Higher-Order Perturbative Corrections
to $b \to c$ Transitions at Zero Recoil

Matthias Neubert

Theory Division, CERN, CH-1211 Geneva 23, Switzerland

Abstract

We estimate the two-loop perturbative corrections to zero-recoil matrix elements of the flavour-changing currents $\bar{c} \gamma^\mu b$ and $\bar{c} \gamma^\mu \gamma_5 b$ by calculating the terms of order $n_f \alpha_s^2$ and substituting the dependence on the number of flavours by the first coefficient of the $\beta$-function. Both for vector and axial vector currents, we find moderate two-loop corrections below 1% in magnitude. Using the Brodsky–Lepage–Mackenzie prescription to set the scale in the order-$\alpha_s$ corrections in the $\overline{\text{MS}}$ scheme, we obtain $\mu_V \simeq 0.92 \sqrt{m_b m_c}$ and $\mu_A \simeq 0.51 \sqrt{m_b m_c}$ in the two cases. These scales are sufficiently large for perturbation theory to be well-behaved. The implications of our results to the extraction of $|V_{cb}|$ are briefly discussed.

(Submitted to Physics Letters B)
1 Introduction

It is now widely accepted that the measurement of the $B \to D^* \ell \nu$ decay rate near zero recoil provides for the most reliable determination of the element $V_{cb}$ of the Cabibbo–Kobayashi–Maskawa matrix. The theoretical description of this process has a solid, model-independent foundation based on the heavy quark expansion, which provides a systematic expansion around the limit $m_b, m_c \to \infty$. In this limit, QCD exhibits a spin–flavour symmetry for hadronic systems containing a heavy quark [1]–[3]. The symmetry-breaking corrections are proportional to powers of $\alpha_s(m_Q)$ or $1/m_Q$, where we use $m_Q$ as a generic notation for $m_b$ and $m_c$. These corrections can be investigated in a systematic way using the heavy quark effective theory (HQET) [4]–[6]. In particular, at zero recoil (i.e. at equal velocities of the heavy mesons) the representation of the flavour-changing currents in the HQET reads

$$\bar{c} \gamma^\mu b \to \eta_V \bar{h}_v^c \gamma^\mu h_v^b + O(1/m_Q^2),$$
$$\bar{c} \gamma^\mu \gamma_5 b \to \eta_A \bar{h}_v^c \gamma^\mu \gamma_5 h_v^b + O(1/m_Q^2),$$

where $h_v^Q$ are the velocity-dependent heavy quark spinors of the HQET. Hadronic matrix elements of the effective current operators are normalized because of heavy quark symmetry. The coefficients $\eta_V$ and $\eta_A$ in (1) take into account finite renormalizations of the currents in the intermediate region $m_b > \mu > m_c$. They can be obtained from an on-shell matching of current matrix elements in QCD with the corresponding matrix elements in the HQET. From a measurement of the $B \to D^* \ell \nu$ decay rate near zero recoil, one can extract the product $|V_{cb} \eta_A (1 + \delta_{1/m^2})|$, where $\delta_{1/m^2}$ stands for non-perturbative power corrections of order $(\Lambda_{QCD}/m_Q)^2$ [3, 7].

As very precise experimental data on this decay mode become available [8], a detailed theoretical analysis of the symmetry-breaking corrections to the heavy quark limit becomes increasingly important. The power corrections $\delta_{1/m^2}$ have recently been the subject of intense interest [9]–[12]. Here we shall focus on the perturbative coefficient $\eta_A$ and its analogue for the vector current, $\eta_V$. At the one-loop order, these coefficients have been known for a long time [3]:

$$\eta_V = 1 + C_F \frac{\alpha_s}{4\pi} \phi(z),$$
\[ \eta_A = 1 + C_F \frac{\alpha_s}{4\pi} [\phi(z) - 2], \tag{2} \]

where \( C_F = \frac{4}{3} \) is a colour factor, \( z = m_c/m_b \), and

\[
\phi(z) = -3 \frac{1 + z}{1 - z} \ln z - 6 = \frac{\ln^2 z}{2} - \frac{\ln^4 z}{120} + \frac{\ln^6 z}{5040} + O(\ln^8 z), \tag{3}
\]

with \( \phi(1) = 0 \). Using \( \mu = \sqrt{m_b m_c} \) for the scale in the running coupling constant, one obtains \( \eta_V \simeq 1.02 \) and \( \eta_A \simeq 0.97 \). Throughout this work, we use the input parameters \( m_b = 4.80 \text{ GeV}, m_c = 1.44 \text{ GeV} \), and \( \Lambda_{\text{QCD}} = 0.11 \text{ GeV} \) in the one-loop expression for the running coupling constant in the \( \overline{\text{MS}} \) scheme (for \( n_f = 4 \)). This gives \( \alpha_s(m_b) \simeq 0.20, \alpha_s(m_c) \simeq 0.29 \), and \( \alpha_s(\sqrt{m_b m_c}) \simeq 0.24 \). The fact that the one-loop corrections are smaller than the naive expectation of \( \alpha_s/\pi \sim 10\% \) makes one suspicious about the importance of higher-order corrections. A renormalization-group improvement of (2) has been performed, which sums logarithms of the type \( (\alpha_s \ln z)^n \), \( \alpha_s (\alpha_s \ln z)^n \) and \( z (\alpha_s \ln z)^n \) to all orders in perturbation theory \( [14] - [18] \).

However, since in the case of \( b \to c \) transitions \( \ln z \) is not a particularly large parameter, one expects that the residual two-loop corrections not included in this procedure are as important as some of the logarithmic terms. Therefore, a complete two-loop calculation seems worth while. In particular, it would help to reduce the scale ambiguity in the above one-loop results. Unfortunately, however, such a calculation appears to be rather tedious for the two-scale problem at hand. In this letter, we derive partial results for the two-loop corrections to \( \eta_V \) and \( \eta_A \), which may be used to obtain an estimate of the size of the full corrections. Moreover, our analysis will allow us to study the convergence of perturbation theory for \( b \to c \) transitions. It is thus of interest even beyond the two-loop order. We emphasize that our main goal is to investigate whether there are indications for large higher-order terms in the perturbative series for \( \eta_V \) and \( \eta_A \), and not so much to obtain predictions for these quantities that are more precise than existing ones. To this end, it would be necessary to perform the complete two-loop calculations.

Let us write the perturbative series for any one of the coefficients \( \eta_V \) and \( \eta_A \) in the form

\[ \eta - 1 = \sum_{n=1}^{\infty} \left( \frac{\alpha_s(\mu)}{4\pi} \right)^n \eta_n(\mu), \tag{4} \]

where \( \alpha_s(\mu) \) is the running coupling constant renormalized at some scale \( \mu \). Since \( \eta \) is renormalization-group invariant, the \( \mu \)-dependence on the right-
hand side must cancel between the expansion coefficients and the running coupling constant. It is useful to make explicit the dependence of the coefficients $\eta_n(\mu)$ on the number of quark flavours. In the case at hand, the first dependence on $n_f$ comes at the two-loop order from diagrams containing a quark loop in a gluon propagator. In general, we may write

$$\eta_n(\mu) = c_{n,n-1}(\mu) \beta_0^{n-1} + c_{n,n-2}(\mu) \beta_0^{n-2} + \ldots + c_{n,0}(\mu); \quad n \geq 1,$$

(5)

where $\beta_0 = 11 - \frac{2}{3} n_f$ is the first coefficient of the $\beta$-function. In particular, at the two-loop order we have

$$\eta - 1 = \frac{\alpha_s(\mu)}{4\pi} c_{1,0}(\mu) + \left(\frac{\alpha_s(\mu)}{4\pi}\right)^2 \left[c_{2,1}(\mu) \beta_0 + c_{2,0}(\mu)\right] + \ldots.$$

(6)

For the case of currents composed of one heavy and one light quark, it has been found in explicit calculations that there are large two-loop coefficients when the scale in the running coupling constant is chosen to be the heavy quark mass $m_Q$, and that these large coefficients are dominated by the term proportional to $c_{2,1} \beta_0$ in (6) [19]. This empirical observation can be understood if one assumes that in these on-shell calculations the relevant scale is much below the “natural” scale $m_Q$, meaning that in loop diagrams virtual momenta below $m_Q$ give a sizeable contribution. Because of the relation

$$\alpha_s(\kappa \mu) = \alpha_s(\mu) \sum_{n=0}^{\infty} \left(\frac{\beta_0 \alpha_s(\mu)}{4\pi}\right)^n (-\ln \kappa^2)^n + \ldots,$$

(7)

using an inadequate scale can induce large higher-order coefficients $c_{n,n-1}$. It is possible to absorb some of these large corrections by using a lower scale. However, in some cases this scale turns out to be too low for perturbation theory to be well-defined. Some of the heavy–light currents considered in Ref. [19] provide an example of this phenomenon. As we will discuss below, another example is provided by inclusive decays of hadrons containing a heavy quark [20]. One of our purposes here is to investigate if something similar happens for currents composed of two heavy quarks.

2 Large-$n_f$ asymptotics of perturbation theory

The above discussion justifies that a calculation of the coefficients $c_{n,n-1}$, and in particular of the two-loop coefficient $c_{2,1}$, is worthwhile. Not only can it
serve as an estimate of the size of the full two-loop correction, but also to choose an appropriate scale in the order-$\alpha_s$ term. Technically, the coefficients $c_{n,n-1}$ can be projected out by considering the formal limit of large $n_f$, in which the series (4) takes the form

$$\eta - 1 = \frac{1}{\beta_0} \sum_{n=1}^{\infty} \left( \frac{\beta_0 \alpha_s(\mu)}{4\pi} \right)^n c_{n,n-1}(\mu) + O(1/n_f^2)$$

Note that $\beta_0$ is of order $n_f$, whereas the product $\beta_0 \alpha_s$ is of order $n_f^0$. A convenient way to analyse this series is by considering its Borel transform with respect to $\ln(\mu^2/\Lambda_{\text{QCD}}^2)$ \cite{21}, which we define as

$$\tilde{F}(u, \mu) = \sum_{n=1}^{\infty} \frac{u^{n-1}}{\Gamma(n)} c_{n,n-1}(\mu).$$

The function $\tilde{F}(u, \mu)$ can be calculated using the renormalon calculus of Beneke and Braun \cite{23, 24}. In Ref. \cite{22}, this technique has been used to calculate the Wilson coefficients of flavour-changing heavy quark currents at arbitrary velocity transfer. It is straightforward to specialize the results to zero recoil to obtain explicit expressions for the Borel transforms of the coefficients $\eta_V$ and $\eta_A$. We find

$$\tilde{F}_{V,A}(u, \mu) = C_F e^{-C_u} \left( \frac{\mu^2}{m_b m_c} \right)^u \frac{\Gamma(u)}{\Gamma(2-u)} \frac{\Gamma(1-2u)}{\Gamma(3-u)} \left\{ \begin{array}{c} 2(1 \pm u) \frac{z^u - z^{1-u}}{1-u} \\
+ 2(1-u) \frac{z^{-u} - z^{1-u}}{1-z} \end{array} \right\}$$

$$+ \frac{2(1-u)}{1+2u} \frac{z^{-u} - z^{1-u}}{1-z} + \frac{1+z}{1-z} \left( z^{-u} - z^u \right),$$

where again $z = m_c/m_b$. The upper (lower) sign in the first term in parenthesis refers to the vector (axial vector) current. $C$ is a scheme-dependent constant, with $C = -5/3$ in the $\overline{\text{MS}}$ scheme. The scheme- and scale-dependence of $\tilde{F}(u, \mu)$ cancels against the scheme- and scale-dependence of the running

\footnote{This definition differs from the one adopted in Ref. \cite{22} by a factor $1/\beta_0$.}
coupling constant when one inverts the Borel transformation using the integral relation

\[ \eta - 1 = \frac{1}{\beta_0} \int_0^\infty du \left( \frac{\Lambda^2_{\text{QCD}}}{\mu^2} \right)^u \tilde{F}(u, \mu) + O(1/n_f^2), \tag{11} \]

since the product

\[ \frac{\Lambda^2_{\text{QCD}}}{\mu^2} e^{-C} \mu^2 = \Lambda^2_{\text{QCD}} e^{-C} = \Lambda^2_{\overline{\text{MS}}} e^{5/3} \tag{12} \]

is scheme- and scale-independent.

According to (9), the coefficients \( c_{n,n-1} \) can be obtained from a expansion of the Borel transform in powers of \( u \). Substituting the result back into (6), we find that the \( \mu \)-dependence indeed cancels. At the two-loop order, we obtain

\[ \eta_V = 1 + \bar{\alpha}_s C_F \phi(z) + \left( \frac{\bar{\alpha}_s}{4\pi} \right)^2 \left[ C_F \left( -C - \frac{3}{2} \right) \phi(z) \beta_0 + c_{2,0}^V(z) \right] + \ldots , \]

\[ \eta_A = 1 + \bar{\alpha}_s C_F \left[ \phi(z) - 2 \right] + \left( \frac{\bar{\alpha}_s}{4\pi} \right)^2 \left\{ C_F \left( -C \left[ \phi(z) - 2 \right] - \frac{5}{6} \phi(z) + 1 \right) \beta_0 + c_{2,0}^A(z) \right\} + \ldots , \tag{13} \]

where \( \bar{\alpha}_s \equiv \alpha_s(\sqrt{m_b m_c}) \). Since by charge conservation the vector current is not renormalized for \( z = 1 \), it follows that \( c_{2,0}^V(1) = 0 \). For \( z = m_c/m_b = 0.3 \), we obtain in the \( \overline{\text{MS}} \) scheme (with \( C = -5/3 \))

\[ \eta_V \simeq 1 + 0.236 \frac{\bar{\alpha}_s}{\pi} + \left( 0.082 + \frac{1}{16} c_{2,0}^V(z) \right) \left( \frac{\bar{\alpha}_s}{\pi} \right)^2 + \ldots , \]

\[ \eta_A \simeq 1 - 0.431 \frac{\bar{\alpha}_s}{\pi} + \left( -1.211 + \frac{1}{16} c_{2,0}^A(z) \right) \left( \frac{\bar{\alpha}_s}{\pi} \right)^2 + \ldots , \tag{14} \]

where we use \( \beta_0 = 25/3 \), corresponding to \( n_f = 4 \), which is appropriate for the intermediate region \( m_b > \mu > m_c \). The partial two-loop corrections that we have computed amount to very moderate effects, which however have the same sign as the one-loop corrections. Numerically, with \( \bar{\alpha}_s \simeq 0.24 \), we obtain \( \delta \eta_V \simeq 5 \times 10^{-4} \) for the corresponding contribution to \( \eta_V \), and \( \delta \eta_A \simeq -7 \times 10^{-3} \) for the contribution to \( \eta_A \). Thus, we find no indication for large two-loop corrections in the case of heavy–heavy currents. This is in stark contrast to the case of heavy–light currents, where the coefficient of the \( (\alpha_s/\pi)^2 \) term is typically of order 10, with \( \frac{1}{16} c_{2,0} \) of order unity \[19\].
3 BLM scale setting

Brody, Lepage and Mackenzie (BLM) have advocated to absorb vacuum polarization effects into the running coupling constant by choosing the scale in the order-$\alpha_s$ correction so that there are no corrections of order $\beta_0 \alpha_s^2$ in an expansion such as (6) [25]. This physically appealing scale-setting prescription usually results in a reasonable perturbative series. Accepting this point of view, one may argue that perturbation theory works well in cases where the BLM scale is sufficiently large, whereas it breaks down if this scale is too low. Recently, it has been shown that the BLM scale to be used in inclusive $\bar{B} \to X \ell \bar{\nu}$ decays is very low, $\mu_{\text{incl}} \approx 0.07 \text{mb}$, indicating a breakdown of perturbation theory [20]. In fact, it had been noted before that the one-loop corrections to the inclusive decay rate exhibit a strong scale dependence [26]. This observation puts severe limitations on the usefulness of inclusive decays for the determination of $|V_{cb}|$. At least, a calculation of the two-loop corrections is necessary before a reliable analysis can be performed. Fortunately, as we will now show, the situation appears to be much better for the exclusive decay $\bar{B} \to D^* \ell \bar{\nu}$.

From (6) and (7), it follows that for a general perturbative series the BLM scale is given by

$$\mu_{\text{BLM}} = \exp \left( -\frac{c_{2,1}(\mu)}{2c_{1,0}(\mu)} \right) \mu,$$

which can be shown to be $\mu$-independent. Note that the BLM scale is not scheme-independent; instead, it is such that the value of $\alpha_s(\mu_{\text{BLM}})$ is scheme-independent. According to (12), this requires that $\mu_{\text{BLM}} \propto e^{C/2}$. Indeed, from our calculation in the previous section, we obtain

$$\mu_V = e^{3/4} e^{C/2} \sqrt{m_b m_c},$$

$$\mu_A = \exp \left\{ \frac{6 - 5\phi(z)}{12(2 - \phi(z))} \right\} e^{C/2} \sqrt{m_b m_c}. \quad (16)$$

For $z = 0.3$ and in the $\overline{\text{MS}}$ scheme (with $C = -5/3$), this yields $\mu_V \approx 0.920 \sqrt{m_b m_c}$ and $\mu_A \approx 0.509 \sqrt{m_b m_c}$. These scales are large enough for perturbation theory to be well-behaved. The corresponding scheme-independent values of the running coupling constant are $\alpha_s(\mu_V) \approx 0.24$ and $\alpha_s(\mu_A) \approx 0.30$. Using these coupling constants instead of $\alpha_s(\sqrt{m_b m_c})$ in the one-loop expressions (2) changes the values of $\eta_V$ and $\eta_A$ by $\delta \eta_V \approx 5 \times 10^{-4}$ and
\[ \delta \eta_A \simeq -9 \times 10^{-3} \]. These changes are practically identical to our estimates of the two-loop corrections at the end of the previous section.

4 Conclusions

We have presented a partial calculation of the two-loop matching corrections to the flavour-changing currents in \( b \to c \) transitions at zero recoil. Both for vector and axial vector currents we find small corrections, which are below 1% in magnitude. Using the Brodsky–Lepage–Mackenzie prescription to set the scale in the order-\( \alpha_s \) corrections, we obtain \( \alpha_s(\mu_V) \simeq 0.24 \) and \( \alpha_s(\mu_A) \simeq 0.30 \) for the relevant coupling constants in the two cases. The fact that the corresponding scales are sufficiently large indicates good convergence of perturbation theory for exclusive \( b \to c \) transitions. This is in contrast to inclusive \( B \) decays, where it was found that the appropriate scale is as low as 350 MeV, indicating a breakdown of perturbation theory \([20]\). These findings support that the exclusive semileptonic decay \( \bar{B} \to D^* \ell \bar{\nu} \) is the “gold-plated” mode for a precision measurement of \( |V_{cb}| \) \([12]\).

It is interesting to compare our estimate of the size of the two-loop corrections with the intrinsic uncertainty in \( \eta_V \) and \( \eta_A \), which results from the necessity to regularize the divergent asymptotic behaviour of the perturbative series for these quantities. In fact, both \( \eta_V \) and \( \eta_A \) are known to contain infrared renormalons, which lead to ambiguities of order \( (\Lambda_{QCD}/m_Q)^2 \). A measure of the resulting intrinsic uncertainty is \([22]\)

\[ \Delta \eta_{V,A} = \frac{3\beta_0}{32} \left[ \Delta m \left( \frac{1}{m_c} \mp \frac{1}{m_b} \right) \right]^2, \]  

where \( \Delta m \sim \Lambda_{QCD} \) is the renormalon ambiguity in the pole mass of a heavy quark \([24, 27]\). As previously, the upper (lower) sign refers to the vector (axial vector) current. Assuming \( \Delta m \simeq 0.1 \) GeV, we obtain \( \Delta \eta_V \simeq 0.2\% \) and \( \Delta \eta_A \simeq 0.6\% \). These numbers match nicely with our estimate of the two-loop corrections, indicating that it is sufficient and adequate to truncate the perturbative series at the two-loop order.

In conclusion, we thus believe that the existing calculations of the matching corrections for heavy quark currents, for instance the one-loop results \( \eta_V \simeq 1.02 \) and \( \eta_A \simeq 0.97 \) obtained from \([2]\), or the values \( \eta_V = 1.025 \pm 0.015 \) and \( \eta_A = 0.985 \pm 0.015 \) obtained by performing a next-to-leading order
renormalization-group improvement \cite{18}, are reliable at the level of a few per cent. There are no indications for unusually large higher-order corrections.

Acknowledgements: It is a pleasure to thank Thomas Mannel for useful discussions and Chris Sachrajda for collaboration on subjects closely related to this work. I am indebted to David Broadhurst and Andrey Grozin for making the results of Ref. \cite{19} available to me prior to publication.

References

[1] M.B. Voloshin and M.A. Shifman, Yad. Fiz. 47, 801 (1988) [Sov. J. Nucl. Phys. 47, 511 (1988)].

[2] N. Isgur and M.B. Wise, Phys. Lett. B 232, 113 (1989); 237, 527 (1990).

[3] For a review, see: M. Neubert, SLAC preprint SLAC–PUB–6263 (1993), to appear in Phys. Rep., and references therein.

[4] E. Eichten and B. Hill, Phys. Lett. B 234, 511 (1990); 243, 427 (1990).

[5] H. Georgi, Phys. Lett. B 240, 447 (1990).

[6] T. Mannel, W. Roberts, and Z. Ryzak, Nucl. Phys. B 368, 204 (1992).

[7] M. Neubert, Phys. Lett. B 264, 455 (1991).

[8] For a summary of the experimental situation, see: R. Patterson, to appear in: Proceedings of the 27th International Conference on High Energy Physics, Glasgow, Scotland, July 1994.

[9] A.F. Falk and M. Neubert, Phys. Rev. D 47, 2965 (1993).

[10] T. Mannel, Phys. Rev. D 50, 428 (1994).

[11] M. Shifman, N.G. Uraltsev, and A. Vainshtein, Minnesota preprint TPI-MINN-94/13-T (1994); I. Bigi, M. Shifman, N.G. Uraltsev, and A. Vainshtein, Minnesota preprint TPI-MINN-94/12-T (1994).
[12] M. Neubert, CERN preprint CERN-TH.7395/94 (1994), to appear in Phys. Lett. B.

[13] J.E. Paschalis and G.J. Gounaris, Nucl. Phys. B 222, 473 (1983); F.E. Close, G.J. Gounaris, and J.E. Paschalis, Phys. Lett. B 149, 209 (1984).

[14] A.F. Falk, H. Georgi, B. Grinstein, and M.B. Wise, Nucl. Phys. B 343, 1 (1990).

[15] X. Ji and M.J. Musolf, Phys. Lett. B 257, 409 (1991).

[16] D.J. Broadhurst and A.G. Grozin, Phys. Lett. B 267, 105 (1991).

[17] A.F. Falk and B. Grinstein, Phys. Lett. B 247, 406 (1990).

[18] M. Neubert, Phys. Rev. D 46, 2212 (1992).

[19] D.J. Broadhurst and A.G. Grozin, Open University preprint OUT-4102-52 (1994).

[20] M. Luke, M.J. Savage, and M.B. Wise, Toronto preprint UTPT 94-24 (1994).

[21] G. 't Hooft, in: The Whys of Subnuclear Physics, Proceedings of the 15th International School on Subnuclear Physics, Erice, Sicily, 1977, edited by A. Zichichi (Plenum Press, New York, 1979), p. 943.

[22] M. Neubert and C.T. Sachrajda, CERN preprint CERN-TH.7312/94 (1994).

[23] M. Beneke, Phys. Lett. B 307, 154 (1993); Nucl. Phys. B 405, 424 (1993).

[24] M. Beneke and V.M. Braun, Nucl. Phys. B 426, 301 (1994).

[25] S.J. Brodsky, G.P. Lepage, and P.B. Mackenzie, Phys. Rev. D 28, 228 (1983).

[26] P. Ball and U. Nierste, Munich preprint TUM-T31-56/94/R (1994).

[27] I.I. Bigi, M.A. Shifman, N.G. Uraltsev, and A.I. Vainshtein, CERN preprint CERN-TH.7171/94 (1994).