The Smash Product of Monoidal Theories

Amar Hadzihasanovic
Department of Software Science
Tallinn University of Technology
Tallinn, Estonia
amar@cs.ioc.ee

Abstract—The tensor product of props was defined by Hackney and Robertson as an extension of the Boardman–Vogt product of operads to more general monoidal theories. Theories that factor as tensor products include the theory of commutative monoids and the theory ofbialgebras. We give a topological interpretation (and vast generalisation) of this construction as a low-dimensional projection of a “smash product of pointed directed spaces”. Here directed spaces are embodied by combinatorial structures called diagrammatic sets, while Gray products replace cartesian products. The correspondence is mediated by a web of adjunctions relating diagrammatic sets, pros, probs, props, and Gray-categories. The smash product applies to presentations of higher-dimensional theories and systematically produces higher-dimensional coherence data.

INTRODUCTION

It is an insight due to Lawvere [1] that algebraic theories can be embodied by cartesian monoidal categories whose objects are freely generated from a set of sorts. In theoretical computer science, it has since been recognised that cartesianness of a monoidal structure amounts to the ability to copy and delete the data on which a theory operates, and can be weakened in order to introduce forms of resource-sensitivity.

Symmetric monoidal theories, embodied by props [2], have become increasingly popular as specifications of generic compositional systems. As a further weakening, monoidal and braided monoidal theories, as pros and probs, can be used to implement topological constraints on access to data.

It is often the case that these theories decompose into smaller units, satisfying certain compatibility constraints. A prolific line of research [3], [4], [5] has been dedicated to methods for constructing the specification of a composite theory from its constituents.

A uniform method for composing symmetric monoidal theories is the tensor product of props, defined by Hackney and Robertson [6]. Theories that factor as tensor products include the theories of bialgebras and of commutative monoids and cmonoids, which feature prominently in algebraic approaches to signal flow graphs [7] and quantum circuits [8], and also all cartesian theories that are free on a prop.

Existing methods have an intrinsic limitation. In a computational setting, we usually do not work with a theory, but with a presentation of a theory by generators and relations. The orientation of the relations can be significant: for example, seen as a rewrite system, the presentation specifies a model of computation, for which the presented theory is only a denotational semantics. While some of the compositional methods may extract an equational presentation, they do not retain any information on the orientation of equations.

In this article, we present a generalisation of the tensor product of props, which not only produces a presentation (with oriented equations) of the tensor product from presentations of its factors, but also systematically produces higher-dimensional coherence data, known as syzygies in homotopical algebra [9], which exhibit confluence at “new” critical branchings created by the product. Coherence data is usually pursued with analytic methods of rewriting theory; our results appear to be a gateway to new synthetic and compositional methods.

Our generalisation takes the form of a smash product of pointed directed spaces, analogous to the classical construction of topological spaces. This can be interpreted via Burroni’s paradigm of rewriting as the study of presented higher-categorical structures [10], [11], which are formally analogous to CW complexes in topology, but have “directed cells”.

The idea of a smash product of monoidal theories was first put forward in [12], but the Burroni–Street theory of polygraphs proved to be an unsuitable foundation. Instead we use a combinatorial model of directed space called a diagrammatic set [13]. In this model, a directed n-cell is roughly the face poset of a regular CW-decomposition of the closed n-ball, together with an orientation subdividing the boundary of each cell into an input and output half, which both determine composable higher-categorical diagrams.

The definitions of the smash products of pointed diagrammatic sets and of pointed spaces are identical, except a non-cartesian monoidal structure, the Gray product, replaces the cartesian product of spaces. There is a realisation functor from diagrammatic sets to spaces which turns Gray products into cartesian products and smash products into smash products.

To prove our main result – the smash product subsumes the tensor product of props – as a first step we reduce the latter to an “external” product of pros which produces a prob. The technical core is the construction of a web of adjunctions relating pros, diagrammatic sets, andhaps. These enable us to embed pros into pointed diagrammatic sets through a nerve functor, then realise the smash product of nerves as a prob. It is then a matter of calculation to verify that the result is naturally isomorphic to the tensor product of pros.

Overall, our results reveal the surprising fact that the tensor product of props and the smash product of pointed spaces are two facets of the same construction.
Structure of the article

Section I recaps the elementary theory of diagrammatic sets and strict $\omega$-categories. Section II introduces categories of pros, props, and props and shows how props are special Gray-categories, a form of semistrict tricategory [14]. Section III constructs the adjunctions that relate diagrammatic sets, pros, and Gray-categories. Section IV defines the tensor product of props and the smash product of pointed diagrammatic sets, then presents the main theorem that relates them. Section V shows through an example how the smash product of two presentations produces higher-dimensional coherence data.

We assume a solid knowledge of category theory and some basic notions of topology and order theory. We use the diagrammatic order $f, g$ for the composition of morphisms $f$ and $g$ in a category, but the “classical” order $GF$ for the composition of functors $F$ and $G$.

I. SOME HIGHER STRUCTURES

We quickly go through the main definitions, and refer the reader to [13] for an in-depth development.

1 (Graded poset). Let $P$ be a finite poset with order relation $\leq$ and let $P_1$ be $P$ extended with a least element $\perp$. We say that $P$ is graded if, for all $x \in P$, all directed paths from $x$ to $\perp$ in the Hasse diagram $\mathcal{H}P_1$, oriented from covering to covered elements, have the same length. If this length is $n+1$, we let $\dim(x) := n$ be the dimension of $x$.

2 (Closed and pure subsets). Let $P$ be a poset and $U \subseteq P$. The closure of $U$ is the subset $\text{cl}U := \{x \in P \mid \exists y \in U \ x \leq y\}$ of $P$. We say that $U$ is closed if $\text{cl}U = U$. If $P$ is graded, the dimension $\dim(U)$ of $U$ is $\max\{\dim(x) \mid x \in U\}$ if $U$ is inhabited, $-1$ otherwise. We say that $U$ is pure if its maximal elements all have dimension $\dim(U)$.

3 (Oriented graded poset). An orientation on a finite poset $P$ is an edge-labelling $o$ of its Hasse diagram with values in $\{+, -\}$. An oriented graded poset is a finite graded poset with an orientation.

4 (Boundaries). Let $P$ be an oriented graded poset and $U \subseteq P$ a closed subset. For all $\alpha \in \{+, -\}$ and $n \in \mathbb{N}$, we let $\Delta^\alpha_n U \subseteq U$ be the subset of elements $x$ such that $\dim(x) = n$ and, if $y \in U$ covers $x$, then $o(y \rightarrow x) = \alpha$.

We let $x \in \partial^\alpha_n U$ if and only if $x \in \text{cl}(\Delta^\alpha_n U)$ or $x \in U$ and for all $y \in U$, if $x \leq y$, then $\dim(y) \leq n$. We write $\partial^\alpha_n U := \partial^\alpha_n U \cup \partial_n^\alpha U$.

We call $\partial_n^\alpha U$ the input $\alpha$-boundary and $\partial^\alpha_n U$ the output $\alpha$-boundary of $U$. For each $x \in P$, we write $\partial^\alpha_n x := \partial^\alpha_n \text{cl}(x)$. We may omit the index $n$ when $n = \dim(U) - 1$.

5 (Atoms and molecules). Let $P$ be an oriented graded poset. We define a family of closed subsets of $P$, the molecules of $P$, by induction on proper subsets. If $U$ is a closed subset of $P$, then $U$ is a molecule if either

- $U$ has a greatest element, in which case we call it an atom, or
- there exist molecules $U_1$ and $U_2$, both properly contained in $U$, and $n \in \mathbb{N}$ such that $U_1 \cap U_2 = \partial^\alpha_n U_1 = \partial^\alpha_n U_2$ and $U = U_1 \cup U_2$.

We define $\subseteq$ to be the smallest partial order relation such that, if $U_1$ and $U_2$ are molecules and $U_1 \cap U_2 = \partial^\alpha_n U_1 = \partial^\alpha_n U_2$, then $U_1, U_2 \subseteq U_1 \cup U_2$. We say $n$-molecule for an $n$-dimensional molecule.

6 (Spherical boundary). An $n$-molecule $U$ in an oriented graded poset has spherical boundary if, for all $k < n$,

$$\partial^\alpha_k U \cap \partial_{k-1}^\alpha U = \partial_{k-1} U.$$ 

7 (Regular directed complex). An oriented graded poset $P$ is a regular directed complex if, for all $x \in P$ and $\alpha, \beta \in \{+, -, \}$,

1) $\text{cl}\{x\}$ has spherical boundary,
2) $\partial^\alpha x$ is a molecule, and
3) $\partial^\alpha(\partial^\beta x) = \partial^\alpha_{n-2} x$ if $n := \dim(x) > 1$.

A map $f : P \rightarrow Q$ of regular directed complexes is a function of their underlying sets that satisfies

$$\partial^\alpha_n f(x) = f(\partial^\alpha_n x)$$

for all $x \in P$, $n \in \mathbb{N}$, and $\alpha \in \{+, -, \}$. We call an injective map an inclusion. With their maps, regular directed complexes form a category $\text{DCpx}^R$.

As shown in [13, Section 1.3], $\text{DCpx}^R$ has an initial object, a terminal object, and pushouts of inclusions.

A regular molecule is a regular directed complex which is a molecule. By [13, Proposition 1.38], if two regular molecules are isomorphic in $\text{DCpx}^R$, they are isomorphic in a unique way, and we can treat them as equal.

8 (Globe). For each $n \in \mathbb{N}$, let $O^n$ be the poset with a pair of elements $k^+, k^-$ for each $k < n$ and a greatest element, with the partial order defined by $j^\alpha \leq k^\beta$ if and only if $j \leq k$. This is a graded poset, with $\dim(O^n) = n$ and $\dim(k^\alpha) = k$ for all $k < n$.

With the orientation $o(y \rightarrow k^\alpha) := \alpha$ if $y$ covers $k^\alpha$, $O^n$ becomes a regular directed complex, in particular a regular atom. We call $O^n$ the $n$-globe.

9 (Pasting of molecules). Let $U_1, U_2$ be regular molecules and suppose that $\partial^\alpha_k U_1$ and $\partial^\beta_k U_2$ are isomorphic in $\text{DCpx}^R$. Given an isomorphic copy $V$ of the two, there is a unique span of inclusions $V \rightarrow U_1$ and $V \rightarrow U_2$ whose images are, respectively, $\partial^\alpha_k U_1$ and $\partial^\beta_k U_2$. We let $U_1 \#_k U_2$ be the pushout

$$\begin{array}{ccc}
V & \xrightarrow{} & U_2 \\
\downarrow & & \downarrow \\
U_1 & \xrightarrow{} & U_1 \#_k U_2
\end{array}$$

in $\text{DCpx}^R$. Then $U_1 \#_k U_2$ is a regular molecule, decomposing as $U_1 \cup U_2$ with $U_1 \cap U_2 = \partial^\alpha_k U_1 = \partial^\beta_k U_2$.

10 (−−construction). Let $U, V$ be regular $n$-molecules with spherical boundary such that $\partial^\alpha U$ is isomorphic to $\partial^\alpha V$ for all $\alpha \in \{+, -, \}$.
Form the pushout $U \cup V$ of the span of inclusions $\partial U \hookrightarrow U$, $\partial U \hookrightarrow V$ whose images are $\partial U$ and $\partial V$, respectively. We define $U \rightarrow V$ to be the oriented graded poset obtained from $U \cup V$ by adjoining a greatest element $\top$ with $\partial^- \top := U$ and $\partial^+ \top := V$. Then $U \Rightarrow V$ is an $(n+1)$-dimensional atom with spherical boundary.

11 (Construction). Let $U$ be a regular molecule with spherical boundary. Then $\partial^- U \Rightarrow \partial^+ U$ is defined, and we denote it by $\langle U \rangle$.

Example 1. There is a unique 0-atom, namely, the 0-globe $1 := O^0$, which is also the terminal object of $\mathbb{D}Cpx^R$.

We define a sequence $\{I_n\}_{n>0}$ of 1-molecules by

$$I_1 := O^1, \quad I_n := I_{n-1} \#_0 O^1 \text{ for } n > 1.$$ 

Every regular 1-molecule is of the form $I_n$ for some $n > 0$.

For each pair $n, m > 0$, let $U_{n, m} := (I_n \Rightarrow I_m)$. Every regular 2-atom is of the form $U_{n, m}$ for some $n, m > 0$. Regular 2-molecules are then generated by $I_1$ and the $U_{n, m}$ under the pasting operations $\#_0, \#_1$.

12 (Diagrammatic set). We write $\bigcirc$ (to be read "atom") for a skeleton of the full subcategory of $\mathbb{D}Cpx^R$ on the atoms of every dimension.

A diagrammatic set is a presheaf on $\bigcirc$. Diagrammatic sets and their morphisms of presheaves form a category $\mathbb{G}Set$.

We identify $\bigcirc$ with a full subcategory $\bigcirc \hookrightarrow \mathbb{G}Set$ via the Yoneda embedding. With this identification, we use morphisms in $\mathbb{G}Set$ as our notation for both elements and structural operations of a diagrammatic set $X$:

- $x \in X(U)$ becomes $x : U \rightarrow X$, and
- for each map $f : V \rightarrow U$ in $\bigcirc$, $X(f)(x) \in X(V)$ becomes $f \circ x : V \rightarrow X$.

As described in [13, §4.4], the embedding $\bigcirc \hookrightarrow \mathbb{G}Set$ extends to an embedding $\mathbb{D}Cpx^R \hookrightarrow \mathbb{G}Set$.

13 (Diagrams and cells). Let $X$ be a diagrammatic set and $U$ a regular molecule. A diagram of shape $U$ in $X$ is a morphism $x : U \rightarrow X$. A diagram is a cell if $U$ is an atom. For all $n \in \mathbb{N}$, we say that $x$ is an $n$-diagram or an $n$-cell when $\dim(U) = n$.

If $U$ decomposes as $U_1 \#_k U_2$, we write $x = x_1 \#_k x_2$ for $x_1 := \iota_i \circ x$; $x_2$, where $\iota_i$ is the inclusion $U_i \hookrightarrow U$ for $i \in \{1, 2\}$. This extends associatively to $n$-ary decompositions for $n > 2$.

Let $k^+_x : \partial^+_k U \rightarrow U$ be the inclusions of the $k$-boundaries of $U$. The input $k$-boundary of $x$ is the diagram $\partial^+_k x := k^+_x \circ x$ and the output $k$-boundary of $x$ is the diagram $\partial^-_k x := \iota_k \circ x$.

We may omit the index $k$ when $k = \dim(U) - 1$.

We write $x : y^{-} \Rightarrow y_{+}$ to express that $\partial^+_k x = y_{\alpha}$ for each $\alpha \in \{+, -, \}$, and say that $x$ is of type $y^{-} \Rightarrow y_{\alpha}$. Two diagrams $x_1, x_2$ are parallel if they have the same type.

Example 2. A 1-cell $a$ in a diagrammatic set has shape $I_1$. A 2-cell $\varphi$ has shape $U_{n, m}$ for some $n, m > 0$, so it is of type

$$a_1 \#_0 \ldots \#_0 a_n \Rightarrow b_1 \#_0 \ldots \#_0 b_m$$

for some 1-cells $a_1, \ldots, a_n, b_1, \ldots, b_m$.

14 (Dual diagrammatic set). Let $U$ be a regular atom. The oriented graded poset $U^\circ \Rightarrow$ with the same underlying poset as $U$ and the opposite orientation $\omega(y \rightarrow x) := -\omega(x \rightarrow y)$ is a regular atom. If $f : U \rightarrow V$ is a map in $\bigcirc$, its underlying function also defines a map $f^\circ : U^\circ \rightarrow V^\circ$.

Let $X$ be a diagrammatic set. Its dual $X^\circ$ is the diagrammatic set defined by $X^\circ(\cdot) := X(\cdot^\circ)$. This determines an involutive endofunctor on $\mathbb{G}Set$.

15 (Reflective $\omega$-graph). Let $O$ be the full subcategory of $\bigcirc$ whose objects are the globes $O^n$. For all $n$ and $k < n$,

- the $k$-boundary inclusions $\iota^+_k, \iota^-_k$ are the only inclusions of $O^k$ into $O^n$,
- the map $\tau : O^n \rightarrow O^k$, defined by $\tau(\#_k) := k$ if $j \geq k$ and $\tau(\#_\alpha) := \#_\alpha$ if $j < k$, is the only surjective map from $O^n$ onto $O^k$.

A reflective $\omega$-graph is a presheaf $X$ on $O$. With their morphisms of presheaves, reflective $\omega$-graphs form a category $\omega\mathbb{G}Ph_{ref}$.

We treat reflective $\omega$-graphs as "restricted" diagrammatic sets, and use for them the same terminology and notation. Because all $n$-cells in a reflective $\omega$-graph $X$ have the same shape $O^n$, we leave it implicit and write $X_n := X(O^n)$.

16 (Units). Let $x$ be a $k$-cell in a reflective $\omega$-graph $X$. For $n > k$, we let $\varepsilon_n x := \tau(x)$ where $\tau$ is the unique surjective map $O^n \rightarrow O^k$. We call $\varepsilon_n x$ a unit on $x$. We may omit the index when $n = k + 1$.

17 (Rank of a cell). Let $x$ be an $n$-cell in a reflective $\omega$-graph. The rank $rk(x)$ of $x$ is defined inductively on $n$ as follows:

- if $n = 0$, then $rk(x) := 0$;
- if $n > 0$, if $x = \varepsilon_y$ for some $y$, then $rk(x) := rk(y)$, otherwise $rk(x) := n$.

18 (Partial $\omega$-category). A partial $\omega$-category is a reflective $\omega$-graph $X$ together with partial $k$-composition operations

$$\#_k : X_n \times X_n \rightarrow X_n$$

for all $n \in \mathbb{N}$ and $k < n$, satisfying the following axioms:

1) for all $n$-cells $x, y$ and all $k < n$ such that $x \#_k y$ is defined,

$$\partial^+_k x = \partial^- y \text{ and } \varepsilon(x \#_k y) = \varepsilon x \#_k \varepsilon y;$$

2) for all $n$-cells $x$ and all $k < n$, the $k$-compositions

$$x \#_k \varepsilon_n(\partial^+_k x) \text{ and } \varepsilon_n(\partial^+_k x) \#_k x$$

are defined and equal to $x$;

3) for all $(n + 1)$-cells $x, y$ and $k < n$, whenever the left-hand side is defined, the right-hand side is defined and

$$\partial^- (x \#_n y) = \partial^- x,$$

$$\partial^+_n (x \#_k y) = \partial^+_n y,$$

$$\partial^+_n (x \#_k y) = \partial^+_n x \#_k \partial^+_n y;$$

4) for all cells $x, y, z$ and all $k$ such that both sides are defined,

$$(x \#_k y) \#_k z = x \#_k (y \#_k z);$$
5) for all cells $x, y, x', y'$, all $n$ and all $k < n$ such that both sides are defined,

$$
(x \#_n x') \#_k (y \#_n y') = (x \#_k y) \#_n (x' \#_k y').
$$

A functor $f : X \to Y$ of partial $\omega$-categories is a morphism of the underlying reflexive $\omega$-graphs such that, for all cells $x, y$ in $X$, if $x \#_n y$ is defined in $X$ then $f(x) \#_n f(y)$ is defined and equal to $f(x \#_n y)$ in $Y$. Partial $\omega$-categories and their functors form a category $\text{p}_{\omega}\text{Cat}$.

We will generally confuse the notation for a $k$-cell and the units on it: if $x$ is an $n$-cell and $y$ a $k$-cell, $k < n$, such that $x \#_m y$ is defined, we will write $x \#_m y := x \#_m \varepsilon_n y$.

19 ($\omega$-Precategory). An $\omega$-precategory is a partial $\omega$-category $X$ such that, for all $n$-cells $x, y$ in $X$, the $k$-composition $x \#_k y$ is defined if and only if $\partial^+_k x = \partial^-_k y$ and $\text{min}\{\text{rk}(x), \text{rk}(y)\} \leq k + 1$. With their functors, $\omega$-precategories form a category $\text{PreCat}$.

20 ($\omega$-Category). An $\omega$-category is a partial $\omega$-category such that, for all $n$-cells $x, y$ in $X$, the $k$-composition $x \#_k y$ is defined if and only if $\partial^+_k x = \partial^-_k y$. With their functors, $\omega$-categories form a category $\text{Cat}$.

The inclusion $\text{Cat} \hookrightarrow \text{p}_{\omega}\text{Cat}$ has a left adjoint $-^\ast : \text{p}_{\omega}\text{Cat} \to \text{Cat}$. By [13, Proposition 1.23], if $P$ is a regular directed complex, there is a partial $\omega$-category $\text{Mot}P$ where

1) the set $\text{Mot}P_n$ of $n$-cells is the set of molecules $U \subseteq P$ with $\text{dim}(U) \leq n$,

2) $\partial^+_k : \text{Mot}P_n \to \text{Mot}P_{n-1} = U \to \partial^+_k U$,

3) $\varepsilon_n : \text{Mot}P_n \to \text{Mot}P_n$ is $U \to U$,

4) $U \#_k V$ is defined if and only if $U \cap V = \partial^+_k U = \partial^-_k V$, and in that case it is equal to $U \cup V$.

The assignment $P \mapsto \text{Mot}P$ extends to a functor $\text{Mot}^\ast : \text{DCpx}^R \to \text{Cat}$ which is faithful and injective on objects.

Remark 1. There is a forgetful functor from $\text{Cat}$ to $\text{PreCat}$ which makes $x \#_k y$ undefined whenever $\text{min}\{\text{rk}(x), \text{rk}(y)\} > k + 1$. This functor is full and faithful, and its image consists of the $\omega$-precategories satisfying

$$
(x \#_{k-1} \partial^+_k y) \#_k (\partial^-_k x \#_{k-1} y) = (\partial^+_k x \#_{k-1} y) \#_k (x \#_{k-1} \partial^-_k y)
$$

for all cells $x, y$ with $\text{min}\{\text{rk}(x), \text{rk}(y)\} = k + 1$ and $\partial^+_k x = \partial^-_k y$.

21 (Skeleton). Let $X$ be an $\omega$-(pre)category, $n \in \mathbb{N}$. The $n$-skeleton $\sigma_{\leq n} X$ of $X$ is the restriction of $X$ to cells of rank $\leq n$. We let $\sigma_{<1} X := \emptyset$, the initial $\omega$-(pre)category.

22 ($n$-Category). An $\omega$-(pre)category is an $n$-(pre)category if it is equal to its $n$-skeleton. An $n$-(pre)category is determined by its restriction to $k$-cells with $k \leq n$.

Let $\text{nPreCat}$ denote the full subcategory of $\text{PreCat}$ and $\text{nCat}$ the full subcategory of $\text{Cat}$ on the $n$-(pre)categories.

In both cases, the inclusion of subcategories has a right adjoint and $\sigma_{\leq n}$ is the comonad induced by the adjunction.

The inclusion also has a left adjoint, inducing a monad $\tau_{\leq n}$: given $X$, the $n$-(pre)category $\tau_{\leq n} X$ is obtained from $\sigma_{\leq n} X$ by identifying all pairs of $n$-cells $x, y$ such that there exists an $(n+1)$-cell $e : x \Rightarrow y$ in $X$.

Remark 2. Both $\text{PreCat}$ and $\text{Cat}$ are categories of algebras for finitary monads on $\text{Gph}_{	ext{fin}}$, a presheaf topos. By the Remark at the end of [15, §2.78], they are locally finitely presentable, and in particular have all small limits and colimits.

23 (Polygraph). Let $\partial O^n := \sigma_{\leq n-1} O^n$.

A $\omega$-(pre)category $X$ is an $\omega$-(pre)category $X$ together with a set $\mathcal{X} = \sum_{n \in \mathbb{N}} \mathcal{X}_n$ of generating cells such that, for all $n \in \mathbb{N}$,

$$
\prod_{x \in \mathcal{X}_n} \partial O^n \hookrightarrow \prod_{x \in \mathcal{X}_n} O^n
$$

is a pushout in $\text{PreCat}$ or $\text{Cat}$.

An $n$-(pre)polygraph is a $(\omega$-(pre)$)$category whose underlying $\omega$-(pre)category is an $n$-(pre)category. In an $n$-(pre)polygraph, $\mathcal{X}_m = \emptyset$ for $m > n$.

In every $(\omega$-pre$)$polygraph $(X, \mathcal{X})$, all cells in $\mathcal{X}_n$ have rank $n$. The set $\mathcal{X}_0$ is the entire set $X_0$ of 0-cells.

Both 1-precategories and 1-categories coincide with small categories; a 1-(pre)polygraph is a category free on a graph.

II. MONOIDAL THEORIES

For technical reasons, we treat pros as a special case of a structure of bicoloured pro, whose relation to pros is the same as the relation of bicategories to monoidal categories.

24 (Bicoloured pro). A bicoloured pro $X$ is a 2-category $X$ together with the structure of a 1-polygraph $(\sigma_{\leq 1} T, \mathcal{F})$ on its 1-skeleton.

A morphism $f : (T, \mathcal{F}) \to (S, \mathcal{G})$ of bicoloured pros is a functor $f : T \to S$ of 2-categories with the property that $f(a) \in \mathcal{G} \cup \mathcal{F}_1$ for all $a \in \mathcal{F}_1$. Bicoloured pros and their morphisms form a category $\text{Pro}_{\text{bi}}$.

25 (Strict monoidal category). A strict monoidal category is a 2-category with a single 0-cell [16, Theorem 4.1].

26 (Pro). A pro is a bicoloured pro with a single 0-cell. We let $\text{Pro}$ denote the full subcategory of $\text{Pro}_{\text{bi}}$ on pros.

If $(T, \mathcal{F})$ is a pro, its 1-skeleton is the free monoid on $\mathcal{F}_1$, whose 1-cells can be identified with finite ordered lists of elements of $\mathcal{F}_1$. Seeing a pro as the embodiment of a monoidal theory, we interpret the elements of $\mathcal{F}_1$ as sorts, and 2-cells

$$
\varphi : (a_1, \ldots, a_n) \Rightarrow (b_1, \ldots, b_m)
$$

as operations taking $n$ inputs of sorts $a_1, \ldots, a_n$ and returning $m$ outputs of sorts $b_1, \ldots, b_m$. 
In particular, if the monoidal theory is one-sorted, then \( T \) is a singleton, the 1-cells of \( T \) are in bijection with natural numbers, and the type of a 2-cell is fixed by the \textit{arity} of its input and its output. In that case, we may write \( \varphi: (n) \Rightarrow (m) \) for an operation with \( n \) inputs and \( m \) outputs.

A model of the monoidal theory \((T, \mathcal{T})\) in a monoidal category \( M \) is a strong monoidal functor from \( T \) to \( M \).

\textbf{Example 3.} The monoid \( \mathbb{N} \) of natural numbers with addition, seen as a strict monoidal category with no rank-2 cells, is a one-sorted pro. This corresponds to the “trivial” theory of objects in a monoidal category.

\textbf{Example 4.} There is a one-sorted pro \( \text{Mon} \) whose 1-cell \((n)\) is identified with the finite ordinal \( \{0 < \ldots < n - 1\} \) for each \( n \in \mathbb{N} \), and 2-cells \( \varphi: (n) \Rightarrow (m) \) are order-preserving maps. The 0-composite of \( \varphi: (n) \Rightarrow (m) \) and \( \psi: (p) \Rightarrow (q) \) is given by “concatenation” of ordinals and maps. Models of \( \text{Mon} \) are monoids in a monoidal category.

\textbf{Example 5.} If \((T, \mathcal{T})\) is a pro, then \((T^{\text{co}}, \mathcal{T}^{\text{co}})\), obtained by reversing the orientation of all 2-cells of \( T \), is also a pro. For example, \( \text{Mon}^{\text{co}} \) is the theory of co-monoids.

\textbf{Proposition 1.} The categories \( \text{Pro}_{\text{un}} \) and \( \text{Pro} \) have all small limits and colimits.

\textbf{27 (Braided strict monoidal category).} A braided strict monoidal category is a strict monoidal category \( X \) together with a family of 2-cells

\[ \sigma_{x,y}: x \neq y \Rightarrow y \neq x \]

called braidings, indexed by 1-cells \( x, y \), satisfying the following axioms:

1) the braidings are invertible, that is, there are unique 2-cells \( \sigma_{x,y}^{-1} \), called inverse braidings, such that \( \sigma_{x,y} \circ \sigma_{x,y}^{-1} = \sigma_{x,y}^{-1} \circ \sigma_{x,y} \) and \( \sigma_{x,y} \# \sigma_{y,z} = \sigma_{x,z} \) are units;

2) they are natural in their parameters, that is, for all 2-cells \( \varphi: x \Rightarrow x' \) and \( \psi: y \Rightarrow y' \),

\[ (\varphi \# y) \# (\psi \# 0) = \sigma_{x,y} \# \psi \circ \psi \# (\varphi \# 0), \]

\[ (x \# \psi) \# (y \# \psi) = \sigma_{x,y} \# \psi \circ (x \# 0), \]

3) they are compatible with 0-composition and units, that is,

\[ \sigma_{x,x} \# x, y = (x \# 0 \sigma_{x,y}^{-1}) \# \sigma_{x,y} \# 0 \]

\[ \sigma_{x,y} \# 0, y = (\sigma_{x,y} \# 0 \# y) \# \sigma_{x,y} \# 0, y \]

\[ \sigma_{x, \text{e}} = \sigma_{x, \text{e}} \]

\[ \text{whenever the left-hand side is defined.} \]

\textbf{A functor} \( f: X \rightarrow Y \) is a functor of the underlying 2-categories that preserves braidings. With their functors, braided strict monoidal categories form a category \( \text{BrMonCat}_{\text{str}} \).

\textbf{28 (Prob).} A \text{prob} is a pro together with a structure of braided strict monoidal category on its underlying strict monoidal category. A \text{morphism} of pros is a morphism of pros that preserves the braidings and their morphisms form a category \( \text{Prob} \).

\textbf{Remark 3.} To determine a unique structure of braided strict monoidal category on a pro it is, in fact, sufficient to give braidings \( \sigma_{a,b} \) for all pairs of generating 1-cells \( a, b \).

\textbf{29 (Dual braided structure).} Let \( X \) be a braided strict monoidal category with braidings \( \{\sigma_{x,y}\} \). The family of 2-cells

\[ \sigma_{x,y}^*: \sigma_{y,x}^{-1} \]

defines a second structure \( X^* \) of braided strict monoidal category on the underlying strict monoidal category of \( X \). This extends to an involutive endofunctor \( ^* \) on \( \text{BrMonCat}_{\text{str}} \), which also induces a duality on \( \text{Prob} \).

\textbf{30 (Symmetric strict monoidal category).} A symmetric strict monoidal category is a braided strict monoidal category \( X \) satisfying \( X = X^* \).

\textbf{31 (Prop).} A \text{prop} is a prob whose underlying braided strict monoidal category is symmetric. We let \( \text{Prop} \) denote the full subcategory of \( \text{Prob} \) on props.

\textbf{Proposition 2.} The categories \( \text{BrMonCat}_{\text{str}}, \text{Prob}, \) and \( \text{Prop} \) have all small limits and colimits.

There is an obvious forgetful functor \( U: \text{Prop} \rightarrow \text{Pro} \) and an inclusion of subcategories \( \text{Prop} \hookrightarrow \text{Prob} \). Both of these have left adjoints:

- the left adjoint \( F: \text{Pro} \rightarrow \text{Prop} \) of \( U \) freely adds braidings \( \sigma_{x,y} \) and inverse braidings \( \sigma_{x,y}^{-1} \) for all pairs of 1-cells \( x, y \) of a prob (or just the generating ones, see Remark 3), then quotients by the axioms of braided strict monoidal categories;

- the reflector \( r: \text{Prop} \rightarrow \text{Prop} \) quotients by the equation \( \sigma_{x,y} = \sigma_{y,x}^* \) for all pairs of 1-cells \( x, y \) of a prob.

\textbf{Example 6.} The free prob \( \mathcal{B} := \mathcal{F}\mathbb{N} \) is the theory of braids. With 1-composition, 2-cells of type \( (n) \Rightarrow (m) \) in \( \mathcal{B} \) form the braid group \( B_n \) on \( n \) strands.

\textbf{Example 7.} The prop reflection \( \mathcal{S} := \mathcal{R}\mathcal{B} \) of the theory of braids is the \textit{theory of permutations}. With 1-composition, 2-cells of type \( (n) \Rightarrow (n) \) in \( \mathcal{S} \) form the symmetric group \( S_n \) on \( n \) elements.

\textbf{Example 8.} Let \( C\text{Mon} \) be defined as \( \text{Mon} \), but 2-cells of type \( (n) \Rightarrow (m) \) are all functions from \( (n) \) to \( (m) \). This is a one-sorted prop with braidings generated by \( \sigma_{1,1}: (2) \Rightarrow (2) \), \( 0 \mapsto 1, 1 \mapsto 0 \). It corresponds to the theory of \textit{commutative monoids} in symmetric monoidal categories. Similarly, models of \( C\text{Mon}^{\text{co}} \) are \textit{commutative comonoids}.

\textbf{32 (Gray-category).} A \textit{Gray-category} is a 3-precategory \( G \) together with a family of 3-cells

\[ \chi_{x,y}: (x \neq 0 \partial^{-} y) \# (x \# 0 \partial^{-} y) \Rightarrow (x \# 0 \partial^{-} y) \# (x \# 0 \partial^{-} y) \]

called \textit{interchangers}, indexed by 2-cells \( x, y \) with \( \partial_{0}^{+} x = \partial_{0}^{-} y \), satisfying the following axioms:

1) the interchangers are invertible, that is, there are unique 3-cells \( \chi_{x,y}^{-1} \) called \textit{inverse interchangers}, such that \( \chi_{x,y}^{-1} \) and \( \chi_{x,y}^{-1} \# 2 \chi_{x,y} \) are units;
2) the interchangers are natural in their parameters, that is, for all 3-cells \( \varphi : x \Rightarrow x' \) and \( \psi : y \Rightarrow y' \) with \( \partial_0^+ \varphi = \partial_0^+ \psi \),
\[
((\varphi \circ_0^0 \varphi) \circ_1^0 (\varphi \#_0^0 y)) \circ_2^0 \chi_{x',y} = \chi_{x,y} \circ_2^0 ((\varphi \circ_0^0 \varphi) \circ_1^0 (\varphi \#_0^0 y)),
\]
\[
((x \circ_0^0 \partial_1^y) \circ_1^y (x \#_0^0 y)) \circ_2^y \chi_{x,y'} = \chi_{x,y} \circ_2^y ((\partial_1^y x \#_0^0 y) \circ_1^y (x \#_0^0 y)).
\]
3) the interchangers are compatible with 1-compositions and units, that is, \( \chi_{x,y} \#_1^0 y \) and \( \chi_{x,y} \) are equal to
\[
((x \circ_0^0 \partial_1^y \#_0^0 y) \#_2^0 (x \#_0^0 y)),
\]
and symmetrically for \( \chi_{x,y} \#_1^0 y \) and \( \chi_{x,y} \);
4) for all pairs of 3-cells \( \varphi, \psi \) with \( \partial_1^+ \varphi = \partial_1^+ \psi \), we have
\[
((\varphi \#_1^0 \varphi \#_0^0 \varphi) \#_2^0 (\varphi \#_0^0 \varphi) \#_2 (\varphi \#_1^0 \varphi \#_0^0 \varphi)) = (\partial_1^y x \#_0^0 y).
\]
A functor \( f : G \to H \) of Gray-categories is a functor of the underlying 3-precategories that preserves the interchangers. With their functors, Gray-categories form a category \( \text{GrayCat} \).

A concise definition is that a Gray-category is a small category enriched over \( \text{2Cat} \) with the “pseudo” Gray tensor product [14, Chapter 5]. As in [17, §1.4], one derives that \( \text{GrayCat} \) is locally finitely presentable, and in particular has all small limits and colimits.

Every 3-category seen as a 3-precategory admits a natural structure of Gray-category with units as interchangers. This determines an embedding \( \text{3Cat} \to \text{GrayCat} \).

33. Given a braided strict monoidal category \( X \), we define a Gray-category \( BX \) as follows. For all \( n \in \mathbb{N} \), let \( BX_{n+1} := X_n \), with the same boundary and unit operators as \( X \) between \( BX_{n+2} \) and \( BX_{n+1} \). We let \( BX_0 := \{ \bullet \} \), with the only possible unit and boundary operators relating it to \( BX_1 \). This defines the underlying reflexive \( \omega \)-graph of \( B \).

To make \( BX \) a 3-precategory, since it has no rank-1 cells, it suffices to define compositions of the form \( x \cdot_k y \) where
\[
\bullet \quad k = 1 \quad \text{and} \quad \min \{ r(k, x), r(k, y) \} = 2, \quad \text{or}
\]
\[
\bullet \quad k = 2 \quad \text{and} \quad r(k, x) = r(k, y) = 3.
\]
In either case, \( x \cdot_{k-1} y \) is defined in \( X \), and we let \( x \cdot_k y \) be equal to it in \( BX \).

Finally, given 2-cells \( x, y \) in \( BX \), we let the interchanger \( \chi_{x,y} \) correspond to the braiding \( \sigma_{x,y} \) in \( X \). It is an exercise to check that this gives \( BX \) the structure of a Gray-category.

This extends to a functor \( B : \text{BrMonCat}_{str} \to \text{GrayCat} \) in the obvious way. By [18, Theorem 2.16], this functor is full and faithful, and its essential image consists exactly of those Gray-categories that have a single 0-cell and a single 1-cell. This is an alternative characterisation of \( \text{BrMonCat}_{str} \) as a full subcategory of \( \text{GrayCat} \).

34. If \( X \) is a braided strict monoidal category, the structure of a 1-polygraph on \( \sigma_{\leq 1} X \) determines a unique structure of 2-prepolygraph on \( \sigma_{\leq 2} BX \), and vice versa. A functor \( f \) of braided strict monoidal categories sends generators to generators if and only if \( B f \) does.

Thus, a prob is equivalently defined as a Gray-category \( T \) with a single 0-cell, a single 1-cell, and the structure of a 2-prepolygraph \( (\sigma_{\leq 2} T, \mathcal{F}) \) on its 2-skeleton. A morphism \( f : (T, \mathcal{F}) \to (S, \mathcal{P}) \) of probs is a functor of Gray-categories such that \( f(a) \in \{ x \bullet \} \cup \mathcal{P}_2 \) for all \( a \in \mathcal{P}_2 \).

We conclude that there is a triangle of functors
\[
\begin{array}{ccc}
U_2 & \text{Prob} & U_3 \\
\downarrow & \uparrow & \downarrow \\
\text{BrMonCat}_{str} & \text{GrayCat} &
\end{array}
\]
commuting up to natural isomorphism, where \( U_2 \) and \( U_3 \) are the forgetful functors associated to the two alternative definitions of prob.

The functor \( U_2 : \text{Prob} \to \text{BrMonCat}_{str} \) is pseudomonic: it is faithful and it reflects and is full on isomorphisms. This captures the fact that a 2-category admits at most one structure of bicoloured pro, a consequence of the general fact that an \( \omega \)-category admits at most one structure of polygraph [19, Section 4, Proposition 8]. Because the composite of a pseudomonic with a full and faithful functor is pseudomonic, \( U_3 : \text{Prob} \to \text{GrayCat} \) is also pseudomonic.

III. PROS AND DIAGRAMMATIC SETS

In this section we define adjunctions relating pros, diagrammatic sets, and Gray-categories. This requires a few combinatorial results about directed complexes.

35. Let \( U \) be a closed subset of a regular directed complex. For each \( n \geq -1 \), \( \mathbb{M}_n U \) is the bipartite directed graph whose vertices are the \( x \in U \) such that
\[
\dim(x) \leq n, \quad \text{or} \quad x \text{ is maximal and } \dim(x) > n,
\]
and there is an edge \( y \to x \) if and only if
\[
\dim(y) \leq n, \quad \dim(x) > n, \quad \text{and } y \in \partial_+^n x \setminus \partial_{n-1}^n x, \quad \text{or} \quad \dim(y) > n, \quad \dim(x) \leq n, \quad \text{and } x \in \partial_+^n y \setminus \partial_{n-1}^n y.
\]
The frame dimension \( \text{frdim}(U) \) of \( U \) is the integer
\[
\max \{ \dim(\{x\} \cap \{y\}) \mid x, y \text{ maximal in } U, x \neq y \}.
\]
36 (Frame acyclicity). A regular directed complex \( P \) is frame-acyclic if, for all molecules \( U \) in \( P \), if \( \text{frdim}(U) = k \), then \( \mathbb{M}_k U \) is acyclic.

Proposition 3. Let \( P \) be a frame-acyclic regular directed complex. Then \( (\text{MoP}, \{ \{x\} \} \in P) \) is a polygraph.

37 (k-Order). Let \( U \) be a regular \( n \)-molecule. For \( k < n \), a \( k \)-order on \( U \) is a linear ordering \( \{x_1, \ldots, x_m\} \) of the set
\[
\{x \in U \mid x \text{ is maximal and } \dim(x) > k\}
\]
with the property that, if there is a path from \( x_i \) to \( x_j \) in \( \mathbb{M}_{k+1} U \), then \( i \leq j \).
Lemma 1. Let $U$ be a frame-acyclic regular molecule, $k \geq \frdim(U)$, and let $(x_1, \ldots, x_m)$ be a k-order on $U$. There exist molecules $V_1, \ldots, V_m$, such that

$$U = V_1 \#_k \ldots \#_k V_m$$

and $x_i \in V_j$ if and only if $i = j$ for all $i, j \in \{1, \ldots, m\}$.

Theorem 1. Let $P$ be a regular directed complex with $\dim(P) \leq 3$. Then $P$ is frame-acyclic.

**38.** Given a regular directed complex $P$ and $n \in \mathbb{N}$, let $\sigma_{\leq n} P \subseteq P$ be the closed subset of elements $x \in P$ with $\dim(x) \leq n$. Then $\mathrm{Mol}(\sigma_{\leq n} P)^*$ and $\sigma_{\leq n} \mathrm{Mol} P^*$ are isomorphic $n$-categories.

By Proposition 3 combined with Theorem 1, for $n \leq 3$ the $n$-category $\sigma_{\leq n} \mathrm{Mol} P^*$ admits the structure of a polygraph with $\{\mathrm{cl}\{x\} \ | \ \dim(x) \leq n\}$ as generating cells. Because, in general, for an $\omega$-category $X$,

$$\sigma_{\leq k} X = \sigma_{\leq k}(\tau_{\leq k} X)$$

the 2-category $\tau_{\leq 2} \mathrm{Mol} P^*$ has the structure of a bicoloured pro with generators $\{\mathrm{cl}\{x\} \ | \ \dim(x) \leq 1\}$.

Moreover, if $\mathit{f}: P \to Q$ is a morphism in $\mathrm{DCpx}^\mathbb{R}$, then $\tau_{\leq 2} \mathrm{Mol} \mathit{f}^*$ sends each generator $\mathrm{cl}\{x\}$ to a generator $\mathrm{cl}\{\mathit{f}(x)\}$, so it is compatible with this structure. This defines a functor $P: \mathrm{DCpx}^\mathbb{R} \to \mathrm{Pro}_{bi}$ that fits into a commutative square

$$\begin{array}{ccc}
\mathrm{DCpx}^\mathbb{R} & \stackrel{P}{\longrightarrow} & \mathrm{Pro}_{bi} \\
\downarrow_{\mathrm{Mol}(-)^*} & & \downarrow_{\mathrm{U}} \\
\omega\mathrm{Cat} & \stackrel{\tau_{\leq 2}}{\longrightarrow} & 2\mathrm{Cat}.
\end{array}$$

Because $\mathrm{DCpx}^\mathbb{R}$ is small, $\emptyset\mathrm{Set}$ is locally small, and by Proposition 1 $\mathrm{Pro}_{bi}$ has all small colimits, by [20, Corollary 6.2.6] the left Kan extension of $P: \mathrm{DCpx}^\mathbb{R} \to \mathrm{Pro}_{bi}$ along the embedding $\mathrm{DCpx}^\mathbb{R} \hookrightarrow \emptyset\mathrm{Set}$ exists. This produces a functor $P: \emptyset\mathrm{Set} \to \mathrm{Pro}_{bi}$.

We may reason as in [13, Proposition 7.10] to show that $P: \mathrm{DCpx}^\mathbb{R} \to \mathrm{Pro}_{bi}$ preserves the colimits that are already in $\mathrm{DCpx}^\mathbb{R}$, and deduce from [Corollary 1.34, \textit{ibid.}] that the left Kan extension of $P$ along $\mathrm{DCpx}^\mathbb{R} \to \emptyset\mathrm{Set}$ is the left Kan extension of its restriction to $\emptyset$ along the Yoned embedding. For standard reasons, $P$ is left adjoint.

**39.** (Diagrammatic nerve of bicoloured pros). The diagrammatic nerve of bicoloured pros is the right adjoint

$$N: \mathrm{Pro}_{bi} \to \emptyset\mathrm{Set}$$

to the functor $P: \emptyset\mathrm{Set} \to \mathrm{Pro}_{bi}$.

**Proposition 4.** The functor $N$ is full and faithful.

**40.** Next, we construct a functor $G: \mathrm{DCpx}^\mathbb{R} \to \mathrm{GrayCat}$, different from the “obvious” one obtained by composing $\mathrm{Mol}(-)^*: \mathrm{DCpx}^\mathbb{R} \to \omega\mathrm{Cat}$ with $\tau_{\leq 3}: \omega\mathrm{Cat} \to 3\mathrm{Cat}$ and then including $3\mathrm{Cat}$ in $\mathrm{GrayCat}$.

Every regular directed complex is the colimit of the diagram of inclusions of its atoms [13, Corollary 1.34]. We impose that $G$ preserve these colimit diagrams. Then it suffices to define $G$ on atoms of increasing dimension. Let $\emptyset\mathrm{Cl}_{\leq n}$ be the full subcategory of $\emptyset$ on the atoms of dimension $\leq n$.

**41.** (\textit{G} in dimension $\leq 2$). On regular atoms of dimension $\leq 2$, we define $G$ to be $\mathrm{Mol}(-)^*: \emptyset\mathrm{Cl}_{\leq 2} \to 3\mathrm{Cat}$ followed by the embedding $3\mathrm{Cat} \hookrightarrow \mathrm{GrayCat}$. We extend $G$ along colimits to all regular directed complexes of dimension $\leq 2$.

**42.** Let $P$ be a 2-dimensional regular directed complex. Then $G(\sigma_{\leq 1} P)$ is equal to (the image under the embedding $3\mathrm{Cat} \hookrightarrow \mathrm{GrayCat}$ of) $\mathrm{Mol} \sigma_{\leq 1} P^*$ and has the structure of a 1-(pre)polygraph with the 1-atoms of $P$ as generators. Now, for all 2-atoms $x \in P$,

$$\begin{array}{ccc}
\partial O^2 & \longrightarrow & O^2 \\
\downarrow & & \downarrow \\
\mathrm{Mol}(\partial x) & \hookrightarrow & \mathrm{Mol}(\mathrm{cl}\{x\})
\end{array}$$

is a pushout both in $\omega\mathrm{PreCat}$ and $\mathrm{GrayCat}$. By the dual of the pullback lemma, the pushout of the span

$$\begin{array}{c}
\prod_{\dim(x) = n} \mathrm{Mol}(\partial x) \\
\prod_{\dim(x) = n} \mathrm{Mol}(\mathrm{cl}\{x\})
\end{array}$$

in $\omega\mathrm{PreCat}$ determines a 2-prepolygraph $(\mathbf{GP})_2$, while in $\mathrm{GrayCat}$ it is equivalent to the construction of $\mathbf{GP}$. The results of [21, Section 1.6] imply that

1) $(\mathbf{GP})_2$ is obtained from $(\mathbf{GP})_2$ by freely attaching some 3-cells (interchange generators) indexed by generating cells of $(\mathbf{GP})_2$, and imposing some equations of 3-cells, so in particular

2) $(\mathbf{GP})_2$ is the 2-skeleton of $\mathbf{GP}$.

In the terminology of Forest and Mimram, $P$ determines a presentation of the 2-precategory $(\mathbf{GP})_2$, which can be completed to a Gray presentation of $\mathbf{GP}$ by freely adding the necessary structural generators.

**Lemma 2.** Let $U$ be a regular 2-molecule. There is a bijective correspondence between

1) cells of rank 2 in $U$, and
2) 2-molecules $V \subseteq U$ together with a 1-order.

More in general, if $P$ is a 2-dimensional regular directed complex, a pair $(V, (x_i)_{i=1}^n)$ of a 2-molecule in $P$ and a 1-order on it determines a unique cell of rank 2 in $\mathbf{GP}$.

**43.** (Totally loop-free molecule). Given a regular molecule $U$, let $\mathcal{H} U$ be the directed graph obtained from $\mathcal{H} U$ by reversing all the edges labelled $-$.

We say that $U$ is \textit{totally loop-free} if $\mathcal{H} U$ is acyclic as a directed graph. If $U$ is totally loop-free, for all $x, y \in U$, we let $x \preceq y$ if and only if there is a path from $x$ to $y$ in $U$.

**Proposition 5.** Let $U$ be a regular molecule. If $\dim(U) \leq 2$, then $U$ is totally loop-free and $\preceq$ is a linear order on $U$, restricting to a 1-order and a 0-order on $U$.

**44.** (Normal 1-order). Let $U$ be a regular 2-molecule. The normal 1-order on $U$ is the 1-order obtained as the restriction of $\preceq$.
Proposition 6. Let $U$ and $V \subseteq U$ be regular 2-molecules and let $(x_i)_{i=1}^m$ and $(x'_i)_{i=1}^m$ be two 1-orders on $V$. Then in $GU$ there is a unique 3-cell from $(V,(x_i)_{i=1}^m)$ to $(V,(x'_i)_{i=1}^m)$.

The proof is based on the methods of [21] for showing that a Gray presentation is coherent; specifically the fact that we can impose a termination order on 2-cells $(V,(x_i)_{i=1}^m)$ based on how distant $(x_i)_{i=1}^m$ is from the normal 1-order.

45 (G in dimension 3). Let $U$ be a regular 3-atom. We define $GU$ to be the pushout

$$
\begin{array}{ccc}
\partial O^3 & \rightarrow & O^3 \\
\downarrow & & \downarrow \\
\partial f & \rightarrow & \partial f \\
G(\partial U) & \rightarrow & GU
\end{array}
$$

in $\text{GrayCat}$, where $f$ sends $2^\alpha$ to $(\partial^\alpha U, \text{normal 1-order})$ for each $\alpha \in \{+, -\}$.

Now every map $f: U \rightarrow V$ in $\mathcal{O}_3$ determines an assignment of generators of $GV$ to generators of $GU$ which is compatible with boundaries, hence extends uniquely to a functor $GF: GU \rightarrow GV$. This defines $G: \mathcal{O}_3 \rightarrow \text{GrayCat}$. We extend $G$ along colimits to all regular directed complexes of dimension $\leq 3$.

By construction, if $P$ is a regular directed complex of dimension 3, we can associate to each 3-atom $U$ of $P$ a 3-cell

$$[[U]]: (\partial^- U, \text{normal 1-order}) \Rightarrow (\partial^+ U, \text{normal 1-order})$$

in $GP$. We will extend this assignment to all 3-molecules.

46 (Substitution). Let $V$ and $W$ be regular n-molecules with spherical boundary, let $U$ be a regular n-molecule, and suppose $V \subseteq U$. Then $U \setminus (V \setminus \partial V)$ is a closed subset of $U$.

Suppose that $\partial^\alpha V$ is isomorphic to $\partial^\alpha W$ for all $\alpha \in \{+, -\}$. From [13, Lemma 2.2] we obtain a unique isomorphism $\iota: \partial U \iso \partial V$. We define $U[W/V]$ to be the pushout

$$
\begin{array}{ccc}
\partial V & \rightarrow & U \setminus (V \setminus \partial V) \\
\downarrow & & \downarrow \\
W & \rightarrow & U[W/V]
\end{array}
$$

in $\text{DCpx}^R$, and call it the substitution of $W$ for $V \subseteq U$. By [Proposition 2.4, ibid] this is an n-molecule with boundaries isomorphic to those of $U$, such that $W \subseteq U[W/V]$.

47. Suppose that $U$ contains a single 3-dimensional element $x$. Then $\partial^- x \subseteq \partial^- U$ for all $\alpha \in \{+, -\}$, the substitution $\partial^\alpha U[(\partial^\alpha x)/\partial^- x]$ is well-defined, and

$$\partial^- U[(\partial^- x)/\partial^- x] = \partial^+ U[(\partial^+ x)/\partial^+ x].$$

Pick a 1-order $(x_i)_{i=1}^m$ on $\partial^\alpha U[(\partial^\alpha x)/\partial^- x]$; then $c\{x_k\}$ is equal to $(\partial^\alpha x)$ for a unique $k \in \{1, \ldots, m\}$. Let $y_1, \ldots, y_p$ be the normal 1-order on $\partial^- x$ and let $z_1, \ldots, z_q$ be the normal 1-order on $\partial^+ x$. Then

$$
\begin{align*}
(x_i)_{i=1}^{m+p-1} &:= (x_1, \ldots, x_{k-1}, y_1, \ldots, y_p, x_{k+1}, \ldots, x_m), \\
(x_i)_{i=1}^{m+q-1} &:= (x_1, \ldots, x_{k-1}, z_1, \ldots, z_q, x_{k+1}, \ldots, x_m)
\end{align*}
$$

are 1-orders on $\partial^- U$ and $\partial^+ U$, respectively.

Now, substituting $[c\{x\}]$ for $c\{x_k\}$ in a decomposition of $\partial^\alpha U[(\partial^\alpha x)/\partial^- x]$ corresponding to the 1-order $(x_i)_{i=1}^m$ as by Lemma 1 yields a valid expression for a 3-cell

$$[c\{x\}]: (\partial^- U, (x_i)_{i=1}^{m+p-1}) \Rightarrow (\partial^+ U, (x_i)_{i=1}^{m+q-1})$$

in GP. By Proposition 6, there are unique 3-cells

$$\chi^-: (\partial^- U, \text{normal 1-order}) \Rightarrow (\partial^- U, (x_i)_{i=1}^{m+p-1}),$$

$$\chi^+: (\partial^+ U, (x_i)_{i=1}^{m+q-1}) \Rightarrow (\partial^+ U, \text{normal 1-order})$$

obtained as composites of interchangers and inverse interchangers, respectively. We define $[[U]]$ to be the composite $\chi^- \circ [c\{x\}] \circ \chi^+$. We can show that this is independent of the choice of 1-order made earlier.

48. Let $U$ be any regular 3-molecule in $P$ and fix a 2-order $(x_1, \ldots, x_m)$ on $U$. Then Lemma 1 combined with Theorem 1 gives a decomposition $U = V_1 \# \ldots \# \# V_m$, where $x_i$ is the only 3-dimensional element of $V_i$ for each $i \in \{1, \ldots, m\}$. We let $[[U]]$ in $GP$ be the composite $[[V_1]] \# \ldots \# \# [[V_m]]$ of the 3-cells

$$[[V_i]]: (\partial^- V_i, \text{normal 1-order}) \Rightarrow (\partial^+ V_i, \text{normal 1-order})$$

defined in §47. We can show that this is independent of the 2-order chosen on $U$.

49 (G in dimension ≥ 4). Let $U$ be a regular 4-atom. We define $GU$ to be the quotient of $G(\partial U)$ by the equation

$$[[\partial^- U]] = [[\partial^+ U]],$$

where the 3-molecules $\partial^\alpha U$ are interpreted in $G(\partial U)$ as by §48.

If $f: U \rightarrow V$ is a map in $\mathcal{O}_4$, its restriction to $\partial U$ determines a functor $G(\partial f): G(\partial U) \rightarrow GV$. If $U$ is a 4-atom, then either $\dim(f(U)) < 4$ and $f(\partial^- U) = f(\partial^+ U)$, or $\dim(f(U)) = 4$, and $f(U) = V$ and $f(\partial^- U) = \partial^+ V$ for each $\alpha \in \{+, -, 0\}$. In either case, $G(\partial f)$ is compatible with the equation $[[\partial^- U]] = [[\partial^+ U]]$, so it factors uniquely through a functor $GF: GU \rightarrow GV$.

This defines $G: \mathcal{O}_4 \rightarrow \text{GrayCat}$, and we extend it along colimits to all regular directed complexes of dimension $\leq 4$.

Finally, if $f: P \rightarrow Q$ is a map of regular directed complexes of arbitrary dimension, it restricts to a map $\sigma_{\leq 4}: \sigma_{\leq 4} P \rightarrow \sigma_{\leq 4} Q$, and we let $GF$ be equal to $G(\sigma_{\leq 4})$.

This defines $G: \text{DCpx}^R \rightarrow \text{GrayCat}$.

Remark 4. By construction $G$ ignores any elements of dimension > 4. The idea is that, while 4-cells can contribute non-trivial equations of 3-cells in a Gray-category, higher-dimensional cells can only contribute trivial "equations of equations" with no visible effect.

50. Because $\text{GrayCat}$ has all small colimits, we are in the conditions of [20, Corollary 6.2.6] and we can define a functor $G: \Omega \text{Set} \rightarrow \text{GrayCat}$.
as the left Kan extension of \( G : \text{DCpx}^R \to \text{GrayCat} \) along the embedding \( \text{DCpx}^R \hookrightarrow \mathcal{S}et \). Since we made sure at every step that \( G \) preserve the colimits in \( \text{DCpx}^R \), this is in fact the same as the left Kan extension of the restriction of \( G \) to \( \mathcal{S}et \) along the Yoneda embedding. For usual reasons, \( G \) is a left adjoint functor.

IV. THE SMASH PRODUCT

A. The tensor product of pros

We reconstruct the tensor product of pros, as defined by Hackney and Robertson, as a reflection of an “external” tensor product of pros producing a prob.

**Lemma 3.** Let \( s \) be a permutation on the set \( \{1, \ldots, n\} \). Then \( s \) is either the identity or admits a unique decomposition

\[
s = s_1 \cdots s_p
\]

with the following properties. For each \( i \in \{1, \ldots, p\} \), let \( s^{(i)} := s_1; \ldots; s_p \). Then

1) \( s_i \) is an element transposition \( (k \ k + 1) \) of two consecutive elements, and
2) \( k \) is the least element of \( \{1, \ldots, n\} \) such that \( s^{(i)}(k + 1) < s^{(i)}(k) \).

**51.** Let \( s \) be a permutation on the set \( \{1, \ldots, n\} \). For all \( 1 \)-cells \( (a_1, \ldots, a_n) \) in a prob \( (T, \mathcal{F}) \), we define an invertible 2-cell

\[
\sigma(s) : (a_1, \ldots, a_n) \Rightarrow (a_{s(1)}, \ldots, a_{s(n)})
\]

in \( T \); the dependence of \( \sigma(s) \) on \( (a_1, \ldots, a_n) \) is left implicit.

- If \( s \) is the identity, we let \( \sigma(s) \) be the unit on \( (a_1, \ldots, a_n) \).
- If \( s \) is an elementary transposition \( (k \ k + 1) \) of two consecutive elements, we let \( \sigma(s) \) be

\[
a_{k + 1} \notin \ldots \notin a_{k - 1} \notin a_k \in \sigma^{-1} \in \sigma(a_{k + 1} \in \sigma^{-1} \sigma(a_k \in \sigma^{-1} \in \sigma(a_{k + 2} \in \cdots \notin a_n).
\]

- In general, if \( s = s_1 \cdots s_p \) is the decomposition of \( s \) given by Lemma 3, we let

\[
\sigma(s) := \sigma(s_1) \#_1 \cdots \#_1 \sigma(s_p).
\]

We also define a second 2-cell of the same type by

\[
\sigma^{-1}(s) := \sigma(s)^{-1}.
\]

**52.** Let \( (T, \mathcal{F}) \) be a prob and let \( \{a_{i, j}\}_{1 \leq i \leq n, 1 \leq j \leq m} \) be a doubly indexed collection of \( 1 \)-cells in \( \mathcal{F} \). We denote by \( (a_{i,j})^{n}_{1 \leq j \leq 1 \cdots 1} \) the 1-cell

\[
(a_1, \ldots, a_{n+1}, a_1, a_2, \ldots, a_n, 1, m, \ldots, a_{m+n}, m)
\]

and by \( (a_{i, j})^{m}_{1 \leq i \leq 1 \cdots 1} \) the 1-cell

\[
(a_1, 1, a_1, m, a_2, 1, m, a_3, 1, m, \ldots, a_{n+1}, 1, m, a_{n+2}, 1, m, \ldots, a_{m+n}, 1, m)
\]

We let the 2-cells \( \sigma : ((a_{i, j})^{n}_{1 \leq j \leq 1 \cdots 1}) \Rightarrow ((a_{i, j})^{n}_{1 \leq j \leq 1 \cdots 1}) \) and \( \sigma : ((a_{i, j})^{m}_{1 \leq i \leq 1 \cdots 1}) \Rightarrow ((a_{i, j})^{m}_{1 \leq i \leq 1 \cdots 1}) \) be equal to \( \sigma(s^{-1}) \) and its inverse \( \sigma^*(s) \), respectively, for the permutation \( s \) implied by the reordering of (3) into (4).

**53** (Tensor product of pros). The tensor product \( (T, \mathcal{F}) \otimes (S, \mathcal{F}) \) of two pros \( (T, \mathcal{F}) \) and \( (S, \mathcal{F}) \) is the prob \( (T \otimes S, \mathcal{F} \otimes \mathcal{F}) \) constructed as follows.

Let \( (\mathcal{F} \otimes \mathcal{F})_0 := \{ \bullet \} \), \( (\mathcal{F} \otimes \mathcal{F})_1 := \{ a \otimes c \}_{a \in \mathcal{F}, c \in \mathcal{F}} \). This determines \( \sigma_{\leq 1} (T \otimes S) \) together with its 1-polygraph structure, which makes it a pro.

Construct the coproducts

\[
\bigoplus_{c \in \mathcal{F}} (T, \mathcal{F}), \bigoplus_{a \in \mathcal{F}} (S, \mathcal{F})
\]

in \( \text{Pro} \). Denote by \( \neg \otimes \neg \) the inclusion of \( (T, \mathcal{F}) \) into the \( d \)-indexed summand and by \( \neg \otimes \neg \) the inclusion of \( (S, \mathcal{F}) \) into the \( b \)-indexed summand. There are morphisms

\[
\sigma_{\leq 1} (T \otimes S) \to \bigoplus_{c \in \mathcal{F}} (T, \mathcal{F}), \quad \sigma_{\leq 1} (T \otimes S) \to \bigoplus_{a \in \mathcal{F}} (S, \mathcal{F})
\]

uniquely determined by the “tautologous” assignments \( a \otimes c \mapsto a \otimes c \). In \( \text{Pro} \), construct the pushout

\[
\bigoplus_{c \in \mathcal{F}} (T, \mathcal{F}) \downarrow \bigoplus_{a \in \mathcal{F}} (S, \mathcal{F}) \quad \bigoplus_{a \in \mathcal{F}} (T, \mathcal{F}) \downarrow \bigoplus_{c \in \mathcal{F}} (S, \mathcal{F})
\]

Finally, construct the free prob \( F(T \square S) \) and quotient it by the following equations: for all \( 2 \)-cells

\[
\varphi : (a_{1,1})^{n}_{1 \leq 1 \cdots 1} \Rightarrow (b_{j,1})^{m}_{1 \leq 1 \cdots 1} \text{ in } T, \quad \psi : (c_{k,1})^{p}_{1 \leq 1 \cdots 1} \Rightarrow (d_{1,1})^{q}_{1 \leq 1 \cdots 1} \text{ in } S,
\]

the 1-composite of

\[
(a_{1,1})^{n}_{1 \leq 1 \cdots 1} \Rightarrow \cdots \Rightarrow (a_{n,1})^{n}_{1 \leq 1 \cdots 1} \Rightarrow (a_n \otimes \psi)_{1 \leq 1 \cdots 1} \Rightarrow \cdots \Rightarrow (a_{1,1})^{n}_{1 \leq 1 \cdots 1}
\]

is equal to the 1-composite of

\[
(a_{1,1})^{n}_{1 \leq 1 \cdots 1} \Rightarrow \cdots \Rightarrow (a_{n,1})^{n}_{1 \leq 1 \cdots 1} \Rightarrow ((a_i \otimes d_i)_{1 \leq 1 \cdots 1})_{1 \leq 1 \cdots 1} \Rightarrow ((a_i \otimes d_i)_{1 \leq 1 \cdots 1})_{1 \leq 1 \cdots 1}
\]

We label this equation \( \varphi \otimes \psi \).

Note that any composite indexed by an empty list must be interpreted as a unit on \( a \) of the appropriate dimension.

If \( f : (T, \mathcal{F}) \to (T', \mathcal{F'}) \) and \( g : (S, \mathcal{F}) \to (S', \mathcal{F'}) \) are morphisms of pros, we can define morphisms

\[
\bigoplus_{c \in \mathcal{F}} (T, \mathcal{F}) \to \bigoplus_{c \in \mathcal{F}} (T', \mathcal{F'}), \quad x \otimes c \mapsto f(x) \otimes g(c),
\]

Taking the transpose morphisms in \( \text{Pro} \), and using the universal property of the pushout (6) which is preserved by \( F \), we obtain a unique morphism \( F(T \square S) \to (T' \square S', \mathcal{F}' \square \mathcal{F'}) \) of pros which is compatible with the \( \varphi \otimes \psi \) equations, hence factors uniquely through a morphism

\[
f \otimes g : (T \otimes S, \mathcal{F} \otimes \mathcal{F}) \to (T' \otimes S', \mathcal{F}' \otimes \mathcal{F'})
\]

This defines a functor \( \neg \otimes \neg : \text{Pro} \times \text{Pro} \to \text{Pro} \).

**Remark 5.** When either \( \varphi \) or \( \psi \) is a unit, the equation \( \varphi \otimes \psi \) holds automatically by the axioms of braiding. So \( \varphi \otimes \psi \) is only non-trivial when both cells have rank 2.
One can derive, as a consequence, that the monoid $\mathbb{N}$ is a “relative unit” for the tensor product, in the sense that the functors $\mathbb{N} \otimes -$ and $- \otimes \mathbb{N}$ are naturally isomorphic to $F : \text{Pro} \to \text{Prob}$.

Example 9. The tensor product $\text{Bialg} := \text{Mon} \otimes \text{Mon}^{\text{co}}$ is one variant of the braided monoidal theory of bialgebras; the other is $\text{Mon}^{\text{co}} \otimes \text{Mon}$. They have the same prop reflection, namely the symmetric monoidal theory of bialgebras [22].

Example 10. The tensor product $\text{BrCMon} := \text{Mon} \otimes \text{Mon}$ is the braided monoidal theory of commutative monoids. Its prop reflection is isomorphic to $\text{CMon}$.

54 (Tensor product of props). The tensor product $(T, \mathcal{T}) \otimes_S (S, \mathcal{S})$ of two props $(T, \mathcal{T})$ and $(S, \mathcal{S})$ is the quotient of $\tau(\text{U}(T, \mathcal{T}) \otimes \text{U}(S, \mathcal{S}))$ by the equations

\[ \sigma_{a,b} \otimes c = \sigma_{a \otimes b,c} ; \quad a \otimes \sigma_{c,d} = \sigma_{a \otimes c,a \otimes d} \quad (7) \]

for all $a, b \in \mathcal{T}_1$ and $c, d \in \mathcal{S}_1$, where $\sigma_{a,b}$ and $\sigma_{c,d}$ are the original bradings of $T$ and $S$.

As shown in [6, Section 3], the tensor product of props is part of a symmetric monoidal closed structure on $\text{Prop}$, whose unit is the theory of permutations $\Sigma$.

Example 11. Given a prop $(T, \mathcal{T})$, the tensor product $(T, \mathcal{T}) \otimes_S \text{Mon}^{\text{co}}$ is the free cartesian prop, also known as Lawvere theory, on $(T, \mathcal{T})$ [23].

Remark 6. The tensor product of props is compatible with the tensor product of pros in the sense that the diagram of functors

\[
\begin{array}{ccc}
\text{Pro} \times \text{Pro} & \longrightarrow & \text{Prob} \\
\tau F \times \tau F \downarrow & & \downarrow \\
\text{Prop} \times \text{Prop} & \longrightarrow & \otimes \\
\end{array}
\]

commutes up to natural isomorphism. The reason why this works is that, when $\varphi$ or $\psi$ is a braiding $\sigma_{a,b}$, the equation $\varphi \otimes \psi$ combined with (7) holds automatically in a prop. It follows that, while $\text{UrF}(T, \mathcal{T}) \otimes \text{UrF}(S, \mathcal{S})$ has additional generators and equations compared to $(T, \mathcal{T}) \otimes (S, \mathcal{S})$, these are all trivialised by the combined action of $\tau$ and (7).

Remark 7. As shown in [6, Proposition 40], the tensor product of props extends the Boardman-Vogt product of symmetric operads [24], in the sense that there is an embedding of the category of symmetric operads into the category of props which is strong monoidal with respect to the two monoidal structures.

Remark 8. The tensor product of props is not symmetric. However there is a natural isomorphism between $(T, \mathcal{T}) \otimes (S, \mathcal{S})$ and $(S, \mathcal{S}) \otimes (T, \mathcal{T})^\ast$, where $-^\ast$ is the duality defined in §29. From this we can recover a symmetry for the tensor product of props.

B. The smash product of pointed diagrammatic sets

55 (Gray product). Let $P, Q$ be regular directed complexes. The Gray product $P \otimes Q$ of $P$ and $Q$ is the cartesian product $P \times Q$ of their underlying posets with the following orientation. Write $x \otimes y$ for a generic element of $P \otimes Q$. For all $x' \text{ covered by } x$ in $P$ and all $y' \text{ covered by } y$ in $Q$,

\[ o(x \otimes y \to x' \otimes y') := o_P(x \to x'), \]

\[ o(x \otimes y \to x \otimes y') := (-)^{\text{dim}(x)} o_Q(y \to y'), \]

where $o_P$ and $o_Q$ are the orientations of $P$ and $Q$, respectively.

As shown in [13, Section 2.2], $P \otimes Q$ is a regular directed complex. If $f : P \to P'$ and $g : Q \to Q'$ are maps of regular directed complexes, let $f \otimes g : P \otimes Q \to P' \otimes Q'$ have the cartesian product of $f$ and $g$ as underlying function. Then $f \otimes g$ is a map of regular directed complexes.

Gray products determine a (non-symmetric) monoidal structure on $\text{DCpx}^R$ whose unit is the terminal object $1$.

56. The monoidal structure on $\text{DCpx}^R$ sends atoms to atoms, so it restricts to a monoidal structure on $\circ$, which, by Day’s theory [25], extends along the Yoneda embedding to a monoidal biclosed structure on $\mathbb{OSet}$. This is compatible with the embedding of $\text{DCpx}^R$ into $\mathbb{OSet}$.

Explicitly, let $X$ and $Y$ be diagrammatic sets. The Gray product $X \otimes Y$ of $X$ and $Y$ is the colimit in $\mathbb{OSet}$ of the diagram

\[
\begin{array}{ccc}
\mathbb{O}/X \times \mathbb{O}/Y & \overset{\text{dom} \times \text{dom}}{\longrightarrow} & \mathbb{O} \times \mathbb{O} \overset{\text{incl}}{\longrightarrow} \mathbb{O} & \longrightarrow & \mathbb{OSet}, \\
\end{array}
\]

where $\mathbb{O}/X$ is the comma category of the Yoneda embedding over the constant functor at $X$.

Remark 9. The dimensions of cells add under the Gray product, that is, if $x$ is an $n$-cell and $y$ is an $m$-cell, then $x \otimes y$ is an $(n + m)$-cell.

The Gray product is not the cartesian product in $\mathbb{OSet}$. However, the monoidal unit is the terminal object, which gives us “projection” morphisms $X \otimes Y \to X$ and $X \otimes Y \to Y$.

57 (Pointed diagrammatic set). A pointed diagrammatic set is a diagrammatic set $X$ together with a distinguished 0-cell $\bullet : 1 \to X$, the basepoint.

A morphism $f : (X, \cdot_X) \to (Y, \cdot_Y)$ of pointed diagrammatic sets is a morphism $f : X \to Y$ such that $f(\cdot_X) = \cdot_Y$. With their morphisms, pointed diagrammatic sets form a category $\mathbb{OSet}_\bullet$.

58 (Wedge sum). The wedge sum of two pointed diagrammatic sets $(X, \cdot_X)$ and $(Y, \cdot_Y)$ is the pointed diagrammatic set $(X \lor Y, \cdot)$ where

1) $X \lor Y$ is the quotient of $X + Y$ by the equation $\cdot_X = \cdot_Y$, and

2) $\cdot$ is the result of the identification of $\cdot_X$ and $\cdot_Y$.

59 (Smash product). Let $(X, \cdot_X)$ and $(Y, \cdot_Y)$ be pointed diagrammatic sets. There is an inclusion $X \lor Y \hookrightarrow X \otimes Y$ defined by $x \mapsto x \otimes \cdot_Y$, $y \mapsto \cdot_X \otimes y$ on cells in $X$ and $Y$, respectively.
The smash product of \((X \cdot X)\) and \((Y \cdot Y)\) is the pointed diagrammatic set \((X \otimes Y, \bullet)\) obtained from the pushout diagram:

\[
\begin{array}{ccc}
X \vee Y & \rightarrow & X \otimes Y \\
\downarrow & & \downarrow \\
1 & \rightarrow & X \otimes Y
\end{array}
\]

in \(\mathbb{G} \text{Set}\) (the “quotient of \(X \otimes Y\) by the subspace \(X \vee Y\)”).

The smash product is part of a biclosed monoidal structure on \(\mathbb{G} \text{Set}_\bullet\), whose unit is the diagrammatic set \(1 + 1\), pointed with one of the coproduct inclusions, and all structural isomorphisms are derived from those of the Gray product.

**Remark 10.** The smash product of pointed diagrammatic sets is a “directed” counterpart to the smash product of pointed topological spaces, with the Gray product playing the rôle of the cartesian product of spaces.

The formal correspondence is made concrete through the geometric realisation of diagrammatic sets [13, §8.38]. This functor \(\mathbb{G} \text{Set} \rightarrow \text{cgHaus}\) sends 0-cells in a diagrammatic set to points in a space, so it lifts to a functor \(\mathbb{G} \text{Set}_\bullet \rightarrow \text{cgHaus}_\bullet\) to the category of pointed compactly generated Hausdorff spaces and pointed maps.

This functor sends smash products in \(\mathbb{G} \text{Set}_\bullet\) to smash products in \(\text{cgHaus}_\bullet\), that is, it is strong monoidal with respect to the two monoidal structures.

**60.** Let \((T, \mathcal{F})\) be a pro. Its diagrammatic nerve \(\text{N}(T, \mathcal{F})\) has a single 0-cell, so it is canonically pointed, and the restriction of \(\text{N}\) to \(\text{Pro}\) lifts uniquely to a functor \(\text{N}: \text{Pro} \rightarrow \mathbb{G} \text{Set}_\bullet\).

We state our main theorem.

**Theorem 2.** The diagram of functors

\[
\begin{array}{ccc}
\text{Pro} \times \text{Pro} & \xrightarrow{- \otimes -} & \text{Prob} \\
\downarrow \text{N} \times \text{N} & & \downarrow \text{U}_3 \\
\mathbb{G} \text{Set}_\bullet \times \mathbb{G} \text{Set}_\bullet & \xrightarrow{- \otimes (-)^\circ} & \mathbb{G} \text{Set}_\bullet
\end{array}
\]

commutes up to natural isomorphism.

The proof is by carefully unfolding the definitions.

**Remark 11.** The form of Theorem 2 does not suggest, at first sight, that the smash product of pointed diagrammatic sets subsumes and generalises the tensor product of props. Nevertheless, we argue that this is the case essentially.

Since \(U_3\) is pseudomonic (§34), if \(\text{GX}\) is isomorphic to \(U_3(T, \mathcal{F})\) for some pro \((T, \mathcal{F})\), then this pro is essentially unique. On the image of \(\text{N} \otimes (\text{N}(-))^\circ\), then, we can lift \(\text{G}\) to a functor with codomain \(\text{Prob}\), and compute the tensor product of two pros through the lower leg of the diagram.

From the tensor product of pros, we can recover the tensor product of props via a universal characterisation in \(\text{Prop}\), independent of the specific construction.

**V. HIGHER-DIMENSIONAL CELLS**

The computational advantage of smash products is that they produce presentations with oriented equations, which furthermore contain higher-dimensional coherence data. We explore this advantage in one example.

**61 (Diagrammatic complex).** For each \(n \in \mathbb{N} + \{-1\}\), the restriction functor \(\mathbb{G} \text{Set} \rightarrow \text{Psh}(\mathcal{G}_n)\) has a left adjoint; let \(\sigma_{\leq n}\) be the comonad induced by this adjunction. The \(n\)-skeleton of a diagrammatic set \(X\) is the counit \(\sigma_{\leq n} X \rightarrow X\). For all \(k \leq n\), the \(k\)-skeletons uniquely through the \(n\)-skeleton of \(X\).

A **diagrammatic complex** is a diagrammatic set \(X\) together with a set \(\mathcal{X} = \sum_{n \in \mathbb{N}} \mathcal{X}_n\) of generating cells such that, for all \(n \in \mathbb{N}\),

\[
\prod_{x \in \mathcal{X}_n} \partial U(x) \leftrightarrow \prod_{x \in \mathcal{X}_n} U(x)
\]

\[
\downarrow
\]

\[
\sigma_{\leq n-1} X \leftrightarrow \sigma_{\leq n} X
\]

is a pushout in \(\mathbb{G} \text{Set}\), where \(U(x)\) denotes the shape of \(x\).

**Remark 12.** Let \((X, \mathcal{X})\) and \((Y, \mathcal{Y})\) be diagrammatic complexes. Then \(X \otimes Y\) is a diagrammatic complex with

\[
(\mathcal{X} \times \mathcal{Y}) := \sum_{k=0}^n \{x \otimes y \mid x \in \mathcal{X}_k, y \in \mathcal{Y}_{n-k}\}.
\]

As a consequence, the smash product \(X \otimes Y\) of pointed diagrammatic complexes \((X, \mathcal{X}, \bullet_X)\) and \((Y, \mathcal{Y}, \bullet_Y)\) is a pointed diagrammatic complex whose generating cells are \(\bullet\) and the pairs \(x \otimes y\) with \(x \in \mathcal{X} \setminus \{\bullet_X\}\) and \(y \in \mathcal{Y} \setminus \{\bullet_Y\}\).

**62 (Diagrammatic presentation).** Let \((T, \mathcal{F})\) be a bicoloured pro. A **presentation** of \((T, \mathcal{F})\) is a diagrammatic complex \((X, \mathcal{X})\) such that \(\text{PX}\) is isomorphic to \((T, \mathcal{F})\).

If \((T, \mathcal{F})\) is a prob, a **presentation** of \((T, \mathcal{F})\) is a diagrammatic complex \((X, \mathcal{X})\) such that \(\text{GX}\) is isomorphic to \(U_3(T, \mathcal{F})\).

For the purpose of computing a tensor product, we can replace the nerves of pros \((T, \mathcal{F})\), \((S, \mathcal{F})\) with a pair of presentations \((X, \mathcal{X}), (Y, \mathcal{Y})\) to obtain a presentation \(X \otimes Y\) of \((T, \mathcal{F})\) \(\otimes (S, \mathcal{F})\).

**Example 12.** The pro of monoids \(\text{Mon}\) admits the following presentation \((X, \mathcal{X})\). To begin, \(\mathcal{X}_0\) contains a single 0-cell \(\bullet\) and \(\mathcal{X}_1\) a single 1-cell \(\cdot\). Next, \(\mathcal{X}_2\) contains a 2-cell \(\mu: 1 \#_1 1 \Rightarrow 1\) and a 2-cell \(\eta: \varepsilon \cdot \Rightarrow 1\), which we picture in string diagrams [26] as

Finally, \(\mathcal{X}_3\) contains 3-cells \(\alpha, \lambda, \rho\) of the form

\[
\begin{array}{ccc}
\alpha & \alpha & \alpha \\
\downarrow & \downarrow & \downarrow \\
\lambda & \lambda & \lambda \\
\downarrow & \downarrow & \downarrow \\
\rho & \rho & \rho
\end{array}
\]
In our pictures, a dotted wire or dotless node indicates a degenerate cell of the form \( p; x \) for a surjective map \( p \) of atoms that strictly decreases dimension.

Now \((X^\circ, \mathcal{X}^\circ)\) is a presentation of \( \text{Mon}^\circ \), so the smash product \( X \otimes X \) is a presentation of the prop \( \text{Bialg} \) of bialgebras (Example 9). To simplify, we employ the following abuse of notation: we represent a 3-diagram \( x \) in \( X \otimes X \) as a 2-dimensional diagram in \( \text{Bialg} \) whose image through \( U_3 \) has the same composite as \( G_r \). This allows us to depict \( n \)-cells in \( X \otimes X \) as if they were \((n-1)\)-cells.

To begin, \( X \otimes X \) has a single generating 0-cell and no generating 1-cells. The only generating 2-cell is \( 1 \otimes 1 \).

The generating 3-cells are \( \mu \otimes 1, \eta \otimes 1, 1 \otimes \mu, \) and \( 1 \otimes \eta \).

With our abuse of notation, these are depicted as

\[
\begin{align*}
\begin{array}{c}
\mu_1 \\quad \eta_1 \quad 1 \mu \quad 1 \eta \\
\end{array}
\end{align*}
\]

There are 10 generating 4-cells which can be subdivided into three groups. Those of the form \( x \otimes 1 \) for \( x \in \mathcal{X}_3 \) have the same representation as \( x \), that is,

\[
\begin{align*}
\begin{array}{c}
\alpha_1 \quad \lambda_1 \quad \rho_1 \\
\end{array}
\end{align*}
\]

Those of the form \( 1 \otimes x \) for \( x \in \mathcal{X}_3 \) have the same representation as \( x^\otimes \):

\[
\begin{align*}
\begin{array}{c}
\lambda \quad \mu \quad \rho \\
\end{array}
\end{align*}
\]

Finally, those of the form \( x \otimes y \) for \( x, y \in \mathcal{X}_2 \) are

\[
\begin{align*}
\begin{array}{c}
\nu \quad \eta \\
\end{array}
\end{align*}
\]

This presentation of \( \text{Bialg} \) contains new critical branchings that do not correspond to critical branchings in the presentations of \( \text{Mon} \) or \( \text{Mon}^\circ \). For example, we have the following critical branching involving \( \alpha \otimes 1 \) and \( \mu \otimes \mu \):

\[
\begin{align*}
\begin{array}{c}
\alpha_1 \quad \mu \mu \\
\end{array}
\end{align*}
\]

There are 12 generating 5-cells of \( X \otimes X \), of the form \( x \otimes y \) where either \( x \in \mathcal{X}_3 \) and \( y \in \mathcal{X}_2 \) or \( x \in \mathcal{X}_2 \) and \( y \in \mathcal{X}_3 \). We observe that these are syzygies exhibiting confluence at these critical branchings. For example, \( \partial^-(\alpha \otimes \mu) \) is

\[
\begin{align*}
\begin{array}{c}
\alpha_1 \quad \mu \mu \\
\end{array}
\end{align*}
\]

while \( \partial^+(\alpha \otimes \mu) \) is

\[
\begin{align*}
\begin{array}{c}
\alpha_1 \quad \mu \mu \\
\end{array}
\end{align*}
\]

which exhibits confluence at the critical branching (11).

As another example, \( \partial^-(\gamma \otimes \lambda) \) and \( \partial^+(\gamma \otimes \lambda) \) are

\[
\begin{align*}
\begin{array}{c}
\eta \quad \lambda \\
\end{array}
\end{align*}
\]

respectively, so \( \gamma \otimes \lambda \) exhibits confluence at a critical branching involving \( \gamma \otimes \eta \) and \( 1 \otimes \lambda \).

These syzygies are oriented, so they can be interpreted as higher-dimensional rewrites creating critical branchings one dimension up. The 9 generating 6-cells of \( X \otimes X \), of the form \( x \otimes y \) where \( x \in \mathcal{X}_3 \) and \( y \in \mathcal{X}_3 \), are higher syzygies exhibiting confluence at these higher branchings.

**Conclusions and Outlook**

We have shown that the smash product of pointed diagrammatic sets extends the tensor product of props in a way that produces computationally relevant data. However, we have so far explored only a small corner of the wide higher-dimensional generalisation that we achieved.

First of all, we would like to better understand how the global computational properties of a smash product are affected by those of its factors. Is it possible to extract confluent, terminating, or coherent presentations of a tensor product from confluent, terminating, or coherent presentations of its factors?

Further still, we may want to leave behind the interpretation of diagrammatic complexes as presentations of pros or props and consider them directly as embodiments of higher-dimensional theories, such as homotopical algebraic theories, but not limited to invertible higher-dimensional data.

For example, we can add generating 4-cells corresponding to Mac Lane’s triangle and pentagon equations to our presentation of \( \text{Mon} \), then take the localisation at \( \{ \alpha, \lambda, \rho \} \) [13, §6.4], which universally weakly inverts these 3-cells, to obtain a presentation of the theory of pseudomonoids [27], whose models include small monoidal categories.

Diagrammatic sets with weak composites [13, §6.1], a model of weak higher categories, can function as semantic universes for such higher-dimensional theories. The biclosed structure on \( \odot \text{Set}_\bullet \) allows us to form spaces of models of theories in a pointed diagrammatic set. In future work we intend to explore this as a context for computationally-relevant higher-dimensional algebra.

We briefly mention other potential developments. Dorn, Reutter, and Vicary have defined a semistrict algebraic model of \( n \)-categories, called associative \( n \)-categories, which is equivalent to Gray-categories for \( n = 3 \) [28], [29]. It is conceivable that the adjunction of Section III relating diagrammatic sets to Gray-categories may generalise to associative \( n \)-categories for \( n > 3 \). We note, however, that our construction uses a property, frame acyclicity, which holds in general
up to dimension 3 but fails in dimension 4 or higher, so it is likely that new ideas will be needed.

The theory of diagrammatic sets is based on simple data structures: an atom $U$ can be encoded as the directed graph $\mathcal{H}^0 U$ of $\$43$ together with a grading of its vertices; the Gray product is then encoded as a cartesian product of directed graphs (with some edges reversed), while the degrees of vertices are summed. We expect that this setup should lend itself to computational formalisation.

This is of particular interest considering that the theory of associative $n$-categories is formalised in the graphical proof assistant homotopy.io: implementing the constructions of Section III would give us access to visualisations of Gray and smash products through this graphical frontend.

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