COBOUNDARY CATEGORIES AND LOCAL RULES

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Abstract. We develop a theory of local rules for coboundary categories and show how this theory applies to crystals. This generalises combinatorial constructions using jeu-de-taquin from tableaux to words in crystals of minuscule representations.

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1. Introduction

Let $C$ be a crystal of a finite type Cartan matrix. For $r \geq 0$, let $\otimes^r C$ be the crystal of words in $C$ of length $r$. Then we have a canonical isomorphism of crystals

$$\otimes^r C \cong \bigsqcup_{\omega} C(\omega) \times B(\omega)$$

where $B(\omega)$ is a set and $C(\omega)$ is a connected crystal.

Taking the particular case in which $C$ is the crystal of the vector representation of $GL(n)$ we recover the Robinson-Schensted correspondence. In this case $B(\omega)$ is the set of standard tableaux of shape $\omega$.

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The main result in [HK06] is the construction of an action of the \(r\)-fruit cactus group, \(C_r\), on each set \(B(\omega)\). Our objective is to understand the combinatorial significance of these actions. This action is defined using several applications of the Lusztig involution. In this paper we develop a more efficient approach based on local rules. In the case \(C\) is the crystal of the vector representation of \(\text{GL}(n)\) these are the local rules introduced by Fomin and used to define evacuation, see [Sta99, Chapter 7: Appendix 1].

The action of the \(r\)-fruit cactus group on the set of standard tableaux of shape \(\omega\) for \(\omega\) a partition of size \(r\) is studied in [Spe14]. These actions encode much of the standard combinatorics of tableaux which is described in [Sag91, Chapter 3]. The main examples are that promotion, evacuation, and dual Knuth moves are each given by the action of a specific element of the cactus group. This definition of these operations generalises to define these operations on the sets \(B(\omega)\) for any crystal \(C\).

In order to make use of the general theory of local rules we need a formula for the local rules. The only known explicit and practical formula for local rules only applies to crystals of minuscule representations. This rule is [vL98, Rule 4.1.1] and was related to the crystal commutor in [Len08, Proposition 4.1].

The non-trivial minuscule representations are known and, for the convenience of the reader, we give the list here:

- **type \(A_n\):** All exterior powers of the vector representation.
- **type \(B_n\):** The spin representation.
- **type \(C_n\):** The vector representation.
- **type \(D_n\):** The vector representation and the two half-spin representations.
- **type \(E_6\):** The two fundamental representations of dimension 27.
- **type \(E_7\):** The fundamental representation of dimension 56.

There are no nontrivial minuscule representations in types \(G_2\), \(F_4\) or \(E_8\).

Taking the exterior powers of the vector representation of \(\text{GL}(n)\) the general construction gives an action of the cactus group on conjugate semistandard tableaux. This gives a novel approach to some known combinatorics, see [CGP16]. The two minuscule representations of exceptional groups are isolated examples. Hence the main applications of our results are expected to be to the remaining minuscule representations of classical groups.

2. COBOUNDARY CATEGORIES

Coboundary categories are monoidal categories with extra structure which implies that taking the tensor product of two objects in the two possible orders gives two objects which are naturally isomorphic. We
assume, for simplicity of exposition, that a monoidal category means a strict monoidal category.

Coboundary categories were first defined in [Dri89] where the motivating example is the representation category of a quantised enveloping algebra. In this section, we recall the definition and show, following [Gur], that coboundary categories can also be defined as algebras over an operad for a groupoid operad constructed from the cactus groups.

2.1. Coboundary categories. The original definition of a coboundary category from [Dri89] is:

**Definition 2.1.** A coboundary category is a monoidal category together with natural maps \( \sigma_{A,B} : A \otimes B \to B \otimes A \) for all objects \( A, B \). These maps are required to satisfy the three conditions

- \( \sigma_{A,I} = 1_A \) and \( \sigma_{I,B} = 1_B \)
- \( \sigma_{A,B} \circ \sigma_{B,A} = 1_{B \otimes A} \)
- the following diagram commutes

\[
\begin{array}{ccc}
A \otimes B \otimes C & \xrightarrow{1_A \otimes \sigma_{B,C}} & A \otimes C \otimes B \\
\sigma_{A,B} \otimes 1_C & & \sigma_{A,C} \otimes B \\
B \otimes A \otimes C & \xrightarrow{\sigma_{B \otimes A,C}} & C \otimes B \otimes A
\end{array}
\]

These conditions imply that the following diagram commutes

\[
\begin{array}{ccc}
C \otimes B \otimes A & \xrightarrow{\sigma_{C \otimes B,A}} & A \otimes C \otimes B \\
\sigma_{C,B} \otimes 1_A & & \sigma_{B,A} \otimes 1_C \\
B \otimes A \otimes C & \xrightarrow{\sigma_{B,A \otimes C}} & A \otimes B \otimes C
\end{array}
\]

2.2. Cactus groups. The finite presentations of the cactus groups were originally given in [Dev99].

**Definition 2.2.** The \( r \)-fruit cactus group, \( \mathfrak{C}_r \), has generators \( s_{p,q} \) for \( 1 \leq p < q \leq r \) and defining relations

- \( s_{p,q}^2 = 1 \)
- \( s_{p,q} s_{k,l} = s_{k,l} s_{p,q} \) if \( [p,q] \cap [k,l] = \emptyset \)
- \( s_{p,q} s_{k,l} = s_{p+q-l,p+q-k} s_{p,q} \) if \( [k,l] \subseteq [p,q] \)

There is a homomorphism \( \mathfrak{C}_r \to \mathfrak{S}_r \) defined by \( s_{p,q} \mapsto \hat{s}_{p,q} \) where \( \hat{s}_{p,q} \) is the permutation

\[
\hat{s}_{p,q}(i) = \begin{cases} 
p + q - i & \text{if } p \leq i \leq q \\
i & \text{otherwise}
\end{cases}
\]

Note that \( \mathfrak{C}_r \) is generated by \( s_{1,q} \) for \( 2 \leq q \leq r \), since

\[
s_{p,q} = s_{1,q} s_{1,q-p+1,q}
\]
A different set of generators are the elements $\sigma_{p,s,q}$ defined by

$$\sigma_{p,s,q} = s_p q s_{s+1,q} s_{r}$$

Let $\tilde{\sigma}_{p,s,q}$ be the image of $\sigma_{p,s,q}$ under the homomorphism $\mathfrak{C}_r \to \mathfrak{S}_r$. Then $\tilde{\sigma}_{p,s,q}$ is the permutation

$$\tilde{\sigma}_{p,s,q}(i) = \begin{cases} 
  i & \text{if } 1 \leq i < p \\
  i + s - p & \text{if } p \leq i \leq s \\
  i + s - q & \text{if } s < i \leq q \\
  i & \text{if } q < i \leq r 
\end{cases}$$

The following relations between the two sets of generators are given in [HK06, Lemma 3].

$$\sigma_{p,q,r} s_{k,l} = \begin{cases} 
  s_{k+q-r,r+q-r} \sigma_{p,q,r} & \text{if } [k : l] \subseteq [p : r] \\
  s_{k+p-r-1,l+p-r-1} \sigma_{p,q,r} & \text{if } [k : l] \subseteq [r + 1 : q] 
\end{cases}$$

$$s_{p,q} = \sigma_{p,q,r} s_{p,r} s_{r+1,q}$$

### 2.3. Cactus operad

Our approach to operads follows [LTV12, Chapter 5]. Let $\mathcal{C}$ be a cocomplete symmetric monoidal category. The basic examples are sets and vector spaces. In this paper we also use topological spaces, groupoids and categories.

A **collection** in $\mathcal{C}$ is a functor $\mathfrak{S}_* \to \mathcal{C}$. There is a functor from collections to endofunctors of $\mathcal{C}$. The endofunctor associated to the collection $F$ is given on objects by

$$F(V) = \bigoplus_{n \geq 0} F[n] \otimes \mathfrak{S}_n \otimes^n V$$

The category of collections has a monoidal structure given by

$$(F \circ G)(V) = F(G(V))$$

Then an **operad** in $\mathcal{C}$ is a monad in the category of collections; and an algebra for the operad is an algebra for the monad.

Next we construct a groupoid operad from the cactus groups and show that coboundary categories are algebras for this operad.

**Definition 2.3.** For $n \geq 0$, the cactus groupoid $\mathfrak{C}_n$ has objects $\mathfrak{S}_n$. A morphism is of the form

$$u \xrightarrow{c} u.\hat{c}$$

for $u \in \mathfrak{S}_n$ and $c \in \mathfrak{C}_n$.

For $n \geq 0$, the groupoid $\mathfrak{C}_n$ has an action of $\mathfrak{S}_n$ given by

$$w : (u \xrightarrow{c} u.\hat{c}) \mapsto (w.u \xrightarrow{c} w.u.\hat{c})$$

Denote the collection in groupoids $n \mapsto \mathfrak{C}_n$ by $\mathfrak{C}_*$.  

**Proposition 2.4.** The collection in groupoids, $\mathfrak{C}_*$, admits the structure of an operad.
Proof. There is a topological operad constructed in [HK06]. This is a variation of the mosaic operad which is a cyclic operad constructed in [Dev99]. The spaces are denoted by $\tilde{\mathcal{M}}_{n+1}^0$.

For $n \geq 0$, the groupoid $\tilde{\mathcal{C}}_n$ is the fundamental groupoid of $\tilde{\mathcal{M}}_{n+1}^0$ relative to a basepoint which has a free and transitive action of $S_n$. □

The operad $\tilde{\mathcal{C}}_\bullet$ gives a monad on categories such that algebras for the monad are algebras for the operad. This monad is given by:

$$C \mapsto \coprod_{r \geq 0} C^r \times_{\tilde{\mathcal{C}}_r} \tilde{\mathcal{C}}_r$$

Example 2.5. The free coboundary category on the category with one morphism is the groupoid $\coprod_{r \geq 0} \mathcal{C}_r$.

The next result is [HK06, Theorem 10] and [Gur, Corollary 4.14].

Proposition 2.6. Any coboundary category is an algebra for the category operad $\tilde{\mathcal{C}}_\bullet$.

Proof. The groupoid operad $\tilde{\mathcal{C}}_\bullet$ defines a monad on categories. The claim is that a coboundary category is an algebra for this monad.

Let $C$ be a coboundary category. We define functors, for $r \geq 0$,

$$C^r \times_{\tilde{\mathcal{C}}_r} \tilde{\mathcal{C}}_r \to C$$

These functors are determined by the definition on the generators $\sigma_{p,s,q}$ by

$$[A_1, A_2, \ldots, A_r] \mapsto (A_1 \otimes \cdots A_{p-1}) \otimes \sigma_{(A_p \otimes \cdots A_s), (A_{s+1} \otimes \cdots A_q)} \otimes (A_{q+1} \otimes \cdots A_r)$$

In particular, for each $r \geq 0$ and each object $A$, there is a natural action of $\mathcal{C}_r$ on $\otimes^r A$.

3. Local rules

First we define local rules in a coboundary category.

Definition 3.1. The local rules, $\tau^A_{B,C}$: $A \otimes B \otimes C \to A \otimes C \otimes B$, are defined to be the composite

$$(10) \quad A \otimes B \otimes C \xrightarrow{\sigma_{A,B} \otimes 1_C} B \otimes A \otimes C \xrightarrow{\sigma_{B,A} \otimes C} A \otimes C \otimes B$$

The coboundary condition is recovered by taking $\sigma_{B,C} = \tau^B_{B,C}$. The coboundary condition can then be rewritten as

$$\tau^C_{A,B} \sigma_{A,C} \otimes 1_B 1_A \otimes \sigma_{B,C} = \sigma_{B,C} \otimes 1_A \tau^B_{A,C} \sigma_{A,B} \otimes 1_C$$

as maps $A \otimes B \otimes C \to C \otimes B \otimes A$.

In the rest of this section we give some further properties of local rules.
Lemma 3.2. For all $A, B, C$, $\tau_{B,C}^A \tau_{C,B}^A = 1_{A \otimes B \otimes C}$.

Proof. The following diagram is commutative

\[
\begin{array}{cccccc}
A \otimes B \otimes C & \xrightarrow{\sigma_{A,B} \otimes 1_C} & B \otimes A \otimes C & \xrightarrow{\sigma_{B,A} \otimes C} & A \otimes C \otimes B \\
1_A \otimes \sigma_{B,C} & \downarrow & \sigma_{B \otimes A} & \downarrow & \sigma_{A,B} \otimes 1_C \\
A \otimes C \otimes B & \xrightarrow{\sigma_{A,C} \otimes B} & C \otimes B \otimes A & \xrightarrow{1_C \otimes \sigma_{B,A}} & C \otimes A \otimes B \\
1_A \otimes \sigma_{C,B} & \downarrow & \sigma_{C,B} \otimes 1_A & \downarrow & \sigma_{B,A} \otimes C \\
A \otimes B \otimes C & \xrightarrow{\sigma_{A,B} \otimes C} & B \otimes C \otimes A & \xrightarrow{\sigma_{B,C} \otimes A} & A \otimes B \otimes C
\end{array}
\]

This completes the proof since the top edge is $\tau_{B,C}^A$, the right edge is $\tau_{C,B}^A$, the left edge is $1_{A \otimes B \otimes C}$ and the bottom edge is $1_{A \otimes B \otimes C}$. □

Lemma 3.3. For all $A, B, C, D$,

\[
\tau_{B,C}^A \otimes D = \tau_{B,D}^A \tau_{B,C}^A \otimes 1_D
\]

Proof. By Lemma 3.3 we have

\[
\tau_{B,D}^A \tau_{B,C}^A \otimes 1_D = (\sigma_{B,A \otimes C \otimes D} \sigma_{A \otimes C,B} \otimes 1_D) \left( \sigma_{B,A \otimes C} \otimes 1_D \sigma_{A,B} \otimes 1_C \otimes D \right) = \sigma_{B,A \otimes C \otimes D} \sigma_{A,B} \otimes 1_C \otimes D = \tau_{B,C \otimes D}^A
\]

This is a rephrasing of [Len08, Lemma 4.3].

Lemma 3.4. For all $A, B, C, D$,

\[
\tau_{A \otimes B, C}^D = \tau_{A \otimes B}^D \tau_{A \otimes B, C}^D \otimes 1_D
\]

Proof. By Lemma 3.3 we have

\[
\tau_{D,B \otimes C}^A = \tau_{D,C}^A \tau_{D,B}^A \otimes 1_C
\]

Taking the inverse of both sides using Lemma 3.2 gives the result. □

The next lemma constructs the commutors $\sigma_{B,\otimes^{r+1}B}$ for $r > 0$ from the local rules $\tau_{B,B}^I$.

Lemma 3.5. For each object $B$ and $r \geq 0$,

\[
\sigma_{B,\otimes^{r+1}B} = \tau_{B,B}^{\otimes r} \tau_{B,B}^{\otimes r-1} \cdots \tau_{B,B}^{\otimes 0} \tau_{B,B}^{I}
\]

Proof. The proof is by induction on $r$. The base of the induction is the case $r = 0$ which is the observation that $\tau_{B,C}^I = \sigma_{B,C}$.

Substitute $A = I$, $C = \otimes^r B$, $D = B$ in Lemma 3.3 3.4. This gives

\[
\sigma_{B,\otimes^{r+1}B} = \tau_{B,B}^{\otimes r} \sigma_{B,\otimes^r B} \otimes 1_B
\]

□
4. Crystals

For each finite type Cartan matrix there is a monoidal category of finite crystals. These were shown to be coboundary categories in [HK06]. In this section we describe the local rules for the full monoidal subcategory generated by the crystals of minuscule representations.

4.1. Crystals. In this section we give a summary of the basic theory of crystals. This is based on [Jos95, Chapter 5].

If $X$ is a set then $X^*$ is the pointed set $X \coprod \{0\}$. A partial function $X \rightarrow Y$ can be identified with a map of pointed sets $X^* \rightarrow Y^*$.

The minimal data for a normal crystal is a finite set $B$ together with partial functions $e_i: B \rightarrow B$ for $i \in I$. Each $e_i$ is injective and nilpotent.

A morphism $B \rightarrow B'$ is a partial function $F: B^* \rightarrow B'^*$ such that $F \circ e_i = e'_i \circ F$ for $i \in I$.

The minimal data for a crystal is usually presented as a directed graph with vertex set $B$ and edges labelled by $I$. There is an $i$-edge from $x \rightarrow y$ if and only if $e_i x = y$. The directed graph of a crystal has no oriented cycles.

The minimal data we have presented determines additional data. In most presentations of the theory of crystals this data is included in the definition.

The first additional data we add are the partial functions $f_i: B \rightarrow B$ for $i \in I$. These are defined by $f_i x = y$ if and only if $e_i y = x$. These are also injective and nilpotent.

Now we define functions $\varepsilon_i, \varphi_i: B \rightarrow \mathbb{N}$ by

$$\varepsilon_i(x) = \max\{k | e_i^k \neq 0\}$$

$$\varphi_i(x) = \max\{k | f_i^k \neq 0\}$$

The weight function is given by

$$\text{wt}_i(x) = \langle \varepsilon(x) - \varphi(x), \alpha_i^\vee \rangle$$

An element $x \in B$ is highest weight if $e_i(x) = 0$, or, equivalently, if $\varepsilon_i(x) = 0$ for all $i \in I$.

A homomorphism between crystals induces a weight-preserving partial function between the highest weight elements and is determined by this partial function.

The tensor product of crystals $B$ and $C$ is constructed as follows:

The set underlying $B \otimes C$ is $B \times C$. The partial functions, $e_i$, for $i \in I$, are defined by

$$e_i(x \otimes y) = \begin{cases} e_i(x) \otimes y & \text{if } \varphi(x) \geq \varepsilon(y) \\ x \otimes e_i(y) & \text{otherwise} \end{cases}$$
The functions $\varepsilon_i$ and $\varphi_i$ are defined by
\[
\varepsilon_i(x \otimes y) = \varepsilon_i(x) + \max\{0, \varepsilon_i(y) - \varphi_i(x)\}
\]
\[
\varphi_i(x \otimes y) = \varphi_i(x) + \max\{0, \varphi_i(x) - \varepsilon_i(y)\}
\]

Then we have $\text{wt}(x \otimes y) = \text{wt}(x) + \text{wt}(y)$.

The highest weight elements of $B \otimes C$ are the elements $x \otimes y$ such that $x \in B$ is highest weight and $\varepsilon_i(y) \leq \varphi_i(x)$ for $i \in I$.

4.2. Words. Let $C$ be a crystal of a finite type Cartan matrix. For $r \geq 0$, let $\otimes^r C$ be the crystal of words in $C$ of length $r$. Then we have a canonical isomorphism of crystals
\[
\otimes^r C \cong \bigsqcup \omega C(\omega) \times B(\omega)
\]
where $B(\omega)$ is a set and $C(\omega)$ is a connected crystal. This is an isomorphism of crystals so for $w \leftrightarrow (P, Q)$ we have $e_i w \leftrightarrow (e_i P, Q)$ and $f_i w \leftrightarrow (f_i P, Q)$ for all $\alpha \in I$.

Let $w \leftrightarrow (P, Q)$ under this correspondence. Then $P$ generalises the insertion tableau and $Q$ generalises the recording tableau. For $w \leftrightarrow (P, Q)$ and $w' \leftrightarrow (P', Q')$; then $P = P'$ means $w$ and $w'$ are in the same position in isomorphic components; and $Q = Q'$ means $w$ and $w'$ are in the same component.

In special cases there is a combinatorial construction of the sets $B(\omega)$ and a corresponding insertion algorithm. In general we can take $B(\omega)$ to be the set of highest weight words of length $r$ and weight $\omega$.

The main result of [HK06] is that the monoidal category of crystals is a coboundary category. This gives a natural action of $\mathfrak{C}_r$ on the set $B(\omega)$ for each dominant weight, $\omega$. Using the equivalence between involutions and matchings, we can describe this action by giving a graph with vertices $B(\omega)$ and an edge labelled $(p, q)$ between $u$ and $v$ if $u \neq v$ and $s_{p,q} : u \leftrightarrow v$. An example is shown in Figure 1.

4.3. Tableaux. Take $C$ to be the crystal of the vector representation of $\text{GL}(n)$. Then the decomposition (14) is the Robinson-Schensted correspondence with $B(\omega)$ the set of standard tableaux of shape $\omega$.

Let $\omega$ be a partition of size $r$. Then we have an action of the $r$-fruit cactus group, $\mathfrak{C}_r$, on the set of standard tableaux of shape $\omega$. The group $\mathfrak{C}_r$ is generated by the elements $s_{1,p}$ for $2 \leq p \leq r$; so the action of $\mathfrak{C}_r$ is determined by the action of these elements. For $2 \leq p \leq r$, the action of $s_{1,p}$ is given by applying evacuation to the subtableau with entries $1, \ldots, p$ leaving the remaining entries fixed.

Example 4.1. Take $C$ to be the crystal of the two dimensional representation of $\text{SL}(2)$. The set $B(0)$ is the set of noncrossing perfect matchings on $r$ points. The action of $s_{1,p}$ is given by the rule that each
pair \((i, j)\) with \(i < j\) gives a pair

\[
\begin{cases}
(p - j + 1, p - i + 1) & \text{if } i < j \leq p \\
(p - i + 1, j) & \text{if } i \leq p < j \\
(i, j) & \text{if } p < i < j
\end{cases}
\]

The case \(r = 6\) is shown in Figure 1.

Then we have the following properties:
- evacuation is given by the action of \(s_{1,r}\)
- promotion is given by the action of \(s_{1,r}s_{2,r}\)
- the elements \(s_{p, p+1}\) act trivially
- the dual Knuth move \(D_i\) is given by the action of \(s_{i, i+2}\)

4.4. Local rules. Fix a crystal \(C\). Then the action of \(\mathfrak{C}_r\) on \(\otimes^r C\) is determined by the involutions \(\tau^B_{C,C}\) where \(B\) is arbitrary. These involutions are crystal homomorphisms and so are determined by the restriction to highest weight elements.

It is convenient to represent a highest weight element \(x \otimes y \in B \otimes C\) by \(\lambda \rightarrow y \mu\) where \(\lambda = \text{wt}(x)\) and \(\mu = \text{wt}(x \otimes y)\).

This notation extends to highest weight words. Given a word \(w = x_1 \otimes x_2 \otimes \cdots \otimes x_r\) define the weights \(\lambda_k\) for \(0 \leq k \leq r\) by \(\lambda_0 = 0\) and \(\lambda_i = \lambda_{i-1} + \text{wt}(x_i)\) for \(1 \leq k \leq r\). Then the word \(w\) is highest weight if and only if \(\varepsilon(\lambda_{k-1}) \leq \varphi_i(x_k)\) for \(i \in I\) and \(1 \leq k \leq r\). This implies that \(\lambda_k\) is dominant for \(0 \leq k \leq r\). The word \(w\) is then represented by

\[
0 \xrightarrow{x_1} \lambda_1 \xrightarrow{x_2} \cdots \xrightarrow{x_{r-1}} \lambda_{r-1} \xrightarrow{x_r} \lambda_r
\]
It is convenient to represent the local rules by a square as shown in Figure 2. The vertices are dominant weights. The crystal $A$ has highest weight $\kappa$. This diagram represents the two following equations which are equivalent by Lemma 3.2:

$$
\tau^A_{B,C}(a, h, v) = (a, v', h') \quad \tau^A_{C,B}(a, v', h') = (a, h, v)
$$

where $a$ is the highest weight element of $A$.

In this notation we have that Lemmas 3.3 and 3.4 say that these are equal and similarly for two squares stacked vertically.

**Definition 4.2.** A representation is minuscule if the Weyl group acts transitively on the weights of the representation.

If $C$ is minuscule then we can safely omit the edge labels as these are determined by the corner labels.

**Definition 4.3.** For each weight $\lambda$, there is a unique weight which is both dominant and in the Weyl group orbit of $\lambda$. Denote this element by $\text{dom}_W(\lambda)$.

**Example 4.4.** For $\text{GL}(n)$, the corner labels are weakly decreasing sequences of integers of length $n$. The Weyl group is $\mathfrak{S}_n$ and $\text{dom}_{\mathfrak{S}_n}$ takes a sequence of integers of length $n$ and rearranges into weakly decreasing order.

**Example 4.5.** For $\text{Sp}(2n)$, the corner labels are partitions of length $n$. The Weyl group is a hyperoctahedral group. The map $\text{dom}_W$ takes a sequence of integers of length $n$, forms the absolute values, and rearranges into weakly decreasing order.

The following interpretation of [vL98, Rule 4.1.1] is given in [Len08, Proposition 4.1].
Proposition 4.6. Using the notation in Figure 2: for minuscule crystals $B$ and $C$ the following are equivalent
\[
\tau_{B,C}^A(b, h, v) = (a, v', h') \quad \mu = \text{dom}_W(\kappa + \nu - \lambda)
\]
\[
\tau_{C,B}^A(b, v', h') = (a, h, v) \quad \lambda = \text{dom}_W(\kappa + \nu - \mu)
\]

5. Growth diagrams

The local rules are used to build growth diagrams. Growth diagrams are used to define operations. The main examples are:

- promotion; two row growth diagrams
- evacuation; triangular growth diagrams
- rectification; rectangular growth diagrams

Define evacuation to be the action of $s_{1,r}$. The growth diagram for evacuation is triangular, see [Sta99, Chapter 7: Appendix 1 Figure A1-13].

Define promotion to be the action of $s_{1,r}s_{2,r}$. The growth diagram for promotion consists of two rows, see [Rub11, Figure 6].

Definition 5.1. Let $B$ be a crystal and $w \in B$. Then the rectification of $w$, $\text{rect}(w)$, is the unique highest weight element in $B$ such that $w$ and $\text{rect}(w)$ are in the same connected component of $B$.

Lemma 5.2. Let $B$ be a crystal and $w \in B$. Choose a crystal $A$ and $u \in A$ such that $u \otimes w \in A \otimes B$ is highest weight. Then $\sigma_{A,B}(u, w) = (\text{rect}(w), u')$.

Proof. Put $\sigma_{A,B}(u, w) = (w', u')$. Then $w'$ is highest weight because $w'u'$ is highest weight. Also $w$ and $w'$ are in the same component because $\sigma$ is a crystal morphism.

This is given in terms of local rules by a rectangular growth diagram, see [Sta99, Chapter 7: Appendix 1 Figure A1-11].

5.1. Evacuation. In this section we give the growth diagram description of evacuation. We show that the result of applying $s_{1,r}$ to a highest weight word, as in (15), can be read off a triangular diagram. The triangular diagram for the case $r = 3$ is shown in Figure 8.

Definition 5.3. A triangular growth diagram is a triangular diagram with vertices labelled by dominant weights and edges labelled by elements of crystals such that the labels of each cell satisfy the local rules as represented in Figure 2. We also require that the first vertex on each row is labelled by the zero weight.

These conditions imply that a triangular growth diagram can be reconstructed from the top edge viewed as a highest weight word.

Proposition 5.4. Given a triangular growth diagram let $w$ be the top edge viewed as a highest weight word and let $w'$ be the right edge viewed as a highest weight word. Then $w' = s_{1,r}(w)$. 

This is an extension of [Whi15, Proposition 3.31].

Proof. The proof is by induction on $r$. The basis of the induction is the case $r = 2$.

The inductive step is based on the relation $s_{1,r+1} = \sigma_{1,r,r+1} s_{1,r}$ which is a special case of (2). The inductive step is given by interpreting this in terms of growth diagrams. The inductive hypothesis gives the growth diagram for $s_{1,r}$ and (16) (with the squares stacked vertically) gives the growth diagram for $\sigma_{1,r,r+1}$. □

Note that it follows from the symmetry of the local rules that the reflection of a triangular growth diagram is also a triangular growth diagram. This shows that $w' = s_{1,r}(w)$ if and only if $w = s_{1,r}(w')$ so evacuation is an involution.

5.2. Cylindrical diagrams. This section follows [Spe14].

The actions of the generators of $C_r$ are defined using cylindrical growth diagrams, following [Spe14, §6] and [Whi15].

Definition 5.5. Let $\mathbb{I} \subset \mathbb{Z}^2$ be the subset consisting of pairs $(i, j)$ such that $0 \leq j – i \leq r$. A cylindrical growth diagram of shape $\lambda/\mu$ is a function $\gamma$ from $\mathbb{I}$ to partitions such that

- $\gamma(i, j) = \mu$ if $j – i = 0$
- $\gamma(i, j) = \lambda$ if $j – i = r$
- Every step north/up adds a box.
- Every step east/right adds a box.
- Every unit square satisfies the local rule.

Example 5.6. This is a cylindrical growth diagram. The rows are standard tableaux.
**Definition 5.7.** A path through $\mathcal{I}$ is a sequence $(i_0, j_0), (i_1, j_1), \ldots, (i_r, j_r)$ of elements of $\mathcal{I}$ such that $i_0 = j_0$ and, for $0 \leq k < r$, the difference $(i_{k+1}, j_{k+1}) - (i_k, j_k)$ is either $(-1, 0)$ or $(0, 1)$.

Given a cylindrical growth diagram, $\gamma$ and a path we can restrict $\gamma$ to the path by taking the sequence $\gamma(i_0, j_0), \gamma(i_1, j_1), \ldots, \gamma(i_r, j_r)$ to get a highest weight vector. For a fixed path this is a bijection between cylindrical growth diagrams and paths. The inverse map is giving by extending using the boundary conditions and the local rules. The conditions on local rules are required to show this is well-defined.

**Definition 5.8.** Define an operator $[p : q]$ on cylindrical growth diagrams of length $r$ and shape $\lambda/\mu$. The action of $[p : q]$ on $\gamma$ satisfies

$$(17) \quad ([p : q]\gamma)(i, j) = \begin{cases} 
\gamma(i, j) & \text{if } [i : j] \cap [p : q] = \emptyset \\
\gamma(i, j) & \text{if } [p : q] \subset [i : j] \\
\gamma(i', j') & \text{if } [i : j] \subset [p : q]
\end{cases}$$

where $i' = p + q - j$ and $j' = p + q - i$. These conditions specify $([p : q]\gamma)$ on the path $(p, j)$, $j \geq p$ and so determine $([p : q]\gamma)$.

**Theorem 5.9.** The map $s_{p,q} \mapsto [p : q]$ extends to an action of $\mathfrak{C}$, on highest weight words of length $r$ and shape $\lambda/\mu$.

This is an extension of the description of wall crossing in [Spe14, § 6.1]. This is illustrated in [Whi15, Figure 4].

**Proof.** This follows from Proposition [5.4] and the identity (5).

**Example 5.10.** This is an example of a cylindrical growth diagram for a symplectic group. The rows are oscillating tableaux.

If we start with the oscillating tableau on the first row

\[
\begin{array}{cccccc}
\emptyset & \square & \square & \square & \square & \emptyset \\
\emptyset & \square & \square & \square & \square & \emptyset \\
\emptyset & \square & \square & \square & \square & \emptyset \\
\emptyset & \square & \square & \square & \square & \emptyset \\
\emptyset & \square & \square & \square & \square & \emptyset \\
\emptyset & \square & \square & \square & \square & \emptyset \\
\emptyset & \square & \square & \square & \square & \emptyset
\end{array}
\]
Promotion is given by the second row

\[
\begin{array}{cccccccc}
\emptyset & \Box & \Box & \Box & \Box & \Box & \Box & \emptyset \\
\emptyset & \Box & \Box & \Box & \Box & \Box & \Box & \emptyset \\
\emptyset & \Box & \Box & \Box & \Box & \Box & \Box & \emptyset \\
\end{array}
\]

Evacuation is given by the column with head \emptyset

\[
\begin{array}{cccccccc}
\emptyset & \Box & \Box & \Box & \Box & \Box & \Box & \emptyset \\
\emptyset & \Box & \Box & \Box & \Box & \Box & \Box & \emptyset \\
\emptyset & \Box & \Box & \Box & \Box & \Box & \Box & \emptyset \\
\end{array}
\]

To obtain the action of \( s_{3,6} \) on this we can start by filling in the third row or the sixth column. Then we complete the diagram using the local rules. This gives:

\[
\begin{array}{cccccccc}
\emptyset & \Box & \Box & \Box & \Box & \Box & \Box & \emptyset \\
\emptyset & \Box & \Box & \Box & \Box & \Box & \Box & \emptyset \\
\emptyset & \Box & \Box & \Box & \Box & \Box & \Box & \emptyset \\
\end{array}
\]

Then we take the oscillating tableaux on the first row.

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