Magnus Force Approach to \(N\)-Vortex Dynamics in an Annular Bose-Einstein Condensate

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(Dated: April 1, 2015)

We consider analytically the dynamics of \(N\) vortices in an annular Bose-Einstein condensate obtaining expressions for the free energy and vortex precession rates to logarithmic accuracy. Both uniform and Thomas-Fermi profiles are discussed, and lower bounds for the lifetime of a single vortex are obtained. We conclude by showing that the Magnus force and the method of images give equivalent results in a homogeneous system, but in a Thomas-Fermi cloud the equivalence is only approximate.

Keywords: vortex dynamics, free energy, Bose-Einstein condensate, precession, ring trap

In ultra-cold quantum gases, the behaviour of collective non-linear excitations such as quantised vortices [1, 2] and dark solitons [3–5] can be distinctively different in non-simply connected geometries [6, 7] as a result of the ‘holed’ topology. These geometries are within experimental reach with many proposals for ring traps [8–10] having been recently suggested, including a genuinely non-trivial ring topology where the central region is occupied by a tapered optical fibre and therefore essentially at infinite potential [11].

The quantised vortices are a hallmark of superfluidity of the atomic Bose-Einstein condensate (BEC) [12], and understanding their dynamics helps explain properties of superconductors [13] and quantum liquids [14]. In annular geometry, many-vortex states such as a circular array of vortex-antivortex pairs can be produced via the snake instability of a ring dark soliton [15], whose subsequent vortex dynamics are not comprehensively studied. Protocols for experimental creation of ring dark solitons have been proposed [16, 17]. Experimentally, it has been possible to excite vortices and persistent currents in an annular condensate by stirring with a narrow blue-detuned optical beam [10], with the microscopic vortex nucleation mechanism in the stirred annulus having been studied theoretically as well [18]. Remarkably, also uniform BECs have been experimentally demonstrated [19], making the study of vortex dynamics in homogeneous condensates possible.

Among common theoretical approaches to vortex dynamics are the method of images [14, 20, 21] and the Magnus force [22–24] wherein it is normally assumed that the spatial vortex separation is much larger than the healing length. The Magnus force appears when a vortex moves with respect to the superfluid. In an infinite homogeneous neutral superfluid at zero temperature (in the absence of any quasi-particle scattering [25]), the Magnus effect arises as a transverse force [26] that relies on the superfluidity and single-valuedness of the condensate order parameter [27], having also a direct connection to the Berry phase [27].

Unlike the Magnus force, the method of images has been used to study vortex dynamics in an annulus [6, 28, 29], and in this paper we apply the Magnus force to vortex dynamics in a ring-shaped BEC. In other settings, the Magnus force and the method of images often give similar results [24, 30], which suggests the possibility of their equivalence under certain conditions. At first sight any equivalence seems ruled out because the method of images concerns the boundaries of the condensate while the Magnus effect has been argued to be a property of the vortex itself [27] (just like the classical Magnus force producing lift is a property of an aerofoil itself [31, 32]), present in an infinite homogeneous system. However, as we will see, the boundaries affect the free energy of a vortex, and when there is no dissipation, the vortex must follow contours of the free energy. Therefore, when the vortex moves, the transverse Magnus force must equal exactly to the force arising from the gradient of the free energy, and thus be affected by the boundaries.

In this work, we construct a theoretical framework to study the dynamics of \(N\) vortices in an annular BEC, which makes it possible to further study the ensuing rich dynamics of various vortex configurations in a non-simply connected geometry. We present analytical results for the free energy of \(N\) vortices and precession rate of a single vortex in both uniform and Thomas-Fermi annular clouds. As an application, these results are used to obtain a lower bound for the lifetime of a single vortex in a ring-shaped condensate. We conclude by showing that the Magnus force and the method of images are equivalent in a homogeneous system, but with a Thomas-Fermi density profile the equivalence is only approximate. Throughout this work we scale our units so that \(\hbar = 2m = 1\), which means we measure time, length, and energy in terms of \(\omega_x^{-1}\), \(a_{osc} = \sqrt{\hbar/2m\omega_x}\), and \(\hbar\omega_x\) respectively, where \(\omega_x\) is the angular frequency of the trap in the \(x\)-direction, and \(m\) is the mass of the atoms in the cloud.

**Free energy and vortex precession in an annulus.** We wish to obtain the free energy \(E\) of having \(N_v\) vortices at \(\{\rho_j\} (j = 1, 2 \ldots N_v)\) in the annulus \(\Omega = \{a < r < b; 0 \leq \varphi < 2\pi\}\). To this end, first focussing on a single vortex at \(\rho \in \Omega\), we will work in terms of a stream function \(\chi_\rho(\mathbf{r})\).
such that the superfluid velocity $v$ is given by

$$v(r) = \frac{\hbar}{m} \hat{z} \times \nabla \chi_{\rho}(r).$$  \hspace{1cm} (1)

It follows by definition of the vorticity $\nabla \times v$ of irrotational flow that the stream function satisfies the Poisson equation. To take into account the physical no-flow requirement $n v \cdot \hat{n} = 0$ and the geometry of the system, where $n$ is the superfluid density and $\hat{n}$ the unit outward normal, we require Dirichlet boundary conditions:

$$\left\{ \begin{array}{ll}
\nabla^2 \chi_{\rho}(r) = 2\pi \sum_{j=1}^{N_v} \delta_2^j(r - \rho_j), \\
\chi_{\rho}(a) = \chi_{\rho}(b) = 0.
\end{array} \right. \hspace{1cm} (2)$$

Using the stream function $\chi_{\rho}[33]$, the free energy $E$ of a vortex (i.e. the additional energy due to the presence of the vortex) at position $\rho$ in a uniform annular BEC of density $n_0$ occupying the area $\Omega$ is given by [34]

$$E = 4\pi n_0 \ln \left( \frac{b}{\xi} \right) + g(\rho, \rho), \hspace{1cm} (3)$$

where $\xi$ is the healing length, and

$$g(\eta_1, \eta_2) = \frac{\ln \left( \frac{\rho_1}{a} \right) \ln \left( \frac{\rho_2}{a} \right)}{\ln \left( \frac{\rho_1}{\rho_2} \right)} + \ln \left[ \frac{a^2 - \eta_1 \eta_2}{b^2 - \eta_1 \eta_2} \right] + 2 \sum_{n=1}^{\infty} \frac{a^{2n}(b^{2n} - \rho_1^{2n})(\rho_2^{2n} - a^{2n})}{2n b^{2n} (\rho_1 \rho_2)^n (b^{2n} - a^{2n})} \cos[n(\phi_{12})], \hspace{1cm} (4)$$

where the bar denotes complex conjugation, $\eta_1 = \rho_1 e^{i\phi_1}$, $\eta_2 = \rho_2 e^{i\phi_2}$, $\{\rho_1, \rho_2\} \in [a, b]$; $\{\phi_1, \phi_2\} \in [0, 2\pi]$, and $\phi_{12} \equiv \phi_1 - \phi_2$. Equation (3) is given to logarithmic accuracy, i.e. we assume $\ln(b/\xi) \gg 1$, and that terms of order $O(\xi^2)$ can be neglected. In the limit as $a \to 0$, Eq. (3) simplifies to the well-known result [35]

$$\lim_{a \to 0} E = 4\pi n_0 \left[ \ln \left( \frac{b}{\xi} \right) + \ln \left( 1 - \frac{\rho^2}{b^2} \right) \right] \hspace{1cm} (5)$$

for a uniform circular condensate of radius $b$ with an off-axis vortex at $r = \rho$.

For a uniform annular condensate with $N_v$ vortices at $\{\rho_j\}$ of charges $\{\kappa_j\}$ respectively, we obtain [34]

$$\frac{E}{4\pi n_0} = N_v \ln \left( \frac{b}{\xi} \right) + \sum_{k=1}^{N_v} g(\rho_k, \rho_k) + \sum_{k<l}^{N_v} \kappa_k \kappa_l \left\{ \begin{array}{l}
\Theta(\rho_k \leq \rho_l) \left[ \ln \left( \frac{b}{|\rho_k - \rho_l|} \right) + g(\eta_k, \eta_l) \right] + \\
\Theta(\rho_k > \rho_l) \left[ \ln \left( \frac{b}{|\rho_k - \rho_l|} \right) + g(\eta_l, \eta_k) \right],
\end{array} \right. \hspace{1cm} (6)$$

where $\Theta$ is the Heaviside step function, and $\phi_{kl} \equiv \phi_k - \phi_l$ is the angle between the $k$th and $l$th vortex.

In the Thomas-Fermi approximation, valid for large strongly-repulsive condensates ($b/\xi \gg 1$), we assume that the spatial variation of the density is so small that the kinetic energy comes only from the velocity, i.e. gradient of the phase. In particular, in this work we assume that $\nabla n = 0$, and the density at the vortex can be replaced by the vortex-free state, as corrections are of order $O(\xi^2/b^2)$, and the vortex does not alter the condensate...
density significantly [36]. Denoting a circle of radius \( \xi \) centered at the vortex at \( \rho \) by \( B_\rho^\xi \), the free energy depends only on surface terms \( \partial B_\rho^\xi \) if we ignore the volume term arising from \( \nabla n \neq 0 \). Therefore, we need only multiply the free energy by the Thomas-Fermi density \( n_{\text{TF}}(\rho) \) at the vortex:

\[
E_{\text{TF}} = \frac{1}{2} \int_{\Omega \setminus B_\rho^\xi} n_{\text{TF}} |\omega|^2 = \left[ \frac{1}{2} \left( \frac{\rho - r_{\text{trap}}}{b} \right)^2 \right] E. \quad (7)
\]

We have plotted the free energy of a single vortex [see Eq. (3)] in Fig. 1(a). The free energy vanishes at the condensate boundaries \( (r = \{a, b\}) \) meaning it is possible to nucleate vortices on both rims of the annulus as the non-linear excitations have no gap there.

As the Gross-Pitaevskii equation is Hamiltonian in structure, at \( T = 0 \) in the absence of dissipation, a vortex will follow equipotential contours of its free energy, which in general depend on density inhomogeneities [see Eq. (7)]. We have plotted contours of the \( N_c \)-vortex free energy [see Eq. (6)] in Fig. 2. They are the trajectories of a single (positive) vortex if the other vortices are held fixed. In general, the \( N_c \) vortices will follow simultaneous trajectories that leave \( E \) time-invariant.

As the free energy of a single vortex depends only on the radial coordinate, the cylindrical symmetry means the vortex will precess around the trap centre, and the rate can be calculated using the Magnus force,

\[
nm(\kappa \times \rho) = \nabla_\rho E, \quad (8)
\]

where \( n \) is the density and \( \kappa \) is the vortex circulation vector. Using Eqs. (3) and (7), in a uniform condensate in the annulus \( \Omega \), an off-axis vortex at \( r = \rho \) will precess at the rate

\[
\omega = -\frac{1}{\rho^2} \ln \left( \frac{\xi}{\rho} \right) - \frac{2}{\rho^2} + \frac{4}{\rho^2 - a^2} + \frac{4b^2}{1 - \frac{\rho^2}{\rho^2}},
\]

while in an annular Thomas-Fermi condensate \( n_{\text{TF}} = n_0 \left[ 1 - \left( \frac{r - r_{\text{trap}}}{b} \right)^2 \right] \), the precession rate is

\[
\omega_{\text{TF}} = \omega + \frac{4}{b^2} \left[ \ln \left( \frac{b}{\xi} \right) + g(\rho, \rho) \right]. \quad (10)
\]

See Fig. 1(b) for plots of Eqs. (9) and (10). It can be seen in Fig. 1(b) that the annular boundary can formally stabilise a vortex (i.e. \( \omega = 0 \)), which is not the case in cylindrical geometry. The precession frequency changes sign at this quasi-stable radius. Similar results have been obtained in Ref. [28] using numerical integration of the Gross-Pitaevskii equation, the method of images, and an approximative analytical solution of the Poisson equation, where the annulus is conformally mapped onto a straight strip. In comparison, our solution for the stream function is an exact Green’s function of the Dirichlet problem in the annulus, and it allows for a generalisation to \( N_c \) vortices using Eq. (6) and the Lagrangian formalism [30], for example. Furthermore, it makes it possible to discuss the equivalence of the Magnus force and the method of images (see below).

**Phonon radiation by a precessing vortex.** In Ref. [37], the power \( P \) radiated by a vortex executing circular motion in an infinite uniform system was obtained to be

\[
P = \frac{\pi q^2 \omega^3 \rho^2}{4c^2}, \quad (11)
\]

where \( q = -\hbar \sqrt{2\pi n_0/m} \), and the speed of sound \( c = \sqrt{\gamma n_0/m} \). Setting \( P = dE/dt = (\partial E/\partial \rho)d\rho/dt \) and using Eqs. (3) and (9), we may obtain a lower bound for the vortex lifetimes \( \tau_{(a,b)} \) in the annulus:

\[
\tau_{(a,b)} = \left[ \int_{r_{\omega} \pm 10\xi}^{(a,b)} \partial \rho \frac{d\rho}{P(\rho)} \right] = \frac{4c^2}{\pi q^2} \left( \int_{r_{\omega} \pm 10\xi}^{(a,b)} \partial \rho \frac{d\rho}{\omega^3 \rho^2} \right), \quad (12)
\]

where \( r_{\omega} \) is the radius corresponding to \( \omega = 0 \). Numerically evaluating the integral (see Fig. 3) shows that while the vortex lifetimes remain small for very narrow condensates, the vortex quickly becomes long-lived as the thickness of the annulus is increased. Decreasing the healing length also makes the vortex lifetime longer.

**Equivalence of the Magnus force and the method of images.** The relevant kinematic boundary condition at the condensate edges is \( n v \cdot \mathbf{n} = 0 \). In general, and in particular if the density is non-zero at the edge, the velocity field can have no outward component, which can be ensured, for example, by invoking the method of images. If we have \( n = 0 \) at the edge, it appears that the images are not needed as the no-flow condition is automatically satisfied. In the Thomas-Fermi approximation of sufficiently large clouds, the stream function will have to have
the same Dirichlet boundary conditions for the continuity equation \( \nabla \cdot (n\mathbf{v}) = n\nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla n = 0 \) to be fulfilled at the condensate boundary [37]. In a uniform condensate, the continuity equation is satisfied automatically.

By the Uniqueness Theorem, any solution to the Poisson equation with given boundary conditions [see Eq. (2)] is unique and equivalent to having the images present. Therefore, the vortex dynamics that arises from the Magnus force, which requires solving the Poisson equation with the given boundary conditions by any method to obtain the free energy, can be seen to be equivalent to the dynamics given by the method of images directly. This equivalence is only approximate if the system has a non-zero density gradient, see Eq. (7).

Conclusions. We have studied vortex dynamics in an annular BEC, obtaining expressions for the free energy, precession rates, and vortex lifetimes. We also showed that the Magnus force and the method of images are equivalent descriptions of vortex dynamics in uniform condensates, but with an inhomogeneous density profile the equivalence is only approximative. The results are important because they make it possible to further study the rich physics of many-vortex configurations in a non-simply connected geometry; for example, the Tkachenko modes of vortex necklaces, which will be studied in a future investigation.

We acknowledge the support of the Jenny and Antti Wihuri Foundation (LAT).

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Appendix A: Green’s function in an annulus

In this Appendix, we derive the Green’s function in an annulus \( \Omega = \{ a < r < b; 0 \leq \varphi < 2\pi \} \) with Dirichlet boundary conditions using the method of images. The location of the images is described in Refs. [1–3], but in addition the limit of infinitely many images must be taken with care when constructing the Green’s function below. By definition, the Green’s function satisfies the Poisson equation

\[
\nabla^2 G_\rho(r) = 2\pi \delta^{(2)}(r - \rho), \tag{A1}
\]

where \( \delta^{(2)} \) is a two-dimensional delta function, and \( \{ r, \rho \} \in \Omega \).

In general, the Green’s function can be decomposed into a radially symmetric singular and a regular part, which we label by \( F_\rho(r) \) and \( H_\rho(r) \) respectively:

\[
G_\rho(r) = F_\rho(r) + H_\rho(r). \tag{A2}
\]

The singular part \( F_\rho(r) \) is the fundamental solution corresponding to system response at a general point \( r \) to a point source at some arbitrary point \( \rho \), and can be written down as

\[
F_\rho(r) = -\frac{1}{2} \ln \left( |r - \rho|^2 \right). \tag{A3}
\]

The regular part \( H_\rho(r) \) is harmonic in \( \Omega \) satisfying Laplace’s equation there. The choice of this analytic harmonic function is asserted by the boundary conditions. Installing \( N \) images at \( \{ \tilde{\rho}_j \} \), all outside of \( \Omega \), we can write

\[
H(\tilde{\rho}_j)(r) = \sum_{j=1}^{N} (\pm) \frac{1}{2} \ln \left( |r - \tilde{\rho}_j|^2 \right), \tag{A4}
\]

where the sign depends on the winding number of the particular image vortex.

Considering a stream function \( \chi_\rho(r) \) such that the superfluid velocity \( \mathbf{v} \) is given by

\[
\mathbf{v}(r) = \frac{\hbar}{m} \mathbf{\hat{z}} \times \nabla \chi_\rho(r), \tag{A5}
\]

it follows by definition of the vorticity \( \nabla \times \mathbf{v} \) of irrotational flow that the stream function satisfies the Poisson equation. To take into account the physical no-flow requirement \( n \mathbf{v} \cdot \mathbf{\hat{n}} = 0 \) and the geometry of the system, where \( n \) is the superfluid density and \( \mathbf{\hat{n}} \) the unit outward normal, we require Dirichlet boundary conditions. In the annulus \( \Omega \) with \( N_v \) vortices at \( \{ \rho_j \} \) \( (j = 1, 2, \ldots N_v) \), we have:

\[
\begin{align*}
\nabla^2 \chi(\rho_j)(r) &= 2\pi \sum_{j=1}^{N_v} \delta^{(2)}(r - \rho_j), \\
\chi(\rho_j)(a) &= \chi(\rho_j)(b) = 0. \tag{A6}
\end{align*}
\]

Inserting the images as per the Dirichlet boundary conditions, we obtain

\[
\begin{align*}
\chi_\rho(r) &= \frac{1}{2} \ln \left( \frac{1}{|r|} \right) + \frac{1}{2} \sum_{j=1}^{\infty} (\pm) \ln \left( \frac{|r - \tilde{\rho}_j|^2}{2} \right) \\
N_v \rho \chi_\rho(r) &= \frac{1}{2} \ln \left( \frac{b^2}{|r|^2} \right) - \frac{1}{2} \ln \left( \frac{b^2}{a^2} \right) \\
-2N_v \rho \chi_\rho(r) &= \ln \left( \frac{a}{\rho} \right) \ln (b) - \ln \left( \frac{b}{a} \rho \right) \\
-\frac{1}{2} \sum_{j=1}^{N_v} \left\{ \ln \left[ \left| r - \left( \frac{a}{\rho} \right)^2 \mathbf{\hat{z}} + \frac{\tilde{\rho}_j}{\rho} \right|^2 \right] + (a \leftrightarrow b) \right\}. \tag{A7}
\end{align*}
\]

Equation (A7) for \( \chi_\rho(r) \) can be regrouped to read

\[
\chi_\rho(r) = \frac{1}{2} \ln \left( \frac{1}{|r|^2} \right) + \frac{1}{2} \sum_{j=1}^{\infty} (\pm) \ln \left( \frac{|r - \tilde{\rho}_j|^2}{2} \right) + (a \leftrightarrow b) \frac{1}{2} \sum_{j=1}^{N_v} \left\{ \ln \left[ \left| r - \left( \frac{a}{\rho} \right)^2 \mathbf{\hat{z}} + \frac{\tilde{\rho}_j}{\rho} \right|^2 \right] + (a \leftrightarrow b) \right\}. \tag{A8}
\]

where \( z = r e^{i\varphi} \), \( \eta = \rho e^{i\phi} \) is the position of the vortex, and \( \Theta \) is the Heaviside step function. Equation (A8) agrees with the equivalent result of obtaining the Green’s function by the method of eigenfunction expansion [4], as expected by the Uniqueness Theorem, and we adhere to it due to its computational friendliness. See Fig. 1 for a few example plots of Eq. (A8).
FIG. 1. Annular BEC with $a = 0.5$, and $b = 4.0$. (a) The stream function $\chi$ [i.e. constant-velocity flow lines, see Eq. (A8)] of a single vortex (*) at $\rho = 2.0$. (b) Two (positive) vortices at $\rho = 2.0$ separated by an angle of $2\pi/9$. (c) Same as (b) but with two opposite vortices. The red circles show the condensate boundaries.

In the limit as $a \to 0$ of a circular trap, the stream function in Eq. (A8) is given by

$$\lim_{a\to0} \chi_{\rho}(r) = \ln \left( \frac{b^2 - r^2 e^{i(\varphi - \phi)}}{b^2 r e^{i(\varphi - \phi)}/\rho^2} \right), \quad (A9)$$

which can be decomposed into the singular and regular parts $\ln \left( b^2 / |r e^{i(\varphi - \phi)}/\rho^2| \right)$ and $\ln \left( \left(1 - r^2 e^{i(\varphi - \phi)} / b^2 \right) \right)$, respectively.

Appendix B: Evaluation of the free energy

1. Uniform Condensate

The free energy of a vortex at position $\rho$ in a uniform annular BEC of density $n_0$ occupying the area $\Omega = \{a < r < b; 0 \leq \varphi < 2\pi\}$ is:

$$E = \frac{1}{2} n_0 \int_{\Omega} |v|^2 = \infty. \quad (B1)$$

The diverging part arises from the vortex core, and does not contain physically interesting information. Let us renormalise the kinetic energy by excising $B^\xi_{\rho}$, a small circle of radius $\xi$ centered at the vortex, where $\xi$ is the healing length. Using Eqs. (A5) and (A2), $\nabla \cdot (n_1 \chi \nabla \chi) = \left( \nabla n \right) \cdot \left( \nabla \chi \right) + n_1 \nabla^2 \chi$, $\nabla^2 \chi = 2\pi \delta^{(2)}(r - \rho)$, and the Divergence Theorem noting that the singularity is not inside the integration domain, the finite kinetic energy thus obtained is

$$E = \frac{1}{2} n_0 \int_{\Omega \setminus B^\xi_{\rho}} |v|^2 = 2n_0 \int_{\Omega \setminus B^\xi_{\rho}} |\nabla \chi_{\rho}(r)|^2$$

$$= -2n_0 \int_{\partial B^\xi_{\rho}} (F_{\rho} + H_{\rho}) \partial_n (F_{\rho} + H_{\rho}), \quad (B2)$$

where $\partial_n$ denotes the outward unit normal derivative on $\partial B^\xi_{\rho}$. Using Eq. (A8), the individual terms are readily evaluated to yield

$$E = 4\pi n_0 \left\{ \ln \left( \frac{b}{\xi} \right) + \ln \left( \frac{\xi}{a} \right) \right\} \quad (B3)$$

Note that this result is given to logarithmic accuracy, i.e. we assume $\ln (b/\xi) \gg 1$, and that terms of order $\mathcal{O}(\xi^2)$ can be neglected.

In the limit as $a \to 0$, Eq. (B3) simplifies to the well-known result [5]

$$\lim_{a \to 0} E = 4\pi n_0 \left[ \ln \left( \frac{b}{\xi} \right) + \ln \left( 1 - \frac{\rho^2}{b^2} \right) \right] \quad (B4)$$

for a uniform circular condensate of radius $b$ with an off-axis vortex at $r = \rho$.

Having multiple vortices in the condensate only adds non-singular interaction terms of the form $\int_{\Omega \setminus (B^\xi_{\rho})} \nabla \chi_{\rho_{\kappa}} \cdot \nabla \chi_{\rho_{\ell}}$, which can be evaluated using similar methods as above. For a uniform annular condensate with $N_{\nu}$ vortices at $\rho = \{\rho_j\}$ of charges $\{\kappa_j\}$ respectively, we obtain

$$E = \frac{4\pi n_0}{N_{\nu}} \left[ \ln \left( \frac{b}{\xi} \right) + \sum_{k=1}^{N_{\nu}} \left\{ \ln \left( \frac{\ell_k}{a} \right) \ln \left( \frac{b}{\ell_k} \right) + \ln \left( \frac{\rho_k^2 - a^2}{\rho_k^2 - \rho^2} \right) \right\} + 2 \sum_{n=1}^{\infty} \frac{a^{2n}(b^{2n} - \rho_k^{2n})(\rho_k^{2n} - a^{2n})}{2nb^{2n}\rho_k^{2n}(b^{2n} - a^{2n})} \right\} + \sum_{k < \ell} \kappa_k \kappa_{\ell} \left\{ \Omega(\rho_k \leq \rho_\ell) \left[ \ln \left( \frac{\ell_k}{a} \right) \ln \left( \frac{\ell_k}{\rho_k} \right) + \ln \left( \frac{a^2 - \rho_k \rho_\ell e^{i\phi_{kl}}}{\rho_k \rho_\ell e^{i\phi_{kl}}} \right) \right\} + \Omega(\rho_k > \rho_\ell) \left[ \ln \left( \frac{\ell_k}{a} \right) \ln \left( \frac{\ell_k}{\rho_k} \right) + \ln \left( \frac{a^2 - \rho_k \rho_\ell e^{i\phi_{kl}}}{\rho_k \rho_\ell e^{i\phi_{kl}}} \right) \right\} + 2 \sum_{n=1}^{\infty} \frac{a^{2n}(b^{2n} - \rho_k^{2n})(\rho_k^{2n} - a^{2n})}{2nb^{2n}(\rho_k \rho_\ell)^n(b^{2n} - a^{2n})} \cos(n\phi_{kl}) \right\} \right\}, \quad (B5)$$

where $\phi_{kl} = \phi_k - \phi_\ell$ is the angle between the $k$th and $l$th vortex. Note the $k \leftrightarrow l$ symmetry.
When $a \rightarrow 0$, Eq. (B5) simplifies to

$$
\frac{E}{4\pi n_0} \stackrel{a \rightarrow 0}{=} N_v \ln \left( \frac{b}{\xi} \right) + \sum_{k=1}^{N_v} \ln \left( 1 - \frac{\rho_k^2}{b^2} \right) + \frac{1}{2} \sum_{k<l} \kappa_k \kappa_l \ln \left[ \frac{b^2 - 2 \rho_k \rho_l \cos(\phi_{kl}) + \left( \frac{\rho_k \rho_l}{b} \right)^2}{\rho_k^2 - 2 \rho_k \rho_l \cos(\phi_{kl}) + \rho_l^2} \right].
$$

(B6)

2. Thomas-Fermi Profile

In the Thomas-Fermi approximation, valid for large strongly-repulsive condensates ($b/\xi \gg 1$, where $b$ is the size of the condensate and $\xi$ the healing length), we assume that the spatial variation of the density is so small that the kinetic energy comes only from the velocity, i.e. gradient of the phase. In particular, in this work we assume that $\nabla n = \mathbf{0}$, and the density at the vortex can be replaced by the vortex-free state, as corrections are of order $O(\xi^2/b^2)$, and the vortex does not alter the condensate density significantly [6]. Equation (B2) depends only on surface terms $\partial B_{\rho}^\xi$ if we ignore the volume term arising from $\nabla n \neq \mathbf{0}$. Therefore, we need only multiply it by the Thomas-Fermi density $n_{\text{TF}}(\rho)$ at the vortex:

$$
E_{\text{TF}} = \frac{1}{2} \int_{\Omega \setminus B_{\rho}^\xi} n_{\text{TF}} |v|^2 = \left[ 1 - \left( \frac{\rho - r_{\text{trap}}}{b} \right)^2 \right] E
$$

\[
\rightarrow 0
- \frac{1}{4} \int_{\Omega \setminus B_{\rho}^\xi} \nabla n_{\text{TF}} \cdot \nabla \left( \chi_{\rho}^2 \right).
\]

(B7)

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