Stable exponential cosmological solutions with two factor spaces in the Einstein–Gauss–Bonnet model with a $\Lambda$-term

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Abstract

We study $D$-dimensional Einstein–Gauss–Bonnet gravitational model including the Gauss–Bonnet term and the cosmological term $\Lambda$. We find a class of solutions with exponential time dependence of two scale factors, governed by two Hubble-like parameters $H > 0$ and $h$, corresponding to factor spaces of dimensions $m > 2$ and $l > 2$, respectively. These solutions contain a fine-tuned $\Lambda = \Lambda(x, m, l, \alpha)$, which depends upon the ratio $h/H = x$, dimensions of factor spaces $m$ and $l$, and the ratio $\alpha = \alpha_2/\alpha_1$ of two constants ($\alpha_2$ and $\alpha_1$) of the model. The master equation $\Lambda(x, m, l, \alpha) = \Lambda$ is equivalent to a polynomial equation of either fourth or third order and may be solved in radicals. The explicit solution for $m = l$ is presented in “Appendix”. Imposing certain restrictions on $x$, we prove the stability of the solutions in a class of cosmological solutions with diagonal metrics. We also consider a subclass of solutions with small enough variation of the effective gravitational constant $G$ and show the stability of all solutions from this subclass.

Keywords Gauss-Bonnet · Cosmology · Variation of $G$

1 Introduction

In this paper we deal with the so-called Einstein–Gauss–Bonnet (EGB) gravitational model in $D$ dimensions, which contains Gauss–Bonnet term and cosmological term $\Lambda$. The so-called Gauss–Bonnet term appeared in (super)string theory as a correction (in appropriate frame) to the effective (super)string action (e.g. heterotic one) [1–4].
It should be noted that at present EGB gravitational model, e.g. with cosmological term $\Lambda$, and its modifications, see [5–28] and refs. therein, are under intensive studies in cosmology, aimed at possible explanation of accelerating expansion of the Universe, which follows from supernovae (type Ia) observational data [29–31].

In this paper we study the cosmological model with diagonal metric governed by $n > 3$ scale factors ($D = n + 1$) depending upon one variable , which is the synchronous time variable. As it is well-known, the multidimensional gravitational model ($D > 4$) with quadratic in curvatures terms governed by the Lagrangian $R - 2\Lambda + \alpha R^2 + \beta R_{mn}R^{mn} + \gamma R_{mnpq}R^{mnpq}$ leads us for generic values of coupling constants $\alpha$, $\beta$ and $\gamma$ to the fourth-order differential equations for the components of the metric, while for the unique choice of the constants $\alpha = \gamma = -4\beta$ (yielding the Gauss–Bonnet term) the equations of motion are of the second order [32]. In cosmological model with the Gauss–Bonnet term the equations of motion are governed by an effective Lagrangian which contains 2-metric (or minisupermetric) $G_{ij}$ and finslerian metric $G_{ijkl}$, see [13,14] for $\Lambda = 0$ and [21,33] for $\Lambda \neq 0$.

Here we consider the cosmological solutions with exponential dependence of scale factors and obtain a class of solutions with two scale factors, governed by two Hubble-like parameters $H > 0$ and $h$, which correspond to factor spaces of dimensions $m > 2$ and $l > 2$, respectively, ($D = 1 + m + l$) and obey relations: $mh + lh \neq 0$ and $H \neq h$. The solutions depend upon $\Lambda$ and $\alpha = \alpha_2/\alpha_1$. The latter is the ratio of two constants of the model: $\alpha_2$ and $\alpha_1$. (In string inspired models $\alpha$ corresponds to Regge slope parameter $\alpha'$ which is inverse proportional to the tension of the string.) Any of these solutions describes an exponential (e.g. accelerated) expansion of 3-dimensional subspace with Hubble parameter $H > 0$ [34]. Here, as in our previous paper [22], we use the Chirkov–Pavluchenko–Toporensky scheme of reduction (of the set of polynomial equations) [17], which gives us a drastical simplification of the search and analysis of the exact solutions under consideration.

The special $m = 3$ case was studied recently in ref. [27]. The analysis of the number of solutions for given $\Lambda$ and $\alpha$ (denoted as $n(\Lambda, \alpha)$), carried out in ref. [27], tells us that exponential solutions exist if and only the following bounds on $\Lambda$ and $\alpha$ are valid:

$$\Lambda \alpha \leq \frac{3l^2 - 7l + 6}{8l(l - 1)} = \frac{3D^2 - 31D + 82}{8(D - 4)(D - 5)} = \lambda_c$$

(1.1)

for $\alpha > 0$ and

$$\Lambda |\alpha| > \frac{(l + 2)(l + 3)}{8l(l + 1)} = \frac{(D - 2)(D - 1)}{8(D - 4)(D - 3)} = |\lambda_a|$$

(1.2)

for $\alpha < 0$. Here $D = 4 + l$.

Relations (1.1) and (1.2) were obtained (independently) in ref. [28], where they were compared with some other relations obtained in physical literature, e.g. in refs.

1 The second relation (1.2) was extended in ref. [28] to $\Lambda |\alpha| \geq |\lambda_a|$ by adding into consideration the case $H = h$ [16,22]. In ref. [28] the cosmological constant $\Lambda_P$ is related to our one as $\Lambda_P = 2\Lambda$ and the internal space dimension $l$ is denoted as $D$. 

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[35] ($\Lambda < 0$, AdS/CFT correspondence), [36] ($\Lambda < 0$, black holes) etc. In this paper we generalize these bounds to $m \geq 3$ case.

We study the stability of the solutions in a class of cosmological solutions with diagonal metrics by using results of refs. [21,22] (see also approach of ref. [19]) and show that the solutions, considered here, are stable if certain restrictions on ratio $h/H$ are imposed.

Here we also study solutions with a small enough variation of the effective gravitational constant $G$ in Jordan frame [37,38] (see also [39,40] and refs. therein) which obey the most severe restrictions on variation of $G$ from ref. [41]. These solutions are shown to be stable.

We note that a class of stable solutions with zero variation of the effective gravitational constant $G$ (either in Jordan or Einstein frames) was considered recently in [24] for $(m, l) \neq (6, 6), (7, 4), (9, 3)$. Two special solutions for $D = 22, 28$ and $\Lambda = 0$ were found earlier in ref. [18]; in ref. [21] it was proved that they are stable. Another six stable exponential solutions (five in dimensions $D = 7, 8, 9, 13$ and two - for $D = 14$) were found recently in [23].

It should be stressed here that the main physical motivation for this paper and numerous other articles devoted to higher dimensional cosmological solutions in EGB gravity is based on (i) a possible solution of “dark energy” problem, i.e. explaining the accelerated expansion of the Universe. Here, as in some previous papers, we try to strengthen the physical counterpart of the investigations [e.g. item (i)] by studying also: (ii) the stability of the obtained solutions and (iii) by considering the solutions with small enough variation of the effective gravitational constant $G$ (in Jordan frame) which satisfy the observational restrictions. At the moment the item (ii) is rarely considered in physical publications on multidimensional EGB cosmology while the item (iii) is mostly studied (to our knowledge) in our papers.

2 The cosmological model

The action of the model reads as follows

$$S = \int_M d^Dz \sqrt{|g|} \left[ \alpha_1 (R[g] - 2\Lambda) + \alpha_2 \mathcal{L}_2[g] \right], \quad (2.1)$$

where $g = g_{MN} dz^M \otimes dz^N$ is the metric defined on the manifold $M$, dim $M = D$, $|g| = |\det(g_{MN})|$, $\Lambda$ is the cosmological term, $R[g]$ is scalar curvature,

$$\mathcal{L}_2[g] = R_{MNPQ}R^{MNPQ} - 4R_{MN}R^{MN} + R^2$$

is the Gauss–Bonnet term and $\alpha_1, \alpha_2$ are nonzero constants.

We consider the following (product) manifold

$$M = \mathbb{R} \times M_1 \times \cdots \times M_n \quad (2.2)$$
with the cosmological metric

\[ g = -dt \otimes dt + \sum_{i=1}^{n} B_i e^{2v_i t} dy^i \otimes dy^i, \]  

(2.3)

where \( B_i > 0 \) are arbitrary constants, \( i = 1, \ldots, n \). In (2.2) \( M_1, \ldots, M_n \) are one-dimensional manifolds, either \( \mathbb{R} \) or \( S^1 \), and \( n > 3 \).

The equations of motion for the action (2.1) leads us the to the set of following polynomial equations \([21,33]\)

\[ G_{ij} v^i v^j + 2\Lambda - \alpha G_{ijkl} v^i v^j v^k v^l = 0, \]  

(2.4)

\[ [2G_{ij} v^j - \frac{4}{3}\alpha G_{ijkl} v^i v^k v^l] \sum_{k=1}^{n} v^k - \frac{2}{3} G_{sj} v^s v^j + \frac{2}{3}\Lambda = 0, \]  

(2.5)

\( i = 1, \ldots, n, \) where \( \alpha = \alpha_2/\alpha_1 \). Here

\[ G_{ij} = \delta_{ij} - 1, \quad G_{ijkl} = G_{ij} G_{ik} G_{jl} G_{jl} G_{kl} \]  

(2.6)

are, respectively, the components of two metrics on \( \mathbb{R}^n [13,14] \): 2-metric and Finslerian 4-metric, respectively. For the case \( n > 3 \) (or \( D > 4 \)) we have a set of forth-order polynomial equations.

For \( \Lambda = 0 \) and \( n > 3 \) the set of Eqs. (2.4) and (2.5) has an isotropic solution \( v^1 = \ldots = v^n = H \) only if \( \alpha < 0 \) \([13,14]\). In \([16]\) an isotropic solution was obtained for \( \Lambda \neq 0 \).

It was proved in ref. \([13,14]\) that there are no more than three different numbers among \( v^1, \ldots, v^n \), when \( \Lambda = 0 \). This is also valid for the case \( \Lambda \neq 0 \) when \( \sum_{i=1}^{n} v^i \neq 0 \) \([22]\).

3 Solutions with two Hubble-like parameters

In this section we find a class of solutions to the set of Eqs. (2.4), (2.5) with the following set of Hubble-like parameters

\[ v = \begin{pmatrix} H, H, H, \ldots, H, H, \ldots, H \end{pmatrix} \]  

(3.1)

where \( H \) is the Hubble-like parameter corresponding to an \( m \)-dimensional factor space with \( m > 2 \), while \( h \) is the Hubble-like parameter corresponding to the \( l \)-dimensional factor space, \( l > 2 \). For future cosmological applications we split the \( m \)-dimensional factor space into the product of two subspaces of dimensions 3 and \( m - 3 \), respectively. The first one is considered as “our” 3d space while the second one is identified with a subspace of \((m - 3 + l)\)-dimensional internal space.
For a description of an accelerated expansion of a 3-dimensional subspace (which may describe our Universe) we put

$$H > 0.$$  \hspace{1cm} (3.2)

It is well-known that the (four-dimensional) effective gravitational constant $G = G_{\text{eff}}$ in the Brans–Dicke–Jordan (or simply Jordan) frame \cite{37,38} is proportional to the inverse volume scale factor of the internal space, see \cite{39} and references therein.

We note, that due to ansatz (3.1) “our” 3d space expands (isotropically) with Hubble parameter $H$ and $(m - 3)$-dimensional part of internal space expands (isotropically) with the same Hubble parameter $H$ too. Here we consider for physical applications (in our epoch) the internal space to be compact one, i.e. we put in (2.2) $M_4 = \cdots = M_n = S^1$. We also put the internal scale factors corresponding to present time to be small enough as compared to the scale factor of “our” space.

Due to ansatz (3.1), the $m$-dimensional factor space is expanding with the Hubble parameter $H > 0$, while the evolution of the $l$-dimensional factor space is described by the Hubble-like parameter $h$.

It was shown earlier in refs. \cite{17,22} that the ansatz (3.1) with two restrictions on parameters $H$ and $h$ imposed

$$mH + lh \neq 0, \quad H \neq h,$$  \hspace{1cm} (3.3)

leads us to a reduction of the relations (2.4) and (2.5) to the following set of two polynomial equations

$$E = mH^2 + lh^2 - (mH + lh)^2 + 2\Lambda - \alpha[m(m - 1)(m - 2)(m - 3)H^4$$
$$+ 4m(m - 1)(m - 2)lh^3h + 6m(m - 1)(l - 1)H^2h^2$$
$$+ 4ml(l - 1)(l - 2)HHh^3 + l(l - 1)(l - 2)(l - 3)h^4] = 0,$$  \hspace{1cm} (3.4)

$$Q = (m - 1)(m - 2)H^2 + 2(m - 1)(l - 1)HH$$
$$+(l - 1)(l - 2)h^2 = -\frac{1}{2\alpha}.$$  \hspace{1cm} (3.5)

Using Eq. (3.5) we get for $m > 2$ and $l > 2$

$$H = (-2\alpha \mathcal{P})^{-1/2},$$  \hspace{1cm} (3.6)

where we denote

$$\mathcal{P} = \mathcal{P}(x, m, l) = (m - 1)(m - 2)$$
$$+ 2(m - 1)(l - 1)x + (l - 1)(l - 2)x^2,$$  \hspace{1cm} (3.7)

$$x = h/H,$$  \hspace{1cm} (3.8)

and

$$\alpha \mathcal{P} < 0.$$  \hspace{1cm} (3.9)
We note that the following "duality" identity is obeyed

\[ \mathcal{P}(x, m, l) = x^2 \mathcal{P}(1/x, l, m), \]  

(3.10)

for \( x \neq 0 \).

Due to restrictions (3.3) we have

\[ x \neq x_d = x_d(m, l) \equiv -m/l, \quad x \neq x_a \equiv 1, \]

(3.11)

where

\[ x_d(m, l) = 1/x_d(l, m). \]

(3.12)

The relation (3.7) is valid only if

\[ \mathcal{P}(x, m, l) \neq 0. \]

(3.13)

We note, that for \( \mathcal{P}(x, m, l) = 0 \) the relation (3.5) is not obeyed.

The substitution of relation (3.6) into (3.4) leads us to following formulas

\[ \Delta \alpha = \lambda = \lambda(x, m, l) \equiv \frac{1}{4}(\mathcal{P}(x, m, l))^{-1}\mathcal{M}(x, m, l) \]

\[ + \frac{1}{8}(\mathcal{P}(x, m, l))^{-2}\mathcal{R}(x, m, l), \]

(3.14)

\[ \mathcal{M}(x, m, l) \equiv m + lx^2 - (m + lx)^2, \]

(3.15)

\[ \mathcal{R}(x, m, l) \equiv m(m - 1)(m - 2)(m - 3) + 4m(m - 1)(m - 2)lx \]

\[ + 6m(m - 1)(l - 1)x^2 + 4ml(l - 1)(l - 2)x^3 \]

\[ + l(l - 1)(l - 2)(l - 3)x^4. \]

(3.16)

In what follows we use the following duality identities

\[ \mathcal{M}(x, m, l) = x^2\mathcal{M}(1/x, l, m), \]

(3.17)

\[ \mathcal{R}(x, m, l) = x^4\mathcal{R}(1/x, l, m) \]

(3.18)

for \( x \neq 0 \). The identities (3.10), (3.17) and (3.18) imply

\[ \lambda(x, m, l) = \lambda(1/x, l, m) \]

(3.19)

for \( x \neq 0 \).

Using (3.13) we obtain

\[ x \neq x_{\pm} \equiv x_{\pm}(m, l) \equiv \frac{-(m - 1)(l - 1) \pm \sqrt{\Delta(m, l)}}{(l - 1)(l - 2)}. \]

(3.20)

\[ \Delta(m, l) \equiv (m - 1)(l - 1)(m + l - 3) = \Delta(l, m). \]

(3.21)
Here $x_{\pm}(m, l)$ are roots of the quadratic equation $\mathcal{P}(x, m, l) = 0$. These roots obey the following relations

$$x_+(m, l)x_-(m, l) = \frac{(m - 1)(m - 2)}{(l - 1)(l - 2)},$$
\hspace{1cm} (3.22)

$$x_+(m, l) + x_-(m, l) = -2 \frac{(m - 1)}{l - 2},$$
\hspace{1cm} (3.23)

which lead us to the inequalities

$$x_-(m, l) < x_+(m, l) < 0.$$  \hspace{1cm} (3.24)

These relations and the duality identity (3.10) imply

$$x_{\mp}(m, l) = \frac{1}{x_{\pm}(l, m)}$$ \hspace{1cm} (3.25)

for all $m > 2$ and $l > 2$.

Using (3.9) and (3.14) we get

$$\Lambda = \alpha^{-1} \lambda(x, m, l),$$ \hspace{1cm} (3.26)

where

$$x_-(m, l) < x < x_+(m, l) \text{ for } \alpha > 0$$ \hspace{1cm} (3.27)

and

$$x < x_-(m, l), \text{ or } x > x_+(m, l) \text{ for } \alpha < 0.$$ \hspace{1cm} (3.28)

For $\alpha < 0$ we have the following limit

$$\lim_{x \to \pm \infty} \lambda(x, m, l) = \lambda_{\infty}(l) \equiv -\frac{l(l + 1)}{8(l - 1)(l - 2)} < 0.$$ \hspace{1cm} (3.29)

Hence

$$\lim_{x \to \pm \infty} \Lambda = \Lambda_{\infty} \equiv -\frac{l(l + 1)}{8\alpha(l - 1)(l - 2)} > 0,$$ \hspace{1cm} (3.30)

$l > 2$. We note that $\Lambda_{\infty}$ does not depend upon $m$. For $x = 0$ we get in agreement with the duality identity (3.19)

$$\Lambda = \Lambda_0 = \alpha^{-1} \lambda(0, m, l) = -\frac{m(m + 1)}{8\alpha(m - 1)(m - 2)} > 0,$$ \hspace{1cm} (3.31)

$m > 2$. We see that $\Lambda_0$ does not depend upon $l$. For $x = 0$ the Hubble-like parameters read

$$H = H_0 = (-2\alpha(m - 1)(m - 2))^{-1/2}, \hspace{1cm} h = 0$$ \hspace{1cm} (3.32)

and due to relations (2.2), (2.3) we get the product of (a part of) $(m + 1)$-dimensional de-Sitter space and $l$-dimensional Euclidean space.
“Master” equation. We rewrite eq. (3.14) in the following form

\[ 2 \mathcal{P}(x, m, l) M(x, m, l) + R(x, m, l) - 8\lambda(P(x, m, l))^2 = 0. \]  

(3.33)

This equation may be called as a master equation, since the solutions under consideration are governed by it. The master equation is of fourth order in \( x \) for \( \lambda \neq \lambda_\infty(l) \) or less (of third order for \( \lambda = \lambda_\infty(l) \)). For any \( m > 2 \) and \( l > 2 \) the equation (3.33) may be solved in radicals, though the general solution has a rather cumbersome form and will not be presented here. It is worth for any given \( m \) and \( l \) to find the solution just by using Maple or Mathematica. An example of explicit (generic) solution for \( m = l = 3 \) is presented in “Appendix A” (for \( m = l = 3 \) see ref. [27]). Several special solutions with \( m = 3 \) and \( l = 4 \) were given in ref. [21].

Now we consider the behaviour of the function \( \lambda(x, m, l) \) in the vicinity of the points \( x_- (m, l) \) and \( x_+ (m, l) \). Here the following proposition takes place.

**Proposition 1** For \( m > 2, l > 2 \)

\[ \lambda(x, m, l) \sim B_\pm(m, l)(x - x_\pm(m, l))^2, \]  

(3.34)

as \( x \to x_\pm = x_\pm(m, l) \), where \( B_\pm(m, l) < 0 \) and hence

\[ \lim_{x \to x_\pm} \lambda(x, m, l) = -\infty. \]  

(3.35)

The Proposition 1 can be proved by using the following lemma.

**Lemma** For all \( m > 2, l > 2 \)

\[ R_\pm(m, l) \equiv R(x_\pm(m, l), m, l) < 0. \]  

(3.36)

The proof of the Lemma is presented in the “Appendix B”.

**Proof of the Proposition 1** Relation (3.34) follows from (3.14), \( \mathcal{P}(x, m, l) = (l-1)(l-2)(x-x_+)(x-x_-) \) and Lemma. Here

\[ B_\pm(m, l) = \frac{R_\pm(m, l)}{8(l-1)^2(l-2)^2(x_+ - x_-)^2} = \frac{R_\pm(m, l)}{32(m-1)(l-1)(m+l-3)} < 0, \]  

(3.37)

where \( m > 2 \) and \( l > 2 \). Relation (3.35) follows from (3.34) and (3.37). Thus, the Proposition 1 is proved.

Now we analyze the behaviour of the function \( \lambda(x, m, l) \), for fixed \( m, l \) and \( x \neq x_\pm(m, l) \). We find the extremum points obeying \( \frac{\partial}{\partial x} \lambda(x, m, l) = 0 \). By straightforward calculations we obtain

\[ \frac{\partial}{\partial x} \lambda(x, m, l) = -f(x, m, l)(\mathcal{P}(x, m, l))^{-3}, \]  

(3.38)
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\[ f(x, m, l) = (l - 1)(m - 1)(x - 1)(lx + m) \times \]
\[ \times [(l - 2)x + m - 1][(l - 1)x + m - 2], \quad (3.39) \]

\( x \neq x_{\pm}(m, l) \). Using these relations we are led to the following extremum points

\[ x_a = 1, \quad (3.40) \]
\[ x_b = x_b(m, l) \equiv -\frac{m - 1}{l - 2} < 0, \quad (3.41) \]
\[ x_c = x_c(m, l) \equiv -\frac{m - 2}{l - 1} < 0, \quad (3.42) \]
\[ x_d = x_d(m, l) \equiv -\frac{m}{l} < 0. \quad (3.43) \]

These points obey the following duality identities

\[ x_d(m, l) = \frac{1}{x_d(l, m)}, \quad (3.44) \]
\[ x_b(m, l) = \frac{1}{x_c(l, m)}. \quad (3.45) \]

We also obtain the inequality

\[ x_b(m, l) < x_c(m, l), \quad (3.46) \]

which is valid since

\[ x_c(m, l) - x_b(m, l) = \frac{m + l - 3}{(l - 1)(l - 2)} > 0 \quad (3.47) \]

for all \( m > 2, l > 2 \).

The points \( x_b, x_c, x_d \) obey the following inclusion

\[ x_i(m, l) \in (x_-(m, l), x_+(m, l)), \quad (3.48) \]

\( i = b, c, d \) for \( m > 2, l > 2 \). This inclusion just follows from relations \( P_i(m, l) = P(x_i(m, l), m, l) < 0, i = b, c, d, \) since

\[ P_b(m, l) = -\frac{(m - 1)(m + l - 3)}{(l - 2)} < 0, \quad (3.49) \]
\[ P_c(m, l) = -\frac{(m - 2)(m + l - 3)}{(l - 1)} < 0, \quad (3.50) \]
\[ P_d(m, l) = -\frac{(l - 2)m^2 + 2lm + (m - 2)l^2}{l^2} < 0, \quad (3.51) \]

for all \( m > 2 \) and \( l > 2 \).
By using relations
\begin{align}
xd - xc &= \frac{m - 2l}{l(l - 1)}, \\
xd - xb &= \frac{2m - l}{l(l - 2)},
\end{align}

we get
\begin{align}
(1) \quad &xb < xc < xd, \quad \text{for } l < m/2, \\
(2) \quad &xb < xd < xc, \quad \text{for } m/2 < l < 2m, \\
(3) \quad &xd < xb < xc, \quad \text{for } l > 2m,
\end{align}

and
\begin{align}
(1a) \quad &xb < xc = xd, \quad \text{if } l = m/2, \\
(3a) \quad &xd = xb < xc, \quad \text{for } l = 2m.
\end{align}

We note that \( m/2 < l < 2m \) may be rewritten as \( l/2 < m < 2l \).

Let us calculate \( \lambda_i = \lambda(x_i, m, l), i = a, b, c, d \). We obtain
\begin{align}
\lambda_a &= -\frac{(m + l - 1)(m + l)}{8(m + l - 3)(m + l - 2)} < 0, \\
\lambda_b &= \frac{lm^2 + (l^2 - 8l + 8)m + l^2 - l}{8(l - 2)(m - 1)(l + m - 3)} > 0, \\
\lambda_c &= \frac{ml^2 + (m^2 - 8m + 8)l + m^2 - m}{8(m - 2)(l - 1)(l + m - 3)} > 0,
\end{align}

and
\begin{align}
\lambda_d &= \frac{ml(m + l)}{8(lm^2 + ml^2 - 2m^2 - 2l^2 + 2lm)} > 0.
\end{align}

In the proof of inequalities (3.60), (3.61) the following relations were used:
\begin{align}
u(m, l) &= lm^2 + (l^2 - 8l + 8)m + l^2 - l > 0, \\
v(m, l) &= ml^2 + (m^2 - 8m + 8)l + m^2 - m > 0 \\
w(m, l) &= m^2(l - 2) + l^2(m - 2) + 2lm > 0
\end{align}

for \( m > 2, l > 2 \). Indeed, for \( m \geq 4, l \geq 4 \) we get \( u(m, l) = ml(m + l - 8) + 8m + l^2 - l > 0 \) and \( u(4, 3) = 26, u(3, 4) = 24, u(3, 3) = 12 \). Thus, the first relation (3.63) is valid. The second one (3.64) just follows from the first one and \( v(m, l) = u(l, m) \).
We also get
\[
\lambda_b - \lambda_c = \frac{(m - l)(m + l - 3)}{4(l - 2)(l - 1)(m - 2)(m - 1)} \begin{cases} 
> 0, & \text{if } m > l, \\
= 0, & \text{if } m = l, \\
< 0, & \text{if } m < l.
\end{cases}
\] (3.66)

and
\[
\lambda_d - \lambda_c = \frac{(m - 1)(m - 2l)^3}{4(l - 1)(m - 2)(m + l - 3)w(m, l)},
\] (3.67)

\[
\lambda_d - \lambda_b = \frac{(l - 1)(l - 2m)^3}{4(m - 1)(l - 2)(m + l - 3)w(m, l)},
\] (3.68)

for \(m > 2, l > 2\). ( \(w(m, l)\) is defined in (3.65).) By using these relations we obtain
\[
\lambda_d - \lambda_c \begin{cases} 
> 0, & \text{if } m > 2l, \\
= 0, & \text{if } m = 2l, \\
< 0, & \text{if } m < 2l.
\end{cases}
\] (3.69)

and
\[
\lambda_d - \lambda_b \begin{cases} 
> 0, & \text{if } l > 2m, \\
= 0, & \text{if } l = 2m, \\
< 0, & \text{if } l < 2m.
\end{cases}
\] (3.70)

Now we study the behaviour of the function \(\lambda(x, m, l)\) with respect to variable \(x\) for fixed integer numbers \(m > 2, l > 2\). We denote by \(n(\Lambda, \alpha)\) the number of solutions (in \(x\)) of the equation \(\Lambda\alpha = \lambda(x, m, l)\). We calculate \(n(\Lambda, \alpha)\) by using inequalities for points of extremum \(x_i\) and \(\lambda_i\) \((i = b, c, d)\) presented above and relations (3.35) and (3.39).

First, we start with the case \(\alpha > 0\) and \(x_- < x < x_+\).

1. \(m > 2l\). We get \(x_b < x_c < x_d\) and \(\lambda_c < \lambda_d < \lambda_b\). Here \(x_b\) and \(x_d\) are points of local maximum (\(x_b\) is a point of maximum on interval \((x_-, x_+)\)) and \(x_c\) is a point of local minimum. We obtain
\[
n(\Lambda, \alpha) = \begin{cases} 
0, & \Lambda\alpha > \lambda_b, \\
1, & \Lambda\alpha = \lambda_b, \\
2, & \lambda_d < \Lambda\alpha < \lambda_c, \\
2, & \Lambda\alpha = \lambda_d, \\
3, & \lambda_c < \Lambda\alpha < \lambda_d, \\
3, & \Lambda\alpha = \lambda_c, \\
2, & \Lambda\alpha < \lambda_c.
\end{cases}
\] (3.71)

Here and in what follows we use \(x \neq x_d\).
We present an example of the function $\lambda(x) = \Lambda(x)\alpha$ for $\alpha > 0$, $m = 12$ and $l = 3$ at Fig. 1. At this and other figures the point $(x_i, \lambda_i)$ is marked by $i$, where $i = a, b, c, d$.

(10) $m = 2l$. We have $x_b < x_c = x_d$ and $\lambda_c = \lambda_d < \lambda_b$. Here $x_b$ is the point of maximum on the interval $(x_- , x_+)$ and $x_c = x_d$ is the point of inflection. We obtain

$$n(\Lambda, \alpha) = \begin{cases} 0, & \Lambda \alpha > \lambda_b, \\ 1, & \Lambda \alpha = \lambda_b, \\ 2, & \lambda_d < \Lambda \alpha < \lambda_b, \\ 1, & \Lambda \alpha = \lambda_d, \\ 2, & \Lambda \alpha < \lambda_d. \end{cases}$$ (3.72)

An example of the function $\lambda(x) = \Lambda(x)\alpha$ for $\alpha > 0$, $m = 12$ and $l = 6$ is depicted at Fig. 2.

(2) $l/2 < m < 2l$. We obtain $x_b < x_d < x_c$ and $\lambda_d < \lambda_c, \lambda_d < \lambda_b$. The points $x_b, x_c$ are points of local maximum and $x_d$ is point of local minimum.

Now we split this case on three subcases: (2$_+^l$) $l < m < 2l$, (2$_0^l$) $m = l$ and (2$_-^l$) $l/2 < m < l$. 
Fig. 2 The function $\lambda(x) = \Lambda(x)\alpha$ for $\alpha > 0$, $m = 12$ and $l = 6$

$(2_+)$ $l < m < 2l$. In this subcase we have $\lambda_d < \lambda_c < \lambda_b$ and hence

$$n(\Lambda, \alpha) = \begin{cases} 
0, & \Lambda\alpha > \lambda_b, \\
1, & \Lambda\alpha = \lambda_b, \\
2, & \lambda_c < \Lambda\alpha < \lambda_b, \\
3, & \Lambda\alpha = \lambda_c, \\
4, & \lambda_d < \Lambda\alpha < \lambda_c, \\
2, & \Lambda\alpha = \lambda_d, \\
2, & \Lambda\alpha < \lambda_d. 
\end{cases}$$

$(3.73)$

An example of the function $\lambda(x) = \Lambda(x)\alpha$ for $\alpha > 0$, $m = 9$ and $l = 7$ is depicted at Fig. 3.

$(2_0)$ $m = l$. In this subcase we find $\lambda_d < \lambda_c = \lambda_b$. Hence

$$n(\Lambda, \alpha) = \begin{cases} 
0, & \Lambda\alpha > \lambda_b = \lambda_c, \\
2, & \Lambda\alpha = \lambda_b = \lambda_c, \\
4, & \lambda_d < \Lambda\alpha < \lambda_b = \lambda_c, \\
2, & \Lambda\alpha = \lambda_d, \\
2, & \Lambda\alpha < \lambda_d. 
\end{cases}$$

$(3.74)$

At Fig. 4 an example of the function $\lambda(x) = \Lambda(x)\alpha$ for $\alpha > 0$ and $m = l = 4$ is presented.
Fig. 3 The function $\lambda(x) = \Lambda(x)\alpha$ for $\alpha > 0$, $m = 9$ and $l = 7$

Fig. 4 The function $\lambda(x) = \Lambda(x)\alpha$ for $\alpha > 0$ and $m = l = 4$
Fig. 5 The function $\lambda(x) = \frac{1}{\Lambda_1(x)}\alpha$ for $\alpha > 0$, $m = 4$ and $l = 6$

$(2_{\ldots}) l/2 < m < l$. In this subcase one gets $\lambda_d < \lambda_b < \lambda_c$ and

$$n(\Lambda, \alpha) = \begin{cases} 0, & \Lambda \alpha > \lambda_c, \\ 1, & \Lambda \alpha = \lambda_c, \\ 2, & \lambda_b < \Lambda \alpha < \lambda_c, \\ 3, & \Lambda \alpha = \lambda_b, \\ 4, & \lambda_d < \Lambda \alpha < \lambda_b, \\ 2, & \Lambda \alpha = \lambda_d, \\ 2, & \Lambda \alpha < \lambda_d. \end{cases}$$ (3.75)

An example of the function $\lambda(x) = \frac{1}{\Lambda(x)}\alpha$ for $\alpha > 0$, $m = 4$ and $l = 6$ is depicted at Fig. 5.

(3) $m < l/2$. We have $x_d < x_b < x_c$ and $\lambda_b < \lambda_d < \lambda_c$. Here $x_c$ and $x_d$ are points of local maximum ($x_c$ is the point of maximum on interval $(x_-, x_+)$) and $x_b$ is a point of local minimum. We find

$$n(\Lambda, \alpha) = \begin{cases} 0, & \Lambda \alpha > \lambda_c, \\ 1, & \Lambda \alpha = \lambda_c, \\ 2, & \lambda_d < \Lambda \alpha < \lambda_c, \\ 3, & \Lambda \alpha = \lambda_d, \\ 4, & \lambda_b < \Lambda \alpha < \lambda_d, \\ 2, & \Lambda \alpha = \lambda_b, \\ 2, & \Lambda \alpha < \lambda_b. \end{cases}$$ (3.76)
An example of the function \( \lambda(x) = \Lambda(x)\alpha \) for \( \alpha > 0 \), \( m = 4 \) and \( l = 12 \) is depicted at Fig. 6.

\[(3.70) \quad m = l/2.\] We get relations \( x_d = x_b < x_c \) and \( \lambda_b = \lambda_d < \lambda_c \). Here \( x_c \) is the point of maximum on interval \((x-, x_+)\) and \( x_b = x_d \) is the point of inflection. We obtain

\[
n(\Lambda, \alpha) = \begin{cases} 0, & \Lambda\alpha > \lambda_c, \\ 1, & \Lambda\alpha = \lambda_c, \\ 2, & \lambda_d < \Lambda\alpha < \lambda_c, \\ 1, & \Lambda\alpha = \lambda_d, \\ 2, & \Lambda\alpha < \lambda_d. \end{cases}
\]

(3.77)

Bounds on \( \Lambda\alpha \) for \( \alpha > 0 \). Summarizing all cases presented above we find that for \( \alpha > 0 \) exact solutions under consideration exist if and only if

\[
\Lambda\alpha \leq \begin{cases} \lambda_b, & \text{for } m \geq l, \\ \lambda_c, & \text{for } m < l, \end{cases}
\]

(3.78)

where \( \lambda_b = \lambda_b(m, l) \) and \( \lambda_c = \lambda_c(m, l) \) are defined in (3.60) and (3.61), respectively. For \( m = 3 \) and \( l \geq 3 \) we are led to relation (1.1).

An example of the function \( \lambda(x) = \Lambda(x)\alpha \) for \( \alpha > 0 \), \( m = 4 \) and \( l = 8 \) is presented at Fig. 7. By using our analysis we find that for \( \alpha > 0 \) and small enough value of \( \Lambda \) there exist at least a pair of solutions \( x_1, x_2 \), obeying \( x_- < x_1 < x_2 < x_+ < 0 \).

Let us consider the case \( \alpha < 0 \). We have \( \Lambda|\alpha| = -\lambda(x) \), where \( x < x_- \) or \( x > x_+ \). Due to to the relations (3.29), (3.39) and Proposition 1 the function \(-\lambda(x)\)
is monotonically increasing in two intervals: i) in the interval \((-\infty, x_-)\) from \(-\lambda_\infty\) to \(+\infty\) and ii) in the interval \((x_a, +\infty)\) from \(-\lambda_a\) to \(-\lambda_\infty\). The function \(-\lambda(x)\) is monotonically decreasing in the interval \((x_+, x_a)\) from \(+\infty\) to \(-\lambda_a\). Here \(x_a\) is a point of local minimum of the function \(\Lambda(x)|\alpha| = -\lambda(x)\), which is excluded from the solution and \(-\lambda_a < -\lambda_\infty\). This inequality may be readily verified: due to relations (3.29), (3.59) we obtain

\[
\lambda_a - \lambda_\infty = \frac{2(m - 1)(2l - 1)m + 2l(l - 2)}{8(l - 1)(l - 2)(m + l - 3)(m + l - 2)} > 0, \tag{3.79}
\]

for \(m > 2\) and \(l > 2\). The functions \(\Lambda(x) = \lambda(x)/\alpha\) for \(\alpha = +1, -1\), respectively, and \(m = l = 4\) are presented at Fig. 8.

For the number of solutions (for \(\alpha < 0\)) we obtain

\[
n(\Lambda, \alpha) = \begin{cases} 
2, & \Lambda|\alpha| > |\lambda_\infty|, \\
1, & \Lambda|\alpha| = |\lambda_\infty|, \\
2, & |\lambda_a| < \Lambda|\alpha| < |\lambda_\infty|, \\
0, & \Lambda|\alpha| \leq |\lambda_a|. 
\end{cases} \tag{3.80}
\]

Here \(x \neq x_a = 1\). Hence, for \(\alpha < 0\) and big enough values of \(\Lambda\) there exist two solutions \(x_1, x_2\): \(x_1 < x_- < 0\) and \(x_2 > x_+\).
The functions $\Lambda(x) = \frac{\lambda(x)}{\alpha}$ for $\alpha = \pm 1$ and $m = l = 4$

**Bounds on $|\Lambda|\alpha| \text{ for } \alpha < 0$**. It follows from (3.80) that for $\alpha < 0$ exact solutions under consideration exist if and only if

$$\Lambda|\alpha| > |\lambda_a| = \frac{(D - 2)(D - 1)}{8(D - 4)(D - 3)}, \quad (3.81)$$

where $\lambda_a$ is defined in (3.59). This relation is valid for all $m > 2, l > 2 (D = m + l + 1)$, e.g. for $m = 3$ (see 1.2).

### 4 Stability analysis

Here, using results of refs. [21,22], we study the stability of exponential solutions (2.3) with non-static total volume factor, i.e. we put

$$S_1(v) = \sum_{i=1}^{n} v^i \neq 0. \quad (4.1)$$

Here, as in [21,22], we impose the following restriction

(R) $\det(L_{ij}(v)) \neq 0 \quad (4.2)$

on the symmetric matrix

$$L = (L_{ij}(v)) = (2G_{ij} - 4\alpha G_{ijks} v^k v^s). \quad (4.3)$$
For general cosmological ansatz with the (diagonal) metric

\[ g = -dt \otimes dt + \sum_{i=1}^{n} e^{2\beta_i(t)} dy_i \otimes dy_i, \quad (4.4) \]

we have the set of (algebraic and differential) equations [13,14]

\[ E = G_{ij} h^i h^j + 2\Lambda - \alpha G_{ijkl} h^i h^j h^k h^l = 0, \quad (4.5) \]

\[ Y_i = \frac{dL_i}{dt} + \left( \sum_{j=1}^{n} h^j \right) L_i - \frac{2}{3} (G_{sj} h^s h^j - 4\Lambda) = 0, \quad (4.6) \]

where \( h^i = \dot{\beta}^i \),

\[ L_i = L_i(h) = 2G_{ij} h^j - \frac{4}{3} \alpha G_{ijkl} h^i h^j h^k h^l, \quad (4.7) \]

\( i = 1, \ldots, n. \)

Earlier, it was proved [22] that a constant solution \((h^i(t)) = (v^i) (i = 1, \ldots, n; \ n > 3)\) to Eqs. (4.5), (4.6) obeying restrictions (4.1), (4.2) is stable under perturbations

\[ h^i(t) = v^i + \delta h^i(t), \quad (4.8) \]

\( i = 1, \ldots, n, \) (as \( t \to +\infty \)) in the following case

\[ S_1(v) = \sum_{k=1}^{n} v^k > 0 \quad (4.9) \]

and it is unstable (as \( t \to +\infty \)) when

\[ S_1(v) = \sum_{k=1}^{n} v^k < 0. \quad (4.10) \]

For our consideration we have \( S_1(v) = mH + lh \) and hence due to \( H > 0 \) the restriction (4.9) may be written in the following form

\[ x > -\frac{m}{l} = x_d, \quad (4.11) \]

while the restriction (4.10) may be written as

\[ x < -\frac{m}{l} = x_d. \quad (4.12) \]

The perturbations \( \delta h^i \) obey (in the linear approximation) the following set of linear equations [21,22]

\[ C_i(v)\delta h^i = 0, \quad (4.13) \]
\[ L_{ij}(v)\delta h^j = B_{ij}(v)\delta h^j. \] (4.14)

Here

\[
C_i(v) = 2v_i - 4\alpha G_{ijks}v^jv^kv^s, \tag{4.15}
\]
\[
L_{ij}(v) = 2G_{ij} - 4\alpha G_{ijks}v^k v^s, \tag{4.16}
\]
\[
B_{ij}(v) = -\left( \sum_{k=1}^{n} v^k \right) L_{ij}(v) - L_i(v) + \frac{4}{3} v_j, \tag{4.17}
\]

where \( v_i = G_{ij}v^j, L_i(v) = 2v_i - \frac{4}{3} \alpha G_{ijks}v^jv^kv^s \) and \( i, j, k, s = 1, \ldots, n \).

In case when restrictions (4.1), (4.2) are imposed, the set of equations on perturbations (4.13), (4.14) has the following solution [22]

\[
\delta h_i = A^i \exp(-S_1(v)t), \tag{4.18}
\]
\[
\sum_{i=1}^{n} C_i(v)A^i = 0, \tag{4.19}
\]

\((A^i \text{ are constants}) i = 1, \ldots, n.\)

It was shown in [22] that for the vector \( v \) from (3.1), obeying relations (3.3), the matrix \( L \) is a block-diagonal one

\[
(L_{ij}) = \text{diag}(L_{\mu\nu}, L_{\alpha\beta}), \tag{4.20}
\]

where

\[
L_{\mu\nu} = G_{\mu\nu}(2 + 4\alpha S_{HH}), \tag{4.21}
\]
\[
L_{\alpha\beta} = G_{\alpha\beta}(2 + 4\alpha S_{hh}) \tag{4.22}
\]

and

\[
S_{HH} = (m - 2)(m - 3)H^2 + 2(m - 2)lHh + l(l - 1)h^2, \tag{4.23}
\]
\[
S_{hh} = m(m - 1)H^2 + 2m(l - 2)Hh + (l - 2)(l - 3)h^2. \tag{4.24}
\]

The matrix (4.20) is invertible only if \( m > 1, l > 1 \) and

\[
S_{HH} \neq -\frac{1}{2\alpha}, \tag{4.25}
\]
\[
S_{hh} \neq -\frac{1}{2\alpha}. \tag{4.26}
\]

We remind (the reader) that the matrices \((G_{\mu\nu}) = (\delta_{\mu\nu} - 1)\) and \((G_{\alpha\beta}) = (\delta_{\alpha\beta} - 1)\) are invertible only if \( m > 1 \) and \( l > 1 \).
Now, we prove that inequalities (4.25), (4.26) are obeyed if

\[ x \neq -\frac{m-2}{l-1} = x_c \]  
(4.27)

and

\[ x \neq -\frac{m-1}{l-2} = x_b \]  
(4.28)

for \( l > 2 \).

Let us suppose that (4.25) does not take place, i.e. \( S_{HH} = -\frac{1}{2\pi} \). Then using (3.5) we obtain

\[ S_{HH} - Q = -2(H-h)((m-2)H + (l-1)h) = 0, \]  
(4.29)

which implies due to \( H-h \neq 0 \) (see 3.3)

\[(m-2)H + (l-1)h = 0. \]  
(4.30)

This relation contradicts to the restriction (4.27). The obtained contradiction proves the inequality (4.25).

Now let us suppose that (4.26) is not valid, i.e. \( S_{hh} = -\frac{1}{2}\alpha \). Then using (3.5) we find

\[ S_{hh} - Q = -2(h-H)((l-2)h + (m-1)H) = 0. \]  
(4.31)

Due to \( H-h \neq 0 \) this implies

\[(l-2)h + (m-1)H = 0, \]  
(4.32)

which is in the contradiction with the restrictions (4.28) and \( H > 0 \). This contradiction lead us to the proof of the inequality (4.26).

Thus, we have proved that relations (4.25) and (4.26) are valid and hence the restriction (4.2) is satisfied for our solutions.

Thus we have proved the following proposition.

**Proposition 2** The cosmological solutions under consideration, which obey \( x = h/H \neq x_i, i = a, b, c, d \), where \( x_a = 1 \), \( x_b = -\frac{m-1}{l-2} \), \( x_c = -\frac{m-2}{l-1} \), \( x_d = -\frac{m}{T} \), are stable if i) \( x > x_d \) and unstable if ii) \( x < x_d \).

Here it should be noted that our anisotropic solutions with non-static volume factor are not defined for \( x = x_a \) and \( x = x_d \). Meanwhile, they are defined when \( x = x_b \) or \( x = x_c \), if \( x \neq x_d \). The stability analysis of these special solutions can not be covered by the equations for perturbations (4.13), (4.14) in the linear approximation. As it was pointed out in ref. [26] this analysis needs a special consideration.

Now we consider the number of non-special stable solutions which are given by Proposition 2 (see item i). We denote this number as \( n_+ (\Lambda, \alpha) \). By using the results from the previous section (e.g. illustrated by figures) we obtain for \( \alpha > 0 \):

\[
(1) \quad (1_0) \quad m \geq 2l \\
\]

\[ n_+ (\Lambda, \alpha) = \begin{cases} 
0, & \Lambda \alpha \geq \lambda_d, \\
1, & \Lambda \alpha < \lambda_d;
\end{cases} \]  
(4.33)
(2) \( \frac{l}{2} < m < 2l \)

\[
n_+(\Lambda, \alpha) = \begin{cases} 
0, & \Lambda\alpha \geq \lambda_c, \\
2, & \lambda_d < \Lambda\alpha < \lambda_c, \\
1, & \Lambda\alpha \leq \lambda_d; 
\end{cases} \tag{4.34}
\]

(3) \( m < \frac{l}{2} \)

\[
n_+(\Lambda, \alpha) = \begin{cases} 
0, & \Lambda\alpha \geq \lambda_c, \\
2, & \lambda_d \leq \Lambda\alpha < \lambda_c, \\
3, & \lambda_b < \Lambda\alpha < \lambda_d, \\
1, & \Lambda\alpha \leq \lambda_b; 
\end{cases} \tag{4.35}
\]

(30) \( m = \frac{l}{2} \)

\[
n_+(\Lambda, \alpha) = \begin{cases} 
0, & \Lambda\alpha \geq \lambda_c, \\
2, & \lambda_d < \Lambda\alpha < \lambda_c, \\
1, & \Lambda\alpha \leq \lambda_d. 
\end{cases} \tag{4.36}
\]

We see, that for \( \alpha > 0 \) and small enough value of \( \Lambda \) there exists at least one stable solution with \( x \in (x_-, x_+) \).

**Bounds on \( \Lambda\alpha \) for stable solutions with \( \alpha > 0 \).** Summarizing all cases presented above we find that for \( \alpha > 0 \) stable exact solutions under consideration exist if and only if

\[
\Lambda\alpha < \begin{cases} 
\lambda_d, & \text{for } m \geq 2l, \\
\lambda_c, & \text{for } m < 2l, 
\end{cases} \tag{4.37}
\]

where \( \lambda_c = \lambda_c(m, l) \) and \( \lambda_d = \lambda_d(m, l) \) are defined in (3.61) and (3.62), respectively.

For \( m = 3 \) and \( l > 2 \) we are led to relation \( \Lambda\alpha < \lambda_c \) instead of (1.1).

In the case \( \alpha < 0 \) we obtain

\[
n_+(\Lambda, \alpha) = \begin{cases} 
1, & \Lambda|\alpha| \geq |\lambda_\infty|, \\
2, & |\lambda_a| < \Lambda|\alpha| < |\lambda_\infty|, \\
0, & \Lambda|\alpha| \leq |\lambda_a|. 
\end{cases} \tag{4.38}
\]

Here the inequality \( x \neq x_a = 1 \) was used. Our analysis tells us that for \( \alpha < 0 \) and big enough value of \( \Lambda \) there exists at least one stable solution governed by \( x \) which obey \( x > x_+ \). We also find that the solution with \( x < x_- \) is unstable.

**Bounds on \( \Lambda|\alpha| \) for stable solutions with \( \alpha < 0 \).** It follows from (4.38) that for \( \alpha < 0 \) stable exact solutions under consideration exist if and only if the relation (3.81) \( (\Lambda|\alpha| > |\lambda_a|) \) is obeyed.

### 5 Solutions describing a small enough variation of \( G \)

Here we analyze the solutions by using the restriction on variation of the effective gravitational constant \( G \), which is inversely proportional (in the Jordan frame) to the volume scale factor of the (anisotropic) internal space \([18,39,40]\) (see also references therein), i.e.

\[ \text{Springer} \]
\[ G = \text{const exp} \left[ -(m - 3) H t - l h t \right]. \] (5.1)

By using (5.1) one can get the following formula for a dimensionless parameter of temporal variation of \( G \) (\( G \)-dot):

\[ \delta \equiv \frac{\dot{G}}{G H} = -(m - 3 + l x), \quad x = h / H. \] (5.2)

Here \( H > 0 \) is the Hubble parameter.

Due to observational data, the variation of the gravitational constant is on the level of \( 10^{-13} \) per year and less. For example, one can use, as it was done in ref. [18], the following bounds on the value of the dimensionless variation of the effective gravitational constant:

\[ -0,65 \cdot 10^{-3} < \delta < 12 \cdot 10^{-3}. \] (5.3)

They come from the most stringent limitation on \( G \)-dot obtained by the set of ephemerides [41] and value of the Hubble parameter (at present) [34] when both are written with 95% confidence level [18].

When the value \( \delta \) is fixed we get from (5.2)

\[ x = x_0(\delta) = x_0(\delta, m, l) \equiv - \frac{(m - 3 + \delta)}{l}. \] (5.4)

We remind (the reader) that our solutions are defined if

\[ P(x_0(\delta, m, l), m, l) \neq 0, \] (5.5)

or, if

\[ x_0(\delta, m, l) \neq x_{\pm}(m, l). \] (5.6)

The substitution of \( x = x_0(\delta, m, l) \) into quadratic polynomial (3.7) gives us

\[ P(x_0(\delta, m, l), m, l) = P(x_0(0, m, l), m, l) - 4 \frac{(l - 1)(m + l - 3)}{l^2} \delta + \frac{(l - 1)(l - 2)}{l^2} \delta^2, \] (5.7)

where (see [24])

\[ P(x_0(0, m, l), m, l) \equiv P_0(m, l) = \frac{1}{l^2} (m + l - 3)(5 - m)(l + 2m - 6). \] (5.8)

We note that equation \( P_0(m, l) = 0 \) implies relation \( l = l_0(m) = \frac{2m - 6}{m - 5} = 2 + \frac{4}{m - 5}, \) \( m \neq 5. \) For \( m > 9 \) we get \( 2 < l_0(m) < 3, \) that means that integer solutions are absent in this interval. For \( 3 \leq m \leq 9 \) and \( m \neq 5, \) the only integer values of \( l_0(m) > 2 \) takes place for \( m = 6, 7, 9 \) and we get a special set of pairs \((m, l)\):

\[ A) \quad (m, l) = (6, 6), (7, 4), (9, 3), \] (5.9)
which was obtained in [24]. In the case A) the restriction (5.5) gives us (see 5.7) \( \delta \neq 0 \) and \(-4(l+m-3)\delta + (l-2)\delta^2 \neq 0\) for \( l > 2\) which lead us to two restrictions: \( \delta \neq 0 \) and \( \delta \neq \frac{4(l+m-3)}{l-2} \) for \((m, l) = (6, 6), (7, 4), (9, 3)\), respectively. But the second one may be omitted due to bounds (5.3).

Let us consider the second case

\[ B) \quad m = 5, \quad l > 2. \quad (5.10) \]

In this case the restriction (5.5) reads

\[ 4(l + 2) - 4(l - 1)(l + 2)\delta + (l - 1)(l - 2)\delta^2 \neq 0, \quad (5.11) \]

\( l > 2\). It may be rewritten as

\[ \delta \neq \delta_{\pm}(5, l) \equiv 2 \frac{(l + 2)}{(l - 2)} \left( 1 \pm \sqrt{\frac{l^2}{(l - 1)(l + 2)}} \right). \quad (5.12) \]

The first restriction \( \delta \neq \delta_{+}(5, l) \), \( l > 2\), may be omitted due to the bounds (5.3) since \( \delta_{+}(5, l) > 2 \frac{(l + 2)}{(l - 2)} > 2 \) for \( l > 2\). So, the only second restriction \( \delta \neq \delta_{-}(5, l) \), \( l > 2\), should be imposed. Since

\[ \delta_{-}(5, l) = \frac{2}{l - 1} \left( 1 + \frac{l^2}{(l - 1)(l + 2)} \right)^{-1} \sim 1/(l - 1) \quad (5.13) \]

as \( l \rightarrow +\infty\), this restriction forbids one value of \( \delta \) obeying the bounds (5.3) for big enough value of \( l \) (e.g., for \( l > 1000\)).

Now we consider the last case

\[ C) \quad (m, l) \text{ do not belong to cases A and B.} \quad (5.14) \]

In the case C) the restriction (5.5) reads

\[ (l + m - 3)(5m - 1)(2m - 6) - 4(l - 1)(l + m - 3)\delta + (l - 1)(l - 2)\delta^2 \neq 0, \quad (5.15) \]

\( l > 2\). It may be rewritten as

\[ \delta \neq \delta_{\pm}(m, l) \equiv 2 \frac{(l + m - 3)}{(l - 2)} \left( 1 \pm \sqrt{\frac{l^2(m - 1)}{4(l - 1)(l + m - 3)}} \right). \quad (5.16) \]

The first restriction \( \delta \neq \delta_{+}(m, l) \) (\( l > 2\)) may be omitted due to the bounds (5.3) since \( \delta_{+}(m, l) > 2 \frac{(l + m - 3)}{(l - 2)} > 2 \) for \( m > 2, l > 2\). So, the only second restriction \( \delta \neq \delta_{-}(m, l) \), should be imposed. Here another equivalent relation may be used
\[ \delta_{-}(m, l) = \frac{(5 - m)l + 2m - 6}{2(l - 1)} \left( 1 + \sqrt{\frac{l^2(m - 1)}{4(l - 1)(l + m - 3)}} \right)^{-1}. \] (5.17)

Thus, for our special values of \( \delta \) obeying the bounds (5.3) the only restriction on \( \delta \) coming from (5.6) or (5.5) are the following ones

\[
\begin{align*}
\delta &\neq 0, \quad \text{in the case A}, \quad (5.18) \\
\delta &\neq \delta_{-}(5, l), \quad \text{in the case B}, \quad (5.19) \\
\delta &\neq \delta_{-}(m, l), \quad \text{in the case C}, \quad (5.20)
\end{align*}
\]

where \( \delta_{-}(5, l) \) is defined in (5.13) and \( \delta_{-}(m, l) \) is defined in (5.17).

Now we analyse the stability of these special solutions. The main condition for stability \( x_0(\delta) > x_d \) is satisfied since

\[ x_0(\delta) - x_d = \frac{3 - \delta}{l} > 0 \] (5.21)

due to our bounds (5.3).

Other three conditions (see Proposition 2): \( x_0(\delta) \neq x_a, x_0(\delta) \neq x_b \) and \( x_0(\delta) \neq x_c \) read

\[
\begin{align*}
\delta &\neq \delta_a = -(m + l - 3), \quad (5.22) \\
\delta &\neq \delta_b = \frac{2(m + l - 3)}{l - 2}, \quad (5.23) \\
\delta &\neq \delta_c = \frac{2(m + l - 3)}{l - 1}. \quad (5.24)
\end{align*}
\]

They are satisfied due to bounds (5.3) and inequalities: \( \delta_a \leq -3, \delta_b > 2 \) and \( \delta_c > 1. \)

Thus, we have shown that all well-defined solutions under consideration, which obey restrictions (5.18), (5.19), (5.20) and the physical bounds (5.3), are stable.

6 Conclusions

We have considered the \( D \)-dimensional Einstein–Gauss–Bonnet (EGB) model with the \( \Lambda \)-term and two non-zero constants \( \alpha_1 \) and \( \alpha_2 \). By dealing with diagonal cosmological metrics, we have found for certain (fine-tuned) \( \Lambda = \Lambda(x, m, l, \alpha) \) with \( \alpha = \alpha_2/\alpha_1 \) a class of solutions with exponential time dependence of two scale factors. This exponential dependence is governed by two Hubble-like parameters \( H > 0 \) and \( h \), corresponding to submanifolds of dimensions \( m > 2 \) and \( l > 2 \), respectively, with \( D = 1 + m + l \). Here \( m > 2 \) is the dimension of the expanding subspace and \( l > 2 \) is the dimension of another one, the dimensionless parameter \( x = h/H \) satisfies the following restrictions: \( x \neq 1, x \neq x_d = -m/l \) and \( (m - 1)(m - 2) + 2(m - 1)(l - 1)x + (l - 1)(l - 2)x^2 \neq 0. \)
Any obtained solution describes an exponential expansion of 3-dimensional subspace (which may be identified with “our” space) with the Hubble parameter $H > 0$ and anisotropic behaviour of $(m - 3 + l)$-dimensional internal space: expanding in $(m - 3)$ dimensions (with Hubble-like parameter $H$) and either contracting, or expanding (with Hubble-like parameter $h$) or stable in $l$ dimensions. The solutions are governed by master equation \( \Lambda(x, m, l, \alpha) = \Lambda \), which may be solved in radicals for all values of \( \Lambda \), since it is equivalent to a polynomial equation of either fourth or third order (depending upon \( \Lambda \)). The analytical solution for \( m = l \) [33] is presented in “Appendix A”.

Here we have obtained the bounds on \( \Lambda \) which guarantee the existence of the exponential cosmological solutions under consideration:

\[
\Lambda \alpha \leq \lambda_*
\]

for \( \alpha > 0 \) and

\[
\Lambda |\alpha| > \frac{(D - 2)(D - 1)}{8(D - 4)(D - 3)}
\]

for \( \alpha < 0 \). In (6.1) we denote: \( \lambda_* = \lambda_b \) for \( m \geq l \) and \( \lambda_* = \lambda_c \) for \( m < l \), where \( \lambda_b \) and \( \lambda_c \) are defined in (3.60) and (3.61), respectively. These bounds generalize the bounds (1.1) and (1.2) for \( m = 3 \) [28]. It should be noted that the bounds (6.1) and (6.2) were obtained here without solving the equations of motion (e.g. the master equation). They were obtained by analyzing the function \( \lambda = \lambda(x, m, l) \) from (3.14) (\( \lambda = \Lambda \alpha \)), e.g. by using the “duality” identity \( \lambda(x, m, l) = \lambda(1/x, l, m) \). (The “duality” transformation \( (x, m, l) \mapsto (1/x, l, m) \) describes just a trivial interchange of factor spaces which corresponds to the replacement \( (H, h, m, l) \mapsto (h, H, l, m) \).)

Using the scheme, which was developed in ref. [22], we have proved that any of these solutions obeying the additional restrictions: \( x \neq -\frac{m-2}{l-1} \) and \( x \neq -\frac{m-1}{l-2} \), is stable (as \( t \to +\infty \)) if \( x > x_d = -m/l \) and unstable if \( x < x_d \).

We have also found that for \( \alpha > 0 \) stable exact solutions exist if and only if:

\[
\Lambda \alpha < \lambda_d
\]

for \( m \geq 2l \) and

\[
\Lambda \alpha < \lambda_c,
\]

for \( m < 2l \), where \( \lambda_d \) is defined in (3.62). For \( \alpha < 0 \) stable exact solutions exist only if the relation (6.2) is obeyed.

It was also shown that all (well-defined) solutions with small enough variation of the effective gravitational constant \( G \) (in the Jordan frame) are stable.

Here an open problem is to extend the cosmological solutions from this paper to the cosmological type solutions in the Lovelock gravitational model [42], e.g. to solutions describing cosmological and static configurations. Another problem is related to search and analysis of the solutions with three factor spaces. These and some other topics may be addressed in our separate publications.

It should be noted here that the results obtained in this paper and its possible extensions to static and other cases may be used in other areas of physics (e.g. chro-
modynamics, condensed matter etc) by applying powerful holographic methods based on AdS/CFT, dS/CFT approaches and its generalizations.

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Appendix

A The analytical solution for $m = l$

For any $m = l > 2$ the master Eq. (3.33) reads

$$Ax^4 + Bx^3 + Cx^2 + Bx + A = 0,$$

where

$$A = 8\lambda(m - 2)^2(m - 1) + m(m + 1)(m - 2),$$
$$B = 32\lambda(m - 2)(m - 1)^2 + 4m(m - 1)^2,$$
$$C = 16\lambda(m - 1)(3m^2 - 8m + 6) + 2m(m - 1)(3m - 4).$$

It may be readily solved in radicals, by using the substitution $y = x + \frac{1}{x}$ [33]. For $A \neq 0$ we obtain

$$x = \frac{1}{4A} \left( -B + \nu_1 \sqrt{E - 2B\nu_2 \sqrt{d}} + \nu_2 \sqrt{d} \right),$$

where $\nu_1 = \pm 1, \nu_2 = \pm 1$ and

$$d = 8A^2 - 4CA + B^2, \quad E = -8A^2 - 4CA + 2B^2.$$

We get

$$d = 16m^2(2m^2 - 7m + 7) - 128m(m - 1)(m - 2)(2m - 3)\lambda,$$
$$E = 1024\lambda^2(m - 2)^2(m - 1)^2(2m - 3)$$
$$+ 128\lambda(m - 2)(m - 1)m(4m - 7)$$
$$- 16m^2(2m^3 - 11m^2 + 15m - 4).$$

For $A = 0$, the solution reads

$$x = \frac{1}{2B} \left( -C \pm \sqrt{C^2 - 4B^2} \right), \quad \text{or} \quad x = 0,$$

where

$$B = -8m(m - 1), \quad C = -\frac{4m}{m - 2}(4m^2 - 10m + 7).$$
The special solution for \( m = 3 \) was considered recently in ref. [27].

### B The proof of the Lemma

Here we give the proof of the Lemma from Sect. 2. The calculations (by using Mathematica) lead us to following relations

\[
\mathcal{R}_{\pm}(m, l) = \mathcal{R}(x_{\pm}(m, l), m, l) = \frac{A(m, l) \pm B(m, l)\sqrt{\Delta(m, l)}}{C(l)} \tag{B.11}
\]

where

\[
A(m, l) = -2(m - 1)(l + m - 3)A_{\ast}(m, l),
\]

\[
A_{\ast}(m, l) = l^2 m^2 + 4l^2 m - 4m^2 + l^3 m - 4l^2 m - 8lm + 8m - 2l^3 + 8l^2 - 4l,
\]

\[
B(m, l) = 8l(m - 1)^2(l + m - 3) > 0,
\]

\[
\Delta(m, l) = (m - 1)(l - 1)(l + m - 3) > 0,
\]

\[
C(l) = (l - 2)^3(l - 1) > 0.
\]

In order to prove \( \mathcal{R}_{-}(m, l) < 0 \) it is sufficient to prove that \( A_{\ast}(m, l) > 0 \) for \( m > 2 \) and \( l > 2 \).

Let \( m \geq 4 \). Then we group \( A_{\ast}(m, l) \) as the sum of the non-negative terms:

\[
A_{\ast}(m, l) = (l^2 m^2 - 4l^2 m)_1 + (4lm^2 - 4m^2 - 8lm)_2 + (l^3 m - 2l^2 m)_3 + (8m)_4 + (8l^2 - 4l)_5;
\]

where

\[
(\cdot)_1 = l^2 m^2 - 4l^2 m = l^2 m(m - 4) \geq 0,
\]

\[
(\cdot)_2 = 4lm^2 - 4m^2 - 8lm = 2(l - 2)m^2 + 2lm(m - 4) > 0,
\]

\[
(\cdot)_3 = l^3 (m - 2) > 0,
\]

\[
(\cdot)_4 > 0,
\]

\[
(\cdot)_5 = 4l(2l - 1) > 0.
\]

Thus, we get \( A_{\ast}(m, l) > 0 \) for \( m \geq 4 \) and \( l > 2 \). For \( m = 3 \) we have \( A_{\ast}(3, l) = l^3 + 5l^2 + 8l - 12 \geq 84 \) (as \( l \geq 3 \)). Thus, \( \mathcal{R}_{-}(m, l) < 0 \) \((m > 2, l > 2)\) is proved.

Now we prove \( \mathcal{R}_{+}(m, l) < 0 \) \((m > 2, l > 2)\). By using the identities (3.18), (3.25) and definitions of \( \mathcal{R}_{\pm}(m, l) \) we obtain

\[
\mathcal{R}_{+}(m, l) = \mathcal{R}(x_{+}(m, l), m, l) = (x_{+}(m, l))^4\mathcal{R}\left(\frac{1}{x_{+}(m, l)}, l, m\right)
\]

\[
= (x_{+}(m, l))^4\mathcal{R}(x_{-}(l, m), l, m) = (x_{+}(m, l))^4\mathcal{R}_{-}(l, m) < 0.
\]

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By this we complete the proof of the Lemma.

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