Protected helical transport in magnetically doped quantum wires: beyond the 1D paradigm

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One-dimensional (1D) quantum wires, which are functionalized by magnetic ad-atoms, can host ballistic helical transport. Helicity protects transport from an undesirable influence of material imperfections and makes the magnetically doped wire very promising elements for nanoelectronics and spintronics. However, fabricating purely 1D conductors is experimentally very challenging and not always feasible. In this paper, we show that the protected helical transport can exist even in quasi-1D wires. We model the quasi-1D magnetically doped wire as two coupled dense 1D Kondo chains. Each chain consists of itinerant electrons interacting with localized quantum magnetic moments - Kondo impurities. We have analyzed the regimes of weak-, intermediate- and strong inter-chain coupling and found conditions necessary for the origin of the aforementioned protected transport. Our results give a push for experimental realizations of the helical states in magnetically doped wires.

One major stepping stone in the development of modern nanoelectronic and spintronic devices is the reduction of destructive effects caused by material imperfections, e.g., backscattering and localization. One dimensional (1D) conductors are especially sensitive to such undesirable effects which suppress ballistic transport [1]. One possibility for protected transport is provided by the helicity of conduction electrons. Helicity, defined as \( h = \text{sgn}(p) \cdot \text{sgn}(\sigma) \), reflects the lock-in relation between the electron’s momentum, \( p \), and spin, \( \sigma \).

Physical mechanisms, which yield helical states, generally fall in two categories. The first category includes topological insulators [2–4]. The archetypal example is a two-dimensional material whose bulk is a topologically nontrivial insulator possessing time-reversal symmetry [5–7]. 1D conducting helical modes appear at the edges of these systems. Modern experiments show also helicity of hinge states in high-order topological insulators [8–10]. Protection of the helical edge transport is expected to be virtually ideal though it is not robust in reality [11–19].

The second category includes systems where the helical states are governed by interactions, for example, the hyperfine interaction between nuclear magnetic moments and itinerant electrons [20–24], the spin-orbit interaction (SOI) in a combination with either magnetic fields [25, 26] or Coulomb interactions [27]. Several experiments confirmed the existence of helical states in interacting systems [25, 26, 28, 29].

Another promising platform for protected helical transport is provided by magnetically doped 1D quantum wires [30–33]. It is somewhat similar to the successful realization of topological superconductivity [34–36]. In spite of the solid theoretical background, experiments demonstrating helical transport in the magnetically doped 1D wires are still missing. The main obstacle hampering these experiments in traditional materials (GaAs or SiGe) is the non-triviality of methods used to

\[ \text{FIG. 1. The quasi-1D doped quantum wire is modeled by two coupled Kondo chains, each consisting of an itinerant electron system (orange tubes) and an array of localized quantum magnetic impurities (blue spins). Electrons can tunnel at every site of the electronic lattice into the neighboring wire, tunneling is indicated by the orange dashed lines.} \]

produce truly 1D conductors [37, 38].

The goal of this Letter is to show that the strict one-dimensionality is not necessary and the protected helical states can emerge also in quasi-1D samples.

Model: Magnetically doped quantum wires can be described by the well-known theoretical model of a Kondo chain (KC) - a 1D array of localized quantum magnetic impurities interacting with 1D itinerant electrons [39–49]. The physics of the KC is governed by two competing, mutually exclusive effects: the Kondo effect and the indirect, Ruderman–Kittel–Kasuya–Yosida (RKKY), exchange interaction between the impurities. The dominant effect can be found from Doniach’s criterion which is based on the comparison of relevant energy scales: the Kondo temperature, \( T_K \), and the RKKY energy, \( E_{\text{RKKY}} \) [50]. If \( T_K > E_{\text{RKKY}} \), the Kondo screening dominates; magnetic impurities are screened individually. This leads to a Kondo insulator at half filling and a heavy fermion phase away from half filling [39, 51]. In the opposite case \( T_K < E_{\text{RKKY}} \), the RKKY interaction dominates and governs inter-impurity correlations. One can translate the above inequality to distances and show that RKKY overwhelms the Kondo effect in dense KCs, where the (mean)
inter-impurity distance $\xi_s$ is smaller than a crossover scale $\xi_c$: $\xi_s \leq \xi_c \propto \xi_0 (\rho_0 J^2/T_K)^{1/2}$ [30–33]. Here $J$, $\rho_0$, and $\xi_0$ are the Kondo coupling, the density of states, and the lattice spacing, respectively. The RKKY dominated regime is typical in 1D systems [32, 52]. Two of us have recently shown that the (partially) protected transport can exist in the dense and incommensurate KC with a small Kondo coupling, which can be anisotropic (easy-plane anisotropy) [30, 31] or isotropic [32, 33]. In both cases, protection of transport results from the (global or local) helicity of localized spins.

To investigate the importance of the one dimensionality for the protected transport we consider the quasi-1D model consisting of two tunneling-coupled dense and incommensurate KCs, Fig.1. This model possesses a non-trivial degree of freedom: The magnetic impurities in different KCs are correlated only via the tunneling and it is a-priori not clear, whether they form a global helical ordering, which is present in each KC and has the same direction of rotation. Such correlations are expected to protect ballistic transport in the quasi-1D system.

The Hamiltonian of our quasi-1D model reads as

$$\hat{H} = \hat{H}^{(1)}_{KC} + \hat{H}^{(2)}_{KC} + \hat{H}_{\text{tun}};$$

where $\hat{H}^{(1,2)}_{KC}$ are the Hamiltonians of the uncoupled KCs:

$$\hat{H}^{(n)}_{KC} = -t \left( c^\dagger_j c^\sigma_{j+1} \right)^{(n)} + J_a \left( c^\dagger_j S^a_j \sigma_a c^\sigma_j \right)^{(n)} + \text{h.c.},$$

$$\hat{H}_{\text{tun}}$$ describes the electron tunneling:

$$\hat{H}_{\text{tun}} = -t_{\perp} \left( c^\dagger_j c^\sigma_{j} \right)^{(2)} + \text{h.c.;}$$

$c_j = (c_j^\uparrow, c_j^\downarrow)^T$ is a spinor and $c^\sigma_j$ annihilates (creates) an electron with spin $\sigma = \uparrow, \downarrow$ at a lattice site $j$ of a given chain marked by $n = 1, 2; t$ ($t_{\perp}$) is the intra (inter) chain hopping strength, $J_a$ is the strength of the Kondo interaction in $a = x, y, z$ direction, $S^a_j$ is a spin $s$ operator of an impurity located at lattice site $j$ of the chain $n$; and $\sigma_a$ are the Pauli matrices [53]. For simplicity, we do not distinguish the crystalline lattice constant $\xi_0$ and $\xi_s$, assume that the individual KCs have same parameters, and focus on low temperatures, $T \to 0$. We explore the case of the easy-plane magnetic anisotropy $J_x = J_y = J \gg J_z \to 0$ with a small coupling constant, $J \ll t$, and incommensurate band fillings. This setup is relevant for the search of protected transport [30, 31].

Three regimes of the tunneling-coupled KC: The non-interacting part of the Hamiltonian (1), $H_0 \equiv H|_{J=0}$, has the spectrum $\varepsilon_{\pm}(k) = -2t \cos(k \xi_0) \mp t_{\perp} - \mu$ [54]. The value of $t_{\perp}$ determines three different regimes: the strong-, intermediate, and weak- interchain tunneling, see Fig.2.

If tunneling is strong, $2t \ll t_{\perp}$, there are two bands separated by a large gap of order $t_{\perp}$. Without loss of generality, we can place the chemical potential, $\mu$, in the lower band and take into account the electron-spin interaction perturbatively by using the smallness of $J/t_{\perp}$ and $t/t_{\perp}$. We will show that such a perturbation yields only small and inessential corrections to the physics of the helical 1D wire described in Refs. [30, 31].

More interesting are the other two cases of the intermediate, $t_{\perp} \lesssim 2t$, or small, $t_{\perp} \ll 2t$, tunneling, which can possess four Fermi points and, therefore, cannot be reduced to the 1D helical wire. In the former case, the Fermi points are well separated and one has to take into account all electron-spin interactions non-perturbatively. If tunneling is weak, the four Fermi points almost coincide in pairs. This allows one to treat the tunneling $t_{\perp}$ as a small perturbation for two decoupled KCs.

Next, we rewrite the Kondo interaction in the eigenbasis of $H_0$:

$$\hat{H}_{\text{int}} = \langle J/2 \rangle \left[ c^\dagger_\nu S^b_{\perp} \sigma_b c_\nu + c^\dagger_\nu S^b_{\perp} \sigma_b c_{-\nu} + \text{h.c.} \right],$$

where $b = x, y$, $S^a_{\pm} = S^{(1)}_{\pm} \pm S^{(2)}_{\pm}$ and $\nu = +(-)$ labels the lower (upper) band fermions. The Kondo interaction enables intra- and interband scattering processes. We switch now to the functional integral formulation of the theory on the imaginary time contour and analyze the three cases shown in Fig. 2 in detail. The localized spins in this approach are conveniently parameterized by a properly normalized vector field [55].

Strong tunneling, $J \ll 2t \ll t_{\perp}$: If $\mu$ belongs to the lower band and $T = 0$, transitions between the bands are virtual and result only in a small renormalization of parameters of the conduction band [56]. To show this, we integrate out the fermions from the upper band perturbatively. This yields a mass term for the propagator of the conduction electrons from the lower band: $\Sigma_- = J^2 (S^b_{\perp})^2 \langle \psi_+ \psi_- \rangle + O(J^2/t_{\perp}^2)$, see Suppl.Mat. (SM) A; $\psi_{\pm}$ are fermionic fields. $\Sigma_-$ governs a shift of $\mu$ and enables a weak spin conserving backscattering. Both effects are parametrically small.
compared to those governed by the intraband Kondo interaction. Therefore, the interband transitions can be neglected and the Lagrangian density of the electrons in the lower band reduces to

\[ \mathcal{L}^{(ST)}_+ \simeq \bar{\psi}_+ \left[ -i \omega + \epsilon_+(k) - \mu + (J \rho_s/2) S^b_+ \sigma_0 \right] \psi_+; \]  

where \( \omega \) is the fermionic Matsubara frequency, and \( \rho_s \) is the spins density. Below, we will change to the continuous limit with \( \rho_s = \text{const} \) and absorb \( \rho_s \) in the coupling constant: \( J' \equiv J \rho_s/2 \). Eq.\( (5) \) describes a single KC where the itinerant electrons interact with the composite spins \( S_+ \). This theory can be studied by using the approach developed in Refs.\( [30, 31] \) for the usual KC. It is to absorb them into spin configurations and find the Lagrangian does not contain 2-body interactions. Therefore, the interband transitions can be compared to those governed by the intraband Kondo interaction.

Intermediate tunneling, \( t_\perp \lesssim 2t \): Let us analyze the case where four Fermi points (two in the lower band and two in the upper band with Fermi momenta \( \pm k_F^{(\pm)} \), respectively) coexist and are well separated, \( \delta k_F \equiv k_F^{(+) - k_F^{(-)} } \approx k_F^{(+) + k_F^{(-)} } / 2 \). The first step of the further analysis is to single out slow modes. We linearize the dispersion relation of the non-interacting system around the Fermi points and introduce smooth left (L) and right (R) moving modes in a standard way. These fermionic modes are described by the Lagrangian \( \mathcal{L}_0 = \bar{R}_\nu \partial_+^{(\nu)} R_\nu + L_\nu \partial_-^{(\nu)} L_\nu \) with \( \partial_\nu^{(\nu)} = \partial_\tau + i v^{(\nu)} \partial_x \) being the chiral derivative. The Fermi velocity depends on the band index: \( v^{(\nu)} = 2t \xi_0 \left( \frac{k_F^{(\nu)} \xi_0}{c} \right) \).

The physics of the dense KCs is governed by backscattering of the fermions \( [30-33] \) described by

\[ \mathcal{L}^{(\nu \nu')}_{bs} = J' R_\nu S^b_\nu \sigma_3 L_{\nu'} e^{2ik_F^{(\nu')^x} x + h.c.} \]  

\( k_F^{(\nu')^x} = \hat{k}_F \) for the interband backscattering, \( \nu = -\nu' \), and \( k_F^{(\nu')^x} = \hat{k}_F + iv_\nu k_F / 2 \) for the intraband one, \( \nu = \nu' \). Backscattering opens a gap in the spectrum of fermions and, thus, reduces the ground state energy of the entire system \( [30-33] \).

We are interested in the low energy physics whose Lagrangian does not contain 2\( k_F \)-oscillations. Our strategy is to absorb them into spin configurations and find the configuration, which minimizes the ground state energy by maximizing backscattering. We decompose the spin variables into slow and fast components (see SM B):

\[ S^{(n)} / s = m_n + \left[ e_1^{(n)} \cos(Qx) + e_2^{(n)} \sin(Qx) \right] \sqrt{1 - m_n^2}, \]

where \( m_n = \sin(\sigma^{(n)}) \left[ e_1^{(n)} \times e_2^{(n)} \right] \). 2\( k_F^{(--)} \leq Q \leq 2k_F^{(++)} \) and \( e_1^{(n)} \) are two orthonormal vectors which lie almost in the plane defined by the magnetic anisotropy ("easy plane"). These two vectors can be parameterized by the in-plane polar angle, \( \psi^{(n)} \), and by another angle describing small out-of-plane fluctuations, \( \theta^{(n)} \). Oscillating terms allow one to absorb 2\( k_F \)-oscillations from the backscattering and, thus, are needed to minimize the ground state energy. The angle \( \sigma^{(n)} \) weigths the zero mode \( m_n \) and has the semiclassical value \( \sigma^{(n)} \rightarrow 0 \). Deviations of \( \alpha^{(n)} \) from this value are small. \( \theta^{(n)} \) and \( \sigma^{(n)} \) are massive variables and they can be integrated out in the Gaussian approximation \( [30, 31] \).

Eq.\( (6) \) contains oscillations with three different wave vectors, \( 2k_F^{(++)} = 2k_F^{(--)}, 2k_F^{(--)} = 2k_F^{(+-)} \), and \( 2k_F = 2|k_F^{(+-)}| \), which are of the same order in the intermediate tunneling regime and correspond to various intra- and inter-band scatterings, see Fig.3. By tuning \( Q \), one can absorb into the spin configuration only one of these three vectors, two others result in fast oscillations which do not contribute to the low energy theory. The remaining smooth part of the backscattering opens the helical gap, see Eq.\( (8) \) below, in the fermionic spectrum. If \( Q = 2k_F^{(\mp)} \), the gap is opened only in one band, either in the lower band or in the upper one. The choice \( Q = 2k_F \) results in doubling the number of gapped fermionic modes. Moreover, it provides the maximal value of all gaps, see SM D. We thus conclude that the ground state energy reaches its minimum at \( Q = 2k_F \).

After inserting this choice into Eq.\( (6) \) and neglecting oscillating terms, we arrive at:

\[ \mathcal{L}^{(\nu \nu')}_{bs} \approx R_\nu \left[ \frac{\Delta^{(1)} - \Delta^{(2)}}{J'} \right] L_{\nu'}, + h.c., \quad \nu \neq \nu', \]

where \( \Delta^{(n)} \) are scattering amplitudes of the respective 1D KCs \( [30-33] \): \( \Delta^{(n)} / J = e^{i \psi^{(n)} / 2} - e^{-i \psi^{(n)} / 2} \), \( \sigma_+ = \cos(\psi^{(n)} / 2) \), \( \sigma_- = \cos(\psi^{(n)} / 2) \), \( \tilde{J} = s \cos(\theta^{(n)}) J' \). Next, we use the classical value \( \psi^{(n)} = 0 \) and look for the classical value of \( \theta^{(n)} \). Based on Refs.\( [30, 31] \), we anticipate that \( \theta^{(n)} = 0 \) or \( \pi \). The gap values (at fixed angles \( \psi^{(n)} \)) are different in the cases \( \theta^{(1)} = \theta^{(2)} \) and \( \theta^{(1)} \neq \theta^{(2)} \). For example, if \( \theta^{(1)} = 0 \) then

\[ \theta^{(2)} = 0: \quad \tilde{m}_- = 2i \tilde{J} e^{-i \psi^{(2)} / 2} \sin(\psi^{(1)}) \sigma_+, \]

\[ \theta^{(2)} = \pi: \quad \tilde{m}_- = -\tilde{J} \left( e^{i \psi^{(2)} / 2} \sigma_- + e^{-i \psi^{(1)} / 2} \sigma_+ \right). \]
Here $\hat{m}_{-} = \hat{\Delta}^{(1)} - \hat{\Delta}^{(2)}$; $\tilde{\psi} = (\psi^{(1)} + \psi^{(2)})/2$ and $\delta\psi = (\psi^{(1)} - \psi^{(2)})/2$. The modulus of the eigenvalue of $\hat{m}$ reaches maximum in Eq. (8) at $\delta\psi = \pm \pi/2$ and becomes twice as large as that in Eq. (9). Therefore, we come across a mode locking of the in-plane spin polar angles which makes the spin configuration $\theta^{(1)} = \theta^{(2)}$ energetically favorable, see SM D. The phase factor $\tilde{\psi}$ in Eq. (8) can be gauged out and, after straightforward calculations, we arrive at the expression for the gain (with respect to the non-interacting case, $J = 0$) of the ground state energy:

$$\delta E^{(TT)} = \left[ 4\xi_0 \tilde{J}^2 / \pi \left( v_F + v_{F'} \right) \right] \log \left( 2t/|\tilde{J}| \right);$$  (10)

see SM C,D. The analysis of the ground state shows that the helical symmetry is spontaneously broken and a gap opens for fermions with a given helicity in both bands. As a result, we find gapless helical fermions with $h = -1$ for $\theta^{(1,2)} = 0$ (or $h = +1$ for $\theta^{(1,2)} = \pi$).

To finalize the derivation of the effective low-energy theory, we reintestate the Wess-Zumino term for the spin variables $[55]$ and integrate out all massive fields approximately, see SM H. This yields the Lagrangian

$$\mathcal{L}^{(TT)} = \mathcal{L}_{\text{LL}} \tilde{\psi} + \sum_{\nu = \pm} \mathcal{L}_0[R_{\nu}, L_{\nu,\tau}].$$  (11)

Here $\mathcal{L}_{\text{LL}} = [(\partial_\tau \tilde{\psi})^2 + (v_\psi \partial_\rho \tilde{\psi})^2]/2\pi K_\psi$ is the Luttinger liquid Lagrangian, which describes the slow, $v_\psi \ll v_F$, collective bosonic helical mode with the effective strong interactions, $K_\psi \ll 1$. Gapless fermionic modes have the same helicity in each band, $\nu = \pm$. This parametrically suppresses Anderson localization which can be induced by an additional (weak) spinless disorder, see SM F. Thus, transport in such a system is ballistic in parametrically long samples.

**Weak tunneling.** $t_\perp \leq J \ll 2t$: If $t_\perp$ is small, the separation between the Fermi points shrinks and they almost coincide in pairs when $\delta k_F \approx 2t_\perp / \tilde{v}_F \ll \tilde{k}_F$. We start again from Eq. (6), however, unlike the intermediate tunneling, $\delta k_F$-oscillations are slow and cannot be neglected in the low energy sector. This makes the number of the gapped fermionic modes independent of the choice of $Q$. We retain $Q = 2\tilde{k}_F$ for convenience and repeat the steps resulting in Eq. (7). Slow $\delta k_F$-oscillations yield now additional intraband scattering terms:

$$\mathcal{L}_{bs}^{\nu\nu} \approx R_{\nu} \left[ \hat{\Delta}^{(1)} + \hat{\Delta}^{(2)} \right] L_{\nu} e^{i\delta k_F p_x} + \text{h.c.}$$  (12)

The slowly oscillating backscattering opens a gap at the energy which is shifted by $\delta k_F v_\psi$, leading to a small number of occupied (or empty) states above (or below) the gap $[32, 33]$. However, these states are energetically split off by the gap and thus do not have any noticeable influence on the dc transport.

Next, we use the value $\alpha_{\text{cl}}^{(n)} = 0$ and look for the optimal spin configuration with $\theta_{\text{cl}}^{(n)} = 0$ or $\pi$. The intraband scattering introduces a new gap structure. Additionally to Eqs. (8,9), we find for $\theta^{(1)} = 0$:

$$\theta^{(2)} = 0 : \quad \hat{m}_{+} = -2\tilde{J} e^{-i\tilde{\psi}} \cos (\delta\psi) \sigma_+,$$

$$\theta^{(2)} = \pi : \quad \hat{m}_{+} = \tilde{J} \left( e^{i\tilde{\psi}^{(2)}} \sigma_- - e^{-i\tilde{\psi}^{(1)}} \sigma_+ \right);$$  (14)

where $\hat{m}_{+} = \hat{\Delta}^{(1)} + \hat{\Delta}^{(2)}$. We integrate out the gapped fermions, expand the result perturbatively in $t_\perp / t \ll 1$, and find the expression for the (relative) ground state energy of the weakly coupled KCs:

$$\delta E^{(WT)} \approx -\left( 2\xi_0 \tilde{J}^2 / \pi \tilde{v}_F \right) \log \left( 2t/|\tilde{J}| \right) \left[ 1 + \left( t_\perp / t \sin (\tilde{k}_F \xi_0) \right)^2 \left( 1 + 2 \cos^2 (\delta\psi) \cot^2 \left( \tilde{k}_F \xi_0 \right) \right) \right],$$

with $\tilde{v}_F = v_F (\tilde{k}_F)$ and $\xi_0 t_\perp / \tilde{v}_F \ll 1$, see SM E. The energy gain due to the mode locking, $\delta\psi \approx 0$ or $\pi$, manifests itself in Eq. (15) starting from the term $O(t_\perp / t)^2$ and guarantees that the helical phase provides the minimum of the ground state energy. The low energy theory looks identical to Eq. (11) for $v_F = v_{F'} = \tilde{v}_F$. We conclude that the transport is protected in weakly coupled KCs.

**Vanishing tunneling:** If $t_\perp \ll J \ll 2t$, the perturbative corrections to the ground state energy in Eq. (15) become beyond the accuracy of calculations and must be neglected. The ground state is degenerate and one cannot determine whether the optimal spin configuration is $\theta^{(1)} = \theta^{(2)} = 0$ or $\theta^{(1)} = 0, \theta^{(2)} = \pi$. The latter configuration corresponds to the phase where gapless fermions have opposite helicity in different wires. Clearly, two channels with opposite helicity form a usual (non-helical) spinful conducting channel where transport is not protected.

To ensure a reliable protection, one needs an additional ingredient of the theory which could remove the degeneracy of $\delta E^{(WT)}$. One possibility can be provided by a weak intrinsic Dresselhaus SOI, which typically exists in Gallium quantum wires [57], see SM G.

**Conclusions:** We have shown that strict onedimensionality is not a necessary prerequisite for the formation of a helical phase with protected transport in nanowires functionalized by magnetic adatoms. To demonstrate this statement, we have studied a theoretical model of two magnetically anisotropic 1D Kondo chains coupled by the interchain tunneling of itinerant electrons. The anisotropy simplifies calculations but it is not expected to be crucial for our conclusions $[32, 33]$. The ground state of this model is manifestly helical when the interchain tunneling is larger- or of the order of the exchange coupling between the itinerant electrons and localized spins. The latter, in turn, must be much smaller than the width of the conduction band. These conditions are natural for experimental setups. If tunneling is extremely small, the stability of the helical phase can
be ensured by, e.g., a weak (much smaller than the band width) Dresselhaus SOI. Our results substantially expand predictions made for purely 1D wires [30, 31] and could facilitate experiments devoted to the study of the protected transport in various magnetically doped nanostructures.

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Supplemental Materials for the manuscript

"Protected helical transport in magnetically doped quantum wires: beyond the 1D paradigm"

by F. Stäbler, A.M. Tsvelik, and O.M. Yevtushenko

Suppl.Mat. A: Derivation of the effective Lagrangian in the strong tunneling limit

Let us consider two strongly coupled 1D KCs, $2t \ll t_\perp$. The non-interacting band structure of the system is shown in the right panel of Fig. 2. Our goal is to integrate out the upper band fermions to obtain an effective action for the lower band fermions, where we place the chemical potential. Next, we switch on a finite, but weak exchange interaction of the impurity spins and the conduction electrons $J \ll 2t$. We define the Green’s function of the lower and upper band.

\begin{align}
-\mathbf{G}^\perp_+ &= -i\omega + \epsilon_0(k) - \mu' + J\mathcal{M}_+, \\
-\mathbf{G}^\perp_- &= -i\omega + \epsilon_0(k) - \mu' - 2t_\perp + J\mathcal{M}_+,
\end{align}

where we redefined $\mu = \mu' - t_\perp$ and $\epsilon_0(k) = \epsilon_\pm(k)|_{\mu=0}$ and $\mathcal{M}_\pm = \sum_{b=x,y} \frac{1}{2} \mathbf{S}_b^\pm \sigma_b$ is a matrix, which contains all the backscattering amplitudes generated by the spin $\mathbf{S}_\pm$. We integrate out the upper band $(\perp)$ fermions using the identity

\begin{equation}
\left\langle \exp \left\{ J \int d\zeta \, c^\dagger_+ \mathcal{M}_- c_- + h.c. \right\} \right\rangle = \exp \left\{ J^2 \int d\zeta_1 \, d\zeta_2 \, c^\dagger_+ \mathcal{M}_- \bigg|_{\zeta_1} \mathcal{G}^{-1} \bigg|_{\zeta_1-\zeta_2} \mathcal{M}_- \bigg|_{\zeta_2} c_+ \right\}
\end{equation}

where $\zeta_n = \{\tau_n, x_n\}$. In the next step, we use the smallness of $\frac{J}{t_\perp}$ and expand $\mathbf{G}_-$ perturbatively.

\begin{equation}
\mathbf{G} = \frac{1}{-i\omega + \epsilon_0(k) + 2t_\perp} \left( 1 + \frac{J\mathcal{M}_-}{-i\omega + \epsilon_0(k) + 2t_\perp} \right)^{-1} \approx \frac{1}{2t_\perp},
\end{equation}

Inserting the expansion in Eq.(18) gives for the Lagrangian of the lower band

\begin{equation}
\mathcal{L} = c^\dagger_+ \left[ -\mathbf{G}^{-1} - \frac{J^2}{2t_\perp} \left( \mathbf{S}_- \right)^2 \right] c_+.
\end{equation}

Suppl.Mat. B: Seperating fast and slow variables

In Eq.(6), we obtained fast oscillating backscattering terms. However, we assume, that the spins $\mathbf{S}_{1,2}$ have fast oscillating components, which can compensate the fastness of the backscattering. To see this, we explicitly separate the fast and the slow variables with a suitable parameterization for the spins. We start from Eq.(6)

\begin{align}
\mathcal{L}_{bs}^{\mu\nu} &= \frac{J_a}{2} \mathbf{R}_\nu^a \mathbf{S}_a^\perp \sigma_a \mathbf{L}_\mu e^{2ik^{(s)}_F x}, \\
\mathcal{L}_{bs'}^{\mu\nu} &= \frac{J_a}{2} \mathbf{R}_\nu^a \mathbf{S}_a^\perp \sigma_a \mathbf{L}_\mu e^{i(k^{(s)}_F + k^{(c)}_F) x}.
\end{align}

We explicitly single out a slow and fast component of the spin around $\bar{k}_F \equiv (k_F^{(s)} + k_F^{(c)})/2$. This procedure does not result in over counting angles, since after integrating out the massive variables, the low energy theory only depends on two angles per spin, thus justifying this approach.

\begin{equation}
\mathbf{S}^{(n)}_s = \left( \mathbf{m}_n + \left[ \mathbf{e}_1 \cos \left( 2\bar{k}_F x \right) + \mathbf{e}_2 \sin \left( 2\bar{k}_F x \right) \right] \sqrt{1 - \mathbf{m}_n^2} \right)^{(n)}_n, \quad \mathbf{m}_n = \sin \left( \alpha^{(n)} \right) \mathbf{e}_3^{(n)}
\end{equation}

Now we parameterize the orthonormal triad $\{\mathbf{e}_1^{(n)}, \mathbf{e}_2^{(n)}, \mathbf{e}_3^{(n)}\}$ by spherical coordinates.
FIG. 4. Decomposition of the individual impurity spins in a fast and slow component.

\[
e_1^{(n)} = \left(-\cos(\theta^{(n)}), -\cos(\theta^{(n)}) \sin(\psi^{(n)}) + \sin(\theta^{(n)}), \sin(\theta^{(n)})\right)^T
\]

\[
e_2^{(n)} = \left(\sin(\psi^{(n)}), -\cos(\psi^{(n)}), 0\right)^T
\]

\[
e_3^{(n)} = \left(\sin(\theta^{(n)}), \sin(\psi^{(n)}) \sin(\theta^{(n)}), \cos(\theta^{(n)})\right)^T
\]

After inserting the new spin parameterization, the back scattering terms take the following form

\[
\mathcal{L}^{\nu\nu}_{bs} = R^{\dagger}_{\nu,\sigma} \left(\Delta^{(1)} + \Delta^{(2)}\right) L_{\nu',\sigma'} e^{i\delta k_F x},
\]

\[
\mathcal{L}^{\nu\nu'}_{bs} = R^{\dagger}_{\nu,\sigma} \left(\Delta^{(1)} - \Delta^{(2)}\right) L_{\nu',\sigma'}.
\]

The scattering amplitudes are given by

\[
\Delta^{(n)} = \tilde{J} \left[e^{i\psi^{(n)}} \sin^2 \left(\frac{\theta^{(n)}}{2}\right) \sigma_- - e^{-i\psi^{(n)}} \cos^2 \left(\frac{\theta^{(n)}}{2}\right) \sigma_+\right],
\]

with \(\tilde{J} = s \rho_s \frac{J}{2} \cos(\alpha^{(n)})\). The amplitudes contain the phase factors \(\psi^{(1)}\) and \(\psi^{(2)}\), which can be partially gauged away. The rest, especially \(\alpha^{(1)}, \alpha^{(2)}, \theta^{(1)}, \theta^{(2)}\) enters the ground state energy equation. The classical values of the latter four angles are thus determined by the configuration which has the minimal ground state energy.

**Suppl. Mat. C: Groundstate energy of gapped 1D Dirac fermions**

We want to calculate the gain in ground state energy of gapped 1D Dirac fermions with respect to the ungapped fermions. Let us consider a gapped fermionic Green’s function of the form

\[
-G^{-1} = \begin{pmatrix}
-i\omega + v_{F_1} k & m \\
- i\omega - v_{F_2} k & -i\omega - v_{F_2} k
\end{pmatrix},
\]
and define

\[-G_0^{-1} = \begin{pmatrix} -i\omega + v_{F_1} k & 0 \\ 0 & -i\omega - v_{F_2} k \end{pmatrix}, \quad \Delta = \begin{pmatrix} 0 & m \\ m^* & 0 \end{pmatrix}.\]  

(31)

The partition function corresponding to Eq. (30) is given by

\[Z = \det (-G^{-1}) = \det (-G_0^{-1} + \Delta) = Z_0 \exp \text{Tr} \log (1 - G_0 \Delta),\]  

(32)

where we used the identity \(\det (A) = \exp \text{Tr} \log (A)\) in the last step. We can compute the free energy \(F = -T \log (Z)\), and expand the free energy in leading order of \(\Delta\). We find

\[F = F_0 - T \text{Tr} \log (1 - G_0 \Delta) \approx F_0 + \frac{T}{2} \text{Tr} G_0 \Delta G_0 \Delta.\]  

(33)

Note that the linear term in the expansion is absent, because of the off diagonal structure of \(\Delta\) and on the other hand reflects the fact, that we expand the ground state energy around its minimum. In the limit \(T \to 0\) we can convert the summation over the Matsubara frequency to an integral and find

\[\delta E = \frac{T}{2} \text{Tr} G_0 \Delta G_0 \Delta \to -\xi_0 \int \frac{d\{\omega, k\}}{(2\pi)^2} \frac{|m|^2}{(-i\omega + v_{F_1} k)(-i\omega - v_{F_2} k)},\]  

(34)

which has poles at \(\omega_1 = -iv_{F_1} k\) and \(\omega_2 = iv_{F_2} k\). We find

\[\delta E = \frac{-\xi_0}{2\pi} \int_{\lambda|m|}^{2\pi} dk \frac{|m|^2}{k (v_{F_1} + v_{F_2})} = -\frac{\xi_0}{2\pi (v_{F_1} + v_{F_2})} |m|^2 \log \left( \frac{2t}{|m|} \right),\]  

(35)

where we used the band width as a high energy cut-off.

**Suppl. Mat. D: Derivation of the ground state energy equation in the intermediate tunneling regime**

Let us consider the gap structure \(\hat{m}_-\) of Eqs. (8) and (9). Our goal is to calculate and compare the ground state energies for both spin configurations. The gap structure \(\hat{m}_-\) plays the role of a mass term which mixes fermions of the \(\nu = \pm\) bands \(\mathcal{L}_{\nu}^{\nu'} \simeq R^\nu L_{\nu'} + \text{h.c.}, \nu \neq \nu'.\)

\[
\begin{align*}
\theta^{(1)} = \theta^{(2)} = 0 & : \hat{m}_- = 2i\tilde{J} e^{-i\tilde{\psi}} \sin (\delta \psi) \sigma_+, \\
\theta^{(1)} = 0, \theta^{(2)} = \pi & : \hat{m}_- = -\tilde{J} \left( e^{i\tilde{\psi}} \sigma_- + e^{-i\tilde{\psi}} \sigma_+ \right).
\end{align*}
\]

(36)

(37)

The common phase factors \(\tilde{\psi}\) and \(\psi^{(1)}, \psi^{(2)}\) in (36) and (37) can be removed by a gauge transformation or by bosonizing the theory and shifting the phases. The phases enter the low energy Lagrangian as a chiral anomaly in the form of a Luttinger liquid Lagrangian \(\mathcal{L}_{\text{LL}}[\Phi, v_{\Phi}] = [\partial_{\phi} \Phi]^2 + (v_{\Phi} \partial_{\phi} \Phi)^2]/2\pi K_{\Phi}, \Phi = \psi, \psi^{(1)}, \psi^{(2)},\) which we will discuss later. We set the common phase factors to zero in the following calculations.

**Spin configuration I: \(\theta^{(1)} = \theta^{(2)} = 0\)**

The gap structure gaps only fermions of helicity \(h = +1\) and is given by

\[
\begin{align*}
\hat{m}_- = m_- \sigma_+, \quad m_- = 2i\tilde{J} \sin (\delta \psi),
\end{align*}
\]

which leads to the following inverse Green’s function.

\[-G^{-1} = \begin{pmatrix}
\partial_{R_+} & 0 & 0 & m_- & 0 & 0 & 0 & 0 & 0 \\
0 & \partial_{L_+} & m_- & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & m_- & \partial_{R_-} & 0 & 0 & 0 & 0 & 0 & 0 \\
m_- & 0 & 0 & \partial_{L_-} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \partial_{R_+} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \partial_{L_+} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \partial_{R_-} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \partial_{L_-} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \partial_{R_+} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \partial_{L_+} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \partial_{R_-} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \partial_{L_-} & 0 \\
\end{pmatrix}.
\]

(39)
where $\partial_{n/L} = \partial_x \mp iv_F \partial_x$ is the chiral derivative for the respective bands and the ordering of the states is indicated to the right of the Green’s function. We focus on the gapped block

$$ -G_m^{-1} = \begin{pmatrix} \partial_{R_+} & 0 & 0 & m_- \ \\
0 & \partial_{L_+} & m_- & 0 \\
0 & m_- & \partial_{R_-} & 0 \\
m_- & 0 & 0 & \partial_{L_-} \end{pmatrix} R_{+\uparrow} \cup \begin{pmatrix} \partial_{R_+} & m_- & 0 & 0 \ \\
m_- & \partial_{L_-} & 0 & 0 \\
0 & 0 & \partial_{R_-} & m_- \\
0 & 0 & m_- & \partial_{L_-} \end{pmatrix} L_{-\uparrow} $$  \hspace{1cm} (40)

Changing the ordering of the states gives us the above matrix in block diagonal form. This allows us to use Eq. (35), see Suppl. Mat. C. We now integrate out all gapped fermions. The ground state energy is the sum of the ground state energy of the two individual blocks in (40), which gives a factor of two compared to (35).

$$ \delta E^{(IT,1)} = -\frac{4\xi_0 J^2 \sin^2 (\delta \psi)}{\pi (v_{F+} + v_{F-})} \log \left( \frac{2t|\tilde{J}|}{|J|} \right); \hspace{1cm} (41) $$

The ground state energy is minimal if there is a mode-locking of the in-plane polar angles $\delta \psi = \pm \frac{\pi}{2}$.

**Spin configuration II:** $\theta^{(1)} = 0, \theta^{(2)} = \pi$

The gap structure now contains gaps for fermions of all helicites in both bands, but the effective size of the gap is reduced by a factor of two.

$$ \hat{m}_- = m_- (\sigma_+ + \sigma_-), \hspace{0.5cm} m_- = -\tilde{J}. $$  \hspace{1cm} (42)

This leads to the following inverse Green’s function.

$$ -G^{-1} = \begin{pmatrix} \partial_{R_+} & 0 & 0 & m_- \ \\
0 & \partial_{L_+} & m_- & 0 \\
0 & m_- & \partial_{R_-} & 0 \\
m_- & 0 & 0 & \partial_{L_-} \end{pmatrix} R_{+\uparrow} \cup \begin{pmatrix} \partial_{R_+} & m_- & 0 & 0 \ \\
m_- & \partial_{L_-} & 0 & 0 \\
0 & 0 & \partial_{R_-} & m_- \\
0 & 0 & m_- & \partial_{L_-} \end{pmatrix} L_{-\uparrow} $$  \hspace{1cm} (43)

It can be block diagonalized in the following form

$$ -G^{-1} = \begin{pmatrix} \partial_{R_+} & m_- & 0 & 0 \ \\
m_- & \partial_{L_-} & 0 & 0 \\
0 & \partial_{R_-} & m_- & 0 \\
0 & 0 & \partial_{L_-} & m_- \end{pmatrix} R_{\uparrow} \cup \begin{pmatrix} \partial_{R_+} & m_- & 0 & 0 \ \\
m_- & \partial_{L_-} & 0 & 0 \\
0 & 0 & \partial_{R_-} & m_- \\
0 & 0 & m_- & \partial_{L_-} \end{pmatrix} L_{\uparrow} $$  \hspace{1cm} (44)

We again integrate out all gapped fermions with the help of Eq. (35) and obtain

$$ \delta E^{(IT,II)} = -\frac{2\xi_0 J^2}{\pi (v_{F+} + v_{F-})} \log \left( \frac{2t}{|J|} \right); \hspace{1cm} (45) $$

Comparing (41) and (45), we find, that a helical phase, where fermions with helicity $h = -1$ remain gapless, is energetically favored $\delta E^{(IT,1)} < \delta E^{(IT,II)}$. The case where the gapless modes have helicity $h = +1$ can be found analogously with $\theta^{(1)} = \theta^{(2)} = \pi$. 

Let us now consider the weak tunneling regime. In contrast to the intermediate tunneling case we find the "off diagonal" gap structure $L_{\nu \nu}^{bs} \simeq R_{\nu}^\dagger \hat{m}_- L_{\nu'} + h.c.$, $\nu \neq \nu'$ of Eqs. (8) and (9) and the "diagonal" gaps $L_{\nu \nu}^{bs} \simeq R_{\nu}^\dagger \hat{m}_+ L_{\nu} + h.c.$ of Eqs. (13) and (14). This makes the weak tunneling case distinct from the intermediate tunneling case. Similar to Suppl.Mat. D, we analyze two different spin configuration and calculate and compare the ground state energies for both spin configurations.

$$\theta^{(1)} = \theta^{(2)} = 0 : \hat{m}_- = 2i \tilde{J} e^{-i\tilde{\psi}} \sin(\delta \psi) \sigma_+,$$  \hspace{1cm} (46)

$$\theta^{(1)} = \theta^{(2)} = 0 : \hat{m}_+ = -2\tilde{J} e^{-i\tilde{\psi}} \cos(\delta \psi) \sigma_+;$$  \hspace{1cm} (47)

$$\theta^{(1)} = 0, \theta^{(2)} = \pi : \hat{m}_- = -\tilde{J} \left( e^{i\psi^{(2)}} \sigma_- + e^{-i\psi^{(1)}} \sigma_+ \right).$$  \hspace{1cm} (48)

$$\theta^{(1)} = 0, \theta^{(2)} = \pi : \hat{m}_+ = \tilde{J} \left( e^{i\psi^{(2)}} \sigma_- - e^{-i\psi^{(1)}} \sigma_+ \right);$$  \hspace{1cm} (49)

The common phase factors $\tilde{\psi}$ and $\psi^{(1)}, \psi^{(2)}$ in Eqs.(46) - (49) can be removed by a gauge transformation or by bosonizing the theory and shifting the phases. The phases enter the low energy Lagrangian in the form of a Luttinger liquid Lagrangian $L_{LL}[\Phi, \nu] = (\partial_\tau \Phi)^2 + (\nu \Phi \partial_x \Phi)^2 / 2\pi K \Phi$, $\Phi = \tilde{\psi}, \psi^{(1)}, \psi^{(2)}$, which we will discuss later. We set the common phase factors to zero in the following calculations.

**Spin configuration I: $\theta^{(1)} = \theta^{(2)} = 0$**

The gap structure gaps only fermions of helicity $h = +1$ and is given by

$$\hat{m}_- = m_+ \sigma_+, \quad m_- = 2i \tilde{J} \sin(\delta \psi),$$  \hspace{1cm} (50)

$$\hat{m}_+ = m_- \sigma_+, \quad m_+ = -2\tilde{J} \cos(\delta \psi),$$  \hspace{1cm} (51)

which leads to the following inverse Green’s function.

$$-G^{-1} = \begin{pmatrix}
\partial_{R_+} & m_+ & 0 & m_- & 0 & 0 & 0 & 0 \\
m_+ & \partial_{L_+} & m_- & 0 & 0 & 0 & 0 & 0 \\
0 & m_- & \partial_{R_-} & m_+ & 0 & 0 & 0 & 0 \\
m_- & 0 & m_+ & \partial_{L_-} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \partial_{R_+} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \partial_{L_+} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \partial_{R_-} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \partial_{L_-}
\end{pmatrix},$$  \hspace{1cm} (52)

where $\partial_{R/L} = \partial_\tau \mp iv_F \partial_x$ is the chiral derivative for the respective bands and the ordering of the states is indicated to the right of the Green’s function. We focus on the gapped block

$$-G_m^{-1} = \begin{pmatrix}
\partial_{R_+} & m_+ & 0 & m_- & 0 & 0 & 0 & 0 \\
m_+ & \partial_{L_+} & m_- & 0 & 0 & 0 & 0 & 0 \\
0 & m_- & \partial_{R_-} & m_+ & 0 & 0 & 0 & 0 \\
m_- & 0 & m_+ & \partial_{L_-} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \partial_{R_+} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \partial_{L_+} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \partial_{R_-} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \partial_{L_-}
\end{pmatrix},$$  \hspace{1cm} (53)

In contrast to the intermediate tunneling case, we cannot block diagonalize this matrix and make use of the Eq. (35) in Suppl.Mat. C. However, we can directly integrate out the gapped fermions analogously to Eq.(33).
Spin configuration II: \( \theta^{(1)} = 0, \theta^{(2)} = \pi \)

The gap structure now contains gaps for fermions of all helicites in both bands, but the effective size of the gap is reduced by a factor of two.

\[
\hat{m}_- = m_- (\sigma_+ + \sigma_-), \quad m_- = -\tilde{J},
\]
\[
\hat{m}_+ = m_+ (\sigma_- - \sigma_+), \quad m_+ = \tilde{J},
\]

which leads to the following inverse Green’s function.

\[
-\mathbf{G}^{-1} = \begin{pmatrix}
\partial_{R_+} & -m_+ & 0 & m_- & 0 & 0 & 0 \\
-m_+ & \partial_{L_+} & m_- & 0 & 0 & 0 & 0 \\
0 & m_- & \partial_{R_-} & -m_+ & 0 & 0 & 0 \\
m_- & 0 & -m_+ & \partial_{L_-} & 0 & 0 & 0 \\
0 & 0 & 0 & \partial_{R_+} & m_+ & 0 & m_- \\
0 & 0 & 0 & m_+ & \partial_{L_+} & m_- & 0 \\
0 & 0 & 0 & 0 & m_- & \partial_{R_-} & m_+ \\
0 & 0 & 0 & 0 & m_+ & \partial_{L_-} & 0
\end{pmatrix}
\]
\[ \delta E^{(WT,II)} = -\frac{\xi_0 J^2}{4\pi} \left[ \frac{2}{v_{F+}} + \frac{2}{v_{F-}} + \frac{8}{\text{Tr} \, G_0 \Delta_+ G_0 \Delta_+} \right] \log \left( \frac{2t}{|J|} \right) = -\frac{\xi_0 J^2}{2\pi} \left[ \frac{(v_{F+} + v_{F-})^2 + 4v_{F+} v_{F-}}{v_{F+} v_{F-} (v_{F+} + v_{F-})} \right] \log \left( \frac{2t}{|J|} \right). \] (61)

We again expand perturbatively and obtain

\[ \delta E^{(WT,II)} = -\frac{2\xi_0 J^2}{\pi v_F} \left[ 1 + 4t^2 \left( \frac{\xi_0^2}{v_F^2} + \frac{\xi_0^4 v_F^2}{v_F^2} \right) \right] \log \left( \frac{2t}{|J|} \right). \] (62)

Similar to the intermediate tunneling case, the mode locking of the in-plane polar angle \( \delta \psi = 0 \) provides the minimum of the ground state energy in the weak tunneling case \( t_\perp \leq J \ll 2t \) and makes spin configuration I more favorable. In the limit of vanishing tunneling \( t_\perp \leq J \ll 2t \), we neglect higher order terms and Spin configuration I and II cannot be distinguished. In Suppl.Mat. G we introduce an additional Dresselhaus SOI and show that it removes the ground state degeneracy in favor of the helical phase. The wires behave essentially uncoupled. Spin configuration II cannot be distinguished. In Suppl.Mat. F we introduce an additional Dresselhaus SOI and show that it removes the ground state degeneracy in favor of the helical phase. The wires behave essentially uncoupled. Spin configuration I resembles a helical wire where all channels support fermions of the same helicity, either \( h = +1 \) or \( h = -1 \). In Spin configuration II, fermions can travel in any direction. Some channels have helicity \( h = +1 \) and others \( h = -1 \). This configuration resembles a non-helical spinful wire.

**Suppl.Mat. F: Spinless disorder**

Let us consider the influence of disorder, modeled by a weak random scalar potential in each wire. On the Hamiltonian level we get the interaction term

\[ \hat{H}_{\text{dis}} = \int dx \, V(x) \left( c^\dagger(x) c(x) \right)^{(n)} = \int dk \int dq \, V(q) \left( c^\dagger(k + q) c(k) \right)^{(n)}, \] (63)

where we defined \( V_q = \int dx \, e^{i q x} V(x) \). We switch from the wire index to the band index using the transformation, which diagonalized the non-interacting Hamiltonian \( c_{1/2} = \frac{1}{\sqrt{2}} (c_+ \pm c_-) \) and obtain

\[ \hat{H}_{\text{dis}} = \int dk \int dq \, V(q) \left( c^\dagger(k + q)_+ c(k)_+ + c^\dagger(k + q)_- c(k)_- \right), \] (64)

Since we are interested in the low energy behavior of the system we focus on the momenta around the Fermi points of the system \( q = \pm 2k_{F\pm} \). This gives

\[ \hat{H}_{\text{dis}} = \int dk \left( \sum_{q = \pm 2k_{F\pm}} V(q) \, c^\dagger(k + q)_+ c(k)_+ + \sum_{q = \pm 2k_{F\pm}} V(q) \, c^\dagger(k + q)_- c(k)_- \right). \] (65)

We now switch to the Lagrangian formulation and introduce the smooth chiral fields \( R_{\pm}(k) = \psi_{k \pm k_{F\pm}} \) and \( L_{\pm}(k) = \psi_{k - k_{F\pm}} \), which are the shifted fermionic fields \( \psi_{\pm} \) of the disorder Lagrangian. We obtain

\[ \mathcal{L}_{\text{dis}} = g_{2k_{F\pm}} R^\dagger_{\pm} L_{\pm} + g_{2k_{F\pm}} R^\dagger_{\pm} L_{\pm} + \text{h.c.}, \] (66)

where \( g_{2k_{F\pm}} = V(\pm 2k_{F\pm}) \). We analyze the case where we find four Fermi points on the level of the chemical potential. It is convenient to analyze the intermediate and weak tunneling regimes individually.

**Weak tunneling regime**
For simplicity, we neglect the difference between the Fermi momenta $2k_{F_\pm} \approx 2k_{F_-}$, since the Fermi points almost coincide in pairs. Furthermore, we assume that the system adapts the spin configuration I: $\theta^{(1)} = \theta^{(2)} = 0$. The Green’s function of the system is given by

$$-G^{-1} = 
\begin{pmatrix}
\partial_{R_+} & m_+ & 0 & m_- & 0 & g_{2k_{F_+}} & 0 & 0 \\
m_+ & \partial_{L_+} & m_+ & 0 & g_{2k_{F_+}} & 0 & 0 & 0 \\
0 & m_- & \partial_{R_-} & m_+ & 0 & 0 & 0 & g_{2k_{F_-}} \\
m_- & 0 & m_+ & \partial_{L_-} & 0 & 0 & 0 & 0 \\
g_{2k_{F_+}} & 0 & 0 & 0 & \partial_{R_+} & 0 & 0 & 0 \\
g_{2k_{F_-}} & 0 & 0 & 0 & 0 & \partial_{L_+} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \partial_{R_-} & 0 \\
0 & 0 & 0 & g_{2k_{F_-}} & 0 & 0 & 0 & \partial_{L_-} \\
\end{pmatrix}
$$

(67)

with $m_+ = -2\tilde{J}\cos(\delta\psi)$, $m_- = 2i\tilde{J}\sin(\delta\psi)$. Our goal is to integrate over the gapped fermions and derive an effective action for the ungapped fermions. We define the gapped spinor $\Psi_g = (R_{+\dagger}, L_{+\dagger}, R_{-\dagger}, L_{-\dagger})^T$ and the ungapped spinors $\Psi_u = (R_{+\dagger}, L_{+\dagger}, R_{-\dagger}, L_{-\dagger})^T$. Straightforward calculation yields

$$\exp \left\{ -\int d(\tau, x) \Psi^\dagger_g \hat{g} \Psi^\dagger_u \right\} = \exp \left\{ \frac{1}{2} \int d(\tau_1, x_1, \tau_2, x_2) \left\{ \hat{g} \Psi^\dagger_u (x_1, \tau_1), \langle \Psi^\dagger_g \Psi_g \rangle_g (x_1, \tau_1) \right\} \right\}.$$  

(68)

$\langle \ldots \rangle_g$ denotes the integration over the gapped fermions and we defined $\hat{g} = (g_{2k_{F_+}} \sigma_+ \sigma_- + g_{2k_{F_-}} \sigma_- \sigma_+) \otimes \tau_5$, where $\sigma_\pm = \sigma_x \pm i\sigma_y$ with $\sigma_x, \sigma_y$ being the first and second pauli matrix in spin and wire band space. The gapped Green’s function can be computed directly by inversion if we insert the results we got earlier, namely, the in-plane modelocking of the form $\delta\psi = 0$ and $v_{F_+} = v_{F_-} = v_F$. Note, that after the modelocking we only find intraband scattering $m_+ = 2\tilde{J}$ and $m_- = 0$. This gives

$$\langle \Psi^\dagger_g \Psi_g \rangle_g = G_m = \frac{1}{4\tilde{J}^2 + v_F^2 k^2 + \omega^2} \begin{pmatrix}
\omega + v_F k & 2\tilde{J} & 0 & 0 \\
2\tilde{J} & \omega - v_F k & 0 & 0 \\
0 & 0 & \omega + v_F k & 2\tilde{J} \\
0 & 0 & 2\tilde{J} & \omega - v_F k \\
\end{pmatrix} \approx \frac{1}{2\tilde{J}} \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1/2J \\
\end{pmatrix},$$

(69)

which we expanded using the smallness of $\frac{\omega}{\tilde{J}}$. Using Eq.(68) gives for the effective action of the ungapped fermions

$$\mathcal{L}_{\text{dis}} \simeq \frac{g^2_{2k_{F_+}}(x)}{4\tilde{J}} R_{+\dagger}^+ L_{+\dagger} \left\{ \frac{g^2_{2k_{F_-}}(x)}{4\tilde{J}} R_{-\dagger}^+ L_{-\dagger} \right\} + \text{h.c.}$$

(70)

If $g_{2k_{F_\pm}}/\tilde{J} \ll 1$, effective backscattering, which is governed by multiparticle scattering processes and localization is suppressed. The transport properties of the gapless modes become protected up to parametrically large sample sizes. In spin configuration II $\theta^{(1)} = 0$, $\theta^{(2)} = \pi$ all fermionic modes are gapped. In this case, there is no ballistic transport.

**Intermediate tunneling regime**

In the intermediate tunneling regime, there are four Fermi points, which are well separated, i.e. the difference between the Fermi momenta is large $\delta k_F \simeq \tilde{k}_F$. Let us start from Eq.(66)

$$\mathcal{L}_{\text{dis}} = g_{2k_{F_+}} R_{+\dagger}^+ L_{-\dagger} + g_{2k_{F_-}} R_{-\dagger}^+ L_{+\dagger} + \text{h.c.}$$

(71)

Furthermore, we assume, that the system is in the energetically more favorable spin configuration I $\theta^{(1)} = \theta^{(2)} = 0$,
with a mode locking of the form \( \delta \psi = \pm \frac{\pi}{2} \). The Green’s function is then given by

\[
-\mathbf{G}^{-1} = \begin{pmatrix}
\partial_{R_+} & 0 & 0 & m_- & 0 & g_{2k_F+} & 0 & 0 \\
0 & \partial_{L_+} & m_-^* & 0 & g_{2k_F+} & 0 & 0 & 0 \\
0 & m_- & \partial_{R_-} & 0 & 0 & 0 & 0 & g_{2k_F-} \\
m_-^* & 0 & 0 & \partial_{L_-} & 0 & 0 & 0 & 0 \\
0 & g_{2k_F+} & 0 & 0 & \partial_{R_+} & 0 & 0 & 0 \\
g_{2k_F+} & 0 & 0 & 0 & \partial_{R_+} & 0 & 0 & 0 \\
0 & 0 & g_{2k_F-} & 0 & 0 & \partial_{R_-} & 0 & 0 \\
0 & 0 & g_{2k_F-} & 0 & 0 & \partial_{R_-} & 0 & 0 \\
0 & 0 & g_{2k_F-} & 0 & 0 & \partial_{L_-} & 0 & 0 \\
\end{pmatrix} \begin{pmatrix}
R_{+\uparrow} \\
L_{+\downarrow} \\
R_{-\uparrow} \\
L_{-\downarrow} \\
R_{+\downarrow} \\
L_{+\uparrow} \\
R_{-\downarrow} \\
L_{-\uparrow} \\
\end{pmatrix}, \tag{72}
\]

We proceed by integrating out the gapped fermions similar to the intermediate tunneling case using Eq.(68). The gapped Green’s function is then given by

\[
\langle \psi_g^\dagger \psi_g \rangle_g = \begin{pmatrix}
i\omega + v_F k \\
4J^2 + \omega^2 + i \omega k (v_F^+ - v_F^-) \\
0 \\
4J^2 + \omega^2 - i \omega k (v_F^+ - v_F^-) \\
0 \\
4J^2 + \omega^2 + i \omega k (v_F^+ - v_F^-) \\
0 \\
-4J^2 + \omega^2 - i \omega k (v_F^+ - v_F^-) \\
\end{pmatrix} \begin{pmatrix}
0 & 0 & 0 & \frac{i}{2J} \\
0 & 0 & -\frac{i}{2J} & 0 \\
0 & \frac{i}{2J} & 0 & 0 \\
-\frac{i}{2J} & 0 & 0 & 0 \\
\end{pmatrix} \begin{pmatrix}
i\omega - v_F k \\
4J^2 + \omega^2 - i \omega k (v_F^+ - v_F^-) \\
0 \\
4J^2 + \omega^2 + i \omega k (v_F^+ - v_F^-) \\
0 \\
4J^2 + \omega^2 - i \omega k (v_F^+ - v_F^-) \\
0 \\
4J^2 + \omega^2 + i \omega k (v_F^+ - v_F^-) \\
\end{pmatrix}, \tag{73}
\]

where we again expanded the Green’s function using the smallness of \( \frac{J}{d} \). Using Eq.(68) gives for the effective action of the ungapped fermions

\[
\mathcal{L}_{\text{dis}} \simeq \frac{i g_{2k_F+}(x) g_{2k_F-}(x)}{4J} \left[ R_{+\downarrow}^\dagger L_{-\uparrow} + R_{-\downarrow}^\dagger L_{+\uparrow} \right] + h.c. \tag{75}
\]

Note, that the effective disorder mixes the band indices of the ungapped fermions. However, similar to the weak tunneling case, this effect is suppressed \( g_{2k_F} / J \ll 1 \) and we expect ballistic transport of the gapless fermions up to parametrically large scales.

**Suppl.Mat. G: Dresselhaus Spin-Orbit interaction**

In the limit of vanishing tunneling, the KCs have almost equal ground state energies, in the sense that perturbative corrections to the uncoupled chains are small and we neglect them. Thus, there is a degenerate ground state. To find the spin configuration which wins in real materials, we introduce an additional Dresselhaus spin-orbit interaction, which is present for example in GaAs quantum wires.

\[
\hat{H}_{\text{SOI}} = d \left( c_i^\dagger \hat{c}_j \sigma_z \right)^{(n)}, \quad n = 1, 2, \tag{76}
\]

where \( \hat{c}_j \) is the momentum operator in the direction of the KCs. We assume, that the spin-orbit interaction is weak \( t_\perp \ll d \ll J \ll 2t \). We do not take into account the tunneling effect, since it is subleading. For the following calculations we set \( t_\perp = 0 \). The band structure of the non-interacting system in the presence of spin orbit interaction is given by

\[
\varepsilon_{1/2}(k) = -2t \cos(k \xi_0) \pm dk, \tag{77}
\]
FIG. 5. Bandstructure in the presence of Dresselhaus SOI $\varepsilon_{x,\pm} = -2t \cos(k\xi_0) \pm dk$. The splitting due to the tunneling is subleading and not shown here. The slope of the linearized dispersion (orange and green), and thus the respective Fermi velocities, is different for $\{R\uparrow, L\downarrow\}$ and $\{R\downarrow, L\uparrow\}$. This allows us to distinguish the degenerate ground states in the weak tunneling limit.

Note, that the spin-orbit interaction now effects different spins in different ways. This is the main difference to the splitting caused by the tunneling. We proceed similar to the case without SOI and place our chemical potential such that we find four Fermi points and single out smooth chiral modes in the following way

\[
c_{\uparrow} = e^{-ik_{F,\uparrow} x} R_\uparrow + e^{ik_{F,\downarrow} x} L_\downarrow, \tag{78}
\]

\[
c_{\downarrow} = e^{-ik_{F,\downarrow} x} R_\downarrow + e^{ik_{F,\uparrow} x} L_\uparrow. \tag{79}
\]

Note, that the Fermi velocities will be different for different helical sectors. $\{R\uparrow, L\downarrow\}$ depend on $v_{F,\uparrow} = 2t\xi_{0} \sin(k_{F,\xi_{0}}) + d$ and $\{R\downarrow, L\uparrow\}$ depend on $v_{F,\downarrow} = 2t\xi_{0} \sin(k_{F,\xi_{0}}) - d$, respectively. The spin-orbit interaction is already diagonal in the wire space. We separe the slow and fast spin degrees of freedom, following the steps presented in Suppl.Mat. B. We explicitly single out a fast $k_{F,12} = \frac{1}{2}(k_{F,\uparrow} + k_{F,\downarrow})$ component of the spins and assume, that the splitting $\delta k_{F,12} = k_{F,\uparrow} - k_{F,\downarrow} = \frac{2}{\xi_{0}} \arcsin\left(\frac{dk_{F,\xi_{0}}}{2v_{F,12}}\right)$ is small. The Kondo interaction is then given by

\[
\mathcal{L}_{\text{bs}}^{h=+1} = -\tilde{J} \left(R^\dagger e^{-i\psi \cos^2\left(\frac{\theta}{2}\right) \sigma_{\uparrow} L}\right)^{(n)} e^{i\delta k_{F,12} x} + \text{h.c.} \tag{80}
\]

\[
\mathcal{L}_{\text{bs}}^{h=-1} = \tilde{J} \left(R^\dagger e^{i\psi \sin^2\left(\frac{\theta}{2}\right) \sigma_{\downarrow} L}\right)^{(n)} e^{-i\delta k_{F,12} x} + \text{h.c.} \tag{81}
\]

which are two copies of the gap structure derived in [30, 31]. The oscillations in Eqs. (80) and (81) are slow and can be gauged away. The gauge transformation leads to a small unimportant shift of the chemical, which we will not discuss here. In the following, we will ignore the oscillations. We can now compare the ground state energies of the two different spin configurations.

**Spin configuration I:** $\theta^{(1)} = \theta^{(2)} = 0$

The Green’s function of the system is given by

\[
-G^{-1} = \begin{pmatrix}
\partial_{R_1} & m & 0 & 0 & 0 & 0 & 0 & 0 \\
m & \partial_{L_1} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \partial_{R_1} & m & 0 & 0 & 0 & 0 \\
0 & 0 & m & \partial_{L_1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \partial_{R_2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \partial_{L_2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \partial_{R_2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \partial_{L_2} & 0 \\
\end{pmatrix}
\]

\[
R_1 \uparrow, L_1 \downarrow, \quad R_2 \uparrow, L_2 \downarrow, \quad R_1 \downarrow, L_1 \uparrow, \quad R_2 \downarrow, L_2 \uparrow,
\]

\[
(82)
\]

where $m = 2\tilde{J}$ and $\partial_{n,l} = \partial_{\tau} \mp iv_{F,\tau} \partial_{x}$, $l = 1, 2$. Note that the gap in the 1D case is twice as large as in the diagonalized basis. This is just a matter of the definition of $\tilde{J}$. We use Eq.(35) from Supp. Mat. C and compute the
ground state energy by integrating out the gapped fermions. We find
\[ \delta E^{(\text{WT, LSOI})} = -\frac{2|\xi_0|J^2}{\pi v_{F_1}} \log \left( \frac{2t}{|J|} \right), \]  \hspace{1cm} (83) \]
Since the spin-orbit interaction is weak, we can expand the Fermi velocity using the smallness of $\frac{d}{t} \ll 1$. We find
\[ \delta E^{(\text{WT, LSOI})} \approx -\frac{2|\xi_0|J^2}{\pi (\tilde{v}_{F_{12}} + d)} \log \left( \frac{2t}{|J|} \right). \]  \hspace{1cm} (84) \]
Note, that if we choose $\theta^{(1)} = \theta^{(2)} = \pi$, Eq.(83) would depend on $\tilde{v}_{F_{12}} - d$ in the denominator. Let us assume, that $d < 0$. This means, that gapping all fermions with helicity $h = -1$ is more favorable than gapping all fermions with helicity $h = -1$. The resulting helical phase will thus always have gapless modes with helicity $h = -1$.

**Spin configuration II**: $\theta^{(1)} = 0$, $\theta^{(2)} = \pi$

The Green’s function of the system is given by
\[ -G^{-1} = \begin{pmatrix} \partial_{R_1}, m & 0 & 0 & 0 & 0 & 0 & 0 \\ m & \partial_{L_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \partial_{R_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \partial_{L_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \partial_{R_2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \partial_{L_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \partial_{L_2} \end{pmatrix}, \]  \hspace{1cm} (85) \]
with $m = 2\tilde{J}$. We follow the same steps as before and find
\[ \delta E^{(\text{WT, LSOI})} \approx -\frac{\xi_0 J^2}{\pi \tilde{v}_{F_{12}}} \left[ \frac{1}{\tilde{v}_{F_{12}} + d} + \frac{1}{\tilde{v}_{F_{12}} - d} \right] \log \left( \frac{2t}{|J|} \right). \]  \hspace{1cm} (86) \]
If we now compare Eqs.(84) and (86) we can see, that for any choice of $d \neq 0$ the fully helical phase will always be energetically more favorable. For $d < 0$ the gapless modes have helicity $h = -1$ and for $d > 0$ the gapless modes have helicity $h = +1$.

**Suppl.Mat. H**: Renormalization of the Luttinger parameter $K_\psi$

**Intermediate tunneling**

After integrating out the gapped fermions we obtain the following low energy Lagrangian
\[ \mathcal{L} = \frac{\delta E}{\xi_0} + \frac{1}{2\pi} \left[ (\partial_\tau \tilde{\psi})^2 + (v_{F_+} \partial_\sigma \tilde{\psi})^2 \right] + \frac{1}{2\pi} \left[ (\partial_\tau \tilde{\psi})^2 + (v_{F_-} \partial_\sigma \tilde{\psi})^2 \right] + \mathcal{L}_{WZ} + \mathcal{L}_0[R_{\pm}, L_{\pm}], \]  \hspace{1cm} (87) \]
where $\mathcal{L}_0$ is the chiral Lagrangian for the gapless fermions and $\mathcal{L}_{WZ}$ is the slow Wess Zumino term derived in [30, 31], which is given by
\[ \mathcal{L}_{WZ} = \frac{i s_{p_+}}{\xi_0} \sin \left( \alpha^{(n)} \right) \cos \left( \theta^{(n)} \right) \partial_\psi \bar{\psi}^{(n)}. \]  \hspace{1cm} (88) \]
For spin configuration I the classical values are $\alpha^{(n)} = 0$ and $\theta^{(n)} = 0$ or $\theta^{(n)} = \pi$. Let us assume the first case $\theta^{(n)} = 0$. The ground state energy equation in the intermediate tunneling regime is given by
\[ \frac{\delta E^{(IT)}}{\xi_0} = \frac{(s_{p_+} J)^2}{\pi (v_{F_+} + v_{F_-})} \log \left( \frac{2t}{|J|} \right) \left[ -4 + 4\delta \psi^2 + 4(\tilde{\alpha}_2^2 + \delta \alpha^2) + 2\left( \tilde{\theta}^2 + \delta \theta^2 \right) \right]; \]  \hspace{1cm} (89) \]
where $\delta \psi' = \delta \psi - \frac{\pi}{2}$, $\tilde{\alpha}/\tilde{\theta} = \frac{1}{2} \left( \alpha/\theta^{(1)} + \alpha/\theta^{(2)} \right)$ and $\delta \alpha/\delta \theta = \frac{1}{2} \left( \alpha/\theta^{(1)} - \alpha/\theta^{(2)} \right)$. In leading order the Wess Zumino term reads as

$$L_{WZ} = \frac{2 i s \rho_s}{\xi_0} \int_D \left[ \tilde{\alpha} \partial_\tau \tilde{\psi} + \delta \alpha \partial_\tau \delta \psi' \right].$$ (90)

Note, that the only massless field in our theory is $\tilde{\psi}$. Integrating out the massive variables will thus lead to a renormalization of the compressibility and velocity of the Luttinger liquid action in (87). Since we are interested only in the low frequency behavior of the system, we neglect the second term in (90), because $\delta \psi'$ is a massive variable $\sim J^2$. After integrating out the massive fields, we obtain the following Lagrangian

$$L = \frac{1}{2 \pi} \left[ (\partial_\tau \tilde{\psi})^2 + (v_F \partial_x \tilde{\psi})^2 \right] + \frac{1}{2 \pi} \left[ (\partial_\tau \tilde{\psi})^2 + (v_F \partial_x \tilde{\psi})^2 \right] + \frac{D^2}{16 C J^2} \left( \partial_\tau \tilde{\psi} \right)^2,$$ (91)

Slightly rewriting this Lagrangian gives the usual LL Lagrangian with renormalized compressibility $K_\psi$

$$L = \frac{1}{K_\psi} L[\tilde{\psi}, v_\psi] + L_0[R_{\pm \downarrow}, L_{\pm \uparrow}],$$ (92)

with $L[\tilde{\psi}, v_\psi] = \frac{1}{2 \pi K_\psi} \left[ (\partial_\tau \tilde{\psi})^2 + (v_F \partial_x \tilde{\psi})^2 \right]$ and $v_\psi = K_\psi \sqrt{v_F^2 + v_F^2}$. The compressibility becomes strongly renormalized and is given by

$$K_\psi = \sqrt{\frac{1}{2 + \frac{D^2}{16 C J^2}}} \approx \frac{4 \sqrt{C}}{D} J \ll 1,$$ (93)

where we used the fact, that $J/t \ll 1$ and expanded in the last step. It was shown in [30, 31], that Eq.(92), upon bosonization, consists of two helical U(1) Luttinger liquids, which couple to charge and spin sources simultaneously. The collective mode $\tilde{\psi}$ becomes strongly renormalized.