Moderate deviation principle for the two-dimensional stochastic Navier-Stokes equations with anisotropic viscosity *

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Abstract

In this paper, we prove a central limit theorem and establish a moderate deviation principle for the two-dimensional stochastic Navier-Stokes equations with anisotropic viscosity. The proof for moderate deviation principle is based on the weak convergence approach.

Keywords: Stochastic Navier-Stokes equations; Anisotropic viscosity; Central limit theorem; Moderate deviation principle; Weak convergence approach

1 Introduction

The main aim of this work is to establish central limit theorem and moderate deviation principle for the stochastic Navier-Stokes equation with anisotropic viscosity. We consider the following stochastic Navier-Stokes equation with anisotropic viscosity on the two dimensional (2D) torus $T^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2$:

$$\begin{align*}
du &= \partial_t^2 u dt - u \cdot \nabla u dt + \sigma(t, u) dW(t) - \nabla p dt, \\
div u &= 0, \\
u(0) &= u_0,
\end{align*}\tag{1}$$

where $u(t, x)$ denotes the velocity field at time $t \in [0, T]$ and position $x \in T^2$, $p$ denotes the pressure field, $\sigma$ is the random external force and $W$ is an $l^2$-cylindrical Wiener process.

Let’s first recall the classical Navier-Stokes (N-S) equation which is given by

$$\begin{align*}
du &= \nu \Delta u dt - u \cdot \nabla u dt - \nabla p dt, \\
div u &= 0, \\
u(0) &= u_0,
\end{align*}\tag{2}$$

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where $\nu > 0$ is the viscosity of the fluid. (2) describes the time evolution of an incompressible fluid. In 1934, J. Leray proved global existence of finite energy weak solutions for the deterministic case in the whole space $\mathbb{R}^d$ for $d = 2, 3$ in the seminar paper [Ler33]. For more results on deterministic N-S equation, we refer to [CKN82], [Tem79], [Tem95], [K101] and reference therein. For the stochastic case, there exists a great amount of literature too. The existence and uniqueness of solutions and ergodicity property to the stochastic 2D Navier-Stokes equation have been obtained (see e.g. [FG95], [MR05], [HM06]). Large deviation principles for the two-dimensional stochastic N-S equations have been established in [CM10] and [SS06]. Moderate deviation principles for the two-dimensional stochastic N-S equations have been established in [WZZ15].

Compared to (2), (1) only has partial dissipation, which can be viewed as an intermediate equation between N-S equation and Euler equation. System of this type appear in in geophysical fluids (see for instance [CDGG06] and [Ped79]). Instead of putting the classical viscosity $-\nu \Delta$ in (2), meteorologist often modelize turbulent diffusion by putting a viscosity of the form: $-\nu_h \Delta_h - \nu_3 \partial^2_x$, where $\nu_h$ and $\nu_3$ are empiric constants, and $\nu_3$ is usually much smaller than $\nu_h$. We refer to the book of J. Pedlovsky [Ped79, Chapter 4] for a more complete discussion. For the 3 dimensional case there is no result concerning global existence of weak solutions.

In the 2D case, [LZZ18] investigates both the deterministic system and the stochastic system (1) for $H^{0,1}$ initial value (For the definition of space see Section 2). The main difference in obtaining the global well-posedness for (1) is that the $L^2$-norm estimate is not enough to establish $L^2([0, T], L^2)$ strong convergence due to lack of compactness in the second direction. In [LZZ18], the proof is based on an additional $H^{0,1}$-norm estimate. In this paper, we want to investigate deviations of stochastic Navier-Stokes equations from the deterministic case.

The large deviation theory concerns the asymptotic behavior of a family of random variables $X_\varepsilon$ and we refer to the monographs [DPZ09] and [Str84] for many historical remarks and extensive references. It asserts that for some tail or extreme event $A$, $P(X_\varepsilon \in A)$ converges to zero exponentially fast as $\varepsilon \to 0$ and the exact rate of convergence is given by the so-called rate function. The large deviation principle was first established by Varadhan in [Var66] and he also studied the small time asymptotics of finite dimensional diffusion processes in [Var67]. Since then, many important results concerning the large deviation principle have been established. For results on the large deviation principle for stochastic differential equations in finite dimensional case we refer to [FW84]. For the extensions to infinite dimensional diffusions or SPDE, we refer the readers to [BDM08], [CM10], [DM09], [Liu09], [LRZ13], [RZ08], [XZ09], [Zha00] and the references therein.

Moderate deviation is the theory filling in the gap between the central limit theorem and the large deviation principle (see Section 2). Moderate deviation estimates arise in the theory of statistical inference. It can provide us with the rate of convergence and a useful method for constructing asymptotic confidence intervals, see [Erm12], [GZ11], [K103], [Kal83] and references therein. For the study of MDP for general Markov process see [Lim95]. Results of MDP for stochastic partial differential equations have been obtained in [WZ14], [BDG16], [DXZZ17] and references therein.

For $\varepsilon > 0$, consider the equation:

$$
\begin{align*}
    du^\varepsilon(t) &= \partial_1^2 u^\varepsilon(t)dt - B(u^\varepsilon(t))dt + \sqrt{\varepsilon}\sigma(t, u^\varepsilon(t))dW(t), \\
    u^\varepsilon(0) &= u_0,
\end{align*}
$$

(3)
where the definition of $B$ will be given in Section 2.

As $\varepsilon \to 0$, $u^\varepsilon$ will converges to the solution to the following deterministic equation:

$$
\begin{align*}
  du^0(t) &= \partial_1^2 u^0(t)dt - B(u^0(t))dt, \\
  u^0(0) &= u_0.
\end{align*}
$$

(4)

We will investigate deviations of $u^\varepsilon$ from the deterministic solution $u^0$. That is, the asymptotic behaviour of the trajectory

$$
\frac{1}{\sqrt{\varepsilon\lambda(\varepsilon)}}(u^\varepsilon - u^0),
$$

where $\lambda(\varepsilon)$ is some deviation scale which strongly influence the behaviour.

1. The case $\lambda(\varepsilon) = \frac{1}{\sqrt{\varepsilon}}$ provides large deviation principle (LDP) estimates, which has been studied in [CZ20].

2. If $\lambda(\varepsilon) = 1$, we are in the domain of the central limit theorem (CLT). For the study of the central limit theorem for stochastic (partial) differential equation, we refer the readers to [WZZ15], [CLWY18] and [WZ14]. We will show that $\frac{u^\varepsilon - u^0}{\sqrt{\varepsilon}}$ converges to a solution of a stochastic equation as $\varepsilon \to 0$ in Section 3.

3. To fill in the gap between the CLT and LDP, we will study the so-called moderate deviation principle (MDP). The moderate deviation principle refines the estimates obtained through the central limit theorem. It provides the asymptotic behaviour for $P(\|u^\varepsilon - u^0\| \geq \delta \sqrt{\varepsilon})$ while CLT gives bounds for $P(\|u^\varepsilon - u^0\| \geq \delta \sqrt{\varepsilon})$. Throughout this paper we may assume

$$
\lambda(\varepsilon) \to \infty, \quad \sqrt{\varepsilon\lambda(\varepsilon)} \to 0 \text{ as } \varepsilon \to 0.
$$

We study the moderate deviations by using the weak convergence approach. This approach is mainly based on a variational representation formula for certain functionals of infinite dimensional Brownian Motion, which was establishment by Budhiraja and Dupuis in [BD00]. The main advantage of the weak convergence approach is that one can avoid some exponential probability estimates, which might be very difficult to derive for many infinite dimensional models. To use the weak convergence approach, we need to prove two conditions in Hypothesis [4.2]. We will use the argument in [WZZ15], in which the authors first establish the convergence in $L^2([0,T], L^2)$ and then by using this and Itô’s formula to obtain $L^\infty([0,T], L^2) \cap L^2([0,T], H^1)$ convergence. As mentioned above, due to the lack of compactness in the second direction, we need to do $H^{0,1}$ estimate for the skeleton equation (13), which requires $H^{0,2}$ estimates of solution to the deterministic equation (11). To obtain this, we use a commutator estimate (see Lemma [A,3]) from [CDGG00]. This also leads to $H^{0,2}$ condition for the initial value.

Organization of the paper

In Section 2, we introduce the basic notation, definition and recall some preliminary results. In Section 3, we will build the central limit theorem. In Section 4, we prove the moderate deviation principle for the the two-dimensional stochastic Navier-Stokes equations with anisotropic viscosity.

## 2 Preliminary

**Function spaces on $\mathbb{T}^2$**
We first recall some definitions of function spaces for the two dimensional torus $\mathbb{T}^2$.

Let $\mathbb{T}^2 = \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}/2\pi\mathbb{Z} = (\mathbb{T}_h, \mathbb{T}_v)$ where $h$ stands for the horizontal variable $x_1$ and $v$ stands for the vertical variable $x_2$. For exponents $p, q \in [1, \infty)$, we denote the space $L^p(\mathbb{T}_h, L^q(\mathbb{T}_v))$ by $L^p_h(L^q_v)$, which is endowed with the norm

$$
\|u\|_{L^p_h(L^q_v)} := \left\{ \int_{\mathbb{T}_h} \left( \int_{\mathbb{T}_v} |u(x_1, x_2)|^q dx_2 \right)^{\frac{p}{q}} dx_1 \right\}^{\frac{1}{p}}.
$$

Similar notation for $L^p_v(L^q_h)$. In the case $p, q = \infty$, we denote $L^\infty$ the essential supremum norm. Throughout the paper, we denote various positive constants by the same letter $C$.

For $u \in L^2(\mathbb{T}^2)$, we consider the Fourier expansion of $u$:

$$
\hat{u}(k) = \frac{1}{(2\pi)^2} \int_{[0,2\pi]^2} u(x)e^{-ik\cdot x} \, dx,
$$

where $\hat{u}(k) := \frac{1}{(2\pi)^2} \int_{[0,2\pi]^2} u(x)e^{-ik\cdot x} \, dx$ denotes the Fourier coefficient of $u$ on $\mathbb{T}^2$.

Define the Sobolev norm:

$$
\|u\|_{H^s}^2 := \sum_{k \in \mathbb{Z}^2} (1 + |k|^2)^s |\hat{u}_k|^2,
$$

and the anisotropic Sobolev norm:

$$
\|u\|_{H^{s,s'}}^2 := \sum_{k \in \mathbb{Z}^2} (1 + |k_1|^2)^s (1 + |k_2|^2)^{s'} |\hat{u}_k|^2,
$$

where $k = (k_1, k_2)$. We define the Sobolev spaces $H^s(\mathbb{T}^2)$, $H^{s,s'}(\mathbb{T}^2)$ as the completion of $C^\infty(\mathbb{T}^2)$ with the norms $\|\cdot\|_{H^s}$, $\|\cdot\|_{H^{s,s'}}$ respectively. The notation $L^p_v(H^s_h)$ is given by

$$
\|u\|_{L^p_v(H^s_h)} := \left( \int_{\mathbb{T}_v} \|u(\cdot, x_2)\|_{H^s(\mathbb{T}_h)}^p \, dx_2 \right)^{\frac{1}{p}}.
$$

Let us recall the definition of anisotropic dyadic decomposition of the Fourier space, which will lead to another representation of $H^{s,s'}$ in the sense of Besov space. For a general introduction to the theory of Besov space we refer to [BCD11], [Tri78], [Tri06].

Let $\chi, \theta \in \mathcal{D}$ be nonnegative radial functions on $\mathbb{R}$, such that

i. the support of $\chi$ is contained in a ball and the support of $\theta$ is contained in an annulus;

ii. $\chi(z) + \sum_{j \geq 0} \theta(2^{-j}z) = 1$ for all $z \in \mathbb{R}$.

iii. $\text{supp}(\chi) \cap \text{supp}(\theta(2^{-j} \cdot)) = \emptyset$ for $j \geq 1$ and $\text{supp}(\theta(2^{-j} \cdot)) \cap \text{supp}(\theta(2^{-j} \cdot)) = \emptyset$ for $|i - j| > 1$.

We call such $(\chi, \theta)$ dyadic partition of unity. The Littlewood-Paley blocks in the vertical variable are now defined as $u = \sum_{j \geq -1} \Delta^v_j u$, where

$$
\Delta^v_{-1} u = \mathcal{F}^{-1}(\chi(|k_2|)\hat{u}) \quad \Delta^v_j u = \mathcal{F}^{-1}(\theta(2^{-j}|k_2|)\hat{u}), \quad k_2 \in \mathbb{Z},
$$

where $\mathcal{F}^{-1}$ is the inverse Fourier transform. The anisotropic Sobolev norm can also be defined as follows:

$$
\|u\|_{H^{s,s'}} = \left( \sum_{j \geq -1} 2^{2js'} \|\Delta^v_j u\|_{L^2_v(H^s(\mathbb{T}^2))}^2 \right)^{\frac{1}{2}}.
$$
To formulate the stochastic Navier-Stokes equations with anisotropic viscosity, we need the following spaces:

\[ H := \{ u \in L^2(\mathbb{T}^2; \mathbb{R}^2); \text{div} \ u = 0 \}, \]
\[ V := \{ u \in H^1(\mathbb{T}^2; \mathbb{R}^2); \text{div} \ u = 0 \}, \]
\[ \tilde{H}^{s,s'} := \{ u \in H^{s,s'}(\mathbb{T}^2; \mathbb{R}^2); \text{div} \ u = 0 \}. \]

Moreover, we use \( \langle \cdot, \cdot \rangle \) to denote the scalar product (which is also the inner product of \( L^2 \) and \( H \))

\[ \langle u, v \rangle = \sum_{j=1}^{2} \int_{\mathbb{T}^2} u^j(x)v^j(x)dx \]

and \( \langle \cdot, \cdot \rangle_X \) to denote the inner product of Hilbert space \( X \) where \( X = l^2, V \) or \( \tilde{H}^{s,s'} \).

Due to the divergence free condition, we need the Larey projection operator \( P_H : L^2(\mathbb{T}^2) \to H \):\]
\[ P_H : u \mapsto u - \nabla \Delta^{-1}(\text{div} \ u). \]

By applying the operator \( P_H \) to (1) we can rewrite the equation in the following form:

\[ du(t) = \partial_t u(t)dt - B(u(t))dt + \sigma(t,u(t))dW(t), \]
\[ u(0) = u_0, \quad (5) \]

where the nonlinear operator \( B(u,v) = P_H(u \cdot \nabla v) \) with the notation \( B(u) = B(u,u) \). Here we use the same symbol \( \sigma \) after projection for simplicity.

For \( u, v, w \in V \), define
\[ b(u,v,w) := \langle B(u,v), w \rangle. \]

We have \( b(u,v,w) = -b(u,w,v) \) and \( b(u,v,v) = 0 \).

We put some estimates of \( b \) in the Appendix.

**Large deviation principle**

We recall the definition of the large deviation principle. For a general introduction to the theory we refer to [DPZ09], [DZ10].

**Definition 2.1 (Large deviation principle).** Given a family of probability measures \( \{ \mu_\varepsilon \}_{\varepsilon > 0} \) on a metric space \((E, \rho)\) and a lower semicontinuous function \( I : E \to [0, \infty] \) not identically equal to \( +\infty \). The family \( \{ \mu_\varepsilon \} \) is said to satisfy the large deviation principle (LDP) with respect to the rate function \( I \) if

(\( U \) for all closed sets \( F \subset E \) we have

\[ \limsup_{\varepsilon \to 0} \varepsilon \log \mu_\varepsilon(F) \leq -\inf_{x \in F} I(x), \]

(\( L \) for all open sets \( G \subset E \) we have

\[ \liminf_{\varepsilon \to 0} \varepsilon \log \mu_\varepsilon(G) \geq -\inf_{x \in G} I(x). \]

A family of random variable is said to satisfy large deviation principle if the law of these random variables satisfy large deviation principle.

Moreover, \( I \) is a good rate function if its level sets \( I_r := \{ x \in E : I(x) \leq r \} \) are compact for arbitrary \( r \in (0, +\infty) \).
Definition 2.2 (Laplace principle). A sequence of random variables \( \{X^\varepsilon \} \) is said to satisfy the Laplace principle with rate function \( I \) if for each bounded continuous real-valued function \( h \) defined on \( E \)

\[
\lim_{\varepsilon \to 0} \varepsilon \log E \left[ e^{-\frac{1}{\varepsilon} h(X^\varepsilon)} \right] = - \inf_{x \in E} \{ h(x) + I(x) \}.
\]

Given a probability space \((\Omega, \mathcal{F}, P)\), the random variables \( \{Z^\varepsilon\} \) and \( \{\overline{Z}^\varepsilon\} \) which take values in \((E, \rho)\) are called exponentially equivalent if for each \( \delta > 0 \),

\[
\lim_{\varepsilon \to 0} \varepsilon \log P(\rho(Z^\varepsilon, \overline{Z}^\varepsilon) > \delta) = -\infty.
\]

Lemma 2.3 ([DZ10, Theorem 4.2.13]). If an LDP with a rate function \( I(\cdot) \) holds for the \( \{Z^\varepsilon\} \), which are exponentially equivalent to \( \{\overline{Z}^\varepsilon\} \), then the same LDP holds for \( \{\overline{Z}^\varepsilon\} \).

**Existence and uniqueness of solutions**

We introduce the precise assumptions on the diffusion coefficient \( \sigma \). Given a complete probability space \((\Omega, \mathcal{F}, P)\) with filtration \( \{\mathcal{F}_t\}_{t \geq 0} \). Let \( L_2(I^2, U) \) denotes the Hilbert-Schmidt norms from \( I^2 \) to \( U \) for a Hilbert space \( U \). We recall the following conditions for \( \sigma \) from [LZZ18]:

(i) **Growth condition**

There exists nonnegative constants \( K'_i, K_i, \tilde{K}_i \) such that for every \( t \in [0, T] \):

- (A0) \( \|\sigma(t, u)\|_{L_2(I^2, H^{-1})}^2 \leq K'_0 + K'_1 \|u\|^2_H \);
- (A1) \( \|\sigma(t, u)\|_{L_2(I^2, H)}^2 \leq K_0 + K_1 \|u\|^2_H + K_2 \|\partial_1 u\|^2_H \);
- (A2) \( \|\sigma(t, u)\|_{L_2(I^2, H^{0,1})}^2 \leq \tilde{K}_0 + \tilde{K}_1 \|u\|^2_{H^{0,1}} + \tilde{K}_2 (\|\partial_1 u\|^2_H + \|\partial_1 \partial_2 u\|^2_H) \);

(ii) **Lipschitz condition**

There exists nonnegative constants \( L_1, L_2 \) such that:

- (A3) \( \|\sigma(t, u) - \sigma(t, v)\|_{L_2(I^2, H)}^2 \leq L_1 \|u - v\|^2_H + L_2 \|\partial_1 (u - v)\|^2_H \).

The following theorem from [LZZ18] shows the well-posedness of equation [3]:

**Theorem 2.4** ([LZZ18, Theorem 4.1, Theorem 4.2]). Under the assumptions (A0), (A1), (A2) and (A3) with \( K_2 < \frac{2}{21}, \tilde{K}_2 < \frac{1}{5}, L_2 < \frac{1}{5} \), equation [2] has a unique probabilistically strong solution \( u \in L^\infty([0, T], \dot{H}^{0,1}) \cap L^2([0, T], \dot{H}^{1,1}) \cap C([0, T], H^{-1}) \) for \( u_0 \in \dot{H}^{0,1} \).

### 3 Central limit theorem

In this section, we will establish the central limit theorem. Let \( u^\varepsilon \) be the solution to [3] and \( u^0 \) the solution to [4]. Then we have the following estimates from Lemma 3.5, Lemma 4.1, Lemma 4.2 and Lemma 4.4 in [LZZ18]:

**Lemma 3.1.** Assume (A0)-(A3) hold with \( K_2 < \frac{2}{21}, \tilde{K}_2 < \frac{1}{5}, L_2 < \frac{1}{5} \), there exists \( \varepsilon_0 > 0 \) such that

\[
\sup_{\varepsilon \in (0, \varepsilon_0)} E \left( \sup_{t \in [0, T]} \|u^\varepsilon(t)\|_{\dot{H}}^2 + \int_0^T \|u^\varepsilon(s)\|_{\dot{H}^{1,0}}^2 ds \right) \leq C.
\]
Particularly,
\[
\sup_{t \in [0,T]} \| u^0(t) \|_{H^0,1}^2 + \int_0^T \| u^0(s) \|_{H^1,1}^2 \, ds \leq C.
\]

We have the following $\tilde{H}^{0,2}$ estimate for $u^0$:

**Lemma 3.2.** Given $u_0 \in \tilde{H}^{0,2}$, the unique solution $u^0$ to (4) satisfies the following estimate:

\[
\sup_{t \in [0,T]} \| u^0(t) \|_{\tilde{H}^{0,2}}^2 + \int_0^T \| u^0(t) \|_{\tilde{H}^{1,2}}^2 \, dt \leq C. \tag{6}
\]

**Proof.** Let’s start by proving a priori estimates for $u^0$. Applying the operator $\Delta_k^v$ and using an $L^2$ energy estimate, we have

\[
\frac{1}{2} \frac{d}{dt} \| u_k^0(t) \|_{H}^2 + \| \partial_1 u_k^0(t) \|_{H}^2 \leq \langle \Delta_k^v(u^0 \cdot \nabla u^0), u_k^0 \rangle,
\]

where we denote by $u_k^0$ the term $\Delta_k^v u^0$. By Lemma [A.3] with $s = 2, s_0 = 1$ and $u = v = u^0$, there exists $d_k \in l^1$ such that

\[
\frac{1}{2} \frac{d}{dt} \| u_k^0(t) \|_{H}^2 + \| \partial_1 u_k^0(t) \|_{H}^2 \leq Cd_k 2^{-4k} \left( \| u^0 \|_{\tilde{H}^{1,2}} \| u^0 \|_{\tilde{H}^{1,2}}, \| \partial_1 u^0 \|_{\tilde{H}^{0,2}} + \| u^0 \|_{\tilde{H}^{1,2}} \| \partial_1 u^0 \|_{\tilde{H}^{0,1}} \right).
\]

Now multiplying by $2^{4k}$ and taking sum over $k$ gives

\[
\frac{1}{2} \frac{d}{dt} \| u^0(t) \|_{\tilde{H}^{0,2}}^2 + \| \partial_1 u^0(t) \|_{\tilde{H}^{0,2}}^2 \leq C \left( \| u^0 \|_{\tilde{H}^{1,2}} \| u^0 \|_{\tilde{H}^{1,2}}, \| \partial_1 u^0 \|_{\tilde{H}^{0,2}} + \| u^0 \|_{\tilde{H}^{1,2}} \| \partial_1 u^0 \|_{\tilde{H}^{0,1}} \right).
\]

By interpolation inequalities (see [BCD11] Theorem 2.80) we have

\[
\| u^0 \|_{\tilde{H}^{k,s}} \leq \| u^0 \|_{\tilde{H}^{0,s}}^{\frac{4}{s}} \| u^0 \|_{\tilde{H}^{1,s}}^{\frac{1}{s}},
\]

where $s = 1, 2$. Thus we infer that

\[
\frac{1}{2} \frac{d}{dt} \| u^0(t) \|_{\tilde{H}^{0,2}}^2 + \| \partial_1 u^0(t) \|_{\tilde{H}^{0,2}}^2 \leq C \left( \| u^0 \|_{\tilde{H}^{1,2}} \| u^0 \|_{\tilde{H}^{1,2}}, \| \partial_1 u^0 \|_{\tilde{H}^{0,2}} + \| u^0 \|_{\tilde{H}^{1,2}} \| \partial_1 u^0 \|_{\tilde{H}^{0,2}} \right)
\]

\[
\leq \alpha \| \partial_1 u^0 \|_{\tilde{H}^{0,2}}^2 + C \| u^0 \|_{\tilde{H}^{1,1}} \| u^0 \|_{\tilde{H}^{1,2}} + \| u^0 \|_{\tilde{H}^{0,2}} + \| u^0 \|_{\tilde{H}^{1,2}} \| \partial_1 u^0 \|_{\tilde{H}^{0,2}}
\]

\[
\leq \alpha \| \partial_1 u^0 \|_{\tilde{H}^{0,2}}^2 + C \| u^0 \|_{\tilde{H}^{1,1}} \| u^0 \|_{\tilde{H}^{1,2}} + \| u^0 \|_{\tilde{H}^{0,2}} + \| u^0 \|_{\tilde{H}^{1,2}} \| \partial_1 u^0 \|_{\tilde{H}^{0,2}}
\]

\[
\leq \alpha \| \partial_1 u^0 \|_{\tilde{H}^{0,2}}^2 + C \| u^0 \|_{\tilde{H}^{1,1}} \| u^0 \|_{\tilde{H}^{1,2}} + \| u^0 \|_{\tilde{H}^{0,2}} + \| u^0 \|_{\tilde{H}^{1,2}} \| \partial_1 u^0 \|_{\tilde{H}^{0,2}}
\]

\[
\leq \alpha \| \partial_1 u^0 \|_{\tilde{H}^{0,2}}^2 + C \| u^0 \|_{\tilde{H}^{1,1}} \| u^0 \|_{\tilde{H}^{1,2}} + \| u^0 \|_{\tilde{H}^{0,2}} + \| u^0 \|_{\tilde{H}^{1,2}} \| \partial_1 u^0 \|_{\tilde{H}^{0,2}}
\]

\[
\leq \alpha \| \partial_1 u^0 \|_{\tilde{H}^{0,2}}^2 + C \| u^0 \|_{\tilde{H}^{1,1}} \| u^0 \|_{\tilde{H}^{1,2}} + \| u^0 \|_{\tilde{H}^{0,2}} + \| u^0 \|_{\tilde{H}^{1,2}} \| \partial_1 u^0 \|_{\tilde{H}^{0,2}}
\]

\[
\leq \alpha \| \partial_1 u^0 \|_{\tilde{H}^{0,2}}^2 + C \| u^0 \|_{\tilde{H}^{1,1}} \| u^0 \|_{\tilde{H}^{1,2}} + \| u^0 \|_{\tilde{H}^{0,2}} + \| u^0 \|_{\tilde{H}^{1,2}} \| \partial_1 u^0 \|_{\tilde{H}^{0,2}}
\]

\[
\leq \alpha \| \partial_1 u^0 \|_{\tilde{H}^{0,2}}^2 + C(1 + \| u^0 \|_{\tilde{H}^{1,1}})(1 + \| u^0 \|_{\tilde{H}^{1,2}}) \| \partial_1 u^0 \|_{\tilde{H}^{0,2}}^2.
\]
where we used Young’s inequality in the third inequality and \( \alpha < \frac{1}{2} \). Then Gronwall’s inequality implies that

\[
\sup_{t \in [0,T]} \| u^0(t) \|^2_{H^{0,2}} + \int_0^T \| \partial_1 u^0(t) \|^2_{H^{0,2}} dt \\
\leq \| u_0 \|^2_{H^{0,2}} \exp \left( C \sup_{t \in [0,T]} (1 + \| u^0(t) \|^2_{H^{0,1}}) \int_0^T (1 + \| u^0(t) \|^2_{H^{1,1}}) dt \right).
\]

Then by Lemma 3.1, we get the result.

The next proposition is about the convergence of \( u^\varepsilon \).

**Proposition 3.3.** Assume (A0)-(A3) hold with \( K_2 < \frac{2}{27}, K_2 < \frac{1}{5}, L_2 < \frac{1}{5} \), then there exists a constant \( \varepsilon_0 > 0 \) such that, for any \( \varepsilon \in (0, \varepsilon_0) \), we have

\[
E \left( \sup_{t \in [0,T]} \| u^\varepsilon(t) - u^0(t) \|^2_H + \int_0^T \| u^\varepsilon(s) - u^0(s) \|^2_{H^{1,0}} ds \right) \leq C \varepsilon. \tag{7}
\]

**Proof.** Applying Itô’s formula to \( \| u^\varepsilon(t) - u^0(t) \|^2_H \), we have

\[
\| u^\varepsilon(t) - u^0(t) \|^2_H \\
= -2 \int_0^t \| \partial_1 (u^\varepsilon - u^0)(s) \|^2_H ds - 2 \int_0^t \langle u^\varepsilon(s) - u^0(s), B(u^\varepsilon(s)) - B(u^0(s)) \rangle ds \\
+ 2 \sqrt{\varepsilon} \int_0^t \langle u^\varepsilon(s) - u^0(s), \sigma(s, u^\varepsilon(s))dW(s) \rangle + \varepsilon \int_0^t \| \sigma(s, u^\varepsilon(s)) \|^2_{L_2(\mathbb{E}, H)} ds.
\]

By Lemma A.1 we have

\[
\| (u^\varepsilon(s) - u^0(s), B(u^\varepsilon(s)) - B(u^0(s))) \| \\
= \| b(u^\varepsilon - u^0, u^\varepsilon - u^0) - b(u^0, u^\varepsilon - u^0) \| \\
= \| b(u^\varepsilon - u^0, u^\varepsilon - u^0) \| \\
\leq \frac{1}{4} \| \partial_1 (u^\varepsilon - u^0) \|^2_H + C(1 + \| u^0 \|^2_{H^{1,1}}) \| u^\varepsilon - u^0 \|^2_H.
\]

By the Burkhölder-Davis-Gundy’s inequality (see [LR15, Appendix D]), we have

\[
2 \sqrt{\varepsilon} E \left( \sup_{s \in [0,t]} \left| \int_0^t \langle u^\varepsilon(s) - u^0(s), \sigma(s, u^\varepsilon(s))dW(s) \rangle \right| \right) \\
\leq 6 \sqrt{\varepsilon} E \left( \int_0^t \| u^\varepsilon(s) - u^0(s) \|^2_H \| \sigma(s, u^\varepsilon(s)) \|^2_{L_2(\mathbb{E}, H)} ds \right)^{\frac{1}{2}} \\
\leq 6 \sqrt{\varepsilon} E \left( \sup_{s \in [0,t]} \| u^\varepsilon(s) - u^0(s) \|^2_H \int_0^t (K_0 + K_1 \| u^\varepsilon(s) \|^2_H + K_2 \| \partial_1 u^\varepsilon(s) \|^2_H ) ds \right)^{\frac{1}{2}} \\
\leq \frac{1}{2} E \left( \sup_{s \in [0,t]} \| u^\varepsilon(s) - u^0(s) \|^2_H \right) + C \varepsilon E \left( \int_0^t (1 + \| u^\varepsilon(s) \|^2_H + \| \partial_1 u^\varepsilon(s) \|^2_H ) ds \right),
\]
where we used (A1) in the last second line. Thus by above estimates and (A1) we deduce that

\[
E \left( \sup_{s \in [0,t]} \| u^\varepsilon(s) - u^0(s) \|_H^2 + \int_0^t \| u^\varepsilon(s) - u^0(s) \|_{H^{1,0}}^2 ds \right)
\]

\[\leq C \int_0^t (1 + \| u^0(s) \|_{H^{1,1}}^2)E( \sup_{l \in [0,s]} \| u^\varepsilon(l) - u^0(l) \|_H^2)ds + C\varepsilon E \left( \int_0^t (1 + \| u^\varepsilon(s) \|_H^2 + \| \partial_1 u^\varepsilon(s) \|_{H}^2)ds \right).
\]

Then Gronwall’s inequality and Lemma 3.1 imply that

\[
E \left( \sup_{s \in [0,T]} \| u^\varepsilon(s) - u^0(s) \|_H^2 + \int_0^T \| u^\varepsilon(s) - u^0(s) \|_{H^{1,0}}^2 ds \right)
\]

\[\leq C\varepsilon E \left( \int_0^T (1 + \| u^\varepsilon(s) \|_H^2 + \| \partial_1 u^\varepsilon(s) \|_{H}^2)ds \right) e^{C \int_0^T (1 + \| u^0(s) \|_{H^{1,1}}^2)ds}
\]

\[\leq C\varepsilon.
\]

\[\square
\]

Let \( V^0 \) be the solution to the following SPDE:

\[
dV^0(t) = \partial_2^2 V^0(t)dt - B(V^0(t), u^0(t))dt - B(u^0(t), V^0(t))dt + \sigma(t, u^0(t))dW(t),
\]

\[V^0(0) = 0.
\]

(8)

**Lemma 3.4.** Assume that \( u^0 \) satisfies (F). Then under the assumptions (A0), (A1), (A2), equation (8) has a unique probabilistically strong solution

\[V^0 \in L^\infty([0, T], \tilde{H}^{0,1}) \cap L^2([0, T], \tilde{H}^{1,1}) \cap C([0, T], H^{-1}).
\]

**Proof.** The proof follows a very similar Galerkin approximation argument as in [LZZ18, Section 4], so we show some key steps here.

Let \( \{e_k, k \geq 1\} \) be an orthonormal basis of \( H \) whose elements belong to \( H^2 \) and orthogonal in \( \tilde{H}^{0,1} \) and \( \tilde{H}^{1,0} \). Let \( \mathcal{H}_n = \text{span}\{e_1, \ldots, e_n\} \) and let \( P_n \) denote the orthogonal projection from \( H \) to \( \mathcal{H}_n \). For \( l^2 \)-cylindrical Wiener process \( W(t) \), let \( W_n(t) = \Pi_n W(t) := \sum_{j=1}^n \psi_j \beta_j(t), \) where \( \beta_j \) is a sequence of independent Brownian motions and \( \psi_j \) is an orthonormal basis of \( l^2 \). Set \( F : H^1 \to H^{-1} \) with \( F(u) = -B(u, u^0) - B(u^0, u) + \partial_1^2 u \).

Fix \( n \geq 1 \) and for \( v \in \mathcal{H}_n \) consider the following equation on \( \mathcal{H}_n \):

\[
d\langle V_n(t), v \rangle = \langle P_n F(V_n), v \rangle dt + \langle P_n \sigma(t, u^0(t))dW_n(t), v \rangle
\]

\[V_n(0) = P_n u_0.
\]

(9)

Then by [LR15, Theorem 3.1.1] there exists unique global strong solution \( V_n \) to (9). Moreover, \( V_n \in C([0, T], \mathcal{H}_n) \).
We first prove a priori estimates. Applying Itô’s formula to $\|V_n\|_{\tilde{H}^{0,1}}^2$, we have

$$\|V_n(t)\|_{\tilde{H}^{0,1}}^2 + 2\int_0^t \|\partial_t V_n(s)\|_{\tilde{H}^{0,1}}^2 ds = \|P_n u_0\|_{\tilde{H}^{0,1}}^2 - 2 \int_0^t \langle B(V_n, u^0), V_n \rangle_{\tilde{H}^{0,1}} ds$$

$$+ 2 \int_0^t \langle \sigma(s, u^0(s)) dW_n(s), V_n(s) \rangle_{\tilde{H}^{0,1}}$$

$$+ \int_0^t \|P_n \sigma(s, u^0(s)) \Pi_n\|_{L_2(\tilde{H}^{0,1})}^2 ds.$$

By Lemma [A.1] and Young’s inequality, we have

$$\|\langle B(V_n, u^0) + B(u^0, V_n), V_n \rangle_{\tilde{H}^{0,1}}\|$$

$$\leq |b(V_n, u^0, V_n)| + |b(\partial_2 V_n, u^0, \partial_2 V_n)| + |b(V_n, \partial_2 u^0, \partial_3 V_n)|$$

$$+ |b(\partial_2 u^0, V_n, \partial_2 V_n)|$$

$$\leq C \left( \|V_n\|_{\tilde{H}^{1,0}} \|u^0\|_{\tilde{H}^{1,1}} \|V_n\|_H + \|\partial_2 V_n\|_{\tilde{H}^{1,0}} \|u^0\|_{\tilde{H}^{1,1}} \|\partial_2 V_n\|_H\right)$$

$$+ \|V_n\|_{\tilde{H}^{1,0}} \|\partial_3 u^0\|_{\tilde{H}^{1,1}} \|\partial_2 V_n\|_H + \|\partial_2 u^0\|_{\tilde{H}^{1,0}} \|V_n\|_{\tilde{H}^{1,1}} \|\partial_2 V_n\|_H$$

$$\leq \alpha \|V_n\|_{\tilde{H}^{1,1}}^2 + C \|u^0\|_{\tilde{H}^{1,2}}^2 \|V_n\|_{\tilde{H}^{0,1}}^2,$$

where $\alpha < \frac{1}{2}$.

The growth condition and Lemma 3.1 imply that

$$\int_0^t \|P_n \sigma(s, u^0(s)) \Pi_n\|_{L_2(\tilde{H}^{0,1})}^2 ds \leq C \int_0^t (1 + \|u^0\|_{\tilde{H}^{1,1}}^2) ds \leq C.$$

Similarly, by the Burkhölder-Davis-Gundy’s inequality, we have

$$2E \left( \sup_{s \in [0,t]} \left| \int_0^t \langle \sigma(s, u^0(s)) dW_n(s), V_n(s) \rangle_{\tilde{H}^{0,1}} \right| \right)$$

$$\leq 6E \left( \int_0^t \|P_n \sigma(s, u^0(s)) \Pi_n\|_{L_2(\tilde{H}^{0,1})}^2 \|V_n(s)\|_{\tilde{H}^{0,1}}^2 ds \right)^{\frac{1}{2}}$$

$$\leq \beta E \left( \sup_{s \in [0,t]} \|V^0(s)\|_{\tilde{H}^{0,1}}^2 \right) + C \int_0^t (1 + \|u^0\|_{\tilde{H}^{1,1}}^2) ds$$

$$\leq \beta E \left( \sup_{s \in [0,t]} \|V^0(s)\|_{\tilde{H}^{0,1}}^2 \right) + C,$$

where $\beta < \frac{1}{2}$.

Then we get

$$E \left( \sup_{s \in [0,t]} \|V_n(s)\|_{\tilde{H}^{0,1}}^2 \right) + E \int_0^t \|V_n(s)\|_{\tilde{H}^{1,1}}^2 ds$$

$$\leq C + C \int_0^t (\|u^0\|_{\tilde{H}^{1,2}}^2 + 1) E \left( \sup_{r \in [0,s]} \|V_n(r)\|_{\tilde{H}^{0,1}}^2 \right) ds.$$
Then by Gronwall’s inequality and (9), we have

$$E \left( \sup_{s \in [0,t]} \|V_n(s)\|_{\dot{H}^{0,1}}^2 \right) + E \int_0^t \|V_n(s)\|_{\dot{H}^{1,1}}^2 ds \leq \exp \left( C \int_0^t \left( \|u^0\|_{\dot{H}^{0,2}}^2 + 1 \right) ds \right) \leq C. \quad (10)$$

The rest part of the existence proof is very similar as in the proof of [LZZ18, Theorem 4.1], we only need to point out that the convergence of $F(V_n)$ holds as $n \to \infty$: From the proof we could obtain that there exists another stochastic basis $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ and random variables $\tilde{V}_n$ with same law of $V_n$ such that $\tilde{V}_n \to \tilde{V}$ in $C([0, T], H^{-1}) \cap L^2([0, T], H)$, $\tilde{P}$-a.s. (in the sense of subsequence). Fix $l \in C^\infty(\mathbb{T}^2)$ with $\text{div} l = 0$. Since $F(V_n)$ is actually linear term, the convergence of $\tilde{V}_n$ in $L^2([0, T], H)$ implies that

$$\int_0^t \langle F(\tilde{V}_n), P_n l \rangle ds \to \int_0^t \langle F(\tilde{V}), l \rangle ds, \tilde{P} \text{-a.s.}$$

For uniqueness, assume $V_1^0, V_2^0$ are two solutions in $L^\infty([0, T], \dot{H}^{0,1}) \cap L^2([0, T], \dot{H}^{1,1}) \cap C([0, T], H^{-1})$ with the same initial condition, let $w = V_1 - V_2$, then $w(0) = 0$ and $w$ satisfies

$$dw(t) = \partial_t^2 w(t)dt - B(w(t), u^0(t))dt - B(u^0(t), w(t))dt.$$

Then similarly as the proof of the uniqueness for the deterministic Navier-Stokes equation with anisotropic viscosity, we know that $w = 0$.

\[\square\]

**Remark 3.5.** Note here we do not need assumption (A3) and $L^4(\Omega)$ estimate of $V_n$ since the drift term $\sigma(t, u^0)$ does not depend on $V_n$.

The main theorem of this section is the following central limit theorem.

**Theorem 3.6.** Assume (A0)-(A3) hold with $K_2 < \frac{2}{21}, \tilde{K}_2 < \frac{1}{5}, L_2 < \frac{1}{5}$, then for $u_0 \in \dot{H}^{0,2}$ we have

$$\lim_{\varepsilon \to 0} E \left( \sup_{t \in [0,T]} \left\| \frac{u^\varepsilon(t) - u^0(t)}{\sqrt{\varepsilon}} - V^0(t) \right\|_{\dot{H}}^2 + \int_0^T \left\| \frac{u^\varepsilon(t) - u^0(t)}{\sqrt{\varepsilon}} - V^0(t) \right\|_{\dot{H}^{1,0}}^2 dt \right) = 0$$

**Proof.** Let $V^\varepsilon = \frac{u^\varepsilon(t) - u^0(t)}{\sqrt{\varepsilon}}$. Then we have

$$dV^\varepsilon(t) = \partial_t^2 V^\varepsilon(t)dt - B(V^\varepsilon(t), u^\varepsilon(t))dt - B(u^0(t), V^\varepsilon(t))dt + \sigma(t, u^\varepsilon(t))dW(t),$$

$$V^\varepsilon(0) = 0,$$

and

$$d(V^\varepsilon - V^0) = \partial_t^2 (V^\varepsilon - V^0)dt - (B(V^\varepsilon, u^\varepsilon) - B(V^0, u^0))dt - B(u^0, V^\varepsilon - V^0)dt + (\sigma(t, u^\varepsilon) - \sigma(t, u^0))dW(t).$$

\[\text{11}\]
By Itô’s formula, we have

\[
\|V^\varepsilon(t) - V^0(t)\|_H^2 + 2 \int_0^t \|\partial_1(V^\varepsilon(s) - V^0(s))\|_H^2 ds
\]

\[=
- 2 \int_0^t \langle B(V^\varepsilon, u^\varepsilon) - B(V^0, u^0), V^\varepsilon - V^0 \rangle ds
\]

\[+ 2 \int_0^t \langle (\sigma(s, u^\varepsilon) - \sigma(s, u^0))dW(s), V^\varepsilon(s) - V^0(s) \rangle
\]

\[+ \int_0^t \|\sigma(s, u^\varepsilon) - \sigma(s, u^0)\|_{L_2(l^2, H)}^2 ds
\]

\[\leq 2 \int_0^t |b(V^\varepsilon - V^0, u^0, V^\varepsilon - V^0)|ds
\]

\[+ 2 \int_0^t |b(V^\varepsilon, u^\varepsilon - u^0, V^\varepsilon - V^0)|ds
\]

\[+ 2 \int_0^t \langle (\sigma(s, u^\varepsilon) - \sigma(s, u^0))dW(s), V^\varepsilon(s) - V^0(s) \rangle
\]

\[+ \int_0^t \|\sigma(s, u^\varepsilon) - \sigma(s, u^0)\|_{L_2(l^2, H)}^2 ds
\]

\[=: I_1 + I_2 + I_3 + I_4.
\]

Taking the supremum and the expectation, we obtain that

\[
E\left( \sup_{s \in [0,t]} \|V^\varepsilon(s) - V^0(s)\|_H^2 + 2 \int_0^t \|\partial_1(V^\varepsilon(s) - V^0(s))\|_H^2 ds \right)
\]

\[\leq E(I_1(t) + I_2(t) + \sup_{s \in [0,t]} I_3(s) + I_4(t)).
\]

By Lemma A.1, we have

\[
EI_1(t) \leq 2E \int_0^t \left( \frac{1}{4} \|V^\varepsilon - V^0\|_{H^{1,0}}^2 + C\|u^0\|^2_{H^{1,1}} \|V^\varepsilon - V^0\|^2_H \right) ds.
\]

By Lemma A.1, we have

\[
EI_2(t) = 2\sqrt{\varepsilon} E \int_0^t |b(V^\varepsilon, V^\varepsilon, V^\varepsilon - V^0)|ds
\]

\[= 2\sqrt{\varepsilon} E \int_0^t |b(V^\varepsilon, V^\varepsilon, V^0)|ds = 2\sqrt{\varepsilon} E \int_0^t |b(V^\varepsilon, V^0, V^\varepsilon)|ds
\]

\[\leq \sqrt{\varepsilon} C E \int_0^t (\|V^\varepsilon\|^2_{H^{1,0}} + \|V^\varepsilon\|^2_H + \|V^0\|^2_{H^{1,1}}) ds.
\]
By the Burkholder-Davis-Gundy inequality and (A3), we have
\[
E \left( \sup_{s \in [0,t]} I_3(s) \right) \leq C E \left( \int_0^t \| \sigma(s, u^\varepsilon) - \sigma(s, u^0) \|_{L^2(I, H)}^2 \| V^\varepsilon - V^0 \|_{H_1}^2 \, ds \right)^{1/2}
\]
\[
\leq C E \left( \sup_{s \in [0,t]} \| V^\varepsilon - V^0 \|_{H_1}^2 \int_0^t \| \sigma(s, u^\varepsilon) - \sigma(s, u^0) \|_{L^2(I, H)}^2 \, ds \right)^{1/2}
\]
\[
\leq \frac{1}{2} E \left( \sup_{s \in [0,t]} \| V^\varepsilon - V^0 \|_{H_1}^2 \right) + C E \left( \int_0^t \| u^\varepsilon - u^0 \|_{H_1}^2 + \| \partial_1 (u^\varepsilon - u^0) \|_{H^1}^2 \, ds \right).
\]

By (A1), we have
\[
EI_4(t) \leq C E \left( \int_0^t \| u^\varepsilon - u^0 \|_{H_1}^2 + \| \partial_1 (u^\varepsilon - u^0) \|_{H_1}^2 \, ds \right).
\]

The above estimates together with Lemma 3.3 and Lemma 3.7 below induce that
\[
E \left( \sup_{s \in [0,t]} \| V^\varepsilon(s) - V^0(s) \|_{H_1}^2 + \int_0^t \| V^\varepsilon(s) - V^0(s) \|_{H_1}^2 \, ds \right)
\]
\[
\leq C E \int_0^t \left( \| u^0(s) \|_{H_1}^2 \sup_{l \in [0,s]} \| V^\varepsilon(l) - V^0(l) \|_{H_1}^2 \right) \, ds
\]
\[
+ \sqrt{\varepsilon} C E \int_0^t \left( \| V^\varepsilon \|_{H_1}^2 \sup_{l \in [0,s]} \| V^\varepsilon \|_{H_1}^2 + \| V^0 \|_{H_1}^2 \right) \, ds
\]
\[
+ CE \left( \int_0^t \| u^\varepsilon - u^0 \|_{H_1}^2 + \| \partial_1 (u^\varepsilon - u^0) \|_{H_1}^2 \, ds \right)
\]
\[
\leq C E \int_0^t \left( 1 + \| u^0(s) \|_{H_1}^2 \right) \sup_{l \in [0,s]} \| V^\varepsilon(l) - V^0(l) \|_{H_1}^2 \right) \, ds + C(\sqrt{\varepsilon} + \varepsilon).
\]

Then by Gronwall’s inequality and Lemma 3.1 we have
\[
E \left( \sup_{s \in [0,t]} \| V^\varepsilon(s) - V^0(s) \|_{H_1}^2 + \int_0^t \| V^\varepsilon(s) - V^0(s) \|_{H_1}^2 \, ds \right)
\]
\[
\leq C(\sqrt{\varepsilon} + \varepsilon) \exp \left( C \int_0^t \left( 1 + \| u^0(s) \|_{H_1}^2 \right) \, ds \right) \leq C(\sqrt{\varepsilon} + \varepsilon).
\]

Let \( \varepsilon \to 0 \), we complete the proof. 

\[ \square \]

It remains to establish the following lemma.

**Lemma 3.7.** Assume (A0)-(A3) hold with \( K_2 < \frac{2}{21}, \bar{K}_2 < \frac{1}{3}, L_2 < \frac{1}{5} \). Let \( V^\varepsilon \) be the solution to (\textbf{11}), then there exists a constant \( \varepsilon_0 > 0 \) such that
\[
\sup_{\varepsilon \in (0, \varepsilon_0)} E \int_0^T \| V^\varepsilon(s) \|_{H_1}^2 \| V^\varepsilon(s) \|_{H_1}^2 \, ds < \infty.
\]
Proof. Applying Itô’s formula to \( \|V^\varepsilon\|^4_H \), we have

\[
d\|V^\varepsilon\|^4_H \leq 2\|V^\varepsilon\|^2_H \left( -2\|\partial_1 V^\varepsilon\|^2_H dt - 2b(V^\varepsilon, u^\varepsilon, V^\varepsilon)dt \\
+ 2\langle \sigma(t, u^\varepsilon)dW(t), V^\varepsilon \rangle + \|\sigma(t, u^\varepsilon)\|^2_{L^2(p, H)}dt \right) + 4\|\sigma(t, u^\varepsilon(t))\|^2_{L^2} dt.
\]

Taking the supremum and the expectation, we have

\[
E \left( \sup_{s \in [0,t]} \|V^\varepsilon(s)\|^4_H + 4 \int_0^t \|V^\varepsilon(s)\|^2_H \|\partial_1 V^\varepsilon(s)\|^2_H ds \right)
\leq 4E \left( \int_0^t \|V^\varepsilon(s)\|^2_H |b(V^\varepsilon(s), u^\varepsilon(s), V^\varepsilon(s))|ds \right)
+ 6E \left( \int_0^t \|V^\varepsilon(s)\|^2_H \|\sigma(s, u^\varepsilon(s))\|^2_{L^2(p, H)}ds \right)
+ 4E \left( \sup_{s \in [0,t]} \left| \int_0^t \|V^\varepsilon(s)\|^2_H \langle \sigma(s, u^\varepsilon(s))dW(s), V^\varepsilon(s) \rangle \right| \right)
=: I_1 + I_2 + I_3.
\]

Recall that \( V^\varepsilon = \frac{u^\varepsilon - u^0}{\sqrt{\varepsilon}} \). By Lemma [A.1] we have

\[
I_1(t) = 4E \left( \int_0^t \|V^\varepsilon(s)\|^2_H |b(V^\varepsilon(s), u^0(s) + \sqrt{\varepsilon}V^\varepsilon(s), V^\varepsilon(s))|ds \right)
= 4E \left( \int_0^t \|V^\varepsilon(s)\|^2_H |b(V^\varepsilon(s), u^0(s), V^\varepsilon(s))|ds \right)
\leq E \left( \int_0^t \|V^\varepsilon(s)\|^2_H (\|\partial_1 V^\varepsilon(s)\|^2_H + C(1 + \|u^0(s)\|^2_{H^{1,1}}))\|V^\varepsilon(s)\|^2_H ds \right)
\leq E \int_0^t \|V^\varepsilon(s)\|^2_H \|\partial_1 V^\varepsilon(s)\|^2_H ds + CE \left( \int_0^t (1 + \|u^0(s)\|^2_{H^{1,1}}) \sup_{l \in [0,s]} \|V^\varepsilon(l)\|^4_H ds \right).
\]

Note that Proposition [3.3] implies the boundedness of \( u^0 \) in \( L^2([0,T], H^{1,1}) \). By (A1) we have

\[
I_2(t) \leq CE \left( \int_0^t \|V^\varepsilon(s)\|^2_H (1 + \|u^\varepsilon(s)\|^2_H + \|\partial_1 u^\varepsilon(s)\|^2_H) ds \right)
\leq CE \left( \int_0^t \|V^\varepsilon(s)\|^2_H (1 + \|u^0(s)\|^2_H + \varepsilon\|V^\varepsilon(s)\|^2_H + \|\partial_1 u^0(s)\|^2_H + \varepsilon\|\partial_1 V^\varepsilon(s)\|^2_H) ds \right)
\leq C + \varepsilon CE \left( \sup_{s \in [0,t]} \|V^\varepsilon(s)\|^4_H \right) + \varepsilon CE \left( \int_0^t \|V^\varepsilon(s)\|^2_H \|\partial_1 V^\varepsilon(s)\|^2_H ds \right).
\]

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By the Burkholder-Davis-Gundy inequality, (A1) and Proposition 3.3 we have

\[
I_3(t) 
\leq CE \left( \int_0^t \left\| V^\varepsilon(s) \right\|_H^2 \left( \int_0^t \left\| V^\varepsilon(s) \right\|_H^2 (1 + \left\| u^\varepsilon(s) \right\|_H^2 + \left\| \partial_1 u^\varepsilon(s) \right\|_H^2) ds \right)^{\frac{1}{2}} \right)
\]

\[
\leq CE \left( \sup_{s \in [0,t]} \left\| V^\varepsilon(s) \right\|_H^2 \left( \int_0^t \left\| V^\varepsilon(s) \right\|_H^2 (1 + \left\| u^\varepsilon(s) \right\|_H^2 + \left\| \partial_1 u^\varepsilon(s) \right\|_H^2 + \varepsilon \left\| \partial_1 V^\varepsilon(s) \right\|_H^2) ds \right)^{\frac{1}{2}} \right)
\]

\[
\leq \frac{1}{2} E \left( \sup_{s \in [0,t]} \left\| V^\varepsilon(s) \right\|_H^4 \right) + CE \left( \int_0^t \left\| V^\varepsilon(s) \right\|_H^2 \left\| \partial_1 V^\varepsilon(s) \right\|_H^2 ds \right)
\]

Combining the above estimates, there exists constants \( C_0 \) and \( C_1 \),

\[
E \left( \frac{1}{2} - C_0 \varepsilon \right) \left( \sup_{s \in [0,t]} \left\| V^\varepsilon(s) \right\|_H^2 + (3 - C_1 \varepsilon) \int_0^t \left\| V^\varepsilon(s) \right\|_H^2 \left\| \partial_1 V^\varepsilon(s) \right\|_H^2 ds \right)
\]

\[
\leq C + CE \left( \int_0^t (1 + \left\| u^0(s) \right\|_{H^{1,1}}^2) \sup_{l \in [0,s]} \left\| V^\varepsilon(l) \right\|_H^4 ds \right).
\]

When \( \varepsilon < \varepsilon_0 := \min \{ \frac{1}{4C_0}, \frac{3}{2C_1} \} \), by Gronwall’s inequality, we have

\[
E \left( \sup_{s \in [0,t]} \left\| V^\varepsilon(s) \right\|_H^4 + \int_0^t \left\| V^\varepsilon(s) \right\|_H^2 \left\| \partial_1 V^\varepsilon(s) \right\|_H^2 ds \right) \leq C \exp \left( \int_0^t (1 + \left\| u^0(s) \right\|_{H^{1,1}}^2) ds \right).
\]

Again by Lemma 3.1 we complete the proof.

\[
\square
\]

4 Moderate deviations

In this section, we will prove that \( Z^\varepsilon := \frac{1}{\sqrt{\varepsilon \lambda(\varepsilon)}} (u^\varepsilon - u^0) \) satisfies LDP on

\[
L^\infty([0,T], H) \cap L^2([0,T], H^{1,0}) \cap C([0,T], H^{-1})
\]

if \( \lambda(\varepsilon) \) satisfies:

\[
\lambda(\varepsilon) \to \infty, \quad \sqrt{\varepsilon} \lambda(\varepsilon) \to 0 \text{ as } \varepsilon \to 0.
\]

We will use the weak convergence approach introduced by Budhiraja and Dupuis in [BD00]. The starting point is the equivalence between the large deviation principle and the Laplace principle. This result was first formulated in [Puk94] and it is essentially a consequence of Varadhan’s lemma [Var66] and Bryc’s converse theorem [Bry90].
Remark 4.1. By [DZ10] we have the equivalence between the large deviation principle and the \L{aplace principle in completely regular topological spaces. In [BD00] the authors give the weak convergence approach on a Polish space. Since the proof does not depend on the separability and the completeness, the result also holds in metric spaces.

Let \( \{W(t)\}_{t \geq 0} \) be a cylindrical Wiener process on \( l^2 \) w.r.t. a complete filtered probability space \((\Omega, \mathcal{F}, \mathcal{F}_t, P)\) (i.e. the path of \( W \) take values in \( C([0,T]; U) \), where \( U \) is another Hilbert space such that the embedding \( l^2 \subset U \) is Hilbert-Schmidt). For \( \varepsilon > 0 \), suppose \( g^\varepsilon : C([0,T], U) \rightarrow E \) is a measurable map. Let

\[
A := \left\{ v : v \text{ is } l^2\text{-valued } \mathcal{F}_t\text{-predictable process and } \int_0^T \|v(s)(\omega)\|^2 ds < \infty \text{ a.s.} \right\},
\]

\[
S_N := \left\{ \phi \in L^2([0,T], l^2) : \int_0^T \|\phi(s)\|^2 ds \leq N \right\},
\]

\[
A_N := \{ v \in A : v(\omega) \in S_N \text{ P-a.s.} \}.
\]

Here we will always refer to the weak topology on \( S_N \) in the following if we do not state it explicitly.

Now we formulate the following sufficient conditions for the \L{aplace principle of \( g^\varepsilon(W(\cdot)) \) as \( \varepsilon \rightarrow 0 \).

**Hypothesis 4.2.** There exists a measurable map \( g^0 : C([0,T], U) \rightarrow E \) such that the following two conditions hold:
1. Let \( \{v^\varepsilon : \varepsilon > 0\} \subset A_N \) for some \( N < \infty \). If \( v^\varepsilon \) converge to \( v \) in distribution as \( S_N \)-valued random elements, then

\[
g^\varepsilon \left( W(\cdot) + \frac{1}{\sqrt{\varepsilon}} \int_0^\cdot v^\varepsilon(s)ds \right) \rightarrow g^0 \left( \int_0^\cdot v(s)ds \right)
\]

in distribution as \( \varepsilon \rightarrow 0 \).
2. For each \( N < \infty \), the set

\[
K_N = \left\{ g^0 \left( \int_0^\cdot \phi(s)ds \right) : \phi \in S_N \right\}
\]

is a compact subset of \( E \).

**Lemma 4.3 ([BD00 Theorem 4.4]).** If \( u^\varepsilon = g^\varepsilon(W) \) satisfies the Hypothesis 4.2, then the family \( \{u^\varepsilon\} \) satisfies the \L{aplace principle (hence large deviation principle) on \( E \) with the good rate function \( I \) given by

\[
I(f) = \inf_{\{\phi \in L^2([0,T], l^2) : f = g^0(\int_0^\cdot \phi(s)ds)\}} \left\{ \frac{1}{2} \int_0^T \|\phi(s)\|^2 ds \right\}.
\]

(12)
Let us introduce the following skeleton equation associated to $Z^\varepsilon = \frac{1}{\sqrt{\varepsilon N(\varepsilon)}}(u^\varepsilon - u^0)$, for $\phi \in L^2([0, T], l^2)$:

$$dX^\phi(t) = \partial^2_t X^\phi(t)dt - B(X^\phi(t), u^0(t))dt - B(u^0(t), X^\phi(t))dt + \sigma(t, u^0(t))\phi(t)dt,$$

$$X^\phi(0) = 0.$$

Define $g^0 : C([0, T], U) \to L^\infty([0, T], H) \cap L^2([0, T], \tilde{H}^{1,0}) \cap C([0, T], H^{-1})$ by

$$g^0(h) := \begin{cases} X^\phi, & \text{if } h = \int_0^\varepsilon \phi(s)ds \text{ for some } \phi \in L^2([0, T], l^2); \\ 0, & \text{otherwise.} \end{cases}$$

Then the rate function can be written as

$$I(g) = \inf \left\{ \frac{1}{2} \int_0^T \|\phi(s)\|^2_{L^2} ds : g = X^\phi, \phi \in L^2([0, T], l^2) \right\},$$

where $g \in L^\infty([0, T], H) \cap L^2([0, T], \tilde{H}^{1,0}) \cap C([0, T], H^{-1})$.

The main result of this section is the following one:

**Theorem 4.4.** Assume (A0)-(A3) hold with $K_2 < \frac{2}{21}, \tilde{K}_2 < \frac{1}{5}, L_2 < \frac{1}{6}$ and $u_0 \in \tilde{H}^{0,2}$, then $Z^\varepsilon$ satisfies a large deviation principle on $L^\infty([0, T], H) \cap L^2([0, T], \tilde{H}^{1,0}) \cap C([0, T], H^{-1})$ with speed $\lambda^2(\varepsilon)$ and with the good rate function $I$ given by (13), more precisely, it holds that

$$\limsup_{\varepsilon \to 0} \frac{1}{\lambda^2(\varepsilon)} \log P \left( \frac{u^\varepsilon - u^0}{\sqrt{\varepsilon \lambda(\varepsilon)}} \in F \right) \leq -\inf_{g \in F} I(g),$$

(L) for all open sets $G \subset L^\infty([0, T], H) \cap L^2([0, T], \tilde{H}^{1,0}) \cap C([0, T], H^{-1})$ we have

$$\limsup_{\varepsilon \to 0} \frac{1}{\lambda^2(\varepsilon)} \log P \left( \frac{u^\varepsilon - u^0}{\sqrt{\varepsilon \lambda(\varepsilon)}} \in G \right) \geq -\inf_{g \in G} I(g).$$

By Lemma [13] we should check that Hypothesis [12] holds with $\varepsilon$ replaced by $\lambda^{-2}$. The proof is divided into the following lemmas. The first lemma is about the solution to (13).

**Proposition 4.5.** Assume (A0)-(A2) hold. For all $u_0 \in \tilde{H}^{0,2}$ and $\phi \in L^2([0, T], l^2)$ there exists a unique solution

$$X^\phi \in L^\infty([0, T], \tilde{H}^{0,1}) \cap L^2([0, T], \tilde{H}^{1,1}) \cap C([0, T], H^{-1})$$

to (13).

**Proof.** We start by giving a priori estimates. Using an $H^{0,1}$ energy estimate, we have

$$\frac{1}{2} \frac{d}{dt} \|X^\phi\|^2_{H^{0,1}} + ||\partial_t X^\phi||^2_{H^{0,1}} = - \langle B(X^\phi, u^0) + B(u^0, X^\phi), X^\phi \rangle_{H^{0,1}} + \langle \sigma(t, u^0(t))\phi(t), X^\phi \rangle_{H^{0,1}}.$$
The first two terms on the right hand side can be dealt by the same calculation as in the proof of Lemma 3.4. For the third term we have

\[ |\langle \sigma(t, u^0(t)) \phi(t), X^\phi \rangle|_{H^{0.1}} \leq \| \sigma(t, u^0) \|_{L^2(\mathbb{R}^2, H^{0.1})} \| \phi(t) \|_{L^2} \| X^\phi(t) \|_{H^{0.1}} \]

\[ \leq \tilde{K}_0 + \tilde{K}_1 \| u \|_{H^{0.1}}^2 + \tilde{K}_2 (\| \partial_1 u \|_H^2 + \| \partial_1 \partial_2 u \|_H^2) + C \| \phi \|_{L^2}^2 \| X^\phi \|_{H^{0.1}}^2 \]

\[ \leq C + C \| \phi \|_{L^2}^2 \| X^\phi \|_{H^{0.1}}^2, \]

where we used (A2) in the second line. Thus we deduce that

\[ \| X^\phi(t) \|_{H^{0.1}}^2 + \int_0^t \| X^\phi(s) \|_{H^{1.1}}^2 ds \]

\[ \leq C + C \int_0^t (1 + \| u^0 \|_{H^{1.2}}^2 + \| \phi \|_{L^2}^2) \| X^\phi \|_{H^{0.1}}^2 ds. \]

By Gronwall’s inequality we have

\[ \| X^\phi(t) \|_{H^{0.1}}^2 + \int_0^t \| X^\phi(s) \|_{H^{1.1}}^2 ds \]

\[ \leq C \exp \left( \int_0^t (1 + \| u^0 \|_{H^{1.2}}^2 + \| \phi \|_{L^2}^2) ds \right) \leq C, \]

where we used Lemma 3.2.

The existence results will be given by compactness arguments (see [LZZ18, Theorem 3.1]). We put them in the following for the use in the proof of next lemma.

Consider the approximate equation:

\[
\begin{aligned}
  dX^\phi(t) &= \partial_1^2 X^\phi(t) dt + \epsilon^2 \partial_2^2 X^\phi(t) dt - B(X^\phi, u^0) dt - B(u^0, X^\phi) dt + \sigma(t, u^0(t)) \phi(t) dt, \\
  X^\phi(0) &= 0.
\end{aligned}
\]  

(15)

It follows from classical theory on Navier-Stokes system that (15) has a unique global smooth solution \( z^\phi_\epsilon \) for any fixed \( \epsilon \). Furthermore, we have

\[ \| X^\phi(t) \|_{H^{0.1}}^2 + \int_0^t \| X^\phi(s) \|_{H^{1.1}}^2 ds \leq C. \]

Then \( \{ X^\phi_\epsilon \}_{\epsilon > 0} \) is uniformly bounded in \( L^\infty([0, T], H^{0.1}) \cap L^2([0, T], H^{1.1}) \), hence bounded in \( L^4([0, T], H^{1.2}) \) (by interpolation) and \( L^4([0, T], L^4(\mathbb{T}^2)) \) (by Sobolev embedding). Thus \( B(X^\phi_\epsilon, u^0) \) and \( B(u^0, X^\phi_\epsilon) \) are uniformly bounded in \( L^2([0, T], H^{-1}) \). Let \( p \in (1, \frac{4}{3}) \), we have

\[ \int_0^T \| \sigma(s, u^0(s)) \|_{H^{-1}} ds \leq \int_0^T \| \sigma(s, u^0(s)) \|_{L^2(\mathbb{R}^2, H^{-1})}^p ds \]

\[ \leq C \int_0^T (1 + \| \sigma(s, u^0(s)) \|_{L^2(\mathbb{R}^2, H^{-1})}^4 + \| \phi(s) \|_{L^2}^2) ds \]

\[ \leq C \int_0^T (1 + \| u^0(s) \|_H^4 + \| \phi(s) \|_{L^2}^2) ds \leq \infty, \]
where we used Young’s inequality in the second line and (A0) in the third line. It comes out
that

\[
\{\partial_t X_\epsilon^\varphi\}_{\epsilon > 0} \text{ is uniformly bounded in } L^p([0, T], H^{-1}).
\]

(16)

Thus by Aubin-Lions lemma (see [LZZ18, Lemma 3.6]), there exists a \( X^\varphi \in L^2([0, T], H) \) such that

\[
X_\epsilon^\varphi \to X^\varphi \text{ strongly in } L^2([0, T], H) \quad \text{as } \epsilon \to 0 \text{ (in the sense of subsequence)}. 
\]

Since \( \{X_\epsilon^\varphi\}_{\epsilon > 0} \) is uniformly bounded in \( L^\infty([0, T], \tilde{H}^{0,1}) \cap L^2([0, T], \tilde{H}^{1,1}) \), there exists a \( \tilde{X} \in L^\infty([0, T], \tilde{H}^{0,1}) \cap L^2([0, T], \tilde{H}^{1,1}) \) such that

\[
X_\epsilon^\varphi \to \tilde{X} \text{ weakly in } L^2([0, T], \tilde{H}^{1,1}) \quad \text{as } \epsilon \to 0 \text{ (in the sense of subsequence)}.
\]

\[
X_\epsilon^\varphi \to \tilde{X} \text{ weakly star in } L^\infty([0, T], \tilde{H}^{0,1}) \quad \text{as } \epsilon \to 0 \text{ (in the sense of subsequence)}.
\]

By the uniqueness of weak convergence limit, we deduce that \( X^\varphi = \tilde{X} \). By (16) and [FG95, Theorem 2.2], we also have for any \( \delta > 0 \)

\[
X_\epsilon^\varphi \to X^\varphi \text{ strongly in } C([0, T], H^{-1-\delta}) \quad \text{as } \epsilon \to 0 \text{ (in the sense of subsequence)}.
\]

Now we use the above convergence to prove that \( X^\varphi \) is a solution to (13). Note that for any \( \varphi \in C^\infty([0, T] \times \mathbb{T}^2) \) with \( \text{div} \varphi = 0 \), for any \( t \in [0, T] \), \( z_\epsilon^\varphi \) satisfies

\[
\langle X^\varphi(t), \varphi(t) \rangle = \int_0^t \langle X_\epsilon^\varphi, \partial_t \varphi \rangle - \langle \partial_1 X_\epsilon^\varphi, \partial_1 \varphi \rangle - \epsilon^2 \langle \partial_2 X_\epsilon^\varphi, \partial_2 \varphi \rangle - \langle B(X_\epsilon^\varphi, u^0) - B(u^0, X_\epsilon^\varphi) + \sigma(s, u^0) \phi, \varphi \rangle ds.
\]

(17)

Let \( \epsilon \to 0 \) in (17), we have \( X^\varphi \in L^\infty([0, T], \tilde{H}^{0,1}) \cap L^2([0, T], \tilde{H}^{1,1}) \) and

\[
\partial_t X^\varphi = \partial_1^2 X^\varphi - B(X^\varphi, u^0) - B(u^0, X^\varphi) + \sigma(t, u^0(t)) \phi.
\]

Since the right hand side belongs to \( L^p([0, T], H^{-1}) \), we deduce that

\[
X^\varphi \in L^\infty([0, T], \tilde{H}^{0,1}) \cap L^2([0, T], \tilde{H}^{1,1}) \cap C([0, T], H^{-1}).
\]

The uniqueness part is exactly the same as in Lemma 3.4.

The following Lemma shows that \( I \) is a good rate function. The proof follows essentially the same argument as in [WZZ15, Proposition 4.5].

**Lemma 4.6.** Assume (A0)-(A2) hold. For all \( N < \infty \), the set

\[
K_N = \left\{ g^0 \left( \int_0^1 \phi(s) ds \right) : \phi \in S_N \right\}
\]

is a compact subset in \( L^\infty([0, T], H) \cap L^2([0, T], \tilde{H}^{1,0}) \cap C([0, T], H^{-1}). \)
Proof. By definition, we have

\[ K_N = \left\{ X^\phi : \phi \in L^2([0, T], l^2), \quad \int_0^T \|\phi(s)\|^2_H ds \leq N \right\}. \]

Let \( \{X_{\phi_n}\} \) be a sequence in \( K_N \) where \( \{\phi_n\} \subset S_N \). Note that \( X_{\phi_n} \) is uniformly bounded in \( L^\infty([0, T], H^{1,0}) \cap L^2([0, T], H^{1,1}) \). Thus by weak compactness of \( S_N \), a similar argument as in the proof of Lemma [4.3] shows that there exists \( \phi \in S_N \) and \( X' \in L^2([0, T], H) \) such that the following convergence hold as \( n \to \infty \) (in the sense of subsequence):

\( \phi_n \to \phi \) in \( S_N \) weakly,
\( X_{\phi_n} \to X' \) in \( L^2([0, T], H^{1,0}) \) weakly,
\( X_{\phi_n} \to X' \) in \( L^\infty([0, T], H) \) weak-star,
\( X_{\phi_n} \to X' \) in \( L^2([0, T], H) \) strongly.
\( X_{\phi_n} \to X' \) in \( C([0, T], H^{-1-\delta}) \) strongly for any \( \delta > 0 \).

Then for any \( \varphi \in C^\infty([0, T] \times \mathbb{T}^2) \) with \( \text{div}\varphi = 0 \) and for any \( t \in [0, T] \), \( X_{\phi_n} \) satisfies

\[ \langle X_{\phi_n}(t), \varphi(t) \rangle = \langle u_0, \varphi(0) \rangle \]
\[ + \int_0^t \langle X_{\phi_n}, \partial_t \varphi \rangle - \langle \partial_t X_{\phi_n}, \varphi \rangle + \langle -B(X_{\phi_n}, u^0) - B(u^0, X_{\phi_n}) + \sigma(s, u^0)\phi_n, \varphi \rangle ds. \] (18)

Let \( n \to \infty \), we deduce that \( X' \) is a solution to (13). By the uniqueness of solution, we deduce that \( X' = X^\phi \).

Our goal is to prove \( X_{\phi_n} \to X^\phi \) in \( L^\infty([0, T], H) \cap L^2([0, T], \dot{H}^{1,0}) \cap C([0, T], H^{-1}) \).

Let \( w^n = X_{\phi_n} - X^\phi \), by a direct calculation, we have

\[ \|w^n(t)\|^2_{\dot{H}} + 2 \int_0^t \|\partial_w w^n(s)\|^2_{\dot{H}} ds \]
\[ = -2 \int_0^t \langle w^n(s), B(X_{\phi_n}(s) - X^\phi(s), u^0(s)) \rangle ds \]
\[ -2 \int_0^t \langle w^n(s), B(u^0(s), X_{\phi_n}(s) - X^\phi(s)) \rangle ds \]
\[ + 2 \int_0^t \langle w^n(s), \sigma(s, u^0(s))(\phi_n(s) - \phi(s)) \rangle ds \]
\[ \leq 2 \int_0^t |b(w^n, u^0, w^n)(s)| ds + 2 \int_0^t \|w^n(s), \sigma(s, u^0(s))(\phi_n(s) - \phi(s))\| ds \]
\[ \leq \int_0^t \|\partial_w w^n(s)\|^2_{\dot{H}} + C(1 + \|u^0(s)\|^2_{\dot{H}^{1,1}})\|w^n(s)\|^2_{\dot{H}} ds \]
\[ + C \int_0^t \|w^n(s)\|_{\dot{H}} \|\phi_n(s) - \phi(s)\|_{L^2} (1 + \|u^0(s)\|^2_{\dot{H}} + \|\partial_w u^0(s)\|^2_{\dot{H}})^{\frac{1}{2}} ds, \]

where we used Lemma [A.1] and (A1) in the last inequality.
Note that $\phi_n, \phi$ are in $\mathcal{S}_N$, we have
\[
\|w^n(t)\|_H^2 + \int_0^t \|\partial_1 w^n(s)\|_H^2 ds
\leq C \int_0^t (1 + \|u^0(s)\|_{H^{1.1}}^2) \|w^n(s)\|_H^2 ds
\]
\[+ C \left( \int_0^t \|w^n(s)\|_H^2 (1 + \|u^0(s)\|_H^2 + \|\partial_1 u^0(s)\|_H^2) ds \right)^{\frac{1}{2}} \left( \int_0^t \|\phi_n(s) - \phi(s)\|_2^2 \right)^{\frac{1}{2}},
\]
\[\leq C \int_0^t (1 + \|u^0(s)\|_{H^{1.1}}^2) \|w^n(s)\|_H^2 ds
\]
\[+ C \sqrt{N} \left( \int_0^t \|w^n(s)\|_H^2 (1 + \|u^0(s)\|_H^2 + \|\partial_1 u^0(s)\|_H^2) ds \right)^{\frac{1}{2}}.
\]
For any $\epsilon > 0$, let
\[A_\epsilon := \{ s \in [0, T]; \|w^n(s)\|_H > \epsilon \}.
\]
Since $X^{\phi_n} \to X^\phi$ in $L^2([0, T], H)$ strongly, we have
\[\int_0^T \|w^n(s)\|_H^2 ds \to 0, \text{ as } n \to \infty
\]
and $\lim_{n \to \infty} \text{Leb}(A_\epsilon) = 0$, where $\text{Leb}(B)$ means the Lebesgue measure of $B \in \mathcal{B}(\mathbb{R})$. Thus we have
\[\int_0^T (1 + \|u^0(s)\|_{H^{1.1}}^2) \|w^n(s)\|_H^2 ds
\]
\[\leq \left( \int_{A_\epsilon} + \int_{[0, T] \setminus A_\epsilon} \right) (1 + \|u^0(s)\|_{H^{1.1}}^2) \|w^n(s)\|_H^2 ds
\]
\[\leq C \epsilon + 2 \int_{A_\epsilon} (1 + \|u^0(s)\|_{H^{1.1}}^2)(\|X^{\phi_n}(s)\|_H^2 + \|X^\phi(s)\|_H^2) ds
\]
\[\leq C \epsilon + C \int_{A_\epsilon} (1 + \|u^0(s)\|_{H^{1.1}}^2) ds
\]
\[\to C \epsilon \text{ as } n \to \infty,
\]
where we used Lemma 3.1 in the last line. A similar argument also implies that
\[\int_0^T (1 + \|u^0(s)\|_H^2 + \|\partial_1 u^0(s)\|_H^2) \|w^n(s)\|_H^2 ds \leq C \epsilon.
\]
Hence we have
\[\sup_{t \in [0, T]} \|w^n(t)\|_H^2 + \int_0^T \|\partial_1 w^n(s)\|_H^2 ds \leq C \epsilon + C \sqrt{\epsilon} \text{ as } n \to \infty.
\]
Since $\epsilon$ is arbitrary, we obtain that
\[X^{\phi_n} \to X^\phi \text{ strongly in } L^\infty([0, T], H) \cap L^2([0, T], H^{1.0}) \cap C([0, T], H^{-1}).
\]
The next step is to check Hypothesis 1. To this end, recall that $Z^\epsilon = \frac{v^\epsilon - u^0}{\sqrt{\epsilon} \lambda^\epsilon}$, then

$$dZ^\epsilon(t) = \partial_t^2 Z^\epsilon(t) dt - B(Z^\epsilon(t), u^0(t) + \sqrt{\epsilon} \lambda^\epsilon(t)) \lambda^\epsilon(t) dt - u^0(t) + \sqrt{\epsilon} \lambda^\epsilon Z^\epsilon(t) dW(t),$$

with initial value $Z^\epsilon(0) = 0$. The uniqueness of solution to (19) is very similar to that of (5). Then it follows from Yamada-Watanabe theorem (See [LR15 Appendix E]) that there exists a Borel-measurable function

$$g^\epsilon : C([0, T], U) \to L^\infty([0, T], H) \cap L^2([0, T], \mathcal{H}^{1,0}) \cap C([0, T], H^{-1})$$

such that $Z^\epsilon = g^\epsilon(W)$ a.s.

Now consider the following equation:

$$dX^\epsilon(t) = \partial_t^2 X^\epsilon(t) dt - B(X^\epsilon(t), u^0(t) + \sqrt{\epsilon} \lambda^\epsilon(t)) v^\epsilon(t) dt - B(u^0(t), X^\epsilon(t)) dt + \sigma(t, u^0(t) + \sqrt{\epsilon} \lambda^\epsilon(t)) v^\epsilon(t) dt + \lambda^{-1}(\epsilon) \sigma(t, u^0(t) + \sqrt{\epsilon} \lambda^\epsilon(t)) dW(t),$$

where $v^\epsilon \in \mathcal{A}_N$ for some $N < \infty$. Here $X^\epsilon$ should have been denoted $X^\epsilon_{v^\epsilon}$ and the slight abuse of notation is for simplicity.

**Lemma 4.7.** Assume $(A0)$-$(A3)$ hold with $K_2 < \frac{2}{21}, \tilde{K}_2 < \frac{1}{2}, L_2 < \frac{1}{3}$ and $v^\epsilon \in \mathcal{A}_N$ for some $N < \infty$. Then $X^\epsilon = g^\epsilon(W(\cdot) + \lambda^\epsilon(\cdot) \int_0^s v^\epsilon(s) ds)$ is the unique strong solution to (20).

**Proof.** Since $v^\epsilon \in \mathcal{A}_N$, by the Girsanov theorem (see [LR15 Appendix I]), $\tilde{W}(\cdot) := W(\cdot) + \lambda^\epsilon(\cdot) \int_0^s v^\epsilon(s) ds$ is an $t^2$-cylindrical Wiener-process under the probability measure

$$d\tilde{P} := \exp \left\{-\lambda(\epsilon) \int_0^T v^\epsilon(s) dW(s) - \frac{1}{2} \lambda^2(\epsilon) \int_0^T ||v^\epsilon(s)||_2^2 ds \right\} dP.$$
By Lemma A.1 there exists constants $\tilde{\alpha} \in (0, 1)$ and $\tilde{C}$ such that

$$|b(W^\varepsilon, u^0 + \sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon, W^\varepsilon)| \leq \tilde{\alpha} \|\partial_1 W^\varepsilon\|^2_H + \tilde{C}(1 + \|u^0 + \sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon\|_{H^{1,1}}^2)\|W^\varepsilon\|^2_H.$$ 

We also have

$$2|\langle \sigma(s, u^0 + \sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon) v^\varepsilon - \sigma(s, u^0 + \sqrt{\varepsilon}\lambda(\varepsilon)\tilde{X}^\varepsilon) v^\varepsilon, W^\varepsilon\rangle|$$

$$\leq 2\|\sigma(s, u^0 + \sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon) - \sigma(s, u^0 + \sqrt{\varepsilon}\lambda(\varepsilon)\tilde{X}^\varepsilon)\|_H\|W^\varepsilon\|_H$$

$$\leq \|\sigma(s, u^0 + \sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon) - \sigma(s, u^0 + \sqrt{\varepsilon}\lambda(\varepsilon)\tilde{X}^\varepsilon)\|^2_{L_2(\mathbb{R}, H)} + \|v^\varepsilon\|^2_H\|W^\varepsilon\|^2_H.$$ 

By (A3), we have

$$\|\sigma(s, u^0 + \sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon) - \sigma(s, u^0 + \sqrt{\varepsilon}\lambda(\varepsilon)\tilde{X}^\varepsilon)\|^2_{L_2(\mathbb{R}, H)}$$

$$\leq \sqrt{\varepsilon}\lambda(\varepsilon)(L_1\|W^\varepsilon\|^2_H + L_2\|\partial_1 W^\varepsilon\|^2_H).$$ 

By the Burkholder-Davis-Gundy’s inequality (see [LR15, Appendix D]), we have

$$2\lambda^{-1}(\varepsilon)|E[\sup_{t \in [0,T]} \int_0^t e^{-q(s)}\langle W^\varepsilon(s), (\sigma(s, u^0 + \sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon) - \sigma(s, u^0 + \sqrt{\varepsilon}\lambda(\varepsilon)\tilde{X}^\varepsilon)\rangle dW(s)]]|$$

$$\leq 6\lambda^{-1}(\varepsilon)E\left(\int_0^t e^{-2q(s)}\|\sigma(s, u^0 + \sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon) - \sigma(s, u^0 + \sqrt{\varepsilon}\lambda(\varepsilon)\tilde{X}^\varepsilon)\|^2_{L_2(\mathbb{R}, H)}\|W^\varepsilon(s)\|^2_H ds\right)^{\frac{1}{2}}$$

$$\leq \sqrt{\varepsilon}E(\sup_{s \in [0,t]} (e^{-q(s)}\|W^\varepsilon(s)\|^2_H)) + 9\sqrt{\varepsilon}E\int_0^t e^{-q(s)}(L_1\|W^\varepsilon(s)\|^2_H + L_2\|\partial_1 W^\varepsilon(s)\|^2_H) ds,$$

where we used (A3). 

Let $k > 2\tilde{C}$ and we may assume $\sqrt{\varepsilon}\lambda(\varepsilon) < 1$, by (A3) we have

$$e^{-q(t)}\|W^\varepsilon(t)\|^2_H + (2 - 2\alpha - L_2\varepsilon\lambda^2(\varepsilon))\int_0^t e^{-q(s)}\|\partial_1 W^\varepsilon(s)\|^2_H ds$$

$$\leq C\int_0^t e^{-q(s)}\|W^\varepsilon(s)\|^2_H ds$$

$$+ 2\lambda^{-1}(\varepsilon)\int_0^t e^{-q(s)}\langle W^\varepsilon(s), (\sigma(s, u^0 + \sqrt{\varepsilon}\lambda(\varepsilon)X^\varepsilon) - \sigma(s, u^0 + \sqrt{\varepsilon}\lambda(\varepsilon)\tilde{X}^\varepsilon)\rangle dW(s).$$

Let $\varepsilon$ be small enough such that $1 - \sqrt{\varepsilon} - L_2\varepsilon\lambda^2(\varepsilon) - 9\sqrt{\varepsilon}L > 0$. Then we have

$$E(\sup_{s \in [0,t]} (e^{-q(s)}\|W^\varepsilon(s)\|^2_H)) \leq CE\int_0^t e^{-q(s)}\|W^\varepsilon(s)\|^2_H ds.$$ 

By Gronwall’s inequality we obtain $W^\varepsilon = 0$ $P$-a.s., i.e. $\tilde{X}^\varepsilon = X^\varepsilon$ $P$-a.s.

Then by the Yamada-Watanabe theorem, we have $X^\varepsilon$ is the unique strong solution to (20). 

\begin{proof}

\end{proof}

**Lemma 4.8.** Assume $X^\varepsilon$ is a solution to (20) with $v^\varepsilon \in \mathcal{A}_N$ and $\varepsilon < 1$ small enough. Then we have

$$E(\sup_{t \in [0,T]} \|X^\varepsilon(t)\|^4_H) + E\int_0^T (\|X^\varepsilon(s)\|^4_H + 1)\|X^\varepsilon(s)\|^2_{H^{1,2}} ds \leq C(N).$$

(21)
Moreover, there exists $k > 0$ such that

\[
E( \sup_{t \in [0,T]} e^{-kg(t)} \|X^\varepsilon(t)\|_{H_{0,1}}^2 ) + E \int_0^T e^{-kg(s)} \|X^\varepsilon(s)\|_{H_{1,1}}^2 ds \leq C(N), \tag{22}
\]

where $g(t) = \int_0^t \| \partial_1 X^\varepsilon(s) \|_{H}^2 ds$ and $C(N)$ is a constant depend on $N$ but independent of $\varepsilon$.

**Proof.** We prove (21) by two steps of estimates. For the first step, applying Itô's formula to $X^\varepsilon(t)$, we have

\[
\|X^\varepsilon(t)\|_{H}^2 + 2 \int_0^t \| \partial_1 X^\varepsilon(s) \|_{H}^2 ds
\]

\[
= -2 \int_0^t b(X^\varepsilon, u^0, X^\varepsilon) ds + 2 \int_0^t \langle X^\varepsilon(s), \sigma(s, u^0 + \sqrt{\varepsilon} \lambda(\varepsilon) X^\varepsilon(s)) v^\varepsilon(s) \rangle ds
\]

\[
+ 2\lambda^{-1}(\varepsilon) \int_0^t \langle X^\varepsilon(s), \sigma(s, u^0 + \sqrt{\varepsilon} \lambda(\varepsilon) X^\varepsilon(s)) dW(s) \rangle
\]

\[
+ \lambda^{-2}(\varepsilon) \int_0^t \| \sigma(s, u^0 + \sqrt{\varepsilon} \lambda(\varepsilon) X^\varepsilon(s)) \|_{L_2(t^2, H)}^2 ds
\]

\[
\leq \int_0^t \left( \frac{1}{2} \| \partial_1 X^\varepsilon(s) \|_{H}^2 + C(1 + \| u^0 \|_{H_{1,1}}^2) \| X^\varepsilon \|_{H}^2 \right) ds
\]

\[
+ \int_0^t (\| X^\varepsilon(s) \|_{H}^2 \| v^\varepsilon(s) \|_{H}^2 + \| \sigma(s, u^0 + \sqrt{\varepsilon} \lambda(\varepsilon) X^\varepsilon(s)) \|_{L_2(t^2, H)}^2 ) ds
\]

\[
+ 2\lambda^{-1}(\varepsilon) \int_0^t \langle X^\varepsilon(s), \sigma(s, u^0 + \sqrt{\varepsilon} \lambda(\varepsilon) X^\varepsilon(s)) dW(s) \rangle
\]

\[
+ \lambda^{-2}(\varepsilon) \int_0^t \| \sigma(s, u^0 + \sqrt{\varepsilon} \lambda(\varepsilon) X^\varepsilon(s)) \|_{L_2(t^2, H)}^2 ds
\]

\[
\leq \int_0^t \left( \frac{1}{2} \| \partial_1 X^\varepsilon(s) \|_{H}^2 + C(1 + \| u^0 \|_{H_{1,1}}^2) \| X^\varepsilon \|_{H}^2 \right) ds + \int_0^t \| X^\varepsilon(s) \|_{H}^2 \| v^\varepsilon(s) \|_{H}^2 ds
\]

\[
+ (1 + \lambda^{-2}(\varepsilon)) \int_0^t (K_0 + K_1 \| u^0 + \sqrt{\varepsilon} \lambda(\varepsilon) X^\varepsilon \|_{H}^2 + K_2 \| \partial_1 (u^0 + \sqrt{\varepsilon} \lambda(\varepsilon) X^\varepsilon) \|_{H}^2 ) ds
\]

\[
+ 2\lambda^{-1}(\varepsilon) \int_0^t \langle X^\varepsilon(s), \sigma(s, u^0 + \sqrt{\varepsilon} \lambda(\varepsilon) X^\varepsilon(s)) dW(s) \rangle,
\]

where we used (A1) in the last inequality.

Note that $v^\varepsilon \in A_N$, by Lemma [3.1] and Gronwall’s inequality,

\[
\|X^\varepsilon(t)\|_{H}^2 + \left( \frac{3}{2} - \varepsilon K_2 - \lambda^2(\varepsilon) \varepsilon K_2 \right) \int_0^t \| \partial_1 X^\varepsilon(s) \|_{H}^2 ds
\]

\[
\leq (C + 2\lambda^{-1}(\varepsilon) \int_0^t \langle X^\varepsilon(s), \sigma(s, u^0 + \sqrt{\varepsilon} \lambda(\varepsilon) X^\varepsilon(s)) dW(s) \rangle)e^{C_1(N)}.
\]

For the term on the right hand side, by the Burkholder-Davis-Gundy inequality we have
\[2\lambda^{-1}(\varepsilon)e^{C_1(N)}E\left(\sup_{0 \leq s \leq t} \left| \int_0^s (X^\varepsilon(r), \sigma(r, u^0 + \sqrt{\varepsilon}\lambda X^\varepsilon(r))dW(r)\right|\right)\]

\[\leq 6\lambda^{-1}(\varepsilon)e^{C_1(N)}E\left(\int_0^t \left\| X^\varepsilon(r) \right\|^2_H \|\sigma(r, u^0 + \sqrt{\varepsilon}\lambda X^\varepsilon(r))\|^2_{L_2(I^2, H)}ds\right)^{\frac{1}{2}}\]

\[\leq \lambda^{-1}(\varepsilon)E\left[\sup_{0 \leq s \leq t} \left(\| X^\varepsilon(s) \|^2_H \right)\right]\]

\[+ 9\lambda^{-2}(\varepsilon)e^{C_1(N)}E\int_0^t \left[ K_0 + K_1 \left\| u^0 + \sqrt{\varepsilon}\lambda X^\varepsilon(s) \right\|^2_H + K_2 \left\| \partial_1 u^0 + \sqrt{\varepsilon}\lambda X^\varepsilon(s) \right\|^2_H \right]ds,\]

where \((9\varepsilon e^{C_1(N)} + \varepsilon^2)K_2 - \frac{3}{2} < 0\) (this can be done since \(\sqrt{\varepsilon}\lambda(\varepsilon) \to 0\)) and we used (A1) in the last inequality. Thus we have

\[E\left[\sup_{s \in [0, t]} \left\| X^\varepsilon(t) \right\|^2_H \right] + E\int_0^t \left\| \partial_1 X^\varepsilon(s) \right\|^2_H ds \leq C(N) + C(N) \int_0^t E\left[\sup_{r \in [0, s]} \left(\| X^\varepsilon(r) \|^2_H \right)\right] ds.\]

Then by Gronwall’s inequality we have

\[E\left(\sup_{0 \leq t \leq T} \| X^\varepsilon(t) \|^2_H \right) + E\int_0^T \| \partial_1 X^\varepsilon(s) \|^2_H ds \leq C(N).\]  \hfill (23)

Now by Itô’s formula we have

\[\| X^\varepsilon(t) \|^4_H = - 4 \int_0^t \| X^\varepsilon(s) \|^2_H \left\| \partial_1 X^\varepsilon(s) \right\|^2_H ds - 4 \int_0^t \| X^\varepsilon(s) \|^2_H b(X^\varepsilon, u^0, X^\varepsilon) ds + 4 \int_0^t \| X^\varepsilon(s) \|^2_H \left\| \sigma(s, u^0 + \sqrt{\varepsilon}\lambda X^\varepsilon(s)) \right\|^2_{L_2(I^2, H)} ds\]

\[+ 2\lambda^{-2}(\varepsilon) \int_0^t \| X^\varepsilon(s) \|^2_H \|\sigma(s, u^0 + \sqrt{\varepsilon}\lambda X^\varepsilon(s))\|_{L_2(I^2, H)}^2 ds\]

\[+ 4\lambda^{-2}(\varepsilon) \int_0^t \left\| \sigma(s, u^0 + \sqrt{\varepsilon}\lambda X^\varepsilon(s))^{*}(X^\varepsilon) \right\|^2_{I^2} ds\]

\[+ 4\lambda^{-1}(\varepsilon) \int_0^t \left\| X^\varepsilon(s) \right\|^2_H \left\| X^\varepsilon(s), \sigma(s, u^0 + \sqrt{\varepsilon}\lambda X^\varepsilon(s))dW(s) \right\|_H \]

\[= - 4 \int_0^t \| X^\varepsilon \|^2_H \left\| \partial_1 X^\varepsilon(s) \right\|^2_H ds + I_0 + I_1 + I_2 + I_3 + I_4.\]

By Lemma A.1

\[|I_0(t)| \leq 4 \int_0^t \| X^\varepsilon \|^2_H \left(\frac{1}{4} \left\| \partial_1 X^\varepsilon \right\|^2_H + C(1 + \| u^0 \|^2_{H_{1,1}}) \left\| X^\varepsilon \right\|^2_H \right) ds.\]
By (A1) we have
\[ I_1(t) \leq 4 \int_0^t \|X^\varepsilon(s)\|^2_{L^2} \sigma(s, u^0 + \sqrt{\varepsilon\lambda}X^\varepsilon(s)) \|L_{L^2(H)}\|v^\varepsilon(s)\|_H \|X^\varepsilon(s)\|_H ds \]
\[ \leq 2 \int_0^t \|X^\varepsilon(s)\|^2_{L^2} (K_0 + K_1 \|u^0 + \sqrt{\varepsilon\lambda}X^\varepsilon(s)\|_{L^2}^2)
+ K_2 \|\partial_1(u^0 + \sqrt{\varepsilon\lambda}X^\varepsilon(s))\|_{L^2}^2 + \|v^\varepsilon(s)\|^2_{L^2} \|X^\varepsilon(s)\|_{L^2}^2 ds, \]
and
\[ I_2 + I_3 \leq 6\lambda^{-2}(\varepsilon) \int_0^t \|\sigma(s, u^0 + \sqrt{\varepsilon\lambda}X^\varepsilon(s))\|_{L^2(L^2, H)}^2 \|X^\varepsilon(s)\|_{L^2}^2 ds \]
\[ \leq 6\lambda^{-2}(\varepsilon) \int_0^t (K_0 + K_1 \|u^0 + \sqrt{\varepsilon\lambda}X^\varepsilon(s)\|_{L^2}^2)
+ K_2 \|\partial_1(u^0 + \sqrt{\varepsilon\lambda}X^\varepsilon(s))\|_{L^2}^2 \|X^\varepsilon(s)\|_{L^2}^2 ds. \]
Thus we have
\[ \|X^\varepsilon(t)\|_{L^2}^4 + (3 - 2\varepsilon\lambda^2(\varepsilon)K_2 - 6\varepsilon K_2) \int_0^t \|X^\varepsilon(s)\|_{L^2}^2 \|\partial_1X^\varepsilon(s)\|_{L^2}^2 ds \]
\[ \leq I_4 + C + C \int_0^t (1 + \|u^0(s)\|_{L^2}^2 + \|v^\varepsilon(s)\|^2_{L^2}) \|X^\varepsilon(s)\|_{L^2}^4 ds. \]
Since \(v^\varepsilon \in A_N\), by Gronwall’s inequality we have
\[ \|X^\varepsilon(t)\|_{L^2}^4 + (3 - 2\varepsilon\lambda^2(\varepsilon)K_2 - 6\varepsilon K_2) \int_0^t \|X^\varepsilon(s)\|_{L^2}^2 \|\partial_1X^\varepsilon(s)\|_{L^2}^2 ds \]
\[ \leq (I_4 + C) e^{C_2(N)}. \]
Then the Burkhoelder-Davis-Gundy inequality, the Young’s inequality and (A1) imply that
\[ E(\sup_{s \in [0,t]} I_4(s)) \leq 12\lambda^{-1}(\varepsilon)E \left( \int_0^t \|\sigma(s, u^0 + \sqrt{\varepsilon\lambda}X^\varepsilon(s))\|^2_{L^2(L^2, H)} \|X^\varepsilon(s)\|_{L^2}^6 ds \right)^{\frac{1}{2}} \]
\[ \leq \lambda^{-1}(\varepsilon)E(\sup_{s \in [0,t]} \|X^\varepsilon(s)\|_{L^2}^4) + 36\lambda^{-1}(\varepsilon)E \int_0^t (K_0 + K_1 \|u^0 + \sqrt{\varepsilon\lambda}X^\varepsilon(s)\|_{L^2}^2)
+ K_2 \|\partial_1(u^0 + \sqrt{\varepsilon\lambda}X^\varepsilon(s))\|_{L^2}^2 \|X^\varepsilon(s)\|_{L^2}^2 ds. \]
Let \(\varepsilon\) small enough such that \(3 - 2\varepsilon\lambda^2(\varepsilon)K_2 - 6\varepsilon K_2 - 36\varepsilon K_2 e^{C_2(N)} > 0\) and \(\lambda^{-1}(\varepsilon) e^{C_2(N)} < 1\). Then the above estimates and (21) imply that
\[ E(\sup_{s \in [0,t]} \|X^\varepsilon(s)\|_{L^2}^4) + \int_0^t \|X^\varepsilon(s)\|^2_{L^2} \|X^\varepsilon(s)\|^2_{L^2,1,0} ds \]
\[ \leq C(N) + C(N)E \int_0^t \|X^\varepsilon(s)\|^4_{L^2} ds, \]
which by Gronwall’s inequality yields that

\[ E(\sup_{s \in [0,t]} \|X^\varepsilon(s)\|^4_H) + \int_0^t \|X^\varepsilon(s)\|^2_H \|X^\varepsilon(s)\|^2_{H_{1,0}} ds \leq C(N). \]

For (22), let \( h(t) = kg(t) + \int_0^t \|v^\varepsilon(s)\|^2 ds \) for some universal constant \( k \). Applying Itô’s formula to \( e^{-h(t)} \|X^\varepsilon(t)\|^2_{\tilde{H}_{0,1}} \) by applying Itô’s formula to its finite-dimension projection first and then passing to the limit, we have

\[
e^{-h(t)}\|X^\varepsilon(t)\|^2_{\tilde{H}_{0,1}} + 2 \int_0^t e^{-h(s)}(\|\partial_1 X^\varepsilon(s)\|^2_H + \|\partial_1 \partial_2 X^\varepsilon(s)\|^2_H) ds \\
= - \int_0^t e^{-h(s)}(k \|\partial_1 X^\varepsilon(s)\|^2_H + \|v^\varepsilon(s)\|^2_H) \|X^\varepsilon(s)\|^2_{\tilde{H}_{0,1}} ds \\
- 2 \int_0^t e^{-h(s)} b(X^\varepsilon, u^0, X^\varepsilon) ds - 2 \int_0^t e^{-h(s)} \langle \partial_2 X^\varepsilon(s), \partial_2 (X^\varepsilon \cdot \nabla (u^0 + \sqrt{\varepsilon} \lambda(\varepsilon) X^\varepsilon))(s) \rangle ds \\
- 2 \int_0^t e^{-h(s)} \langle \partial_2 X^\varepsilon(s), \partial_2 (u^0 \cdot \nabla X^\varepsilon)(s) \rangle ds \\
+ 2 \int_0^t e^{-h(s)} \langle X^\varepsilon(s), \sigma(s, u^0 + \sqrt{\varepsilon} \lambda(\varepsilon) X^\varepsilon(s)) v^\varepsilon(s) \rangle_{\tilde{H}_{0,1}} ds \\
+ 2 \lambda^{-1}(\varepsilon) \int_0^t e^{-h(s)} \|\sigma(s, u^0 + \sqrt{\varepsilon} \lambda(\varepsilon) X^\varepsilon(s))\|^2_{L^2(\tilde{H}_{0,1})} ds.
\]

By Lemma [A.1] we have

\[ 2|b(X^\varepsilon, u^0, X^\varepsilon)| \leq \alpha \|\partial_1 X^\varepsilon\|^2_H + C(1 + \|u^0\|^2_{H_{1,1}}) \|X^\varepsilon\|^2_H, \]

where \( \alpha < \frac{1}{3} \). By Lemma [A.2] there exists \( C_1 \),

\[ 2\sqrt{\varepsilon} \lambda(\varepsilon)|\langle \partial_2 X^\varepsilon, \partial_2 (X^\varepsilon \cdot \nabla X^\varepsilon) \rangle| \leq \alpha \|\partial_1 \partial_2 X^\varepsilon\|^2_H + C_1(1 + \|\partial_1 X^\varepsilon\|^2_H) \|\partial_2 X^\varepsilon\|^2_H. \]

By Lemma [A.1] we have

\[ 2|\langle \partial_2 X^\varepsilon, \partial_2 (X^\varepsilon \cdot \nabla u^0) \rangle| \leq 2|b(\partial_2 X^\varepsilon, u^0, \partial_2 X^\varepsilon)| + 2|b(X^\varepsilon, \partial_2 u^0, \partial_2 X^\varepsilon)| \\
\leq \alpha (\|X^\varepsilon\|^2_{H_{1,0}} + \|\partial_2 X^\varepsilon\|^2_{H_{1,0}}) + C \|u^0\|^2_{H_{2,1}} \|\partial_2 X^\varepsilon\|^2_H. \]

Similarly,

\[ |\langle \partial_2 X^\varepsilon(s), \partial_2 (u^0 \cdot \nabla X^\varepsilon)(s) \rangle| = |b(\partial_2 u^0, X^\varepsilon, \partial_2 X^\varepsilon)| \leq \alpha \|X^\varepsilon\|^2_{H_{1,1}} + C \|u^0\|^2_{H_{1,1}} \|\partial_2 X^\varepsilon\|^2_H. \]

By Young’s inequality,

\[ 2|\langle X^\varepsilon(s), \sigma(s, u^0 + \sqrt{\varepsilon} \lambda(\varepsilon) X^\varepsilon) v^\varepsilon(s) \rangle_{\tilde{H}_{0,1}}| \leq \|X^\varepsilon\|^2_{H_{0,1}} \|v^\varepsilon\|^2_2 + \|\sigma(s, u^0 + \sqrt{\varepsilon} \lambda(\varepsilon) X^\varepsilon)\|^2_{L^2(\tilde{H}_{0,1})}. \]
Choosing $k > 2C_1 \sqrt{\varepsilon \lambda(\varepsilon)}$, we have

$$e^{-h(t)}\|X^\varepsilon(t)\|_{\tilde{H}^{0,1}}^2 + (2 - 3\alpha) \int_0^t e^{-h(s)}\|X^\varepsilon(s)\|_{\tilde{H}^{1,1}}^2 ds$$

$$\leq C \int_0^t e^{-h(s)}(1 + \|u^0\|_{\tilde{H}^{1,2}}^2)\|X^\varepsilon(s)\|_{\tilde{H}^{0,1}}^2 ds$$

$$+ (1 + \lambda^{-2}(\varepsilon)) \int_0^t e^{-h(s)}\|\sigma(s, u^0 + \sqrt{\varepsilon \lambda(\varepsilon)}X^\varepsilon)\|_{L_2(\tilde{H}^{0,1})}^2 ds$$

$$+ 2\lambda^{-1}(\varepsilon) \int_0^t e^{-h(s)}\langle X^\varepsilon(s), \sigma(s, u^0 + \sqrt{\varepsilon \lambda(\varepsilon)}X^\varepsilon)dW(s)\rangle_{\tilde{H}^{0,1}}.$$

By (A2) we have

$$(1 + \lambda^{-2}(\varepsilon))\|\sigma(s, u^0 + \sqrt{\varepsilon \lambda(\varepsilon)}X^\varepsilon)\|_{L_2(\tilde{H}^{0,1})}^2 \leq C(1 + \|u^0\|_{\tilde{H}^{1,1}}^2)$$

$$+ (1 + \lambda^{-2}(\varepsilon)) \left( \tilde{K}_0 + \tilde{K}_1 \varepsilon^2(\varepsilon)\|X^\varepsilon\|_{\tilde{H}^{0,1}}^2 + \tilde{K}_2 \varepsilon^3(\varepsilon)(\|\partial_1 X^\varepsilon\|_{\tilde{H}^{1}}^2 + \|\partial_1 \partial_2 X^\varepsilon\|_{\tilde{H}^{2}}^2) \right).$$

By the Burkhölder-Davis-Gundy inequality we have

$$2\lambda^{-1}(\varepsilon)E\left( \sup_{s \in [0,t]} | \int_0^s e^{-h(r)}\langle X^\varepsilon(r), \sigma(r, u^0 + \sqrt{\varepsilon \lambda(\varepsilon)}X^\varepsilon) dW(r) \rangle_{\tilde{H}^{0,1}} | \right)$$

$$\leq 6\lambda^{-1}(\varepsilon)E\left( \int_0^t e^{-2h(s)}\|X^\varepsilon(s)\|_{\tilde{H}^{0,1}}^2 \|\sigma(s, u^0 + \sqrt{\varepsilon \lambda(\varepsilon)}X^\varepsilon)\|_{L_2(\tilde{H}^{0,1})}^2 ds \right)^{\frac{1}{2}}$$

$$\leq \lambda^{-1}(\varepsilon)E[ \sup_{s \in [0,t]} (e^{-h(s)}\|X^\varepsilon(s)\|_{\tilde{H}^{0,1}}^2)] + \lambda^{-1}(\varepsilon)C \int_0^t e^{-h(s)}(1 + \|u^0\|_{\tilde{H}^{1,1}}^2) ds$$

$$+ 9\varepsilon \lambda(\varepsilon)E \int_0^t e^{-h(s)}[\tilde{K}_1 \|X^\varepsilon(s)\|_{\tilde{H}^{0,1}}^2 + \tilde{K}_2(\|\partial_1 X^\varepsilon(s)\|_{\tilde{H}^{1}}^2 + \|\partial_1 \partial_2 X^\varepsilon(s)\|_{\tilde{H}^{2}}^2)] ds,$$

where we choose $\varepsilon$ small enough such that $(9\varepsilon \lambda(\varepsilon) + \varepsilon^2(\varepsilon) + \varepsilon)\tilde{K}_2 < 1 - 3\alpha$ and we used (A2) in the last inequality.

Combine the above estimates, we have

$$E( \sup_{s \in [0,t]} e^{-h(s)}\|X^\varepsilon(s)\|_{\tilde{H}^{0,1}}^2 ) + E \int_0^t e^{-h(s)}\|X^\varepsilon(s)\|_{\tilde{H}^{1,1}}^2 ds$$

$$\leq C + CE \left( \int_0^t e^{-h(s)}(1 + \|u^0(s)\|_{\tilde{H}^{1,2}}^2)\|X^\varepsilon(s)\|_{\tilde{H}^{0,1}}^2 ds \right).$$

Then Gronwall’s inequality and (25) imply that

$$E( \sup_{0 \leq t \leq T} \|X^\varepsilon(t)\|_{\tilde{H}^{0,1}}^2 ) + E \int_0^T e^{-h(s)}\|X^\varepsilon(s)\|_{\tilde{H}^{1,1}}^2 ds \leq C.$$
Similar as [LZZ18, lemma 4.3], we have the following tightness lemma:

**Lemma 4.9.** Assume $X^\varepsilon$ is a solution to (20) with $v^\varepsilon \in \mathcal{A}_N$ and $\varepsilon$ small enough. There exists $\varepsilon_0 > 0$, such that $\{X^\varepsilon\}_{\varepsilon \in (0, \varepsilon_0)}$ is tight in the space

$$\chi = C([0, T], H^{-1}) \bigcap L^2([0, T], H) \bigcap L^2_{w}([0, T], H^{1,1}) \bigcap L^\infty_{w^*}([0, T], H^{0,1}),$$

where $L^2_w$ denotes the weak topology and $L^\infty_{w^*}$ denotes the weak star topology.

**Proof.** Note that the law of $Z_\varepsilon^i$ is defined on the path space $C([0, T], H^{-1})$. First we should point out that it can be restricted to $\chi$. We denote the space $C([0, T], H^{-1})$ by $X$ with Borel $\sigma$-algebra $\mathcal{B}(X)$.

For $N \in \mathbb{N}$, let

$$Y_N := \{w \in L^2([0, T], \tilde{H}^{1,1}) : \|w\|_{L^2([0, T], \tilde{H}^{1,1})} \leq N\},$$

equipped with the weak topology on $L^2([0, T], \tilde{H}^{1,1})$. Then $Y_N$ is compact and metrizable, hence separable and complete.

Similarly, let

$$Z_N := \{w \in L^\infty([0, T], \tilde{H}^{0,1}) : \|w\|_{L^\infty([0, T], \tilde{H}^{0,1})} \leq N\},$$

equipped with the weak star topology on $L^\infty([0, T], \tilde{H}^{0,1})$. Then $Z_N$ is compact and metrizable, hence separable and complete.

Define

$$\chi_N = C([0, T], H^{-1}) \bigcap L^2([0, T], H) \bigcap Y_N \bigcap Z_N := X_1 \cap X_2 \cap X_3 \cap X_4,$$

where $X_i$ are complete separable metric spaces with metric $d_i$, $i = 1, 2, 3, 4$. Let $\chi_N$ be equipped with the metric $d = \max\{d_1, d_2, d_3, d_4\}$. Then $\chi_N$ is separable. To show that $\chi_N$ is complete, it is enough to show that if $w_k \in \chi_N$, $k \in \mathbb{N}$ and $w_k \rightarrow w^{(i)} \in X_i$ in $d_i$ for every $1 \leq i \leq 4$, then $w^{(1)} = w^{(2)} = w^{(3)} = w^{(4)}$. This is true since obviously we have the continuous embedding

$$X_i \subset \mathcal{M}([0, T], H^{-2}), \quad 1 \leq i \leq 4,$$

where $\mathcal{M}$ denotes the space of Radon measures. Hence $(\chi_N, d)$ is a complete separable metric space. Furthermore, the following embeddings are continuous and hence measurable:

$$(\chi_N, d) \subset X.$$

Therefore by Kuratowski’s theorem we have for the Borel $\sigma$-algebra $\mathcal{B}(\chi_N)$ of $(\chi_N, d)$,

$$\chi_N \in \mathcal{B}(X), \quad \mathcal{B}(\chi_N) = \mathcal{B}(X) \cap \chi_N.$$

Consequently, $\chi = \cup \chi_N \in \mathcal{B}(X)$.

Note that $\chi_N$ is a $\tau_\chi$-closed subset of $\chi$. Let $A \subset \chi$ be $\tau_\chi$-closed. Then $A \cap \chi_N$ is $\tau_\chi$-closed too, hence

$$A \cap \chi_N \in \mathcal{B}(\chi_N) = \mathcal{B}(X) \cap \chi_N = \{B \in \mathcal{B}(X) : B \subset \chi_N\} \subset \{B \in \mathcal{B}(X) : B \subset \chi\} \subset \mathcal{B}(X) \cap \chi.$$

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Hence
\[ A = \bigcup_{N=1}^{\infty} A \cap \chi_N \in \mathcal{B}(X) \cap \chi \]
and
\[ \mathcal{B}(\tau_\chi) \subset \mathcal{B}(X) \cap \chi. \]

Since \( \chi \subset X \) continuously, hence measurably, we have \( \mathcal{B}(X) \cap \chi \subset \mathcal{B}(\tau_\chi) \). Then
\[ \mathcal{B}(\tau_\chi) = \mathcal{B}(X) \cap \chi. \]

Thus any probability measure on \( X \) can be restricted on \( \chi \).

Let \( k \) be the same constant as in the proof of (22) and let
\[
K_R := \left\{ u \in C([0, T], H^{-1}) : \sup_{t \in [0, T]} \| u(t) \|_H^2 + \int_0^T \| u(t) \|_{H^{1,0}}^2 dt + \| u \|_{C^{\frac{1}{2}}([0, T], H^{-1})}^2 \right\},
\]
where \( C^{\frac{1}{2}}([0, T], H^{-1}) \) is the Hölder space with the norm:
\[
\| f \|_{C^{\frac{1}{2}}([0, T], H^{-1})} = \sup_{0 \leq s < t \leq T} \frac{\| f(t) - f(s) \|_{H^{-1}}}{|t - s|^\frac{1}{2}}.
\]

Then from the proof of [LZZ18] Lemma 4.3, we know that for any \( R > 0 \), \( K_R \) is relatively compact in \( \chi \).

Now we only need to show that for any \( \delta > 0 \), there exists \( R > 0 \), such that \( P(X^\varepsilon \in K_R) > 1 - \delta \) for any \( \varepsilon \in (0, \varepsilon_0) \), where \( \varepsilon_0 \) is the constant such that Lemma 4.8 hold.

By Lemma 4.8 and Chebyshev inequality, we can choose \( R_0 \) large enough such that
\[
P \left( \sup_{t \in [0, T]} \| X^\varepsilon(t) \|_H^2 + \int_0^T \| X^\varepsilon(t) \|_{H^{1,0}}^2 dt > \frac{R_0}{3} \right) < \frac{\delta}{4},
\]
and
\[
P \left( \sup_{t \in [0, T]} e^{-k} f_0^t \| \partial_1 X^\varepsilon(s) \|_{H^2}^2 ds \| X^\varepsilon(t) \|_{H^{0,1}}^2 + \int_0^T e^{-k} f_0^t \| \partial_1 X^\varepsilon(s) \|_{H^2}^2 ds \| X^\varepsilon(t) \|_{H^{1,1}}^2 dt \geq \frac{R_0}{3} \right) < \frac{\delta}{4},
\]
where \( k \) is the same constant as in (22).

Fix \( R_0 \) and let
\[
\hat{K}_{R_0} = \left\{ u \in C([0, T], H^{-1}) : \sup_{t \in [0, T]} \| u(t) \|_H^2 + \int_0^T \| u(t) \|_{H^{1,0}}^2 dt \leq \frac{R_0}{3} \text{ and }
\right. 
\sup_{t \in [0, T]} e^{-k} f_0^t \| \partial_1 u(s) \|_{H^2}^2 ds \| u(t) \|_{H^{0,1}}^2 + \int_0^T e^{-k} f_0^t \| \partial_1 u(s) \|_{H^2}^2 ds \| u(t) \|_{H^{1,1}}^2 dt \leq \frac{R_0}{3} \},
\]

Then \( P(X^\varepsilon \in C([0, T], H^{-1}) \setminus \hat{K}_{R_0}) < \frac{\delta}{2}. \)
Now for \( X^\varepsilon \in \hat{K}_{R_0} \), we have \( \partial_t^2 X^\varepsilon \) is uniformly bounded in \( L^2([0,T], H^{-1}) \). Similar as in Lemma 4.2, \( X^\varepsilon \) is uniformly bounded in \( L^4([0,T], H^2) \) and \( L^4([0,T], L^4(T^2)) \), thus \( B(X^\varepsilon, u^0 + \sqrt{\varepsilon} \lambda(\varepsilon)X^\varepsilon) \) and \( B(u^0, X^\varepsilon) \) are uniformly bounded in \( L^2([0,T], H^{-1}) \). By Hölder’s inequality, we have

\[
\sup_{s,t \in [0,T], s \neq t} \frac{\left\| \int_s^t \partial_t^2 X^\varepsilon (r) + B(X^\varepsilon, u^0 + \sqrt{\varepsilon} \lambda(\varepsilon)X^\varepsilon) + B(u^0, X^\varepsilon) \, dr \right\|_{H^{-1}}^2}{|t-s|} \\
\leq \int_0^T \left\| \partial_t^2 X^\varepsilon (r) + B(X^\varepsilon, u^0 + \sqrt{\varepsilon} \lambda(\varepsilon)X^\varepsilon) + B(u^0, X^\varepsilon) \right\|_{H^{-1}}^2 \, dr \leq C(R_0),
\]

where \( C(R_0) \) is a constant depend on \( R_0 \). For any \( p \in (1, \frac{4}{3}) \), by Hölder’s inequality, we have

\[
\sup_{s,t \in [0,T], s \neq t} \frac{\left\| \int_s^t \sigma(r, u^0 + \sqrt{\varepsilon} \lambda(\varepsilon)X^\varepsilon (r)) v^\varepsilon (r) \, dr \right\|_{H^{-1}}^p}{|t-s|^{p-1}} \\
\leq \int_0^T \left\| \sigma(r, u^0 + \sqrt{\varepsilon} \lambda(\varepsilon)X^\varepsilon (r)) v^\varepsilon (r) \right\|_{H^{-1}}^p \, dr \\
\leq \int_0^T \left\| \sigma(r, u^0 + \sqrt{\varepsilon} \lambda(\varepsilon)X^\varepsilon (r)) \right\|_{L^2([0,T], H^{-1})}^p \, |v^\varepsilon (r)|_t^p \, dr \\
\leq C \int_0^T (1 + \left\| u^0 + \sqrt{\varepsilon} \lambda(\varepsilon)X^\varepsilon (r) \right\|_H^4 + \left\| v^\varepsilon (r) \right\|_t^4) \, dr \\
\leq C(R_0),
\]

where we used Young’s inequality and (A0) in the third inequality. Moreover, for any \( 0 \leq s \leq t \leq T \), by Hölder’s inequality we have

\[
E \left\| \int_s^t \sigma(r, u^0 + \sqrt{\varepsilon} \lambda(\varepsilon)X^\varepsilon (r)) \, dW(r) \right\|_{H^{-1}}^4 \\
\leq C E \left( \int_s^t \left\| \sigma(r, u^0 + \sqrt{\varepsilon} \lambda(\varepsilon)X^\varepsilon (r)) \right\|_{L^2([0,T], H^{-1})}^2 \, dr \right)^2 \\
\leq C |t-s| E \int_s^t \left\| \sigma(r, u^0 + \sqrt{\varepsilon} \lambda(\varepsilon)X^\varepsilon (r)) \right\|_{L^2([0,T], H^{-1})}^4 \, dr \\
\leq C |t-s|^2 \left( 1 + E \left( \sup_{t \in [0,T]} \left\| u^0 + \sqrt{\varepsilon} \lambda(\varepsilon)X^\varepsilon (t) \right\|_H^4 \right) \right) \\
\leq C |t-s|^2,
\]

where we used (A0) in the third inequality and (21) in the last inequality. Then by Kolmogorov’s continuity criterion, for any \( \alpha \in (0, \frac{1}{4}) \), we have

\[
E \left( \sup_{s,t \in [0,T], s \neq t} \frac{\left\| \int_s^t \sigma(r, u^0 + \sqrt{\varepsilon} \lambda(\varepsilon)X^\varepsilon (r)) \, dW(r) \right\|_{H^{-1}}^4}{|t-s|^{2\alpha}} \right) \leq C.
\]

Choose \( p = \frac{8}{7}, \alpha = \frac{1}{8} \) in the above estimates, we deduce that there exists \( R > R_0 \) such that

\[
P \left( \left\| X^\varepsilon \right\|_{C^{\frac{1}{p}}((0,T], H^{-1})} > \frac{R}{3}, X^\varepsilon \in \hat{K}_{R_0} \right) \leq \frac{E \left( \sup_{s,t \in [0,T], s \neq t} \frac{\left\| X^\varepsilon (t) - X^\varepsilon (s) \right\|_{H^{-1}}}{|t-s|^{2\alpha}} \right)_{H^{-1}}}{\frac{R}{3}} 1_{\{X^\varepsilon \in \hat{K}_{R_0}\}} \leq \frac{\delta}{2}.
\]
Combining the fact that \( P(X^\varepsilon \in C([0, T], H^{-1}) \setminus \tilde{K}_{R_0}) < \frac{\delta}{2} \), we finish the proof.

\[ \square \]

**Lemma 4.10.** Let \( \{v^\varepsilon\} \varepsilon > 0 \subset A_N \) for some \( N < \infty \). Assume \( v^\varepsilon \) converge to \( v \) in distribution as \( S_N \)-valued random elements, then

\[
g^\varepsilon \left( W(\cdot) + \lambda(\varepsilon) \int_0 \varepsilon^2(s)ds \right) \to g^0 \left( \int_0 v(s)ds \right)
\]

in distribution as \( \varepsilon \to 0 \).

**Proof.** The proof follows essentially the same argument as in [WZZ15, Proposition 4.7].

By Lemma 4.7, we have

\[
\lim_{\varepsilon \to 0} \left[ E \sup_{t \in [0, T]} \|Y^\varepsilon(t)\|_H^2 + E \int_0^T \|Y^\varepsilon(t)\|_{H^{1,0}}^2 dt \right] = 0,
\]

and

\[
\lim_{\varepsilon \to 0} \left[ E \sup_{t \in [0, T]} (e^{-kg(t)} \|Y^\varepsilon(t)\|_{H^{0,1}}^2) + E \int_0^T e^{-kg(t)} \|Y^\varepsilon(t)\|_{H^{1,1}}^2 dt \right] = 0,
\]

where \( g(t) = f_0 \|\partial_s X^\varepsilon(s)\|_H ds \) and \( k \) are the same as in (22).

Set

\[
\Xi := \left( \chi, S_N, L^\infty([0, T], H) \right) \bigcap L^2([0, T], H^{1,0}) \bigcap C([0, T], H^{-1})
\]

The above limit implies that \( Y^\varepsilon \to 0 \) in \( L^\infty([0, T], H \bigcap L^2([0, T], H^{1,0}) \bigcap C([0, T], H^{-1}) \) almost surely as \( \varepsilon \to 0 \) (in the sense of subsequence). By Lemma 4.9 the family \( \{(X^\varepsilon, v^\varepsilon)\} \varepsilon \in (0, \varepsilon_0) \) is tight in \( (\chi, S_N) \). Let \( (X_v, v, 0) \) be any limit point of \( \{(X^\varepsilon, v^\varepsilon, Y^\varepsilon)\} \varepsilon \in (0, \varepsilon_0) \). Our goal is to show that \( X_v \) has the same law as \( g^0 \left( \int_0 v(s)ds \right) \) and \( X^\varepsilon \) convergence in distribution to \( X_v \) in the space \( L^\infty([0, T], H \bigcap L^2([0, T], H^{1,0}) \bigcap C([0, T], H^{-1}) \).

By Jakubowski-Skorokhod’s representation theorem (see [Jak98 or LZZ18, Theorem 4.3]), there exists a stochastic basis \( (\hat{\Omega}, \hat{F}, \{\hat{\mathcal{F}}_t\} \varepsilon \in [0,T], \hat{P}) \) and, on this basis, \( \Xi \)-valued random variables \( (\tilde{X}_v, \tilde{v}, 0), (\tilde{X}^\varepsilon, \tilde{v}^\varepsilon, \tilde{Y}^\varepsilon) \), such that \( (\tilde{X}^\varepsilon, \tilde{v}^\varepsilon, \tilde{Y}^\varepsilon) \) (respectively \( (\tilde{X}_v, \tilde{v}, 0) \)) has the same law as \( (X^\varepsilon, v^\varepsilon, Y^\varepsilon) \) (respectively \( (X_v, v, 0) \)), and \( (\tilde{X}_v, \tilde{v}^\varepsilon, \tilde{Y}^\varepsilon) \to (\tilde{X}_v, \tilde{v}, 0), \hat{P} \)-a.s.

We have

\[
d(\tilde{X}^\varepsilon(t) - \tilde{Y}^\varepsilon(t)) = \partial^2_t (\tilde{X}^\varepsilon(t) - \tilde{Y}^\varepsilon(t))dt - B(\tilde{X}^\varepsilon, u^0 + \sqrt{\varepsilon} \lambda(\varepsilon) \tilde{X}^\varepsilon)dt \]

\[ - B(u^0, \tilde{X}^\varepsilon)dt + \sigma(t, u^0 + \sqrt{\varepsilon} \lambda(\varepsilon) \tilde{X}^\varepsilon(t)) \tilde{v}^\varepsilon(t)dt, \]

\[ \tilde{X}^\varepsilon(0) - \tilde{Y}^\varepsilon(0) = 0, \]

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and
\[ P(\tilde{X}^\varepsilon - \tilde{Y}^\varepsilon \in L^\infty([0, T], H) \cap L^2([0, T], H^1) \cap C([0, T], H^{-1})) = P(X^\varepsilon - Y^\varepsilon \in L^\infty([0, T], H) \cap L^2([0, T], H^1) \cap C([0, T], H^{-1})) = 1. \]

Let \( \tilde{\Omega}_0 \) be the subset of \( \tilde{\Omega} \) such that for \( \omega \in \tilde{\Omega}_0, \)
\[ (\tilde{X}^\varepsilon, \tilde{v}^\varepsilon, \tilde{Y}^\varepsilon)(\omega) \to (\tilde{X}_v, \tilde{v}, 0)(\omega) \text{ in } \Xi, \]
and
\[ e^{-k} \int_0^T \|\tilde{X}^\varepsilon(\omega, s)\|_{L^2}^2 ds \to 0 \text{ in } L^\infty([0, T], H^{0,1}) \cap L^2([0, T], H^{1,1}) \cap C([0, T], H^{-1}), \]
then \( P(\tilde{\Omega}_0) = 1. \) For any \( \omega \in \tilde{\Omega}_0, \) fix \( \omega, \)
we have \( \sup_\varepsilon \int_0^T \|\tilde{X}^\varepsilon(\omega, s)\|_{L^2}^2 ds < \infty, \)
then we deduce that
\[ \lim_{\varepsilon \to 0} \left( \sup_{t \in [0, T]} \|\tilde{Y}^\varepsilon(\omega, t)\|_{H^{0,1}} + \int_0^T \|\tilde{Y}^\varepsilon(\omega, t)\|_{H^{1,1}}^2 dt \right) = 0. \quad (27) \]

Now we show that
\[ \sup_{t \in [0, T]} \|\tilde{X}^\varepsilon(\omega, t) - \tilde{X}_v(\omega, t)\|_{L^2}^2 + \int_0^T \|\tilde{X}^\varepsilon(\omega, t) - \tilde{X}_v(\omega, t)\|_{H^{1,0}}^2 dt \to 0 \text{ as } \varepsilon \to 0. \quad (28) \]

Let \( U^\varepsilon = \tilde{X}^\varepsilon(\omega) - \tilde{Y}^\varepsilon(\omega), \)
then by (26) we have
\[ dU^\varepsilon(t) = \partial^2 U^\varepsilon(t) dt - B(U^\varepsilon + \tilde{Y}^\varepsilon, u^0 + \sqrt{\varepsilon} \lambda(\varepsilon)(U^\varepsilon + \tilde{Y}^\varepsilon)) dt - B(u^0, U^\varepsilon + \tilde{Y}^\varepsilon) + \sigma(t, u^0 + \sqrt{\varepsilon} \lambda(\varepsilon)(U^\varepsilon(t) + \tilde{Y}^\varepsilon(t)))\tilde{v}^\varepsilon(t) dt. \quad (29) \]

Since \( U^\varepsilon(\omega) \to \tilde{X}_v(\omega) \) in \( \chi, \)
by a very similar argument as in Lemma 4.6 we deduce that
\( \tilde{X}_v = X^\tilde{v} = g^0 \left( \int_0^\infty \tilde{v}(s) ds \right). \)
Moreover, note that \( \tilde{X}^\varepsilon(\omega) \to X^\tilde{v}(\omega) \) weak star in \( L^\infty([0, T], H^{0,1}), \)
then the uniform boundedness principle implies that
\[ \sup_{\varepsilon} \sup_{t \in [0, T]} \|\tilde{X}^\varepsilon(\omega)\|_{H^{0,1}} < \infty. \quad (30) \]

Let \( w^\varepsilon = U^\varepsilon - X^\tilde{v}, \)
then we have
\[ \|w^\varepsilon(t)\|_{L^2}^2 + 2 \int_0^t \|\partial_1 w^\varepsilon(s)\|_{L^2}^2 ds \\
= -2 \int_0^t \langle w^\varepsilon(s), B(U^\varepsilon + \tilde{Y}^\varepsilon, u^0 + \sqrt{\varepsilon} \lambda(\varepsilon)(U^\varepsilon + \tilde{Y}^\varepsilon)) - B(X^\tilde{v}, u^0) \rangle ds \\
-2 \int_0^t \langle w^\varepsilon(s), B(u^0, u^\varepsilon + \tilde{Y}^\varepsilon) \rangle ds \\
+2 \int_0^t \langle w^\varepsilon(s), \sigma(s, u^0 + \sqrt{\varepsilon} \lambda(\varepsilon)(U^\varepsilon + \tilde{Y}^\varepsilon))\tilde{v}^\varepsilon(s) - \sigma(s, u^0)\tilde{v}(s) \rangle ds \\
= I_1 + I_2 + I_3. \]
By Lemma [A.1], we have

\[ |I_1 + I_2| \]
\[ = \left| \int_0^t b(w^\varepsilon, u^0 + \sqrt{\varepsilon} \lambda (\varepsilon)(X^{\tilde{v}} + \tilde{Y}^{\varepsilon}), w^\varepsilon) + b(\tilde{Y}^{\varepsilon}, u^0, w^\varepsilon) \right. \\
\left. + \sqrt{\varepsilon} \lambda (\varepsilon) b(X^{\tilde{v}} + \tilde{Y}^{\varepsilon}, X^{\tilde{v}} + \tilde{Y}^{\varepsilon}, w^\varepsilon) + b(u^0, \tilde{Y}^{\varepsilon}, w^\varepsilon) ds \right| \]
\[ \leq \int_0^t \left[ \frac{1}{2} \| \partial_t w^\varepsilon (s) \|^2_{\tilde{H}} + C(1 + \| u^0 (s) \|^2_{\tilde{H}^1, 1} + \| X^{\tilde{v}} (s) \|^2_{\tilde{H}^1, 1} + \| \tilde{Y}^{\varepsilon} (s) \|^2_{\tilde{H}^1, 1} \| w^\varepsilon (s) \|^2_{\tilde{H}} \right] ds \]
\[ + \int_0^t \left[ \| \tilde{Y}^{\varepsilon} (s) \|^2_{\tilde{H}^1, 1} + C(1 + \| u^0 (s) \|^2_{\tilde{H}^1, 1} + \| \tilde{Y}^{\varepsilon} (s) \|^2_{\tilde{H}^1, 1} \| w^\varepsilon (s) \|^2_{\tilde{H}} \right] ds \]
\[ + \sqrt{\varepsilon} \lambda (\varepsilon) \int_0^t \left[ \| X^{\tilde{v}} (s) \|^2_{\tilde{H}^1, 1} + \| \tilde{Y}^{\varepsilon} (s) \|^2_{\tilde{H}^1, 1} + \| X^{\tilde{v}} (s) \|^2_{\tilde{H}^1, 1} + \| \tilde{Y}^{\varepsilon} (s) \|^2_{\tilde{H}^1, 1} \| w^\varepsilon (s) \|^2_{\tilde{H}} \right] ds \]
\[ \leq \int_0^t \left[ \frac{1}{2} \| \partial_t w^\varepsilon (s) \|^2_{\tilde{H}} + C(1 + \| u^0 (s) \|^2_{\tilde{H}^1, 1} + \| X^{\tilde{v}} (s) \|^2_{\tilde{H}^1, 1}) \| w^\varepsilon (s) \|^2_{\tilde{H}} \right] ds \]
\[ + C \int_0^t \| \tilde{Y}^{\varepsilon} (s) \|^2_{\tilde{H}^1, 1} ds + \sqrt{\varepsilon} \lambda (\varepsilon) \int_0^t \| X^{\tilde{v}} (s) \|^2_{\tilde{H}^1, 0} ds. \]

where we used the fact that by (27) and (30) \( w^\varepsilon \) are uniformly bounded in \( L^ \infty([0, T], H) \) in the last inequality. By (A1) and (A3) we have

\[ |I_3(t)| = \int_0^t \langle w^\varepsilon (s), (\sigma (s, u^0 + \sqrt{\varepsilon} \lambda (\varepsilon)(U^{\varepsilon} + \tilde{Y}^{\varepsilon})) - \sigma (s, u^0)) \tilde{v}^\varepsilon (s) \rangle ds \]
\[ + \int_0^t \langle w^\varepsilon (s), \sigma (s, u^0) (\tilde{v}^\varepsilon (s) - \tilde{v} (s)) \rangle ds \]
\[ \leq C(\sqrt{\varepsilon} \lambda (\varepsilon))^{\frac{1}{2}} \int_0^t \left( \| w^\varepsilon (s) \|_{H} \| \tilde{v}^\varepsilon (s) \|_{L^2} (\| w^\varepsilon (s) \|^2_{\tilde{H}^1, 0} + \| X^{\tilde{v}} (s) \|^2_{\tilde{H}^1, 0} + \| \tilde{Y}^{\varepsilon} (s) \|^2_{\tilde{H}^1, 0}) \right) \frac{1}{2} ds \]
\[ + \int_0^t \left( \| w^\varepsilon (s) \|_{H} \| \tilde{v}^\varepsilon (s) - \tilde{v} (s) \|_{L^2} (K_0 + K_1 \| u^0 (s) \|^2_{H} + K_2 \| \partial_1 u^0 (s) \|^2_{H}) \frac{1}{2} ds \right) \]
\[ \leq (\sqrt{\varepsilon} \lambda (\varepsilon))^{\frac{1}{2}} \left( C N + C_1 \int_0^t (\| w^\varepsilon (s) \|^2_{\tilde{H}^1, 0} + \| X^{\tilde{v}} (s) \|^2_{\tilde{H}^1, 0} + \| \tilde{Y}^{\varepsilon} (s) \|^2_{\tilde{H}^1, 0}) ds \right) \]
\[ + C N^{\frac{1}{2}} \left( \int_0^t (\| u^\varepsilon (s) \|^2_{H} (K_0 + K_1 \| u^0 (s) \|^2_{H} + K_2 \| \partial_1 u^0 (s) \|^2_{H}) ds \right) \]

where we used the fact that \( w^\varepsilon \) are uniformly bounded in \( L^ \infty([0, T], H) \) and that \( \tilde{v}^\varepsilon, \tilde{v} \) are in \( \mathcal{A}_N \). Note here \( C_1 \) is a positive constant. Thus choose \( \varepsilon \) small enough such that \( \frac{1}{2} + (\sqrt{\varepsilon} \lambda (\varepsilon))^{\frac{1}{2}} C_1 < 1, \)

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Lemma A.1. For smooth functions $u, v, w$ from $\mathbb{T}^2$ to $\mathbb{R}^2$ with divergence free condition, we have

$$\|b(u, v, w)\| \leq C\|u\|_{H^{1,0}}\|v\|_{H^{1,1}}\|w\|_{L^2}.$$ 

Proof.

$$|b(u, v, w)| \leq (\|u\|_{L^8_{\infty}(L^2)}\|\partial_1 v\|_{L^2(\mathbb{T}^2)} + \|u^2\|_{L^8_{\infty}(L^2)}\|\partial_2 v\|_{L^2(\mathbb{T}^2)})\|w\|_{L^2}$$

$$\leq C\left(\|u\|_{L^2}\|u\|_{L^2}^{\frac{1}{2}} + \|u\|_{L^2}^{\frac{1}{2}}\right)^{\frac{1}{2}}\|\partial_1 v\|_{L^2}\|\partial_2 v\|_{L^2} + \|\partial_1 v\|_{L^2}^{\frac{1}{2}}\|w\|_{L^2}$$

$$+ \left(\|u^2\|_{L^2}\|\partial u^2\|_{L^2} + \|u^2\|_{L^2}^{\frac{1}{2}}\right)^{\frac{1}{2}}\|\partial_2 v\|_{L^2}\|\partial_2 v\|_{L^2} + \|\partial_2 v\|_{L^2}^{\frac{1}{2}}\right)^{\frac{1}{2}}\|w\|_{L^2}$$

$$\leq C\|u\|_{H^{1,0}}\|v\|_{H^{1,1}}\|w\|_{L^2},$$

where we used the divergence free condition to deal with the term $\partial_2 u^2$ in the last inequality.  

Lemma A.2. For smooth function $u$ form $\mathbb{T}^2$ to $\mathbb{R}^2$ with divergence free condition, we have

$$|\langle \partial_2 u, \partial(u \cdot \nabla u)\rangle| \leq a\|\partial_1 \partial_2 u\|_{L^2}^2 + C(1 + \|\partial_1 u\|_{L^2}^2)\|\partial_2 u\|_{L^2}^2,$$

where $a > 0$ is a constant small enough.
The following estimates are obtained by [CDGG00] in dimension 3, we now present its 2-dimension version.

**Lemma A.3 ([CDGG00] Lemma 3).** For any real number $s_0 > \frac{1}{2}$ and $s \geq s_0$, for any vector fields $u$ and $w$, with divergence free condition, there exists constants $C$ and $d_k(u, w)$ such that

$$
|\langle \Delta_0^v(u \cdot \nabla w), \Delta_0^v w \rangle| \leq Cd_k 2^{-2ks} \|w\|_{H^{s+\frac{1}{2}, s}} (\|u\|_{H^{s+\frac{1}{2}, s}} \|\partial_1 w\|_{H^{s, \infty}} + \|u\|_{H^{s+\frac{1}{2}, s}} \|\partial_1 w\|_{H^{s, \infty}}
+ \|\partial_1 u\|_{H^{s, \infty}} \|w\|_{H^{s+\frac{1}{2}, s}} + \|\partial_1 u\|_{H^{s, \infty}} \|w\|_{H^{s+\frac{1}{2}, s}}),
$$

where $\sum_k d_k = 1$.

**Proof.** Define

$$
F_k^h = \Delta_0^v(u^1 \partial_1 w) \quad \text{and} \quad F_k^w = \Delta_0^v(u^2 \partial_2 w).
$$

Let us start by proving the result for $F_k^h$. Recall the Bony decomposition (see [BCD11]) in vertical variables for tempered distributions $a, b$:

$$
ab = T_a^u b + T_b^u a + R^u(a, b),
$$

with

$$
T_a^u b = \sum_j S^u_{j-1}a \Delta_j^v b \quad \text{and} \quad R^v(a, b) = \sum_{|k-j| \leq 1} \Delta_k^v a \Delta_j^v b,
$$

where $S^u_{j-1}a = \sum_{j' \leq j-2} \Delta_{j'}^v a$.

Then we have by Hölder’s inequality and Sobolev embedding $H^{\frac{1}{2}}(\mathbb{T}) \hookrightarrow L^4(\mathbb{T})$

$$
\langle \Delta^v_k(u^1 \partial_1 w), \Delta^v_k w \rangle \leq \|\Delta^v_k(u^1 \partial_1 w)\|_{L^2(\mathbb{T})} \|\Delta^v_k w\|_{L^2(\mathbb{T})}
\leq C \|\Delta^v_k(T_{u^1}^0 \partial_1 w + T_{\partial_1 w}^0 u^1 + R^v(u^1, \partial_1 w))\|_{L^2(\mathbb{T})} \|\Delta^v_k w\|_{L^2(\mathbb{T})}
\leq C \|\Delta^v_k(T_{u^1}^k \partial_1 w + T_{\partial_1 w}^k u^1 + R^v(u^1, \partial_1 w))\|_{L^2(\mathbb{T})} \|2^{-ks} c_k \|w\|_{H^{s+\frac{1}{2}, s}}
$$

(32)

where $c_k = \frac{2^{ks}\|\Delta^v_k w\|_{L^2(\mathbb{T})}}{\|w\|_{H^{s+\frac{1}{2}, s}}} \in l^2$. For the first term of the third line, we have

$$
\|\Delta^v_k(T_{u^1}^k \partial_1 w)\|_{L^2(\mathbb{T})} \leq \sum_{|k-k'| \leq N_0} \|S^v_{k'-1}u^1 \Delta^v_{k'} \partial_1 w\|_{L^2(\mathbb{T})} \leq \sum_{|k-k'| \leq N_0} \|S^v_{k'-1}u^1 \|_{L^\infty(\mathbb{T})} \|\Delta^v_{k'} \partial_1 w\|_{L^2(\mathbb{T})}
\leq C \sum_{|k-k'| \leq N_0} \|u^1\|_{H^{s+\frac{1}{2}, s}} 2^{-ks} b_k \|\partial_1 w\|_{H^{s, \infty}} \leq C b_k^{(1)} 2^{-ks} \|u^1\|_{H^{s+\frac{1}{2}, s}} \|\partial_1 w\|_{H^{s, \infty}},
$$

where $b_k = \frac{2^{ks}\|\Delta^v_k \partial_1 w\|_{L^2(\mathbb{T})}}{\|\partial_1 w\|_{H^{s, \infty}}} \in l^2$ and $b_k^{(1)} = 2^{ks} \sum_{|k-k'| \leq N_0} 2^{-k's} b_k' \in l^2$. Note here $N_0$ depends on the choice of Dyadic partition. For the second term, similarly we have

$$
\|\Delta^v_k(T_{\partial_1 w}^k u^1)\|_{L^2(\mathbb{T})} \leq \sum_{|k-k'| \leq N_0} \|S^v_{k'-1} \partial_1 w \|_{L^\infty(\mathbb{T})} \|\Delta^v_{k'} u^1\|_{L^2(\mathbb{T})}
\leq C \sum_{|k-k'| \leq N_0} \|\partial_1 w\|_{H^{s+\frac{1}{2}, s}} 2^{-ks} a_k' \|u\|_{H^{s+\frac{1}{2}, s}} \leq C a_k^{(1)} 2^{-ks} \|\partial_1 w\|_{H^{s, \infty}} \|u\|_{H^{s+\frac{1}{2}, s}}.
$$
where $a_k = \frac{2^{ks}\|\Delta_k^v u\|_{L^2(\mathbb{T}^d)}}{\|u\|_{H^{\frac{d}{2}-s}}} \in l^2$ and $a_k^{(1)} = 2^{ks} \sum_{|k-k'| \leq N_0} 2^{-k's} \tilde{c}_k \in l^2$.

$$||\Delta_k^v R^v(\bar{u}^1, \partial_1 w)||_{L^2(L^d_k)} \leq \sum_{k'-|k| \leq 1, k' \geq k-N_0} ||\Delta_k^v u^1||_{L^2(L^d_k)} ||\Delta_k^v \partial_1 w||_{L^\infty(L^d_k)}$$

$$\leq C \sum_{k' \geq k-N_0} 2^{-k's} a_{k'} ||u||_{H^{\frac{d}{2}-s}} ||\partial_1 w||_{H^{0,s}}$$

$$\leq Ca_k^{(2)} 2^{-k's} ||u||_{H^{\frac{d}{2}-s}} ||\partial_1 w||_{H^{0,s}},$$

where $a_k^{(2)} = 2^{ks} \sum_{k' \geq k-N_0} 2^{-k's} a_{k'} = \sum_{k' \in \mathbb{N}} I\{k' \leq N_0\} 2^{ks} a_{k-k'}$ and by Young’s convolution inequality

$$\|a(2)\|_2 \leq \|I\{k' \leq N_0\} 2^{ks} \|_1 \|a\|_2 < \infty.$$  

This implies that

$$|\langle F_k^v, \Delta_k^v w \rangle| \leq Cc_k(b_k^{(1)} + a_k^{(1)} + a_k^{(2)}) 2^{-2ks} \|w\|_{H^{\frac{d}{2}-s}} (||u||_{H^{\frac{d}{2}-s}} ||\partial_1 w||_{H^{0,s}} + ||u||_{H^{\frac{d}{2}-s}} ||\partial_1 w||_{H^{0,s}}),$$

where $c_k(b_k^{(1)} + a_k^{(1)} + a_k^{(2)}) \in l^1$.

To estimate the term $\langle F_k^v, \Delta_k^v u \rangle$, write $\Delta_k^v(u^2 \partial_2 w) = F_k^{v,1} + F_k^{v,2}$ with

$$F_k^{v,1} = \Delta_k^v \sum_{k' \geq k-N_0} S_{k'+2}^w \partial_2 w \Delta_k^v u^2$$

and

$$F_k^{v,2} = \Delta_k^v \sum_{|k-k'| \leq N_0} S_{k'-1}^w u^2 \Delta_k^v \partial_2 w.$$

For $F_k^{v,1}$, again we have by Hölder’s inequality and Sobolev embedding,

$$\|F_k^{v,1}\|_{L^2(L^d_k)} \leq \sum_{k' \geq k-N_0} \|S_{k'+2}^w \partial_2 w\|_{L^\infty(L^d_k)} \|\Delta_k^v u^2\|_{L^2(L^d_k)}$$

$$\leq C \sum_{k' \geq k-N_0} 2^{k'} \|S_{k'+2}^w \partial_2 w\|_{L^\infty(L^d_k)} 2^{-k'} \|\Delta_k^v \partial_2 u^2\|_{L^2(L^d_k)}$$

$$\leq C \sum_{k' \geq k-N_0} \|w\|_{H^{\frac{d}{2}-s}} 2^{-k's} \tilde{c}_k \|\partial_1 u\|_{H^{0,s}}$$

$$\leq C 2^{-k's} \tilde{c}_k \|w\|_{H^{\frac{d}{2}-s}} ||\partial_1 u||_{H^{0,s}},$$

where we use Bernstein’s inequality twice in the second inequality and divergence free condition in the third inequality. Note here $\tilde{c}_k = \frac{2^{ks}\|\Delta_k^v \partial_1 u\|_{L^2(L^d_k)}}{\|\partial_1 u\|_{H^{0,s}}} \in l^2$ and $\tilde{c}_k^{(2)} = 2^{ks} \sum_{k' \geq k-N_0} 2^{-k's} \tilde{c}_k \in l^2$.

Then similar as [32] we have

$$|\langle F_k^{v,1}, \Delta_k^v w \rangle| \leq Cc_k \tilde{c}_k^{(2)} 2^{-2ks} \|w\|_{H^{\frac{d}{2}-s}} ||w||_{H^{\frac{d}{2}-s}} ||\partial_1 u||_{H^{0,s}}.$$  

The last term $F_k^{v,2}$ requires commutator estimates. Following a computation in [CL92], we have

$$\langle F_k^{v,2}, \Delta_k^v w \rangle = \langle S_{k-1}^w u^2 \Delta_k^v \partial_2 w, \Delta_k^v w \rangle + R_k(u, w)$$

with

$$R_k(u, v) = \sum_{|k-k'| \leq N_0} \langle [\Delta_k^v, S_{k'-1}^w u^2] \Delta_k^v \partial_2 w, \Delta_k^v w \rangle$$

$$- \sum_{|k-k'| \leq N_0} \langle (S_{k-1}^w - S_{k'-1}^w) u^2 \Delta_k^v \Delta_k^v \partial_2 w, \Delta_k^v w \rangle.$$
Using an integration by parts and divergence free condition, we have
\[
\left| \langle S_{k-1}^u u^2 \Delta^v_k \partial_2 w, \Delta^v_k w \rangle \right| = \frac{1}{2} \left| \langle S_{k-1}^u u^2 \Delta^v_k \partial_2 w, \Delta^v_k w \rangle \right| = \frac{1}{2} \left| \langle S_{k-1}^u \partial_1 u^1 \Delta^v_k w, \Delta^v_k w \rangle \right|
\leq C \| S_{k-1}^u \partial_1 u^1 \|_{L^\infty_{H^k_0}} \| \Delta^v_k w \|_{L^2_{H^k_0}}^2
\leq C c_k^2 2^{-2k} \| \partial_1 u^1 \|_{H^{n_0},0} \| \| w \|_{H^{k+1}}^2.
\]

Note that the Fourier transform of \((S_{k-1}^u - S_{k'-1}^u) u^2\) is supported in \(2^k A\) since \(|k - k'| \leq N_0\) where \(A\) is an annulus. We have by Bernstein’s inequality
\[
\| \sum_{|k' - k| \leq N_0} (S_{k-1}^u - S_{k'-1}^u) u^2 \Delta^v_k \Delta^v_{k'} \partial_2 w \|_{L^2_{H^k_0}} \leq C \sum_{|k' - k| \leq N_0} \| (S_{k-1}^u - S_{k'-1}^u) u^2 \|_{L^\infty_{H^k_0}} \| \Delta^v_k \Delta^v_{k'} \partial_2 w \|_{L^2_{H^k_0}} \leq C \sum_{|k' - k| \leq N_0} \| \partial_1 u^1 \|_{H^{n_0},0} \| 2^{-k} \| \| w \|_{H^{k+1}}^2.
\]

This similar as (32) implies that
\[
\left| \langle \sum_{|k' - k| \leq N_0} (S_{k-1}^u - S_{k'-1}^u) u^2 \Delta^v_k \Delta^v_{k'} \partial_2 w, \Delta^v_k w \rangle \right| \leq C c_k^2 2^{-2k} \| \partial_1 u^1 \|_{H^{n_0},0} \| \| w \|_{H^{k+1}}^2.
\]

To estimate the term \(\langle [\Delta^v_k, S_{k'-1}^u u^2] \Delta^v_{k'} \partial_2 w, \Delta^v_k w \rangle\), we have for any function \(f\),
\[
[\Delta^v_k, S_{k'-1}^u u^2] f(x_1, x_2) = 2^k \int_{T_v} h(2^k y_2)(S_{k'-1}^u u^2(x_1, x_2) - S_{k'-1}^u u^2(x_1, x_2 - y_2)) f(x_1, x_2 - y_2) dy_2
\]
\[
= \int_{T_v \times [0,1]} h_1(2^k y_2)(S_{k'-1}^u \partial_2 u^2)(x_1, x_2 + (t - 1)y_2) f(x_1, x_2 - y_2) dy_2 dt
\]
\[
= - \int_{T_v \times [0,1]} h_1(2^k y_2)(S_{k'-1}^u \partial_1 u^1)(x_1, x_2 + (t - 1)y_2) f(x_1, x_2 - y_2) dy_2 dt,
\]
where \(h = \mathcal{F}^{-1} \chi^{(1)}, (k = -1)\) or \(h = \mathcal{F}^{-1} \theta^{(1)}, (k \geq 0)\), \(h_1(z) = zh(z)\) and we use divergence free condition in the last line. This implies
\[
\| [\Delta^v_k, S_{k'-1}^u u^2] f(\cdot, x_2) \|_{L^4_{H^k_0}} \leq C \int h_1(2^k y_2) \| S_{k'-1}^u \partial_1 u^1 \|_{L^\infty_{L^2}} \| f(\cdot, x_2 - y_2) \|_{L^4_{H^k}} dy_2
\]

Then we get
\[
\| [\Delta^v_k, S_{k'-1}^u u^2] f \|_{L^4_{L^2(H^k_0)}} \leq C 2^{-k} \| S_{k'-1}^u \partial_1 u^1 \|_{L^\infty_{L^2}} \| f \|_{L^2_{H^k}^4}.
\]

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Hence
\[ \sum_{|k-k'| \leq N_0} |\langle \Delta_k^u, S_{k-1}^u u^2 \rangle \Delta_k^w \partial_2 w, \Delta_k^w w \rangle| \]
\[ \leq C 2^{-k} \sum_{|k-k'| \leq N_0} \|S_{k-1} \partial_1 u \|_{L^\infty} 2^{k'} \|\Delta_{k'} w \|_{L^2(H_t)} \|\Delta_k^w w \|_{L^2(H_t)} \]
\[ \leq C \sum_{|k-k'| \leq N_0} \|\partial_1 u \|_{H^{0,0}} 2^{-k}s_c k' \|w\|_{H_1^s} 2^{-k} s_c k \|w\|_{H_1^s}, \]
\[ \leq C c_k c_k(1) 2^{-2ks} \|\partial_1 u \|_{H^{0,0}} \|w\|_{H_1^s} \|w\|_{H_1^s}, \]

where \( c_k(1) = 2^{ks} \sum_{|k-k'| \leq N_0} 2^{-k's_c k'} \in l^2 \)

Combining all the term together, let
\[ d'_k = c_k(b_k(1) + a_k(1) + a_k(2) + c_k(2) + c_k + c_k(1)) \in l^1 \quad \text{and} \quad d_k = \frac{d'_k}{\|d'_k\|_{l^1}} \]

then the result holds. \[ \square \]

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