Ideal-magnetohydrodynamic theory of low-frequency Alfvén waves in the H-1 Heliac

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Abstract

A part analytical, part numerical ideal-MHD analysis of low-frequency Alfvén wave physics in the H-1 stellarator is given. The three-dimensional, compressible ideal spectrum for H-1 is presented and it is found that despite the low $\beta$ of H-1 plasmas ($\beta \approx 10^{-4}$), significant Alfvén-acoustic interactions occur at low frequencies. Several quasi-discrete modes are found with the three-dimensional linearized ideal-MHD eigenmode solver CAS3D, including beta-induced Alfvén eigenmode-type modes in beta-induced gaps. The strongly shaped, low-aspect ratio magnetic geometry of H-1 presents a challenging numerical problem for wave-mode physics codes. For CAS3D, convergence requires the inclusion of many Fourier harmonics for the parallel component of the fluid displacement eigenvector even for shear wave motions. Finally, an explanation is offered for the measured configurational frequency dependence in terms of the scaling of the highest beta-induced gap with local temperature at the mode resonant surface. If validated it suggests a technique for temperature inference at the resonant surface of a mode, thereby extending the tools of MHD spectroscopy beyond $q$ profile inference.

(Some figures may appear in colour only in the online journal)

1. Introduction

A rich low-frequency wave phenomenology showing Alfvénic characteristics has been observed in plasmas generated on the H-1 heliac [1–3]. Previous ideal-MHD studies of these waves in cylindrical geometry has shown that cylindrical shear Alfvén continuum turning points reproduce important features of measured magnetic frequencies [1, 2], and cylindrical global Alfvén eigenmode (GAE) modelling produces plausible radial eigenmode profiles and magnetic attenuation predictions [3]. However, it is well known that Alfvén waves existing independently in a cylindrical plasma will couple in a toroid due to magnetic field-strength modulations, and the same is true for coupling between acoustic waves and Alfvén waves at low frequencies [4]. Since these coupling effects are expected to feature prominently in low aspect-ratio, strongly shaped plasmas like those created on H-1, this paper aims to provide a geometrically accurate treatment of H-1 wave physics within the theoretical framework of ideal MHD.

Two significant drivers of Alfvén wave activity studies in the magnetic confinement fusion context are the direct effects such wave activity may have on plasma confinement, through energetic particle interactions for example, and the diagnostic utility wave activity provides as proxy for determining underlying plasma parameters [4]. In particular, low-frequency Alfvén-acoustic waves have drawn increased attention in recent years as they are generally poorly understood and are excited in tokamaks and stellarators [5, 6]. Even under the simplifying assumptions of ideal MHD, the low-frequency domain is difficult to treat theoretically because, in the presence of geodesic curvature, coupling between Alfvén and sound branches introduces considerable complexity to the linearized eigenfrequency spectrum. Furthermore, ideal MHD provides a poor description of compressible motions [7], and it omits thermal ion interactions, such as ion Landau damping, which become significant in this range of frequencies [8]. However, ideal MHD has the important advantage that it
is simple enough to allow for an accurate treatment of the geometry, providing a good first model for understanding the low-frequency domain [8–10].

In this work we present the three-dimensional, compressible ideal spectrum for H-1, based on numerical simulations with the three-dimensional continuous spectrum code CONTI [11] in combination with the three-dimensional ideal-MHD linear eigenmode solver CAS3D [12, 13]. Significant coupling of eigenmode Fourier harmonics in both toroidal and poloidal directions are induced by a substantial mirror term (inherent in a helical axis, low aspect-ratio configuration) in combination with a variety of strong field-strength modulations within a cross-section. We have found that the conditions for convergence of CAS3D in Fourier space are quite stringent for H-1 configurations, in the sense that shear Alfvén frequencies dip sharply near the core unless the component of fluid motion along the field lines has a Fourier basis including at least two toroidal side bands, even for high-frequency modes with negligible acoustic interaction. An explanation for this finding is given.

The first two helical Alfvén eigenmode (HAE) gaps and the toroidal Alfvén eigenmode (TAE) gap are identified. Significant beta-induced gap structure is found, starting at around 35 kHz. Consequently, H-1 fluctuations, which typically lie near or below 35 kHz, may be considered sub-GAM modes [5]. Based on analytic estimates, it is proposed that H-1 fluctuations may include beta-induced Alfvénic eigenmodes (BAEs) partially reproducing characteristic ‘whale-tail’ structures in configuration space due to hollow temperature profiles. Low-frequency discrete modes are presented.

The wider impact of this work is multifold: this is one of the most complete treatments of beta-induced gap mode physics in a heliac, complementing studies on other types for stellarator [11] and tokamak configurations. We have found that the heliac configuration, because it features a high mirror term, presents a strong convergence test for wave-mode physics codes. We have also offered an explanation for frequency scaling with configuration mediated by the temperature profile. If validated, it suggests a technique for temperature inference at the resonant surface of a mode, thereby extending the tools of MHD spectroscopy beyond q profile inference.

The outline of the paper is as follows. Section 2 provides an overview of low-frequency H-1 wave activity. Section 3 discusses H-1 equilibrium and magnetic geometry. Section 4 gives a summary of low-frequency stellarator Alfvén wave physics as well as a discussion of CAS3D convergence requirements for H-1. Section 5 presents the ideal spectrum and low-frequency discrete modes. In section 6 we show that, in the presence of hollow temperature profiles, the highest BAE gap frequency reproduces parts of the configuration-space whale-tails. We conclude in section 7.

2. Overview of low-frequency waves in H-1 plasmas

In this section we discuss low-frequency wave activity on H-1 and give a brief overview of previous studies. H-1 is a ‘flexible’ heliac, which means that a variety of magnetic geometries can be accessed by adjusting $\kappa_h$, a parameter measuring relative coil currents (held constant during a discharge), where $0 \leq \kappa_h \leq 1.2$. Different values of $\kappa_h$ correspond to different plasma parameters, flux-surface geometries and rotational transform $\tau = 1/q$ profiles (see [1] for details). Perhaps the most important of these is the $\tau$ dependence on $\kappa_h$, shown in figure 1.

The configurational dependence of fluctuation frequencies is our main focus as H-1 discharges are usually dominated by a single coherent steady-frequency fluctuation present in a time interval with limited variation in equilibrium magnetic field strength and plasma density [1,3]. Figure 2 shows the frequencies measured by poloidal Mirnov coils at different times within discharges for a selection of discharges [2].

It has been shown [1, 2] that the ‘V’ structures (which we term ‘whale-tails’) near $\kappa_h = 0.4$ and $\kappa_h = 0.75$ in figure 2 qualitatively follow the turning points of the cylindrical shear Alfvén frequency $\omega_A = |n - m| B_0 / R \mu_0 \rho$ for $(m, n) = (4, 5)$ and $(m, n) = (3, 4)$, respectively (compare with low-order rationals in figure 1), where the turning points of $\omega_A$ depend on $\kappa_h$ through $\tau$ and $\rho$ (the mass density), with the average magnetic field strength $B_0 \approx 0.46 \text{T}$ held constant during a discharge. Mode-number assignments for these two regions are obtained from poloidal Mirnov array phase signatures [1, 2] and initial analysis of helical Mirnov array measurements.
where \( m \) and \( n \) are the azimuthal and axial mode numbers, respectively. These are dominant mode number assignments as multiple mode-numbers have not yet been resolved from the available data. Assuming the best information available for mass density and using H-1’s average major radius \( R = 1 \, \text{m} \) in the formula for \( \omega_A \), predicted frequencies were found to be a factor of \( \lambda \approx 3.7 \times 10^5 \) times larger than measured frequencies. The discrepancy factor is reduced to \( \lambda \approx 3 \) if the helical extension of the arclength of the magnetic axis is accounted for by setting \( R = 1.25 \, \text{m} \). This discrepancy was not the same for each of the four branches of the two whale-tails and mode localization assumptions were made for some configurations. Consequently, the value \( \lambda \approx 3 \) should be taken as a coarse measure of the theory-measurement frequency discrepancy.

The frequency scaling behaviour, which is qualitatively analogous to frequency sweeping of reversed-shear Alfven eigenmodes (RSAE) as \( q \) changes in tokamak discharges [4], suggests that the low-frequency modes are GAEs and non-conventional GAEs (NGAEs) at local minima and maxima in \( \omega_A \), respectively. The low toroidal currents of H-1 favour the existence of NGAEs, whereas RSAEs have only been observed in tokamaks during ‘Alfvén cascades’ in which several waves with different mode numbers sweep through a broad frequency range during the course of a discharge, unlike the steady, solitary H-1 modes [14]. However, no robust explanations have been found for the quantitative discrepancy. While significant impurities are likely to be present in H-1 plasmas, cylindrical frequencies are expected to depend weakly on this effect as \( \omega_A \) is proportional to the inverse square root of the effective mass. Collisions with neutrals are likewise not thought to be able to explain such a large discrepancy [1].

Cylindrical ideal-MHD modelling of GAE eigenmodes under H-1 conditions has shown agreement with magnetic attenuation and optical emission measurements [3]. However, as that study assumed cylindrical geometry, no comparisons were made with modes other than GAEs. Furthermore, that study highlighted the multiply peaked nature of the Fourier spectrum of magnetic phase measurements, pointing to the importance of mode-number coupling in H-1. We will return to this ‘GAE hypothesis’ in view of the three-dimensional spectrum.

We are not aware of other reports of any high resolution observations of the dependence of frequency on \( \epsilon \), and the associated whale-tail structure. We expect that such observations are only practicable in devices that are flexible and have low shear, so that the \( \epsilon \) value is adjustable and well defined. The heliac is well suited to this measurement, as \( \epsilon \) is easily variable from \( N/3 \) to \( N/2 \), where \( N = 3 \) for H-1, and shear is low to medium. One of us has analysed data from the TJ-II heliac without success, and the Heliotron-J device with limited success. In both cases, analysis has been constrained by the limited number of significantly different \( \epsilon \) values used for normal machine operations.

3. H-1 equilibrium with VMEC

We use a Boozer coordinate representation \((s, \vartheta, \zeta)\) for equilibrium, where \( s \) denotes the normalized toroidal flux \( \psi / \psi_{\text{edge}} \), \( 2\pi \psi \) is the toroidal flux, and \( \vartheta, \zeta \) are poloidal and toroidal Boozer angles, respectively. The Variational Moments Equilibrium Code (VMEC) [15] solves the ideal equilibrium equation \( \nabla \varrho = (\nabla \times \mathbf{B}) \times \mathbf{B} \) in three dimensions (assuming nested flux surfaces) in its own coordinate system [16], and further computation is required to transform from the lean VMEC coordinate Fourier representation of the magnetic field to the larger Boozer one.

We have opted to give VMEC predetermined \( \varrho \) profiles since the profiles shown in figure 1, which are vacuum field profiles, are considered reliable when a plasma is present due to the fact that H-1 operates at low \( \beta (\beta \approx 10^{-4}) \) [17]. Pressure profiles are obtained from electron density measurements using the ideal gas law \( p = n_e k_B T_e + n_i k_B T_i = 2 n_e k_B T_e \) assuming \( n = n_e \) and constant plasma temperatures \( T_i = T_e = 20 \, \text{eV} \) [2]. VMEC is run in fixed-boundary mode, with boundary surfaces obtained using DESCUR [16] which generates a VMEC coordinate representation of a magnetic surface from a field-line trace. The field-line trace used has field periodicity of \( N_p = 3 \) (i.e. all equilibrium quantities are three-periodic in the toroidal direction) and is stellarator symmetric (symmetry-breaking terms arising from the response of the magnetic field coils to the applied field are ignored).

Transformation to Boozer coordinates is performed with the MC3D code (part of the CAS3D package). This transformation gives poorly reconstructed pressure profiles (from \( \nabla \varrho = (\nabla \times \mathbf{B}) \times \mathbf{B} \) where \( \mathbf{B} \) is reconstructed in the Boozer coordinate Fourier basis) for H-1 configurations unless a VMEC equilibrium with a large number magnetic surfaces \( N_s \) is supplied. Figure 3(a) shows the improvement of the agreement between the prescribed VMEC pressure gradient and the MC3D-reconstructed pressure gradient with increasing \( N_s \). The reconstructed pressure for the \( N_s = 500 \) case is shown in figure 3(b) along with the prescribed VMEC pressure and electron density inversions from a \( k_B = 0.54 \) discharge. Singular behaviour at the core is a consequence of the \( s = 0 \) coordinate singularity arising from the vanishing of the poloidal flux gradient, which affects the VMEC equilibrium as well as the Boozer coordinate transformation [15]. The \( \nabla \varrho \) discrepancy near the edge may be due to the difficulty of representing \( \mathbf{B} \) on the more strongly shaped outer magnetic surfaces, even though the geometry of the flux surfaces is accurately reconstructed. The largest \( N_s = 500 \) Fourier table used before memory requirements became prohibitive was \( 0 < m < 40 \) and \( -30 < n < 30 \). Figure 3(a) shows little improvement moving from \( N_s = 350 \) to \( N_s = 500 \), although agreement is harder to obtain for higher \( \beta \) values (for example \( \beta = 5 \times 10^{-4} \) compared with the present \( \beta = 1 \times 10^{-4} \)) where \( N_s = 500 \) flux surfaces or more are necessary (larger \( \beta \) values are expected with the recently upgraded H-1 heating system [18]).

Stellarator symmetry simplifies the Fourier representation of equilibrium quantities by permitting sine-only or cosine-only expansions. The Boozer coordinate Fourier expansion of \( \mathbf{B} = |\mathbf{B}| \) on a field period can be written as [14, 15]

\[
B(s, \vartheta, \zeta) = \sum_{mn} B_{mn}(s) \cos (m \vartheta + n N_p \zeta)
\]

\[
= B_0 \left( 1 + \sum_{mn} b_{mn}(s) \cos (m \vartheta + n N_p \zeta) \right).
\]
Figure 3. (a) Prescribed VMEC pressure gradient compared with the MC3D-reconstructed pressure gradient for increasing numbers of flux surfaces \( N_s = 100, 200, 350, 500 \). (b) Pressure reconstructed from the \( N_s = 500 \) case in (a), compared with the prescribed VMEC pressure as fitted to electron density inversions [19]. All pressures normalized to core fitted pressure. This is a \( \kappa_B = 0.54 \) configuration.

Figure 4. (a) Fourier components of the magnetic field strength \( B = |B| \) in Boozer coordinates on a field period. The largest four components shown with thick lines are \( (m, n) = (0, 0) \)—red (with \( B_0 = 0.46 \) subtracted), \( (1, 0) \)—blue, \( (1, −1) \)—green, \( (1, −2) \)—orange. All other components in thin black. (b) Fourier components of the contravariant metric component \( g^{ss} \equiv \nabla s \cdot \nabla s \) in magnetic coordinates. The largest three components shown with thick lines are \( (m, n) = (0, 0) \)—red, \( (2, −2) \)—yellow, \( (3, −3) \)—brown. All other components in thin black. This is a \( \kappa_B = 0.54 \) configuration.

Figure 4(a) gives the flux-surface dependence of the \( B_{mn} \) for \( \kappa_B = 0.54 \) in magnetic coordinates showing that \( m = 1 \) Fourier components dominate the poloidal field-strength modulation. Similarly, the Boozer coordinate Fourier representation of the contravariant metric component \( g^{ss} \equiv \nabla s \cdot \nabla s \) can be written as

\[
g^{ss}(s, \theta, \zeta) = \sum_{mn} \tilde{g}^{ss}_{mn}(s) \cos(m\theta + nN_{fp}\zeta)
\]

\[
= \tilde{\delta}_0 \frac{2\psi B_0}{s_{edge}} \left( 1 + \sum_{mn} \tilde{g}^{ss}_{mn}(s) \cos(m\theta + nN_{fp}\zeta) \right),
\]

where the last expression on the right defines \( \tilde{g}^{ss}_{mn}(s) \) and \( \tilde{\delta}_0 = \delta_0(\psi) \) which is related to the elongation of flux-surface cross-sections [14]. The flux-surface dependence of the \( g_{mn}^{ss} \) is shown in Figure 4(b), illustrating the strong ellipticity \( (m = 2) \) and triangularity \( (m = 3) \) of the bean-shaped cross-sections.

4. Wave theory preliminaries

4.1. Mode families

The toroidal \( N_{fp} \)-periodicity of stellarators leads to the existence of independent mode-families, with interaction between mode numbers \( n \) and \( n' \) when

\[
n + n' \in N_{fp}Z \quad \text{or} \quad n - n' \in N_{fp}Z, \quad (3)
\]

where \( N_{fp}Z = \{ \ldots, −2N_{fp}, −N_{fp}, 0, N_{fp}, 2N_{fp}, \ldots \} \) [12]. This represents the effect of interactions between toroidal mode numbers separated by multiples of \( N_{fp} \) due to the toroidal field-periodicity combined with interactions between \( n \) and \( −n \) due to poloidal 'tokamak' magnetic field modulations.

For H-1 configurations, with \( N_{fp} = 3 \), there are two mode-families: \( N = 0 \) with \( n \in \{ \ldots, −6, −3, 0, 3, 6, \ldots \} \) and \( N = 1 \) with \( n \in \{ \ldots, −5, −4, −2, −1, 1, 2, 4, 5, \ldots \} \). We will focus on the \( N = 1 \) mode family since it contains both of the dominant mode-number pairs \( (4, 5) \) and \( (3, 4) \) for the two whale-tails (see section 2).

4.2. Continuous spectrum

The continuous ideal-MHD spectrum for a general three-dimensional toroid can be obtained by CONTI [11] which solves the coupled continuum equations [20]

\[
\left[ \frac{\omega^2 \rho |\nabla \psi|^2}{B^2} + (B \cdot \nabla) \left( \frac{|\nabla \psi|^2}{B^2} (B \cdot \nabla) \right) \right] \xi_s + \gamma \rho \kappa_s (\nabla \cdot \xi) = 0,
\]

where \( \kappa_s = \kappa_B \), \( \gamma \rho \) and \( \xi_s \) are the pitch, density, and poloidal velocity, respectively. The poloidal/azimuthal and toroidal/meridional mode numbers are denoted by \( m \) and \( n \), respectively. The isovector field \( \psi \) is defined as the solution of the ideal-MHD equations.
\[ \kappa_s \xi_s + \left[ \frac{\gamma \rho + B^2}{B^2} + \frac{\gamma \rho}{\alpha_s^2 \beta^2} (B \cdot \nabla) \frac{1}{B^2} (B \cdot \nabla) \right] (\nabla \cdot \xi) = 0, \tag{5} \]

where \( \gamma \) is the adiabatic index, \( \omega \) is the angular frequency, \( \xi \) is the component of the plasma displacement and \( \kappa_s = 2 \kappa_b (B \times \nabla \psi)^/B^2 \) is the geodesic curvature, which is proportional to the surface component \((\nabla \psi \times B)/\kappa_b \) of the magnetic field-line curvature \( \kappa \equiv (b \cdot V b) \) where \( b = B/B \). For a given flux surface these equations give the condition for the existence of non-square-integrable eigenfunctions of the linearized ideal-MHD equations with a spatial singularity at \( \psi \) [20].

Coupling between Fourier components of the shear Alfvén spectrum arises due to modulation in \(|\nabla \psi|/B\), where figure 4 shows the numerator (up to a constant factor) and denominator in this expression. Additionally, for nonzero \( p \) and \( \k_i \) there is coupling between equations (4) and (5), describing the shear Alfvén (compressional and acoustic) and compressional Alfvén branches of the ideal-MHD continuum, respectively. Setting aside the high-frequency compressional Alfvén waves, it can be seen from the identity

\[ \nabla \cdot \xi = b \cdot \nabla \xi_\parallel + \nabla \cdot \xi_\perp \tag{6} \]

that this coupling involves two effects [21]. The first term on the right represents coupling between the acoustic motion parallel to the magnetic field lines and the surface displacement \( \xi_\parallel \). The second term represents the compressional response due to motion across the equilibrium field lines \( v_\perp = B \times E/B^2 \) in the presence of geodesic curvature [9]. If the coupling is strong enough, deviations from the incompressible shear Alfvén spectrum include the formation of Alfvén-sound gaps due to the first effect, and in some cases, immediately above these, \( \beta \)-induced gaps due to the second. The upper limit of the \( \beta \)-induced gaps, located near the surface where \( n - \imath m = 0 \), are called ‘geodesic-acoustic frequencies’, denoted \( \omega_G \).

An approximate expression for the Alfvén resonance ignoring Alfvén–Alfvén interactions (equation (48) in [14]) can be used to obtain estimates for \( \omega_G \) by taking the limit \( n - \imath m \to 0 \) yielding for H-1

\[ \frac{b_{1,0}^2}{\omega^2 - (2N_{\text{ip}} - \imath)^2 c_s^2 / R^2} + \frac{b_{1,-1}^2}{\alpha_s^2 - (2N_{\text{ip}} - \imath)^2 c_s^2 / R^2} \]

\[ + \frac{b_{1,-2}^2}{\alpha_s^2 - (2N_{\text{ip}} - \imath)^2 c_s^2 / R^2} = \frac{2 \psi \delta_0}{B_0 c_s^2}, \tag{7} \]

where all flux-surface quantities are evaluated at the surface where \( n - \imath m = 0 \), \( c_s = \sqrt{\gamma p/\rho} \) is the adiabatic sound speed, \( \delta_0 \sim 1.5 \) and only the largest harmonics identified in figure 4 with thick lines are considered. The geodesic frequencies \( \omega_G \) form a subset of the solutions \( \omega \) to equation (7) [14].

A criterion for estimating the strength of the Alfvén-acoustic interaction based on equation (7) has been proposed, given in terms of the ‘sound parameter’

\[ \Theta = \frac{\epsilon^2 \delta_0 \rho c_s^2}{2 B_{10}^2}, \tag{8} \]

where \( \epsilon = \sqrt{\gamma p / \rho} / B \) is a measure of the inverse aspect ratio and \( b_{10} \) and \( \delta_0 \geq 1 \) are defined in equations (1) and (2), respectively [14]. Strong interaction is predicted when \( \Theta < 1 \) and weak interaction when \( \Theta \gg 1 \) [14]. For H-1 the sound parameter range from \( \Theta \approx 0.75 \) at \( \kappa_b = 0.3 \) to \( \Theta \approx 1.1 \) at \( \kappa_b = 0.9 \) suggesting non-trivial Alfvén-acoustic interactions at low frequencies. H-1 is unusual in this regard, since \( \epsilon/b_{10} > 1 \) for most stellarators (H-1 has \( \epsilon/b_{10} < 1 \), and therefore it is often the case that \( \Theta \gg 1 \) for stellarators with \( r > 1 \) [14].

Note that \( \Theta \) is a geometric parameter. Acoustic frequencies and solutions to equation (7) are proportional to \( c_s \sim \sqrt{\beta} v_A \), where \( v_A = B / \sqrt{} \mu_0 \rho \) is the Alfvén speed. Thus, even if the geometric conditions for strong Alfvén-acoustic interactions are satisfied, smaller \( c_s \) implies smaller gaps and lower frequencies at which these gaps appear. H-1 plasmas have low \( \beta \), but \( c_s \) only depends on \( \sqrt{\beta} \), and the low H-1 plasma density implies \( v_A \approx 5 \times 10^6 \text{ m s}^{-1} \) which is sufficient to bring solutions of equation (7) into the tens of kilohertz.

### 4.3. Discrete spectrum and CAS3D convergence

The full ideal-MHD linearized eigenmode spectrum, including discrete or quasi-discrete modes, can be found with CAS3D [12,13]. CAS3D solves the variational eigenvalue problem [22] \( \delta S/\delta \xi = 0 \), where

\[ S = a^2 K - \delta W, \tag{9} \]

the kinetic energy is \( K = \frac{1}{2} \int \rho |\xi|^2 \, dV \) and the potential energy \( \delta W \) can be written as [12]

\[ \delta W = \frac{1}{2} \int |C|^2 - D(\xi \cdot \nabla s) + \gamma p(\nabla \cdot \xi)^2 \, dV, \tag{10} \]

where

\[ C = \nabla \times (\xi \times B) + J \times \nabla s / |\nabla s|^2 \cdot \nabla s, \tag{11} \]

\[ D = 2 (J \times \nabla s) \cdot (B \cdot \nabla) \nabla s / |\nabla s|^4, \tag{12} \]

and the plasma is assumed to be surrounded by a perfectly conducting wall.

For H-1 configurations, CAS3D produces poorly converged spectra with shear Alfvén frequencies dipping sharply near the core unless the parallel displacement \( \xi_\parallel \) is represented by many Fourier harmonics. Some insight into this problem can be obtained by rearranging equation (6) and solving in magnetic coordinates to give

\[ \xi_{\parallel mn} = -i B/\sqrt{r} (\nabla \cdot \xi - \nabla \cdot \xi_\perp \), \tag{13} \]

where the Jacobian is \( \sqrt{r} = [\nabla (\psi) \cdot (\nabla b \times \nabla \xi)]^{-1} \). For a physical divergence \( \nabla \cdot \xi \) that needs to be numerically approximated, it can be seen that the spectral width of \( \xi_\parallel \) must be larger than that of the other displacement components due to the modulation in \( B/\sqrt{r} \). CAS3D uses Boozer coordinates for which \( B/\sqrt{r} \propto 1/B \).

Inadequate representation of \( \xi_\parallel \) has deleterious effects well outside of the frequency domain in which parallel ion-acoustic motion is physically important. For simplicity, let us consider the case of approximately ‘pure’ shear Alfvén waves which
should have $\nabla \cdot \xi \approx 0$ and $\xi_\parallel \approx 0$. The parallel displacement $\xi_\parallel$ appears only in the fluid compression term $\gamma p |\nabla \cdot \xi|^2$ and the parallel component of the kinetic energy $K_\parallel = \frac{1}{2} \int \rho |\xi_\parallel|^2 \, \mathrm{d}r$. If $\xi_\parallel$ is not represented with a sufficient number of Fourier harmonics, both $\nabla \cdot \xi \approx 0$ and $\xi_\parallel \approx 0$ will not be well satisfied, and both $\int \gamma p (\nabla \cdot \xi)^2 \, \mathrm{d}r$ and $K_\parallel$ will be larger than they should be. Although $\gamma p |\nabla \cdot \xi|^2$ will still be small relative to the magnetic terms in $\delta W$ when $\beta$ is small, an inflated $K_\parallel$ will cause significant distortions to the shear Alfvén spectrum.

This effect has been documented for unstable modes with the three-dimensional ideal spectral code TERPSICHORE [23]. For CAS3D with W7-X equilibria it was found that a one-third larger Fourier table for the parallel displacement was sufficient to avoid these problems [13]. For H-1, due to a significant mirror-term modulation of around 20%, CAS3D Alfvén spectra are grossly distorted near the core where poloidal couplings are small and $\nabla \cdot \xi \approx 0$ cannot locally be satisfied, and both $\int \gamma p (\nabla \cdot \xi)^2 \, \mathrm{d}r$ and $K_\parallel$ will be larger than they should be. Although $\gamma p |\nabla \cdot \xi|^2$ will still be small relative to the magnetic terms in $\delta W$ when $\beta$ is small, an inflated $K_\parallel$ will cause significant distortions to the shear Alfvén spectrum.

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![Figure 5](image.png)

**Figure 5.** (a) CONTI continuous spectrum for $N = 1$, $\kappa_h = 0.30$ with $p$ and $\rho$ linear in $s$ and constant temperature. Sound branches in grey, Alfvén branches in colour. (b) Continuous spectrum for $N = 1$, $\kappa_h = 0.54$ with $p$ and $\rho$ quadratic in $s$ as in figure 3(b) and constant temperature. Modes classified as acoustic not shown, Alfvén branches in yellow apart from Alfvén branches in colour. (with multiples of 3 removed to exclude $N = 0$).

**5. Numerical results**

This section starts with a description of the broad features of the H-1 linearized spectrum before moving on to a more detailed study of the low-frequency part of the spectrum. Configuration space can be divided into four regions of interest corresponding to the left and right sides of the two whale-tails. On the left side of the low-$\kappa_h$ whale-tail, the $(m, n) = (4, 5)$ shear Alfvén continuum intersects the low-frequency domain because there is a flux surface in the plasma with $t = 5/4$ whereas on the right side of the low-$\kappa_h$ whale-tail there is no surface for which $t = 5/4$ and the $(m, n) = (4, 5)$ shear Alfvén continuum rapidly moves to high frequencies with increasing $\kappa_h$. The same pattern repeats itself for the $(m, n) = (3, 4)$ shear Alfvén continuum and the high-$\kappa_h$ whale-tail. Consequently, we focus our attention on $\kappa_h = 0.30$ as representative of low-frequency Alfvén-acoustic interactions on H-1.

**5.1. Alfvén spectrum**

The shear Alfvén gap structure for a $\kappa_h = 0.30$ H-1 configuration obtained with CONTI can be seen in figure 5(a). Two HAE gaps are present, a large one at around 150 kHz corresponding to a coupling displacement of $(\Delta m, \Delta n) = (2, 3)$ (i.e. coupling between $(m, n)$ and $(m + \Delta m, n + \Delta n))$, and above it, a small gap corresponding to $(\Delta m, \Delta n) = (3, 3)$. Above both of these gaps lies the TAE gap at around 350 kHz. This overall gap structure is found to be qualitatively the same for all $\kappa_h$, with significant quantitative differences when there are large changes in $\rho$ near $\kappa_h = 0.4$ and $\kappa_h = 0.75$ [1].
gaps are far too high to offer an explanation for the observed low-frequency wave activity.

Figure 5(b) shows a comparison of the \((4, 5)\) and \((3, 4)\) Alfvén branches with their cylindrical equivalents for \(\kappa_h = 0.54\) (at the overlap of the two whale-tails). This shows that apart from the appearance of gap structure, at frequencies greater than 50 kHz the three-dimensional shear Alfvén frequency turning points are about a factor of 0.85 lower than corresponding cylindrical frequency turning points, implying a reduction of the GAE-measurement frequency discrepancy to \(\lambda \approx 2.5\) (see section 2). Geometric effects therefore alter GAE frequencies more than previously thought \([1, 2]\), but are not sufficient to bring them in line with measurements.

5.2. Alfvén-acoustic spectrum

We now turn our attention to the low-frequency domain. Figure 6(a) shows the low-frequency details of figure 5, where the \((m, n) = (4, 5)\) branch has been identified, confirming the significant Alfvén-acoustic interactions predicted in section 4.2. Several Alfvén-acoustic gaps are formed, with large \(\beta\)-induced gaps appearing above some of these notably at 9 and 12 kHz. Another smaller \(\beta\)-induced gap is found at 30 kHz which is more easily seen in figure 6(b). The positive solutions to equation (7) at \(s = 0.55\) are \(\omega \approx 9.4, 13.2, 31.1\) kHz which are therefore all seen to be approximate geodesic frequency solutions.

Figure 7(a) shows the full \(\kappa_h = 0.3\) low-frequency ideal-MHD spectrum produced by CAS3D, where a small mode-number table (adjacent poloidal mode numbers and two toroidal side bands) has been used to make the gap structure clear. The CAS3D spectrum differs from the CONTI spectrum in that it shows a strong geodesic cut-off near 35 kHz and the lower sections of the shear Alfvén spectrum are absent.

Several ‘quasi-discrete’ modes are found below 35 kHz, meaning that the mode spatial structures have large sections
apparently free from continuum interactions. Figure 8(a) shows an example BAE using a small Fourier basis in order to suppress acoustic resonances. Even with the inclusion of many sound branches, eigenmode clusters with shear Alfvén characteristics are found as shown in figure 7(b) using the potential energy classification scheme. Quasi-discrete modes are found in some of these clusters, such as the mode shown in figure 8(b), found near the cluster circled in figure 7(b). As these are fixed-boundary simulations, the fluid displacement orthogonal to the flux surfaces $\xi^s$ must vanish at the edge, which it does due to a sharp gradient discontinuity near the edge presumably due to continuum interactions. Modes with a very similar structure but the discontinuity appearing further inside the plasma are also found. The modes shown in figure 8 may therefore experience significant continuum damping.

Low-frequency quasi-discrete modes dominated by $(m, n) = (3, 4)$ are also found for the left branch of the high-$\kappa_h$ whale-tail. Two $\kappa = 0.60$ quasi-discrete modes are shown in figures 9(a) and (b), with frequencies of 38 kHz and 34 kHz, respectively. The mode in figure 9(b) shows clearly a typical example of the aforementioned near-edge resonances. On the right sides of the whale-tails, near $\kappa = 0.50$ and $\kappa = 0.90$, we have not been able to find low-frequency Alfvénic discrete modes dominated by $(m, n) = (4, 5)$ and $(m, n) = (3, 4)$, respectively.

6. Alfvén-acoustic whale-tails?

Since $\beta$-induced gaps below 50 kHz and quasi-discrete modes with reduced continuum damping exist in these gaps, we offer a partial alternative explanation of H-1 wave activity in terms of BAEs or $\beta$-induced Alfvén-acoustic eigenmodes (BAAEs).

Figure 10(a) shows the $\kappa_h$ dependence of the $(m, n) = (4, 5)$ cylindrical shear Alfvén continuum $\omega_A$ near $\kappa = 0.4$ (the low-$\kappa$ whale-tail) and figure 10(b) shows the same for $(m, n) = (3, 4)$ near $\kappa = 0.75$ (the high-$\kappa$ whale-tail). The left branches of both of these whale-tails coincide with the inward migration of the outermost surface at which the cylindrical shear Alfvén continuum vanishes $s_{\text{res}}$. The geodesic-acoustic minimum of the three-dimensional compressible shear Alfvén spectrum is located near $s_{\text{res}}$ (compare figure 6(a) with the $\kappa = 0.30$ curve in figure 10(a)), and it will migrate inwards with $s_{\text{res}}$.

There are indications that H-1 electron temperature profiles increase steeply towards the plasma edge and also that they ‘hollow out’ as $\kappa_s = 0.4$ and $\kappa_s = 0.75$ are...
approached. Consider the left side of the low-κₜₕ whale-tail covering the range 0.3 ≤ κₜₕ ≤ 0.4. In the absence of ion temperature measurements or accurate spatially resolved electron temperature profiles for these values of κₜₕ, we postulate a total temperature profile \( T(\kappa, s) \) that evolves linearly from \( T = 27 \text{ eV} \) at \( \kappaₜₕ = 0.3 \) and \( s_{\text{res}} \approx 0.55 \) to \( T = 8 \text{ eV} \) at \( \kappaₜₕ = 0.38 \) and \( s_{\text{res}} \approx 0.25 \) where the values \( s_{\text{res}} \) are taken from the \((m, n) = (4, 5)\) Alfvén branch (compare with figure 10). We emphasize: this temperature profile is not intended to represent the temperature for a single discharge, it represents local temperatures in \((s, \kappaₜₕ)\) space postulating a simple linear profile consistent with preliminary spatially resolved electron temperature reconstructions [19], assuming \( T_e = T_i \). These temperatures can be used in equation (7) to estimate how the \( \approx 32 \text{ kHz} \) branch of \( \omega_G \) evolves with \( \kappaₜₕ \) for the \((m, n) = (4, 5)\) shear Alfvén branch in the range 0.3 ≤ κₜₕ ≤ 0.4, which is shown in figure 11. Comparison of figure 11 with figure 2 shows that BAEs or BAAEs excited just below \( \omega_G \) will show a very similar configurational frequency dependence with the left side of the low-κ whale-tail.

Unfortunately the available electron temperature measurements do not extend to the outer flux surfaces \((s > 0.6)\) which would be necessary for an explicit comparison of \( \omega_G \) in the range 0.6 ≤ κₜₕ ≤ 0.75 with the left side of the high-κ whale-tail. It is clear though that a similar frequency dependence on κₜₕ will be observed provided that the temperature increases accordingly; that it increases to some degree is fairly certain.

7. Discussion and conclusions

The numerical results presented in section 5 supplement the most detailed numerical study of low-frequency Alfvén-acoustic spectra in stellarators to date [11], which did not consider heliac configurations. That study could not determine whether quasi-discrete modes were present on an HSX configuration (which they considered in detail) due to ubiquitous continuum resonances at low frequencies in the ideal-MHD description. While we have found many reasonably smooth low-frequency quasi-discrete modes for H-1 configurations with the same numerical tools, these modes will probably experience enhanced continuum interactions in the presence of the non-constant temperatures postulated in section 6. Whether these interactions will outweigh the relevant driving mechanisms is an important question that requires more detailed physics than ideal-MHD alone provides [10].

The presence of \( \beta \)-induced gaps and partial gaps, the existence of discrete modes in these gaps and the predicted configurational dependence of the highest BAE gap frequency in the presence of H-1’s hollow temperature profiles can account for important features of wave frequency measurements, namely the left sides of the V-shaped ‘whale-tail’ frequency structures observed in magnetic configuration space. Thus, a potentially complete explanation of the left sides of both whale-tails at both \( \kappaₜₕ \) values has been provided, which does not require the arbitrary scale factor of roughly 3 required for an explanation in terms of GAE eigenmodes. The possibility that BAE-type modes are being observed on H-1 is intriguing because the operating parameters of H-1 are quite dissimilar from the tokamak conditions in which the BAE has been previously documented [6]. There is some experimental evidence of the geodesic-acoustic nature of the fluctuations observed at around 10 kHz (at the centre of the whale-tails), with the fluctuations tending to become more electrostatic in
nature and showing a marked reduction in poloidal variation with phase [1]. It is interesting that low-frequency oscillations associated with $E \times B$ flow (which is closely connected with the formation of $\beta$-induced gaps) have been documented in TJ-II, the other large heliac experiment [24]. It is also possible that BAAE modes are being observed, which, if true, may prove to be of some interest since it remains unclear how BAAEs are able to avoid strong ion Landau damping. H-1 physics may be able shed some light on this matter since the observed low-frequency modes are a robust feature of H-1 plasmas [5, 10, 11].

This ‘BAE/BAAE hypothesis’ does not, however, provide an explanation for the right sides of the whale-tails as the relevant resonant surface will leave the plasma and the Alfvén continuum will rapidly approach higher frequencies. For example, at $\kappa_b = 0.54$ the (4, 5) minimum frequency (see figure 5(b)) is more than twice as large as measurements. Furthermore, no discrete modes were found below the (4, 5) and (3, 4) shear Alfvén continuum minima near $\kappa = 0.45$ and $\kappa = 0.9$ (the right-hand sides of the whale-tails). Nevertheless, as quasi-discrete Alfvénic modes with dominant mode numbers (4, 5) and (3, 4) could not be found at low frequencies for $\kappa = 0.45$ and $\kappa = 0.9$, respectively, there do not appear to be any ideal-MHD candidates for the right sides of the whale-tails apart from GAEs that have had their frequencies lowered by non-ideal effects, or by an effective mass $m_{\text{eff}}$ around six times greater near the plasma core than our impurity-free assumption of $m_{\text{eff}} = 2.5m_p$ where $m_p$ is the proton mass.

We have also shown that the spectral width of the magnetic coordinate Fourier representation of the parallel fluid displacement is larger than that of the perpendicular fluid displacement unless $\sqrt{\rho} \propto 1/B$. (Magnetic coordinate systems with $\sqrt{\rho} \propto 1/B$ are usually avoided in applications as they do not have continuous gridlines [25].) Consequently, codes that solve the variational formulation of the linearized ideal-MHD equations (usually in Boozer coordinates) for high mirror-term stellarators will encounter the problems described section 4.3, namely eigenfrequencies erroneously dropping off to zero in the vicinity of the core, unless a substantially enlarged Fourier representation is used for the parallel fluid displacement.

In terms of future work, it is important that uncertainty in the effective mass and its radial dependence be reduced, as this parameter affects both shear Alfvén and acoustic frequency predictions. If accurate spatially resolved measurements of the relative plasma density fluctuations can be obtained, comparison with Mirnov array measurements would enable calculation of mode polarization which would help one to classify the observed modes. We also plan to obtain energetic particle drive estimates for CAS3D eigenfunctions and extend equilibrium and spectral calculations to incorporate a free boundary and vacuum wave field solutions.

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