Properties of the zeros of generalized basic hypergeometric polynomials

*Oksana Bihun*\(^1\) and +**Francesco Calogero**\(^2\)

\(^1\)Department of Mathematics, Concordia College
901 8th Str. S, Moorhead, MN 56562, USA, +1-218-299-4396

\(^2\)Physics Department, University of Rome “La Sapienza”
p. Aldo Moro, I-00185 ROMA, Italy, +39-06-4991-4372

Abstract

We define the generalized basic hypergeometric polynomial of degree \(N\) as follows:

\[
P_N(\alpha_1, \ldots, \alpha_r; \beta_1, \ldots, \beta_s; q; z) = \sum_{m=0}^{N} \left[ \frac{(q^{-N}; q)_m}{(q; q)_m} \left( \frac{\alpha_1; q)_m \cdots (\alpha_r; q)_m}{(\beta_1; q)_m \cdots (\beta_s; q)_m} \left[ (-1)^m q^{m(m-1)/2} \right]^{s-r} z^m \right]
\]

Here \(N\) is an arbitrary positive integer, \(r\) and \(s\) are arbitrary nonnegative integers, the \(r + s\) parameters \(\alpha_j\) and \(\beta_k\) are arbitrary (“generic”, possibly complex) numbers, \((\alpha; q)_m\) is the \(q\)-Pochhammer symbol and \(r+1\phi_s(q^{-N}, \alpha_1, \ldots, \alpha_r; \beta_1, \ldots, \beta_s; q; z)\) is the generalized basic hypergeometric function. In this paper we obtain a set of \(N\) nonlinear algebraic equations satisfied by the \(N\) zeros \(\zeta_n = \zeta_n(\alpha; \beta; q; \gamma)\) of this polynomial. We moreover identify an \((N \times N)\)-matrix \(M = M(\alpha; \beta; \gamma; q; N)\) featuring the \(N\) eigenvalues \(\mu_n = -q^{(s-r)(N-n)}(q^{-n} - 1) \prod_{j=1}^{r} (\alpha_j q^{-n} - 1)\), \(n = 1, 2, \ldots, N\). These \(N\) eigenvalues depend only on the \(r\) parameters \(\alpha_j\) (besides \(q\) and \(N\)), implying that the \((N \times N)\)-matrix \(M\) is isospectral for variations of the \(s\) parameters \(\beta_k\); and they clearly are rational numbers if \(q\) and the \(r\) parameters \(\alpha_j\) are themselves rational numbers: a nontrivial Diophantine property.

Keywords: basic hypergeometric polynomials, Diophantine properties, isospectral matrices, special functions.

1 Introduction

We define the generalized basic hypergeometric polynomial of degree \(N\) as follows:

\[
P_N(\alpha_1, \ldots, \alpha_r; \beta_1, \ldots, \beta_s; q; z) = \sum_{m=0}^{N} \left[ \frac{(q^{-N}; q)_m}{(q; q)_m} \left( \frac{\alpha_1; q)_m \cdots (\alpha_r; q)_m}{(\beta_1; q)_m \cdots (\beta_s; q)_m} \left[ (-1)^m q^{m(m-1)/2} \right]^{s-r} z^m \right]
\]

\[
= r+1\phi_s(q^{-N}, \alpha_1, \ldots, \alpha_r; \beta_1, \ldots, \beta_s; q; z).
\]

Here \(N\) is an arbitrary positive integer, \(r\) and \(s\) are arbitrary nonnegative integers, the \(r + s\) parameters \(\alpha_j\) and \(\beta_k\) are arbitrary (“generic”, possibly complex) numbers, \((\gamma; q)_m\) is the \(q\)-Pochhammer symbol,

\[
(\gamma; q)_0 = 1, \quad (\gamma; q)_m = (1 - \gamma) \cdots (1 - \gamma q^{m-1}) \quad \text{for} \quad m = 1, 2, 3, \ldots,
\]

and \(r+1\phi_s(\alpha_0, \alpha_1, \ldots, \alpha_r; \beta_1, \ldots, \beta_s; q; z)\) is the generalized basic hypergeometric function (see [3] below for its definition, of course consistent with [1]; and, for instance, [1] for its properties). We actually prefer to work hereafter with the monic version of this polynomial, \(p_N(\alpha; \beta; q; z)\), defined of course as follows:

\[
p_N(\alpha; \beta; q; z) = p_N(\alpha; \beta; q; z) = \frac{(q; q)_N}{(q^{-N}; q)_N} \left[ \frac{(\alpha_1; q)_N \cdots (\alpha_r; q)_N}{(\beta_1; q)_N \cdots (\beta_s; q)_N} \left[ (-1)^N q^{N(N-1)/2} \right]^s P_N(\alpha; \beta; q; z),
\]

\[(3a)\]
which is clearly characterized by the property
\[
\lim_{z \to \infty} \left[ p_N \left( \alpha; \beta; q; z \right) \right] = 1 .
\] (3b)

In this paper we report properties satisfied by the \( N \) zeros \( \zeta_n = \zeta_n (\alpha; \beta; q; N) \) of this polynomial \( p_N (z) \) (or, equivalently, \( P_N (z) \)), namely by the \( N \) roots \( \zeta_n \) defined (up to permutations) by the formula
\[
p_N (\alpha; \beta; q; \zeta_n) = 0 , \quad n = 1, 2, \ldots, N ,
\] (4a)
so that
\[
p_N (\alpha; \beta; q; z) = \prod_{n=1}^{N} \left[ z - \zeta_n (\alpha; \beta; q; N) \right].
\] (4b)

**Notation 1.1.** Let us confirm that throughout this paper the parameters \( r, s \) and \( N \) are 3 arbitrary nonnegative integers (except when their values are explicitly assigned, see for instance the subsections at the end of next section). Above (and hereafter) the short-hand notations \( \alpha, \beta \) respectively \( \zeta \) denote the unordered sets of the \( r, s \) respectively \( N \) numbers \( \alpha_j, \beta_k \) respectively \( \zeta_n \). The \( N \) zeros \( \zeta_n \) are of course functions of the \( r + s + 2 \) parameters \( \alpha_j, \beta_k, q \neq 1 \) and \( N \), i. e. \( \zeta = \zeta (\alpha; \beta; q; N) \). Note that we occasionally omit to indicate explicitly the dependence on some, or on all, of these parameters (including, systematically, the dependence on \( r \) and \( s \), the values of which are considered fixed throughout this paper). Generally (unless otherwise specified) the indices \( n, m, \ell \) run over the positive integers from 1 to \( N \), while the index \( j \) runs from 1 to \( r \) and the index \( k \) from 1 to \( s \). Upper-case underlined letters denote \( N \times N \) matrices: for instance the matrix \( M \) has the \( N^2 \) elements \( M_{mn} \). We moreover assume the parameters \( \alpha_j, \beta_k, q \neq 1 \) to have generic (possibly complex) values, and the \( N \) zeros \( \zeta_n \) to be all different among themselves, \( \zeta_n \neq \zeta_m \) for \( n \neq m \), which is of course the generic case; on the understanding that otherwise our formulas might have to be understood *cum grano salis*, via appropriate limiting processes. □

In this paper we firstly obtain a set of algebraic equations satisfied by the \( N \) zeros \( \zeta_n \) of the *generalized basic hypergeometric polynomial* of order \( N \), additional to the algebraic equation defining these zeros, see (4a). This result—reported below as **Proposition 2.1** because we were unable to find any mention of it in the literature—does not seem to us to be particularly significant, but it is instrumental to prove what we consider our main finding: the identification of an \( (N \times N) \)-matrix \( M \) that features the \( N \) eigenvalues
\[
\mu_n = -q^{(s-r)(N-n)} (q^{-n} - 1) \prod_{j=1}^{r} (\alpha_j q^{N-n} - 1) , \quad n = 1, 2, \ldots, N .
\] (5)
This matrix \( M \) is explicitly defined below in terms of the \( r + s + 2 \) parameters \( \alpha_j, \beta_k, q \neq 1 \) and \( N \) characterizing the *generalized basic hypergeometric polynomial* \( p_N (z) \), and of its \( N \) zeros \( \zeta_n \) (which themselves depend on course on the \( r + s + 2 \) parameters \( \alpha_j, \beta_k, q \neq 1 \) and \( N \)). While the \( N \) eigenvalues \( \mu_n \) clearly depend only on the \( r + 2 \) parameters \( \alpha_j, q \neq 1 \) and \( N \), implying that the \( (N \times N) \)-matrix \( M \) is *isospectral* for variations of the \( s \) parameters \( \beta_k \); and *all* these eigenvalues are clearly *rational* numbers if \( q \) and the \( r \) parameters \( \alpha_j \) are themselves *rational* numbers: a nontrivial *Diophantine* property.

The findings outlined above are detailed in the following Section 2, and our main finding is proven in Section 3. A terse Section 4 (“Outlook”) outlines possible future developments. The definitions and some standard properties of the generalized basic hypergeometric polynomials are reported in the Appendix, for the convenience of the reader and also to specify our notation; the reader is advised to glance through this Appendix before reading the next section, and then to return to it whenever appropriate.

For a terse review of analogous results for *generalized hypergeometric polynomials* the interested reader is referred to [2]; of course the results reported in this reference [2] can be obtained (up to obvious notational changes) from those reported in the present paper by taking appropriately the \( q \to 1 \) limit. For results analogous to those obtained here, but for polynomials belonging to the Askey and \( q \)-Askey schemes, see [3] [4]; and let us recall in this connection that the polynomials belonging to the Askey, respectively the \( q \)-Askey, schemes (see for instance [5]) are also special cases of generalized hypergeometric, respectively basic hypergeometric, functions, with indices \( r \leq 3 \) and \( s \leq 3 \), in which the polynomial variable is however related to *parameters* of the hypergeometric function rather than to its *argument*; hence the results reported in [3] [4] are not special cases of those obtained and reported in [2] and below.

Finally, let us mention that the properties of the zeros of polynomials are a core problem of mathematics to which, over time, an immense number of investigations have been devoted. Nevertheless new findings in this area continue to emerge, see, for instance [6] [7] [8] [9] [10] [11] (and of course [2] [3] [4]).
2 Results

The generalized basic hypergeometric function \( r + 1 \phi_s \left( \alpha_0, \alpha_1, \ldots, \alpha_r; \beta_1, \ldots, \beta_s; q; z \right) \) is defined as follows (see for instance eq. (1.2.22) in [1], or eq. (0.4.2) in [5]):

\[
r + 1 \phi_s \left( \alpha_0, \alpha_1, \ldots, \alpha_r; \beta_1, \ldots, \beta_s; q; z \right) = \sum_{p=0}^{\infty} \frac{\left( \alpha_0; q \right)_p \left( \alpha_1; q \right)_p \cdots \left( \alpha_r; q \right)_p}{\left( q; q \right)_p \left( \beta_1; q \right)_p \cdots \left( \beta_s; q \right)_p} \left( -1 \right)^p q^{p(p-1)/2} z^p,
\]

where the \( q \)-Pochhammer symbol \( (\alpha; q)_p \) is defined above, see (2), and the rest of the notation is, we trust, self-evident. Clearly if one of the \( r + 1 \) parameters \( \alpha_j \) coincides with a negative integer power of \( q \), say (without loss of generality) \( \alpha_0 = q^{-N} \), and all the other \( r + s \) parameters \( \alpha_j \) and \( \beta_k \) have generic (possibly complex) values, the series in the right-hand side of the definition (6) of the generalized hypergeometric function terminates at \( j = N \) (since clearly \( (q^{-N}; q)_p = 0 \) for \( p = N + 1, N + 2, \ldots \), see (2)). This leads to the definition (11) of the generalised basic hypergeometric polynomial \( P_N \left( z \right) \) (of degree \( N \) in \( z \); below, we rather work with the monic version \( p_N \left( z \right) \) of this polynomial, see (3)).

The first result of this paper—proven in the Appendix—is the following

**Proposition 2.1.** The \( N \) zeros \( \zeta_n \equiv \zeta_n \left( \alpha; \beta; q; N \right) \) of the generalized basic hypergeometric (monic) polynomial \( p_N \left( \alpha; \beta; q; z \right) \), see [1], (3a) and (4a), satisfy the following set of \( N \) algebraic equations:

\[
- \sum_{m=1}^{N} \left( \zeta_n q - \zeta_m \right) + \sum_{k=1}^{s} \left( -1 \right)^k b_k \left( \beta \right) \prod_{m=1}^{N} \left( \zeta_n q^k - \zeta_m \right) - \prod_{m=1}^{N} \left( \zeta_n q^{k+1} - \zeta_m \right) = 0,
\]

\[
- \left( -1 \right)^{r-s} \zeta_n \sum_{m=1}^{N} \left( \zeta_n q^{s-r} - \zeta_m \right) - q^{-N} \prod_{m=1}^{N} \left( \zeta_n q^{s-r+1} - \zeta_m \right) = 0,
\]

where the \( s \) quantities \( b_k \left( \beta \right) \), respectively the \( r \) quantities \( a_j \left( \alpha \right) \), are defined, in terms of the \( s \) parameters \( \beta_k \) respectively the \( r \) parameters \( \alpha_j \), by (47) respectively (49). Note that the formula (7) becomes a bit simpler in the “balanced” case (i.e., if \( r = s \)). Also note that, if instead \( s > r \), some terms in the sum over the index \( j \) have a vanishing value (for \( j = r - s \) or \( j = s - r - 1 \)).

Our main finding—proven in Section 3—is the following

**Proposition 2.2.** Let the \( \left( N \times N \right) \)-matrix \( M \) be defined, componentwise, as follows in terms of the \( r + s + 2 \) parameters \( \alpha_j, \beta_k, q, N \) characterizing the generalized basic hypergeometric polynomial and of its \( N \) zeros \( \zeta_n \equiv \zeta_n \left( \alpha; \beta; q; N \right) \), see [1], (3a) and (4a):

\[
M_{nn} = \left( -1 \right)^s \left\{ \left( q - 1 \right)^2 g_n \left( 1, \zeta \right) + \sum_{k=1}^{s} b_k \left( \beta \right) \frac{\left( -1 \right)^k}{q^k} \left[ \left( q^{k+1} - 1 \right)^2 g_n \left( k + 1, \zeta \right) - \left( q^k - 1 \right)^2 g_n \left( k, \zeta \right) \right] \right\}
\]

\[
\left\{ \left( -1 \right)^{r+1} \zeta_n \left[ \left( q^{-N} \right) \left( q^{s-r+1} - 1 \right)^2 g_n \left( s - r + 1, \zeta \right) - \left( q^{s-r} - 1 \right)^2 g_n \left( s - r, \zeta \right) \right] \right. 
\]

\[
+ \sum_{j=1}^{r} a_j \left( \alpha \right) \left( -1 \right)^j \left[ \left( q^{-N} \right) \left( q^{j+s+1-r} - 1 \right)^2 g_n \left( j + s + 1 - r, \zeta \right) - \left( q^{j+s-r} - 1 \right)^2 g_n \left( j + s - r, \zeta \right) \right] \}
\]

\[
\left. \left\{ \left( -1 \right)^{r} \left[ q^{-N} \left( q^{s-r+1} - 1 \right) f_n \left( s - r + 1, \zeta \right) - \left( q^{s-r} - 1 \right) f_n \left( s - r, \zeta \right) \right] \right. 
\]

\[
+ \sum_{j=1}^{r} a_j \left( \alpha \right) \left( -1 \right)^j \left[ q^{-N} \left( q^{j+s+1-r} - 1 \right) f_n \left( j + s + 1 - r, \zeta \right) - \left( q^{j+s-r} - 1 \right) f_n \left( j + s - r, \zeta \right) \right] \right) 
\],

where \( n = 1, 2, \ldots, N \).
\[ M_{nm} = (-1)^{s+1} \frac{\zeta_n}{(\zeta_n - \zeta_m)^2} \left\{ (q - 1)^2 f_{nm}(1, \zeta) \right. \\
+ \sum_{k=1}^{s} b_k(\beta) \frac{(-1)^k}{q^k} \left[ (q^{k+1} - 1)^2 f_{nm}(k + 1, \zeta) - (q^k - 1)^2 f_{nm}(k, \zeta) \right] \right\} \\
+ (-1)^r \frac{\zeta_n^2}{(\zeta_n - \zeta_m)^2} \left\{ q^{-N}(q^{s-r+1} - 1)^2 f_{nm}(s - r + 1, \zeta) - (q^{r-r} - 1)^2 f_{nm}(s - r, \zeta) \right. \\
+ \sum_{j=1}^{r} a_j(q)(-1)^j \left[ q^{-N}(q^{j+s+1-r} - 1)^2 f_{nm}(j + s + 1 - r, \zeta) - (q^{j+s-r} - 1)^2 f_{nm}(j + s - r, \zeta) \right] \right\}, \]
\[ n, m = 1, 2, ..., N, \quad n \neq m, \tag{8b} \]

where the quantities \( f_{nm}(p, \zeta) \) and \( g_n(p, \zeta) \) are defined by \([40]\). Then the \( N \) eigenvalues \( \mu_n \) of this matrix are given by the following neat formulas:
\[ \mu_n = -q^{(s-r)(N-n)}(q^{n-1} - 1) \prod_{j=1}^{r} (\alpha_j q^{N-n} - 1), \quad n = 1, 2, ..., N. \tag{9} \]

The following corollary, which is an immediate consequence of Proposition 2.2, yields, via the definition \([8]\) of the \((N \times N)\)-matrix \( M \) and the expression \([9]\) of its \( N \) eigenvalues \( \mu_n \), a number of algebraic formulas satisfied by the \( N \) zeros \( \zeta_n \) of the generalized basic hypergeometric polynomial of order \( N \), see \([11]\) or \([55]\) and \([4a]\).

**Corollary 2.1.**

\[ \text{Trace } [M^p] = \sum_{n=1}^{N} [(\mu_n)^p], \quad p = 1, 2, 3, ..., \tag{10a} \]
\[ \text{Det } M = \prod_{n=1}^{N} (\mu_n). \tag{10b} \]

**Remark 2.1.** For fixed \( q \) and \( N \), the \( N \) eigenvalues \( \mu_n \) of the \((N \times N)\)-matrix \( M \) (see \([8]\)) depend only on the \( r \) parameters \( \alpha_j \) (see \([9]\)), while the matrix \( M \) depends on the \( s + r \) parameters \( \beta_k \) and \( \alpha_j \) via the dependence of the parameters \( b_k \) respectively \( a_j \) on \( \beta_k \) respectively \( \alpha_j \) (see \([17]\) respectively \([19]\)) and via the dependence of the \( N \) zeros \( \zeta_n \) on the parameters \( \beta_k \) and \( \alpha_j \). Hence the \((N \times N)\)-matrix \( M \) is isospectral for variations of the \( s \) parameters \( \beta_k \). And note moreover that the \( N \) eigenvalues \( \mu_m \) are rational numbers if the \( r \) parameters \( \alpha_j \) are themselves rational numbers, and \( q \) is also a rational number: a nontrivial Diophantine property of the \((N \times N)\)-matrix \( M \). \( \square \)

**Remark 2.2.** All the above results are of course true as written only provided the \( N \) zeros \( \zeta_n \) are all different among themselves; but they remain valid by taking appropriate limits whenever this restriction does not hold. \( \square \)

**Remark 2.3.** Immediate generalizations—whose explicit formulations can be left to the interested reader—of Propositions 2.1 and 2.2 obtain from these two propositions via the special assignment \( \alpha_{\tilde{s}+p} = \beta_{\tilde{z}+p} \) for \( p = 1, ..., u \) with \( u \) an arbitrary nonnegative integer such that both \( \tilde{r} = r - u \) and \( \tilde{s} = s - u \) are positive integers. These propositions refer then to the \( N \) zeros of the generalized basic hypergeometric polynomial \( \tilde{P}_N(\alpha_1, ..., \alpha_\tilde{r}; \beta_1, ..., \beta_\tilde{s}; q; z) = P_N(\alpha_1, ..., \alpha_r, ..., \alpha_\tilde{s}; \beta_1, ..., \beta_\tilde{s}, ..., \beta_\tilde{z}; q; z) \) which depend additionally to \( N \) and \( q \)—only on the \( \tilde{r} + \tilde{s} = r + s - 2u \) parameters \( \alpha_j \) with \( j = 1, ..., \tilde{r} = r - u \) and \( \beta_k \) with \( k = 1, ..., \tilde{s} = s - u \), but feature quantities \( b_k \) and \( a_j \) (see \([17]\) and \([49]\)) that depend on the \( 2 + r + s \) parameters \( N, q, \alpha_j \) with \( j = 1, ..., r \) and \( \beta_k \) with \( k = 1, ..., s \). \( \square \)

Let us end this Section 2 by displaying explicitly the above results for small values of the integers \( r, s \) and \( u \) (see Remark 2.3).

### 2.1 The case \( r = s = 1, u = 0 \)

If \( r = s = 1 \) and \( u = 0 \), the generalized basic hypergeometric polynomial \([11]\) is given by
\[ P_N(\alpha_1; \beta_1; q; z) = \sum_{m=0}^{N} \frac{(q^{-N}; q)_m (\alpha_1; q)_m}{(q; q)_m (\beta_1; q)_m} z^m. \tag{11} \]
Let $\zeta_n$, where $n = 1, 2, \ldots, N$, be the zeros of $P_N(\alpha_1; \beta_1; q; z)$. Then, by Proposition 2.1,

$$
\left(1 - \zeta_n q^{-N} + \frac{\beta_1}{q} - \alpha_1 \zeta_n \right) \prod_{m=1}^{N} (\zeta_n q - \zeta_m) + \left( -\frac{\beta_1}{q} + \alpha_1 \zeta_n q^{-N} \right) \prod_{m=1}^{N} (\zeta_n q^2 - \zeta_m) = 0, \quad n = 1, 2, \ldots, N.
$$

(12)

Define an $(N \times N)$-matrix $M$ componentwise as follows:

$$
M_{nn} = (q - 1)^2 g_n(1, \zeta) \left[ -1 - \frac{\beta_1}{q} + \zeta_n (q^{-N} + \alpha_1) \right] + (q^2 - 1)^2 g_n(2, \zeta) \left[ \frac{\beta_1}{q} - \zeta_n \alpha_1 q^{-N} \right]
+ (q - 1) f_n(1, \zeta) [-q^{-N} - \alpha_1] + (q^2 - 1) f_n(2, \zeta) \alpha_1 q^{-N}, \quad n = 1, 2, \ldots, N,
$$

(13a)

$$
M_{nm} = \frac{\zeta_n}{(\zeta_n - \zeta_m)^2} \left\{ (q - 1)^2 f_{nm}(1, \zeta) \left[ 1 + \frac{\beta_1}{q} - \zeta_n (q^{-N} + \alpha_1) \right]
+ (q^2 - 1)^2 f_{nm}(2, \zeta) \left[ -\frac{\beta_1}{q} + \zeta_n \alpha_1 q^{-N} \right] \right\}, \quad n, m = 1, 2, \ldots, N, \quad n \neq m,
$$

(13b)

where the quantities $f_{nm}(p, \zeta)$ and $g_n(p, \zeta)$ are defined by [40]. By Proposition 2.2, the $N$ eigenvalues $\mu_n$ of this matrix are given by

$$
\mu_n = -(q^{-n} - 1)(\alpha_1 q^{N-n} - 1), \quad n = 1, 2, \ldots, N.
$$

(14)

By Corollary 2.1, the trace of the matrix $M$ is given by

$$
\text{Tr}(M) = -\alpha_1 \frac{q^{N+2}}{q^2 - 1} (1 - q^{-2N-2}) + \frac{q + \alpha_1 q^{N+1}}{q - 1} (1 - q^{-N-1}) - N - 1
$$

(15a)

and the determinant of $M$

$$
\det(M) = (-1)^N \prod_{n=1}^{N} \left[ (q^{-n} - 1)(\alpha_1 q^{N-n} - 1) \right].
$$

(15b)

### 2.2 The case $r = 2, s = 1, u = 0$

If $r = 2, s = 1$ and $u = 0$, the generalized basic hypergeometric polynomial (11) is given by

$$
P_N(\alpha_1, \alpha_2; \beta_1; q; z) = \sum_{m=0}^{N} \frac{(q^{-N}; q)_m (\alpha_1; q)_m (\alpha_2; q)_m}{(q; q)_m (\beta_1; q)_m} (-1)^m q^{m(1-m)/2} z^m.
$$

(16)

Let $\zeta_n$, where $n = 1, 2, \ldots, N$, be the zeros of $P_N(\alpha_1, \alpha_2; \beta_1; q; z)$ and let $a_1 = a_1 (\alpha) = \alpha_1 + \alpha_2, a_2 = a_2 (\alpha) = \alpha_1 \alpha_2$ (see [19]). Then, by Proposition 2.1,

$$
\left[ -1 - \frac{\beta_1}{q} + \zeta_n (a_2 + q^{-N} a_1) \right] \prod_{m=1}^{N} (\zeta_n q - \zeta_m) + \left[ \frac{\beta_1}{q} - \zeta_n q^{-N} a_2 \right] \prod_{m=1}^{N} (\zeta_n q^2 - \zeta_m)
+ \zeta_n \prod_{m=1}^{N} (\zeta_n q^{-1} - \zeta_m) = 0, \quad n = 1, 2, \ldots, N.
$$

(17)

Moreover, by Proposition 2.2, the $N \times N$ matrix $\underline{M}$ defined componentwise by

$$
M_{nn} = (q - 1)^2 g_n(1, \zeta) \left[ -1 - \frac{\beta_1}{q} + \zeta_n (a_1 q^{-N} + a_2) \right] + (q^2 - 1)^2 g_n(2, \zeta) \left[ \frac{\beta_1}{q} - \zeta_n a_2 q^{-N} \right]
+ (q - 1) f_n(1, \zeta) [-a_1 q^{-N} + a_2] + (q^2 - 1) f_n(2, \zeta) a_2 q^{-N}, \quad n = 1, 2, \ldots, N,
$$

(18a)
\[ M_{nm} = \frac{\zeta_n}{(\zeta_n - \zeta_m)^2} \left( (q - 1)^2 f_{nm}(1, \zeta) \left[ 1 + \frac{\beta_1}{q} - \zeta_n (a_1 q^{-N} + a_2) \right] \right. \\
+ (q^2 - 1)^2 f_{nm}(2, \zeta) \left[ - \frac{\beta_1}{q} + \zeta_n a_2 q^{-N} \right] \\
\left. -(q^{-1} - 1)^2 f_{nm}(-1, \zeta) \zeta_n \right), \quad n, m = 1, 2, \ldots, N, n \neq m, \quad (18b) \]

has the eigenvalues
\[ \mu_n = -q^{-N+n}(q^{-n}-1)(\alpha_1 q^{N-n}-1)(\alpha_2 q^{N-n}-1), \quad n = 1, 2, \ldots, N. \quad (19) \]

By Corollary 2.1, the trace of the matrix \( M \) is given by
\[ \text{Tr}(M) = \frac{q^{-N}}{q^2 - 1} \left\{ -N(q^2-1) \left[ 1+q^N(\alpha_1+\alpha_2) \right] + (q^N-1) \left[ q^2+\alpha_1+\alpha_2-\alpha_1\alpha_2+q^{1+N}\alpha_1\alpha_2+q(1+\alpha_1+\alpha_2) \right] \right\} \quad (20a) \]
and the determinant of \( M \)
\[ \text{det}(M) = (-1)^N \prod_{n=1}^{N} \left( q^{-N+n}(q^{-n}-1)(\alpha_1 q^{N-n}-1)(\alpha_2 q^{N-n}-1) \right). \quad (20b) \]

2.3 The case \( r = s = 2, u = 0 \)

If \( r = s = 2 \) and \( u = 0 \), the generalized basic hypergeometric polynomial \( (1) \) is given by
\[ P_N(\alpha_1, \alpha_2; \beta_1, \beta_2; q; z) = \sum_{m=0}^{N} \frac{(q^{-N}; q)_m (\alpha_1; q)_m (\alpha_2; q)_m}{(q; q)_m (\beta_1; q)_m (\beta_2; q)_m} z^m. \quad (21) \]

Let \( \zeta_n \), where \( n = 1, 2, \ldots, N \), be the zeros of \( P_N(\alpha_1, \alpha_2; \beta_1, \beta_2; q; z) \) and let \( a_1 = \alpha_1(\alpha) = \alpha_1 + \alpha_2, a_2 = \alpha_2(\alpha) = \alpha_1\alpha_2, \) (see \( (49) \)), \( b_1 = b_1(\beta) = \beta_1 + \beta_2, b_2 = b_2(\beta) = \beta_1\beta_2 \) (see \( (17) \)). Then, by Proposition 2.1,
\[ \left[ -1 - \frac{b_1}{q} + \zeta_n(q^{-N} + a_1) \right] \prod_{m=1}^{N} (\zeta_n q - \zeta_m) \\
+ \left[ \frac{b_1}{q} + \frac{b_2}{q^2} - \zeta_n(q^{-N} a_1 + a_2) \right] \prod_{m=1}^{N} (\zeta_n q^2 - \zeta_m) \\
+ \left[ -\frac{b_2}{q^2} + \zeta_n q^{-N} a_2 \right] \prod_{m=1}^{N} (\zeta_n q^3 - \zeta_m) = 0, \quad n = 1, 2, \ldots, N. \quad (22) \]

Moreover, by Proposition 2.2, the \( N \times N \) matrix \( M \) defined component-wise by
\[ M_{nn} = (q - 1)^2 g_n(1, \zeta) \left[ 1 + \frac{b_1}{q} - \zeta_n(q^{-N} + a_1) \right] \\
+ (q^2 - 1)^2 g_n(2, \zeta) \left[ - \frac{b_1}{q} - \frac{b_2}{q^2} + \zeta_n(q^{-N} a_1 + a_2) \right] \\
+ (q^3 - 1)^2 g_n(3, \zeta) \left[ \frac{b_2}{q^2} - \zeta_n a_2 q^{-N} \right] + (q - 1) f_n(1, \zeta) \left[ q^{-N} + a_1 \right] \\
+ (q^2 - 1) f_n(2, \zeta) \left[ a_1 q^{-N} - a_2 \right] + (q^3 - 1) f_n(3, \zeta) a_2 q^{-N}, \quad n = 1, 2, \ldots, N, \quad (23a) \]
\[ M_{nm} = \frac{\zeta_n}{(\zeta_n - \zeta_m)^2} \left\{ (q - 1)^2 f_{nm}(1, \zeta) \left[ -1 - \frac{b_1}{q} + \zeta_n(q^{-N} + a_1) \right] \\
+ (q^2 - 1)^2 f_{nm}(2, \zeta) \left[ \frac{b_1}{q} + \frac{b_2}{q^2} - \zeta_n(a_1 q^{-N} + a_2) \right] \\
+ (q^3 - 1)^2 f_{nm}(3, \zeta) \left[ -\frac{b_2}{q^2} + \zeta_n a_2 q^{-N} \right] \right\}, \quad n, m = 1, 2, \ldots, N, n \neq m, \quad (23b) \]
has the eigenvalues

\[ \mu_n = -(q^{-n} - 1)(\alpha_1 q^{N-n} - 1)(\alpha_2 q^{N-n} - 1), \quad n = 1, 2, \ldots, N. \]  

(24)

By Corollary 2.1, the trace of the matrix \( M \) is given by

\[ \text{Tr}(M) = -\sum_{n=1}^{N} \left[ (q^{-n} - 1)(\alpha_1 q^{N-n} - 1)(\alpha_2 q^{N-n} - 1) \right] \]  

(25a)

and the determinant of \( M \)

\[ \det(M) = (-1)^N \prod_{n=1}^{N} \left[ (q^{-n} - 1)(\alpha_1 q^{N-n} - 1)(\alpha_2 q^{N-n} - 1) \right]. \]  

(25b)

### 2.4 The case \( r = s = 2, \ u = 1 \)

If \( r = s = 2 \) and \( u = 1 \), then \( \beta_2 = \alpha_2 \) and the generalized basic hypergeometric polynomial (11) is given by

\[ \tilde{P}_N^{(1,1)}(\alpha_1, \alpha_2; \beta_1; q; z) = P_N(\alpha_1, \alpha_2; \beta_1, \alpha_2; q; z) = \sum_{m=0}^{N} \frac{(q^{-N}; q)_m (\alpha_1; q)_m}{(q; q)_m (\beta_1; q)_m} z^m. \]  

(26)

Let \( \zeta_n \), where \( n = 1, 2, \ldots, N \), be the zeros of \( \tilde{P}_N^{(1,1)}(\alpha_1, \alpha_2; \beta_1; q; z) \) and let \( a_1 = \alpha_1 + \alpha_2, \ a_2 = \alpha_1 \alpha_2, \) (see (10)), \( b_1 = \beta_1 + \alpha_2, \ b_2 = \beta_1 \alpha_2 \) (see (17)). Then, by Proposition 2.1 and Corollary 2.1, the zeros \( \zeta \) satisfy algebraic relations (22) and the matrix \( \tilde{M} \) defined in terms of the zeros \( \zeta \) by formulas (??) has the eigenvalues

\[ \mu_n = -(q^{-n} - 1)(\alpha_1 q^{N-n} - 1)(\alpha_2 q^{N-n} - 1), \quad n = 1, 2, \ldots, N, \]  

(27)

the trace

\[ \text{Tr}(\tilde{M}) = -\sum_{n=1}^{N} \left[ (q^{-n} - 1)(\alpha_1 q^{N-n} - 1)(\alpha_2 q^{N-n} - 1) \right] \]  

(28a)

and the determinant

\[ \det(\tilde{M}) = (-1)^N \prod_{n=1}^{N} \left[ (q^{-n} - 1)(\alpha_1 q^{N-n} - 1)(\alpha_2 q^{N-n} - 1) \right]. \]  

(28b)

### 3 Proof of Proposition 2.2

Let the \( t \)-dependent monic polynomial, of degree \( N \) in \( z \), be characterized by its \( N \) zeros \( z_n(t) \),

\[ \psi_N(z, t) = \prod_{n=1}^{N} \left[ z - z_n(t) \right], \]  

(29a)

and by the \( N \) coefficients \( c_m(t) \) of its expansion in powers of \( z \),

\[ \psi_N(z, t) = z^N + \sum_{m=1}^{N} c_m(t) z^{N-m}. \]  

(29b)

It is plain that these two representations, (29a) and (29b), are consistent, implying a uniquely defined expression of the \( N \) coefficients \( c_m(t) \) in terms of the \( N \) zeros \( z_n(t) \), and an expression—unique up to permutations and of course explicitly known only for \( N \leq 4 \)—of the \( N \) zeros \( z_n(t) \) in terms of the \( N \) coefficients \( c_m(t) \).

The next step is to assume that this \( t \)-dependent polynomial \( \psi_N(z, t) \) satisfy the following linear Differential q-Difference Equation (DqDE):

\[
\frac{\partial \psi_N(z, t)}{\partial t} = z^{-1} \left[ \Delta_1 \prod_{k=1}^{s} \left( \Delta_{\beta_k/q} \right) \right] \psi_N(z, t) \\
- \left[ \Delta_{q^{-N}} \prod_{j=1}^{r} \left( \Delta_{\alpha_j} \right) \right] \psi_N \left( q^{s-r} z, t \right),
\]  

(30)
where the operator $\Delta_\gamma$, acting on functions of the variable $z$, is defined as follows:

\[
\Delta_\gamma = (\gamma \delta - 1) \quad \text{with} \quad \delta f(z) = f(qz)
\]  

(31a) (see (14b)). Note that this implies

\[
\Delta_\gamma z^p = (\gamma q^p - 1) z^p.
\]  

(31b)

Let us first of all check whether this is consistent with the fact that $\psi_N(z,t)$ is a polynomial of degree $N$ in $z$. Since the operator $\Delta_\gamma$, when applied to a power of $z$, does not change that power but only multiplies it by a number, see (31b), it is plain that this is guaranteed by the fact that $\Delta q^N z^N = (1 - 1) z^N = 0$ (see (31a)); while of course clearly $\Delta_c c = 0$ (where $c$ indicates any constant, i.e. a quantity independent of the variable $z$).

Next, let us see what the $DqDE$ (30) implies for the coefficients $c_m(t)$, see (29b). Via (29b) and (31b) it clearly amounts to the following autonomous linear system of ODEs:

\[
c_m(t) = \left[ (q^{n-m+1} - 1) \prod_{k=1}^s (\beta_k q^{n-m} - 1) \right] c_{m-1}(t) \\
- \left[ q^{(s-r)(N-m)} (q^{-m} - 1) \prod_{j=1}^r (\alpha_j q^{N-m} - 1) \right] c_m(t),
\]  

\[m = 1, 2, ..., N, \quad \text{with} \quad c_0 = 1.
\]  

(32a)

Of course here and below a superimposed dot denotes a $t$-differentiation.

Note the formal consistency of this system of evolution equations with the assignment $c_0 = 1$, hence with the fact that $\psi_N(z,t)$ is a monic polynomial of degree $N$ in $z$, see (29b).

It is plain that the solution of this system reads

\[
c_m(t) = \sum_{n=0}^N \left[ \eta_n u_n^{(m)} \exp(\mu_m t) \right],
\]  

(32b)

where the $N$-vectors $u^{(m)}$, respectively the $N$ numbers $\mu_n$, are the $N$ eigenvectors, respectively the corresponding $N$ eigenvalues, of the $(N \times N)$-matrix $C$ with elements

\[
C_{nn} = -q^{(s-r)(N-n)} (q^{-n} - 1) \prod_{j=1}^r (\alpha_j q^{N-n} - 1), \quad n = 1, 2, ..., N,
\]  

(32c)

\[
C_{n,n-1} = (q^{N-n+1} - 1) \prod_{k=1}^s (\beta_k q^{N-n} - 1), \quad n = 2, ..., N,
\]  

(32d)

and all other elements vanishing:

\[
C u^{(m)} = \mu_n u^{(m)};
\]  

(32e)

while the $N$ ($t$-independent) numbers $\eta_n$ can be arbitrarily assigned—getting thereby the general solution of system (32a)—or can be adjusted to fit the initial data $c_m(0)$ so that

\[
c_m(0) = \sum_{n=1}^N \left[ \eta_n u_n^{(m)} \right], \quad m = 1, ..., N
\]  

(32f)

—getting thereby the solution of the initial-value problem for system (32a). The triangular character of the $(N \times N)$-matrix $C$, see (32c) and (32d), implies that its $N$ eigenvalues $\mu_n$ coincide with its diagonal elements:

\[
\mu_n = -q^{(s-r)(N-n)} (q^{-n} - 1) \prod_{j=1}^r (\alpha_j q^{N-n} - 1), \quad n = 1, 2, ..., N.
\]  

(32g)

Our next task is to discuss the $t$-evolution of the $N$ zeros $z_n(t)$ of $\psi(z,t)$, see (38a), implied by the $DqDE$ (30).
The first observation is that the equilibrium—i. e., \( t \)-independent—solution \( \bar{\psi}(z) \) of DqDE (30) is the generalized basic hypergeometric polynomial \( p_N(z) = p_N(\alpha; \beta; q; z) \), see (3a),

\[
\bar{\psi}(z) = p_N(z) \, .
\] (33)

This is implied by the \( q \)-difference equation satisfied by \( p_N(z) \), see (15), which clearly implies that the right-hand side of the DqDE (30) vanishes for \( \psi(z, t) = \bar{\psi}(z) = p_N(z) \) (and note the consistency implied by the fact that \( \bar{\psi}(z) \) and \( p_N(z) \) are both monic polynomials of degree \( N \)). Hence the equilibrium—i. e., \( t \)-independent—configuration of the \( N \) zeros \( z_n(t) \) of \( \psi(z, t) \), see (38a), is

\[
z_n(t) = z_n(0) = \bar{z}_n = \zeta_n \, , \quad n = 1, 2, ..., N \, ,
\] (34)

where the \( N \) numbers \( \zeta_n \) are the \( N \) zeros of the (monic) generalized basic hypergeometric polynomial \( p_N(z) \), see (3a).

Next, let us reformulate DqDE (30) as follows:

\[
\left(\frac{\partial}{\partial t}\right) \psi_N(z, t) = \text{RHS}(z, t) \, , \tag{35a}
\]

\[
\text{RHS}(z, t) = (-1)^s z^{-1}\left\{\psi_N(qz, t) - \psi_N(z, t) + \sum_{k=1}^{s} b_k(\beta) \frac{(-1)^k}{q^k} \left[\psi_N(q^{k+1}z, t) - \psi_N(q^kz, t)\right]\right\}
\]

\[
-(-1)^s \left\{ q^{-N} \psi_N(q^{s+r+1}z, t) - \psi_N(q^{s-r}z, t) \right\}
\]

\[
+ \sum_{j=1}^{r} a_j(\alpha)(-1)^j \left[ q^{-N} \psi_N(q^{j+s+1-r}z, t) - \psi_N(q^{j+s-r}z, t)\right]\, , \tag{35b}
\]

by repeating, on the right-hand side of DqDE (30), the same development that led, in the Appendix, from (44a) to (50c); this of course implies that the quantities \( b_k(\beta) \) respectively \( a_j(\alpha) \) are, here and hereafter, defined as in the Appendix in terms of the parameters \( \beta_k \) respectively \( \alpha_j \), see (4) and (9).

Next, let us insert in this DqDE, (35), the representation (29a) of the monic polynomial \( \psi(z, t) \) via its zeros \( z_n(t) \). The left-hand side of this DqDE then reads (by logarithmic \( t \)-differentiation of (29a))

\[
\left(\frac{\partial}{\partial t}\right) \psi_N(z, t) = -\psi_N(z, t) \sum_{m=1}^{N} \left[ \frac{\dot{z}_m(t)}{z - z_m(t)} \right] = -\sum_{m=1}^{N} \left\{ \dot{z}_m(t) \prod_{\ell=1, \ell \neq m}^{N} [z - \zeta_\ell(t)] \right\} \, , \tag{36a}
\]

implying, for \( z = z_n(t) \),

\[
\left. \left(\frac{\partial}{\partial t}\right) \psi_N(z, t) \right|_{z=z_n(t)} = -\dot{z}_n(t) \prod_{\ell=1, \ell \neq n}^{N} [z_n(t) - \zeta_\ell(t)] \, . \tag{36b}
\]

While the right-hand side (35b) of (35a) clearly reads

\[
\text{RHS}(z, t) = (-1)^s z^{-1}\left\{ \prod_{\ell=1}^{N} [qz - \zeta_\ell(t)] - \prod_{\ell=1}^{N} [z - \zeta_\ell(t)] \right\}
\]

\[
+ \sum_{k=1}^{s} b_k(\beta) \frac{(-1)^k}{q^k} \left\{ \prod_{\ell=1}^{N} [q^{k+1}z - \zeta_\ell(t)] - \prod_{\ell=1}^{N} [q^kz - \zeta_\ell(t)] \right\}
\]

\[
-(-1)^s \left\{ q^{-N} \prod_{\ell=1}^{N} [q^{s+r+1}z - \zeta_\ell(t)] - \prod_{\ell=1}^{N} [q^{s-r}z - \zeta_\ell(t)] \right\}
\]

\[
+ \sum_{j=1}^{r} a_j(\alpha)(-1)^j \left[ q^{-N} \prod_{\ell=1}^{N} [q^{j+s+1-r}z - \zeta_\ell(t)] - \prod_{\ell=1}^{N} [q^{j+s-r}z - \zeta_\ell(t)]\right]\, , \tag{37a}
\]
implying, for \( z = z_n(t) \),

\[
RHS (z_n(t), t) = (-1)^s z_n(t)^{-1} \left\{ \prod_{\ell=1}^{N} (q z_n(t) - z_\ell(t)) \right\}
+ \sum_{k=1}^{s} b_k (\beta) (-1)^k \left[ \prod_{\ell=1}^{N} [q^{k+1} z_n(t) - z_\ell(t)] - \prod_{\ell=1}^{N} [q^{k} z_n(t) - z_\ell(t)] \right] 
- (-1)^r \left\{ q^{-N} \prod_{\ell=1}^{N} [q^{s-r+1} z_n(t) - z_\ell(t)] - \prod_{\ell=1}^{N} [q^{s-r} z_n(t) - z_\ell(t)] \right\}
+ \sum_{j=1}^{r} a_j (\omega) (-1)^j \left\{ q^{-N} \prod_{\ell=1}^{N} [q^{j+s+1-r} z_n(t) - z_\ell(t)] - \prod_{\ell=1}^{N} [q^{j+s-r} z_n(t) - z_\ell(t)] \right\}.
\]

(37b)

It is thus seen that the equations of motion characterizing the \( t \)-evolution of the \( N \) zeros \( z_n(t) \) of \( \psi(z,t) \) read as follows (of course below a superimposed dot denotes a \( t \)-differentiation, and we omit for notational simplicity to display the \( t \)-dependence of the zeros):

\[
\dot{z}_n = (-1)^{s+1} \left\{ (q-1) f_n(1, \bar{z}) + \sum_{k=1}^{s} b_k (\beta) (-1)^k \left[ (q^{k+1} - 1) f_n(k+1, \bar{z}) - (q^{k} - 1) f_n(k, \bar{z}) \right] \right\}
+ (-1)^r z_n \left\{ q^{-N} (q^{s-r+1} - 1) f_n(s-r+1, \bar{z}) - (q^{s-r} - 1) f_n(s-r, \bar{z}) \right\}
+ \sum_{j=1}^{r} a_j (\omega) (-1)^j \left[ q^{-N} (q^{j+s+1-r} - 1) f_n(j+s+1-r, \bar{z}) - (q^{j+s-r} - 1) f_n(j+s-r, \bar{z}) \right]\right\},
\]

(38a)

where

\[
f_n(p, \bar{z}) = f_n(p, z_1, \ldots, z_N) = \prod_{\ell=1, \ell \neq n}^{N} \left( \frac{q^p z_n - z_\ell}{z_n - z_\ell} \right), \quad n = 1, 2, \ldots, N.
\]

(38b)

This is an interesting dynamical system, a complete investigation of which is beyond the scope of the present paper. But before proceeding with our task, let us pause and recall that the first idea to relate the zeros of polynomials to the equilibria of a dynamical system goes back to Stieltjes and Szegö [12], was resuscitated in [13] to identify “solvable” many-body problems (see also the extended treatment of this approach in [14]), and then extensively used to obtain results concerning the zeros of the classical polynomials and of Bessel functions, see the paper [15] where several such findings are derived and reviewed. For more recent developments along somewhat analogous lines see, for instance, [16, 17, 18, 19, 20].

Here we need to focus only on the behavior of this dynamical system. In the immediate neighborhood of its equilibrium configuration \( z_n(t) = \bar{z}_n = \zeta_n \), see (34). To this end we set

\[
z_n(t) = \zeta_n + \varepsilon \xi_n(t), \quad n = 1, \ldots, N,
\]

(39a)

implying

\[
\dot{z}_n(t) = \varepsilon \dot{\xi}_n(t), \quad n = 1, \ldots, N,
\]

(39b)

with \( \varepsilon \) infinitesimal. To aid the task of linearizing system (38a), we introduce the quantities

\[
f_{nm}(p, \bar{z}) = f_{nm}(p, z_1, \ldots, z_N) = \prod_{\ell=1, \ell \neq n,m}^{N} \left( \frac{q^p z_n - z_\ell}{z_n - z_\ell} \right)
\]

(40a)

(implying \( f_{nm}(p, \bar{z}) = f_n(p, \bar{z}) \left[ (z_n - z_m) / (q^p z_n - z_m) \right] \); see (38b)), and

\[
g_n(p, \bar{z}) = g_n(p, z_1, \ldots, z_N) = \sum_{k=1, k \neq n}^{N} \left[ f_{nk}(p, \bar{z}) \frac{z_k}{(z_n - z_k)^2} \right];
\]

(40b)

and we note that

\[
\frac{\partial f_n}{\partial z_n}(p, \bar{z}) = (1 - q^p) g_n(p, \bar{z})
\]

(40c)
\[ \frac{\partial f_n}{\partial z_m}(p, z) = (q^p - 1) f_{nm}(p, z) \frac{z_n}{(z_n - z_m)^2}, \] (40d)

where \( n, m = 1, 2, \ldots, N \) and \( n \neq m \).

The insertion of ansatz (39) in the equations of motion (38a) is, to order \( \varepsilon^0 = 1 \), clearly consistent via Proposition 2.1. To order \( \varepsilon \), it yields the linearized system of ODEs

\[ \dot{\xi} = M \xi; \quad \dot{\xi}_n = \sum_{m=1}^{N} (M_{nm} \xi_m), \quad n = 1, \ldots, N, \] (41)

with the \((N \times N)\)-matrix \( M \) defined, componentwise, by (8) (the diligent reader might wish to check, using the notation and the formulas in (38b) and (40), the relevant computation, which is standard but somewhat cumbersome—although certainly doable without the help of a computer). Here and hereafter the notation \( \xi = \xi(t) \) denotes the \( N \)-vector of components \( \xi_n = \xi_n(t) \). As for the implications to orders \( \varepsilon^p \) with \( p = 2, 3, \ldots \) of the insertion of the ansatz (39) in the equations of motion (38a), they consist of additional systems of algebraic equations satisfied by the zeros \( \zeta_n \) of the generalized basic hypergeometric polynomial \( P_N(z) \) or \( p_N(z) \), see (1), (3a) and (4a), the explicit display of which is left to the interested reader (to get them it might be expedient to take advantage of appropriate computer packages such as Mathematica or Maple).

The general solution of the system of linear ODEs (41) reads of course as follows:

\[ \xi(t) = \sum_{m=1}^{N} \left[ \eta_m \exp(\tilde{\mu}_m t) \ v^{(m)} \right], \] (42)

where the \( N \) (t-independent) parameters \( \eta_m \) can be arbitrarily assigned (or adjusted to satisfy the \( N \) initial conditions \( \xi_n(0) \)), while the numbers \( \tilde{\mu}_m \) respectively the \( N \)-vectors \( v^{(m)} \) are clearly the \( N \) eigenvalues respectively the \( N \) eigenvectors of the matrix \( M \),

\[ M v^{(m)} = \tilde{\mu}_m v^{(m)}, \quad m = 1, \ldots, N. \] (43)

But the behavior of the dynamical system (41) in the immediate vicinity of its equilibria cannot differ from its general behavior, which is characterized by the \( N \) exponentials \( \exp(\mu_m t) \), as implied by the relation between the \( N \) zeros \( \zeta_n(t) \) and the coefficients \( c_m(t) \) of the monic polynomial (of degree \( N \) in \( z \)) \( \psi_N(z, t) \), see (29), and by the explicit formula, see (32), detailing the t-evolution of the \( N \) coefficients \( c_m(t) \). Hence the (set of) eigenvalues \( \tilde{\mu}_m \) of the matrix \( M \), see (3), must coincide with the (set of) eigenvalues \( \mu_m \), see (32g), of the matrix \( C \), see (32). Proposition 2.2 is thereby proven.

### 4 Outlook

A follow-up to the present paper might explore the dynamical system (38a), as well as other dynamical systems obtained by analogous methods, also connected with hypergeometric or basic hypergeometric polynomials, but featuring more interesting equations of motion, for instance of Newtonian type—allowing their interpretation as “many-body models”.

Another possible direction of further investigation might try and extend the approach and findings, reported in this paper for the \( N \) zeros of basic hypergeometric polynomials of order \( N \), to the, generally infinite, set of zeros of (nonpolynomial) generalized basic hypergeometric functions.

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6 Appendix A: Properties of the generalized basic hypergeometric polynomials

In this appendix we report some properties of the generalized basic hypergeometric polynomial \( p_N(z) \equiv p_N(\alpha; \beta; q; z) \) (see (11) and (58) and Notation 1.1); and we prove Proposition 2.1. It is understood that, throughout this Appendix, the \( r + s + 2 \) parameters \( \alpha_1, \beta_k, q \neq 1 \) and \( N \) have fixed values; the indication of the dependence upon them is omitted whenever this simplifies the notation at no cost in terms of clarity.

Our starting point is the following \( q \)-difference equation satisfied by the generalized basic hypergeometric function \( r + 1 \phi_s(\alpha_0, \alpha_1, \ldots, \alpha_r; \beta_1, \ldots, \beta_s; z) \) (see Exercise 1.31 on page 27 of [1], and note the replacement of \( \gamma \) with \( \beta + 1 \) and of \( \alpha_0 \) with \( q^{-N} ) \):

\[
\Delta_1 \left[ \prod_{k=1}^{s} (\Delta_{\beta_k/q}) \right] r + 1 \phi_s \left( q^{-N}, \alpha_1, \ldots, \alpha_r; \beta_1, \ldots, \beta_s; q; z \right)
\]

\[
-z \Delta_{q^{-N}} \left[ \prod_{j=1}^{r} (\Delta_{\alpha_j}) \right] r + 1 \phi_s \left( q^{-N}, \alpha_1, \ldots, \alpha_r; \beta_1, \ldots, \beta_s; q; zq^{s-r} \right) = 0 , \tag{44a}
\]

where the difference operators \( \Delta_\gamma \) and \( \delta \) operate as follows on functions \( f(z) \) of the variable \( z \):

\[
\Delta_\gamma f(z) = \gamma f(z) - f(z) , \quad \Delta_\delta f(z) = \delta f(z) = f(qz) . \tag{44b}
\]

Note that the operators \( \Delta_\gamma \) and \( \Delta_\delta \) commute,

\[
\Delta_\gamma \Delta_\delta f(z) = \gamma_1 \gamma_2 f(q^2 z) - (\gamma_1 + \gamma_2) f(qz) + f(z) = \Delta_\delta \Delta_\gamma f(z) . \tag{44c}
\]

Via (11) and (58) this implies that the generalized basic hypergeometric polynomial of order \( N \) satisfies the following \( q \)-difference equation:

\[
\Delta_1 \left[ \prod_{k=1}^{s} (\Delta_{\beta_k/q}) \right] p_N(\alpha; \beta; q; z) - z \Delta_{q^{-N}} \left[ \prod_{j=1}^{r} (\Delta_{\alpha_j}) \right] p_N(\alpha; \beta; q; zq^{s-r}) = 0 . \tag{45}
\]

It is now convenient to introduce the identities

\[
\prod_{k=1}^{s} (\Delta_{\beta_k/q}) = \prod_{k=1}^{s} \left( \beta_k \frac{\delta}{q} - 1 \right) = (-1)^s \prod_{k=1}^{s} \left( 1 - \beta_k \frac{\delta}{q} \right)
\]

\[
= (-1)^s \left\{ 1 + \sum_{k=1}^{s} \left[ b_k(\beta) \left( -\frac{\delta}{q} \right)^k \right] \right\} , \tag{46}
\]

where

\[
\prod_{k=1}^{s} (1 + \beta_k x) \equiv 1 + \sum_{k=1}^{s} [b_k(\beta) x^k] , \tag{47a}
\]

so that the quantities \( b_k(\beta) \) are defined as follows,

\[
b_1(\beta) = \sum_{k=1}^{s} (\beta_k) , \tag{47b}
\]

\[
b_2(\beta) = \sum_{k_1, k_2=1, k_1 \neq k_2}^{s} (\beta_{k_1} \beta_{k_2}) , \tag{47c}
\]

\[
b_3(\beta) = \sum_{k_1, k_2, k_3=1, k_1 \neq k_2, k_2 \neq k_3, k_3 \neq k_1}^{s} (\beta_{k_1} \beta_{k_2} \beta_{k_3}) , \tag{47d}
\]

and so on, up to

\[
b_s(\beta) = \prod_{k=1}^{s} (\beta_k) ; \tag{47e}
\]
as well as the analogous identities

\[
\prod_{j=1}^{r} (\Delta_{\alpha_j}) = \prod_{k=1}^{r} (\alpha_j \delta - 1) = (-1)^r \prod_{j=1}^{r} (1 - \alpha_j \delta)
\]

\[
= (-1)^r \left\{ 1 + \sum_{j=1}^{r} \left[ a_j (\mathbf{\alpha}) (-\delta)^j \right] \right\},
\]

(48)

where

\[
\prod_{j=1}^{r} (1 + \alpha_j x) \equiv 1 + \sum_{j=1}^{r} \left[ a_j (\mathbf{\alpha}) x^j \right],
\]

(49a)

so that the quantities \( a_j (\mathbf{\alpha}) \) are defined as follows,

\[
a_1 (\mathbf{\alpha}) = \sum_{k=1}^{r} (\alpha_j),
\]

(49b)

\[
a_2 (\mathbf{\alpha}) = \sum_{j_1, j_2=1, j_1 \neq j_2}^{r} (\alpha_{j_1}, \alpha_{j_2}),
\]

(49c)

\[
a_3 (\mathbf{\alpha}) = \sum_{j_1, j_2, j_3=1, j_1 \neq j_2, j_2 \neq j_3, j_3 \neq j_1}^{r} (\alpha_{j_1}, \alpha_{j_2}, \alpha_{j_3}),
\]

(49d)

and so on, up to

\[
a_r (\mathbf{\beta}) = \prod_{j=1}^{r} (\alpha_j).
\]

(49e)

Hence the \( q \)-difference equation \( \text{(4a)} \) satisfied by the \textit{generalized basic hypergeometric polynomial} \( p_N (z) \) (see \( \text{(3a)} \) and \( \text{(II)} \)) reads as follows (see \( \text{(44b)} \)):

\[
(1 - \delta) \left\{ 1 + \sum_{k=1}^{s} \left[ b_k (\mathbf{\beta}) \left( -\frac{r}{a} \right)^k \right] \right\} p_N (z)
\]

\[
- z \left( 1 - q^{-N} \delta \right) (-1)^{r-s} \left\{ 1 + \sum_{j=1}^{r} \left[ a_j (\mathbf{\alpha}) (-\delta)^j \right] \right\} p_N (z q^{s-r}) = 0,
\]

(50a)

hence

\[
\left\{ 1 - \delta + \sum_{k=1}^{s} \left[ (-q)^{-k} b_k (\mathbf{\beta}) \left( \delta^k - \delta^{k+1} \right) \right] \right\} p_N (z)
\]

\[
- (-1)^{r-s} z \left\{ 1 - q^{-N} \delta + \sum_{j=1}^{r} \left[ (-1)^j a_j (\mathbf{\alpha}) \left( \delta^j - q^{-N} \delta^{j+1} \right) \right] \right\} p_N (z q^{s-r}) = 0,
\]

(50b)

hence

\[
p_N (z) - p_N (z q) + \sum_{k=1}^{s} \left[ (-q)^{-k} b_k (\mathbf{\beta}) \left[ p_N (z q^k) - p_N (z q^{k+1}) \right] \right.
\]

\[
- (-1)^{r-s} z \left[ p_N (z q^{s-r}) - q^{-N} p_N (z q^{s-r+1}) \right]
\]

\[
- (-1)^{r-s} z \left\{ \sum_{j=1}^{r} \left[ (-1)^j a_j (\mathbf{\alpha}) \left[ p_N (z q^{s-r+j}) \right] - q^{-N} \sum_{j=1}^{r} \left[ (-1)^j a_j (\mathbf{\alpha}) p_N (z q^{s-r+j+1}) \right] \right] \right\} = 0.
\]

(50c)

Next, let us look at this formula—which is a polynomial equation in \( z \) of degree \( N + 1 \)—at \( z = \zeta_n \), where \( \zeta_n \) is one of the \( N \) zeros of the polynomial \( p_N (z) \), see \( \text{(13)} \). It is then plain that there hold the \( N \) algebraic
\[ -p_N(\zeta_n q) + \sum_{k=1}^{s} (-q)^{-k} b_k(\beta) \left[ p_N(\zeta_n q^k) - p_N(\zeta_n q^{k+1}) \right] \]

\[ -(-1)^{r-s} \zeta_n \left[ p_N(\zeta_n q^{s-r}) - q^{-N} p_N(\zeta_n q^{s-r+1}) \right] \]

\[ -(-1)^{r-s} \zeta_n \left\{ \sum_{j=1, j \neq r-s}^{r} \left[ (-1)^j a_j(\alpha) p_N(\zeta_n q^{s-r+j}) \right] \right\} = 0, \]

\( n = 1, 2, \ldots, N. \)  

(51)

And it is then immediately seen that the insertion in these formulas of the expression (4b) of the monic polynomial \( p_N(z) \) in terms of its zeros yields (7). Proposition 2.1 is thereby proven.

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