On the Scattering Method for the $\bar{\partial}$-equation and Reconstruction of Convection Coefficients

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September 28, 2018

Abstract: In this paper we reconstruct convection coefficients from boundary measurements. We reduce the Beals and Coifman formalism from a linear first order system to a formalism for the $\bar{\partial}$-equation.

1 Introduction

The pioneering work of Nachman and Ablowitz [16], Sylvester and Uhlmann [21], Nachman [17] and Henkin and Novikov [10] introduced inverse scattering methods to the parameter identification problems. In their work, the linear Schrödinger equation in the physical space is paired with a pseudo-analytic equation in the complex space of the parameter. Another method, due to Beals and Coifman [2], pairs a first order $\bar{\partial}$ system in the physical space with a pseudo-analytic matrix equation in the parameter space. Sung analyzed lower regularity assumptions in [22, 23, 24]. This method was ingeniously used by Brown and Uhlmann [4] in unique identification of the conductivity $\sigma \in \nabla \cdot \sigma \nabla u = 0$ and by Cheng and Yamamoto [5], [6] in proving unique determination of the convection coefficients $b_1$ and $b_2$ in $\Delta u + b_1 u_x + b_2 u_y = 0$.

We consider here the scattering problem for $\bar{\partial}$-equations (theorems 1.1 and 1.2 below). Here $\bar{\partial} = (\partial_x + i\partial_y)/2$ is the Cauchy-Riemann operator. This can be seen as a diagonal version of the formalism in Beals and Coifman, see lemma 2.1. Due to the symmetry between the scattered solutions in the physical space and the ones in the parameter space, we are able to present a non-linear analog of the Fourier inversion formula (compare (2) and (7) below).

As an application, we revisit the inverse problem proposed in [6] and present a reconstruction procedure. The method is based on solving a singular boundary integral equation in the Hardy space of functions in the exterior of the disc. This method was first introduced by Knudsen and Tamasan in connection with the electrical impedance tomography problem in [12]. The method presented here can be seen as its generalization.

I was informed recently about the reconstruction step being obtained independently by Tong, Cheng and Yamamoto [7]. I thank them for letting me know about their new result. The main difference of the method presented here from their method is the formalism of inverse scattering.

For $k \in \mathbb{C}$ arbitrarily fixed, we say that $u$ behaves like $e^{izk}$ (written $u \sim e^{izk}$) in $L^r(\mathbb{R}^2)$ for large $z$, if $u(z,k)e^{-izk} - 1 \in L^r(\mathbb{R}^2)$. We use the notation $\langle k \rangle = (1 + |k|^2)^{1/2}$. The scattering method is the content of the following two theorems.

∗This work was done during the author’s visit at IPAM-UCLA in the Fall 2003
Theorem 1.1 (Forward Scattering). Assume that $q \in L^\tilde{p}_c(\mathbb{R}^2_+)$, $\tilde{p} > 2$ has compact support. For each $k \in \mathbb{C}$, the equation
\[
\frac{\partial \Psi}{\partial z}(z) + q(z)\Psi(z) = 0, \quad z \in \mathbb{C},
\] has unique solutions $\Psi_r(z,k) \sim e^{izk}$ and $\Psi_i(z,k) \sim ie^{izk}$ in $L^\tilde{p}$ for large $z$, and the scattering transform
\[
t(k) = -\frac{i}{\pi} \int_{\mathbb{R}^2} e^{izk}\Psi(z) (\Psi_r(z,k) - i\Psi_i(z,k)) \, d\mu(z)
\]
is well defined. Moreover, if $q \in W^\varepsilon_{\tilde{p}'}(\mathbb{R}^2_+)$ for some $\varepsilon > 0$ and $k \in \mathbb{C} - \{0\}$, we have
\[
\|\Psi_r(z,k)e^{-izk} - 1\|_{L^\tilde{p}(\mathbb{R}^2_+)} + \|\Psi_i(z,k)e^{-izk} - i\|_{L^\tilde{p}(\mathbb{R}^2_+)} \leq C \langle k \rangle^{-\varepsilon}
\]
and
\[
\|[\Psi_r(z,k) - i\Psi_i(z,k)]e^{-izk} - 2\|_{W^{1,\tilde{p}}(\mathbb{R}^2_+)} \leq C \langle k \rangle^{-\varepsilon},
\]
and then $t \in L^r(\mathbb{R}^2_+)$ for each $r > 2/(\varepsilon + 1)$. In particular $t \in L^r(\mathbb{R}^2_+ \cap L^{r'}(\mathbb{R}^2_+) \cap L^{r\tilde{r}}(\mathbb{R}^2_+))$ for some $r < 2$, where $r^{-1} = r^{-1} - 1/2$ and $r^{r^{-1}} + r^{-1} = 1$.

Theorem 1.2 (Inverse Scattering). Let $q$, $\Psi_r$, $\Psi_i$ and $t(k)$ and $r$, $r'$, $\tilde{r}$ be as given in the forward scattering. Then the equation
\[
\frac{\partial \Phi}{\partial k}(k) + t(k)\Phi(k) = 0, \quad k \in \mathbb{C},
\]
has unique solutions $\Phi_r \sim e^{izk}$ and $\Phi_i \sim ie^{izk}$ in $L^\tilde{r}(\mathbb{R}^2_k)$ for large $k \in \mathbb{C}$. Moreover, $\Psi$‘s and $\Phi$‘s are related by
\[
\begin{align*}
\text{Re } \Phi_j &= -\text{Im } \Psi_j, & \text{Re } \Phi_r &= \text{Re } \Psi_r, \\
\text{Im } \Phi_j &= \text{Im } \Psi_j, & \text{Im } \Phi_r &= -\text{Re } \Psi_i,
\end{align*}
\]
in particular $\Phi_r - i\Phi_i = \Psi_r - i\Psi_i$ and
\[
q(z) = -\frac{i}{\pi} \int_{\mathbb{R}^2} e^{izk}t(k) (\Phi_r(z,k) - i\Phi_i(z,k)) \, d\mu(k).
\]

Let $\Omega \subset \mathbb{R}^2$ be a bounded, simple connected domain with Lipschitz boundary and $\tilde{p} > 2$. For $b_1, b_2 \in L^\tilde{p}(\Omega)$ and $g \in W^{2-1/\tilde{p},\tilde{p}}(\partial\Omega)$, let $u \in W^{2,\tilde{p}}$ be the unique solution of the boundary value problem
\[
\begin{align*}
\Delta u(x) + b_1(x)\frac{\partial u(x)}{\partial x_1} + b_2(x)\frac{\partial u(x)}{\partial x_2} &= 0, & x \in \Omega, \\
u(x) = g(x), & x \in \partial\Omega.
\end{align*}
\]
The Dirichlet to Neumann map $\Lambda_{b_1, b_2} : W^{2-1/\tilde{p},\tilde{p}}(\partial\Omega) \to W^{1-1/\tilde{p},\tilde{p}}(\partial\Omega)$ is given by
\[
\Lambda_{b_1, b_2} g(x) = \nu_1(x)\frac{\partial u(x)}{\partial x_1} + \nu_2(x)\frac{\partial u(x)}{\partial x_2}, \quad x \in \partial\Omega,
\]
where $(\nu_1(x), \nu_2(x))$ is the outer normal at $x$ on the boundary. Cheng and Yamamoto proved that $\Lambda_{b_1, b_2}$ uniquely determines $b_1$ and $b_2$ in $L^\tilde{p}(\Omega)$.

Working with the equation in the whole plane and using the inverse scattering for $\dbar$-equations allows us to go beyond uniqueness and present a method of reconstruction. We prove the following result.
**Theorem 1.3.** Let $\Omega \subset \mathbb{R}^2$ be bounded, simple connected domain with Lipschitz boundary, and $b_1, b_2 \in W^1_p(\Omega)$, with support inside $\Omega$ for some $\varepsilon > 0$. Then $b_1, b_2$ can be reconstructed from $\Lambda_{b_1, b_2}$.

The fact that they vanish on the boundary is not a severe restriction as one can always extended the coefficients across the boundary, preserving the regularity, and then have them vanish outside a ball. The Dirichlet-to-Neumann map can be pushed to an outside boundary as shown by Nachman in [17], see also [13]. While $\hat{L}^p(\Omega)$ is enough regularity to prove unique determination of $b_1, b_2$, we assume here $\varepsilon$-extra regularity and provide a reconstruction method.

In the end we point out the connection with the first order $\overline{\partial}$ system and characterize its Cauchy data in terms of the Dirichlet-to-Neumann map of a related second order elliptic equation, thus answering a question of Uhlmann in [25].

2 Proof of the theorems 1.1 and 1.2

We identify a point in $\mathbb{R}^2$ with a point in the complex plane by $x_1 + ix_2 = z$. By $\overline{\partial}^{-1}$ we denote the solid Cauchy transform

$$\overline{\partial}^{-1} f(z) = \frac{1}{\pi} \int_{\mathbb{R}^2} \frac{f(\zeta)}{z - \zeta} d\mu(\zeta), \quad (9)$$

where $d\mu(\zeta)$ is the Lebesgue area. We also denote by $e(z, k) = \exp(i zk + \bar{z}k)$.

We look for solutions of (1) of the form $\Psi_r = \psi_re^{\pm zk}$ and $\Psi_i = i\psi_ie^{\pm zk}$ with $\psi_r, \psi_i \in 1 + \hat{L}^p(\mathbb{R}^2)$. The equations for $\psi_r$ respectively $\psi_i$ are

$$\begin{align*}
\partial_{\bar{z}} \psi_r + qe(z, -k)\overline{\psi_r} &= 0, \\
\partial_{\bar{z}} \psi_i - qe(z, -k)\overline{\psi_i} &= 0.
\end{align*} \quad (10)$$

A key ingredient is the Hardy-Littlewood Sobolev inequality which yields $\overline{\partial}^{-1} : L^p(\mathbb{R}^2) \to \hat{L}^p(\mathbb{R}^2)$ is bounded (see Stein [20]) for $p$ and $\hat{p}$ related by

$$\frac{1}{\hat{p}} = \frac{1}{p} - \frac{1}{2}. \quad (11)$$

Since $q \in L^p_{\text{loc}}(\mathbb{R}^2) \subset L^2(\mathbb{R}^2)$ and $L^\hat{p}(\mathbb{R}^2) : L^2(\mathbb{R}^2) \subset L^p(\mathbb{R}^2)$ we have $\overline{\partial}^{-1}(q) : L^p t(\mathbb{R}^2) \to \hat{L}^p(\mathbb{R}^2)$ is a bounded operator. Since $q$ has compact support we can use Rellich imbedding to conclude that $\overline{\partial}^{-1}(q) : L^p(\mathbb{R}^2) \to \hat{L}^p(\mathbb{R}^2)$ is compact. Then we can apply Fredholm’s alternative in $\hat{L}^p(\mathbb{R}^2)$ to the equivalent integral equation

$$\begin{align*}
\{I + \overline{\partial}^{-1}[q(\cdot)e(\cdot, -k)\overline{\cdot})]\}(\psi_r(z) - 1) &= \overline{\partial}^{-1}[qe(\cdot, -k)], \\
\{I - \overline{\partial}^{-1}[q(\cdot)e(\cdot, -k)\overline{\cdot})]\}(\psi_i(z) - 1) &= -\overline{\partial}^{-1}[qe(\cdot, -k)].
\end{align*} \quad (12)$$

The fact that the homogeneous equation has only the null solution comes from Liouville’s theorem for pseudo-analytic functions with coefficients in $L^\hat{p}(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)$ shown by Vekua [V62]. Since we integrate in (2) over the support of $q$, together with the imbedding $L^\hat{p}_{\text{loc}} \subset L^p_{\text{loc}}$, gives a pointwise well defined $t(k)$.
For $k \in \mathbb{C}$ let $(\overline{\partial} - ik)^{-1}$ be defined by $e(z, k)\overline{\partial}^{-1}(e(z, -k) \cdot)$ and let the indexes $\hat{p}$ and $p$ be related by (11). An interpolation (with $\epsilon$ being the interpolation parameter) between the estimates of Nachman [18] $||((\overline{\partial} - ik)^{-1} f ||_{L^p} \leq C ||f||_{L^p(\mathbb{R}^2)}$ and $||((\overline{\partial} - ik)^{-1} ||_{L^p} \leq (C/|k|)||f||_{W^{1,p}(\mathbb{R}^2)}$ gives

$$||((\overline{\partial} - ik)^{-1} f ||_{L^p} \leq \frac{C}{|k|^\epsilon} ||f||_{W^{\epsilon,p}(\mathbb{R}^2)}.$$  \hspace{3em} (13)

See Proposition 2.3 in [13] for details. Since $|e(z, k)| = 1$ the last estimate implies that

$$||\overline{\partial}^{-1}(e(\cdot, -k)q)||_{L^p(\mathbb{R}^2)} \leq C ||\langle k \rangle^{-\epsilon}||q||_{W^{\epsilon,p}(\mathbb{R}^2)}.$$  \hspace{3em}

The decay rate in (3) follows from the (uniform in $k$) bounded-ness of the map $[I - \overline{\partial}^{-1}qe(\cdot, -k)\cdot]^{-1}$ from $L^p(\mathbb{R}^2)$ to $L^p(\mathbb{R}^2)$ as explained above. The further regularity property for the combination $\Psi_r + i\Psi_i$ in (11) will be shown in Lemma 2.1. For now we assumed it holds.

Brown and Uhlmann [4] showed that $q \in L^p(\mathbb{R}^2)$ implies $t \in L^2(\mathbb{R}^2)$. While this is good enough for existence, for reconstruction we need $t \in L^r$ for some index $r < 2$. This is ensured by extra regularity imposed in $q$ as was shown by Knudsen and the author in [13]. For completeness we repeat the arguments. The main ingredient is an $L^2$ bounded-ness property for pseudo-differential operators with non smooth symbol (see Coifman and Meyer [8] or Brown and Uhlmann [3]). If $Mq$ is defined by

$$Mq(k) = \frac{i}{\pi} \int_{\mathbb{R}^2} e(z, k)\overline{q}(z)a(z, k)d\mu(z),$$

where $a$ has compact support in $z$ and $||a(\cdot, k)||_{H^s(\mathbb{R}^2)} \leq C ||\langle k \rangle^{-\epsilon}||q||_{W^{\epsilon,p}(\mathbb{R}^2)}$, then $M : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ is bounded.

Rewrite now

$$t(k) = -2i\mathcal{F}(\overline{q})(-2k_1, 2k_2) + T(k) \hspace{3em} (14)$$

$$T(k) = \frac{i}{\pi} \int_{\mathbb{R}^2} e(z, k)\overline{q}(z)[\psi_r(z, k) + \psi_i(z, k) - 2]d\mu(z), \hspace{3em} (15)$$

where $\mathcal{F}$ is the Fourier transform. Since $q \in L^p(\mathbb{R}^2) \subset L^r(\mathbb{R}^2)$ for $1 \leq r \leq 2$ we get $\mathcal{F}(\overline{q}) \in L^s(\mathbb{R}^2)$ for $s > 2/(1 + \epsilon)$.

Let $M$ be the operator defined by $a(z, k) = \chi(z)[\psi_r(z, k) + \psi_i(z, k) - 2] \in W^{1,\hat{p}}(\mathbb{R}^2) \subset H^1(\mathbb{R}^2)$, where $\chi$ is a cut-off function equal to 1 on the support of $q$. The following chain of inequalities for $0 < \delta < \epsilon$ give the result

$$||T||_{L^s(\mathbb{R}^2)} = ||\langle k \rangle^{-\delta} M\overline{q}||_{L^s(\mathbb{R}^2)} \leq C ||\langle k \rangle^{-\delta}||_{L^{1/(s-1/2)-1}} ||q||_{L^2(\mathbb{R}^2)},$$

for $\delta > 2(1/s - 1/2)$ or equivalently $s > 2/(\delta + 1) > 2/(\epsilon + 1)$.

In order to exhibit the relation with the old formalism, we prove theorem [12] by reducing it to the former. Let us define $m_1(z, k)$ and $m_2(z, k)$ in terms of the $\psi$’s by

$$m_1(z, k) = \frac{1}{2}(\psi_r(z, k) + \psi_i(z, k)) \hspace{3em} (16)$$

$$m_2(z, k) = \frac{1}{2}e(z, -k)(\overline{\psi}_r(z, k) - \overline{\psi}_i(z, k))$$

The simple result below shows that $(m_1, m_2)^t$ is the first column of the Jost matrix in the complex geometrical optic solutions of Beals and Coifman.
Lemma 2.1. Let $m_1$ and $m_2$ defined in (13). Then $m_1(\cdot,k) - 1, m_2(\cdot,k) \in L^\tilde{p}(\mathbb{R},)$, and they satisfy
\[
\overline{\partial} m_1 = q m_2
\]
\[
(\partial + i k) m_2 = \overline{\partial} m_1.
\]
Moreover, the following estimates hold,
\[
||m_1(\cdot,k) - 1||_{W^{1,\tilde{p}}(\mathbb{R},)} \leq C(k)^{-\epsilon}
\]
\[
||m_2(\cdot,k)||_{L^\tilde{p}(\mathbb{R},)} \leq C(k)^{-\epsilon}.
\]

Proof. From their definition $m_1(\cdot,k) - 1, m_2(\cdot,k) \in L^\tilde{p}(\mathbb{R},)$ since $\psi_r, \psi_i \in L^\tilde{p}(\mathbb{R},)$ and $|e(z,k)| = 1$. The fact that they solve the system (17) comes from a straightforward calculation and the equations (10). The $L^\tilde{p}$ estimates of decay in $k$ for not $m_1$ and $m_a$ come from the estimates (8) for $\psi_r$ and $\psi_i$ proven above. We are left to justify the extra smoothness gained by $m_1$. From the first equation we have that $m_1 - 1 = \overline{\partial}^{-1}(qm_2)$. Since $q \in L^\tilde{p}(\mathbb{R},) \subset L^2(\mathbb{R},)$ and $L^2 \cdot L^\tilde{p} \subset L^p$ we have $\overline{\partial}^{-1}(qm_2) \in W^{1,\tilde{p}}(\mathbb{R},)$ with an imbedding constant which depends on the support of $q$ but it is independent of $k$. We have the following chain of inequalities.
\[
||m_1(\cdot,k) - 1||_{W^{1,\tilde{p}}(\mathbb{R},)} = ||\overline{\partial}^{-1}(qm_2)||_{W^{1,\tilde{p}}(\mathbb{R},)} \leq C||q(\cdot)m_2(\cdot,k)||_{L^p(\mathbb{R},)}
\]
\[
\leq C||q||_{L^2(\mathbb{R},)} ||m_2(\cdot,k)||_{L^\tilde{p}(\mathbb{R},)} \leq C < k >^{-\epsilon}.
\]

This also completes the proof the theorem (14).

Formulate the inverse scattering formalism of Beals and Coifman only in terms of the first column of Jost matrix, see Knudsen and Tamasan [13] for details. For the analysis with $q \in L^1(\mathbb{R},) \cap L^\infty(\mathbb{R},)$ see Sung [22], or Brown and Uhlmann [4] for $q \in L^\tilde{p}(\mathbb{R},)$.

Theorem 2.2 (Beals & Coifman scattering method). Let $q \in W^{\ell,\tilde{p}}(\mathbb{R},)$. For any $z \in C$, the system (17) has a unique solution $m_1(z,k), m_2(z,k)$ with $(m_1(\cdot,k) - 1, m_2(\cdot,k)) \in L^\tilde{p}(\mathbb{R},)$. Furthermore, the map $k \rightarrow m(\cdot,k)$ is differentiable (in the norm topology) with values in $W^{\ell,\tilde{p}}(dx)$ and satisfies pointwise in $z \in \mathbb{C}$ the system
\[
\frac{\partial}{\partial k} m_1(z,k) = t(k)e(z,-k)m_2(z,k),
\]
\[
\frac{\partial}{\partial k} m_2(z,k) = t(k)e(z,-k)m_1(z,k),
\]
where
\[
t(k) = -\frac{i}{\pi} \int_{\mathbb{R}} e(z,k) \gamma(z)m_1(z,k) d\mu(z).
\]

Look for solutions of (6) in the form $\Phi(z,k) = ie^{izk}\phi_r(z,k)$ respectively $\Phi_i = e^{izk}\phi_i(z,k)$. As in the forward problem, they must satisfy an integral formulation analogous to (12) where the role of $k$ and $z$ is reversed. Since $t(k) \in L^q(\mathbb{R},) \cap L^2(\mathbb{R},)$ we have existence and uniqueness for their solution in $L^\tilde{p}(\mathbb{R},)$, where $\tilde{r}^{-1} = r^{-1} - 1/2$. Using the equations (20) it is easy to check that
\[
\frac{\partial}{\partial k}(m_1 - m_2)(z,k) = -t(k)e(z,-k)m_1 - m_2(z,k)
\]
\[
\frac{\partial}{\partial k}(m_1 + m_2)(z,k) = t(k)e(z,-k)m_1 + m_2(z,k).
\]
By the uniqueness result we must have
\begin{align}
\phi_i(z, k) &= m_1(z, k) + m_2(z, k) \\
\phi_r(z, k) &= m_1(z, k) - m_2(z, k).
\end{align}
(23)

The following equalities show the relation between solutions of the forward and inverse equation.
\[
\Phi_i = i e^{izk} \phi_i = i e^{izk} (m_1 + m_2) = \frac{i e^{izk}}{2} (\psi_r + \psi_i) + \frac{i e^{izk}}{2} e(z, -k)(\overline{\psi_i} - \overline{\psi_r})
\]
\[
= \frac{i}{2} \Psi_r + \frac{1}{2} \Psi_i - \frac{1}{2} \frac{i}{2} \frac{\overline{\Psi_r}}{\Psi_i} = -\text{Im} \Psi_r + i \text{Im} \Psi_i.
\]

Similarly, \( \Phi_r = \text{Re} \Psi_r - i \text{Re} \Psi_i \). These prove the identities (6). Formula (7) is due to a symmetry argument as follows. Starting with \( q \) produce \( \psi_r \) and \( \psi_i \) by solving (1). Via (3) produce \( \phi_r \) and \( \phi_i \) and then \( t(k) \) as in (17). Take this \( t(k) \) and do now forward scattering starting from the \( k \)-space, i.e. produce \( \Phi_r \) and \( \Phi_i \) by solving (15) and via (14) produce \( \Psi_r \) and \( \Psi_i \). Define a potential \( q_1 \) using (7) for the \( z \)-space. In particular we know that for any \( k \in \mathbb{C} \) we have \( \partial \Psi_r + q \overline{\Psi_r} = 0 \) since we started that way, but also now we have \( \partial \Psi_r + q_1 \overline{\Psi_r} = 0 \). In particular we have \( (q(z) - q_1(z)) \overline{\Psi_r}(z, k) = 0 \) for all \( k \in \mathbb{C} \). Hence \( q = q_1 \).

\section{Reconstructing convection coefficients}

In this section we apply the above scattering method to reconstruction of the convection coefficients \( b_1, b_2 \) in
\[
\Delta u(x) + b_1 \frac{\partial u}{\partial x}(x) + b_2(x) \frac{\partial u}{\partial x}(x) = 0, \quad x \in \Omega
\]
(24)
from the Dirichlet-to-Neumann map \( \Lambda_{b_1, b_2} \). Here \( \Omega \subset \mathbb{R}^2 \) is a bounded, simply connected domain with Lipschitz boundary.

We assume here that \( b_1, b_2 \in W^{\varepsilon, \tilde{p}}(\Omega) \), \( \tilde{p} > 2 \) are real valued maps with compact support in \( \Omega \) and set \( b = (b_1 + ib_2)/4 \).

The following result from Vekua \cite{V62} makes the reduction of (24) to a \( \overline{\partial} \)-equation. If \( u \) is a solution of (24) then \( w = \partial u \) solves
\[
\overline{\partial} w(z) + b(z) \overline{w}(z) + b(z) w(z) = 0.
\]
(25)

\begin{lemma}
Let \( \Omega \) be simply connected with Lipschitz boundary. If \( u \in W^{2, \tilde{p}}(\Omega) \) is a solution of (24), then \( w = \partial u \in W^{1, \tilde{p}}(\Omega) \) is a solution of (25). Conversely, if \( w \in W^{1, \tilde{p}}(\Omega) \) is a solution of (25) then there exists an \( u \in W^{2, \tilde{p}}(\Omega) \) solution of (24) and such that \( \partial u = w \) in \( \Omega \).
\end{lemma}

\begin{proof}
By Sobolev imbedding we have \( u \in C^{1+\alpha}(\Omega) \) with \( \alpha = 1 - 1/\tilde{p} \) and \( w \in C^\alpha(\Omega) \). As a direct consequence of the Poincaré lemma, notice that if \( \overline{\partial} w \) is real valued, then \( w = \partial u \) for some real valued \( u \). Indeed \( 2 \overline{\partial} w = (\partial_x + i \partial_y)(f + ig) = (\partial_x f - \partial_y g) + i(\partial_y f + \partial_x g) \). By assumption \( \partial_x g = -\partial_y f \), from where the one-form \( gdy - f dx \) is exact. Therefore, there exists a real valued \( F \) such that \( dF = (-f) dx + gdy \). We have \( w = f + ig = \partial_x (w) - i \partial_y (w) = \partial (-2F) \). The equivalence is now apparent.
\end{proof}

Now we extend \( b \in W^{\varepsilon, \tilde{p}}(\Omega) \) by zero outside \( \Omega \). Its extension denoted also by \( b \) preserves regularity \( b \in W^{\varepsilon, \tilde{p}}(\mathbb{R}^2) \). From now on we shall work with solutions of (25) in the whole plane.
Lemma 3.2. The equation \(25\) has unique solutions in the whole plane \(W_r(z, k) \sim e^{izk}\) respectively \(W_i(z, k) \sim ie^{izk}\) in \(L^\beta(R^2)\) for large \(z\). Moreover, \(e^{-izk}W_r - 1, e^{-izk}W_i - i \in W^{1, \beta}(R^2)\) and \(W_r(\cdot, k), W_i(\cdot, k) \in W^{1, \beta}\)\(_{loc}\)(\(R^2\)).

Proof. As in the proof of theorem 1.1, we look for solutions \(W(z, k) = e^{izk}w(z, k)\) with \(w - 1 \in L^\beta\). The equation for \(w\) is

\[
\overline{\partial}(w(z) - 1) + b(z)(w(z) - 1) + e(z, -k)b(z)(\overline{w(z)} - 1) = -b(z) - e(z, -k)b.
\]

(26)

Using the fact that \(\overline{\partial}^{-1} : f \in L^\beta_c(R^2) \rightarrow W^{1, \beta}(R^2)\) together with \(b\) of compact support we get \(\overline{\partial}^{-1}(b) : L^\beta(R^2) \rightarrow L^\beta(R^2)\) is a compact operator. We apply Fredholm’s alternative in \(L^\beta(R^2)\) to the equivalent integral equation

\[
\{I + \overline{\partial}^{-1}[b(\cdot) + e(\cdot, -k)b(\cdot)]\}(w(z) - 1) = -\overline{\partial}^{-1}[b + e(\cdot, -k)b].
\]

Uniqueness comes from Liouville’s theorem for the \(\overline{\partial}\)-equation with coefficients in \(L^\beta(R^2) \cap L^p(R^2)\), see Vekua [V62]. By construction we already have that \(g = w_r - 1 \in W^{1, \beta}(R^2)\). Then \(W_r(z, k) = e^{izk}(g + 1) \in L^\beta_{loc}(R^2)\), \(\partial W_r = ike^{izk}g(z, k) + ike^{izk} + e^{izk}g(z, k) \in L^\beta_{loc}(R^2)\) and \(\partial W_r = e^{izk}g(z, k) \in L^\beta_{loc}(R^2)\). Similar relations hold for \(W_i\).

To simplify notations, let

\[
q(z) = b(z)e^{-1\overline{\partial}}w(z) - \overline{\partial}^{-1}\overline{b}(z)
\]

(27)

denote a new potential and notice that if \(w\) is a solution of \(25\) then \(v = e^{1\overline{\partial}}w\) is a solution of

\[
\overline{\partial}v + q = 0.
\]

(28)

Since \(b \in L^\beta(R^2) \cap L^p(R^2)\) we have that \(\overline{\partial}^{-1}b \in L^\infty(R^2) \cap C^\frac{1}{2}R^2\), see Vekua [V62]. Then \(e^{-\overline{\partial}^{-1}b} \in L^\infty(R^2)\) and so \(q \in L^\beta(R^2) \cap L^p(R^2)\).

The next theorem relates scattering solutions of \(25\) to scattering solutions of \(28\) and gives the behavior in \(k\) of \(W_r(z, k)\) and \(W_i(z, k)\).

Proposition 3.3. Let \(b \in W^c\_\beta_{c}(R^2)\), for some \(\varepsilon > 0\). Let \(W_r\) and \(W_i\) be the scattering solutions for \(25\) as given by the lemma above, and let \(\Psi_r\) and \(\Psi_i\) be the scattering solutions for \(28\) as given by the theorem 1.1. Then \(W_r = e^{-\overline{\partial}^{-1}b}\Psi_r\), \(W_i = e^{-\overline{\partial}^{-1}b}\Psi_i\) and

\[
\|W_r(z, k)e^{-izk} - e^{-\overline{\partial}^{-1}b}\|_{L^\beta(R^2)} + \|W_i(z, k)e^{-izk} - ie^{-\overline{\partial}^{-1}b}\|_{L^\beta(R^2)} \leq C(k)^{-\varepsilon},
\]

\[
\|W_r(z, k) - iW_i(z, k)e^{-izk} - 2e^{-\overline{\partial}^{-1}b}\|_{W^{1, \beta}(R^2)} \leq C(k)^{-\varepsilon}.
\]

(29)

Proof. The fact that \(W_r\) and \(W_i\) solve \(25\) is trivial. Uniqueness result of lemma 3.2 ensures that they are the scattering solutions of \(25\). The estimates follow directly from the estimates for \(\psi_r\) and \(\psi_i\) in \(3\) and \(4\) and from the fact that \(e^{-\overline{\partial}^{-1}b} \in L^\infty(R^2)\) as noticed before. Again, the imbedding \(W^{1, \beta}(R^2) \subset C^{1-\frac{1}{2}, \beta}(R^2)\) shows that the estimates \(29\) hold pointwise in \(z \in C\).

We have now all the ingredients necessary for reconstruction. Since \(q\) in \(27\) has compact support in \(\Omega\), the scattering transform depends only on the traces on \(\partial\Omega\) of the scattering solutions \(\Psi_r\) and \(\Psi_i\). Let \(v = v_1 + iv_2\) be the complex-normal to the boundary. Then

\[
t(k) = -\frac{i}{\pi} \int_{\Omega} e^{izk}q(z) (\Psi_r(z, k) - i\Psi_i(z, k)) d\mu(z) = \frac{i}{\pi} \int_{\Omega} e^{izk}i (\partial\Psi_r(z, k) - i\partial\Psi_i(z, k)) d\mu(z)
\]

\[
= \frac{i}{2\pi} \int_{\partial\Omega} e^{izk}q(z) (\overline{\Psi_r(z, k)} - i\overline{\Psi_i(z, k)}) d\sigma(z),
\]

(30)
The last equality uses the fact that $\partial(e^{itk}) = 0$.

Next we show how to reconstruct traces of $\Psi_r$ and $\Psi_i$ to $\partial\Omega$ from the Dirichlet to Neumann map $\Lambda_{b_1, b_2}$. First we reconstruct traces of $W_r$ and $W_i$ to $\partial\Omega$.

As in Knudsen and Tamasan [13], we consider the single layer potential operator $S_k : C^\alpha(\partial\Omega) \to C^\alpha(\partial\Omega)$, $\alpha = 1 - 2/\tilde{p}$, defined by

$$S_k f(z) = \frac{1}{2\pi i} p.v. \int_{\partial\Omega} f(\zeta) g_k(\zeta - z) d\zeta, \quad z \in \partial\Omega,$$

where $g_k(z) = e^{-izk}/(\pi z)$ is a Cauchy kernel for the Dirichlet problem. We notice already that $\partial_z S_k f$ is a bounded operator (e.g. see Muskhelishvili [15]). Since $q$ has compact support we have that $W_r$ and $W_i$ are analytic outside $\Omega$ and behaves like $e^{izk}$ at infinity. Traces of such functions will satisfy a singular boundary equations involving $S_k$. Inside $\Omega$ they satisfy a pseudo-analytic equation. This will impose constrains (in terms of $\Lambda_{b_1, b_2}$) on their trace. We will prove that these two conditions are sufficient to determine the traces.

We notices already that $W_r(\cdot, k), W_i(\cdot, k) \in C^\alpha(\mathbb{R}^2)$ with $\alpha = 1 - 2/\tilde{p}$, whence their traces on $\partial\Omega$ are in $C^\alpha(\partial\Omega)$. Let

$$C^\alpha_0(\partial\Omega) := \{ h \in C^\alpha(\partial\Omega) : \int_{\partial\Omega} h(s) ds = 0 \}.$$

Define now a right inverse of the tangential vector field $\partial_s$ (here $s$ is the arc length) on $\partial\Omega$ by

$$\partial_s^{-1} f(t) = \int_0^t f(s) ds, \quad (31)$$

for $f \in C^\alpha(\partial\Omega)$. In the above integral we fixed an arbitrary point on $\partial\Omega$ from where we measure the arc length counter-clockwise. Notice that $\partial_s^{-1} : C^\alpha_0(\partial\Omega) \to C^{1+\alpha}(\partial\Omega)$ is a well defined (independent of the reference point) bounded operator. The following result defines a Hilbert transform for the pseudo-analytic maps.

**Lemma 3.4.** $\mathcal{H}_b \equiv -\Lambda_{b_1, b_2} \partial_s^{-1} : C^\alpha_0(\partial\Omega) \to C^\alpha(\partial\Omega)$ is a bounded operator.

**Proof.** Let $g = \partial_s^{-1} f \in C^{\alpha+1}(\partial\Omega) \subset W^{2-1/\tilde{p}}, b$ Classical theory of PDE (e.g. see Gilbarg and Trudinger [9]) gives that the boundary value problem

$$\Delta u(x) + b_1 \frac{\partial u}{\partial x}(x) + b_2(x) \frac{\partial u}{\partial x}(x) = 0, \quad x \in \Omega$$

$$u|_{\partial\Omega}(x) = g(x), \quad x \in \partial\Omega$$

has a unique solution up to a constant in $W^{2,\tilde{p}}(\Omega)$ and $\|u\|_{W^{2,\tilde{p}}(\Omega)} \leq C\|g\|_{W^{2-1/\tilde{p}},\tilde{p}}(\partial\Omega)$. Using the mapping properties of the Dirichlet to Neumann map we have

$$\|\mathcal{H}_b f\|_{C^\alpha(\partial\Omega)} \leq \|\Lambda_{b_1, b_2} g\|_{W^{1,\tilde{p}}(\partial\Omega)} \leq \|\nabla u\|_{W^{1,\tilde{p}}(\Omega)} \leq \|u\|_{W^{2,\tilde{p}}(\Omega)}$$

$$\leq C\|g\|_{W^{2-1/\tilde{p},\tilde{p}}(\partial\Omega)} \leq C\|g\|_{C^{1+\alpha}(\partial\Omega)} \leq C\|f\|_{C^\alpha(\partial\Omega)}.$$

Next we show that $\mathcal{H}_b$ reconstructs traces of the exponentially growing solutions on $\partial\Omega$.  

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Theorem 3.5 (Trace theorem). Let $b \in W_c^{rac{\alpha}{2}}(\Omega)$. Consider the class of functions

$$\mathcal{B} = \{ h \in C^\alpha(\partial \Omega) : \text{Im}(\nu h) \in C^\alpha(\partial \Omega) \}.$$

Then, for each $k \in \mathbb{C}$ arbitrarily fixed, the traces $h_r = W_r(\cdot, k)|_{\partial \Omega}$, respectively $h_i = W_i(\cdot, k)|_{\partial \Omega}$ are the unique solution in $\mathcal{B}$ of the systems

$$\begin{align*}
(I - i \mathcal{S}_k) h_r(z) &= 2 e^{izk}, \quad z \in \partial \Omega, \\
\mathcal{H}_b(\text{Im}(\nu h_r))(z) &= \text{Re}(\nu h_r)(z), \quad z \in \partial \Omega, \\
\end{align*}$$

respectively,

$$\begin{align*}
(I - i \mathcal{S}_k) h_i(z) &= 2 ie^{izk}, \quad z \in \partial \Omega, \\
\mathcal{H}_b(\text{Im}(\nu h_i))(z) &= \text{Re}(\nu h_i)(z), \quad z \in \partial \Omega. \\
\end{align*}$$

Proof. We argue only for $W_r$, the arguments for $W_i$ are similar.

We prove first the necessity. The arguments for (33) are identical to the ones in [13] reason for which we only sketch them here. Fix $k \in \mathbb{C}$ and suppress the $k$ dependence, we have $W_r(\cdot) = W_r(\cdot, k)$ is analytic outside $\Omega$ and $e^{-izk} W_r - 1 \in L^p(\mathbb{C} - \Omega)$. The Green-Gauss formula for $z \in \mathbb{C} - \Omega$ gives

$$e^{-izk} W_r(z) - 1 = -\frac{1}{2\pi i} \int_{\partial \Omega} \frac{e^{-iz\zeta} W_r(\zeta)}{\zeta - z} d\zeta. \tag{35}$$

Now let $z$ approach (from the exterior) a boundary point $z_0$ and use Plemelj formula (see Muskhelishvili [15]).

$$\lim_{z \to z_0} \frac{1}{2\pi i} \int_{\partial \Omega} \frac{f(\zeta) d\zeta}{\zeta - z} = -\frac{1}{2} f(z_0) + \frac{1}{2\pi i} \text{p.v.} \int_{\partial \Omega} \frac{d\zeta}{\zeta - z_0}$$

to get (33).

Next we prove the necessity of (34). Recall from lemma 3.2 that $W_r(z) = \partial u(z)$ for some $u \in W^{2,\bar{p}}(\Omega)$ which solve the equation (24). Therefore

$$h_r = W_r|_{\partial \Omega} = \frac{1}{2}(\partial_x + i \partial_y) u|_{\partial \Omega}. \tag{36}$$

For $z \in \partial \Omega$ let $(\nu_1(z), \nu_2(z))$ be the unit outer normal, we also let $\nu(z) = \nu_1(z) + i \nu_2(z)$. Next we express the partial derivatives for points on the boundary in terms of the tangent $\partial_s$ and the normal $\partial_r$ derivatives

$$\nabla u(x) = \begin{pmatrix}
-\nu_2 & \nu_1 \\
\nu_2 & -\nu_1
\end{pmatrix}
\begin{pmatrix}
\partial_s u \\
\Lambda_{b_1, b_2} u
\end{pmatrix}, \tag{37}
$$

where we recall $\partial_r u = \Lambda_{b_1, b_2} u$. Therefore $2h_r = (\partial_x - i \partial_y) u = -i \nu \partial_s u + \partial_r u$, or, using $\nu \overline{\nu} = 1$,

$$2\nu h_r = \Lambda_{b_1, b_2} u - i \partial_s u. \tag{38}$$

Note that $\text{Im}(\nu h_r) = -\partial_s u/2$ and thus $h_r \in \mathcal{B}$ and $\partial_s^{-1}(\text{Im}(\nu h_r))$ makes perfect sense. Identifying the real part in (35) gives (34). Notice not only that we proved necessity but also we provided existence of solutions for (34) and (33).

Conversely, let $h \in \mathcal{B}$ be a solution of the system (34) and (33). We extend $h$ inside $\Omega$ by the following procedure. Inspired by (38) define $g = -\partial_s^{-1} \text{Im}(2\nu h) \in C^\alpha(\partial \Omega)$ then uniquely solve the boundary value problem (5) for $u \in W^{2,\bar{p}}(\Omega)$. Notice $g$ is real valued hence $u$ has also real values.
Define $W_r(z) = \partial u(z)$ inside $\Omega$ and notice that $\partial u|_{\partial \Omega} \in C^\alpha(\partial \Omega)$. Now check that $\partial u|_{\partial \Omega} = h$. Indeed, as before, $2\partial u = -i\nabla \sigma u + \nabla \Lambda_{b_1,b_2} u = i\nabla \text{Im}(2\nu h) + \nabla \text{Re}(2\nu h)$. The last equality used the fact that $h$ is a solution of (34). Multiplication by $\nu$ gives $\partial u = h$.

Inspired by (35) define $W_r$ analytically outside $\Omega$ by

$$W_r(z) = e^{izk} - \frac{1}{2\pi i} \int_{\partial \Omega} \frac{e^{-i(\zeta-z)k}h(\zeta)d\zeta}{\zeta - z} \quad z \in \mathbb{C} - \Omega. \quad (39)$$

The fact that $h$ solves (33) implies that $\lim_{z \to z_0 \in \partial \Omega} W_r(z) = h(z_0)$. Thus $W_r$ is an outside continuous extension of $h$. Moreover, $e^{-izk} W_r - 1 = O(1/z)$ for $z$ large, hence $W_r \in L^p(\mathbb{C} - \Omega)$.

We produced a continuous map in $\mathbb{R}^2$ which solves (25) both inside and outside $\Omega$ and behaves like $e^{izk}$ for $z$ large. We need to check that it solves the equation (25) across the boundary. Since $b$ has compact support inside $\Omega$ we have that $W_r$ is in fact analytic in both sides of the boundary and continuous across. Morera’s theorem asserts that $W_r$ must be in fact analytic across. Therefore $W_r$ solves (25) in the whole plane and has the right behavior at infinity. Uniqueness in lemma 3.2 concludes the proof. □

Immediate consequence to the proposition 3.3 and to the pointwise estimates (29) we can determine the traces on $\partial \Omega$ of $\Psi_r$ and $\Psi_i$. Moreover by formula (30) we determine the scattering transform.

**Corollary 3.6 (Reconstruction of the scattering transform).** Under the assumptions of the proposition 3.3 we have

$$e^{-\overline{\sigma}^{-1}b}(z) = \lim_{k \to \infty} W_r(z,k), \quad z \in \partial \Omega. \quad (40)$$

and for any $k \in \mathbb{C}$ we recover

$$\Psi_r(z,k) = e^{\overline{\sigma}^{-1}b}(z)W_r(z,k), \quad z \in \partial \Omega, \quad (41)$$

$$\Psi_i(z,k) = e^{\overline{\sigma}^{-1}b}(z)W_i(z,k), \quad z \in \partial \Omega. \quad (42)$$

Moreover,

$$t(k) = \frac{i}{2\pi} \int_{\partial \Omega} e^{i\overline{\sigma}^{-1}(z)(\overline{\Psi}(z,k) - i\overline{\Psi}(z,k))} d\sigma(z), \quad (43)$$

is a function in $L^r(\mathbb{R}^2) \cap L^{\tilde{r}}(\mathbb{R}^2) \cap L^{r'}(\mathbb{R}^2)$ for some $r < 2$, $\tilde{r}^{-1} = r^{-1} - 1/2$ and $r^{-1} + r'^{-1} = 1$.

Now we use the inverse scattering method of theorem 1.2 to reconstruct $q$.

**Corollary 3.7.** Let $\Phi_r \sim e^{izk}$ and $\Phi_i \sim ie^{izk}$ in $L^p$ for large $k \in \mathbb{C}$ be the unique solutions

$$\frac{\partial \Phi}{\partial k}(k) + t(k)\overline{\Phi}(k) = 0, \quad k \in \mathbb{C}. \quad (44)$$

Then

$$q(z) = -\frac{i}{\pi} \int_{\mathbb{R}^2} e^{izk}(\Phi_r(z,k) - i\Phi_i(z,k)) d\mu(k). \quad (45)$$

Knowing $q$ we also know $|b|$ since from (27) we have $|q| = |b|$. Next we show how to determine its argument by solving (27) to recover $b$.

The following result is due to Cheng and Yamamoto [5]. For the sake of completeness we sketch its proof.
Lemma 3.8. If \( q \in L^\tilde{p}(\Omega) \) then there exist at most one solution \( b \in L^\tilde{p}(\Omega) \) of the equation

\[
\overline{\partial}(z)e^{\overline{\partial}^{-1}b(z) - \overline{\partial}^{-1}d(z)} = q(z), \quad z \in \Omega. \tag{46}
\]

Proof. Assume there are two solutions \( b_1, b_2 \in L^\tilde{p}(\Omega) \) and let \( d = \overline{\partial}^{-1}(b_2 - b_1) \in W^1,\tilde{p}(\Omega) \subset H^1(\Omega) \).

From (40) we have \( d = 0 \) on \( \partial \Omega \). Hence \( d \in H^1_0(\Omega) \). Since both solve (46) we have \( b_1(z) = b_2(z)e^{d(z) - d(z)} \) from where

\[
|\partial d| = |b_2| \cdot |e^{d(z) - d(z)} - 1| = |q| \cdot |e^{d(z) - d(z)} - 1| \leq |q| \cdot |d - d| \leq 2|q||d|. \tag{47}
\]

By Carleman estimates for \( d \in H^1(\mathbb{R}^2) \) of compact support (see Hörmander [11], Prop. 17.2.3) we have

\[
\int_\Omega \Delta \varphi |d|^2 e^{2\varphi} dx \leq 4 \int_\Omega |\overline{\partial}d|^2 e^{2\varphi} dx, \tag{48}
\]

for some strictly convex function \( \varphi \in C^2(\Omega) \). Approximate a \( \varphi \in H^2,\tilde{p}(\Omega) \) solution of \( \Delta \varphi = 17|q|^2 \) in \( L^2(\Omega) \) by a smooth sequence \( \varphi_n \to \varphi \) uniformly on \( \overline{\Omega} \). Then

\[
\int_\Omega \Delta \varphi_n |d|^2 e^{2\varphi_n} dx \leq 16 \int_\Omega |q|^2 |d|^2 e^{2\varphi_n} dx.
\]

For \( n \) sufficiently large the reverse inequality holds. Hence \( d = 0 \) and \( b_1 = b_2 \).

We are left to find the unique solution of (46).

Lemma 3.9 (Phase unwrapping). Let \( v \in 1 + L^\tilde{p}(\mathbb{R}^2) \) be the unique solution of

\[
\overline{\partial}v = qv \tag{49}
\]

then \( v \) vanishes on a set of measure zero. Define \( b = \overline{\partial}v/v \) on the set where \( v \) does not vanish, else we can set \( b = q \). Then \( b \) is the unique solution of (46) in \( L^\tilde{p}(\Omega) \).

Proof. Existence and uniqueness of \( v \) follows from the Fredholm alternative as before. It is known from Vekua [V62] that the set of zeroes of pseudo-analytic functions has measure zero. Since \( \overline{\partial}v = qv \) we have that \( v \) also solves \( \overline{\partial}v = bv \) in the whole plane. Equivalently \( \overline{\partial}(e^{-\overline{\partial}^{-1}b}v) = 0 \). Thus \( e^{-\overline{\partial}^{-1}b}v \) is analytic and also goes to 1 as \( |z| \to \infty \). By Liouville’s theorem we have \( v = e^{\overline{\partial}^{-1}b} \). From its definition we have

\[
b = \overline{\partial}e^{-\overline{\partial}^{-1}b}e^{\overline{\partial}^{-1}b}.
\]

4 Concluding Remarks

In order to solve the inverse problem one finds first the traces of the exponentially behaving solutions. It is easy to show that any solution of (33) outside a disk can be represented as a series

\[
W(z, k) = e^{izk} \sum_{n=0}^{\infty} \frac{a_n}{z^n},
\]

with \( a_n \)'s unknown coefficients. We determine them by solving the singular boundary integral equations [34]. This step is severely ill posed and regularization techniques are necessary, truncation
The second step consists in constructing the scattering transform \( t(k) \) via the formulae of corollary 3.6. Next we solve the weakly singular integral equations (5) in the \( k \)-space. This part is stable. It is here that we need the \( \epsilon \)-extra regularity. One needs \( t \in L^r(\mathbb{R}^2_k) \) for some \( r < 2 \) in order to solve (5). If \( q \) is only in \( L^p_\Xi \) then \( t \in L^2(\mathbb{R}^2) \) (according to Sung [23] as corrected by Brown and Uhlmann [4]) and this suffices for uniqueness. This would recover the uniqueness result of Cheng and Yamamoto. It is not clear how to find solutions of (5) when \( t \in L^2(\mathbb{R}^2_k) \).

Reconstruct \( q \) from the formula (7). Notice that we have estimates of decay in \( k \) for \( t \in L^r(\mathbb{R}^2_k) \) as well as for \( e^{-izk}(\Phi_r - i\Phi_i) - 2 \) as given in [4]. These can lead to estimates of the truncation error in the integral in (7).

One of the questions in [25] concerned the characterization of traces of exponentially behaving solutions in the first order system in \( \Omega: \bar{\partial}v = qw \) and \( \partial w = \bar{q}v \). A partial answer was given by Knudsen and the author in [13] for \( q \) of the special form \( q = \partial f \) with \( f \) real valued. We can give now the answer for a general \( q \). Note that \( v \pm \bar{w} \) solves the \( \bar{\partial} \)-equation \( \bar{\partial}u + q\bar{w} = 0 \), and that we characterized the traces on the boundary of such solutions in terms of a Hilbert transform.

Acknowledgement

I would like to thank Professor A. Nachman for his generous sharing of ideas in the inverse scattering theory, and the organizers, Professors J. McLaughlin and H. Engl, for the invitation in the special semester on Inverse Problems at IPAM-UCLA.

References

[1] J. A. Barceló, T. Barceló and A. Ruiz 2001 Stability for the conductivity equation in the plane for less regular conductivities, J. Differential Equations, 173 231 – 270

[2] R. Beals and R. R. Coifman, 1988 The spectral problem for the Davey-Stewartson and Ishimori hierarchies, Nonlinear evolution equations: Integrability and spectral methods, Manchester University Press, Manchester, 15–23

[3] R. M. Brown and G. Uhlmann, 1996 Uniqueness in the inverse conductivity problem with less regular conductivities in two dimensions, Private notes

[4] R. M. Brown and G. Uhlmann, 1997 Uniqueness in the inverse conductivity problem for nonsmooth conductivities in two dimensions, Comm. Partial Differential Equations 22 1009–1027

[5] J. Cheng and M. Yamamoto 1998 Unique Determination of two convection coefficients from Dirichlet to Neumann map, University of Tokyo preprint UTMS 98-31, to appear in SIAM J. Math. Anal.

[6] J. Cheng and M. Yamamoto 2000 The global uniqueness for determining two convection coefficients from Dirichlet to Neumann map in two dimensions Inverse Problems 16 L25–L35

[7] C. Tong J. Cheng and M. Yamamoto 2004 Reconstruction of convection coefficients of an elliptic equation in the plane by the Dirichlet to Neumann map, electronic preprint arxiv.org

[8] R. R. Coifman and Y. Meyer 1978 Au delà des opérateurs pseudodifférentiels, Astérisque 57, Société Mathématique de France, Paris
[9] D. Gilbarg and N. Trudinger 1983 *Elliptic partial differential equations of second order*, Springer-Verlag, Berlin

[10] G. M. Henkin and R. G. Novikov 1987 $\overline{\partial}$-equation in the multidimensional inverse scattering problem, *Russian Math. Surveys* **42** 109–180

[11] L. Hörmander 1985 *The Analysis of Linear Partial Differential Operators*, vol 3, Springer

[12] K. Knudsen and A. Tamasan 2001 Reconstruction of less regular conductivities in the plane, MSRI preprint 2001-035.

[13] K. Knudsen and A. Tamasan 2003 Reconstruction of less regular conductivities in the plane, MaPhySto preprint 1398-2699, in print in *Comm. Partial Differential Equations* **29**

[14] K. Knudsen 2003A new direct method for reconstructing isotropic conductivities in the plane, *Physiol. Meas.* **24** 391–401

[15] M. I. Muskhelishvili 1953 *Singular integral equations. Boundary problems of function theory and their application to mathematical physics*, P. Noordhoff N. V., Groningen

[16] A. I. Nachman, M. J. Ablowitz 1984 A multidimensional inverse-scattering method, *Stud. Appl. Math.* **71** 243–250 and 251–262.

[17] A. I. Nachman 1988 Reconstructions from boundary measurements *Ann. of Math.* **128** 531–576

[18] A. I. Nachman 1996 Global uniqueness for a two-dimensional inverse boundary value problem, *Ann. of Math.* **143** 71–96

[19] L. Nirenberg and H. F. Walker 1973 The null spaces of elliptic partial differential operators in $\mathbb{R}^n$ *J. Math. Anal. Appl.* **42** 271–301.

[20] E. M. Stein 1970 *Singular integrals and differentiability properties of functions*, Princeton University Press, Princeton, N.J.

[21] J. Sylvester and G. Uhlmann 1987 A global uniqueness theorem for an inverse boundary value problem, *Ann. of Math.* **125** 153–169

[22] L. Sung 1994 An inverse scattering transform for the Davey-Stewartson II equations. I *J. Math. Anal. Appl.* **183** 121–154

[23] L. Sung 1994 An inverse scattering transform for the Davey-Stewartson II equations. II, *J. Math. Anal. Appl.* **183** 289–325

[24] L. Sung 1994 An inverse scattering transform for the Davey-Stewartson II equations. III *J. Math. Anal. Appl.* **183** 477–494

[25] G. Uhlmann 2003 Inverse Boundary Problems in two Dimensions *Function Spaces, Differential Operators and Nonlinear Analysis- The Hans Triebel Anniversary Volume*, eds. D. Haroske et al. Birkhäuser, Basel-Boston-Berlin 183 – 203

[V62] I.N. Vekua 1962 *Generalized analytic functions*, Pergamon Press, London

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