Extended Dijkstra algorithm and Moore-Bellman-Ford algorithm

Cong-Dian Cheng

School of Intelligence Technology, Geely University of China, Chengdu 641400, PR China
E-mail: zhiyang918@163.com.

Abstract Study the general single-source shortest path problem. Firstly, define a path function on a set of some path with same source on a graph; and develop a kind of general single-source shortest path problem (GSSSP) on the defined path function. Secondly, following respectively the approaches of the well known Dijkstra’s algorithm and Moore-Bellman-Ford algorithm, design an extended Dijkstra’s algorithm (EDA) and an extended Moore-Bellman-Ford algorithm (EMBFA) to solve the problem GSSSP under certain given conditions. Thirdly, introduce a few concepts, such as order-preserving in last road (OPLR) of path function, and so on. And under the assumption that the value of related path function for any path can be obtained in \( M(n) \) time, prove respectively the algorithm EDA solving the problem GSSSP in \( O(n^2)M(n) \) time and the algorithm EMBFA solving the problem GSSSP in \( O(mn)M(n) \) time. Finally, some applications of the designed algorithms are shown with a few examples. What we done can improve both the researches and the applications of the shortest path theory.

Keywords graph · network · path function · shortest path · algorithm

Mathematics Subject Classification (2010) MSC 90C35(Primary) · MSC 90C27 (Secondary)

1 Introduction

The shortest path problems are one kind of best known combinatorial optimization problems and have been extensively studied for more than half a century, which have many applications in network, electrical routing, transportation, robot motion planning, critical path computation in scheduling, quick response to urgent relief, etc; and can also unify framework for many optimization problems such as knapsack, sequence alignment in molecular biology, inscribed polygon construction, and length-limited Huffman-coding, etc. For the basic knowledge of the shortest path problems, please refer to the chapter 7 of [1] and the other literatures afterwards.

The classical single-source shortest path problem of network, denoted by CSSSP, is the most famous one of the shortest path problems, and a lot of works have been done to study and solve this kind of shortest path problem. Among many algorithms for the problem CSSSP, Dijkstra’s Algorithm (DA) and Moore-Bellman-Ford Algorithm (MBFA) are two well-known and most fundamental, which have now been come the core technique to solve many optimization problems. As we all know, the first one can solve the problem
CSSSP with nonnegative edge weights in $O(n^2)$ time and the second one can deal with the problem CSSSP with arbitrary conservative weights in $O(nm)$ time. Here $n$ and $m$ denote respectively the number of vertices and the number of edges on the graph. See, e.g., the chapter 7 of [1] and literatures [2-5].

Note the facts when a graph with nonnegative edge weights and its edges model respectively the transportation system of a city and the roads of the city, some of the edges of the graph may be blocked at certain times, and the traveler only observes that upon reaching an adjacent site of the blocked edge. Xiao et al. [6] (2009) introduce the definition of the risk of pathes, which is really a function on the set of all the pathes with a same source on the graph; and introduce also the anti-risk problem (ARP) of finding a path such that it has minimum risk, which, on the one hand, is a kind of single-source shortest path problem, and on the other hand, is different from the classical single-source shortest path problem. They also show the problem ARP can be solved in $O(nm + n^2 \log n)$ time suppose that at most one edge may be blocked. Afterwards, Mahadeokar and Saxena [7] (2014) propose a faster algorithm to solve the problem ARP.

Motivated by the stated background of research above, the present work develops a general single-source shortest path problem (GSSSP), which include the classical problem CSSSP and the problem ARP as its special cases; and tries to design an extended Dijkstra's algorithm (EDA) and an extended Moore-Bellman-Ford Algorithm (EMBFA) to solve the problem GSSSP under certain conditions, which respectively reduce to Dijkstra’s algorithm and Moore-Bellman-Ford Algorithm while the problem GSSSP is the classical problem CSSSP; and make some other related studies.

The rest of this article is organized as follows. Some preliminaries are presented in Section 2. Section 3 formulates the problem GSSSP. Section 4 is specially devoted to designing the algorithms EDA and EMBFA respectively. Section 5 makes the analyses of EDA and EMBFA. Section 6 shows the applications of EDA and EMBFA with a few instances of problem GSSSP. Finally, the paper is concluded with the Section 7.

2 Preliminaries

This section provides some preliminaries for our sequel research.

Suppose $V$ is a set of $n (>1)$ points.

Let $u, v \in V$. We use $[u, v]$ to denote an edge connecting two points $u$ and $v$. And use $(u, v)$ to denote a road on the edge $[u, v]$ from $u$ to $v$ ($v$ to $u$). (Note: $[u, v] = [v, u]$, while $(u, v) \neq (v, u)$.)

When there are more than one edge (road) between $u$ and $v$ (from $v$ to $u$), $[u, i, v]$ may be used to denote the $i$th edge (road). However, to simplify in notation afterwards, $[u, i, v]$ is denoted as $[u, v]$ and $(u, i, v)$ is denoted as $(u, v)$ when $i$ needn’t be indicated.

A edge $[u, v]$ is called undirected (directed) if there are two roads $(u, v)$ and $(v, u)$ (there is only one road $(u, v)$ or $(v, u)$) on it.
Let $E$ be all the edges and $R$ be all the roads. The triple $(V, E, R)$ is called as graph, which is also denoted by the tuple $(V, E)$ ( $(V, R)$ ) when all the roads $R$ (edges $E$ ) needn’t be indicated for clearness and brevity.

A graph is called as a undirected graph (directed graph\mixed graph) if it has only undirected edges (has only directed edges) has both undirected edges and directed edges).

Suppose $G = (V, E, R)$ is a graph.

When $[v_{(i-1)}, v_i] \in E$, $i = 1, 2, \cdots, k$, the combination of edges $\{[v_0, v_1], [v_1, v_2], \cdots, [v_{k-1}, v_k]\}$ is called as a chain connecting $v_0$ and $v_k$, denoted by $C[v_0, v_k]$. (Note: $C[v_0, v_k] = C[v_k, v_0]$)

When $(v_{(i-1)}, v_i) \in R$, $i = 1, 2, \cdots, k$, the combination of roads $\{(v_0, v_1), (v_1, v_2), \cdots, (v_{k-1}, v_k)\}$ is called as a path from $v_0$ to $v_k$, denoted by $P(v_0, v_k)$. (Note: $P(v_0, v_k) \neq P(v_k, v_0)$.) We also use $\{v_0, v_1, \cdots, v_{k-1}, v_k\}$, $(v_0, v_1) + (v_0, v_1) + \cdots + (v_{k-1}, v_k)$, $(v_0, v_1, \cdots, v_{k-1} + (v_{k-1}, v_k))$ denote $P(v_0, v_k)$. That is,

$$P(v_0, v_k) = \{(v_0, v_1), (v_1, v_2), \cdots, (v_{k-1}, v_k)\} = (v_0, v_1, \cdots, v_{k-1}, v_k) = (v_0, v_1, \cdots, v_{k-1}) + (v_{k-1}, v_k).$$

$v_0$ and $v_k$ are respectively called the source and terminal of $P$, denoted by $s(P)$ and $t(P)$; and $(v_{(i-1)}, v_i)$ is called a road of $P$, denoted by $(u_i, v_i)$ in $P$. We call $v_i$ an ancestor of $v_k$ resp. $v_k$ as a descendant of $v_i$ in path $P$, when $i < k$. And we also call $v_k$ can be reached from $v_0$. A path $P = (v_0, v_1, \cdots, v_{k-1}, v_k)$ is called a no circles if $v_i \neq v_j$ while $i \neq j$.

A mapping $w : E(R) \to (-\infty, \infty)$ is called as a weight of edges (roads). And the triple $(V, w, E)$ $(V, w, R)$ is called as a network with edge (road) weight. The tuple $(G, w)$ is used to represent both networks $(V, w, E)$ and $(V, w, R)$ when $w((u, v)) = w((u, v)), \forall u, v \in V$.

Let $|\delta(v)|$ be the degree of point $v$, see e.g. the chapter 2 of [1]. Define $\Delta(G) = \max \{|\delta(v)| | v \in V\} < \infty$, called as the maximum degree of graph $G$.

Let $s \in V$ (called source point). All the paths (or the paths without circles) of $G$ with the same source point $s$ is called a path system on $[G, s]$. To be convenient and clear, we stipulate $(s, s)$ is a special path with the source point $s$.

Suppose $\mathcal{P}$ is a path system on $[G, s]$.

We put $\mathcal{P}(u) = \{P \in \mathcal{P} | t(P) = u\} \mathcal{P}(s) = \{(s, s)\}$, $V(\mathcal{P}) = \{u | \mathcal{P}(u) \neq \emptyset\}$, $R(\mathcal{P}) = \{(u, v) \in P | P \in \mathcal{P}\}$ and $E(\mathcal{P}) = \{(u, v) | (u, v) \in R(\mathcal{P})\}$. And let $\mathcal{P}_{nc}(u)$ be all the paths without circles and in $\mathcal{P}(u)$.

When $P = (v_0, v_1, \cdots, v_{k-1}) \in \mathcal{P}$, $P' = (v_0, v_1, \cdots, v_{k-1}, v_k) = P + (v_{k-1}, v_k) \in \mathcal{P}$ and $v_{k-1} \neq v_k$, we call $P$ as the farther of $P'$, denoted by $FP'$, and $P'$ as a son of $P$, denoted by $SP$. We also call $v_{k-1}$ as the farther of $v_k$ in the path $P'$ and $v_k$ is the son of $v_{k-1}$ in the path $P'$. 

∀P, P′ ∈ ℙ, define P′ ⪯ P if and only if P = (v₀, v₁, · · · , vₖ₋₁, vₖ), P′ = (v₀, v₁, · · · , vₖ₋₁, vₖ′), 0 ≤ k′ < k; and P′ ⪯ P if and only if P′ < P or P′ = P.

A mapping f : ℙ → (−∞, ∞) is called as a path function on ℙ.

Finally, ∀v ∈ V(ℙ), define m⊥(v) = inf{f(P)|P ∈ ℙ(v)}. And P is called a minimum path on f if f(P) = m⊥(t(P)).

**Definition 1** Let ℙ be a path system on [G, s] and f be a path function on ℙ. (i) f is said to be non-decreasing if and only if ∀P′ ∈ ℙ, provided P′ ≤ P, we have f(P′) ≤ f(P). f is said to be non-decreasing in shortest path (NDSP) if and only if ∀v ∈ V(ℙ), provided P ∈ ℙ(v) and f(P) = m⊥(v), we have f(P′) ≤ f(SP) for any son SP of P. (ii) f is said to be increasing if and only if ∀P′, P ∈ ℙ, provided P′ < P, we have f(P′) < f(P). f is said to be increasing in shortest path (INSP) if and only if ∀P, P′ ∈ ℙ(v), we have f(P) < f(SP) for any son SP of P. (iii) f is said to be weak order-preserving (WOP) if and only if ∀v ∈ ℙ(v), provided ∀P, P′ ∈ ℙ(u), P + (u, v), P′ + (u, v) ∈ ℙ and f(P) < f(P′), we have f(P + (u, v)) < f(P′ + (u, v)). f is said to be order-preserving in shortest path (OPSP) if and only if f is WOP, and ∀u, v ∈ V, provided ∀P, P′ ∈ ℙ(u), P + (u, v), P′ + (u, v) ∈ ℙ and f(P) = f(P′), we have f(P + (u, v)) = f(P′ + (u, v)). (iv) f is said to be semi-order-preserving (SOP) if and only if f is WOP and ∀u, v ∈ V, provided ∀P, P′ ∈ ℙ(u), P + (u, v), P′ + (u, v) ∈ ℙ and f(P) ≤ f(P′), we have f(P + (u, v)) ≤ f(P′ + (u, v)). f is said to be semi-order-preserving in shortest path (SOPSP) if and only if ∀v ∈ V, provided ∀P, P′ ∈ ℙ(u), P + (u, v), P′ + (u, v) ∈ ℙ(v), f(P) ≤ f(P′) and f(P) = m⊥(u), we have f(P + (u, v)) ≤ f(P′ + (u, v)).

**Definition 2** Let ℙ be a path system on [G, s] and f be a path function on ℙ. f is said to have no negative (resp. non-positive) circles if and only if ∀v ∈ V, provided ∀P ∈ ℙ(v), C = (v, v₁, · · · , vₖ, v) and ∀P = P + C ∈ ℙ(v), we have f(P′) − f(P) ≥ 0 (resp. f(P′) − f(P) > 0).

**Definition 3** Let ℙ be a path system on [G, s] and f be a path function on ℙ. f is said to be weak inherited on shortest path (WISP) (inherited on shortest path (ISP)) if ∀v ∈ V(ℙ) \ {s}, there is a path P = (v₀, v₁, · · · , vₖ) ∈ ℙ(v) such that Pᵢ = (vᵢ, vᵢ₊₁, · · · , vᵢ₊ₖ) is the shortest path, i = 1, 2, · · · , k (if ∀P ∈ ℙ(n, s), provided P is the shortest path, then FP must be the shortest path.)

In terms of Definition 1, Definition 2 and Definition 3, we can immediately obtain the following statements.

**Proposition 1** Let f be a path function. If f is non-decreasing (increasing, WOP, OP, SOP), then it must be NDSP (INSP, WOPSP, OPSP, SOPSP).

**Proposition 2** Let ℙ be a path system. Then ∀v ∈ V, |℘ₐ(v)| is finite.

**Proposition 3** Let f be a path function. If f is non-decreasing, then it must have no negative circles. If f is increasing, then it must have no non-
positive circles. If \( f \) has no non-positive circles, then it must have no negative circles.

**Proposition 4** Let \( f \) be a path function. \( f \) is WISP if and only if \( \forall v \in \{P(v) \setminus \{s\} \}, \) there is a path \( P = (v_0, v_1, \ldots, v_{k-1}, v_k) \in P(v) \) such that it has no circles and \( P_i = (v_0, v_1, \ldots, v_i) \) is the shortest path for any \( i = 1, 2, \ldots, k. \)

**Lemma 1** Let \( f \) be INSP and the shortest path \( P = (v_0, v_1, \ldots, v_k, v_1^*, \ldots, v_i^*, v_0^*) \) such that \( P_i = (v_0, v_1, \ldots, v_k, v_1^*, \ldots, v_i^*), i = 1, \ldots, l, \) are all the shortest paths. Then \( v_i^* \neq v_j^* \) when \( i \neq j. \)

**Proof** Assume \( l \geq i > j \geq 0 \) and \( v_i^* = v_j^* = v. \) Then we have \( m_f(v) = f(P_i) \leq f(P_{i-1}) \leq \cdots \leq f(P_0) = m_f(v). \) Hence Lemma 1 is true.

**Proposition 5** Let \( \mathcal{P} \) be a path system on \([G, s]\) and \( f \) be a path function without negative circles on \( \mathcal{P}. \) Then: (i) \( \forall v \in V(\mathcal{P}), m_f(v) = \min\{f(P)|P \in \mathcal{P}_c(v)\}; \) (ii) \( f \) is ISP if \( f \) is OPSP and INSP.

**Proof** Let \( \forall v \in V(\mathcal{P}), \) let \( P \in \mathcal{P}(v) \) have a circle. Then there is a path \( P' = (v_0, v_1, \ldots, v_k, v_k) \in \mathcal{P}, 0 \leq k \leq n \) such that \( P = P' + (v_0^*, v_1^*, \ldots, v_k^*). \) That is, \( f(P) = \min\{f(P)|P \in \mathcal{P}_c(v)\}; \) (ii) \( f \) is ISP if \( f \) is OPSP and INSP.

**Proposition 6** Let \( \mathcal{P} \) be a path system on \([G, s]\) and \( f \) be a path function without non-positive circles on \( \mathcal{P}. \) Then: (i) \( \forall v \in V(\mathcal{P}) \setminus \{s\}, \) the shortest path from \( s \) to \( v \) has no circles; (ii) \( f \) is WISP if \( f \) is SOPSP.

**Proof** \( \forall v \in V(\mathcal{P}) \setminus \{s\}, \) let \( P \in \mathcal{P}(v) \) have a circle. Then there is a path \( P' = (v_0, v_1, \ldots, v_k, v_k) \in \mathcal{P}(v), 0 \leq k \leq n - 1, \) such that \( P = P' + \)}
Problem

Let \( P \) be all the shortest path. Assume \( v_k \neq v \). Let also \( P' = (v_0, v_1', \ldots, v_k', v_k) \) be a shortest path and \( v_0' \neq v_k \). For \( f \) is SOPSP, we have \( f(P' + (v_k, v)) \leq f(P' + (v_k, v)) = f(P) \), where \( P'' = (v_0, v_1, \ldots, v_k) \). That is, \( P' + (v_k, v) \) is also a shortest path. Set \( v_0'' = v, v_1'' = v_k \), and newly put \( v_1 = v_1', v_2 = v_2', \ldots, v_k = v_k', k = k' \). \( P_0'' = (v_0, v_1, \ldots, v_k, v_1', v_0') \). \( P_1'' = (v_0, v_1, \ldots, v_k, v_1') \). Then \( P_0'', P_1'' \) are all the shortest path, \( v_k \neq v_1'' \) and \( v_0'' \neq v_1'' \) since the shortest path has no circles. If \( v_k \neq v_0 \), repeating the above process, we can also obtain a shortest path \( P_0'' = (v_0, v_1, \ldots, v_k, v_2', v_1', v_0') \) such that \( P_1'' = (v_0, v_1, \ldots, v_k, v_2', v_1') \). \( P_2'' = (v_0, v_1, \ldots, v_k, v_1', v_0) \) are all the shortest paths, \( v_k \neq v_2'' \) and \( v_1'' \neq v_i'' \) whenever \( i \neq j \) for all \( 0 \leq i, j \leq 2 \). For \( n \) is finite, this implies that there is a shortest path \( P'' = (v_0, v_1, \ldots, v_k, v_1', v_0) \) such that \( P_i'' = (v_0, v_1', \ldots, v_i') \), \( 0 \leq i \leq l \), are all the shortest path, \( v_0 \neq v_1'' \) and \( v_1'' \neq v_i'' \) whenever \( i \neq j \). So (ii) holds.

3 Problem

Definition 4 Let \( \mathcal{P} \) be a path system on \([G, s]\) and \( f \) be a path functional on \( \mathcal{P} \). The problem to find a path \( P \in \mathcal{P}(v) \) such that \( f(P) = \min_f(v) \) for all \( v \in V(\mathcal{P}) \) is called as general single-source shortest path problem (GSSSP) on \([G, s, \mathcal{P}, f]\).

It is clear that the problem GSSSP is just the problem CSSSP when \( G \) is graph with weight \( w \) and \( f \) is the path functional \( d \) of example 1, see Section 5. It is also clear that the problem ARP (see literature [6] or example 2) is an instance of the problem GSSSP. The two facts show that the problem GSSSP is really generalization of the problem CSSSP.

Theorem 1 Let \( \mathcal{P} \) be a path system on \([G, s]\) and \( f \) be a path function on \( \mathcal{P} \). If \( f \) has no negative circles, then the problem GSSSP can be solved. That is, \( \forall v \in V(\mathcal{P}) \), there is a path \( P \in \mathcal{P}(v) \) such that \( f(P) = \min_f(v) \), namely \( P \) is a shortest path from \( s \) to \( v \).

Proof From Proposition 5, Theorem 1 is true.

4 Algorithm

For problem GSSSP, following the approaches of the algorithms DA and MBFA, an extended Dijkstra’s algorithm (EDA) and an extended Moore-Bellman-Ford algorithm (EMBFA) can be respectively designed to solve it under certain conditions. We accomplish the tasks in this section.

Extended Dijkstra’s Algorithm (EDA)

Input: graph \( G = (V, R) \), point \( s \in V \), an appointed path system \( \mathcal{P} \) on \([G, s]\) and a path function \( f \) on \( \mathcal{P} \), which is SOPSP, WISP and NDSP.

Output: a spanning tree \( T \) of graph \((V(\mathcal{P}), E(\mathcal{P}))\), which is an arborescence rooted at point \( s \) and satisfies: \( \forall v \in V(\mathcal{P}) \), if \( P_T(v) = (s, v_1, \ldots, v_k, v) \),
1 \leq k \leq n - 2$, is the path from $s$ to $v$ on the tree $T$, then $P_T(v) \in \mathcal{P}$ and $f(P_T(v)) = m_f(v)$.

**Process:**
1. Put $\mathcal{C} \leftarrow \{s]\}, k \leftarrow 0, v(k) = s$.
2. For each road $(u, v) \in (V \setminus \mathcal{C})$, $P_T(u) + (u, v) \in \mathcal{P}$ such that $f(P_T(u) + (u, v)) \leq \min\{f(P_T(u) + (u, v)) | u \in \mathcal{C}, v \in (V \setminus \mathcal{C}), P_T(u) + (u, v) \in \mathcal{P}\}$.

Put
\[
\mathcal{C} \leftarrow (\mathcal{C} \cup \{v'\}), k \leftarrow (k + 1), P_T(v') = P_T(u') + (u', v')(, v(k) = v'),
\]
$V(T) \leftarrow (V(T) \cup \{v\}), R(T) \leftarrow (R(T) \cup (u', v'))$,
$\mathcal{P}(T) \leftarrow (\mathcal{P}(T) \cup \{P_T(v')\})$.

If $\{u, v\} \in \mathcal{C}, v \in (V \setminus \mathcal{C}), P_T(u) + (u, v) \in \mathcal{P} \neq \emptyset$, return to step 2. Otherwise, implement the next step.

3. Put $T = (V(T), R(T))$. Output the graph $T$. Then return.

**Extended Moore-Bellman-Ford Algorithm (EMBA)**

**Input:** graph $G = (V, R)$, point $s \in V$, an appointed path system $\mathcal{P}$ on $[G, s]$ and a path function $f$ on $\mathcal{P}$, which has no negative circles and is OP.

**Output:** a spanning tree $T$ of a graph $(V(\mathcal{P}), E(\mathcal{P}))$, which is an arborescence rooted at point $s$ and satisfies: $\forall v \in V(\mathcal{P})$, if $P_T(v) = (v_0, v_1, \cdots, v_k, v), 0 \leq k \leq n - 2$, is the path from $s$ to $v$ on the tree $T$, then $P_T(v) \in \mathcal{P}$ and $f(P_T(v)) = m_f(v)$.

**Process:**
1. Put $P_T(s) = (s, s), f(P_T(s)) = 0; P_T(v) = (s, \infty, v), f(P_T(v)) = \infty, \forall v \in [V(\mathcal{P}) \setminus \{s\}]; \mathcal{P}(T) = \{P_T(v) | v \in V(\mathcal{P})\}$.
2. For $i = 1, 2, \cdots, n$, do:
   for each road $(u, v) \in [R(\mathcal{P}) \setminus (s, s)]$, if $f(P_T(v)) > f(P_T(u) + (u, v))$, then set $P_T(v) \leftarrow P_T(u) + (u, v)$.
3. Put $T = (V(\mathcal{P}), E(\mathcal{P}(T)))$. Output the graph $T$. Then return.

In order to simplify the analytical process of algorithm EDA, we also propose the next algorithm STA.

**Spanning Tree Algorithm (STA)**

**Input:** a graph $G = (V, R)$ and point $s \in V$, which can reach any point $v$ of $G$, namely, $\forall v \in V$, there is a path from $s$ to $v$.

**Output:** a spanning tree $T$ of $G$, which is also an arborescence rooted at point $s$.

**Process:**
1. Put $\mathcal{C} \leftarrow \{s\}, R(T) \leftarrow \emptyset$. (Put $k \leftarrow 0, v(k) = s$.)
2. Find a $u \in \mathcal{C}$ and a $v \in (V \setminus \mathcal{C})$ such that $(u, v) \in R$. Then put
\[
\mathcal{C} \leftarrow \mathcal{C} \cup \{v\}, R(T) \leftarrow R(T) \cup \{(u, v)\}.
\]
5 Analysis of algorithms

Lemma 2 (i) Algorithm STA works well. (ii) The output of STA $T$ is a spanning tree of $G$. (iii) $T$ is an arborescence rooted at $s$. (iv) The complexity of STA is $O(n^2)$.

Proof When $C \neq V$, there must be a $u \in (V - C)$ and a $v \in C$ such that $(u, v) \in R$ since $s$ can reach any point of $G$. Note that $n$ is finite. We can easily know that (i) is obvious.

In terms of the process of algorithm STA and (i), it is obvious that $T$ is connected and $V(T) = V$. So, in order to show (ii) is true, we need only to prove $T$ has no circles. Assume that $T$ has a circle $C = (v(k_1), v(k_2), \cdots, v(k_l))$ and $v(k_1) = v(k_l)$. Then $k_1 \neq k_l$. This is impossible for one index corresponds only one point. Hence (ii) is true.

By the process of algorithm STA, we can easily know that $T$ is a directed graph, $|\delta^- (s)| = 0$ and $|\delta^- (v)| = 1$ for any $v \in (V - \{s\})$. Here $|\delta^- (v)|$ is the in-degree of $v$ in the directed graph $T$, see e.g. 2.1 of [1]. Hence (iii) is true.

Finally (iv) is obvious. This completes the proof.

Theorem 2 (i) Algorithm EDA works well. (ii) The output $T$ is a spanning tree of the subgraph of $G$ induced by $V(P)$. (iii) $T$ is an arborescence rooted at $s$. (iv) $\forall v \in [V(T) \setminus \{s\}], f(P_T(v)) = m_T(v)$. (v) The complexity is $M(n)\Delta O(n^2)$ provided $f(P)$ can be obtained in $M(n)$ time for any $P \in P$. Here $n = |V|$, $\Delta = \Delta(G)$.

Proof. Note that $V$ and $R$ are all the finite sets. By observing the process of algorithm EDA, we can easily know that (i) holds. (ii) and (iii) can be immediately obtained from Lemma 2. Note that $f(P)$ can be obtained in $M(n)$ time for any $P \in P$ and $\Delta$ is the maximum degree of the points of graph $G$. Observe also the process of algorithm EDA. Following the approach to analyse the complexity of algorithm DA, we can easily know (v) is true. Next we focus to prove (iv).

It is obvious that (iv) is true for $k = 0$. Assume (iv) is not true for $k = 1$. Then $m_T(v(1)) < f(P_T(v(1)))$. That is, there is a path $P = (v_0, v_1, \cdots, v_i, v(1))$ such that $f(P) = m_T(v(1)) < f(P_T(v(1)))$; and $i \geq 1$, namely there are at least two roads in the path $P$. Since $f$ is WISP, we can also make $P$ satisfy: $f(P) = m_T(v(1)), l = 1, \cdots, i, P_l = (v_0, v_1, \cdots, v_l)$. Since $f$ is also NDSP, this implies $f((v_0, v_1)) = f(P_1) \leq f(P_2) \leq \cdots \leq f(P_l) \leq f(P) = m_T(v(1)) < f(P_T(v(1)))$, which contradicts to the definition of $v(1)$ in algorithm EDA. Hence (iv) is true for $k \leq 1$.

Assume now (iv) is true for $l \leq k < n$ with $k \geq 1$. Then we can easily verify that it is also true for $l \leq (k + 1)$. In fact, suppose (iv) is not true for $(k + 1)$. Then $m_T(v(k + 1)) < f(P_T(v(k + 1)))$. Since $f$ is WISP, there is a
path 

\[ P = (v_0, v_1, \cdots, v_i, v_{i+1}, \cdots, v_{k+1}) \]

with \( i \geq 0, j \geq 0 \) such that \( v_i \in C_k, v_{i+1} \in (V - C_k) \), and 

\[
f(P) = m_f(v(k + 1)) < f(Pr(v(k + 1))); \quad (1)
\]

\[
f(P) = m_f(v_l), \forall l = 1, \cdots, i + j, \quad (2)
\]

where \( C_k = \{v(0), v(1), \cdots, v(k)\}, P_1 = (v_0, v_1, \cdots, v_l) \). Since \( f \) is NDSP, we have 

\[
f(P) \geq f(P_{i+1}) \geq \cdots \geq f(P_{i+1}). \quad (3)
\]

For \( v_i \in C_k \), by the hypothesis of induction, we have \( f(Pr(v_i)) = m_f(v_i) \). Since also \( f \) is SOP, we have 

\[
f(P_{i+1}) = f(P_i + (v_i, v_{i+1})) \geq f(Pr(v_i) + (v_i, v_{i+1})). \quad (4)
\]

Combining (1), (3) and (4), we obtain: 

\[
f(Pr(v_i) + (v_i, v_{i+1})) < f(Pr(v(k + 1))).
\]

This contradicts to the definition of \( v(k + 1) \). Hence (iv) is true for \((k+1)\).

Finally, by the induction principle, we know that (iv) is true. The proof completes.

**Theorem 3**

(i) Algorithm EMBFA works well. (ii) The output \( T \) is a spanning tree of the subgraph of \( G \) induced by \( V(P) \). (iii) \( T \) is an arborescence rooted at \( s \). (iv) \( \forall v \in [V(T) \setminus \{s\}], f(Pr(v)) = m_f(v) \). (v) The complexity of EMBFA is \( M(n)\Delta O(nm) \) provided \( f(P) \) can be obtained in \( M(n) \) time for any \( P \in \mathcal{P} \). Here \( n = |V|, m = |E|, \Delta = \Delta(G) \).

**Proof.**

Note that \( V \) and \( R \) are all the finite sets. We can easily know (i) holds.

We now show that (ii) is true. Firstly, it is obvious that \( V(Pr) = V(P) \). Secondly, \( \forall v \in [V(P) \setminus \{s\}], Pr(v) \) is a path from \( s \) to \( v \). In fact, \( v \) can be reached from \( s \). Let \( P = (s, v_1, \cdots, v_{k-1}, v) \in \mathcal{P} \) be a no circles path, then \( k \leq n - 1 \). This implies that \( Pr \) must be a path from \( s \) to \( v \) after the \( k \) times of iteration, which can be strictly proved by induction. Hence \( T \) is connected. Thirdly, \( \forall v \in [V(P) \setminus \{s\}], \) any update of \( Pr(v) \) can not creates circles. In fact, if a circle is created by an update of \( Pr(v) \), then there must be an \( u \in V(P) \) such that \( Pr(u) = (u, v, v') \), and \( f(Pr(v)) > f(Pr(u) + (u, v)) \). This implies that \( f \) has negative circles. So \( \forall v \in [V(P) \setminus \{s\}], \) any update of \( Pr(v) \) can not creates circles. For the statement, any member of \( \mathcal{P} \) has no circles. Hence, assume \( T \) has a circle \( C \). Then there must be two path \( P_1, P_2 \in \mathcal{P} \) such that \( [(V(P_1) \cap V(P_2)) \setminus \{s\}] \neq \emptyset \), where \( V(P) \) denotes all the points in the path \( P \). That is, there must be a point \( t \in [(V(P_1) \cap V(P_2)) \setminus \{s\}] \) such that \( P_1 = (s, v_1, \cdots, v_k, t), P_2 = (s, u_1, \cdots, u_t, t) \in \mathcal{P} \) and \( P_1 \neq P_2 \). This implies \( |\delta_T(t)| \geq 2 \). However, on the other hand, from the process of EMBFA, we can easily know \( |\delta_T(t)| \leq 1 \).

Hence \( T \) has no circles. Finally, comibing the three statements above, we know (ii) is true.
From the process of EMBFA, we can easily know that $T$ is a directed graph, $|\delta_T(s)| = 0$ and $|\delta_T(v)| = 1$ for any $v \in (V(T) \setminus \{s\})$. Hence (iii) is true.

Note that $f(P)$ can be obtained in $M(n)$ time for any $P \in \mathcal{P}$ and $\Delta$ is the maximum degree of the points of graph $G$. Following the approach to analyse the complexity of algorithm MBFA, we can easily know (v) is true by the process of algorithm EMBFA. Next we focus to prove (iv).

We only to show the following statement is true.

forall $v \in [V(\mathcal{P}) \setminus \{s\}]$, if there is a path $P \in \mathcal{P}$ such that $P$ is shortest path from $s$ to $v$ and $P = (s, v_1, \cdots, v_k)$, $v_k = v$, $1 \leq k \leq n - 1$, then $P_T(v)$ must be shortest path from $s$ to $v$ after $k$ iterations of EMBFA.

Firstly, the statement is obviously true for $k = 1$. In fact, assume $v \in [V(\mathcal{P}) \setminus \{s\}]$ and $P = (s,v) \in \mathcal{P}$ is the shortest path from $s$ to $v$. Then $f(P_T(v)) \leq f((s,v))$ after $(s,v)$ is addressed in the 1th iteration. That is, $P_T(v)$ must be the shortest path from $s$ to $v$ after the 1th iteration. So the statement is true for $k = 1$.

Secondly, assume it is true for a $k < n - 2$ is true. $\forall v \in [V(\mathcal{P}) \setminus \{s\}]$, let $P = (s,v_1,\cdots,v_k,v_{k+1}) \in \mathcal{P}(v)$ be a shortest path from $s$ to $v$. Then, for $f$ has no negative circles and is OP, $FP = (s,v_1,\cdots,v_k)$ is also a shortest path from the term (ii) of proposition 5. Hence, by the hypothesis of induction, we have

$$f(P_T(v_k)) \leq f((s,v_1,\cdots,v_k)).$$

(5)

On the other hand, after $(k+1)$ iterations, we have

$$f(P_T(v)) \leq f(P_T(v_k) + (v_k,v)).$$

(6)

Note $f$ is OP. By combining (5) and (6), we can obtain

$$f(P_T(v)) \leq f(P_T(v_k) + (v_k,v)) \leq f((s,v_1,\cdots,v_k) + (v_k,v)) = f(P).$$

(7)

Therefore $P_T(v)$ must be the shortest path from $s$ to $v$ after the $(k+1)$th iteration. That is, the statement is true for $(k+1)$.

Finally, noting $f$ has no negative circles, we can know that $\forall v \in [V(\mathcal{P}) \setminus \{s\}]$, there is a shortest path $P = (s,v_1,\cdots,v_k,v)$ from $s$ to $v$ such that $k \leq n - 2$. Hence (iv) is true. The proof completes.

6 Applications

This section shows the application of algorithms EDA and EMBFA by providing few instances.

Example 1. Given a connected network with nonnegative weight $(G,w)$ and source point $s \in V$, let $d(P) = \sum_{i=1}^{k} w(v_{i-1},v_i)$ for any path $P = (v_0,v_1,\cdots,v_k)$ of the network. Let $\mathcal{P}$ be the set of all the pathes with $s(P) = s$. Then it is obvious that $\mathcal{P}$ is a path system on $[G,s]$ and $d$ is a path function on $\mathcal{P}$. The problem to find a path $P^* \in \mathcal{P}(v)$ such that $d(P^*) = m_s(v)$ for any $v \in [V \setminus \{s\}]$ is called classical single-source shortest path problem with
nonnegative weight (CSSSP-NW). See e.g. 7.1 of literature [1]. Note that \( w \) is nonnegative. We can easily know that \( d \) is nondecreasing and OP. So, in terms of Theorem 2 and Theorem 3, the problem CSSSP-NW can be effectively solved by the algorithms EDA and EMBFA respectively.

**Remark 1.** For the example 1, we should note the following facts. (i) EDA and EMBFA respectively reduces to DA and MBFA. (ii) Let \( P, P' \in \mathcal{P} \) and \( SP = P+(u, v), SP' = P'+(u, v) \in \mathcal{P} \). Then \( d(SP) - d(P) = d(SP') - d(P') = w((u, v)) \) for the function \( d \). However, \( f(SP) - f(P) = f(SP') - f(P') \) may not hold for a general path function \( f \) on \( \mathcal{P} \). That is, we may not find a weight of graph \( G \) such that \( f(P) = \sum_{i=1}^{k} w(v_{i-1}, v_i), \forall P = (v_0, v_1, \ldots, v_{k-1}, v_k) \in \mathcal{P} \) for an general path function \( f \) on \( \mathcal{P} \).

**Example 2.** For the example 1, change the nonnegative weight \( w \) as a conservative weight, see e.g. Definition 7.1 of [1], then we call the problem to find a path \( P^* \in \mathcal{P}(v) \) such that \( d(P^*) = m_d(v) \) as classical single-source shortest path problem with conservative weight (CSSSP-CW). From \( w \) is a conservative weight, we can easily know that \( d \) has no negative circles. On the other hand, we can also easily know that \( d \) is OP. So, the problem CSSSP-GW can be effectively solved by the algorithm EMBFA.

**Example 3.** For the example 1, let also \( \mathcal{P}' \) be all the paths of graph \( G \). Define \( d(u, v) = \min\{d(P)|s(P) = u, t(P) = v, P \in \mathcal{P}'\} \) for any two points \( u, v \in V \), called the distance between \( u \) and \( v \) on the network \((V, w)\). Define \( d_G((u', v'))(u, v) \) as the distance between \( u \) and \( v \) on the network \((V, R - \{(u', v')\}, w)\) for any two points \( u, v \in V \) and road \((u', v') \in R \), called the detour distance between \( u \) and \( v \) on the case that the road \((u', v')\) is blocked. Here \( G((u', v')) \) denotes the graph \((V, R - \{(u', v')\})\).

Define

\[
r(P) = \max\{d(P), d_G((v_{k-1}, v_k))(s, v_k), d(P_i), d_G((v_{i-1}, v_i))(s, v_i)\}
\]

for any \( P = (v_0, \cdot, v_{k-1}, v_k) \in \mathcal{P} \). We call \( r(P) \) as the risk of \( P \). It is obvious that \( r \) is a path function on \( \mathcal{P} \). The problem to find a path \( P \in \mathcal{P}(v) \) such that \( r(P) = m_r(v) \) for any \( v \in [V(P) \setminus \{s\}] \) is called the anti-risk path problem (ARP) of finding a path such that it has minimum risk. See the literature [6].

Let \( P = (s, v_1, \cdot, v_{k-1}, v_k) \in \mathcal{P} \) and \( SP = (s, v_1, \cdot, v_{k-1}, v_k, v_{k+1}) \in \mathcal{P} \). Then we have

\[
\begin{align*}
r(SP) &= \max\{d(SP), d_G((v_{k+1}, v_k))(s, v_k), d(SP_i), d_G((v_{i-1}, v_i))(s, v_i)\} \\
SP_i &= (v_1, \cdot, v_k, v_{k+1}), 1 \leq i \leq k \}
\end{align*}
\]

\[
\begin{align*}
&= \max\{w(v_k, v_{k+1}) + d(SP), d_G((v_{k+1}, v_k))(s, v_k), w(v_k, v_{k+1}) + d(SP_i), d_G((v_{i-1}, v_i))(s, v_i)\} \\
&= \max\{d_G((v_{k+1}, v_k))(s, v_k), w(v_k, v_{k+1}) + d(SP) + d_G((v_{i-1}, v_i))(s, v_i)\} \\
&\quad |P_i = (v_1, \cdot, v_k), 1 \leq i \leq k - 1 \}
\end{align*}
\]

\[
\begin{align*}
&= \max\{d_G((v_{k+1}, v_k))(s, v_k), w(v_k, v_{k+1}) + \max\{d(P), d_G((v_{i-1}, v_i))(s, v_i)\} \\
&\quad |P_i = (v_1, \cdot, v_k), 1 \leq i \leq k - 1 \}
\end{align*}
\]

\[
\begin{align*}
&= \max\{d_G((v_{k+1}, v_k))(s, v_k), w(v_k, v_{k+1}) + r(P)\} \geq r(P).
\end{align*}
\]
This shows $r$ is nondecreasing. Let also $P' \in \mathcal{P}(v_k), SP' = P' + (v_k, v_{k+1}) \in \mathcal{P}(v_{k+1})$. Suppose $r(P) \geq r(P')$. Then, from (8), we have
\[
\begin{align*}
  r(SP) &= \max\{d_{G \setminus \{(v_k, v_{k+1})\}}(s, v_{k+1}), w(v_k, v_{k+1}) + r(P)\} \\
  &\geq \max\{d_{G \setminus \{(v_k, v_{k+1})\}}(s, v_{k+1}), w(v_k, v_{k+1}) + r(P')\} = r(SP').
\end{align*}
\]
This shows $r$ is SOP.

For $r$ is nondecreasing, to solve the problem ARP, we need only to consider the path function on the set of all the paths without circles.

Let $\mathcal{P}_1$ be the system of all the paths of source $s$ and without circles.

For $\mathcal{P}_1$ has no circles, $r$ must has no non-positive circles. Since also $r$ is SOP, we can easily know that it is WISP from Proposition 3 and Proposition 6. Note that $r$ is SOP, WISP and nondecreasing. By Theorem 2, ARP can be effectively solved by the algorithm EDA. The end of example 3.

**Remark 2.** (i) Due to the cause of symmetry, in order to conveniently understand the examples, we can interpret $P = (s, v_1, \cdots, v_k, v, v')$ as the path from $v$ to $s$. (ii) Xiao et al. [6] (2009) introduce the definition of the risk of a path, and the anti-risk path problem (ARP) to finding a path such that it has no non-positive circles. Since also $r$ is SOP, WISP and nondecreasing. Moreover, $c$ is nondecreasing. To solve the problem ARP, we need only to consider the path function on the set of all the paths without circles.

**Example 4.** In the understructure of example 1, let $\mathcal{P}$ be the system of all the paths of source $s$ and without circles. Assume $p \in (0, 1)$. Let first $c((s, s)) = 0$. Then $\forall P = (v_0, v_1, \cdots, v_k) \in \mathcal{P}, k \geq 0$, provided $SP = (v_0, v_1, \cdots, v_k, v_{k+1}) \in \mathcal{P}$, let $c(SP) = pd_{G \setminus \{(v_k, v_{k+1})\}}(v_0, v_{k+1}) + w(v_k, v_{k+1}) + c(P)$. Then $c$ is a path function on $\mathcal{P}, \forall v \in V$, assume $P = (v_0, v_1, \cdots, v_{k-1}, v_k) \in \mathcal{P}(v)$. Then $\forall SP = (v_0, \cdots, v_k, v_{k+1}) \in \mathcal{P}$, we have
\[
\begin{align*}
  c(SP) &= pd_{G \setminus \{(v_k, v_{k+1})\}}(v_k, v_{k+1}) + w(v_k, v_{k+1}) + c(P) \\
  &\geq c(P).
\end{align*}
\]
This shows $c$ is nondecreasing. Moreover, $\forall P' = (v_0, v_1', \cdots, v_{l-1}', v_l') \in \mathcal{P}(v)$, and $SP' = P' + (v_k, v_{k+1}) \in \mathcal{P}$, assume $c(P) \geq c(P')$. We have
\[
\begin{align*}
  c(SP) &= pd_{G \setminus \{(v_k, v_{k+1})\}}(v_k, v_{k+1}) + w(v_k, v_{k+1}) + c(P) \\
  &\geq pd_{G \setminus \{(v_k, v_{k+1})\}}(v_k, v_{k+1}) + w(v_k, v_{k+1}) + c(P') \\
  &= c(SP').
\end{align*}
\]
This shows $c$ is SOP.

Since $\mathcal{P}$ has no circles, $c$ has no non-positive circles. Note also $c$ is SOP. We know that $c$ is WISP from the term (ii) of Proposition 6. Hence, by Theorem 2, the problem GSSSP with path function $c$ can be effectively solved by the algorithm EDA. The end of example 4.
Remark 3. The path function $e$ can be interpret as follows. Suppose that at most one edge may be blocked. $\forall P \in \mathcal{P}$, $e(P) = pd_{G \setminus (v_{k-1}, v_k)}(v_{k-1}, v_k) + w(v_{k-1}, v_k) + c(FP)$ denotes the cost that one goes to the point $s$ from the point $v_k$ by train (or ship, or plane), among which, the term $w(v_{k-1}, v_k) + c(FP)$ is the normal cost, while the term $pd_{G \setminus (v_{k-1}, v_k)}(v_{k-1}, v_k)$ is the additional cost due that one needs to change route when the road $(v_{k-1}, v_k)$ is blocked. To some extent, $p$ is the probability that the road $(v_{k-1}, v_k)$ may be blocked.

Example 5 In the understructure of example 1, let $\mathcal{P}$ be the system of all the pathes of source $s$ and without circles. Assume $p \in (0, 1)$. Let first $e((s, s)) = 0$. Then $\forall P = (v_0, v_1, \cdots, v_{k-1}, v_k) \in \mathcal{P}, 0 \leq k \leq n - 1$, provided $SP = (v_0, \cdots, v_k, v_{k+1}) \in \mathcal{P}$, let $e(SP) = pd_{G \setminus (v_k, v_{k+1})}(v_0, v_1) + (1 - p)[w(v_k, v_{k+1}) + c(SP)]$, $\forall v \in [V \setminus \{s\}]$, let $P = (v_0, v_1, \cdots, v_{k-1}, v_k) \in \mathcal{P}(v), P' = (v_0, v_1', \cdots, v_{k-1}', v_k') \in \mathcal{P}(v)$. Provide $SP = P + (v_k, v_{k+1}) \in \mathcal{P}, SP' = P' + (v_k, v_{k+1}) \in \mathcal{P}$ and $e(P) < e(P')(e(P) = e(P'))$. Then we have

\[
e(SP) = pd_{G \setminus (v_k, v_{k+1})}(v_k, v_{k+1}) + (1 - p)[w(v_k, v_{k+1}) + e(P)] \\
< pd_{G \setminus (v_k, v_{k+1})}(v_k, v_{k+1}) + (1 - p)[w(v_k, v_{k+1}) + e(P')] \\
(= pd_{G \setminus (v_k, v_{k+1})}(v_k, v_{k+1}) + (1 - p)[w(v_k, v_{k+1}) + e(P')]) \\
= e(SP')(e(SP')).
\]

This shows that $e$ is OP. Hence, by Theorem 3, the problem GSSSP with path function $e$ can be solve by the algorithm EMBFA. The end of example 5.

Remark 4. The path function $e$ can be interpret as follows. Suppose that at most one edge may be blocked. $\forall P \in \mathcal{P}$, $e(P) = pd_{G \setminus (v_{k-1}, v_k)}(v_{k-1}, v_k) + (1 - p)[w(v_{k-1}, v_k) + c(FP)]$ denotes the mean expense that one goes to the point $s$ from the point $v_k$ by train (or ship, or plane), among which, the term $(1 - p)[w(v_{k-1}, v_k) + c(FP)]$ is the expense through the normal route with road $w(v_{k-1}, v_k)$ from $v_k$ to $s$; the term $pd_{G \setminus (v_{k-1}, v_k)}(v_{k-1}, v_k)$ is the expense through the deviate route that is the shortest path from $v_k$ to $s$ on the graph $G \setminus (v_{k-1}, v_k); p$ is the probability that the road $(v_{k-1}, v_k)$ may be blocked, or one choose the deviate route.

7 Concluding remarks

The present work have introduced a kind of general single-source shortest path problem (GSSSP) and have designed two algorithms (EDA and EMBFA) to solve it under certain conditions. It is an interesting topic for further research in the future to explore other efficient algorithms and the applications of the problem GSSSP. Finally, cordially hope the present work can improve the development of the researches and applications of the shortest path problems.

Acknowledgements The authors cordially thank the anonymous referees for their valuable comments which lead to the improvement of this paper.
References

1. Korte B, Vygen J. Combinatorial optimization theory and algorithms. Springer-Verlag, Berlin (2000)
2. Dijkstra E W. A note on two problems in connexion with graphs. Numer Math, 1(4): 269-271 (1959)
3. Ford, L R. Network flow theory, The Rand Corporation, Santa Monica (1956)
4. Bellman, R E. On a routing problem. Quarterly of Applied Mathematics, 16: 87-90 (1958)
5. Moore, E F. The shortest path through a maze. Proceedings of the international Symposium on the Theory of Switching, Part II, 285-292, Harvard University Press (1959)
6. Xiao P, Xu Y, Su B. Finding an anti-risk path between two nodes in undirected graphs. J Comb Optim, 17: 235-246 (2009)
7. Mahadeokar J, Saxena S. Faster algorithm to find anti-risk path between two nodes of an undirected graph. J Comb Optim, 27: 798-807 (2014)
8. Hershelberger J, Suri S. Vickrey prices and shortest paths: what is an edge worth?. In: Proceedings of the 42nd annual IEEE symposium on foundations of computer science, 252-259 (2001)
9. Hershelberger J, Suri S, Bhosle A. On the difficulty of some shortest path problems. In: Proceedings of the 20th symposium on theoretical aspects of computer science, 343-354 (2003)
10. Zhang H L, Xu Y F, Wen X G. Optimal shortest path set problem in undirected graphs. J Comb Optim, 29: 511-520 (2015)
11. Nip K, Wang Z, Nobihon F T, Leus R. A combination of flow shop scheduling and the shortest path problem. J Comb Optim, 29: 36-52 (2015)
12. Feng G. Finding k Shortest Simple Paths in Directed Graphs: A Node Classification Algorithm. NETWORKS: 6-17 (2014)
13. Du D Z, Graham R L, Pardalos P M, Wan P J, Wu W I, Zhao W B. Analysis of Greedy Approximations with Nonsubmodular Potential Functions. In: Proceedings of the Nineteenth Annual ACM-SIAM Symposium on Discrete Algorithms, 167-175 (2008)
14. Murota K, Shioura A. Dijkstra’s algorithm and L-concave function maximization. Math Program, Ser. A, 145: 163-177 (2014)