Quantum algebra of the Hamiltonian constraint in reduced 4-dimensional gravity

Eyo Eyo Ita III

February 7, 2022

Department of Applied Mathematics and Theoretical Physics
Centre for Mathematical Sciences, University of Cambridge, Wilberforce Road
Cambridge CB3 0WA, United Kingdom
eei20@cam.ac.uk

Abstract

In this paper we demonstrate closure of the quantum algebra of Hamiltonian constraints in a theory directly related to a certain sector of general relativity reduced to diagonal variables.
1 Introduction

In [1] an action was presented which is related to a certain sector of reduced gravity, and it was demonstrated that the associated system is Dirac consistent at the classical level as a stand-alone action. By reduced gravity, we mean that the theory has three degrees of freedom per point at the kinematic level, namely the level prior to implementation of the Hamiltonian constraint. In this theory there are no Gauss’ law and no diffeomorphism constraints. Additionally, there are six distinct sectors, referred to as quantizable configurations $\Gamma_q$, which exhibit the same features as outlined in [1]. The results of the present paper will also apply to these configurations $\Gamma_q$.

The purpose of the present paper will be to verify closure of the algebra of constraints for the action written down in [1] at the quantum level. In this section we will introduce the action, and in section 2 we carry out the computation of the quantum constraints algebra.

Let us consider a system with configuration and momentum space variables $\Gamma_{Kin} = (X,Y,T)$ and $P_{Kin} = (\Pi_1,\Pi_2,\Pi)$ defined on a 4-dimensional spacetime manifold of topology $M = \Sigma \times R$, where $\Sigma$ is a 3-dimensional spatial hypersurface. The variables are in general complex, and the configuration space variables take on the ranges $-\infty < |X|, |Y|, |T| < \infty$. The mass dimensions of all variables have been chosen to be

$$[\Pi_1] = [\Pi_2] = [\Pi] = 1; \ [X] = [Y] = [T] = 0. \quad (1)$$

These variables define the following kinematic phase space action for a totally constrained system

$$I_{Kin} = -\frac{i}{G} \int dt \int_\Sigma d^3x (\Pi_1 \dot{X} + \Pi_2 \dot{Y} + \Pi \dot{T}) - iH[N]. \quad (2)$$

The field $N$ is an auxiliary field smearing a phase space function $H$, such that the Hamiltonian density is given by

$$H[N] = \int_\Sigma d^3x NUe^{-T/2}\Phi. \quad (3)$$

The quantities in (3) are defined as follows. First we have $\Phi$, given by

$$\Phi = \sqrt{\Pi(\Pi + \Pi_1)(\Pi + \Pi_2)} \left[ k + e^T \left( \frac{1}{\Pi + \Pi_1} + \frac{1}{\Pi + \Pi_2} \right) \right] \quad (4)$$

where $k$ is a numerical constant.\footnote{For the reduced sector of gravity, we must have $k = \Lambda a_0^{-3}$, where $\Lambda$ is the cosmological constant and $a_0$ is a numerical constant of mass dimension $[a_0] = 1$.} There are no spatial derivatives in any of the quantities in (4), and all spatial derivatives in the theory (2) are confined
to the quantity $U$, given by
\[
U = \left[ 1 + e^{-T} \left( (\partial_2 Z)(\partial_2 X)(\partial_1 Y) - (\partial_3 Y)(\partial_1 Z)(\partial_2 X) \right) + e^{-2X}(\partial_1 Y)(\partial_1 Z) + e^{-2Y}(\partial_2 Z)(\partial_2 X) + e^{-2Z}(\partial_3 X)(\partial_3 Y) \right]^{1/2}
\] (5)
with $Z = T - X - Y$. We have defined
\[
\partial_1 = \frac{\partial}{\partial y^1}; \quad \partial_2 = \frac{\partial}{\partial y^2}; \quad \partial_3 = \frac{\partial}{\partial y^3},
\] (6)
where $y^1$, $y^2$ and $y^3$ are dimensionless spatial coordinates in $\Sigma$.

The canonical structure of (2) yields the following fundamental Poisson brackets
\[
\{X(x,t), \Pi_1(y,t)\} = \{Y(x,t), \Pi_2(y,t)\} = \{T(x,t), \Pi(y,t)\} = -iG\delta^{(3)}(x,y).
\] (7)

In this paper we will check for closure of the quantum constraints algebra of (2). But prior to proceeding with the algebra, it is worthwhile to present a short background of the significance of the action (2).

### 1.1 Relation to an antecedent of the CDJ action

The significance of the action (2) is that it can be obtained from a restricted sector of an action for general relativity, which appears in [2] as an intermediate step in obtaining the CDJ pure spin connection formulation for gravity,\(^2\) as we will demonstrate. Consider the following change of variables of the action (2)
\[
\Pi = a_0^3 e^T \lambda_3; \quad \Pi + \Pi_1 = a_0^3 e^T \lambda_1; \quad \Pi + \Pi_2 = a_0^3 e^T \lambda_2
\] (8)
for the momentum space variables $P_{\text{Kin}}$, and
\[
X = \ln\left(\frac{a_1}{a_0}\right); \quad Y = \ln\left(\frac{a_2}{a_0}\right); \quad T = \ln\left(\frac{a_1 a_2 a_3}{a_0^2}\right)
\] (9)
for the configuration space variables $\Gamma_{\text{Kin}}$, where $a_0$ is a numerical constant of mass dimension $[a_0] = 1$. Let us also make the definitions
\[
x^1 = \frac{y^1}{a_0}; \quad x^2 = \frac{y^2}{a_0}; \quad x^3 = \frac{y^3}{a_0}
\] (10)
with $y^1$, $y^2$ and $y^3$ the dimensionless spatial coordinates in $\Sigma$. This implies that $[x^1] = [x^2] = [x^3] = -1$, namely that the coordinates $x^1$, $x^2$ and $x^3$

\(^2\)The initial CDJ refer to Capovilla, Dell and Jacobson, who developed a nonmetric formulation for gravity written almost completely in terms of the spin connection.
have dimensions of length. Substitution of (9) and (10) into (5) yields

\[ U = (a_1 a_2 a_3)^{-1} \left[ (a_1 a_2 a_3)^2 + (\partial_2 a_3)(\partial_3 a_1)(\partial_1 a_2) 
- (\partial_3 a_2)(\partial_1 a_3)(\partial_2 a_1) + a_2 a_3 (\partial_1 a_2)(\partial_1 a_3) + a_3 a_1 (\partial_2 a_3)(\partial_2 a_1) 
+ a_1 a_2 (\partial_3 a_1)(\partial_3 a_2) \right]^{1/2} = (\det A)^{-1}(\det B)^{1/2}, \tag{11} \]

from which one recognizes \( U \) as the square root of the determinant of the magnetic field \( B_{\mu}^a \) for a diagonal connection \( A_{\mu}^a = \text{diag}(a_1, a_2, a_3) \), with the leading order term in \( (\det A) \) factored out. In matrix form this is given by

\[
a_i^0 = \begin{pmatrix}
a_1 & 0 & 0 \\
0 & a_2 & 0 \\
0 & 0 & a_3 \\
\end{pmatrix}; \quad b_i^a = \begin{pmatrix}
a_2 a_3 & -\partial_3 a_2 & \partial_2 a_3 \\
\partial_3 a_1 & a_3 a_1 & -\partial_1 a_3 \\
-\partial_2 a_1 & \partial_1 a_2 & a_1 a_2 \\
\end{pmatrix}.
\]

Substitution of (8), (9) and (11) into (2) yields

\[ I = -\frac{i}{G} \int dt \int_\Sigma d^3 x \left( \lambda_1 a_2 a_3 \dot{a}_1 + \lambda_2 a_3 a_1 \dot{a}_2 + \lambda_3 a_1 a_2 \dot{a}_3 
- iN(\det b)^{1/2} \sqrt{\lambda_1 \lambda_2 \lambda_3} \left( \Lambda + \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} \right) \right), \tag{12} \]

Let us make the following definitions for the magnetic field and the temporal component of the curvature

\[
b_i^a = \frac{1}{2} \epsilon^{ijk} f_{jk}^a; \quad f_{0i}^a = \dot{a}_i^a - D_i a_0^a, \tag{13} \]

where \( D_i \) is the SO(3,C) covariant derivative with respect to the spatial connection \( a_i^a \). Then the integrand of the canonical one form of (12) can be written as

\[
\lambda_g b_i^a \dot{a}_i^a = \frac{1}{2} \lambda_g \epsilon^{ijk} f_{jk}^a f_{0i}^a + \lambda_g b_i^a D_i a_0^a, \tag{14} \]

where \( a_0^a \) is the temporal component of the connection \( a_i^a \). Then defining \( \epsilon^{ijk} = \epsilon^{0ijk} \) and using the symmetries of the 4-D epsilon symbol \( \epsilon^{\mu\nu\rho\sigma} \), then (14) is given by

\[
\frac{1}{8} \lambda_g f_{\mu\nu}^g f_{\rho\sigma}^g \epsilon^{\mu\nu\rho\sigma} - a_0^a b_i^a D_i \lambda_g. \tag{15} \]

Using equation (15) to replace the canonical one form in (12), we get the action

\[ I = -\frac{i}{G} \int_M d^4 x \left[ \frac{1}{8} \lambda_g f_{\mu\nu}^g f_{\rho\sigma}^g \epsilon^{\mu\nu\rho\sigma} 
- iN(\det b)^{1/2} \sqrt{\lambda_1 \lambda_2 \lambda_3} \left( \Lambda + \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} \right) \right] + \int dt \int_\Sigma d^3 x a_0^a b_i^a D_i \lambda_g, \tag{16} \]
where $\eta = (\det b)^{1/2} \sqrt{\lambda_2 \lambda_2 \lambda_3}$. Equation (16) is none other than the CDJ antecedent appearing in [2] with the following caveats: (i) The Gauss’ law constraint is missing. This is the last term on the right hand side of (16), which cancels the same quantity from the curvature squared term. (ii) Equation (16) is the restriction of the aforementioned action to diagonal variables.

2 Quantum constraints algebra of the Hamiltonian constraint

Upon quantization of (2), the dynamical variables become promoted to quantum operators satisfying equal-time commutation relations

$$[\hat{X}(x,t), \hat{\Pi}_1(y,t)] = [\hat{Y}(x,t), \hat{\Pi}_2(y,t)] = [\hat{T}(x,t), \hat{\Pi}(y,t)] = (hG)\delta^{(3)}(x,y), \quad (17)$$

with all other commutators vanishing. The smeared Hamiltonian constraint (3) becomes promoted to a composite operator constraint

$$\hat{H}[N] = \int_{\Sigma} d^3x N(x) \hat{\eta}(x) \hat{\Phi}(x), \quad (18)$$

where we have made the following definitions

$$\hat{\Phi} = \sqrt{\hat{\Pi}(\hat{\Pi} + \hat{\Pi}_1)(\hat{\Pi} + \hat{\Pi}_2)} \left[ k + \left( \frac{1}{\hat{\Pi}} + \frac{1}{\hat{\Pi} + \hat{\Pi}_1} + \frac{1}{\hat{\Pi} + \hat{\Pi}_2} \right) e^{\hat{T}} \right], \quad (19)$$

with the operator ordering as indicated. The physical states are defined as those states $|\psi\rangle \in |\psi_{phys}\rangle$ such that $\hat{H}[\psi] = 0$ with $\hat{\Phi}$ appearing to the right. For the quantum constraints algebra to be consistent in the Dirac sense, the algebra must close with $\hat{\Phi}$ appearing to the right.

References [4] and [5]) state that the quantization of theories containing operator products evaluated at the same point results in infinities which need to be regularized. A possible regularization prescription is to individually smear each operator appearing in the operator product. We will show that such regularization procedures are not necessary for the case presented in this paper, since the smearing of the constraints automatically eliminates any infinites. For the quantum constraints algebra we will use the following operator identity for composite operators

$$[\hat{A}\hat{B}, \hat{C}\hat{D}] = \hat{C}[\hat{A}, \hat{D}]\hat{B} + \hat{A}[\hat{B}, \hat{C}]\hat{D} + [\hat{A}, \hat{C}]\hat{B}\hat{D} + \hat{C}\hat{A}[\hat{B}, \hat{D}], \quad (20)$$
where \( \hat{A}, \hat{B}, \hat{C}, \) and \( \hat{D} \) are bosonic operators. Using equation (20), the quantum constraints algebra of the Hamiltonian (18) is given by

\[
[\hat{H}[M], \hat{H}[N]] = \int_{\Sigma} d^3 x \int_{\Sigma} d^3 y M(x) N(y) [\hat{\eta}(x) \hat{\phi}(x), \hat{\eta}(y) \hat{\phi}(y)]
\]

\[
= \int_{\Sigma} d^3 x \int_{\Sigma} d^3 y M(x) N(y) \left[ \hat{\eta}(y) [\hat{\eta}(x), \hat{\phi}(x)] + \hat{\eta}(y) [\hat{\hat{\phi}}(x), \hat{\eta}(y)] \hat{\phi}(y) + [\hat{\eta}(x), \hat{\eta}(y)] \hat{\phi}(x) \hat{\phi}(y) \right]. \tag{21}
\]

We must now analyse each term appearing in (21). The third term on the right hand side of (21) vanishes since it is a commutator purely between configuration space variables. The fourth term of (21) vanishes, which can be seen as follows. Make the following definitions

\[
\hat{W}(x, y) = \hat{\eta}(x) \hat{\eta}(y); \quad \hat{\phi}(x) \equiv \hat{\chi}(x) + \hat{\hat{S}}(x) e^{\hat{T}(x)} \tag{22}
\]

where \( \chi \) and \( S \) depend only on momentum space variables, whose specific form can be read off from (19). Note that the following relations hold

\[
[\hat{\chi}(x), e^{\hat{T}(y)}] = - [e^{\hat{T}(y)}, \hat{\chi}(x)] = \left( \frac{\partial \chi}{\partial \Pi} \right) e^{\hat{T}(y)} \delta^{(3)}(x, y). \tag{23}
\]

This is a consequence of (17), where \( \Pi \) is the only variable with nonvanishing relations with \( T \). We are now ready to proceed with the fourth term of (21), which is given by

\[
\int_{\Sigma} d^3 x \int_{\Sigma} d^3 y \hat{W}(x, y) \left[ \hat{\chi}(x) + \hat{\hat{S}}(x) e^{\hat{T}(x)}, \hat{\chi}(y) + \hat{\hat{S}}(y) e^{\hat{T}(y)} \right] \tag{24}
\]

where we have used the definitions (22). Expansion of (24) leads to the following four terms

\[
\int_{\Sigma} d^3 x \int_{\Sigma} d^3 y \hat{W}(x, y) \left[ \hat{\hat{\chi}}(x), \hat{\chi}(y) \right] + \int_{\Sigma} d^3 x \int_{\Sigma} d^3 y \hat{W}(x, y) \hat{\hat{S}}(x) \left[ \hat{\chi}(x), e^{\hat{T}(y)} \right] \]

\[
+ \int_{\Sigma} d^3 x \int_{\Sigma} d^3 y \hat{W}(x, y) \hat{\hat{S}}(x) \left[ e^{\hat{T}(x)}, \hat{\chi}(y) \right] + \int_{\Sigma} d^3 x \int_{\Sigma} d^3 y \hat{W}(x, y) \left[ \hat{\hat{S}}(x) e^{\hat{T}(x)}, \hat{\hat{S}}(y) e^{\hat{T}(y)} \right]. \tag{25}
\]

which we will in turn analyse. The first term of (25) vanishes due to vanishing commutation relations between momentum space variables. Using the results of (23), the middle two terms of (25) combine into

\[
\int_{\Sigma} d^3 x \int_{\Sigma} d^3 y \hat{W}(x, y) \left[ - \left( \frac{\partial \hat{\chi}}{\partial \Pi} \right)_x e^{\hat{T}(y)} + \left( \frac{\partial \hat{\chi}}{\partial \Pi} \right)_y e^{\hat{T}(x)} \right] \delta^{(3)}(x, y)
\]

\[
= \int_{\Sigma} d^3 x \hat{W}(x, x) \left[ - \left( \frac{\partial \hat{\chi}}{\partial \Pi} \right)_x e^{\hat{T}(x)} + \left( \frac{\partial \hat{\chi}}{\partial \Pi} \right)_x e^{\hat{T}(x)} \right] = 0 \tag{26}
\]
which vanishes after integration of the delta function, leaving the remaining fourth term of (25). Application of the identity (20) to this term yields

\[ \int d^3x \int d^3y \hat{W}(x,y) [\hat{S}(x)e^{\hat{T}(x)}, \hat{S}(y)e^{\hat{T}(y)}] = \int d^3x \int d^3y \hat{W}(x,y) [\hat{S}(x), e^{\hat{T}(x)}] e^{\hat{T}(y)} + \int d^3x \int d^3y \hat{W}(x,y) [\hat{S}(x), \hat{S}(y)] e^{T(x)} e^{T(y)} + \int d^3x \int d^3y \hat{W}(x,y) \hat{S}(x) \hat{S}(y) [e^{T(x)}, e^{T(y)}]. \]  

(27)

The third term of (27) vanishes due to vanishing commutation relations between momentum space variables, and the fourth term vanishes due to vanishing commutation relations between configuration space variables \( T \). Using (23), the first and second term of (27) combine into

\[ \int d^3x \int d^3y \hat{W}(x,y) \left[ -\hat{S}(y) \left( \frac{\partial \hat{S}}{\partial \Pi} \right)_x e^{\hat{T}(x)} + \hat{S}(x) \left( \frac{\partial \hat{S}}{\partial \Pi} \right)_y e^{\hat{T}(y)} \right] \delta^{(3)}(x,y) = \int d^3x \int d^3y \hat{W}(x,y) \left[ -S \left( \frac{\partial S}{\partial \Pi} \right)_x e^{T} + S \left( \frac{\partial S}{\partial \Pi} \right)_y e^{T} \right] = 0 \]  

(28)

which also vanishes. We have shown that the third and fourth terms on the right hand side of (21) both vanish, which leaves us with the first and second terms. Make the definitions

\[ \frac{\delta \Phi(x)}{\delta \Pi_1(y)} = Q^1(x)\delta^{(3)}(x,y); \quad \frac{\delta \Phi(x)}{\delta \Pi_2(y)} = Q^2(x)\delta^{(3)}(x,y); \quad \frac{\delta \Phi(x)}{\delta \Pi(y)} = Q^3(x)\delta^{(3)}(x,y), \]  

(29)

where \( Q^i(x) \) are functions on \( \Omega_{K_{in}} \) whose specific form will not be needed for what follows. Likewise make the definitions

\[ \frac{\delta \eta(x)}{\delta X(y)} = \xi^1(x)\delta^{(3)}(x,y) \left( \frac{\partial}{\partial x^3} \right); \]

\[ \frac{\delta \eta(x)}{\delta Y(y)} = \xi^2(x)\delta^{(3)}(x,y) \left( \frac{\partial}{\partial x^2} \right); \]

\[ \frac{\delta \eta(x)}{\delta T(y)} = \xi^3(x)\delta^{(3)}(x,y) \left( \frac{\partial}{\partial x^1} \right). \]  

(30)

The notation in (30) signifies that the partial derivatives will act on all objects with \( x \) dependence which multiply the terms that the derivatives originally came from. Using (29) and (30), we have the following operator relations

\[ [\hat{\zeta}(x), \hat{\Phi}(y)] = \xi^1(y)\hat{Q}^1(y)\delta^{(3)}(x,y) \left( \frac{\partial}{\partial x^3} \right) \]

\[ [\hat{\Phi}(x), \hat{\zeta}(y)] = -\xi^2(y)\hat{Q}^2(x)\delta^{(3)}(x,y) \left( \frac{\partial}{\partial y^2} \right). \]  

(31)
Hence it is apparent from (31) that $\frac{\partial}{\partial x^i}$ acts on objects containing $x$ dependence, and $\frac{\partial}{\partial y^i}$ acts on objects containing $y$ dependence. So continuing from (21) and using (31), we have

$$[\hat{H}[M], \hat{H}[N]] =$$

$$\int_\Sigma d^3x \int_\Sigma d^3y M(x)N(y) \left[ \hat{\eta}(y) \hat{\zeta}^i_j(x) \hat{Q}^I(y) \left( \frac{\partial}{\partial x^j} \right) \hat{\Phi}(x) - \hat{\eta}(x) \hat{\zeta}^i_j(y) \hat{Q}^I(y) \left( \frac{\partial}{\partial x^j} \right) \hat{\Phi}(y) \right] \delta^{(3)}(x, y)$$

$$= \int_\Sigma d^3x \int_\Sigma d^3y \left[ N(y) \hat{\eta}(y) \frac{\partial}{\partial x^j}(M(x) \hat{\zeta}^i_j(x) \hat{Q}^I(y) \hat{\Phi}(x)) - M(x) \hat{\zeta}(x) \frac{\partial}{\partial y^j}(N(y) \hat{\zeta}^i_j(y) \hat{Q}^I(x) \hat{\Phi}(y)) \right] \delta^{(3)}(x, y).$$

Integration with respect to $y$ collapses the delta function, which yields

$$\int_\Sigma d^3x \left[ N\hat{\zeta} \frac{\partial}{\partial x^j}(M\hat{\zeta}^i_j \hat{Q}^I \hat{\Phi}) - M\hat{\zeta} \frac{\partial}{\partial x^j}(N\hat{\zeta}^i_j \hat{Q}^I \hat{\Phi}) \right]$$

$$= \int_\Sigma d^3x (N\partial_iM - M\partial_iN) \hat{\eta} \hat{\zeta}^i_j \hat{Q}^I \hat{\Phi},$$

whence the operator $\hat{\Phi}$ appears to the right. Since $\hat{\Phi}$ is proportional to the Hamiltonian constraint it follows that

$$[\hat{H}[M], \hat{H}[N]] \mid \psi \rangle = \hat{H}[M, N] \mid \psi \rangle,$$

namely that the commutator of two Hamiltonian constraints is a Hamiltonian constraint with the constraint appearing to the right. The quantum algebra of the Hamiltonian constraint closes with structure functions, and it closes in direct analogy to its classical counterpart in [1] when one makes the identification $q^I \eta^j_I \rightarrow \hat{\eta} \hat{\zeta}^i_j \hat{Q}^I$. Moreover, the algebra closes with the proper ordering taken into account with the Hamiltonian constraint operator to the right. For these reasons we conclude that the quantum constraints algebra is Dirac consistent and is free of anomalies.

### 3 Conclusion

The main result of this paper has been to verify the closure of the quantum constraints algebra for a theory of ‘reduced’ gravity introduced in [1]. Future directions of research will be to investigate the Hilbert space structure of the resulting theory.
References

[1] Eyo Ita ‘Dirac consistency of the algebra of Hamiltonian constraints in reduced 4-D general relativity’ Hadronic Journal Vol. 33, No. 5 (2010) pp.637-654

[2] Richard Capovilla, John Dell and Ted Jacobson ‘A pure spin-connection formulation of gravity’ Class. Quantum. Grav. 8 (1991) 59-73

[3] Paul Dirac ‘Lectures on quantum mechanics’ Yeshiva University Press, New York, 1964

[4] N. Kontoleonard and D.C. Wiltshire ‘Operator ordering and consistency of the wave function of the universe’ Phys. Rev. D59 (1999) 063513

[5] N.C. Tsamis and R.P. woodard. ‘The factor-ordering problem must be regulated’ Phys. Rev. D36 (1987) 3641