Fate of higher-order topological insulator under Coulomb interaction

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In this article, we study the influence of long-range Coulomb interaction on three-dimensional second-order topological insulator (TI) by renormalization group theory. We find that both the analysis method and conclusions in the recent Letter Phys. Rev. Lett. \textbf{127}, 176601 (2021) are unreliable. There are two problems in this Letter. Firstly, the characteristic described by the RG flows \( m \rightarrow \infty \) and \( D \rightarrow 0 \) can not be used as the criterion for transition from second-order TI to TI, since this characteristic could be essentially not induced by Coulomb interaction but only results from the trivial power counting contribution of fermion action. Indeed, this characteristic is satisfied even for free second-order TI. Second, the flow of \( B \) is not paid attention, which is very important and should be seriously studied. In this article, we analyze carefully the corrections for the flows of the model parameters induced by Coulomb interaction. We find that the sign of \( m \) changes but the sign of \( B \) holds if the initial Coulomb strength is large enough, while the sign of \( m \) holds but the sign of \( B \) changes if the initial Coulomb strength takes small values. These results indicate that second-order TI is unstable to trivial band insulator not only under strong Coulomb interaction but also under weak Coulomb interaction. We also study the effects of disorder scattering in second-order TI by renormalization group theory. According to the criterion in Phys. Rev. Lett. \textbf{127}, 176601 (2021), weak disorder drives second-order TI to TI. However, we find that second-order TI is robust against weak disorder, since weak disorder does not give qualitative modification for second-order TI. This result is consistent with recent studies based on other methods. Interplay of Coulomb interaction and disorder in second-order TI is also investigated.

\section{I. INTRODUCTION}

The studies about topological materials including topological insulators (TIs), topological superconductors, and various topological semimetals, have become one of the most important fields in condensed matter physics \textsuperscript{[1–11]}. Topological materials have critical potential applications, including quantum computation, thermoelectric devices \textit{etc.}, due to their fantastic properties. Dirac, Weyl, and Majorana fermions have been observed in some topological materials \textsuperscript{[1–11]}. These fermion excitations resemble the elementary particles in high-energy physics. Thus, topological materials could provide a platform to simulate the concepts and phenomena in high-energy physics. In some topological materials, there are also unusual fermion excitations, such as semi-Dirac fermions, double-Weyl fermions, triple-Weyl fermions, multi-fold degenerate fermions \textit{etc.} which have not counterparts in high-energy physics \textsuperscript{[8–11]}. These unusual fermion excitations could result in novel physical behaviors.

Recently, higher-order topological materials attracted a lot of interest \textsuperscript{[12–30]}. For \( d \) dimensional TI, the system has \( d - 1 \) gapless Dirac edge states. For \( d \) dimensional second-order TI, the system hosts \( d - 2 \) gapless edge states. Concretely, three-dimensional (3D) second-order TI hosts 1D hinge states, and 2D second-order TI has 0D corner states.

Study about correlated interaction effects in topological materials is an important direction and attracted particular interest \textsuperscript{[31–77]}. For example, the theoretical studies showed that long-range Coulomb interaction induces singular fermion velocity renormalization for 2D Dirac fermions, which has been observed experimentally \textsuperscript{[31–53]}. Recently, Zhao \textit{et al.} studied the influence of long-range Coulomb interaction on 3D second-order TIs by renormalization group (RG) theory \textsuperscript{[73]}. They concluded that 3D second-order TIs are always unstable under Coulomb interaction. They showed that there are two types of transitions: second-order TI to TI and second-order TI to trivial band insulator.

However, after careful studies, we find that both of the analysis method and conclusions in Ref. \textsuperscript{[73]} are unreliable. There are two problems in Ref. \textsuperscript{[73]}. First, the characteristic described by the RG flows \( m \rightarrow \infty \) and \( D \rightarrow 0 \) can not be used as the criterion for transition from second-order TI to TI, since this characteristic could be essentially not induced by Coulomb interaction but only results from the trivial power counting contribution of fermion action. Actually, this characteristic is satisfied even for free second-order TI. Second, the flow of \( B \) is not paid attention, which is very important and should be seriously studied.

In the recent Comment \textsuperscript{[74]}, Lee and Yang have also pointed the problems in Ref. \textsuperscript{[73]} and indicated the conclusions are misleading. The problems in Ref. \textsuperscript{[73]} pointed by Lee and Yang and the ones pointed by us are similar to each other.
Whereas, there are also differences between the studies in the Comment [24] and the studies by us. Based on further calculations, they concluded that second-order TI is robust against weak Coulomb interaction. However, we find that the sign of $m$ holds but the sign of the parameter $B$ changes for weak Coulomb interaction. Namely $mB$ changes under the weak Coulomb interaction. It represents that second-order TI is unstable to trivial band insulator under weak Coulomb interaction.

We also study the influence of disorder on second-order TI. According to the criterion in Ref. [24], weak disorder drives second-order TI to TI. However, we find that weak disorder does not give qualitative modification for second-order TI through RG analysis. Namely, second-order TI is robust against weak disorder. Our result is consistent with recent studies about disorder effects in second-order TI based on other methods [24, 25]. The interplay of long-range Coulomb interaction and disorder is also investigated.

The rest of paper is structured as follows. The model for second-order TI with long-range Coulomb interaction is defined in Sec. I. In Sec. II we present the RG analysis for influence of Coulomb interaction on second-order TI based on numerical and analytical calculations. We compare interaction effects in related systems and give discussions for some related questions in Sec. III. In Sec. IV we analyze the effects of disorder in second-order TI. The interplay of long-range Coulomb interaction and disorder in second-order TI is studied in Sec. V. A brief summary is given in Sec. VI. The detailed calculations and derivations are presented in Appendices.

II. MODEL

The free action of fermions is

$$S_{\Psi} = \int \frac{dk_0}{2\pi} \frac{d^3k}{(2\pi)^3} \bar{\Psi}(k_0, k) (ik_0\gamma_0 + \mathcal{H}_f) \Psi(k_0, k),$$

(1)

where $\Psi$ is four-component spinor and $\bar{\Psi} = \Psi^\dagger \gamma_0$. The fermion Hamiltonian density takes the form

$$\mathcal{H}_f = i \left[ v(k_x\gamma_x + k_y\gamma_y) + v_z k_z\gamma_z + D(k_x^2 - k_y^2) \gamma_5 \right] + m - B_\perp k_\perp^2 - B_z k_z^2,$$

(2)

$v, v_z, D, B_\perp, B_z$ are model parameters. If $mB_\perp > 0$ and $D \neq 0$, it corresponds to second-order TI. If $mB_\perp, v_z > 0$ and $D = 0$, it corresponds to TI. If $mB_\perp < 0$, it corresponds to trivial band insulator. The matrices $\gamma_0, \gamma_x, \gamma_y, \gamma_z$ and $\gamma_5$ satisfy the anticommuting relation $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$.

The long-range Coulomb interaction between fermions can be written as

$$H_C = \frac{1}{4\pi} \int d^3x d^3x' \rho(x) e^2 \frac{1}{\epsilon|x - x'|} \rho(x'),$$

(3)

where $\rho(x) = \Psi^\dagger(x)\Psi(x)$ is the fermion density operator, $e$ electric charge, and $\epsilon$ dielectric constant. The Coulomb interaction can be decoupled by introducing a bosonic field $\phi$ through Hubbard-Stratonovich transformation. Accordingly, the Coulomb interaction between fermions can be described by the action of fermion-boson coupling as

$$S_{\Psi\phi} = ig \int d\tau d^3x \bar{\Psi}\gamma_0\Psi \phi,$$

(4)

where $g = e/\sqrt{\epsilon}$. In energy-momentum space, it takes the form

$$S_{\Psi\phi} = ig \int \frac{dk_0}{2\pi} \frac{d^3k}{(2\pi)^3} \bar{\Psi}(k_0, k) \gamma_0 \times \Psi(k_0, k) \phi(k_0 - k_0, k_1 - k_2).$$

(5)

The free action of boson field $\phi$ can be written as

$$S_\phi = \int \frac{dk_0}{2\pi} \frac{d^3k}{(2\pi)^3} \phi(k_0, k) \left(k_x^2 + \eta k_z^2\right) \phi(k_0, k),$$

(6)

where $\eta$ is used to describe the anisotropy of $\phi$.

III. RG RESULTS

In this article, we study the influence of long-range Coulomb interaction on second-order TI through RG method [78]. After detailed calculations and derivations shown in Appendices, we obtain the RG equations as following.

$$\frac{dv}{d\ell} = \alpha R_v v,$$

(7)

$$\frac{dv_z}{d\ell} = \alpha R_v v_z,$$

(8)

$$\frac{dm}{d\ell} = m + \alpha R_m,$$

(9)

$$\frac{dB_\perp}{d\ell} = -B_\perp + \alpha R_{B_\perp},$$

(10)

$$\frac{dB_z}{d\ell} = -B_z + \alpha R_{B_z},$$

(11)

$$\frac{D}{d\ell} = -D + \alpha R_D D,$$

(12)

$$\frac{d\alpha}{d\ell} = -\alpha^2 R_\alpha,$$

(13)

$$\frac{d\zeta}{d\ell} = \alpha R_\zeta \zeta,$$

(14)

where $\ell$ is the RG running parameter. The strength of Coulomb interaction $\alpha$ and the parameter $\zeta$ are defined as

$$\alpha = \frac{g^2}{4\pi^2 v^2 \eta},$$

(15)

$$\zeta = \frac{v_z}{v \sqrt{\eta}}.$$  

(16)

The transformations

$$\frac{m}{v\Lambda} \to m,$$

(17)
\[ \frac{B \Lambda}{v} \rightarrow B, \]  
\[ \frac{B}{\eta} \rightarrow B_z, \]  
\[ \frac{D \Lambda}{v} \rightarrow D, \]

have been utilized. \( R_r, R_m, R_{B_r}, R_{B_m}, R_B, R_m, \) and \( R \) are functions of \( m, B_{\perp}, B_z, D, \) and \( \zeta, \) whose concrete expressions can be found in Appendix I.

For free fermions, the RG equations become

\[
\begin{align*}
\frac{dv_f}{d\ell} &= 0, \\
\frac{dv_f}{d\ell} &= 0, \\
\frac{dm_f}{d\ell} &= m, \\
\frac{dB_{\perp}^f}{d\ell} &= -B_{\perp}, \\
\frac{dB_{z}^f}{d\ell} &= -B_z, \\
\frac{DD_f^f}{d\ell} &= -D_f^f,
\end{align*}
\]

where the superscript \( f \) corresponds to free fermions. The corresponding solutions are

\[
\begin{align*}
v_f(\ell) &= v_0, \\
v_{\perp f}(\ell) &= v_{\perp 0}, \\
m_f(\ell) &= m_0 e^{-\ell}, \\
B_{\perp f}(\ell) &= B_{\perp 0} e^{-\ell}, \\
B_{z f}(\ell) &= B_{z 0} e^{-\ell}, \\
D_f(\ell) &= D_0 e^{-\ell},
\end{align*}
\]

where \( v_0, v_{\perp 0}, B_{\perp 0}, B_{z 0}, \) and \( D_0 \) are initial values. It is easy to find that \( v_f(\ell) \) and \( v_{\perp f}(\ell) \) are fixed, and

\[
\begin{align*}
m_f(\ell) &\rightarrow \infty, \\
B_{\perp f}(\ell) &\rightarrow 0, \\
B_{z f}(\ell) &\rightarrow 0, \\
D_f(\ell) &\rightarrow 0,
\end{align*}
\]

in the lowest energy limit \( \ell \rightarrow \infty \). These results are directly related to the fact that \( v \) and \( v_z \) are coefficients of linear terms of momentum components, \( m \) is coefficient of zero power of momentum components, and \( B_{\perp}, B_z, D \) are coefficients of quadratic terms of momentum components.

The physical meaning of these results shown in Eqs. (27)-(32) and Eqs. (33)-(36) is: Taking the linear terms of Hamiltonian as reference energy, the zero power term becomes larger and larger, and the quadratic terms become smaller and smaller, with lowering of momentum. Namely,

\[ m/E_{\text{linear}}(k) \rightarrow \infty, \]  

in the limit \( k \rightarrow 0 \).

In Ref. [23], the characteristic described by the RG flows \( m(\ell) \rightarrow \infty \) and \( D(\ell) \rightarrow 0 \) in the limit \( \ell \rightarrow \infty \) are used as the criterion for transition from second-order TI to TI. It is clear that using this characteristic as the criterion for transition from second-order TI to TI is invalid. This characteristic is even satisfied for free second-order TI.

The flows of \( v(\ell), v_{\perp}(\ell), m(\ell), \alpha(\ell), D(\ell), \) and \( D(\ell)/D^f(\ell) \) with different initial values of Coulomb strength are presented in Figs. 1(a)-1(f) respectively. We find that \( v(\ell)/v_0 \) and \( v_{\perp}(\ell)/v_{\perp 0} \) flow to positive constants. Thus, Coulomb interaction results in qualitative corrections for \( v \) and \( v_{\perp} \). According to Figs. 1(c) and 1(d),

\[
\begin{align*}
\alpha(\ell) &\rightarrow \alpha^*, \\
\zeta(\ell) &\rightarrow \zeta^*,
\end{align*}
\]

where \( \alpha^* \) and \( \zeta^* \) are positive constants. As shown in Fig. 1(e), \( D(\ell) \) approaches to zero quickly with lowering of energy scale. From Fig. 1(f), we find that \( D(\ell)/D^f(\ell) \) flows to a positive constant value in the low energy regime. It indicates that \( D(\ell) \) only acquires quantitative correction in presence of Coulomb interaction, and takes qualitatively same behavior as \( D^f(\ell) \). It means that qualitative behaviors of the term \( D(k^2_z - k^2_y) \) are not changed under Coulomb interaction. As the term
with lowering of energy scale, where Figs. 2(c) and 2(e),

values of Coulomb strength in Figs. 2(a)-2(f) respectively. 

\[ B \]

obtain the qualitatively same behavior as

\[ m \]

where

\[ B \]

We presented the flows of \( m \) in Figs. 2(a)-2(f) with different initial values of Coulomb strength. Blue, red, green, black, and magenta lines correspond to the initial values \( \alpha_0 = 0.1, 0.2, 0.3, 0.4, 0.5 \) respectively: \( m_0 = 0.1, B_{10} = 1, B_{20} = 1, D_0 = 1, \zeta_0 = 0.1 \) are taken.

\( D(k_x^2 - k_y^2) \) is not changed qualitatively by Coulomb interaction, the transition from second-order TI to TI stated in Ref. 23 does not exist.

We presented the flows of \( m(\ell) \), \( m(\ell)/m^f(\ell) \), \( B_{1}(\ell) \), \( B_{2}(\ell) \), \( B_{1}(\ell)/B_{1}^f(\ell) \), \( B_{2}(\ell) \), \( B_{2}(\ell)/B_{2}^f(\ell) \) with different initial values of Coulomb strength in Figs. 2(a)-2(f) respectively. If the initial Coulomb strength is small, we find that

\[ m(\ell)/m^f(\ell) \to c_{m}^*, \quad \text{with} \quad c_{m}^* > 0, \]  

(44)

where \( c_{m}^* \) is a positive constant. It means that \( m(\ell) \) takes the qualitatively same behavior as \( m^f(\ell) \). According to Figs. 2(c) and 2(e),

\[ B_{1}(\ell) \to c_{B_{1}}^*, \quad \text{with} \quad c_{B_{1}}^* < 0, \]  

(45)

\[ B_{2}(\ell) \to c_{B_{2}}^*, \quad \text{with} \quad c_{B_{2}}^* < 0, \]  

(46)

with lowering of energy scale, where \( c_{B_{1}}^* \) and \( c_{B_{2}}^* \) are negative constants. We can find that signs of \( B_{1}(\ell) \) and \( B_{2}(\ell) \) change. In the low energy regime, it is easy to obtain

\[ B_{1}(\ell)/B_{1}^f(\ell) \sim \frac{c_{B_{1}}^*}{B_{10}} e^\ell \to -\infty, \]  

(47)

\[ B_{2}(\ell)/B_{2}^f(\ell) \sim \frac{c_{B_{2}}^*}{B_{20}} e^\ell \to -\infty. \]  

(48)

These results indicate that the behaviors of \( B_{1} \) and \( B_{2} \) are obviously modified by Coulomb interaction. These results can be also noticed from Figs. 2(d) and 2(f). The behaviors shown in Eqs. 47 and 48 reveal that the positive quadratic terms \( B_{1} k_x^2 \) and \( B_{2} k_x^2 \) become negative linear terms of momentum components induced by Coulomb interaction.

If the initial Coulomb strength is large enough, we notice that

\[ m(\ell)/m^f(\ell) \to c_{m}^*, \quad \text{with} \quad c_{m}^* < 0, \]  

(49)

where \( c_{m}^* \) is a negative constant. The sign of \( m(\ell) \) changes in this case. As shown in Figs. 2(c) and 2(e),

\[ B_{1}(\ell) \to c_{B_{1}}^*, \quad \text{with} \quad c_{B_{1}}^* > 0, \]  

(50)

\[ B_{2}(\ell) \to c_{B_{2}}^*, \quad \text{with} \quad c_{B_{2}}^* > 0, \]  

(51)

where \( c_{B_{1}}^* \) and \( c_{B_{2}}^* \) are positive constants. The signs of \( B_{1}(\ell) \) and \( B_{2}(\ell) \) hold. We can further get

\[ B_{1}(\ell)/B_{1}^f(\ell) \sim \frac{c_{B_{1}}^*}{B_{10}} e^\ell \to \infty, \]  

(52)

\[ B_{2}(\ell)/B_{2}^f(\ell) \sim \frac{c_{B_{2}}^*}{B_{20}} e^\ell \to \infty, \]  

(53)

which can be also viewed from Figs. 2(d) and 2(f). The flows as shown in Eq. 52 and 53 mean that the positive quadratic terms \( B_{1} k_x^2 \) and \( B_{2} k_x^2 \) are modified to positive linear terms of momentum components by Coulomb interaction.

In order to better understanding this question, we give analytical calculation for the asymptotic behaviors of \( B_{1}(\ell) \) and \( B_{2}(\ell) \). We have find that \( \alpha(\ell) \to \alpha^*, m(\ell) \to \infty \) for weak Coulomb interaction and \( m(\ell) \to -\infty \) for strong enough Coulomb interaction. Accordingly, in the low energy regime, the RG equations for \( B_{1} \) and \( B_{2} \) can be approximated by

\[ \frac{dB_{1}}{d\ell} \sim -B_{1} - \frac{1}{3} \alpha^* \frac{m}{|m|} \sim -B_{1} - \frac{1}{3} \alpha^* \text{sgn}(m), \]  

(54)

\[ \frac{dB_{2}}{d\ell} \sim -B_{2} - \frac{1}{3} \alpha^* \frac{m}{|m|} \sim -B_{2} - \frac{1}{3} \alpha^* \text{sgn}(m). \]  

(55)

Accordingly, we can find that

\[ B_{1}(\ell) \to \frac{1}{3} \alpha^* \text{sgn}(m), \]  

(56)

\[ B_{2}(\ell) \to -\frac{1}{3} \alpha^* \text{sgn}(m), \]  

(57)

in the limit \( \ell \to \infty \). The numerical results of \( B_{1}(\ell)/(\alpha(\ell)/3) \) and \( B_{2}(\ell)/(\alpha(\ell)/3) \) are shown in Fig. 3. We can find these numerical results verify the asymptotic behaviors as shown in Eqs. 55 and 57.

Therefore, we can find that the sign of \( m \) holds but the signs of \( B_{1} \) and \( B_{2} \) change for weak Coulomb interaction, while the sign of \( m \) changes but the signs of \( B_{1} \) and \( B_{2} \) hold if Coulomb interaction is strong enough. Namely, signs of \( m B_{1} \), \( m B_{2} \) usually always change. Thus, second-order TI usually becomes a trivial band insulator not only under strong Coulomb interaction but also under weak Coulomb interaction.

Taking proper initial conditions, \( m, B_{1}, B_{2} \) may flow to zero simultaneously. In this case, the system may become a second-order Dirac semimetal (DSM).
In order to avoid possible misunderstanding, it should be strength that result $m(\ell) \to \infty$ does not mean the physical mass becomes to infinity, but represents that the physical mass becomes larger and larger comparing to the linear terms of Hamiltonian with lowering of momentun. The physical mass for free second-order TI is a constant. The physical mass for second-order TI with weak Coulomb interaction only receives quantitative correction and is also a constant with same sign. Thus, we can find that the RG results shown in above will not break down at the scale where $m(\ell)$ is order of one. Actually, the RG results are valid in the limit $\ell \to \infty$, namely $k \to 0$.

IV. RESULTS FOR OTHER RELATED SYSTEMS AND SOME DISCUSSIONS

A. Second-order Dirac semimetal

Taking the initial conditions $m_0 = 0$, $B_{\perp 0} = 0$, and $B_{\parallel 0} = 0$, we can obtain the RG equations for second-order DSM. The corresponding numerical results are shown in Fig. 4. We notice that $\alpha$ flows to zero slowly, and $\nu$ and $\nu_\perp$ increase slowly with lowering of energy scale. As shown in Fig. 4(b), $D(\ell)/Df(\ell)$ decreases slowly with lowering of energy scale. These results imply that the terms $v_k \perp$, $v_k \perp$, $D(k_x^2 - k_y^2)$ receive weak logarithmic-like corrections of momentum components under Coulomb interaction. It indicates that the observable quantities such as density of states, specific heat, compressibility acquire logarithmic-like corrections of energy or temperature, and second-order DSM is robust against Coulomb interaction. These results are consistent with recent studies in Ref. [?].

B. 3D TI

Taking the initial value $D_0 = 0$, we get the RG equations for TI. From numerical and analytical calculations, we also find that $\alpha$ approaches to constant value $\alpha^*$, and

$$B_{\perp}(\ell) \to -\frac{1}{3} \alpha^* \text{sgn}(m),$$

$$B_{\parallel}(\ell) \to -\frac{1}{3} \alpha^* \text{sgn}(m).$$

The flows of $\alpha(\ell)$, $\nu(\ell)$, $B_{\perp}(\ell)$, $B_{\parallel}(\ell)/D(\ell)/3$, $B_{\perp}(\ell)/D(\ell)/3$ for TI are presented in Fig. 4. These results indicate that TI is also usually unstable to trivial band insulator not only under strong Coulomb interaction but also under weak Coulomb interaction.

In the study by Goswami et al. [38], the RG equations for the TI with long-range Coulomb interaction are given. However, the RG flow of the parameter $B$ is not analyzed carefully and changing of sign of $B$ is not noticed in Ref. [38]. Through numerical and analytical calculations for the RG equations shown in Ref. [38], we obtain that $\alpha$ flows to a constant value $\alpha^*$, and

$$B(\ell) \to \frac{\pi}{3} \alpha^* \text{sgn}(m),$$

in the limit $\ell \to \infty$ in their notations. It should be noticed that the definition of $B$ in Ref. [38] is different.
through the RG equations in Ref. [38] are completely
through RG calculation, we find that
is also interesting. For short-range interactions with the
take the behaviors
which are same as the free system, and \( m \) will not change
sign under weak short-range four-fermion interactions. Thus, second-order TI is robust against weak short-range
four-fermion interactions. If the strength of short-range
four-fermion interaction is large enough, some quantum
phase transitions could be triggered. The detailed stud-
ies about second-order TI under short-range four-fermion
interactions are left in further studies.

The important point we should strength here is that
weak long-range Coulomb interaction is different from
weak short-range four-fermion interactions. Weak long-
range Coulomb interaction drives second-order TI to trivial
band insulator, whereas second-order TI is robust
against weak short-range four-fermion interactions.

D. Comments for the arguments in a recent
reference

In a recent reference, Liu et al. give some argu-
ments for our results about sign change of \( B_{\perp,z} \), and
they believe that second-order TI is robust against weak
Coulomb interaction [77]. We should indicate that their
arguments are unfounded. First, as shown in Sec. IIII
through numerical and analytical calculations, we have
clearly showed that \( B_{\perp,z} \) indeed change sign under weak
long-range Coulomb interaction. Second, the Coulomb
strength \( \alpha(\ell) \) decreases from the initial value \( \alpha_0 \) gradually
and approaches to a small finite value \( \alpha^* \) in the
lowest energy limit \( \ell \to \infty \). Thus, the RG is controlled
and the corresponding results are valid in the lowest
energy regime. Their argument that RG becomes invalid
in large \( \ell \) (low energy regime) is incorrect. Third, their
arguments about the high order terms of momenta are
invalid. Liu et al. considered that \( B_{\perp}k_z^2 \), \( B_zk_z^2 \) terms
come form the expansion of \( \sum_i t_i \cos(k_i a) \) as
\[
\sum_i t_i \cos(k_i a) = \text{const} - B_{\perp}k_z^2 - B_zk_z^2 + O(k^4, k^6),
\]
and higher order terms beyond \( k^2 \) terms exist in general.
They stated that even if \( B_{\perp,z} \) change sign, then the high
order terms \( k^4, k^6 \) can not be ignored. However, we
should indicate that their arguments are unfounded. It
is easy to find that \( \sum_i t_i \cos(k_i a) \) can be expand as
\[
\sum_i t_i \cos(k_i a) = \text{const} - B_{\perp}k_z^2 - B_zk_z^2 \]
\[
+ B_{\perp(4)}(k_x^4 + k_y^4) + B_{z(4)}k_z^4

- B_{\perp(6)}(k_x^6 + k_y^6) - B_{z(6)}k_z^6

\ldots + (-1)^n [B_{\perp(2n)}(k_x^{2n} + k_y^{2n}) + B_{z(2n)}k_z^{2n}]
\ldots
\]
(67)
where the coefficients \( B_{\perp, z, B_{\perp(4)}, B_{z(4)}, B_{\perp(6)}, B_{z(6)}} \) etc. are all positive. We can find the terms \( k_z^{2n} \) with \( n \) being odd number are negative, whereas the terms \( k_z^{2n} \) with \( n \) being even number are positive. These terms could be
modified by long-range Coulomb interaction. From the
numerical and analytical calculations shown in Sec. IIII
we have showed that \( B_{\perp} \) and \( B_z \) change sign under the
weak Coulomb interaction. The high order terms \( k_z^4, k_z^6 \)
could be also modified by long-range Coulomb interaction. Through tedious calculations, we find that the RG equations for $B_{\perp(4)}$, $B_{z(4)}$, $B_{\perp(6)}$, and $B_{z(6)}$ in the low energy regime can be approximately written as

$$
\frac{dB_{\perp(4)}}{dt} \sim -3B_{\perp(4)} + \frac{1}{5} \alpha^* \text{sgn}(m),
$$

(68)

$$
\frac{dB_{z(4)}}{dt} \sim -3B_{z(4)} + \frac{1}{5} \alpha^* \text{sgn}(m),
$$

(69)

$$
\frac{dB_{\perp(6)}}{dt} \sim -5B_{\perp(6)} - \frac{1}{7} \alpha^* \text{sgn}(m),
$$

(70)

$$
\frac{dB_{z(6)}}{dt} \sim -5B_{z(6)} - \frac{1}{7} \alpha^* \text{sgn}(m),
$$

(71)

where the transformations

$$
B_{\perp(4)}\Lambda^3 \rightarrow B_{\perp(4)},
$$

(72)

$$
B_{z(4)}\lambda^3 \rightarrow B_{z(4)},
$$

(73)

$$
B_{\perp(6)}\lambda^5 \rightarrow B_{\perp(6)},
$$

(74)

$$
B_{z(6)}\lambda^5 \rightarrow B_{z(6)},
$$

(75)

have been utilized. Solving Eqs. (68)-(71), we obtain

$$
B_{\perp(4)}(\ell) \rightarrow \frac{1}{15} \alpha^* \text{sgn}(m),
$$

(76)

$$
B_{z(4)}(\ell) \rightarrow \frac{1}{15} \alpha^* \text{sgn}(m),
$$

(77)

$$
B_{\perp(6)}(\ell) \rightarrow \frac{1}{35} \alpha^* \text{sgn}(m),
$$

(78)

$$
B_{z(6)}(\ell) \rightarrow \frac{1}{35} \alpha^* \text{sgn}(m),
$$

(79)

in the lowest energy limit $\ell \rightarrow \infty$. We can find that for weak Coulomb interaction $B_{\perp(4)}$ and $B_{z(4)}$ are still positive, whereas $B_{\perp(6)}$ and $B_{z(6)}$ become to negative. Further, we notice that the RG equations for the coefficients of high order terms can be generally expressed by

$$
\frac{dB_{\perp(2n)}}{dt} \sim -(2n-1)B_{\perp(2n)} + \frac{(-1)^n}{2n+1} \alpha^* \text{sgn}(m),
$$

(80)

$$
\frac{dB_{z(2n)}}{dt} \sim -(2n-1)B_{z(2n)} + \frac{(-1)^n}{2n+1} \alpha^* \text{sgn}(m),
$$

(81)

where the transformations

$$
B_{\perp(2n)}\Lambda^{2n-1} \rightarrow B_{\perp(2n)},
$$

(82)

$$
B_{z(2n)}\lambda^{2n-1} \rightarrow B_{z(2n)},
$$

(83)

have been employed. From Eqs. (80) and (81), we get

$$
B_{\perp(2n)}(\ell) \rightarrow (1)^n \frac{1}{(2n+1)(2n-1)} \alpha^* \text{sgn}(m),
$$

(84)

$$
B_{z(2n)}(\ell) \rightarrow (1)^n \frac{1}{(2n+1)(2n-1)} \alpha^* \text{sgn}(m),
$$

(85)

in the limit $\ell \rightarrow \infty$. Thus, under weak Coulomb interaction, $B_{\perp(2n)}$ change sign if $n$ is odd number, but hold sign if $n$ is even number. From the above results we know that the original negative terms $k_{2n}^n$ with $n$ being odd number become positive, and the original positive terms $k_{2n}^n$ with $n$ being even number are still positive, under weak Coulomb interaction. Therefore, we find that the system becomes trivial insulator under the influence of weak long-range Coulomb interaction even if the high order terms $k_{2n}^1$, $k_{2n}^6$ etc. are considered.

V. INFLUENCE OF DISORDER SCATTERING

In Ref. \cite{72}, Zhao et al. also studied the influence of disorder on 3D second-order TI based on RG method. Their studies about disorder effects in second-order TI have several problems. First, they used the invalid criterion for transition from second-order TI to TI to study the influence of disorder on second-order TI. According to their criterion, considering only disorder, weak disorder drives second-order TI to TI. However, recent studies about disorder effects in 3D second-order TI through other methods showed that 3D second-order TI is robust against weak disorder \cite{24, 25}.

Second, generation new type of disorder by one type of disorder is not considered in Ref. \cite{72}. If the fermion Hamiltonian of a system satisfies

$$
\mathcal{H}_f(k) + \mathcal{H}_f(-k) = 0,
$$

(86)

one type of disorder can exist solely. For examples, in DSM or Weyl semimetal (WSM), the Hamiltonian satisfies Eq. (86), thus one type of disorder can exist solely \cite{45, 79, 84}. However, if the Hamiltonian of a system satisfies

$$
\mathcal{H}_f(k) + \mathcal{H}_f(-k) \neq 0.
$$

(87)

Some type of disorder can not exist solely and generate other types of disorder. For examples, the generation effect of new disorder results from Eq. (87) was considered in the studies about disorder effects in 2D semi-DSM \cite{83}, 3D anisotropic WSM \cite{80, 82}, 3D double-WSM \cite{56, 89}, and Luttinger semimetal \cite{93, 91}, in which the Hamiltonian satisfies Eq. (87). We can find that the Hamiltonian of second-order TI as shown in Eq. (2) satisfies Eq. (87). Therefore, the generation of new type of disorder by one type of disorder should be carefully studied.

Third, generation of new type of disorder by two types of disorder was not considered in Ref. \cite{72}. If two types of disorder exist initially, third type of disorder usually is generated. This effect could be quite important. For example, for 2D DSM, if any two types of disorder among random chemical potential, random vector potential, and random mass exist initially, the third one is generated.
and the system becomes compressible diffusive metal \[80, 81\]. In Ref. \[73\], they considered random chemical potential and random mass. Other types of disorder could be generated by random chemical potential and random mass. Whereas, this effect was not considered in Ref. \[73\].

In order to avoid the second and third problems above-mentioned, we find that two kinds of disorder should be considered for second-order TIs. The corresponding action for fermion-disorder coupling reads

\[
S_{\text{dis}} = \int d\tau d^3x \Psi \left[ V_C(x) \gamma_0 + V_M(x) \mathbb{1}_{4 \times 4} + V_{SO} (x) \gamma_0 \gamma_5 + \sum_{j=x,y} V_{SO \perp} (x) \gamma_0 \gamma_j + V_{AMN \perp} (x) \gamma_0 \gamma_5 + V_{AMN \perp} (x) \sum_{j=x,y} \gamma_j \right] \Psi. \tag{88}
\]

\(V_C(x)\) corresponds to random chemical potential. \(V_M(x)\) stands for random mass. \(V_{AC}(x)\) represents random axial chemical potential. \(V_{SO} (x)\) and \(V_{SO \perp} \) correspond to random spin-orbital scattering within \(xy\) plane and along \(z\) axis. \(V_{PM}(x)\) stands for random pseudo mass. \(V_{AMN \perp}\) and \(V_{AMN \perp}\) represent random magnetization within \(xy\) plane and along \(z\) axis. \(V_{CR \perp}\) and \(V_{CR \perp}\) stand for random current within \(xy\) plane and along \(z\) axis.

The quenched random field \(V_j\) is taken as a Gaussian white noise distribution that satisfies \(\langle V_j (x) \rangle = 0\) and \(\langle V_j (x) V_j (x') \rangle = \delta_j (x - x')\). The disorder can be treated by the replica method. The effective action in replica formalism can be written as

\[
S_{\text{dis}} = -\frac{\Delta C}{2} \int d\tau d\tau' d^3x \left[ (\bar{\Psi}_a \gamma_0 \Psi_a)_\tau \left( (\bar{\Psi}_b \gamma_0 \Psi_b)_\tau \right) - \frac{\Delta M}{2} \int d\tau d\tau' d^3x \left( (\bar{\Psi}_a \gamma_0 \gamma_5 \Psi_a)_\tau \left( (\bar{\Psi}_b \gamma_0 \gamma_5 \Psi_b)_\tau \right) - \frac{\Delta SO_{\perp}}{2} \sum_{j=x,y} \int d\tau d\tau' d^3x \left( (\bar{\Psi}_a \gamma_0 \gamma_j \Psi_a)_\tau \left( (\bar{\Psi}_b \gamma_0 \gamma_j \Psi_b)_\tau \right) - \frac{\Delta SO_{\perp}}{2} \sum_{j=x,y} \int d\tau d\tau' d^3x \left( (\bar{\Psi}_a \gamma_0 \gamma_j \Psi_a)_\tau \left( (\bar{\Psi}_b \gamma_0 \gamma_j \Psi_b)_\tau \right) + \sum_{j=x,y} \int d\tau d\tau' d^3x \left[ (\bar{\Psi}_a \gamma_j \gamma_j \Psi_a)_\tau \left( (\bar{\Psi}_b \gamma_j \gamma_j \Psi_b)_\tau \right) + \sum_{j=x,y} \int d\tau d\tau' d^3x \left( (\bar{\Psi}_a \gamma_j \gamma_j \Psi_a)_\tau \left( (\bar{\Psi}_b \gamma_j \gamma_j \Psi_b)_\tau \right) - \frac{\Delta AMN_{\perp}}{2} \sum_{j=x,y} \int d\tau d\tau' d^3x \left( (\bar{\Psi}_a \gamma_j \gamma_j \Psi_a)_\tau \left( (\bar{\Psi}_b \gamma_j \gamma_j \Psi_b)_\tau \right) - \frac{\Delta AMN_{\perp}}{2} \sum_{j=x,y} \int d\tau d\tau' d^3x \left( (\bar{\Psi}_a \gamma_j \gamma_j \Psi_a)_\tau \left( (\bar{\Psi}_b \gamma_j \gamma_j \Psi_b)_\tau \right) \right), \tag{89}
\]

with \(a, b = 1, 2, ..., n\) being the replica indices. At the end of the calculation, the limit \(n \to 0\) will be taken.

Through the detailed calculations shown in the Appendices, we get the RG equations as following

\[
\begin{align*}
\frac{dv}{d\ell} &= -C^\text{dis}_{0} v, \tag{90} \\
\frac{dv_z}{d\ell} &= -C^\text{dis}_{0} v_z, \tag{91} \\
\frac{dm}{d\ell} &= m + C^m_{\text{dis}} m, \tag{92} \\
\frac{dB_1}{d\ell} &= -B_1, \tag{93} \\
\frac{dB_2}{d\ell} &= -B_2, \tag{94} \\
\frac{dD}{d\ell} &= -D, \tag{95} \\
\frac{d\Delta C}{d\ell} &= -\Delta C + 2\Delta C (\Delta C + \Delta M + \Delta AC + 2\Delta SO_{\perp} + \Delta SO_{\perp} + \Delta PM + 2\Delta MN_{\perp} + \Delta MN_{\perp} + 2\Delta AMN_{\perp}) \tag{96}
\end{align*}
\]
\[ + \Delta_{AMNz} + 2 \Delta_{CRL} + \Delta_{CRz} \left( g_1^D + g_1^P + g_1^m \right) \]
\[ + 4 \left[ \left( \Delta c \Delta_{Dz} + \Delta_{PM} \Delta_{MNz} + \Delta_{AMNz} \Delta_{CRz} + \Delta_{AMNz} \Delta_{CRz} \right) g_1^1 \right] \]
\[ + \left( \Delta c \Delta_{MP} + 2 \Delta_{SOz} \Delta_{AMNz} + \Delta_{SOz} \Delta_{AMNz} \right) g_1^D \]
\[ + \left( \Delta c \Delta_{M} + 2 \Delta_{MNz} \Delta_{AMNz} + \Delta_{MNz} \Delta_{AMNz} \right) g_1^m \],
\[ (96) \]

\[ \frac{d \Delta_{AC}}{d \ell} = - \Delta_{AC} + 2 \Delta_{AC} \left( \Delta c + \Delta c + \Delta c + 2 \Delta_{SOz} + \Delta_{SOz} + \Delta_{SOz} + \Delta_{PM} + 2 \Delta_{MNz} + \Delta_{MNz} \right) \]
\[ - 2 \Delta_{AMNz} + \Delta_{AMNz} - 2 \Delta_{CRz} - 2 \Delta_{CRz} \left( - g_1^i + g_1^D + g_1^m \right) \]
\[ + 4 \left[ \left( \Delta_{SOz} \Delta_{SOz} + \Delta_{MNz} + \Delta_{MNz} \Delta_{AMNz} + \Delta_{MNz} \Delta_{CRz} \right) g_1^i \right] \]
\[ + 2 \left( \Delta_{SOz} + \Delta_{2 MNz} + \Delta_{SOz} \right) \left( \Delta_{AC} \Delta_{PM} + 2 \Delta_{SOz} \Delta_{CRz} + \Delta_{SOz} \Delta_{CRz} \right) g_1^D \]
\[ + \left( \Delta_{AC} \Delta_{AC} + \Delta_{MNz} \Delta_{CRz} + 2 \Delta_{MNz} \Delta_{CRz} \right) g_1^m \],
\[ (97) \]

\[ \frac{d \Delta_{SOz}}{d \ell} = - \Delta_{SOz} + 2 \Delta_{SOz} \left( \Delta c + \Delta c + \Delta c + 2 \Delta_{SOz} - \Delta_{SOz} - \Delta_{PM} - 2 \Delta_{MNz} + \Delta_{MNz} \right) \]
\[ + 2 \Delta_{AMNz} + \Delta_{AMNz} - 2 \Delta_{CRz} + 2 \Delta_{CRz} \left( - g_1^i + g_1^D + g_1^m \right) \]
\[ + 4 \left[ \left( \Delta_{AC} \Delta_{SOz} + \Delta_{PM} \Delta_{CRz} + 4 \Delta_{MNz} \Delta_{AMNz} + \Delta_{MNz} \Delta_{AMNz} \right) g_1^i \right] \]
\[ + 2 \left( \Delta_{AC} \Delta_{AMNz} + \Delta_{PM} \Delta_{MNz} + \Delta_{AC} \Delta_{CRz} + \Delta_{SOz} \Delta_{PM} \right) g_1^D \]
\[ + \left( \Delta_{AC} \Delta_{MNz} + \Delta_{MNz} \Delta_{CRz} + \Delta_{AC} \Delta_{CRz} + \Delta_{SOz} \Delta_{PM} \right) g_1^m \],
\[ (99) \]

\[ \frac{d \Delta_{PM}}{d \ell} = - \Delta_{PM} + 2 \Delta_{PM} \left( \Delta c + \Delta c + \Delta c - 2 \Delta_{SOz} - \Delta_{SOz} - \Delta_{PM} + 2 \Delta_{MNz} + \Delta_{MNz} \right) \]
\[ - 2 \Delta_{AMNz} + \Delta_{AMNz} + 2 \Delta_{CRz} + 2 \Delta_{CRz} \left( g_1^i + g_1^D + g_1^m \right) \]
\[ + 4 \left[ \left( \Delta_{C} \Delta_{MNz} + \Delta_{AMNz} + \Delta_{SOz} \Delta_{CRz} + \Delta_{SOz} \Delta_{CRz} \right) g_1^i \right] \]
\[ + \left( \Delta_{C} \Delta_{MNz} + \Delta_{MNz} + 2 \Delta_{SOz} \Delta_{CRz} + 2 \Delta_{SOz} \Delta_{CRz} \right) g_1^D \]
\[ + \left( \Delta_{C} \Delta_{PM} + 2 \Delta_{SOz} \Delta_{MNz} + \Delta_{SOz} \Delta_{MNz} \right) g_1^m \],
\[ (100) \]

\[ \frac{d \Delta_{MNz}}{d \ell} = - \Delta_{MNz} + 2 \Delta_{MNz} \left( \Delta c + \Delta c + \Delta c - \Delta_{AC} + \Delta_{SOz} - \Delta_{PM} - 2 \Delta_{MNz} + \Delta_{AMNz} \right) \]
\[ \times \left( g_1^i + g_1^D + g_1^m \right) \]
\[ + 2 \left[ \left( \Delta_{C} \Delta_{PM} + \Delta_{C} \Delta_{CRz} + \Delta_{AC} \Delta_{MNz} + 4 \Delta_{SOz} \Delta_{AMNz} + \Delta_{SOz} \Delta_{AMNz} \right) g_1^i \right] \]
\[ + 2 \left( \Delta_{C} \Delta_{CRz} + \Delta_{AC} \Delta_{MNz} + \Delta_{SOz} \Delta_{AMNz} + \Delta_{SOz} \Delta_{AMNz} \right) g_1^D \]
\[ + 2 \left( \Delta_{C} \Delta_{PM} + 2 \Delta_{SOz} \Delta_{MNz} + \Delta_{SOz} \Delta_{MNz} \right) g_1^m \],
\[ (102) \]
\[
\frac{d\Delta_{MNz}}{dl} = -\Delta_{MNz} + 2\Delta_{MNz}(\Delta_C + \Delta_M - \Delta_{AC} + 2\Delta_{SO\perp} - \Delta_{SOz} - \Delta_{PM} - 2\Delta_{MN\perp} + \Delta_{MNz}) \\
-2\Delta_{AM\perp} + \Delta_{AMz} + 2\Delta_{CR\perp} - \Delta_{CRz}) (G_1^+ - G_1^+ - G_1^D + G_1^m) \\
+ 4\left(\Delta_M \Delta_{CR\perp} + \Delta_{AC} \Delta_{MN\perp} + \Delta_{SO\perp} \Delta_{AMz} + \Delta_{SOz} \Delta_{AMz}\right) G_1^+ \\
+ (\Delta_C \Delta_{PM} + 2\Delta_{SO\perp} \Delta_{AMz} + \Delta_{SOz} \Delta_{AMz}) G_1^+ \\
+ (\Delta_{AC} \Delta_{AMz} + \Delta_{MNz} + \Delta_{AC} \Delta_{CRz} + \Delta_{SOz} \Delta_{PM}) G_1^m \right], (103)
\]

\[
\frac{d\Delta_{AM\perp}}{dl} = -\Delta_{AM\perp} + 2\Delta_{AM\perp}(\Delta_C + \Delta_M - \Delta_{AC} - 2\Delta_{SO\perp} + \Delta_{SOz} - \Delta_{PM} - 2\Delta_{MN\perp} + \Delta_{MNz} - \Delta_{AMz} - \Delta_{CRz}) \\
\times (-G_1^+ + G_1^D + G_1^m) \\
+ 2\left(\Delta_M \Delta_{PM} + \Delta_{AC} \Delta_{AM\perp} + 4\Delta_{SO\perp} \Delta_{MN\perp} + \Delta_{SOz} \Delta_{MNz}\right) G_1^+ \\
+ 2\left(\Delta_C \Delta_{AM\perp} + \Delta_{AC} \Delta_{AM\perp} + 4\Delta_{SO\perp} \Delta_{MN\perp} + \Delta_{SOz} \Delta_{MNz}\right) G_1^+ \\
+ (\Delta_M \Delta_{SO\perp} + 2\Delta_{SO\perp} \Delta_{AM\perp} + \Delta_{SOz} \Delta_{MNz} + \Delta_{MNz} \Delta_{CR\perp}) G_1^D \\
+ (\Delta_{AC} \Delta_{SOz} + 2\Delta_{SO\perp} \Delta_{AM\perp} + 2\Delta_{MN\perp} \Delta_{CR\perp}) G_1^m \right], (104)
\]

\[
\frac{d\Delta_{AMz}}{dl} = -\Delta_{AMz} + 2\Delta_{AMz}(\Delta_C + \Delta_M - \Delta_{AC} - 2\Delta_{SO\perp} + \Delta_{SOz} + \Delta_{PM} - 2\Delta_{MN\perp} + \Delta_{MNz} + \Delta_{AMz} + \Delta_{CRz}) \\
\times (G_1^+ + G_1^D + G_1^m) \\
+ 2\left(\Delta_M \Delta_{AMz} + \Delta_{AC} \Delta_{CRz} + \Delta_{SOz} \Delta_{PM}\right) G_1^+ \\
+ 2\left(\Delta_C \Delta_{AM\perp} + \Delta_{AC} \Delta_{CRz} + 4\Delta_{SO\perp} \Delta_{PM}\right) G_1^+ \\
+ (\Delta_M \Delta_{SO\perp} + \Delta_{PM} \Delta_{CR\perp} + \Delta_{MNz} \Delta_{AMz} + \Delta_{MNz} \Delta_{AM\perp}) G_1^D \\
+ (\Delta_{AC} \Delta_{SOz} + \Delta_{PM} \Delta_{CRz} + 2\Delta_{MN\perp} \Delta_{AMz}) G_1^m \right], (105)
\]

\[
\frac{d\Delta_{CR\perp}}{dl} = -\Delta_{CR\perp} + 2\Delta_{CR\perp}(\Delta_C + \Delta_M - \Delta_{AC} - 2\Delta_{SO\perp} + \Delta_{SOz} + \Delta_{PM} - 2\Delta_{MN\perp} + \Delta_{MNz}) \\
\times (G_1^+ + G_1^D + G_1^m) \\
+ 2\left(\Delta_M \Delta_{AMz} + \Delta_{AC} \Delta_{CRz} + \Delta_{SOz} \Delta_{PM}\right) G_1^+ \\
+ 2\left(\Delta_C \Delta_{AM\perp} + \Delta_{AC} \Delta_{CRz} + 4\Delta_{SO\perp} \Delta_{PM}\right) G_1^+ \\
+ (\Delta_M \Delta_{SO\perp} + \Delta_{PM} \Delta_{CR\perp} + \Delta_{MNz} \Delta_{AMz} + \Delta_{MNz} \Delta_{AM\perp}) G_1^D \\
+ (\Delta_{AC} \Delta_{SOz} + \Delta_{PM} \Delta_{CRz} + 2\Delta_{MN\perp} \Delta_{AMz}) G_1^m \right], (106)
\]

\[
\frac{d\Delta_{CRz}}{dl} = -\Delta_{CRz} + 2\Delta_{CRz}(\Delta_C + \Delta_M - \Delta_{AC} - 2\Delta_{SO\perp} + \Delta_{SOz} + \Delta_{PM} - 2\Delta_{MN\perp} + \Delta_{MNz}) \\
\times (G_1^+ + G_1^D + G_1^m) \\
+ 2\left(\Delta_M \Delta_{AMz} + \Delta_{AC} \Delta_{CRz} + \Delta_{SOz} \Delta_{PM}\right) G_1^+ \\
+ 2\left(\Delta_C \Delta_{AM\perp} + \Delta_{AC} \Delta_{CRz} + 4\Delta_{SO\perp} \Delta_{PM}\right) G_1^+ \\
+ (\Delta_M \Delta_{SO\perp} + \Delta_{PM} \Delta_{CR\perp} + \Delta_{MNz} \Delta_{AMz} + \Delta_{MNz} \Delta_{AM\perp}) G_1^D \\
+ (\Delta_{AC} \Delta_{SOz} + \Delta_{PM} \Delta_{CRz} + 2\Delta_{MN\perp} \Delta_{AMz}) G_1^m \right], (107)
\]

The transformations
\[
\Delta_j = \frac{\lambda}{2\pi^2 v^2 \sqrt{\eta}} \rightarrow \Delta_j \quad (108)
\]

have been used. The expressions of \(C_0^{\text{dis}}, C_{\phi}^{\text{dis}}, G_1^+, G_1^-, G_1^P, \) and \(G_1^m\) can be found in Appendix E.

It is easy to find that
\[
B_{1}(l) = B_{1,0} e^{-\ell}, \quad (109)
\]
\[
B_{2}(l) = B_{2,0} e^{-\ell}, \quad (110)
\]
\[
D(l) = D_0 e^{-\ell}, \quad (111)
\]

which take the same behaviors as the clean second-order TI.

Considering random chemical potential initially with different initial values \(\Delta_{C0}\), the flows of different disorder coupling parameters are shown in Figs. (a)-(d). The flows of \(m(l)\) and \(\ell(l)/\ell(l)\) are depicted in Figs. (m) and (n). If \(\Delta_{C0}\) is small, \(\Delta\ell(l)\) approaches to zero in the lowest energy limit \(l \rightarrow \infty\), other disorder coupling parameters \(\Delta_M, \Delta_{AC}, \Delta_{SO\perp}, \Delta_{SOz}, \Delta_{PM}, \Delta_{MN\perp}, \Delta_{MNz}, \Delta_{AM\perp}, \Delta_{AMz}, \Delta_{CR\perp},\) and \(\Delta_{CRz}\) increase from zero at beginning and also approach to zero in the lowest energy limit. In this case, as shown in Fig. (m), \(m(l) \rightarrow \infty\) in the lowest energy limit.

In Ref. 73, the behaviors \(D(l) \rightarrow 0\) and \(m(l) \rightarrow \infty\) in the lowest energy limit are used as the criterion for
the transition from second-order TI to TI. According to their criterion, weak disorder drives second-order TI to TI. However, recent studies based on other methods showed that second-order TI is robust against weak disorder. This contradiction is just due to that the criterion used in Ref. 73 is invalid.

As depicted in Fig. 6(m), \( m(\ell)/m^f(\ell) \) approaches to a positive constant value in the lowest energy limit if \( \Delta C_0 \) is small. This indicates that \( m(\ell) \) takes the qualitatively same behavior as the clean second-order TI. Additionally, \( B_\perp(\ell) \), \( B_c(\ell) \), \( D(\ell) \) take the same behaviors as clean second-order TI. These behaviors clearly reflect that weak disorder does not induce qualitative modification for second-order TI. Namely, second-order TI is robust against weak disorder.

If \( \Delta C_0 \) is large enough, we can find that \( \Delta C \) approaches to infinity at a finite running parameter \( \ell_c \). \( \Delta M, \Delta AC, \Delta SO_1, \Delta SO_2, \Delta PM, \Delta MN_1, \Delta MN_2, \Delta AMN_1, \Delta AMN_2, \Delta CR_1, \) and \( \Delta CR_2 \) grow from zero and also flow to infinity at \( \ell_c \). As shown in Figs. 6(m) and 6(n), we notice that \( m(\ell) \) approaches to finite value when \( \ell \to \ell_c \). These behaviors represent that second-order TI is driven to a diffusive metal phase.

Considering random mass initially with different initial values \( \Delta M_0 \), the flows of disorder coupling parameters \( \Delta M, \Delta AC, \Delta PM, \Delta AMN_1, \) and \( \Delta AMN_2 \) are displayed in Figs. 7(a)-7(e). \( \Delta C, \Delta SO_1, \Delta SO_2, \Delta MN_1, \Delta MN_2, \Delta CR_1, \Delta CR_2 \) are not generated, and equal to the initial value zero. Thus, the flows of these seven disorder coupling parameters are not shown in Fig. 7. The flow of \( m(\ell)/m^f(\ell) \) is shown in 7(f). If \( \Delta M_0 \) is small enough, \( \Delta M \) flows to zero in the lowest energy limit \( \ell \to \infty \). \( \Delta AC, \Delta PM, \Delta AMN_1, \Delta AMN_2 \) increase from zero at the beginning, but become to decrease when \( \ell \) is large enough and finally approach to zero in the lowest energy limit. \( m(\ell)/m^f(\ell) \) approaches to a positive constant which is greater than 1. It represents that disorder only induces quantitative increment but does not result in qualitative modification for \( m \). Thus, second-order TI is stable in this case.

If \( \Delta M_0 \) is large enough, \( \Delta M \) flows to infinity when \( \ell \) approaches to a critical value \( \ell_c \). \( \Delta M, \Delta AC, \Delta PM, \Delta AMN_1, \) and \( \Delta AMN_2 \) are generated from zero and finally also approach to infinity when \( \ell \to \ell_c \). \( m(\ell)/m^f(\ell) \) flows to a finite value in the limit \( \ell \to \ell_c \). These behaviors should represent that the system is driven to a diffusive metal state. Whereas, it was shown that the system can not be driven to diffusive metal phase even \( \Delta M_0 \) is large in Ref. 73. This is due to that the generation of new types of disorder by \( \Delta M \) is not considered in Ref. 73.

In proper initial conditions, we notice that the second-order TI is driven to trivial band insulator with intermediate disorder strength. For example, considering random axial chemical initially with different values of \( \Delta AC_0 \), the flows of \( \Delta M, \Delta AC, \Delta PM, \Delta AMN_1, \) and \( \Delta AMN_2 \) are shown in Figs. 8(a)-8(e). If \( \Delta AC_0 \) is
small, the disorder coupling parameters $\Delta_M$, $\Delta_{AC}$, $\Delta_{PM}$, $\Delta_{AMN\perp}$, and $\Delta_{AMNz}$ all flow to zero in the lowest energy limit $\ell \to \infty$. According to Fig. S(f), $m(\ell)/m^f(\ell)$ flows to a positive constant value which is smaller than one. It represents that $m$ only decreases quantitatively but not receive qualitative modification. Thus, the system is still in second-order TI phase with small $\Delta_{AC0}$. If $\Delta_{AC0}$ is large enough, $\Delta_M$, $\Delta_{AC}$, $\Delta_{PM}$, $\Delta_{AMN\perp}$, and $\Delta_{AMNz}$ all approach to infinity at a finite running parameter $\ell_c$. $m(\ell)/m^f(\ell)$ approaches to finite value at same $\ell_c$. In this case, the system is driven to diffusive metal phase. If $\Delta_{AC0}$ takes intermediate values, $\Delta_M$, $\Delta_{AC}$, $\Delta_{PM}$, $\Delta_{AMN\perp}$, and $\Delta_{AMNz}$ all flow to zero in the lowest energy limit $\ell \to \infty$. Whereas, $m(\ell)/m^f(\ell)$ flows to a negative constant value. It represents that the system becomes a trivial band insulator.

VI. INTERPLAY OF COULOMB INTERACTION AND DISORDER

In this section, we study the interplay of long-range Coulomb interaction and disorder. The RG equations for the model parameters considering both of Coulomb interaction and disorder can be found in Appendix A. Considering initially both of long-range Coulomb interaction and random chemical potential, we show the flows of $\alpha(\ell)$, $\Delta_C(\ell)$, $\Delta_C(\ell)/\alpha(\ell)$, $D(\ell)$, $m(\ell)/m^f(\ell)$, and $B_{\perp}(\ell)/(\alpha(\ell)/3)$ in Figs. 9 and 10 which can be obtained with different initial conditions. The flow of $B_{\perp}(\ell)/(\alpha(\ell)/3)$ is not shown since it take the qualitatively same behavior as $B_{\perp}(\ell)/(\alpha(\ell)/3)$. If $\Delta_{C0}$ and $\alpha_0$ both are small, $\alpha(\ell)$ flows to a constant value $\alpha^*$, $\Delta_C(\ell)$ approaches to zero, $D(\ell)/D^f(\ell)$ approaches to a positive constant value, $m(\ell)/m^f(\ell)$ flows to a positive constant value, and $B_{\perp}(\ell)/(\alpha(\ell)/3) \to -1$ in the lowest energy limit. These behaviors represent that second-order TI is driven to trivial band insulator. For a given $\Delta_{C0}$, if $\alpha_0$ is large enough, we can find that $\alpha(\ell)$ flows to a constant value $\alpha^*$, $\Delta_C(\ell)$ approaches to zero, $D(\ell)/D^f(\ell)$ approaches to a positive constant value, $m(\ell)/m^f(\ell)$ flows to a nega-
If $\alpha$ is large enough, we notice that flow to a finite values at a finite running parameter $\Delta C\alpha$ to diffusive metal phase. These behaviors represent that second-order TI is driven disorder effects in second-order TI through other methods. This result is consistent with recent studies about indicates that second-order TI is robust against weak disorder. According to the criterion adopted in Ref. [73], we find that weak disorder does not result in qualitative order TI. According to the criterion adopted in Ref. [73], is invalid. Second, the flow of $C$ Coulomb interaction but also under weak Coulomb interaction is unstable to trivial band insulator not only under strong interaction and (b) disorder. The solid line represents the fermion action. We also study the effects of disorder in second-order TI considering both of long-range Coulomb interaction and disorder.

For very strong Coulomb interaction, the particle-hole pairs may be formed, and second-order TI may become an excitonic insulator. In this article, we do not consider this possibility. This possibility could be studied by self-consistent Dyson-Schwinger equations.

\[ G_0 (k_0, k) = \frac{1}{i k_0 \gamma_0 + \mathcal{H}_f}, \]

where \[ \mathcal{H}_f = i [v (k_x \gamma_x + k_y \gamma_y) + v_z \gamma_z + D (k^2_x - k^2_y) \gamma_3] + m - B_1 k^2_x - B_z k^2_z. \]

with $k_1^2 = k^2_x + k^2_y$. The propagator of boson field $\phi$ can be written as

\[ D_0 (k_0, k) = \frac{1}{k^2_1 + \eta k^2_z}. \]

\section*{Appendix A: The propagators of fermion and boson}

\section*{Appendix B: The self-energy of fermion}

\begin{figure}[h]
  \centering
  \includegraphics[width=\textwidth]{fig10.png}
  \caption{(a)-(f): Flows of parameters $\alpha(\ell)$, $\Delta C(\ell)$, $\Delta C(\ell)/\alpha(\ell)$, $D(\ell)$, $m(\ell)/m^0(\ell)$, and $B_1(\ell)/M(\ell)$ initially considering both of long-range Coulomb interaction and random chemical potential. Blue, red, green, black, and magenta lines correspond to the initial values $\Delta C_0 = 0.05, 0.1, 0.2, 0.5, 1.0$ respectively. Initial values of other disorder parameters are taken to be zero. $\alpha_0 = 0.2, m_0 = 0.1, B_{1,0} = 1, B_{2,0} = 1, D_0 = 1$ are taken.}
\end{figure}

\section*{VII. SUMMARY}

In this article, we study the influence of Coulomb interaction on second-order TI by RG theory. We show that both the analysis method and conclusions in recent studies in Ref. [73] are unreliable. First, the criterion for transition from second-order TI to TI adopted in Ref. [73] is invalid. Second, the flow of $B$ is important, but is not paid attention in Ref. [73]. Through analyzing the corrections for flows of model parameters induced by Coulomb interaction, we find that second-order TI is unstable to trivial band insulator not only under strong Coulomb interaction but also under weak Coulomb interaction. We also study the effects of disorder in second-order TI. According to the criterion adopted in Ref. [73], weak disorder will drive second-order TI to TI. Whereas, we find that weak disorder does not result in qualitative modification for the Hamiltonian of second-order TI. It indicates that second-order TI is robust against weak disorder. This result is consistent with recent studies about disorder effects in second-order TI through other methods [24,25]. We also obtain the behaviors of second-order TI considering both of long-range Coulomb interaction and disorder.

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\section*{Appendix A: The propagators of fermion and boson}

The propagator of fermion is given by

\[ G_0 (k_0, k) = \frac{1}{i k_0 \gamma_0 + \mathcal{H}_f}, \]

where

\[ \mathcal{H}_f = i [v (k_x \gamma_x + k_y \gamma_y) + v_z \gamma_z + D (k^2_x - k^2_y) \gamma_3] + m - B_1 k^2_x - B_z k^2_z. \]

with $k_1^2 = k^2_x + k^2_y$. The propagator of boson field $\phi$ can be written as

\[ D_0 (k_0, k) = \frac{1}{k^2_1 + \eta k^2_z}. \]

\section*{Appendix B: The self-energy of fermion}

\begin{figure}[h]
  \centering
  \includegraphics[width=\textwidth]{fig11.png}
  \caption{Self-energy of fermions due to (a) Coulomb interaction and (b) disorder. The solid line represents the fermion propagator, and the wavy line stands for the boson propagator that is equivalent to the Coulomb interaction function. The dashed line denotes disorder scattering.}
\end{figure}

1. **The self-energy of fermion induced by Coulomb interaction**

As shown in Fig. [11], the self-energy of fermion induced by Coulomb interaction is given by

\[ \Sigma (k_0, k) = -g^2 \int_{-\infty}^{+\infty} \frac{dq_0}{2\pi} \int d^3q G_0(q_0, q) \gamma_0 \]

\[ \times \left[ -g^2 \int_{-\infty}^{+\infty} \frac{dq_0}{2\pi} \int d^3q G_0(q_0, q) \gamma_0 \right]. \]
\[ \Sigma(k_0, k) = -\frac{g^2}{2} \int' \frac{d^2 q}{(2\pi)^3} \frac{m - B_k q_1^2 - B_z q_2^2}{E_q(q_1^2 + \eta q_2^2)} \]

Substituting Eqs. (A1) and (A3) into Eq. (B1), to the leading order, we obtain

\[ \Sigma(k_0, k) = -\frac{g^2}{2} \int' \frac{d^2 q}{(2\pi)^3} \frac{m - B_k q_1^2 - B_z q_2^2}{E_q(q_1^2 + \eta q_2^2)} \]

where

\[ C_m = \alpha \frac{v A}{m} \left[ \frac{m}{v A} (F_0^+ + F_0^-) - \frac{B_k A}{v} F_0^+ \right. \]

\[ C_v = \alpha F_0^+, \]

\[ C_{B_\perp} = 2 \alpha F_0^+ \]

\[ C_B = \alpha \frac{v}{B_k A} \left[ \frac{m}{v A} (F_1^+ + F_1^-) - \frac{B_k A}{v} F_1^+ \right. \]

\[ C_D = \alpha F_1^D, \]

\[ \alpha = \frac{g^2}{4\pi^2 v \sqrt{\eta}}. \]

The field amplitudes \( F_{0,1} \), \( F_{0,1}^\perp \), \( F_{1,1} \), \( F_{1,1}^\perp \), and \( F_{1,1}^D \) are given by

\[ F_0^+ = \frac{1}{4\pi} \int_0^\pi \sin(\phi) d\phi \int_0^{2\pi} d\theta \frac{\sin^2(\phi)}{\Xi}, \]

\[ F_0^- = \frac{1}{4\pi} \int_0^\pi \sin(\phi) d\phi \int_0^{2\pi} d\theta \frac{\sin^2(\phi)}{\Xi}, \]

\[ F_1^+ = \frac{1}{4\pi} \int_0^\pi \sin(\phi) d\phi \int_0^{2\pi} d\theta \frac{-\sin^2(\phi) \cos(2\phi)}{\Xi}, \]

\[ F_1^- = \frac{1}{4\pi} \int_0^\pi \sin(\phi) d\phi \int_0^{2\pi} d\theta \frac{-\sin^2(\phi) \cos(2\phi)}{\Xi}, \]

\[ F_1^D = \frac{1}{2\pi} \int_0^\pi \sin(\phi) d\phi \int_0^{2\pi} d\theta \frac{-\sin^4(\phi) \cos^2(2\phi)}{\Xi}. \]

and employing the RG scheme

\[ \int \frac{d^3 q'}{(2\pi)^3} = \frac{1}{8\pi^2} \int_0^\pi \sin(\phi) d\phi \int_0^{2\pi} d\theta \int_{b\Lambda}^{\Lambda} dq' q'^2. \]

where \( b = e^{-\ell} \) with \( \ell \) being RG running parameter and \( \Lambda \) the momentum cutoff, we finally obtain

\[ \Sigma(k_0, k) = -m C_m - iv_k (k_0 \gamma_x + k_y \gamma_y) C_v \ell \]

\[-iv_k k_0 \gamma_z C_v \ell - B_k \left( k_1^2 + k_2^2 \right) C_{B_\perp} \ell \]

\[-B_z k_1^2 C_{B_\perp} \ell - i D \left( k_z^2 - k_2^2 \right) \gamma_5 C_D \ell, \]

where

\[ \Xi = \left[ \sin^2(\phi) + \left( \frac{v_\perp}{v \sqrt{\eta}} \right)^2 \cos^2(\phi) \right. \]

\[ + \frac{D^2}{v^2 \Lambda^2} \sin^4(\phi) \cos^2(2\theta) \]

\[ + \left. \left( \frac{m}{v A} - B_k A \sin^2(\phi) - B_z A \cos^2(\phi) \right)^2 \right]. \]
2. The self-energy of fermion induced by disorder scattering

As depicted in Fig. [11] the self-energy of fermions induced by disorder scattering is expressed by

\[ \Sigma_{\text{dis}}(k_0) = \sum_j \Delta_j \int \frac{d^3k}{(2\pi)^3} \Gamma_j G_0(k_0, k) \Gamma_j. \] (B21)

Substituting Eq. (A1) into Eq. (B21), we obtain

\[ \Sigma_{\text{dis}}(k_0) = -i k_0 \gamma_0 c_{\text{dis}}^\dagger \ell - m c_{\text{dis}}^\dagger \ell, \] (B22)

where

\[ C_0^{\text{dis}} = (\Delta_C + \Delta_M + \Delta_{AC} + 2\Delta_{SOL} + \Delta_{SO} + \Delta_{PM} + 2\Delta_{MN} + 2\Delta_{AMN} + 2\Delta_{CMN} + 2\Delta_{CRM} \right) \]

\[ C_m^{\text{dis}} = -(\Delta_C + \Delta_M - \Delta_{AC} - 2\Delta_{SOL} - \Delta_{SO} - \Delta_{PM} + 2\Delta_{MN} + \Delta_{AMN}) \right). \] (B23)

Appendix C: The self-energy of boson

As shown in Fig. [12] the self-energy of boson is defined as

\[ \Pi(k_0, k) = g^2 \int \frac{dq_0}{2\pi} \int' \frac{d^3q}{(2\pi)^3} Tr\left[ \gamma_0 G_0(q_0, q) \gamma_0 G_0(k_0 + q_0, k, q) \right]. \] (C1)

Substituting Eq. (A1) into Eq. (C1), we arrive

\[ \Pi(k_0, k) = 4g^2 \int \frac{dq_0}{2\pi} \int' \frac{d^3q}{(2\pi)^3} \left( \frac{1}{q_0^2 + E_0^2} \right) \left\{ -q_0 (k_0 + q_0) + v^2 q_z (k_x + q_x) + v^2 q_y (k_y + q_y) + v^2 q_z (k_x + q_x) - (k_y + q_y)^2 \right\} \]

\[ + (m - B_1 q_0^2) \left[ m - B_1 (k_1 + q_1)^2 \right] \right\}. \] (C2)

Taking \( k_0 = 0 \), and performing further simplification, we get

\[ \Pi(k) = -g^2 \frac{1}{2} \int' \frac{d^3q}{(2\pi)^3} F_1(k, q) + g^2 \frac{1}{2} \int' \frac{d^3q}{(2\pi)^3} F_2(k, q) \] (C3)

where

\[ F_1(k, q) = (2v^2 q_z + 4D^2 q_x - 4D^2 q_z q_y - 4mB_1 q_x + 4B_1 q_z q_y + 4B_1 q_z q_y) k_x \]
\[ + (2v^2 q_x + 4D^2 q_y - 4D^2 q_x q_z - 4mB_1 q_y + 4B_1 q_z q_y + 4B_1 q_z q_y) k_y \]
\[ + (2v^2 q_z - 4mB_1 q_z + 4B_1 q_z q_y + 4B_1 q_z q_y) k_z \]
\[ + (v^2 + 6D^2 q_x - 2D^2 q_y + 4B_1^2 q_x - 2mB_1 + 2B_1 q_y + 2B_1 q_z)^2) k_z \]
\[ + (v^2 + 6D^2 q_y - 2D^2 q_x + 4B_1^2 q_y - 2mB_1 + 2B_1 q_y + 2B_1 q_z)^2) k_y \]
\[ + (v^2 + 4B_1^2 q_z - 2mB_1 + 2B_1 q_z q_y + 2B_1^2 q_z)^2) k_z \]
\[ + (-8D^2 q_x q_y + 8B_1^2 q_x q_y) k_x k_y + 8B_1 q_x q_y + 8B_1 q_z q_y + 8B_1 q_x q_y k_y k_z \], \] (C4)
and

\[
F_2(k, q) = (v^2q_x + 2D^2q_x^3 - 2D^2q_xq_y^2 - 2mB_\perp q_x + 2B_\perp^2q_x^2 + 2B_\perp^2q_y^2)k_x \\
+ (v^2q_y + 2D^2q_y^3 - 2D^2q_yq_x^2 - 2mB_\perp q_y + 2B_\perp^2q_y^2 + 2B_\perp^2q_x^2)k_y \\
+ (v^2q_z - 2mB_\perp q_z + 2B_\perp^2q_z^2 + 2B_\perp^2q_y^2)k_z \\
+ (D^2q_x^2 - D^2q_y^2 - mB_\perp + B_\perp^2q_z^2 + B_\perp^2q_y^2)k_x^2 \\
+ (-D^2q_x^2 + D^2q_y^2 - mB_\perp + B_\perp^2q_z^2 + B_\perp^2q_y^2)k_y^2 \\
+ (-mB_z + B_\perp^2q_z^2 + B_\perp^2q_y^2)k_y^2.
\]  

(C5)

Retaining the quadratic order of \(k_i\), \(\Pi(k)\) can be further written as

\[
\Pi(k) = k^2 \frac{1}{2} \gamma^2 \int' \frac{d^3q}{(2\pi)^3} \left\{ \frac{1}{E_q^3} \left[ (v^2 + 4D^2q_x^2 + 4B_\perp^2q_x^2) + \frac{1}{E_q} \left[ (v^2 + 2D^2(q_x^2 - q_y^2) - 2mB_\perp + 2B_\perp^2q_x^2 + 2B_\perp^2q_y^2)q_x^2 \right] \right] \right\}^{-1}
\]

\[
+ k^2 \frac{1}{2} \gamma^2 \int' \frac{d^3q}{(2\pi)^3} \left\{ \frac{1}{E_q^3} \left[ (v^2 + 4D^2q_y^2 + 4B_\perp^2q_y^2) + \frac{1}{E_q} \left[ (v^2 + 2D^2(q_y^2 - q_x^2) - 2mB_\perp + 2B_\perp^2q_y^2 + 2B_\perp^2q_x^2)q_y^2 \right] \right] \right\}^{-1}
\]

\[
+ k^2 \frac{1}{2} \gamma^2 \int' \frac{d^3q}{(2\pi)^3} \left\{ \frac{1}{E_q^3} \left[ (v^2 + 4D^2q_z^2) + \frac{1}{E_q} \left[ (v^2 - 2mB_\perp + 2B_\perp^2q_z^2 + 2B_\perp^2q_y^2)q_z^2 \right] \right] \right\}^{-1}.
\]  

(C7)

Adopting the transformation as shown in Eq. (B4), and utilizing the RG scheme as shown in Eq. (B5), we obtain

\[
\Pi(k) = -C_x k_x^2 \ell - C_y k_y^2 \ell - \eta C_z k_z^2 \ell,
\]  

(C8)

where

\[
C_x = \alpha F_x^z,
\]

\[
C_y = \alpha F_y^z,
\]

\[
C_z = \alpha F_z^z.
\]  

(C9) (C10) (C11)

\(F_x^z, F_y^z, F_z^z\) are expressed by

\[
F_x^z = \frac{1}{4\pi} \int_0^\pi \sin(\varphi) d\varphi \int_0^{2\pi} d\theta \left\{ \frac{1 + 4 \left( (\frac{D\Delta}{v})^2 + (\frac{B\perp \Delta}{v})^2 \right) \sin^2(\varphi) \cos^2(\theta)}{\Xi^3} \right\}
\]

\[
- \left[ 1 + 2 \left( \frac{D\Delta}{v} \right)^2 \sin^2(\varphi) \cos(2\theta) - \frac{mB_\perp}{vB_\perp} + 2 \left( \frac{B\perp \Delta}{v} \right)^2 \sin^2(\varphi) + 2 \frac{B_\perp \Delta}{B_\perp v_B} \cos^2(\varphi) \right] \sin^2(\varphi) \cos^2(\theta) \right\}
\]  

(C12)

\[
F_y^z = \frac{1}{4\pi} \int_0^\pi \sin(\varphi) d\varphi \int_0^{2\pi} d\theta \left\{ \frac{1 + 4 \left( (\frac{D\Delta}{v})^2 + (\frac{B\perp \Delta}{v})^2 \right) \sin^2(\varphi) \sin^2(\theta)}{\Xi^3} \right\}
\]

\[
- \left[ 1 - 2 \left( \frac{D\Delta}{v} \right)^2 \sin^2(\varphi) \cos(2\theta) - \frac{mB_\perp}{vB_\perp} + 2 \left( \frac{B\perp \Delta}{v} \right)^2 \sin^2(\varphi) + 2 \frac{B_\perp \Delta}{B_\perp v_B} \cos^2(\varphi) \right] \sin^2(\varphi) \sin^2(\theta) \right\}
\]  

(C13)

\[
F_z^z = \frac{1}{4\pi} \int_0^\pi \sin(\varphi) d\varphi \int_0^{2\pi} d\theta \left\{ \left( \frac{mB_\perp}{v\sqrt{\eta}} \right)^2 + 4 \left( \frac{B\perp \Delta}{v\sqrt{\eta}} \right)^2 \cos^2(\varphi) \right\}
\]

\[
- \left( \frac{mB_\perp}{v\sqrt{\eta}} \right)^2 - 2 \frac{mB_\perp^2}{vB_\perp v_B} + 2 \left( \frac{B\perp \Delta}{v\sqrt{\eta}} \right)^2 \cos^2(\varphi) + 2 \frac{B_\perp \Delta}{B_\perp v_B} \sin^2(\varphi) \right\} \cos^2(\varphi)
\]  

(C14)

where \(\Xi\) is given by Eq. (B19). It easy to verify that

and \(F_2^z = F_2^y\). Then we can define

\[
F_2^\perp = F_2^z = F_2^y\]

(C15)

\[C_\perp = \alpha F_2^\perp\]

(C16)
Accordingly, the polarization can be written as
\[ \Pi(k) = -C_1 k_+^2 \ell - \eta C_2 k_0^2 \ell. \] (C17)

Appendix D: The corrections to fermion-boson coupling vertex

As shown in Fig. 13(a), the correction to fermion-boson coupling vertex induced by Coulomb interaction takes the form
\[ \Gamma^{(1)} = -i g^3 \int_{-\infty}^{+\infty} dq_0 \int_0^{q_0} d^3q \gamma_0 G_0(q_0, \mathbf{q}) \times \gamma_0 G_0(q_0, \mathbf{q}) \gamma_0 D_0(q_0, \mathbf{q}). \] (D1)
Substituting the expressions of fermion propagator and boson propagator into Eq. (D1), we find
\[ \Gamma^{(1)} = 0, \] (D2)
which indicates
\[ \delta g^{(1)} = 0. \] (D3)

The correction to fermion-boson coupling induced by disorder scattering as shown in Fig. 13(b) can be written as
\[ \Gamma^{(2)} = i g \sum_j \delta \gamma_j \int_0^{q_0} d^3k \gamma_0 \Gamma_j G_0(0, \mathbf{k}) \Gamma_j G_0(0, \mathbf{k}) \gamma_0 \] (D4)
Substituting Eq. (A1) into Eq. (D4), one can get
\[ \Gamma^{(2)} = i g \gamma_0 G_0^{\text{dis}} \ell, \] (D5)
where \( G_0^{\text{dis}} \) is given by Eq. (128). Thus,
\[ \delta g^{(2)} = g G_0^{\text{dis}} \ell. \] (D6)

Then we obtain
\[ \delta g = \delta g^{(1)} + \delta g^{(2)} = g G_0^{\text{dis}} \ell. \] (D7)

Appendix E: Corrections to fermion-disorder coupling vertex

The correction induced by disorder scattering from Fig. 14(a) can be written as
\[ W^{(1)} = \sum_i W_i^{(1)}, \] (E1)
where
\[ W_i^{(1)} = \Delta_i \sum_j \Gamma_j (\bar{\Psi}_a \Gamma_i \Psi) \int \frac{d^3k}{(2\pi)^3} \left[ \bar{\Psi}_b \Gamma_j G_0(0, \mathbf{k}) \Gamma_i \times G_0(0, \mathbf{k}) \Gamma_j \Psi_b \right]. \] (E2)

The correction generated by disorder scattering from Figs. 14(b) and 14(c) takes the form
\[ W^{(2)+(3)} = \sum_i \sum_{i \leq j} W_{ij}^{(2)+(3)}, \] (E3)
where
\[ W_{ij}^{(2)+(3)} = \Delta_i \Delta_j \int \frac{d^3k}{(2\pi)^3} \left[ \bar{\Psi}_a \Gamma_i G_0(0, \mathbf{k}) \Gamma_j \Psi_a \right] \times \left\{ \bar{\Psi}_b \left[ \Gamma_j G_0(0, \mathbf{k}) \Gamma_i + \Gamma_j G_0(0, -\mathbf{k}) \Gamma_j \Psi_b \right] \right\}. \] (E4)

Fig. 14(d) induces the correction
\[ W^{(4)} = \sum_i W_i^{(4)}, \] (E5)
where
\[ W_i^{(4)} = -\Delta_i g^2 \left( \bar{\Psi}_a \Gamma_i \Psi_a \right) \int \frac{d^3q}{2\pi} \int \frac{d^3q}{(2\pi)^3} \left[ \bar{\Psi}_b \gamma_0 \times G_0(q_0, \mathbf{q}) \Gamma_i G_0(0, \mathbf{k}) \Gamma_j \Psi_b \right] D_0(q_0, \mathbf{q}). \] (E6)

The contribution from Fig. 14(e) is given by
\[ W_i^{(5)} = \Delta_i g^2 \left( \bar{\Psi}_a \Gamma_i \Psi_a \right) \int \frac{d^3q}{2\pi} \int \frac{d^3q}{(2\pi)^3} \text{Tr} \left[ \Gamma_i G_0(q_0, \mathbf{q}) \times \gamma_0 G_0(q_0, \mathbf{k} + \mathbf{q}) \right] D_0(\mathbf{k}) \left( \bar{\Psi}_b \gamma_0 \Psi_b \right). \] (E7)
1. Correction induced by disorder scattering from Feynman diagram \(14\) (a)

Substituting the Eq. (A1) into Eq. (E2), we arrive

\[
W_i(1) = \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} \ell \Delta_i \left( \bar{\Psi}_a \Gamma_i \Psi_a \right) \sum_j \Delta_j
\]

\[
\times \left\{ \bar{\Psi}_b \Gamma_j \left[ -\frac{1}{2} (\gamma_x \Gamma_i \gamma_x + \gamma_y \Gamma_i \gamma_y) G_i^\dagger - \gamma_z \Gamma_i \gamma_z G_i^\dagger - \gamma_\perp \Gamma_i \gamma_\perp G_i^D + \Gamma_i G_i^m \right] \Gamma_j \Psi_b \right\} \tag{E8}
\]

with

\[
G_i^D = \frac{1}{4\pi} \int_0^{\pi} d\varphi \sin(\varphi) \int_0^{2\pi} d\theta \frac{\sin^2(\varphi)}{\Xi^4}, \tag{E9}
\]

\[
G_i^m = \frac{1}{4\pi} \int_0^{\pi} d\varphi \sin(\varphi) \int_0^{2\pi} d\theta \frac{\cos^2(\varphi)}{\Xi^4}, \tag{E10}
\]

where \(\Xi\) is expressed by Eq. (B19).

\[W_i(1)\] can be further written as

\[
W_i(1) = \frac{\delta \Delta(1)}{2} \left( \bar{\Psi}_a \gamma_\perp \gamma_\perp \Psi_a \right) \left( \bar{\Psi}_b \gamma_\perp \gamma_\perp \Psi_b \right), \tag{E13}
\]

where

\[\delta \Delta(1) = 2\Delta_C (\Delta_C + \Delta_M + \Delta_{AC} + 2\Delta_{SO\perp} + \Delta_{SOz} + \Delta_{PM} + 2\Delta_{MN\perp} + \Delta_{MNz} + 2\Delta_{AMN\perp} + \Delta_{AMNz})\]

\[+ 2\Delta_{CR1} + \Delta_{CR2} \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} \left( G_i^D + G_i^m \right) \ell, \tag{E14}\]

\[\delta \Delta(1) = 2\Delta_M (\Delta_C + \Delta_M - \Delta_{AC} - 2\Delta_{SO\perp} - \Delta_{SOz} - \Delta_{PM} + 2\Delta_{MN\perp} + \Delta_{MNz} + 2\Delta_{AMN\perp} + \Delta_{AMNz})\]

\[- 2\Delta_{CR1} - 2\Delta_{CR2} \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} \left( -G_i^D + G_i^m \right) \ell, \tag{E15}\]

\[\delta \Delta(1) = 2\Delta_{AC} (-\Delta_C + \Delta_M + \Delta_{AC} + 2\Delta_{SO\perp} + \Delta_{SOz} + \Delta_{PM} + 2\Delta_{MN\perp} + \Delta_{MNz} - 2\Delta_{AMN\perp} - \Delta_{AMNz})\]

\[- 2\Delta_{CR1} - 2\Delta_{CR2} \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} \left( -G_i^D + G_i^m \right) \ell, \tag{E16}\]

\[\delta \Delta(1) = 2\Delta_{SO\perp} (-\Delta_C + \Delta_M + \Delta_{AC} + \Delta_{SOz} - \Delta_{PM} - \Delta_{MNz} + \Delta_{AMNz} - \Delta_{CR2}) \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} \]

\[\times \left( -G_i^D + G_i^m \right) \ell, \tag{E17}\]

\[\delta \Delta(1) = 2\Delta_{SOz} (-\Delta_C + \Delta_M + \Delta_{AC} + 2\Delta_{SO\perp} - \Delta_{SOz} - \Delta_{PM} - 2\Delta_{MN\perp} + \Delta_{MNz} + 2\Delta_{AMN\perp} - \Delta_{AMNz})\]

\[- 2\Delta_{CR1} + \Delta_{CR2} \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} \left( G_i^D + G_i^m \right) \ell, \tag{E18}\]

\[\delta \Delta(1) = 2\Delta_{PM} (-\Delta_C + \Delta_M + \Delta_{AC} - 2\Delta_{SO\perp} - \Delta_{SOz} - \Delta_{PM} + 2\Delta_{MN\perp} + \Delta_{MNz} - 2\Delta_{AMN\perp} - \Delta_{AMNz})\]

\[+ 2\Delta_{CR1} + \Delta_{CR2} \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} \left( G_i^D + G_i^m \right) \ell, \tag{E19}\]

\[\delta \Delta(1) = 2\Delta_{MN\perp} (\Delta_C + \Delta_M - \Delta_{AC} + \Delta_{SOz} - \Delta_{PM} - \Delta_{MNz} - \Delta_{AMNz} - \Delta_{CR2}) \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} \]

\[\times \left( -G_i^D + G_i^m \right) \ell, \tag{E20}\]

\[\delta \Delta(1) = 2\Delta_{MN\perp} (\Delta_C + \Delta_M - \Delta_{AC} + 2\Delta_{SO\perp} - \Delta_{SOz} - \Delta_{PM} - 2\Delta_{MN\perp} + \Delta_{MNz} - 2\Delta_{AMN\perp} + \Delta_{AMNz})\]

\[+ 2\Delta_{CR1} - \Delta_{CR2} \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} \left( G_i^D - G_i^m \right) \ell, \tag{E21}\]

\[\delta \Delta(1) = 2\Delta_{AMN\perp} (\Delta_C + \Delta_M + \Delta_{AC} - \Delta_{SOz} + \Delta_{PM} - \Delta_{MNz} - \Delta_{AMNz} - \Delta_{CR2}) \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} \]

\[\times \left( -G_i^D + G_i^m \right) \ell. \tag{E22}\]
\[ \delta \Delta_{AMNz}^{(1)} = 2 \Delta_{AMNz}(\Delta_C + \Delta_M + \Delta_{AC} - 2\Delta_{SO\perp} + \Delta_{SOz} + \Delta_{PM} - 2\Delta_{MN\perp} + \Delta_{MNz} - 2\Delta_{AMN\perp} + \Delta_{AMNz} \]
\[ -2\Delta_{CR\perp} + \Delta_{CRz} \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} (-G_1^+ + G_1^- + G_1^D + G_1^m) \ell, \] (E23)

\[ \delta \Delta_{CR\perp}^{(1)} = 2 \Delta_{CR\perp} (-\Delta_C + \Delta_M - \Delta_{AC} - \Delta_{SOz} + \Delta_{PM} - 2\Delta_{MN\perp} + \Delta_{MNz} + \Delta_{CRz}) \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} \]
\[ \times (G_1^+ + G_1^- + G_1^D) \ell, \] (E24)

\[ \delta \Delta_{CRz}^{(1)} = 2 \Delta_{CRz} (-\Delta_C + \Delta_M - \Delta_{AC} - 2\Delta_{SO\perp} + \Delta_{SOz} + \Delta_{PM} - 2\Delta_{MN\perp} + \Delta_{MNz} + 2\Delta_{AMN\perp} - \Delta_{AMNz} \]
\[ + 2\Delta_{CR\perp} - \Delta_{CRz}) \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} (G_1^+ - G_1^- + G_1^D + G_1^m) \ell. \] (E25)

It should be noticed that
\[ G_1^+ + G_1^- + G_1^D + G_1^m = G_0^+ + G_0^- . \] (E26)

2. Correction induced by disorder scattering from Feynman diagrams [14b] and [14c]

Substituting Eq. (A1) into Eq. (E4), we obtain

\[ W_{i,j}^{(2)+(3)} = -\frac{1}{2} \Delta_i \Delta_j \left( \psi_a \Gamma_i \gamma \Gamma_j \psi_a \right) \left[ \psi_b (\Gamma_j \gamma \Gamma_i - \Gamma_i \gamma \Gamma_j) \psi_b \right] \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} G_1^+ \]
\[ -\frac{1}{2} \Delta_i \Delta_j \left( \psi_a \Gamma_i \gamma \Gamma_j \psi_a \right) \left[ \psi_b (\Gamma_j \gamma \Gamma_i - \Gamma_i \gamma \Gamma_j) \psi_b \right] \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} G_1^+ \]
\[ -\Delta_i \Delta_j \left( \Psi_a \Gamma_i \gamma \Gamma_j \Psi_a \right) \left[ \Psi_b (\Gamma_j \gamma \Gamma_i - \Gamma_i \gamma \Gamma_j) \Psi_b \right] \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} G_1^+ \]
\[ -\Delta_i \Delta_j \left( \Psi_a \Gamma_i \gamma \Gamma_j \Psi_a \right) \left[ \Psi_b (\Gamma_j \gamma \Gamma_i - \Gamma_i \gamma \Gamma_j) \Psi_b \right] \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} G_1^D \]
\[ + \Delta_i \Delta_j \left( \psi_a \Gamma_i \gamma \Gamma_j \psi_a \right) \left[ \psi_b (\Gamma_j \gamma \Gamma_i + \Gamma_i \gamma \Gamma_j) \psi_b \right] \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} G_1^m. \] (E27)

Concretely, we find that

\[ W_{CC}^{(2)+(3)} = 2 \Delta_C \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} G_1^D \ell \left( \psi_a \Gamma_i \gamma \Psi_a \right) \left( \psi_b \Gamma_i \gamma \Psi_b \right) + 2 \Delta^2 \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} G_1^m \ell \left( \psi_a \Psi_a \right) \left( \psi_b \Psi_b \right), \] (E28)

\[ W_{MM}^{(2)+(3)} = 2 \Delta_M \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} G_1^D \ell \left( \psi_a \Gamma_i \gamma \Psi_a \right) \left( \psi_b \Gamma_i \gamma \Psi_b \right) + 2 \Delta^2 \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} G_1^m \ell \left( \psi_a \Psi_a \right) \left( \psi_b \Psi_b \right), \] (E29)

\[ W_{AC,AC}^{(2)+(3)} = 2 \Delta_2 \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} G_1^D \ell \left( \psi_a \Gamma_i \gamma \Psi_a \right) \left( \psi_b \Gamma_i \gamma \Psi_b \right) + 2 \Delta^2 \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} G_1^m \ell \left( \psi_a \Psi_a \right) \left( \psi_b \Psi_b \right), \] (E30)

\[ W_{SO\perp,SO\perp}^{(2)+(3)} = 4 \Delta^2 \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} G_1^{m} \ell \left( \psi_a \Gamma_i \gamma \Psi_a \right) \left( \psi_b \Gamma_i \gamma \Psi_b \right) + 4 \Delta^2 \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} G_1^{D} \ell \left( \psi_a \Gamma_i \gamma \Psi_a \right) \left( \psi_b \Gamma_i \gamma \Psi_b \right) \]
\[ + 4 \Delta^2 \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} G_1^{m} \ell \left( \psi_a \Gamma_i \gamma \Psi_a \right) \left( \psi_b \Gamma_i \gamma \Psi_b \right), \] (E31)

\[ W_{SO\perp,SO\perp}^{(2)+(3)} = 2 \Delta^2 \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} G_1^{D} \ell \left( \psi_a \Gamma_i \gamma \Psi_a \right) \left( \psi_b \Gamma_i \gamma \Psi_b \right) + 2 \Delta^2 \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} G_1^{m} \ell \left( \psi_a \Gamma_i \gamma \Psi_a \right) \left( \psi_b \Gamma_i \gamma \Psi_b \right), \] (E32)

\[ W_{PM,PM}^{(2)+(3)} = 2 \Delta^2 \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} G_1^{D} \ell \left( \psi_a \Gamma_i \gamma \Psi_a \right) \left( \psi_b \Gamma_i \gamma \Psi_b \right) + 2 \Delta^2 \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} G_1^{m} \ell \left( \psi_a \Gamma_i \gamma \Psi_a \right) \left( \psi_b \Gamma_i \gamma \Psi_b \right), \] (E33)

\[ W_{MN\perp,MN\perp}^{(2)+(3)} = 2 \Delta^2 \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} G_1^{D} \ell \left( \psi_a \Gamma_i \gamma \Psi_a \right) \left( \psi_b \Gamma_i \gamma \Psi_b \right) + 4 \Delta^2 \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} G_1^{D} \ell \left( \psi_a \Gamma_i \gamma \Psi_a \right) \left( \psi_b \Gamma_i \gamma \Psi_b \right) \]
\[ + 4 \Delta^2 \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} G_1^{m} \ell \left( \psi_a \Gamma_i \gamma \Psi_a \right) \left( \psi_b \Gamma_i \gamma \Psi_b \right), \] (E34)

\[ W_{MN\perp,MN\perp}^{(2)+(3)} = 2 \Delta^2 \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} G_1^{D} \ell \left( \psi_a \Gamma_i \gamma \Psi_a \right) \left( \psi_b \Gamma_i \gamma \Psi_b \right) \]
\[ + 4 \Delta^2 \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} G_1^{m} \ell \left( \psi_a \Gamma_i \gamma \Psi_a \right) \left( \psi_b \Gamma_i \gamma \Psi_b \right). \]
\[ W_{AMN, AMN}^{(2)+(3)} = 4\Delta_{AMN}^2 \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} G_1^D \left( \bar{\Psi}_a \gamma_5 \Psi_a \right) \left( \bar{\Psi}_b \gamma_5 \Psi_b \right) + 4\Delta_{AMN}^2 \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} G_1^D \left( \bar{\Psi}_a \gamma_5 \Psi_a \right) \left( \bar{\Psi}_b \gamma_5 \Psi_b \right) \]
\[ + 4\Delta_{AMN}^2 \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} G_1^m \left( \bar{\Psi}_a \Psi_a \right) \left( \bar{\Psi}_b \Psi_b \right), \quad (E35) \]

\[ W_{AMN, AMNz}^{(2)+(3)} = 2\Delta_{AMN}^2 \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} G_1^D \left( \bar{\Psi}_a \gamma_5 \Psi_a \right) \left( \bar{\Psi}_b \gamma_5 \Psi_b \right) + 2\Delta_{AMN}^2 \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} G_1^m \left( \bar{\Psi}_a \Psi_a \right) \left( \bar{\Psi}_b \Psi_b \right), \quad (E36) \]

\[ W_{CRL, CRL}^{(2)+(3)} = 4\Delta_{CRL}^2 \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} G_1^D \left( \bar{\Psi}_a \gamma_5 \Psi_a \right) \left( \bar{\Psi}_b \gamma_5 \Psi_b \right) + 4\Delta_{CRL}^2 \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} G_1^D \left( \bar{\Psi}_a \gamma_5 \Psi_a \right) \left( \bar{\Psi}_b \gamma_5 \Psi_b \right) \]
\[ + 4\Delta_{CRL}^2 \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} G_1^m \left( \bar{\Psi}_a \Psi_a \right) \left( \bar{\Psi}_b \Psi_b \right), \quad (E37) \]

\[ W_{CRL, CRz}^{(2)+(3)} = 2\Delta_{CRL}^2 \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} G_1^D \left( \bar{\Psi}_a \gamma_5 \Psi_a \right) \left( \bar{\Psi}_b \gamma_5 \Psi_b \right) + 2\Delta_{CRL}^2 \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} G_1^m \left( \bar{\Psi}_a \Psi_a \right) \left( \bar{\Psi}_b \Psi_b \right), \quad (E38) \]

\[ W_{C,M}^{(2)+(3)} = \Delta C \Delta M \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} G_{\ell}^D \left( \bar{\Psi}_a \gamma_0 \gamma_j \Psi_a \right) \left( \bar{\Psi}_b \gamma_0 \gamma_j \Psi_b \right) \]
\[ + 2\Delta C \Delta M \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} G_{\ell}^D \left( \bar{\Psi}_a \gamma_0 \gamma_j \Psi_a \right) \left( \bar{\Psi}_b \gamma_0 \gamma_j \Psi_b \right) \]
\[ + 2\Delta C \Delta M \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} G_{\ell}^m \left( \bar{\Psi}_a \gamma_0 \Psi_a \right) \left( \bar{\Psi}_b \gamma_0 \Psi_b \right), \quad (E39) \]

\[ W_{C, AC}^{(2)+(3)} = 0, \quad (E40) \]

\[ W_{C, SO\perp}^{(2)+(3)} = 2\Delta C \Delta S_{\perp} \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} G_{\ell}^D \left( \bar{\Psi}_a \Psi_a \right) \left( \bar{\Psi}_b \Psi_b \right) \]
\[ + 2\Delta C \Delta S_{\perp} \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} G_{\ell}^D \left( \bar{\Psi}_a \gamma_5 \gamma_j \Psi_a \right) \left( \bar{\Psi}_b \gamma_5 \gamma_j \Psi_b \right), \quad (E41) \]

\[ W_{C, SOz}^{(2)+(3)} = 2\Delta C \Delta SO_{z} \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} G_{\ell}^D \left( \bar{\Psi}_a \Psi_a \right) \left( \bar{\Psi}_b \Psi_b \right) \]
\[ + 2\Delta C \Delta SO_{z} \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} G_{\ell}^D \left( \bar{\Psi}_a \gamma_5 \gamma_j \Psi_a \right) \left( \bar{\Psi}_b \gamma_5 \gamma_j \Psi_b \right), \quad (E42) \]

\[ W_{C, PM}^{(2)+(3)} = \Delta C \Delta PM \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} G_{\ell}^D \left[ \left( \bar{\Psi}_a \gamma_5 \gamma_j \Psi_a \right) \left( \bar{\Psi}_b \gamma_5 \gamma_j \Psi_b \right) \right. \]
\[ + 2\Delta C \Delta PM \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} G_{\ell}^D \left( \bar{\Psi}_a \gamma_5 \gamma_j \Psi_a \right) \left( \bar{\Psi}_b \gamma_5 \gamma_j \Psi_b \right) \]
\[ + 2\Delta C \Delta PM \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} G_{\ell}^m \left( \bar{\Psi}_a \gamma_0 \Psi_a \right) \left( \bar{\Psi}_b \gamma_0 \Psi_b \right), \quad (E43) \]

\[ W_{C, MN\perp}^{(2)+(3)} = 2\Delta C \Delta MN_{\perp} \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} G_{\ell}^D \left( \bar{\Psi}_a \gamma_5 \Psi_a \right) \left( \bar{\Psi}_b \gamma_5 \Psi_b \right) \]
\[ + 2\Delta C \Delta MN_{\perp} \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} G_{\ell}^m \left( \bar{\Psi}_a \gamma_5 \gamma_j \Psi_a \right) \left( \bar{\Psi}_b \gamma_5 \gamma_j \Psi_b \right), \quad (E44) \]

\[ W_{C, MNz}^{(2)+(3)} = 2\Delta C \Delta MN_{z} \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} G_{\ell}^D \left( \bar{\Psi}_a \gamma_5 \Psi_a \right) \left( \bar{\Psi}_b \gamma_5 \Psi_b \right) \]
\[ + 2\Delta C \Delta MN_{z} \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} G_{\ell}^m \left( \bar{\Psi}_a \gamma_5 \gamma_j \Psi_a \right) \left( \bar{\Psi}_b \gamma_5 \gamma_j \Psi_b \right), \quad (E45) \]

\[ W_{C, AMN\perp}^{(2)+(3)} = 2\Delta C \Delta AMN_{\perp} \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} G_{\ell}^D \left( \bar{\Psi}_a \gamma_5 \Psi_a \right) \left( \bar{\Psi}_b \gamma_5 \Psi_b \right) \]
\[ + 2\Delta C \Delta AMN_{\perp} \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} G_{\ell}^m \left( \bar{\Psi}_a \gamma_5 \gamma_j \Psi_a \right) \left( \bar{\Psi}_b \gamma_5 \gamma_j \Psi_b \right), \quad (E46) \]
\[\begin{align*}
W_{(2)+(3)}^{(2)+(3)}_{C,AMNz} &= \Delta_{C,AMNz} \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} \left\langle G_1^m \sum_{j=x,y} \left( \bar{\Psi}_a i \gamma_j \Psi_b \right) \left( \bar{\Psi}_b i \gamma_j \Psi_b \right) \right\rangle + 2 \Delta_{C,AMNz} \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} \left\langle G_1^m \left[ \bar{\Psi}_a i \gamma_y \gamma_z \Psi_a \left( \bar{\Psi}_b i \gamma_y \gamma_z \Psi_b \right) + \bar{\Psi}_a i \gamma_z \gamma_x \Psi_a \left( \bar{\Psi}_b i \gamma_z \gamma_x \Psi_b \right) \right] \right\rangle, \\
W_{(2)+(3)}^{(2)+(3)}_{C,CRz} &= \Delta_{C,CRz} \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} \left\langle G_1^m \sum_{j=x,y} \left( \bar{\Psi}_a i \gamma_j \gamma_z \Psi_a \right) \left( \bar{\Psi}_b i \gamma_j \gamma_z \Psi_b \right) \right\rangle + 2 \Delta_{C,CRz} \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} \left\langle G_1^m \left[ \bar{\Psi}_a i \gamma_y \gamma_z \gamma_y \Psi_a \left( \bar{\Psi}_b i \gamma_y \gamma_z \gamma_y \Psi_b \right) + \bar{\Psi}_a i \gamma_z \gamma_x \gamma_z \Psi_a \left( \bar{\Psi}_b i \gamma_z \gamma_x \gamma_z \Psi_b \right) \right] \right\rangle, \\
\end{align*}\]

\[\begin{align*}
W_{(2)+(3)}^{(2)+(3)}_{C,AC} &= \Delta_{C,AC} \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} \left\langle G_1^m \left( \bar{\Psi}_a \gamma_0 \gamma_5 \Psi_a \left( \bar{\Psi}_b \gamma_0 \gamma_5 \Psi_b \right) \right) \right\rangle, \\
W_{(2)+(3)}^{(2)+(3)}_{C,SOz} &= \Delta_{C,SOz} \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} \left\langle G_1^m \left( \bar{\Psi}_a \gamma_0 \gamma_5 \Psi_a \left( \bar{\Psi}_b \gamma_0 \gamma_5 \Psi_b \right) \right) \right\rangle + 2 \Delta_{C,SOz} \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} \left\langle G_1^m \left[ \bar{\Psi}_a i \gamma_y \gamma_z \gamma_y \gamma_z \Psi_a \left( \bar{\Psi}_b i \gamma_y \gamma_z \gamma_y \gamma_z \Psi_b \right) + \bar{\Psi}_a i \gamma_z \gamma_x \gamma_z \gamma_x \Psi_a \left( \bar{\Psi}_b i \gamma_z \gamma_x \gamma_z \gamma_x \Psi_b \right) \right] \right\rangle, \\
W_{(2)+(3)}^{(2)+(3)}_{M,PM} &= \Delta_{M,PM} \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} \left\langle G_1^m \sum_{j=x,y} \left( \bar{\Psi}_a i \gamma_j \gamma_z \Psi_a \right) \left( \bar{\Psi}_b i \gamma_j \gamma_z \Psi_b \right) \right\rangle + 2 \Delta_{M,PM} \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} \left\langle G_1^m \left[ \bar{\Psi}_a i \gamma_y \gamma_z \gamma_y \gamma_z \Psi_a \left( \bar{\Psi}_b i \gamma_y \gamma_z \gamma_y \gamma_z \Psi_b \right) + \bar{\Psi}_a i \gamma_z \gamma_x \gamma_z \gamma_x \Psi_a \left( \bar{\Psi}_b i \gamma_z \gamma_x \gamma_z \gamma_x \Psi_b \right) \right] \right\rangle, \\
W_{(2)+(3)}^{(2)+(3)}_{M,MNz} &= \Delta_{M,MNz} \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} \left\langle G_1^m \sum_{j=x,y} \left( \bar{\Psi}_a i \gamma_j \gamma_z \Psi_a \right) \left( \bar{\Psi}_b i \gamma_j \gamma_z \Psi_b \right) \right\rangle + 2 \Delta_{M,MNz} \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} \left\langle G_1^m \left[ \bar{\Psi}_a i \gamma_y \gamma_z \gamma_y \gamma_z \Psi_a \left( \bar{\Psi}_b i \gamma_y \gamma_z \gamma_y \gamma_z \Psi_b \right) + \bar{\Psi}_a i \gamma_z \gamma_x \gamma_z \gamma_x \Psi_a \left( \bar{\Psi}_b i \gamma_z \gamma_x \gamma_z \gamma_x \Psi_b \right) \right] \right\rangle, \\
\end{align*}\]
\[ W_{AMN \perp}^{(2)+(3)} = 2 \Delta M \Delta_{AMN} \frac{\Lambda}{2\pi v^2 \sqrt{\eta}} G^D_1 \big( \bar{\Psi}_a \gamma^0 \gamma_z \Psi_a \big) \big( \bar{\Psi}_b \gamma^0 \gamma_z \Psi_b \big), \]

\[ W_{AMNZ}^{(2)+(3)} = 2 \Delta M \Delta_{AMNz} \frac{\Lambda}{2\pi v^2 \sqrt{\eta}} G^D_1 \bar{\Psi}_a \gamma^0 \gamma_z \Psi_a \big( \bar{\Psi}_b \gamma^0 \gamma_z \Psi_b \big), \]

\[ W_{CR \perp}^{(2)+(3)} = 2 \Delta M \Delta_{CR} \frac{\Lambda}{2\pi v^2 \sqrt{\eta}} G^D_1 \big( \bar{\Psi}_a \gamma^0 \gamma_z \Psi_a \big) \big( \bar{\Psi}_b \gamma^0 \gamma_z \Psi_b \big), \]

\[ W_{CRz}^{(2)+(3)} = \Delta M \Delta_{CRz} \frac{\Lambda}{2\pi v^2 \sqrt{\eta}} G^D_1 \big( \bar{\Psi}_a \gamma^0 \gamma_z \Psi_a \big) \big( \bar{\Psi}_b \gamma^0 \gamma_z \Psi_b \big), \]

\[ W_{SO \perp}^{(2)+(3)} = 2 \Delta M \Delta_{SO} \frac{\Lambda}{2\pi v^2 \sqrt{\eta}} G^D_1 \big( \bar{\Psi}_a \gamma^0 \gamma_z \Psi_a \big) \big( \bar{\Psi}_b \gamma^0 \gamma_z \Psi_b \big), \]

\[ W_{SOz}^{(2)+(3)} = \Delta M \Delta_{SOz} \frac{\Lambda}{2\pi v^2 \sqrt{\eta}} G^D_1 \big( \bar{\Psi}_a \gamma^0 \gamma_z \Psi_a \big) \big( \bar{\Psi}_b \gamma^0 \gamma_z \Psi_b \big), \]

\[ W_{PM}^{(2)+(3)} = 2 \Delta M \Delta_{PM} \frac{\Lambda}{2\pi v^2 \sqrt{\eta}} G^D_1 \big( \bar{\Psi}_a \gamma^0 \gamma_z \Psi_a \big) \big( \bar{\Psi}_b \gamma^0 \gamma_z \Psi_b \big), \]

\[ W_{MN \perp}^{(2)+(3)} = 2 \Delta M \Delta_{MN} \frac{\Lambda}{2\pi v^2 \sqrt{\eta}} G^D_1 \big( \bar{\Psi}_a \gamma^0 \gamma_z \Psi_a \big) \big( \bar{\Psi}_b \gamma^0 \gamma_z \Psi_b \big), \]

\[ W_{MNz}^{(2)+(3)} = \Delta M \Delta_{MNz} \frac{\Lambda}{2\pi v^2 \sqrt{\eta}} G^D_1 \big( \bar{\Psi}_a \gamma^0 \gamma_z \Psi_a \big) \big( \bar{\Psi}_b \gamma^0 \gamma_z \Psi_b \big), \]

\[ W_{AMN \perp}^{(2)+(3)} = 2 \Delta M \Delta_{AMN} \frac{\Lambda}{2\pi v^2 \sqrt{\eta}} G^D_1 \big( \bar{\Psi}_a \gamma^0 \gamma_z \Psi_a \big) \big( \bar{\Psi}_b \gamma^0 \gamma_z \Psi_b \big), \]
\[ W^{(2) + (3)}_{AC,AMNz} = \Delta_{AC} \Delta_{AMNz} \frac{\Lambda}{2 \pi^2 v^2} \sum_{j=x,y} G^j \sum_{j=x,y} (\bar{\Psi}_a i \gamma_5 \gamma_j \Psi_a) (\bar{\Psi}_b i \gamma_5 \gamma_j \Psi_b), \]

\[ W^{(2) + (3)}_{AC,CRz} = 2 \Delta_{AC} \Delta_{CRz} \frac{\Lambda}{2 \pi^2 v^2} \sum_{j=x,y} G^j \sum_{j=x,y} (\bar{\Psi}_a i \gamma_j \Psi_a) (\bar{\Psi}_b i \gamma_j \Psi_b) \]

\[ W^{(2) + (3)}_{AC,CRz} = 2 \Delta_{AC} \Delta_{CRz} \frac{\Lambda}{2 \pi^2 v^2} \sum_{j=x,y} G^j \sum_{j=x,y} (\bar{\Psi}_a \gamma_0 \gamma_j \Psi_a) (\bar{\Psi}_b \gamma_0 \gamma_j \Psi_b) \]

\[ W^{(2) + (3)}_{SO_{1},SO_z} = 2 \Delta_{SO_{1}} \Delta_{SO_z} \frac{\Lambda}{2 \pi^2 v^2} \sum_{j=x,y} G^j \sum_{j=x,y} (\bar{\Psi}_a \gamma_0 \gamma_j \Psi_a) (\bar{\Psi}_b \gamma_0 \gamma_j \Psi_b), \]

\[ W^{(2) + (3)}_{SO_{1},PM} = 2 \Delta_{SO_{1}} \Delta_{PM} \frac{\Lambda}{2 \pi^2 v^2} \sum_{j=x,y} G^j \sum_{j=x,y} (\bar{\Psi}_a \gamma_0 \gamma_j \Psi_a) (\bar{\Psi}_b \gamma_0 \gamma_j \Psi_b) \]

\[ W^{(2) + (3)}_{SO_{1},MN_{\perp}} = 4 \Delta_{SO_{1}} \Delta_{MN_{\perp}} \frac{\Lambda}{2 \pi^2 v^2} \sum_{j=x,y} G^j \sum_{j=x,y} (\bar{\Psi}_a i \gamma_5 \gamma_j \Psi_a) (\bar{\Psi}_b i \gamma_5 \gamma_j \Psi_b) \]

\[ W^{(2) + (3)}_{SO_{1},MN_{z}} = 2 \Delta_{SO_{1}} \Delta_{MN_{z}} \frac{\Lambda}{2 \pi^2 v^2} \sum_{j=x,y} G^j \sum_{j=x,y} (\bar{\Psi}_a \gamma_0 \gamma_j \Psi_a) (\bar{\Psi}_b \gamma_0 \gamma_j \Psi_b) \]

\[ W^{(2) + (3)}_{SO_{1},AMN_{\perp}} = 4 \Delta_{SO_{1}} \Delta_{AMN_{\perp}} \frac{\Lambda}{2 \pi^2 v^2} \sum_{j=x,y} G^j \sum_{j=x,y} (\bar{\Psi}_a \gamma_0 \gamma_j \Psi_a) (\bar{\Psi}_b \gamma_0 \gamma_j \Psi_b) \]
\[ W^{(2)+(3)}_{SO_{1},AMNz} = 2\Delta SO_1 \Delta AMNz \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} \left( G^\perp_1 \left( \Psi_a i\gamma_5 \Psi_a \right) \left( \bar{\Psi}_b i\gamma_5 \Psi_b \right) + 2\Delta SO_1 \Delta AMNz \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} \left[ (\Psi_a i\gamma_5 \gamma_5 \Psi_a) (\bar{\Psi}_b i\gamma_5 \gamma_5 \Psi_b) + (\Psi_a i\gamma_5 \gamma_5 \Psi_a) (\bar{\Psi}_b i\gamma_5 \gamma_5 \Psi_b) \right] \right), \]  

\[ W^{(2)+(3)}_{SO_{1},CRz} = 2\Delta SO_1 \Delta CRz \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} \left( G^\perp_1 \left( \Psi_a i\gamma_5 \Psi_a \right) \left( \bar{\Psi}_b i\gamma_5 \Psi_b \right) + 2\Delta SO_1 \Delta CRz \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} \sum_{j=x,y} \left( \Psi_a i\gamma_5 \gamma_j \Psi_a \right) \left( \bar{\Psi}_b i\gamma_5 \gamma_j \Psi_b \right) \right), \]  

\[ W^{(2)+(3)}_{SO_{1},CMz} = 4\Delta SO_1 \Delta CM \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} \left( G^\perp_1 \left( \Psi_a i\gamma_5 \Psi_a \right) \left( \bar{\Psi}_b i\gamma_5 \Psi_b \right) + 4\Delta SO_1 \Delta CM \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} \left[ (\Psi_a i\gamma_5 \gamma_5 \Psi_a) (\bar{\Psi}_b i\gamma_5 \gamma_5 \Psi_b) \right] \right), \]  

\[ W^{(2)+(3)}_{SO_{1},PM} \]  

\[ W^{(2)+(3)}_{SO_{1},MNz} = 2\Delta SO_1 \Delta MNz \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} \left( G^\perp_1 \left( \Psi_a i\gamma_5 \gamma_5 \Psi_a \right) \left( \bar{\Psi}_b i\gamma_5 \gamma_5 \Psi_b \right) + 2\Delta SO_1 \Delta MNz \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} \sum_{j=x,y} \left( \Psi_a i\gamma_5 \gamma_j \Psi_a \right) \left( \bar{\Psi}_b i\gamma_5 \gamma_j \Psi_b \right) \right), \]  

\[ W^{(2)+(3)}_{SO_{1},AMNz} = 2\Delta SO_1 \Delta AMNz \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} \left( G^\perp_1 \left( \Psi_a i\gamma_5 \gamma_5 \Psi_a \right) \left( \bar{\Psi}_b i\gamma_5 \gamma_5 \Psi_b \right) + 2\Delta SO_1 \Delta AMNz \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} \left[ (\Psi_a i\gamma_5 \gamma_5 \Psi_a) (\bar{\Psi}_b i\gamma_5 \gamma_5 \Psi_b) \right] \right), \]  

\[ W^{(2)+(3)}_{SO_{1},MN} \]  

\[ W^{(2)+(3)}_{SO_{1},AM} \]  

\[ W^{(2)+(3)}_{SO_{1},CR} \]  

\[ W^{(2)+(3)}_{SO_{1},PM} \]  

\[ W^{(2)+(3)}_{SO_{1},MN} \]
\[ W_{SOz,\text{CRL}}^{(2)+(3)} = 2\Delta SO_z \Delta_{AMNz} \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} \int G^D \left( \Psi_{a\gamma_0 \gamma_z} \Psi_a \right) \left( \Psi_{b\gamma_0 \gamma_b} \right) \]
\[ + 2\Delta SO_z \Delta_{CRL} \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} \int G^D \left( \Psi_{a\gamma_0 \gamma_z} \Psi_a \right) \left( \Psi_{b\gamma_0 \gamma_b} \right) \]
\[ + 2\Delta SO_z \Delta_{CRL} \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} \int G^D \sum_{j=x,y} \left( \Psi_{a\gamma_0 \gamma_j \gamma_z} \Psi_a \right) \left( \Psi_{b\gamma_0 \gamma_j \gamma_b} \right), \] (E83)

\[ W_{SOz,\text{Crz}}^{(2)+(3)} = 2\Delta SO_z \Delta_{Crz} \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} \int G^D \left( \Psi_{a\gamma_0 \gamma_z} \Psi_a \right) \left( \Psi_{b\gamma_0 \gamma_b} \right), \] (E84)

\[ W_{PM,\text{MNz}}^{(2)+(3)} = 2\Delta PM \Delta_{MNz} \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} \int G^D \left( \Psi_{a\gamma_0 \gamma_z} \Psi_a \right) \left( \Psi_{b\gamma_0 \gamma_b} \right) \]
\[ + 2\Delta PM \Delta_{MNz} \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} \int G^D \left( \Psi_{a\gamma_0 \gamma_z} \Psi_a \right) \left( \Psi_{b\gamma_0 \gamma_b} \right) \]
\[ + 2\Delta PM \Delta_{MNz} \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} \int G^D \sum_{j=x,y} \left( \Psi_{a\gamma_0 \gamma_j \gamma_z} \Psi_a \right) \left( \Psi_{b\gamma_0 \gamma_j \gamma_b} \right), \] (E85)

\[ W_{PM,\text{MNNz}}^{(2)+(3)} = 2\Delta PM \Delta_{MNNz} \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} \int G^D \left( \Psi_{a\gamma_0 \gamma_z} \Psi_a \right) \left( \Psi_{b\gamma_0 \gamma_b} \right) \]
\[ + 2\Delta PM \Delta_{MNNz} \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} \int G^D \left( \Psi_{a\gamma_0 \gamma_z} \Psi_a \right) \left( \Psi_{b\gamma_0 \gamma_b} \right) \]
\[ + 2\Delta PM \Delta_{MNNz} \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} \int G^D \sum_{j=x,y} \left( \Psi_{a\gamma_0 \gamma_j \gamma_z} \Psi_a \right) \left( \Psi_{b\gamma_0 \gamma_j \gamma_b} \right), \] (E86)

\[ W_{PM,\text{AMNNz}}^{(2)+(3)} = 2\Delta PM \Delta_{AMNNz} \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} \int G^D \left( \Psi_{a\gamma_0 \gamma_z} \Psi_a \right) \left( \Psi_{b\gamma_0 \gamma_b} \right) \]
\[ + 2\Delta PM \Delta_{AMNNz} \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} \int G^D \left( \Psi_{a\gamma_0 \gamma_z} \Psi_a \right) \left( \Psi_{b\gamma_0 \gamma_b} \right) \]
\[ + 2\Delta PM \Delta_{AMNNz} \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} \int G^D \sum_{j=x,y} \left( \Psi_{a\gamma_0 \gamma_j \gamma_z} \Psi_a \right) \left( \Psi_{b\gamma_0 \gamma_j \gamma_b} \right), \] (E87)

\[ W_{PM,\text{AMNZ}}^{(2)+(3)} = 2\Delta PM \Delta_{AMNZ} \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} \int G^D \left( \Psi_{a\gamma_0 \gamma_z} \Psi_a \right) \left( \Psi_{b\gamma_0 \gamma_b} \right) \]
\[ + 2\Delta PM \Delta_{AMNZ} \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} \int G^D \left( \Psi_{a\gamma_0 \gamma_z} \Psi_a \right) \left( \Psi_{b\gamma_0 \gamma_b} \right) \]
\[ + 2\Delta PM \Delta_{AMNZ} \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} \int G^D \sum_{j=x,y} \left( \Psi_{a\gamma_0 \gamma_j \gamma_z} \Psi_a \right) \left( \Psi_{b\gamma_0 \gamma_j \gamma_b} \right), \] (E88)

\[ W_{PM,\text{CRz}}^{(2)+(3)} = 2\Delta PM \Delta_{CRz} \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} \int G^D \left( \Psi_{a\gamma_0 \gamma_z} \Psi_a \right) \left( \Psi_{b\gamma_0 \gamma_b} \right) \]
\[ + 2\Delta PM \Delta_{CRz} \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} \int G^D \sum_{j=x,y} \left( \Psi_{a\gamma_0 \gamma_j \gamma_z} \Psi_a \right) \left( \Psi_{b\gamma_0 \gamma_j \gamma_b} \right), \] (E89)

\[ W_{PM,\text{Chz}}^{(2)+(3)} = 2\Delta PM \Delta_{Chz} \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} \int G^D \sum_{j=x,y} \left( \Psi_{a\gamma_0 \gamma_j \gamma_z} \Psi_a \right) \left( \Psi_{b\gamma_0 \gamma_j \gamma_b} \right) \]
\[ + 2\Delta PM \Delta_{Chz} \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} \int G^D \left( \Psi_{a\gamma_0 \gamma_z} \Psi_a \right) \left( \Psi_{b\gamma_0 \gamma_b} \right), \] (E90)

\[ W_{MNz,AMNNz}^{(2)+(3)} = 2\Delta MNz,\Delta_{MNNz} \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} \int G^D \left( \Psi_{a\gamma_0 \gamma_z} \Psi_a \right) \left( \Psi_{b\gamma_0 \gamma_b} \right), \] (E91)

\[ W_{MNz,AMNNz}^{(2)+(3)} = 4\Delta MNz,\Delta_{MNNz} \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} \int G^D \sum_{j=x,y} \left( \Psi_{a\gamma_0 \gamma_j \gamma_z} \Psi_a \right) \left( \Psi_{b\gamma_0 \gamma_j \gamma_b} \right) \]
\[ + 4\Delta MNz,\Delta_{MNNz} \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} \int G^D \left( \Psi_{a\gamma_0 \gamma_z} \Psi_a \right) \left( \Psi_{b\gamma_0 \gamma_b} \right), \]
\[ W_{\text{MIN}_{\perp}, \text{AMN}_{z}}^{(2)+(3)} = 2 \Delta_{\text{MIN}_{\perp}, \text{AMN}_{z}} \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} G_1^z \left( \Psi_a \gamma_0 \Psi_a \right) \left( \Psi_b \gamma_0 \Psi_b \right) + 2 \Delta_{\text{MIN}_{\perp}, \text{AMN}_{z}} \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} G_1^x \left( \Psi_a \gamma_0 \gamma_j \Psi_a \right) \left( \Psi_b \gamma_0 \gamma_j \Psi_b \right) + 2 \Delta_{\text{MIN}_{\perp}, \text{AMN}_{z}} \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} G_1^D \left( \Psi_a i \gamma_j \Psi_a \right) \left( \Psi_b i \gamma_j \Psi_b \right), \]  
(E92)
\[ W_{\text{AMNz},\text{CRz}}^{(2)+(3)} = 2 \Delta_{\text{AMNz}} \Delta_{\text{CRz}} \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} \left( \bar{\Psi} \right) \Delta_{\text{AMNz}} \Delta_{\text{CRz}} \left( \bar{\Psi} \right) \]

\[ W_{\text{CRz},\text{CRz}}^{(2)+(3)} = 2 \Delta_{\text{CRz}} \Delta_{\text{CRz}} \frac{\Lambda}{2\pi^2 v^2 \sqrt{\eta}} \left( \bar{\Psi} \right) \Delta_{\text{AMNz}} \Delta_{\text{CRz}} \left( \bar{\Psi} \right) \]

\[ W^{(2)+(3)} \text{ can be written as} \]

\[ W^{(2)+(3)} = \sum_{i \leq j} \sum_{i,j} W_{i,j}^{(2)+(3)} = \frac{\delta W^{(2)+(3)}}{2} (\bar{\Psi} \Gamma_i \Psi_a) (\bar{\Psi} \Gamma_j \Psi_b), \]

where

\[ \delta W_{\text{AMNz},\text{CRz}}^{(2)+(3)} = 4 \left[ (\Delta_M \Delta_{\text{SOz}} + \Delta_{\text{PM}} \Delta_{\text{MNz}} + \Delta_{\text{AMNz}} \Delta_{\text{CRz}} + \Delta_{\text{AMNz}} \Delta_{\text{CRz}}) \right] \bar{G}_1 \left( \bar{\Psi} \right) + (\Delta_M \Delta_{\text{SOz}} + \Delta_{\text{PM}} \Delta_{\text{MNz}} + \Delta_{\text{AMNz}} \Delta_{\text{CRz}}) \bar{G}_1 \]

\[ \delta W_{\text{CRz},\text{CRz}}^{(2)+(3)} = 2 \left[ (\Delta_M \Delta_{\text{SOz}} + \Delta_{\text{PM}} \Delta_{\text{MNz}} + \Delta_{\text{AMNz}} \Delta_{\text{CRz}} + \Delta_{\text{AMNz}} \Delta_{\text{CRz}}) \right] \bar{G}_1 \]

\[ \delta W^{(2)+(3)} \text{ can be written as} \]

\[ W^{(2)+(3)} = \sum_{i \leq j} \sum_{i,j} W_{i,j}^{(2)+(3)} = \frac{\delta W^{(2)+(3)}}{2} (\bar{\Psi} \Gamma_i \Psi_a) (\bar{\Psi} \Gamma_j \Psi_b), \]

where

\[ \delta W_{\text{AMNz},\text{CRz}}^{(2)+(3)} = 4 \left[ (\Delta_M \Delta_{\text{SOz}} + \Delta_{\text{PM}} \Delta_{\text{MNz}} + \Delta_{\text{AMNz}} \Delta_{\text{CRz}} + \Delta_{\text{AMNz}} \Delta_{\text{CRz}}) \right] \bar{G}_1 \]

\[ \delta W_{\text{CRz},\text{CRz}}^{(2)+(3)} = 2 \left[ (\Delta_M \Delta_{\text{SOz}} + \Delta_{\text{PM}} \Delta_{\text{MNz}} + \Delta_{\text{AMNz}} \Delta_{\text{CRz}} + \Delta_{\text{AMNz}} \Delta_{\text{CRz}}) \right] \bar{G}_1 \]
\[+ \Delta^2_{PM} + 2 \Delta^2_{MN \perp} + \Delta^2_{MN \parallel} + 2 \Delta^2_{AMN \perp} + 2 \Delta^2_{AMN \parallel} + 2 \Delta^2_{\sigma t \epsilon_{RL}} + \Delta^2_{\epsilon CRL} \] 
\[\frac{\Lambda}{2 \pi^2 v^2} \sqrt{\eta} \ell, \quad (E112)\]

\[\delta \Delta^{(2) + (3)}_{MN \perp} = 2 \left[ (\Delta C_{\Delta PM} + \Delta M_{\Delta CR} + \Delta AC_{\Delta AMN \perp} + 4 \Delta_{SO1 \Delta AMN \perp} + \Delta_{SO2 \Delta AMN \parallel} \right] \frac{\Lambda}{2 \pi^2 v^2} \sqrt{\eta} \ell, \quad (E113)\]

\[\delta \Delta^{(2) + (3)}_{MN \parallel} = 4 \left[ (\Delta M_{\Delta CR} + \Delta AC_{\Delta AMN \perp} + \Delta_{SO1 \Delta AMN \perp} + \Delta_{SO2 \Delta AMN \parallel} \right] \frac{\Lambda}{2 \pi^2 v^2} \sqrt{\eta} \ell, \quad (E114)\]

\[\delta \Delta^{(2) + (3)}_{AMN \perp} = 4 \left[ (\Delta C_{\Delta CR} + \Delta M_{\Delta CR} + \Delta AC_{\Delta AMN \perp} + \Delta_{SO1 \Delta AMN \perp} + \Delta_{SO2 \Delta AMN \parallel} \right] \frac{\Lambda}{2 \pi^2 v^2} \sqrt{\eta} \ell, \quad (E115)\]

\[\delta \Delta^{(2) + (3)}_{AMN \parallel} = 4 \left[ (\Delta C_{\Delta CR} + \Delta M_{\Delta CR} + \Delta AC_{\Delta AMN \perp} + \Delta_{SO1 \Delta AMN \perp} + \Delta_{SO2 \Delta AMN \parallel} \right] \frac{\Lambda}{2 \pi^2 v^2} \sqrt{\eta} \ell, \quad (E116)\]

\[\delta \Delta^{(2) + (3)}_{CR \perp} = 2 \left[ (\Delta C_{\Delta AMN z} + \Delta M_{\Delta AMN z} + \Delta AC_{\Delta CR} + \Delta_{SO2 \Delta PM} \right] \frac{\Lambda}{2 \pi^2 v^2} \sqrt{\eta} \ell, \quad (E117)\]

\[\delta \Delta^{(2) + (3)}_{CR \parallel} = 4 \left[ (\Delta C_{\Delta AMN \perp} + \Delta M_{\Delta AMN \perp} + \Delta AC_{\Delta CR \perp} + \Delta_{SO1 \Delta AMN \perp} + \Delta_{SO2 \Delta AMN \parallel} \right] \frac{\Lambda}{2 \pi^2 v^2} \sqrt{\eta} \ell, \quad (E118)\]

3. Correction induced by long-range Coulomb interaction from Feynman diagram [13] (d)

Substituting the expressions of fermion and boson propagators into Eq. [E6], we get

\[W_i = \Delta_i \left( \bar{\psi}_{a} \Gamma_{i} \psi_{a} \right) \left( \bar{\psi}_{b} \Gamma_{i} \psi_{b} \right) \frac{1}{2} \left( F_0^\dagger + F_0 \right) \ell \]

\[+ \Delta_i \left( \bar{\psi}_{a} \Gamma_{i} \psi_{a} \right) \left( \bar{\psi}_{b} \Gamma_{a} \gamma_{5} \gamma_{0} \psi_{b} \right) \alpha F_0^\dagger \ell \]

\[+ \Delta_i \left( \bar{\psi}_{a} \Gamma_{a} \psi_{a} \right) \left( \bar{\psi}_{b} \Gamma_{a} \gamma_{5} \gamma_{0} \psi_{b} \right) \alpha F_0^\dagger \ell \]

\[+ \Delta_i \left( \bar{\psi}_{a} \Gamma_{i} \psi_{a} \right) \left( \bar{\psi}_{b} \Gamma_{a} \gamma_{5} \gamma_{0} \psi_{b} \right) \alpha F_0 \ell \]

\[+ \Delta_i \left( \bar{\psi}_{a} \Gamma_{a} \psi_{a} \right) \left( \bar{\psi}_{b} \Gamma_{a} \gamma_{5} \gamma_{0} \psi_{b} \right) \alpha F_0 \ell \]

\[- \Delta_i \left( \bar{\psi}_{a} \Gamma_{i} \psi_{a} \right) \left( \bar{\psi}_{b} \Gamma_{a} \gamma_{5} \gamma_{0} \psi_{b} \right) \alpha F_3^\dagger \ell, \quad (E119)\]

where

\[F_3^\dagger = \frac{1}{16 \pi} \int_{0}^{\pi} \sin(\phi) d\phi \int_{0}^{2\pi} d\theta \frac{\sin^2(\phi)}{\Xi^3}, \quad (E120)\]

\[F_3^\dagger = \frac{1}{8 \pi} \int_{0}^{\pi} \sin(\phi) d\phi \int_{0}^{2\pi} d\theta \times \left( \frac{\sqrt{\pi \gamma}}{\Xi^3} \right)^2 \cos^2(\phi), \quad (E121)\]

\[F_3^D = \frac{1}{8 \pi} \int_{0}^{\pi} \sin(\phi) d\phi \int_{0}^{2\pi} d\theta \times \frac{D^2 \alpha^2 \sin^2(\phi) \cos^2(2\theta)}{\Xi^3}, \quad (E122)\]
\[ F_3^m = \frac{1}{8\pi} \int_0^\pi \sin(\varphi) d\varphi \int_0^{2\pi} d\theta \times \left( \frac{\frac{m}{\sin^2(\varphi)} - \frac{B_A}{n} \cos^2(\varphi)}{\Xi^3} \right)^2 \]  

\( \Xi \) is given by Eq. (B19).

\[ W_i^{(4)} \] can be also further written as

\[ W_i^{(4)} = \frac{\delta \Delta C}{2} (\bar{\Psi}_a \Gamma_1 \Psi_a) (\bar{\Psi}_b \Gamma_1 \Psi_b), \]  

where

\[ \delta \Delta C^{(4)} = 2 \Delta C \alpha \left[ \frac{1}{2} (F_0^+ + F_0^-) - 2F_3^+ - F_3^- \right] \ell, \]  

\( \delta \Delta M^{(4)} = 2 \Delta M \alpha \left[ \frac{1}{2} (F_0^+ + F_0^-) + 2F_3^+ + F_3^- \right] \ell, \]  

\[ \delta \Delta AC^{(4)} = 2 \Delta AC \alpha \left[ \frac{1}{2} (F_0^+ + F_0^-) - 2F_3^+ - F_3^- \right] \ell, \]  

\[ \delta \Delta SO^{(4)} = 2 \Delta SO \alpha \left[ \frac{1}{2} (F_0^+ + F_0^-) - 2F_3^+ + F_3^- \right] \ell, \]  

\[ \delta \Delta PM^{(4)} = 2 \Delta PM \alpha \left[ \frac{1}{2} (F_0^+ + F_0^-) + 2F_3^+ + F_3^- \right] \ell, \]  

\[ \delta \Delta MN^{(4)} = 2 \Delta MN \alpha \left[ \frac{1}{2} (F_0^+ + F_0^-) - 2F_3^+ + F_3^- \right] \ell, \]  

\[ \delta \Delta AMN^{(4)} = 2 \Delta AMN \alpha \left[ \frac{1}{2} (F_0^+ + F_0^-) + 2F_3^+ - F_3^- \right] \ell, \]  

\[ \delta \Delta AMN^z = 2 \Delta AMN \alpha \left[ \frac{1}{2} (F_0^+ + F_0^-) + 2F_3^+ - F_3^- \right] \ell, \]  

\[ \delta \Delta CR^{(4)} = 2 \Delta CR \alpha \left[ \frac{1}{2} (F_0^+ + F_0^-) + 2F_3^+ + F_3^- \right] \ell, \]  

\[ \delta \Delta CR^z = 2 \Delta CR \alpha \left[ \frac{1}{2} (F_0^+ + F_0^-) + 2F_3^+ - F_3^- \right] \ell, \]  

\[ \delta \Delta C = 0. \]  

4. Correction induced by long-range Coulomb interaction from Feynman diagram 14 (e)

Substituting Eqs. (A11) and (A32) into Eq. (E7), one can obtain

\[ W_C^{(5)} = \Delta C (\bar{\Psi}_a \gamma_0 \Psi_a) \Pi(k) D_0(k) (\bar{\Psi}_b \gamma_0 \Psi_b) \]

\[ = \Delta C (\bar{\Psi}_a \gamma_0 \Psi_a) (\bar{\Psi}_b \gamma_0 \Psi_b) \]  

It is approximated as

\[ W_C^{(5)} \approx -\Delta C \left( \frac{C_1 + C_2}{2} \right) (\bar{\Psi}_a \gamma_0 \Psi_a) (\bar{\Psi}_b \gamma_0 \Psi_b) \]

It is easy to verify that

\[ W_M^{(5)} = 0, \]  

\[ W_{AC}^{(5)} = 0, \]  

\[ W_{SO}^{(5)} = 0, \]  

\[ W_{SOz}^{(5)} = 0, \]  

\[ W_{PM}^{(5)} = 0, \]  

\[ W_{MN}^{(5)} = 0, \]  

\[ W_{AMN}^{(5)} = 0, \]  

\[ W_{CR}^{(5)} = 0, \]  

\[ W_{CRz}^{(5)} = 0. \]  

\[ W_i^{(5)} \] can be written as

\[ W_i^{(5)} = \frac{\delta \Delta I^{(5)}}{2} (\bar{\Psi}_a \Gamma_1 \Psi_a) (\bar{\Psi}_b \Gamma_1 \Psi_b), \]

where

\[ \delta \Delta C^{(5)} = -\Delta C \alpha (\bar{\Psi}_2 \gamma_0 \Psi_2) \ell. \]
\[ \delta \Delta_{M}^{(5)} = 0, \quad \delta \Delta_{AC}^{(5)} = 0, \quad \delta \Delta_{SO}^{(5)} = 0, \quad \delta \Delta_{PM}^{(5)} = 0, \quad \delta \Delta_{MNz}^{(5)} = 0, \quad \delta \Delta_{AMNz}^{(5)} = 0, \quad \delta \Delta_{CRL}^{(5)} = 0, \quad \delta \Delta_{CFz}^{(5)} = 0. \] (E154)

5. Summary of the corrections

The total contribution is given by

\[ \delta \Delta_{i} = \delta \Delta_{i}^{(1)} + \delta \Delta_{i}^{(2)+(3)} + \delta \Delta_{i}^{(4)} + \delta \Delta_{i}^{(5)}. \] (E165)

The concrete expressions of \( \Delta_{i} \) can be obtained through Eqs. (E114)-(E125), Eqs. (E126)-(E136), (E143)-(E144) into Eq. (E165).

Appendix F: Derivation of RG equations

The free action of fermions is

\[ S_{\Psi} = \int \frac{dk_{0}}{2\pi} \frac{d^{3}k}{(2\pi)^{3}} \bar{\Psi}(k_{0}, k) \left\{ i \left[ k_{0} \gamma_{0} + v \left( k_{x} \gamma_{x} + k_{y} \gamma_{y} \right) + v_{z} k_{z} \gamma_{z} + D \left( k_{x}^{2} - k_{y}^{2} \right) \gamma_{5} \right] + m - B_{\perp} k_{\perp}^{2} \right\} \Psi(k_{0}, k). \] (F1)

Considering the corrections of fermion self-energies induced by Coulomb interaction and disorder, the action of fermions becomes

\[ S_{\Psi} = \int \frac{dk_{0}}{2\pi} \frac{d^{3}k}{(2\pi)^{3}} \bar{\Psi}(k_{0}, k) \left\{ i \left[ k_{0} \gamma_{0} + v \left( k_{x} \gamma_{x} + k_{y} \gamma_{y} \right) + v_{z} k_{z} \gamma_{z} + D \left( k_{x}^{2} - k_{y}^{2} \right) \gamma_{5} \right] + m - B_{\perp} k_{\perp}^{2} \right\} \Psi(k_{0}, k). \] (F2)

Employing the transformations

\[ k_{x} = k_{x}' e^{-\ell}, \quad k_{y} = k_{y}' e^{-\ell}, \quad k_{z} = k_{z}' e^{-\ell}, \quad k_{\perp} = k_{\perp}' e^{-\ell}, \quad \Psi = \Psi' e^{\frac{1}{2} \left( -k_{0}'^{2} - 2k_{\perp}' \right)}, \quad v = v' e^{-\left( C_{m} - C_{a} \right) \ell}, \quad v_{z} = v_{z}' e^{-\left( C_{m} - C_{a} \right) \ell}, \quad m = m' e^{-\left( 1 + C_{m} + C_{a} \right) \ell}, \quad B_{\perp} = B_{\perp}' e^{\left( 1 + C_{m} + C_{a} \right) \ell}, \quad B_{z} = B_{z}' e^{\left( 1 + C_{m} + C_{a} \right) \ell}, \quad D = D' e^{\left( 1 - C_{m} + C_{a} \right) \ell}. \] (F3)

the action of fermions can be further written as

\[ S_{\Psi'} = \int \frac{dk_{0}'}{2\pi} \frac{d^{3}k'}{(2\pi)^{3}} \bar{\Psi}'(k_{0}', k') \left\{ i \left[ k_{0}' \gamma_{0} + v' \left( k_{x}' \gamma_{x} + k_{y}' \gamma_{y} \right) + k_{z}' \gamma_{z} e^{C_{a} \ell} \right] + D' \left( k_{x}'^{2} - k_{y}'^{2} \right) \gamma_{5} \right] + m' - B_{\perp}' k_{\perp}'^{2} - B_{z}' k_{z}'^{2} \right\} \Psi'(k_{0}', k'). \] (F4)

which recovers the original form of fermion action.

The free action of boson \( \phi \) is

\[ S_{\phi} = \int \frac{dk_{0}}{2\pi} \frac{d^{3}k}{(2\pi)^{3}} \phi(k_{0}, k) \left( k_{\perp}^{2} + \eta k_{z}^{2} \right) \phi(k_{0}, k). \] (F5)

Including the correction to self-energy of boson, the action of \( \phi \) becomes

\[ S_{\phi} = \int \frac{dk_{0}}{2\pi} \frac{d^{3}k}{(2\pi)^{3}} \phi(k_{0}, k) \left( k_{\perp}^{2} + \eta k_{z}^{2} - \Pi(0, k) \right) \times \phi(k_{0}, k), \] (F6)

\[ \approx \int \frac{dk_{0}}{2\pi} \frac{d^{3}k}{(2\pi)^{3}} \phi(k_{0}, k) \left( k_{\perp}^{2} e^{C_{a} \ell} + \eta k_{z}^{2} e^{C_{a} \ell} \right) \times \phi(k_{0}, k). \] (F7)

Using the transformations as shown in Eqs. (E3)- (E6), and

\[ \phi = \phi' e^{i \left( \gamma_{-} - C_{a} \ell \right)}, \quad \eta = \eta' e^{i \left( \gamma_{-} - C_{a} \ell \right)}, \] (F8)

the action of boson can be written as

\[ S_{\phi'} = \int \frac{dk_{0}'}{2\pi} \frac{d^{3}k'}{(2\pi)^{3}} \phi'(k_{0}', k') \left( k_{\perp}'^{2} + \eta' k_{z}'^{2} \right) \times \phi(k_{0}', k'). \] (F9)

which has the same form as the original boson action.

The action of fermion-boson coupling is

\[ S_{\Psi \phi} = ig \int \frac{dk_{0,1}}{2\pi} \frac{d^{3}k_{1}}{(2\pi)^{3}} \frac{dk_{0,2}}{2\pi} \frac{d^{3}k_{2}}{(2\pi)^{3}} \bar{\Psi}(k_{0,1}, k_{1}) \times \gamma_{0} \Psi(k_{0,2}, k_{2}) \phi(k_{0,1} - k_{0,2}, k_{1} - k_{2}). \] (F10)
Considering the vertex correction, the action has the form
\[ S_{\phi} = i (g + \delta g) \int \frac{dk_{0,1}}{2\pi (2\pi)^3} \frac{dk_{0,2}}{2\pi (2\pi)^3} \frac{dk_2}{2\pi (2\pi)^3} \bar{\Psi}(k_{0,1}, k_1) \times \gamma_0 \Psi(k_{0,2}, k_2) \phi(k_{0,1} - k_{0,2}, k_1 - k_2) \]
\[ \approx i g e^{C_{\phi}} e^{i \ell} \int \frac{dk_{0,1}}{2\pi (2\pi)^3} \frac{dk_{0,2}}{2\pi (2\pi)^3} \frac{dk_2}{2\pi (2\pi)^3} \bar{\Psi}(k_{0,1}, k_1) \times \gamma_0 \Psi(k_{0,2}, k_2) \phi(k_{0,1} - k_{0,2}, k_1 - k_2). \]
Applying the transformations as shown in Eqs. (F3)-(F7), (F17), and
\[ g = g' e^{C_{\phi}} e^{i \ell}, \]
the action can be written as
\[ S_{\phi'} = i g' \int \frac{dk_{0,1}^d k_1^d}{(2\pi)^{11}} \frac{dk_{0,2}^d k_2^d}{(2\pi)^{11}} \bar{\Psi}'(k_{0,1}', k_1') \times \gamma_0 \Psi'(k_{0,2}', k_2') \phi(k_{0,1}' - k_{0,2}', k_1' - k_2'). \]
which recovers the original form of the action of fermion-boson coupling. The action of fermion-disorder coupling is given by
\[ S_{dis} = -\sum_{j=0}^{3} \frac{\Delta_j + \delta \Delta_j}{2} \int \frac{dk_{0,1}' dk_{0,2}' dk_1' dk_2'}{(2\pi)^{11}} \bar{\Psi}_a(k_{0,1}', k_1') \Psi_a(k_{0,1}, k_1) \Gamma_j \]
\[ \times \bar{\Psi}_b(k_{0,2}', k_2') \Psi_b(k_{0,2}, k_2) \Gamma_j \]
\[ \times \Psi_b(k_{0,2}, k_1 + k_2 + k_3). \]
Including the corrections to one-loop order, the action of fermion-disorder coupling becomes
\[ S_{dis} = -\sum_{j=0}^{3} \frac{\Delta_j + \delta \Delta_j}{2} \int \frac{dk_{0,1}' dk_{0,2}' dk_1' dk_2'}{(2\pi)^{11}} \bar{\Psi}_a(k_{0,1}', k_1') \Psi_a(k_{0,1}, k_1) \Gamma_j \]
\[ \times \bar{\Psi}_b(k_{0,2}', k_2') \Psi_b(k_{0,2}, k_2) \Gamma_j \]
\[ \times \Psi_b(k_{0,2}, k_1 + k_2 + k_3). \]
Using the transformations as shown in Eqs. (F3)-(F7), we obtain
\[ S_{dis} = -\sum_{j=0}^{3} \frac{\Delta_j + \delta \Delta_j}{2} e^{(1 - 2C_{\phi}) \ell} \]
\[ \times \int \frac{dk_{0,1}' dk_{0,2}' dk_1' dk_2'}{(2\pi)^{11}} \bar{\Psi}_a(k_{0,1}', k_1') \Gamma_j \]
\[ \times \bar{\Psi}_a(k_{0,1}, k_1) \Psi_a(k_{0,1}', k_1') \Gamma_j \]
\[ \times \Psi_b(k_{0,2}, k_1 + k_2 + k_3). \]
\[ \approx -\sum_{j=0}^{3} \frac{[\Delta_j + \Delta_j (1 - 2C_{\phi}) \ell + \delta \Delta_j]}{2} \]
\[ \times \int \frac{dk_{0,1}' dk_{0,2}' dk_1' dk_2'}{(2\pi)^{11}} \bar{\Psi}_a(k_{0,1}', k_1') \Gamma_j \]
\[ \times \bar{\Psi}_a(k_{0,1}, k_1) \Psi_a(k_{0,1}', k_1') \Gamma_j \]
\[ \times \Psi_b(k_{0,2}, k_1 + k_2 + k_3). \]
Adopting the transformation
\[ \Delta'_j = \Delta_j + \Delta_j (1 - 2C_{\phi}^0) \ell + \delta \Delta_j, \]
the action for fermion-disorder coupling can be written as
\[ S_{dis} = -\sum_{j=0}^{3} \frac{\Delta'_j}{2} \int \frac{dk_{0,1}' dk_{0,2}' dk_1' dk_2'}{(2\pi)^{11}} \bar{\Psi}_a'(k_{0,1}', k_1') \Psi_a'(k_{0,1}, k_1) \Gamma_j \]
\[ \times \bar{\Psi}_b'(k_{0,2}', k_2') \Psi_b'(k_{0,2}, k_2) \Gamma_j \]
\[ \times \Psi_b(k_{0,2}, k_1 + k_2 + k_3). \]
which recovers the original form of the action of fermion-disorder coupling. From Eqs. (F8)-(F13), (F18), (F22), and (F27), we obtain the RG equations
\[ \frac{dv}{dl} = (C_v - C_{\phi}^0) v, \]
\[ \frac{dv_z}{dl} = (C_{v_z} - C_{\phi}^0) v_z, \]
\[ \frac{dm}{dl} = (1 + C_m + C_{\phi}^0 - C_{\phi}^0) m, \]
\[ \frac{dB_v}{dl} = (-1 - C_{\phi}^0) B_v, \]
\[ \frac{dB_z}{dl} = (1 + C_D - C_{\phi}^0) B_z, \]
\[ \frac{dD}{dl} = (C_z - C_{\phi}^0) D, \]
\[ \frac{d\eta}{dl} = (C_z - C_{\phi}^0) \eta, \]
\[ \frac{dg}{dl} = -\frac{C_{\phi}^0}{2} g, \]
\[ \frac{d\alpha}{dl} = -\alpha \left[ C_v - C_{\phi}^0 + \frac{1}{2} (C_z + C_{\phi}^0) \right], \]
\[ \frac{d\zeta}{dl} = \left[ C_v - C_{\phi}^0 - \frac{1}{2} (C_z + C_{\phi}^0) \right], \]
\[ \frac{d\Delta_j}{dl} = \Delta_j (1 - 2C_{\phi}^0) + \frac{d\Delta_j}{dl}, \]
where \( \zeta \) is defined as
\[ \zeta = \frac{v_z}{v \sqrt{\eta}}. \]
Adopting the transformations
\[ \frac{m}{v \Lambda} \rightarrow m, \]
\[ \frac{B_{\perp} A}{v} \rightarrow B_{\perp}, \]
\[ \frac{B_{\perp} A}{v \Lambda} \rightarrow B_z, \]
\[ \frac{DA}{v} \rightarrow D, \]
the RG equations can be further compactly written as

\[
\frac{d\Delta C}{d\ell} = -\Delta C + 2\Delta C (\Delta C + \Delta M + \Delta AC + 2\Delta SO\perp + \Delta SOz + \Delta PM + 2\Delta MN\perp + \Delta MNz + 2\Delta AMN\perp
\]
\[
+ \Delta AMNz + 2\Delta CR\perp + \Delta CRz) \left(G_1^f + G_1^D + G_1^m\right)
\]
\[
+ 4 \left(\Delta C\Delta SO\perp + \Delta PM\Delta AMN\perp + \Delta MN\perp \Delta CRz + \Delta AMNz \Delta CRz + \Delta MNz \Delta CRz\right) G_1^f + (\Delta C \Delta SOz + \Delta PM \Delta MNz
\]
\[
+ 2\Delta AMN\perp \Delta CRz) G_1^f + (\Delta C \Delta PM + 2\Delta SO\perp \Delta AMNz + \Delta SOz \Delta AMNz) G_1^D
\]
\[
+ (\Delta C \Delta M + 2\Delta MN\perp \Delta AMNz + \Delta MNz \Delta AMNz) G_1^m\right]
\]
\[
- \Delta C\alpha \left(2F_0^3 + \frac{1}{2}F_2^3 + \frac{3}{2}F_3^3\right),
\]

\[
\frac{d\Delta M}{d\ell} = -\Delta M + 2\Delta M (\Delta C + \Delta M - \Delta AC - 2\Delta SO\perp - \Delta SOz - \Delta PM + 2\Delta MN\perp + \Delta MNz + 2\Delta AMN\perp
\]
\[
+ \Delta AMNz - 2\Delta CR\perp - 2\Delta CRz) \left(-G_1^f - G_1^D + G_1^m\right)
\]
\[
+ 4 \left(\Delta C\Delta SO\perp + \Delta PM\Delta AMN\perp + \Delta MN\perp \Delta CRz + \Delta AMNz \Delta CRz + \Delta MNz \Delta CRz\right) G_1^f + (\Delta C \Delta SOz + \Delta PM \Delta MNz
\]
\[
+ 2\Delta AMN\perp \Delta CRz) G_1^f + (\Delta C \Delta PM + 2\Delta SO\perp \Delta AMNz + \Delta SOz \Delta AMNz) G_1^D
\]
\[
+ (\Delta C \Delta M + 2\Delta MN\perp \Delta AMNz + \Delta MNz \Delta AMNz) G_1^m\right]
\]
\[
+ 2\Delta M\alpha \left[\frac{1}{2}(-F_0^f + F_0^D) + \frac{1}{4}F_2^f - \frac{1}{2}F_2^D - F_3^f + F_3^D - F_3^m\right],
\]

\[
\frac{d\Delta AC}{d\ell} = -\Delta AC + 2\Delta AC (\Delta C + \Delta M - \Delta AC + 2\Delta SO\perp + \Delta SOz + \Delta PM + 2\Delta MN\perp + \Delta MNz + 2\Delta AMN\perp
\]
\[
- 2\Delta AMN\perp - \Delta AMNz - 2\Delta CR\perp - \Delta CRz) \left(-G_1^f - G_1^D + G_1^m\right)
\]
\[
+ 4 \left(\Delta SO\perp \Delta SOz + \Delta MN\perp \Delta AMNz + \Delta AMN\perp \Delta CRz + \Delta CRz \Delta CRz\right) G_1^f
\]
\[
+ 2 \left(\Delta SO\perp^2 + \Delta MN\perp^2 + \Delta AMN\perp^2 + \Delta CRz \Delta CRz\right) G_1^f + (\Delta AC \Delta PM + 2\Delta SO\perp \Delta CRz + \Delta SOz \Delta CRz) G_1^D
\]
\[
+ (\Delta AC \Delta AC + \Delta AMNz \Delta CRz + 2\Delta MN\perp \Delta CRz) G_1^m\right]
\]
\[
+ 2\Delta AC\alpha \left[\frac{1}{2}(-F_0^f + F_0^D) + \frac{1}{4}F_2^f - \frac{1}{2}F_2^D - F_3^f + F_3^D + F_3^m\right],
\]

\[
\frac{d\Delta SO\perp}{d\ell} = -\Delta SO\perp + 2\Delta SO\perp (\Delta C + \Delta M + \Delta AC + \Delta SOz - \Delta PM - \Delta MNz + \Delta AMNz - \Delta CRz)
\]
\[
\times \left(-G_1^f - G_1^D + G_1^m\right)
\]
\[
+ 2 \left(\Delta C \Delta M + \Delta AC \Delta SO\perp + \Delta PM \Delta CRz + 4\Delta MN\perp \Delta AMNz + \Delta MNz \Delta AMNz\right) G_1^f
\]
\[
+ 2 \left(\Delta AC \Delta SO\perp + \Delta PM \Delta CRz + \Delta MN\perp \Delta AMNz + \Delta MNz \Delta AMNz\right) G_1^D
\]
\[
+ 2 \left(\Delta C \Delta AMNz + \Delta M \Delta MN\perp + \Delta AC \Delta CRz + \Delta SO\perp \Delta PM\right) G_1^m
\]
\[
+ 2 \left(\Delta M \Delta SO\perp + \Delta PM \Delta MN\perp + \Delta AMNz \Delta CRz + \Delta AMNz \Delta CRz\right) G_1^m\right]
\[
\frac{d\Delta_{SOz}}{dt} = -\Delta_{SOz} + 2\Delta_{SOz}(-\Delta C + \Delta M + \Delta AC + 2\Delta SOz - \Delta PM - 2\Delta MNz + \Delta MNz)
+ 2\Delta_{AMNz} - \Delta_{AMNZ} - 2\Delta_{CRz} + \Delta_{CRz}) (-G_1^t + G_1^t - G_1^P + G_1^m)
+ 4 \left( (\Delta C \Delta_{SOz} + \Delta PM \Delta_{CRz} + \Delta MNz \Delta_{AMNZ} + \Delta_{MNz} \Delta_{AMNZ}) G_1^t
+ (\Delta C \Delta_{SOz} + 2\Delta_{MNz} \Delta_{AMNZ} + \Delta_{MNz} \Delta_{AMNZ}) G_1^t
+ (\Delta C \Delta_{AMNZ} + \Delta_{MNz} \Delta_{AMNZ} + \Delta AC \Delta_{CRz} + \Delta SOz \Delta PM) G_1^P
+ (\Delta M \Delta_{SOz} + \Delta PM \Delta_{MNz} + 2\Delta_{AMNZ} \Delta_{CRz}) G_1^m \right)
+ 2\Delta_{SOz} \alpha \left[ \frac{1}{2} (-F_0^t + F_0^z) + \frac{1}{4} F_2^t - \frac{1}{4} F_2^z - 2F_3^t + F_3^d + F_3^m \right],
\] (F59)

\[
\frac{d\Delta_{PM}}{dt} = -\Delta_{PM} + 2\Delta_{PM}(-\Delta C + \Delta M + \Delta AC - 2\Delta SOz - \Delta PM + 2\Delta MNz + \Delta_{MNz})
- 2\Delta_{AMNZ} - \Delta_{AMNZ} + 2\Delta_{CRz} + \Delta_{CRz}) (G_1^t + G_1^t - G_1^P + G_1^m)
+ 4 \left( (\Delta C \Delta_{MNz} + \Delta M \Delta_{AMNZ} + \Delta SOz \Delta_{CRz} + \Delta SOz \Delta_{CRz}) G_1^t
+ (\Delta C \Delta_{MNz} + \Delta M \Delta_{AMNZ} - 2\Delta SOz \Delta_{CRz}) G_1^t
+ (\Delta C \Delta_{SOz} + \Delta PM \Delta_{MNz} + \Delta_{AMNZ} \Delta_{CRz} + \Delta_{AMNZ} \Delta_{CRz}) G_1^P
+ (\Delta M \Delta_{SOz} + \Delta PM \Delta_{MNz} + 2\Delta_{AMNZ} \Delta_{CRz}) G_1^m \right)
+ 2\Delta_{PM} \alpha \left[ \frac{1}{2} (-F_0^t + F_0^z) + \frac{1}{4} F_2^t - \frac{1}{4} F_2^z + 2F_3^t + F_3^d + F_3^m \right],
\] (F60)

\[
\frac{d\Delta_{MNz}}{dt} = -\Delta_{MNz} + 2\Delta_{MNz}(-\Delta C + \Delta M + \Delta AC + \Delta SOz - \Delta PM - \Delta_{MNz} - \Delta_{AMNZ} + \Delta_{CRz})
\times (G_1^t - G_1^P + G_1^m)
+ 2 \left( (\Delta C \Delta_{PM} + \Delta M \Delta_{CRz} + \Delta AC \Delta_{MNz} + 4\Delta SOz \Delta_{AMNZ} + \Delta SOz \Delta_{AMNZ}) G_1^t
+ (\Delta M \Delta_{CRz} + \Delta AC \Delta_{MNz} + \Delta SOz \Delta_{AMNZ} + \Delta SOz \Delta_{AMNZ}) G_1^t
+ (\Delta C \Delta_{SOz} + \Delta PM \Delta_{MNz} + 2\Delta_{AMNZ} \Delta_{CRz}) G_1^P
+ (\Delta C \Delta_{AMNZ} + \Delta M \Delta_{MNz} + \Delta_{AMNZ} \Delta_{CRz} + \Delta SOz \Delta PM) G_1^m \right)
+ 2\Delta_{MNz} \alpha \left[ \frac{1}{2} (-F_0^t + F_0^z) + \frac{1}{4} F_2^t - \frac{1}{4} F_2^z - 2F_3^d + F_3^d - F_3^m \right],
\] (F62)

\[
\frac{d\Delta_{MNz}}{dt} = -\Delta_{MNz} + 2\Delta_{MNz}(-\Delta C + \Delta M + \Delta AC - \Delta SOz - \Delta PM - 2\Delta_{MNz} + \Delta_{MNz})
- 2\Delta_{AMNZ} + \Delta_{AMNZ} - 2\Delta_{CRz} - \Delta_{CRz}) (G_1^t + G_1^t - G_1^P + G_1^m)
+ 4 \left( (\Delta C \Delta_{CRz} + \Delta M \Delta_{SOz} + \Delta AC \Delta_{MNz} + 4\Delta SOz \Delta_{AMNZ} + \Delta SOz \Delta_{MNz}) G_1^t
+ (\Delta C \Delta_{SOz} + \Delta AC \Delta_{MNz} + \Delta SOz \Delta_{AMNZ} + \Delta SOz \Delta_{AMNZ}) G_1^t
+ (\Delta C \Delta_{SOz} + \Delta PM \Delta_{MNz} + \Delta_{MNz} \Delta_{CRz} + \Delta_{MNz} \Delta_{CRz}) G_1^P
+ (\Delta_{AMNZ} + \Delta M \Delta_{MNz} + \Delta_{AMNZ} \Delta_{CRz} + \Delta_{AMNZ} \Delta_{CRz}) G_1^m \right)
+ 2\Delta_{MNz} \alpha \left[ \frac{1}{2} (-F_0^t + F_0^z) + \frac{1}{4} F_2^t - \frac{1}{4} F_2^z - 2F_3^d + F_3^d - F_3^m \right],
\] (F63)

\[
\frac{d\Delta_{AMNZ}}{dt} = -\Delta_{AMNZ} + 2\Delta_{AMNZ}(-\Delta C + \Delta M + \Delta AC - \Delta SOz + \Delta PM - \Delta_{MNz} - \Delta_{AMNZ} - \Delta_{CRz})
\times (-G_1^t + G_1^P + G_1^m)
+ 2 \left( (\Delta C \Delta_{CRz} + \Delta M \Delta_{SOz} + \Delta AC \Delta_{AMNZ} + 4\Delta SOz \Delta_{MNz} + \Delta SOz \Delta_{MNz}) G_1^t
+ (\Delta C \Delta_{CRz} + \Delta AC \Delta_{AMNZ} + \Delta SOz \Delta_{MNz} + \Delta SOz \Delta_{MNz}) G_1^t
+ (\Delta C \Delta_{SOz} + \Delta PM \Delta_{AMNZ} + \Delta_{MNz} \Delta_{CRz} + \Delta_{MNz} \Delta_{CRz}) G_1^P
+ (\Delta_{AMNZ} + \Delta M \Delta_{MNz} + \Delta_{AMNZ} \Delta_{CRz} + \Delta_{AMNZ} \Delta_{CRz}) G_1^m \right)
+ 2\Delta_{AMNZ} \alpha \left[ \frac{1}{2} (-F_0^t + F_0^z) + \frac{1}{4} F_2^t - \frac{1}{4} F_2^z + 2F_3^d + F_3^d - F_3^m \right],
\] (F64)
\[
\frac{d\Delta_{AMNz}}{d\ell} = -\Delta_{AMNz} + 2\Delta_{AMNz} (\Delta C + \Delta M + \Delta AC - 2\Delta SO\perp + \Delta SOz + \Delta PM - 2\Delta_{MN\perp} + \Delta_{MNz} \\
-2\Delta_{AMN\perp} + \Delta_{AMNz} - 2\Delta_{CR\perp} - \Delta_{CRz}) \ (-G_i^1 + G_i^D + G_i^m) \\
+4 \left[ (\Delta C \Delta_{CR\perp} + \Delta AC \Delta_{AMN\perp} + \Delta SO\perp \Delta_{MNz} + \Delta SOz \Delta_{MN\perp}) G_i^1 \\
+ (\Delta M \Delta_{PM} + \Delta SO\perp \Delta_{MN\perp} + \Delta SOz \Delta_{MNz}) G_i^D \\
+ (\Delta C \Delta_{SOz} + \Delta PM \Delta_{AMNz} + 2\Delta_{MN\perp} \Delta_{CR\perp}) G_i^m \right] \\
+2\Delta_{AMNz} \alpha \left[ \frac{1}{2} (-F_0^1 + F_0^z) + \frac{1}{4} F_2^1 - \frac{1}{4} F_2^z + 2 F_3^z - F_3^D - F_3^m \right],
\]

(F65)

\[
\frac{d\Delta_{CR\perp}}{d\ell} = -\Delta_{CR\perp} + 2\Delta_{CR\perp} (-\Delta C + \Delta M - \Delta AC - \Delta SO\perp + \Delta SOz + \Delta PM - 2\Delta_{MN\perp} + \Delta_{MNz} \\
+2\Delta_{AMN\perp} - \Delta_{AMNz} + 2\Delta_{CR\perp} - \Delta_{CRz}) \ (G_i^1 - G_i^D + G_i^m) \\
+4 \left[ (\Delta C \Delta_{AMN\perp} + \Delta M \Delta_{MNz} + \Delta AC \Delta_{CR\perp} + \Delta SO\perp \Delta_{PM}) G_i^1 \\
+ (\Delta AC \Delta_{SOz} + \Delta PM \Delta_{CR\perp} + 2\Delta_{MN\perp} \Delta_{AMNz}) G_i^D \\
+ (\Delta M \Delta_{CRz} + \Delta AC \Delta_{MNz} + 2\Delta_{SO\perp} \Delta_{AMNz}) G_i^m \right] \\
+2\Delta_{CR\perp} \alpha \left[ \frac{1}{2} (-F_0^1 + F_0^z) + \frac{1}{4} F_2^1 - \frac{1}{4} F_2^z + 2 F_3^z + F_3^D + F_3^m \right],
\]

(F66)

\[
\frac{d\Delta_{CRz}}{d\ell} = -\Delta_{CRz} + 2\Delta_{CRz} (-\Delta C + \Delta M - \Delta AC - 2\Delta SO\perp + \Delta SOz + \Delta PM - 2\Delta_{MN\perp} + \Delta_{MNz} \\
+2\Delta_{AMN\perp} - \Delta_{AMNz} + 2\Delta_{CR\perp} - \Delta_{CRz}) \ (G_i^1 + G_i^D + G_i^m) \\
+4 \left[ (\Delta C \Delta_{AMN\perp} + \Delta M \Delta_{MNz} + \Delta AC \Delta_{CR\perp} + \Delta SO\perp \Delta_{PM}) G_i^1 \\
+ (\Delta AC \Delta_{SOz} + \Delta PM \Delta_{CR\perp} + 2\Delta_{MN\perp} \Delta_{AMNz}) G_i^D \\
+ (\Delta M \Delta_{CRz} + \Delta AC \Delta_{MNz} + 2\Delta_{SO\perp} \Delta_{AMNz}) G_i^m \right] \\
+2\Delta_{CRz} \alpha \left[ \frac{1}{2} (-F_0^1 + F_0^z) + \frac{1}{4} F_2^1 - \frac{1}{4} F_2^z + 2 F_3^z - F_3^D - F_3^m \right],
\]

(F67)

where

\[
\begin{align*}
\mathcal{R}_v &= F_0^1, \\
\mathcal{R}_v^z &= 2 F_0^z, \\
\mathcal{R}_m &= m F_0^1 - B_\perp F_0^1 - B_z F_0^z, \\
\mathcal{R}_B_\perp &= -m (F_1^1 + F_1^z) + B_\perp (F_1^1 - F_0^1) + B_z F_1^z, \\
\mathcal{R}_B_z &= -m (F_0^1 + F_0^z - 2 (F_1^1 + F_1^z)) + B_\perp (F_0^1 - 2 F_1^1) + B_z (-F_0^1 + F_0^z - 2 F_1^1 - F_2^1 + F_2^z), \\
\mathcal{R}_D &= (F_0^D - F_0^z), \\
\mathcal{R}_D^z &= (F_2^D - F_2^z), \\
\mathcal{R}_\theta &= \frac{1}{2} F_2^z, \\
\mathcal{R}_\alpha &= F_0^1 + \frac{1}{2} (F_2^1 + F_2^z), \\
\mathcal{R}_C &= 2 F_0^z - F_0^1 - \frac{1}{2} (F_2^1 - F_2^z), \\
C_{0}^{\text{dis}} &= (\Delta C + \Delta M + \Delta AC + 2\Delta SO\perp + \Delta SOz + \Delta PM + 2\Delta_{MN\perp} + \Delta_{MNz} + 2\Delta_{AMN\perp}) \\
&\quad \times (G_0^1 \pm G_0^D) +\Delta_{AMNz} + 2\Delta_{CR\perp} + \Delta_{CRz}) \times (G_0^1 \pm G_0^D), \\
\end{align*}
\]

(F68)

\[
C_{0}^{\text{dis}} = -(\Delta C + \Delta M - \Delta AC - 2\Delta SO\perp - \Delta SOz - \Delta PM + 2\Delta_{MN\perp} + \Delta_{MNz} + 2\Delta_{AMN\perp}) \\
&\quad \times \frac{1}{m} \left[ m (G_0^1 \pm G_0^D) - B_\perp G_0^1 - B_z G_0^z \right],
\]

(F79)

with

\[
\begin{align*}
F_0^1 &= \frac{1}{4\pi} \int_0^\pi \sin(\varphi) d\varphi \int_0^{2\pi} d\theta \frac{\sin^2(\varphi)}{\Xi}, \\
F_0^z &= \frac{1}{4\pi} \int_0^\pi \sin(\varphi) d\varphi \int_0^{2\pi} d\theta \frac{\cos^2(\varphi)}{\Xi}, \\
F_1^1 &= -\frac{1}{4\pi} \int_0^\pi \sin(\varphi) d\varphi \int_0^{2\pi} d\theta \frac{\sin^2(\varphi) \cos(2\varphi)}{\Xi}, \\
F_1^z &= -\frac{1}{4\pi} \int_0^\pi \sin(\varphi) d\varphi \int_0^{2\pi} d\theta \\
&\quad \times \frac{\sin^2(\varphi) \cos(2\varphi)}{\Xi},
\end{align*}
\]

(F80)
\[ F_1^D = \frac{1}{2\pi} \int_0^{2\pi} \sin(\varphi) d\varphi \int_0^{2\pi} d\theta \]

\[ F_2^i = \frac{1}{4\pi} \int_0^{2\pi} \sin(\varphi) d\varphi \int_0^{2\pi} d\theta \left\{ \frac{1 + 4(D^2 + B^2)}{\Xi^3} \sin^2(\varphi) \cos^2(\theta) \right\} \]

\[ F_3^i = \frac{1}{8\pi} \int_0^{2\pi} \sin(\varphi) d\varphi \int_0^{2\pi} d\theta \left\{ \frac{1 + 4(D^2 + B^2)}{\Xi^3} \sin^2(\varphi) \sin^2(\theta) \right\} \]

\[ F_2^b = \frac{1}{4\pi} \int_0^{2\pi} \sin(\varphi) d\varphi \int_0^{2\pi} d\theta \left\{ \frac{\zeta^2 + 4B^2 \cos^2(\varphi)}{\Xi^3} - \frac{(\zeta^2 - 2mBz + 2B^2 \cos^2(\varphi) + 2BzB \sin^2(\varphi))^2}{\Xi^5} \right\} \]

\[ F_3^b = \frac{1}{8\pi} \int_0^{2\pi} \sin(\varphi) d\varphi \int_0^{2\pi} d\theta \left\{ \frac{\zeta^2 + 4B^2 \cos^2(\varphi)}{\Xi^3} - \frac{(\zeta^2 - 2mBz + 2B^2 \cos^2(\varphi) + 2BzB \sin^2(\varphi))^2}{\Xi^5} \right\} \]

\[ F_3^1 = \frac{1}{16\pi} \int_0^{2\pi} \sin(\varphi) d\varphi \int_0^{2\pi} d\theta \frac{\sin^2(\varphi)}{\Xi^3} \]

\[ F_3^2 = \frac{1}{8\pi} \int_0^{2\pi} \sin(\varphi) d\varphi \int_0^{2\pi} d\theta \frac{\sin^2(\varphi)}{\Xi^3} \]

\[ F_3^3 = \frac{1}{8\pi} \int_0^{2\pi} \sin(\varphi) d\varphi \int_0^{2\pi} d\theta \frac{\sin^2(\varphi)}{\Xi^3} \]

\[ F_3^4 = \frac{1}{8\pi} \int_0^{2\pi} \sin(\varphi) d\varphi \int_0^{2\pi} d\theta \frac{\sin^2(\varphi)}{\Xi^3} \]

\[ G_0^1 = \frac{1}{4\pi} \int_0^{2\pi} \sin(\varphi) d\varphi \int_0^{2\pi} d\theta \frac{\sin^2(\varphi)}{\Xi^2} \]

\[ G_0^2 = \frac{1}{4\pi} \int_0^{2\pi} \sin(\varphi) d\varphi \int_0^{2\pi} d\theta \frac{\cos^2(\varphi)}{\Xi^2} \]

\[ G_1^1 = \frac{1}{4\pi} \int_0^{2\pi} \sin(\varphi) d\varphi \int_0^{2\pi} d\theta \frac{\sin^2(\varphi)}{\Xi^2} \]

\[ G_1^2 = \frac{1}{4\pi} \int_0^{2\pi} \sin(\varphi) d\varphi \int_0^{2\pi} d\theta \frac{\cos^2(\varphi)}{\Xi^2} \]

\[ G_0^D = \frac{1}{4\pi} \int_0^{2\pi} \sin(\varphi) d\varphi \int_0^{2\pi} d\theta \]

\[ G_1^D = \frac{1}{4\pi} \int_0^{2\pi} \sin(\varphi) d\varphi \int_0^{2\pi} d\theta \]

\[ G_0^m = \frac{1}{4\pi} \int_0^{2\pi} \sin(\varphi) d\varphi \int_0^{2\pi} d\theta \frac{\sin^2(\varphi)}{\Xi^2} \]

Employing the relation

\[ F_1^1 + F_1^b = F_0^1 - F_0^b \]

we can find that \( R_{Bz} \) can be also written as

\[ R_{Bz} = -m \left( -F_0^1 + 3F_0^b \right) + B_\perp (F_0^1 - 2F_1^1) + B_z \left( -F_1^1 - 3F_1^2 - F_1^b + F_1^D \right) \]

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