Applications of iterated curve blowup to set theoretic complete intersections in \(\mathbb{P}^3\)

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Introduction

We describe some new results on the set-theoretic complete intersection problem for projective space curves. Fix an algebraically closed ground field \( k \). Let \( S, T \subset \mathbb{P}^3 \) be surfaces. Suppose that \( S \cap T \) is set-theoretically a smooth curve \( C \) of degree \( d \) and genus \( g \). For purposes of the introduction, we label the main results as A, B, Q, X, I, II, and III.\(^\dag\) The results I, II, and III are more technical than A, B, Q, and X.

Suppose that \( S \) and \( T \) have no common singular points. We discover that this requirement imposes severe limitations. Indeed, theorem (X) asserts that if \( C \) is not a complete intersection, then \( \deg(S), \deg(T) < 2d^4 \). Fixing \( (d, g) \), one can in fact form a finite list of all possible pairs \( (\deg(S), \deg(T)) \), which is much shorter than the list implied by theorem (X). For instance, when \( (d, g) = (4, 0) \), and assuming for simplicity that \( \deg(S) \leq \deg(T) \), we find that

\[
(\deg(S), \deg(T)) \in \{(3, 4), (3, 8), (4, 4), (4, 7), (6, 26), (9, 48), (10, 28), \]
\[
(12, 18), (13, 16), (17, 220), (18, 118), (19, 84), (20, 67), (22, 50), (28, 33)\}.
\]

Very little is known about which of these degree pairs actually correspond to surface pairs \( (S, T) \).

Suppose that \( S \) and \( T \) have only rational singularities, and that the ground field \( k \) has characteristic zero. We continue to assume that \( S \) and \( T \) have no common singular points. Under these conditions, we prove (A) that \( d \leq g + 3 \). (The actual statement is somewhat stronger.)

Suppose that \( S \) is normal, and that \( d > \deg(S) \). Make no assumptions about how the singularities of \( S \) and \( T \) meet. Assume that \( \text{char}(k) = 0 \). We show (Q) that \( C \) is linearly normal. In particular, it follows by Riemann-Roch that \( d \leq g + 3 \).

Suppose that \( S \) is a quartic surface having only rational singularities. Allow \( T \) to be an arbitrary surface, and make no assumptions about how the singularities of \( S \) and \( T \) meet. Assume that \( \text{char}(k) = 0 \). Under these conditions, we prove (B) that \( C \) is linearly normal.

In other papers [17], [18], we have proved the following complementary results (in characteristic zero): if \( S \) is has only ordinary nodes as singularities, \(^1\)

\(^1\)The actual numbering in the text is A = 12.2, B = 11.11, Q = 11.8, X = 13.1, I = 10.1, II = 10.3, III = 11.4.
or is a cone, or has degree \( \leq 3 \), then \( d \leq g + 3 \). It is conceivable (in characteristic zero) that this inequality is valid without any restrictions whatsoever on \( S \) and \( T \), or even that \( C \) is always linearly normal. Examples of smooth set-theoretic complete intersection curves in \( \mathbb{C}P^3 \) have been constructed by Gallarati [8], Catanese [3], Rao ([27] prop. 14), and the author [19].

To explain the results (I), (II), and (III), and to describe the methods by which we prove (A), (B), and (X), there are two key ideas which must be discussed. Both of these ideas have to do with the iterated blowing up of curves.

The first idea has to do with certain invariants \( p_i = p_i(S, C) \) \( (i \in \mathbb{N}) \) which we associate to a pair \((S, C)\) consisting of an abstract surface \( S \) and a smooth curve \( C \) on \( S \) such that \( C \not\subset \text{Sing}(S) \). Let \( \pi : \tilde{S} \to S \) be the blowup along \( C \). Then \( p_1(S, C) \) is the sum of the multiplicities of the exceptional curves. (See §2 for details.) Moreover, \( \pi \) admits a unique section \( \tilde{C} \) over \( C \), so we can define \( p_2(S, C) = p_1(\tilde{S}, \tilde{C}) \), \( p_3(S, C) = p_2(\tilde{S}, \tilde{C}) \), and so forth. We refer to the sequence \((p_1, p_2, \ldots)\) as the type of \((S, C)\). It is a sum of local contributions, one for each singular point of \( S \) along \( C \), and it is a rather mysterious measure of how singular \( S \) is along \( C \). The type depends not only on the particular species of singular points of \( S \) which lie on \( C \), but also on the way in which \( C \) passes through those points. For example, the local contribution to the type coming from an \( A_3 \) singularity is either \((1, 1, 1, 0, \ldots)\) or \((2, 0, \ldots)\), depending on how \( C \) passes through the singular point.

The second idea is the following construction. For this we assume (as in the first paragraph) that \( C = S \cap T \) (in \( \mathbb{P}^3 \)) and that \( C \not\subset \text{Sing}(S), C \not\subset \text{Sing}(T) \). Other than this, no restrictions are necessary on the singularities of \( S \) and \( T \). Let \( Y_1 \) denote the blowup of \( \mathbb{P}^3 \) along \( C \). Let \( S_1, T_1 \subset Y_1 \) denote the strict transforms of \( S \) and \( T \) respectively. Let \( E_1 \subset Y_1 \) be the exceptional divisor, which is a ruled surface over \( C \). Then \( S_1 \cap E_1 \) is a curve \( C_1 \) (mapping isomorphically onto \( C \)), together with some rulings. The total number of rulings, counted with multiplicities, is \( p_1(S, C) \). Now let \( Y_2 \) be the blowup of \( Y_1 \) along \( C_1 \). Let \( S_2, T_2, E_2 \subset Y_2 \) be as above. Then \( S_2 \cap E_2 \) is a curve \( C_2 \) plus \( p_2(S, C) \) rulings. Iterate this construction \( n \) times, where \( n \) is the multiplicity of intersection of \( S \) and \( T \) along \( C \). Then \( S_n \cap T_n \) is a union of strict transforms of rulings. This fact leads us to theorems (I) and (II), which are statements about the numbers \( p_i \). Theorem (III) is also

\[\text{We also give an alternate proof of the key ingredient of (X), which is independent of the main machine of this paper.}\]
such a statement, but it does not depend on the construction we have just described.

We describe theorems (I), (II), and (III). These depend on the data $(s, t, d, g)$, where $s = \deg(S)$, $t = \deg(T)$. To make this description as simple as possible, we restrict our attention here to the special case where $(s, t, d, g) = (4, 4, 4, 0)$.

Theorem (I) has the hypothesis that $\text{Sing}(S) \cap \text{Sing}(T) = \emptyset$. Its conclusion (applied to our special case) is that:

$$p_1 = p_2 = p_3 = 8.$$ 

Theorem (I) is used in the proofs of (A) and (X).

Theorem (II) has no additional hypotheses. Its conclusion (applied to our special case) is that:

$$p_1 \geq 8;$$
$$2p_1 + p_2 \geq 24;$$
$$8p_1 + 3p_2 + p_3 \geq 96.$$ 

Theorem (II) is not used in the proofs of (A) or (B).

Theorem (III) has the hypotheses that $S$ has only rational singularities, and that $\text{char}(k) = 0$. Its conclusion (applied to our special case) is that:

$$\frac{1}{2}p_1 + \frac{1}{6}p_2 + \frac{1}{12}p_3 + \cdots + \frac{1}{k(k+1)}p_k + \cdots \geq 6.$$ 

Theorem (III), or actually a minor variant of it, is used in the proof of (B).

We now mention some open problems and possible ways to improve upon the results in this paper.

1. Let $(S, C)$ be the local scheme $S$ of a normal surface singularity, together with a smooth curve $C$ on $S$. There are three fundamental invariants of $(S, C)$ which are utilized in this paper. Firstly, there is the type of $(S, C)$. Secondly, there is the order of $(S, C)$, i.e. the smallest positive integer $n$ such that $\mathcal{O}_S(nC)$ is Cartier. Thirdly, there is $\Delta(S, C)$, which we describe in §1. What relationships exist between these three invariants? What is their relationship to the Milnor fiber?

2. We suspect that (A), (B), and (III) are valid over an arbitrary algebraically closed field. There are significant difficulties in proving this which we have not explored fully. The proofs of (I) and (II) do not depend on the characteristic.
3. The proofs of (A) and (B) use a bound (11.3) on the number of exceptional curves in a minimal resolution for a surface \( S \subset \mathbb{CP}^3 \) having only rational singularities. Formulate and prove a suitable generalization for arbitrary normal surfaces.

4. Construct examples of surfaces \( S, T \subset \mathbb{CP}^3 \), having only rational singularities, meeting set-theoretically along a smooth curve \( C \), such that \( \text{Sing}(S) \cap \text{Sing}(T) \neq \emptyset \). The only example we know of is where \( \text{deg}(S) = \text{deg}(T) = 2 \), and \( C \) is a line.

5. The generic hypothesis that \( C \not\subset \text{Sing}(T) \) can probably be eliminated.

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Conventions

(1) We fix an algebraically closed field \( k \).

(2) A curve [resp. surface] [resp. three-fold] is an excellent \( k \)-scheme such that every maximal chain of irreducible proper closed subsets has length one [resp. two] [resp. three]. We make the following additional assumptions:

- all curves are reduced and irreducible;
- in part III and the introduction, all surfaces are reduced and irreducible.

(3) A surface embeds in codimension one if it can be exhibited as an effective Cartier divisor on a regular three-fold.

(4) A variety is an integral separated scheme of finite-type over \( k \).

(5) If \( X \) and \( Y \) are schemes, then the notation \( X \subset Y \) carries the implicit assumption that \( X \) is a closed subscheme of \( Y \).

(6) In several situations, we use bracketed exponents to denote repetition in sequences, and we drop trailing zeros, where appropriate. For example,

\[
(2, 1^4) = (2, 1, 1, 1, 1, 0, \ldots)
\]

and

\[
(3^{[\infty]}) = (3, 3, \ldots).
\]
(7) We use the Grothendieck convention regarding projective space bundles.

(8) For any variety \( V \), we let \( A^k(V) \) denote the group of codimension \( k \) cycles on \( V \), modulo algebraic equivalence. When \( d = \dim(V) \) and \( V \) is complete, we identify \( A^d(V) \) with \( \mathbb{Z} \).

Part I

Local geometry of smooth curves on singular surfaces

1 Definitions

We define the category of surface-curve pairs. (Sometimes, we use the shorthand term pair for a surface-curve pair.) An object \((S, C)\) in this category consists of a surface \( S \), together with a curve \( C \subset S \), such that \( C \) is a regular scheme and \( C \not\subset \text{Sing}(S) \). A morphism \( f : (S', C') \to (S, C) \) is a pair \((S' \to S, C' \to C)\) of morphisms of \( k \)-schemes, such that the diagram:

\[
\begin{array}{ccc}
C' & \to & C \\
\downarrow & & \downarrow \\
S' & \to & S
\end{array}
\]

commutes, and such that if \( \phi : C' \to C \times_S S' \) is the induced map, then \( \phi \times_S \text{Spec} \mathcal{O}_{S,C} \) is an isomorphism. Such a morphism \( f \) is cartesian if \( \phi \) is an isomorphism. Most properties of morphisms of schemes also make sense as properties of morphisms in this category: the properties are to be interpreted as properties of the morphism \( S' \to S \).

Let \((S, C)\) be a surface-curve pair. We say that:

- \((S, C)\) is geometric if \( S \) is a variety;
- \((S, C)\) is local if \( S \) is a local scheme;
• \((S, C)\) is local-geometric if \(S\) is a local scheme, essentially of finite type over \(k\).

To give a local surface-curve pair \((S, C)\) is equivalent to giving the data \((A, p)\), consisting of an excellent local \(k\)-algebra \(A\), of pure dimension two, together with a height one prime \(p \subset A\) such that \(A_p\) and \(A/p\) are regular. We write \((S, C) = \text{Spec}(A, p)\) to denote this correspondence.

There are two operations on surface-curve pairs which we will be using. Firstly, if \((S, C)\) is a local surface-curve pair, then the completion \((\hat{S}, \hat{C})\) makes sense and is also a local surface-curve pair. Indeed, if \((S, C) = \text{Spec}(A, p)\), then \(A/p\) is regular, and so \(\hat{A}/\hat{p}\) is regular, since it equals \(\hat{A}/\hat{p}\), and the completion of a regular local ring is regular. The reader may also check easily that \(\hat{A}/\hat{p}\) is regular. Moreover, \(\hat{A}\) is excellent, since any noetherian complete local ring is excellent. Note also: there is a canonical morphism \((S, C) \to (\hat{S}, \hat{C})\).

Secondly, for any surface-curve pair \((S, C)\), one can define the blowup \((\tilde{S}, \tilde{C})\) of \((S, C)\). This is done by letting \(\pi : \tilde{S} \to S\) be the blowup of \(S\) along \(C\), and by letting \(\tilde{C}\) be the unique section of \(\pi\) over \(C\), which exists e.g. by ([11] 7.3.5). There is a canonical morphism \((\tilde{S}, \tilde{C}) \to (S, C)\).

Two local surface-curve pairs are analytically isomorphic if their completions are isomorphic.

If \((S, C)\) is a surface-curve pair, and \(p \in C\), we let \((S, C)_p\) denote the corresponding local surface-curve pair. A configuration is an element of the free abelian monoid on the set of analytic isomorphism classes of local-geometric pairs.

Let \((S, C)\) be a geometric surface-curve pair. We may associate the configuration:

\[
\sum_{p \in \text{Sing}(S) \cap C} [(S, C)_p]
\]

to \((S, C)\). On occasion, we shall identify \((S, C)\) with the associated configuration.

We are interested in invariants of a geometric surface-curve pair \((S, C)\) which depend only on the associated configuration. There are four such invariants which we shall consider:

1. The order of \((S, C)\) is the smallest \(n \in \mathbb{N}\) such that \(O_S(nC)\) is Cartier, or else \(\infty\) if \(O_S(nC)\) is not Cartier for all \(n \in \mathbb{N}\). If \(\text{Sing}(S) \cap C =\)
{p_1, \ldots, p_k}, then
\[ \text{order}(S, C) = \text{lcm}\{\text{order}(S, C)_{p_1}, \ldots, \text{order}(S, C)_{p_k}\}, \]
so the computation of the order is a purely local problem. Moreover, at least if \( S \) is normal, the order depends only on the associated configuration. Indeed, in that case, if \((S, C)\) is a local-geometric pair, then \( S \) is excellent, so \( \hat{S} \) is normal, and so by ([6] 6.12) one knows that the canonical map \( \text{Cl}(S) \to \text{Cl}(\hat{S}) \) is injective. Hence \( \text{order}(S, C) = \text{order}(\hat{S}, \hat{C}) \).

(2) The type of \((S, C)\), which is the sequence \((p_i)_{i \in \mathbb{N}}\) discussed in the introduction, and studied in §2.

(3) Assume that \( S \) is normal. We define an invariant \( \Delta(S, C) \in \mathbb{Q} \). Let \( \pi : \tilde{S} \to S \) be a minimal resolution, and let \( E_1, \ldots, E_n \subset \tilde{S} \) be the exceptional curves. Let \( \tilde{C} \subset \tilde{S} \) be the strict transform of \( C \). According to ([26] p. 241), there is a unique \( \mathbb{Q} \)-divisor \( E = \sum a_i E_i \) such that \((\tilde{C} + E) \cdot E_i = 0\) for all \( i \). We define \( \Delta(S, C) = -E^2 \). Then \( \Delta(S, C) \) is independent of \( \pi \). If \( S \) is projective, then \( \Delta(S, C) = C^2 - \tilde{C}^2 \), where \( C^2 \) is defined in ([26] p. 241).

(4) Assume that \( S \) has only rational double points along \( C \). Let \( \Sigma(S, C) \) equal the number of exceptional curves in the minimal resolution of those singularities of \( S \) which lie on \( C \).

2 The type of a surface-curve pair

We define the type of a surface-curve pair, and show that it is an analytic invariant, at least when the surface embeds in codimension one.

**Definition.** Let \((S, C)\) be a surface-curve pair. Let \((\tilde{S}, \tilde{C})\) be the blowup of \((S, C)\). Let \( E_1, \ldots, E_n \subset \tilde{S} \) be the (reduced) exceptional curves. We define numbers \( p_i(S, C) \), for each \( i \in \mathbb{N} \). Define:

\[ p_1(S, C) = \sum_{i=1}^{n} \text{length} \mathcal{O}_{\pi^{-1}(C), E_i}, \]

where \( \pi : \tilde{S} \to S \) is the blowup map. For \( i \geq 2 \), recursively define \( p_i(S, C) \) by:

\[ p_{i+1}(S, C) = p_i(\tilde{S}, \tilde{C}). \]
The type of \((S, C)\) is the sequence \((p_1, p_2, \ldots)\).

It is clear that the computation of the \(p_i\) may be reduced to the computation of the \(p_i\) when \((S, C)\) is a local pair.

**Remark 2.1** Let \((S, C)\) be a surface-curve pair, and assume that \(S\) embeds in codimension one. Then we have \(S \subset T\) for some smooth three-fold \(T\). Let \(\pi_S: \tilde{S} \to S\) and \(\pi_T: \tilde{T} \to T\) be the blowups of \(S\) and \(T\) along \(C\). Then \(\pi_S^{-1}(C) \cong \tilde{S} \cap E\), where \(E \subset \tilde{T}\) is the exceptional divisor. This fact plays an absolutely central role in our type computations.

**Remark 2.2** We do not know for which \((S, C)\) we have \(p_1(S, C) \geq p_2(S, C)\), and hence that \(p_k(S, C) \geq p_{k+1}(S, C)\) for all \(k \geq 1\). Conceivably, these inequalities may hold whenever \(S\) embeds in codimension one, or even whenever \(S\) is Cohen-Macaulay. By explicit calculation, we shall find in (3.2) that the inequalities hold if \(S\) has only rational double points along \(C\). However, as (3.2) shows, for some \((S, C)\) one has \(p_1(S, C) < p_2(S, C)\).

**Remark 2.3** We consider the following general question. Let \((S, C)\) be a local-geometric pair. Assume that \(S\) is not smooth. Let \(p \in S\) be the closed point. Let \((\tilde{S}, \tilde{C})\) be the blowup of \(S\) along \(C\). Let \(\pi: \tilde{S} \to S\) be the blowup map. What is the structure of \(X = \pi^{-1}(p)_{\text{red}}\)? If \(S\) embeds in codimension one, then \(X\) will be a \(\mathbb{P}^1\). Weird things can happen if \(S\) is not Cohen-Macaulay. For example, in (3.2), \(X\) is isomorphic to \(\text{Proj } \mathbb{C}[s, t, u]/(s^3 - t^2u)\), which is a rational curve with a cusp. In (3.3), \(X\) is the disjoint union of a point and several copies of \(\mathbb{P}^1\), which do not meet \(\tilde{C}\). The isolated point of \(X\) is the unique point of \(\tilde{C}\) lying over \(p\). Assuming only that \(S\) is Cohen-Macaulay, we do not know if \(X\) is always isomorphic to \(\mathbb{P}^1\), or even if it is always connected. However: if \(S\) embeds in codimension two, then \(X\) embeds in \(\mathbb{P}^2\).

We will prove (2.9) that the type of a local-geometric pair \((S, C)\) is an analytic invariant, provided that \(S\) embeds in codimension one. There are some preliminaries.

**Lemma 2.4** Let \(f: A \to B\) be a flat, formally smooth homomorphism of Artin local rings. Assume that \(A\) contains a field. Then \(\text{length}(A) = \text{length}(B)\).
Proof. Let $K$ and $L$ be the residue fields of $A$ and $B$. Let $i : K \to L$ be the induced map. Let $m$ be the maximal ideal of $A$. Then the map $K = A/m \to B/mB$ is formally smooth, so $B/mB = L$ and so $i$ is formally smooth. A theorem of Cohen ([23] 28.J) implies that $A$ contains a coefficient field, which we also denote by $K$. Since $i$ is formally smooth, $L/K$ is a separable field extension. It follows by the cited theorem that we may find a coefficient field $L$ for $B$ which contains $f(K)$.

Let $A = A \otimes_K L$. Then $f$ factors as:

$$A \xrightarrow{h} A \xrightarrow{g} B.$$ 

Since $i$ is formally smooth, so is $h$. Since both $h$ and $g \circ h$ are formally smooth it follows by ([12] 17.1.4) that $g$ is formally smooth.

Clearly $B$ is a finite $A$-module. In particular, $g$ is of finite-type, so $g$ is smooth. Since $g$ is smooth of relative dimension zero, $g$ is étale. By ([12] 18.1.2), the obvious functor:

$$\langle \text{étale } A\text{-schemes} \rangle \longrightarrow \langle \text{étale } L\text{-schemes} \rangle$$

is an equivalence of categories, so $g$ is an isomorphism. Hence $B \cong A \otimes_K L$. Hence length($A$) = length($B$).

Corollary 2.5 Let $f : X' \to X$ be a flat, formally smooth morphism of irreducible noetherian schemes. Assume that $X$ is defined over a field. Let $\eta$ and $\eta'$ be the generic points of $X$ and $X'$. Then:

$$\text{length } \mathcal{O}_{X',\eta'} = \text{length } \mathcal{O}_{X,\eta}.$$ 

Remark 2.6 We do not know if (2.4) and (2.5) are true without the hypothesis of being “defined over a field”.

Lemma 2.7 Let $f : (S_1,C_1) \to (S_2,C_2)$ be a flat morphism of surface-curve pairs. Then $f$ is cartesian.

Proof. We must show that the induced map $\phi : C_1 \to C_2 \times_{S_1} S_2$ is an isomorphism. It suffices to show that $C_1 = C_2 \times_{S_1} S_2$ as closed subschemes of $S_2$. Let $\pi : C_2 \times_{S_1} S_2 \to C_2$ be the projection map. Because $\pi$ is flat, any irreducible component of $C_2 \times_{S_1} S_2$ must dominate $C_2$. (See e.g. [14] III 9.7.) But $\phi \times_{S_2} \text{Spec } \mathcal{O}_{S_2,C_2}$ is an isomorphism, so it follows that $C_2 \times_{S_1} S_2$ is irreducible. Since $C_1 = C_2 \times_{S_1} S_2$ at their generic points, they are equal as closed subschemes of $S_2$. \hfill \qed
**Proposition 2.8** Let \( f : (S_1, C_1) \to (S_2, C_2) \) be a formally smooth, flat morphism of surface-curve pairs. Assume that the induced map \( C_1 \to C_2 \) is bijective. Assume that \( S_1 \) and \( S_2 \) embed in codimension one. Then \((S_1, C_1)\) and \((S_2, C_2)\) have the same type.

**Proof.** The subscript \( i \) will always vary through the set \( \{1, 2\} \). Because of our hypothesis on the map \( C_1 \to C_2 \), we may assume that \( S_1, S_2 \) are local schemes and that \( f \) is a local morphism. Let \( \pi_i : (\tilde{S}_i, \tilde{C}_i) \to (S_i, C_i) \) be the blowup maps. By the universal property of blowing up, and because \( f \) is cartesian by (2.7), we obtain a map \( \tilde{f} : \tilde{S}_1 \to \tilde{S}_2 \) which makes the diagram:

\[
\begin{array}{ccc}
\tilde{S}_1 & \xrightarrow{\tilde{f}} & \tilde{S}_2 \\
\downarrow^{\pi_1} & & \downarrow^{\pi_2} \\
S_1 & \xrightarrow{f} & S_2
\end{array}
\]

commute. Furthermore, using the flatness of \( f \), we see that this diagram is cartesian and as a consequence that \( \tilde{f} \) is flat and formally smooth. Since \( S_1 \) and \( S_2 \) embed in codimension one, so do \( \tilde{S}_1 \) and \( \tilde{S}_2 \). Since \( p_{k+1}(S_i, C_i) = p_k(\tilde{S}_i, \tilde{C}_i) \) for all \( k \geq 1 \), the proof of the proposition will follow if we can show that \( p_1(S_1, C_1) = p_1(S_2, C_2) \).

Let \( x_i \in S_i \) be the unique closed points. It is clear that \( \pi_1^{-1}(x_1) \) maps onto \( \pi_2^{-1}(x_2) \). Moreover, \( \tilde{f}(\tilde{C}_1) = \tilde{C}_2 \). Let \( \tilde{x}_i = \pi_i^{-1}(x_i) \cap \tilde{C}_i \). Then \( \tilde{f}(\tilde{x}_1) = \tilde{x}_2 \). Let \( P_i = C_i \times_S \tilde{S}_i \). A little thought shows that there is a cartesian diagram:

\[
\begin{array}{ccc}
P_1 & \to & P_2 \\
\downarrow & & \downarrow \\
\tilde{S}_1 & \to & \tilde{S}_2.
\end{array}
\]

Since \( S_1 \) and \( S_2 \) embed in codimension one, \( P_i = \tilde{C}_i \cup E_i \), where \( E_i \cong \mathbb{P}^1 \) and \( \tilde{C}_i \cap E_i = \tilde{x}_i \). The equality \( p_1(S_1, C_1) = p_1(S_2, C_2) \) can then be deduced from (2.8).

If \((S, C)\) is a local-geometric pair, then \( S \) is excellent, so the completion map \( \hat{S} \to S \) is formally smooth. Hence we have:

**Corollary 2.9** If two local-geometric surface-curve pairs embed in codimension one and are analytically isomorphic, then they have the same type.

**Remark 2.10** We do not know if (2.8) and (2.9) are true without the hypothesis of “embedding in codimension one”.

11
3 Examples

We give three examples which illustrate type computations and pathological aspects of blowing up. Cf. (5.2), where rational double points are dealt with.

The first example illustrates a general conjecture which we cannot yet make precise: amongst surfaces of given degree in $\mathbb{P}^3$, those which occur in positive characteristic can have "larger" type than those which occur in characteristic zero. Of course, the type also depends on the choice of a curve on the surface.

More specifically, the example shows that in characteristic two, a quartic surface (together with a suitably chosen curve) can have $p_1 = p_2 = p_3 = 8$. We expect that this cannot happen in characteristic zero. If so, it would follow from (10.1 = "I") that a smooth quartic rational curve $C \subset \mathbb{C}P^3$ cannot be the set-theoretic complete intersection of two quartic surfaces, unless $C$ is contained in the singular locus of one of the surfaces.

On the other hand, Hartshorne [15] and Samuel [30] have shown that in positive characteristic, the monomial rational quartic curve $C \subset P^3$ is a set-theoretic complete intersection. See (10.2) for additional comments.

The second two examples have to do with pairs $(S,C)$ in which $S$ is not Cohen-Macaulay. These seem to be of some intrinsic interest, but have no direct relevance to the problem of set-theoretic complete intersections in $P^3$. Example two might be viewed as a statement about the properties of the singularity at the vertex of the cone over a space curve. It would be very nice to understand better the connection between this singularity and the properties of the space curve.

Proposition 3.1 Let $k$ be an algebraically closed field of characteristic two. Let $S \subset \mathbb{P}^3$ be the cuspidal cone given by $y^4 - x^3w = 0$. Let $C \subset S$ be the smooth rational quartic curve given by

$$(s,t) \mapsto (x,y,z,w) = (s^4, s^3t, st^3, t^4).$$

Then $C$ meets $\text{Sing}(S)$ at the unique point $(0,0,0,1)$, and the type of $(S,C)$ is $(8,8,8)$.

Proof. We will calculate in the category of affine varieties, so we will replace $S$ by an affine variety, and when we refer to a blowup, we will actually mean
a correctly chosen affine piece of the blowup. We let \((S_n, C_n)\) denote the \(n\)th iterated blowup of \((S, C)\).

The assertion that \(C \cap \text{Sing}(S) = \{(0, 0, 0, 1)\}\) is easily checked. Taking the affine piece at \(w = 1\), we find that \(S\) is given by \(y^4 = x^3\) and that \(C\) is given by \(x = yz\) and \(y = z^3\). Making the change of variable \(x \mapsto x + yz\), followed by \(y \mapsto y + z^3\), we obtain the new equation:

\[
y^4 + z^{12} = (x + yz + z^4)^3
\]

for \(S\) and the equation \(x = y = 0\) for \(C\).

Blow up \(S\) along \(C\), formally substituting \(xy\) for \(x\). Then \(S_1\) is given by:

\[
y^3 = x^3y^2 + x^2y^2z + x^2yz^2 + xy^2z^2 + y^3z^3 + y^6z + xz^8 + z^9.
\]

Intersecting with the exceptional divisor, as in (2.1), corresponds to setting \(y = 0\). We obtain \(z^8(z + x) = 0\), which tells us that \(p_1(S, C) = 8\), and that \(C_1\) is given by \(y = 0, z + x = 0\). Making the change of variable \(z \mapsto z - x\), we obtain the new equation:

\[
y^3 = y^3z^3 + y^6z^6 + x^4yz^2 + x^8z + z^9
\]

for \(S_1\), and the equation \(x = y = 0\) for \(C_1\).

Blow up \(S_1\) along \(C_1\), formally substituting \(zy\) for \(y\). Then \(S_2\) is given by:

\[
y^2 = y^4z^3 + y^6z^6 + x^4y^2z^2 + y^8z^9 + x^8z.
\]

Setting \(y = 0\), we obtain \(x^8z = 0\), which tells us that \(p_2(S, C) = 8\), and that \(C_2\) is given by \(y = z = 0\).

Blow up \(S_2\) along \(C_2\), formally substituting \(zy\) for \(z\). Then the blown up surface \(S_3\) is given by:

\[
y = y^6z^3 + y^{11}z^6 + x^4y^3z^2 + y^{16}z^9 + x^8z
\]

Setting \(y = 0\), we obtain \(x^8z = 0\), which tells us that \(p_3(S, C) = 8\), and that the new curve \(C_3\) is given by \(y = z = 0\). One can check that \(S_3\) is smooth along \(C_3\), so \(p_k(S, C) = 0\) for all \(k > 3\).

\[\text{Proposition 3.2} \quad \text{Let} \ S \subset \mathbb{A}^4 = \text{Spec} \mathbb{C}[x, y, z, w] \text{ be the cone over the monomial quartic curve:}
\]

\[
(s, t) \mapsto (x, y, z, w) = (s^4, s^3t, st^3, t^4)
\]
Let $C \subset S$ be the ruling given by $y = z = w = 0$. Then $\text{type}(S, C) = (1, 2)$. Moreover, if $\pi : S_1 \to S$ denotes the blowup along $C$, and $p \in S$ denotes the unique singular point, then $\pi^{-1}(p)_\text{red} \cong \text{Proj} \mathbb{C}[s, t, u](s^3 - t^2 u)$.

**Proof.** One sees that $S$ is given by the equations $yz = xw$, $x^2 z = y^3$, $z^3 = yw^2$, and $y^2 w = xz^2$. The blowup $S_1$ of $S$ along $C$ is obtained by formally substituting $z = sy$, $w = ty$. Then $S_1$ is given by $sy = tx$, $sx^2 = y^2$, $s^3 = t^2$, and $ty = s^2 x$. Then:

$$\pi_1^{-1}(C) \cong \text{Spec} \mathbb{C}[x, y, s, t]/(y, tx, sx^2, s^3 - t^2, s^2 x).$$

Set-theoretically,

$$\pi_1^{-1}(C) = V(s, t, y) \cup V(x, y, s^3 - t^2).$$

We have $C_1 = V(s, t, y)$. Thus:

$$p_1(S, C) = \text{length} \mathbb{C}[x, s, t]/(tx, sx^2, s^3 - t^2, s^2 x)_{(x, s^3 - t^2)},$$

which equals one.

The blowup $S_2$ of $S_1$ along $C_1$ is obtained by formally substituting $s = ay$, $t = by$. Then $S_2$ is given by $ax^2 = y$ and $b = a^2 x$. Let $\pi_2 : S_2 \to S_1$ be the blowup map. Then:

$$\pi_2^{-1}(C_1) \cong \text{Spec} \mathbb{C}[a, b, x]/(ax^2, b - a^2 x) \cong \text{Spec} \mathbb{C}[a, x]/(ax^2).$$

This implies that $p_2(S, C) = 2$. Since $S_2$ is smooth, we see that $\text{type}(S, C)$ is as claimed.

**Example 3.3** Let $S$ be a smooth surface, which is a variety. Fix $n \geq 2$, and let $p_1, \ldots, p_n \in S$ be distinct (closed) points. Let $\pi : S \to \overline{S}$ be the morphism which pinches $p_1, \ldots, p_n$ together, yielding $p \in \overline{S}$. Let $C \subset S$ be a smooth curve passing through $p_1$ but not through $p_2, \ldots, p_n$. Then $\overline{C} = \pi(C)$ is smooth. Let $f : X \to \overline{S}$ be the blowup along $\overline{C}$. Then $X$ is obtained from $S$ by blowing up $p_2, \ldots, p_n$. Hence $f^{-1}(\overline{C})$ is isomorphic to the disjoint union of $\overline{C}$ with $n - 1$ copies of $\mathbb{P}^1$. The type of $(S, C)$ is $(n - 1)$.

---

\[3\text{In this situation, where } S \text{ does not embed in codimension one, it is apparently necessary to look at all of the affine pieces of the blowup. The details of this are left to the reader. These calculations are greatly facilitated by the use of a computer program such as Macaulay.}\]
4 Classification of rational double point pairs

In this section, we assume that \( k \) has characteristic zero. We describe a classification of local-geometric pairs \((S, C)\), up to analytic isomorphism, where \( S \) is the local scheme of a rational double point singularity.

Let \((S, C)\) be a local-geometric pair corresponding to a rational double point. As is well-known, such objects \( S \) are classified (up to analytic isomorphism) by A-D-E Dynkin diagrams. Let \( \tilde{S} \) be the minimal resolution of \( S \), and let \( \tilde{C} \subset \tilde{S} \) be the strict transform of \( C \). Let \( E_1, \ldots, E_n \subset \tilde{S} \) be the exceptional curves, numbered as in ([18] p. 167). Then \( \tilde{C} \) meets a unique exceptional curve \( E_k \), and we have \( \tilde{C} \cdot E_k = 1 \). Moreover, there are some restrictions on \( k \), depending on \( S \). (See [18] 2.2.)

In this way, we are able to define certain local-geometric pairs \( A_{n,k} \), \( D_{n,k} \), and \( E_{n,k} \). In fact, one can show [20] that these pairs are well-defined, up to analytic isomorphism. We have:

**Theorem 4.1** Let \((S, C)\) be a local-geometric pair, where \( S \) is the local scheme of a rational double point singularity. Then \((S, C)\) is analytically isomorphic to a unique member of the following list of local pairs:

- \( A_{n,k} \) (for some positive integers \( n, k \) with \( k \leq (n + 1)/2 \));
- \( D_{n,1} \) (for some integer \( n \geq 4 \));
- \( D_{n,n} \) (for some integer \( n \geq 5 \));
- \( E_{6,1} \);
- \( E_{7,1} \).

**Remark 4.2** Equations for these pairs may be found in the proof of (5.2).

5 Invariants of rational double point configurations

In this section, we assume that \( k \) has characteristic zero. We will calculate the type of \((S, C)\) in the case where \( S \) is the local scheme of a rational double point singularity. This depends not only on \( S \), but also on
C. Note that if $S$ is the local scheme of any rational singularity, and $S$ embeds in a nonsingular three-fold, then $S$ “is” a rational double point.

For each pair of positive integers $(n, k)$ with $k \leq n$, we define a sequence $\phi(n, k)$ of integers, via the following recursive definition:

$$
\phi(n, k) = \begin{cases} 
\phi(n, n - k + 1), & \text{if } k > \frac{n+1}{2}; \\
(k), & \text{if } k = \frac{n+1}{2}; \\
(k, \phi(n - k, k)), & \text{if } k < \frac{n+1}{2}.
\end{cases}
$$

**Examples.**

1. $\phi(n, 1) = (1^{[n]})$ for all $n \geq 1$;
2. $\phi(rk, k) = (k^{[r-1]}, 1^{[k]})$ for all $k \geq 1$, $r \geq 1$ (generalizing 1);
3. $\phi(rk - 1, k) = (k^{[r-1]})$ for all $r \geq 2$, $k \geq 1$ (also generalizing 1);
4. $\phi(10, 4) = (4, 3, 1^{[3]})$.

Let $a, b \in \mathbb{N}$. For each integer $n \geq 0$, we define the $n^{th}$ iterated remainder on division of $a$ by $b$, denoted $\text{rem}_n(a, b)$. Let $\text{rem}_0(a, b) = b$, and let $\text{rem}_1(a, b)$ be the usual remainder. For $n \geq 2$, define:

$$
\text{rem}_n(a, b) = \begin{cases} 
\text{rem}_1(\text{rem}_{n-2}(a, b), \text{rem}_{n-1}(a, b)), & \text{if } \text{rem}_{n-1}(a, b) \neq 0; \\
0, & \text{if } \text{rem}_{n-1}(a, b) = 0.
\end{cases}
$$

Let $a, b \in \mathbb{N}$. For each integer $n \geq 1$, we define the $n^{th}$ iterated quotient of $a$ by $b$, denoted $\text{div}_n(a, b)$. Let $\text{div}_1(a, b) = \lfloor a/b \rfloor$. For $n \geq 2$, define:

$$
\text{div}_n(a, b) = \begin{cases} 
\text{div}_1(\text{rem}_{n-2}(a, b), \text{rem}_{n-1}(a, b)), & \text{if } \text{rem}_{n-1}(a, b) \neq 0; \\
0, & \text{if } \text{rem}_{n-1}(a, b) = 0.
\end{cases}
$$

**Proposition 5.1** Fix $k, n \in \mathbb{N}$ with $k \leq (n + 1)/2$. Let $t$ be the largest integer such that $\text{rem}_t(n - k + 1, k) \neq 0$. Let $r_i = \text{rem}_i(n - k + 1, k)$, $d_i = \text{div}_i(n - k + 1, k)$, for various $i$. Then:

$$
\phi(n, k) = (r_0^{[d_1]}, r_1^{[d_2]}, \ldots, r_t^{[d_{t+1}]}).
$$
Sketch. Define \( r_{-1} = n - k + 1 \). One shows that for all \( p \geq 0 \),

\[
\phi(r_{p-1} + r_p - 1, r_p) = \begin{cases} 
(r_p^{[d_p+1]}), & \text{if } r_{p+1} \neq 0; \\
(r_p^{[d_p+1]}), & \text{if } r_{p+1} = 0.
\end{cases}
\]

The result then follows by induction.

Proposition 5.2 The type of \( A_{n,k} \) is \( \phi(n,k) \). The type of \( D_{n,1} \) is \( (2) \). We have:

\[
\text{type}(D_{n,n}) = \begin{cases} 
\frac{n}{2}, & \text{if } n \text{ is even}; \\
\left(\frac{n-1}{2}, 1^{[n-1]}\right), & \text{if } n \text{ is odd}.
\end{cases}
\]

The type of \( E_{6,1} \) is \( (2,2) \). The type of \( E_{7,1} \) is \( (3) \).

Proof. We let \((S,C)\) correspond to the given pair. The comments in the first paragraph of the proof of (3.1) apply equally well here. We make use of the explicit resolutions of rational double points given in the appendix to [25].

First we consider the \( A_{n,k} \) case. (We allow \( 1 \leq k \leq n \).) Then \( S \) is given by \( xy - z^{n+1} = 0 \), and \( C \) is given parametrically by \( x = u^k, y = u^{n-k+1}, z = u \). In terms of the notation used in [25], this may be seen as the image of \( V(u_k = 1) \subset W_k \). After making the change of variable \( x \mapsto x + z^k \) and \( y \mapsto y + z^{n-k+1} \), we find that \( S \) is given by:

\[
xy + yz^k + xz^{n-k+1} = 0, \quad (*)
\]

and that \( C \) is given by \( x = y = 0 \). From now on, we assume that \( k \leq \frac{n+1}{2} \).

Blow-up along \( C \), formally substituting \( yx \) for \( y \). Then \( S_1 \) is given by:

\[
xy + yz^k + z^{n-k+1} = 0.
\]

Intersecting with the exceptional divisor, as in (2.1), corresponds to setting \( x = 0 \). We obtain \( z^k(y + z^{n-2k+1}) = 0 \). This tells us that \( p_1(S,C) = k \) and that \( C_1 \) is given by \( x = 0 \) and \( y + z^{n-2k+1} = 0 \). After making the change of variable \( y \mapsto y - z^{n-2k+1} \), and hence \( y \mapsto -y \), we obtain the equation:

\[
xy + yz^k + xz^{n-2k+1} = 0 \quad (**)
\]

for \( S_1 \), and the equation \( x = y = 0 \) for \( C_1 \). If \( n - 2k + 1 = 0 \), then \( S_1 \) is smooth along \( C_1 \), and we are done. Otherwise, compare (*) with (**), to complete the \( A_{n,k} \) case.
Now we deal with the case $D_{n,1}$. In terms of the notation used in [25], $C$ is the image of $V(v_0 = 0) \subset W_0$. Following [25], we would have two cases ($n$ even, $n$ odd), but in fact these two cases are identical in this situation, after interchanging variables ($x \leftrightarrow y$). We find that $S$ is given by:

$$x^2z + y^2 - z^{n-1} = 0,$$

and that $C$ is given by $y = z = 0$. Blow up along $C$, formally substituting $zy$ for $z$. Then $S_1$ is given by:

$$x^2z + y - y^{n-2}z^{n-1} = 0.$$

Setting $y = 0$, we obtain $x^2z = 0$. This tells us that $p_1(S,C) = 2$ and that $C_1$ is given by $y = z = 0$. An easy calculation shows that $S_1$ is smooth. The result for $D_{n,1}$ follows.

Now we deal with the case $D_{n,n}$. In terms of the notation used in [25], $C$ is the image of $V(u_n = 0) \subset W_n$. We may take the same equation for $S$ as we did in the case $D_{n,1}$. There are two cases:

Case I: $n$ is even. Then $C$ is given parametrically by $x = u^{(n-2)/2}$, $y = 0$, $z = u$. After making the change of variable $x \mapsto x + z^{(n-2)/2}$, we find that $S$ is given by:

$$x^2z + y^2 + 2xz^{n/2} = 0,$$

and that $C$ is given by $x = y = 0$. Blow up along $C$, formally substituting $xy$ for $x$. Then $S_1$ is given by:

$$x^2yz + y + 2xz^{n/2} = 0.$$

Setting $y = 0$, we obtain $xz^{n/2} = 0$. This tells us that $p_1(S,C) = n/2$ and that $C_1$ is given by $x = y = 0$. On checks that $S_1$ is smooth.

Case II: $n$ is odd. Then $C$ is given parametrically by $x = 0$, $y = u^{(n-1)/2}$, $z = u$. (In this case, $x$ and $y$ are interchanged from the notation in [25].) After making the change of variable $y \mapsto y + z^{(n-1)/2}$, we find that $S$ is given by:

$$x^2z + y^2 + 2yz^{(n-1)/2} = 0,$$

and that $C$ is given by $x = y = 0$. Blow up along $C$, formally substituting $yx$ for $y$. Then $S_1$ is given by:

$$xz + xy^2 + 2yz^{(n-1)/2} = 0.$$

(***)

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Setting $x = 0$, we obtain $yz^{(n-1)/2} = 0$. This tells us that:

$$p_1(S, C) = (n - 1)/2$$

and that $C_1$ is given by $x = y = 0$. Now blow-up along $C_1$, formally substituting $xy$ for $x$. Then the blow-up $S_2$ is given by:

$$xz + xy^2 + 2z^{(n-1)/2} = 0.$$ Setting $y = 0$, we obtain $z(x+2z^{(n-1)/2-1}) = 0$. This tells us that $p_2(S, C) = 1$ and that $C_2$ is given by $y = 0$ and $x + 2z^{(n-1)/2-1} = 0$. After making the change of variable $x \mapsto x - 2z^{(n-1)/2-1}$, we find that $S_2$ is given by:

$$xz + xy^2 - 2y^2z^{(n-1)/2-1} = 0,$$

and that $C_2$ is given by $x = y = 0$.

Now blow up along $C_2$, formally substituting $xy$ for $x$. Then the blown up surface $S_3$ is given by:

$$xz + xy^2 - 2y^2z^{(n-1)/2-1} = 0,$$

$C_3$ is given by $x = y = 0$, and $p_3(S, C) = 1$. Replacing $y$ by $-y$, we may assume that $S_3$ is given by:

$$xz + xy^2 + 2yz^{(n-1)/2-1} = 0.$$ This looks like (***) except that $n$ is now replaced by $n - 2$. Note that if $n = 1$, then (*** is smooth. A little thought shows that the asserted type of $D_{n,n}$ is correct. A posteriori, we see that $(S_1, C_1) = A_{n-1,1}$. A direct proof of this assertion would of course simplify the proof.

For both $E_{6,1}$ and $E_{7,1}$, we may choose any smooth curve for $C$.

For $E_{6,1}$, $S$ is given by $x^2 - y^3 - z^4 = 0$, and $C$ is given by $y = 0$, $x + z^2 = 0$. After making the change of variable $x \mapsto x - z^2$, we obtain the new equation $x^2 - 2xz^2 - y^3 = 0$ for $S$. Then $C$ is given by $x = y = 0$. Blow up along $C$, substituting $xy$ for $x$. The equation for $S_1$ is:

$$x^2y - 2xz^2 - y^2 = 0.$$ Setting $y = 0$, we obtain $xz^2 = 0$. Hence $p_1(S, C) = 2$, and $C_1$ is given by $x = y = 0$. Blow up $S_1$ along $C_1$, substituting $xy$ for $x$. The equation for
\[ S_2 \text{ is } x^2y^2 - 2xz^2 - y = 0. \text{ Substituting } y = 0, \text{ we obtain } xz^2 = 0. \text{ Hence } p_2(S, C) = 2. \text{ One checks that } S_2 \text{ is smooth, so } p_k(S, C) = 0 \text{ for all } k > 2. \]

For \( E_{7,1} \), \( S \) is given by \( x^2 + y^3 - yz^3 = 0 \), and \( C \) is given by \( x = y = 0 \). Blow up along \( C \), substituting \( y x \) for \( y \). The equation for \( S_1 \) is \( x + x^2y^3 - yz^3 = 0 \). Setting \( x = 0 \), we obtain \( yz^3 = 0 \). Hence \( p_1(S, C) = 3 \). As \( S_1 \) is smooth, we see that the type of \( E_{7,1} \) is as claimed.

**Warning!** Amongst the rational double point local-geometric pairs, those of the kind \((S, C) = D_{n,n}\) with \( n \) odd \((n \geq 5)\) are highly atypical. The following phenomena happen only for these special pairs:

1. \( \Sigma(S, C) < \sum_{i=1}^{\infty} p_i(S, C) \);
2. \( p_r(S, C) \neq 0 \) for some \( r > \text{order}(S, C) \): see (5.6).

The calculation in the proposition allows one to compute not just the type of a rational double point, but also the precise sequence of (analytic equivalence classes of) local surface-curve pairs which arise under successive blowups:

- \( \text{blowup}(A_{n,k}) \left\{ \begin{array}{ll} \text{is smooth,} & \text{if } k = \frac{n+1}{2}; \\ A_{n-k,n-2k+1}, & \text{if } \frac{n-k+1}{2} < k < \frac{n+1}{2}; \\ A_{n-k,k}, & \text{if } k \leq \frac{n-k+1}{2}; \end{array} \right. \)
- \( \text{blowup}(D_{n,1}) \) is smooth;
- \( \text{blowup}(D_{n,n}) \) is smooth, if \( n \) is even;
- \( \text{blowup}(D_{n,n}) = A_{n-1,1}, \) if \( n \) is odd;
- \( \text{blowup}(E_{6,1}) = A_{3,2}; \)
- \( \text{blowup}(E_{7,1}) \) is smooth.

We now calculate \( \text{order}(S, C) \), where \( S \) is the local scheme of a rational double point.

**Proposition 5.3** The order of \( A_{n,k} \) is the order of \( \overline{k} \) in \( \mathbb{Z}/(n+1)\mathbb{Z} \). The order of \( D_{n,1} \) is 2. The order of \( D_{n,n} \) is 2 if \( n \) is even, and it is 4 if \( n \) is odd. The order of \( E_{6,1} \) is 3. The order of \( E_{7,1} \) is 2.
Proof. Let \((S, C)\) correspond to the given pair. Some of the orders \((E_{6,1}, E_{7,1}, D_{n,1} \text{ (n even)}\) and \(D_{n,n} \text{ (n even)}\)) can be computed immediately if one knows the abstract group \(\text{Cl}(S)\). A list of these groups may be found in ([22] p. 258). We do not use this approach.

Let \(\tilde{S}\) be the minimal resolution of \(S\). Let \(E_1, \ldots, E_n \subset \tilde{S}\) be the exceptional curves. Let \(\tilde{C} \subset \tilde{S}\) denote the strict transform of \(C\). There is a unique \(\mathbb{Q}\)-divisor \(E = a_1E_1 + \cdots + a_nE_n\) such that \(\tilde{C} \cdot E_i = -E \cdot E_i\) for all \(i\). (See [26].) The total transform \(\tilde{C} \subset \tilde{S}\) of \(C\) is integral (i.e. \(E\) is integral, i.e. \(a_1, \ldots, a_n \in \mathbb{Z}\)) if and only if “\(C\) is locally analytically equivalent to zero”.

Since \(S\) is a rational double point, this is equivalent to \([C] = 0\) in \(\text{Cl}(S)\). As this discussion applies not just to \(C\), but also to positive integer multiples of \(C\), we see that the order of \((S, C)\) is the least positive integer \(N\) such that

\[
N(a_1, \ldots, a_n) \in \mathbb{Z}^n.
\]

Let \(M\) be the inverse of the self-intersection matrix of the \(E_i\). Then \((a_1, \ldots, a_n)\) is the \(k\)th column of \(M\). This may be computed from an explicit formula for \(M\), which one may find in ([18] p. 169).

In case \((S, C) = A_{n,k}\), one finds that:

\[
a_i = \begin{cases} \\
-k(n-i+1)/(n+1), & \text{if } i \geq k; \\
ki/(n+1) - i, & \text{if } i \leq k.
\end{cases}
\]

From this we calculate that \(a_1 = k/(n+1) - 1\). The proof for \(A_{n,k}\) follows.

In case \((S, C) = D_{n,1}\), one finds that:

\[
a_i = \begin{cases} \\
-1, & \text{if } i \leq n-2; \\
-1/2, & \text{if } n-1 \leq i \leq n.
\end{cases}
\]

Hence order\((D_{n,1}) = 2\).

In case \((S, C) = D_{n,n}\), one finds that:

\[
a_i = \begin{cases} \\
-i/2, & \text{if } i \leq n-2; \\
-(n-2)/4, & \text{if } i = n-1; \\
n/4, & \text{if } i = n.
\end{cases}
\]

Hence order\((D_{n,n})\) is as claimed.

We now deal with the two exceptional cases. The inverses of the self-intersection matrices do not appear in [18], and we omit them here for lack of space. In case \((S, C) = E_{6,1}\), one finds that:

\[
(a_1, \ldots, a_n) = (-\frac{4}{3}, -\frac{5}{3}, -2, -1, -\frac{4}{3}, -\frac{2}{3}),
\]

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and in case \((S,C) = E_{7,1}\), one finds that:

\[
(a_1, \ldots, a_n) = \left(-\frac{3}{2}, -2, -\frac{5}{2}, -3, -\frac{3}{2}, -2, -1\right).
\]

It is interesting to note that for \(D_{n,n}\) (\(n\) odd), one has \(p_r(S,C) \neq 0\) for some \(r > \text{order}(S,C)\). This does not occur for the other rational double point pairs, as we shall see in \((5.6)\).

**Lemma 5.4** Let \(k\) and \(N\) be positive integers, with \(k < N\). Assume that \(k \nmid N\). Then:

\[
\left\lfloor \frac{N}{k} \right\rfloor \leq \frac{(N-k)}{\gcd(k,N)}.
\]

**Proof.** First suppose that \(k > N/2\). Then \(\left\lfloor \frac{N}{k} \right\rfloor = 1\), so we must show that \(\gcd(k,N) \leq N - k\). Indeed, if \(x|k\) and \(x|N\), then \(x|(N-k)\), so this is clear.

Hence we may assume that \(k \leq N/2\). Since \(k \nmid N\), \(\gcd(k,N) \leq k/2\). Therefore it suffices to show that \((k/2)(N/k) \leq N - k\). This follows from \(k \leq N/2\).

**Corollary 5.5** Let \(k\) and \(N\) be positive integers, with \(k \leq N\). Let \(t\) be the smallest positive integer such that \(\text{rem}_t(N,k) = 0\). Let \(d_i = \text{div}_i(N,k)\), for \(i = 1, \ldots, t\). Then:

\[
d_1 + \cdots + d_t \leq \frac{N}{\gcd(k,N)}.
\]

**Proof.** The case \(k = N\) is clear, so we may assume that \(k < N\). If \(t = 1\), the result is clear. Let \(r_1 = \text{rem}_1(N,k)\). We may assume that \(r_1 \neq 0\). By induction on \(t\), we may assume that:

\[
d_2 + \cdots + d_t \leq k/\gcd(r_1,k).
\]

Therefore it suffices to show that:

\[
d_1 + \frac{k}{\gcd(r_1,k)} \leq \frac{N}{\gcd(k,N)}.
\]

One sees that \(\gcd(r_1,k) = \gcd(k,N)\). Therefore it suffices to show that \(d_1 \leq (N-k)/\gcd(k,N)\). This follows from \((5.4)\).
Proposition 5.6 Let \((S, C)\) be a local-geometric pair corresponding to a rational double point. Assume that \((S, C) \neq D_{n,n}\) for any odd integer \(n \geq 5\). Then \(p_r(S, C) = 0\) for all \(r \geq \text{order}(S, C)\).

Proof. We utilize (5.2) and (5.3). The only nontrivial case is \((S, C) = A_{n,k}\). We may assume that \(k \leq (n + 1)/2\). For any \(a, b \in \mathbb{N}\), let \(o(a, b)\) denote the order of \(\bar{a}\) in \(\mathbb{Z}/b\mathbb{Z}\). In the notation of (5.4), we must show that:

\[ d_1 + \cdots + d_{t+1} < o(k, n + 1). \]

Translating to the notation of (5.3) \((N = n + 1)\), both \(d_1\) and \(t\) change by 1. The statement we need is:

\[ (d_1 - 1) + d_2 + \cdots + d_t < o(k, N). \]

Since \(o(k, N) = N/\gcd(k, N)\), this does follow from (5.3).

The content of the following proposition may be found in ([18] proof of 2.3, pp. 169-170).

Proposition 5.7 We have:

\[
\begin{align*}
\Delta(A_{n,k}) &= k(n - k + 1)/(n + 1) \\
\Delta(D_{n,1}) &= 1 \\
\Delta(D_{n,n}) &= n/4 \\
\Delta(E_{6,1}) &= 4/3 \\
\Delta(E_{7,1}) &= 3/2.
\end{align*}
\]

6 Technical lemmas on rational double points

In this section we assume that \(k\) has characteristic zero. We prove various technical relationships between the invariants of rational double point local-geometric pairs. We use these results in part III. The result (6.3) appears to be of intrinsic interest.

Proposition 6.1 Let \((S, C)\) be a local-geometric pair corresponding to a rational double point. Write:

\[
\text{type}(S, C) = (n_1^{[k_1]}, \ldots, n_r^{[k_r]})
\]

with \(n_1 > \cdots > n_r \geq 1\) and \(k_i \geq 1\) for each \(i\). Assume that \(r > 1\). Then \(k_r > 1\) and \(n_r|n_{r-1}\).
Proof. We use (5.2). The proposition is clear if \((S, C)\) is of species \(D\) or \(E\). Therefore we may assume that \((S, C)\) is of species \(A\). In the notation of (5.1), we may write:

\[
\text{type}(S, C) = (r_0^{[d_1]}, \ldots, r_t^{[d_{t+1}]}).
\]

Since \(r_{t+1} = 0\), we have \(r_t = r_t d_{t+1}\). Hence \(r_t | r_{t-1}\). Hence \(n_r | n_{r-1}\). Since \(r_{t-1} > r_t, d_{t+1} > 1\). Hence \(k_r > 1\).

Proposition 6.2 If \(\text{type}(A_{n, k}) = \text{type}(A_{n', k'})\), where \(k \leq (n + 1)/2\) and \(k' \leq (n' + 1)/2\), then \(n = n'\) and \(k = k'\).

Proof. We use (5.3). Write \(\text{type}(A_{n, k}) = (r_0^{[d_1]}, \ldots, r_t^{[d_{t+1}]}),\) as in (5.1). Then \(n = n' = r_0(d_1 + 1) + r_1 - 1\), and \(k = k' = r_0\).

Lemma 6.3 Fix positive integers \(k\) and \(N\) with \(k \leq N/2\). Let \(t\) be the smallest positive integer such that \(\text{rem}_t(N, k) = 0\). Let \(r_i = \text{rem}_i(N, k), d_i = \text{div}_i(N, k),\) for various \(i\). Then:

(i) If \(t = 1\), then \(r_0 d_1^{-1} = k^2/N\).

(ii) If \(t = 2\), then \((r_0 - r_1)d_1^{-1} + r_1(d_1 + d_2)^{-1} \leq k^2/N\).

(iii) If \(t \geq 3\), then:

\[
(r_0 - r_1)d_1^{-1} + (r_1 - r_2)(d_1 + d_2)^{-1} + r_2(d_1 + d_2 + 1)^{-1} \leq k^2/N.
\]

Proof. First suppose that \(t = 1\). Then \(N = d_1 k\). Hence \(k^2/N = k/d_1 = r_0 d_1^{-1}\). This proves (i).

Now suppose that \(t = 2\). Then \(N = r_0 d_1 + r_1\) and \(r_0 = r_1 d_2\). We must show that:

\[
\frac{r_0 - r_1}{d_1} + \frac{r_1}{d_1 + d_2} \leq \frac{r_0^2}{r_0 d_1 + r_1}.
\]

Substitute \(r_0 = r_1 d_2\), and cancel out \(r_1\). We must show:

\[
\frac{d_2 - 1}{d_1} + \frac{1}{d_1 + d_2} \leq \frac{d_1^2}{d_1 d_2 + 1}.
\]

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Eliminating denominators, we find that we must show:

\[(d_1 - 1)(d_2 - 1) \geq 0,\]

which is certainly true.

Finally, suppose that \( t \geq 3. \) Then \( N = r_0d_1 + r_1 \) and \( r_0 = r_1d_2 + r_2. \) We must show that:

\[
\frac{r_0 - r_1}{d_1} + \frac{r_1 - r_2}{d_1 + d_2} + \frac{r_2}{d_1 + d_2 + 1} \leq \frac{r_0^2}{r_0d_1 + r_1}.
\]

Substitute \( r_0 = r_1d_2 + r_2. \) We must show that:

\[
\frac{r_1d_2 + r_2 - r_1}{d_1} + \frac{r_1 - r_2}{d_1 + d_2} + \frac{r_2}{d_1 + d_2 + 1} \leq \frac{(r_1d_2 + r_2)^2}{r_1d_1d_2 + r_2d_1 + r_1}.
\]

Now cancel denominators. (This is best done with the aid of a computer.) We must show:

\[
d_2r_1^2 - d_1^2d_2r_1^2 - d_1d_2^2r_1^2 + d_1^2d_2^2r_1^2 - d_2^3r_1^2 + d_1d_2^3r_1^2 - d_2^2r_1r_2 - d_2r_1r_2
\]

\[
- d_1d_2r_1r_2 + 2d_1^2d_2r_1r_2 - d_2^2r_1r_2 + d_1d_2d_2r_1r_2 + d_1^2r_2 \geq 0.
\]

Equivalently, we must show that:

\[
r_1^2d_2^2[2(d_1 - 1) + d_1(d_1d_2 - d_1 - d_2) + 1]
\]

\[
+ r_1r_2[d_1^2(d_2 - 1) + d_2^2(d_1 - 1) + d_2^2(d_1^2 - d_1 - 1)] + d_2^2r_2 \geq 0.
\]

If \( d_1 \geq 2 \) and \( d_2 \geq 2, \) this is clear. Since \( k \leq N/2, \) we have \( d_1 \geq 2. \) Suppose that \( d_2 = 1. \) Then the needed inequality simplifies to:

\[
r_1r_2(d_1^2 - 2) + d_2^2r_2 \geq 0,
\]

which is true.

From (6.3), we obtain the following weaker statement, which we use in (6.2):

**Corollary 6.4** Fix positive integers \( k \) and \( N, \) with \( k \leq N/2. \) Let \( t \) be the smallest positive integer such that \( \text{rem}_t(N,k) = 0. \) Let \( r_i = \text{rem}_i(N,k), \)

\( d_i = \text{div}_i(N,k), \) for various \( i. \) Then:

\[
(r_0 - r_1)d_1^{-1} + \cdots + (r_{t-1} - r_t)(d_1 + \cdots + d_t)^{-1} \leq k^2/N.
\]
Warning! When we use (6.4), the symbol $d_1$ will appear to have two different values, differing by 1: in the application (6.5), $d_1$ will be smaller by 1.

Proposition 6.5 Let $(S, C)$ correspond to a rational double point singularity. Let $p_k = p_k(S, C)$, for each $k \in \mathbb{N}$. Then:

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} p_k \geq \Delta(S, C).$$

Proof. We use (5.7) and (5.2). If $(S, C)$ is not of species $A$, then the proposition is proved by the following table:

| singularity | $\sum_{k=1}^{\infty} \frac{1}{k(k+1)} p_k$ | $\Delta(S, C)$ |
|-------------|---------------------------------|----------------|
| $D_{n,1}$   | 1                               | 1              |
| $D_{n,n}$ (n even) | $n/4$ | $n/4$ |
| $D_{n,n}$ (n odd) | $n/4 + \sum_{k=2}^{n} \frac{1}{k(k+1)}$ | $n/4$ |
| $E_{6,1}$   | 4/3                             | 4/3            |
| $E_{7,1}$   | 3/2                             | 3/2            |

Suppose that $(S, C) = A_{n,k}$ for some $n, k$. We may assume that $k \leq (n + 1)/2$. We must show that:

$$\sum_{j=1}^{\infty} \frac{1}{j(j+1)} \phi(n, k)_j \geq k(n - k + 1)/(n + 1). \quad (*)$$

Let $t$ be the largest integer such that $\text{rem}_i(n - k + 1, k) \neq 0$. For $i = 0, \ldots, t$, let $r_i = \text{rem}_i(n - k + 1, k)$, $d_{i+1} = \text{div}_i(n - k + 1, k)$. By (5.1), we see that (*) is equivalent to:

$$r_0 \sum_{j=1}^{d_1} \frac{1}{j(j+1)} + r_1 \sum_{j=d_1+1}^{d_1+d_2} \frac{1}{j(j+1)} + \cdots + r_t \sum_{j=d_1+\cdots+d_t+1}^{d_1+\cdots+d_{t+1}} \frac{1}{j(j+1)} \geq \frac{k(n - k + 1)}{n + 1}.$$
Note that for any $a, b \in \mathbb{N}$ with $a \leq b$,
\[
\sum_{j=a+1}^{b} \frac{1}{j(j + 1)} = \frac{b}{b + 1} - \frac{a}{a + 1}.
\]
Hence (*) is equivalent to:
\[
(r_0 - r_1) \left( \frac{d_1}{d_1 + 1} \right) + \cdots + (r_{t-1} - r_t) \left( \frac{d_1 + \cdots + d_t}{d_1 + \cdots + d_t + 1} \right) + r_t \left( \frac{d_1 + \cdots + d_{t+1}}{d_1 + \cdots + d_{t+1} + 1} \right) \geq k(n - k + 1)/(n + 1).
\]
This is equivalent to:
\[
r_0 - (r_0 - r_1)(d_1 + 1)^{-1} - \cdots - (r_{t-1} - r_t)(d_1 + \cdots + d_t + 1)^{-1} - r_t(d_1 + \cdots + d_{t+1} + 1)^{-1} \geq k(n - k + 1)/(n + 1).
\]
Since $r_0 = k$, this is equivalent to:
\[
(r_0 - r_1)(d_1 + 1)^{-1} + \cdots + (r_{t-1} - r_t)(d_1 + \cdots + d_t + 1)^{-1}
\]
\[
+ r_t(d_1 + \cdots + d_{t+1} + 1)^{-1} \leq k^2/(n + 1).
\]
Let $N = n + 1$. Then this follows from (5.4), and thence completes the proof.

**Definition.** Let $(S, C)$ be a local-geometric pair corresponding to a rational double point. Then the deficiency of $(S, C)$ is:
\[
def(S, C) = \Sigma(S, C) - \sum_{i=1}^{\infty} p_i(S, C).
\]
One always has $\def(S, C) \geq 0$, except for $D_{n,n}$, with $n$ odd, $n \geq 5$.

**Part II**

**Iterated curve blowups**

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7 Intersection ring of a blow up

In this section we describe (without proof) the intersection ring of the blow-up of a nonsingular variety along a nonsingular subvariety, following the statements given in ([7] 6.7, 8.3.9). There are two differences between the assertions we make and the assertions made in [7]. Firstly, we work with cycles modulo algebraic equivalence, rather than modulo rational equivalence. Secondly, we have adjusted the signs to reflect our convention regarding projective space bundles.

Let $X$ be a nonsingular closed subvariety of a nonsingular variety $Y$. Let $d = \text{codim}(X, Y)$, and assume that $d \geq 2$. Let $N$ be the normal bundle of $X$ in $Y$.

Let $\tilde{Y}$ be the blow-up of $Y$ along $X$. The exceptional divisor is isomorphic to $\mathbb{P}N^*$. We use the following diagram to fix notation:

\[
\begin{array}{ccc}
\mathbb{P}N^* & \xrightarrow{j} & \tilde{Y} \\
\downarrow{g} & & \downarrow{f} \\
X & \xrightarrow{i} & Y
\end{array}
\]

Let $F = \text{Ker}[g^*(N^*) \xrightarrow{\text{can}} \mathcal{O}_{\mathbb{P}N^*}(1)]$. For each $k$, there is a canonically split exact sequence:

\[
0 \longrightarrow A^{k-d}(X) \xrightarrow{\delta} A^{k-1}(\mathbb{P}N^*) \oplus A^k(Y) \xrightarrow{\beta} A^k(\tilde{Y}) \longrightarrow 0
\]

of cycle groups modulo algebraic equivalence. The maps are given by:

\[
\delta(x) = (c_{d-1}(F) \cdot g^*(x), i_*(x))
\]

and

\[
\beta(\tilde{x}, y) = j_*(\tilde{x}) + f^*(y).
\]

This describes $A^*(\tilde{Y})$ as an abelian group. The ring structure is described by the following rules:

\[
(f^*y) \cdot (f^*y') = f^*(y \cdot y')
\]

\[
(j_*\tilde{x}) \cdot (j_*\tilde{x}') = -j_*(c_1(\mathcal{O}_{\mathbb{P}N^*}(1)) \cdot \tilde{x} \cdot \tilde{x}')
\]

\[
(f^*y) \cdot (j_*\tilde{x}) = j_*((g^*i^*y) \cdot \tilde{x}).
\]
8 The intersection ring of an iterated curve blow-up

The result of this section is:

**Theorem 8.1** Let $Y_0 = \mathbb{P}^3$. Let $C_0 \subset Y_0$ be a nonsingular curve of degree $d$ and genus $g$. Let $Y_1$ be the blow-up of $Y_0$ along $C_0$. Choose a smooth curve $C_1$ which lies on the exceptional divisor $E_1 \subset Y_1$ and which meets each ruling on $E_1$ exactly once. Let $Y_2$ be the blow-up of $Y_1$ along $C_1$. Iterate this process: $Y_{k+1}$ is obtained by blowing up a smooth curve $C_k \subset E_k \subset Y_k$. We assume that $C_k$ meets each ruling on $E_k$ exactly once and that for all $k \geq 2$, $C_k \neq E_k \cap E_{k-1}$, where $E_{k-1} \subset Y_k$ denotes the strict transform of $E_{k-1}$. Let $H \subset Y_0$ be a plane. Let $h = [H] \in A^1(Y_0)$. Let $e_k = [E_k] \in A^1(Y_k)$. Let $r_k \in A^1(E_k)$ denote the class of a ruling, which we identify with its image in $A^2(Y_k)$. Identify $h$, $e_k$ and $r_k$ with their images in the intersection ring $A^*(Y_n)$ of the $n$th iterated blow-up $Y_n$. Then $A^k(Y_n)$ has as a basis:

$$[Y_n] \ (k = 0); \ h, e_1, \ldots, e_n \ (k = 1); \ h^2, r_1, \ldots, r_n \ (k = 2); \ 1 \ (k = 3).$$

This information, together with the following multiplication rules, completely describe $A^*(Y_n)$ as a graded ring: $h^3 = 1$, $h \cdot r_k = 0$, $h^2 \cdot e_k = 0$, $e_i \cdot r_j = -\delta_{i,j}$, $h \cdot e_k = d_r k$, $e_i \cdot e_j = -\beta_{i,j}$ (if $i < j$),

$$e_k^2 = -dh^2 - \alpha_{k-1}k - \sum_{i=1}^{k-1} \beta_{i}r_i,$$

where $\alpha_k$ is determined by $[C_k] = c_1O_{E_k}(1) - \alpha_k r_k$ in $A^1(E_k)$, for $k \geq 1$, $\alpha_0 = 2 - 2g - 4d$, and $\beta_k = \alpha_{k-1} - \alpha_k$, for each $k \geq 1$.

We note the following generalization and conceptual reformulation of (8.1), whose proof is omitted. It will not be used again.

**Theorem 8.2** Let $Y_0$ be a nonsingular complete three-fold. Let $C_0 \subset Y_0$ be a nonsingular curve. Let $Y_1$ be the blow-up of $Y_0$ along $C_0$. Choose a smooth curve $C_1$ which lies on the exceptional divisor $E_1 \subset Y_1$ and which meets each ruling on $E_1$ exactly once. Let $Y_2$ be the blow-up of $Y_1$ along $C_1$. Iterate this process: $Y_{k+1}$ is obtained by blowing up a smooth curve $C_k \subset E_k \subset Y_k$. We assume that $C_k$ meets each ruling on $E_k$ exactly once and that for all $k \geq 2$, 

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$C_k \neq E_k \cap E_{k-1}$, where $E_{k-1,k} \subset Y_k$ denotes the strict transform of $E_{k-1}$.
Let $e_k = [E_k] \in A^1(Y_k)$. Let $r_k \in A^1(E_k)$ denote the class of a ruling, which we identify with its image in $A^2(Y_k)$. Identify $e_k$ and $r_k$ with their images in the intersection ring $A^*(Y_n)$ of the $n^{th}$ iterated blow-up $Y_n$. Then $A^*(Y_n)$ is the graded $A^*(Y_0)$-algebra generated by $e_1, \ldots, e_n$ (degree 1) and $r_1, \ldots, r_n$ (degree 2), modulo the relations: $A^1(Y_0) \cdot r_k = 0$, $A^2(Y_0) \cdot e_k = 0$, $e_i \cdot r_j = -\delta_{i,j}$, $h \cdot e_k = (h \cdot C_0) r_k$ (for all $h \in A^1(Y_0)$), $e_i \cdot e_j = -\beta_{i,j}$ (if $i < j$),

$$e_k^2 = -[C_0] - \alpha_{k-1} r_k - \sum_{i=1}^{k-1} \beta_i r_i,$$

where $\alpha_k$ is determined by $[C_k] = c_1 O_{E_k}(1) - \alpha_k r_k$ in $A^1(E_k)$, for $k \geq 1$, $\alpha_0 = \deg(N_{C_0}^*)$, and $\beta_k = \alpha_{k-1} - \alpha_k$, for each $k \geq 1$.

The remainder of this section breaks up into two parts. First we introduce various notations and conventions which we will use in the proof and in subsequent sections. Then we prove (8.1).

There are group homomorphisms $A^*(E_k) \rightarrow A^{i+1}(Y_k)$ and injective ring homomorphisms:

$$A^*(Y_0) \rightarrow A^*(Y_1) \rightarrow \cdots \rightarrow A^*(Y_n).$$

We systematically identify various elements with their images, via these maps. Since the latter maps are ring homomorphisms, it is not necessary to distinguish between multiplication in $A^*(Y_i)$ and $A^*(Y_j)$, for any $i, j$. On the other hand, since the maps $A^i(E_k) \rightarrow A^{i+1}(Y_k)$ are not ring homomorphisms, it is necessary to distinguish between multiplication in $A^*(Y_k)$ and $A^*(E_k)$. We do this by using a dot ($\cdot$) to denote multiplication in $A^*(Y_k)$ and brackets ($\langle, \rangle$) to denote multiplication in $A^*(E_k)$. No problems are introduced by the fact that $k$ does not occur explicitly in the bracket notation.

Let $c_k = [C_k]$. This is an element of $A^2(Y_k)$, and it is an element of $A^1(E_k)$ if $k \geq 1$. Let $d_k = c_1 O_{E_k}(1)$. It is an element of $A^1(E_k)$.

For $k \leq n$, let $E_{k,n} \subset Y_n$ denote the strict transform of $E_k$. In $A^1(Y_n)$ we have $e_k = [E_{k,n}] + \cdots + [E_{n,n}]$. (This depends on our assumption that $C_k \neq E_k \cap E_{k-1}$.) In particular, the reader should observe the following insidious source of error: $e_k \neq [E_{k,n}]$. This same sort of error applies to other cycles which we shall discuss.

In this section we do not fix a particular ruling $R_k \subset E_k$. We do so in the next section. Having made such a choice, one can then discuss the strict transform $R_{k,n} \subset Y_n$ of $R_k$. 

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Let $N_k$ be the normal bundle of $C_k$ in $Y_k$. Then $E_k \cong \mathbb{P}(N_{k-1}^*)$. For each $k = 0, \ldots, n$, we let $\alpha'_k = \deg(N_k^*)$. For each $k = 1, \ldots, n$, we let $\beta'_k = \alpha'_{k-1} - \alpha_k$. (We will show that $\alpha_k = \alpha'_k$ and hence that $\beta_k = \beta'_k$.)

**Proof (of 8.1).** We make repeated use of the results of §7, without explicitly referring to them. The abelian group structure of $A^*(Y_n)$ and the assertions that $h^3 = 1$, $h \cdot r_k = 0$, $h^2 \cdot e_k = 0$, and $e_i \cdot r_j = -\delta_{i,j}$ are left to the reader.

We compute $h \cdot e_i$. Let $\mu_i = h \cdot c_i$. Note that:

$$h \cdot e_i = \mu_i - 1 r_i.$$

We show that $\mu_k$ is independent of $k$, and in fact equals $d$. First one checks that $h \cdot c_0 = d$. Now we have:

$$\mu_i = h \cdot c_i = \langle h \cdot e_i, c_i \rangle = \langle \mu_{i-1} r_i, c_i \rangle = \langle \mu_{i-1} r_i, d_i - \alpha_i r_i \rangle = \mu_{i-1}.$$

Hence $\mu_k = d$ for all $k$. Hence $h \cdot c_i = d$ and $h \cdot e_i = d r_i$.

We now work on showing that $\alpha_k = \alpha'_k$. In the process we calculate $e_i \cdot c_j$ for all $i \leq j$, a result we shall need later. We have:

$$\langle d_i, d_i \rangle = c_1(N_{i-1}^*) = \alpha'_{i-1}.$$

Further:

$$e_i \cdot c_i = -\langle d_i, c_i \rangle = -\langle d_i, d_i - \alpha_i r_i \rangle = -(\alpha'_{i-1} - \alpha_i) = -\beta'_i.$$

Using this we find:

$$e_i \cdot c_{i+1} = \langle (e_i \cdot c_i) r_{i+1}, c_{i+1} \rangle = e_i \cdot c_i.$$

Continuing in this manner, the reader may verify that for $i \leq j$, $e_i \cdot c_j = -\beta'_i$. 

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The class of the canonical divisor on $Y_i$ is given by:

$$[K_{Y_i}] = -4h + e_1 + \cdots + e_i.$$  

(This may be computed from the formula for the canonical divisor of a blowup – see [9] p. 608.)

For all $i \geq 0$, we have:

$$\alpha_i' = c_1N_i^* = -c_1 \det(N_i) = -[[K_{C_i}] - [K_{Y_i}]|_{C_i}] = -[[K_{C_i}] - [K_{Y_i}] \cdot c_i] = -[2g - 2 - (4h + e_1 + \cdots + e_i) \cdot c_i] = -[2g - 2 + 4d + \beta_1' + \cdots + \beta_i'] .$$

From this, and from the definition of the $\beta$’s and the $\alpha$’s, we conclude:

$$\alpha'_k = \alpha_k \text{ for all } k \geq 0.$$  

For $i < j$,

$$e_i \cdot e_j = (e_i \cdot c_{j-1})r_j ,$$

so we obtain the formula $e_i \cdot e_j = -\beta_i r_j$.

We proceed to calculate $e_k^2$. By the definition of the map $\delta$ given in §7, we have:

$$d_i = \alpha_{i-1}r_i + c_{i-1} \quad (i \geq 1).$$

Continuing to calculate, we find:

$$c_i = d_i - \alpha_i r_i \quad (i \geq 1)$$

$$d_i = \alpha_{i-1} r_i + d_{i-1} - \alpha_{i-1} r_{i-1} \quad (i \geq 2)$$

$$d_i - d_{i-1} = \alpha_{i-1} r_i - \alpha_{i-1} r_{i-1} \quad (i \geq 2)$$

$$d_1 = a_0 r_1 + dh^2$$

$$d_k = dh^2 + \left( \sum_{i=1}^{k-1} (\alpha_{i-1} - \alpha_i) r_i \right) + \alpha_{k-1} r_k \quad (k \geq 1)$$

$$e_k^2 = -d_k \quad (k \geq 1)$$

$$= - \left[ dh^2 + \left( \sum_{i=1}^{k-1} \beta_i r_i \right) + \alpha_{k-1} r_k \right] .$$
9 The strict transform of a ruling

The results of this section will be used in the proof of theorem II (10.3). The notations introduced in §8 remain in effect in this section.

Fix a particular ruling $R_k \subset E_k$, where $1 \leq k \leq n$. We compute the class of $R_{k,n}$ in $A^2(Y_n)$. A priori, this is a $\mathbb{Z}$-linear combination of $h^2, r_1, \ldots, r_n$, which depends on the particular choice of $R_k$.

We use the term graph to mean an undirected graph, which we shall formally view as a reflexive, symmetric relation. By an augmented graph, we shall mean a graph, together with a mapping from the set of vertices of that graph to $\mathbb{Z}$. If the augmentation map is injective, we shall refer to the graph as a labeled graph, with the obvious connotations.

Let $\Gamma$ be a labeled graph, which we suppose has a maximum vertex $m$. We define various labeled graphs, coming from $\Gamma$, with maximum vertex $m + 1$.

First we define a labeled graph $\Gamma^+$ by $\text{vertices}(\Gamma^+) = \text{vertices}(\Gamma) \cup \{m + 1\}$ and $\text{edges}(\Gamma^+) = \text{edges}(\Gamma) \cup \{\text{edge}(m, m + 1)\}$.

Now suppose that $\text{edge}(l, m) \in \Gamma$. We define a graph $\Gamma^l$ by $\text{vertices}(\Gamma^l) = \text{vertices}(\Gamma) \cup \{m + 1\}$ and

$$\text{edges}(\Gamma^l) = \text{edges}(\Gamma) \cup \{\text{edge}(l, m + 1), \text{edge}(m, m + 1)\} - \{\text{edge}(l, m)\}.$$ Intuitively, this construction may be thought of as adding a vertex $(m + 1)$ “in the middle” of the edge from $l$ to $m$.

**Definition.** A standard operation is an operation on a labeled graph of the form $\Gamma \mapsto \Gamma^+$ or $\Gamma \mapsto \Gamma^l$ for some $l$. A standard labeled graph is a labeled graph obtained from a one-vertex labeled graph by a finite sequence of standard operations.

It is not hard to see that given a standard labeled graph, one may compute the last standard operation which was performed, and thence undo that operation. It follows that:

**Proposition 9.1** Let $G$ be a standard labeled graph. Then there is a unique sequence of standard operations which gives rise to $G$.

Fix integers $k$ and $m$ with $1 \leq k \leq m \leq n$. Let $R_k \subset E_k$ be a ruling. We will show how to associate a certain standard labeled graph $\Gamma_m(R_k)$ to $R_k$, in such a way that $[R_{k,m}] \in A^2(Y_m)$ depends only on $\Gamma_m(R_k)$.
To do this, consider the set of all curves $H \subset Y_m$ which are the strict transforms of some ruling $R_l$ on $E_l$, for some $l$ with $k \leq l \leq m$. To each such $H$, we may associate an integer, namely $l$. It may be that $H \subset E_{l'}$, for some $l'$ with $l' \neq l$ and $k \leq l' \leq m$, but this does not matter to us. The set of all such curves $H$ may be viewed as the vertices of a graph $\Gamma_m(k)$: two distinct vertices are connected by an edge if and only if the corresponding two curves on $Y_m$ meet. There is an augmentation on $\Gamma_m(k)$ given by $H \mapsto l$ as above.

Define $\Gamma_m(R_k)$ to be the maximal connected subgraph of $\Gamma_m(k)$ which contains $R_{k,m}$. The augmentation on $\Gamma_m(k)$ induces an augmentation on $\Gamma_m(R_k)$. We shall prove shortly (9.4) that $\Gamma_m(R_k)$ is a labeled graph, and that in fact it is a standard labeled graph.

Lemma 9.2 If two distinct curves $H_1, H_2 \in \Gamma_m(k)$ meet, then they meet at a unique point, and they meet transversally.

Proof. What we need to show is that if $p \leq q$ are integers ($k \leq p, q \leq m$), and if $R_p \subset E_p$ and $R_q \subset E_q$ are rulings, and if $R_{p,m}$ meets $R_{q,m}$ (but $R_{p,m} \neq R_{q,m}$), then in fact $R_{p,m}$ meets $R_{q,m}$ at a unique point and they do so transversally. It suffices to show that $R_{p,q}$ meets $R_q$ in this way. We may assume that $p < q$.

Indeed if $R_{p,q}$ met $R_q$ at more than one point, or if they did not meet transversally, then the image of $R_{p,q}$ under the map $Y_q \to Y_p$ would be singular, because this map contracts $R_q$.

Lemma 9.3 No three distinct curves $H_1, H_2, H_3 \in \Gamma_m(k)$ meet at a common point.

Proof. We may reduce to showing the following: if $p < q < r$ ($k \leq p, q, r \leq m$) and $R_p \subset E_p$, $R_q \subset E_q$, and $R_r \subset E_r$ are rulings, then $R_{p,r} \cap R_{q,r} \cap R_r = \emptyset$. We proceed by contradiction: let $x \in R_{p,r} \cap R_{q,r} \cap R_r$. We may assume that $r$ is minimal with respect to this assertion.

Let $y$ be the image of $x$ under the map $Y_r \to Y_{r-1}$. Then $y \in R_{p,r-1} \cap R_{q,r-1} \cap C_{r-1}$. If $q < r - 1$, then for some ruling $R_{r-1} \subset E_{r-1}$, we have $y \in R_{p,r-1} \cap R_{q,r-1} \cap R_{r-1}$, thereby contradicting the minimality of $r$. Hence we may assume that $q = r - 1$.

To prove the lemma, it suffices to show that $T_y(R_{p,r-1}) + T_y(R_{q,r-1}) + T_y(C_{r-1}) = T_y(Y_{r-1})$. Since by (1.2) $R_{p,r-1}$ meets $R_{q,r-1}$ at a unique point,
this will imply that \( R_{p,r} \cap R_{q,r} = \emptyset \), thereby yielding a contradiction. Substituting \( q = r - 1 \), we must show:

\[
T_y(R_{p,r-1}) + T_y(R_{r-1}) + T_y(C_{r-1}) = T_y(Y_{r-1}).
\]

The curves \( R_{r-1} \) and \( C_{r-1} \) meet transversally at \( y \), tangentially spanning \( T_y(E_{r-1}) \). Therefore, to prove (*), and hence the lemma, it suffices to show that \( R_{p,r-1} \) meets \( E_{r-1} \) transversally. This may be deduced by repeated application of the following two facts, applied to integers \( t \) with \( p \leq t \leq r - 2 \):

- if \( R_{p,t} \) meets \( C_t \) transversally (on \( Y_t \)), then \( R_{p,t+1} \) meets \( E_{t+1} \) transversally (on \( Y_{t+1} \));
- if \( R_{p,t} \) meets \( E_t \) transversally (on \( Y_t \)), then \( R_{p,t} \) meets any smooth curve on \( E_t \) transversally (if at all).

\[\text{Proposition 9.4}\]

Let \( k, m \in \mathbb{Z} \), with \( 1 \leq k \leq m \leq n \). Let \( R_k \subset E_k \) be a ruling. Let \( \Gamma = \Gamma_m(R_k) \). Then \( \Gamma \) is a standard labeled graph with vertices \( [k,m] \cap \mathbb{Z} \), and provided that \( m < n \), \( \Gamma_{m+1}(R_k) \) is obtained from \( \Gamma \) by a single standard operation.

Proof. By induction, we may assume that \( \Gamma \) is a standard labeled graph with vertices \( [k,m] \cap \mathbb{Z} \). For each \( q \) between \( k \) and \( m \), let \( R_q \subset E_q \) be the ruling corresponding to the vertex \( q \in \Gamma \).

First we show (*) that if \( l \) is such that \( k \leq l < m \) and \( R_{l,m} \) meets \( C_m \), then in fact \( R_{l,m} \), \( R_m \), and \( C_m \) meet at a common point. Suppose otherwise: \( R_{l,m} \cap R_m \cap C_m = \emptyset \). We will obtain a contradiction. We may choose \( m \) to be as small as possible. There are two cases.

Case (a). We have \( l = m - 1 \). Since \( R_{m-1} \) meets \( C_m \) transversally at a single point, \( R_{m-1,m} \) meets \( E_m \) at a single point. Since \( \Gamma \) is a standard labeled graph, it is clear that \( R_{m-1,m} \) meets \( R_m \). Since \( R_{m-1,m} \) meets \( C_m \), we see that \( R_{m-1,m} \) meets \( E_m \) at two distinct points: contradiction. This proves case (a).

Case (b). We have \( l < m - 1 \). Since \( R_{l,m} \) meets \( C_m \) (and a fortiori \( R_{l,m} \) meets \( E_m \)), it follows that \( R_{l,m-1} \) meets \( C_{m-1} \). By the minimality of \( m \), \( R_{l,m-1} \cap R_{m-1} \cap C_{m-1} \neq \emptyset \). It follows that \( R_{l,m} \) and \( R_{m-1,m} \) meet a common ruling on \( E_m \). Since \( R_{m-1,m} \) meets \( R_m \), it is clear that this ruling must be \( R_m \). Hence \( R_{l,m} \) meets \( R_m \). Thus \( R_{l,m} \) meets both \( R_m \) and \( C_m \), but the
three curves do not meet at a common point. Hence $R_{l,m}$ meets two distinct rulings on $E_m$. Hence $R_{l,m-1}$ meets $C_{m-1}$ at $\geq 2$ distinct points, so $R_l$ meets $C_l$ at $\geq 2$ distinct points: contradiction. This proves case (b), and hence ($\ast$).

We now proceed with the proof of the proposition. There are two cases.

Case I. For no $l$ (with $k \leq l < m$) is it true that $R_{l,m}$, $R_m$ and $C_m$ have a point in common. We claim that $\Gamma_{m+1}(R_k) = \Gamma^+$. It suffices to show that $R_m$ is the unique curve in $\Gamma$ which meets $C_m$. This follows from ($\ast$).

Case II. For some $l$ (with $k \leq l < m$), $R_{l,m}$, $R_m$ and $C_m$ have a point (say $x$) in common. We claim that $\Gamma_{m+1}(R_k) = \Gamma^l$. To prove this, we need to prove two things:

(i) for any $q$ such that $k \leq q < m$ and $q \neq l$, $R_{q,m}$ does not meet $C_m$;

(ii) $T_x(R_{l,m}) + T_x(R_m) + T_x(C_m) = T_x(Y_m)$.

The first assertion follows immediately from ($\ast$) and from (9.3). The second assertion follows from the proof of (9.3).

Lemma 9.5 Let $G$ be a standard labeled graph, constructed from the single vertex graph $\{k\}$ by a sequence of standard operations $+, k^{[p]}, o_1, \ldots, o_r$, for some $p, r \geq 0$, such that $o_1 \neq k$. Then $G - \{k\}$ is a standard labeled graph, which can be constructed from the single vertex graph $\{k+1\}$ by the sequence of standard operations:

$$
\begin{cases}
+ [p], o_1, \ldots, o_r, & \text{if } p \geq 1; \\
+, o_2, \ldots, o_r, & \text{if } p = 0 \text{ and } r \geq 1; \\
\emptyset, & \text{if } p = r = 0.
\end{cases}
$$

The proof of this lemma is left to the reader.

Let $G$ be a standard labeled graph, with smallest vertex $k$, having at least two vertices. It is clear that there is a unique $r > k$ such that edge$(k, r)$ is in $G$. We define the order of $G$ to be $r - k$. Moreover, if $G$ has order $p$, then $G$ is constructed from the single vertex graph $\{k\}$ by a sequence of operations which begins with $+, k^{[p-1]}$, and whose next operation (if any) is not $k$.

Let $G$ be any standard labeled graph, with vertices $k, \ldots, m$. We associate a function $\mu_G : G \to \mathbb{N}$, defined by inducting on $G$: if $G$ is a single vertex graph, then $\mu_G(k) = 1$. If $G$ is any standard labeled graph, then $\mu_{G^+}(j) = \mu_G(j)$ and $\mu_{G^l}(j) = \mu_G(j)$ for all $j$ with $k \leq j \leq m$, and $\mu_{G^+(m+1)} = \mu_G(m)$, $\mu_{G^l(m+1)} = \mu_G(m) + \mu_G(l)$. The fact that $\mu_G$ is well-defined follows from (9.1).
Proposition 9.6  Let \( G \) be a standard labeled graph, with smallest vertex \( k \), having at least two vertices. Then:

\[
\mu_G = \mu_{\{k\}} + \sum_{i=1}^{\text{ord}(G)} \mu_{G-\{k,\ldots,k+i-1\}},
\]

where the functions on the right hand side are viewed as functions on \( G \), via extension by zero.

Sketch. Use (9.5). If \( p = \text{ord}(G) \), then

\[ G \leftrightarrow +, k^{[p-1]}, * \]

where * is a sequence of standard operations (possibly empty), not beginning with \( k \). The case \( p = 1 \) is left to the reader. For \( p \geq 2 \):

\[ G - \{k\} \leftrightarrow +^{[p-1]}, * \]

\[ G - \{k, k+1\} \leftrightarrow +^{[p-2]}, * \]

and so forth:

\[ G - \{k, \ldots, k + p - 2\} \leftrightarrow +, * \]

\[ G - \{k, \ldots, k + p - 1\} \leftrightarrow *' \]

where *' can be determined from (9.3). We compute \( \mu \) in a special case, namely when * is empty. Then:

\[ G \leftrightarrow (1, 1, 2, 3, \ldots, p) \]

\[ G - \{k\} \leftrightarrow (0^{[1]}, 1^{[p]}) \]

\[ \ldots \]

\[ G - \{k, \ldots, k + p - 1\} \leftrightarrow (0^{[p]}, 1^{[1]}) \]

where the sequences on the right are \((\mu(k), \ldots, \mu(k + p))\). In this case the proposition is clear. The general case is left to the reader.

Corollary 9.7  Fix an integer \( k \) with \( 1 \leq k \leq n \). Let \( R_k \subset E_k \) be a ruling. If \( k = n \) then \([R_{k,n}] = r_n\), and if \( k < n \), then there exists an integer \( l \), with \( k < l \leq n \), such that

\[
[R_{k,n}] = r_k - \sum_{i=k+1}^{l} r_i.
\]
SKETCH. For each integer $m$ with $k \leq m \leq n$, let $R_m \subset E_m$ be the ruling which enters into $\Gamma_n(R_k)$. For each integer $l$ with $k \leq l \leq n$, write:

$$\mu_{\Gamma_n(R_l)} = (b_l, \ldots, b_n).$$

By considering the scheme-theoretic inverse image of $R_l$ under the map $Y_n \to Y_l$, one can show that:

$$r_l = b_l[R_{l,n}] + \cdots + b_n[R_{n,n}].$$

The result then follows from (9.6).

**Corollary 9.8** Let $H \subset Y_n$ be a cycle which is a sum of strict transforms of rulings. Then $[H]$ is a positive $\mathbf{Z}$-linear combination of the classes:

$$r_n, (r_{n-1} - r_n), (r_{n-2} - r_{n-1} - r_n), \ldots, (r_1 - r_2 - \cdots - r_n).$$

**Corollary 9.9** Let $H \subset Y_n$ be a cycle which is a sum of strict transforms of rulings. Then there exists integers $a_1, \ldots, a_n$ such that $[H] = a_1r_1 + \cdots + a_nr_n$ and for each integer $k$ with $1 \leq k \leq n$, we have:

$$\left(\sum_{i=1}^{k-1} 2^{k-i-1}a_i\right) + a_k \geq 0.$$

**Part III**

**Application to set-theoretic complete intersections**

**10 Theorems I and II**

Let $S, T \subset \mathbf{P}^3$ be surfaces of degrees $s$ and $t$, respectively. Write $S_0 = S$, $T_0 = T$. Assume that $S \cap T$ is set-theoretically a smooth curve $C = C_0$. Let $d = \text{deg}(C)$. Then $d|st$. Let $n = st/d$. Assume that $C \not\subset \text{Sing}(S)$ and that $C \not\subset \text{Sing}(T)$. Let $Y_0 = \mathbf{P}^3$. Let $Y_1$ be the blow-up of $Y_0$ along $C_0$. 

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Let $S_1 \subset Y_1$ be the strict transform of $S$. There is a unique curve $C_1 \subset S_1$ which maps isomorphically onto $C_0$. Let $Y_2$ be the blow-up of $Y_1$ along $C_1$. Iterate this process. This puts us in the situation of (8.1). Let $p_i = p_i(S, C)$, for $i = 1, \ldots, n - 1$.

For each $k = 1, \ldots, n$, let $S_k$ and $T_k$ denote the strict transforms of $S$ and $T$ on $Y_k$. Since $C \not\subset \text{Sing}(S)$ and $C \not\subset \text{Sing}(T)$, it follows that for each $k$ with $0 \leq k \leq n$, $S_k$ meets $T_k$ along $C_k$ with multiplicity $n - k$. As consequences of this, we see that $S_n \cap T_n$ is a union of strict transforms of rulings, and that $[S_k] = s h - \sum_{i=1}^k e_i$, $[T_k] = t h - \sum_{i=1}^k e_i$.

First we derive the formula:

$$\beta_k = ds + (2 - 4d - 2g) - p_k. \quad (*)$$

We have $[S_k] \cdot e_k = c_k + p_k r_k$. (See 2.1) Combining this with $[S_k] = s h - \sum_{i=1}^k e_i$, $h \cdot e_k = d r_k$ (from 8.1), and $c_k = d_k - \alpha_k r_k$ (from p. 32), we obtain:

$$d s r_k - \sum_{i=1}^k (e_i \cdot e_k) = d_k - \alpha_k r_k + p_k r_k.$$ 

Combine this with $e_i \cdot e_k = -\beta_i r_k$ (if $i < k$) (from 8.1) and $e_k^2 = -d_k$ (from p. 32) to obtain:

$$d s r_k + (\beta_1 + \cdots + \beta_{k-1}) r_k = -\alpha_k r_k + p_k r_k.$$ 

Combine this with the formula:

$$\alpha_k = (2 - 4d - 2g) - (\beta_1 + \cdots + \beta_k) \quad \text{(**)}$$

from p. 32 to obtain (*).

Since $S_n \cap T_n$ is a union of strict transforms of rulings, it follows from (9.9) that:

$$(s h - \sum_{i=1}^n e_i)(t h - \sum_{i=1}^n e_i) = \sum_{l=1}^n a_l r_l, \quad (***)$$

for some integers $a_l$ such that for each $k$ with $1 \leq k \leq n$, we have:

$$\left(\sum_{m=1}^{k-1} 2^{k-m-1} a_m\right) + a_k \geq 0.$$ 

We proceed to analyze the consequences of this. The left hand side of (***) equals:

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\[ st h^2 - d(s + t)(\sum_{i=1}^{n} \mathbf{r}_i) - \sum_{1 \leq i < j \leq n} \beta_i \mathbf{r}_j - \sum_{1 \leq j < i \leq n} \beta_j \mathbf{r}_i \]

\[ - \sum_{k=1}^{n} \left[ d h^2 + \left( \sum_{i=1}^{k-1} \beta_i \mathbf{r}_i \right) + \alpha_{k-1} \mathbf{r}_k \right]. \]

Then for each \( m \) with \( 1 \leq m \leq n \):

\[ -a_m = d(s + t) + 2 \sum_{i=1}^{m-1} \beta_i + (n - m) \beta_m + \alpha_{m-1}. \]

Substituting \( \alpha_k = (2 - 4d - 2g) - (\beta_1 + \ldots + \beta_k) \), we obtain:

\[ -a_m = d(s + t) + \sum_{i=1}^{m-1} \beta_i + (n - m) \beta_m + (2 - 4d - 2g). \]

In the special case where \( \text{Sing}(S) \cap \text{Sing}(T) = \emptyset \), we have \( a_m = 0 \) for all \( m \), with \( 1 \leq m \leq n \). Now substitute \( \beta_i = ds + (2 - 4d - 2g) - p_i \). We obtain:

\[ \left( \sum_{i=1}^{m-1} p_i \right) + (n - m)p_m = d[n(s - 4) + t] + (2 - 2g)n. \]

This implies:

**Theorem 10.1 ("I")** Let \( C \subset \mathbb{P}^3 \) be a smooth curve of degree \( d \) and genus \( g \). Suppose that \( C = S \cap T \), where \( S \) and \( T \) are surfaces of degree \( s \) and \( t \) respectively. Assume that \( \text{Sing}(S) \cap \text{Sing}(T) = \emptyset \). Let \( n = st/d \). Let \( p_i = p_i(S,C) \), for each \( i = 1, \ldots, n - 1 \), as defined in \S 2. Then:

\[ p_1 = \cdots = p_{n-1} = \frac{1}{n-1} \left\{ d[n(s - 4) + t] + (2 - 2g)n \right\}. \]

**Example 10.2** If \( s = t = 4, d = 4, g = 0 \), we obtain \( p_1 = p_2 = p_3 = 8 \). This can occur in characteristic two, at least. Indeed, let \( (S,C) \) be as in (3.1), and let \( T \) be given by \( z^4 - xw^3 = 0 \).

We now return to the general case. For each \( k \) with \( 1 \leq k \leq n \), we have:
\[
\left( \sum_{m=1}^{k-1} 2^{k-m-1} [d(s + t) + \sum_{i=1}^{m-1} \beta_i + (n - m)\beta_m + (2 - 4d - 2g)] \right) \\
+ [d(s + t) + \sum_{i=1}^{k-1} \beta_i + (n - k)\beta_k + (2 - 4d - 2g)] \leq 0.
\]

A simplification yields:

\[
2^{k-1} [d(s + t - 4) + 2 - 2g] + \left( \sum_{i=1}^{k-1} 2^{k-i-1}(n - i + 1)\beta_i \right) + (n - k)\beta_k \leq 0.
\]

Note that:

\[
\sum_{i=1}^{k-1} 2^{k-i-1}(n - i + 1) = (n - 1)2^{k-1} + k - n.
\]

Substitute \( \beta_i = ds + (2 - 4d - 2g) - p_i \). We obtain:

**Theorem 10.3 ("II")** Let \( C \subset \mathbb{P}^3 \) be a smooth curve of degree \( d \) and genus \( g \). Suppose that \( C = S \cap T \), where \( S \) and \( T \) are surfaces of degree \( s \) and \( t \) respectively. Assume that \( C \notin \text{Sing}(S) \) and \( C \notin \text{Sing}(T) \). Let \( n = st/d \). Let \( p_i = p_i(S, C) \), for each \( i = 1, \ldots, n - 1 \), as defined in §4. Then for each \( k = 1, \ldots, n - 1 \), we have:

\[
\sum_{i=1}^{k-1} 2^{k-i-1}(n - i + 1)p_i + (n - k)p_k \geq 2^{k-1} \{ dt + n[d(s - 4) + 2 - 2g] \}.
\]

**Examples.**

- \( s = 2, t = 3, d = 3, g = 0 \): then the theorem yields the single inequality \( p_1 \geq 1 \);
- \( s = 2, t = 2, d = 1, g = 0 \): as above the theorem yields \( p_1 \geq 1 \);
- \( s = 4, t = 4, d = 4, g = 0 \): the theorem yields three inequalities:
  
  (i) \( p_1 \geq 8 \);
  (ii) \( 2p_1 + p_2 \geq 24 \);
  (iii) \( 8p_1 + 3p_2 + p_3 \geq 96 \).
11 Theorems III, Q, and B

Theorem 11.1 ("III") Let $C \subset \mathbb{CP}^3$ be a smooth curve of degree $d$ and genus $g$ which is the set-theoretic complete intersection of two surfaces $S, T$ of degrees $s, t$, respectively. Let $n = st/d$. Let $p_k = p_k(S, C)$, for each $k \in \mathbb{N}$. Assume that $S$ has only rational singularities. Assume that $C \not\subset \text{Sing}(T)$. Then:

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} p_k \geq \frac{d^2}{s} + d(s-4) + 2 - 2g.$$ 

Proof. By ([18] 1.1), we know that $\Delta(S, C) = \frac{d^2}{s} + d(s-4) + 2 - 2g$. Apply (6.3).

Remark 11.2 In the statement of (11.1), we do not know if $\sum_{k=1}^{\infty}$ can be replaced by $\sum_{k=1}^{n-1}$. From (5.6), we see that this can at least be done if $(S, C)$ does not contain any singularities of type $D_{n,n}$ (with $n$ odd).

Lemma 11.3 Let $S \subset \mathbb{CP}^3$ be a surface of degree $s$ having only rational singularities. Let $\pi : \tilde{S} \to S$ be a minimal resolution. Let $N$ be the number of exceptional curves on $\tilde{S}$. Then:

$$N \leq \frac{s}{3}(2s^2 - 6s + 7) - 1.$$ 

Proof. Clearly $N \leq \text{rank} \text{NS}(\tilde{S}) - 1$. Also rank $\text{NS}(\tilde{S}) \leq h^{1,1}(\tilde{S})$, so it suffices to show that $h^{1,1}(\tilde{S}) = \frac{s}{3}(2s^2 - 6s + 7)$. By simultaneous resolution of rational double points [2], and deformation invariance of Hodge numbers, we may reduce to showing that $h^{1,1}(S) = \frac{s}{3}(2s^2 - 6s + 7)$ if $S$ is itself nonsingular. We have:

$$h^{1,1}(S) = h^2(S, \mathcal{Q}) - 2h^2(S, \mathcal{O}_S) = [\chi_{\text{top}}(S) + 4h^1(S, \mathcal{O}_S) - 2] - 2h^2(S, \mathcal{O}_S).$$

Using the fact that the top Chern class of the tangent bundle equals the Euler characteristic (see e.g. [1] 11.24, 20.10.6), and using Riemann-Roch, we find:

$$\chi_{\text{top}}(S) = c_2(S) = 12\chi(S) - c_1^2(S) = 12(1 - h^1(S, \mathcal{O}_S) + h^2(S, \mathcal{O}_S)) - (4-s)^2s.$$
The formula for $h^{1,1}$ follows from $h^1(S, O_S) = 0$ and

$$h^2(S, O_S) = \binom{s - 1}{3}.$$ 

**Remark 11.4** We do not know if the lemma remains valid if $C$ is replaced by an algebraically closed field of positive characteristic.

**Lemma 11.5** Let $(p_k)_{k \in \mathbb{N}}$ be a sequence of nonnegative integers. Let $n$ be a nonnegative integer. Assume that:

(i) $p_1 \leq 9 - \frac{2}{5}n$

(ii) $p_1 \geq p_2 \geq p_3 \geq \cdots$

(iii) $\sum_{k=1}^{\infty} p_k \leq 19 - n$

(iv) $n/4 + \sum_{k=1}^{\infty} \frac{1}{k(k+1)} p_k \geq 6.$

Then $n = 0$ and $(p_k) \in \{(9, 8, 2), (9, 9), (9, 9, 1)\}$.

**Proof.** Constraints (i), (iii), and (iv) imply that

$$n/4 + \frac{1}{2} \floor{9 - \frac{2}{5}n} + \frac{1}{6}(19 - n - \floor{9 - \frac{2}{5}n}) \geq 6.$$ 

It follows that $n \in \{0, 2\}$.

Suppose that $n = 2$. Then the same constraints imply that

$$1/2 + \frac{1}{3}p_1 + \frac{1}{6}(17 - p_1) \geq 6,$$

so $p_1 = 8$. Now we see that the left hand side of (iv) is maximized when $(p_k) = (8, 8, 1)$. In that case, the left hand side of (iv) is $5\frac{11}{12}$: contradiction. Hence $n = 0$. Then

$$\frac{1}{3}p_1 + \frac{1}{6}(19 - p_1) \geq 6,$$

so $p_1 = 9$. If $p_2 \leq 7$, then the sum in (iv) is bounded by the sum obtained when $(p_k) = (9, 7, 3)$. This sum is $< 6$, so $p_2 \leq 7$. Hence $p_2 \in \{8, 9\}$. Etc. \(\blacksquare\)
Proposition 11.6 Let $C \subset \mathbb{CP}^3$ be a smooth curve of degree $d$ and genus $g$, which lies on a surface $S \subset \mathbb{CP}^3$ of degree $s$. Assume that $C \not\subset \text{Sing}(S)$. Let $p_1 = p_1(S, C)$. Let $N$ be the normal bundle of $C$ in $\mathbb{CP}^3$, and let $l$ be the maximum degree of a sub-line-bundle of $N$. Let $k = 3d + (2g - 2) - l$. Then $p_1 \leq d(s - 1) - k$.

Proof. We use the notation of (8.1). We also use various facts from §10, which although apparently dependent on another surface $T$, actually make sense in this context. We have $\langle c_1, c_1 \rangle \geq \deg(N) - 2l$. Since $\deg(N) = 4d + 2g - 2$ and $k = 3d + (2g - 2) - l$, we have:

$$\langle c_1, c_1 \rangle \geq -2d + 2 - 2g + 2k. \quad (\dagger)$$

Now $c_1 = d_1 - \alpha_1 r_1$ in $A^1(E_1)$, and $\langle d_1, d_1 \rangle = 2 - 2g - 4d$, so:

$$\langle c_1, c_1 \rangle = \langle d_1, d_1 \rangle - 2\alpha_1 = 2 - 2g - 4d - 2\alpha_1.$$

Combining this with (\dagger), we obtain $d + \alpha_1 \leq -k$. The formulas (*) and (**) from §11 imply that $\alpha_1 = p_1 - ds$. Hence $p_1 \leq d(s - 1) - k$.

Remark 11.7 This result (11.6) is a strengthening of the very elementary fact that:

$$|\text{Sing}(S) \cap C| \leq d(s - 1).$$

Theorem 11.8 ("Q") Let $C \subset \mathbb{CP}^3$ be a curve. Assume that $C = S \cap T$ set-theoretically for some surfaces $S$ and $T$. Assume that $S$ is normal. Assume that $\deg(C) > \deg(S)$. Then $C$ is linearly normal.

Proof. To any Weil divisor $E$ on a normal surface $S$, one can associate a reflexive $\mathcal{O}_S$-module $\mathcal{O}_S(E)$. We recall the following result of Sakai from [29], which is a slightly less general version of theorem 5.1 of that paper:

Let $S$ be a normal projective surface. Let $D$ be a nef Weil divisor on $S$ with $D^2 > 0$. Then $H^1(S, \mathcal{O}_S(-D)) = 0$.

Since the canonical map $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1)) \to H^0(S, \mathcal{O}_S(1))$ is surjective, it suffices to show that the canonical map $H^0(S, \mathcal{O}_S(1)) \to H^0(S, \mathcal{O}_C(1))$ is
surjective. Let $H$ be a hyperplane section of $S$. From the long exact sequence coming from

$$0 \longrightarrow \mathcal{O}_S(H - C) \longrightarrow \mathcal{O}_S(H) \longrightarrow \mathcal{O}_C(H) \longrightarrow 0,$$

we see that it is sufficient to show that $H^1(S, \mathcal{O}_S(H - C)) = 0$.

Let $d = \deg(C)$. Let $s = \deg(S)$, $t = \deg(T)$, and let $n$ be the multiplicity of intersection of $S$ with $T$ along $C$, $n = st/d$. Since $d > s$, we have $t > n$. Hence $(t - n)H$ is a very ample Cartier divisor. Since $n(C - H) \sim (t - n)H$, the theorem follows from Sakai’s result.

Corollary 11.9 Let $C \subset \mathbf{CP}^3$ be a smooth curve. Assume that $C$ is the set theoretic complete intersection of two normal surfaces $S$ and $T$, with multiplicity $\leq 3$. Then $C$ is linearly normal.

Proof. Using the notation of the proof of (11.8), we are done if either $s$ or $t$ is bigger than $n$. Otherwise, $d \leq 3$, and so $C$ is linearly normal anyway.

Remark 11.10 For the case of multiplicity 4, we must have $C$ linearly normal, except possibly for the case where $C$ is a rational quartic, which is the set-theoretic complete intersection of two normal quartic surfaces. It is not known if this is possible.

Theorem 11.11 (“B”) Let $S, T \subset \mathbf{CP}^3$ be surfaces. Assume that $S \cap T$ is set-theoretically a smooth curve. Assume that $\deg(S) = 4$ and that $S$ has only rational singularities. Then $C$ is linearly normal.

Proof. Let $C$ have degree $d$ and genus $g$. By (11.8), we may assume that $d = 4$ and $g = 0$. By [18], we may assume that $\deg(T) \geq 4$. Since $\deg(S) \leq \deg(T)$, we may assume that $C \not\subset \text{Sing}(T)$, as follows. Suppose that $C \subset \text{Sing}(T)$. Write $S = V(f)$, $T = V(g)$. Choose $h$ so that $\deg(fh) = \deg(g)$, and so that $C \not\subset V(h)$. Then $C \not\subset \text{Sing}(V(fh + g))$. Hence we may replace $T$ by $V(fh + g)$.

Write $(S, C) = (S', C') + (S'', C'')$, where

$$(S'', C'') = D_{n_1,n_1} + \cdots + D_{n_r,n_r},$$

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$n_1, \ldots, n_r$ are odd integers $\geq 5$, and $(S', C')$ is a configuration which does not involve any such singularities. Let $p_i = p_i(S', C')$. Let $n = n_1 + \cdots + n_r$.

We show that the hypotheses of (11.5) are satisfied.

We apply (11.6), using that fact [5] that $l = 7$, concluding that $p_1(S, C) \leq 9$. We have $p_1(\sum D_{n_i, n_i}) = \sum (n_i - 1)/2$ by (5.2), and $n_i - 1 \geq \frac{4}{5}n_i$, so $p_1(\sum D_{n_i, n_i}) \geq \frac{2}{5}n$. Thus hypothesis (i) is satisfied. Hypothesis (ii) holds.

Hypothesis (iii) follows from (11.3) and from the fact that $(S', C')$ contains no $D_{m,m}$ pairs with $m$ odd, $m \geq 5$, so that def $(S', C') \geq 0$. To prove hypothesis (iv), we would like to use (11.1 = “III”), but that is not quite good enough. By (6.5),

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)}p_k \geq \Delta(S', C').$$

Let $s = \deg(S) = 4$. By ([18] 1.1), we know that:

$$\Delta(S, C) = d^2/s + d(s - 4) + 2 - 2g = 6.$$

Then $\Delta(S', C') = \Delta(S, C) - \Delta(S'', C'')$. By (5.7), $\Delta(S'', C'') = n/4$. Hypothesis (iv) follows.

By (11.3), we conclude that $n = 0$ and that:

$$\text{type}(S, C) \in \{(9, 8, 2), (9, 9), (9, 9, 1)\}.$$

We will use (2.2), (6.1), and (6.2).

Suppose that $\text{type}(S, C) = (9, 9, 1)$. Then for some $p \in C$, $\text{type}(S, C)_p = (r, 1, 1)$ for some $r \geq 1$. The case $r > 1$ is impossible because $\text{type}(S, C) - \text{type}(S, C)_p = (9 - r, 9 - 1)$ and $9 - r \geq 9 - 1$. Hence $\text{type}(S, C)_p = (1, 1, 1)$. Hence $(S, C)_p = A_{3,1}$. Since $\Sigma(S, C) \leq 19$ by (11.3), and since $9 + 9 + 1 = 19$, we have $\text{def}(S, C) = 0$. Therefore, since the other singularities of $S$ along $C$ have type $(k, k)$ for some $k \leq 8$, we see that the other singularities must be $A_{3k-1,k}$ for some $k \in \{1, \ldots, 8\}$, depending on the singular point. Amongst these, only $A_{2,1}$ has deficiency zero. Hence $(S, C) = 8A_{2,1} + A_{3,1}$. Hence $\Delta(S, C) = 8(\frac{2}{3}) + \frac{2}{4} \neq 6$: contradiction.

Now suppose that $\text{type}(S, C) = (9, 8, 2)$. Then for some $p \in C$,

$$\text{type}(S, C)_p \in \{(1, 1, 1), (2, 1, 1), (2, 2, 2)\}.$$

These types are realized by the singularities $A_{3,1}, A_{4,2},$ and $A_{7,2}$, respectively, and by no others. Since $A_{7,2}$ has nonzero deficiency, it can be excluded. Hence
If $(S, C)_p = A_{3,1}$, then we may assume that $(S, C) = 2A_{3,1} +$ other, where the “other” part must have type $(7, 6)$. The only zero-deficiency rational double point configuration which realizes this type is $A_{1,1} + 6A_{2,1}$. Hence $(S, C) = A_{1,1} + 6A_{2,1} + 2A_{3,1}$. By [24], the sum of the contributions of the singularities must not exceed $(2/3) \text{deg}(S)(\text{deg}(S) - 1)^2 = 24$, where each singularity $p$ contributes $e(E) - 1/|G|$, $e(E)$ is the topological Euler characteristic of the exceptional fiber in the minimal resolution of $p$, and $G$ is the order of the group which defines $p$ as a quotient singularity. In particular, an $A_n$ singularity contributes $(n + 1) - (n + 1)^{-1}$. Then the sum of the contributions is 25: contradiction.

Suppose that type$(S, C) = (9, 9)$. Then each singularity of $S$ along $C$ must have type $(k, k)$ for some $k$, depending on the singular point. Hence $(S, C)$ must be built up from $E_{6,1}$ and $A_{3k-1,k}$ for various $k$. Since $\text{def}(E_{6,1}) = 2$, we may rule out that case. In fact, there are only two configurations with deficiency $\leq 1$: either $(S, C) = 9A_{2,1}$ or else $(S, C) = 7A_{2,1} + A_{5,2}$. In both cases, order$(S, C) = 3$. Hence we may assume that $\text{deg}(T) = 3$. By [18], we know that this is impossible.

12 Theorem A

Lemma 12.1 Let $s, t, d, g \in \mathbb{Z}$. Assume that $t \geq s \geq 4$, $d \geq 1$, and $g \geq 0$. Assume that $d|st$. Let $n = st/d$. Assume that $n \geq 2$. Let $r = d[n(s - 4) + t] + (2 - 2g)n$. Assume that $r \leq \frac{s}{3}(2s^2 - 6s + 7) - 1$. Then $d \leq g + 3$.

Proof. We assume that $d \geq g + 4$, working toward a contradiction. We have $2 - 2g \geq 10 - 2d$, so:

$$d[n(s - 4) + t] + (10 - 2d)n \leq \frac{s}{3}(2s^2 - 6s + 7) - 1.$$  \hspace{1cm} (*)

First suppose that $s = 4$. Then $t \geq 4$, $t \geq d/2$, and $dt + (10 - 2d)n \leq 19$. Substituting $n = st/d = 4t/d$ and simplifying, we obtain:

$$(d^2 - 8d + 40)t \leq 19d.$$ \hspace{1cm} (†)

Since $t \geq d/2$, we have $(d^2 - 8d + 40)(1/2) \leq 19$. It follows that $d \leq 7$. Hence
$d \in \{4, 5, 6, 7\}$. In each case, (†) gives us an upper bound $t_{\max}$ for $t$:

| $d$ | $t_{\max}$ |
|-----|-------------|
| 4   | 3           |
| 5   | 3           |
| 6   | 4           |
| 7   | 4           |

The cases $d \in \{4, 5\}$ contradict $t \geq 4$. In case $d \in \{6, 7\}$, we have $t = 4$, which contradicts our assumption that $d|st$. Hence $s > 4$.

Now suppose that $s = 5$. Then $t \geq 5$, $t \geq 2d/5$, and $d(t + n) + (10 - 2d)n \leq 44$. Substituting $n = st/d = 5t/d$ and simplifying, we obtain:

$$t \leq 44d/(d^2 - 5d + 50).$$

This implies that $t < 5$: contradiction. Hence $s \geq 6$.

Since $n = st/d \geq s^2/d$, it follows from (*) that:

$$d \left[ \frac{s^2}{d} (s - 6) + s \right] + 10 \frac{s^2}{d} \leq \frac{s}{3} (2s^2 - 6s + 7).$$

This implies that:

$$s(s - 6) + d + 10 \frac{s}{d} \leq \frac{1}{3} (2s^2 - 6s + 7) \tag{**}$$

and in particular that:

$$s(s - 6) \leq \frac{1}{3} (2s^2 - 6s + 7).$$

It follows that $s \leq 12$. Hence $6 \leq s \leq 12$. If $s = 12$, (**) implies that $d + 120/d \leq 2\frac{1}{3}$. This is absurd. In a similar manner, one may eliminate the cases where $6 \leq s \leq 11$.

**Theorem 12.2 (“A”)** Let $S, T \subset \mathbb{CP}^3$ be surfaces. Assume that $S$ has only rational singularities. Assume that $\deg(S) \leq \deg(T)$. Assume that $S \cap T$ is set-theoretically a smooth curve $C$ of degree $d$ and genus $g$. Assume that $\text{Sing}(S) \cap \text{Sing}(T) = \emptyset$. Then $d \leq g + 3$. 

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Proof. Let $s = \deg(S)$, $t = \deg(T)$, $n = st/d$. Let $p_i = p_i(S, C)$. By (10.1 = “I”), we have:

\[ p_1 = \cdots = p_{n-1} = \frac{1}{n-1} \{d[n(s-4) + t] + (2-2g)n\}. \]

We show that $S$ has no singularities of type $D_{t,t}$ (with $t$ odd, $t \geq 5$), lying on $C$. There are two cases. If $n = 2$, then order$(S, C)|2$. But by (5.3), the order of $D_{t,t}$ (as above) is 4. Hence $n > 2$. Hence $p_1 = p_2$. But $p_1(D_{t,t}) > p_2(D_{t,t})$ (by 5.2), so “$D_{t,t} \notin (S, C)$” for $t$ odd, $t \geq 5$.

From this, it follows that $p_1 + \cdots + p_{n-1} \leq \Sigma(S, C)$. Let $r = d[n(s-4) + t] + (2-2g)n$. By (11.3), we conclude that $r \leq \frac{s}{2}(2s^2 - 6s + 7) - 1$.

The case $n = 1$ corresponds to a complete intersection, and the theorem is easily verified in this case. Therefore we may assume that $n \geq 2$. By [18], it follows that if $s \leq 3$, then $d \leq g + 3$. Hence we may assume that $s \geq 4$. Therefore (12.1) applies, and we conclude that $d \leq g + 3$.

Corollary 12.3 Let $S, T \subset \mathbb{CP}^3$ be surfaces. Assume that $S$ and $T$ have only rational singularities. Assume that $S \cap T$ is set-theoretically a smooth curve $C$ of degree $d$ and genus $g$. Assume that $\text{Sing}(S) \cap \text{Sing}(T) = \emptyset$. Then $d \leq g + 3$.

13 Theorem X

As a corollary of theorem (I), we show:

Theorem 13.1 (“X”) Let $C \subset \mathbb{P}^3$ be a smooth curve. Assume that $C$ is not a complete intersection. Suppose that $C = S \cap T$ as sets, where $S, T \subset \mathbb{P}^3$ are surfaces. Assume that $\text{Sing}(S) \cap \text{Sing}(T) = \emptyset$. Then:

\[ \deg(S), \deg(T) < 2 \cdot \deg(C)^4. \]

First we make a few remarks.

(1) The proof of theorem (X) depends primarily on the fact that the numbers $p_k$ in theorem (I) must be integers.
(2) The importance of theorem (X) is that an upper bound is given for the degrees of $S$ and $T$, that this bound is computable, and that this bound depends only on the degree of $C$. In the proof, we give the better bounds $\deg(S) < 2 \cdot \deg(C)^2$, $\deg(T) < 2 \cdot \deg(C)^4$, provided that $\deg(S) \leq \deg(T)$.

(3) Via the bounds in theorem (X), it becomes a computer triviality to find all possible degrees for $S$ and $T$ which are consistent with the integrality of the numbers $p_k$ in theorem (I).

(4) Doing this when $\deg(C) = 4$, genus($C$) = 0, and assuming for efficiency that $\deg(S) \leq \deg(T)$, we find:

$$(\deg(S), \deg(T)) \in \{(3, 4), (3, 8), (4, 4), (4, 7), (6, 26), (9, 48), (10, 28), (12, 18), (13, 16), (17, 220), (18, 118), (19, 84), (20, 67), (22, 50), (28, 33)\}.$$

[We have excluded the cases where $\deg(S)$ is 1 or 2, which cannot occur.]

(5) We do not know which of these pairs of integers can be realized by pairs of surfaces, as in theorem (X). All that we know is that $(3, 4)$ and $(3, 8)$ cannot be realized in characteristic zero, and that $(4, 4)$ can be realized in characteristic two.

(6) Theorem (X) is false without the hypothesis that $C$ is a complete intersection. Counterexample: for any $s \in \mathbb{N}$, one can find a smooth curve $D \subset \mathbb{P}^2$ of degree $s$ and a line $L \subset \mathbb{P}^2$ such that $D \cap L$ is a single point, set-theoretically. Let $S$ and $T$ be cones over $D$ and $L$, with the same vertex. Then $S \cap T$ is a line, set-theoretically. Theorem (X) is also false without the hypothesis that $\text{Sing}(S) \cap \text{Sing}(T) = \emptyset$.

Before proceeding with the proof of theorem (X), we need the following lemma, which was known in characteristic zero, and for the smooth case, was known in all characteristics. (See proof for references.)

**Lemma 13.2** Let $S \subset \mathbb{P}^3$ be a normal surface. Then $\text{Pic}(S)/\text{Pic}(\mathbb{P}^3)$ is torsion-free.
Before proceeding with the proof, we recall some standard material on differentials for which we do not have a good reference. First of all, for any scheme $X$, there is a map of sheaves of abelian groups:

$$d\log : \mathcal{O}_X^* \rightarrow \Omega_X$$

given by $f \mapsto df/f$. (All sheaves we shall discuss are sheaves on the Zariski site.)

Now suppose that $X$ is a normal proper variety, defined over an algebraically closed field $k$ of positive characteristic $p$. Then we have an exact sequence:

$$0 \rightarrow \mathcal{O}_X^* \xrightarrow{F} \mathcal{O}_X^* \xrightarrow{d\log} \Omega_X$$

of sheaves of abelian groups on $X$, where $F$ denotes the Frobenius map. The exactness in the middle depends on normality, and may be deduced e.g. from ([16] I 4.2). Let $\mathcal{D}$ be the image of $d\log$. Since $X$ is proper, $H^0(X, \mathcal{O}_X^*) = k^*$, so $H^0(X, F)$ is an isomorphism, and we obtain an isomorphism $H^0(X, \mathcal{D}) \cong \ker H^1(X, F)$. We have $\ker H^1(X, F) \cong p \text{Pic}(X)$. Composing with the canonical injection $H^0(X, \mathcal{D}) \rightarrow H^0(X, \Omega_X)$, we obtain an injective group homomorphism:

$$\psi_X : p \text{Pic}(X) \rightarrow H^0(X, \Omega_X).$$

**Proof** (of [13.2]. First we show ($\ast$) that $\text{Pic}(S)/\text{Pic}(\mathbb{P}^3)$ has no torsion, except possibly for $p$-torsion, when the ground field has positive characteristic $p$. These arguments are very similar to those given by Lang [21]. The methods were invented by Grothendieck ([10] Exposé XI), and further studied by Hartshorne ([13] §4.3). We refer the reader to [21] or [13] for details.

Let $S_n$ be the $n^{th}$ infinitesimal neighborhood of $S$ in $\mathbb{P}^3$. Then:

$$\text{Pic}(\mathbb{P}^3) \cong \lim_{\leftarrow} \text{Pic}(S_n).$$

Moreover, for each $n$ there is an exact sequence of abelian groups:

$$0 \rightarrow \text{Pic}(S_{n+1}) \rightarrow \text{Pic}(S_n) \rightarrow H^2(S, \mathcal{J}^n/\mathcal{J}^{n+1}),$$

where $\mathcal{J}$ is the ideal sheaf of $S$ in $\mathbb{P}^3$. Since the $H^2$ term is a vector space, ($\ast$) follows.

From now on we may assume that the ground field has positive characteristic $p$. A standard calculation shows that $H^0(S, \Omega_S) = 0$. Up to now,
we have not used the hypothesis that $S$ is normal. We now use this hypothesis. Via the map $\psi_S$, defined immediately above this proof, we see that $p\text{Pic}(S) = 0$. (This argument is essentially that used in [21].)

Finally, to complete the proof, we must show that $O_S(1)$ does not have a $p^{th}$ root in Pic$(S)$. The argument given here is essentially the argument given in ([4] 1.8). For any variety $X$, there is a natural group homomorphism $H^1(d\log) : \text{Pic}(X) \to H^1(X, \Omega_X)$. Consider this map when $X = S$ and when $X = P^3$. A standard calculation shows that the map $H^1(P^3, \Omega_{P^3})\to H^1(S, \Omega_S)$ is injective. Moreover, one knows that the image of $[O_{P^3}(1)]$ in $H^1(P^3, \Omega_{P^3})$ is not zero. (See e.g. [14] Chapter 3, exercise 7.4.) Hence the image of $[O_S(1)]$ in $H^1(S, \Omega_S)$ is not zero. Hence $[O_S(1)]$ does not have a $p^{th}$ root in Pic$(S)$.

Remark 13.3 Over an algebraically closed field of positive characteristic, let $S \subset P^3$ be a surface, not necessarily normal. We do not know if Pic$(S)/\text{Pic}(P^3)$ is torsion-free, or even if Pic$(S)$ is torsion-free. Answers to these questions might be obtained from a general structure theorem for Ker$[\text{Pic}(S) \to \text{Pic}(S_{\text{nor}})]$, where $S$ is an arbitrary projective variety.

Corollary 13.4 In $P^3$, suppose that $C = S \cap T$ as sets, where $C$ is a curve, and $S, T$ are surfaces. Assume that $C$ does not meet Sing$(S)$. Then there exists a surface $T' \subset P^3$ such that $C = S \cap T'$, scheme-theoretically.

Proof. Since $T \cap \text{Sing}(S) = \emptyset$, $S$ is normal. By (13.2), Pic$(S)/\text{Pic}(P^3)$ is torsion-free. Hence $[O_S(C)] = 0$ in Pic$(S)/\text{Pic}(P^3)$. Hence $O_S(C) \cong O_S(t)$ for some $t \in N$. Since the canonical map $H^0(P^3, O_{P^3}(t)) \to H^0(S, O_S(t))$ is surjective, it follows that there exists a surface $T'$ of degree $t$ as claimed. 

Remark 13.5 Robbiano [28] proved this in the case where $S$ is smooth and the ground field has characteristic zero.

Proof (of theorem X). Let $d = \deg(C)$, $g = \text{genus}(C)$, $s = \deg(S)$, $t = \deg(T)$. We may assume that $s \leq t$. We show that $s < 2d^2$ and $t < 2d^4$. Let $n = st/d$. By theorem (I), we know that:

$$(n - 1) \mid \{d[n(s - 4) + t] + (2 - 2g)n\}.$$
A proof of this fact, independent of (I), is given at the end of this paper. By ([13.4]), we know that \( C \) meets Sing\( (S) \). Hence \( p_1(S, C) > 0 \). Hence the right hand side is positive. Write \( d = d_s d_t \), where \( d_s, d_t \in \mathbb{N}, d_s | s, \) and \( d_t | t \). Let \( s_1 = s/d_s, \) \( t_1 = t/d_t \). Then \( n = s_1 t_1 \), so:

\[
(s_1 t_1 - 1) \{d[s_1 t_1 (d_s s_1 - 4) + d_t t_1] + (2 - 2g) s_1 t_1\}.
\]

The right hand side is divisible by \( t_1 \), and \( \gcd(s_1 t_1 - 1, t_1) = 1 \), so:

\[
(s_1 t_1 - 1) \{d[s_1 (d_s s_1 - 4) + d_t] + (2 - 2g) s_1\}.
\]

Thus for some \( k \in \mathbb{N} \), we have:

\[
(s_1 t_1 - 1)k = d[s_1 (d_s s_1 - 4) + d_t] + (2 - 2g) s_1.
\]

Reorganizing, we find:

\[
(s_1 t_1 - 1)k = (d d_s) s_1^2 + (2 - 2g - 4d) s_1 + dd_t.
\]

Now we have \( t \geq s \), so \( t_1 \geq (d_s/d_t) s_1 \). Hence:

\[
\left[s_1^2 \left(\frac{d_s}{d_t}\right) - 1\right] k \leq (d d_s) s_1^2 + (2 - 2g - 4d) s_1 + dd_t.
\]

It is conceivable that the left hand side of this inequality is negative. This will not effect the following argument. Suppose that \( k \geq dd_t \). After a short calculation, one finds that \( s_1 \leq dd_t/(2d + g - 1) \), and hence that \( s \leq d^2/(2d + g - 1) \). This implies that \( s < 2d^2 \). Hence, in order to prove our assertion that \( s < 2d^2 \), we may assume that \( k < dd_t \).

From (**), we obtain:

\[
(dd_s) s_1^2 + (2 - 2g - 4d - t_1 k) s_1 + (k + dd_t) = 0.
\]

Hence \( s_1 | (k + dd_t) \). Hence \( s_1 \leq k + dd_t \). Hence \( s_1 < 2dd_t \). Hence \( s < 2d^2 \).

To complete the proof, we must show that \( t < 2d^4 \). The right hand side of (*) is nonzero, so:

\[
s_1 t_1 - 1 \leq d[s_1 (d_s s_1 - 4) + d_t] + (2 - 2g) s_1.
\]

Dividing by \( s_1 \) and isolating \( t_1 \), we find:

\[
t_1 \leq d[s_1 (d_s s_1 - 4 + d_t s_1^{-1})] + 2 - 2g + s_1^{-1}.
\]

Taking account of \( t_1 = td_t^{-1} \) and \( s_1 = sd_s^{-1} \), we obtain:

\[
t \leq d_t [d(s - 4 + ds^{-1})] + 2 - 2g + ds^{-1}.
\]

Since \( s < 2d^2 \), it follows (with a little work) that \( t < 2d^4 \).
Remark 13.6 We give here an alternate proof of the main ingredient of the proof of (X), namely that

\[(n-1) \mid \{d[n(s-4)+t] + (2 - 2g)n\} \quad (\dagger)\]

Let \(\tilde{C}\) be the scheme-theoretic complete intersection of \(S\) and \(T\). Let \(\mathcal{J}\) be the ideal sheaf of \(C\) in \(\tilde{C}\). Let \(p\) be a closed point of \(C\). If \(\mathcal{O}_{S,p}\) is regular, then near \(p\), \(\tilde{C}\) and \(C\) are Cartier divisors on \(S\), with \(\tilde{C} = nC\). Choose an isomorphism \(\mathcal{O}_{S,p} \cong k[[x,y]]\), such that \(C\) corresponds to \(V(x)\). Then \(\tilde{C}\) corresponds to \(V(x^n)\). Therefore the algebra of conormal invariants

\[ \mathcal{A} = \mathcal{O}_{\tilde{C}}/\mathcal{J} \oplus \mathcal{J}/\mathcal{J}^2 \oplus \mathcal{J}^2/\mathcal{J}^3 \oplus \cdots \]

is a locally free \(\mathcal{O}_C\)-module near \(p\). But we similarly get the same conclusion if \(\mathcal{O}_{T,p}\) is regular, so in fact \(\mathcal{A}\) is locally free since \(\text{Sing}(S) \cap \text{Sing}(T) = \emptyset\).

Moreover, there is a line bundle \(\mathcal{L}\) on \(C\) such that \(\mathcal{A} \cong \mathcal{O}_C \oplus \mathcal{L} \oplus \mathcal{L}^2 \oplus \cdots \oplus \mathcal{L}^{n-1}\). Hence \(\chi(\mathcal{A}) = n(1-g) + \binom{n}{2} \deg(\mathcal{L})\). On the other hand, \(\chi(\mathcal{A}) = \chi(\mathcal{O}_{\tilde{C}})\), which (via \(\tilde{C} = S \cap T\)) is easily computed to be \(st(4-s-t)/2\).

Hence

\[ n(1-g) + \binom{n}{2} \deg(\mathcal{L}) = \frac{st(4-s-t)}{2}. \]

Hence

\[ \binom{n}{2} \bigg| \frac{st(4-s-t)}{2} - n(1-g). \]

It is not difficult to verify that this is equivalent to \((\dagger)\).

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