On comatrix corings and bimodules

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Abstract

To any bimodule which is finitely generated and projective on one side one can associate a coring, known as a comatrix coring. A new description of comatrix corings in terms of data reminiscent of a Morita context is given. It is also studied how properties of bimodules are reflected in the associated comatrix corings. In particular it is shown that separable bimodules give rise to coseparable comatrix corings, while Frobenius bimodules induce Frobenius comatrix corings.

1 Introduction

One of the first and most fundamental examples of corings is provided by the canonical coring of Sweedler [16] which can be associated to any ring extension \( B \to A \). The structure of the canonical coring detects whether such an extension is separable, split or Frobenius. Comodules of this coring provide one with an equivalent description of the descent theory for an extension \( B \to A \). In recent paper [10] it has been realised that Sweedler’s canonical corings are special examples of more general class of corings termed comatrix corings. A comatrix \( A \)-coring can be associated to any \((B, A)\)-bimodule \( M \) provided \( M \) is a finitely generated projective right \( A \)-module. It is natural to expect that such a coring should reflect properties of module \( M \) in a way similar to the relationship between properties of ring extensions and those of corresponding canonical corings.

The aim of this paper it to study properties of comatrix corings in relation to properties of bimodules. In particular we show that the dual \((A, B)\)-bimodule \( M^* \) is a separable bimodule if and only if the corresponding comatrix coring is a cosplit coring. On the other hand if \( M \) is a separable (resp. Frobenius) bimodule then the comatrix coring is coseparable (resp. Frobenius) coring. The converse holds provided

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certain faithful flatness condition is satisfied. Since for any \((B, A)\)-bimodule \(M\) one can consider a ring extension \(B \to S\), where \(S\) is the right endomorphism ring of \(M\), there is also associated canonical Sweedler’s coring. We study how the above properties of a comatrix coring are reflected by the properties of corresponding Sweedler’s coring. This coring formulation of properties of modules can shed new light on module-theoretic conjectures such as the Caenepeel-Kadison conjecture on biseparable and Frobenius extensions [17] (cf. [5] for a coring formulation of the problem).

The paper is organised as follows. In Section 2 we give a new formulation of comatrix corings in terms of algebraic data which are very similar (semi-dual) to Morita contexts. This formulation of comatrix corings puts them in a broader perspective of established algebraic theories and can suggest new applications to, for instance, K-theory. In Section 3 we study the properties of a comatrix coring associated to a \((B, A)\)-bimodule \(M\) in relation to module properties of \(M\).

**Notation and preliminaries.** Throughout the paper, \(A\) and \(B\) are associative rings with 1. For modules we use the standard module theory notation, for example a \((B, A)\)-bimodule \(M\) is often denoted by \(B M_A\), \(\text{Hom}_{−, −}(−, −)\) denotes the Abelian group of right \(A\)-module maps, \(\text{Hom}_{B−}(−, −)\) denotes left \(B\)-module maps etc. The left endomorphisms of \(M\) are denoted by \(\text{End}_{B−}(M)\) and their ring structure is provided by opposite composition of maps. Similarly, the right endomorphisms of \(M\) are denoted by \(\text{End}_{−A}(M)\) and their ring structure is provided by composition of maps. The dual of a right \(A\)-module \(M\) is denoted by \(M^*\), while the dual of a left \(A\)-module \(N\) is denoted by \(^*N\). The identity morphism of \(M\) is denoted also by \(M\).

An \((A, A)\)-bimodule \(C\) is called an \(A\)-coring provided there exist \((A, A)\)-bimodule maps \(Δ_C : C → C ⊗_A C\) and \(ε_C : C → A\) such that

\[
(Δ_C ⊗_A C) ◦ Δ_C = (Δ_C ⊗_C C) ◦ Δ_C, \quad (ε_C ⊗_C C) ◦ Δ_C = (C ⊗_C ε_C) ◦ Δ_C = C.
\]

The map \(Δ_C\) is known as a coproduct or comultiplication, while \(ε_C\) is called a counit. To indicate the action of \(Δ_C\) we use the Sweedler sigma notation, i.e., for all \(c ∈ C\),

\[
Δ_C(c) = \sum c_{(1)} ⊗ c_{(2)}, \quad (Δ_C ⊗ C) ◦ Δ_C(c) = (C ⊗ Δ_C) ◦ Δ_C(c) = \sum c_{(1)} ⊗ c_{(2)} ⊗ c_{(3)},
\]

etc. A morphism of \(A\)-corings is an \((A, A)\)-bimodule map \(ϕ : C → D\) such that \((ϕ ⊗ ϕ) ◦ Δ_C = Δ_D ◦ ϕ\) and \(ε_D ◦ ϕ = ε_C\). Given any \(A\)-coring \(C\), its left dual \(^*C\) is a ring with the multiplication

\[
(ϕϕ')(c) = \sum ϕ(c_{(1)})ϕ'(c_{(2)})), \quad \forall ϕ, ϕ' ∈ ^*C, c ∈ C,
\]

and the unit \(ε_C\).

Given any ring extension \(B → A\), the \((A, A)\)-bimodule \(C = A ⊗_B A\) is an \(A\)-coring with the coproduct

\[
Δ_C : A ⊗_B A → A ⊗_B A ⊗_A A ⊗_B A \cong A ⊗_B A ⊗_B A, \quad a ⊗ a' → a ⊗ 1_A ⊗ a',
\]

and the counit \(ε_C : A ⊗_B A → A, a ⊗ a' → aa'\) [16]. Through the natural identification \(\text{Hom}_{A−}(A ⊗_B A, A) \cong \text{End}_{B−}(A)\), the left dual ring of \(C\) is anti-isomorphic to the
endomorphism ring \( \text{End}_{B-}(A) \). The coring \( \mathcal{C} \) is known as a **Sweedler’s \( A \)-coring** associated to a ring extension \( B \to A \). This is the most fundamental example of a coring, thus it is often termed a **canonical coring**. For other examples of corings and further details about their structure and properties we refer to [3, 10] and to forthcoming monograph [6].

2 Comatrix corings from contexts

Let \( M \) be a \((B, A)\)-bimodule such that \( M_A \) is finitely generated and projective module. Denote by \( M^* = \text{Hom}_-(M, A) \) is the dual \((A, B)\)-bimodule and let \( \{e_i \in M, e_i^* \in M^*\}_{i \in I} \) be a finite dual basis of \( M \). Then the \((A, A)\)-bimodule \( M^* \otimes_B M \) is an \( A \)-coring with the coproduct

\[
\Delta_{M^* \otimes_B M} : M^* \otimes_B M \to M^* \otimes_B M \otimes_A M^* \otimes_B M, \quad \varphi \otimes m \mapsto \sum_{i \in I} \varphi \otimes e_i \otimes e_i^* \otimes m,
\]

and the counit

\[
\varepsilon_{M^* \otimes_B M} : M^* \otimes_B M \to A, \quad \varphi \otimes m \mapsto \varphi(m).
\]

The coring \( M^* \otimes_B M \) is known as a **comatrix \( A \)-coring** [10]. Note that the definition of the coproduct does not depend on the choice of a dual basis (cf. [10, Remark 1]). In this section we show that comatrix corings can be understood in terms of data very reminiscent of Morita contexts in the classical module theory.

**Definition 2.1.** Given a pair of algebras \( A, B \), a **comatrix coring context** consists of an \((A, B)\)-bimodule \( N \), a \((B, A)\)-bimodule \( M \) and a pair of bimodule maps

\[
\sigma : N \otimes_B M \to A, \quad \tau : B \to M \otimes_A N,
\]

such that the following diagrams commute. A comatrix coring context is denoted by \((A, B, A N_B, B M_A, \sigma, \tau)\).

**Example 2.2.** Suppose that \( A N_B \) and \( B M_A \) together with bimodule maps \( \sigma : N \otimes_B M \to A, \quad \hat{\tau} : M \otimes_A N \to B \) form a Morita context. Suppose that \( \hat{\tau} \) is surjective. By standard arguments in Morita theory (cf. [2, Ch. II.3]) one proves that \( \hat{\tau} \) is bijective, and let \( \tau \) be the inverse of \( \hat{\tau} \). Then \((A, B, A N_B, B M_A, \sigma, \tau)\) is a comatrix context.

**Proof.** This follows immediately from the definition of a Morita context. \( \square \)

Example 2.2 justifies the use of the term **context** in Definition 2.1. The following example explains the appearance of words **comatrix** and **coring**.

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Example 2.3. Let $M$ be a $(B, A)$-bimodule such that $M_A$ is finitely generated and projective module with a dual basis $\{e_i, e_i^*\}_{i \in I}$. Define $\tau : B \to M \otimes_A M^*$ by $b \mapsto \sum_{i \in I} b e_i \otimes e_i^* = \sum_{i \in I} e_i \otimes e_i^* b$. Then

$$(A, B, A M^*_B, B M_A, \varepsilon_{M^* \otimes B M}, \tau)$$

is a comatrix coring context.

Proof. This follows immediately from the properties of a dual basis. \(\square\)

Example 2.3 has the following converse, which constitutes the main result of this section.

Theorem 2.4. Let $(A, B, A N_B, B M_A, \sigma, \tau)$ be a comatrix coring context. Let $e = \tau(1_B) \in M \otimes_A N$. Then

(1) $M$ is a finitely generated and projective right $A$-module and $A N_B$ is isomorphic to $M^*$.

(2) $\mathcal{C} = N \otimes_B M$ is an $A$-coring with the coproduct

$$\Delta_\mathcal{C} : \mathcal{C} \to \mathcal{C} \otimes_A \mathcal{C}, \quad n \otimes m \mapsto n \otimes e \otimes m,$$

and counit $\varepsilon_\mathcal{C} = \sigma$.

(3) The coring $\mathcal{C}$ is isomorphic to the comatrix coring $M^* \otimes_B M$.

Proof. (1) Write $e = \sum_{i \in I} m_i \otimes n_i$. Since $e = \tau(1_B)$, the second of the diagrams in Definition 2.1 implies that for all $m \in M$, $m = \sum_i m_i \sigma(n_i \otimes m)$, i.e., $M_A$ has a finite dual basis $\{m_i, \sigma(n_i \otimes -)\}_{i \in I}$. Therefore $M_A$ is a finitely generated and projective module. Furthermore, the $(A, B)$-bimodule map $\chi : N \to M^*$, $n \mapsto \sigma(n \otimes -)$ is an $(A, B)$-bimodule isomorphism with the inverse $\chi^{-1} : M^* \to N$, $\varphi \mapsto \sum_i \varphi(m_i)n_i$. Indeed, take any $n \in N$ and $m \in M$ and use the first of the diagrams in Definition 2.1 to compute

$$(\chi^{-1} \circ \chi)(n \otimes m) = \sum_i \sigma(n \otimes m_i) n_i \otimes m = n \otimes m.$$

Similarly take any $\varphi \in M^*$ and $n \in M$, and use the facts that $\sigma$ is $(A, A)$-bilinear and that $\{m_i, \sigma(n_i \otimes -)\}$ is a dual basis to compute

$$(\chi \circ \chi^{-1})(\varphi \otimes m) = \sum_i \sigma(\varphi(m_i)n_i \otimes -) \otimes m = \sum_i \varphi(m_i) \sigma(n_i \otimes -) \otimes m = \varphi \otimes m,$$

as required.

(2) Note that $\Delta_\mathcal{C}$ is well-defined since the fact that $\tau$ is a $(B, B)$-bimodule map implies that $e$ is $B$-invariant, i.e., $e \in (M \otimes_A N)^B = \{x \in M \otimes_A N \mid \forall b \in B, bx = xb\}$. $\Delta_\mathcal{C}$ is coassociative directly from its definition. Finally, the counit properties of $\varepsilon_\mathcal{C} = \sigma$ follow from the commutative diagrams in Definition 2.1. For example

$$(\varepsilon_\mathcal{C} \otimes \varepsilon_\mathcal{C}) \circ \Delta_\mathcal{C}(n \otimes m) = n \otimes (M \otimes \sigma)(e \otimes m) = n \otimes (M \otimes \sigma)(\tau(1) \otimes m) = n \otimes m,$$

be the second of these diagrams. This proves that $\mathcal{C}$ is an $A$-coring.
(3) Write $e = \sum_i m_i \otimes n_i$, and let $\chi$ be the $(A, B)$-bimodule isomorphism constructed in (1). The induced map $\vartheta = \chi \otimes M : N \otimes_B M \to M^* \otimes_B M$, $n \otimes m \mapsto \sigma(n \otimes -) \otimes m$ is then an isomorphism of $(A, A)$-bimodules. Note that for all $m \in M$ and $n \in N$, $\varepsilon_M \otimes_B M(\vartheta(n \otimes m)) = \sigma(n \otimes m) = \varepsilon_M(n \otimes m)$. Furthermore,

\[ (\vartheta \otimes \vartheta)(\Delta_M(n \otimes m)) = \sum_i \vartheta(n \otimes m_i) \otimes \vartheta(n_i \otimes m) \]

\[ = \sum_i \sigma(n \otimes -) \otimes m_i \otimes \sigma(n_i \otimes -) \otimes m = \Delta_M \otimes_B M(\vartheta(n \otimes m)), \]

since the definition of the coproduct in a comatrix coring does not depend on the choice of a dual basis. Thus we conclude that $\vartheta$ is a morphism of $A$-corings.

The inverse of $\vartheta$ is given by $\chi^{-1} \otimes_B M$ and comes out as

\[ \vartheta^{-1} : M^* \otimes_B M \to N \otimes_B M, \quad \varphi \otimes m \mapsto \sum_i \varphi(m_i) n_i \otimes m. \]

The fact that $\sigma$ is an $(A, A)$-bimodule map and the second of the diagrams in Definition 2.4 facilitate the following calculation for all $\varphi \in M^*$ and $m \in M$

\[ \sigma(\sum_i \varphi(m_i) n_i \otimes m) = \varphi(\sum_i m_i \sigma(n_i \otimes m)) = \varphi(m). \]

This means that $\varepsilon_M \circ \vartheta^{-1} = \varepsilon_M \otimes_B M$. Furthermore

\[ (\vartheta^{-1} \otimes \vartheta^{-1})(\Delta_M \otimes_B M(\varphi \otimes m)) = \sum_i \vartheta^{-1}(\varphi \otimes m_i) \otimes \vartheta^{-1}(\sigma(n_i \otimes -) \otimes m) \]

\[ = \sum_{i,j,k} \varphi(m_j) n_j \otimes m_i \otimes \sigma(n_i \otimes m_k) n_k \otimes m \]

\[ = \sum_{i,j} \varphi(m_j) n_j \otimes m_i \otimes n_i \otimes m \]

\[ = \Delta_M(\vartheta^{-1}(\varphi \otimes m)), \]

where the third equality follows from the first of the diagrams in Definition 2.4. Thus $\vartheta^{-1}$ is also an $A$-coring morphism. Consequently, $\vartheta$ is an $A$-coring isomorphism and we conclude that the coring $C$ is isomorphic to the comatrix coring $M^* \otimes_B M$ as asserted.

In view of Example 2.3, Theorem 2.4 asserts that comatrix coring contexts provide one with an equivalent description of comatrix corings. As an immediate consequence of Theorem 2.4, we also obtain the following description of a left dual ring of the coring associated to a comatrix coring context.

**Corollary 2.5.** Let $(A, B, A N_B, B M_A, \sigma, \tau)$ be a comatrix coring context and let $C = N \otimes_B M$ be the associated $A$-coring. Then the ring $^*C$ is anti-isomorphic to the endomorphism ring $\text{End}_{B^-(M)}$.

**Proof.** Since the coring $C$ is isomorphic to the comatrix coring $M^* \otimes_B M$ there is a ring isomorphism $^*C \cong (M^* \otimes_B M)$. The latter ring is anti-isomorphic to $\text{End}_{B^-(M)}$ by [10] Proposition 1. □
3 Comatrix corings of separable and Frobenius bimodules

This section is devoted to studies of relationship between properties of comatrix corings and the following two notions from the classical module theory. Let $M$ be a $(B, A)$-bimodule. Following Sugano [15] (cf. [7]), $M$ is called a separable bimodule or $B$ is said to be $M$-separable over $A$ provided the evaluation map

$$M \otimes_A^* M \to B, \quad m \otimes \varphi \mapsto \varphi(m),$$

is a split epimorphism of $(B, B)$-bimodules. Following [1] and [12] a bimodule $BM_A$ is said to be Frobenius if both $BM$ and $M_A$ are finitely generated and projective and $^*M \cong M^*$ as $(A, B)$-bimodules.

These properties of a bimodule $BM_A$ lead to corresponding properties of the ring extension $B \to S = \text{End}_{-A}(M)$, $b \mapsto [m \mapsto bm]$. As shown in [15] (cf. [13] Theorem 3.1), if $M$ is a separable bimodule, then $B \to S$ is a split extension, i.e., there exists a $B$-bimodule map $s : S \to B$ such that $s(1_S) = 1_B$. Conversely, if $BM_A$ is such that $M_A$ is finitely generated projective, and $B \to S$ is a split extension, then $BM_A$ is a separable bimodule. Furthermore, the endomorphism ring theorem (cf. [12] Theorem 2.5) asserts that if $BM_A$ is a Frobenius bimodule, then $B \to S$ is a Frobenius extension, i.e., $S_B$ is finitely generated projective and $S^* \cong S$ as $(B, S)$-bimodules.

Before we begin the discussion of the relationship of module properties of $M$ and the properties of the corresponding comatrix coring we make the following clarifying

Remark 3.1. Let $BM_A$ be a bimodule with $M_A$ finitely generated and projective, and consider its right endomorphism ring $S = \text{End}_{-A}(M)$. Then there is a canonical isomorphism of $S$-bimodules $M \otimes_A M^* \cong S$ which sends a simple tensor $m \otimes \varphi \in M \otimes_A M^*$ to the endomorphism $x \mapsto m\varphi(x)$. Its inverse is given by the assignment $s \mapsto \sum_i s(e_i) \otimes e_i^*$, where $\{e_i, e_i^*\}_{i\in I}$ is a finite dual basis. From now on, we always identify $M \otimes_A M^*$ and $S$. With this identification, the product in the ring $S$ (the composition) obeys the following rules: given $s \in S, m \otimes \varphi, m' \otimes \varphi' \in M \otimes_A M^*$,

$$s(m \otimes \varphi) = s(m) \otimes \varphi,$$

$$(m \otimes \varphi)s = m \otimes \varphi s,$$

$$(m \otimes \varphi)(m' \otimes \varphi') = m\varphi(m') \otimes \varphi' = m \otimes \varphi(m') \varphi'.$$

In this case we can consider a comatrix $A$-coring $M^* \otimes_B M$. Furthermore, since there is a ring map $B \to S$, there is also canonical Sweedler’s $S$-coring $S \otimes_B S$. These are the corings which reflect the structure of $M$, and thus they will be of special interest in this section.

An $A$-coring $C$ is said to be cosplit provided there exists an $(A, A)$-bimodule section of the counit, i.e., iff $\varepsilon_C$ is a split epimorphism of $A$-bimodules (cf. [6] 26.14). The following theorem provides one with the complete description of separability of the dual module $AM_B^*$. 


**Theorem 3.2.** Let $BM_A$ be a bimodule such that $MA$ is a finitely generated projective, and let $S = \text{End}_A(M)$. Then

1. $AM_B$ is a separable bimodule if and only if the comatrix coring $M^* \otimes_B M$ is a cosplit $A$-coring.

2. If the comatrix coring $M^* \otimes_B M$ is a cosplit $A$-coring then Sweedler’s coring $S \otimes_B S$ is a cosplit $S$-coring.

**Proof.**

(1) $M^*$ is a separable bimodule if and only if the evaluation map $M^* \otimes_B \ast(M^*) \to A$ is a split epimorphism of $A$-bimodules. Using the natural isomorphism $\ast(M^*) \cong M$ the evaluation map coincides with the counit $\varepsilon_{M^* \otimes_B M} : M^* \otimes_B M \to A$.

(2) Since $M^* \otimes_B M$ is a cosplit coring, there is an $A$-bimodule map $e : A \to M^* \otimes_B M$ such that $e \ast_{M^* \otimes_B M} = A$. Now use the correspondence between $S$ and $M \otimes_A M^*$ discussed in Remark 3.1 and define $\bar{e} : S \to S \otimes_B S$ as the composite

$$S = M \otimes_A M^* \cong M \otimes_A A \otimes A M^* \xrightarrow{M \otimes \ast M^*} M \otimes_A M^* \otimes_B M \otimes_A M^* = S \otimes_B S.$$

Clearly $\bar{e}$ is an $S$-bimodule map. We need to prove that $\bar{e}$ is a splitting of the counit of the canonical coring $S \otimes_B S$. Recall that the counit $\varepsilon_{S \otimes B S}$ is simply the multiplication map $S \otimes_B S \to S$. Write $e(1_A) = \sum_{e_{\alpha}} w_{\alpha} \otimes w_{\alpha} \in M^* \otimes_B M$, and note that $\sum_{e_{\alpha}} w_{\alpha}(w_{\alpha}) = 1_A$. Identify $1_S$ with $\sum_{e_{\alpha}} e_{i} \otimes A e_{i}^*$. Then $\bar{e}(1_S) = \sum_{e_{\alpha}} e_{i} \otimes A e_{i}^* \otimes w_{\alpha} \otimes e_{i}^*$ and, therefore, the multiplication map evaluated at $\bar{e}(1_S)$ gives

$$\sum_{i, \alpha} (e_{i} \otimes w_{\alpha}^*) (w_{\alpha} \otimes e_{i}^*) = \sum_{i, \alpha} e_{i} w_{\alpha} (w_{\alpha}) \otimes e_{i}^* = \sum_{i} e_{i} \otimes e_{i}^* = 1_S.$$

Since $\bar{e}$ is an $S$-bimodule map, we deduce that it splits the counit of $S \otimes_B S$, i.e., $S \otimes_B S$ is a cosplit $S$-coring.

**Definition 3.3.** Given an $A$-coring $C$, an $A$-bimodule map $\gamma : C \otimes_A C \to A$ such that for all $c, c' \in C$,

$$\sum c(1) \gamma(c(2) \otimes c') = \sum \gamma(c \otimes c') c'(1)$$

is called a pre-cointegral.

**Lemma 3.4.** If the comatrix coring $M^* \otimes_B M$ has a pre-cointegral $\gamma$, then the composite map $\tilde{\gamma}$ given by

$$M \otimes_A M^* \otimes_B M \otimes_A M^* \otimes_B M \otimes_A M^* \xrightarrow{M \otimes \gamma \otimes M^*} M \otimes_A A \otimes_A M^* \cong M \otimes_A M^*$$

is a pre-cointegral for $S \otimes_B S$.

**Proof.** Note that we implicitly identify $S$ with $M \otimes_A M^*$ as in Remark 3.1. Obviously, $\tilde{\gamma} : S \otimes_B S \otimes_B S \to S$ is an $S$-bimodule map. Furthermore for all $s, s', s'' \in S$ we
compute
\[ s \otimes \tilde{\gamma}(1_S \otimes s' \otimes s'') = \sum_{i,k} s \otimes \tilde{\gamma}(e_i \otimes e_i^* \otimes s' \otimes s''(e_k) \otimes e_k^*) \]
\[ = \sum_{i,k} s \otimes e_i \gamma(e_i^* \otimes s' \otimes s''(e_k)) \otimes e_k^* \]
\[ = \sum_{i,j,k} s(e_j) \otimes e_j^* \otimes e_i \gamma(e_i^* \otimes s' \otimes s''(e_k)) \otimes e_k^* \]
\[ = \sum_{i,j,k} s(e_j) \otimes e_j^* \otimes s'' \otimes e_i \gamma(e_i^* \otimes s' \otimes e_i^*) \]
\[ = \sum_{i,j} \tilde{\gamma}(s(e_j) \otimes e_j^* \otimes s' \otimes e_i \otimes e_i^*) \otimes s'' = \tilde{\gamma}(s \otimes s' \otimes 1_S) \otimes s''. \]

Here the identification in Remark 3.1 has been used in derivation of the first, third, fifth and seventh equalities. The fourth equality follows from the fact that \( \gamma \) is a pre-cointegral. In view of the definition of a coproduct in Sweedler’s coring this proves that \( \gamma \) is a pre-cointegral.

A pre-cointegral \( \gamma \) is called a **cointegral** provided \( \gamma \circ 1 = \varepsilon_C \). Following [11], an \( A \)-coring \( C \) is called a **coseparable coring** provided its coproduct \( 1_C \) is a split monomorphism of \((C, C)\)-bicomodules. Equivalently, by [3, Theorem 3.5, Corollary 3.6] an \( A \)-coring is a coseparable coring provided it has a cointegral. Coseparable corings turn out to correspond to separable bimodules.

**Theorem 3.5.** Let \( B M_A \) be a bimodule such that \( M_A \) is a finitely generated projective, and let \( S = \text{End}_{-A}(M) \). Then

1. If \( M \) is a separable bimodule, then the comatrix coring \( M^* \otimes_B M \) is a coseparable \( A \)-coring.
2. If a comatrix coring \( M^* \otimes_B M \) is a coseparable \( A \)-coring then \( S \otimes_B S \) is a coseparable \( S \)-coring.

**Proof.** (1) Since \( B M_A \) is separable, \( B \to S \) is a split extension (cf. [13, Theorem 3.1]). Let \( s : S \to B \) be a \( B \)-bimodule splitting of the unit map. With the identification in Remark 3.1, this means that \( s(\sum e_i \otimes e_i^*) = 1_B \). Define \( \gamma : M^* \otimes_B M \otimes_A M^* \otimes_B M \to A \) as the composite map

\[ M^* \otimes_B M \otimes_A M^* \otimes_B M \xrightarrow{M^* \otimes_B s \otimes_B M} M^* \otimes_B B \otimes_B M \cong M^* \otimes_B M \xrightarrow{\varepsilon M^* \otimes_B M} A. \]

Clearly, \( \gamma \) is a homomorphism of \( A \)-bimodules. We need to prove that \( \gamma \) is a cointegral
for $M^* \otimes_B M$. Given $\varphi \otimes m, \varphi' \otimes m' \in M^* \otimes_B M$,
\[
\sum_i (\varphi \otimes e_i)\gamma(e_i^* \otimes m \otimes \varphi' \otimes m') = \sum_i (\varphi \otimes e_i)e_i^*(s(m \otimes \varphi')(m')) \\
= \varphi \otimes_B s(m \otimes \varphi')(m') \\
= \varphi s(m \otimes \varphi') \otimes_B m' \\
= \sum_i \varphi(s(m \otimes \varphi')(e_i))(e_i^* \otimes m') \\
= \sum_i \gamma(\varphi \otimes m \otimes \varphi' \otimes e_i)(e_i^* \otimes m'),
\]
where the second equality follows from the dual basis property. Furthermore
\[
(\gamma \circ \Delta_{M^* \otimes_B M})(\varphi \otimes m) = \sum_i \gamma(\varphi \otimes e_i \otimes e_i^* \otimes m) = \sum_i \varphi(s(e_i \otimes e_i^*))(m)) \\
= \varphi(m) = \varepsilon_{M^* \otimes_B M}(\varphi \otimes m).
\]
Thus $\gamma$ is a cointegral in $M^* \otimes_B M$, i.e., the comatrix coring $M^* \otimes_B M$ is coseparable as required.

(2) Suppose that $M^* \otimes_B M$ is coseparable, and let $\gamma$ be the corresponding cointegral. We aim to show that the pre-cointegral $\tilde{\gamma}$ in the canonical coring $S \otimes_B S$ constructed in Lemma 3.1 is a cointegral. In view of the definition of the coproduct and counit in Sweedler’s coring this is equivalent to showing that for all $s, s' \in S$, $\tilde{\gamma}(s \otimes 1_S \otimes s') = ss'$. We freely use the identification of $S$ with $M \otimes_A M^*$ described in Remark 3.1 to compute
\[
\tilde{\gamma}(s \otimes 1_S \otimes s') = \sum_{i,j,k} \tilde{\gamma}(s(e_j) \otimes e_j^* \otimes e_i \otimes e_i^* \otimes s'(e_k) \otimes e_k^*) \\
= \sum_{i,j,k} s(e_j)\gamma(e_j^* \otimes e_i \otimes e_i^* \otimes s'(e_k)) \otimes e_k^* \\
= \sum_{j,k} s(e_j)\gamma(\Delta_{M^* \otimes_B M}(e_j^* \otimes s'(e_k))) \otimes e_k^* \\
= \sum_{j,k} s(e_j)e_j^*(s'(e_k)) \otimes e_k^* = \sum_k s(s'(e_k)) \otimes e_k = ss',
\]
as required. Note that the fourth equality follows from the fact that the pre-cointegral $\gamma$ is a cointegral. Therefore we conclude that $\tilde{\gamma}$ is a cointegral for $S \otimes_B S$, i.e., the canonical coring is coseparable as asserted.

Note that Theorem 3.5 implies in particular that if $BMA$ is a separable bimodule, then $S \otimes_B S$ is a coseparable coring. This also follows from [13 Theorem 3.1(1)] and [6 26.10] (the latter is a refinement of [3 Corollary 3.7]). Theorem 3.5 leads to a more complete description of the relationship between separable bimodules and coseparable comatrix corings in the case of a faithfully flat extension $B \to S$.

**Corollary 3.6.** Let $BMA$ be a bimodule such that $MA$ is a finitely generated projective, and let $S = \text{End}_{-A}(M)$. If either $BS$ or $SB$ is faithfully flat then the following statements are equivalent
(a) $M$ is a separable bimodule.
(b) The comatrix coring $M^* \otimes_B M$ is a coseparable $A$-coring.
(c) $S \otimes_B S$ is a coseparable $S$-coring.

Proof. The implications $(a) \Rightarrow (b) \Rightarrow (c)$ are contained in Theorem 3.5. Suppose that $S \otimes_B S$ is a coseparable $S$-coring. In view of the faithful flatness, $B \to S$ is a split extension by [3, Corollaries 3.6, 3.7]. Since $M_A$ is finitely generated projective, [13, Theorem 3.1(2)] implies that $M$ is a separable bimodule. This proves the implication $(c) \Rightarrow (a)$, and completes the proof of the corollary.

Note that $S$ is a faithfully flat left $B$-module if $M$ is a faithfully flat left $B$-module.

Recall from [14] that a ring extension $B \to S$ is a Frobenius extension if and only if the restriction of scalars functor has the same right and left adjoint (cf. [14]).

Following this observation a functor is called a Frobenius functor in case it has the same right and left adjoint (cf. [8], [9]). Motivated by this correspondence between Frobenius extensions and Frobenius functors one says that an $A$-coring $\mathcal{C}$ is Frobenius provided the forgetful functor from the category of right $\mathcal{C}$-comodules to the category of right $A$-modules is Frobenius. Equivalently, $\mathcal{C}$ is a Frobenius coring if and only if there exist an invariant $e \in \mathcal{C}^A = \{c \in \mathcal{C} \mid \forall a \in A, \ ac = ca\}$ and a pre-integral $\gamma : \mathcal{C} \otimes_A \mathcal{C} \to A$ such that for all $c \in \mathcal{C}$, $\gamma(c \otimes_A e) = \gamma(e \otimes_A c) = \varepsilon_\mathcal{C}(c)$. The pair $(\gamma, e)$ is called a reduced Frobenius system for $\mathcal{C}$ [4].

Theorem 3.7. Let $B M_A$ be a bimodule such that $M_A$ is a finitely generated projective, and let $S = \text{End}_{-A}(M)$. Then

(1) If $M$ is a Frobenius bimodule, then $M^* \otimes_B M$ is a Frobenius $A$–coring.
(2) If $M^* \otimes_B M$ is a Frobenius $A$–coring, then $S \otimes_B S$ is a Frobenius $S$–coring.

Proof. (1) Let $\mathcal{C} = M^* \otimes_B M$, and denote by $R$ the opposite ring of $\text{^*} \mathcal{C}$, i.e. $R = (\text{^*} \mathcal{C})^{opp}$. In view of [3, Theorem 4.1], to prove that $M^* \otimes_B M$ is a Frobenius coring suffices it to construct an $(A, R)$-bimodule isomorphism $\mathcal{C} \cong R$. On the other hand, by [10, Proposition 1], there is a ring isomorphism $R \cong \text{End}_{B-}(M)$, such that the right $R$-module structure on $\mathcal{C}$ is given by $(\varphi \otimes m) \cdot r = \varphi \otimes r(m)$ for $\varphi \otimes m \in \mathcal{C}$ and $r \in R$ viewed as an element of $\text{End}_{B-}(M)$. Suppose that $M$ is a Frobenius bimodule and let $\theta : M^* \to M^*$ be the (defining) Frobenius $(A, B)$-bimodule isomorphism. Define an $(A, B)$-bimodule isomorphism $\iota : \mathcal{C} \cong R$ as the composite

$$M^* \otimes_B M \overset{\theta \otimes M}{\longrightarrow} *M \otimes_B M \cong \text{End}_{B-}(M),$$

where the last isomorphism follows from the fact that $B M$ is a finitely generated projective module. The isomorphism $\iota$ explicitly comes out as $\iota(\varphi \otimes m)(x) = \theta(\varphi)(x)m$ for $\varphi \otimes m \in \mathcal{C}$, $x \in M$. A routine calculation verifies that $\iota$ is an $(A, R)$-bimodule map.

(2) Suppose that $M^* \otimes_B M$, and let $\gamma : M^* \otimes_B M \otimes_A M^* \otimes_B M \to A$ and $e = \sum_a w^*_a \otimes w_a \in (M^* \otimes_B M)^A$ be a reduced Frobenius system. This means that $\gamma$ is a pre-cointegral and for all $\varphi \otimes m \in M^* \otimes_B M$

$$\sum_a \gamma(\varphi \otimes m \otimes w^*_a \otimes w_a) = \sum_a \gamma(w^*_a \otimes w_a \otimes \varphi \otimes m) = \varphi(m).$$

(*)
Consider the pre-cointegral $\tilde{\gamma} : S \otimes_B S \otimes_B S \to S$ constructed in Lemma 3.4 and define $\tilde{e} = \sum_{i,a} e_i \otimes w_a^* \otimes w_a \otimes e_i^* \in (S \otimes_B S)^S$. In this definition and throughout the rest of the proof we freely use the identification of $S$ with $M \otimes A M^*$ described in Remark 3.1. We need to check that $(\tilde{\gamma}, \tilde{e})$ is a Frobenius system for $S \otimes_B S$, i.e., that

$$\tilde{\gamma}(s \otimes s' \tilde{e}) = ss' = \tilde{\gamma}(\tilde{e}s \otimes s'),$$

for all $s, s' \in S$. This is carried out by the following explicit computations. First,

$$\tilde{\gamma}(s \otimes s' \tilde{e}) = \sum_{i,a} \tilde{\gamma}(s \otimes s'(e_i) \otimes w_a^* \otimes w_a \otimes e_i^*)$$

$$= \sum_{i,k,a} \tilde{\gamma}(s(e_k) \otimes e_k^* \otimes s'(e_i) \otimes w_a^* \otimes w_a \otimes e_i^*)$$

$$= \sum_{i,k} s(e_k) \gamma(e_k^* \otimes s'(e_i) \otimes w_a^* \otimes w_a) \otimes e_i^*$$

$$= \sum_{i,k} s(e_k) e_k^*(s'(e_i)) \otimes e_i^* = \sum_{i,k} (s(e_k) \otimes e_k^*)(s'(e_i) \otimes e_i^*) = ss',$$

as required. Note that the first equality follows from the definition of $\tilde{e}$ and already incorporates the formula for the product in $S$ in terms of elements of $M \otimes A M^*$ as explained in Remark 3.1. Remark 3.1 is also used to derive the second and fifth equalities. The penultimate equality follows from equation (*). Second

$$ss' = \sum_k s(e_k) \otimes e_k^* s' = \sum_{i,k} e_i e_i^* s(e_k) \otimes e_k^* s'$$

$$= \sum_{i,k} e_i \otimes e_i^* s(e_k) e_k^* s' = \sum_{i,k} e_i \otimes \gamma(w_a^* \otimes w_a \otimes e_i^* s \otimes e_k) e_k^* s'$$

$$= \sum_{i,k} \tilde{\gamma}(e_i \otimes w_a^* \otimes w_a \otimes e_i^* s \otimes e_k \otimes e_k^* s') = \tilde{\gamma}(\tilde{e}s \otimes s'),$$

as required. Here Remark 3.1 is used in derivation of the first two and the last equalities, while the fourth equality follows from equation (*). Thus we have proven that $(\tilde{\gamma}, \tilde{e})$ is a Frobenius system for $S \otimes_B S$, so that $S \otimes_B S$ is a Frobenius coring as asserted.

\[ \square \]

Note that Theorem 3.7 implies in particular that if $BM_A$ is a Frobenius bimodule, then $S \otimes_B S$ is a Frobenius coring. This also follows from the endomorphism ring theorem [12, Theorem 2.5] and [4, Theorem 2.7]. As was the case for coseparable comatrix coalgebras, Theorem 3.7 leads to a more complete description of the relationship between Frobenius bimodules and Frobenius comatrix coalgebras in the case of a faithfully flat extension $B \to S$ which in addition satisfies a weak version of Williard’s condition.

**Corollary 3.8.** Let $BM_A$ be a bimodule such that both $M_A$ and $BM$ are finitely generated projective, and let $S = \operatorname{End}_{-A}(M)$. Suppose that either $BS$ or $SB$ is faithfully flat and that $\operatorname{Hom}_{-S}(M, S) \cong \operatorname{Hom}_{-A}(M, A)$ as $(A,B)$-bimodules. Then the following statements are equivalent

11
(a) $M$ is a Frobenius bimodule.
(b) The comatrix coring $M^* \otimes_B M$ is a Frobenius $A$-coring.
(c) $S \otimes_B S$ is a Frobenius $S$-coring.

Proof. The implications $(a) \Rightarrow (b) \Rightarrow (c)$ are contained in Theorem 3.7. Suppose that $S \otimes_B S$ is a Frobenius $S$-coring. In view of the faithful flatness, $B \to S$ is a Frobenius extension by [4, Theorem 2.7]. Since $\text{Hom}_{S-}(M, S) \cong \text{Hom}_{-A}(M, A)$ as $(A, B)$-bimodules, the converse of the endomorphism ring theorem [12, Theorem 2.8] implies that $M$ is a Frobenius bimodule. This proves the implication $(c) \Rightarrow (a)$, and completes the proof of the corollary. 

As noted in [12, Section 2.3], the condition $\text{Hom}_{S-}(M, S) \cong \text{Hom}_{-A}(M, A)$ as $(A, B)$-bimodules is in particular satisfied when $M_A$ is a generator module.

Remark 3.9. The central idea of this paper is that properties of a bimodule $BMA$ imply analogous properties of the endomorphism ring $S = \text{End}_{-A}(M)$. These in turn lead to corresponding properties of the Sweedler $S$-coring associated to the extension $B \to S$. The comatrix $A$-coring built with $M$ can be thought of as a dual of the endomorphism ring, and thus can be envisioned as lying in between a bimodule $M$ and the Sweedler coring associated to $B \to S$. Thus, combining the results of the present paper with that of existing literature, the situation can be summarised in terms of the following deductive diagrams.

In case $M_A$ is finitely generated and projective,

$\begin{align*}
M^* \text{ separable bimodule} & \quad \overset{\text{Th. 3.2}}{\longrightarrow} \quad M^* \otimes_B M \text{ cosplit coring} \\
B \to S \text{ separable extension} & \quad \overset{\text{[12]}}{\downarrow} \\
S \otimes_B S \text{ cosplit coring} & \quad \overset{\text{[3, Cor. 3.4]}}{\downarrow} \\
M \text{ separable bimodule} & \quad \overset{\text{Th. 3.5}}{\longrightarrow} \quad M^* \otimes_B M \text{ coseparable coring} \\
B \to S \text{ split extension} & \quad \overset{\text{[12]}}{\downarrow} \\
S \otimes_B S \text{ coseparable coring} & \quad \overset{\text{[3, Cor. 3.7]}}{\downarrow} \quad \overset{\text{faithful flatness}}{\downarrow} \\
\end{align*}$

In case $B M$ and $M_A$ are finitely generated and projective,

$\begin{align*}
M \text{ Frobenius bimodule} & \quad \overset{\text{Th. 3.7}}{\longrightarrow} \quad M^* \otimes_B M \text{ Frobenius coring} \\
B \to S \text{ Frobenius extension} & \quad \overset{\text{[12, Th. 2.7]}}{\downarrow} \quad \overset{\text{faithful flatness}}{\downarrow} \\
S \otimes_B S \text{ Frobenius coring} & \quad \overset{\text{[12]} \text{ Willard’s condition}}{\downarrow} \\
\end{align*}$
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