ON NEWTON EQUATIONS WHICH ARE TOTALLY INTEGRABLE AT INFINITY

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Abstract. In this paper Hamiltonian system of time dependent periodic Newton equations is studied. It is shown that for dimensions 3 and higher the following rigidity results holds true: If all the orbits in a neighborhood of infinity are action minimizing then the potential must be constant. This gives a generalization of the previous result [2], where it was required all the orbits to be minimal. As a result we have the following application: Suppose that for the time-1 map of the Hamiltonian flow there exists a neighborhood of infinity which is filled by invariant Lagrangian tori homologous to the zero section. Then the potential must be constant. Remarkably, the statement is false for $n=1$ case and remains unknown to the author for $n=2$.

1. Introduction and the result

In this note we consider the system of Newton equations (1) with time dependent periodic potential $u(q, t)$:

$$u : T^n \times S^1 \to \mathbb{R}.$$  

Here $T^n = \mathbb{R}^n/\Gamma$ is an $n$-torus, $S^1 = \mathbb{R}/\mathbb{Z}$ and $q = (q_1, \ldots, q_n)$ and $t$ stand for standard coordinates on $\mathbb{R}^n$ and $\mathbb{R}$ respectively. We fix a Riemannian metric $g$ on the torus and study the system of Newton equations determined by $g$:

$$\nabla_q \dot{q} = -\nabla_q u$$

Here and later $\nabla, | \cdot |$ are computed in terms of the Riemannian metric $g$.

Classically, the extremals of the action functional

$$\int \left(\frac{1}{2} |\dot{q}|^2 - u(q, t)\right) dt$$

coincide with the solutions of (1) and are described in Hamiltonian formalism by means of the Hamiltonian flow of

$$H : T^*T^n \times S^1 \to \mathbb{R}, \quad H(p, q, t) = \frac{1}{2} |p|^2 + u(q, t).$$

Let $\phi$ denotes the time-1 map of the Hamiltonian flow of $H$.

The main object of our study is the set

$$\mathcal{M} \subseteq T^*T^n \times S^1$$

Date: 20 May 2015.

2010 Mathematics Subject Classification. Primary:37J50;53C24.

Key words and phrases. Total integrability, Minimal orbits, Hopf rigidity, Conjugate points.
swept by those infinite orbits of the Hamiltonian flow \((p(t), q(t), t)\) such that the corresponding extremals \((q(t))\) have no conjugate points, or, in other words which are local minima of the action functional between any two of its points. By definition, \(\mathcal{M}\) is a closed invariant subset of the phase space \(T^*T^n \times S^1\).

The purpose of this note is to prove the following generalization of rigidity theorem discovered in [2]:

**Theorem 1.1.** Let \(n \geq 3\). Assume that the set \(\mathcal{M}\) contains a neighborhood of infinity of \(T^*T^n \times S^1\):

\[
\mathcal{M} \supset \{|p| > R\},
\]

for some positive constant \(R\). Then

1. The metric \(g\) must be Euclidean.
2. The potential \(u\) does not depend on \(q\).

Theorem 1.1 implies the following corollary for phase portraits of integrable Hamiltonians. We shall say that the system \((1)\) is totally integrable at infinity if there exists a neighborhood of infinity of \(T^*T^n\) filled by Lagrangian tori homologous to the zero section which stay invariant under the time-1 map \(\phi\).

**Theorem 1.2.** Let \(n \geq 3\), and suppose that the Hamiltonian system \((2)\) is totally integrable at infinity. Then the Riemannian metric \(g\) must be Euclidean and the potential \(u\) does not depend on \(q\).

Theorem 1.2 follows from Theorem 1.1 applying the so called generalized Birkhoff theorem. This theorem states that every Lagrangian torus homologous to the zero section which is invariant under \(\phi\) is a graph, and therefore consists of minimal orbits. The generalized Birkhoff theorem was proved in series of joint papers with L.Polterovich (see [3] for the case most suitable for this paper) where an extra dynamical assumption was imposed. Nowadays, it is known to be true without this assumption due to the final solution obtained in [1] by a different methods.

Let me point out that for Euclidean \(g\) and arbitrary periodic potential \(u\), by KAM theory, there are lots of invariant tori of the Hamiltonian system \((2)\) lying near \(\infty\). All of them consist of minimal orbits.

The statements of Theorems 1.1, 1.2 in the case \(n = 1\) are false as the following example shows.

**Example 1.** Let \(n = 1\). Consider any non-constant autonomous periodic potential \(u(q)\) so that \(H(p, q, t) = \frac{1}{2}p^2 + u(q)\). As the energy constant \(h\) varies the energy level curves \(\{H = h\}\) undergo perestroika on the phase cylinder and it is easy to see that

\[
\mathcal{M} = \bigcup_{h \geq \max u} \left\{ p = \pm \sqrt{2(h-u)} \right\}.
\]

Therefore, for \(n = 1\) any autonomous potential \(u\) satisfies the assumption of the theorem. Another example is the system \((2)\) with periodic potential of the form \(u = u(mq + nt)\), which is integrable and has a similar phase portrait as above.
Questions and Remarks.

1) In [2] the case \( M = T^*T^n \times S^1 \) is considered for any \( n \geq 1 \). Let me remark that for \( n = 2 \) the method described below does not work so it is unclear if the result of the theorems remains valid in this case. Obviously, one can not create counter-examples by taking direct products of known Hamiltonians.

2) It is an interesting question if the statement allows further refinement. For instance does the result hold true assuming that the complement of the set \( M \) has finite measure?

3) Another open question for the case \( n = 1 \): is it true that the systems described in the example are the only ones which are totally integrable at infinity? Are there integrable systems [2] which are not totally integrable at infinity?

The strategy of the proofs of Theorem 1.1 is similar to that of [2]. In this paper in order to handle the non-compactness of the phase space we need to introduce suitably chosen cutoff function in the phase space and then to stretch it to become concentrated closer and closer to infinity.

The paper is organized as follows: in Section 2 we use Burago-Ivanov theorem to prove that the metric \( g \) has to be Euclidean; in Section 3 we get an inequality applying Hopf method, and finally in section 4 we use the stretching of the cutoff function to prove the reverse inequality unless \( \nabla_q u \) vanishes identically. Combination of the results of these sections yields the proof.

2. Flatness of the metric \( g \)

In this section we show that the argument of [2] applies in our situation. So assume that \( \mathcal{M} \) contains a neighborhood of infinity. We have to show that the metric \( g \) is Euclidean. In the opposite case it follows from [4] that there are geodesics of \( g \) which have conjugate points. Let \( q(t), t \in [0; T] \) be a segment of such a geodesic so that \( q(0) \) and \( q(T_*) \) for some \( 0 < T_* < T \) are conjugate. Let \( \gamma = (p(t), q(t)) \) be the corresponding orbit of the Hamiltonian flow of the metric \( g \), \( |p(t)| = 1 \). Consider now the orbit \( \tilde{\gamma} = (\tilde{p}(t), \tilde{q}(t)), t \in [0; T] \) of the Hamiltonian flow with the perturbed function

\[
H_\epsilon = \frac{1}{2} |p|^2 + \epsilon^2 u(q, \epsilon t)
\]

with the same initial conditions as \( \gamma \), and for \( \epsilon \) small enough. Then by continuous dependence, the extremal \( (\tilde{q}(t)) \) also has conjugate points somewhere on the segment \([0; T]\).

Moreover notice, that there is a correspondence between the orbits \((p(t), q(t))\) of \( H \) and \((\epsilon p(\epsilon t), q(\epsilon t))\) of \( H_\epsilon \) for any \( \epsilon \).

Using this correspondence we get the orbit \( \Gamma = (\tilde{p}(\frac{1}{\epsilon} t), \tilde{q}(\frac{1}{\epsilon} t)), t \in [0; T] \) of \( H_\epsilon \) also has conjugate points. The last thing is to see that we started with a geodesic segment on the energy level \(|p| = 1\), so for \( \epsilon \) small enough the constructed segment \( \Gamma \) lies in \{|p| > R\} \subset \mathcal{M} \). This contradiction proves the claim.
3. E. Hopf method and cutoff function

From now on we shall assume that the Riemannian metric $g$ is standard Euclidean. This is in fact the main case of this note.

First of all notice that since only gradient of the potential $u$ is involved in the equations of the system (1) we are free to add any function of $t$ to $u$, so we shall assume everywhere in the sequel that for some constant $M > 0$:

$$0 \leq u(q, t) \leq M, \ \forall (q, t).$$

It then follows that high energy levels of $H$ lie entirely in $\mathcal{M}$:

$$\forall h > \frac{R^2}{2} + M \implies \{ H = h \} \subset \{ |p| > R \} \subset \mathcal{M}.\tag{3}$$

We shall denote by $\rho$ a non-negative smooth function of one variable (which will be composed with the Hamiltonian $H$ later on) with compact support so that

$$\text{supp}(\rho) \subset \left( \frac{R^2}{2} + M; +\infty \right).$$

Let me denote by $\mu$ the invariant measure $d\mu = dpdqdt$. We have:

**Theorem 3.1.** Suppose that the set $\mathcal{M}$ contains the neighborhood of infinity $\{|p| > R\}$. Then for any function $\rho$ defined above one has the inequality:

$$D = \int (\rho'(H))^2 u_t^2 d\mu + \frac{1}{n} \int (\rho^2(H)) (\nabla_q u)^2 d\mu \geq 0.$$

**Proof.** We proceed as in the original Hopf method [6], [5] and also [7], [2] and construct a measurable matrix function

$$A : \mathcal{M} \rightarrow \mathbb{R},$$

satisfying the matrix Riccati equation:

$$L_v A + A^2 + \text{Hess}(u) = 0,$$

where $L_v$ denotes the Lie derivative along the vector field

$$v = \partial_t + \sum_{i=1}^{n} (p_i \partial q_i - u_i \partial p_i).$$

In addition $A$ and $L_v A$ are uniformly bounded on the whole $\mathcal{M}$. Denoting $a = \text{Tr} A$ and using the inequality for the trace $\text{Tr} A^2 \geq \frac{1}{n}(\text{Tr} A)^2$, we have the following inequality for $a$:

$$L_v a + \frac{1}{n} a^2 + \Delta_q u \leq 0.\tag{4}$$

Next, we multiply the inequality (4) by the function $\rho^2 \circ H$:

$$\rho^2(H) L_v a + \frac{1}{n} \rho^2(H) a^2 + \rho^2(H) \Delta_q u \leq 0.$$

Or equivalently:

$$L_v (\rho^2(H) a) - a L_v (\rho^2(H)) + \frac{1}{n} \rho^2(H) a^2 + \rho^2(H) \Delta_q u \leq 0.$$
Since $L_vH = u_t$ we have
\begin{equation}
L_v(\rho^2(H)a) - 2a\rho'(H)\rho(H)u_t + \frac{1}{n}\rho^2(H)a^2 + \rho^2(H)\Delta_q u \leq 0.
\end{equation}

Then we integrate this inequality over the invariant set $\mathcal{M}$ with respect to invariant measure $d\mu = dpdqdt$. Since the support of $\rho \circ H$ lies entirely in $\mathcal{M}$, by (3), we can extend the integration to the whole $T^\ast T^n \times S^1$:
\begin{equation}
-2\int a\rho'(H)\rho(H)u_t dqdt + \frac{1}{n}\int \rho^2(H)a^2 dqdt + \int \rho^2(H)\Delta_q u dqdt \leq 0,
\end{equation}
where we used the fact that the integral of the first term of (5) vanishes, since the flow of the field $v$ preserves the measure $\mu$.

Integrating by parts the last term of (6) and applying Cauchy-Schwartz inequality to the first term of (6) we get:
\begin{equation}
-2\left(\int (\rho'(H))^2 u_t^2 dqdt\right)^{\frac{1}{2}} \left(\int \rho^2(H)a^2 dqdt\right)^{\frac{1}{2}} + 
+ \frac{1}{n}\int \rho^2(H)a^2 dqdt - \int (\rho^2(H)(\nabla_q u)^2 dqdt \leq 0.
\end{equation}
Notice that (7) is a quadratic inequality in the quantity $\left(\int \rho^2(H)a^2 dqdt\right)^{\frac{1}{2}}$ Thus the discriminant $D$ must be non-negative:
\begin{equation}
D = \int (\rho'(H))^2 u_t^2 dqdt + \frac{1}{n}\int (\rho^2(H)(\nabla_q u)^2 dqdt \geq 0.
\end{equation}
This proves the claim. 

4. Estimating $D$ from above

In what follows we shall stretch the function $\rho$ of the previous section with the help of a small parameter $0 < \alpha < 1$ in the following way:
\begin{equation}
\rho_\alpha(x) := \rho(\alpha x).
\end{equation}
Then for any $0 < \alpha < 1$ we have:
\begin{equation}
\text{supp}(\rho_\alpha) = \frac{1}{\alpha} \text{supp}(\rho) \subset \left(\frac{1}{\alpha}(R^2/2 + M); +\infty\right) \subset \left(R^2/2 + M; +\infty\right).
\end{equation}
Thus theorem 3.1 applies to every such $\rho_\alpha$ and we have
\begin{equation}
D_\alpha = \int (\rho_\alpha'(H))^2 u_t^2 dqdt + \frac{1}{n}\int (\rho_\alpha^2(H)(\nabla_q u)^2 dqdt \geq 0.
\end{equation}
In this section we prove:

**Theorem 4.1.** Let $n \geq 3$. If $\int (\nabla_q u)^2 dqdt > 0$ then there exists an $\alpha \in (0; 1)$ such that $D_\alpha < 0$.

**Proof.** Let me denote the first and the second integrals of $D_\alpha$ in (8) by $A$ and $B$ respectively. We need to estimate each of them from above.
To estimate $A$, use Foubini theorem, then pass to spherical coordinates $(r = |p|, \omega)$ in the fibers and then to the energy instead of $|p|$ as follows:
\begin{equation}
A = \int \left(\int (\rho_\alpha'(H))^2 r^{n-1} drd\omega\right) u_t^2 dqdt =
\end{equation}
\[ \omega_n \int \left( \int (\rho_\alpha^2(H))^2 r^{n-2} \left( \frac{r^2}{2} + u \right) \right) u^2 \, dq \, dt = \]
\[ \omega_n \int \left( \int (\rho_\alpha'(H))\left(2(\alpha H - u)\right)^{n-2} \right) u^2 \, dq \, dt \leq \omega_n \int u^2 \, dq \, dt \left( \int (\rho_\alpha'(H))^2 (2\alpha H)^{n-2} \, dH \right), \]

where we used \( 0 \leq u \) in the last line of the estimate. In the last integral we replace \( \rho_\alpha(H) \) by \( \rho(\alpha H) \) and change the integration variable \( H \rightarrow \alpha H \). In this case we have:

\[ A \leq \omega_n \int u^2 \, dq \, dt \left( \alpha^{\frac{n-4}{2}} \omega_n \int u^2 \, dq \, dt \right) = C_1 \alpha^{\frac{n-4}{2}} \omega_n \int u^2 \, dq \, dt, \]

where \( \omega_n \) is the volume of the unite \((n-1)\)-sphere and \( C_1 \) is the following constant:

\[ C_1 = \int \rho(x)^2 (2x)^{\frac{n-4}{2}} \, dx. \]

Estimating \( B \) we proceed in a similar manner as for \( A \):

\[ B = \frac{1}{n} \int (\rho_\alpha^2)'(H) (\nabla_q u)^2 \, d\mu = \]
\[ = \frac{1}{n} \int \left( \int (\rho_\alpha^2(H))' r^{n-1} \, dr \, d\omega \right) \left| \nabla_q u \right|^2 \, dq \, dt = \]
\[ = \frac{\omega_n}{n} \int \left( \int (\rho_\alpha^2(H))' r^{n-2} \left( \frac{r^2}{2} + u \right) \right) \left| \nabla_q u \right|^2 \, dq \, dt = \]
\[ = \frac{\omega_n}{n} \int \left( \int (\rho_\alpha^2(H))' (\alpha H - u)^{n-2} \right) \left| \nabla_q u \right|^2 \, dq \, dt. \]

Integrating by parts in the inner integral we have

\[ B = -\frac{\omega_n(n-2)}{n} \int \left( \int (\rho_\alpha^2(H))(2(\alpha H - u)^{n-2}) \right) \left| \nabla_q u \right|^2 \, dq \, dt. \]

Notice that the exponent \( \frac{n-4}{2} \) in (10) can change sign therefore we need to split into three cases:

1) Case \( n = 3 \). In this case from (11) we have

\[ B = -\frac{\omega_3}{3} \int \left( \int (\rho_\alpha^2(H))(2(\alpha H - u)^{1-2}) \right) \left| \nabla_q u \right|^2 \, dq \, dt \leq \]
\[ \leq -\frac{\omega_3}{3} \int (\rho_\alpha^2(H))(2H)^{1-2} \, dH \int |\nabla_q u|^2 \, dq \, dt, \]

since \( u \geq 0 \).

Changing variable \( H \rightarrow \alpha H \) in the integral in brackets we have:

\[ B \leq -\frac{\omega_3}{3} C_2 \alpha^{-\frac{2}{2}} \int |\nabla_q u|^2 \, dq \, dt, \]

where the constant \( C_2 \) equals

\[ C_2 = \int \rho^2(x)(2x)^{-\frac{1}{2}} \, dx. \]
Therefore in this case we have for $D_\alpha$ from (9)(11):

$$D_\alpha = A + B \leq \omega_3 C_1 \alpha^{\frac{1}{2} - \frac{1}{2}} \int u_t^2 dq dt - \frac{\omega_3}{3} C_2 \alpha^{\frac{1}{2} - \frac{1}{2}} \int |\nabla u|^2 dq dt.$$

Since $\int |\nabla u|^2 dq dt > 0$ then the right hand side tends to $-\infty$ as $\alpha$ tends to zero. This proves the theorem for the first case.

2) Case $n = 4$. In this case we compute from (10) (12)

$$B = \frac{\omega_4}{2} \left( \int (\rho_\alpha^2(H)) dH \right) \int |\nabla u|^2 dq dt = \frac{\omega_4}{2} C_2 \alpha^{-1} \int |\nabla u|^2 dq dt.$$

where $C_2 = \int \rho^2(x) dx$. So in this case we have for $D_\alpha$ from (9)(12):

$$D_\alpha \leq C_1 \omega_4 \int u_t^2 dq dt - \frac{\omega_4}{2} C_2 \alpha^{-1} \int |\nabla u|^2 dq dt.$$

In this case again the right hand side tends to $-\infty$ as $\alpha \to 0$.

3) Case $n \geq 5$. In this case since $u \leq M$ we have:

$$B = -\frac{\omega_n(n-2)}{n} \left( \int (\rho_\alpha^2(H))(2(\alpha H - u))^{\frac{n-4}{2}} dH \right) \int |\nabla u|^2 dq dt \\ \leq -\frac{\omega_n(n-2)}{n} \left( \int (\rho_\alpha^2(H))(2(\alpha H - M))^{\frac{n-4}{2}} dH \right) \int |\nabla u|^2 dq dt.$$

Changing the variable in the integral in brackets $H \to \alpha H$ we get:

$$B \leq -\frac{\omega_n(n-2)}{n} \left( \alpha^{\frac{2-n}{2}} \int (\rho_\alpha^2(\alpha H))(2(\alpha H - \alpha M))^{\frac{n-4}{2}} dH \right) \int |\nabla u|^2 dq dt \leq \\ -\frac{\omega_n(n-2)}{n} \left( \alpha^{\frac{2-n}{2}} \int (\rho_\alpha^2(x))(2(x - M))^{\frac{n-4}{2}} dx \right) \int |\nabla u|^2 dq dt,$$

where we used $\alpha M < M$.

Therefore we have

$$B \leq -\frac{\omega_n(n-2)}{n} C_2 \alpha^{\frac{2-n}{2}} \int |\nabla u|^2 dq dt,$$

where

$$C_2 = \int (\rho^2(x))(2(x - M))^{\frac{n-4}{2}} dx.$$

Thus we have from (9)(13) the estimate for $D_\alpha$:

$$D_\alpha = A + B \leq \omega_n C_1 \alpha^{\frac{1}{2} - \frac{n}{2}} \int u_t^2 dq dt - \frac{\omega_n(n-2)}{n} C_2 \alpha^{\frac{2-n}{2}} \int |\nabla u|^2 dq dt = \\
\omega_n \alpha^{\frac{2-n}{2}} \left( C_1 \alpha \int u_t^2 dq dt - \frac{(n-2)}{n} C_2 \int |\nabla u|^2 dq dt \right).$$

Thus also in this case the limit of the right hand side is $-\infty$ when $\alpha \to 0$. This completes the proof in all the cases.

**Remark.** Notice that the case $n = 2$ is excluded in this method, because for $n = 2$ by (11) gives $B = 0$ and the argument breaks down, i.e $D_\alpha$ is indeed non-negative. I don’t know if this is the artifact of the method or the statement of the main theorem fails in this case.
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