ON THE PERIODIC AND ASYMPTOTICALLY PERIODIC NONLINEAR
HELMHOLTZ EQUATION

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Abstract. In the first part of this paper, the existence of infinitely many $L^p$-standing wave solutions for the nonlinear Helmholtz equation

$$-\Delta u - \lambda u = Q(x)|u|^{p-2}u \quad \text{in } \mathbb{R}^N$$

is proven for $N \geq 2$ and $\lambda > 0$, under the assumption that $Q$ be a nonnegative, periodic and bounded function and the exponent $p$ lies in the Helmholtz subcritical range. In a second part, the existence of a nontrivial solution is shown in the case where the coefficient $Q$ is only asymptotically periodic.

1. Introduction

In this paper, we consider for $N \geq 2$ the semilinear equation

$$-\Delta u - \lambda u = Q(x)|u|^{p-2}u, \quad x \in \mathbb{R}^N,$$

where $\lambda > 0$, $Q$ is a bounded and nonnegative function, and the exponent $p$ lies in the subcritical range $2_* < p < 2^*$, where

$$2_* := \frac{2(N+1)}{N-1} \quad \text{and} \quad 2^* = \begin{cases} \frac{2N}{N-2} & \text{if } N \geq 3 \\ \infty & \text{if } N = 2. \end{cases}$$

We study existence of real-valued solutions $u: \mathbb{R}^N \to \mathbb{R}$ that decay to zero at infinity. Such solutions correspond, via the Ansatz

$$\psi(t,x) := e^{i\sqrt{\lambda}t}u(x),$$

to weakly spatially decaying, standing wave solutions of the Nonlinear Wave Equation

$$\partial_t \psi - \Delta \psi = Q(x)|\psi|^2\psi, \quad t \in \mathbb{R}, \ x \in \mathbb{R}^N.$$ 

Under general assumptions on $Q$, the problem cannot be handled using direct variational methods, since its solutions (if any) are not expected to decay faster than $O(|x|^{-\frac{N+1}{N}})$ at infinity and will therefore not belong to the space $L^2(\mathbb{R}^N)$. Recently, a dual method has been set up, which allows to study this problem variationally. Using it, the existence of nontrivial solutions lying in $L^p(\mathbb{R}^N)$ for all $q \geq p$ and admitting an expansion of the form

$$\lim_{R \to \infty} \frac{1}{R} \int_{B_R} \left| u(x) + 2 \left( \frac{2\pi}{\sqrt{\lambda}|x|} \right)^{\frac{N-1}{2}} \text{Re} \left[ e^{i\sqrt{\lambda}|x|-i\frac{(N-1)}{4}g_u \left( \frac{x}{|x|} \right)} \right] \right|^2 \ dx = 0,$$

where $g_u(\xi) = -\frac{i}{4} \left( \frac{2\pi}{\sqrt{\lambda}} \right)^{\frac{N-2}{2}} \mathcal{F} \left( Q|u|^{p-2}u \right)(\sqrt{\lambda}\xi), \ \xi \in S^{N-1},$ was proven (see [17, 18]).

Here, $\mathcal{F}$ denotes the Fourier transform. More precisely, infinitely many nontrivial bounded $W^{2,p}(\mathbb{R}^N)$-solutions were obtained under the assumption $\lim_{|x| \to \infty} Q(x) = 0$ (see also [15] where

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more general nonlinearities were considered). For periodic $Q$, only the existence of a non-trivial solution (pair) was proven, and one of the main goals of the present paper is to show that $\| \$ in fact possesses infinitely many geometrically distinct solutions in $W^{2,p}(\mathbb{R}^N)$.

In the case where $\lambda < 0$, or more generally when $-\lambda$ is replaced by a bounded and periodic function $V$ satisfying $\inf V > 0$, results giving the existence of infinitely many solutions for $\| \$ go back to the work of Coti Zelati and Rabinowitz $[12]$. Shortly after, Alama and Li $[5]$ using a dual method, and Kryszewski and Szulkin $[19]$ using a linking argument and a new degree theory, extended this result to the case where $0$ lies in a spectral gap of the Schrödinger operator $-\Delta + V(x)$. More recently, multiplicity results were given for the periodic Schrödinger equation with more general nonlinearities (see e.g. $[1] [2] [11] [14] [23] [22]$ and the references therein).

In the present paper, we show that the proof scheme developed by Szulkin and Weth in $[24]$ Theorem 1.2 for the periodic Schrödinger equation and used recently by Squassina and Szulkin $[22]$ in the case of a logarithmic nonlinearity, can be adapted to work in combination with the dual variational method. In particular, using the framework of $[17]$, we obtain the following result.

**Theorem 1.1.** Let $2_+ < p < 2^*$ and consider $Q \in L^\infty(\mathbb{R}^N) \setminus \{0\}$ nonnegative and $\mathbb{Z}^N$-periodic. Then $\| \$ has infinitely many geometrically distinct pairs of strong solutions $\pm u_n \in W^{2,p}(\mathbb{R}^N)$, $p \leq q < \infty$.

The proof (see Section 3) is based on the fact that, assuming that only finitely many geometrically distinct solutions exist, one is able to show the discreteness of Palais-Smale sequences for the energy functional and then apply a deformation argument to get a contradiction. The main technical point, when passing to the dual method is to find suitable sequences for the energy functional and then apply a deformation argument to get a contradiction. The main technical point, when passing to the dual method is to find suitable sequences for the energy functional and then apply a deformation argument to get a contradiction. The main technical point, when passing to the dual method is to find suitable sequences for the energy functional and then apply a deformation argument to get a contradiction.

Among one of the problems most extensively studied in the context of nonlinear Schrödinger equation is the case of asymptotically periodic or asymptotically autonomous nonlinearity. Starting with the work by Ding and Ni $[13]$ based on Lions’ concentration-compactness principle, many authors studied this case (see e.g. $[7] [10] [21]$ and the references therein). In Section 3 of the present paper, we give an analogous existence result for the nonlinear Helmholtz equation. We namely study $\| \$ in the case where the coefficient $Q$ satisfies $|Q(x) - Q_\infty(x)| \to 0$ as $|x| \to \infty$ for some periodic function $Q_\infty$. Under assumptions close to those of Ding and Ni $[13]$, we obtain the existence of a nontrivial solution (see Theorem 1.3). Our proof uses the fibered method applied to the dual energy functional, and we show that the dual ground-state is attained under these assumptions.

Let us mention that the above results can be extended to slightly more general nonlinearities. Indeed, using a dual variational method in Orlicz spaces (as discussed e.g. in $[15]$ it can be shown that the problem

$$-\Delta u - \lambda u = \sum_{i=1}^m A(x)^{p_i} Q_i(x) |u|^{p_i-2} u, \quad x \in \mathbb{R}^N$$

with $\lambda > 0$ has infinitely many geometrically distinct solutions under the assumptions:

(i) $2_* < p_1 \leq p_2 \leq \ldots \leq p_m < 2^*$,
(ii) $Q_1, \ldots, Q_m \in L^\infty(\mathbb{R}^N) \setminus \{0\}$ are $\mathbb{Z}^N$-periodic with $0 < \inf_{\mathbb{R}^N} Q_i \leq \sup_{\mathbb{R}^N} Q_i < \infty$ for all $1 \leq i \leq m$, and
(iii) $A \in L^\infty(\mathbb{R}^N) \setminus \{0\}$ is nonnegative and $\mathbb{Z}^N$-periodic.
The outgoing radial fundamental solution of the Helmholtz operator given by
\[ \Phi(x) = \frac{i}{4}(2\pi|x|)^{N-2}H_2^{(1)}(|x|), \quad x \neq 0, \]
where \( H_2^{(1)} \) denotes the first Hankel function (or Bessel function of the third kind, see \([20]\)), and we denote by \( \Psi \) its real part. As a consequence of results by Kenig, Ruiz and Sogge \([18]\), the mapping \( R: S(\mathbb{R}^N) \rightarrow S'(\mathbb{R}^N) \),
\[ Rf := \Psi * f = \text{Re}(\Phi) * f, \quad f \in S(\mathbb{R}^N), \]
where \( S(\mathbb{R}^N) \) and \( S'(\mathbb{R}^N) \) denote respectively the Schwartz space and the space of tempered distributions, can be extended in a unique way to a continuous operator \( R: L^{p'}(\mathbb{R}^N) \rightarrow L^p(\mathbb{R}^N) \) for each \( 2_* \leq p \leq 2^* \) (resp. \( p < \infty \) in the case \( N = 2 \)). Here and in the following, \( p' = \frac{p}{p-1} \) denotes the conjugate exponent to \( p \). Moreover, for every \( f \in L^{p'}(\mathbb{R}^N) \), the function \( u = Rf \) belongs to \( W^{2,p'}(\mathbb{R}^N) \) and solves the Helmholtz equation \( -\Delta u - u = f \) in the strong sense.

Let \( Q \in L^\infty(\mathbb{R}^N) \) be a nonnegative function, \( Q \neq 0 \) and define for \( 2_* \leq p \leq 2^* \) with \( p < \infty \), the Birman-Schwinger type operator \( K: L^{p'}(\mathbb{R}^N) \rightarrow L^p(\mathbb{R}^N) \) by setting
\[ Ku := \frac{Q}{Q^*}R(Q^*v), \quad v \in L^{p'}(\mathbb{R}^N). \]
This operator is symmetric, i.e., \( \int_{\mathbb{R}^N} wK(u) \, dx = \int_{\mathbb{R}^N} wK(v) \, dx \) holds for \( v, w \in L^{p'}(\mathbb{R}^N) \). Moreover, when \( p < 2^* \) it is locally compact in the sense that for every bounded and measurable set \( B \subset \mathbb{R}^N \), the operator \( 1_B K: L^{p'}(\mathbb{R}^N) \rightarrow L^p(\mathbb{R}^N) \) is compact (see \([17\) Lemma 4.1]). Here and in the sequel \( 1_B \) will stand for the characteristic function of the set \( B \).

Consider the energy functional \( J: L^{p'}(\mathbb{R}^N) \rightarrow \mathbb{R} \)
\[ J(v) = \frac{1}{p'} \int_{\mathbb{R}^N} |v|^{p'} \, dx - \frac{1}{2} \int_{\mathbb{R}^N} Q(x)^{\frac{1}{p'}}v(x)R(Q^*v)(x) \, dx = \frac{1}{p'}\|v\|_{p'}^p - \frac{1}{2} \int_{\mathbb{R}^N} vK(v) \, dx. \]
We note that \( J \) is of class \( C^1 \) with
\[ J'(w) = \int_{\mathbb{R}^N} \left( |v|^{p'-2}v - K(v) \right) w \, dx \quad \text{for all } v, w \in L^{p'}(\mathbb{R}^N). \]
Furthermore, every critical point of \( J \) corresponds to a solution of \( (1) \). More precisely, \( v \in L^{p'}(\mathbb{R}^N) \) satisfies \( J'(v) = 0 \) if and only if it solves the integral equation
\[ |v|^{p'-2}v = Q^\frac{1}{p'}R(Q^*v). \]
Setting \( u = R(Q^{\frac{1}{p}}v) \in L^p(\mathbb{R}^N) \), it follows that
\[
u = R(Q|u|^{p-2}u),
\]
and \( u \in W^{2,q}(\mathbb{R}^N) \), \( q \geq p \), is therefore a strong solution of (1) (see [17] Lemma 4.3 and Theorem 4.4) and [16, Theorem 1.1] concerning the regularity and asymptotic behavior of \( u \). We note also that \( u = 0 \) if and only if \( v = 0 \).

The geometry of the functional \( J \) is of mountain pass type (cf. the original paper by Ambrosetti and Rabinowitz [6]):

**Lemma 2.1.**

(i) There exist \( \alpha > 0 \) and \( 0 < \rho < 1 \) such that \( J(v) \geq \alpha > 0 \) for all \( v \in L^p(\mathbb{R}^N) \) with \( \|v\|_p = \rho \).

(ii) For every \( m \in \mathbb{N} \), there exists an \( m \)-dimensional subspace \( W_m \subset C^\infty(\mathbb{R}^N) \) and some \( R = R(W_m) > 0 \) such that \( J(v) \leq 0 \) for every \( v \in W_m \) with \( \|v\|_{p'} \geq R \).

The proof of these results can be found in [17] Lemma 4.2 and Lemma 5.1 and [16] Proof of Theorem 1.3(ii)).

We mention a useful property of the Palais-Smale sequences for \( J \) which can be deduced from the results in [17]. For the reader’s convenience, we give a short proof.

**Lemma 2.2.** Suppose \( 2_* \leq p < 2^* \), and let \( (v_n)_n \subset L^p(\mathbb{R}^N) \) be a Palais-Smale sequence for \( J \). Then \( (v_n)_n \) is bounded in \( L^p(\mathbb{R}^N) \) and (up to a subsequence) there exists \( v \in L^p(\mathbb{R}^N) \) such that \( J(v) = 0 \), \( v_n \rightharpoonup v \) weakly in \( L^p(\mathbb{R}^N) \) and \( J(v) \leq \liminf_{n \to \infty} J(v_n) \). Moreover, for any bounded and measurable set \( B \subset \mathbb{R}^N \) there holds \( 1_B v_n \to 1_B v \) strongly in \( L^p(\mathbb{R}^N) \) and \( 1_B |v_n|^{p-2} v_n \to 1_B |v|^{p-2} v \) strongly in \( L^p(\mathbb{R}^N) \).

**Proof.** To prove the boundedness of the sequence, we choose \( C > 0 \) such that \( J(v_n) \leq C \) and \( \|J'(v_n)\| \leq C \) for all \( n \in \mathbb{N} \). If \( \|v_n\|_{p'} \geq 1 \) we can write
\[
C \geq \frac{J(v_n)}{\|v_n\|_{p'}} - \frac{1}{2} J'(v_n) \frac{v_n}{\|v_n\|_{p'}} = \left( \frac{1}{p} - \frac{1}{2} \right) \|v_n\|_{p'}^{p'-1}.
\]
As a consequence, we find for all \( n \), \( \|v_n\|_{p'}^{p'-1} \leq \max\{1, \left( \frac{2}{p'} - \frac{1}{2} \right)^{-1} C \} \), since \( 1 < p' < 2 \), thus showing that \( (v_n)_n \) is bounded in \( L^p(\mathbb{R}^N) \). Hence, going if necessary to a subsequence, we may assume that \( v_n \rightharpoonup v \) weakly in \( L^p(\mathbb{R}^N) \) for some \( v \in L^p(\mathbb{R}^N) \). Let \( B \subset \mathbb{R}^N \) be bounded and measurable. For \( m, n \in \mathbb{N} \) and \( \varphi \in L^p(\mathbb{R}^N) \) we find that
\[
\int_B \left( |v_n|^{p'-2} v_n - |v_n|^{p-2} v_n \right) \varphi \, dx \leq \|J'(v_n) - J'(v_m)||\varphi||_{p'} + \|1_B K(v_n - v_m)||\varphi||_{p'}.
\]
Since \( 1_B K \) is compact, we infer from the above estimate that \( (1_B |v_n|^{p-2} v_n)_n \) is a Cauchy sequence in \( L^p(\mathbb{R}^N) \). Hence, there exists \( z \in L^p(\mathbb{R}^N) \) such that \( 1_B |v_n|^{p-2} v_n \to z \) strongly in \( L^p(\mathbb{R}^N) \) and therefore
\[
\begin{align*}
\int_{\mathbb{R}^N} |1_B v_n - |z|^{p-2} z|^{p'} \, dx & \leq C \int_{\mathbb{R}^N} \left( |v_n|^{p'-1} + |z| \right)^{(p-2)p'} |1_B |v_n|^{p-2} v_n - z|^{p'} \, dx \\
& \leq C \left( \|v_n\|_{p'}^{p'-1} + \|z\|_{p'}^{p'} \right) \|1_B |v_n|^{p-2} v_n - z\|_{p'}.
\end{align*}
\]
As a consequence, we obtain \( 1_B v_n \to |z|^{p-2} z \) as \( n \to \infty \), strongly in \( L^p(\mathbb{R}^N) \). Moreover, by uniqueness of the weak limit, there holds \( z = 1_B |v|^{p-2} v \), and this gives the desired strong local convergence.
As a consequence, we find that for every $\varphi \in C_\infty^0(\mathbb{R}^N)$,
\[
J'(v) \varphi = \int_{\text{supp } \varphi} |v|^{p'-2}v \varphi \, dx - \int_{\mathbb{R}^N} \varphi \mathcal{K}(v) \, dx \\
= \lim_{n \to \infty} \int_{\text{supp } \varphi} |v_n|^{p'-2}v_n \varphi \, dx - \int_{\mathbb{R}^N} \varphi \mathcal{K}(v_n) \, dx = \lim_{n \to \infty} J'(v_n) \varphi = 0,
\]
since $v_n \to v$. Hence $J'(v) = 0$, and since the norm $\| \cdot \|_{p'}$ is weakly lower sequentially continuous, we infer that
\[
J(v) = J(v) - \frac{1}{2}J'(v)v = \left( \frac{1}{p'} - \frac{1}{2} \right) \int_{\mathbb{R}^N} |v|^{p'} \, dx \\
\leq \liminf_{n \to \infty} \left( \frac{1}{p'} - \frac{1}{2} \right) \int_{\mathbb{R}^N} |v_n|^{p'} \, dx = \liminf_{n \to \infty} J(v_n) - \frac{1}{2}J'(v_n)v_n = \liminf_{n \to \infty} J(v_n),
\]
and this concludes the proof. 

In the case where $2_* < p < 2^*$ and the coefficient $Q$ is $\mathbb{Z}^N$-periodic, we proved in [17] the existence of a nontrivial critical point for $J$. A crucial ingredient in our proof was the following nonvanishing property (see [17] Theorem 3.1 and [16] Theorem 3.1).

**Theorem 2.3.** Let $2_* < p < 2^*$ and consider a bounded sequence $(v_n)_n \subset L^{p'}(\mathbb{R}^N)$ satisfying
\[
\limsup_{n \to \infty} \int_{\mathbb{R}^N} v_n R v_n \, dx > 0.
\]
Then there exists $R > 0$, $\zeta > 0$ and a sequence $(x_n)_n \subset \mathbb{R}^N$ such that, up to a subsequence,
\[
\int_{B_R(x_n)} |v_n|^{p'} \, dx \geq \zeta \quad \text{for all } n.
\]

3. **Existence of infinitely many solutions in the periodic setting**

In this section, we prove the multiplicity result, Theorem 1.2 announced in the introduction. We assume throughout that $Q$ is bounded, nonnegative, $\neq 0$ and $\mathbb{Z}^N$-periodic, and let $\mathcal{K} := \{ u \in L^{p'}(\mathbb{R}^N) : J'(u) = 0 \}$. Moreover, we denote by $\mathcal{A}$ a symmetric subset of $\mathcal{K}$ (i.e. $\mathcal{A} = -\mathcal{A}$) which contains exactly one element from each orbit $\mathcal{O}(w) = \{ w(\cdot - y) : y \in \mathbb{Z}^N \}$. From now on, let us suppose by contradiction that the set $\mathcal{A}$ is finite.

We follow below the ideas of the proof of [22] Theorem 1.2 and [22] Theorem 1.1.

**Lemma 3.1.** $\kappa := \inf \{ \| v - w \|_{p'} : v, w \in \mathcal{K}, v \neq w \} > 0.$

**Proof.** Let $(v_n)_n, (w_n)_n \subset \mathcal{K}$ be sequences such that $v_n \neq w_n$ for all $n$ and $\| v_n - w_n \|_{p'} \to \kappa$, as $n \to \infty$. Then there are sequences $(y_n)_n, (z_n)_n \subset \mathbb{Z}^N$ for which $\tilde{v}_n := v_n(\cdot + y_n) \in \mathcal{A}$ and $\tilde{w}_n := w_n(\cdot + z_n) \in \mathcal{A}$ for all $n$. Since $\mathcal{A}$ is finite, there holds (up to a subsequence) $\tilde{v}_n \to \tilde{v} \in \mathcal{A}, \tilde{w}_n \to \tilde{w} \in \mathcal{A}$ for all $n$, and either $y_n - z_n = y_0 \in \mathbb{Z}^N$ for all $n$, or $|y_n - z_n| \to \infty$.

In the case where $y_n - z_n = y_0 \in \mathbb{Z}^N$ for all $n$, we find
\[
0 < \| \tilde{v}(\cdot - y_0) - \tilde{w}_n(\cdot - z_n) \|_{p'} = \| \tilde{v}_n(\cdot - y_n) - \tilde{w}_n(\cdot - z_n) \|_{p'} = \| v_n - w_n \|_{p'} = \kappa,
\]
since the orbits $\mathcal{O}(\tilde{v})$ and $\mathcal{O}(\tilde{w})$ are distinct. In the second case, we may take without loss of generality $\tilde{v} \neq 0$ and we remark that $\tilde{w}(\cdot + y_0 - z_0) \to 0$ in $L^{p'}(\mathbb{R}^N)$. This gives
\[
\kappa = \lim_{n \to \infty} \| \tilde{v}_n(\cdot - y_n) - \tilde{w}_n(\cdot - z_n) \|_{p'} = \lim_{n \to \infty} \| \tilde{v} - \tilde{w}(\cdot + y_0 - z_0) \|_{p'} \geq \| \tilde{v} \|_{p'} > 0,
\]
and thus concludes the proof. 

\[\square\]
Lemma 3.2. Let $2^* < p < 2^*$ and consider two Palais-Smale sequences $(v_n)_n, (w_n)_n \subset L^{p'}(\mathbb{R}^N)$ for $J$. Then, either $\|v_n - w_n\|_{p'} \to 0$, as $n \to \infty$, or $\limsup_{n \to \infty} \|v_n - w_n\|_{p'} \geq \kappa$, where $\kappa$ is given by Lemma 3.1.

Proof. First note that the sequences $(v_n)_n, (w_n)_n$ are bounded. We distinguish two cases.

Case 1: If $\int_{\mathbb{R}^N} (v_n - w_n)K(v_n - w_n) \, dx \to 0$, as $n \to \infty$, it follows that

\[
\int_{\mathbb{R}^N} \left( |v_n|^{p'-2}v_n - |w_n|^{p'-2}w_n \right) (v_n - w_n) \, dx
= \left( J'(v_n) - J'(w_n) \right) (v_n - w_n) + \int_{\mathbb{R}^N} (v_n - w_n)K(v_n - w_n) \, dx \to 0,
\]
as $n \to \infty$. Moreover, since $1 < p' < 2$, there holds for every $a, b > 0$:

\[
(a^{p'-1} - b^{p'-1})(a - b) = (p' - 1)(a - b) \int_b^a t^{p'-2} \, dt \geq (p' - 1)(a - b)^2(a + b)^{p'-2}.
\]

Using the Reverse Hölder inequality (see [3, Theorem 2.12]) we obtain

\[
\int_{\mathbb{R}^N} \left( |v_n|^{p'-2}v_n - |w_n|^{p'-2}w_n \right) (v_n - w_n) \, dx \geq (p' - 1) \int_{\mathbb{R}^N} (v_n - w_n)^2 (|v_n| + |w_n|)^{p'-2} \, dx
\geq (p' - 1) \left( \int_{\mathbb{R}^N} |v_n - w_n|^{p'} \, dx \right)^{\frac{2}{p'}} \left( \int_{\mathbb{R}^N} (|v_n| + |w_n|)^{p'} \, dx \right)^{\frac{1}{p'} - \frac{2}{p'}}.
\]

Using (6) and the boundedness of $(v_n)_n$ and $(w_n)_n$, we deduce that $\|v_n - w_n\|_{p'} \to 0$ as $n \to \infty$.

Case 2: If $\limsup_{n \to \infty} \|\int_{\mathbb{R}^N} (v_n - w_n)K(v_n - w_n) \, dx\| > 0$, the nonvanishing property (see Theorem 2.3) gives the existence of $R, \zeta > 0$ and a sequence $(x_n)_n \subset \mathbb{R}^N$ such that (up to a subsequence)

\[
\int_{B_R(x_n)} |v_n - w_n|^{p'} \, dx \geq \zeta \quad \text{for all } n.
\]

We may even assume (making $R$ larger if necessary) that $x_n \in \mathbb{Z}^N$ for all $n$. Setting $\tilde{v}_n = v_n(+x_n)$ and $\tilde{w}_n = w_n(+x_n)$ for each $n$, we find, using the $\mathbb{Z}^N$-translation invariance of $J$, that $(\tilde{v}_n)_n$ and $(\tilde{w}_n)_n$ are bounded Palais-Smale sequences for $J$. From Lemma 2.2 there exist $\tilde{v}, \tilde{w} \in \mathcal{K}$ satisfying $\tilde{v}_n \rightharpoonup \tilde{v}$ and $\tilde{w}_n \rightharpoonup \tilde{w}$ weakly in $L^{p'}(\mathbb{R}^N)$ as well as $1_{B_R}\tilde{v}_n \to 1_{B_R}\tilde{v}$ and $1_{B_R}\tilde{w}_n \to 1_{B_R}\tilde{w}$ strongly in $L^p(\mathbb{R}^N)$. The property (7) then implies $\tilde{v} \neq \tilde{w}$, and we conclude that

\[
\limsup_{n \to \infty} \|v_n - w_n\|_{p'} \geq \liminf_{n \to \infty} \|\tilde{v}_n - \tilde{w}_n\|_{p'} \geq \|\tilde{v} - \tilde{w}\|_{p'} \geq \kappa.
\]

\[\square\]

Let $H : L^{p'}(\mathbb{R}^N) \setminus \mathcal{K} \to L^p(\mathbb{R}^N)$ be a locally Lipschitz continuous and $\mathbb{Z}_2$-equivariant pseudo-gradient vector field for $J$, i.e.,

\[\|H(v)\|_{p'} < 2 \min\{\|J'(v)\|, 1\}\quad\text{and}\quad J'(v)H(v) > \min\{\|J'(v)\|, 1\}\|J'(v)\|, \quad v \in L^p(\mathbb{R}^N)\]

(see [23, Definition II.3.1]). We consider for $v \in L^{p'}(\mathbb{R}^N) \setminus \mathcal{K}$ the flow $\eta$ given by

\[
\begin{cases}
\frac{\partial}{\partial t} \eta(t, v) = -H(\eta(t, v)), \\
\eta(0, v) = v,
\end{cases}
\]

and denote by $(T^-(v), T^+(v))$ the maximal existence time interval for the trajectory $t \mapsto \eta(t, v)$. 

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Lemma 3.3. For every \( v \in L^{p'}(\mathbb{R}^N) \setminus \mathcal{K} \), the following alternative holds: either \( \lim_{t \to T^+(v)} \eta(t, v) \) exists and is a critical point of \( J \), or \( \lim_{t \to T^+(v)} J(\eta(t, v)) = -\infty \). In the latter case, \( T^+(v) = \infty \).

Proof. Let \( v \in L^{p'}(\mathbb{R}^N) \) and let us first remark that \( J \) is bounded on bounded sets and that it is strictly decreasing along trajectories of \( \eta \). In particular, if \( t \mapsto J(\eta(t, v)) \) becomes unbounded as \( t \to T^+(v) \), then \( \|\eta(t, v)\|_{p'} \to \infty \) as \( t \to T^+(v) \) and the boundedness of \( H \) implies \( T^+(v) = \infty \) and \( \lim_{t \to T^+(v)} J(\eta(t, v)) = -\infty \).

Let us now assume that \( \ell := \inf\{J(\eta(t, v)) : 0 \leq t < T^+(v)\} > -\infty \), and observe that \( \ell = \lim_{t \to T^+(v)} J(\eta(t, v)) \) holds, since \( J \) is strictly decreasing along the flow. If \( T^+(v) < \infty \), we can write for \( 0 \leq s < t < T^+(v) \):

\[
\|\eta(t, v) - \eta(s, v)\|_{p'} \leq \int_s^t \|H(\eta(\tau, v))\|_{p'} d\tau < 2(t - s).
\]

Hence the limit \( \lim_{t \to T^+(v)} \eta(t, v) \) exists and is a critical point, since otherwise the trajectory could be continued beyond \( T^+(v) \). It remains to consider the case where \( T^+(v) = \infty \). In order to prove the existence of \( \lim_{t \to \infty} \eta(t, v) \) in this case, we show that for every \( \varepsilon > 0 \) there exists \( t_\varepsilon > 0 \) such that \( \|\eta(t_\varepsilon, v) - \eta(t, v)\|_{p'} < \varepsilon \) for all \( t \geq t_\varepsilon \). Assume by contradiction that this property does not hold. Then for some \( 0 < \varepsilon < \frac{\kappa}{2} \) there is an increasing sequence \( (t_n)_n \subset (0, \infty) \) satisfying \( t_n \to \infty \) and \( \|\eta(t_n, v) - \eta(t_{n+1}, v)\|_{p'} = \varepsilon \) for all \( n \). Choosing the smallest \( t^*_n \in (t_n, t_{n+1}] \) such that \( \|\eta(t_n, v) - \eta(t^*_n, v)\|_{p'} = \varepsilon \) and setting \( \kappa_n := \min_{s \in [t_n, t^*_n]} \|J'(\eta(s, v))\| \), we obtain from the properties of the pseudo-gradient field that

\[
\frac{\varepsilon}{3} = \|\eta(t_n, v) - \eta(t^*_n, v)\|_{p'} \leq \int_{t_n}^{t^*_n} \|H(\eta(s, v))\|_{p'} ds
\]

\[
\leq \frac{2}{\kappa_n} \int_{t_n}^{t^*_n} J'(\eta(s, v))H(\eta(s, v)) ds = \frac{2}{\kappa_n} (J(\eta(t_n, v)) - J(\eta(t^*_n, v))).
\]

Since \( J(\eta(t_n, v)) - J(\eta(t^*_n, v)) \to \ell - \ell = 0 \) as \( n \to \infty \), we obtain \( \lim_{n \to \infty} \kappa_n = 0 \). Therefore, we find \( s^1_n \in [t_n, t^*_n] \) such that \( J'(\eta(s^1_n, v)) \to 0 \) as \( n \to \infty \). Choosing the largest \( t^*_n \in (t_n, t_{n+1}) \) for which \( \|\eta(t^*_n, v) - \eta(t_{n+1}, v)\|_{p'} = \frac{\varepsilon}{3} \), we find similarly some \( s^2_n \in [t^*_n, t_{n+1}] \) such that \( J'(\eta(s^2_n, v)) \to 0 \) as \( n \to \infty \). Hence, \( (\eta(s^1_n, v))_n \) and \( (\eta(s^2_n, v))_n \) are Palais-Smale sequences for \( J \) and satisfy for all \( n \):

\[
\frac{\varepsilon}{3} \leq \|\eta(s^1_n, v) - \eta(s^2_n, v)\|_{p'} \leq 2\varepsilon < \kappa,
\]

which contradicts Lemma 3.2. Thus, \( v^* := \lim_{t \to \infty} \eta(t, v) \) exists and the estimate

\[
\int_0^\infty \min\{\|J'(\eta(s, v))\|_{p'}^2, 1\} ds \leq J(v) - J(v^*)
\]

implies that it is a critical point. \(\square\)

We note that, since \( \mathcal{A} \) is finite, there is for each fixed \( d > 0 \) some \( \varepsilon_0 > 0 \) such that \( J^d_{d-2\varepsilon_0} \cap \mathcal{K} = \mathcal{K}_d \). Here and in the sequel, we use the following notation for super- and sublevel sets of \( J \): for \( a, b \in \mathbb{R} \), we let \( J_a := \{u \in L^p(\mathbb{R}^N) : J(u) \geq a\} \), \( J^b := \{u \in L^p(\mathbb{R}^N) : J(u) \leq b\} \) and \( J^a_a := J_a \cap J^b \). In addition, we set for \( d \in \mathbb{R} \), \( \mathcal{K}_d := \{u \in \mathcal{K} : J(u) = d\} \). For
given $\delta > 0$ we shall denote, in the following, the open $\delta$-neighborhood of a set $B \subset L^p(\mathbb{R}^N)$ by $U_\delta(B)$.

**Lemma 3.4.** Let $d > 0$ and consider $\varepsilon_0 > 0$ such that $J^{d+2\varepsilon_0} \cap K = K_d$. For every $\delta > 0$, there exists $0 < \varepsilon < \varepsilon_0$ and a continuous 1-parameter family of odd homeomorphisms of $L^p(\mathbb{R}^N)$, $\{\tilde{\eta}(t, \cdot)\}_{t \in \mathbb{R}}$, such that

(i) $t \mapsto J(\tilde{\eta}(t, v))$ is nonincreasing, $\forall v \in L^p(\mathbb{R}^N)$;

(ii) For every $v \in J^{d+\varepsilon} \cap J(\tilde{\eta}(K_d))$, there exists $T \geq 0$ for which $J(\tilde{\eta}(T, v)) \leq d - \varepsilon$.

**Proof.** Taking without loss of generality $\delta < \kappa$, we first claim that

$$\tau := \inf \{\|J'(v)\| : v \in J^{d+\varepsilon} \cap (U_\delta(K_d) \setminus U_{\frac{\delta}{2}}(K_d))\} > 0.$$ 

Assuming by contradiction that $\tau = 0$ holds, we can find a sequence $(v_n)_n \subset U_\delta(K_d) \setminus U_{\frac{\delta}{2}}(K_d)$ such that $J(v_n) \in [d - 2\varepsilon_0, d + 2\varepsilon_0]$ for all $n$ and $J'(v_n) \to 0$ as $n \to \infty$. Using the $\mathbb{Z}^N$-invariance of $J$ and the finiteness of $A$, we find that (up to a subsequence) $(v_n)_n \subset U_\delta(\{w_0\}) \setminus U_{\frac{\delta}{2}}(\{w_0\})$ for some $w_0 \in K_d$. This implies that

$$\frac{\delta}{2} \leq \limsup_{n \to \infty} \|v_n - w_0\|_{p'} \leq \delta < \kappa,$$

which contradicts Lemma 3.2. Hence, $\tau > 0$ must hold.

Without loss of generality we may assume $\tau < 1$. Choosing $0 < \varepsilon < \min\{\varepsilon_0, \frac{\delta}{2}\}$, we claim that the following property holds for every $v \in J^{d+\varepsilon} \cap J(\tilde{\eta}(K_d))$:

$$\lim_{t \to J^-(v)} J(\tilde{\eta}(t, v)) < d - \varepsilon \quad (9)$$

$$\eta(t, v) \notin U_{\frac{\delta}{2}}(K_d) \cap J_d - \varepsilon \quad \text{for all } 0 \leq t < T^+(v). \quad (10)$$

Assuming by contradiction that at least one of the above properties does not hold, we find by Lemma 3.3 and the fact that $J^{d+2\varepsilon_0} \cap K = K_d$, some $v \in J^{d+\varepsilon} \cap J(\tilde{\eta}(K_d))$ and some $0 < t_0 < T^+(v)$ such that $\eta(t_0, v) \in U_{\frac{\delta}{2}}(K_d) \cap J_d - \varepsilon$. Moreover, since $\delta < \kappa$, there is, according to Lemma 3.1, $w \in K_d$ such that $\eta(t_0, v) \in U_{\frac{\delta}{2}}(\{w\}) \cap J_d - \varepsilon$. Setting

$$t_1 := \sup\{t \in [0, t_0) : \eta(t, v) \notin U_\delta(\{w\})\} \quad \text{and} \quad t_2 := \inf\{t \in (t_1, T^+(v)) : \eta(t, v) \in U_{\frac{\delta}{2}}(\{w\})\}$$

we obtain $t_2 \leq t_0$ and

$$\frac{\delta}{2} \leq \|\eta(t_1, v) - \eta(t_2, v)\|_{p'} \leq \int_{t_1}^{t_2} \|H(\eta(s, v))\|_{p'} \, ds \leq 2(t_2 - t_1).$$

Consequently,

$$J(\eta(t_0, v)) \leq J(\eta(t_2, v)) = J(\eta(t_1, v)) - \int_{t_1}^{t_2} J'(\eta(s, v))H(\eta(s, v)) \, ds \leq (d + \varepsilon) - t^2(t_2 - t_1) < d - \varepsilon,$$

contradicting the fact that $\eta(t_0, v) \in J_d - \varepsilon$. This contradiction proves (9) and (10).

Let us now choose a locally Lipschitz continuous function $\chi: L^p(\mathbb{R}^N) \to [0, 1]$ such that $\chi(-v) = \chi(v)$ for all $v$, $\chi = 1$ on $J^{d+\varepsilon} \cap J(\tilde{\eta}(K_d))$ and $\chi = 0$ on $J_d + 2\varepsilon_0 \cup J^{d-2\varepsilon_0} \cup (U_{\frac{\delta}{2}}(K_d) \cap J_d - 2\varepsilon_0)$. For every $v \in L^p(\mathbb{R}^N)$, the Cauchy problem

$$\begin{cases} \frac{\partial}{\partial t}\tilde{\eta}(t, v) &= -\chi(\tilde{\eta}(t, v))H(\tilde{\eta}(t, v)), \\ \tilde{\eta}(0, v) &= v, \end{cases} \quad (11)$$
has a unique solution \( \tilde{\eta}(t, v) \) defined on \( \mathbb{R} \), and the flow \( \tilde{\eta} \) is continuous on \( \mathbb{R} \times L^p(\mathbb{R}^N) \). By the semigroup property and the fact that \( \chi H \) is odd, \( \tilde{\eta}(t, \cdot) \) is an odd homeomorphism for every \( t \in \mathbb{R} \). Also, since \( \chi \geq 0 \) we obtain from the properties of the pseudogradient field \( H \), that \( J \) is nonincreasing along the trajectories of \( \tilde{\eta} \).

Let now \( v \in J^{d+\varepsilon}_d \cup U_p(K_d) \) be chosen. If \( J(v) < d - \varepsilon \), then (ii) holds with \( T = 0 \). Otherwise, \( v \in J^{d+\varepsilon}_d \cup U_p(K_d) \) and by (10), we obtain \( \chi(\eta(t, v)) = 1 \) for all \( 0 \leq t < T^+(v) \) such that \( J(\eta(t, v)) \geq d - \varepsilon \). The uniqueness of the flow therefore implies \( \tilde{\eta}(t, v) = \eta(t, v) \) for all such \( t \), and using (9), we find \( 0 \leq T < T^+(v) \) for which \( J(\tilde{\eta}(T, v)) = J(\eta(T, v)) = d - \varepsilon \). This shows that \( \tilde{\eta} \) has the properties (i) and (ii) and concludes the proof. \( \square \)

In order to obtain a contradiction to the assumption that \( \mathcal{A} \) is finite, we will need the following variant of Benci’s pseudoindex (see [9], [8]). Let \( \Sigma \) denote the family of all compact and symmetric subsets of \( L^p(\mathbb{R}^N) \), and consider \( \rho > 0 \) as given by Lemma 2.1(i), i.e., small enough that for some \( \alpha > 0 \), \( J(v) \geq \alpha \) for all \( v \in \mathbb{S}_{\rho}(0) = \{ v \in L^p(\mathbb{R}^N) : \| v \|_{L^p} = \rho \} \). For \( A \in \Sigma \), set
\[
i^*(A) := \inf_{A \in \Sigma} \sup_{v \in A} J(A). \]
where \( \gamma \) denotes the Kronecseiskii genus and \( H := \{ h : L^p(\mathbb{R}^N) \to L^p(\mathbb{R}^N) \text{ odd homeomorphism with } J(h(v)) \leq J(v) \text{ for all } v \in L^p(\mathbb{R}^N) \} \).

We note that according to [9], Proposition 1.6, \( \iota^* \) is a pseudoindex in the sense of [9], Definition 1.2. Moreover, using Lemma 2.1, we find that sets of arbitrarily large pseudoindex exist. Indeed, arguing as in [22, Lemma 2.1], we find that for every \( m \), \( \iota^*(\mathcal{W}_m \cap B_R(0)) \geq m \), where \( \mathcal{W}_m \) and \( R = R(\mathcal{W}_m) \) are as in Lemma 2.1(ii).

We can now give the proof of the multiplicity result.

Proof of Theorem 1.1. For \( k \in \mathbb{N} \), let
\[
d_k := \inf_{A \in \Sigma} \sup_{v \in A} J(A). \]
Since there exist sets of arbitrarily large pseudoindex, \( d_k \) is well-defined for all \( k \). Moreover our choice of \( \rho \) gives \( d_k \geq \alpha, \forall k \geq 1 \). We shall prove that for every \( k \),
\[
(12) \quad \mathcal{K}_{d_k} \neq \emptyset \quad \text{and} \quad d_k < d_{k+1}. \]

Let \( k \in \mathbb{N} \) and consider \( d = d_k \). First remark that by Lemma 3.1, \( \mathcal{K} \) is a countable discrete set and therefore \( \gamma(K_d) = 0 \) or 1. Taking \( \delta > 0 \) and \( U := U_d(K_d) \) such that \( \gamma(U) = \gamma(K_d) \), we consider corresponding \( \varepsilon > 0 \) and \( \{ \tilde{\eta}(t, \cdot) \}_{t \geq 0} \) given by Lemma 3.4.

Let \( A \in \Sigma \) be chosen in such a way that \( \iota^*(A) \geq k \) and \( \sup_{v \in A} J(A) \leq d + \varepsilon \). Since \( A \) is compact, the property (ii) of the family \( \{ \tilde{\eta}(t, \cdot) \}_{t \geq 0} \), implies the existence of some common \( T \geq 0 \) such that \( J(\tilde{\eta}(T, v)) \leq d - \varepsilon \) for all \( v \in A \setminus U \). From the properties of the pseudoindex (see [9], Definition 1.2) and since \( \tilde{\eta}(T, \cdot) \in \mathcal{H} \), we obtain
\[
k \leq \iota^*(A) \leq \iota^*(A \setminus U) + \gamma(U) \leq \iota^*(\tilde{\eta}(T, A \setminus U)) + \gamma(K_d) \leq k - 1 + \gamma(K_d). \]

Hence \( \gamma(K_d) \geq 1 \), and therefore \( K_d \neq \emptyset \). Moreover, if \( d_k = d_{k+1} \) would hold, then we could choose \( A \) such that \( \iota^*(A) \geq k + 1 \) in the above argument and this would give \( \gamma(K_d) \geq 2 \), contradicting the fact that \( \gamma(\mathcal{K}) \leq 1 \). Consequently, \( d_k < d_{k+1} \) holds for all \( k \) and (12) is proved. But this gives the existence of infinitely many distinct critical levels, contradicting the assumption that \( \mathcal{A} \) is finite.

Hence, the functional \( J \) has infinitely many geometrically distinct critical points, and by [17] Lemma 4.3] and [16] Lemma 2.4], these give rise to geometrically distinct strong solutions of (1). This concludes the proof. \( \square \)
4. The asymptotically periodic problem

In this section, we give sufficient conditions for the existence of a (nontrivial) solution to (1) in the case where

\[
\lim_{R \to \infty} \text{ess sup}_{|x| \geq R} |Q(x) - Q_\infty(x)| = 0,
\]

for some bounded, nonnegative and \(\mathbb{Z}^N\)-periodic function \(Q_\infty \neq 0\).

In order to show the existence of a critical point for \(J\), we shall compare its behavior with that of the limit energy functional:

\[
J_\infty(v) = \frac{1}{p'} \int_{\mathbb{R}^N} |v|^{p'} \, dx - \frac{1}{2} \int_{\mathbb{R}^N} Q_\infty^{\frac{1}{2}} v R(Q_\infty^{\frac{1}{2}} v) \, dx, \quad v \in L^{p'}(\mathbb{R}^N).
\]

Considering the minimax levels

\[
c = \inf_{v \neq 0} \sup_{t > 0} J(tv) \quad \text{and} \quad c_\infty = \inf_{v \neq 0} \sup_{t > 0} J_\infty(tv),
\]

we first remark that \(0 < c, c_\infty < \infty\), as follows from Lemma 2.1. In addition, we notice that for each \(v \in U^+: = \left\{ u \in L^p(\mathbb{R}^N) : \int_{\mathbb{R}^N} Q^{\frac{1}{2}} u R(Q^{\frac{1}{2}} u) \, dx > 0 \right\}\) there is a unique \(t = t_v > 0\), given by

\[
t_v^{2-p'} = \frac{\int_{\mathbb{R}^N} |v|^{p'} \, dx}{\int_{\mathbb{R}^N} Q^{\frac{1}{2}} v R(Q^{\frac{1}{2}} v) \, dx},
\]

for which \(J(tv) > J(sv)\) holds for all \(s > 0, \ s \neq t_v\). We can therefore write

\[
c = \inf_{v \in U^+} J(t_v v) = \inf_{v \in U^+} \left( \frac{1}{p'} - \frac{1}{2} \right) t_v^{p'} \int_{\mathbb{R}^N} |v|^{p'} \, dx,
\]

since \(\sup_{t>0} J(tv) = \infty\) for \(v \notin U^+ \cup \{0\}\). Similarly,

\[
c_\infty = \inf_{v \in U^\infty_+} J_\infty(t_v^\infty v) = \inf_{v \in U^\infty_+} \left( \frac{1}{p'} - \frac{1}{2} \right) (t_v^\infty)^{p'} \int_{\mathbb{R}^N} |v|^{p'} \, dx,
\]

with corresponding definitions for \(U^\infty_+\) and \(t_v^\infty\).

**Lemma 4.1.** Let \(2_* \leq p < 2^*\) and \(Q, Q_\infty \in L^\infty(\mathbb{R}^N) \setminus \{0\}\) be nonnegative functions. Consider the mountain-pass level

\[
b = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t)), \quad \text{where} \quad \Gamma = \left\{ \gamma \in C([0,1], L^{p'}(\mathbb{R}^N)) : \gamma(0) = 0 \text{ and } J(\gamma(1)) < 0 \right\}.
\]

Then

(i) \(c = b\).

(ii) If \(\text{(13)}\) holds, then \(c \leq c_\infty\).

**Proof.** (i) For each \(v \in U^+\), we have \(\lim_{t \to \infty} J(tv) = -\infty\). Using \(\text{(15)}\), we obtain \(c = \inf_{v \in U^+} \sup_{t > 0} J(tv) \geq b\). For the converse inequality, we note that since \(p' < 2\) there exists \(\eta > 0\) such that

\[
J'(v)v = \int_{\mathbb{R}^N} |v|^{p'} \, dx - \int_{\mathbb{R}^N} Q^{\frac{1}{2}} v R(Q^{\frac{1}{2}} v) \, dx > 0, \quad \text{for all } 0 < \|v\|_{p'} \leq \eta.
\]

Also, if \(\gamma \in \Gamma\), then \(0 > J(\gamma(1)) = \frac{1}{p'} \int_{\mathbb{R}^N} |\gamma(1)|^{p'} \, dx - \frac{1}{2} \int_{\mathbb{R}^N} Q^{\frac{1}{2}} \gamma(1) R(Q^{\frac{1}{2}} \gamma(1)) \, dx\) and this gives \(J'(\gamma(1))\gamma(1) < 0\). Since \(\gamma(0) = 0\) and \(\|\gamma(1)\|_{p'} > \eta\), there is some \(0 < \bar{t} < 1\) such that
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with \( v = \gamma(\tilde{t}) \), there holds \( J'(v) = 0 \) and \( v \neq 0 \). In particular, \( v \in U^+ \) and \( t_\gamma = 1 \), which implies that

\[
\max_{t \in [0,1]} J(t) = J(t_\gamma) \geq J(v) \geq c,
\]

and since \( \gamma \in \Gamma \) was chosen arbitrarily, we obtain \( b \geq c \).

(ii) As a consequence of \([13]\), the following holds for every \( v \in L^p(\mathbb{R}^N) \) and every sequence \((y_n)_n \subset \mathbb{R}^N \) such that \( \lim_{n \to \infty} |y_n| = \infty \):

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} Q_\infty^p \frac{1}{p} v(\cdot - y_n) R(Q_\infty^p v(\cdot - y_n)) \, dx = \lim_{n \to \infty} \int_{\mathbb{R}^N} Q_\infty^p (\cdot + y_n) R(Q_\infty^p (\cdot + y_n)v) \, dx
\]

\[
= \int_{\mathbb{R}^N} Q_\infty^p v R(Q_\infty^p v) \, dx.
\]

Consider \( v \in U^+ \) and let \( v_n = v(\cdot - y_n) \) for some sequence \((y_n)_n \subset \mathbb{R}^N \) such that \( \lim_{n \to \infty} |y_n| = \infty \). Then for \( n \) large enough, the above property gives \( v_n \in U^+ \) and from \([14]\), \( \lim_{n \to \infty} t_{v_n} = t_v^\infty \).

Consequently,

\[
c \leq \lim_{n \to \infty} \left( \frac{1}{p'} - \frac{1}{2} \right) t_{v_n}^p \int_{\mathbb{R}^N} |v_n|^p \, dx = \left( \frac{1}{p'} - \frac{1}{2} \right) \left( t_v^\infty \right)^p \int_{\mathbb{R}^N} |v|^p \, dx,
\]

and since \( v \in U^+ \) was arbitrarily chosen, we conclude that \( c \leq c_\infty \). \( \square \)

**Proposition 4.2.** Let \( 2^* \leq p < 2^* \), \( Q, Q_\infty \in L^\infty(\mathbb{R}^N) \setminus \{0\} \) be nonnegative functions such that \([13]\) holds. If \( c < c_\infty \), then \( J \) has a (nontrivial) critical point at level \( c \).

**Proof.** Since \( c = b \) and \( b \) is the mountain-pass level associated to \( J \), a standard deformation argument (see e.g. \([17]\) Lemma 6.1) gives the existence of a Palais-Smale sequence \((v_n)_n \subset L^{p'}(\mathbb{R}^N) \) for \( J \) at level \( c \). According to Lemma \([2,2]\), we can assume (going to a subsequence) that \( v_n \rightharpoonup v \) weakly in \( L^{p'}(\mathbb{R}^N) \) where \( J'(v) = 0 \) and \( J(v) \leq \liminf_{n \to \infty} J(v_n) = c \).

If \( v \neq 0 \), then we are done, since \( J'(v) = 0 \) gives \( v \in U^+ \) and \( t_\gamma = 1 \), from which \( J(v) \geq c \) follows. Assuming by contradiction that \( v_n \rightharpoonup 0 \) holds, we write

\[
\int_{\mathbb{R}^N} Q_\infty^p v_n R(Q_\infty^p v_n) \, dx = \int_{\mathbb{R}^N} Q_\infty^p v_n R(Q_\infty^p v_n) \, dx
\]

\[
+ \int_{\mathbb{R}^N} (Q_\infty^p - Q_\infty^p) v_n R(Q_\infty^p + Q_\infty^p v_n) \, dx
\]

and see that for \( r > 0 \),

\[
\left| \int_{\mathbb{R}^N} (Q_\infty^p - Q_\infty^p) v_n R(Q_\infty^p + Q_\infty^p v_n) \, dx \right|
\]

\[
\leq \left( ||Q_\infty||_\infty + ||Q||_\infty \right) \left[ C ||v_n||_{L^p} \right] \left[ \text{ess sup}_{|x| > r} |Q_\infty(x)^\frac{1}{p} - Q(x)^\frac{1}{p}| \right]
\]

\[
+ ||v_n||_{L^p} ||1_{B_r(0)} R(Q_\infty^p + Q_\infty^p v_n)||_{L^p}.
\]

Since by assumption \([13]\) we have \( \text{ess sup}_{|x| > r} |Q_\infty(x)^\frac{1}{p} - Q(x)^\frac{1}{p}| \to 0 \), as \( r \to \infty \), and since the operator \( 1_{B_r(0)} R: L^{p'}(\mathbb{R}^N) \to L^p(\mathbb{R}^N) \) is compact for every \( r > 0 \) (see \([17]\) Lemma 4.1), we
Then, according to the above remark, there exists 

\[ b_0 \] 

has no higher energy than the mountain-pass level 

\[ w_2.2 \), we obtain that the nontrivial critical point 

\[ \text{Proof.} \]

Let 

\[ \text{Theorem 4.3.} \]

\[ Q \]

holds for the resolvent 

\[ v \]

\[ \text{We now restrict to the case} \quad p > 2, \text{in which the nonvanishing property (Theorem 2.3)} \]

\[ \text{In this case, we note that the level} \quad c \text{ corresponding to the periodic functional} \quad J_\infty \text{ is attained and coincides with the least-energy level, i.e.,} \]

\[ c_\infty = \inf \{ J_\infty(v) : v \in L^{p'}(\mathbb{R}^N), v \neq 0 \text{ and } J'_\infty(v) = 0 \}. \]

Indeed, using the weak lower semicontinuity of \( J_\infty \) along Palais-Smale sequences (see Lemma 2.2), we obtain that the nontrivial critical point \( w \) of \( J_\infty \) given by the proof of [17] Theorem 6.2 has no higher energy than the mountain-pass level \( b_\infty \). Since by Lemma 4.1(i), \( c_\infty = b_\infty \), and since every critical point \( v \) of \( J_\infty \) satisfies \( J_\infty(v) \geq c_\infty \), we find that \( J_\infty(w) = c_\infty \).

As a consequence, we obtain the following existence result.

**Theorem 4.3.** Let \( 2 < p < 2^* \), \( Q, Q_\infty \in L^\infty(\mathbb{R}^N) \setminus \{0\} \) be nonnegative functions such that \( Q_\infty \) is \( \mathbb{Z}^N \)-periodic. If \( Q_\infty \leq Q \) a.e. on \( \mathbb{R}^N \) and [13] is satisfied, then \( J \) has a nontrivial critical point at level \( c \) and there holds \( c = \inf \{ J(v) : v \in L^{p'}(\mathbb{R}^N), v \neq 0 \text{ and } J'(v) = 0 \} \).

**Proof.** If \( c < c_\infty \), the conclusion follows from Proposition 1.2. Assume now that \( c = c_\infty \). Then, according to the above remark, there exists \( w \in L^{p'}(\mathbb{R}^N) \) such that \( J'_\infty(w) = 0 \) and \( J_\infty(w) = c_\infty \). Since \( 0 \leq Q_\infty \leq Q \) a.e. in \( \mathbb{R}^N \), by assumption, we may consider 

\[ v := \left( \frac{Q_\infty}{Q} \right)^{\frac{1}{p'}} w \in L^{p'}(\mathbb{R}^N). \]

There holds 

\[ \int_{\mathbb{R}^N} Q^{\frac{1}{p'}} v R(Q^{\frac{1}{p'}} v) dx = \int_{\mathbb{R}^N} Q^{\frac{1}{p'}} w R(Q^{\frac{1}{p'}} w) dx = \int_{\mathbb{R}^N} |w|^{p'} dx > 0, \]

and this gives \( v \in U^+ \). As a consequence, we find 

\[ c \leq J(t_v v) = \frac{t_v^{p'}}{p'} \int_{\mathbb{R}^N} \left( \frac{Q_\infty}{Q} \right)^{p'-1} |w|^{p'} dx - \frac{t_v^2}{2} \int_{\mathbb{R}^N} Q^{\frac{1}{p'}} w R(Q^{\frac{1}{p'}} w) dx \]

\[ \leq J_\infty(t_v w) \leq J_\infty(w) = c_\infty, \]

and since \( c = c_\infty \) by assumption, we conclude that \( J_\infty(t_v w) = J_\infty(w) \), and therefore \( t_v = t_0^\infty = 1 \). Hence, \( J(v) = c \) and \( J'(v)v = 0 \).
In order to show that $J'(v) = 0$, we proceed as follows: Consider $\varphi \in L^p(\mathbb{R}^N)$. Since $v \in U^+$ and $U^+$ is an open set, there is $\delta > 0$ such that $v + s\varphi \in U^+$ for all $|s| < \delta$. Using the characterization (13) of $c$, and since $J(v) = c$, $t_v = 1$, we can write for $|s| < \delta$:

$$0 \leq J(t_s(v + s\varphi)) - J(v) \leq J(t_s(v + s\varphi)) - J(t_sv) = st_s J'(t_s(v + s\varphi)) \varphi,$$

where $t_s = t_{v+s\varphi}$ and where $0 \leq \tau = \tau(s) \leq 1$ is given by the mean-value theorem. It follows that

$$J'(t_{-s}(v - s\varphi)) \varphi \leq 0 \leq J'(t_s(v + s\varphi)) \varphi$$

for all $0 < s < \delta$. Letting $s \to 0$ we obtain $J'(v) \varphi = 0$, since the functions $s \mapsto t_{v+s\varphi}$ are continuous on $(-\delta, \delta)$ and $t_v = 1$. Therefore, $v$ is a nontrivial critical point for $J$ at level $c$. □

**Remark 4.4.** The preceding result gives conditions ensuring the existence of a nontrivial critical point for $J$, and one might wonder whether or not, similar to the periodic case, the existence of infinitely many solutions can be proven under the assumption (13).

It is a fact that the dual functional $J$ associated to (1) bears many similarities with the one considered by Alama and Li in [4] and associated to a NLS equation with frequency in a spectral gap. In particular, a splitting property holds for Palais-Smale sequences of the dual functional, and the arguments in [2] can be used with only minor changes to give the existence of infinitely many critical points for $J$, assuming that (13) holds and that there exist $0 < a < C_\infty$ and $v_0 \in \mathcal{K}_c^\infty$ such that $v_0$ is isolated in $\mathcal{K}_c^{\infty+a}$.

It is however not clear whether this last assumption can be verified for some class of periodic potentials $Q_\infty$.

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