Multifractal Analysis for the Overlapping Windows Method of a Time Series of Synthetic Earthquakes obtained with a Spring-Block Model

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Abstract. We work with the model proposed by Olami, Feder and Christensen (OFC) to build time series with 10 000 000 data of synthetic earthquakes magnitudes, and conservation value $\beta = 0.2$. In this time series, we located the earthquakes with greater magnitude. We built 160 windows before and after these great earthquakes. The windows have 1024 data with an overlap of 896 points. For each windows, the values of the multifractality degree $\Delta \alpha$ and the parameter of symmetry were calculated. We follow the evolution of these parameters before and after the big earthquakes. The multifractal spectra have larger widths before than after the earthquake. The multifractal spectra of the 320 windows are left skewed, but the multifractal spectra of the time series before the earthquake are more biased towards the left than the multifractal spectra after the earthquake. Finally, we measured the curvature $K$ around the maximum of the multifractal spectra, the $K$ value is bigger before than after the earthquake. This suggests that there is apparently a process of preparation for the earthquake and that the dynamics after the earthquake is very different.

1. Introduction

Earthquakes are one of the most devastating catastrophes in the nature and Mexico is one of countries with highest seismicity, mainly because it is surrounded by four plates, North America, Pacific, Cocos and Rivera. Seismologists can explain the seismicity phenomenon, however, they cannot predict the earthquake occurrence because they don’t know all the properties that govern them.

The energy released during the earthquake is believed to increase exponentially with the earthquake magnitude according to the Gutenberg-Richter law and it can be interpreted as a manifestation of the self-organized critical behaviour of the earth dynamics. Concerning synthetic seismicity Bak and Tang [1] introduced a new self-organized critical (SOC) model to mimic the earthquake dynamics [1]. Olami, Feder and Christensen (OFC) introduced a generalized, continuous and non-conservative cellular automaton model that displays SOC to obtain time series of synthetic earthquakes [2, 3]. These time series have multifractal behaviour [4] according to the results obtained from the algorithm of Chhabra and Jensen [5]. In this work, the OFC model is used to generate a time series with level of conservation $\beta=0.2$ of 10 million of data. We located the earthquakes of great magnitude in the time series and built
160 windows before and after each one of these big earthquakes, the windows have 1024 data and have an overlapping of 896 data. We calculated the multifractal spectra for each window with the Chhabra and Jensen method and we observed that the multifractal spectra have larger widths before than after the big earthquakes. We calculated the symmetry parameter $r$ and found that the multifractal spectra of all windows are skewed to the left. We noticed that the multifractal spectra of the windows before the earthquake are more skewed to the left than the multifractal spectra after the earthquake. And finally, we measured the curvature $K$ parameter around the maximum of the multifractal spectra. We observed that the $K$ value is bigger before than after the earthquake. This suggests that there is apparently a process of preparation for earthquakes because the dynamics before and after great earthquakes is very different.

2. Methods

The dynamics of earthquake faults may provide a physical realization of the recently proposed idea of SOC. BTW introduced the concept of self-organized criticality, dynamical many-body systems reach a critical state without the need to fine-tune the system parameters [6]. Earthquakes are probably the most relevant paradigm of self-organized criticality [7]. Gutenberg and Richter realized that the rate of occurrence of earthquakes of magnitude $M$ greater than $m$ is given by the relation

$$\log_{10} N(M > m) = a - bm$$

(1)

This is the Gutenberg-Richter law and essentially it is a power law. In fact, power laws are quite common in nature. Thus the Gutenberg-Richter law can be interpreted as a manifestation of the self-organized critical behaviour of the earth dynamics. Bak and Tang indicated that the simple conservative SOC models can serve as a framework for explaining the power-law behaviour [6]. OFC introduced a new generalized, continuous, non-conservative cellular automaton spring-block model of earthquakes that displays SOC in 1992 [2, 3]. This model is a two-dimensional version of the famous Burridge-Knopoff spring-block model for earthquakes [9].

Spring-block model is a two-dimensional dynamical system of blocks interconnected by springs. Each block is connected to the four nearest neighbours. Additionally, each block is connected to a single rigid driving plate by another set of springs as well as connected frictionally to a fixed rigid plate [see Figure 1]. The blocks are driven by the relative movement of the two rigid plates. When the force on one of the blocks is larger than some threshold value $F_{th}$ (the maximal static friction), the block slips. We assume that the moving block will slip to the zero-force position. This assumption is not essential for the behaviour of the model as will become evident later on. The slip of one block will redefine the forces on its nearest neighbours. This can result in further slips and a chain reaction can evolve [1].

![Figure 1. The two-dimensional system of blocks connected by springs. The strain of the blocks increases uniformly as a response to the relative movement of the rigid plates.](image)

For the purpose of mapping the spring-block model into a cellular automaton model we define an $L \times L$ array of blocks by $(i, j)$, where $i, j$ are integers restricted to the interval between 1 and $L$. The displacement of each block from its relaxed position on the lattice is defined as $d_{i,j}$. The total force exerted by the springs on a given block $(i,j)$ is expressed by

$$F_{i,j} = K_1[2dx_{i,j} - dx_{i-1,j} - dx_{i+1,j}] + K_2[2dx_{i,j} - dx_{i,j-1} - dx_{i,j+1}] + K_L dx_{i,j}$$

(2)
where $K_1$, $K_2$, and $K_L$ denote the elastic constants. When the two rigid plates move relative to each other the total force on each block increases uniformly (with a rate proportional to $KLV$, where $V$ is the relative velocity between the two rigid plates) until one site reaches the threshold value and the process of relaxation begins (an earthquake is triggered). It can easily be shown that the redistribution of strain after a local slip at the position $(i, j)$ is given by the relation

$$F_{i,j} \rightarrow F_{i,j} + \delta F_{i,j},$$

$$F_{i,j+1} \rightarrow F_{i,j+1} + \delta F_{i,j+1},$$

$$F_{i,j} \rightarrow 0$$

(3)

where the increases in the nearest-neighboring forces are

$$\delta F_{i,j} = \frac{K_1}{2K_1 + 2K_2 + K_L} F_{i,j} = \beta_1 F_{i,j}$$

$$\delta F_{i,j+1} = \frac{K_2}{2K_1 + 2K_2 + K_L} F_{i,j+1} = \beta_2 F_{i,j+1}$$

(4)

For simplicity we denote the elastic ratios by $\beta_1$ and $\beta_2$, respectively. Notice that this relaxation rule is very similar to the BTW model. However, the redistribution of the force is non-conservative [2].

This model is restricted to the isotropic case, $K_1 = K_2$ ($\beta_1 = \beta_2 = \beta$). The boundary condition of the model is rigid, implying that $F = 0$ on the boundary. The time interval between earthquakes is much larger than the actual duration of an earthquake. Thus, the mapping of the spring-block model into a continuous, non-conservative cellular automaton modelling earthquake is described by the following algorithm [2].

(1) Initialize all sites to a random value between 0 and $F_{th}$.

(2) If any $F_{i,j} \geq F_{th}$ then redistribute the force on $F_{i,j}$ to its neighbours according to the rule

$$F_{n,n} \rightarrow F_{n,n} + \alpha F_{i,j},$$

$$F_{i,j} \rightarrow 0$$

(5)

where $F_{n,n}$ are the strains for the four-nearest neighbours. An earthquake is evolving.

(3) Repeat step 2 until the earthquake is fully evolved.

(4) Locate the block with the largest strain, $F_{max}$ Add $F_{th} - F_{max}$, to all sites (global perturbation) and return to step 2.

This model reproduces the actual dynamical process associated with earthquake faults and the Gutenberg-Richter law. The fractal geometric distribution and the earthquake dynamics are the spatial and temporal signatures of the same phenomenon [2-3, 7]. Olami, Feder and Christensen also considered the anisotropic case, when $\beta_1 \neq \beta_2$, but they obtained the same behavior [2, 3], it means that it is not necessary to consider the anisotropic case, because we obtain the same results.

Many time series obtained from measurements carried out in complex systems have multifractal features. This is because complex systems are composed of many components that interact with each other in a non-linear way. Time series of different dynamical complex systems have monofractal or multifractal behaviour. Monofractal signals are homogeneous and they are characterized by a single global exponent, while multifractal signals require many exponents to fully characterize their scaling properties. They are intrinsically more complex and inhomogeneous [4]. The multifractality degree is a measure of the complexity of the system and provides information about on what such non-linear and steady is not the same. However, the multifractal spectra of real systems are difficult to manage, in the sense that is difficult to get information from them.

Fractal and multifractal methods have been used extensively in the description of time series [10]. Earthquake occurrence may also have fractal like characteristics. It is thought that earthquakes occur on faults which are essentially a fracture in the earth’s crust. A simple clean cut in a three-dimensional object would have a dimension of two. However, consider the situation where small faults branch off larger faults, and from these smaller faults, even smaller faults are found. And this replication is repeated many times to a finer and finer level. However, within that fault network, there are certain areas that
will be much more active than others; i.e., have a greater probability of an earthquake event. As such, we could think of the set of possible locations of where an earthquake could occur to be a fractal set, but on that fractal set is a probability measure which describes the likelihood of an event. Usually this probability distribution is extremely irregular to the extent that it does not have a density probability [4].

The multifractal distributions can also be described by the graphic of fractal dimension \( f(\alpha) \) versus Hölder exponent \( \alpha \), it is named multifractal spectrum. To calculate the multifractal spectra, we use the method proposed by Chhabra and Jensen [4]. This method considers time series as a singular measure \( P(x) \) if we normalize it. We obtain the fractal dimension \( f(\alpha) \) covering the measure with boxes of length \( L = 2^{-n} \) and computing the probabilities \( \mu_i(L) \) in each of the boxes. We then construct a one-parameter family of normalized measures \( \mu_i(q, L) \), where the probabilities in the boxes of size \( L \) are [9]:

\[
\mu_i(q, L) = \frac{[P_i(L)]^q}{\sum_i[P_i(L)]^q}
\]  

(6)

The fractal dimension is

\[
f(q) = \lim_{L \to 0} \frac{\sum_i \mu_i(q, L) \log[\mu_i(q, L)]}{\log L},
\]  

(7)

and the singularity strength is

\[
\alpha(q) = \lim_{L \to 0} \frac{\sum_i \mu_i(q, L) \log[P_i(L)]}{\log L}.
\]  

(8)

Equations (7) and (8) provide a relationship between the dimension \( f(\alpha) \) and the mean singularity strength \( \alpha \) as implicit functions of the parameter \( q \). To obtained multifractal spectra, for each \( q \) value we evaluate the numerators on the right-hand sides of the equations (6) and (7), for decreasing box sizes (increasing \( n \)) and we graph these results versus \( \log L \). The graphics obtained are straight line and we calculate slope’s graphics and so we obtain a pair of point \( f(q) \) and \( \alpha(q) \) for each \( q \). With these pair of values we built multifractal spectra with points with coordinates \((\alpha(q), f(q))\).

The parameter \( q \) provides a microscope for exploring different regions of the singular measure. For \( q>1 \), \( \mu(q) \) amplifies the more singular regions of \( P \), while for \( q<1 \) it accentuates the less singular regions, and for \( q=1 \) the measure \( \mu(1) \) replicates the original measure [4].

We characterize the multifractal spectra by its width and its asymmetry. \( \Delta\alpha = \alpha_{\text{max}} - \alpha_{\text{min}} \) is a measure of how wide is the range of fractal exponents found in the signal; and, thus, it measures the time series degree of multifractality [4]; \( \alpha_0 \) corresponds to the maximum of \( f(\alpha) \). The asymmetry depends on \( \alpha_{\text{max}}, \alpha_{\text{min}} \) and \( \alpha_0 \). We define \( \Delta\alpha_{\text{right}} = \alpha_{\text{max}} - \alpha_0 \) and \( \Delta\alpha_{\text{left}} = \alpha_0 - \alpha_{\text{min}} \). If \( \Delta\alpha_{\text{right}} = \Delta\alpha_{\text{left}} \), the spectrum is symmetric, and if \( \Delta\alpha_{\text{right}} \neq \Delta\alpha_{\text{left}} \) it is asymmetric. If \( \Delta\alpha_{\text{right}} > \Delta\alpha_{\text{left}} \), the spectrum is biased to the right and if \( \Delta\alpha_{\text{right}} < \Delta\alpha_{\text{left}} \) it is asymmetric. If \( \Delta\alpha_{\text{right}} \gg \Delta\alpha_{\text{left}} \), the spectrum is biased to the right.

We introduce the symmetry parameter \( r \), to better quantify the symmetry,

\[
r = \frac{\alpha_{\text{max}} - \alpha_0}{\alpha_0 - \alpha_{\text{min}}} = \frac{\Delta\alpha_{\text{right}}}{\Delta\alpha_{\text{left}}}
\]  

(9)

If \( r = 1 \) then the spectrum is symmetric, if \( r > 1 \) the spectrum is right skewed and if \( r < 1 \) then it is left skewed. If \( r \ll 1 \), the multifractal spectrum is sharply skewed towards the left. If \( r \gg 1 \), the multifractal spectrum is sharply skewed towards the right.
Figure 2. We show the different possible values of the symmetry parameter. If $r = 1$ then the spectrum is symmetric, if $r > 1$ the spectrum is right skewed and if $r < 1$ it is left skewed. If $r << 1$, the multifractal spectrum is more skewed towards the left. If $r >> 1$, the multifractal spectrum is more skewed towards the right.

The change in shape of the curve the multifractal spectrum around the maximum give us information about it. For this reason, we calculated the curvature around the maximum of the multifractal spectrum. The curvature of a curve is often denoted by a single letter $K$ and it can be calculated by

$$K = \frac{\left| \frac{d^2y}{dx^2} \right|}{\left[ 1 + \left( \frac{dy}{dx} \right)^2 \right]^{3/2}}$$

(10)

3. Results

In this work, the OFC model is used to generate a time series with level of conservation $\beta = 0.2$ of 10 million of data (Figure 3). We chose the magnitude as the base $n$ logarithm of the number of blocks that relaxed. In this case the matrix that represents the seismic fault has size $L \times L$ with $L = 100$ and we found the biggest number of blocks relaxed is 6561. We defined the magnitude, as the base logarithm 3 of 6561 and found that is 8 in this case. For this reason, we studied all earthquakes bigger or equal than 6561 relaxed blocks. Based on the earthquakes of magnitude 8 we built 160 windows before and after each of them with 1024 data and overlap of 896 points.

Figure 3. The catalogue of synthetic seismicity is a time series of $1 \times 10^6$ synthetic earthquakes. We show in this plot the magnitude as the number of blocks that were relaxed in each earthquake.
We calculated the multifractal spectra of all windows with the algorithm of Chhabra and Jensen. In Figure 4, the multifractal spectra of window number 160 before and window number 1 after the earthquake are shown.

We calculated the width $\Delta \alpha$ for each multifractal spectrum. Figure 5 shows the different values of $\Delta \alpha$ for the 160 windows before (blue) and 160 windows after (pink) of the earthquake. The black lines are the average of the $\Delta \alpha$ values. We could appreciate in the graphic that there is more complexity before the earthquake than after it.

The values of $\Delta \alpha_{\text{right}}$ (blue), $\Delta \alpha_{\text{left}}$ (pink) were measured to know the bias of the multifractal spectra. In Figure 6 we noticed that the multifractal spectra are skewed to the left. Nevertheless, the 160 windows before the earthquake are more asymmetrical than they after the earthquake. The blue lines are the averages of the $\Delta \alpha_{\text{right}}$ and $\Delta \alpha_{\text{left}}$ values.

We calculated $r$ for all the multifractal spectra to know where the multifractal spectra are skewed. Figure 7 shows all $r$ different values for 320 windows. It is appreciated that although all the multifractal spectra are skewed to the left, which is something we had already noticed, the 160 windows before the earthquake have less average values than the windows after it. This means the multifractal spectra before to the earthquake are so biased to the left that the after it. The black lines are the averages of the $r$ values.

Finally, we measured $K$ around the maximum of the all multifractal spectra. In Figure 8 we observed that the $K$ values are bigger before than after the earthquake. The black lines are the averages of the $K$ values.

![Figure 4](image_url)

**Figure 4.** Multifractal spectra of the data in the first window before (a) and the first window after (b) the earthquake.

![Figure 5](image_url)

**Figure 5.** We show the different values of $\Delta \alpha$ for the 160 windows before (blue) and 160 windows after (pink) of the earthquake. We notice a difference between the windows before that after of the earthquake. The black lines are the averages of the $\Delta \alpha$ values.
Figure 6. We show the values of $\Delta \alpha_{\text{right}}$ (blue points) and $\Delta \alpha_{\text{left}}$ (pink points). The multifractal spectra are skewed to left. The blue lines are the averages $\Delta \alpha$ values.

Figure 7. We show different $r$ values for 320 windows. The multifractals spectra before of the earthquake are more skewed to the left that the after it. The black lines are the averages of the $r$ values.

Figure 8. We show the $K$ values for 320 windows. The multifractals spectra before of the earthquake have $K$ values bigger than after it. The black lines are the averages of the $K$ values.
4. Conclusions
We worked with the OFC model to generate a time series of synthetic earthquakes with 10 million events for $\beta = 0.2$, because this model reproduces the Gutenberg-Richter graphs for the different values around $\beta$. In this series, we located earthquakes with biggest magnitude and based on this value we built 160 windows before and after each earthquake. All windows have 1024 data with an overlap of 896 points. We did multifractal analysis of all windows and calculated parameters $\Delta \alpha, \Delta \alpha_{\text{right}}, \Delta \alpha_{\text{left}}, r$ and $K$.

We noted that these parameters are good indicators to show what happens before and after earthquakes. This suggests that apparently exist a process of preparation of the earthquake and that the dynamics after the earthquake is very different. We found in this time series other big earthquake but we reported one case because the results are similar, but it is necessary to make a complete window of the interval between two or more earthquakes of the same magnitude to be completely conclusive.

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5. References
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