TIGHT BOUNDS FOR MULTI-FREQUENCY DIFFERENTIAL INCLUSIONS
APPLIED TO CONTROL SYSTEMS

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Abstract. We present new tight bounds for averaging differential inclusions, which we apply to multifrequency inclusions consisting of a sum of time-periodic set-valued functions. Specifically we establish estimates of order $O(\epsilon)$ on the approximation error for this averaging problem. These results are then applied to control systems consisting of a sum of time-periodic functions.

1. Introduction

The averaging of differential inclusions seeks to approximate the solution-set of a time varying inclusion with small amplitude (or, equivalently, by change of variables a highly oscillatory systems), by the solution of an auxiliary averaged time-independent differential inclusion, in a finite but large time domain. The averaged inclusion is obtained by computing the time average, and an estimates on the difference in the Hausdorff distance between the solution sets of both systems is sought. The time-independent system is amenable to analysis and applications of averaging in stabilization and optimality can be found in Gama and Smirnov [6].

In this paper, we consider the approximation of $S_{[0, \epsilon^{-1}]} (\epsilon F, x_0)$ the solution-set in the domain $[0, \epsilon^{-1}]$ of the differential inclusion

$$\dot{x} \in \epsilon F(t, x), \ x(0) = x_0,$$

where we focus on the case where

$$F(t, x) = F_1(\omega_1 t, x) + \cdots + F_m(\omega_m t, x)$$

and each $F_j(t, x)$ is periodic in $t$ with period 1. The solution is approximated by $S_{[0, \epsilon^{-1}]} (\epsilon \bar{F}, x_0)$ the solution-set of the averaged differential inclusion

$$\dot{y} \in \epsilon \bar{F}(y), \ y(0) = x_0$$

in $[0, \epsilon^{-1}]$, where

$$\bar{F}(x) = \lim_{T \to \infty} \frac{1}{T} \int_0^T \bar{F}(s, x)ds.$$ 

The integral considered is the Aumann integral (see, Aumann [3]) and the convergence is in the Hausdorff distance and the supremum norm.

Our main result establishes an $O(\epsilon)$ estimation of the approximation error, i.e., the Hausdorff distance between the solution sets $S_{[0, \epsilon^{-1}]} (\epsilon F, x_0)$ and $S_{[0, \epsilon^{-1}]} (\epsilon \bar{F}, x_0)$. Namely, for every solution of (1.1) there is a solution of (1.2) which is $\epsilon$ close to it, and vice-versa. This result extends the classical bound of $O(\epsilon)$ for time periodic inclusion ($m = 1$) by Plotnikov [8] to multi frequency inclusions. We also establish extend this tight estimate when each coordinate of $F(t, x)$ is periodic, and provide new estimates for non-periodic inclusions.

These results are applied to the averaging of control systems of the form

$$\dot{x} = \epsilon g(t, x, u), \ x(0) = x_0$$

where

$$g(t, x, u) = g_1(\omega_1 t, x, u) + \cdots + g_m(\omega_m t, x, u)$$
and every \( g_i(t, x, u) \) is periodic in \( t \) with period 1 (Note that the same control appears in all terms). The averaged equation corresponds to the chattering limit

\[
\dot{y} \in \mathcal{C}\bar{G}(y) = \lim_{T \to \infty} \frac{1}{T} \int_0^T \{ g(s, x, u) \mid u \in U \} \; ds,
\]

and not to the trivial time average. An equivalent definition of \( \mathcal{G}(y) \), which follows from the study of singularly perturbed control system, can be obtained when all the \( g_j(t, x, u) \) are continuous in \( t \) and the set of frequencies \( \omega_1, \omega_2, \ldots, \omega_m \) is linearly independent over the integers. Then

\[
\left\{ \int_{[0,1]^m} \sum_{j=1}^m g_j(y, \phi_j, u(\phi)) \; d\phi \mid u : [0,1]^m \to U \text{ is measurable} \right\},
\]

where \( \phi = (\phi_1, \phi_2, \ldots, \phi_m) \in [0,1]^m \).

We establish that the approximation error for this multifrequency system is \( O(\epsilon) \), extending the result in [5] as well as that by Bombrun and Pomet [4, Theorem 3.7] for linear systems where each term shared the same period but had a different control, and was of the form \( u_j g_j(\omega_0 t, x) \). This bound also improves a previous bound of order \( O(\sqrt{\epsilon}) \) presented by Artstein, for the case \( m > 1 \).

Applying a change of variables \( \tau = \epsilon^{-1} t \) we obtain the fast time-varying system; Equations (1.1) and (1.2) reduce to

\[
x' \in F(\tau/\epsilon, x), \quad y' \in \bar{F}(y) \quad x(0) = y(0) = x_0,
\]

and Equations (1.4) and (1.5) transform to

\[
x' = g(\tau/\epsilon, x, u), \quad y' \in \bar{G}(y) \quad x(0) = y(0) = x_0.
\]

Clearly, our bounds can be applied to these systems in the time interval \([0,1]\).

Averaging differential inclusions generalizes the classical averaging method of averaging differential equations. For a reference in the averaging method of ordinary differential equations, the reader is referred to the book of Sanders, Verhulst and Morduck [9] for a reference on this subject and to works of Artstein [11] and Bright [13] for a modern treaty and improved estimates on the error, the line of which we follow in this paper. For a reference to results in differential inclusions refer to the review papers Klymchuk, Plotnikov and Skripnik [7] and to [3].

To the best of the author’s knowledge, except for bounds for periodic differential inclusion, these results are the first quantitate results for averaging differential inclusions. We believe that methods may also be applied to the much needed field of the quantitative analysis of averaging singularly perturbed inclusions and control systems, where qualitative bounds are sparse. For a review on averaging in singularly control systems Artstein [2] and the references within.

The structure of this paper is as follows: Section 2 presents the assumption and notations used throughout this paper. Section 3 presents the a the key lemma, which estimates the error when the averaged equation considered is computed over a finite time interval. In Section 3 results for differential inclusion are presented, and in the last section they are applied to control systems.

2. Notations and Assumptions

In what follows, we use the following notions. We denote the d-dimensional Euclidean space by \( \mathbb{R}^d \), a vector by \( x \in \mathbb{R}^d \) and its Euclidean norm by \( |x| \). The Euclidean ball centered at \( x \) with radius \( r \) is denoted by \( B(x,r) \subseteq \mathbb{R}^d \). Given two sets \( A_1, A_2 \subseteq \mathbb{R}^d \) their Minkowski sum is denoted by \( A_1 + A_2 = \{ x_1 + x_2 \mid x_1 \in A_1, x_2 \in A_2 \} \). We endow the set of continuous function defined on \([0,\epsilon^{-1}]\) with the supremum norm, defined by \( \| y(\cdot) \| = \sup_{t \in [0,\epsilon^{-1}]} |y(t)| \). Given a normed vector space \( (\mathcal{X}, \| \cdot \|) \) the distance between a point \( x \in \mathcal{X} \) and a set \( A \subseteq \mathcal{X} \) is denoted by \( d(x, A) = \inf \{ \| x - y \| \mid y \in A \} \) and the Hausdorff distance between two sets \( A_1, A_2 \subseteq \mathcal{X} \) by

\[
d_H(A_1, A_2) = \max \{ \sup \{ d(y, A_2) \mid y \in A_1 \}, \sup \{ d(y, A_1) \mid y \in A_2 \} \}.
\]

We denote by \( S_{[0,T]}(G, x_0) \) the solution-set of the differential inclusion \( \dot{x} \in G(t, x) \), with initial condition \( x(0) = x_0 \) in the domain \([0,T]\). We use the notation \( \dot{x}(\cdot) = \frac{d}{dt} x(\cdot) \) for the time derivative.
We consider solutions of differential equations of the form \(\dot{x} \in F(t, x)\) in the domain \(\Omega \subset \mathbb{R}^d\), satisfying the following conditions.

**Assumption 2.1.** The set-valued function \(F(t, x) : \mathbb{R} \times \Omega \rightrightarrows \mathbb{R}^d\) satisfies the following conditions:

1. The values of \(F(t, x)\) are non-empty, closed and convex in its domain.
2. For every \(t, x\) in its domain \(F(t, x) \subset B(0, M)\).
3. \(F(t, x)\) is measurable in \(t\).
4. \(F(t, x)\) is uniformly Lipschitz continuous in \(x\) with a Lipschitz constant \(K\), namely,
   \[
   d_H(F(t, x_1), F(t, x_2)) \leq K |x_1 - x_2|
   \]
   for all \(t \in \mathbb{R}, x_1, x_2 \in \Omega\).
5. The time average function \(\bar{F}(x)\), defined in (1.3), exists.

Note that we can always extend \(F(t, x)\) to \(\mathbb{R} \times \mathbb{R}^d\) so that it satisfies the conditions of Assumption (2.1) for \(\Omega = \mathbb{R}^d\).

Throughout this paper we assume the following assumption on the solutions of (1.1) and (1.2).

**Assumption 2.2.** All the solutions of (1.1) and (1.2) are contained in \(\Omega\).

**Remark 2.3.** The requirement that \(F(t, x)\) is convex valued can be relaxed, as by Filippov theorem the solution set of \(F(t, x)\) is dense in the solution set of the inclusion obtained by replacing the right hand side by the convex hull of \(F(t, x)\), also the average of both mapping is equal.

**Remark 2.4.** The Lipschitz regularity of \(F(t, x)\) can be relaxed so that the Lipschitz constant, \(k(t)\), depends on \(t\). In this case the same results hold as long as there exists \(K\) so that \(\epsilon \int_0^1 k(s) ds \leq K\) holds for every \(T \in [0, \epsilon^{-1}]\).

The following lemma easily follows from the assumptions above.

**Lemma 2.5.** If \(F(t, x)\) satisfies Assumption (2.1) then so does \(\bar{F}(x)\).

Assumption (2.1) imply the existence of Filippov solutions to both (1.1) and (1.2) in any finite time interval, as well as the validity of the Filippov-Gronwall inequality stated below.

**Theorem 2.6.** Let \(F(t, x)\) satisfy the conditions of Assumption (2.1) and \(y : [0, T] \rightarrow \Omega\) be an absolutely continuous function satisfying \(y(0) = x(0)\). There exist a solution \(x^*(\cdot)\) of (1.1) such that

\[
\sup_{t \in [0, T]} |x^*(t) - y(t)| \leq e^{KT} \int_0^T d(\dot{y}(t), \epsilon F(t, y(t))) dt.
\]

### 3. Key Lemma

In this section we study the effect of a finite-time averaging, or partial average, on the solution-set of a differential inclusion. The bound we obtain is used in the following section to estimate the averaging approximation error.

**Definition 3.1.** Given a set-valued mapping \(F(t, x)\) and \(T > 0\) we define

\[
F_T(t, x) = \frac{1}{T} \int_0^T F(t + s, x) ds.
\]

Notice that when \(F(t, x)\) is periodic in \(t\) with period \(T\) then \(F_T(t, x) = \bar{F}(x)\).

We shall define by \(S_{[0, \epsilon^{-1}]}(\epsilon F_T, x_0)\) the solution set of the equation

\[
\dot{z} \in F_T(t, z), \quad z(0) = x_0
\]

in \([0, \epsilon^{-1}]\).

**Lemma 3.2.** If \(F(t, x)\) satisfies Assumption (2.1) then so does \(F_T(t, x)\), for every \(T > 0\).
Proposition 3.3. Suppose $F(t,x)$ satisfies Assumption (T) and $T > 0$ then

$$d_H\left(S_{[0,t]}(eF(x_0)), S_{[0,t]}(eF_T, x_0)\right) \leq eMT \left(1 + \frac{3}{2}Ke^K\right).$$

Proof. Let $x^*(\cdot)$ be an arbitrary solution of (1.1) in $[0, e^{-1}]$ which we extend, in an arbitrary manner, to a solution of (1.1) in $[0, e^{-1} + T]$. We approximate $x^*(\cdot)$ in $[0, e^{-1}]$ by $\hat{x}(t) = \frac{1}{T} \int_0^t x^*(t+s) \, ds$, where for every $t \in [0, e^{-1}]$

$$|\hat{x}(t) - x^*(t)| \leq \frac{1}{T} \int_0^T |x^*(t+s) - x^*(t)| \, ds \leq \frac{e}{T} \int_0^T Msds \leq \frac{1}{2}eM.\tag{2.25}$$

Since $\hat{x}(t) = \frac{1}{T} \left( x^*(t+T) - x^*(t) \right) = \frac{1}{T} \int_0^T \dot{x}^*(t+s) \, ds$ the triangle inequality and the Lipschitz continuity of $F(t,x)$ imply that for every $t \in [0, e^{-1}]$

$$d\left(\hat{x}(t), eF_T(t, x^*(t))\right) \leq e\int_0^T F(t+s, x^*(t+s)) \, ds, \quad d\left(eF_T(t, x^*(t)), eF_T(t, \hat{x}(t))\right) \leq \frac{eK}{T} \int_0^T |x^*(t+s) - x^*(t)| \, ds \leq \frac{1}{2}e^2KMT.$$

Thus,

$$d\left(\hat{x}(t), eF_T(t, \hat{x}(t))\right) \leq d\left(\hat{x}(t), eF_T(t, x^*(t))\right) + d_H\left(eF_T(t, x^*(t)), eF_T(t, \hat{x}(t))\right) \leq \frac{1}{2}e^2KMT + \frac{1}{2}e^2KMT = e^2KMT,$$

for every $t \in [0, e^{-1}]$, and

$$\int_0^{e^{-1}} d\left(\hat{x}(t), eF_T(t, \hat{x}(t))\right) < eKMT.$$

By the Filippov-Gronwall inequality there exists $z^*(\cdot)$ a solution of (3.1) which is $eKMTe^K$ close to $\hat{x}(\cdot)$, hence, $eMT \left(\frac{1}{2} + Ke^K\right)$ close to $x^*(\cdot)$ in $[0, e^{-1}]$.

On the other hand, let $z^*(\cdot)$ be an arbitrary solution of (3.1) defined on $[0, e^{-1}]$. Now for every $t \in [0, e^{-1}]$

$$z^*(t) \in \int_0^t eF_T(s, z^*(s)) \, ds = \frac{1}{T} \int_0^T \int_0^T eF(s_1 + s_2, z^*(s_1)) \, ds_2 \, ds_1.$$

Let $A = \{(s_1, s_2) \in \mathbb{R}^2 | s_1 \in [0, e^{-1}], s_2 \in [s_1 + s_1 + T]\}$, and $u(s_1, s_2) \in eF(s_2, z^*(s_1))$ be a measurable selection defined in $A$, so that for every $t \in [0, e^{-1}]$

$$z^*(t) = x_0 + \frac{1}{T} \int_0^t \int_{s_1 + s_2} u(s_1, s_2) \, ds_2 \, ds_1.$$

To generate an approximation of $z^*(\cdot)$ we extend it to $[-T, 0]$ by setting $z^*(t) = x_0$, and then extending $u(s_1, s_2)$ to $[-T, 0] \times [0, T]$ by choosing an arbitrary measurably selection $u(s_1, s_2) \in eF(s_2, x_0)$. Then we approximate $z^*(t)$ by

$$\tilde{z}(t) = x_0 + \frac{1}{T} \int_0^t \int_{s_2 - T}^{s_2} u(s_1, s_2) \, ds_1 \, ds_2.$$

Setting

$$A_1^t = \{(s_1, s_2) \in \mathbb{R}^2 | s_1 \in [0, t], s_2 \in [s_1, s_1 + T]\}$$

and

$$A_2^t = \{(s_1, s_2) \in \mathbb{R}^2 | s_2 \in [0, t], s_1 \in [s_2 - T, s_2]\}$$

we observe that

$$z^*(t) = x_0 + \frac{1}{T} \int_0^t u(s_1, s_2) \, ds_1, s_2.$$
and
\[ \ddot{z} (t) = x_0 + \frac{1}{T} \int_{A_l^1} u(s_1, s_2) \, d(s_1, s_2). \]

Thus, for any \( t \in [0, \epsilon^{-1}] \) their difference is
\[
|\ddot{z}(t) - z^*(t)| = \frac{1}{T} \left| \int_{A_l^1} u(s_1, s_2) \, d(s_1, s_2) - \int_{A_l^2} u(s_1, s_2) \, d(s_1, s_2) \right| \\
= \frac{1}{T} \left| \int_{A_l^1 \setminus A_l^2} u(s_1, s_2) \, d(s_1, s_2) - \int_{A_l^2 \setminus A_l^1} u(s_1, s_2) \, d(s_1, s_2) \right| \leq \epsilon MT,
\]
since the measure of the set \((A_l^1 \setminus A_l^2) \cup (A_l^2 \setminus A_l^1)\) is bounded by \(T^2\).

To apply the Filippov-Gronwall inequality we need to bound
\[
(3.2) \quad d(\ddot{z}(t), \epsilon F(t, \ddot{z}(t))) \leq d(\ddot{z}(t), \epsilon F(t, z^*(t))) + \epsilon d_H(F(t, z^*(t)), F(t, \ddot{z}(t))).
\]

The second term above is bounded by \(\epsilon^2 KMT\) and the first term by
\[
d(\ddot{z}(t), \epsilon F(t, z^*(t))) = \epsilon d_H \left( \frac{1}{T} \int_0^T F(t, z^*(t) - s) \, ds, F(t, z^*(t)) \right) \\
\leq \frac{\epsilon K}{T} \int_0^T |z^*(t - s) - z^*(t)| \, ds \leq \frac{1}{2} \epsilon^2 KMT,
\]
where we use the fact that
\[
\ddot{z}(t) = \frac{1}{T} \int_{t-T}^t u(s, t) \, ds \in \frac{1}{T} \int_{t-T}^t \epsilon F(t, z^*(s)) \, ds = \frac{1}{T} \int_0^T \epsilon F(t, z^*(t - s)) \, ds.
\]

This bounds (3.2) by \(\frac{3}{2} \epsilon^2 KMT\), and establishes the existence of a solution \(x^{**} (\cdot)\) of (1.1) which is \(\frac{3}{2} \epsilon KMT e^K\) far from \(\ddot{z}(\cdot)\) in \([0, \epsilon^{-1}]\), and thus \(\epsilon MT (1 + \frac{3}{2} Ke^K)\) far from \(z^*(\cdot)\), which completes the proof. \(\square\)

4. Averaging Differential Inclusions

In this section we establish new estimates for the averaging of differential inclusions, in the general case, which we apply to obtain sharp bounds for multifrequency differential inclusion. Specifically, we consider two types of inclusions of the form (1.1), where \(F(t, x)\) is either of the form \(F(t, x) = F_1(t, x) + F_2(t, x) + \cdots + F_m(t, x)\) and each \(F_j(t, x)\) is periodic in \(t\) with period \(T_j\), or when each entry of \(F(t, x)\) is periodic, namely, \(F(t, x) = (F_1(t, x), F_2(t, x), \ldots, F_m(t, x))\), and \(F_j(t, x)\) is periodic in \(t\) with period \(T_j\).

Artstein\(^{10}\) presented a new approach for estimating the approximation error in the study of averaging ordinary differential equations, which uses quantitative information on the local fluctuations of the time-dependent vector field. He extended this idea to control systems and differential inclusions in a series of talks, and in this paper we provide a proof for this theorem, as well as new tight bounds.

We start by presenting Artstein’s gauge, and then establish its additivity and verify our main result on multi-frequency differential equations.

**Theorem 4.1.** Suppose \(F(t, x)\) satisfies Assumption \(2.1\) and there exists \((\Delta(\epsilon), \eta(\epsilon))\) satisfying
\[
(4.1) \quad d_H \left( \frac{\epsilon}{\Delta(\epsilon)} \int_{s_0}^{s_0 + \frac{\Delta(\epsilon)}{3}} F(s, x) \, ds, \bar{F}(x) \right) \leq \eta(\epsilon),
\]
for all \(s_0 \geq 0\) and \(x \in \Omega\), then the approximation error is bounded by
\[
M \left( 1 + \frac{3}{2} Ke^K \right) \Delta(\epsilon) + e^K \eta(\epsilon)
\]
in the time interval \([0, \epsilon^{-1}]\). In particular the approximation error is of order \(O(\max(\Delta(\epsilon), \eta(\epsilon)))\).
From the aforementioned observations we conclude that for every functions satisfy $F_j (t, x)$ has a well defined average $\bar{F}_j (x)$. If for every $j = 1, \ldots, m$ there exist $(\Delta_j (\epsilon), \eta_j (\epsilon))$ satisfying

$$
\left| \frac{\epsilon}{\Delta_j (\epsilon)} \int_{s_0}^{s_0 + \Delta (\epsilon)} F_j (s, x) \, ds, \bar{F}_j (x) \right| \leq \eta_j (\epsilon),
$$

for all $s_0 \geq 0$ and $x \in \Omega$. Then the averaging estimation of the system $[11]$ in the time interval $[0, \epsilon^{-1} \mathbf{]}$ is

$$
M \left( 1 + \frac{3}{2} Ke^K \right) \sum_{j=1}^{m} \Delta_j (\epsilon) + e^K \sum_{j=1}^{m} \eta_j (\epsilon).
$$

Proof. To verify this theorem we shall use the following two observations. Suppose $G(t, x)$ satisfies Assumption $[2.4]$ and $|d_H (G(s, x), G(x))| \leq \alpha$, then $|d_H (G_T (s, x), G(x))| \leq \alpha$ for every $T > 0$. Also, Fubini’s theorem implies that

$$
\frac{1}{T_1} \int_{t}^{t + T_1} G_T (s, x) \, ds = \frac{1}{T_1} \int_{t}^{t + T_1} G_T (s, x) \, ds,
$$

for every $T_1, T_2 > 0, x \in \Omega$ and $t \in \mathbb{R}$.

For every $j = 1, \ldots, m$ let $T_j = \frac{\Delta_j (\epsilon)}{\epsilon}$. We define a sequence of set-valued mapping, setting $F_j^0 (t, x) = F_j (t, x)$ and

$$
F_j^j (t, x) = \frac{1}{T_j} \int_{t}^{t + T_j} F_j^{j-1} (s, x) \, ds,
$$

for $j = 1, \ldots, m$. We also set $F_j^0 (t, x) = F_j (t, x)$ and $F_j^i (t, x) = \sum_{j=1}^{m} F_j^j (t, x)$. These set-valued functions satisfy $F_j^i (t, x) = F_j^{i-1} (t, x)$. Now by the triangle inequality

$$
d_H \left( S_{[0, \epsilon^{-1}]} (\epsilon F, x_0), S_{[0, \epsilon^{-1}]} (\epsilon \bar{F}, x_0) \right) \leq d_H \left( S_{F^m, x_0}, S_{[0, \epsilon^{-1}]} (\epsilon \bar{F}, x_0) \right) + \sum_{j=1}^{m} d_H \left( S_{F^{j-1}, x_0}, S_{F^j, x_0} \right).
$$

From the aforementioned observations we conclude that for every $j = 1, \ldots, m$ and $t \in [0, \epsilon^{-1}]$ we have that

$$
d_H \left( F_j^m (t, x), \bar{F}_j (x) \right) \leq \frac{1}{T_j} \int_{s_0}^{s_0 + T_j} F_j (s, x) \, ds, \bar{F}_j (x) \right) \leq \eta_j (\epsilon),
$$

Proof. Set $T = \frac{\Delta (\epsilon)}{\epsilon}$. The triangle inequality bounds $d_H \left( S_{[0, \epsilon^{-1}]} (\epsilon F, x_0), S_{[0, \epsilon^{-1}]} (\epsilon \bar{F}, x_0) \right)$ by

$$
d_H \left( S_{[0, \epsilon^{-1}]} (\epsilon F, x_0), S_{[0, \epsilon^{-1}]} (\epsilon \bar{F}, x_0) \right) + d_H \left( S_{[0, \epsilon^{-1}]} (\epsilon \bar{F}, x_0), S_{[0, \epsilon^{-1}]} (\epsilon \bar{F}, x_0) \right).
$$

The first term above is bounded using Lemma $3.2$ by $M \left( 1 + \frac{3}{2} Ke^K \right) \Delta (\epsilon)$ and the second term is bounded by the Filippov-Gronwall inequality by $\eta (\epsilon) e^K$, since $[1.1]$ implies that $d_H (F_T (t, x), \bar{F} (x)) \leq \eta (\epsilon)$.

Applying this theorem to a periodic differential inclusion implies the classical result of Plotnikov $[8]$.
and, thus
\[ d_H \left( F^m (t, x), \bar{F} (x) \right) \leq \sum_{j=1}^{m} \eta_j (\epsilon) \] and by the Filippov-Gronwall inequality
\[ d_H \left( S_{[0, \epsilon^{-1}]} (\epsilon F^m, x_0), S_{[0, \epsilon^{-1}]} (\epsilon \bar{F}, x_0) \right) \leq \sum_{j=1}^{m} \eta_j (\epsilon). \]

Applying Lemma 3.2 \(m\) times we conclude that (4.4) is bounded by
\[ \sum_{j=1}^{m} \left( M \left( 1 + \frac{3}{2} Ke^K \right) \Delta_j (\epsilon) + e^K \eta_j (\epsilon) \right). \]

This latter theorem implies one of our main results.

**Corollary 4.4.** Suppose \( F(t, x) = F_1(t, x) + F_2(t, x) + \cdots + F_m(t, x) \) satisfies Assumption 2.1 and for every \( j = 1, \ldots, m \) the set-valued function \( F_j(t, x) \) is periodic in \( t \) with period \( T_j \). Then, the estimation error is
\[ (4.5) \]
\[ \epsilon M \left( 1 + \frac{3}{2} Ke^K \right) \sum_{j=1}^{m} T_j, \]

In particular the estimation is of order \( O(\epsilon) \).

**Proof.** For every \( j = 1, \ldots, m \) set \( \Delta_j (\epsilon) = \epsilon T_j \) and \( \eta_j (\epsilon) = 0 \), and apply Theorem 4.3.

We now extend our result to multifrequency differential inclusions where \( F(t, x) \) is of the form
\[ F(t, x) = (F_1(t, x), F_2(t, x), \ldots, F_d(t, x)), \]
where each of its components \( (F_j(t, x)) \) satisfies a bound of the form (4.2). This extension is crucial in the following section where our results are applied to control systems.

Note that, Theorem 4.3 cannot be applied to such a function since these functions may not be expressed as a sum of set periodic set-valued mappings as can be seen in the following example.

**Example 4.5.** Consider the set-valued mapping
\[ F(t, x) = F(t) = \{(7 \cos t + \cos \pi t + \sin u) \in \mathbb{R}^2 | u \in [0, 2\pi]\}. \]

It is clear that it is not the Minkovski sum of its components, namely, that
\[ F(t, x) \neq \{(7 \cos t + \cos u, 0) | u \in [0, 2\pi]\} + \{(0, 7 \sin \pi t + \sin u) | u \in [0, 2\pi]\}. \]

Applying the same line of proof as in Theorem 4.3 we conclude the following result.

**Theorem 4.6.** Suppose \( F(t, x) = (F_1(t, x), F_2(t, x), \ldots, F_d(t, x)) \) satisfies Assumption 2.1. If for every \( j = 1, \ldots, m \) there exists \((\Delta_j (\epsilon), \eta_j (\epsilon))\) satisfying (4.2) Then the estimation error is given by
\[ M \left( 1 + \frac{3}{2} Ke^K \right) \sum_{j=1}^{m} \Delta_j (\epsilon) + e^K \sqrt{\sum_{j=1}^{m} (\eta_j (\epsilon))^2}. \]

Form \( m = d \). In particular the estimation is of order \( O(\epsilon) \).

**Proof.** The proof follows from the proof of Theorem 4.3 with the exception that in this case we bound
\[ d_H \left( F^m (t, x), \bar{F} (x) \right) \leq \sum_{j=1}^{d} d_H \left( F_j^m (t, x), \bar{F}_j (x) \right) \leq \sqrt{\sum_{j=1}^{m} (\eta_j (\epsilon))^2}. \]

**Corollary 4.7.** Suppose \( F(t, x) = (F_1(t, x), F_2(t, x), \ldots, F_d(t, x)) \) satisfies Assumption 2.1 and that for every \( j = 1, \ldots, m \) its \( j \)th entry \( F_j(t, x) \) is periodic in \( t \) with period \( T_j \). Then the estimation error is given by (4.3) with \( m = d \). In particular the estimation is of order \( O(\epsilon) \).
Our main result for this section is as follows.

We assume our system satisfies the following conditions.

Assumption 5.1. We assume that \( \mathbb{U} \subset \mathbb{R}^k \) is compact and that for every \( j = 1, \ldots, m \) the following conditions hold:

1. \( g_j(t, x, u) : \mathbb{R} \times \Omega \times \mathbb{U} \rightarrow \mathbb{R}^d \) is bounded in norm by \( M_j \).
2. \( g_j(t, x, u) \) is measurable in \( t \) and \( u \).
3. \( g_j(t, x, u) \) satisfies Lipschitz conditions in \( x \) uniformly in \( t \) and \( u \), with a Lipschitz constant \( K_j \).

Our assumptions imply that \( G(t, x) \) satisfies Assumption (2.1), with \( M = \sum_{j=1}^{m} M_j \) and \( K = \sum_{j=1}^{m} K_j \), where the periodicity of the functions \( g_j(t, x, u) \) implies that the average of \( G(t, x) \) exists.

Our main result for this section is as follows.

Theorem 5.2. Suppose \( g(t, x, u) = g_1(t, x, u) + g_2(t, x, u) + \cdots + g_m(t, x, u) \) satisfies the condition of Assumption (2.1) and for every \( j = 1, \ldots, m \) the function \( g_j(t, x, u) \) is periodic in \( t \) with period \( T_j \). Then the approximation error is

\[
\epsilon \sqrt{m M_H \left( 1 + \frac{3}{2} K_H \epsilon^2 \right) \sum_{j=1}^{N} T_j},
\]

where \( M_H = \sqrt{\sum_{j=1}^{m} M_j^2} \) and \( K_H = \sqrt{m \sum_{j=1}^{m} K_j^2} \). In particular, it is of order \( O(\epsilon) \).
Although each \( g_j (t, x, u) \) is periodic, the fact they all imply the same control \( u \), implies that one cannot necessarily write \( G (t, x) \) as a sum of periodic set-valued mapping (see Example \ref{example:product_system}), and we cannot trivially apply our results from the previous section. To verify Theorem \ref{theorem:main_result} we, essentially, “decouple” the periods in the system, by splitting the multi-frequency system to a system of \( m \) coupled periodic equations, each having a different period, then apply our bounds. By this we, essentially, decouple the different periods of the system, introducing a control system of dimension \( \mathbb{R}^{md} \) containing \( m \) subsystem of dimension \( \mathbb{R}^d \), each having a periodic vector field. In order that this system represents the solution of the original equation, we must couple all the the new variables in the following manner.

We represent a vector in \( \mathbb{R}^{md} \) by \( z = (z_1, z_2, \ldots, z_m) \in \mathbb{R}^{md} \) where \( z_j \in \mathbb{R}^d \), and we define the linear map

\[
\Phi (z) = \sum_{j=1}^{m} z_j,
\]

which is Lipschitz continuous with a Lipschitz constant \( \sqrt{m} \). With this notation we defined the auxiliary system \( \dot{z} = h (t, z, u) \), \( z (0) = z^0 \) by

\[
\dot{z} = \begin{bmatrix}
\dot{z}_1 \\
\dot{z}_2 \\
\vdots \\
\dot{z}_m
\end{bmatrix} = \begin{bmatrix}
g_1 (t, z_1 + \cdots + z_m, u) \\
g_2 (t, z_1 + \cdots + z_m, u) \\
\vdots \\
g_m (t, z_1 + \cdots + z_m, u)
\end{bmatrix}
= \begin{bmatrix}
g_1 (t, \Phi (z), u) \\
g_2 (t, \Phi (z), u) \\
\vdots \\
g_m (t, \Phi (z), u)
\end{bmatrix} = h (t, z, u),
\]

where \( z^0 = x_0 \) and \( z^0_j = 0 \in \mathbb{R}^d \) for \( j = 2, \ldots, m \). This system is constructed so that \( g (t, \Phi (z), u) = \Phi (h (t, z, u)) \). This also holds for the corresponding averaged systems and \( \bar{G} (\Phi (z)) = \Phi (\bar{H} (z)) \), where

\[
\bar{H} (x) = \lim_{T \to \infty} \frac{1}{T} \int_0^T \{ h (s, x, u) | u \in U \} \, ds.
\]

In particular, these equalities imply that when \( x^* (\cdot) \) is a solution of \ref{equation:original_system} with control \( u^* (\cdot) \), then applying the same control to the auxiliary system, we obtain a solution \( z^* (\cdot) \) satisfying

\[
x^* (t) = \Phi (z^* (t)) = z^*_1 (t) + \cdots + z^*_m (t),
\]

for every \( t \in [0, \epsilon^{-1}] \) and vice-versa. Thus we conclude that \( d_H (G, x_0, \Phi (S_H, x_0)) = 0 \), and similarly that \( d_H (\bar{G}, x_0, \Phi (S_{\bar{H}}, x_0)) = 0 \).

From this we conclude that it suffices to study the approximation error of averaging the auxiliary system \ref{equation:auxiliary_system}.

**Lemma 5.3.** The approximation error of Equation \ref{equation:auxiliary_system} is

\[
\epsilon M_H \left( 1 + \frac{3}{2} K_H \epsilon^2 K_H \right) \sum_{j=1}^{m} T_j,
\]

where \( M_H = \sqrt{\sum_{j=1}^{m} M_j^2} \) and \( K_H = \sqrt{m \sum_{j=1}^{m} K_j^2} \).

**Proof.** It is clear that the function \( h (t, z, u) \) is bounded in norm by \( M_H \). To bound its Lipschitz condition we observe that

\[
|g_j (t, \Phi (z^1), u) - g_j (t, \Phi (z^2), u)| \leq K_j |\Phi (z^1) - \Phi (z^2)| \leq \sqrt{m} K_j |z^1 - z^2|,
\]

for arbitrary \( z^1, z^2 \in \Omega \) thus the Lipschitz constant of \( h (t, z, u) \) is \( K_H \), and the lemma follows from Corollary \ref{corollary:Lipschitz_bound}.

We are now ready to prove the main result of this section.

**Proof of Theorem \ref{theorem:approximation_error}.** The triangle inequality bounds

\[
d_H (G, x_0, S_{\bar{G}, x_0}) \leq d_H (G, x_0, \Phi (S_H, x_0)) + d_H (\Phi (S_H, x_0), \Phi (S_{\bar{H}}, x_0)) + d_H (\Phi (S_{\bar{H}}, x_0), S_{\bar{G}, x_0}).
\]
While the first and third terms above equal zero, we bound the second term using the Lipschitz constant of $\Phi$ and by Lemma 5.3
\[
d_H \left(S_{G,x_0}, S\bar{G}_{x_0}\right) \leq d_H \left(\Phi(S_{H,x_0}), \Phi(S\bar{G}_{H,x_0})\right) \leq \sqrt{m}d_H \left(S_{H,x_0}, S\bar{H}_{x_0}\right)
\leq \epsilon\sqrt{m}M_H \left(1 + \frac{3}{2}K_H^2\right)\sum_{j=1}^{m}T_j.
\]

The latter theorem can be extended in a similar manner to Corollary 4.9 when each entry of $g_j(t,x,u)$ has a different period.

**Theorem 5.4.** Suppose $g(t,x,u) = g_1(t,x,u) + g_2(t,x,u) + \cdots + g_m(t,x,u)$ satisfies the condition 1-3 of Assumption 5.1 and for every $j = 1, \ldots, m$ the $i$'th entry of $g_j(t,x,u)$ is periodic in $t$ with period $T_{j,i}$. If
\[
\{T_{i,j} | j = 1, \ldots, m, i = 1, \ldots, d\} \subset \{T_1, \ldots, T_N\}
\]
then the approximation error is $\epsilon\sqrt{m}M_H \left(1 + \frac{3}{2}K_H^2\right)\sum_{j=1}^{m}T_j$, where $M_H = \sqrt{\sum_{j=1}^{m}M_j^2}$ and $K_H = \sqrt{m}\sum_{j=1}^{m}K_j^2$. In particular, it is of order $O(\epsilon)$.

The following is an application of our results.

**Example 5.5.** Consider the control system given by
\[
\dot{x} = \epsilon g(t,x,u) = \epsilon x + \epsilon u \left(\cos(2\pi t) + \cos(2t)\right), \quad x(0) = 0.
\]

where $U = [-1, 1]$. The averaged equation in this case can be expressed according to (1.4) by
\[
y = \epsilon \bar{G} \left(y\right) = \left\{\epsilon y + \frac{1}{(2\pi)^2} \int_{[0,2\pi]^2} \epsilon u(\phi_1,\phi_2) \left(\cos(\phi_1) + \cos(\phi_2)\right) d(\phi_1,\phi_2) | u : [0,2\pi]^2 \rightarrow [-1, 1]\right\}.
\]

The set $\bar{G} \left(y\right)$ is convex and by symmetry we conclude that $\bar{G} \left(y\right) = [y - \alpha, y + \alpha]$, where
\[
\alpha = \frac{1}{(2\pi)^2} \int_{[0,2\pi]^2} |\cos(\phi_1) + \cos(\phi_2)| d\phi \approx 0.815
\]
was computed analytically. So in the the domain $\Omega = [-2, 2]$ we have that $M_H = \sqrt{10}$, $K_H = \sqrt{2}$ and our theorem implies that the estimation error is bounded by
\[
\epsilon\sqrt{20} \left(1 + \frac{3}{2}\sqrt{2e\sqrt{2}}\right) \left(1 + \pi^{-1}\right).
\]

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