The Parity Hamiltonian Cycle Problem

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Abstract

Motivated by a relaxed notion of the celebrated Hamiltonian cycle, this paper investigates its variant, parity Hamiltonian cycle (PHC): A PHC of a graph is a closed walk which visits every vertex an odd number of times, where we remark that the walk may use an edge more than once. First, we give a complete characterization of the graphs which have PHCs, and give a linear time algorithm to find a PHC, in which every edge appears at most four times, in fact. In contrast, we show that finding a PHC is $NP$-hard if a closed walk is allowed to use each edge at most $z$ times for each $z = 1, 2, 3$ (PHC$_z$ for short), even when a given graph is two-edge connected. We then further investigate the PHC$_3$ problem, and show that the problem is in $P$ when an input graph is four-edge connected. Finally, we are concerned with three (or two)-edge connected graphs, and show that the PHC$_3$ is in $P$ for any $C_{\geq 5}$-free or $P_6$-free graphs. Note that the Hamiltonian cycle problem is known to be $NP$-hard for those graph classes.

Keywords: Hamiltonian cycle problem, $T$-join, graph factor

1 Introduction

It is said that the graph theory has its origin in the seven bridges of Königsberg settled by Leonhard Euler [2]. An Eulerian cycle, named after him in modern terminology, is a cycle which uses every edge exactly once, and now it is well-known that a connected undirected graph has an Eulerian cycle if and only if every vertex has an even degree. A Hamiltonian cycle (HC), a similar but completely different notion, is a cycle which visits every vertex exactly once. In contrast to the clear characterization of an Eulerian graph, the question if a given graph has a Hamiltonian cycle is a celebrated NP-complete problem due to Karp [16]. The HC problem is widely interested in computer science or mathematics, and has been approached with several variant or related problems. The traveling salesman problem (TSP) in a graph is a problem to minimize the length of a walk which visits every vertex at least once, instead of exactly once. Another example may be a two-factor (in cubic graphs), which is a collection of cycles covering every vertex exactly once, meaning that the connectivity of the HC is relaxed (cf. [13, 14, 3, 4]).

It could be a natural idea for the HC problem to modify the condition on the visiting number keeping the connectivity condition. This paper proposes the parity Hamiltonian cycle (PHC),
which is a variant of the Hamiltonian cycle: a PHC is a closed walk which visits every vertex an odd number of times (see Section 2 for more rigorous description). Note that a closed walk is allowed to use an edge more than once. The PHC problem is to decide if a given graph has a PHC. We remark that another version of the problem which is to find a closed walk visiting each vertex an even number of times is trivial: find a spanning tree and trace it twice.

It may not be trivial if the PHC problem is in \( \mathcal{NP} \), since the length of a PHC is unbounded in the problem. This paper firstly shows in Section 3.1 that the PHC problem is in \( \mathcal{P} \), in fact. Precisely, we give a complete characterization of the graphs which have PHCs. Furthermore, we show that if a graph has a PHC then we can find a PHC in linear time, where PHC for a positive integer \( z \) denotes a PHC which uses each edge at most \( z \) times. In contrast, Section 3.2 shows that the PHC problem is \( \mathcal{NP} \)-complete for each \( z = 1, 2, 3 \) (see Table 1). We then further investigate the PHC\(_3\) problem. In precise, the PHC3 problem is \( \mathcal{NP} \)-complete for two-edge connected graphs, while Section 3.3 shows that it is solved in polynomial time for four-edge connected graphs. The complexity of the PHC\(_3\) for three-edge connected graphs remains unsettled (see Table 2).

As an approach to the PHC\(_3\) problem for three-edge connected graphs, we in Section 4 utilize the celebrated ear-decomposition, which is actually a well-known characterization of two-edge connected graphs. Then, Section 4 shows that the PHC3 problem is in \( \mathcal{P} \) for any two-edge connected \( C_{\geq 5} \)-free or \( P_6 \)-free graphs (see Section 2.2 for the graph classes). The classes of \( C_{\geq 5} \)-free or \( P_6 \)-free contain some important graph classes such as chordal, chordal bipartite and cograph. We remark that it is known that the Hamiltonian cycle is \( \mathcal{NP} \)-complete for \( C_{\geq 4} \)-free graphs, as well as \( P_5 \)-free graphs (cf. [5]).

In precise, we in Section 4 introduce a stronger notion of all-roundness (and bipartite all-roundness) of a graph, which is a sufficient condition that a graph has a PHC. Catlin [7] presented a similar notion of collapsible in the context of spanning Eulerian subgraphs, and the all-roundness is an extended notion of the collapsible. Then, we show that any two-edge connected \( C_{\geq 5} \)-free or \( P_6 \)-free graphs are all-round or bipartite all-round. We conjecture that any two-edge connected

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1 Notice that those hardness results are independent, e.g., “\( z = 3 \) is hard” does not imply “\( z = 2 \) is hard,” and vice versa.
$C_{\geq 7}$-free graphs are all-round, while it seems not true for $C_{\geq 8}$-free nor $P_7$-free.

Section 5 is for a miscellaneous discussion. Section 5.1 extends the results for two-edge connected graphs in Section 4 to (one-edge) connected graphs. In Section 5.2, we remark that a dense graph is also all-round using some techniques in Section 4. Before closing the paper, Section 5.3 briefly discusses the connection between the PHC and other problems, such as Hamiltonian cycle or Eulerian cycle, regarding a generalized problem described in Section 4.

Related works Here, we refer to the work by Brigham et al. [6], which investigated a similar (or, essentially the same) problem. Brigham et al. [6] showed that any connected undirected graph has a parity Hamiltonian path or cycle (see Section 3.1.2). Their proof was constructive, and they gave a linear time algorithm based on the depth first search. As far as we know, it is the uniquely known result on the problem. To be precise, we remark that their argument does not imply that the PHC problem is in $P$: for the purpose, we need an argument when a graph does not contain a parity Hamiltonian cycle.

Notice that the condition that a HC visits each vertex $1 \in \mathbb{R}$ times is replaced by $1 \in \mathbb{GF}(2)$ times in the PHC. Modification of the field is found in some graph problems, such as group-labeled graphs or nowhere-zero flows [15, 17]. It was shown that the extension complexity of the TSP is exponential [24, 9, 10], while it is an interesting question if the PHC has an efficient (extended) formulation over $\mathbb{GF}(2)$.

2 Preliminary

This section introduces graph terminology of the paper. First, we define the parity Hamiltonian cycle in Section 2.1 with an almost minimal introduction of a graph terminology. Then, we in Section 2.2 give some other terminology, such as $T$-join, ear decomposition, graph classes, which are commonly used in the graph theory.

Very recently, we have investigated a PHC on directed graphs in [21] after this paper.
2.1 Parity Hamiltonian cycle

2.1.1 Definition

An undirected simple graph (simply we say “graph”) \( G = (V, E) \) is given by a vertex set \( V \) (or we use \( V(G) \)) and an edge set \( E \subseteq \binom{V}{2} \) (or \( E(G) \)). Let \( \delta_G(v) \) denote the set of incident edges to \( v \), and let \( d_G(v) \) denote the degree of a vertex \( v \) in \( G \), i.e., \( d_G(v) = |\delta_G(v)| \). We simply use \( \delta(v) \) and \( d(v) \) without a confusion.

A walk is an alternating sequence of vertices and edges \( v_0e_1v_1\cdots v_{\ell-1}e_{\ell}v_{\ell} \) with an appropriate \( \ell \in \mathbb{Z}_{\geq 0} \), such that \( e_i = \{v_{i-1}, v_i\} \in E \) for each \( i \) (1 \( \leq i \leq \ell \)). Note that each vertex or edge may appear more than once in a walk. A walk is closed if \( v_\ell = v_0 \). A graph \( G \) is connected if there exists a walk from \( u \) to \( v \) for any pair of vertices \( u, v \in V \).

For a closed walk \( w = v_0e_1\cdots e_{\ell}v_{\ell} \), the visit number of \( v \in V \), denoted by \( \text{visit}(v) \), is the number that \( v \) appears in the walk except for \( v_0 \) (since \( v_0 = v_\ell \)). A parity Hamiltonian cycle (PHC for short) of a graph \( G \) is a closed walk in which \( \text{visit}(v) \equiv 1 \pmod{2} \) holds for each \( v \in V \). Remark again that an edge may appear more than once in a PHC \( w \). Clearly, a graph must be connected to have a PHC, and this paper is basically concerned with connected graphs.

An edge count vector \( x \in \mathbb{Z}_{\geq 0}^E \) of a closed walk \( w \) is an integer vector where \( x_e \) for \( e \in E \) counts the number of occurrence of \( e \) in \( w \). Remark that

\[
\text{visit}(v) = \frac{1}{2} \sum_{e \in \delta(v)} x_e
\]

holds for any closed walk. Thus, we see that a PHC is a closed walk whose edge count vector \( x \in \mathbb{Z}_E \) satisfies the parity condition

\[
\sum_{e \in \delta(v)} x_e \equiv 2 \pmod{4}
\]

for each \( v \in V \).

2.1.2 PHC as an Eulerian cycle of a multigraph

As given an arbitrary integer vector \( x \in \mathbb{Z}_E \), the parity condition (2) is a necessary condition that \( x \) is an edge count vector of a PHC. In fact, the following easy but important observation provides an if-and-only-if condition.

**Proposition 2.1.** Let \( G = (V, E) \) be an undirected simple graph and let \( x \in \mathbb{Z}_E \) be an arbitrary integer vector. Let \( F = \{e \in E \mid x_e > 0\} \), then, \( x \) is an edge count vector of a PHC in \( G \) if and only if \( x \) satisfies (2) and the subgraph \( H = (V, F) \) of \( G \) is connected.

As a preliminary of the proof of Proposition 2.1, we introduce an Eulerian cycle of a multigraph. For a simple graph \( G = (V, E) \) and any nonnegative integer vector \( x \in \mathbb{Z}_E \), let \( E_x \) be a multiset such that \( e \in E \) appears \( x_e \) times in \( E_x \). Then, let \( (G, x) \) represent a multigraph with a vertex set \( V \) and a multiedge set \( E_x \). Note that \( (G, 1) = G \) where \( 1 \in \mathbb{Z}_E \) denotes the all one vector. We say \( (G, x) \) is connected if a simple graph \( H = (V, F) \) is connected where \( F = \{e \in E \mid x_e > 0\} \). An Eulerian cycle of \( (G, x) \) is a closed walk which uses each element of the multiset \( E_x \) exactly once. It is celebrated fact due to Euler [3] that \( (G, x) \) has an Eulerian cycle if and only if \( (G, x) \) is connected and \( x \) satisfies the Eulerian condition

\[
\sum_{e \in \delta(v)} x_e \equiv 0 \pmod{2}
\]
holds for any $v \in V$.

**Proof of Proposition 2.1.** The ‘only if’ part is easy from the definition. We prove the ‘if’ part. Note that $x$ satisfies (3) since $x$ satisfies (2). Since $H$ is connected by the hypothesis, the multigraph $(G, x)$ has an Eulerian cycle $w$. Considering (1), it is not hard to see that $w$ is a PHC since $x$ satisfies (2).

For convenience, we say $x \in \mathbb{Z}^E_{\geq 0}$ admits a PHC in $G$ if $x$ is an edge count vector of a PHC in $G$. In summary, Proposition 2.1 implies the following.

**Corollary 2.2.** Let $G = (V, E)$ be an undirected simple graph and let $x \in \mathbb{Z}^E_{\geq 0}$ be an arbitrary integer vector. Then, $x$ admits a PHC in $G$ if and only if $(G, x)$ is connected and $x$ satisfies (2).

### 2.1.3 PHC with an edge constraint

As we repeatedly remarked, a PHC may use an edge more than once. For convenience, let PHC$_z$ for $z \in \mathbb{Z}_{>0}$ denote a PHC using each edge at most $z$ times.

### 2.2 Other graph terminology

This subsection introduces some other graph terminology which we will use in this paper.

#### 2.2.1 Fundamental notations

A simple path is a walk $w = v_0e_1v_1e_2 \cdots e_{\ell}v_{\ell}$ which visits every vertex (and hence every edge) at most once, where $\ell \geq 0$ is the length of the path $w$. Similarly, a simple cycle is a closed walk $w = v_0e_1v_1e_2 \cdots e_{\ell}v_0$ which visits every vertex at most once, where $\ell \geq 0$ is the length of the cycle $w$. An odd cycle is a simple cycle of odd length.

Let $G = (V, E)$ be a graph. For an edge subset $F \subseteq E$, let $G - F$ denote a graph $H = (V, E \setminus F)$. For a vertex subset $S \subseteq V$, let $G - S$ denote the subgraph induced by $V \setminus S$, i.e., $G - S$ is given by deleting from $G$ all vertices of $S$ and all edges incident to $S$. For convenience, we simply use $G - e$ for $e \in E$ instead of $G - \{e\}$, and $G - v$ for $v \in V$ as well. For a pair of graphs $G$ and $H$, let $G + H = (V(G) \cup V(H), E(G) \cup E(H))$.

#### 2.2.2 T-join

Let $G = (V, E)$ be a graph and let $T$ be a subset of $V$ such that $|T|$ is even. Then, $J \subseteq E$ is a $T$-join if the graph $H = (V, J)$ satisfies

$$d_H(v) \equiv \begin{cases} 1 \pmod{2} & \text{if } v \in T, \\ 0 \pmod{2} & \text{otherwise,} \end{cases}$$

for any $v \in V$. Notice that a graph $H' = (T, J)$ may not be connected, in general. A $T$-join is a generalized notion of a matching, meaning that $J$ is a matching when all edges in $J$ are disjoint.

**Theorem 2.3** (cf. [18]). For any connected simple graph $G = (V, E)$ and for any $T \subseteq V$ satisfying that $|T|$ is even, $G$ contains a T-join.

A T-join is found in $O(|V| + |E|)$ time (see Section A).
2.2.3 Edge connectivity

A graph is \( k \)-edge connected for a positive integer \( k \) if the graph remains connected after removing arbitrary \( k - 1 \) edges. A \( k \)-edge connected component \( H \) of \( G \) is a maximal induced subgraph of \( G \) such that \( H \) is \( k \)-edge connected.

The ear decomposition is a celebrated characterization of two-edge connected graphs. An ear \( w = v_0e_1v_1 \cdots e_{\ell}v_{\ell} \) of a graph \( G \) is a simple path (\( v_0 = v_{\ell} \) may hold) of length at least one where \( v_0 \) and \( v_{\ell} \) are in the same two-edge connected component of \( G - w \) and \( d(v_i) = 2 \) for each \( i = 1, \ldots, \ell - 1 \). A cycle graph, which consists of a simple cycle only, is two-edge connected. It is not difficult to see that any two-edge connected graph, except for a cycle graph, has an ear. By the definition, a graph deleting an ear \( w \) except for \( v_0 \) and \( v_{\ell} \) from a two-edge connected graph \( G \) is again two-edge connected unless \( G \) is a cycle graph. Recursively deleting ears from a two-edge connected graph \( G \), we eventually obtain a cycle graph. The sequence of ears in the operation is called ear decomposition of \( G \). The following fact is well-known.

**Theorem 2.4** (cf. [23]). A graph \( G \) is two-edge connected if and only if \( G \) has an ear decomposition.

2.2.4 Graph classes

Let \( P_n \) \((n \geq 2)\) denote a graph consisting of a simple path with \( n \) vertices. Notice that the length of the path \( P_n \) is \( n - 1 \). Let \( C_n \) \((n \geq 3)\) denote a cycle graph with \( n \) vertices. A graph is \( P_k \)-free (resp. \( C_k \)-free) if it does not contain \( P_k \) (resp. \( C_k \)) as an induced subgraph. A graph is \( C_{\geq k} \)-free if \( G \) is \( C_{k'} \)-free for all \( k' \geq k \). Clearly, any \( P_k \)-free graph is also \( P_{k+1} \)-free, as well as any \( C_{\geq k} \)-free graph is also \( C_{\geq k+1} \)-free. We can also observe that any \( P_k \)-free graph is \( C_{\geq k+1} \)-free. However, any \( C_{\geq k} \)-free is not included in \( P_l \)-free for any \( l \), since a tree, clearly \( C_{\geq 3} \)-free, admits a path of any length.

Many important graph classes are known to be characterized as \( P_k \)-free or \( C_{\geq k} \)-free. For instance, cographs is equivalent to \( P_4 \)-free, chordal is equivalent to \( C_{\geq 4} \)-free, and chordal bipartite is \( C_{\geq 6} \)-free bipartite (cf. [5]).

3 Computational Complexity of The PHC Problems

It may not be trivial if the PHC problem is in \( \text{NP} \), since the length of a closed walk is unbounded in the problem. Section 3.1 completely characterizes the graphs which have PHCs, and shows that the PHC problem is in \( \text{P} \). Furthermore, if a graph has a PHC then we can find a PHC in linear time. In contrast, Section 3.2 shows that the PHC problems is \( \text{NP} \)-hard for each \( z = 1, 2, 3 \). Section 3.3 further investigates the PHC problem, and shows that the problem is in \( \text{P} \) for four-edge connected graphs.

3.1 The characterization of the graphs which have PHCs

To begin with, we give an if-and-only-if characterization of graphs which have PHCs.

**Theorem 3.1.** A connected graph \( G = (V, E) \) contains a PHC if and only if the order \( |V| \) is even or \( G \) is non-bipartite.

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\(^4\) \( C_{\geq k} \)-free is often denoted by \( C_{n+k} \)-free [5].

\(^5\) Here, we omit the definitions of cograph, chordal and chordal bipartite. This paper requires the properties of \( P_k \)-free or \( C_{\geq k} \)-free, only.
Proof. We show the ‘if’ part in a constructive way. First, we are concerned with the case that $|V|$ is even. Let $J \subseteq E$ be a $V$-join of $G$, which always exists by Theorem 2.3. Let $x \in \mathbb{Z}_{\geq 0}^E$ be given by
\[
x_e = \begin{cases} 
2 & \text{if } e \in J, \\
4 & \text{otherwise}.
\end{cases}
\] (5)
Then, $x$ satisfies the parity condition (2), that is $\sum_{e \in \delta(v)} x_e \equiv 2 \pmod{4}$, for each vertex $v \in V$ since $J$ is a $V$-join of $G$. Clearly, $(G, x)$ is connected since $G$ is connected and $x_e \geq 1$ for any $e \in E$. By Corollary 2.2, $x$ admits a PHC in $G$.

Next, we are concerned with the case that $|V|$ is odd and $G$ is non-bipartite. As a preliminary step, let $C$ be an arbitrary odd cycle of $G$. Let $T = V \setminus V(C)$, where notice that $|T|$ is even. Using Theorem 2.3 let $J \subseteq E$ be a $T$-join. Let $x' \in \mathbb{Z}_{\geq 0}^E$ be given by
\[
x'_e = \begin{cases} 
2 & \text{if } e \in J, \\
4 & \text{otherwise}.
\end{cases}
\] (6)
Then, $x'$ satisfies the parity condition exactly for each $v \in T$, i.e.,
\[
\sum_{e \in \delta(v)} x'_e \equiv \begin{cases} 
2 & \text{if } v \in T, \\
0 & \text{otherwise,}
\end{cases} \pmod{4}
\]
hold since $J$ is a $T$-join of $G$. Recalling that $V \setminus T = V(C)$, we modify $x'$ to $x''$ by adding $E(C)$, i.e.,
\[
x''_e = \begin{cases} 
x'_e + 1 & \text{if } e \in E(C), \\
x_e & \text{otherwise}.
\end{cases}
\] (7)
This modification increases the visit number of each vertex of $V \setminus T$ exactly by one, and hence $x''$ satisfies the parity condition (2) for each $v \in V$. Clearly, the multigraph $(G, x'')$ is connected, and $x''$ admits a PHC by Corollary 2.2.

Finally, we show the ‘only if’ part. Suppose that $G = (U, V; E)$ is a bipartite graph with an odd order. Without loss of generality, we may assume that $|U|$ is odd, and hence $|V|$ is even. Assume for a contradiction that $G$ has a PHC. Since visit($v$) of a PHC is odd for each $v \in U \cup V$, $\sum_{v \in U} \text{visit}(v)$ is odd and $\sum_{v \in V} \text{visit}(v)$ is even, respectively. On the other hand, any closed walk of $G$ satisfies that $\sum_{u \in U} \text{visit}(u) = \sum_{v \in V} \text{visit}(v)$ since $G$ is bipartite. Contradiction.
3.1.1 The PHC4 problem

The proof of Theorem 3.1 implies a PHC using every edge at most five times: if $G$ is even order then $x$ given by (5) uses every edge at most four times, while if $G$ is odd order and nonbipartite then $x''$ given by (6) and (7) uses every edge at most five times. An enhancement of Theorem 3.1 is given as follows.

**Theorem 3.2.** A connected graph $G = (V, E)$ contains a PHC if and only if the order $|V|$ is even or $G$ is non-bipartite.

**Proof.** Notice that (6) and (7) imply that $x''_e \in \{2, 3, 4, 5\}$. Then, we modify $x''$ to $x''' \in \{1, 2, 3, 4\}$ by setting

$$x'''_e = 1$$

for each $e \in E$ satisfying $x''_e = 5$. This modification preserves the parity condition (2) for each $v \in V$. Clearly, $x_e \geq 1$ for each $e \in E$, meaning that $(G, x)$ is connected. By Corollary 2.2 we obtain the claim.

We remark that the construction in the proofs of Theorems 3.1 and 3.2 implies the following fact, since a $T$-join is found in linear time by Theorem 2.3.

**Corollary 3.3.** The PHC$_z$ problem for any $z \geq 4$ is solved in linear time.

3.1.2 A related topic: a maximization version of the PHC problem

For a graph which does not have a PHC, a reader may be interested in a maximization version of the problem. The following theorem answers the question.

**Theorem 3.4 (cf. [6]).** Every connected graph has a closed walk visiting all vertices such that the walk visits at least $|V| - 1$ vertices an odd number of times and uses each edge at most four times.

Theorem 3.4 is suggested by Brigham et al. [6], in which they gave a linear time algorithm based on the depth first search. We here show a slightly generalized claim using a $T$-join, in an approach different from [6]. Given a graph $G = (V, E)$ and $S \subseteq V$, an $S$-odd walk is a closed walk of $G$ which visits every vertex of $S$ an odd number of times and visits every other vertex an even number of times. Clearly, a $V$-odd walk is a PHC of $G$.

**Theorem 3.5.** For any graph $G$ and any $S \subseteq V(G)$, $G$ contains an $S$-odd walk if and only if $|S|$ is even or $G$ is non-bipartite. Furthermore, we can find an $S$-odd walk which uses each edge at most four times.

**Proof of Theorem 3.5.** The ‘only-if’ part is (essentially) the same as Theorem 3.1. The ‘if’ part is also similar to Theorems 3.1 and 3.2 as follows. When $|S|$ is even, let $J$ be a $S$-join of $G$. Let $x \in \mathbb{Z}_{\geq 0}^E$ be given by

$$x_e = \begin{cases} 2 & \text{if } e \in J \\ 4 & \text{otherwise,} \end{cases}$$

then $(G, x)$ has an Eulerian cycle, which in fact an $S$-odd walk since $J$ is a $S$-join of $G$. 

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When \(|S|\) is odd and \(G\) is non-bipartite, let \(C\) be an odd cycle of \(G\). Let
\[ T = \{v \in V(G) \setminus V(C) \mid v \in S\} \cup \{v \in V(C) \mid v \notin S\}, \]
and let \(J'\) be a \(T\)-join of \(G\). Let \(x' \in \mathbb{Z}_{\geq 0}^E\) be given by
\[
x'_e = \begin{cases} 
2 & \text{if } e \in J \text{ and } e \notin E(C), \\
3 & \text{if } e \in J \text{ and } e \in E(C), \\
4 & \text{if } e \notin J \text{ and } e \notin E(C), \\
1 & \text{if } e \notin J \text{ and } e \in E(C),
\end{cases}
\]
then \(\text{visit}(v)\) is odd for each vertex in \(S\) since \(J'\) is a \(T\)-join, while \(\text{visit}(v)\) is even for others. \(\square\)

### 3.2 The PHC\(_z\) problems when \(z = 1, 2, 3\)

In Section 3.1 we have established that the PHC\(_z\) problem for any \(z \geq 4\) is polynomial time solvable. This section shows that the PHC\(_z\) problem is \(\mathcal{NP}\)-complete for each \(z \in \{1, 2, 3\}\). Remark that the following Theorems 3.6, 3.7, and 3.8 are independent, e.g., the fact that PHC\(_3\) is \(\mathcal{NP}\)-complete does not imply the fact that PHC\(_2\) is \(\mathcal{NP}\)-complete, and vice versa.

**Theorem 3.6.** The PHC\(_1\) problem is \(\mathcal{NP}\)-complete.

**Proof.** It is known that the HC problem is \(\mathcal{NP}\)-complete even when a given graph is three-edge connected planar cubic \([11]\). It is not difficult to see that the PHC problem with \(z = 1\) for a cubic graph is exactly same as the HC problem. \(\square\)

**Theorem 3.7.** The PHC\(_2\) problem is \(\mathcal{NP}\)-complete, even when a given graph is two-edge connected.

**Proof.** We reduce the HC problem for cubic graphs to the PHC\(_2\) problem. Let \(G\) be a cubic graph, which is an input of the HC problem. Then we construct a graph \(H\) as an input of the PHC\(_2\) problem, as follows (see also Figure 3):

- Subdivide every edge \(e = \{v, u\} \in E(G)\) into a path of length three, i.e., remove \(e\) and add vertices \(v_e, u_e\) and edges \(\{v, v_e\}, \{v_e, u_e\}, \{u_e, u\}\).
- For each vertex \(v \in V(G)\), attach a cycle of length four, i.e., add vertices \(w_{v_1}, w_{v_2}, w_{v_3}\) and edges \(\{v, w_{v_1}\}, \{w_{v_1}, w_{v_2}\}, \{w_{v_2}, w_{v_3}\}, \{w_{v_3}, v\}\).

For convenience, let \(V\) denote the set of original vertices, i.e., \(V = V(G)\), let \(V_s\) denote the set of vertices \(u_e, v_e\) added by subdivision, i.e., \(|V_s| = 2|E(G)|\), and let \(V_c\) denote the set of vertices \(w_{v_1}, w_{v_2}, w_{v_3}\) in attached cycles, i.e., \(|V_c| = 3|V(G)|\), and hence \(V(H) = V \cup V_s \cup V_c\). Then, we show that \(G\) has a HC if and only if \(H\) has a PHC\(_2\).

If \(G\) has a HC, we claim that a PHC\(_2\) is in \(H\). Suppose that \(C \subseteq E(G)\) is a HC of \(G\). For a path \(v \{v_e, v_e\} v_e \{u_e, u_e\} u_e \{v_e, u_e\} u\), set \(x_{v, v_e} = x_{u_e, u} = x_{v_e, u_e} = 1\) if \(e \in C\), otherwise set \(x_{v, v_e} = x_{u_e, u} = 2\) and \(x_{v_e, u_e} = 0\). For a cycle attached to \(v \in V(G)\), set \(x_{v, w_{v_1}} = x_{w_{v_1}, w_{v_2}} = x_{w_{v_2}, w_{v_3}} = x_{w_{v_3}, v} = 1\) (see Figure 3 right). It is not difficult to see that \(x\) indicates a connected closed walk in \(H\), since \(\sum_{e' \in \delta_H(v')} x'_{e'}\) is even for each \(v' \in V(H)\), and HC \(C\) is connected in \(G\). It is also not difficult to see that every vertex is visited an odd number of times; the visit number is three for each vertex in \(V\) and one for each vertex in \(V_s \cup V_c\). Hence \(x\) admits a PHC\(_2\) of \(H\).

For the converse, assuming that \(H\) has a PHC\(_2\), we claim that \(G\) has a HC. Let \(x\) be the edge count vector of the PHC\(_2\) of \(H\). Notice that \(d_H(v') = 2\) for \(v' \in V_s \cup V_c\), and it implies that any PHC\(_2\) in \(H\) visits every vertex of \(V_s \cup V_c\) exactly since any PHC\(_2\) is allowed to use every edge at...
most twice. Then it is not difficult to observe that \( x_e = 1 \) for every edge of the attached cycles, that is, \( x_{\{v,w_1\}} = x_{\{w_1,w_2\}} = x_{\{w_2,w_3\}} = x_{\{w_3,v\}} = 1 \). Furthermore, there are three possible assignments of \( (x_{\{v,w\}}, x_{\{v\},u}, x_{\{u\},w}) \), that is, \( (1,1,1) \), \( (2,0,2) \) or \( (0,2,0) \), where \( (0,2,0) \) is inadequate because a PHC must be connected. Now, noting that \( d_H(v) = 5 \) holds for each \( v \in V \) since \( G \) is cubic, let \( a, b, c, \{v, w\}, \{v, w_1\}, \{v, w_3\} \) be the edges incident to \( v \). Then, any assignments \( x_a, x_b, x_c, x_{\{v, w_1\}}, x_{\{v, w_3\}} \) of a PHC must satisfy

\[
x_a + x_b + x_c + x_{\{v,w_1\}} + x_{\{v,w_3\}} \equiv 2 \pmod{4}
\]

by the parity condition on \( v \). As we saw, \( x_{\{v,w_1\}} = x_{\{v,w_3\}} = 1 \) holds, which implies

\[
x_a + x_b + x_c \equiv 0 \pmod{4}.
\]

Since each value of \( x_a, x_b, x_c \) is at most two and none of them is equal to zero, exactly two of \( x_a, x_b, x_c \) must be assigned to one and the other must be assigned to two; exactly two of the three subdivided paths incident to \( v \) are assigned to \( (1,1,1) \)'s and the remaining one is assigned to \( (2,0,2) \). Now, let \( C' \subset E(G) \) be the set of edges corresponding to \( (1, 1, 1) \) paths in a PHC in \( H \). From the connectedness of PHC in \( H \) it is clear that \( C' \) is a HC in \( G \).

In a similar way to Theorem 3.7 but more complicated, we can show the hardness of the PHC problem.

**Theorem 3.8.** The PHC problem is \( \text{NP}-\)complete, even when a given graph is two-edge connected.

**Proof.** We reduce an instance \( G \) of the HC problem for cubic graphs to an instance \( H \) of the PHC problem exactly in the same way as the proof of Theorem 3.7 (see Fig. 3). We here reuse the notations in the proof of Theorem 3.7 If \( G \) has an HC, we obtain a PHC of \( H \) by Theorem 3.7 where we remark that a PHC is a PHC.

The opposite direction is similar to Theorem 3.7. Let \( \mathbf{x} \) be the edge count vector of the PHC of \( H \). Since PHC visits the vertices in \( V_c \) an odd number of times, the assignment of the edges \( \{v, w_1\}, \{w_1, w_2\}, \{w_2, w_3\}, \{w_3, v\} \) are valued to either all ones or all threes. Meanwhile, the possible assignments of \( (x_{\{v,w\}}, x_{\{v\},u}, x_{\{u\},w}) \) are

\[
(1,1,1), (2,0,2), (3,3,3), (0,2,0),
\]

where \( (0,2,0) \) is inadequate since any PHC is connected.
Let \( a, b, c \in E(H) \) denote three edges incident to \( v \) other than \( \{ v, w_{v1} \} \) and \( \{ v, w_{v3} \} \). Then, any assignments \( x_a, x_b, x_c, x_{\{v,w_{v1}\}}, x_{\{v,w_{v3}\}} \) must satisfy
\[
x_a + x_b + x_c + x_{\{v,w_{v1}\}} + x_{\{v,w_{v3}\}} \equiv 2 \pmod{4}
\]
by the parity condition of \( v \). As we saw, \( x_{\{v,w_{v1}\}} = x_{\{v,w_{v3}\}} = 1 \) or 3, both of which implies
\[
x_a + x_b + x_c \equiv 0 \pmod{4}.
\]
Since each of \( x_a, x_b, x_c \) is at most three and none of them are equal to zero by (9), the possible assignments are either
\[
(x_a, x_b, x_c) = (1, 1, 2) \text{ or } (3, 3, 2).
\]
In the former assignment, the assignment for the three subdivided paths incident to \( v \) consists of two \((1, 1, 1)\)s and one \((2, 0, 2)\), and two \((3, 3, 3)\)s and one \((2, 0, 2)\) in the latter case.

Let \( C' \in E(G) \) be the set of edges which are corresponding to paths in \( H \) assigned as \((1, 1, 1)\) and \((3, 3, 3)\). Since \( \text{PHC}_3 \) is connected, \( C' \) clearly suggests an HC of \( G \). \( \square \)

### 3.3 The \text{PHC}_3 problem for four-edge connected graphs

The \( \text{PHC}_3 \) problem is \( \mathcal{NP} \)-complete for two-edge connected graphs, as we have shown in Theorem 3.8. This subsection establishes the following.

**Theorem 3.9.** A four-edge connected graph \( G = (V, E) \) contains a \( \text{PHC}_3 \) if and only if the order \( |V| \) is even or \( G \) is non-bipartite.

To prove Theorem 3.9 we use the following celebrated theorem.

**Theorem 3.10 ([19, 12]).** Every four-edge connected graph has two edge disjoint spanning trees.

**Proof of Theorem 3.10.** The ‘only-if’ part follows that of Theorem 3.1. We show the ‘if’ part, in a constructive way. Suppose that \( G \) is four-edge connected. Then, let \( \tau \) and \( \tau' \) be a pair of edge disjoint spanning trees of \( G \), implied by Theorem 3.9. Intuitively, we construct a connected closed walk on \( \tau \), and control the parity condition using edges in \( \tau' \), then we obtain a \( \text{PHC}_3 \).

Let \( x \in \mathbb{Z}_{\geq 0}^E \) be given by
\[
x_e = \begin{cases} 
2 & \text{if } e \in E(\tau), \\
0 & \text{otherwise}. 
\end{cases}
\]  
(10)

Then, \((G, x)\) is connected, and has an Eulerian cycle, say \( w \). Let \( S \) be the set of vertices with even degree in \( \tau \), i.e., \( S \) is the entire set of vertices each of which \( w \) visits an even number of times. We also remark that \( |V \setminus S| \) is even, by the shake-hands-theorem. In the following, we consider two cases whether \( |V| \) is even or odd.

If \( |V| \) is even, then \( |S| \) is even. Let \( J \subseteq E(\tau') \) be an \( S \)-join in the tree \( \tau' \). Then, let \( x' \in \mathbb{Z}_{\geq 0}^E \) be defined by
\[
x'_e = \begin{cases} 
x_e + 2 & \text{if } e \in J, \\
x_e & \text{otherwise}. 
\end{cases}
\]  
(11)

It is easy to see that \( x' \) satisfies the parity condition (2) for each vertex of \( V \) since \( J \) is a \( S \)-join. Clearly \((G, x')\) is connected, and \( x' \) admits a \( \text{PHC} \) by Corollary 2.2. Notice that \( J \subseteq E(\tau') \) is disjoint with \( E(\tau) \), meaning that (10) and (11) imply that \( x'_e \leq 2 \) for each \( e \in E \). We obtain the claim in the case.
If $|V|$ is odd, then $|S|$ is odd. By the hypothesis, $G$ is non-bipartite and hence $G$ contains an odd cycle, say $C$. Then, let $x'' \in \mathbb{Z}_{ \geq 0}^E$ be defined by
\[
x''_e = \begin{cases} 
  x_e + 1 & \text{if } e \in E(C), \\
  x_e & \text{otherwise},
\end{cases}
\] (12)
i.e., increase the value of $x_e$ for every $e \in E(C)$ by one. We can observe from the construction that $(G, x'')$ again has an Eulerian cycle, say $w'$. Let $S' = S \oplus V(C)$, then $S'$ is the entire set of vertices each of which $w'$ visits an even number of times. Since $|S|$ is odd and $|V(C)|$ is odd, $|S'|$ is even. Let $J \subseteq E(\tau')$ be a $S'$-join of $\tau'$, and let $x''' \in \mathbb{Z}_{ \geq 0}^E$ be defined by
\[
x'''_e = \begin{cases} 
  x''_e + 2 & \text{if } e \in J, \\
  x''_e & \text{otherwise},
\end{cases}
\] (13)
Now, it is easy to see that $x'''$ satisfies the parity condition (2) for each vertex of $V$ since $J$ is a $S'$-join. Clearly $(G, x''')$ is connected, and $x'''$ admits a PHC. Considering the fact that $J \subseteq E(\tau')$ and $E(\tau)$ are disjoint, we observe from (10), (12) and (13) that
\[
x'''_e = \begin{cases} 
  3 & \text{if } e \in (E(\tau) \cup J) \cap E(C), \\
  2 & \text{if } e \in (E(\tau) \cup J) \setminus E(C), \\
  1 & \text{if } e \in E(C) \setminus (E(\tau) \cup J), \\
  0 & \text{otherwise},
\end{cases}
\] (14)
hold, meaning that $x'''_e \leq 3$ for each $e \in E$. We obtain the claim. \square

It is known (cf. [23]) that edge disjoint spanning trees $\tau$ and $\tau'$ in four-edge connected graph are found in polynomial time (e.g., $O(|E(G)|^2)$ time, due to Roskind, Tarjan [22]). Thus, the proof of Theorem 3.9 also implies a polynomial time (e.g., $O(|E(G)|^2)$) algorithm to find a PHC$_3$ in a four-edge connected graph $G$.

4 The PHC$_3$ Problem for Two-Edge Connected Graphs

The PHC$_3$ problem is $NP$-complete for two-edge connected graphs (Theorem 3.8), while it is solved in polynomial time for any four-edge connected graph (Theorem 3.9). As an approach to three-edge connected graphs, this section further investigates the PHC$_3$ problem using the celebrated ear decomposition. This section establishes the following theorem.

**Theorem 4.1.** Suppose that a two-edge connected graph $G$ is $P_6$-free or $C_{\geq 5}$-free. Then, $G = (V, E)$ contains a PHC$_3$ if and only if the order $|V|$ is even or $G$ is non-bipartite.

Notice that $C_{\geq 5}$-free contains some important graph classes such as chordal (equivalent to $C_{\geq 4}$-free), chordal bipartite (equivalent to $C_{\geq 5}$-free bipartite), and cograph (equivalent to $P_4$-free). We also remark that the Hamiltonian cycle problem is $NP$-complete for $C_{\geq 4}$-free graphs, as well as $P_5$-free graphs (cf. [5]).

4.1 Preliminary

For the purpose, we introduce the notion of the all-roundness of a graph in Section 4.1.
4.1.1 Generalized problem

Section 4 is actually concerned with the following problem, slightly generalizing the PHC$_3$ problem.

**Problem 1.** Given a graph $G = (V, E)$ and a map $f : V \to \{0, 1, 2, 3\}$, find $x \in \{0, 1, 2, 3\}^E$ satisfying the conditions that

$$\sum_{e \in \delta(v)} x_e \equiv f(v) \pmod{4} \quad \text{for any } v \in V,$$

(15)

$(G, x)$ is connected. (16)

Clearly, PHC$_3$ is given by setting $f(v) = 2$ for any $v \in V$ (recall Corollary 2.2). For convenience, we call $x \in \{0, 1, 2, 3\}^E$ a mod-4 $f$-factor of $G$ if $x$ satisfies (15). A mod-4 $f$-factor $x \in \{0, 1, 2, 3\}^E$ is connected if it satisfies (16), i.e., Problem 1 is a problem to find a connected mod-4 $f$-factor. We remark the following two facts.

**Proposition 4.2.** A graph $G = (V, E)$ has a mod-4 $f$-factor only when the map $f$ satisfies that

$$\sum_{v \in V} f(v) \text{ is even.}$$

(17)

**Proof.** Summing up (15) over $V$, we obtain

$$\sum_{v \in V} f(v) \equiv \sum_{v \in V} \sum_{e \in \delta(v)} x_e \pmod{4}.$$ (18)

It is not difficult to see that

$$\sum_{v \in V} \sum_{e \in \delta(v)} x_e = 2 \sum_{e \in E} x_e$$

(19)

holds, in an analogy with the handshaking lemma, that is $\sum_{v \in V} \sum_{e \in \delta(v)} 1 = \sum_{e \in E} 2$. By (18) and (19), we obtain that

$$\sum_{v \in V} f(v) \equiv 2 \sum_{e \in E} x_e \pmod{4}$$

which implies the claim. 

We will later show in Lemma 4.11 that (17) is also sufficient for any connected non-bipartite graph to have a mod-4 $f$-factor. For bipartite graphs, we need an extra necessary condition on $f$.

**Proposition 4.3.** A bipartite graph $G = (U, V; E)$ has a mod-4 $f$-factor only when the map $f$ satisfies that

$$\sum_{v \in U} f(v) \equiv \sum_{v \in V} f(v) \pmod{4}.$$ (20)

**Proof.** Since $G = (U, V; E)$ is bipartite,

$$\sum_{v \in U} \sum_{e \in \delta(v)} x_e = \sum_{v \in V} \sum_{e \in \delta(v)} x_e$$

(21)

is required. Summing up (15) over $U$ and $V$, respectively, we obtain

$$\sum_{v \in U} f(v) \equiv \sum_{v \in U} \sum_{e \in \delta(v)} x_e \pmod{4}, \quad \text{and}$$

$$\sum_{v \in V} f(v) \equiv \sum_{v \in V} \sum_{e \in \delta(v)} x_e \pmod{4}$$

hold, which with (21) implies the claim. 

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Figure 4: $K_3$ is all-round.

Notice that the condition (20) implies (17). We will show in Lemma 4.11 that (20) is also sufficient for any connected bipartite graph.

4.1.2 All-roundness

Then, we introduce the following two notions.

Definition 4.4. A graph $G$ is all-round if $G$ has a connected mod-4 $f$-factor for any map $f$ satisfying (17).

Definition 4.5. A bipartite graph $G$ is bipartite all-round if $G$ has a connected mod-4 $f$-factor for any map $f$ satisfying (20).

It is not difficult to see that Definitions 4.4 and 4.5 are (too strong) sufficient condition that a graph contains a PHC$_3$. In the rest of Section 4, we will show the following theorems, which immediately implies Theorem 4.1.

Theorem 4.6. Every two-edge connected $C_{\geq 5}$-free graph is either all-round or bipartite all-round.

Theorem 4.7. Every two-edge connected $P_6$-free graph, except for $C_5$, is either all-round or bipartite all-round.

4.1.3 All-roundness of small graphs

As a preliminary step of the proof, we remark the following facts, which are confirmed by brute force search.

Proposition 4.8. $C_3$ (i.e., $K_3$) is all-round (See Figure 4).

Proposition 4.9. $C_4$ and $C_6$ are bipartite all-round, respectively.

The following fact may be counterintuitive.

Proposition 4.10. $C_5$ is not all-round.

Figure 5 shows an example of $f$ for which Problem 1 does not have a solution. Notice that $C_5$ clearly has a PHC$_3$. 
4.2 All-roundness of $C_{\geq 5}$-free: Proof of Theorem 4.6

This section shows Theorem 4.6, presenting some useful idea on a mod-4 $f$-factor of a graph, and all-roundness or bipartite all-roundness. To begin with, we show for any appropriate map $f$ that it is easy to find a mod-4 $f$-factor, which may be disconnected.

Lemma 4.11. Any connected non-bipartite graph has a mod-4 $f$-factor for any map $f$ satisfying (17). Any connected bipartite graph has a mod-4 $f$-factor for any map $f$ satisfying (20).

Proof. We give a constructive proof. Let $T := \{v \in V \mid f(v) \text{ is odd}\}$. We remark that $|T|$ is even since $\sum_{v \in V} f(v)$ is even by (17). Then, let $J \subseteq E$ be a $T$-join, and let $x \in \{0, 1, 2, 3\}$ be given by

$$x_e = \begin{cases} 1 & \text{if } e \in J \\ 0 & \text{otherwise.} \end{cases}$$

Let $f' : V \to \{0, 1, 2, 3\}$ be

$$f'(v) = \left( f(v) - \sum_{e \in \delta(v)} x_e \right) \mod 4. \quad (22)$$

Remark that $f'(v)$ is even for any $v \in V$, i.e., $f'(v) = 0$ or 2 for any $v \in V$. Let $S = \{v \in V \mid f'(v) = 2\}$. If $|S|$ is even, then let $J'$ be a $S$-join and let $x' \in \{0, 1, 2, 3\}$ be defined by

$$x'_e = \begin{cases} x_e + 2 & \text{if } e \in J' \\ x_e & \text{otherwise.} \end{cases}$$

It is not difficult to observe that $x'$ satisfies (15), and $x'_e \leq 3$ holds for any $e \in E$. Thus, we obtain the claim in the case. Here we remark that if $G$ is bipartite then $|S|$ is even, since (20) implies

$$\sum_{v \in U \cup V} f'(v) = \sum_{v \in U} f'(v) + \sum_{v \in V} f'(v) \equiv \sum_{v \in U} f'(v) - \sum_{v \in U} f'(v) \equiv 0 \pmod{4}.$$

If $|S|$ is odd, we need an extra process. Notice that $G$ is non-bipartite in the case, since the above argument. Let $C$ be an odd cycle of $G$ and let $x'' \in \{0, 1, 2, 3\}^{E(G)}$ be

$$x''_e = \begin{cases} x_e + 1 & \text{if } e \in E(C) \\ x_e & \text{otherwise.} \end{cases}$$
Let \( f'': V \to \{0, 1, 2, 3\} \) be
\[
f''(v) = \left( f(v) - \sum_{e \in \delta(v)} x''_e \right) \mod 4.
\] (23)

Let \( S' = \{ v \in V(G) \mid f''(v) = 2 \} \). Then, \( S' = S \oplus V(C) \) holds, which implies \( |S'| \) is even since \( |S| \) and \( |V(C)| \) are respectively odd. Let \( J' \subseteq E(G) \) be an \( S' \)-join and let \( x''' \in \{0, 1, 2, 3\}^{E(G)} \) be
\[
x'''_e = \begin{cases} x''_e + 2 & \text{if } e \in J' \\ x''_e & \text{otherwise.} \end{cases}
\]

Then we obtain a mod-4 \( f \)-factor.

To obtain a connected mod-4 \( f \)-factor, the notions of all-roundness and bipartite all-roundness play an important role. The following lemma is an easy observation from the definition.

Lemma 4.12 (connecting lemma). Let \( x \) be a mod-4 \( f \)-factor of \( G \), and let \( H_1 \) and \( H_2 \) be a distinct pair of connected components of \((G, x)\). Suppose that there is a connected subgraph \( H \) of \( G \) such that \( V(H) \) intersects both \( V(H_1) \) and \( V(H_2) \), and that \( H \) is all-round or bipartite all-round. Then, \( G \) has another mod-4 \( f \)-factor \( x' \) such that \( H_1 \) and \( H_2 \) are connected in \((G, x')\) where other connected components are respectively kept being connected.

Proof. As given \( x \in \{0, 1, 2, 3\}^{E(G)} \) and \( H \) described in the hypothesis, we define a map \( f_H : V(H) \to \{0, 1, 2, 3\} \) by
\[
f_H(v) = \sum_{e \in \delta_H(v)} x_e \mod 4.
\]

Let \( y \in \{0, 1, 2, 3\}^{E(H)} \) be defined by \( y_e = x_e \) for each \( e \in E(H) \), then clearly \( y \) is a mod-4 \( f_H \)-factor of \( H \). Proposition 4.2 implies that \( f_H \) satisfies (17), as well as Proposition 4.3 implies that \( f_H \) satisfies (20) if \( H \) is bipartite. The hypothesis that \( H \) is all-round or bipartite all-round implies that there is a connected mod-4 \( f_H \)-factor \( y' \in \{0, 1, 2, 3\}^{E(H)} \) of \( H \). Let \( x' \in \{0, 1, 2, 3\}^{E(G)} \) be
\[
x'_e = \begin{cases} y'_e & \text{if } e \in E(H), \\ x_e & \text{otherwise,} \end{cases}
\]
then, it is not difficult to observe that \( x' \) is a desired mod-4 \( f \)-factor of \( G \) (See also Figure 6). 

It is not difficult to see that Lemmas 4.11 and 4.12 imply the following useful lemma.
Lemma 4.13. Let $G$ be a graph. Suppose that for any edge $e \in E(G)$ there exists a subgraph $H$ of $G$ such that $e \in E(H)$ and $H$ is all-round or bipartite all-round. Then, $G$ is all-round, or bipartite all-round.

Proof. For any appropriate $f$, meaning that $f$ satisfies (17), and (20) if $G$ is bipartite, Lemma 4.11 implies a mod-$4$ $f$-factor $x$. Since the hypothesis, we can obtain a connected mod-$4$ $f$-factor by iteratively applying the connecting lemma (Lemma 4.12) to $x$.

In fact, the all-roundness of $C_{\geq 5}$-free (Theorem 4.6) is immediate from Lemma 4.13.

Proof of Theorem 4.6. Let $G$ be a two-edge connected $C_{\geq 5}$-free graph. Then, any edge $e$ of $G$ is contained in $C_3$ or $C_4$. Since $C_3$ is all-round (Proposition 4.8), and $C_4$ is bipartite all-round (Proposition 4.9), Lemma 4.13 implies that $G$ is all-round or bipartite all-round.

We will use Lemma 4.13 again in the proof of Theorem 4.7, with some extra arguments. We also show another example in Section 5.2 where we apply Lemma 5.2 to dense graphs.

4.3 All-roundness of $P_6$-free: Proof of Theorem 4.7

We cannot directly apply Lemma 4.13 to a $P_6$-free graph $G$, since $G$ may contain $C_5$ that is NOT all-round (recall Proposition 4.10). To prove Theorem 4.7 we investigate the all-roundness (or bipartite all-roundness) of two-edge connected graphs, considering ear-decomposition in Section 4.3.1.

4.3.1 Ear-decomposition for mod-4 all-round graphs

This section presents three lemmas, which claim a short ear preserves the all-roundness or bipartite all-roundness.

Lemma 4.14. Let $G$ and $G'$ be two-edge connected non-bipartite graphs, where $G'$ is given by adding to $G$ an ear of length at most seven. If $G$ is all-round, then $G'$ is all-round.

Proof. The proof idea is similar to Lemma 4.13 construct a mod-$4$ $f$-factor which may not be connected (Lemma 4.11), and let it be connected using the connecting lemma (Lemma 4.12) on assumption that $G$ is all-round. A technical issue of the proof idea is that the ear is not bipartite all-round. We show that a desired mod-$4$ $f$-factor always exist if the ear is short.

Let $p = v_0v_1v_2e_2 \cdots e_{\ell}v_{\ell}$ denote the ear added to $G$, and let $P = \{\{v_0, v_1, \ldots, v_\ell\}, \{e_1, \ldots, e_\ell\}\}$, where $\ell \leq 7$. Remark that $v_0, v_\ell \in V(G)$. As given an arbitrary map $f: V(G') \to \{0, 1, 2, 3\}$ satisfying (17), let $x \in \{0, 1, 2, 3\}^{E(G')}$ be a mod-$4$ $f$-factor implied by Lemma 4.11. For convenience, let $x^{(0)} = (x_1, \ldots, x_\ell) \in \{0, 1, 2, 3\}^{E(P)}$ where $x_i$ denotes $x_{e_i}$ for simplicity. We also define $x^{(a)} = \{0, 1, 2, 3\}^{E(P)}$ for each $a \in \{1, 2, 3\}$ by

$$x^{(a)}_i = \begin{cases} (x_i + a) \mod 4 & \text{if } i \text{ is odd}, \\ (x_i - a) \mod 4 & \text{if } i \text{ is even}. \end{cases}$$

Notice that $x^{(a)}_i$ for each $a \in \{0, 1, 2, 3\}$ is a mod-$4$ $f_p^{(a)}$-factor for a map $f_p^{(a)}: V(P) \to \{0, 1, 2, 3\}$ given by

$$f_p^{(a)}(v) = \begin{cases} (x_v + a) \mod 4 & \text{if } v = v_0, \\ (x_v + x_{v+1}) \mod 4 = f(v) & \text{if } v \in \{v_1, \ldots, v_{\ell-1}\}, \\ (x_v + (-1)^a) \mod 4 & \text{if } v = v_\ell. \end{cases}$$
At the same time, let \( f_G^{(a)} : V(G) \to \{0, 1, 2, 3\} \) be defined for each \( a \in \{0, 1, 2, 3\} \) by

\[
f_G^{(a)}(v) = \begin{cases} 
\left( f(v) - f_P^{(a)}(v) \right) \mod 4 & \text{if } v \in \{v_0, v_\ell\}, \\
\frac{f(v)}{4} & \text{otherwise},
\end{cases}
\]

then \( G \) has a connected mod-4 \( f_G^{(a)} \)-factor \( y^{(a)} \in \{0, 1, 2, 3\}^{E(G)} \) since \( G \) is all-round by the hypothesis. If \((P, x^{(a)})\) consists of at most two connected component, i.e., \( x^{(a)} \) holds for each \( i \in \{1, 2, \ldots, \ell\} \), then \( x^{(a)} \) implies a connected mod-4 \( f \)-factor of \( G' \).

Then, we claim that there is \( a \in \{0, 1, 2, 3\} \) such that \( (P, x^{(a)}) \) consists of at most two connected component. Remark that \( \{x^{(0)}_i, x^{(1)}_i, x^{(2)}_i, x^{(3)}_i\} = \{0, 1, 2, 3\} \) holds for each \( i \in \{1, 2, \ldots, \ell\} \); otherwise the multiset \( \{x^{(a)}_j \mid a \in \{0, 1, 2, 3\}, j \in \{1, 2, \ldots, \ell\}\} \) contains 8 or more 0’s. The \( x^{(a)} \) is the desired mod-4 \( f_P^{(a)} \)-factor for \( P \), and we obtain the claim. \( \square \)

The following lemma is a version of Lemma 4.14 for bipartite graphs. The proof is essentially the same, and we omit it.

**Lemma 4.15.** Let \( G \) and \( G' \) be two-edge connected bipartite graphs, where \( G' \) is given by adding to \( G \) an ear of length at most seven. If \( G \) is bipartite all-round, then \( G' \) is bipartite all-round. \( \square \)

Finally, we show the following lemma, which claims a connection between a bipartite all-round graph and an all-round graph.

**Lemma 4.16.** Let \( G \) be a two-edge connected bipartite graph, and let \( G' \) be a two-edge connected non-bipartite graph given by adding to \( G \) an ear of length at most three. If \( G \) is bipartite all-round, then \( G' \) is all-round.

**Proof.** The proof is similar to Lemma 4.14. Let \( p = v_0 e_1 v_1 e_2 \cdots e_\ell v_\ell \) denote the ear added to \( G \) where \( \ell \leq 3 \), and let \( P = (\{v_0, v_1, \ldots, v_\ell\}, \{e_1, \ldots, e_\ell\}) \). Remark that \( v_0, v_\ell \in V(G) \). As given an arbitrary map \( f: V(G') \to \{0, 1, 2, 3\} \) satisfying (17), let \( x \in \{0, 1, 2, 3\}^{E(G')} \) be a mod-4 \( f \)-factor implied by Lemma 4.11. For convenience, let \( x^{(0)} = (x_1, \ldots, x_\ell) \in \{0, 1, 2, 3\}^{E(P)} \) where \( x_i \) denotes \( x_{e_i} \) for simplicity. We also define \( x^{(2)} \in \{0, 1, 2, 3\}^{E(P)} \) by

\[
x^{(2)}_i = (x_i + 2) \mod 4.
\]

Notice that \( x^{(2)}_i \) is also a mod-4 \( f_P^{(2)} \)-factor for a map \( f^{(2)}_P : V(P) \to \{0, 1, 2, 3\} \) given by

\[
f^{(2)}_P(v) = \begin{cases} 
(x_i + 2) \mod 4 & \text{if } v = v_0, \\
(x_i + x_{i+1}) \mod 4 = f(v) & \text{if } v \in \{v_1, \ldots, v_{\ell-1}\}, \\
(x_\ell + 2) \mod 4 & \text{if } v = v_\ell.
\end{cases}
\]

At the same time, let \( f^{(a)}_G : V(G) \to \{0, 1, 2, 3\} \) be defined for each \( a \in \{0, 2\} \) by

\[
f^{(a)}_G(v) = \begin{cases} 
\left( f(v) - f^{(a)}_P(v) \right) \mod 4 & \text{if } v \in \{v_0, v_\ell\}, \\
f(v) & \text{otherwise},
\end{cases}
\]

then \( f^{(0)}_G \) clearly satisfies (20), and \( f^{(2)}_G \) as well. Thus, \( G \) has a connected bipartite mod-4 \( f^{(a)}_G \)-factor \( y^{(a)} \in \{0, 1, 2, 3\}^{E(G)} \) for each \( a \in \{0, 2\} \) since \( G \) is bipartite all-round by the hypothesis. If \( (P, x^{(a)}) \)
consists of at most two connected component, i.e., \( x_i^{(a)} = 0 \) holds at most one \( i \in \{1, 2, \ldots, \ell\} \), then \( x^{(a)} \) and \( y^{(a)} \) implies a connected mod-4 \( f \)-factor of \( G' \).

Then, we claim that \((P, x^{(0)})\) or \((P, x^{(2)})\) consists of at most two connected component. Remark that \( \{x_i^{(0)}, x_i^{(2)}\} = \{0, 2\} \) or \( \{1, 3\} \) holds for each \( i \in \{1, 2, \ldots, \ell\} \). Since \( \ell \leq 3 \), it is not difficult to observe the claim.

\[ \square \]

### 4.3.2 Proof of Theorem 4.7

Now, we show Theorem 4.7. In fact, we prove the following lemma, which with Lemma 4.13 immediately implies Theorem 4.7

**Lemma 4.17.** Let \( G = (V, E) \) be a two-edge connected \( P_6 \)-free graph such that \( G \neq C_5 \). For any edge \( e \in E \), there is a subgraph \( H \) of \( G \) such that \( H \) contains \( e \) and it is all-round or bipartite all-round.

**Proof.** Suppose that an edge \( e \) is contained in a cycle \( C \) of length \( \ell \). If \( \ell = 3, 4, 6 \), then \( C \) itself is all-round or bipartite all-round, and we obtain the claim. Thus, we will show the cases of \( \ell = 5 \) and \( \ell \geq 7 \).

First, we show the case \( \ell \geq 7 \). To begin with, we remark that \( P_6 \)-free implies that the cycle \( C \) has at least two chords, say \( f \) and \( h \), which do not share their end vertices; otherwise, suppose they share a vertex \( v \) then the induced subgraph removing \( v \) contains \( P_6 \) (or longer). For convenience, we say a chord separates \( C \) into \( (a, \ell - a) \) \( (a \in \{2, 3, \ldots, \lfloor \ell/2 \rfloor \}) \) if the chord connects vertices in the distance \( a \) along \( C \). The proof idea is an induction on \( \ell \), i.e., we show a shorter cycle containing \( e \). Recall that \( C_3 \) is all-round (Proposition 4.8), and \( C_4 \) and \( C_6 \) are bipartite all-round (Proposition 4.9), while \( C_5 \) is not all-round (Lemma 4.10).

In the case of \( \ell = 7 \). If a chord separates \( C \) into \((2, 5)\), then \( e \) is contained in \( C_3 \) or \( C_6 \), and we obtain the claim. If both chords \( f \) and \( g \) separates \( C \) into \((3, 4)\), then we claim that \( H = C + f + g \) is all-round. Note that \( H \) is isomorphic to one of two graphs in Figure 7. In the left graph, we can find an ear decomposition consisting of \( C_4 (1-5-6-7-1, \text{in Figure 7}) \) and ears of length 3 (1-2-3-7) and 2 (3-4-5). Let \( H' \) denote the graph consisting of \( C_4 (1-5-6-7-1) \) and an ear of length 3 (1-2-3-7), then Lemma 4.15 implies that \( H' \) is bipartite all-round. Since \( H \) is given by \( H' \) and an ear of length two, Lemma 4.16 implies that \( H \) is all-round. In the right graph, we can find an ear decomposition consisting of \( C_4 (1-5-6-7-1) \), an ear of length 2 (5,4,7), and an ear of length 3 (1,2,3,4). Then, \( H \) is also all-round by a similar argument.

In the case of \( \ell = 8 \). Then, a chord possibly separates \( C \) into \((2, 6)\), \((3, 5)\) or \((4, 4)\). The cases of \((2, 6)\) and \((3, 5)\) are easy; In the case of \((2, 6)\), \( e \) is contained in \( C_5 \) or \( C_7 \), where the latter case is reduced to the above case of \( \ell = 7 \). In the case of \((3, 5)\), \( e \) is contained in \( C_4 \) or \( C_6 \). Finally, in the case that both \( f \) and \( g \) separates \( C \) into \((4, 4)\), then \( H = C + f + g \) is isomorphic one of two graphs in Figure 8. One graph (upper one) consists of \( C_4 (1-5-4-8-1) \) and two ears of length 3 (1-2-3-4 and 5-6-7-8), and it is all-round by Lemma 4.16. The other consists of \( C_6 (1-5-4-3-7-8-1) \) and ears of length 2 (1-2-3 and 5-6-7), and it is bipartite all-round, by Lemma 4.15. Thus we obtain the claim in the case.

In the case of \( \ell = 9 \). Then, a chord possibly separates \( C \) into \((2, 7)\), \((3, 6)\) or \((4, 5)\). In the case of \((2, 6)\) and \((3, 5)\) are easy, since \( e \) is contained in \( C_5, C_7, C_4 \) or \( C_6 \). If both \( f \) and \( g \) separates \( C \) into \((4, 5)\), then \( H = C + f + g \) is isomorphic one of three graphs in Figure 9. We can observe that each graph has an ear decomposition consisting of \( C_6 (1-2-3-4-5-6-1) \) with ears of length two and three. Then, each graph is all-round by Lemmas 4.14, 4.15 and 4.16.

In the case of \( \ell \geq 10 \). Suppose that a chord separates \( C \) into \((a, \ell - a)\) where \( a \leq \ell - a \), then \( e \) is in \( C_{a+1} \) or \( C_{\ell-a+1} \). Unless \( e \in C_5 \), i.e., \( a = 4 \) and \( e \in C_{a+1} \), the case is reduced to a shorter
cycle. Suppose $e \in C_{a+1}$ where $a = 4$. Then, $\ell - a + 1 \geq 7$ implies that $C_{\ell-a+1}$ contains another chord. Using the chord, we can reduce the case to a shorter cycle.

Next, we show the remaining case of $\ell = 5$. Suppose that $e$ is contained in a cycle $C$ of length five. We consider two cases: $C$ is the unique cycle which contains $e$, or there is another cycle containing $e$.

First, we consider the former case. Since there is no cycle containing $e$ other than $C$, $C$ has no chord. Furthermore, since $G \neq C_5$, there exists a vertex $v \in V(C)$ that is contained in another cycle, which implies there exists an edge $f \in \delta(v) \setminus E(C)$. Since $G$ is a two-edge connected $P_6$-free graph, $f$ is contained in a $C_3$; Otherwise $G$ has a $P_6$ as an induced subgraph. Then, Lemma 4.14 implies that $C + C_3$ is all-round, and we obtain the claim in the case.

Next, we consider the second case. If there exists another cycle containing $e$ and its length is not five, the argument for $\ell \neq 5$ establishes Lemma 4.17 for $e$. Suppose every cycle containing $e$ has length five. Let $C''$ be one of the cycles containing $e$ other than $C$. Then $C$ and $C''$ has common edges. Let $k$ be the number of the common edges. If $k = 2$ or $3$, Lemma 4.16 implies that $C + C''$ is all-round. If $k = 1$, $C + C'' - e$ is a cycle of length eight. Then $C + C'' - e$ has a chord $f \neq e$ because $G$ is $P_6$-free. By the argument of the case $\ell = 8$, $C + C'' + f$ is all-round. \qed
5 Miscellaneous Discussions

This section remarks three related topics. Section 5.1 extends the arguments in Section 4 to graphs with bridges, and presents Theorem 5.4, which extends Theorem 4.1 about two-edge connected graphs to any connected. Section 5.2 remarks an all-roundness of dense graphs applying Lemma 4.13. Section 5.3 briefly explains a connection between the PHC problem and other problems such as the HC problem.

5.1 All-roundness of graphs with bridges

This subsection is concerned with graphs with bridges.

Theorem 5.1. For a connected graph $G = (V, E)$, let $B \subseteq E$ denote the set of bridges, and let $S \subseteq V$ denote the set of isolated vertices in $G - B$. If $S = \emptyset$ and every two-edge connected component in $G - B$ is all-round then $G$ is all-round.

Proof. Let $C_1, \ldots, C_l$ be the two-edge connected components of $G - B$, and let $H$ be a graph obtained by contracting every $C_i$ of $G$. Notice that $E(H) = B$. Let $u_1, \ldots, u_l$ be the vertices of $H$ where each $u_i$ corresponds to $C_i$. For each $C_i$, let $T = \{ u_i \mid \sum_{v \in C_i} f(v) \equiv 1 \pmod{2} \}$ and let $J$ be a $T$-join of $H$. Then, let $x \in \{0, 1, 2, 3\}^{E(H)}$ be defined by

$$x_e = \begin{cases} 1 & \text{if } e \in J, \\ 2 & \text{if } e \notin J. \end{cases}$$

Let a map $f' : V \to \{0, 1, 2, 3\}$ be defined by

$$f'(v) = \left( f(v) - \sum_{e \in \delta_H(v)} x_e \right) \mod 4$$

for each $v \in V(G)$. Notice that $f'(v) = 0$ for $v \in S$, since $x$ is a mod-4 $f_H$-factor. Since each $C_i$ is all-round, $G$ has a mod-4 $f'$-factor $y$ which are connected in each two-edge connected component.
Let $x' \in \{0, 1, 2, 3\}^{E(G)}$ be defined by

$$x'_e = \begin{cases} x_e & \text{if } e \in B, \\ y_e & \text{otherwise,} \end{cases}$$

then we obtain a connected mod-4 $f$-factor.

Now, we are concerned with the PHC3 problems again. The following two propositions are easy observations from the fact that any closed walk must pass each edge of $\delta(v)$ for any $v \in S$ an even number of times since they are bridges.

**Proposition 5.2.** For a connected graph $G = (V, E)$, let $B \subseteq E$ denote the set of bridges and let $S \subseteq V$ denote the set of isolated vertices in $G - B$. If $G$ contains a PHC3, then $d_G(v)$ is odd for any $v \in S$.

**Proposition 5.3.** For a connected graph $G = (V, E)$, let $B \subseteq E$ denote the set of bridges and let $S \subseteq V$ denote the set of isolated vertices in $G - B$. Suppose that $d_G(v)$ is odd for any $v \in S$ and that every connected component of $G - S$ is all-round, then $G$ has a PHC3.

By Theorem 5.1 and Propositions 5.2 and 5.3, we obtain the following theorem.

**Theorem 5.4.** Suppose that $G = (V, E)$ is a connected $P_6$-free, or $C_{\geq 5}$-free graph. Let $S \subseteq V$ denote the set of isolated vertex in $G - B$. Note that $S = \emptyset$ if $G$ is two-edge connected, while the reverse is not true. Then, $G$ has a PHC3 if and only if $d_G(v)$ is odd for every $v \in S$.

### 5.2 All-roundness of dense graphs

This section shows another application of Lemma 4.13.

**Proposition 5.5.** Let $G = (V, E)$ be a connected graph where $|V| \geq 3$ and the minimum degree of $G$ is at least $|V|/3$. Then, $G$ is all-round, or bipartite all-round if $G$ is bipartite.

**Proof.** We show that every edge $e = uv \in E$ is contained in a cycle of length three or four, then Lemma 4.13 implies that $G$ is all-round. If $|V|$ is odd, the degree of each $u$ and $v$ is strictly greater than $|V|/2$ by the hypothesis. This implies that $u$ and $v$ has a common neighbor $w$ by the pigeon hole principle. Thus any edge is contained in a cycle of length three.

Suppose $|V|$ is even. If $u$ and $v$ has a common neighbor, then we obtain the claim. If it is not the case, we can observe that $|N(u) \setminus \{v\}| = |N(v) \setminus \{u\}| = |V|/2 - 1$ holds. Let $w \in N(u) \setminus \{v\}$, then $d(w) \geq |V|/2$ implies that $w$ is connected to a vertex in $N(v) \setminus \{u\}$ by the pigeon hole principle. Thus we obtain a cycle of the length four in the case.

In fact, we can show the following stronger theorem with some complicated arguments (see Appendix B for the proof).

**Theorem 5.6.** Let $G = (V, E)$ be a connected graph where $|V| \geq 4$ and the minimum degree of $G$ is at least $|V|/3$. Then, $G$ is all-round, or bipartite all-round if $G$ is bipartite.
5.3 Connection of Parity Hamiltonian, Hamiltonian, Eulerian

We remark the connection between the PHC problem and the HC problem, or other related topics. In fact, we are concerned with the following generalized version of Problem 1.

**Problem 2** (connected mod-$d$ $f$-factor with edge capacity constraints). Given a graph $G = (V, E)$, a map $z: E \to \mathbb{Z}_{\geq 0}$, a positive integer $d$, and a map $f: V \to \mathbb{Z}_{\geq 0}$, find $x \in \mathbb{Z}^E$ satisfying the conditions that

$$
\sum_{e \in \delta(v)} x_e \equiv f(v) \pmod{d} \quad \text{for any } v \in V,
$$

$$
(G, x) \text{ is connected.}
$$

$$
x_e \leq z(e) \quad \text{for any } e \in E.
$$

The PHC problem is given by setting $d = 4$ and $f(v) = 2$ for any $v \in V$ with an appropriate capacity constraint. Problem 1 is given by setting $d = 4$ and $z(e) = 3$ for any $e \in E$. The Hamiltonian cycle problem is represented by setting $d = n$, $f(v) = 2$ for any $v \in V$, and $z(e) = 1$ for any $e \in E$. We remark that the Hamiltonian cycle problem is also given (in its original form) by setting $d = \infty$, $f(v) = 2$ for any $v \in V$, and removing (25) (or setting $z(e) = \infty$ for any $e \in E$). The Eulerian cycle problem is given by setting $d = 2$, $f(v) = 0$ for any $v \in V$, and replacing (26) with $x_e = 1$ for any $e \in E$.

A two-factor plays a key role in the arguments of the HC problem in cubic graphs, where the connectivity constraint is relaxed [13, 14, 3, 4]. Motivated by a connected “factor,” this paper has investigated connected mod-4 factors. A mod-$d$ factor for prime $d$ is an interesting future work.

6 Concluding Remarks

In this paper, we have introduced the parity Hamiltonian cycle problem. We have shown that the problem is in $P$ when $z \geq 4$, while $NP$-complete when $z \leq 3$. Then, we are involved in the case $z = 3$, and showed some graph classes for which the problem is in $P$. It is open if the PHC$_3$ problem is in $P$ for three-edge connected graphs.

More sophisticated arguments on the connection between the PHC problem and related topics, such as HC, $T$-join, even-factors, extended complexities, jump systems, etc., are significant future works.

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5 The argument of this section might be appropriate appearing in the section of Concluding Remarks. However, it is too long to put there, and we discuss just before the concluding remark.
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A Finding A $T$-Join in Linear Time

In this section we describe an linear-time algorithm to find a $T$-join of a graph. The algorithm is described in Algorithm A.1.

### Algorithm A.1 Linear Time Algorithm to Find a $T$-join

```
MAKE-T-JOIN ($G, T$)
for all $v \in V(G)$ do
  $v.color \leftarrow$ WHITE
end for
exchange_flg $\leftarrow$ FALSE
$J \leftarrow \emptyset$
for all $v \in V(G)$ do
  if $v.color =$ WHITE then
    MAKE-T-JOIN-REC ($G, T, v$)
    if exchange_flg $= TRUE$ then
      return FALSE
    end if
  end if
end for
return $J$
```

The input of Algorithm A.1 is a pair of an undirected graph $G$ and a vertex set $T \subseteq V$. Algorithm A.1 outputs a $T$-join $J$ if it exists, or FALSE otherwise.

Algorithm A.1 essentially runs the depth first search, determining an edge should be picked for or removed from $J$ while tracing edges. Algorithm A.1 uses two variables, color and exchange_flg. color is an attribute of each vertex $v$ and it takes value of WHITE or BLACK. $v.color =$ WHITE indicates that $v$ is not visited by the algorithm, otherwise $v$ is already visited. exchange_flg is a boolean variable which determines an edge the algorithm is looking now should be picked for $J$ or not. If exchange_flg is TRUE the algorithm does the following: if $e \notin J$ then picks $e$ for $J$, otherwise removes $e$ from $J$. If exchange_flg is FALSE the algorithm does nothing for $e$.

Since Algorithm A.1 runs the depth first search with constant overheads, it clearly terminates in linear time. We show that the algorithm works correctly.

**Proposition A.1.** Algorithm A.1 returns a $T$-join of $G$ if it exists, or FALSE otherwise.

---

6 Here we does not assume that $G$ is connected.
**Algorithm A.2 Body of the Algorithm**

MAKE-T-JOIN-REC \((G, T, v)\)

\(v.color \leftarrow \text{BLACK}\)

if \(v \in T\) then

\(\text{exchange}_flg \leftarrow \text{exchange}_flg\)

end if

for all \(u \in N(v)\) do

if \(u.color = \text{WHITE}\) then

if \(\text{exchange}_flg = \text{TRUE}\) then

\(J \leftarrow J \triangle \{u, v\}\)

end if

MAKE-T-JOIN-REC \((G, T, u)\)

if \(\text{exchange}_flg = \text{TRUE}\) then

\(J \leftarrow J \triangle \{u, v\}\)

end if

end if

end for

**Proof.** It suffices to show that Algorithm A.1 works correctly on connected graphs; if \(G\) has more than one connected component we can apply the correctness proof for each connected component. Suppose \(G\) does not have a \(T\)-join, i.e., \(|T|\) is odd. Since the number of inversion of \(\text{exchange}_flg\) is equal to \(|T|\), the value of \(\text{exchange}_flg\) is inverted an odd number of times. Then Algorithm A.2 ends the recursion with \(\text{exchange}_flg\) being TRUE, Algorithm A.1 returns FALSE.

Suppose \(G\) has a \(T\)-join, i.e., \(|T|\) is even. Let \(H\) be a graph obtained by doubling every edge in \(G\). We can regard the depth first search on \(G\) as a connected closed walk of \(H\) which traces each edge exactly once. Let \(e_1, e_2 \in E(H)\) be the copies of \(e \in E(G)\), and let \(J' \subseteq E(H)\) be a set of edges which are passed when \(\text{exchange}_flg\) is TRUE. Then \(e\) is a member of the output of Algorithm A.1 \(J\) if and only if exactly one of \(e_1\) and \(e_2\) is a member of \(J'\), that is,

\[ J = \{e \in E(G) \mid (e_1 \in J') \oplus (e_2 \in J')\}. \tag{27} \]

The inversion of \(\text{exchange}_flg\) occurs if and only if the algorithm visits \(v \in T\) first time, which causes \(|\delta_H(v) \cap J'|\) to be odd for each \(v \in T\). On the other hand, it holds that \(|\delta_H(v) \cap J'|\) is even for every \(v \notin T\), since \(\text{exchange}_flg\) is not inverted when the algorithm visits \(v\). By (27), \(|\delta_H(v) \cap J'| \equiv |\delta_G(v) \cap J| \pmod{2}\) for each \(v \in V(G)\); \(|\delta_G(v) \cap J| \equiv 1 \pmod{2}\) for \(v \in T\) and \(|\delta_G(v) \cap J| \equiv 0 \pmod{2}\) for \(v \notin T\). Thus \(J\) is a \(T\)-join of \(G\).

**B Proof of Theorem 5.6**

To prove Theorem 5.6, we need some more lemmas in the following.

**Lemma B.1.** Let \(I\) be a graph given by connecting \(G\) and \(H\) by an edge, where \(G\) and \(H\) are all-round. Then, \(I\) is all-round.

**Proof.** Let \(e\) be the edges connecting \(G\) and \(H\). Notice that \(\sum_{v \in V(G)} f(v) \equiv \sum_{v \in V(H)} f(v) \pmod{2}\) since \(f\) satisfies (17). Let \(x_e = 1\) if \(\sum_{v \in V(G)} f(v) \equiv 1 \pmod{2}\), otherwise let \(x_e = 2\). Let \(f'\) be defined by (22), then \(f'\) satisfies (17). Since \(G\) and \(H\) are all-round, \(I\) has a connected mod-4 \(f\)-factor.

\(\square\)
Lemma B.2. Let $I$ be a graph given by connecting $G$ and $H$ by two edges, where $G$ is all-round and $H$ is bipartite all-round. Then, $I$ is all-round.

Proof. Let $e_1, e_2$ be the edges connecting $G$ and $H$. Set the value of $x_{e_1}$ and $x_{e_2}$ to satisfy the following two conditions:

$$\sum_{v \in V(G)} f'(v) \equiv 0 \pmod{2}$$

and

$$\sum_{v \in U(H)} f'(v) \equiv \sum_{v \in V(H)} f'(v) \pmod{4}$$

where $f'$ is defined by (22) and $U(H)$ and $V(H)$ is the color classes of $H$. Since $G$ is all-round and $H$ is bipartite all-round, $G \cup H$ has a connected mod-4 $f'$-factor $x^*$. By a combination of $x^*$ and $x_{e_1}, x_{e_2}$, we obtain a connected mod-4 $f$-factor of $G$. $\square$

Lemma B.3. Let $I$ be a graph given by connecting $G$ and $H$ by two edges, where $G$ and $H$ are bipartite all-round. Then, $I$ is all-round or bipartite all-round.

Proof. Let $e_1, e_2$ be the edges connecting $G$ and $H$, and let $G = (A_1, B_1, E(G)), H = (A_2, B_2, E(H))$. There are three cases of which components $e_1$ and $e_2$ connect (see Figures 10, 11 and 12).

Case 1. If both $e_1$ and $e_2$ connect $A_1$ and $B_2$, set the value of $x_{e_1}, x_{e_2}$ to satisfy

$$\sum_{v \in A_1} f(v) - \sum_{v \in B_1} f(v) \equiv x_{e_1} + x_{e_2} \pmod{4}$$

(28)

and at least one of $x_{e_1}$ or $x_{e_2}$ is not equal to zero.

Case 2. If $e_1$ connects $A_1$ and $B_2$, and $e_2$ connects $A_2$ and $B_1$, set the value of $x_{e_1}, x_{e_2}$ to satisfy

$$\sum_{v \in A_1} f(v) - \sum_{v \in B_1} f(v) \equiv x_{e_1} - x_{e_2} \pmod{4}$$

(29)

and at least one of $x_{e_1}$ or $x_{e_2}$ is not equal to zero.

Case 3. If $e_1$ connects $A_1$ and $B_1$, and $e_2$ connects $A_1$ and $B_2$, set the value of $x_{e_1}, x_{e_2}$ to satisfy both

$$\sum_{v \in A_1} f(v) - \sum_{v \in B_1} f(v) \equiv x_{e_1} + x_{e_2} \pmod{4}$$

(30)

and

$$\sum_{v \in A_2} f(v) - \sum_{v \in B_2} f(v) \equiv x_{e_1} - x_{e_2} \pmod{4},$$

(31)

and at least one of $x_{e_1}$ or $x_{e_2}$ is not equal to zero.

Let $f'$ be defined by (22), $G \cup H$ has a connected mod-4 $f'$-factor $x^*$. By combining $x^*$ and $x_{e_1}, x_{e_2}$, we obtain a connected mod-4 $f$-factor of $G$. $\square$

We also use the following two lemmas (we omit the proofs).

Lemma B.4. Let $G = (V, E)$ be a connected graph where $|V| \geq 8$ and the minimum degree of $G$ is at least $|V|/3$. Then, every vertex in $G$ belongs to a triangle or a square.
Lemma B.5. Let $G = (V, E)$ be a connected graph where $|V| \geq 4$ and the minimum degree of $G$ is at least $|V|/3$. Let $C_1, \ldots, C_k$ $(k \geq 3)$ be induced subgraphs of $G$ such that $\bigcup_{i=1}^{k} V(C_i) = V$ and $V(C_i) \cap V(C_j) = \emptyset$ for any $i \neq j$. Then there exists a pair $C_i, C_j (i \neq j)$ such that $|\delta(V(C_i)) \cap \delta(V(C_j))| \geq 2$.

Now we prove Theorem 5.6.

Proof of Theorem 5.6. Let $n = |V|$. We can check Theorem 5.6 is true for any graph with $n \leq 7$ by the brute force search. Suppose $n \geq 8$. We consider two cases whether $G$ has a bridge or not.

Case 1. $G$ has a bridge.

Let $e$ be the bridge and $C_1, C_2$ be the two components of $G - e$ (Note that neither $C_1$ nor $C_2$ is bipartite since $\delta(G) \geq n/3$). Then $|V(C_i)| \geq n/3 + 1$ for each $i = 1$ and 2, therefore $n/3 + 1 \leq |V(C_i)| \leq (2/3)n - 1$ ($i = 1, 2$).

If $|V(C_i)| \leq (2/3)n - 2$ for both $i = 1, 2$,

$$\delta(C_i) \geq \frac{n}{3} - 1 = \frac{1}{2} \left( \frac{2}{3}n - 2 \right) \geq \frac{1}{2} |V(C_i)|$$

holds for each $i = 1, 2$, Theorem 5.5 implies the all-roundness of $C_1$ and $C_2$. Thus, by Lemma B.1. $G$ is all-round.

Suppose $|V(C_1)| = (2/3)n - 1$. Then $|V(C_2)| = (1/3)n + 1 \leq (2/3)n - 2$, therefore $C_2$ is all-round as same as the previous case. Let $v \in V(C_1)$ be an end vertex of $e$. Then we have

$$\delta(C_1 - v) \geq \frac{n}{3} - 1 = \frac{1}{2} \left( \frac{2}{3}n - 2 \right) \geq \frac{1}{2} |V(C_1 - v)|,$$

hence $C_1 - v$ is all-round. Since $C_1$ is obtained by adding ears of length at most two to $C_1 - v$, $C_1$ is all-round by Lemma 4.11. Thus $C_1$ and $C_2$ are both all-round, by Lemma B.1. $G$ is all-round.

Case 2. $G$ has no bridges.

If $G$ has no bridges, $G$ is two-edge connected. By Lemma B.4, every vertex of $G$ belongs to a triangle or a square. Let $C_1, \ldots, C_k$ be the triangles and squares, all-round (or bipartite-all-round) subgraphs of $G$. If $k = 2$, There are at least two edges connecting $C_1, C_2$ since $G$ is two-edge

\footnote{$G$ has at most one bridge since $\delta(G) \geq n/3$.}
connected, thus $G$ is all-round by Lemmas B.1, B.2 and B.3. If $k \geq 3$, we can find a pair $C_i, C_j$ which has at least two connecting edges between them by Lemma B.5. By combining $C_i$ and $C_j$ together with the connecting edges, we obtain a larger subgraph $C_i'$ which is all-round or bipartite all-round, by Lemmas B.1, B.2 and B.3. This operation reduces the number of the subgraphs $k$ by one, by repeatedly taking this operation then we can reduce the case to $k = 2$. \qed