Magnonic Casimir Effect in Ferrimagnets

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Quantum fluctuations are the key concepts of quantum mechanics. Quantum fluctuations of quantum fields induce a zero-point energy shift under spatial boundary conditions. This quantum phenomenon, called the Casimir effect, has been attracting much attention beyond the hierarchy of energy scales, ranging from elementary particle physics to condensed matter physics together with photonics. However, the application of the Casimir effect to spintronics has not yet been investigated enough, particularly to ferrimagnetic thin films, although yttrium iron garnet (YIG) is one of the best platforms for spintronics. Here we fill this gap. Using the lattice field theory, we investigate the Casimir effect induced by quantum fields for magnons in insulating magnets and find that the magnonic Casimir effect can arise not only in antiferromagnets but also in ferrimagnets including YIG thin films. Our result suggests that YIG, the key ingredient of magnon-based spintronics, can serve also as a promising platform for manipulating and utilizing Casimir effects, called Casimir engineering. Microfabrication technology can control the thickness of thin films and realize the manipulation of the magnonic Casimir effect. Thus, we pave the way for magnonic Casimir engineering.

Introduction.—Toward efficient transmission of information that goes beyond what is offered by conventional electronics, the last two decades have seen a significant development of magnon-based spintronics \cite{1}, called magnonics. The main aim of this research field is to use the quantized spin waves, magnons, as a carrier of information in units of the reduced Planck constant $\hbar$. A promising strategy for this holy grail is to explore insulating magnets. Thanks to the complete absence of any conducting metallic elements, insulating magnets are free from drawbacks of conventional electronics, such as substantial energy loss due to Joule heating. This is the advantage of insulating magnets. Thus, exploring quantum functionalities of magnons in insulating magnets is a central task in the field of magnonics.

Quantum fluctuations of photon fields induce a zero-point energy shift, called the Casimir energy, under spatial boundary conditions. This Casimir effect is a fundamental phenomenon of quantum physics, and the original platform for the Casimir effect was the photon field \cite{2–4}, which is described by quantum electrodynamics. The concept can be extended to various fields such as scalar, vector, tensor, and spinor fields. Nowadays, the Casimir effect has been attracting much attention beyond the hierarchy of energy scales, ranging from elementary particle physics to condensed matter physics \cite{5, 6}. As an example, see Refs. \cite{7–14} for Casimir effects in magnets \cite{15}. However, the application of the Casimir effect to spintronics has not yet been studied enough, particularly to ferrimagnetic thin films (see Fig. 1), although yttrium iron garnet (YIG) \cite{16} has been playing a central role in spintronics.

Here we fill this gap. In terms of the lattice field theory, we investigate the Casimir effect induced by quantum fields for magnons in insulating magnets and refer to it as the magnonic Casimir effect. We study the behavior of the magnonic Casimir effect with a particular focus on the thickness dependence of thin films, which can be experimentally controlled by microfabrication technology \cite{17, 18}. Then, we show that the magnonic Casimir effect can arise not only in antiferromagnets (AFMs) but also in ferrimagnets with realistic model parameters for YIG thin films. Our study indicates that YIG, an ideal platform for magnonics, can serve also as a key ingredient of Casimir engineering \cite{19} which aims at exploring quantum-mechanical functionalities of nanoscale devices.

Antiferromagnets.—We consider insulating AFMs described by the quantum Heisenberg Hamiltonian which has $U(1)$ symmetry about the quantization axis and

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig1.png}
\caption{Schematic of the ferrimagnetic thin film for the magnonic Casimir effect. Two kinds of magnons (circles) arise from the alternating structure of up and down spins (arrowed lines). Wavy lines represent quantum-mechanical behaviors of magnons in the discrete energy.}
\end{figure}
study the behavior of the magnonic Casimir effect with a focus on the thickness dependence. The AFM is a two-sublattice system, and the ground state has the Néel magnetic order [20]. From the spin-wave theory with the Bogoliubov transformation, elementary excitations are two kinds of magnons designated by the index \( \sigma = \pm \) having the spin angular momentum \( \sigma \hbar \). Owing to the \( U(1) \) symmetry, the Hamiltonian can be recast into the diagonal form with the magnon energy dispersion for the wave number \( \mathbf{k} = (k_x, k_y, k_z) \) as \( \epsilon_{\sigma, \mathbf{k}} \) where the total spin angular momentum of magnons is conserved. Two kinds of magnons \((\sigma = \pm)\) are in degenerate states. Hence, we study the low-energy magnon dynamics of the insulating AFM by using the quantum field theory of complex scalar fields, i.e., the complex Klein-Gordon field theory [21, 22]. Then, we can see that there exists a zero-point energy \([23]\). This is the origin of the Casimir effect.

Note that throughout this study, we focus on clean insulating magnets and work under the assumption that the total spin along the quantization direction is conserved and thus remains a good quantum number.

Through the lattice regularization, the Casimir energy \( E_{\text{Cas}} \) is defined as the difference between the zero-point energy \( E_{0}^{\text{int}} \) for the continuous energy \( \epsilon_{\sigma, \mathbf{k}} \) and the one \( E_{0}^{\text{sum}} \) for the discrete energy \( \epsilon_{\sigma, \mathbf{k}, n} \) with \( n \in \mathbb{Z} \). In two-sublattice systems, such as AFMs and ferrimagnets (see Fig. 1), the wave numbers on the lattice are replaced by \((k_j a)^2 \rightarrow [2(1 - \cos(k_j a))] \) in the \( j \) direction for \( j = x, y, z \), where \( a \) is the length of a magnetic unit cell. Here, by taking the Brillouin zone (BZ) into account, we set the boundary condition for the \( z \) axis in wave space \( \mathbf{k} = (k_x, k_y, k_z) \) as \( k_z \rightarrow \pi n/L_z \), i.e., \( k_z a \rightarrow \pi n/N_z \), where \( L_z := aN_z \) is the thickness of magnets, \( N_z \in \mathbb{N} \) is the number of magnetic unit cells in the \( j \) direction for \( j = x, y, z \), and \( n = 1, 2, ..., 2N \) for \( N \in \mathbb{N} \). The number of unit cells on the \( xy \) plane is \( 4N_x N_y \), and that of magnetic unit cells is \( N_x N_y \). Thus, the magnonic Casimir energy per the number of magnetic unit cells \( N_x N_y \) on the surface for \( N_z = N \) is described as \([24–28]\)

\[
E_{\text{Cas}} := E_{0}^{\text{sum}}(N) - E_{0}^{\text{int}}(N), \tag{1a}
\]

\[
E_{0}^{\text{sum}}(N) = \sum_{\sigma = \pm} \int_{\text{BZ}} d^2(k_{\perp}) \left( \frac{1}{2} \sum_{n=1}^{2N} |\epsilon_{\sigma, \mathbf{k}, n}| \right), \tag{1b}
\]

\[
E_{0}^{\text{int}}(N) = \sum_{\sigma = \pm} \int_{\text{BZ}} d^2(k_{\perp}) \left( \frac{k_{\perp}}{2\pi} \right)^2 \int_{\text{BZ}} d(k_{\parallel}) \left( \frac{\epsilon_{\sigma, \mathbf{k}}}{2\pi} \right), \tag{1c}
\]

where \( k_{\perp} := \sqrt{k_x^2 + k_y^2} \), \( d^2(k_{\perp}) = d(k_x a) d(k_y a) \), the integral is over the first BZ, and the factor \( 1/2 \) arises from the zero-point energy for the scalar field. Since the constant terms which are independent of the wave number do not contribute to the Casimir energy, we drop them throughout this study. To see the dependence of the Casimir energy \( E_{\text{Cas}} \) on the thickness of magnets \( L_z := aN_z \), it is convenient to introduce the rescaled Casimir energy \( C_{\text{Cas}} \) in terms of \( N_z^b \) for \( b \in \mathbb{R} \) as

\[
C_{\text{Cas}}^{[b]} := E_{\text{Cas}} \times N_z^b. \tag{2}
\]

Then, we refer to \( C_{\text{Cas}}^{[b]} \) as the magnonic Casimir coefficient in the sense that \( E_{\text{Cas}} = C_{\text{Cas}}^{[b]} N_z^b \).

Here, we consider magnons in AFMs on a cubic lattice with the energy dispersion \( \epsilon_{\sigma, \mathbf{k}} = \epsilon_{\sigma, \mathbf{AFM}} \) [29]:

\[
\epsilon_{\sigma, \mathbf{k}} = \hbar \omega_0 \sqrt{\delta + (\frac{k a}{2})^2}, \tag{3a}
\]

\[
\hbar \omega_0 := 2\sqrt{3} JS, \tag{3b}
\]

\[
\delta := 3 \left[ \left( \frac{K_{\delta}}{6J} \right)^2 + 2 \left( \frac{K_{\delta}}{6J} \right) \right], \tag{3c}
\]

where \( k := |k| \), \( J > 0 \) parametrizes the exchange interaction between the nearest-neighbor spins of the spin quantum number \( S \), and \( K \geq 0 \) is the easy-axis anisotropy, which provides the magnon energy gap and ensures the Néel magnetic order. Two kinds of magnons \((\sigma = \pm)\) are in degenerate states. In the absence of the spin anisotropy, \( K = 0 \), the energy gap vanishes \( \delta = 0 \), and the gapless magnon mode behaves like a relativistic particle with the linear energy dispersion. From the results obtained in Refs. [30–32], we roughly estimate the model parameter values for \( \text{Cr}_2\text{O}_3 \), as an example, as follows [29]: \( J = 15 \text{ meV}, \ S = 3/2, \ K = 0.03 \text{ meV}, \) and \( a = 0.49607 \text{ nm} \). These parameter values provide \( \hbar \omega_0 \sim 77.94 \text{ meV} \) and \( \delta \sim 2 \times 10^{-3} \).

Figure 2 shows that the magnonic Casimir effect arises in the thin film of the AFM. The magnonic Casimir energy \( E_{\text{Cas}} \) of the magnitude \( O(10^{-2}) \) meV is induced for \( N_z \geq 2 \). Even in the presence of the magnon energy gap \( \delta \neq 0 \), the absolute value amounts to \( O(10^{-2}) \) meV and decreases as the magnon energy gap increases. Thus, the magnonic Casimir energy takes a maximum

![Figure 2](image-url)
absolute value in the gapless mode $\delta = 0$, where the magnon behaves like a relativistic particle with the linear energy dispersion. We remark that in the case of the gapped magnon modes, the absolute value of the magnonic Casimir coefficient $C_{\text{Cas}}^{[3]} = E_{\text{Cas}} \times N^3$ decreases as the thickness of the thin film increases. This behavior is similar to the Casimir effect known for massive degrees of freedom [33, 34]. In the case of the gapless mode, the magnonic Casimir coefficient $C_{\text{Cas}}^{[3]}$ approaches asymptotically to a constant value, and the magnonic Casimir energy exhibits the behavior of $E_{\text{Cas}} \propto 1/N^3$ as the thickness increases. The asymptotic value of $C_{\text{Cas}}^{[3]}$ for the gapless magnon mode $\delta = 0$ given in the numerical result (see Fig. 2) is estimated approximately as $(-\pi^2/720) \times (\hbar \omega_0/2) \sim -0.5341$ meV from an analytical calculation. The factor of $-\pi^2/720$ is well known as the analytic solution for the conventional Casimir effect of a massless complex scalar field in continuous space [34]. Thus, although the magnonic Casimir effect is realized on the lattice, it is qualitatively and quantitatively analogous to the continuous counterpart, except for $a$-dependent lattice effects.

Ferrimagnets.—We develop the study of AFMs into ferrimagnets where the ground state has an alternating structure of up and down spins on a cubic lattice (see Fig. 1). In contrast to AFMs, the spin quantum number on the two-sublattice is different from each other in ferrimagnets. Hence, the degeneracy for two kinds of magnons ($\sigma = \pm$) is intrinsically lifted. In ferrimagnetic thin films, dipolar interactions due to the nonzero magnetization play a key role. Still, at low temperatures where the magnon-magnon interaction of the fourth order in magnon operators is negligibly small [35], the number of magnons and the total spin angular momentum are conserved, and the Hamiltonian for the ferrimagnetic thin film can be diagonalized with the magnon energy dispersion $e_{\sigma, \mathbf{k}} = e_{\text{ferri}, \mathbf{k}}$.

Here, we consider magnons in the thin film of clean insulating ferrimagnets on a cubic lattice subjected to in-plane magnetic fields at such low temperatures. Still, due to the competition between dipolar and exchange interactions, the minimum energy point shifts from the zero mode of magnons, $\mathbf{k} = 0$, to a finite wave number mode which is characterized by the thickness of the thin film $L_z = aN_z$ (see Fig. 1).

The magnon energy dispersion along the in-plane direction is provided in Refs. [36, 37], whereas the dispersion along the $z$ axis in the thin film has not yet been established [38]. Hence, taking into account the competition between dipolar and exchange interactions in the thin film, we phenomenologically assume the behavior that the power of $k_z$, $l \in \mathbb{R}$, approaches asymptotically to $l = 2$ in the bulk limit, whereas it slightly differs from $l = 2$ as long as we consider the thin film (see Fig. 1). Using this assumption and Refs. [36, 37], the magnon energy dispersions are

$$
e_{\sigma, \mathbf{k}}^{\text{ferri}} = \sqrt{\sigma H_0 + \Delta_{\sigma} + \frac{D_z}{a^2} (k_{\perp} a)^2 + \frac{D_z}{a^2} (|k_z| a)^l} \quad (4a)$$

$$\times \sqrt{\sigma H_0 + \Delta_{\sigma} + \frac{D_z}{a^2} (k_{\perp} a)^2 + \frac{D_z}{a^2} (|k_z| a)^l + \sigma \hbar \omega_M \mathcal{F}_k}.$$  

$$\mathcal{F}_k := \mathcal{P}_{k_{\perp}} (1 - \mathcal{P}_{k_{\perp}}) \frac{\sigma \hbar \omega_M}{\sigma H_0 + \Delta_{\sigma} + \frac{D_z}{a^2} (|k_z| a)^l} \left( \frac{k_{\perp}}{k_{\perp}} \right)^2,$$  

$$+ 1 - \mathcal{P}_{k_{\perp}} \left( \frac{k_y}{k_{\perp}} \right)^2,$$  

$$\mathcal{P}_{k_{\perp}} := 1 - \frac{1 - e^{-k_{\perp} L_z}}{k_{\perp} L_z}, \quad (4c)$$

where the external magnetic field is applied along the $y$ axis (see Fig. 1) and $\sigma H_0$ represents the resulting Zeeman energy, $\Delta_{\sigma} \geq 0$ is the (intrinsic) magnon energy gap in ferrimagnets, $D_z(a) > 0$ is the spin stiffness constant, $k_{\perp} := \sqrt{k_x^2 + k_y^2}$, $\hbar \omega_M := 4\pi\gamma M_s$ with the saturated magnetization density $M_s$ and the gyromagnetic ratio $\gamma$, and the term $\mathcal{F}_k$ is responsible for the shift of the minimum energy point from the zero mode to a finite wave number mode due to the competition between dipolar and exchange interactions in the ferrimagnetic thin film: The first term of $\mathcal{F}_k$ [see Eq. (4b)] reproduces the Damon-Eshbach magnetostatic surface mode [39], and the last term reproduces the backward volume magnetostatic mode [39].

From the results obtained in Refs. [40–42], the model parameter values for YIG thin films are estimated as follows: $D/a^2 \sim 3.37645$ meV [35] with $a = 1.2376$ nm [43], $H_0 \sim 8.10373$ $\mu$eV [35], and $\hbar \omega_M \sim 20.3369$ $\mu$eV [35]. Then, we estimate the magnon energy gap $\Delta_{\sigma}$ as $\Delta_{\sigma} = \Delta_{\sigma=0} - \Delta_{\sigma=\pm} \sim 39.848$ $81$ meV with $\Delta_{\sigma=\pm} \sim 2.1319$ meV and $\Delta_{\sigma=\pm} \sim 41.98072$ meV, which satisfy the con-
Under the phenomenological assumption that the value of $l$ [see Eq. (4a)] approaches asymptotically to $l = 2$ in the bulk limit, in this work focusing on the thin film, we study the behavior of the magnonic Casimir effect by changing the value slightly from $l = 2$. As examples, we consider the cases of $l = 2.1$, $l = 1.99$, and $l = 1.9$. Figure 3 shows that the magnonic Casimir effect arises in the ferrimagnetic thin film. There is the magnonic Casimir energy $E_{\text{Cas}}$ of the magnitude $O(1)$, $O(10)$, and $O(10) \mu$eV for $l = 1.99$, $l = 1.9$, and $l = 2.1$, respectively, in $N_z \geq 2$. As the thickness increases, the magnonic Casimir coefficient $C_{\text{Cas}}^{\sigma}$ approaches asymptotically to a constant value, and the magnonic Casimir energy exhibits the behavior of $E_{\text{Cas}} \propto 1/N_z^2$. We also find from Fig. 3 that the sign of the magnonic Casimir coefficient and energy for $l = 2.1$ is positive $C_{\text{Cas}}^{[2,1]} = E_{\text{Cas}} \times N_z^2 > 0$ in $N_z \geq 2$, whereas that for $l = 1.9$ is negative $C_{\text{Cas}}^{[1,9]} = E_{\text{Cas}} \times N_z^2 < 0$. This means that the Casimir force works in the opposite direction.

Note that even if $l = 2$, the magnonic Casimir effect arises in the ferrimagnetic thin film. We have numerically confirmed that although the value is small, there does exist the magnonic Casimir energy of the magnitude $|E_{\text{Cas}}| \leq O(0.1) \text{meV for } l = 2$. This strong suppression of the Casimir energy is a general property of the Casimir effect for quadratic dispersions on the lattice \[28\], and the survival values originate from the dipolar interaction in the ferrimagnetic thin film.

We remark that as long as we consider thin films, the value of $D_z$ can differ from $D$. Even in that case, the magnonic Casimir effect arises. When the value of $D_z$ changes from $D$ to $0.8D$ as an example, the magnonic Casimir energy $E_{\text{Cas}}$ increases approximately by 0.8 times. For more details about its dependence on the parameters $D_z$ and $l$, see the Supplementary Material.

Proposal for experimental observation.—The magnonic Casimir energy of the ferrimagnetic thin film depends strongly on external magnetic fields through Zeeman coupling as in Eq. (4a) and contributes to magnetization of magnets, whereas the photon and phonon Casimir effects \[46\] do not usually. On the other hand, in the presence of magnetostriction \[47–50\], the phonon Casimir effect is influenced by magnetostriction, and its correction for the phonon Casimir energy depends on magnetic fields and contributes to magnetization. However, such a contribution to magnetization from the phonon Casimir effect should be negligibly small by the factor of $10^{-6}$ compared with that from the magnonic Casimir effect of ferrimagnets because the magnetostriction constant (i.e., the correction for the lattice constant) is known to be $10^{-6}$ for YIG \[47, 48\]. Hence, even in the presence of magnetostriction, the magnonic Casimir effect can be distinguished from the others. Thus, we expect that our theoretical prediction, the magnonic Casimir effect in ferrimagnets, can be experimentally observed through measurement of magnetization and its film thickness dependence. For more details, see the Supplemental Material.

For observation, a few comments are in order. First, we remark on edge/surface magnon modes. The magnonic Casimir effect in our setup (see the thin film of Fig. 1) is induced by magnon fields with wave numbers $k_z$ discretized by small $N_z$, and its necessary condition is a $k_z$-dependent dispersion relation under the discretization of $k_z$. Throughout this study, we consider thin films of $N_z \ll N_x, N_y$. Even if additional edge/surface magnon modes exist as well as the Damon-Eshbach magnetostatic surface mode and the backward volume magnetostatic mode [see Eq. (4b)], they are confined only on the $x$-$y$ plane. Then, their wave number in the $z$ direction is always zero, i.e., $k_z = 0$, and its energy dispersion relation is independent of $k_z$. Since a $k_z$-independent dispersion relation cannot induce the Casimir effect, the edge/surface modes cannot contribute to the magnonic Casimir effect. In this sense, our magnonic Casimir effect is not affected by the existence of edge/surface magnon modes.

Note that details of the edge condition, such as the presence or absence of disorder, may change the boundary condition for the wave function of magnons, but the existence of the magnonic Casimir effect remains unchanged. Even if there is a change in the spectrum near the edge, the magnonic Casimir effect is little influenced as long as one does not assume an ultrathin film such as $N_z = 1, 2, 3$. In this sense, we expect that the following size of thin films is appropriate for observation of our prediction: $N_z \sim 10$, i.e., the film thickness of YIG is $L_z = aN_z \sim 12.376 \text{nm}$. Note that microfabrication technology \[17, 18\] can control the thickness of thin films and realize the manipulation of the magnonic Casimir effect.

Next, we remark on the magnon band structure. Since our magnonic Casimir effect is induced by the $k_z$-dependent dispersion, its Casimir energy of ferrimagnets is mainly characterized by the $D_z$-term in Eq. (4a), i.e., $D_z/(|k_z|a)^4$. Hence, we have investigated its dependence on both $l$ and $D_z$ (see Fig. 3 and the Supplemental Material). Even if the magnon band structure is affected due to some reasons, the magnonic Casimir effect of ferrimagnets is little influenced by other details of the magnon band structure except for $l$ and $D_z$.

Lastly, we remark on thermal effects. At nonzero temperature, thermal contributions to the Helmholtz free en-
nmag | 2001-1-25 | (1a) is dominant.

**Conclusion.**—We have shown that the magnonic Casimir effect can arise not only in antiferromagnets but also in ferrimagnets with realistic model parameters for YIG. Since the lifetime of magnons in YIG thin films is the longest among known materials, and magnons exhibit long-distance transport over centimeter distances \[52\], YIG is the key ingredient of magnonics \[1, 16\], which has already realized the magnon transistor \[53\]. Our result suggests that YIG can serve also as a promising platform for Casimir engineering \[19\]: Because the magnonic Casimir effect contributes to the internal pressure of thin films, it will provide the new principles of nanoscale devices such as highly sensitive pressure sensors and magnon transistors. Thus, our study paves the way for magnonic Casimir engineering.

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[44] The magnon energy dispersions and their temperature dependences in YIG were measured by inelastic neutron scattering \[59–65\]. We estimate the magnon energy gap
\( \Delta \sigma \) by applying the model \([66]\) of the effective block spins to YIG \([35, 63]\). The theoretical estimate for the value, \( \Delta \sigma = \Delta \sigma + \sim 39.848 \text{ meV} \), is consistent with the experimental data \([62]\).

[45] Higher energy bands than those of Eq. (4a) also contribute to the magnonic Casimir energy. However, the contribution becomes smaller as the shape of the bands is flatter. Numerical calculations of Refs. \([63–65]\) show that higher energy bands tend to be flat. Thus, we expect that the magnonic Casimir energy is dominated by the two bands of Eq. (4a).

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Supplemental Material

In this Supplemental Material, first, we provide some details about the dependence of the magnonic Casimir effect on the parameters \( l \) and \( D_z \) in ferrimagnetic thin films. Next, we remark on its film thickness dependence. Then, we provide another point of view for its robustness against disorder effects. Lastly, we comment on the distinction between the Casimir effect and the thermal Casimir effect.

I. THE PARAMETER \( l \)- AND \( D_z \)-DEPENDENCE

In the main text, we have studied the magnonic Casimir energy \( E_{\text{Cas}} \) and the coefficient \( C_{\text{Cas}}^{[l]} = E_{\text{Cas}} \times N_z^l \) for \( l = 2.1, l = 2.0, l = 1.9, \) and \( l = 1.9 \) in the ferrimagnetic thin film by using the model parameter values for YIG with fixed \( D_z = D \). Here, we provide more details about its dependence on the parameters \( l \) and \( D_z \).

First, we consider the cases of \( l = 1.5 \) and \( l = 1.0 \) with setting \( D_z = D \). Figure S1 shows that the magnonic Casimir effect still arises in the ferrimagnetic thin film. There is the magnonic Casimir energy \( E_{\text{Cas}} \) of the magnitude \( O(10^{-2}) \) meV, \( O(10^{-1}) \) meV, and \( O(10^{-1}) \) meV for \( l = 1.9, l = 1.5, \) and \( l = 1.0 \), respectively, in \( N_z \geq 2 \). As the value of \( l \) decreases from \( l = 2 \) and approaches to \( l = 1 \), the magnitude of the magnonic Casimir energy increases. Note that it amounts to \( O(10^{-1}) \) meV even in \( N_z = O(10) \) for \( l = 1.0 \). As the thickness increases, the magnonic Casimir coefficient \( C_{\text{Cas}}^{[l]} \) approaches asymptotically to a constant value and the magnonic Casimir energy exhibits the behavior of \( E_{\text{Cas}} \propto 1/N_z^l \).

Then, we consider the cases of \( D_z/D = 0.3, D_z/D = 0.5, \) and \( D_z/D = 0.8 \) by fixing \( l = 1.99 \). Figure S2 shows that the magnonic Casimir effect still arises in the ferrimagnetic thin film. When the value of \( D_z \) changes from \( D \) to \( 0.8D \) as an example, the magnonic Casimir energy \( E_{\text{Cas}} \) increases approximately by 0.8 times. Thus, the value of \( E_{\text{Cas}} \) is approximately proportional to \( D_z \).

II. REMARKS ON THE THICKNESS DEPENDENCE OF MAGNETIZATION

In the main text, we have remarked that our prediction, the magnonic Casimir effect in ferrimagnets, can be observed through measurement of magnetization and its film thickness dependence. Here, we add an explanation about it. At zero temperature, the Helmholz free energy of magnon fields in thin films (i.e., the sum over discrete \( k_z \)) is \( E_{\text{sum}}(N_z)N_xN_y \), and that under the bulk approximation (i.e., the integral with respect to continuous \( k_z \)) is \( E_{\text{int}}^0(N_z)N_xN_y \) [see Eqs. (1a)-(1c)]. The difference between them is characterized by the magnonic Casimir energy \( E_{\text{Cas}} \) as \( E_{\text{Cas}} = E_{\text{sum}}(N_z)N_xN_y = E_{\text{int}}^0(N_z)N_xN_y + E_{\text{Cas}}N_xN_y \), where the magnon energy dispersion of ferrimagnets (i.e., magnets including dipolar interactions) is Eq. (4a). Note that the magnetic-field derivative (i.e., \( H_0 \)-derivative) of the Helmholz free energy is magnetization. Then, magnetization of thin films consists of two parts: The bulk component and the magnonic Casimir energy. Since \( E_{\text{int}}^0(N_z) \propto N_z \), whereas \( E_{\text{Cas}} \propto 1/(N_z)^2 \), magnetization of thin films exhibits a different \( N_z \)-dependence from the bulk component, and its difference is characterized by the magnonic Casimir energy. In other words, magnetization of thin films exhibits a different film thickness dependence from the bulk component due to the magnonic Casimir effect. Hence, our prediction, the magnonic Casimir effect in ferrimag-
netic thin films (i.e., magnetic thin films including dipolar interactions), can be observed through measurement of magnetization and its film thickness dependence.

Note that if dipolar interactions are relevant also in antiferromagnets, its low-energy magnon dynamics is essentially described by Eq. (4a) given for ferrimagnets. The only difference is that the spin quantum number for each sublattice is identical in antiferromagnets, where the (intrinsic) magnon energy gap for each mode $\sigma = \pm$ can be identical $\Delta_{\sigma=\pm} = \Delta_{\sigma=-}$ [see Eq. (4a)]. In this sense, its Casimir effect exhibits qualitatively the same behavior as Fig. 3.

### III. REMARKS ON DISORDER EFFECTS

In the main text, we have remarked that details of the edge condition, such as the presence or absence of disorder, may change the boundary condition for the wave function of magnons, but the existence of the magnonic Casimir effect remains unchanged. Here, we add a comment on disorder effects. Since the magnonic Casimir energy does not depend on the Bose-distribution function [see Eqs. (1a)-(1c)], not only the low-energy magnon mode ($\sigma = +$) but also its high-energy mode ($\sigma = -$) in ferrimagnets contributes to the magnonic Casimir effect. Therefore, it can be expected that as long as disorder effects on the bulk are weak enough that the high-energy mode is little influenced, the existence of the magnonic Casimir effect in ferrimagnets remains unchanged.

### IV. THERMAL CASIMIR EFFECT

In the main text, we have explained that thermal contributions to the Helmholtz free energy arise at nonzero temperature. Here, we add a remark on it. Although its thermal contribution is called the “thermal Casimir energy”, there is a crucial distinction between the Casimir effect and the thermal Casimir effect: The zero-point energy, which is the key concept of quantum mechanics and plays a crucial role in the Casimir effect, is absent in the thermal Casimir effect. It should be noted that the zero-point energy is one of the most striking phenomenon of quantum mechanics in the sense that there are no classical analogs. The Casimir effect arises from the zero-point energy due to quantum fluctuations and is not affected by temperatures, whereas the thermal Casimir effect arises from thermal fluctuations and is exponentially suppressed at low temperatures. The thermal Casimir effect vanishes at zero temperature, whereas the Casimir effect does exist even at zero temperature. Thus, there is a significant distinction between the Casimir effect and the thermal Casimir effect.