OPTIMALITY CONDITIONS OF THE FIRST EIGENVALUE OF A FOURTH ORDER STEKLOV PROBLEM

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Abstract. In this paper we compute the first and second general domain variation of the first eigenvalue of a fourth order Steklov problem. We study optimality conditions for the ball among domains of given measure and among domains of given perimeter. We show that in both cases the ball is a local minimizer among all domains of equal measure and perimeter.

1. Introduction. We consider open bounded domains in $\mathbb{R}^n$, which are at least of class $C^{4,\alpha}$ for $0 < \alpha \leq 1$. Let

$$
\mathcal{R}(u, \Omega) := \frac{\int_{\Omega} |\Delta u|^2 \, dx}{\int_{\partial\Omega} |\partial_{\nu} u|^2 \, dS}
$$

(1)
on the space $H^2 \cap H^1_0(\Omega)$. We set $\mathcal{R} = \infty$ for $u \in H^2_0(\Omega)$. Further we define

$$
d_1(\Omega) = \inf \{ \mathcal{R}(u, \Omega) : u \in H^2 \cap H^1_0(\Omega) \}.
$$

(2)

Any minimizer $u$ solves the fourth order Steklov eigenvalue problem

$$
\Delta^2 u = 0 \quad \text{in} \quad \Omega
$$

(3)

$$
u = 0 \quad \text{on} \quad \partial\Omega
$$

(4)

$$
\Delta u - d_1(\Omega)\partial_{\nu} u = 0 \quad \text{on} \quad \partial\Omega.
$$

(5)

Note that the eigenvalue appears in the boundary condition. Let us summarize some properties which are well known by now. We refer to [1, 3, 4, 5] and [6] for more details.

- The minimizer $u \in [H^2 \cap H^1_0(\Omega)] \setminus H^2_0(\Omega)$ is positive in $\Omega$ and unique up to a multiplicative constant. Thus the first eigenvalue $d_1$ is simple (see Theorem 1 in [3]).
- Among all convex domains in $\mathbb{R}^n$ having the same measure or perimeter as the unit ball, there exists an optimal one, minimizing $d_1$ (see Theorem 4.6 in [5] and Theorem 5 in [1]).
- Bucur, Ferrero and Gazzola showed that the eigenvalue is differentiable for the perturbed unit ball (see p. 19 in [4]). In complete analogy we can show that the first eigenvalue is differentiable with respect to smooth perturbations.

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Moreover, (3) - (5) admits infinitely many (countable) eigenvalues. The only
eigenfunction of one sign is the one corresponding to the first eigenvalue $d_1(\Omega)$.
All eigenfunctions are smooth inside $\Omega$, and up to the boundary, they are as
smooth as the boundary permits (i.e. $u \in C^{4,\alpha}(\Omega)$) (see Theorem 1 in [3] and
Theorem 1 in [6]).

In a recent paper Raulot and Savo studied the Steklov problem on compact
manifolds with boundary (see [10]). They obtained estimates for the first
Steklov eigenvalue.

The case of higher Steklov eigenvalues $(\lambda_k)_k$ is less investigated. However in
a recent paper Liu proved, among other results, some asymptotic formulas
for $\lambda_k$ as $k \to \infty$ (see [9]). In particular the author investigated asymptotic
formula for rectangular parallelepips.

We are interested in the domain dependence of $d_1(\Omega)$. This problem has recently
drawn some attention, since it is known that the ball is not the absolute minimizer
of $d_1(\Omega)$ among all domains of given measure. Kuttler [8] could show - in two
dimensions - that a square has a first eigenvalue $d_1$, which is strictly smaller than
the one of a disc. Ferrero, Gazzola and Weth (see [6] (1.14) and (1.15)) could
improve Kuttler's inequality. More recently Antunes and Gazzola [1] gave numerical
evidence that the optimal planar shape is the regular pentagon.

We are interested in clarifying the role of the ball. Bucur, Ferrero and Gazzola
[4] could show that the ball is a critical point of $d_1$ among all measure preserving
smooth perturbations. One of our main results is:

**Theorem 1.1.** The ball is a strict local minimizer among all domains of equal
measure.

Antunes and Gazzola [1] suggest that among all convex planar domains with a
fixed perimeter $d_1$ is minimized by the disk. We could also show that the ball is a
local minimizer for perturbations which are perimeter preserving in n dimensions.

**Theorem 1.2.** The ball is a strict local minimizer among all domains of equal
perimeter.

The paper is organized as follows. In Section 2 we introduce some notation
needed throughout the paper. From Section 3 on we consider any domain $\Omega$ of
class $C^{4,\alpha}$ at least. Domains of this class will be referred to as smooth. Such a
domain will be imbedded into a family of domains $(\Omega_t)_t$, where $\Omega_t$ is smooth as
well and $\Omega := \Omega_0$. The construction of $\Omega_t$ is given in Section 3. Throughout the
paper we will refer to $\Omega_t$ as a "smoothly perturbed" domain. There will be two
families under consideration. The first one has prescribed measure in the sense that $|\Omega_t| = |\Omega| + o(t^2)$ (see Lemma 3.1). The second family has prescribed perimeter
in the sense that $|\partial \Omega_t| = |\partial \Omega| + o(t^2)$ (Lemma 3.2). Finally we compute the first
domain variation of $d_1$ among domains of given measure and among domains of
given perimeter (see Theorem 3.7). From Section 4 on the unperturbed domain $\Omega$
is the unit ball B. We compute also the first and second variation of $d_1(B)$ among
domains of equal measure and among all domains of equal perimeter. Using
results of the previous sections we provide a formula for $d_1''(B)$

- in the measure preserving case (Theorem 4.5)
- in the perimeter preserving case (Theorem 5.2).
In particular the second variation in both cases is a quadratic form in $u'$ alone. Then we discuss the sign of this quadratic form and derive the strict positivity (Theorem 4.6 Theorem 5.3).

2. Notation. The outer unit normal vector field will be denoted by $\nu$. For smooth vector fields $\theta, \eta : \Omega \to \mathbb{R}^n$ we write

$$\theta = (\theta_1(x), \theta_2(x), \ldots, \theta_n(x)) \quad \text{and} \quad \eta = (\eta_1(x), \eta_2(x), \ldots, \eta_n(x)).$$

Let $\vartheta : \Omega \to \mathbb{R}^n$ be a smooth vectorfield, then

$$\eta \cdot \vartheta := \sum_{i=1}^{n} \eta_i \vartheta_i$$

and

$$\eta \cdot (D\vartheta) := \sum_{i,j=1}^{n} \eta_j \partial_i \vartheta_j.$$

Furthermore we will write

$$D\vartheta : D\vartheta := \sum_{i,j=1}^{n} \partial_i \vartheta_j \vartheta_i.$$

Let $(\tau_i)_i$ the basis of the tangent space in a point in $\partial \Omega$. Then for any smooth vector field $\theta$ we have the decomposition

$$\theta = \theta^\tau + (\theta \cdot \nu) \nu \quad \text{where} \quad \theta^\tau = \sum_{i=1}^{n} (\theta \cdot \tau_i) \tau_i$$

(6)

on $\partial \Omega$. We denote by

$$\nabla f = \nabla f - \nu \partial_{\nu} f$$

(7)

the tangential gradient of a smooth function $f : \Omega \to \mathbb{R}$.

For a smooth vector field $\theta : \Omega \to \mathbb{R}^n$ we define

$$\text{div} \theta = \text{div} \theta - \nu \cdot (D\theta)$$

on $\partial \Omega$

as the tangential divergence. Here $\text{div} \theta$ denotes the usual divergence of $\theta$ in $\mathbb{R}^n$.

The mean curvature of $\Gamma$ is defined as

$$(n-1)H := \text{div} \nu.$$

In particular we have $H = \frac{1}{R}$ if $\Omega$ is a ball of radius $R$.

We will frequently use the Gauss theorem on surfaces.

**Lemma 2.1.** Let $f \in C^1(\partial \Omega)$ and $\theta \in C^1(\Omega, \mathbb{R}^n)$, then

$$\int_{\partial \Omega} f \text{div} \theta \, dS = -\int_{\partial \Omega} \theta \cdot \nabla f \, dS + (n-1) \int_{\partial \Omega} f \, H \cdot \nu \, dS.$$  

(8)

By [7] (Proposition 5.4.12) we know

**Lemma 2.2.** Let $\Omega$ and $u \in C^2(\Omega)$ then

$$\Delta u = \Delta_{\Gamma} u + (n-1)H \partial_{\nu} u + \partial_{\nu}^2 u$$

on $\partial \Omega$,

(9)

where $\partial_{\nu}^2 u = \nu \cdot (D^2 u \nu)$ and where $\Delta_{\Gamma} u$ denotes the Laplace-Beltrami operator on $\partial \Omega$.

**Remark 1.** If $u = 0$ on $\partial \Omega$, then

$$\Delta u = \partial_{\nu}^2 u + (n-1)H \partial_{\nu} u$$

on $\partial \Omega$.

(10)

We also recall the representation of the Laplace operator in polar coordinates

$$\Delta u = \partial^2_{\nu} u + \frac{n-1}{r} \partial_r u + \frac{1}{r^2} \Delta_{\Gamma} u.$$  

(11)
3. Domain perturbation.

3.1. Measure and perimeter preserving domains. Let \( \Omega_t \) be a family of perturbations of the domain \( \Omega \subset \mathbb{R}^n \) of the form

\[
\Omega_t := \{ y = x + t\theta(x) + \frac{t^2}{2}\eta(x) + o(t^2) : x \in \Omega \},
\]

where \( o(t^2) \) collects all terms such that \( \frac{o(t^2)}{t^2} \to 0 \) for \( t \to 0 \). For \( t \in (-t_0, t_0) \) and \( t_0 > 0 \) sufficiently small, \( \Phi(t, \cdot) : \Omega \to \Omega_t \) is a diffeomorphism.

By Jacobi’s formula we have for small \( t \)

\[
\det (D\Phi) = 1 + t \text{div} \theta + \frac{t^2}{2} \left( (\text{div} \theta)^2 - D\theta : D\theta + \text{div} \eta \right) + o(t^2).
\]  
(13)

In the sequel we will consider perturbation which are measure resp. perimeter preserving. A perturbation will be measure preserving, iff

\[
|\Omega_t| = |\Omega| + o(t^2),
\]

where \( |\Omega| \) is the \( n \)-dimensional Lebesgue measure of \( \Omega \). In particular we have

\[
\left. \frac{d}{dt} \right|_{t=0} |\Omega_t| = 0 \quad \text{and} \quad \left. \frac{d^2}{dt^2} \right|_{t=0} |\Omega_t| = 0.
\]  
(14)

In view of (13) this implies

**Lemma 3.1.** A measure preserving perturbation satisfies

(i)

\[
\int_{\partial\Omega} \theta \cdot \nu \, dS = 0.
\]  
(15)

and

(ii)

\[
\int_{\Omega} ((\text{div} \theta)^2 - D\theta : D\theta + \text{div} \eta) \, dx = 0.
\]  
(16)

By Lemma 2.1 in [2] we get the following equivalence.

**Remark 2.** Let \( \theta, \eta \in C^{1,1}(\Omega, \mathbb{R}^n) \). Then

\[
\int_{\Omega} ((\text{div} \theta)^2 - D\theta : D\theta + \text{div} \eta) \, dx = 0
\]

is equivalent to

\[
\int_{\partial\Omega} (\theta \cdot \nu) \text{div} \theta \, dS - \int_{\partial\Omega} \theta \cdot (D\theta \nu) \, dS + \int_{\partial\Omega} \eta \cdot \nu \, dS = 0.
\]  
(17)

Similarly we define the class of perimeter preserving perturbations. A perturbation will be perimeter preserving, iff

\[
|\partial\Omega_t| = |\partial\Omega| + o(t^2).
\]

Thus we have

\[
\left. \frac{d}{dt} \right|_{t=0} |\partial\Omega_t| = 0 \quad \text{and} \quad \left. \frac{d^2}{dt^2} \right|_{t=0} |\partial\Omega_t| = 0,
\]

where \( |\partial\Omega| \) is the \((n-1)\)-dimensional Hausdorff measure of \( \partial\Omega \).

We get the following result for perimeter preserving perturbations (see also [2] Section 2.2.1):
Lemma 3.2. (i) A perimeter preserving perturbation satisfies
\[(n - 1) \int_{\partial \Omega} H \theta \cdot \nu \, dS = 0.\] (18)

(ii) For the unit ball a perimeter preserving perturbation satisfies
\[\int_{\partial B} |\nabla \Gamma(\theta \cdot \nu)|^2 - (n - 1)(\theta \cdot \nu)^2 \, dS\]
\[+ (n - 1) \int_{\partial B} (\theta \cdot \nu) \text{div} \theta - \theta \cdot (D\theta \nu) + \eta \cdot \nu \, dS = 0.\] (19)

3.2. Properties of the variation of the outer normal vector. Let \( \nu_t \) be the outer normal on \( \partial \Omega_t \), then we set
\[\nu'(0, x) = \left. \frac{\partial \nu(t, \Phi(t, x))}{\partial t} \right|_{t=0} \quad \text{and} \quad \nu''(0, x) = \left. \frac{\partial^2 \nu(t, \Phi(t, x))}{\partial t^2} \right|_{t=0}.\]

For brevity, we write \( \nu' \) and \( \nu'' \) rather than \( \nu'(0, x) \) and \( \nu''(0, x) \). Let \( x \in \overline{\Omega} \), then we define \( \delta \) as the distance function to the boundary
\[\delta(x) := \inf \{|x - z|; z \in \partial \Omega\}.\]

For a sufficiently small tubular neighbourhood of \( \partial \Omega \) we define \( \tilde{\nu} := \nabla \delta(x) \) as a smooth extension of \( \nu \). The extension of \( \nu \) implies also an extension of \( \nu' \) into this tubular neighbourhood.

A direct consequence of this extension is the formula
\[\nu D\nu = \sum_{i=1}^{n} \partial_j \nu_i \nu_i = \frac{1}{2} \nabla(|\nu|^2) = 0.\] (20)

Indeed, this follows from the fact that \( |\nabla \delta(x)| = 1 \) for all \( x \) in a sufficiently small tubular neighbourhood. This also implies
\[\text{div} \, \Gamma \nu = \text{div} \nu.\] (21)

Since \( \nu_t \cdot \nu = 1 \) on \( \partial \Omega_t \) we have
\[1 = \nu(t, \Phi(t, x)) \cdot \nu(t, \Phi(t, x)) \quad \forall x \in \partial \Omega\]
for all sufficiently small \( t \). Thus
\[0 = \frac{d}{dt} \left[ \nu(t, \Phi(t, x)) \cdot \nu(t, \Phi(t, x)) \right] \bigg|_{t=0} = 2\partial_t \nu \cdot \nu + 2(D\nu \theta) \cdot \nu.\]

By (20) this implies
\[\nu' \cdot \nu = 0.\] (22)

Similarly we have
\[\nu \cdot \nu'' = -(\nu')^2.\] (23)

Since \( u = 0 \) on \( \partial \Omega \), the decomposition of \( \nabla u|_{\partial \Omega} \) gives
\[\nabla u \cdot \nu' = (\nabla u \cdot \tau_i) \tau_i \cdot \nu' + (\nabla u \cdot \nu) \nu \cdot \nu' = 0 \quad \text{on} \ \partial \Omega,\] (24)

where summation over repeated indices is understood. Here \( \tau_i \) denotes again an orthonormal basis of the tangent space in some point of \( \partial \Omega \).

Finally we like to mention that
\[\nu' = -\nabla \Gamma(\theta \cdot \nu)\] (see e.g. [7] Proposition 5.4.14).
Lemma 3.3. Let $\partial \Omega$ be smooth and $\nu$ and $\nu'$ as above. Then

$$\nu \cdot (D\nu') = 0 \quad \text{on} \partial \Omega.$$  

Proof. First we use (25) and (7) then we get by differentiating

$$\nu_i \partial_i \nu_j' \nu_j = -\nu_i \partial_i [\partial_j (\theta \cdot \nu) - \nu_j \nu_k \partial_k (\theta \cdot \nu)]\nu_j$$

$$= -\nu_i \partial_i \partial_j (\theta \cdot \nu) \nu_j + \nu_i [\partial_i \nu_j \nu_k \partial_k (\theta \nu)]\nu_j$$

$$+ \nu_i [\nu_j \partial_i \nu_k \partial_k (\theta \cdot \nu)]\nu_j + \nu_i [\nu_j \nu_k \partial_i \partial_k (\theta \cdot \nu)]\nu_j$$

$$= + \nu_i [\partial_i \nu_j \nu_k \partial_j (\theta \cdot \nu)]\nu_j + \nu_i [\nu_j \partial_i \nu_k \partial_k (\theta \cdot \nu)]\nu_j.$$

Then (20) gives the claim. \qed

3.3. General first domain variation. A. Henrot and M. Pierre ([7] Chapter 5) proved a general formula for the first variation of a domain functional.

Theorem 3.4. Let $\theta$ and $\Omega$ be as in (12). For $t \in (0, 1]$, let $g(t) \in L^1(\Omega_t)$ be differentiable in $t = 0$ with $g(0) \in W^{1,1}(\Omega)$. Then the first variation is

$$\frac{d}{dt} \int_{\Omega_t} g(t) \, dy = \int_{\Omega} \partial_t g(0) \, dx + \int_{\partial \Omega} g(0) \theta(x) \cdot \nu \, dS.$$

For the first domain variation for boundary integrals (see [7] p. 191), we get the following result:

Theorem 3.5. Let $\theta$ and $\Omega$ be as in (12). For $t \in (0, 1]$, let $g(t) \in L^1(\Omega_t)$ be differentiable in $t = 0$ with $g(0) \in W^{1,1}(\Omega)$. Then we get

$$\frac{d}{dt} \int_{\partial \Omega_t} g(t) \, dS(y) = \int_{\partial \Omega} [\partial_t g(0) + \theta \cdot \nu \partial_r g(0) + (n - 1)H \theta \cdot \nu g(0)] \, dS.$$

With these general first domain variations we can recover Lemma 3.1 (i) resp. Lemma 3.2 (i) by setting $g(t) = 1$.

3.4. First domain variation. We apply the previous formulas to our Steklov eigenvalue problem. For that we consider the Steklov eigenvalue

$$d_1(\Omega_t) = \inf \{ \mathcal{R}(u, \Omega_t) : u \in [H^2 \cap H_0^1(\Omega_t)] \setminus H_0^2(\Omega_t) \}$$

for a family ($\Omega_t$)$_t$ of perturbed domains in the sense of Section 3.1. We set

$$d_1'(\Omega) := \frac{d}{dt} \bigg|_{t=0} d_1(\Omega_t) \quad \text{and} \quad d_1''(\Omega) := \frac{d^2}{dt^2} \bigg|_{t=0} d_1(\Omega_t)$$

as the first resp. the second domain variation.

Further, let $u(t, y)$ be the corresponding eigenfunction of $d_1(\Omega_t)$ with $y \in \Omega_t$. We will abbreviate and write

$$u(t) := u(t, y).$$

Then $u(t)$ solves the Euler Lagrange equation

$$\Delta^2 u(t) = 0 \quad \text{in} \quad \Omega_t \quad \text{(26)}$$

$$u(t) = 0 \quad \text{in} \quad \partial \Omega_t \quad \text{(27)}$$

$$\Delta u(t) - d_1(\Omega_t) \partial_r u(t) = 0 \quad \text{in} \quad \partial \Omega_t. \quad \text{(28)}$$

Let us now introduce the shape derivative $u'$ as

$$\frac{\partial u(t, \Phi(t, x))}{\partial t} \bigg|_{t=0}.$$
First we investigate the behavior of the shape derivative on $\Omega$ and on $\partial \Omega$.

**Lemma 3.6.** Let $u(t)$ be a solution of (26)-(28). The shape derivative $u'$ satisfies

\[
\Delta^2 u' = 0 \quad \text{in} \quad \Omega \quad \text{(29)}
\]

\[
u' \cdot \nabla u = 0 \quad \text{in} \quad \partial \Omega \quad \text{(30)}
\]

\[
\Delta u' - d_1(\Omega)\partial_{\nu'} u' - d_1(\Omega)\nu \cdot (D^2 u \theta) = - (\nabla \Delta u) \cdot \theta + d_1'(\Omega)\partial_{\nu} u \quad \text{in} \quad \partial \Omega. \quad \text{(31)}
\]

**Proof.** We give the proof of (31). (29) and (30) follows with the same technique. First we note that $y = x + t\theta(x)$ for $x \in \partial \Omega$. Thus we may rewrite (28) as equation on $\partial \Omega$ for all $|t| < t_0$:

\[
\Delta u(t, \Phi(t,x)) - d_1(\Omega)\nu(t, \Phi(t,x)) \cdot \nabla u(t, \Phi(t,x)) = 0.
\]

We differentiate this equation with respect to $t$ and get

\[
0 = \partial_t \Delta u(t, \Phi(t,x)) + \nabla(\Delta u(t, \Phi(t,x)) \cdot \partial_t \Phi(t,x))
\]

\[
- \frac{d}{dt} \bigg|_{t=0} d_1(\Omega_t)\nu(t, \Phi(t,x)) \cdot \nabla u(t, \Phi(t,x))
\]

\[
- d_1(\Omega_t) \partial_t \nu(t, \Phi(t,x)) \cdot \nabla u(t, \Phi(t,x))
\]

\[
- d_1(\Omega_t) [D\nu(t, \Phi(t,x)) \partial_t \Phi(t,x)] \cdot \nabla u(t, \Phi(t,x))
\]

\[
- d_1(\Omega_t) \nu(t, \Phi(t,x)) \cdot \partial_t \nabla u(t, \Phi(t,x))
\]

\[
- d_1(\Omega_t) \nu(t, \Phi(t,x)) \cdot [D^2 u(t, \Phi(t,x)) \partial_t \Phi(t,x)].
\]

In $t = 0$ this gives

\[
0 = \Delta u' + (\nabla \Delta u) \cdot \theta - \frac{d}{dt} \bigg|_{t=0} d_1(\Omega_t)\partial_{\nu'} u
\]

\[
- d_1(\Omega)\nabla u \cdot \nu' - d_1(\Omega)(D\nu \theta) \cdot \nabla u - d_1(\Omega)\partial_{\nu} u' - d_1(\Omega)(D^2 u \theta) \cdot \nu.
\]

Since $u = 0$ on $\partial \Omega$, $\nabla u = \partial_{\nu} u$ on $\partial \Omega$. So we get

\[
(D\nu \theta) \cdot \nabla u = \partial_{\nu} u \nu_k \partial_j \nu_i \theta_j.
\]

By (20) this expression and by (24) $d_1(\Omega)\nabla u \cdot \nu'$ is zero. This proves Lemma 3.6. $\square$

**Theorem 3.7.** 1. Let $\Omega_t$ be a family of measure preserving perturbations of $\Omega$ as described in (12). Then $\Omega$ is a critical point of the energy $d_1(\Omega_t)$, i.e. $d_1'(\Omega) = 0$, if and only if

\[
(\Delta u)^2 - (n-1)d_1(\Omega)(\partial_{\nu} u)^2 H - 2(\partial_{\nu} u)\partial_{\nu} \Delta u = \text{const} \quad \text{on} \quad \partial \Omega. \quad \text{(33)}
\]

2. Let $\Omega_t$ be a family of perimeter preserving perturbations of $\Omega$ as described in (12) and $c \in \mathbb{R}$. Then $\Omega$ is a critical point of the energy $d_1(\Omega_t)$, if and only if

\[
\frac{1}{(n-1)}(\Delta u)^2 - d_1(\Omega)(\partial_{\nu} u)^2 - \frac{2}{(n-1)}(\partial_{\nu} u)\partial_{\nu} \Delta u = cH \quad \text{on} \quad \partial \Omega. \quad \text{(34)}
\]

**Proof.** By Theorem 3.4 the numerator integral of the Rayleigh quotient in (1) becomes

\[
\frac{d}{dt} \bigg|_{t=0} \int_{\Omega_t} [\Delta u(t,y)]^2 \, dy = 2 \int_{\Omega} \Delta u \Delta u' \, dx + \int_{\partial \Omega} |\Delta u|^2 \theta \cdot \nu \, dS
\]

\[
= 2 \int_{\partial \Omega} \partial_{\nu} u \Delta u' \, dS + \int_{\partial \Omega} |\Delta u|^2 \theta \cdot \nu \, dS.
\]

For the last equality we used the biharmonicity of $u'$ and $u = 0$ on $\partial \Omega$. 

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From (28) we deduce that
\[
\frac{d}{dt}\bigg|_{t=0} \int_{\partial \Omega} |\partial_{\nu} u(t)|^2 \, dS(t)
\]
\[
= \int_{\partial \Omega} 2(\partial_{\nu} u)(\partial_{\nu} u' + \nabla u \cdot \nu') + (\theta \cdot \nu)[\nabla(\partial_{\nu} u)^2 + (\partial_{\nu} u)^2\text{div}\nu] \, dS
\]
\[
= 2 \int_{\partial \Omega} \partial_{\nu} u \partial_{\nu} u' \, dS + 2 \int_{\partial \Omega} \theta \cdot \nu \partial_{\nu} u \partial_{\nu}^2 u \, dS + (n - 1) \int_{\partial \Omega} H \theta \cdot \nu (\partial_{\nu} u)^2 \, dS.
\]

Note that the eigenfunction is unique up to scaling, so we can normalize \( \int_{\partial \Omega} |\partial_{\nu} u|^2 \, dS = 1 \) and get
\[
\frac{d}{dt}\bigg|_{t=0} d_1(\Omega_t) = \int_{\partial \Omega} 2\partial_{\nu} u \Delta u' \, dS + \int_{\partial \Omega} |\Delta u|^2 \theta \cdot \nu \, dS - d_1(\Omega) \int_{\partial \Omega} 2(\partial_{\nu} u)(\partial_{\nu} u')
\]
\[
- d_1(\Omega) \int_{\partial \Omega} (\theta \cdot \nu)[2(\partial_{\nu} u)\partial_{\nu}^2 u + (n - 1)(\partial_{\nu} u)^2 H] \, dS.
\]

Next we use (31). Then we get
\[
\frac{d}{dt}\bigg|_{t=0} d_1(\Omega_t) = 2 \int_{\partial \Omega} d_1(\Omega) \partial_{\nu} u \nu \cdot (D^2 u \theta) - \partial_{\nu} u \theta \cdot \nabla(\Delta u) \, dS
\]
\[
+ \int_{\partial \Omega} |\Delta u|^2 \theta \cdot \nu \, dS - 2d_1(\Omega) \int_{\partial \Omega} \partial_{\nu}^2 u \partial_{\nu} u (\theta \cdot \nu) \, dS
\]
\[
- (n - 1)d_1(\Omega) \int_{\partial \Omega} |\partial_{\nu} u|^2 H \theta \cdot \nu \, dS.
\]

We consider the first integral on the right hand side. Since \( \theta^\tau = \theta - (\theta \cdot \nu)\nu \) (see (6)) we get
\[
d_1(\Omega) \partial_{\nu} u \nu \cdot (D^2 u \theta) - \partial_{\nu} u \theta \cdot \nabla(\Delta u) = d_1(\Omega) \partial_{\nu} u \nu \cdot (D^2 u \theta^\tau) - \partial_{\nu} u \theta^\tau \cdot \nabla(\Delta u)
\]
\[
+ (\theta \cdot \nu) (d_1(\Omega) \partial_{\nu} u \partial_{\nu}^2 u - \partial_{\nu} u \partial_{\nu} u(\Delta u)) .
\]

From (28) we deduce that
\[
d_1(\Omega) \partial_{\nu} u \nu \cdot (D^2 u \theta^\tau) - \partial_{\nu} u \theta^\tau \cdot \nabla(\Delta u) = 0
\] on \( \partial \Omega \). From this we derive
\[
\frac{d}{dt}\bigg|_{t=0} d_1(\Omega_t) = \int_{\partial \Omega} |\Delta u|^2 \theta \cdot \nu \, dS - (n - 1)d_1(\Omega) \int_{\partial \Omega} H (\partial_{\nu} u)^2 \theta \cdot \nu \, dS
\]
\[
- 2 \int_{\partial \Omega} \partial_{\nu} u(\theta \cdot \nu)\partial_{\nu} u \, dS
\]
\[
= \int_{\partial \Omega} [|\Delta u|^2 - (n - 1)d_1(\Omega) H (\partial_{\nu} u)^2 - 2\partial_{\nu} u \partial_{\nu} u(\Delta u)] \theta \cdot \nu \, dS.
\]

So far we did not use the fact that the vector fields are either measure or perimeter preserving. Assume we consider vector fields which are measure preserving in the sense of (15). Assume \( d_1'(\Omega) = 0 \). Then this is equivalent to the fact that (33) holds. This is a consequence of the fundamental lemma of the calculus of variations. Similarly assume the vector fields are perimeter preserving in the sense of (18). Then \( d_1'(\Omega) = 0 \) for all such vector fields implies (34).

**Remark 3.** We observe that the first variation does not depend on \( \eta \) or on the tangential components of the vector field \( \theta \).
4. Domain variation for the unit ball.

4.1. First variation for the ball. From now on we consider a family $B_t$ of perturbations of the unit ball $B$ of the form as in (12).

Some particular properties hold for the unit ball. Notice that the mean curvature simplifies to $H = 1$. In [3] it was shown that the first eigenfunction $u$ is

$$u(x) = 1 - |x|^2.$$ 

We easily compute

$$\frac{d}{dt}u(B_t) = n, \quad \Delta u = -2n, \quad \nabla u = -2\nu,$$

$$\partial_i\partial_j u = -2\delta_{ij} \quad \text{and} \quad D\nu = \left(\delta_{ij} - \nu_i\nu_j\right)$$ 

(35)

For the shape derivative we get $\Delta^2 u' = 0$ in $B$. The boundary conditions (30) and (31) are with the help of (35)

$$u' = 2\theta \cdot \nu \quad \text{and} \quad \Delta u' + n \partial_\nu u' + 2n\theta \cdot \nu = -2d_1'(B) \quad \text{in} \quad \partial B.$$ 

(37)

We recall the mean value formula for harmonic functions on a ball.

**Lemma 4.1.** Let $w$ be harmonic in a ball $B_1$. Then

$$\int_{\partial B_R} w \, dS - \frac{n}{R} \int_{B_R} w \, dx = 0$$

for all $0 < R \leq 1$.

**Proof.** Any harmonic function $w \in C^0(B_1)$ satisfies the mean value formula

$$\omega_n w(0) = \frac{1}{R^n} \int_{B_R} w \, dx \quad \text{for all} \quad 0 < R < 1.$$

We differentiate this expression with respect to $R$. This gives the claim for $0 < R < 1$. By continuity of $w$ this holds for $0 < R \leq 1$. \qed

**Theorem 4.2.** Let $(B_t)_t$ for $|t| \leq t_0$ be a family of perturbations of the unit ball of given measure or given perimeter. Then the unit ball $B_0 = B$ is a critical point, i.e. $d_1'(B_t)|_{t=0} = 0$ holds.

**Proof.** We integrate the second equation in (37) and get

$$2|\partial B| d_1'(B) = \int_{\partial B} -\Delta u' + n \partial_\nu u' - 2n\theta \cdot \nu \, dS.$$

We apply Lemma 4.1 for $\Delta u'$. Then the first variation $d_1'(B)$ is

$$2|\partial B| d_1'(B) = -2n \int_{\partial B} \theta \cdot \nu \, dS.$$ 

(38)

The right hand side is zero if either the perturbations are measure preserving (see (15)) or if they are perimeter preserving (see (18)) since $H = 1$. \qed

Now (37) takes the following form

$$u' = 2\theta \cdot \nu \quad \text{and} \quad \Delta u' = n \partial_\nu u' - n u' \quad \text{in} \quad \partial B.$$ 

(39)
4.2. Second variation for the ball. We compute the second domain variation to investigate local properties of the critical point $B$. Two strategies are possible. Starting with the Rayleigh quotient (1) we could derive formulas as in Theorem 3.4 and in Theorem 3.5 for the second derivation of the numerator and denominator in (1). Alternatively we can use (28) where $d_1(B,t)$ appears explicitly in the boundary condition. It turns out that the second choice is simpler.

We first use (28) in order to have an explicit expression for $d''(B)$. For that we use (32) and differentiate it again with respect to $t$. In $t = 0$ we then get

$$0 = \Delta u'' + 2\theta \cdot \nabla \Delta u' + \eta \cdot \nabla \Delta u + (D^2(\Delta u)) \cdot \theta - d''_1(B) \partial_t u$$

$$- d_1(B) (\nu'' + 2\theta \nu' + \eta \nu + (D^2\nu) \cdot \theta) \cdot \nabla u$$

$$- 2d_1(B) (\nu' + \theta \nu) \cdot (\nabla u' + D^2u\theta)$$

$$- d_1(B) (\partial_t u'' + 2(D^2u\theta) \cdot \nu + (D^2\eta) \cdot \nu + \nu \cdot ((D^2\nabla u\theta)) \cdot \theta.$$

The explicitly known eigenfunction $u$ (see (35)) simplifies this expression to

$$0 = \Delta u'' + 2\theta \cdot \nabla \Delta u' + 2d''_1(B)$$

$$+ 2n (\nu'' + 2\theta \nu') \cdot \nu + (\eta D\nu) \cdot \nu + ((D^2\nu) \cdot \theta) \cdot \nu$$

$$- 2n (\nu' \cdot \nabla u' + (\theta D\nu) \cdot \nabla u' - 2\theta \cdot \nu' - 2(\theta D\nu) \cdot \theta)$$

$$- n (\partial_t u'' + 2(D^2u\theta) \cdot \nu - 2\eta \cdot \nu).$$

We compute some of the terms in this sum explicitly for the ball.

1. By (23), (25) and then (39) we have

$$\nu'' \cdot \nu = -|\nabla_r (\theta \cdot \nu)|^2 = -\frac{1}{4} |\nabla_r u'|^2.$$

2. By (6)

$$4n(\theta D\nu') \cdot \nu + 4n\theta \cdot \nu'$$

$$= 4n(\theta^T D\nu') \cdot \nu + 4n\theta^T \cdot \nu + 4n\theta \cdot \nu (\nu D\nu') \cdot \nu + 4n(\theta \cdot \nu) \nu \cdot \nu'.$$

The third term in the sum vanishes by Lemma 3.3 and the fourth vanishes by (22). For the first two terms note that by (35)

$$4n(\theta^T D\nu') \cdot \nu + 4n\theta^T \cdot \nu$$

$$= 4n\theta^T \cdot \nabla (\nu \cdot \nu') - 4n(\theta^T D\nu') \cdot \nu' + 4n\theta^T \cdot \nu'$$

$$= 4n\theta^T \cdot \nabla (\nu \cdot \nu') - 4n\theta^T \cdot \nu' + 4n\theta^T \cdot \nu \nu' + 4n\theta^T \cdot \nu'$$

Since $\nu \cdot \nu' = 0$ in $\partial B$ the tangential derivative $\theta^T \cdot \nabla (\nu \cdot \nu') = 0$. The second and fourth term cancel and the third term vanishes since $\theta^T \cdot \nu = 0 = \nu \cdot \nu'$. So

$$4n(\theta D\nu') \cdot \nu + 4n\theta \cdot \nu' = 0.$$

3. By (20) we get $(\eta D\nu) \cdot \nu = \eta_j \partial_j \nu \cdot \nu' = 0.$

4. By (36) we get

$$((D^2\nu) \cdot \theta) \cdot \nu = \partial_j \nu \partial_j \theta \cdot \nu_k \nu_k = -|\theta|^2 - (\theta \cdot \nu)^2 - (\theta \cdot \nu)^2 + 3(\theta \cdot \nu)^2 = -|\theta|^2.$$

5. By (25) we have $\nu' \cdot \nabla u' = -\nabla_r (\theta \cdot \nu) \cdot \nabla_r u'$. (39) states that $u' = 2\theta \cdot \nu$.

Hence $\nabla_r (\theta \cdot \nu) = \frac{1}{2} \nabla_r u'$. Thus

$$\nu' \cdot \nabla u' = -\frac{1}{2} |\nabla_r u'|^2.$$
We rewrite (44) as an equation for $\theta$:

$$0 = \Delta \theta'' - n \partial_\nu \theta'' + 2 \theta \cdot \nabla \Delta u' - 2 n (D^2 u' \theta) \cdot \nu + 2 d''_1(B) + \frac{n}{2} |\nabla \Gamma u'|^2$$

(41)

Finally we use (9) to replace $\partial_\nu^2 u'$. This gives

$$2 \theta \cdot \nabla \Gamma u' = 2 n \theta \cdot (D^2 u' \nu) + 2 n \theta \cdot \nabla u' - 2 n \theta \cdot \nu u'$$

(43)

Then we apply (39) to replace $\Delta u'$ on the right hand side of (43). This gives the claim.

We insert (42) into (41). After some cancellation we get

$$0 = \Delta \theta'' - n \partial_\nu \theta'' + 2 n^2 \theta \cdot \nu u' + 2 n \theta \cdot (\theta \cdot \nu) \Delta u' + 2 \theta \cdot \nu \partial_\nu \Delta u'$$

$$+ 2 d''_1(B) + \frac{n}{2} |\nabla \Gamma u'|^2 + 2 n |\theta|^2 - 2 n \eta \cdot \nu - 2 n \theta \cdot \nabla u'.$$

(44)

We rewrite (44) as an equation for $2d''_1(B)$. By (39) we get

$$2d''_1(B) = - (\Delta u'' - n \partial_\nu u'') - n^2 u'^2 - nu' \Delta u' - u' \partial_\nu \Delta u'$$

$$- \frac{n}{2} |\nabla \Gamma u'|^2 - 2 n |\theta|^2 - 2 n \eta \cdot \nu - 2 n \theta \cdot \nabla u'.$$

(45)

Next we integrate over $\partial B$. Recall that $\Delta u''$ is harmonic (see (29)). Thus

$$\int_B \Delta u'' \, dS - n \int_{\partial B} \partial_\nu u'' \, dS = \int_{\partial B} \Delta u'' \, dS - n \int_B \Delta u'' \, dS$$

(46)

vanishes because of Lemma 4.1.

After integration of (45), the fourth term on the right hand side of (45) can be integrated by parts (see (8))

$$- n \int_{\partial B} u' \Delta u' \, dS = n \int_{\partial B} |\nabla \Gamma u'| \, dS.$$
This and (46) simplify (45) and we get
\[ 2d''_1(B)\partial B = -n^2 \int_{\partial B} u''^2 \, dS - \int_{\partial B} u' \partial_\nu \Delta u' \, dS + \frac{n}{2} \int_{\partial B} \nabla_1 u''^2 \, dS \\
- 2n \int_{\partial B} |\theta'|^2 \, dS - 2n \int_{\partial B} \eta \cdot \nu \, dS + 2n \int_{\partial B} \theta \cdot \nabla u' \, dS. \]
(47)

The following integral identity was proved in Lemma 5 from [2].

**Lemma 4.4.** For measure preserving perturbations there holds
\[ \int_{\partial B} |\theta'|^2 \, dS = -2 \int_{\partial B} \theta' \cdot (D\theta\nu) \, dS + (n - 1) \int_{\partial B} (\theta \cdot \nu)^2 \, dS + \int_{\partial B} \eta \cdot \nu \, dS. \]

We rewrite this formula for our case. Note that the first integrand on the right hand side can be written with the help of (35) and (39) as
\[ 2\theta' \cdot (D\theta\nu) = 2\theta' \cdot \nabla (\theta\nu) - 2\theta' \cdot (\theta D\nu) = \theta' \cdot \nabla u' - 2|\theta'|^2. \]
(48)

Thus
\[ -2n \int_{\partial B} |\theta'|^2 \, dS + 2n \int_{\partial B} \theta' \cdot \nabla_1 u' \, dS - 2n \int_{\partial B} \eta \cdot \nu \, dS = \frac{n(n - 1)}{2} \int_{\partial B} u''^2 \, dS. \]

We insert this into (47) and use (39) again. So we get
\[ 2d''_1(B)\partial B = -n^2 \int_{\partial B} u''^2 \, dS - \int_{\partial B} u' \partial_\nu \Delta u' \, dS + \frac{n}{2} \int_{\partial B} \nabla_1 u''^2 \, dS \\
+ \frac{n(n - 1)}{2} \int_{\partial B} u''^2 \, dS + n \int_{\partial B} u' \partial_\nu u' \, dS \]
(49)

and can formulate the main result of this section.

**Theorem 4.5.** Assume that \( d_1(B) \) is the first fourth order Steklov eigenvalue of the unit ball \( B \). Let \( u' \) be the biharmonic shape derivative with boundary conditions given by (39). Then for measure preserving perturbations the second variation \( d''_1(B) \) is given by
\[ 2d''_1(B)\partial B = \int_{\partial B} |\Delta u'|^2 \, dx - n \int_{\partial B} |\partial_\nu u'|^2 \, dS + 2n \int_{\partial B} \partial_\nu u' \, u' \, dS \\
- \frac{1}{2} n(n + 1) \int_{\partial B} u''^2 \, dS + \frac{n}{2} \int_{\partial B} |\nabla_1 (u')|^2 \, dS. \]
(50)

Proof. Since \( u' \) is biharmonic we get by partial integration
\[ -\int_{\partial B} u' \partial_\nu \Delta u' \, dS = \int_{B} |\Delta u'|^2 \, dx - \int_{\partial B} \partial_\nu u' \, \Delta u' \, dS. \]

Then (49) and (39) give the claim. \( \square \)

It is a remarkable property of the model that \( d''_1(B) \) only depends on \( u' \) and does not depend on \( u'' \) or on \( \eta \). Thus \( d''_1(B) \) is a quadratic functional of \( u' \). We set
\[ J_{vol}(u') := \int_{B} |\Delta u'|^2 \, dx - n \int_{\partial B} |\partial_\nu u'|^2 \, dS + 2n \int_{\partial B} \partial_\nu u' \, u' \, dS \\
- \frac{1}{2} n(n + 1) \int_{\partial B} u''^2 \, dS + \frac{n}{2} \int_{\partial B} |\nabla_1 (u')|^2 \, dS \]
(51)

for any shape derivative \( u' \). Recall that \( u' \in H^{2,2}(B) \) is a shape derivative iff \( u' \) is biharmonic and satisfies (39) for any domain variation that is measure preserving. In the next subsection we will show that \( J_{vol} > 0 \) for all such shape derivatives.
4.3. Discussion of the sign of the second variation of the unit ball. To determine the sign of $d''_1(B)$ resp. $J_{vol}$ we expand $u'$ and $\theta \cdot \nu$ with respect to spherical harmonics.

Let $s \in \mathbb{N} \cup \{0\}$ and $i = 1, \ldots, d_s$ for $d_s = (2s + n - 2)(s + n - 2)!/(s - 2)!$. The function $Y_{s,i}^{}(\xi)$ denotes the $i$-th spherical harmonic eigenfunction of order $s$. The $Y_{s,i}^{}(\xi)$ form an orthonormal basis of $L^2(\partial B)$. In particular we have

$$\Delta r Y_{s,i}^{}(\xi) + s(s + n - 2)Y_{s,i}^{}(\xi) = 0, \quad \xi \in \partial B. \quad (52)$$

Then

$$u'(x) = \sum_{s=0}^{\infty} \sum_{i=1}^{d_s} A_{s,i}^{}(r) Y_{s,i}^{}(\xi), \quad \xi \in \partial B \quad (53)$$

for $x \in B$, $r = |x|$ and $\xi = \frac{x}{|x|}$. We also write

$$\theta \cdot \nu|_{\partial B} = \sum_{s=0}^{\infty} \sum_{i=1}^{d_s} c_{s,i}^{} Y_{s,i}^{}(\xi), \quad \xi \in \partial B. \quad (54)$$

**Remark 4.** For $s = 0$ the spherical harmonic $Y_{0,0}^{}$ is the constant function. It is excluded by (15) and (18), so the sum in (53) and (54) begins with $s = 1$.

We have to solve $\Delta^2 u'(x) = 0$. We write it as a system $\Delta u'(x) = v(x)$ and $\Delta v(x) = 0$. First we set

$$v(x) = \sum_{s=1}^{\infty} \sum_{i=1}^{d_s} X_{s,i}^{}(r) Y_{s,i}^{}(\xi). \quad (55)$$

With (11) the equation $\Delta v(x) = 0$ is equivalent to

$$\Delta v(x) = \sum_{s=1}^{\infty} \sum_{i=1}^{d_s} Y_{s,i}^{}(\xi) X_{s,i}''(r) + \frac{n-1}{r} X_{s,i}^{}(r) Y_{s,i}^{}(\xi) + \frac{X_{s,i}^{}(r)}{r^2} + \Delta r Y_{s,i}^{}(\xi) = 0. \quad (56)$$

We use (52) and get

$$\sum_{s=1}^{\infty} \sum_{i=1}^{d_s} r^2 X_{s,i}''(r) + (n-1) r X_{s,i}^{}(r) = \sum_{s=1}^{\infty} \sum_{i=1}^{d_s} s(n+s-2) X_{s,i}^{}(r). \quad (57)$$

A solution of (57) is given by

$$\sum_{s=1}^{\infty} \sum_{i=1}^{d_s} X_{s,i}^{}(r) = \sum_{s=1}^{\infty} \sum_{i=1}^{d_s} a_{s,i}^{} r^s + b_{s,i}^{} r^{n-s+2}. \quad (58)$$

Note that the solution $r^{n-s+2}$ is singular, thus it will be neglected and the solution of $\Delta v(x) = 0$ is

$$v(x) = \sum_{s=1}^{\infty} \sum_{i=1}^{d_s} a_{s,i}^{} r^s Y_{s,i}^{}(\xi). \quad (59)$$

Now we can solve $\Delta u'(x) = v(x)$. By (53), (11) and (59) we get
\[
\sum_{s=1}^{\infty} \sum_{i=1}^{d_s} A'_{s,i}(r)Y_{s,i}(\xi) + \frac{n-1}{r} A_{s,i}(r)Y_{s,i}(\xi) + \frac{A_{s,i}(r)}{r^2} \Delta r Y_{s,i}(\xi)
= \sum_{s=1}^{\infty} \sum_{i=1}^{d_s} a_{s,i} r^{s} Y_{s,i}(\xi).
\]

Neglecting singular solutions again we obtain
\[
A_{s,i}(r) = b_{s,i} r^{s} + a_{s,i} r^{2+s}.
\] (60)

Hence (53) can be rewritten as
\[
u'(x) = \sum_{s=1}^{\infty} \sum_{i=1}^{d_s} (a_{s,i} r^{s} + b_{s,i} r^{s+2}) Y_{s,i}(\xi), \quad \xi \in \partial B
\] (61)

and (54) as
\[
\theta \cdot \nu |_{\partial B} = \sum_{s=1}^{\infty} \sum_{i=1}^{d_s} c_{s,i} Y_{s,i}(\xi), \quad \xi \in \partial B.
\] (62)

Recall that the shape derivative \( u' \) also solves \( \Delta u' = n \partial_r u' - nu' \) on \( \partial B \). By (61) and (11) we compute
\[
\Delta u' = \sum_{s=1}^{\infty} \sum_{i=1}^{d_s} [(2n+4s)b_{s,i} r^{s}] Y_{s,i}(\xi)
\]
and
\[
\partial_r u' = \sum_{s=1}^{\infty} \sum_{i=1}^{d_s} (s a_{s,i} r^{s-1} + (s+2)b_{s,i} r^{s+1}) Y_{s,i}(\xi).
\]

For \( r = 1 \) we then get
\[
(2n+4s) b_{s,i} = ns a_{s,i} + n(s+2) b_{s,i} - na_{s,i} - nb_{s,i}.
\]

The boundary condition \( u' = 2 \theta \cdot \nu \) and \( 2 \theta \cdot \nu = \sum_{s=1}^{\infty} \sum_{i=1}^{d_s} c_{s,i} Y_{s,i}(\xi) \) determine the coefficients \( a_{s,i} \) and \( b_{s,i} \):
\[
a_{s,i} = - \frac{(n-4)s}{2s} c_{s,i},
\]
\[
b_{s,i} = \frac{n(s-1)}{2s} c_{s,i}.
\]

Further we calculate
\[
\int_B |\Delta u'|^2 \, dx = (n+2s)^{n(s-1)} c_{s,i}^2,
\]
\[
-n \int_{\partial B} |\partial_r u'|^2 \, dS = -n \left( \frac{n(n-4)s}{2} + \frac{n(s-1)(s+2)}{2s} \right)^2 c_{s,i}^2,
\]
\[
2n \int_{\partial B} \partial_r u' u' \, dS = \frac{4n(2ns + 4s^2 - 2n)}{2s},
\]
\[
-\frac{n}{2} (n+1) \int_{\partial B} u'^2 \, dS = -2n(n+1)c_{s,i}^2,
\]
and
\[
\frac{n}{2} \int_{\partial B} |\nabla r(u')|^2 \, dS = 2ns(s+n-2)c_{s,i}^2.
\]
We insert all these expressions into (51) and get $\mathcal{J}_{\text{vol}}$ as
\[
4s^2\mathcal{J}_{\text{vol}} := -2n^2 + (14n^2 - 8n)s + (8n^2 - 4n)s^2 + (8n^2 + 16n)s^3 + 8ns^4.
\]
Note that the second variation only depends on the dimension $n$ and $s$. Now we can prove our first main result (see also Theorem 1.1):

**Theorem 4.6.** The ball is a strict local minimizer among all domains of equal measure.

**Proof.** The second variation $\mathcal{J}_{\text{vol}}(u')$ in (51) depends on $u'$. With the help of spherical harmonics $\mathcal{J}_{\text{vol}}$ is given by (up to a constant multiplicative factor)
\[
\mathcal{J}_{\text{vol}}(n,s) := -2n^2 + (14n^2 - 8n)s + (8n^2 - 4n)s^2 + (8n^2 + 16n)s^3 + 8ns^4.
\]
For all $n \geq 2$ and $s \geq 1$ the second domain variation $\mathcal{J}_{\text{vol}}(n,s)$ is strictly positive.

5. **Perimeter constraint.** In this section we change the constraint. Now we consider domains with the same surface area rather than the same measure. Thus we consider the class of perturbations of the ball for which $\theta$ and $\eta$ satisfy (18) and (19). The computation for $d''(B)$ is the same as in Section 4. Thus we recall (47).
\[
2d''(B)\mathcal{J}_{\text{vol}} = -n^2 \int_{\partial B} u'^2 dS - \int_{\partial B} u'\partial_{n} \Delta u' dS + \frac{n}{2} \int_{\partial B} |\nabla u'|^2 dS
\]
\[
- 2n \int_{\partial B} |\theta'|^2 dS - 2n \int_{\partial B} \eta \cdot \nu dS + 2n \int_{\partial B} \theta \cdot \nabla u' dS.
\]
We need to find a new characterization for $\int_{\partial B} |\theta'|^2 dS$. This is provided by the following Lemma, which replaces Lemma 4.4.

**Lemma 5.1.** For measure preserving perturbations there holds
\[
-2n \int_{\partial B} |\theta'|^2 dS + 2n \int_{\partial B} \theta \cdot \nabla u' dS - 2n \int_{\partial B} \eta \cdot \nu dS = \frac{n}{2(n-1)} \int_{\partial B} |\nabla u'|^2 dS + \frac{n}{2(n-2)} \int_{\partial B} u'^2 dS.
\]

**Proof.** The proof is similar to the proof of Lemma 5 in [2], however, we apply (19) instead of (17). In fact by (6) and (35) we get
\[
\int_{\partial B} |\theta'|^2 dS = \int_{\partial B} \theta \cdot (D\nu \theta) dS = \int_{\partial B} \theta \cdot (D\nu) dS.
\]
We integrate by parts (see (8))
\[
\int_{\partial B} \theta \cdot (D\nu \theta) dS
\]
\[
= -\int_{\partial B} \text{div} r \theta^\tau (\theta \cdot \nu) dS - \int_{\partial B} \theta^\tau \cdot (D\theta \nu) dS
\]
\[
= -\int_{\partial B} \text{div} r \theta^\tau (\theta \cdot \nu) dS - \int_{\partial B} \theta \cdot (D\theta \nu) dS + \int_{\partial B} \nu \cdot (D\theta \nu)(\theta \cdot \nu) dS.
\]
We use the fact that
\[
\text{div} r \theta^\tau = \text{div} r \theta - \text{div} r (\nu(\theta \cdot \nu)) = \text{div} \theta - \nu \cdot (D\theta \nu) - (\theta \cdot \nu) \text{div} r \nu,
\]
If we compare this quadratic form with Theorem 5.2. Assume that 

This implies the following theorem.

Note that the large bracket on the right hand side is zero if and only if the domain deformation consists of translations, i.e. \( \theta \cdot \nu = a \cdot x \) for some vector \( a \in \mathbb{R}^n \). Otherwise it is positive (see Lemma 2 - Lemma 3 [2]). This gives our second main result (see also Theorem 1.2).
Theorem 5.3. The ball is a strict local minimizer among all domains of equal perimeter.

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