KURTOSIS AS A NON–GAUSSIAN SIGNATURE
OF THE LARGE–SCALE VELOCITY FIELD

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Abstract

We discuss the non–linear growth of the kurtosis of the smoothed peculiar velocity field (along an arbitrary direction), in an Einstein–de Sitter universe, induced by Gaussian primordial density fluctuations. Applying the perturbative theory, we show that, for different cosmological models, a departure from the original Gaussian distribution is gravitationally induced only on small scales ($\lesssim 20 - 30$ Mpc). Models with scale–free power spectrum $P(k) \propto k^n$, with $-1 < n \leq 1$, are also considered. When the fluid particles move according to the Zel’dovich approximation the probability distribution of the peculiar velocity field remains unaltered during the evolution.

Subject headings: Cosmology – Galaxies: clustering – large–scale structure of the Universe

1 Introduction

In the past decade astronomers and cosmologists devoted a great effort to measure large–scale deviations from the Hubble flow and to interpret the cosmological inferences. Redshift–independent distance estimators and all–sky catalogs of galaxy redshifts allowed to estimate the (radial) peculiar motions of galaxies. Undertaking detailed maps of the peculiar velocity field allows for example to make dynamical estimates of the density parameter $\Omega_0$, to test the primordial density fluctuation power spectrum $P(k)$, and to measure directly the underlying total (luminous plus dark) mass distribution, once the gravitational instability picture is assumed.

The statistical properties of the large–scale motions may be described in terms of the peculiar velocity correlation tensor (Górski et al. 1989; Groth, Juszkiewicz, & Ostriker 1989; Tormen et al. 1993) and/or the central moments of the velocity probability distribution function (pdf) $p(v)$ (Kofman et al. 1994; Bernardeau 1994). Assuming that the very early density pdf is Gaussian, it results that, during the linear regime, $p(v)$ is a Gaussian too, with mean $\langle v \rangle = 0$ and variance $\sigma_v^2 = (H_0^2 \Omega_0^{1.2}/2\pi^2) \int_0^\infty dk P(k)$, $H_0$ being the Hubble constant. Even if the primordial density field is Gaussian distributed, the non–linear time evolution will ensure that the mass density fluctuations $\delta$ become highly non–Gaussian (Peebles 1980; Fry 1984; Goroff et al. 1986; Juszkiewicz, Bouchet, & Colombi 1993; Catelan & Moscardini 1994), implying a modification of the original Gaussian pdf $p(v)$. However, due to the isotropy of the cosmological velocity field, it is expected that all the odd moments of $p(v)$ remain zero during the growth of the density fluctuations. A particular case is the third central moment, the velocity skewness, discussed for example in Ruamsuwan & Fry (1992). Thus, any gravitationally induced departure from the Gaussian distribution may be sought only in the fourth central moment of $p(v)$, the velocity kurtosis, as well as in higher order even central moments (Grinstein et al. 1987; Kofman et al. 1994; Bernardeau 1994). These have been also analyzed in the framework of the global texture model by Scherrer (1992) and Catelan & Scherrer (1994).

In this work, we study the non–Gaussian content of the velocity pdf in terms of the velocity kurtosis as induced by the gravitational growth of the initially Gaussian density fluctuations. The velocity kurtosis describes important features such as sharpness of the velocity pdf and the extent of its rare–event tail. Taking advantage of the exact perturbative technique (Fry 1984; Goroff et al. 1986) and the Zel’dovich approximation (see Grinstein & Wise 1987), we estimate the kurtosis
of the peculiar velocity along a direction $\hat{a}$, namely the parameter $K_v \equiv [(v_\alpha^4) - 3(v_\alpha^2)^2]/(v_\alpha^2)^2$, after smoothing with a Gaussian filter. In particular we study the dependence on both the scale in the context of several cosmological scenarios and the primordial spectral index, when an initial (scale–free) power spectrum $P(k) \propto k^n$ is assumed.

The layout of this paper is the following. In Section 2 the exact perturbative theory and the (Eulerian version of the) Zel’dovich approximation are reviewed. In Section 3 we discuss the gravitationally induced velocity kurtosis parameter $K_v$ of an initially Gaussian peculiar velocity field. Our results and conclusions are presented in Section 4.

## 2 Non–Linear Time Evolution

We assume that present–day structures formed by gravitational instability from Gaussian fluctuations $\delta$ in a pressureless fluid with matter density $\rho = \rho_b[1 + \delta]$, where $\rho_b$ is the background mean density. The density fluctuation field $\delta$ may be written as a Fourier integral, $\delta(x, t) = (2\pi)^{-3} \int dk \tilde{\delta}(k, t) e^{ikx}$, where $x$ and $k$ are the comoving Eulerian coordinate and wavevector, $t$ is the cosmic time. The power spectrum $P(k)$ fully determines the statistics of the primordial Gaussian density field, whose variance is $\sigma^2 = (1/2\pi^2) \int_0^\infty dk k^2 P(k)$. We filter the field $\delta$ by means of a Gaussian window function $W_R(x) = (2\pi R^2)^{-3/2} \exp(-x^2/2R^2)$. The mass variance on scale $R$, $\sigma^2_R$, is related to $P(k)$ by $\sigma^2_R = (1/2\pi^2) \int_0^\infty dk k^2 P(k) [\tilde{W}_R(k)]^2$, where $\tilde{W}_R(k)$ is the Fourier transform of $W_R(x)$.

### 2.1 Equations of Motion: Perturbative Theory

The time evolution equations for the matter density fluctuation $\delta(x, t)$ and the peculiar velocity field $v(x, t)$ are the Euler equation and the continuity equation, i.e.

$$\begin{align*}
\partial_t \mathbf{v} + \frac{1}{a} (\mathbf{v} \cdot \nabla) \mathbf{v} + \frac{\dot{a}}{a} \mathbf{v} &= \mathbf{g} , \\
\partial_t \delta + \frac{1}{a} \nabla \cdot (1 + \delta) \mathbf{v} &= 0 .
\end{align*}$$

(1)

Here $\partial_t \equiv \partial/\partial t$ and spatial derivatives are with respect to $x$. The density contrast $\delta$ is related to the Newtonian gravitational potential $\triangle(x, t) \equiv - (4\pi)^{-1} \int dx' \delta(x', t)/|x' - x|$ via the Poisson equation, $\nabla^2 \triangle = \delta$. In terms of $\triangle$ the peculiar gravitational acceleration is defined as $\mathbf{g} = - 4\pi G \rho_b a \nabla \triangle$. We analyze these equations assuming an Einstein–de Sitter universe with no cosmological constant. In such a model, the scale factor $a$ is proportional to $t^{2/3}$ during the matter dominated epoch, and the adiabatic expansion implies that $6\pi G \rho_b t^2 = 1$.

The first–order solution for $\delta$ has the well–known self–similar form, namely, considering only the growing mode, $\delta^{(1)}(x, t) = D(t) \delta_1(x)$, where $D(t) \propto a(t)$ is the time growth factor of the mass fluctuations. In the linear regime, the peculiar velocity field is proportional to the gravitational acceleration

$$\mathbf{v} = -a \frac{D}{\dot{D}} \nabla \triangle .$$

(3)

This relation shows that the linear velocity field is irrotational; its growing mode corresponds to the growing mode of the density field and, for a flat universe, the classical law $\mathbf{v} = \mathbf{g} t \sim t^{1/3}$ is
recovered. According to Eq.(3), the particles of the gravitating fluid move along the direction of the gravitational force.

It may be useful to give explicitly the Fourier transform of Eq.(3), i.e., for the component along a fixed direction $\hat{\alpha}$,

$$\tilde{v}_\alpha(k,t) = i a \frac{\dot{D}}{D} \frac{k_\alpha}{k^2} \tilde{\delta}(k,t).$$  (4)

Higher order approximations of the density solution may be recovered if one expands the mass density fluctuation field $\delta(x,t)$ about the background solution $\delta = 0$, namely $\delta = \sum_n \delta^{(n)}$ with $\delta^{(n)} = O(\delta^n)$, then solving the differential equation for any $\delta^{(n)}$ (Peebles 1980; Fry 1984). The perturbative expansion for $\delta$ reads (e.g. Goroff et al. 1986): $\delta(x,t) = \sum_{n=1}^{\infty} [D(t)]^n \delta_n(x)$. The first term of the expansion corresponds to the linear approximation. We see that the scale factor $D(t)$ acts as a coupling constant, since $\delta^{(n)} \propto D^n$.

In a similar fashion, expanding $v$ about the solution $v = 0$, one obtains

$$v(x,t) = \frac{2}{3} t \sum_{n=1}^{\infty} [D(t)]^n v_n(x),$$  (5)

where $v_n = O(v_1^n)$ and $-\nabla \cdot v_1 = \delta_1$. We assume that at any order $\nabla \wedge v_n = 0$ (Kelvin circulation theorem).

Here we review the exact perturbative technique to solve approximately the equations of motion (1) and (2) explicitly up to third order in the peculiar velocity field. In particular, we use the third–order solution to compute the fourth–order moment (namely the kurtosis $K_v$). We adopt the same notation of Fry (1984) and Catelan & Moscardini (1994).

(i) Second–Order Velocity Solution

The second–order peculiar velocity $v^{(2)}$ is a solution of the equations

$$\partial_0 \left( a v^{(2)} \right) + \left( v^{(1)} \cdot \nabla \right) v^{(1)} = a \mathbf{g}^{(2)},$$  (6)

$$\partial_0 \delta^{(2)} + a^{-1} \nabla \cdot \left( \mathbf{v}^{(2)} + \delta^{(1)} \mathbf{v}^{(1)} \right) = 0,$$  (7)

where the second–order peculiar acceleration is $\mathbf{g}^{(2)} = -4\pi G \rho_b a \nabla \Delta^{(2)}$ and $\nabla^2 \Delta^{(2)} \equiv \delta^{(2)}$. The second–order density contribution $\delta^{(2)}$ has been derived by Peebles (1980). Since $\delta^{(2)} \propto D^2$, it results that $\mathbf{g}^{(2)} \propto \rho_b a D^2$ and the second–order velocity solution reads

$$\mathbf{v}^{(2)} = - a \frac{\dot{D}}{D} \left[ 2 \nabla \Delta^{(2)} - \delta^{(1)} \nabla \Delta^{(1)} \right] + \mathbf{F}_2,$$  (8)

where $\mathbf{F}_2$ is a divergenceless vector such that $\nabla \wedge \mathbf{v}^{(2)} = 0$ (Catelan et al. 1994). The Fourier transformed $\tilde{v}^{(2)}_\alpha$ may be directly obtained from the solution (8),

$$\tilde{v}^{(2)}_\alpha(k,t) = i a \frac{\dot{D}}{D} \frac{k_\alpha}{k^2} \int \frac{d\mathbf{k}'}{(2\pi)^3} K^{(2)}(k',k-k') \tilde{\delta}^{(1)}(k',t) \tilde{\delta}^{(1)}(k-k',t),$$  (9)

where we have defined the kernel

$$K^{(2)}(k_1,k_2) = \frac{3}{7} + \frac{k_1 \cdot k_2}{k_2^2} + \frac{4}{7} \left( \frac{k_1 \cdot k_2}{k_1 k_2} \right)^2.$$  (10)
It is straightforward to show that \( \langle \mathbf{v}^{(2)} \rangle = 0 \). In an Einstein–de Sitter universe \( \mathbf{v}^{(2)} \sim t \), and it grows slower than \( \delta^{(2)} \sim t^{4/3} \). We stress the fact that \( \mathbf{v}^{(2)} \) is not parallel to the second–order acceleration \( [\alpha - \nabla \delta^{(2)}] \); this is a consequence of non–locality. Thus, the gravitational field changes direction and the particles are not accelerated in a fixed direction, unlike the linear regime. The density–velocity relation in the quasi–linear regime and the cosmological implications of non–locality have been recently explored by Nusser et al. (1991) and Gramann (1993a). The solution (9) is also derived by Gramann (1993b), who applies a second–order Lagrangian perturbative technique.

(ii) Third–Order Velocity Solution

The third–order approximation \( \mathbf{v}^{(3)} \) is a solution of the differential equations

\[
\partial_\mu \left( a \mathbf{v}^{(3)} \right) + \left( \mathbf{v}^{(1)} \cdot \nabla \right) \mathbf{v}^{(2)} + \left( \mathbf{v}^{(2)} \cdot \nabla \right) \mathbf{v}^{(1)} = a \mathbf{g}^{(3)},
\]

\[
\partial_\mu \delta^{(3)} + a^{-1} \nabla \cdot \left( \mathbf{v}^{(3)} + \delta^{(1)} \mathbf{v}^{(2)} + \delta^{(2)} \mathbf{v}^{(1)} \right) = 0.
\]

Here \( \mathbf{g}^{(3)} = -4\pi G \rho_0 a \nabla \delta^{(3)} \) and \( \nabla^2 \delta^{(3)} \equiv \delta^{(3)} \). The third–order density solution has been obtained by Fry (1984). Since \( \mathbf{g}^{(3)} \propto \rho_0 a D^3 \), and using the results of the previous subsection, the velocity \( \mathbf{v}^{(3)} \) may be written as

\[
\mathbf{v}^{(3)} = -a \frac{\dot{D}}{D} \left[ 3 \nabla \delta^{(3)} + \delta^{(1)} \nabla \delta^{(2)} - 2 \delta^{(1)} \nabla \delta^{(1)} - \delta^{(2)} \nabla \delta^{(1)} \right] + \mathbf{F}_3.
\]

Again the additive term \( \mathbf{F}_3 \) is such that \( \nabla \cdot \mathbf{F}_3 = 0 \). The Fourier transform of the previous expression is

\[
\tilde{\mathbf{v}}^{(3)}(k, t) = \frac{ia}{D} \int \frac{d^3k_1 d^3k_2 d^3k_3}{(2\pi)^6} \delta_D \left( \sum_{h=1}^{3} k_h - k \right) K^{(3)}(k_1, k_2, k_3) \tilde{\delta}^{(1)}(k_1, t) \tilde{\delta}^{(1)}(k_2, t) \tilde{\delta}^{(1)}(k_3, t),
\]

where the third–order kernel is

\[
K^{(3)}(k_1, k_2, k_3) = 3 J^{(3)}(k_1, k_2, k_3) - \frac{k \cdot k_1}{k_1^2} J^{(2)}(k_2, k_3) - \frac{k \cdot (k_1 + k_2)}{(k_1 + k_2)^2} K^{(2)}(k_1, k_2),
\]

and the functions \( J^{(2)} \) and \( J^{(3)} \), corresponding to the second– and third–order density solutions (see e.g. Fry 1984; Catelan & Moscardini 1994), read respectively

\[
J^{(2)}(k_1, k_2) \equiv \frac{5}{7} + \frac{k_1 \cdot k_2}{k_2^2} + \frac{2}{7} \left( \frac{k_1 \cdot k_2}{k_1 k_2} \right)^2,
\]

\[
J^{(3)}(k_1, k_2, k_3) \equiv J^{(2)}(k_2, k_3) \left[ \frac{1}{3} + \frac{1}{3} \frac{k_1 \cdot (k_2 + k_3)}{(k_2 + k_3)^2} + \frac{4}{9} \frac{k \cdot k_1 \cdot k \cdot (k_2 + k_3)}{(k_2 + k_3)^2} \right]
\]

\[
- \frac{2}{9} \frac{k \cdot k_1 \cdot k \cdot (k_2 + k_3) \cdot (k_2 + k_3) \cdot k_3}{k_2^2} + \frac{1}{9} \frac{k \cdot k_2 \cdot k \cdot k_3}{k_2^2}.
\]

It is not difficult to show that \( \langle \mathbf{v}^{(3)} \rangle = 0 \). In an Einstein–de Sitter universe, \( \mathbf{v}^{(3)} \sim t^{5/3} \).
As Goroff et al. (1986) have shown, the (Fourier transformed) $n$–th order velocity solution may be represented in integral form as
\[
\tilde{v}_n(k) = i \frac{k}{k^2} \left\{ \prod_{h=1}^{n} \int \frac{dk_h}{(2\pi)^3} \tilde{\delta}(k_h) \right\} \left[ (2\pi)^3 \delta_D(\sum_{j=1}^{n} k_j - k) \right] K^{(n)}(k_1, \ldots, k_n).
\]

The presence of the Dirac delta function comes from momentum conservation in Fourier space. The kernels $K^{(n)}$ are homogeneous (with degree 0) functions of the wavevector $s k_1, \ldots, k_n$, describing the effects of non–linear collapse (tidal and shear effects). In general the $K^{(n)}$ are very complicated for $n > 3$. [A discussion of the properties of the kernels $K^{(n)}$ is given in Wise (1988). Explicit recursion relations with their Feynman diagrammatic representations are given by Goroff et al. (1986) and Wise (1988).]

### 2.2 Zel’dovich Approximation

In the Zel’dovich approximation (Zel’dovich 1970) the motion of particles from the initial comoving (Lagrangian) positions $q$ is approximated by straight paths. The Eulerian position at time $t$ is then given by the uniform motion
\[
x(q, t) = q + D(t) S(q),
\]

where $D(t)$ is the growth factor of linear density perturbations and $S(q)$ is the displacement vector related to the primordial velocity field. Grinstein & Wise (1987) give an Eulerian representation of the Zel’dovich approximation by a diagrammatic perturbative approach similar to that of the previous section. They showed that the $n$–th order perturbative corrections $\delta_n(x)$, when the density fluctuation field $\delta$ is evolved according to the Zel’dovich approximation, are such that
\[
\delta(x, t) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} [D(t)]^n \sum_{[h_n]=1}^{3} \frac{\partial}{\partial x_{h_1}} \cdots \frac{\partial}{\partial x_{h_n}} [S_{h_1} \cdots S_{h_n}].
\]

Here $\sum_{[h_n]} \equiv \sum_{h_1} \cdots \sum_{h_n}$. Note that the first term recovers the linear approximation, in that $S = v_1$, where $\delta_1(x) = -\nabla \cdot v_1$. This expansion for $\delta$ corresponds to different symmetric kernels $K^{(n)}_{ZA}$, which can be written in the following compact form
\[
K^{(n)}_{ZA}(k_1, \ldots, k_n) = \frac{1}{n!} \prod_{h=1}^{n} \frac{k \cdot k_h}{k_h^2},
\]

where $k \equiv \sum_{h=1}^{n} k_h$. The kernels $K^{(n)}_{ZA}$ are the same obtained in the expansion of the density contrast (see Catelan & Moscardini 1994) and are symmetric by construction.

### 3 Kurtosis of the Velocity Field

In this section, we compute the gravitationally induced kurtosis $K_v$ of an initial Gaussian velocity field in a flat universe. We restrict the calculation to the velocity along a chosen direction $\hat{\alpha}$. The lowest order non–zero reduced contribution to $K_v$ is
\[
\langle v^{(1)}_{\hat{\alpha}} \rangle^2 \langle v^{(2)}_{\hat{\alpha}} \rangle^2 K_v \equiv 6 \langle v^{(1)}_{\hat{\alpha}} \rangle^2 \langle v^{(2)}_{\hat{\alpha}} \rangle^2_c + 4 \langle v^{(1)}_{\hat{\alpha}} \rangle^3 \langle v^{(3)}_{\hat{\alpha}} \rangle_c,
\]

\[\tag{22}
\]

\[\]
where the subscript $c$ indicates the connected part of the four–point velocity correlation.

It is not difficult to verify that $K_v$ depends on the normalization of the power spectrum. Furthermore it varies with time. The related quantity which is independent both of the power spectrum normalization and of time is $S_{4,v} = K_v / \langle \delta \rangle$: due to these properties, observational estimates of $S_{4,v}$ would allow one to detect *intrinsic* features of the evolved large–scale velocity field.

From the perturbative results in Eqs.(9), (14) and (22), one finally gets the integral expression of the kurtosis of the smoothed velocity field $v_R$:

$$K_v(R) = \frac{24}{\sigma^2_{v,R}} \left( \frac{dD}{D} \right)^4 \int \frac{dk_1 dk_2 dk_3}{(2\pi)^9} \frac{k_1^4 k_2^2 k_3^3}{k_1^2 k_2^2 k_3^2} \left| k_1 + k_2 + k_3 \right|^2 \times$$

$$P(k_1) P(k_2) P(k_3) \left[ P(k_3) K^{(3)}_{s}(k_1, k_2, k_3) + 2 K^{(2)}_{s}(-k_2, k_2 + k_3) K^{(2)}_{s}(k_1, k_2 + k_3) P(|k_2 + k_3|) \right],$$

(23)

where $\sigma^2_{v,R}$ is the variance of smoothed velocity field. The kernels $K^{(n)}_s$ are obtained by complete symmetrization of the kernels $K^{(n)}$. The analogous expression in the Zel’dovich approximation is obtained by replacing the kernels $K^{(n)}_s$ with the corresponding $K^{(n)}_{ZA}$. Finally, note that $P(k)$ completely describes the process of growth of the higher order velocity moments from Gaussian initial conditions.

### 4 Discussion and Conclusions

We calculate the previous integrals, by an Adaptive Multidimensional Monte Carlo Integration subroutine, in the framework of several different cosmological models. In particular, we consider:

1. standard cold dark matter model (SCDM), i.e. with $b = 1$ for the linear biasing parameter;
2. a biased (BCDM) version of the same model, with $b = 1.5$;
3. tilted cold dark matter model (TCDM), with spectral index $n = 0.7$ and $b = 2.0$;
4. a ‘mixed’ model (MDM), with 60% cold and 30% hot dark matter, $b = 1.5$. All transfer functions have been taken from Holtzmann (1989), with the Hubble constant $H_0 = 50 \text{ km s}^{-1} \text{ Mpc}^{-1}$.

The results for $K_v$ are shown in Fig.1a. The associate relative uncertainty estimates from the Monte Carlo Integration (not shown in the figure for clarity) are always smaller than 5%. For any model the velocity kurtosis is a decreasing function of the smoothing scale $R$. On large scales $K_v$ is always consistent with zero, thus the distribution of the peculiar velocity field is very close to a Gaussian one. However, a strong non–Gaussian signature – i.e. $K_v \gtrsim 1$ – is induced on scales smaller than $\lesssim 20 – 30$ Mpc, where nevertheless observations are at the moment affected by large uncertainties. Furthermore, even if the general trend is similar for all considered models, the departure from Gaussianity is larger for models with low biasing parameter: TCDM gives always the lowest values of $K_v$.

A previous estimate of the kurtosis of the velocity field was obtained by Kofman et al. (1994), who used the smoothed velocity fields of a $N$–body simulation of the standard cold dark matter model with box–size of 400 Mpc to study the velocity distribution $p(v)$. In order to have a more direct comparison, in this case we calculated $K_v$ using their choice for the CDM transfer function.
(Davis et al. 1985) and limiting the numerical integration of the expression (23) in the range of wavevectors spanned by the $N$–body simulation: from the frequency corresponding to the box–size to the Nyquist one. In Fig.1b we show both our perturbative evaluations and the Kofman et al.’s data: the agreement is quite good, except for the small scale (12 Mpc) result. It might seem surprising that the numerical simulations, which are supposed to better describe the highly nonlinear regime, miss the departure from Gaussianity of the velocity field: in fact, for the density field, the perturbative evaluations of higher moments underestimate the $N$–body results. A possible explanation is that the multiple filtering (a trilinear interpolation plus a small–scale smoothing) necessary in the simulations to reconstruct the velocity field from the particle distribution might smooth the small–scale non–Gaussian signal. Moreover, a further “numerical smoothing” can appear due to the intrinsic gridding present in the particle–mesh code. In any case, it is necessary to be cautious in the measurements of higher order moments in $N$–body simulations because they are strongly affected by the high–value tails, which characterize the particular realization, size of the box and other numerical problems (Kofman, private communication): this is also shown by the large error bars in the $N$–body results.

We also calculate the velocity kurtosis in terms of the intrinsic parameter $S_{4,v}$. In Fig.2a, $S_{4,v}$ is plotted for the same models previously considered: in this case, due to the independence of $S_{4,v}$ on the power spectrum normalization, the biased and the standard CDM originate the same curve. All models present a similar behavior and the differences are always inside the error bars, shown for clarity only for SCDM. In Fig.2b, the dependence of $S_{4,v}$ on the primordial spectral index $n$ is shown for scale–free power spectra $P(k) \propto k^n$, with $n$ in the range $-1 < n \leq 1$, for both the perturbative and Zel’dovich approximations. Due to the assumed scale–invariance, $S_{4,v}$ only depends on the primordial spectral index $n$, and not on the scale $R$. It may be noted that in the Zel’dovich approximation, unlike the perturbative case, the velocity kurtosis is practically constant and consistent with zero for any value of $n$. Thus, we confirm the results of Kofman et al. (1994), who demonstrate that the Eulerian Gaussian one–point pdf $p(v)$ is time–invariant as long as the Zel’dovich approximation holds. This is essentially due to the simple time scaling of the particle Eulerian position $x(q,t)$ in Eq.(19).

Finally we want to stress that one has to be careful about making quantitative comparisons between our results, directly related to the underlying (dark plus luminous) mass distribution, and observational data. In particular, Grinstein et al. (1987) suggest that when point–like luminous objects, like galaxies, are used to sample the peculiar flow within a region of the sky, it is the volume average of $n(x)v(x)$ which is actually measured instead of the volume average of $v(x)$, $n(x)$ being the number density of luminous tracers. If the objects are biased tracers of the underlying mass distribution, then nonlinear effects on small–scales may preclude any direct comparison of the observed velocity moments with ensemble expectations. A further analysis of this problem is in progress.

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Figure captions

**Figure 1.** The kurtosis ratio $K_v$ of the velocity field. Left panel: the behavior versus the scale $R$ for Gaussian filter and different cosmological models (parameters are in the text): standard cold dark matter (solid line); biased cold dark matter (dotted line); tilted cold dark matter (dashed line); mixed dark matter (dotted–dashed line). Right panel: comparison with results from $N$–body (Kofman et al. 1994, filled squares) in the case of standard cold dark matter. Filled triangles and solid line refer to the perturbative estimates when the integration is limited in the interval ranging from the box–size frequency to the Nyquist one; dotted line refers to the perturbative estimates when the integration is over all frequencies.

**Figure 2.** The intrinsic parameter $S_{4,v}$. Left panel: the behavior versus the scale $R$ for different cosmological models: standard cold dark matter (triangles), tilted cold dark matter (squares), mixed dark matter (circles). Error bars, shown only for SCDM, refer to the associated uncertainty estimate from the Monte Carlo Integration. Right panel: the behavior versus the primordial spectral index $n$ for power–law spectra $P(k) \propto k^n$ and Gaussian filter, for both the perturbative (triangles and solid line) and Zel’ dovich approximations (squares and dotted line).