On maximal inequalities for purely discontinuous $L_q$-valued martingales

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Abstract

We prove maximal inequalities for $L_q$-valued martingales obtained by stochastic integration with respect to compensated random measures. A version of these estimates for integrals with respect to compensated Poisson random measures were first obtained by Dirksen [2] using arguments based on inequalities for sum of independent Banach-space-valued random variables, geometric properties of Banach spaces, and decoupling inequalities. Our proofs are completely different and rely almost exclusively on classical stochastic analysis for real semimartingales.

1 Introduction

Our purpose is to prove maximal inequalities for purely discontinuous martingales taking values in $L_q$ spaces, with bounds expressed in terms of predictable elements only. In particular, if $\mu$ is a random measure with compensator $\nu$ and $\bar{\mu} := \mu - \nu$, the main result takes the form

$$\left( E \sup_{t \geq 0} \| (g * \bar{\mu})_t \|_{L_q}^p \right)^{1/p} \simeq_{p,q} \| g \|_{I_{p,q}} \quad \forall p, q \in ]1, \infty[, $$

where the integrand $g$ takes values in an $L_q$ space, and the norm on the right-hand side depends, roughly speaking, on integrals of functionals of $g$ with respect to $\nu$ only.

Estimates of this type were obtained for the first time, assuming that $\mu$ is a Poisson random measure, by Dirksen [2], using an abstract (and very elegant) approach. Namely, he first obtains suitable vector-valued generalizations of Rosenthal’s inequality for sums of independent random variables, and then deduces from them inequalities for stochastic integrals of step processes with respect to compensated Poissonian measures by means of decoupling techniques. His proofs rely on several sophisticated arguments, pertaining to the interplay between the geometry of Banach spaces and estimates for series of random variables on them, as well as, as already mentioned, to (vector-valued extensions of) decoupling inequalities.

Our approach is completely different and uses essentially only stochastic calculus for real semimartingales. Our interest in this problem arose while working on [8], where we

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derived a very special case of the above maximal inequality (namely the case $p = q \geq 2$), by a rather elementary integration in space of a corresponding inequality for real-valued processes (apparently) due to Novikov \cite{9}. The latter inequality has been known for almost 40 years (cf. \cite{7} for an extensive review and a brief historical account). However, it seems to be generally believed that, as common wisdom suggests, this naive approach is doomed to fail if one wants to estimate the $p$-th moment of the $L_q$-norm, with $p \neq q$. This is actually one of the main reasons, at least from the point of view of someone interested in stochastic PDEs, why developing stochastic integration and, more generally, stochastic calculus on Banach spaces is a worthy endeavor (see e.g. \cite{10} and references therein for recent developments in this direction, when the integrator is a Wiener process). One of the main messages of the present work is that the common belief just described is exaggerated, in the sense that one can indeed get very far using only pointwise estimates, integration, and classical results of stochastic calculus. In a figurative way, thinking to a scale of abstraction’s level, one could say that Dirksen’s results are obtained “from above”, and ours are obtained “from below”.

2 Main result

Let $(X, \mathcal{A}, n)$ be a measure space, and denote $L_q$ spaces on $X$ simply by $L_q$, for any $q \geq 1$. All random elements will be defined on a fixed filtered probability space $(\Omega, \mathcal{F}, F, \mathbb{P})$, $\mathcal{F} := (\mathcal{F}_t)_{t \geq 0}$. We shall use the notation $\| \cdot \|_{L_p}$ to denote $(\mathbb{E} | \cdot |^p)^{1/p}$. Similarly, given a normed space $E$ and an $E$-valued random variable $\xi$, we shall use the notation $\| \xi \|_{L_p(E)} := (\mathbb{E} \| \xi \|^p)^{1/p}$, sometimes omitting the parentheses around $E$ if the meaning is clear. A completely similar convention will be in place also for other integrals on other spaces. Let $\mu$ be a random measure on $\mathbb{R}^+ \times Z$, $\nu$ be its dual predictable projection (compensator), and $\bar{\mu} := \mu - \nu$ be the corresponding compensated random measure. Integrals over $\mathbb{R}^+ \times Z$ will be denoted simply by an integration sign. Let $g : \Omega \times \mathbb{R}^+ \times Z \times X \to \mathbb{R}$ be such that $(\omega, t, z) \mapsto g(\omega, t, z, x)$ is predictable for each $x$ and $(\omega, t, z) \mapsto \|g(\omega, t, z, \cdot)\| \in L_q$ for all $(\omega, t, z)$.

For any $p_1, p_2, p_3 \in [1, \infty]$, introduce the spaces

$$L_{p_1, p_2, p_3} := L_{p_1} L_{p_2} (\nu) L_{p_3}, \quad \bar{L}_{p_1, p_2} := L_{p_1} L_{p_2} L_2 (\nu),$$

where $L_p (\nu) := L_p (\mathbb{R}^+ \times Z, \nu)$ and the above convention about $L_p$ spaces with mixed norms is in place. In particular, this means that, for instance,

$$\xi \in L_{p_2} (\nu) L_{p_3} \iff \| \xi \|_{L_{p_2} (\nu) L_{p_3}} := \left( \int_{\mathbb{R}^+ \times Z} \| \xi \|_{L_{p_3} (\nu)}^{p_2} d\nu \right)^{1/p_3} < \infty.$$

The (proof of the) following theorem, whose original formulation is due to Dirksen \cite{2}, is our main result.

**Theorem 2.1.** Let $p, q \in [1, \infty[$. One has

$$\left\| \sup_{t \geq 0} \| (g \ast \bar{\mu})_t \|_{L_q} \right\|_{L_p} \approx_{p, q} \| g \|_{I_{p, q}}.$$
where

\[ \mathcal{I}_{p,q} := \begin{cases} 
L_{p,p,q} + L_{p,q,q} + \tilde{L}_{p,q}, & 1 < p \leq q \leq 2, \\
(L_{p,p,q} \cap L_{p,q,q}) + \tilde{L}_{p,q}, & 1 < q \leq p \leq 2, \\
L_{p,p,q} \cap (L_{p,q,q} + \tilde{L}_{p,q}), & 1 < q < 2 \leq p, \\
(L_{p,p,q} + L_{p,q,q}) \cap \tilde{L}_{p,q}, & 2 \leq p \leq q, \\
L_{p,p,q} \cap L_{p,q,q} \cap \tilde{L}_{p,q}, & 2 \leq q \leq p. 
\end{cases} \]  
(2.1)

The proof of theorem, which follows by a series of Lemmata and Propositions, is in Section 4 below.

An explicit description of the norms of the spaces appearing above is as follows:

\[ \|g\|_{L_{p,p,q}} = \left( \mathbb{E} \left( \int \|g\|^p_{L_q} \, d\nu \right)^{1/p} \right), \]
\[ \|g\|_{L_{p,q,q}} = \left( \mathbb{E} \left( \left( \int \|g\|^q_{L_q} \, d\nu \right)^{p/q} \right)^{1/p} \right), \]
\[ \|g\|_{\tilde{L}_{p,q}} = \left( \mathbb{E} \left( \left( \int |g|^2 \, d\nu \right)^{1/2} \right)^{1/p} \right). \]

3 Preliminaries and auxiliary results

The maximal inequalities in the following theorem are known, and their proofs can be found e.g. in [5, 6, 7].

**Theorem 3.1.** Let \( g \) take values in a Hilbert space \( H \). Then one has

\[ \mathbb{E} \sup_{t \geq 0} \| (g \ast \bar{\mu})_t \|_H^p \lesssim_p \mathbb{E} \left( \int \|g\|^2_H \, d\nu \right)^{1/p} \quad \forall p \in [0,2], \]  
(3.1)

\[ \mathbb{E} \sup_{t \geq 0} \| (g \ast \tilde{\mu})_t \|_H^p \lesssim_p \mathbb{E} \left( \int \|g\|^2_H \, d\nu \right)^{1/2} \quad \forall p \in [1,2], \]  
(3.2)

\[ \mathbb{E} \sup_{t \geq 0} \| (g \ast \tilde{\mu})_t \|_H^p \lesssim_p \mathbb{E} \left( \mathbb{E} \left( \|g\|^2_H \, d\nu \right)^{1/2} \right)^{1/p} + \mathbb{E} \left( \int \|g\|^2_H \, d\nu \right)^{1/2} \quad \forall p \in [2,\infty]. \]  
(3.3)

Throughout the rest of the paper, unless otherwise stated, we shall use the notation

\[ M := g \ast \bar{\mu}, \]

as well as

\[ [M, M]_t := \int_0^t \int_Z |g|^2 \, d\mu, \quad \langle M, M \rangle_t := \int_0^t \int_Z |g|^2 \, d\nu. \]

Moreover, for notational compactness, we shall write, for any \( L_q \)-valued process \( Y \),

\[ \|Y\|_{L_{p,L_q}}^\infty := \left( \mathbb{E} \sup_{t \geq 0} \|Y_t\|^p_{L_q} \right)^{1/p}. \]

The following estimate plays a crucial role throughout the paper.
Theorem 3.2. One has, for any \( p, q \in ]1, \infty[ \),
\[
\| [M, M]_{1/2}^1 \|_{L^p L^q} \lesssim_{p, q} \| M^* \|_{L^p L^q} \lesssim_{p, q} \| [M, M]_{1/2}^1 \|_{L^p L^q}.
\tag{3.4}
\]
Moreover, the upper bound also holds for \( p = q = 1 \).

Proof. The map \( M \mapsto [M, M]_{1/2} \) is sublinear and bounded on \( L_p L_p \) and on \( L_p L_2 \) for all \( p \in ]1, \infty[ \). The inequality on the left then follows by the extension of Riesz-Thorin interpolation due to Benedek and Panzone \([1]\), coupled with the linearization theorem by Janson \([3]\).

To prove the inequality on the right, we use a duality argument: we have
\[
\| M_\infty \|_{L_p L_q} = \sup_{\zeta \in B(0_{\ell^q}, L_q)} \mathbb{E}(M_\infty, \zeta),
\]
where \((\cdot, \cdot)\) stands for the duality form between \( L_p \) and \( L_{p'} \). Now take a martingale \( N \) with final value \( N_\infty = \zeta \), and note
\[
\mathbb{E}(M_\infty, \zeta) = \mathbb{E}(M_\infty, N_\infty) = \mathbb{E}[M_\infty, N_\infty]
\leq \| [M, M]_{1/2}^1 \|_{L_p L_q} \| [N, N]_{1/2}^1 \|_{L_{p'} L_q}
\lesssim_{p, q} \| [M, M]_{1/2}^1 \|_{L_p L_q} \| N_\infty \|_{L_{p'} L_q} \lesssim \| [M, M]_{1/2}^1 \|_{L_p L_q},
\]
from which the second inequality follows immediately.

The following simple estimate will be used repeatedly.

Lemma 3.3. Let \((X, A, m)\) be a measure space and \( p > r \geq 1 \). If \( f \in L_r(X, m) \cap L_p(X, m) \), then \( f \in L_q(X, m) \) for all \( q \in ]r, p[ \), and
\[
\| f \|_{L_q}^\alpha \leq \| f \|_{L_r}^\alpha + \| f \|_{L_p}^\alpha \quad \forall \alpha > 0.
\]
Proof. Let \( \theta \in [0, 1[ \) be such that \( q = r\theta + p(1-\theta) \). By Lyapunov’s inequality one has, after raising to the power \( \alpha \),
\[
\| f \|_{L_q}^\alpha \leq \| f \|_{L_r}^{\alpha\theta} \| f \|_{L_p}^{(1-\theta)},
\]
Young’s inequality \( ab \leq a^s/s + b^{s'}/s' \) with conjugate exponents \( s = 1/\theta \) and \( s' = 1/(1-\theta) \) yields
\[
\| f \|_{L_q}^\alpha \leq \theta \| f \|_{L_r}^\alpha + (1-\theta)\| f \|_{L_p}^\alpha \leq \| f \|_{L_r}^\alpha + \| f \|_{L_p}^\alpha.
\]

We shall also use several times the following inequality between norms of functions in \( L_p \)-spaces with mixed norms, which sometimes goes under the name of Hölder-Minkowski’s inequality:
\[
\| f \|_{L_p(L_q)} \leq \| f \|_{L_q(L_p)} \quad \forall p \geq q.
\tag{3.5}
\]
4 Proof of the main result

It is enough to prove only the upper bounds
\[ \|(g \ast \bar{\mu})_t^\ast\|_{L_p L_q} \lesssim_{p,q} \|g\|_{I_{p,q}}, \]
as the lower bounds will follow by duality (in fact, the dual of \(I_{p,q}\) is \(I_{p',q'}\) – cf. [2]).

It should be noted that in many cases we prove in fact more general results than those needed to obtain the above upper bound.

4.1 Case \(1 < p \leq q \leq 2\)

The upper bound in Theorem 2.1 with parameters \(p\) and \(q\) such that \(1 < p \leq q \leq 2\) is a consequence of the next three Propositions.

**Proposition 4.1.** Let \(1 \leq q \leq 2, 0 < p \leq q\). One has
\[ \mathbb{E}\sup_{t \geq 0} \| (g \ast \bar{\mu})_t \|_{L_q}^p \lesssim_{p,q} \mathbb{E} \left( \int \|g\|_{L_q}^q \, d\nu \right)^{p/q}. \]

**Proof.** Inequality (3.2) with exponent \(1 \leq q \leq 2\) and \(H = \mathbb{R}\) yields
\[ \mathbb{E}\sup_{t \geq 0} | (g \ast \bar{\mu})_t |^q \lesssim_q \mathbb{E} \int |g|^q \, d\nu, \]
hence also, by Fatou’s lemma and Tonelli’s theorem,
\[ \mathbb{E}\sup_{t \geq 0} \| (g \ast \bar{\mu})_t \|_{L_q}^q \lesssim_q \mathbb{E} \int \|g\|_{L_q}^q \, d\nu. \]

Let \(T\) be any stopping time. Replacing \(g\) with \(g1_{[0,T]}(t)\), the previous inequality implies
\[ \mathbb{E}\sup_{t \leq T} \| (g \ast \bar{\mu})_t \|_{L_q}^q \lesssim_q \mathbb{E} \int_0^T \int \|g\|_{L_q}^q \, d\nu. \]

Lenglart’s domination inequality finally gives
\[ \mathbb{E}\sup_{t \geq 0} \| (g \ast \bar{\mu})_t \|_{L_q}^p \lesssim_{p,q} \mathbb{E} \left( \int \|g\|_{L_q}^q \, d\nu \right)^{p/q} \]
for any \(0 < p < q\). \(\square\)

**Remark 4.2.** The localization step spelled out in the previous proof, which is needed to apply Lenglart’s domination inequality, will be implicitly assumed in the proofs to come.

**Proposition 4.3.** Let \(0 < p \leq q \leq 2\). One has
\[ \mathbb{E}\sup_{t \geq 0} \| (g \ast \bar{\mu})_t \|_{L_q}^p \lesssim_{p,q} \mathbb{E} \left( \int |g|^2 \, d\nu \right)^{1/2} \left\| \left( \int |g|^2 \, d\nu \right)^{1/2} \right\|_{L_q}^p. \]
Proof. Inequality (5.1) with exponent \( q \in [0, 2] \) and \( H = \mathbb{R} \) yields

\[
\mathbb{E} \sup_{t \geq 0} \left| (g \ast \bar{\mu})_t \right|^q \lesssim q \mathbb{E} \left( \int |g|^2 \, d\nu \right)^{q/2}.
\]

Integrating over \( X \), taking into account Fatou’s lemma and Tonelli’s theorem, one obtains

\[
\mathbb{E} \sup_{t \geq 0} \left| (g \ast \bar{\mu})_t \right|^q \lesssim q \mathbb{E} \left( \int_0^T |g|^2 \, d\nu \right)^{q/2},
\]

which in turn yields, appealing to Lenglart’s domination inequality,

\[
\mathbb{E} \sup_{t \geq 0} \left| (g \ast \bar{\mu})_t \right|^p \lesssim p,q \mathbb{E} \left( \int_0^T |g|^2 \, d\nu \right)^{p/2}.
\]

\[
\square
\]

**Proposition 4.4.** Let \( 1 < p \leq q \leq 2 \). One has

\[
\mathbb{E} \sup_{t \geq 0} \left| (g \ast \bar{\mu})_t \right|^p \lesssim p,q \mathbb{E} \int \left| g \right|^p \, d\nu.
\]

**Proof.** By Theorem 3.2, one has

\[
\left\| (g \ast \bar{\mu})_t^* \right\|_{L^p_q} \lesssim p,q \left\| [M, M]_{1}^{1/2} \right\|_{L^p_q} = \left\| \Delta M \right\|_{L^p_q},
\]

Since \( p \leq 2 \), one has \( \left\| \Delta M \right\|_{\ell^2} \leq \left\| \Delta M \right\|_{\ell^p} \), hence, by inequality (5.5),

\[
\left\| \Delta M \right\|_{L^p_q \ell^2} \leq \left\| \Delta M \right\|_{L^p_q \ell^p} \leq \left\| \Delta M \right\|_{L^p_q \ell^p} = \mathbb{E} \int \left| g \right|^p \, d\mu = \mathbb{E} \int \left| g \right|^p \, d\nu.
\]

\[
\square
\]

**4.2 Case \( 1 < q \leq p \leq 2 \)**

The upper bound in Theorem 2.1 with parameters \( p \) and \( q \) such that \( 1 < q \leq p \leq 2 \) is a consequence of the next two Propositions.

**Proposition 4.5.** Let \( 1 < q \leq p \leq 2 \). Then one has

\[
\mathbb{E} \sup_{t \geq 0} \left| (g \ast \bar{\mu})_t \right|^p \lesssim p,q \mathbb{E} \int \left| g \right|^q \, d\nu + \mathbb{E} \left( \int \left| g \right|^q \, d\nu \right)^{p/q}.
\]

**Proof.** Appealing to Theorem 3.2, one has

\[
\mathbb{E} \sup_{t \geq 0} \left| (g \ast \bar{\mu})_t \right|^p \lesssim p,q \mathbb{E} \left\| [M, M]_{1}^{1/2} \right\|^p_{L^q} = \left\| \Delta M \right\|^p_{L^p_q \ell^2},
\]

where, since \( q \leq 2 \),

\[
\left\| \Delta M \right\|^p_{L^p_q \ell^2} \leq \left\| \Delta M \right\|^p_{L^p_q \ell^q} = \mathbb{E} \left( \int \left| g \right|^q \, d\mu \right)^{p/q}.
\]

Writing

\[
\mathbb{E} \left( \int \left| g \right|^q \, d\mu \right) = \int \left| g \right|^q \, d\mu + \int \left| g \right|^q \, d\nu,
\]

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the previous two inequalities yield
\[ \mathbb{E} \sup_{t \geq 0} \|(g \ast \hat{\mu})\|_{L_q}^p \lesssim_{p,q} \mathbb{E} \left( \int \|g\|_{L_q}^q \, d\hat{\mu} \right)^{p/q} + \mathbb{E} \left( \int \|g\|_{L_q}^q \, d\nu \right)^{p/q}, \]
where we have used inequality (3.3) with exponent \( p/q \leq 2 \).

**Proof of Proposition 4.6.** Let \( 1 < q \leq p \leq 2 \). Then one has
\[ \mathbb{E} \sup_{t \geq 0} \|(g \ast \hat{\mu})t\|_{L_q}^p \lesssim_{p,q} \mathbb{E} \left( \int |g|^2 \, d\nu \right)^{1/2} \left\| M \right\|_{L_q}^p. \]

For the proof we need the following estimate.

**Lemma 4.7.** Let \( 2 \leq p \leq q \). Then
\[ \left\| \langle M, M \rangle^{1/2} \right\|_{L_p L_q} \lesssim_{p,q} \left\| M \right\|_{L_p L_q}. \]

**Proof.** One has
\[ \mathbb{E} \left\| \langle M, M \rangle^{1/2} \right\|_{L_q}^p = \mathbb{E} \left\| \langle M, M \rangle \right\|_{L_q/2}^{p/2} = \mathbb{E} \left\| \int |g|^2 \, d\nu \right\|_{L_q/2}^{p/2} \lesssim \mathbb{E} \left\| \int |g|^2 \, d\hat{\mu} \right\|_{L_q/2}^{p/2} + \mathbb{E} \left\| \int |g|^2 \, d\mu \right\|_{L_q/2}^{p/2}, \]
where, thanks to Theorem 3.2 and Doob’s inequality,
\[ \mathbb{E} \left\| \int |g|^2 \, d\mu \right\|_{L_q/2}^{p/2} = \mathbb{E} \left\| [M, M] \right\|_{L_q/2}^{p/2} = \mathbb{E} \left\| [M, M] \right\|_{L_q}^{p/2} \lesssim_{p,q} \mathbb{E} \left\| M \right\|_{L_q}^p. \]

Moreover, setting \( N := |g|^2 \ast \hat{\mu} \), one has, again by Theorem 3.2
\[ \mathbb{E} \left\| \int |g|^2 \, d\hat{\mu} \right\|_{L_q/2}^{p/2} \lesssim_{p,q} \mathbb{E} \left\| [N, N]^{1/2} \right\|_{L_q/2}^{p/2} = \mathbb{E} \left( \int |g|^4 \, d\mu \right)^{1/2} \left\| M \right\|_{L_q}^p \lesssim_{p,q} \mathbb{E} \left\| M \right\|_{L_q}^p. \]

**Proof of Proposition 4.6.** We use a duality argument: let us write
\[ \left\| M \right\|_{L_q L_q} = \sup_{N \in B_1} \mathbb{E} \int_X M \, d\nu. \]
where \( B_1 \) stands for the unit ball of \( L_{p'}L_{q'} \). Introduce the martingale \( N \) defined by \( N_t = \mathbb{E}[N_{t|\mathcal{F}_t}] \) for all \( t \geq 0 \). The identity \( \mathbb{E}M_{\infty} N_{\infty} = \mathbb{E} \langle M, N \rangle_{\infty} \), Kunita-Watanabe’s inequality, Hölder’s inequality, and the previous Lemma imply

\[
\mathbb{E} \int_X M_{\infty} N_{\infty} \leq \left\| \langle M, M \rangle_{\infty}^{1/2} \right\|_{L_p L_q} \left\| \langle N, N \rangle_{\infty}^{1/2} \right\|_{L_{p'}L_{q'}} \\
\lesssim_{p,q} \left\| \langle M, M \rangle_{\infty}^{1/2} \right\|_{L_p L_q} \left\| N_{\infty} \right\|_{L_{p'}L_{q'}} \leq \left\| \langle M, M \rangle_{\infty}^{1/2} \right\|_{L_p L_q},
\]

whence the conclusion, because

\[
\langle M, M \rangle_{\infty} = \int |g|^2 \, d\nu.
\]

### 4.3 Case \( 1 < p \leq 2 \leq q \)

The upper bound in Theorem 2.1 with parameters \( p \) and \( q \) such that \( 1 < p \leq 2 \leq q \) follows by the next two Propositions.

**Proposition 4.8.** Let \( q \geq 2 \), \( 0 < p \leq q \). Then one has

\[
\mathbb{E} \sup_{t \leq T} \left\| (g * \hat{\mu})_t \right\|_{L_q}^p \leq \frac{1}{q} \mathbb{E} \left( \int |g|^p \, d\nu \right) + \mathbb{E} \left( \int |g|^2 \, d\nu \right)^{1/2}.
\]

**Proof.** Inequality (3.3), with exponent \( q \geq 2 \) and \( H = \mathbb{R} \), and integration over \( X \) yield

\[
\mathbb{E} \sup_{t \leq T} \left\| (g * \hat{\mu})_t \right\|_{L_q}^q \leq \mathbb{E} \left( \int |g|^q \, d\nu \right) + \mathbb{E} \left( \int |g|^2 \, d\nu \right)^{1/2},
\]

therefore, by Lenglart’s domination inequality,

\[
\mathbb{E} \sup_{t \leq T} \left\| (g * \hat{\mu})_t \right\|_{L_q}^p \leq \frac{1}{q} \mathbb{E} \left( \int |g|^p \, d\nu \right) + \mathbb{E} \left( \int |g|^2 \, d\nu \right)^{1/2}.
\]

**Proposition 4.9.** Let \( 1 < p \leq 2 \leq q \). Then one has

\[
\mathbb{E} \sup_{t \geq 0} \left\| (g * \hat{\mu})_t \right\|_{L_q}^p \leq \mathbb{E} \int |g|^p \, d\nu.
\]

**Proof.** By Theorem 3.2 and Hölder-Minkowski’s inequality (3.5), one has

\[
\left\| (g * \hat{\mu})_t \right\|_{L_p L_q} \leq |M|_{\infty}^{1/2} \left\| M \right\|_{L_p L_q} \leq \left\| \Delta M \right\|_{L_p L_q} \leq \left\| \Delta M \right\|_{L_{p'}L_{q'}},
\]

where

\[
\left\| \Delta M \right\|_{L_{p'}L_{q'}} = \mathbb{E} \sum \left\| \Delta M \right\|_{L_q}^p = \mathbb{E} \int |g|_L^p \, d\mu = \mathbb{E} \int |g|_L^p \, d\nu.
\]
4.4 Case $1 < q \leq 2 \leq p$

The upper bound in Theorem 2.1 with parameters $p$ and $q$ such that $1 < q \leq 2 \leq p$ is a consequence of the next two Propositions.

**Proposition 4.10.** Let $1 < q \leq 2 \leq p$. Then one has

$$\mathbb{E} \sup_{t \geq 0} \| (g * \bar{\mu})_t \|_{L_q}^p \lesssim_{p,q} \mathbb{E} \left( \int \| g \|_{L_q}^q \, d\nu \right)^{p/q} + \mathbb{E} \int \| g \|_{L_q}^p \, d\nu.$$ 

**Proof.** Proceeding as in the proof of Proposition 4.5 one obtains

$$\mathbb{E} \left( \int \| g \|_{L_q}^q \, d\mu \right)^{p/q} \lesssim_{p,q} \mathbb{E} \left( \int \| g \|_{L_q}^q \, d\bar{\mu} \right)^{p/q} + \mathbb{E} \left( \int \| g \|_{L_q}^q \, d\nu \right)^{p/q}. $$

If $p \leq 2q$, i.e. if $p/q \leq 2$, the second inequality in Theorem 3.1 (with $H = \mathbb{R}$) yields

$$\mathbb{E} \left( \int \| g \|_{L_q}^q \, d\nu \right)^{p/q} \lesssim_{p,q} \mathbb{E} \int \| g \|_{L_q}^p \, d\nu,$$

as desired. Otherwise, if $p > 2q$, i.e. if $p/q > 2$, the third inequality in Theorem 3.1 (with $H = \mathbb{R}$) yields

$$\mathbb{E} \left( \int \| g \|_{L_q}^q \, d\bar{\mu} \right)^{p/q} \lesssim_{p,q} \mathbb{E} \int \int \| g \|_{L_q}^p \, d\nu + \mathbb{E} \left( \int \| g \|_{L_q}^{2q} \, d\nu \right)^{p/(2q)}$$

$$= \mathbb{E} \int \| g \|_{L_q}^p \, d\nu + \mathbb{E} \left( \int \| g \|_{L_q}^{2q} \, d\nu \right)^{p/q}.$$

Since we have $q < 2q < p$, by Lemma 3.3 one immediately gets

$$\mathbb{E} \left\| g \right\|_{L_q}^p \leq \mathbb{E} \left\| g \right\|_{L_q}^p + \mathbb{E} \left\| g \right\|_{L_p(\nu)}^p$$

$$= \mathbb{E} \left( \int \| g \|_{L_q}^q \, d\nu \right)^{p/q} + \mathbb{E} \int \| g \|_{L_q}^p \, d\nu,$$

and the proof is concluded. \hfill \Box

**Proposition 4.11.** Let $1 \leq q \leq 2 \leq p$. Then one has

$$\mathbb{E} \sup_{t \geq 0} \| (g * \bar{\mu})_t \|_{L_q}^p \lesssim_{p,q} \mathbb{E} \left( \int \| g \|_{L_q}^q \, d\nu \right)^{1/2} + \mathbb{E} \int \| g \|_{L_q}^p \, d\nu.$$ 

For the proof we need the following result by Lenglart, Lepingle and Pratelli (see [4, Lemma 1.1]).

**Lemma 4.12.** Let $A$ and $B$ be increasing adapted processes. If there exist $r > 0$ and $\alpha > 0$ such that

$$\mathbb{E}(A_T - A_S)^r \leq \alpha \mathbb{E}B_T^r \mathbf{1}_{\{S < T\}}$$

for all stopping times $S$ and $T$ such that $S < T$, then one has, for any moderate function $F$,

$$\mathbb{E}F(A_\infty) \lesssim_{r,\alpha,F} \mathbb{E}F(B_\infty).$$
Proof of Proposition 4.11. We shall proceed in several steps.

Step 1. We introduce the Davis decomposition of \( M := g \ast \bar{\mu} \): define the real-valued process \( S_t := \sup_{s \leq t} \| \Delta M_s \|_{L_q}, \) and set

\[
K_1^1 := \sum \Delta M_s, \quad K_1^2 := \tilde{K}_1^1, \quad K := K^1 - K_1^2.
\]

Then there exists an \( L_q \)-valued predictable process \( g' \) such that \( K^1 = g' \ast \mu, \) hence \( K^2 = g' \ast \nu \) and \( K = g' \ast \bar{\mu}. \) Now set \( L = M - K = (g - g') \ast \bar{\mu}. \)

Step 2. Denoting the total variation by \( \| \cdot \|_{TV} \), one has

\[
M^*_{\infty} \leq K^*_{\infty} + L^*_{\infty} \leq \| K \|_{TV} + L^*_{\infty},
\]

hence, for any \( p \geq 1, \)

\[
E \sup_{t \geq 0} \| M_t \|_{L_q}^p \lesssim_p E \| K \|_{TV(L_q)}^p + E \sup_{t \geq 0} \| L_t \|_{L_q}^p,
\]

where

\[
E \| K \|_{TV}^p \lesssim_p E \| K \|_{TV}^p + \tilde{E} \| K \|_{TV}^p \lesssim \tilde{E} \| K \|_{TV}^p,
\]

and \( \| K \|_{TV(L_q)} \lesssim S_{\infty} \) implies

\[
E \| K \|_{TV}^p \lesssim_p ES_{\infty}^p.
\]

Step 3. Let \( S, T \) be any stopping times. Setting

\[
(L^{S,T})_t := (L_{(S+t) \wedge T} - L_{S-}) 1_{\{S<T\}},
\]

we are going to show that

\[
L^*_T - L^*_{S-} \leq (L^{S,T})_{\infty} 1_{\{S<T\}}.
\]

Since \( t \mapsto L^*_t \) is increasing and the right-hand side is positive, we can assume \( S < T \) without loss of generality. If \( L^*_T = L^*_{S-}, \) the inequality is obviously true, hence we can assume \( L^*_T > L^*_{S-}, \) thus also

\[
L^*_T \leq \sup_{S \leq s \leq T} \| L_s \|_{L_q}.
\]

We therefore have, writing \( \| \cdot \| \) instead of \( \| \cdot \|_{L_q} \) for compactness of notation,

\[
L^*_T - L^*_{S-} \leq \left( \sup_{s \in [S,T]} \| L_s \| - \sup_{s < S} \| L_s \| \right) 1_{\{S<T\}}
\]

\[
\leq \left( \sup_{s \in [S,T]} \| L_s \| - \| L_{S-} \| \right) 1_{\{S<T\}}
\]

\[
\leq \left( \sup_{s \in [S,T]} \| L_s - L_{S-} \| \right) 1_{\{S<T\}} = (L^{S,T})_{\infty} 1_{\{S<T\}},
\]

where we have used the obvious estimates \( \sup_{s < S} \| L_s \| \geq \| L_{S-} \| \) and \( \| x \| - \| y \| \leq \| x - y \|. \)
Step 4. The previous step immediately implies
\[ \mathbb{E}(L_{T-}^* - L_{S-}^*)^q \lesssim \mathbb{E}((L^{S,T})_t^*)^q \mathbf{1}_{\{S<T\}}, \]
By Theorem 3.2 we have
\[ \mathbb{E}((L^{S,T})_t^*)^q \mathbf{1}_{\{S<T\}} \lesssim_q \mathbb{E}\|[L^{S,T}, L^{S,T}]_\infty^{1/2}\|^q_{L_q}, \]
where
\[ [L^{S,T}, L^{S,T}]_\infty \leq [L, L]_T + \|\Delta L_S\|^2 \leq [L, L]_{T-} + \|\Delta L_S\|^2 + \|\Delta L_T\|^2, \]
hence
\[ \|[L^{S,T}, L^{S,T}]_\infty^{1/2}\|^q_{L_q} \leq \|[L, L]_{T-}^{1/2}\|^q_{L_q} + \|\Delta L_S\|_{L_q} + \|\Delta L_T\|_{L_q} \lesssim \|[L, L]_{T-}^{1/2}\|^q_{L_q} + S_{T-}, \]
therefore also, recalling that \( q \leq 2 \),
\[ \mathbb{E}(L_{T-}^* - L_{S-}^*)^q \lesssim_q \mathbb{E}\|[L, L]_{T-}^{1/2}\|^q_{L_q} + \mathbb{E}S_{T-}^q \lesssim_q \mathbb{E}\|[L, L]_{T-}^{1/2}\|^p_{L_q} + \mathbb{E}S_{T-}^p. \]
We can now apply the previous Lemma to obtain
\[ \mathbb{E}(L_{\infty}^*)^p \lesssim_{p, q} \mathbb{E}\|[L, L]_{\infty}^{1/2}\|^p_{L_q} + \mathbb{E}S_{\infty}^p. \]

Step 5. Recalling the estimate on \( K \), we are left with
\[ \mathbb{E}(M_{\infty}^*)^p \lesssim \mathbb{E}\|[L, L]_{\infty}^{1/2}\|^p_{L_q} + \mathbb{E}S_{\infty}^p, \]
where, since \( L = (g - g') \ast \tilde{\mu} \) and \(|g'| \leq |g|\) pointwise (in fact by construction \( g' \) represents some jumps of \( M \) only),
\[ \langle L, L \rangle_{\infty} = \int |g - g'|^2 d\nu \lesssim \int |g|^2 d\nu, \]
and
\[ \mathbb{E}S_{\infty}^p = \mathbb{E} \sup_{a \geq 0} \|\Delta M_a\|^p_{L_q} \lesssim \mathbb{E} \sum \|\Delta M\|^p_{L_q} = \mathbb{E} \int \|g\|^p_{L_q} d\mu = \mathbb{E} \int \|g\|^p_{L_q} d\nu. \]

4.5 Case \( 2 \leq p \leq q \)

The upper bound in Theorem 2.1 with parameters \( p \) and \( q \) such that \( 2 \leq p \leq q \) follows by Proposition 4.8 and by the next Proposition.

Proposition 4.13. Let \( 2 \leq p \leq q \). Then one has
\[ \mathbb{E} \sup_{t \geq 0} \| (g \ast \tilde{\mu})_t \|^p_{L_q} \lesssim_{p, q} \mathbb{E} \int \|g\|^p_{L_q} d\nu + \mathbb{E} \left( \int |g|^2 d\nu \right)^{1/2} \|L_q \|_{L_q}. \]
Proof. Recalling Theorem 3.2 one has

$$\mathbb{E}[\|g * \tilde{\mu}\|_{L_q}^p] \leq_{p,q} \mathbb{E}[\|M, M\|_{L_q}^{1/2}]^p = \mathbb{E}\left(\int |g|^2 d\mu\right)^{1/2} p_{L_q}.$$ 

If $p = 2$, it holds

$$\mathbb{E}\left(\int |g|^2 d\mu\right)^{1/2} p_{L_q} = \mathbb{E}\left(\int |g|^2 d\mu\right)_{L_{q/2}} \leq \mathbb{E}\int |g|^2 d\mu = \mathbb{E}\int |g|^2 d\nu = \mathbb{E}\int |g|^2 d\nu,$$

that is the claim is proved in the case $p = 2$. Therefore we assume, for the rest of the proof, that $p > 2$. One has

$$\mathbb{E}\left(\int |g|^2 d\mu\right)^{1/2} p_{L_q} \leq \mathbb{E}\left(\int |g|^2 d\mu\right)^{1/2} p_{L_q} + \mathbb{E}\left(\int |g|^2 d\nu\right)^{1/2} p_{L_q},$$

where

$$\mathbb{E}\left(\int |g|^2 d\mu\right)^{1/2} p_{L_q} = \mathbb{E}\left(\int |g|^2 d\mu\right)^{p/2}_{L_{q/2}}.$$

If $2 < p \leq 4$, i.e. if $1 < p/2 \leq 2$, Propositions 4.4 and 4.9 imply that

$$\mathbb{E}\left(\int |g|^2 d\mu\right)^{p/2}_{L_{q/2}} \leq_{p,q} \mathbb{E}\int |g|^2 d\nu = \mathbb{E}\int |g|^2 d\nu.$$

We have thus proved that the claim of the Proposition is true for any $p, q$ such that $2 < p \leq 4$ and $q \geq p$. We now proceed by induction, i.e. we assume the claim is true for $2 < p \leq 2^n$ (which we have just verified for $n = 2$), and we show that the claim is true for $2 < p \leq 2^{n+1}$. In fact, we have just seen that

$$\mathbb{E}[\|g * \tilde{\mu}\|_{L_q}^p] \leq_{p,q} \mathbb{E}\left(\int |g|^2 d\mu\right)^{p/2}_{L_{q/2}} + \mathbb{E}\left(\int |g|^2 d\nu\right)^{1/2} p_{L_q},$$

where, since $p/2 \leq 2^n$, the inductive assumption implies

$$\mathbb{E}\left(\int |g|^2 d\mu\right)^{p/2}_{L_{q/2}} \leq_{p,q} \mathbb{E}\left(\int |g|^2 d\nu\right)^{1/2} p_{L_{q/2}} + \mathbb{E}\left(\int |g|^2 d\nu\right)^{p/2}_{L_{q/2}} d\nu$$

$$= \mathbb{E}\left(\int |g|^4 d\nu\right)^{1/4} p_{L_q} + \mathbb{E}\int |g|^p d\nu.$$

Note that, by Lemma 3.3, since $4 < p$, one has

$$\left(\int |g|^4 d\nu\right)^{1/2} = \|g\|_{L_4(d\nu)} \leq \|g\|_{L_2(d\nu)} + \|g\|_{L_p(d\nu)},$$

hence

$$\mathbb{E}\left(\int |g|^2 d\nu\right)^{1/4} p_{L_q} \leq \mathbb{E}\left(\int |g|^2 d\nu\right)^{1/2} p_{L_q} + \mathbb{E}\int |g|^p d\nu.$$
The proof is finished by observing that, since \( q \geq p \), Hölder-Minkowski’s inequality \((3.5)\) yields
\[
\mathbb{E}\left\| g \right\|_{L_p(\nu)}^p \leq \mathbb{E}\left\| g \right\|_{L_q}^p = \mathbb{E} \int \left\| g \right\|_{L_q}^p \, d\nu.
\]

4.6 Case \( 2 \leq q \leq p \)

The upper bound in Theorem \( 2.1 \) with parameters \( p \) and \( q \) such that \( 2 \leq p \leq q \) is proved in the following Proposition.

**Proposition 4.14.** Let \( 2 \leq q \leq p \). Then one has
\[
\mathbb{E} \sup_{t \geq 0} \| (g * \bar{\mu})_t \|_{L_q} \lesssim_{p,q} \mathbb{E} \int \| g \|_{L_q}^q \, d\nu + \mathbb{E} \left( \int \| g \|_{L_q}^q \, d\nu \right)^{p/q} + \mathbb{E} \left( \int \| g \|_{L_q}^2 \, d\nu \right)^{1/2} \left\| g \right\|_{L_q}^p.
\]

**Proof.** The claim is certainly true if \( q = 2 \), by Theorem \( 3.1 \). We shall therefore assume \( q > 2 \) from now on. In view of Theorem \( 3.2 \), it is enough to estimate the \( L_pL_q \) norm of \( [M,M]_{\infty}^{1/2} \). We use once again the decomposition
\[
[M,M]_{\infty}^{1/2} = \left( \int |g|^2 \, d\mu \right)^{1/2} \leq \left( \int |g|^2 \, d\bar{\mu} \right)^{1/2} + \left( \int |g|^2 \, d\nu \right)^{1/2},
\]
and its immediate consequence
\[
\left\| [M,M]_{\infty}^{1/2} \right\|_{L_pL_q} \leq \left\| \int |g|^2 \, d\bar{\mu} \right\|_{L_pL_q}^{1/2} + \left\| \left( \int |g|^2 \, d\nu \right)^{1/2} \right\|_{L_pL_q}.
\]
We are thus left with the task of estimating the first term on the right-hand side. In particular, writing
\[
\mathbb{E} \left\| \int |g|^2 \, d\bar{\mu} \right\|_{L_q}^{1/2} \left\| g \right\|_{L_q}^p = \mathbb{E} \left\| \int |g|^2 \, d\bar{\mu} \right\|_{L_q}^{p/2},
\]
we observe that, assuming \( 2 \leq q \leq 4 \), i.e. \( 1 < q/2 \leq 2 \), Propositions \( 4.5 \) and \( 4.10 \) imply
\[
\mathbb{E} \left\| \int |g|^2 \, d\bar{\mu} \right\|_{L_q}^{1/2} \left\| g \right\|_{L_q}^p \lesssim_{p,q} \mathbb{E} \left( \int \| g \|_{L_q}^q \, d\nu \right)^{p/q} + \mathbb{E} \int \| g \|_{L_q}^2 \, d\nu + \mathbb{E} \int |g|^4 \, d\nu.
\]
We have thus proved that the claim is true for any \( q \in [2,4] \). We proceed by induction, showing that if the claim is true for \( 2 \leq q \leq 2^n \) (which is indeed the case with \( n = 2 \)), then the claim remains true for \( 2 \leq q \leq 2^{n+1} \). In view of the reasoning at the beginning of the proof, it suffices to estimate the term on the right-hand side of \( (4.2) \):
\[
\mathbb{E} \left\| \int |g|^2 \, d\bar{\mu} \right\|_{L_q}^{p/2} \lesssim_{p,q} \mathbb{E} \left( \int \| g \|_{L_q}^q \, d\nu \right)^{p/q} + \mathbb{E} \int \| g \|_{L_q}^2 \, d\nu + \mathbb{E} \left( \int |g|^4 \, d\nu \right)^{1/4} \left\| g \right\|_{L_q}^p.
\]
Since $4 < q$, Lemma 3.3 yields
\[
\left( \int |g|^4 \, d\nu \right)^{1/4} = \| g \|_{L_q(\nu)} \leq \| g \|_{L_4(\nu)} + \| g \|_{L_q(\nu)},
\]

hence also
\[
\mathbb{E} \left[ \left( \int |g|^4 \, d\nu \right)^{1/4} \right]^{p} \leq \mathbb{E} \left[ \| g \|_{L_2(\nu)} \right]^{p} + \mathbb{E} \left[ \| g \|_{L_q(\nu)} \right]^{p}
\]

\[
= \mathbb{E} \left[ \left( \int |g|^2 \, d\nu \right)^{1/2} \right]^{p} + \mathbb{E} \left[ \| g \|_{L_q(\nu)} \right]^{p}
\]

\[
= \mathbb{E} \left[ \left( \int |g|^2 \, d\nu \right)^{1/2} \right]^{p} + \mathbb{E} \left( \int |g|^q \, d\nu \right)^{p/q},
\]

thus concluding the proof. \qed

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