An Integrable Model for Density-Modulated Quantum Condensates

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An integrable model possessing inhomogeneous ground states is proposed. The model is a higher-order analog of the nonlinear Schrödinger equation. The inverse scattering theory with the elliptic-functional background is formulated, and the N-soliton solution is obtained as a particular solution. We provide the exact bosonic and fermionic quasiparticle eigenstates and reveal various kinds of dynamics such as dark soliton billiards, dislocations, gray solitons, and envelope solitons.

Introduction.—Spatially inhomogeneous quantum condensates have been attracting a lot of attention for a long time. For bosonic condensates, the supersolid phase, which was originally discussed four decades ago [1, 2], has received a renewed interest since the torsional oscillator experiments of $^4$He [3, 4]. While the most recent work [5] has concluded the absence of supersolidity, the candidate of supersolid is now also proposed in Rydberg matters [6]. For fermionic condensates, the realization and observation of Fulde-Ferrell (FF) [7] and Larkin-Ovchinnikov (LO) [8] states have been a long-standing topic. Within a framework of self-consistent Bogoliubov-de Gennes (BdG) formalism, the LO state is shown to be a ground state in the presence of a magnetic field or a population imbalance [9, 10]. Experimental candidates are, for example, CeCoIn$_5$ in condensed matters [11, 12] and the spin-imbalanced superfluid $^6$Li in ultracold atoms [13, 14].

To study the quantum condensates, the nonlinear Schrödinger (NLS) type equation and its generalizations are often used, and referred to as the Gross-Pitaevskii or the Ginzburg-Landau (GL) equation for bosonic or fermionic systems. Though Gorkov’s original derivation ensures that the GL description is good only near $T = T_c$, the recent studies show that the gap function obeys the NLS equation with higher-order corrections even near $T = 0$ [18, 20].

Many theoretical studies have established a common and model-independent understanding for the mechanism of modulation, the properties of ground states, and low-energy excitations around them. Compared to stationary states, however, the nature of nonlinear excitations such as solitons or vortices passing through modulated condensates has not been fully investigated yet, because of the difficulty to treat time-dependent phenomena. The knowledge for solitons is also important to understand transport phenomena past an obstacle in non-stationary regimes [21]. To investigate these issues, an integrable model would play a prominent role, since we can access the various kinds of dynamics exactly. In this Letter we propose an integrable model of non-uniform quantum condensates. Solving it by the inverse scattering method (ISM), we obtain an N-soliton solution and reveal its behavior. We also provide eigenstates of the BdG operators for both bosonic and fermionic systems, which are essential for the physics of quasiparticles and Nambu-Goldstone (NG) modes.

Note that the solitons given here are different from gap solitons. The system forms a pattern not by a periodic external force but by itself, and hence the modulated background and the solitons influence each other.

The energy functional of the model proposed in this Letter is given by

$$H = c_3 I_3 + c_5 I_5,$$

where $I_3 := \int dx (|\psi|^2 + |\psi|^4)$ and $I_5 := \int dx [(|\psi|^4)^2 + 6(|\psi|^4)|\psi|^2 + (\nabla^2 |\psi|^4)]^2 + 2|\psi|^6$ are the third and fifth conserved quantities in the NLS hierarchy [22, 23]. How to find this model as a modulated condensate is explained below. The corresponding partial differential equation is given by $i\dot{\psi} = \frac{\delta H}{\delta \psi^*}$. For convenience, we henceforth consider the equation for $q := |\psi|^2$. The equation is then given by

$$iq_t = -\mu q + c_3 (-q_{xx} + 2|q|^2 q) + c_5 [q_{xxxx} - 2(|q|^2)^2 q - 3q^2 q_{xx} + 6|q|^4 q],$$

where the subscripts $t$ and $x$ denote the differentiation. $\mu$ is a chemical potential if $\psi$ is regarded as a bosonic condensate. This is a special case of the AKNS$_3$ equation [24].

The idea of model construction.—Before beginning the analysis of Eq. (3), let us see how to construct an integrable model of modulated condensates. We first show that the model with a non-local interaction such as soft-core bosons [25, 29], which are used as a model of supersolid, can be approximated by a higher-order differential equation. Let us demonstrate it by using the Gaussian-like two-body interaction $V(x) = \frac{V_0(x)}{2a \sqrt{\pi}} e^{-a x^2/(4a^2)}$, where $V_0(x)$ is a slowly-varying even function. If $a \approx 0$, we can show $\frac{1}{2e \sqrt{\pi}} e^{-a x^2/(4a^2)} = \delta(x) + a^2 \delta''(x) + \frac{a^4}{6} \delta^{(6)}(x) + \cdots$. Using this, the NLS equation for the soft-core boson model $i\dot{\psi}(x,t) = -\nabla^2 \psi(x,t) + \int dy V(x-y)\psi(y,t) \psi(x,t)$ can be approximated up to $O(a^3)$ as follows:

$$iq_t = -\psi_{xx} + [\tilde{V}_0 |\psi|^2 + \tilde{V}_2 (|\psi|^2)^2 + \tilde{V}_4 (|\psi|^2)]_{xxxx} \psi,$$
where $\tilde{V}_0 = V_0(0) + a^2 V''_0(0) + a^4 V^{(4)}_0(0)$, $\tilde{V}_3 = a^3 V_0(0) + 3a^2 V''_0(0)$, and $\tilde{V}_4 = \frac{a^4}{7} V_0(0)$. Even though Eq. (3) is a too rough approximation for the original non-local model, it exhibits a roton minimum in the Bogoliubov spectrum and has an inhomogeneous ground state under certain parameter regions, as similar to Ref. [27]. So it can be used as a model of supersolid. It is reasonable that the higher-order derivative can induce a spatial order, because the energy of the system should have a minimum at a non-zero momentum, and the simplest such example is given by $E(p) \sim -p^2 + p^4$. In this regard, we mention that many pattern-forming models in various fields have higher-order derivatives, such as the convective instability [30], the magnetic fluids [31], and the generalized GL theory [32].

Unfortunately, however, Eq. (3) is not integrable in general. Thus the next problem is how to construct an integrable model including higher-order derivatives. This is solved as follows. The integrable model generally has infinitely many conserved quantities $I_1, I_2, \ldots$, and the model of their linear combination is also solvable by the ISM. Since the even-number $I_n$ in the NLS hierarchy breaks a parity symmetry [23], the minimal model including higher-order derivatives is given by $H = c_1 I_3 + c_5 I_5$, that is the model (1).

The system is unstable if $c_5 < 0$ since the dispersion of the linearized operator becomes $\epsilon \sim -k^4$. We can also confirm that if both $c_1$ and $c_5$ are positive, the ground state is always a trivial uniform state. Thus, the non-trivial physics arises when $c_3 < 0$ and $c_5 > 0$. Henceforth we consider this case.

Density-modulated ground state.—Let us begin the analysis of the model (1) in detail. We first determine the static ground state. Although the most general stationary solution to Eq. (1) is a quasi-periodic solution written by the Riemann theta function with genus $g$ [24, 33], here we assume that higher genus solutions are energetically unfavored, and we only consider the following two candidates; the FF state and the LO state

$$q_{\text{FF}}(x, \tilde{\rho}) = \sqrt{\rho} \psi^x, \quad q_{\text{LO}}(x, \tilde{\rho}, \mathbf{m}) = \sqrt{\rho} \alpha \text{sn}(a(x)m), \quad \alpha := \sqrt{\rho}/Q(m),$$

where $Q(m) := 1 - \frac{\varepsilon}{K(m)}$ with $K(m)$ and $E(m)$ being the complete elliptic integral of the first and second kind, respectively [34]. $\tilde{\rho}$ is the average of particle number density, and $p$ and $m$ are variational parameters to be chosen to minimize the energy. We can check that these states solve Eq. (4) if we set $\mu = c_1 (p^2 + 2 \tilde{\rho}) + c_5 (p^4 + 12 \tilde{\rho} p^2 + 6\tilde{\rho}^2)$ for $q_{\text{FF}}$ and $\mu = c_1 (m^2 + 1)a^2 + c_5 (m^2 + 4m + 1)a^4$ for $q_{\text{LO}}$. Let $E_{\text{FF}}(\tilde{\rho}, p)$ and $E_{\text{LO}}(\tilde{\rho}, m)$ be the energies per particle for FF and LO states, and let $p = p_{\text{opt}}(\tilde{\rho})$ and $m = m_{\text{opt}}(\tilde{\rho})$ be the values which minimize $E_{\text{FF}}$ and $E_{\text{LO}}$ under fixed $\tilde{\rho}$, respectively. We also define $p_{\text{opt}}(m)$ as an inverse function of $m_{\text{opt}}(\tilde{\rho})$. The evaluation of them is summarized in the Supplemental Material [35]. Figure 2 shows the plot of $E_{\text{FF}}(\tilde{\rho}, p_{\text{opt}}(\tilde{\rho}))$ and $E_{\text{LO}}(\tilde{\rho}, m_{\text{opt}}(\tilde{\rho}))$ and corresponding peridods. From Fig. 2 we can observe the two facts: (i) If the number density becomes small ($\tilde{\rho} < \frac{c_3}{\bar{c}_3}$), the LO state has a lower energy than the uniform state. (ii) The LO state always has a lower energy than the FF state, except for the dilute limit $\tilde{\rho} \to 0$. Thus the density-modulated LO state is shown to be the lowest-energy state for small particle densities. As shown later, the LO state is linearly stable.

AKNS form.—Equation (2) enjoys the Ablowitz-Kaup-Newell-Seur (AKNS) or the zero-curvature representation [23, 36],

$$\partial_x f = U(x, \tilde{\rho}), \quad \partial_t f = V(x, \tilde{\rho}),$$

where $\tilde{\rho}$ is a spectral parameter and $f$ is a two-component vector. The matrices $U$ and $V$ for Eq. (2) are given by [23, 24]

$$U = \left( \begin{array}{cc} -i\lambda q & q' \\ q & i\lambda \end{array} \right), \quad V = -\mu q^{(1)} + c_3 q^{(3)} + c_5 q^{(5)},$$

where $V^{(a)} = \sum_{j=0}^{a-1} (-2)\lambda^{a-j-1}M^{(j)}$ and the explicit forms of $M^{(j)}$ are given in Ref. [37] with setting $r = q^2$. We can check that the compatibility condition $U_1 - V_1 + [U, V] = 0$ yields Eq. (2). Note that the first equation of Eq. (6) is equivalent to the linearized fermionic BdG equation for the quasiparticle wavefunctions with eigenenergy $-\lambda$:}

$$\left( \begin{array}{cc} i\lambda \Delta & i\Delta \partial_x \\ -i\partial_x & \Delta \end{array} \right) f = -\lambda f, \quad \Delta = iq. \quad (8)$$

Let us define $V_3 := V_{ij=1, c_3=0}$, which yields the AKNS$_1$ equation $iq = -\mu q - q(x) + 2iq^2q$. Using $V_3$, the associated Riemann surface for the LO state (5) is given by

$$\omega^2 = \det V_{ij=q_{\text{LO}}} = 4\lambda^2 - 2\lambda^2(1+m)^2 + \frac{q^4}{4}a^4(1-m)^2. \quad (9)$$

The condition $\omega^2 > 0$ determines the continuous spectrum of the BdG operator [37], which is given by $|q| < \frac{a(1-\sqrt{m})}{2}$ and $|\lambda| > \frac{a(1+\sqrt{m})}{2}$. This two-gap structure has been known in both the condensed matter and the algebro-geometric context [33, 38, 39]. We also define

$$\omega_{\text{LO}} := \det V_{ij=q_{\text{LO}}} = [4c_5\lambda^2 + c_3 + c_5a^2(m+1)]^2 \omega^2, \quad (10)$$

FIG. 1. (Color online) (upper) The energies per particle for FF and LO states. We set $c_1 = -1$ and $c_5 = 1$. For reference, we also show the energy for the uniform state $q = \sqrt{\rho}$. (lower) The periods of FF and LO states, which are given by $2\pi/\rho$ and $4K/\alpha$, respectively.
which will become necessary to describe the time evolution of transition coefficients.

Eigenstates of fermionic BdG operator.— We parametrize the Riemann surface [2] by the uniformization variable \( z \): \[ -\lambda(z) = \frac{ia}{2} \left( \frac{dn(z|m)}{sn(z|m)} - \frac{dn(z|\bar{m})}{sn(z|\bar{m})} \right). \] \[ \omega(z) = \alpha \lambda'(z) = \frac{a^2}{2} \left[ \frac{dn(z|K'-z|m)}{sn(z|m)} - \frac{dn(z|\bar{m})}{sn(z|\bar{m})} \right]. \] The relation between \( z \) and \(-\lambda\) is shown in Fig. 2. \( \omega_5 \) is also parametrized in the same way: \( \omega_5(z) = \{4c_5, \lambda(z)^2 + c_3 + c_5 a^2(m+1)\} \omega(z) \). \( \lambda(z) \) and \( \omega(z) \) have the following periodicity, parity, and the involution relation: \( \lambda(z) = \lambda(z+2nK'+2ilK) = \lambda(K'-z) = \lambda(z^*) \) and \( \omega(z) = \omega(z+2nK'+2ilK) = -\omega(K'-z) = \omega(z^*) \) with \( n \) and \( \pi \) integers. \( \omega_5(z) \) has the same property with \( \omega(z) \).

Henceforth we use the shorthand notation \( z' = K'-z \) unless otherwise noted. The two linearly independent solutions for Eq. (5) with \( \lambda = \lambda(z) \) and \( q = q_{1,0}(x, \tilde{p}, m) \) are given by \[ f_0(x, z) = \frac{i\sigma_2(0)\sigma_3(0)e^{i\bar{q}(x)}}{\sigma_3(0)\sigma_4(X)} \left( \theta_1(X-iZ)/\theta_4(iZ) \right) \left( \theta_3(X-iZ)/\theta_1(iZ) \right) \] and \( f_0(x, z') \), where \( \theta_{\alpha}(u) = \theta_{\alpha}(u, q) \) is a theta function with the nome \( q = e^{-\pi K'/K}, X = \frac{m}{2\sigma_1}, Z = \frac{m}{2\sigma_1}, \) and \( k_0(z) = k(z) - \frac{k_0}{2} \left( \frac{1}{\sqrt{1-x^2}} - \frac{x}{2} \right) \). Here \( \Pi(n|m) \) is the complete elliptic integral of the third kind and \( \mathbf{1} \) is a floor function [44]. Note that \( k(z) \) is meromorphic in the whole \( z \)-plane, since the discontinuity of \( \Pi \) and \( \mathbf{1} \) cancels out. \( k(z) \) satisfies \( k(z) = k(z+2nK'+2ilK) + \frac{m}{k} = -k(z') = k(z^*) \). \( f_0(x, z) \) has the periodicity \( f_0(x, z) = (-1)^f_0(x,z+2nK'+2ilK) \) and the involution relation \( f_0(x, z') = \sigma_1 f_0(x, z)^* \). Note that the solution [15] is also found in Ref. [40] with a slightly different convention.

The time evolution of \( f_0(x, z) \) by Eq. (6) with \( \lambda = \lambda(z) \) is described using \( \omega_5 \). If we choose the initial condition at \( t = 0 \) as \( f = A f_0(x, z) + B f_0(x, z') \), the state at \( t \) is given by \( f = A f_0(x, z)e^{-i\omega(z)t} + B f_0(x, z')e^{-i\omega(z)^*t} \).

Eigenstates of bosonic Bogoliubov operator and linear stability.—In the previous paragraph, we have obtained the spectrum of the fermionic BdG operator by regarding \( q \) as a fermionic condensate. Next, let us derive the bosonic Bogoliubov spectrum by regarding \( q \) as a bosonic condensate. The bosonic Bogoliubov equation can be obtained by linearization of Eq. (2) [41], and we can solve it by the squared eigenfunction [42, 43]. More specifically, let \( f = (\varphi_{\text{Fermi}}, \varphi_{\text{Fermi}}') \) be the solution of Eq. (6), and \( (\varphi_{\text{Bose}}, \varphi_{\text{Bose}}') = (\varphi_{\text{Fermi}}, -\varphi_{\text{Fermi}}') \) solves the bosonic Bogoliubov equation. See the Supplemental Material for technical details [45]. Using this fact and the explicit solution [14], we can readily obtain the dispersion relation as \( (2k(z), -2\omega_5(z)) \). See Fig. 3. Since this bosonic condensate breaks the U(1)-gauge and the translational symmetry, two NG modes appear, i.e., the Bogoliubov phonon and the lattice-vibrating phonon. Figure 3 also automatically proves that there is no negative or complex eigenvalue. Thus the LO state is stable [44].

ISM with LO background.—From this point onwards, we formulate the ISM in the presence of the LO background, and construct the soliton solution. The corresponding problem for the KdV equation is solved in Ref. [45] and generalized to the finite-gap potential case in Ref. [40]. Here we construct the corresponding theory for the NLS hierarchy. In this Letter we only show the main result and sketch of derivations. Detailed discussions will be published elsewhere.

Let us assume that \( q \) satisfies the boundary condition \[ \Phi(x) \rightarrow \begin{cases} q_{1,0}(x) & (x \rightarrow -\infty) \\ q_{1,0}(x-x_0)e^{2ip_0} & (x \rightarrow +\infty) \end{cases}, \] where we omit the arguments \( \tilde{p} \) and \( m. \varphi_0 \) and \( x_0 \) denote the phase and density shift due to solitons and ripple waves. We define the right Jost solution by the asymptotic form \[ f_+(x) \rightarrow \begin{cases} a(z) f_0(x, z) + b(z) f_0(x, z') & (x \rightarrow -\infty) \\ e^{i\varphi(x)} f_0(x-x_0, z) & (x \rightarrow +\infty). \end{cases} \]
This expression also defines the transition coefficients \( a(z) \) and \( b(z) \). The other right Jost solution is given by \( f_+(x, z') \). Calculating the Wronskian \( W = f_+(x, z')' \langle i \sigma_2 \rangle f_+(x, z) \), the relation \( a(z)a(z') - b(z)b(z') = 1 \) follows. The left Jost solution is defined by \( f_-(x, z) \xrightarrow{x \to \infty} f_0(x, z') \). The right and left Jost solutions are related as follows:

\[
\begin{align*}
\left( f_+(x, z) & f_+(x, z') \right) = \left( f_-(x, z') f_-(x, z) \right) \begin{pmatrix} a(z) & b(z') \\ b(z') & a(z) \end{pmatrix}.
\end{align*}
\]

(16)

\( f_\pm \) has the same periodicity and involution relation with \( f_0 \).

The transition coefficients satisfy \( a(z') = a(z')^* \) and \( b(z') = b(z')^* \) and have the same periodicity with \( \lambda(z) \).

Discrete eigenvalues for bound states are given by the zeros of \( a(z) \). Since the BdG operator is self-adjoint, bound states appear only for real \( \lambda \), which corresponds to \( \text{Re} \ z = \pm \frac{1}{2} \), as shown in Fig. 2. Let us assume that there are \( N \) zeros and write them as \( z_1, \ldots, z_N \). We can prove that the normalization constant for the \( j \)-th bound state defined by \( C_j := \int_{-\infty}^{\infty} \text{dx} f_+(x, z_j)^* f_+(x, z_j) \) is given by \( C_j^2 = 2i a(z_j) b(z_j) \).

We introduce the triangular representation for the left Jost solution:

\[
\begin{align*}
&f_+(x, z) = f_0(x, z) + \int_{-\infty}^{x} \text{dx} \Gamma(x, y) f_0(y, z) \tag{17}
\end{align*}
\]

where \( \Gamma(x, y) = a(x) b(y)^* - b(x) a(y)^* \).

The \( k \)-th column of Eq. (16) is given by \( f_0(x, z)^T \sigma_1 \), where \( R \) is a rectangular contour shown in Fig. 2. We obtain the Gel’fand-Levitan-Marchenko (GLM) equation

\[
\begin{align*}
\Gamma(x, y) + \Omega(x, y) + \int_{-\infty}^{x} \text{dw} \Gamma(x, w) \Omega(w, y) = 0, \quad (y < x),
\end{align*}
\]

\[
\begin{align*}
\Omega(x, y) := \int_{R} \frac{dz}{4 \pi i a(z)} b(z) f_0(x, z') f_0(y, z')^T \sigma_1
\end{align*}
\]

(17)

with \( z'_j := K' - z_j \).

\( N \)-soliton solution.— Let us solve the GLM equation for the reflectionless case \( b(z) = 0 \). It can be solved by the following ansatz: \( \Gamma(x, y) = \sum_{j=1}^{N} C_j \frac{g_j(x, y)}{g_j(x, z'_j)} f_0(y, z'_j) \sigma_1 \). Substitution of this ansatz into the GLM equation yields the linear equations

\[
\begin{align*}
g_j(x) + \alpha C_j u_0(x, z'_j) + \alpha \sum_{j=1}^{N} C_j C_j^* M(x, z'_j, z'_j) g_j(x) = 0, \quad (j = 1, \ldots, N), \tag{18}
\end{align*}
\]

for \( g_j(x)'s \):

\[
\begin{align*}
g_j(x) + \alpha C_j u_0(x, z'_j) + \alpha \sum_{j=1}^{N} C_j C_j^* M(x, z'_j, z'_j) g_j(x) = 0, \quad (j = 1, \ldots, N), \tag{18}
\end{align*}
\]

where \( u_0(x, z) = \text{first component of } f_0(x, z) \) and

\[
\begin{align*}
M(x, z_j, z_l) := -2a \frac{\partial \eta_j (0) \partial \eta_l (0) \eta_l (X + iz_j - iz_l)}{\partial \eta_j (0) \eta_l (X + iz_j - iz_l)} e^{-i [k(z_j) + k(z_l)] x},
\end{align*}
\]

with \( Z_j = \frac{\pi i}{M x} \) and \( Z'_l = \frac{\pi i}{M x} \). Using \( g_j(x)'s \), we obtain the \( N \)-soliton solution \( q(x) = q_{LO}(x) + 2 \sum_j C_j g_j(x) u_0(x, z'_j) \).

The time-dependence of \( C_j \) is given by \( C_j(t) = C_j(0) e^{-2i \omega_0 (z'_j) t} \), and the velocity of each soliton becomes \( v_j = \frac{\text{Im} \omega_0 (z'_j)}{\text{Im} \omega_0 (z'_j)} \).

In particular, the one-soliton solution is given by

\[
\begin{align*}
q(x, t) = q_{LO}(x) - \frac{2 \omega_0 C_j(0)^2 e^{-2i \omega_0 (z'_j) t} u_0(x, K' - z_j)^2}{1 + \alpha C_j(0)^2 e^{-2i \omega_0 (z'_j) t} M(x, z_j, z_j)}, \quad (j = 1, \ldots, N). \tag{19}
\end{align*}
\]

This one-soliton solution shows various behaviors dependent on the choice of parameters. When \( m = 1 \), we can broadly classify it into three categories by its velocity, that is, dark solitons, stationary dislocations, and gray solitons. See Fig. 4. In this case, the soliton propagation can be understood approximately as a successive collision between the moving soliton and the array of static dark solitons. Fig. 5(a) shows an example of the dark soliton billiard. We note that the LO background experiences a density shift after the passing of the soliton, as opposed to conventional gap
solitons. When $m = 0$, the LO background is almost trigonometric. In this case the distinction between billiards and gray solitons becomes obscure, and solitons with any velocity can be observed as envelope solitons (Figure 5b). This behavior is very similar to the solitons observed in the soft-core bosons [29]. See also the ancillary animated GIF files for the other examples. The soliton parameters for these animations are noted in the Supplemental Material [35].

Summary.—In summary, we have introduced an integrable model of density-modulated quantum condensates and have provided an $N$-soliton solution. We have shown that the model can be constructed as a linear combination of conserved quantities in the NLS hierarchy and that density-modulated states have lower energy than uniform states. The obtained exact soliton solutions contain various novel dynamics, such as soliton billiards, stationary dislocations, gray solitons, and envelope solitons. We believe that our results are universal and will be useful to understand nonequilibrium and transport phenomena in non-uniform quantum matters. Finally we note several future works, such as the extension to the self-focusing case ($r = -q^2$), the determination of self-consistent soliton solutions in B"{o}dG systems [47], and an exact theory of spontaneous translational-symmetry breaking in the corresponding quantum model.

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Supplemental Material

I. ENERGY EVALUATION FOR FF AND LO STATES

The energy density $h(x)$ at a position $x$ is defined by the integrand of Eq. (1) of the main article. The energy per particle is then defined by $E = \int_0^L dx h(x)$, where $L$ is a period and given by $L = 2\pi/p$ for the FF state and $L = 4K/k_0$ for the LO state, respectively. After a little calculation, we obtain

$$E_{\text{FP}}(\hat{p}, p) = c_5(p^2 + \hat{p} + c_5(p^4 + 6p^2\hat{p} + 2\hat{p}^2)), \quad (S1)$$

$$E_{\text{LO}}(\hat{m}, m) = c_5\hat{m}(m + (m + 1)Q(m)) + c_4\hat{p}^2[2m(m + 1) + (m^2 + 4m + 1)Q(m)]/5Q(m)^3. \quad (S2)$$

Let $p = p_s(\hat{p})$ and $m = m_s(\hat{p})$ be the value which minimize the above energy density $E_{\text{FP}}$ and $E_{\text{LO}}$, respectively. Then, they are determined as follows:

$$p_s(\hat{p}) = \begin{cases} 0 & (\hat{p} > \frac{c_5}{6c_3}) \\ \pm \sqrt{c_3 + 6c_3}\hat{p}/(2c_3) & (\hat{p} < \frac{c_5}{6c_3}) \end{cases} \quad (S3)$$

$$m_s(\hat{p}) = \begin{cases} 1 & \text{inverse function of } \hat{p}_s(m) \quad (\hat{p} > \frac{5c_5}{18c_3}) \\ \frac{5c_3 - 2m + (1 + m)Q(m)}{6c_5[3m(1 + m) + (1 + 4m + m^2)Q(m)]} & (\hat{p} < \frac{5c_5}{18c_3}) \end{cases} \quad (S4)$$

$$\hat{p}_s(m) = \frac{-5c_3[-2m + (1 + m)Q(m)]Q(m)}{6c_5[3m(1 + m) + (1 + 4m + m^2)Q(m)]}. \quad (S5)$$

Here we have assumed $c_3 < 0$ and $c_5 > 0$.

II. SOLUTIONS FOR BOSONIC BOGOLIUBOV EQUATION

The bosonic Bogoliubov equation can be obtained by linearizing Eq. (2) of the main article. More specifically, setting $q = q + \delta q$ in Eq. (2) and cutting higher order terms with respect to $\delta q$, we obtain

$$i\delta q_t = -\mu\delta q + c_3[-\delta q_{xx} + 2(2|q|^2\delta q + q^2\delta q^*)] + c_5[\delta q_{xxxx} - 2(q^*\delta q + q\delta q^*)_{xx}q - 2(q^2)_{xx}\delta q - 3(q^2)_{x}\delta q^* - 6q^*(q\delta q)_{xx} + 6|q|^2(3|q|^2\delta q + 2q^2\delta q^*)]. \quad (S6)$$

Substituting $(\delta q, \delta q^*) = (u, -v)$ into Eq. (S6) and c.c. of Eq. (S6), we obtain the Bogoliubov equation:

$$iu_t = -\mu u + c_3[-u_{xx} + 2(2|q|^2u^* - q^2v) + c_5[u_{xxxx} - 2(q^*u + qv)_{xx}q - 2(q^2)_{xx}u + 3(q^2)_{x}v \quad (S7)$$

$$- 6q^*(qu)_{xx} + 6|q|^2(3|q|^2u - 2q^2v)],$$

$$iv_t = \mu v - c_3[ - v_{xx} + 2(2|q|^2v - q^2u)] - c_5[v_{xxxx} - 2(qv - q^*u)_{xx}q - 2(q^2)_{xx}v + 3(q^2)_{x}u \quad (S8)$$

$$- 6q^*(qv)_{xx} + 6|q|^2(3|q|^2v - 2q^2u)].$$

The stationary Bogoliubov equation with the eigenenergy $\epsilon$ can be obtained by the substitution $(u(x, t), v(x, t)) = (u(x), v(x))e^{-i\epsilon t}$. As stated in the main article, we can solve the above equation by the squared eigenfunction. If $f = (u_{\text{Fermi}}, v_{\text{Fermi}})$ is a solution of Eq. (6) of the main article,

$$(u_{\text{Bose}}, v_{\text{Bose}}) = (u_{\text{Fermi}}, -v_{\text{Fermi}}) \quad (S9)$$

becomes a solution of the bosonic Bogoliubov equation (S7) and (S8). Since the time-evolution of the solution $(u_{\text{Fermi}}, v_{\text{Fermi}})$ with the spectral parameter $\lambda = \lambda(z)$ is given by $e^{i\lambda(z)\tau}$, the time-dependence of the corresponding bosonic solution is given by $e^{2i\lambda(z)\tau}$, which means that the eigenenergy of the eigenstate labeled by $z$ is given by $-2\omega_5(z)$. For the same reason, the crystal momentum is given by $2k(z)$. Thus we obtain Fig. 3 of the main article.

III. PARAMETERS FOR THE ANIMATED GIF FILES

- animation1.gif: Dark soliton billiard. The parameters are the same as Fig. 5(a).
- animation2.gif: Gray soliton. $z_1 = -0.5K' + 0.05iK$ ($\leftrightarrow \lambda = 0.471$) and $C_1(0) = 1.23$. The velocity is given by $v_1 = -0.469$. The other parameters are the same as Fig. 5(a).
- animation3.gif: Static dislocation. $z_1 = -0.5K' + 0.1066iK$ ($\leftrightarrow \lambda = 0.333$) and $C_1(0) = 0.0463$. The velocity is given by $v_1 = 0$. The other parameters are the same as Fig. 5(a).
- animation4.gif: Example of 3-soliton solution. $z_1 = -0.5K' + 0.3iK, z_2 = -0.5K' + 0.1066iK, z_3 = -0.5K' + 0.05iK, C_1(0) = 0.643, C_2(0) = 0.0463, and C_3(0) = 243$. The other parameters are the same as Fig. 5(a).
- animation5.gif: Envelope soliton. The parameters are the same as Fig. 5(b).
- animation6.gif: Another envelope soliton. $z_1 = -0.5K' + 0.34iK$ ($\leftrightarrow \lambda = 0.357$) and $C_1(0) = 5.17$. The velocity becomes $v_1 = -0.0477$. The other parameters are the same as Fig. 5(b).