On $N$-differential graded algebras

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September 12, 2018

Abstract

We introduce the concept of $N$-differential graded algebras (N-dga), and study the moduli space of deformations of the differential of an N-dga. We prove that it is controlled by what we call the (M,N)-Maurer-Cartan equation.

Introduction

The goal of this paper is to take the first step towards finding a generalization of Homological Mirror Symmetry (HMS) [11] to the context of $N$-homological algebra [5]. In [7] Fukaya introduced HMS as the equivalence of the deformation functor of the differential of a differential graded algebra associated with the holomorphic structure, with the deformation functor of an $A_\infty$-algebra associated with the symplectic structure of a Calabi-Yau variety. This idea motivated us to define deformation functors of the differential of an $N$-differential graded algebra. An $N$-dga is a graded associative algebra $A$, provided with an operator $d : A \to A$ of degree 1 such that $d(ab) = d(a)b + (-1)^a ad(b)$ and $d^N = 0$. A nilpotent differential graded algebra (Nil-dga) will be an $N$-dga for some integer $N \geq 2$. Theorem 10 endows the category of Nil-differential graded algebras with a symmetric monoidal structure. We remark that such a monoidal structure cannot be constructed in a natural way for a fixed $N$ (except for $N = 2$), not even using the $q$-deformed Leibniz rule, see [13].

In Section 2 we consider deformations of a 2-dga into an $N$-dga. By deforming 2-dgas one is able to construct a plethora of examples of $N$-dgas. Roughly speaking Theorem 16 tell us that a derivation of a 2-dga $d_A + e$ is an $N$-differential iff

\[(d_{End}(e) + e^2)^{\frac{N-1}{2}}(d_A + e) = 0 \quad \text{for } N \text{ odd,}\]

\[(d_{End}(e) + e^2)^{\frac{N}{2}} = 0 \quad \text{for } N \text{ even.}\]

*Work partially supported by IVIC.
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In Section 3 we introduce a general formalism for discrete quantum mechanics. We introduce these models since they turn out, in a totally unexpected way, to be relevant in the problem of deforming an \( M \)-differential into an \( N \)-differential with \( N \geq M \). Section 4 contains our main result, Theorem 19 which provides an explicit identity called the \((M, N)\)-Maurer-Cartan equation that controls deformations of an \( M \)-complex into an \( N \)-complex. The construction of the \((M, N)\)-Maurer-Cartan equation is based on an explicit description of coefficients \( c_k \) such that

\[
(d_A + e)^N = \sum_{k=0}^{N-1} c_k d_A^k,
\]

where \( c_k \) depends on \( d_A \) and \( e \). In Section 5 we define a functional \( cs_{2,2N} \) whose critical points are naturally determined by the \((2, 2N)\)-Maurer-Cartan equation.

In conclusion in this paper we introduce the moduli space of deformations of the differential of an \( N \)-dga and prove that it is controlled by a generalized Maurer-Cartan equation. We point out that our methods and ideas can be applied in a wide variety of contexts. Examples of \( N \)-dga’s coming from differential geometry are developed in [1]. A \( q \)-analogue, for \( q \) a primitive \( N \)-th root of unity, of our main result Theorem 19 is provided in [2]. In [3] we state an \( N \)-generalized Deligne’s principle and use the constructions of this paper to study \( A_\infty \)-algebras of depth \( N \).

1 \( N \)-differential graded algebras and modules

Throughout this paper we shall work with the abelian category of \( k \)-modules over a commutative ring \( k \) with unit. We will denote by \( A^\bullet \) \( \mathbb{Z} \)-graded \( k \)-modules \( \oplus_{i \in \mathbb{Z}} A^i \). We let \( \bar{a} \in \mathbb{Z} \) denote the degree of the element \( a \in A^\bar{a} \). The following definition is taken from [9].

**Definition 1** Let \( N \geq 1 \) an integer. An \( N \)-complex is a pair \((A^\bullet, d)\), where \( A^\bullet \) is a \( \mathbb{Z} \)-graded object and \( d : A^\bullet \rightarrow A^\bullet \) is a morphism of degree 1 such that \( d^N = 0 \).

Clearly an \( N \)-complex is a \( P \)-complex for all \( P \geq N \). If \( k \) is a field, then an \( N \)-complex \((A^\bullet, d)\) is referred as an \( N \)-differential graded vector space \((N\text{-dgvect})\). An \( N \)-complex \((A^\bullet, d)\) such that \( d^{N-1} \neq 0 \) is said to be a proper \( N \)-complex. Let \((A^\bullet, d_A)\) be an \( M \)-complex and \((B^\bullet, d_B)\) be an \( N \)-complex, a morphism \( f : (A^\bullet, d_A) \rightarrow (B^\bullet, d_B) \) is a morphism \( f : A^\bullet \rightarrow B^\bullet \) of \( k \)-modules such that \( d_B f = f d_A \).

**Lemma 2** Let \((A^\bullet, d_A)\) be a proper \( M \)-complex, \((B^\bullet, d_B)\) be a proper \( N \)-complex and \( f : (A^\bullet, d_A) \rightarrow (B^\bullet, d_B) \) be a morphism, then (1) If \( \text{Ker}(f) = 0 \), then \( M \leq N \); (2) If \( \text{Im}(f) = B^\bullet \), then \( M \geq N \) and (3) If \( \text{Ker}(f) = 0 \) and \( \text{Im}(f) = B^\bullet \), then \( M = N \).
Proof. 1. Assume that $N < M$ and let $a \in A^i$ then $f(d_N^A(a)) = d_B^N(f(a)) = 0$. This implies that $d_N^A(a) \in \text{Ker}(f) = 0$, and therefore $d_N^A(a) = 0$ which is in contradiction with the fact that $(A^*, d_A)$ is a proper $M$-complex. The proof of 2. is analogous to 1., 3. follows from 1. and 2.♦

Example 3 Consider $V = \mathbb{C}<e_1, e_2, e_3>$ the complex vector space generated by $e_1, e_2, e_3$. We endow $V$ with a $\mathbb{Z}$-graduation declaring $\bar{e}_1 = 0$, $\bar{e}_2 = 1$ and $\bar{e}_3 = 2$. Define the linear map $d : V \to V$ on generators by

$$d(e_1) = e_2, \quad d(e_2) = e_3, \quad \text{and} \quad d(e_3) = 0.$$ 

$(V, d)$ is a proper 3-complex.

Definition 4 Let $(A^*, d)$ be an $N$-complex, we say that an element $a \in A^i$ is $p$-closed if $d^p(a) = 0$ and is $p$-exact if there exists an element $b \in A^{i-N+p}$ such that $d^{N-p}(b) = a$, for $1 \leq p < N$ fixed. The cohomology groups of are the $k$-modules

$$pH^i(A) = \frac{\text{Ker}\{d^p : A^i \to A^{i+p}\}}{\text{Im}\{d^{N-p} : A^{i-N+p} \to A^i\}},$$

where $i \in \mathbb{Z}$, $p = 1, 2, ..., N-1$. We set $kH^*(A) = 0$ for $k \geq N$.

Notice that a 2-complex $A^*$ is just a complex in the usual sense and in this case $p$ is necessarily equal to 1 and $1H^i(A)$ agrees with $H^i(A)$ for all $i \in \mathbb{Z}$.

Definition 5  
(a) Let $N \geq 1$ an integer. An $N$-differential graded algebra or $N$-dga over $k$, is a triple $(A^*, m, d)$ where $m : A^k \otimes A^l \to A^{k+l}$ and $d : A^k \to A^{k+1}$ are $k$-modules homomorphisms satisfying

1) The pair $(A^*, m)$ is a graded associative algebra.

2) For all $a, b \in A^*$, $d$ satisfies the graded Leibniz rule $d(ab) = d(a)b + (-1)^{\bar{a}}ad(b)$.

3) $d^N = 0$, i.e., $(A^*, d)$ is an $N$-complex.

(b) A nilpotent differential graded algebra (Nil-dga) is an $N$-dga for some integer $N \geq 2$.

A 1-dga is a graded associative algebra. A 2-dga is a differential graded algebra.

Lemma 6 Let $(A^*, m, d)$ be an $N$-dga, then if $a$ is $p$-closed and $b$ is $q$-closed then $ab$ is $(p+q-1)$-closed.
Proof. The Lemma follows from the identity

\[ d^n(ab) = \sum_{i=0}^{n} \binom{n}{i} d^i(a)d^{n-i}(b), \]

where \( \binom{n}{i} = (-1)^i \), and for \( j \geq 1 \), \( \binom{n+1}{j} = \binom{n}{j-1} + (-1)^{i+j} \binom{n}{j} \).

When \( n = p + q - 1 \), since \( d^i(a) = 0 \) for \( i \geq p \), we only consider the case \( i < p \), then

\[ n - i = p + q - 1 - i > q - 1 \] and \( d^{n-i}(b) = 0 \), because \( d^j(b) = 0 \) for \( j \geq q \). Thus either \( d^i(a) = 0 \) or \( d^{n-i}(b) = 0 \) for all \( i \), and we have \( ab \) is \((p + q - 1)\)-closed. ♦

Definition 7 Let \((A^*, m_A, d_A)\) be an \( M \)-dga and \((B^*, m_B, d_B)\) be an \( N \)-dga. A morphism \( f : A^* \to B^* \) is a linear map such that \( f m_A = m_B(f \otimes Id) + m_B(Id \otimes f) \) and \( d_B f = f d_A \).

A morphism \( f : A^* \to B^* \) such that \( f(A^i) \subset B^{i+k} \) is said to be a morphism of degree \( k \). A pair of morphisms \( f, g : A^* \to B^* \) of \( N \)-dga are homotopic, if there exist \( h : A^* \to B^* \) of degree \( N - 1 \) such that

\[ f - g = \sum_{i=0}^{N-1} d_B^{N-1-i} h d_A^i. \]

We remark that if two morphisms \( f, g : A^* \to B^* \) of \( \text{Nil-dga} \) are homotopic then they induce the same maps in cohomology.

Let \((A^*, m_A, d_A)\) and \((B^*, m_B, d_B)\) be an \( M \)-dga and an \( N \)-dga, respectively. Define \( d_{A \otimes B} = d_A \otimes Id + Id \otimes d_B \), the identity

\[ d_{A \otimes B}^n(a \otimes b) = \sum_{k=0}^{n} (-1)^k a^{(n-k)} d_A^k(a) \otimes d_B^{n-k}(b) \]
implies,

Proposition 8 The triple \((A^* \otimes B^*, m_{A \otimes B}, d_{A \otimes B})\) is an \((M+N-1)\)-dga, where \( m_{A \otimes B} = m_A \otimes m_B \).

Example 9 Let \((V, d)\) be the 3-complex of in Example 3. On the space \( V \otimes V^* \) consider the base given by \( E_{ij} = e_i \otimes e_j^* \), \( i, j = 1, 2, 3 \), and define

\[ D(E_{ij}) = E_{(i+1)j} + (-1)^{i+j} E_{i(j-1)}, \]

by Proposition 8 and since \( D^4(E_{13}) \neq 0 \), then \((V \otimes V^*, D)\) is a proper 5-dga.
Theorem 10 The category Nil-dgvec is a symmetric monoidal category. Nil-dga is the category of monoids in Nil-dgvec. Nil-dga inherits a symmetric monoidal structure from Nil-dgvec.

Let $V^\bullet$ be an $N$-dga. By Proposition (V$^\bullet$)$^\otimes 2$ is a $(2N-1)$-dga, (V$^\bullet$)$^\otimes 3$ is a $(3N-2)$-dga and in general (V$^\bullet$)$^\otimes k$ is a $[k(N-1)+1]$-dga.

Definition 11 Let $(A^\bullet,m_A,d_A)$ be an $N$-dga and $M^\bullet$ a graded $k$-module. Let $K \geq 2$ an integer. A $K$-differential graded module $(K$-dgm) over $(A^\bullet,m_A,d_A)$, is a triple $(M^\bullet,m_M,d_M)$ with $m_M : A^k \otimes M^l \to M^{k+l}$ and $d_M : M^k \to M^{k+1}$, $k$-modules morphisms satisfying the following properties

1. For all $a,b \in A^\bullet$ and $m \in M^\bullet$, $m_M(a,m_M(b,m)) = m_M(m_A(a,b),m)$. If no confusion arises, we denote $m_M(a,m)$ by am.

2. For all $a \in A^\bullet$ and $m \in M^\bullet$, $d_M(am) = d_A(a)m + (-1)^a ad_M(m)$.

3. The pair $(M^\bullet,d_M)$ is a $K$-complex, $d_M^K = 0$.

Let $(M^\bullet,m_M,d_M)$ be a $K$-dgm and $(N^\bullet,m_N,d_N)$ be an $L$-dgm both over an $N$-dga $(A^\bullet,m_A,d_A)$. A morphism $f : M^\bullet \to N^\bullet$ of degree $k$ is a linear map such that $f(m_M(a,b)) = (-1)^a f m_N(a,f(b))$ and $d_M(f(b)) = f(d_N(b))$, for all $a \in A^\bullet$ and $b \in M^\bullet$. Now let $(M^\bullet,m_M,d_M)$ be a $K$-dgm over an $M$-dga $(A^\bullet,m_A,d_A)$ and $(N^\bullet,m_N,d_N)$ an $L$-dgm over an $N$-dga $(B^\bullet,m_B,d_B)$. The triple $(M \otimes N,m_{M \otimes N},d_{M \otimes N})$ turns out to be a $(K+L-1)$-dgm over $(A \otimes B,m_{A \otimes B},d_{A \otimes B})$, where $m_{M \otimes N}$ and $d_{M \otimes N}$ are defined as before.

Definition 12 The space of endomorphisms of degree $k$ of $M^\bullet$ is $\text{End}^k(M) = \prod_{i \in \mathbb{Z}} \text{Hom}(M^i,M^{i+k})$, this is, $\text{End}^k(M)$ consist of maps $f : M^\bullet \to M^\bullet$ of degree $k$ which are linear in regard to the action of $A^\bullet$ but which does not satisfy necessarily the relation $d_M f = (-1)^f f d_M$.

There are operators $\circ_M : \text{End}(M) \otimes M^\bullet \to M^\bullet$ and $\circ_E : \text{End}(M) \otimes \text{End}(M) \to \text{End}(M)$. Similarly to Proposition (V$^\bullet$)$^\otimes 2$ Proposition (V$^\bullet$)$^\otimes 3$ below provides the natural algebraic structure on $\text{End}(M)$.

Proposition 13 Define $d_{\text{End}}(f) := d_M(f) - (-1)^f f (d_M)$, for $f \in \text{End}(M)$. The triple $(\text{End}(M),\circ_E,d_{\text{End}})$ is a $(2N-1)$-dga, and $(M^\bullet,\circ_M,d_M)$ is an $N$-dgm over $(\text{End}(M),\circ_E,d_{\text{End}})$. 


Proof. Associativity of $\circ_E$ follows from the associativity morphisms composition. The Leibniz rule for $d_{End}$ is a consequence of the Leibniz rule for $d_M$. From the definition of $d_{End}$ we obtain the identity

$$d^n_{End}(f) = \sum_{k=0}^{n} (-1)^f(n-k) d^k_m \circ f \circ d^{n-k}_m$$

which can be proved by induction and holds for all $n \geq 1$. Let $n = 2N - 1$ if $k < N$ then $N - 1 < n - k$ and thus $d^{n-k}_M = 0$. Similarly if $n - k < N$ then $d^k_M = 0$. ♦

2 Deformation theory of 2-dgas into N-dgas

Let $k$ a field and consider the category $\text{Artin}$ of finite dimensional local $k$-algebras. If $R \in \text{Ob}(\text{Artin})$ with maximal ideal $R_+$ then $k \cong R/R_+$ ($R = k[[t]]$ and $R_+ = tk[[t]]$ are examples to keep in mind). Since $k \cong R/R_+$ then $R \cong k \oplus R_+$ as vector spaces. We study deformation theory using the formalism which considers deformations as functors from Artin algebras to Sets for later convenience.

**Definition 14** Let $A^\bullet$ be an $M$-dga, an $N$-deformation of $A^\bullet$ over $R$ is an $N$-dga $A^\bullet_R$ over $R$, with $N \geq M$, such that $A^\bullet_R/R_+A^\bullet_R$ is isomorphic to $A^\bullet$ as $N$-dga. Two $N$-deformations $A^\bullet_R$ and $B^\bullet_R$ are said to be isomorphic if there exists an isomorphism $\Phi : A^\bullet_R \rightarrow B^\bullet_R$ of $N$-dgas such that the induced isomorphism $\bar{\Phi} : A^\bullet_R/R_+A^\bullet_R \rightarrow B^\bullet_R/R_+B^\bullet_R$ satisfies $i_B \bar{\Phi} = i_A$, where $i_A$ and $i_B$ are the isomorphism $i_A : A^\bullet_R/R_+A^\bullet_R \rightarrow A^\bullet$ and $i_B : B^\bullet_R/R_+B^\bullet_R \rightarrow A^\bullet$.

The core of Definition 14 is to require that $d_{A_R}$ reduces to $d_A$, and $m_{A_R}$ reduces to $m_A$ under the natural projection $\pi : A^\bullet_R \rightarrow A^\bullet_R/R_+A^\bullet_R \cong A^\bullet$. Assume that $A^\bullet_R = A^\bullet \otimes R$ as graded algebras. We have the following decomposition

$$A^\bullet_R = A^\bullet \otimes R = A^\bullet \otimes (k \oplus R_+) = (A^\bullet \otimes k) \oplus (A^\bullet \otimes R_+) = A^\bullet \oplus (A^\bullet \otimes R_+).$$

Thus, since $d_{A_R}$ reduces to $d_A$ under the projection $\pi$, we must have

$$d_{A_R} = d_A + e$$

where $e \in \text{Der}(A^\bullet \otimes R_+)$ has degree 1. Moreover, the fact that $d^n_{A_R} = 0$ implies that $e$ is required to satisfy an identity which we call the $(M,N)$-Maurer-Cartan equation. Next proposition is well known and considers the classical case, that is, the $(2,2)$-Maurer-Cartan equation.
Proposition 15 Let $A^\bullet$ be a 2-dga and $A^\bullet_\mathcal{R} = A^\bullet \otimes \mathcal{R}$ be a 2-deformation over $\mathcal{R}$, $d_{A_\mathcal{R}} = d_A + e$ where $e \in \text{Der}(A^\bullet \otimes \mathcal{R}_+)$, then $e$ satisfies the (2,2)-Maurer-Cartan equation given by

$$d_{\text{End}}(e) + e^2 = 0.$$ 

**PROOF.** We have

$$d_{A_\mathcal{R}}^2(a) = (d_A + e)(d_A + e)(a) = d_A^2(a) + d_A(e(a)) + e(d_A(a)) + e^2(a) = d_{\text{End}}(e)(a) + e^2(a), \quad \text{for all } a \in A^\bullet.$$

Suppose that $N = 2k + n$, $n \in \{0, 1\}$ and $k \in \mathbb{N}$, then

$$d_{A_\mathcal{R}}^N = d_{A_\mathcal{R}}^{2k+n} = (d_{A_\mathcal{R}}^2)^k d_{A_\mathcal{R}}^n = (d_{\text{End}}(e) + e^2)^k d_{A_\mathcal{R}}^n, \quad \text{thus}$$

**Theorem 16** Let $A^\bullet$ be a 2-dga. $A^\bullet_\mathcal{R} = A^\bullet \otimes \mathcal{R}$ is an $N$-deformation over $\mathcal{R}$ with $d_{A_\mathcal{R}} = d_A + e$ where $e \in \text{End}(A^\bullet \otimes \mathcal{R}_+)$ of degree 1, iff $e$ satisfies

$$(d_{\text{End}}(e) + e^2)^\frac{N-1}{2}(d_A + e) = 0 \quad \text{for } N \text{ odd},$$

$$(d_{\text{End}}(e) + e^2)^\frac{N}{2} = 0 \quad \text{for } N \text{ even}.$$ 

Theorem 16 can be easily extended to study deformations of the differential of a 2-dgm $M^\bullet$ over a 2-dga $A^\bullet$ as follows.

**Theorem 17** Let $M^\bullet$ be a 2-dgm over a 2-dga $A^\bullet$. Then $M^\bullet_\mathcal{R} = M^\bullet \otimes \mathcal{R}$ is an $N$-deformation over $\mathcal{R}$ with $d_{M_\mathcal{R}} = d_A + e$ where $e \in \text{End}(M^\bullet \otimes \mathcal{R}_+)$ has degree 1, iff $e$ satisfies

$$(d_{\text{End}}(e) + e^2)^\frac{N-1}{2}(d_M + e) = 0 \quad \text{for } N \text{ odd},$$

$$(d_{\text{End}}(e) + e^2)^\frac{N}{2} = 0 \quad \text{for } N \text{ even}.$$ 

Let $M$ be a 3-dimensional smooth manifold. The space $(\Omega^\bullet(M), d)$ of differential forms on $M$ is a differential graded algebra with $d$ the de Rham differential. Let $\pi : E \to M$ be a vector bundle, the space $(\Omega^\bullet(M, E), d_E)$ of $E$-valued forms is a differential graded module over $(\Omega^\bullet(M), d)$, where $d_E$ is the differential induced by $d$. Let $A \in \Omega^1(M)$ and consider the endomorphism $e_A$ induced by $A$, defined by $e_A(\omega) = A \wedge \omega$ for all $\omega \in \Omega^\bullet(M, E)$. The pair $(\Omega^\bullet(M, E), d_E + e_A)$ is a 4-dgm for any $A$. Moreover, according to Theorem 17, $(\Omega^\bullet(M, E), d + e_A)$ is a 3-dgm if and only if for all $\omega$

$$d_{\text{End}}(e_A)(d + e_A) \omega = 0.$$ 

Since $d_{\text{End}}(e_A)(d + e_A)$ is an operator of degree 3, the identity $d_{\text{End}}(e_A)(d + e_A) \omega = 0$ holds for any $k$-form $\omega$, $k \geq 1$. Thus $(\Omega^\bullet(M, E), d + e_A)$ is a 3-dgm if and only if for any 0-form $\omega$

$$d_{\text{End}}(e_A)(d + e_A) \omega = d(A) \wedge (d_E(\omega) + A \wedge \omega) = 0.$$ 

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Similarly, it is easy to deduce from Theorem 17 that if $M$ is an $n$-dimensional smooth manifold and $n < m$, then $(\Omega^\bullet(M, E), d + e_A)$ is an $m$-complex. Let now $M$ be a $2n$-dimensional smooth manifold. Using local coordinates the 2-form $d_{\text{End}}(e_A)$ can be written as $F_{ij}dx^i \wedge dx^j$ where $F_{ij} = \partial_i A_j - \partial_j A_i$. Furthermore,

$$(F_{ij}dx^i \wedge dx^j)^n = \left( \sum_{\alpha \in P(2n)} \prod_{i=1}^n \text{sign}(\alpha)F_{a_i,b_i} \right) dx^1 \wedge \ldots \wedge dx^{2n},$$

where $P(2n)$ is the set of ordered pairings of $[2n] = \{1, \ldots, 2n\}$. Recall that a ordered pairing $\alpha \in P(2n)$ is a sequence $\{(a_i, b_i)\}_{i=1}^n$ such that $[2n] = \bigsqcup_{i=1}^n \{a_i, b_i\}$ and $a_i < b_i$.

By Theorem 17 $(\Omega^\bullet(M, E), d + e_A)$ is a $2n$-complex if and only if the 2-form $F_{ij}dx^i \wedge dx^j$ satisfies

$$\sum_{\alpha \in P(2n)} \text{sign}(\alpha) \prod_{i=1}^n F_{a_i,b_i} = 0.$$

Let $M$ be a complex manifold and consider the differential graded algebra $(\Omega(M), \wedge, \bar{\partial})$, where $\bar{\partial}$ is the Dolbault differential. Let $\pi : E \to M$ be a complex vector bundle, we consider $\Omega(M, E)$ the forms with values in $E$. Recall that a holomorphic structure on $E$ is given by a left differential graded module structure $(\Omega(M, E), \wedge_E, \bar{\partial}_E)$ over the 2-dga $(\Omega(M), \wedge, \bar{\partial})$. Suppose that on $(\Omega(M, E), \wedge_E, \bar{\partial}_E)$ there is a left $N$-differential graded module structure over the 2-dga $(\Omega(M), \wedge, \bar{\partial})$, then in this case we say that $E$ carries an $N$-holomorphic structure.

### 3 Discrete quantum theory

Generally speaking the following data constitute the basic set up for a (non-relativistic) quantum mechanical system: A finite dimensional Riemannian manifold $M$ which is thought as the configuration space of the quantum system; A Lagrangian function $L : TM \to \mathbb{R}$ which assigns weights to points in phase space.

Associated to this data is the Hilbert space $\mathcal{H}$ of quantum states which is usually taken to be $L^2(M)$, the space of square integrable functions on $M$. The dynamics of the quantum system is determined by operators $U_t : \mathcal{H} \to \mathcal{H}$, where $t \in \mathbb{R}$ represents time. The kernel $\omega_t$ of $U_t$ is such that

$$(U_t f)(y) = \int_M \omega_t(y, x) f(x) dx.$$

The key insight of Feynman is that $\omega_t(y, x)$ admits an integral representation

$$\omega_t(y, x) = \int e^{i \int_0^t L(\gamma, \dot{\gamma}) dt} D(\gamma).$$
The integral above runs over all paths $\gamma : [0, t] \to M$ such that $\gamma(0) = x$ and $\gamma(t) = y$. Making rigorous sense of this integral is the main obstacle in turning quantum mechanics a fully rigorous mathematical theory. Recall that a directed graph $\Gamma$ is given by: i) A set $V_\Gamma$ called the set of vertices, ii) A set $E_\Gamma$ called the set of edges and iii) A map $(s, t) : E_\Gamma \to V_\Gamma \times V_\Gamma$. Following the pattern above, one may define a discrete quantum mechanical system as being given by the following data

1. A directed graph $\Gamma$ (finite or infinite) which plays the role of configuration space.

2. A map $L : E_\Gamma \to \mathbb{R}$ called the Lagrangian map of the system.

The associated Hilbert space is $\mathcal{H} = \mathbb{C}^{V_\Gamma}$. The operators $U_n : \mathcal{H} \to \mathcal{H}$, where $n \in \mathbb{Z}$ represents discretized time are given by

$$(U_nf)(y) = \sum_{x \in V_\Gamma} \omega_n(y, x)f(x),$$

where the discretized kernel $\omega_n(y, x)$ admits the following representation

$$\omega_n(y, x) = \sum_{\gamma \in P_n(\Gamma, x, y)} \prod_{e \in \gamma} e^{iL(e)}.$$

Here $P_n(\Gamma, x, y)$ denotes the set of length $n$ paths in $\Gamma$ from $x$ to $y$, i.e., sequences $(e_1, \ldots, e_n)$ of edges in $\Gamma$ such that $s(e_i) = x$, $t(e_i) = s(e_{i+1})$, $i = 1, \ldots, n - 1$ and $t(e_n) = y$.

In Section 4 we show that the generalized Maurer-Cartan equation controlling deformations of $N$-dgas is determined by the kernel of a discrete quantum mechanical system $L$ which we proceed to introduce. Let us first explain our notation and conventions which generalize those introduced in [4].

For $s = (s_1, \ldots, s_n) \in \mathbb{N}^n$ we set $l(s) = n$, the length of the vector $s$, and $|s| = \sum_i s_i$. For $1 \leq i < n$, $s_{>i}$ denotes the vector given by $s_{>i} = (s_{i+1}, \ldots, s_n)$, for $1 \leq i \leq n$, $s_{<i}$ stands for $s_{<i} = (s_1, \ldots, s_{i-1})$, we also set $s_{>n} = s_{<1} = \emptyset$. $\mathbb{N}^{(\infty)}$ denotes the set $\bigcup_{n=0}^{(\infty)} \mathbb{N}^n$, where by convention $\mathbb{N}^{(0)} = \{\emptyset\}$.

We define maps $\delta_i, \eta_i : \mathbb{N}^n \to \{0, 1\}$, for $1 \leq i \leq n$, as follows

$$\delta_i(s) = \begin{cases} 1 & \text{if } s_i = 0, \\ 0 & \text{otherwise}. \end{cases} \quad \eta_i(s) = \begin{cases} 1 & \text{if } s_i \geq 1, \\ 0 & \text{otherwise}. \end{cases}$$

For an $M$-dga $A^\bullet$ and $e \in \text{End}(A^\bullet)$ and $s \in \mathbb{N}^n$ we define $e^{(s)} = e^{(s_1)} \cdots e^{(s_n)}$, where $e^{(l)} = d^l_{\text{End}}(e)$ if $l \geq 1$, $e^{(0)} = e$ and $e^{0} = 1$. In the case that $e_a \in \text{End}(A^\bullet)$ is given by

$$e_a(\phi) = a\phi, \quad \text{for } a \in A^1 \text{ fixed and all } \phi \in A^\bullet,$$
then \( e_a^{(l)} = d_{End}(e_a) \) reduces to \( e_a^{(l)} = e_{d^l(a)} \), thus

\[
e_a^{(s)} = e_a^{(s_1)} \cdots e_a^{(s_n)} = e_{d^{s_1}(a)} \cdots e_{d^{s_n}(a)}.
\]

Where \([k]\) denotes the set \( \{1, 2, \ldots, k\} \). For \( N \in \mathbb{N} \) we define \( E_N = \{s \in \mathbb{N}^{(\infty)} : |s| + l(s) \leq i\} \) and for \( s \in E_N \) we define \( N(s) \in \mathbb{Z} \) by \( N(s) = N - |s| - l(s) \).

We introduce the discrete quantum mechanical system \( L \) by

1. \( V_L = \mathbb{N}^{(\infty)} \).
2. There is a unique directed edge in \( L \) from vertex \( s \) to \( t \) if and only if \( t \in \{(0, s), (s + e_i)\} \) where \( e_i = (0, \ldots, 1_{i-th}, \ldots, 0) \in \mathbb{N}^{(s)} \), in this case we set \( \text{source}(e) = s \) and \( \text{target}(e) = t \).
3. Edges in \( L \) are weighted according to the following table

| source(e) | target(e) | weight(e) |
|-----------|-----------|-----------|
| s         | (0, s)    | 1         |
| s         | s         | \((-1)^{|s|+l(s)}\) |
| s         | (s + e_i) | \((-1)^{|s|+l(s)}\) |

The set \( P_N(\emptyset, s) \) consists of all paths \( \gamma = (e_1, \ldots, e_N) \), such that \( \text{source}(e_1) = \emptyset \), \( \text{target}(e_N) = s \) and \( \text{source}(e_{i+1}) = \text{target}(e_i) \). For \( \gamma \in P_N(\emptyset, s) \) we define the weight \( \omega(\gamma) \) of \( \gamma \) as

\[
\omega(\gamma) = \prod_{i=1}^{N} \omega(e_i).
\]

4 The \((M, N)\)-Maurer-Cartan equation

**Lemma 18** Let \( A^\ast \) be an \( M \)-dga and \( R \in Ob(\text{Artin}) \). We define \( d_{A_R} = d_A + e \) where \( e \in \text{Der}(A^\ast \otimes R_+) \) has degree 1, then

\[
(d_{A_R})^N = \sum_{s \in E_N} c(s, N)e^{(s)}d_A^{N(s)},
\]

where the coefficient \( c(s, N + 1) \) is equal to

\[
\delta_1(s)c(s_{>1}, N) + (-1)^{|s|+l(s)}c(s, N) + \sum_{i=1}^{l(s)} \eta_i(s)(-1)^{|s|+i-1}c(s - e_i, N),
\]

and \( c(0, 1) = c(0, 1) = 1 \).
Proof. We use an induction on $N$. For $N = 1$, since $E_1 = \{s = \emptyset, s = 0\}$

$$d_{AR} = \sum_{s \in E_1} c(s, 1)e^{s}d_{A}^{N(s)} = c(\emptyset, 1)e^{(0)}d_{A}^{1-|\emptyset|-l(\emptyset)} + c(0, 1)e^{(0)}d_{A}^{1-|0|-l(0)}$$

$$= c(\emptyset, 1)d_{A} + c(0, 1)e.$$

Suppose our formula holds for $N$ and let us check it for $N + 1$

$$(d_{AR})^{N+1} = (d_{A} + e)(d_{AR})^{N}$$

$$= (d_{A} + e)\left(\sum_{s \in EN} c(s, N)e^{s}d_{A}^{N(s)}\right)$$

$$= d_{A}\left(\sum_{s \in EN} c(s, N)e^{s}d_{A}^{N(s)}\right) + e\left(\sum_{s \in EN} c(s, N)e^{s}d_{A}^{N(s)}\right)$$

$$= \sum_{s \in EN} c(s, N)d_{A}(e^{s}d_{A}^{N(s)}) + \sum_{s \in EN} c(s, N)e^{s}d_{A}^{N(s)}.$$ \hspace{1cm} (2)

Consider the second term of the right hand side of (2)

$$\sum_{s \in EN} c(s, N)e^{s}d_{A}^{N(s)} = \sum_{s \in EN} c(s, N)e^{(0)}e^{s}d_{A}^{(s)}$$

$$= \sum_{t \in EN_{+1}} c(t_{>1}, N)e^{(t)}d_{A}^{N-|t_{>1}|-l(t_{>1})}$$ \hspace{1cm} (3)

$$= \sum_{s \in EN_{+1}} \delta_{1}(s)c(s_{>1}, N)e^{s}d_{A}^{N(s)+1}.$$ \hspace{1cm} (4)

In (3) we put $t = (0, s)$ thus $|t| = |s|$ and $l(t) = l(s) + 1$ and (4) is obtained by rewriting and changing $t$ by $s$. 

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Now consider the first term of the right hand side of (2)

\[ \sum_{s \in E_N} c(s, N)d_A(e^{(s)})d_A^{N(s)} = \sum_{\substack{s \in E_N \\ 1 \leq i \leq l(s)}} (-1)^{|s|+i-1}c(s, N)e^{(s+e_i)}d_A^{N(s)} \]

\[ + \sum_{s \in E_N} (-1)^{|s|+l(s)}c(s, N)e^{(s)}d_A^{N(s)+1} \quad (5) \]

\[ = \sum_{t \in E_{N+1}} \sum_{i=1}^{l(t)}(\sum_{s \in E_N} (-1)^{|s|+i-1}c(t-e_i, N)e^{(t)}d_A^{N-|t-e_i|-l(t)} \]

\[ + \sum_{s \in E_N} (-1)^{|s|+l(s)}c(s, N)e^{(s)}d_A^{N(s)+1} \quad (6) \]

\[ = \sum_{s \in E_{N+1}} \sum_{i=1}^{l(s)} \eta_i(s)(\sum_{s \in E_N} (-1)^{|s|+i-1}c(s-e_i, N)e^{(s)}d_A^{N(s)+1} \]

\[ + \sum_{s \in E_N} (-1)^{|s|+l(s)}c(s, N)e^{(s)}d_A^{N(s)+1}. \quad (7) \]

Putting \( t = s + e_i \) in the first term of (5) we obtain (6) and rewriting and changing \( t \) by \( s \) we obtain (7). Finally collecting similar terms in (4) and (7), and using the recurrence formula we get

\[ (d_A + e)^{N+1} = \sum_{s \in E_{N+1}} c(s, N+1)e^{(s)}d_A^{N(s)+1}, \]

thus the proof is completed. ♦

The following result generalizes Theorem 16. It provides an explicit formulae for the coefficients of the generalized Maurer-Cartan equation introduced below.

**Theorem 19** We have,

\[ (d_{A_R})^N = \sum_{k=0}^{N-1} c_k d_A^k, \]

where

\[ c_k = \sum_{\substack{s \in E_N \\ N(s) = k \\ s_i < M}} c(s, N)e^{(s)} \quad \text{and} \quad c(s, N) = \sum_{\gamma \in P_N(\emptyset, s)} \omega(\gamma). \]

**Proof.** One checks that the coefficients \( c(s, N) = \sum_{\gamma \in P_N(\emptyset, s)} \omega(\gamma) \) satisfy the recurrence formula of Lemma 18. For this one checks that \( P_{N+1}(\emptyset, s) \) is naturally partitioned in three blocks. The first block contains paths that are the composition of a path \( \gamma : \emptyset \to s \) in \( P_N(\emptyset, s_{>1}) \) with an edge \( s_{>1} \to (0, s_{>1}) \) and corresponds with the first term in (1). The second block consists of paths that are the composition of a path \( \gamma : \emptyset \to s \) in \( P_N(\emptyset, s) \)
with an edge $s \to s$ and corresponds with the second term in (1), finally the last block consists of paths that are the composition of a path $\gamma : \emptyset \to s - e_i$ in $P_N(\emptyset, s - e_i)$ with an edge $s - e_i \to s$ and corresponds with the last term of (1).

Let $A^\bullet$ be an $M$-dga and $A^\bullet_\mathcal{R}$ an $N$-deformation over $\mathcal{R}$ with $A^\bullet_\mathcal{R} = A^\bullet \otimes \mathcal{R}$. For $a \in A^1 \otimes \mathcal{R}_+$ we define $e_a : A^\bullet_\mathcal{R} \to A^\bullet_\mathcal{R}$ by

$$e_a(b) = ab - (-1)^{|b|}ba.$$ 

We are assuming that the product is not graded commutative. It is easy to see that $e_a$ is a derivation of degree 1 on $A^\bullet \otimes \mathcal{R}_+$. Then $d_{A_\mathcal{R}} = d_A + e_a$ is an $N$-deformation of $d_A$ iff $e_a$ satisfies the equation

$$\sum_{s \in E_N, s_i < M} c(s, N)e_a(s)d_A^{N-|s|-l(s)} = 0. \quad (8)$$

Equation (8) will be called the $(M, N)$-Maurer-Cartan equation. We closed this section by formally introducing the $(M, N)$-Maurer-Cartan functor $MC^N_M(A)$ which controls deformations of the differential $d_A$ of an $N$-dga $A^\bullet$.

**Definition 20** For $N \geq M$, $a \in A^1 \otimes \mathcal{R}_+$ is said to be an $(M, N)$-Maurer-Cartan element of $A^\bullet \otimes \mathcal{R}$ if $e_a$ satisfies the $(M, N)$-Maurer-Cartan equation (8). We say that $a$ is homotopic to $a'$, if $e_a$ is homotopic to $e_{a'}$ as morphisms of $N$-dgas.

**Definition 21** We define the $(M, N)$-Maurer-Cartan functor $MC^N_M(A) : \text{Artin} \to \text{Set}$ for each $M$-dga $A^\bullet$ over $k$. Functor $MC^N_M(A)$ is given by

1. Let $\mathcal{R}$ be an object of $\text{Artin}$. $MC^N_M(A)(\mathcal{R})$ is the set of homotopy classes of all $(M, N)$-Maurer-Cartan elements of $A^\bullet \otimes \mathcal{R}$.

2. If $\varphi : \mathcal{R} \to \mathcal{R}'$ be a morphism of the category $\text{Artin}$ and $a$ is an $(M, N)$-Maurer-Cartan element of $A^\bullet \otimes \mathcal{R}$, then $(1 \otimes \varphi)(a)$ is an $(M, N)$-Maurer-Cartan elements of $A^\bullet \otimes \mathcal{R}'$. Thus we obtain a map $\varphi_* : MC^N_M(A)(\mathcal{R}) \to MC^N_M(A)(\mathcal{R}')$.

Deformation theory of $K$-dgms over an $M$-dga can be defined similarly.

5 **Chern-Simons actions**

Let $(A^\bullet, m_A, d_A)$ be a 2-dga over $k$ and let $(M^\bullet, m_M, d_M)$ be a 2-dgm over $(A^\bullet, m_A, d_A)$, consider its $2K$-Maurer-Cartan equation, that is the equation that arises when we deform the 2-dgm $(M^\bullet, m_M, d_M)$ into a $2K$-dgm, $MC_{2K}(a) = (d_{End}(a) + a^2)^K = 0$, where $a \in End(M^\bullet)$ has degree 1. Let us assume that there exists a linear functional $\int : End(M^\bullet) \to k$ of degree $2K+1$, (i.e., $\int b = 0$ if $\bar{b} \neq 2K + 1$) satisfying the following conditions:
1. $\int$ is non degenerate, that is, $\int ab = 0$ for all $a$, then $b = 0$.

2. $\int d(a) = 0$ for all $a$, where $d = d_{\text{End}(M^*)}$.

3. $\int$ is cyclic, this is $\int a_1a_2 \cdots a_n = (-1)^{\delta_1+\delta_2+\cdots+\delta_n} \int a_2 \cdots a_na_1$.

We define the **Chern-Simons** functional $cs_{2,2K} : \text{End}(M^*) \to k$ by

$$cs_{2,2K}(a) = 2K \int \pi(#^{-1}(a(d_{\text{End}}(a) + a^2)^K)),$$

where

1. $k<a, d(a)>$ denotes the free $k$-algebra generated by symbols $a$ and $d(a)$.

2. $\#: k<a, d(a)> \to k<a, d(a)>$ is the linear map defined by

$$\#(a^{i_1}d(a)^{j_1} \cdots a^{i_k}d(a)^{j_k}) = (i_1 + \cdots + i_k + j_1 + \cdots + j_k)a^{i_1}d(a)^{j_1} \cdots a^{i_k}d(a)^{j_k}.$$

3. $\pi : k<a, d(a)> \to \text{End}(M^*)$ is the canonical projection.

For $K = 1$ we have that $cs_{2,2}(a)$ is equal to

$$2 \int \pi(#^{-1}(a(d(a) + a^2))) = 2 \int \pi(#^{-1}(a(d(a) + a^3))) = \int ad(a) + \frac{2}{3}a^3,$$

which is the Chern-Simons functional. In general we have the following result

**Theorem 22** Let $K \geq 1$ be an integer. The Chern-Simons functional $cs_{2,2K}$ is a Lagrangian for the $2K$-Maurer-Cartan equation, i.e., $a \in \text{End}^1(M^*)$ is a critical point of $cs_{2,2K}$ if and only if $(d(a) + a^2)^K = 0$.

**Proof.** We check that $\frac{\partial}{\partial \varepsilon}cs_{2,2K+2}(a + b\varepsilon) |_{\varepsilon=0} = (2K + 2) \int bMC_{2K+2}(a)$.

\[
\frac{\partial}{\partial \varepsilon}cs_{2,2K+2}(a + b\varepsilon) \bigg|_{\varepsilon=0} = \frac{\partial}{\partial \varepsilon}(2K + 2) \pi \int (#^{-1}((a + b\varepsilon)MC_{2K+2}(a + b\varepsilon)) |_{\varepsilon=0})
\]

\[
= (2K + 2) \int \pi(#^{-1}(\frac{\partial}{\partial \varepsilon}(a + b\varepsilon)MC_{2K}(a + b\varepsilon))MC_2(a + b\varepsilon)) |_{\varepsilon=0}
\]

\[
= (2K + 2) \int \pi(#^{-1}(\frac{\partial}{\partial \varepsilon}(a + b\varepsilon)MC_{2K}(a + b\varepsilon)) |_{\varepsilon=0} MC_2(a)
\]

\[+ (2K + 2) \int \pi(#^{-1}(aMC_{2K}(a) \frac{\partial}{\partial \varepsilon}MC_2(a + b\varepsilon))) |_{\varepsilon=0}. \]
By degree reasons the second term of (9) vanishes, the inductive hypothesis yields
\[
\frac{\partial}{\partial \varepsilon} cs_{2,2c+2}(a + b\varepsilon) \bigg|_{\varepsilon = 0} = (2K + 2) \int bMC_{2K}(a)MC_{2}(a)
\]
\[
= (2K + 2) \int bMC_{2K+2}(a).\leftarrow
\]

For \(K = 2, 3\) the Chern-Simons functional \(cs_{2,2K}(a)\) is given by
\[
\begin{align*}
cs_{2,4}(a) &= \int \frac{4}{3}a(d(a))^2 + 2a^3d(a) + \frac{4}{5}a^5. \\
\cs_{2,6}(a) &= \int \frac{3}{2}a(d(a))^3 + \frac{12}{5}a^3(d(a))^2 + \frac{6}{5}ad(a)a^2d(a) + 3a^5d(a) + \frac{6}{7}a^7.
\end{align*}
\]

Acknowledgement

We thank Nicolás Andruskiewitsch, Edmundo Castillo, Eddy Pariguan, Sylvie Paycha and Jim Stasheff for helpful suggestions. Thanks also to an anonymous referee for precise corrections.

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