FAMILIES OF ARTINIAN AND LOW DIMENSIONAL DETERMINANTAL RINGS.

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Abstract. Let GradAlg(H) be the scheme parameterizing graded quotients of $R = k[x_0, \ldots, x_n]$ with Hilbert function $H$ (it is a subscheme of the Hilbert scheme of $\mathbb{P}^n$ with constant Hilbert function $H$ if we restrict to quotients of positive depth). A graded quotient $A = R/I$ of codimension $c$ is called standard determinantal if the ideal $I$ can be generated by the $t \times t$ minors of a homogeneous $t \times (t + c - 1)$ matrix $(f_{ij})$. Given integers $a_0 \leq a_1 \leq \ldots \leq a_{t+c-2}$ and $b_1 \leq \ldots \leq b_t$, we denote by $W_s(b; a) \subset \text{GradAlg}(H)$ the stratum of determinantal rings where $f_{ij} \in R$ are homogeneous of degrees $a_j - b_i$.

In this paper we extend previous results on the dimension and codimension of $W_s(b; a)$ in GradAlg(H) to artinian determinantal rings, and we show that GradAlg(H) is generically smooth along $W_s(b; a)$ under some assumptions. For zero and one dimensional determinantal schemes we generalize earlier results on these questions. As a consequence we get that the general element of a component $W$ of the Hilbert scheme of $\mathbb{P}^n$ is glicci provided $W$ contains a standard determinantal scheme satisfying some conditions. We also show how certain ghost terms disappear under deformation while other ghost terms remain and are present in the minimal resolution of a general element of GradAlg(H).

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1. Introduction

The main goal of this paper is to generalize previous results on maximal families of determinantal schemes, notable to cover maximal families of artinian determinantal $k$-algebras. Recall that a $k$-algebra $A \simeq R/I$, $R = k[x_0, \ldots, x_n]$, of codimension $c$ is called determinantal if the ideal $I$ can be generated by the $r \times r$ minors of a homogeneous $p \times q$ matrix $(f_{ij})$ with $c = (p-r+1)(q-r+1)$. $A$ is called standard (resp. good) determinantal.

Let GradAlg(H) be the “Hilbert scheme of constant Hilbert function”, i.e. the scheme parameterizing graded quotients $A$ of $R$ of depth $A \geq \min(1, \dim A)$ and with Hilbert function $H$. Given integers $a_0 \leq a_1 \leq \ldots \leq a_{t+c-2}$ and $b_1 \leq \ldots \leq b_t$, we denote by $W_s(b; a)$ (resp. $W_s(b; a)$) the stratum in GradAlg(H) consisting of good (resp. standard) determinantal $k$-algebras where $f_{ij}$ are homogeneous polynomials of degrees $a_j - b_i$. Then $W_s(b; a)$ is irreducible, $W_s(b; a) \neq \emptyset$ if $a_i - 1 > b_i$ for $1 \leq i \leq t$, and the closures $\overline{W_s(b; a)}$ and $\overline{W_s(b; a)}$ are equal if $n \geq c$ (Proposition 2.4).

In this paper we focus on the following problems.

1. Determine when $\overline{W_s(b; a)}$ is an irreducible component of GradAlg(H).
2. Find the codimension of $W_s(b; a)$ in GradAlg(H) if its closure is not a component.
(3) Determine when GradAlg($H$) is generically smooth along $W_s(b; a)$.

These questions have been considered in several papers, and in [28] we solve all these problems completely provided $n-c>1$ and $a_0>b_i$. In this case the closure of $W_s(b; a)$ is a generically smooth irreducible component of the usual Hilbert scheme Hilb($\mathbb{P}^n$), as well as of GradAlg($H$) (see Theorem 4.3, i.e. [31] Conjecture 4.2) holds, and $\dim W_s(b; a)$ is determined (equal to $\lambda$ in Theorem 4.11 whence [31] Conjecture 4.1) holds for $n-c>0$), see also [10] for a somewhat different approach to these problems. So we only need to consider the case $n+1-c\in\{0,1,2\}$. We concentrate on artinian determinantal $k$-algebras $(n+1-c=0)$ since we have not treated this case previously, and we prove the main Theorems 4.1 and 4.4 under conditions that allow $c$ to be $n+1$. In addition we prove a new result (Theorem 4.6, extending Theorem 4.4) which applies when $\dim W_s(b; a)\neq \lambda$. This theorem implies that the general element of an irreducible component $W$ of the Hilbert scheme of $\mathbb{P}^n$ is glicci (in the Gorenstein liaison class of a complete intersection) provided $W$ contains a standard determinantal scheme satisfying some conditions (Corollary 4.12). For an introduction to glicciness, see [29], and see [8] and its references for further developments. Finally, in Sec. 6, we generalize and complete several results of [26, 30] for families of determinantal schemes of dimension 0 or 1, some of which occurred in [29, 30], and we slightly extend a result of [31].

We have used two different strategies to attach the problems (1) to (3). Indeed in [29, 30, 26, 31] we successively deleted columns of the $t\times (t+c-1)$ matrix $A$ associated to a determinantal scheme $X:=\text{Proj}(A)$ to get a nest ("flag") of closed subschemes $X=X_c\subset X_{c-1}\subset \ldots \subset X_2\subset \mathbb{P}^n$ and we proved our results inductively by considering the smoothness of the Hilbert flag scheme of pairs and its natural projections to Hilbert schemes. On the other hand, in [28] we compared deformations of $A$ with deformation of the cokernel $M$ of the map $\bigoplus_{j=0}^{t+c-2}R(-a_j)\rightarrow \bigoplus_{i=1}^{t}R(-b_i)$ induced by $A$. This latter approach turned out to be quite successful, and it indeed solved problem (1)-(3) for $n-c>1$. It is this approach that we generalize to the artinian case, only introducing an extra assumption ($\partial\text{Hom}_A(M,M)\simeq k$) in the theorems. In fact we show that the main results of [28] hold, whence partially solving problems (1) to (3) also for $n=c-1$.

For $c=2$ all assumptions of the theorems are fulfilled. We even replace $R$ by any Cohen-Macaulay quotient $\overline{R}$ of $R$ and solve the problems above for determinantal quotients of codimension $c=2$ of $\overline{R}$ (Theorem 4.13). For $c>2$ we need to verify that certain Ext$^j$-groups vanish to apply our results when $n+1-c\in\{0,1,2\}$. In this paper we accomplish this by using Macaulay 2 ([12]). We give many examples, supported by Macaulay 2 computations (over finite fields), to illustrate the theorems in the artinian case. In Sec. 5 we consider ghost terms, i.e. common free summands in consecutive terms of the minimal $R$-free resolution of $A$, and we show that some of them disappear, others remain unchanged under suitable generizations (deformations to a more general algebra) in GradAlg($H$). If $A$ is general in GradAlg($H$) and the assumptions of Theorem 4.6 are fulfilled (e.g. $\dim A\geq 3$ and $a_0\geq b_i$), one may easily describe all ghost terms in its minimal free resolution while it seems hard to get improved results when $\dim A\leq 2$ (Proposition 5.1 Remark 5.2).

In the proofs we use the exactness of the Buchsbaum-Rim complex ([4, 7]) and the 5-term exact sequence associated to the spectral sequence $E^{p,q}_2:=\text{Ext}^{p+q}_A(Tor^R_q(A,M),M)\Rightarrow$
examples of such families. Indeed this study of families of artinian determinantal rings started some time ago during a visit by A. Iarrobino and M. Boij at Oslo University College where we concretely studied

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Notation: Throughout \( \mathbb{P} := \mathbb{P}^n \) is the projective \( n \)-space over an algebraically closed field \( k \), \( R = k[x_0, x_1, \ldots, x_n] \) is a polynomial ring and \( \mathfrak{m} = (x_0, \ldots, x_n) \). If \( X \subset Y \) are closed subschemes of \( \mathbb{P}^n \), we denote by \( \mathcal{I}_{X/Y} \) (resp. \( \mathcal{N}_{X/Y} \)) the ideal (resp. normal) sheaf of \( X \) in \( Y \), and we omit \( /Y \) if \( Y = \mathbb{P}^n \). Let \( I_X = H^0(\mathcal{I}_X) \subset R \) be the saturated homogeneous ideal of \( X \subset \mathbb{P}^n \). When we write \( X = \text{Proj}(A) \) we take \( A := R/I_X \) and \( K_A = \text{Ext}^c_{R}(A, R)(-n - 1) \) for the canonical module of \( A \) or \( X \) where \( c = \text{codim}_X X := n - \dim X \). Note that by the codimension, \( \text{codim}_X X \), of an irreducible \( X \) in a not necessarily equidimensional scheme \( Y \) we mean \( \dim O_{Y,x} - \dim X \), where \( x \) is a general \( k \)-point of \( X \). Moreover we denote by \( \text{hom}(\mathcal{F}, \mathcal{G}) = \dim_k \text{Hom}(\mathcal{F}, \mathcal{G}) \) the dimension of the group of morphisms between coherent \( O_X \)-modules and we use small letters for the \( k \)-vector space dimension of similar groups.

2. Preliminaries

2.1. Hilbert schemes and Hilbert function strata. We denote the Hilbert scheme with the Hilbert polynomial \( p \in \mathbb{Q}[s] \) by \( \text{Hilb}^p(\mathbb{P}^n) \), cf. \cite{13} for existence and \cite{36} for the local theory. Similarly \( \text{GradAlg}^H(R) \), or \( \text{Hilb}^H(\mathbb{P}^n) \) when \( \text{dim} A > 0 \), is the representing object of the functor that parameterizes flat families of graded quotients \( A \) of \( R \) of depth \( \text{depth}_m A \geq \text{min}(1, \text{dim} A) \) and with Hilbert function \( H \), \( H(i) = \text{dim} A_i \) (\cite{23, 24, 14}). We allow calling it “the postulation Hilbert scheme” (\cite{25}, \S 1.1) even though it may be different from the parameter space studied by Gotzmann and Iarrobino (\cite{11, 18}) who study the “same” scheme with the reduced scheme structure. We let \( (A) \), or \( (X) \) where \( X = \text{Proj}(A) \), denote the point of \( \text{GradAlg}(H) := \text{GradAlg}^H(R) \) that corresponds to \( A \). Note that if \( \text{depth}_m A \geq 1 \) and \( 0\text{Hom}_R(I_X, H^1_{(A)}(m)) = 0 \), then

\[
\text{GradAlg}(H) \simeq \text{Hilb}^p(\mathbb{P}^n) \text{ at } (X),
\]

and hence we have an isomorphism \( 0\text{Hom}(I_X, A) \simeq H^0(\mathcal{N}_X) \) of their tangent spaces (cf. \cite{0} for the case \( \text{depth}_m A \geq 2 \), and (9) of \cite{24} for the general case). If (\ref{2.1}) holds and \( X \) is generically a complete intersection, then \( 0\text{Ext}_A^1(I_X/I^2_X, A) \) is an obstruction space of \( \text{GradAlg}(H) \) and hence of \( \text{Hilb}^p(\mathbb{P}^n) \) at \( (X) \) \cite{24}, \S 1.1]. By definition \( X \) (resp. \( A \)) is called unobstructed if \( \text{Hilb}^p(\mathbb{P}^n) \) (resp. \( \text{GradAlg}(H) \)) is smooth at \( (X) \) (resp. \( (A) \)).

We say that \( X \) is general in some irreducible subset \( W \subset \text{Hilb}^p(\mathbb{P}^n) \) if \( (X) \) belongs to a sufficiently small open subset \( U \) of \( W \) such that any \( (X) \) in \( U \) has all the openness properties that we want to require. We define ”\( A \) general in \( \text{GradAlg}(H) \)” similarly.

2.2. Determinantal rings and schemes. We mainly maintain the notions and notations from \cite{26, 28}, but we need to extend some results to artinian determinantal
For a more general background of determinantal rings and schemes, see [2,7,11].

Indeed let

$$\varphi : F = \bigoplus_{i=1}^{t} R(b_i) \rightarrow G := \bigoplus_{j=0}^{t+c-2} R(a_j)$$

be a graded morphism of free $R$-modules, $A = (f_{ij})_{i=1}^{t}, j=0,...,t+c-2$, deg $f_{ij} = a_j - b_i$, a matrix which represents the dual $\varphi^* := \text{Hom}_R(\varphi, R)$ and let $I(A) := I_t(A)$ be the ideal of $R$ generated by the maximal minors of $A$. We always suppose

$$c \geq 2, \quad t \geq 2, \quad b_1 \leq \ldots \leq b_t \quad \text{and} \quad a_0 \leq a_1 \leq \ldots \leq a_{t+c-2}.$$  

A codimension $c$ quotient $A = R/I$ (resp. subscheme $X \subset \mathbb{P}^n$) is called standard determinantal if $I := I(A)$ (resp. $I_X := I(\mathcal{A})$) for some homogeneous $t \times (t+c-1)$ matrix $\mathcal{A}$ as above. They are good determinantal if additionally, the ideal $I_{t-1}(\mathcal{A})$ of submaximal minors has at least codimension $c+1$ in $R$, i.e. $R/I(\mathcal{A})$ is a generic complete intersection of $R$.  

Given integers $b_i$ and $a_j$ satisfying (2.3) we let $W_s(b; a)$ (resp. $W(b; a)$) be the stratum in GradAlg($H$) consisting of standard (resp. good) determinantal quotients, cf. (2.1). Note that we do not require $\mathcal{A}$ to be minimal (i.e. $f_{ij} = 0$ when $b_i = a_j$) for $R/I_t(\mathcal{A})$ to belong to $W_s(b; a)$. In examples, however, we usually consider determinantal rings with positive degree matrix (i.e. for every $i$, $j$, $b_i < a_j$, and $a_j - b_i = 1$ is called the linear case).

**Proposition 2.1.** The closure of $W_s(b; a)$ in GradAlg($H$) is irreducible, and

$$W_s(b; a) \neq \emptyset \quad \Leftrightarrow \quad a_{i-1} \geq b_i \quad \text{for all} \quad i \quad \text{and} \quad a_{i-1} > b_i \quad \text{for some} \quad i.$$  

Moreover if $n-c \geq 0$, then $W_s(b; a) = W(b; a)$. And if $n-c = -1$, then every $A = R/I$ of $W(b; a)$ is a complete intersection (c.i.) of $R$.

**Proof.** For the non-empty statement we refer to (2.2) of [31] whose arguments carry over to the artinian case. The text after [31, (2.2)] shows also that $W_s(b; a) = W(b; a)$ and that this locus is irreducible in the non-artinian case ($n \geq c$). To see that $W_s(b; a)$ is irreducible also when $W_s(b; a)$ parametrizes artinian $k$-algebras we consider the affine scheme $\mathbb{V} = \text{Hom}_R(G^*, F^*)$ whose rational points correspond to $t \times (t+c-1)$ matrices. Since the vanishing of Ext$^1_R(R/I_t(\mathcal{A}), R)$ is an open property, the subset $U$ of $\mathbb{V}$ of matrices such that $I_t(\mathcal{A})$ has maximal codimension in $R$ is open and irreducible. Then, since there is an obvious morphism from $U$ onto $W_s(b; a)$, it follows that $W_s(b; a)$ is irreducible.

In the following let $A = R/I_t(\mathcal{A})$ be standard determinantal, i.e. $(A) \in W_s(b; a)$ and let $X = \text{Proj}(A)$ if dim $A > 0$. Then $R$-free resolutions of $A$ and $M := M_\mathcal{A} := \text{coker} \varphi^*$ are given by the Eagon-Northcott and Buchsbaum-Rim complexes respectively, see [4, 6, 7]. These resolutions are minimal if $\mathcal{A}$ is minimal. For the latter we have

$$0 \rightarrow \wedge^{t+c-1}G^* \otimes S_{c-2}(F) \otimes \wedge^t F \rightarrow \cdots \rightarrow \wedge^{t+i+1}G^* \otimes S_i(F) \otimes \wedge^t F$$

$$\rightarrow \cdots \rightarrow \wedge^{t+1}G^* \otimes S_0(F) \otimes \wedge^t F \rightarrow G^* \xrightarrow{\varphi^*} F^* \rightarrow M \rightarrow 0.$$

(2.4)
The matrix $\mathcal{A}_i$ obtained by deleting the last $c-i$ columns defines a morphism

\[(2.5) \quad \varphi_i : F = \bigoplus_{i=1}^t R(b_i) \longrightarrow G_i := \bigoplus_{j=0}^{t+i-2} R(a_j)\]

Then $\varphi_2$ and hence all $\varphi_i$ are injective. Letting $B_i = \text{coker} \, \varphi_i$, $M_i = \text{coker} \, \varphi_i^* := \text{Hom}_R(\varphi_i, R)$ and $(-)^* = \text{Hom}_R(-, R)$, we have an exact sequence

\[(2.6) \quad 0 \rightarrow B_i^* \rightarrow G_i^* \xrightarrow{\varphi_i^*} F^* \rightarrow M_i \cong \text{Ext}^1_R(B_i, R) \rightarrow 0.\]

If $D_i \cong R/I_{D_i}$ is the $k$-algebra given by the maximal minors of $\mathcal{A}_i$ and $X_i = \text{Proj}(D_i)$ (so $R \rightarrow D_2 \rightarrow D_3 \rightarrow \ldots \rightarrow D_c = A$), then $M_i$ is a $D_i$-module and there is an exact sequence

\[(2.7) \quad 0 \rightarrow D_i \rightarrow M_i(a_{t+i-1}) \rightarrow M_{i+1}(a_{t+i-1}) \rightarrow 0\]

in which $D_i \rightarrow M_i(a_{t+i-1})$ is a regular section that defines $D_{i+1}$ (also when $i + 1 = c$ and $D_c$ is artinian [33, Lem. 3.6]). Hence

\[(2.8) \quad 0 \rightarrow I_{D_{i+1}/D_i} \cong \text{Hom}_{D_i}(M_i(a_{t+i-1}), D_i) \rightarrow D_i \rightarrow D_{i+1} \rightarrow 0.\]

By (2.4) $M_i$ is a maximal Cohen-Macaulay $D_i$-module and so is $I_{D_{i+1}/D_i}$ by (2.8). Moreover by e.g. (2.4), $K_{D_i}(n+1) \cong S_{i-1} M_i(\ell_i)$ where $\ell_i := \sum_{j=0}^{t+i-2} a_j - \sum_{q=1}^{t} b_q$.

**Lemma 2.2.** With the above notation (and $M = M_c$), there are exact sequences

\[0 \rightarrow \text{Hom}_R(M_i, M) \rightarrow F \otimes_R M \rightarrow G_i \otimes_R M \rightarrow B_i \otimes_R M \rightarrow 0, \quad \text{and} \]

\[0 \rightarrow \text{Hom}_R(B_i, F) \rightarrow \text{Hom}_R(B_i, G_i) \rightarrow \text{Hom}_R(B_i, B_i) \rightarrow \text{Hom}_R(M_i, M_i) \rightarrow 0.\]

**Proof.** (cf. [30, Lem. 3.1 and 3.10]). We apply $\text{Hom}_R(B_i, -)$ to

\[(2.9) \quad 0 \rightarrow F \rightarrow G_i \rightarrow B_i \rightarrow 0\]

and using $M_i \cong \text{Ext}^1_R(B_i, R)$ we deduce the exact sequence

\[0 \rightarrow \text{Hom}(B_i, F) \rightarrow \text{Hom}(B_i, G_i) \rightarrow \text{Hom}(B_i, B_i) \rightarrow F \otimes_R M_i \rightarrow G_i \otimes_R M_i.\]

Hence we get the lemma by applying $\text{Hom}(-, M_i)$ to (2.6) and $(-) \otimes_R M$ to (2.9). \qed

**Proposition 2.3.** Set $K_i := \ominus \text{hom}(B_{i-1}, R(a_{t+i-2}))$ and $\ell_i := \sum_{j=0}^{t+i-2} a_j - \sum_{q=1}^{t} b_q$. Letting $h_i := 2a_{t+i-2} - \ell_i + n$ for $3 \leq i \leq c$ and $\binom{n}{a} = 0$ if $a < n$ we have

\[K_{i+3} = \sum_{r+s+t \geq 0} \sum_{\begin{smallmatrix}0 \leq i, j, k \leq t+i \cr r, s \geq 0 \cr 1 \leq j_1 \leq \ldots \leq j_s \leq t\end{smallmatrix}} (-1)^i r \binom{h_i + a_{i+1} + \ldots + a_{i+r} + b_{j_1} + \ldots + b_{j_s}}{n}\]

for $0 \leq i \leq c-3$.

E.g. $K_3 = \binom{h_0}{n}$ and $K_4 = \sum_{j=0}^{t+1} \binom{h_1 + a_j}{n} - \sum_{i=1}^{t} \binom{h_i + b_i}{n}$. Moreover if $\ominus \text{hom}(M, M) = 1$, then

\[\text{aut}(B_c) := \ominus \text{hom}(B_c, B_c) = 1 + K_3 + K_4 + \ldots + K_c.\]

**Proof.** We include the main ingredients of a proof. Indeed combining (2.4) and (2.6) we get a minimal resolution of $B_i^*$, whence $K_i$ coincides with the sum of binomials appearing in the Proposition by the definition of $K_i$ (see [30, p.2882] for details).

Now dualizing the exact sequence $0 \rightarrow R(a_{t+c-2}) \rightarrow B_c \rightarrow B_{c-1} \rightarrow 0$, we get

\[0 \rightarrow \text{Hom}(B_{c-1}, R) \rightarrow \text{Hom}_R(B_c, R) \rightarrow R(-a_{t+c-2}) \rightarrow M_{c-1} \rightarrow M_c \rightarrow 0\]
by (2.6). Combining with (2.7) we get the exact sequence
\[ 0 \to \text{Hom}(B_{c-1}, R) \to \text{Hom}_R(B_c, R) \to I_{D_{c-1}}(-a_{t+c-2}) \to 0 \]
which implies the vanishing of the lower downarrows in the following commutative diagram
\[
\begin{array}{ccc}
0\text{Hom}(B_{c-1}, F) & \longrightarrow & 0\text{Hom}(B_{c-1}, G_c) \\
\downarrow & & \downarrow \\
0\text{Hom}(B_c, F) & \longrightarrow & 0\text{Hom}(B_c, G_c) \\
\downarrow & & \downarrow \\
0\text{Hom}(R(a_{t+c-2}), F) & \longrightarrow & 0\text{Hom}(R(a_{t+c-2}), G_c) .
\end{array}
\]
It follows that the upper downarrows are bijections. Since \( \text{Hom}(B_{c-1}, -) \) is exact on \( 0 \to R(a_{t+c-2}) \to G_c \to G_{c-1} \to 0 \) (because \( \text{Ext}^1(B_{c-1}, -) \cong M_{c-1} \otimes (-) \)) we get
\[ 0\text{hom}(B_c, G_c) - 0\text{hom}(B_c, F) = 0\text{hom}(B_{c-1}, G_{c-1}) - 0\text{hom}(B_{c-1}, F) + K_c . \]
By Lemma 2.2 and the assumption \( 0\text{hom}(M_c, M_c) = 1 \),
\[ \text{aut}(B_c) = 1 + 0\text{hom}(B_c, G_c) - 0\text{hom}(B_c, F) , \]
and since we may suppose a corresponding expression for \( \text{aut}(B_{c-1}) \) we have proved
\[ \text{aut}(B_c) = K_c + \text{aut}(B_{c-1}) . \]
Now, we conclude by induction using that \( \text{aut}(B_2) = 0\text{hom}(I_{D_2}, I_{D_2}) = 1 \). \( \square \)

Finally recall that if \( J = I_{t-1}(A) \) and \( \dim A > 0 \) then \( X = \text{Proj}(A) \) is a local complete intersection (l.c.i.) in \( X - V(J) \). In the following we always take \( Z \supset V(J) \) and \( U = X - Z \), i.e. so that \( X \hookrightarrow \mathbb{P}^n \) is an l.c.i. in \( U \). Since the 1st Fitting ideal of \( M \) is equal to \( J \), we get that \( \tilde{M} \) and \( I/I^2 \) are locally free on \( X - V(J) \), cf. [37, Lem. 1.8] and [1, Lem. 1.4.8]. Note that depth, \( A \geq \text{codim}_X \text{Sing}(X) \) which we use in the following.

Remark 2.4. Let \( X = X_c, Y = X_{c-1} \) and let \( \alpha \) be a positive integer. If \( X \) is general in \( W(\mathbb{k}; a) \neq \emptyset \) and \( a_{i-\min(0, t)} - b_i \geq 0 \) for \( \min(\alpha, t) \leq i \leq t \), then
\[ (2.10) \quad \text{codim}_X \text{Sing}(X_j) \geq \text{min}\{2\alpha - 1, j + 2\} \quad \text{for} \quad 2 \leq j \leq c , \]
\[ \text{cf. the argument of [30, Rem. 2.7], which uses [5].} \]
In particular if we let \( \alpha = 3 \), we get that \( X \hookrightarrow \mathbb{P}^n \) (resp. \( Y \hookrightarrow \mathbb{P}^n \) and \( X \hookrightarrow Y \)) are l.c.i.’s outside a subset \( Z \subset X \) of codimension at least \( \min(5, c + 2) \) (resp. \( \text{min}(4, c) \)) in \( X \). Indeed we may take \( Z = V(I_{t-1}(A_{c-1})) \) for the statement involving \( X \) and \( Y \) because (2.8) imply that \( I_{X/Y} \) is locally free on \( Y - Z \), noting that \( Z \subset V(I_t(A)) = X \). Moreover if \( a_{i-2} - b_i \geq 0 \) for \( 2 \leq i \leq t \), then \( X \hookrightarrow \mathbb{P}^n \) (resp. \( Y \hookrightarrow \mathbb{P}^n \) and \( X \hookrightarrow Y \)) are l.c.i.’s outside a subset \( Z \subset X \) of codimension at least \( \min(3, c + 2) \) (resp. \( \text{min}(2, c) \)). Notice that we interpret \( I(Z) \) as \( m \) if \( Z = \emptyset \).

2.3. The dimension of the determinantal locus. In [28] we proved that the dimension of a non-empty \( W(\mathbb{k}; a) \) in the case \( a_{i-2} - b_i \geq 0 \) for \( i \geq 2 \) and \( n - c \geq 1 \) is given by
\[ (2.11) \quad \dim W_s(\mathbb{k}; a) = \dim W(\mathbb{k}; a) = \lambda_c + K_3 + K_4 + \ldots + K_c , \]
where \( K_i \) is previously defined and \( \lambda_c \) is defined by
\[ (2.12) \quad \lambda_c := \sum_{i,j} \left( \frac{a_j - b_i + n}{n} \right) + \sum_{i,j} \left( \frac{b_i - a_j + n}{n} \right) - \sum_{i,j} \left( \frac{a_i - a_j + n}{n} \right) - \sum_{i,j} \left( \frac{b_i - b_j + n}{n} \right) + 1 . \]
where the indices belonging to \( a_j \) (resp. \( b_i \)) range over \( 0 \leq j \leq t + c - 2 \) (resp. \( 1 \leq i \leq t \)). Using the Hilbert function, \( H_M(\cdot) \), of \( M \), we may alternatively write (2.11) as

\[
(2.13) \quad \dim W_s(\mathbf{b}; \mathbf{a}) = \dim W(\mathbf{b}; \mathbf{a}) = \sum_{j=0}^{t+c-2} H_M(a_j) - \sum_{i=1}^t H_M(b_i) + 1.
\]

For zero-dimensional determinantal schemes \((n - c = 0)\) we have the following.

**Remark 2.5.** (i) Assume \( a_0 > b_t \). Then (2.11) hold provided \( 3 \leq c \leq 5 \) (resp. \( c > 5 \)) and \( a_{t+c-2} > a_{t-2} \) (resp. \( a_{t+3} > a_{t-2} \)) by [31] Thm. 3.2, see Sec. 5 for a generalization.

(ii) If \( \mathcal{A} \) is a linear \( 2 \times (c + 1) \) matrix, we showed in [26] Ex. 3.3 that \( \dim W(\mathbf{b}; \mathbf{a}) \) is strictly smaller that the right-hand side of (2.11) for every \( c > 2 \). To our knowledge this is the only known case where the equality of (2.11) fails to hold when \( n = c \).

2.4. The determinantal locus as components of Hilbert schemes. It was shown in [28] that \( \overline{W}(\mathbf{b}; \mathbf{a}) \), for \( W(\mathbf{b}; \mathbf{a}) \neq \emptyset \), is an irreducible component of \( \text{Hilb}^p(\mathbb{P}^n) \) provided \( n - c \geq 2 \) and \( a_i - \min(3, t) - b_i \geq 0 \) for \( \min(3, t) \leq i \leq t \). If \( n - c \leq 1 \) then \( \overline{W}(\mathbf{b}; \mathbf{a}) \) may fail to be an irreducible component. Indeed \( \overline{W}(0, 0; 1, 1, ..., 1, 2) \) is not an irreducible component of \( \text{GradAlg}(H) \) for every \( c \geq 3 \) both when \( n = c \) and \( n = c + 1 \), see [26] Ex. 4.1. In both cases the \( h \)-vector of a general \( A \) is \((1, c, c)\), e.g. \((\dim A_i)_{i=0}^\infty = (1, c + 1, 2c + 1, 2c + 1, ...)\) if \( n = c \).

The reason for \( \overline{W}(\mathbf{b}; \mathbf{a}) \) to be not a component is that there exists determinantal rings allowing “deformations that do not come from deforming the matrix \( \mathcal{A} \)”. Let us recall the definition of this notion. Here we briefly say “\( T \) a local ring” (resp. “\( T \) artinian”) for a local \( k \)-algebra \((T, m_T)\) essentially of finite type over \( k \) (resp. of finite type satisfying \( m_T^r = 0 \) for some \( r \in \mathbb{Z} \)) such that \( k = T/m_T \). The local deformation functor \( \text{Def}_{A/R} \) is defined for each artinian \( T \) as the set of graded (\( T \)-flat) deformations \( A_T \) of \( A \) to \( T \) (i.e. \( A_T \otimes_T k = A \)). Moreover we say “\( T \to S \) is a small artinian surjection” provided there is a morphism \((T, m_T) \to (S, m_S)\) of local artinian \( k \)-algebras whose kernel \( a \) satisfies \( a \cdot m_T = 0 \).

If \( T \) is a local ring, we denote by \( A_T = (f_{ij}:T) \) a matrix of homogeneous polynomials belonging to the graded polynomial algebra \( R_T := R \otimes_k T \), satisfying \( f_{ij}:T \otimes_T k = f_{ij} \) and \( \deg f_{ij}:T = a_j - b_i \) for all \( i, j \). Note that all elements from \( T \) are considered to be of degree zero. Once having such a matrix \( A_T \), it induces a morphism

\[
(2.14) \quad \varphi_T : F_T := \bigoplus_{i=1}^t R_T(b_i) \to G_T := \bigoplus_{j=0}^{t+c-2} R_T(a_j).
\]

**Lemma 2.6.** If \( A = R/I(\mathcal{A}) \) is standard determinantal, then \( A_T := R_T/I_T(A_T) \) and \( M_T := \text{coker } \varphi_T^* \) are graded deformations of \( A \) and \( M \) respectively for every choice of \( A_T \) as above. Moreover every graded deformation of \( M \) is of the form \( M_T \) for some \( A_T \).

This mainly follows from functoriality and the fact that all maps in the Eagon-Northcott and Buchsbaum-Rim complexes are defined in terms of \( \mathcal{A} \), cf. [26] for details. Note that the final statement is clear because \( M = \text{coker } \varphi^* \). This really means that “every graded deformation of \( M \) to an artinian \( T \) comes from deforming \( \mathcal{A} \)”. This may not hold for \( A \).

**Definition 2.7.** Let \( A = R/I(\mathcal{A}) \). We say “every deformation of \( A \) (or \( X \) if \( \dim A > 1 \), see (2.11)) comes from deforming \( \mathcal{A} \)” if for every local ring \( T \) and every graded deformation \( R_T \to A_T \) of \( R \to A \) to \( T \), then \( A_T \) is of the form \( A_T = R_T/I_T(A_T) \) for some \( A_T \) as above.
Lemma 2.8. Let $A = R/I_t(A)$ be a standard determinantal ring, $(A) \in W_s(b;a)$. If every deformation of $A$ comes from deforming $A$, then $A$ are unobstructed (i.e. $\text{Def}_{A/R}$ are smooth). Moreover $W_s(b;a)$ is an irreducible component of $\text{GradAlg}(H)$.

Proof. See [26, Lem. 4.4], only replacing $\text{Hilb}(\mathbb{P}^n)$ by $\text{GradAlg}(H)$ in its proof. \hfill $\square$

Remark 2.9. By these lemmas we get $T$-flat determinantal schemes by just parameterizing the polynomials of $A$ over a local ring $T$, see Rem. 4.5 of [26] and Laksov’s papers [33, 34] for somewhat similar results for more general determinantal schemes.

3. DEFORMATIONS OF MODULES AND DETERMINANTAL SCHEMES

The main goal of this section is to generalize to $A$ artinian the close relationship between the local deformation functor, $\text{Def}_{M/R}$, of the graded $R$-module $M = \text{coker } \varphi^*$ and the corresponding local functor, $\text{Def}_{A/R}$, of deforming the standard determinantal ring $A = R/I_t(A)$ as a graded $k$-algebra. Note that $I := I_t(A) = \text{ann}(M)$ by [3]. In [28] these functors were shown to be isomorphic (resp. the first a natural subfunctor of the latter) provided $\dim X \geq 2$ (resp. $\dim X = 1$) and $X = \text{Proj}(A)$ general and good determinantal. The comparison between these deformation functors relied on understanding the spectral sequence $E_2^{p,q} := \text{Ext}_A^p(\text{Tor}_R^q(A, M), M) \Rightarrow \text{Ext}_{R/I_t(A)}^{p+q}(M, M)$ and its induced 5-term exact sequence

\begin{equation}
0 \to \text{Ext}_A^1(M, M) \to \text{Ext}_R^1(M, M) \to \delta \to E_2^{0,1} \to \text{Ext}_A^2(M, M) \to \text{Ext}_R^2(M, M) \to .
\end{equation}

Indeed $\text{Tor}_R^q(A, M) \simeq I_X \otimes M$ implies $E_2^{0,1} \simeq \text{Hom}_R(I_X, \text{Hom}_R(M, M))$ in general, and since $\text{Hom}_A(M, M) \simeq A$ provided $A$ is good determinantal by [30, Lem. 3.2], it follows that the morphism $\delta$ of (3.1) induces a natural map

\begin{equation}
0\text{Ext}_R^1(M, M) \longrightarrow (E_2^{0,1})_0 \simeq 0\text{Hom}_R(I_X, A)
\end{equation}

between the tangent spaces of $\text{Def}_{M/R}$ and $\text{Def}_{A/R}$. Indeed recall that there is a natural map $e_M(T) : \text{Def}_{M/R}(T) \to \text{Def}_{A/R}(T)$, $T$ artinian, obtained by taking a matrix $A_T$ whose corresponding morphism has $M_T$ as cokernel and letting $A_T := R_T/I_t(A_T)$ (see Lemma 2.6). Since matrices inducing the same $M_T$ also define the same ideal of maximal minors (Fitting’s lemma, [7], Cor. 20.4), this morphism is well-defined.

The main difficulty in generalizing the comparison above to $A$ artinian is that $A \to \text{Hom}_A(M, M)$ is no longer an isomorphism (for $A$ not a c.i.) which implies that the tangent map of $\text{Def}_{M/R} \to \text{Def}_{A/R}$ may not be the map in (3.2), i.e. $(E_2^{0,1})_0 \simeq 0\text{Hom}_R(I_X, A)$ may fail. The morphism $\text{Def}_{M/R} \to \text{Def}_{A/R}$ is, however, well-defined also in the artinian case, and since $A \to \text{Hom}_A(M, M)$ is injective and the functors are pro-representable under the assumption $0\text{Hom}_A(M, M) \simeq k$ we are able to generalize the comparison.

Definition 3.1. Let $A = R/I_t(A)$ be a standard determinantal ring and let $\ell$ be the category of artinian $k$-algebras (cf. the text before (2.14)). Then the local deformation functor $\text{Def}_{A \in W_s(b;a)}$, defined on $\ell$, is the subfunctor of $\text{Def}_{A/R}$ given by:

$$\text{Def}_{A \in W_s(b;a)}(T) = \{A_T \in \text{Def}_{A/R}(T) | A_T = R_T/I_t(A_T) \text{ for some matrix } A_T \text{ lifting } A \text{ to } T\}.$$  

With this definition it is obvious that the natural map $e_M : \text{Def}_{M/R} \to \text{Def}_{A/R}$ defined above for any $T \in \text{ob}(\ell)$ factors via $\text{Def}_{A \in W_s(b;a)} \hookrightarrow \text{Def}_{A/R}$. Moreover we have
Lemma 3.2. If $D := k[ε]/(ε^2)$ then the degree zero part, $δ_0(M)$, of the connecting homomorphism $δ$ of (3.1) factors through $e_M(D) : \text{Def}_{M/R}(D) → \text{Def}_{A/R}(D)$, i.e. $δ_0(M)$ is the composition

$$0\text{Ext}_R^1(M, M) \xrightarrow{e_M(D)} 0\text{Hom}_R(I, A) \xrightarrow{i_{A,M}} 0\text{Hom}_R(I, \text{Hom}_A(M, M)) \simeq E^{0,1}_2$$

where the map in the middle is induced by $A \hookrightarrow \text{Hom}_A(M, M)$, $1 \in A \mapsto id_M$.

Proof. We know that $e_M(D) : 0\text{Ext}_R^1(M, M) → 0\text{Hom}_R(I, A)$ is well-defined. To describe it, take any $η ∈ 0\text{Ext}_R^1(M, M)$ and let $η' ∈ 0\text{Hom}(G^*, M)$ represent $η$ and $η ∈ 0\text{Hom}(G^*, F^*)$ map to $η'$, cf. (3.3). A maximal minor $f_i$ corresponds to choosing $t$ columns of $A$ which we for simplicity suppose is obtained by deleting the last $c−1$ columns of $A$. With notations as in Sec. 2, $f_i$ is the determinant of $G_1^* = \bigoplus_{j=0}^{c-1} R(-a_j) → F^* = \bigoplus_{j=0}^{c-1} R(b_j)$. By definition $e_M(D)(η)$ is given by the ideal generated by maximal minors of $φ^* + εη$. Let $I_1$ be the principal ideal generated by $f_i$. Since it is easy to compute $\det(φ^* + εη_1)$ where $η_1$ is the composition, $G_1^* ← G^* \xrightarrow{η} F^*$, of the natural inclusion $G_1^* ← G^*$ by $η$, we get that the image of $e_M(D)(η)$ in $0\text{Hom}_R(I_1, A)$ is $A(δ_{f_1})$ via the natural map $I_1 → I$ is $tr(φ_1^* \cdot η_1) ⊗ R 1$ where $1 ∈ A$ is the unity, $φ_1^*$ is the adjoint of $φ_1^*$ and $tr$ the trace map.

Let $M_1 = \text{coker } φ_1^*$ and let $π : M_1 → M$ be the canonical surjection. To see that $δ_0(M) = i_{A,M} \cdot e_M(D)$ it suffices to check that the image of $δ_0(M)(η)$ via the composition

$$0\text{Hom}_R(I, \text{Hom}(M, M)) → 0\text{Hom}_R(I, \text{Hom}(M, M)) → 0\text{Hom}_R(I, \text{Hom}(M, M))$$

of three connecting homomorphisms. If $A_1 = R/I_1$ then $δ_0(M_1)(η_1) = tr(φ_1^* \cdot η_1) ⊗ R id_{M_1}$ by (21) Prop. 2], whence maps to $tr(φ_1^* \cdot η_1) ⊗ R π$ via the right uparrow. Here $η_1 ∈ 0\text{Ext}_M^1(M_1, M_1)$ is represented by the composition $G_1^* \xrightarrow{η_1} F^* → M_1$. Since the elements $η ∈ 0\text{Ext}_R^1(M, M)$ and $η_1 ∈ 0\text{Ext}_M^1(M_1, M_1)$ map to the same element in $0\text{Ext}_R^1(M_1, M)$ (the one represented by the composition $G_1^* \xrightarrow{η} F^* → M$) the proof is complete.

First we will compute the dimension of the pro-representing object of $\text{Def}_{M/R}$, i.e. we need to generalize [28] Thm. 3.2] by weakening its conditions so that it applies to an artinian $A$ and we sketch a proof.

Theorem 3.3. Let $A = R/I_1(A)$ be standard determinantal and let $M = \text{coker } φ^*$. Then $M$ is unobstructed, i.e. $\text{Def}_{M/R}$ is formally smooth. Moreover if $\text{Hom}_A(M, M) ≅ k$, then
Def$_{M/R}$ is pro-representable, and the pro-representing object $H(M/R)$ of Def$_{M/R}$ satisfies
\[ \dim H(M/R) = \dim \text{Ext}^1_{R}(M, M) = \lambda + K^3 + K^4 + \ldots + K_c. \]
Moreover we also have
\[ \dim \text{Ext}^i_{R}(M, M) = \sum_{j=0}^{i+1} (a_j - \sum_{i=1}^{t} H_M(b_i)) + 1. \]

Proof. For the unobstructedness of $M$, see [28 Thm. 3.1] or Ile’s PhD thesis [19 ch. 6].

To see the dimension formula we claim that there is an exact sequence
\[ (3.3) \quad 0 \to \text{Hom}_R(M, M) \to \text{Hom}_R(F^*, M) \to \text{Hom}_R(G^*, M) \to \text{Ext}^1_{R}(M, M) \to 0. \]
Indeed the map $d_1 : \wedge^{i+1}G^* \otimes S_0^*(F) \otimes \wedge^t F \to G^*$ appearing in the Buchsbaum-Rim complex (2.2), takes an element of $\wedge^{i+1}G^* \otimes S_0^*(F) \otimes \wedge^t F$ to a linear combination of maximal minors with coefficients in $G^*$ because $I_i(A) = \text{im}(\wedge^{i}G^* \otimes S_0^*(F) \otimes \wedge^t F \to R)$ is generated by maximal minors, whence $\text{Hom}_R(d_1, M) = 0$ because $I = \text{ann}(M)$. So if we apply $\text{Hom}_R(-, M)$ to (2.4), we get (3.3) by the definition of $\text{Ext}^i_{R}(M, M)$. Counting dimensions in (3.3) and using (2.5) and (3.4) of Theorem 3.3 provided we can prove $\text{Hom}_R(F^*, -)$ and $\text{Hom}_R(G^*, -)$ onto (2.6) with $i = c$ we get
\[ \text{hom}_R(G^*, M) - \text{hom}_R(F^*, M) = \lambda + 1 + \text{hom}_R(G^*, B^*) - \text{hom}_R(F^*, B^*) \]
by using the definition (2.12) of $\lambda$. Hence we get the dimension formula (i.e. the rightmost equality) of Theorem 3.3 provided we can prove
\[ \text{hom}_R(B, G) - \text{hom}_R(B, F) = K^3 + \ldots + K_c. \]
This follow from Proposition 2.3 and the second exact sequence of Lemma 2.2.

Finally the condition $\text{hom}_M(A, M) \simeq k$ allows us to lift automorphisms, i.e. Def$_{M/R}$ is pro-representable, cf. [15 Thm. 19.2], whence $\dim H(M/R) = \dim \text{Ext}^1_{R}(M, M)$ by the smoothness of Def$_{M/R}$.

Let $D := k[e]/(e^2)$ be the dual numbers and denote the dimension in Theorem 3.3 by
\[ (3.4) \quad \lambda := \dim \text{Ext}^1_{R}(M, M) = \lambda + K^3 + K^4 + \ldots + K_c. \]
Recalling that $W_a(k; g)$ is a certain quotient of an open irreducible set in the affine scheme $\mathcal{V} = \text{Hom}_R(G^*, F^*)$ parameterizing determinantal $k$-algebras (Proposition 2.1), we get

Theorem 3.4. Let $A = R/I$, $I = I_1(A)$, be a standard determinantal $k$-algebra and let $M = \text{coker } \varphi^*.$

(i) If $\text{Hom}_A(M, M) \simeq k$ and $\text{Ext}^1_{A}(M, M) = 0$ then
\[ \text{Def}_{A \in W_a(k; g)} \simeq \text{Def}_{M/R}. \]
Hence $\text{Def}_{A \in W_a(k; g)}$ is a formally smooth pro-representable functor and the pro-representing object has dimension
\[ \dim W_a(k; g) = \lambda + K^3 + K^4 + \ldots + K_c. \]
Moreover the tangent space of $\text{Def}_{A \in W_a(k; g)}$ is the subvector space of $\text{Hom}_R(I, A)$ that corresponds to graded deformation $R_D \to A_D$ of $R \to A$ to $D$ of the form $A_D = R_D/I_1(A_D)$ for some matrix $A_D$ which lifts $A$ to $D$. 
(ii) If in addition $\text{Ext}^2_A(M, M) = 0$, then $\text{Def}_{M/R} \simeq \text{Def}_{A \in W_s(\lambda; \underline{a})} \simeq \text{Def}_{A/R}$ and $\text{Def}_{A/R}$ is formally smooth. Moreover every deformation of $X$ comes from deforming $A$ (cf. Definition 2.7).

Proof. We already know that $\text{Def}_{M/R}(T) \to \text{Def}_{A \in W_s(\lambda; \underline{a})}(T)$ is well defined (Fitting’s lemma) and obviously surjective. The injectivity follows from the first part of the proof of [28 Thm. 5.2], replacing in the proof there $\text{Hom}_A(M, M) \simeq A$ by $\text{Hom}_A(M, M) \simeq k$. Indeed if $(\mathcal{A}_T)_1$ and $(\mathcal{A}_T)_2$ are two matrices lifting $A$ to $T$ and such that $I_i((\mathcal{A}_T)_1) = I_i((\mathcal{A}_T)_2)$, then the cokernels of the morphisms determined by $(\mathcal{A}_T)_i$ define two graded deformations $M_1$ and $M_2$ of the $R$-modules $M$ to $R_T$ which by $\text{Ext}^1_A(M, M) = 0$ define the same graded deformation of $M$ to $A_T := R_T/I_i((\mathcal{A}_T)_1)$.

Moreover using $\text{Def}_{A \in W_s(\lambda; \underline{a})} \simeq \text{Def}_{M/R}$ and Theorem 3.3 we get that $\text{Def}_{A \in W_s(\lambda; \underline{a})}$ is formally smooth, whence its pro-representing object $V$ satisfies $H \simeq (H/M(R))$ and since $H$ and its “universal family” $A_H$ are algebraizable by the proof of [28 Thm. 5.2] (or use the 2nd paragraph of the proof of Theorem 4.1 of this paper to see that we get a morphism $\mathcal{O}_{\text{GradAlg}(H)_s(A)} \to S$, $H \simeq S$ with “universal family” $A_S := R_S/I_s(A_S)$, which extends to open subsets $V \subset U$, of $W_s(\lambda; \underline{a})$ and GradAlg($H$) respectively with $(A) \in V$ and $\mathcal{O}_V(A) = S$), we get $\dim H = \dim W_s(\lambda; \underline{a})$ by Theorem 3.3. The description of its tangent space follows from (3.1) and Definition 3.1 since $\text{Hom}_R(I, A)$ is the tangent space of $\text{Def}_{A/R}$.

If $\text{Ext}_A^2(M, M) = 0$ then $\text{Ext}_A^2(M, M) \simeq (E^2_{A}^1)_0 \simeq \text{Ext}_R^1(I, \text{Hom}_A(M, M))$ by (3.1). Hence the first part of the proof and $A \hookrightarrow \text{Hom}_A(M, M)$ imply that all the injections in (3.5) $\text{Def}_{M/R}(D) \hookrightarrow \text{Def}_{A/R}(D) \simeq \text{Hom}_R(I, A) \hookrightarrow \text{Hom}_R(I, \text{Hom}_A(M, M))$

are isomorphisms of finite dimensional vector spaces. It follows that $\text{Def}_{M/R} \to \text{Def}_{A/R}$ is an isomorphism since it is bijective on tangent spaces and $\text{Def}_{M/R}$ is formally smooth. Then we conclude the proof by arguing as in the proof of [28 Thm. 5.2].

Remark 3.5. The theorems of this section admit substantial generalizations with respect to $R$ being a polynomial ring. Indeed we may let $R$ be any graded quotient of a polynomial ring. The main reason for this is that the spectral sequence of this section, cf. (3.1), Fitting’s lemma and the exactness of the Buchsbaum-Rim complex are valid with almost no assumption on $R$ (but we need to replace the binomials defining $\lambda$ and $K$ with their Hilbert functions; the final formula of Theorems 3.3 is, however, valid as stated).

4. The locus of determinantal $k$-algebras

In this section we generalize two of the main theorems (Theorems 5.5 and 5.8) of [28] to cover the artinian case. Indeed using that Theorem 3.4 extends [28 Thm. 5.2] by only assuming $\text{Hom}_A(M, M) \simeq k$ and $A$ standard determinantal instead of good determinantal, the generalizations in Theorems 4.1 and 4.4 are rather immediate. In this section we also prove a new result (Theorem 4.1) to cover cases where $\dim W_s(\lambda; \underline{a}) \neq \lambda$.

In the first theorem we let, as in [28],

$$\text{ext}_A^2(M, M) := \dim \ker(\text{Ext}_A^2(M, M) \to \text{Ext}_R^2(M, M))$$

Clearly $\text{ext}_A^2(M, M) \leq \text{ext}_A^2(M, M)$, and note that we below may replace GradAlg($H$) with Hilb$^p(\mathbb{P}^n)$ if $n-c \geq 1$, cf. the text accompanying (2.1) for explanations and notations.
Theorem 4.1. Let $A = R/I$, $I = I_t(A)$ be a standard determinantal $k$-algebra, i.e. $(A) \in W_s(b;\alpha)$, let $M = \coker \varphi^*$ and suppose $\vartheta \Hom_{A}(M, M) \simeq k$ and $\vartheta \Ext^1_A(M, M) = 0$. Then

$$\dim W_s(b;\alpha) = \lambda := \lambda_e + K_3 + K_4 + ... + K_c .$$

Moreover, for the codimension of $W_s(b;\alpha)$ in GradAlg($H$) in a neighborhood of $(A)$ we have

$$\dim_{(A)} \text{GradAlg}(H) - \dim W_s(b;\alpha) \leq \dim \vartheta \Hom(I, A) - \lambda \leq \ext^2(M, M) ,$$

where the inequality is an equality if and only if GradAlg($H$) is smooth at $(A)$. In particular these conclusions hold if $\dim A \geq 3 + \dim R/I_{t-1}(A)$, or if $\dim A \geq 2$ and $\dim R/I_{t-1}(B) = 0$ where $B$ is obtained from $A$ by deleting some column of $A$ (e.g. if $\dim A \geq 2$, $a_{i-2} \geq b_i$ for $2 \leq i \leq t$ and $A$ is general).

Proof. Since $\depth_{I_{t-1}(A)} A = \dim A - \dim R/I_{t-1}(A)$, this follows from Theorem 3.4 [28, Theorem 5.5] and Remark 2.4 except for the statements on the codimension. However, since $\dim_{(A)} \text{GradAlg}(H) \leq \dim \vartheta \Hom_R(I, A)$ with equality if and only if $A$ is unobstructed, and since the injectivity of $A \hookrightarrow \vartheta \Hom_A(M, M)$, (3.1) and (3.4) imply that $\vartheta \Hom_R(I, A) - \lambda \leq \dim (E_2^{0,1})_0 - \lambda = \ext^2(M, M)$, we get all conclusions of Theorem 4.1. \qed

There are many artinian determinantal rings satisfying the conditions $\vartheta \Hom_A(M, M) \simeq k$ and $\vartheta \Ext^1_A(M, M) = 0$ of Theorem 4.1. After having computed many examples using Macaulay 2 the general picture when $a_0 > b_t$ seems to be that the only artinian determinantal rings $A = R/I_t(A)$ that do not satisfy these conditions are those with matrix $A$ that is linear except in one column $v \in R_m^{\oplus t}$ where the degree is $m \geq 2$, so $A$ is of the form $[B, v]$ where $B$ is linear. The following example avoids this case, but $B$ is otherwise quite close to being linear (the case where $B$ is linear will be considered in Example 4.9).

Example 4.2. (Determinantal artinian quotients of $R$, using Theorem 4.1)

(i) Let $R = k[x_0, x_1, x_2]$ and let $A = [B, v]$ be a general $2 \times 4$ matrix with linear (resp. quadratic) entries in the first and second (resp. third) column and where the entries of $v$ are polynomials of the same degree $m$. The vanishing of all $2 \times 2$ minors defines an artinian ring with $h$-vector $(1, 3, 5, 5, ..., 5, 5, 3)$, where the number of 5’s is $m - 1$. For $m \geq 2$ one verifies that the first conditions of Theorem 4.1 i.e. that all conditions of Theorem 3.4(i) hold, and it follows that $W_s(b;\alpha)$ is an irreducible subset of GradAlg($H$) of dimension $\lambda_3 + K_3$ which is 14 for $m \geq 3$ (or one may use (2.13) to find the dimension). For $m \geq 5$ one shows that $\vartheta \Ext^1_A(I/I^2, A) = 0$ and $\vartheta \Hom_A(I/I^2, A) = 16$, whence $A$ is unobstructed, and we get that the codimension of $W_s(b;\alpha)$ in GradAlg($H$) is 2 by Theorem 4.1 and that GradAlg($H$) is generically smooth along $W_s(b;\alpha)$. The computations for the $\vartheta \Ext^i$-groups above are accomplished by Macaulay 2, but strictly speaking only for $m \leq 10$ from which we clearly see the general pattern also for $m > 10$.

(ii) Let $R = k[x_0, x_1, x_2, x_3]$ and let $A = [B, v]$ be a general $2 \times 5$ matrix with linear (resp. quadratic) entries in the first, second and third (resp. fourth) column and where the entries of $v$ are polynomials of the same degree $m$. The vanishing of all $2 \times 2$ minors defines an artinian ring with $h$-vector $(1, 4, 7, 7, ..., 7, 4)$, where the number of 7’s is $m - 1$. For $m \geq 2$ one verifies that all conditions of Theorem 3.4(i) hold, and we get that $W_s(b;\alpha)$
is an irreducible subset of GradAlg(H) of dimension $\lambda_4 + K_3 + K_4$, which is equal to 25 if $m \geq 3$ (or one may use (2.13) to find the dimension). For $m \geq 5$, $\text{Ext}^1_A(I/I^2, A) = 0$ and $\text{Hom}_A[I/I^2, A] = 33$, whence $A$ is unobstructed, and we get that the codimension of $W_s(\overline{b} ; \overline{a})$ in GradAlg(H) is 8 by Theorem 4.1 and that GradAlg(H) is generically smooth along $W_s(\overline{b} ; \overline{a})$. The computations for the $\text{Ext}^1$-groups above are verified by Macaulay 2 for $m \leq 10$, but their dimensions hold also for $m > 10$.

**Example 4.3.** (determinantal artinian quotients of $R = k[x_0, x_1, x_2]$, using Theorem 4.1)

(i) Let $A$ be a general $2 \times 4$ matrix with quadratic (resp. linear) entries in the first (resp. second) row. The vanishing of all $2 \times 2$ minors defines an artinian ring with $h$-vector $(1, 3, 6, 4, 1)$. Using Macaulay 2 one shows that all conditions of Theorem 3.4(i) hold (and that $\text{ext}_A^2(M, M) = 10$), and it follows that $W_s(\overline{b} ; \overline{a})$ is an irreducible subset of GradAlg(H) of dimension $\lambda_3 + K_3 = 16$. Since $\text{Ext}_A^1(I/I^2, A) = 0$ and $\text{Hom}_A[I/I^2, A] = 20$, we get that $A$ is unobstructed and by Theorem 4.1 that $\text{codim}_{\text{GradAlg}(H)} W(\overline{b} ; \overline{a}) = 4$.

(ii) Again $R = k[x_0, x_1, x_2]$, but now $A$ be a general $3 \times 5$ matrix with quadratic (resp. linear) entries in the first (resp. second and third) row. The vanishing of all $3 \times 3$ minors defines an artinian ring with $h$-vector $(1, 3, 6, 10, 5, 1)$. One verifies that all conditions of Theorem 3.4(i) hold, and we get that $W_s(\overline{b} ; \overline{a})$ is an irreducible subset of GradAlg(H) of dimension $\lambda_3 + K_3 = 25$. Since $\text{Ext}_A^3(I/I^2, A) = 0$ and $\text{Hom}_A[I/I^2, A] = 40$, it follows from Theorem 4.1 that $\text{codim}_{\text{GradAlg}(H)} W(\overline{b} ; \overline{a}) = 15$.

Using Theorem 3.4(ii), we have the following generalization of [28, Theorem 5.8]. Note that we below may replace GradAlg(H) with Hilb$^p(\mathbb{P}^n)$ if dim $A \geq 2$, cf. (2.1).

**Theorem 4.4.** Let $A = R/I$, $I = I_t(A)$ be a standard determinantal $k$-algebra, i.e. $(A) \in W_s(\overline{b} ; \overline{a})$, let $M = \text{coker} \varphi^*$ and suppose $\text{Hom}_A(M, M) \simeq k$ and $\text{Ext}^1_A(M, M) = 0$ for $i = 1$ and 2. Then GradAlg(H) is smooth at $(A)$ and

$$\dim_{(A)} \text{GradAlg}(H) = \lambda_c + K_3 + K_4 + \ldots + K_c .$$

Moreover $W_s(\overline{b} ; \overline{a}) \subset \text{GradAlg}(H)$ is an irreducible component, and every deformation of $A$ comes from deforming $A$. In particular these conclusions hold if $\dim A \geq 4 + \dim R/I_{t-1}(A)$, or if $\dim A \geq 3$ and $\dim R/I_{t-1}(\mathcal{B}) = 0$ where $\mathcal{B}$ is obtained from $A$ by deleting some column of $A$ (e.g. if $\dim A \geq 3$, $a_i - \text{min}(3, t) \geq b_i$ for $\text{min}(3, t) \leq i \leq t$ and $A$ is general).

**Proof.** Since $\text{depth}_{I_{t-1}(A)} A = \dim A - \dim R/I_{t-1}(A)$, this follows from Theorem 3.4 [28, Theorem 5.8] and Remark 2.4.

The condition $\text{Ext}^3_A(M, M) = 0$ of Theorem 4.4 seems harder to satisfy for artinian rings, but it holds quite often if $A$ is not too close to the linear case. Here is an example:

**Example 4.5.** (determinantal artinian quotients of $R = k[x_0, x_1, x_2]$, using Theorem 4.4)

(i) Let $A = [\mathcal{B}, v]$ be a general $2 \times 4$ matrix with linear (resp. cubic) entries in the first and second (resp. third) column and where the entries of $v$ are polynomials of the same degree $m$, $m \geq 3$. The vanishing of all $2 \times 2$ minors defines an artinian ring with $h$-vector $(1, 3, 5, 7, 7, \ldots, 7, 5, 3)$, where the number of $7's$ is $m - 2$. For $m \geq 5$ one verifies that all conditions of Theorem 3.4(ii), i.e. the first conditions of Theorem 4.4 hold, and
it follows that \( \overline{W}(b; a) \) is a generically smooth irreducible component of \( \text{GradAlg}(H) \) of dimension \( \lambda_3 + K_3 = 18 \). The computations for the 0-Ext1-groups above are verified by Macaulay 2 for \( m \leq 10 \), but their dimensions hold also for \( m > 10 \). One also verifies that \( \ Ho^\text{Ext}_1^1(I/I^2, A) = 2 \) for \( m \geq 6 \), so both the generic smoothness along \( \overline{W}(b; a) \) and the conclusion that \( \overline{W}(b; a) \) is an irreducible component of \( \text{GradAlg}(H) \) would be hard to see without using Theorem 4.4.

(ii) Let \( A \) be a general \( 2 \times 4 \) matrix with cubic (resp. linear) entries in the first (resp. second) row. The vanishing of all \( 2 \times 2 \) minors defines an artinian ring with \( h \)-vector \( (1, 3, 6, 10, 9, 7, 3, 1) \). Using Macaulay 2 one verifies that all conditions of Theorem 3.4(ii) hold, (i.e. also \( Ho^2^2(A, M) = 0 \)), and we get that \( \overline{W}(b; a) \) is a generically smooth irreducible component of \( \text{GradAlg}(H) \) of dimension \( \lambda_3 + K_3 = 29 \). In this example \( Ho^\text{Ext}_1^3(I/I^2, A) \neq 0 \) and \( Ho^\text{hom}_A(I/I^2, A) = 29 \), so it is not straightforward to see that \( A \) is unobstructed, but it is, due to Theorem 4.4 which also contains additional information.

As indicated it seems that the only artinian determinantal rings \( A = R/I_1(A) \) when \( a_0 > b_i \) that do not satisfy the conditions of Theorem 3.4(i) are those with a matrix \( A \) that is linear except in one column \( v \in R^c_m \) where the degree is \( m \geq 2 \). Moreover if \( m \geq 3 \) we may even have

\[
_0\text{Hom}_A(M, M) \simeq k, \quad _0\text{Ext}^1_A(M, M) \neq 0 \quad \text{and} \quad _0\text{Ext}^2_A(M, M) = 0,
\]

and the deformation functors of these rings are fully determined by our next Theorem 4.6.

For artinian linear determinantal rings (i.e. \( m = 1 \) above) all conditions of Theorem 3.4(ii) seem to hold and even more, they are rigid in several senses: \( _0\text{Hom}_R(I, A) = 0 \), as well as \( _0\text{Ext}^1_R(M, M) = 0 \), whence the linear case may be not so interesting. Recalling (3.1), i.e. \( \lambda := \dim _0\text{Ext}^1_R(M, M) = \lambda_c + K_3 + K_4 + \cdots + K_c \), we get

Theorem 4.6. Let \( A = R/I, I = I_1(A) \) be a standard determinantal \( k \)-algebra. If \( _0\text{Hom}_A(M, M) \simeq k \) and the map \( _0\text{Ext}^1_A(M, M) \to _0\text{Ext}^2_A(M, M) \) of (3.1) is injective, then the natural morphism of functors \( e_M : \text{Def}_{M/R} \to \text{Def}_{A/R} \) is smooth. Moreover \( \text{Def}_{M/A} \) is smooth and its pro-representing object \( H(M/A) \) satisfies

\[
\dim H(M/A) = \dim _0\text{Ext}^1_A(M, M) = \lambda - \dim W_s(b; a).
\]

Furthermore \( \text{Def}_{A_{\text{st}}} \simeq \text{Def}_{A/R} \) are isomorphic smooth functors, \( W_s(b; a) \subset \text{GradAlg}(R) \) is an irreducible component of dimension \( \lambda - \dim _0\text{Ext}^1_A(M, M) \) and every deformation of \( A \) comes from deforming \( A \). In particular these conclusions hold if \( \dim A \geq 3 + \min \{ \dim R/I_{t-1}(A) + 1, \dim R/I_{t-1}(B) \} \) where \( B \) is obtained from \( A \) by deleting some column of \( A \) (e.g. if \( \dim A \geq 3, a_{i-\min(3, t)} \geq b_i \) for \( \min(3, t) \leq i \leq t \) and \( A \) is general).

Remark 4.7. We may consider \( \text{Def}_{M/A} \) as a fiber functor of \( e_M : \text{Def}_{M/R} \to \text{Def}_{A/R} \). For a description of the obstruction maps of \( \text{Def}_{M/A} \) for arbitrary \( M \), see [37, 20].

Proof. Let \( T \to S \) be a small artinian surjection with kernel \( a \). To prove the (formal) smoothness of \( e_M : \text{Def}_{M/R} \to \text{Def}_{A/R} \), we must by definition show that the map

\[
\text{Def}_{M/R}(T) \to \text{Def}_{M/R}(S) \times_{\text{Def}_{A/R}(S)} \text{Def}_{A/R}(T)
\]

is surjective. Let \( M_S \) be a deformation of \( M \) to \( S \) inducing \( R_S/I_S \in \text{Def}_{A/R}(S) \) and let \( R_T/I_T \) be a deformation of \( R_S/I_S \) to \( T \). Since \( M \) is unobstructed, there exists a
deformation $M'_T$ of $M_S$ to $T$ inducing a deformation $R_T/I'_T$ of $R_S/I_S$. The difference $R_T/I_T - R_T/I'_T$ sits in $\Hom_R(I, A) \otimes_k a$ and since the composed map

$$(4.1) \quad 0\Ext^1_R(M, M) \to \Hom_R(I, A) \to \Hom_R(I, \Hom_A(M, M)) \simeq E^{0,1}_2$$

of Lemma 3.2 is surjective by (3.1) and assumption, we get that $0\Ext^1_R(M, M) \otimes_k a \to \Hom_R(I, A) \otimes_k a$ is surjective. Thus there exists an element in $0\Ext^1_R(M, M) \otimes_k a$ that we can “add” to $M'_T$ to get a deformation $M_T$ of $M_S$ that induces $R_T/I_T$, i.e. $e_M$ is smooth. It follows that $\Def_{A/R}$ is smooth. We also get that $\Def_{A \in W_s(\mathbb{Q})} \to \Def_{A/R}$ is smooth and injective on tangent spaces, whence an isomorphism, and since $\Def_{M/A}$ is a fiber functor of $\Def_{M/R} \to \Def_{A/R}$, it follows that $\Def_{M/R} \to \Def_{A/R}$ is smooth. Hence $\dim H(M/A) = \dim 0\Ext^1_A(M, M)$, and we get the dimension formulas of Theorem 4.6 by (3.1) and the injectivity of $0\Ext^2_A(M, M) \to 0\Ext^2_R(M, M)$ provided $\dim W_s(b; \bar{a}) = \Hom_R(I, A)$.

If we compare $\Def_{A \in W_s(\mathbb{Q})} \simeq \Def_{A/R}$ with Definition 3.1 we get that every deformation $A_T$ of $A$ to an artinian $T$ comes from deforming $A$. To see that we may replace “an artinian $T$” by “$T$ a local ring” in this statement (cf. Definition 2.7) we pick $d := \Hom_R(I, A)$ elements of $0\Ext^1_R(M, M)$ that map to linearly independent elements of $\Hom_R(I, A)$ via the tangent map $0\Ext^1_R(M, M) \to \Hom_R(I, A)$ of $e_M$, and we let $\eta_1, ..., \eta_d \in \Hom(G^*, F^*)$ with presentation matrices $A_1, ..., A_d$ correspond to the $d$ elements we picked in $0\Ext^1_R(M, M)$, cf. the beginning of the proof of Lemma 3.2 for a similar set-up. Let $A_S := A + s_1A_1 + ... + s_dA_d$ and $S := k[s_1, ..., s_d]_{(s_1, ..., s_d)}$ where $k[s_1, ..., s_d]$ is a polynomial ring, and let $O := \mathcal{O}_{\GradAlg(H), (A)}$ be the local ring of $\GradAlg(H)$ at the $k$-point $(A) \in \GradAlg(H)$, $k = \mathbb{F}$. Then the algebraic family $A_S := R_S/I_S(A_S)$ is $S$-flat by Lemma 2.6 and e.g. using the explicit description of the pro-representing object $H(A/R)$ of $\Def_{A/R}$ appearing in the proof of Thm. 4.2.4 of [36] to see that any deformation of $A_S \otimes_S S/(s_1, ..., s_d)^2$ to $H(A/R)$ may serve as a versal lifting, we get that $H(A/R)$ equals $S$ with “universal object” $A_S$ (up to completion, but notice that the versal lifting is defined by polynomials, not power series, in the $s_i$’s). Hence, by the universal property of the representable functor that corresponds to $\GradAlg(H)$, there is a morphism $O \to S$ whose completions are isomorphic because both pro-represents $\Def_{A/R}$ on $\ell$ (4.15 (2.1) and (2.8))). Thus $O \to S$ is injective, and since $O/m^n_O \simeq S/m^n_S$ for every $n$, it follows that $O \to S$ is an isomorphism, having $A_O := A_S \otimes_S O$ as the pullback (considered as scheme) of the universal object of $\GradAlg(H)$ to $\Spec(O)$. This shows that “every deformation of $A$ comes from deforming $A$” because the universal property implies that all deformations are given by pullback to $\Spec(T)$ (i.e. by taking tensor product of $O \to A_O$ via $O \to T$). Then Lemma 2.8 implies that $W_s(b; \bar{a}) \subset \GradAlg(H)$ is a generically smooth irreducible component of dimension $d$.

The argument for the statement in the final sentence is as in the proof of Theorem 4.4 provided $\dim A \geq 4 + \dim R/I_{t-1}(A)$. If, however, $\dim A \geq 3 + \dim R/I_{t-1}(B)$ then $\text{depth}_{I_{t-1}(B)} R/I_t(B) \geq 4$ and we get $0\Ext^2_A(M, M) = 0$ by [28] Thm. 4.5] and we are done. 

As a corollary of the proof, we have

**Corollary 4.8.** Let $A = R/I_t(A)$ be a standard determinantal $k$-algebra. If every deformation $A_T$ of $A$ to any local artinian $k$-algebra $(T, m_T)$ (of finite type such that $(m_T)^r = 0$
for some \( r \in \mathbb{Z} \) and \( k = A/m_r \) comes from deforming \( A \), then every deformation of \( A \) comes from deforming \( A \) (see Definition \( \ref{def:deform} \)).

Proof. Using Lemma \( \ref{lem:unobstructedness} \) the unobstructedness of \( A \) is straightforward. Then we can use the proof in the 2\textsuperscript{nd} paragraph of Theorem \( \ref{thm:main} \) to get the conclusion (note that \( S \) is a regular local ring, and that we do not use the assumption \( _0\text{Hom}_A(M,M) \simeq k \) in this part of the proof). \( \square \)

Let us consider two particular cases of artinian rings:

Example 4.9. (determinantal artinian quotients of \( R \), using Theorem \( \ref{thm:main} \))

(i) Let \( R = k[x_0, x_1, x_2] \) and let \( A = [B, v] \) be a general 2 \times 4 matrix where \( B \) is linear and the entries of \( v \) are polynomials of the same degree \( m \geq 2 \). The vanishing of all 2 \times 2 minors defines an artinian ring with \( h \)-vector \( (1,3,3,...,3) \) where the number of 3's is \( m \). Using Macaulay 2 for \( m \geq 3 \) one verifies that all conditions of Theorem \( \ref{thm:main} \) holds and that \( _0\text{ext}^1_A(M,M) = 2 \), and it follows that \( W_s(b; a) \) is a generically smooth irreducible component of \( \text{GradAlg}(H) \) of dimension \( \lambda_3 + K_3 - _0\text{ext}^1_A(M,M) = 6 \). If \( m = 2 \), Theorem \( \ref{thm:main} \) does not apply because \( _0\text{Ext}^2_A(M,M) \neq 0 \). In fact using Macaulay 2 one may show that all inclusions of \( \left(\begin{array}{c}3 \\ 3\end{array}\right) \) are strict, i.e. non-isomorphisms.

(ii) Let \( R = k[x_0, x_1, x_2, x_3] \) and let \( A = [B, v] \) be a general 2 \times 5 matrix where \( B \) is linear and the entries of \( v \) are polynomials of the same degree \( m \geq 2 \). The vanishing of all 2 \times 2 minors defines an artinian ring with \( h \)-vector \( (1,4,4,..,4) \), where the number of 4's is \( m \). Using Macaulay 2 for \( m \geq 3 \) one verifies that all conditions of Theorem \( \ref{thm:main} \) holds and that \( _0\text{ext}^1_A(M,M) = 4 \). By Proposition \( \ref{prop:regular} \) and \( \ref{prop:homology} \), \( K_3 = 0, K_4 = 4 \) and \( \lambda_4 = 12 \), and it follows that \( W_s(b; a) \) is a generically smooth irreducible component of \( \text{GradAlg}(H) \) of dimension \( \lambda_3 + K_3 + K_4 - _0\text{ext}^1_A(M,M) = 12 \).

Remark 4.10. Both for (i) and (ii) in Example \( \ref{ex:artinian} \) \( _0\text{Ext}^1_A(I/I^2, A) \) seems to vanish for \( m \neq 3 \), and even though the generic smoothness of \( W_s(b; a) \) could be found by using this fact, the conclusion that \( W_s(b; a) \) is an irreducible component of \( \text{GradAlg}(H) \) may not be so easy to see without using Theorem \( \ref{thm:main} \).

For determinantal zero-schemes we showed in \( \ref{cor:dim-formula} \) that the dimension formula \( \ref{eq:dim-formula} \) fails when (and only when?) \( A \) is a linear matrix consisting of two rows, cf. Remark \( \ref{rem:art} \). In our next example we use Theorem \( \ref{thm:main} \) to treat this case.

Example 4.11. (determinantal quotients of \( R = k[x_0, x_1, \cdots, x_c] \) of dimension one)

Let \( A \) be a general \( 2 \times (c + 1) \) matrix of linear entries. The vanishing all 2 \times 2 minors defines a reduced scheme \( X = \text{Proj}(A) \) of \( c + 1 \) general points in \( \mathbb{P}^c \). Since we may suppose \( A \) is good determinantal, \( \text{Hom}_A(M,M) \simeq A \) by \( \ref{lem:homology} \) Lem. 3.2], and we have \( _0\text{ext}^1_R(M,M) = c(c+1) + c - 2 \) by Theorem \( \ref{thm:homology} \). In the range \( 3 \leq c \leq 10 \) we use Macaulay 2 to see \( _0\text{ext}^1_A(M,M) = c - 2 \) and \( _0\text{Ext}^2_A(M,M) = 0 \). For \( c \leq 10 \) we get from Theorem \( \ref{thm:main} \) that \( W(0,0;1,1,\cdots,1) \) is a generically smooth irreducible component of \( \text{GradAlg}(H) \) of dimension \( c(c+1) \) and that every deformation of \( A \) comes from deforming \( A \) (this is not very surprising since the corresponding component of \( \text{Hilb}^p(\mathbb{P}^c) \) of has dimension \( c(c+1) \)).

Theorem \( \ref{thm:main} \) admits a nice consequence concerning glicciness. Indeed since glicciness is not necessarily an open property, the following result may be useful.
Corollary 4.12. Let $X = \text{Proj}(A)$, $I_X = I_t(A)$, be a standard determinantal scheme and suppose that $\partial \text{Hom}_R(I_X, H^1_m(A)) = 0$ and that the map $\partial \text{Ext}_A^2(M, M) \to \partial \text{Ext}_A^2(M, M)$ of (3.1) is injective. Then the Hilbert scheme $\text{Hilb}^p(\mathbb{P}^n)$ is smooth at $(X)$ and $(X)$ belongs to a unique irreducible component of $\text{Hilb}^p(\mathbb{P}^n)$ whose general element $\tilde{X} \subset \mathbb{P}^n$ is glci. In particular this conclusion holds if $\dim X \geq 2 + \min\{\dim R/I_{t-1}(A) + 1, \dim R/I_{t-1}(B)\}$ where $B$ is obtained from $A$ by deleting some column of $A$.

Proof. Indeed $\tilde{X}$ is standard determinantal by Theorem [4.6] and (2.1) and since standard determinantal schemes are glci by [29] Thm. 3.6, we are done. \qed

Finally we remark that [28] Thm. 5.16, see the theorem below, generalizes to cover the artinian case. Indeed its proof enlighten the differences between the cases $c = 2$ and $c > 2$. Here $\mathcal{R}$ is a Cohen-Macaulay quotient of a polynomial ring (i.e. $\text{Proj}(\mathcal{R}) \subset \mathbb{P}^N_k$ is ACM) and $A$ is a standard determinantal quotient of $\mathcal{R}$. The main observation to make is that if $c = 2$ then $M \simeq K_A(s)$ is a twist of the canonical module $K_A$ of $A$. Since it is well known that $\partial \text{Ext}_A^i(M, M) = 0$ for $i > 0$ and $\text{Hom}_A(M, M) \simeq A$, we get $\text{Def}_M/\mathcal{R} \simeq \text{Def}_A/\mathcal{R}$ by Theorem [3.3]. Then Theorem [3.3] and Remark [3.5] lead to the theorem below noting that the assumption $k = \overline{k}$ is eventually included since we define $W_s(b; a)$ as a certain locus (of $k$-points) of $\text{GradAlg}^H(\mathcal{R})$. Notice also that we now deal with standard determinantal quotients $A$ of codimension 2 of $\mathcal{R} \simeq k[x_0, \ldots, x_N]/I_A$; they are usually not determinantal quotients of $k[x_0, \ldots, x_N]$.

With $b, a$ as in (2.3) and $(A) \in W_s(b; a)$ we get (where we for $\dim A \geq 2$ may replace $\text{GradAlg}^H(\mathcal{R})$ by $\text{Hilb}^p(\text{Proj}(\mathcal{R}))$ since (2.4) extends to hold in this generality by [22], Thm. 3.6 and Rem. 3.7), cf. [16], Thm’s 2.9, 2.12 and 3.13 for the irreducibility and dimension of $\overline{W_s(b; a)}$ when $\mathcal{R} = k[x_0, x_1]$.

Theorem 4.13. Let $P = \text{Proj}(\mathcal{R}) \subset \mathbb{P}^N_k$ be an ACM scheme where $k$ is any field and let $X = \text{Proj}(A) \subset P$, $A = R/I_t(A)$, be any standard determinantal scheme of codimension 2 in $P$. Then $\text{GradAlg}^H(\mathcal{R})$ is smooth at $(A)$ and $\dim_{(A)} \text{GradAlg}^H(\mathcal{R}) = \lambda(\mathcal{R})_2$ where

$$\lambda(\mathcal{R})_2 := \sum_{i,j} \dim \mathcal{R}_{(a_j-b_i)} + \sum_{i,j} \dim \mathcal{R}_{(b_i-a_j)} - \sum_{i,j} \dim \mathcal{R}_{(a_i-b_j)} - \sum_{i,j} \dim \mathcal{R}_{(b_i-b_j)} + 1.$$ 

Moreover every deformation of $A$ comes from deforming $A$. In particular if $k = \overline{k}$, then $\text{GradAlg}^H(\mathcal{R})$ is smooth along $W_s(b; a)$ and the closure $\overline{W_s(b; a)}$ in $\text{GradAlg}^H(\mathcal{R})$ is an irreducible component of dimension $\lambda(\mathcal{R})_2$.

Indeed there are no singular points of $\text{GradAlg}^H(\mathcal{R})$ at $(A) \in W_s(b; a)$ when $c = 2$ while singular points of $\text{GradAlg}^H(\mathcal{R})$ for $c > 2$ are quite common (see [31] Rem. 3.6 and [40]).

5. GHOST TERMS

Let $N$ be an graded $R$-module with a minimal $R$-free resolution $0 \to P_n \to N$. By a ghost term of $P_n$ we mean a direct free summand that appears in consecutive terms of $P_n$. As in the minimal resolution conjecture ([38] [43], and see [40] [39] [42] and its references for other contributions), one expects that a general element $R/I$ of $\text{GradAlg}(H)$ contains “as few ghost terms as possible”, while ghost terms for special elements of $\text{GradAlg}(H)$ are more common. In this section we shall see that some ghost terms of a determinantal ring
R/I easily disappear under generizations (deformation to a more general element) while other ghost terms do not. Letting \( a_j = a_0, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{t+c-2} \) and \( b_j = b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_t \) we have the following result.

**Proposition 5.1.** Suppose \( a_j = b_i \) for some \( j \in \{0, t + c - 2\} \) and \( i \in \{1, t\} \). Then we have an inclusion \( W_s(b_i; a_j) \subset W_s(b_j; a_i) \) of open irreducible subsets of \( \text{GradAlg}(H) \). Moreover if \( W_s(b_i; a_j) \setminus W_s(b_j; a_i) \neq \emptyset \), then every \( R/I \) of \( W_s(b_i; a_j) \setminus W_s(b_j; a_i) \) admits a generization \((R/I) \in W_s(b_i; a_j)\) removing exactly all ghost terms in the Eagon Northcott resolution coming from \( a_j = b_i \); in particular a resolution of \( R/I \) of \( W_s(b_i; a_j) \) is given by its Eagon Northcott resolution.

**Proof.** As mainly observed in the proof of Lemma 2.1, the vanishing of \( \text{Ext}^i_R(R/I_t(A), R) \) is an open property. This holds also for the elements \( R/I_t(A) \) of \( \text{GradAlg}(H) \), i.e. the subset of \( \text{GradAlg}(H) \) such that \( I_t(A) \) has maximal codimension in \( R \) (this subset is \( W_s(b_i; a_j) \)) is open. Since it is irreducible by Lemma 2.1 we have proved the open irreducible property stated in Proposition 5.1 and it remains to see the inclusion \( W_s(b_i; a_j) \subset W_s(b_j; a_i) \) and the existence of generizations.

We claim that the elements \( R/I_t(A) \) of \( W_s(b_i; a_j) \) whose matrix \( A \) contains a unit of the field \( k \) at the \((i, j)\)-entry, belong to \( W_s(b_i; a_j) \). Indeed \( A \) is a presentation matrix of \( M = \text{coker} \varphi^* \) and by rearranging the direct summands of the source and target of the morphism \( \varphi^* \), we may assume \((i, j) = (1, 0) \). By elementary row operations we transform \( A \) to a matrix with only zero’s (and one 1) in the first column, i.e. there is an invertible \( t \times t \) matrix \( C \) such that the cokernel of the map induced by \( C \cdot A \) is isomorphic to \( M \). By Fitting’s lemma, \( I_t(A) = I_t(C \cdot A) \). Moreover the \((t-1) \times (t+c-2) \) matrix \( A' \) obtained by deleting the first row and column of \( C \cdot A \) satisfies \( I_{t-1}(A') = I_t(C \cdot A) \) and defines a determinantal ring of \( W_s(b_i; a_j) \), and we have proved the inclusion and the whole claim (but taking any \( R/I_t(A') \) of \( W_s(b_i; a_j) \) and putting \( A = (1, 0, A') \) we get \( (R/I_t(A)) \in W_s(b_i; a_j) \) which is an easier argument for the inclusion).

Finally to see the existence of the generization of \( R/I \) of \( W_s(b_i; a_j) \setminus W_s(b_j; a_i) \), \( I = I_t(A) \), we may assume that \( A \) has a 0 at the \((i, j)\)-entry by the proven claim. Then we apply Lemma 2.6 to \( T = k[u], \varphi = (u) \), letting \( u = 0 \) correspond to \( A \) and \( A_T \) to a matrix obtained by replacing the 0 of the \((i, j)\)-entry by \( u \) and all other \((i', j')\)-entries by polynomials of \( R \otimes k[u] \) of degree \( a_{j'} - b_{j'} \) with coefficients in \( k[u] \) such that the choice \( u = 1 \) makes the corresponding matrix general enough. Indeed with notations as above, i.e. with \((i, j) = (1, 0) \) etc., we may choose the coefficients of the polynomials of the 1 column such that the choice \( u = 1 \) makes the entries equal to 0 for all \((i, 0) \) with \( i > 1 \) and the entries of \( A' \) general. In some open subset \( U \ni \{0, 1\} \) of \( \text{Spec}(k[u]) \) the matrices given by \( A_u \) for \( u \in U \setminus \{0\} \) has maximal codimension in \( R \) and we are done. \( \square \)

**Remark 5.2.** The ghost terms treated in Proposition 5.1 are removable under generization, while all other ghost terms appearing in the Eagon-Northcott resolution of \( R/I_t(A) \) may sometimes probably be removed, but not always. Indeed the latter can not be removed under generization provided every graded deformation of \( R/I_t(A) \) comes from deforming \( A \) because, in this case, \( W_s(b_i; a_j) \) is an irreducible component of \( \text{GradAlg}(H) \), whence the minimal resolution of its general element is given by some Eagon-Northcott resolution of \( R/I_t(A') \) with \( A' \) minimal, cf. Example 5.4 below. Note that Theorem 4.4 and
Theorem 4.6 give conditions under which every graded deformation of \( R/I_t(A) \) comes from deforming \( A \). In particular if \( \dim R/I_t(A) \geq 3 \) and \( a_{i-\min(3,t)} \geq b_i \) for \( \min(3,t) \leq i \leq t \), only ghost terms as in Proposition 5.1 are removable under generization!

**Example 5.3.** (removing all ghost terms using Proposition 5.1)

(i) Let \( R = k[x_0, x_1, x_2] \), let \( B \) be a general 2 \times 3 matrix with linear (resp. quadratic) entries in the first (resp. second) row and let \( A = (w^B) \) where \( u \in k \), \( v \) a row of general linear forms and \( w \) a column whose transpose is \((0, x_0)\). The vanishing of all 3 \times 3 minors defines an artinian ring with \( h \)-vector \((1, 2, 3, 1)\) for every \( u \in k \). If \( I \) (resp. \( I_g \)) is the ideal given by all 3 \times 3 minors of \( A \) with \( u = 0 \) (resp. \( u = 1 \)), then \( (R/I) \in W_s(-2, -1; -1, 0, 0, 0) \) and \( (R/I_g) \in W_s(-2, -1; 0, 0, 0) \) and using Macaulay2 we find minimal resolutions

\[
0 \rightarrow R(-5) \oplus R(-4)^2 \rightarrow R(-4) \oplus R(-3)^3 \rightarrow I \rightarrow 0,
0 \rightarrow R(-5) \oplus R(-4) \rightarrow R(-3)^3 \rightarrow I_g \rightarrow 0.
\]

Here \( R/I_g \) is a generalization of \( R/I \) in \( \text{GradAlg}(H) \).

(ii) Let \( R = k[x_0, x_1, x_2] \), let \( B \) be a general 2 \times 4 matrix with linear (resp. cubic) entries in the first (resp. second) row and let \( A = (w^B) \) where \( u \in k \), \( v \) a row of general linear forms and \( w \) the column whose transpose is \((0, x_0^2)\). Using Macaulay2 we get that the vanishing of all 3 \times 3 minors defines an artinian ring with \( h \)-vector \((1, 3, 6, 10, 9, 7, 3)\). If \( I \) (resp. \( I_g \)) is the ideal given by the 3 \times 3 minors of \( A \) with \( u = 0 \) (resp. \( u = 1 \)), then \( (R/I) \in W_s(-3, -1, -1; -1, 0, 0, 0) \) and \( (R/I_g) \in W_s(-3, -1; 0, 0, 0) \) and we have minimal resolutions

\[
0 \rightarrow R(-10) \oplus R(-8) \oplus R(-6) \rightarrow R(-7)^4 \oplus R(-5)^4 \rightarrow R(-4)^6 \rightarrow I \rightarrow 0,
0 \rightarrow R(-10) \oplus R(-8)^2 \oplus R(-6)^3 \rightarrow R(-8) \oplus R(-7)^4 \oplus R(-6)^2 \oplus R(-5)^8 \rightarrow R(-5)^4 \oplus R(-4)^6 \rightarrow I.
\]

**Example 5.4.** (removing some ghost terms using Proposition 5.1)

Let \( R = k[x_0, x_1, x_2] \) and let \( A = (w^B) \) be as in Example 5.3 (ii) only replacing cubic by quadratic and \( w^T \) by \((0, x_0)\). Using Macaulay2 we get that the vanishing of all 3 \times 3 minors defines an artinian ring with \( h \)-vector \((1, 3, 6, 4, 1)\). If \( I \) (resp. \( I_g \)) is the ideal generated by the 3 \times 3 minors of \( A \) for \( u = 0 \) (resp. \( u = 1 \)), then \( (R/I) \in W_s(-2, -1; -1, 0, 0, 0) \) and \( (R/I_g) \in W_s(-2, -1; 0, 0, 0) \) and we have minimal resolutions \( \text{(5.1)} \)

\[
0 \rightarrow R(-7) \oplus R(-6) \oplus R(-5) \rightarrow R(-5)^4 \oplus R(-4)^4 \rightarrow R(-3)^6 \rightarrow I_g \rightarrow 0,
\]

\( \text{(5.2)} \)

\[
0 \rightarrow R(-7) \oplus R(-6)^2 \oplus R(-5)^3 \rightarrow R(-6) \oplus R(-5)^6 \oplus R(-4)^8 \rightarrow R(-4)^4 \oplus R(-3)^6 \rightarrow I \rightarrow 0.
\]

Observe that there is still a ghost term in the resolution of \( I_g \). In the artinian case, however, we have seen in Example 4.3 that \( W_s(-2, -1; 0, 0, 0, 0) \) is not an irreducible component of \( \text{GradAlg}(H) \). So there is a possibility for removing this ghost under a generalization to a non-determinantal artinian ring! Indeed M. Boij pointed out in a note he sent me that a general artinian algebra with Hilbert function \((1, 3, 6, 4, 1)\) can be seen as a type two algebra \( A \) given by the inverse system \((f, g)\) where \( \deg f = 4 \) and \( \deg g = 3 \). Following Boij’s note, this algebra has a minimal resolution as in \( \text{(5.1)} \) with the ghost term \( R(-5) \) removed and \( A \) is probably a generalization of \( R/I_g \).

On the other hand, we may very well consider the determinantal algebras \( R/I \) and \( R/I_g \) described above in the polynomial ring \( R = k[x_0, x_1, \ldots , x_n] \) for \( n \geq 5 \). This leads
to determinantal rings of dimension greater or equal to 3 with minimal resolutions exactly as in (5.1) and (5.2). Thanks to Theorem 4.1, see Remark 5.2, $W_5(-2, -1; 0, 0, 0, 0)$ is in this case an irreducible component of $\text{GradAlg}(H)$ (as well as of $\text{Hilb}^p(\mathbb{P}^n)$, up to closure), whence $R/I_9$ has no generization to non-determinantal rings and the ghost term $R(-5)$ of (5.1) remains a ghost term for the general element of the component.

Finally suppose $c = 2$. Then repeated use of Proposition 5.1 will remove all ghost terms in the minimal resolution of $I$ since the transpose of $A$ is the Hilbert-Burch matrix. This is rather well known (cf. [9] Thm. 2(iii))); indeed even more precise results are true, see [17] which finds the codimension of all “ghost terms strata” when $\dim A$ is small and $A$ is Cohen-Macaulay. Note that any codimension-2 Cohen-Macaulay quotient $R/I$ is determinantal, so Proposition 5.1 applies to such $R/I$. If $A = R/I$ is the homogeneous coordinate ring of a codimension-2 scheme $X$ in $\mathbb{P}^n$, then $A$ is Cohen-Macaulay for $\dim A = 1$ and there are no ghost terms in the minimal resolution of $I := I_X$ for $A$ general. If $X$ is a locally Cohen-Macaulay (lCM) curve in $\mathbb{P}^3$ and $A$ is not necessarily Cohen-Macaulay, then we can generalize the removal of ghost terms above for codimension-2 quotients to the following.

**Proposition 5.5.** [27] Thm. 2.8 Let $X \subset \mathbb{P}^3$ be any lCM curve and let $0 \to L_4 \xrightarrow{\sigma_3} L_3 \to \cdots \to M \to 0$ be a minimal R-free resolution of the Rao module $M := H^1_*(I_X)$. By [44] there is a minimal R-free resolution of the following form

$$0 \to L_4 \xrightarrow{\sigma_0} L_3 \oplus P_2 \to P_1 \to I_X \to 0$$

(i.e. the composition $L_4 \xrightarrow{\sigma_0} L_3 \oplus P_2 \to P_2$ is zero). If $P_1$ and $P_2$ have a common free summand; $P_2 = P_2' \oplus R(-i)$, $P_1 = P_1' \oplus R(-i)$, then there is a generization $X'$ of $X$ in $\text{GradAlg}(H)$ with minimal resolution

$$0 \to L_4 \xrightarrow{\sigma_0} L_3 \oplus P_2' \to P_1' \to I_X' \to 0.$$  

Note that the statement “a generization $X'$ of $X$ in $\text{Hilb}^p(\mathbb{P}^3)$ with constant postulation” in [27] Thm. 2.8 really means “a generization $X'$ of $X$ in $\text{GradAlg}(H)$”. This result applies to a codimension-2 Cohen-Macaulay quotient $R/I_X$ by letting $M = 0$, which implies $L_3 = L_4 = 0$, coinciding with Proposition 5.1 in this case.

### 6. Upgrading of previous results

In this section we generalize main theorems from [26, 31] by using Theorems 4.1 and 4.4 or more precisely we use that the two conjectures in [31] mentioned in the Introduction now are theorems (for $n - c > 0$). Indeed the previous proofs relied on the part of the conjectures which was proved in [30] at the time these papers were written, but we can, using mainly the “same” proofs as in [26, 31], get stronger results (i.e. the same conclusions under weaker assumptions). To better understand the proofs presented here, it may be a good idea to consider the corresponding proofs in [26] simultaneously.

We start, however, with [31]. The following theorem was proved in [31] Thm. 3.2 under some assumptions (mainly “$a_0 > b_i$, $a_{i+3} > a_{i-2}$ and $n - c \geq 0$”) and it was in [28] Cor. 5.7 shown to be true in general for $n - c \geq 1$ (under elsewhere weaker assumptions: $W(b, a) \neq \emptyset$ and $a_{i-2} \geq b_i$ for $i \geq 2$). For $n = c$, [31] Thm. 3.2 is still the best result we have with assumptions only on $a_j$ and $b_i$, and we can immediately generalize it to
Theorem 6.1. Let $X \subset \mathbb{P}^n$, $(X) \in W(\mathfrak{b}; a)$, be a general determinantal scheme and suppose $a_0 > b_t$ and $a_{t+c-2} > a_t-2$. Then
\[ \dim W(\mathfrak{b}; a) = \lambda_c + K_3 + K_4 + \cdots + K_c. \]

Proof. This follows from [31, Prop. 3.1], see Proposition 6.3(i) below, and [28, Cor. 5.7]. □

Now to generalize the main results of [26], we first recall [24, Prop. 3.4]. With notations as in Sec. 2 (i.e. we delete the last column of $A$ to get $A_{c-1}$ and the map $\varphi_{c-1}$ induced by the transpose of $A_{c-1}$), let $B := A_{c-1}$, $B := R/I(B)$ and $N := \text{coker } \varphi_{c-1}$. Since $A = R/I(A)$, we have $(A) \in W(\mathfrak{b}; a)$, $a = a_0, a_1, ..., a_{t+c-2}$ and $(B) \in W(\mathfrak{b}; a')$ where $a' = a_0, a_1, ..., a_{t+c-3}$ and we set $X = \text{Proj}(A)$ and $Y = \text{Proj}(B)$ in the following.

Proposition 6.2. Let $c \geq 3$, let $(X) \in W(\mathfrak{b}; a)$ and suppose $\dim W(\mathfrak{b}; a') \geq \lambda_{c-1} + K_3 + K_4 + \cdots + K_c$ (and $\geq \lambda_2$ if $c = 3$) and $\text{depth}_{t-1}(B) \geq 2$ for a general $Y = \text{Proj}(B)$ of $W(\mathfrak{b}; a')$. Then
\[ \text{hom}_R(I_Y, I_{X/Y}) \leq \sum_{j=0}^{t+c-3} (a_j - a_{t+c-3} + n) \]
then $\dim W(\mathfrak{b}; a) = \lambda_c + K_3 + K_4 + \cdots + K_c$. We also get equality in (6.1), as well as
\[ \dim W(\mathfrak{b}; a) = \dim W(\mathfrak{b}; a') + \dim_k N(a_{t+c-3})_0 - 1 - \text{hom}_R(I_Y, I_{X/Y}). \]

See [26, Prop. 3.4] for a proof. Then [31, Prop. 3.1] is mainly (i) in the following.

Proposition 6.3. (i) If $a_0 > b_t$ and $a_{t+c-2} > a_t-2$, then (6.1) holds for $X$ general.

(ii) If $\text{Ext}^i_B(N, B(-a_{t+c-2})) = 0$ for $i = 1$ and 2, then (6.1) holds. In particular (6.1) holds provided $\text{depth}_{t-1}(B) \geq 3$ (and $\text{Ext}^1_B(N, B) = 0$ holds provided $\text{depth}_{t-1}(B) \geq 2$).

Proof. (ii) The arguments we apply in the proof are very similar to those needed in (3.1), (3.2) and (3.3) to get related statements, except for we now use the spectral sequence $E^{p,q}_2 := \text{Ext}^p_B(\text{Tor}^R_q(B, N), B) \Rightarrow \text{Ext}^{p+q}(N, B)$ and its corresponding 5-term exact sequence. Indeed due to assumptions and (2.8) we get isomorphisms
\[ \text{Ext}^1_R(N, B) \simeq (E^{0,1}_2)_v \simeq \text{Hom}_R(I_Y, \text{Hom}_B(N, B)) \simeq \text{Hom}_R(I_Y, I_{X/Y}(a_{t+c-3})), \]
for $v = -a_{t+c-2}$. Moreover as in (3.3), cf. (2.15), we get for this $v$ the exact sequence
\[ a_{t+c-3} \rightarrow \text{Hom}_R(N, B) \rightarrow \text{Hom}_R(F^*, B) \rightarrow \text{Hom}_R(G_{c-1}^*, B) \rightarrow \text{Ext}^1_R(N, B) \rightarrow 0 \]
where $\text{Hom}_R(F, B) \simeq \bigoplus_{i=1}^t B_{(b_{i+v})} = 0$ and $\text{Hom}_R(G_{c-1}^*, B) \simeq \dim(\bigoplus_{j=0}^{t+c-3} B_{(a_j+v)}) = \sum_{j=0}^{t+c-3} (a_j - a_{t+c-3} + n)$. This implies (6.1).

The above arguments are used in [32, Rem. 3.4] and the proof of [32, Prop. 3.5] from which we get (6.1) under the assumption $\text{depth}_{t-1}(B) \geq 4$. To see that we can weaken the depth assumption to $\text{depth}_{t-1}(B) \geq 3$, we really need to refine the argument. Following, however, the proof of [28, Thm. 4.5] exactly as in the paragraph after (4.3) in [28], we get $\text{Ext}^i_B(N, B) = 0$ for $i = 1$ and 2 (resp. for $i = 1$) under the assumption $\text{depth}_{t-1}(B) \geq 3$ (resp. $\text{depth}_{t-1}(B) \geq 2$), and we are done. □
Remark 6.4. Using Propositions 6.2 and 6.3(ii) one may reprove the dimension formula $\dim W(b; a) = \lambda_c + K_3 + K_4 + \cdots + K_c$ of Theorem 4.11 in the case $n - c \geq 1$, $a_{i-1} \geq b_i$ for $i \geq 2$ and $W(b; a) \neq \emptyset$ by using the recursive strategy of successively deleting columns of $A$ from the right-hand side, see the Introduction. In [32] we pointed out that [32, Cor. 3.19] shows that the recursive strategy also applies to reprove the generic smoothness of $W(b; a)$ of Theorem 4.4 in the case $n - c \geq 2$ and $a_0 \geq b_t$ (due to Remark 2.4) the latter assumption may here be weakened to $a_{i-\min(3,t)} \geq b_t$ for $\min(3,t) \leq i \leq t$). This means that we have two rather different proofs for the two conjectures of [31].

With the above propositions we start by considering determinantal curves. In this and later results we denote by $\tau_{X/Y}$ the following morphism induced by $I_{X/Y} \hookrightarrow B$:

$$\tau_{X/Y} : \Ext^1_B(I_Y/I^3_Y, I_{X/Y}) \to \Ext^1_B(I_Y/I_Y, B).$$

A main result of [26] (Thm. 4.6) shows that if $\ker \tau_{X/Y} = 0$, $\mathrm{depth}_{I_{X/Y}} B \geq 3$ and $X$ good determinantal, then the property “every deformation of a determinantal scheme comes from deforming the matrix” is transferred from $Y$ to $X$. Recalling that $\dim W(b; a) = \lambda_c + K_3 + \cdots + K_c$ is now a theorem when $n - c = 1$, the following result generalizes [26, Prop. 4.15] to arbitrary $c \geq 3$. (For $c = 2$ we have a more complete result with stronger conclusions in Theorem 4.13).

Proposition 6.5. Let $X = \Proj(A)$, $A = R/I(A)$, be general in $W(b; a)$ and suppose $a_{i-\min(3,t)} \geq b_t$ for $\min(3,t) \leq i \leq t$, $c \geq 3$ and $\dim X = n - c = 1$. If $Y = \Proj(B)$ is defined by the vanishing of the maximal minors of the matrix obtained by deleting the last column of $A$, then the following statements hold:

1. If $\tau_{X/Y} : \Ext^1_B(I_Y/I^3_Y, I_{X/Y}) \to \Ext^1_B(I_Y/I_Y, B)$ is injective, then $X$ is unobstructed and $W(b; a)$ is a generically smooth irreducible component of $\Hilb^p(\mathbb{P}^n)$. Moreover every deformation of $X$ comes from deforming $A$.
2. If $\Ext^1_A(I_X/I^3_X, A) = 0$, then $X$ is unobstructed, and

$$\codim_{\Hilb^p(\mathbb{P}^n)} W(b; a) = \dim \ker \tau_{X/Y} = \Ext^1_B(I_{X/Y}, A).$$

3. We always have

$$\codim_{\Hilb^p(\mathbb{P}^n)} W(b; a) \leq \dim \ker \tau_{X/Y}.$$  \hspace{1cm} (6.3)

Moreover if $\Ext^1_B(I_{X/Y}, A) = 0$, then we have equality in (6.3) if and only if $X$ is unobstructed.

Proof. Thanks to Proposition 6.3(ii) the proof is the “same” as for [26, Prop. 4.15]. Indeed for (i) we apply Theorem 4.4 onto $W(b; a') \ni (Y)$ instead of the corresponding result of [26] which required $c - 1 \leq 4$ and we get (i) from [26, Thm. 4.6] and Lemma 2.8. For (ii) and (iii) we need the final dimension formula of Proposition 6.2 to use the proof of [26]. Using Proposition 6.3(ii) we get the mentioned dimension formula and we are done. \hfill \Box

Now we consider zero dimensional determinantal schemes ($n - c = 0$). In this case we also need to consider the morphism

$$\rho^1 := \rho^1_{X/Y} : \Ext^1_B(I_{X/Y}, I_{X/Y}) \to \Ext^1_B(I_{X/Y}, B).$$


induced by \( I_{X/Y} \to B \). In addition to \cite[Thm. 4.6]{26} which states that \( \ker \tau_{X/Y} = 0 \), \( \ker \rho^1 = 0 \) and \( \depth_{t_{r-1}(B)} B = 2 \) transfer the property “every deformation of a determinantal scheme comes from deforming the matrix” from \( Y \to X \), we need a variation of \cite[Thm. 4.6]{26} which don’t require \( \ker \rho^1 = 0 \) (see \cite[Prop. 4.13]{26} for the details). Using these results the proof for our next theorem is “the same” as the proof \cite[Thm. 4.19]{26} since we already have generalized \cite[Prop. 4.15]{26} to Proposition 6.5 of this paper to cover determinantal curves of arbitrary codimension \( c \geq 3 \). Note that we use Proposition 6.2 to get \( \dim W(\rho; a) = \lambda_c + K_3 + ... + K_c \) since Theorem 4.1 applies to \( (B) \in W(\rho; a') \) to find \( \dim W(\rho; a') \) where \( a' = a_0, a_1, ..., a_{t+c-3} \). Below we have denoted GradAlg\( H \) by Hilb\( H(\mathbb{P}^n) \) since \( \dim A = 1 \), cf. the text before (2.1). Indeed we have the following.

**Theorem 6.6.** Let \( X = \Proj(A) \), \( A = R/I(A) \), be general in \( W(\rho; a) \) and let \( Y = \Proj(B) \) and \( V = \Proj(C) \) be defined by the vanishing of the maximal minors of \( B \) and \( C \) respectively where \( B \) (resp. \( C \)) is obtained by deleting the last column of \( A \) (resp. \( B \)). Suppose \( \dim X = n - c = 0 \), \( a_{i-min(3,t)} \geq b_i \) for \( \min(3,t) \leq i \leq t \) and suppose that (6.1) (cf. Proposition 6.3) holds. Moreover suppose

either \( c = 3 \) or \([ c \geq 4 \) and \( \ker \tau_{Y/V} = 0 \].

Then \( \dim W(\rho; a) = \lambda_c + K_3 + ... + K_c \) and the following statements are true:

(i) If both \( \rho^1 : 0\Ext^1_B(I_{X/Y}, I_{X/Y}) \to 0\Ext^1_B(I_{X/Y}, B) \) and \( \tau_{X/Y} : 0\Ext^1_B(I_{Y/I}^2, I_{X/Y}) \to 0\Ext^1_B(I_Y, I_{Y/2}, I_{X/Y}) \) are injective, then \( A \) is unobstructed and \( W(\rho; a) \) is a generically smooth irreducible component of Hilb\( H(\mathbb{P}^n) \). Moreover every deformation of \( X \) (or \( A \)) comes from deforming \( A \).

(ii) If \( 0\Ext^1_B(I_{X/Y}, A) = 0 \) and \( \ker \tau_{X/Y} = 0 \), then \( W(\rho; a) \) belongs to a unique generically smooth irreducible component \( Q \) of Hilb\( H(\mathbb{P}^n) \) and the codimension of \( W(\rho; a) \) in Hilb\( H(\mathbb{P}^n) \) is \( \dim \ker \rho^1 \). Indeed \( A \) is unobstructed and

\[
\dim Q = \lambda_c + K_3 + ... + K_c + \dim \ker \rho^1.
\]

(iii) If \( 0\Ext^1_A(I_X/I_X^2, A) = 0 \), then \( A \) is unobstructed and

\[
\text{codim}_{\text{Hilb}_H(\mathbb{P}^n)} W(\rho; a) = \dim \ker \rho^1 + \dim \ker \tau_{X/Y} - 0\Ext^1_B(I_{X/Y}, A).
\]

(iv) We always have \( \text{codim}_{\text{Hilb}_H(\mathbb{P}^n)} W(\rho; a) \leq \dim \ker \rho^1 + \dim \ker \tau_{X/Y} \).

Suppose \( 0\Ext^1_B(I_{X/Y}, A) = 0 \). Then we have

\[
\text{codim}_{\text{Hilb}_H(\mathbb{P}^n)} W(\rho; a) = \dim \ker \rho^1 + \dim \ker \tau_{X/Y}
\]

if and only if \( A \) is unobstructed.

**Remark 6.7.** Looking at the proofs we see that we don’t need to suppose (6.1) to get (the part of the) conclusions of (i) and (ii) that don’t involve dimension and codimension formulas. Moreover note the overlap in (ii) and (iv) of the theorem.

Using again the variation \cite[Prop. 4.13]{26} we generalize \cite[Prop. 4.24]{26} to the following.

**Proposition 6.8.** With notations as in the first sentence of Theorem 6.6, suppose \( c \geq 4 \), \( \dim X = n - c = 0 \) and \( X \) general, \( a_{i-min(3,t)} \geq b_i \) for \( \min(3,t) \leq i \leq t \) and suppose that (6.1) holds. Then \( \dim W(\rho; a) = \lambda_c + K_3 + ... + K_c \) and the following statements are true:
(i) If $\langle 0 \rangle^1_{\mathbb{P}^c}(I_{X/Y}, A) = 0$, $\langle 0 \rangle^1_{\mathbb{P}^c}(I_Y / I_X^2, I_{X/Y}) = 0$ and $\langle 0 \rangle^1_{\mathbb{P}^c}(I_Y / I_X^2, B) = 0$ then $A$ is unobstructed. Moreover $W^H(\mathbb{P}^c)$ is contained in a unique generically smooth irreducible component of $\text{Hilb}^H(\mathbb{P}^c)$ of codimension $\dim \ker \rho^{1}_{X/Y} + \dim \ker \tau^1_{X/Y} + \dim \ker \tau^1_{Y/V}$.

(ii) We always have $\text{codim}_{\text{Hilb}^H(\mathbb{P}^c)} W^H(\mathbb{P}^c) = \dim \ker \rho^{1}_{X/Y} + \dim \ker \tau^1_{X/Y} + \dim \ker \tau^1_{Y/V}$. Suppose $\langle 0 \rangle^1_{\mathbb{P}^c}(I_{X/Y}, A) = 0$ and $\langle 0 \rangle^1_{\mathbb{P}^c}(I_Y / I_X^2, B) = 0$. Then we have

$$\text{codim}_{\text{Hilb}^H(\mathbb{P}^c)} W^H(\mathbb{P}^c) = \dim \ker \rho^{1}_{X/Y} + \dim \ker \tau^1_{X/Y} + \dim \ker \tau^1_{Y/V}$$

if and only if $A$ is unobstructed (e.g. $\langle 0 \rangle^1_{\mathbb{P}^c}(I_X / I_X^2, A) = 0$).

Proof. Again we have $\dim W^H(\mathbb{P}^c) = \lambda + K_3 + ... + K_c$ by Proposition 6.2. Since Proposition 6.5 applies for every $c \geq 3$ we can use the proof of [26, Prop. 4.24] to get the Proposition 6.8. Indeed we use Proposition 6.5(ii) and [26, Prop. 4.13] to get (i) above while we use the proof of Proposition 6.5(iii) to get (ii) (see [26] for details). □

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