Finite element method for radially symmetric solution of a multidimensional semilinear heat equation

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This study aims to present the error and numerical blow up analyses of a finite element method for computing the radially symmetric solutions of semilinear heat equations. In particular, this study establishes optimal order error estimates in $L^\infty$ and weighted $L^2$ norms for the symmetric and nonsymmetric formulation, respectively. Some numerical examples are presented to validate the obtained theoretical results.

Key words: numerical analysis, finite element method, semilinear parabolic equation, radially symmetric solution.

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1. Introduction

This study aims to investigate the convergence of a finite element method (FEM) applied to a parabolic equation with singular coefficients for the function $u = u(x,t)$, $x \in \mathcal{T} = [0,1]$ and $t \geq 0$, as expressed in

\begin{align}
    u_t &= u_{xx} + \frac{N-1}{x} u_x + f(u), \quad x \in I = (0,1), \ t > 0, \quad (1a) \\
    u_x(0,t) &= u(1,t) = 0, \quad t > 0, \quad (1b) \\
    u(x,0) &= u^0(x), \quad x \in I, \quad (1c)
\end{align}

where $f$ is a given locally Lipschitz continuous function, $u^0$ is a given continuous function, and

\[ N \geq 2 \quad \text{integer} \quad (2) \]

is a given parameter.

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In the study of an $N$ dimensional semilinear heat equation, the following problem arises:

\begin{align}
U_t &= \Delta U + f(U), \quad x \in \Omega, \ t > 0 \\
U &= 0, \quad x \in \partial\Omega, \ t > 0, \\
U(0,x) &= U^0(x), \quad x \in \Omega,
\end{align}

where $\Omega$ denotes a bounded domain $\mathbb{R}^N$. If one is concerned with the radially symmetric solution $u(|x|) = U(x)$ in the $N$ dimensional sphere $\Omega = \{x \in \mathbb{R}^N \mid |x| = |x|_{\mathbb{R}^N} \leq 1\}$, then (3) implies (1), where $x = |x|$ and $u^0(x) = U_0(x)$.

For a linear case where $f(u) = 0$ is replaced by a given function $f(x,t)$,\[3, 8\] studied the convergence of the FEM to (1) along with the elliptic equation, and proposed two schemes: the symmetric scheme, wherein they established the optimal order error estimate in the weighted $L^2$ norm, and the nonsymmetric scheme, wherein they proved the $L^\infty$ error estimate. Herein, both schemes are applied to the semilinear equation (1) to derive various error estimates.

Moreover, this study includes a discussion of discrete positivity conservation properties, which previous studies \[3, 8\] failed to embrace, but are actually important in the study of diffusion type equations.

Our focus is on the FEM because we are able to use non-uniform partitions of the space variable; therefore, the method deems useful for examining highly concentrated solutions. On this connection, we present our motivation for this study. Till date, the critical phenomenon appearing in the semilinear heat equation of the form

\[U_t = \Delta U + U^{1+\alpha}, \quad \alpha > 0\]

in a multidimensional space has attracted considerable attention since the pioneering work of Fujita \[4\]. According to him, the equation is in the whole $N$ dimensional space, and any positive solution blows up in a finite time if $\alpha \leq 2/N$, whereas a solution is smooth at any time for a small initial value if $\alpha > 2/N$. Therefore, the expression $p_c = 1 + 2/N$ is known as the Fujita’s critical exponent (see \[7, 2\] for critical exponents of other equations). Generally, similar critical exponents can be found for an initial-boundary value problem for the semilinear heat equation (see \[5\] for example); however, the concrete values of those critical conditions seem to be unknown yet. Therefore, we found it interesting to study the numerical methods for computing the solutions of nonlinear partial equations in an $N$ dimensional space. However, computing the non-stationary four-space dimensional problem was difficult even in modern computers. In effect, we agreed to consider the FEM to solve the one space dimensional equation (1). However, we faced another difficulty in dealing with the singular coefficient $(N-1)/x$, which the FEM reasonably simplified, as will be explained later.

Notably, the finite difference method for (1) has been studied and its optimal order convergence has been proved in \[1\], whose finite difference scheme uses a special approximation around the origin to consequently assume a uniform spatial mesh.

This paper comprises of six sections. Section 2 presents our finite element schemes. Well-posedness and positivity conservation are examined in Section 3. Section 4 presents the error estimates and their proofs. Blow up analysis is presented in Section 5. Finally, Section 6 reports some numerical examples that validate our theoretical results.
2. Finite element method

First, we derive two alternate weak formulations of (1). Unless otherwise stated explicitly, we assume that \( f \) is a locally Lipschitz continuous function so that

\[
\forall \mu > 0, \exists M_\mu > 0 : \| f(s) - f(s') \| \leq M_\mu |s - s'| \quad (s, s' \in \mathbb{R}, |s|, |s'| \leq \mu).
\]  

(1)

Let \( \chi \in H^1 = \{v \in H^1(I) | v(1) = 0\} \) be arbitrary. Multiplying both sides of (1a) by \( x^{N-1}\chi \) and using integration by parts over \( I \), we obtain

\[
\int_I x^{N-1}u_t \chi \, dx + \int_I x^{N-1}u_x \chi_x \, dx = \int_I x^{N-1}f(u)\chi \, dx.
\]

(4)

Otherwise, if we multiply both sides of (1a) by \( x\chi \) instead of \( x^{N-1}\chi \) and integrate it over \( I \), we have

\[
\int_I xu_t \chi \, dx + \int_I [xu_x \chi_x + (2 - N)u_x \chi] \, dx = \int_I xf(u)\chi \, dx.
\]

(5)

We call (4) the symmetric weak form due to the symmetric bilinear form associated with the differential operator \( u_{xx} + \frac{N-1}{x}u_x \). On the contrary, (5) is the nonsymmetric weak form. Both forms are identical at \( N = 2 \).

We will now establish the finite element schemes based on these identities. For a positive integer \( m \), we introduce node points

\[
0 = x_0 < x_1 < \cdots < x_{j-1} < x_j < \cdots < x_{m-1} < x_m = 1,
\]

and set \( I_j = (x_{j-1}, x_j) \) and \( h_j = x_j - x_{j-1} \), where \( j = 1, \ldots, m \). The granularity parameter is defined as \( h = \max_{1 \leq j \leq m} h_j \). Let \( \mathcal{P}_k(J) \) be the set of all polynomials in an interval \( J \) of degree \( \leq k \).

We define the P1 finite element space as follows:

\[
S_h = \{ v \in H^1(I) | v \in \mathcal{P}_1(I_j) \ (j = 1, \ldots, m), \ v(1) = 0 \}.
\]

(6)

whose standard basis function \( \phi_j \), \( j = 0, 1, \ldots, m \) is defined as

\[
\phi_j(x_i) = \delta_{ij},
\]

where \( \delta_{ij} \) denotes the Kronecker delta.

For the time discretization, we introduced the non-uniform partitions

\[
t_0 = 0, \quad t_n = \sum_{j=0}^{n-1} \tau_j \ (n \geq 1),
\]

where \( \tau_j > 0 \) denotes the time increments.

Generally, we write \( \partial_{\tau_n} u^{n+1}_h = (u^{n+1}_h - u^n_h) / \tau_n \).

(Sym) Find \( u^{n+1}_h \in S_h, \ n = 0, 1, \ldots \), such that

\[
(\partial_{\tau_n} u^{n+1}_h, \chi) + A(u^{n+1}_h, \chi) = (f(u^n_h), \chi) \quad (\chi \in S_h, \ n = 0, 1, \ldots),
\]

(7)

where \( u^0_h \in S_h \) is assumed to be given. Hereinafter, we set

\[
(w, v) = \int_I x^{N-1}wv \, dx, \quad \|w\|^2 = (w, w) = \int_I x^{N-1}w^2 \, dx,
\]

(8a)

and

\[
A(w, v) = \int_I x^{N-1}w_xv_x \, dx.
\]

(8b)
(Non-Sym) Find $u_h^{n+1} \in S_h$, $n = 0, 1, \ldots$, such that
\[
\langle \partial_t u_h^{n+1}, \chi \rangle + B(u_h^{n+1}, \chi) = \langle f(u_h^n), \chi \rangle \quad (\chi \in S_h, \ n = 0, 1, \ldots),
\] (9)
where
\[
\langle w, v \rangle = \int_I x w v \, dx, \quad |||w|||^2 = \langle w, w \rangle = \int_I x w^2 \, dx,
\]
and
\[
B(w, v) = \int_I x w_x v_x \, dx + (2 - N) \int_I w_x v \, dx.
\] (10a)

It is noteworthy that $B(\cdot, \cdot)$ is coercive in $H^1$ such that
\[
B(w, w) = \langle w_x, w_x \rangle + (2 - N) \int_I w_x w \, dx = |||w_x|||^2 + \frac{N - 2}{2} w(0)^2 \geq |||w|||^2. \quad (11)
\]

3. Well-posedness and positivity conservation

In this section, we will prove the following theorems.

**Theorem 3.1** (Well-posedness of (Sym)). For a given $u_h^n \in S_h$ with $n \geq 0$, the scheme (Sym) admits a unique solution $u_h^{n+1} \in S_h$.

**Theorem 3.2** (Positivity of (Sym)). In addition to the basic assumption \[\text{f1}\], assume that $f$ is a non-decreasing function with $f(0) \geq 0$. \[\text{f2}\]

Let $n \geq 0$ and $u_h^n \geq 0$, and assume that
\[
\tau_n \geq \frac{1}{4} h^2. \quad (12)
\]

Then, the solution $u_h^{n+1}$ of (Sym) satisfies $u_h^{n+1} \geq 0$.

**Theorem 3.3** (Comparison principle for (Sym)). Let $n \geq 0$ and assume that $u_h^n, \tilde{u}_h^n \in S_h$ satisfies $u_h^n \leq \tilde{u}_h^n$ in $I$. Furthermore, we assume that \[\text{f1}\] and \[\text{f2}\] are satisfied. Likewise $u_h^{n+1}, \tilde{u}_h^{n+1} \in S_h$ be the solutions of (Sym) with $u_h^n, \tilde{u}_h^n$, respectively, using the same time increment $\tau_n$. Moreover, assume that (12) is satisfied. Then, we obtain $u_h^{n+1} \leq \tilde{u}_h^{n+1}$ in $I$, and the equality holds true if and only if $u_h^n = \tilde{u}_h^n$ in $I$.

**Theorem 3.4** (Well-posedness of (Non-Sym)). For a given $u_h^n \in S_h$ with $n \geq 0$, the scheme (Non-Sym) admits a unique solution $u_h^{n+1} \in S_h$.

To prove these theorems, we conveniently rewrite (7) into a matrix form. That is, we introduce
\[
M = (\mu_{i,j})_{0 \leq i, j \leq m - 1} \in \mathbb{R}^{m \times m}, \quad \mu_{i,j} = (\phi_j, \phi_i),
\]
\[
A = (a_{i,j})_{0 \leq i, j \leq m - 1} \in \mathbb{R}^{m \times m}, \quad a_{i,j} = A(\phi_j, \phi_i),
\]
\[
u^n = (u^n_j)_{0 \leq j \leq m - 1} \in \mathbb{R}^m, \quad u^n_j = u_h^n(x_j),
\]
\[
F^n = (F^n_j)_{0 \leq j \leq m - 1} \in \mathbb{R}^m, \quad F^n_j = f(u_h^n, \phi_j),
\]
and express (7) as
\[
(M + \tau_n A) u^{n+1} = M u^n + \tau_n F^n \quad (n = 0, 1, \ldots), \quad (13)
\] where $u^n_m = u_h^n(x_m)$ is understood as $u^n_m = 0$.

**Theorem 3.1** is a direct consequence of the following result.
Lemma 3.5. \( M \) and \( A \) are both tri-diagonal and positive-definite matrices.

We state the following proofs.

**Proof of Theorem** 3.2. We use the representative matrix (13) instead of (7) and set
\[
C = (c_{i,j})_{0 \leq i,j \leq m-1} = M + \tau_n A, \quad c_{i,j} = \mu_{i,j} + \tau_n a_{i,j}.
\]
If \( C^{-1} \geq O \), then we get
\[
u^{n+1} = C^{-1} (M \nu^n + \tau_n F^n) \geq 0,
\]
since \( M \geq O \) and \( F^n \geq 0 \) in view of (f2). The proof that \( C^{-1} \geq O \) is true under (12) is divided into three steps, each described as below.

**Step 1.** We show that
\[
\sum_{j=0}^{m-1} c_{i,j} > 0 \quad (0 \leq i \leq m-1).
\]
Letting \( 1 \leq i \leq m - 2 \), we calculate
\[
\sum_{j=0}^{m-1} c_{i,j} = \sum_{j=0}^{i+1} \mu_{i,j} + \tau_n \sum_{j=0}^{i+1} a_{i,j}
\]
\[
= \sum_{j=0}^{i+1} \mu_{i,j} + \tau_n \int_{x_{i-1}}^{x_{i+1}} x^{N-1}(\phi_{i-1} + \phi_i + \phi_{i+1})dx
\]
\[
= \sum_{j=0}^{i+1} \mu_{i,j} > 0,
\]
since \( \phi_{i-1} + \phi_i + \phi_{i+1} \equiv 1 \) in \((x_{i-1}, x_{i+1})\). The cases \( i = 0 \) and \( i = m - 1 \) are verified similarly.

**Step 2.** We show that, if
\[
\tau_n \geq -\frac{\mu_{i,i+1}}{a_{i,i+1}}, \quad -\frac{\mu_{i,i-1}}{a_{i,i-1}} \quad (i = 0, 1, \cdots, m - 1),
\]
then \( C^{-1} \geq O \). First, (15) implies that \( c_{i,i-1}, c_{i,i+1} \leq 0 \) for \( 0 \leq i \leq m - 1 \), since \( a_{i,i-1}, a_{i,i+1} \leq 0 \). Matrix \( C \) is decomposed as \( C = D(I - E) \), where \( D = (d_{i,j})_{0 \leq i,j \leq m-1} \) and \( E = (e_{i,j})_{0 \leq i,j \leq m-1} \) are defined as
\[
d_{i,j} = \begin{cases} c_{i,i} & (i = j) \\ 0 & (i \neq j) \end{cases}, \quad e_{i,j} = \begin{cases} 0 & (i = j) \\ -\frac{c_{i,i}}{c_{i,j}} & (i \neq j) \end{cases},
\]
and \( I \) is the identity matrix. Apparently, \( I - E \) is non-singular and \( D \geq O \). Using (14), we deduce
\[
\|E\|_{\infty} = \max_{0 \leq i \leq m-1} \left(-\frac{c_{i,i-1}}{c_{i,i}}, \frac{c_{i,i+1}}{c_{i,i}}\right) < 1.
\]
Therefore, matrix \( I - E \) is non-singular and \( (I - E)^{-1} = \sum_{k=0}^{\infty} E^k \geq O \). Consequently, we have \( C^{-1} = (I - E)^{-1} D^{-1} \geq O \).

**Step 3.** Finally, we show that (12) implies (15). We calculate
\[
\mu_{i,i+1} = \int_{x_i}^{x_{i+1}} x^{N-1} \frac{1}{h_{i+1}^2}(x - x_i)(x_{i+1} - x) \, dx \leq \frac{1}{4} h_{i+1}^2 \int_{x_i}^{x_{i+1}} x^{N-1} \, dx,
\]
\[
-a_{i,i+1} = \int_{x_i}^{x_{i+1}} x^{N-1} \frac{1}{h_{i+1}^2} \, dx.
\]
Therefore, we deduce
\[
\frac{\mu_{i,i+1}}{a_{i,i+1}} \leq \frac{1}{4} h_{i+1}^2. \tag{\text{□}}
\]
Proof of Theorem 3.3. Since \( f(\tilde{u}_h^n) - f(u_h^n) \geq 0 \) in \( I \), the proof follows exactly the same manner as in the proof of Proposition 3.2.

Hence, we proceed to the result for (Non-Sym):

\[
\mathcal{M}' = (\mu_{i,j}')_{0 \leq i, j \leq m-1} \in \mathbb{R}^{m \times m}, \quad \mu_{i,j}' = \langle \phi_j, \phi_i \rangle, \\
\mathcal{B} = (b_{i,j})_{0 \leq i, j \leq m-1} \in \mathbb{R}^{m \times m}, \quad b_{i,j} = B(\phi_j, \phi_i), \\
\mathcal{G}^n = (G^n_j)_{0 \leq j \leq m-1} \in \mathbb{R}^m, \quad G^n_j = \langle f(u_h^n), \phi_j \rangle,
\]

and express (9) as

\[
(\mathcal{M}' + \tau_n \mathcal{B})u^{n+1} = \mathcal{M}'u^n + \tau_n \mathcal{G}^n \quad (n = 0, 1, \ldots).
\]

In view of (11), we deduce the following result.

Lemma 3.6. \( \mathcal{M}' \) and \( \mathcal{B} \) are both tri-diagonal and positive-definite matrices.

4. Convergence and error analysis

4.1. Results

Our convergence results for (Sym) and (Non-Sym) are stated under a smoothness assumption on the solution \( u \) of (1). That is, given \( T > 0 \) and setting \( Q_T = [0,1] \times [0,T] \), we suppose that \( u \) is sufficiently smooth such that

\[
\kappa_\nu(u) = \sum_{j=0}^{2} \|\partial_x^j u\|_{L^\infty(Q_T)} + \sum_{j=1}^{2+\nu} \|\partial_t^j u\|_{L^\infty(Q_T)} + \sum_{k=1}^{1+\nu} \|\partial_t^k \partial_x^2 u\|_{L^\infty(Q_T)} < \infty,
\]

where \( \nu \) is either 0 or 1.

The partition \( \{x_i\}_{i=0}^m \) of \( \bar{I} = [0,1] \) is assumed to be quasi uniform, in which there exists a positive constant \( \beta \) independent of \( h \) such that

\[
h \leq \beta \min_{1 \leq j \leq m} h_j.
\]

Finally, the approximate initial value \( u_h^0 \) is chosen as

\[
\|u_h^0 - u^0\| \leq C_0 h^2
\]

for a positive constant \( C_0 \).

Moreover, for \( k = 1, 2, \ldots \), we express the positive constans \( C_k = C_k(\gamma_1, \gamma_2, \ldots) \) and \( h_k = h_k(\gamma_1, \gamma_2, \ldots) \) according to the parameters \( \gamma_1, \gamma_2, \ldots \). In particular, \( C_k \) and \( h_k \) are independent of \( h \) and \( \tau \).

Now, we will state the following theorems.

Theorem 4.1 (Convergence for (Sym) in \( \| \cdot \|, I \)). Assume that \( f \) is a globally Lipschitz continuous function; assume (11) and

\[
M = \sup_{\mu > 0} M_\mu < \infty.
\]
Suppose that, for \( T > 0 \), the solution \( u \) of (1) is sufficiently smooth so that (17) for \( \nu = 0 \) holds true. Moreover, assume that (18) and (19) are satisfied. Then, there exists an \( h_1 = h_1(N, \beta) \) such that, for any \( h \leq h_1 \), we have

\[
\sup_{0 \leq t_n \leq T} \| u^n_h - u(\cdot, t_n) \| \leq C_1(h^2 + \tau),
\]

where \( C_1 = C_1(T, M, \kappa_0(u), C_0, N, \beta) \) and \( u^n_h \) is the solution of (Sym).

For the \( L^\infty \) error estimates, we need to further assume that \( u^0_h \) is chosen as

\[
A(u^n_h - u^0, v_h) = 0 \quad (v_h \in S_h). \tag{20}
\]

**Theorem 4.2** (Convergence for (Sym) in \( \| \cdot \|_{L^\infty(\sigma, 1)} \), I). In addition to the assumption of Theorem 4.1, assume that (20) is satisfied. Furthermore, let \( \sigma \in (0, 1) \) be arbitrary. Then, there exists an \( h_2 = h_2(N, \beta) \) such that, for any \( h \leq h_2 \), we have

\[
\sup_{0 \leq t_n \leq T} \| u^n_h - u(\cdot, t_n) \|_{L^\infty(\sigma, 1)} \leq C_2 \left( h^2 \log \frac{1}{h} + \tau \right),
\]

where \( C_2 = C_2(T, M, \kappa_0(u), C_0, N, \beta, \sigma) \) and \( u^n_h \) is the solution of (Sym).

The restriction that \( f \) is a globally Lipschitz continuous function with (f3) can be removed in the manner described as follows.

**Theorem 4.3** (Convergence of (Sym) in \( \| \cdot \|_{L^\infty(\sigma, 1)} \), II). Given \( T > 0 \) and only (11) is satisfied, we suppose that (17) with \( \nu = 0 \), (18), and (19) are satisfied. Furthermore, assume that \( N \leq 3 \) and that there exist positive constants \( c_1 \) and \( \sigma \) such that

\[
\tau h^{-N/2} \leq c_1 h^\sigma. \tag{21}
\]

Then, there exists an \( h_3 = h_3(T, \kappa_0(u), C_0, N, \beta) \) such that, for any \( h \leq h_3 \), we have

\[
\sup_{0 \leq t_n \leq T} \| u^n_h - u(\cdot, t_n) \| \leq C_2(h^2 + \tau),
\]

where \( C_3 = C_3(T, \kappa_0(u), C_0, N, \beta) \) and \( u^n_h \) is the solution of (Sym).

**Theorem 4.4** (Convergence for (Sym) in \( \| \cdot \|_{L^\infty(\sigma, 1)} \), II). Given \( T > 0 \) and (11) is satisfied, we suppose that (17) with \( \nu = 0 \), (18), (19), (20) and (21) are satisfied. Then, there exists an \( h_4 = h_4(T, \kappa_0(u), C_0, N, \beta) \) such that, for any \( h \leq h_4 \), we have

\[
\sup_{0 \leq t_n \leq T} \| u^n_h - u(\cdot, t_n) \|_{L^\infty(\sigma, 1)} \leq C_4 \left( h^2 \log \frac{1}{h} + \tau \right),
\]

where \( C_4 = C_4(T, \kappa_0(u), C_0, N, \beta) \) and \( u^n_h \) is the solution of (Sym).

Subsequently, let us proceed to the error estimates for (Non-Sym). For the approximate initial value \( u^0_h \), we choose

\[
B(u^n_h - u^0, v_h) = 0 \quad (v_h \in S_h). \tag{22}
\]

Quasi-uniformity is required for the time partition; therefore, there exists a positive constant \( \gamma > 0 \) such that

\[
\tau \leq \gamma \tau_{\min} \tag{23}
\]

where \( \tau_{\min} = \min_{n \geq 0} \tau_n \). Moreover, we set

\[
\delta = \sup_{t_k \in [0, T]} |\tau_k - \tau_{k+1}|. \tag{24}
\]
Theorem 4.5 (Convergence for (Non-Sym), I). Let $f$ be a $C^1$ function satisfying

$$M_1 = \sup_{s \in \mathbb{R}} |f'(s)| < \infty, \quad M_2 = \sup_{s \neq s' \in \mathbb{R}} \frac{|f'(s) - f'(s')|}{|s - s'|} < \infty.$$  \hfill (f4)

Given $T > 0$, we suppose that the solution $u$ of (1) is sufficiently smooth such that (17) for $\nu = 1$ holds true. Furthermore, we assume that (18), (22) and (23) are satisfied. Then, there exists an $h_5 = h_5(T, \kappa_1(u), M_1, M_2, \gamma, N, \beta)$ such that, for any $h \leq h_5$, we have

$$\sup_{0 \leq t_n \leq T} \|u_h^n - u(\cdot, t_n)\|_{L^\infty(I)} \leq C_5 \left( \log \frac{1}{h} \right)^{\frac{2}{3}} \left( h^2 + \tau + \frac{\delta}{\tau_{\min}} \right),$$

where $C_5 = C_5(T, \kappa_1(u), M_1, M_2, \gamma, N, \beta) > 0$ and $u_h^n$ is the solution of (Non-Sym).

Finally, we state the error estimates for non-globally Lipschitz continuous function $f$. To avoid unessential complexity, we only deal with the power nonlinearity $f(s) = s|s|^\alpha$.

Theorem 4.6 (Convergence for (Non-Sym), II). Let $f(s) = s|s|^\alpha$ for $s \in \mathbb{R}$, where $\alpha \geq 1$. Given $T > 0$, we suppose that (17) with $\nu = 1$, (18), (22) and (23) are satisfied. Then, there exists an $h_6 = h_6(T, \kappa_1(u), \gamma, N, \beta)$ such that, for any $h \leq h_6$, we have

$$\sup_{0 \leq t_n \leq T} \|u_h^n - u(\cdot, t_n)\|_{L^\infty(I)} \leq C_6 \left( \log \frac{1}{h} \right)^{\frac{2}{3}} (h^2 + \tau),$$

where $C_6 = C_6(T, \kappa_1(u), \gamma, N, \beta)$ and $u_h^n$ is the solution of (Non-Sym).

4.2. Proof of Theorems 4.1 and 4.2

We use the projection operator $P_A$ of $\dot{H}^1 \to S_h$ associated with $A(\cdot, \cdot)$, defined for $w \in \dot{H}^1$ as

$$P_A w \in S_h, \quad A(P_A w - w, \chi) = 0 \quad (\chi \in S_h). \hfill (25)$$

In [3] and [6], the error estimates that follow are proved.

Lemma 4.7. Letting $w \in C^2(\bar{I}) \cap \dot{H}^1$, and (18) be satisfied, for $h \leq h_7 = h_7(N, \beta)$, we obtain

$$\|P_A w - w\| \leq C h^2 \|w_{xx}\|, \hfill (26)$$

$$\|P_A w - w\|_{L^\infty(I)} \leq C \left( \log \frac{1}{h} \right) h^2 \|w_{xx}\|_{L^\infty(I)}, \hfill (27)$$

where $C$ is a positive constant depending only on $N$ and $\beta$.

Proof of Theorem 4.1. Using $P_A u$, we distribute the error in the form

$$u_h^n - u(t_n) = (u_h^n - P_A u(t_n)) + (P_A u(t_n) - u(t_n)) = \rho_h^n.$$  \hfill (28)

We know from (26) that

$$\|ho_h^n\| \leq C h^2 \|u_{xx}(t_n)\| \leq C h^2 \|u_{xx}\|_{L^\infty(Q_T)}, \hfill (29)$$

where $C$ is a positive constant.
Now, we derive the estimation for $\theta^n$. By considering the symmetric weak form at $t = t_{n+1}$, we get

$$(\partial_{\tau_n}u(t_{n+1}), \chi) + A(P_Au(t_{n+1}), \chi) = (f(u(t_{n})), \chi) + (f(u(t_{n+1})) - f(u(t_{n})), \chi) + (\partial_{\tau_n}u(t_{n+1}) - u(t_{n+1}), \chi)$$

which, together with (7), implies that

$$(\partial_{\tau_n}\theta^{n+1}, \chi) + A(\theta^{n+1}, \chi) = (f(u^n_h) - f(u(t_{n})), \chi) - (f(u(t_{n+1})) - f(u(t_{n})), \chi) - (\partial_{\tau_n}u(t_{n+1}) - u(t_{n+1}), \chi) - (\partial_{\tau_n}\rho^{n+1}, \chi). \quad (29)$$

Substituting this for $\chi = \theta^{n+1}$, we obtain

$$\frac{1}{\tau_n} \left\{ \|\theta^{n+1}\|^2 - \|\theta^n\| \cdot \|\theta^{n+1}\| \right\} \leq M\|\theta^n + \rho^n\| \cdot \|\theta^{n+1}\|
+ M\tau_n\|u_t\|_{L^\infty(Q_T)} \cdot \|\theta^{n+1}\| + C\tau_n\|u_{tt}\|_{L^\infty(Q_T)} \cdot \|\theta^{n+1}\| + \|\partial_{\tau_n}\rho^{n+1}\| \cdot \|\theta^{n+1}\|.$$ 

Correspondingly, since

$$\partial_{\tau_n}\rho^{n+1} = P_A \left( \frac{u(t_{n+1}) - u(t_{n})}{\tau_n} \right) - \frac{u(t_{n+1}) - u(t_{n})}{\tau_n},$$

we provide an estimate

$$\|\partial_{\tau_n}\rho^{n+1}\| \leq Ch^2 \left\| \frac{u_{xx}(t_{n+1}) - u_{xx}(t_{n})}{\tau_n} \right\| \leq Ch^2\|u_{xxt}\|_{L^\infty(Q_T)}. \quad (30)$$

To sum up, we obtain

$$\|\theta^{n+1}\| - \|\theta^n\| \leq \tau_n M\|\theta^n\| + Ch^2 M\tau_n + CM\tau^2 + C\tau_n^2 + Ch^2\tau_n$$

and, therefore,

$$\|\theta^n\| \leq e^{MT} \|u^n_h - P_Au^0\| + C e^{MT} \frac{1}{M}(\tau + h^2)
\leq e^{MT}(\|u^n_h - u^0\| + \|u^0 - P_Au^0\|) + C e^{MT} \frac{1}{M}(\tau + h^2)
\leq C'(\tau + h^2), \quad (31)$$

where $C' = C'(T, \kappa_0(u), M, N, \beta, C_0) > 0$. Combining this expression with (28), we deduce the desired error estimate. \hfill \square

**Proof of Theorem 4.3.** We use the same error decomposition process as in the previous proof where $u^n_h - u(t_n) = \theta^n + \rho^n$, and apply (27) to estimate $\|\rho^n\|_{L^\infty(I)}$. Since

$$\|\theta^n\|_{L^\infty(\sigma, I)} \leq \|\theta^n_x\|_{L^1(\sigma, I)} \leq C(\sigma, N)\|\theta^n_x\|,$$ 

we perform an estimation for $\|\theta^n_x\|$. Substituting (29) for $\chi = \partial_{\tau_n}\theta^{n+1}$, we have

$$\|\partial_{\tau_n}\theta^{n+1}\|^2 + A(\theta^{n+1}, \partial_{\tau_n}\theta^{n+1}) \leq M\|\theta^n\| \cdot \|\partial_{\tau_n}\theta^{n+1}\|
+ M\|\rho^n\| \cdot \|\partial_{\tau_n}\theta^{n+1}\| + M\tau_n\|u_t\|_{L^\infty(Q_T)} \cdot \|\partial_{\tau_n}\theta^{n+1}\|
+ \|u_{tt}\|_{L^\infty(Q_T)}\tau_n \|\partial_{\tau_n}\theta^{n+1}\| + \|\partial_{\tau_n}\rho^{n+1}\| \cdot \|\partial_{\tau_n}\theta^{n+1}\|.$$
Correspondingly, we apply an elementary identity

\[
A(\theta^{n+1}, \partial_\tau \theta^{n+1}) = \frac{1}{2} A(\theta^{n+1} - \theta^n + \theta^{n+1}, \partial_\tau \theta^{n+1}) \\
\geq \frac{1}{2 \tau_n} \left[ A(\theta^{n+1}, \theta^{n+1}) - A(\theta^n, \theta^n) \right]
\]

along with Young’s inequality, to obtain

\[
\frac{1}{2 \tau_n} \left[ A(\theta^{n+1}, \theta^{n+1}) - A(\theta^n, \theta^n) \right] \leq \frac{1}{2} \frac{M^2}{\delta_0^2} \|\theta^n\|^2 + \frac{1}{2} \frac{\delta_0^2}{\delta_0^2} \|\partial_\tau \theta^{n+1}\|^2 \\
+ \frac{1}{2} \frac{M^2}{\delta_1^2} \|\rho^n\|^2 + \frac{1}{2} \delta_1^2 \|\partial_\tau \theta^{n+1}\|^2 \\
+ \frac{1}{2} \frac{C^2}{\delta_2^2} \tau_n^2 + \frac{1}{2} \delta_2^2 \|\partial_\tau \theta^{n+1}\|^2 \\
+ \frac{1}{2} \|\partial_\tau \rho^{n+1}\|^2 + \frac{1}{2} \|\partial_\tau \theta^{n+1}\|^2 - \|\partial_\tau \theta^{n+1}\|^2,
\]

where \(\delta_0, \delta_1, \delta_2 > 0\) are constants. After setting \(\delta_0^2 + \delta_1^2 + \delta_2^2 = 1\), we get

\[
A(\theta^{n+1}, \theta^{n+1}) - A(\theta^n, \theta^n) \leq \tau_n \left[ \frac{C^2}{\delta_0^2} \|\theta^n\|^2 + \frac{C^2}{\delta_1^2} \|\rho^n\|^2 + \|\partial_\tau \rho^{n+1}\|^2 + \frac{C^2}{\delta_2^2} \tau_n^2 \right].
\]

Therefore,

\[
A(\theta^n, \theta^n) \leq A(\theta^0, \theta^0) + C^2 \tau_n \sup_{1 \leq k \leq n} \left[ \|\theta^{k-1}\|^2 + \|\rho^{k-1}\|^2 + \|\partial_\tau \rho^{k}\|^2 + \tau_n^2 \right].
\]

Consequently, using (20), (30), and (31), we deduce

\[
\|\theta^n\| \leq C t_n \left( \tau + h^2 \right).
\]

This, together with (27) and (32), implies the desired estimate. \(\square\)

4.3. Proof of Theorems 4.3 and 4.4

For the proof, we utilize the inverse inequality that follows.

**Lemma 4.8** (Inverse inequality). Under condition (18),

\[
\|v_h\|_{L^\infty(I)} \leq C^* h^{-\frac{N}{2}} \|v_h\| \quad (v_h \in S_h),
\]

where \(C^*\) is a positive constant depending only on \(N\) and \(\beta\).

**Proof.** Let \(v_h \in S_h\) be arbitrary. From the norm equivalence in \(\mathbb{R}^2\), we know that

\[
\|v_h\|_{L^\infty(I)} \leq C_{**} h_1^{-1/2} \|v_h\|_{L^2(I)} ,
\]

\[
\|v_h\|_{L^\infty(I_j)} \leq C_{**} h_j^{-1/2} \|v_h\|_{L^2(I_j)} \quad (j = 2, \ldots, m),
\]

10
where $C_{**}$ denotes the absolute positive constant. Given that $\|v_h\|_{L^\infty(I)} = \|v_h\|_{L^\infty(I_1)}$, the expression can be calculated as

$$\|v_h\|_{L^\infty(I_1)} \leq C_{**} h_1^{-1} \int_{h_1/2}^{h_1} x^{-(N-1)} x^{N-1} v_h^2 \, dx$$

$$\leq C_{**} h_1^{-1} \left( \frac{h_1}{2} \right)^{(N-1)} \int_{h_1/2}^{h_1} x^{N-1} v_h^2 \, dx$$

$$\leq C_{**} 2^{N-1} h^{-N} \left( \frac{h_1}{h} \right)^{-N} \int_{h_1/2}^{h_1} x^{N-1} v_h^2 \, dx$$

$$\leq C_{**} 2^N \|v_h\|^2.$$ 

The case $\|v_h\|_{L^\infty(I)} = \|v_h\|_{L^\infty(I_j)}$ with $j = 2, \ldots, m$ is examined similarly. □

**Proof of Theorem 4.3** Consider (1) and (Sym) with replacement $f(s)$ in

$$\tilde{f}(s) = \begin{cases} f(\mu) & (s \geq \mu) \\ f(s) & (-\mu \leq s \leq \mu) \\ f(-\mu) & (s \leq -\mu), \end{cases}$$

where $\mu > 0$ is determined later. Then, $\tilde{f}$ satisfies condition (3) in Theorem 4.1 such that

$$\sup_{s, s' \in \mathbb{R}, s \neq s'} \frac{|\tilde{f}(s) - \tilde{f}(s')|}{|s - s'|} \leq M \equiv \sup_{|\lambda| \leq \mu} M_\lambda < \infty.$$

Let $\tilde{u}$ and $\tilde{u}_h^n$ be the solutions of (1) and (Sym) with $\tilde{f}$, respectively, such that

$$\|\tilde{u}_h^n\|_{L^\infty(I)} \leq \|\theta^n\|_{L^\infty(I)} + \|P_A \tilde{u}(t_n)\|_{L^\infty(I)},$$

where $\theta^n = \tilde{u}_h^n - P_A \tilde{u}(t_n)$ and $\rho^n = P_A \tilde{u}(t_n) - \tilde{u}(t_n)$. Applying Theorem 4.1 to $\tilde{u}$ and $\tilde{u}_h^n$, we obtain

$$\sup_{0 \leq t_n \leq T} \|\tilde{u}_h^n - \tilde{u}(t_n)\| \leq C_2 (h^2 + \tau), \quad (33)$$

where $C_2 = C_2(T, \kappa_0(\tilde{u}), \mu, C_0, N, \beta)$. Moreover, estimate (31) for $\theta^n$ is available. In view of Lemmas 4.7 and 4.8 we determine the estimates

$$\|\theta^n\|_{L^\infty(I)} \leq C_3 \tilde{h}^{-\frac{N}{2}} \|\theta^n\| \leq C_3 \tilde{h}^{-\frac{N}{2}} (h^2 + \tau),$$

$$\|\rho\|_{L^\infty(I)} \leq C_4 \left( h^2 \log \frac{1}{h} \right) \|\tilde{u}_{xx}(t_n)\|_{L^\infty(I)},$$

where $C_3 = C_3(T, \kappa_0(\tilde{u}), \mu, C_0, N, \beta)$ and $C_4 = C_4(N, \beta)$. Therefore, we have

$$\|P_A \tilde{u}(t_n)\|_{L^\infty(I)} \leq \|\tilde{u}(t_n)\|_{L^\infty(I)} + C_5 \left( h^2 \log \frac{1}{h} \right) \|\tilde{u}_{xx}(t_n)\|_{L^\infty(I)}$$

and

$$\|\tilde{u}_h^n\|_{L^\infty(I)} \leq C_3 (h^2 - \frac{N}{2} + h^{-\frac{N}{2}} \tau) + \|\tilde{u}(t_n)\|_{L^\infty(I)} + C_4 \left( h^2 \log \frac{1}{h} \right) \|\tilde{u}_{xx}(t_n)\|_{L^\infty(I)}.$$
At this stage, we set \( \mu = 1 + \|u\|_{L^\infty(Q_T)} \) to obtain \( u = \tilde{u} \) in \( Q_T \) by uniqueness. Moreover, since \( N < 4 \), we can take a very small \( h \) such that

\[
C_6(h^{2 - \frac{N}{2}} + h^{- \frac{N}{2}}) \leq \frac{1}{2}, \quad C_5\left(h^2 \log \frac{1}{h}\right) \|u_{xx}(t_n)\|_{L^\infty(I)} \leq \frac{1}{2}.
\]

Consequently, \( \|\tilde{u}_h^n\|_{L^\infty(I)} \leq \mu \) and, by the uniqueness, \( u_h^n = \tilde{u}_h^n \). Therefore, (33) implies the desired conclusion.

Proof of Theorem 4.4. The proof follows the exact same manner as for Theorem 4.3 using Theorem 4.2 instead of Theorem 4.1.

4.4. Proof of Theorems 4.5 and 4.6

We use the projection operator \( P_B \) of \( \dot{H}^1 \rightarrow S_h \) associated with \( B(\cdot, \cdot) \):

\[
B(P_B w - w, \chi) = 0 \quad (\chi \in S_h).
\] (34)

In [3], the following error estimates are proved.

Lemma 4.9. Letting \( w \in C^2(\bar{I}) \cap \dot{H}^1 \) and (18) is satisfied, for \( h \leq h_8 = h_8(N, \beta) \) we obtain

\[
\|P_B w - w\|_{L^\infty(I)} \leq C_8 h^2 \|w_{xx}\|_{L^\infty(I)},
\] (35)

where \( C_8 = C_8(N, \beta) \).

We also use a version of Poincaré’s inequality (see [8, Lemma 18.1]).

Lemma 4.10. We have

\[
\|||w||| \leq |||w_x||| \quad (w \in \dot{H}(I)).
\] (36)

We now can state the proof that follows.

Proof of Theorem 4.5. Using \( P_B u(t) \in S_h \), we decompose the error into

\[
u_h^n - u(t_n) = (u_h^n - P_B u(t_n)) + (P_B u(t_n) - u(t_n)).
\]

We know from (35) that

\[
\|||\rho^n||| \leq |||\rho^n|||_{L^\infty(I)} \leq C h^2 \|w_{xx}\|_{L^\infty(Q_T)},
\] (37a)

\[
\|||\partial_r \rho^{n+1}||| \leq |||\partial_r \rho^{n+1}|||_{L^\infty(I)} \leq C h^2 \|w_{xx}\|_{L^\infty(Q_T)}.
\] (37b)

Hence, we will focus on estimating \( |||\theta^n_x||| \), since we are aware that

\[
|||\chi|||_{L^\infty(I)} \leq |||\chi_x|||_{L^1(I)} \leq C \left(\log \frac{1}{h}\right)^{\frac{1}{2}} |||\chi_x||| \quad (\chi \in S_h).
\]

Furthermore, [5] and [9] give

\[
\langle \partial_r \theta^{n+1} + \partial_r \rho^{n+1}, \chi \rangle + B(\theta^{n+1}, \chi) = \langle f(u_h^n) - f(u(t_n)), \chi \rangle - \langle f(u(t_{n+1})) - f(u(t_n)), \chi \rangle - \langle \partial_r u(t_{n+1}) - u(t_{n+1}), \chi \rangle
\] (38)
for $\chi \in S_h$. Substituting this for $\chi = \theta^{n+1}$, we have

$$\langle \partial_{t_n} \theta^{n+1}, \theta^{n+1} \rangle + B(\theta^{n+1}, \theta^{n+1}) = \langle f(u^n_h) - f(u(t_n)), \theta^{n+1} \rangle - \langle f(u(t_{n+1})) - f(u(t_n)), \theta^{n+1} \rangle$$

$$- \langle \partial_{t_n} u(t_{n+1}) - u(t_{n+1}), \theta^{n+1} \rangle - \langle \partial_{t_n} \rho^{n+1}, \theta^{n+1} \rangle.$$ (39)

This, together with (11), implies that

$$\tau_{n+1} < \tau_n$$ holds true by uniqueness. Hence, we can apply Theorem 4.5 to obtain

$$\tau_{n+1} = \tau_n + \frac{1}{\alpha}.$$ (41a)

These estimates actually hold; nevertheless, their proof will be postponed for Appendix A.

$$\langle \partial_{t_n} \theta^{n+1}, \theta^{n+1} \rangle \leq M|||u^n_h - u(t_n)||| \cdot |||\theta^{n+1}||| + M|||u(t_{n+1}) - u(t_n)||| \cdot |||\theta^{n+1}|||$$

$$+ |||\partial_{t_n} u(t_{n+1}) - u(t_{n+1})||| + |||\partial_{t_n} \theta^{n+1}|||,$$ (39)

These estimates actually hold; nevertheless, their proof will be postponed for Appendix A.

$$|||\theta^{n+1}||| \leq C(h^2 + \tau),$$ (41a)

$$|||\partial_{t_n} \theta^{n+1}||| \leq C(h^2 + \tau + \frac{\delta}{\tau}).$$ (41b)

Using (37a), (37b), (41a), and (41b), we deduce

$$|||\theta^{n+1}||| \leq C(h^2 + \tau + \frac{\delta}{\tau}).$$

which completes the proof of Theorem 4.5.

Finally, we state the subsequent proof.

**Proof of Theorem 4.6.** Consider problems (1) and (9) with replacement $f(s) = s|s|^\alpha$ by

$$\tilde{f}(s) = \begin{cases} s|s|^\alpha & (|s| \leq \mu) \\ [(1 + \alpha)\mu^\alpha s - \alpha \mu^{1+\alpha}] \text{sgn}(s) & (|s| \geq \mu), \end{cases}$$

where $\mu > 0$ is determined later. Then, $\tilde{f}$ is a $C^1$ function and the corresponding values of $\tilde{M}_1$ and $\tilde{M}_2$ in (4) are expressed as $\tilde{M}_1 = (1 + \alpha)\mu^\alpha$ and $\tilde{M}_2 = (1 + \alpha)\alpha \mu^{\alpha-1}$.

Now, let $\tilde{u}$ and $\tilde{u}^n_h$ be the solutions of (1) and (9) with $\tilde{f}$, respectively. If $\mu \geq \kappa_1(u)$, then $u = \tilde{u}$ holds true by uniqueness. Hence, we can apply Theorem 4.5 to obtain

$$|||\tilde{u}^n_h - u(t_n)|||_{L^\infty(I)} \leq C \left( \log \frac{1}{h} \right)^\frac{1}{2} (h^2 + \tau),$$ (42)

where $C = C(T, \kappa_1(u), \gamma, N, \beta)$. At this juncture, we apply small $h$ and $\tau$ such that $C \left( \log \frac{1}{h} \right)^\frac{1}{2} (h^2 + \tau) < 1$, and set $\mu = \kappa_1(u) + 1$. As $|||\tilde{u}^n_h|||_{L^\infty(I)} \leq \kappa_1(u) + 1 = \mu$, we obtain $\tilde{u}^n_h = u^n_h$ by the uniqueness theorem. Therefore, (42) implies the desired estimate. 

\[\square\]
5. Blow-up analysis

Throughout this section, we set
\[ f(s) = s|s|^{\alpha}, \quad \alpha > 0. \]

Generally, the solution of (1) blows up for a sufficiently large initial data \( u_0 \), and the blow up is controlled by the energy functional associated with (1). Herein, we study whether or not the numerical solution behaves similarly by initially defining some properties of the solution \( u \) of (1).

We define the energy functional associated with (1) as
\[ J(t) = \frac{1}{2}\|u_x\|^2 - \frac{1}{\alpha + 2} \int_I x^{N-1}|u|^{\alpha+2} \, dx. \]

Propositions 5.1 and 5.2 are well-known for the semilinear heat equation in a bounded domain. Herein, we state the proofs, since we can find no explicit reference to the radially symmetric case.

**Proposition 5.1.** \( J(t) \) is a non-increasing function of \( t \).

**Proof.** Substituting \( \chi = u_t \) for (4), we get
\[ \|u_t\|^2 = -\frac{d}{dt} \int_I \frac{1}{2} x^{N-1} u^2 x + \frac{d}{dt} \int_I \frac{1}{\alpha + 2} x^{N-1}|u|^{\alpha+2} \, dx = -\frac{d}{dt} J(t). \]  \hspace{1cm} (43)

**Proposition 5.2.** Suppose that \( u^0(x) \geq 0, \neq 0 \) for \( x \in I \). Then, the following statements are equivalent:

(i) There exists a certain \( T > 0 \) such that \( u \) blows up at \( t = T \) as
\[ \lim_{t \to T} \|u(t)\| = \infty. \]  \hspace{1cm} (44)

(ii) There exists \( t_0 \geq 0 \) such that \( J(t_0) < 0 \).

**Remark 5.3.** Blow up time \( T \) is estimated by
\[ T \leq t_0 + \frac{\alpha + 2}{\alpha^2} \|u(t_0)\|^{-\alpha}. \]

**Proof.** We initially assume (i). Using (43), we calculate
\[ \|u(t)\|^2 = \int_I x^{N-1} \left| \int_0^t \frac{\partial u}{\partial t}(x,s) \, ds + u(0,x) \right|^2 \, dx \]
\[ \leq 2 \int_I x^{N-1} \left[ \int_0^t \left| \frac{\partial u}{\partial t}(x,s) \right|^2 + |u(0,x)|^2 \right] \, dx \]
\[ \leq 2 \int_I x^{N-1} \left[ \left( \int_0^t ds \right) \left( \int_0^t \left( \frac{\partial u}{\partial t}(x,s) \right)^2 \, ds + |u(0,x)|^2 \right) \right] \, dx \]
\[ = 2t \int_0^t \|u_t\|^2 \, ds + 2\|u_0\|^2 \]
\[ = 2t \int_0^t -\frac{d}{dt} J(s) \, ds + 2\|u^0\|^2 \]
\[ = 2t [-J(t) + J(0)] + 2\|u^0\|^2. \]
The left-hand side blows up at $t = T$; therefore, there should be a certain $0 < t_0 < T$ such that $J(t) < 0$ for $t > t_0$.

On the contrary, we assume (ii). Without the loss of generality, $t_0 = 0$. Substituting $\chi = u$ for (4), we obtain

$$\frac{1}{2} \frac{d}{dt} \|u\|^2 = -\|u_x\|^2 + \int_I x^{N-1} |u|^\alpha + 2 \, dx. \tag{45}$$

Applying (ii) and the Hölder’s inequality, we deduce

$$\frac{1}{2} \frac{d}{dt} \|u\|^2 \geq -\frac{2\alpha}{\alpha + 2} \int_I x^{N-1} |u|^\alpha + 2 \, dx \geq -\frac{\alpha}{\alpha + 2} \left( \int_I x^{N-1} \|u\|^2 \right)^\frac{\alpha + 2}{2}. \tag{46}$$

At this point, we see that $y(t)$ solves the differential inequality after setting $y(t) = \|u(t)\|^2$,

$$\frac{dy(t)}{dt} \geq \frac{2\alpha}{\alpha + 2} y(t)^\frac{\alpha + 2}{2}, \quad y(t) \geq 0 \quad (t \geq 0).$$

Therefore, the solution $u$ cannot exist beyond the time

$$T \leq t_0 + \frac{\alpha + 2}{\alpha^2} \|u(t_0)\|^{-\alpha}. \tag{47}$$

For the result that follows, let us consider the discrete energy functional

$$J_h(n) = \frac{1}{2} \|(u^n_h)_x\|^2 - \frac{1}{\alpha + 2} \int_I x^{N-1} |u^n_h|^\alpha + 2 \, dx$$

for the solution $u^n_h$ of (Sym).

**Proposition 5.4.** $J_h(n)$ is a non-increase sequence of $n$.

**Proof.** Substituting $\chi = \partial_{\tau_n} u^{n+1}_h$ for (7), we have

$$\|\partial_{\tau_n} u^{n+1}_h\|_h^2 = -\left( (u^{n+1}_h)_x, (u^{n+1}_h - u^n_h)_x \right) + \left( u^n_h |u^n_h|^{\alpha}, \frac{u^{n+1}_h - u^n_h}{\tau_n} \right).$$

Therefore, for the conditions

$$\left( (u^{n+1}_h)_x, (u^{n+1}_h - u^n_h)_x \right) \geq \frac{1}{2} \left( (u^n_h)_x + (u^{n+1}_h)_x, \frac{u^{n+1}_h - u^n_h}{\tau_n} \right), \tag{46a}$$

$$\left( u^n_h |u^n_h|^{\alpha}, \frac{u^{n+1}_h - u^n_h}{\tau_n} \right) \leq \frac{1}{\tau_n (\alpha + 2)} \left[ \int_I x^{N-1} (|u^{n+1}_h|^\alpha + 2 - |u^n_h|^\alpha + 2) \, dx \right], \tag{46b}$$

we obtain

$$\|\partial_{\tau_n} u^{n+1}_h\|_h^2 \leq -\frac{1}{\tau_n} (J_h(n + 1) - J_h(n)), \tag{47}$$

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which implies that \( J_h(n+1) \leq J_h(n) \).

We can validate (46a) and (46b); (46a) is derived readily. To prove (46b), we set \( g(s) = \frac{1}{\alpha+2} |s|^{\alpha+2} \), and apply the mean value theorem to deduce

\[
g(u_h^{n+1}) - g(u_h^n) = w |w|^{\alpha} \frac{(u_h^{n+1} - u_h^n)}{\tau_n},
\]

where \( w = w(x) = u_h^n + \sigma(u_h^{n+1} - u_h^n) \) and \( \sigma = \sigma(x) \in (0,1) \). Consequently,

\[
J \equiv \frac{1}{\tau_n(\alpha+2)} \left[ \int_I x^{N-1}(|u_h^{n+1}|^{\alpha+2} - |u_h^n|^{\alpha+2}) \, dx \right] - \int_I x^{N-1} u_h^n |u_h^{n+1} - u_h^n| \, dx
\]

\[
= \frac{1}{\tau_n} \int_I x^{N-1} [w|w|^{\alpha} - u_h^n |u_h^{n+1} - u_h^n|] \, dx.
\]

We repeat the mean value theorem to resolve

\[
w|w|^{\alpha} - u_h^n |u_h^{n+1}|^{\alpha} = (\alpha+1)|\tilde{w}|^{\alpha}(w - u_h^n) = (\alpha+1)|\tilde{w}|^{\alpha}\tilde{\sigma}(u_h^{n+1} - u_h^n),
\]

where \( \tilde{w} = u_h^n + \tilde{\sigma}(w - u_h^n) \) and \( \tilde{\sigma} = \tilde{\sigma}(x) \in (0,1) \). Therefore,

\[
J = \frac{1}{\tau_n} \int_I x^{N-1}(\alpha+1)|\tilde{w}|^{\alpha}\tilde{\sigma}(u_h^{n+1} - u_h^n)^2 \, dx \geq 0,
\]

which gives (46b). \( \square \)

**Proposition 5.5.** Let \( u^0 \geq 0, \neq 0 \) for \( x \in I \). Moreover, assume that

\[
T' = \sum_{n=0}^{\infty} \tau_n < \infty.
\]

If the solution \( u_h^n \) of (Sym) blows up in the sense that

\[
\|u_h^n\| \to \infty \quad (n \to \infty),
\]

then there exists an \( N_0 \in \mathbb{N} \) such that \( J_h(N_0) < 0 \).

**Proof.** Writing \( u_h^n(x) = u_h^0(x) + \sum_{i=0}^{n-1} (u_h^{i+1}(x) - u_h^i(x)) \), we deduce

\[
\|u_h^n\|^2 \leq 2 \int_I x^{N-1} \left( |u_h^0(x)|^2 + \left| \sum_{i=0}^{n-1} (u_h^{i+1}(x) - u_h^i(x)) \right|^2 \right) \, dx
\]

\[
\leq 2\|u_h^0\|^2 + 2 \int_I x^{N-1} \left( \sum_{i=0}^{n-1} \tau_i \cdot \left( \sum_{j=0}^{n-1} \frac{(u_h^{j+1}(x) - u_h^j(x))^2}{\tau_j} \right) \right) \, dx
\]

\[
= 2\|u_h^0\|^2 + 2T' \sum_{j=0}^{n-1} \|\partial_{\tau_j} u_h^{j+1} \|^2 \tau_j
\]

\[
= 2\|u_h^0\|^2 + 2T'[-J_h(n) + J_h(0)].
\]

Since the left hand side blows up as \( n \to \infty \), there should be an \( N_0 \) such that \( J_h(n) < 0 \) for \( n \geq N_0 \). \( \square \)
6. Numerical examples

This section provides a few numerical examples to validate our theoretical results.

First, we compared the shapes of both solutions of (Sym) and (Non-Sym), as shown in Fig. 1 for $N = 5$, $\alpha = \frac{4}{3}$ and $u(0, x) = \cos \frac{\pi}{2} x$. We used the uniform space mesh $x_j = jh$ $(j = 0, \ldots, m)$ and $h = 1/m$ with $m = 50$. For the time increment, we considered

$$
\tau_n = \tau \cdot \min \left\{ 1, \frac{1}{\|u_n^h\|_p} \right\}, \quad \left\| u_n^h - u_n^{h'} \right\|_2 = \sum_{j=0}^{m-1} hx_{j+1}^N (u_n^h(x_j))^2,
$$

(49)

where $\tau = \lambda h^2$ and $\lambda = 1/2$. We continuously computed until $t_n \geq T = 0.2$ or $\|u_n^h\|_2^{-1} < \epsilon = 10^{-8}$, wherein both solutions exist globally in time and get close to 0 uniformly in $I$ as $t \to \infty$. No obvious differences were observed in Figs. 1(a) and (b). Subsequently, we took Fig. 2 to the case where the initial value was $u(0, x) = 13 \cos \frac{\pi}{2} x$ and the rest of the parameters are the same. At this point, the solutions of (Sym) and (Non-Sym) blew up after $x = 0.06$ with the distinct observation that the solution of the former blew up earlier than that of the latter. Furthermore, the solution of (Non-Sym) had negative values while that of (Sym) was always positive.

Then, we examined the error estimates of the solutions for the same uniform space mesh $x_j = jh$ $(j = 0, \ldots, m)$ and $h = 1/m$, and regarded the numerical solution with $h' = 1/480$ as the exact solution. The following quantities were compared:

$$
L^1_{\text{err}} = \|u_n^{h'} - u_n^h\|_{L^1(I)}; \\
L^2_{\text{err}} = \|u_n^{h'} - u_n^h\|_{L^2(I)}; \\
L^\infty_{\text{err}} = \|u_n^{h'} - u_n^h\|_{L^\infty(I)}.
$$

Fig. 3 shows the results for $N = 3$, $\alpha = \frac{4}{3}$ and $u(0, x) = \cos \frac{\pi}{2} x$. We used the uniform time increment $\tau_n = \tau = \lambda h^2$ $(n = 0, 1, \ldots)$ with $\lambda = 1/2$ and computed until $t \leq T = 0.005$. For (Sym), we observed the theoretical convergence rate $h^2 + \tau$ in the $\|\cdot\|$ norm (see Theorem 4.3) whereas the rate in the $L^\infty$ norm slightly deteriorated. For (Non-Sym), we observed the second-order convergence in the $L^\infty$ norm, which supports the results in Theorem 4.5.
Figure 2: $N = 5$, $\alpha = \frac{4}{3}$ and $u(0, x) = 13 \cos \frac{\pi}{2} x$.

Figure 3: Errors. $N = 3$, $\alpha = \frac{4}{3}$ and $u(0, x) = \cos \frac{\pi}{2} x$. 
Moreover, we considered the case for $N = 4$, which is not supported in Theorem 4.3 for (Sym), and chose $\alpha = 4$ and $u(0, x) = 3\cos \frac{\pi}{2}x$ for this case. Fig. 4(d) displays the shape of the solution, which blew up at approximately $T = 0.0035$. Furthermore, we computed errors until $T = 0.0011, 0.0022$, and $0.0033$ using the uniform meshes $x_j$ and $\tau_n$ with $\lambda = 0.11$. From Fig. 4 we observed the second-order convergence in the $\|\cdot\|$ norm, suggesting the possibility of removing assumption $N \geq 3$.

Figure 4: Errors. $N = 4$, $\alpha = 4$ and $u(0, x) = 3\cos \frac{\pi}{2}x$.

Finally, we confirmed the non-increasing property of the energy functional for (Sym) by considering $N = 3$, $\alpha = \frac{4}{3}$, and $u(0, x) = \cos \frac{\pi}{2}x$, $13\cos \frac{\pi}{2}x$. We determined the time increment $\tau_n$ through (49), for the uniform space mesh $x_j = jh$ with $h = 1/m$ and $m = 10$. Fig. 5 gives the results, which support that of Proposition 5.4.
A. Proof of (41a) and (41b)

Proofs of (41a) and (41b) are stated in this appendix, through the same notations used in Section 4.

Proof of (41a). Applying (39), (37a), and (37b), we derived the expression

$$
\frac{1}{\tau_n} (\|\theta^{n+1}\|^2 - \|\theta^n\| \cdot \|\theta^n\|) \leq M (\|\theta^n\| + Ch^2 \|u_{xx}\|_{L^\infty(Q_T)}) \cdot \|\theta^{n+1}\| \\
+ M \tau_n \|u_t\|_{L^\infty(Q_T)} \cdot \|\theta^{n+1}\| + \tau_n \|u_{tt}\|_{L^\infty(Q_T)} \cdot \|\theta^{n+1}\| \\
+ Ch^2 \|u_{xxt}\|_{L^\infty(Q_T)} \cdot \|\theta^{n+1}\|.
$$

Consequently, we have

$$\|\theta^{n+1}\| \leq (1 + \tau_n M)\|\theta^n\| + C\tau_n (h^2 + \tau_n).$$

Therefore, in the same way as the derivation of (31), we obtain from (22) the expression

$$\|\theta^n\| \leq C (h^2 + \tau),$$

to complete the proof.

Proof of (41b). First, we prove the case \(n = 0\). Substituting (38) for \(n = 0\) and \(\chi = \theta^1\), we obtain

$$
\left< \frac{\theta^1 - \theta^0}{\tau_0}, \theta^1 \right> + B(\theta^1, \theta^1) \leq \left< f(u^0_n) - f(u^0), \theta^1 \right> \\
- \left< f(u(t_1)) - f(u^0), \theta^1 \right> - \left< \partial_{\tau_0} u(t_1) - u_t(t_1), \theta^1 \right> - \left< \frac{\rho^1 - \rho^0}{\tau_0}, \theta^1 \right>.
$$

Since \(\theta^0 = 0\), we apply (37b), to get

$$
\frac{1}{\tau_0} \|\theta^1\|^2 \leq M \|\rho^0\| \cdot \|\theta^1\| + M \tau_0 \|u_t\|_{L^\infty(Q_T)} \|\theta^1\| \\
+ \tau_0 \|u_{tt}\|_{L^\infty(Q_T)} \|\theta^1\| + \|\partial_{\tau_0} \rho^1\| \cdot \|\theta^1\| \\
\leq C (\tau_0 + h^2) \|\theta^1\|.
$$
Repeatedly using $\theta^0 = 0$, we obtain

$$\|\partial_{\tau_0} \theta^1\| \leq C(\tau_0 + h^2).$$  \hfill (50)

Now we assume $n \geq 0$ and $t_{n+2} \leq T$. Thus, from (38), we derive

$$\langle \partial_{\tau_{n+1}} \theta^{n+2} - \partial_{\tau_n} \theta^{n+1}, \chi \rangle + B(\theta^{n+2} - \theta^{n+1}, \chi)
= \langle f(u_h^{n+1}) - f(u(t_{n+1})) - f(u_h^n) + f(u(t_n)), \chi \rangle$$

$$- \langle f(u(t_{n+2})) - f(u(t_{n+1})) - f(u(t_{n+1})) + f(u(t_n)), \chi \rangle$$

$$- \langle \partial_{\tau_{n+1}} u(t_{n+2}) - u(t_{n+2}) - \partial_{\tau_n} u(t_{n+1}) + u(t_{n+1}), \chi \rangle$$

$$- \langle \partial_{\tau_{n+1}}^p \theta^{n+2} - \partial_{\tau_n}^p \theta^{n+1}, \chi \rangle$$  \hfill (51)

for any $\chi \in S_h$. Substituting this for $\chi = \partial_{\tau_{n+1}} \theta^{n+2}$, we get

$$\|\partial_{\tau_{n+1}} \theta^{n+2}\|^2 - \|\partial_{\tau_n} \theta^{n+1}\| \cdot \|\partial_{\tau_{n+1}} \theta^{n+2}\| \leq \|\partial_{\tau_{n+1}} \theta^{n+2}\| \sum_{j=1}^4 \|J_j\|.$$

For the time being, we admit the following estimates:

$$\|J_1\| \leq C\tau_n(1 + \tau_n)\|\partial_{\tau_n} \theta^{n+1}\| + C\tau_n(h^2 + \tau_n + \tau_n h^2),$$  \hfill (52a)

$$\|J_2\|, \|J_3\| \leq C\tau_{n+1}(\tau_{n+1} + \tau_n) + C|\tau_{n+1} - \tau_n|,$$  \hfill (52b)

$$\|J_4\| \leq C(\tau_{n+1} + \tau_n)h^2.$$  \hfill (52c)

In view of the quasi-uniformity of time partition (23), we have

$$\tau_{n+1} = \frac{\tau_{n+1}}{\tau_n} \leq \gamma \tau_n.$$

Summing up, we deduce

$$b_{n+1} - b_n \leq C\tau_n b_n + C\tau_n \left(h^2 + \tau + \frac{\delta}{\tau_{\min}}\right),$$  \hfill (53)

where $b_n = \|\partial_{\tau_n} \theta^{n+1}\|$. Therefore,

$$b_n \leq e^{CT}b_0 + C(e^{CT} - 1) \left(h^2 + \tau + \frac{\delta}{\tau_{\min}}\right),$$

which, together with (50), implies the desired inequality (41b).

We now prove (52a)--(52c).

**Estimation for** $J_1$. We apply Taylor’s theorem to obtain

$$J_1 = f'(s_1)(u_h^{n+1} - u_h^n) - f'(s_2)(u(t_{n+1}) - u(t_n))$$

$$= f'(s_1)[(\theta^{n+1} + \rho^{n+1}) - (\theta^n + \rho^n)] + \frac{f'(s_1) - f'(s_2)}{s_1 - s_2}(s_1 - s_2)(u(t_{n+1}) - u(t_n)).$$
where \( s_1 = u_{n+1}^n - \mu_1(u_n^{n+1} - u_n^n) \) and \( s_2 = u(t_{n+1}) - \mu_2[u(t_{n+1}) - u(t_n)] \) for some \( \mu_1, \mu_2 \in [0, 1] \). In view of (37a), (37b), and (40a), we find the estimates

\[
\|J_1\| \leq \tau_n M \|\partial_r \theta^{n+1}\| + \tau_n M \|\partial_r \rho^{n+1}\| + \left\| \frac{f'(s_1) - f'(s_2)}{s_1 - s_2} (s_1 - s_2) \right\| \cdot \|\tau_n u_t\|_{L^\infty(Q_T)},
\]

\[
\leq \tau_n M \|\partial_r \theta^{n+1}\| + C\tau_n Mh^2 \|u_{txx}\|_{L^\infty(Q_T)} + \left\| \frac{f'(s_1) - f'(s_2)}{s_1 - s_2} (s_1 - s_2) \right\| \cdot \|\tau_n u_t\|_{L^\infty(Q_T)},
\]

and

\[
\left\| \frac{f'(s_1) - f'(s_2)}{s_1 - s_2} (s_1 - s_2) \right\| \leq M_2 \|\theta^{n+1} + \rho^{n+1} - \mu_1(\theta^{n+1} + \rho^{n+1} - \theta^n - \rho^n) + (\mu_2 - \mu_1)(u(t_{n+1}) - u(t_n))\|
\leq M_2 \{ \|\partial_r \theta^{n+1}\| + \|\partial_r \rho^{n+1}\| + \tau_n \|\partial_r \theta^{n+1}\| + \tau_n \|\partial_r \rho^{n+1}\| + \tau_n \|u_t\|_{L^\infty(Q_T)} \}
\leq M_2 \{ C(h^2 + \tau) + C\tau h^2 \|u_{xx}\|_{L^\infty(Q_T)} + \tau_n \|\partial_r \theta^{n+1}\| + C\tau \rho^2 \|u_{txx}\|_{L^\infty(Q_T)} + \tau_n \|u_t\|_{L^\infty(Q_T)} \}.
\]

**Estimation for J_2.** We begin with

\[
J_2 = f'(s_3)(u(t_{n+2}) - u(t_{n+1})) - f'(s_4)(u(t_{n+1}) - u(t_n))
\]

\[
= \frac{f'(s_3) - f'(s_4)}{s_3 - s_4} (s_3 - s_4) \tau_{n+1} u_t(\eta_1) + f'(s_4)(\tau_{n+1} u_t(\eta_1) - \tau_n u_t(\eta_2))
\]

\[=
\frac{f'(s_3) - f'(s_4)}{s_3 - s_4} (s_3 - s_4) \tau_{n+1} u_t(\eta_1)
\]

\[+ f'(s_4)\tau_{n+1}(u_t(\eta_1) - u_t(\eta_2)) + f'(s_4)(\tau_{n+1} - \tau_n) u_t(\eta_2),
\]

where \( s_3 = u(t_{n+1}) + \mu_3(u(t_{n+2}) - u(t_{n+1})) \) and \( s_4 = u(t_{n+1}) + \mu_4(u(t_n) - u(t_{n+1})) \) for some \( \mu_3, \mu_4 \in [0, 1] \), \( \eta_1 \in [t_{n+1}, t_{n+2}] \), and \( \eta_2 \in [t_n, t_{n+1}] \). Next, we obtain the estimate

\[
\|J_2\| \leq \tau_{n+1} \left\| \frac{f'(s_3) - f'(s_4)}{s_3 - s_4} (s_3 - s_4) \right\| \cdot \|u_t\|_{L^\infty(Q_T)}
\]

\[+ M_1 \tau_{n+1}(\tau_{n+1} + \tau_n) \|u_{tt}\|_{L^\infty(Q_T)} + M_1 |\tau_{n+1} - \tau_n| \cdot \|u_t\|_{L^\infty(Q_T)};
\]

\[
\left\| \frac{f'(s_3) - f'(s_4)}{s_3 - s_4} (s_3 - s_4) \right\| \leq CM_2 (\tau_{n+1} + \tau_n) \|u_t\|_{L^\infty(Q_T)}.
\]

**Estimation for J_3.** We express J_3 as

\[
J_3 = \frac{\tau_{n+1} u_t(t_{n+2}) - \frac{1}{2} \tau_{n+1} u_{tt}(s_5)}{\tau_{n+1} - \frac{1}{2} \tau_{n+1} u_{tt}(s_5)} - u_t(t_{n+2})
\]

\[
- \left( \frac{\tau_n u_t(t_{n+1}) - \frac{1}{2} \tau_n u_{tt}(s_6)}{\tau_n} - u_t(t_{n+1}) \right)
\]

\[=
- \frac{1}{2} \tau_{n+1} u_{tt}(s_5) + \frac{1}{2} \tau_n u_{tt}(s_6)
\]

\[= \frac{1}{2} \tau_{n+1}(u_{tt}(s_6) - u_{tt}(s_5)) - \frac{1}{2} (\tau_{n+1} - \tau_n) u_{tt}(s_6)
\]

\[= \frac{1}{2} \tau_{n+1} u_{ttt}(s_7)(s_5 - s_6) - \frac{1}{2} (\tau_{n+1} - \tau_n) u_{ttt}(s_6)
\]

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for some $s_5 \in [t_{n+1}, t_{n+2}]$, $s_6 \in [t_n, t_{n+1}]$ and $s_7 \in [s_6, s_5] \subset [t_n, t_{n+2}]$. Therefore,
\[
||| J_3 ||| \leq \frac{1}{2} \tau_{n+1} (\tau_{n+1} + \tau_n) \| u_{ttt} \|_{L^\infty(Q_T)} + \frac{1}{2} |\tau_{n+1} - \tau_n| \cdot \| u_{tt} \|_{L^\infty(Q_T)}.
\]

**Estimation for $J_4$.** For some $s_8 \in [t_{n+1}, t_{n+2}]$, $s_9 \in [t_n, t_{n+1}]$ and $s_{10} \in [s_9, s_8]$, we get the expression
\[
\frac{\rho_{n+2} - \rho_{n+1}}{\tau_{n+1}} - \frac{\rho_{n+1} - \rho_n}{\tau_n} = \rho_t(s_8) - \rho_t(s_9) = (s_8 - s_9) \rho_{tt}(s_{10})
\]
Therefore, using (35),
\[
||| J_4 ||| \leq C(\tau_{n+1} + \tau_n) h^2 \| u_{ttxx} \|_{L^\infty(Q_T)}.
\]

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**References**

[1] Y. G. Chen. Blow-up solutions to a finite difference analogue of $u_t = \Delta u + u^{1+\alpha}$ in $N$-dimensional balls. *Hokkaido Math. J.*, 21(3):447–474, 1992.

[2] K. Deng and H. A. Levine. The role of critical exponents in blow-up theorems: the sequel. *J. Math. Anal. Appl.*, 243(1):85–126, 2000.

[3] K. Eriksson and V. Thomée. Galerkin methods for singular boundary value problems in one space dimension. *Math. Comp.*, 42(166):345–367, 1984.

[4] H. Fujita. On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$. *J. Fac. Sci. Univ. Tokyo Sect. I*, 13:109–124 (1966), 1966.

[5] M. Ishiwata. On the asymptotic behavior of unbounded radial solutions for semilinear parabolic problems involving critical Sobolev exponent. *J. Differential Equations*, 249(6):1466–1482, 2010.

[6] D. Jespersen. Ritz-Galerkin methods for singular boundary value problems. *SIAM J. Numer. Anal.*, 15(4):813–834, 1978.

[7] H. A. Levine. The role of critical exponents in blowup theorems. *SIAM Rev.*, 32(2):262–288, 1990.

[8] V. Thomée. *Galerkin finite element methods for parabolic problems*. Springer Verlag, Berlin, second edition, 2006.