AdS spacetimes from wrapped D3-branes

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1. Introduction

A supersymmetric solution of $D = 10$ or $D = 11$ supergravity theory with an anti-de Sitter (AdS) factor is expected to be dual to a supersymmetric conformal field theory (SCFT). It is interesting to elucidate and study the geometrical structures underpinning such solutions for several reasons. For example, the results provide a good starting point for constructing explicit solutions. More generally, a precise global characterization of the relevant geometry is the first step in attempting to obtain existence theorems. Another application is to find geometrical analogues of general properties of classes of SCFTs as in [1, 2]. Finally, possibly after analytic continuation, the AdS solutions can give rise to classes of ‘bubble’ solutions corresponding to certain chiral primaries in SCFTs [3] or to solutions that describe supersymmetric defects,
including Wilson lines, [4–10], all of which are interesting objects to study in the AdS/CFT correspondence.

It is by now well established that $G$-structure techniques [11, 12] are very useful in determining necessary and sufficient conditions for a class of geometries to give supersymmetric solutions. A key observation is that the isotropy group of the Killing spinor(s) defines a canonical $G$ structure, which can, for example, be characterized by certain bi-linears built from the spinors. The Killing spinor equations then impose restrictions on the intrinsic torsion of the $G$ structure and/or relate it to the flux.

To classify supersymmetric $AdS_{d+1}$ solutions, one can start by assuming that the metric is a general warped product of $AdS_d$ space, with its maximally symmetric metric, and a Riemannian manifold $N$:

$$ds^2 = \lambda^{-1} ds^2(AdS_d) + ds^2(N)$$

where the warp factor $\lambda$ depends on the coordinates of $N$. This ansatz is clearly invariant under the $SO(d, 2)$ isometries of the $AdS$ space. One also considers the most general $SO(d, 2)$ invariant ansatz for the matter fields (‘fluxes’) and then analyzes the $G$-structures as just described. Some care is required in order to obtain a precise global statement about the geometry: proceeding naively, say with bi-linears built from the Killing spinors, one might be working with $G$ structures that are only locally defined.

An alternative strategy is to write the $AdS_{d+1}$ metric in Poincaré co-ordinates and to consider the solution, locally, as a special case of a supersymmetric solution with a $d$-dimensional Minkowski space factor. Thus if one has an understanding of the geometry of spacetimes with Minkowski factors in terms of $G$ structures, then one can extract out the geometry underlying the solutions with $AdS$ factors [16]. It turns out that it is in fact not necessary to consider the most general Minkowski solutions: it was first shown in [16] and then subsequently in [17] that in many cases one can consider classes of solutions with Minkowski factors that were called ‘wrapped-brane’ solutions. The name arises because this class of Minkowski solutions, by definition, preserve Killing spinors that satisfy the same projections as for those of a probe brane wrapping a calibrated cycle in a special holonomy manifold, or, equivalently, to a configuration of intersecting branes. From a physical point of view, this is in accord with our expectation that the SCFTs dual to the $AdS$ solutions should live on such wrapped or intersecting branes.

Thus a strategy to classify $AdS$ solutions is to first classify wrapped-brane solutions, which is an interesting result in itself, and then extract out the necessary and sufficient conditions for there to be an $AdS$ solution. A nice feature of this approach is that it provides a neat global description of the relevant geometry arising in the $AdS$ solution in terms of the $G$ structure of the wrapped-brane solution [17, 18]. Typically, if we consider branes wrapping calibrated cycles in manifolds with special holonomy $G$, the associated wrapped-brane solution will have a globally defined $G$-structure but with non-trivial intrinsic torsion.

The approach has now been used to classify $AdS$ solutions associated with wrapped $M5$ branes [17, 18] and wrapped $M2$ branes [19]. In this paper we will focus on type IIB $AdS$ solutions associated with wrapped $D3$ branes. In particular, we will classify solutions that are a warped product of a Minkowski spacetime with an internal space that are associated with probe $D3$-brane wrapping associative three cycles, special Lagrangian (SLAG) three cycles,
Table 1. Wrapped $D_3$-brane geometries and their supersymmetry.

| wrapped brane | manifold | world-volume | susy  | $R$-symmetry |
|---------------|----------|--------------|-------|--------------|
| Associative   | $G_2$    | $\mathbb{R}$ | $\mathcal{N} = 2$ | $U(1)$       |
| SLAG three cycle | $CY_3$  | $\mathbb{R}$ | $\mathcal{N} = 4$ | $SU(2)$      |
| Kähler two cycle | $CY_4$ | $\mathbb{R}^{1,1}$ | $\mathcal{N} = (0, 2)$ | $U(1)$       |
| Kähler two cycle | $CY_3$ | $\mathbb{R}^{1,1}$ | $\mathcal{N} = (2, 2)$ | $U(1) \times U(1)$ |
| Kähler two cycle | $CY_2$ | $\mathbb{R}^{1,1}$ | $\mathcal{N} = (4, 4)$ | $SO(4) \times U(1)$ |

and holomorphic two cycles in Calabi–Yau ($CY$) two, three and four folds. Furthermore, we assume that the only non-trivial flux is the self-dual five-form. We mentioned above that it is known that in many cases the procedure that we will adopt leads to the most general classes of $AdS$ solutions of the type under consideration and we expect that to be the case here also. Our analysis will include some cases that have already been studied before which will provide some confirmation that this expectation is correct. It is certainly the case that in all cases the conditions that we derive are sufficient for having a supersymmetric $AdS$ solution. Rather than insisting at all points on complete generality, our goal is to define a tractable framework to explore the interplay of anti-de Sitter and wrapped-brane geometry.

The cases that we shall consider in this paper are summarized in table 1. We have listed the different types of calibrated cycles a probe $D_3$ brane can wrap inside the given special holonomy manifold. We have also listed the unwrapped world-volume of the $D_3$-brane, with $\mathbb{R}^{1,1}$ representing two-dimensional Minkowski spacetime and $\mathbb{R}$ referring to a time direction, along with the amount of supersymmetry on this space. The final column indicates the $R$-symmetry that arises in the corresponding CFTs; this manifests itself as isometries in the classes of $AdS$ solutions that we shall derive.

In section 2 of this paper we will present and discuss the wrapped-brane geometries. The wrapped-brane geometries corresponding to $D_3$-brane wrapping associative three cycles and Kähler two cycles in $CY_4$ can be obtained as special cases of a more general classification of type IIB geometries with five-form flux that was carried out recently using $G$-structure techniques in [20]. We will obtain the wrapped-brane geometries for all other cases by exploiting the fact that the geometries must admit multiple copies of these basic $G$-structures along with another assumption which we will explain in section 2. This exactly parallels what was done in [17] where it was shown in that context that this indeed does give the most general wrapped-brane geometries. Since we strongly suspect that this is also true here, in the sequel we will refer to these geometries as wrapped-brane geometries.

In section 3, starting from the wrapped-brane geometries in section 2, we determine the extra conditions that need to be imposed in order to obtain $AdS$ spacetimes. The $AdS_2$ geometries for the associative and SLAG three cycles that we derive are new. The $AdS_3$ geometries (or the corresponding bubble solutions that we discuss in a moment) for the cases associated with $D_3$-brane wrapping holomorphic two cycles in $CY_2$, $CY_3$ and $CY_4$ have all been considered before but the derivation from wrapped-brane geometries is new. It is satisfying that we find results in agreement with [3, 23].

It is sometimes possible to analytically continue $AdS$ solutions to obtain other classes of BPS solutions. For example the classes of $AdS_3$ solutions arising from $D_3$-brane wrapping holomorphic two cycles in $CY_2$, $CY_3$ and $CY_4$ give rise to known BPS ‘bubble’ solutions with $\mathbb{R} \times SO(4) \times SO(4)$, $\mathbb{R} \times SO(4) \times U(1)$ and $\mathbb{R} \times SO(4)$ symmetry, that have 1/2, 1/4 and 1/8 supersymmetry, respectively [3, 21–23] (for a unified discussion, differing from that given in
Another class of BPS bubble geometries in $D = 11$ supergravity can be found in [25] and further examples can be easily obtained from the results of [17, 19] (see [26] for an alternative derivation of some of the cases considered in [17, 19]). Here we will see that an analytic continuation of the $AdS_2$ geometries associated with $D3$-brane wrapped on associative three cycles leads to an interesting new class of $1/8$ BPS solutions with $\mathbb{R} \times SU(2)$ symmetry.

In section 4, we show that two explicit $AdS$ solutions that were first constructed using gauged supergravity [27–29], do indeed satisfy our conditions. This provides a very good check on our calculations, the details of which we mostly omit. Using these results also allows us to construct an ansatz for the wrapped-brane geometries that could describe solutions that interpolate from a special holonomy manifold to the explicit $AdS$ solutions (this should be contrasted with the interpolating solutions that correspond to a ‘flow across dimensions’ [27]). In particular, we show that the ansatz includes singular special holonomy manifolds with a calibrated cycle that provide a local model for probe $D3$-brane wrapping the calibrated cycle.

In section 5 we briefly conclude.

### 2. Wrapped $D3$-brane geometries

In this section, we will discuss the wrapped-brane geometries associated with $D3$-brane wrapping calibrated cycles in manifolds with the special holonomy $G$. By definition, these geometries are warped products of $d$-dimensional Minkowski spacetime with a $10 - d$ dimensional Riemannian manifold $M_{10-d}$

$$ds^2 = L^{-1} ds^2(\mathbb{R}^{1,d-1}) + ds^2(M_{10-d})$$

(2.1)

where the Minkowski spacetime should be viewed as the unwrapped part of the $D3$-brane. Thus for $D3$-brane wrapping two cycles we have $d = 2$ and for $D3$-brane wrapping three cycles we have $d = 1$. Both the warp factor $L$ and the metric on $M_{10-d}$ are independent of the coordinates of the Minkowski factor. Only the self-dual five-form flux is non-zero and it is also taken to be invariant under the symmetries of the Minkowski factor. We will write

$$F_5 = \Theta + *_{10} \Theta.$$  

(2.2)

In all cases, $M_{10-d}$ will admit a globally defined $G$-structure with non-trivial torsion, related to the five-form flux, and will preserve 1/2 as much supersymmetry as the type IIB solution with special holonomy $G$ and vanishing five-form flux. We also expect that all of the classes of solutions that we consider will admit Killing spinors that satisfy the same projections\(^5\) as those of a probe $D3$-brane wrapping a special holonomy manifold.

We first discuss the case associated with $D3$-brane wrapping associative three cycles in manifolds with $G_2$ holonomy. The relevant wrapped-brane geometry can simply be obtained by making a restriction on the more general classification that appeared in [20]. From this case we then derive the wrapped-brane geometry associated with $D3$-brane wrapping SLAG three cycles, making clear what we assume in the derivation. Similarly, the case associated with $D3$-brane wrapping Kähler cycles two cycles in $CY_4$ can be obtained from the more general classification that appeared in [20] and we then derive the cases associated with $D3$-branes wrapping Kähler two cycles in $CY_3$ and $CY_2$. A nice consistency check is that the latter wrapped-brane geometry can also be derived from the case associated with $D3$-brane wrapping SLAG three cycles.

\(^5\) For a more precise discussion of this point, we refer to section 2 of [17].
2.1. Associative geometry

We first define this geometry, and then discuss some of its features. For this case the metric and five-form flux can be written
\[ ds^2 = -L^{-1} dt^2 + ds^2(M_7) + L ds^2(\mathbb{R}^2) \]
\[ \Theta = d(e^0 \wedge \varphi), \]
(2.3)
where \( \partial_t \) is Killing and \( e^0 = L^{-1/2} dt \). There is a globally defined (no-where vanishing) \( G_2 \) structure that is specified by an associative three-form \( \varphi \) and co-associative four-form \( \ast_7 \varphi \) defined on \( M_7 \) and compatible with the metric \( ds^2(M_7) \). Furthermore, both \( \varphi \) and the warp factor \( L \) can depend on the coordinates of \( M_7 \) and \( \mathbb{R}^2 \). We require that the intrinsic torsion of the \( G_2 \) structure is determined as follows:
\[ d(e^0 \wedge \text{Vol}_7) = 0, \quad \text{Vol}[\mathbb{R}^2] \wedge d \ast_7 \varphi = 0, \quad \varphi \wedge d \varphi = 0. \]
(2.4)
Finally we require that the Bianchi identity is satisfied,
\[ d \ast_{10} \Theta = 0. \]
(2.5)

With a hopefully obvious choice of frame, we take the ten-dimensional orientation to be positive with respect to
\[ \text{Vol}_{10} = e^0 \wedge \text{Vol}[M_7] \wedge e^8 \wedge e^9, \quad \text{Vol}[M_7] = \frac{1}{4} \varphi \wedge \ast_7 \varphi. \]
(2.6)

Having defined associative geometry we now begin to discuss its features. Every solution of these equations will, by definition, admit two Killing spinors, which satisfy the same orthonormal-frame projections as those of a probe \( D3 \)-brane wrapping an associative three-cycle in a \( G_2 \) manifold. This is because we have obtained the torsion conditions as a special case of those of the more general class of geometries called \( G_2 \) backgrounds in [20]. By the construction of [20], the associative geometries then admit two Killing spinors, satisfying the appropriate algebraic constraints. In order to get the wrapped-brane geometry of interest here, we set \( m = Y_\pm = 0 \) in section 6.1 of [20]. With some work one can recast this restriction of the conditions of [20] in the more transparent way given above\(^6\). For these geometries, supersymmetry plus the Bianchi identity implies all the equations of motion of IIB supergravity are satisfied: this can be shown by studying the integrability conditions for type IIB [30] (see also [31]) and generalizing an argument presented in [12].

The wrapped-brane geometry should be able to describe back-reacted \( D3 \)-brane wrapping associative three cycles (or alternatively an appropriate configuration of intersecting \( D3 \)-branes) and it has several intuitive features. As we have noted before, the Killing time direction in (2.3) corresponds to the unwrapped part of the \( D3 \)-brane, while the two ‘overall transverse’ directions, i.e. transverse to the probe \( D3 \)-brane world-volume and to the \( G_2 \) holonomy manifold, are visible as the \( \mathbb{R}^2 \) factor in (2.3). Recall that the \( \mathbb{R}^{1,2} \times X_7 \) solution of type IIB supergravity where \( X_7 \) has \( G_2 \) holonomy preserves four supersymmetries and obviously has a globally defined \( G_2 \). In the associative wrapped-brane geometry there is still a globally defined \( G_2 \) structure, but it now has non-trivial torsion which leads to the preservation of two supersymmetries.

As somewhat of an aside let us discuss how some of the conditions on the geometry can be understood in terms of generalized calibrations [32] (for further discussion and references, see section 4 of [17]). In particular, the expression for the flux in (2.3) reveals that \( \varphi \) is a generalized calibration: it arises because the geometries can describe the back-reacted geometry of \( D3 \)-brane wrapping associative three cycles. Some of the other conditions in (2.4) have a similar interpretation. The first condition in (2.4) states that in associative geometry, we can also wrap

\(^6\) Up to an irrelevant factor of \(-1/4\) in the definition of the five-form.
a probe $D7$-brane over the entire $G_2$ structure manifold while preserving supersymmetry. The second condition in (2.4) seems to be related to probe $D5$-branes that are calibrated by $*\gamma\psi$ and one of the overall transverse directions. However, this condition is not equivalent to
\[
d(e^0 \wedge *\gamma\psi \wedge e^s) = 0, \\
d(e^0 \wedge *\gamma\psi \wedge e^o) = 0
\]
as one might naively expect from such an interpretation ((2.4) allows $*\gamma\psi$ to have non-trivial dependence on the coordinates of $\mathbb{R}^2$, whereas (2.7) does not). Furthermore the last condition in (2.4) does not have any obvious interpretation in terms of generalized calibrations. This example shows that one needs to use the intuition obtained from generalized calibrations with care.

2.2. SLAG-3 geometry

In this subsection, we define SLAG-3 geometry. The metric and the flux are given by
\[
d\tilde{s}^2 = -L^{-1} dt^2 + ds^2(\mathcal{M}_6) + L \, ds^2(\mathbb{R}^3), \\
\Theta = -d(e^3 \wedge \text{Im} \, \Omega),
\]
where $\partial_t$ is Killing and $e^0 = L^{-1/2} dt$. We require the existence of a globally defined $SU(3)$ structure, given by an everywhere non-vanishing (1,1) form $J_6$ and an everywhere non-vanishing (3,0) form $\Omega_6$, defined on $\mathcal{M}_6$ and compatible with $ds^2(\mathcal{M}_6)$. The $SU(3)$ structure and the warp factor $L$ can depend on the coordinates of both $\mathcal{M}_6$ and $\mathbb{R}^3$. The $SU(3)$ structure satisfies the following conditions on the intrinsic torsion:
\[
dJ_6 = 0, \\
\text{Vol}[\mathbb{R}^3] \wedge d(e^3 \wedge \text{Re} \, \Omega_6) = 0, \\
\text{Im} \, \Omega_6 \wedge d \text{Im} \, \Omega_6 = 0.
\]
We also demand that $d*_{10} \Theta = 0$.

These geometries admit four Killing spinors (half-maximal for an $SU(3)$ structure), satisfying the appropriate algebraic constraints. This is because we have derived the above class of geometries by requiring two independent associative structures, each of which implies the existence of two Killing spinors. If we write
\[
\psi = \pm J_6 \wedge e^7 - \text{Im} \, \Omega_6, \\
*\gamma\psi = \frac{1}{2} J_6 \wedge J_6 \pm \text{Re} \, \Omega_6 \wedge e^7,
\]
and set $e^7 = L^{1/2} dy$, after substituting into (2.3) and (2.4), we get torsion conditions given in the definition. The requirement that we have two independent associative structures is exactly what is wanted for the geometries to preserve Killing spinors satisfying the same algebraic constraints as probe $D3$-brane wrapping SLAG three cycles in the sense of [17]. Note that our assumption that $e^7 = L^{1/2} dy$ implies that there is an $\mathbb{R}^3$ factor in (2.8) corresponding to the three overall transverse directions of a probe $D3$-brane wrapping a SLAG three-cycle. Based on the results of [17], we strongly suspect that this assumption is actually implied by demanding that the class of geometries admit Killing spinors satisfying the same projections as wrapped probe branes. In this paper, we will be content to just assume this condition, and hence it effectively becomes part of our definition of a SLAG-3 geometry. Similar comments will apply to other geometries that we discuss below.

As in the associative case some of these conditions can be interpreted, if somewhat imprecisely, in terms of generalized calibrations. The expression for the flux says that $-\text{Im} \, \Omega_6$ is a generalized calibration corresponding to the fact that the geometries can describe the back-reacted geometry of $D3$-brane wrapping SLAG three cycles. The first condition in (2.9), which can be written $d(e^0 \wedge J_6 \wedge e^s) = 0$, where $e^s$ is a frame direction in any of the three overall transverse directions, corresponds to probe $D3$-brane wrapping a cycle calibrated by $J_6$ and any one of the overall transverse directions. The second condition in (2.9) corresponds to probe $D5$-branes that are calibrated by $\text{Re} \, \Omega$ and any two of the three overall transverse directions, but as in the associative case, only imprecisely. The last condition in (2.9) does not have any obvious generalized calibration interpretation.
2.3. Kähler-2 in CY4 geometry

The metric and flux for this geometry take the form:
\[ ds^2 = L^{-1}ds^2(\mathbb{R}^{1,1}) + ds^2(\mathcal{M}_8), \quad \Theta = \text{Vol}[\mathbb{R}^{1,1}] \wedge d(L^{-1}J_6). \]  
(2.11)

We now require the existence of a globally defined $SU(4)$ structure, specified by everywhere-non-zero forms $J_6, \Omega_8$ on $\mathcal{M}_8$ and compatible with the metric $ds^2(\mathcal{M}_8)$. The intrinsic torsion conditions can be written
\[ d(L^{-1}J_6 \wedge J_6 \wedge J_6) = 0, \quad d(L^{-1}\Omega_8) = 0, \]  
(2.12)
and we also demand that $d \ast_{10}\Theta = 0$. Positive orientation is defined with respect to
\[ \text{Vol}_{10} = e^0 \wedge e^1 \wedge \text{Vol}[\mathcal{M}_8], \quad \text{Vol}[\mathcal{M}_8] = \frac{1}{8!} J_6 J_6 \wedge J_6 \wedge J_6. \]  
(2.13)

We have obtained the torsion conditions from the $SU(4) \ltimes \mathbb{R}^8$ case of [20]; they were also derived in [33]. It follows from the construction of [20] that these geometries admit two Killing spinors, half-maximal for an $SU(4)$ structure. In an orthonormal frame these satisfy the same algebraic projections as for probe $D3$-brane wrapping a Kähler two-cycle in a CY4. The Killing spinors are pure. Note that for this case there are no overall transverse directions.

The expression for the flux says that $J_6$ is a generalized calibration corresponding to the fact that the geometries can describe back-reacted $D3$-brane wrapping holomorphic two cycles. The first condition in (2.12) corresponds to probe $D7$ branes that wrap a six-cycle calibrated by $J_6^2/3!$, while the second condition corresponds to probe $D5$-brane wrapping a four-cycle calibrated by the real or imaginary part of $\Omega_8$.

2.4. Kähler-2 in CY3 geometry

We next consider the wrapped-brane geometries associated with $D3$-brane wrapping holomorphic two cycles in a Calabi–Yau three fold. The metric and the flux are given by
\[ ds^2 = L^{-1}ds^2(\mathbb{R}^{1,1}) + ds^2(\mathcal{M}_6) + L \, ds^2(\mathbb{R}^{2}), \quad \Theta = \text{Vol}[\mathbb{R}^{1,1}] \wedge d(L^{-1}J_6). \]  
(2.14)
We demand the existence of a globally defined $SU(3)$ structure, with everywhere non-vanishing structure forms $J_6, \Omega_8$ defined on $\mathcal{M}_6$ and compatible with $ds^2(\mathcal{M}_6)$. The warp factor $L$ and $J_6, \Omega_6$ depend on the coordinates of both $\mathcal{M}_6$ and $\mathbb{R}^2$. We require the following torsion conditions
\[ \text{Vol}[\mathbb{R}^{2}] \wedge d(J_6 \wedge J_6) = 0, \quad d(L^{-1/2}\Omega_6) = 0, \]  
(2.15)
and also demand that $d \ast_{10}\Theta = 0$.

The torsion conditions may be derived by assuming a pair of $SU(4)$ structures satisfying the Kähler-2 in CY4 condition of the last subsection. To see this, write
\[ J_6 = J_6 \pm e^7 \wedge e^8, \quad \Omega_8 = \Omega_8 \wedge (e^7 \pm ie^8), \]  
(2.16)
together with $e^7 = L^{1/2}dx^7, e^8 = L^{1/2}dx^8$. Substituting both these structures into (2.11) and (2.12) produces (2.14) and (2.15).

The expression for the flux says that $J_6$ is a generalized calibration corresponding to the fact that the geometries can describe back-reacted $D3$-brane wrapping holomorphic two cycles. The first condition in (2.15) corresponds to probe $D7$-branes that wrap a four-cycle calibrated by $J_6^2/2$ and two of the overall transverse directions (the $\mathbb{R}^2$ factor). The second

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7 A potentially confusing point is that the isotropy group of the Killing spinors in the geometry of [20] is $SU(4) \ltimes \mathbb{R}^8$. However, the assumption that we have a warped product of $\mathbb{R}^{1,1}$ with an eight-dimensional manifold reduces this to an $SU(4)$ structure.
condition corresponds to probe $D5$-brane wrapping a three-cycle calibrated by the real or imaginary part of $\Omega_4$ and one of the two overall transverse directions. To see this, we note that we can rewrite the condition as $d(e^0 \wedge e^1 \wedge \Omega_6 \wedge e^\prime) = 0$, for arbitrary $e^\prime$ on the transverse space. Furthermore, we note that the last condition implies that $d(L^{-1}\text{Vol}([M_6])) = 0$ which corresponds to probe $D7$-brane wrapping the whole $CY_3$.

2.5. Kähler-2 in $CY_2$ geometry

Finally, we consider the wrapped-brane geometry corresponding to $D3$-brane wrapping holomorphic two cycles in a Calabi–Yau two fold. The metric and flux are given by

$$dx^2 = L^{-1} dx^2(\mathbb{R}^{1,1}) + ds^2(M_4) + L \, dx^2(\mathbb{R}^4),$$

$$\Theta = \text{Vol}([\mathbb{R}^{1,1}] \wedge d(L^{-1} J_4)).$$

There is a globally defined $SU(2)$ structure, with nowhere-vanishing structure forms $J_4, \Omega_4$ defined on $M_4$ and compatible with the metric $ds^2(M_4)$. The warp factor $L$ and $J_4, \Omega_4$ depend on the coordinates of both $M_4$ and $\mathbb{R}^4$. We require the torsion conditions

$$\text{Vol}([\mathbb{R}^4] \wedge d(L J_4)) = 0, \quad d\Omega_4 = 0,$$

and that $d_{10} \Theta = 0$.

These conditions imply the existence of eight Killing spinors, half-maximal supersymmetry for an $SU(2)$ structure. To see this, observe that they imply the existence of two $SU(3)$ structures satisfying the constraints of the previous subsection. The $SU(3)$ structures are

$$J_6 = J_4 \pm e^5 \wedge e^6, \quad \Omega_6 = \Omega_4 \wedge (e^5 \pm ie^6),$$

with $e^5 = L^{1/2} dx^5$ and $e^6 = L^{1/2} dx^6$. Requiring that both these structures satisfy the torsion conditions of the previous subsection, we get (2.17) and (2.18).

The expression for the flux says that $J_4$ is a generalized calibration corresponding to the fact that the geometries describe back-reacted $D3$-brane wrapping holomorphic two cycles. The first condition in (2.18) corresponds to probe $D7$-branes that wrap a two-cycle calibrated by $J_4$ and four overall transverse directions. The second condition corresponds to probe $D5$-branes wrapping a two-cycle calibrated by the real or imaginary part of $\Omega_4$ and two of the four overall transverse directions: to see this we note that the condition can be equivalently $d(e^0 \wedge e^1 \wedge \Omega_4 \wedge e^\prime \wedge e^\prime) = 0$ for arbitrary $e^\prime, e^\prime$ on the overall transverse space.

We can also obtain these conditions starting from the SLAG-3 case, which provides a nice consistency check. In particular, if we decompose the $SU(3)$ structure $J_6, \Omega_6$ as

$$J_6 = J^{(3)} \pm e^5 \wedge e^6, \quad \Omega_6 = (J^{(2)} + i J^{(1)}) \wedge (e^5 \pm i e^6),$$

set $e^5 = L^{1/2} dx^5, e^6 = L^{1/2} dx^6$, and substitute into the SLAG-3 geometry conditions (2.8) and (2.9) we recover (2.17) and (2.18), provided that we identify $J^{(3)}$ and $J^{(3)} + i J^{(2)}$ with $J_4$ and $\Omega_4$, respectively.

Finally, we point out that we strongly suspect that (2.17) and (2.18) do not constitute the most general wrapped-brane spacetimes in this class. In particular, by analogy with a similar case that was studied in [19], we expect that the derivation that we have used misses the possibility that the overall transverse space, $\mathbb{R}^4$ in (2.17) and (2.18) can be replaced with an arbitrary $CY_2$ metric. It would be interesting to verify this. On the other hand, we suspect that in order to obtain an $AdS_5$ limit, as we do in the next section, it is necessary to take the $CY_2$ metric to be flat $\mathbb{R}^4$; again, this would be interesting to verify explicitly.
3. AdS and bubble geometries

In the previous section we have defined classes of wrapped-brane solutions of type IIB supergravity. In all cases the metric has the form
\[ ds^2 = L^{-1} ds^2(\mathbb{R}^{1,d-1}) + ds^2(M_{10-d}). \] (3.1)
For all cases, except the Kähler-2 in the CY4 case, there are at least two overall transverse directions and we can write
\[ ds^2(M_{10-d}) = ds^2(M_G) + L [dz^2 + z^2 ds^2(S^q)] \] (3.2)
where \( ds^2(S^q) \) is the round metric on a \( q \) sphere and the cases \( q = 1, 2 \) and 3 appear.

In this section, we determine the extra conditions that need to be placed on these geometries in order to extract an AdS solution of the form
\[ ds^2 = \lambda^{-1} ds^2(AdS_d+1) + ds^2(N_{9-d}) \] (3.3)
where in the second line we have written the unit radius AdS space in Poincaré coordinates.

We require that \( \partial r \) is a Killing vector for \( ds^2(N_{9-d}) \). Clearly to obtain this metric from any of the wrapped-brane geometries we must insist that the warp factor takes the form
\[ L = e^{2r\lambda}. \] (3.4)
For the Kähler-2 in the CY4 case, we must demand that \( ds^2(M_8) \) is a cone in order to extract out the AdS radial direction, as we shall explain later (this case is very analogous to the case of Sasaki–Einstein manifolds). For the other cases, following [17], the unit radial one-form can be written
\[ \lambda^{-1/2} dr = \sin \theta \hat{u} + \cos \theta \hat{v}, \] (3.5)
where \( \hat{u} \) is a unit one-form in \( M_G \) and \( \hat{v} \) is a unit one-form in the overall transverse space. We will make the assumption that \( \hat{v} \) is given by
\[ \hat{v} = L^{1/2} dz, \] (3.6)
and so lies along the radial direction \( dz \) of the conformally flat overall transverse space. In addition, we will assume that the rotation angle \( \theta \) must be independent of the AdS radial coordinate. It seems likely that these assumptions can be relaxed (see [17]) but we shall not investigate this issue further here.

We next introduce the orthogonal combination
\[ \hat{\rho} = \cos \theta \hat{u} - \sin \theta \hat{v}. \] (3.7)
It is also convenient to introduce a new coordinate \( \rho \) via
\[ \cos \theta = \lambda \rho, \quad \sin \theta = \sqrt{1 - \lambda^2 \rho^2}. \] (3.8)
Then using the fact that \( dz \) is closed, and \( \theta \) is independent of \( r \), we find
\[ \hat{\rho} = \frac{\lambda^{1/2}}{\sqrt{1 - \lambda^2 \rho^2}} d\rho, \quad \hat{u} = \lambda^{-1/2} \sqrt{1 - \lambda^2 \rho^2} dr + \frac{\lambda^{3/2} \rho}{\sqrt{1 - \lambda^2 \rho^2}} d\rho. \] (3.9)
In addition, we also have
\[ z = -e^{-\tau} \rho. \] (3.10)

We now write
\[ ds^2(M_{10-d}) = ds^2(M_G) + L [dz^2 + z^2 ds^2(S^q)] \] (3.11)
\[ = ds^2(M_G') + (\hat{u})^2 + (\hat{v})^2 + L z^2 ds^2(S^q) \]
where the $G'$ structure on $\mathcal{M}_G$ is a reduction of the $G$-structure on $\mathcal{M}_G$ defined by picking out the particular one form $\tilde{u}$. Given the above formulae, we thus conclude that

$$d\bar{x}^2(N_{0-d}) = d\bar{x}^2(\mathcal{M}_G) + (\bar{\rho})^2 + \lambda \rho^2 d\bar{x}^2(S^6).$$  \hfill (3.12)

Given the supersymmetry conditions on the original space $\mathcal{M}_{10-d}$ it is then straightforward to take (3.3) with $d\bar{x}^2(N_{0-d})$ given by (3.12), demand that the flux has no components along the $AdS$ radial direction, and hence derive the supersymmetry conditions for an $AdS_5$ geometry in terms of the $G'$ structure. We shall present the results of these calculations, which can be technically involved, in the following sub-sections. It is worth emphasizing that, unlike the $G$ structure, this $G'$ structure is, in general, only locally defined, since there can be points where $\sin \theta = 0$ and hence the vector $\hat{u}$ is ill defined.

The discussion thus far has been for the generic case where $dr$ lies partly in $\mathcal{M}_G$ and partly in the overall transverse space. It is not hard to see that it is inconsistent for $dr$ to lie entirely in $\mathcal{M}_G$. One can also consider the possibility that $dr$ lies entirely in the overall transverse space. For cases where the torsion conditions imply a constraint of the form

$$d(L^mVol[\mathcal{M}_G]) = 0,$$  \hfill (3.13)

for some $m \neq 0$, it is also inconsistent. This leaves this possibility open for two classes, SLAG limits, as described above, for each wrapped-brane geometry.

### 3.1. $AdS_2$ from associative

Writing $d\bar{x}^2(\mathcal{M}_T) = d\bar{x}^2(\mathcal{M}_6) + (\bar{u})^2$, after the frame rotation we find that the metric and flux are given by

$$d\bar{x}^2 = \frac{1}{\lambda} \left[ d\bar{x}^2(AdS_2) + \frac{\lambda^2}{1 - \lambda^2 \rho^2} \, d\rho^2 + \lambda^2 \rho^2 d\bar{x}^2(S^1) \right] + d\bar{x}^2(\mathcal{M}_6),$$  \hfill (3.14)

$$\Theta = Vol[AdS_2] \wedge [-d(\lambda^{-1} \sqrt{1 - \lambda^2 \rho^2} J_6) + \lambda^{-1/2} Im \Omega_6 - \lambda^{1/2} \rho J_6 \wedge \hat{\rho}].$$

Here $\mathcal{M}_6$ has an $SU(3)$ structure $J_6$, $\Omega_6$. We find that the $S^1$ direction is Killing, leaving both the $SU(3)$ structure and the warp factor $\lambda$ invariant. In addition the $SU(3)$ structure must satisfy the conditions:

$$d(\lambda^{-1/2} Im \Omega_6 - \lambda^{1/2} \rho J_6 \wedge \hat{\rho}) = 0,$$

$$d \left( \frac{1}{2} J_6 \wedge J_6 + \frac{1}{\lambda \rho} Re \Omega_6 \wedge \hat{\rho} \right) = 0.$$  \hfill (3.15)

The result of a long calculation gives

$$*_{10} \Theta = Vol[S^1] \wedge d(\lambda^{-1/2} \sqrt{1 - \lambda^2 \rho^2} Re \Omega_6),$$  \hfill (3.16)

which shows that the Bianchi identity is satisfied.

These geometries are dual to SCQM with two supersymmetries. The $U(1)$ isometry corresponds to the $U(1)$-$R$-symmetry of the dual theory. An example of this geometry was constructed in [28] (see also [29]) and we shall verify this directly in the next section.

It is interesting to note that we can analytically continue these solutions to obtain a new class of $1/8$ BPS solutions with $\mathbb{R} \times SU(2)$ symmetry. In particular, we take $d\bar{x}^2(AdS_2) \rightarrow -d\bar{x}^2(S^2)$, $Vol[AdS_2] \rightarrow i Vol[S^2]$ and $\lambda \rightarrow -\lambda$ to get

$$d\bar{x}^2 = \frac{1}{\lambda} \left[ d\bar{x}^2(S^2) + \frac{\lambda^2}{\lambda^2 \rho^2 - 1} \, d\rho^2 - \lambda^2 \rho^2 dT^2 \right] + d\bar{x}^2(\mathcal{M}_6),$$

$$\Theta = Vol[S^2] \wedge \left[ -d(\lambda^{-1} \sqrt{\lambda^2 \rho^2 - 1} J_6) + \lambda^{-1/2} Im \Omega_6 + \frac{1}{\sqrt{\lambda^2 \rho^2 - 1}} \lambda \rho J_6 \wedge d\rho \right].$$  \hfill (3.17)
where the time coordinate $T$ was originally a coordinate on the $S^1$. The torsion conditions are now
\[
    d \left( \lambda^{-1/2} \Im \Omega_6 + \frac{\lambda \rho}{\sqrt{\lambda^2 \rho^2 - 1}} J_6 \wedge d\rho \right) = 0,
\]
\[
    d \left( \frac{1}{2} J_6 \wedge J_6 + \frac{1}{\lambda^{1/2} \rho \sqrt{\lambda^2 \rho^2 - 1}} \Re \Omega_6 \wedge d\rho \right) = 0.
\] (3.18)

It would be interesting to study this class of solutions further.

3.2. $AdS_2$ from SLAG-3

$AdS$ radial direction from the overall transverse space. For this case, it is possible for the radial direction to come from the overall transverse space. One finds that $\lambda$ must be a constant, which we take to be 1, and that the solution is simply the well known $AdS$ radial direction from the overall transverse space.

\[
    d\Theta = \text{Vol}[AdS_2] \wedge \Im \Omega_6.
\] (3.19)

$AdS$ radial direction from frame rotation. Alternatively, we can have $d\tau$ point partially in the overall transverse direction and partially in the direction of $M_6$. Writing $d\Theta(M_6) = d\Theta(AdS_2) + (e^5)^2 + (\tilde{u})^2$, after the frame rotation we find that the metric and flux are
\[
    d\Theta = \text{Vol}[AdS_2] \wedge [d(\lambda^{-1/2} \sqrt{1 - \lambda^2 \rho^2} \hat{J}^3) + \lambda^{1/2} \hat{\rho} \hat{J}^3 \wedge \hat{\rho} + \lambda^{-1/2} \hat{J}^2 \wedge e^5].
\] (3.20)

Here $M_4$ has an $SU(2)$ structure $J^i$, $i = 1, 2, 3$, with $J^i J^j = -\delta^{ij} + e^{ijk} J^k$. We find that the $S^2$ directions are Killing, preserve the $SU(2)$ structure and that in addition
\[
    d(\lambda^{-1/2} \sqrt{1 - \lambda^2 \rho^2} e^5) = 0,
\]
\[
    dJ^1 = -\lambda \rho d\log \left( \frac{\lambda^2}{1 - \lambda^2 \rho^2} \right) \wedge e^5 \wedge \hat{\rho},
\]
\[
    d(\lambda^{1/2} \hat{\rho} J^3 \wedge e^5) = d(\lambda^{-1/2} J^2 \wedge \hat{\rho}),
\]
\[
    d(\lambda^{1/2} \rho J^3 \wedge \hat{\rho}) = -d(\lambda^{-1/2} J^2 \wedge e^5).
\] (3.21)

Taking the Hodge dual of the $\Theta$, a long calculation leads to
\[
    *\Theta = -\text{Vol}[S^2] \wedge [d(\rho \sqrt{1 - \lambda^2 \rho^2} J^2) - \lambda^{-1/2} J^2 \wedge \hat{\rho} + \lambda^{1/2} \rho J^3 \wedge e^5] \quad (3.22)
\]
and we see that the Bianchi identity is again implied by the torsion conditions.

These geometries are dual to SCQM with four supersymmetries. The $SU(2)$ isometry of these manifolds is to be identified with the $R$-symmetry of the dual quantum mechanics. We are unaware of any explicit solutions in this class.

At first glance it would seem that we could obtain a new class of $AdS_2$ geometries by making the analytic continuation $\lambda \to -\lambda$, $ds^2(S^2) \leftrightarrow ds^2(AdS_2)$. However, in the new solution if we make a further redefinition $\lambda \to 1/\lambda \rho^2$, along with $J^2 \leftrightarrow J^3$, we find that the solution is exactly the same as that above.
3.3. AdS\textsubscript{3} from Kähler-2 in CY\textsubscript{3}

Writing $\text{d}s^2(\mathcal{M}_4) = \text{d}s^2(\mathcal{M}_4) + (e^5)^2 + (\hat{u})^2$, after the frame rotation we find that the metric and flux are

$$\text{d}s^2 = \frac{1}{\lambda} \left[ \text{d}s^2(\text{AdS}_3) + \frac{\lambda^2}{1 - \lambda^2 \rho^2} \text{d}\rho^2 + \lambda^2 \rho^2 \text{d}^2(S^1) \right] + \text{d}s^2(\mathcal{M}_4) + e^5 \otimes e^5, \tag{3.23}$$

$$\Theta = \text{Vol}[\text{AdS}_3] \wedge \{d(\lambda^{-1/2} \sqrt{1 - \lambda^2 \rho^2} \text{J}_4) - 2\lambda^{-1/2} \rho \text{J}_4 - 2\rho e^5 \wedge \hat{\rho} \}.$$

$\mathcal{M}_4$ has an $SU(2)$ structure $J_4$, $\Omega_4$. We find that the $S^1$ direction preserves the $SU(2)$ structure and is Killing. In addition we must have

$$\text{d}(\lambda^{-1} J_4 + \rho e^5 \wedge \hat{\rho}) = 0,$$

$$\text{d}(\lambda^{-1} \sqrt{1 - \lambda^2 \rho^2} \Omega_4) = i\lambda^{-1/2} \Omega_4 \wedge e^5 - \lambda^{1/2} \rho \Omega_4 \wedge \hat{\rho}. \tag{3.24}$$

and that $d\lambda$ has no component in the $e^5$ direction. These conditions imply that we can introduce a coordinate $\psi$ such that $e^3 = A(d\psi + B)$ with $\partial_\psi$ a Killing vector and $A = \lambda^{1/2} \sqrt{1 - \lambda^2 \rho^2}$. The Killing vector $\partial_\psi$ preserves $J_4$, though $\Omega_4$ has non-zero charge under it. In these coordinates, the Hodge dual of $\Theta$ is

$$*_{10} \Theta = -\text{Vol}[S^1] \wedge \text{d}(\lambda^{-1/2} \sqrt{1 - \lambda^2 \rho^2} J_4 \wedge e^5), \tag{3.25}$$

and we see that the Bianchi identity is again implied by the torsion conditions.

These geometries are dual to two-dimensional SCFTs with $(2, 2)$ supersymmetry. These have a $U(1) \times U(1)$-symmetry which is dual to the two $U(1)$ Killing vectors associated with the $S^1$ and $\partial_\psi$. An example of this geometry can be found in [27] (see also [29]): we will verify that this is indeed a solution of the torsion conditions in the next section.

After analytic continuation we obtain BPS bubbles with $1/4$ supersymmetry and $\mathbb{R} \times SO(4) \times U(1)$ symmetry. These conditions should be equivalent to those of [22], but we have not verified this.

3.4. AdS\textsubscript{3} from Kähler 2 in CY\textsubscript{2}

AdS radial direction from overall transverse space. For this case it is possible for the radial direction to come from the overall transverse space. One finds that $\lambda$ must be a constant, which we take to be 1, and that the solution is the well known $AdS_3 \times S^3 \times CY_2$ solution:

$$\text{d}s^2 = \text{d}s^2(AdS_3) + \text{d}s^2(S^3) + \text{d}s^2(M_4),$$

$$\Theta = \text{Vol}[AdS_3] \wedge J_4. \tag{3.26}$$

AdS radial direction from frame rotation. Alternatively, carrying out the general frame rotation and writing $\text{d}s^2(\mathcal{M}_4) = (e^1)^2 + (e^2)^2 + (e^3)^2 + (\hat{u})^2$ we instead find that the metric and flux are

$$\text{d}s^2 = \frac{1}{\lambda} \left[ \text{d}s^2(\text{AdS}_3) + \lambda^2 \left( \frac{1}{1 - \lambda^2 \rho^2} \text{d}\rho^2 + \rho^2 \text{d}^2(S^1) \right) \right] + e^1 \otimes e^1 + e^2 \otimes e^2 + e^3 \otimes e^3. \tag{3.27}$$

$$\Theta = \text{Vol}[AdS_3] \wedge \{d(\lambda^{-3/2} \sqrt{1 - \lambda^2 \rho^2} e^3) - 2(\lambda^{-1} \rho e^3 \wedge \hat{\rho}) \}.$$

The torsion conditions are that the $S^1$ directions are Killing, $d\lambda$ has no $e^3$ component, together with

$$e^{1,2} = \frac{\lambda^{1/2}}{\sqrt{1 - \lambda^2 \rho^2}} \text{d}x_{1,2}, \quad \text{d} \left( \frac{\lambda^{1/2}}{\sqrt{1 - \lambda^2 \rho^2}} \rho^3 \right) = -\frac{1}{\rho} \star_3 \text{d} \left( \frac{1}{\sqrt{1 - \lambda^2 \rho^2}} \rho^3 \right). \tag{3.28}$$
where $*_{3}$ denotes the Hodge dual on the three-manifold with metric and orientation given by
\[
ds^2 = dx_1^2 + dx_2^2 + d\rho^2, \quad \text{Vol} = dx_1 \wedge dx_2 \wedge d\rho,
\]
along with $d \; *_{10} \Theta = 0$. These conditions imply that we can introduce a coordinate $\psi$ such $e^3 = \lambda^{-1/2} \sqrt{1 - \lambda^2 \rho^2} (d\psi + B)$ with $\partial_\psi$ a Killing vector. These conditions are equivalent to those of LLM [3]. The Bianchi identity is implied by the torsion conditions; the ten-dimensional Hodge dual of $\Theta$ is given by
\[
*_{10} \Theta = -\text{Vol}[S^3] \wedge [d(\lambda^{1/2} \rho^2 \sqrt{1 - \lambda^2 \rho^2} e^3) + 2(\lambda^3 \rho^2 e^{1/2} + \rho e^3 \wedge \hat{\rho})].
\]
These geometries are dual to two-dimensional SCFTs with $(4,4)$ supersymmetry and the $SO(4) \times U(1)$ symmetry of the solution is dual to the $R$ symmetry. We are unaware of any explicit $AdS$ examples in this class. After analytic continuation we recover the $1/2$ BPS LLM bubbling solutions with $\mathbb{R} \times SO(4) \times SO(4) \times U(1)$ symmetry.

3.5. $AdS_3$ from Kähler-2 in $CY_4$

This case is different from the previous cases in that the wrapped-brane spacetime does not have any overall transverse directions. It was first derived from a wrapped-brane geometry in [33]. In the notation of this paper, if we write\(^8\) $L = \lambda e^{2r}$, $e^8 = \lambda^{-1/2}dr$, then equations (2.11) and (2.12) lead to a metric and flux given by
\[
ds^2 = \frac{1}{\lambda} ds^2(AdS_3) + ds^2(M_6) + e^7 \otimes e^7, \quad \Theta = \text{Vol}[AdS_3] \wedge [d(\lambda^{-3/2}e^7) - 2J_6],
\]
with the $SU(3)$ structure satisfying
\[
d(\lambda^{-1} J_6) = 0, \quad J_6^2 \wedge d(\lambda^{1/2} e^7) = \frac{1}{2} \lambda J_6^3, \quad d(\lambda^{-3/2} \Omega_6) = 2\lambda^{-1} e^7 \wedge \Omega_6.
\]
Observe that in this case not all the torsion modules are fixed by the above conditions. As a result, in this case it is necessary to impose $d \; *_{10} \Theta = 0$ as an extra condition. One can show that these are equivalent to the conditions of [23]. In particular we note that these conditions imply that we can introduce a coordinate $\psi$ such $e^7 = \lambda^{-1/2}(d\psi + B)$ with $\partial_\psi$ a Killing vector that preserves $J_6$ and $\lambda$ but not $\Omega_6$. Also the metric $\lambda^{-1} ds^2(M_6)$ is Kähler.

These geometries are dual to two-dimensional SCFTs with $(0,2)$ supersymmetry and the $U(1)$ Killing vector is dual to the $R$-symmetry. A rich set of examples of solutions of these equations can be found in [29, 34–36].

4. Explicit examples

In this section we will study two explicit solutions in detail—the $AdS_3$ solution of [27] that is dual to a SCFT with $\mathcal{N} = (2,2)$ supersymmetry and the $AdS_2$ solution of [28] dual to a SCQM with two supercharges. These arise from $D3$-brane wrapping Kähler two cycles in CY three-folds and associative three-cycles, respectively. In each case, the first part of our

\(^8\) Note that this means that the conformally rescaled metric $Ld\tau^2(M_6)$ is a cone $Ld\tau^2(M_6) = e^2r^2 dr^2 + e^{\tau_2} (\lambda d\tau^2(M_6) + e^\tau \otimes e^\tau)$ and so this case is rather analogous to the Sasaki–Einstein case.
investigation will be to verify that these solutions satisfy our $AdS$ conditions, by making their $G$ structure manifest. This serves as a rigid consistency check of our conditions.

In the second part of our investigation we will frame-rotate back to the canonical Minkowski frame. In other words we will write the solutions in a way in which the wrapped-brane $G$ structure of section 2, defined by half of the Killing spinors, is manifest. Inspired by the form of the metrics when re-written in this fashion, we can construct an ansatz for a more general class of wrapped-brane geometries that could describe an interpolation from a special holonomy metric to the $AdS$ fixed point. Given that the $AdS$ solutions describe the near horizon limit of $D3$-brane wrapping calibrated cycles, such interpolating solutions should exist. We show that our ansatz does indeed include singular special holonomy metrics that have a calibrated cycle of the form that appears in the $AdS$ solutions. The partial differential equations that need to be solved in order to construct interpolating solutions are involved and we have not managed to find any solutions.

4.1. $AdS_3$ from Kähler-2 in $CY_3$

An explicit solution of this type was first constructed in [27]. The solution was first constructed in gauged supergravity and then uplifted to type IIB. It describes the near horizon limit of a $D3$-brane wrapping a holomorphic $H^2$ in a $CY_3$. The $H^2$ can also be replaced with a discrete quotient, $H^2//\Gamma$ and hence a compact Riemann surface with genus $g > 1$. The metric can be written

$$ds^2 = \frac{1}{\lambda} \left[ ds^2(AdS_3) + ds^2(H^2) + \frac{\lambda^2}{1 - \lambda^2 \rho^2} d\rho^2 + \lambda^2 \rho^2 ds^2(S^1) + (1 - \lambda^2 \rho^2)(ds^2(S^2) + (d\psi + P - P')^2) \right],$$

(4.1)

where

$$\lambda^2 = \frac{8}{1 + 4\rho^2}, \quad dP = \text{Vol}(S^2), \quad dP' = \text{Vol}(H^2),$$

(4.2)

with $\rho \in [0, 1/2]$. Observe that the trivial $S^1$ fibre can be taken to smoothly degenerate at $\rho = 0$, while an $S^3$ smoothly degenerates at $\rho = 1/2$ if $\psi$ has period $4\pi$.

We define the frame

$$e^1 + ie^2 = \frac{1}{\lambda^{1/2}} e^{i\psi/2} (d\mu + i \sinh \mu d\beta),$$

$$e^3 + ie^4 = \frac{\sqrt{1 - \lambda^2 \rho^2}}{\lambda^{1/2}} e^{i\psi/2} (d\theta + i \sin \theta d\phi),$$

$$e^5 = \frac{\sqrt{1 - \lambda^2 \rho^2}}{\lambda^{1/2}} (d\psi - \cos \theta d\phi - \cosh \mu d\beta),$$

(4.3)

where $\mu, \beta$ are coordinates for $H^2$ and $\theta, \phi$ are coordinates for $S^2$. We can then define an $SU(2)$ structure by

$$J_4 = e^{12} + e^{34}, \quad \Omega_4 = (e^1 + ie^2) \wedge (e^3 + ie^4).$$

(4.4)

It may be verified explicitly that this is a solution of the conditions that were presented in section 3.3.

We now introduce an ansatz that could describe an interpolating solution from a $CY_3$ metric to the above $AdS_3$ solution. To do this it is illuminating to first identify the $SU(3)$ structure of the $AdS_3$ solution, regarded as a solution with a two-dimensional Minkowski
factor. We therefore rotate back from the AdS to the Minkowski frame, using the formulae of section 3. We find that the basis one-form $e^6$ of the Minkowski frame is given by

$$e^6 = L^{1/2} e^{-r/2} \left( -2 e^{-r/2} \sqrt{1 - 4 \rho^2 / 8} \right).$$

(4.5)

Defining the Minkowski-frame coordinate

$$u = -2 e^{-r/2} \sqrt{1 - 4 \rho^2 / 8},$$

(4.6)

the metric may be written as

$$ds^2 = L^{-1} [ds^2(\mathbb{R}^{1,1}) + F^2 ds^2(H^2)] + L \left[ F^{-1} \left( du^2 + u^2 \left[ ds^2(S^2) + (d\psi + P - P')^2 \right] \right) + dr^2 + r^2 ds^2(S^1) \right],$$

(4.7)

where

$$F = e^\rho = -\frac{u^2}{4 \zeta} + \frac{1}{4 \zeta} \sqrt{u^4 + 4 \zeta^2}.$$

We also find that $L = 2 F^2 / \sqrt{1 - u^2 F^2}$. The $SU(3)$ structure is given by the standard form

$$J_0 = e^{12} + e^{34} + e^{56}, \quad \Omega_6 = (e^{1} + ie^{2}) \wedge (e^{3} + ie^{4}) \wedge (e^{5} + ie^{6}),$$

(4.9)

with the Minkowski frame following from the AdS frame given by

$$e^1 + ie^2 = L^{-1/2} F e^{u \rho} (d\mu + i \sin \mu d\beta),$$

$$e^3 + ie^4 = -\frac{L^{1/2} u}{2F^{1/2}} e^{u \rho} (d\theta + i \sin \theta d\phi),$$

$$e^5 = -\frac{L^{1/2} u}{2F^{1/2}} (d\psi + P - P'),$$

$$e^6 = L^{1/2} F^{-1/2} du.$$

(4.10)

By construction this structure satisfies (2.14) and (2.15), and the Bianchi identity.

We can now make the following ansatz for the interpolating solution:

$$ds^2 = L^{-1} [ds^2(\mathbb{R}^{1,1}) + F_1 F_2 ds^2(H^2)] + L \left[ F_1^{-1} \left( du^2 + u^2 (d\psi + P - P')^2 \right) + F_2^{-1} \frac{u^2}{4} ds^2(S^2) + dr^2 + r^2 ds^2(S^1) \right],$$

(4.11)

with $L$ and $F_{1,2}$ as arbitrary functions of $u, z$. We impose as a boundary condition that this metric smoothly matches on to (4.7) in the AdS limit. Now we wish to determine the other boundary condition, by finding the most general special holonomy metric of the form (4.11). For special holonomy, we must have that $L = 1$, that $F_{1,2}$ are functions of $u$ only, and that $J_0, \Omega_6$, with the obvious frame, are closed. It is easy to verify that $\Omega_6$ is closed for any choice of $F_1, F_2$. Closure of $J_0$ then implies the equations

$$\partial_u (F_1 F_2) + \frac{u}{2 F_1} = 0, \quad \partial_u \left( \frac{u^2}{4 F_2} \right) - \frac{u}{2 F_1} = 0.$$

(4.12)

Adding and integrating, we find

$$F_2 = \frac{a^2 + \sqrt{a^2 - F_1 u^2}}{2F_1},$$

(4.13)
for some positive constant $a^2$. Inserting this into one of the remaining equations, making the substitution $F_1 = a^4 u^{-2} \cos^2 \xi$ and integrating, we get

$$-\frac{1}{3} \sin^3 \xi + \sin \xi = b \mp \frac{u^4}{4a^6}.$$  \quad (4.14)

By sending $\xi \to -\xi$, $b \to -b$, we can choose the upper sign in (4.13) and (4.14). Furthermore in order to obtain a smooth degeneration when $u = 0$ we will choose $b = 2/3$. A final change of coordinates

$$\sin \xi = 1 - \frac{r^2}{3a^2}$$  \quad (4.15)

allows us to cast the metric in the form,

$$ds^2 = \frac{6a^2 - r^2}{6} ds^2(H^2_3) + \frac{r^2}{6} ds^2(S^2) + \kappa^{-1} dr^2 + \frac{\kappa r^2}{9} (d\psi + P - P')^2,$$  \quad (4.16)

where

$$\kappa = \frac{9a^2 - r^2}{6a^2 - r^2}. \quad (4.17)$$

In this form it is clear that the metric is a hyperbolic analogue of the well-known metric on the resolved conifold constructed in [37, 38], which has a holomorphic $S^2$. Note that at $r = 0$, provided that $\psi$ has period $4\pi$ (as in the $AdS_3$ solution), an $S^3$ smoothly degenerates leaving a holomorphic $H^2$. At $r^2 = 6a^2$, however, the metric is singular (unlike the resolved conifold metric). It is natural to view (4.16) as a good local model of a holomorphic $H^2$, for which we can consider wrapping $D3$ branes. The gauged supergravity solution then describes the smooth back-reacted geometry in the near horizon limit\(^9\).

Upon substituting the ansatz (4.11) into the conditions for a wrapped brane geometry, we obtain some complicated p.d.e.’s for $L, F_1$ and $F_2$ which we will not write down. We conjecture that they admit a solution interpolating between the above special holonomy metric and the $AdS$ solution.

### 4.2. $AdS_2$ from associative

An explicit solution of this type was constructed in five-dimensional gauged supergravity in [28] and then uplifted to IIB supergravity. It describes the near-horizon limit of a $D3$-brane wrapping an associative hyperbolic three space, $H^3$. We can also replace $H^3$ with a compact discrete quotient $H^3/\Gamma$ without breaking supersymmetry. Correcting the expression of [28] upon lifting to ten dimensions\(^10\), we find that the metric of the IIB solution is given by

$$ds^2 = \frac{1}{\lambda} \left[ ds^2(AdS_2) + 4 ds^2(H^3) + \frac{\lambda^2}{1 - \lambda^2 \rho^2} d\rho^2 + \lambda^2 \rho^2 ds^2(S^1) + 4(1 - \lambda^2 \rho^2) \tilde{\mu}^a \tilde{\mu}^a \right].$$  \quad (4.18)

Here $ds^2(H^3)$ is the maximally symmetric metric on $H^3$ (with Ricci scalar equal to $-6$). If we introduce left-invariant one-forms $\sigma^a$ on $S^3$ satisfying $d\sigma^a = \frac{1}{2} \epsilon^{abc} \sigma^b \wedge \sigma^c$, and the spin connection $\omega_{ab}$ for $ds^2(H^3)$, then

$$\tilde{\mu}^a = \sigma^a - \frac{1}{2} \epsilon^{abc} \omega_{bc}. \quad (4.19)$$

\(^9\) Following [39] we might also try to interpret the gauged supergravity solution as describing the back-reacted geometry of $D3$-brane wrapping a singular holomorphic two-cycle at $r^2 = 6a^2$.

\(^10\) Our expression differs from [28] by a factor two in the radius of the five-sphere.
In addition
\[ \lambda^2 = \frac{64}{1 + 48\rho^2}, \]  
and we take \( \rho \in [0, 1/4] \). The \( S^1 \) smoothly degenerates at \( \rho = 0 \), while at \( \rho = 1/4 \) the \( S^3 \) smoothly degenerates. Defining the frame
\[ e^a = \frac{2}{\lambda^{1/2}} e^a, \quad \mu^a = \frac{2}{\lambda^{1/2}} \sqrt{1 - \lambda^2 \rho^2} \hat{\mu}^a, \]  
where \( \hat{e}^a \) is a basis for \( ds^2(H^3) \) (and hence \( d\hat{e}^a + \omega^a_b \hat{e}^b = 0 \)), the \( SU(3) \) structure of this solution is given by
\[ J_6 = \mu^a \wedge e^a, \quad \Omega_6 = \frac{1}{2} \epsilon^{abc} (\mu^a + i e^a) \wedge (\mu^b + i e^b) \wedge (\mu^c + i e^c). \]  
Using equations (9.64)–(9.69) of [17], it is easy to verify that this is an exact solution of the torsion conditions and Bianchi identity that we derived in section 3.1.

We now discuss an ansatz that could describe an interpolation from a \( G_2 \) holonomy metric to this \( AdS_5 \) solution. As in the previous subsection, we first obtain the Minkowski \( G_2 \) structure for the \( AdS_5 \) solution. We find that the one-form \( e^7 \) in the Minkowski frame is given by
\[ e^7 = L^{1/2} e^{-\nu/4} d(-\frac{1}{2} e^{-\nu/4} \sqrt{1 - 16\rho^2}). \]  
Defining the Minkowski frame coordinate
\[ u = -\frac{1}{2} e^{-\nu/4} \sqrt{1 - 16\rho^2}, \]  
the metric of the \( AdS_5 \) solution can be written
\[ ds^2 = L^{-1} [-d\tilde{T}^2 + 4 F ds^2(H^3)] + L \left[ F^{-3/4} \left( du^2 + \frac{u^2}{4} \tilde{\mu}^a \tilde{\mu}^a \right) + dz^2 + z^2 d\phi^2 \right], \]  
where
\[ F = e^{2\nu}, \]  
and \( \nu \) is given in terms of \( u \) and \( z \) by a (positive-signature metric inducing) root of the quartic
\[ 256 e^{2\nu} - 32 z^2 e^{2\nu} - 16 u^2 e^{2\nu} + 1 = 0. \]  
We also find that \( L = 4 F/(1 - 3u^2 F^{1/4})^{1/2} \). By construction, the \( G_2 \) structure given by
\[ \psi = J \wedge e^7 - \text{Im} \Omega, \quad \ast\ast\psi = \frac{1}{2} J \wedge J + \text{Re} \Omega \wedge e^7, \]  
with \( e^7 \) given by (4.23), satisfies equations (2.4), including the Bianchi identity.

Based on this result, we can now make the following ansatz for an interpolating solution:
\[ ds^2 = L^{-1} [-d\tilde{T}^2 + F_1^3 ds^2(H^3)] + L \left[ F_2^3 du^2 + F_2^3 \tilde{\mu}^a \tilde{\mu}^a + dz^2 + z^2 d\phi^2 \right], \]  
with \( L, F_{1,2,3} \) arbitrary functions of \( u \) and \( z \), and a boundary condition on the interpolating solution such that (4.29) smoothly matches on to (4.25) in the \( AdS \) limit. The second boundary condition on the interpolating solution is that it should smoothly match onto a \( G_2 \) metric, which we now determine. Requiring that (4.29) is a metric of \( G_2 \) holonomy implies that \( L \) is a constant, which we set to 1, that \( F_{1,2,3} \) are functions of \( u \) only, and that \( \psi \) and \( \ast\ast\psi \), with the obvious frame, are closed. Since \( F_3 \) is a function of \( u \) alone we can choose it to be 1. Then imposing closure of \( \psi \) we find the conditions
\[ \frac{1}{2} \partial_u F_1^3 - F_1 F_2 = 0, \quad \partial_u (F_1 F_2^2) + F_1 F_2 = 0. \]  
Closure of \( \ast\ast\psi \) produces the condition
\[ \partial_u (F_1^2 F_2^2) = F_2^3 - F_1^2 F_2. \]
which is implied by (4.30). It is straightforward to integrate (4.30); adding, we immediately obtain
\[ F_2 = \sqrt{\frac{\alpha}{F_1} - \frac{1}{3}}, \quad (4.32) \]
for some constant \( \alpha \). Then we have
\[ \partial_u F_1 = -\sqrt{\frac{\alpha}{F_1^2} - \frac{1}{3}}. \quad (4.33) \]
Defining a new coordinate \( x \) according to
\[ \partial_u \tilde{\alpha} = \sqrt{\frac{\alpha}{F_1^2} - \frac{1}{3}} \partial_x, \quad (4.34) \]
we get
\[ F_1 = x + \beta. \quad (4.35) \]
The constant \( \beta \) may be eliminated by a shift in \( x \). Finally, defining \( x = (3\alpha)^{1/3} R \), then dropping the tildes together with an overall scale factor of \( 3(3\alpha)^{2/3} \), the \( G_2 \) metric is
\[ ds^2 = \frac{dR^2}{R^2 - 1} + \frac{R^2}{3} ds^2(H^3) + \frac{R^2}{9} \left( \frac{1}{R^3} - 1 \right) \tilde{\mu}^a \tilde{\mu}^a. \quad (4.36) \]
Observe that this metric is the hyperbolic analogue of the well-known \( G_2 \) metric on an \( R^4 \) bundle over \( S^3 \) [40, 41], which has an associative \( S^3 \). In particular, at \( R = 1 \) the metric is smooth and describes an associative \( H^3 \). Unlike the metric in [40, 41], however, the metric (4.36) is singular at \( R = 0 \). It is natural to view (4.36) as a good local model of an associative \( H^3 \), for which we can consider wrapping \( D3 \) branes. The gauged supergravity solution then describes the smooth back-reacted geometry in the near horizon limit\(^{11}\).

Upon substituting the ansatz (4.29) into the conditions for a wrapped brane geometry we obtain some complicated p.d.e.’s for \( L, F_1, F_2 \) and \( F_3 \) which we will not write down. We conjecture that they admit a solution interpolating between the above special holonomy metric and the \( AdS \) solution.

5. Conclusions

In this paper we have derived an interesting class of supersymmetric geometries of type IIB supergravity with Minkowski factors and five-form flux that are associated with \( D3 \)-brane wrapping calibrated cycles in special holonomy manifolds (or configurations of intersecting \( D3 \)-branes). Using these wrapped-brane geometries we determined the extra conditions that are required in order to obtain a supersymmetric solution with an \( AdS \) factor. The \( AdS_3 \) conditions we have derived for the cases of \( D3 \)-brane wrapping associative or SLAG three-cycles, are new. Although the \( AdS_3 \) conditions for the cases of \( D3 \)-brane wrapping holomorphic two cycles in \( CY_2 \), \( CY_3 \) and \( CY_4 \) has been derived before, here we make an explicit link between these geometries and the wrapped-brane geometries.

By analytic continuation of the \( AdS \) metrics and torsion conditions, one obtains the conditions defining a class of supersymmetric geometries containing spheres. The class of BPS geometries with the \( AdS_3 \) replaced by \( S^3 \) has been classified before, but we found a new

\(^{11}\) Following [39] we might also try to interpret the gauged supergravity solution as describing the back-reacted geometry of \( D3 \)-brane wrapping a singular associative three-cycle at \( R = 0 \).
class with an $S^2$ factor by replacing $AdS_2$ with $S^2$ for the case associated with associative three-cycles. These geometries preserve 1/8 supersymmetry and have $\mathbb{R} \times SU(2)$ symmetry. It would be interesting to study them further.

For two explicit $AdS$ solutions, we have verified that they satisfy the appropriate torsion conditions, by explicitly obtaining their structures. This serves as a strong overall consistency check of our results. For these solutions we also constructed a more general ansatz for the corresponding wrapped-brane geometries which could describe solutions that interpolate between a special holonomy metric and the $AdS$ solution. In particular, we show that the ansatz admits singular special holonomy metrics that have calibrated cycles of the appropriate type. It would be very interesting to construct explicit interpolating solutions and to study how the singularity of the special holonomy metric gets resolved in the $AdS$ limit.

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