REMARKS ON KATO’S EULER SYSTEMS FOR ELLIPTIC CURVES WITH ADDITIVE REDUCTION

CHAN-HO KIM AND KENTARO NAKAMURA

Abstract. Extending the former work for the good reduction case, we provide a numerical criterion to verify a large portion of the “Iwasawa main conjecture without $p$-adic $L$-functions” for elliptic curves with additive reduction at an odd prime $p$ over the cyclotomic $\mathbb{Z}_p$-extension. We also deduce the corresponding $p$-part of the Birch and Swinnerton-Dyer formula for elliptic curves of rank zero from the same numerical criterion. We give explicit examples at the end and specify our choice of Kato’s Euler system in the appendix.

1. Introduction

1.1. Overview. This article is a generalization of the numerical criterion for the verification of the Iwasawa main conjecture for modular forms at a good prime [KKS] to the additive reduction case. For the multiplicative reduction case, the main conjecture follows from the good ordinary case and the use of Hida theory ([Ski16]). This criterion also has an application to the $p$-part of the Birch and Swinnerton-Dyer formula for elliptic curves of rank zero (c.f. [JSW17, Theorem 7.2.1]). Since there are only finitely many bad reduction primes for elliptic curves, the criterion can be practically used to check the full Birch and Swinnerton-Dyer formula for an elliptic curve of rank zero and not necessarily square-free conductor. All the known non-CM examples have square-free conductors ([Wan18, Appendix]). We put some numerical examples of elliptic curves with additive reduction at the end.

Let $p$ be an odd prime and $E$ be an elliptic curve of conductor $N$ over $\mathbb{Q}$. Throughout this article, we assume that $E$ has additive reduction at $p$. In other words, $p^2$ divides $N$.

The construction of Kato’s Euler systems [Kat04] and the formulation of the Iwasawa main conjecture without $p$-adic $L$-functions [Kat04, Conjecture 12.10] are insensitive to the reduction type of elliptic curves. See [Jac18] for the $D_{dR}$-valued $p$-adic $L$-function of elliptic curves with additive reduction and also [Del98] and [Del02] for different attempts to understand the additive reduction case.

We expect that the reader has some familiarity with [Rub00] and [MR04].

1.2. Working assumptions and Kurihara numbers. Let

- $\text{Tam}(E)$ be the Tamagawa number of $E$,
- $N_{\text{st}}$ be the product of split multiplicative reduction primes of $E$, and
- $N_{\text{ns}}$ be the product of non-split multiplicative reduction primes of $E$.

We assume the following conditions throughout this article.

Assumption 1.1 (Working assumptions).

1. $p$ does not divide $\text{Tam}(E) \cdot \prod_{\ell \nmid N_{\text{st}}} (\ell - 1) \cdot \prod_{\ell \mid N_{\text{ns}}} (\ell + 1)$. (c.f. Remark 1.8.(4)).

2. The mod $p$ Galois representation $\overline{\rho} : G_{\mathbb{Q}} \to \text{Aut}_{\mathbb{F}_p}(E[p])$ is surjective. (Thus, $E$ is non-CM.)

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(3) The Manin constant is prime to $p$. (It is expected to be true always.)

Let $f \in S_2(\Gamma_0(N))$ be the newform attached to $E$ by [BCDT01, Theorem A]. For $\frac{a}{b} \in \mathbb{Q}$, we define $\left[\frac{a}{b}\right]^+$ by

$$2\pi \int_0^\infty f(\frac{a}{b} + iy)\,dy = \left[\frac{a}{b}\right]^+ \cdot \Omega_E^+ + \left[\frac{a}{b}\right]^\cdot \Omega_E^-$$

where $\Omega_E^\pm$ are the Néron periods of $E$. Then it is well-known that $\left[\frac{a}{b}\right]^+ \in \mathbb{Q}$. The following theorem due to G. Stevens yields the $p$-integrality of the value.

**Theorem 1.2** ([Ste89, §3]). Under (2) and (3) of Assumption 1.1, we have $\left[\frac{a}{b}\right]^+ \in \mathbb{Z}_p$ for $\frac{a}{b} \in \mathbb{Q}$.

A prime $\ell$ is a Kolyvagin prime for $(E,p)$ if $(\ell,Np) = 1$, $\ell \equiv 1 \pmod p$, and $a_\ell(E) \equiv \ell + 1 \pmod p$. We define the Kurihara number for $(E,p)$ at $n$ by

$$\overline{\delta}_n := \sum_{a \in (\mathbb{Z}/n\mathbb{Z})^\times} \left(\frac{a}{n}\right)^+ \cdot \prod_{\ell | n} \log_{E,p}(a) \in \mathbb{F}_p$$

where $n$ is the square-free product of Kolyvagin primes, $\left[\frac{a}{n}\right]^+$ is the mod $p$ reduction of $\left[\frac{a}{n}\right]^+$, and $\log_{E,p}(a)$ is the mod $p$ reduction of the discrete logarithm of $a$ modulo $\ell$ with a fixed primitive root modulo $\ell$. The number $\overline{\delta}_n$ itself is not well-defined, but its non-vanishing question is well-defined.

1.3. **The Iwasawa main conjecture à la Kato.** Let $\mathbb{Q}_\infty$ be the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$ and $\mathbb{Q}_n$ be the subextension of $\mathbb{Q}$ in $\mathbb{Q}_\infty$ of degree $p^n$. Let $\Lambda := \mathbb{Z}_p[\text{Gal}(\mathbb{Q}_\infty/\mathbb{Q})]$ be the Iwasawa algebra. Let $T = \mathcal{T}_pE$ be the $p$-adic Tate module of $E$ and $j : \text{Spec}(\mathbb{Q}_n) \to \text{Spec}(\mathcal{O}_{\mathbb{Q}_n}[1/p])$ be the natural map. Then we define the $i$-th Iwasawa cohomology of $E$ by

$$H^i(T) := \lim_n H^i_{\text{ét}}(\text{Spec}(\mathcal{O}_{\mathbb{Q}_n}[1/p]), j_*T)$$

where $H^i_{\text{ét}}(\text{Spec}(\mathcal{O}_{\mathbb{Q}_n}[1/p]), j_*T)$ is the étale cohomology group. See Appendix A for the full cyclotomic extension.

**Theorem 1.3** ([Kat04, Theorem 12.4.(1) and (3)]). The following statements hold.

1. $H^2(T)$ is a finitely generated torsion module over $\Lambda$.
2. $H^1(T)$ is free of rank one over $\Lambda$ under Assumption 1.1.(2).

We recall the Iwasawa main conjecture without $p$-adic zeta functions à la Kato.

**Conjecture 1.4** (Kato’s IMC, [Kat04, Conjecture 12.10], [Kur02, Conjecture 6.1]).

$$\text{char}_\Lambda \left(\frac{H^1(T)}{\Delta z_{\text{Kato}}}\right) = \text{char}_\Lambda \left(H^2(T)\right)$$

where $z_{\text{Kato}}$ is Kato’s zeta element (Definition A.13).

**Remark 1.5.**

1. Following the argument of [Kur02, §6], [Kob03, Theorem 7.1.ii], if $E(\mathbb{Q}_\infty,p)[p^\infty]$ is finite, then $\text{Sel}_{\text{str}}(\mathbb{Q}_\infty,E[p^\infty])$ and $H^2(T)$ are pseudo-isomorphic as $\Lambda$-modules where $\text{Sel}_{\text{str}}(\mathbb{Q}_\infty,E[p^\infty])$ is the $p$-strict (“fine”) Selmer group of $E$ over $\mathbb{Q}_\infty$.
2. Indeed, due to the argument of [Kat04, §13.13], if $E$ has potentially good reduction at $p$, then $E(\mathbb{Q}_\infty,p)[p^\infty]$ is finite.
(3) The following statements are equivalent:
   (a) $E$ has potentially good reduction at $p$.
   (b) The corresponding local automorphic representation at $p$ is not (a twist of) Steinberg. (See [Roh94, §15].)
   (c) The $j$-invariant of $E$ is $p$-integral. (See [Liu06, Proposition 10.2.33].)

1.4. Main Theorems.

**Theorem 1.6** (The main conjecture). Let $E$ be an elliptic curve with additive reduction at $p > 7$ satisfying Assumption 1.1. If

$$\tilde{\delta}_n \neq 0 \in \mathbb{F}_p$$

for some square-free product of Kolyvagin primes $n$, then we have

$$\text{char}_A \left( \frac{\mathbb{H}^1(T)}{\Lambda_{Kato}} \right) = \text{char}_A \left( \text{Sel}_{str}(\mathbb{Q}_{\infty}, E[p^{\infty}])^\vee \right).$$

If we further assume that $E$ has potentially good reduction at $p$, then Kato’s IMC (Conjecture 1.4) holds, i.e.

$$\text{char}_A \left( \frac{\mathbb{H}^1(T)}{\Lambda_{Kato}} \right) = \text{char}_A \left( \mathbb{H}^2(T) \right).$$

**Theorem 1.7** (The $p$-part of BSD formula). Let $E$ be an elliptic curve with additive reduction at $p > 7$ satisfying Assumption 1.1. Suppose that $L(E, 1) \neq 0$. If

$$\tilde{\delta}_n \neq 0 \in \mathbb{F}_p$$

for some square-free product of Kolyvagin primes $n$, then the $p$-part of Birch and Swinnerton-Dyer formula for $E$ holds, i.e.

$$\text{ord}_p \left( \# \text{III}(E/\mathbb{Q})[p^{\infty}] \right) = \text{ord}_p \left( \frac{L(E, 1)}{\Omega_E^+} \right).$$

**Remark 1.8.**

(1) Even in the $p \leq 7$ case, Theorem 1.6 and Theorem 1.7 hold if $(E, p)$ does not satisfy Assumption 2.5.

(2) It is expected that there always exists a square-free product of Kolyvagin primes $n$ such that $\tilde{\delta}_n \neq 0 \in \mathbb{F}_p$. Practically, it is easy to find such $n$’s. (c.f. [Kur14].)

(3) We do not know whether Theorem 1.6 directly implies Theorem 1.7 or not since there is neither a Mazur-Greenberg style main conjecture nor a control theorem for the additive reduction case.

(4) If we replace $L(E, 1)$ by the $N$-imprimitive $L$-value $L^{(N)}(E, 1)$ in Theorem 1.7, then we can weaken Assumption 1.1.(1) by $p \nmid \text{Tam}(E)$. (c.f. [MR04, Theorem 6.2.4].)

2. Computing the integral lattice

The goal of this section is to extend [Rub00, Proposition 3.5.1] to the additive reduction case over unramified extensions of $\mathbb{Q}_p$. More precisely, we compute the image of the logarithm map of the unramified local points of an elliptic curve with additive reduction.

The main idea is to replace a given elliptic curve with additive reduction by a different elliptic curve with good reduction over a ramified extension of degree 6 such that their generic fibers are isomorphic over the ramified extension. Then we apply the theory of formal groups to the elliptic curves with good reduction. The restriction $p > 7$ appears here.

We do not claim any originality of most argument in this section and closely follow [Pan] and [KP]. We give a full proof for the $p > 7$ case here only for the completeness of the argument and refer [Pan] and [KP] for the $p \leq 7$ case. See also [SD13, Lemma 1].
2.1. The integral lattice. Let $K$ be a finite unramified extension of $\mathbb{Q}_p$ and $k$ be the residue field. Since $K/\mathbb{Q}_p$ is unramified, the base change to $K$ does not change the reduction type of $E$ ([Sil09, Proposition VII.5.4.(a)]).

For a given elliptic curve $E/\mathbb{Q}$ with additive reduction, we consider an elliptic curve $E/\mathbb{Q}_p$ by taking the base change. Let $E$ be a minimal Weierstrass model of $E$ over $\mathbb{Z}_p$ given by a Weierstrass equation

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6$$

where $a_i \in \mathbb{pZ}_p$ for each $i$. By the coordinate change, it is not difficult to see that such a minimal Weierstrass model always exists. See [Pan, Lemma 9] and [KP, Lemma 9], for example. Note that $E(O_K) = E(K) = E(\mathbb{Q})$ is projective. Let $\widetilde{E}/\mathbb{F}_p$ be the reduction of $E$ modulo $p$ and $\bar{E}_{ns}/\mathbb{F}_p$ be the nonsingular locus of $\widetilde{E}/\mathbb{F}_p$. Let $E_0(K) \subset E(K)$ be the inverse image of $\bar{E}_{ns}(k)$ and $E_1(K) \subset E(K)$ be the inverse image of the identity of $\bar{E}_{ns}(k)$, i.e. the kernel of the reduction map. They can also be explicitly written as

$$E_0(K) = \{(x,y) \in E(K) : v_K(x) \leq 0, v_K(y) \leq 0\} \cup \{\infty\},$$

$$E_1(K) = \{(x,y) \in E(K) : v_K(x) \leq -2, v_K(y) \leq -3\} \cup \{\infty\}$$

where $v_K$ is the normalized valuation of $K$.

**Theorem 2.1** ([Pan, Theorem 1], [KP, Theorem 1, Corollary 2]). Let $E/\mathbb{Q}_p$ be an elliptic curve with additive reduction and $K/\mathbb{Q}_p$ be a finite unramified extension. Then the extension of the formal logarithm map on $E$ induces an isomorphism

$$\log_E : E_0(K) \otimes \mathbb{Z}_p \simeq O_K.$$

if $(E,p)$ does not satisfies Assumption 2.5.

**Proposition 2.2.** Let $E/\mathbb{Q}_p$ be an elliptic curve with additive reduction at an odd prime $p$ and $K/\mathbb{Q}_p$ be a finite unramified extension. Then $p$ does not divide $[E(K) : E_0(K)]$.

**Proof.** It is well-known that $[E(K) : E_0(K)] \leq 4$ ([Sil09, Theorem VII.6.1]). Thus, the statement is automatic if $p \geq 5$. For $p = 3$, [Sil99, Table 4.1, Page 365] shows that

$$p \nmid [E(\mathbb{Q}_p^\ur) : E_0(\mathbb{Q}_p^\ur)].$$

Since the formation of Néron models commutes with unramified (étale) base change ([BLR90, Proposition 2.(c), §1.2, Chapter 1]), we have

$$p \nmid [E(K) : E_0(K)]$$

following the argument in [BS12, Proof of Proposition 14].

**Corollary 2.3.** The images of $E(K) \otimes \mathbb{Z}_p$ and $E_0(K) \otimes \mathbb{Z}_p$ under the logarithm map coincide as $O_K$.

Due to [Sil99, Corollary IV.9.1], the identity component of the Néron model of $E$ over $\mathbb{Z}_p$ is isomorphic to $E_0$ over $\mathbb{Z}_p$. Thus, we can choose a Néron differential $\omega_E$ as a basis of the cotangent space of $E$ over $O_K$ (up to a $p$-adic unit). Also, $\omega_E$ corresponds to $f(z)dz$ up to a $p$-adic unit via the modular parametrization under Assumption 1.1.(3).

Let $V = T \otimes \mathbb{Q}_p$ and $D_{dr}(V)/D_{dr}^0(V)$ be the tangent space of $E$ over $K$ and $D_{dr}^0(V)$ be the cotangent space of $E$ over $K$. Then we have a perfect pairing

$$\text{Tr} \circ \langle -,- \rangle_{dr} : D_{dr}^0(V) \times D_{dr}(V)/D_{dr}^0(V) \to K \to \mathbb{Q}_p$$

where the first map is induced from the local Tate duality and the second map is the trace map. Let $\omega_E^* \in D_{dr}(V)/D_{dr}^0(V)$ be the dual basis to $\omega_E$ such that $\langle \omega_E^*, \omega_E \rangle_{dr} = 1$. The map

$$\hat{E}(O_K) \xrightarrow{\log_E} O_K \xrightarrow{\omega_E} O_K \omega_E^* \subseteq D_{dr}(V)/D_{dr}^0(V)$$
yields a natural integral structure on $D^0_{dR}(V)$. By the local Tate duality, we have the following statement.

**Corollary 2.4.** The image of $H^1(K,T)/H^1_J(K,T)$ under $\exp^*$ is $\mathcal{O}_K \omega_E$ where $H^1_J(K,T)$ is the image of the Kummer map of $E(K) \otimes \mathbb{Z}_p$.

2.2. Formal groups of elliptic curves. Let $\hat{E}$ be the formal group of elliptic curve $E$ over $\mathbb{Z}_p$ and $\mathcal{F}_E(X,Y)$ be the corresponding formal group law (as a two variable power series over $\mathbb{Z}_p$). Considering $a_i$’s in Equation (2.1) as variables, we can regard

$$\mathcal{F}_E(X,Y) \in \mathbb{Z}_p[a_1, a_2, a_3, a_4, a_6][X,Y].$$

We also regard a $\mathbb{Z}_p[a_1, a_2, a_3, a_4, a_6]$ as a weighted ring with weight function $\text{wt}(a_i) = i$ for each $i$. Then the coefficients of $\mathcal{F}_E(X,Y)$ in degree $n$ are homogeneous of weight $n - 1$ by [Sil09, Proposition IV.1.1.(c)]. We also have an explicit isomorphism of topological groups

$$\psi_{K'} : E_1(K') = E_1(K') \simeq \hat{E}(m_{K'})$$

defined by $(x, y) \mapsto -\frac{x}{y}$ for any finite extension $K'$ of $\mathbb{Q}_p$ ([Sil09, Proposition VII.2.2]).

2.3. Exceptional cases.

**Assumption 2.5** (Exceptional cases). Following the notation of Equation (2.1), if one of the following conditions

1. $p = 2$ and $a_1 + a_3 \equiv 2 \pmod{4}$,
2. $p = 3$ and $a_2 \equiv 6 \pmod{9}$,
3. $p = 5$ and $a_4 \equiv 10 \pmod{25}$, or
4. $p = 7$ and $a_6 \equiv 14 \pmod{49}$,

holds, then we call it an **exceptional case**.

In the exceptional cases, $E_0(K) = E_0(K')$ admits non-trivial $p$-torsions and we only have

$$\log_E : E_0(K) \otimes \mathbb{Z}_p \to \mathfrak{m}_K$$

as described in [Pan, Theorem 1] and [KP, Theorem 1, Corollary 2]. We do not consider the exceptional cases in this article.

2.4. General case: $p > 7$. From now on, we assume $p > 7$.

**Definition 2.6** (Extended convergence of formal groups). Let $\mathcal{F}_E(X,Y)$ be the formal group law for $E$ ([Sil09, Chapter IV]). Then $\mathcal{F}_E(X,Y)$ converges to an element of $\mathcal{O}_K$ for all $X, Y \in \mathcal{O}_K$ since all $a_i \in p\mathbb{Z}_p$. Thus, we can extend the formal group structure on $\hat{E}(\mathfrak{m}_K)$ to $\hat{E}(\mathcal{O}_K)$.

Note that the weighted ring structure of the formal group law in §2.2 is used in the definition.

**Proof of Theorem 2.1.** Let $L := K(\sqrt[p]{p})$ and $\varpi \in L$ be an element of $L$ satisfying $\varpi^6 = p$, and $\mathcal{C}$ be a smooth projective curve over $\mathcal{O}_L$ embedded in $\mathbb{P}^2_{\mathcal{O}_L}$ defined by the following Weierstrass equation

$$y^2 + \frac{a_1}{\varpi}xy + \frac{a_3}{\varpi^3}y = x^3 + \frac{a_2}{\varpi^2}x^2 + \frac{a_4}{\varpi^4}x^4 + \frac{a_6}{\varpi^6}.$$ 

We also define $\mathcal{C}_0$ and $\mathcal{C}_1$ in the same manner. Note that all the coefficients lie in $\mathcal{O}_L$ due to $\varpi^6 = p$. We denote the generic fiber of $\mathcal{C}$ over $L$ by $C$. Then there exists a birational map $\phi$ over $\mathcal{O}_L$

$$\mathcal{E} \times_{\text{Spec}(\mathcal{O}_K)} \text{Spec}(\mathcal{O}_L) \to \mathcal{C}$$

given by $\phi(x, y) = (\frac{x}{\varpi^2}, \frac{y}{\varpi^3})$ pointwisely. Note that $\phi$ would not be defined at the singular point in the special fiber. However, $\phi$ becomes an isomorphism on the generic fibers over $L$ and sends the point at infinity of $\mathcal{E}$ to that of $C$ by definition. Thus, $\mathcal{E}(L)$ and $\mathcal{C}(L)$ are isomorphic as topological groups. Furthermore, $\phi$ induces a set-theoretic bijection between
$E_0(L)$ and $C_1(L)$ due to Equation (2.2), so it is an isomorphism of topological groups. We have the following commutative diagram (a priori of sets)

$$
\begin{array}{ccc}
E_0(K) & \longrightarrow & E_0(L) \\
\Psi_K & \longrightarrow & \psi_L \\
\hat{E}(O_K) & \longrightarrow & \hat{E}(O_L) = \varpi \cdot \hat{C}(m_L)
\end{array}
$$

where $\Psi_K$ and $\Psi_L$ are defined by the assignment $(x, y) \mapsto -\frac{x}{y}$.

Let $\hat{F}$ be the formal group law of $C$ over $O_L$. By looking at the Weierstrass equations and the corresponding formal group laws, it is easy to see that

$$
\varpi \cdot \hat{F}_C(X, Y) = \hat{F}_C(\varpi \cdot X, \varpi \cdot Y),
$$

where $\hat{F}_C = \hat{F}_C \otimes \mathbb{Z}_p \otimes O_L$.

Thus, $\phi$, $\varpi \cdot x$, and $\psi_L$ in the diagram are all isomorphisms of topological groups, so $\Psi_L$ is also an isomorphism of topological groups. Thus, $E_0(K) \simeq \text{Im}(\Psi_K) \subseteq \hat{E}(O_K)$. Also, we can easily see $\hat{E}(O_K) = \text{Im}(\Psi_K)$ by considering $\Psi_K : (x, y) \mapsto -\frac{x}{y}$. Thus, $\Psi_K$ is also an isomorphism of topological groups.

Since we assume $p > 7$, we have $v_L(p) = 6v_K(p) = 6e(K/\mathbb{Q}_p) < p - 1$. Thus, the formal logarithm on $C$ induces an isomorphism

$$
\log_C : \hat{C}(m_L) \simeq m_L.
$$

Since $E_0(K)$ embeds into $\hat{C}(m_L)$, $E_0(K)$ is also torsion-free. Since the formal logarithm map extends to $\log \otimes \mathbb{Q}_p$:

$$
\begin{array}{ccc}
0 & \longrightarrow & E_1(K) \\
\simeq & \psi_K & \simeq \psi_K \\
0 & \longrightarrow & \hat{E}(m_K) \\
\simeq \log_C & \simeq \log_C \\
0 & \longrightarrow & O_K
\end{array}
$$

and it proves Theorem 2.1. \hfill \Box

3. **The main conjecture: review of [KKS]**

In this section, we present a slightly different computation from that of [KKS] due to a different normalization of Kato’s Euler system. Let $E$ be an elliptic curve over $\mathbb{Q}$ with additive reduction at $p$; especially, the Euler factor at $p$ of $L$-function $L(E, s)$ is 1.

3.1. **Kato’s Euler systems and modular symbols.** Let $c_{\mathbb{Q}(\mu_n)}(1) \in H^1(\mathbb{Q}(\mu_n), T)$ be Kato’s Euler system for $T$ at $\mathbb{Q}(\mu_n)$ such that

$$
\exp^*(c_{\mathbb{Q}}(1)) = \frac{L^{(Np)}(E, 1)}{\Omega_E^+}
$$

where $L^{(Np)}(E, \chi, 1)$ is the $Np$-imprimitive $L$-value of $E$ at $s = 1$ twisted by $\chi$. See Definition A.20 for the precise choice of $c_{\mathbb{Q}(\mu)}(1)$. Following [Rub00, Theorem 3.5.1] and [Kat04, Theorem 6.6 and Theorem 9.7], we have the following zeta value formula.
Theorem 3.1. Let $E$ be an elliptic curve over $\mathbb{Q}$ with additive reduction at $p$. Let $\chi$ be a Dirichlet character of $p$-power exponent and conductor $n$ with $(n,Np) = 1$. Then we have

$$\sum_{c \in (\mathbb{Z}/n\mathbb{Z})^\times} \chi(c) \cdot \left\langle \omega_E, \exp^* \left( L_{\mathbb{C}}(\chi^{-1}(1))^{\sigma_c} \right) \right\rangle_{d\mathbb{R}}$$

(3.1)

$$= \left( \prod_{q \mid Np} (1 - q^{-1} \chi(q)) \right) \cdot \left( \prod_{q \mid Nn} (1 + q^{-1} \chi(q)) \right) \cdot \frac{L^{(p)}(E, \chi, 1)}{\Omega_E^{\chi^{-1}}}$$

where $c \chi^{-1}(1)$ lies in the eigenspace with respect to the complex conjugation with eigenvalue $\chi(-1)$ and $L^{(p)}(E, \chi, 1)$ is the $p$-imprimitive $L$-value of $E$ at $s = 1$ twisted by $\chi$.

Remark 3.2. See §A.3 for the interpolation formula for such Dirichlet characters.

We rewrite the last term in Equation (3.1) in terms of modular symbols. Following [Pol14, (2.2)], we have

$$L(E, \chi, 1) = \frac{\tau(\chi)}{n} \cdot \sum_{a \in (\mathbb{Z}/n\mathbb{Z})^\times} \chi(a) \cdot 2\pi i \cdot \int_{c_\infty}^{a/n} f(z) dz$$

(3.2)

where $\tau(\chi)$ is the Gauss sum of $\chi$. Let $G_n = (\mathbb{Z}/n\mathbb{Z})^\times$, $G_n^p$ be the $p$-part of $G_n$, and $G_n^p$ be the prime-to-$p$ part of $G_n$. Expanding the Gauss sum in Equation (3.2), we have

$$\chi(-1) \cdot \frac{L(E, \chi, 1)}{\Omega_E^{\chi^{-1}}} = \frac{1}{n} \cdot \sum_{c \in (\mathbb{Z}/n\mathbb{Z})^\times} \chi(c) \cdot \sigma_c \cdot \left( \sum_{a \in (\mathbb{Z}/n\mathbb{Z})^\times} \zeta_n^a \cdot \left\lfloor -\frac{a}{n} \right\rfloor \chi^{-1} \right)$$

$$= \frac{1}{n} \cdot \sum_{c_1 \in G_n^p} \sum_{c_2 \in G_n^p} \chi(c_1 c_2) \cdot \sigma_{c_1} \cdot \sigma_{c_2} \cdot \left( \sum_{a \in (\mathbb{Z}/n\mathbb{Z})^\times} \zeta_n^a \cdot \left\lfloor -\frac{a}{n} \right\rfloor \chi^{-1} \right)$$

$$= \frac{1}{n} \cdot \sum_{c_1 \in G_n^p} \sum_{c_2 \in G_n^p} \chi(c_1) \cdot \sigma_{c_1} \cdot \sigma_{c_2} \cdot \left( \sum_{a \in (\mathbb{Z}/n\mathbb{Z})^\times} \zeta_n^a \cdot \left\lfloor -\frac{a}{n} \right\rfloor \chi^{-1} \right).$$

Let $\mathbb{Q}(n)$ be the maximal $p$-subextension of $\mathbb{Q}$ in $\mathbb{Q}(\mu_n)$ and $\mathcal{O}_{\mathbb{Q}(n)}$ the ring of integers of $\mathbb{Q}(n)$. We define the values

$$c_{\mathbb{Q}(n)}^{an,\pm} := \pm \frac{1}{n} \cdot \left( \prod_{q \mid Np} (1 - q^{-1} \sigma_q^{-1}) \right) \cdot \left( \prod_{q \mid Nn} (1 + q^{-1} \sigma_q^{-1}) \right) \cdot \sum_{c_2 \in G_n^p} \sum_{a \in (\mathbb{Z}/n\mathbb{Z})^\times} \zeta_n^a \cdot \left\lfloor -\frac{a}{n} \right\rfloor \in \mathbb{Z}_p \otimes \mathcal{O}_{\mathbb{Q}(n)}$$

in order to have

$$\sum_{c_1 \in G_n^p} \left( \sigma_{c_1} \left( c_{\mathbb{Q}(n)}^{an,\chi^{-1}} \right) \right) \cdot \chi(c_1) = \frac{L^{(Np)}(E, \chi, 1)}{\Omega_E^{\chi^{-1}}}.$$

(3.3)

Indeed, $c_{\mathbb{Q}(n)}^{an,\pm} = 0$ since $\mathbb{Q}(n)$ is totally real.

Comparing the coefficients of (3.1) and (3.3), we can easily observe the following statement.

Proposition 3.3. For any $c_1 \in G_n^p$, we have

$$\left\langle \omega_E, \exp^* \left( L_{\mathbb{C}}(\chi^{-1}(1))^{\sigma_c} \right) \right\rangle_{d\mathbb{R}} = \sigma_{c_1} \left( c_{\mathbb{Q}(n)}^{an,\chi^{-1}} \right)$$

where $c_{\mathbb{Q}(n)}^{\chi^{-1}(1)}$ is Kato’s Euler system at $\mathbb{Q}(n)$. 
3.2. Kolyvagin derivatives on modular symbols and Kurihara numbers. Let \( \ell \) be a Kolyvagin prime and \( \eta_\ell \) be a primitive root modulo \( \ell \). Let

\[
D_\ell := \sum_{i=0}^{\ell-2} i\sigma_{\eta_\ell}^i \in \mathbb{Z}[\text{Gal}(\mathbb{Q}(\mu_\ell)/\mathbb{Q})]
\]

be the Kolyvagin derivative operator for \( \text{Gal}(\mathbb{Q}(\mu_\ell)/\mathbb{Q}) \) with respect to \( \eta_\ell \) and \( D_n = \prod_{\ell \mid n} D_\ell \).

Let \( p^k \mid (\ell - 1) \). Then we let

\[
D_\ell := \sum_{i=0}^{p^k - 1} i\sigma_{\eta_\ell}^{p^k i} \in \mathbb{Z}[\text{Gal}(\mathbb{Q}(\ell)/\mathbb{Q})]
\]

also be the Kolyvagin derivative operator for \( \text{Gal}(\mathbb{Q}(\ell)/\mathbb{Q}) \) with respect to \( \eta_\ell^{p^k} \) and \( D_n = \prod_{\ell \mid n} D_\ell \). Let \( a, b \in \mathbb{Z} \) such that

\[
\frac{\ell - 1}{p^k} \cdot a + p^k \cdot b = 1.
\]

By rewriting

\[
D_\ell = \sum_{i=0}^{\ell-2} \left( \frac{\ell - 1}{p^k} a + p^k b \right) \cdot i \cdot \sigma_{\eta_\ell}^{(\frac{\ell - 1}{p^k} a + p^k b)i},
\]

the natural quotient map

\[
\mathbb{Z}[\text{Gal}(\mathbb{Q}(\mu_\ell)/\mathbb{Q})] \to \mathbb{Z}[\text{Gal}(\mathbb{Q}(\ell)/\mathbb{Q})]
\]

yields

\[
D_\ell \mapsto \left( \frac{\ell - 1}{p^k} \right) \cdot \left( \sum_{i=0}^{\ell-2} a \cdot i \cdot \sigma_{\eta_\ell}^{\frac{\ell - 1}{p^k} ai} \right) + p^k \cdot \left( \sum_{i=0}^{\ell-2} b \cdot i \cdot \sigma_{\eta_\ell}^{\frac{\ell - 1}{p^k} bi} \right)
\]

since \( \sigma_{\eta_\ell}^{p^k} \) maps to 1. We also have

\[
\sum_{i=0}^{\ell-2} a \cdot i \cdot \sigma_{\eta_\ell}^{p^k ai} \equiv \left( \frac{\ell - 1}{p^k} \right) \cdot \sum_{i=0}^{p^k - 1} a \cdot i \cdot \sigma_{\eta_\ell}^{\frac{\ell - 1}{p^k} ai} \pmod{p}
\]

in \( \mathbb{F}_p[\text{Gal}(\mathbb{Q}(\ell)/\mathbb{Q})] \). Since \( a \) is non-zero mod \( p \), we can replace \( a \cdot i \) by \( i \) in the RHS of (3.4).

Repeating this argument for each Kolyvagin prime dividing \( n \), we have the following lemma, which compares Kolyvagin derivatives for \( \text{Gal}(\mathbb{Q}(\mu_n)/\mathbb{Q}) \) and \( \text{Gal}(\mathbb{Q}(n)/\mathbb{Q}) \).

Lemma 3.4.

\[
D_n \equiv u \cdot D_n
\]

for some \( u \in \mathbb{F}_p^\times \) in \( \mathbb{F}_p[\text{Gal}(\mathbb{Q}(n)/\mathbb{Q})] \).

Considering the mod \( p \) Taylor expansion of Mazur-Tate elements at \( \prod_{i=1}^{s} (\sigma_{\eta_{1i}} - 1) \) where \( n = \prod_{i=1}^{s} \ell_i \), we observe the following statement.

Proposition 3.5 ([KKS, Theorem 7.5]). Let \( n \) be a square-free product of Kolyvagin primes and \( D_n \) be “the” Kolyvagin derivative operator in \( \mathbb{Z}_p[\text{Gal}(\mathbb{Q}(\mu_n)/\mathbb{Q})] \). We have the following equalities in \( \mathbb{F}_p \)

\[
D_n \left( \sum_{a \in (\mathbb{Z}/n\mathbb{Z})^\times} \zeta_n^{a'} \left[ \frac{a}{n_f} \right] \pm \right) \equiv \sum_{a \in (\mathbb{Z}/n\mathbb{Z})^\times} \left( \prod_{\ell \mid n} \log_{\ell}(a) \right) \cdot \left[ \frac{a}{n_f} \right] \pm \pmod{p}
\]

where \( a' = \pm a \).
Remark 3.6. In Proposition 3.5, the primitive roots modulo primes dividing $n$ for $D_n$ and $\prod_{\ell | n} \log F_{\ell}$ are chosen to match up. Thus, we can say “the” Kolyvagin derivative. The congruence still works up to multiplication by an element in $F_p^\times$ even if we do not match up the choices.

3.3. Proof of Theorem 1.6. We give a proof of Theorem 1.6. By [MR04, Theorem 5.3.10.(iii)], it suffices check two conditions:

1. the non-triviality of $\kappa_1^{\infty} \in H^1(Q_\infty, T)$.
2. the $\Lambda$-primitivity of the $\Lambda$-adic Kolyvagin system $\kappa^{\infty}$.

The non-triviality of $\kappa_1^{\infty}$ comes from the generic non-vanishing of cyclotomic twists of $L$-values (even when $p$ divides $N$) following [Roh88] and the dual exponential map. Note that there is no relevant Coleman reduction case yet.

Also, it is not difficult to observe that the $\Lambda$-primitivity of the $\Lambda$-adic Kolyvagin system $\kappa^{\infty}$ follows from the primitivity of the Kolyvagin system $\kappa$. See [KKS, Proposition 4.20] for detail.

Since $Q(n)$ is totally real, we have $c_{Q(n)}(1) = c_{Q(n)}^+(1)$. Consider the following commutative diagram

$$
\begin{array}{cccc}
H^1(Q(n), T) & \xrightarrow{\overline{D_n}} & H^1(Q(n), T) & \xrightarrow{\langle \omega_E^*, \exp^*(\text{loc}_p -) \rangle_{dR}} & \mathbb{Z}_p \otimes \mathcal{O}_{Q(n)} \\
\downarrow & & \downarrow & & \downarrow \\
\langle H^1(Q(n), T) \rangle_{\text{Gal}(Q(n)/Q)} & \xrightarrow{\langle \omega_E^*, \exp^*(\text{loc}_p -) \rangle_{dR}} & \mathbb{F}_p \otimes \mathcal{O}_{Q(n)} \\
\end{array}
$$

where $\langle \omega_E^*, \exp^*(\cdot) \rangle_{dR}$ is the induced reduction of $\langle \omega_E^*, \exp^*(\cdot) \rangle_{dR}$ modulo $p$. Because $\overline{D_n} \in \mathbb{Z} \langle \text{Gal}(Q(n)/Q) \rangle$ commutes with loc$_p$, exp$, and $\langle \omega_E^*, - \rangle_{dR}$, we have

$$
\langle \omega_E^*, \exp^*(\text{loc}_p \overline{D_n} c_{Q(n)}^+(1)) \rangle_{dR} = \overline{D_n} \langle \omega_E^*, \exp^*(\text{loc}_p c_{Q(n)}^+(1)) \rangle_{dR}.
$$

Theorem 3.7. If $\kappa_n = 0 \pmod p$, then $\overline{d}_n = 0$.

The proof is identical with that given in [KKS].

Proof. Suppose that $\kappa_n \pmod p = 0 \in H^1_{F(n)}(Q, T/pT) \otimes G_n$. Then, following [MR04, Appendix A], a straightforward computation yields

$$
d_n^+ = 0 \in H^1(Q(n), T)/pH^1(Q(n), T).
$$

Thus, $\overline{D_n} c_{Q(n)}^+(1) \in pH^1(Q(n), T)$. By Proposition 3.3, if $\overline{D_n} \langle \omega_E^*, \exp^*(\text{loc}_p c_{Q(n)}^+(1)) \rangle_{dR} \in p\mathbb{Z}_p \otimes \mathcal{O}_{Q(n)}$, then we have

$$
\overline{D_n} c_{Q(n)}^+ \in p\mathbb{Z}_p \otimes \mathcal{O}_{Q(n)}.
$$

Due to Lemma 3.4 and Proposition 3.5, (3.6) is equivalent to

$$
\left( \prod_{q | N_p} (1 - q^{-1} \sigma_q^{-1}) \right) \cdot \left( \prod_{q | N_{is}} (1 + q^{-1} \sigma_q^{-1}) \right) \cdot \overline{d}_n = 0
$$

in \( F_p \otimes O_{\mathbb{Q}(n)} \) and indeed, \( \delta_n \in F_p \); thus, it is equivalent to
\[
\left( \prod_{q \mid N_{ap}} (1 - q^{-1}) \right) \cdot \left( \prod_{q \mid N_{an}} (1 + q^{-1}) \right) \cdot \delta_n = 0 \in F_p.
\]
Due to Assumption 1.1.(1), it is equivalent to
\[
\delta_n = 0 \in F_p.
\]
\[\square\]

As a result of Theorem 3.7, we obtain an analogue of Conjecture 1.4 with \( z_{Kato}^{(N)} \) (defined in (A.3)). By Lemma A.21, Conjecture 1.4 follows.

4. THE \( p \)-PART OF THE BIRCH AND SWINNERTON-DYER FORMULA FOR THE RANK ZERO CASE

The goal of this section is to prove Theorem 1.7, which generalizes [MR04, Theorem 6.2.4] to the additive reduction case when the corresponding Kolyvagin system is primitive.

**Assumption 4.1 (Hyp(\( \mathbb{Q}, T \)), [Rub00, §2.1]).**

1. There exists an element \( \tau \in G_{\mathbb{Q}} \) such that
   - \( \tau \) acts trivially on \( \mu_p^{\infty} \), and
   - \( T/(\tau - 1)T \) is free of rank one over \( \mathbb{Z}_p \).

2. \( \overline{p} \) is irreducible.

We recall the “error term” in the Euler system argument. Let \( W = E[p^{\infty}] \) and \( \mathbb{Q}(W) \) be the smallest extension of \( \mathbb{Q} \) where \( G_{\mathbb{Q}(W)} \) acts trivially on \( E[p^{\infty}] \). Let
\[
\begin{align*}
n_{E[p^{\infty}]} &:= \text{length}_{\mathbb{Z}_p} \left( H^1(\mathbb{Q}(W)/\mathbb{Q}, E[p^{\infty}]) \cap \text{Sel}_{rel}(\mathbb{Q}, E[p^{\infty}]) \right), \\
n^*_{{E[p^{\infty}]}(1)} &:= \text{length}_{\mathbb{Z}_p} \left( H^1(\mathbb{Q}(W)/\mathbb{Q}, E[p^{\infty}]) \cap \text{Sel}_{str}(\mathbb{Q}, E[p^{\infty}]) \right).
\end{align*}
\]

By [Rub00, Proposition 3.5.8.(ii)], if the \( p \)-adic representation \( \rho : G_{\mathbb{Q}} \to \text{Aut}_{\mathbb{Z}_p}(T) \) is surjective, then
\[
H^1(\mathbb{Q}(W)/\mathbb{Q}, E[p^{\infty}]) \simeq H^1(\text{GL}_2(\mathbb{Z}_p), (\mathbb{Q}_p/\mathbb{Z}_p)^2) = 0,
\]
so \( n_{E[p^{\infty}]} = n^*_{E[p^{\infty}]} = 0 \). Furthermore, if \( \overline{p} \) is surjective with \( p > 3 \), then \( \rho \) is surjective since the determinant of \( \rho \) is the cyclotomic character by [SD86, Lemma 1, §1].

The machinery of Euler and Kolyvagin systems yields the following theorem.

**Theorem 4.2.** Suppose that \( p > 2 \) and \( T \) satisfies Hyp(\( \mathbb{Q}, T \)). Let \( c(1) \) be an Euler system for \( T \) and \( \kappa \) be the corresponding Kolyvagin system with \( c_\mathbb{Q}(1) = \kappa_1 \). Assume that \( \text{loc}_p^*(c_\mathbb{Q}(1)) = \text{loc}_p^*(\kappa_1) \neq 0 \).

1. \( \text{length}_{\mathbb{Z}_p} (\text{Sel}_{str}(\mathbb{Q}, E[p^{\infty}])) \) is finite if and only if \( \kappa_1 \neq 0 \).
2. \( \text{length}_{\mathbb{Z}_p} (\text{Sel}_{str}(\mathbb{Q}, E[p^{\infty}])) \leq \delta(0)(\kappa) := \max\{j : \kappa_1 \in p^j\text{Sel}_{rel}(\mathbb{Q}, T)\}\).
3. If \( \kappa \) is primitive, then
   \[
   \text{length}_{\mathbb{Z}_p} (\text{Sel}_{str}(\mathbb{Q}, E[p^{\infty}])) = \delta(0)(\kappa).
   \]

**Proof.** See [Rub00, Theorem 2.2.2] and [MR04, Corollary 5.2.13] for detail. \( \square \)

**Lemma 4.3** ([CS10, 1.9. Proposition]). If \( \text{Sel}(\mathbb{Q}, E[p^{\infty}]) \) is finite, then we have the following statements:

1. \( H^2(\mathbb{Q}_{\Sigma}/\mathbb{Q}, E[p^{\infty}]) = 0; \)
2. if we further assume \( E(\mathbb{Q})[p] \) is trivial, then the global-to-local map defining Selmer groups is surjective.
Indeed, the vanishing of $H^2(Q_\Sigma/Q, E[p^\infty])$ is equivalent to the finiteness of the $p$-strict Selmer group. See [Hac10, Lemma 3.2].

Using (the Pontryagin dual of) the second sequence of [PR00, A.3.2] with the $p$-strict local condition for $B$ and the $p$-relaxed local condition for $A$, we have the following exact sequence (see also [Kob03, (7.18)])

\begin{equation}
H^2(Q_\Sigma/Q, E[p^\infty])^\vee \to \text{Sel}_{\text{rel}}(Q, T) \to \frac{H^1(Q_p, T)}{E(Q_p) \otimes \mathbb{Z}_p} \to \text{Sel}(Q, E[p^\infty])^\vee \to \text{Sel}_{\text{str}}(Q, E[p^\infty])^\vee \to 0.
\end{equation}

Note that here we use Lemma 4.3.(2) to have the third and the fourth terms.

Furthermore, under the finiteness of the Selmer group (due to Lemma 4.3.(1)), Sequence (4.1) becomes

\begin{equation}
0 \to \text{Sel}_{\text{rel}}(Q, T) \to \frac{H^1(Q_p, T)}{E(Q_p) \otimes \mathbb{Z}_p} \to \text{Sel}(Q, E[p^\infty])^\vee \to \text{Sel}_{\text{str}}(Q, E[p^\infty])^\vee \to 0.
\end{equation}

Taking the quotient by the $\mathbb{Z}_p$-module generated by $c_Q(1)$, Kato’s Euler system at $Q$, we have

\begin{equation}
0 \to \frac{\text{Sel}_{\text{rel}}(Q, T)}{\mathbb{Z}_p c_Q(1)} \to \frac{H^1(Q_p, T)}{\mathbb{Z}_p \text{loc}_p c_Q(1)} \to \text{Sel}(Q, E[p^\infty])^\vee \to \text{Sel}_{\text{str}}(Q, E[p^\infty])^\vee \to 0
\end{equation}

where $H^1_s(Q_p, T) = \frac{H^1(Q_p, T)}{E(Q_p) \otimes \mathbb{Z}_p}$.

The following theorem is the generalized and sharpened version of [Rub00, Theorem 2.2.10.(i)].

**Theorem 4.4.** Suppose that $p > 2$ and $T$ satisfies Hyp($Q, T$). Let $c(1)$ be an Euler system for $T$ and $\kappa$ be the corresponding Kolyvagin system with $c_Q(1) = \kappa_1$. Assume that $\text{loc}_p^s(\kappa_1) \neq 0$ and $\kappa$ is primitive. Then $\text{Sel}(Q, E[p^\infty])$ is finite and

$$\text{length}_{\mathbb{Z}_p} (\text{Sel}(Q, E[p^\infty])) = \text{length}_{\mathbb{Z}_p} H^1_s(Q_p, T)/\text{loc}_p^s(\kappa_1).$$

**Proof.** The finiteness of Selmer groups is already proved in [Rub00, Theorem 2.2.10.(ii)]. We have

$$\partial^{(0)}(\kappa) = \text{length}_{\mathbb{Z}_p} \left( \text{loc}_p^s (\text{Sel}_{\text{rel}}(Q, T))/\mathbb{Z}_p \text{loc}_p^s(\kappa_1) \right) (\text{Sequence (4.2)})$$

$$< \infty. \quad (\text{loc}_p^s(\kappa_1) \neq 0)$$

By Theorem 4.2.(3), we have

$$\text{length}_{\mathbb{Z}_p} (\text{Sel}_{\text{str}}(Q, E[p^\infty])) = \text{length}_{\mathbb{Z}_p} \left( \text{loc}_p^s (\text{Sel}_{\text{rel}}(Q, T))/\mathbb{Z}_p \text{loc}_p^s(\kappa_1) \right) < \infty.$$

Then, by Sequence (4.2) again,

$$\text{length}_{\mathbb{Z}_p} (\text{Sel}(Q, E[p^\infty])) = \text{length}_{\mathbb{Z}_p} \left( H^1_s(Q_p, T)/\mathbb{Z}_p \text{loc}_p^s(\kappa_1) \right) < \infty.$$

Applying Theorem 3.1, we extend [Rub00, Theorem 3.5.11] to the additive reduction case.

**Theorem 4.5.** Let $E$ be a non-CM elliptic curve over $Q$ with additive reduction at an odd prime $p > 7$. Assume that $L(E, 1) \neq 0$, $\overline{p}$ is surjective, $p \nmid \text{Tam}(E)$ and $\kappa$ is primitive. Then

$$\text{ord}_p \left( \#\text{III}(E/Q[p^\infty]) \right) = \text{ord}_p \left( \frac{L(E, 1)}{\Omega_E^+} \right).$$

**Proof.** By Corollary 2.4, we have

$$\langle \omega_E^*, \exp^* (H^1_s(Q_p, T)) \rangle = \mathbb{Z}_p \subseteq \mathbb{Q}_p, \quad \langle \omega_E^*, \exp^* (\text{loc}_p^s(\kappa_1)) \rangle = \frac{L(E, 1)}{\Omega_E^+} \mathbb{Z}_p.$$

Since $L(E, 1) \neq 0$, we have $\text{loc}_p^s(\kappa_1) \neq 0$ via Theorem 3.1. Also, the finiteness of $\text{Sel}(Q, E[p^\infty])$ and the surjectivity of $\overline{p}$ show that

$$\#\text{Sel}(Q, E[p^\infty]) = \#\text{III}(E/Q[p^\infty]).$$
By Theorem 4.4, we obtain
\[ \text{length}_{\mathbb{Z}_p} \left( \text{Sel}(\mathbb{Q}, E[p^\infty]) \right) = \text{length}_{\mathbb{Z}_p} H^1_{\text{s}}(\mathbb{Q}_p, T) / \text{loc}^s_p(\kappa_1). \]
Thus, the conclusion follows. \( \square \)

5. Examples

In this section, we provide four new examples of the main conjecture and the \( p \)-part of the BSD formula. The following SAGE code generates the Kolyvagin primes for \((E, p)\) up to 10,000.

```sage
p = PRIME
E = EllipticCurve('LABEL')
for l in prime_range(10000):
    if mod(l-1,p) == 0:
        if mod(E.ap(l) - l - 1,p) == 0:
            l
```

The following SAGE code computes \( \tilde{\delta}_n \) for elliptic curves when \( n \) is a product of two Kolyvagin primes.

```sage
E = EllipticCurve('LABEL')
f = E.modular_symbol()
p = PRIME
n = PRODUCT OF TWO KOLYVAGIN PRIMES
R1 = Integers(factor(n)[0][0])
g1 = R1.multiplicative_generator()
R2 = Integers(factor(n)[1][0])
g2 = R2.multiplicative_generator()
S = mod(0,p)
for a in range(n):
    if gcd(a,n) == 1:
        aa1 = mod(a,factor(n)[0][0])
        aa2 = mod(a,factor(n)[1][0])
        S = S + mod(aa1.log(g1),p)* mod(aa2.log(g2),p)* mod(f(a/n),p)
S
```

5.1. Elliptic curve of conductor 56144. Let \( p = 11 \) and \( E \) be the elliptic curve over \( \mathbb{Q} \) defined by
\[ y^2 = x^3 - 584551x - 172021102 \]
with conductor 56144 = \( 2^4 \cdot 11^2 \cdot 29 \) as in [LMF17, Elliptic Curve 56144.w1]. The SAGE computation yields the following facts:
- The mod 11 representation is surjective;
- The Tamagawa factor of \( E \) is prime to 11;
- \( a_{29}(E) = 1 \), so 11 does not divide 28 = 29 - 1.
- \( \tilde{\delta}_{397 \cdot 859} \neq 0 \) where 397 and 859 are (the smallest) Kolyvagin primes for \((E, p)\).
Thus, Theorem 1.6 and Theorem 1.7 for \((E, p)\) hold. Note that these theorems do not directly follow from the one divisibility since \( \#\text{III}(E/\mathbb{Q})[11^\infty] \) is 121.

5.2. Elliptic curve of conductor 84700. Let \( p = 11 \) and \( E \) be the elliptic curve over \( \mathbb{Q} \) defined by
\[ y^2 = x^3 - 235390375x - 1409480751250 \]
with conductor 84700 = \( 2^2 \cdot 5^2 \cdot 7 \cdot 11^2 \) as in [LMF17, Elliptic Curve 84700.bi1]. The SAGE computation yields the following facts:
- The mod 11 representation is surjective;
The Tamagawa factor is prime to 11;
a_7(E) = -1, so 11 does not divide 8 = 7 + 1;
\(\tilde{\delta}_{23 \cdot 2113} \neq 0\) where 23 and 2113 are (the smallest) Kolyvagin primes for \((E,p)\).
 Thus, Theorem 1.6 and Theorem 1.7 for \((E,p)\) hold. Note that these theorems do not directly follow from the one divisibility since \(\#\mathcal{M}(E/\mathbb{Q})[11^{\infty}]\) is 121.

5.3. Elliptic curve of conductor 84100. Let \(p = 5\) and \(E\) be the elliptic curve over \(\mathbb{Q}\) defined by

\[y^2 = x^3 + x^2 - 28033x + 1232688\]

with conductor 84100 = \(2^2 \cdot 5^2 \cdot 29^2\) as in [LMF17, Elliptic Curve 84100.b3]. The SAGE computation yields the following facts:

- The mod 5 representation is surjective;
- \(a_4 = 28033 \not\equiv 10 \pmod{25}\), so it is not an exceptional case (Assumption 2.5);
- The Tamagawa factor is prime to 5;
- \(\tilde{\delta}_{191 \cdot 331} \neq 0\) where 191 and 331 are (the smallest) Kolyvagin primes for \((E,p)\).

Considering Remark 1.8.(1), Theorem 1.6 and Theorem 1.7 hold for \((E,p)\). Note that these theorems do not directly follow from the one divisibility since the rank of \(E\) is two.

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Appendix A. The choice of Kato’s Euler system

The goal of this appendix is to give an explicit characterization of the “optimal” Kato’s Euler system

\[c_F \in H^1(F,T(-1))\]

where \(F\) runs over abelian extensions of \(\mathbb{Q}\) such that

\[\exp^*(c_Q(1)) = \frac{L^{(Np)}(E,1)}{\Omega_E^+}\]

where \(c_Q(1)\) is the image of \(c_Q\) in \(H^1(\mathbb{Q},T)\) under the map induced from the Tate twist as described in [Kat04, Theorem 12.5]. This choice of Kato’s Euler system is used in many literatures; however, it seems a bit difficult to find the explicit choice from the original construction of Kato in the literatures. Note that the Euler system in [Kat04] is constructed for \(T(-1)\) due to the cohomological convention of the Galois representation ([Kat04, §14.10]).

Following [Rub98, Corollary 7.2] and [Sch98], there exists an Euler system \(\{c'_F(1)\}_F\) for \(T\) such that

\[\exp^*(c'_Q(1)) = r_E \cdot \frac{L^{(Np)}(E,1)}{\Omega_E^+}\]

for some positive integer \(r_E\). The goal of this appendix is to show that we can take \(r_E = 1\) following [Kat04, Theorem 6.6] and [Del08, Remark in Appendix A (Page 254)] choosing the “right” Kato’s Euler system.

We recall the convention for Kato’s Euler systems. We assume some familiarity with [Kat04] here. We sometimes may use the notation therein from without any caution.
A.1. The cohomology classes. Let \( f \in S_k(\Gamma_1(N), \psi) \) be a newform with Hecke field \( F \) over \( \mathbb{Q} \) and \( S(f) \) be the rank one Hecke module generated by \( f \) over \( F \) as in [Kat04, §6.3]. Let \( \lambda \) be the place of \( F \) compatible with a choice of embedding \( \iota_p : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p \). Let \( m \geq 1 \) be an integer.

**Definition A.1.** Following [Kat04, §5.1], we define \( \xi \) and \( S \) as follows:

1. \( \xi \) is a symbol \( a(A) \) where \( a, A \in \mathbb{Z}, A \geq 1 \) and \( S \) is a non-empty finite set of primes containing \( \text{prime}(mA) \), or
2. \( \xi \) is an element of \( \text{SL}_2(\mathbb{Z}) \) and \( S \) is a non-empty finite set of primes containing \( \text{prime}(mN) \).

**Definition A.2.** Following [Kat04, (5.2.1) and (5.2.2)], we define integers \( r, r' \) and positive integers \( c, d \) as follows:

1. \( 1 \leq r \leq k - 1, 1 \leq r' \leq k - 1 \), and at least one of \( r, r' \) is \( k - 1 \).
2. \( c \) and \( d \) are positive integers with \( \text{prime}(cd) \cap S = \emptyset \), and \( (d, N) = 1 \).

**Definition A.3.** Let

\[
c,dz_m(f, r, r', \xi, S) \in S(f) \otimes \mathbb{Q}(\mu_m)
\]

be the zeta modular form in [Kat04, §6.3] where \( c, d, r, r', \xi, \) and \( S \) satisfy Definition A.1 and Definition A.2.

For any \( x \in V_F(f) \), \( x^\pm := \frac{1}{2}(1 \pm i)x \) where \( i \) is induced from the complex conjugation. Let \( V_F(f)^\pm := V_F(f)^{\pm \pm} \) and then \( \dim_F V_F(f)^\pm = 1 \), respectively. Let \( \text{per}_f : S(f) \to V_F(f) \otimes_F \mathbb{C} \) be the period map in [Kat04, §6.3]. Let \( f^* \) be the complex conjugation of \( f \) (“the dual modular form”).

**Theorem A.4** ([Kat04, Theorem 6.6]). Let \( \chi \) be a character on \( (\mathbb{Z}/m\mathbb{Z})^\times \). Let \( \pm = (-1)^{k-r-1} \). \( \chi(-1) \) and

\[
(u, v) := \begin{cases} 
  (r + 2 - k, r) & \text{if } r' = k - 1 \\
  (k - r', r') & \text{if } r = k - 1
\end{cases}
\]

following [Kat04, (4.2.4)]. We add the following condition.

* If \( \xi \notin \text{SL}_2(\mathbb{Z}) \), then assume \( c \equiv d \equiv 1 \) (mod \( N \)).

Then we have

\[
\sum_{b \in (\mathbb{Z}/m\mathbb{Z})^\times} \chi(b) \cdot \text{per}_f \left( \sigma_b \left( c,dz_m(f, r, r', \xi, S) \right) \right)^\pm = L(S)^{(f^*, \chi, r) \cdot (2\pi i)^{k-r-1} \cdot \gamma^\pm}
\]

where

\[
\gamma = c^2d^2\delta(f, r', a(A)) - c^2d^2(\chi(c)\delta(f, r', ac(A)) \]

\[
- c^2d^2\psi(d)\delta(f, r', "a/d"(A)) + c^2d^2\psi(cd)\psi(d)\delta(f, r', "ac/d"(A))
\]

if \( \xi = a(A) \), and

\[
\gamma = (c^2 - c^2\chi(c))(d^2 - d^2\chi(d))\delta(f, r', \xi)
\]

if \( \xi \in \text{SL}_2(\mathbb{Z}) \).

**Definition A.5.** Let

\[
c,dz_m^{(p)}(f, r, r', \xi, S) \in H^1(\mathbb{Z}[1/p, \zeta_m], V_{\mathcal{O}_\lambda}(f)(k - r))
\]

be the zeta element in [Kat04, (8.1.3)] where \( p \in S \) as well as all the “variables” satisfy Definition A.1 and Definition A.2.

By [Kat04, Proposition 8.12], the cohomology classes

\[
c,dz_m^{(p)}(f, r, r', \xi, S) \in H^1(\mathbb{Z}[1/p, \zeta_m], V_{\mathcal{O}_\lambda}(f)(k - r))
\]
satisfy the Euler system relation except at primes dividing $cdN$. See also [Kat04, §13.3]. By [Kat04, Theorem 9.7], under Definition A.2.(1), we have the following assignment

$$c, d z_m^{(p)}(f, r, r', \xi, S) \mapsto c, d z_m(f, r, r', \xi, S)$$

under the dual exponential map. In other words, zeta modular forms (Definition A.3) and zeta elements (Definition A.5) match in the critical range, i.e. $L$-values at $s = 1, \ldots, k - 1$.

### A.2. Kato’s modification in the $p$-power direction

Following [Kat04, Theorem 12.5 and §13.9], we recall Kato’s modification of zeta elements (in the $p$-power direction) in order to specify $z_{\mathcal{K}_\gamma}$ in Conjecture 1.4.

#### A.2.1. The elements

Let $G_\infty := \text{Gal}(\mathbb{Q}(\zeta_{p^{\infty}})/\mathbb{Q})$ and $j_{p^n} : \text{Spec}(\mathbb{Q}(\zeta_{p^n})) \to \text{Spec}(\mathbb{Z}[\zeta_{p^n}, 1/p])$ be the natural map. Then we define a $\mathbb{Z}_p[G_\infty]$-module

$$H^i(V_{\mathcal{O}_\lambda}(f)) := \lim_{\substack{\to \n}} H^i_{\text{ét}}(\text{Spec}(\mathbb{Z}[\zeta_{p^n}, 1/p], j_{p^n}, V_{\mathcal{O}_\lambda}(f)))$$

where $H^i_{\text{ét}}(\text{Spec}(\mathbb{Z}[\zeta_{p^n}, 1/p], j_{p^n}, V_{\mathcal{O}_\lambda}(f))$ is the étale cohomology group.

**Choice A.6.** Fix elements $\alpha_1, \alpha_2 \in \text{SL}_2(\mathbb{Z})$ and integers $j_1, j_2$ such that

- $1 \leq j_i \leq k - 1 \ (i = 1, 2)$, and
- $\delta(f, j_1, \alpha_1)^+ \neq 0$ and $\delta(f, j_2, \alpha_2)^- \neq 0$ ([Kat04, (13.6)].)

Let $\gamma \in V_{F_\lambda}(f)$. Then we have

$$\gamma = b_1 \cdot \delta(f, j_1, \alpha_1)^+ + b_2 \cdot \delta(f, j_2, \alpha_2)^- \neq 0$$

for some $b_1, b_2 \in \mathbb{Q}_{F_\lambda}$.

**Choice A.7.** Fix $c, d \in \mathbb{Z}$ such that

1. $(cd, 6p) = 1$,
2. $c \equiv d \equiv 1 \pmod{N}$ (needed for [Kat04, Theorem 6.6]), and
3. $c^2 \neq 1$ and $d^2 \neq 1$.

For a commutative ring $R$, let $Q(R)$ be the total quotient ring of $R$.

**Definition A.8.** We define

$$z_{\gamma}^{(p)} := \left\{ \mu(c, d, j_1)^{-1} \cdot b_1 \cdot \left( c, d z^{(p)}_{\gamma}(f, k, j_1, \alpha_1, \text{prime}(pN)) \right)_{n \geq 1} \right\}^-$$

$$+ \left\{ \mu(c, d, j_2)^{-1} \cdot b_2 \cdot \left( c, d z^{(p)}_{\gamma}(f, k, j_2, \alpha_2, \text{prime}(pN)) \right)_{n \geq 1} \right\}^+ \in H^3(V_{\mathcal{O}_\lambda}(f)) \otimes Q(\mathbb{Z}_p[G_\infty])$$

where

$$\mu(c, d, j) := (c^2 - c^{k+1-j} \cdot \sigma_c) \cdot (d^2 - d^{j+1} \cdot \sigma_d) \cdot \prod_{\ell | N} \left( 1 - a_\ell(f) \ell^{-k} \sigma_\ell^{-1} \right) \in \mathbb{Z}_p[G_\infty]$$

for $j \in \mathbb{Z}$ and $\mu(c, d, j)$ is not a zero divisor in $\mathbb{Z}_p[G_\infty]$ for all $j$.

By [Kat04, §13.12], indeed, it is known that $z_{\gamma}^{(p)} \in H^3(V_{\mathcal{O}_\lambda}(f)) \otimes \mathbb{Q}_p$.

#### A.2.2. The submodule inverting $p$

Let $Z(f)$ be the $\mathcal{O}_\lambda[G_\infty] \otimes \mathbb{Q}_p$-submodule of $H^3(V_{\mathcal{O}_\lambda}(f)) \otimes \mathbb{Q}_p$ generated by $z^{(p)}_{\gamma}$ for all $\gamma \in V_{F_\lambda}(f)$. Since $H^3(V_{\mathcal{O}_\lambda}(f)) \otimes \mathbb{Q}_p$ is free of rank one over $\mathcal{O}_\lambda[G_\infty] \otimes \mathbb{Q}_p$ (by [Kat04, Theorem 12.4.(2)]), we easily observe the following independence result considering the evaluation at all the finite order characters on $G_\infty$.

**Proposition A.9.** The cohomology class $z_{\gamma}^{(p)}$ is independent of the choices of $\alpha_1, \alpha_2, j_1, j_2, c, d$ as above, i.e. Choice A.6 and Choice A.7.
Furthermore, since $O_\lambda[G_\infty] \otimes \mathbb{Q}_p$ is a product of PIDs, there exists an element $\gamma_0 \in V_f$ such that $z_{\gamma_0}^{(p)}$ generates $Z(f)$ over $O_\lambda[G_\infty] \otimes \mathbb{Q}_p$. Thus, any non-zero $z_\gamma^{(p)}$ generates $Z(f)$ over $O_\lambda[G_\infty] \otimes \mathbb{Q}_p$.

A.2.3. The submodule without inverting $p$. We assume the following condition and it is satisfied in most cases as described in [Kat04, Remark 12.8].

**Assumption A.10.** Suppose that there exists a $O_\lambda$-lattice $T_f \subseteq V_{F_\lambda}(f)$ such that the image of $\rho_{T_f} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\mu_p^{\infty})) \rightarrow \text{GL}O_\lambda(T_f)$ contains $\text{SL}_2(\mathbb{Z}_p)$.

Let $Z(f, T_f)$ be the $\mathbb{Z}[G_\infty]$-submodule of $H^1(T_f) \otimes \mathbb{Q}_p = H^1(V_{O_\lambda}(f)) \otimes \mathbb{Q}_p$ generated by $z_\gamma^{(p)}$ for all $\gamma \in T_f \subseteq V_{F_\lambda}(f)$. Then by [Kat04, Theorem 12.5.(4)], it is known that $Z(f, T_f) \subseteq H^1(T_f)$ under Assumption 1.1.(2).

Suppose that $T_f = V_{O_\lambda}(f)$. Let $Z$ be the $\mathbb{Z}_p[G_\infty]$-submodule of $Z(f, V_{O_\lambda}(f))$ generated by

$$c,dz_\gamma^{(p)}(f,k,j,a(A), \text{prime}(pA))_{n \geq 1} \in H^1(V_{O_\lambda}(f))$$

where $1 \leq j \leq k - 1$, $A \in \mathbb{Z}$ with $A \geq 1$, $c, d \in \mathbb{Z}$ such that $(c, 6pA) = (d, 6pN) = 1$, and

$$c,dz_\gamma^{(p)}(f,k,j,\alpha, \text{prime}(pN))_{n \geq 1} \in H^1(V_{O_\lambda}(f))$$

where $1 \leq j \leq k - 1$, $\alpha \in \text{SL}_2(\mathbb{Z})$, $c, d \in \mathbb{Z}$ such that $(c, 6pA) = (d, 6pN) = 1$. Then $Z \subseteq Z(f, V_{O_\lambda}(f))$ of finite index due to [Kat04, Theorem 12.6].

**Remark A.11.** Indeed, Assumption A.10 is satisfied for all lattices if it is satisfied for one lattice for non-CM elliptic curves. See [Kat04, Remark 12.8].

**Proposition A.12.** Under Assumption A.10, the submodule $Z(f, T_f) \subseteq H^1(T_f)$ is generated by one element over $O_\lambda[G_\infty]$.

**Proof.** By Remark A.11, we may assume $V_{O_\lambda}(f) = T_f$. Let $\alpha : V_{O_\lambda}(f) \rightarrow H^1(V_{O_\lambda}(f))$ be the map defined by $\gamma \mapsto z_\gamma^{(p)}$ as in [Kat04, Theorem 12.5.(1)]. Consider the $O_\lambda[G_\infty]$-linear map

$$\beta : V_{O_\lambda}(f)^+ \otimes_{O_\lambda} O_\lambda[G_\infty]^- \otimes V_{O_\lambda}(f)^- \otimes_{O_\lambda} O_\lambda[G_\infty]^+ \rightarrow H^1(V_{O_\lambda}(f))$$

defined by $\gamma^+ \otimes 1 + \gamma^+ \otimes 2 \mapsto \lambda_1 \cdot z_\gamma^{(p)} + \lambda_2 \cdot z_\gamma^{(p)}$ and then it satisfies $O_\lambda[G_\infty] \cdot \text{Im}(\alpha) = \text{Im}(\beta)$. The map $\beta$ is well-defined due to the equation

$$z_\gamma^{(p)}(\alpha) = -\sigma_1(z_\gamma^{(p)}).$$

Since $V_{O_\lambda}(f)^+ \otimes_{O_\lambda} O_\lambda[G_\infty]^-$ and $V_{O_\lambda}(f)^- \otimes_{O_\lambda} O_\lambda[G_\infty]^+$ are free of rank one over $O_\lambda[G_\infty]^-$ and $O_\lambda[G_\infty]^+$, respectively, $\text{Im}(\beta)$ is free or rank one over $O_\lambda[G_\infty]$. Thus, $O_\lambda[G_\infty] \cdot \text{Im}(\alpha)$ is also free or rank one over $O_\lambda[G_\infty]$; hence, the conclusion holds.

**Definition A.13** (Kato’s zeta element for elliptic curves). Assume that $f$ corresponds to an elliptic curve over $\mathbb{Q}$ and $T$ is its $p$-adic Tate module. Let $z_{\gamma_0}^{(p)}$ be a generator of $Z(f, V_{O_\lambda}(f)) \subseteq H^1(V_{O_\lambda}(f))$ over $O_\lambda[G_\infty]$. Then we define $z_{Kato} \in H^1(T)$ in Conjecture 1.4 by the image of $z_{\gamma_0}^{(p)}$ under the map

$$H^1(V_{O_\lambda}(f)) \otimes_{(\mu_p^{\infty})} H^1(V_{O_\lambda}(f)(1)) \longrightarrow \text{core}, H^1(T)$$

where the corestriction map is from $\mathbb{Q}(\mu_p^{\infty})$ to $\mathbb{Q}_\infty$.

**Remark A.14.** Note that $\gamma_{\gamma_0}^{\pm}$ corresponds to the Néron periods of the elliptic curve since $T$ comes from the $p$-adic Tate module of $E$. 

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A.3. The Euler system: an extension to the tame direction. The goal of this section is to extend the interpolation property of Kato’s modification $ζ^{(p)}_r$ obtained from
\[ \left( c_dζ_r^{(p)}(f, k, j_1, α_1, \text{prime}(pN)) \right)_{n≥1} \text{ and } \left( c_dζ_r^{(p)}(f, k, j_2, α_2, \text{prime}(pN)) \right)_{n≥1} \]
to the tame direction very slightly. The motivation of the extension is the following question.

**Question A.15.** Do we have an integral Euler system for the triple $(V_{O_λ}(f)(k-r), F_λ, \text{prime}(cdpN))$ modifying $c_dζ_m^{(p)}(f, r, j, ξ, \text{prime}(mN)) \in H^1(Z[1/p, z_m], V_{O_λ}(f)(k-r))$ such that
1. it is independent of the choices of $(c, d)$ and $(j, ξ)$ (Proposition A.9) and
2. it satisfies the interpolation property for all finite order characters on $Gal(Q^{ab}/Q)$ (Theorem A.4)?

We do not have the answer to this question yet. In [Rub98, Corollary 7.2], when $k = 2$, $r = 1$ and $j_1 = j_2 = 1$, Rubin could modify the Euler system
\[ c_dζ_m^{(p)}(f, 1, 1, ξ, \text{prime}(mpN)) \]
by choosing $c$ and $d$ more carefully, i.e. $c \neq 1 \pmod{p}$ and $d \neq 1 \pmod{p}$ in order to have
\[ c \cdot d \cdot (c - σ_c) \cdot (d - σ_d) ∈ \Lambda^X. \]
Thus, Rubin’s modification yields both the integrality and the interpolation property for finite order characters on $Gal(Q^{∞}/Q)$. We extend Rubin’s strategy to the case we need for Theorem 3.1 using the following lemma.

**Lemma A.16.** Assume that
1. $(*)$ $j = k - 1$, $k < p$, and $r ≠ 2$.

Then there exist infinitely many integers $c$ and $d$ such that
\[ (c^2 - c^{r+1-j} \cdot σ_c) \cdot (d^2 - d^{1+r-k} \cdot σ_d) ∈ Z_p[Gal(F/Q)]^X. \]

where $F$ is a cyclic extension of $Q$ of $p$-power degree ramified only at a prime $ℓ$ with $ℓ \equiv 1 \pmod{p}$ and $(ℓ, cdp) = 1$.

**Proof.** Since
\[ (c^2 - c^{r+1-j} \cdot σ_c) \cdot (d^2 - d^{1+r-k} \cdot σ_d) \]
\[ = c^{r+1-j} \cdot (c^{1-r+j} - 1 + 1 - σ_c) \cdot d^{1+r-k} \cdot (d^{1-j-r+k} - 1 + 1 - σ_d) \]
\[ = c^{r+1-k+1} \cdot (c^{1-r+k-1} - 1 + 1 - σ_c) \cdot d^{k-1+r+k} \cdot (d^{1-k+1-r+k} - 1 + 1 - σ_d) \]
\[ = c^{r-k+2} \cdot (c^{k-r} - 1 + 1 - σ_c) \cdot d^r \cdot (d^{2-r} - 1 + 1 - σ_d), \]
under Assumption $(*)$, it suffices to choose $c$ and $d$ such that their reductions mod $p$ are primitive roots mod $p$. □

**Remark A.17.** Assumption $(*)$ is already observed in [Kat04, (5.2.3)]. If $k = 2$, Assumption $(*)$ in Lemma A.16 is automatic.

Following [Kat04, §13.3], we consider the following Euler system
\[ c_dζ_m^{(p)}(f, r, k-1, ξ, \text{prime}(mpN)) \]
in $H^1(Z[1/p, z_m], V_{O_λ}(f)(k-r))$ for the triple $(V_{O_λ}(f)(k-r), F_λ, \text{prime}(cdpN))$ where $ξ ∈ SL_2(Z)$.

**Remark A.18.**
(1) Note that we have the following equality at the level of Iwasawa cohomology
\[
\left( c_d \zeta_{mp^n}^{(p)}(f, r, k - 1, \xi, \text{prime}(mpN)) \right)_n = \left( c_d \zeta_{mp^n}^{(p)}(f, k, k - 1, \xi, \text{prime}(mpN)) \right)_n \otimes \left( \zeta_{p^n}^{\otimes k-r} \right)_n
\]
in \( \lim_{n \to \infty} H^1(\mathbb{Z}[1/p, \zeta_{mp^n}], \mathcal{V}_{\lambda}(f)(k - r)) \).

(2) Since we do not expect
\[
\prod_{\ell \mid N} (1 - a_{\ell}(f) \ell^{-r} \sigma_{\ell}^{-1})
\]
is invertible in general, we do not invert these Euler factors at bad primes.

Under Assumption (** in Lemma A.16, let \( \nu(c, d, k-1, k-r) \) be any element of \( \mathbb{Z}_p[\text{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})] \) which restricts to
\[
\left( (c^2 - c^{r+1-j} \cdot \sigma_c) \cdot (d^2 - d^{j+1+r-k} \cdot \sigma_d) \right)^{-1} \in \mathbb{Z}_p[\text{Gal}(\bar{F}/\mathbb{Q})]
\]
where \( \bar{F} \) is the compositum of all cyclic extensions of \( \mathbb{Q} \) of \( p \)-power degree ramified only at a prime \( \ell \) with \( (\ell, pcdN) = 1 \) and \( \mathbb{Q}_\infty \). Then we define
\[
z_{\gamma, m}^{(Np)}(k-r) := \left\{ \nu(c, d, k-1, k-r) \cdot b_1 \cdot c_d \zeta_{m}^{(p)}(f, r, k - 1, \alpha_1, \text{prime}(pNm)) \right\}^{-1}
+ \left\{ \nu(c, d, k-1, k-r) \cdot b_2 \cdot c_d \zeta_{m}^{(p)}(f, r, k - 1, \alpha_2, \text{prime}(pNm)) \right\}^{+}
\]
in \( H^1(\mathbb{Z}[1/p, \zeta_{m}], \mathcal{V}_{\lambda}(f)(k-r)) \) and then \( z_{\gamma, m}^{(Np)}(k-r) \) obviously forms an integral Euler system and interpolates \( Np \)-imprimitive \( L \)-values of \( f \) at \( s = r \) twisted by finite order characters on \( \text{Gal}(\bar{F}/\mathbb{Q}) \). Since the choice of \( \nu(c, d, k-1, k-r) \) is arbitrary, we do not know the precise interpolation formula for general finite order characters on \( \text{Gal}(\mathbb{Q}^{ab}/\mathbb{Q}) \).

A.4. Optimization of periods. The goal here is to make a good choice of \( \gamma \) in \( z_{\gamma, m}^{(Np)}(k-r) \).

We recall that the period map appeared in §A.1 induces the Shimura isomorphism
\[
\text{per}_f : (S(f) \otimes \mathbb{C})^2 \simeq V_F(f) \otimes \mathbb{C}
\]
defined by \( (f, f) \mapsto \text{per}_f(f) + \iota \text{per}_f(f) \) where \( \iota(x \otimes y) = x \otimes \bar{y} \) for \( x \in V_F(f), y \in \mathbb{C} \). See [Kat04, (7.13.2)] for detail.

A.4.1. Néron periods. Assume that \( f \) corresponds to an elliptic curve \( E \) over \( \mathbb{Q} \). Let \( \omega_f \in S(f) \) be the element corresponding to the Néron differential \( \omega_E \) of \( E \) under the modular parametrization. Then we have
\[
\text{per}_f(\omega_f) = \Omega_E^+ \cdot \gamma^+ + \Omega_E^- \cdot \gamma^-
\]
for some non-zero \( \gamma^\pm \in V_Z(f) \). Here, \( \gamma^\pm \in V_Z(f) \subseteq T(-1) \) is due to Theorem 1.2 and [Kat04, Theorem 13.6] (c.f. [ASS86]).

Choice A.19. We choose \( b_1, b_2 \in \mathbb{Q}(= \mathbb{Q}_{f, \lambda}) \) such that
\[
\gamma^+ = b_1 \cdot \delta(f, 1, \alpha_1)^+, \quad \gamma^- = b_2 \cdot \delta(f, 1, \alpha_2)^-
\]
in Equation (A.1).

Definition A.20. Due to Lemma A.16 with Remark A.17, the Euler system in §3.1 can be chosen by
\[
c_{\mathbb{Q}(\mu_m)}(1) := z_{\gamma, m}^{(Np)}(1)
\]
where \( f \) is the modular form corresponding to \( E \), \( k = 2, j_1 = j_2 = 1, \) and \( \alpha_1, \alpha_2 \) and \( \gamma^\pm \) follows Choice A.19.
We also consider the Iwasawa cohomology version. Let
\[
z_{\gamma}^{(Np)} := \left\{ \nu(c, d, 1, 1) \cdot b_1 \cdot \left( c_d z_{p^n}^{(Np)}(f, 1, 1, \alpha_1, \text{prime}(pNm)) \right)_{n \geq 1} \right\}^{-}
\]
\[
+ \left\{ \nu(c, d, 1, 1) \cdot b_2 \cdot \left( c_d z_{p^n}^{(Np)}(f, 1, 1, \alpha_2, \text{prime}(pNm)) \right)_{n \geq 1} \right\}^+.
\]
Then we define an \(N\)-imprimitive analogue of \(z_{Kato}\) (Definition A.13) for elliptic curves by
\[(A.3) \quad z_{Kato}^{(N)} := \text{cores}_{\mathbb{Q}(\mu_{p^n})/\mathbb{Q}} \left( z_{j_0}^{(Np)}(1) \right).
\]

**Lemma A.21.** Under Assumption 1.1.(1), we have
\[
\text{char}_\Lambda \left( \frac{H^1(T)}{\Lambda z_{Kato}^{(N)}} \right) = \text{char}_\Lambda \left( \frac{H^1(T)}{\Lambda z_{Kato}} \right).
\]

**Proof.** By definition, we have
\[
z_{Kato}^{(N)} = \prod_{\ell|N_{st}} (1 - \ell^{-1}\sigma^{-1}_{\ell}) \cdot \prod_{\ell|N_{ns}} (1 + \ell^{-1}\sigma^{-1}_{\ell}) \cdot z_{Kato},
\]
and, indeed,
\[
\prod_{\ell|N_{st}} (1 - \ell^{-1}\sigma^{-1}_{\ell}) \cdot \prod_{\ell|N_{ns}} (1 + \ell^{-1}\sigma^{-1}_{\ell})
\]
is invertible in \(\Lambda\) under Assumption 1.1.(1). \(\Box\)

**A.4.2. Canonical periods.** We fix an isomorphism \(\iota : \overline{\mathbb{Q}}_p \simeq \mathbb{C}\). Let \(f \in S_k(\Gamma_1(N), \psi)\) and \(\overline{\rho} = \overline{\rho}_f\) be the mod \(p\) representation associated to \(f\). We assume the following property.

**Assumption A.22.** The localized Hecke algebra at the maximal ideal corresponding to \(\rho\) is Gorenstein.

**Theorem A.23.** Assume that \(2 \leq k < p\). If
1. \(p\) does not divide \(N\), or
2. \(p\) exactly divides \(N\) and \(\overline{\rho}\) is \(p\)-distinguished,
then Assumption A.22 holds.

**Proof.** See [Vat99, Theorem 1.13]. \(\Box\)

Under Assumption A.22, we can add a canonical \(p\)-integral structure on the Shimura isomorphism (A.2) following [Vat13, §3]
\[
S(f) \otimes \mathbb{Q}_p \otimes \iota \mathbb{C} \xrightarrow{\text{per}_f} V_{\mathcal{O}\lambda}(f) \otimes_{\mathcal{O}\lambda} F_{\mathcal{O}\lambda} \otimes \iota \mathbb{C}
\]
\[
\xrightarrow{\text{per}_f} \Omega_f^+ : V_{\mathcal{O}\lambda}(f)^+ + \Omega_f^- : V_{\mathcal{O}\lambda}(f)^-
\]
where \(\omega_f\) is an \(\mathcal{O}_\lambda\)-basis of \(S(f) \otimes \mathbb{Q}_p\) and it corresponds to the normalized newform \(f\). Here, \(\Omega_f^\pm\) are the **integral canonical periods of** \(f\). Under the period map, we have
\[(A.4) \quad \text{per}_f(\omega_f) = \Omega_f^+ \cdot \gamma^+ + \Omega_f^- \cdot \gamma^-
\]
for some non-zero \(\gamma^\pm \in V_{\mathcal{O}\lambda}(f)^\pm\). By making the same choice of \(\gamma\) as Choice A.19 with Equation (A.4), we obtain an “optimal” Euler system \(z_{\gamma,m}^{(Np)}(k-r)\).
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(C.-H. Kim) School of Mathematics, Korea Institute for Advanced Study, 85 Hoegiro, Dongdaemun-gu, Seoul 02455, Republic of Korea

E-mail address: chanho.math@gmail.com

(K. Nakamura) Department of Mathematics,Saga University, 1 Honjo-machi, Saga 840-8502, Japan

E-mail address: nkentaro@cc.saga-u.ac.jp