Quantitative statistical stability and convergence to equilibrium.
An application to maps with indifferent fixed points.

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Abstract

We show a general relation between fixed point stability of suitably perturbed transfer operators and convergence to equilibrium (a notion which is strictly related to decay of correlations).
We apply this relation to deterministic perturbations of a large class of maps with indifferent fixed points. It turns out that the $L^1$ dependence of the a.c.i.m. on small suitable deterministic changes for these kind of maps is Hölder, with an exponent which is explicitly estimated.

1 Introduction

The statistical stability of dynamical systems is well understood in the uniformly hyperbolic case (see e.g. [19] and related references). In this case, quantitative estimations are available, proving the Lipschitz or even differentiable dependence of the physical measure under perturbations. For systems having a non uniformly expanding/hyperbolic behavior, much less is known. Qualitative results and some quantitative (providing precise information on the modulus of continuity) ones are known under different assumptions (see. e.g. the survey [3], the papers [1, 2, 9, 5, 4, 21, 12] and [7], for a recent survey, on differentiable dependence, where also some result beyond the uniformly expanding case are discussed). While these results do not give a quantitative information for the general case of deterministic perturbations of maps with indifferent fixed points, some very recent result give quantitative information (and even differentiability) for some deterministic perturbation of certain maps, leading to families of intermittent maps ([8, 10, 16]).

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In this note we show a general quantitative relation between speed of convergence to equilibrium and statistical stability. This relation gives nontrivial information for systems having different speed of convergence to equilibrium. We show the flexibility of this approach applying it to perturbations of a class of maps with indifferent fixed points having an absolutely continuous invariant probability measure. These are systems whose speed of convergence to equilibrium is bounded by a power law. We show that this implies the Hölder continuity of the absolutely continuous invariant measure under (suitable) deterministic perturbations of the system.

The paper is structured as follows: in Section 2 we show a general result estimating quantitatively the stability of operator’s fixed points under certain perturbations which is suitable to be applied to estimate the stability of physical measures. In Section 3 we apply this result to a class of maps with indifferent fixed points under a large set deterministic perturbations. We remark that our results applies to perturbations of actual intermittent maps and not only to perturbations of expanding maps leading to intermittent behavior after perturbation.

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2 Quantitative fixed point stability and convergence to equilibrium.

Let us consider a dynamical system \((X, T)\) where \(X\) is a metric space and the space \(SM(X)\) of Borel measures with sign on \(X\). The dynamics \(T\) naturally induces a function \(L : SM(X) \to SM(X)\) which is linear and is called transfer operator. If \(\nu \in SM(X)\) then \(L[\nu] \in SM(X)\) is defined by

\[
L[\nu](B) = \nu(T^{-1}(B))
\]

for every measurable set \(B\). If \(X\) is a manifold, the measure is absolutely continuous: \(d\nu = f \, dm\) (here \(m\) represents the Lebesgue measure) and \(T\) is nonsingular, the operator induces another operator \(\tilde{L} : L^1(m) \to L^1(m)\) acting on the measure densities \((\tilde{L}f = \frac{d(L[f \, m]1)}{d\, m})\). By a small abuse of notation we will still indicate by \(L\) this operator.

An invariant measure is a fixed point for the transfer operator. Let us now see a quantitative measure statement for these fixed points under suitable perturbations of the operator.

Let us consider a certain system having a transfer operator \(L_0\) for which we know the speed of convergence to equilibrium (see \(\Omega\) below). Consider a ”nearby” system \(L_1\) having suitable properties which will be specified below: suppose there are two normed vector spaces of measures with sign \(B_s \subseteq B_w \subseteq\)
SM(X) (the strong and weak space) with norms \(|f|_w \leq |f|_s\) and suppose the operators \(L_0\) and \(L_1\) preserve the spaces: \(L_i(B_s) \subset B_s\) and \(L_i(B_w) \subset B_w\) with \(i \in \{0, 1\}\).

Let us consider \(V_s := \{f \in B_s, f(X) = 0\}\) the space of zero average measures in \(B_s\). The speed of convergence to equilibrium of a system will be measured by the speed of contraction to 0 of this space by the iterations of the transfer operator.

**Definition 2** Let \(\phi(n)\) be a real sequence converging to zero. We say that the system has convergence to equilibrium with respect to norms \(|f|_w, |f|_s\) and speed \(\phi\) if for all \(g \in V_s\)

\[
|L^n_0(g)|_w \leq \phi(n)|g|_s.
\]

Suppose \(f_0, f_1 \in B_s\) are fixed probability measures of \(L_0\) and \(L_1\). The following statement gives a quantitative way to estimate the statistical stability of \(L_0\) under perturbation if the speed of convergence to equilibrium is known.

**Theorem 3** Suppose we have estimations on the following aspects of the system:

1. (speed of convergence to equilibrium) there is \(\phi \in C^0(\mathbb{R})\), \(\phi(t)\) decreasing to 0 as \(t \to \infty\) such that \(L_0\) has convergence to equilibrium with respect to norms \(|f|_w, |f|_s\) and speed \(\phi\).

2. (control on the norms of the invariant measures) There is \(M \geq 0\) such that

\[
\max(|f_1|_s, |f_0|_s) \leq M;
\]

3. (iterates of the transfer operator are bounded for the weak norm) there is \(\hat{C} \geq 0\) such that for each \(n\),

\[
|L^n_0|_{B_w \to B_w} \leq \hat{C}.
\]

4. (control on the size of perturbation in the strong-weak norm) Denote

\[
\sup_{|f|_s \leq 1} \|(L_1 - L_0)f\|_w := \epsilon
\]

consider the decreasing function \(\psi\) defined as \(\psi(x) = \frac{\phi(x)}{x}\), then

\[
|f_1 - f_0|_w \leq (2M + M\hat{C})\epsilon(\psi^{-1}(\epsilon) + 1).
\]

\(^1\)Similar quantitative stability statements are used in [13] and [14] to support rigorous computation of invariant measures and get quantitative estimations for the statistical stability of Lorenz like maps.
**Proof.** The proof is a direct computation from the assumptions

\[
\|f_1 - f_0\|_w \leq \|L_1^N f_1 - L_0^N f_0\|_w \\
\leq \|L_1^N f_1 - L_0^N f_1\|_w + \|L_0^N f_1 - L_0^N f_0\|_w \\
\leq \|L_0^N (f_1 - f_0)\|_w + \|L_1^N f_1 - L_0^N f_1\|_w.
\]

Since \(f_1 - f_0 \in V_s\), \(\|f_1 - f_0\|_s \leq 2M\),

\[
\|f_1 - f_0\|_w \leq 2M \phi(n) + \|L_1^N f_1 - L_0^N f_1\|_w
\]

but

\[
(L_0^N - L_1^N) = \sum_{k=1}^{N} L_0^{N-k}(L_0 - L_1)L_1^{k-1}
\]

hence

\[
-(L_1^N - L_0^N) f_1 = \sum_{k=1}^{N} L_0^{N-k}(L_0 - L_1)L_1^{k-1} f_1
\]

\[
= \sum_{k=1}^{N} L_0^{N-k}(L_0 - L_1) f_1
\]

then

\[
\|f_1 - f_0\|_w \leq 2M \phi(N) + \epsilon MN \tilde{C}.
\]

Now consider the function \(\psi\) defined as \(\psi(x) = \frac{\phi(x)}{x}\), chose \(N\) such that \(\psi^{-1}(\epsilon) \leq N \leq \psi^{-1}(\epsilon) + 1\), in this way \(\frac{\phi(N)}{N} \leq \epsilon \leq \frac{\phi(N-1)}{N-1}\),

\[
\|f_1 - f_0\|_w \leq (2M + M \tilde{C}) \epsilon (\psi^{-1}(\epsilon) + 1).
\]

**Remark 4** If \(\phi(x) = Cx^{-\alpha}\) then \(\psi(x) = Cx^{-\alpha-1}\) and \(\epsilon (\psi^{-1}(\epsilon) + 1) = C \epsilon (\frac{1}{\sqrt{\epsilon - \sqrt{\epsilon}} + 1}) \sim \epsilon^{1-\frac{1}{2\alpha}}\) and we have the estimation for the modulus of continuity

\[
\|f_1 - f_0\|_w \leq K_1 \epsilon^{1-\frac{1}{2\alpha}}
\]

where the constant \(K_1\) depends on \(M, \tilde{C}, C\) and not on the distance of the operators measured by \(\epsilon\).

\[\blacksquare\]

### 3 Maps with indifferent fixed points

In this section we introduce a family of maps with indifferent fixed point; we then adapt some known results for these kind of maps to the form required to apply Theorem 3.
Definition 5 Let $0 < \alpha < 1$ and let us consider a map $T : [0,1] \to [0,1]$ we say that $T$ is in the set $M(\alpha,c,C,d)$ if:

1. $T(0) = 0$ and there is a point $\overline{d} \in (d,1)$ such that the two branches $T|_{(0,\overline{d})} := T_1$ and $T|_{[\overline{d},1)} := T_2$ are continuous and onto.

2. Each branch $T_i$ of $T$ is increasing, weakly convex and can be extended to a $C^1$ function;

3. $T''$ is continuous everywhere but in the points 0 and $\overline{d}$; $T'' > 1$ for all $x \in (0,\overline{d}) \cup (\overline{d},1)$ and $T''(0) = 1$.

4. There are $\overline{\alpha}, \overline{C}, \overline{c} \in (0,\infty)$, $\overline{\alpha} \leq \alpha, \overline{C} \leq C, \overline{c} \leq c$ such that

\[
|T'(x)| = 1 + \overline{c}x^{\overline{\alpha}} + o(x^{\overline{\alpha}}) \quad (2)
\]

\[
|T''(x)| \leq \overline{C}x^{\overline{\alpha}-1},
\]

\[
|T'(x)| \leq \overline{C}.
\]

We say that $T$ is in the set $M_0(\alpha,c,C,d)$ if moreover $T$ has an absolutely continuous invariant probability measure with density $h$ such that

\[
\gamma_0 \leq h(x)
\]

for some $0 < \gamma_0 < 1$. We indicate with $M_0$ the set of maps that are in $M_0(\alpha,c,C,d)$ for some value of the parameters.

Remark 6 We remark that the above conditions are verified for maps of the following kind: $T(x) = \begin{cases} x(1 + 2^\alpha x^\alpha) & \forall x \in [0,1/2) \\ 2x - 1 & \forall x \in [1/2,1] \end{cases}$, in particular this map is in $M_0$ (see [18]).

Remark 7 If a map $T$ is in $M_0(\alpha,c,C,d)$, the following condition is then satisfied: there is a constant $C_3 \in (0,\infty)$ such that

\[
T(x) \geq x + C_3x^{1+\alpha}. \quad (3)
\]

Notation 8 Let us denote by $M(\alpha,c,C,C_3,d)$ the set of maps in $M(\alpha,c,C,d)$ satisfying (3) for a certain $C_3$.

For such a map, we can apply a useful estimation on the asymptotic behavior of $f$ which can be found in [20] (Proposition 1.1, Theorem 1, Equation 3).

Proposition 9 Let us consider the transfer operator $L_T$ associated to $T \in M(\alpha,c,C,C_3,d)$ and the following cone of decreasing functions

\[
C_A = \{g \in L^1|g \geq 0, g \text{ decreasing}, \int_0^1 g \, dm = 1, \int_0^x g \, dm \leq Ax^{1-\alpha}\}.
\]

Let $A_+ = ((1-\alpha)C_3^{-2+\alpha})^{-1}$, if $A \geq A_+$ then $L(C_A) \subseteq C_A$. Moreover the unique invariant density $f$ of $T$ is in $C_{A_+}$.
Remark 10  We remark ([20], lemma 2.1) that if \( f \in C_A \) then \( f(x) \leq Ax^{-\alpha} \).

Now we estimate the speed of convergence to equilibrium. The following is proved in [18]

**Theorem 11**  Let us consider \( T \in M_0(\alpha, c, C, d) \), if \( g \in L^\infty, f \in C^1, \int f \, dx = 0, \) \( 0 < \gamma < \frac{1}{\alpha} - 1 \) there is \( C_f \) depending on \( f \) such that

\[
| \int f g \circ T^n \, dm | \leq C_f \| g \|_\infty n^{-\gamma}. \tag{4}
\]

Similarly to what is done in [11], Appendix B, for the decay of correlations, we show that by the uniform boundedness principle it is possible to see that a statement like (4) implies a statement on the convergence to equilibrium which is uniform also in \( f \):

**Proposition 12**  Let us consider a system satisfying a non uniform convergence to equilibrium estimation, as in (4), then there is a constant \( C_2 \) such that for each \( g \in L^\infty, f \in C^1, \int f \, dx = 0, \) \( 0 < \gamma, \) then

\[
| \int f g \circ T^n \, dm | \leq C_2 \| f \|_{C^1} \| g \|_\infty n^{-\gamma} \tag{5}
\]

for some constant \( C_2 \).

**Proof.**  Let \( B_{L^\infty} \) be the unit ball in \( L^\infty \). For any \(( n, g ) \in \mathbb{N} \times B_{L^\infty} \), let us consider the family of non-negative functions defined by

\[
q_{(n,g)}(f) = n^{-\gamma} | \int f g \circ T^n \, dm |.
\]

For any \( f_1, f_2, f_3 \in C^1 \) it holds

\[
q_{(n,g)}(f_1 + f_2) \leq q_{(n,g)}(f_1) + q_{(n,g)}(f_2)
\]

and

\[
q_{(n,g)}(-f_3) = q_{(n,g)}(f_3).
\]

Moreover it follows from (4) that for each \( f \in C^1 \)

\[
\sup_{\mathbb{N} \times B_{L^\infty}} q_{(n,g)}(f) \leq C_f < \infty.
\]

By the uniform boundedness principle then (see [15], Theorem 1.29, page 139 ) it follows

\[
\sup_{\mathbb{N} \times B_{L^\infty}, \| f \|_{C^1} \leq 1} q_{(n,g)}(f) := C_2 < \infty.
\]

Which implies\( \blacksquare \)
Remark 13 The above proof generalizes to any speed of convergence to equilibrium. Moreover, since for any Lipschitz function $f$ there is $f \in C^1$ such that $||f||_{C^1} \leq ||f||_{lip}$ (where $|| \cdot ||_{lip}$ denotes the Lipschitz norm) and $||f - f||_1 \leq \epsilon$ ($\in L^\infty$, thus approximating $f$ in $L^1$ is enough to approximate the integral of the product) then the above theorem also holds for Lipschitz functions $f$ with the Lipschitz norm:

$$|\int f \circ T^n \ dm| \leq C_2 ||f||_{lip} ||g||_{\infty} n^{-\gamma}.$$ 

3.1 Application of Theorem 3 to maps with indifferent fixed points

To apply Theorem 3 we need some technical work to verify the required assumptions and provide the required estimations. This will be done in this section and in the following ones. Let us consider the following norm:

$$|||f|||_\alpha = \max(\text{esssup}_{x \in (0,1]} |x^\alpha f(x)|, \text{esssup}_{x \in (0,1]} |x^{\alpha+1} f'(x)|).$$

We show now how Theorem 3 can be applied to our class of maps, considering $||| \cdot |||_\alpha$ and $|| \cdot ||_1$ as strong and weak norms.

The first step is to extend the result of Proposition 12 from Lipschitz observables to a class including functions with finite $||| \cdot |||_\alpha$ norm.

3.1.1 Convergence to equilibrium for the $||| \cdot |||_\alpha$ norm (item 1 of Theorem 3)

In the following proposition we show how a relation like Proposition 12 implies an estimation of the convergence to equilibrium, as Definition 2 with respect to $|| \cdot ||_1$ and the strong norm $||| \cdot |||_\alpha$.

Proposition 14 If for a nonsingular function $T$ and some $\gamma > 0$, $C_2 \geq 0$, it holds that for each $g \in L^\infty$ and $f \in Lip[0,1]$, with $\int f \ dx = 0$,

$$|\int f \ g \circ T^n \ dm| \leq C_2 ||f||_{lip} ||g||_{\infty} n^{-\gamma};$$

then there is $C_4 \geq 0$ such that

$$||L^n f||_1 \leq C_4 n^{-\frac{\gamma}{2}(1-\alpha)} ||f||_\alpha$$

where $L$ is the transfer operator associated to $T$.

Proof. Suppose $||g||_{\infty} \leq 1$. There is no loss of generality in supposing $f(0) \geq 0$. Let $q \in (0,1]$ small enough such that $f|_{(0,q]} \geq 0$,

$$|\int f \ g \circ T^n \ dm| \leq |\int_0^q f \ g \circ T^n \ dm + \int_q^1 f \ g \circ T^n \ dm| \leq ||f1_{[0,q]}||_1 + \frac{1}{2} q^{\alpha+1} ||f||_\alpha + |\int_0^1 f_q \ g \circ T^n \ dm|$$
where \( f_q(x) = \begin{cases} q^{-1}f(q)x & \text{for } x \leq q \\ \frac{q}{f(x)} & \text{for } x \geq q \end{cases} \). Since we have convergence to equilibrium for functions of zero average we consider \( \hat{f}_q(x) = f_q(x) - \int f_q(x) \, dm \).

Since \( \int f \, dm = 0 \) we have that \( \int f_q(x) \, dm \leq ||f1_{[0,q]}||_1 + \frac{q^{-(\alpha + 1)}}{2} ||f||_\alpha \).

Hence
\[
||L^n f||_1 \leq \sup_{||g||_\infty \leq 1} \int f \circ T^n \, dm \\
\leq 2||f1_{[0,q]}||_1 + q^{-\alpha} ||f||_\alpha + ||L^n \hat{f}_q||_1 \\
\leq ||f||_\alpha \frac{2 - \alpha}{1 - \alpha} q^{-\alpha + 1} + ||L^n f_q||_1.
\]

Since \( f_q(x) \) is Lipschitz and \( ||\hat{f}_q||_{lip} \leq ||f||_\alpha q^{-\alpha - 1} \), by \( 6 \)
\[
||L^n \hat{f}_q||_1 \leq C_2 ||\hat{f}_q||_{Lip} n^{-\gamma} \\
\leq C_2 ||f||_\alpha q^{-\alpha - 1} n^{-\gamma}.
\]

and then
\[
||L^n f||_1 \leq ||f||_\alpha \frac{2 - \alpha}{1 - \alpha} q^{-\alpha + 1} + C_2 ||f||_\alpha q^{-\alpha - 1} n^{-\gamma} \\
\leq ||f||_\alpha (2 - \frac{\alpha}{1 - \alpha} q^{-\alpha + 1} + C_2 q^{-\alpha - 1} n^{-\gamma})
\]

choosing \( q = \sqrt{n^{-\gamma}} \), then
\[
||L^n f||_1 \leq C_4 ||f||_\alpha n^{-2(1 - \alpha)}
\]

\[\blacksquare\]

\[3.1.2 \text{ Applying Theorem 3}\]

We can now apply Theorem 3 for suitable perturbations, obtaining that the physical invariant measure varies continuously, with a Holder continuity modulus. In this we will consider the norm \( ||| \cdot |||_{\alpha} \) as a strong norm and the \( L^1 \) norm as a weak one.

We remark that the transfer operators are \( L^1 \) contractions then the constant at item 3 of Theorem 3 satisfies \( \tilde{C} = 1 \). By Theorem 3 it follows:

**Proposition 15** Let us suppose that \( T_0 \in M_0(\alpha, c, C, d) \) at beginning of Section 3, let \( L_0 \) be its transfer operator and \( f_0 \) its absolutely continuous invariant measure. Let \( T_1 \in M(\alpha, c, C, d) \), let \( L_1 \) be its transfer operator and \( f_1 \) its absolutely continuous invariant measure. If \( T_0 \) and \( T_1 \) are such that
\[
\max(||f_1||_\alpha, ||f_0||_\alpha) \leq M; \quad (7)
\]

and \( L_1 \) is a small perturbation of \( L_0 \), in the following sense:
\[
\sup_{||f||_\alpha \leq 1} ||(L_1 - L_0)f||_1 \leq \epsilon \quad (8)
\]
then for each $0 < \gamma < \frac{1}{\alpha} - 1$ there is $K_2$, depending on $M, \check{C} = 1$ (see Remark 4) and $T_0$, but not on $\epsilon$ such that
\[
\| f_1 - f_0 \|_1 \leq K_2 \epsilon^{\frac{1}{\gamma + 1}}.
\]

We now show how assumptions (7) and (8) can be naturally supposed for deterministic perturbations in our class of maps, and we get an explicit estimation of $M$.

3.1.3 Bound on the strong norm of the invariant measures (item 2 of Theorem 3)

In this section we show how it is possible to have an estimation of the $\| \|_\alpha$ norm of the absolutely continuous invariant measures of maps in the set $M(\alpha, c, C, d)$.

To estimate the $\| \|_\alpha$ norm of this invariant density, we need to have an estimation on its derivative. In [18] the following is proved:

**Lemma 16** Let $T \in M(\alpha, c, C, d)$, and $h$ is its a.c.i.m; let $c_T = \| T' \|_\infty$ consider $K_T, a_T, b_T \in \mathbb{R}$ such that:
\[
K_T = \sup_x (x^{\alpha - 1} T(x)); \quad (9)
\]
\[
a_T > \sup_x \left( \frac{4 C K_T}{|D_x T|^2} \right); \quad (10)
\]
\[
b_T > a_T \sup_x \left( \frac{2 c_T T(x) - x |D_x T|}{(|D_x T| - 2 c_T T(x))} \right). \quad (11)
\]

Then the cone
\[
C_0 = \{ f \in C^1([0,1]) \mid 0 \leq f(x) \leq 2 h(x) \int_0^1 f \, dx ; \ |f'(x)| \leq \frac{a_T + b_T x}{x} f(x) \}
\]
is invariant for the transfer operator.

**Proof.** In [18] (Lemma 5.1) is proved that, if $f \in C_0$ then
\[
|(Lf)'(x)| \leq \frac{a - bx}{x} Lf(x) \sup_{y \in [0,1]} \left[ \frac{2 C y^{\alpha - 1} T(y)}{|D_y T|^2 a + b T(y)} + \frac{T(y)}{y |D_y T|} \frac{a + b y}{a + b T(y)} \right].
\]

The result follows by remarking that for $a, b$ big enough
\[
\sup_{y \in [0,1]} \left[ \frac{2 C y^{\alpha - 1} T(y)}{|D_y T|^2 a + b T(y)} + \frac{T(y)}{y |D_y T|} \frac{a + b y}{a + b T(y)} \right] < 1.
\]

We need to have an explicit estimation of $a, b$ in function of $T$. Since $T(x) = x + \frac{1}{\alpha + 1} x^{\alpha + 1} + o(x^{\alpha + 1})$ then $y^{\alpha - 1} T(y) = y^\alpha + \frac{1}{\alpha + 1} y^{2\alpha} + o(y^{2\alpha})$ is bounded.
Suppose \( y^{\alpha-1}T(y) \leq K \), then
\[
\frac{2C}{|D_yT|^2} \frac{y^{\alpha-1}T(y)}{a + bT(y)} \leq \frac{2C}{|D_yT|^2} \frac{K}{a + bT(y)} + \frac{T(y)}{y|D_yT|} \frac{a + by}{a + bT(y)}.
\]

Now let us find a sufficient condition for which the two summands are less or equal than \( \frac{1}{2} \) and \( \frac{1}{2c_T} \) (which is \( \leq \frac{1}{2} \)). We will obtain the conditions 9, 10, 11.

For the first summand
\[
\frac{2C}{|D_yT|^2} \frac{K}{a + bT(y)} \leq \frac{1}{2}
\]
\[
\frac{4CK}{|D_yT|^2} \leq a + bT(y)
\]
hence \( \frac{4CK}{|D_yT|^2} \leq a \) is a sufficient condition. For the second summand
\[
\frac{T(y)}{y|D_yT|} \frac{a + by}{a + bT(y)} \leq \frac{1}{2c_T}
\]
\[
2c_T T(y)a + 2c_T T(y)by \leq y|D_yT|a + y|D_yT|bT(y)
\]
\[
2c_T T(y)a - y|D_yT|a \leq y|D_yT|bT(y) - 2c_T T(y)by
\]
\[
a_s(2c_T T(y) - y|D_yT|) \leq \frac{b}{a}(y|D_yT|T(y) - 2c_T T(y))
\]
\[
\frac{2c_T T(y) - y|D_yT|}{(|D_yT| - 2c_T T(y)y) \leq \frac{b}{a}.
\]

By Remark 10 we have the following

**Proposition 17** Let \( T \in M(\alpha, c, C, C_3, d) \), suppose \( T \) satisfies the assumptions of Lemma 16. Then the unique a.c.i.m \( h \) is such that
\[
||h||_\alpha \leq \max(A_*, A_*(a_T + b_T)).
\]

**Proof.** The condition \( ||h||_\alpha \leq A_* \) follows from \( h(x) \leq A_* x^{-\alpha} \) (see Remark 10). By Lemma 16 we have
\[
|h'(x)| \leq \frac{a_T + b_T x}{x} h(x) \leq (a_T + b_T x) A_* x^{-\alpha-1}.
\]

**3.1.4 Mixed norm for the difference of the operators for deterministic perturbations** (item 4 of Theorem 3)

In this section we see a family of perturbations of maps with an indifferent fixed point, for which the difference \( ||(L_\delta - L)f||_1 \) can be estimated for each
Let us consider two maps $T_0, T_1 \in M(\alpha, c, C, d)$, let $L_0, L_1$ be their transfer operators. Let $T_{0,1}, T_{0,2}, T_{1,1}, T_{1,2}$ be the inverse branches of the two maps. Suppose that $T_1$ is a small perturbation of $T_0$ in the following sense:

N1 for $i \in \{1, 2\}, x \in [0, 1], x^{\alpha - 1}|T_{0,i}^{-1}(x) - T_{1,i}^{-1}(x)| \leq \epsilon$;

N2 for all $x \in [0, 1], ||T_0^\prime(x) - T_1^\prime||_\infty \leq \epsilon$;

then there is $C_6 \geq 0$ not depending on $T_1$ such that

$$||L_0 - L_1||_1 \leq C_6 ||f||_\alpha.$$

Proof. Writing the transfer operators explicitly:

$$L_0 f(x) = \frac{f(T_{0,1}^{-1}(x))}{T_0^\prime(T_{0,1}(x))} + \frac{f(T_{0,2}^{-1}(x))}{T_0^\prime(T_{0,2}(x))}, L_1 f(x) = \frac{f(T_{1,1}^{-1}(x))}{T_1^\prime(T_{1,1}(x))} + \frac{f(T_{1,2}^{-1}(x))}{T_1^\prime(T_{1,2}(x))}$$

$$||L_0 f(x) - L_1 f(x)||_1 \leq \left|\left| \frac{f(T_{0,1}^{-1}(x))}{T_0^\prime(T_{0,1}(x))} + \frac{f(T_{0,2}^{-1}(x))}{T_0^\prime(T_{0,2}(x))} - \frac{f(T_{1,1}^{-1}(x))}{T_1^\prime(T_{1,1}(x))} - \frac{f(T_{1,2}^{-1}(x))}{T_1^\prime(T_{1,2}(x))} \right|\right|_1$$

$$\leq \left|\left| \frac{f(T_{0,1}^{-1}(x))}{T_0^\prime(T_{0,1}(x))} - \frac{f(T_{1,1}^{-1}(x))}{T_1^\prime(T_{1,1}(x))} \right|\right|_1 + \left|\left| \frac{f(T_{0,2}^{-1}(x))}{T_0^\prime(T_{0,2}(x))} - \frac{f(T_{1,2}^{-1}(x))}{T_1^\prime(T_{1,2}(x))} \right|\right|_1$$

$$\quad + \left|\left| \frac{f(T_{1,1}^{-1}(x))}{T_0^\prime(T_{0,1}(x))} - \frac{f(T_{1,2}^{-1}(x))}{T_1^\prime(T_{1,1}(x))} \right|\right|_1 + \left|\left| \frac{f(T_{1,2}^{-1}(x))}{T_0^\prime(T_{0,2}(x))} - \frac{f(T_{1,2}^{-1}(x))}{T_1^\prime(T_{1,2}(x))} \right|\right|_1$$

$$= I + II + III + IV$$

Now

$$I = \frac{||f(T_{0,1}^{-1}(x)) - f(T_{1,1}^{-1}(x))||}{T_0^\prime(T_{0,1}(x))} \leq ||f(T_{0,1}^{-1}(x)) - f(T_{1,1}^{-1}(x))||$$

$$\leq \int_0^1 |f(T_{0,1}^{-1}(x)) - f(T_{1,1}^{-1}(x))| \, dx$$

$$\leq \int_0^1 ||f||_{C^\beta} x^{\beta - 1} ||T_{0,1}^{-1}(x) - T_{1,1}^{-1}(x)|| \, dx$$
by Lagrange theorem, where

\[
\min(T_{0,1}^{-1}(x), T_{1,1}^{-1}(x)) \leq \xi \leq \max(T_{0,1}^{-1}(x), T_{1,1}^{-1}(x)).
\]

But since \( T_i \in M(\alpha, c, C, d) \)

\[
C^{-1} x \leq \min(T_{0,1}^{-1}(x), T_{1,1}^{-1}(x)) \leq \max(T_{0,1}^{-1}(x), T_{1,1}^{-1}(x)) \leq C x
\]

hence

\[
\int_0^1 |f(T_{0,1}^{-1}(x)) - f(T_{1,1}^{-1}(x))| \, dx \leq \int_0^1 ||f||_\beta C^{\beta+1} x^{-\beta-1} |T_{0,1}^{-1}(x) - T_{1,1}^{-1}(x)| \, dx
\]

\[
\leq C^{\beta+1} ||f||_\beta.
\]

For the summand \( II \) the estimation is easier, since \( \min(T_{0,2}^{-1}(x), T_{1,2}^{-1}(x)) \geq d \geq 0 \).

\[
\frac{||f(T_{0,2}^{-1}(x)) - f(T_{1,2}^{-1}(x))||}{T_0(T_{0,2}(x))} \leq \int_0^1 |f(T_{0,2}^{-1}(x)) - f(T_{1,2}^{-1}(x))| dx
\]

\[
\leq \int_0^1 ||f||_\beta d^{-\beta-1} |T_{0,1}^{-1}(x) - T_{1,1}^{-1}(x)| \, dx
\]

\[
\leq d^{-\beta-1} ||f||_\beta.
\]

For the summand \( III \) :

\[
III = \left| \frac{f(T_{1,1}^{-1}(x))}{T_0(T_{1,1}(x))} - \frac{f(T_{1,1}^{-1}(x))}{T_1(T_{1,1}(x))} \right|_1
\]

\[
\leq \left| \frac{f(T_{1,1}^{-1}(x))}{T_0(T_{1,1}(x))} - \frac{f(T_{1,1}^{-1}(x))}{T_0(T_{0,1}(x))} \right|_1 + \left| \frac{f(T_{1,1}^{-1}(x))}{T_1(T_{1,1}(x))} - \frac{f(T_{1,1}^{-1}(x))}{T_0(T_{1,1}(x))} \right|_1
\]

(12)

The last two summands can be estimated as follows

\[
\left| \frac{f(T_{1,1}^{-1}(x))}{T_0(T_{0,1}(x))} - \frac{f(T_{1,1}^{-1}(x))}{T_0(T_{1,1}(x))} \right|_1 \leq \int |f(T_{1,1}^{-1}(x))| \, dx \left| \frac{1}{T_0(T_{0,1}(x))} - \frac{1}{T_0(T_{1,1}(x))} \right|_\infty
\]

\[
\leq ||f||_\alpha \int (T_{1,1}^{-1}(x))^{-\alpha} \, dx \left| \frac{1}{T_0(T_{0,1}(x))} - \frac{1}{T_0(T_{1,1}(x))} \right|_\infty
\]

\[
\leq \text{Const}_1 ||f||_\alpha \frac{T_0''}{(T_0')^2}(\xi) (T_{0,1}^{-1}(x) - T_{1,1}^{-1}(x)) ||_\infty
\]

for some \( \xi \) such that

\[
C^{-1} x \leq \min(T_{0,1}^{-1}(x), T_{1,1}^{-1}(x)) \leq \xi \leq \max(T_{0,1}^{-1}(x), T_{1,1}^{-1}(x)) \leq C x
\]

as before. Since \( |T_{0,1}^{-1}(x) - T_{1,1}^{-1}(x)| \leq c x^{\alpha+1} \) and \( |T_0''(x)| \leq C x^{\alpha-1} \), then

\[
\sup \left| \frac{T_0''}{(T_0')^2}(\xi)(T_{0,1}^{-1}(x) - T_{1,1}^{-1}(x)) \right| \leq C^2 \epsilon
\]
\[
\left\| \frac{f(T^{-1}_{1,1}(x))}{T'_0(T^{-1}_{1,1}(x))} - \frac{f(T^{-1}_{1,1}(x))}{T'_0(T^{-1}_{1,1}(x))} \right\|_1 \leq \text{Const}_2. \epsilon \|f\|_\alpha.
\]

For the other summand in (12)

\[
\left\| \frac{f(T^{-1}_{1,1}(x))}{T'_0(T^{-1}_{1,1}(x))} - \frac{f(T^{-1}_{1,1}(x))}{T'_0(T^{-1}_{1,1}(x))} \right\|_1 \leq \text{Const.} \|f\|_\alpha |T'_0 - T'_1|_\infty \leq \text{Const.} \epsilon \|f\|_\alpha.
\]

IV can be treated similarly. ■

By what is proved above we can state the following quantitative statistical stability under deterministic perturbations result:

**Theorem 19** Let us consider a family of maps \( T_s \), \( s \in [0,1) \), such that \( T_0 \in M_0(\alpha,c,C,d) \) and \( T_s \in M(\alpha,c,C,C_3,d) \). Suppose that each \( T_s \) satisfy the assumptions (9), (10), (11) of Lemma 16 uniformly (with uniform constants for each \( s \)). Let us suppose that also assumptions \( N1,N2 \) are satisfied, i.e:

- for each \( i \in \{0,1\}, s \in [0,1], x \in [0,1] \), \( x^{-\alpha-1}|T^{-1}_{0,i}(x) - T^{-1}_{s,i}(x)| \leq \epsilon_s; \)
- for all \( s \in [0,1], ||T'_0(x) - T'_s(x)||_\infty \leq \epsilon_s. \)

Let \( f_s \) the absolutely continuous invariant measures of \( T_s \), then for each \( 0 < \gamma < \frac{1}{\alpha} - 1 \)

\[
||f_s - f_0||_1 \leq O(\epsilon_s^{1 - \frac{1}{2(1-\alpha)+1}}).
\]

**Remark 20** We remark that this statement gives Holder continuity if \( 1 - \frac{1}{2(1-\alpha)+1} > 0 \). This is always verified for \( 0 \leq \alpha < 1. \)

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