Abstract—This paper investigates the hybrid precoding design for millimeter wave (mmWave) multiple-input multiple-output (MIMO) systems with finite-alphabet inputs. The precoding problem is a joint optimization of analog and digital precoders, and we treat it as a matrix factorization problem with power and constant modulus constraints. Our work presents three main contributions: First, we present a sufficient condition and a necessary condition for hybrid precoding schemes to realize unconstrained optimal precoders exactly when the number of data streams \( N_s \) satisfies \( N_s = \min \{ \text{rank}(H), N_{\text{rf}} \} \), where \( H \) represents the channel matrix and \( N_{\text{rf}} \) is the number of radio frequency (RF) chains. Second, we show that the coupled power constraint in our matrix factorization problem can be removed without loss of optimality. Third, we propose a Broyden-Fletcher-Goldfarb-Shanno (BFGS)-based algorithm to solve our matrix factorization problem using gradient and Hessian information. Several numerical results are provided to show that our proposed algorithm outperforms existing hybrid precoding algorithms.

Index Terms—Hybrid precoding, finite-alphabet inputs, matrix factorization, nonconvex optimization.

I. INTRODUCTION

Millimeter wave (mmWave) multiple-input multiple-output (MIMO) communication is a promising technology for future generation cellular systems to address the wireless spectrum crunch. It makes use of the mmWave band from 30 GHz to 300 GHz, which implies a much wider bandwidth than current cellular systems operating in microwave bands. Moreover, a short wavelength of radio signals in the mmWave band enables large number of antennas to be equipped in transceivers, and this allows for applying massive multiple-input multiple-output (MIMO) technique in mmWave communication systems.

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For conventional MIMO systems, linear precoding is utilized to maximize the data rate, and it is implemented in the digital domain by the unconstrained optimal precoder. However, the implementation of unconstrained optimal precoders requires one radio frequency (RF) chain per antenna, which will result in prohibitive cost and power consumption in mmWave MIMO systems. To address this issue, a hybrid precoding scheme has been proposed for mmWave MIMO systems to reduce the number of RF chains \([1]-[7]\). This scheme divides the linear precoder into analog and digital precoders, which are implemented in analog and digital domains, respectively. The digital precoder is realized by a small amount of RF chains, and the analog precoder is realized by phase shifters. Due to the property of phase shifters, each entry of the analog precoder satisfies the constant modulus constraint. These nonconvex constant modulus constraints form a major barrier for hybrid precoding design.

Several hybrid precoding algorithms have been proposed for mmWave MIMO systems \([1]-[7]\). The work in \([1]\) first formulated the hybrid precoding problem as a matrix factorization problem, and then applied the orthogonal matching pursuit (OMP) algorithm to find near-optimal analog and digital precoders. In \([3]\), the authors utilized the formulation proposed in \([1]\), and then employed a manifold based alternating minimization algorithm to design hybrid precoders. References \([5]\) and \([7]\) introduced and analyzed low complexity hybrid precoding algorithms based on the matrix factorization. There were also some studies on how to achieve the performance of unconstrained optimal precoders with hybrid precoding schemes \([2], [4]\), yet requiring the number of RF chains to be twice as much as the number of data streams.

Most existing works on hybrid precoding assume Gaussian inputs, which are rarely realized in practice. It is well known that practical systems utilize finite-alphabet inputs, such as phase-shift keying (PSK) or quadrature amplitude modulation (QAM). Furthermore, it has been shown that precoding designs under Gaussian inputs are quite suboptimal for practical systems with finite-alphabet inputs \([8]-[14]\). A unified framework for linear precoding design under finite-alphabet inputs has been proposed in \([15]\). Recently, the authors in \([5]\) presented an iterative gradient ascent algorithm for mmWave MIMO systems with finite-alphabet inputs. In each iteration, the gradient ascent algorithm updated the unconstrained precoder using gradient information, and then it employed a heuristic way to partition the unconstrained precoder into analog and digital precoders. Simulation results illustrated that the gradient ascent algorithm can achieve up to 0.4 bps/Hz gains.
compared to the Gaussian inputs scenario.

A. Contributions

In this paper, we investigate the hybrid precoding design for mmWave MIMO systems with finite-alphabet inputs. The contributions of this paper are summarized as follows:

- We first provide a sufficient condition under which hybrid precoding schemes can realize any unconstrained optimal precoders exactly. When the sufficient condition does not hold, we also present a necessary condition for hybrid precoding to achieve the performance of unconstrained optimal precoders.
- We prove that the power constraint in the hybrid precoding problem (10) can be removed without loss of local and/or global optimality. This result greatly simplifies the precoding design, and it enable us to design an efficient algorithm for the hybrid precoding problem.
- We present closed form expressions for gradient and Hessian of the hybrid precoding problem. Then we utilize this model, the channel matrix H be written as

\[ y = H F_{RF} F_{BB} x + n \]

where \( H \in \mathbb{C}^{N_t \times N_t} \) is the channel matrix; \( F_{RF} \in \mathbb{C}^{N_r \times N_t} \) is the analog precoder with \( F \) being the constant modulus set; \( F_{BB} \in \mathbb{C}^{N_t \times N_r} \) is the digital precoder; \( x \in \mathbb{C}^{N_t \times 1} \) is the input data vector and \( n \in \mathbb{C}^{N_t \times 1} \) is independent and identically distributed (i.i.d.) circularly symmetric complex Gaussian noise with zero-mean and covariance \( \sigma^2 I \).

Suppose that the channel \( H \) is known at both the transmitter and receiver, and each entry of the input data vector \( x \) is uniformly distributed from a given constellation set with cardinality \( M \). Then the input-output mutual information is given by

\[ I(x; y) = N_t \log M - \frac{1}{M N_t} \sum_{m=1}^{M N_t} E_n \left\{ \log \sum_{k=1}^{M N_t} e^{-d_{mk}} \right\} \]

where \( d_{mk} = \sigma^{-2} (\| H F_{RF} F_{BB} (x_m - x_k) + n \|^2 - \| n \|^2) \), with \( x_m \) and \( x_k \) being two possible input data vectors from \( x \).

B. Notations

The following notations are adopted throughout the paper:

- Boldface lowercase letters, boldface uppercase letters, and calligraphic letters are used to denote vectors, matrices and sets, respectively. The real and complex number fields are denoted by \( \mathbb{R} \) and \( \mathbb{C} \), respectively.
- The superscripts \((\cdot)^T\), \((\cdot)^*\) and \((\cdot)^H\) stand for transpose, conjugate, and conjugate transpose operations, respectively. \( \text{tr}(\cdot) \) is the trace of a matrix; \( \| \cdot \| \) denotes the Euclidean norm of a vector; \( \| \cdot \|_F \) represents the Frobenius norm of a matrix; \( E_x(\cdot) \) represents the statistical expectation with respect to \( x \); \( X_{kl} \) represents the \((k, l)\)-th element of \( X \); \( \mathbf{I} \) and \( \mathbf{0} \) denote an identity matrix and a zero matrix, respectively, with appropriate dimensions; \( X \succeq 0 \) denotes a positive semidefinite matrix; \( \otimes \) and \( \circ \) are Kronecker and Hadamard matrix products, respectively; \( \mathcal{I}(\cdot) \) represents the mutual information; \( \Re \) and \( \Im \) are the real and image parts of a complex value; \( \log(\cdot) \) is used for the base two logarithm.

II. System Model and Problem Formulation

In this section, we present system and channel models for mmWave MIMO systems, and then formulate the hybrid precoding design as a matrix factorization problem. Finally, we briefly introduce a few notations on complex matrix derivatives.

A. System Model

Consider a point-to-point mmWave MIMO system, where a transmitter with \( N_t \) antennas sends \( N_s \) data streams to a receiver with \( N_r \) antennas. The number of RF chains at the transmitter is \( N_{rf} \), which satisfies \( N_s \leq N_{rf} \leq N_t \). We consider the hybrid precoding scheme, where \( N_s \) data streams are first precoded using a digital precoder, and then shaped by an analog precoder. The received baseband signal \( y \in \mathbb{C}^{N_r \times 1} \) can be written as

\[ y = H F_{RF} F_{BB} x + n \]

where \( H \in \mathbb{C}^{N_t \times N_t} \) is the channel matrix; \( F_{RF} \in \mathbb{C}^{N_r \times N_t} \) is the analog precoder with \( F \) being the constant modulus set; \( F_{BB} \in \mathbb{C}^{N_t \times N_r} \) is the digital precoder; \( x \in \mathbb{C}^{N_t \times 1} \) is the input data vector and \( n \in \mathbb{C}^{N_r \times 1} \) is independent and identically distributed (i.i.d.) circularly symmetric complex Gaussian noise with zero-mean and covariance \( \sigma^2 I \).
{θ_{i,t}} are drawn independently from the uniform distribution. Similarly, \( A_r \) and \( \text{diag}(α) \) are also full rank matrices with probability one. Therefore, the rank of \( H \) is given by
\[
\text{rank}(H) = \min\{L, N_r, N_t\}. \tag{8}
\]

C. Problem Formulation

A fundamental approach for hybrid precoding design is to maximize the input-output mutual information under the power and constant modulus constraints. Suppose that the mmWave receiver can optimally decode data using the received signal \( y \), then the hybrid precoding problem is formulated as
\[
\begin{align*}
\text{maximize} & \quad I(x; y) \\
\text{subject to} & \quad \text{tr}(F_{\text{RF}}^H F_{\text{RF}} F_{\text{BB}}) \leq P
\end{align*} \tag{9}
\]
where \( I(x; y) \) is given in (3). \( P \) is the transmit power constraint and \( \mathcal{U} = \mathbb{F}^{N_r \times N_t} \) is the feasible set of analog precoders. It is challenging to solve problem (9) directly due to two reasons: First, problem (9) is nonconvex because both \( I(x; y) \) and \( \mathcal{U} \) are neither convex nor concave with respect to \((F_{\text{RF}}, F_{\text{BB}})\). Second, iterative algorithms for problem (9) have to evaluate the objective function \( I(x; y) \) in each iteration, which can be very costly because \( I(x; y) \) has no closed form expressions.

To mitigate these difficulties and simplify the precoding design, we adopt the following matrix factorization formulation \([1]\), where hybrid precoders \((F_{\text{RF}}, F_{\text{BB}})\) are found by approximating the unconstrained optimal precoder \( F_{\text{opt}} \), i.e.,
\[
\begin{align*}
\text{minimize} & \quad \|F_{\text{opt}} - F_{\text{RF}} F_{\text{BB}}\|^2_F \\
\text{subject to} & \quad \text{tr}(F_{\text{RF}}^H F_{\text{RF}} F_{\text{BB}}) \leq P
\end{align*} \tag{10}
\]
The unconstrained optimal precoder \( F_{\text{opt}} \) is given by \([2]\), \([3]\).
\[
F_{\text{opt}} = \text{maximize}_{F \in \mathcal{F}} I(x; y) \tag{11}
\]
where \( \mathcal{F} = \{F \mid \text{tr}(F^H F) \leq P \} \).

D. Preliminaries on Complex Matrix Derivatives

The problems investigated in this paper are nonlinear optimization with complex matrix variables, thus we briefly introduce a few definitions on complex matrix derivatives. For a univariate function \( f(x) : \mathbb{C} \to \mathbb{R} \), the definition of the complex derivative is given in \([17]\):
\[
\frac{\partial f}{\partial x} \triangleq \frac{1}{2} \left[ \frac{\partial f}{\partial \text{Re}(x)} - j \frac{\partial f}{\partial \text{Im}(x)} \right]. \tag{12}
\]
\[
\frac{\partial f}{\partial x^*} \triangleq \frac{1}{2} \left[ \frac{\partial f}{\partial \text{Re}(x)} + j \frac{\partial f}{\partial \text{Im}(x)} \right]. \tag{13}
\]
For a multivariate function \( f(X) : \mathbb{C}^{n \times r} \to \mathbb{R} \), the partial derivatives with respect to \( X \) and \( X^* \) are matrices
\[
\frac{\partial f}{\partial X} \triangleq \begin{bmatrix} \frac{\partial f}{\partial X_{kl}} \end{bmatrix} \text{ and } \frac{\partial f}{\partial X^*} \triangleq \begin{bmatrix} \frac{\partial f}{\partial X_{kl}^*} \end{bmatrix}\tag{14}
\]
where \( X_{kl} \) denotes the \((k, l)\)-th element of \( X \). In addition, the complex gradient matrix \( \nabla_X f(X) \) is defined as
\[
\nabla_X f(X) \triangleq \frac{\partial f}{\partial X^*}. \tag{15}
\]
Let \( X_1 \in \{X, X^*\} \) and \( X_2 \in \{X, X^*\} \), then the complex Hessian of \( f(X) \) with respect to \( X_1 \) and \( X_2 \) is defined in \([17]\):
\[
\mathcal{H}_{X_1, X_2} f \triangleq \frac{\partial}{\partial \text{vec}^T(X_1)} \left[ \frac{\partial f}{\partial \text{vec}(X_2)} \right]^T. \tag{16}
\]

III. STRUCTURES OF THE HYBRID PRECODING PROBLEM

In this section, we first present a sufficient condition and a necessary condition, under which hybrid precoding schemes can realize any unconstrained optimal precoder exactly. Then we prove that the power constraint \( \text{tr}(F_{\text{RF}}^H F_{\text{RF}} F_{\text{BB}}) \leq P \) in problem (10) can be removed without loss of local and/or global optimality.

A. Optimality of Hybrid Precoding Schemes

The hybrid precoding scheme offers a tradeoff between performance gain and hardware complexity, and its performance is bounded by the unconstrained optimal precoder. When the hybrid precoding scheme can realize any unconstrained optimal precoder exactly, it is an optimal scheme. Then a fundamental question arises:

- Question 1: under what conditions can hybrid precoding schemes realize unconstrained optimal precoders exactly?

In other words, we want to find necessary and/or sufficient conditions, under which there exist \((F_{\text{RF}}, F_{\text{BB}})\) such that \( F_{\text{RF}} \in \mathcal{U} \) and \( F_{\text{opt}} = F_{\text{RF}} F_{\text{BB}} \). The best known result related to this question was shown in \([2]\) and \([3]\). It states that when the number of data streams \( N_s \) satisfies \( N_s \leq \frac{1}{2} N_{tt} \), we can construct analog and digital precoders to realize any unconstrained optimal precoder with dimensions \( N_t \times N_s \). However, this result sacrifices the number of data streams to satisfy \( F_{\text{opt}} = F_{\text{RF}} F_{\text{BB}} \). In order to achieve the maximum degree of freedom, we should transmit \( \min\{\text{rank}(H), N_{tt}\} \) data streams rather than \( \frac{1}{2} N_{tt} \) data streams. This motivates us to reconsider Question 1 under \( N_s = \min\{\text{rank}(H), N_{tt}\} \).

First, we transform Question 1 into another existence problem through the following proposition.

**Proposition 1:** Suppose \( F_{\text{RF}} \) is a full rank matrix, then the following two statements are equivalent:

1. There exists \((F_{\text{RF}}, F_{\text{BB}})\) such that \( F_{\text{RF}} \in \mathcal{U} \) and \( F_{\text{opt}} = F_{\text{RF}} F_{\text{BB}} \).
2. There exists a full rank square matrix \( S \in C^{N_{tt} \times N_{tt}} \) such that \( U_{RF} S \in \mathcal{U} \).

Here \( U_{RF} \in C^{N_{tt} \times N_{tt}} \) is a semi-unitary matrix whose columns are left singular vectors of \( F_{\text{opt}} \).

**Proof:** See Appendix A. \( \square \)

Based on Proposition 1, our original problem is equivalent to the existence problem of a full rank square matrix \( S \) satisfying \( U_{RF} S \in \mathcal{U} \). By exploiting the inherent structure of the mmWave MIMO channel, we provide a sufficient condition to guarantee the existence of such full rank matrix \( S \). The main idea is similar to Theorem 1 of \([13]\).

**Proposition 2:** When the number of paths \( L \) satisfies \( L \leq \min\{N_r, N_t, N_{tt}\} \), there exists a full rank matrix \( S \) satisfying \( A_4 = U_{RF} S \in \mathcal{U} \), where \( A_4 \) is the array steering matrix given in \([7]\).
Combining Propositions 1 and 2, we conclude that when \( L \leq \min\{N_t, N_r, N_{rf}\} \), hybrid precoding schemes can realize any unconstrained optimal precoder \( \mathbf{F}_{\text{opt}} \) exactly. However, the sufficient condition in Proposition 2 does not always hold in practice because the number of paths may be greater than the number of RF chains. In the rest of this subsection, we propose a necessary condition for the existence of \( \mathbf{S} \) satisfying \( \mathbf{U}_F \mathbf{S} \in \mathcal{U} \), and the proposed necessary condition is independent of \( L, N_{rf}, N_r \) and \( N_t \).

We first rewrite \( \mathbf{U}_F \mathbf{S} \in \mathcal{U} \) as
\[
\left[ \mathbf{U}_F \mathbf{s}_F \mathbf{U}_F^H \right]_{kk} = \frac{1}{N_t}, \quad k = 1, \ldots, N_t, \ell = 1, \ldots, N_{rf}
\]  
(17)
where \( \mathbf{s}_F \) is the \( \ell \)th column of \( \mathbf{S} \). Combining condition (17) and \( \text{rank}(\mathbf{S}) = N_{rf} \), the original problem is equivalent to the existence of \( N_{rf} \) linear independent solutions \( \{\mathbf{s}_F\}_{i=1}^{N_{rf}} \) to the following system of quadratic equations:
\[
\left[ \mathbf{U}_F \mathbf{s} \mathbf{U}_F^H \right]_{kk} = \frac{1}{N_t}, \quad k = 1, \ldots, N_t.
\]  
(18)
Unfortunately, problem (18) is intractable because checking the existence of solutions to a general quadratic system is \( \mathcal{N}P \)-hard [19]. Instead, we investigate necessary conditions for the existence of solutions to (18).

The main idea is to transform (18) into a linear system by semidefinite programming. Define \( \mathbf{Z} = N_t \mathbf{s}_F \mathbf{U}_F^H \), the quadratic system (18) can be written as
\[
\left[ \mathbf{U}_F \mathbf{Z} \mathbf{U}_F^H \right]_{kk} = 1, \quad k = 1, \ldots, N_t, \mathbf{Z} \succeq 0, \quad \text{rank}(\mathbf{Z}) = 1.
\]  
(19)
Furthermore, according to
\[
\text{vec}(\mathbf{U}_F \mathbf{Z} \mathbf{U}_F^H) = (\mathbf{U}_F^\otimes \mathbf{U}_F) \text{vec}(\mathbf{Z})
\]  
(20)
equations (19) is expressed more compactly as
\[
\mathbf{K}_F \text{vec}(\mathbf{Z}) = 1, \quad \mathbf{Z} \succeq 0, \quad \text{rank}(\mathbf{Z}) = 1
\]  
(21)
where the \( \ell \)th row of \( \mathbf{K}_F \) is chosen as the \( \ell \)th row of \( \mathbf{U}_F^\otimes \mathbf{U}_F \). Through some standard algebraic manipulations, we can express \( \mathbf{K}_F \) as
\[
\mathbf{K}_F = \left[ \text{diag}(\mathbf{u}_t^*) \mathbf{U}_F, \ldots, \text{diag}(\mathbf{u}_{N_t}^*) \mathbf{U}_F \right]
\]  
(22)
where \( \mathbf{u}_t \) represents the \( \ell \)th column of \( \mathbf{U}_F \).

The main idea for solving equations (21) is the nonlinear constraints \( \mathbf{Z} \succeq 0 \) and \( \text{rank}(\mathbf{Z}) = 1 \), which restrict solutions of \( \mathbf{K}_F \text{vec}(\mathbf{Z}) = 1 \) with a certain structure. Therefore, we first relax the nonlinear constraints and focus on the linear system \( \mathbf{K}_F \text{vec}(\mathbf{Z}) = 1 \). Clearly, if equations (21) has \( N_{rf} \) linear independent solutions, then \( \mathbf{K}_F \text{vec}(\mathbf{Z}) = 1 \) should have at least \( N_{rf} \) linear independent solutions. Based on this observation, the following proposition provides a necessary condition for the existence of a full rank \( \mathbf{S} \) such that \( \mathbf{U}_F \mathbf{S} \in \mathcal{U} \).

**Proposition 3:** If there exist a full rank square matrix \( \mathbf{S} \) satisfying \( \mathbf{U}_F \mathbf{S} \in \mathcal{U} \), then
\[
\text{rank}(\mathbf{K}_F) \leq N_{rf}^2 - N_{rf} + 1
\]  
(23)
Proof: See Appendix A.

Note that we can compute \( \text{rank}(\mathbf{K}_F) \) without the knowledge of \( \mathbf{F}_{\text{opt}} \) because its left singular vectors \( \mathbf{U}_F \) can always be chosen as the first \( N_{rf} \) columns of \( \mathbf{V}_H \), with \( \mathbf{V}_H \in \mathcal{C}^{N_r \times N_t} \) being the right singular vectors of \( \mathbf{H} \) [12, Proposition 2]. Therefore, when the transmitter has perfect channel state information, it can construct \( \mathbf{K}_F \) and check whether \( \text{rank}(\mathbf{K}_F) \leq N_{rf}^2 - N_{rf} + 1 \) holds. If the necessary condition does not hold, then hybrid precoding schemes cannot realize unconstrained optimal precoders exactly.

When the sufficient condition in Proposition 2 does not hold, \( \mathbf{K}_F \) is usually a full rank matrix. In this case, we derive the minimum number of RF chains required for hybrid precoding to achieve the performance of unconstrained optimal precoders.

**Corollary 1:** When \( \mathbf{K}_F \) is a full rank matrix, it requires at least \( \sqrt{N_t - \frac{3}{4} + \frac{1}{2}} \) RF chains for hybrid precoding schemes to realize unconstrained optimal precoders exactly.

**Proof:** Since \( \mathbf{K}_F \) is a full rank matrix, \( \text{rank}(\mathbf{K}_F) = \min\{N_t, N_{rf}^2\} \). Inserting \( \text{rank}(\mathbf{K}_F) \) into \( \text{rank}(\mathbf{K}_F) \leq N_{rf}^2 - N_{rf} + 1 \) and using quadratic formula, we obtain
\[
N_{rf} \geq \sqrt{N_t - \frac{3}{4} + \frac{1}{2}}.
\]  
(24)
This completes the proof.

### B. Structures of the Matrix Factorization Formulation

Given the unconstrained optimal precoder \( \mathbf{F}_{\text{opt}} \), the matrix factorization problem (10) belongs to the class of polynomial optimization: The objective function \( ||\mathbf{F}_{\text{opt}} - \mathbf{F}_R \mathbf{F}_B||_F^2 \) is a convex quartic function with respect to matrix variables \( \mathbf{F}_R, \mathbf{F}_B \), the power constraint \( \text{tr}(\mathbf{F}_B^H \mathbf{F}_R \mathbf{F}_R \mathbf{F}_B) \leq P \) is a convex quartic constraint, and the constant modulus constraints \( \mathcal{U} \) are nonconvex quadratic equality constraints. Such a problem is nonconvex due to the nonconvexity of \( \mathcal{U} \), and theoretical challenges of problem (10) are listed as follows:

1) The optimization variables \( \mathbf{F}_R \) and \( \mathbf{F}_B \) are coupled through the power constraint. Therefore, we cannot deploy the alternating minimization approach which requires separate variables in constraints. If we jointly optimize \( \mathbf{F}_R, \mathbf{F}_B \), the difficulty also lies in handling the coupled feasible region of problem (10).

2) More importantly, the bilinear mapping \( \mathbf{F}_R, \mathbf{F}_B \rightarrow \mathbf{F}_B \mathbf{F}_R \) is not a one-to-one mapping, thus \( \mathbf{F}_B \mathbf{F}_R \) and \( \langle \mathbf{F}_B, \mathbf{F}_R \mathbf{F}_B \rangle \) result in the same objective value, where \( \mathbf{S} \) is a diagonal matrix with unit modulus diagonal entries to ensure \( \mathbf{F}_R \mathbf{S} \in \mathcal{U} \). In other words, we should expect problem (10) to have infinite number of local minima and saddle points.

The first issue is fully addressed by the following theorem, which shows the equivalence between problems (10) and the following relaxed problem:

\[
\minimize_{\mathbf{F}_R \in \mathcal{U}, \mathbf{F}_B} ||\mathbf{F}_{\text{opt}} - \mathbf{F}_R \mathbf{F}_B||_F^2.
\]  
(25)

**Theorem 1:** If \( (\mathbf{F}_R, \mathbf{F}_B) \) is a KKT point of problem (25), then it satisfies \( \text{tr}(\mathbf{F}_B^H \mathbf{F}_R \mathbf{F}_R \mathbf{F}_B) \leq P \).

**Proof:** See Appendix A.
According to Theorem 1, any KKT point of problem (25) satisfies \( \text{tr}(RF_{BB}^H FRF_{BB}) \leq P \), thus the power constraint can be removed without loss of local and global optimality.

The rest of this paper focuses on solving problem (25). Problem (28) is an unconstrained least square problem where a given matrix \( F_{RF} \) is factorized into two complex matrices \( (RF, F_{BB}) \) under constant modulus constraints on \( RF \). Since \( (RF, F_{BB}) \rightarrow F_{RF}F_{BB} \) is not a one-to-one mapping, problem (28) has infinite number of saddle points, and this issue will be addressed in Section IV.

IV. CONSTANT MODULUS MATRIX FACTORIZATION

A. Problem Reformulation

First, we observe that for any given \( FRF \), problem (25) is a least square problem

\[
\minimize_{\hat{F}_{BB}} \|F_{opt} - FRF_{BB}\|^2_F. \tag{26}
\]

Suppose that \( FRF \) has full column rank, then the optimal solution of problem (26) is

\[
F_{BB} = F_{RF}^+ F_{opt} \tag{27}
\]

where \( F_{RF}^+ = (F_{RF}^H F_{RF})^{-1} F_{RF}^H \) is the Moore-Penrose pseudoinverse of \( FRF \). Inserting (27) into problem (25), \( F_{BB} \) is eliminated and we obtain the modified problem:

\[
\minimize_{F_{RF} \in U} f(F_{RF}) = \|F_{opt} - FRF F_{opt}\|^2_F. \tag{28}
\]

The following theorem guarantees that problems (25) and (28) are equivalent.

\textbf{Theorem 2:} If \( \hat{F}_{RF} \) is a KKT point of problem (28) and \( \hat{F}_{BB} = F_{RF}^+ F_{opt} \), then \( F_{RF}, F_{BB} \) is a KKT point of problem (25). Furthermore, \( F_{RF} \) is a globally optimal solution of problem (28) if and only if \( (F_{RF}, F_{BB}) \) is a globally optimal solution of problem (25).

\textbf{Proof:} See Appendix B.

The benefit of this reformulation is that problem (28) can be solved more efficiently because its search space is reduced from \( (RF, F_{BB}) \) to \( FRF \).

Problem (28) involves minimizing a polynomial with non-convex constant modulus constraints, which is difficult to handle. Note that the constant modulus constraints imply that only the phase of \( FRF \) can be changed. Therefore, instead of using \( FRF \) as the optimization variable, it is more convenient to optimize the phase of \( FRF \) directly. Let the phase of \( FRF \) be \( \Phi_{RF} \), i.e., \( FRF = \sqrt{\text{det}(\Phi_{RF})} e^{j\Phi_{RF}} \). Using \( \Phi_{RF} \) as the optimization variable and rewriting \( FRF \) as \( FRF(\Phi_{RF}) \), we can reformulate problem (28) as the following unconstrained minimization problem

\[
\minimize_{\Phi_{RF}} \psi(\Phi_{RF}) = \|F_{opt} - FRF(\Phi_{RF}) F_{RF}^+ (\Phi_{RF}) F_{opt}\|^2_F. \tag{29}
\]

Although (29) is a unconstrained problem, it is still not recommended to solve this problem directly because the objective function \( \psi(\Phi_{RF}) \) is ill-behaved: First, \( \psi(\Phi_{RF}) = \psi(\Phi_{RF} + S) \) for any rank one real matrix \( S \). Thus problem (29) has infinite number of local minima and saddle points; Second, the Hessian of \( \psi(\Phi_{RF}) \) at any point \( \Phi_{RF} \) is a singular matrix.

To show this, we expand \( \psi(\Phi_{RF} + S) \) at \( \Phi_{RF} \) using Taylor’s theorem:

\[
\psi(\Phi_{RF} + S) = \psi(\Phi_{RF}) + \nabla \psi(\Phi_{RF})^T vec(S) + \frac{1}{2} vec(S)^T \nabla^2 \psi(\Phi_{RF}) vec(S) + o(\|vec(S)\|^2) \tag{30}
\]

where \( \nabla \psi(\Phi_{RF}) \) and \( \nabla^2 \psi(\Phi_{RF}) \) are the gradient and Hessian of \( \psi(\Phi_{RF}) \) respectively, and \( o(\|vec(S)\|^2) \) is the Peano’s form of the remainder. For any nonzero rank one real matrix \( S \), we have \( \psi(\Phi_{RF} + S) = \psi(\Phi_{RF}) \), which implies

\[
\psi(\Phi_{RF} + S) = \psi(\Phi_{RF}) \tag{31}
\]

Therefore, \( \nabla^2 \psi(\Phi_{RF}) \) is a singular matrix.

We address these two issues by restricting the first row of \( FRF \) being a zero vector. Note that \( FRF \) can be partitioned into two blocks

\[
FRF = FRF - 1r = \begin{bmatrix} r \\ R \end{bmatrix} \tag{32}
\]

such that the first row of \( FRF \) is a zero vector, and \( \psi(F_{RF}) = \psi(\Phi_{RF}) \). Therefore, we can optimize \( \psi(\Phi_{RF}) \) over a special class of \( FRF \) satisfying

\[
FRF = \begin{bmatrix} 0 \\ \Phi \end{bmatrix} \tag{33}
\]

where \( \Phi \in \mathbb{R}^{(N-1) \times N_M} \). Using \( \Phi \) as the optimization variable, problem (29) is further reformulated as

\[
\minimize_{\Phi} \varphi(\Phi) = \psi\left\{ \begin{bmatrix} 0 \\ \Phi \end{bmatrix} \right\}. \tag{34}
\]

B. Gradient and Hessian

In this subsection, we derive the gradient and Hessian of \( \varphi(\Phi) \), which are the foundation for developing numerical algorithms to solve problem (28). Since the gradient and Hessian of \( \varphi(\Phi) \) depend on those of \( f(F_{RF}) \), we first provide the gradient and Hessian of \( f(F_{RF}) \) in the following lemma.

\textbf{Lemma 1:} The complex gradient and Hessian matrices of \( f(F_{RF}) \) are given by

\[
\nabla_{F_{RF}} f(F_{RF}) \triangleq \frac{\partial f(F_{RF})}{\partial F_{RF}} = -Z_1 F_{opt} Z_2^H \tag{35}
\]

\[
CH_{F_{RF}} f(F_{RF}) \triangleq \begin{bmatrix} H_{FRF,F_{RF}} f(F_{RF}) & H_{FRF,F_{RF}} f(F_{RF}) & H_{FRF,F_{RF}} f(F_{RF}) \end{bmatrix} \nabla_{F_{RF}} f(F_{RF}) = H_{FRF,F_{RF}} f(F_{RF}) \tag{36}
\]

where \( Z_1 = I - F_{RF} F_{RF}^+ \), \( Z_2 = F_{RF} F_{opt} \), and

\[
H_{FRF,F_{RF}} f(F_{RF}) = (Z_2 Z_1^H)^T \otimes Z_1 - [F_{RF} F_{RF}^+)^{-1}]^T \otimes Z_1 F_{opt} F_{opt}^+ Z_1^H
\]
where $G$ are given by $\ell_{N_1,N_\ell}$ hold for general nonconvex problems. In order to ensure the theorem, we construct the corresponding $\Phi_{RF}$ in (43). Then $\nabla^2 \Phi_{RF}$ is obtained by deleting the first row of $\nabla \Phi_{RF}$, and $\nabla^2 \Phi_{RF}$ is obtained by deleting the $(N_\ell + 1)\text{th}$ rows and columns of $\nabla^2 \Phi_{RF}$, with $\ell = 0, 1, \ldots, N_{\ell - 1}$. The gradient and Hessian of $\Phi_{RF}$ are given in the following theorem.

**Theorem 3:** The gradient and Hessian matrices of $\Phi_{RF}$ are given by

\[
\nabla \Phi_{RF} = 2[ G ]
\]
\[
\nabla^2 \Phi_{RF} = 2[ M ] - 2 \text{diag} \left( \text{vec}(R [ G ]) \right)
\]

where $G = \nabla F_{\ell} f (F_{RF})$ and $\nabla^2 \Phi_{RF}$ is obtained by deleting the first row of $\nabla \Phi_{RF}$, and $\nabla^2 \Phi_{RF}$ is obtained by deleting the $(N_\ell + 1)\text{th}$ rows and columns of $\nabla^2 \Phi_{RF}$, with $\ell = 0, 1, \ldots, N_{\ell - 1}$. The gradient and Hessian of $\Phi_{RF}$ are given in the following theorem.

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\]

where $G = \nabla F_{\ell} f (F_{RF})$ and $\nabla^2 \Phi_{RF}$ is obtained by deleting the first row of $\nabla \Phi_{RF}$, and $\nabla^2 \Phi_{RF}$ is obtained by deleting the $(N_\ell + 1)\text{th}$ rows and columns of $\nabla^2 \Phi_{RF}$, with $\ell = 0, 1, \ldots, N_{\ell - 1}$. The gradient and Hessian of $\Phi_{RF}$ are given in the following theorem.

The BFGS-based Algorithm

In this subsection, we propose a Broyden-Fletcher-Goldfarb-Shanno (BFGS)-based method to solve problem (34). The BFGS method is a well-known quasi-Newton algorithm for unconstrained optimization problems. It updates the current solution $\Phi_n$ to $\Phi_{n+1}$ by the following rule:

\[
\Phi_{n+1} = \Phi_n + \rho_n S_n
\]

where $S_n$ is the descent direction, and $\rho_n > 0$ is the stepsize. The descent direction $S_n$ is given by

\[
\text{vec}(S_n) = -B_n \text{vec}[\nabla \varphi(\Phi_n)].
\]

Here $B_n$ is a symmetric positive definite matrix which approximates the inverse of $\nabla^2 \varphi(\Phi_n)$. Note that the positive definiteness of $B_n$ ensures that $S_n$ is a descent direction, i.e.,

\[
\text{tr}[\nabla \varphi(\Phi_n) S_n] = -\text{vec}[\nabla \varphi(\Phi_n)]^T B_n \text{vec}[\nabla \varphi(\Phi_n)] < 0.
\]

The matrix $B_n$ is usually updated by the inverse BFGS formula

\[
B_{n+1} = \left( I - \frac{s_n y_n^T}{y_n^T s_n} \right) B_n \left( I - \frac{s_n y_n^T}{y_n^T s_n} \right)^T + \frac{s_n s_n^T}{y_n^T s_n}
\]

where $s_n = \text{vec}(\Phi_{n+1} - \Phi_n)$ and $y_n = \text{vec}(\nabla \varphi(\Phi_{n+1}) - \nabla \varphi(\Phi_n))$. Clearly, $B_{n+1}$ will inherit the positive definiteness of $B_n$ as long as $y_n^T s_n > 0$. However, the condition $y_n^T s_n > 0$ does not hold for general nonconvex problems. In order to ensure the positive definiteness of $B_{n+1}$, a cautious update rule for $B_n$ is proposed

\[
B_{n+1} = \begin{cases} \begin{aligned} B_n & \quad \text{if } \frac{y_n^T s_n}{\|s_n\|^2 \|\nabla \varphi(\Phi_n)\|_F} > \eta_{\text{bfgs}} \\ B_n & \quad \text{otherwise} \end{aligned} \end{cases}
\]

where $\eta_{\text{bfgs}} = 10^{-6}$ is a small constant. The update rule in (44) guarantees that $B_n$ is a positive definite matrix in each iteration, and thus $S_n$ should be a descent direction. However, due to the roundoff error, sometimes the direction generated by (41) may not be a descent direction. To address this numerical issue, we choose $S_n$ as

\[
\text{vec}(S_n) = \begin{cases} -B_n \text{vec}[\nabla \varphi(\Phi_n)] & \text{if } \xi_n > \delta_{\text{bfgs}} \\ -\text{vec}[\nabla \varphi(\Phi_n)] & \text{otherwise} \end{cases}
\]

where $\xi_n = \text{vec}[\nabla \varphi(\Phi_n)]^T B_n \text{vec}[\nabla \varphi(\Phi_n)]$ and $\delta_{\text{bfgs}} = 10^{-6}$ is a small constant.

After obtaining the descent direction $S_n$, we need to determine the stepsize $\rho_n$ such that the objective function is decreasing in each iteration. We propose a modified backtracking line search method, which is usually more efficient than the classic backtracking line search [21]. The main idea is to use $\rho_{n-1}$ as the initial guess of $\rho_n$, and then either increases or decreases it to find the largest $\rho_n \in G_n$ such that

\[
\nabla \varphi(\Phi_n + \rho_n S_n) \leq \nabla \varphi(\Phi_n) + \rho_n \beta_{\text{bfgs}} \text{tr}[\nabla \varphi(\Phi_n) S_n].
\]

where $\beta_{\text{bfgs}} \in [0, 0.5]$ is a constant to control the stepsize. Specifically, the stepsize $\rho_n$ is set as

\[
\rho_n = \begin{cases} \frac{2K_1 - 1}{2} \rho_{n-1} & \text{if } \rho_{n-1} \in G_n \\ \frac{1}{2} \rho_{n-1} & \text{if } \rho_{n-1} \notin G_n \end{cases}
\]

where $K_1 \geq 0$ is the smallest integer such that $2K_1 - 1 \rho_{n-1} \notin G_n$, and $K_2 \geq 0$ is the smallest integer such that $\frac{1}{2}K_2 - 1 \rho_{n-1} \in G_n$. The details of our BFGS-based algorithm is summarized in Algorithm 1.

**Algorithm 1** BFGS-based algorithm

1. Inputs: $F_{\text{opt}}$, $\Phi_0$ and $B_0$. Set $\rho_0 = 1$, $\beta_{\text{bfgs}} = 0.5$, and $\epsilon = 10^{-4}$.
2. For $n = 0, 1, 2, \ldots$ (outer iterations)
   - Determine the descent direction $S_n$ by (44).
   - Compute the stepsize $\rho_n$ via (46).
   - Update $\Phi_n$ to $\Phi_{n+1}$ according to (40).
   - If $\min \left\{ \frac{\|\nabla \varphi(\Phi_{n+1}) - \nabla \varphi(\Phi_n)\|_F}{\|\nabla \varphi(\Phi_{n+1})\|_F} \right\} < \epsilon$, stop.
   - Update $B_n$ to $B_{n+1}$ by (43).
3. Outputs: $F_{RF} = \frac{1}{\sqrt{N_T}} e^{j\Phi_{\text{RF}}}$, $F_{BB} = F_{RF}^\dagger F_{\text{opt}}^\dagger$.

According to [20], the BFGS-based algorithm proposed in Algorithm 1 can converge to a stationary point of problem (29), i.e., the limit of $\nabla \varphi(\Phi_n)$ satisfies

\[
\lim_{n \to \infty} \|\nabla \varphi(\Phi_n)\|_F = 0.
\]

The performance and convergence speed of Algorithm 1 depends on $\Phi_0$ and $B_0$. Here a good choice for the initial analog precoder phase is $\angle U_F$, where $\angle U_F$ is the phase of $U_F$. Then the corresponding $\Phi_0$ is set as

\[
\Phi_0 = [\angle U_F]_{2N_T}, -1[\angle U_F]_{1-N_T}.
\]
The initial inverse Hessian approximation $B_0$ will greatly affect the efficiency of Algorithm 1, thus we need to design it carefully. Let the eigendecomposition of $\nabla^2 \psi(\Phi_0)$ be

$$
\nabla^2 \psi(\Phi_0) = U_0 \Sigma_0 U_0^T
$$

(49)

where $U_0 \in \mathbb{C}^{(N_t-1)N_r \times (N_t-1)N_r}$ is a unitary matrix, and $\Sigma_0 \in \mathbb{R}^{(N_t-1)N_r \times (N_t-1)N_r}$ is a diagonal matrix with eigenvalues arranged in decreasing order. Then $B_0$ is given by

$$
B_0 = U_0 \hat{\Sigma}_0^{-1} U_0^T
$$

(50)

where $\hat{\Sigma}_0$ is a diagonal matrix with the $k$-th diagonal entry being

$$
[\hat{\Sigma}_0]_{k,k} = \begin{cases} 
|\Sigma_0|_{k,k} & \text{if } |\Sigma_0|_{k,k} \geq \delta_{\min} \\
\delta_{\min} & \text{otherwise.} 
\end{cases}
$$

(51)

Here the small constant $\delta_{\min}$ is set as $\delta_{\min} = 10^{-4}$. Since $\hat{\Sigma}_0^{-1}$ is a diagonal matrix with positive diagonal entries, the positive definiteness condition of $B_0$ is satisfied.

### D. Complexity Analysis

In this subsection, we discuss the per-iteration complexity of the proposed BFGS-based algorithm. Typically, the most time consuming operation in Algorithm 1 is evaluating $\psi(\Phi)$ and $\nabla \psi(\Phi)$. Therefore, it is important to analyze the complexity for $\psi(\Phi)$ and $\nabla \psi(\Phi)$. Given $\Phi$, we construct the corresponding analog precoder phase $\Phi_{RF}$ satisfying (33) and the analog precoder $F_{RF} = \frac{1}{\sqrt{N_t}} \psi(\Phi_{RF})$. Then we decompose $F_{RF}$ by QR decomposition

$$
F_{RF} = Q_{RF} R_{RF}
$$

(52)

where $Q_{RF} \in \mathbb{C}^{N_t \times N_t}$ is a unitary matrix, and $R_{RF} \in \mathbb{C}^{N_t \times N_t}$ is an invertible upper triangle matrix. In this way, we can compute $\psi(\Phi)$ efficiently as

$$
\psi(\Phi) = \|F_{opt}\|_F^2 - \|Q_{RF}^H F_{opt}\|_F^2.
$$

(53)

The QR decomposition requires $O(N_t N_r^2)$ flops, and computing $\|Q_{RF}^H F_{opt}\|_F^2$ requires $O(N_t N_r N_t)$ flops. Therefore, the complexity for computing $\psi(\Phi)$ is about $O(N_t N_r^2 + N_t N_r N_t)$.

The gradient matrix $\nabla \psi(\Phi)$ can be expressed as

$$
\nabla \psi(\Phi) = [\nabla \psi(\Phi_{RF})]_{2:N_t, \bullet}
$$

(54)

where $\nabla \psi(\Phi_{RF})$ can be expressed using QR decomposition

$$
\nabla \psi(\Phi_{RF}) = 2 \Im \left( [Q_{RF} Z_{RF} - F_{opt}] Z_{RF}^H (R_{RF}^{-1})^H Q_{RF} \right).
$$

(55)

Here $Z_{RF} = Q_{RF}^H F_{opt}$. Then the complexity for computing $\nabla \psi(\Phi)$ is about $O(N_t N_r^2 + N_t^3 + N_s N_r^2 + N_t N_r N_t N_s)$. Finally, since $B_n \in \mathbb{R}^{(N_t-1)N_r \times (N_t-1)N_r}$, the updating rule in (25) requires $O([N_t - 1]^2 N_r^2)$ flops. Then the per-iteration complexity of Algorithm 1 is given by

$$
O(N_t N_r^2 + N_t^3 + N_s N_r^2 + N_t N_r N_t + [N_t - 1]^2 N_r^2).
$$

(56)

## V. Simulation Results

### A. Average Euclidean Error Evaluation

The proposed BFGS-based algorithm solves a general constant modulus matrix factorization problem

$$
\text{minimize}_{F_{RF} \in \mathbb{C}^{N_t \times N_r}, F_{BB} \in \mathbb{C}^{N_r \times N_s}} \|F_{opt} - F_{RF} F_{BB}\|_F^2.
$$

(57)

Therefore, it is of interest to evaluate the performance of our proposed algorithm for arbitrary given matrix $F_{opt}$.

We generate $N$ independent samples $F_{opt}^{(i)} \in \mathbb{C}^{N_t \times N_r}$, $i = 1, 2, ..., N$ with i.i.d. zero-mean unit-variance complex Gaussian entries. Each sample is then normalized to satisfy

$$
\|F_{opt}^{(i)}\|_F^2 = N_r, \quad i = 1, 2, ..., N.
$$

(58)

Subsequently, we evaluate the performance of our proposed algorithm by the average Euclidean error, given by

$$
\frac{1}{N} \sum_{i=1}^{N} \|F_{opt}^{(i)} - F_{RF} F_{BB}^{(i)}\|_F^2.
$$

(59)

where $F_{RF}^{(i)} \in \mathbb{C}^{N_t \times N_r}$ and $F_{BB}^{(i)} \in \mathbb{C}^{N_r \times N_s}$ are outputs of Algorithm 1 with the given input $F_{opt}^{(i)}$.

We make head-to-head comparisons between our proposed BFGS-based algorithm and three existing algorithms, namely the manifold optimization based alternating minimization (MO–AltMin) [3], the iterative matrix decomposition (IMD) [7] and the hybrid design by alternating minimization (HD–AM) [5]. To the best of our knowledge, these three algorithms are the best existing algorithms based on the matrix factorization approach. Note that the authors in [3] and [7] claim that their proposed algorithms have significant performance gains over other existing algorithms, and the authors in [5] claim that the HD–AM algorithm provides the best solution among four different hybrid precoding algorithms proposed in [5]. Therefore, if the proposed BFGS-based algorithm can beat these algorithms, we believe it outperforms other existing algorithms based on the matrix factorization approach.

The matrix factorization based algorithms [3, 5, 7] involve a normalization procedure to ensure $\|F_{RF} F_{BB}\|_F^2 = P$. Since the mutual information is monotonically increasing...
with respect to \( \| F_{RF} F_{BB} \|_F \), this procedure will increase the achievable rate. However, when we choose the Euclidean error as the performance metric, the normalization procedure will decrease the overall performance because these algorithms and our proposed BFGS-based algorithm are designed to solve problem (57) without the equality power constraint. Therefore, for the sake of fairness, we do not execute the normalization for all algorithms in this subsection.

We set the number of samples as \( N = 500 \), and \( N_{rf} \) and \( N_s \) are restricted to be \( N_{rf} = N_s = 4 \). The initial analog precoders for these four algorithms are set as \( \mathcal{F}_{\text{opt}}^{(i)} \). The average Euclidean error and average running time of four algorithms are presented in Fig. 1 and Table I. From Fig. 1 and Table I, we have the following remarks:

1) The proposed BFGS-based algorithm and the MO-AltMin algorithm are guaranteed to converge to the stationary point of problem (57), while the HD–AM and IMD algorithms may not achieve this goal.

2) The proposed BFGS-based algorithm significantly outperforms the HD–AM, IMD and MO–AltMin algorithms in the whole range of \( N_T \). In addition, it consumes much lower computational time than the MO–AltMin algorithm.

3) The phenomenon that the BFGS-based algorithm outperforms the MO-AltMin algorithm can be explained as follows. For nonconvex problem (57), its stationary points can be local minimum (positive definite Hessian), local maximum (negative definite Hessian), or saddle point (indefinite Hessian). Most stationary points are saddle points in high dimensional space, and the objective value at the saddle point is usually worse than that at the local optimum [22]. In order to decrease the possibility for converging to the saddle point, we can 1) decrease the dimensions of the search space; 2) use Hessian information to avoid converging to the indefinite Hessian point [22], [24]. Since the proposed BFGS-based algorithm utilize these two techniques to avoid saddle points, its performance is better than that of the MO-AltMin algorithm.

B. Average Mutual Information Evaluation With Gaussian Inputs

We consider a \( 4 \times 72 \) MIMO system with \( N_{rf} = N_s = 4 \). The number of physical propagation paths is set as \( L = 8 \), and the signal-to-noise ratio (SNR) is defined as \( \text{SNR} = \frac{P}{\sigma^2} \). We generate \( N = 1000 \) channel realizations by (3), and evaluate the system performance by the following average mutual information with Gaussian inputs:

\[
\frac{1}{N} \sum_{i=1}^{N} \log \det \left( I + \sigma^{-2} H_i Q_i H_i^H \right)
\]

where \( H_i \) is the \( i \)th channel realization, and \( Q_i = F_{RF}^{(i)} F_{BB}^{(i)} H_{RF}^{(i)} H_{BB}^{(i)} \) with \( (F_{RF}^{(i)}, F_{BB}^{(i)}) \) being the analog and digital precoder solution corresponding to \( H_i \).

We set the performance of unconstrained optimal precoder as a benchmark, and then compare our proposed BFGS-based algorithm with the IMD algorithm, the HD–AM algorithm and the MO–AltMin algorithm. The unconstrained optimal precoder \( F_{\text{opt}} \) under Gaussian inputs can be obtained by the waterfilling (WF) algorithm, and all hybrid precoding algorithms in this subsection use the same \( F_{\text{opt}} \) to design analog and digital precoders. Moreover, the initial analog precoders of these algorithms are set as \( F_{RF} = \frac{1}{\sqrt{N_{rf}}} e^{jV_i \ast 1:N_{rf}} \), where \( [V_i]_{1:N_{rf}} \) is the first \( N_{rf} \) right singular vectors of \( H \).

Table II demonstrates the average mutual information with Gaussian inputs versus SNR for various algorithms. From Table II, we have the following remarks:

1) The proposed BFGS-based algorithm has about 10% performance gain over HD–AM and IMD algorithms in low SNR regimes because HD–AM and IMD algorithms are designed for full rank \( F_{\text{opt}} \). However, the unconstrained optimal precoder \( F_{\text{opt}} \) is not a full rank matrix in low SNR regimes. In addition, the HD–AM and IMD algorithms can be applied only when \( N_{rf} = N_s \), while our proposed BFGS-based algorithm and the MO-AltMin algorithm can work for arbitrary \( N_{rf} \) and \( N_s \).

2) When the unconstrained optimal precoder is obtained by WF algorithm and the performance metric is chosen as the average mutual information, the gain of our proposed BFGS-based algorithm over the MO-AltMin algorithm is not very significant compared with Fig. 1. However, as shown in Table I, our proposed BFGS-based algorithm is much faster than the MO-AltMin algorithm. Therefore, our proposed BFGS-based algorithm also has advantages over the MO-AltMin algorithm.

C. Average Mutual Information Evaluation With Finite-Alphabet Inputs

We first consider a \( 64 \times 64 \) MIMO system with \( N_{rf} = N_s = 4 \). The number of physical propagation paths is set as \( L = 6 \). The input signal is drawn from QPSK modulation, and SNR is defined as \( \text{SNR} = \frac{P}{\sigma^2} \). In addition, the system performance is measured by the average mutual information, which is averaged over 1000 channel realizations generated by (3).

![Fig. 2: Average mutual information versus SNR for different algorithms in a 64 × 64 MIMO system with N_{rf} = N_s = 4.](image-url)
Table I: Average running time (in secs.) versus $N_T$ with 500 randomly generated full rank $F_{opt}$.

| $N_T$ | Proposed BFGS-based algorithm | HD–AM algorithm | IMD algorithm | MO–AltMin algorithm |
|-------|-------------------------------|----------------|-------------|---------------------|
| 32    | 0.014s                        | 0.008s         | 0.012s      | 0.349s              |
| 48    | 0.021s                        | 0.009s         | 0.013s      | 0.696s              |
| 64    | 0.034s                        | 0.014s         | 0.019s      | 1.226s              |
| 80    | 0.033s                        | 0.017s         | 0.020s      | 1.924s              |
| 96    | 0.149s                        | 0.022s         | 0.022s      | 5.429s              |

Table II: Average mutual information with Gaussian inputs versus SNR for various algorithms.

Among these algorithms, our proposed BFGS-based algorithm and the gradient ascent algorithm are designed for finite-alphabet inputs, and the remaining three algorithms are designed under Gaussian inputs. Specifically, the HD–AM and MO–AltMin algorithms decompose the WF optimal precoder into digital and analog precoders, and then evaluate the corresponding mutual information under finite-alphabet inputs.

Fig. 2 demonstrates the average mutual information versus SNR for different algorithms. The results in Fig. 2 imply three observations. First, our proposed BFGS-based algorithm has the potential to achieve the performance of unconstrained optimal precoders. Second, our algorithm has about 0.2 bps/Hz improvement compared to the gradient ascent algorithm. Since mmWave provide very large bandwidths, a gain of 0.2 bps/Hz would translate to a large increase in the effective data rate. Third, the proposed BFGS-based algorithm has about 3dB gain over the HD–AM and MO–AltMin algorithms. This is mainly because the unconstrained optimal precoder designed under Gaussian inputs will lead to significant performance loss when applying to finite-alphabet inputs.

Next, we consider a $32 \times 80$ MIMO system with $L = 8$, $N_{rf} = 6$ and $N_s = 4$. The input signal is drawn from QPSK modulation. In this case, the gradient ascent and HD–AM algorithms cannot work because they assume $N_s = N_{rf}$. Therefore, we only compare our proposed BFGS-based algorithm with the MO-AltMin Algorithm. The simulation result is shown in Fig. 3. Based on the results in Fig. 3, we have the following remarks:

- The proposed BFGS-based algorithm and the MO–AltMin Algorithm are more general than the gradient ascent and HD–AM algorithms because they can work when $N_s < N_{rf}$.
- Our proposed algorithm can achieve the performance of unconstrained optimal precoder in whole SNR regimes. In addition, the MO–AltMin algorithm with WF optimal precoder has about 2–3dB performance loss compared with the our proposed BFGS-based algorithm.

VI. Conclusion

This paper considers the hybrid precoding design for mmWave MIMO systems with finite-alphabet inputs. The precoding problem has been formulated as a matrix factorization problem with constant modulus constraints. We first proposed a sufficient and a necessary condition for the hybrid precoding scheme to achieve the performance of unconstrained optimal precoders. Next, we decoupled the constant modulus matrix factorization problem by showing that the power constraint can be removed without loss of local and/or global optimality. Then we proposed a BFGS-based method to solve the constant modulus matrix factorization problem. Numerical results have demonstrated the effectiveness of our proposed algorithm for hybrid precoding designs in mmWave MIMO systems.
APPENDIX A
PROOFS OF PROPOSITIONS 1–3 AND THEOREM 1

Proof of Proposition 1: If there exists a full rank square matrix S such that $U_F S \in \mathcal{U}$, we can construct $F_{RF}$ and $F_{BB}$ as

$$F_{RF} = U_F S, \quad F_{BB} = S^{-1} \Sigma_F V_F^H$$

where $\Sigma_F$ is a diagonal matrix with singular values of $F_{opt}$ arranged in decreasing order, and $V_F$ is a unitary matrix with right singular vectors of $F_{opt}$. Then

$$F_{RF} \in \mathcal{U}, \quad F_{RF} F_{BB} = U_F \Sigma_F V_F^H = F_{opt}.$$  \hspace{1cm} (61)

Conversely, if there exists $(F_{RF}, F_{BB})$ such that $F_{opt} = F_{RF} F_{BB}$, $C(F_{opt})$ is a subspace of $C(F_{RF})$, where $C(\cdot)$ represents the space spanned by columns of a matrix. Moreover, according to $F_{opt} = U_F \Sigma_F V_F^H$, the first rank($F_{opt}$) columns of $U_F$ form an orthogonal basis of $C(F_{opt})$. Since $C(F_{opt})$ is a subspace of $C(F_{RF})$, we can use the Gram-Schmidt algorithm to construct the remaining $N_{rf} - \text{rank}(F_{opt})$ columns of $U_F$ such that the columns of $U_F$ form an orthogonal basis of $C(F_{RF})$. Then there exists a full rank matrix $S$ satisfying $F_{RF} = U_F S \in \mathcal{U}$. This completes the proof.

Proof of Proposition 2: Let the SVD of $H$ be

$$H = U_H \Sigma_H V_H^H$$

where $U_H \in \mathbb{C}^{N_t \times \text{rank}(H)}$ is a unitary matrix with left singular vectors, $\Sigma_H \in \mathbb{C}^{\text{rank}(H) \times \text{rank}(H)}$ is a diagonal matrix with singular values arranged in decreasing order, and $V_H \in \mathbb{C}^{N_t \times \text{rank}(H)}$ is a unitary matrix with right singular vectors. Based on equation (5), when $L \leq \min(N_r, N_t), \text{rank}(H) = L$. Then the columns of $V_H$ form an orthogonal basis of $C(H^H)$. Moreover, since $H = A_t \text{diag}(\alpha) A_t^H$ and $\text{rank}(A_t) = L$, the columns of $A_t$ also form a basis of $C(H^H)$. Therefore, there exists a full rank square matrix $S \in \mathbb{C}^{L \times L}$ such that $A_t = V_H S \in \mathcal{U}$. The semi-unitary matrix $V_H$ has a close connection with the left singular vectors of $F_{opt}$. Specifically, the left singular vectors of $F_{opt}$ can always be chosen as the first $N_s$ columns of $V_H$ [12] Proposition 2, i.e.,

$$U_F = [V_H]_{*,1:N_s}.$$  \hspace{1cm} (64)

Therefore, when $L = N_s = \min\{L, N_{rf}\} \leq \min(N_r, N_t)$, we have $A_t = V_H S = U_F S \in \mathcal{U}$. Finally, $L = \min\{L, N_{rf}\} \leq \min(N_r, N_t)$ holds if and only if $L \leq \min(N_r, N_t, N_{rf})$. This completes the proof.

Proof of Proposition 3: We first rewrite the solutions of $K_F \text{vec}(Z) = 1$ as

$$\text{vec}(Z) = \xi_0 + \sum_{i=1}^{l} \alpha_i \xi_i.$$  \hspace{1cm} (65)

Here $\xi_0$ is a particular solution to $K_F \text{vec}(Z) = 1$, $\{\alpha_i \}_{i=1}^{l}$ are complex numbers, and $\{\xi_i \}_{i=1}^{l}$ is a basis of $\mathcal{N}(K_F)$, where $\mathcal{N}(\cdot)$ represents the null space of a matrix. Since the nonlinear equations

$$K_F \text{vec}(Z) = 1, \quad Z \succeq 0, \quad \text{rank}(Z) = 1$$

have $N_{rf}$ linear independent solutions, the dimension of $\mathcal{N}(K_F)$ should be at least $N_{rf} - 1$, which implies

$$\dim[\mathcal{N}(K_F)] = N_{rf} - \text{rank}(K_F) \geq N_{rf} - 1.$$  \hspace{1cm} (66)

This completes the proof.

Proof of Theorem 1: If $(\hat{F}_{RF}, \hat{F}_{BB})$ is a KKT point of problem (28), then it satisfies the following KKT conditions:

$$-(F_{opt} - \hat{F}_{RF} \hat{F}_{BB}) \hat{F}_{BB}^H + \Upsilon \circ \hat{F}_R = 0$$

$$F_{RF}^* (F_{opt} - \hat{F}_{RF} \hat{F}_{BB}) = 0$$

$$F_{RF}^* \circ \hat{F}_R = \frac{1}{N_t} I$$  \hspace{1cm} (68), (69), (70)

where $\Upsilon_{ij}$ is the Lagrangian multiplier associated with the equality constraint $[\hat{F}_{RF}]_{ij} [\hat{F}_{BB}]_{ij} = 1/N_t$. Suppose $F_{RF}$ has full column rank, then equation (69) becomes

$$F_{BB} = F_{RF}^+ F_{opt}.$$  \hspace{1cm} (71)

Conversely, if there exists $(\hat{F}_{RF}, \hat{F}_{BB})$ such that $F_{opt} = \hat{F}_{RF} \hat{F}_{BB}$, $C(F_{opt})$ is a subspace of $C(F_{RF})$, where $C(\cdot)$ represents the space spanned by columns of a matrix. Moreover, according to $F_{opt} = U_F \Sigma_F V_F^H$, the first rank($F_{opt}$) columns of $U_F$ form an orthogonal basis of $C(F_{opt})$. Since $C(F_{opt})$ is a subspace of $C(F_{RF})$, we can use the Gram-Schmidt algorithm to construct the remaining $N_{rf} - \text{rank}(F_{opt})$ columns of $U_F$ such that the columns of $U_F$ form an orthogonal basis of $C(F_{RF})$. Then there exists a full rank matrix $S$ satisfying $F_{RF} = U_F S \in \mathcal{U}$. This completes the proof.

APPENDIX B
PROOFS OF THEOREM 2–3 AND LEMMA 1

Proof of Theorem 2: The KKT conditions of problem (28) are given by

$$-(F_{opt} - \hat{F}_{RF} \hat{F}_{BB}) \hat{F}_{BB}^H + \Upsilon \circ \hat{F}_R = 0$$

$$F_{RF}^* (F_{opt} - \hat{F}_{RF} \hat{F}_{BB}) = 0$$

$$F_{RF}^* \circ \hat{F}_R = \frac{1}{N_t} I$$  \hspace{1cm} (75), (76), (77)

where $\Upsilon_{ij}$ is the Lagrangian multiplier associated with the equality constraint $[\hat{F}_{RF}]_{ij} [\hat{F}_{BB}]_{ij} = 1/N_t$. Suppose $F_{RF}$ is a KKT point of problem (28) and $\hat{F}_{BB} = F_{RF}^+ F_{opt}$, $(\hat{F}_{RF}, \hat{F}_{BB})$
satisfies equations (76) and (77). Moreover, \( \hat{F}_{RF} \) satisfies the following stationarity condition of problem (25):

\[-(I - \hat{F}_{RF}F_{RF}^+)^H F_{opt}^H + \mathbf{Y} \circ \hat{F}_{RF} = 0 \]  

(78)

where \(- (I - \hat{F}_{RF}F_{RF}^+)^H F_{opt}^H + \hat{F}_{RF}^H \) is the complex gradient of \( f(F_{RF}) \), and \( \mathbf{Y} \) is the lagrangian multiplier. Inserting \( \hat{F}_{BB} = F_{RF}^+ F_{opt} \) into equation (78), it becomes

\[-(\hat{F}_{opt} - \hat{F}_{RF} \hat{F}_{BB}) \hat{F}_{BB}^H + \mathbf{Y} \circ \hat{F}_{RF} = 0 \]  

(79)

which is exactly the stationarity condition of problem (25) given in equation (73). Therefore, the KKT point of problem (25) satisfies equations (75)-(77) and it is a KKT point of problem (25).

Suppose that \( \hat{F}_{RF} \) is a globally optimal solution of problem (25) and \( F_{BB} = F_{RF}^+ F_{opt} \), then

\[ r(\hat{F}_{RF}, \hat{F}_{BB}) = f(\hat{F}_{RF}) \]  

(80)

where \( r(F_{RF}, F_{BB}) = \|F_{opt} - F_{RF} F_{BB}\|^2_F \). We further assume \( (\hat{F}_{RF}, \hat{F}_{BB}) \) is not a globally optimal solution of problem (25), i.e., there exists a feasible solution \( (F_{RF}, F_{BB}) \) such that \( r(F_{RF}, F_{BB}) < r(\hat{F}_{RF}, \hat{F}_{BB}) \). Since for any given \( F_{BB} \), \( f(F_{RF}) \leq r(F_{RF}, F_{BB}) \), we have

\[ f(F_{RF}) \leq r(F_{RF}, F_{BB}) < r(\hat{F}_{RF}, \hat{F}_{BB}) = f(\hat{F}_{RF}) \]  

(81)

which is a contradiction to the fact that \( \hat{F}_{RF} \) is a globally optimal solution of problem (25). Therefore, \( (\hat{F}_{RF}, \hat{F}_{BB}) \) is a globally optimal solution of problem (25).

Conversely, suppose that \( (\hat{F}_{RF}, \hat{F}_{BB}) \) is a globally optimal solution of problem (25), then

\[ r(\hat{F}_{RF}, \hat{F}_{BB}) = f(\hat{F}_{RF}) \]  

(82)

Similarly, we assume \( \hat{F}_{RF} \) is not a globally optimal solution of problem (25), i.e., there exists a feasible \( \hat{F}_{RF} \) such that \( f(F_{RF}) < f(\hat{F}_{RF}) \). Let \( F_{BB} = F_{RF}^+ F_{opt} \), then

\[ f(F_{RF}) = r(F_{RF}, F_{BB}) < f(\hat{F}_{RF}, \hat{F}_{BB}) = r(F_{RF}, F_{BB}) \]  

(83)

which is a contradiction to the fact that \( (\hat{F}_{RF}, \hat{F}_{BB}) \) is a globally optimal solution of problem (25). Therefore, \( \hat{F}_{RF} \) is a globally optimal solution of problem (25). This completes the proof.

**Proof of Lemma 1:** We first compute the complex gradient matrix \( \nabla_{F_{RF}} f(F_{RF}) \). Note that \( f(F_{RF}) \) can be rewritten as

\[ f(F_{RF}) = \|F_{opt}\|^2_F - \text{tr}(F_{RF}^+ F_{opt} F_{opt}^H F_{RF}). \]  

(84)

Then the differential of \( f(F_{RF}) \) is given by

\[ df(F_{RF}) = -\text{tr}(dF_{RF}^+ F_{opt} F_{opt}^H dF_{RF}) - \text{tr}(F_{RF}^+ F_{opt} F_{opt}^H dF_{RF}). \]  

(85)

The differential of \( F_{RF}^+ = (F_{RF}^H F_{RF})^{-1} F_{RF}^H \) in equation (85) can be computed as follows:

\[ df_{RF} = df(F_{RF}) = (F_{RF}^H F_{RF})^{-1} F_{RF}^H - F_{RF}^+ df_{RF} \]  

(86)

where the second equality in (86) holds due to the following equation

\[ d(A^{-1}) = -A^{-1} dA A^{-1}. \]  

(87)

Inserting (86) into (85), we have

\[ df(F_{RF}) \triangleq \text{tr}(df_{RF}^H \nabla_{F_{RF}} f(F_{RF}) + \nabla_{F_{RF}} f(F_{RF})^H df_{RF}) \]

\[ = -\text{tr}(df_{RF}^H Z_1 F_{opt} Z_2^H) - \text{tr}(Z_2 F_{opt}^H Z_1^H dF_{RF}) \]  

(88)

where \( Z_1 = I - F_{RF} F_{RF}^+ \) and \( Z_2 = F_{RF}^+ F_{opt} \). Thus the complex gradient matrix of \( f(F_{RF}) \) is \( \nabla_{F_{RF}} f(F_{RF}) = -Z_1 F_{opt} Z_2^H \).

Next, we compute the Hessian matrix \( \nabla^2_{F_{RF}} f(F_{RF}) \). Since \( \nabla^2_{F_{RF}} f(F_{RF}) \) contains four blocks, we first determine \( \nabla^2_{F_{RF}, F_{RF}} f(F_{RF}) \) and \( \nabla^2_{F_{RF}, F_{opt}} f(F_{RF}) \). According to the definition

\[ \text{vec} [d\nabla_{F_{RF}} f(F_{RF})] = \hat{H}_{F_{RF}} F_{opt} f(F_{RF}) \text{vec}(dF_{RF}) \]

(90)

we obtain \( \hat{H}_{F_{RF}, F_{RF}} f(F_{RF}) \) and \( \hat{H}_{F_{RF}, F_{opt}} f(F_{RF}) \) through computing the differential of \( \nabla_{F_{RF}} f(F_{RF}) \):

\[ \text{vec}[d\nabla_{F_{RF}} f(F_{RF})] = -dZ_1 F_{opt} F_{opt} Z_2^H + Z_1 F_{opt} Z_2^H. \]  

(91)

Then we vectorize \( d\nabla_{F_{RF}} f(F_{RF}) \) using the formula

\[ \text{vec}(AXB) = (B^T \otimes A) \text{vec}(X) \]

\[ = \text{vec}((Z_1 F_{opt} Z_2^H)^T \otimes (F_{RF}^H)^T) \text{vec}(dF_{RF}) \]  

(94)

\[ = \text{vec}((Z_1 F_{opt} Z_2^H)^T \otimes (F_{RF}^H)^T) \text{vec}(dF_{RF}) \]  

(95)

\[ = \text{vec}((Z_1 F_{opt} Z_2^H)^T \otimes (F_{RF}^H)^T) \text{vec}(dF_{RF}) \]  

(96)

where \( K_{N_t, N_t} \) is the commutation matrix such that \( \text{vec}(dF_{RF}) = K_{N_t, N_t} \text{vec}(dF_{RF}) \). Then we can obtain

\[ \hat{H}_{F_{RF}, F_{RF}} f(F_{RF}) = (Z_1 Z_2^H)^T \otimes Z_1 \]

\[ = (Z_1 Z_2^H)^T \otimes Z_1. \]  

(98)

\[ \hat{H}_{F_{RF}, F_{RF}} f(F_{RF}) = (Z_1 Z_2^H)^T \otimes (F_{RF}^H)^T K_{N_t, N_t} \]

\[ + (F_{RF}^H)^T \otimes Z_1. \]  

(99)
The remaining two blocks \( \mathcal{H}_{F_{RF}} f (F_{RF}) \) and \( \mathcal{H}_{F_{RF}} f (F_{RF}) \) can be obtained via \( \mathcal{H}_{F_{RF}} f (F_{RF}) \) and \( \mathcal{H}_{F_{RF}} f (F_{RF}) \). Since
\[
\begin{align*}
\frac{\partial f (F_{RF})}{\partial F_{RF}} &= \mathcal{H}_{F_{RF}} f (F_{RF}) \text{vec}(dF_{RF}) \text{vec}(dF_{RF}) \\
&\quad + \mathcal{H}_{F_{RF}} f (F_{RF}) \text{vec}(dF_{RF}) + \mathcal{H}_{F_{RF}} f (F_{RF}) \text{vec}(dF_{RF}) \text{vec}(dF_{RF}) \text{vec}(dF_{RF})
\end{align*}
\]
can be expressed as
\[
\text{vec}(\frac{\partial f (F_{RF})}{\partial F_{RF}}) = \mathcal{H}_{F_{RF}} f (F_{RF}) \text{vec}(dF_{RF})
\]
This completes the proof.

Proof of Theorem 3: We first rewrite \( \psi (\Phi_{RF}) \) as the composition of \( f(F_{RF}) \) and \( F_{RF}(\Phi_{RF}) \), i.e.,
\[
\psi (\Phi_{RF}) = f(F_{RF}(\Phi_{RF})).
\]
Using the chain rule in differentiation, the differential of \( \psi (\Phi_{RF}) \) is
\[
d[\psi (\Phi_{RF})] = \text{tr}[\nabla_{F_{RF}} f (F_{RF})^H dF_{RF}(\Phi_{RF})]
\]
Inserting \( dF_{RF}(\Phi_{RF}) = jF_{RF} \circ d\Phi_{RF} \), \( d[\psi (\Phi_{RF})] \) is expressed as
\[
d[\psi (\Phi_{RF})] = j \text{tr}[\nabla_{F_{RF}} f (F_{RF})^H (F_{RF} \circ d\Phi_{RF})]
\]
where (107) holds due to the following equality
\[
\text{tr}[A^T (B \circ C)] = \text{tr}[(A \circ B)^T C].
\]
Then the gradient of \( \psi (\Phi_{RF}) \) can be obtained from (107):
\[
\nabla \psi (\Phi_{RF}) = j \nabla_{F_{RF}} f (F_{RF}) \circ F_{RF} - j \nabla_{F_{RF}} f (F_{RF}) \circ F_{RF}^*
\]
Next, we compute the Hessian of \( \psi (\Phi_{RF}) \). According to the definition
\[
\text{vec}[d^2 \psi (\Phi_{RF})] = \nabla^2 \psi (\Phi_{RF}) \text{vec}(d\Phi_{RF})
\]
we can obtain \( \nabla^2 \psi (\Phi_{RF}) \) by computing the differential of \( \text{vec}[\nabla \psi (\Phi_{RF})] \):
\[
\text{vec}[d^2 \psi (\Phi_{RF})] = 2 \text{tr} \{ \text{vec}[\nabla_{F_{RF}} f (F_{RF}) \circ F_{RF}^*] \}.
\]
Using the product rule in differentiation, \( \text{d} (\text{vec}[\nabla_{F_{RF}} f (F_{RF}) \circ F_{RF}^*]) \) is given by
\[
\text{d} (\text{vec}[\nabla_{F_{RF}} f (F_{RF}) \circ F_{RF}^*]) = \text{vec}[d \nabla_{F_{RF}} f (F_{RF})] \circ \text{vec}(F_{RF}^*)
\]
\[
\quad + \text{vec}[\nabla_{F_{RF}} f (F_{RF})] \circ \text{vec}(dF_{RF}^*)
\]
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