Algorithmic learning of probability distributions from random data in the limit *

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Abstract. We study the problem of identifying a probability distribution for some given randomly sampled data in the limit, in the context of algorithmic learning theory as proposed recently by Vinanyi and Chater [25]. We show that there exists a computable partial learner for the computable probability measures, while by Bienvenu, Monin and Shen [6] it is known that there is no computable learner for the computable probability measures. Our main result is the characterization of the oracles that compute explanatory learners for the computable (continuous) probability measures as the high oracles. This provides an analogue of a well-known result of Adleman and Blum [1] in the context of learning computable probability distributions. We also discuss related learning notions such as behaviorally correct learning and other variations of explanatory learning, in the context of learning probability distributions from data.

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1 Introduction

We are interested in the following informally stated general problem, which we study in the context of formal language identification and algorithmic learning theory:

Given a probability distribution $\mathcal{P}$ and a sufficiently large sample of randomly chosen data from the given distribution, learn or estimate a probability distribution with respect to which the sample has been randomly sampled. \hfill (1)

Problem (1) has a long history in statistics (e.g. see [24]) and has more recently been approached in the context of computational learning, in particular the probably approximately correct (PAC) learning model, starting with [13]. The same problem was recently approached in the context of Algorithmic Learning Theory, in the tradition of Gold [12], and Kolmogorov complexity by Vinanyi and Chater in [25].

The learning concepts discussed in Vinanyi and Chater [25] are very similar in nature to the classic concepts of algorithmic learning which are motivated by the problem of language learning in the limit (see [21]) but they differ in two major ways. In the classic setting, one starts with a class of languages or functions which have a finite description (e.g. they are computable) and the problem is to find an algorithm (often called a learner) which can infer, given a sufficiently long text from any language in the given class, or a sufficiently long segment of the characteristic sequence of any function in the given class, a description of the language or function in the form of a grammar or a program. More precisely, the desired algorithm makes successive predictions given longer and longer segments of the input sequence, and is required to converge to a correct grammar or program for the given infinite input.

If we apply the concept of identification in the limit to Problem (1), according to Vinanyi and Chater [25], one starts with a class $\mathcal{V}$ of finitely describable probability distributions (say, the computable measures on the Cantor space) and we have the following differences with respect to the classic setting:

- the inputs on which the learner is supposed to succeed in the limit are random sequences with respect to some probability distribution in the given class $\mathcal{V}$, and not elements of $\mathcal{V}$;
- success of the learner $\mathcal{L}$ on input $X$ means that $\mathcal{L}(X \upharpoonright n)$ converges, as $n \to \infty$, to a description of some element of $\mathcal{V}$ with respect to which $X$ is random.

First, note that just as in the context of computational learning theory, here too we need to restrict the probability distributions in Problem (1) to a class of ‘feasible’ distributions, which in our case means computable distributions in the Cantor space. Second, in order to specify the learning concept we have described, we need to define what we mean by random inputs $X$ with respect to a computable distribution $\mathcal{P}$ in the given class $\mathcal{V}$ on which the learner is asked to succeed. Vinanyi and Chater [25] ask the learner to succeed on every real $X$ which is algorithmically random, in the sense of Martin-Löf [18], with respect to some

\footnote{Probabilistic methods and learning concepts in formal language and algorithmic learning theory have been studied long before [25], see [22] and the survey [2]. However most of this work focuses on identifying classes of languages or functions using probabilistic strategies, rather than identifying probability distributions as Problem (1) asks. Bienvenu and Monin [5, Section IV] do study a form of (1) through a concept that they call layerwise learnability of probability measures in the Cantor space, but this is considerably different than Vinanyi and Chater in [25] and the concepts of Gold [12], the most important difference being that it refers to classes of probability measures that are not necessarily contained in the computable probability measures.}
computable probability measure. Then the interpretation of Problem (1) through the lenses of algorithmic learning theory and in particular the ideas of Vinanyi and Chater [25] is as follows:

Given a computable measure $\mu$ and an algorithmically random stream $X$ with respect to $\mu$, learn in the limit (by reading the initial segments of $X$) a computable measure $\mu'$ with respect to which $X$ is algorithmically random.

This formulation invites many different formalizations of learning concepts which are parallel to the classic theory of algorithmic learning, and although we will comment on some of them later on, this article is specifically concerned with EX-learning (explanatory learning, one of the main concepts in Gold [12]), which means that in (2) we require the learner to eventually converge to a specific description of the computable measure with the required properties.

Formally, a learner is a computable function $L$ from the set of binary strings $2^{<\omega}$ to $2^{<\omega}$.

**Definition 1.1** (Success of learners on measures). A learner $L$, EX-succeeds on a measure $\mu$ if for every $\mu$-random real $X$ the limit of $L(X \upharpoonright n)$ as $n \to \infty$ exists and is an index of a computable measure $\nu$ such that $X$ is $\nu$-random.

Vinanyi and Chater [25] observed for any uniformly computable class $C$ of computable measures there exists a computable learner which is successful on all of them, in the sense that it correctly guesses appropriate computable measures for every stream which is $\mu$-random with respect to some $\mu \in C$. Then Bienvenu, Monin and Shen [6] showed that the class of computable measures is not learnable in this way. This result can be viewed as an analogue of the classic theorem in Gold [12] that the class of computable functions is not EX-learnable. In relation to the latter, Adleman and Blum [1] showed that the the oracles that EX-learn all computable functions are exactly the oracles $A$ whose jump computes the jump of the halting problem ($\emptyset'' \leq_T A'$), i.e. the oracles that can decide in the limit the totality of partial computable functions.

In this article we show that an oracle $A$ can learn the class of computable measures (in the sense that it computes the required learner) if and only if it is high, i.e. $A' \geq_T \emptyset'$. We prove the following form of this statement, taking into account the characterization of high oracles from Martin [17] as the ones that can compute a function which dominates all computable functions.

**Theorem 1.2.** If a function dominates all computable functions, then it computes a learner which EX-succeeds on all computable measures. Conversely, if a learner EX-succeeds on all computable (continuous) measures then it computes a function which dominates all computable functions.

This provides an analogue of the result of Adleman and Blum [1] in the context of learning of probability measures, and is the first oracle result in this topic. In particular, it shows that the computational power required for learning all computable reals in the sense of Gold [12] (identification by explanation) is the same power that is required for learning all computable measures in the framework of Vinanyi and Chater [25]. Our methods differ from Bienvenu, Monin and Shen [6], and borrow some ideas from Adleman and Blum [1], but in the context of sets of positive measure instead of reals. Moreover our arguments show a

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2From this point on we will use the term (probability) measure instead of distribution, since the literature in algorithmic randomness that we are going to use is mostly written in this terminology.

3EX-learning, BC-learning, BC$^*$-learning etc. In Odifreddi [19, Chapter VII.5] the reader can find a concise and accessible introduction to these basic learning concepts and results.

4as opposed to, for example, eventually giving different indices of the same measure, or even different measures all of which satisfy the required properties.
stronger version of Theorem 1.2, which we detail in Section 4, and which roughly says that the theorem holds also for fixed positive probability of EX-success on all measures.

On the positive side, Osherson, Stob and Weinstein [20] introduced the notion of partial learning for computable sequences, and showed that there is a computable learner which partially learns all computable binary sequences. We introduce the corresponding notion for measures and show an analogous result.

We say that a learner $L$ partially succeeds on measure $\mu$ if for all $\mu$-random $X$ there exists a $j_0$ such that

- there are infinitely many $n$ with $L(X \upharpoonright n) = j_0$;
- if $j \neq j_0$ then there are only finitely many $n$ with $L(X \upharpoonright n) = j$;
- $\mu_{j_0}$ is a computable measure such that $X$ is $\mu_{j_0}$-random.

**Theorem 1.3.** There exists a computable learner which partially succeeds on all computable measures.

We give the proof of Theorem 1.3 in Section 2.3.

Behaviorally correct learning, or BC-learning, is another standard notion in algorithmic learning theory, and requires that for all computable $X$, there exists some $n_0$ such that for all $n > n_0$ the learner on $X \upharpoonright n$ predicts an index of a computable function with characteristic sequence $X$ (instead of converging to a single such index as in explanatory learning). Bienvenu, Figueira, Monin and Shen [3] considered the analogue of BC-learning for measures and showed that there exists no computable learner which BC learns all computable measures. They also considered the analogue of $BC^*$-learning (which is the same as BC but ignoring finite differences of the functions) for measures and showed that there exists a computable learner which $BC^*$-learns all computable measures, hence giving an analogue of a theorem of Harrington who proved the same in the classical setting. In Section 4 we discuss additional facts about EX and BC learning that one may try to establish using the methods developed in the present paper.

## 2 Background facts and the the easier proofs

Consider the Cantor space $2^\omega$, which is the set of all infinite binary sequences which we call reals. This is a topological space generated by the basic open sets $\llbracket \sigma \rrbracket = \{\sigma * X \mid X \in 2^\omega\}$ for all binary strings $\sigma$, where $*$ denotes concatenation. Then the open sets can be represented by sets of strings $Q$ and we use $\llbracket Q \rrbracket$ to denote the set of reals which have a prefix in $Q$. We may identify each Borel probability measure on $2^\omega$ by its measure representation, i.e. a function $\mu : 2^{<\omega} \rightarrow [0,1]$ (determining its values on the basic open sets) with the property $\mu(\sigma) = \mu(\sigma * 0) + \mu(\sigma * 1)$ for each $\sigma \in 2^{<\omega}$, which maps the empty string to 1. Given set of strings $C$, we let $\mu(C)$ denote the measure of the corresponding open set in the Cantor space, which equals $\sum_{\sigma \in C} \mu(\sigma)$ in the particular case that $C$ is prefix-free.

Let us fix a universal enumeration $(\mu_i)$ of all partial computable measures which we view as the partial computable functions $\mu$ with the property that (a) they are defined on the empty string and equal 1, and (b) for all $\sigma$, if $\mu(\sigma * i) \downarrow$ for some $i \in \{0,1\}$ then $\mu(\rho) \downarrow$ for all strings of length at most $|\sigma| + 1$ and $\mu(\sigma) = \mu(\sigma * 0) + \mu(\sigma * 1)$. Then clearly $(\mu_i)$ contains all computable measures. We use the suffix ‘$[s]$’ to

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3Harrington’s result is reported in [8]. The computable learner $L$ with the stated property, given $\sigma$ outputs an index of the following program, where $\varphi$ is a standard list of all partial computable functions: on input $n$, search for the least $e \leq n$ such that $\varphi_e(n)$ extends $\sigma$; if such exists, output $\varphi_e(n)$; otherwise output 0. It is not hard to see that this learner has the required properties.
denote the state of an object after $s$ steps of computation. Given a prefix-free set $C$ of strings and $i$, we say that $\mu_i(C)[s] \downarrow$ if $\mu_i(\sigma)[s] \downarrow$ for all $\sigma \in C$. Without loss of generality, in our universal enumeration $(\mu_i)$ we assume that if $\mu_i(\sigma)[s] \downarrow$ then $\mu_i(\tau)[s] \downarrow$ for all strings $\tau$ of length at most $|\sigma|$. Each $\mu_i$ has a time-complexity (possibly partial) function which maps each $n$ to the least stage $s$ such that $\mu_i(2^{sx})[s] \downarrow$. Generally speaking we are interested in continuous measures i.e. measures $\mu$ such that $\mu((X)) = 0$ for each real $X$.

### 2.1 Algorithmic randomness with respect to computable measures

Bienvenu and Merkle [4] contains an informative presentation of algorithmic randomness with respect to computable measures. Here we recall the basic concepts and facts on this topic that are directly related to our arguments. A Martin-Löf $\mu$-test is a uniformly computably enumerable sequence $(U_i)$ of sets of strings such that $\mu(U_i) \leq 2^{-i}$ for each $i$. We say that $X$ is $\mu$-random for a computable measure $\mu$ if $X \notin \cap_i \downarrow U_i$ for all Martin-Löf $\mu$-tests $(U_i)$. In the case where a computable measure $\mu$ is continuous (i.e. it does not have atoms) the theory of Martin-Löf $\mu$-randomness is entirely similar to the theory of Martin-Löf randomness with respect to the uniform measure. For example, by Levin [16] we have the following characterization in terms of the prefix-free initial segment Kolmogorov complexity $n \mapsto K(X \upharpoonright_n)$:

\[
\text{Given a computable measure } \mu, \text{ a real } X \text{ is Martin-Löf } \mu\text{-random if and only if } \exists c \forall n K(X \upharpoonright_n) \geq -\log \mu(X \upharpoonright_n) - c. \quad (3)
\]

An important concept for the proof of both of the clauses of Theorem 1.2 is the randomness deficiency of a real $X$ with respect to a computable measure $\mu$. There are different definitions of this notion, but most of them are effectively equivalent (in a way to be made precise in the following) and are based on the same intuition:

- $X$ is $\mu$-random if and only if it has bounded (i.e. finite) $\mu$-randomness deficiency;
- the more $\mu$-randomness deficiency $X$ has, the less $\mu$-random $X$ is.

It will help the uniformity of our treatment to deal with the partial computable measures and regard totality as a special case. In this respect, we define the randomness deficiency functions as a uniform sequence of partial computable functions $(d_e)$ corresponding to $(\mu_e)$ as follows:

\[
d_e(\sigma) = -\log \mu_e(\sigma) - K(\sigma) \quad \text{for each } e, \sigma.
\]

Then we can also define the randomness deficiency functions on reals as the sequence $(d_e)$ defined as:

\[
d_e(X) = \sup_n d_e(X \upharpoonright_n) \quad \text{for each } e, X
\]

where the supremum is taken over the $n$ such that $d_e(X \upharpoonright_n) \downarrow$ (hence, at least $n = 0$). In this way, the $\mu$-randomness deficiency of $\sigma$ if the amount that $\sigma$ can be compressed by the underlying universal machine, compared to its default code-length $-\log \mu_e(\sigma)$ which is chosen according to $\mu_e$. Similarly, the $\mu$-randomness deficiency of $X$ is the maximum amount by which the initial segments of $X$ are compressible.

Alternatively, we could have defined the $\mu$-randomness deficiency of $X$ as the least $i$ such that $X \notin \uparrow U_i$, where $(U_i)$ is a fixed universal Martin-Löf $\mu$-test; this definition can also be made uniform in the indices of the partial computable measures. The intuition behind this alternative deficiency notion is that effectively


producing a $\mu$-small open neighborhood of $X$ increases the randomness deficiency of $X$. This interpretation will be crucial in Section 3, and we will later observe that it is essentially equivalent to the definition of $d_\epsilon(X)$ that we gave in terms of Kolmogorov complexity.

2.2 Proof of the first clause of Theorem 1.2 (easier)

Recall that for any oracle $A$ we have $A' \supseteq \emptyset''$ if and only if $\Pi^0_2 \subseteq \Delta^0_2(A)$, which means that the answers to any uniformly $\Pi^0_2$ sequence of questions, such as the totality of partial computable functions, can be approximated by a function which is computable in $A$. Hence, also in view of the domination result of Martin mentioned earlier, the first clause of Theorem 1.2 can be stated as follows:

Let $f$ be a function with binary values such that for each $e$ we have $\lim_i f(e, s) = 1$, if $\mu_e$ is total and $\lim_i f(e, s) = 0$ if $\mu_e$ is partial. Then there exists a learner $L$ which is computable in $f$ and which EX-succeeds on all computable measures.

The main idea is to first observe that given a uniformly computable sequence $(\lambda_i)$ of total measures, we can define the randomness deficiency functions $(d_i)$ as in Section 2.1 (but with respect to $(\lambda_i)$ instead of $(\mu_i)$) and these will be total. Hence we can define the computable learner which monitors the deficiencies along each real $X$ and at each step $n$ predicts the index $i$ which minimizes the cost $d_i(X \upharpoonright_n)[n] + i$. It is easy to check that this learner succeeds in all of the measures $(\lambda_i)$.\footnote{Since Definition 1.1 refers to the universal indexing $(\mu_i)$, at this point the reader may be concerned with the difference in the indexing $(\lambda_i)$. However this is not an issue, since there is a computable map from the indices in special list $(\lambda_i)$ to the corresponding items in the universal list $(\mu_i)$ that we fixed.}

Second, with an oracle which decides the randomness deficiency functions $(d_i)$ as in Section 2.1, at this point the reader may be concerned with the difference in the indexing $(\lambda_i)$ instead of $(\mu_i)$. Hence, also in view of the domination result of Martin mentioned earlier, the first clause of Theorem 1.2 can be stated as follows:

Let $f$ be a function with binary values such that for each $e$ we have $\lim_i f(e, s) = 1$, if $\mu_e$ is total and $\lim_i f(e, s) = 0$ if $\mu_e$ is partial. Then there exists a learner $L$ which is computable in $f$ and which EX-succeeds on all computable measures.

The main idea is to first observe that given a uniformly computable sequence $(\lambda_i)$ of total measures, we can define the randomness deficiency functions $(d_i)$ as in Section 2.1 (but with respect to $(\lambda_i)$ instead of $(\mu_i)$) and these will be total. Hence we can define the computable learner which monitors the deficiencies along each real $X$ and at each step $n$ predicts the index $i$ which minimizes the cost $d_i(X \upharpoonright_n)[n] + i$. It is easy to check that this learner succeeds in all of the measures $(\lambda_i)$.\footnote{Since Definition 1.1 refers to the universal indexing $(\mu_i)$, at this point the reader may be concerned with the difference in the indexing $(\lambda_i)$. However this is not an issue, since there is a computable map from the indices in special list $(\lambda_i)$ to the corresponding items in the universal list $(\mu_i)$ that we fixed.}

Second, with an oracle which decides the totality of partial computable functions in the limit, we can implement a similar learner for the universal list $(\mu_i)$, by eventually identifying and ignoring the partial members of $(\mu_i)$ in our calculations of the costs $d_i(X \upharpoonright_n)[n] + i$.

Formally, for each $\sigma$ define

$$\text{cost}(\sigma, e)[s] = e + \max\{d_\epsilon(\sigma \upharpoonright_n)[s] \mid n \leq |\sigma| \land d_\epsilon(\sigma \upharpoonright_n)[s] = 0\}$$

where the maximum of the empty set is defined by default to be 0. Then for each $\sigma$

$$L(\sigma) = \min\{i \leq |\sigma| \mid f(i, |\sigma|) = 1 \land \text{cost}(\sigma, i)[|\sigma|] = \min_j \text{cost}(\sigma, j)[|\sigma|]\}$$

i.e. $L(\sigma)$ is the least $i \leq |\sigma|$ which minimizes the cost of $\sigma$.

Clearly $L$ is computable in $f$, so it remains to show that for every $X$ which is $\mu_j$-random for some total $\mu_j$, the limit $\lim_n L(X \upharpoonright_n)$ exists and is a number $i$ such that $\mu_i$ is total and $X$ is $\mu_i$-random. Consider the least number $e$ which minimizes the expression $d_\epsilon(X) + e$ amongst the indices of total computable measures. It remains to show that $\lim_n L(X \upharpoonright_n) = e$. Note that by our hypothesis about $X$, $d_\epsilon(X) + e$ is a finite number.

Let $s_0$ be a stage such that

(a) $f(j, s) = \lim_i f(j, t)$ for all $j \leq d_\epsilon(X) + e$

(b) $j + \max_{i \leq s} d_j(X \upharpoonright_i)[s] \geq \max_{i \leq s} d_\epsilon(X \upharpoonright_i)[s] + e$ for all $s \geq s_0$ and $j \leq d_\epsilon(X) + e$ such that $\mu_j$ is total;

(c) $j + \max_{i \leq s} d_j(X \upharpoonright_i)[s] > \max_{i \leq s} d_\epsilon(X \upharpoonright_i)[s] + e$ for all $s \geq s_0$ and $j < e$ such that $\mu_j$ is total.
By the choice of $X$ and the definition of $d_e(X)$ we have $\mathcal{L}(X \upharpoonright n) \leq d_e(X) + e$ for all $n$. If $\mathcal{L}(X \upharpoonright n) = j$ for some $n > s_0$, then by the choice of $s_0$ the measure $\mu_j$ is total, so by clause (b) above, and the minimality in the definition of $\mathcal{L}$ on $X \upharpoonright n$, we must have $j \leq e$. Then by clause (c) and the definition of $\mathcal{L}$ it is not possible that $j < e$, so $j = e$ and this shows that $\mathcal{L}(X \upharpoonright n) = e$ for all $n > s_0$, which concludes the proof of (4).

2.3 Proof of Theorem 1.3 about partial learning

Let $\ell_i[s]$ be the largest number $\ell$ such that $\mu_i(2^{\ell}f)[s] \downarrow$. A stage $s$ is called $i$-expansionary if $\ell_i[t] < \ell_i[s]$ for all $i$-expansionary stages $t < s$. By the padding lemma let $p$ be a computable function such that for each $i$, $j$ we have $\mu_{p(i,j)} = \mu_i$ and $p(i, j) < p(i, j + 1)$.

At stage $s$, we define $\mathcal{L}(\sigma)$ for each $\sigma$ of length $s$ as follows. For the definition of $\mathcal{L}(\sigma)$ find the least $i$ such that $\sigma$ is $i$-expansionary and $d_i(\sigma)[s] \leq i$. Then let $j$ be the least such that $p(i, j)$ is larger than any $k$-expansionary stage $t < |\sigma|$ for any $k < i$ such that $d_k(\sigma \upharpoonright t)[t] \leq k$, and define $\mathcal{L}(\sigma_i) = p(i, j)$.

Let $X$ be a real. Note that $\mathcal{L}(X \upharpoonright n) = x$ for infinitely many $n$, then $x = p(i, j)$ for some $i$, $j$, which means that $\mu_i = \mu_x$ is total and there are infinitely many $x$-expansionary stages as well as infinitely many $i$-expansionary stages. This implies that there are at most $x$ many $y$-expansionary stages $t$ for any $y < x$ with $d_x(\sigma \upharpoonright y)[t] \leq y$. Moreover for each $z > x$ there are at most finitely many $n$ such that $\mathcal{L}(X \upharpoonright n) = z$. Indeed, for each $z$ if $n_0$ is an $i$-expansionary stage then $\mathcal{L}(X \upharpoonright n_0) \neq z$ for all $n > n_0$. Moreover if $\mathcal{L}(X \upharpoonright n) = x$ for infinitely many $n$, then $d_x(X) = d_x(X) \leq i$ and $\mu_i$ is total, so $X$ is $\mu_i$-random. We have shown that for each $X$ there exists at most one $x$ such that $\mathcal{L}(X \upharpoonright n) = x$ for infinitely many $n$, and in this case $\mu_x$ is total and $X$ is $\mu_x$-random.

It remains to show that if $X$ is $\mu$-random for some computable $\mu$, then there exists some $x$ such that $\mathcal{L}(X \upharpoonright n) = x$ for infinitely many $n$. If $X$ is $\mu_i$-random for some $i$ such that $\mu_i$ is total, let $i$ be the least such number with the additional property that $d_i(X) \leq i$ (which exists by the padding lemma). Also let $j$ be the least number such that $p(i, j)$ is larger than any stage $t$ which is $k$-expansionary for any $k < i$ with $d_k(\sigma \upharpoonright k)[t] \leq k$. Then the construction will define $\mathcal{L}(X \upharpoonright n) = p(i, j)$ for each $i$-expansionary stage $n$ after the last $k$-expansionary stage $t$ for any $k < i$ with $d_k(\sigma \upharpoonright k)[t] \leq k$. We have shown that $\mathcal{L}$ partially succeeds on every $\mu$-random $X$ for any computable measure $\mu$.

3 Proof of the second clause of Theorem 1.2 (harder)

In order to make $\mathcal{L}$ compute a function $f$ which dominates every computable function, the idea is to use the convergence times of the current guesses (e.g., for the strings of length $s$) of $\mathcal{L}$ in order to produce the large number $f(s)$. The immediate problem is that some of the current guesses may point to partial measures $\mu_i$, so the search of some convergence times may be infinitely long. Although we cannot decide at stage $s$ which of these guesses are such, we know that they are erroneous guesses, and they cannot be maintained with positive probability, with respect to any computable probability measure $\mu$. Hence for each such guess $\mu_i$ (on a string $\sigma$ of length $s$) we can wait for either the convergence of $\mu_i \upharpoonright s$ or the change of the $\mathcal{L}$-prediction in a sufficiently ‘large’ set of extensions of $\sigma$. In order to make this idea work, we would need to argue that

for each computable function $h$, the failure of $f \leq_T \mathcal{L}$ to dominate $h$ means that for some computable measure $\lambda_h$, the learner $\mathcal{L}$ fails to give correct predictions in the limit for a set of reals of positive $\lambda_h$-measure. 

(5)
In order to make these failures concrete, in Section 3.1 we show that without loss of generality we may assume that \( \mathcal{L} \) does not only predict a measure \( \mu_i \) along each real \( X \), but also an upper bound on the \( \mu_i \)-randomness deficiency of \( X \). Then the crucial lemma which allows the above argument for the domination of \( h \) from \( f \leq_T \) to succeed is the following fact, which we prove in Section 3.2:

for every computable function \( h \) there exists a computable measure \( \lambda_h \) and a \( \lambda_h \)-large class of reals \( \mathcal{V}_h \) such that for any \( X \in \mathcal{V}_h \) and \( \mu_i \) that may be the suggested hypothesis of \( \mathcal{L} \) along \( X \),

\[
\text{either the time-complexity of } \mu_i \text{ dominates } h \text{ or } X \text{ has large } \mu_i \text{-deficiency (above the guess of } \mathcal{L} \).
\]

We will also need to make sure that the measure \( \lambda_h \) that we design from \( h \) is relatively identical to the uniform measure for all strings in long intervals of lengths, compared to the growth of \( h \). This feature will allow us to know how long to wait for the convergence of \( \mu_i \upharpoonright_s \) for the guess \( \mu_i \) of \( \mathcal{L} \) on some \( \sigma \) at stage \( s \), i.e., on ‘how many’ reals extending \( \sigma \) does \( \mathcal{L} \) have to change its guess before we give-up on the convergence of \( \mu_i \upharpoonright_s \). Indeed, although this size of reals is supposed to be with respect to \( \lambda_h \), the definition of \( f \) should not depend directly on \( \lambda_h \) since the totality of \( h \) and hence of \( \lambda_h \) cannot be determined by \( \mathcal{L} \) at each stage \( s \).

Then the crucial positive \( \lambda_h \)-measure set in the main argument (5) will be a subset of \( \mathcal{V}_h \), namely all the reals in \( \mathcal{V}_h \) except for the open sets \( \mathcal{M}_h(s) \) of reals for which we did not weight long enough at the various stages \( s \) of the definition of \( f \), before we give up waiting for the convergence of the current guesses. The tricky part of the construction of \( f \), which we present and verify in Section 3.4, is to ensure that a positive \( \lambda_h \)-measure remains in \( \mathcal{V}_h \), despite the fact that \( \lambda \) is not available in the construction in order to directly measure the sets \( \mathcal{M}_h(s) \) that need to be removed from \( \mathcal{V}_h \).

### 3.1 Randomness deficiency and learning

The proof of the second clause of Theorem 1.2 will be based on the fact that

if a learner learns a computable measure \( \mu \) along a real \( X \), such that \( X \) is \( \mu \)-random, then it can also learn an upper bound on the randomness deficiency of \( X \) with respect to \( \mu \).

\[
\text{(7)}
\]

We call this notion **strong EX-learning along } X \text{** and by the padding lemma (the fact that one can effectively produce arbitrarily large indices of any given computable measure) we can formulate it as follows.

**Definition 3.1** (Strong EX-learning). Given a class of computable measures \( C \), a learner \( F \) and a real \( X \), we say that the learner strongly EX-succeeds on \( X \) if \( \lim_n F(X \upharpoonright_n) \) exists and equals an index \( i \) of some \( \nu \in C \) such that the \( \nu \)-randomness deficiency of \( X \) is bounded above by \( i \). Given \( \mu \in C \) we say that \( F \) strongly EX-succeeds on \( \mu \) if it strongly EX-succeeds on every \( \mu \)-random real \( X \).

Then we can write (7) as follows.

**Lemma 3.2.** Given a class of computable measures \( C \) and a learner \( F \), there exists a learner \( F^* \) which strongly EX-succeeds on every real \( X \), on which the given learner EX-succeeds.

**Proof.** Let \( g \) be a computable function such that for each \( i, t \) the value \( g(i, t) \) is an index of \( \mu_i \) and \( g(i, t) > t \). Recall the definition of \( d_i(\sigma)[s] \) from Section 2.1. Define \( F^*(\sigma) = g(F(\sigma), d(F(\sigma), \sigma)[|\sigma|]) \). Given \( X \), suppose that \( \lim_n F(X \upharpoonright_n) = i \) and \( X \) is \( \mu_i \)-random. Then \( \lim_n d_i(X \upharpoonright_n)[n] < \infty \) so \( \lim_m F^*(X \upharpoonright_m) \) exists and is an index \( j \) such that \( \mu_j = \mu_i \). Moreover by the definition of \( g \) we have that \( j > d_j(X) \) which show that \( F^* \) strongly EX-succeeds on \( X \). \( \square \)
In the crucial lemma (6) that we prove in the next section, we will need to increase the randomness deficiency of some reals, which we do through tests. Recall we have fixed a certain effective sequence \((\mu_e)\) of partial computable measures which includes all computable measures. For each \(e\) we define a clopen \(\mu_e\)-test to be a partial computable function \(i \rightarrow D_i\) from integers to finite sets of strings (described explicitly) with the properties that for each \(i\), if \(D_i \downarrow\) then \(D_j \downarrow\), \(\mu_e(D_j) \downarrow\) and \(\mu_e(D_j) \leq 2^{-j}\) for each \(j \leq i\). Note that we incorporate partiality in the definition of clopen tests. In this way, there exists an effective universal list of all clopen \(\mu_e\)-tests for each \(e\), so we may refer to an index of a clopen test, in relation the universal list. The following fact is folklore.

Uniformly in a randomness deficiency bound \(k\), an index of a partial computable measure \(\mu_i\) and the index of a \(\mu_i\)-clopen test, we can compute the index of a member of the test which, if defined, the strings in it have deficiency exceeding \(k\).

In combination with the recursion theorem, (8) says that when we construct a \(\mu\)-clopen test for some partial computable measure \(\mu\), we can calculate a lower bound on the indices of the members test that guarantees sufficiently high (larger than the prescribed value \(k\)) \(\mu\)-randomness deficiency of all of the strings in them.

We state and prove the version of (8) which will be used in the argument of Section 3.2. A uniform sequence of clopen tests \((G^i)\) is a uniformly computable sequence of clopen tests \(G^i = (G^i_j)_{j \in \mathbb{N}}\) such that for each \(i\), \(G^i\) is a clopen \(\mu_i\)-test. As before, since we incorporated partiality in the definition of clopen tests, each uniform sequence of clopen tests are indexed in a fixed universal enumeration of all uniform sequences of clopen tests. The main argument in Section 3.2 will be the construction of a \(\mu_i\)-test for each \(i\), hence uniform sequence of clopen tests, which control the randomness deficiencies of a set of reals through the following fact.

**Lemma 3.3** (Uniformity of randomness deficiency). There is a computable function \(q(t,e,k)\) which takes any index \(t\) of a uniform sequence of clopen tests \(G^t = (G^t_j)\) and any number \(k\), and for each \(e\) we have \(d_e(\sigma) > k\) for all \(\sigma \in G^e_{q(t,e,k)}\), provided that \(\mu_e(G^e_{q(t,e,k)})\) is defined.

**Proof.** We construct a prefix-free machine \(M\) and a function \(q\), and by the recursion theorem we may use the index \(x\) of \(M\) in the definition of \(M,q\).\(^7\) We define \(q(t,e,k) = t + e + k + x + 5\) and the machine \(M\) as follows. For each index \(t\) we let \(G^t = (G^t_j)\) be the uniform sequence of clopen tests with index \(t\) and for each \(e,k\) we wait until \(\mu_e(G^e_{q(t,e,k)}) \downarrow\); if and when this happens we compress all strings \(\sigma \in G^e_{q(t,e,k)}\) using \(M\), so that \(K_M(\sigma) = -\log \mu_e(\sigma) - k - x\) (and stop these compressions if and when the compression cost exceeds \(1\)). Since \(\mu_e(G^e_{q(t,e,k)}) < 2^{-t-e-x-k-5}\) the cost of this operation is \(2^{-t-e-k-x-5+k+x} = 2^{-t-e-5}\) and the total cost is \(\sum_{t,e} 2^{-t-e-5} < 1\) so the compression of \(M\) as we described it will not be stopped due to an excess of the compression cost. Then by the definition of \(M\), for each \(t,e\) such that \(\mu_e(G^e_{q(t,e,k)}) \downarrow\) and each \(\sigma \in G^e_{q(t,e,k)}\),

\[
K(\sigma) \leq K_M(\sigma) + x = -\log \mu_e(\sigma) + x - k - x = -\log \mu_e(\sigma) - k
\]

so \(d_e(\sigma) \geq k\) which concludes the proof. \(\square\)

\(^7\)Formally, we run the construction with a free parameter \(y\) which is treated as if it is the index of the machine that we are constructing, although it may not be; thus producing a computable function \(h\) which gives the uniform sequence of machines \(M_{h(y)}\) (given a universal enumeration \(M_i\) of all prefix-free machines). In the argument here, the weight of each \(M_{h(y)}\) is explicitly forced to be at most 1, thereby stopping any further computation if this bound is reached. This ensures that for each \(y\) the process defines a prefix-free machine, even if \(y\) is not an index of the machine constructed. Then by the recursion theorem we choose \(x\) such that \(M_{h(y)} = M_x\), and this is the desired index for which the construction correctly assumes that the input number is an index of the machine being constructed.
Informally, the computable function $p$ of Lemma 3.3 will tell the construction of Section 3.2, how long it needs to build the $\mu_e$-test $G^e$ for each $e$, in order to achieve the required $\mu_e$-randomness deficiency.

3.2 The domination lemma

For the verification of the dominating function from the learner, in Section 3.4, we need a lemma which says, roughly speaking, that for each computable function $h$ there exists a computable measure whose time-complexity is higher than $h$ and which resembles the uniform measure except for a very sparse set of strings.\(^8\) Before we state the lemma, we need the following definitions.

Definition 3.4 (Relative equality of measures in intervals). Given two measures $\mu, \lambda$ we say that $\mu$ is relatively equal (or identical) to $\lambda$ in an interval $[a, b]$ if for all strings $\sigma$ of length in $[a, b]$ and each $j \in \{0, 1\}$ we have $\mu(\sigma \ast j) / \mu(\sigma) = \lambda(\sigma \ast j) / \lambda(\sigma)$.

Note that if $\mu$ is relatively identical to $\lambda$ in $[a, b]$ then for each $s, t \in [a, b]$ with $s < t$, each $\sigma \in 2^s$ and each prefix-free set $D$ of strings in $2^{2s}$ extending $\sigma$, we have $\mu_\sigma(D) = \lambda_\sigma(D)$, where $\mu_\sigma(D) := \mu(D) / \mu(\sigma)$ is the conditional $\mu$-measure with respect to $\sigma$, and similarly with $\lambda_\sigma$.

Definition 3.5 (Sparse measures). Given an increasing function $h$, a sequence $(n_i)$ is $h$-sparse if $h(n_i) < n_{i+1}$ for each $i$. A measure $\lambda$ is $h$-sparse if there exists an $h$-sparse sequence $(n_i)$ such that for each $i$, $\lambda$ is relatively identical to the uniform measure in $(n_i, n_{i+1}]$.

The following crucial lemma is exactly what we need for the argument of Section 3.4.

Lemma 3.6 (Time-complexity of computable measures). Given computable functions $h, g, p$ there exists a $p$-sparse computable measure $\lambda$ and a $\Pi^0_1$ class of reals $C$ such that $\lambda(C) = 1$ and for every $X \in C$ and every index $i$ of a computable measure $\mu_i$ such that the $\mu_i$-deficiency of $X$ is $\leq g(i)$, the time-complexity of $\mu_i$ dominates $h$.

We can easily derive Lemma 3.6 from the following technical lemma, whose proof we give in Section 3.3.

Lemma 3.7. Given any computable functions $h, g, p$ there exists a $p$-sparse computable measure $\lambda$, a $\Pi^0_1$ class of reals $C$ such that $\lambda(C) = 1$ and a uniform sequence of clopen tests $G^i = (G^i_j)$ such that, if $\ell_i$ is the length of $G^i$, we have:

(a) $\mu_i(G^i_j) \leq 2^{-j}$ and $C \subseteq \ll G^i_j \gg$ for all $j < \ell_i$;

(b) if the time-complexity of $\mu_i \upharpoonright n$ does not dominate $h$ then the length of $G^i$ is $g(i)$.

In particular, $G^i$ is a $\mu_i$-test for each $i$ such that $\mu_i$ is total, and $\mu_i(V) \leq 2^{-g(i)}$ if in addition clause (b) holds.

Note that the hypothesis in clause (b) above implies that $\mu_i$ is total.

We prove Lemma 3.6 so fix $h_0, g_0, p_0$ in place of $h, g, p$ in the statement of the lemma. In order to obtain the desired $C, \lambda$ corresponding to the given $h_0, g_0, p_0$, we are going to apply Lemma 3.7 on $h = h_0, g = g_1, p = p_0$ where $g_1$ is an appropriate function related to $g_0$, which we are going to obtain as a fixed-point. If we fix $h = h_0, p = p_0$ and regard (the index of) $g$ as a variable in Lemma 3.7, we get a total effective index-map $j \mapsto v(j)$ from any index of a function $g$, to an index $v(i)$ of a uniform sequence of clopen tests

\(^8\)A similar tool was used in [1], namely a version of the fact from [7] that given any computable function $h$ there exists a computable function such that any implementation of it converges in time exceeding $h$.  

10
\( \mathcal{G}' \), such that if the function \( g \) with index \( j \) is total, the stated properties hold. Now recall the function \( q \) of Lemma 3.3. By the recursion theorem and the s-m-n theorem we can choose \( j_0 \) such that the function with index \( j_0 \) is the same as the function \( e \mapsto q(v(j_0), e, g_0(e)) \). In particular, this function it total (because \( e \mapsto q(v(j), e, g_0(e)) \) is total for every \( j \)) so we can define \( g_1 \) to be the function with index \( j_0 \). Then the properties of \( q \) and the uniform sequence of clopen tests indexed by \( v(j_0) \) according to Lemma 3.7, shows that Lemma 3.7 applied to \( h = h_0, g = g_1, p = p_0 \) produces \( \lambda, C \) satisfying the properties of Lemma 3.6 for \( h = h_0, g = g_0, p = p_0 \).

### 3.3 Proof of Lemma 3.7

Given \( h, g, p \) we produce a sequence \((C_s)\) of finite sets of strings such that the strings in \( C_s \) have length \( s \) and every string in \( C_{s+1} \) has a prefix in \( C_s \). Then we define \( C \) to be the set of reals which have a prefix in each of the sets \( C_s \), i.e. \( C = \cap_s \mathbb{P} \mathbb{C}_s \) which is clearly a \( \Pi^0_1 \) set, and ensure that \( C \) is \( p \)-sparse. Without loss of generality we assume that \( p \) is increasing.

We also define a uniform sequence of clopen tests \((G'_i)\) whose members will be defined to be \( C_s \) for certain \( s \). In particular, if stage \( s \) acts for \( i \) then the next member of \( G' \) will be defined equal to \( C_s \). In this sense, certain members of \((C_s)\) become members of the tests \( G_i \) in a one-to-one fashion (in the sense that each \( C_s \) is assigned to at most one \( G_i \)). In order to make \( \lambda \) a \( p \)-sparse measure, if stage \( s \) acts for some \( i \) (hence, it is an active stage) the next \( p(s) \) many stages of the construction are suspended, in the sense that no action is allowed on those stages. Define \( \lambda \) on the empty string equal to 1.

At each stage \( s \) some \( i < s \) may require attention, in which case \( s \) will be an \( i \)-stage for the least such \( i \), and a new member of \( G' \) will be defined equal to \( C_s \). We say that \( i < s + 1 \) requires attention at stage \( s + 1 \) if it has received attention less than \( g(i) \) many times and

- \( \mu_i \uparrow_s [h(s + 1)] \downarrow; \)
- for all \( j < i \) and all \( n \leq (s + 1, f(s + 1)] \) we have \( \mu_j \uparrow_n [h(n)] \uparrow \) or \( j \) has acted \( g(j) \) times.

Intuitively, for \( i \) to require attention at stage \( s + 1 \) we require first that it has received attention less than \( g(i) \) many times, second that we can determine the values of \( \mu_i \) for strings up to length \( s + 1 \) (which are the strings from which we are about to choose the members of \( C_{s+1} \)) and third, that in none of the future stages that are about to become suspended as a result to \( \mu_i \) receiving attention at this stage, some \( \mu_j \) with \( j < i \) might require attention.

**Construction of \( \lambda \) and \((C_s)\).** At stage \( s + 1 \), if this is a suspended stage or no \( i \leq s \) requires attention,

- let \( C_{s+1} = \{ \sigma * 0, \sigma * 1 \mid \sigma \in C_s \} \);
- for each \( \sigma \in C_s \) and \( j \in \{0, 1\} \) let \( \lambda(\sigma * j) = \lambda(\sigma)/2; \)

and go to stage \( s + 2 \). Otherwise, consider the least \( i \leq s \) which requires attention and for each string \( \sigma \) of length \( s + 1 \) do the following:

- define \( j_\sigma \) to be the least such that \( \mu_i(\sigma * j_\sigma) \leq \mu_i(\sigma * (1 - j_\sigma)) \)
- for each \( \sigma \in C_s \) define \( \lambda(\sigma * j_\sigma) = 0; \) and \( \lambda(\sigma * (1 - j_\sigma)) = \lambda(\sigma); \)
- for each \( \sigma \in 2^s - C_\sigma \) define \( \lambda(\sigma * 0) = \lambda(\sigma * 1) = \lambda(\sigma)/2; \)
• define $C_{s+1} = [\sigma \ast (1 - j_{\sigma}) | \sigma \in C_s]$.

Declare stage $s + 1$ active (acting on $i$) and the next $p(s + 1)$ stages suspended. Also let $j$ be the least such that $G_j^i \uparrow$ and define $G_j^i = C_{s+1}$.

**Lemma 3.8.** The function $\lambda$ is a continuous computable $p$-sparse measure and $\lambda(C) = 1$.

**Proof.** Inductively it follows that $\lambda$ is a measure representation and since the construction is effective, it is also computable. Also if $(n_i)$ are the active stages, the construction shows that $\lambda$ is relatively identical to the uniform measure in each of the intervals $[n_i, n_{i+1}]$, since it only differs from the uniform measure on the strings of lengths $n_i, i \in \mathbb{N}$. Moreover by construction $(n_i)$ is $p$-sparse, which shows that $\lambda$ is also $p$-sparse. Clearly there are infinitely many stages which are either suspended or no $i$ requires attention, so $\lambda$ is continuous. Finally at each step that $[C_{s+1}]$ is smaller than $[C_s]$, all the $\lambda$-measure of $[C_s]$ is transferred to $[C_{s+1}]$. Hence $\lambda(C_s) = 1$ for all $s$, which shows that $\lambda(C) = 1$. □

Note that the sequence $G_j^i$ is uniformly partial computable, in the sense that the function $(i, j) \mapsto G_j^i$ is partial computable.

**Lemma 3.9.** For each $i, j$, if $G_{j+1}^i \downarrow$ then $G_j^i, \mu_i(G_{j+1}^i), \mu_i(G_j^i)$ are defined, $C \subseteq [G_{j+1}^i]$ and $\mu_i(G_{j+1}^i) \leq \mu_i(G_j^i)/2$. In particular, for each $i$, $G_i$ is a $\mu_i$-test.

**Proof.** By the construction, if $G_{j+1}^i \downarrow$ then necessarily $G_j^i$ is defined. Moreover, in this case, if $G_{j+1}^i$ was defined at some stage $s$, we have $G_{j+1}^i = C_s$ and stage $s$ acted on $i$, which means that $\mu_i(C_s)[s] \downarrow$ so $\mu_i(G_{j+1}^i)[s] \downarrow$. By the definition of $C$ as the intersection of all $C_t, t \in \mathbb{N}$, we also have $C \subseteq [G_{j+1}^i]$. Also since the strings in $C_s$ are longer than the strings in any $C_t, t < s$, by the same argument we have $\mu_i(G_j^i)[s] \downarrow$. Now let $t < s$ such that $G_{j+1}^i = C_t$ and note that by construction we have $\mu_i(C_{s-1}) \leq \mu_i(C_s)/2$. Since $t \leq s - 1$ we have $[C_t] \subseteq [C_{s-1}]$ and

$$\mu_i(G_j^i) = \mu_i(C_t) \leq \mu_i(C_{s-1}) \leq \mu_i(C_s)/2 = \mu_i(G_j^i)/2$$

which concludes the proof of the lemma. □

It remains to show clause (b) of Lemma 3.7. If $\mu_i$ is not total then clearly the time-complexity of $\mu_i \uparrow_n$ will be equal to infinity co-finitely often, hence dominating $h$. hence it suffices to show the following fact by a finite injury argument.

**Lemma 3.10.** If $\mu_i$ is total and its time-complexity does not dominate $h$, then the length of $G_i$ is $g(i)$.

**Proof.** By the definition of ‘requiring attention’, each $j$ will receive attention at most finitely many times. We argue that each $j$ will also require attention at most $g(i)$ many times. Indeed, otherwise there would be a least $j$ which requires attention infinitely many times, and by the construction it will receive attention infinitely many times, which is a contradiction.

Let $s_0$ be a stage after which no $\mu_j, j \leq i$ requires attention, and let $s_1 = f(s_0)$. It suffices to show that the length of $G_i[s_1]$ is $g(i)$. Note that each time that $i$ receives attention, the length of (the current state of) $G_i$ increases by 1, so the length of $G_i$ equals the number of times that $i$ receives attention. Hence the length of $G_i$ is at most $g(i)$. For a contradiction, assume that the length of $G_i$ is less than $g(i)$. By the hypothesis...
about $\mu_i$, there will be a least stage $s + 1 > s_1$ such that $\mu_i \uparrow_{s+1} [h(s + 1)] \downarrow$. Since the length of $G^i[s]$ is less than $g(i)$, by the choice of $s_0, s_1$, no $j < i$ requires attention at stage $s + 1$.

We also argue that $s + 1$ is not a suspended stage. Indeed, $s + 1$ cannot be suspended by some $\mu_j, j < i$ because all of these indices last acted before stage $s_0$, which means that they cannot suspend any stage after $p(s_0) = s_1$. If $s + 1$ was suspended by some $j > i$ then this must have happened at some stage $t < s + 1$ such that $s + 1 \in (t, p(t))$. But according to the construction, $j$ cannot require attention at stage $t$ because $i < j$ and $\mu_i \uparrow_{s+1} [h(s + 1)] \downarrow$ while $s + 1 \leq p(t)$. We may conclude that stage $s + 1$ is not suspended.

In order to show that $i$ requires attention at stage $s + 1$ it remains to show that for each $j < i$ either $j$ has acted $g(j)$ many times or for all $n \leq (s + 1, f(s + 1)]$ we have $\mu_j \uparrow_n [h(n)] \uparrow$. Indeed, if $j$ had acted less than $g(j)$ many times and the latter condition did not hold, $j$ would require attention at stage $s + 1$, which contradicts our choice of $s_0$. Therefore, according to the conditions for index $i$ to require attention, $i$ will require attention at stage $s + 1$ and since no $j < i$ requires attention, it will receive attention at stage $s + 1$. This again contradicts the choice of $s_0$, and concludes the proof that at stage $s_1$ the length of $G^i$ will be $g(i)$.

This concludes the proof of Lemma 3.7.

### 3.4 The dominating function of the learner

Given a learner $L$ which EX-succeeds on all computable measures, we construct a function $f \leq L$ which dominates all computable functions. Note that

$$
\text{for each } \sigma, \text{ letting } e := L(\sigma), \text{ there exists some } n_{\sigma} > 2|\sigma| \text{ such that either } \mu_e \uparrow_{|\sigma|} [n_{\sigma}] \downarrow \text{ or the proportion of extensions } \tau \text{ of } \sigma \text{ of length } n_{\sigma} \text{ such that } L(\rho) = L(\sigma) \text{ for all } \rho \in [\sigma, \tau] \text{ is } < 2^{-|\sigma|-5}.
$$

(9)

This is because otherwise, $L$ would give a partial measure (hence a wrong prediction) for a positive measure (with respect to the uniform measure) class of reals.

For each $t$ define $f(t) = \max_{\sigma \in 2^t} F(\sigma)$ where

$$
F(\sigma) = \max \{n_{\sigma}, n_\tau \mid n \in ([\sigma], n_{\sigma}) \land \tau \in \text{EXT}(\sigma, n)\}
$$

and $\text{EXT}(\sigma, n)$ denotes the strings of length $n$ which extend $\sigma$.

It remains to show that $f$ dominates all computable functions, so fix a computable function $h$. Without loss of generality we assume that $h$ is increasing. Apply Lemma 3.6 in order to obtain $\lambda_h$ and $C_h$ with the stated properties. Since the remaining of the argument refers to $h$, we drop the subscripts in $\lambda_h$ and $C_h$, and denote them by $\lambda$ and $C$ respectively. We also get an increasing sequence $(m_i)$ of active h-stages such that $h(m_i) < m_{i+1}$ for each $i$, and inside the intervals $[m_i, m_{i+1}]$ the measure $\lambda$ is relatively identical to the uniform measure. We are going to define a sequence $D_t$ of sets of strings such that $\lambda(D_t) < 2^{-t-5}$ for each $t$. Then we define $C^* = C - \cup_t [D_t]$ and show that if $f$ does not dominate $h$ then $L$ fails for $\lambda$ on the set $C^*$. This is a contradiction since $\lambda(C^*) \geq 1/2$.

**Definition 3.11 (Definition of $D_t$).** We define $D_t$ by following the definition of $f(t)$. For each $\sigma \in 2^t$ let $e_\sigma = L(\sigma)$ and enumerate extensions of $\sigma$ into $D_t$ according to the first applicable clause:

(a) if $f(t) > h(t)$ or $\mu_e \uparrow_t [n_{\sigma}] \downarrow$ then do not enumerate any extension of $\sigma$ into $D_t$;
(b) otherwise, if there is no $h$-active stage in $(t, n_\tau)$ then we enumerate into $D_t$ all extensions $\tau$ of length $n_\tau$ such that $L(\rho) = L(\sigma)$ for all $\rho \in \{\sigma, \tau\}$;

(c) otherwise, let $n_\sigma^*$ be the least $h$-active stage in $[\sigma, n_\sigma]$ and for each string in $\tau \in \text{EXT}(\sigma, n_\sigma^*)$, if $\mu_e \uparrow n_\sigma^* \uparrow [n_\tau] \uparrow$ where $e_\tau := L(\tau)$, enumerate into $D_t$ all extensions $\rho$ of $\tau$ of length $n_\tau$ such that $L(\eta) = L(\tau)$ for all $\eta \in [\tau, \rho]$.

**Lemma 3.12** (Bounds on the $\lambda$-measure of $D_t$). For each $t$ we have $\lambda(D_t) < 2^{-t-5}$.

**Proof.** Given $t$, it suffices to show that for each $\sigma$ of length $t$, we enumerate into $D_t$ a set of extensions of $\sigma$ of $\lambda$-measure at most $2^{-t-5} \cdot \lambda(\sigma)$. Then putting together these bounds for all $\sigma$ of length $t$, we get that $\lambda(D_t) < 2^{-t-5}$.

So fix $\sigma$. In case (a) we do not enumerate any extensions of $\sigma$ into $D_t$, so there is nothing to prove. In case (b) we have $\mu_e \uparrow [n_\sigma] \uparrow$, so by the definition of $n_\sigma$ in (9) the proportion of extensions of $\sigma$ of length $n_\sigma$ that are enumerated into $D_t$ is less than $2^{-t-5}$. But what we really need here is to show that their $\lambda$-measure relative to $\sigma$ is bounded in this way. In case (b) we also have that there is no $h$-active stage in $[t, n_\sigma]$ so $\lambda$ is relatively identical to the uniform measure in $[t, n_\sigma]$. Hence, since the uniform measure of the strings enumerated in $D_t$ relative to $\sigma$ is less than $2^{-t-5}$ the same is true of their $\lambda$-measure. In other words, we get the bound $2^{-|\sigma|-5} \cdot \lambda(\sigma)$ for these strings as desired.

We can get the same bound in case (c) as follows. Since $n_\sigma^*$ is an $h$-active stage and $h$-active stages are $h$-sparse, it follows that there is no $h$-active stage in $(n_\sigma^*, h(n_\sigma^*))$. If $n_\tau \geq h(n_\sigma^*)$ for some $\tau$ of length $n_\tau$ extending $\sigma$, then by the definition of $f$ and the monotonicity of $h$ we would have $f(t) > h(t)$, contrary to our assumption that clause (a) does not apply. Hence for each $\tau \in \text{EXT}(\sigma, n_\sigma^*)$ we have $n_\tau < h(n_\sigma^*)$ so that there is no $h$-active stage in $(n_\sigma^*, n_\tau]$. Now we can use the same argument that we used in case (b) above, but with respect to each $\tau \in \text{EXT}(\sigma, n_\sigma^*)$. If $\mu_e \uparrow [n_\sigma^* \uparrow [n_\tau] \uparrow$ where $e_\tau := L(\tau)$ then by the definition of $n_\tau$ we have the that proportion of the extensions of $\tau$ of length $n_\tau$ which are enumerated into $D_t$ is less than $2^{-t-5}$, and since there is no $h$-active stage in $(n_\sigma^*, n_\tau]$ the $\lambda$-measure of these strings is less than $2^{-t-5} \cdot \lambda(\tau)$. On the other hand if $\mu_e \uparrow [n_\sigma^* \uparrow [n_\tau] \downarrow$ no extensions of $\tau$ are enumerated into $D_t$ so this bound holds trivially. Putting together these bounds for all $\tau$ of length $n_\sigma^*$ extending $\sigma$, we get the bound $2^{-|\sigma|-5} \cdot \lambda(\sigma)$ on the $\lambda$-measure of the strings in $D_t$ extending $\sigma$, as required. \qed

**Lemma 3.13** (Domination or leaning failure). If $f$ does not dominate $h$ then $L$ fails for $\lambda$ on each $X \in C^*$.

**Proof.** Assume that $f$ does not dominate $h$ and let $X \in C^*$. For a contradiction assume that $L$ does not fail for $\lambda$ on $X$. As a consequence $\lim_n L(X \uparrow_n)$ exists and is an index $e$ of a computable measure $\mu_e$ such that $X$ is $\mu_e$-random. Note that by the discussion of Section 3.1 we can assume that $L$ strongly EX-succeeds for $\lambda$ on $X$. Hence we can pick some $n_0$ such that for all $n \geq n_0$:

(i) $L(X \uparrow_n) = e$ and $d_e(X \uparrow_n) \leq e$;

(ii) if $d_e(X) \leq e$ then $\mu_e \uparrow [h(n)] \uparrow$;

where the existence of $n_0$ such that the second clause is met for all $n \geq n_0$ follows by Lemma 3.6 applied to $h$, $g(m) := m + 2$, $p(m) := h(m) + 1$ and $i := e$.

Now pick some $t > n_0$ such that $f(t) \leq h(t)$, which exists by our hypothesis. We trace the definition of $f(t)$ in order to derive our contradiction. Let $\sigma = X \uparrow_i$ and recall the definition of $n_\sigma$ in (9) (where $e$ has the same meaning as here). If $\mu_e \uparrow [n_\sigma] \downarrow$, then $\mu_e \uparrow [f(t)] \downarrow$ since $f(t) > n_\sigma$ and $\mu_e \uparrow [h(t)] \downarrow$ by the choice
of \( t \). Then by clause (ii) in the choice of \( n_0 \) we have \( d_e(X) > e \) which contradicts clause (i) of the definition of \( n_0 \), since by definition \( d_e(X \upharpoonright n) \leq d_e(X) \) for all \( n \).

Hence we must conclude that \( \mu_e \upharpoonright t [n_\tau] \uparrow \), which means that clause (a) in the definition of \( D_t \) with respect to \( \sigma = X \upharpoonright t \) does not apply. If clause (b) applies for \( \sigma = X \upharpoonright t \) in the definition of \( D_t \), then since \( X \notin [D_t] \) we have \( L(X \upharpoonright t) \neq L(X \upharpoonright s) \) for some \( t' > t \), which contradicts clause (i) of the definition of \( n_0 \). Hence we must conclude that clause (c) in the definition of \( D_t \) with respect to \( \sigma = X \upharpoonright t \) applies. Now consider \( n_\tau^* \) as in Definition 3.11 and fix \( \tau = X \upharpoonright n_\tau^* \). Note that by the choice of \( n_0 \) we have \( L(X \upharpoonright t) = L(X \upharpoonright n_\tau^*) \) so \( e_\tau = e \). If \( \mu_e \upharpoonright n^*_\tau [n_\tau] \downarrow \) then \( \mu_e \upharpoonright n^*_\tau [h(n^*_\tau)] \downarrow \) since \( n_\tau < f(t) \leq h(t) < h(n^*_\tau) \). In this case, by clause (ii) in the definition of \( n_0 \), we have \( d_e(X) > e \) which contradicts clause (i) of the definition of \( n_0 \). Hence we can assume that \( \mu_e \upharpoonright n^*_\tau [n_\tau] \uparrow \), and consider the enumeration of extensions of \( \tau = X \upharpoonright n_\tau^* \) that occurs in \( D_t \). Since \( X \notin [D_t] \), by clause (c) of the definition of \( D_t \) we have that \( L(X \upharpoonright n^*_\tau) \neq L(X \upharpoonright s) \) for some \( s \in (n^*_\tau, n_\tau) \).

However this contradicts clause (i) of the definition of \( n_0 \). From this final contradiction we can conclude that \( L \) does fail for \( \lambda \) on \( X \in C^* \), which concludes our proof. \( \square \)

Putting everything together, we show that \( f \) dominates every computable function \( h \) or fails for a computable measure \( \lambda \) on a set of \( \lambda \)-measure > 1/2. Given a computable \( h \) we may assume that \( h \) is increasing and consider \( \lambda, C \) from Lemma 3.6 and the sets \( (D_t) \) from Definition 3.11. Considering \( C^* := C - \cup_i [D_i] \), by Lemma 3.12 we have that \( \lambda(C^*) > 1/2 \) and by Lemma 3.13 we have that if \( f \) does not dominate \( h \) then it fails on every real in \( C^* \). This concludes the proof of Theorem 1.2.

4 Concluding remarks and directions for further research

Our main result was that a learner \( L \) can EX-succeed on all computable (continuous) measures if and only if it computes a function that dominates all computable functions. The harder part of this equivalence was to show that a learner with the above learning power computes a function that dominates all computable functions. In fact, our argument in Section 3.4 shows a stronger result, with respect to a learning notion that was also considered by Bienvenu, Monin and Shen [6].

**Definition 4.1.** A learner \( L \), EX-succeeds on a measure \( \mu \) with probability \( q > 0 \) if for a set of reals \( X \) of \( \mu \)-measure at least \( q \), the limit of \( L(X \upharpoonright n) \) as \( n \to \infty \) exists and is an index of a computable measure \( \nu \) such that \( X \) is \( \nu \)-random.

In the argument of Section 3.4 we can clearly make the \( \lambda \)-measure of \( \cup_i D_i \) as small as we like, which means that \( \lambda C^* \) can be made as close to 1 as we might require. This means that we have actually proved the following, stronger statement.

**Theorem 4.2.** If a learner EX-succeeds on all computable (continuous) measures with fixed positive probability \( q > 0 \), then it computes a function which dominates all computable functions.

A variation of EX-learning with oracles from the literature in algorithmic learning is when we only allow finitely many queries to the oracle when trying to guess a suitable measure for any specific \( X \). This notion is often called EX[*]-learning and in Fortnow et.al. [10, 9] it was shown that an oracle \( A \) can EX[*]-learn all computable functions if and only if \( A \oplus \emptyset' \geq_T \emptyset'' \). This notion has a direct analogue in the context of learning measures, so we use the same notation. In collaboration with Fang Nan, we have proved the following.
Theorem 4.3 (with Fang Nan). An oracle $A$ can EX[*]-learn all (continuous) computable measures if and only if $A \oplus \emptyset' \geq_T \emptyset''$.

In the classic setting, there is no concrete characterization of the oracles that BC-learn all computable functions (recall the notion of BC-learning from the discussion in the end of Section 1).\footnote{However note that by Stephan and Kummer \cite{14, 15}, in the c.e. sets an oracle can BC-learn all computable functions if and only if it can EX-learn all computable functions.} Hence for behavioral learning, we propose the following conjecture.

**Conjecture.** An oracle $A$ can BC-learn all computable functions if and only if it can BC-learn all computable (continuous) measures.

One can also study the oracles that are useless for learning, in the sense that any collection of computable measures that are learned with queries to $A$ can also be learned by a computable learner. In the classic learning setting these oracles were characterized by Slaman and Solovay \cite{23} as the 1-generic sets which are computable from the halting problem.

Finally we wish to suggest that one can study restricted classes of computable measures and get similar results. For example, Bienvenu, Monin and Shen \cite{6} stated and proved their theorem in terms of a general computable metric space of measures, using the framework developed by Gács \cite{11}. As a result, they where able to draw more general conclusions, such that the fact that there is no computable learner which EX or BC learns all computable Bernoulli measures. We conjecture that An oracle $A$ can EX-learn all computable Bernoulli measures if and only if $A' \geq_T \emptyset''$.

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