Schwinger-Dyson Equations and Chiral Symmetry Breaking in 2D Induced Gravity

E. Elizalde
Department ECM and IFAE, Faculty of Physics, University of Barcelona, Diagonal 647, 08028 Barcelona, and Center for Advanced Studies, CSIC, Camí de Sta Bàrbara, 17300 Blanes, Catalonia, Spain,

S.D. Odintsov and A. Romeo
Department ECM, Faculty of Physics, University of Barcelona, Diagonal 647, 08028 Barcelona, Catalonia, Spain,

and
Yu.I. Shil’nov
Department of Theoretical Physics, Kharkov State University, Svobody sq. 4, Kharkov 310077, Ukraine.

Abstract

The Schwinger-Dyson equations in the ladder approximation for 2D induced gravity coupled to fermions on a flat background are obtained in conformal gauge. A numerical study of these equations shows the possibility of chiral symmetry breaking in this theory.
The Schwinger-Dyson equations provide a convenient way to study dynamical symmetry breaking and dynamical mass generation in quantum field theory. However, they constitute an infinite set of integral equations and surely some truncation scheme is necessary in order to be able to solve it.

In pioneering works [1, 2], the approach to dynamical fermion mass generation in quantum electrodynamics based on some truncated version of the Schwinger-Dyson equations (sometimes called ladder approximation) was developed, and it was shown the possibility of chiral symmetry breaking in QED, and also the existence of a critical coupling constant in the Landau gauge (for a review see [3]). During the last fifteen years there has been much activity in developing and further extending the approach of refs. [1, 2] to dynamical symmetry breaking in QED (for a review of the current status of this subject, see the proceedings [5]). However, if dynamical mass is generated, the nonperturbative Ward-Takahashi identity in QED is not satisfied. That is why the dynamically generated fermionic mass, as well as the critical coupling constant in QED, are highly gauge-dependent. That fact was confirmed recently by the study of dynamical symmetry breaking in QED in an arbitrary covariant gauge [4].

Currently, there is no doubt that the Schwinger-Dyson equations are powerful tools for the study of non-perturbative effects in field theory. However, perhaps some new approaches to truncation of the Schwinger-Dyson equations should be developed, particularly in order to solve the gauge-dependence problem in the dynamical symmetry breaking of QED. It may also be useful for this purpose to study the standard Schwinger-Dyson equations [1, 2] in other models (albeit more complicated) like quantum gravity. Some time ago, a study along this line was done in Ref. [6] for 4D Einstein quantum gravity coupled to fermions on a flat background. As a result, the possibility of chiral symmetry breaking in such model was shown [6].

The purpose of the present letter is to study the Schwinger-Dyson equations and the problem of chiral symmetry breaking in 2D induced gravity [7] coupled to fermions on a flat background. Notice that, recently, 2D induced gravity has been a popular
subject of study as a toy model for more realistic 4D gravity, which still does not exist as a self-consistent theory.

We shall start from the following action

\[ S = S_g + S_f, \]
\[ S_g = -\frac{1}{2\gamma} \int d^2x \sqrt{-g} \left( R \frac{1}{\Delta} R + \Lambda \right), \]
\[ S_f = \int d^2x \sqrt{-g} i \bar{\Psi} \gamma^\mu D_\mu \Psi, \]

(1)

where \( R \) is the two-dimensional curvature, \( \Psi \) the 2D spinor, \( \Lambda \) the cosmological constant, and \( D_\mu \) is the 2D covariant derivative for spinors. In the standard approach to 2D induced gravity, working in the conformal gauge \[8, 9\]

\[ g_{\mu\nu} = e^{\phi} \eta_{\mu\nu}, \]

(2)

one can start from a matter theory (like \( S_f \)) and induce \( S_g \) from it, by integrating over matter. After that, one has an exactly solvable theory with the action \( S_g \).

Instead, we will adopt here another approach, namely, to start from the action (1), where gravitational and spinor fields are supposed to be quantum. Using the conformal gauge (2) and working in the \textit{vierbein} formalism on a flat background, we consider \( \gamma \) as some given constant and do not integrate over \( \Psi \). Then, the term of the action corresponding to the interaction between \( \Psi \) and \( \phi \) is given by

\[ S_{\text{int}} = \frac{i}{2} \int d^2x \bar{\Psi} \left[ \phi \partial \Psi + \frac{1}{2} (\partial \phi) \Psi \right]. \]

(3)

Let us now calculate the effective potential for composite fields \[10\] in the ladder approximation \[1\]-\[3\]:

\[ V_{\text{eff}} = -i \text{Sp}[\ln S_0^{-1} S - S_0^{-1} S + 1] + V_2, \]

(4)

where the free fermion propagator is given by

\[ S_0(p) = \frac{1}{p}, \]

(5)
and the exact fermion propagator is

\[ S(p) = \frac{1}{\mathcal{A}(p^2)p - B(p^2)}, \]  

(6)

\(\mathcal{A}\) and \(\mathcal{B}\) being, for the moment, unknown functions.

Note that \(V_2\) corresponds (as it follows from the structure of \(S_{\text{int}}\)) to a two-particle irreducible diagram, which is similar to the analogous diagram in QED. Here, in \(V_2\) the fermion propagator is the exact one, while the vertex and the \(\varphi\) (graviton) field propagator

\[ \Gamma(k, p) = \frac{1}{2}k + \frac{1}{4}k, \]

\[ G(k) = \frac{\gamma}{k^2 - \Lambda/2}. \]  

(7)

are tree-level quantities (ladder approximation). Hence, \(V_2\) is given by

\[ V_2 = -\frac{i}{2} \int \frac{d^2p}{(4\pi)^2} \int \frac{d^2q}{(4\pi)^2} \text{Tr} \left[ \Gamma(p - q, q)S(q)\Gamma(q - p, p)G(p, q) \right]. \]  

(8)

Using (3) and (6)-(8) for calculating the effective potential (4), one can get, after performing Wick’s rotation and the angular integration (we drop the details of these calculations):

\[ V_{\text{eff}} = -\frac{N_f M^2}{8\pi} \left\{ \int_0^1 dx \left[ \ln \left( A^2(x) + \frac{B^2(x)}{x} \right) - 2\frac{A(x)(A(x) - 1)x + B^2(x)}{xA^2(x) + B^2(x)} \right] 
\right. 
+ g \int_0^1 \frac{dx}{xA^2(x) + B^2(x)} \int_0^1 \frac{dy}{yA^2(y) + B^2(y)} \left[ A(x)A(y)K_A(x, y) + B(x)B(y)K_B(x, y) \right] \right\}, \]  

(9)

where \(N_f\) is the dimension of the fermion representation, \(M\) is the momentum cutoff, \(x = p^2/M^2\), \(y = q^2/M^2\), \(A(x) = \mathcal{A}(p^2)\), \(B(x) = \mathcal{B}(p^2)/M\) and

\[ K_A(x, y) = \frac{-4xy + x + y + l - \sqrt{(x + y + l)^2 - 4xy}}{2\sqrt{(x + y + l)^2 - 4xy}}, \]

\[ K_B(x, y) = \frac{2(x + y) + l - \sqrt{(x + y + l)^2 - 4xy}}{\sqrt{(x + y + l)^2 - 4xy}}, \]  

(10)
with the notations
\[ g = \frac{\gamma}{64\pi}, \tag{11} \]
and
\[ l = \frac{\Lambda}{2M^2}. \tag{12} \]

The Schwinger-Dyson equation which corresponds to the effective potential of the composite fields (8), in the ladder approximation takes the form
\[ i(S^{-1}(p) - S_0^{-1}(p)) = \int \frac{d^2q}{(4\pi)^2} \left[ \Gamma(p - q, q) S(q) \Gamma(q - p, p) G(p, q) \right]. \tag{13} \]

Bearing in mind (8), integrating over the angles and doing the same changes of variable as before in (13), one can show that the functions \( A \) and \( B \) must obey integral equations of the following form
\[
A(x) = 1 + g \int_0^1 dy \frac{A(y)}{y A^2(y) + B^2(y)} \frac{1}{x} K_A(x, y), \\
B(x) = g \int_0^1 dy \frac{B(y)}{y A^2(y) + B^2(y)} K_B(x, y). \tag{14}
\]

Of course, it is not possible to solve these equations analytically. However, we will obtain their numerical solution by a standard iterative method at some region of the theory parameters. First, we fix the values of the parameters \( g \) and \( l \) appearing in (12) and prepare two types of trial functions (the procedure is quite similar to the one used in ref. [6]):

(a) \( A^0(x) = c_1, \quad B^0(x) = 0, \)
(b) \( A^0(x) = c_1, \quad B^0(x) = c_2, \)

where \( c_1 \) and \( c_2 \) are constants between 0 and 1. The functions \( A^0(x) \) and \( B^0(x) \) are the starting point of a self-consistent iterative calculation in which one finds successive pairs of functions \( A^i(x) \) and \( B^i(x) \). To be more precise, each pair is obtained from the previous one by means of the recurrence
\[
A^{i+1}(x) = 1 + g \int_0^1 dy \frac{A^i(y)}{y A^2(y) + B^2(y)} \frac{1}{x} K_A(x, y), \\
B^{i+1}(x) = g \int_0^1 dy \frac{B^i(y)}{y A^2(y) + B^2(y)} K_B(x, y). \tag{15}
\]
The sequences formed by the \( \{A^i(x)\} \) and \( \{B^i(x)\} \) are expected to converge into the functions \( A(x) \) and \( B(x) \), respectively, which are the desired solutions of (14). In practice, we judge the degree of convergence of these series by the smallness of the squared norms of the differences \( A^{i+1} - A^i \) and \( B^{i+1} - B^i \). In our calculations, we have set bounds of orders \( 10^{-4} - 10^{-6} \). If for the given \( g \) and \( l \) there are solutions of both types, (a) and (b), only the most stable of both by \( V_{\text{eff}} \) must be chosen as the one corresponding to the true vacuum.

We have executed this algorithm to solve (15), starting from trial functions (a) and (b) for fixed \( l = 0.5 \) and varying \( g \). For very small \( g \)'s, both types lead to curves close to \( A(x) = 1, B(x) = 0 \), i.e. the chiral symmetric solution, as should be expected. Moreover, their respective potentials are practically indistinguishable. As \( g \) increases, the \( V_{\text{eff}} \) for the (a)-type solution appears to be just slightly higher than for the (b)-type one, thus selecting the latter as the physical vacuum. This happens until some specific value of \( g \), around 0.1, is reached, for which the (a)-solution looks rather reluctant to converge. Before this happens, \( V_{\text{eff}} \) is marginally higher than for the (b)-solution. This seems to indicate that the solution ‘lost’ was not physical.

Typical curves representing the \( A \) and \( B \) functions obtained are shown in Fig. 1. All of them correspond to (b)-solutions (i.e. chiral non-symmetric solutions) found for different values of \( g \), and for \( l = 0.5 \). The starting constants where \( c_1 = c_2 = 0.5 \), but their precise value has no influence on the final form of \( A \) and \( B \).

As is clear, the value of \( V_{\text{eff}} \) corresponding to a solution obtained for a given \( g \) changes as we vary \( g \). \( V_{\text{eff}} \) vanishes as \( g \) approaches zero, and increases as \( g \) grows. There is a specific \( g \) for which it stops growing, and shows a slow decrease for larger \( g \)'s. However, this should be regarded cautiously, since the larger \( g \) is, the closer we are to regions where perturbative methods fail.

Hence, as we see from our numerical analysis, there is a possibility of chiral symmetry breaking in 2D induced gravity coupled to fermions. Our study has been done in the ‘physical’ conformal gauge which, in some sense, may be considered as
the analogue of the ‘physical’ Landau gauge in QED. The results of these numerical studies should be gauge dependent, particularly because we do not pay attention to the nonperturbative Ward-Takahashi identity resulting from general covariance. (Note that this identity is not so easy to obtain explicitly in the noncovariant conformal gauge). Hence, for a complete proof that chiral symmetry breaking in $2D$ induced gravity is a real physical phenomenon one has to repeat the same study in some covariant gauge (of harmonic type [11], for example). This is left to be done elsewhere.

**Acknowledgements**

S.D.O. would like to thank T. Muta for helpful discussions. This work has been supported by CIRIT (Generalitat de Catalunya) and by DGICYT (Spain), project no. PB90-0022.
References

[1] T. Maskawa and H. Nakajima, *Progr. Theor. Phys.* **52** (1974) 1326; **54** (1975) 860.

[2] R. Fukuda and T. Kugo, *Nucl. Phys.* **B117** (1976) 250.

[3] P.I. Fomin, V.P. Gusynin, V.A. Miransky and Yu. A. Sitenko, *Rivista Nuovo Cimento* **6** (1983) 1.

[4] K.I. Aoki, M. Bando, T. Kugo, K. Hasebe and H. Nakatani, *Progr. Theor. Phys.* **81** (1989) 1866.
   D.C. Curtis and M.K. Pennington, *Phys. Rev.* **D 48** (1993) 4933.

[5] T. Muta and K. Yamawaki (eds.), *Proceedings of the Workshop on Dynamical Symmetry Breaking*, Nagoya, 1990.
   W.A. Bardeen, J. Kodaira and T. Muta (eds.), *Proceedings of the International Workshop on Electroweak Symmetry Breaking*, World Sci., Singapore, 1992.

[6] O. Abe, *Progr. Theor. Phys.* **73** (1985) 1560.

[7] A.M. Polyakov, *Phys. Lett.* **B103** (1981) 207.

[8] F. David, *Mod. Phys. Lett.* **A3** (1988) 1651.

[9] J. Distler and H. Kawai, *Nucl. Phys.* **B 321** (1989) 500.

[10] J.M. Cornwall, R. Jackiw and E. Tomboulis, *Phys. Rev.* **D10** (1974) 2428.

[11] S.D. Odintsov and I.L. Shapiro, *Phys. Lett.* **B 263** (1991) 183; *Int. J. Mod. Phys.* **D1** (1993) 571.
Figure Caption

Fig. 1.

Plot of the functions $A$ and $B$ obtained as the (b)-type solutions for $g = 0.1, 0.2$ and $0.25$, keeping $l = 0.5$ fixed. Notice how $B$ deviates more and more from the $g = 0$ solution ($B(x) = 0$) as $g$ increases. Although not shown in the figure, the curve keeps going up for larger values of $g$. 
This figure "fig1-1.png" is available in "png" format from:

http://arxiv.org/ps/hep-th/9402087v1