Metric Characterisation of Unitaries in JB*-Algebras

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Abstract. Let \( M \) be a unital JB*-algebra whose closed unit ball is denoted by \( B_M \). Let \( \partial_e(B_M) \) denote the set of all extreme points of \( B_M \). We prove that an element \( u \in \partial_e(B_M) \) is a unitary if and only if the set
\[
\mathcal{M}_u = \{ e \in \partial_e(B_M) : \| u \pm e \| \leq \sqrt{2} \}
\]
contains an isolated point. This is a new geometric characterisation of unitaries in \( M \) in terms of the set of extreme points of \( B_M \).

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1. Introduction

We know from a celebrated result of R.V. Kadison that the extreme points of the closed unit ball of a C*-algebra \( A \) are precisely the maximal partial isometries in \( A \), that is, the elements \( u \) in \( A \) such that \((1 - uu^*)(1 - u^*u) = \{0\} \) (see [14]). Every unitary in \( A \) is an extreme point of its closed unit ball, but the reciprocal implication is not always true. In 2002, C.A. Akemann and N. Weaver searched for a characterisation of partial isometries, unitaries, and invertible elements in a unital C*-algebra \( A \) in terms of the Banach space structure of certain subsets of \( A \), the dual space, \( A^* \), or the predual, \( A_* \), when \( A \) is a von Neumann algebra (cf. [1]). The resulting characterisations are called geometric because only the Banach space structure of \( A \) is employed. It should be noted that the geometric characterisation of partial isometries in a C*-algebra was subsequently extended to a geometric characterisation of tripotents in a general JB*-triple (see, [6,7]). The geometric characterisation of unitaries actually relies on a good knowledge on the set of states of a Banach space \( X \) relative to an element \( x \) in its unit sphere, \( S(X) \), defined by
\[
S_x := \{ \varphi \in X^* : \varphi(x) = \| \varphi \| = 1 \}.
\]
The element $x$ is called a vertex of the closed unit ball of $X$ (respectively, a geometric unitary of $X$) if $S_x$ separates the points of $X$ (respectively, spans $X^*$).

Akmann and Weaver proved that a norm-one element $x$ in a C*-algebra $A$ is (an algebraic) unitary (i.e. $xx^* = x^*x = 1$) if and only if $S_x$ spans $A^*$. In a von Neumann algebra $W$ an analogous characterisation holds when one uses the predual, $W_*$, in lieu of the dual space and the set of normal states relative to $x$, $S^x = \{ \varphi \in W_* : \varphi(x) = ||\varphi|| = 1 \}$, in place of $S_x$ (cf. [1, Theorem 3]).

Appropriate versions of the just commented result in the setting of JB*-algebras and JB*-triples were established by A. Rodríguez Palacios in [23] (see Sect. 2 for the missing notions). We recall that a complex (respectively, real) Jordan algebra $M$ is a (not-necessarily associative) algebra over the complex (respectively, real) field whose product is abelian and satisfies $(a \circ b) \circ c = a \circ (b \circ c^2) \circ a^2$ ($a, b \in M$). A normed Jordan algebra is a Jordan algebra $M$ equipped with a norm, $||.||$, satisfying $||a \circ b|| \leq ||a|| \cdot ||b||$. A Jordan Banach algebra is a normed Jordan algebra whose norm is complete. Every real or complex associative Banach algebra is a real Jordan Banach algebra with respect to the product $a \circ b := \frac{1}{2}(ab + ba)$.

An element $a$ in a unital Jordan Banach algebra $J$ is called invertible whenever there exists $b \in J$ satisfying $a \circ b = 1$ and $a^2 \circ b = a$. The element $b$ is unique and it will be denoted by $a^{-1}$ (cf. [10, 3.2.9]).

A JB*-algebra is a complex Jordan Banach algebra $M$ equipped with an algebra involution $^*$ satisfying $||\{a, a, a\}|| = ||a||^3$, $a \in M$ (we recall that $\{a, a, a\} = 2(a \circ a^*) \circ a - a^2 \circ a^*$). A $JB$-algebra is a real Jordan Banach algebra $J$ in which the norm satisfies the following two axioms for all $a, b \in J$

(i) $||a^2|| = ||a||^2$;
(ii) $||a^2|| \leq ||a^2 + b^2||$.

The hermitian part, $M_{sa}$, of a JB*-algebra, $M$, is always a JB-algebra. A celebrated theorem due to J.D.M. Wright asserts that, conversely, the complexification of every JB-algebra is a JB*-algebra (see [25]). We refer to the monographs [10] and [5] for the basic notions and results in the theory of JB- and JB*-algebras.

Every C*-algebra $A$ is a JB*-algebra when equipped with its natural Jordan product $a \circ b = \frac{1}{2}(ab + ba)$ and the original norm and involution. Norm-closed Jordan *-subalgebras of C*-algebras are called JC*-algebras.

Given an element $a$ in a JB*-algebra $M$, the symbol $U_a$ will stand for the linear operator defined by $U_a(x) = 2(a \circ x) \circ a - a^2 \circ x$ ($x \in M$). Let us observe that if a C*-algebra $A$ is regarded as a JB*-algebra, we have $U_a(x) = axa$ for all $x \in A$.

Two elements $a, b$ in a Jordan algebra $M$ are said to operator commute if

$$(a \circ c) \circ b = a \circ (c \circ b),$$

for all $c \in M$. By the centre of $M$ we mean the set of all elements of $M$ which operator commute with any other element in $M$. 

We recall that an element \( u \) in a unital JB*-algebra \( M \) is a unitary if it is invertible and its inverse coincides with \( u^* \). An element \( s \) in a unital JB-algebra \( J \) is called a symmetry if \( s^2 = 1 \). The set of all symmetries in \( J \) will be denoted by \( \text{Symm}(J) \). If \( M \) is a JB*-algebra, we shall write \( \text{Symm}(M) \) for \( \text{Symm}(M_{sa}) \).

The geometric characterisation of unitaries in JB*-algebras reads as follows: For a norm-one element \( u \) in a JB*-algebra \( M \), the following conditions are equivalent:

1. \( u \) is a unitary in \( M \);
2. \( u \) is a geometric unitary in \( M \);
3. \( u \) is a vertex of the closed unit ball of \( M \),

(see [23, Theorem 3.1] and [5, Theorem 4.2.24], where the result is proved in the more general setting of JB*-triples).

Surprisingly, as shown by C.-W. Leung, C.-K. Ng, N.-C. Wong in [18], the case of JB-algebras differs slightly from the result stated for JB*-algebras. Suppose \( x \) is a norm-one element in a JB-algebra \( J \), then the following statements are equivalent:

(a) \( x \) is a geometric unitary in \( J \);
(b) \( x \) is a vertex of the closed unit ball of \( J \);
(c) \( x \) is an isolated point of \( \text{Symm}(J) \) (endowed with the norm topology);
(d) \( x \) is a central unitary in \( J \);
(e) The multiplication operator \( M_x : z \mapsto x \circ z \) satisfies \( M_x^2 = \text{id}_J \),

(see [18, Theorem 2.6] or [5, Proposition 3.1.15]).

Except perhaps statements (c), (d) and (e) above, the previous characterisations rely on the set of states \( S_x \) of the underlying Banach space at an element \( x \) in the unit sphere, that is, they are geometric characterisations in which the structure of the whole dual space plays an important role.

From a completely independent setting, the different attempts to solve the problem of extending a surjective isometry between the unit spheres of two Banach spaces to a surjective real linear isometry between the spaces (known as Tingley’s problem) have produced a substantial collection of new ideas and devices which are, in most of cases, interesting by themselves (cf., for example, [2, 4, 20–22]). Let us borrow some words from [4] “...it is really impressive the development of machinery and technics that this problem (Tingley’s problem) has led to.”. We shall place our focus on the next result, included by M. Mori in [20], which provides a new characterisation of unitaries in a unital C*-algebra.

From now on, the closed unit ball of a Banach space \( X \) will be denoted by \( \mathcal{B}_X \). The set of all extreme points of a convex set \( C \) will be denoted by \( \partial_e(C) \).

**Theorem 1.1** [20, Lemma 3.1]. Let \( A \) be a unital C*-algebra, and let \( u \in \partial_e(\mathcal{B}_A) \). Then the following statements are equivalent:

(a) \( u \) is a unitary (i.e., \( uu^* = u^*u = 1 \));
(b) The set \( \mathcal{A}_u = \{ e \in \partial_e(\mathcal{B}_A) : \|u \pm e\| = \sqrt{2} \} \) contains an isolated point.
The advantage of the previous result is that it characterises unitaries among extreme points of the closed unit ball of a unital C*-algebra \( A \) in terms of the subset of all points in \( \partial_e(\mathcal{B}_A) \) at distance \( \sqrt{2} \) from the element under study. We do not need to deal with the dual of \( A \).

The purpose of this note is to explore the validity of this characterisation in the setting of JB*-algebras. In a first result we prove that for each tripotent \( u \) in a JB*-triple \( E \) the equality

\[
\{ e \in \text{Trip}(E_2(u)) : \| u + e \| \leq \sqrt{2} \} = \{ i(p - q) : p, q \in \mathcal{P}(E_2(u)) \text{ with } p \perp q \}
\]

holds true, where given a JB*-triple \( E \), the symbol \( \text{Trip}(E) \) stands for the set of all tripotents in \( E \). Furthermore, if \( u \) is unitary in \( E \), then

\[
\mathcal{E}_u = \left\{ e \in \partial_e(\mathcal{B}_E) : \| u + e \| \leq \sqrt{2} \right\} = i\text{Symm}(E_2(u))
\]

and the elements \( \pm iu \) are isolated in \( \mathcal{E}_u \) (Corollary 3.3).

After some technical results inspired from recent achievements by J. Hamhalter, O. F. K. Kalenda, H. Pfitzner, and the second author of this note in [9], we arrive to our main result in Theorem 3.8, where we prove the following: Let \( u \) be an extreme point of the closed unit ball of a unital JB*-algebra \( M \). Then the following statements are equivalent:

(a) \( u \) is a unitary tripotent;
(b) The set \( \mathcal{M}_u = \{ e \in \partial_e(\mathcal{B}_M) : \| u + e \| \leq \sqrt{2} \} \) contains an isolated point.

### 2. Background on JB*-Algebras and JB*-Triples

Suppose \( A \) is a unital C*-algebra whose set of projections (i.e. symmetric idempotents) will be denoted by \( \mathcal{P}(A) \). It is known that the distance from 1 to any projection in \( \mathcal{P}(A) \setminus \{ 1 \} \) is 1, that is, \( \| 1 - q \| \in [0, 1] \) for all \( q \in \mathcal{P}(A) \). Suppose \( p \) is a central projection in \( A \). In this case, \( A \) writes as the orthogonal sum of \( pAp \) and \( (1 - p)A(1 - p) \), and every projection \( q \) in \( A \) is of the form \( q = q_1 + q_2 \), where \( q_1 \leq p \) and \( q_2 \leq 1 - p \). Then it easily follows that \( \| p - q \| = \max\{ \| p - q_1 \|, \| q_2 \| \} \in [0, 1] \) for each \( q \in \mathcal{P}(A) \), which shows that \( p \) is isolated (in the norm topology) in \( \mathcal{P}(A) \). An easy example of a non-isolated projection can be given with 2 by 2 matrices. It is known that every rank one projection in \( M_2(\mathbb{C}) \) can be written in the form

\[
p = \left( \begin{array}{cc} t & \gamma \sqrt{t(1 - t)} \\ \gamma \sqrt{t(1 - t)} & 1 - t \end{array} \right), \quad \gamma \in \mathbb{C} \text{ with } |\gamma| = 1 \text{ and } t \in [0, 1].
\]

The mapping \( q : [0, 1] \to \mathcal{P}(M_2(\mathbb{C})) \), \( q(s) = \left( \begin{array}{cc} s & \gamma \sqrt{s(1 - s)} \\ \gamma \sqrt{s(1 - s)} & 1 - s \end{array} \right) \) is continuous and shows that \( p \) is non-isolated in \( \mathcal{P}(M_2(\mathbb{C})) \). The natural question is whether \( p \) being isolated in \( \mathcal{P}(A) \) implies that \( p \) is central in \( A \). This question has been explicitly treated by M. Mori in [20, Proof of Lemma 3.1]. The argument is as follows, suppose \( p \) is isolated in \( \mathcal{P}(A) \), for each \( a = a^* \) in \( A \), we consider the mapping \( \omega : \mathbb{R} \to \mathcal{P}(A) \), \( \omega(t) := e^{ita}pe^{-ita} \), which is differentiable with \( \omega(0) = p \). We deduce from the assumption on \( p \) that \( \omega \) must be constant, and thus taking derivative at \( t = 0 \) we get \( iap - ipa = 0 \),
which implies that \( p \) commutes with every hermitian element in \( A \). That is every isolated projection in \( \mathcal{P}(A) \) is central in \( A \). We gather this information in the next result.

**Proposition 2.1.** Let \( p \) be a projection in a unital \( C^* \)-algebra \( A \). Then the following statements are equivalent:

(a) \( p \) is (norm) isolated in \( \mathcal{P}(A) \);
(b) \( p \) is a central projection in \( A \);
(c) \( 1-2p \) is (norm) isolated in \( \text{Symm}(A) \).

**Proof.** The implication \((a) \Rightarrow (b)\) is proved in [20, Proof of Lemma 3.1], while \((b) \Rightarrow (a)\) has been commented before. Finally it is easy to see that a sequence \((q_n) \subseteq \mathcal{P}(A)\{p\}\) converges in norm to \( p \) if and only if the sequence \( (1-2q_n) \subseteq \text{Symm}(A)\{1-2p\}\) converges in norm to \( 1-2p \). \( \square \)

Let us observe that in case that \( A \) is a von Neumann algebra, the equivalence of \((a)\) and \((b)\) in the above Proposition was proved by Y. Kato in [15].

A Jordan version of Proposition 2.1 was considered by J.D.M. Wright and M.A. Youngson in [26]. Before going into details, let us note that the lacking of associativity for the product of a JB*-algebra makes invalid the arguments presented above, and specially the use of products of the form \( e^{ita}pe^{-ita} \) is not always possible in the Jordan analogue of \((a) \Rightarrow (b)\).

In our approach to the Jordan setting, JB*-algebras and JB-algebras
will be regarded as JB*-triples and real JB*-triples, respectively. According to the original definition, introduced by W. Kaup in [16], a JB*-triple is a complex Banach space \( E \) equipped with a continuous triple product \( \{.,.,.\} : E \times E \times E \to E \), \((a,b,c) \mapsto \{a,b,c\}\), which is bilinear and symmetric in \((a,c)\) and conjugate linear in \( b \), and satisfies the following axioms for all \( a,b,x,y \in E \):

(a) \( L(a,b)L(x,y) = L(x,y)L(a,b) + L(L(a,b)x,y) - L(x,L(b,a)y) \), where \( L(a,b) : E \to E \) is the operator defined by \( L(a,b)x = \{a,b,x\} \);
(b) \( L(a,a) \) is a hermitian operator with non-negative spectrum;
(c) \( \|\{a,a,a\}\| = \|a\|^3 \).

Examples of JB*-triples include all \( C^* \)-algebras and JB*-algebras with triple products of the form

\[ \{x,y,z\} = \frac{1}{2}(xy^*z + zy^*x), \tag{1} \]

and

\[ \{x,y,z\} = (x \circ y^*) \circ z + (z \circ y^*) \circ x - (x \circ z) \circ y^*, \tag{2} \]

respectively.

Given an element \( x \) in a JB*-triple \( E \), we shall write \( x^{[1]} := x \), \( x^{[3]} := \{x,x,x\} \), and \( x^{[2n+1]} := \{x,x,x^{[2n-1]}\} \), \((n \in \mathbb{N})\).

Like real \( C^* \)-algebras are defined as real norm closed Hermitian subalgebras of \( C^* \)-algebras (cf. [19]), a real closed subtriple of a JB*-triple is called a real JB*-triple (see [11]). Every JB*-triple is a real JB*-triple when
it is regarded as a real Banach space. In particular every JB-algebra is a real JB*-triple with the triple product defined in (2) (see [11]).

An element \( e \) in a real or complex JB*-triple \( E \) is said to be a tripotent if \( \{e, e, e\} = e \). Each tripotent \( e \in E \), determines a decomposition of \( X \), known as the Peirce decomposition associated with \( e \), in the form

\[
E = E_2(e) \oplus E_1(e) \oplus E_0(e),
\]

where \( E_j(e) = \{x \in E : \{e, e, x\} = \frac{j}{2} x\} \) for each \( j = 0, 1, 2 \).

Triple products among elements in the Peirce subspaces satisfy the following Peirce arithmetic: \( \{E_i(e), E_j(e), E_k(e)\} \subseteq E_{i-j+k}(e) \) if \( i-j+k \in \{0, 1, 2\} \), and \( \{E_i(e), E_j(e), E_k(e)\} = \{0\} \) otherwise, and

\[
\{E_2(e), E_0(e), E\} = \{E_0(e), E_2(e), E\} = 0.
\]

Consequently, each Peirce subspace \( E_j(e) \) is a real or complex JB*-subtriple of \( E \).

The projection \( P_k(e) \) of \( E \) onto \( E_k(e) \) is called the Peirce \( k \)-projection. It is known that Peirce projections are contractive (cf. [8, Corollary 1.2]) and determined by the following identities \( P_2(e) = Q(e)^2 \), \( P_1(e) = 2(L(e, e) - Q(e)^2) \), and \( P_0(e) = \text{Id}_E - 2L(e, e) + Q(e)^2 \), where \( Q(e) : E \to E \) is the conjugate or real linear map defined by \( Q(e)(x) = \{e, x, e\} \). A tripotent \( e \in E \) is called unitary (respectively, complete or maximal) if \( E_2(e) = E \) (respectively, \( E_0(e) = \{0\} \)). This definition produces no contradiction because unitary elements in a unital JB*-algebra \( M \) are precisely the unitary tripotents in \( M \) when the latter is regarded as a JB*-triple (cf. [3, Proposition 4.3]). A tripotent \( e \) in \( E \) is called minimal if \( E_2(e) = \mathbb{C} e \neq \{0\} \). The set of all tripotents (respectively, of all complete tripotents) in a JB*-triple \( E \) will be denoted by Trip\( (E) \) (respectively, Trip\( \text{max} (E) \)).

It is worth remarking that if \( E \) is a complex JB*-triple, the Peirce 2-subspace \( E_2(e) \) is a unital JB*-algebra with unit \( e \), product \( x \circ_e y := \{x, e, y\} \) and involution \( x^* := \{e, x, e\} \), respectively.

Let us recall that a couple of elements \( a, b \) in a real or complex JB*-triple \( E \) are called orthogonal (written \( a \perp b \)) if \( L(a, b) = 0 \). It is known that \( a \perp b \Longleftrightarrow \{a, a, b\} = 0 \Longleftrightarrow \{b, b, a\} = 0 \Longleftrightarrow b \perp a \). If \( e \) is a tripotent in \( E \), it follows from Peirce rules that \( a \perp b \) for every \( a \in E_2(e) \) and every \( b \in E_0(e) \). Two projections \( p, q \) in a JB*-algebra are orthogonal if and only if \( p \circ q = 0 \). An additional geometric property of orthogonal elements shows that \( ||a \pm b|| = \max\{||a||, ||b||\} \) whenever \( a \) and \( b \) are orthogonal elements in a real or complex JB*-triple (cf. [8, Lemma 1.3]).

Henceforth the set, Trip\( (E) \), of all tripotents in a JB*-triple \( E \), will be equipped with the natural partial order defined by \( u \leq e \) in Trip\( (E) \) if \( e - u \) is a tripotent in \( E \) with \( e - u \perp u \), equivalently, if \( u \) is a projection in the JB*-algebra \( E_2(e) \).

One of the useful geometric properties of a real or complex JB*-triple, \( E \), asserts that the extreme points of its closed unit ball, \( \mathcal{B}_E \), are precisely the complete tripotents in \( E \), that is,

\[
\partial_e(\mathcal{B}_E) = \text{Trip}_{\text{max}}(E)
\]
Let $a$ be a hermitian element in a JB*-algebra $M$, the spectral theorem [10, Theorem 3.2.4] assures that the JB*-subalgebra of $M$ generated by $a$ is isometrically JB*-isomorphic to a commutative $C^*$-algebra. In particular, we can write $a$ as the difference of two orthogonal positive elements in $M_{sa}$. By applying this result it can be seen that every tripotent in $M_{sa}$ is the difference of two orthogonal projections in $M$, and furthermore, when $M$ is unital we obtain

$$\partial_e(\mathcal{B}_{M_{sa}}) = \text{Symm}(M) = \{ s \in M_{sa} : s^2 = 1 \}$$

(cf. [26] or [5, Proposition 3.1.9]). As in the associative case, the symbol $\mathcal{P}(M)$ will stand for the set of all projections (i.e., self-adjoint idempotents) in a JB*-algebra $M$.

The next result, which is a Jordan version of Proposition 2.1, was originally established in [12, Proposition 1.3], and a new proof can be consulted in [5, Proposition 3.1.24 and Remark 3.1.25]. An alternative proof, based on the structure of real JB*-triples, is included here for the sake of completeness.

**Proposition 2.2** [12, Proposition 1.3], [5, Proposition 3.1.24]. Let $p$ be a projection in a unital JB*-algebra $M$. Then the following statements are equivalent:

(a) $p$ is (norm) isolated in $\mathcal{P}(M)$;
(b) $p$ is a central projection;
(c) $1 - 2p$ is (norm) isolated in $\text{Symm}(M)$.

**Proof.** The equivalence $(c) \iff (a)$ follows by the same arguments employed in the case of $C^*$-algebras.

$(c) \Rightarrow (b)$ Suppose $1 - 2p$ is (norm) isolated in $\text{Symm}(M)$. We consider $M_{sa}$ as a real JB*-triple. Given $a, b \in M_{sa}$, by the axioms in the definition of JB*-triples, the mapping

$$\Phi_{t}^{a,b} = \exp(t(L(a,b) - L(b,a))) = \sum_{n=0}^{\infty} \frac{t^n}{n!} (L(a,b) - L(b,a))^n : M \to M$$

is a surjective linear isometry for all $t \in \mathbb{R}$, and clearly maps $M_{sa}$ into itself. Since $1 - 2p$ is an extreme point of the closed unit ball of $M_{sa}$, we deduce that $\Phi_{t}^{a,b}(1 - 2p)$ must be an extreme point of the closed unit ball of $M_{sa}$, and hence a complete tripotent in $M_{sa}$, or equivalently, a symmetry in $M$. Therefore the mapping $\omega : \mathbb{R} \to \text{Symm}(M), t \mapsto \omega(t) = \Phi_{t}^{a,b}(1 - 2p)$ is differentiable with $\omega(0) = 1 - 2p$. Since $1 - 2p$ is isolated in $\text{Symm}(M)$, the mapping $\omega(t)$ must be constant in a neighborhood of 0, and thus, by taking derivative at $t = 0$ we get

$$0 = (L(a,b) - L(b,a))(1 - 2p) = \{ a,b,1 - 2p \} - \{ b,a,1 - 2p \},$$

equivalently,

$$((1 - 2p) \circ a) \circ b = ((1 - 2p) \circ b) \circ a,$$

for all $a, b \in M_{sa}$ (and for all $a, b \in M$). This shows that $1 - 2p$ (and hence $p$) lies in the centre of $M$ as desired.
(b) ⇒ (a) If $p$ is a central projection in $M$, we know from [10, Lemma 2.5.5] that $M = U_p(M) \oplus U_{1-p}(M)$, where for each $z \in M$, $U_z(x) = \{z, x^*, z\}$ ($\forall x \in M$). We further know that every element in $U_p(M)$ is orthogonal to every element in $U_{1-p}(M)$. Arguing as in the associative case (see Proposition 2.1 above), we prove that for each projection $q$ in $M$ we have $\|p-q\| \in \{0,1\}$, which concludes the proof.

3. Metric Characterisation of Unitaries

Let us revisit some of the arguments in the proof of [20, Lemma 3.1] under the point of view of Jordan algebras.

**Proposition 3.1.** Let $e$ be a maximal partial isometry in a unital $C^*$-algebra $A$, and let $l = ee^*$ and $r = e^*e$ denote the left and right projections of $e$. Suppose we can find two orthogonal projections $p, q \in A$ such that $l = p + q$. Then the element $y = i(p-q)e$ lies in $A_e = \{y \in \mathcal{B}_e(A) : \|e \pm y\| = \sqrt{2}\}$, and for each $\theta \in \mathbb{R}$ the element

$$y_\theta := P_2(e^*)(y) + \cos(\theta)P_1(e^*)(y) + \sin(\theta)P_1(e^*)(1)$$

is a maximal partial isometry in $A$.

If we further assume that $p$ and $q$ are central projections in $lAl$, the following statements hold:

(a) The elements $p' = epe^*$ and $q' = ege^*$ are two orthogonal central projections in $rAr$, with $r = p' + q'$;

(b) Suppose that $e$ is not unitary in $A$, and take $y = i(p-q)e$. Then $y$ lies in $A_e$, and for each $\theta \in \mathbb{R}$ the element $y_\theta := P_2(e^*)(y) + \cos(\theta)P_1(e^*)(y) + \sin(\theta)P_1(e^*)(1)$ is a maximal partial isometry in $A$ with $\|e \pm y_\theta\| = \sqrt{2}$, i.e. $y_\theta$ lies in $A_e$ (actually, $\frac{e \mp y_\theta}{\sqrt{2}}$ is a maximal partial isometry in $A$), and $y_\theta \neq y$ for all $\theta$ in $\mathbb{R}\backslash \{2\pi\mathbb{Z} \cup \pi \frac{1+4\mathbb{Z}}{2}\}$. Furthermore, $\|y - P_2(y)(y_\theta)\| \leq 1 - \cos(\theta)$, and hence $P_2(y)(y_\theta)$ is invertible in $A_2(y)$ for $\theta$ close to zero.

**Proof.** Let us prove the first statement. Clearly, $y = i(p-q)e$ lies in $A_e$. By [9, Lemma 6.1] there exist a complex Hilbert space $H$ and an isometric unital Jordan *-monomorphism $\Psi : A \to B(H)$ such that $\Psi(e)^*\Psi(e) = 1$. Let us denote $v = \Psi(e)$, $z = \Psi(y)$, and $z_\theta = \Psi(y_\theta)$. We observe that

$$z_\theta = P_2(v^*)(z) + \cos(\theta)P_1(v^*)(z) + \sin(\theta)P_1(v^*)(1),$$

because $\Psi$ is a unital Jordan *-monomorphism, and hence it preserves triple products and involution. Clearly, $v = \Psi(e)$ is a maximal partial isometry (actually, an isometry $v^*v = 1$) in $B(H)$. We shall write $B$ for $B(H)$. Having the above properties in mind we can rewrite $z_\theta$ in the form

$$z_\theta = v^*vzv + \cos(\theta)((1-v^*v)zv + v^*vz(1-vv^*)) + \sin(\theta)((1-v^*v)1v + v^*v(1-vv^*)) = zv^* + \cos(\theta)z(1-vv^*) + \sin(\theta)(1-vv^*).$$

Let us observe that the latter expression already appears in the proof of [20, Lemma 3.1].
The elements $y_1 = ipe$ and $y_2 = iqe$ are two orthogonal partial isometries in $A$ with $y = y_1 - y_2$ and $e = (-i)(y_1 + y_2)$. Therefore $z_1 = \Psi(y_1)$ and $z_2 = \Psi(y_2)$ are two orthogonal partial isometries in $B(H)$ with $z = z_1 - z_2$ and $v = -i(z_1 + z_2)$. We can easily conclude that

$$z^*z = (z_1^* - z_2^*)(z_1 - z_2) = z_1^* z_1 + z_2^* z_2 = v^* v = 1,$$

and similarly, $z^* z = vv^*$. \hfill (5)

Let us examine the element $z_θ$ more closely. Having in mind (5) and the equality $z = (z^* z) z = (vv^*)z$, it follows from the properties commented above that

$$z_θ^* z_θ = vv^* z^* z v v^* + \cos(\theta) v v^* z^* z (1 - vv^*) + \sin(\theta) v v^* z^* z (1 - vv^*)$$

$$+ \cos(\theta)(1 - vv^*) z^* z v v^* + \cos^2(\theta)(1 - vv^*) z^* z (1 - vv^*)$$

$$+ \cos(\theta) \sin(\theta)(1 - vv^*) z^* z (1 - vv^*) + \sin(\theta)(1 - vv^*) z v v^*$$

$$+ \sin(\theta) \cos(\theta)(1 - vv^*) z (1 - vv^*) + \sin^2(\theta)(1 - vv^*)$$

$$= vv^* + \cos^2(\theta)(1 - vv^*) + \sin^2(\theta)(1 - vv^*) = 1,$$

witnessing that $z_θ$ is an isometry in $B$. It then follows from the properties of $\Psi$ that $y_θ = \Psi^{-1}(\Psi(y_θ)) \in \partial_1(B_A)$ is a complete tripotent in $A$.

Concerning the second statement, let us analyze the element $w = e ± y_θ$. As before, up to an application of [9, Lemma 6.1], we can suppose that $r = e^* e = 1$. We set $l = ee^*$. Assuming that $e$ is not unitary, the projection $1 - l = 1 - ee^*$ is not zero. We therefore have

$$w = e ± y_θ = (e ± y) l + (e ± \cos(\theta) y)(1 - l) + \sin(\theta)(1 - l),$$

and we shall compute $w^* w$.

(a) Let us make some observations. The mappings $Φ_1 : l A l → r A r$, $x → e^* x e$ and $Φ_2 : r A r → l A l$, $y → e y e$ are well defined, linear, and contractive. It is easy to see that $x = l x l = e(e^* x e)e^* = Φ_2 Φ_1(x)$ and $y = e(e^* y e)e = Φ_1 Φ_2(y)$, for all $x ∈ l A l$ and $y ∈ r A r$. Therefore $Φ_2$ and $Φ_1$ are linear bijections and inverses each other. Furthermore, for all $x, z ∈ l A l$, we have $Φ_1(x) Φ_1(z) = (e^* x e)(e^* z e^*) = e(xz)e^* = Φ_1(xz)$, and $Φ_1(x)^* = (e^* x e)^* = e^* x^* e = Φ_1(x^*)$, for all $x ∈ l A l$, which shows that the first mapping is a unital *-isomorphism. Then the elements $p' = Φ_1(p)$ and $q' = Φ_1(q)$ are two orthogonal central projections in $r A r = A$ with $1 = r = p' + q'$.

(b) We derive from the above conclusions that $pe = e p'$, and $qe = e q'$. Consequently,

$$y = i(p - q)e = i e(p' - q'), \quad e ± y = e(\mu ± p' + \overline{\mu} ± q'),$$

and $e ± \cos(\theta) y = e(\lambda ± p' + \overline{\lambda} ± q')$. 

where \( \mu_{\pm} = 1 \pm i \), and \( \lambda_{\pm} = 1 \pm i \cos(\theta) \). We study next all summands involved in the product \( w^*w \):

\[
((e \pm y)l)^*((e \pm y)l) = l(e \pm y)^*(e \pm y)l = l(\mu_{\pm}p' + \mu_{\pm}q')^{e*(\mu_{\pm}p' + \mu_{\pm}q')}
\]

\[
= 2l(p' + q')l = 2l;
\]

\[
\sin(\theta)((e \pm y)l)^*(1 - l) = \sin(\theta)l(\mu_{\pm}p' + \mu_{\pm}q')e^*(1 - l) = 0;
\]

\[
((e \pm y)l)^*(e \pm \cos(\theta)y)(1 - l) = l(\mu_{\pm}p' + \mu_{\pm}q')e^*e((\lambda_{\pm}p' + \lambda_{\pm}q'))(1 - l)
\]

\[
= l(\lambda_{\pm}\mu_{\pm}p' + \lambda_{\pm}\mu_{\pm}q')(1 - l);
\]

\[
(1 - l)(e \pm \cos(\theta)y)^*(e \pm \cos(\theta)y)(1 - l)
\]

\[
= (1 - l)\left(\overline{\lambda_{\pm}p'} + \lambda_{\pm}q'\right)e^*e(\mu_{\pm}p' + \overline{\mu_{\pm}q'})l
\]

\[
= (1 - l)(\overline{\lambda_{\pm}\mu_{\pm}p'} + \lambda_{\pm}\overline{\mu_{\pm}q'})l;
\]

\[
(1 - l)(e \pm \cos(\theta)y)^*(e \pm \cos(\theta)y)(1 - l)
\]

\[
= (1 - l)(\overline{\lambda_{\pm}p'} + \lambda_{\pm}q')e^*e\left(\lambda_{\pm}p' + \overline{\lambda_{\pm}q'}\right)(1 - l)
\]

\[
= (1 - l)(\overline{\lambda_{\pm}p'} + \lambda_{\pm}q')\left(\lambda_{\pm}p' + \overline{\lambda_{\pm}q'}\right)(1 - l)
\]

\[
= |\lambda_{\pm}|^2(1 - l)(p' + q')(1 - l)
\]

\[
= (1 + \cos^2(\theta))(1 - l);
\]

\[
((e \pm \cos(\theta)y)(1 - l))^*(\sin(\theta)(1 - l))
\]

\[
= \sin(\theta)(1 - l)(\overline{\lambda_{\pm}p'} + \lambda_{\pm}q')^{e^*(1 - l)} = 0;
\]

\[
(\sin(\theta)(1 - l))^*(x \pm y)l = \sin(\theta)(1 - l)e(\mu_{\pm}p' + \overline{\mu_{\pm}q'})l = 0;
\]

\[
(\sin(\theta)(1 - l))^*(e \pm \cos(\theta)y)(1 - l) = \sin(\theta)(1 - l)e\left(\lambda_{\pm}p' + \overline{\lambda_{\pm}q'}\right)(1 - l) = 0;
\]

\[
(\sin(\theta)(1 - l))^*(\sin(\theta)(1 - l)) = \sin^2(\theta)(1 - l).
\]

By adding the previous nine identities, and having in mind that \( p' \) and \( q' \) are central projections, we get

\[
\frac{w^*w}{2} = l + \frac{1}{2}(1 + \cos^2(\theta))(1 - l) + \frac{1}{2}\sin^2(\theta)(1 - l)
\]

\[
+ \frac{1}{2}l(\alpha p' + \overline{\alpha}q')(1 - l) + \frac{1}{2}(1 - l)(\alpha q' + \alpha q')l = 1,
\]

which proves that \( \frac{w}{\sqrt{2}} = \frac{e \pm y_\theta}{\sqrt{2}} \) is an isometry, and consequently, \( \|e \pm y_\theta\| = \sqrt{2} \).

Let us now check that \( y_\theta \neq y \) for all \( \theta \) in \( \mathbb{R} \setminus (2\pi \mathbb{Z} \cup \pi \mathbb{Z} + \frac{\pi + 2\pi}{2}) \). Note that \( l \neq 1 \). Since

\[
(y - y_\theta)^*(y - y_\theta) = (1 - \cos(\theta))^2(1 - l)y^*y(1 - l) + \sin^2(\theta)(1 - l)
\]

\[
- (1 - \cos(\theta))\sin(\theta)(1 - l)(y + y^*)(1 - l)
\]

\[
= 2(1 - \cos(\theta))(1 - l) - 2(1 - \cos(\theta))\sin(\theta)a
\]

\[
= 2(1 - \cos(\theta))(1 - l) - \sin(\theta)a \neq 0,
\]

where \( a = (1 - l)\frac{y + y^*}{2}(1 - l) \) is a hermitian element in the closed unit ball of \( (1 - l)A(1 - l) \), and hence \( \|\sin(\theta)a\| \leq |\sin(\theta)| < 1 \).

Finally, the identity

\[
P_2(y)(y_\theta) = lylr + \cos(\theta)ly(1 - l)r + \sin(\theta)(l(1 - l)r = lyl + \cos(\theta)ly(1 - l)
\]
allows us to conclude that $\|y - P_2(y)(y_\theta)\| = \|(1 - \cos(\theta))l y(1 - l)\| \leq 1 - \cos(\theta)$, which finishes the proof. \hfill \Box

Our goal in this section is to establish a similar characterisation of unitaries to that given in Theorem 1.1 in the setting of JB*-algebras and JB*-triples. It should be noted that the characterisation of unitaries in the case of JB*-algebras is far from being a consequence of the result in the associative case. We begin by describing the set of partial isometries at distance smaller than or equal to $\sqrt{2}$ from the unit of a JB*-algebra. As observed by Mori in [20], in the easiest case $A = \mathbb{C}$, for $u \in \partial_e(B_A) = \{ z \in \mathbb{C} : |z| = 1 \}$, we have $A_u = \{ e \in \partial_e(B_C) : \| u \pm e \| = \sqrt{2} \} = \{ iu, -iu \}$. But we can also add that $A_u = \{ e \in \partial_e(B_C) : \| u \pm e \| \leq \sqrt{2} \}$.

Let us recall a property valid for every C*-algebra $A$:

$U_p(a) = pap = 0$ with $a \geq 0$ and $p$ a projection in $A$ implies $ap = pa = 0$. \hfill (6)

Indeed, let us write $a = yy^*$, for some $y \in A$. By hypothesis $0 = pap = pyy^*p = (py)(py)^*$, and thus the Gelfand-Naimark axiom gives $py = y^*p = 0$, and consequently $pa = pyy^* = 0 = yy^*p = ap$, as desired.

Lemma 3.2. Let $M$ be a unital JB*-algebra. Let $e$ be a tripotent in $M$ satisfying $\|1 \pm e\| \leq \sqrt{2}$. Then there exist two orthogonal projections $p, q$ in $M$ such that $e = i(p - q)$. Consequently, \begin{equation} \{ e \in \text{Trip}(M) : \|1 \pm e\| \leq \sqrt{2} \} = \{ i(p - q) : p, q \in \mathcal{P}(M) \text{ with } p \perp q \}. \end{equation} \hfill (7)

Proof. Let $N$ denote the JB*-subalgebra of $M$ generated by $1, e$ and $e^*$. It follows from the Shirshov-Cohn theorem [10, Theorems 2.4.14 and 7.2.5], combined with Wright’s theorem [25, Corollary 2.2 and subsequent comments], that $N$ is special, that is, there exists a unital C*-algebra $A$ containing $N$ as unital JB*-subalgebra. The C*-algebra $A$ contains $1$ and the partial isometry $e$, and we have $\|1 \pm e\| \leq \sqrt{2}$. Let us write $l = ee^*$ and $r = e^*e$ for the left and right projections of $e$ in $A$. Then, it follows that

$$0 \leq \frac{1}{2}(1 + l \pm (e + e^*)) = \frac{1}{2}(1 \pm e)(1 \pm e^*) \leq \frac{1}{2}\|(1 \pm e)(1 \pm e^*)\|1 = \frac{1}{2}\|1 \pm e\|^2 \leq 1. \hfill (7)$$

By evaluating the positive map $2U_l : x \mapsto 2lxl$ at the elements in the above list of inequalities we get

$2l \pm U_l(e + e^*) \leq 2l.$

Therefore $U_l(e + e^*) = 0$. We shall next show that $e + e^* = 0$. Namely, since $U_l(1 - l) = 0$, it is clear that $U_l(1 - l \pm (e + e^*)) = 0$. Moreover, it follows from the definition of $l$ that $U_{1-l}(e) = U_{1-l}(e^*) = 0$, and consequently,

$$e + e^* = (l + (1 - l))(e + e^*)(l + (1 - l)) = U_l(e + e^*) + U_{1-l}(e + e^*)(1 - l) + (1 - l)(e + e^*) = e(1 - l) + (1 - l)e^*.$$

Back to (7) we get

$2l + (1 - l) \pm e(1 - l) \pm (1 - l)e^* = 1 + l \pm (e + e^*) \leq 2 1 = 2l + 2(1 - l),$
and thus 
\[ \pm(e(1 - l) + (1 - l)e^*) \leq 1 - l. \]
Clearly, \( a = 1 - l + (e(1 - l) + (1 - l)e^*) \geq 0 \) with \( U_l(a) = 0 \). It follows from (6) that 
\[ e(1 - l) = le(1 - l) = la = 0 = al = (1 - l)e^*l = (1 - l)e^*. \]
We observe that, trivially, \((1 - l)e = e^*(1 - l) = 0\). Summarizing, we have shown that 
\[ e + e^* = U_l(e + e^*) + (1 - l)(e + e^*)l + l(e + e^*)(1 - l) + U_{1-l}(e + e^*) = 0,\]
that is \( e = -e^* \) is a skew symmetric partial isometry in \( A \), and thus there exist two orthogonal projections \( p, q \) in \( A \) such that \( e = i(p - q) \). Since 
\( e = i(p - q) \in M \), it follows that \( e^2 = -p - q \) and \( p - q \) both belong to \( M \), and consequently, \( p, q \in M \), which concludes the proof. \( \square \)

Given a tripotent \( u \) in a JB*-triple \( E \), the Peirce 2-subspace \( E_2(u) \) is a unital JB*-algebra with unit \( u \) (see page 5). So, the first statement in the next corollary is a straight consequence of our previous lemma.

**Corollary 3.3.** Let \( u \) be a tripotent in a JB*-triple \( E \). Then 
\[ \{e \in \text{Trip}(E_2(u)) : \|u e\| \leq \sqrt{2}\} = \{i(p - q) : p, q \in \mathcal{P}(E_2(u)) \text{ with } p \perp q\}. \]
Furthermore, if \( u \) is unitary in \( E \), then 
\[ \mathcal{E}_u = \{e \in \partial_e(B_E) : \|u e\| \leq \sqrt{2}\} = i\text{Symm}(E_2(u)) \]
\[ = \{i(p - q) : p, q \in \text{Trip}(E), p, q \leq u, p \perp q, p + q = u\} \tag{8} \]
and the elements \( \pm i u \) are isolated in \( \mathcal{E}_u \).

**Proof.** The first statement is a consequence of Lemma 3.2. If \( u \) is unitary the equality \( E = E_2(u) \) holds. Having in mind that \( \partial_e(B_E) = \text{Trip}_{\max}(E) \), we deduce from the first statement that 
\[ \mathcal{E}_u \subseteq \{i(p - q) : p, q \in \text{Trip}(E), p, q \leq u, p \perp q\}. \]
But every \( e = i(p - q) \in \mathcal{E}_u \) must be also a complete tripotent in \( E \), which forces \( p + q = u \), otherwise \( r = u - p - q \) would be a non-zero element in \( E_0(e) \), which is impossible, so (8) is clear. It is obvious that \( \pm i u \in \mathcal{E}_u \) and for any \( i(p - q) \in \mathcal{E}_u \backslash \{\pm i u\} \) we have 
\[ \|i u \pm i(p - q)\| = \|i(1 \pm 1) p + i(1 \mp 1) q\| = \max\{\|i(1 \pm 1) p\|, \|i(1 \mp 1) q\|\} = 2. \]
This proves that \( \pm i u \) are isolated in \( \mathcal{E}_u \). \( \square \)

The Jordan version of Theorem 1.1(a) \( \Rightarrow \) (b) has been established in Corollary 3.3 even in the setting of JB*-triples. For the reciprocal implication we shall first prove a technical result which also holds for JB*-triples.

**Proposition 3.4.** Let \( u \) be a tripotent in a JB*-triple \( E \), and let 
\[ \mathcal{E}_u = \{e \in \partial_e(B_E) : \|u e\| \leq \sqrt{2}\}. \]
Then every element \( y \in \mathcal{E}_u \) with \( P_1(u)(y) \neq 0 \) or \( P_0(u)(y) \neq 0 \) is non-isolated in \( \mathcal{E}_u \). Consequently, every isolated element \( y \in \mathcal{E}_u \) belongs to \( i\text{Symm}(E_2(u)) \).
Proof. Let us take \( y \in \mathcal{E}_u \) with \( P_1(u)(y) \neq 0 \) or \( P_0(u)(y) \neq 0 \). By [8, Lemma 1.1] for each \( \lambda \in \mathbb{C} \) with \( |\lambda| = 1 \) the mapping \( S_{\lambda}(u) = \lambda^2 P_2(u) + \lambda P_1(u) + P_0(u) = \lambda^2 P_2(u) + \lambda P_1(u) + P_0(u) \) is an isometric triple isomorphism on \( E \). Therefore the mapping \( R_{\lambda}(u) = \lambda^2 S_{\lambda}(u) = P_2(u) + \lambda P_1(u) + \lambda^2 P_0(u) \) is an isometric triple isomorphism on \( E \) for all \( \lambda \) in the unit sphere of \( \mathbb{C} \). Let us observe that

\[
y - R_{\lambda}(u)(y) = (1 - \lambda)P_1(u)(y) + (1 - \lambda^2)P_0(u)(y).
\]

Since Peirce projections are contractive, we get

\[
|\lambda - 1||P_1(u)(y)| = |(1 - \lambda)P_1(u)(y)| = |P_1(u)(y - R_{\lambda}(u)(y))| \\
\leq |y - R_{\lambda}(u)(y)|,
\]

and thus

\[
|\lambda - 1||P_1(u)(y)| = |(1 - \lambda^2)P_0(u)(y)| = |P_0(u)(y - R_{\lambda}(u)(y))| \\
\leq |y - R_{\lambda}(u)(y)|,
\]

and thus

\[
|\lambda - 1||P_1(u)(y)| = |(1 - \lambda^2)P_0(u)(y)| = |P_0(u)(y - R_{\lambda}(u)(y))| \\
\leq |y - R_{\lambda}(u)(y)|,
\]

and thus

\[
|\lambda - 1||P_1(u)(y)| = |(1 - \lambda^2)P_0(u)(y)| = |P_0(u)(y - R_{\lambda}(u)(y))| \\
\leq |y - R_{\lambda}(u)(y)|,
\]

for all \( \lambda \in \mathbb{T} \setminus \{\pm 1\} \). Clearly, \( R_{\lambda}(u)(y) = \lambda \frac{y - 1}{|\lambda|} y \) in norm.

On the other hand, \( R_{\lambda}(u)(u) = u \) for all \( |\lambda| = 1 \). Since \( R_{\lambda}(u) \) is an isometric triple automorphism on \( E \) and \( y \in \partial E \) we deduce that \( R_{\lambda}(u)(y) \in \partial E \), and

\[
||u \pm R_{\lambda}(u)(y)|| = ||R_{\lambda}(u)(u) \pm R_{\lambda}(u)(y)|| = ||R_{\lambda}(u)(u \pm y)|| = ||u \pm y|| \leq \sqrt{2},
\]

for all \( |\lambda| = 1 \). Therefore \( y \) is non-isolated in \( \mathcal{E}_u \), which concludes the proof of the first statement.

For the last statement, let us assume that \( y \in \mathcal{E}_u \) is an isolated point.

It follows from the first statement that \( P_1(u)(0) = P_0(u)(0) \). That is, \( y \in \partial E \cap E_2(u) \) with \( |u \pm y|| \leq \sqrt{2} \). We conclude from Corollary 3.3 that \( y \in i\text{Symm}(E_2(u)) \). \( \square \)

Remark 3.5. The arguments given in the proof of Proposition 3.4 are valid to establish the following: Let \( u \) be a tripotent in a JB*-triple \( E \), and let

\[
\tilde{\mathcal{E}}_u = \{ e \in \text{Trip}(E) : |u \pm e| \leq \sqrt{2} \}.
\]

Then every element \( y \in \tilde{\mathcal{E}}_u \) with \( P_1(u)(y) \neq 0 \) or \( P_0(u)(y) \neq 0 \) is non-isolated in \( \tilde{\mathcal{E}}_u \).

We continue gathering the tools and results needed in our characterisation of unitaries in JB*-algebras. One of the most successful tools in the theory of Jordan algebras is the Shirshov-Cohn theorem, which affirms that the JB*-subalgebra of a JB*-algebra generated by two symmetric elements (and possibly the unit element) is a JC*-algebra, that is, a JB*-subalgebra of some \( B(H) \) (cf. [10, Theorem 7.2.5] and [25, Corollary 2.2]). The next lemma is an appropriate version of the Shirshov-Cohn theorem.

Lemma 3.6. Let \( u_1 \) and \( u_2 \) be two orthogonal tripotents in a unital JB*-algebra \( M \). Then the JB*-subalgebra \( N \) of \( M \) generated by \( u_1, u_2 \) and the unit element is a JC*-algebra, that is, there exists a complex Hilbert space \( H \)
satisfying that \( N \) is a JB*-subalgebra of \( B(H) \), we can further assume that the unit of \( N \) coincides with the identity on \( H \).

**Proof.** Let us fix \( t \in (0, 1) \). We consider the element \( e = u_1 + tu_2 \). Let \( N_0 \) denote the JB*-subalgebra of \( M \) generated by \( e, e^* \) and the unit element. It follows from the Shirshov-Cohn theorem that \( N_0 \) is a JC*-algebra. We observe that \( N_0 \) is a JB*-subtriple of \( M \), therefore the element \( e^{[2n-1]} \) belongs to \( N_0 \) for all natural \( n \). Now, applying that \( u_1 \) and \( u_2 \) are two orthogonal tripotents, we can deduce that

\[
e^{[2n-1]} = u_1 + t^{(2n-1)}u_2.
\]

The sequence \((e^{[2n-1]})_n = (u_1 + t^{(2n-1)}u_2)_n\) converges in norm to \( u_1 \), and thus \( u_1 \) lies in \( N_0 \). Consequently, \( u_1 \) and \( u_2 \) both belong to \( N_0 \).

Similarly, \( u_1^* \) and \( u_2^* \) belong \( N_0 \), hence \( N = N_0 \) is a JC*-algebra.

The final statement can be obtained as in the proof of [9, Lemma 6.2].

\( \square \)

The next result is inspired by [9, Lemmata 6.2 and 6.3].

**Proposition 3.7.** Let \( u_1 \) and \( u_2 \) be two orthogonal tripotents in a unital JB*-algebra \( M \) satisfying the following properties:

(a) \( u = u_1 + u_2 \) is a complete tripotent in \( M \);
(b) \( u_1, u_2 \) are central projections in the JB*-algebra \( M_2(u) \).

Let \( N \) denote the JB*-subalgebra of \( M \) generated by \( u_1, u_2 \) and the unit element. Then \( N \) is a JC*-subalgebra of some \( C^* \)-algebra \( B \), and \( u \) is a complete tripotent in the \( C^* \)-subalgebra \( A \) of \( B \) generated by \( N \). Moreover, the elements \( u_1, u_2 \) are central projections in the JB*-algebra \( A_2(u) \).

**Proof.** Lemma 3.6 guarantees that \( N \) is a JB*-subalgebra of a unital \( C^* \)-algebra \( B \), and we can also assume that \( N \) contains the unit of \( B \). Clearly, \( u, u_1 \) and \( u_2 \) are partial isometries in \( A \). Let \( l_i = u_iu_i^* \) and \( r_i = u_i^*u_i \) denote the left and right projections of \( u_i \) in \( A \) (\( i = 1, 2 \)). We shall also write \( l = uu^* = l_1 + l_2 \) and \( r = u^*u = r_1 + r_2 \), for the left and right projections of \( u \) in \( A \), respectively. Let us note that \( l_1 \perp l_2 \) and \( r_1 \perp r_2 \).

By hypothesis, \( u_1, u_2 \) are central projections in the JB*-algebra \( M_2(u) \), and hence in \( N_2(u) \). It then follows that the identity

\[
lnr = n_2(u) = n_2(u_1) \oplus \infty n_2(u_2) = l_1nr_1 \oplus \infty l_2nr_2
\]

holds. Having in mind that \( 1 \in N \), we deduce that \( lr = l_1r_1 + l_2r_2, l_1r_2 + l_2r_1 = 0 \), and so \( l_1r = l_1r_1 \) which proves that \( l_1r_2 = 0 \), and hence \( l_2r_1 = 0 \).

Recall that \( A \) is the \( C^* \)-subalgebra of \( B \) generated by \( N \). We shall next show that \( u \) is a complete tripotent in \( A \). We know that \( u \) is a complete tripotent in \( M \), and hence in \( N \). Clearly \( u \) is a tripotent in \( A \). The Peirce 0-projection on \( A \) is given by \( P_0(u)(x) = (1-l)x(1-r) \ (x \in A) \). We therefore know that \((1-l)x(1-r) = 0 \), for all \( x \in N \). We shall prove that \((1-l)x(1-r) = 0 \) for all \( x \in A \). For this purpose we shall adapt some techniques from the proof of [9, Lemma 6.2].


Finally, if \( x \) with the JB MJOM Unitaries in JB∗-Algebras

\[ l_1(u_1^*)^n(1 - r) = l(u_1^*)^n(1 - r) = (u_1^*)^n(1 - r) \]
\[ l_2(u_2^*)^n(1 - r) = l(u_2^*)^n(1 - r) = (u_2^*)^n(1 - r), \]

where in the first two equalities we applied that \( l_1r_2 = l_2r_1 = 0 \).

Fix \( t \in (0, 1) \). We have shown in the proof of Lemma 3.6 that \( N \) coincides with the JB∗-subalgebra of \( M \) generated by \( e = u_1 + tu_2 \) and 1. Let \( A_0 \) denote the set of all finite products of \( e, e^* \) and 1. Since \( A \) is the closed linear span of \( A_0 \) we only need to prove that \( (1 - l)x(1 - r) = 0 \), for all \( x \in A_0 \).

We say that an element \( x \in A \) satisfies property (\( \diamond \)) if

\[ x(1 - r) = 0, \text{ or } x(1 - r) = (1 - r), \text{ or } x(1 - r) = (u_1^*)^n(1 - r) + t^m(u_2^*)^n(1 - r), \]

for some \( n, m \in \mathbb{N} \).

Let us fix an element \( x \in A \) satisfying property (\( \diamond \)). If \( x(1 - r) = 0 \), we have \( e^*x(1 - r) = 0 \), and \( ex(1 - r) = 0 \). If \( x(1 - r) = (1 - r) \), it follows that

\[ e^*x(1 - r) = e^*(1 - r) = u_1^*(1 - r) + te_2^*(1 - r), \text{ and } ex(1 - r) = e(1 - r) = 0. \]

If \( x(1 - r) = (u_1^*)^n(1 - r) + t^m(u_2^*)^n(1 - r) \), for some \( n, m \in \mathbb{N} \), it can be seen that

\[ e^*x(1 - r) = e^*(u_1^*)^n(1 - r) + t^m e^*(u_2^*)^n(1 - r) = (u_1^*)^{n+1}(1 - r) + t^{m+1}(u_2^*)^{n+1}(1 - r), \]

where we applied that \( u_1 \perp u_2, l_1r_2 = 0 \), and \( l_2r_1 = 0 \). This shows that \( e^*x \) satisfies property (\( \diamond \)).

In the latter case, by applying \( u_1 \perp u_2, l_1r_2 = 0 \), and \( l_2r_1 = 0 \), we also have

\[ ex(1 - r) = e(u_1^*)^n(1 - r) + t^me(u_2^*)^n(1 - r) = u_1((u_1^*)^n(1 - r) + t^{m+1}u_2(u_2^*)^n(1 - r) = (u_1u_2^*)(u_2^*)^{n-1}(1 - r) + t^{m+1}(u_2u_2^*)(u_2^*)^{n-1}(1 - r) = l_1(u_1^*)^{n-1}(1 - r) + t^{m+1}l_2(u_2^*)^{n-1}(1 - r) = (by (9)) = (u_1^*)^{n-1}(1 - r) + t^{m+1}(u_2^*)^{n-1}(1 - r), \]

witnessing that \( ex \) satisfies property (\( \diamond \)).

We have proved that if \( x \) satisfies property (\( \diamond \)), then \( ex \) and \( e^*x \) both satisfy property (\( \diamond \)). It is not hard to check that \( 1, e, \text{ and } e^* \) satisfy property (\( \diamond \)). We can thus conclude that every element in \( A_0 \) satisfies property (\( \diamond \)). So, for each \( x \in A_0 \) we have \( (1 - l)x(1 - r) = 0 \) if \( x(1 - r) = 0 \). If \( x(1 - r) = (1 - r) \), it follows from the fact that \( 1 \in N \) and \( u \) is complete in \( N \), that

\[ (1 - l)x(1 - r) = (1 - l)(1 - r) = (1 - l)1(1 - r) = 0. \]

Finally, if \( x(1 - r) = (u_1^*)^n(1 - r) + t^m(u_2^*)^n(1 - r) \), for some \( n, m \in \mathbb{N} \), we easily check that

\[ (1 - l)x(1 - r) = (1 - l)(u_1^*)^n(1 - r) + t^m(1 - l)(u_2^*)^n(1 - r) = 0, \]
where in the last equality we applied that \((u_1^*)^n, (u_2^*)^n \in N\) and \(u\) is a complete tripotent in \(N\). This proves that \((1 - l)A_0(1 - r) = \{0\}\), and hence \(u\) is complete in \(A\).

It remains to prove that \(u_1\) and \(u_2\) are central projections in \(A_2(u)\). We claim that

\[
l_1 Ar_2 = l_2 Ar_1 = \{0\}.
\]

Indeed, it is enough to prove that

\[
l_1(x_1 \cdots x_m) r_2 = l_2(x_1 \cdots x_m) r_1 = 0,
\]

for all natural \(m\) and \(x_1, \ldots, x_m \in \{e, e^*\}\) because \(N\) is the JB*-subalgebra of \(M\) generated by \(e, e^*\) and the unit. We shall prove (11) by induction on \(m\). We know from the hypotheses that \(l_1 Nr_2 = l_2 Nr_1 = \{0\}\), so the case, \(m = 1\) is clear.

The case \(m = 2\) is worth to be treated independently. The products of two elements are the following: \(e^2, (e^*)^2, ee^*\) and \(e^*e\). The elements \(e^2\) and \((e^*)^2\) belong to \(N\), and thus \(l_1 e^2 r_2 = l_2 e^2 r_1 = l_1 (e^*)^2 r_2 = l_2 (e^*)^2 r_1 = 0\). By the properties seen in the above paragraphs we have

\[
l_1 ee^* r_2 = er_1 e^* r_2 = ee^* l_1 r_2 = 0.
\]

Since \(e \circ e^* \in N\), it follows that \(l_1 (ee^* + e^*e) r_2 = 0\). The last two equalities together give

\[
l_1 ee^* r_2 = l_1 e^* er_2 = 0.
\]

Similar arguments show that

\[
l_2 ee^* r_1 = l_2 e^* er_1 = 0.
\]

Suppose, by the induction hypothesis, that (11) holds for all natural numbers \(2 \leq m \leq m_0\). Let us make an observation, for any natural \(k \leq m_0 - 1\) it follows from the induction hypothesis that

\[
l_1(x_1 \cdots x_k) l_2e = l_1 x_1 \cdots x_k er_2 = 0;
\]

therefore

\[
0 = (l_1(x_1 \cdots x_k) l_2e)(l_1(x_1 \cdots x_k) l_2e)^* = l_1(x_1 \cdots x_k) l_2ee^* l_2(x_k^* \cdots x_1^*) l_1
\]

\[
= l_1(x_1 \cdots x_k) l_2 l_2(x_k^* \cdots x_1^*) l_1 = (l_1(x_1 \cdots x_k) l_2)(l_1(x_1 \cdots x_k) l_2)^*;
\]

witnessing that

\[
l_1(x_1 \cdots x_k) l_2 = 0, \text{ for all natural } k \leq m_0 - 1.
\]

We deal next with the case \(m_0 + 1\). We pick \(x_1, \ldots, x_{m_0}\), \(x_{m_0 + 1} \in \{e, e^*\}\). Since \(e^{m+1}, (e^*)^{m+1} \in N\), the desired conclusion is clear for \(x_1 = \ldots = x_{m+1} = e\) and \(x_1 = \ldots = x_{m+1} = e^*\). We can therefore assume the existence of \(j \in \{1, \ldots, m_0\}\) such that \(x_j x_{j+1} = e^*e = 1\) or \(x_j x_{j+1} = ee^*\). In the first case

\[
l_1 x_1 \cdots x_{m_0 + 1} r_2 = l_1 x_1 \cdots x_{j-1} x_{j+1} \cdots x_{m_0 + 1} r_2 = 0,
\]
by the induction hypothesis. In the second case we have

\[ l_1 x_1 \cdots x_{m_0+1} r_2 = l_1 x_1 \cdots x_{j-1} l x_{j+1} \cdots x_{m_0+1} r_2 \]

\[ = l_1 x_1 \cdots x_{j-1} l x_{j+1} \cdots x_{m_0+1} r_2 + l x_1 \cdots x_{j-1} l x_{j+1} \cdots x_{m_0+1} r_2 = 0, \]

where in the last equality we applied (12) and the induction hypothesis.

Similar ideas to those we gave above are also valid to establish

\[ l_2 x_1 \cdots x_m r_1 = 0, \]

for all \( m \in \mathbb{N}, x_1 \cdots x_m \in \{e, e^*\}. \)

This finishes the induction argument and the proof of the claim in (10). It follows from (10) that \( u_1 \) and \( u_2 \) are central projections in \( A_2(u) \). \( \square \)

The desired characterisation of unitaries in a unital JB*-algebra is now established in our main result.

**Theorem 3.8.** Let \( u \) be an extreme point of the closed unit ball of a unital JB*-algebra \( M \). Then the following statements are equivalent:

(a) \( u \) is a unitary tripotent;

(b) The set \( \mathcal{M}_u = \{e \in \partial_e(B_M) : \|u \pm e\| \leq \sqrt{2}\} \) contains an isolated point.

**Proof.** Corollary 3.3 gives (a) \( \Rightarrow \) (b).

(b) \( \Rightarrow \) (a) We shall show that if \( u \) is not a unitary tripotent then every point \( y \in \mathcal{M}_u \) is non-isolated. We therefore assume that \( u \) is not a unitary tripotent. Let us fix \( y \in \mathcal{M}_u \). If \( P_1(u)(y) \neq 0 \), Proposition 3.4 implies that \( y \) is non-isolated in \( \mathcal{M}_u \). We can therefore assume that \( P_1(u)(y) = 0 \), and hence \( y = P_2(u)(y) \). So, \( y \) and \( u \) lie in the JB*-algebra \( M_2(u) \) (we observe that the latter need not be a JB*-subalgebra of \( M \)). Since \( y \) also is an extreme point of the closed unit ball of \( M_2(u) \) and \( \|u \pm y\| \leq \sqrt{2} \), Corollary 3.3 implies that \( y \) lies in \( i \text{Symm}(M_2(u)) \), therefore, there exist orthogonal tripotents \( u_1, u_2 \in M \) with \( u_1, u_2 \leq u, u_1 + u_2 = u \) and \( y = i(u_1 - u_2) \).

If \( u_2 \) is non-isolated in \( \mathcal{P}(M_2(u)) \), then there exists a sequence \( (q_n)_n \subseteq \mathcal{P}(M_2(u)) \) with \( q_n \neq u_2 \), for all \( n \), converging to \( u_2 \) in norm. In this case the sequence \( (i(u_2 - 2q_n))_n \) is contained in \( \mathcal{M}_u \setminus \{y = i(u_1 - u_2)\} \) (let us observe that \( u - 2q_n \) is a symmetry in \( M_2(u) \) and since \( u \in \partial_e(B_M) \), [24, Lemma 4] implies that \( i(u_2 - 2q_n) \in \partial_e(B_M) \) for all \( n \in \mathbb{N} \), and clearly \( \|u \pm i(u_2 - 2q_n)\| = \sqrt{2} \) and converges to \( y \) in norm. We have therefore shown that \( y \) is non-isolated in \( \mathcal{M}_u \).

We finally assume that \( u_2 \) is isolated in \( \mathcal{P}(M_2(u)) \). In this case Proposition 2.2 proves that \( u_2 \) (and hence \( u_1 \)) is a central projection in \( M_2(u) \). We are in position to apply Proposition 3.7 to the tripotents \( u_1, u_2 \) and \( u = u_1 + u_2 \) in \( M \). Let \( N \) denote the JB*-subalgebra of \( M \) generated by \( u_1, u_2 \) and the unit element. By the just quoted proposition, \( N \) is a JC*-subalgebra of some C*-algebra \( B, u \) is a complete tripotent in the C*-subalgebra \( A \) of \( B \) generated by \( N \), and the elements \( u_1, u_2 \) are central projections in the JB*-algebra \( A_2(u) \). Let us observe that \( u \) and \( y \) both belong to \( N \) (and to \( A \)). Proposition 3.1, applied to \( A, u, p = u_1u_1^*, q = u_2u_2^*, \) and \( y \), implies that for each \( \theta \in \mathbb{R} \) the element

\[ y_\theta := P_2(u^*)(y) + \cos(\theta)P_1(u^*)(y) + \sin(\theta)P_1(u^*)(1) \]
is a maximal partial isometry in $A$ with $\|u + y_\theta\| = \sqrt{2}$, and $y_\theta \neq y$ for all $\theta$ in $\mathbb{R} \setminus (2\pi\mathbb{Z} \cup \pi \mathbb{Z})$ because $u$ is not unitary in $N$ nor in $A$. We further know from the just quoted proposition that $\|y - P_2(y)(y_\theta)\| \leq 1 - \cos(\theta)$, and hence $P_2(y)(y_\theta)$ is invertible in $N_2(y)$ for $\theta$ close to zero. Since $y \in \partial_c(B_M)$, it follows from [13, Lemma 2.2] that $y_\theta$ is Brown-Pedersen quasi-invertible in the terminology of [13], which combined with the fact that $y_\theta$ is a tripotent in $N$ (and hence in $M$), trivially implies that $y_\theta \in \partial_c(B_M)$. Therefore, for $\theta$ close to zero, $y_\theta \in M_u \setminus \{y\}$ and $y_\theta \to y$ in norm when $\theta \to 0$, witnessing that $y$ is non-isolated in $M_u$. \hfill \Box

Let us conclude this note with some afterthoughts on JB$^*$-triples. Let $E$ be a JB$^*$-triple with dimension at least 2. Suppose $u$ is a complete tripotent in $E$ which is not unitary. In view of Corollary 3.3 and Theorem 3.8, a natural topic remains to be studied: Does the set $E_u = \{ e \in \partial_c(B_E) : \|u \pm e\| \leq \sqrt{2}\}$ contains no isolated points?

Every JB$^*$-triple $E$ admitting a unitary element is a unital JB$^*$-algebra with Jordan product and involution given in (2). Actually, there is a one-to-one (geometric) correspondence between the class of unital JB$^*$-algebras and the class of JB$^*$-triples admitting a unitary element. The next corollary is thus a rewording of our Theorem 3.8.

**Corollary 3.9.** Let $E$ be a JB$^*$-triple admitting a unitary element. Suppose $u$ is an extreme point of the closed unit ball of $E$. Then the following statements are equivalent:

(a) $u$ is a unitary tripotent;
(b) The set $E_u = \{ e \in \partial_c(B_E) : \|u \pm e\| \leq \sqrt{2}\}$ contains an isolated point.

A typical example of a JB$^*$-triple admitting no unitary tripotents is a rectangular Cartan factor of type 1 of the form $C = B(H,K)$, of all bounded linear operators between two complex Hilbert spaces $H$ and $K$, with dim($H$) $>$ dim($K$).

In the simplest case $K = \mathbb{C}$, and hence $C = H$ is a Hilbert space with triple product $\{a,b,c\} = \frac{1}{2}(\langle a|b\rangle c + \langle c|b\rangle a)$ ($a, b, c \in H$). Every norm-one element in $C$ is an extreme point of its closed unit ball, that is, $\partial_c(B_C) = S(C)$. Let us fix $u \in S(C)$. By assuming dim($C$) $\geq 2$ it is not hard to see that

$$C_u = \{ e \in \partial_c(B_C) : \|u \pm e\| \leq \sqrt{2}\} = \{ i t u + x : t \in \mathbb{R}, x \in C, \langle e, x \rangle = 0, t^2 + \|x\|^2 = 1\},$$

is pathwise-connected.

In the case in which dim($K$) $\geq 2$, every complete tripotent in $C$ must be a partial isometry $u$ satisfying $uu^* = id_K$ (and clearly, $u^*u \neq id_H$). Let us take $y \in C_u = \{ e \in \partial_c(B_C) : \|u \pm e\| \leq \sqrt{2}\}$. We shall see that $y$ is non-isolated in $C_u$. By Corollary 3.3 and Proposition 3.4 we can assume that $y \in \mathbb{S}(C_2(u))$, that is, there exist two orthogonal tripotents $u_1, u_2$ with $u_1, u_2 \leq u$, $u_1 + u_2 = u$, and $y = i(u_1 - u_2)$. We may assume that $u_2 \neq 0$. Let us take a minimal tripotent $e$ such that $e \leq u_2$, that is, $u_2 = (u_2 - e) + e$ with $(u_2 - e) \perp e$. In this case $e = \xi \otimes \eta : \zeta \mapsto \langle \zeta, \eta \rangle \xi$ with $\eta \in S(H)$, $\xi \in S(K)$. 
Since $u^*u \neq \text{id}_H$, we can pick $\tilde{\eta} \in S(H)$ with $\langle \tilde{\eta}, u^*u(H) \rangle = \{0\}$. The element $\tilde{e} = \xi \otimes \tilde{\eta}$ is a minimal tripotent in $C$ with $\tilde{e} \perp u_1, u_2 - e$. It is not hard to check that, for each real $\theta$, the element $y_\theta := i(u_1 - (u_2 - e) - \cos(\theta)e + \sin(\theta)\tilde{e})$ is a complete tripotent in $C$, by orthogonality and from the fact that $\|\alpha e + \beta \tilde{e}\|^2 = |\alpha|^2 + |\beta|^2$ for all $\alpha, \beta \in \mathbb{C}$, we can deduce that

$$\|u \pm y_\theta\| = \max\{\|(1 \pm i)u_1\|, \|(1 \mp i)(u_2 - e)\|, \|(1 \pm i \cos(\theta))e \pm \sin(\theta)\tilde{e}\|\} = \sqrt{2}.$$

Since $y \neq y_0 \to y$ for $\theta \to 0$, we conclude that $y$ is non-isolated in $C_u$ as claimed.

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