The Universal Tutte Polynomial

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- Joint work with -
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1. Generalizing the Tutte polynomial to polymatroids.

\[ T_P(x, y) \]
Outline

1. Generalizing the Tutte polynomial to polymatroids.

\[ T_P(x, y) \]

matroids

polymatroids

hypergraphs

graphs

new

known
1. Generalizing the Tutte polynomial to polymatroids.

\[ T_P(x, y) \]

2. Universal Tutte polynomial

\[ T_n(x, y; z) \]
Tutte polynomial of polymatroids
Tutte polynomial of a graph

Def: For a connected graph $G = (V, E)$, the **Tutte polynomial** is

$$T_G(x, y) = \sum_{S \subseteq E} (x - 1)^{\text{cork}(S)}(y - 1)^{\text{null}(S)},$$

where

- $\text{cork}(S) = \#$ edges to add in order to get a connected subgraph,
- $\text{null}(S) = \#$ edges to delete to in order to get an acyclic subgraph.
Def: For a connected graph \( G = (V, E) \), the **Tutte polynomial** is

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Example:

\[
T_G(x, y) = x^3 + 2x^2 + 2xy + y^2 + x + y.
\]
Tutte polynomial of a graph

\[ T_G(x, y) \] captures a lot of enumerative information about \( G \).

# spanning trees, # forests, # connected subgraphs,
# acyclic orientations, # totally cyclic orientations,
Chromatic polynomial, Potts polynomial,
\( G \)-parking functions by degree, Reliability polynomial...

(Tutte polynomial is “universal for linear graph invariants.”)
Tutte polynomial of a graph

Def. Internal/External activities:
Fix a total order $\prec$ on $E$. For a spanning tree $T \subseteq E$,

$$IA(T) = \{ e \in T \mid \not\exists e' \prec e \text{ such that } T - e + e' \text{ is a tree} \}$$

$$EA(T) = \{ e \notin T \mid \not\exists e' \prec e \text{ such that } T + e - e' \text{ is a tree} \}$$
Tutte polynomial of a graph

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Example:

$T$

IA($T$) = {1}
EA($T$) = {3}
Tutte polynomial of a graph

**Thm** [Tutte/Crapo 47/67].

\[ T_G(x, y) = \sum_{T \text{ spanning tree}} x^{|IA(T)|} y^{|EA(T)|}. \]
Tutte polynomial of a graph

**Thm** [Tutte/Crapo 47/67].

\[ T_G(x, y) = \sum_{T \text{ spanning tree}} x^{|\text{IA}(T)|} y^{|\text{EA}(T)|}. \]

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Relation between the two expressions of $T_G(x, y)$?

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Example:

$G$

“Crapo’s interval partition”

$T_G(x, y) = x^2 + x + y.$
Matroids are abstraction of a graph: the matroid tells you which subsets of edges form a spanning tree.
Matroids

Def. A **matroid** on a set \( E \) is a set \( M \subseteq 2^E \) of **bases** satisfying:

**Exchange Axiom:** \( \forall A, B \in M, \forall i \in A \setminus B, \exists j \in B \setminus A \) such that

\[
A \cup \{j\} \setminus \{i\} \in M \quad \text{and} \quad B \cup \{i\} \setminus \{j\} \in M.
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Example. The following is a matroid on $E = [5]$: $M = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 3, 5\}, \{1, 4, 5\}, \{2, 3, 5\}, \{2, 4, 5\}\}$
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Prop. For any connected graph \( G = (V, E) \),
\[
M_G := \{T \subseteq E \text{ spanning tree}\}
\]
is a matroid on \( E \).
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Tutte polynomial of matroids

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Thm [Tutte/Crapo]

$$T_M(x, y) = \sum_{A \text{ basis}} x^{\lvert \text{IA}(A) \rvert} y^{\lvert \text{EA}(A) \rvert},$$

where

- $\text{IA}(A) = \{ e \in A \mid \not\exists e' \prec e \text{ such that } A - e + e' \text{ is a basis} \}$
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Polymatroids

\[ P \subseteq \mathbb{Z}^n \]

\[ M \subseteq \{0,1\}^n \]

matroids

polymatroids

hypergraphs

graphs
Polymatroids

Notation. \( \{e_1, \ldots, e_n\} \) canonical basis of \( \mathbb{Z}^n \).
\( a \in \mathbb{Z}^n \) has coordinates \( a = (a_1, \ldots, a_n) \).
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Def. A polymatroid on \( E = [n] \) is a finite set \( P \subseteq \mathbb{Z}^n \) satisfying

**Exchange Axiom:** \( \forall a, b \in P, \forall i \text{ s.t. } a_i > b_i, \exists j \text{ s.t. } b_j > a_j \) and
\( a + e_j - e_i \in P \) and \( b + e_i - e_j \in P \).
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Example. The following is a polymatroid on \( E = [3] \):

\[
\begin{align*}
a_1 + a_2 + a_3 & = 4 \\
1 & \quad 2 \\
3 & \quad \uparrow
\end{align*}
\]
Polymatroids

Notation. \( \{e_1, \ldots, e_n\} \) canonical basis of \( \mathbb{Z}^n \).
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Remark. For any matroid \( M \) on \( E = [n] \),

\[
P(M) := \left\{ a = \sum_{i \in A} e_i \mid A \in M \right\} \subseteq \{0, 1\}^n
\]

is a polymatroid. "Base polytope"

In fact, matroids \( \iff \) polymatroids contained in \( \{0, 1\}^n \).
Polymatroids from hypergraphs

Def: A hypergraph on a set $V$, is a multiset $E$ of subsets of $V$.

Example:
Polymatroids from hypergraphs

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Example:

![Hypergraph Diagram]

Remark: Graphs correspond to hypergraphs where every hyperedge $e \in E$ has size 2.
Polymatroids from hypergraphs

Def: Let $H = (V, E)$ be a hypergraph. Let $B_H$ be the corresponding bipartite graph. A **spanning hypertree** of $H$ is a function $\delta : E \rightarrow \mathbb{Z}_{\geq 0}$ such that there exists a spanning tree $T$ of $B_H$ such that

$\forall e \in E, \quad \delta(e) = \deg_T(e) - 1$. 
Polymatroids from hypergraphs

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**Example:**

\[
\begin{array}{ccc}
\text{H} & \text{has 3 hypertrees} \\
\begin{array}{c}
\text{2} \\
\text{0} \\
\text{1} \\
\text{1} \\
\text{0} \\
\text{2}
\end{array}
\end{array}
\]
Polymatroids from hypergraphs

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$$\forall e \in E, \quad \delta(e) = \deg_T(e) - 1.$$ 

**Remark:** If a hypergraph $H$ corresponds to a graph $G$, then the spanning hypertrees of $H$ are in bijection with the spanning trees of $G$. 

$G$ $B_H$
Polymatroids from hypergraphs

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---

**Diagram:**

- $G$
- $T$
- $B_H$
- $\delta$
Polymatroids from hypergraphs

**Prop:** For any hypergraph $H = (V, E)$, the set of spanning hypertrees of $H$ forms a polymatroid $P_H$ on $E$.

**Example:**

\[ H = \]

\[ P_H \]
Tutte polynomial of polymatroids?
Tutte polynomial of polymatroids?

**Tentative definition:** Let $P \subseteq \mathbb{Z}^n$ be a polymatroid. For $a \in P$, let

- $IA(a) = \{i \in [n] \mid \forall j < i \text{ such that } a - e_i + e_j \in P\}$,
- $EA(a) = \{i \in [n] \mid \forall j < i \text{ such that } a + e_i - e_j \in P\}$.

Then

$$TP(x, y) = \sum_{\text{a basis}} x^{|IA(a)|} y^{|EA(a)|}.$$
Tutte polynomial of polymatroids?

Tentative definition: Let $P \subseteq \mathbb{Z}^n$ be a polymatroid. For $a \in P$, let

\begin{align*}
IA(a) &= \{i \in [n] \mid \not\exists j < i \text{ such that } a - e_i + e_j \in P\}, \\
EA(a) &= \{i \in [n] \mid \not\exists j < i \text{ such that } a + e_i - e_j \in P\}.
\end{align*}

$$TP(x, y) = \sum_{a \text{ basis}} x^{|IA(a)|} y^{|EA(a)|}.$$ 

Does not work! Not invariant under reordering of $[n]$.

However $TP(x, 1)$ and $TP(1, y)$ are invariant under reordering of $[n]$. [Kalman 13, Kalman & Postnikov 17]
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For $a \in P$

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$T_P(x, y) = \sum_{a \text{ basis}} x^{|IA(a) \setminus EA(a)|} y^{|EA(a) \setminus IA(a)|} (x + y - 1)^{|IA(a) \cap EA(a)|}$.
Tutte polynomial of polymatroids?

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**Example:**

\[
T_P(x, y) = (x + y - 1)(x^2 + 2xy + y^2 + 2y + 3x + 2y + 2)
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T_P(x, y) = \sum_{a \text{ basis}} x^{\text{IA}(a) \setminus \text{EA}(a)} y^{\text{EA}(a) \setminus \text{IA}(a)} (x + y - 1)^{\text{IA}(a) \cap \text{EA}(a)}.
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T_P(x, y) = \sum_{a \text{ basis}} x^{|\text{IA}(a)\setminus\text{EA}(a)|} y^{|\text{EA}(a)\setminus\text{IA}(a)|} (x + y - 1)^{|\text{IA}(a)\cap\text{EA}(a)|}.
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**Thm [BKP]** This polynomial is invariant under reordering of $[n]$. 

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\]

**Thm [BKP]** This polynomial is invariant under reordering of $[n]$.

Moreover, for any matroid $M$ of rank $d$ on $E = [n]$,
\[
T_{P(M)}(x, y) = x^{n-d} y^d T_M \left( \frac{x + y - 1}{y}, \frac{x + y - 1}{x} \right).
\]
Tutte polynomial of polymatroids.

\[ T_P(x, y) = \sum_{a \in \mathbb{Z}^b} ??? \]

Interval partition?
Tutte polynomial of polymatroids.

Def. Let $P \subseteq \mathbb{Z}^n$ be a polymatroid. For $a \in P$ we define the cone

$$C(a) = a + \sum_{i \in IA(a) \setminus EA(a)} \mathbb{Z}_{\leq 0} e_i + \sum_{i \in EA(a) \setminus IA(a)} \mathbb{Z}_{\geq 0} e_i + \sum_{i \in IA(a) \cap EA(a)} \mathbb{Z} e_i.$$
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Tutte polynomial of polymatroids.

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Thm [BKP] For any polymatroid $P \subseteq \mathbb{Z}^n$,

$$\bigcup_{a \in P} C(a) = \mathbb{Z}^n.$$  

Moreover,

$$T_P \left( \frac{1}{1-u}, \frac{1}{1-v} \right) = \sum_{c \in \mathbb{Z}^n} u^{\text{cork}(c)} v^{\text{null}(c)},$$

where $\text{cork}(c) = \min(|b| | c + b \geq a \in P)$, $\text{null}(c) = \min(|b| | c - b \leq c \in P)$.  

Relation with Cameron-Fink’s invariant

**Def:** The **Cameron-Fink invariant** for a polymatroid $P \subseteq \mathbb{Z}^n$ is the unique polynomial $Q_P(x, y)$ such that $\forall k, \ell \in \mathbb{Z}_{\geq 0}$,

$$Q_P(k, \ell) = |(P + k\nabla + \ell\Delta) \cap \mathbb{Z}^n|,$$

where $\Delta = \text{conv}(e_i, i \in [n])$ and $\nabla = \text{conv}(-e_i, i \in [n])$. 
Relation with Cameron-Fink’s invariant

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Q_P(k, \ell) = |(P + k \nabla + \ell \Delta) \cap \mathbb{Z}^n|,
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Example:
Relation with Cameron-Fink’s invariant

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where $\Delta = \text{conv}(e_i, \ i \in [n])$ and $\nabla = \text{conv}(-e_i, \ i \in [n])$.

Prop [BKP]:

$$Q_P(x, y) = \sum_{i,j} c_{i,j} \binom{x}{i} \binom{y}{j},$$

where $c_{i,j} = [x^i y^j] \frac{T_P(x + 1, y + 1)}{x + y + 1}$. 
Universal Tutte Polynomial
Polymatroids as generalized permutahedra

**Def.** A polytope $P \subseteq \mathbb{R}^n$ is a **generalized permutahedron** if every edge has direction of the form $e_i - e_j$ for some $i, j \in [n]$.

**Example.**
Polymatroids as generalized permutahedra

Def. A polytope $P \subseteq \mathbb{R}^n$ is a **generalized permutahedron** if every edge has direction of the form $e_i - e_j$ for some $i, j \in [n]$.

Example.

Prop. $P \subseteq \mathbb{Z}^n$ is a polymatroid if and only if $P = P \cap \mathbb{Z}^n$ for some generalized permutahedron $P$ with vertices in $\mathbb{Z}^n$. 
The space of polymatroids

Def. A function $f : 2^{[n]} \to \mathbb{R}$ is submodular if $f(\emptyset) = 0$ and

$$\forall A, B \subseteq [n], \quad f(A) + f(B) \geq f(A \cup B) + f(A \cap B).$$
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**Prop [Edmonds 70].** For a submodular function $f : 2^{[n]} \rightarrow \mathbb{Z}$, the set

$$P_f := \{a \in \mathbb{Z}^n \mid \sum_{i \in [n]} a_i = f([n]) \text{ and } \forall I \subseteq [n], \sum_{i \in I} a_i \leq f(I)\}$$

is a polymatroid.
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\]
is a polymatroid.

Conversely, for any polymatroid $P \subseteq \mathbb{Z}^n$ there exists a unique
submodular function $f : 2^{[n]} \to \mathbb{Z}$ such that $P = P_f$.

We call $f$ the **rank function** of $P$. 
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\]
is a polymatroid.

Conversely, for any polymatroid $P \subseteq \mathbb{Z}^n$ there exists a unique submodular function $f : 2^{[n]} \to \mathbb{Z}$ such that $P = P_f$.

We call $f$ the **rank function** of $P$.

**Conclusion.** The set of polymatroids on $[n]$ is indexed by the lattice points in the following polyhedron of dimension $2^{n-1}$:
\[
\{(f_I)_{I \subseteq [n]} \in \mathbb{R}^{2^{[n]}} \mid f_{\emptyset} = 0 \text{ and } \forall A, B \subseteq [n], \quad f_A + f_B \geq f_{A \cup B} + f_{A \cap B}\}
\]
Universal Tutte polynomial

Thm [BKP]. The Tutte polynomial is polynomial in the rank function.
Universal Tutte polynomial

**Thm [BKP].** Let $n \in \mathbb{Z}_{>0}$, and let $z = (z_I)_{\emptyset \neq I \subseteq [n]}$ be variables. There exists a unique polynomial $T_n(x, y; z)$ such that for all submodular function $f : 2^{[n]} \to \mathbb{Z}$,

$$T_{Pf}(x, y) = T_n(x, y; z)|_{z_I = f(I)}.$$
Universal Tutte polynomial

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\[
T_{Pf}(x, y) = T_n(x, y; z)|_{z_I = f(I)}.
\]

**Example.**

\[
\frac{T_3(x, y; z)}{x + y - 1} = x^2 + 2xy + y^2 \\
+ (z_1 + z_2 + z_3 - z_{123} - 2) x \\
+ (z_{12} + z_{13} + z_{23} - 2z_{123} - 2) y \\
+ \frac{1}{2} (z_{123}^2 - z_{12}^2 - z_{13}^2 - z_{23}^2 - z_1^2 - z_2^2 - z_3^2) \\
- z_{123}(z_1 + z_2 + z_3) \\
+ (z_1z_{12} + z_1z_{13} + z_2z_{12} + z_2z_{23} + z_3z_{13} + z_3z_{23}) \\
+ \frac{1}{2} (3z_{123} - z_{12} - z_{13} - z_{23} - z_1 - z_2 - z_3) + 1.
\]
Proof:
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**Uniqueness:**
Space $\Omega$ of polymatroids contains a cone of dimension $2^{n-1}$. 
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Space $\Omega$ of polymatroids contains a cone of dimension $2^{n-1}$.

**Existence:**
- In the bulk of $\Omega$: activity constant in the interior of each face, and number of points in each face is polynomial in the $z_I$. 

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![Diagram showing the space $\Omega$ with arrows indicating directions and labels for $z_1, z_2, z_3, z_{1,2}, z_{1,3}, z_{2,3}, z_{1,2,3}$]
Proof:

Uniqueness:
Space $\Omega$ of polymatroids contains a cone of dimension $2^{n-1}$.

Existence:
- In the bulk of $\Omega$: activity constant in the interior of each face, and number of points in each face is polynomial in the $z_I$.
- At the boundary of $\Omega$ the contribution of “collapsing” faces behaves polynomially.

$$T_2(x, y; z) = (x + y - 1)x + (x + y - 1)y + (z_1 + z_2 - z_{1,2} - 1)(x + y - 1)$$
Application: Brylawsky’s identities

**Easy consequence:** For any polymatroid $P \subseteq \mathbb{Z}^n$,

$$[x^i y^{n-i}] T_P(x, y) = [x^i y^{n-i}] T_n(x, y; 0) = \binom{n}{i}.$$
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**Cor:**[Brylawski 72] For any matroid $M \subseteq 2^{[n]}$ the coefficients $t_{i,j} = [x^i y^j] T_M(x, y)$ satisfy

$$\forall p < n, \quad \sum_{i=0}^p \sum_{j=0}^i \binom{p-i}{j} (-1)^j t_{i,j} = 0.$$
Explicit formula for $T_n$

Def:[Postnikov] $(d_I)_{\emptyset \neq I \subseteq [n]} \in \mathbb{Z}_{\geq 0}^{2^n}$ is draconian if

$$\forall I_1, \ldots, I_k \subseteq [n], \quad d_{I_1} + \cdots + d_{I_k} \leq |I_1 \cup \cdots \cup I_k| - 1,$$

and

$$\sum_{I \subseteq [n]} d_I = n - 1.$$
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**Def:** [Postnikov] $(d_I)_{\emptyset \neq I \subseteq [n]} \in \mathbb{Z}^2_{\geq 0}$ is **draconian** if

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The **dragon polynomial** is the following polynomial in $t = (t_I)_{\emptyset \neq I \subseteq [n]}$

$$D_n(t) = \sum_{(d_I) \text{ draconian}} \binom{t_{[n]} - 1}{d_{[n]}} \prod_{\emptyset \neq I \subseteq [n]} \binom{t_I}{d_I},$$

where $\binom{t}{d} := \frac{t(t-1) \cdots (t-d+1)}{d!}$. 


Explicit formula for $T_n$

**Thm [BKP]:** Let $\hat{T}_n(x, y; t) = T_n(x, y; z)|_{z_I=\sum J \subseteq [n], \ J \cap I \neq \emptyset} t_J$.

Then

$$\hat{T}_n(x, y; t) = (x + y - 1) \sum_{B=(B_1, \ldots, B_{\ell})} (-1)^{\ell-1} D_n(t^B) x^{lr(B)-1} y^{rl(B)-1},$$

where

- $t^B = (t^B_I)$ with $t^B_I = \sum_{J \subseteq \bigcup_{i<k} B_i} t_{I \cup J}$ if $I \subseteq B_k$ for some $k$, 0 otherwise,

- $lr(B)$ is the number of left-to-right minima of $B$,
- $rl(B)$ is the number of right-to-left minima of $B$. 
Some explanation/intuition for the formula:

\[ \hat{T}_n(x, y; t) = (x + y - 1) \sum_{B=(B_1, \ldots, B_\ell)} (-1)^{\ell-1} D_n(t^B) x^{lr(B)-1} y^{rl(B)-1}, \]

\[ \mathcal{P} = \sum_{I \subseteq [n]} t_I \Delta_I, \]

- Change of variables \( z \rightarrow t \):
  The tuple \( z = (z_I) \) given by \( z_I = \sum_{J \subseteq [n] : J \cap I \neq \emptyset} t_J \) is the rank function of \( P = \sum_{I \subseteq [n]} t_I \Delta_I \), where \( \Delta_I = \text{conv}(e_i, i \in I) \).
Some explanation/intuition for the formula:

\[ \hat{T}_n(x, y; t) = (x + y - 1) \sum_{B=(B_1, \ldots, B_\ell)} (-1)^{\ell-1} D_n(t^B) x^{lr(B)-1} y^{rl(B)-1}, \]

where \( B = \{B_1, \ldots, B_\ell\} \) \( \cup B_k = [n] \).

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- The partitions \( B \) index the faces of a generic permutahedron. The tuple \( t^B \) gives the rank function of the face.
Some explanation/intuition for the formula:

$$\widehat{T}_n(x, y; t) = (x + y - 1) \sum_{B = (B_1, \ldots, B_{\ell}) \cup B_k = [n]} (-1)^{\ell - 1} D_n(t^B) x^{lr(B) - 1} y^{rl(B) - 1},$$

- Change of variables $z \rightarrow t$:
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- The partitions $B$ index the faces of a generic permutahedron. The tuple $t^B$ gives the rank function of the face.

- The dragon polynomial $D_n(t)$ gives the number of lattice points in the interior of a permutahedron [Postnikov 06].
The draconian sequences correspond to the hypertrees of the complete hypergraph $H_n$ on $[n]$ having one hyperedge for each $I \subseteq [n]$. 

̂$T_n(x, y; t) = (x+ y−1) \sum_{B=(B_1,\ldots,B_{\ell}) \cup B_k=[n]} (-1)^{\ell−1} D_n(t^B) x^{lr(B)−1} y^{rl(B)−1},$
Cor [BKP]: The classical permutahedron

\[ P_n = \text{conv}\{(\pi(1), \pi(2), \ldots, \pi(n)), \pi \in S_n\} \cap \mathbb{Z}^n \]

has Tutte polynomial

\[ T_{P_n}(x, y) = \sum_{F \text{ forest on } [n]} (x + y - 1)^\# \text{ connected components}. \]

Example. 

\[ T_{P_3}(x, y) = (x + y - 1)^3 + 3(x + y - 1)^2 + 3(x + y - 1). \]
Thanks.