Application of a $\mathbb{Z}_3$-orbifold construction to the lattice vertex operator algebras associated to Niemeier lattices

Daisuke Sagaki
Institute of Mathematics, University of Tsukuba, Tennodai 1-1-1, Tsukuba, Ibaraki 305-8571, Japan
(e-mail: sagaki@math.tsukuba.ac.jp)

Hiroki Shimakura
Graduate School of Information Sciences, Tohoku University, Aramaki aza Aoba 6-3-09, Aoba-ku, Sendai 980-8579, Japan
(e-mail: shimakura@m.tohoku.ac.jp)

Abstract

By applying Miyamoto’s $\mathbb{Z}_3$-orbifold construction to the lattice vertex operator algebras associated to Niemeier lattices and their automorphisms of order 3, we construct holomorphic vertex operator algebras of central charge 24 whose Lie algebras of the weight one spaces are of types $A_6^{123}$, $E_6G_2^{31}$, and $A_5^3D_4^3A_1^{31}$, which correspond to No. 6, No. 17, and No. 32 on Schellekens’ list, respectively.

1 Introduction.

The classification of holomorphic vertex operator algebras (VOAs for short) is a fundamental problem in the VOA theory. By Zhu’s theory (see [Z]), their central charges are divisible by 8. We know from [DMI] that a holomorphic VOA of central charge 8 or 16 is isomorphic to a lattice VOA. Thus the next problem is to classify the holomorphic VOAs of central charge 24. In [S], Schellekens gave a list of possible 71 Lie algebra structures of the weight one spaces of holomorphic VOAs of central charge 24 (it is mysterious that the number 71 appears here; for, 71 is the largest prime factor of the order of the Monster simple group). So the first step of the classification would be:

For each Lie algebra on Schellekens’ list, construct a holomorphic VOA of central charge 24 whose Lie algebra of the weight one space is isomorphic to it.

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It is well-known that the lattice VOA associated to a Niemeier lattice (i.e., unimodular, positive-definite, even lattice of rank 24) is a holomorphic VOA of central charge 24. Because there exist exactly 24 Niemeier lattices (see, e.g., [CS, Chapter 16, Table 16.1]), we can obtain 24 holomorphic VOAs of central charge 24 in this manner. Also, we can obtain 15 holomorphic VOAs of central charge 24 by applying the \( \mathbb{Z}_2 \)-orbifold construction to the lattice VOAs associated to Niemeier lattices and the \((-1)\)-isometry (see [FLM, DGM]); the Moonshine VOA is included in these 15 holomorphic VOAs. In [La, LS1, LS2], Lam and the second named author proved that there exist exactly 56 holomorphic framed VOAs of central charge 24, including 39 (= 24 + 15) holomorphic VOAs mentioned above. According to Schellekens’ list, there should exist at least 15 (= 71 − 56) non-framed holomorphic VOAs of central charge 24, which should correspond to:

| No. in [S] | Dimension of the weight one space | Lie algebra structure of the weight one space |
|------------|-----------------------------------|---------------------------------------------|
| 3          | 36                                | \( D_{4,12}A_{2,6} \)                         |
| 4          | 36                                | \( C_{4,10} \)                               |
| 6          | 48                                | \( A_{2,3}^6 \)                              |
| 8          | 48                                | \( A_{5,6}C_{2,3}A_{1,2} \)                  |
| 9          | 48                                | \( A_{4,5}^2 \)                              |
| 11         | 48                                | \( A_{6,7} \)                                |
| 14         | 60                                | \( F_{4,6}A_{2,2} \)                         |
| 17         | 72                                | \( A_{5,3}D_{4,3}A_{1,1}^3 \)                |
| 20         | 72                                | \( D_{6,5}A_{1,1}^2 \)                       |
| 21         | 72                                | \( C_{5,3}G_{2,2}A_{1,1} \)                  |
| 27         | 96                                | \( A_{8,3}A_{2,1}^2 \)                       |
| 28         | 96                                | \( E_{6,4}C_{2,1}A_{2,1} \)                  |
| 32         | 120                               | \( E_{6,3}G_{2,1}^3 \)                       |
| 34         | 120                               | \( D_{7,3}A_{3,1}G_{2,1} \)                  |
| 45         | 168                               | \( E_{7,3}A_{5,1} \)                         |

Here, \( X_{m,n} \) denotes the simple Lie algebra of type \( X_m \) whose “level” is equal to \( n \) (for the definition of the level, see [2.3]).

In [Mi], Miyamoto established a \( \mathbb{Z}_3 \)-orbifold construction to a lattice VOA and a lattice automorphism of order 3 satisfying the condition that the rank of the fixed point lattice is divisible by 6 (see [2.3]). Then he constructed a new (non-framed) holomorphic VOA of central charge 24 whose Lie algebra of the weight one space is of type \( E_{6,3}G_{2,1}^2 \), by applying his \( \mathbb{Z}_3 \)-orbifold construction to the lattice VOA associated to the Niemeier lattice \( \text{Ni}(E_6^1) \) (with \( E_6^1 \) the root lattice) and its automorphism \( \sigma_6 \) of order 3 (see [Mi, §5.2] and also Appendix in this paper); this holomorphic VOA corresponds to No. 32 on Schellekens’ list. Also, he obtained a holomorphic VOA whose weight one space is identical to \( \{0\} \), by applying his
$\mathbb{Z}_3$-orbifold construction to the Leech lattice VOA and a fixed-point-free automorphism $\sigma_7$ of order 3 of the Leech lattice; this holomorphic VOA is conjecturally isomorphic to the Moonshine VOA $V^\natural$.

In this paper, we also construct some new holomorphic VOAs as a further application of Miyamoto’s $\mathbb{Z}_3$-orbifold construction; in Theorem 3.2.2 (resp., Theorems 4.2.1, 5.2.1, 6.2.1, 7.2.1), we obtain a holomorphic VOA whose Lie algebra of the weight one space is of type $A_{6,2}$ (resp., $A_{6,2}$, $E_{6,3}G_{2,1}$, $A_{5,3}D_{4,3}A_{1,1}$, $A_{5,3}D_{4,3}A_{1,1}$), by applying the $\mathbb{Z}_3$-orbifold construction to the lattice VOA associated to the Niemeier lattice $\text{Ni}(A_{12}^2)$ (resp., $\text{Ni}(D_4^6)$, $\text{Ni}(D_4^6)$, $\text{Ni}(D_4^6)$, $\text{Ni}(A_4^4D_4)$) and its automorphism $\sigma_1$ (resp., $\sigma_2$, $\sigma_3$, $\sigma_4$, $\sigma_5$) of order 3, which corresponds to No. 6 (resp., No. 6, No. 32, No. 17, No. 17) on Schellekens’ list.

Remark. The Lie algebras of the weight one spaces in the holomorphic VOAs obtained in Theorems 3.2.2 and 4.2.1 (resp., Theorem 5.2.1 and [Mi, §5.2], Theorems 6.2.1 and 7.2.1) are isomorphic. However, we do not know whether or not these holomorphic VOAs are isomorphic (as VOAs).

In [ISS], we will classify, up to conjugation, all lattice automorphisms of order 3 of Niemeier lattices, satisfying the condition that the ranks of the fixed point lattices are divisible by 6 (i.e., those to which we can apply the $\mathbb{Z}_3$-orbifold construction), and prove that if such a lattice automorphism of a Niemeier lattice is not conjugate to any of the $\sigma_1$, $\sigma_2$, $\sigma_3$, $\sigma_4$, $\sigma_5$, $\sigma_6$, $\sigma_7$ above, then the holomorphic VOA obtained by the $\mathbb{Z}_3$-orbifold construction is isomorphic to the lattice VOA associated to a Niemeier lattice. In other word, a non-framed holomorphic VOA of central charge 24 which can be obtained by the $\mathbb{Z}_3$-orbifold construction is one of those obtained in this paper and [Mi].

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2 Review.

2.1 Lattice VOAs and their automorphisms. In this subsection, we review the definition of a lattice vertex operator algebra (VOA for short); for the details, see, e.g., [LL, §6.4 and §6.5].

Let $L$ be a positive-definite, even lattice with $\mathbb{Z}$-bilinear form $\langle \cdot, \cdot \rangle$. Regard $\mathfrak{h} := L \otimes_\mathbb{Z} \mathbb{C}$ as an abelian Lie algebra, and define its affinization to be the Lie algebra $\hat{\mathfrak{h}} := \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}]$ with Lie bracket given by:

$$[x \otimes t^m, y \otimes t^n] = \delta_{m+n,0} \langle x, y \rangle k \quad \text{for } x, y \in \mathfrak{h} \text{ and } m, n \in \mathbb{Z},$$

$$[\mathfrak{h}, k] = \{0\};$$
for simplicity of notation, we denote $h \otimes t^m$ by $h(m)$ for $h \in \mathfrak{h}$ and $m \in \mathbb{Z}$. The Lie subalgebra $\mathfrak{b} := \mathfrak{h} \otimes \mathbb{C}[t] \oplus Ck \subset \hat{\mathfrak{h}}$ acts on the one-dimensional vector space $\mathbb{C}$ as follows: for $c \in \mathbb{C}$,

$$h(m) \cdot c = 0 \quad \text{for all } h \in \mathfrak{h} \text{ and } m \in \mathbb{Z}_{\geq 0}, \quad k \cdot c = c.$$ 

Then we define

$$M(1) := \text{Ind}^{\hat{\mathfrak{h}}}_{\mathfrak{b}} \mathbb{C}.$$ 

Fix a positive even integer $s \in 2\mathbb{Z}_{>0}$. Let us define $c_0 : L \times L \to \mathbb{Z}/s\mathbb{Z}$ by:

$$c_0(\alpha, \beta) = \frac{s}{2} \langle \alpha, \beta \rangle + s\mathbb{Z},$$

which is an alternating $\mathbb{Z}$-bilinear map. Let $\varepsilon_0 : L \times L \to \mathbb{Z}/s\mathbb{Z}$ be a 2-cocycle corresponding to $c_0 : L \times L \to \mathbb{Z}/s\mathbb{Z}$, normalized as: $\varepsilon_0(\alpha, 0) = \varepsilon_0(0, \alpha) = 0$ for all $\alpha \in L$. Let $\langle \kappa \rangle$ be the cyclic group of order $s$. We define a product on

$$\hat{L} := \{(\kappa^a, e_\alpha) \mid a \in \mathbb{Z}/s\mathbb{Z}, \alpha \in L\}$$

as follows: for $a, b \in \mathbb{Z}/s\mathbb{Z}$ and $\alpha, \beta \in L$,

$$(\kappa^a, e_\alpha) \cdot (\kappa^b, e_\beta) := (\kappa^{a+b+s\varepsilon_0(\alpha, \beta)}, e_{\alpha+\beta}).$$

Then, $\hat{L}$ is a group with $(\kappa^0, e_0)$ the identity element, and is the central extension of $L$ by the cyclic group $\langle \kappa \rangle$ of order $s$ with $c_0 : L \times L \to \mathbb{Z}/s\mathbb{Z}$ the commutator map. The cyclic group $\langle \kappa \rangle$ acts on the one-dimensional space $\mathbb{C}$ by: $\kappa \cdot c = \xi c$ for $c \in \mathbb{C}$, where $\xi \in \mathbb{C}$ is a primitive $s$-th root of unity. Define

$$\mathbb{C}\{L\} := \mathbb{C}[\hat{L}] \otimes_{\langle \kappa \rangle} \mathbb{C},$$

where $\mathbb{C}[\hat{L}]$ denotes the group ring of the group $\hat{L}$; remark that $\{e^\alpha := (\kappa^0, e_\alpha) \otimes 1 \mid \alpha \in L\}$ is a basis of $\mathbb{C}\{L\}$. 

Now, set

$$V_L := M(1) \otimes \mathbb{C}\{L\}.$$ 

Then, $V_L$ admits a VOA structure whose central charge is equal to the rank of the lattice $L$ (which is independent of the choices of $s$, $\varepsilon_0$, and $\xi$). Recall that the weight of $h_k(\langle\alpha,\alpha\rangle) \in V_L$ is given by:

$$n_k + \cdots + n_1 + \frac{\langle\alpha,\alpha\rangle}{2} \in \mathbb{Z}_{\geq 0}.$$ 

In particular, the weight one space $(V_L)_1$ of $V_L$ is spanned by

$$\{h(-1)1 \otimes e^0 \mid h \in \mathfrak{h}\} \cup \{1 \otimes e^\alpha \mid \alpha \in \Delta\}.$$
if necessary, we may assume that

\[ \epsilon \in \langle \text{condition that} \rangle \]

where \( \Delta = \Delta(L) := \{ \alpha \in L \mid \langle \alpha, \alpha \rangle = 2 \} \), the set of roots in \( L \). Denote by

\[ Y(\cdot, z) : V_L \rightarrow (\text{End}_C V_L)[[z, z^{-1}]], \quad a \mapsto Y(a, z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1} \]

the vertex operator for \( V_L \). For latter use, let us recall the definition of \( Y(a, z) \) for some special \( a \in V_L \). First, the Lie algebra \( \hat{h} \) acts on \( V_L = M(1) \otimes \mathbb{C}\{L\} \) as follows: \( k \in \hat{h} \) acts as the identity, and for \( h \in \hat{h} \) and \( n \in \mathbb{Z} \),

\[ h(n)(u \otimes e^\beta) = \begin{cases} (h, \beta)(u \otimes e^\beta) & \text{if } n = 0, \\ (h(n)u) \otimes e^\beta & \text{if } n \neq 0, \end{cases} \quad \text{for } u \in M(1) \text{ and } \beta \in L. \]

Also, for \( \alpha \in L \), we define \( z^\alpha \in \text{Hom}_C(V_L, V_L[z, z^{-1}]) \) by

\[ z^\alpha(u \otimes e^\beta) := z^{(\alpha, \beta)}(u \otimes e^\beta) \quad \text{for } u \in M(1) \text{ and } \beta \in L. \]

In addition, the group \( \hat{L} \) acts on \( V_L = M(1) \otimes \mathbb{C}\{L\} \) as follows:

\[ g \cdot (u \otimes v) = u \otimes (g \cdot v) \quad \text{for } g \in \hat{L} \text{ and } u \in M(1), \quad v \in \mathbb{C}\{L\}, \]

where the \( g \cdot v \) above is given by the natural action of \( \hat{L} \) on \( \mathbb{C}\{L\} \). In particular,

\[ (\kappa^0, e_\alpha) \cdot (u \otimes e^\beta) := u \otimes (\xi^{e_\alpha \beta} e^{\alpha+\beta}) \quad \text{for } \alpha, \beta \in L \text{ and } u \in M(1). \]

We have

\[ Y(h(-1)1 \otimes e^0, z) = \sum_{n \in \mathbb{Z}} h(n) z^{-n-1} \quad \text{for } h \in \hat{h}, \]

\[ Y(1 \otimes e^\alpha, z) = E^-(-\alpha, z) E^+(\alpha, z) \left( \sum_{\kappa \in L} (\kappa^0, e_\alpha) z^\alpha \right) \quad \text{for } \alpha \in L, \]

where

\[ E^\pm(-\alpha, z) := \exp \left( \sum_{n \in \mathbb{Z}, \pm n > 0} -\alpha(n) z^{-n} \right). \]

### 2.2 Twisted modules over lattice VOAs.

Keep the notation in §2.1. An automorphism of the lattice \( L \) is, by definition, a \( \mathbb{Z} \)-module automorphism \( \sigma \) of \( L \) satisfying the condition that \( \langle \sigma \alpha, \sigma \beta \rangle = \langle \alpha, \beta \rangle \) for all \( \alpha, \beta \in L \). Denote by \( \text{Aut}(L) \) the group of all lattice automorphisms of \( L \). Let \( \sigma \in \text{Aut}(L) \) be of odd order, and set \( s := 2|\sigma| \) (with notation in §2.1), where \( |\sigma| \) denotes the order \( \sigma \). Replacing \( \varepsilon_0 : L \times L \rightarrow \mathbb{Z}/s\mathbb{Z} \) with \( (\alpha, \beta) \mapsto \sum_{r=0}^{|\sigma|-1} \varepsilon_0(\sigma^r \alpha, \sigma^r \beta) \) for \( \alpha, \beta \in L \)

if necessary, we may assume that \( \varepsilon_0 \) is \( \sigma \)-invariant. Then we deduce that the lattice automorphism \( \sigma \in \text{Aut}(L) \) naturally induces a VOA automorphism of \( V_L \); by abuse of notation, we denote this VOA automorphism also by \( \sigma \in \text{Aut}(V_L) \). Remark that

\[ \sigma(h_k(-n_k) \cdots h_1(-n_1)1 \otimes e^\alpha) = (\sigma h_k)(-n_k) \cdots (\sigma h_1)(-1)1 \otimes e^{\sigma \alpha}. \]
Now, we recall a construction of σ-twisted modules over the lattice VOA $V_L$ from [DL] and [Le] in the case that σ is of odd order $p$; in fact, a lattice automorphism mainly treated in this paper is of order 3. Set $s := 2|\sigma| = 2p$ as above, and set $\zeta := \xi^2 \in \mathbb{C}$, which is a primitive $p$-th root of unity (recall that $\xi$ is a primitive $s$-th root of unity).

For $n \in \mathbb{Z}$, define $\mathfrak{h}_{(n)} := \{ h \in \mathfrak{h} \mid \sigma h = \zeta^n h \}$; note that $\mathfrak{h}_{(n)} = \mathfrak{h}_{(n+pk)}$ for all $n, k \in \mathbb{Z}$, and $\mathfrak{h} = \mathfrak{h}_{(0)} \oplus \mathfrak{h}_{(1)} \oplus \cdots \oplus \mathfrak{h}_{(p-1)}$. Let us define the $\sigma$-twisted affine Lie algebra associated to the abelian Lie algebra $\mathfrak{h}$ to be

$$\hat{\mathfrak{h}}[\sigma] := \bigoplus_{n \in (1/p)\mathbb{Z}} \mathfrak{h}_{(pn)} \otimes \mathbb{C}t^n \oplus \mathbb{C}k,$$

with Lie bracket

$$[x \otimes t^m, y \otimes t^n] = \delta_{m+n, 0} m(x, y)k \quad \text{for } m, n \in (1/p)\mathbb{Z} \text{ and } x \in \mathfrak{h}_{(pm)}, y \in \mathfrak{h}_{(pn)},$$

for simplicity of notation, we denote $h \otimes t^m$ by $h(m)$ for $m \in (1/p)\mathbb{Z}$ and $h \in \mathfrak{h}_{(pm)}$. The Lie subalgebra

$$\hat{\mathfrak{b}}[\sigma] := \bigoplus_{n \in (1/p)\mathbb{Z}_{\geq 0}} \mathfrak{h}_{(pn)} \otimes \mathbb{C}t^n \oplus \mathbb{C}k \subset \hat{\mathfrak{h}}[\sigma]$$

acts on the one-dimensional vector space $\mathbb{C}$ as follows: for $c \in \mathbb{C}$,

$$h(m) \cdot c = 0 \quad \text{for all } m \in (1/p)\mathbb{Z}_{\geq 0} \text{ and } h \in \mathfrak{h}_{(pm)}, \quad k \cdot c = c.$$

Then we define

$$M(1)[\sigma] := \text{Ind}_{\hat{\mathfrak{b}}[\sigma]}^{\hat{\mathfrak{h}}[\sigma]} \mathbb{C}.$$

Define an alternating $\mathbb{Z}$-bilinear map $\varepsilon_0^\sigma : L \times L \to \mathbb{Z}/s\mathbb{Z}$ by:

$$\varepsilon_0^\sigma(\alpha, \beta) = \sum_{r=0}^{p-1} (s/2 + sr/p) \langle \sigma^r \alpha, \beta \rangle + s\mathbb{Z}, \quad \text{(2.2.1)}$$

and then define $\varepsilon_0^\sigma : L \times L \to \mathbb{Z}/s\mathbb{Z}$ by (see [DL] Remarks 2.1 and 2.2])

$$\varepsilon_0^\sigma(\alpha, \beta) := \varepsilon_0(\alpha, \beta) + \sum_{0 < r < p/2} (s/2 + sr/p) \langle \sigma^r \alpha, \beta \rangle + s\mathbb{Z} \quad \text{for } \alpha, \beta \in L.$$

It can be easily checked that $\varepsilon_0^\sigma$ is a $\sigma$-invariant, normalized 2-cocycle corresponding to $\varepsilon_0^\sigma$. We define another product $\ast$ on $\hat{L} = \{(\kappa^a, e_{\alpha}) \mid a \in \mathbb{Z}/s\mathbb{Z}, \alpha \in L\}$ as follows: for $a, b \in \mathbb{Z}/s\mathbb{Z}$ and $\alpha, \beta \in L$,

$$(\kappa^a, e_{\alpha}) \ast (\kappa^b, e_{\beta}) := (\kappa^{a+b} + \varepsilon_0^\sigma(\alpha, \beta), e_{\alpha+\beta}).$$

Then, $(\hat{L}, \ast)$ is a group with $(\kappa^0, e_0)$ the identity element; we denote this group by $\hat{L}_\sigma$. It can be easily seen that $\hat{L}_\sigma$ is the central extension of $L$ by the cyclic group $\langle \kappa \rangle$ of order $s$ with
\( e_0^\sigma : L \times L \rightarrow \mathbb{Z}/s\mathbb{Z} \) the commutator map. Because \( e_0^\sigma \) is \( \sigma \)-invariant, the lattice automorphism \( \sigma \in \text{Aut}(L) \) induces a group automorphism \( \sigma \in \text{Aut}(\hat{L}_\sigma) \) defined by: \( \sigma(\kappa^a, e_\alpha) = (\kappa^a, e_{\sigma\alpha}) \) for \( a \in \mathbb{Z}/s\mathbb{Z} \) and \( \alpha \in L \).

For \( h \in \mathfrak{h} \) and \( n \in \mathbb{Z} \), denote by \( \tau_{(n)} \) the image of \( h \) under the projection \( \mathfrak{h} \rightarrow \mathfrak{h}_{(n)} \); note that \( \tau_{(0)} = (1/p) \sum_{r=0}^{n-1} \sigma^r h \) for \( h \in \mathfrak{h} \). Set

\[
N := \{ \alpha \in L \mid \langle \alpha, \mathfrak{h}_{(0)} \rangle = \{0\} \} = \{ \alpha \in L \mid \alpha_{(0)} = 0 \};
\]

the (second) equality follows from the fact that \( \langle \sigma^r \alpha, h \rangle \) for all \( r \in \mathbb{Z} \) and all \( \alpha \in L \) and \( h \in \mathfrak{h}_{(0)} \), and the fact that \( \langle \cdot, \cdot \rangle \) is nondegenerate on \( \mathfrak{h}_{(0)} \). Set

\[
R := \{ \alpha \in N \mid c_0^\sigma(\alpha, N) = 0 \}, \quad M := (1 - \sigma)L. \tag{2.2.3}
\]

Then, \( M \subset R \subset N \). Also, because

\[
M \otimes_{\mathbb{Z}} \mathbb{C} = (1 - \sigma) \underbrace{(L \otimes_{\mathbb{Z}} \mathbb{C})}_{= \mathfrak{h}} = \mathfrak{h}_{(1)} \oplus \mathfrak{h}_{(2)} = \{ h \in \mathfrak{h} \mid \langle h, \mathfrak{h}_{(0)} \rangle = \{0\} \} = N \otimes_{\mathbb{Z}} \mathbb{C}, \tag{2.2.4}
\]

it follows immediately that \( N/M \) is a finite group, and hence so is \( N/R \).

Now, for a sublattice \( Q \) of \( L \), we set \( \hat{Q}_\sigma := \{ (\kappa^a, e_\alpha) \mid a \in \mathbb{Z}/s\mathbb{Z}, \alpha \in Q \} \) (which is a subgroup of \( \hat{L}_\sigma \)). Then, \( \hat{M}_\sigma \subset \hat{R}_\sigma \subset \hat{\mathbb{C}}^{N\sigma} \). It is known from \[\text{Le}] \text{ Propositions 6.1 and 6.2} \) (see also \[\text{DL}] \text{ Remark 4.2}) that there exists a finite-dimensional irreducible \( \hat{\mathbb{C}}^{N\sigma} \)-module \( T \) of dimension \( |N/R|^{1/2} \) on which \( \hat{M}_\sigma \) acts according to a group homomorphism \( \tau : \hat{M}_\sigma \rightarrow \langle \xi \rangle \subset \mathbb{C}^\times \), i.e., \( g \cdot t = \tau(g)t \) for \( g \in \hat{M}_\sigma \) and \( t \in T \). Set

\[
U_T := \text{Ind}_{\hat{\mathbb{C}}^{N\sigma}}^{\hat{L}_\sigma} T.
\]

\textbf{Remark 2.2.1} (see \[\text{Le}] \text{ Section 7}). Since \( \tau(\hat{M}_\sigma) \subset \langle \xi \rangle \), it follows immediately that \( \hat{M}_\sigma/\ker \tau \) is a finite group. Also, since \( N/M \) is a finite group as seen above, so is \( \hat{\mathbb{C}}^{N\sigma}/\hat{M}_\sigma \). Thus, \( \hat{\mathbb{C}}^{N\sigma}/\ker \tau \) is a finite group. Since every element in \( \ker \tau \) acts on \( T \) as the identity, we conclude that each \( g \in \hat{\mathbb{C}}^{N\sigma} \) acts on \( T \) as a linear automorphism of finite order. In particular, the action of each \( g \in \hat{\mathbb{C}}^{N\sigma} \) on \( T \) is semisimple.

Now, we define

\[
V_L^T := M(1)[\sigma] \otimes_U T.
\]

Then we know from \[\text{DL}] \text{ Theorem 7.1} \) that \( V_L^T \) admits a \( \sigma \)-twisted \( V_L \)-module structure. Note that \( V_L^T = M(1)[\sigma] \otimes_U T \) is spanned by the elements of the form: \( h_k(-n_k) \cdots h_1(-n_1)1 \otimes (g \cdot t) \) with \( n_1, \ldots, n_k \in (1/p)\mathbb{Z}_{>0}, h_1 \in \mathfrak{h}_{(-n_1)}, \ldots, h_k \in \mathfrak{h}_{(-n_k)} \), and \( g \in \hat{L}_\sigma, t \in T \). The weight of an element of the form above is given by

\[
n_k \cdot \cdots \cdot n_1 \cdot \rho + \frac{1}{2} \langle \mathfrak{g}_{(0)}, \mathfrak{g}_{(0)} \rangle \in \rho + (1/p)\mathbb{Z}_{\geq 0}, \tag{2.2.5}
\]
where
\[ \rho := \frac{1}{4p^2} \sum_{r=1}^{p-1} r(p-r) \dim \mathfrak{h}_{(r)}, \] (2.2.6)
and where for \( g = (\kappa^\alpha, e^\alpha) \in \widehat{L_\sigma} \), we set \( \bar{g} := \alpha \in L \); notice that for \( g \in \widehat{L_\sigma} \),
\[ \langle \bar{g}_{(0)}, \bar{g}_{(0)} \rangle = 0 \iff \bar{g}_{(0)} = 0 \iff \bar{g} \in N \iff g \in \widehat{N_\sigma}, \]
and hence
\[ (V^T_L)_\rho = \mathbb{C}1 \otimes T \quad \text{with} \quad \dim (V^T_L)_\rho = |N/R|^{1/2}. \] (2.2.7)

Denote by
\[ Y_{\sigma}(\cdot, z) : V_L \to (\text{End}_\mathbb{C} V^T_L)[[z^{1/p}, z^{-1/p}]], \quad a \mapsto Y_{\sigma}(a, z) = \sum_{n \in (1/p)\mathbb{Z}} a_n z^{-n-1} \]
the \( \sigma \)-twisted vertex operator for \( V^T_L \). For latter use, let us recall the definition of \( Y_{\sigma}(a, z) \) for some special \( a \in V_L \). First, the Lie algebra \( \hat{\mathfrak{h}}[\sigma] \) acts on \( V^T_L = M(1)[\sigma] \otimes U_T \) as follows: \( \mathfrak{k} \) acts as the identity, and for \( n \in (1/p)\mathbb{Z} \) and \( h \in \mathfrak{h}_{(pn)} \),
\[ h(n)(u \otimes (g \cdot t)) = \begin{cases} \langle h, \bar{g}_{(0)} \rangle(u \otimes (g \cdot t)) & \text{if } n = 0, \\ (h(n)u) \otimes (g \cdot t) & \text{if } n \neq 0 \end{cases} \] (2.2.8)
for \( u \in M(1)[\sigma] \) and \( g \in \widehat{L_\sigma}, t \in T \). For \( \alpha \in L \), define \( z^{\alpha(0)} \in \text{Hom}_\mathbb{C}(V^T_L, V^T_L[z^{1/p}, z^{-1/p}]) \) by
\[ z^{\alpha(0)}(u \otimes (g \cdot t)) := z^{(\alpha(0), \bar{g}(0))}(u \otimes (g \cdot t)) \quad \text{for } u \in M(1)[\sigma] \text{ and } g \in \widehat{L_\sigma}, t \in T. \]
In addition, the group \( \widehat{L_\sigma} \) acts on \( V^T_L \) as follows:
\[ g \cdot (u \otimes w) = u \otimes (g \cdot w) \quad \text{for } x \in \widehat{L_\sigma} \text{ and } u \in M(1)[\sigma], w \in U_T. \]
Now, we deduce from [DL] (4.40) and (4.45) that
\[ Y_{\sigma}(h(-1)1 \otimes e^0, z) = \sum_{n \in (1/p)\mathbb{Z}} h_{(pn)}(n)z^{-n-1}; \] (2.2.9)
observe that \( \Delta_z(h(-1)1 \otimes e^0) = 0 \), where \( \Delta_z \) is defined as [DL] (4.42)]. Also, we know from [DL] (4.34) and (4.39)] that for \( \alpha \in L \), the vertex operator \( Y_{\sigma}(1 \otimes e^\alpha, z) \) is equal, up to a specified constant multiple (which depends only on \( \alpha \)), to
\[ E^-_{\sigma}(-\alpha, z)E^+_{\sigma}(-\alpha, z) (\kappa^\alpha, e^\alpha) z^{\alpha(0) + \{\alpha(0), \bar{g}(0)\} - \langle \alpha, \alpha \rangle}/2, \] (2.2.10)
where
\[ E^\pm_{\sigma}(-\alpha, z) := \exp \left( \sum_{n \in (1/p)\mathbb{Z} \geq 0} \frac{-\alpha_{(pn)}(n)}{n} z^{-n} \right). \]
Recall that for every \( a \in (V_L)_1 \), the 0-th operator \( a_0 \in \text{End}_\mathbb{C}(V^T_L) \) (i.e., the coefficient of \( z^{-1} \) in \( Y_{\sigma}(a, z) \)) is weight-preserving. In particular, the top weight space \( (V^T_L)_\rho \) is stable under the action of \( a_0 \).
Lemma 2.2.2. (1) For every \( h \in \mathfrak{h} \), the 0-th operator \((h(-1)1 \otimes e^0)_0 \in \text{End}_{\mathbb{C}}(V_L^T)\) of \( h(-1)1 \otimes e^0 \in (V_L)_1 \) acts on the top weight space \((V_L^T)_\rho\) trivially.

(2) If \( \alpha \in \Delta \setminus N \), then the 0-th operator \((1 \otimes e^\alpha)_0 \in \text{End}_{\mathbb{C}}(V_L^T)\) of \( 1 \otimes e^\alpha \in (V_L)_1 \) acts on the top weight space \((V_L^T)_\rho\) trivially.

(3) If \( \alpha \in \Delta \cap N \), then the action of the 0-th operator \((1 \otimes e^\alpha)_0 \in \text{End}_{\mathbb{C}}(V_L^T)\) of \( 1 \otimes e^\alpha \in (V_L)_1 \) on \((V_L^T)_\rho\) is semisimple.

Proof. (1) By (2.2.9), we have \((h(-1)1 \otimes e^0)_0 = h_{(0)}(0)\). Because \((V_L^T)_\rho = \mathbb{C}1 \otimes T\) by (2.2.7), it follows immediately from (2.2.8) that \((h(-1)1 \otimes e^0)_0 = h_{(0)}(0)\) acts on the top weight space \((V_L^T)_\rho\) trivially.

(2), (3) Let \( \alpha \in \Delta \), and let \( 1 \otimes t \in (V_L^T)_\rho = \mathbb{C}1 \otimes T\). Set \( v := (\kappa^0, e_\alpha) \cdot t \in U_T\), and \( d := \langle \alpha_{(0)}, \alpha_{(0)} \rangle/2 \in (1/p)\mathbb{Z}\); remark that \( d = 0 \) if and only if \( \alpha \in N \). By (2.2.10), \((1 \otimes e^\alpha)_0(1 \otimes t)\) is a scalar multiple of the coefficient of \( z^{-1}\) in

\[
E^-_\sigma(-\alpha, z)E^+_\sigma(-\alpha, z)(\kappa^0, e_\alpha)z^{\alpha_{(0)} + \langle \alpha_{(0)}, \alpha_{(0)} \rangle/2 - \langle \alpha, \alpha \rangle/2}(1 \otimes t) = E^-_\sigma(-\alpha, z)E^+_\sigma(-\alpha, z)(1 \otimes v)z^{-1+d} = E^-_\sigma(-\alpha, z)(1 \otimes v)z^{-1+d} \]

since \( \alpha_{(p\alpha)}(n)1 = 0 \) for all \( n \in (1/p)\mathbb{Z}_{>0} \).

\[
(1 \otimes v)z^{-1+d} + \text{(higher terms)}.
\]

If \( \alpha \notin N \), then the coefficient of \( z^{-1}\) in \( Y_\sigma(1 \otimes e^\alpha, z)(1 \otimes t)\) is equal to 0 (since \( d > 0 \)), which implies that \((1 \otimes e^\alpha)_0(1 \otimes t) = 0\). Thus we have proved part (2). Assume that \( \alpha \in N \). Then, \((\kappa^0, e_\alpha) \in \widehat{N}_\sigma\), and \((1 \otimes e^\alpha)_0\) sends \( 1 \otimes t \in (V_L^T)_\rho = \mathbb{C}1 \otimes T\) to a scalar multiple of \( 1 \otimes v = 1 \otimes (\langle \kappa^0, e_\alpha \rangle \cdot t)\). Therefore, by Remark 2.2.1, \((1 \otimes e^\alpha)_0\) is semisimple on \((V_L^T)_\rho\). Thus we have proved part (3). \( \square \)

The following lemma may be known, but we give a proof for completion.

Lemma 2.2.3. The \( \sigma\)-twisted \( V_L\)-module \( V_L^T = M(1)[\sigma] \otimes U_T\) is irreducible.

Proof. Let \( W \subset V_L^T\) be a nonzero \( \sigma\)-twisted \( V_L\)-submodule. First, we show that \( W\) contains a nonzero element of the form: \( u \otimes (g \cdot t)\) for some \( u \in M(1)[\sigma]\) and \( g \in \widehat{\mathcal{L}}_\sigma\), \( t \in T\). Take a complete set \( \{ g_i \mid i \in I \} \subset \widehat{\mathcal{L}}_\sigma\) of representatives for cosets in \( \widehat{\mathcal{L}}_\sigma/\widehat{N}_\sigma\). Then,

\[
U_T = \bigoplus_{i \in I} g_i \cdot T.
\]

Let \( w \in W\), \( w \neq 0\). There exists a finite subset \( J \) of \( I \) such that \( w = \sum_{i \in J} u_i \otimes (g_i \cdot t_i)\) with some \( u_i \in M(1)[\sigma]\) and \( t_i \in T\) for \( i \in J\). For each \( h \in \mathfrak{h}_{(0)}\), we have

\[
h(0)^k w = \sum_{i \in J} \langle h, g_{(0)} \rangle^k (u_i \otimes (g_i \cdot t_i)) \quad \text{for } 0 \leq k \leq |J| - 1. \tag{2.2.11}
\]
Because \( \overline{g_i(0)} \), \( i \in I \), are all distinct (notice that \( \overline{g_i(0)} = \overline{g_j(0)} \iff g_i - g_j \in N \iff g_i g_j^{-1} \in \overline{N}_\sigma \)), we can take \( h \in h(0) \) in such a way that \( \langle h, \overline{g_i(0)} \rangle \), \( i \in J \), are all distinct. Then the coefficient matrix of equation system (2.2.11) (which is a Vanderspande type matrix) is invertible. Therefore, for each \( i \), the coefficient matrix of equation system (2.2.11) (which is a Vanderspande type matrix) is invertible. Therefore, for each \( i \in J \), \( u_i \otimes (g_i \cdot t_i) \) can be written as a linear combination of \( h(0)^k w \) for \( 0 \leq k \leq |J| - 1 \). Since \( h(0) = (h(-1) 1 \otimes e^0)_0 \) by (2.2.11), and since \( W \) is a \( \sigma \)-twisted \( \mathcal{V}_L \)-submodule by assumption, we get \( u_i \otimes (g_i \cdot t_i) \in W \) for every \( i \in J \).

Next, let us show that \( W \) includes the top weight space \( (V_L^T)^\rho = \mathbb{C}1 \otimes U_T \). Take \( g \in \hat{\mathcal{L}}_\sigma \) and \( t \in T \) such that

\[
W \cap (M(1)[\sigma] \otimes (g \cdot t)) \neq \{0\}.
\]

By virtue of (2.2.8) and (2.2.11), both of \( W \) and \( M(1)[\sigma] \otimes (g \cdot t) \) are \( \hat{\mathfrak{h}}[\sigma] \)-modules. Furthermore, we deduce, by standard argument (as for the Fock space over the Heisenberg algebra) that \( M(1)[\sigma] \otimes (g \cdot t) \) is an irreducible \( \hat{\mathfrak{h}}[\sigma] \)-module. Thus, \( W \cap (M(1)[\sigma] \otimes (g \cdot t)) = M(1)[\sigma] \otimes (g \cdot t) \), which implies that

\[
W \supset M(1)[\sigma] \otimes (g \cdot t).
\]

In particular, \( W \supset \mathbb{C}1 \otimes (g \cdot t) \). Now, for \( g' = (\kappa_\alpha, e_\alpha) \in \hat{\mathcal{L}}_\sigma \), we deduce, as in the proof of part (2), (3) of Lemma 2.2.2, that

\[
Y(1 \otimes e^\alpha, z)(1 \otimes (g \cdot t)) = C(1 \otimes (g' g \cdot t))z^d + \text{(higher terms)}
\]

for some \( C \in \mathbb{C}^\times \) and \( d \in (1/p)\mathbb{Z} \); recall that \( (\kappa, e_0) \in \hat{\mathcal{L}}_\sigma \) acts on \( U_T \) as a scalar multiple by \( \xi \). Hence, \( 1 \otimes (g' g \cdot t) \in W \) for every \( g' \in \hat{\mathcal{L}}_\sigma \), which implies that \( 1 \otimes U_T \subset W \).

Finally, since \( W \) is an \( \hat{\mathfrak{h}}[\sigma] \)-module as mentioned above, it follows immediately that \( M(1)[\sigma] \otimes U_T \subset W \), and hence \( W = M(1)[\sigma] \otimes U_T \). We have thus proved the lemma.

\[ \square \]

### 2.3 Miyamoto’s \( \mathbb{Z}_3 \)-orbifold construction.

Keep the notation in the previous subsections 2.1 and 2.2. Assume, in addition, that

(i) \( L \) is unimodular, and \( \sigma \in \text{Aut}(L) \) is of order 3;

(ii) \( \rho = \frac{1}{18}(\dim h(1) + \dim h(2)) \in (1/3)\mathbb{Z} \) (see [DLM2 §5]).

Since \( L \) is unimodular by assumption (i), the lattice VOA \( V_L \) is holomorphic. Also we know from [DLM2 Theorem 10.3] that for each \( r = 1, 2 \), there exists a unique irreducible \( \sigma^r \)-twisted \( \mathcal{V}_L \)-module, which we denote by \( V_L(\sigma^r) \); by Lemma 2.2.3 these twisted \( \mathcal{V}_L \)-modules can be obtained by the method (due to Dong and Lepowsky) mentioned in 2.2. Recall from (2.2.3) that \( V_L(\sigma) \) and \( V_L(\sigma^2) \) decompose as follows:

\[
V_L(\sigma) = \bigoplus_{n \in (1/3)\mathbb{Z}_{\geq 0}} V_L(\sigma)_{\rho + n}, \quad V_L(\sigma^2) = \bigoplus_{n \in (1/3)\mathbb{Z}_{\geq 0}} V_L(\sigma^2)_{\rho + n}.
\]

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Remark 2.3.1. We see from [DLM1, Lemma 3.7] that the restricted dual

\[ V_L(\sigma)' := \bigoplus_{n \in (1/3)\mathbb{Z}_{\geq 0}} V_L(\sigma)^*_{p+n} \]

of \( V_L(\sigma) \) admits an irreducible \( \sigma^2 \)-twisted \( V_L \)-module structure. By uniqueness, \( V_L(\sigma^2) \) is isomorphic to \( V_L(\sigma)' \).

Recall that \( \rho \in (1/3)\mathbb{Z}_{\geq 0} \) by assumption (ii). Set

\[ V_L(\sigma)_Z = \bigoplus_{n \in \mathbb{Z}} V_L(\sigma)_n, \quad V_L(\sigma^2)_Z = \bigoplus_{n \in \mathbb{Z}} V_L(\sigma^2)_n, \]

and then

\[ \tilde{V}_L^\sigma := V_L^\sigma \oplus V_L(\sigma)_Z \oplus V_L(\sigma^2)_Z, \]

where \( V_L^\sigma \) is the fixed point subVOA of \( V_L \) under \( \sigma \in \text{Aut}(V_L) \).

Theorem 2.3.2 ([M1, §5]). We can give \( \tilde{V}_L^\sigma \) a VOA structure whose central charge is equal to the rank of the lattice \( L \). Furthermore, the VOA \( \tilde{V}_L^\sigma \) is \( C_2 \)-cofinite and holomorphic.

Remark 2.3.3. In fact, the VOA \( \tilde{V}_L^\sigma \) is a \((\mathbb{Z}/3\mathbb{Z})\)-graded simple current extension of \( V_L^\sigma \); for the definition and properties of simple current extensions, see, e.g., [LY1, §2].

Denote by

\[ \tilde{Y}(\cdot, z) : \tilde{V}_L^\sigma \to (\text{End}_C \tilde{V}_L^\sigma)[[z, z^{-1}]], \quad a \mapsto \tilde{Y}(a, z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1} \]

the vertex operator for the VOA \( \tilde{V}_L^\sigma \). Then, for \( a \in V_L^\sigma \),

\[ \tilde{Y}(a, z) = \begin{cases} Y(a, z) & \text{on } V_L^\sigma, \\ Y_0(a, z) & \text{on } V_L(\sigma)_Z, \\ Y_{\sigma^2}(a, z) & \text{on } V_L(\sigma^2)_Z. \end{cases} \] (2.3.1)

2.4 Lie algebra of the weight one space. First, let us recall the following basic fact: Let \( V = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} V_n \) be a (general) VOA with \( a \in V \mapsto Y(a, z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1} \in (\text{End}_C V)[[z, z^{-1}]] \) the vertex operator. If \( \dim V_0 = 1 \), then the weight one space \( V_1 \) of \( V \) carries a Lie algebra structure with Lie bracket given by: \([a, b] = a_0 b \) for \( a, b \in V_1 \).

Remark 2.4.1. Keep the notation and setting in (2.3) We see that the VOA \( \tilde{V}_L^\sigma \) in Theorem 2.3.2 satisfies the condition that \( \dim(\tilde{V}_L^\sigma)_0 = 1 \). The Lie algebra \( (\tilde{V}_L^\sigma)_1 \) of the weight one space decomposes as:

\[ (\tilde{V}_L^\sigma)_1 = (V_L^\sigma)_1 \oplus V(\sigma)_1 \oplus V(\sigma^2)_1. \]

This decomposition makes \( (\tilde{V}_L^\sigma)_1 \) a \((\mathbb{Z}/3\mathbb{Z})\)-graded Lie algebra, where \( (V_L^\sigma)_1 \) (resp., \( V(\sigma)_1 \), \( V(\sigma^2)_1 \)) is the subspace of degree \( \mathfrak{u} \in \mathbb{Z}/3\mathbb{Z} \) (resp., \( \mathfrak{u}, \mathfrak{z} \in \mathbb{Z}/3\mathbb{Z} \)). In particular, \( (V_L^\sigma)_1 \) is a Lie subalgebra of \( (\tilde{V}_L^\sigma)_1 \), and each of \( V_L(\sigma)_1 \) and \( V_L(\sigma^2)_1 \) is a module over \( (V_L^\sigma)_1 \) via the adjoint action (see also (2.3.1)).
The next proposition follows immediately from [DM1, Theorem 3].

**Proposition 2.4.2.** Let \( V = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} V_n \) be a \( C_2 \)-cofinite, holomorphic VOA of central charge 24 satisfying the condition that \( \dim V_0 = 1 \) (such as \( \tilde{V}_L^m \) in Theorem 2.3.2). Then the Lie algebra \( V_1 \) of the weight one space is \( \{0\} \), abelian (of dimension 24), or semisimple.

Keep the setting in Proposition 2.4.2. When the Lie algebra \( V_1 \) of the weight one space is semisimple, we define the level of a simple component of \( V_1 \) as follows (see also, e.g., [DM1, §3] and [DM2, §3]): Let \( s \subset V_1 \) be a simple component of \( V_1 \), and \( (\cdot, \cdot)_s \) the nondegenerate, symmetric, invariant bilinear form on \( s \), normalized so that the squared length of a long root of \( s \) is equal to 2. Then, for \( x, y \in s \) and \( m, n \in \mathbb{Z} \), we have

\[
[x_m, y_n] = (x_0 y)_{m+n} + k_s m \delta_{m+n, 0} (x, y)_s \text{id}_V \quad \text{on } V
\]

for some \( k_s \in \mathbb{Z}_{\geq 1} \) (see [DM2, Theorem 3.1 (c)]); we call \( k_s \) the level of \( s \), and say that \( s \) is of type \( X_{\ell,k_s} \) if \( s \) is the simple Lie algebra of type \( X_{\ell} \). We know from [DM1, (1.1)] that

\[
\frac{h_s^\vee}{k_s} = \frac{\dim V_1 - 24}{24}
\]

(2.4.1)

for every simple component \( s \) of \( V_1 \), where \( h_s^\vee \) is the dual Coxeter number of the simple Lie algebra \( s \).

In the following, we denote by \( g_{X_{\ell}} \) the simple Lie algebra of type \( X_{\ell} \).

**Remark 2.4.3.** A Niemeier lattice is, by definition, a unimodular, positive-definite, even lattice of rank 24; for the list of all Niemeier lattices, see [CS, Chapter 16, Table 16.1]. Let \( L \) be a Niemeier lattice. Then the lattice VOA \( V_L \) is \( C_2 \)-cofinite, holomorphic, and of central charge 24. Let \( Q \) be the root lattice of \( L \), i.e., the sublattice of \( L \) generated by the roots \( \Delta(L) \). If \( Q = \{0\} \), then \( (V_L)_1 \) is an abelian Lie algebra of dimension 24. Assume that \( Q \) is not identical to \( \{0\} \), and decomposes into the direct sum of indecomposable root lattices as follows: \( Q = Q_1 \oplus \cdots \oplus Q_m \). Then, \( (V_L)_1 \cong s_1 \oplus \cdots \oplus s_m \) as Lie algebras, where \( s_k, 1 \leq k \leq m, \) is the simple Lie algebra whose type is same with that of \( Q_k \). In addition, the level of \( s_k \) is equal to 1 for every \( 1 \leq k \leq m \).

**3 Construction of a holomorphic VOA (1).**

**3.1 Root lattice \( A_n \).** Following [CS] Chapter 4, §6.1, we set

\[
A_n := \{(x_0, x_1, \ldots, x_n) \in \mathbb{Z}^{n+1} \mid x_0 + x_1 + \cdots + x_n = 0\},
\]

\[
\Delta(A_n) := \{\alpha \in A_n \mid (\alpha, \alpha) = 2\} \quad \text{(the set of roots in } A_n)\),
\]

\[
[\ell] := \frac{1}{n+1}(\ell, \ldots, \ell, \underbrace{\ell - n - 1, \ldots, \ell - n - 1}_{\ell \text{ times}}) \in A_n^* \quad \text{for } \ell = 0, 1, \ldots, n.
\]
where \( A_n^* \subset A_n \otimes \mathbb{Z} \mathbb{R} \) denotes the dual lattice of \( A_n \). If we set \([\ell] := [\ell] + A_n\) for \( \ell = 0, 1, \ldots, n \), then \( A_n^*/A_n = \{ [\ell] \mid \ell = 0, 1, \ldots, n \} \), and there exists an isomorphism of additive groups from \( A_n^*/A_n \) to \( \mathbb{Z}/(n+1)\mathbb{Z} = \{ [\ell] \mid \ell = 0, 1, \ldots, n \} \) that maps \([\ell]\) to \( \ell \) for \( \ell = 0, 1, \ldots, n \). Furthermore, we define an action of the ring \( \mathbb{Z}/(n+1)\mathbb{Z} \) on \( A_n^*/A_n \) in such a way that the isomorphism \( A_n^*/A_n \cong \mathbb{Z}/(n+1)\mathbb{Z} \) above is also an isomorphism of \( \mathbb{Z}/(n+1)\mathbb{Z} \)-modules. Namely, we set
\[
\ell_1 \cdot [\ell_2] := \ell_1 \ell_2 \mod n + 1 \quad \text{for } \ell_1, \ell_2 = 0, 1, \ldots, n.
\]

3.2 Niemeier lattice \( \text{Ni}(A_2^{12}) \) and its automorphism \( \sigma_1 \) of order 3. In this subsection, we recall the definition of the Niemeier lattice \( \text{Ni}(A_2^{12}) \) with \( A_2^{12} \) the root lattice, and define a lattice automorphism \( \sigma_1 \in \text{Aut}(\text{Ni}(A_2^{12})) \) of order 3.

Define \( Q \) to be the direct sum \( A_2^{12} \) of 12 copies of the root lattice \( A_2 \); following [CS] Chapter 10, §1.5, we use \( \Omega := \{ \infty, 0, 1, \ldots, 10 \} \) for the index set of the coordinate for \( Q \); that is, \( Q = \{ (\alpha_i)_{i \in \Omega} \mid \alpha_i \in A_2 \text{ for } i \in \Omega \} \). Since \( Q^* = (A_2^*)^{12} \), it follows from §3.1 that \( Q^*/Q = \{ ([\ell_i])_{i \in \Omega} \mid \ell_i = 0, 1, 2 \text{ for each } i \in \Omega \} \), which we can identify with the 12-dimensional vector space \( \mathbb{F}_3^{12} \) over the field \( \mathbb{F}_3 := \mathbb{Z}/3\mathbb{Z} \) of three elements. For \( [\ell] \in A_2^*/A_2 \) and \( i \in \Omega \), define \( [\ell]^{(i)} \) to be the element \( ([\ell_i])_{i \in \Omega} \in Q^*/Q \) with \( \ell_i = \ell \) and \( \ell_j = 0 \) for all \( j \in \Omega \), \( j \neq i \). Then, \( \{ [1]^{(i)} \mid i \in \Omega \} \) forms a basis of \( Q^*/Q \), which we identify with the standard orthonormal basis of \( \mathbb{F}_3^{12} \).

Denote by \( \mathfrak{S}(\Omega) \) the symmetric group of \( \Omega \); each element \( g \in \mathfrak{S}(\Omega) \) acts linearly on the vector space \( Q^*/Q \) by: \( g \cdot [\ell]^{(i)} = [\ell]^{(g(i))} \) for \( i \in \Omega \). Define \( \nu \in \mathfrak{S}(\Omega) \) by: \( \nu = (\infty)(X9876543210) \), where \( X \) denotes 10, that is,
\[
\nu(i) = \begin{cases} 
10 & \text{if } i = 0, \\
n & \text{if } 1 \leq i \leq 10, \\
\infty & \text{if } i = \infty.
\end{cases}
\]

Set \( \Theta := \{ 0, 1, 3, 4, 5, 9 \} \subset \Omega \), and define
\[
w_0 := \sum_{i \in \Omega \setminus \Theta} [1]^{(i)} - \sum_{j \in \Theta} [1]^{(j)} = \sum_{i \in \Omega} [1]^{(i)} + 2 \sum_{j \in \Theta} [1]^{(j)},
\]
\[
w_i := \nu^i \cdot w_0 \quad \text{for } 0 \leq i \leq 10, \quad w_\infty := \sum_{i \in \Omega} [1]^{(i)}.
\]

**Theorem 3.2.1** ([CS] Chapter 10, Theorems 2 and 3). (1) Define \( C_{12} \) to be the subspace of \( Q^*/Q \) spanned by \( \{ w_i \mid i \in \Omega \} \). Then, \( C_{12} \) is isomorphic to the ternary Golay code in \( \mathbb{F}_3^{12} \).

(2) The subspace \( C_{12} \) is 6-dimensional with \( \{ w_\infty, w_1, w_3, w_4, w_5, w_9 \} \) a basis.

(3) The subspace \( C_{12} \) is stable under the action of \( \nu \in \mathfrak{S}(\Omega) \).
(4) Define $\delta \in \mathcal{G}(\Omega)$ by: $\delta = (\infty)(0)(1)(2X)(34)(59)(67)(8)$. Then, $C_{12}$ is stable under the action of $\delta \in \mathcal{G}(\Omega)$.

(5) Set $\sigma' := \nu^{-1} \circ \delta$. Then, $\sigma' = (\infty)(4)(7)(012)(35X)(689)$. In particular, the order of $\sigma'$ is equal to 3.

We now set (see [CS, Chapter 18, §4, II])

$$(Q \subset) \quad \text{Ni}(A_2^{12}) := \bigcup_{C \in C_{12}} C \quad (\subset Q^*)$$

From now on, we arrange the coordinate of $Q^* = (A_2^*)^{12}$ as follows:

$$(\mu_i)_{i \in \Omega} = \left( \mu_{\infty}, \mu_4, \mu_7 \mid \mu_0, \mu_3, \mu_6 \mid \mu_1, \mu_5, \mu_8 \mid \mu_2, \mu_{10}, \mu_9 \right) \quad e \in (A_2^*)^3$$

We first define an automorphism $\sigma'$ of $Q^*$ of order 3 by:

$$(\mu_{\infty^{\sigma'}} \mid \mu_{036} \mid \mu_{158} \mid \mu_{2X9}) \quad \longrightarrow \quad \left( \mu_{\infty^{\sigma'}} \mid \mu_{2X9} \mid \mu_{036} \mid \mu_{158} \right)$$

for $\mu_{\infty^{\sigma'}}$, $\mu_{036}$, $\mu_{158}$, $\mu_{2X9} \in (A_2^*)^3$. Then we deduce from Theorem 3.2.1 (5) that $\sigma'$ stabilizes the Niemeier lattice $\text{Ni}(A_2^{12})$, and hence $\sigma' \in \text{Aut}(\text{Ni}(A_2^{12}))$. Next, let $\varphi = \varphi_{A_2}$ denote the lattice automorphism of $A_2$ defined by: $(x_0, x_1, x_2) \mapsto (x_2, x_0, x_1)$; note that $\varphi([\ell]) = [\ell]$ for every $\ell = 0, 1, 2$. So, if we define an automorphism $\sigma''$ of $Q^*$ (of order 3) by:

$$(\mu_{\infty}, \mu_4, \mu_7 \mid \mu_{036} \mid \mu_{158} \mid \mu_{2X9}) \quad \longrightarrow \quad (\varphi(\mu_{\infty}), \varphi(\mu_4), \varphi(\mu_7) \mid \mu_{036} \mid \mu_{158} \mid \mu_{2X9})$$

for $\mu_{\infty}$, $\mu_4$, $\mu_7 \in A_2^*$ and $\mu_{036}$, $\mu_{158}$, $\mu_{2X9} \in (A_2^*)^3$, then $\sigma''$ stabilizes the Niemeier lattice $\text{Ni}(A_2^{12})$, and hence $\sigma'' \in \text{Aut}(\text{Ni}(A_2^{12}))$. Set

$$\sigma_1 := \sigma' \circ \sigma''$$

it is obvious that $\sigma_1$ is of order 3. The fixed point subspace $\mathfrak{h}(0)$ of $\mathfrak{h} = \text{Ni}(A_2^{12}) \otimes \mathbb{C}$ under the $\sigma_1$ above is identical to

$$\{ (0 \mid h \mid h \mid h) \mid h \in A_2^3 \otimes \mathbb{C} = (A_2 \otimes \mathbb{C})^3 \},$$

where 0 denotes the zero vector in $A_2^3 \otimes \mathbb{C} = (A_2 \otimes \mathbb{C})^3$. Thus, $\dim \mathfrak{h}(0) = 2 \times 3 = 6$, and hence

$$\rho = \frac{1}{18}(\dim \mathfrak{h}(1) + \dim \mathfrak{h}(2)) = \frac{1}{18}(24 - \dim \mathfrak{h}(0)) = 1. \quad (3.2.1)$$

Also, remark that

$$\mathfrak{h}(1) \oplus \mathfrak{h}(2) = N \otimes_\mathbb{Z} \mathbb{C} = \{ (h \mid h_1 \mid h_2 \mid -h_1 - h_2) \mid h, h_1, h_2 \in A_2^3 \otimes \mathbb{C} \}; \quad (3.2.2)$$

for the definition of $N$, see (2.2.2).

By (3.2.1), we can apply Theorem 2.3.2 to $L = \text{Ni}(A_2^{12})$ and the $\sigma_1$ above, and obtain a $C_2$-cofinite, holomorphic VOA $\tilde{V}_L^{\sigma_1}$ of central charge 24. We are ready to state the main result in this section.
Theorem 3.2.2. Keep the notation and setting above. For \( L = \text{Ni}(A_2^{12}) \) and the \( \sigma_1 \) above, the Lie algebra \((V_L^{\sigma_1})_1\) is of type \(A_{2,3}^0\). Therefore, \(V_L^{\sigma_1}\) corresponds to No. 6 on Schellekens’ list [8, Table 1].

3.3 Proof of Theorem [3.2.2]. First, let us determine the Lie algebra structure of \((V_L^{\sigma_1})_1\). For \( h \in A_2 \otimes \mathbb{C} \), define \( h(i) \) to be \((h_i)_{i \in \Omega} \in \mathfrak{h} = L \otimes \mathbb{C} \) with \( h_i = h \) and \( h_j = 0 \) for all \( j \in \Omega \), \( j \neq i \), and set

\[
\begin{align*}
  h^{(012)} := h^{(0)} + h^{(1)} + h^{(2)} &= (0, 0, 0 | h, 0, 0 | h, 0, 0 | h, 0, 0), \\
  h^{(35X)} := h^{(3)} + h^{(5)} + h^{(10)} &= (0, 0, 0 | h, 0, 0 | h, 0, 0 | h, 0, 0), \\
  h^{(689)} := h^{(6)} + h^{(8)} + h^{(9)} &= (0, 0, 0 | h, 0, 0 | h, 0, 0 | h, 0, 0);
\end{align*}
\]

note that \( h^{(012)} = h^{(0)} + \sigma_1 h^{(0)} + \sigma_1^2 h^{(0)} \), and so on. Observe that the set \( \Delta = \Delta(L) \) of roots in \( L \) is identical to \( \{ \alpha(i) | \alpha \in \Delta(A_2), i \in \Omega \} \). We see that \( \Delta \) is stable under \( \sigma_1 \in \text{Aut}(L) \), and the action of \( \sigma_1 \) on \( \Delta \) is fixed point free. Define \( \mathfrak{g}^{(012)} \) to be the subspace spanned by \( \{ h^{(012)}(-1) \otimes e^{\alpha} | h \in A_2 \otimes \mathbb{C} \} \) and \( \{ 1 \otimes e^{\alpha(0)} + 1 \otimes e^{\alpha(1)} + 1 \otimes e^{\alpha(2)} | \alpha \in \Delta(A_2) \} \), and define \( \mathfrak{g}^{(35X)} \) and \( \mathfrak{g}^{(689)} \) in exactly the same way as \( \mathfrak{g}^{(012)} \) with 0, 1, 2 replaced by 3, 5, 10 and 6, 8, 9, respectively. Also, define

\[
\mathfrak{a} := \text{Span}_{\mathbb{C}} \{ 1 \otimes e^{\alpha(i)} + 1 \otimes e^{\sigma(\alpha)(i)} + 1 \otimes e^{\sigma^2(\alpha)(i)} | \alpha \in \Delta(A_2), i = \infty, 4, 7 \};
\]

note that \( \dim \mathfrak{a} = 2 \times 3 = 6 \) since \( |\Delta(A_2)/\varphi| = 2 \).

Lemma 3.3.1. As a vector space,

\[
(V_L^{\sigma_1})_1 = \mathfrak{g}^{(012)} \oplus \mathfrak{g}^{(35X)} \oplus \mathfrak{g}^{(689)} \oplus \mathfrak{a},
\]

and hence \( \dim(V_L^{\sigma_1})_1 = 8 \times 3 + 6 = 30 \). Furthermore, \( \mathfrak{g}^{(012)} \), \( \mathfrak{g}^{(35X)} \), and \( \mathfrak{g}^{(689)} \) are ideals of the (whole) Lie algebra \((V_L^{\sigma_1})_1\) isomorphic to the simple Lie algebra of type \(A_2\). In addition, \( \mathfrak{a} \) is an abelian Lie subalgebra, and the (adjoint) actions of an element of \( \mathfrak{a} \) on \( V_L(\sigma_1)_1 \) and \( V_L(\sigma_2)_1 \) are semisimple.

Proof. We can easily check (3.3.1). Let us show the assertion for \( \mathfrak{g}^{(012)} \); we can show the assertions for \( \mathfrak{g}^{(35X)} \) and \( \mathfrak{g}^{(689)} \) similarly. It follows immediately from the definition of the vertex operator for \( V_L \) (see (2.1.1) and (2.1.2)) that \( \mathfrak{g}^{(012)} \) is a Lie subalgebra isomorphic to the simple Lie algebra of type \(A_2\), and

\[
[\mathfrak{g}^{(012)}, \mathfrak{g}^{(35X)}] = [\mathfrak{g}^{(012)}, \mathfrak{g}^{(689)}] = [\mathfrak{g}^{(012)}, \mathfrak{a}] = \{0\}.
\]

Let us show that \([a, b] = a_0 b = 0\) for all \( a \in \mathfrak{g}^{(012)} \) and \( b \in V_L(\sigma_1)_1 \). Assume that \( a = h^{(012)}(-1) \otimes e^0 \) for some \( h \in A_2 \otimes \mathbb{C} \). Since \( V_L(\sigma_1)_1 \) is the top weight space of \( V_L(\sigma_1) \) by
it follows immediately from Lemma 2.2.2(1), along with (2.3.1), that $a_0b = 0$ for all $b \in V_L(\sigma_1)$. Assume that $a = 1 \otimes e^{\alpha(0)} + 1 \otimes e^{\alpha(1)} + 1 \otimes e^{\alpha(2)}$ for some $\alpha \in \Delta(A_2)$. Notice that $\alpha(i), i = 0, 1, 2,$ is not contained in $N$ by (3.2.2). Thus we see from Lemma 2.2.2(2), along with (2.3.1), that $a_0b = 0$ for all $b \in V_L(\sigma_1)$. Similarly, we can show that $[a, b] = a_0b = 0$ for all $a \in g^{(012)}$ and $b \in V_L(\sigma_1^2)$. Thus we have proved that $g^{(012)}$ is an ideal of $(\tilde{V}_L^\sigma)_1$.

Now, it can be easily checked by the definition of the vertex operator for $V_L$ (see (2.1.2)) that $a$ is an abelian Lie subalgebra. Let (and fix) $\alpha \in \Delta(A_2)$, and $i = \infty, 4, 7$. Note that $\varphi^r(\alpha(i)) \in N$ by (3.2.2). Hence we see from Lemma 2.2.2(3) that $(1 \otimes e^{\varphi^r(\alpha(i))})_0$ is semisimple on $V(\sigma_1)$ for each $r = 0, 1, 2$. Since $\langle \varphi^r(\alpha), \varphi^{r+1}(\alpha) \rangle = -1$ for every $r = 0, 1, 2$, it follows immediately from the definition (2.2.1) of the commutator map $c_0^i$ for $\hat{L}_{\sigma_1}$ that $c_0^i(\varphi^r(\alpha(i)), \varphi^{r+1}(\alpha(i))) = 0$ for every $r = 0, 1, 2$. Thus, $(\kappa^0, e_{\varphi^r(\alpha(i))}) \in \hat{L}_{\sigma_1}, r = 0, 1, 2$, commute with each other, and hence so are $(1 \otimes e^{\varphi^r(\alpha(i))})_0 \in \text{End}_C(V(\sigma_1)_1), r = 0, 1, 2$, because $(1 \otimes e^{\varphi^r(\alpha(i))})_0$ is identical to a scalar multiple of the action of $(\kappa^0, e_{\varphi^r(\alpha(i))})$ on $V(\sigma_1)_1$ (see the proof of Lemma 2.2.2(3)). Therefore we conclude that $(1 \otimes e^{\varphi^r(\alpha(i))})_0 + (1 \otimes e^{\varphi^r(\alpha(i))})_0 + (1 \otimes e^{\varphi^r(\alpha(i))})_0$ is also semisimple on $V(\sigma_1)_1$. Similarly, we can show that $(1 \otimes e_{\varphi^r(\alpha(i))})_0 + (1 \otimes e^{\varphi^2(\alpha(i))})_0 + (1 \otimes e^{\varphi^2(\alpha(i))})_0$ is semisimple also on $V(\sigma_2^3)_1$. This completes the proof of the lemma.

We see from Proposition 2.4.2 and Lemma 3.3.1 that the Lie algebra $(\tilde{V}_L^\sigma)_1$ is semisimple. Also, we deduce from the definition that the levels of the simple components $g^{(012)}, g^{(35, X)}$, and $g^{(689)}$ of $(\tilde{V}_L^\sigma)_1$ are all equal to 3. Since the dual Coxeter number of $A_2$ is equal to 3, it follows immediately from (2.3.1) that

$$\frac{3}{3} = \frac{\dim(\tilde{V}_L^\sigma)_1 - 24}{24},$$

and hence $\dim(\tilde{V}_L^\sigma)_1 = 48$.

Therefore we have $\dim V(\sigma_1)_1 = \dim V(\sigma_2^3)_1 = 9$ (see Remark 2.3.1).

Remark 3.3.2. We see from (2.2.7) that $\dim V(\sigma_1)_1 = |N/R|^{1/2}$. Also it can be shown that $R = M = (1 - \sigma_1)L$. Using these facts, we can determine the dimension of $V(\sigma_1)_1$ also by lattice theoretic method.

Set

$$g := a \oplus V(\sigma_1)_1 \oplus V(\sigma_2^3)_1 \subset (\tilde{V}_L^\sigma)_1.$$

By Remark 2.4.1 and the argument above, we see that $g$ is an ideal of the semisimple Lie algebra $(\tilde{V}_L^\sigma)_1$ satisfying the following conditions:

(i) $g$ is a $(\mathbb{Z}/3\mathbb{Z})$-graded, semisimple Lie algebra of dimension 24;

(ii) $a$ is an abelian subalgebra of dimension 6 whose (adjoint) actions on $g_1$ and $g_2$ are both semisimple.
Proposition 3.3.3. The Lie algebra \( \mathfrak{g} \) above is isomorphic to the semisimple Lie algebra of type \( A_2^3 \).

Proof. We see by (ii) and [K, Lemma 8.1 b)] that the centralizer \( \mathfrak{z} = \{ x \in \mathfrak{g} \mid [x, h] = 0 \text{ for all } h \in \mathfrak{a} \} \) of \( \mathfrak{a} \) is a Cartan subalgebra of \( \mathfrak{g} \). Notice that

\[
\mathfrak{z} = \mathfrak{a} \oplus (\mathfrak{z} \cap \mathfrak{g}_1) \oplus (\mathfrak{z} \cap \mathfrak{g}_2).
\]

Suppose that \( \mathfrak{z} \cap \mathfrak{g}_1 \neq \{0\} \). Let \( h \in \mathfrak{z} \cap \mathfrak{g}_1 \), and let \( \alpha \) be a root of \( \mathfrak{g} \) such that \( \alpha(h) \neq 0 \). Take a nonzero root vector \( x \in \mathfrak{g} \) with respect to \( \alpha \), and write it as: \( x = x_0 + x_1 + x_2 \) with \( x_0 \in \mathfrak{a} \) and \( x_i \in \mathfrak{g}_i, i = 1, 2 \); note that \( [h, x_0] = 0 \). Then,

\[
\alpha(h)(x_0 + x_1 + x_2) = \alpha(h)x = [h, x] = [h, x_0 + x_1 + x_2] = [h, x_0] + [h, x_1] + [h, x_2].
\]

Since \( \alpha(h) \neq 0 \), we get \( x_1 = 0 \). Also, since \( 0 = [h, x_1] = \alpha(h)x_2 \), and \( \alpha(h) \neq 0 \), we have \( x_2 = 0 \). Similarly, \( x_0 = 0 \). Thus we obtain \( x = 0 \), which is a contradiction. Hence, \( \mathfrak{z} \cap \mathfrak{g}_1 = \{0\} \). Similarly, we can show that \( \mathfrak{z} \cap \mathfrak{g}_2 = \{0\} \). Therefore we conclude that \( \mathfrak{a} \) is a Cartan subalgebra of \( \mathfrak{g} \).

Since the rank of \( \mathfrak{g} \) is equal to \( \dim \mathfrak{a} = 6 \), we deduce from (i) that \( \mathfrak{g} \) is isomorphic to the semisimple Lie algebra of type \( A_2^3 \) or \( B_2A_2A_1^2 \). Suppose that \( \mathfrak{g} \) is of type \( B_2A_2A_1^2 \). Since \( \mathfrak{g}_1 \) and \( \mathfrak{g}_2 \) are stable under the adjoint action of the Cartan subalgebra \( \mathfrak{a} \), it follows that \( \mathfrak{g}_1 \) and \( \mathfrak{g}_2 \) are direct sums of root spaces of \( \mathfrak{g} \) with respect to the Cartan subalgebra \( \mathfrak{a} \), and each root space of \( \mathfrak{g} \) is contained in either \( \mathfrak{g}_1 \) or \( \mathfrak{g}_2 \). Let \( \mathfrak{s} \) be the ideal of \( \mathfrak{g} \) isomorphic to the simple Lie algebra \( \mathfrak{g}_{B_2} \) of type \( B_2 \), and let \( \alpha_1, \alpha_2 \) be the simple roots for \( \mathfrak{s} \cong \mathfrak{g}_{B_2} \) such that \( \theta := 2\alpha_1 + \alpha_2 \) is the highest root of \( \mathfrak{s} \cong \mathfrak{g}_{B_2} \). We suppose that \( \mathfrak{s}_\theta \subset \mathfrak{g}_1 \) (resp., \( \mathfrak{s}_\theta \subset \mathfrak{g}_2 \)). If the root space \( \mathfrak{s}_{-\alpha_1} \) is contained in \( \mathfrak{g}_2 \) (resp., \( \mathfrak{g}_1 \)), then we have

\[
\{0\} \neq \mathfrak{s}_{\alpha_1 + \alpha_2} = [\mathfrak{s}_{\theta}, \mathfrak{s}_{-\alpha_1}] \subset \mathfrak{a}
\]

by (i), which is a contradiction. If \( \mathfrak{s}_{-\alpha_1} \subset \mathfrak{g}_1 \) (resp., \( \mathfrak{s}_{-\alpha_1} \subset \mathfrak{g}_2 \)), then we have

\[
\{0\} \neq \mathfrak{s}_{\alpha_2} = [[\mathfrak{s}_{\theta}, \mathfrak{s}_{-\alpha_1}], \mathfrak{s}_{-\alpha_1}] \subset \mathfrak{a}
\]

by (i), which is also a contradiction. Thus we have proved that \( \mathfrak{g} \) is not of type \( B_2A_2A_1^2 \), thereby completing the proof of the proposition. \( \square \)

Theorem 3.2.2 follows immediately from Lemma 3.3.1, Proposition 3.3.3 and (2.4.1).

4 Construction of a holomorphic VOA (2).
4.1 Root lattice $D_4$ and its fixed-point-free automorphism of order 3. Following [CS, Chapter 4, §7.1], we set

$$D_4 := \{(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 \mid x_1 + x_2 + x_3 + x_4 \in 2\mathbb{Z}\};$$

$$\Delta(D_4) := \{ \alpha \in D_4 \mid \langle \alpha, \alpha \rangle = 2 \} \quad \text{(the set of roots in } D_4),$$

$$[0] := (0, 0, 0, 0) \in D_4^\ast, \quad [1] := (1/2, 1/2, 1/2, 1/2) \in D_4^\ast,$n

$$[2] := (0, 0, 0, 1) \in D_4^\ast, \quad [3] := (1/2, 1/2, 1/2, -1/2) \in D_4^\ast,$n

where $D_4^\ast \subset D_4 \otimes \mathbb{R}$ denotes the dual lattice of $D_4$. If we set $[\ell] := [\ell] + D_4$ for $\ell = 0, 1, 2, 3$, then we have $D_4^\ast/D_4 = \{ [\ell] \mid \ell = 0, 1, 2, 3 \}$, and the additive group $D_4^\ast/D_4$ is naturally isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$; in fact, we have $[\ell] + [\ell] = [0]$ for all $\ell = 0, 1, 2, 3$, and $[1] + [2] = [3]$.

We define a linear automorphism $\varphi = \varphi_{D_4}$ of $D_4 \otimes \mathbb{R}$ by:

$$(1, 0, 0, 0) \mapsto \frac{1}{2}(-1, 1, 1, 1), \quad (0, 1, 0, 0) \mapsto \frac{1}{2}(-1, -1, 1, -1),$$

$$(0, 0, 1, 0) \mapsto \frac{1}{2}(-1, -1, -1, 1), \quad (0, 0, 0, 1) \mapsto \frac{1}{2}(-1, 1, -1, -1);$$

we deduce that the restriction of $\varphi$ to $D_4$ (resp., $D_4^\ast$) is a lattice automorphism of order 3 of $D_4$ (resp., $D_4^\ast$), which is fixed point free on $D_4$ (resp., $D_4^\ast$). Remark that

$$\varphi([0]) = [0], \quad \varphi([1]) = [2], \quad \varphi([2]) = [3], \quad \varphi([3]) = [1]. \quad (4.1.1)$$

4.2 Niemeier lattice $\text{Ni}(D_4^6)$ and its automorphism $\sigma_2$ of order 3. The Niemeier lattice $L = \text{Ni}(D_4^6)$ with $Q = D_4^6$ the root lattice is, by definition (see [CS, Chapter 16, Table 16.1]), the sublattice of $Q^\ast = (D_4^\ast)^6$ generated by $Q$, $[111111]$, $[222222]$, and

$$0(023322) = \{(0023322), [023320], [033202], [032023], [020233]\},$$

where $[a_1 \cdots a_6] := ([a_1], \ldots, [a_6]) \in Q^\ast = (D_4^\ast)^6$.

Let us define $\sigma_2 : Q^\ast \to Q^\ast$ by: $\sigma_2 = \varphi^{\otimes 6}$, that is, $\sigma_2(\gamma_1, \ldots, \gamma_6) = (\varphi(\gamma_1), \ldots, \varphi(\gamma_6))$. Then, $L \subset Q^\ast$ is stable under the action of $\sigma_2$. Indeed, it is obvious that $\sigma_2(Q) \subset Q$. Also, $\sigma_2([111111]) \in [222222] + Q \subset L$, and $\sigma_2([222222]) \in [333333] + Q = [111111] + [222222] + Q \subset L$. In addition,

$$\sigma_2([0023322]) \in [003113] + Q = [020233] + [023320] + Q \subset L.$$

1There seems to be a typo in the row of $D_4^6$ on [CS, Chapter 16, Table 16.1]; $[222222]$ should be added to the column “Generators for glue code”.

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Similarly, we can show that \( \sigma_2([0(02332)]) \subset L \). Therefore, \( \sigma_2 \) is a lattice automorphism of \( L \) of order 3. Because \( \varphi \) is fixed point free on \( D_4^* \), it follows that \( \sigma_2 \) is fixed point free on \( L \), which implies that rank \( L^\sigma_2 = 0 \), and hence

\[
\rho = \frac{1}{18}(\dim \mathfrak{h}(1) + \dim \mathfrak{h}(2)) = \frac{1}{18}(24 - 0) = \frac{4}{3}, \tag{4.2.1}
\]

Thus we can apply Theorem 2.3.2 to \( L = \text{Ni}(D_4^4) \) and the \( \sigma_2 \) above, and obtain a \( C_2 \)-cofinite, holomorphic VOA \( \tilde{V}_L^\sigma_2 \) of central charge 24; note that \( \dim(\tilde{V}_L^\sigma_2)_0 = \dim(V_L^\sigma_2)_0 = 1 \). We are ready to state the main result in this section.

**Theorem 4.2.1.** Keep the notation and setting above. For \( L = \text{Ni}(D_4^4) \) and the \( \sigma_2 \) above, the Lie algebra \( (\tilde{V}_L^\sigma_2)_1 \) is of type \( A_{2,3}^6 \). Therefore, \( \tilde{V}_L^\sigma_2 \) corresponds to No. 6 on Schellekens’ list \([5, Table 1]\).

**Proof.** By (4.2.1) and (2.2.5), we see that both of the top weights of \( V(\sigma_2) \) and \( V(\sigma_2^2) \) are equal to 4/3. Hence, \( (\tilde{V}_L^\sigma_2)_1 = (V_L^\sigma_2)_1 \). So, let us determine the Lie algebra structure of \( (V_L^\sigma_2)_1 \). For each \( h \in D_4 \otimes \mathbb{Z} \mathbb{C} \) and \( 1 \leq i \leq 6 \), we define \( h^{(i)} \) to be the element \( (h_1, \ldots, h_6) \in \mathfrak{h} = L \otimes \mathbb{Z} \mathbb{C} = (D_4 \otimes \mathbb{Z} \mathbb{C})^6 \) with \( h_i = h \) and \( h_j = 0 \) for all \( 1 \leq j \leq 6, j \neq i \). Define \( \mathfrak{g}^{(i)} \) to be the Lie subalgebra of \( (V_L)_1 \) generated by \( \{h^{(i)}(-1) \otimes e^0 \mid h \in D_4 \otimes \mathbb{Z} \mathbb{C}\} \) and \( \{1 \otimes e^{\alpha^{(i)}} \mid \alpha \in \Delta(D_4)\} \). Then we have

\[
(V_L)_1 \cong \mathfrak{g}^{(1)} \oplus \cdots \oplus \mathfrak{g}^{(6)} \quad \text{with} \quad \mathfrak{g}^{(i)} \cong \mathfrak{g}_{D_4} \quad \text{for each} \ 1 \leq i \leq 6.
\]

It can be easily seen that \( \sigma_2 \in \text{Aut}(V_L) \) preserves each simple component \( \mathfrak{g}^{(i)} \cong \mathfrak{g}_{D_4} \) of \( (V_L)_1 \); indeed, for each \( 1 \leq i \leq 6 \), we see that

\[
\begin{align*}
\sigma_2(h^{(i)}(-1) \otimes e^0) &= (\varphi(h))^{(i)}(-1) \otimes e^0 \quad \text{for} \ h \in D_4 \otimes \mathbb{Z} \mathbb{C}; \\
\sigma_2(1 \otimes e^{\alpha^{(i)}}) &= 1 \otimes e^{\varphi(\alpha^{(i)})} \quad \text{for} \ \alpha \in \Delta(D_4).
\end{align*}
\]

Thus the restriction of \( \sigma_2 \) to a simple component \( \mathfrak{g}^{(i)} \cong \mathfrak{g}_{D_4} \) is identical to the Lie algebra automorphism, denoted also by \( \varphi \), of \( \mathfrak{g}_{D_4} \) induced from \( \varphi \in \text{Aut}(D_4) \). Therefore,

\[
(\tilde{V}_L^\sigma_2)_1 = (V_L^\sigma_2)_1 = (\mathfrak{g}^{(1)})^{\sigma_2} \oplus \cdots \oplus (\mathfrak{g}^{(6)})^{\sigma_2} \cong (\mathfrak{g}_{D_4}^{\varphi})^{\oplus 6}.
\]

Because \( \varphi \) is fixed point free on the root lattice \( D_4 \), we see that \( \dim \mathfrak{g}_{D_4}^{\varphi} = 0 + 24/3 = 8 \) (recall that the number of roots of \( D_4 \) is equal to 24). Thus we have \( \dim(\tilde{V}_L^\sigma_2)_1 = 8 \times 6 = 48 \). Therefore it follows from Proposition 2.4.2 that \( (\tilde{V}_L^\sigma_2)_1 \cong (\mathfrak{g}_{D_4}^{\varphi})^{\oplus 6} \) is semisimple, and hence so is \( \mathfrak{g}_{D_4}^{\varphi} \). Observe that \( \mathfrak{g}_{A_2} \) is a unique semisimple Lie algebra of dimension 8. Thus, \( \mathfrak{g}_{D_4}^{\varphi} \) is of type \( A_2 \), and hence \( (\tilde{V}_L^\sigma_2)_1 \) is of type \( A_2^6 \). Since \( \dim(\tilde{V}_L^\sigma_2)_1 = 48 \) as seen above, we see from (2.4.1) that \( (\tilde{V}_L^\sigma_2)_1 \) is of type \( A^6_{2,3} \), as desired. \( \square \)
5 Construction of a holomorphic VOA (3).

5.1 Dynkin diagram automorphism for $D_4$. We use the notation and setting for the root lattice $D_4$ introduced in §4.1. Let $\omega$ be a Dynkin diagram automorphism of $D_4$ of order 3. Then, $\omega$ acts on $D_4$ as follows:

$$\omega(1, -1, 0, 0) = (0, 0, 1, -1), \quad \omega(0, 0, 1, -1) = (0, 0, 1, 1),$$
$$\omega(0, 0, 1, 1) = (1, -1, 0, 0), \quad \omega(0, 1, -1, 0) = (0, 1, -1, 0);$$

we can easily check that

$$\omega([0]) = [0], \quad \omega([1]) = [2], \quad \omega([2]) = [3], \quad \omega([3]) = [1].$$

5.2 Niemeier lattice $\text{Ni}(D_4^6)$ and its automorphism $\sigma_3$ of order 3. Keep the notation and setting for the Niemeier lattice $L = \text{Ni}(D_4^6)$ with $Q = D_4^6$ the root lattice in §4.2. Let us define $\sigma_3 : Q^* \rightarrow Q^*$ by:

$$\sigma_3(\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6) = (\varphi(\gamma_1), \varphi(\gamma_2), \varphi(\gamma_3), \omega(\gamma_4), \omega(\gamma_5), \omega(\gamma_6)).$$

We can show in exactly the same way as for $\sigma_2$ in §4.2 that $L \subset Q^*$ is stable under the action of $\sigma_3$. Therefore, $\sigma_3$ gives a lattice automorphism of $L$ of order 3; since rank $D_4^6 = 0$ and rank $D_4^6 = 2$, it follows that rank $L^{\sigma_3} = 2 \times 3 = 6$, and hence

$$\rho = \frac{1}{18}(\dim h_{(1)} + \dim h_{(2)}) = \frac{1}{18}(24 - 6) = 1.$$

(5.2.1)

Thus we can apply Theorem 2.3.2 to $L = \text{Ni}(D_4^6)$ and the $\sigma_3$ above, and obtain a $C_2$-cofinite, holomorphic VOA $\tilde{V}_L^{\sigma_3}$ of central charge 24; note that $\dim(\tilde{V}_L^{\sigma_3})_0 = \dim(V_L^{\sigma_3})_0 = 1$. We are ready to state the main result in this section.

Theorem 5.2.1. Keep the notation and setting above. For $L = \text{Ni}(D_4^6)$ and the $\sigma_3$ above, the Lie algebra $(\tilde{V}_L^{\sigma_3})_1$ is of type $E_{6,3}G_{2,1}^3$. Therefore, $\tilde{V}_L^{\sigma_3}$ corresponds to No. 32 on Schellekens’ list [3, Table 1].

Proof of Theorem 5.2.1. First, let us determine the Lie algebra structure of $(V_L^{\sigma_3})_1$. We use the notation in the proof of Theorem 4.2.1 recall that

$$(V_L)_1 \cong \mathfrak{g}^{(1)} \oplus \cdots \oplus \mathfrak{g}^{(6)} \text{ with } \mathfrak{g}^{(i)} \cong \mathfrak{g}_{D_4} \text{ for every } 1 \leq i \leq 6.$$

It can be easily seen that $\sigma_3 \in \text{Aut}(V_L)$ preserves each simple component $\mathfrak{g}^{(i)} \cong \mathfrak{g}_{D_4}$ of $(V_L)_1$; indeed, for each $i = 1, 2, 3$,

$$\left\{\begin{array}{ll}
\sigma_3(h^{(i)}(-1)1 \otimes e^0) = (\varphi(h))^{(i)}(-1)1 \otimes e^0 & \text{for } h \in D_4 \otimes_{\mathbb{C}} \mathbb{C}; \\
\sigma_3(1 \otimes e^{a^{(i)}}) = 1 \otimes e^{\varphi(a^{(i)})} & \text{for } a \in \Delta(D_4),
\end{array}\right.$$
and for each $i = 4, 5, 6$,

$$\begin{cases} 
\sigma_3(h^{(i)}(-1)1 \otimes e^0) = (\omega(h))^{(i)}(-1)1 \otimes e^0 \quad \text{for } h \in D_4 \otimes_{\mathbb{C}} \mathbb{C}; \\
\sigma_3(1 \otimes e^{\alpha^{(i)}}) = 1 \otimes e^{\omega(\alpha^{(i)})} \quad \text{for } \alpha \in \Delta(D_4).
\end{cases}$$

Thus the restriction of $\sigma_3$ to a simple component $g_i \cong g_{D_4}$ for $i = 1, 2, 3$ (resp., for $i = 4, 5, 6$) is identical to the Lie algebra automorphism, denoted also by $\varphi$ (resp., $\omega$), of $g_{D_4}$ induced from $\varphi \in \text{Aut}(D_4)$ (resp., the Dynkin diagram automorphism $\omega$). Therefore,

$$(V_L^{\sigma_3})_1 = (g^{(1)})^{\sigma_3} \oplus \cdots \oplus (g^{(6)})^{\sigma_3} \cong (g_{D_4}^\varphi)^{\oplus 3} \oplus (g_{D_4}^\omega)^{\oplus 3}.$$  

We see from the proof of Theorem 4.2.1 that $g_{D_4}^\varphi$ is isomorphic to the simple Lie algebra $g_{A_2}$ of type $A_2$. Also, it is well-known (see, e.g., [K, p.128, Case 4]) that $g_{D_4}^\omega$ is isomorphic to the simple Lie algebra $g_{G_2}$ of type $G_2$.

Now, fix $i = 4, 5, 6$. The Lie algebra $(g^{(i)})^{\sigma_3} \cong g_{D_4}^\omega \cong g_{G_2}$ is generated by:

$$\{h^{(i)}(-1)1 \otimes e^0 \mid h \in D_4 \otimes_{\mathbb{C}} \mathbb{C}, \omega(h) = h\},$$

$$\{1 \otimes e^{\alpha^{(i)}} + 1 \otimes e^{\omega(\alpha^{(i)})} + 1 \otimes e^{\omega^2(\alpha^{(i)})} \mid \alpha \in \Delta(D_4), \omega(\alpha) \neq \alpha\},$$

and

$$\{1 \otimes e^{\alpha^{(i)}} \mid \alpha \in \Delta(D_4), \omega(\alpha) = \alpha\}.$$

Because $\alpha^{(i)} \in \Delta \setminus N$ for all $\alpha \in \Delta(D_4)$, it follows from Lemma 2.4.2 (1) and (2), along with 2.3.1, that $[a, b] = a_0b = 0$ for all $a \in (g^{(i)})^{\sigma_3}$ and $b \in V(\sigma_3)_1$. Similarly, we can check that $[a, b] = a_0b = 0$ for all $a \in (g^{(i)})^{\sigma_3}$ and $b \in V(\sigma_3)_1$. Thus, $(g^{(i)})^{\sigma_3} \cong g_{G_2}$ is a (simple) ideal of $(\tilde{V}_L^{\sigma_3})_1$. Because $\{1 \otimes e^{\alpha^{(i)}} \mid \alpha \in \Delta(D_4), \omega(\alpha) = \alpha\}$ correspond to the root vectors for long roots of $G_2$, we deduce that the level of $(g^{(i)})^{\sigma_3}$ is equal to 1. Therefore, $(g^{(i)})^{\sigma_3}$ is of type $G_{2,1}$. Since the dual Coxeter number of $G_2$ is equal to 4, we see by 2.4.4 that

$$\frac{4}{1} = \dim(\tilde{V}_L^{\sigma_3})_1 - 24, \quad \text{and hence } \dim(\tilde{V}_L^{\sigma_3})_1 = 120.$$  

Set

$$g := \left( (g^{(1)})^{\sigma_3} \oplus (g^{(2)})^{\sigma_3} \oplus (g^{(3)})^{\sigma_3} \right) \oplus V(\sigma_3)_1 \oplus V(\sigma_3^2)_1 \subset (\tilde{V}_L^{\sigma_3})_1.$$  

By Remark 2.4.1, 2.4.4, and the argument above, we see that $g$ is an ideal of $(\tilde{V}_L^{\sigma_3})_1$ satisfying the following conditions:

(i) $g$ is a $(\mathbb{Z}/3\mathbb{Z})$-graded, semisimple Lie algebra of dimension 78;

(ii) the dual Coxeter number of a simple component of $g$ is contained in 4$\mathbb{Z}$.

Hence, $g$ is of type

$$E_6, \quad A_7A_3, \quad \text{or } C_6^3A_3.$$  

We will show that $g$ is of type $E_6$.  

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Define a linear automorphism \( f : \mathfrak{g} \to \mathfrak{g} \) by: \( f(x) = \zeta^k x \) for \( x \in \mathfrak{g}_k, k = 0, 1, 2 \). Then, \( f \) is a Lie algebra automorphism of \( \mathfrak{g} \). It follows from [K] Proposition 8.1] that there exist an element \( h \) of a Cartan subalgebra \( \mathfrak{z} \) of \( \mathfrak{g} \) and a Dynkin diagram automorphism \( \kappa \) of \( \mathfrak{g} \) preserving \( \mathfrak{z} \) such that \( f \in \text{Aut}(\mathfrak{g}) \) is conjugate to

\[
f' := \kappa \exp \left( \frac{2\pi \sqrt{-1}}{3} h \right).
\]

(Observe that [K] Proposition 8.1] is valid for a semisimple Lie algebra. In this case, a Dynkin diagram automorphism may permute some isomorphic components.) Because \( f \) is of order 3, and hence so is \( f' \), it follows immediately that \( \kappa^3 \) is equal to the identity map. We first claim that

**Claim 1.** The Dynkin diagram automorphism \( \kappa \) is equal to the identity map.

**Proof of Claim [1].** Suppose that \( \kappa \) is not equal to the identity map. Then, \( \kappa \) is of order 3. Because neither of \( E_6 \) nor \( A_7A_3 \) has a Dynkin diagram automorphism of order 3, \( \mathfrak{g} \) should be of type \( C_3^3A_3 \), and \( \kappa \) should be the Dynkin diagram automorphism of the Dynkin diagram of type \( C_3^3A_3 \) that permutes the (three) \( C_3 \)-components, and acts on the \( A_3 \)-component trivially. Write \( \mathfrak{g} \) as: \( \mathfrak{g} = \mathfrak{g}^1 \oplus \mathfrak{g}^2 \oplus \mathfrak{g}^3 \oplus \mathfrak{g}^4 \), where \( \mathfrak{g}^1 \cong \mathfrak{g}^2 \cong \mathfrak{g}^3 \cong \mathfrak{g}_{C_3} \), and \( \mathfrak{g}^4 \cong \mathfrak{g}_{A_3} \); we may assume that \( \kappa \) maps \( (x_1, x_2, x_3, x_4) \in \mathfrak{g} \) to \( (x_2, x_3, x_1, x_4) \in \mathfrak{g} \). Following this decomposition, write the \( h \) as: \( h = (h_1, h_2, h_3, h_4) \). Then we see that

\[
f'(x_1, x_2, x_3, x_4) = (e_2x_2, e_3x_3, e_1x_1, e_4x_4),
\]

where for simplicity of notation, we set \( e_j := \exp \left( \frac{2\pi \sqrt{-1}}{3} h_j \right) \in \text{Aut}(\mathfrak{g}_j) \) for \( j = 1, 2, 3, 4 \).

Define \( g \in \text{Aut}(\mathfrak{g}) \) by: \( g(x_1, x_2, x_3, x_4) = (x_1, e_2x_2, e_1^{-1}x_3, x_4) \). It follows by direct computation that

\[
gf'g^{-1}(x_1, x_2, x_3, x_4) = (x_2, x_3, x_1, e_4x_4).
\]

We deduce that the fixed point Lie subalgebra of \( \mathfrak{g} \) under \( gf'g^{-1} \in \text{Aut}(\mathfrak{g}) \) contains an ideal isomorphic to the simple Lie algebra of type \( C_3 \). Because \( f \in \text{Aut}(\mathfrak{g}) \) is conjugate to \( f' \), and hence to \( gf'g^{-1} \), it follows that the fixed point Lie subalgebra \( \mathfrak{g}_0 \) of \( \mathfrak{g} \) under \( f \) also contains an ideal isomorphic to the simple Lie algebra of type \( C_3 \). However, this is a contradiction, since \( \mathfrak{g}_0 \) is of type \( A_2^3 \) by [5.2.2]. Thus we have proved Claim 1. \( \blacksquare \)

Because \( \kappa \) is equal to the identity map, we see that the fixed point Lie subalgebra of \( \mathfrak{g} \) under \( f' \in \text{Aut}(\mathfrak{g}) \) contains the Cartan subalgebra \( \mathfrak{z} \) of \( \mathfrak{g} \). Because \( f \in \text{Aut}(\mathfrak{g}) \) is conjugate to \( f' \), the fixed point Lie subalgebra \( \mathfrak{g}_0 \) of \( \mathfrak{g} \) under \( f \) also contains a Cartan subalgebra of \( \mathfrak{g} \). Thus the rank of \( \mathfrak{g} \) is less than or equal to the rank of \( \mathfrak{g}_0 \), which is equal to 6. Thus we conclude by (5.2.3) that \( \mathfrak{g} \) is of type \( E_6 \). This completes the proof of the theorem. \( \square \)
6 Construction of a holomorphic VOA (4).

6.1 Automorphism of order 3 on the root lattice $D_4$. We use the notation for the root lattice $D_4$ introduced in §4.1. We define a lattice automorphism $\psi = \psi_{D_4} \in \text{Aut}(D_4)$ by:

$$(x_1, x_2, x_3, x_4) \mapsto (x_2, x_3, x_1, x_4).$$

Observe that $\psi$ acts on the set $\Delta(D_4)$ of roots fixed-point-freely. Set

$\beta_1 := (1, -1, 0, 0) \in \Delta(D_4), \quad \beta_2 := (1, 0, 0, -1) \in \Delta(D_4),$

$\beta_3 := (1, 1, 0, 0) \in \Delta(D_4), \quad \beta_4 := (1, 0, 0, 1) \in \Delta(D_4).$

Then, $\{\pm \beta_j \mid 1 \leq j \leq 4\}$ is a complete set of representatives of $\psi$-orbits in $\Delta(D_4)$. Also, we should remark that $\psi([\ell]) = [\ell]$ for every $\ell = 0, 1, 2, 3$. (6.1.1)

6.2 Niemeier lattice $\text{Ni}(D_6^6)$ and its automorphism $\sigma_4$ of order 3. Keep the notation and setting for the Niemeier lattice $L = \text{Ni}(D_6^6)$ with $Q = D_6^6$ the root lattice in §4.2. Let us define $\sigma_4 : Q^* \to Q^*$ by:

$$\sigma_4(\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6) = (\psi(\gamma_1), \varphi(\gamma_2), \varphi^{-1}(\gamma_3), \gamma_6, \varphi^{-1}(\gamma_4), \varphi(\gamma_5)),
$$

where $\varphi = \varphi_{D_4}$ is the lattice automorphism of $D_4^6$ introduced in §4.1. Then we deduce from (4.1.1) and (6.1.1) that $L \subset Q^*$ is stable under the action of $\sigma_4$; for example, we have, modulo $Q$,

$$[111111] \mapsto [123132] \equiv [032023] + [111111],$$

$$[222222] \mapsto [231213] \equiv [023320] + [032023] + [222222],$$

$$[002332] \mapsto [001221] \equiv [033202] + [032023],$$

and so on. Therefore, $\sigma_4$ gives a lattice automorphism of $L$ of order 3; since rank $D_4^5 = 2$ and rank $D_4^6 = 0$, it follows that rank $L_{\sigma_4} = 2 + 4 = 6$, and hence

$$\rho = \frac{1}{18}(\dim h_{(1)} + \dim h_{(2)}) = \frac{1}{18}(24 - 6) = 1.$$

(6.2.1)

Thus we can apply Theorem 2.3.2 to $L = \text{Ni}(D_6^6)$ and the $\sigma_4$ above, and obtain a $C_2$-cofinite, holomorphic VOA $\tilde{V}_L^{\sigma_4}$ of central charge 24. We are ready to state the main result in this section.

**Theorem 6.2.1.** Keep the notation and setting above. For $L = \text{Ni}(D_6^6)$ and the $\sigma_4$ above, the Lie algebra $(\tilde{V}_L^{\sigma_4})_1$ is of type $A_{5,3}D_{4,3}A_{1,1}$. Therefore, $\tilde{V}_L^{\sigma_4}$ corresponds to No. 17 on Schellekens’ list [8, Table 1].

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Proof. We will use the notation introduced in the proof of Theorem 4.2.1. Recall that

\[(V_L)_1 \cong g^{(1)} \oplus \ldots \oplus g^{(6)} \quad \text{with} \quad g^{(i)} \cong g_{D_4} \quad \text{for every } 1 \leq i \leq 6.\]

Then, \(\sigma_4\) acts on \(g^{(1)}\) (resp. \(g^{(2)}, g^{(3)}\)) as the Lie algebra automorphism induced from \(\psi\) (resp., \(\varphi, \varphi^{-1}\)). Also, we can easily check that the fixed point subalgebra of \(g^{(4)} \oplus g^{(5)} \oplus g^{(6)}\) under \(\sigma_4\) is isomorphic to \(g_{D_4}\). Thus it follows that

\[(V_L^{\sigma_4})_1 \cong g_{D_4}^\psi \oplus g_{D_4}^{\varphi} \oplus g_{D_4}^{\varphi^{-1}} \oplus g_{D_4}.\]  \hfill (6.2.2)

In addition, by the same way as in the proof of Theorem 3.2.2, we deduce that \(g_{D_4}\), the last component of the right-hand side above, is a simple ideal of type \(D_{4,3}\) of the whole Lie algebra \(\tilde{V}_L^{\sigma_4}\). Since the dual Coxeter number of \(D_4\) is equal to 6, it follows immediately from (2.4.1) that

\[
\frac{6}{3} = \frac{\dim(\tilde{V}_L^{\sigma_4})_1 - 24}{24}, \quad \text{and hence} \quad \dim(\tilde{V}_L^{\sigma_4})_1 = 72.
\]

Now, let us determine the Lie algebra structure of \(g_{D_4}^\psi\), the first component of the right-hand side of (6.2.2). Observe that \(g_{D_4}^\psi\) is spanned by the following:

\[
\begin{align*}
\{ & 1 \otimes e^{\pm \beta_j^{(1)}} + 1 \otimes e^{\pm \psi(\beta_j^{(1)})} + 1 \otimes e^{\pm \psi^2(\beta_j^{(1)})} \mid j = 1, 2, 3, 4 \} \quad (8 \text{ vectors}); \\
\{ & h^{(1)}(-1) \otimes e^0 \mid h \in D_4 \otimes \mathbb{C}, \psi(h) = h \} \quad (2\text{-dimensional});
\end{align*}
\]
in particular, \(\dim g_{D_4}^\psi = 10\). Let \(s\) be the Lie subalgebra of \(g_{D_4}^\psi\) generated by:

\[
\begin{align*}
\{ & 1 \otimes e^{\pm \beta_j^{(1)}} + 1 \otimes e^{\pm \psi(\beta_j^{(1)})} + 1 \otimes e^{\pm \psi^2(\beta_j^{(1)})} \mid j = 2, 3, 4 \} \quad (6 \text{ vectors}); \\
\{ & h^{(1)}(-1) \otimes e^0 \mid h \in D_4 \otimes \mathbb{C}, \psi(h) = h \} \quad (2\text{-dimensional}).
\end{align*}
\]

Then we deduce that \(s\) is an ideal of \(g_{D_4}^\psi\); indeed, it suffices to show that \(s\) is stable under the adjoint action of \(1 \otimes e^{\pm \beta_j^{(1)}} + 1 \otimes e^{\pm \psi(\beta_j^{(1)})} + 1 \otimes e^{\pm \psi^2(\beta_j^{(1)})}\). This follows from the facts that \(\langle h, \beta_j \rangle = 0\) for all \(h \in D_4 \otimes \mathbb{C}\) such that \(\psi(h) = h\), and that for any \(0 \leq r, s \leq 2\) and \(j = 2, 3, 4\),

\[
[1 \otimes e^{\pm \psi^r(\beta_j^{(1)})}, 1 \otimes e^{\pm \psi^s(\beta_j^{(1)})}]
\]
is equal to either 0 or a scalar multiple of \(1 \otimes e^{\pm \psi^t(\beta_k^{(1)})}\) for some \(0 \leq t \leq 2\) and \(k = 2, 3, 4\). Also, because \(\pm \psi^r(\beta_j^{(1)}) \in \Delta \setminus N\) for all \(0 \leq r \leq 2\) and \(j = 2, 3, 4\), it follows from Lemma 2.2.2(1) and (2) that \(s\) is also an ideal of the whole Lie algebra \((\tilde{V}_L^{\sigma_4})_1\); in particular, \(s\) is a semisimple Lie algebra. Because \(8 \leq \dim s \leq 10 = \dim g_{D_4}^\psi\), and because the dual Coxeter number of a simple component of \(s\) is an even integer by (2.4.1), we deduce that \(s \cong g_{A_1}^{s_3}\). Therefore the Lie algebra \(g_{D_4}^\psi\) is a reductive Lie algebra of the form: \(\mathfrak{a} \oplus g_{A_1}^{s_3}\), where \(\mathfrak{a}\) is a 1-dimensional (abelian) Lie algebra.
Set
\[ g := (a \oplus g_{D_4}^\sigma) \oplus V(\sigma_4)_1 \oplus V(\sigma_2^2)_1. \]

By Remark 2.4.1, and the argument above, we see that \( g \) is an ideal of \( (\tilde{V}_L^\sigma)_1 \) satisfying the following conditions:

(i) \( g \) is a \((\mathbb{Z}/3\mathbb{Z})\)-graded, semisimple Lie algebra of dimension 35;

(ii) the dual Coxeter number of a simple component of \( g \) is an even integer.

Hence, \( g \) is of type
\[ A_5 \quad \text{or} \quad C_3G_2; \quad (6.2.3) \]
in particular, \( g \) is of rank 5. We will show that \( g \) is of type \( A_5 \).

We see from the proof of Theorem 3.2.1 that both of \( g_{D_4}^\sigma \) and \( g_{D_4}^{\sigma^{-1}} \) are isomorphic to \( g_{A_2} \) (and hence \( g_0 \cong a \oplus g_{A_2}^\circ \)). Let \( t_1 \) and \( t_2 \) be the Cartan subalgebras of \( g_{D_4}^\sigma \cong g_{A_2} \) and \( g_{D_4}^{\sigma^{-1}} \cong g_{A_2} \), respectively. Also, let \( t \) be the maximal subalgebra of \( g_0 \), containing \( t_1 \oplus t_2 \), such that the adjoint action of each element of \( t \) on \( g \) is semisimple. We see from [K, Lemma 8.1 b)] that the centralizer \( h \) of \( t \) in \( g \) is a Cartan subalgebra of \( g \). It can be easily checked that \( h \) contains \( a \oplus t_1 \oplus t_2 \). Since \( g \) is of rank 5, we have \( g = a \oplus t_1 \oplus t_2 \). Thus we deduce that each root space of \( g \) (with respect to \( h \)) is contained in exactly one of the graded spaces \( g_0 \), \( g_1 \), and \( g_2 \). In particular, there exists a simple ideal \( u \) of \( g \) that contains \( g_{D_4}^\sigma \cong g_{A_2} \).

Assume that the ratio of the squared length of a long root of \( u \) to the squared length of a root of \( g_{D_4}^\sigma \cong g_{A_2} \) is \( r \) to 1; note that \( r \in \{1, 2, 3\} \). We deduce from the proof of Theorem 3.2.2 that if we normalize the bilinear form \((\cdot, \cdot)_u\) on \( u \) in such a way that the squared norm of a long root of \( u \) is equal to 2, then for \( x, y \in g_{D_4}^\sigma \) and \( m, n \in \mathbb{Z} \),
\[ [x_m, y_n] = (x_0y)_{m+n} + (3/r)m\delta_{m+n, 0}(x, y)_u \text{id}_{\tilde{V}_L^\sigma} \quad \text{on } \tilde{V}_L^\sigma. \]
Because the level of a simple component is a (positive) integer, the level of \( u \) is equal to either 1 or 3. If \( u \) were of type \( C_3 \) or \( G_2 \), then the level of \( u \) would be equal to 2, which is a contradiction. Thus, by \( 3.2.3 \), we conclude that \( g \) is of type \( A_5 \). This completes the proof of Theorem 6.2.1. \( \square \)

7 Construction of a holomorphic VOA (5).

7.1 Automorphism of order 3 on the root lattice \( A_5 \). We use the notation introduced in 3.1 (with \( n = 5 \)). Define a lattice automorphism \( \psi = \psi_{A_5} \in \text{Aut}(A_5) \) by:
\[ (x_0, x_1, x_2, x_3, x_4, x_5) \mapsto (x_2, x_0, x_1, x_5, x_3, x_4). \]
Observe that \( \psi \) acts on the set \( \Delta(A_5) \) of roots fixed-point-freely. Set
\[ \beta_1 := (1, -1, 0, 0, 0, 0) \in \Delta(A_5), \quad \beta_2 := (0, 0, 0, 1, 0, -1) \in \Delta(A_5), \]
\[ \beta_3 := (1, 0, 0, -1, 0, 0) \in \Delta(A_5), \quad \beta_4 := (1, 0, 0, 0, -1, 0) \in \Delta(A_5), \]
\[ \beta_5 := (1, 0, 0, 0, 0, -1) \in \Delta(A_5). \]

Then, \( \{ \pm \beta_j \mid 1 \leq j \leq 5 \} \) is a complete set of representatives of \( \psi \)-orbits in \( \Delta(A_5) \). Also, it can be easily checked that

\[ \psi([\ell]) = [\ell] \quad \text{for every } \ell = 1, 2, \ldots, 5. \quad (7.1.1) \]

### 7.2 Niemeier lattice \( \text{Ni}(A_4^3 D_4) \) and its automorphism \( \sigma_5 \) of order 3

The Niemeier lattice \( L = \text{Ni}(A_4^3 D_4) \) with \( Q = A_4^3 D_4 \) the root lattice is, by definition (see [CS, Chapter 16, Table 16.1]), the sublattice of \( Q^* = (A_4^3)^4 D_4^* \) generated by \( Q, [33001], [30302], [30033], \) and \( [2(024)0] = \{ [20240], [22400], [24020] \} \), where \( [a_1 \cdots a_5] := ([a_1], \ldots, [a_5]) \in Q^* = (A_4^3)^4 D_4^* \).

Let us define \( \sigma_5 : Q^* \to Q^* \) by:

\[ \sigma_5(\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5) = (\psi(\gamma_1), \gamma_4, \gamma_2, \gamma_3, \psi(\gamma_5)), \]

where \( \varphi = \varphi_{D_4} \) is the lattice automorphism of \( D_4^* \) introduced in §4.4. Then we deduce from (4.1.1) and (7.1.1) that \( L \subset Q^* \) is stable under the action of \( \sigma_5 \). Therefore, \( \sigma_5 \) gives a lattice automorphism of \( L \) of order 3; since rank \( A_5^3 \) is 1 and rank \( D_4^* \) is 0, it follows that rank \( L^{\sigma_5} = 1 + 5 = 6 \), and hence

\[ \rho = \frac{1}{18}(\dim h_1 + \dim h_2) = \frac{1}{18}(24 - 6) = 1. \quad (7.2.1) \]

Thus we can apply Theorem 2.3.2 to \( L = \text{Ni}(A_4^3 D_4) \) and the \( \sigma_5 \) above, and obtain a \( C_2 \)-cofinite, holomorphic VOA \( \hat{V}_L^{\sigma_5} \) of central charge 24. We are ready to state the main result in this section.

**Theorem 7.2.1.** Keep the notation and setting above. For \( L = \text{Ni}(A_4^3 D_4) \) and the \( \sigma_5 \) above, the Lie algebra \( \hat{V}_L^{\sigma_5} \) is of type \( A_{5,3} D_{4,3} A_{1,1}^3 \). Therefore, \( \hat{V}_L^{\sigma_5} \) corresponds to No. 17 on Schellekens’ list [5, Table 1].

**Proof.** We see that

\[ (V_L)_1 \cong \mathfrak{g}^{(1)} \oplus \cdots \oplus \mathfrak{g}^{(5)} \]

with \( \mathfrak{g}^{(i)} \cong \mathfrak{g}_{A_5} \) for every \( 1 \leq i \leq 4 \), and \( \mathfrak{g}^{(5)} \cong \mathfrak{g}_{D_4} \).

Then, \( \sigma_5 \) acts on \( \mathfrak{g}^{(1)} \cong \mathfrak{g}_{A_5} \) (resp., \( \mathfrak{g}^{(5)} \cong \mathfrak{g}_{D_4} \)) as the Lie algebra automorphism induced from \( \psi = \psi_{A_5} \) (resp., \( \varphi = \varphi_{D_4} \)). Also, we can easily check that the fixed point subalgebra of \( \mathfrak{g}^{(2)} \oplus \mathfrak{g}^{(3)} \oplus \mathfrak{g}^{(4)} \) under \( \sigma_5 \) is isomorphic to \( \mathfrak{g}_{A_5} \). Thus it follows that

\[ (V_L^{\sigma_5})_1 \cong \mathfrak{g}_{A_5}^0 \oplus \mathfrak{g}_{A_5} \oplus \mathfrak{g}_{D_4}^0. \quad (7.2.2) \]
In addition, by the same way as in the proof of Theorem 3.2.2, we deduce that \( \mathfrak{g}_{A_5} \), the second component of the right-hand side above, is a simple ideal of type \( A_{5,3} \) of the whole Lie algebra \( (\tilde{V}_L^{\sigma_5})_1 \). Since the dual Coxeter number of \( A_5 \) is equal to 6, it follows immediately from (2.4.1) that

\[
\frac{6}{3} = \dim(\tilde{V}_L^{\sigma_5})_1 - 24, \quad \text{and hence} \quad \dim(\tilde{V}_L^{\sigma_5})_1 = 72.
\]

Now, let us determine the Lie algebra structure of the first component \( \mathfrak{g}^{\psi}_{A_5} \). Observe that \( \mathfrak{g}^{\psi}_{A_5} \) is spanned by the following:

\[
\{ 1 \otimes e^{\pm \beta_j} + 1 \otimes e^{\pm \psi_5(\beta_j)} + 1 \otimes e^{\pm \psi_2(\beta_j)} \mid 1 \leq j \leq 5 \} \quad \text{(10 vectors)};
\]

\[
\{ h^{(1)}(-1) \otimes e^0 \mid h \in A_5 \otimes \mathbb{Z} \mathbb{C}, \psi(h) = h \} \quad \text{(1-dimensional)};
\]

in particular, \( \dim \mathfrak{g}^{\psi}_{A_5} = 11 \). Let \( \mathfrak{s} \) be the Lie subalgebra of \( \mathfrak{g}^{\psi}_{A_5} \) generated by:

\[
\{ 1 \otimes e^{\pm \beta_j} + 1 \otimes e^{\pm \psi_5(\beta_j)} + 1 \otimes e^{\pm \psi_2(\beta_j)} \mid 3 \leq j \leq 5 \} \quad \text{(6 vectors)};
\]

\[
\{ h^{(1)}(-1) \otimes e^0 \mid h \in A_5 \otimes \mathbb{Z} \mathbb{C}, \psi(h) = h \} \quad \text{(1-dimensional)}.
\]

Then we can show in exactly the same way as for \( \mathfrak{s} \) in the proof of Theorem 6.2.1 that \( \mathfrak{s} \) is an ideal of type \( A_{3,1} \) of the whole Lie algebra \( (\tilde{V}_L^{\sigma_5})_1 \). Therefore the Lie algebra \( \mathfrak{g}^{\psi}_{A_5} \) is a reductive Lie algebra of the form: \( \mathfrak{a} \oplus \mathfrak{g}^{\psi}_{A_5} \), where \( \mathfrak{a} \) is a 2-dimensional (abelian) Lie algebra.

Set

\[
\mathfrak{g} := (\mathfrak{a} \oplus \mathfrak{g}^{\psi}_{D_4}) \oplus V(\sigma_5)_1 \oplus V(\sigma_5^2)_1.
\]

By Remark 2.4.1 (2.4.1), and the argument above, we see that \( \mathfrak{g} \) is an ideal of \( (\tilde{V}_L^{\sigma_5})_1 \) satisfying the following conditions:

(i) \( \mathfrak{g} \) is a \((\mathbb{Z}/3\mathbb{Z})\)-graded, semisimple Lie algebra of dimension 28;

(ii) the dual Coxeter number of a simple component of \( \mathfrak{g} \) is an even integer.

Hence, \( \mathfrak{g} \) is of type \( D_4 \) or \( G_2^2 \) (7.2.3).

in particular, \( \mathfrak{g} \) is of rank 4. We will show that \( \mathfrak{g} \) is of type \( D_4 \).

Recall from the proof of Theorem 4.2.1 that \( \mathfrak{g}^{\sigma_2}_{D_4} \) is isomorphic to the simple Lie algebra of type \( A_2 \). Let \( \mathfrak{t} \) be a Cartan subalgebra of \( \mathfrak{g}^{\sigma_2}_{D_4} \cong \mathfrak{g}_{A_2} \). By the same way as in Theorem 6.2.1 we can show that \( \mathfrak{h} = \mathfrak{a} \oplus \mathfrak{t} \) is a Cartan subalgebra of \( \mathfrak{g} \). Thus each root space of \( \mathfrak{g} \) is contained in exactly one of the graded spaces \( \mathfrak{g}_0, \mathfrak{g}_1, \) and \( \mathfrak{g}_2 \). In particular, there exists a simple ideal \( \mathfrak{u} \) of \( \mathfrak{g} \) that contains \( \mathfrak{g}^{\sigma_2}_{D_4} \cong \mathfrak{g}_{A_2} \). Then we can prove in exactly the same way as in Theorem 6.2.1 that the level of \( \mathfrak{u} \) is equal to either 1 or 3. If \( \mathfrak{u} \) were of type \( G_2 \) (see (7.2.3)), then the level of it would be equal to 2, which is a contradiction. Thus we conclude that \( \mathfrak{g} \) is of type \( D_4 \).

This completes the proof of Theorem 7.2.1. \( \square \)
A Appendix.

In [Mi, §5.2], Miyamoto proved that the VOA obtained by applying Theorem 2.3.2 to the Niemeier lattice $L = \mathrm{Ni}(E_6^4)$ with $Q = E_6^4$ the root lattice and a specified automorphism $\sigma_6$ of order 3 corresponds to No. 32 on Schellekens’ list [S]. In Appendix, after reviewing the definitions of $\mathrm{Ni}(E_6^4)$ and $\sigma_6$, we give another proof for this fact, based on the Dong-Lepowsky construction of twisted modules (see §2.2) and the formula (2.4.1) due to Dong and Mason.

A.1 Niemeier lattice $\mathrm{Ni}(E_6^4)$ and its automorphism of order 3. Let $\{\alpha_i \mid 1 \leq i \leq 6\}$ be the simple roots for the root lattice $E_6 = \bigoplus_{i=1}^6 \mathbb{Z}\alpha_i$;

$$\begin{array}{c}
\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4 \quad \alpha_5 \quad \alpha_6
\end{array}$$

We set

$$[0] := 0, \quad [1] := \frac{1}{3}(\alpha_1 - \alpha_2 + \alpha_4 - \alpha_5), \quad [2] := \frac{1}{3}(-\alpha_1 + \alpha_2 - \alpha_4 + \alpha_5),$$

and

$$\overline{[0]} := [0] + E_6, \quad \overline{[1]} := [1] + E_6, \quad \overline{[2]} := [2] + E_6.$$

Then, $E_6^* / E_6 = \{[\ell] \mid \ell = 0, 1, 2\}$, and the additive group $E_6^* / E_6$ is naturally isomorphic to $\mathbb{Z}/3\mathbb{Z}$. Denote by $\Delta(E_6)$ the set of roots in $E_6$. Let $\varphi_E$ denote the lattice automorphism of $E_6$ defined by: $\varphi = r_1r_2r_3r_4r_5r_6$, where $r_i$ is the simple reflection with respect to the simple root $\alpha_i$ for $1 \leq i \leq 6$, and $r_6$ is the reflection with respect to the highest root of $E_6$; note that $\varphi$ is of order 3, and $\varphi$ acts on $E_6 \otimes \mathbb{C}$ fixed-point-freely. Also we see that $\varphi([\ell]) = [\ell]$ for every $\ell = 0, 1, 2$.

The Niemeier lattice $L = \mathrm{Ni}(E_6^4)$ with $Q = E_6^4$ the root lattice is, by definition (see [CS, Chapter 16, Table 16.1]), the sublattice of $Q^* = (E_6^4)^4$ generated by $Q$ and $[1(012)] = \{(1012), [1120], [1201]\}$, where $[a_1 \cdots a_4] := ([a_1], \ldots, [a_4]) \in Q^* = (E_6^4)^4$. Let us define $\sigma_6 : Q^* \to Q^*$ by:

$$(\gamma_1, \gamma_2, \gamma_3, \gamma_4) \mapsto (\varphi(\gamma_1), \gamma_4, \gamma_2, \gamma_3).$$

Then we deduce that $L \subset Q^*$ is stable under the action of $\sigma_6$. Therefore, $\sigma_6$ gives a lattice automorphism of $L$ of order 3. Since rank $E_6^d = 0$, it follows immediately that rank $L^{\sigma_6} = 6$, and hence

$$\rho = \frac{1}{18}(\dim h_{(1)} + \dim h_{(2)}) = \frac{1}{18}(24 - 6) = 1. \quad (A.1.1)$$

Thus we can apply Theorem 2.3.2 to $L = \mathrm{Ni}(E_6^4)$ and the $\sigma_6$ above, and obtain a $C_2$-cofinite, holomorphic VOA $\tilde{V}_L^{\sigma_6}$ of central charge 24. The following theorem is the main result in [Mi §5.2].
Theorem A.1.1. Keep the notation and setting above. For $L = \text{Ni}(E_6^3)$ and the $\sigma_6$ above, the Lie algebra $(\tilde{V}_L^{\sigma_6})_1$ is of type $E_{6,3}G_{2,1}^3$. Therefore, $\tilde{V}_L^{\sigma_6}$ corresponds to No. 32 on Schellekens’ list [S, Table 1].

Proof. We deduce in exactly the same way as Lemma 3.3.1 that $(V_L^{\sigma_6})_1 = \mathfrak{g}_{E_6}^{\varphi} \oplus \mathfrak{g}^{(234)}$, where $\mathfrak{g}^{(234)}$ is isomorphic to the simple Lie algebra $\mathfrak{g}_{E_6}$ of type $E_6$, and $\mathfrak{g}_{E_6}^{\varphi}$ is the fixed point subalgebra of $\mathfrak{g}_{E_6}$ under the Lie algebra automorphism induced from $\varphi \in \text{Aut}(E_6)$; we deduce that $\mathfrak{g}_{E_6}^{\varphi}$ is isomorphic to the semisimple Lie algebra $\mathfrak{g}_{E_6}^{3}$. Furthermore, we see by Lemma 2.2.1 that $\mathfrak{g}^{(234)}$ is a simple ideal of the (whole) Lie algebra $(\tilde{V}_L^{\sigma_6})_1$ whose level is equal to 3. Since the dual Coxeter number of $E_6$ is equal to 12,

$$\frac{12}{3} = \frac{\dim(\tilde{V}_L^{\sigma_6})_1 - 24}{24},$$

and hence $\dim(\tilde{V}_L^{\sigma_6})_1 = 120$.

Set

$$\mathfrak{g} := \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \quad \text{with} \quad \mathfrak{g}_0 := \mathfrak{g}_{E_6}^{\varphi}, \quad \mathfrak{g}_1 := V(\sigma_6)_1, \quad \mathfrak{g}_2 := V(\sigma_6^3)_1.$$

Then we see from the argument above, along with Proposition 2.4.2 and (2.4.1), that $\mathfrak{g}$ is an ideal of $(\tilde{V}_L^{\sigma_6})_1$ satisfying the following conditions:

(i) $\mathfrak{g}$ is a $(\mathbb{Z}/3\mathbb{Z})$-graded, semisimple Lie algebra of dimension 42;

(ii) the dual Coxeter number of a simple component of $\mathfrak{g}$ is contained in $4\mathbb{Z}$.

Thus, $\mathfrak{g}$ is isomorphic to the semisimple Lie algebra of type $G_2^3$ or $C_3^2$; in particular, the rank of $\mathfrak{g}$ is equal to 6. Hence the Cartan subalgebra of $\mathfrak{g}_{E_6}^{\varphi} \cong \mathfrak{g}_{A_2}^{3}$ is also a Cartan subalgebra of $\mathfrak{g}$. Thus the root system of $\mathfrak{g}$ contains the root system of $A_2^2$, which implies that $\mathfrak{g}$ is of type $G_2^3$. Thus we have proved the proposition.

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