A Global Compact Result for a Fractional Elliptic Problem with Critical Sobolev-Hardy Nonlinearities on $\mathbb{R}^N$ *

Lingyu Jin, Shaomei Fang
College of Science, South China Agriculture University,
Guangzhou 510642, P. R. China

Abstract

In this paper, we are concerned with the following type of elliptic problems:

$$\begin{cases} (-\Delta)^\alpha u + a(x)u = \frac{|u|^{2^*_s-2}u}{|x|^s} + k(x)|u|^{q-2}u, \\ u \in H^\alpha(\mathbb{R}^N), \end{cases} \quad (*)$$

where $2 < q < 2^*$, $0 < \alpha < 1$, $0 < s < 2\alpha$, $2^*_s = 2(N - s)/(N - 2\alpha)$ is the critical Sobolev-Hardy exponent, $2^* = 2N/(N - 2\alpha)$ is the critical Sobolev exponent, $a(x), k(x) \in C(\mathbb{R}^N)$. Through a compactness analysis of the functional associated to $(*)$, we obtain the existence of positive solutions for $(*)$ under certain assumptions on $a(x), k(x)$.

Key words and phrases. fractional Laplacian, compactness, positive solution, unbounded domain, Sobolev-Hardy nonlinearity.

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1 Introduction

We consider the following nonlinear elliptic equations:

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\[
\begin{cases}
(-\Delta)^\alpha u + a(x)u = \frac{|u|^{2^*-2}u}{|x|^s} + k(x)|u|^{q-2}u, 
& x \in \mathbb{R}^N, \\
u \in H^\alpha(\mathbb{R}^N),
\end{cases}
\tag{1.1}
\]
where \(2 < q < 2^*, 0 < \alpha < 1, 0 < s < \alpha, 2^*_s = 2(N-s)/(N-2\alpha)\) is the critical Sobolev exponent, \(2^* = 2N/(N-2\alpha)\) is the critical Sobolev exponent, \(a(x), k(x) \in C(\mathbb{R}^N)\).

In the case \(\alpha = 1\), problem (1.1) with the Sobolev-Hardy term has been extensively studied (see [3], [4], [5], [7], [11]). For \(0 < \alpha < 1\) the nonlocal operator \((-\Delta)^\alpha\) in \(\mathbb{R}^N\) is defined on the Schwarz class through the Fourier form or via the Riesz potential. Recently the fractional and more general non-local operators of elliptic type have been widely studied, both for their interesting theoretical structure and concrete applications in many fields such as optimization, finance, phase transitions, stratified materials, anomalous diffusion and so on (see [9], [10], [14], [19], [20], [21]). When \(s = 0\), (1.1) are the elliptic equation involving the nonlocal operator and the critical Sobolev nonlinearity. Abundant results have been accumulated (see [9], [10], [19], [20], [21]). When \(0 < s < 2\alpha\), (1.1) has the Sobolev-Hardy nonlinearity. In particular, recently Yang etc. in [25], [26] considered the existence of solutions for (1.1) (\(0 < s < 2\alpha\) or \(s = 2\alpha\)) in a bounded domain. Motivated by it, we consider the compactness analysis and thereby obtain the existence of the solutions for problem (1.1) in \(\mathbb{R}^N\). Compare with Yang’s work, the new difficulty of this problem that emerges here is the lack of compactness caused by the unbounded domain \(\mathbb{R}^N\). As is well known, the translation invariance of \(\mathbb{R}^N\) and the scaling invariance of critical exponents are typical difficulties in the study of elliptic equations. Indeed, such invariance disables the compactness of the embeddings. To overcome the difficulties caused by the lack of compactness, we carry out a non-compactness analysis which can distinctly express all the parts which cause non-compactness. As a result, we are able to obtain the existence of nontrivial solutions of the elliptic problem including the critical nonlinear term on unbounded domain by getting rid of these noncompact factors. To be more specific, for the Palais-Smale sequences of the variational functional corresponding to (1.1) we first establish a complete noncompact expression which includ all the blowing up bubbles caused by critical Sobolev-Hardy and unbounded domains. Then by applying the noncompact expression, we derive the existence of positive solutions for (1.1). Our methods base on some techniques of [3], [12], [13], [15], [17], [18], [22], [23], [24].

Before introducing our main results, we give some notations and assumptions.

**Notations and assumptions:**

Let \(N \geq 1\), \(u \in L^2(\mathbb{R}^N)\), let the Fourier transform of \(u\) be

\[
\hat{u}(\xi) = \mathcal{F}(\xi) := \frac{1}{(2\pi)^{\frac{N}{2}} \cdot \mathbb{R}^N} \int_{\mathbb{R}^N} e^{-i\xi \cdot x} u_0(x) dx.
\]
For $\alpha > 0$, the Sobolev space $H^\alpha(\mathbb{R}^N)$ is defined as the completion of $C_0^\infty(\mathbb{R}^N)$ with the norm

$$
\|u\|_{H^\alpha(\mathbb{R}^N)} = \|\hat{u}\|_{L^2(\mathbb{R}^N)} + \|\xi^{\alpha}\hat{u}\|_{L^2(\mathbb{R}^N)}.
$$

Let $\dot{H}^\alpha(\mathbb{R}^N)$ be the homogeneous version as the completion of $C_0^\infty(\mathbb{R}^N)$ under the norm

$$
\|u\|_{\dot{H}^\alpha(\mathbb{R}^N)} = \|\xi^{\alpha}\hat{u}\|_{L^2(\mathbb{R}^N)}.
$$

We define the operator $(-\Delta)^\alpha u$, $\alpha \in \mathbb{R}$ by the Fourier transform

$$
(-\Delta)^\alpha u(\xi) = |\xi|^{2\alpha} \hat{u}(\xi), \quad \forall u \in C_0^\infty(\mathbb{R}^N).
$$

Then we have

$$
\|\xi^{\alpha}\hat{u}\|_{L^2(\mathbb{R}^N)}^2 = \int_{\mathbb{R}^N} |(-\Delta)^{\alpha/2} u|^2 dx.
$$

By the Parseval identity, we also have

$$
\|u\|^2_{\dot{H}^\alpha(\mathbb{R}^N)} = \|u\|^2_{L^2(\mathbb{R}^N)} + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2\alpha}} dxdy = \|u\|^2_{L^2(\mathbb{R}^N)} + \int_{\mathbb{R}^N} |(-\Delta)^{\alpha/2} u|^2 dx.
$$

Denote $c$ and $C$ as arbitrary constants. Let $B(x, r)$ denote a ball centered at $x$ with radius $r$, $B(r)$ denote a ball centered at $0$ with radius $r$ and $B(x, r)^C = \mathbb{R}^N \setminus B(x, r)$.

In this paper we assume that:

(a) $0 \leq a(x) \in C(\mathbb{R}^N)$, $0 \leq k(x) \in C(\mathbb{R}^N)$;

(b) $\lim_{|x| \to \infty} a(x) = \bar{a} > 0$, $\lim_{|x| \to \infty} k(x) = \bar{k} > 0$, $\inf_{x \in \mathbb{R}^N} a(x) = \bar{a} > 0$, $\inf_{x \in \mathbb{R}^N} k(x) = \bar{k} > 0$.

In the following, we assume that $a(x), k(x)$ always satisfy (a) and (b). The energy functional associated with (1.1) is

$$
I(u) = \frac{1}{2} \int_{\mathbb{R}^N} \left( |(-\Delta)^{\alpha/2} u|^2 + a(x)u^2 \right) dx - \frac{1}{2s} \int_{\mathbb{R}^N} \frac{|u|^s}{|x|^s} dx - \frac{1}{q} \int_{\mathbb{R}^N} k(x)|u|^q dx, \quad u \in H^\alpha(\mathbb{R}^N).
$$

We next present some problems associated to (1.1) as the follows.

The limit equation of (1.1) at infinity is

$$
\begin{cases}
(-\Delta)^\alpha u + \bar{a} u = \bar{k} |u|^{q-2} u, \\
u \in H^\alpha(\mathbb{R}^N),
\end{cases}
$$

(1.2)

and its corresponding variational functional is

$$
I^\infty(u) = \frac{1}{2} \int_{\mathbb{R}^N} \left( |(-\Delta)^{\alpha/2} u|^2 + \bar{a}|u|^2 \right) dx - \frac{1}{q} \int_{\mathbb{R}^N} \bar{k}|u|^q dx, \quad u \in H^\alpha(\mathbb{R}^N).
$$
The limit equation of (1.1) related to the Sobolev-Hardy critical nonlinear term is

$$
\begin{align*}
(-\Delta)^{\alpha} u &= \frac{|u|^{2^*-2}u}{|x|^s}, \\
 u &\in \dot{H}^\alpha(\mathbb{R}^N),
\end{align*}
$$

(1.3)

and the corresponding variational functional is

$$
I_s(u) = \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\alpha/2} u|^2 \, dx - \frac{1}{2^*_s} \int_{\mathbb{R}^N} \frac{|u|^{2^*_s}}{|x|^s} \, dx, \\
 u &\in \dot{H}^\alpha(\mathbb{R}^N).
$$

In [24] Chen and Yang proved that all the positive solutions of (1.3) are of the form

$$
U_\varepsilon(x) = \varepsilon^{\frac{2\alpha-N}{2}} U(x/\varepsilon),
$$

(1.4)

where $C_2 > C_1 > 0$ are constants. These solutions are also minimizers for the quotient

$$
S_{\alpha,s} = \inf_{u \in \dot{H}^\alpha(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} \frac{|u|^{2^*_s}}{|x|^s} \, dx}{\left( \int_{\mathbb{R}^N} \frac{|u|^{2^*_s}}{|x|^s} \, dx \right)^{\frac{2^*_s}{2^*_s}}}.
$$

which is associated with the fractional Sobolev-Hardy inequality

$$
\left( \int_{\mathbb{R}^N} \frac{|u|^{2^*_s}}{|x|^s} \, dx \right)^{\frac{2^*_s}{2^*_s}} \leq S_{\alpha,s}^{-1} \int_{\mathbb{R}^N} |(-\Delta)^{\alpha/2} u|^2 \, dx.
$$

Define

$$
D_0 = \int_{\mathbb{R}^N} \left( \frac{1}{2} |(-\Delta)^{\alpha/2} U|^2 - \frac{1}{2^*_s} \frac{|U|^{2^*_s}}{|x|^s} \right) \, dx = \frac{2\alpha - s}{2(N-s)} S_{\alpha,s}^{\frac{N-s}{2^*_s}},
$$

$$
\mathcal{N} = \{ u \in H^\alpha(\mathbb{R}^N) \setminus \{0\} | \int_{\mathbb{R}^N} \left( |(-\Delta)^{\alpha/2} u|^2 + a|u|^2 - k|u|^q \right) \, dx = 0 \},
$$

and

$$
J^\infty = \inf_{u \in \mathcal{N}} I^\infty(u).
$$

It is known that $\mathcal{N} \neq \emptyset$ since problem (1.2) has at least one positive solution if $N > 2\alpha$ (see [16]) for $1 < q < 2^*$. The main result of our paper is as follows:

**Theorem 1.1.** Suppose $a(x), k(x)$ satisfy (a) (b), $2 < q < 2^*, N > 2\alpha$. Assume that $\{u_n\}$ is a positive Palais-Smale sequence of I at level $d \geq 0$, then there exist two sequences $\{R_i\} \subset \mathbb{R}^+ (1 \leq j \leq l_1)$ and $\{y_i\} \subset \mathbb{R}^N (1 \leq j \leq l_2)$, $0 \leq u \in H^\alpha(\mathbb{R}^N)$, and $0 < u_j \in H^\alpha(\mathbb{R}^N) (1 \leq j \leq l_2)$, $(l_1, l_2 \in \mathbb{N})$ such that up to a subsequence:
\begin{itemize}
  \item $I'(u) = 0$, $I^\infty'(u_j) = 0$ (1 \leq j \leq l_2);
  \item $R^j_n \to 0$ (1 \leq j \leq l_1) as $n \to \infty$;
  \item $|y^j_n| \to \infty$ (1 \leq j \leq l_2) as $n \to \infty$;
  \item $d = I(u) + l_1D_0 + \sum_{j=1}^{l_2} I^\infty(u_j)$;
\end{itemize}

\[ \|u_n - u - \sum_{j=1}^{l_1} U^R_n - \sum_{j=1}^{l_2} u_j(x - y^j_n)\|_{H^\alpha(\mathbb{R}^N)} = o(1) \text{ as } n \to \infty. \quad (1.5) \]

In particular, if $u \not\equiv 0$, then $u$ is a weakly solution of (1.1). Note that the corresponding sum in (1.2) will be treated as zero if $l_i = 0$ ($i = 1, 2$).

**Remarks:**

1) Similar as Corollary 3.3 in [17], one can show that any Palais-Smale sequence for $I$ at a level which is not of the form \( m_1D + m_2J^\infty \), $m_1, m_2 \in \mathbb{N} \cup \{0\}$, gives rise to a non-trivial weak solution of equation (1.1).

2) In our non-compactness analysis, we prove that the blowing up positive Palais-Smale sequences can bear exactly two kinds of bubbles. Up to harmless constants, they are either of the form

\[ U^R_n(x), \ |R_n| \to 0 \text{ as } n \to \infty, \]

or

\[ u(x - y_n) \in H^1(\mathbb{R}^N), \ |y_n| \to \infty, \text{ as } n \to \infty, \]

where $u$ is the solution of (1.2). For any Palais-Smale sequence $u_n$ for $I$, ruling out the above two bubbles yields the existence of a non-trivial weak solution of equation (1.1).

Using above compact results and the Mountain Pass Theorem [1] we prove the following corollary.

**Corollary 1.1.** Assume that $2 < q < 2^*$ for $N \geq 4\alpha$. If $a(x), k(x)$ satisfy (a) and (b), and

\[ \bar{a} \geq a(x), k(x) \geq \bar{k} > 0, \ k(x) \not\equiv \bar{k}. \quad (1.6) \]

Then (1.1) has a nontrivial solution $u \in H^\alpha(\mathbb{R}^N)$ which satisfies

\[ I(u) < \min\{\frac{2\alpha - s}{2(N - s)} S_{\alpha,s}^{\frac{N-s}{2}}, J^\infty\}. \]

This paper is organized as follows. In Section 2, we give some preliminary lemmas. In Section 3, we prove Theorem 1.1 by carefully analyzing the features of a positive Palais-Smale sequence for $I$. Corollary 1.1 is proved at the end of Section 3 by applying Theorem 1.1 and the Mountain Pass Theorem.
2 Some preliminary lemmas

In order to prove our main theorem, we give the following Lemmas.

**Lemma 2.1.** (Lemma 2.1, [22]) Let \( \{\rho_n\}_{n \geq 1} \) be a sequence in \( L^1(\mathbb{R}^N) \) satisfying
\[
\rho_n \geq 0 \quad \text{on} \quad \mathbb{R}^N, \quad \lim_{n \to \infty} \int_{\mathbb{R}^N} \rho_n \, dx = \lambda > 0, \tag{2.1}
\]
where \( \lambda > 0 \) is fixed. Then there exists a subsequence \( \{\rho_{n_k}\} \) satisfying one of the following two possibilities:

(i) (Vanishing):
\[
\lim_{k \to \infty} \sup_{y \in \mathbb{R}^N} \int_{y + B_R} \rho_{n_k} (x) \, dx = 0, \quad \text{for all} \quad R < +\infty. \tag{2.2}
\]

(ii) (Nonvanishing): \( \exists \alpha > 0, R < +\infty \) and \( \{y_k\} \subset \mathbb{R}^N \) such that
\[
\lim_{k \to +\infty} \int_{y_k + B_R} \rho_{n_k} (x) \, dx \geq \alpha > 0.
\]

**Lemma 2.2.** (Lemma 2.3, [8]) Let \( 1 \leq p < \infty \), with \( p \neq \frac{2N}{N-2\alpha} \). Assume that \( u_n \) is bounded in \( L^p(\mathbb{R}^N) \), \((-\Delta)^{\alpha/2}u \) is bounded in \( L^2(\mathbb{R}^N) \) and
\[
\sup_{y \in \mathbb{R}^N} \int_{y + B_R} |u_n|^p \, dx \to 0 \quad \text{for some} \quad R > 0 \quad \text{as} \quad n \to \infty.
\]
Then \( u_n \to 0 \) in \( L^q(\mathbb{R}^N) \), for \( q \) between \( p \) and \( \frac{2N}{N-2\alpha} \).

**Lemma 2.3.** Suppose that \( 0 < s < 2\alpha \) and \( N > 2\alpha \). Then there exists \( C > 0 \) such that for any \( u \in \dot{H}^\alpha(\mathbb{R}^N) \),
\[
\left( \int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_s}}{|x|^s} \, dx \right)^\frac{2}{2^*_s} \leq C \|u\|^2_{\dot{H}^\alpha(\mathbb{R}^N)} \tag{2.3}
\]
a.e.,
\[
\dot{H}^\alpha(\mathbb{R}^N) \hookrightarrow L^{2^*_s}(\mathbb{R}^N, |x|^{-s})
\]
is continuous. In addition, the inclusion
\[
\dot{H}^\alpha(\mathbb{R}^N) \hookrightarrow L^p_{\text{loc}}(\mathbb{R}^N, |x|^{-s}),
\]
is compact if \( 2 \leq p < 2^*_s \).

**Proof.** The proof of (2.3) is similar to that of Lemma 3.1 in [23]. Now we prove the compact impeding if \( 2 \leq p < 2^*_s \). Let \( \{u_n\} \) be a bounded sequence in \( \dot{H}^\alpha(\mathbb{R}^N) \), then up to a subsequence (still denoted by \( \{u_n\} \)),
\[
u_n \rightharpoonup u \text{ in } \dot{H}(\mathbb{R}^N).
\]
Denote \( v_n = u_n - u \), then for any \( B(0, r) \),

\[
v_n \rightharpoonup 0 \text{ in } \dot{H}(\mathbb{R}^N), \quad v_n \to 0 \text{ in } L^q(B(0, r)), \quad 2 \leq q < 2^* = \frac{2N}{N-2\alpha}.
\]

Fix \( r > 0 \), since \((p - \frac{s}{\alpha})(\frac{2\alpha}{2\alpha - s}) < 2^*\), it follows

\[
\int_{B(0,r)} \frac{|v_n|^p}{|x|^s} \, dx = \int_{B(0,r)} \frac{|v_n|^{s/\alpha}}{|x|^s} |v_n|^{p-s/\alpha} \, dx \\
\leq \left( \int_{B(0,r)} \frac{|v_n|^2}{|x|^{2\alpha}} \, dx \right)^{\frac{s}{2\alpha}} \left( \int_{B(0,r)} |v_n|^{(p-s/\alpha)} \, dx \right)^{\frac{2\alpha-s}{2\alpha}} \quad (2.4)
\]

\[
\leq c \|(-\Delta)^{\alpha/2} v_n\|_{L^2(\mathbb{R}^N)} \left( \int_{B(0,r)} |v_n|^{(p-s/\alpha)} \, dx \right)^{\frac{2\alpha-s}{2\alpha}} \to 0,
\]

and

\[
\int_{B(0,r)^c} \frac{|v_n|^p}{|x|^s} \, dx \leq \int_{B(0,r)^c} \frac{|v_n|^p}{r^s} \, dx \leq \frac{1}{r^s} \|v_n\|^p_{L^p(\mathbb{R}^N)} \quad (2.5)
\]

Letting \( r \to \infty \), collecting (2.4) and (2.5), it implies that

\[
u_n \to u \text{ in } L^p_{loc}(\mathbb{R}^N, |x|^{-s}).
\]

This completes the proof. \(\square\)

**Lemma 2.4.** Let \( \{u_n\} \) be a Palais-Smale sequence of \( I \) at level \( d \in \mathbb{R} \). Then \( d \geq 0 \) and \( \{u_n\} \subset H^\alpha(\mathbb{R}^N) \) is bounded. Moreover, every Palais-Smale sequence for \( I \) at a level zero converges strongly to zero.

**Proof.** Since \( a(x) \geq 0 \), \( \bar{a} > 0 \), \( \inf_{\mathbb{R}^N} a(x) = \bar{a} > 0 \), we have

\[
\|u_n\|^2_{H^\alpha(\mathbb{R}^N)} + \int_{\mathbb{R}^N} a(x)|u_n|^2 \, dx \geq c\|u_n\|^2_{H^\alpha(\mathbb{R}^N)},
\]

and hence for \( q \leq 2^* \)

\[
d + 1 + o(\|u_n\|) \geq I(u_n) - \frac{1}{q}\langle I'(u_n), u_n \rangle \\
= \left(\frac{1}{2} - \frac{1}{q}\right) \int_{\mathbb{R}^N} \langle(-\Delta)^{\alpha/2} u_n, a(x)u_n \rangle dx \\
+ \left(\frac{1}{q} - \frac{1}{2^*}\right) \int_{\mathbb{R}^N} |u_n|^{2^*} |x|^s dx \\
\geq C\|u_n\|^2_{H^\alpha(\mathbb{R}^N)},
\]

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for $2^* < q < 2^*$,

$$d + 1 + o(\|u_n\|) \geq I(u_n) - \frac{1}{2^*} \langle I'(u_n), u_n \rangle$$

$$= \left( \frac{1}{2} - \frac{1}{2^*} \right) \int_{\mathbb{R}^N} \left( |(-\Delta)^{\alpha/2} u_n|^2 + a(x)|u_n|^2 \right) dx$$

$$+ \left( \frac{1}{2^*} - \frac{1}{q} \right) \int_{\mathbb{R}^N} |u_n|^q dx$$

$$\geq C \|u_n\|_{H^\alpha(\mathbb{R}^N)}^2.$$

(2.7)

It follows from (2.6) and (2.7) that $\{u_n\}$ is bounded in $H^\alpha(\mathbb{R}^N)$ for $2 < q < 2^*$. Since

$$d = \lim_{n \to \infty} I(u_n) - \max \left\{ \frac{1}{q} - \frac{1}{2^*} \right\} \langle I'(u_n), u_n \rangle \geq C \limsup_{n \to \infty} \|u_n\|_{H^\alpha(\mathbb{R}^N)}^2,$$

we have $d \geq 0$. Suppose now that $d = 0$, we obtain from the above inequality that

$$\lim_{n \to \infty} \|u_n\|_{H^\alpha(\mathbb{R}^N)} = 0.$$

Let $\{u_n\}$ be a Palais-Smale sequence. Up to a subsequence, we assume that

$$u_n \rightharpoonup u_0 \quad \text{in} \quad H^\alpha(\mathbb{R}^N) \quad \text{as} \quad n \to \infty.$$

Obviously, we have $I'(u_0) = 0$. Let $v_n = u_n - u_0$, from Lemma 2.3 as $n \to \infty$,

$$v_n \to 0 \quad \text{in} \quad H^\alpha(\mathbb{R}^N),$$

$$v_n \to 0 \quad \text{in} \quad L^p_{\text{loc}}(\mathbb{R}^N, |x|^{-s}) \quad \text{for all} \quad 2 \leq p < 2^*,$$

$$v_n \to 0 \quad \text{in} \quad L^q_{\text{loc}}(\mathbb{R}^N) \quad \text{for all} \quad 2 < q < 2^*.$$

(2.8) (2.9) (2.10)

As a consequence, we have the following Lemma.

**Lemma 2.5.** $\{v_n\}$ is a Palais-Smale sequence for $I$ at level $d_0 = d - I(u_0)$.

**Proof.** By the Brézis-Lieb Lemma in [2] and $v_n \to 0$ in $H^\alpha(\mathbb{R}^N)$, as $n \to \infty$, we have

$$\int_{\mathbb{R}^N} |v_n|^q dx = \int_{\mathbb{R}^N} |u_n|^q dx - \int_{\mathbb{R}^N} |u_0|^q dx + o(1) \quad \text{for all} \quad 2 \leq q \leq 2^*;$$

(2.11)

$$\int_{\mathbb{R}^N} \frac{|v_n|^{2^*}}{|x|^s} dx = \int_{\mathbb{R}^N} \frac{|u_n|^{2^*}}{|x|^s} dx - \int_{\mathbb{R}^N} \frac{|u_0|^{2^*}}{|x|^s} dx + o(1);$$

(2.12)
a test function and from Lemma 2.3, in Lemma 2.6. Let \{v_n\} \subset H^\alpha(\mathbb{R}^N) be a Palais-Smale sequence of I at level d and \(v_n \to 0\) in \(H^\alpha(\mathbb{R}^N)\) as \(n \to \infty\). If there exists a sequence \(\{r_n\} \subset \mathbb{R}^+\), with \(r_n \to 0\) as \(n \to \infty\) such that \(\tilde{v}_n(x) := r_n^{-N/2} v_n(r_n x)\) converges weakly in \(\dot{H}^\alpha(\mathbb{R}^N)\) and almost everywhere to some \(0 \neq v_0 \in \dot{H}^\alpha(\mathbb{R}^N)\) as \(n \to \infty\), then \(v_0\) solves (1.3) and the sequence \(z_n := v_n - v_0(x/r_n)r_n^{-\frac{N-\alpha}{2}}\) is a Palais-Smale sequence of I at level d \(- I_\mu(v_0)\).

**Proof.** First, we prove that \(v_0\) solves (1.3) and \(I(z_n) = I(v_0) - I_\mu(v_0)\). Fix a ball \(B(0,r)\) and a test function \(\phi \in C_0^\infty(B(0,r))\). Since

\[
\tilde{v}_n(x) \to v_0 \text{ in } \dot{H}^\alpha(\mathbb{R}^N),
\]

applying Lemma 2.3, it implies

\[
\langle \phi, I'_\mu(v_0) \rangle = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(v_0(x) - v_0(y))(\phi(x) - \phi(y))}{|x-y|^{N+2\alpha}} \, dx \, dy - \int_{\mathbb{R}^N} \frac{|v_0|^{2\alpha-2}v_0\phi}{|x|^{\alpha}} \, dx
\]

\[
= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(\tilde{v}_n(x) - \tilde{v}_n(y))(\phi(x) - \phi(y))}{|x-y|^{N+2\alpha}} \, dx \, dy - \int_{\mathbb{R}^N} \frac{|\tilde{v}_n|^{2\alpha-2}\tilde{v}_n\phi}{|x|^{\alpha}} \, dx
\]

\[
+ r_n^{-\frac{N-\alpha}{2}} \int_{\mathbb{R}^N} a(r_n x)\phi \tilde{v}_n \, dx - r_n^{-\frac{N-\alpha}{2}} \int_{\mathbb{R}^N} k(r_n x)\phi |\tilde{v}_n|^{q-2}\tilde{v}_n \, dx + o(1) \quad (2.16)
\]

\[
= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(v_n(x) - v_n(y))(\phi_n(x) - \phi_n(y))}{|x-y|^{N+2\alpha}} \, dx \, dy - \int_{\mathbb{R}^N} \frac{|v_n|^{2\alpha-2}v_n\phi_n}{|x|^{\alpha}} \, dx
\]

\[
+ \int_{\mathbb{R}^N} a(x)\phi_n v_n \, dx - \int_{\mathbb{R}^N} k(x)\phi_n |v_n|^{q-2}v_n \, dx + o(1) = o(1) \text{ as } n \to \infty,
\]

Hence \(I(v_n) = I(u_n) - I(u_0) + o(1) = d - I(u_0) + o(1)\).

For \(v \in C_0^\infty(\mathbb{R}^N)\), there exists a \(B(0,r)\) such that \(\text{supp} v \subset B(0,r)\). Then as \(n \to \infty,\)

\[
\left| \int_{\mathbb{R}^N} k(x)|v_n|^{q-2}v_n v \, dx \right| \leq c \int_{B(0,r)} |v_n|^{q-2}v_n v \, dx = o(1),
\]

and from Lemma 2.3

\[
\left| \int_{\mathbb{R}^N} \frac{|v_n|^{2\alpha-2}v_n v}{|x|^{\alpha}} \, dx \right| \leq \int_{|x| \leq r} \frac{|v_n|^{2\alpha-2}v_n v}{|x|^{\alpha}} \, dx \leq c \int_{|x| \leq r} \frac{|v_n|^{2\alpha-1}}{|x|^{\alpha}} \, dx = o(1).
\]

By (2.8), (2.14) and (2.15), we have \(\langle v, I'(v_n) \rangle = o(1)\) as \(n \to \infty\). \(\square\)
where \( \phi_n(x) = r_n^{-\frac{2\alpha - N}{2}} \phi(x/r_n). \) The last equality in (2.16) holds since \( \|\phi\|_{H^{\alpha}(\mathbb{R}^N)} = \|\phi_n\|_{H^{\alpha}(\mathbb{R}^N)} + o(1) \) as \( n \to \infty. \) Thus \( v_0 \) solves (1.3). Let

\[ z_n(x) = v_n(x) - r_n^{-\frac{2\alpha - N}{2}} v_0\left(\frac{x}{r_n}\right) \in H^{\alpha}(\mathbb{R}^N), \]

then

\[ \bar{z}_n = r_n^{-\frac{N-2\alpha}{2}} z_n(r_n x) = \bar{v}_n - v_0(x). \]

Obviously \( z_n \to 0 \) in \( H^{\alpha}(\mathbb{R}^N) \) as \( n \to \infty. \) Now we prove that \( \{z_n\} \) is a Palais-Smale sequence of \( I \) at level \( d - I_s(v_0). \)

Since

\[ \int_{\mathbb{R}^N} |v_0|^{\frac{2\alpha - N}{2}} p dx = r_n^{\frac{N-2\alpha}{2}} \|v_0\|_{L^p(\mathbb{R}^N)}^p \to 0, \quad \text{as} \quad n \to \infty, \quad \text{for all} \quad 1 \leq p < 2^{*}_{\alpha}, \quad (2.17) \]

by the Brézis-Lieb Lemma and the weak convergence, similar to Lemma 2.5, we can prove have

\[ I(z_n) = I(v_n) - I_s(v_0), \]

\[ \langle I'(z_n), \phi \rangle = o(1) \]

as \( n \to \infty. \) It completes the proof. \( \square \)

**Lemma 2.7.** Let \( 0 < \alpha < N/2, \) \( 0 < s < 2\alpha, \) \( \{u_n\} \subset \dot{H}^{\alpha}(\mathbb{R}^N) \) be a bounded sequence such that

\[ \inf_{n \in \mathbb{N}} \int_{\mathbb{R}^N} \frac{|u_n|^2}{|x|^s} dx \geq c > 0. \quad (2.18) \]

Then, up to subsequence, there exist a family of positive numbers \( \{r_n\} \subset \mathbb{R}^N \) such that

\[ \bar{u}_n \rightharpoonup w \neq 0 \text{ in } \dot{H}^{\alpha}(\mathbb{R}^N), \quad (2.19) \]

where \( \bar{u}_n = r_n^{-\frac{N-2\alpha}{2}} u_n(r_n x). \)

**Proof.** For the proof of (2.19), refer to the proof of Theorem 1.3 in [24]. Here we Omit it. \( \square \)

### 3 Non-compactness analysis

In this section, we prove Theorem 1.1 by Concentration-Compactness Principle and a delicate analysis of the Palais-Smale sequences of \( I. \)

**Proof of Theorem 1.1.** By Lemma 2.4, we can assume that \( \{u_n\} \) is bounded. Up to a subsequence, let \( n \to \infty, \) we assume that

\[ u_n \rightharpoonup u \text{ in } H^{\alpha}(\mathbb{R}^N), \quad (3.1) \]
\[ u_n \to u \text{ in } L^p_{\text{loc}}(\mathbb{R}^N) \text{ for } 1 < p < 2^*_\alpha, \] (3.2)
\[ u_n \to u \text{ a.e. in } \mathbb{R}^N. \] (3.3)

Denote \( v_n = u_n - u \), then \( \{v_n\} \) is a Palais-Smale sequence of \( I \) and \( v_n \to 0 \) in \( H^\alpha(\mathbb{R}^N) \). Then by Lemma 2.5 we know that
\[ I(v_n) = I(u_n) - I(u) + o(1), \text{ as } n \to \infty, \] (3.4)
\[ I'(v_n) = o(1), \text{ as } n \to \infty, \] (3.5)
\[ \|v_n\|_{H^\alpha(\mathbb{R}^N)} = \|u_n\|_{H^\alpha(\mathbb{R}^N)} - \|u\|_{H^\alpha(\mathbb{R}^N)} + o(1), \text{ as } n \to \infty. \] (3.6)

Without loss of generality, we may assume that
\[ \|v_n\|^2_{H^\alpha(\mathbb{R}^N)} \to l > 0 \text{ as } n \to \infty. \]

In fact if \( l = 0 \), Theorem 1.1 is proved for \( l_1 = 0, l_2 = 0 \).

**Step 1:** Getting rid of the blowing up bubbles caused by the Hardy term.

Suppose there exists \( 0 < \delta < \infty \) such that
\[ \int_{|x|<R} \frac{|v_n|^2}{|x|^s} \, dx \geq \delta > 0, \text{ for some } 0 < R < \infty. \] (3.7)

It follows from Lemma 2.7 that there exist a positive sequence \( \{r_n\} \subset \mathbb{R} \) such that
\[ \bar{v}_n = r_n^{\frac{N-2\alpha}{2}} v_n(r_n x) \to v_0 \neq 0 \text{ in } \dot{H}^\alpha(\mathbb{R}^N) \]

Now we claim that \( r_n \to 0 \) as \( n \to \infty \). In fact there exist \( R_1 > 0 \) such that
\[ \int_{B(0,R_1)} |v_0|^p \, dx = \delta_1 > 0, \text{ for } 1 < p < 2^*_\alpha. \] (3.8)

From the Sobolev compact embedding and (3.1)–(3.2), we have that
\[ v_n \to 0 \text{ in } L^p(B(0, r)) \text{ for all } 1 < p < 2^*_\alpha, \]
\[ \bar{v}_n \to v_0 \text{ in } L^p(B(0, r)) \text{ for all } 1 < p < 2^*_\alpha, \]
\[ 0 \neq \|v_0\|^2_{L^2(B(0, r))} + o(1) = \int_{B(0, r)} |\bar{v}_n|^2 \, dx = r_n^{-2\alpha} \int_{B(0, r_n R)} |v_n|^2 \, dx. \] (3.9)

If \( r_n \to r_0 > 0 \), then
\[ r_n^{-2\alpha} \int_{B(0, r_n R_1)} |v_n|^2 \, dx \leq c r_0^{-2\alpha} \|v_n\|^2_{L^2(B(0, c R_1))} \to 0; \]
if \( r_n \to \infty \), then
\[
r_n^{-2\alpha} \int_{B(0,r_n,R_1)} |v_n|^2 \, dx \leq r_n^{-2\alpha} \|v_n\|_{H^\alpha(\mathbb{R}^N)}^2 \to 0.
\]
A contradiction to (3.9). Thus we have \( r_n \to 0 \).

Define \( z_n = v_n - v_0(\frac{x}{r_n})^\frac{2\alpha-N}{2} \), then \( z_n \to 0 \) in \( H^\alpha(\mathbb{R}^N) \). It follows from Lemma 2.6 that \( \{z_n\} \) is a Palais-Smale sequence of \( I \) satisfying
\[
I(z_n) = I(v_n) - I_s(v_0) + o(1), \quad \text{as} \quad n \to \infty.
\]
(3.10)

If still there exists a \( \delta > 0 \), such that
\[
\int_{|x| < R} \frac{|v_n|^2}{|x|^s} \, dx \geq \delta > 0,
\]
(3.12)
then repeat the previous argument. The iteration must stop after finite times. And we will have a new Palais-Smale sequence of \( I \), (without loss of generality) denoted by \( \{v_n\} \), such that
\[
\int_{|x| < R} \frac{|v_n|^2}{|x|^s} \, dx = o(1), \quad \text{as} \quad n \to \infty, \quad \text{for any} \quad 0 < R < \infty,
\]
(3.11)
and \( v_n \to 0 \) in \( H^\alpha(\mathbb{R}^N) \) as \( n \to \infty \).

**Step 2**: Getting rid of the blowing up bubbles caused by unbounded domains.

Suppose there exists \( 0 < \delta < \infty \) such that
\[
\|v_n\|_{H^\alpha(\mathbb{R}^N)}^2 \geq c \int_{\mathbb{R}^N} |v_n|^q \, dx \geq \delta > 0.
\]
(3.12)
By Lemma 2.1, there exists a subsequence still denoted by \( \{v_n\} \), such that one of the following two cases occurs.

i) Vanish occurs.
\[
\forall 0 < R < \infty, \quad \sup_{y \in \mathbb{R}^N} \int_{y + B_R} (|(-\Delta)^{\alpha/2} v_n|^2 + |v_n|^2) \, dx \to 0 \quad \text{as} \quad n \to \infty.
\]
By the Sobolev inequality and Lemma 2.2 we have
\[
\int_{\mathbb{R}^N} |v_n|^p \, dx \to 0 \quad \text{as} \quad n \to \infty, \quad \forall \ 1 < p < 2^*_s,
\]
which contradicts (3.12).

ii) Nonvanish occurs.
There exist \( \beta > 0, \ 0 < \bar{R} < \infty, \ \{y_n\} \subset \mathbb{R}^N \) such that
\[
\liminf_{n \to \infty} \int_{y_n + B_{\bar{R}}} (|(-\Delta)^{\alpha/2} v_n|^2 + |v_n|^2) \, dx \geq \beta > 0.
\]
(3.13)
We claim \(|y_n| \to \infty\) as \(n \to \infty\). Otherwise, \(\{v_n\}\) is tight, and thus \(\|v_n\|_{L^n(R^N)} \to 0\) as \(n \to \infty\). This contradicts (3.12).

Fo proceed, we first construct the Palais-Smale sequences of \(I^\infty\).

Denote \(\bar{v}_n = v_n(x + y_n)\). Since \(\|\bar{v}_n\|_{H^\alpha(R^N)} = \|v_n\|_{H^\alpha(R^N)} \leq c\), without loss of generality, we assume that as \(n \to \infty\),

\[
\bar{v}_n \to v_0 \text{ in } H^\alpha(R^N),
\]

\[
\bar{v}_n \to v_0 \text{ in } L^p_{loc}(R^N), \text{ for any } 1 < p < 2^*.
\]

By (3.11), we have \(\forall \phi \in C^\infty_0(R^N),\)

\[
\int_{R^N} \frac{|v_n|^{2^*} \phi}{|x + y_n|^s} dx = \int_{R^N} \frac{|v_n|^{2^*} \phi_n}{|x|^s} dx
\]

\[
= \int_{|x| > r} \frac{|v_n|^{2^*} \phi_n}{|x|^s} dx + o(1)
\]

\[
\leq \frac{1}{r^s} \left( \int_{R^N} |v_n|^{(2^* - 1)/2} dx \right)^{2^* - 1} \left( \int_{R^N} |\phi_n|^{2^*} dx \right)^{1/2} + o(1) \text{ as } n \to \infty,
\]

where \(\phi_n = \phi(x - y_n)\). Let \(r \to \infty\), since \(\frac{(2^* - 1)/2}{2^* - 1} < 2^*\), we have

\[
\int_{R^N} \frac{|\bar{v}_n|^{2^*} \phi}{|x + y_n|^s} dx = o(1) \text{ as } n \to \infty.
\] (3.14)

Similarly we have

\[
\int_{R^N} \frac{|\bar{v}_n|^{2^*}}{|x + y_n|^s} dx = o(1) \text{ as } n \to \infty.
\] (3.15)

Since \(v_n \to 0\) in \(H^\alpha(R^N)\) and \(\lim_{n \to \infty} a(x + y_n) = \bar{a}\), we have as \(n \to \infty\),

\[
\int_{R^N} a(x)v_n\phi_n dx = \int_{R^N} \bar{a}\bar{v}_n\phi dx + \int_{R^N} [a(x + y_n) - \bar{a}]\bar{v}_n\phi dx
\]

and

\[
|\int_{R^N} [a(x + y_n) - \bar{a}]\bar{v}_n\phi dx| \leq c\int_{R^N} |a(x + y_n) - \bar{a}|^2 \phi^2 dx)^{1/2} = o(1),
\]

that is,

\[
\int_{R^N} \bar{a}\bar{v}_n\phi dx = o(1) = \int_{R^N} a(x)v_n\phi_n dx \text{ as } n \to \infty.
\] (3.16)

Similarly we have

\[
\int_{R^N} k(x)|v_n|^{q^*} \phi_n dx = \int_{R^N} \bar{k}|\bar{v}_n|^{q^*} \bar{v}_n\phi dx = o(1) \text{ as } n \to \infty.
\] (3.17)

Recall that \(v_n\) is a Palais-Smale sequence of \(I\), by (3.14)–(3.17) we have

\[
\langle I'(v_n), \phi_n \rangle + o(1) = \langle I^\infty(\bar{v}_n), \phi \rangle = o(1), \text{ as } n \to \infty.
\] (3.18)
This shows that $\bar{v}_n$ is a nonnegative Palais-Smale sequence of $I^\infty(u)$, and $v_0$ is a weak solution of (1.2).

We claim that $v_0 \neq 0$. From (3.12), we may assume there exists a sequence $\{y_n\}$ satisfying (3.13) and

$$
\int_{B(y_n,R)} |v_n|^q dx = b + o(1) > 0, \text{ as } n \to \infty,
$$

where $b > 0$ is a constant.

If $v_0 = 0$, we have $\int_{B(\bar{R})} |\bar{v}_n|^q dx = \int_{B(y_n,R)} |v_n|^q dx = o(1)$ as $n \to \infty$ for $0 < R < \infty$ which contradicts (3.19).

Denote $z_n = \bar{v}_n - v_0$. Since

$$
I(v_n) = \frac{1}{2} \int_{\mathbb{R}^N} \left( \left| (-\Delta)^{\alpha/2} v_n \right|^2 + a(x) |v_n|^2 \right) dx - \frac{1}{2^*_s} \int_{\mathbb{R}^N} |v_n|^{2^*_s} dx - \frac{1}{q} \int_{\mathbb{R}^N} k(x) |v_n|^q dx
$$

$$
= \frac{1}{2} \int_{\mathbb{R}^N} \left( \left| (-\Delta)^{\alpha/2} \bar{v}_n \right|^2 + a(x + y_n) |\bar{v}_n|^2 \right) dx - \frac{1}{2^*_s} \int_{\mathbb{R}^N} |\bar{v}_n|^{2^*_s} dx
$$

$$
- \frac{1}{q} \int_{\mathbb{R}^N} k(x + y_n) |\bar{v}_n|^q dx
$$

$$
= \frac{1}{2} \int_{\mathbb{R}^N} \left( \left| (-\Delta)^{\alpha/2} \bar{v}_n \right|^2 + \bar{a} |\bar{v}_n|^2 \right) dx - \frac{1}{q} \int_{\mathbb{R}^N} \bar{k} |\bar{v}_n|^q dx + o(1),
$$

where the last equality is a result of (3.15), therefore, as $n \to \infty$,

$$
\|z_n\|_{H^\alpha(\mathbb{R}^N)} = \|\bar{v}_n\|_{H^\alpha(\mathbb{R}^N)} - \|\bar{v}_0\|_{H^\alpha(\mathbb{R}^N)} + o(1),
$$

$$
I(z_n) = I^\infty(\bar{v}_n) - I^\infty(\bar{v}_0) + o(1) = I(v_n) - I^\infty(v_0) + o(1). \tag{3.21}
$$

Hence $z_n \to 0$ in $H^\alpha(\mathbb{R}^N)$ as $n \to \infty$, and $z_n$ is a Palais-Smale sequences of $I$. If $\|z_n\|_{L^q(\mathbb{R}^N)} \to c > 0$ as $n \to \infty$, then one can repeat Step 2 for finite times, since the amount of sequences satisfying (3.13) is finite.

Thus we obtain a new Palais-Smale sequence of $I$, without loss of generality still denoted by $v_n$, such that

$$
\|v_n\|_{L^q(\mathbb{R}^N)} \to 0, \quad \int_{\mathbb{R}^N} \frac{|v_n|^{2^*_s}}{|x|^s} dx \to 0
$$

as $n \to \infty$. Then we have

$$
\|v_n\|^2_{H^\alpha(\mathbb{R}^N)} \leq c \int_{\mathbb{R}^N} \left( \left| (-\Delta)^{\alpha/2} v_n \right|^2 + a(x) |v_n|^2 \right) dx \leq c \left( \|v_n\|^q_{L^q(\mathbb{R}^N)} + \int_{\mathbb{R}^N} \frac{|v_n|^{2^*_s}}{|x|^s} dx \right) \to 0
$$

as $n \to \infty$. The proof of Theorem 1.1 is complete.

Now we are ready to prove corollary 1.1 by Mountain Pass Theorem and Theorem 1.1.
Proof of Corollary 1.1

From

\[
I(tu) = \frac{t^2}{2} \left[ \int_{\mathbb{R}^N} |(-\Delta)^{\alpha/2} u|^2 \, dx + \int_{\mathbb{R}^N} a(x) u^2 \, dx \right] - \frac{|t|^{2^*_s}}{2^*_s} \int_{\mathbb{R}^N} \frac{|u|^{2^*_s}}{|x|^s} \, dx - \frac{|t|^q}{q} \int_{\mathbb{R}^N} k(x) |u|^q \, dx,
\]

we deduce that for a fixed \( u \neq 0 \) in \( H^\alpha(\mathbb{R}^N) \), \( I(tu) \to -\infty \) if \( t \to \infty \). Since

\[
\int_{\mathbb{R}^N} |u|^q \, dx \leq C \|u\|_{H^\alpha(\mathbb{R}^N)}^q, \quad \int_{\mathbb{R}^N} \frac{|u|^{2^*_s}}{|x|^s} \, dx \leq C \|u\|_{H^{\alpha}(\mathbb{R}^N)}^{2^*_s},
\]

we have

\[
I(u) \geq c \|u\|_{H^{\alpha}(\mathbb{R}^N)}^2 - C(\|u\|_{H^\alpha(\mathbb{R}^N)}^q + \|u\|_{H^{\alpha}(\mathbb{R}^N)}^{2^*_s}).
\]

Hence, there exists \( r_0 > 0 \) small such that \( I(u) \mid_{\partial B(0, r_0)} \geq \rho > 0 \) for \( q, 2^*_s > 2 \).

As a consequence, \( I(u) \) satisfies the geometry structure of Mountain-Pass Theorem. Now define

\[
c^* =: \inf_{\gamma \in \Gamma} \sup_{t \in [0, 1]} I(\gamma(t)),
\]

where \( \Gamma = \{ \gamma \in C([0, 1], H^\alpha(\mathbb{R}^N)) : \gamma(0) = 0, \gamma(1) = \psi_0 \in H^\alpha(\mathbb{R}^N) \} \) with \( I(t \psi_0) \leq 0 \) for all \( t \geq 1 \).

To complete the proof of Corollary 1.1, we need to verify that \( I(u) \) satisfies the local Palais-Smale conditions. According to Remarks 2), we only need to verify that

\[
c^* < \min \left\{ \frac{2\alpha - s}{2(N - s)} S^{\frac{N - s}{2\alpha - s}}_{\alpha, s}, J^\infty \right\}. \quad (3.22)
\]

Set \( v_\varepsilon(x) = \frac{U_\varepsilon}{\int_{\mathbb{R}^N} \frac{|U_\varepsilon|^2}{|x|^s} \, dx}^{1/2^*_s} \). We claim

\[
\max_{t > 0} I(tv_\varepsilon) < \frac{2\alpha - s}{2(N - s)} S^{\frac{N - s}{2\alpha - s}}_{\alpha, s}. \quad (3.23)
\]

In fact, from (1.4) it is easy to calculate the following estimates

\[
\|v_\varepsilon\|_{H^\alpha(\mathbb{R}^N)} = S_{\alpha, s},
\]

\[
\int_{\mathbb{R}^N} |v_\varepsilon|^2 \, dx \leq c e^{2\alpha - N} \int_{\mathbb{R}^N} \frac{1}{(1 + |x|^2)^{N - 2\alpha}} \, dx \leq \begin{cases} O(\varepsilon^{2\alpha}), & N \geq 4\alpha; \\
O(\varepsilon^{2\alpha} \log \varepsilon), & N = 4\alpha; \end{cases} \quad (3.25)
\]

\[
\int_{\mathbb{R}^N} |v_\varepsilon|^q \, dx \geq O(\varepsilon^{(2\alpha - N)q/2 + N}). \quad (3.26)
\]
Since $2^* > q > 2$ we have
\[ O(\varepsilon^{2\alpha}) = o(\varepsilon^{(2\alpha-Nq)/2+N}), \ O(\varepsilon^{2\alpha|\log\varepsilon|}) = o(\varepsilon^{(2\alpha-Nq)/2+N}). \] (3.27)

Denote $t_\varepsilon$ be the attaining point of $\max_{t>0} I(tv_\varepsilon)$, we can prove that $t_\varepsilon$ is uniformly bounded (see [6]). Hence, for $\varepsilon > 0$ sufficient small,
\[
\max_{t>0} I(tv_\varepsilon) = I(t_\varepsilon v_\varepsilon)
\leq \max_{t>0} \left\{ \frac{t^2}{2} \int_{\mathbb{R}^N} \left| (-\Delta)^{\alpha/2} v_\varepsilon \right|^2 dx - \frac{t^{2^*}}{2^{2^*}} \int_{\mathbb{R}^N} \frac{|v_\varepsilon|^{2^*}}{|x|^s} dx \right. \\
- O(\varepsilon^{(2\alpha-Nq)/2+N}) + \begin{cases} \ O(\varepsilon^{2\alpha}), & N > 4\alpha, \\ \ O(\varepsilon^{2\alpha|\log\varepsilon|}), & N = 4\alpha; \end{cases} \\
\left. < \frac{2\alpha - s}{2(N-s)} S_{\alpha,s}^{N-s} \right) \text{ (by (3.27))}.
\]

This completes the proof of (3.23). By the definition of $c^*$, we have $c^* < \frac{2\alpha - s}{2(N-s)} S_{\alpha,s}^{N-s}$.

Next we verify
\[ c^* < J^\infty. \] (3.28)

Let $\{u_0\}$ be the minimizer of $J^\infty$, $I^\infty(u_0) = J^\infty$ and
\[
\int_{\mathbb{R}^N} \left| (-\Delta)^{\alpha/2} u_0 \right|^2 + \bar{a} u_0^2 dx = \int_{\mathbb{R}^N} \bar{k} |u_0|^q dx.
\]

Let
\[
g(t) = I^\infty(tu_0) = \frac{1}{2} t^2 \int_{\mathbb{R}^N} \left| (-\Delta)^{\alpha/2} u_0 \right|^2 + \bar{a} u_0^2 dx - \frac{t^q}{q} \int_{\mathbb{R}^N} \bar{k} |u_0|^q dx,
\]
\[
g'(t) = t \int_{\mathbb{R}^N} \left| (-\Delta)^{\alpha/2} u_0 \right|^2 + \bar{a} u_0^2 dx - t^{q-1} \int_{\mathbb{R}^N} \bar{k} |u_0|^q dx.
\]

Thus $g'(t) \geq 0$ if $t \in (0, 1)$; $g'(t) \leq 0$ if $t \geq 1$. Then
\[
g(1) = I^\infty(u_0) = \max_l I^\infty(u);
\]
where $l = \{tu_0, t \geq 0, u_0 \text{ fixed}\}$. (3.29)

Since there exists a $t_0 > 0$ such that $\sup_{t \geq 0} I(tu_0) = I(t_0 u_0)$, from (3.29) and the assumptions of $a(x)$ and $k(x)$, we have
\[ J^\infty = I^\infty(u_0) \geq I^\infty(t_0 u_0) > I(t_0 u_0) = \sup_{t \geq 0} I(tu_0). \]

It proves (3.28). By (3.23) and (3.28) we have (3.22). Then the proof is completed.
References

[1] Ambrosetti A, Rabinowitz P H. Dual variational methods in critical point theory and applications. J. Funct. Anal, 1973, 14: 349-381.

[2] Brézis H. and Nirenberg L. Positive solutions of nonlinear elliptic equations involving critical exponents. Comm. Pure. Appl. Math, 1983, 36: 437-477.

[3] Cao D, Peng S. A global compactness result for singular elliptic problems involving critical Sobolev exponent. Proc. Amer. Math. Soc, 2003, 131: 1857-1866.

[4] Cao D, Peng S. A note on the sign-changing solutions to elliptic problem with critical Sobolev and Hardy terms. J. Diff. Equats, 2003, 193: 424-434.

[5] Chabrowski J. On the nonlinear Neumann problem involving the critical Sobolev exponent and Hardy potential. Rev. Mat. Complut, 2004, 17: 195-227.

[6] Deng Y, Guo Z, Wang G. Nodal solutions for p-Laplace equations with critical growth. Nonlinear Analysis, 2003, 54: 1121-1151.

[7] Deng Y, Jin L, Peng S. A Robin boundary problem with Hardy potential and critical nonlinearities[J]. Journal d’Analyse Mathmatique, 2008, 104(1): 125-154.

[8] dAvenia P, Siciliano G, Squassina M. On fractional Choquard equations[J]. Mathematical Models and Methods in Applied Sciences, 2014, 25(8): 1447-1476.

[9] Fiscella A, Bisci G M, Servadei R. Bifurcation and multiplicity results for critical non-local fractional Laplacian problems. Bulletin Des Sciences Mathmatiques, 2015, 140(1): 14-35.

[10] Felmer P, Quaas A, Tan J. Positive solutions of the nonlinear Schrödinger equation with the fractional Laplacian. Proceedings of the Royal Society of Edinburgh: Section A Mathematics, 2012, 142(06): 1237-1262.

[11] Jannelli E. The role played by space dimension in elliptic critical problems. J. Diff. Equats, 1999, 156: 407-426.

[12] Lions P L. The concentration-compactness principle in the calculus of variations, The locally compact cases, Part I. Ann. Inst. Henri Poincare, Analyse Nonlinéaire, 1984, 1: 109-145.

[13] Lions P L. The concentration-compactness principle in the calculus of variations, The locally compact cases, Part II. Ann. Inst. Henri Poincare, Analyse Nonlinéaire, 1984, 1: 223-283.
[14] Nezza E D, Palatucci G, Valdinoci E. Hitchhikers guide to the fractional Sobolev spaces. Bulletin Des Sciences Mathmatiques, 2011, 136(5): 521-573.

[15] Palatucci G, Pisante A. Improved Sobolev embeddings, profile decomposition, and concentration-compactness for fractional Sobolev spaces[J]. Calculus of Variations and Partial Differential Equations, 2014, 50(3-4): 799-829.

[16] Secchi S. Ground state solutions for nonlinear fractional Schrödinger equations in $R^N$. Journal of Mathematical Physics, 2013, 54(3): 056108-305.

[17] Smets D. Nonlinear Schrödinger equations with Hardy potential and critical nonlinearities. Tran. Amer. Math. Soc, 2005, 357: 2909-2938.

[18] Struwe M. A global compactness result for elliptic boundary value problems involving limiting nonlinearities. Math. Z, 1984, 187: 511-517.

[19] Servadei R, Valdinoci E. The Brezis-Nirenberg result for the fractional Laplacian[J]. Transactions of the American Mathematical Society, 2015, 367(1): 67-102.

[20] Servadei R, Valdinoci E. Fractional Laplacian equations with critical Sobolev exponent[J]. Revista Matemtica Complutense, 2015, 28(3): 1-22.

[21] Shang X, Zhang J, Yang Y. Positive solutions of nonhomogeneous fractional Laplacian problem with critical exponent[J]. Communications on Pure and Applied Analysis, 2014, 13(2): 567-584.

[22] Zhu X, Cao D. The concentration-compactness principle in nonlinear elliptic equations. Acta Mathematica Scientia, 1989, 9: 307-323.

[23] Yang J. Fractional Sobolev-Hardy inequality in $R^N$. Nonlinear Analysis, 2015, 119: 179-185.

[24] Chen X, Yang J F. Weighted Fractional Sobolev Inequality in $R^N$, N[J]. Advanced Nonlinear Studies, 2016, 16(3): 623-641.

[25] Yang J, Yu X. Fractional Hardy-Sobolev elliptic problems. arXiv preprint arXiv: 1503.00216, 2015.

[26] Wang X, Yang J. Singular critical elliptic problems with fractional Laplacian[J]. Electronic Journal of Differential Equations, 2015, 297, 1-12.