STABLE ULRICH BUNDLES ON FANO THREEFOLDS WITH PICARD NUMBER 2

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ABSTRACT. In this paper, we consider the existence problem of rank one and two stable Ulrich bundles on imprimitive Fano 3-folds obtained by blowing-up one of $\mathbb{P}^3$, $Q$ (smooth quadric in $\mathbb{P}^4$), $V_3$ (smooth cubic in $\mathbb{P}^4$) or $V_4$ (complete intersection of two quadrics in $\mathbb{P}^5$) along a smooth irreducible curve. We prove that the only class which admits Ulrich line bundles is the one obtained by blowing up a genus 3, degree 6 curve in $\mathbb{P}^3$. Also, we prove that there exist stable rank two Ulrich bundles with $c_1 = 3H$ on a generic member of this deformation class.

1. Introduction

The existence of Ulrich bundles on smooth projective varieties is related to a number of geometric questions. For instance, the existence of rank 1 or rank 2 Ulrich bundles on a hypersurface is related to the representation of that hypersurface as a determinant or Pfaffian ([1]). Another question of interest is the Minimal Resolution Conjecture (MRC) ([24], [15]). In [7], the existence problem of Ulrich bundles on del Pezzo surfaces was related to the MRC for a general smooth curve in the linear system of the first Chern class of the Ulrich bundle. Also, in [10], it is proved that the cone of cohomology tables of vector bundles on a $k$-dimensional scheme $X \subset \mathbb{P}^N$ is the same as the cone of cohomology tables of vector bundles on $\mathbb{P}^k$ if and only if there exists an Ulrich bundle on $X$.

It was conjectured in [11] that on any variety there exist Ulrich bundles. Although it is known that any projective curve ([9]), hypersurfaces and complete intersections ([19]), cubic surfaces ([5]), abelian surfaces ([2]), Veronese varieties ([11]) admit Ulrich bundles, such a general existence result is not known. The finer question of determining the minimal rank of Ulrich bundles (which do not contain bundles of lower rank as direct summands) on a given variety seems to be a quite difficult problem.

The problem that has attracted the most attention is the existence of stable Ulrich bundles with given rank and Chern classes. Stable Ulrich bundles are particularly interesting as they are the building blocks of all Ulrich bundles: Every Ulrich bundle is semistable, and the Jordan-Hölder factors are stable Ulrich bundles.

There are very few results on Ulrich bundles over Fano 3-folds with Picard number higher than one like [15]. In this paper, we studied the construction of stable...
Ulrich bundles on imprimitive Fano 3-folds obtained by blowing-up one of $\mathbb{P}^3$, $Q$ (smooth quadric in $\mathbb{P}^4$), $V_3$ (smooth cubic in $\mathbb{P}^4$) or $V_4$ (complete intersection of two quadrics in $\mathbb{P}^5$) along a smooth irreducible curve. There are 36 deformation classes of Fano 3-folds with Picard number $\rho = 2$ and 27 of these are imprimitive ([22, Table 12.3]). Among all imprimitive Fano 3-folds of Picard number $\rho = 2$, 21 deformation classes are obtained by blowing-up one of $\mathbb{P}^3$, $Q$, $V_3$ or $V_4$ along a smooth irreducible curve. We focus on these 3-folds and we consider rank 1 and 2 stable Ulrich bundles.

Throughout the paper, we use basically Riemann Roch computations, positivity results, Leray spectral sequence, projection formula, the package RandomSpace-Curve of Macaulay2, Casanellas-Hartshorne extension method and computations of local dimension of Quot Scheme.

First, we prove that the only class which admits rank 1 Ulrich bundles is the one obtained by blowing up a genus 3, degree 6 curve in $\mathbb{P}^3$, which is [22, No:12 in Table 12.3]. These varieties admit two classes of rank 1 Ulrich bundles $L_1$ and $L_2$ (Theorem 3.9).

The next step is to construct rank 2 stable Ulrich bundles on these varieties. To do this, we first construct rank 2 simple Ulrich bundles (Theorem 4.8). For this, we use extensions of rank 1 Ulrich bundles $L_1$ and $L_2$:

$$0 \to L_1 \to E \to L_2 \to 0$$

or

$$0 \to L_2 \to E \to L_1 \to 0.$$

Then $E$ is Ulrich and simple; and it has first Chern class $3H$.

Then, to determine whether there exists a stable Ulrich bundle of rank 2 with $c_1 = 3H$, we use the Quot scheme. It is known that stable vector bundles are simple. We consider the local dimension of the Quot scheme at the simple Ulrich bundle with first Chern class $3H$ and find a lower bound to this dimension (Theorem 4.19). Then we find an upper bound to the dimension of the subset parametrizing the non-stable Ulrich bundles (Proposition 4.20 and Proposition 4.21). The latter dimension is strictly smaller than the former; that is, there are stable, rank 2 Ulrich bundles with first Chern class $3H$ (Theorem 4.22).

1.1. Notations and Conventions. We work over an algebraically closed field $\mathbb{K}$ of characteristic 0.

- $X$: Smooth projective variety of degree $c$ and dimension $k$ in $\mathbb{P}^N$.
- $H_X$: Hyperplane class of $X$.
- $K_X$: Canonical divisor of $X$.
- $E(t)$: The vector bundle $E \otimes O_X(tH_X)$ where $E$ is a vector bundle on $X$, and $t \in \mathbb{Z}$.
- $C$: Smooth, irreducible curve of degree $d$ and genus $g$.
- $Q$: Smooth quadric in $\mathbb{P}^4$
- $V_3$: Smooth cubic in $\mathbb{P}^4$
- $V_4$: Complete intersection of two quadrics in $\mathbb{P}^5$
- $\tilde{X}$: Blow-up of $X$ along $C$.
- $\tilde{Y}$: Non-hyperelliptic Fano 3-fold which is obtained by blowing up one of $\mathbb{P}^3$, $Q$, $V_3$ or $V_4$ along $C$. 
\[ Y: \text{Deformation class of Fano 3-folds which is obtained by blowing up } \mathbb{P}^3 \text{ along a smooth irreducible space curve of degree 6 and genus 3, which is scheme theoretic intersection of cubics.} \]

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2. Preliminaries

2.1. Fano Varieties.

Definition 2.1. A smooth projective variety \( X \) is called a Fano variety if its anticanonical divisor \( -K_X \) is ample.

Definition 2.2. A Fano 3-fold is imprimitive if it is isomorphic to the blow-up of a Fano 3-fold along a smooth irreducible curve.

The classification of Fano 3-folds with \( \rho = 2 \) has been completed and it can be found in [22, Table 12.3]. In this paper, we consider the question of existence of Ulrich bundles on \( \tilde{Y} \).

Upon blowing-up \( X \) along \( C \), we have the following commutative diagram:

\[
\begin{array}{ccc}
E & \xrightarrow{j} & \tilde{X} \\
\downarrow{g} & & \downarrow{f} \\
C & \xrightarrow{i} & X \\
\end{array}
\]

the map \( f \) is the blow-down map and \( E = \mathbb{P}N \) is the exceptional divisor, where \( N \) is the normal bundle of \( C \) in \( X \). Recall that \( \tilde{X} \) stands for \( \tilde{\mathbb{P}}^3 \). Let \( h \) be the class of a plane in \( A^1(X) \), and let \( l = h^2 \) be the class of a line in \( A^2(X) \). We will denote \( \tilde{h} \) and \( \tilde{l} \) for the pullbacks of \( h \) and \( l \) to \( \tilde{X} \) respectively; and \( e \) denotes the class of the exceptional divisor. Also for any divisor \( D \in Z^1(C) \), we denote by \( F_D = g^*D \in Z^1(C) \) the corresponding linear combination of fibers \( E \to C \), and similarly for divisor classes.

Theorem 2.3. Let \( D = a\tilde{h} - b \) be a divisor on \( \tilde{Y} = \tilde{\mathbb{P}}^3 \), where \( a, b \in \mathbb{Z} \). Let \( D(t) = D + tH_{\tilde{Y}} \). Then

\[
\chi(\tilde{Y}, \mathcal{O}(D(t))) = \frac{1}{6}[62 - 8d + 2g]t^3 \\
+ \frac{1}{6}[(48 - 3d)a + (6g - 12d - 6)b - 12d + 3g + 93]t^2 \\
+ \frac{1}{6}[12a^2 + (6g - 6)b^2 - 6dab + (48 - 3d)a + (6g - 6 - 12d)b \\
+ 43 - 4d + g]t \\
+ \frac{1}{6}[a^3 + (4d + 2g - 2)b^3 + 6a^2 + (3g - 3da - 3)b^2 + 11a \\
+ (g - 3da - 4d - 1)b + 6].
\]
Proof. It is well-known that
\[ K_{\tilde{P}_3} = (-3 - 1)\tilde{h} = -4\tilde{h} \]
and
\[ c(T_{\tilde{P}_3}) = (1 + h)^{1+3} = 1 + 4h + 6h^2 + 4h^3. \]
So by [14, Example 15.4.3], we have
\[
c_1(T_{\tilde{Y}}) = f^* c_1(T_{\tilde{P}_3}^3) + (1 - 2)[E]
= f^* (4h) - e
= 4\tilde{h} - e
\]
\[
c_2(T_{\tilde{P}_3}) = f^* c_2(T_{\tilde{P}_3}^3) + f^* i_*[C] - f^* c_1(T_{\tilde{P}_3}^3)[E]
= f^* (6h^2) + d\bar{l} - f^* (4h)e
= (6 + d)\bar{h}^2 - 4\bar{h}e
\]
\[
K_{\tilde{P}_3} = f^* K_{\tilde{P}_3} + (2 - 1)[E]
= f^* (-4h) + e
= -4\tilde{h} + e.
\]
Then using [25, Lemma 2.1], we obtain
\[ \bar{h}^3 = 1 \]
\[ e^3 = -(-K_{\tilde{P}_3} \cdot C) + 2 - 2g = -(4h \cdot C) + 2 - 2g = -4d - 2g + 2 \]
\[ e^2 \cdot (-K_{\tilde{P}_3}) = 2g - 2 \Rightarrow e^2 (4\tilde{h} - e) = 2g - 2 \]
\[ \Rightarrow 4\tilde{h}e^2 - e^3 = 2g - 2 \]
\[ \Rightarrow 4\tilde{h}e^2 - (4d - 2g + 2) = 2g - 2 \]
\[ \Rightarrow \tilde{h}e^2 = -d \]
\[ e \cdot (-K_{\tilde{P}_3})^2 = (-K_{\tilde{P}_3} \cdot C) + 2 - 2g \Rightarrow e (4\tilde{h} - e)^2 = (4h \cdot C) + 2 - 2g \]
\[ \Rightarrow 16\tilde{h}^2e - 8\tilde{h}e^2 + e^3 = 4d + 2 - 2g \]
\[ \Rightarrow 16\tilde{h}^2e + 8d - 4d - 2g + 2 = 4d + 2 - 2g \]
\[ \Rightarrow \tilde{h}^2e = 0. \]
Since \( \tilde{Y} = \tilde{P}^3 \) is non-hyperelliptic Fano,
\[ H_{\tilde{P}_3} = -K_{\tilde{P}_3} = 4\tilde{h} - e. \]

Let \( D = a\tilde{h} - be \) be a divisor class on \( \tilde{Y} \). Then
\[ D(t) = D + tH_{\tilde{Y}} = (a\tilde{h} - be) + t(4\tilde{h} - e) \]
\[ = (a + 4t)\tilde{h} - (b + te). \]

Then, we apply Riemann-Roch theorem for line bundles on 3-folds to obtain
\[
\chi(\tilde{Y}, \mathcal{O}(D(t))) = \frac{1}{6}(D(t))^3 + \frac{1}{4} c_1(T_{\tilde{Y}}) \cdot (D(t))^2 + \frac{1}{12} (c_1^2(T_{\tilde{Y}}) + c_2(T_{\tilde{Y}})) \cdot (D(t)) + \frac{1}{24} c_1(T_{\tilde{Y}}) \cdot c_2(T_{\tilde{Y}}).
\]
Then we have
\[
\chi(\overline{Y}, \mathcal{O}(D(t))) = \frac{1}{6}[(a + 4t)\tilde{h} - (b + t)e]^3 + \frac{1}{4}[4\tilde{h} - e][(a + 4t)\tilde{h} - (b + t)e]^2
+ \frac{1}{12}[(4\tilde{h} - e)^2 + (6 + d)\tilde{h}^2 - 4\tilde{h}e][(a + 4t)\tilde{h} - (b + t)e]
+ \frac{1}{24}[4\tilde{h} - e][6\tilde{h}^2 + d\tilde{h} - 4\tilde{h}e]
\]
\[
= \frac{1}{6}[(a + 4t)^3\tilde{h}^3 - 3(a + 4t)^2(b + t)\tilde{h}^2 e + 3(a + 4t)(b + t)^2\tilde{h}e^2
- (b + t)^3e^3]
+ \frac{1}{4}[4\tilde{h} - e][(a + 4t)^2\tilde{h}^2 - 2(a + 4t)(b + t)\tilde{h}e + (b + t)^2e^2]
+ \frac{1}{12}(16\tilde{h}^2 - 8he + e^2 + (6 + d)\tilde{h}^2 - 4\tilde{h}e][(a + 4t)\tilde{h} - (b + t)e]
+ \frac{1}{24}[4\tilde{h} - e][(6 + d)\tilde{h}^2 - 4\tilde{h}e].
\]

Then,
\[
\chi(\overline{Y}, \mathcal{O}(D(t))) = \frac{1}{6}[(a + 4t)^3 - 3d(a + 4t)(b + t)^2 - (-4d - 2g + 2)(b + t)^3]
+ \frac{1}{4}[4(a + 4t)^2 - 4d(b + t)^2 - 2d(a + 4t)(b + t)
- (-4d - 2g + 2)(b + t)^2]
+ \frac{1}{12}[22(a + 4t) - 12d(b + t) - (-4d - 2g + 2)(b + t)]
+ \frac{1}{24}[24 + 4d - 4d].
\]

Now, by expanding, we obtain
\[
\chi(\overline{Y}, \mathcal{O}(D(t))) = \frac{1}{6}[a^3 + 12at^2 + 48at^2 + 64t^3 - 3dab^2 - 6dabt - 12db^2t
- 3dat^2 - 24dbt^2 - 12dt^3 + 4db^3 + 12dbt^2 + 4dt^3
+ 2g^3 + 6gb^2t + 6gdb^2 + 2gt^3 - 2b^3 - 6bt^2 - 6b^2t - 2t^3]
+ \frac{1}{4}[4a^2 + 32at + 64t^2 - 4db^2 - 8dt - 4dt^2 - 2dab - 2dat
- 8dbt - 8dt^2 + 4db^2 + 8dbt + 4dt^2 + 2g^2 + 4g^2t + 4g + 2gt^3
- 2b^2 - 4bt - 2t^2]
+ \frac{1}{12}[22a + 88t - 12db - 12dt + 4db + 4dt + 2gb + 2gt
- 2b - 2t]
+ \frac{1}{24}24.
\]

Then, collecting the terms with same powers of \(t\)
\[
\chi(\overline{Y}, \mathcal{O}(D(t))) = \frac{1}{24}[256 - 48d + 16d + 8g - 8t^3]
+ \frac{1}{24}[192a - 12da - 96db + 48db + 24gb - 24b + 384 - 24d].
\]
Finally,

\[ \chi(\tilde{Y}, \mathcal{O}(D(t))) = \frac{1}{6} [62 - 8d + 2g]t^3 \]

\[ + \frac{1}{6} (48 - 3d)a + (6g - 12d - 6)b - 12d + 3g + 93]t^2 \]

\[ + \frac{1}{6} [12a^2 + (6g - 6)b^2 - 6dab + (48 - 3d)a + (6g - 6 - 12d)b + 43 - 4d + g]t \]

\[ + \frac{1}{6} [a^3 + (4d + 2g - 2)b^3 + 6a^2 + (3g - 3da - 3)b^2 + 11a + (g - 3da - 4d - 1)b + 6]. \]

\[ \square \]

**Theorem 2.4.** Let \( D = a\tilde{h} - be \) be a divisor on \( \tilde{Y} = \tilde{Q} \), where \( a, b \in \mathbb{Z} \). Let \( D(t) = D + tH_{\tilde{Q}} \). Then

\[ \chi(\tilde{Y}, \mathcal{O}(D(t))) = \frac{1}{24} [208 - 24d + 8g]t^3 \]

\[ + \frac{1}{24} [(216 - 12d)a + (24g - 36d - 24)b - 36d + 12g + 312]t^2 \]

\[ + \frac{1}{24} [72a^2 + (24g - 24)b^2 - 24dab + (216 - 12d)a + (24g - 24 - 36d)b + 152 - 6d + 4g]t \]

\[ + \frac{1}{24} [8a^3 + (12d + 8g - 8)b^3 + 36a^2 + (12g - 12da - 12)b^2 + 52a + (4g - 12da - 12d - 4)b + 24 + 3d]. \]

**Proof.** It follows same pattern in proof of Theorem 2.3 with minor computational changes. \( \square \)

**Theorem 2.5.** Let \( D = a\tilde{h} - be \) be a divisor on \( \tilde{Y} = \tilde{V}_3 \), where \( a, b \in \mathbb{Z} \). Let \( D(t) = D + tH_{\tilde{V}_3} \). Then

\[ \chi(\tilde{Y}, \mathcal{O}(D(t))) = \frac{1}{12} [44 - 8d + 4g]t^3 \]

\[ + \frac{1}{12} [(72 - 6d)a + (12g - 12d - 12)b - 12d + 6g + 66]t^2 \]

\[ + \frac{1}{12} [36a^2 + (12g - 12)b^2 - 12dab + (72 - 6d)a + (12g - 12 - 12d)b + 46 + 2g]t \]
\[ \frac{1}{12}[6a^3 + (4d + 4g - 4)b^3 + 18a^2 + (6g - 6da - 6)b^2 + (24 + 2d)a + (2g - 6da - 4d - 2)b + 12 + 2d]. \]

**Proof.** It follows same pattern in proof of Theorem 2.3 with minor computational changes. \(\square\)

**Theorem 2.6.** Let \( D = a\tilde{h} - b \) be a divisor on \( \tilde{Y} = \tilde{V}_4 \), where \( a, b \in \mathbb{Z} \). Let \( D(t) = D + tH_{\tilde{V}_4} \). Then
\[
\chi(\tilde{Y}, \mathcal{O}(D(t))) = \frac{1}{12}[60 - 8d + 4g]t^3
+ \frac{1}{12}[(96 - 6d)a + (12g - 12d - 12)b - 12d + 6g + 90]t^2
+ \frac{1}{12}[48a^2 + (12g - 12)b^2 - 12dab + (96 - 6d)a + (12g - 12 - 12d)b + 54 + 2g + 2dt]
+ \frac{1}{12}[8a^3 + (4d + 4g - 4)b^3 + 24a^2 + (6g - 6da - 6)b^2 + (28 + 3d)a + (2g - 6da - 4d - 2)b + 12 + 3d].
\]

**Proof.** It follows same pattern in proof of Theorem 2.3 with minor computational changes. \(\square\)

**Theorem 2.7** (Leray Spectral Sequence). Suppose \( \pi : X_1 \to X_2 \) is a morphism of varieties. Then for any \( \mathcal{O}_{X_2} \)-module \( \mathcal{F} \), there is a spectral sequence with \( E_2 \) term given by \( H^p(X_2, R^q \pi_* \mathcal{F}) \) abutting to \( H^{p+q}(X_1, \mathcal{F}) \).

**Corollary 2.8.** Let \( L \) be a line bundle on \( \tilde{Y} \) and \( p + q = k \). Then
\begin{itemize}
  \item \( H^k(\tilde{Y}, L) = 0 \) if \( H^p(Y, R^q f_* L) = 0 \) for all possible \( p \) and \( q \)
  \item \( H^k(\tilde{Y}, L) \cong H^p(Y, R^q f_* L) \) if \( H^p(Y, R^q f_* L) = 0 \) except the tuple \((p, q) = (r, s)\).
\end{itemize}

**Proof.** It is a direct consequence of Theorem 2.7. \(\square\)

### 2.2 Ulrich Bundles

The general references for this section are [7] and [21].

**Definition 2.9.** Let \( \mathcal{E} \) be a vector bundle on a nonsingular projective variety \( X \). Then \( \mathcal{E} \) is said to be **semistable** if for every nonzero subbundle \( \mathcal{F} \) of \( \mathcal{E} \) we have the inequality
\[
\frac{P_\mathcal{F}}{\text{rank}(\mathcal{F})} \leq \frac{P_\mathcal{E}}{\text{rank}(\mathcal{E})},
\]
where \( P_\mathcal{F} \) and \( P_\mathcal{E} \) are the respective Hilbert polynomials and comparison is based on the lexicographic order. It is **stable** if one always has strict inequality above.

**Definition 2.10.** Let \( \mathcal{E} \) be a vector bundle on a nonsingular projective variety \( X \). The **slope** \( \mu(\mathcal{E}) \) of \( \mathcal{E} \) is defined as \( \text{deg}(\mathcal{E})/\text{rank}(\mathcal{E}) \). We say that \( \mathcal{E} \) is \( \mu \)-semistable if for every subbundle \( \mathcal{F} \) of \( \mathcal{E} \) with \( 0 < \text{rank}(\mathcal{F}) < \text{rank}(\mathcal{E}) \), we have \( \mu(\mathcal{F}) \leq \mu(\mathcal{E}) \). We say \( \mathcal{E} \) is **\( \mu \)-stable** if strict inequality always holds above.

**Lemma 2.11.** The two definitions are related as follows:
\[
\mu - \text{stable} \Rightarrow \text{stable} \Rightarrow \text{semistable} \Rightarrow \mu - \text{semistable}.
\]

**Proof.** See [21] 1.2.13. \(\square\)
Definition 2.12. A vector bundle $E$ on $X$ is called ACM (arithmetically Cohen-Macaulay) if $H^i(E(t)) = 0$ for all $t \in \mathbb{Z}$ and $0 < i < k$.

Definition 2.13. Let $E$ be a vector bundle of rank $r$ on $X$. Then $E$ is Ulrich if for some linear projection $\pi : X \to \mathbb{P}^k$ we have $\pi_*E \cong \mathcal{O}_{\mathbb{P}^k}^{\oplus r}$.

Proposition 2.14. Let $E$ be a vector bundle of rank $r$ on $X$. Then $E$ is Ulrich if and only if it is ACM with Hilbert polynomial $cr(t + k)$.

Proof. See [7, Proposition 2.3]. □

Theorem 2.15. Let $\tilde{Y}$ be one of the following Fano 3-folds:

1. the blow-up of $\mathbb{P}^3$ along an intersection of two cubics,
2. the blow-up of $\mathbb{P}^3$ along a curve of degree 7 and genus 5 which is an intersection of cubics,
3. the blow-up of $\mathbb{P}^3$ along a curve of degree 6 and genus 3 which is an intersection of cubics,
4. the blow-up of $\mathbb{P}^3$ along the intersection of a quadric and a cubic,
5. the blow-up of $\mathbb{P}^3$ along an elliptic curve which is an intersection of two quadrics,
6. the blow-up of $\mathbb{P}^3$ along a twisted cubic,
7. the blow-up of $\mathbb{P}^3$ along a plane cubic,
8. the blow-up of $\mathbb{P}^3$ along a conic,
9. the blow-up of $\mathbb{P}^3$ along a line.

Then Ulrich line bundles can exist only on the class (3).

Proof. Let $D = a\tilde{h} - b\tilde{e}$ be a divisor class on $\tilde{Y}$. We can compute Hilbert polynomial of $\mathcal{O}_{\tilde{Y}}(D)$ by Theorem 2.3. By Proposition 2.14 this must be equal to $\deg(\tilde{Y})(t + k)^3$. We will equate the coefficients of these two polynomials and try to find integer solutions for $a$ and $b$ in each case separately.

(1) Since $C$ is an intersection of two cubics, $d = 9$. By the adjunction formula, $g = 10$. Then $k = H^3 = 10$. Now, equate the coefficients of $t^2$:

$$\frac{10.6}{6} t^2 = \frac{1}{6}[(48 - 3.9)a + (6.10 - 12.9 - 6)b - 12.9 + 3.10 + 93]t^2$$

which gives

$$a = \frac{18b + 15}{7}.\n$$

Next, equate the coefficients of $t$ and use the above relation between $a$ and $b$ to get

$$\frac{10.11}{6} t = \frac{1}{6}(12\left(\frac{18b + 15}{7}\right))^2 + (6.10 - 6)b^2 - 6.9(\frac{18b + 15}{7})b$$

$$+ (48 - 3.9)(\frac{18b + 15}{7}) + (6.10 - 6 - 12.9)b + 43 - 4.9 + 10)t$$

which gives

$$b = \frac{3}{2} + \frac{7}{30}\sqrt{65}.$$\n
There is no integer solution for $a$ and $b$, so there exists no Ulrich line bundle.

For the other items except (3), proof follows same pattern in proof of item (1) with minor computational changes.
(3) (This case is \[22\text{ No.12 in Table 12.3}\].) It is given that \(d = 6\) and \(g = 3\).

Then \(k = H^3 = 20\). Then equate the coefficients of \(t^2\):

\[
\frac{20}{6}t^2 = \frac{1}{6}(48 - 3.6)a + (6.3 - 12.6 - 6)b - 12.6 + 3.3 + 93|t^2
\]

which gives

\[a = 2b + 3.\]

Next, equate the coefficients of \(t\) and use the above relation between \(a\) and \(b\) to get

\[
\frac{20.11}{6}t = \frac{1}{6}[12(2b + 3)^2 + (6.3 - 6)b^2 - 6.6(2b + 3)b + (48 - 3.6)(2b + 3) + (6.3 - 6 - 12.6)b + 43 - 4.6 + 3|t
\]

which gives

\[b = 0 \text{ or } b = 3.\]

Then we have \((a, b) = (3, 0)\) or \((a, b) = (9, 3)\). Both of these solutions satisfy also the equality of coefficients of \(t^2\) and constant terms. So the divisors \(3\tilde{h}\) and \(9\tilde{h} - 3e\) yield possible Ulrich line bundles. (We note that to be Ulrich, they must also satisfy the ACM condition.)

\[\square\]

**Theorem 2.16.** Let \(\tilde{Y}\) be one of the following Fano 3-folds:

1. the blow-up of \(Q\) along the intersection of two divisors from \(|O_Q(2)|\),
2. the blow-up of \(Q\) along a curve of degree 6 and genus 2,
3. the blow-up of \(Q\) along an elliptic curve of degree 5,
4. the blow-up of \(Q\) along a twisted quartic,
5. the blow-up of \(Q\) along an intersection of two divisors from \(|O_Q(1)|\) and \(|O_Q(2)|\),
6. the blow-up of \(Q\) along a conic,
7. the blow-up of \(Q\) along a line.

Then Ulrich line bundles can not exist on none of them.

**Proof.** It follows same pattern in proof of Theorem 2.15 with minor computational changes. \(\square\)

**Theorem 2.17.** Let \(\tilde{Y}\) be one of the following Fano 3-folds:

1. the blow-up of \(V_3\) along a plane cubic,
2. the blow-up of \(V_3\) along a line.

Then Ulrich line bundles can not exist on none of them.

**Proof.** It follows same pattern in proof of Theorem 2.16 with minor computational changes. \(\square\)

**Theorem 2.18.** Let \(\tilde{Y}\) be one of the following Fano 3-folds:

1. the blow-up of \(V_4\) along an elliptic curve which is an intersection of two hyperplane sections,
2. the blow-up of \(V_4\) along a conic,
3. the blow-up of \(V_4\) along a line.
Then Ulrich line bundles can not exist on non of them.

Proof. It follows same pattern in proof of Theorem 2.15 with minor computational changes. □

3. ULRICH LINE BUNDLES ON Y

We recall that $Y$ is the Fano 3-fold which is obtained as the blow-up of $\mathbb{P}^3$ along a curve $C$ of degree 6 and genus 3.

We also recall the following commutative diagram as in ‘Preliminaries’ section:

$$
\begin{array}{ccc}
E & \xrightarrow{j} & Y \\
\downarrow{g} & & \downarrow{f} \\
C & \xrightarrow{i} & \mathbb{P}^3
\end{array}
$$

**Proposition 3.1.** The canonical map $O_C \to g_* O_E$ is an isomorphism.

Proof. Note that $g : E \to C$ is a ruled surface. Then the result follows from [17] Lemma 2.1 of Chapter V]. □

**Corollary 3.2.** $f_* (O_{\mathbb{P}^3}(-mE)) = I_C^m$ and $R^i f_* O_{\mathbb{P}^3}(-me) = 0$ for $m \geq 0$ and $i > 0$.

Proof. See [23] Lemma 4.3.16]. □

**Lemma 3.3.** $f_* O_E(mE) = 0$ for $m > 0$.

Proof. Note that $g : E \to C$ is a ruled surface. So, by [17] Proposition 8.20 of Chapter II], we have

$$
w_E \cong w_Y \otimes O_Y(E) \otimes O_E \implies w_E \cong O_E(E)(-1) \implies O_E(E) \cong w_E(1) \implies O_E(E) \cong O_E(K_E + H_E).
$$

Then we have

$$
O_E(mE) \cong O_Y(mE) \otimes O_E \\
\cong O_Y(E)^\otimes m \otimes O_E \\
\cong [O_Y(E) \otimes O_E]^\otimes m \\
\cong O_E(E)^\otimes m \\
\cong O_E(K_E + H_E)^\otimes m \\
\cong O_E(m(K_E + H_E))
$$

where $D = mK_E + mH_Y$.

Also, by [18] Lemma 2.10 in Chapter V], we know that

$$
K_E \cong -2C_0 + D_C \cdot F
$$

where $C_0$ is a section of the map $g$, $F$ is the fiber of $g$ and $D_C$ is a divisor class on $C$.

Then

$$
D \cdot F = (-2mC_0 + mD_C \cdot F + mH_E) \cdot F \\
= -2mC_0 \cdot F + mD_C \cdot F^2 + mH_E \cdot F
$$
\[ D \cdot F = -2m + 0 + m = -m. \]

Hence \( D \cdot F \) is negative. So, following the proof of [13, Lemma 2.1 in Chapter V], one can easily show that

\[ f_*\mathcal{O}_E(D) = f_*\mathcal{O}_E(mE) = 0. \]

\[ \square \]

**Proposition 3.4.** \( f_*\mathcal{O}_{\tilde{P}^3}(mE) = \mathcal{O}_{P^3} \) for \( m > 0 \).

**Proof.** We have the exact sequence

\[ 0 \to \mathcal{O}_{\tilde{P}^3}(-E) \to \mathcal{O}_{\tilde{P}^3} \to \mathcal{O}_E \to 0. \]

Now twist this exact sequence by \( E \) and get

\[ 0 \to \mathcal{O}_{\tilde{P}^3} \to \mathcal{O}_{\tilde{P}^3}(E) \to \mathcal{O}_E(E) \to 0. \]

Then consider the long exact sequence

\[ 0 \to f_*\mathcal{O}_{\tilde{P}^3} \to f_*\mathcal{O}_{\tilde{P}^3}(E) \to f_*\mathcal{O}_E(E) \to \cdots. \]

By Lemma 3.3 \( f_*\mathcal{O}_E(E) = 0 \). So

\[ f_*\mathcal{O}_{\tilde{P}^3}(E) \simeq f_*\mathcal{O}_{\tilde{P}^3} \simeq \mathcal{O}_{P^3}. \]

Similarly, now twist the exact sequence (*) by \( 2E \) and get

\[ 0 \to \mathcal{O}_{\tilde{P}^3}(E) \to \mathcal{O}_{\tilde{P}^3}(2E) \to \mathcal{O}_E(2E) \to 0. \]

Then consider the exact sequence

\[ 0 \to f_*\mathcal{O}_{\tilde{P}^3}(E) \to f_*\mathcal{O}_{\tilde{P}^3}(2E) \to f_*\mathcal{O}_E(2E) \to \cdots. \]

Again by Lemma 3.3 \( f_*\mathcal{O}_E(2E) = 0 \). Therefore

\[ f_*\mathcal{O}_{\tilde{P}^3}(2E) \simeq f_*\mathcal{O}_{\tilde{P}^3}(E) \simeq \mathcal{O}_{P^3}. \]

Hence, by induction on \( m \), we have \( f_*\mathcal{O}_{\tilde{P}^3}(mE) = \mathcal{O}_{P^3} \) for \( m > 0 \). \[ \square \]

**Lemma 3.5.** Let \( \mathcal{E} \) be an Ulrich bundle of rank \( r \) on \( \tilde{Y} \). Then \( \mathcal{E}^\vee(3) \) is also Ulrich.

**Proof.** We use Proposition 2.14.

First,

\[ (*) \quad H^i(\tilde{Y}, \mathcal{E}^\vee(3)(t)) = H^i(\tilde{Y}, \mathcal{E}^\vee(3 + t)) \]

\[ = H^{3-i}(\tilde{Y}, \mathcal{E}(-3 - t) \otimes K_{\tilde{Y}})^\vee \quad (\text{Serre Duality}) \]

\[ = H^{3-i}(\tilde{Y}, \mathcal{E}(-3 - t) \otimes (-H))^\vee \quad (\tilde{Y} \text{ is Fano}) \]

\[ = H^{3-i}(\tilde{Y}, \mathcal{E}(-4 - t))^\vee. \]

But we know that \( \mathcal{E} \) is Ulrich, so it is ACM by Proposition 2.14. Then the middle cohomologies of all twists of \( \mathcal{E} \) vanish; so \( H^{3-i}(\tilde{Y}, \mathcal{E}(-4 - t)) \) vanishes for \( i = 1, 2 \) and \( t \in \mathbb{Z} \).
Hence $H^1(\tilde{Y}, \mathcal{E}^\vee(3)(t)) = 0$; that is, $\mathcal{E}^\vee(3)$ is ACM.

Second,

\[
\begin{align*}
\chi(\tilde{Y}, \mathcal{E}^\vee(3)(t)) &= \sum_{i=0}^{3} (-1)^i h^i(\tilde{Y}, \mathcal{E}^\vee(3)(t)) \\
&= \sum_{i=0}^{3} (-1)^i h^{3-i}(\tilde{Y}, \mathcal{E}(-4-t)) \quad \text{(by \text{(*)})} \\
&= -cr \left(\frac{-4-t+3}{3}\right) \\
&= -\frac{(t+1)(t+2)(t+3)}{6} \\
&= cr \left(\frac{t+3}{3}\right)
\end{align*}
\]

Therefore $\mathcal{E}^\vee(3)$ is Ulrich by Proposition 2.14.

\[
\square
\]

**Lemma 3.6.** Let $C$ be a curve cut out scheme-theoretically in $\mathbb{P}^3$ by cubic hypersurfaces. Then

\[
H^i(\mathbb{P}^3, I_C(3)) = 0 \quad \text{for} \quad i \geq 1 \quad \text{provided} \quad k \geq 3a.
\]

**Proof.** This is a special case of [3, Proposition 1].

**Lemma 3.7.** If $C$ is an ACM curve in $\mathbb{P}^3$ with $d = 6$ and $g = 3$, then its ideal sheaf $I_C$ in $\mathbb{P}^3$ has the minimal free resolution:

\[
0 \to O_{\mathbb{P}^3}(-4) \to O_{\mathbb{P}^3}(-3) \to I_C \to 0.
\]

**Proof.** Since $C$ is ACM, by [12, p.2], it has a minimal free resolution of the form:

\[
0 \to \bigoplus_{j=1}^{k-1} O_{\mathbb{P}^3}(-n_j) \to \bigoplus_{l=1}^{k} O_{\mathbb{P}^3}(-m_l) \to I_C \to 0.
\]

Since $I_C(3)$ is generated by global sections [18, Ex. 8.7(c)], $m_l = 3$ for all $l$ and we have:

\[
(\text{*)} \quad 0 \to \bigoplus_{j=1}^{k-1} O_{\mathbb{P}^3}(-n_j) \to \bigoplus_{l=1}^{k} O_{\mathbb{P}^3}(-3) \to I_C \to 0.
\]

We know that $h^0(I_C(3)) = 4$ and $h^1(I_C(3-i)) = 0$ for all $i > 0$ by [18, Ex. 8.7(c)]. Since $h^1(I_C(3-i)) = 0$ for all $i > 0$, we have $h^2(I_C(1)) = 0$.

Then $\sum_{j=1}^{k-1} h^3(-n_j + 1) = \sum_{j=1}^{k-1} h^0(n_j - 1 - 4) = 0$. Then $n_j \leq 4$. But, since (\text{*)} is a minimal free resolution, we have $n_j \geq 4$. So $n_j = 4$. Since $h^0(I_C(3)) = 4$, we have $k = 4$.

\[
\square
\]

**Proposition 3.8** (Yusuf Mustopa, written in private communication). If $C$ is an ACM space curve with $d = 6$ and $g = 3$, then $H^1(I_C(5)) = 0$ for all $i > 0$.

**Proof.** Twisting the sequence

\[
0 \to I_C \to O_{\mathbb{P}^3} \to O_C \to 0
\]
by $I_C(5)$ yields the long exact sequence
\[ 0 \to \text{Tor}_1^{C_3}(I_C(5), \mathcal{O}_C) \to I_C \otimes I_C(5) \to I_C(5) \to \mathcal{N}_{C_3}^*(5) \to 0. \]
This can be broken into two short exact sequences, one of which is
\[ 0 \to \text{Tor}_1^{C_3}(I_C(5), \mathcal{O}_C) \to I_C \otimes I_C(5) \to I_C^2(5) \to 0. \]
Since $\text{Tor}_1^{C_3}(I_C(5), \mathcal{O}_C)$ has at most 1-dimensional support, we have
\[ H^t(\text{Tor}_1^{C_3}(I_C(5), \mathcal{O}_C)) = 0 \text{ for all } i > 1. \]
It then suffices to show the vanishing of $H^t(I_C \otimes I_C(5))$ for all $i > 0$.

We know, by Lemma 3.7, that $I_C$ has a minimal free resolution of the form
\[ 0 \to \mathcal{O}_{p_3}^{\oplus 3}(-4) \to \mathcal{O}_{p_3}^{\oplus 4}(-3) \to I_C \to 0. \]
As before, we consider the twist by $I_C(5)$. Then we have a long exact sequence
\[ 0 \to \text{Tor}_1^{C_3}(I_C, I_C(5)) \to I_C(1)^{\oplus 3} \to I_C(2)^{\oplus 4} \to I_C \otimes I_C(5) \to 0. \]
Given that $\text{Tor}_1^{C_3}(I_C, I_C(5))$ has at most 1-dimensional support and is a subsheaf of the torsion-free sheaf $I_C(1)^{\oplus 3}$, it is equal to 0; so we have
\[ 0 \to I_C(1)^{\oplus 3} \to I_C(2)^{\oplus 4} \to I_C \otimes I_C(5) \to 0. \]
But, we know that, by Lemma 3.7, $I_C$ has a minimal free resolution of the form
\[ 0 \to \mathcal{O}_{p_3}^{\oplus 3}(-4) \to \mathcal{O}_{p_3}^{\oplus 4}(-3) \to I_C \to 0. \]
So $H^t(I_C(k)) = 0$ for all $i > 0$ and $k > 0$. Then $H^t(I_C \otimes I_C(5)) = 0$ for all $i > 0$; so the result follows.

**Theorem 3.9.** Suppose that $C$ is ACM. Then there are only two Ulrich line bundles $L_1$ and $L_2$, and they correspond to divisors $D_1 = 9h - 3e$ and $D_2 = 3h$ on $Y$.

**Proof.** We will use Proposition 2.14 to show that $L_1$ and $L_2$ are Ulrich line bundles. In Theorem 2.14, we showed that $L_1$ and $L_2$ satisfy the Hilbert polynomial condition. So, it remains to show that $L_1$ and $L_2$ are ACM; i.e, to show that $H^1(Y, L_1(t)) = H^2(Y, L_1(t)) = 0$ and $H^1(Y, L_2(t)) = H^2(Y, L_2(t)) = 0$ for all $t \in \mathbb{Z}$.

Consider $L_1$ first.
- $t \geq 0$;

Then
\[ L_1(t) = \mathcal{O}_Y(9h - 3e + t(4h - e)) = \mathcal{O}_Y((4t + 9)h + (-t - 3)e). \]
Since $-t - 3 < 0$, by the projection formula and Corollary 3.2, we have
\[ f_*L_1(t) = f_*\mathcal{O}_Y((4t + 9h + (-t - 3)e) = \mathcal{O}_{p_3}(4t + 9) \otimes f_*\mathcal{O}_Y((4t + 9)h) = I_C^{t+3} \otimes \mathcal{O}_{p_3}(4t + 9). \]
So $H^1(\mathbb{P}^3, f_*L_1(t)) = H^1(\mathbb{P}^3, I_C^{t+3}(4t + 9))$ and it is 0 by Lemma 3.6 since $4t + 9 \geq 3(t + 3)$.
Now we consider $H^0(\mathbb{P}^3, R^1 f_*L_1(t))$. By projection formula, we have
\[ H^0(\mathbb{P}^3, R^1 f_*L_1(t)) = H^0(\mathbb{P}^3, R^1 f_*\mathcal{O}_{p_3}(4t + 9) \otimes \mathcal{O}_{p_3}(4t + 9)). \]
But \( R^1 f_* O_{\mathbb{P}^3} ((-t - 3)e) = 0 \) by Corollary 3.2 since \(-t - 3 \leq 0\); and so
\[ H^0(\mathbb{P}^3, R^1 f_* L_1(t)) = 0. \]
Then, by Corollary 2.8 we have
\[ H^1(Y, L_1(t)) = 0. \]

Now consider \( H^0(\mathbb{P}^3, R^2 f_* L_1(t)) \) and \( H^1(\mathbb{P}^3, R^1 f_* L_1(t)) \). Note that they are 0 by Corollary 3.2. Also \( H^2(\mathbb{P}^3, f_* L_1(t)) = H^2(\mathbb{P}^3, \mathcal{P}(4t + 3)) = 0 \) again by Lemma 3.6. Then, by Corollary 2.8 we have
\[ H^2(Y, L_1(t)) = 0. \]

- \( t < -4 \):

Then
\[
H^1(Y, L_1(t)) = H^2(Y, L_1^\vee (-t) \otimes K_Y)^\vee = H^2(Y, \mathcal{O}_{\mathcal{Y}}((-4t - 9)\tilde{h} + (t + 3)e) \otimes \mathcal{O}_{\mathcal{Y}}(-4\tilde{h} + e))^\vee = H^2(Y, \mathcal{O}_{\mathcal{Y}}((-4t - 13)\tilde{h} + (t + 4)e))^\vee.
\]

Similarly,
\[
H^2(Y, L_1(t)) = H^1(Y, \mathcal{O}_{\mathcal{Y}}((-4t - 13)\tilde{h} + (t + 4)e))^\vee.
\]

So, if \( H^i(Y, \mathcal{O}_{\mathcal{Y}}((-4t - 13)\tilde{h} + (t + 4)e)) \) for \( i = 1, 2 \) vanishes, the result will follow.
Since \( t + 4 < 0 \), by the projection formula and Corollary 3.2 we have
\[
f_* \mathcal{O}_{\mathcal{Y}}((-4t - 13)\tilde{h} + (t + 4)e) = O_{\mathbb{P}^3}(-4t - 13) \otimes f_* \mathcal{O}_{\mathcal{Y}}((t + 4)e) = I_C^{t-4} \otimes O_{\mathbb{P}^3}(-4t - 13) = I_C^{t-4}(-4t - 13).
\]

So \( H^1(\mathbb{P}^3, f_* \mathcal{O}_{\mathcal{Y}}((-4t - 13)\tilde{h} + (t + 4)e)) = H^1(\mathbb{P}^3, I_C^{t-4}(-4t - 13)) \) and it is 0 by Lemma 3.6 since \(-4t - 13 \geq 3(-t - 4)\).

Now consider \( H^0(\mathbb{P}^3, R^1 f_* \mathcal{O}_{\mathcal{Y}}((-4t - 13)\tilde{h} + (t + 4)e)) \). By the projection formula, we have
\[
H^0(\mathbb{P}^3, R^1 f_* \mathcal{O}_{\mathcal{Y}}((-4t - 13)\tilde{h} + (t + 4)e)) = H^0(\mathbb{P}^3, R^1 f_* O_{\mathbb{P}^3}((t + 4)e) \otimes O_{\mathbb{P}^3}(-4t - 13)).
\]

But \( R^1 f_* O_{\mathbb{P}^3}((t + 4)e) = 0 \) by Corollary 3.2 since \( t + 4 \leq 0 \). So, by Corollary 2.8
\[ H^1(Y, \mathcal{O}_{\mathcal{Y}}((-4t - 13)\tilde{h} + (t + 4)e)) = 0. \]

Hence
\[ H^2(Y, L_1(t)) = 0. \]

Now consider \( H^0(\mathbb{P}^3, R^2 f_* \mathcal{O}_{\mathcal{Y}}((-4t - 13)\tilde{h} + (t + 4)e)) \) and
\[ H^1(\mathbb{P}^3, R^1 f_* \mathcal{O}_{\mathcal{Y}}((-4t - 13)\tilde{h} + (t + 4)e)), \]
and note that they are 0 by Corollary 3.2.
Also \( H^2(\mathbb{P}^3, f_* \mathcal{O}_{\mathcal{Y}}((-4t - 13)\tilde{h} + (t + 4)e)) = H^2(\mathbb{P}^3, I_C^{t-4}(-4t - 13)) = 0 \) again by Lemma 3.6. So, by Corollary 2.8
\[ H^2(Y, \mathcal{O}_{\mathcal{Y}}((-4t - 13)\tilde{h} + (t + 4)e)) \] vanish. So
\[ H^1(Y, L_1(t)) = 0. \]

- \( t = -4 \):
Then
\[ H^i(Y, L_1(-4)) = H^i(Y, O_Y(-7\tilde{h} + e)). \]

Then by [3] Lemma 1.4, we have
\[ H^i(Y, O_Y(-7\tilde{h} + e)) = H^i(\mathbb{P}^3, O_{\mathbb{P}^3}(7\tilde{h})). \]
So,
\[ H^3(Y, L_1(-4)) = H^2(Y, L_1(-4)) = 0. \]

• \( t = -3: \)

Then
\[ H^i(Y, L_1(-3)) = H^i(Y, O_Y(-3\tilde{h})). \]

Then by [3] Lemma 1.4, we have
\[ H^i(Y, O_Y(-3\tilde{h})) = H^i(\mathbb{P}^3, O_{\mathbb{P}^3}(3h)). \]
So,
\[ H^1(Y, L_1(-3)) = H^2(Y, L_1(-3)) = 0. \]

So far, we showed that \( H^i(Y, L_1(t)) = H^j(Y, L_1(t)) = 0 \) for all \( t \) except \( t = -1, -2. \)

For the remaining two values of \( t \), we assume that \( C \) is ACM.

• \( t = -1: \)

Again by Corollary 2.8 if \( H^i(\mathbb{P}^3, f_*L_1(-1)) \) for \( i = 1, 2, \)
\( H^j(\mathbb{P}^3, R^1f_*L_1(-1)) \) for \( j = 0, 1 \) and \( H^0(\mathbb{P}^3, R^2f_*L_1(-1)) \) vanishes,
\( H^1(Y, L_1(-1)) \) and \( H^2(Y, L_1(-1)) \) vanish.

Note that \( H^i(\mathbb{P}^3, f_*L_1(-1)) = H^i(\mathbb{P}^3, I_C^3(5)) \) for \( i = 1, 2 \) by the projection formula
and Corollary 3.2, and we know that \( H^i(\mathbb{P}^3, I_C^3(5)) = 0 \) for \( i = 1, 2 \) by Proposition 3.8.
Also, we know that \( H^i(\mathbb{P}^3, R^jf_*L_1(-1)) = 0 \) for \( j = 0, 1 \) and
\( H^0(\mathbb{P}^3, R^2f_*L_1(-1)) = 0 \) by the projection formula and Corollary 3.2.
So,
\[ H^1(Y, L_1(-1)) = H^2(Y, L_1(-1)) = 0. \]

• \( t = -2: \)

By Corollary 2.8 if all of \( H^i(\mathbb{P}^3, f_*L_1(-2)) \) for \( i = 1, 2, \)
\( H^j(\mathbb{P}^3, R^1f_*L_1(-2)) \) for \( j = 0, 1 \) and \( H^0(\mathbb{P}^3, R^2f_*L_1(-2)) \) vanish,
\( H^1(Y, L_1(-2)) \) and \( H^2(Y, L_1(-2)) \) vanish.

Note that \( H^i(\mathbb{P}^3, f_*L_1(-2)) = H^i(\mathbb{P}^3, I_C(1)) \) for \( i = 1, 2 \) by the projection formula
and Corollary 3.2. But \( H^i(\mathbb{P}^3, I_C(1)) = 0 \) for \( i = 1, 2 \) since \( C \) is ACM and by
Lemma 3.7 \( I_C \) has a minimal free resolution
\[ 0 \to O_{\mathbb{P}^3}(4) \to O_{\mathbb{P}^3}(3) \to I_C \to 0. \]

Also, we know that \( H^j(\mathbb{P}^3, R^1f_*L_1(-2)) = 0 \) for \( j = 0, 1 \) and
\( H^0(\mathbb{P}^3, R^2f_*L_1(-2)) = 0 \) by projection formula and Corollary 3.2.
So,
\[ H^1(Y, L_1(-2)) = H^2(Y, L_1(-2)) = 0. \]

Hence \( H^1(Y, L_1(t)) = H^2(Y, L_1(t)) = 0 \) for all \( t \in \mathbb{Z} \), and the result follows for \( L_1. \)

Consider \( L_2 \) next.
\( L_2 \) is Ulrich by Lemma 3.3 since
\[ L_1^\vee(3) = (-(9\tilde{h} + 3e) + 3(4\tilde{h} - e) = 3\tilde{h} = L_2. \]
Remark 3.10. We know that $H_{6,3,3}$, which is the open subscheme of the Hilbert Scheme parametrizing the smooth irreducible curves of $d = 6$ and $g = 3$ in $\mathbb{P}^3$, is irreducible by [3, Theorem 4]. Also, we know that the property of being an ACM sheaf is an open condition by [3]. Hence, if we assume $C$ is ACM, then the line bundles $L_1$ and $L_2$ exist on a generic element of the deformation class $Y$.

4. Rank 2 Ulrich Bundles on $Y$

Let $E$ be a vector bundle of rank $r$, and $L$ a line bundle on $X$. Then, by [14, Ex. 3.2.2], for all $p \geq 0$,

$$c_p(E \otimes L) = \sum_{i=0}^{p} \binom{p}{i} c_i(E) \cdot c_{i}^{p-i}(L).$$

Then, if $E$ is a rank 2 vector bundle, we have

$$c_1(E \otimes O_X(tH)) = \sum_{i=0}^{1} \left( \begin{array}{c} 2-i \\ 1-i \end{array} \right) c_i(E) \cdot c_1^{1-i}(O_X(tH)) = 2c_0(E) \cdot c_1(O_X(tH)) + c_1(E) = c_1(E) + 2tH$$

and

$$c_2(E \otimes O_X(tH)) = \sum_{i=0}^{2} \left( \begin{array}{c} 2-i \\ 2-i \end{array} \right) c_i(E) \cdot c_1^{2-i}(O_X(tH)) = \sum_{i=0}^{2} c_i(E) \cdot c_1^{2-i}(O_X(tH)) = c_0(E) \cdot c_1^2(O_X(tH)) + c_1(E) \cdot c_1(O_X(tH)) + c_2(E) + (tH)^2 + tc_1(E) \cdot H + c_2(E).$$

Theorem 4.1. Let $(\tilde{Y}, H)$ be a Fano threefold which is the blow-up of $\mathbb{P}^3$ along a smooth, irreducible curve of degree $d$ and genus $g$. If $E$ is a rank 2 Ulrich bundle on $\tilde{Y}$, then we have

1. $H^2 \cdot c_1(E) = 3H^3$,
2. $H \cdot c_2(E) = \frac{1}{2} H \cdot c_1^2(E) - 2H^3 + 4$,
3. $2c_1^3(E) - 6c_1(E) \cdot c_2(E) + c_1(E) \cdot c_2(K_{\tilde{Y}}) = 9H^3$.

Proof. Let $c_i = c_i(E)$ and $d_i = c_i(K_{\tilde{Y}})$. Then, by Riemann-Roch theorem

$$\chi(\tilde{Y}, E(t)) = \frac{1}{6}[(2tH + c_1)^3 - 3(2tH + c_1)((tH)^2 + tc_1 \cdot H + c_2)] + \frac{1}{4}H[(2tH + c_1)^2 - 2((tH)^2 + tc_1 \cdot H + c_2)] + \frac{1}{12}(H^2 + d_2)(2tH + c_1) + \frac{1}{12}Hd_2$$

$$= \frac{1}{6}(8H^3t^3 + 12H^2 \cdot c_1t^2 + 6H \cdot c_1^2t + c_1^3 - 6H^3t^3 - 6H^2 \cdot c_1t^2 - 3H^2 \cdot c_2t - 3H \cdot c_1^2t - 3c_1 \cdot c_2)$$

$$+ \frac{1}{4}(4H^3t^2 + 4H^2 \cdot c_1t + H \cdot c_1^2 - 2H^3t^2 - 2H^2 \cdot c_1t - 2H \cdot c_2)$$
\[ + \frac{1}{12} (2H^3 t + H^2 \cdot c_1 + 2H \cdot d_2 t + c_1 \cdot d_2) + \frac{1}{12} (H \cdot d_2) \]
\[ = \frac{1}{3} H^3 t^3 + \left( \frac{1}{2} H^2 \cdot c_1 + \frac{1}{2} H^3 \right) t^2 \]
\[ + \left( \frac{1}{2} H \cdot c_1^2 - H \cdot c_2 + \frac{1}{2} H^2 \cdot c_1 + \frac{1}{6} H^3 + \frac{1}{6} H \cdot d_2 \right) t \]
\[ + \left( \frac{1}{6} c_1^3 - \frac{1}{2} c_1 \cdot c_2 + \frac{1}{4} H \cdot c_1^2 - \frac{1}{2} H \cdot c_2 + \frac{1}{12} H^2 \cdot c_1 \right. \]
\[ \left. + \frac{1}{12} c_1 \cdot d_2 + \frac{1}{12} H \cdot d_2 \right). \]

Since \( E \) is a rank 2 Ulrich bundle, by Proposition 2.14, we have
\[ \chi(\widetilde{Y}, E(t)) = 2 H^3 \left( t + \frac{3}{3} \right) = H^3 \left( t^3 + 6t^2 + 11t + 6 \right). \]

So, if we equate coefficients of \( t^2 \), we get
\[ \frac{1}{2} H^2 \cdot c_1 + \frac{1}{2} H^3 = 2H^3 \]
\[ \Rightarrow H^2 \cdot c_1 = 3H^3. \]

If we equate coefficients of \( t \), we get
\[ \frac{1}{2} H \cdot c_1^2 - H \cdot c_2 + \frac{1}{2} H^2 \cdot c_1 + \frac{1}{6} H^3 + \frac{1}{6} H \cdot d_2 = \frac{11}{3} H^3 \]
\[ \Rightarrow H \cdot c_2 = \frac{1}{2} H \cdot c_1^2 - 2H^3 + \frac{1}{6} H \cdot d_2 \]
\[ (by \ part \ (1)) \]
\[ \Rightarrow H \cdot c_2 = 1 \frac{1}{2} H \cdot c_1^2 - 2H^3 + 4. \]

If we equate constant terms, we get
\[ \frac{1}{6} c_1^3 - \frac{1}{2} c_1 \cdot c_2 + \frac{1}{4} H \cdot c_1^2 - \frac{1}{2} H \cdot c_2 + \frac{1}{12} H^2 \cdot c_1 + \frac{1}{12} c_1 \cdot d_2 + \frac{1}{12} H \cdot d_2 = 2H^3 \]
\[ \Rightarrow 2c_1^3 - 6c_1 \cdot c_2 + c_1 \cdot d_2 = 9H^3. \]

\[ \square \]

**Theorem 4.2.** Let \( E \) be a rank 2 Ulrich bundle on \( Y \) with \( c_1(Y) = x \tilde{h} - ye \). Then there are 7 possibilities for \( c_1(Y) \), which are

- \( 6\tilde{h} \),
- \( 8\tilde{h} - e \),
- \( 10\tilde{h} - 2e \),
- \( 12\tilde{h} - 3e = 3H \),
- \( 14\tilde{h} - 4e \),
- \( 16\tilde{h} - 5e \),
- \( 18\tilde{h} - 6e \).

**Proof.** We know that \( H_Y = 4\tilde{h} - e \). By Theorem 1.1
\[ (4\tilde{h} - e)^2 (x \tilde{h} - ye) = 3(4\tilde{h} - e)^3 \]
\[ \Rightarrow 16x + 8y(-6) + x(-6) - y(-28) = 3.20 \]
\[ (Theorem 2.3) \]
\[ \Rightarrow x = 2y + 6. \]
Since $E$ is Ulrich, it is $\mu$-semistable by Theorem 4.13. So, we can apply Bogomolov’s Inequality [21, Theorem 7.3.1] and get
\[(2.2c_2(E) - (2 - 1)c_1^2(E))H \geq 0\]
\[\Rightarrow 4Hc_2(E) - Hc_1^2(E) \geq 0\]
\[\Rightarrow 4(Hc_2(E) - 2H^3 + 4) - Hc_1^2(E) \geq 0\]
\[\Rightarrow Hc_2(E) - 2H^3 + 16 \geq 0\]
\[\Rightarrow (4\tilde{h} - e)(x\tilde{h} - ye)^2 - 8.20 + 16 \geq 0\]
\[\Rightarrow 4x^2 + 4y^2(-6) + 2xy(-6) - y^2(-28) - 144 \geq 0\]
\[\Rightarrow 4(2y + 6)^2 - 24y^2 - 12y(2y + 6) + 28y^2 - 144 \geq 0\]
\[\Rightarrow -4y^2 + 24y \geq 0\]
\[\Rightarrow 0 \leq y \leq 6.\]

4.1. Simple Ulrich Bundles on $Y$ with $c_1 = 3H$.

**Proposition 4.3.** Let $X$ be projective variety of dimension $k$ in $\mathbb{P}^N$ and $I_X$ be the ideal sheaf of $X$ in $\mathbb{P}^N$. Then $H^i(\mathbb{P}^N, I_X^n(t))$ is upper semi-continuous for $i > k$.

**Proof.** Twisting the sequence
\[0 \to I_X \to O_{\mathbb{P}^N} \to O_X \to 0\]
by $I_X^{\otimes n-1}(t)$ yields the long exact sequence
\[0 \to \text{Tor}_1^{O_{\mathbb{P}^N}}(I_X^{\otimes n-1}(t), O_X) \to I_X^{\otimes n}(t) \to I_X^{\otimes n-1}(t) \to O_X \otimes I_X^{\otimes n-1}(t) \to 0\]
This can be broken into two short exact sequences, one of which is
\[0 \to \text{Tor}_1^{O_{\mathbb{P}^N}}(I_X^{\otimes n-1}(t), O_X) \to I_X^{\otimes n}(t) \to I_X^n(t) \to 0\]
Since $\text{Tor}_1^{O_{\mathbb{P}^N}}(I_X^{\otimes n-1}(t), O_X)$ has at most $k$-dimensional support, we have $H^i(\mathbb{P}^N, \text{Tor}_1^{O_{\mathbb{P}^N}}(I_X^{\otimes n-1}(t), O_X)) = 0$ for all $i > k$. So, by long exact sequence of cohomology, we get $H^i(\mathbb{P}^N, I_X^{\otimes n}(t)) = H^i(\mathbb{P}^N, I_X^n(t))$ for all $i > k$. Since left hand side is upper semi-continuous, the right hand side is upper semi-continuous. \qed

**Theorem 4.4.** Let $C$ be an smooth ACM space curve with $d = 6$ and $g = 3$. Then $h^2(\mathbb{P}^3, I_C^{\mathbb{P}^3}(6)) = 0$ and $h^2(\mathbb{P}^3, I_C^{\mathbb{P}^3}(2)) \leq 8$ for a generic such $C$.

**Proof.** Use Macaulay2 [16] for computations:

```
i1 : k = ZZ/32647; R = k[x,y,z,w];
i2 : load"RandomSpaceCurves.m2";
i3 : J=ideal (-2215x^2+10620x^2y+2508xy^2-15048y^3-5453x^2z-2767xyz
+886y^2+2222xz^2+1759yz^2-9499z^3+3014x^2w+12412xyw
-1419y^2-11910xzw-3506yzw-831x^2w-1546xw^2+4414yw^2
-10576xw^2+15049y^3-6292x^3+10864x^2y+5626xy^2+8024y^3
+10837x^2y^2+9956y^2x-9501x^2-9538yz^2+9745z^3
+15665x^2w-3220xyw-12116yw^2+1148xzw-3392yzw-1539z^2w
```

Then load the package RandomSpaceCurves [4] to produce explicit example of smooth ACM space curve $C$ of $d = 6$ and $g = 3$ with ideal $J$:
-3915xw²-5992yw²+15589zw²+7309w³, 870x³+9582x²y -172xy²+1923xyz+13352y²z+7141xz² -13354yz²+15747z³+1042x²w+1494xyw-11584y²w+7730xzw -4628yzw+9837z²w-4220xw²+4893yw²-15379zw²-13719w³,
-15941x³-8361x²y-16223xy²+12866y³-4501x²z+13591xyz -11196y²z-6043xz²-7842yz²+11284z³+1057x²w-2552xyw +6508y²w+15994xzw-2374yzw-10280z²w+7766xw²+15317yw² -10555zw²+7241w³)

Then check whether $C'$ is a smooth ACM space curve of $d=6$ and $g=3$:

Then compute $h^2(J^3_C(6))$ and $h^2(J^2_C(2))$:

But, we know that these cohomologies are upper semi-continuous functions by Proposition 4.3. Hence, we have $h^2(\mathbb{P}^3, I^3_C(6)) = 0$ and $h^2(\mathbb{P}^3, I^2_C(2)) \leq 8$ for a generic element of all smooth ACM space curves of $d=6$ and $g=3$. 

**Remark 4.5.** Since cohomology is an upper semi-continuous function, as stated in the proof of Theorem 1.21 smooth ACM space curves of $d=6$ and $g=3$ satisfying $h^2(J^3_C(6)) = 0$ form an open subset of all smooth ACM space curves of $d=6$ and $g=3$. Also by Remark 3.10 we know that $H_{6,3,3}$ is irreducible and smooth ACM space curves of $d=6$ and $g=3$ form an open subset in $H_{6,3,3}$. So, smooth ACM space curves of $d=6$ and $g=3$ satisfying $h^2(\mathbb{P}^3, I^3_C(6)) = 0$ form an open subset of all smooth space curves of $d=6$ and $g=3$. Hence, $h^2(\mathbb{P}^3, I^3_C(6)) = 0$ for a generic element of the deformation class $Y$. By a similar argument, $h^2(\mathbb{P}^3, I^2_C(2)) \leq 8$ for a generic element of the deformation class $Y$.

**Corollary 4.6.** For a generic element of the deformation class of $Y$, we have $\text{ext}^1(L_2, L_1) = 8$.

**Proof.** We know that

$$\text{ext}^1(L_2, L_1) = h^1(Y, L^\vee_2 \otimes L_1),$$

where $L^\vee_2 \otimes L_1 = \mathcal{O}_Y(-3\delta + (9\delta - 3e)) = \mathcal{O}_Y(9\delta - 3e)$.

By Theorem 2.23 $\chi(Y, L^\vee_2 \otimes L_1) = -8$. So, we have

$$h^0(L^\vee_2 \otimes L_1) - h^1(L^\vee_2 \otimes L_1) + h^2(L^\vee_2 \otimes L_1) - h^3(L^\vee_2 \otimes L_1) = -8$$
Corollary 2.8 to compute \( R^H \text{ext} Y \). But, \( L \)

For a generic element of deformation class \( Y \), \( H^2(\mathbb{P}^3, R^1 f_* L^\vee_2 \otimes L_1) = 0 \).

Also by the projection formula, we know that

\[
\begin{align*}
L(h_{20} \otimes \mathbb{O}) & = L(h_{20} \otimes \mathbb{O}) = 0 \\
L(h_{20} \otimes \mathbb{O}) & = L(h_{20} \otimes \mathbb{O}) = 0.
\end{align*}
\]

So, \( H^0(\mathbb{P}^3, R^2 f_* L^\vee_2 \otimes L_1) = 0 \).

Similarly, \( H^1(\mathbb{P}^3, R^1 f_* L^\vee_2 \otimes L_1) = 0 \).

Also by the projection formula, we know that

\[
\begin{align*}
L(h_{20} \otimes \mathbb{O}) & = L(h_{20} \otimes \mathbb{O}) = 0 \\
L(h_{20} \otimes \mathbb{O}) & = L(h_{20} \otimes \mathbb{O}) = 0.
\end{align*}
\]

So \( H^2(\mathbb{P}^3, f_* L^\vee_2 \otimes L_1) = H^2(\mathbb{P}^3, I^3_C(6)). \) Hence, by Corollary 2.8

\[
H^2(Y, L^\vee_2 \otimes L_1) = H^2(\mathbb{P}^3, I^3_C(6)).
\]

So

\[
h^1(Y, L^\vee_2 \otimes L_1) = h^2(\mathbb{P}^3, I^3_C(6)) + 8.
\]

But, \( h^2(\mathbb{P}^3, I^3_C(6)) = 0 \) by Remark 1.5 for a generic element of deformation class \( Y \). Hence, \( \text{ext}^1(L_2, L_1) = 8 \) for a generic element of deformation class \( Y \).

\begin{corollary}
For a generic element of deformation class \( Y \), \( \text{ext}^1(L_1, L_2) \leq 8. \)
\end{corollary}

\begin{proof}
We know that

\[
\text{ext}^1(L_1, L_2) = h^1(Y, L^\vee_1 \otimes L_2) = h^2(Y, L^\vee_2 \otimes L_1 \otimes K_Y)
\]

where \( L^\vee_2 \otimes L_1 \otimes K_Y = \mathcal{O}_Y(-(3h) + (9\bar{h} - 3e) + (-4h + e)) = \mathcal{O}_Y(2\bar{h} - 2e) \). Use Corollary 2.8 to compute \( h^2(Y, L^\vee_2 \otimes L_1 \otimes K_Y) \).

We know that \( H^0(\mathbb{P}^3, R^2 f_* L^\vee_2 \otimes L_1 \otimes K_Y) = H^0(\mathbb{P}^3, R^2 f_* \mathcal{O}_Y(-2e) \otimes \mathcal{O}_{\mathbb{P}^3}(2)) \) by the projection formula and \( R^2 f_* \mathcal{O}_Y(-2e) \otimes \mathcal{O}_{\mathbb{P}^3}(2) \) by Corollary 3.2

So, \( H^0(\mathbb{P}^3, R^2 f_* L^\vee_2 \otimes L_1 \otimes K_Y) = 0 \).

Similarly, \( H^1(\mathbb{P}^3, R^1 f_* L^\vee_2 \otimes L_1 \otimes K_Y) = H^1(\mathbb{P}^3, R^1 f_* \mathcal{O}_Y(-2e) \otimes \mathcal{O}_{\mathbb{P}^3}(2)) \) by the projection formula and \( R^1 f_* \mathcal{O}_Y(-3e) \otimes \mathcal{O}_{\mathbb{P}^3}(2) \) by Corollary 3.2

So, \( H^1(\mathbb{P}^3, R^1 f_* L^\vee_2 \otimes L_1 \otimes K_Y) = 0 \).

Also, by the projection formula, we know that

\[
\begin{align*}
f_* (L^\vee_2 \otimes L_1 \otimes K_Y) & = f_* \mathcal{O}_Y(2\bar{h} - 2e) \\
& = \mathcal{O}_{\mathbb{P}^3}(2) \otimes f_* \mathcal{O}_Y(-2e) \\
& = I^2_C(2) \otimes \mathcal{O}_{\mathbb{P}^3}(2) \\
& = I^2_C(2).
\end{align*}
\]

So \( H^2(\mathbb{P}^3, f_* L^\vee_2 \otimes L_1 \otimes K_Y) = H^2(\mathbb{P}^3, I^2_C(2)). \) Hence, by Corollary 2.8

\[
H^2(Y, L^\vee_2 \otimes L_1 \otimes K_Y) = H^2(\mathbb{P}^3, I^2_C(2)).
\]

But, \( h^2(\mathbb{P}^3, I^2_C(2)) \leq 8 \) by Remark 1.5 for a generic element of deformation class \( Y \).

Hence, \( \text{ext}^1(L_1, L_2) \leq 8 \) for a generic element of deformation class \( Y \).
\end{proof}
Theorem 4.8. Let $\mathcal{E}$ be a rank 2 vector bundle on $Y$ obtained by a non-split extension

\[ 0 \to L_1 \to \mathcal{E} \to L_2 \to 0 \]

or

\[ 0 \to L_2 \to \mathcal{E} \to L_1 \to 0 \]

where $L_1 = \mathcal{O}_Y(9\tilde{h} - 3e)$ and $L_2 = \mathcal{O}_Y(3\tilde{h})$. Then $\mathcal{E}$ is a simple Ulrich bundle with $c_1(\mathcal{E}) = 12\tilde{h} - 3e$ and $c_2(\mathcal{E}) = 27\tilde{h}^2 - 9\tilde{h}e$.

Proof. By Theorem 3.9, $L_1$ and $L_2$ are Ulrich line bundles. Since they are Ulrich, they have the same slope by Proposition 2.14. Since they are line bundles, they are trivially stable. Clearly, they are non-isomorphic. Hence $\mathcal{E}$ is a simple vector bundle by [5, Lemma 4.2].

Since $L_1$ and $L_2$ are Ulrich bundles, $\mathcal{E}$ is an Ulrich bundle by [7, Proposition 2.8]. Moreover, we have

\[
c_1(\mathcal{E}) = c_1(L_1) + c_1(L_2) = (9\tilde{h} - 3e) + (3\tilde{h}) = 12\tilde{h} - 3e
\]

and

\[
c_2(\mathcal{E}) = c_1(L_1)c_1(L_2) = (9\tilde{h} - 3e)(3\tilde{h}) = 27\tilde{h}^2 - 9\tilde{h}e.
\]

\[\square\]

Theorem 4.9. Let $\mathcal{E}$ be a rank 2 simple Ulrich bundle on $Y$ with $c_1(\mathcal{E}) = 12\tilde{h} - 3e$ and $c_2(\mathcal{E}) = 27\tilde{h}^2 - 9\tilde{h}e$. Then $h^1(\mathcal{E} \otimes \mathcal{E}^\vee) - h^2(\mathcal{E} \otimes \mathcal{E}^\vee) = 15$.

Proof. Note that the Chern polynomial of $\mathcal{E}$ is

\[
c_t(\mathcal{E}) = (1 + (9\tilde{h} - 3e)t)(1 + (3\tilde{h})t) = \prod_{i=1}^2 (1 + a_i t)
\]

where $a_1 = 9\tilde{h} - 3e$ and $a_2 = 3\tilde{h}$. Also,

\[
c_1(\mathcal{E}^\vee) = (-1)^1 c_1(\mathcal{E}) = -12\tilde{h} + 3e
\]

and

\[
c_2(\mathcal{E}^\vee) = (-1)^2 c_2(\mathcal{E}) = 27\tilde{h}^2 - 9\tilde{h}e.
\]

Then the Chern polynomial of $\mathcal{E}^\vee$ is

\[
c_t(\mathcal{E}^\vee) = (1 + (-9\tilde{h} + 3e)t)(1 + (-3\tilde{h})t) = \prod_{i=1}^2 (1 + b_i t)
\]
where \( b_1 = -(9\tilde{h} - 3e) \) and \( b_2 = -3\tilde{h} \). Then we have

\[
c_t(E \otimes E') = \prod_{i,j=1}^{2} (1 + (a_i + b_j)t)
\]

\[
= (1 + 0t)(1 + (6\tilde{h} - 3e)t)(1 + (-6\tilde{h} + 3e)t)(1 + 0t)
\]

\[
= 1 + 0t + (-36\tilde{h}^2 + 36\tilde{h}e - 9e^2)t^2 + 0t^3 + 0t^4.
\]

So, \( c_2(E \otimes E') = -36\tilde{h}^2 + 36\tilde{h}e - 9e^2 \) and \( c_i(E \otimes E') = 0 \) for \( i = 1, 3, 4 \).

By Theorem 2.3 we have

\[
c_1(T_Y) = 4\tilde{h} - e
\]

\[
c_2(T_Y) = 12\tilde{h}^2 - 4\tilde{h}e
\]

and

\[
\deg(\tilde{h}^3) = 1
\]

\[
\deg(\tilde{h}^2e) = 0
\]

\[
\deg(\tilde{h}e^2) = -6
\]

\[
\deg(\tilde{e}^3) = -28.
\]

Apply the Riemann-Roch theorem for \( E \otimes E' \) on \( Y \) if \( c_i = c_i(E \otimes E') \) and \( d_i = c_i(T_Y) \):

\[
\chi(Y, E \otimes E') = \frac{1}{6} \left( c_1^2 - 3c_1c_2 + 3c_3 + \frac{1}{4}d_1(c_1^2 - 2c_2) + \frac{1}{12}(d_1^2 + d_2)c_1 + \frac{4}{24}d_1d_2 \right)
\]

\[
= \frac{1}{4}(4\tilde{h} - e)(-2(-36\tilde{h}^2 + 36\tilde{h}e - 9e^2)) + \frac{4}{24}(4\tilde{h} - e)(12\tilde{h}^2 - 4\tilde{h}e)
\]

\[
= \frac{1}{4}(-72) + \frac{1}{6}(24)
\]

\[
= -14.
\]

Then we have

\[
h^0(E \otimes E') - h^1(E \otimes E') + h^2(E \otimes E') - h^3(E \otimes E') = -14
\]

\[
\Rightarrow h^1(E \otimes E') - h^2(E \otimes E') = 14 + h^0(E \otimes E') - h^3(E \otimes E')
\]

\[
= 14 + \text{hom}(E, E) - \text{hom}(E(1), E)
\]

where \( \text{hom}(E, E) = 1 \) since \( E \) is simple. So

\[
h^1(E \otimes E') - h^2(E \otimes E') = 14 + 1 - \text{hom}(E(1), E)
\]

where \( \text{hom}(E(1), E) = 0 \) by [21 Proposition 1.2.7]. So

\[
h^1(E \otimes E') - h^2(E \otimes E') = 14 + 1 - 0
\]

\[
= 15.
\]
4.2. **Quot Scheme.** The general reference for this section is [21, Section 2.2].

The Quot scheme $\text{Quot}_X(F, P)$ parametrizes quotient sheaves of a given $\mathcal{O}_X$-module $F$ with Hilbert polynomial $P$. In this subsection, we briefly review some properties of the Quot scheme, including properties about its local dimension.

Let $\kappa$ be a field, $S$ be a $\kappa$-scheme of finite type and $\text{Sch}/S$ be the category of $S$-schemes. Let $\phi : X \to S$ be a projective morphism and $\mathcal{O}_X(1)$ an $\phi$-ample line bundle on $X$. Let $\mathcal{H}$ be a coherent $\mathcal{O}_X$-module and $P \in \mathbb{Q}[z]$ a polynomial. The functor

$$Q := \text{Quot}_{X/S} : (\text{Sch}/S)^{op} \to (\text{Sets})$$

is defined as follows:

If $T \to S$ is an object in $\text{Sch}/S$, let $Q(T)$ be the set of all $T$-flat coherent quotient sheaves $H_T = \mathcal{O}_T \otimes \mathcal{H} \to F$ with Hilbert polynomial $P$. And if $h : T' \to T$ is an $S$-morphism, let $Q(h) : Q(T) \to Q(T')$ be the map that sends $H_T \to F$ to $H_{T'} \to h^*F$.

**Theorem 4.10.** The functor $\text{Quot}_{X/S}(\mathcal{H}, P)$ is represented by a projective $S$-scheme $\pi : \text{Quot}_{X/S}(\mathcal{H}, P) \to S$.

**Proof.** See [21, Theorem 2.2.4].

**Proposition 4.11.** Let $X$ be a projective scheme over a field $\kappa$ and $H$ a coherent sheaf on $X$. Let $[q : \mathcal{H} \to F] \in \text{Quot}(\mathcal{H}, P)$ be a $\kappa$-rational point and $K = \ker(q)$. Then

$$\text{hom}(K, F) \geq \text{dim}_{[q]} \text{Quot}(\mathcal{H}, P) \geq \text{hom}(K, F) - \text{ext}^1(K, F).$$

If equality holds at the second place, $\text{Quot}(\mathcal{H}, P)$ is a local complete intersection near $[q]$. If $\text{ext}^1(K, F) = 0$, then $\text{Quot}(\mathcal{H}, P)$ is smooth at $[q]$.

**Proof.** See [21, Proposition 2.2.8].

4.3. **Stable Ulrich Bundles on $Y$ with $c_1 = 3H$.** We review some well-known facts.

**Proposition 4.12.** Let $E$ be a stable bundle on $X$. Then $E$ is simple; i.e., $\text{End}(E) \cong \mathbb{K}$.

**Proof.** Since $\mathbb{K}$ is algebraically closed, it follows from [21, Corollary 1.2.8].

**Theorem 4.13.** Let $E$ be an Ulrich bundle of rank $r$ on a nonsingular projective variety $X$. Then,

- $E$ is semistable and $\mu$-semistable,
- If $E$ is stable, then it is also $\mu$-stable.

**Proof.** See [5, Theorem 2.9].

Hence, (semi)stability and $\mu$-(semi)stability are equivalent for an Ulrich bundle $E$ by Lemma 2.11 and Theorem 4.13.

**Proposition 4.14.** Let $E$ be an Ulrich bundle of rank $r$ on a nonsingular projective variety $X$. Then $E$ is globally generated.

**Proof.** See [7, Corollary 2.5].
Lemma 4.15. Let $E$ be an Ulrich bundle on $X$. Then for any Jordan-Hölder filtration

$$0 = E_0 \subseteq E_1 \subseteq \cdots \subseteq E_{m-1} \subseteq E_m = E$$

we have that $E_i$ is an Ulrich bundle for $1 \leq i \leq m$. In particular, if $E$ is a strictly semistable Ulrich bundle of rank $r \geq 2$, then there exist a subbundle $F$ of $E$ having rank $s < r$ which is Ulrich.

Proof. See [6] Lemma 2.15. □

Definition 4.16. Let $E$ be a nontrivial locally free sheaf on $X$. The trace map $tr : \text{End}(E) \to \mathcal{O}_X$ induces $tr^i : \text{Ext}^i(E, E) \cong H^i(\text{End}(E)) \to H^i(\mathcal{O}_X)$. These homomorphisms are surjective. Let $\text{Ext}^i(E, E)_o$ denote the kernel of $tr^i$.

Proposition 4.17. If $E$ is locally free sheaf on $Y$, then $\text{Ext}^i(E, E)_o = \text{Ext}^i(E, E)$ for $0 < i < 3$.

Proof. Note that $H^i(Y, \mathcal{O}_Y) = 0$ for $0 < i < 3$. So the kernel of $tr^i$ is $\text{Ext}^i(E, E)$ for $0 < i < 3$. □

We want to analyze the local dimension of Quot scheme. For this, we will follow the discussion and the notation of [21 Section 4.3]. Let $F$ be semistable sheaf on $X$. Let $m$ be a sufficiently large integer such that $F(m)$ is globally generated, $V$ be a vector space of dimension $P_X(m)$ and $H := V \otimes_k \mathcal{O}_X(-m)$. Let $R \subset \text{Quot}(H, P)$ be the open subscheme of those quotients $[\rho : H \to E]$ where $V \to H^0(E)$ is an isomorphism.

Proposition 4.18. $H^1(Y, \mathcal{O}_Y) \cong 0$ for $i > 0$.

Proof. See [20] p. 153. □

Theorem 4.19. Let $E$ be a rank $2$ simple Ulrich bundle on $Y$, with $c_1(E) = 12h - 3e$ and $c_2(E) = 27h^2 - 9eh$. Then $\dim_{[\rho]} R \geq 1614$ for a fixed $[\rho : H \to E]$.

Proof. We will follow the construction in [21 p.115]. First, note that $E$ is semistable by Theorem 4.13. Second, $E$ is globally generated by Proposition 4.13.

So $V$ is a vector space of dimension $40$, since $P_Y(0) = 20 \cdot 2(3+0) = 40$.

Then $H := V \otimes_k \mathcal{O}_Y = \mathcal{O}_Y^{\oplus 40}$.

Fix $[\rho : H \to E] \in R$.

1. Let $K$ be the kernel of $\rho$; that is, we have

$$0 \to K \to H \to E \to 0.$$ 

Then we have the long exact sequence of cohomology

$$0 \to H^0(Y, K) \to H^0(Y, H) \to H^0(Y, E) \to H^1(Y, K) \to H^1(Y, H) \to H^1(Y, E) \to H^2(Y, K) \to H^2(Y, H) \to H^2(Y, E) \to H^3(Y, K) \to H^3(Y, H) \to H^3(Y, E) \to 0.$$ 

Since $H = \mathcal{O}_Y^{\oplus 40}$ and $E$ is globally generated by Proposition 4.13.

$H^0(Y, H) \cong H^0(Y, E)$. So $H^0(Y, K) \cong 0$. Then, since $\text{Hom}(H, K) \cong \text{Hom}(\mathcal{O}_Y, K)^{\oplus 40} \cong H^0(Y, K)^{\oplus 40}$, $\text{Hom}(H, K) \cong 0$.

Since $H^1(Y, H) \cong H^1(Y, \mathcal{O}_Y)^{\oplus 40} \cong 0$ by Proposition 4.18 and $H^0(Y, H) \cong$
Consider the short exact sequence

\[ 0 \to K \to \mathcal{H} \to \mathcal{E} \to 0. \]

Then take the functor \( \text{Hom}(\mathcal{H}, -) \)

\[ 0 \to \text{Hom}(\mathcal{H}, K) \to \text{Hom}(\mathcal{H}, \mathcal{H}) \to \text{Hom}(\mathcal{H}, \mathcal{E}) \]

\[ \to \text{Ext}^1(\mathcal{H}, K) \to \text{Ext}^1(\mathcal{H}, \mathcal{H}) \to \text{Ext}^1(\mathcal{H}, \mathcal{E}) \]

\[ \to \text{Ext}^2(\mathcal{H}, K) \to \text{Ext}^2(\mathcal{H}, \mathcal{H}) \to \text{Ext}^2(\mathcal{H}, \mathcal{E}) \]

\[ \to \text{Ext}^3(\mathcal{H}, K) \to \text{Ext}^3(\mathcal{H}, \mathcal{H}) \to \text{Ext}^3(\mathcal{H}, \mathcal{E}) \to 0. \]

By step (1), we know that \( \text{Hom}(\mathcal{H}, K) \cong 0 \) and \( \text{Ext}^i(\mathcal{H}, K) \cong 0 \) for \( i > 0 \). So, \( \text{Hom}(\mathcal{H}, \mathcal{H}) \cong \text{Hom}(\mathcal{H}, \mathcal{E}) \) and \( \text{Ext}^i(\mathcal{H}, \mathcal{H}) \cong \text{Ext}^i(\mathcal{H}, \mathcal{E}) \) for \( i > 0 \). On the other hand, \( \text{Ext}^i(\mathcal{H}, \mathcal{H}) \cong \text{Ext}^i(\mathcal{O}_Y^{\oplus 40}, \mathcal{O}_Y^{\oplus 40}) \cong H^i(Y, \mathcal{O}_Y)^{\oplus 1600} \) for \( i > 0 \). Since \( H^1(Y, \mathcal{O}_Y) \cong 0 \) for \( i > 0 \) by Proposition 4.18 \( \text{Ext}^i(\mathcal{H}, \mathcal{H}) \cong 0 \). Hence \( \text{Hom}(\mathcal{H}, \mathcal{H}) \cong \text{Hom}(\mathcal{H}, \mathcal{E}) \) and \( \text{Ext}^i(\mathcal{H}, \mathcal{E}) = 0, i > 0 \).

(3) Again consider the short exact sequence

\[ 0 \to K \to \mathcal{H} \to \mathcal{E} \to 0. \]

Then take the functor \( \text{Hom}(-, \mathcal{E}) \)

\[ 0 \to \text{Hom}(\mathcal{E}, \mathcal{E}) \to \text{Hom}(\mathcal{H}, \mathcal{E}) \to \text{Hom}(K, \mathcal{E}) \]

\[ \to \text{Ext}^1(\mathcal{E}, \mathcal{E}) \to \text{Ext}^1(\mathcal{H}, \mathcal{E}) = 0 \to \ldots \]

leads to equality \( \text{hom}(K, \mathcal{E}) = \text{hom}(\mathcal{H}, \mathcal{E}) + \text{ext}^1(\mathcal{E}, \mathcal{E}) - \text{hom}(\mathcal{E}, \mathcal{E}) \).

Since \( \text{Ext}^i(\mathcal{H}, \mathcal{E}) = 0 \) for \( i > 0 \), \( \text{Ext}^i(K, \mathcal{E}) \cong \text{Ext}^{i+1}(\mathcal{E}, \mathcal{E}) \) for \( i > 0 \).

(4) The boundary map \( \text{Ext}^1(\mathcal{K}, \mathcal{E}) \to \text{Ext}^2(\mathcal{E}, \mathcal{E}) \) maps the obstruction to extend \( [\rho] \) onto the obstructions to extend \( [\mathcal{E}] \) (see 21.2.A.8). The latter is contained in the subspace \( \text{Ext}^2(\mathcal{E}, \mathcal{E})_0 \). This gives the dimension bound, using Proposition 4.11

\[ \dim_{[\rho]} R \geq \text{hom}(K, \mathcal{E}) - \text{ext}^2(\mathcal{E}, \mathcal{E})_0. \]

Then, by step (3), we have

\[ \dim_{[\rho]} R \geq \text{hom}(\mathcal{H}, \mathcal{E}) + \text{ext}^1(\mathcal{E}, \mathcal{E}) - \text{hom}(\mathcal{E}, \mathcal{E}) - \text{ext}^2(\mathcal{E}, \mathcal{E})_0. \]

Then, by Proposition 4.17

\[ \dim_{[\rho]} R \geq \text{hom}(\mathcal{H}, \mathcal{E}) + \text{ext}^1(\mathcal{E}, \mathcal{E}) - \text{hom}(\mathcal{E}, \mathcal{E}) - \text{ext}^2(\mathcal{E}, \mathcal{E}). \]

Then, by step (2), we have

\[ \dim_{[\rho]} R \geq \text{hom}(\mathcal{H}, \mathcal{H}) + \text{ext}^1(\mathcal{E}, \mathcal{E}) - \text{hom}(\mathcal{E}, \mathcal{E}) - \text{ext}^2(\mathcal{E}, \mathcal{E}). \]

Since \( \mathcal{E} \) is simple and \( \mathcal{H} = \mathcal{O}_Y^{\oplus 40} \), we have

\[ \dim_{[\rho]} R \geq 1600 + \text{ext}^1(\mathcal{E}, \mathcal{E}) - 1 - \text{ext}^2(\mathcal{E}, \mathcal{E}). \]
Then, by Theorem 4.9 and the equality $h^i(\mathcal{E} \otimes \mathcal{E}^\vee) = ext^i(\mathcal{E}, \mathcal{E})$, we have $\dim_{[\rho]} R \geq 1600 - 1 + 15 = 1614$. □

Let $R' \subset Quot(\mathcal{H}, P)$ be the subset parametrizing the quotients $[\rho : H \to E]$ where $E$ is obtained as an extension of $L_2$ by $L_1$.

**Proposition 4.20.** $\dim_{[\rho]} R' = 1606$ for a fixed $[\rho : H \to E]$.

**Proof.** The projectivization of $Ext^1(L_2, L_1)$ has dimension $8 - 1 = 7$ by Corollary 4.6. $R'$ is the union of all orbits of extensions of $L_2$ by $L_1$ under the action of $PGL(V)$, so around each fixed $[\rho : H \to E]$, $\dim_{[\rho]} R' = 1599 + 7 = 1606$. □

Let $R'' \subset Quot(\mathcal{H}, P)$ be the subset parametrizing the quotients $[\rho : H \to E]$ where $E$ is obtained as an extension of $L_1$ by $L_2$.

**Proposition 4.21.** $\dim_{[\rho]} R'' \leq 1606$ for a fixed $[\rho : H \to E]$.

**Proof.** The projectivization of $Ext^1(L_1, L_2)$ has dimension $\leq 8 - 1 = 7$ by Corollary 4.7. $R''$ is the union of all orbits of extensions of $L_1$ by $L_2$ under the action of $PGL(V)$, so around each fixed $[\rho : H \to E]$, $\dim_{[\rho]} R'' \leq 1599 + 7 = 1606$. □

**Theorem 4.22.** There exist rank 2 stable Ulrich bundles with $c_1(E) = 12h - 3e$ on a generic element of the deformation class $Y$.

**Proof.** By Theorem 4.8 there are rank 2 simple Ulrich bundle $E$ with the given Chern classes.

We know that the property of being Ulrich is an open condition. So there is an open subset $U$ of $R$ around $[\rho : H \to E]$ containing Ulrich bundles. By Theorem 4.19 $U$ has dimension at least 1614.

We also know that every Ulrich bundle is semistable by Theorem 4.13. If all elements of $U$ were strictly semistable, then by Lemma 4.15 and [7, Proposition 2.8], they would be extensions of Ulrich line bundles. But there are only two Ulrich line bundles $L_1$ and $L_2$ on $Y$. So they would be extensions of $L_2$ by $L_1$ or extensions of $L_1$ by $L_2$.

However, the dimension of $R'$ at the points that are extensions of $L_2$ by $L_1$ is 1606 by Proposition 4.20 and the dimension of $R''$ at the points that are extensions of $L_1$ by $L_2$ is at most 1606 by Proposition 4.21. Since both these dimensions are strictly smaller than 1614, not all Ulrich bundles with the given Chern classes are obtained by extensions. In other words, not all Ulrich bundles in $U$ are strictly semistable.

Hence there are rank 2 stable Ulrich bundles with $c_1(E) = 12h - 3e$. □

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