Finding Common Ground for Incoherent Horn Expressions

Ana Ozaki\textsuperscript{1}, Anum Rehman\textsuperscript{1}, Philip Turk\textsuperscript{1}, and Marija Slavkovik\textsuperscript{1}

\textsuperscript{1}University of Bergen, \{ana.ozaki, marija.slavkovik\}@uib.no

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Abstract

Autonomous systems that operate in a shared environment with people need to be able to follow the rules of the society they occupy. While laws are unique for one society, different people and institutions may use different rules to guide their conduct. We study the problem of reaching a common ground among possibly incoherent rules of conduct. We formally define a notion of common ground and discuss the main properties of this notion. Then, we identify three sufficient conditions on the class of Horn expressions for which common grounds are guaranteed to exist. We provide a polynomial time algorithm that computes common grounds, under these conditions. We also show that if any of the three conditions is removed then common grounds for the resulting (larger) class may not exist.

1 Introduction

Systems capable of some level of autonomous operation should be built to respect the moral norms and values of the society in which they operate [Bremner \textit{et al.}, 2019; Dignum, 2019; Moor, 2006; Wallach and Allen, 2008; Winfield \textit{et al.}, 2019]. If there are multiple stakeholders supplying different rules, how should possible inconsistencies among them be resolved?

The question of whether or not moral conflicts do really exist has been long argued in moral philosophy [Donagan, 1984]. It has also been argued that people do not tend to follow moral theories, but rules of thumb when choosing what to do in a morally sensitive situation [Hory, 1994]. Numerous conflicts do arise among rules. A normative conflict is a situation in which an agent ought to perform two actions that cannot both be performed [Horty, 2003].

There is an entire field that studies the resolution of normative conflicts [Santos \textit{et al.}, 2018]. Baum [2020] and Rahwan [2018] argue that some form of a social choice approach is needed to decide and Botan \textit{et al.} [2021] consider judgment aggregation as a method. Noothigattu \textit{et al.} [2018] consider learning the moral preferences of people and using them to vote on what is the right thing for an autonomous system to do. Liao \textit{et al.} [2018] and Liao \textit{et al.} [2019] propose an argumentation based approach.
In this paper we do not aim to solve the problem of moral normative conflicts, moral social choice or deontic conflict resolution. We are interested in exploring a very practical, admittedly limited, way of reaching a common ground on sets of rules. We assume that each stakeholder with an interest in governing the behavior of an autonomous system contributes a set of rules which the system should implement. However, these rules are considered to be under-specified. Namely, we assume that the stakeholder does not exclude that exceptions to the rules exist. We design an algorithm that “corrects” the rules supplied by one stakeholder with exceptions raised by another stakeholder.

We only consider rules expressed in Horn logic. This is motivated primarily by the availability of tools that handle this logic, when compared with deontic logics and formalisms based on nonmonotonic reasoning, which are two other natural choices to represent behavior rules. We study a notion of common ground and under which conditions it exists.

To illustrate the main ideas and approaches we follow in building our algorithm, consider as an example a police robot\footnote{Similar examples can be found in [Bjørgen et al., 2018]} that detects a potentially illegal activity in a supermarket.

**Example 1.** Assume that the police robot has detected smoke and a child who is smoking. The robot has made the following deduction:

A child is smoking in a forbidden to smoke area.

Now assume we have stakeholders, which we may also refer to as “agents”, with the following recommendations. The first agent puts forward the rule: “if there is an illegal activity, the police should be informed”. In symbols:

\[
\text{illegalActivity} \rightarrow \text{policeCall} \quad (1)
\]

The second agent points out that it is a child who is smoking, and so, the parents are the ones who should decide if the police should be informed. Police officers do exercise an independent judgement as part of best serving the public and we would not want to have a more oppressive society with robots that completely eliminate this practice. More specifically, we have the rule “if there is an illegal activity done by a child then their parents should be called”. In symbols:

\[
(\text{illegalActivity} \land \text{child}) \rightarrow \text{parentsAlert} \quad (2)
\]

The agents agree that smoking in a forbidden to smoke area (e.g., bus stop) is an illegal activity and that, if this happens, either the police or the parents should be called. They agree that not both should be called, that is,

\[
(\text{parentsAlert} \land \text{policeCall}) \rightarrow \bot.
\]

Calling both parents and the police is somewhat pointless since the police is obliged to call the parents of the minor.
In this example, there is a clear incoherence but not necessarily a conflict between the two agents. We consider this an incoherence because the second agent reasons using a more specific rule than the first. Rules could be even further specialized and take into account the case in which the child is not under parental supervision (e.g., alone in the supermarket).

Consider the rule: “if there is an illegal activity done by a child who is unsupervised (by an adult) then the police should be called”. In symbols:

$$(\text{illegalActivity} \land \text{child} \land \text{unsupervised}) \rightarrow \text{policeCall}$$

(3)

A common ground between (1) and (2) can be reached by transforming (1) into

$$(\text{illegalActivity} \land \text{adult}) \rightarrow \text{policeCall}$$

(4)

What is interesting about (4) is that it is coherent with both (2) and (3). When there are two applicable rules but one is more specialized, legal theory prescribes the use of the *lex specialis derogat legi generali* principle meaning “special law repeals general laws”. We explore the possibility of applying this principle as a basis for reaching common grounds.

What makes one set of rules a common ground? We consider the scenario in which all stakeholders are equally important. A set of rules, represented as a Horn expression, can only be a common ground for their behaviour rules, given as input, if it is coherent with them. We formally define the notion of a common ground for a given set of Horn expressions and discuss the main properties of it. Common grounds do not always exist. We identify a large class of Horn expressions for which common grounds are guaranteed to exist and provide a polynomial time algorithm that computes common grounds for Horn expressions in this class. We assume a common background knowledge for the stakeholders that expresses mutually exclusive conditions, for example, being an adult or a child. Our main contributions are

- the formalisation and discussion of a notion of common ground for incoherent rules (Section 2);
- the proposal of a polynomial time algorithm for finding common grounds based on the *lex specialis derogat legi generali* legal principle (Section 3); and
- an analysis of when a common ground for incoherent rules is guaranteed to exist (Section 3).

In Section 4 we discuss related work. In Section 5 we outline our conclusions and discuss directions for future work.

## 2 Rules, Incoherences, and Common Ground

We represent rule recommendations using propositional Horn logic. In the following, we briefly introduce the syntax and semantics of propositional Horn expressions and provide basic notions used in this paper. Then, we formally introduce our notions of *coherence*, *conflict* and *common ground*. 
2.1 Rules

Syntax  An atom is a boolean variable. A literal is an atom $p$ or its negation $\neg p$. A Horn clause is a disjunction of literals where at most one is positive. It is definite if it has exactly one positive literal. For a given definite Horn clause $\phi$, we define $\text{ant}(\phi)$ to be the set of all atoms such that their negation occurs in $\phi$, while $\text{con}(\phi)$ is the positive literal in $\phi$. A definite Horn clause is non-trivial if $\text{con}(\phi) \notin \text{ant}(\phi)$. We may treat a set of formulas and the conjunction of its elements interchangeably. Also, we often write Horn clauses as rules of the form

$$(p_1 \land \ldots \land p_n) \rightarrow q \text{ or } (p_1 \land \ldots \land p_n) \rightarrow \bot$$

the latter are for non-definite clauses. A Horn expression is a (finite) set of Horn clauses. It is called definite if all clauses in it are definite.

Semantics  The semantics is given by interpretations. We denote by $\text{true}(I)$ the set of variables assigned to true in an interpretation $I$. We say that $I$ satisfies a Horn clause $\phi$ if

- $\text{ant}(\phi) \not\subseteq \text{true}(I)$ or,
- in the case $\phi$ is definite, $\text{con}(\phi) \in \text{true}(I)$.

It satisfies a Horn expression $\mathcal{F}$ if it satisfies all clauses in $\mathcal{F}$. A Horn expression $\mathcal{F}$ entails a clause $\phi$, written $\mathcal{F} \models \phi$, if every interpretation that satisfies $\mathcal{F}$ also satisfies $\phi$.

Scenario  We consider a scenario with multiple stakeholders, or agents. Each stakeholder $i$ is associated with its own set of behaviour rules, e.g., “if there is an illegal activity then call the police”. The stakeholders also share background knowledge. Their background knowledge contains basic constraints about the world, such as “a person cannot be an adult and a child at the same time”. Since stakeholders may diverge in their behaviour, the goal is to find a Horn expression $\mathcal{F}$ that is a representative of such behaviours that also respects the constraints about the world; we call $\mathcal{F}$ a common ground.

We represent behaviour rules with definite Horn expressions, denoted $\mathcal{F}_i$ for each stakeholder $i$, and the background knowledge with a set of non-definite Horn clauses. This way of representing behaviour rules and background knowledge simplifies the presentation of the technical results while capturing a large class of scenarios. Also, as discussed earlier, Horn logic is a convenient formalism because there are several tools for performing automated reasoning.

The background knowledge $\mathcal{B}$ for $\mathcal{F}_1, \ldots, \mathcal{F}_n$ is defined as a set of non-definite Horn clauses, built from atoms occurring in $\mathcal{F}_1, \ldots, \mathcal{F}_n$, expressing pairwise disjointness constraints (e.g., a person cannot be a child and an adult, or a child and a teenager). For a given atom $p$ occurring in $\mathcal{B}$, we define

$$\mathcal{P}_\mathcal{B} = \{ q \mid \mathcal{B} \models (p \land q) \rightarrow \bot \}.$$
We may omit writing ‘for $F_1, \ldots, F_n$’ and the subscript $\cdot_B$ when this is clear from the context. Elements of $\overline{p}$ are called *excludents* of $p$. We assume that for every $p$ occurring in $F$ the set $\overline{p}$ is not empty. Whenever we write $\overline{p}$ in a rule, we assume it refers to a representative of an atom in $\overline{p}$, which means “not $p$”. Having a symbol that represents the exclusion of an atom can be seen as a weak form of negating it which is computationally efficient, since we remain in the Horn logic.

**Example 2.** The background knowledge of the stakeholders in Example 1 can be modeled as:

\[
(\text{parentsAlert} \land \text{policeCall}) \rightarrow \bot, \quad (\text{child} \land \text{adult}) \rightarrow \bot,
\]
\[
(\text{child} \land \text{teen}) \rightarrow \bot, \quad (\text{supervised} \land \text{unsupervised}) \rightarrow \bot.
\]

In our notation, $\overline{\text{child}} = \{\text{adult}, \text{teen}\}$.  

Given a Horn clause $\phi$ with $p \in \text{ant}(\phi)$, we denote by $\phi^{\setminus p}$ the result of replacing $p \in \text{ant}(\phi)$ by $q$. Also, we denote by $\phi^{-p}$ the result of removing $p$ from $\text{ant}(\phi)$; and we denote by $\phi^{+p}$ the result of adding $p$ to $\text{ant}(\phi)$.

### 2.2 Coherence and Conflict

To capture the semantics of the background knowledge on behaviour rules of stakeholders, we define two concepts: coherence and conflict. Coherence is a property of a definite Horn clause w.r.t. a Horn expression or of a Horn expression. Intuitively, in a “well behaved” set of rules, one should not be able to infer an atom $p$ and an element of $\overline{p}$. Such Horn expressions will be called incoherent. More specifically, a clause $\phi$ is incoherent with a Horn expression when adding the antecedent of this clause to the set makes it possible for both the clause’s consequent, $\text{con}(\phi)$, and an element of $\overline{\text{con}(\phi)}$ to be inferred from this union (see Example 3).

**Example 3.** Assume that

\[
(\text{parentsAlert} \land \text{policeCall}) \rightarrow \bot \in \mathcal{B}.
\]

Consider the Horn expression $\{\text{illegalActivity} \rightarrow \text{policeCall}\}$ and the clause $\text{illegalActivity} \land \text{child} \rightarrow \text{parentsAlert}$. The union of the antecedent of the clause, the clause itself, and the Horn expression implies both policeCall and parentsAlert, which should not happen according to $\mathcal{B}$. In symbols, $(\text{parentsAlert} \in \text{policeCall}$ and vice-versa).

Having an incoherent set of rules means that there might be a situation in which the set would not be able to offer any guidance as to what to do. For simplicity, we may omit referring explicitly to the background knowledge since we assume one group of stakeholders at a time with a unique background knowledge.

Let $\phi$ and $\psi$ be definite Horn clauses and let $\mathcal{F}$ be a definite Horn expression. A derivation of $\phi$ w.r.t. $\psi$ and $\mathcal{F}$ is a sequence $\phi_1, \ldots, \phi_n$ of clauses in $\mathcal{F}$ such that

- $\phi_1 = \psi, \phi_n = \phi$,
- $\text{ant}(\phi_{i+1}) \subseteq \bigcup_{1 \leq j \leq i} \text{ant}(\phi_j) \cup \text{con}(\phi_j)$, and,
• for all \(1 < i < n\), there is \(j\) such that \(i < j \leq n\) and \(\text{con}(\phi_i) \in \text{ant}(\phi_j)\).

We write \(\psi \Rightarrow F \phi\) if there is a derivation of \(\phi\) w.r.t. \(\psi\) and \(F\). Assuming \(\phi, \psi \in F\), we have that

\[
\psi \Rightarrow F \phi \iff F \cup \text{ant}(\psi) \models \text{ant}(\phi).
\]

Some examples of derivations can be seen in Figure 1.

The following proposition is useful for proving our results\(^2\).

**Proposition 4.** Given definite Horn clauses \(\phi\) and \(\psi\) and a definite Horn expression \(F\), one can decide in linear time on the number of literals in \(F\) whether there is a derivation of \(\phi\) w.r.t. \(\psi\) and \(F\).

We are now ready for our definition of coherence.

**Definition 5** (Coherence). A definite Horn clause \(\phi\) is coherent with a Horn expression \(F\) if \(F\} \{\phi\} \neq \phi\) and

- there is no \(\psi \in F\) such that \(\psi \Rightarrow F \phi\) or \(\phi \Rightarrow F \psi\) while \(\text{con}(\psi) \in \text{con}(\phi)\) (note that \(\text{con}(\psi) \in \text{con}(\phi)\) implies \(\text{con}(\phi) \in \text{con}(\psi)\)).

The set \(F\) is coherent if all \(\phi \in F\) are coherent with \(F\) (and incoherent otherwise).

Conflict is a property of a set of definite Horn clauses. The notion of conflict is stronger than the notion of coherence. Intuitively, a conflict is an incoherence that cannot be easily resolved. The incoherence in Example 3 is due to the fact that \(\text{illegalActivity} \land \text{child}\) implies both \(\text{parentsAlert}\) and \(\text{policeCall}\) while the background knowledge states that both cannot be true. These rules are not in conflict and incoherence can be resolved as follows. The rule

\[(\text{illegalActivity} \land \text{child}) \rightarrow \text{parentsAlert}\]

can be considered as an exception to the more general rule

\[\text{illegalActivity} \rightarrow \text{policeCall}.
\]

Then, all we have to do to restore coherence is to change the latter rule into a more specific one that says:

- unless the exceptional case (e.g., it is a child) has occurred, take this action (e.g., call the police).

A set can be in conflict when we cannot find a “suitable” atom to add to the antecedent of an incoherent rule, as a way to further specify it and avoid incoherence. The “suitable” atoms are chosen from the excludents of the atoms in the antecedent of the more specific rule involved in the incoherence.

**Definition 6** (Conflict). Let \(F\) be a definite Horn expression. We say that \(F\) is in conflict if

\(^2\)All of our proofs are given in detail in the Appendix
• there are $\phi, \psi \in F$ s.t. $\phi \Rightarrow F \psi$ and $\text{con}(\phi) \in \overline{\text{con}(\psi)}$ (i.e., $F$ is incoherent); and

• there is no $r \in \text{ant}(\phi) \setminus \text{ant}(\psi)$ with $q \in \tau$ s.t. $\psi^+q$ is coherent with $F \setminus \{\psi\}$.

Example 7. Consider $B$ in Example 3 and the rules:

(1) illegalActivity $\rightarrow$ parentsAlert,
(2) illegalActivity $\rightarrow$ policeCall,
(3) (illegalActivity $\land$ child) $\rightarrow$ parentsAlert.

The set with the first two rules, (1) and (2), is in conflict, while the set with the last two rules, (2) and (3), is not.

2.3 Common Ground

We are now ready to provide the notion of a common ground.

Definition 8 (Common Ground). Let $F_1, \ldots, F_n$ be definite Horn expressions, each associated with a stakeholder $i \in \{1, \ldots, n\}$. Let $B$ be a set describing background knowledge. A formula $F$ is a common ground for $F_1, \ldots, F_n$ and $B$ if it satisfies each of the following postulates:

(P1) $F$ is coherent;

(P2) if $\bigcup_{i=1}^{n} F_i$ is coherent, then $F \equiv \bigcup_{i=1}^{n} F_i$;

(P3) for all $i \in \{1, \ldots, n\}$ and all $\phi \in F_i$, we have that $F \not\models \text{ant}(\phi) \rightarrow p$ with $p \in \text{con}(\phi)$;

(P4) for each $\phi \in F$, there is $\psi \in \bigcup_{i=1}^{n} F_i$ with $\{\psi\} \models \phi$;

(P5) for all $i \in \{1, \ldots, n\}$ and all $\phi \in F_i$, there is (a non-trivial) $\psi \in F$ such that $\{\phi\} \models \psi$; and

(P6) for all $\phi \in F$, if there is $p \in \text{ant}(\phi)$ such that, for all $q \in \tau$, $F \cup \{\phi \setminus p\}$ is coherent and there is $i \in \{1, \ldots, n\}$ such that $F_i \models \phi \setminus p$ then $F_i \not\models \phi^+p$.

We now discuss and motivate the postulates that characterize a common ground.

(P1) The first postulate is intuitive: the learned set of rules should be coherent with the background knowledge. If they were not so, the theory would recommend, for example, two mutually exclusive courses of action for the same situation (described here in terms of rule antecedents).

(P2) The common ground should have as much of the input rules supplied by the stakeholders as possible, hence (P2) ensures that if the union of rules provided by stakeholders is coherent, then this should be the common ground.

(P3) The motivation for (P3) we find in Hare [1972]: “the essence of morality is to treat the interests of others as of equal weight with ones own”, which we here interpret as a requirement that all agent’s rules are considered equally informative and should not be entirely overridden. (P3) ensures that a rule that is in strict opposition with what a stakeholder recommends is not in the common ground.
We also do not want that some rules “sneak in” in the common ground, without being explicitly supported by a stakeholder. This is operationalized by (P4) that guarantees that a rule in a common ground can always be “traced back” to a rule from a stakeholder.

(P5) The fifth postulate ensures that some part of a stakeholder’s rule is in a common ground, though, in a “weaker” form. In other words, a non-trival part of each stakeholder’s rules should be in a common ground.

(P6) The sixth is the most “tricky” postulate to explain, however, it is essential for Definition 8 because it avoids that unintended rules become part of the common ground. We illustrate this with the following example.

Example 9. Consider $B$ in Example 3 and

$$F_1 = \{ \phi = \text{illegalActivity} \rightarrow \text{policeCall},$$
$$\psi = \text{lowBattery} \rightarrow \text{charge}\}$$

$$F_2 = \{ \varphi = (\text{illegalActivity} \land \text{child}) \rightarrow \text{parentsAlert}\}.$$  

To resolve the incoherence in $F_1 \cup F_2$, one can replace $\phi$ with $(\text{illegalActivity} \land \text{child}) \rightarrow \text{policeCall}$. Without (P6), the rules

$$(\text{illegalActivity} \land \text{lowBattery}) \rightarrow \text{policeCall},$$
$$(\text{illegalActivity} \land \text{charge}) \rightarrow \text{policeCall}$$

could also be used to replace $\phi$ as they satisfy (P1)-(P5). Though, these rules are unintended since lowBattery and charge are unrelated with the incoherence in $F_1 \cup F_2$.  

3 Finding Common Grounds

We investigate the problem of finding a common ground for rules supplied by stakeholders, considering that they have a basic background knowledge, as described in Section 2. We use the notion of non-redundant and acyclic Horn expressions, defined as follows. A Horn expression $F$ is non-redundant if for all $\phi \in F$ it is not the case that $F \setminus \{\phi\} = \phi$. It is acyclic if there is no sequence of clauses $\phi_1, \ldots, \phi_n \in F$ such that $\text{con}(\phi_i) \in \text{ant}(\phi_{i+1})$, for all $1 \leq i < n$, and $\phi_1 = \phi_n$.

In particular, we show that for non-redundant, not in conflict, and acyclic Horn expressions, a common ground is guaranteed to exist and can be computed in polynomial time (Theorem 15). Our result is tight in the sense that if any of these three conditions is removed then common grounds may not exist. We first show the negative results, stated in Theorem 10.

Theorem 10. Consider the class of non-redundant, not in conflict, and acyclic Horn expressions. If we extend this class by removing any of the three conditions (while still keeping the remaining two) a common ground for Horn expressions in the extended class may not exist.
The rest of this section is devoted to show that if \( \bigcup_{i=1}^{n} F_i \) is an acyclic, non-redundant, and not in conflict Horn expression then a common ground always exists (Theorem 15). Our proof strategy consists in showing that Algorithm 1 returns a common ground for Horn expressions \( F_1, \ldots, F_n \), if the three mentioned conditions are satisfied. We also show that Algorithm 1 terminates in in the size of \( F_1, \ldots, F_n \) and the size of the background knowledge.

Before we explain the algorithm, we introduce some notions. Common grounds are found by modifying incoherent clauses, in particular, by adding atoms to the antecedent of a clause \( \phi \). As we shall see later, at most one atom is added. The resulting clause \( \phi' \) is such that \( \{ \phi \} \models \phi' \) but \( \{ \phi' \} \not\models \phi \). We may refer to \( \phi' \) as the result of ‘weakening’ \( \phi \) by adding some atom to its antecedent, or simply say that \( \phi' \) is a ‘weaker’ version of \( \phi \).

**Definition 11.** Let \( \phi \) and \( \psi \) be definite Horn clauses and let \( F \) be a definite Horn expression\(^3\). The (incoherence) dependency graph of \( F \) is the directed graph \((V, E)\), where

- \( V \) is the set of all pairs \((\psi, \phi)\) such that \( \psi \Rightarrow F \phi \) and \( \text{con}(\psi) \in \text{con}(\phi) \), and,
- \( E \) is the set of all \(( (\psi', \phi'), (\psi, \phi) ) \) such that \( \phi' \neq \phi \) and \( \phi' \) occurs in a derivation of \( \phi \) w.r.t. \( \psi \) and \( F \).

We say that \( v' \in V \) is a parent for \( v \in V \) if \( (v', v) \in E \).

We say that the pair \((\psi, \phi)\) is safe for \( F \) if \((\psi, \phi)\) has no parent in the dependency graph of \( F \).

Figure 1 illustrates an example where there is no pair of safe clauses. For this example, one can also see that there is no common ground (cf. Definition 8) for the Horn expressions.

By Lemma 12 stated in the following, if the theory is incoherent and the Horn expression is acyclic (which also means avoiding cycles in the dependency graph) then there is a safe pair of clauses in it. We use this property in Algorithm 1.

**Lemma 12.** Let \( F \) be an acyclic Horn expression. If \( F \) is incoherent then there are \( \psi, \phi \in F \) such that \((\psi, \phi)\) is safe (Definition 11). Moreover, one can find \( \psi, \phi \in F \) such that \((\psi, \phi)\) is safe in quadratic time in the size of \( F \) (and \( B \)).

Algorithm 1 receives as input the background knowledge \( B \) and a finite list of definite Horn expressions \( F_1, \ldots, F_n \). It verifies that \( F = \bigcup_{i=1}^{n} F_i \) is not in conflict, not redundant, and not acyclic. At each iteration of the “while” loop (Line 5), Algorithm 1 first selects clauses \( \psi, \phi \in F \) such that \((\psi, \phi)\) is safe (Line 6). By Lemma 12, at least one safe pair is guaranteed to exist. Then, in Line 7, it resolves incoherences by replacing \( \phi \) with all weaker versions of this clause that are coherent with the Horn expression being constructed. As we formally state later on in Theorem 15, Algorithm 1 outputs a common ground for \( F \) (and \( B \)).

**Example 13.** Consider \( F_1 \ldots F_7 \) as in Figure 2. We have that \( F = \bigcup_{i=1}^{7} F_i \) is not in conflict, not redundant, acyclic, but incoherent. The nodes \((\phi_2, \phi_1)\) and \((\phi_5, \phi_4)\) have

\(^3\)We may omit ‘for \( F \)’ if this is clear from the context.
\[ F_1 = \{ p \rightarrow q \}, \quad F_2 = \{ (r \land s) \rightarrow \overline{q} \}, \quad F_3 = \{ r \rightarrow p \}, \quad F_4 = \{ (q \land t) \rightarrow p \}, \quad F_5 = \{ q \rightarrow r \}, \quad F_6 = \{ (p \land u) \rightarrow \overline{r} \}. \]

\[
\phi_2 \Rightarrow F \phi_1 : (r \land \overline{s}) \rightarrow \overline{q}, r \rightarrow p, p \rightarrow q, \\
\phi_4 \Rightarrow F \phi_3 : (q \land \overline{t}) \rightarrow \overline{p}, q \rightarrow r, r \rightarrow p \\
\phi_6 \Rightarrow F \phi_5 : (p \land u) \rightarrow \overline{r}, p \rightarrow q, q \rightarrow r
\]

Figure 1: A dependency graph.

Algorithm 1: Building coherent \( F \)

**Input:** Horn expression sets \( F_1, \ldots, F_n \) and \( B \).

**Output:** A common ground for \( F_1, \ldots, F_n \) and \( B \) or \( \emptyset \)

1. \( F := F_1 \cup \cdots \cup F_n \)
2. if \( F \) is cyclic or redundant or in conflict then \( \text{return } \emptyset \) (A common ground may not exist by Th. 10)
3. end if
4. while \( F \) is incoherent do
5. Find \( \psi, \phi \in F \) such that \( (\psi, \phi) \) is safe
6. Replace \( \phi \) by all \( \phi' \in \{ \phi^+ \} \) coherent with \( F \setminus \{ \phi \} \)
7. end while
8. return \( F \)

\( F^* \) for \( F_1, \ldots, F_7 \), which is the union of:

\[
F_1^* = \{ (p \land u) \rightarrow s \}, \quad F_2^* = \{ (p \land u) \rightarrow \overline{s} \}, \\
F_3^* = \{ (t \land \overline{q}) \rightarrow \overline{s} \}, \quad F_4^* = \{ (t \land u) \rightarrow p \}, \\
F_5^* = \{ (t \land \overline{p}) \rightarrow \overline{s} \}, \quad F_6^* = \{ s \rightarrow q \}, \\
F_7^* = \{ (t \land u) \rightarrow \overline{q} \}.
\]

We assume that Algorithm 1 selects clauses following some fixed but arbitrary order (e.g. lexicographic) if there are multiple safe pairs of clauses.

The condition that \( F \setminus \{ \phi \} \not\models \phi \) in Definition 5 is important to ensure that if the input
of Algorithm 1 is non-redundant then the output is also non-redundant. We illustrate this with the following example.

**Example 14.** Consider

\[ F_1 = \{ (q \land r) \rightarrow s, (p \land q \land r) \rightarrow \overline{s} \}, \]
\[ F_2 = \{ (p \land u) \rightarrow s \}, \]
\[ F_3 = \{ (t \land \overline{q}) \rightarrow s \}, \]
\[ F_4 = \{ t \rightarrow p \}, \]
\[ F_5 = \{ (t \land \overline{u}) \rightarrow \overline{p} \}, \]
\[ F_6 = \{ s \rightarrow q \}, \]
\[ F_7 = \{ (p \land t) \rightarrow \overline{q} \}. \]

Algorithm 1 would resolve the incoherence in this case by replacing the clause in \( F_2 \) by \((p \land q) \rightarrow s\). The latter rule is redundant because it is implied by \( F_1 \).

We point out that conflict, redundancy, and acyclicity are all conditions that can be determined in polynomial time, since the number of iterations is linear on the size of the input, it can be determined in polynomial time whether the input of Algorithm 1 is as expected. We are now ready to state our main theorem.

**Theorem 15.** The output of Algorithm 1 with an acyclic, non-redundant, not in conflict Horn expression \( \bigcup_{i=1}^{n} F_i \) and \( B \) as input is a common ground for \( \bigcup_{i=1}^{n} F_i \) and \( B \). Algorithm 1 terminates in \( O((|\bigcup_{i=1}^{n} F_i| + |B|)^4) \).

The safe condition in Line 6 of Algorithm 1 avoids that the algorithm introduces rules that violate (P6). We illustrate this in the following example.

**Example 16.** Assume that the safe condition is not in Line 6 of Algorithm 1 and that the algorithm can choose any \( \psi, \phi \in \mathcal{F} \) such that \( \psi \Rightarrow \mathcal{F} \phi \) and \( \text{con}(\psi) \in \text{con}(\phi) \). Let

\[ \mathcal{F} := \{ p \rightarrow s, (t \land \overline{q}) \rightarrow \overline{s}, t \rightarrow p, (t \land \overline{u}) \rightarrow \overline{p} \}. \]
Then, our algorithm could select the first two clauses and replace $p \rightarrow s$ by some clauses, $(p \land q) \rightarrow s$ being one of them. The last two clauses are still incoherent. Selecting them means that $t \rightarrow p$ is replaced by $(t \land u) \rightarrow p$. Now $(p \land q) \rightarrow s$ violates (P6) because $(p \land \lnot t) \rightarrow s$ is replaced by $(p \land q) \rightarrow s$ because $(p \land q) \rightarrow s$ (in other words, $\mathcal{F} \models (p \land q) \rightarrow s$).

So far we have referred to ‘a common ground’. Example 17 illustrates that common grounds may not be unique.

Example 17. Consider

$$\mathcal{F}_1 = \{(t \land \lnot p) \rightarrow r, (p \land s) \rightarrow r\}, \quad \mathcal{F}_2 = \{p \rightarrow r\}.$$  

Replacing the clause in $\mathcal{F}_2$ by either $(p \land \lnot t) \rightarrow r$ or $(p \land \lnot s) \rightarrow r$ would yield a common ground.

A way to deal with non-uniqueness could be by adding all possible weakening options in the common ground. In the next sections, we relate our work with the literature and discuss our results.

4 Related Work

Our common ground postulates resemble the integrity constraints (IC) postulates in belief merging (BM) [Konieczny and Pino Pérez, 2002; Schwind and Marquis, 2018]. However it is not trivial to liken the two sets of postulates, even if we liken background knowledge to the integrity constraints in BM. The BM postulates are constructed to capture the idea of minimal change in belief merging and to align merging with belief revision (with the consensus postulate from Schwind and Marquis [2018] considered in addition). Our (P3-P5) pose requirements that are more exacting than the consensus postulate of Schwind and Marquis [2018], while (P6) does not have an intuitive counterpart in BM.

Resolving deontological conflicts has a long tradition [Santos et al., 2018]. In deontic logic our example becomes: you ought to call the police when you witness illegal behavior. A moral dilemma occurs when two excludent propositions are obligatory in the same situation: in our case it is both obligatory to call the police and it is obligatory to not call the police. Standard deontic logic interprets obligations as a necessity operator - $O\text{policeCall}$ represents the police ought to be called. Deontic conflicts occur when two or more norms cannot be enacted at the same time. Norms are typically expressed as formulas of modal (deontic) logic and the problem of conflict resolution is considered as part of deontic logic reasoning. An overview of the state of the art is given in [Santos et al., 2018]. Lellmann and Ciabattoni [2020] consider sequent rules for conflict resolution in dyadic deontic logics using the principle of specialization we consider here, but they propose setting priorities between norms rather than changing them as we do.

Our example can also be given as a of non-monotonic statement: normally you call the police when you witness illegal behavior, but when the perpetrator is a child you

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4This is why we avoid calling our formulas norms.
make an exception and call the parents. Conflicts among rules in non-monotonic reasoning have been considered in for example Reiter and Criscuolo [1981] and Delgrande and Schaub [1997].

Horty [1994] proposed a non-monotonic approach to deontic logic by introducing the concept of conditional obligations and defining when one deontic rule overrules another. Our example of parents that should be called when a child is involved in illegal behaviour becomes a conditional obligation in Horty’s system, that overrides the obligation to call the police when illegal activity is detected. Further, Horty introduces the concept of conditional extension, that is the set of conclusions of the not overridden rules. When there are conflicting obligations, these are placed in separate extensions.

Hare [1952] argues that we experience a process of learning to act properly by getting right the qualifications of the rules we use. As our abilities increase we modify the initial rules in a way that yields a more complex set of rules which handle exceptions. When discussing his own approach to dealing with underspecified rules (see also in Section 4), Horty [1994] brings up the views of Hare to argue that Hare’s strategy of encoding the exceptions explicitly into rules is problematic. He points to the conclusions of Touretzky [1984] that since any knowledge based system must be able to accommodate updates in a simple way, if rules were to be continually modified in order to reflect new exceptions that are being introduced, this would make the update operation too difficult and the resulting default unwieldy. One way to look at our work is that we show that “unwieldy” is not the case for our Horn rules, under the circumstances we specify.

5 Discussion

Our algorithm is resolves incoherences following the *lex specialis derogat legi generali* legal principle. We show that for non-redundant, acyclic, and not in conflict Horn expressions, it produces a common ground in polynomial time (we leave optimisations of the algorithm as future work). If any of the three conditions is removed then we prove that there is no algorithm that is guaranteed to produce a common ground.

Our approach does not solve the larger problem of how to reason with underspecified rules in general. This problem cannot be handled by explicitly adding new exceptions as their number is unlimited. Furthermore, people do not think of exceptions ahead of time, they dynamically change their rules of thumb when they become aware of an exception. Nevertheless, the algorithm we propose can be used to effectively compute a common ground for incoherent rules.

We interpret conflicts as an indication that the stakeholders are not yet ready to be engaged in finding a common ground. The pre-step then would be to signal to the stakeholders that they should consider whether there are exceptions or qualifiers to some of the suggested rules, which they may not have considered. The requirement of only considering non-redundant clauses is not particularly limiting as the clauses can be pre-processed to remove redundancies. Recommendation rules tend to be such that the actions (which appear in the right side of the rule) are not among the preconditions (which appear in the left side of the rule), as in Example 1. Thus, they can still be represented in the acyclic fragment of the Horn logic.

As future work, it would be interesting to investigate the case of conflicting rules
but allowing the stakeholders to express preferences over the rules they propose. The moral machine experiment Awad et al. [2018] provides an interesting starting point for applying our algorithm: the choice of each user to a scene, e.g., “if the passenger is a dog and a cat and the pedestrian is an elderly woman then crash into a barrier” can be naturally represented as a Horn rule \((p \land q) \rightarrow r\). It would be interesting to analyse, e.g., what would be a common ground for a country or a group of countries.

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A  Proofs for Section 3

Proposition 4. Given definite Horn clauses $\phi$ and $\psi$ and a definite Horn expression $F$, one can decide in linear time on the number of literals in $F$ whether there is a derivation of $\phi$ w.r.t. $\psi$ and $F$.

Proof. We define $S_1$ as the set of all clauses in $F$ with the antecedent contained in $\text{ant}(\psi) \cup \text{con}(\psi)$. For $i > 1$, let $S_i$ be the set of clauses $\varphi \in F \setminus \left( \bigcup_{1 \leq j \leq i-1} S_j \right)$ such that $\text{ant}(\varphi) \subseteq \{ p \mid p \text{ occurs in a clause in } \left( \bigcup_{1 \leq j \leq i-1} S_j \right) \}$. If, for some $j$,

$$\text{ant}(\varphi) \subseteq \{ p \mid p \text{ occurs in a clause in } \left( \bigcup_{1 \leq j \leq i-1} S_j \right) \}$$

then, by definition of a derivation, there is a derivation of $\phi$ w.r.t. $\psi$ and $F$. Since $j$ can be at most $|F|$, this can be computed in polynomial time in the size of $F$, $\phi$, and $\psi$. \qed

Theorem 10. Consider the class of non-redundant, not in conflict, and acyclic Horn expressions. If we extend this class by removing any of the three conditions (while still keeping the remaining two) a common ground for Horn expressions in the extended class may not exist.

Proof. This theorem following directly from Theorems 18, 19, and 20 together. \qed

Theorem 18. There are non-redundant, not in conflict, but cyclic Horn expressions for which no common ground exists.

Proof. Consider

$$\begin{align*}
F_1 &= \{ p \rightarrow q \}, \\
F_2 &= \{ (r \land \bar{r}) \rightarrow \bar{\eta} \}, \\
F_3 &= \{ r \rightarrow p \}, \\
F_4 &= \{ (q \land \bar{r}) \rightarrow \bar{p} \}, \\
F_5 &= \{ q \rightarrow r \}, \text{ and} \\
F_6 &= \{ (p \land u) \rightarrow \tau \}.
\end{align*}$$

We can see that $\bigcup_{i=1}^{6} F_i$ is not in conflict and not redundant. Moreover, there exists a cycle:

$$q \rightarrow r (F_5), \ r \rightarrow p (F_3), \ p \rightarrow q (F_1).$$

The $\bigcup_{i=1}^{6} F_i$ is incoherent. Each of the clauses in $F_2, F_4$ and $F_6$ is incoherent with $\bigcup_{i=1}^{6} F_i$.

Assume a common ground $F$ for $F_1, \ldots, F_6$ exists. A common ground satisfies (P1)-(P6). We give a proof by contradiction. First, we show that $q \rightarrow r$ and $p \rightarrow q$ are not in $F$. Then, we show that at least one of $q \rightarrow r$ and $p \rightarrow q$ must be in $F$. \qed
**Assume that** \( q \rightarrow r \) **is in** \( F \). Since (P4) holds, the clause \((q \land \overline{t}) \rightarrow p (F_4)\) or a weaker version of it has to be in \( F \). As a consequence, \( r \rightarrow p (F_3) \) cannot be in \( F \) since that would make \( F \) not coherent and in violation of (P1). It follows that \( r \rightarrow p (F_3) \) must be weakened, adding variables to its antecedent. By (P6), the only atoms that can be added to the antecedent of \( r \rightarrow p \) are either \( t \) or \( \overline{t} \), but not \( \overline{t} \). Consider \( p \rightarrow q (F_1) \) which by (P4) should either be in \( F \) or have a weaker version of it in \( F \). By (P6), there is no clause in \( F \) that is the result of weakening \( p \rightarrow q \) (adding either \( \overline{r} \) or \( s \) to the antecedent does not make \( F \) coherent given what we know about it so far). Since (P5) must hold, it follows that \( p \rightarrow q \in F \). However, by definition of \( F_5 \), if \( p \rightarrow q \in F \) (and, by assumption, \( q \rightarrow r \in F \)) then \( F \) is not coherent. Therefore \( q \rightarrow r \not\in F \).

**Assume that** \( p \rightarrow q \not\in F \). If \( r \rightarrow p \in F \) then \( F \) is not coherent because, by (P5), there is \( \phi \in F \) such that \( F_2 \models \phi \). Thus, \( r \rightarrow p \not\in F \). We also have that \( q \rightarrow r \not\in F \), otherwise, since \( p \rightarrow q \in F \) and, again by (P5), there is \( \phi \in F \) such that \( F_6 \models \phi \), \( F \) would not be coherent. Similar to the argument in the previous paragraph, one can see that, by (P4), \( r \rightarrow p \) is coherent with \( F \). By (P5), there is \( \phi \in F \) that is the result of weakening \( r \rightarrow p \). However, since \( r \rightarrow p \) is coherent with \( F \), any such \( \phi \in F \) violates (P6). Therefore \( p \rightarrow q \not\in F \).

**Assume that** \( p \rightarrow q \not\in F \) **and** \( q \rightarrow r \not\in F \). We argue this is also a contradiction. If \( p \rightarrow q \not\in F \), then one can show, with arguments similar to the ones above, that \( q \rightarrow r \) is coherent with \( F \). If \( q \rightarrow r \not\in F \) then, by (P5), there is \( \phi \in F \) that is the result of weakening \( q \rightarrow r \). However, since \( q \rightarrow r \) is coherent with \( F \), any such \( \phi \in F \) violates (P6).

**Theorem 19.** There are acyclic, non-redundant, but in conflict Horn expressions for which no common ground exists.

**Proof.** Consider

\[
\begin{align*}
F_1 &= \{(p \land q) \rightarrow r\}, \\
F_2 &= \{(p \land \overline{q}) \rightarrow r\}, \text{ and} \\
F_3 &= \{p \rightarrow \overline{r}\}.
\end{align*}
\]

The formula \( F = \bigcup_{i=1}^3 F_i \) is in conflict because

- there are \( \phi, \psi \in F \) s.t. \( \phi \Rightarrow F \psi \) and \( \text{con}(\phi) \in \text{con}(\psi) \), namely, if we consider \( \phi = (p \land q) \rightarrow r \) and \( \psi = p \rightarrow \overline{r} \);

- and there is no \( q \in \text{ant}(\phi) \setminus \text{ant}(\psi) \) with \( \psi^+ \overline{r} \) coherent with \( F \setminus \{\psi\} \) (we abuse the notation and take \( \{\overline{r}\} \) as \( \overline{r} \)).

Since \( F \cup \{p, q\} \models r \land \overline{r} \) and the clause in \( F_3 \) cannot be ‘weakened’ we can see that a common ground does not exist.

**Theorem 20.** There are acyclic, not in conflict, but redundant Horn expressions for which no common ground exists.
Proof. Consider
\[ \mathcal{F}_1 = \{ p \rightarrow q \} \] and
\[ \mathcal{F}_2 = \{ (p \land s) \rightarrow q, (p \land s \land r) \rightarrow q \}. \]
We have that \( \mathcal{F}_1 \cup \mathcal{F}_2 \) is acyclic, not in conflict, and incoherent with \( \mathcal{F}_1 \cup \mathcal{F}_2 \). It is redundant because
\[ \{ p \rightarrow q \} \models (p \land s) \rightarrow q. \]
Suppose there is a common ground \( \mathcal{F} \) for \( \mathcal{F}_1, \mathcal{F}_2 \).

Assume \((p \land s \land \psi) \rightarrow q \notin \mathcal{F}\). By (P4) the common ground formula \( \mathcal{F} \) for \( \mathcal{F}_1, \mathcal{F}_2 \) satisfies the following statement:

- for every \( \phi \in \mathcal{F} \), there is \( \psi \in \mathcal{F}_1 \cup \mathcal{F}_2 \) such that \( \{ \psi \} \models \phi \).

We can then assume that, for every \( \phi \in \mathcal{F} \), either \( \phi \in \mathcal{F}_1 \cup \mathcal{F}_2 \) or \( \phi \) is a weaker version of a clause in \( \mathcal{F}_1 \cup \mathcal{F}_2 \). We have that \((p \land s \land r) \rightarrow \bar{q}\) is in \( \mathcal{F} \) because there is no atom that can be used to weaken this clause in a coherent and non-trivial way. This means that \((p \land s) \rightarrow q\) (and also \(p \rightarrow q\)) would be incoherent with \( \mathcal{F} \). By (P5), \( \mathcal{F} \) has a weaker version of \((p \land s) \rightarrow q\) (instead of \((p \land s) \rightarrow q\)). The only weaker versions of \((p \land s) \rightarrow q\) that are coherent with a formula containing \( (p \land s \land r) \rightarrow \bar{q} \) are \((p \land s \land \psi) \rightarrow q\) and \((p \land s \land q) \rightarrow q\). Thus, \((p \land s \land \psi) \rightarrow q \in \mathcal{F}\).

Assume \((p \land s \land \psi) \rightarrow q \in \mathcal{F}\). We have that \((p \land s \land \psi) \rightarrow q\) is coherent with \( \mathcal{F} \) because, by (P4), \((p \land s \land r) \rightarrow \bar{q}\) is the only clause entailed by \( \mathcal{F} \) with \( \bar{q} \) in the consequent of the clause. Then, \( \mathcal{F} \) does not satisfy (P6) because for \( s \in \text{ant}((p \land s \land \psi) \rightarrow q) \) we have that \( \mathcal{F}_1 \models (p \land \psi) \rightarrow q \). Thus, it cannot be a common ground.

We have then reached a contradiction and can thus conclude that there is no common ground for \( \mathcal{F}_1, \mathcal{F}_2 \).

\[ \square \]

Lemma 12. Let \( \mathcal{F} \) be an acyclic Horn expression. If \( \mathcal{F} \) is incoherent then there are \( \psi, \phi \in \mathcal{F} \) such that \( (\psi, \phi) \) is safe (Definition 11). Moreover, one can find \( \psi, \phi \in \mathcal{F} \) such that \( (\psi, \phi) \) is safe in quadratic time in the size of \( \mathcal{F} \) (and \( \mathcal{B} \)).

Proof. Let \((V, E)\) be the dependency graph for \( \mathcal{F} \).

Claim 21. There is \( v \in V \) such that \( v \) is safe.

Proof of Claim 21 We show that there is \( v \in V \) without any parent, which, by Definition 11, implies that there is \( v \in V \) such that \( v \) is safe. Take an arbitrary \( v \in V \) and assume there is a sequence \( v_1, \ldots, v_n \) with \( n > 1 \) such that \( v_1 = v_n = v \) and \((v_i, v_{i+1}) \in E\) for all \( 1 \leq i < n \). For all \( 1 \leq i < n \), let \( v_i = (\psi_i, \phi_i) \), \( v_{i+1} = (\psi_{i+1}, \phi_{i+1}) \). This means that \( \phi_{i+1} \) occurs in a derivation of \( \phi_i \) w.r.t. \( \psi_i \), for all \( 1 \leq i < n \), and \( \phi_1 \) occurs in a derivation of \( \phi_n \) w.r.t. \( \psi_n \). This contradicts the fact that \( \mathcal{F} \) is acyclic. Thus, there is \( v \in V \) without any parent. This finishes the proof of Claim 21.

Claim 21 directly implies the first statement of this lemma. One can determine construct the dependency graph in quadratic time (since there are quadratic many
possible pairs) and check if there is a derivation using the strategy in the proof of Proposition 4.

Lemmas 22-24 are used to prove our main result (Theorem 15). In the following, we denote by $F^n$ the formula $F$ at the beginning of the $n$-th iteration of Algorithm 1. Lemma 22 states that the clauses in a safe pair can only be the ones given as input to the algorithm, not their ‘weakened’ versions. This means that the number of iterations of Algorithm 1 is polynomially bounded on the size of its input.

**Lemma 22.** In each iteration $n$ of Algorithm 1, if there are $\phi, \psi \in F^n$ s.t. $(\psi, \phi)$ is safe for $F^n$ then $\phi, \psi \in F^1$.

*Proof.* At the first iteration the lemma holds trivially. Suppose that for $n > 1$ there are $\psi, \phi \in F^n$ s.t. $(\psi, \phi)$ is safe for $F^n$. We first argue that $\psi$ must be in $F^1$. Then, we argue that this also needs to be the case for $\phi$.

**Assume** $\psi \notin F^1$. This means that $\psi$ has been added in Line 7 at iteration $k$ with $1 < k < n$. Then $\psi$ is coherent with $F^k \setminus \{\psi_r\}$ where $\psi_r$ is the clause being replaced. To show that $\psi$ is coherent with $F^{k+1}$ we need to argue that $\psi$ is coherent with $F^k \setminus \{\psi_r\}$ s and the clauses added in Line 7, which are coherent with $F^k \setminus \{\psi_r\}$. We can see that this holds because the consequent of all such clauses is the same as $\psi$. Thus, $\psi$ is coherent with $F^{k+1}$. By assumption, $(\psi, \phi)$ is safe for $F^n$, which means that $\psi \Rightarrow_{F^n} \phi$. That is,

$$F^n \cup \text{ant}(\psi) \models \text{ant}(\phi).$$

For all $m \leq n$, we have that $F^m \models F^n$. Since $k + 1 \leq n$, we have, in particular, that $F^{k+1} \models F^n$. Then,

$$F^{k+1} \cup \text{ant}(\psi) \models \text{ant}(\phi).$$

Either $\phi \in F^{k+1}$ or there is $\phi' \in F^{k+1}$ such that $\text{ant}(\phi') \subset \text{ant}(\phi)$. In other words, $\phi$ is a weakened version of $\phi'$, meaning that $\text{con}(\phi) = \text{con}(\phi')$. So either $\psi \Rightarrow_{F^{k+1}} \phi$ or $\psi \Rightarrow_{F^{k+1}} \phi'$, with $\text{con}(\psi) \in \text{con}(\phi)$. In both cases, this contradicts the fact that $\psi$ is coherent with $F^{k+1}$. Thus, we have that $\psi \in F^1$ as required.

**Assume** $\phi \notin F^1$. Let $k > 1$ be minimal s.t. $\phi \in F^k$. Then, $\phi$ was added at iteration $k - 1$, which means that $\phi$ is coherent with $F^{k-1} \setminus \{\phi_r\}$, where $\phi_r$ is the clause that was replaced. In fact, since the other clauses added in Line 7 at iteration $k - 1$ have the same consequent as $\phi$, we have that $\phi$ is coherent with $F^{k}$. We have already argued that $\psi$ is in $F^1$. Since $\psi \in F^n$ and $n \geq k$ we have that $\psi \in F^k$. As $F^k \models F^n$, $\psi \Rightarrow_{F^n} \phi$ implies $\psi \Rightarrow_{F^k} \phi$. The assumption that $(\psi, \phi)$ is safe implies that $\text{con}(\psi) \in \text{con}(\phi)$, which contradicts the fact that $\phi$ is coherent with $F^k$. Thus, $\phi \in F^1$.

**Lemma 23.** In all iterations of Algorithm 1 the set

$$\{\phi^+ p \mid p \in I, l \in \text{ant}(\psi) \setminus \text{ant}(\phi)\}$$

in Line 7 is not empty.
Proof. In the first iteration, the lemma holds because of the assumption that \( F^1 \) is not in conflict. Assume that \( F^n \) at the beginning of iteration \( n \) is not in conflict. We show that \( F^{n+1} \) at the beginning of iteration \( n+1 \) is not in conflict, which means that the set in Line 7 is not empty. Suppose this is not the case. That is,

- there are \( \psi_1, \psi_2 \in F^{n+1} \) s.t. \( \psi_1 \Rightarrow_{F^{n+1}} \psi_2 \), \( \text{con} (\psi_1) \subseteq \text{con}(\psi_2) \), and
- there is no \( r \in \text{ant}(\psi_1) \setminus \text{ant}(\psi_2) \) with \( q \in \tau \) s.t. \( \psi_1^r \) is coherent with \( F^{n+1} \setminus \{\psi_2\} \).

By the inductive hypothesis, this can only be because, at iteration \( n \), there are \( \psi, \phi \in F^n \) such that \( (\psi, \phi) \) is safe (for \( F^n \)) and \( \phi \) is replaced by

\[
\{ \phi^p \mid p \in I, l \in \text{ant}(\psi) \setminus \text{ant}(\phi) \},
\]

and now, after the update, \( F^{n+1} \) is in conflict. For all \( \phi' \) in such set, \( \text{con}(\phi) = \text{con}(\phi') \) and \( \text{ant}(\phi) \subseteq \text{ant}(\phi') \). So \( F^n \models F^{n+1} \). Moreover, every such \( \phi' \) is coherent with \( F^{n+1} \). This means that \( \psi_1, \psi_2 \) above are in \( F^n \). If there is no \( r \in \text{ant}(\psi_1) \setminus \text{ant}(\psi_2) \) with \( q \in \tau \) s.t. \( \psi_1^r \) is coherent with \( F^{n+1} \setminus \{\psi_2\} \) then, since \( F^n \models F^{n+1} \), the same happens with \( F^n \setminus \{\psi_2\} \). This means that \( F^n \) is in conflict which contradicts our initial assumption that this is not the case.

Lemma 24 shows that Algorithm 1 satisfies (P3).

**Lemma 24.** Let \( F \) be the output of Algorithm 1 with \( \bigcup_{i=1}^n F_i \) not in conflict as input. For all \( i \in \{1, \ldots, n\} \) and all \( \phi \in F_i \), \( F \not\models (\text{ant}(\phi) \rightarrow p) \) with \( p \in \text{con}(\phi) \).

**Proof.** Assume there is \( i \in \{1, \ldots, n\} \) and \( \phi \in F_i \) such that

\[ F \models \text{ant}(\phi) \rightarrow p \text{ with } p \in \text{con}(\phi). \]

By Lemma 23, for all \( i \in \{1, \ldots, n\} \) and all \( \psi \in F_i \) there is \( \psi' \in F \) such that \( \{\psi\} \models \psi' \). Let \( \phi' \in F \) be such that \( \{\phi\} \models \phi' \). If

\[ F \models \text{ant}(\phi) \rightarrow p \text{ with } p \in \text{con}(\phi) \]

then there is \( \phi^* \in F \) such that \( \phi' \Rightarrow_F \phi^* \) and \( \text{con}(\phi^*) = p \). Indeed, if \( F \models \text{ant}(\phi) \rightarrow p \)
then there is a minimal subset \( \{\phi_1, \ldots, \phi_k\} \) of \( F \) such that

\[ \text{con}(\phi_k) = p, \]

\[ \{\phi_1, \ldots, \phi_k\} \models \text{ant}(\phi) \rightarrow p, \]

and

\[ F \cup \text{ant}(\phi) \models \text{ant}(\phi_i), \]

for all \( 1 \leq i \leq k \).

As \( \{\phi\} \models \phi' \) and \( \phi \) is satisfiable (because it is a definite Horn clause), \( \text{ant}(\phi) \subseteq \text{ant}(\phi') \), so \( F \cup \text{ant}(\phi') \models \text{ant}(\phi_i) \). For all \( 1 \leq i \leq k \). In particular, \( F \cup \text{ant}(\phi') \models \text{ant}(\phi_k) \).

This means that \( \phi' \Rightarrow_F \phi_k \) and \( \phi^* = \phi_k \) is as required. This contradicts the condition in the main loop of Algorithm 1. \( \square \)
Lemma 25. Let $\mathcal{F}$ be the output of Algorithm 1 with $\bigcup_{i=1}^{n} \mathcal{F}_i$ not redundant as input. For all $\phi \in \mathcal{F}$, if there is $p \in \text{ant}(\phi)$ such that, for all $q \in \overline{p}$, $\mathcal{F} \cup \{\phi \land p\}$ is coherent and there is $i \in \{1, \ldots, n\}$ such that $\mathcal{F}_i \models \phi \land p$ then $\mathcal{F}_i \not\models \phi \lor \neg p$.

Proof. To show our lemma we use the following claim.

Claim 26. Let $(\psi, \phi)$ be the safe pair chosen at iteration $n$ of the ‘while loop’ by Algorithm 1. For all iterations $k \geq n$ and all $\phi \lor p$ used to replace $\phi$ in Line 7 of Algorithm 1 (at iteration $n$), there is $q \in \overline{p}$ such that $\phi \lor p$ is incoherent with $\mathcal{F}^k$.

The proof of this claim follows from the fact that, since $(\psi, \phi)$ is safe, it has no parent node in the dependency graph of $\mathcal{F}^n$. This means that

- $\psi \Rightarrow_{\mathcal{F}^k} \phi$, $\text{con}(\psi) \subseteq \text{con}(\phi)$, and

- there is no pair $(\psi', \phi')$ (in the dependency graph of $\mathcal{F}^n$) such that $\phi' \neq \phi$ and $\phi'$ occurs in a derivation of $\phi$ w.r.t. $\psi$ and $\mathcal{F}$.

This means that, for all $k \geq n$, we have that $\psi \Rightarrow_{\mathcal{F}^k} \phi$, and $\text{con}(\psi) \subseteq \text{con}(\phi)$ (because none of the rules involved in the derivation are incoherent, so the algorithm will not change them). In other words, for all iterations $k \geq n$ and all $\phi \lor p$ used to replace $\phi$ in Line 7 of Algorithm 1 (at iteration $n$), there is $q \in \overline{p}$ such that $\phi \lor p$ is incoherent with $\mathcal{F}^k$. Indeed, we can take $q = l \in \text{ant}(\psi)$ (see Line 7 of Algorithm 1).

To finish the proof of this lemma we make a case distinction. Let $\mathcal{F}$ be the Horn expression returned by Algorithm 1. Either a rule in $\mathcal{F}$ has been added at some iteration in Line 7 (and never replaced again by Lemma 22) or it belongs to $\mathcal{F}^1$. In the former case, the claim ensures that the rule satisfies the statement of this lemma, which corresponds to (P6) in Definition 8. In the latter, since $\bigcup_{i=1}^{n} \mathcal{F}_i$ is not redundant, it cannot be that $\mathcal{F}_i \models \phi \lor p$ for some $i \in \{1, \ldots, n\}$. $\square$

Theorem 15. The output of Algorithm 1 with an acyclic, non-redundant, not in conflict Horn expression $\bigcup_{i=1}^{n} \mathcal{F}_i$ and $\mathcal{B}$ as input is a common ground for $\bigcup_{i=1}^{n} \mathcal{F}_i$ and $\mathcal{B}$. Algorithm 1 terminates in $O((\|\bigcup_{i=1}^{n} \mathcal{F}_i\| + \|\mathcal{B}\|)^4)$.

Proof. We first argue that Algorithm 1 terminates in $O((\|\bigcup_{i=1}^{n} \mathcal{F}_i\| + \|\mathcal{B}\|)^4)$. We first point out that checking whether $\bigcup_{i=1}^{n} \mathcal{F}_i$ is cyclic, redundant, and in conflict can all be performed in $O((\|\bigcup_{i=1}^{n} \mathcal{F}_i\| + \|\mathcal{B}\|)^2)$. Lemma 22 bounds the number of iterations of the main loop to $\|\bigcup_{i=1}^{n} \mathcal{F}_i\|$ because it implies that only clauses in $\mathcal{F}^1 := \bigcup_{i=1}^{n} \mathcal{F}_i$ can be replaced. Since safe pairs can only come from clauses in $\mathcal{F}^1$ (Lemma 22), one does not need to compute incoherence in each iteration of the main loop (after the first computation, it is enough to keep track of the incoherent clauses in safe pairs). By Lemma 12, Line 6 can be computed in quadratic time. Line 7 can be computed in cubic time.

We now argue that the output $\mathcal{F}$ of Algorithm 1 is a common ground for $\bigcup_{i=1}^{n} \mathcal{F}_i$ and $\mathcal{B}$.

(P1) It is satisfied as this is the condition of the “while” loop.

(P2) It is also satisfied because, if $\bigcup_{i=1}^{n} \mathcal{F}_i$ is coherent, the algorithm does not enter in the “while” loop and simply returns $\bigcup_{i=1}^{n} \mathcal{F}_i$. 21
(P3) By Lemma 24, (P3) is satisfied.

(P4) By Lemma 22, only clauses in $\bigcup_{i=1}^{n} F_i$ are weakened. So, for all $\phi \in \mathcal{F}$, either $\phi \in \bigcup_{i=1}^{n} F_i$ or it results from adding an atom $p$ to the antecedent of a clause $\phi'$ in $\bigcup_{i=1}^{n} F_i$. This means that (P4) is satisfied.

(P5) We also have (P5) since, by Lemma 23, at least one weakened version of a replaced clause remains in $\mathcal{F}$.

(P6) By Lemma 25, (P6) is satisfied.

$\square$