On solvability of the matrix equation $AXB = C$
over a principal ideal domain

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Abstract—In this paper we present conditions of solvability of
the matrix equation $AXB = B$ over a principal ideal domain. The
necessary and sufficient conditions of solvability of equation $AXB$
$= B$ in terms of the Smith normal forms and in terms of the Hermi-
tfe normal forms of matrices constructed in a certain way by
using the coefficients of this equation are proposed. If a solution
of this equation exists we propose the method for its construction.

Keywords — matrix equation, solution, domain of principal
ideal

I. INTRODUCTION

Let $K$ denote an integer domain with an identity $e \neq 0$ .
Further, let $K_{m,n}$ be the set of $m \times n$ matrices over $K$ . Denote
by $I_n$ the identity matrix of dimension $n$ and by $0_{m,n}$ the zero
$m \times n$ matrix. For any matrix $A \in K_{m,n}$ rank $A$ and $A'$
denote the rank and the transpose matrix of $A$ respectively.
We will denote by $GL(m, K)$ the set of invertible matrices in
$K_{m,m}$.

Consider the matrix equation

$$AXB = C,$$  \hspace{1cm} (1)

where $A \in K_{m,n}$, $B \in K_{k,l}$, $C \in K_{m,l}$ and $X$ is unknown
$n \times k$ matrix over $K$ . This equation is one of the best known
matrix equations in matrix theory and its applications. The
problem of solvability of equation (1) has drawn the attention
of many mathematicians. Many authors addressed the question
when the equation (1) (over the set of real numbers $R$, the set of
complex number $C$ or the set the quaternion skew field $H$)
has a solution belonging to a special class of matrices. They
are given necessary and sufficient conditions (using genera-
lized inverses) for the existence of the Hermitian, skew-Hermitian,
reflective, anti-reflective, positive and real-positive solutions,
and the general solutions. More details on this problem and many references to the original literature can be found in [1–6], [8–15], [18–22].

Many authors consider the classical systems of matrix
equations over fields, commutative rings and a skew field. Mitra [13] proposed conditions for the existence of common
solutions of the linear matrix equations $A_1XB_1 = C_1$ and
$A_2XB_2 = C_2$ over a field. In [15] conditions for the existence
of a common solution of these equations over a principal ideal
domain were given. Similar problems were investigated in [22]
for equations over a regular ring with identity.

Let $K = F$ be a field and let $A$ be a nonzero matrix over
$F$ . A generalized inverse of $A$ denoted by $A^-$ is a matrix
which satisfies the equation $AA^-A = A$ . It may be noted that
the generalized inverse of a matrix over a commutative ring
$K$ with identity not always exists. The solvability criterion
for equation (1) is written in the form. A necessary and
sufficient condition for the solution of the equation $AXB = C$
is $AA^-CB^-B = C$ and in this case the general solution is
$X = A^-CB^- + U + AA^-UB^-B$ , where $U$ is arbitrary $n \times k$
matrix over the field $F$ (see [2], [13]).

On the other hand, it is well know (see [16]) that the
equation (1) has a solution over a field $F$ if and only if each of
the equations $AY = C$ and $ZB = C$ has $ZB = C$ a solution.

Consider the example. Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and
$C = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$ be matrices over the integer number ring $Z$. It is
easily verified that $Y_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the solution of the equation
$AY = C$ and $Z_0 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is the solution of the equation
$ZB = C$ . Since $Y = XB$ , we have $XB = Y_0$ . It is easily verified that $XB = Y_0$ has no solution over ring $Z$. Therefore,
the solvability criterion of equation (1) cannot be transferred
to rings. On the other hand, there is a little information on the
solvability conditions of equation (1) over commutative rings
in the literature (see [4], [15], [18], [19]).

The paper is organized as follows. In Section 2, we pre-
sent necessary and sufficient conditions for solvability of the
matrix equation $AXB = C$ over a principal ideal domain. In
Section 3 we investigate a special case of the matrix equation
$AXB = C$.
II. MAIN RESULTS

Further $K = R$ is a principal ideal domain with an iden-
tity element. Let $A \in R_{m,n}$ be a matrix of rank $r$ over a
principal ideal ring $R$. For $A$ there exist matrices
$U \in GL(m,R)$ and $V \in GL(n,R)$ such that

$$U_A AV_A = S_A = \text{diag} (a_1, a_2, \ldots, a_r, 0, \ldots, 0)$$

is a diagonal matrix, where $a_1, a_2, \ldots, a_r$ are all nonzero and
$a_i | a_{i+1}$ (divides) for all $i = 1, 2, \ldots, r - 1$. The matrix $S_A$ is
called the Smith normal form of the matrix $A$. The matrix
$S_A$ can be written in the form.

$$S(A) = \begin{bmatrix} S(A) & 0_{r,n-r} \\ 0_{m-r,r} & 0_{m-r,n-r} \end{bmatrix}$$

where $S(A) = \text{diag} (a_1, a_2, \ldots, a_r) \in R_{r,r}$.

**Theorem 1.** Let $A \in R_{m,n}$, $B \in R_{k,l}$, $C \in R_{m,l}$ and let

$$S_A = U_A AV_A = \text{diag} (a_1, a_2, \ldots, a_p, 0, \ldots, 0)$$

$$S_B = U_B BV_B = \text{diag} (b_1, b_2, \ldots, b_q, 0, \ldots, 0)$$

be Smith normal forms of matrices $A$ and $B$ respectively, where
$U_A \in GL(m,R)$, $V_A \in GL(n,R)$, $U_B \in GL(k,R)$ and
$V_B \in GL(l,R)$. The matrix equation $AXB = C$ is solvable over
$R$ if and only if $U_A CV_B = \begin{bmatrix} D & 0_{p,l-q} \\ 0_{m-p,q} & 0_{m-p,l-q} \end{bmatrix}$, where
$D \in R_{p,q}$ and $D = \text{diag} (a_1, a_2, \ldots, a_p) G \text{diag} (b_1, b_2, \ldots, b_q)$, where $G \in R_{p,q}$.

**Proof.** Let $X_0 \in M_{n,k}(R)$ be a solution of the equation
$AXB = C$. From the equality $AX_0B = C$ we obtain

$$U_A AV_A V_A^{-1} X_0 U_B^{-1} U_B BV_B = U_A CV_B . \tag{2}$$

Put $V_A^{-1} X_0 U_B^{-1} = G = \begin{bmatrix} G_{12} \\ G_{21} \end{bmatrix}$, where $G \in R_{p,q}$ and

$$U_A CV_B = G \begin{bmatrix} D \\ D_{21} \\ D_{22} \end{bmatrix}$$

where $D \in R_{p,q}$. We rewrite equality (2) in the form

$$\begin{bmatrix} S(A) & 0_{p,n-p} \\ 0_{m-p,p} & 0_{m-p,n-p} \end{bmatrix} \begin{bmatrix} G \\ G_{12} \\ G_{21} \end{bmatrix} \begin{bmatrix} S(B) \\ 0_{q,l-q} \\ 0_{k-q,l-p} \end{bmatrix} = \begin{bmatrix} D \\ D_{21} \\ D_{22} \end{bmatrix} .$$

From this we have

$$D_{12} = 0_{p,l-q}, \quad D_{21} = 0_{m-p,q}, \quad D_{22} = 0_{m-p,l-q},$$

and $\text{diag} (a_1, a_2, \ldots, a_p) G \text{diag} (b_1, b_2, \ldots, b_q) = D$.

Conversely, let matrices $U_A \in GL(m,R)$, $V_A \in GL(n,R)$, $U_B \in GL(k,R)$ and $V_B \in GL(l,R)$ such that

$$U_A AV_A = S_A = \text{diag} (a_1, a_2, \ldots, a_p, 0, \ldots, 0)$$

and

$$U_B BV_B = S_B = \text{diag} (b_1, b_2, \ldots, b_q, 0, \ldots, 0).$$

Further, let $U_A CV_B = \begin{bmatrix} D & 0_{p,l-q} \\ 0_{m-p,q} & 0_{m-p,l-q} \end{bmatrix}$, where

$$D = \text{diag} (a_1, a_2, \ldots, a_p) G \text{diag} (b_1, b_2, \ldots, b_q)$$

and $G \in R_{p,q}$.

From the last equality we have

$$C = U_A^{-1} \begin{bmatrix} S(A) & 0_{p,l-q} \\ 0_{m-p,q} & 0_{m-p,l-q} \end{bmatrix} V_B^{-1} = U_A^{-1} S_A V_A^{-1} B = AX_0B,$$

where

$$S(A) = \text{diag} (a_1, a_2, \ldots, a_p), \quad S(B) = \text{diag} (b_1, b_2, \ldots, b_q)$$

and

$$V_A = \begin{bmatrix} G \\ 0_{m-p,q} \end{bmatrix} \begin{bmatrix} 0_{p,l-q} \\ 0_{m-p,l-q} \end{bmatrix} U_A = X_0.$$

Thus, the matrix $X_0$ is a solution of the matrix equation
$AXB = C$ and the proof of Theorem 1 is complete.

**Corollary 1.** Let the matrix equation $AXB = C$ be solvable over
$R$. Then $S_C = S_A \Phi$ and $S_C = \Psi S_B$.

**Theorem 2.** Let $A \in R_{m,n}$, $B \in R_{k,l}$ and $C \in R_{m,l}$. Further,
let $U_A \in GL(m,R)$, $V_A \in GL(n,R)$, $U_B \in GL(k,R)$ and
$V_B \in GL(l,R)$ such that

$$U_A AV_A = S_A = \text{diag} (a_1, \ldots, a_p, 0, \ldots, 0)$$

$$U_B BV_B = S_B = \text{diag} (b_1, \ldots, b_q, 0, \ldots, 0)$$

be Smith normal forms of matrices $A$ and $B$ respectively. If

$$U_A CV_B = \begin{bmatrix} \text{diag} (a_1, \ldots, a_p) G \text{diag} (b_1, \ldots, b_q) & 0_{p,l-q} \\ 0_{m-p,q} & 0_{m-p,l-q} \end{bmatrix},$$

where $G \in R_{p,q}$, then for arbitrary matrices $T_{12} \in R_{p,k-q}$, $T_{21} \in R_{n-p,q}$ and $T_{22} \in R_{n-p,k-q}$ the matrix

$$X_T = V_A \begin{bmatrix} G \\ T_{21} \\ T_{22} \end{bmatrix} U_B \in R_{n,k}$$

is a general solution of the equation $AXB = C$.

**Proof.** By Theorem 1 the matrix

$$X_0 = V_A \begin{bmatrix} G \\ 0_{n-p,q} \end{bmatrix} \begin{bmatrix} 0_{p,k-q} \\ 0_{n-p,k-q} \end{bmatrix} U_A$$
is a solution of the equation $AXB = C$. Let $T_{12} \in R_{p,k-q}$, $T_{21} \in R_{n-p,q}$ and $T_{22} \in R_{n-p,k-q}$ be arbitrary matrices over a principal ideal domain $R$. Consider the matrix

$$X_T = \begin{bmatrix} 0_{p,q} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}.$$ 

It is clear that $S_A X_T S_B = 0_{m,l}$. From this equality it follows

$$U_A^{-1} S_A X_T S_B V_B^{-1} = U_A^{-1} S_A V_A^{-1} V_A X_T U_B U_B^{-1} S_B V_B^{-1} =$$

$$= AV_A X_T U_B B = AX_H B = 0_{m,l}.$$ 

Thus, the matrix $X_H = V_A X_T U_B = V_A \begin{bmatrix} 0_{p,q} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} U_B$ is a solution of the homogeneous matrix equation $AXB = 0_{m,l}$. In this connection it should be pointed out that

$$AX_0 B + AX_T B = A(X_0 + X_T) = C.$$ 

Hence, the matrix $X_q = X_0 + X_T$ is a general solution of the matrix equation $AXB = C$. The proof is completed.

Let $A \in R_{m,n}$ be a non-zero matrix with rank $A = r$ in which the first $k$ rows are zero, i.e., $A = \begin{bmatrix} 0_{k,n} \\ A_1 \end{bmatrix}$ and the first row of the matrix $A_1$ is non-zero, then, for $A$, there exists a matrix $W \in GL(n,R)$ such that

$$AW = H_A = \begin{bmatrix} 0_{k,n} \\ H_1 & 0_{m,n-r} \\ \vdots & \vdots \\ H_r & 0_{m,n-r} \end{bmatrix},$$

where $H_1 = \begin{bmatrix} a_{11} \\ \vdots \\ a_{n1} \end{bmatrix} \in R_{m,1}$, $H_2 = \begin{bmatrix} h_{21} & a_{21} \\ \vdots & \vdots \\ h_{r_1} & a_{r_1} \end{bmatrix} \in D_{m,2}$, ..., $H_r = \begin{bmatrix} h_{1r} & \cdots & h_{r-1}r, a_r \end{bmatrix} \in D_{m,r}$, and $k + m_1 + \ldots + m_r = m$.

The lower block-triangular matrix $H_A$ is called the (right) Hermite normal form of the matrix $A$ and it is uniquely defined for $A$ (see [7]).

The Kronecker product of the matrices $A = \begin{bmatrix} a_{ij} \end{bmatrix} \in R_{m,n}$ and $B \in R_{k,l}$ is the $mk \times nl$ matrix expressible in partitioned form as

$$A \otimes B = \begin{bmatrix} a_{11} B & \ldots & a_{1n} B \\ \vdots & \ddots & \vdots \\ a_{m1} B & \ldots & a_{mn} B \end{bmatrix} \in R_{mk,nl}.$$ 

The operator vector for any matrix $C = \begin{bmatrix} c_{ij} \end{bmatrix} \in R_{m,l}$ is defined in the following way (see [12, Chapter 12])

$$\text{vec}(C) = \begin{bmatrix} c_{11} & \ldots & c_{1l} \\ c_{21} & \ldots & c_{2l} \\ \vdots & \ddots & \vdots \\ c_{ml} & \ldots & c_{ml} \end{bmatrix}^T,$$

i.e. the entries of $C$ are stacked columnwise forming a vector of length $ml$.

Let $A \in R_{m,n}$, $B \in R_{k,l}$ and $C \in R_{m,l}$. Use the Kronecker product we will write the equation $AXB = C$ as the vector equation $A \otimes B^T \text{vec}(X) = \text{vec}(C)$, where $B^t$ is the transpose matrix of $B$ (see [12, Chapter 12, Theorem 12.3.1]). Thus, applying Theorem 1 in [17] to this system of linear equations, we have the following result.

**Theorem 3.** Let $A \in R_{m,n}$, $B \in R_{k,l}$ and $C \in R_{m,l}$. The matrix equation $AXB = C$ is consistent if and only if the Hermite normal forms of the matrices

$$\begin{bmatrix} A \otimes B^t & 0_{ml,1} \end{bmatrix} \text{ and } \begin{bmatrix} A \otimes B^t \ \text{vec}(C) \end{bmatrix}$$

coincide.

**Corollary 2.** Let $A_i \in R_{m,n}$, $B_i \in R_{k,l}$ and $C_i \in R_{m,l}$, $i = 1, 2$. The matrix equations $A_1 X B_1 = C_1$ and $A_2 X B_2 = C_2$ have a common solution over $R$ if and only if the Hermite normal forms of the matrices

$$\begin{bmatrix} A_1 \otimes B_1^t & 0_{ml,1} \end{bmatrix} \text{ and } \begin{bmatrix} A_1 \otimes B_1^t \ \text{vec}(C_1) \end{bmatrix}$$

$$\begin{bmatrix} A_2 \otimes B_2^t & 0_{ml,1} \end{bmatrix} \text{ and } \begin{bmatrix} A_2 \otimes B_2^t \ \text{vec}(C_2) \end{bmatrix}$$

coincide.

### III. Applications

Let $A \in R_{m,n}$ be a nonzero matrix. Special case of matrix equation (1) is the following matrix equation $AXA = A$, where $X$ is unknown $n \times m$ matrix over $R$. Any solution of this equation is called generalized inverse and is denoted by $A^-$. We note that there exist matrices over $R$ which do not have generalized inverses. The problem when generalized inverse exists for every matrix over a commutative ring $K$ with an identity element was study by many authors (see [4], [19] and references therein). One of the applications of theorems 1 and 2 is the following proposition.

**Theorem 4.** Let $A \in R_{m,n}$ be a matrix of rank $A = r$. The equation $AXA = A$ has a solution over $R$ if and only if $S_A = UAV = \begin{bmatrix} I_r & 0_{r,m-r} \\ 0_{n-r,r} & 0_{n-r,m-r} \end{bmatrix}$, where $U \in GL(m,R)$ and $V \in GL(n,R)$.

If equation $AXA = A$ is consistent then for arbitrary matrices $P_{12} \in R_{r,m-r}$, $P_{21} \in R_{n-r,r}$ and $P_{22} \in R_{n-r,m-r}$ the matrix $X_P = V_A \begin{bmatrix} 0 & I^t_r \\ P_{12} & P_{21} \end{bmatrix} U_B$ is its general solution.

**Proof.** Let $X_0 \in R_{m,n}$ be a solution of the matrix equation $AXA = A$. Further, let $U \in GL(m,R)$ and $V \in GL(n,R)$ such that $U_A V_A A = S_A = \text{diag}(a_1, \ldots, a_r, 0, \ldots, 0)$. By Theorem 1 we have $S_A V^{-1} X_0 U^{-1} S_A = S_A$.

Put $V^{-1} X_0 U^{-1} = \begin{bmatrix} D & D_{21} \\ D_{21} & D_{22} \end{bmatrix}$, where $D \in R_{r,r}$. From the equality $S_A V^{-1} X_0 U^{-1} S_A = S_A$ we find that
S(ADA) = S(A),
where \( S(A) = \text{diag}(a_1, \ldots, a_n) \) is a nonsingular matrix.

We can now easily show that \( S(A) \in GL(r, R) \). So, we can assume that \( S(A) = I_r \). Thus, \( S_A = \begin{bmatrix} I_r & 0_{r,n-r} \\ 0_{m-r, r} & 0_{m-r,n-r} \end{bmatrix} \).

Conversely, if \( S_A = UAV \begin{bmatrix} I_r & 0_{r,m-r} \\ 0_{n-r, r} & 0_{n-r,m-r} \end{bmatrix} \), where \( U \in GL(n,R) \) and \( V \in GL(m,R) \), then by Theorem 1 we have that the matrix \( X_0 = V \begin{bmatrix} I_r & 0_{r,m-r} \\ 0_{n-r, r} & 0_{n-r,m-r} \end{bmatrix} U \) is a solution of the equation \( AXA = A \). By Theorem 2 for arbitrary matrices \( P_{12} \in R_{r,m-r} \), \( P_{21} \in R_{n-r,r} \) and \( P_{22} \in R_{n-r,m-r} \) the matrix
\[
X_P = V_A \begin{bmatrix} G \\ P_{21} \end{bmatrix} P_{22} U_B
\]
is a general solution of the equation \( AXA = A \). This completes the proof of Theorem 3.

**Corollary 3.** Let \( A \in R_{m,n} \) be a matrix with the Smith normal form \( S_A = \begin{bmatrix} I_r & 0_{r,m-r} \\ 0_{n-r, r} & 0_{n-r,m-r} \end{bmatrix} \). Then for every solution \( X_0 \) of the equation \( AXA = A \) both matrices \( X_0A \) and \( AX_0 \) are idempotent matrices of rank \( r \).

**Proof.** Let a matrix \( X_0 \) be a solution of the equation \( AXA = A \). From equality \( AX_0A = A \) it follows that \( AX_0 \) and \( X_0A \) are nonzero matrices. Thus,
\[
AX_0A = (AX_0)^2 = AX_0.
\]

Similarly, \( X_0AX_0 = (X_0A)^2 = X_0A \) and the proof of the Corollary is complete.

**IV. CONCLUSIONS**

Necessary and sufficient conditions for existence and expression of a solution of the matrix equation \( AXB = C \) over a principal ideal domain are derived. Some results are true for this matrix equation over domains of elementary divisors and Bezout domains.

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