2-REFLECTIVE MODULAR FORMS: A JACOBI FORMS APPROACH

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Abstract. We give an explicit formula to express the weight of 2-reflective modular forms. We prove that there is no 2-reflective lattice of signature (2, n) when \( n \geq 15 \) and \( n \neq 19 \) except the even unimodular lattices of signature (2, 18) and (2, 26). As applications, we give a simple proof of Looijenga’s theorem that the lattice \( 2U \oplus 2E_8(-1) \oplus \langle -2n \rangle \) is not 2-reflective if \( n > 1 \). We also classify reflective modular forms on lattices of large rank and the modular forms with the simplest reflective divisors.

1. Introduction

Let \( M \) be an even lattice of signature (2, n). A non-constant holomorphic modular form for \( M \) is called reflective if the support of its divisor is actually contained in the union of quadratic divisors determined by reflective vectors of \( M \). This type of modular forms first appeared in the works of Borcherds [1, 2] and Gritsenko-Nikulin [18]. They have many applications in various related topics, such as the classification of Lorentzian Kac-Moody algebras [19, 20, 22, 23, 28, 29]; search of hyperbolic reflection groups [3]; the theory of moduli spaces [16, 17, 24, 26].

Reflective modular forms seem to be exceptional and very rare. The classification of reflective modular forms has been widely studied by several mathematicians. In 1996, Gritsenko and Nikulin first conjectured that the number of lattices possessing reflective modular forms is finite [21] and gave a complete classification for \( n = 3 \) [20, 22]. Scheithauer gave a complete classification of reflective modular forms of singular weight on lattices of prime level [28, 29, 30, 31]. Looijenga [24] proved one part of the arithmetic mirror symmetry conjecture formulated in [21], which might give a new approach to classify reflective modular forms. Recent work of Ma [26] showed that there are only finitely many lattices of signature (2, n) which carry a strongly reflective modular form that vanishes of order one along the reflective divisors when \( n \geq 4 \).

The aim of this paper is to investigate 2-reflective modular forms which are the most basic class of reflective modular forms. A non-constant holomorphic modular form for \( M \) is called 2-reflective if the support of its zero divisor is contained in the Heegner divisor defined by the \((-2)\)-vectors in \( M \). The lattice \( M \) is called 2-reflective if it admits a 2-reflective modular form. In 2017, Ma [25] showed that there are only finitely many 2-reflective lattices of signature (2, n) with \( n \geq 7 \) and there is no 2-reflective lattice when \( n \geq 26 \) except the even unimodular lattice \( II_{2,26} \) of signature (2, 26). In this paper, we prove the following main theorem.

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Theorem 1.1. There is no 2-reflective lattice of signature $(2, n)$ when $n \geq 15$ and $n \neq 19$ except the even unimodular lattices of signature $(2, 18)$ and $(2, 26)$.

Our approach is based on the Gritsenko-Nikulin representation of Borcherds products in terms of Jacobi forms [14, 20]. When $M$ contains two integral hyperbolic planes (i.e. $M = 2U \oplus L(-1)$), every 2-reflective modular form can be represented as a Borcherds product [5]. By means of the isomorphism between vector valued modular forms and Jacobi forms, there exists a weakly holomorphic Jacobi form of weight 0 and index $L$ such that its Borcherds product gives the above 2-reflective modular form. Then the identity (Lemma 2.3) related to $q^0$-term of Jacobi forms of weight 0 yields a formula expressing the weight of 2-reflective modular forms (Theorem 3.2). Furthermore, we can construct holomorphic Jacobi forms of weight 6 and weight 7 from the above Jacobi form of weight 0 by using the weight raising differential operators (Lemma 2.2). The existence of such Jacobi forms implies the non-existence of 2-reflective modular forms with respect to lattices of large rank. Our main theorem gives us a necessary condition for a lattice of signature $(2, 19)$ being 2-reflective (Theorem 3.6). From this, we deduce that the interesting lattices $T_n := 2U \oplus 2E_8(-1) \oplus \langle -2n \rangle$ are not 2-reflective for $n \geq 2$ (Theorem 3.9). Note that this question has been investigated in [22, 27] and the above result was first proved by Looijenga in [24].

The above arguments can also be used to classify reflective modular forms. In section 4, we study reflective modular forms on lattices of prime level and show that such modular forms do not exist when the rank of lattice is big enough (Theorem 4.2, Theorem 4.6). In [25], Ma showed that there is no reflective lattice of signature $(2, n)$ with $n \geq 26$ except the even unimodular lattice of signature $(2, 26)$. As an extension of Ma’s result, we conclude that there is no reflective lattice of signature $(2, n)$ when $n \geq 23$ except the scaling of $I_{2,26}$ (Theorem 4.8).

Our approach can be applied to some other questions. For instance, it provides us a straightforward way to classify the modular forms with the simplest reflective divisors, i.e. the dd-modular forms (see Section 5). This gives a generalization of main results in [7, 15] (see Theorem 5.2).

The paper is organized as follows. In section 2 we introduce briefly Jacobi forms and differential operators. In section 3 we define 2-reflective modular forms and prove our main theorem. An application is also presented. In section 4 we classify reflective modular forms. Section 5 is devoted to the classification of dd-modular forms.

2. Preliminaries: Jacobi forms

In this section, some standard facts about Jacobi forms are reviewed. We refer to [8, 11] for more details. From now on, $L$ always denotes an even positive-definite lattice with bilinear form $(\cdot, \cdot)$ and dual lattice $L^\vee$. The rank of $L$ is denoted as $\text{rank}(L)$. We define the Jacobi forms in the following way.

Definition 2.1. Let $\varphi : \mathbb{H} \times (L \otimes \mathbb{C}) \to \mathbb{C}$ be a holomorphic function and $k \in \mathbb{Z}$. If $\varphi$ satisfies the functional equations

\begin{align*}
(2.1) \quad & \varphi \left( \frac{a\tau + b}{c\tau + d}, \frac{3}{c\tau + d} \right) = (c\tau + d)^k e^{i\pi \frac{c(1,3)}{c\tau + d}} \varphi(\tau, 3), \\
(2.2) \quad & \varphi(\tau, 3 + x\tau + y) = e^{-i\pi((x,x)\tau + 2(x,3))} \varphi(\tau, 3)
\end{align*}
for any \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \) and any \( x, y \in L \) and \( \varphi \) admits a Fourier expansion as

\[
\varphi(\tau, z) = \sum_{n \geq n_0} \sum_{l \in L^\vee} f(n, l) q^n \zeta^l
\]

where \( n_0 \in \mathbb{Z} \), \( q = e^{2\pi i \tau} \) and \( \zeta^l = e^{2\pi i (l, z)} \), then \( \varphi \) is called a weakly holomorphic Jacobi form of weight \( k \) and index \( L \). If \( \varphi \) further satisfies the condition

\[
f(n, l) \neq 0 \implies n \geq 0
\]

then \( \varphi \) is called a weak Jacobi form. If \( \varphi \) further satisfies the stronger condition

\[
f(n, l) \neq 0 \implies 2n - (l, l) \geq 0
\]

then \( \varphi \) is called a holomorphic Jacobi form. We denote by \( J^{w.h.}_{k, L} \) (resp. \( J^w_{k, L} \), \( J^h_{k, L} \)) the vector space of weakly holomorphic Jacobi forms (resp. weak Jacobi forms, holomorphic Jacobi forms) of weight \( k \) and index \( L \).

We recall the following weight raising differential operator, which will be used later. Such technique can also be found in [9] for the general case or in [10] for classical Jacobi forms.

**Lemma 2.2.** Let \( \psi(\tau, z) = \sum a(n, l) q^n \zeta^l \) be a weakly holomorphic Jacobi form of weight \( k \) and index \( L \). Then \( H_k(\psi) \) is a weakly holomorphic Jacobi form of weight \( k + 2 \) and index \( L \), where

\[
H_k(\psi) = H(\psi) + (2k - \text{rank}(L)) G_2(\tau) \psi
\]

\[
H(\psi)(\tau, z) = \frac{1}{2} \sum_{n \in \mathbb{Z}} \sum_{l \in L^\vee} [2n - (l, l)] a(n, l) q^n \zeta^l,
\]

and \( G_2(\tau) = -\frac{1}{12} + \sum_{n \geq 1} \sigma(n) q^n \) is the Eisenstein series of weight 2.

The next lemma plays a key role in our discussions of the weight of 2-reflective modular forms and in the classification of dd-modular forms. It is a particular case of [14, Proposition 2.2]. We next give it a simple proof.

**Lemma 2.3.** Assume that \( \phi \) is a weakly holomorphic Jacobi form of weight 0 and index \( L \) with the Fourier expansion

\[
\phi(\tau, z) = a q^{-1} + \sum_{l \in L^\vee} c(0, l) \zeta^l + O(q).
\]

Then the following identity is valid

\[
\sum_{l \in L^\vee} c(0, l) - \frac{12}{\text{rank}(L)} \sum_{l \in L^\vee} c(0, l)(l, l) - 24a = 0.
\]

**Proof.** From Lemma 2.2 it follows that \( H_0(\phi) \) is a weakly holomorphic Jacobi form of weight 2. Therefore \( H_0(\phi)(\tau, 0) \) is a nearly holomorphic modular form of weight 2 for the full modular group \( \text{SL}_2(\mathbb{Z}) \). By [11, Lemma 9.2], \( H_0(\phi)(\tau, 0) \) has zero constant term, which establishes the desired formula. \( \square \)
3. 2-REFLECTIVE MODULAR FORMS

3.1. Non-existence of 2-reflective modular forms. Let $M$ be an even integral lattice of signature $(2,n)$, $n \geq 3$, and let

$$\mathcal{D}(M) = \{ [\omega] \in \mathcal{P}(M \otimes \mathbb{C}) : (\omega, \omega) = 0, (\omega, \bar{\omega}) > 0 \}^+$$

be the associated Hermitian symmetric domain of type IV (here $+$ denotes one of its two connected components). Let us denote the index 2 subgroup of the orthogonal group $O(M)$ preserving $\mathcal{D}(M)$ by $O^+(M)$. We define

$$\mathcal{H} := \bigcup_{l \in M, (l,l) = -2} l^+ \cap \mathcal{D}(M)$$

as the Heegner divisor of $\mathcal{D}(M)$ generated by the $(-2)$-vectors in $M$.

Definition 3.1. Let $F$ be a non-constant holomorphic modular form on $\mathcal{D}(M)$ with respect to a finite-index subgroup $\Gamma < O^+(M)$ and a character (of finite order) $\chi : \Gamma \to \mathbb{C}$. $F$ is called 2-reflective if the support of its zero divisor is contained in $\mathcal{H}$. The lattice $M$ is called 2-reflective if it admits a 2-reflective modular form.

Following [25], we consider the decomposition (3.3) of the $(-2)$-Heegner divisor $\mathcal{H}$. Let $A_M := M^\vee/M$ be the discriminant group of $M$. We denote the most important subgroup of $O^+(M)$ acting trivially on $A_M$ by $\tilde{O}^+(M)$. The invariances of $\tilde{O}^+(M)$-orbit of a primitive vector $l \in M$ are the norm $(l,l)$ and the image $l/\text{div}(l) \in A_M$, where $\text{div}(l)$ is the natural number generating the ideal $(l,M)$. From this point of view, we choose the following notations. For $\lambda \in A_M$ and $m \in \mathbb{Q}$ we put

$$\mathcal{H}(\lambda, m) = \bigcup_{l \in M, (l,l) = 2m} l^+ \cap \mathcal{D}(M)$$

as the Heegner divisor of discriminant $(\lambda, m)$. In particular, $\mathcal{H} = \mathcal{H}(0,-1)$. Let $\pi_M \subset A_M$ be the subset of elements of order 2 and norm $-1/2$. For each $\mu \in \pi_M$ we abbreviate $\mathcal{H}(\mu, -1/4)$ to $\mathcal{H}_\mu$. Also, we set

$$\mathcal{H}_0 = \bigcup_{l \in M, (l,l) = -2, \text{div}(l) = 1} l^+ \cap \mathcal{D}(M).$$

Then we have the following decomposition

$$\mathcal{H} = \mathcal{H}_0 + \sum_{\mu \in \pi_M} \mathcal{H}_\mu.$$  

By [25 Lemma 2.2], if $M$ admits a 2-reflective modular form with respect to some $\Gamma_0 < O^+(M)$, then $M$ also has a 2-reflective modular form with respect to any other finite-index subgroup $\Gamma < O^+(M)$. Thus, the lattice $M$ is 2-reflective if and only if it admits a 2-reflective modular form with respect to $\tilde{O}^+(M)$. Throughout the section, we only consider 2-reflective modular forms with respect to $\tilde{O}^+(M)$.

Next, we assume that the lattice $M$ contains $2U$ and $M = 2U \oplus L(-1)$, where $L$ is a positive-definite even lattice. In this case, each $\mathcal{H}_\mu$ is an $\tilde{O}^+(M)$-orbit of a single quadratic divisor $l^+ \cap \mathcal{D}(M)$ and it is irreducible. We can write each element of $\pi_M$ in the form $\mu = (0, n_\mu, \mu_0/2, 1, 0)$, where $n_\mu \in \mathbb{Z}$, $\mu_0 \in L$ and
2nµ − \frac{1}{4}(µ_0, µ_0) = -\frac{1}{2}. If M admits a 2-reflective modular form F of weight k, then its divisor can be written as

\text{div}(F) = β_0 \mathcal{H}_0 + \sum_{µ ∈ π_M} β_µ \mathcal{H}_µ \tag{3.4}

where β_µ are non-negative integers. By \cite[Theorem 5.12]{34} or \cite[Theorem 1.2]{35}, there exists a nearly holomorphic vector valued modular form \( f \) of weight \( -\text{rank}(L)/2 \) with respect to the Weil representation \( ρ_M \) of \( Mp_2(\mathbb{Z}) \) on the group ring \( \mathbb{C}[A_M] \) with principal part

\[ β_0 q^{-1} e_0 + \sum_{µ ∈ π_M} (β_µ - β_0) q^{-1/4} e_µ, \]

such that \( F \) is the Borcherds product of \( f \). In view of the isomorphism between Jacobi forms and vector valued modular forms, there exists a weakly holomorphic Jacobi form \( φ_L \) of weight 0 and index L with singular Fourier coefficients of the form (see \cite{36})

\[ \text{sing}(φ_L) = β_0 \sum_{r ∈ L} q^{(r, r)/2} \zeta^r + \sum_{µ ∈ π_M} (β_µ - β_0) \sum_{s ∈ L + µ_0/2} q^{(s, s)/2 - 1/4} e_s \tag{3.5} \]

where \( \zeta^r := e^{2πi(r, L)} \). Thus, we have

\[ φ_L(τ, z) = β_0 q^{-1} + β_0 \sum_{r ∈ R(L)} \zeta^r + 2k + \sum_{u ∈ π_M} (β_µ - β_0) \sum_{s ∈ R_µ(L)} \zeta^s + O(q) \tag{3.6} \]

here and subsequently, \( R(L) \) denotes the set of 2-roots in \( L \) and

\[ R_µ(L) := \{ s ∈ L’ : 2s ∈ R(L), s - µ_0/2 ∈ L \}. \tag{3.7} \]

With the help of equation (3.6) and Lemma \cite[23]{38} we conclude the following theorem.

**Theorem 3.2.** Let \( L \) be a positive-definite even lattice and \( M = 2U ⊕ L(-1) \). Suppose that \( F \) is a 2-reflective modular form of weight \( k \) with the divisor of the form (3.4). Then the weight \( k \) of \( F \) is given by the following formula

\[ k = β_0 \left[ 12 + |R(L)| \left( \frac{12}{\text{rank}(L)} - \frac{1}{2} \right) \right] \tag{3.8} \]

\[ + \left( \frac{3}{\text{rank}(L)} - \frac{1}{2} \right) \sum_{µ ∈ π_M} (β_µ - β_0)|R_µ(L)|. \]

**Remark 3.3.** From the above theorem, we have

(1) If \( R(L) \) is empty, then the weight of 2-reflective modular form is \( 12β_0 \).

(2) When \( \text{rank}(L) ≥ 6 \), there is no 2-reflective modular form with \( β_0 = 0 \). This fact can be also proved by Riemann–Roch theorem as the proof of \cite[Proposition 6.1]{31}.

(3) When \( \text{rank}(L) = 6 \), the weight of 2-reflective modular form is \( β_0(12 + \frac{3}{2}|R(L)|) \) and the modular form is not of singular weight.

We next study modular forms with the complete 2-divisor (i.e. \( \text{div}(F) = \mathcal{H} \)), which are the simplest 2-reflective modular forms.
Theorem 3.4. If there exists a modular form with the complete 2-divisor for \( M := 2U \oplus L(-1) \), then either \( \text{rank}(L) \leq 8 \), or \( L \) is a unimodular lattice of rank 16 or 24. Moreover, the weight of the corresponding modular form is

\[
k = 12 + |R(L)| \left( \frac{12}{\text{rank}(L)} - \frac{1}{2} \right).
\]

Proof. Firstly, the above formula is a direct result of Theorem 3.2. We note the modular form with the complete 2-divisor by \( F \). Then there exists a weakly holomorphic Jacobi form of weight 0 and index \( L \)

\[
\phi(\tau, z) = q^{-1} + \sum_{r \in R(L)} \zeta^r + 2k + O(q)
\]

whose singular Fourier coefficients are

\[
sing(\phi) = \sum_{n \geq -1} \sum_{l \in L \setminus (l,l) = 2n+2} q^n \zeta^l,
\]

such that Borch(\( \phi \)) = \( F \). By [25], it is known that \( \text{rank}(L) < 24 \) or \( L \) is a unimodular lattice of rank 24. Further assume \( \text{rank}(L) \leq 23 \), and we construct following two Jacobi forms by the differential operators. For simplicity of notations, we write \( R = |R(L)| \) and \( n_0 = \text{rank}(L) \).

\[
f_2 := \frac{24}{n_0 - 24} H_0(\phi) = q^{-1} + \sum_{r \in R(L)} \zeta^r - R + O(q) \in \mathcal{J}_{2,w,h}.
\]

\[
f_4 := \frac{24}{n_0 - 28} H_2(f_2) = q^{-1} + \sum_{r \in R(L)} \zeta^r - \frac{(R + 24)(n_0 - 4)}{n_0 - 28} + O(q) \in \mathcal{J}_{4,w,h}.
\]

It is easy to see that

\[
g := \frac{n_0 - 28}{48} (E_4 \phi - f_4) = R \left( 1 - \frac{14}{n_0} \right) + 6(n_0 - 26) + O(q) \in \mathcal{J}_{4,L}
\]

\[
h := E_6 \phi - E_4 f_2 = \frac{24R}{n_0} - 720 + O(q) \in \mathcal{J}_{6,L}
\]

are holomorphic Jacobi forms of weight 4 and 6, respectively. Since the singular weight of holomorphic Jacobi form of index \( L \) is \( \frac{n_0}{2} \), \( g = 0 \) if \( n_0 > 8 \) and \( h = 0 \) if \( n_0 > 12 \). By direct calculations, we have

- when \( R = 0 \), \( g \neq 0 \) if \( n_0 < 24 \).
- when \( R > 0 \), \( g \neq 0 \) if \( n_0 \leq 14 \).

When \( n_0 = 16 \), from \( h = 0 \), it follows that the Fourier coefficients of \( \phi \) satisfy: \( c(n,l) = 0 \) if \( 2n - (l,l) = 0 \) and \( l \notin L \). Therefore the following Jacobi form of singular weight 8

\[
E_8 \phi - E_6 f_2 = 1728 + \sum_{n \geq 0, l \in L^\vee : 2n = (l,l)} a(n,l) q^n \zeta^l \in \mathcal{J}_{8,L}
\]

satisfies the same condition: \( a(n,l) = 0 \) if \( 2n - (l,l) = 0 \) and \( l \notin L \). We then obtain

\[
E_8 \phi - E_6 f_2 = 1728 \sum_{l \in L} q^{\frac{2l(l)}{l(l)} \zeta^l}
\]

and \( L \) has to be unimodular. The proof is completed. \( \square \)
Remark 3.5. It is worth pointing out that there exist lattices $L$ which admit a modular form with the complete 2-divisor when $1 \leq \text{rank}(L) \leq 8$ (see [23]). The Igusa form $\chi_{35}$ is a modular form with the complete 2-divisor.

We are going to generalize our method to prove the non-existence of 2-reflective modular forms in higher dimensions.

Theorem 3.6. Suppose that $M := 2U \oplus L(-1)$ is a 2-reflective lattice satisfying $\text{rank}(L) \geq 13$. Then either $\text{rank}(L) = 17$, or $L$ is a unimodular lattice of rank 16 or 24. Furthermore, when $\text{rank}(L) = 17$, the weight of the corresponding 2-reflective modular form is $75\beta_0$, where $\beta_0$ is the multiplicity of the divisor $\mathcal{H}_0$.

Proof. If $M$ has a 2-reflective modular form $F$ of weight $k$ with the divisor of the form (3.4), then there exists a weakly Jacobi form $\phi$ of weight 0 and index $L$ with the singular Fourier coefficients of the form (3.5). Let us assume that $\text{rank}(L) \leq 23$. We next construct a holomorphic Jacobi form of weight 6 from $\phi$.

We write $\phi = S_1 + d + S_2 + \cdots$, where $S_1$ and $S_2$ are the first and second terms in (3.5), respectively, and $d = 2k$. It is clear that $\phi - S_1 - S_2$ does not have term with negative hyperbolic norm. We can construct ($n_0 = \text{rank}(L)$)

$$f_2 := \frac{24}{n_0 - 24} H_0(\phi) = S_1 + d_1 + c_1 S_2 + \cdots \in J_{2,L}^{w,h},$$

$$f_4 := \frac{24}{n_0 - 28} H_2(f_2) = S_1 + d_2 + c_2 S_2 + \cdots \in J_{4,L}^{w,h},$$

$$f_6 := \frac{24}{n_0 - 32} H_4(f_4) = S_1 + d_3 + c_3 S_2 + \cdots \in J_{6,L}^{w,h},$$

where

$$d_1 = \frac{n_0(d - 24\beta_0)}{n_0 - 24}, \quad c_1 = \frac{n_0 - 6}{n_0 - 24},$$

$$d_2 = \frac{(n_0 - 4)(d_1 - 24\beta_0)}{n_0 - 28}, \quad c_2 = \frac{n_0 - 10}{n_0 - 28},$$

$$d_3 = \frac{(n_0 - 8)(d_2 - 24\beta_0)}{n_0 - 32}, \quad c_3 = \frac{n_0 - 14}{n_0 - 32}.$$  

It is obvious that

$$\varphi_6 := (c_1 - c_3) E_6 \phi + (c_3 - 1) E_4 f_2 + (1 - c_1) f_6 = u + O(q) \in J_{6,L}.$$  

is a holomorphic Jacobi form of weight 6, where

$$u = (d - 504\beta_0)(c_1 - c_3) + (d_1 + 240\beta_0)(c_3 - 1) + d_3(1 - c_1).$$  

In view of the singular weight, $\varphi_6 = 0$ if $\text{rank}(L) \geq 13$. From Remark 3.3 we know that if $F$ exists then $d = 2k \geq \text{rank}(L)$ and $\beta_0 > 0$ when $\text{rank}(L) \geq 6$. By direct calculations, when $n_0 = 13$ or 14, $u \neq 0$, which is impossible.

We next assume that $15 \leq \text{rank}(L) \leq 23$. We construct

$$g = E_4 \phi - f_4 = (d + 240\beta_0) - d_2 + (1 - c_2) S_2 + \cdots \in J_{4,L}^{w,h}.$$  

Note that $\eta^6 g$ is a holomorphic Jacobi form of weight 7 and index $L$ with character. In view of the singular weight, we have $\eta^6 g = 0$ and then $g = 0$. If $S_2 = 0$, then $L$ is a unimodular lattice of rank 16 by Theorem 3.4. If $S_2 \neq 0$, then $1 - c_2 = 0$, which gives $n_0 = 17$. By $(d + 240\beta_0) - d_2 = 0$ and $n_0 = 17$, we get $d = 150\beta_0$. We hence complete the proof. \qed
Remark 3.7. Firstly, there exist 2-reflective lattices when $1 \leq \text{rank}(L) \leq 8$. When $\text{rank}(L) = 11, 12$, we do not know if there exists 2-reflective lattice. When $\text{rank}(L) = 9, 10, 17$, there exist 2-reflective lattices. They are constructed as follows:

- $L = E_8 \oplus A_1$: $\text{Borch}(E_4 E_{4,1} \otimes \vartheta_{E_8}/\Delta)$ is a 2-reflective modular form of weight 195.
- $L = E_8 \oplus 2A_1$: $\text{Borch}(E_4 \otimes E_{4,1} \otimes \vartheta_{E_8}/\Delta)$ is a 2-reflective modular form of weight 138.
- $L = 2E_8 \oplus A_1$: $\text{Borch}(E_4 \otimes \vartheta_{E_8} \otimes \vartheta_{E_8}/\Delta)$ is a 2-reflective modular form of weight 75.

Here, we construct Borcherds products from weakly holomorphic Jacobi forms of weight 0 (see [14]). $\vartheta_{E_8}$ is the theta series for the root lattice $E_8$, which is a holomorphic Jacobi form of weight 4 and index $E_8$. $E_{4,1}$ is the Jacobi-Eisenstein series of weight 4 and index 1 introduced in [10].

We now consider the general case that $M$ does not necessarily contain $2U$.

**Theorem 3.8.** There is no 2-reflective lattice of signature $(2, n)$ when $n \geq 15$ and $n \neq 19$ except the even unimodular lattices $\Pi_{2,18}$ and $\Pi_{2,26}$.

**Proof.** The proof is similar to the proof of [25, Proposition 3.1]. By [25] and [26], we know

1) If $M$ has a 2-reflective modular form, then any even overlattice $M'$ of $M$ has a 2-reflective modular form too.

2) One can choose an even overlattice $M'$ of $M$ such that $M'$ contains $2U$.

We thus complete the proof by the above theorem.

3.2. Application: moduli of K3 surfaces. As an application, we consider the lattices

$$T_n = U \oplus U \oplus E_8(-1) \oplus E_8(-1) \oplus \langle -2n \rangle$$

where $n \in \mathbb{N}$. The modular variety $\tilde{O}^+(T_n) \backslash \mathcal{D}(T_n)$ is the moduli space of polarized K3 surfaces of degree $2n$. The subset

$$\text{Discr} = \bigcup_{l \in T_n} \langle t, l \rangle = -2 \cap \mathcal{D}(T_n)$$

is the discriminant of the moduli space. Nikulin [27] posed the question whether the discriminant is equal to the set of zeros of certain automorphic form. This question is equivalent to whether $T_n$ is 2-reflective. Nikulin showed that for any $N$ there exists $n > N$ such that $T_n$ is not 2-reflective. Gritsenko and Nikulin [22] proved that the lattices $T_n$ are not 2-reflective for big $n$. Finally, Looijenga [24] demonstrated that $T_n$ is not 2-reflective if $n \geq 2$. As a direct consequence of Theorem 3.2 and Theorem 3.6 we present a pretty simple proof of the result.

**Theorem 3.9.** The lattice $T_n$ is 2-reflective if and only if $n = 1$.

**Proof.** By Remark 3.7, $T_1$ is 2-reflective. By contradiction, we assume that $T_n$ is 2-reflective with $n \geq 2$. On the one hand, by Theorem 3.2 the weight of the corresponding 2-reflective modular form is

$$k = \beta_0 \left[ 12 + 480 \left( \frac{12}{17} - \frac{1}{2} \right) \right] > 110\beta_0.$$
On the other hand, Theorem 3.6 tells that \( k = 75 \beta_0 \), which leads to a contradiction. Hence \( T_n \) is not 2-reflective when \( n \geq 2 \).

\[ \blacksquare \]

4. Non-existence of reflective modular forms

Let \( M \) be an even lattice of signature \((2, n)\), \( n \geq 3 \). The level of \( M \) is the smallest positive integer \( N \) such that \( N(r, r) \in 2\mathbb{Z} \) for all \( r \in M^\perp \). A primitive vector \( l \in M \) of negative norm is called reflective if the reflection

\begin{equation}
\sigma_l(x) = x - \frac{2(l, x)l}{(l, l)}, \quad x \in M
\end{equation}

is in \( O^+(M) \). A non-constant holomorphic modular form for \( M \) is called reflective if the support of its divisor is set-theoretically contained in the union of quadratic divisors \( l^\perp \cap \mathcal{D}(M) \) determined by reflective vectors \( l \) of \( M \). The lattice \( M \) is called reflective if it admits a reflective modular form. A primitive vector \( l \in M \) with \( (l, l) = -2d \) is reflective if and only if \( \text{div}(l) = 2d \) or \( d \). Let us fix \( \lambda = [l/\text{div}(l)] \in A_M \). Then

\begin{equation} \label{eq:reflective_vectors}
l^\perp \cap \mathcal{D}(M) \text{ is contained in } H(\lambda, -1/(4d)) \text{ in the first case, and is contained in } \end{equation}

\[ \mathcal{H}(\lambda, -1/d) - \sum_{2\nu = \lambda} \mathcal{H}(\nu, -1/(4d)) \]

in the second case.

Reflective modular forms of singular weight on lattices of prime level were completely classified by Scheithauer in [29, 31]. As another application of our arguments in the previous section, we attempt to classify reflective modular forms on lattices of prime level and large rank.

Let \( M = 2U \oplus L(-1) \) be an even lattice of prime level \( p \) and \( F \) be a non-constant reflective modular form of weight \( k \) with respect to \( \bar{O}^+(M) \). By [31 Section 6], the divisor of \( F \) can be represented as

\begin{equation} \label{eq:divisor_of_F}
\text{div}(F) = \beta_0 \mathcal{H}_0 + \sum_{\gamma \in \pi_{M, p}} \beta_\gamma \mathcal{H}(\gamma, -1/p),
\end{equation}

where \( \pi_{M, p} \subset A_M \) is the subset of elements of norm \(-2/p\). By [3], there exists a nearly holomorphic modular form with principal part

\[ \beta_0 q^{-1} e_0 + \sum_{\gamma \in \pi_{M, p}} \beta_\gamma q^{-1/p} e_\gamma. \]

Then there exists a weakly Jacobi form of weight 0 with singular Fourier coefficients

\begin{equation} \label{eq:singular_fourier_coefficients}
\text{sing}(\psi_L) = \beta_0 \sum_{r \in L} q^{(r, r)/2-1} \zeta^r + \sum_{\gamma \in \pi_{M, p}} \beta_\gamma \sum_{s \in L+\gamma} q^{(s, s)/2-1/p} \zeta^s.
\end{equation}

Then the \( q^0 \)-term of \( \psi_L \) can be written as

\[ \psi_L(\tau, z) = \beta_0 q^{-1} + \sum_{r \in R(L)} \zeta^r + 2k + \sum_{\gamma \in \pi_{M, p}} \beta_\gamma \sum_{s \in C_\gamma(L)} \zeta^s + O(q), \]

where

\begin{equation} \label{eq:subset_of_L_prime}
C_\gamma(L) := \{ s \in L^\perp : (s, s) = 2/p, s - \gamma \in L \}.
\end{equation}
Thus, we get a formula related to the weight of the above reflective modular form

\[
k = \beta_0 \left[ 12 + |R(L)| \left( \frac{12}{\text{rank}(L)} - \frac{1}{2} \right) \right] + \left( \frac{12}{p \cdot \text{rank}(L)} - \frac{1}{2} \right) \sum_{\gamma \in \pi_{M,p}} \beta_\gamma |C_\gamma(L)|.
\]

(4.5)

It is possible to find a similar formula on the weight of reflective modular forms for general lattices \(2U \oplus L(-1)\).

**Remark 4.1.** Let \(M = 2U \oplus L(-1)\) be an even lattice of prime level \(p\) and \(F\) be a reflective modular form of weight \(k\) for \(M\). From (4.3), \(k = \beta_0(12 + \frac{1}{2}|R(L)|)\) and \(F\) is not of singular weight when \(\text{rank}(L) = 12\) and \(p = 2\). When \(\text{rank}(L) = 8\) and \(p = 3, k = \beta_0(12 + |R(L)|)\) and \(F\) is not of singular weight.

By [25, Proposition 3.2], when reflective modular form \(F\) exists, we have that either \(\text{rank}(L) \leq 23\) or \(L\) is a unimodular lattice of rank \(24\). We next give a finer classification of reflective modular forms on lattices of prime level.

**Theorem 4.2.** Let \(M = 2U \oplus L(-1)\) be an even lattice of prime level \(p\). If \(M\) admits a reflective modular form of weight \(k\) for \(\mathcal{O}^+(M)\), then we have

1. when \(p = 2\), either \(\text{rank}(L) \leq 16\) or \(\text{rank}(L) = 20\) and \(k = 24\beta_0\).
2. when \(p = 3\), either \(\text{rank}(L) \leq 13\) or \(\text{rank}(L) = 18\) and \(k = 48\beta_0\).
3. when \(p \geq 5\), \(\text{rank}(L) \leq 8 + 24/(p + 1)\).

**Proof.** Similar to the proof of Theorem 3.6 there exists a weakly Jacobi form \(\phi\) of weight \(0\) and index \(L\) with the singular Fourier coefficients of the form (4.3). Assume that \(\text{rank}(L) \leq 23\). We write \(\phi = S_1 + d + S_2 + \cdots\), where \(S_1\) and \(S_2\) are the first and second terms in (4.3), respectively, and \(d = 2k\). We can construct \((n_0 = \text{rank}(L))\) and \(a = 24/p\)

\[
f_2 := \frac{24}{n_0 - 24} H_0(\phi) = S_1 + d_1 + c_1 S_2 + \cdots \in J_{2,L}^{w,h},
\]

\[
f_4 := \frac{24}{n_0 - 28} H_4(f_2) = S_1 + d_2 + c_2 S_2 + \cdots \in J_{4,L}^{w,h},
\]

\[
f_6 := \frac{24}{n_0 - 32} H_4(f_4) = S_1 + d_3 + c_3 S_2 + \cdots \in J_{6,L}^{w,h},
\]

where

\[
d_1 = \frac{n_0(d - 24\beta_0)}{n_0 - 24}, \quad c_1 = \frac{n_0 - a}{n_0 - 24},
\]

\[
d_2 = \frac{(n_0 - 4)(d_1 - 24\beta_0)}{n_0 - 28}, \quad c_2 = \frac{n_0 - a - 4}{n_0 - 28},
\]

\[
d_3 = \frac{(n_0 - 8)(d_2 - 24\beta_0)}{n_0 - 32}, \quad c_3 = \frac{n_0 - a - 8}{n_0 - 32}.
\]

It is obvious that

\[
\varphi_6 := (c_1 - c_3)E_6\phi + (c_3 - 1)E_4f_2 + (1 - c_1)f_6 = u + O(q) \in J_{6,L}
\]

is a holomorphic Jacobi form of weight 6, where

\[
u = (d - 504\beta_0)(c_1 - c_3) + (d_1 + 240\beta_0)(c_3 - 1) + d_3(1 - c_1).\]
We also construct
\[ g = E_4\phi - f_4 = (d + 240\beta_0) - d_2 + (1 - c_2)S_2 + \cdots \in J^{uc,h}_4. \]

By Theorem 3.4 we have \( S_2 \neq 0 \) when \( n_0 > 8 \). By direct calculations, we get
\begin{equation}
(4.6) \quad c_2 = 1 \iff \text{rank}(L) = 14 + \frac{12}{p}.
\end{equation}

Therefore, when \( p = 2 \), \( c_2 = 1 \) if and only if \( n_0 = 20 \); When \( p = 3 \), \( c_2 = 1 \) if and only if \( n_0 = 18 \); When \( p > 3 \), \( c_2 = 1 \) if and only if \( n_0 = 18 \); \( c_2 = 1 \) if \( n_0 = 18 \) and \( d = 48\beta_0 \);
\begin{itemize}
  \item when \( p = 2 \), if \( g = 0 \) then \( n_0 = 20 \) and \( d = 48\beta_0 \);
  \item when \( p = 3 \), if \( g = 0 \) then \( n_0 = 18 \) and \( d = 96\beta_0 \);
  \item when \( p \geq 5 \), \( g \neq 0 \).
\end{itemize}

Suppose \( g \neq 0 \). Then the weakly Jacobi form \( g \) corresponds to a nearly holomorphic vector valued modular form
\[ F = F_0e_0 + \sum_{\gamma \in A_M} F_\gamma e_\gamma \]
of weight \( 4 - n_0/2 \). It is clear that \( F_0 \neq 0 \) is a nearly holomorphic modular form of weight \( 4 - n_0/2 \) with respect to \( \Gamma_0(p) \). As in the proof of [31, Proposition 6.1], the Riemann-Roch theorem applied to \( F_0 \) gives
\[ -1 \leq \nu_0(F_0) + \nu_\infty(F_0) \leq \left( 4 - \frac{n_0}{2} \right) \frac{p + 1}{12}. \]

This implies
\[ n_0 \leq 8 + \frac{24}{p + 1}. \]

It remains to prove that \( M \) is not reflective if \( n_0 = 14 \) and \( p = 3 \). But in this case, \( u \neq 0 \) and then \( \varphi_6 \neq 0 \), which gives a contradiction. The proof is completed. \( \square \)

Note that when \( \text{rank}(L) = 16 \) and \( p = 2 \), we have \( u \equiv 0 \). Therefore, our argument cannot determine the weight of the corresponding reflective modular form.

**Remark 4.3.** The rank of an even positive-definite lattice of level 2 is known to be divisible by 4 (see [32]). If \( L \) is an even positive-definite lattice of level 3 and of rank \( n \) and with determinant \( \det(L) = |L^\vee/L| = 3^r \), then either \( r \in \{0, n\} \) and \( 8|n \), or \( 0 \leq r \leq n \) and \( 2r \equiv n \mod 4 \). Thus, by the above theorem, the rank of \( L \) can only take 4, 8, 12, 16, 20 when \( p = 2 \). The rank of \( L \) is even when \( p = 3 \). We note that there exist reflective lattices of such ranks. When \( L = D_4, 2D_4, E_8 \oplus D_4, E_8 \oplus 2D_4 \) or \( 2E_8 \oplus D_4, M = 2U \oplus L(-1) \) is of level 2 and admits a reflective modular form. When \( L = A_2, 2A_2, 3A_2, 4A_2, E_8 \oplus A_2, E_8 \oplus 2A_2 \) or \( 2E_8 \oplus A_2, M = 2U \oplus L(-1) \) is of level 3 and admits a reflective modular form.

By the above theorem and the weight formula (4.5), it is easy to prove the following criterions.

**Corollary 4.4.** Suppose that the even lattice \( M = 2U \oplus L(-1) \) of prime level \( p \) is reflective.
\begin{itemize}
  \item[(1)] When \( \text{rank}(L) = 20 \) and \( p = 2 \), we have \( |R(L)| \geq 120 \).
  \item[(2)] When \( \text{rank}(L) = 18 \) and \( p = 3 \), we have \( |R(L)| \geq 216 \).
\end{itemize}
The above result can be used to judge whether a given lattice is reflective or not. For instance, we see at once that $2U \oplus L(-1)$ is not reflective when $L = E_6(2) \oplus 3D_4$ or $2E_6 \oplus 3A_2$.

We next extend the above classification results to the general case. The following lemma introduced in [26, Corollary 3.2] is very useful for our purpose.

**Lemma 4.5.** Let $M$ be a lattice of signature $(2, n)$ with $n \geq 11$. There exists a lattice $M_1$ on $M \otimes \mathbb{Q}$ such that $O^+ (M) \subset O^+ (M_1)$ and that $M_1$ is a scaling of an even lattice $M_2$ containing $2U$.

**Theorem 4.6.** There is no reflective lattice of signature $(2, n)$ and of level 2 when $n > 22$ except $II_{2,26}(2)$; There is no reflective lattice of signature $(2, n)$ and of level 3 when $n \geq 16$ and $n \neq 20$ except $II_{2,18}(3)$ and $II_{2,26}(3)$; There is no reflective lattice of signature $(2, n)$ and of prime level $p \geq 5$ when $n > 10 + 24/(p + 1)$ except $II_{2,18}(p)$ and $II_{2,26}(p)$.

**Proof.** Conversely, suppose that $M$ is a reflective lattice of signature $(2, n)$ and of prime level $p$. Let $F$ be a reflective modular form for $M$. In our case, we have $n \geq 11$. By Lemma 4.5, there exists an even overlattice $M_1$ of $M$ such that $O^+ (M) \subset O^+ (M_1)$ and that $M_1$ is a scaling of an even lattice $M_2$ containing $2U$. Then either $M_2 = M_1$ or $M_1 = M_2(p)$ and $M_2$ is unimodular.

In the first case, if $M_1$ is not unimodular, then $M_1$ is of level $p$. Since $O^+ (M) \subset O^+ (M_1)$, $F$ is a reflective modular form for $M$, contrary to Theorem 4.2. Next assume that $M_1$ is unimodular. By Theorem 3.4 $F$ is not a 2-reflective modular form for $M$. Therefore, there exists a primitive vector $l \in M$ satisfying $(l, l) = -2p$ and $\text{div}(l) = p$. Then the reflection $\sigma_l$ belongs to $O^+ (M)$ and it also belongs to $O^+ (M_1)$. Since $M_1$ is unimodular and $l$ is primitive in $M_1$, there exists a vector $l_1 \in M_1$ such that $(l, l_1) = 1$. Then $\sigma_l(l_1) = l_1 + \frac{l}{p} \not\in M_1$, a contradiction.

In the second case, there exists an even lattice $M_0 < M_2$ such that $M = M_0(p)$. As $M$ is of level $p$, it follows that $M_0$ is unimodular. Thus, the theorem is proved.

**Remark 4.7.** [31, Theorem 6.5] gives bounds on the signature for reflective modular forms which are not invariant under $O(A_M)$. But the bounds do not hold in the symmetric case. However, the above result gives bounds in the symmetric case.

**Theorem 4.8.** There is no reflective lattice of signature $(2, n)$ when $n \geq 23$ except the scaling of even unimodular lattice of signature $(2, 26) : II_{2,26}(m)$, $m \in \mathbb{N}$.

**Proof.** Let $M$ be an even lattice of signature $(2, n)$ with $n \geq 23$. Firstly, we assume that $M$ contains $2U$. On the contrary, suppose that $M$ is reflective. Similarly, there exists a weakly Jacobi form $\phi$ of weight 0 and we can construct a weakly Jacobi form $f_4$ of weight 4 from $\phi$. We define $g = E_4 \phi - f_4$. From (4.16), we claim $g \neq 0$. Then $\eta^{12}g$ is a holomorphic Jacobi form of weight 10 with character, which leads to a contradiction.

Now consider the general case that $M$ does not necessarily contain $2U$. We can assume that $M$ is not a scaling of any even lattice and $F$ is a reflective modular form for $M$. By Lemma 4.5, there exists an even overlattice $M_1$ of $M$ such that $M_1$ contains $2U$ and $O^+(M) \subset O^+(M_1)$. Then $F$ is a reflective modular form for $M_1$. From the first step, it follows $M_1 = II_{2,26}$. If $F$ is a Borcherds product of a nearly holomorphic modular form $f$, then $\Delta f$ is a holomorphic modular form of weight 0 and hence must be an $M_{p_2}(\mathbb{Z})$-invariant vector in $\mathbb{C}[A_M]$. Then we get $M = II_{2,26}$.
because $\Delta f$ does not transform correctly under $S$ when $|A_M| \neq 1$. This completes the proof.

Remark 4.9. The above proof does not cover the case: $M$ is a finite-index sublattice of $I_{2,26}$. $O^+(M) \subset O^+(I_{2,26})$, all reflective modular forms for $M$ are not Borcherds products. We believe that there is no such $M$. By [5], $M$ is not of type $U \oplus U(N) \oplus L(-1)$. Therefore, the above case is insignificant for application. In fact, Scheithauer defined reflective modular forms as Borcherds products with reflective divisors in his series of articles.

Remark 4.10. When $1 \leq \text{rank}(L) \leq 20$ and $\text{rank}(L) \neq 15$ or 19, there exist reflective lattices $2U \oplus L(-1)$, such as $A_n$ for $1 \leq n \leq 7$, $D_4$, $E_8 \oplus A_1$, $E_8 \oplus A_2$, $E_8 \oplus A_2 \oplus A_1$, $E_8 \oplus D_4$, $E_8 \oplus D_4 \oplus A_1$, $E_8 \oplus D_4 \oplus A_2$, $E_8 \oplus 2D_4$, $2E_8 \oplus A_1$, $2E_8 \oplus A_2$, $2E_8 \oplus D_4$. But we do not know if there exists reflective lattice $2U \oplus L(-1)$ with $\text{rank}(L) = 15$ or 19.

Questions 4.11. We formulate many unsolved questions related to this paper.

1. Are there 2-reflective lattices of signature $(2,13)$ or $(2,14)$?
2. Are there reflective lattices of signature $(2,17)$ or $(2,21)$?
3. Let $M = 2U \oplus L(-1)$ be a 2-reflective lattice of signature $(2,19)$. Is $L$ equal to $2E_8 \oplus A_1$ up to isomorphism? Classify 2-reflective lattices of signature $(2,19)$, reflective lattices of signature $(2,22)$ and of level 2, reflective lattices of signature $(2,20)$ and of level 3.

5. A further application: dd-modular forms

Our arguments in the previous section are also applicable to some other questions. In this section we use the similar arguments to classify the modular forms with the simplest reflective divisors, i.e. the dd-modular forms defined in [7].

Let $nA_1$ denote the lattice of $n$ copies of $A_1 = (2)$, $n \in \mathbb{N}$. Let $\{e_1, ..., e_n\}$ denotes the standard basis of $\mathbb{R}^n$ with standard scalar product $(\cdot, \cdot)$. We choose the following model for the lattice $nA_1(m)$:

\[(5.1) \quad ((e_1, ..., e_n)z, 2m(\cdot, \cdot))\]

and set $z_n = \sum_{i=1}^{n} z_i e_i \in nA_1 \oplus \mathbb{C}$, $\zeta_i = e^{2\pi i z_i}$, for $1 \leq i \leq n$. We define

\[(5.2) \quad \Gamma_{n,m} := O^+(2U \oplus nA_1(-m))\]

and the definition of dd-modular forms is as follows.

Definition 5.1. A holomorphic modular form with respect to $\Gamma_{n,m}$ is called a dd-modular form if it vanishes exactly along the $\Gamma_{n,m}$-orbit of the diagonal $\{z_n = 0\}$. The $\Gamma_{n,m}$-orbit of the diagonal $\{z_n = 0\}$, denoted by $\Gamma_{n,m}\{z_n = 0\}$, is called the diagonal divisor.

It is well known that the Igusa form $\Delta_5$ which is the product of the ten even theta constants vanishes precisely along the diagonal divisor $\{z = 0\}$. Therefore, the dd-modular form is a natural generalization of $\Delta_5$. Gritsenko and Hulek [13] proved that the dd-modular form exists for the lattice $A_1(m)$ if and only if $1 \leq m \leq 4$. Cléry and Gritsenko [7] developed the arguments in [15] and gave the full classification of the dd-modular forms with respect to the Hecke subgroups of the Siegel paramodular groups. But their approach is hard to generalize to higher dimensions. dd-modular forms are crucial in determining the structure of
the fixed space of modular forms and have applications in physics. As an important application of our arguments, we prove the following classification results for all dd-modular forms.

**Theorem 5.2.** The dd-modular form exists if and only if the pair \((n, m)\) takes one of the eight values

\[(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (3, 1), (4, 1).\]

**Proof.** Suppose that \(F_c\) is a modular form of weight \(k\) with respect to \(\Gamma_{n, m}\) with the divisor \(c \cdot \Gamma_{n, m}\{z_n = 0\}\), where \(c\) is the multiplicity of the diagonal divisor and it is a positive integer. The diagonal divisor \(\Gamma_{n, m}\{z_n = 0\}\) is the union of the primitive Heegner divisors \(\mathcal{P}(\pm e_i/(2m), -1/(4m)), 1 \leq i \leq n\), where the primitive Heegner divisor of discriminant \((\mu, d)\) is defined as

\[
\mathcal{P}(\mu, d) = \bigcup_{M + \mu \not\equiv 0 \text{ primitive}} l^+ \cap D(M).
\]

It is clear that we have

\[
\mathcal{P}(\mu, y) = \mathcal{H}(\mu, y) - \sum_{d > y} x_d \mathcal{H}(\lambda_d, d),
\]

where \(x_d\) are integers and \(\lambda_d \in M^\vee\) (we refer to [3] Lemma 4.2] for an explicit formula). For arbitrary Heegner divisor \(\mathcal{H}(\lambda, d)\) with \(\lambda = (0, n_1, \lambda_0, n_2, 0) \in A_M\), the principal part of the corresponding nearly holomorphic modular form of weight \(-\text{rank}(L)/2\) with respect to the Weil representation \(\rho_M\) of \(\text{Mp}_2(\mathbb{Z})\) is \(q^d \mathbf{e}_\lambda\). Hence the singular Fourier coefficients of the corresponding weakly holomorphic Jacobi form of weight 0 are represented as

\[
\sum_{r \in L + \lambda_0} q^{(r, r)/2 + d} e^{2\pi i (r, z)}.
\]

Since \(\{\pm e_i/(2m) : 1 \leq i \leq n\}\) is the set of vectors in \(nA_1(m)^\vee\) with the minimum norm \(1/(2m)\) in \(nA_1(m)^\vee/nA_1(m)\), through the previous explanations, there exists a weak Jacobi form \(f_{nA_1, m}\) of weight 0 and index \(nA_1(m)\) satisfying

\[
f_{nA_1, m} = c \cdot \sum_{1 \leq i \leq n} \zeta^{\pm 1} + 2k + O(q)
\]

such that \(F_c\) is the Borcherds product of \(f_{nA_1, m}\). By Lemma [23], we get

\[
m(2nc + 2k) = 12c,
\]

then \(nm \leq 5\). It is not hard to show that a weak Jacobi form for \(nA_1(m)\) has integral Fourier coefficients if its \(q^0\)-term is integral when \(nm \leq 5\).

When \(m \leq 4, 2k\) is integral if \(c = 1\). Hence the existence of \(F_c\) is equivalent to the existence of \(F_1\). In view of \(k \geq n/2\), then the triplet \((m, n, k)\) can only take one of the eight values

\[(1, 1, 5), (1, 2, 4), (1, 3, 3), (1, 4, 2), (2, 1, 2), (2, 2, 1), (3, 1, 1), (4, 1, \frac{1}{2}).\]

When \(m = 5\), we only need to consider the case of \(c = 5\), and we obtain the unique solution \((5, 1, 1)\). But the unique weak Jacobi form of weight 0 and index 5 for \(A_1\) is \(\psi^{(1)}_{0, 5} = 5c^{\pm 1} + 2 + q(-\zeta^5 + \cdots)\) (see [12]). The corresponding Borcherds product is not holomorphic, that is, \(F_c\) does not exist in the case. We have thus proved the theorem. \(\square\)
Remark 5.3. Similarly, we can define dd-modular forms with respect to the lattices $A_n(m)$ or $D_n(m)$. Using the same methodology, we can easily classify these dd-modular forms. In fact, dd-modular forms with respect to the lattices $L(m)$, where $L = A_n, n \geq 2$ or $L = D_n, n \geq 4$, exist if and only if the pair $(L, m)$ takes one of the following fifteen values

\[
(A_2, 1) \quad (A_3, 1) \quad (A_4, 1) \quad (A_5, 1) \quad (A_6, 1) \quad (A_7, 1) \quad (A_2, 2) \quad (A_2, 3) \\
(A_3, 2) \quad (D_4, 1) \quad (D_5, 1) \quad (D_6, 1) \quad (D_7, 1) \quad (D_8, 1) \quad (D_4, 2).
\]

Note that all dd-modular forms in Theorem 5.2 and in the above list do exist and can be found in [7, 13, 20, 23].

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