $u_0$ defined by $u_0^\circ = u_0 - r^{-\mu} P_0(1/r) \Phi_0.$
Proof The rest of this subsection is devoted to the proof of this proposition. The proof consists of three parts: Step one, we first show that any element in $P(R)$ can be written as in (10). Step two, we show any initial data given by (10) is indeed contained in $P(R)$. Finally in Step three we prove the identities and inequalities in the proposition.

**Step one** Let us consider

$$u_k(r, t) = r^{-\nu_k} \int_{S^{d-1}} u(r\theta, t) \Phi_k(\theta) d\theta.$$  

Let $\Box = \partial_t^2 - \partial_r^2 - \frac{d+2\nu_k-1}{r} \partial_r$. A basic calculation shows

$$\Box u_k = \left(\Box - r^{-\nu_k}\right) \int_{S^{d-1}} u(r\theta, t) \Phi_k(\theta) d\theta + r^{-\nu_k} \int_{S^{d-1}} \Box u(r\theta, t) \Phi_k(\theta) d\theta = 0.$$  

Thus if $u_k$ is viewed as a radial function defined on $\mathbb{R}^{d+2\nu_k}$, it satisfies the free wave equation

$$\partial_t^2 u_k - \Delta_{\mathbb{R}^{d+2\nu_k}} u_k = 0, \quad |x| > 0.$$  

In addition, we have

$$\int_{R^+ | t |} \left( |\partial_t u_k(r, t)|^2 + |\partial_r u_k(r, t)|^2 \right) r^{d+2\nu_k-1} dr \leq \int_{R^+ | t |} \left( \int_{S^{d-1}} |\partial_r u(r\theta, t)|^2 d\theta + \int_{S^{d-1}} |\partial_t u(r\theta, t)|^2 d\theta \right) r^{d-1} dr \lesssim \int_{|x| > R^+ | t |} |\nabla_{x,t} u(x, t)|^2 dx.$$  

Thus $u_k$ is also a weakly non-radiative solution. According to the explicit expression of radial non-radiative solutions, there exist constants $A_{k,k_1}$ and $B_{k,k_2}$, so that

$$u_k(r, 0) = \sum_{1 \leq k_1 \leq \frac{\nu_k + 1}{2}} A_{k,k_1} r^{-d-2\nu_k+2k_1} = r^{-\nu_k} P_k(1/r);$$

$$\partial_t u_k(r, 0) = \sum_{1 \leq k_2 \leq \frac{\nu_k + 1}{2}} B_{k,k_2} r^{-d-2\nu_k+2k_2} = r^{-\nu_k-1} Q_k(1/r).$$

Here $P_k(z)$ and $Q_k(z)$ are polynomials as given in Proposition 2.1. Therefore we have

$$\int_{S^{d-1}} u_0(r\theta) \Phi_k(\theta) d\theta = r^{-\nu_k} P_k(1/r);$$

$$\int_{S^{d-1}} u_1(r\theta) \Phi_k(\theta) d\theta = r^{-\nu_k-1} Q_k(1/r).$$
Next we show the polynomials satisfy the inequalities in in Proposition 2.4. We have
\[
\int_{\mathbb{S}^{d-1}} \nabla_\theta u_0(r\theta) \nabla_\theta \Phi_k(\theta) d\theta = - \int_{\mathbb{S}^{d-1}} u(r\theta, 0) \Delta_{\mathbb{S}^{d-1}} \Phi_k(\theta) d\theta
= \nu_k (d - 2 + \nu_k) r^{-\mu} P_k(1/r).
\]
Since \( \nabla_\theta \Phi_k \) are orthogonal to each other with \( L^2(\mathbb{S}^{d-1}) \) norm \( \sqrt{\nu_k (d - 2 + \nu_k)} \), we have
\[
\sum_{k=1}^{\infty} \nu_k (d - 2 + \nu_k) r^{-2\mu} |P_k(1/r)|^2 \leq \| \nabla_\theta u_0(r\theta) \|_{L^2(\mathbb{S}^{d-1})}^2
\]
By the inequality \( \| \nabla_\theta u_0(r\theta)/r \|_{L^2(\{x: |x| > R\})} = \| \nabla u_0 \|_{L^2(\{x: |x| > R\})} \leq \| \nabla u \|_{L^2(\{x: |x| > R\})} \), we have
\[
\sum_{k=1}^{\infty} \nu_k (d - 2 + \nu_k) \frac{1}{R} \int_0^{1/R} |P_k(z)|^2 \, dz \lesssim \| \nabla u_0 \|_{L^2(\{x: |x| > R\})}^2 < +\infty.
\]
Similarly
\[
\sum_{k=1}^{\infty} \int_0^{1/R} |Q_k(z)|^2 \, dz = \| u_1 \|_{L^2(\{x: |x| > R\})}^2 < +\infty.
\]
Next we differentiate \( u_0 \) in \( r \) and obtain
\[
\int_{\mathbb{S}^{d-1}} \partial_r u_0(r\theta) \Phi_k(\theta) d\theta = r^{-\mu - 1} (-\mu P_k(1/r) - (1/r) P_k'(1/r)).
\]
Following the same argument as above, we obtain
\[
\sum_{k=0}^{\infty} \int_0^{1/R} |\mu P_k(z) + z P_k'(z)|^2 \, dz \leq \| \partial_r u_0 \|_{L^2(\{x: |x| > R\})}^2 < +\infty.
\]
Combining this inequality with 4, we have
\[
\sum_{k=0}^{\infty} \int_0^{1/R} \left( \nu_k (d - 2 + \nu_k) |P_k(z)|^2 + |z P_k'(z)|^2 \right) \, dz < +\infty.
\]
Since \( \Phi_k(\theta) \) is a Hilbert basis, we may finally write \( (u_0, u_1) \) in the following form by 2 and 3. (These infinite sums are understood as convergence in \( L^2(\mathbb{S}^{d-1}) \) and \( L^2(\{x: |x| > R\}) \) respectively.)
\[
u_0(r, \theta) = \sum_{k=0}^{\infty} r^{-\mu} P_k(1/r) \Phi_k(\theta) \quad u_1(r, \theta) = \sum_{k=0}^{\infty} r^{-\mu - 1} Q_k(1/r) \Phi_k(\theta).
\]

**Step two** Let us assume \( P_k(z) \) and \( Q_k(z) \) satisfy the conditions given in the proposition. Now we show \( (u_0, u_1) \in P(R) \). We start by proving \( (u_0, u_1) \in H^1 \times L^2(\{x: |x| > R\}) \). This is clear that the series
\[
\sum_{k=0}^{\infty} r^{-\mu - 1} Q_k(1/r) \Phi_k(\theta)
\]
converges in the space \( L^2(\{x: |x| > R\}) \) and
\[
\| u_1 \|_{L^2(\{x: |x| > R\})} = \sum_{k=0}^{\infty} \int_0^{1/R} |Q_k(z)|^2 \, dz < +\infty.
\]
Next we show \( u_0 \in \dot{H}^1(\{x: |x| > R\}) \). We need the following technical lemma, whose proof is put in the Appendix.
Lemma 2.2. Let \( L \geq 2l > 0 \) and \( P(z) \) be a polynomial of degree \( \kappa \). Then we have

\[
\max_{z \in [0,L]} |P(z)|^2 \leq \frac{(\kappa + 1)^2}{L} \int_0^L |P(z)|^2 dz;
\]

\[
\int_0^L |zP'(z)|^2 dz \leq \frac{2\kappa(\kappa + 1)l}{L} \int_0^L |P(z)|^2 dz.
\]

As a result, we have

\[
\left\| \sum_{k=N}^{\infty} r^{-\mu} P_k(1/r) \Phi_k(\theta) \right\|_{L^2(S^{d-1})} = r^{-\mu} \left( \sum_{k=N}^{\infty} |P_k(1/r)|^2 \right)^{1/2}
\]

\[
\lesssim r^{-\mu} \left( \sum_{k=N}^{\infty} \frac{(\mu + \nu_k)^2 R}{2} \int_0^{1/R} |P_k(z)|^2 dz \right)^{1/2}.
\]

converges to zero uniformly in \( r \in [R, +\infty) \) as \( N \to +\infty \). Thus the series

\[
\sum_{k=0}^{\infty} r^{-\mu} P_k(1/r) \Phi_k(\theta)
\]

converges to \( u_0 \) in \( C([R, \infty); L^2(S^{d-1})) \). Next we show

\[
\nabla_x u_0(r, \theta) = \sum_{k=0}^{\infty} \nabla_x \left( r^{-\mu} P_k(1/r) \Phi_k(\theta) \right)
\]

\[
= \sum_{k=0}^{\infty} r^{-\mu-1} \left( P_k(1/r) \nabla_x \Phi_k(\theta) - [\mu P_k(1/r) + (1/r) P'_k(1/r)] \Phi_k(\theta) \right). \tag{6}
\]

Our assumption on \( P_k(z) \), as well as the orthogonality of \( \{ \nabla_x \Phi_k \}_{k \geq 0} \) and \( \{ \Phi_k \}_{k \geq 0} \), guarantee that the series in the right hand side converges in \( L^2([R, \infty) \times S^{d-1}; r^{d-1} dr d\theta) \), or equivalently in \( L^2(\{ x : |x| > R \}) \). Given any \( \varphi \in C_0^\infty(\{ x : |x| > R \}) \), we have

\[
\int_{|x| > R} \left( \sum_{k=0}^{N} r^{-\mu} P_k(1/r) \Phi_k(\theta) \right) \nabla_x \varphi(r, \theta) dr d\theta = -\int_{|x| > R} \varphi(x) \sum_{k=0}^{N} \nabla_x \left( r^{-\mu} P_k(1/r) \Phi_k(\theta) \right) dx.
\]

By the convergence of series we make \( N \to +\infty \) and obtain

\[
\int_{|x| > R} u_0(x) \nabla_x \varphi(x) dx = -\int_{|x| > R} \varphi(x) \sum_{k=0}^{\infty} \nabla_x \left( r^{-\mu} P_k(1/r) \Phi_k(\theta) \right) dx.
\]

This verifies (6). Please note that we always have \( \nabla_x \Phi_k \cdot \theta = 0 \), thus (6) is actually an orthogonal decomposition. This immediately gives

\[
\| \partial_r u_0(r, \theta) \|_{L^2(\{ x : |x| > R \})} = \sum_{k=0}^{\infty} \int_0^{1/R} |\mu P_k(z) + z P'_k(z)|^2 dz < +\infty; \tag{7}
\]

\[
\| \nabla u_0 \|_{L^2(\{ x : |x| > R \})}^2 = \sum_{k=1}^{\infty} \nu_k (d - 2 + \nu_k) \int_0^{1/R} |P_k(z)|^2 dz < +\infty. \tag{8}
\]

In summary, we have \( (u_0, u_1) \in \dot{H}^1 \times L^2(\{ x : |x| > R \}) \). For completeness we may define \( u_0, u_1 \) in the region \( \{ x : |x| \leq R \} \) so that \( (u_0, u_1) \in \dot{H}^1 \times L^2(\mathbb{R}^d) \). Next we show that \( S_L(u_0, u_1) \) is a
Thus we may combine the two limits above and obtain that the inequality (9) holds. By finite speed of energy propagation, we also have that $u$ satisfies the equation $(\partial_t^2 - \Delta) u = (\partial_t^2 - \frac{d-1}{r} \partial_r) f(r, t) = - \frac{\nu (d+\nu_k-2)}{r^2} f(r, t)$. In fact these constant can be determined inductively. Therefore $v = f(r, t) \Phi_k(\theta)$ solves the equation
\begin{equation}
(\partial_t^2 - \Delta) v = (\partial_t^2 - \frac{d-1}{r} \partial_r - \frac{\Delta_{\theta}}{r^2}) v = 0, \quad |x| = r > 0.
\end{equation}
with initial data $(r^{-d-\nu_k+2k_1} \Phi_k(\theta), 0)$. A basic calculation shows that
\begin{equation}
\lim_{t \to \pm \infty} \int_{|x| > |t| + R} |\nabla_{x,t} v(x, t)|^2 dx = 0.
\end{equation}
Similarly we may find a non-radiative solution $v$ to (9) with initial data $(0, r^{-d-\nu_k+1+2k_2} \Phi_k(\theta))$. By linearity, we may find a non-radiative solution $v_N$ to (9) with initial data $v_{0,N}, v_{1,N}$ so that
\begin{align*}
v_{0,N}(r, \theta) &= \sum_{k=0}^N r^{-\mu} P_k(1/r) \Phi_k(\theta); \quad v_{1,N}(r, \theta) = \sum_{k=0}^N r^{-\mu} Q_k(1/r) \Phi_k(\theta).
\end{align*}
By a standard centre cut-off and finite speed of propagation we obtain initial data $(u_{0,N}, u_{1,N}) \in H^1 \times L^2$ and corresponding free wave $u_N = S_L(u_{0,N}, u_{1,N})$ so that
\begin{align*}
v_{0,N}(r, \theta) &= \sum_{k=0}^N r^{-\mu} P_k(1/r) \Phi_k(\theta); \quad v_{1,N}(r, \theta) = \sum_{k=0}^N r^{-\mu} Q_k(1/r) \Phi_k(\theta); \quad r > R
\end{align*}
and
\begin{align*}
\lim_{t \to \pm \infty} \int_{|x| > |t| + R} |\nabla_{x,t} u_N(x, t)|^2 dx = 0.
\end{align*}
By finite speed of energy propagation, we also have that $u = S_L(u_0, u_1)$ satisfies
\begin{align*}
\limsup_{t \to \pm \infty} \int_{|x| > |t| + R} |\nabla_{x,t} (u(x, t) - u_N(x, t))|^2 dx &\leq \int_{|x| > R} \left( |\nabla u_0 - \nabla u_{0,N}|^2 + |u_1 - u_{1,N}|^2 \right) dx.
\end{align*}
We may combine the two limits above and obtain that the inequality
\begin{align*}
\limsup_{t \to \pm \infty} \int_{|x| > |t| + R} |\nabla_{x,t} u(x, t)|^2 dx \leq \int_{|x| > R} \left( |\nabla u_0 - \nabla u_{0,N}|^2 + |u_1 - u_{1,N}|^2 \right) dx.
\end{align*}
holds for all $N \geq 1$. Finally we make $N \to +\infty$ and conclude that $(u_0, u_1) \in P(R)$.

**Step three** Now we show that the identities and inequalities given in (i), (ii) and (iii) hold. Part (i) and (ii) have been proved in step two, see [9], [10], [11] and [12]. Now we consider part (iii). We have
\begin{align*}
\partial_r u^*_0 &= \sum_{k=1}^\infty r^{-\mu - 1} (-\mu P_k(1/r) - (1/r) P'_k(1/r)) \Phi_k(\theta)
\end{align*}
Thus
\begin{align*}
\int_{|x| > R_1} |\partial_r u^*_0|^2 dx &= \sum_{k=1}^{\infty} \int_0^{1/R_1} |\mu P_k(z) + z P'_k(z)|^2 dz \leq \sum_{k=1}^{\infty} \int_0^{1/R_1} |P_k(z)|^2 + |z P'_k(z)|^2 dz.
\end{align*}
We then apply Lemma 2.2 and obtain

\[ \int_{|x| > R_1} |\partial_r u_0|^2 dx \lesssim_d \sum_{k=1}^\infty (\mu + \nu_k)^2 \frac{R}{R_1} \int_0^{1/R} |P_k(z)|^2 dz \lesssim_d (R/R_1) \int_{|x| > R} |\nabla u_0(x)|^2 dx. \]

In order to find an upper bound of \( \|\partial_r u_0\|_{L^2} \), we also need to consider the radial part \( r^{-\mu} P_0(1/r) \Phi_0(\theta) \). In this case \( \nu_0 = 0 \) and \( \Phi_0 \) is simply a constant. We may follow the same argument above and obtain

\[ \int_{|x| > R_1} |\partial_r u_0|^2 dx = \int_{|x| > R_1} |\partial_r u_0|^2 dx + \int_{|x| > R_1} |\partial_r [r^{-\mu} P_0(1/r) \Phi_0(\theta)]|^2 dx \]

\[ \lesssim_d \frac{R}{R_1} \int_{|x| > R} |\nabla u_0(x)|^2 dx. \]

This completes the proof of Proposition 2.1.

2.2 Even dimensions

In this subsection we generalize our result on weakly non-radiative solutions to the even dimensions.

**Proposition 2.3.** Assume that \( d \geq 2 \) is an even integer and \( \mu = d/2 \). Then \( (u_0, u_1) \in P(R) \) is equivalent to saying that there exist two sequences of polynomials \( \{P_k(z)\}_{k \geq 0} \) and \( \{Q_k(z)\}_{k \geq 0} \) of the following form

\[ P_k(z) = \sum_{1 \leq k_1 \leq \left\lfloor \frac{d-1}{2} \right\rfloor} A_{k,k_1} z^{\mu+\nu_k-2k_1}; \quad Q_k(z) = \sum_{1 \leq k_2 \leq \left\lfloor \frac{d-1}{2} \right\rfloor} B_{k,k_2} z^{\mu+\nu_k-1-2k_2}. \]

with

\[ \sum_{k=0}^\infty \int_0^{1/R} z \left( \nu_k (d-2+\nu_k) |P_k(z)|^2 + |z P'_k(z)|^2 \right) dz < +\infty; \quad \sum_{k=0}^\infty \int_0^{1/R} z |Q_k(z)|^2 dz < +\infty. \]

so that

\[ u_0(r, \theta) = \sum_{k=0}^\infty r^{-\mu} P_k(1/r) \Phi_k(\theta) \quad u_1(r, \theta) = \sum_{k=0}^\infty r^{-\mu-1} Q_k(1/r) \Phi_k(\theta). \]  

Here the first identity holds for every \( r > R \) in the sense of \( L^2(S^{d-1}) \) convergence. The second one holds in the sense of \( L^2(\{x : |x| > R\}) \) convergence. In addition, we have
Lemma 2.4. The main difference is that we rely on a slightly modified version of the technical lemma. The proof in the even dimensions is almost the same as in the odd dimensions thus we omit it.

(i) The derivative of $u_0$ can be given by

$$\nabla_x u_0(r, \theta) = \sum_{k=0}^\infty r^{-\mu - 1} \{ P_k(1/r) \nabla \Theta_k(\theta) - [\mu P_k(1/r) + (1/r) P_k'(1/r)] \Theta_k(\theta) \}.$$ 

This identities holds in the sense of $L^2(\{x : |x| > R\})$ convergence. Here $\nabla \Theta_k(\theta)$ is in the tangent space of $S^{d-1}$ at the point $\theta$ thus can be naturally embedded into $\mathbb{R}^d$.

(ii) The norms of $(u_0, u_1)$ can be determined by $P_k(z)$ and $Q_k(z)$'s:

$$\|\nabla u_0\|_{L^2(\{|x| > R\})}^2 = \sum_{k=1}^\infty \nu_k (d - 2 + \nu_k) \int_0^{1/R} z |P_k(z)|^2 dz;$$
$$\|u_1\|_{L^2(\{|x| > R\})}^2 = \sum_{k=0}^\infty \int_0^{1/R} z |Q_k(z)|^2 dz;$$
$$\|\partial_\theta u_0\|_{L^2(\{|x| > R\})}^2 = \sum_{k=0}^\infty \int_0^{1/R} z |P_k'(z)|^2 dz < +\infty.$$

(iii) The derivative $\partial_\theta u_0$ satisfies the following decay estimates ($R_1 \geq 2R$)

$$\int_{|x| > R_1} |\partial_\theta u_0(x)|^2 dx \lesssim (R/R_1) \int_{|x| > R} |\nabla u_0(x)|^2 dx;$$
$$\int_{|x| > R_1} |\partial_\theta u_0^0(x)|^2 dx \lesssim (R/R_1) \int_{|x| > R} |\nabla u_0(x)|^2 dx.$$

Here $u_0^0$ is the non-radial part of $u_0$ defined by $u_0^0 = u_0 - r^{-\kappa} P_0(1/r) \Theta_0$.

The proof in the even dimensions is almost the same as in the odd dimensions thus we omit it here. The main difference is that we rely on a slightly modified version of the technical lemma about polynomials, which is given below and proved in the appendix.

Lemma 2.4. Let $L \geq 2l > 0$ and $P(z)$ be a polynomial of degree $\kappa$. Then we have

$$\max_{z \in [0, L]} z |P(z)|^2 \leq \frac{2(\kappa + 1)^2}{L} \int_0^L z |P(z)|^2 dz;$$
$$\int_0^L z |P'(z)|^2 dz \leq \frac{2\kappa(\kappa + 2)l}{L} \int_0^L z |P(z)|^2 dz.$$

3 Non-linear Non-radiative Solutions

In this section we show that non-radiative solutions to a wide range of nonlinear wave equations in the three-dimensional case share the same asymptotic behaviour as non-radiative free waves, without the radial assumption.

Assumptions We consider the energy-critical non-linear wave equation in $\mathbb{R}^3$

$$\partial_t^2 u - \Delta u = F(x, t, u), \quad (x, t) \in \mathbb{R}^3 \times \mathbb{R}.$$ 

Here the nonlinear term $F(x, t, u)$ satisfies

$$|F(x, t, u)| \leq C|u|^5; \quad |F(x, t, u_1) - F(x, t, u_2)| \leq C(|u_1|^4 + |u_2|^4)|u_1 - u_2|.$$ 

This covers both the defocusing ($F(x, t, u) = -|u|^4 u$) and focusing ($F(x, t, u) = |u|^4 u$) wave equations, which have been extensively studied in the past decades.
3.1 Preliminary results

We first give a few preliminary results and introduce a few notations.

Radiation fields Radiaton field describes the asymptotic behaviour of free waves as time tends to infinity. In its earlier history radiation field was mainly a conception in mathematical physics. See Friedlander [10] [11], for instance. The following modern version is given in [8].

Theorem 3.1 (Radiation fields). Assume that \( d \geq 3 \) and let \( u \) be a solution to the free wave equation \( \partial_t^2 u - \Delta u = 0 \) with initial data \((u_0, u_1) \in \dot{H}^1 \times L^2(\mathbb{R}^d)\). Then \((u_t \) is the derivative in the radial direction\)

\[
\lim_{t \to \pm \infty} \int_{\mathbb{R}^d} \left( |\nabla u(x, t)|^2 - |u_r(x, t)|^2 + \frac{|u(x, t)|^2}{|x|^2} \right) dx = 0
\]

and there exist two functions \( G_{\pm} \in L^2(\mathbb{R} \times S^{d-1}) \) so that

\[
\begin{align*}
\lim_{t \to \pm \infty} \int_0^\infty \int_{S^{d-1}} \left| \frac{d}{dr} u(r\theta, t) - G_{\pm}(r \mp t, \theta) \right|^2 d\theta dr = 0; \\
\lim_{t \to \pm \infty} \int_0^\infty \int_{S^{d-1}} \left| \frac{d}{dr} u(r\theta, t) \pm G_{\pm}(r \mp t, \theta) \right|^2 d\theta dr = 0.
\end{align*}
\]

In addition, the maps \((u_0, u_1) \to \sqrt{2}G_{\pm}\) are bijective isometries from \(\dot{H}^1 \times L^2(\mathbb{R}^d)\) to \(L^2(\mathbb{R} \times S^{d-1})\).

We call \(G_{\pm}\) radiation fields associated to the free wave \(u\). Throughout this section we utilize the notations \(T_{\pm}\) for the linear map from the initial data \((u_0, u_1)\) to the corresponding radiation fields \(G_{\pm}\). It immediately follows the theorem that

\[
\lim_{t \to \pm \infty} \int_{|x| > R + |t|} |\nabla_t x u(x, t)|^2 dx = 2 \int_R^{\infty} \int_{S^2} |G_{\pm}(s, \theta)|^2 d\theta ds.
\]

In addition, the map between \(G_{\pm}\) is an isometry given explicitly by

\[
G_{\pm}(s, \theta) = \begin{cases} 
( -1 )^{\frac{d-1}{2}} G_{\mp}(s, \theta), & \text{if } d \text{ is odd; } \\
( -1 )^{\frac{d}{2}} (H G_{\mp})(s, \theta), & \text{if } d \text{ is even. }
\end{cases}
\]

This can proved in different methods. Please refer to Côte-Laurent [1], Duyckaerts-Kenig-Merle [4] and Li-Shen-Wei [16], for examples. As a result, the following identity holds for all odd dimensions \(d \geq 3\):

\[
\sum_{\pm} \lim_{t \to \pm \infty} \int_{|x| > R + |t|} |\nabla_t x u(x, t)|^2 dx = 2 \int_R^{\infty} \int_{S^2} |G_{\mp}(s, \theta)|^2 d\theta ds. \tag{12}
\]

As a result, the \(L^2\) decay rate of radiation field \(G_{\mp}(s, \theta)\) near the infinity indicates to what extent the free wave \(u\) looks like a non-radiative solution.

Decay of linear non-radiative solutions Another important ingredient of our estimate on non-linear non-radiative solutions is the corresponding decay estimates of linear non-radiative solutions. We claim that given any constant \(\kappa \in (0, 1/5)\), the following inequality holds

\[
\|u\|_{L^\infty_t L^\infty_r (\{x: |x| > R + |t|\})} \lesssim_{\kappa} (R/r)^\kappa E^{1/2}, \tag{13}
\]

for any \(R > 0\) and \(R\)-weakly non-radiative linear wave \(u\) with a finite energy \(E\), i.e. a finite-energy solution to the homogeneous linear wave equation \(\partial_t^2 u - \Delta u = 0\) so that

\[
\lim_{t \to \pm \infty} \int_{|x| > R + |t|} |\nabla_t x u(x, t)|^2 dx = 0.
\]
In fact, it was prove in Li-Shen-Wang [17] that any $R$-weakly non-radiative linear wave $u$ satisfies that inequality
\[ \|u\|_{L^p_t L^q_x([x,|x|>r+|t|])} \lesssim (R/r)^{1/3} E^{1/2}. \] (14)

We may interpolate it with a regular Strichartz estimate (see Ginibre-Velo [13])
\[ \|u\|_{L^p_t L^q_x(\mathbb{R} \times \mathbb{R}^3)} \lesssim p,q E^{1/2} \]
with $p = 2^+$ and $q = \infty^-$ and conclude that the inequality [13] holds for any $\kappa \in (0,1/5)$.

### 3.2 Statement and Proof

**Proposition 3.2.** Let $u$ be an $R$-weakly non-radiative solution to the non-linear wave equation
\[ \begin{cases} \partial_t^2 u - \Delta u = F(x, t, u), & (x, t) \in \mathbb{R}^3 \times \mathbb{R}; \\ (u, u_t)|_{t=0} = (u_0, u_1) \in \dot{H}^1 \times L^2(\mathbb{R}^3). \end{cases} \]

Here the nonlinear term satisfies (11). Then we have

(a) Given any $\kappa \in (0,1/5)$, the radiation field $G_-(s, \omega)$ associated to the linear wave $S_L(u_0, u_1)$ satisfies a decay estimate
\[ \|G_-\|_{L^2((s,|s|>r) \times \mathbb{S}^2)} \lesssim r^{-5\kappa}, \quad \forall r \gg R. \]

It is equivalent to saying (see [12])
\[ \lim_{t \to \pm \infty} \int_{|x|>|t|+|t|} |\nabla_{t,x} S_L(u_0, u_1)(x, t)|^2 \, dx \lesssim r^{-10\kappa}, \quad \forall r \gg R. \]

(b) The initial data $u_0$ satisfy the decay estimate
\[ \int_{|x|>r} |\partial_r u_0(x)|^2 \, dx \lesssim r^{-1}, \quad \forall r \gg R. \]

(c) We also the decay estimate
\[ \sup_{t \in \mathbb{R}} \int_{|x|>|t|+|t|} |u(x, t)|^6 \, dx \lesssim r^{-2}, \quad \forall r \gg R. \]

**Proof.** Let us first introduce a notation for convenience. We define
\[ S(r) = \|G_-\|_{L^2((s,|s|>r) \times \mathbb{S}^2)} = \left( \int_{|s|>r} \int_{\mathbb{S}^2} |G_-(s, \omega)|^2 \, d\omega \, ds \right)^{1/2}. \]

Given any $r \gg r_1 \gg R$, we may break $G_-$ into two parts
\[ G_1(s, \omega) = \begin{cases} G_-(s, \omega), & |s| \leq r_1; \\ 0, & |s| > r_1; \end{cases} \quad G_2(s, \omega) = \begin{cases} 0, & |s| \leq r_1; \\ G_-(s, \omega), & |s| > r_1. \end{cases} \]

Therefore we have
\[ (u_0, u_1) = T_{-1}^r G_1 + T_{-1}^r G_2. \] (15)

We also define $\chi_r(x, t)$ to be the characteristic function of the exterior region $\Omega(r) = \{(x, t) : |x| > |t| + r \}$ and
\[ \|v\|_{Y(r)} = \|\chi_r(x, t)u\|_{L^5_t L^{10}(\mathbb{R} \times \mathbb{R}^3)} = \|v\|_{L^5_t L^{10}((x,|x|>r+|t|)}; \]

11
Next we give a reasonable upper bound of \(\|S_L(u_0, u_1)\|_{Y(\tau)}\) by our decay estimate assumption. In fact we have

\[
\|S_L(u_0, u_1)\|_{Y(\tau)} \leq \|S_L T^{-1} G_1\|_{Y(\tau)} + \|S_L T^{-1} G_2\|_{Y(\tau)} \\
\lesssim (r_1/r)^\alpha |G_1|_{L^2} + |G_2|_{L^2} \\
\lesssim (r_1/r)^\alpha + S(r_1). \tag{16}
\]

Here we utilize the fact that \(G_1\) is supported in \([-r_1, r_1] \times \mathbb{S}^2\) thus the linear free wave \(S_L T^{-1} G_1\) with radiation field \(G_1\) is an \(r_1\)-weakly non-radiative free wave. We then apply (13) on the \(G_1\) part and the classic Strichartz estimate on the \(G_2\) part. Now we consider a modified non-linear wave equation

\[
\begin{aligned}
\begin{cases}
\partial_t^2 v - \Delta v = \chi_r(x, t) F(x, t, v), & (x, t) \in \mathbb{R}^3 \times \mathbb{R}; \\
(v, v_1)|_{t=0} = (u_0, u_1) \in H^1 \times L^2(\mathbb{R}^3).
\end{cases}
\end{aligned}
\tag{17}
\]

First of all, the following inequalities hold by our assumption on the non-linear term \(F\).

\[
\begin{aligned}
||x, F(x, t, v)||_{L^1 L^2(\mathbb{R}^3 \times \mathbb{R})} & \lesssim ||v||_{Y(\tau)}^4; \\
||\chi_r F(x, t, v_1) - \chi_r F(x, t, v_2)||_{L^1 L^2(\mathbb{R}^3 \times \mathbb{R})} & \lesssim ((||v_1||_{Y(\tau)}^4 + ||v_2||_{Y(\tau)}^4)) ||v_1 - v_2||_{Y(\tau)}.
\end{aligned}
\]

We also recall the classic Strichartz estimate (see [13]): if \(w\) solves the 3D linear wave equation \(\partial_t^2 w - \Delta w = F\) with initial data \((w_0, w_1)\), then

\[
||w||_{L^1 L^2 u(\mathbb{R}^3 \times \mathbb{R}^3)} + ||(w, \partial_t w)||_{C(\mathbb{R}; H^1 \times L^2)} \lesssim ((||w_0||_{H^1}, ||w_1||_{L^2})) ||F||_{L^1 L^2(\mathbb{R}^3 \times \mathbb{R})}.
\]

We may combine all these inequalities, apply a standard fixed-point argument of contraction map and conclude that as long as \(||S_L(u_0, u_1)||_{Y(\tau)}\) is sufficiently small, which holds under our assumption \(r \gg r_1 \gg R\) by (16), the equation (17) always has a global-in-time solution \(v\), so that

\[
||v||_{Y(\tau)} \leq 2||S_L(u_0, u_1)||_{Y(\tau)}. \tag{18}
\]

More details about the fixed-point argument of this kind can be found, for instance, in Pecher [13]. Furthermore, we may write \(v\) as a sum of two terms

\[
v = v_1 + v_2.
\]

They are the linear propagation part and the contribution of non-linear term, respectively:

\[
v_1 = S_L(u_0, u_1); \quad v_2 = \int_0^t \sin(t - \tau) \sqrt{-\Delta} (\chi_r F(\cdot, \tau, v(\cdot, \tau))) d\tau. \tag{19}
\]

The triangle inequality in \(L^2\) space gives

\[
\left( \int_{|x| > r + |t|} |\nabla_{t,x} v|^2 dx \right)^{1/2} \geq \left( \int_{|x| > r + |t|} |\nabla_{t,x} v_1|^2 dx \right)^{1/2} - \left( \int_{|x| > r + |t|} |\nabla_{t,x} v_2|^2 dx \right)^{1/2}
\]

for any given time \(t\). A comparison of our modified non-linear wave equation (17) with the original one shows that \(u(x, t) \equiv v(x, t)\) in the exterior region \(\Omega(\tau)\) by finite speed of propagation. Therefore our non-radiative assumption on \(u\) also applies on \(v\) in the exterior region \(\Omega(\tau)\). This gives

\[
\lim_{t \to \pm \infty} \int_{|x| > r + |t|} |\nabla_{t,x} v|^2 dx = 0.
\]

Therefore we have

\[
\liminf_{t \to \pm \infty} \int_{|x| > r + |t|} |\nabla_{t,x} v_2(x, t)|^2 dx \geq \lim_{t \to \pm \infty} \int_{|x| > r + |t|} |\nabla_{t,x} v_1(x, t)|^2 dx.
\]
We then recall the property of radiation field and obtain
\[
\sum_{\pm} \lim_{t \to \pm \infty} \int_{|x| > r+|t|} |\nabla_{t,x} v_1(x,t)|^2 dx = 2 \int_{|x| > r} \int_{\mathbb{R}^3} |G_-(s,\omega)|^2 d\omega ds = 2S^2(r).
\]
We may also find an upper bound of the integral about \(v_2\) by Strichartz estimates
\[
\int_{|x| > r+|t|} |\nabla_{t,x} v_2(x,t)|^2 dx \leq \int_{\mathbb{R}^3} |\nabla_{t,x} v_2(x,t)|^2 dx \leq \|\chi\tau F(x,t,v)\|_{L^1 L^{2,\infty}(\mathbb{R} \times \mathbb{R}^3)} \lesssim \|v\|_{Y(\mathbb{R})}^{10}.
\]
Combining these inequalities we obtain \(S(r) \lesssim \|v\|_{Y(\mathbb{R})}^5\). We then utilize the upper bound given in (18) and obtain
\[
S(r) \lesssim \|S_L(u_0, u_1)\|_{Y(\mathbb{R})}^5, \quad r \gg R.
\] (20)
A combination of this inequality with (10) immediately gives a recursion formula when \(r \gg r_1 \gg R\).
\[
S(r) \lesssim \frac{r^5}{r^6} + S^5(r_1).
\]
We then apply Lemma 4.3 whose statement and proof is postponed to the appendix, and conclude that given any \(\beta \in (0, 5\kappa)\), the following estimate holds if \(r \geq R_0(u, \kappa, \beta)\) is sufficiently large
\[
S(r) \leq r^{-\beta}.
\]
Next we give a more detailed estimate of \(\|S_L(u_0, u_1)\|_{Y(\mathbb{R})}\) as \(r \to +\infty\). We fix a constant \(\beta \in (\kappa, 5\kappa)\), choose \(R_0 = R_0(u, \kappa, \beta)\) accordingly as above and define
\[
G_0(s, \omega) = \begin{cases} G_-(s, \omega), & |s| \leq R_0; \\ 0, & |s| > R_0; \end{cases} \quad G_j(s, \omega) = \begin{cases} G_-(s, \omega), & 2^{j-1}R_0 < |s| \leq 2^j R_0; \\ 0, & \text{otherwise}; \end{cases} \quad j \geq 1.
\]
Thus we have
\[
(u_0, u_1) = \sum_{j=0}^{\infty} T^{-1} G_j.
\]
If \(r \in [2^n R_0, 2^{n+1} R_0]\) for an integer \(n \geq 0\), then we have
\[
\|S_L(u_0, u_1)\|_{Y(\mathbb{R})} \lesssim \sum_{j=0}^{n} \left\| S_L T^{-1} G_j \right\|_{Y(\mathbb{R})} + \left\| S_L T^{-1} \left( \sum_{j=n+1}^{\infty} G_j \right) \right\|_{Y(\mathbb{R})} \lesssim \sum_{j=0}^{n} \left( 2^j R_0 / r \right)^\kappa \left\| G_j \right\|_{L^2} + \left\| \sum_{j=n+1}^{\infty} G_j \right\|_{L^2} \lesssim r^{-\kappa} + \sum_{j=1}^{n} 2^{\kappa j} r^{-\kappa} S(2^{j-1} R_0) + S(2^n R_0) \lesssim r^{-\kappa} + \sum_{j=1}^{n} 2^{\kappa j} r^{-\kappa} (2^{j-1} R_0)^{-\beta} + (2^n R_0)^{-\beta} \lesssim r^{-\kappa}.
\]
We apply the decay estimate (13) and use the upper bound \(S(r) \leq r^{-\beta}\) here. Finally we recall (20) and conclude that the inequality \(S(r) \lesssim r^{-5\kappa}\) holds if \(r \geq R_1\) is sufficiently large. This finishes the proof of part (a). The proof of part (b) is similar the final stage of proof for part
(a). We first fix a constant $\kappa \in (1/10,1/5)$. According to part (a), there exists $R_1 > R$ so that $S(r) \lesssim r^{-5\kappa}$ holds for $r \geq R_1$. We define

$$G_0(s, \omega) = \begin{cases} G_-(s, \omega), & |s| \leq R_1; \\ 0, & |s| > R_1; \end{cases} \quad G_j(s, \omega) = \begin{cases} G_-(s, \omega), & 2^{j-1}R_1 < |s| \leq 2^jR_1; \\ 0, & \text{otherwise}; \end{cases} \quad j \geq 1;$$

and

$$(u_0, u_1) = \sum_{j=0}^\infty (u_{0,j}, u_{1,j}), \quad (u_{0,j}, u_{1,j}) = T^-_j G_j.$$

Since $(u_{0,j}, u_{1,j}) \in P(2^j R_1)$, if $r > 2^j R_1$, then we may apply Proposition 2.1 and obtain

$$\left( \int_{|x| > r} |\partial_r u_{0,j}(x)|^2 dx \right)^{1/2} \lesssim (2^j R_1/r)^{1/2} \left( \int_{|x| > 2^j R_1} |\nabla u_{0,j}(x)|^2 dx \right)^{1/2} \lesssim (2^j R_1/r)^{1/2} \| G_j \|_{L^2}.$$

Furthermore, if we also have $j \geq 1$, then we may use the upper bound of $S(r)$ and obtain $\|G_j\|_{L^2} \lesssim S(2^{j-1} R_1) \lesssim (2^{j-1} R_1)^{-5\kappa}$. As a result, we have

$$\left( \int_{|x| > r} |\partial_r u_{0,j}(x)|^2 dx \right)^{1/2} \lesssim (2^j R_1/r)^{1/2} (2^{j-1} R_1)^{-5\kappa} \lesssim (2^j R_1)^{1/2 - 5\kappa r^{-1/2}}.$$
As a result, if $2^{n-1} R_1 < r < 2^n R_1$, then we have

$$
\sup_{t \in \mathbb{R}} \left( \int_{|x| > r + |t|} |S_L(u_0, u_1)|^6 \, dx \right)^{1/6} \leq \sum_{j=0}^{n-1} \sup_{t \in \mathbb{R}} \left( \int_{|x| > r + |t|} |S_L(u_{0,j}, u_{1,j})|^6 \, dx \right)^{1/6}
+ \sup_{t \in \mathbb{R}} \left( \int_{|x| > r + |t|} \left| S_L \left( \sum_{j=n}^{\infty} (u_{0,j}, u_{1,j}) \right) (x,t) \right|^6 \, dx \right)^{1/6}
\lesssim \sum_{j=0}^{n-1} (2^{j} R_1/r)^{1/6} \|G_j\|_{L^2} + \left\| \sum_{j=n}^{\infty} G_j \right\|_{L^2}.
$$

We then apply the $L^2$ decay estimate of $G_j$ given in part (a) and obtain ($r > R_1$)

$$
\sup_{t \in \mathbb{R}} \left( \int_{|x| > r + |t|} |S_L(u_0, u_1)|^6 \, dx \right)^{1/6} \lesssim (R_1/r)^{1/6} + \sum_{j=1}^{n-1} (2^j R_1/r)^{1/6} (2^{j-1} R_1)^{-5\kappa} + (2^{n-1} R_1)^{-5\kappa}
\lesssim r^{-1/3} + \sum_{j=1}^{n-1} (2^j R_1)^{1/6-5\kappa} r^{-1/3} + r^{-5\kappa}
\lesssim r^{-1/3}.
$$

Next we recall that if we let $v$ solves (17) and define $v_1, v_2$ accordingly as in (19), then

$$
u(x,t) = v(x,t) = v_1(x,t) + v_2(x,t)
$$

holds in the exterior region $\{(x,t) : |x| > r + |t|\}$. Our argument above has already given $L^6$ upper bound of $v_1 = S_L(u_0, u_1)$. It suffices to consider the upper bound of $v_2$. By the Strichartz estimates, we have

$$
\sup_{t \in \mathbb{R}} \|v_2(\cdot, t)\|_{L^6(\mathbb{R}^3)} \lesssim \sup_{t \in \mathbb{R}} \|v_2(\cdot, t)\|_{H^1(\mathbb{R}^3)} \lesssim \|F(x,t,v)\|_{L^4 L^2(\mathbb{R} \times \mathbb{R}^3)} \lesssim \|v\|_{Y(r)}^5.
$$

Finally we recall (13) and the estimate $\|S_L(u_0, u_1)\|_{Y(r)} \lesssim r^{-\kappa}$ given in part (a), if $r$ is sufficiently large, and obtain $\|v\|_{Y(r)} \lesssim r^{-\kappa}$. Combining this with the inequality above we have

$$
\sup_{t \in \mathbb{R}} \|v_2(\cdot, t)\|_{L^6(\mathbb{R}^3)} \lesssim r^{-5\kappa}, \quad r \gg R.
$$

We collect upper bounds of $v_1 = S_L(u_0, u_1)$ and $v_2$ to conclude the proof of part (c).

$$
\sup_{t \in \mathbb{R}} \left( \int_{|x| > r + |t|} |u(x,t)|^6 \, dx \right)^{1/6} \lesssim r^{-1/3}, \quad r \gg R.
$$

4 Appendix

In this section we prove a few technical lemmata. The authors believe that these results are probably previously known. For completeness we still give their proof.
Polynomial estimates. We start by Lemma 2.2. By change of variables \( x = 2z/L - 1 \), we may rewrite this technical lemma as below.

**Lemma 4.1.** Let \( 0 < \delta \leq 1 \) and \( P(x) \) be a polynomial of degree \( \kappa \). Then we have

\[
\max_{x \in [-1,1]} |P(x)|^2 \leq \frac{\kappa + 1}{2} \int_{-1}^{1} |P(x)|^2 dx;
\]

\[
\int_{-1}^{-1+\delta} |(x+1)P'(x)|^2 dx \leq \kappa(\kappa + 1)\delta \int_{-1}^{1} |P(x)|^2 dx.
\]

**Proof.** Let us recall Legendre polynomials \( P_n \) defined by

\[
P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n.
\]

It is well known that \( \{P_n\}_{n=0,1,2,...} \) are orthogonal to each other in \( L^2([-1,1]) \) with norm \( \|P_n\|_{L^2} = \frac{2}{2n+1} \). In addition, these polynomials satisfy

\[
\frac{d}{dx} \left[ (1-x^2) \frac{d}{dx} P_n(x) \right] + n(n+1)P_n(x) = 0.
\]

More details about the properties of Legendre polynomials can be found, for instance, in Folland [12]. We consider the orthogonal decomposition of \( P(x) \):

\[
P(x) = \sum_{n=0}^{\kappa} a_n P_n(x) \quad \Rightarrow \quad \int_{-1}^{1} |P(x)|^2 dx = \sum_{n=0}^{\kappa} \frac{2|a_n|^2}{2n+1}.
\]

This immediately gives

\[
\max_{x \in [-1,1]} |P(x)|^2 \leq \left( \sum_{n=0}^{\kappa} |a_n| \right)^2 \leq \left( \sum_{n=0}^{\kappa} \frac{2n+1}{2} \right) \left( \sum_{n=0}^{\kappa} \frac{2|a_n|^2}{2n+1} \right) = \frac{(\kappa + 1)^2}{2} \int_{-1}^{1} |P(x)|^2 dx.
\]

We also have

\[
\int_{-1}^{-1+\delta} |(x+1)P'(x)|^2 dx \leq \delta \int_{-1}^{1} (1-x^2)|P'(x)|^2 dx \leq \delta \int_{-1}^{1} (1-x^2)|P'(x)|^2 dx
\]

We then integrate by parts, use the differential equation above and obtain

\[
\int_{-1}^{1} (1-x^2)|P'(x)|^2 dx = -(1-x^2)P(x) \cdot \frac{d}{dx} [(1-x^2)P'(x)] dx
\]

\[
= \int_{-1}^{1} \left( \sum_{n=0}^{\kappa} a_n P_n(x) \right) \left( \sum_{n=0}^{\kappa} n(n+1)a_n P_n(x) \right) dx
\]

\[
= \sum_{n=0}^{\kappa} \frac{2n(n+1)|a_n|^2}{2n+1}
\]

\[
\leq \kappa(\kappa + 1) \int_{-1}^{1} |P(x)|^2 dx.
\]

Combining these two inequalities, we finish the proof. \(\square\)

We also need a similar lemma, where \( dx \) is substituted by \( (x+1)dx \). This immediately gives Lemma 2.4 by a change of variables \( x = 2z/L - 1 \).
Lemma 4.2. Let $0 < \delta \leq 1$ and $P(x)$ be a polynomial of degree $\kappa$. Then we have

$$\max_{x \in [-1,1]} (x+1)|P(x)|^2 \leq (\kappa+1)^2 \int_{-1}^{1} (x+1)|P(x)|^2 dx; \quad (21)$$

$$\int_{-1}^{-1+\delta} (x+1)^3|P'(x)|^2 dx \leq \kappa(\kappa+2)\delta \int_{-1}^{1} (x+1)|P(x)|^2 dx. \quad (22)$$

Proof. We define $Q_n(x)$ to be the modified Legendre polynomial of degree $n$:

$$Q_n(x) = \frac{1}{2n+1(n+1)!} d^{n+1} \left[ (x+1)^n (x-1)^{n+1} \right] = \frac{(2n+1)!}{2^{n+1}n!(n+1)!} x^n + \cdots.$$  

If $n \geq m$ are nonnegative integers, then we may apply integration by parts and obtain

$$\int_{-1}^{1} (x+1)Q_n(x)Q_m(x)dx = \frac{(-1)^{n+1}}{2^{n+1}(n+1)!} \int_{-1}^{1} (x+1)^n(x-1)^{n+1} \frac{d^{n+1}}{dx^{n+1}} [(x+1)Q_m(x)] dx.$$  

A basic calculation shows

$$\frac{d^{n+1}}{dx^{n+1}} [(x+1)Q_m(x)] = \left\{ \begin{array}{ll} \frac{(2n+1)!}{2^{n+1}m!} & \text{if } m = n; \\ 0 & \text{if } m < n. \end{array} \right.$$  

Therefore $\{Q_n(x)\}_{n \geq 0}$ are orthogonal to each other in the Hilbert space $L^2([-1,1];(x+1)dx)$ and the norms of these polynomials are given by

$$\|Q_n\|_{L^2([-1,1];(x+1)dx)}^2 = \frac{1}{2(2n+1)}.$$  

In addition, these polynomials satisfy a similar differential equation to Legendre polynomials.

$$\frac{d}{dx} \left[ (x+1)(1-x^2) \frac{d}{dx} Q_n(x) \right] + n(n+2)(x+1)Q_n(x) = 0. \quad (23)$$  

In order to prove this identity, we observe that $\frac{d}{dx} \left[ (x+1)(x^2-1) \frac{d}{dx} Q_n(x) \right]$ is a polynomial of degree $n+1$ and contain a factor of $x+1$. Thus we may write

$$\frac{d}{dx} \left[ (x+1)(x^2-1) \frac{d}{dx} Q_n(x) \right] = \sum_{j=0}^{n} a_j (x+1)Q_j(x).$$  

We multiply both sides by $Q_j(x)$, integrate from $x = -1$ to $x = 1$ and apply integration by parts

$$\frac{a_j}{2(j+1)} = \int_{-1}^{1} Q_j(x) \frac{d}{dx} \left[ (x+1)(x^2-1) \frac{d}{dx} Q_n(x) \right] dx$$

$$= \int_{-1}^{1} Q_n(x) \frac{d}{dx} \left[ (x+1)(x^2-1) \frac{d}{dx} Q_j(x) \right] dx$$

$$= \frac{(-1)^{n+1}}{2^{n+1}(n+1)!} \int_{-1}^{1} (x+1)^n(x-1)^{n+1} \frac{d^{n+1}}{dx^{n+1}} [x+1)(x^2-1) \frac{d}{dx} Q_j(x) ] dx.$$  

A direct calculation shows

$$\frac{d^{n+2}}{dx^{n+2}} \left[ (x+1)(x^2-1) \frac{d}{dx} Q_j(x) \right] = \left\{ \begin{array}{ll} \frac{n(n+2)(2n+1)!}{2^{n+1}m!} & \text{if } j = n; \\ 0 & \text{if } j < n. \end{array} \right.$$  

Thus we have $a_j = 0$ if $j < n$ and $a_n = n(n+2)$. This gives (23). Now we are ready to prove Lemma 4.2. We first prove the second inequality (22). Let $P(x)$ be a polynomial of degree $\kappa$. We may write

$$P(x) = \sum_{n=0}^{\kappa} a_n Q_n(x).$$  

17
We have
\[ \int_{-1}^{-1+\delta}(x+1)^3|P'(x)|^2dx \leq \delta \int_{-1}^{-1+\delta}(x+1)(1-x^2)|P'(x)|^2dx \leq \delta \int_{-1}^{1}(x+1)(1-x^2)|P'(x)|^2dx. \]
We then integrate by parts, use the differential equation and orthogonality of \( \{Q_n\} \).
\[
\int_{-1}^{1}(x+1)(1-x^2)|P'(x)|^2dx = - \int_{-1}^{1}P(x) \frac{d}{dx}[(x+1)(1-x^2)P'(x)] dx
\]
\[
= \int_{-1}^{1} \left( \sum_{n=0}^{\kappa} a_n Q_n(x) \right) \left( \sum_{n=0}^{\kappa} n(n+2)a_n(x+1)Q_n(x) \right) dx
\]
\[
= \sum_{n=0}^{\kappa} \frac{n(n+2)|a_n|^2}{2(n+1)}
\]
\[
\leq \kappa(\kappa + 2) \int_{-1}^{1}(x+1)|P(x)|^2dx.
\]
Combining these two inequalities, we finish the proof of (22). We then prove the first inequality (21). First of all, we have
\[
\max_{x \in [0,1]} |P(x)|^2 \leq (\kappa + 1)^2 \int_{0}^{1}|P(x)|^2 dy \leq (\kappa + 1)^2 \int_{-1}^{1}(x+1)|P(x)|^2 dy.
\]
Here we apply Lemma 2.2. This deals with the case \( x \in [0,1] \). Next we observe that if \( x \in (-1,0) \), then we may apply a translated-version of Lemma 2.2 and obtain
\[
|P(x)|^2 \leq \max_{y \in [x,1]} |P(y)|^2 \leq \frac{(\kappa + 1)^2}{1-x} \int_{x}^{1}|P(y)|^2 dy \leq \frac{(\kappa + 1)^2}{1-x^2} \int_{x}^{1}(1+y)|P(y)|^2 dy.
\]
This immediately gives
\[
(1+x)|P(x)|^2 \leq \frac{(\kappa + 1)^2}{1-x} \int_{x}^{1}(1+y)|P(y)|^2 dy \leq (\kappa + 1)^2 \int_{-1}^{1}(1+y)|P(y)|^2 dy, \quad x \in (-1,0).
\]
Finally we combine this with the upper bound (24) for \( x \in [0,1] \) to finish the proof of (21). \( \square \)

**Decay by recursion** Finally we prove a lemma giving polynomial decay by a suitable recursion formula.

**Lemma 4.3.** Assume that \( l > 1 \) and \( \alpha > 0 \) are constants. Let \( S : [R, +\infty) \to [0, +\infty) \) be a function satisfying
- \( S(r) \to 0 \) as \( r \to +\infty \);
- The recursion formula \( S(r_2) \lesssim (r_1/r_2)^\alpha + S^l(r_1) \) holds when \( r_2 \gg r_1 \gg R \).

Then given any constant \( \beta \in (0, (1-1/l)\alpha) \), the decay estimate \( S(r) \leq r^{-\beta} \) holds as long as \( r > R_0 \) is sufficiently large.

**Proof.** Without loss of generality, we may assume the recursion formula
\[
S(r_2) \leq \frac{1}{2}(r_1/r_2)^\alpha + \frac{1}{2}S^l(r_1)
\]
holds for \( r_2 \gg r_1 \gg r \). Otherwise we may slightly reduce the values of \( l \) and \( \alpha \). We first find a small constant \( \gamma > 0 \) so that \( S(r) \leq r^{-\gamma} \) for large \( r \), then plug this estimate back in the
recursion formula and slightly enlarge the value of \( \gamma \), finally iterate our argument to finish the proof. We start by recalling the assumption on the limit of \( S(r) \) at the infinity and choosing a large constant \( M > R \) so that

\[
S(r) < 1/2, \quad \forall r \in [M, M^l].
\]

This implies that we may choose a sufficiently small constant \( \gamma \in (0, (1 - 1/l)\alpha) \) so that

\[
S(r) < r^{-\gamma}, \quad \forall r \in [M, M^l].
\]

Next we prove that \( S(r) \leq r^{-\gamma} \) holds for any \( r \geq M \) by induction. It suffices to shows that this inequality holds for \( r \in [M^{k}, M^{k+1}] \) if it holds for \( r \in [M^{k-1}, M^{k}] \). In fact, if \( r \in [M^{k}, M^{k+1}] \), then we have

\[
S(r) \leq \frac{1}{2} (r^{1/l}/r)^\alpha + \frac{1}{2} S^l(r^{1/l}) \leq \frac{1}{2} r^{-(1-1/l)\alpha} + \frac{1}{2} r^{-\gamma} \leq r^{-\gamma}.
\]

Here we utilize induction hypothesis on \( S(r^{1/l}) \). Next we plug in \( r_1 = r^{\alpha/(\alpha + \gamma l)} \) and \( r_2 = r \) in the recursion formula, use the already known upper bound \( S(r_1) \leq r_1^{-\gamma} \), for sufficiently large \( r \), then obtain

\[
S(r) \leq \frac{1}{2} (r^{\alpha/(\alpha + \gamma l)}/r)^\alpha + \frac{1}{2} S^l(r^{\alpha/(\alpha + \gamma l)}) \leq r^{-\alpha \gamma l/(\alpha + \gamma l)}.
\]

We may iterate this argument and conclude that

\[
S(r) \leq r^{-\gamma_k}, \quad \forall r \geq r_k.
\]

Here \( \gamma_k \in (0, (1 - 1/l)\alpha) \) are defined by the induction formula

\[
\gamma_0 = \gamma; \quad \gamma_{k+1} = \frac{\alpha \gamma_k l}{\alpha + \gamma_k l}, \quad k \geq 0.
\]

In order to finish the proof, we only need to show \( \gamma_k \to (1 - 1/l)\alpha \) as \( k \to +\infty \). In fact, we may rewrite the induction formula in the form of

\[
(1 - 1/l)\alpha - \gamma_{k+1} = \frac{\alpha}{\alpha + \gamma_k l} \cdot [(1 - 1/l)\alpha - \gamma_k].
\]

Thus \( \gamma_k \in (0, (1 - 1/l)\alpha) \) increases as \( k \to +\infty \). This implies

\[
(1 - 1/l)\alpha - \gamma_{k+1} \leq \frac{\alpha}{\alpha + \gamma_k l} \cdot [(1 - 1/l)\alpha - \gamma_k] \quad \Rightarrow \quad (1 - 1/l)\alpha - \gamma_k \to 0^+.
\]

\[\Box\]

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