Space, Matter and Interactions in a Quantum Early Universe

Part I : Kac-Moody and Borcherds Algebras

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Abstract

We introduce a quantum model for the Universe at its early stages, formulating a mechanism for the expansion of space and matter from a quantum initial condition, with particle interactions and creation driven by algebraic extensions of the Kac-Moody Lie algebra $e_9$. We investigate Kac-Moody and Borcherds algebras, and we propose a generalization that meets further requirements that we regard as fundamental in quantum gravity.
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1 Introduction

This is the first of two papers - see also [1] - describing an algebraic model of quantum gravity.
We believe it is a fresh and original attempt to discuss quantum gravity, based only on the fundamental principles of quantum physics and on well tested experimental evidence, without being biased by any other model or theory. In this first paper, we provide an introduction to the foundations of the model, and we start investigating the mathematical structures that suit our purpose. In the second paper, [1], we will deal with a physical model relying on a particular infinite dimensional algebra.

The Lie algebra at the core of our model has the following features and interpretation:

1. it is an infinite dimensional Lie algebra extending $e_8$, that is regarded as the internal-quantum-number subalgebra, meaning that the $e_8$ roots represent charges and spin of elementary particles;
2. its root lattice is Lorentzian;
3. the subspace of the lattice complementary to that of $e_8$ is interpreted as momentum space.

Remark 1.1. The Lie algebra $e_8$ has been considered by many as a possible algebra for grand-unification as well as for quantum gravity. It has then been considered not suitable after the no-go theorem by Distler and Garibaldi, [2]. We will show in section 2.3 how $e_8$ fulfills the requirements for standard model degrees of freedom and algebras, which seems to contradict the thesis of Distler and Garibaldi. We underline here that it does not, since the hypothesis denoted TOE1 by the authors of [2], in particular the fact that the algebra of the standard model centralizes $sl(2, \mathbb{C})$, not only does not apply, but actually needs not to do so, as it will become obvious in the development of section 2.3.

Algebraic methods are extensively used and successfully exploited in string theory and conformal field theory in two dimensions, through the concept of vertex operator algebras, [3]–[5], in order to describe the interactions between different strings, localized at vertices, analogously to the Feynman diagram vertices. Mathematically, the underlying concept of a vertex algebra was introduced by Borcherds, [6]–[8], in order to prove the Monstrous Moonshine conjecture, [9].

The algebras used in this paper may be regarded as vertex operator algebras in a broader sense, since they are characterized by interaction operators that look like generators of a Lie algebra, and whose product depends upon parameters related to the spacetime creation and expansion. The Lie algebra acts locally, but it is immersed in a wider, vertex-type algebra by means of a mechanism which creates a discrete quantum spacetime.

The Pauli exclusion principle is fulfilled by turning the algebra into a Lie superalgebra, using the Grassmann envelope.

The resulting model is thus intrinsically relativistic, both because of the way spacetime expands and because the Poincaré group acts locally on the Lie algebra. Furthermore, the conservation of charge and momentum is a consequence of the Lie product, and in this respect they are treated at the same level.
1.1 $\mathfrak{g}_u$, a Lie Algebra for Quantum Gravity

At a very fundamental level, we make the following assumptions on quantum gravity, founded on the current theoretical and experimental knowledge in physics.

QG.1) gravity is a characteristic of spacetime;

QG.2) spacetime is *dynamical* and related to matter. Therefore, we assume that it emerges from the existence of particles and their interactions. There is no way of defining distances and time lapses without interactions, so that the creation and expansion of spacetime is itself a rule followed by particle interactions;

QG.3) a suitable mathematical structure at the core of the description of quantum gravity is that of an algebra, which we will henceforth denote by $\mathfrak{g}_u$, whose generators represent the particles and whose product yields the building blocks of the interactions (let us call them *elementary interactions*). As a consequence, the interactions are endowed with a tree structure, thus opening the way for a description of scattering amplitudes in terms of what we would call *gravitahedra*, providing a generalization of the associahedra and permutahedra in the current theory of scattering, [10]-[17]. The structure constants of the algebra determine the quantum amplitudes of the elementary interactions; in particular, we assume $\mathfrak{g}_u$ to be a Lie algebra, because it enables to derive the fundamental conservation laws observed in physics directly from the action of the generators as derivations (Jacobi identity); as in the theory of fields, the interactions may only occur locally, point by point in the expanding spacetime, that can therefore be viewed as a parameter on which the algebra product depends;

QG.4) in agreement with the theory of a big bang, strongly supported by the current observations, we assume the existence of an initial quantum state, mathematically represented by an element of the universal enveloping algebra of $\mathfrak{g}_u$. Such an element is made of generators that can all interact among themselves, thus yielding the first geometrical interpretation: that of a point where particles may interact;

QG.5) a particle has a certain probability amplitude to interact but also not to interact, in which case it expands as described in section 1.2;

QG.6) particles are quantum objects, hence their existence through interactions occurs with certain amplitudes. Therefore, *spacetime acquires a quantum structure*: a point in space and time is where particles are present with a certain amplitude and may interact. The amplitude related to the quantum spacetime point is the sum of the amplitudes for particles to be there. Consequently, the fact that gravitation appears as an attractive force has to be explained through amplitudes and their interference;

QG.7) there is a universal clock, which provides the ordering of the finite and discrete number of interactions. The expansions are also countable, hence discrete: the structure of spacetime that emerges is discrete and finite, as is the Universe and the quantum theory describing it. There is no divergence of any sort: quantum field theory in the continuum, with its divergences and related renormalizations, is an approximation that may be useful for calculations long after the big bang;
the finiteness of the expanding Universe, and thus the absence of spacetime beyond it, affects the quantum initial state of particles, which are not free to move on the spacetime stage but are bound as if they were surrounded by infinitely high barriers. The steady state of such a particle is a **superposition of states with opposite 3-momenta**, representing an object that moves simultaneously in opposite directions, where by 3-momentum we denote the spatial component of 4-momentum.

### 1.2 Expansion

The assignment of opposite 3-momenta is inherent to the quantum behavior of a particle in a box, in which the square of the momentum, but not the momentum itself, has a definite value in a stationary state. In standard relativistic and non-relativistic quantum mechanics, the ground state is a superposition of *generalized states* with opposite momenta $e^{i(kx+a)}$ and $e^{-i(kx+a)}$. We maintain the same energy and start enlarging the box on opposite sides along the direction of $\vec{p}$ in steps of $|\vec{p}|/E$ in Planck units, so that a massless particle travels at velocity $c$ and a massive one slower than that. We get a wave proportional to $\sin(\pi/n(x/a+n))$, for $a = 1 = |\vec{p}|/E$ and $n = 1, 2, 3, ...$.

The wave function for the first four expansions is shown in Fig. [1]. We take the discretized picture of the sine function maxima and minima, the dots, with cosmological time $t = n - 1$.

![Figure 1: Expansion of a particle (blue dot) along $\vec{p}$ (x-axis) with amplitudes (y-axis)](image)

The amplitude acquires also a time dependent phase $e^{iEt}$ that makes it complex.

### 1.3 Fermions and Bosons

The Lie algebras considered in this paper contain $\mathfrak{e}_8$, and thus $\mathfrak{d}_8$. Under the adjoint action of $\mathfrak{d}_8$, the generators of $\mathfrak{e}_8$ split into spinorial and non-spinorial ones, providing the algebra with a 2-graded structure. We give the spinorial generators the physical meaning of fermions, whereas the non-spinorial generators will be given the physical meaning of bosons, in order to automatically comply with the addition of angular momenta.
On the other hand, the Pauli exclusion principle is embodied in the Grassmann envelope that turns the 2-graded algebra into a Lie superalgebra. The degrees of freedom of the spin-1/2 fermions originate from the superposition of opposite 3-momenta and the corresponding change of helicity caused by the reflection at the space boundary. The Poincaré group is then naturally emerging as a group of transformations of the local algebra $\mathfrak{g}_u$ leaving the charges invariant.

All these topics will be treated in the companion paper [1].

1.4 Quantum Quasicrystal

The expansion of the space that we propose has two fundamental features:

1. a space point may exist with a certain probability amplitude, this latter being the sum of the amplitudes for some particles - matter or radiation - to be there: no space point can possibly be empty;

2. space is a quantum object, that expands according to algebraic rules.

Because of these two features our model of the Universe can be conceived as a quantum quasicrystal, [18][19][20].

2 $\mathfrak{e}_8$, the Charge/Spin Subalgebra

In our treatment, we use the following labels for the Dynkin diagram of $\mathfrak{e}_8$:

\[
\alpha_6 \quad \alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4 \quad \alpha_5 \quad \alpha_7 \quad \alpha_8
\]  

(2.1)

A way of writing the simple roots of $\mathfrak{e}_8$ in the orthonormal basis \(\{k_1, ..., k_8\}\) of \(\mathbb{R}^8\) is:

\[
\begin{align*}
\alpha_1 & = k_1 - k_2; \\
\alpha_2 & = k_2 - k_3; \\
\alpha_3 & = k_3 - k_4; \\
\alpha_4 & = k_4 - k_5; \\
\alpha_5 & = k_5 - k_6; \\
\alpha_6 & = k_6 - k_7; \\
\alpha_7 & = k_6 + k_7; \\
\alpha_8 & = -\frac{1}{2} (k_1 + k_2 + k_3 + k_4 + k_5 + k_6 + k_7 + k_8).
\end{align*}
\]  

(2.2)

The whole root system $\Phi_8$ of $\mathfrak{e}_8$ (obtained from the simple roots by Weyl reflections) can be written as follows:

\[
\begin{align*}
\Phi_8 & = \Phi_O \cup \Phi_S \text{ (240 roots)}; \\
\Phi_O & := \{\pm k_i \pm k_j | 1 \leq i < j \leq 8\}, \\
\Phi_S & := \left\{ \frac{1}{2} (\pm k_1 \pm k_2 \pm ... \pm k_7 \pm k_8), \text{ even # of } + \right\}, \quad 4\binom{9}{2} = 112; \\
& 2^7 = 128.
\end{align*}
\]  

(2.3)

The first set $\Phi_O$ of 112 roots is the set of roots of $\mathfrak{d}_8 \simeq \mathfrak{so}(16, \mathbb{C})$. The set $\Phi_S$ is a Weyl spinor of $\mathfrak{d}_8$ with respect to the adjoint action (every orthogonal Lie algebra
in even dimension $d_n \simeq \mathfrak{so}(2n, \mathbb{C})$ has a Weyl spinor representation of dimension $2^{n-1}$.

If $\alpha$ is a root, there is a unique way of writing it as $\alpha = \sum \lambda_i \alpha_i$ where the $\lambda_i$’s are simple (in fact, all $\lambda_i$’s are positive for positive roots, and negative for negative roots). The sum $ht(\alpha) := \sum \lambda_i$ is called the **height** of $\alpha$.

The fact that the roots of $\Phi_O$ are the roots of a subalgebra and those of $\Phi_S$ correspond to a representation of it can be seen by noticing that: $\Phi_O + \Phi_S \subset \Phi_O$, $\Phi_O + \Phi_S \subset \Phi_S$. Moreover, $\Phi_S + \Phi_S \subset \Phi_O$ implies that $\Phi_O$ is embedded into $\mathfrak{e}_8$ in a symmetric way.

Thus, one can consistently define a non-Cartan generators $x_\alpha$ bosonic if $\alpha \in \Phi_O$ and fermionic if $\alpha \in \Phi_S$. We also call fermionic or bosonic the root $\alpha$ associated to a fermionic (resp. bosonic) non-Cartan generator $x_\alpha$. A Cartan generator $h_\alpha$ is always bosonic for any $\alpha$ since $h_\alpha = [x_\alpha, x_\alpha]$. The roots of $\mathfrak{e}_8$ split into 128 fermions (F) and 112 bosons (B).

### 2.1 Algebraic Structure

The $\mathfrak{e}_8$ algebra can be defined from its root system $\Phi_8$, [21][22][23], over the complex field extension $\mathbb{C}$ of the rational integers $\mathbb{Z}$ in the following way:

a) we select the set of simple roots $\Delta_8$ of $\Phi_8$;

b) we select a basis $\{h_1, ..., h_8\}$ of the 8-dimensional vector space $\mathfrak{h}$ over $\mathbb{C}$ and set $h_\alpha = \sum_{i=1}^8 \lambda_i h_i$ for each $\alpha \in \Phi_8$ such that $\alpha = \sum_{i=1}^8 \lambda_i \alpha_i$;

c) we associate to each $\alpha \in \Phi_8$ a one-dimensional vector space $L_\alpha$ over $\mathbb{C}$ spanned by $x_\alpha$;

d) we define $\mathfrak{e}_8 = \mathfrak{h} \bigoplus_{\alpha \in \Phi_8} L_\alpha$ as a vector space over $\mathbb{C}$;

e) we give $\mathfrak{e}_8$ an algebraic structure by defining the following multiplication on the basis $\{h_1, ..., h_8\} \cup \{x_\alpha \mid \alpha \in \Phi_8\}$, by linearity to a bilinear multiplication $\mathfrak{e}_8 \times \mathfrak{e}_8 \to \mathfrak{e}_8$:

\[
\begin{align*}
[h_i, h_j] &= 0, \quad 1 \leq i, j \leq 8 \\
[h_i, x_\alpha] &= -[x_\alpha, h_i] = (\alpha, \alpha_i) x_\alpha, \quad 1 \leq i \leq 8, \quad \alpha \in \Phi_8 \\
x_\alpha, x_{-\alpha} &= -h_\alpha \\
x_\alpha, x_\beta &= 0 \text{ for } \alpha, \beta \in \Phi_8 \text{ such that } \alpha + \beta \notin \Phi_8 \text{ and } \alpha \neq -\beta \\
x_\alpha, x_\beta &= \varepsilon(\alpha, \beta) x_{\alpha+\beta} \text{ for } \alpha, \beta \in \Phi_8 \text{ such that } \alpha + \beta \in \Phi_8
\end{align*}
\]

where $\varepsilon(\alpha, \beta)$ is the **asymmetry function**, introduced in [24] as in Definition 2.1, see also [23].

**Definition 2.1.** Let $\mathbb{L}_{\mathfrak{e}_8}$ denote the lattice of all linear combinations of the simple roots with integer coefficients

\[
\mathbb{L}_{\mathfrak{e}_8} = \left\{ \sum_{i=1}^8 c_i \alpha_i \mid c_i \in \mathbb{Z}, \ \alpha_i \in \Delta_8 \right\}
\]

the asymmetry function $\varepsilon(\alpha, \beta) : \mathbb{L}_{\mathfrak{e}_8} \times \mathbb{L}_{\mathfrak{e}_8} \to \{-1, 1\}$ is defined by:

\[
\varepsilon(\alpha, \beta) = \prod_{i,j=1}^8 \varepsilon(\alpha_i, \alpha_j)^{\ell_i m_j} \text{ for } \alpha = \sum_{i=1}^8 \ell_i \alpha_i, \ \beta = \sum_{j=1}^8 m_j \alpha_j
\]
where $\alpha_i, \alpha_j \in \Delta_8$ and

$$
\varepsilon(\alpha_i, \alpha_j) = \begin{cases} 
-1 & \text{if } i = j \\
-1 & \text{if } \alpha_i + \alpha_j \text{ is a root and } \alpha_i < \alpha_j \\
+1 & \text{otherwise}
\end{cases}
$$

(2.7)

We recall the following standard result on the roots of $e_8$ (normalized to 2), [21, 22]:

**Proposition 2.2.** For each $\alpha, \beta \in \Phi_8$ the scalar product $(\alpha, \beta) \in \{\pm 2, \pm 1, 0\}$; $\alpha + \beta$ (respectively $\alpha - \beta$) is a root if and only if $(\alpha, \beta) = -1$ (respectively +1); if both $\alpha + \beta$ and $\alpha - \beta$ are not in $\Phi_8 \cup \{0\}$ then $(\alpha, \beta) = 0$. For $\alpha, \beta \in \Phi_8$ if $\alpha + \beta$ is a root then $\alpha - \beta$ is not a root.

The following properties of the asymmetry function follow from its definition, [23].

**Proposition 2.3.** The asymmetry function $\varepsilon$ satisfies, for $\alpha, \beta, \gamma \in \Lambda_{e_8}$:

1. $\varepsilon(\alpha + \beta, \gamma) = \varepsilon(\alpha, \gamma)\varepsilon(\beta, \gamma)
2. \varepsilon(\alpha, \beta + \gamma) = \varepsilon(\alpha, \beta)\varepsilon(\alpha, \gamma)
3. $\varepsilon(\alpha, \alpha) = (-1)^{\frac{1}{2}(\alpha, \alpha)} \Rightarrow \varepsilon(\alpha, \alpha) = -1$ if $\alpha \in \Phi_8
4. $\varepsilon(\alpha, \beta)\varepsilon(\beta, \alpha) = (-1)^{(\alpha, \beta)} \Rightarrow \varepsilon(\alpha, \beta) = -\varepsilon(\beta, \alpha)$ if $\alpha, \beta, \gamma \in \Phi_8
5. $\varepsilon(0, \beta) = \varepsilon(\alpha, 0) = 1
6. $\varepsilon(-\alpha, \beta) = \varepsilon(\alpha, \beta)^{-1} = \varepsilon(\alpha, \beta)
7. $\varepsilon(\alpha, -\beta) = \varepsilon(\alpha, \beta)^{-1} = \varepsilon(\alpha, \beta)

Property iv) shows that the product in (2.4) is indeed antisymmetric.

### 2.2 $e_8$ Charges and the Magic Star

There are four orthogonal $a_2$'s in $e_8$, where orthogonal means that the planes on which their root systems lie are orthogonal to each other.

We denote one of them $a_2^f$ for color, one $a_2^c$ for flavor, the other two $a_2^{(1)}$ and $a_2^{(2)}$:

- $a_2^f : k_i - k_j , i \neq j , i, j \in \{1, 2, 3\}
- a_2^c : k_i - k_j , i \neq j , i, j \in \{4, 5, 6\}
- a_2^{(1)} : \pm(k_7 + k_8), \pm\frac{1}{2}(k_1 + k_2 + k_3 + k_4 + k_5 + k_6 - k_7 - k_8)
  - \pm\frac{1}{2}(k_1 + k_2 + k_3 + k_4 + k_5 + k_6 + k_7 + k_8)
- a_2^{(2)} : \pm(k_7 - k_8), \pm\frac{1}{2}(-k_1 + k_2 - k_3 + k_4 - k_5 + k_6 - k_7 + k_8)
  - \pm\frac{1}{2}(-k_1 + k_2 - k_3 + k_4 - k_5 + k_6 + k_7 - k_8)

The generators of $a_2^f$ and $a_2^c$ are bosonic.

The Magic Star (MS) of $e_8$ shown in Fig. 3 is obtained by projecting its roots on the plane of $a_2^f$, [23]. The pair of integers $(r, s)$ are the (Euclidean) scalar products $r := (\alpha, k_1 - k_2)$ and $s := (\alpha, k_1 + k_2 - 2k_3)$, for each root $\alpha$. The fermions on the
Figure 2: The Magic Star (MS): $g_0 = e_6$ in the MS of $e_8$, $g_0 = a_2^{(1)} \oplus a_2^{(2)}$ in the MS of $e_6$; the triangles represent the $3$ and $\overline{3}$ representations of the $a_2$ with roots in the external hexagon.

The tips of the MS are quarks since they are acted upon by $a_2^*_2$: they are colored. The fermions within the center of the MS are leptons: they are colorless. A similar MS of $e_6$ within $e_8$ is obtained by projecting the roots in the center of the MS of $e_8$ on the plane of $a_2^*$. Notice that in each tip of the MS of $e_8$ we get 27 roots, 11 of which are bosonic and 16 fermionic; this corresponds to the following decomposition of the irrepr. $27$ of $e_6$:

$$e_6 \supset d_5 \oplus \mathbb{C},$$

$$27 = 1_{-4} \oplus 10_2 \oplus 16_{-1}. \quad (2.8)$$

On the other hand, within $e_6$ we have 9 roots in each tip of the MS, 5 of which are bosonic and 4 fermionic; this corresponds to the following decomposition of the repr. $(3, 3)$ of $a_2 \oplus a_2$:

$$a_2 \oplus a_2 \supset a_1 \oplus a_1 \oplus \mathbb{C}_I \oplus \mathbb{C}_{II} \supset a_1 \oplus a_1 \oplus \mathbb{C}_{I+II},$$

$$(3, 3) = (1, 1)_{-2} \oplus (2, 2)_{1,1} \oplus (2, 1)_{1,-2} \oplus (1, 2)_{-2,1} \quad (2.9)$$

Tables 1, 2, 3, 4 describe the content of these MS, root by root.

The magic of the MS is that each tip $\{r, s\}$ of the star, $\{r, s\} \in \{\{0, \pm 2\}, \{\pm 1, \pm 1\}\}$, both in the case of $e_8$ and of $e_6$, can be viewed as a cubic (simple) Jordan algebra $J_{\{r,s\}}$, over the octonions and the complex field respectively, and each pair of opposite tips with respect to the center of the star has a natural algebraic structure of a Jordan pair. The algebra in the center of the star is the derivation algebra of the Jordan pair; when the Jordan pair is made of a pair of Jordan algebras, its derivations also define the Lie algebra of the structure group of the Jordan algebra itself. [26] [27] [28] [29].

2.3 The Standard Model

In this section, we relate the $e_8$ charges to the degrees of freedom of the Standard Model (SM) of elementary particle physics. It is not our aim to carry through a detailed analysis, in particular we do not consider symmetry breaking, nor Higgs mechanism, nor chirality and parity violation by weak interactions in the fermionic
sector. We do however focus on spin as an internal degree of freedom, and this will be instrumental for the treatment of the Poincaré action on our algebra, which we will investigate in the companion paper, [1].

The first important step, after splitting the roots into colored and colorless as in the previous section, is to find the electromagnetic $$u(1)^{em}$$ that gives the right charges to quarks and leptons. We select the one generated by

$$h_{\gamma} = -i \left( \frac{1}{3} h_{a_1} + \frac{2}{3} h_{a_2} + h_{a_3} \right),$$

(2.10)

giving to $$x_\alpha$$, where $$\alpha = \sum \lambda_i k_i$$, the charge

$$q_{e.m.}(\alpha) := (\alpha, q_\gamma) = -\frac{1}{3} (\lambda_1 + \lambda_2 + \lambda_3) + \lambda_4,$$

(2.11)

where

$$q_\gamma := -\frac{1}{3} \alpha_1 - \frac{2}{3} \alpha_2 - \alpha_3 = -\frac{1}{3} (k_1 + k_2 + k_3) + k_4.$$  

(2.12)

The second column of Tables 1, 2, 3, 4 shows the charges of the $$e^s$$ where

$$s$$ algebra. The spin-1 particle within $$(\frac{1}{2}, \frac{1}{2})$$ subalgebra group of rotations in the internal space. For this purpose we select the spin (diagonal) charges to quarks and leptons. We select the one generated by in the previous section, is to find the electromagnetic be instrumental for the treatment of the Poincaré action on our algebra, which we do however focus on spin as an internal degree of freedom, and this will

$$\rho \pm$$ to the roots $$\rho$$ of (1, 1, 1) corresponds to the 6 components as a rank 2 antisymmetric tensor in 4 dimensions, with selfdual and antidselfdual parts (1, 0) and (0, 1), respectively. Notice that all fermions have half-integer spin $$(\frac{1}{2}, 0)$$ or $$(0, \frac{1}{2})$$, whereas all bosons of type $$x_\alpha$$ have integer spin.

In order to define the action of the Poincaré group in [1], we need the covering group of rotations in the internal space. For this purpose we select the spin (diagonal) subalgebra $$su(2)^{spin} \in a_1^{(1)} \oplus a_1^{(2)}$$ as the compact (real) form with generators

$$R^+ + R^-, \quad i \left( R^+ - R^- \right), \quad i H_R, \quad \text{where}\quad R^+ := x_{\rho_1} + x_{\rho_2}, \quad R^- := x_{-\rho_1} + x_{-\rho_2}, \quad H_R := \frac{1}{2} (h_{\rho_1} + h_{\rho_2}).$$

(2.14)

The $$(\frac{1}{2}, \frac{1}{2})$$ representation splits into a scalar and a vector under this rotation subalgebra. The spin-1 particle within $$(\frac{1}{2}, \frac{1}{2})$$ is the linear span of the generators $$x_{k_i \pm k_5}$$ with z-component of spin $$s_z := \frac{1}{2} (\rho_1 + \rho_2, k_i \pm k_5) = (k_5, k_i \pm k_5) = \pm 1$$ and

$$\frac{1}{2} (\varepsilon (\rho_1, k_i - k_5) x_{k_i - k_5} + \varepsilon (\rho_2, k_i - k_5) x_{k_i + k_5})$$

with $$s_z = 0$$; the corresponding scalar is

$$\frac{1}{2} (\varepsilon (\rho_1, k_i - k_5) x_{k_i - k_5} - \varepsilon (\rho_2, k_i - k_5) x_{k_i + k_5}),$$

as it can be easily verified.

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2.3.1 $W^\pm$

Let us now consider the $W^\pm$ bosons. There are not many choices for them: indeed, they must be colorless vectors with respect to $\mathfrak{su}(2)^{\text{spin}}$ and have electric charge $\pm 1$. The $W^\pm$ bosons are therefore the generators associated to $\pm(k_4 - k_5)$ (within $\mathfrak{a}_2$ mentioned in section 2.2), $\pm(k_4 + k_5)$, the electric charge $\pm 1$ given by the presence of $k_4$; they change flavor to both quarks and leptons. The above analysis suggests that the extra degree of freedom needed, say for $W^+$, to become massive, from the 2 degree of freedom of the massless helicity-1 state, is $\varepsilon(p_1, k_4 + k_5)x_{k_4+k_5} + \varepsilon(p_2, k_4 - k_5)x_{k_4-k_5}$, as a part of the Higgs mechanism - that we will not discuss any further in this paper.

**Remark 2.4.** We could have made other equivalent choices for the $\mathfrak{d}_2 \simeq \mathfrak{so}(4, \mathbb{C}) \simeq \mathfrak{a}_1 \oplus \mathfrak{a}_1$ subalgebra, which has to act non-trivially on $W^\pm$: once passing to real forms, it cannot possibly commute with the weak interaction $\mathfrak{su}(2)^L$. For what concerns the no-go theorem by Distler Garibaldi, the hypothesis TOE1 of [2] cannot possibly apply, as outlined in the introduction, see remark [1.1]. We also emphasize that, contrary to [2], we are dealing with the complex form of $\mathfrak{e}_8$ because we want complex phases for the particle states.

Using the properties of the asymmetry function and the ordering of the simple roots $\alpha_i < \alpha_{i+1}$ we get:

$$\varepsilon(p_1, k_4 - k_5) = \varepsilon(p_2, k_4 - k_5) = 1,$$

hence the massive $W^\pm$ is described by three components:

$$W^\pm_1 := x_{\pm k_4+k_5} \quad (s_z = 1);$$
$$W^\pm_0 := \frac{1}{2}(x_{\pm k_4-k_5} + x_{\pm k_4+k_5}) \quad (s_z = 0);$$
$$W^\pm_{-1} := x_{\pm k_4-k_5} \quad (s_z = -1).$$

Moreover, using the notation

$$R_x := \frac{1}{2}(x_{p_1} + x_{p_2} + x_{-p_1} + x_{-p_2});$$
$$R_y := \frac{1}{2}(x_{p_1} + x_{p_2} - x_{-p_1} - x_{-p_2});$$
$$R_z := \frac{1}{2}(h_{p_1} + h_{p_2}),$$

we obtain

$$[R_x, W^\pm_{-1}] = W^\pm_0, \quad [R_y, W^\pm_1] = W^\pm_0, \quad [R_y, W^\pm_{-1}] = -i W^\pm_0, \quad [R_z, W^\pm_1] = -\frac{1}{2}(W^\pm_1 + W^\pm_{-1}),$$

$$[R_z, W^\pm_{-1}] = i s_z W^\pm_{s_z}, \quad s_z \in \{1, 0, -1\}. \quad (2.19)$$

These commutation relations correspond to the action of the rotation matrices:

$$J_x := \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad J_y := \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_z := i \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad (2.20)$$

on the vectors $v = v_+ e_+ + v_- e_+ + v_- e_- \in \mathbb{R}^3$ in the spherical basis $\{e_+, e_-, e_-\}$ corresponding, for angular momentum 1, to the spherical harmonic basis for the
irreducible representations of $SO(3)$. With respect to the same vector $v = v_x e_x + v_y e_y + v_z e_z \in \mathbb{R}^3$ in the standard orthogonal basis $\{e_x, e_y, e_z\}$, we have:

$$
e_+ = \frac{1}{\sqrt{2}} (e_x + i e_y), \quad e_- = \frac{1}{\sqrt{2}} (e_x - i e_y),$$

$$v_+ = \frac{1}{\sqrt{2}} (v_x - i v_y), \quad v_- = \frac{1}{\sqrt{2}} (v_x + i v_y). \quad (2.21)$$

The transformation between (column) vectors in the two bases is represented by the unitary matrix $U$:

$$
\begin{pmatrix}
v_+

v_x

v_z

v_-
\end{pmatrix} = U
\begin{pmatrix}
v_x

v_y

v_z
\end{pmatrix}, \quad U := \begin{pmatrix}
\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\
0 & 1 & 0 \\
\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}
\end{pmatrix}. \quad (2.22)
$$

The correspondence with $R$'s and $W$'s is:

$$
R_x \leftrightarrow J_x, \quad R_y \leftrightarrow J_y, \quad R_z \leftrightarrow J_z,$$

$$W_1^\pm \leftrightarrow v_+, \quad W_{-1}^\pm \leftrightarrow v_-, \quad W_0^\pm \leftrightarrow \frac{1}{\sqrt{2}} v_z. \quad (2.23)
$$

**Remark 2.5.** We have an interesting relationship between weak and rotation generators in the internal space (spin generators) by noticing that

$$
\begin{align*}
[W_1^+ + W_{-1}^+, W_0^-] &= [W_1^- + W_{-1}^-, W_0^+] = R_x \\
[W_1^+ - W_{-1}^+, iW_0^-] &= [W_1^- - W_{-1}^-, iW_0^+] = R_y.
\end{align*} \quad (2.24)
$$

and consequently, the relation with $R_z = [R_x, R_y]$.

### 2.3.2 $Z^0$

We associate the $Z^0$ boson with spin $s_z = 0$, denoted by $Z_0^0$, to the vector orthogonal to $q_7$ in the plane of $(k_4 - k_5)$ and $q_7$, hence it is, up to a scalar, the Cartan generator $Z_0^0 = \frac{1}{4}(h_{a_1} + 2h_{a_2} + 3h_{a_3} + 4h_{a_4})$; it interacts with left-handed neutrinos and right-handed antineutrinos, contrary to the photon; it does not allow for flavor changing neutral currents.

Notice that the generator of hypercharge $u(1)^Y$ of the standard model is in this setting compact Cartan generator $i h_Y$ where $h_Y := -\frac{1}{8}(2h_{a_1} + 4h_{a_2} + 6h_{a_3} + 3h_{a_4})$. The Weinberg angle $\theta_W$ is the angle between the axis representing the photon and that representing the hypercharge, therefore $\theta_W = \pi/2 - \phi$ where $\phi$ is the angle between $q_7$ and $k_4 - k_5$; we get $\sin^2\theta_W = 3/8$.

Since $\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 = k_1 + k_2 + k_3 + k_4 - 4k_5$ hence $\frac{1}{4}(\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4, x_{\pm 1, 2}) = -(k_5, \pm \rho_1) = -(k_5, \pm \rho_2) = \mp 1$, we have the following commutation relations:

$$
\begin{align*}
[x_{\rho_1} + x_{\rho_2} + x_{-\rho_1} + x_{-\rho_2}, Z_0^0] &= (x_{\rho_1} + x_{\rho_2} - x_{-\rho_1} - x_{-\rho_2}); \\
[x_{\rho_1} + x_{\rho_2} - x_{-\rho_1} - x_{-\rho_2}, Z_0^0] &= (x_{\rho_1} + x_{\rho_2} + x_{-\rho_1} + x_{-\rho_2}),
\end{align*} \quad (2.25)
$$

that is

$$
\begin{align*}
[R_x, Z_0^0] &= -i R_y, & [R_y, Z_0^0] &= i R_x, & [R_z, Z_0^0] &= 0. \quad (2.26)
\end{align*}
$$

We want $Z^0$, as a spin-1 particle, to obey the same commutation relations with the rotation generators as $W^\pm$. We can so define the spin $\pm 1$ components of $Z^0$ by comparison with the last commutator in each row of (2.18):

$$
\begin{align*}
Z_0^0 &= - [R_x, Z_0^0] + i [R_y, Z_0^0] = -R_x + i R_y = -R_+; \\
Z_{-1}^0 &= - [R_x, Z_0^0] - i [R_y, Z_0^0] = R_x + i R_y = R_–. \quad (2.27)
\end{align*}
$$

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(see (2.14) for the definition of $R_{\pm}$).

By looking at Table 3, we notice that $Z_0$ interacts, for instance, with $\nu'$ with spin $-\frac{1}{2}$ to give $\nu_e'$ with spin $\frac{1}{2}$, and similarly for other leptons and for quarks. In particular there are no flavor changing neutral currents.

2.3.3 The Tables at a Glance

From the Tables 1, 2, 3, 4 one can deduce all the standard model charges (in particular we have denoted with a prime possible mixings in Tables 3, 4). We have:

SM.1) the color charges are denoted by the pair \{rc, sc\} and one can associate colors to them, say blue = \{1, 1\}, green = \{-1, 1\} and red = \{0, -2\}, and similarly for the anti-colors;

SM.2) the quarks are the fermions in Table 1 with a certain color; they come in 3 color families and anti-quarks have anti-colors and opposite electric charges $-\frac{2}{3}, \frac{1}{3}$ with respect to quarks;

SM.3) the gluons are the generators of $a_2^c$, change color to the quarks on which they act as on a $\mathbf{3}$ or $\overline{\mathbf{3}}$ representation and their electric charge is 0;

SM.4) the leptons are in the center of the MS in Table 1 and are the fermions in Table 2;

SM.5) the leptons have integer electric charge in \{-1, 0, 1\};

SM.6) there are 4 flavor families; we have used the notation $\chi, \nu\chi$ for the fourth lepton family and $T, B$ for the fourth quark family;

SM.7) the fourth column of Tables 3 and 4 shows the component of spin along the axis specified by the spin generator $R_z := \frac{i}{2}(h_{p_1} + h_{p_2})$. Obviously a rotation by $2\pi$ of quarks and leptons changes their sign, whether it leaves vector bosons invariant.

The consequences of this classification with respect to the Poincaré action on $g_u$ will be discussed in the companion paper [1].

3 The Kac-Moody Algebras $e_9, e_{10}, e_{12}, de_{12}$

Let $g(A)$ denote the Kac-Moody algebra associated to the $n \times n$ Cartan matrix $A$, with Cartan subalgebra $\mathfrak{h}$. For all algebras in this paper, $A$ is symmetric; its entries are denoted by $a_{ij}$. We denote the Chevalley generators by $E_i$, associated to the simple root $\alpha_i$, and by $F_i$, associated to the root $-\alpha_i$. Let $n_+$ (resp. $n_-$) denote the subalgebra of $g(A)$ generated by \{E_1, ..., E_n\} (resp. \{F_1, ..., F_n\}). By Theorem 1.2 a), e) in [24], the following triangular decomposition holds:

$$g(A) = n_+ \oplus \mathfrak{h} \oplus n_- \quad \text{(direct sum of vector spaces).} \quad (3.1)$$

Note that for a root $\alpha > 0$ (resp. $\alpha < 0$) we have $\alpha \in \mathfrak{h}^*$, the dual of $\mathfrak{h}$, and the vector space $g_\alpha = \{x \in g(A) \mid [h, x] = \alpha(h)x \ \forall h \in \mathfrak{h}\}$ is the linear span of the elements of the form $[...[E_{i_1}, E_{i_2}], E_{i_3}][...E_{i_s}]$ (resp. $[...[F_{i_1}, F_{i_2}], F_{i_3}][...F_{i_s}]$), such that $\alpha_{i_1} + ... + \alpha_{i_s} = \alpha$ (resp. $= -\alpha$). The multiplicity $m_\alpha$ of a root $\alpha$ is defined as $m_\alpha := \dim g_\alpha$ ($m_{\alpha_i} = m_{-\alpha_i} = 1$ for each simple root $\alpha_i$).
Kac-Moody algebras, [21, 22], can be tackled in terms of simple roots and their (extended) Dynkin diagram, or equivalently their Cartan matrix, without any reference to root coordinates. Some physical features or interpretations may however be more explicit when roots are expressed in an orthonormal basis rather than in a simple root basis. This is the case of this paper, in which some root coordinates, except for the case of $e_8$, are interpreted as momentum coordinates. We recall that the metric is Euclidean for $e_8$ but Lorentzian in the case of $e_9, e_{10}, e_{12}, de_{12}$.

Our notation for the simple roots is shown in the Dynkin diagram of $e_{12}$:

$$
\alpha_1 \quad \alpha_0 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4 \quad \alpha_5 \quad \alpha_6 \quad \alpha_7 \quad \alpha_8
$$

Analogous diagrams are those of $e_{10}$ (without $\alpha_{-1}, \alpha_{0^v}$), of $e_9$ (without $\alpha_{-1}, \alpha_{0^v}, \alpha_{0^v}$) and of $e_8$ (without $\alpha_{-1}, \alpha_{0^v}, \alpha_{0^v}, \alpha_0$).

### 3.1 The Simple Roots of $e_9$ and $e_{10}$

We introduce the following set of simple roots $\alpha_{-1}, \alpha_0, \alpha_1, ..., \alpha_8$ of $e_{10}$ in terms of the basis vectors $k_1, k_0, k_1, ..., k_8$ spanning the Lorentzian space $\mathbb{R}^{9,1}$, with $(k_1, k_1) = -1$ and $(k_i, k_i) = 1$, for $0 \leq i \leq 8$:

\[
\begin{align*}
\alpha_{-1} &= \frac{1}{2} (k_{-1} - 3k_0); \\
\alpha_0 &= \frac{1}{2} (k_{-1} + k_0) - k_1 + k_8; \\
\alpha_1 &= k_1 - k_2; \\
\alpha_2 &= k_2 - k_3; \\
\alpha_3 &= k_3 - k_4; \\
\alpha_4 &= k_4 - k_5; \\
\alpha_5 &= k_5 - k_6; \\
\alpha_6 &= k_6 - k_7; \\
\alpha_7 &= k_6 + k_7; \\
\alpha_8 &= -\frac{1}{2} (k_1 + k_2 + k_3 + k_4 + k_5 + k_6 + k_7 + k_8).
\end{align*}
\]

All these roots have norm 2, with respect to the scalar product $(\cdot, \cdot)$, and the corresponding Cartan matrix is the Gram matrix of the $e_{10}$ even unimodular Lorentzian lattice $II_{9,1}$ made of all the vectors in $\mathbb{R}^{9,1}$ whose components are all in $\mathbb{Z}$ or all in $\mathbb{Z} + \frac{1}{2}$ and have integer scalar product with $\frac{1}{2} (k_1 + k_0 + k_1 + k_2 + k_3 + k_4 + k_5 + k_6 + k_7 + k_8)$, as it can be easily checked.

The *affine* Kac-Moody algebra $e_9$ is obtained by eliminating the root $\alpha_{-1}$. Notice that

\[
\alpha_0 = \alpha_9 + \delta, \quad \delta := \frac{1}{2} (k_1 + k_0), \quad \text{(hence } \langle \delta, \delta \rangle = 0); \\
\alpha_9 := -2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 5\alpha_4 + 6\alpha_5 + 3\alpha_6 + 4\alpha_7 + 2\alpha_8),
\]

where $\alpha_9 = -k_1 + k_8$ is the lowest height root of $e_8$ and $\delta$ is a lightlike vector.

### 3.2 $e_9$

The Cartan subalgebra of $e_9$ is the span of 10 generators, two new ones with respect to $e_8$. We write

\[
h = \text{span}\{K, d, h_i \mid i = 1, ..., 8\},
\]
where
\[ K := h_\delta, \quad \delta := \frac{1}{2} (k_{-1} + k_0); \]
\[ d := h_\rho, \quad \rho := -k_{-1} + k_0, \] (3.6)
and \( K \) is a central element.

Let \( \Phi_8 \) be the root system of \( e_8 \), \( \mathfrak{h} \) its Cartan subalgebra, \( h_\alpha \) (resp. \( e_\alpha \)) a Cartan (resp. non-Cartan) generator associated to the root \( \alpha \), \( h_i \) for \( 1 \leq i \leq 8 \) a Cartan generator associated to the simple root \( \alpha_i \), \( X, Y \) either Cartan or non-Cartan generators of \( e_8 \). It is shown by Kac, \[24\], that:

1. The root system \( \Phi_9 \) of \( e_9 \) is
\[ \Phi_9 := \{ \alpha + m\delta \mid \alpha \in \Phi_8, \ m \in \mathbb{Z} \} \cup \{ m\delta \mid m \in \mathbb{Z}\backslash\{0\} \}. \] (3.7)

2. \( e_8 \) is determined by the following commutation relations:
\[
\begin{align*}
[h, h'] &= 0 \quad \text{if} \ h, h' \in \mathfrak{h}; \\
[h, E_\alpha] &= (h^*, \alpha) E_\alpha \quad \text{if} \ h \in \mathfrak{h}, \ \alpha \in \Phi_8; \\
[E_\alpha, E_{-\alpha}] &= -h_\alpha \quad \text{if} \ \alpha \in \Phi_8; \\
[E_\alpha, E_\beta] &= 0 \quad \text{if} \ \alpha, \beta \in \Phi_8, \ \alpha + \beta \notin \Phi_8 \cup \{0\}; \\
[E_\alpha, E_\beta] &= \varepsilon (\alpha, \beta) E_{\alpha+\beta} \quad \text{if} \ \alpha, \beta, \alpha + \beta \in \Phi_8,
\end{align*}
\] (3.8)
where \( \varepsilon \) is Kac’s asymmetry function, see section 2.1.

3. The commutation relations in \( e_9 \) are the same as those for the central extended loop algebra of \( e_8 \) plus derivations:
\[
[t^m \otimes X \oplus \lambda K \oplus \mu d, t^n \otimes Y \oplus \lambda t K \oplus \nu d] = \\
(t^{m+n} \otimes [X, Y] - m\lambda t^m \otimes X + n\lambda t^n \otimes Y) \oplus m\delta_{m,-n}(X|Y) \ K
\] (3.9)
with the following correspondence
\[
\begin{align*}
t^m \otimes E_\alpha &\leftrightarrow x_{\alpha+m\delta}; \\
t^m \otimes H_\alpha &\leftrightarrow x^\alpha_{\alpha+m\delta} \quad \text{if} \ m \neq 0; \\
t^0 \otimes H_\alpha &\leftrightarrow h_\alpha; \\
K, d &\leftrightarrow h_\delta, h_\rho, \ (\text{see }3.6)
\end{align*}
\] (3.10)
and with the invariant non-degenerate symmetric bilinear form \((\cdot, \cdot)\) defined by:
\[
(X|Y) := \begin{cases} 
(\alpha, \beta) & \text{if} \ X = H_\alpha, \ Y = H_\beta; \\
0 & \text{if} \ X = H_\alpha, \ Y = E_\beta; \\
-\delta_{\alpha,-\beta} & \text{if} \ X = E_\alpha, \ Y = E_\beta.
\end{cases}
\] (3.11)

For \( \alpha, \beta \in \Phi_8 \) (roots of \( e_8 \)) and the letter \( h \) referring to a Cartan generator, the commutation relations, with no reference to the loop algebra, are:
\[
\begin{align*}
[h, h'] &= 0 \\
[h_\alpha, x^\beta_{m\delta}] &= 0 \quad \text{if} \ m \neq 0 \\
[h_\delta, x^\alpha_{m\delta}] &= 0 \quad \text{if} \ m \neq 0 \\
[h_\rho, x^\alpha_{m\delta}] &= m x^\alpha_{m\delta} \quad \text{if} \ m \neq 0 \\
[h_\beta, x_{\alpha+m\delta}] &= (\beta, \alpha) x_{\alpha+m\delta} \\
[h_\delta, x_{\alpha+m\delta}] &= 0 \\
[h_\rho, x_{\alpha+m\delta}] &= m x_{\alpha+m\delta} \\
x^{\alpha}_{m\delta}, x^{\beta}_{n\delta} &= m \delta_{m,-n}(\alpha, \beta) h_\delta \quad \text{if} \ m, n \neq 0 \\
[x^{\alpha}_{m\delta}, x^{\beta}_{n\delta}] &= (\alpha, \beta) x_{\beta+(m+n)\delta} \quad \text{if} \ m \neq 0
\end{align*}
\] (3.12)
\[ [x_{\alpha+m\delta}, x_{\beta+n\delta}] = \begin{cases} 
0 & \text{if } \alpha + \beta \notin \Phi_8 \cup \{0\} \\
\varepsilon(\alpha, \beta) x_{\alpha+\beta+(m+n)\delta} & \text{if } \alpha + \beta \in \Phi_8 \\
-x_{\alpha}^{(m+n)\delta} & \text{if } \alpha + \beta = 0 \text{ and } m + n \neq 0 \\
h_{\alpha+m\delta} & \text{if } \alpha + \beta = 0 \text{ and } m + n = 0 
\end{cases} \]  
(3.13)

**Remark 3.1.** The commutation relations of \(e_9\) are essentially determined by those of \(e_8\), whose main ingredient for explicit calculations is the asymmetry function.

**Remark 3.2.** The second correspondence in (3.10) shows why the so called imaginary roots \(m\delta\) are 8-fold degenerate: the space of generators associated to each root \(m\delta\) is indeed an 8-dimensional space isomorphic to the span of \(\{h_i, i = 1, \ldots, 8\}\), namely the Cartan subalgebra of \(e_8\).

### 3.3 \(e_9\) in a 1 + 1 dimensional toy model

The explicit construction presented here, using a realization of the roots in terms of the orthonormal basis \(\{k_i, i = -1, 0, 1, \ldots, 8\}\) of \(\mathbb{R}^{9,1}\), suggests to let the coordinates \(k_1, \ldots, k_9\) relate to charge/spin degrees of freedom, and to interpret the coordinates \(k_8, k_9\) as 2-momentum coordinates with Lorentzian signature.

A crucial step in our model, describing a Universe that expands from an initial quantum state, is to restrict the particles forming that state, hence their interactions, to lie in the subalgebra \(n_+\) of the triangular decomposition (3.11) of \(e_9\) (the reason will be explained in item TM.3 below). The restriction to \(n_+\) has the following consequences:

**TM.1** The only commutation relations within (3.12) and (3.13) occurring in \(n_+\) are, for \(\alpha, \beta \in \Phi_8\):

\[
\begin{align*}
[x_{\alpha}^{m\delta}, x_{\beta}^{n\delta}] &= 0 & m, n &\neq 0 \\
[x_{\alpha}^{m\delta}, x_{\beta}^{\alpha+\beta+(m+n)\delta}] &= (\alpha, \beta) x_{\beta+(m+n)\delta} & m > 0, & n \geq 0 \\
[x_{\alpha+m\delta}, x_{\beta+n\delta}] &= \begin{cases} 
0 & \text{if } \alpha + \beta \notin \Phi_8 \cup \{0\} \\
\varepsilon(\alpha, \beta) x_{\alpha+\beta+(m+n)\delta} & \text{if } \alpha + \beta \in \Phi_8 \\
-x_{\alpha}^{(m+n)\delta} & \text{if } \alpha + \beta = 0 
\end{cases} & & \text{if } \alpha + \beta = 0 \text{ and } m + n = 0 
\end{align*}
\]  
(3.14)

We remark that \(\alpha + \beta = 0\) in the last commutation relation implies that either \(\alpha\) or \(\beta\) is negative, hence \(m + n \neq 0\), being \(m, n \geq 0\) and in particular \(m > 0\) in \(x_{\alpha+m\delta}\) whenever \(\alpha\) is a negative root of \(e_8\).

**TM.2** The roots involved in the interactions are not only real; we have for instance that \(\alpha_0, -\alpha_9\) are positive roots and \((\alpha_0, -\alpha_9) = -2\), therefore, by Proposition 5.1 of [24], at least \(\alpha_0 + (-\alpha_9) = \delta\) and \(\alpha_0 + 2(-\alpha_9)\) are roots; the outgoing generator \(x_{\alpha}^{\alpha_0}\) of the interaction between \(x_{\alpha_0} = x_{\alpha_0+\delta}\) and \(x_{-\alpha_0}\) is similar to the Cartan generator \(h_{\alpha_0}\), except that it carries a momentum \(\delta = \frac{1}{2}(k_1 + k_9)\). This yields the interesting consideration that neutral radiation fields like the photon are not associated to Cartan elements of infinite dimensional Kac-Moody algebras but to imaginary roots, a feature that is not present in Yang-Mills theories. It is also worth noticing that the second equation in (3.14) implies that the neutral radiation field keeps memory of the particle-antiparticle pair which produced it, represented in the equation by the root \(\alpha\).

**TM.3** The fact that all particles are in \(n_+\) ensures that their energy is always positive, even though they may be related to both positive and negative roots of \(e_8\),
as revealed by the fact that \( \alpha_0 \) is the negative root of lowest height. In other words we get both particles and antiparticles, and all of them do have positive energy.

TM.4) The momenta given to each particle by the interactions are lightlike. Energy momentum is conserved because the outgoing particle in an elementary interaction is associated to a root which is the sum of the roots of the incoming particles. \textit{All particles are massless}, since momenta add up in the unique spatial direction.

TM.5) We give fermionic particles helicity \( 1/2 \).

So far everything runs smooth and seems physically plausible, but:

TM.6) The initial quantum state of the 2-dimensional toy model under consideration is to be a superposition of states with momenta in opposite space directions and opposite helicity.

TM.7) Since \( \tilde{\delta} := \frac{1}{2}(k_1 - k_0) \) is not a root of \( e_9 \), we need to introduce the \textit{auxiliary} roots \( \alpha + m\tilde{\delta} \)

yielding the needed superposition of momenta.

We will use the notation:

\[
\tilde{p} := m\tilde{\delta} \quad \text{for} \quad p = m\delta;
\]

\[
\tilde{x}_{\alpha+p} := x_{\alpha+p} + \eta_\alpha x_{\alpha+\tilde{\delta}}, \quad \text{where} \quad \eta_\alpha^d = 1;
\]

so that \( \{x_{\alpha+p} \to \eta_\alpha x_{\alpha+\tilde{\delta}}, \quad x_p \to \eta_\alpha x^\alpha_p\} \) is an isomorphism \( e_9 \to \tilde{e}_9 \) and \( x_\alpha \to \eta_\alpha x_\alpha \) is an \( e_8 \) involution, up to a sign. The coefficient \( \eta_\alpha \in \{\pm 1, \pm i\} \) has been introduced to have the freedom to vary it depending on the spin of the generator related to \( \alpha \).

The commutation relations (3.14) become, for \( \alpha, \beta \in \Phi_8 \) and \( p_1, p_2 \) linear combination with positive integer coefficients of \( \delta, \tilde{\delta} \):

\[
\begin{align*}
[x^\alpha_{p_1}, x^\beta_{p_2}] & = 0 \quad \quad \quad \quad \quad \text{if} \quad \alpha + \beta = \delta \neq \tilde{\delta} \\
[x^\alpha_{p_1}, x^\beta_{p_2}] & = (\alpha, \beta)x_{\beta+p_1+p_2} \quad \quad \text{if} \quad \alpha + \beta \neq \Delta_8 \cup \{0\} \\
[x_{\alpha+p_1}, x_{\beta+p_2}] & = \begin{cases} 
0 & \text{if} \quad \alpha + \beta \in \Phi_8 \\
\varepsilon(\alpha, \beta)x_{\alpha+\beta+p_1+p_2} & \text{if} \quad \alpha + \beta \neq \Delta_8 \\
-x^\alpha_{p_1+p_2} & \text{if} \quad \alpha + \beta = 0
\end{cases}
\end{align*}
\]

(3.17)

\( \alpha + \beta = 0 \) in the last commutation relation implies \( p_1 + p_2 \neq 0 \), as remarked in item [TM.1]. Moreover \( (p_1 + p_2)^2 \geq 0 \) and we still have positive energy associated to all particles. It is no longer true, however, that particles are necessarily massless, as we immediately realize by the fact that \( \delta + \tilde{\delta} = k_{-1} \) represents a mass at rest. We also notice that the product is to be antisymmetric, therefore

\[
x^-p = -x^p \quad \text{and} \quad [x^\alpha_{p_1}, x^\beta_{p_2}] = -[x^\alpha_{p_1}, x^\beta_{p_2}]
\]

(3.18)

Moreover, for consistency

\[
x^{\alpha + \beta} = x^\alpha_p + x^\beta_p
\]

(3.19)
We will prove in the forthcoming paper \textsuperscript{1}, as a particular case of a more general statement, that the algebra so defined is a Lie algebra.

TM.8) We lack two spatial dimensions. This suggests a further extension to \textbf{e}_{12} or \textbf{de}_{12}, or to analogous Borcherds (or generalized Kac-Moody) algebras, as we will investigate in the next sections.

TM.9) Our toy model still lacks three features, which urges to a further extension of the algebra (investigated in the companion paper \textsuperscript{1}):

(a) \textit{locality}, i.e. spacetime related multiplication rules that immerse the algebra into a vertex-type algebra;
(b) \textit{space expansion} within the vertex algebra;
(c) \textit{Pauli exclusion principle}, that, as we will see, requires an extension to Lie superalgebra.

The above considerations are hinting to the fact that the extension of \textbf{e}_8 to Kac-Moody, or even beyond, to Generalized Kac-Moody (Borcherds), algebras is very appealing to particle physics, not only to 2-dimensional conformal field theory, \textsuperscript{30}.

3.4 \textbf{e}_{12} and \textbf{de}_{12}

The Dynkin diagram of \textbf{e}_{12} is shown in (3.2). We use the same indices for the simple roots $\alpha_1, \alpha_{0'}, \alpha_{0'}, \alpha_0, \alpha_1, ... , \alpha_8$ and the orthonormal basis vectors $k_1, k_{0'}, k_{0'}, k_0, k_1, ... , k_8$ of the Lorentzian space $\mathbb{R}^{11,1}$, with $(k_i, k_i) = 1$, for $i = 0', 0', 0, 1, ... , 8$, and $(k_1, k_1) = -1$.

It is here worth presenting a different choice for the set of simple roots, with respect to (3.3), in which the $\textbf{e}_8$ simple roots are all fermionic:

\begin{align*}
\alpha_{-1} &= -\frac{\sqrt{3}}{2}k_{0'} + \frac{\sqrt{2}}{2}k_{0'} + \sqrt{3}\delta; \\
\alpha_{0'} &= \sqrt{2}k_{0'} + \delta; \\
\alpha_{0'} &= \sqrt{2}k_{0'} - 2k_0 + 2\delta; \\
\alpha_0 &= \frac{1}{2}(-k_1 + k_2 + k_3 - k_4 - k_5 - k_6 - k_7 - k_8); \\
\alpha_1 &= \frac{1}{2}(k_1 - k_2 - k_3 - k_4 + k_5 + k_6 + k_7 - k_8); \\
\alpha_2 &= \frac{1}{2}(k_1 - k_2 + k_3 + k_4 - k_5 - k_6 - k_7 + k_8); \\
\alpha_3 &= \frac{1}{2}(-k_1 + k_2 - k_3 - k_4 - k_5 + k_6 + k_7 + k_8); \\
\alpha_4 &= \frac{1}{2}(k_1 + k_2 - k_3 - k_4 + k_5 - k_6 - k_7 + k_8); \\
\alpha_5 &= \frac{1}{2}(-k_1 - k_2 + k_3 + k_4 + k_5 - k_6 + k_7 - k_8); \\
\alpha_6 &= \frac{1}{2}(k_1 + k_2 + k_3 - k_4 - k_5 + k_6 + k_7 + k_8); \\
\alpha_7 &= \frac{1}{2}(-k_1 - k_2 - k_3 - k_4 - k_5 + k_6 - k_7 + k_8); \\
\alpha_8 &= \frac{1}{2}(k_1 + k_2 + k_3 + k_4 + k_5 + k_6 - k_7 - k_8),
\end{align*}

where $\delta = \frac{1}{2}(k_0 + k_{-1})$.

The corresponding Cartan matrix is the Gram matrix of the $\textbf{e}_{12}$ lattice in $\mathbb{R}^{11,1}$ which is not unimodular. We interpret the coordinates $(k_{0'}, k_{0'}, k_0, k_1)$ as 4-momentum coordinates with Lorentzian signature.
where \( \delta \) is:

Several difficulties in proceeding with our program arise with Kac-Moody algebras:

4 Beyond Kac-Moody explicit computer calculations, which for the case of de

einate between simple roots of simplifications, due to the presence of only one irrational number, \( R \).

P.3) However, the most important issue comes from physics: in \( e_{12} \) and \( de_{12} \) three simple roots (namely, \( \alpha_{-1}, \alpha_{0}\prime \), and \( \alpha_{0}\prime \)) have *tachyon-like* momenta, due to
their positive norm. The interpretation of such tachyonic momenta, as well as the investigation of their impact on the interactions among charged particles, is beyond the scope of the present paper; computer calculations, starting from an initial state, may reveal the scenario that tachyonic simple roots may yield to.

It is our opinion that these are good motivations for focusing our investigation on Generalized Kac-Moody (Borcherds) algebras, where two of the three difficulties listed above disappear.

5 Borcherds \( \mathcal{B}_{12} \)

Borcherds algebras are a generalization of Kac-Moody algebras obtained by releasing the condition on the diagonal elements of the Cartan matrix, which are then allowed to be non-positive, as well as by restricting the Serre relations to the generators associated to positive norm simple roots, \cite{31, 32}.

A generalized Kac-Moody (or Borcherds) algebra \( \mathfrak{B} \) is constructed as follows.

Let \( H \) be a real vector space with a symmetric bilinear inner product \((\cdot,\cdot)\), and with elements \( h_i \) indexed by a countable set \( I \), such that \((h_i, h_j) \leq 0 \) if \( i \neq j \) and \( 2(h_i, h_j)/(h_i, h_i) \) is an integer if \( (h_i, h_i) \) is positive. The matrix \( A \) with entries \( a_{ij} := (h_i, h_j) \) is called the symmetrized Cartan matrix of \( \mathfrak{B} \).

The generalized Kac-Moody (or Borcherds) algebra \( \mathfrak{B} \) associated to \( A \) is defined to be the Lie algebra generated by \( H \) and elements \( e_i \) and \( f_i \), for \( i \in I \), with the following relations:

1. The (injective) image of \( H \) in \( \mathfrak{B} \) is commutative.
2. If \( h \) is in \( H \), then \([h, e_i] = (h, h_i)e_i \text{ and } [h, f_i] = -(h, h_i)f_i.\)
3. \([e_i, f_j] = \delta_{ij}h_i.\)
4. If \( a_{ii} > 0 \) and \( i \neq j \), then \( \text{ad}(e_i)^n e_j = \text{ad}(f_i)^n f_j = 0, \) where \( n = 1 - 2a_{ij}/a_{ii}.\)
5. If \( a_{ij} = 0, \) then \([e_i, e_j] = [f_i, f_j] = 0.\)

If \( a_{ii} > 0 \) for all \( i \in I \), then \( \mathfrak{B} \) is the Kac-Moody algebra with Cartan matrix \( A \). In general, \( \mathfrak{B} \) has almost all the properties of a Kac-Moody algebra, the only major difference being that \( \mathfrak{B} \) is allowed to have imaginary simple roots.

The root lattice \( \mathbb{L} \) is the free Abelian group generated by elements \( \alpha_i \) for \( i \in I \), called simple roots, and \( \mathbb{L} \) has a real-valued bilinear form defined by \((\alpha_i, \alpha_j) = a_{ij}.\)

The Lie algebra \( \mathfrak{B} \) is then graded by \( \mathbb{L} \) with \( H \) in degree 0, \( e_i \) (resp \( f_i \)) in degree \( \alpha_i \) (resp. \( -\alpha_i \)). A root is a nonzero element \( \alpha \) of \( \mathbb{L} \) such that there are elements of \( \mathfrak{B} \) of degree \( \alpha \). A root \( r \) is called real if \( (r, r) > 0 \), otherwise it is called imaginary. A root \( r \) is positive if it is a sum of simple roots, and negative if \( -r \) is positive. Notice that every root is either positive or negative, \cite{31}.

We build the following symmetrized Cartan matrix for a Borcherds algebra of
rank 12, that we denote by \( B_{12} \):

\[
\begin{pmatrix}
-1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \\
\end{pmatrix}.
\] (5.1)

Notice that, for \( \delta := \alpha_0 + 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 5\alpha_4 + 6\alpha_5 + 3\alpha_6 + 4\alpha_7 + 2\alpha_8 \), a 4-momentum vector can be written as

\[
p := p_0\alpha_1 + p_1 (\alpha_{0''} - \alpha_1) + p_2 (\alpha_{0'} - \alpha_1) + p_3 (\delta - \alpha_1).
\] (5.2)

Using the Cartan Matrix \( [5.1] \) we get indeed:

\[
\begin{align*}
(\alpha_1, \alpha_1) &= -1 \\
(\alpha_{0''} - \alpha_1, \alpha_{0''} - \alpha_1) &= (\delta - \alpha_1, \delta - \alpha_1) = 1; \\
(\alpha_1, \alpha_{0''} - \alpha_1) &= (\alpha_1, \delta - \alpha_1) = 0; \\
(\alpha_{0''} - \alpha_1, \alpha_{0'} - \alpha_1) &= (\alpha_{0'} - \alpha_1, \delta - \alpha_1) = 0,
\end{align*}
\] (5.3)

hence the Lorentzian scalar product:

\[
(p, p') = -p_0p_0' + p_1p_1' + p_2p_2' + p_3p_3'.
\] (5.4)

Let us restrict to positive roots \( r = \sum_I \lambda_i \alpha_i, \; \lambda_i \in \mathbb{N} \cup \{0\} \), and let us denote by \( B^+ \) the corresponding subalgebra of \( B_{12} \). The physical motivation for restricting to \( B^+ \) is that, given a positive root \( r = \sum_I \lambda_i \alpha_i \), its 4-momentum is

\[
p = (p_0, p_1, p_2, p_3) = (\lambda_{-1} + \lambda_{0''} + \lambda_{0'} + \lambda_0, \lambda_{0''}, \lambda_{0'}, \lambda_0),
\] (5.5)

with \( \lambda_{-1}, \lambda_{0''}, \lambda_{0'}, \lambda_0 \geq 0 \), implying \( m^2 := -p^2 \geq 0 \), namely \( p \) either lightlike or timelike. In particular:

\[
p^2 = (\lambda_{-1}^2 + 2\lambda_{-1}\sum_{i\neq -1} \lambda_i + \sum_{i\neq j, i,j \neq -1} \lambda_i \lambda_j), \; i, j \in \{-1, 0'', 0', 0\}
\]

\[
\begin{aligned}
&= 0 \quad \text{if } \lambda_{-1} = 0 \text{ and at most one } \lambda_i \neq 0, i \neq -1 \\
&= -1 \quad \text{if } \lambda_{-1} = 1 \text{ and all } \lambda_i = 0, i \neq -1 \\
&\leq -2 \quad \text{otherwise}
\end{aligned}
\] (5.6)

**Remark 5.1.** Notice that the mass of a particle cannot be arbitrary small, since there is a lower limit \( m \geq 1 \).

For \( r = \sum_I \lambda_i \alpha_i, \; \lambda_i \in \mathbb{N} \cup \{0\} \) we introduce the notation

\[
r = \alpha + p \\
\alpha := \lambda_0 \alpha_0 + \lambda_1 \alpha_1 + \ldots + \lambda_8 \alpha_8
\]

\[
= (\lambda_1 - 2\lambda_0) \alpha_1 + (\lambda_2 - 3\lambda_0) \alpha_2 + (\lambda_3 - 4\lambda_0) \alpha_3
\]

\[
+ (\lambda_4 - 5\lambda_0) \alpha_4 + (\lambda_5 - 6\lambda_0) \alpha_5 + (\lambda_6 - 3\lambda_0) \alpha_6
\]

\[
+ (\lambda_7 - 4\lambda_0) \alpha_7 + (\lambda_8 - 2\lambda_0) \alpha_8
\] (5.7)
Thus, $\alpha$ is in the lattice $\mathbb{L}_{e_8}$ of $e_8$, and a precise physical meaning is assigned to positive real and imaginary roots when $\alpha \in \mathbb{L}_{e_8}\backslash\{0\}$:

**Proposition 5.2.** A generator in $B^+$, associated to a positive root $r = \alpha + p$, with $\alpha \in \mathbb{L}_{e_8}\backslash\{0\}$ and momentum $p \neq 0$, is massive if and only if $\alpha + p$ is an imaginary root; it is massless if and only if $r$ is real, in which case it is a positive real root of $e_9 \subset B_{12}$.

**Proof.** The proof consists of the following steps:

1. from Proposition 2.1. of [31], it holds that every positive root $r = \alpha + p$ is conjugate under the Weyl group to a root $r_0 = \alpha' + p'$, such that either $r_0$ is a simple real root $\alpha_i$, $i \in \{0, 1, ..., 8\}$, or it is a positive root in the Weyl chamber (namely $(r_0, \alpha_i) \leq 0$ for all simple roots $\alpha_i$);

2. since $r$ and $r_0$ are conjugate under the Weyl group, then $(r, r) = (r_0, r_0)$;

3. if $r_0$ is a real simple root, then it is a root of $e_9$ and $p'^2 = 0$; $r_0$ is real and so is $r$. Since the Weyl group is generated by the reflections $\rho - (\rho, \alpha_i)\alpha_i$, where the $\alpha_i$ simple roots are real, hence $\alpha_i \in e_9 \subset B_{12}$, it coincides with the Weyl group of $e_9$. By applying to $r_0$ Weyl reflections we stay within $e_9$, since every Kac-Moody algebra is invariant under the Weyl group; therefore, $r$ is a real root of $e_9$, namely $r = \alpha + m\delta$, $\alpha \in e_8$ and $m\delta$ is lightlike;

4. if $r_0$ is in the Weyl chamber, then $(r, r) = (r_0, r_0) = \sum \lambda_i(r_0, \alpha_i) \leq 0$, $i \in I$, since all $\lambda_i$ are positive being $r_0$ is a positive root; thus, $r$ is imaginary;

5. since $(r, r) = \alpha^2 + p^2 \leq 0$ with $\alpha^2 \geq 2$, then $m^2 \geq \alpha^2 \geq 2$, and the particle associated to $r$ is massive.

\[ \square \]

**Remark 5.3.** In the massive case, the lower limit of the mass grows with the norm of $\alpha$: if $\alpha \in \mathbb{L}_{e_8}\backslash\{0\}$ is not a root of $\Phi_8$, then the mass is certainly bigger than the lower mass a particle corresponding to a root of $\Phi_8$ may have. We also notice that charged massless particles ($\alpha \neq 0$ in the root $\alpha + p$) are quite peculiar, since their momentum can only be in 1 direction. The photon is not in this class, since it has $\alpha = 0$, but the (non-virtual) gluons are. A non-virtual photon can be produced in a decay process, [1].

**Remark 5.4.** We emphasize that two of the three problems listed in section[4] about Kac-Moody algebras vanish in the Borcherds algebra $B_{12}$. These are obviously [P.1] and [P.3]. But [P.2] still remains, [1].

The companion paper [1], based on the treatment and considerations of this paper, will focus on a particular rank-12 algebra $g_u$, in order to build a model for quantum gravity. In particular, we will turn $g_u$ into a Lie superalgebra, and we will discuss scattering processes and decays.

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| roots                                      | q.e.m. | # | \{r_c, s_c\} |
|-------------------------------------------|--------|---|---------------|
| ±(k_1 - k_2)                             | 0      | 2 | ±\{2, 0\}    |
| ±(k_2 - k_3)                             | 0      | 2 | ±\{-1, 3\}   |
| ±(k_1 - k_3)                             | 0      | 2 | ±\{1, 3\}    |
| ±k_i ± k_j                               | 5 \leq i \leq 7 | 0 | 24            |
| k_4 ± k_i                                | 5 \leq i \leq 8 | 1 | 8             |
| -k_4 ± k_i                               | 5 \leq i \leq 7 | -1| 8             |
| \frac{1}{2}(±(k_1 + k_2 + k_3 + k_4) ± \ldots ± k_8)\ even # of + | 0 | 16           |
| \frac{1}{2}(-(k_1 + k_2 + k_3) + k_4 ± \ldots ± k_8)\ even # of + | 1 | 8            |
| \frac{1}{2}((k_1 + k_2 + k_3) - k_4 ± \ldots ± k_8)\ even # of + | -1 | 8          |
| -k_2 - k_3, k_1 + k_4                    | 2/3    | 2 | ±\{1, 1\}    |
| k_1 - k_4                                | -4/3   | 1 | ±\{1, 1\}    |
| \frac{1}{2}(k_1 - k_2 - k_3 + k_4 ± \ldots ± k_8)\ even # of + | 2/3 | 8          |
| \frac{1}{2}(k_1 - k_2 - k_3 - k_4 ± \ldots ± k_8)\ even # of + | -1/3 | 8       |
| +k_2 + k_3, -k_1 - k_4                   | -2/3   | 2 | ±\{-1, -1\}  |
| -k_1 + k_4                               | 4/3    | 1 | ±\{-1, -1\}  |
| \frac{1}{2}(k_1 + k_2 + k_3 - k_4 ± \ldots ± k_8)\ even # of + | -2/3 | 8          |
| \frac{1}{2}(k_1 - k_2 + k_3 + k_4 ± \ldots ± k_8)\ even # of + | -1/3 | 8       |
| -k_1 - k_3, k_2 + k_4                    | 2/3    | 2 | ±\{-1, 1\}   |
| k_2 - k_4                                | -4/3   | 1 | ±\{-1, 1\}   |
| \frac{1}{2}(k_1 - k_2 - k_3 + k_4 ± \ldots ± k_8)\ even # of + | 2/3 | 8          |
| \frac{1}{2}(k_1 - k_2 + k_3 - k_4 ± \ldots ± k_8)\ even # of + | -1/3 | 8       |
| k_1 + k_3, -k_2 - k_4                    | -2/3   | 2 | ±\{1, -1\}   |
| -k_2 + k_4                               | 4/3    | 1 | ±\{1, -1\}   |
| \frac{1}{2}(k_1 - k_2 + k_3 - k_4 ± \ldots ± k_8)\ even # of + | -2/3 | 8          |
| \frac{1}{2}(k_1 - k_2 + k_3 + k_4 ± \ldots ± k_8)\ even # of + | 1/3 | 8       |
| -k_1 - k_2, k_3 + k_4                    | 2/3    | 2 | ±\{0, -2\}   |
| k_3 - k_4                                | -4/3   | 1 | ±\{0, -2\}   |
| \frac{1}{2}(k_1 - k_2 + k_3 + k_4 ± \ldots ± k_8)\ even # of + | 2/3 | 8          |
| \frac{1}{2}(k_1 - k_2 - k_3 - k_4 ± \ldots ± k_8)\ even # of + | -1/3 | 8       |
| k_1 + k_2, -k_3 - k_4                    | -2/3   | 2 | ±\{0, 2\}    |
| -k_3 + k_4                               | 4/3    | 1 | ±\{0, 2\}    |
| \frac{1}{2}(k_1 + k_2 - k_3 - k_4 ± \ldots ± k_8)\ even # of + | -2/3 | 8          |
| \frac{1}{2}(k_1 + k_2 - k_3 - k_4 ± \ldots ± k_8)\ even # of + | 1/3 | 8       |

Table 1: The Magic Star of e_8: q.e.m.(\alpha) = (\alpha, -\frac{1}{3}(k_1 + k_2 + k_3) + k_4); r_c(\alpha) = (\alpha, k_1 - k_2), s_c(\alpha) = (\alpha, k_1 + k_2 - 2k_3).
| roots                          | q.e.m. | # of roots | \{r_f, s_f\} |
|-------------------------------|-------|------------|---------------|
| ±(k_4 - k_5)                  | ±1    | 2          | ±\{(2, 0)\}  |
| ±(k_5 - k_6)                  | 0     | 2          | ±\{-1, 3\}   |
| ±(k_4 - k_6)                  | ±1    | 2          | ±\{1, 3\}    |
| ±k_2 ± k_8                    | 0     | 4          |               |
| \(\frac{1}{2}(±(k_1 + k_2 + k_3 + k_4 + k_5 + k_6) ± k_7 ± k_8)\) | even # of + | 0 | 4 |
| \(\frac{1}{2}(-k_1 - k_2 - k_3 + k_4 + k_5 + k_6 ± k_7 ± k_8)\) | even # of + | 1 | 2 |
| \(\frac{1}{2}(k_1 + k_2 + k_3 - k_4 - k_5 - k_6 ± k_7 ± k_8)\) | even # of + | -1 | 2 |
| \(\frac{1}{2}(-k_1 - k_2 - k_3 + k_4 - k_5 - k_6 ± k_7 ± k_8)\) | even # of + | 1 | 2 |
| -k_5 - k_6                    | 0     | 1          |               |
| k_4 ± k_6                     | i = 7, 8 | 1 | 4 |
| \(\frac{1}{2}(k_1 + k_2 + k_3 + k_4 - k_5 - k_6 ± k_7 ± k_8)\) | even # of + | 0 | 2 |
| \(\frac{1}{2}(-k_1 - k_2 - k_3 + k_4 - k_5 - k_6 ± k_7 ± k_8)\) | even # of + | 1 | 2 |
| k_5 ± k_6                     | 0     | 1          |               |
| -k_4 ± k_4                    | i = 7, 8 | -1 | 4 |
| \(\frac{1}{2}(k_1 + k_2 + k_3 - k_4 + k_5 + k_6 ± k_7 ± k_8)\) | even # of + | -1 | 2 |
| \(\frac{1}{2}(-k_1 - k_2 - k_3 + k_4 + k_5 + k_6 ± k_7 ± k_8)\) | even # of + | 0 | 2 |
| k_4 ± k_5                     | 1     | 1          |               |
| -k_4 ± k_5                    | i = 7, 8 | 0 | 4 |
| \(\frac{1}{2}(k_1 + k_2 + k_3 - k_4 + k_5 + k_6 ± k_7 ± k_8)\) | even # of + | 0 | 2 |
| \(\frac{1}{2}(-k_1 - k_2 - k_3 + k_4 - k_5 + k_6 ± k_7 ± k_8)\) | even # of + | 1 | 2 |
| -k_4 ± k_5                    | 0     | 1          |               |
| k_4 ± k_5                     | i = 7, 8 | 0 | 4 |
| \(\frac{1}{2}(k_1 + k_2 + k_3 + k_4 - k_5 + k_6 ± k_7 ± k_8)\) | even # of + | 0 | 2 |
| \(\frac{1}{2}(-k_1 - k_2 - k_3 + k_4 + k_5 - k_6 ± k_7 ± k_8)\) | even # of + | 1 | 2 |

Table 2: The Magic Star of \(e_6\): \(q_{e.m.}(\alpha) = (\alpha, -\frac{1}{3}(k_1 + k_2 + k_3) + k_4); r_f(\alpha) = (\alpha, k_4 - k_5), s_f(\alpha) = (\alpha, k_4 + k_5 - 2k_6).\)
| roots $[k := k_1 + k_2 + k_3]$ | $q_{e.m.}$ | lepton | $\sigma_z := \frac{1}{2} (\sigma_1 + \sigma_2)$ | $\{r_f, s_f\}$ |
|---------------------------------|----------|--------|---------------------------------|----------------|
| $\frac{1}{2} (k + k_4 - k_5 - k_6 + k_7 + k_8)$ | 0 | $\nu_\tau^\prime$ | $-1/2$ | $\{1, 1\}$ |
| $\frac{1}{2} (k + k_4 - k_5 - k_6 - k_7 - k_8)$ | 0 | $\nu_\tau^\prime$ | $-1/2$ | $\{-1, -1\}$ |
| $\frac{1}{2} (-k + k_4 - k_5 + k_6 + k_7 + k_8)$ | 1 | $e^+$ | $-1/2$ | $\{-1, 1\}$ |
| $\frac{1}{2} (-k + k_4 - k_5 - k_6 - k_7 - k_8)$ | 1 | $\mu^+$ | $-1/2$ | $\{0, 2\}$ |

**Table 3:** The lepton families with their spin-$z$.  

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| roots | \[ k' := k_3 - k_2 - k_3 \] | q.c.m. \[ \sigma_z := \frac{1}{2} (\alpha + \rho_2) \] | \( \{ r_f, s_f \} \) |
|-------|---------------------------------|-----------------|-----------------|
| \( \frac{1}{2} (k' + k_4 - k_5 - k_6 + k_7 + k_8) \) | \( \frac{1}{3} \) | \( \tilde{b}' \) | \( -1/2 \) | \( \{1, 1\} \) |
| \( \frac{1}{2} (k' + k_4 - k_5 - k_6 - k_7 - k_8) \) | \( \frac{1}{3} \) | \( \tilde{B}' \) | \( -1/2 \) |
| \( -\frac{1}{2} (k' + k_4 - k_5 - k_6 + k_7 + k_8) \) | \( -\frac{2}{3} \) | \( u \) | \( -1/2 \) |
| \( -\frac{1}{2} (k' + k_4 - k_5 - k_6 - k_7 - k_8) \) | \( -\frac{2}{3} \) | \( c \) | \( -1/2 \) |
| \( -\frac{1}{2} (k' + k_4 - k_5 - k_6 + k_7 + k_8) \) | \( -\frac{2}{3} \) | \( b' \) | \( 1/2 \) |
| \( -\frac{1}{2} (k' + k_4 - k_5 - k_6 - k_7 - k_8) \) | \( -\frac{2}{3} \) | \( B' \) | \( 1/2 \) | \( \{ -1, -1 \} \) |
| \( \frac{1}{2} (k' - k_4 + k_5 - k_6 + k_7 + k_8) \) | \( -\frac{2}{3} \) | \( \bar{t} \) | \( 1/2 \) | \( \{ -1, 1 \} \) |
| \( \frac{1}{2} (k' - k_4 + k_5 - k_6 - k_7 - k_8) \) | \( -\frac{2}{3} \) | \( \bar{T} \) | \( 1/2 \) |
| \( -\frac{1}{2} (k' - k_4 + k_5 - k_6 + k_7 - k_8) \) | \( -\frac{1}{3} \) | \( d' \) | \( 1/2 \) |
| \( -\frac{1}{2} (k' - k_4 + k_5 - k_6 - k_7 - k_8) \) | \( -\frac{1}{3} \) | \( s' \) | \( 1/2 \) |
| \( -\frac{1}{2} (k' - k_4 + k_5 + k_6 + k_7 + k_8) \) | \( 1/3 \) | \( \tilde{t} \) | \( -1/2 \) | \( \{ 0, -2 \} \) |
| \( -\frac{1}{2} (k' - k_4 + k_5 + k_6 - k_7 - k_8) \) | \( 1/3 \) | \( \tilde{T} \) | \( -1/2 \) |
| \( -\frac{1}{2} (k' - k_4 + k_5 + k_6 + k_7 - k_8) \) | \( -\frac{1}{2} \) | \( d' \) | \( -1/2 \) |
| \( -\frac{1}{2} (k' - k_4 + k_5 + k_6 - k_7 - k_8) \) | \( -\frac{1}{2} \) | \( s' \) | \( -1/2 \) |
| \( -\frac{1}{2} (k' - k_4 - k_5 + k_6 + k_7 + k_8) \) | \( 2/3 \) | \( \tilde{t} \) | \( 1/2 \) | \( \{ 0, 2 \} \) |
| \( -\frac{1}{2} (k' - k_4 - k_5 + k_6 - k_7 - k_8) \) | \( 2/3 \) | \( \tilde{T} \) | \( 1/2 \) |
| \( -\frac{1}{2} (k' - k_4 - k_5 - k_6 + k_7 + k_8) \) | \( 1/3 \) | \( \bar{d}' \) | \( 1/2 \) |
| \( -\frac{1}{2} (k' - k_4 - k_5 - k_6 - k_7 - k_8) \) | \( 1/3 \) | \( \bar{s}' \) | \( 1/2 \) |
| \( \frac{1}{2} (k' - k_4 - k_5 - k_6 + k_7 - k_8) \) | \( -\frac{2}{3} \) | \( \bar{c} \) | \( -1/2 \) | \( \{ 0, 0 \} \) |
| \( \frac{1}{2} (k' - k_4 - k_5 - k_6 - k_7 + k_8) \) | \( -\frac{2}{3} \) | \( \bar{u} \) | \( -1/2 \) |
| \( \frac{2}{3} (k' - k_4 - k_5 - k_6 - k_7 + k_8) \) | \( -\frac{2}{3} \) | \( \bar{B}' \) | \( -1/2 \) |
| \( \frac{1}{2} (k' - k_4 - k_5 - k_6 - k_7 - k_8) \) | \( -\frac{2}{3} \) | \( \bar{b}' \) | \( -1/2 \) |
| \( \frac{1}{2} (k' - k_4 - k_5 - k_6 + k_7 - k_8) \) | \( 2/3 \) | \( c \) | \( 1/2 \) |
| \( \frac{1}{2} (k' - k_4 - k_5 - k_6 - k_7 + k_8) \) | \( 2/3 \) | \( u \) | \( 1/2 \) |
| \( -\frac{1}{2} (k' - k_4 - k_5 - k_6 + k_7 - k_8) \) | \( 1/3 \) | \( \tilde{B}' \) | \( 1/2 \) |
| \( -\frac{1}{2} (k' - k_4 - k_5 - k_6 - k_7 + k_8) \) | \( 1/3 \) | \( \tilde{b}' \) | \( 1/2 \) |

Table 4: The flavor families of blue (\( \{1,1\} \)) quarks with their spin-z.