GRÖBNER BASES OF SYZYGIES AND STANLEY DEPTH

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Abstract. Let \( F \) be a free resolution of a \( \mathbb{Z}^n \)-graded submodule of a free module over the polynomial ring \( K[x_1, \ldots, x_n] \). We show that for a suitable term order on \( F \), the initial module of the \( p \)'th syzygy module \( Z_p \) is generated by terms \( m_i e_i \) where the \( m_i \) are monomials in \( K[x_{p+1}, \ldots, x_n] \). Also for a large class of free resolutions \( F \), encompassing Eliahou-Kervaire resolutions, we show that a Gröbner basis for \( Z_p \) is given by the boundaries of generators of \( F_p \).

We apply the above to give lower bounds for the Stanley depth of the syzygy modules \( Z_p \). We also show that if \( I \) is any squarefree ideal in \( K[x_1, \ldots, x_n] \), the Stanley depth of \( I \) is at least of order \( \sqrt{2n} \).

Introduction

Let \( K \) be a field and \( S = K[x_1, \ldots, x_n] \) the polynomial ring in \( n \) variables over \( K \). We study Gröbner bases of syzygies of finitely generated \( \mathbb{Z}^n \)-graded modules over \( S \), and apply this to give lower bounds for the Stanley depth of syzygy modules.

Fix any monomial order \( < \) on \( S \) and let \( F \) be a free \( \mathbb{Z}^n \)-graded \( S \)-module with a homogeneous basis \( F = e_1, \ldots, e_m \). We define a monomial order on \( F \) by setting \( u e_i > v e_j \) if \( i < j \), or \( i = j \) and \( u > v \), where \( u \) and \( v \) are monomials of \( S \). If \( M \) is a \( \mathbb{Z}^n \)-graded submodule of \( F \), a basic observation is that the initial module in \( (M) \) does not depend on the monomial order \( < \) on \( S \) but only on the basis \( F \). Therefore we denote the initial module of \( M \) with respect to this monomial order by \( \text{in}_F(M) \). We have \( \text{in}_F(M) = \bigoplus_{j=1}^m I_j e_j \), where each \( I_j \) is a monomial ideal.

We call the basis \( F \) lex-refined, if \( \deg(e_1) \geq \deg(e_2) \geq \ldots \geq \deg(e_m) \) in the lexicographical order. Our first main result, Theorem 1.1, shows that the initial modules of syzygy modules, when choosing a lex-refined basis, have a simple and natural property: let \( M \) be a \( \mathbb{Z}^n \)-graded submodule of a free module \( F_0 \) with free resolution \( F_1 \to F_2 \to \cdots \to F_p \to \cdots \to F_1 \to M \to 0 \). For \( 0 \leq p \leq n \) let \( Z_p \subseteq F_p \) be the \( p \)'th syzygy module. Then the initial module in \( F(Z_p) \) is \( \bigoplus_{j=1}^m I_j e_j \), where the minimal set of monomial generators of each \( I_j \) belongs to \( K[x_{p+1}, \ldots, x_n] \).

This theorem may remind the reader to a well-known result of F.-O. Schreyer, see Section 5.5 of [5], who showed that for any finitely generated module \( M \) one can find a free resolution and suitable monomial orders on the free modules of the resolution such that the initial modules of the syzygies enjoy the same nice property as described above. The point here is that no assumption is made on the \( \mathbb{Z}^n \)-graded resolution on \( M \). In particular, the theorem is valid for the graded minimal free resolution of \( M \).

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In general of course it is not so easy to compute the initial module of a syzygy module in a free resolution $F$ of a module $M$. But for certain classes of resolutions this may be done in a pleasant way. We say that the resolution has boundary Gröbner bases if for each $p$ there exists a basis $\mathcal{F}_p$ of $F_p$ such that $\text{in}_{\mathcal{F}_p}(Z_p(F))$ is generated by the initial terms of $\partial_{p+1}(e_i)$ where $\partial_i$ denotes the differential of $F$, and $e_i$ ranges over $\mathcal{F}_{p+1}$. If $F$ has such bases, the initial modules of the syzygies can easily be read off from the matrices describing $\partial_i$ with respect to these bases. We show that the Taylor resolution as well as the Eliahou–Kervaire resolution have boundary Gröbner bases.

We then apply the first result on syzygies to give lower bounds for the Stanley depth of syzygies. A *Stanley decomposition* of a $\mathbb{Z}^n$-graded $S$-module $M$ is a direct sum decomposition $M = \bigoplus_{i=1}^n u_iK[Z_i]$ of $M$ as a $\mathbb{Z}^n$-graded $K$-vector space, where each $u_i$ is a homogeneous element of $M$, $K[Z_i]$ is a polynomial ring is a set of variables $Z_i \subset \{x_1, \ldots, x_n\}$, and each $u_iK[Z_i]$ is a free $K[Z_i]$-submodule of $M$. The minimum of the numbers $|Z_i|$ is called the Stanley depth of this decomposition. The *Stanley depth of* $M$, denoted $\text{sdepth} M$, is the maximal Stanley depth of a Stanley decomposition of $M$. In his paper [1] Stanley conjectured that $\text{sdepth} M \geq \text{depth} M$. This conjecture is widely open. In the papers listed in the references in this paper and the references therein, the reader can inform himself you about the present status of the conjecture.

Naively one could expect, that like for the ordinary depth, the Stanley depth of the first syzygy module $Z_1(M)$ of a $\mathbb{Z}^n$-graded module $M$ is one more than that of $M$, as along as $M$ is not free. This of course would immediately imply Stanley’s conjecture. However this is not the case. For example, if we let $m = (x_1, \ldots, x_n)$ be the graded maximal ideal of $S$. Then $\text{sdepth} S/m = 0$, while $\text{sdepth} m = \lceil n/2 \rceil$, as shown in [1]. Nevertheless it might be true that one always has $\text{sdepth} Z_1(M) \geq \text{sdepth} M$. But at the moment even the inequality $\text{sdepth} I \geq \text{sdepth} S/I$ for a monomial ideal $I$ is unknown. However as one of the main results of this paper, we show in Theorem 2.2 that if $M$ is a $\mathbb{Z}^n$-graded free submodule of free $S$-module, then for $p < n$ the $p'$th syzygy module of $M$ with respect to any (not necessarily minimal) $\mathbb{Z}^n$-graded free resolution of $M$ has Stanley depth at least $p + 1$.

One problem in proving such a result as stated in Theorem 2.2 is the fact that at present no method is known to compute the Stanley depth of a $\mathbb{Z}^n$-graded module in a finite number of steps. So far this can be done only for modules of the form $I/J$ where $J \subset I \subset S$ are monomial ideals, see [8]. It seems not even to be known that for a monomial ideal $I \subset S$ one has $\text{sdepth} I \oplus S = \text{sdepth} I$, as one would expect. The only method known to get a lower bound for the Stanley depth of a $\mathbb{Z}^n$-graded module $M$ is to find a suitable filtration of the module whose factors are of the form $I$ or $S/I$ where $I$ is a monomial ideal. The Stanley depth of $M$ is then just the minimum of the Stanley depth of the factors of the filtration. This enables us to give lower bounds for the Stanley depth of a syzygy module by using that if the initial module in $\mathcal{F}(Z_p)$ is $\bigoplus_{j=1}^m I^j e_j$, then the monomial ideals $I^j$ are the factors of a suitable filtration of $Z_p$. Therefore the Stanley depth of $M$ is greater or equal to the minimum of the Stanley depths of the $I^j$.

We apply the second result on Gröbner basis of syzygies to show that when $I \subset S$ is a monomial complete intersection minimally generated by $m$ elements, the $p'$th syzygy
module of $S/I$ has Stanley depth at least $n - \lfloor \frac{m-p}{2} \rfloor$, see Proposition 2.4. This indicates that our general lower bounds for the Stanley depth of syzygy modules are far from being optimal.

Somewhat independently of the above, we show that if $I$ is a squarefree ideal over the polynomial ring in $n$ variables, the Stanley depth of $M$ is at least of order $2 \sqrt{n}$, Theorem 3.4. This is quite in contrast to ordinary monomial ideals. In fact it is known by Cimpoeaş [3] that sufficiently high powers of $m$ have Stanley depth 1. The proof of Theorem 3.4 is based on a construction of interval partitions [7] by M.Keller et. al. which is further refined M.Ge et. al. in [6]. Applying this we give lower bounds, Theorem 2.6, for the Stanley depth of syzygy modules of a squarefree submodule of a free module. This bound is considerably better than what we have for arbitrary $\mathbb{Z}^n$-graded submodules.

The organisation of the paper is as follows. In Section 1 we consider resolutions of $\mathbb{Z}^n$-graded $S$-modules and initial modules of their syzygy modules determined by choosing ordered multihomogeneous bases for the terms in the resolutions. We first prove that for lex-refined orders, the generators of the initial module of the $p$'th syzygy module does not involve the first $p$ variables. We then give classes of resolutions which have boundary Gröbner bases. In Section 2 we give the lower bounds on Stanley depth of syzygies. These are a consequence of the results in Section 1, and in the squarefree case, a consequence of the result in the next Section 3. In this last section, we show that the Stanley depth of any squarefree monomial ideal in $n$ variables is at least of order $2 \sqrt{n}$.

1. **Gröbner basis of syzygies of multigraded modules**

We consider term orders on $\mathbb{Z}^n$-graded free modules over a polynomial ring in $n$ variables which are determined by fixing a multihomogeneous basis $e_1, \ldots, e_m$ of the free module and comparing terms $ue_i$ and $ve_j$ by first comparing their basis elements. For such term orderings the initial term of any multihomogeneous element will be determined solely by the ordering of the $e_i$'s, and so also the initial module of any multihomogeneous submodule.

A natural ordering of the $e_i$'s is by lexicographic ordering of their multidegrees. We show then that the syzygies of a free resolution of a $\mathbb{Z}^n$-graded module have the nice and natural property that for each successive syzygy module we miss an extra variable in their generating set.

Let $K$ be a field, $S = K[x_1, \ldots, x_n]$ the polynomial ring over $K$ in $n$ indeterminates, $M$ a $\mathbb{Z}^n$-graded $S$-module and

$$F_1 : \cdots \rightarrow F_p \rightarrow F_{p-1} \rightarrow \cdots \rightarrow F_1 \rightarrow M \rightarrow 0$$

a $\mathbb{Z}^n$-graded (not necessarily minimal) free $S$-resolution of $M$ with all $F_i$ finitely generated. Let $Z_p \subset F_p$ be the $p$'th syzygy module of $M$ (with respect to this resolution). We are interested in the initial module of the syzygy module $Z_p$ with respect to a monomial order on $F_p$. In this paper we restrict our attention to monomial orders of the following type: we denote by $\text{Mon}(S)$ the set of monomials of $S$, fix a multihomogeneous basis $F = e_1, \ldots, e_m$ of $F_p$ and a monomial order $<$ on $S$, and define a monomial order on $F_p$ by setting

$$ue_i > ve_j, \quad \text{if } i < j, \text{ or } i = j \text{ and } u > v.$$
Here $u, v \in \text{Mon}(S)$. We denote by $\text{in}_<(Z_p)$ the monomial submodule of $F_p$ which is generated by all the initial monomials of elements of $Z_p$. Notice that

\begin{equation}
\text{in}_<(Z_p) = \bigoplus_{j=1}^{m} I_j e_j,
\end{equation}

where each $I_j$ is a monomial ideal.

Since $Z_p$ is $\mathbb{Z}^n$-graded, $\text{in}_<(Z_p)$ is generated by the initial monomials of multihomogeneous elements of $Z_p$. Let $z = \sum_{i=1}^{m} f_i e_i$ be a multihomogeneous element in $Z_p$. Then each $f_i$ is a term $a_i u_i$ with $a_i \in K$ and $u_i$ a monomial. Thus $\text{in}_<(z) = u_j e_j$, where $j$ is the smallest number such that $a_i \neq 0$. This consideration shows that for the above monomial order, the initial module of a syzygy module of a monomial ideal depends only on the given basis not on the chosen monomial order on $S$. Thus we write $\text{in}_<(Z_p)$ to denote the initial module of $Z_p$ with respect to this monomial order induced by $\mathcal{F}$.

In general, for a given resolution $F$, there are several equally natural choices of multihomogeneous bases for the $F_p$. Here we will choose for each $F_p$ a basis compatible with the lexicographical order of the multidegrees of the basis elements. We call such a basis of $F_p$ lex-refined. Thus a basis $\mathcal{F} = e_1, \ldots, e_m$ of $F_p$ of multihomogeneous elements is lex-refined, if $\deg(e_1) \geq \deg(e_2) \geq \ldots \geq \deg(e_m)$ in the lexicographical order.

**Theorem 1.1.** Suppose $M$ is a submodule of a free module $F_0$. Let $0 \leq p \leq n$ be an integer, and let $\mathcal{F}$ be a lex-refined basis of $F_p$. Then $\text{in}_<(Z_p) = \bigoplus_{j=1}^{m} I_j e_j$, where the minimal set of monomial generators of each $I_j$ belongs to $K[x_{p+1}, \ldots, x_n]$.

This theorem is an immediate consequence of the following

**Proposition 1.2.** Let $\varphi: F \to G$ be a homomorphism of finitely generated $\mathbb{Z}^n$-graded free $S$-modules with $M = \text{Im} \varphi$ and $N = \text{Ker} \varphi$. Let $G = e'_1, \ldots, e'_m$ be a lex-refined basis of $G$ and $\mathcal{F} = e_1, \ldots, e_n$ a lex-refined basis of $F$, and assume that for some $p \leq n$, $\text{in}_<(M)$ is generated by monomials not divisible by $x_1, \ldots, x_{p-1}$. Then $\text{in}_<(N)$ is generated by monomials not divisible by $x_1, \ldots, x_p$.

**Proof.** 1. Let $s \in N$ be a multihomogeneous element, and let $x_r$ be the variable with least index such that $x_r$ divides $\text{in}(s)$. To demonstrate the desired property of $\text{in}_<(N)$, it will be sufficient to show that there exists an element $\tilde{s} \in N$ such that

1. $\text{in}(s) = \text{in}(\tilde{s})$;
2. $x_r$ divides $\tilde{s}$, if $r \leq p$.

Indeed, if (1) and (2) are satisfied, then $t = \tilde{s}/x_r \in N$ and $\text{in}(s) = x_r \text{in}(t)$. Thus we may delete $\text{in}(s)$ as a generator of $\text{in}_<(N)$.

2. In order to show the existence of $\tilde{s}$ with these properties, we write $s = s' + s''$, where $s'$ is the sum of all terms of $s$ which are not divisible by one of the variables $x_1, \ldots, x_{r-1}$, and $s''$ the sum of the other terms in $s$.

Let $s = \sum_{i=1}^{c} a_i u_i e_i$ with $a_i \in K$ and $u_i \in \text{Mon}(S)$. For simplicity we may assume that $\text{in}(s) = u_1 e_1$. Then, since $s$ is multihomogeneous and since $\mathcal{F}$ is a lex-refined basis of $F$, it follows that $u_1 \leq u_j$ in the lexicographical order for all $i \leq j$ in the support of $s$. Since $\text{in}(s') = \text{in}(s) = u_1 e_1$, it follows that $u_1 \leq u_i$ for all $i \in \text{supp}(s')$. The monomial $u_1$ is
divisible by \( x_r \), and no \( u_i \) with \( i \in \text{supp}(s') \) is divisible by any \( x_j \) for \( j < r \). Hence the inequality \( u_1 \leq u_i \) implies that \( x_r \) divides all \( u_i \) with \( i \in \text{supp}(s') \). In other words, \( x_r \) divides \( s' \).

3. Since \( s \in N \), it follows that \( \varphi(s') = -\varphi(s'') \). We denote this element by \( z \). Since \( z = \varphi(s') \), we see that \( x_r \) divides \( z \), and since \( z = -\varphi(s'') \) it follows that each of the terms of \( z \) is divisible by at least one \( x_i \) with \( i < r \).

Let \( \text{in}(z/x_r) = w e'_k \). Since \( \deg(z/x_r) = \deg(s) - \varepsilon_r \), where \( \varepsilon_r \) is the \( r \)th canonical basis vector of \( \mathbb{Z}^n \), it follows that \( \deg(e'_k) \leq \deg(s) \) for \( j = 1, \ldots, r - 1 \), and since each term of \( z/x_r \) is divisible at least one \( x_j \) with \( j < r \), we conclude that \( \deg(e'_k) < \deg(s) \) for at least one \( j < r \).

4. Now we may write \( z/x_r = \sum b_i v_i g_i \) with \( b_i \in K \) and \( v_i \in \text{Mon}(S) \), where the \( g_i \) form a reduced Gröbner basis of \( M \) (with respect to the monomial order induced by \( \mathcal{G} \)) and with the additional property that \( \text{in}(v_i g_i) \leq \text{in}(z/x_r) = w e'_k \) for all \( i \). Let \( \text{in}(g_i) = w_j e'_j \). Then \( j_i \geq k \), and hence \( \deg(e'_j) \leq \deg(e'_k) \) in the lexicographic order. Thus for all \( i \) it follows that \( \deg(e'_j) \leq \deg(s) \) for \( j = 1, \ldots, r - 1 \) with strict inequality for at least one \( j \).

5. According to our hypothesis, we may assume that none of the variables \( x_1, \ldots, x_{p-1} \) divides any of the \( w_j \). Thus, together with what we have shown in the last paragraph we see that for all \( i \) we have that \( \deg(g_i) \leq \deg(s) \) for \( j = 1, \ldots, r - 1 \), but with strict inequality for at least one \( j < r \). We may now lift each \( g_i \) to an element \( f_i \) of \( F_p \) with the same multidegree, and set

\[
 s'' = \sum b_i v_i f_i \quad \text{and} \quad s = s' - x_r s''.
\]

Obviously, \( x_r \) divides \( s \) and \( s \in N \). We claim that \( \text{in}(x_r s'') < \text{in}(s') \). Since \( \text{in}(s) = \text{in}(s') \), the claim will then imply that \( \text{in}(s) = \text{in}(s') \), as desired.

In order to prove the claim, assume to the contrary that \( \text{in}(x_r s'') \geq \text{in}(s') \) for some \( i \). Since \( \text{in}(v_i f_i) \geq \text{in}(x_r s'') \) for some \( i \), it follows for this index \( i \) that \( \text{in}(x_r v_i f_i) \geq u_1 e_1 \) which implies that \( \text{in}(f_i) = v e_1 \) for some \( v \in \text{Mon}(S) \). In particular, \( \deg(e'_j) \leq \deg(f_i) \) for all \( j \). Since \( \deg(f_i) = \deg(g_i) \), and since there exists \( j < r \) such that \( \deg(g_i) j < \deg(s) j = \deg(u_1 e_1 j) \), it follows that \( \deg(e'_j) j < \deg(u_1 e_1 j) \) for some \( j < r \). This implies that \( x_j \) divides \( u_1 \), a contradiction. □

Let \( F \) be a finitely generated \( \mathbb{Z}^n \)-graded \( S \)-module with multigraded basis \( \mathcal{F} = e_1, \ldots, e_m \), and \( M \) a \( \mathbb{Z}^n \)-graded submodule of \( F \). In general it is not easy to compute \( \text{in}_F(M) \) explicitly. In the following we describe resolutions where for suitable bases the initial modules of the syzygies can be simply determined. These bases are however not necessarily lex- refined.

**Definition 1.3.** Let \( F \) be the resolution \( \square \) with differential \( \partial \). It has boundary Gröbner bases if for each \( F_p \) there exists a basis \( \mathcal{F}_p \) such that

\[
 \text{in}_{\mathcal{F}_p}(Z_p(F_i)) = (\text{in}_{\mathcal{F}_p}(\partial_{p+1}(e_i)): e_i \in \mathcal{F}_{p+1}).
\]

Resolutions with boundary Gröbner bases have the pleasant property that the initial modules of the syzygies can be immediately read off from the matrices describing the differential maps with respect to these bases.
The resolutions we have in mind arise as iterated mapping cones. So let \( I \subset S \) be a monomial ideal with monomial generators \( u_1, \ldots, u_m \). The iterated mapping cone resolution is constructed inductively by using induction on the number of generators. For \( m = 1 \) it is just the complex \( F_{i-1}^1 : 0 \to S(\mathbf{a}_1) \to S \to S/(u_1) \to 0 \), where \( \mathbf{a}_1 \) is the multidegree of \( u_1 \) and the differential of the complex is multiplication by \( u_1 \). Let \( I_j \) be the ideal generated by \( u_1, \ldots, u_j \), and suppose for some \( j < m \) we have already constructed the resolution \( F_{i-1}^j \) of \( S/I_j \). Consider the exact sequence

\[
0 \to I_{j+1}/I_j \to S/I_j \to S/I_{j+1} \to 0.
\]

Observe that

\[
I_{j+1}/I_j \cong (S/(I_j : (u_{j+1})))(-\mathbf{a}_{j+1}),
\]

where \( \mathbf{a}_{j+1} = \deg u_{j+1} \). Let \( G_{j}^{(j)} \) be a \( \mathbb{Z}^n \)-graded free \( S \)-resolution of this cyclic module and \( \varphi^{(j)} : G_{j}^{(j)} \to F_{j}^{(j)} \) a complex homomorphism of \( \mathbb{Z}^n \)-graded complexes extending the inclusion map \( I_{j+1}/I_j \to S/I_j \). Then we define \( F_{j+1}^{(j+1)} \) as the mapping cone of \( \varphi^{(j)} \).

The free resolution obtained by iterated mapping cones is not at all unique. It depends on the choice of the free resolutions \( G_{j}^{(j)} \) as well as on the complex homomorphisms \( \varphi^{(j)} \).

Here are a few prominent examples of resolutions which arise as iterated mapping cones.

**Examples 1.4.** (a) The Taylor complex (cf. [5]) is an iterated mapping cone. Let \( u_1, \ldots, u_m \) be a sequence of monomials. Assuming the Taylor complex for any sequence of monomials of length \( m - 1 \) is already constructed, one constructs the Taylor complex for \( u_1, \ldots, u_m \) by choosing for \( F_{i}^{(m-1)} \) the Taylor complex of the sequence \( u_1, \ldots, u_{m-1} \), and for \( G_{i}^{(m-1)} \) one takes the Taylor complex for the sequence

\[
u_1/\gcd(u_1, u_m), \ldots, u_{m-1}/\gcd(u_{m-1}, u_m)\]

which is a system of generators of \( (u_1, \ldots, u_{m-1}) : (u_m) \). The map \( \varphi_{m-1} \) can be defined in a canonical way.

The Taylor complex provides a \( \mathbb{Z}^n \)-graded free \( S \)-resolution of \( S/(u_1, \ldots, u_m) \) (which in general is not minimal). In case \( u_1, \ldots, u_m \) is a regular sequence, the Taylor complex coincides with the Koszul complex of this sequence and is minimal.

(b) A monomial ideal \( I \subset S \) is said to have linear quotients, if \( I \) is generated by homogeneous polynomials \( u_1, \ldots, u_m \) with \( \deg u_1 \leq \deg u_2 \leq \cdots \leq \deg u_m \) such that each of the colon ideals \( L_j = (u_1, \ldots, u_j) : (u_{j+1}) \) is generated by a subset of the variables. For such an ideal we can use in the construction of the iterated mapping cone for each \( j \) the Koszul complex as \( G_{j}^{(j)} \) to resolve \( S/L_j \). Considering the degrees of the resolutions at each step we see that \( \varphi_j(G_{j}^{(j)}) \subset mF_{j}^{(j)} \), so that in this case the iterated mapping cone provides a minimal free \( \mathbb{Z}^n \)-graded resolution of \( S/I \).

An important special case is that of a stable ideal. Recall that a monomial ideal \( I \) is called stable if for all monomial \( u \in I \) the monomial \( x_j(u/x_{m(u)}) \in I \) for all \( j < m(u) \). Here \( m(u) \) is the largest index with the property that \( x_{m(u)} \) divides \( u \). Let the minimal set of monomial generators \( u_1, \ldots, u_m \) of \( I \) be ordered in such a way that for \( i < j \) either \( \deg u_i < \deg u_j \), or \( \deg u_i = \deg u_j \) and \( u_j < u_i \) in the lexicographic order. Then with respect to this sequence of generators, \( I \) has linear quotients. The corresponding
iterated mapping cone yields the so-called Eliahou-Kervaire resolution of $S/I$, provided the complex homomorphisms $\varphi_j$ at each step are chosen properly.

The next lemma is a direct consequence of the following two observations:

(i) Let $F$ be a finitely generated $\mathbb{Z}^n$-graded $S$-module with multigraded basis $\mathcal{F} = e_1, \ldots, e_m$, and $M$ a $\mathbb{Z}^n$-graded submodule of $F$ with $\operatorname{in}_\mathcal{F}(M) = \bigoplus_{j=1}^m I_j e_j$. For $j = 1, \ldots, m$ set $F(j) = \bigoplus_{i=j}^m S e_i$ and let $F(m+1) = 0$. Then for the factors of the induced filtration

$$M = M \cap F(1) \supset M \cap F(2) \supset \cdots \supset M \cap F(m) \supset M \cap F(m+1) = 0$$

we have

$$M \cap F(j)/M \cap F(j+1) \cong I_j(- \deg(e_j)).$$

(ii) Let $M$ be a finitely generated $\mathbb{Z}^n$-graded module and $N \subset M$ a $\mathbb{Z}^n$-graded submodule of $M$, $F_i$ a $\mathbb{Z}^n$-graded free $S$-resolution of $M$, $G_i$ a $\mathbb{Z}^n$-graded free $S$-resolution of $N$ and $\varphi_i : G_i \to F_i$ a $\mathbb{Z}^n$-graded complex homomorphism which extends the inclusion map $N \to M$. The mapping cone $C_i$ of $\varphi_i$ is a $\mathbb{Z}^n$-graded free resolution of $M/N$. For the syzygies of these complexes we have for all $i \geq 0$ the following exact sequences

$$0 \to Z_i(F_i) \to Z_i(C_i) \to Z_{i-1}(G_i) \to 0,$$

where we set $Z_{-1}(G_i) = N$.

**Lemma 1.5.** With the notation introduced in (ii), let $\mathcal{G} = g_1, \ldots, g_r$ be a basis of $G_{i-1}$, $\mathcal{F} = f_1, \ldots, f_s$ a basis of $F_i$ and $\mathcal{C}$ the basis of $C_i$ which is obtained by composing $\mathcal{G}$ with $\mathcal{F}$, that is, $\mathcal{C} = g_1, \ldots, g_r, f_1, \ldots, f_s$. Then

$$\operatorname{in}_\mathcal{C} Z_i(C_i) = \operatorname{in}_\mathcal{G} Z_{i-1}(G_i) \oplus \operatorname{in}_\mathcal{F} Z_i(F_i).$$

In the following corollary we consider for the syzygy modules appearing in the preceding lemma, Gröbner bases with respect to monomial orders induced by the given bases.

**Corollary 1.6.** A Gröbner basis of $Z_i(C_i)$ is obtained by composing a Gröbner basis of $Z_i(F_i)$ with the preimages of the elements of a Gröbner bases of $Z_{i-1}(G_i)$ with respect to the epimorphism $Z_i(C_i) \to Z_{i-1}(G_i)$. In particular if $F_i$ and $G_i$ have boundary Gröbner bases, then $C_i$ has boundary Gröbner bases.

Now Corollary 1.6 yields

**Corollary 1.7.** Let the resolution $F_i$ of $S/(u_1, \ldots, u_n)$ be an iterated mapping cone by resolutions $G^{(j)}$ of $\mathcal{J}$ for $j = 1, \ldots, m-1$. If each $G^{(j)}$ has boundary Gröbner bases, then $F_i$ has boundary Gröbner bases.

As an application of these observations we obtain

**Proposition 1.8.** The Taylor complex and the iterated mapping cone of an ideal with linear quotients have boundary Gröbner bases. In particular, the Koszul complex attached to a regular sequence as well as the Eliahou-Kervaire resolution for stable ideals have boundary Gröbner bases.
For each $p$ with $M$ one has $sdepth F$ with already observed, for a short exact sequence of $\mathbb{Z}^n$-graded modules.

**Proof.** Let $F_p$ be the Taylor complex on a sequence of monomials of length $m$. The complexes $G^{(j)}$ are Taylor complexes on sequences of length $m-1$. Thus by using induction on $m$, it follows from Corollary 1.7 that $F_p$ has boundary Gröbner bases. On the other hand, if $F_p$ is an iterated mapping cone for an ideal with linear quotients, then all $G^{(j)}$ are Koszul complexes, which are special Taylor complexes, so that all $G^{(j)}$ have boundary Gröbner bases. Hence the desired result follows again by applying Corollary 1.7.

To be more concrete let $T_p$ be the Taylor complex attached with the sequence $u_1, \ldots, u_m$. For each $p$, $T_p$ has the following basis: $e_F = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_p}$ with $F = 1 \leq i_1 < i_2 < \cdots < i_p \leq m$, and the differential is given by

$$\partial_p(e_F) = \sum_{j=1}^{p} (-1)^{j+1} \frac{u_{F \setminus \{i_j\}}}{u_{F \setminus \{i_j\}}} e_{F \setminus \{i_j\}},$$

where for any subset $G \subset [n]$ we let $u_G$ be the least common multiple of the monomials $u_i$ with $i \in G$. If we order the basis elements iteratively as described in Lemma 1.5, then $e_m > e_{m-1} > \cdots > e_1$ and more generally $e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_p} > e_{i_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_p}$ if for some $k$ one has $i_p = j_p, \ldots, i_{k+1} = j_{k+1}$ and $i_k > j_k$. With this order, the elements $e_F$ with $F \subset [n]$ and $|F| = p$ form boundary Gröbner bases. Thus we obtain

$$\text{in}(Z_p(T_p)) = \bigoplus_{F \subset [m], |F| = p} I_F e_F,$$

with

$$(8) \quad I_F = \left( \frac{u_{F \cup \{i\}}}{u_F} \right)_{i \in [m], i < \min(F)}.$$

2. **Stanley depth of syzygies**

In this section we consider lower bounds for the Stanley depth of syzygies. First we give a lower bound in general for syzygies of $\mathbb{Z}^n$-graded submodules of free modules. Then, in the case of squarefree modules we can give a considerably better bound. The lower bounds have a form which is natural for syzygies. They essentially increase by one for each successive syzygy. However the actual behavior of Stanley depth of successive syzygy modules is probably far from the lower bound.

Our tool to obtain lower bounds for the Stanley depth is the following simple observation.

**Lemma 2.1.** Let $F$ be a finitely generated $\mathbb{Z}^n$-graded $S$-module with multigraded basis $F = e_1, \ldots, e_m$, $M$ a $\mathbb{Z}^n$-graded submodule of $F$, and $\text{in}_F(M) = \bigoplus_{j=1}^{m} I_j e_j$. Then

$$\text{sdepth } M \geq \min\{\text{sdepth } I_1, \ldots, \text{sdepth } I_m\}.$$

**Proof.** Let $M = M_0 \supset M_1 \supset \cdots \supset M_r = 0$ be any $\mathbb{Z}^n$-graded filtration of $M$. Since, as we already observed, for a short exact sequence of $\mathbb{Z}^n$-graded modules

$$0 \to M' \to M \to M'' \to 0$$

one has $\text{sdepth } M \geq \min\{\text{sdepth } M', \text{sdepth } M''\}$, we deduce that

$$\text{sdepth } M \geq \max_i \{\text{sdepth } M_i/M_{i+1}\}.$$
Applying this general fact to the filtration \( F \) induced by \( F \), the result follows from \( (6) \).

Now we present our main results concerning Stanley depth of syzygies.

**Theorem 2.2.** Let \( M \) be a \( \mathbb{Z}^n \)-graded submodule of a free module, and let \( F \) be a free resolution as in \( (1) \). Then for \( p \geq 0 \) the \( p \)'th syzygy module \( Z_p \) has Stanley depth greater than or equal to \( p + 1 \), or it is a free module.

**Proof of Theorem 2.2.** Let \( F \) be a lex-refined basis for \( F_p \). If \( p \geq n \) then \( Z_p \) is free, so suppose \( p < n \). By Theorem 1.1, in \( F(Z_p) = \bigoplus_{j=1}^m I_j e_j \), where the minimal set of monomial generators of each of the monomial ideals \( I_j \) belongs to \( K[x_{p+1}, \ldots, x_n] \). But then \( \text{sdepth} I_j \geq p + 1 \). In fact, Cimpoeaş [4, Corollary 1.5] showed that the Stanley depth of any \( \mathbb{Z}^n \)-graded torsionfree \( S \)-module is at least 1. Hence the asserted inequality for the Stanley depth of \( I_j \) follows from [8, Lemma 3.6]. Now the desired inequalities for the Stanley depths of the syzygy modules follow from \( (2.1) \).

**Remark 2.3.** In general the lower bound \( (p + 1) \) is probably far too small. W.Bruns, C.Krattenthaler, and J.Uliczka consider in [2] syzygies of the Koszul complex. They conjecture that the last half of these syzygies always have Stanley depth equal to \( n - 1 \).

On the other hand, the bound is sharp for the first syzygy module of any monomial ideal \( I \subset S = K[x_1, x_2, x_3] \) with \( \dim S/I = 0 \). Indeed, the predicted Stanley depth of \( Z_1(I) \) is at least 2. It cannot be three, because otherwise \( Z_1(I) \) would be free.

That indeed our lower bound for the Stanley depth is in general far too small can be seen in the following special case.

**Proposition 2.4.** Let \( I \subset S \) be a monomial complete intersection minimally generated by \( m \) elements, and \( Z_p \) the \( p \)'th syzygy module of \( S/I \). Then either \( Z_p \) is free or

\[
\text{sdepth} Z_p \geq n - \left\lfloor \frac{m - p}{2} \right\rfloor.
\]

**Proof.** Let \( u_1, \ldots, u_m \) be the regular sequence generating \( I \). The Taylor complex associated with this sequence, which is the Koszul complex, is a minimal free resolution of \( S/I \). With the notation of (8) we have \( I_F = (u_i)_{i \in [m], i < \min(F)} \), so that \( \text{sdepth} Z_p \geq \min\{\text{sdepth}(u_i; \ i < \min(F))\} \). By a result of Shen [9] one has \( \text{sdepth} J = n - \left\lfloor \frac{m}{2} \right\rfloor \) for a monomial ideal \( J \subset K[x_1, \ldots, x_n] \) generated by a regular sequence of length \( m \). This yields the desired conclusion.

In the case where \( M \) is a squarefree ideal or more generally a squarefree module, we also get better bounds. Recall that a \( \mathbb{Z}^n \)-graded \( S \)-module \( M \) is squarefree (defined by K.Yanagawa [11]), if it fulfils the following.

- \( M_a \) is nonzero only if \( a \in \mathbb{N}^n \).
- When \( a \) is in \( \mathbb{N}^n \), with nonzero \( i \)'th coordinate, and \( \varepsilon_i \) is the \( i \)'th unit coordinate vector, the multiplication map

\[
M_a \xrightarrow{x_i} M_a + \varepsilon_i
\]

is an isomorphism of vector spaces.
Squarefree modules form an abelian category with squarefree projective covers. In particular kernels of morphisms of squarefree modules are squarefree, and so syzygies modules in a squarefree resolution of a squarefree module, are squarefree.

**Lemma 2.5.** Let $M$ be a squarefree submodule of a free module $F$. Then for any term ordering on $F$, the initial module $\text{in}(M)$ is a squarefree module.

**Proof.** Let $g_1, \ldots, g_p$ be a basis for $M$, such that the $\text{in}(g_i) = u_r e_i$, are a basis for $\text{in}(M)$, and suppose $a_i \neq 0$. Then $x_i g_1, \ldots, x_i g_p$ are a basis for $M + e_i$, and their initial terms are the $x_i u_r e_i$, which form a basis for $\text{in}(M + e_i)$. □

In the last section we show that for squarefree ideals there is a considerably better lower bound for the Stanley depth than 1, which we use in Theorem 2.2. Using this we get the following.

**Theorem 2.6.** Let $M$ be a squarefree submodule of a free module, with $d+1$ the smallest degree of a generator of $M$. Let $s$ be the largest integer such that $(2s + 1)(s + 1) \leq n + 1 - d - p$. Then for $p \geq 1$ the $p$'th syzygy module in a squarefree resolution of $M$ is either free, or it has Stanley depth greater or equal to $2s + 1 + d + p$. Explicitly this is

$$2 \left\lceil \frac{\sqrt{2n - 2d - 2p} + 2.25 + 0.5}{2} \right\rceil + d + p - 1.$$

**Proof.** Use a term order as in (2) on the $p$'th term $F_p$ in the resolution. We get

$$\text{in}(Z_p) = \bigoplus_{j=1}^{m} I_j e_j,$$

where $I_j$ is a squarefree monomial ideal. Since $\text{in}(Z_p)$ is a squarefree module, each generator $g_j e_j$ of $I_j e_j$ is a squarefree term.

Suppose first that the resolution is minimal. Then the total degree of $e_j$ is at least $d + p$ where $d + p \leq n$. Hence the generators of $I_j$ involves no more than $n - d - p$ variables, corresponding to the coordinates of the multidegree of $e_j$ which are zero. The result then follows from Lemma 2.1 and Theorem 3.4.

In the case that the resolution is not necessarily minimal, the syzygies $Z'_p$ of such a resolution differ from the syzygies $Z_p$ of the minimal free resolution by a free summand, that is, $Z'_p = Z_p \oplus F$, so either $Z'_p$ is free or it has Stanley depth greater or equal to the Stanley depth of $Z_p$. □

3. **Stanley depth of squarefree monomial ideals**

In this section we show that the Stanley depth of any squarefree monomial ideal in $n$ variables, is bounded below by a bound of order $\sqrt{2n}$. This is quite in contrast to ordinary depth where for instance the maximal ideal $(x_1, \ldots, x_n)$ has depth one.

Our argument is based on a construction of interval partitions by M.Keller et al. which is further refined M.Ge, J.Lin, and Y.-H.Shen in [6]. The argument is an application of Proposition 3.5 in [6].

We recall the construction of [7]. Let $[n] = \{1, 2, \ldots, n\}$. For subsets $A$ and $B$ of $[n]$, the interval $[A, B]$ consists of the subsets $C$ of $[n]$ such that $A \subseteq C \subseteq B$. We think of the
elements of $[n]$ arranged clockwise around the circle and for $i, j$ in $[n]$ let the block $[i, j]$ be the set of points starting with $i$, going clockwise, and ending with $j$. Now given $A \subseteq [n]$, and a real number $\delta \geq 1$, called a density, the block structure of $A$ with respect to $\delta$ is a partition of the elements of $[n]$ into connected blocks $B_1, G_1, B_2, G_2, \ldots, B_p, G_p$ fulfilling the following.

- The first (going clockwise) element of $b_i$ of $B_i$ is in $A$.
- Each $G_i$ is disjoint from $A$.
- For each $B_i$ we have
  \[ \delta \cdot |A \cap B_i| - 1 < |B_i| \leq \delta \cdot |A \cap B_i|. \]
- For each $y$ such that $[b_i, y]$ is a proper subset of $B_i$, we have
  \[ |[b_i, y]| + 1 \leq \delta \cdot |[b_i, y] \cap A|. \]

For $1 \leq \delta \leq \frac{(n-1)}{n}$ the block structure for a subset $A$ exists and is unique by Lemma 2.7 in [7]. Let $\mathcal{G}_\delta$ be the union $G_1 \cup \cdots \cup G_p$. We then define the set $f_\delta(A)$ to be $A \cup \mathcal{G}_\delta (A)$. The intervals we shall study will now be of the form $[A, f_\delta(A)]$ or closely related.

For certain values of $n$ and cardinalities of $A$, the intervals fulfil some very nice properties. The following are basic facts from [7]. It is a synopsis of Lemma 3.1, Lemma 3.2 and Lemma 3.5 there.

**Lemma 3.1.** Let $n = as + a + s$.

1. If $A \subseteq [n]$ is an $a$-set, then $f_{s+1}(A)$ is an $(a + s)$-set.
2. The intervals $[A, f_{s+1}(A)]$ are disjoint when $A$ varies over the $a$-sets.

We are interested in getting disjoint intervals, but we need a way to adopt the above lemma to the case of arbitrary $n$, and to be able to vary $a$ and $s$. The following still fixes $s$ and $a$ but allows $n$ to be arbitrary above a bound. It is Proposition 3.3 in [6].

**Proposition 3.2.** Let $n \geq as + a + s$ and $A \subseteq [n]$ an $a$-set. Consider $\tilde{A} = A \cup \{n + 1, \ldots, n + n - a\}$ as a subset of $[ns + n + s]$.

1. The intersection $f_{s+1}(\tilde{A}) \cap [n]$ is an $(a + s)$-set.
2. The intervals $[A, f_{s+1}(\tilde{A}) \cap [n]]$ are disjoint as $A$ varies over the $a$-sets.

Finally we need to be more flexible with $a$ and $s$, and still have disjoint intervals. The following is Proposition 3.5 of [6] specialized to the case when $d = 1, d + q = a$ and $d + l = b$.

**Proposition 3.3.** Let $A$ and $B$ be subsets of $[n]$ of cardinalities $a \leq b$. Suppose $s' \leq s$ are non-negative integers such that

\[ n + 1 \geq (b + 1)(s' + 1) \geq (a + 1)(s + 1). \]

Consider $\tilde{A}$ as a subset of $[ns + n + s]$ and $\tilde{B}$ as a subset of $[ns' + n + s']$. Then if $B$ is not in $[A, f_{s+1}(\tilde{A}) \cap [n]]$, this interval is disjoint from the interval $[B, f_{s' + 1}(\tilde{B}) \cap [n]]$.

We are now ready to prove our theorem.
Theorem 3.4. Let $s$ be the largest integer such that $n + 1 \geq (2s + 1)(s + 1)$. Then the Stanley depth of any squarefree monomial ideal in $n$ variables is greater or equal to $2s + 1$. Explicitly this lower bound is

$$2 \left\lfloor \frac{\sqrt{2n + 2.25} + 0.5}{2} \right\rfloor - 1.$$ 

Remark 3.5. For $n = 5$, the above bound says that the Stanley depth is greater than or equal to $3$ which is best possible, since this is the Stanley depth of the maximal ideal $(x_1, \ldots, x_5)$.

Remark 3.6. In [4] it is shown that the Stanley depth of the squarefree Veronese ideal generated by squarefree monomials of degree $d$ has Stanley depth less or equal to $\left\lfloor \frac{n + 1}{d + 1} \right\rfloor + d - 1$. With $d + 1$ approximately $\sqrt{n} + 1$ this is approximately $2\sqrt{n} + 1 - 2$. Thus our lower bound is right up to a constant.

Proof of Theorem 3.4. The squarefree ideal $I$ corresponds to an order filter $P_I$ of the poset consisting of subsets of $[n]$, by taking supports of the squarefree monomials in $I$. By J.Herzog et.al [8], if we have a partition $P$ of $P_I$, the Stanley depth of $I$ is greater or equal to the minimum cardinality of any subset $B$ of $[n]$ such that $[A, B]$ is an interval in $P$. We shall therefore construct a suitable partition to give the lower bound.

Given positive integers $n, r$ and $s$ with $r > s$ and

$$(n + 1) \geq (r + 1)(s + 1).$$

Define the sequence $\sigma : [r] \rightarrow \mathbb{Z}$ by

$$\sigma(i) = \begin{cases} s, & i \geq s + 1 \\ s + k, & i = s + 1 - k \leq s + 1 \end{cases}.$$ 

Note that if $u \leq v$ then $(u + 1)v \geq u(v + 1)$. Therefore the expression $(i + 1)(\sigma(i) + 1)$ weakly decreases as $i$ decreases, enabling us to apply Proposition 3.3.

We now construct a partition $\mathcal{P}$ of $P_I$ as follows. Let $\mathcal{P}_1$ consist of all intervals

$$\{\{i\}, f_{\sigma(1)+1}([i]) \cap [n]\}$$

where $\{i\}$ is in $P_I$. Let $\mathcal{P}_2$ consist of all intervals

$$\{\{i_1, i_2\}, f_{\sigma(2)+1}([i_1, i_2]) \cap [n]\}$$

where $\{i_1, i_2\}$ is in $P_I$ but not in any of the intervals in $\mathcal{P}_1$. By Proposition 3.3, $\mathcal{P}_2$ will consist of disjoint intervals. Having constructed $\mathcal{P}_{n-1}$ we construct $\mathcal{P}_n$ by adding to $\mathcal{P}_{n-1}$ all intervals $[A, f_{\sigma(a)+1}(\tilde{A})] \cap [n]$ where $A$ is an $a$-set in $P_I$ not in any interval of $\mathcal{P}_{n-1}$. Having reached $\mathcal{P}_r$ we obtain $\mathcal{P}$ by adding all trivial intervals $[B, B]$ where $B$ is in $P_I$ but not in any of the intervals in $\mathcal{P}_r$. Note that each such $B$ has cardinality greater or equal to $r + 1$.

The Stanley depth of the partition $\mathcal{P}$ will be the smallest of the numbers $r + 1$ and $i + \sigma(i)$ for $i = 1, \ldots, r$, which is the minimum of $r + 1$ and $2s + 1$. Choose $r = 2s$ and $s$
to be the largest number such that \( n + 1 \geq (2s + 1)(s + 1) \). Then the Stanley depth of \( P \) is \( 2s + 1 \). We get

\[
(2s + 1)(s + 1) \leq n + 1 \\
4s^2 + 6s + 2 \leq 2n + 2 \\
(2s + 1.5)^2 \leq 2n + 2.25 \\
2s + 2 \leq \sqrt{2n + 2.25} + 0.5
\]

which gives

\[
s \leq \left\lfloor \frac{\sqrt{2n + 2.25} + 0.5}{2} \right\rfloor - 1
\]

\[
2s + 1 \leq 2 \left\lfloor \frac{\sqrt{2n + 2.25} + 0.5}{2} \right\rfloor - 1.
\]

The largest value of \( 2s + 1 \) is then given by the right side above. \( \square \)

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