Seepage consolidation under elastic body’s deformation under normal load

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Abstract. The process of seepage consolidation of an elastic saturated body under the action of a normal load that is instantly applied to its surface is considered. The fluid and the skeleton grains are assumed to be incompressible and the volume skeleton strains are related to grain repacking. To the well-known spatial consolidation scheme is added the equality obtained using the conditions of compatibility of deformations. It is shown that the sum of effective normal stresses satisfies the heat equation and with a known boundary condition can be found as a solution to the corresponding boundary value problem. On the surface of the body for pressure is adopted the condition «high-permeability piston». A pressure-related auxiliary function that satisfies the Laplace equation is introduced. The boundary condition for it is determined by the boundary condition for the above sum. The proposed scheme for studying the consolidation of an elastic body is illustrated by the example of uniform normal loading of the surface of an elastic porous sphere. The formulations of the corresponding boundary value problems are given. In the analytical form the pressure of the fluid, the total and effective normal stresses of the skeleton, the displacement of points of the sphere and its surface in the process of consolidation were found. It is shown that the pressure of the fluid at each fixed point inside the sphere decreases with increasing time.

1. Introduction
The formation and development of seepage consolidation theory is related to the work of K. Terzaghi, N.M. Gersevanov, V.A. Florin and others. A general mathematical model for seepage consolidation and analytical methods for its study were proposed by M. Bio. A bibliography was given in [1, 2]. In [3] a mathematical model of consolidation under plane deformation of elastic half-space under the action of normal vertical load with the use of the compatibility equation was proposed. The analytical representation for fluid pressure and total normal stresses of elastic porous skeleton is obtained.

In the present study the approach used in [3] is implemented for the development of an appropriate mathematical model for consolidation of an elastic saturated porous body under the action of an arbitrary normal load on its surface. As an example the consolidation of an elastic saturated porous sphere under the action of the surface normal uniformly distributed load is studied.

2. Mathematical model of consolidation of elastic saturated body
The process of seepage consolidation of a fluid-saturated elastic porous body under the action of an instantaneously applied normal load \( \Pi \) on its surface \( S \) is considered.

The mathematical model of consolidation includes the complete equation of motion (phase quasi-equilibrium), the continuity (mass balance) equations, the filtration law, rheological relations for the
porous skeleton, initial conditions [4]. These equations are supplemented by the equality obtained using the conditions of compatibility of deformations [5]:

$$\Delta p = \frac{1 - \nu}{1 + \nu} \Delta J^f,$$

(1)

where $p$ is the fluid pressure, $J^f = \sigma_{11}^f + \sigma_{22}^f + \sigma_{33}^f$, $\sigma_{11}^f$, $\sigma_{22}^f$, $\sigma_{33}^f$ is the components of effective normal stresses [1], $\nu$ is Poisson's ratio. From the results of [4] with considering (1) follows equation for $J^f$

$$\frac{\partial J^f}{\partial t} = \kappa \Delta J^f,$$

(2)

where $t$ is the time, $\kappa = kE(1-\nu)/[\mu_0(1-\nu-2\nu^2)]$, $k$ is the skeleton permeability, $E$ is Young's modulus, $\mu_0$ is the fluid viscosity.

The initial condition for the function $J^f$ is [4]

$$J^f(x_1, x_2, x_3, 0) = 0.$$

Here, $x_1$, $x_2$, $x_3$ are Cartesian coordinates. The boundary condition for the function $J^f$ coincides with the boundary condition for the function $J = \sigma_{11} + \sigma_{22} + \sigma_{33}$, where $\sigma_{11}$, $\sigma_{22}$, $\sigma_{33}$ is the components of total normal stresses. The formulation of the corresponding boundary value problem requires the specification of the last boundary condition.

On the boundary for the pressure we impose the condition of the type of «high-permeability piston» [1]

$$p(x_1, x_2, x_3, t) = 0, \ x_i \subset S, \ i = 1 \div 3.$$

(3)

We introduce the following auxiliary function

$$F = p - \frac{1 - \nu}{1 + \nu} J^f,$$

(4)

in accordance with (1) satisfying the equation

$$\Delta F = 0.$$

(5)

With a known boundary condition for the function $J^f$, the boundary condition for the function $F$ with using the formulas (3), (4) is also known.

3. Consolidation of an elastic saturated sphere under a uniform normal load

Suppose that at the time $t = 0$ a uniform load $\Pi$ is instantly applied to the surface of the sphere (see Figure 1). We introduce spherical coordinates $r$, $\varphi$, $\theta$. In the center of the sphere $r = 0$, its radius is equal $R$.

Imagine the total normal stresses in the form

$$\sigma_r = \sigma_r^f - p, \ \sigma_\varphi = \sigma_\varphi^f - p, \ \sigma_\theta = \sigma_\theta^f - p.$$

(6)
where \( \sigma_f^f \), \( \sigma_\phi^f \), \( \sigma_\theta^f \) is the effective stresses in the skeleton of sphere.

**Figure 1.** The elastic saturated sphere under the action of a normal load on its surface

Find the sum of the effective normal stresses \( J^f = \sigma_f^f + \sigma_\phi^f + \sigma_\theta^f \). The initial and boundary conditions for \( J^f \) are

\[
J^f(r,0) = 0, \quad J^f(R,t) = -3\Pi.
\]  
(7)

The boundary problem (2), (7) for the sphere has the following solution [6]

\[
J^f(r,t) = -3\Pi \left[ 1 - \frac{2R}{\pi r} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{\pi n r}{R} \exp \left( -\frac{\kappa \pi^2 n^2 t}{R^2} \right) \right].
\]  
(8)

The boundary condition for the function \( F = F(r) \) satisfying equation (5), in accordance with (3), (4), (7), is:

\[
F(R) = \frac{3\Pi(1-\nu)}{1+\nu}.
\]  
(9)

The boundary value problem (5), (9) for the sphere has the trivial solution

\[
F(r) = \frac{3\Pi(1-\nu)}{1+\nu}.
\]  
(10)

When \( t = 0 \) \( J^f = 0 \), \( \nu = 1/2 \), and from (4), (10) it follows that \( p(r,0) = \Pi \). From (4), (8), (10) is the fluid pressure \( p = p(r,t) \). Let

\[
T = \frac{\kappa \pi^2 t}{R^2}, \quad P(r,T) = \frac{p(r,t)}{\Pi}.
\]
Then

\[ P(r, T) = \frac{6(1 - \nu)}{1 + \nu} R \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{\pi n r}{R} \exp(-Tn^2). \tag{11} \]

From the relations (6), (8), (11), can be determined the sum of total normal stresses

\[ J(r, T) = -3\Pi \left[ 1 + \frac{4(1 - 2\nu)}{1 + \nu} R \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{\pi n r}{R} \exp(-Tn^2) \right] \]

and the normal stresses [7]

\[ \sigma_r (r, T) = \sigma_\phi (r, T) = \frac{J(r, T)}{3}. \]

Next we find the normal displacement \( u_r = u_r (r, T) \). The sum of effective normal stresses \( J^f = \sigma_r^f + \sigma_\phi^f + \sigma_\theta^f \) – the first stress invariant, is related to the volume skeleton strains \( \Theta = \varepsilon_r + \varepsilon_\phi + \varepsilon_\theta \) by the relation

\[ J^f = \frac{E}{1 - 2\nu} \Theta. \tag{12} \]

It is known [8] that

\[ \Theta = \frac{1}{r^2} \frac{d}{dr} \left( r^2 u_r \right). \tag{13} \]

From (12), (13) follows the equation

\[ \frac{1}{r^2} \frac{d}{dr} \left( r^2 u_r \right) = \frac{1 - 2\nu}{E} J^f (r, T), \tag{14} \]

the right part of which is represented by formula (8).

The function \( u_r = u_r (r, T) \) satisfying equation (14), (8) and the condition \( u_r (0, T) = 0 \), has the form

\[ u_r (r, T) = -\frac{\Pi(1 - 2\nu)}{E} \left( r - \frac{6R^3}{\pi^3 r^2} \sum_{n=1}^{\infty} (-1)^{n+1} \left[ \sin \frac{\pi n r}{R} n^3 - \frac{\pi r}{R} \cos \frac{\pi n r}{R} n^2 \right] \exp(-Tn^2) \right). \]

We introduce dimensionless quantities
\[ U_r(r, T) = \frac{E u_r(r, T)}{\pi R}, \quad f(T) = 1 - \frac{6}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \exp(-Tn^2). \]

The displacement of the surface of the sphere on which \( r = R \) is described by the formula

\[ U_r(T) = -(1 - 2\nu) f(T). \]

4. Conclusions

It is shown that the study of the process of consolidation of an elastic saturated body under the action of a load applied instantaneously to its surface reduces to solution of standard boundary-value problems for heat and Laplace equations. With a uniform application of normal load to the surface of the sphere, the time-dependent fluid pressures, the total normal stresses and the displacement of the surface of the sphere were found.

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