Four-pion production

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Starting from the low-energy structure derived from QCD, we extend the amplitudes for four-pion production in \( e^+e^- \) annihilation and \( \tau \) decays up to invariant four-pion masses of 1 GeV. Cross sections and branching ratios \( BR(\rho^0 \rightarrow 4\pi) \) are compared with available data.

1. INTRODUCTION

The production of four pions in \( e^+e^- \) annihilation and \( \tau \) decays is interesting in its own right but it also represents a non-negligible component of hadronic vacuum polarization. In fact, almost 5 % of the lowest-order hadronic contribution to the anomalous magnetic moment of the muon \( a_\mu \) is due to four pions [1]. At the level of accuracy required for a comparison of the standard model prediction for \( a_\mu \) with the recent BNL measurement [2], better knowledge of this contribution would be welcome (see the discussion in Ref. [1]).

For the determination of the fine structure constant at \( s = M_Z^2 \), the relative importance of the four-pion contribution is even bigger but in this case the available precision is sufficient for the time being.

In this talk, I report on the work of Ref. [3] where we have constructed the relevant amplitudes up to invariant four-pion masses of about 1 GeV. This is not enough for the purpose of calculating \( a_\mu \), but it is a first step in this direction. Even if it is impossible to calculate the amplitudes directly from QCD our aim was to construct amplitudes that are at least consistent with QCD. This program turned out to have some surprises in store.

The procedure of Ref. [3] starts from a calculation to next-to-leading order in the low-energy expansion of QCD, employing the methods of chiral perturbation theory (CHPT) [4]. The corresponding low-energy amplitudes cannot be used directly in the physical region because the four-pion threshold of 560 MeV is already close to the resonance region. But the low-energy amplitudes contain nontrivial information how to continue to higher energies. At next-to-leading order, the traces of \( \rho \) and scalar meson exchange appear in the amplitudes. Supplemented by \( \omega, a_1 \) and double \( \rho \) exchange, the resulting \( e^+e^- \) cross sections describe the available experimental data very well up to cms energies of about 1 GeV.

2. SYMMETRIES AND LOW-ENERGY LIMIT

There are altogether four different channels accessible in \( e^+e^- \) annihilation and \( \tau \) decays into four pions. In the isospin limit that we assume throughout, the amplitudes of either \( e^+e^- \rightarrow 2\pi^0\pi^+\pi^- \) or \( \tau^- \rightarrow \nu_\tau 2\pi^-\pi^+\pi^0 \) are sufficient to determine all four amplitudes. One important advantage of the chiral approach is that not only chiral symmetry but also charge conjugation invariance, Bose symmetry and electromagnetic gauge invariance are manifest at each stage of the calculation.

At leading order in the low-energy expansion, the amplitude is completely determined by “virtual” bremsstrahlung. From the diagrams in Fig. 1, the matrix element of the electromagnetic current governing the amplitude for \( e^+e^- \rightarrow 2\pi^0\pi^+\pi^- \) is found to be

\[
\langle \pi^0(p_1)\pi^0(p_2)\pi^+(p_3)\pi^-(p_4)|J^{\mu}_{elm}(0)|0 \rangle = \frac{s - M_\pi^2}{F_\pi^2} \left( \frac{2p_3^{\mu}}{2p_3 \cdot q - q^2} - \frac{2p_4^{\mu}}{2p_4 \cdot q - q^2} \right),
\]
where \( s = (p_1 + p_2)^2 \), \( q = p_1 + p_2 + p_3 + p_4 \) and \( F_\pi = 92.4 \text{ MeV} \) is the pion decay constant. This matrix element has a very suggestive structure: \((s - M_\pi^2)/F_\pi^2 \) is the (lowest-order) scattering amplitude for \( \pi^0 \pi^0 \rightarrow \pi^+ \pi^- \) and the second term reduces to the usual bremsstrahlung factor for real photons \( (q^2 \rightarrow 0) \).

Although the amplitude (1) is by itself not of phenomenological relevance we can use the isospin relations (5) and calculate also the leading-order \( \tau \) decay amplitudes. In the chiral limit \( (M_\tau = 0) \), those amplitudes should coincide with those of Ref. (6). Unfortunately, there are two (identical) misprints in Ref. (6) that have meanwhile propagated into some of the subsequent literature (7 and references therein). Although the structure is correct, the normalization is not: the correct amplitude (1) and the corresponding \( \tau \) decay amplitudes are smaller by a factor \( \sqrt{2}/(3\sqrt{3}) \) (or 1/13.5 in rate). This normalization error also affects the so-called CLEO current (13) in the Monte Carlo package TAUOLA (8).

The “current algebra” amplitude (1) is important for checking the low-energy limit of QCD but it is not a realistic approximation in the physical region. In order to see the traces of meson resonance exchange, we have to go at least to next-to-leading order.

3. Resonance Exchange

At next-to-leading order in the chiral expansion, the amplitude consists of two parts: a loop amplitude (9) and a tree amplitude containing the (renormalized) coupling constants of the chiral Lagrangian of \( O(p^4) \) (14). With the standard values of those constants, one arrives at cross sections that are still unrealistic. As indicated by the dotted curves in Figs. 2,3, the theoretical cross sections are significantly smaller than the measured ones.

The seeds of meson resonance exchange appear first in the coupling constants of \( O(p^4) \). In fact, those constants are known to be saturated by meson resonance exchange to a large extent (1,2). This saturation makes the matching between the strictly chiral amplitude to \( O(p^4) \) and a more realistic meson resonance exchange amplitude almost trivial. Using the standard chiral resonance Lagrangian (1), the resonance amplitudes are guaranteed to exhibit the correct low-energy behaviour to \( O(p^4) \).

In four-pion production, only the \( \rho \) and the (isoscalar) scalar mesons contribute at \( O(p^4) \). As could have been expected, \( \rho \) exchange dominates by far. The overall contribution from scalar exchange turns out to be very small so that the controversial structure of the scalar sector is not relevant in practice.

The modified amplitudes with \( \rho \) and scalar exchange are definitely more realistic than the chiral low-energy amplitudes. However, except in the vicinity of the \( \rho \) pole, the resulting cross sections are still too small (not shown in Figs. 2,3). The obvious lesson is that important ingredients of the amplitudes are still missing that only show up at orders \( p^6 \) or higher in the chiral expansion.

The most important missing degrees of freedom are easily found: both quantum number considerations and experimental information (13,14) indicate that \( \omega \) and \( a_1 \) exchange must be incorporated. Whereas the lowest-order coupling of the \( \omega \) to pions is unique, there is some ambiguity in the \( a_1 \rho \pi \) couplings (to be resolved eventually by studies of three-pion production (9)). With a simplifying assumption for those couplings (9), including double \( \rho \) exchange that also comes in at \( O(p^6) \) and performing a resummation of some terms making up the \( \rho \)-dominated pion form factor, the amplitudes assume their final form. Except for the ambiguity in the \( a_1 \rho \pi \) couplings, all resonance couplings can be determined from the respective decay widths.
4. COMPARISON WITH DATA

The two main assets of our amplitudes are:

- They contain the relevant degrees of freedom for describing four-pion production up to energies of about 1 GeV.
- They exhibit the correct low-energy behaviour to $O(p^4)$ by construction.

For energies below 1 GeV, annihilation data are available for the channel $2\pi^+2\pi^-$ mainly. In Fig. 2, the theoretical cross sections are compared with the most recent (and most precise) data from the CMD-2 Collaboration [14] (see Ref. [3] for the full data set). The cross section for our model is shown as the full curve. The dashed curve corresponds to omitting the loop amplitude of $O(p^4)$ [10] (except for the contribution to the width of the $\rho$ meson). The obvious conclusion is that the amplitude is completely dominated by resonance exchange. Although a possible enhancement of the cross section in the region between 800 and 900 MeV could not be explained with our amplitude the gross features of the data can be reproduced over a range of two orders of magnitude with almost no free parameters. In retrospect, the simplifying assumption for the $a_1\rho\pi$ couplings (that actually has a theoretical basis [15]) is justified by comparison with experiment: other choices for the couplings could not reproduce the data. Note that $\omega$ exchange does not contribute in this channel: the cross section near 1 GeV is completely dominated by $a_1$ exchange.

The experimental situation is less satisfactory for the other annihilation channel $2\pi^0\pi^+\pi^-$. The most precise experiment [13] has measured the cross section at only two energies below 1.05 GeV. The comparison between theory and experiment is shown in Fig. 3. The loop contribution is even less relevant in this case. The theoretical cross section increases by three orders of magnitude from the $\rho$ resonance to match the two data points. For this channel, the cross section near 1 GeV is dominated by $\omega$ exchange. The theoretical prediction is therefore less sensitive to assumptions about the $a_1\rho\pi$ couplings.

Our amplitudes and the corresponding cross sections cannot be extended to the phenomenologically most interesting region above 1 GeV without further input. The reason is that the amplitudes do not satisfy the high-energy constraints of QCD. In fact, the theoretical cross sections exceed the data soon above 1 GeV of cms energy. Additional higher-mass states must be included to access the region up to 2 GeV and to ensure a proper high-energy behaviour. Resummations similar to the pion form factor may also be necessary.

From our amplitudes we can also extract the branching ratios for the four-pion decay modes of the $\rho^0$. For $q^2 = M_\rho^2$ several contributions to the amplitudes are of comparable size, with partly destructive interference. In this way, uncertainties in the resonance couplings are enhanced. We therefore quote predictions for the branching ra-
tios with a 40 % uncertainty:

\[
BR(\rho^0 \to 2\pi^+ 2\pi^-) = (6.7 \pm 2.7) \times 10^{-6} \quad (2)
\]
\[
BR(\rho^0 \to 2\pi^0 \pi^+ \pi^-) = (5.0 \pm 2.0) \times 10^{-6} \quad (3)
\]

For comparison, the Particle Data Group lists

\[
BR(\rho^0 \to 2\pi^+ 2\pi^-) = (1.8 \pm 0.9) \times 10^{-5} \quad (4)
\]
\[
BR(\rho^0 \to 2\pi^0 \pi^+ \pi^-) < 4 \times 10^{-5} \quad (5)
\]

5. CONCLUSIONS

The following features of our model for four-pion production are worth repeating:

- The amplitudes exhibit the correct low-energy structure to \(O(p^4)\) in the chiral expansion.

- All symmetries of the transitions are manifest in the QFT framework of CHPT: (broken) chiral symmetry, gauge invariance, Bose symmetry and charge conjugation.

- In addition to \(\rho\) (and the less important scalar) exchange, \(\omega\) and \(a_1\) exchange are crucial for understanding the experimental results already at energies below 1 GeV.

- Good agreement with available data is obtained, covering several orders of magnitude in cross sections.

- To extend the amplitudes to energies above 1 GeV, the correct high-energy behaviour still needs to be implemented. For the same reason, comparison with \(\tau\) decay data is postponed.

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