Abstract

We study the long-time behavior of the Cesaro means of fundamental solutions for fractional evolution equations corresponding to random time changes in the Brownian motion and other Markov processes. We consider both stable subordinators leading to equations with the Caputo-Djrbashian fractional derivative and more general cases corresponding to differential-convolution operators, in particular, distributed order derivatives.
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1 Introduction

Let \( \{ X_t, t \geq 0; P_x, x \in E \} \) be a strong Markov process in a phase space \( \mathbb{R}^d \). Denote by \( T_t \) its transition semigroup (in an appropriate Banach space) and by \( L \) the generator of this semigroup. Let \( S_t, t \geq 0 \) be a subordinator (i.e., a non-decreasing real-valued Lévy process) with \( S_0 = 0 \) and the Laplace exponent \( \Phi \):

\[
E[e^{-\lambda S_t}] = e^{-t\Phi(\lambda)} \quad t, \lambda > 0.
\]

We assume that \( S_t \) is independent of \( X_t \).

Denote by \( E_t, t > 0 \) the inverse subordinator, and introduce the time-changed process \( Y_t = X_{E_t} \). A general aim is to analyze the properties of \( Y_t \) depending on the given Markov process \( X_t \) and the particular choice of subordinator \( S_t \). There is interest in this kind of problem in diverse disciplines. In addition to purely stochastic motivations, a transform of the Markov process \( X_t \) in the non-Markov one \( Y_t \) implies the presence of effects in the corresponding dynamics. This feature is important in a number of physical models. In particular, progress in the understanding of this process may lead to the realization of useful models of biological time in the evolution of species and ecological systems. Currently, there exist rather complete studies of such
problems in the case of so-called stable subordinators [BM01, MS04] and in special examples of initial processes $X_t$ (see, e.g., [MS15], [LSS07], [Mim16]).

As a basic characteristic of the new process $Y_t$, we may study the time evolution

$$u(t, x) = E^x[f(Y_t)]$$

for a given initial data $f$.

As it was pointed out in several works, see e.g. [Toa15], [Che17] and references therein, $u(t, x)$ is the unique strong solution (in some appropriate sense) to the following Cauchy problem

$$\mathbb{D}_t^{(k)} u(t, x) = Lu(t, x) \quad u(0, x) = f(x). \quad (1.1)$$

Here we have a generalized fractional derivative (GFD for short)

$$\mathbb{D}_t^{(k)} \phi(t) = \frac{d}{dt} \int_0^t k(t - s)(\phi(s) - \phi(0)) ds$$

with a kernel $k$ uniquely defined by $\Phi$.

Let $u_0(t, x)$ be the solution to a similar Cauchy problem but with ordinary time derivative

$$\frac{\partial}{\partial t} u(t, x) = Lu(t, x) \quad u(0, x) = f(x). \quad (1.2)$$

In stochastic terminology, it is the solution to the forward Kolmogorov equation corresponding to the process $X_t$. Under quite general assumptions there is a convenient and essentially obvious relation between these evolutions that is known as the subordination principle:

$$u(t, x) = \int_0^\infty u_0(\tau, x) G_t(\tau) d\tau,$$

where $G_t(\tau)$ is the density of $E_t$.

A similar relation holds for fundamental solutions (or heat kernels in another terminology) $v(x, t)$ and $v^E(x, t)$ of equations (1.2) and (1.1), respectively. For certain classes of a priori bounds for fundamental solutions $v(x, t)$, the properties of the subordinated kernels were studied in [CKKW18]. The main technical tool used in this work involves a scaling property assumed for $\Phi$ [CKKW18] that is a global condition on the Lévy characteristic $\Phi(\lambda)$. It is nevertheless difficult to give an interpretation of this scaling assumption in terms of the subordinator.
The aim of the present work is to extend the class of random times for which one may obtain information about the time asymptotic of \( v^E(x,t) \). We consider the following three classes of admissible kernels \( k \in L^1_{\text{loc}}(\mathbb{R}_+) \), characterized in terms of the Laplace transforms \( K(\lambda) \) as \( \lambda \to 0 \) (i.e., as local conditions):

\[ K(\lambda) \sim \lambda^{\theta-1}, \quad 0 < \theta < 1. \quad (C1) \]

\[ K(\lambda) \sim \lambda^{-1} L \left( \frac{1}{\lambda} \right), \quad L(y) := \mu(0) \log(y)^{-1}. \quad (C2) \]

\[ K(\lambda) \sim \lambda^{-1} L \left( \frac{1}{\lambda} \right), \quad L(y) := C \log(y)^{-1-s}, \quad s > 0, \quad C > 0. \quad (C3) \]

For each kernel of this type, we establish the asymptotic properties of subordinated heat kernels from different classes of a priori bounds. It is important to stress that in working with much more general random times (i.e., without the scaling property), a price must be paid for such an extension, namely the replacement of \( v^E(x,t) \) by its Cesaro mean. This is the key technical observation that underlies the analysis of several different model situations.

2 Preliminaries

Let \( S = \{S(t), \ t \geq 0\} \) be a stationary stochastic process with independent non-negative increments and \( P(S(0) = 0) = 1 \). Such processes \( S \) are known as subordinators. They form a special class of Lévy processes taking values in \([0, \infty)\) and their sample paths are non-decreasing. In addition we assume that \( S \) has no drift (see [Ber96] for more details). The infinite divisibility of the law of \( S \) implies that its Laplace transform can be expressed in the form

\[ \mathbb{E}(e^{-\lambda S_t}) = e^{-\Phi(\lambda)}, \quad \lambda \geq 0, \]

where \( \Phi : [0, \infty) \to [0, \infty) \), called the Laplace exponent (or cumulant), is a Bernstein function. The associated Lévy measure \( \sigma \) has support in \([0, \infty)\) and fulfills

\[ \int_{(0,\infty)} (1 \wedge \tau) \, d\sigma(\tau) < \infty \quad (2.1) \]

such that the Laplace exponent \( \Phi \) can be expressed as

\[ \Phi(\lambda) = \int_{(0,\infty)} (1 - e^{-\lambda \tau}) \, d\sigma(\tau), \quad (2.2) \]
which is known as the Lévy-Khintchine formula for the subordinator \( S \). For given Lévy measure \( \sigma \), we define the function \( k \) by
\[
k : (0, \infty) \longrightarrow (0, \infty), \ t \mapsto k(t) := \sigma((t, \infty))
\]  
(2.3)
and denote its Laplace transform by \( K \); i.e., for any \( \lambda \geq 0 \) one has
\[
K(\lambda) := \int_0^\infty e^{-\lambda t} k(t) \, dt.
\]  
(2.4)
We note that by the Fubini theorem, the function \( K \) is given in terms of the Laplace exponent. Specifically,
\[
K(\lambda) = \int_0^\infty e^{-\lambda t} \int_0^t d\sigma(s) \, dt = \int_0^\infty \int_s^\infty e^{-\lambda t} \, dt \, d\sigma(s) = \frac{1}{\lambda} \Phi(\lambda),
\]
i.e.,
\[
\Phi(\lambda) = \lambda K(\lambda), \ \forall \lambda \geq 0. \tag{2.5}
\]
Given the inverse process \( E \) of the subordinator \( S \), namely
\[
E(t) := \inf\{s \geq 0 : S(s) \geq t\} = \sup\{s \geq 0 : S(t) \leq s\}.
\]  
(2.6)
the marginal density of \( E(t) \) will be denoted by \( G_t(\tau) \), \( t, \tau \geq 0 \), more explicitly
\[
G_t(\tau) \, d\tau = \partial_\tau P(E(t) \leq \tau) = \partial_\tau P(S(\tau) \geq t) = -\partial_\tau P(S(\tau) < t).
\]

**Example 1** (\( \theta \)-stable subordinator and Gamma processes). 1. Let \( S \) be a \( \theta \)-stable subordinator \( \theta \in (0, 1) \) with Laplace exponent
\[
\Phi_\theta(\lambda) = \lambda^\theta = \frac{\theta}{\Gamma(1-\theta)} \int_0^\infty (1 - e^{-\lambda \tau}) \tau^{-1-\theta} \, d\tau,
\]
from which it follows that the Lévy measure \( \sigma \) is given by
\[
d\sigma(\tau) = \frac{\theta}{\Gamma(1-\theta)} \tau^{-1-\theta} \, d\tau.
\]
We have \( K(\lambda) = \lambda^{\theta-1} \) and \( k(t) = t^{-\theta}/\Gamma(1-\theta) \). The corresponding GFD \( \mathbb{D}_t^{(k)} \) is the Caputo-Djrbashian fractional derivative \( \mathbb{D}_t^{(\theta)} \) of order \( \theta \in (0, 1) \).

\(^1\)The restriction \( \theta \in (0, 1) \) and not \( \theta \in (0, 2) \) is due to the requirement (2.1). The boundary \( \theta = 1 \) corresponds to a degenerate case since \( S(t) = t \).
2. The Gamma process $Y^{(a,b)}$ with parameters $a, b > 0$ is given by its Laplace exponent $\Phi_{(a,b)}$ as

$$
\Phi_{(a,b)}(\lambda) = a \log \left( 1 + \frac{\lambda}{b} \right) = \int_0^\infty (1 - e^{-\lambda \tau}) a \tau^{-1} e^{-b \tau} d\tau,
$$

where the second equality is known as the Frullani integral [AdR90]. The corresponding Lévy measure is given by

$$
d\sigma(\tau) = a \tau^{-1} e^{-b \tau} d\tau.
$$

We have $K(\lambda) = \lambda^{-1} a \log (1 + \frac{\lambda}{b})$ and $k(t) = a \Gamma(0, bt)$. The corresponding GFD is given by

$$
(D_t^{(a,b)} f)(t) = \frac{d}{dt} \int_0^t \Gamma(0, b(t - s))(f(s) - f(0)) ds.
$$

The following question evokes an important characteristic is the Laplace transform of the density $G_t(\tau)$. Can the $\tau$-Laplace (or $t$-Laplace) transform of $G_t(\tau)$ be computed for an arbitrary subordinator? Explicitly, can we compute

$$
\int_0^\infty e^{-\lambda \tau} G_t(\tau) d\tau \quad \text{or} \quad \int_0^\infty e^{-\lambda t} G_t(\tau) dt.
$$

The answer for the $t$-Laplace transform is affirmative and the result is given in (2.13) below. On the other hand, for the $\tau$-Laplace transform a partial answer has been given for the class of $\theta$-stable processes; namely

**Example 2** (cf. Prop. 1(a) in [Bin71]). If $S$ is a $\theta$-stable process, then the inverse process $E(t)$ has the Mittag-Leffler distribution, as follows.

$$
\mathbb{E}(e^{-\lambda E(t)}) = \sum_{n=0}^{\infty} \frac{(-\lambda t^\theta)^n}{\Gamma(n\theta + 1)} = E_\theta(-\lambda t^\theta). \quad (2.7)
$$

It follows from the asymptotic behavior of the Mittag-Leffler function $E_\theta$ that

$$
\mathbb{E}(e^{-\lambda E(t)}) \sim \frac{C}{t^\theta}, \quad \text{as } t \to \infty.
$$

In addition, using the fact that

$$
E_\theta(-x) = \int_0^\infty e^{-x \tau} M_\theta(\tau) d\tau, \quad \forall x \geq 0, \quad (2.8)
$$
where $M_\theta$ is the so-called $M$-Wright (cf. [MMP10] for more details and properties), it follows that

$$
\mathbb{E}(e^{-\lambda E(t)}) = \int_0^\infty e^{-\lambda t_\theta M_\theta(\tau)} d\tau = \int_0^\infty e^{-\lambda t\theta M_\theta(\tau t^{-\theta})} d\tau
$$

from which we obtain the density of $E(t)$ explicitly as

$$
G_t(\tau) = t^{-\theta} M_\theta(\tau t^{-\theta}).
$$

On the other hand, for a general subordinator, the following lemma determines the $t$-Laplace transform of $G_t(\tau)$, with $k$ and $K$ given in (2.3) and 2.4, respectively.

**Lemma 3.** The $t$-Laplace transform of the density $G_t(\tau)$ is given by

$$
\int_0^\infty e^{-\lambda t} G_t(\tau) dt = K(\lambda) e^{-\tau k(\lambda)}.
$$

In addition, the double $(\tau, t)$-Laplace transform of $G_t(\tau)$ is given by

$$
\int_0^\infty \int_0^\infty e^{-\lambda t} e^{-\tau \theta} G_t(\tau) dt d\tau = \frac{K(\lambda)}{\lambda K(\lambda) + p}.
$$

**Proof.** The maps $\lambda \mapsto \lambda K(\lambda)$ are complete Bernstein functions, and for any $s \geq 0$, $\tau \mapsto e^{-\tau t}$ is a completely monotone function. The composition $\lambda \mapsto e^{-s\lambda K(\lambda)}$ is then a completely monotone function (cf. [SSV12, Thm. 3.7]). By Bernstein’s theorem, for any $\tau \geq 0$ there exists a probability measure $\eta_\tau$ on $\mathbb{R}_+$ such that

$$
e^{-\tau \lambda K(\lambda)} = \int_0^\infty e^{-\lambda s} d\eta_\tau(s).
$$

Defining

$$
g(\lambda, \tau) := K(\lambda) e^{-\tau \lambda K(\lambda)}, \quad \tau, \lambda > 0
$$

and

$$
G_t(\tau) := \int_0^t k(t - s) d\eta_\tau(s),
$$

we then have

$$
\int_0^\infty e^{-\lambda t} G_t(\tau) dt = g(\lambda, \tau) = K(\lambda) e^{-\tau \lambda K(\lambda)}.
$$
In fact, by definition and using Fubini’s theorem we may obtain
\[
\int_0^\infty e^{-\lambda t} G_t(\tau) \, dt = \int_0^\infty \int_t^\infty e^{-\lambda t} k(t-s) \, d\eta_r(s) \, dt
\]
\[
= \int_0^\infty \int_0^\infty e^{-\lambda t} k(t-s) \, dt \, d\eta_r(s)
\]
\[
= K(\lambda) \int_0^\infty e^{-\lambda s} \, d\eta_r(s)
\]
\[
= g(\lambda, \tau).
\]

In addition, it follows easily from (2.12) that
\[
\int_0^\infty g(\lambda, \tau) \, d\tau = \frac{1}{\lambda}
\]
so that (2.13) may be written as
\[
\int_0^\infty e^{-\lambda t} \, dt \int_0^\infty G_t(\tau) \, d\tau = \frac{1}{\lambda}
\]
which implies that \(G_t(\tau)\) is a \(\tau\)-density on \(\mathbb{R}_+\):
\[
\int_0^\infty G_t(\tau) \, d\tau = 1.
\]

Finally, the double \((\tau, t)\)-Laplace transform follows from
\[
\int_0^\infty \int_0^\infty e^{-p\tau} e^{-\lambda t} G_t(\tau) \, dt \, d\tau = \int_0^\infty e^{-p\tau} g(\lambda, \tau) \, d\tau
\]
\[
= K(\lambda) \int_0^\infty e^{-p\tau} e^{-\tau\lambda K(\lambda)} \, d\tau
\]
\[
= \frac{K(\lambda)}{\lambda K(\lambda) + p}.
\]

3 Subordinated Heat Kernel

In this section, we investigate the long-time behavior of the fundamental solutions for fractional evolution equations corresponding to random time changes in the Brownian motion by the inverse process \(E(t)\). We consider
three classes of time change, namely those corresponding to the $\theta$-stable subordinator, the distributed order derivative, and the class of Stieltjes functions. Henceforth $L$ always denotes a slowly varying function (SVF) (see for instance [BGT87] and [SSV12]), while $C$, $C'$ are constants whose values are unimportant, and which may change from line to line.

Let $v(x,t)$ be the fundamental solution of the heat equation

$$
\begin{cases}
\frac{\partial u(x,t)}{\partial t} = \Delta u(x,t) \\
u(x,0) = \delta(x),
\end{cases}
$$

(3.1)

where $\Delta$ denotes de Laplacian in $\mathbb{R}^d$. It is well known that the solution $v(x,t)$ of (3.1), called heat kernel (also known as Green function), is given by

$$
v(x,t) = \frac{1}{(2\pi t)^{d/2}} e^{-\frac{|x|^2}{4t}}
$$

(3.2)

and the associated stochastic process is classical Brownian motion in $\mathbb{R}^d$. We recall that the heat kernel $v(x,t)$ has the following long-time behavior

$$
v(x,t) \sim Ct^{-d/2}, \text{ as } t \to \infty.
$$

We are interested in studying the long-time behavior of the subordination of the solution $v(x,t)$ by the density $G_t(\tau)$, in terms of the function $v^E(x,t)$ defined, for any $x \in \mathbb{R}^d$ and $t \geq 0$, by

$$
v^E(x,t) := \int_0^\infty v(x,\tau)G_t(\tau) \, d\tau = \frac{1}{(2\pi t)^{d/2}} \int_0^\infty \tau^{-d/2} e^{-\frac{|x|^2}{4\tau}} G_t(\tau) \, d\tau.
$$

(3.3)

Then $v^E(x,t)$ is the fundamental solution of the general fractional time differential equation, that is,

$$
\begin{cases}
\mathbb{D}_t^{(k)} u(x,t) = \Delta u(x,t) \\
u(x,0) = \delta(x).
\end{cases}
$$

(3.4)

Here the $\mathbb{D}_t^{(k)}$ are differential-convolution operators defined, for any nonnegative kernel $k \in L^1_{\text{loc}}(\mathbb{R}_+)$, by

$$
(\mathbb{D}_t^{(k)} u)(t) := \frac{d}{dt} \int_0^t k(t-\tau)u(\tau) \, d\tau - k(t)u(0), \quad t > 0.
$$

(3.5)
In order to study the time evolution of $v^E(x, t)$, one possibility is to define its Cesaro mean

$$M_t(v^E(x, t)) := \frac{1}{t} \int_0^t v^E(x, s) \, ds,$$

which may be written as

$$M_t(v^E(x, t)) = \int_0^\infty v(x, \tau) M_t\left(G_t(\tau)\right) d\tau.$$  \hspace{1cm}(3.6)

The long-time behavior of the Cesaro mean $M_t(v^E(x, t))$ was investigated in [KKdS19, Sec. 3] for the three classes of admissible kernels and $d \geq 3$. The method was based on the ratio Tauberian theorem from [LSS07]. In the next section we use an alternative method to find the long-time behavior of the Cesaro mean $M_t(v^E(x, t))$.

4 Alternative Method for Subordinated Heat Kernel

The Laplace transform method is based on the result of Lemma 3 wherein the $t$-Laplace transform of the subordination $v^E(x, t)$ is explicitly given by

$$(\mathcal{L} v^E(x, \cdot))(\lambda) = C \int_0^\infty \tau^{-d/2} e^{-\frac{|x|^2}{4\tau}} (\mathcal{L} G(\tau))(\lambda) \, d\tau = CK(\lambda) \int_0^\infty \tau^{-d/2} e^{-\frac{|x|^2}{4\tau} - \tau \lambda K(\lambda)} \, d\tau.$$ \hspace{1cm}(4.1)

The integral in (4.1) is computed according to the formula,

$$\int_0^\infty \tau^{-d/2} e^{-\frac{|x|^2}{4\tau} - \tau b} \, d\tau = \begin{cases} \sqrt{\pi} e^{-2\sqrt{ab}/\sqrt{b}}, & d = 1, \\ 2K_0\left(2\sqrt{ab}\right), & d = 2, \\ 2 \left(\frac{x}{b}\right)^{(2-d)/4} K_{d/2-1}\left(2\sqrt{ab}\right), & d \geq 3. \end{cases}$$

(see for instance [EMOT54, page 146, eqs. (27), (29)]), where $a = \frac{|x|^2}{4\tau}$, $b = \lambda K(\lambda)$, and $K_\nu(z)$ is the modified Bessel function of the second kind [AS92].
The asymptotic of the Bessel function $K_0(z)$ as $z \to 0$ is well known (e.g., see [AS92, Eqs. (9.6.8) and (9.6.9)]) and is given by

\[ K_0(z) \sim -\ln(z), \quad (4.2) \]

\[ K_\nu(z) \sim \frac{1}{2} \Gamma(\nu) \left(\frac{z}{2}\right)^{-\nu} \sim Cz^{-\nu}, \quad \Re(\nu) > 0. \quad (4.3) \]

With these explicit formulas, we study each class (C1), (C2), and (C3) separately.

(C1). For this class, $K(\lambda) = \lambda^{\theta-1}, 0 < \theta < 1$.

1. For $d = 1$, we obtain

\[ (\mathcal{L} v^E(x, \cdot))(\lambda) = C\lambda^{-1+\theta/2}e^{-\sqrt{2}|x|\lambda^{\theta/2}} = \lambda^{-(1-\theta/2)}L\left(\frac{1}{\lambda}\right), \]

where $L(y) = Ce^{-\sqrt{2}|y|y^{-\theta/2}}$ is a slowly varying function (SVF). Accordingly, application of the Karamata-Tauberian theorem [BGT87] and [Sen76, Sec. 2.2] gives

\[ M_t(u(x, t)) \sim Ct^{-\theta/2}e^{-\sqrt{2}|x|t^{-\theta/2}} \sim Ct^{-\theta/2}, \quad t \to \infty. \]

2. For $d = 2$, we have

\[ (\mathcal{L} v^E(x, \cdot))(\lambda) = C\lambda^{-(1-\theta)}K_0(\sqrt{2}|x|\lambda^{\theta/2}) = \lambda^{-(1-\theta)}L\left(\frac{1}{\lambda}\right), \]

where $L(y) = CK_0(\sqrt{2}|y|y^{-\theta/2})$ is a SVF. Invoking the Karamata-Tauberian theorem and (4.2) yields, for $t \to \infty$,

\[ M_t(u(x, t)) \sim Ct^{-\theta}K_0(\sqrt{2}|x|t^{-\theta/2}) \sim Ct^{-\theta}L(\sqrt{2}|x|t^{-\theta/2}). \]

3. For $d \geq 3$, the Laplace transform of $v^E(x, t)$ has the form

\[ (\mathcal{L} v^E(x, \cdot))(\lambda) = C|x|^{(2-d)/2}\lambda^{-(1-\theta)} \left(\frac{1}{\lambda}\right)^{(2-d)/4} K_{\frac{d}{2}-1}(\sqrt{2}|x|\lambda^{\theta/2}) \]

\[ \quad = \lambda^{-(1-\theta)}L\left(\frac{1}{\lambda}\right), \]

where $L(y) = C|x|^{(2-d)/2}y^{\theta(2-d)/4}K_{\frac{d}{2}-1}(\sqrt{2}|x|y^{-\theta/2})$ is a SVF. It then follows from the Karamata-Tauberian theorem and (4.3) that

\[ M_t(u(x, t)) \sim Ct^{-\theta}L(t) \sim C|x|^{(\theta+1)(2-d)/2}t^{-\theta}, \quad t \to \infty. \]
Here we have $K(\lambda) \sim \lambda^{-1}L(\lambda^{-1})$ as $\lambda \to 0$, where $L(y) = \mu(0)\log(y)^{-1}$, $\mu(0) \neq 0$. Again we study the three different cases $d = 1$, $d = 2$ and $d \geq 3$.

1. For $d = 1$, the $t$-Laplace transform of $v^E(x,t)$ can be written, for $\lambda \to 0$, as

$$
(\mathcal{L}v^E(x,\cdot))(\lambda) = C\lambda^{-1} \log(\lambda^{-1})^{-1/2} e^{-\sqrt{2\mu(0)}|x| \log(\lambda^{-1})^{-1/2}}
$$

$$
= \lambda^{-1}L\left(\frac{1}{\lambda}\right),
$$

where $L(y) = C\log(y)^{-1/2} e^{-\sqrt{2\mu(0)}|x| \log(y)^{-1/2}}$ is a SVF. Application of the Karamata-Tauberian theorem then gives

$$
M_t(u(x,t)) \sim CL(t) \sim C\log(t)^{-1/2} e^{-\sqrt{2\mu(0)}|x| \log(t)^{-1/2}}, \quad t \to \infty.
$$

2. If $d = 2$, we have

$$
(\mathcal{L}v^E(x,\cdot))(\lambda) = C\lambda^{-1} \log(\lambda^{-1})^{-1/2} K_0\left(\sqrt{2\mu(0)}|x| \log(\lambda^{-1})^{-1/2}\right)
$$

$$
= \lambda^{-1}L\left(\frac{1}{\lambda}\right),
$$

where $L(y) = C\log(y)^{-1} K_0\left(\sqrt{2\mu(0)}|x| \log(y)^{-1/2}\right)$ is a SVF. As $t \to \infty$ the Karamata-Tauberian theorem and (4.2) then yield

$$
M_t(u(x,t)) \sim CL(t) \sim C\log(t)^{-1} \ln\left(\sqrt{2\mu(0)}|x| \log(t)^{-1/2}\right).
$$

3. For $d \geq 3$, it follows that, as $\lambda \to 0$,

$$
(\mathcal{L}v^E(x,\cdot))(\lambda) = C|x|^{(2-d)/2} \lambda^{-1} \log(\lambda^{-1})^{-1+(2-d)/4}
\times K_{\frac{d}{2}-1}\left(C'|x| \log(\lambda^{-1})^{-1/2}\right)
$$

$$
= \lambda^{-1}L\left(\frac{1}{\lambda}\right),
$$

where $L(y) = C|x|^{(2-d)/2} \log(y)^{-1+(2-d)/4} K_{\frac{d}{2}-1}\left(C'|x| \log(y)^{-1/2}\right)$ is a SVF. To verify that $L(y)$ is a SVF, one may note that $\log(y)^{-1+(2-d)/4}$ as well as is $K_{\frac{d}{2}-1}\left(C'|x| \log(y)^{-1/2}\right)$ according to (4.3); the stated
result then follows from Prop. 1.3.6 in [BGT87]. It follows in turn from the Karamata-Tauberian theorem and (4.3) that

\[ M_t(u(x,t)) \sim C L(t) \sim C|x|^{2-d} \log(t)^{-1}, \quad t \to \infty. \]

(C3) We now have \( \mathcal{K}(\lambda) \sim C \lambda^{-1} L(\lambda^{-1})^{-1-s} \), as \( \lambda \to 0 \) and \( s > 0 \), \( C > 0 \).

1. For \( d = 1 \), the \( t \)-Laplace transform of \( v^E(x,t) \) can be written, for \( \lambda \to 0 \), as

\[
(\mathcal{L} v^E(x,\cdot))(\lambda) = C \lambda^{-1} \log(\lambda^{-1})^{-(1+s)/2} e^{-C' \sqrt{2} |x| \log(\lambda^{-1})^{-(1+s)/2}} \\
= \lambda^{-1} L \left( \frac{1}{\lambda} \right),
\]

where \( L(y) = C \log(y)^{-(1+s)/2} e^{-C' \sqrt{2} |x| \log(y)^{-(1+s)/2}} \) is a SVF, as is easily seen. Application of the Karamata-Tauberian theorem gives, as \( t \to \infty \)

\[
M_t(u(x,t)) \sim C L(t) \sim C \log(t)^{-(1+s)/2} e^{-C' \sqrt{2} |x| \log(t)^{-(1+s)/2}}.
\]

2. For \( d = 2 \), we have

\[
(\mathcal{L} v^E(x,\cdot))(\lambda) = C \lambda^{-1} \log(\lambda^{-1})^{-1-s} K_0 \left( C' |x| \log(\lambda^{-1})^{-(1+s)/2} \right) \\
= \lambda^{-1} L \left( \frac{1}{\lambda} \right),
\]

where

\[
L(y) = C \log(y)^{-1-s} K_0 \left( C' |x| \log(y)^{-(1+s)/2} \right)
\]

is a SVF. Use of the Karamata-Tauberian theorem and (4.2) now yield the behavior

\[
M_t(u(x,t)) \sim C L(t) \sim C \log(t)^{-1-s} \ln \left( C' |x| \log(t)^{-(1+s)/2} \right).
\]

as \( t \to \infty \).

3. For \( d \geq 3 \), it follows that

\[
(\mathcal{L} v^E(x,\cdot))(\lambda) = C |x|^{(2-d)/2} \lambda^{-1} \log(\lambda^{-1})^{-(1+s)(1-(2-d)/4)} \\
\times K_{\frac{d}{2}-1} \left( C' \sqrt{2} |x| \log(\lambda^{-1})^{-(1+s)/2} \right) \\
= \lambda^{-1} L \left( \frac{1}{\lambda} \right),
\]

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as \( t \to \infty \), where
\[
L(y) = C|x|^{(2-d)/2} \log(y)^{-(1+s)(2+d)/4} K_{\frac{d}{2}-1} \left( C' \sqrt{2} |x| \log(y)^{-(1+s)/2} \right)
\]
is a SVF. We note that \( L(y) \) is the product of two SVF’s which is in turn a SVF (see Prop. 1.3.6 in [BGT87]). It follows from the Karamata-Tauberian theorem and (4.3) that
\[
M_t(u(x, t)) \sim C L(t) \sim C|x|^{2-d} \log(t)^{-1-s}, \quad t \to \infty.
\]

**Remark 4 (Gaussian convolution kernel).** We consider the nonlocal operator \( \mathcal{L} \) on functions \( u : \mathbb{R}^d \to \mathbb{R} \) defined in integral form by
\[
(\mathcal{L}u)(x) := (a * u)(x) - u(x) = \int_{\mathbb{R}^d} a(x - y)[u(y) - u(x)] dy,
\]
where the convolution kernel \( a \) is non-negative, symmetric, bounded, and integrable, i.e.,
\[
a(x) \geq 0, \quad a(x) = a(-x), \quad a(x) \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d).
\]
In addition, the kernel \( a \) is a density in \( \mathbb{R}^d \) with finite second moment; explicitly
\[
\langle a \rangle := \int_{\mathbb{R}^d} a(x) dx = 1, \quad \int_{\mathbb{R}^d} |x|^2 a(x) dx < \infty.
\]
Since \( \mathcal{L} \) is a bounded operator in \( L^2(\mathbb{R}^d) \), its heat semigroup \( e^{t\mathcal{L}} \) can be easily computed by using the exponential series according to
\[
e^{t\mathcal{L}} = e^{-t} e^{ta} = e^{-t} \sum_{k=0}^{\infty} \frac{t^k a^{*k}}{k!} = e^{-t} \text{Id} + e^{-t} \sum_{k=1}^{\infty} \frac{t^k a^{*k}}{k!}.
\]
By removing the singular part \( e^{-t} \text{Id} \) of the heat semigroup, we obtain the regularized heat kernel
\[
v(x, t) = e^{-t} \sum_{k=1}^{\infty} \frac{t^k a^{*k}(x)}{k!},
\]
with the source at the origin. In other words, for any \( f \in L^2(\mathbb{R}^d) \), a solution to the nonlocal Cauchy problem
\[
\begin{cases}
\frac{\partial u(x, t)}{\partial t} = \mathcal{L}u(x, t), \\
u(x, 0) = f(x),
\end{cases}
\]
(4.8)
has the form \( u(x,t) = e^{-t}f(x) + (v * f)(x,t) \) with \( v \) given by (4.7). In particular, the fundamental solution of the problem (4.8) is

\[
u(x,t) = e^{-t} \delta(x) + v(x,t).
\]

For any \( r > 0 \), if \(|x| \leq rt^{1/2}\), then the following asymptotic for \( v(x,t) \) holds as \( t \to \infty \)

\[
v(x,t) = \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|x|^2}{4t}} (1 + o(t^{-1/4})),
\]

see [GKPZ18, Thm. 2.1]. If we denote by \( v^E(x,t) \) the subordination of \( v(x,t) \) by the density \( G_t(\tau) \), then the Cesaro mean of \( v^E(x,t) \) has long time behavior given as above for the different classes of admissible kernels \( k \). A similar result is also true if the kernel \( a \) has a generic light tails, see Remark 12

5 Other Heat Kernel Estimates

5.1 Stable Subordinator

5.1.1 The Local Dirichlet Form

Assume that \( X \) is a \( m \)-symmetric Hunt process associated with a regular local Dirichlet form \((\mathcal{E}, \mathcal{F})\) on \( L^2(\mathbb{R}^d, m) \) and that it has a heat kernel \( v(x,t) \) with respect to the measure \( m \). Then it follows from [GK08, Thm. 4.1] that the heat kernel \( v(x,t) \) satisfies the following estimate, with \( \alpha \geq 2 \),

\[
v(x,t) \leq Ct^{-d/\alpha} \exp \left(-|x|^\alpha/(\alpha-1)t^{-1/(\alpha-1)} \right).
\]

**Theorem 5.** Let \( v^E(x,t) \) be the subordination of \( v(x,t) \) by the kernel \( G_t(\tau) \) corresponding to the \( \theta \)-stable subordinator. The long-time behavior of the Cesaro mean of \( v^E(x,t) \) as \( t \to \infty \) is then given by

\[
M_t(v^E(x,t)) \sim \begin{cases} 
Ct^{-\theta d/\alpha}, & d < \alpha, \\
Ct^{-\theta} \log \left(2(|x|^\theta t^{-\theta/\alpha})^{-1}\right), & d = \alpha, \\
Ct^{-\theta} |x|^\alpha - d, & d > \alpha.
\end{cases}
\]

**Proof.** Recall the expression of the subordination, i.e.,

\[
v^E(x,t) = \int_0^\infty v(x,\tau)G_t(\tau) \, d\tau.
\]
Computing its $t$-Laplace transform and using (5.1) yields
\[
(L v^E(x, \cdot))(\lambda) = K(\lambda) \int_0^\infty v(x, \tau) e^{-\tau\lambda K(\lambda)} d\tau \\
\leq C K(\lambda) \int_0^\infty \tau^{-\mu} e^{-\alpha \tau - \nu \tau} d\tau \\
= C K(\lambda) a^{(1-\mu)/\nu} \int_0^\infty \tau^{-\mu} e^{-\tau^{1-\alpha} a^{1/\nu} b^\tau} d\tau
\] (5.2)

where $a = |x|^{\alpha \nu}$, $b = \lambda K(\lambda)$, $\nu = 1/(\alpha - 1)$, and $\mu = d/\alpha$. The last equality is obtained by a change of variable $\tau = a^{1/\nu} z$. Noting that since $\alpha \geq 2$ we have $e^{-\tau^{1-\nu}} \leq e^{-\tau^{1-1}}$, the integral in (5.2) may be estimated by
\[
\int_0^\infty \tau^{-\mu} e^{-\tau^{1-\alpha} a^{1/\nu} b^\tau} d\tau \leq \int_0^\infty \tau^{-\mu} e^{-\tau^{1-1} a^{1/\nu} b^\tau} d\tau.
\] (5.3)
The integral on the right hand side can be computed according to the ratio $\mu = d/\alpha$, with the results
\[
\int_0^\infty \tau^{-\mu} e^{-\tau^{1-1} a^{1/\nu} b^\tau} d\tau = \begin{cases} (a^{1/\nu} b)^{\mu-1} \Gamma(1 - \mu), & d < \alpha, \\
2K_0 \left(2(a^{1/\nu} b)^{1/2}\right), & d = \alpha, \\
2(a^{1/\nu} b)^{\mu^2} K_{\mu-1} \left(2(a^{1/\nu} b)^{1/2}\right), & d > \alpha.
\end{cases}
\]
Here $K_\zeta$ is the modified Bessel function of the second kind. We thus find a bound for $(L v^E(x, \cdot))(\lambda)$, namely
\[
(L v^E(x, \cdot))(\lambda) \leq \begin{cases} C \lambda^{\mu-1} K(\lambda)^{\mu}, & d < \alpha, \\
C K(\lambda) K_0 \left(2(|x|^\alpha \lambda K(\lambda))^{1/2}\right), & d = \alpha, \\
C|x|^{(\alpha-d)/2} K(\lambda)^{(\mu+1)/2} K_{\mu-1} \left(2(|x|^\alpha \lambda K(\lambda))^{1/2}\right), & d > \alpha.
\end{cases}
\] (5.4)

We now study each case separately.

**|$d < \alpha|$.** It follows that
\[
(L v^E(x, \cdot))(\lambda) \leq C \lambda^{-(1-\theta \mu)} = \lambda^{-\rho} L \left(\frac{1}{\lambda}\right), \quad \lambda \to 0,
\]
where $\rho = 1 - \theta \mu$ and $L(y) = C$ is a trivial SVF. Therefore, by the Karamata-Tauberian theorem it follows that
\[
M_t(v^E(x, t)) \sim Ct^{-\theta \mu} L(t) = Ct^{-\theta d/\alpha}.
\]
We have
\[(\mathcal{L}v^E(x, \cdot))(\lambda) \leq C\lambda^{-(1-\theta)}K_0 \left(2(|x|^\alpha \lambda^{\theta})^{1/2}\right) = \lambda^{-\rho}L \left(\frac{1}{\lambda}\right),\]
where \(\rho = 1 - \theta\) and \(L(y) = 2K_0 \left(2(|x|^\alpha y^{-\theta})^{1/2}\right)\) is a SVF. Then by the Karamata-Tauberian theorem it follows that
\[M_t(v^E(x, t)) \sim Ct^{-\theta} \log \left((2(|x|^{-\theta/\alpha})^{\alpha/2})^{-1}\right) \sim Ct^{-\theta} \log \left((2(|x|^{-\theta/\alpha})^{1/2})\right).

\[d = \alpha\] We may write
\[(\mathcal{L}v^E(x, \cdot))(\lambda) \leq 2|x|^{(\alpha-d)/2} \lambda^{-(1-\theta)}\lambda^{\theta(\mu-1)/2}K_{\mu-1} \left(2(|x|^\alpha \lambda^{\theta})^{1/2}\right) = \lambda^{-\rho}L \left(\frac{1}{\lambda}\right),\]
where \(\rho = 1 - \theta\) and \(L(y) = 2|x|^{(\alpha-d)/2}y^{-\theta(\mu-1)/2}K_{\mu-1} \left(2(|x|^\alpha y^{-\theta})^{1/2}\right)\) is a SVF. It follows from the Karamata-Tauberian theorem and (4.3) that
\[M_t(v^E(x, t)) \sim Ct^{-\theta}L(t) \sim Ct^{-\theta}|x|^{\alpha-d}.\]

Remark 6. The result of Theorem 5 is in agreement with those obtained in [CKKW18, Thm. 1.3].

5.1.2 The Pure Jump Case
Let us now assume that the Dirichlet form \((\mathcal{E}, \mathcal{F})\) is of pure jump type, then the heat kernel \(v(x, t)\) satisfies the estimate
\[v(x, t) \leq Ct^{-d/\alpha} \left(1 + |x|^{-1/\alpha}\right)^{-(d+\alpha)},\]
with \(\alpha \geq 2\) (cf. [GK08, Thm. 4.1]).

Theorem 7. Let \(v^E(x, t)\) be the subordination of \(v(x, t)\) by the kernel \(G_t(\tau)\) corresponding to the \(\theta\)-stable subordinator. Then for the long-time behavior of the Cesaro mean of \(v^E(x, t)\) we have:

1. Assume that \(|x|^{-\theta/\alpha} \leq 1\); then
   \[M_t(v^E(x, t)) \sim \begin{cases} Ct^{-d/\alpha}, & d < \alpha, \\ Ct^{-\theta} \log \left(2(|x|^{-\theta/\alpha})^{-1}\right), & d = \alpha, \\ C|x|^{\alpha-d}t^{-\theta}, & d > \alpha. \end{cases}\]
2. If $|x|t^{-\theta/\alpha} \geq 1$, then

$$M_t(v^E(x, t)) \sim C|x|^{-d-\alpha}t^\theta.$$  

Proof. It follows that the Laplace transform of $v^E(x, t)$ is given by

$$(Lv^E(x, \cdot))(\lambda) = \mathcal{K}(\lambda) \int_0^\infty v(x, \tau)e^{-\tau\mathcal{K}(\lambda)}d\tau$$

$$\leq C\mathcal{K}(\lambda) \int_0^\infty \tau^{-d/\alpha}(1 + |x|\tau^{-1/\alpha})^{-d-\alpha}e^{-\tau\mathcal{K}(\lambda)}d\tau$$

$$= C|x|^{\alpha-d}\mathcal{K}(\lambda) \int_0^\infty \tau^{-d/\alpha}(1 + \tau^{-1/\alpha})^{-d-\alpha}e^{-\tau|x|^{\alpha\lambda\mathcal{K}(\lambda)}}d\tau,$$

where the last equality is obtained with the change of variables $\tau = |x|^{\alpha}z$.

For $\mathcal{K}(\lambda) = \lambda^{\theta-1}$, the Laplace transform $(Lv^E(x, \cdot))(\lambda)$ has the form

$$(Lv^E(x, \cdot))(\lambda) = C|x|^{\alpha-d}\lambda^{-(1-\theta)} \int_0^\infty \tau^{-d/\alpha}(1 + \tau^{-1/\alpha})^{-d-\alpha}e^{-\tau|x|^{\theta/\alpha}\lambda}d\tau.$$  

(5.6)

1. Let us assume that $|x|t^{-\theta/\alpha} \leq 1$ as $t \to \infty$, which is equivalent to $|x|\lambda^{\theta/\alpha} \leq 1$ as $\lambda \to 0$. The integral in the right hand side of (5.6) is written as

$$\int_0^1 \tau^{-d/\alpha}(1 + \tau^{-1/\alpha})^{-d-\alpha}e^{-\tau|x|^{\theta/\alpha}\lambda}d\tau$$

$$+ \int_1^\infty \tau^{-d/\alpha}(1 + \tau^{-1/\alpha})^{-d-\alpha}e^{-\tau|x|^{\theta/\alpha}\lambda}d\tau$$

$$=: I_1(x, \lambda) + I_2(x, \lambda).$$

For the integral $I_1(x, \lambda)$ we have the estimate

$$I_1(x, \lambda) \leq \int_0^1 \tau^{-d/\alpha}(1 + \tau^{-1/\alpha})^{-d-\alpha}d\tau \leq \int_0^1 \tau d\tau = \frac{1}{2}.$$  

(5.7)

For the integral $I_2(x, \lambda)$ we distinguish three cases:

(a) For $d > \alpha$, $I_2(x, \lambda)$ is finite, more precisely

$$I_2(x, \lambda) \leq C \int_1^\infty \tau^{-d/\alpha}(1 + \tau^{-1/\alpha})^{-d-\alpha}d\tau$$

$$= C\frac{\alpha_2F_1(d - \alpha, d + \alpha; d - \alpha + 1; -1)}{d - \alpha} < \infty,$$  

(5.8)
where \( _2F_1 \) is the hypergeometric function \([OLBC10, \text{Ch. 15}]\). Therefore, for \( d > \alpha \) we have

\[
(L^\nu^E(x, \cdot))(\lambda) \leq C |x|^{\alpha-d} \lambda^{-(1-\theta)} = \lambda^{-\rho} L \left( \frac{1}{\lambda} \right),
\]

from which follows that

\[
M_t(v^E(x, t)) \sim C |x|^{\alpha-d} t^{-\theta}.
\]

(b) For \( d < \alpha \), we drop the term which has no influence on the asymptotic and write

\[
\int_0^\infty \tau^{-d/\alpha} (1 + \tau^{-1/\alpha})^{-d-\alpha} e^{-\tau |x|^{\theta/\alpha} \alpha} \, d\tau \leq \int_0^\infty \tau^{-d/\alpha} e^{-\tau |x|^{\theta/\alpha} \alpha} \, d\tau
\]

\[
= |x|^{-\alpha} \int_0^\infty \tau^{-d/\alpha} e^{-\tau \lambda^\theta} \, d\tau
\]

\[
= |x|^{-\alpha} \lambda^{\theta(d/\alpha-1)} \Gamma(1 - d/\alpha),
\]

where the second from the last equality is obtained by the change of variables \( z = |x|^{-\alpha} \tau \). We then find that the Laplace transform \((L^\nu^E(x, \cdot))(\lambda)\) allows the estimate

\[
(L^\nu^E(x, \cdot))(\lambda) \leq C \lambda^{-(1-\theta d/\alpha)} = \lambda^{-\rho} L \left( \frac{1}{\lambda} \right),
\]

and we have

\[
M_t(v^E(x, t)) \sim C t^{-\theta d/\alpha}.
\]

(c) For \( d = \alpha \) the integral \( I_1(x, \lambda) \) is finite, as seen from (5.7). For the integral \( I_2(x, \lambda) \) we obtain

\[
I_2(x, \lambda) \leq C \int_1^\infty \tau^{-1} e^{-\tau |x|^{\theta/\alpha} \alpha} \, d\tau
\]

\[
= C \int_{|x|^{\alpha} \lambda^{\theta}}^\infty \tau^{-1} e^{-\tau} \, d\tau
\]

\[
= -C \text{Ei}(-|x|^{\alpha} \lambda^{\theta}),
\]

where \( \text{Ei} \) is the exponential integral \([GR14, 8.211]\). It follows from the properties of \( \text{Ei} \) \([GR14, 8.214]\) that

\[
I_2(x, \lambda) \leq C \log(|x|^{\theta/\alpha}), \quad \lambda \to 0,
\]

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which implies the estimate
\[
(\mathcal{L} v^E(x, \cdot))(\lambda) \leq C \lambda^{-(1-\theta)} \left( \frac{1}{2} + \log(|x|^{\theta/\alpha}) \right)
\]
\[
= \lambda^{-(1-\theta)} L \left( \frac{1}{\lambda} \right).
\]
for the Laplace transform, where \( L(y) = C \left( \frac{1}{2} + \log(|x|^{\theta/\alpha}) \right) \) is a SVF. Thus, it follows from the Karamata-Tauberian theorem that
\[
M_t(v^E(x, t)) \sim C t^{-\theta} \left( \frac{1}{2} + \log(|x|^{\theta/\alpha}) \right).
\]
\[
\sim C t^{-\theta} \log \left( 2(|x|^{\theta/\alpha})^{-1} \right)
\]

2. If \(|x|^{\theta/\alpha} \geq 1\) as \(t \to \infty\) which implies that \(|x|^{\theta/\alpha} \geq 1\) as \(\lambda \to 0\), we have
\[
(\mathcal{L} v^E(x, \cdot))(\lambda) = |x|^{\alpha-d} \lambda^{-(1-\theta)} \int_0^\infty \tau^{-d/\alpha} \left(1 + \tau^{-1/\alpha}\right)^{-d-\alpha} e^{-\tau(|x|^{\theta/\alpha})} \, d\tau
\]
and observe that as \(\tau \to 0\) we have
\[
\tau^{-d/\alpha} \left(1 + \tau^{-1/\alpha}\right)^{-d-\alpha} \sim \tau = \tau^{\frac{2+1}{\alpha}}.
\]
It follows from the Watson lemma [Olv74, Ch. 3] that
\[
\int_0^\infty \tau^{-d/\alpha} \left(1 + \tau^{-1/\alpha}\right)^{-d-\alpha} e^{-\tau(|x|^{\theta/\alpha})} \, d\tau \sim |x|^{-2\alpha \lambda - 2\theta}, \ \lambda \to 0
\]
which implies
\[
(\mathcal{L} v^E(x, \cdot))(\lambda) = |x|^{-\alpha-d} \lambda^{-(1+\theta)} = \lambda^{-\rho L \left( \frac{1}{\lambda} \right)}, \ \lambda \to 0.
\]
Accordingly, we arrive at the long-time behavior for the Cesaro mean of \(v^E(x, t)\), namely
\[
M_t(v^E(x, t)) \sim C |x|^{-d-\alpha t^\theta}.
\]

Remark 8. The results of Theorem 7 are in agreement with those obtained by [CKKW18, Thm. 1.3].
5.2 General Case

We investigate the long time behavior of the subordinated solution $v(x,t)$ by the kernel $G_t(\tau)$ for a general class of subordinators $S_t$, $t > 0$ having Laplace exponent $\Phi(\lambda) = \lambda \mathcal{K}(\lambda)$ with $\mathcal{K}$ defined in (2.4). The following condition is to be satisfied:

(H) The function $\mathcal{K}(\lambda)$ exists for all $\lambda > 0$ and belongs to the Stieltjes class $\mathcal{S}$ (or equivalently, the function $\Phi$ belongs to the complete Bernstein function class $\mathcal{CBF}$ (see [SSV12] for definitions), and

$$\mathcal{K}(\lambda) \to \infty \text{ as } \lambda \to 0, \quad \mathcal{K}(\lambda) \to 0 \text{ as } \lambda \to \infty, \quad (5.9)$$

$$\Phi(\lambda) \to 0 \text{ as } \lambda \to 0, \quad \Phi(\lambda) \to \infty \text{ as } \lambda \to \infty. \quad (5.10)$$

In addition, assume that $\Re+ \ni \lambda \mapsto \Phi(\lambda^{-1}) = \lambda^{-1}\mathcal{K}(\lambda^{-1})$ is a SVF.

Remark 9. An example of a function $\mathcal{K}$ satisfying the condition (H) is given by the distributed order derivative, which, as $\lambda \to 0$, behaves as

$$\mathcal{K}(\lambda) \sim \lambda^{-1} \log \left( \frac{1}{\lambda} \right)^{-1}$$

or

$$\mathcal{K}(\lambda) \sim C\lambda^{-1} \log \left( \frac{1}{\lambda} \right)^{-1-s}, \quad s, C > 0.$$}

For more details, see [Koc08].

5.2.1 The Local Dirichlet Form

We recall that in this case the Laplace transform of the subordination $v^E(x,\cdot)$ has the estimate

$$(\mathcal{L}v^E(x,\cdot))(\lambda) \leq C\mathcal{K}(\lambda) \int_0^\infty \tau^{-\mu}e^{-|x|^\alpha \tau^{\nu-\nu-\tau} \Phi(\lambda)} d\tau, \quad (5.11)$$

where $\mu = d/\alpha$ and $\nu = 1/(\alpha - 1)$, $\alpha \geq 2$.

Theorem 10. Let (H) be satisfied and $v^E(x,t)$ the function that verifies (5.11). Then the long-time behavior of the Cesaro mean of $v^E(x,t)$ is given by

$$M_t(v^E(x,t)) \sim \begin{cases} 
C\Phi(t^{-1})^{d/\alpha}, & d < \alpha, \\
C\Phi(t^{-1}) \log \left( \frac{2}{|x|\Phi(t^{-1})^{1/\alpha}} \right), & d = \alpha, \\
C|x|^{\alpha-d}\Phi(t^{-1}), & d > \alpha.
\end{cases}$$
Proof. 1. Case $d < \alpha$. It follows from (5.11) with a change of variable $\tau = |x|^\alpha z$ that

$$\mathcal{L} v^E(x, \cdot)(\lambda) = C|x|^{\alpha - d} K(\lambda) \int_0^\infty \tau^{-\mu} e^{-\tau^\mu - |x|^\alpha \lambda K(\lambda)} d\tau$$

$$\leq C|x|^{\alpha - d} K(\lambda) \int_0^\infty \tau^{-\mu} e^{-|x|^\alpha \lambda K(\lambda)} d\tau$$

and the integral on the right hand side is given explicitly by

$$\int_0^\infty \tau^{-\mu} e^{-|x|^\alpha \lambda K(\lambda)} d\tau = \Gamma(\mu - 1) (|x|^\alpha \lambda K(\lambda))^{\mu - 1}$$

Thus, we obtain

$$\mathcal{L} v^E(x, \cdot)(\lambda) \leq C \lambda^{-1} \Phi(\lambda) = \lambda^{-1} L \left( \frac{1}{\lambda} \right),$$

where $L(y) = C \Phi(y^{-1})^\mu$ is a SVF using (H) and Prop. 1.3.6 in [BGT87]. Therefore, by the Karamata-Tauberian theorem we have

$$M_t(v^E(x, t)) \sim C L(t) = C \Phi(t^{-1})^{d/\alpha}, \ t \to \infty.$$

2. Case $d > \alpha$. We may estimate $(\mathcal{L} v^E(x, \cdot))(\lambda)$ by

$$\mathcal{L} v^E(x, \cdot)(\lambda) = C K(\lambda) \int_0^\infty \tau^{-\mu} e^{-|x|^\alpha \tau - \tau^\mu \lambda K(\lambda)} d\tau$$

$$\leq C K(\lambda) \int_0^\infty \tau^{-\mu} e^{-|x|^\alpha \tau} d\tau$$

and the integral on the right hand side is equal to

$$\frac{1}{\nu} \Gamma \left( \frac{\mu - 1}{\nu} \right) (|x|^\alpha)^{(1-\mu)/\nu}.$$

Therefore, we obtain

$$\mathcal{L} v^E(x, \cdot)(\lambda) \leq C|x|^{\alpha - d} \lambda^{-1} \Phi(\lambda) = \lambda^{-1} L \left( \frac{1}{\lambda} \right),$$

where $L(y) = C|x|^{\alpha - d} \Phi(y^{-1})$ is a SVF. As a consequence of the Karamata-Tauberian theorem we derive

$$M_t(v^E(x, t)) \sim C L(t) = C L(t) = C|x|^{\alpha - d} \Phi(t^{-1}), \ t \to \infty.$$
3. Case $d = \alpha$. We have

\[
(\mathcal{L} v^E(x, \cdot))(\lambda) = C\mathcal{K}(\lambda) \int_0^\infty \tau^{-1} e^{-|x|^\alpha \tau^{-\nu} - \tau \lambda \mathcal{K}(\lambda)} \, d\tau
\]

\[
= C\mathcal{K}(\lambda) \int_0^\infty \tau^{-1} e^{-\tau^{-\nu} - \tau |x|^\alpha \lambda \mathcal{K}(\lambda)} \, d\tau
\]

\[
\leq C\mathcal{K}(\lambda) \int_0^\infty \tau^{-1} e^{-\tau^{-1} - \tau |x|^\alpha \lambda \mathcal{K}(\lambda)} \, d\tau,
\]

where the last integral is computed according to

\[
\int_0^\infty \tau^{-1} e^{-\tau^{-1} - \tau |x|^\alpha \lambda \mathcal{K}(\lambda)} \, d\tau = 2K_0(2(|x|^\alpha \Phi(\lambda))^{1/2}).
\]

Using (4.2) and (5.10) we obtain the following bound as $\lambda \to 0$:

\[
(\mathcal{L} v^E(x, \cdot))(\lambda) \leq C\mathcal{K}(\lambda) \log \left(2(|x|^\alpha \Phi(\lambda))^{1/2}\right)
\]

\[
= C\lambda^{-1} \Phi(\lambda) \log \left(2(|x|^\alpha \Phi(\lambda))^{1/2}\right)
\]

\[
= \lambda^{-1} L \left(\frac{1}{\lambda}\right),
\]

where $L(y) = C\Phi(y^{-1}) \log \left(2(|x|^\alpha \Phi(y^{-1}))^{1/2}\right)$. We have assumed that $\Phi(y^{-1}) = y^{-1}K(y^{-1})$ is a SVF, which from Prop. 1.3.6 in [BGT87] guarantees that $L(y)$ is also a SVF. Application of the Karamata-Tauberian theorem gives

\[
M_t(v^E(x, t)) \sim CL(t) = C\Phi(t^{-1}) \log \left(2(|x|^\alpha \Phi(t^{-1}))^{1/2}\right)
\]

\[
\sim C\Phi(t^{-1}) \log \left(\frac{2}{|x|^\alpha \Phi(t^{-1})^{1/\alpha}}\right), \quad t \to \infty.
\]

5.2.2 The Pure Jump Case

We are now interested in studying the long-time behavior of the subordination $v^E(x, t)$ when the solution $v(x, t)$ satisfies (5.5), that is

\[
v(x, t) \leq C t^{-d/\alpha} \left(1 + |x| t^{-1/\alpha}\right)^{-(d+\alpha)}, \quad \alpha \geq 2.
\]
Theorem 11. Assume that the condition \((H)\) holds and \(v^E(x,t)\) the function such that
\[
(\mathcal{L}v^E(x,\cdot))(\lambda) \leq C|x|^{\alpha-d}\lambda^{-1}\Phi(\lambda) \int_0^\infty \tau^{-d/\alpha}(1-\tau^{-1/\alpha})^{-d-\alpha}e^{-\tau|x|^{\alpha}\Phi(\lambda)} \, d\tau.
\]
(5.12)

Then we have two alternatives:

1. Assume that \(|x|\Phi(t^{-1})^{1/\alpha} \leq 1\), then
   \[
   M_t(v^E(x,t)) \sim \begin{cases} 
   C\Phi(t^{-1})^{d/\alpha}, & d < \alpha, \\
   C\Phi(t^{-1}) \log \left( \frac{x}{\Phi(t^{-1})^{1/\alpha}} \right), & d = \alpha, \\
   C|x|^{\alpha-d}\Phi(t^{-1}), & d > \alpha.
   \end{cases}
   \]

2. If \(|x|\Phi(t^{-1})^{1/\alpha} \geq 1\), then
   \[
   M_t(v^E(x,t)) \sim \frac{C}{|x|^{d+\alpha}\Phi(t^{-1})}.
   \]

Proof. 1. Assume that \(|x|\Phi(t^{-1})^{1/\alpha} \leq 1\) as \(t \to \infty\) or equivalently \(|x|\Phi(\lambda)^{1/\alpha} \leq 1\) as \(\lambda \to 0\). The integral on the right hand side of (5.12) can be written as
\[
\int_0^1 \tau^{-d/\alpha}(1-\tau^{-1/\alpha})^{-d-\alpha}e^{-\tau|x|^{\alpha}\Phi(\lambda)} \, d\tau + \int_1^\infty \tau^{-d/\alpha}(1-\tau^{-1/\alpha})^{-d-\alpha}e^{-\tau|x|^{\alpha}\Phi(\lambda)} \, d\tau \\
= I_1(x,\lambda) + I_2(x,\lambda).
\]
For the integral \(I_1(x,\lambda)\) the following estimate easily follows,
\[
I_1(x,\lambda) \leq C \int_0^1 \tau^{-d/\alpha}(1-\tau^{-1/\alpha})^{-d-\alpha} \, d\tau \leq \int_0^1 \tau \, d\tau = \frac{1}{2}, \quad (5.13)
\]
On the other hand, we distinguish three cases for the integral \(I_2(x,\lambda)\):
(a) For \(d > \alpha\), \(I_2(x,\lambda)\) is bounded; see (5.8). Thus we have
\[
(\mathcal{L}v^E(x,\cdot))(\lambda) \leq C|x|^{\alpha-d}\lambda^{-1}\Phi(\lambda) = \lambda^{-1}L\left(\frac{1}{\lambda}\right),
\]
where \(L(y) = C|x|^{\alpha-d}\Phi(\lambda^{-1})\) is a SVF. Therefore we obtain the long-time behavior
\[
M_t(v^E(x,t)) \sim CL(t) = C|x|^{\alpha-d}\Phi(t^{-1}), \quad t \to \infty.
\]
(b) Assume that \( d < \alpha \). Then we have the following estimate for \((\mathcal{L} v^E(x, \cdot))(\lambda)\),
\[
(\mathcal{L} v^E(x, \cdot))(\lambda) \leq C|x|^{\alpha-d} \lambda^{-1} \Phi(\lambda) \int_{0}^{\infty} \tau^{-d/\alpha} e^{-\tau|x|^\alpha \Phi(\lambda)} \, d\tau
= C|x|^{\alpha-d} \lambda^{-1} \Phi(\lambda)(|x|^\alpha \Phi(\lambda))^{d/\alpha-1}
= C \lambda^{\alpha-1} \Phi(\lambda)^{d/\alpha},
\]
and it follows that
\[
(\mathcal{L} v^E(x, \cdot))(\lambda) \leq \lambda^{-1} L \left( \frac{1}{\lambda} \right),
\]
where \( L(y) = C \Phi(y^{-1})^{d/\alpha} \) is a SVF due to (H) and Prop. 1.3.6 in [BGT87]. Therefore, we obtain
\[
M_t(v^E(x, t)) = C \Phi(t^{-1})^{d/\alpha}.
\]
(c) For \( d = \alpha \), the integral \( I_2(x, \lambda) \) is finite, see (5.13). We estimate the integral \( I_2(x, \lambda) \) as follows:
\[
I_2(x, \lambda) \leq C \int_{1}^{\infty} \tau^{-1} e^{-\tau|x|^\alpha \Phi(\lambda)} \, d\tau = C \int_{|x|^\alpha \Phi(\lambda)}^{\infty} \tau^{-d/\alpha} e^{-\tau} \, d\tau
= -C \text{Ei} (-|x|^\alpha \Phi(\lambda)).
\]
From the properties of the function \( \text{Ei} \) (cf. [GR14, 8.214]), we have
\[
I_2(x, \lambda) \leq C \log (|x|^\alpha \Phi(\lambda)) = C \log (|x| \Phi(\lambda)^{1/\alpha}), \quad \lambda \to 0,
\]
which leads to the estimate
\[
(\mathcal{L} v^E(x, \cdot))(\lambda) \leq C \lambda^{-1} \Phi(\lambda) \left( \frac{1}{2} + \log (|x| \Phi(\lambda)^{1/\alpha}) \right)
= \lambda^{-1} L \left( \frac{1}{\lambda} \right),
\]
where \( L(y) = C \Phi(y^{-1}) \left( \frac{1}{2} + \log (|x| \Phi(y^{-1})^{1/\alpha}) \right) \) is a SVF based on (H) and Prop. 1.3.6 in [BGT87]. It follows from the Karamata-Tauberian theorem that
\[
M_t(v^E(x, t)) \sim C L(t) = C \Phi(t^{-1}) \left( \frac{1}{2} + \log (|x| \Phi(t^{-1})^{1/\alpha}) \right)
\sim C \Phi(t^{-1}) \log \left( \frac{2}{|x| \Phi(t^{-1})^{1/\alpha}} \right).
\]
2. If \(|x|\Phi(t^{-1})^{1/\alpha} \geq 1\) as \(t \to \infty\), which implies that \(|x|\Phi(\lambda)^{1/\alpha} \geq 1\) as \(\lambda \to 0\), we have

\[
(\mathcal{L}v^E(x, \cdot))(\lambda) \leq C|x|^{\alpha-d}\lambda^{-1}\Phi(\lambda) \int_0^\infty \tau^{-d/\alpha}(1 - \tau^{-1/\alpha})^{d-\alpha} e^{-\tau(x|\Phi(\lambda)^{1/\alpha})^\alpha} \, d\tau
\]

and observe that as \(\tau \to 0\) we obtain

\[
\tau^{-d/\alpha}(1 + \tau^{-1/\alpha})^{-d-\alpha} \sim \tau^{\frac{d+1}{\alpha}}.
\]

It follows from the Watson lemma [Olv74, Ch. 3] that

\[
\int_0^\infty \tau^{-d/\alpha}(1 + \tau^{-1/\alpha})^{-d-\alpha} e^{-\tau(x|\Phi(\lambda)^{1/\alpha})^\alpha} \, d\tau \sim |x|^{-2\alpha}\Phi(\lambda)^{-2}, \lambda \to 0
\]

which implies

\[
(\mathcal{L}v^E(x, \cdot))(\lambda) \leq C|x|^{-\alpha-d}\lambda^{-1}\Phi(\lambda)^{-1} = \lambda^{-1}L \left( \frac{1}{\lambda} \right), \lambda \to 0,
\]

where \(L(y) = C|x|^{-\alpha-d}\Phi(y^{-1})^{-1}\) is a SVF. Thus we arrive at the long-time behavior for the Cesaro mean of \(v^E(x, t)\), namely

\[
M_t(v^E(x, t)) \sim C|x|^{-d-\alpha}\Phi(t^{-1})^{-1} = \frac{C}{|x|^{d+\alpha}\Phi(t^{-1})}.
\]

**Remark 12.** Let us consider the linear nonlocal diffusion

\[
\frac{\partial v(x, t)}{\partial t} = \mathcal{L}v(x, t) := \int_{\mathbb{R}^d} a(x-y)(v(y, t) - v(x, t)) \, dy, \quad v(x, 0) = \delta(x)
\]

with a jump kernel \(a\) that is positive, symmetric, normalized via \(\langle a \rangle := \int_{\mathbb{R}^d} a(x) \, dx = 1\), with light tails, i.e., \(a(x) \leq Ce^{-b|x|}, C, b > 0\).

1. It was shown in Lem. 6.2 in [KMV17] that there exist constants \(C, \alpha < \infty\) such that the transition density \(v(x, t)\) admits the following estimate

\[
v(x, t) \leq Cte^{\alpha^2t-\varepsilon|x|} \leq C, e^{(\alpha^2+\delta)t-\varepsilon|x|} \quad x \neq 0,
\]

for small enough \(\varepsilon \geq 0\) and \(\delta > 0\). This situation is realized, for instance, in the kinetic limit of the spatial contact model in the supercritical regime [FFK10, KKP08].
2. The bound (5.14) (which is not optimal as shown in [GKPZ18]) was used in the study of the front propagation of a population density governed by the operator \( \mathcal{L} \) (see [KMV17] for more details).

3. We may consider the subordination of \( v(x, t) \) by the kernel \( G_t(\tau) \) as before in obtaining the long-time behavior. It was shown in [KK17, Thm. 4.1] that

\[
A(\lambda, t) = \int_0^\infty e^{\lambda \tau} G_t(\tau) \, d\tau, \quad \lambda, t > 0
\]

has the following long-time behavior as \( t \to \infty \),

\[
A(\lambda, t) = \frac{\lambda}{\Phi'(p_0(\lambda)) p_0(\lambda)} e^{p_0(\lambda)t} + o(e^{p_0(\lambda)t}),
\]

where \( p_0(\lambda) \) is a superadditive function (cf. [Øst06]). Therefore the long-time behavior of the subordination \( v^E(x, t) \) is given as \( t \to \infty \) by

\[
v^E(x, t) = C \frac{\zeta e^{-|x|}}{\Phi'(p_0(\zeta)) p_0(\zeta)} e^{p_0(\zeta)t} + o(e^{p_0(\zeta)t}), \quad \zeta := \alpha \varepsilon^2 + \delta.
\]

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