Sub-Poissonian phononic population in a nanoelectromechanical system

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Abstract. The properties of the phononic distribution of a mechanical oscillator coupled to a single-electron transistor are investigated in the sequential tunnelling regime. It is shown that for not too strong electron–phonon interaction the electrical current may induce a distribution of phonons with sub-Poissonian statistics, which is characterized by a selective population of few phonon states. Depending on the choice of parameters, such a sub-Poissonian phonon distribution can be accompanied either by a super- or a sub-Poissonian electronic Fano factor.

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1. Introduction

Nanodevices where mechanical motion couples directly to electron transport through a mesoscopic conductor, have recently received a lot of attention, both from the theoretical and the experimental point of view [1, 2]. Realizations of these nanoelectromechanical systems (NEMS) have been obtained with single oscillating molecules [3], semiconductor beams coupled to normal and superconducting single-electron transistors (SET) [4]–[6], suspended carbon nanotubes [7] and, very recently, with suspended graphene sheets [8]. NEMS are interesting dynamical systems, not only because they are ideally suited for fast and ultrasensitive detection, but also because they are expected to show several peculiar transport features ranging from shuttling instability [9, 10] to avalanche-like transport [11, 12].

Lately, attention has also been given to the mechanical properties of NEMS, as it appears now that experiments are close enough to the quantum limit [5, 6] to test theoretically predicted quantum features in the vibrational motion [13, 14]. Of particular interest are those associated with the discrete energy states of the oscillator and, indeed, several proposals have been put forward to measure discrete number states [15]–[17]. Furthermore, it is well known that the distribution of oscillation quanta (phonons) in NEMS is strongly affected by the transport of electrons [18, 19], so that one may ask whether there exist regimes in which the current may induce non-classical features in the phonon distribution.

The study of non-classical boson states has a long history, see, e.g. photon antibunching in resonance fluorescence [20] and squeezed states [21]. Of particular interest are systems in which the emission of non-classical radiation is caused by the electronic transport such as, for example, in a quantum point contact, where the electronic shot noise may be a source of antibunched photons [22], or in a two-level quantum dot, where emission of antibunched phonons is expected when transport is characterized by bunching of tunnelling electrons [23].

In the context of NEMS, one is generally not interested in the emission statistics but rather in the properties of the phononic population which often have thermal-like features [18, 24]. Nevertheless, the existence of non-classical phonon states has been recently predicted for a mechanical resonator coupled to a superconducting SET tuned to the Josephson quasiparticle resonance [25], the presence of such states being signalled by a sub-Poissonian population of phonon with variance smaller than the average phonon number, i.e. \( \text{Var} l < \langle l \rangle \).

The system we study in this paper consists of a harmonic oscillator coupled to a normal SET. This particular realization of a NEMS has attracted a lot of attention in recent years due to its experimental feasibility; existing literature is mainly concerned with electron transport properties, with several reports of phonon signatures in the current noise either in the frequency domain or at \( \omega = 0 \), (see, e.g. [26]–[30]), usually in the form of a noise increase with respect to Poissonian transport. Other authors have addressed the dynamics of the oscillator, showing that the resonator is damped and its frequency is shifted because of the coupling with the SET [31].

Here, the main focus is on the population of phonons induced by electric transport. We show that, for not too strong interaction and in the presence of asymmetry of the tunnelling barriers, it is possible to achieve a selective population of few phonon states such that the distribution of the phonon number \( l \) displays a sub-Poissonian behaviour. Moreover, we consider the zero-frequency current noise and show that the fluctuations of both the phonon distribution and the electron current can be enhanced or reduced with respect to Poissonian statistics one independently of the other.
2. Model and methods

The system we consider is a gated SET coupled to leads and to a harmonic oscillator. The SET Hamiltonian is described within the standard constant-interaction model for spinless particles. In particular, the charging-energy term is

\[ H_c = E_c (n - n_g)^2, \]

where \((n - n_g)\) is the effective number of electrons on the SET and \(n_g\) is proportional to the charge induced by the gate. The oscillator and coupling terms are

\[ H_{ph} = \omega_0 l + \sqrt{\lambda} \omega_0 (b^+ + b) (n - n_g), \]

where \(l = b^+ b\) is the phonon number operator and \(b^+\) creates vibrational excitations with energy \(\omega_0\). The dimensionless parameter \(\sqrt{\lambda}\) defines the strength of electromechanical interaction between the position of the oscillator and the effective charge on the SET. Such a term can be induced by an oscillating gate capacitively coupled to the dot [4]. The leads are Fermi liquids with Hamiltonian

\[ H_{leads} = \sum_{k,\alpha=L,R} \varepsilon_{k,\alpha} c_{k,\alpha}^+ c_{k,\alpha}, \]

and chemical potentials \(\mu_{L,R} = \mu_0 \pm eV/2\), where \(V\) is the applied bias voltage, \(\mu_0\) the equilibrium chemical potential, and we have assumed symmetric voltage drops at the tunnelling barriers.

In the limit \(\omega_0, eV, k_B T \ll E_c, \Delta E\), where \(\Delta E\) is the average single-particle level spacing, the SET excess occupancy is limited to 0 and we can focus on the lowest unoccupied single-particle level \(\xi\). This regime corresponds to an infinite Hubbard-type electron–electron interaction. In this case, the total Hamiltonian can be written as

\[ H = \epsilon n + H_{ph} + H_{leads} + H_{tunn}, \]

where \(\epsilon = \xi + 2E_c(1/2 - n_g)\) and \(n = d^+d\) are respectively the energy and the occupation number of the single level, and \(H_{tunn}\) is the tunnelling term

\[ H_{tunn} = \sum_{k,\alpha=L,R} (t_{\alpha} c_{k,\alpha}^+ d + h.c.). \]

Here, \(t_{\alpha}\) are the tunnelling amplitudes, with asymmetry \(A = |t_R|^2/|t_L|^2\).

Being interested in the weak-tunnelling limit, we treat \(H_{tunn}\) as a perturbation. It is convenient to perform a canonical transformation to make the unperturbed Hamiltonian diagonal in the electron and phonon number operators \(n, l\). The desired transformation is the Lang–Firsov polaron transformation \(\hat{O} = U \hat{O} U^\dagger\) with \(U = \exp \eta (b - b^+)\) and \(\eta = \sqrt{\lambda} (n - n_g)\) [18]. The transformed Hamiltonian is given by \(\hat{H} = \tilde{\epsilon} n + \omega_0 l + H_{leads} + \tilde{H}_{tunn},\) where \(\tilde{\epsilon} = \epsilon - \lambda \omega_0\) and

\[ \tilde{H}_{tunn} = \sum_{k,\alpha=L,R} (t_{\alpha} c_{k,\alpha}^+ d e^{-\sqrt{\lambda}(b^+ - b)} + h.c.). \]

In the polaron picture, coherences between states with different phonon number can be neglected as long as the level broadening \(\gamma\) induced by tunnelling is the smallest energy scale being considered [18], i.e. \(\gamma \ll \omega_0, k_B T\). In this limit, the reduced density matrix \(\tilde{\rho}\) of the SET + oscillator system in the polaron picture is diagonal both in the electron and phonon...
the electronic and phononic expectation values can be evaluated in the polaron picture as 

\[ \langle \bar{O} \rangle = \text{Tr} \left[ \bar{O} \rho \right] = \sum_{\alpha \beta} \langle \bar{O} | \rho | \alpha \beta \rangle P_{\alpha \beta} \]

where \( P_{\alpha \beta} \) are the total rates for tunnelling in (out of) the level, and are given by

\[ \Gamma_{\alpha \beta}^{il} = 2\pi \nu |t_a|^2 X^{il} f_a \left( \omega_0 (l' - l) \right), \]

\[ \Gamma_{\alpha \beta}^{il'} = 2\pi \nu |t_a|^2 X^{il'} \left[ 1 - f_a \left( \omega_0 (l - l') \right) \right], \]

where \( f_a(x) = f(x + \xi - \mu_a) \) and \( f(x) \) is the Fermi function. The coefficients \( X^{il} = \left| \langle \bar{l}' | e^{-\sqrt{2}b^\dagger b} | l \rangle \right|^2 \) are the Franck–Condon (FC) factors [11, 18] and \( \nu \) is the density of states of the leads at the Fermi level. In the following, we assume \( \mu_0 = \xi - \lambda \omega_0 \) so that \( n_g = 1/2 \) defines on-resonance conditions. We focus on the regimes of weak (\( \lambda \ll 1 \)) and intermediate (\( \lambda \approx 1 \)) phonon coupling, where cotunnelling is negligible out of the Coulomb-blockaded regions [12].

Electronic and phononic expectation values can be evaluated in the polaron picture as 

\[ \langle O \rangle = \text{Tr} [ \bar{O} \rho ] = \sum_{\alpha \beta} \langle O | \rho | \alpha \beta \rangle P_{\alpha \beta}. \]

It is useful to define also a ‘hybrid’ average \( \langle O \rangle_{\bar{\rho}} = \text{Tr} [ \bar{O} \bar{\rho} ] \). In terms of \( \langle \cdot \rangle_{\bar{\rho}} \), we can write

\[ \langle l \rangle = \langle l \rangle_{\bar{\rho}} + \langle \eta^2 \rangle_{\bar{\rho}}, \]

\[ \langle l^2 \rangle = \langle l^2 \rangle_{\bar{\rho}} + 4 \langle \eta^2 l \rangle_{\bar{\rho}} + \langle \eta^4 \rangle_{\bar{\rho}}, \]

where we have used the fact that \( \bar{b} = b - \eta \) and that \( \bar{\rho} \) is diagonal in the considered weak-tunnelling limit. Note that for operators like \( n \), which are left unchanged by the canonical transformation, it is \( \langle n \rangle_{\bar{\rho}} = \langle n \rangle \). In order to characterize the phonon distribution, we introduce the phonon Fano factor [19, 25]

\[ F_{\text{ph}} = \frac{\text{Var} l}{\langle l \rangle}. \]

It can be directly calculated in terms of equations (8) and (9), exploiting the stationary solution of the rate equations (6). It is useful to define also the parameter \( Q = \text{Var} l - \langle l \rangle \), which is the numerator of the Mandel parameter commonly employed in quantum optics [32].

The electronic Fano factor is instead defined as

\[ F = \frac{S}{2 \varepsilon \langle I \rangle}, \]

where \( S = 2 \int \text{d}t [\langle I(t) I(0) \rangle - \langle I \rangle^2] \) is the zero-frequency current noise and \( \langle I \rangle \) is the stationary current, which we evaluate following [33].

### 3. Results

#### 3.1. Phonon properties

We first consider the properties of the phonon distribution induced by tunnelling. In particular, we are interested in understanding whether there are regimes in which the electron transport through the dot may induce a sub-Poissonian behaviour \( F_{\text{ph}} < 1 \).
Figure 1. Phonon Fano factor $F_{\text{ph}}$ as a function of $V$ and $n_g$ for $\lambda = 0.7$ and two different values of the asymmetry: (a) $A = 0.1$, (b) $A = 0.01$. In both panels: $k_B T = 0.01 \omega_0$, $E_c = 10 \omega_0$.

Theoretically, the super- (sub-)Poissonian character of the phonon distribution is more easily studied in terms of the parameter $Q$, being $F_{\text{ph}} < 1$ only if $Q < 0$. Taking into account equations (8) and (9), $Q$ can be written as

$$Q = Q_\rho + 2\lambda \langle l \rangle_\rho (n_g - 1)^2 + \lambda^2 (2n_g - 1)^2 (n \rangle_\rho (1 - \langle n \rangle_\rho)$$

$$+ 2\lambda (2n_g - 1)\langle l \rangle_\rho \langle n \rangle_\rho - 2\langle ln \rangle_\rho,$$

(12)

where $Q_\rho = \langle l^2 \rangle_\rho - \langle l \rangle_\rho^2 - \langle l \rangle_\rho$ is solely determined by the phonon distribution in the polaron frame $\tilde{P}_l = \tilde{P}_{0,l} + \tilde{P}_{1,l}$. Note that one always has $Q \geq 0$ for phonons obeying a thermal distribution $\tilde{P}_l = e^{-\beta \omega_0 l}(1 - e^{-\beta \omega_0})$. However, when one considers the non-equilibrium population of phonons associated with the stationary solution of equation (6), the sign of $Q$ is not a priori defined.

By solving numerically the rate equations (6) for a wide range of parameters we have found that, even if the phonon distribution induced by tunnelling is typically super-Poissonian, in the presence of asymmetry of the tunnelling barriers it may exhibit sub-Poissonian behaviour ($F_{\text{ph}} < 1$) with certain voltage conditions. Such a situation is depicted for example in figure 1, which shows a density plot of the phonon Fano factor as a function of voltage $V$ and gate $n_g$ for $\lambda = 0.7$ and two different values of the asymmetry. The voltage ranges which correspond to region-I in figure 1(a) are the most favourable for having $F_{\text{ph}} < 1$ with $A < 1$. A sub-Poissonian $F_{\text{ph}}$ can also be obtained in region-II, but only for much stronger asymmetry, see figure 1(b).

It is worth noticing that, as it was observed by Mitra et al [18], the phonon distribution $\tilde{P}_l$ is much narrower off-resonance (region-I) than on-resonance (region-II). This fact can be qualitatively understood observing that, in region-I, at low temperatures $k_B T \ll \omega_0$ the only energetically allowed transitions which increase the phonon number are of the type $|0, l \rangle \rightarrow |1, l + 1 \rangle$, see equation (7). As a consequence, the phononic excitations can be populated only via a series of subsequent tunnelling events such as

$$|0, 0 \rangle \rightarrow |1, 1 \rangle \rightarrow |0, 1 \rangle \rightarrow |1, 2 \rangle \rightarrow \cdots \rightarrow |0, l - 1 \rangle \rightarrow |1, l \rangle.$$

(13)

Note that, in the original frame, the explicit form of the thermal distribution is diagonal on the basis formed by the eigenstates $U^\dagger |n, l \rangle$ of the original Hamiltonian.

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Figure 2. (a) Density plot of $F_{ph}$ as a function of $A$ and $\lambda$ for $eV = 2\omega_0$, $n_g = 0.525$ (centre of region-I in figure 1(a)) and $k_B T = 0.01 \omega_0$, $E_c = 10 \omega_0$. The black line represents the contour $F_{ph} = 1$. (b) $\bar{P}_l$ (black) and $P_l$ (grey) as a function of $l$ for $eV = 2\omega_0$, $n_g = 0.525$, $k_B T = 0.01 \omega_0$, $E_c = 10 \omega_0$ and $\lambda = 0.9$, $A = 0.01$. Inset: set of six states relevant for the dynamics of the system in the case $A \ll 1$. States are labelled as $|n, l\rangle$, arrows represent the energetically allowed transitions in region-I for $k_B T \ll \omega_0$.

This means that any excited phonon state $l$ requires at least $2l - 1$ subsequent tunnelling events to be populated. Vice versa, in region-II the transitions $|1, l\rangle \rightarrow |0, l + 1\rangle$ are also allowed and therefore the $l$th phonon state can be reached by a sequence such as $|0, 0\rangle \rightarrow |1, 1\rangle \rightarrow |0, 2\rangle \rightarrow \ldots$, which consists of $l$ steps only. In other words, populating the phonon excitations is more difficult off-resonance than on-resonance, and this results in a narrower phonon distribution.

The fact that $F_{ph} < 1$ can be observed solely at low voltages and preferably off-resonance suggests that a sub-Poissonian phonon Fano factor requires a slender phonon distribution.

For quantitative discussions we then focus on region-I, which is limited by the conditions $\omega_0 \leq eV \leq 3\omega_0$ and $1/2 \leq n_g \leq 1/2 + \omega_0 / 2E_c$. Analogous results hold specularly for region-III, given that one lets $A \rightarrow 1/A$. As shown in figure 2(a), in region-I the phonon Fano factor exhibits a crossover between super- and sub-Poissonian behaviour depending on the values of $A$ and $\lambda$. Note that, even if $F_{ph}$ depends parametrically on $n_g$, such a dependence is very weak and can be essentially neglected as long as $n_g$ belongs to region-I. The results of figure 2(a) are thus representative of any point in region-I, apart for those very close to the borders. While for symmetric barriers ($A = 1$) it is always $F_{ph} \geq 1$, the phonon Fano factor can be turned to sub-Poissonian values by strong asymmetries $A \ll 1$, if $\lambda$ is smaller than a certain threshold value.

The role played by the asymmetry and the existence of a threshold for $\lambda$ can be qualitatively understood by analysing the limit $A \ll 1$. Numerically, we have found that, in region-I, $\bar{P}_l$ scales as

$$\bar{P}_{l \geq 1} \propto A^{(l-1)}, \quad \text{for} \quad A \ll 1.$$
This power law may be interpreted as follows. For \( A \ll 1 \), the rates for tunnelling-out of the dot are suppressed by a factor of \( A \) with respect to those for tunnelling-in. On the other hand, populating the \( j \)th phononic excitation requires \emph{at least} \( (l - 1) \) tunnelling-out events, see, e.g. equation (13). Therefore, one can expect the leading contribution to \( \bar{\mathcal{P}}_{l \geq 1} \) to scale as equation (14).

As a consequence of equation (14), for \( A \ll 1 \), to lowest order in \( A \) the dynamics of the system reduces to the six states with \( l \leq 2 \), i.e. to the states represented in figure 2(b) (inset). In this case one can solve analytically the rate equations equation (6) in the stationary limit, obtaining

\[
\bar{\mathcal{P}}_0 = 1 - \bar{\mathcal{P}}_1 + \mathcal{O}[A],
\]

\[
\bar{\mathcal{P}}_1 = \frac{2 + \lambda(2 - 2\lambda + \lambda^2)}{(2 - 2\lambda + \lambda^2)(5 + \lambda^2)} + \mathcal{O}[A],
\]

\[
\bar{\mathcal{P}}_2 = \mathcal{O}[A].
\]

Moreover, by taking into account that for \( A \ll 1 \) the dot is prevalently full so that \( \langle n \rangle_{\bar{\rho}} \approx 1 \) and \( \langle nl \rangle_{\bar{\rho}} \approx \langle l \rangle_{\bar{\rho}} \approx \bar{\mathcal{P}}_1 \), it is easy to see that equation (12) reduces to

\[
Q = \bar{\mathcal{P}}_1[2\lambda(n_{\bar{g}} - 1)^2 - \bar{\mathcal{P}}_1] + \mathcal{O}[A],
\]

which can be either positive or negative depending on the value of \( \bar{\mathcal{P}}_1 \). Substituting equation (16) into (18), one can show that for strong asymmetries the phonon distribution is sub-Poissonian for any \( \lambda < \lambda_{cr} \), where \( \lambda_{cr} \approx 1/4(n_{\bar{g}} - 1)^2 \). Note that \( \lambda_{cr} \approx 1 \), being \( n_{\bar{g}} \) limited to region-I.

The existence of a threshold value \( \lambda_{cr} \) can be qualitatively understood observing that a strong electron–phonon coupling is unfavourable for \( Q < 0 \) in two respects: on one hand, it increases the weight of the positive term \( \propto (n_{\bar{g}} - 1)^2 \); on the other hand, it is well known that \( \bar{\mathcal{P}}_0 \to 1 \) as \( \lambda \) is increased [18], so that \( \langle l \rangle_{\bar{\rho}} \to 0 \).

The above analysis suggests that the sub-Poissonian phonon Fano factor is induced by a phonon distribution \( \bar{\mathcal{P}}_l \) in which only the first few phonon states are populated and yet the occupation probability of the excited states is comparable with the one of the ground state. We refer to such a situation as a \emph{selective population} of phonon states which, as we saw, can be realized only for not too strong electron–phonon interaction.

From the numerical solution of the problem, we find that for any \( \lambda < \lambda_{cr} \) a finite threshold \( A_{\lambda} < 1 \) exists such that the phonon distribution is sub-Poissonian for \( A < A_{\lambda} \) (see figure 2(a)). Importantly, for intermediate values of the electron–phonon coupling \( \lambda \approx 1 \) it is possible to obtain \( \bar{F}_{\text{ph}} < 1 \) for asymmetries which are experimentally feasible, i.e. \( A_{\lambda} \lesssim 1 \). This is because for \( \lambda \approx 1 \) the phonon distribution \( \bar{\mathcal{P}}_l \) is already narrow for symmetric barriers. In fact the transition \( |1, 1 \rangle \to |0, 1 \rangle \), which forms the second step of the sequence in equation (13), is suppressed for \( \lambda = 1 \) due to the vanishing of the corresponding FC factor \( X^{11} = e^{-\lambda}(1 - \lambda)^2 \). In this case, the dynamics of the system is frozen to states with \( l \leq 1 \) and can be solved analytically for any value of the asymmetry. One then obtains

\[
F_{\text{ph}}|_{\lambda=1} = 1 + \frac{3 - 8n_{\bar{g}} + 4n_{\bar{g}}^2 + 2A(3 - 8n_{\bar{g}} + 5n_{\bar{g}}^2)}{(2 + A)[n_{\bar{g}}^2(A + 2) - 4n_{\bar{g}} + 3]},
\]

which gives, e.g. \( F_{\text{ph}} = 0.96 \) for \( n_{\bar{g}} = 0.54 \) and \( A = 0.1 \). In this case, the asymmetry plays the role of enhancing the occupation of the first excited state \( l = 1 \).
Finally, some comments are in order. Firstly, we recall that $\bar{P}_l$ is the phonon distribution in the polaron picture. The intrinsic phonon distribution is $P_l = \sum_n P_{n,l}$ where $P_{n,l} = \langle n, l | \rho | n, l \rangle$ and $\rho = U^\dagger \bar{\rho} U$ is the density matrix in the original picture. Note that, in terms of $P_l$, the average phonon number reads $\langle l \rangle = \sum_l l P_l$, and similarly $\langle l^2 \rangle = \sum_l l^2 P_l$. We can evaluate $P_l$ taking into account that $P_{n,l} = \sum_n X_n^{\dagger l} \bar{P}_{n,l}$, where $X_n^{\dagger l} = |\langle n, l | U | n, l' \rangle|^2$ are generalized FC-factors. Such a relationship is a consequence of $\bar{\rho}$ being diagonal in the weak-tunnelling limit. Comparing $P_l$ and $\bar{P}_l$, it is clear that one can speak of selective population only in the polaron picture (see figure 2(b)). Then the presence of a selective population of $\bar{P}_l$ induces an intrinsic phonon distribution $P_l$ with sub-Poissonian behaviour, signalled by $F_{\text{ph}} < 1$. We want to stress that this sub-Poissonian phonon distribution is not merely a result of the transformation to the original frame that we perform in this work. In fact, having $Q_{\bar{\rho}} < 0$ does not guarantee that $Q < 0$: on the contrary, we have checked that generally one has $Q > Q_{\bar{\rho}}$. This observation is supported by the general tendency to have distributions $P_l$ in the original frame broader than $\bar{P}_l$ as shown prototypically in figure 2(b).

Secondly, a suppressed phonon Fano factor $F_{\text{ph}} < 1$ has recently been predicted for an oscillator driven by a superconducting SET in the limit $\gamma \sim \omega_0$, and this has been interpreted as signature of a number-squeezed state [25]. According to Mandel and Wolf [32], a numbersqueezed state is a state with sub-Poissonian Fano factor which exhibits squeezing in one of its quadratures. In our case, we obtain instead a sub-Poissonian distribution without squeezing. In fact, given the generalized quadrature phase operators $X = b \phi e^{i\phi} + b^* e^{-i\phi}$ and $P = i(b \phi e^{i\phi} - b^* e^{-i\phi})$, where $\phi$ is an arbitrary phase factor [32], it is easy to check that, as long as one assumes the reduced density matrix $\bar{\rho}$ in the polaron picture to be diagonal both in the electron and phonon numbers, it is $\text{Var} X \geq 1$ and $\text{Var} P \geq 1$ for every value of $\phi$, i.e. there is no squeezing. In other words, the absence of squeezing is ultimately a consequence of the loss of coherence in the weak-tunnelling limit $\gamma \ll \omega_0$.

Lastly, we comment about the stability of the sub-Poissonian phonon distribution against phonon relaxation. The latter will eventually drive $F_{\text{ph}}$ towards super-Poissonian values, since $F_{\text{ph}} < 1$ is due to a strongly out-of-equilibrium phonon distribution. We have investigated the phonon Fano factor in region-I against an increasing phonon relaxation rate $\omega_{\text{rel}}$ using the approach in [34]. Interestingly, we found that up to moderate $\omega_{\text{rel}} \lesssim 0.2 \Gamma_0$, where $\Gamma_0 = 2\pi \nu |t_L|^2$, a region of sub-Poissonian $F_{\text{ph}}$ survives for $A \approx 1$ and $\lambda \approx 1$.

3.2. Current Fano factor

Up to now, we have considered solely the characteristics of the phonon distribution induced by tunnelling. However, it is well known that the transport properties of the system are in turn strongly affected by phonons. Signatures of this interplay are especially visible in the current Fano factor, which is very sensitive to the electron–phonon interaction [12, 34]. For example, a giant enhancement of $F$ has been predicted as a fingerprint of strong electron–phonon coupling [11]. Here, we consider intermediate coupling and we focus on the study of the (sub-) super-Poissonian character of $F$ with respect to that of $F_{\text{ph}}$. We find that in region-I all the possible combinations of $F$, $F_{\text{ph}} \lesssim 1$ can be obtained by tuning $A$ and $\lambda$ (see figure 3(a)). This is possible because the super- and sub-Poissonian character of $F$ and $F_{\text{ph}}$ have different physical origins. In fact, while $F_{\text{ph}} < 1$ presumes a selective population of phonon states, $F > 1$ is induced by a bunching of tunnelling events [35, 36].
Figure 3. (a) Phase diagram of the possible combinations of $F, F_{\mathrm{ph}}$ in region-I ($eV = 2\omega_0, n_g = 0.525$) depending on $\lambda$ and $A$ (see text for discussion). (b) Plots of $F_{\mathrm{ph}}$ and $F$ as a function of $V$ for $n_g = 0.525$ and $\lambda = 0.5, A = 0.01$ corresponding to the asterisk in panel $a$ for $eV = 2\omega_0$. Boxes: exact numerical solutions; lines: analytical results obtained by solving the six-state model of figure 2(b) (inset). In both panels $k_B T = 0.01\omega_0, E_c = 10\omega_0$.

A simple explanation of this mechanism can be given considering again the case $\lambda \approx 1$. In this case, since the transition $|1, 1\rangle \rightarrow |0, 1\rangle$ is suppressed by the vanishing of $X_{11}$, the dynamics of the system reduces to the three states $|1, 1\rangle \leftrightarrow |0, 0\rangle \leftrightarrow |1, 0\rangle$, and one can obtain the following expression for the current Fano factor:

$$F = \frac{4 + A^2}{(2 + A)^2} + \frac{4 + 2A + 14A^2 + 9A^3 - A^4}{(1 + A)^2(2 + A)^3}(1 - \lambda)^2. \quad (20)$$

Note that the transitions $|0, 0\rangle \leftrightarrow |1, 1\rangle$ and $|0, 0\rangle \leftrightarrow |1, 0\rangle$ act as two competing transport channels, whose relative weight is determined by the ratio $X_{01}^0 / X_{00}^0 = \lambda$. It follows that for $\lambda = 1$ the system behaves as a spin degenerate single level so that it is always $F \leq 1$ [37]. Vice versa, for $\lambda < 1$ the state $|1, 1\rangle$ is a trap state and blocks the transport through the other more conducting channel $|0, 0\rangle \leftrightarrow |1, 0\rangle$. In the presence of asymmetry, such a dynamical channel blockade [35] leads to bunching of tunnelling events and to super-Poissonian current noise $F > 1$. However, as the difference between the two competing transport channels is fairly weak, $F$ is only slightly above 1 (see figure 3(b)). The same mechanism occurs for $\lambda > 1$ but, in this case, it is transport through the excited state that is blocked by the occupation of $|1, 0\rangle$.

It is then clear why super-Poissonian current noise and sub-Poissonian phonon distribution can occur simultaneously only for $\lambda < 1$, when the trapping mechanism responsible for $F > 1$ also favours the selective population of the phonon states. Vice versa, for $\lambda > 1$ the occupation of the vibrational ground state is strongly favoured since $|1, 0\rangle$ is the trap state, and the phonon distribution is mainly super-Poissonian. Note that, outside region-I, the six-state model of figure 2(b) is no longer sufficient to describe the dynamics of the system, which exhibits $F \leq 1$ and $F_{\mathrm{ph}} > 1$ as commonly expected of a fermionic and bosonic system, respectively.
3.3. Discussion

We now briefly comment on possible experimental realizations of the system we have been discussing. The validity of the rate equation approach relies on the condition $\gamma \ll k_B T \ll \omega_0$. Such an energy hierarchy is satisfied in suspended carbon nanotubes \[7, 38\], whose vibrational energies range from hundreds of micro-electron-volt for the longitudinal phonons \[38\] up to multi-electron-volt for the radial breathing mode \[7\]. These frequencies are several orders of magnitude larger than the thermal energy ($k_B T \sim 8.6 \mu eV$) associated with typical experimental temperatures ($T \sim 0.1$ K), which in turn is larger than the level broadening $\gamma$ due to tunnelling. This can be estimated in terms of the average tunnelling rate, which is in the order of $0.4 \mu eV$ for a current of about 100 pA. Moreover, suspended carbon nanotubes are typically characterized by slow vibrational relaxation rates, asymmetric tunnel barriers \[7, 38\] and values of the electron–phonon coupling of order unity \[38\]. Another system which satisfies the energy hierarchy described above is the suspended phonon cavity \[39\], with currents in the order of few hundreds of pico-ampere, electronic temperatures $\sim 0.1$ K and vibrational energy $\omega_0 \approx 100 \mu eV$. In \[39\], however, the oscillator damping seems to be quite strong, therefore the possibility of observing $F_{ph} < 1$ could be hampered.

In order to provide a clearer picture of our results, let us now summarize all the conditions that have to be met in order to observe $F_{ph} < 1$, assuming the energy hierarchy discussed above. As discussed in section 3.1, a selective phonon distribution for tunnel barriers asymmetry $A < 1$ is best obtained for $V$ and $n_g$ belonging to region-I, the blue diamond-shaped region in figure 1(a). Assuming a symmetric voltage drop, this corresponds to maximal bounds of $\omega_0 < eV < 3\omega_0$ and $0.5 < n_g < 0.55$. In the case of $A > 1$, results analogous to those described in section 3.1 are obtained in region-III, which is bound by $\omega_0 < eV < 3\omega_0$ and $0.45 < n_g < 0.5$. In view of the typical resonator frequencies outlined above, it should be possible to easily identify these parameter regions in experiments.

Concentrating on the case $A < 1$ with the system operating in region-I, a summary of the results for $F_{ph}$ and $F$ is shown in figure 3(a). Since there is no unique relationship linking $F$ to $F_{ph}$, it is difficult to exploit the transport properties of the system to globally obtain unambiguous information about the phonon distribution. However, with a suitable restriction of the parameters space it is possible to link electronic and phonon Fano factor. Consider as an example the case $\lambda < 1$: it is clear (see figure 3(a)) that upon tuning $A \ll 1$, one eventually obtains a weak super-Poissonian electronic Fano factor and correspondingly $F_{ph} < 1$. Therefore, in this regime it is indeed possible to exploit electronic properties to gain qualitative information on the vibrational dynamics. As an example, for the system described in \[38\], with the reasonable assumption of asymmetric tunnel barriers we can expect a super-Poissonian current Fano factor associated with sub-Poissonian phononic population.

4. Conclusions

In conclusion, we have analysed the phonon distribution induced by tunnelling in a mechanical resonator coupled to a SET. We have shown the existence of regimes where the current produces a selective population of few phonon states with sub-Poissonian statistics. The role played by the asymmetry of the tunnelling barriers in affecting the phonon Fano factor has been discussed, as well as the existence of a maximum critical value for the electron–phonon interaction $\lambda_{cr}$ to observe $F_{ph} < 1$. We have also shown that for moderate values of the electron–phonon

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interaction $\lambda \approx 1$, a sub-Poissonian phonon Fano factor can be obtained already for asymmetries which are experimentally feasible, $A \lesssim 1$. We have checked that $F_{\text{ph}} < 1$ is stable up to moderate phonon relaxation for $A \approx 1$ and $\lambda \approx 1$. Finally, we have considered the electronic noise, finding different combinations of sub- and super-Poissonian electron and phonon Fano factors depending on the asymmetry and on the strength of the electron–phonon coupling. In selected regions of the parameters space, as in region-I and for $\lambda < 1$, measuring $F > 1$ would allow to infer $F_{\text{ph}} < 1$. This may be seen as an indirect probe of phonon properties by means of the NEMS transport, in a complementary way with respect to the recently proposed direct measurement schemes [15]–[17] involving external probes. However, in the general case no unequivocal relationship between shot noise and phonon Fano factor can be found.

In order to extract additional information about the distribution of states of the resonator exploiting transport properties, one could extend the analysis to the frequency-resolved shot noise $S(\omega)$ [27, 40, 41]. Indeed $S(\omega)$ exhibits peaks at multiples of the (detuned) fundamental frequency of the oscillator. The height and width of these peaks are strongly affected by the mechanical properties of the resonator [41]. It is therefore possible that a careful analysis of these noise peaks could bring additional insight into the phonon distribution. Another interesting possibility could be to study the full counting statistics in the search for possible connections with the phonon distribution Fano factor.

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