A New Bound for Kloosterman Sums

By Jason Fulman

Version of June 20, 2001

Stanford University (until 8/1/01)

US Airways (on 8/1/01)

[University of Pittsburgh (after 8/1/01)]

Department of Mathematics

Building 380, MC 2125

Stanford, CA 94305-2125

email: fulman@math.stanford.edu
Abstract

We give generating functions for Gauss sums for finite general linear and unitary groups. For the general linear case only our method of proof is new, but we deduce a bound on Kloosterman sums which is sometimes sharper than Deligne’s bound from algebraic geometry.

1 Introduction

The problem of bounding exponential sums such as the Kloosterman sum

$$K(c, d) = \sum_{x \in \mathbb{F}_p} e^{2\pi i (cx + d)}$$

is mathematically central, with applications to coding theory [Hu], spectral graph theory [T], and modular forms [Sa]. A main method for bounding such sums is to use deep results from algebraic geometry.

Let $\lambda$ be a nontrivial additive character for the finite field $\mathbb{F}_q$, and let $\mathbb{F}_q^*$ denote the non-zero elements of $\mathbb{F}_q$. Deligne [D] (see also the exposition [Se]) proves that for $x \in \mathbb{F}_q^*$

$$\left| \sum_{\alpha_1, \ldots, \alpha_{n-1} \in \mathbb{F}_q^*} \lambda(\alpha_1 + \cdots + \alpha_{n-1} + \frac{x}{\alpha_1 \cdots \alpha_{n-1}}) \right| \leq n \sqrt{q^{n/2}}.$$

As Section 2 indicates, it is a simple consequence of Fourier analysis that

$$\left| \sum_{\alpha_1, \ldots, \alpha_{n-1} \in \mathbb{F}_q^*} \lambda(\alpha_1 + \cdots + \alpha_{n-1} + \frac{x}{\alpha_1 \cdots \alpha_{n-1}}) \right| \leq (1 - \frac{1}{q-1})q^{n/2} + \frac{1}{q-1},$$

which is sometimes a stronger bound (though useless for the case $n = 2$).

The main point of this note is to discuss the relationship of the second bound with the finite classical groups. Section 2 derives a formula for the exponential sum

$$\sum_{g \in \text{GL}(n, q)} \chi(\det(g))\lambda(\text{tr}(g))$$

where $\chi$ is a multiplicative character of $\mathbb{F}_q^*$, $\lambda$ is an additive character of $\mathbb{F}_q$, $\det$ denotes determinant and $\text{tr}$ denotes trace. This result is known ([E], [Ki1], [Ko], [La]), but the proof given here uses the cycle index generating function of $GL(n, q)$ which has the attractive point of involving “Euler products” over irreducible polynomials.

Section 2 then derives the second bound on Kloosterman sums two paragraphs back by studying characteristic polynomials of random elements of $SL(n, q)$. Although the derivation of the bound
does not require the study of characteristic polynomials of random matrices over finite fields, such a
derivation is of interest given that Deligne’s work is related to characteristic polynomials of random
matrices from compact Lie groups [Ka], [KaS]. It would be very interesting to go directly from finite
classical groups to compact Lie groups; p-adic groups may give the bridge.

We note that the link between Kloosterman sums and finite classical groups underlying the
second bound is due to Kim [Ki1], who used the Bruhat decomposition of $SL(n, q)$ with respect
to a maximal parabolic subgroup. Thus Section 2 gives a different explanation of this link, using
conjugacy classes (and Fourier analysis) rather than a Bruhat decomposition. Section 3 closes by
applying our method to find a simple generating function for the Kloosterman sum

$$\sum_{g \in GL(n, q)} \lambda(xtr(g) + ytr(g^{-1}))$$

which was evaluated by Kim [Ki4]. Section 3 derives a generating function for the exponential sum

$$\sum_{g \in U(n, q) \subseteq GL(n, q^2)} \chi(det(g))\lambda(tr(g))$$

where $\chi$ is a multiplicative character of $F_{q^2}^*$, and $\lambda$ is a nontrivial additive character of $F_{q^2}$. Kim
[Ki2], [Ki3] found quite different formulas for this exponential sum as involved sums of Kloosterman
sums.

As a final remark, our motivation for trying to bound Kloosterman sums by studying character-
istic polynomials of random matrices arose from card shuffling [F3], [F4]. Two methods of shuffling,
“affine shuffles” and “shuffles followed by cuts” lead to exceptionally close distributions on permu-
tations, exactly equal up to lower order terms involving Ramanujan sums (a type of exponential
sum). The cycle type of a permutation after a $q$ affine shuffle on $S_n$ has the same distribution as
the cycle type of a random degree $n$ monic polynomial over $F_q$ with constant term 1 and the cycle
type of a permutation after a $q$ riffle shuffle followed by a cut has the same distribution as the cycle
type of a random degree $n$ polynomial over $F_q$ with non-zero constant term. It would be interesting
to find physically natural shuffles whose distribution on cycle types agrees with that arising from
characteristic polynomials of random matrices.

2 Gauss Sums for $GL(n, q)$

Let $\chi$ be a multiplicative character of $F_q^*$ and let $\lambda$ be an additive character of $F_q$. Given a
polynomial $P = x^n + c_{n-1}x^{n-1} + \cdots + c_0$ over $F_q$ we define $\chi(P) = \chi((-1)^nc_0)$ for $c_0 \neq 0$ and
\( \chi(P) = 0 \) if \( c_0 = 0 \). We define \( \lambda(P) \) as \( \lambda(-c_{n-1}) \). Note that \( \chi(P_1 P_2) = \chi(P_1) \chi(P_2) \) and that \( \lambda(P_1 P_2) = \lambda(P_1) \lambda(P_2) \). Also \( G(\chi, \lambda) \) denotes the Gauss sum

\[
\sum_{x \in F_q^*} \chi(x) \lambda(x).
\]

Before proceeding we recall an elementary and well-known lemma. In what follows \( \phi(z) \) denotes a monic non-constant irreducible polynomial over \( F_q \). For completeness we include a proof for the case that \( \lambda \) is non-trivial.

**Lemma 1**

\[
\prod_{\phi \neq z} \left( \frac{1}{1 - \chi(\phi) \lambda(\phi) u^{\deg(\phi)} / q^{\deg(\phi)}} \right) = \begin{cases} 
\frac{1-u/q^i}{1-u/q^{i-1}} & \text{if } \lambda, \chi \text{ trivial} \\
1 & \text{if } \lambda \text{ trivial, } \chi \text{ nontrivial} \\
1 + u G(\chi, \lambda)/q^i & \text{if } \lambda \text{ nontrivial}
\end{cases}
\]

**Proof:** Let \( P \) denote a monic polynomial (not necessarily irreducible) over \( F_q \). The left hand side is equal to

\[
\sum_{P: P(0) \neq 0} u^{\deg(P)} \chi(P) \lambda(P) / q^{\deg(P)}.
\]

The coefficient of \( u^n \) (\( n \geq 2 \)) vanishes because given \( c_0 \) the distribution of \( c_{n-1} \) for a random monic degree \( n \) polynomial is uniform over all \( q \) possible values. \( \Box \)

The first part of Theorem 1 is known ([E], [Ko], [Ki1], [La]) but the method of proof we give is new. The second part of Theorem 1 is simply a generating function version of the first part.

**Theorem 1**

1.

\[
1 + \sum_{n=1}^{\infty} \frac{u^n}{|GL(n, q)|} \sum_{g \in GL(n, q)} \chi(\det(g)) \lambda(\text{tr}(g)) = \begin{cases} 
\frac{1}{1-u} & \text{if } \lambda, \chi \text{ trivial} \\
1 & \lambda \text{ trivial, } \chi \text{ nontriv.} \\
\prod_{i \geq 1} (1 + \frac{u G(\chi, \lambda)}{q^i}) & \text{if } \lambda \text{ nontrivial}
\end{cases}
\]

2. For \( n \geq 1 \),

\[
\sum_{g \in GL(n, q)} \chi(\det(g)) \lambda(\text{tr}(g)) = \begin{cases} 
|GL(n, q)| & \text{if } \lambda, \chi \text{ trivial} \\
0 & \text{if } \lambda \text{ trivial, } \chi \text{ nontrivial} \\
q^{\binom{n}{2}} G(\chi, \lambda)^n & \text{if } \lambda \text{ nontrivial}
\end{cases}
\]

**Proof:** To prove Theorem 1 we recall some work of Stong [St] on the cycle index of \( GL(n, q) \) (a survey of applications of cycle indices of the finite classical groups can be found in [F2]). As the
textbook [He] explains in the section on rational canonical forms, the conjugacy classes of $GL(n, q)$
correspond to the following combinatorial data: to each monic non-constant irreducible polynomial
$\phi$ over $F_q$, associate a partition (perhaps the trivial partition) $\nu_\phi$ of a non-negative integer $|\nu_\phi|$.
We write $\nu \vdash j$ if $\nu$ is a partition of the integer $j$. The only restrictions necessary for this data to
represent a conjugacy class are

1. $|\nu_z| = 0$
2. $\sum_\phi |\nu_\phi| \text{deg}(\phi) = n.$

The size of the conjugacy class corresponding to the data $\nu_\phi$ is equal to
$$|GL(n, q)| \prod_\phi c_\phi(\nu_\phi),$$
where $c_\phi(\nu_\phi)$ is a function of $\nu$ and $\phi$ which depends on $\phi$ only through the degree of $\phi$. Define $a_{\phi, \nu}(g)$ to be one
if $\nu$ is the partition corresponding to $\phi$ in the rational canonical form of $g$ and to be zero otherwise.

It follows that

$$1 + \sum_{n=1}^{\infty} \frac{u^n}{|GL(n, q)|} \sum_{g \in GL(n, q)} \chi(\text{det}(g)) \lambda(\text{tr}(g)) = \prod_{\phi \neq z} \left( 1 + \sum_{j=1}^{\infty} \sum_{\nu \vdash j} \frac{(\chi(\phi) \lambda(\phi) u^{\text{deg}(\phi)})^j}{c_\phi(\nu_\phi)} \right).$$

We now use the fact (derived in [St]) that

$$1 + \sum_{j=1}^{\infty} \sum_{\nu \vdash j} \frac{(u)^j {\nu}^\text{deg}(\phi)}{c_\phi(\nu_\phi)} = \prod_{i \geq 1} \left( 1 + \frac{u^{\text{deg}(\phi)}}{q^i \cdot \text{deg}(\phi)} \right).$$

Other elementary derivations can be found in [F2]. (There are similar factorizations for all irre-
ducible polynomials for all finite classical groups).

Consequently

$$1 + \sum_{n=1}^{\infty} \frac{u^n}{|GL(n, q)|} \sum_{g \in GL(n, q)} \chi(\text{det}(g)) \lambda(\text{tr}(g)) = \prod_{\phi \neq z} \prod_{i \geq 1} \left( 1 + \frac{u^{\text{deg}(\phi)}}{q^i \cdot \text{deg}(\phi)} \right).$$
Part 1 of Theorem \ref{thm_part1} now follows from Lemma \ref{lem_part2}. Part 2 follows from Part 1 using Euler’s identity

\[
\prod_{i \geq 1} (1 + \frac{u}{q^i}) = \sum_{n=0}^{\infty} \frac{u^n}{(q^n - 1) \cdots (q - 1)}.
\]

\[\square\]

Next we consider the Kloosterman sum

\[f_\lambda(x) = \sum_{\alpha_1, \cdots, \alpha_{n-1} \in F_q^*} \lambda(\alpha_1 + \cdots + \alpha_{n-1} + \frac{x}{\alpha_1 \cdots \alpha_{n-1}})\]

where \(x \in F_q^*\). From pages 46-7 of \cite{Ka}, the Fourier transform of the function \(f_\lambda\) at the multiplicative character \(\chi\) is equal to \(G(\lambda, \chi)^n\). Fourier inversion implies that

\[f_\lambda(x) = \frac{1}{q-1} \sum_{\chi} \bar{\chi}(x) G(\lambda, \chi)^n.\]

Since \(|G(\lambda, \chi)| = q^{n/2}\) for \(\lambda, \chi\) non-trivial \cite{LiN}, it follows that

\[
\left| \sum_{\alpha_1, \cdots, \alpha_{n-1} \in F_q^*} \lambda(\alpha_1 + \cdots + \alpha_{n-1} + \frac{x}{\alpha_1 \cdots \alpha_{n-1}}) \right| \leq (1 - \frac{1}{q-1}) q^{n/2} + \frac{1}{q-1}.
\]

This proves the bound stated in the introduction; furthermore the technique clearly works for the other sums listed on page 47 of \cite{Ka}.

Corollary \ref{corollary_1} derives the same bound by studying the trace of characteristic polynomials of elements of \(GL(n, q)\) with determinant \(x\). (The first version of this paper only stated the bound of Corollary \ref{corollary_1} for \(x = 1\); Robin Chapman asked us if the proof could be extended for \(x \neq 1\)).

**Corollary 1** Let \(\lambda\) be a non-trivial additive character of \(F_q\). Then for \(x \in F_q^*\),

\[
\left| \sum_{\alpha_1, \cdots, \alpha_{n-1} \in F_q^*} \lambda(\alpha_1 + \cdots + \alpha_{n-1} + \frac{x}{\alpha_1 \cdots \alpha_{n-1}}) \right| \leq (1 - \frac{1}{q-1}) q^{n/2} + \frac{1}{q-1}.
\]

**Proof:** The paper \cite{Ki1} proves (see the remark on page 303) by very elementary means (using only the Bruhat decomposition of \(SL(n, q)\) with respect to a parabolic subgroup) that

\[
q^{\binom{n}{2}} \sum_{\alpha_1, \cdots, \alpha_{n-1} \in F_q^*} \lambda(\alpha_1 + \cdots + \alpha_{n-1} + \frac{x}{\alpha_1 \cdots \alpha_{n-1}}) = \sum_{g \in GL(n, q) \atop \det(g) = x} \lambda(tr(g)).
\]

Clearly

\[
\sum_{g \in GL(n, q) \atop \det(g) = x} \lambda(tr(g)) = \frac{1}{q-1} \sum_{\chi} \bar{\chi}(x) \sum_{g \in GL(n, q)} \chi(\det(g)) \lambda(tr(g))
\]
where the sum is over all multiplicative characters of $F_q^*$. From Theorem 1 and the equation $|G(\chi, \lambda)| = q^{1/2}$ for non-trivial $\chi, \lambda$ it follows that

$$\frac{1}{q^{(n)}} \left| 1 - \sum_{\chi} \bar{\chi}(x) \sum_{g \in \text{GL}(n,q)} \chi(\text{deg}(g))\lambda(\text{tr}(g)) \right| \leq \frac{q-2}{q-1}q^{n/2} + \frac{1}{q-1}.$$

Note that the second term on the right-hand side arises from $\chi$ trivial. $\square$

**Remark:** We observe that Fourier analysis combined with Theorem 1 gives a proof of Kim’s relation

$$q^{(n)} \sum_{\alpha_1, \cdots, \alpha_{n-1} \in F_q^*} \lambda(\alpha_1 + \cdots + \alpha_{n-1} + \frac{x}{\alpha_1 \cdots \alpha_{n-1}}) = \sum_{g \in \text{GL}(n,q)} \lambda(\text{tr}(g))$$

which avoids the Bruhat decomposition of $SL(n, q)$ with respect to a parabolic subgroup. Indeed, Fourier analysis gives that the left-hand side is equal to

$$\frac{1}{q-1} q^{(n)} \sum_{\chi} \bar{\chi}(x) G(\lambda, \chi)^n.$$

Theorem 1 implies that this is equal to

$$\sum_{g \in \text{GL}(n,q) \atop \det(g) = x} \lambda(\text{tr}(g)).$$

To close this section we find a generating function for

$$\sum_{g \in \text{GL}(n,q)} \lambda(x\text{tr}(g) + y\text{tr}(g^{-1}))$$

with $x, y \neq 0$. The sum was evaluated by Kim (page 64 of [Ki4]) using induction and the Bruhat decomposition.

Let $K_\lambda(x, y)$ with $x, y \neq 0$ denote the Kloosterman sum $\sum_{\alpha \in F_q^*} \lambda(x\alpha + \frac{y}{\alpha})$. Define $\tau(P) = \lambda(x\text{tr}(P) + y\text{tr}(P^{-1}))$ where $\text{tr}(P)$ is the sum of the roots of $P$ and $\text{tr}(P^{-1})$ the sum of the reciprocals of the roots of $P$. To be explicit, given $P = x^n + c_{n-1}x^{n-1} + \cdots + c_0$, define $\tau(P)$ to be $\lambda(-xc_{n-1} - yc_1/c_0)$ for $\text{deg}(P) \geq 2$ and to be $\lambda(-xc_0 - y/c_0)$ for $\text{deg}(P) = 1$. Note that $\tau(P_1P_2) = \tau(P_1)\tau(P_2)$.

**Lemma 2** For $x, y \neq 0$,

$$\prod_{\phi \neq z} \left( 1 - \frac{1}{1 - \tau(\phi)u^{\text{deg}(\phi)} / q^{\text{deg}(\phi)}} \right) = 1 + \frac{uK_\lambda(x, y)}{q^i} + \frac{qu^2}{q^{2i}}.$$
Proof: The left hand side is equal to
\[ \sum_{P : P(0) \neq 0} \frac{u^{\deg(P)} \tau(P)}{q^{i \cdot \deg(\phi)}}. \]

The terms corresponding to \( \deg(P) \geq 3 \) all vanish. The degree 1 term is equal to \( \frac{uK_\lambda(x,y)}{q^i} \). The degree 2 term is equal to
\[ \frac{u^2}{q^{2i}} \sum_{c_0 \in F_q^*} \sum_{c_1 \in F_q} \lambda(-xc_1)\lambda(-yc_1/c_0) \]
which simplifies to \( \frac{wu^2}{q^{2i}} \). \( \square \)

Given Lemma 2, Theorem 2 is proved exactly as Theorem 1.

Theorem 2
For \( \lambda \) a non-trivial additive character of \( F_q \), and \( x, y \) nonzero elements of \( F_q \),
\[ 1 + \sum_{n=1}^{\infty} \frac{u^n}{|GL(n,q)|} \sum_{g \in GL(n,q)} \lambda(x \text{tr}(g) + y \text{tr}(g^{-1})) = \prod_{i \geq 1} \left( 1 + \frac{uK_\lambda(x, y)}{q^i} + \frac{wu^2}{q^{2i}} \right). \]

3 Gauss Sums for \( U(n, q) \)

This section derives a generating function for
\[ \sum_{g \in U(n, q) \subseteq GL(n, q^2)} \chi(\text{det}(g))\lambda(\text{tr}(g)) \]
where \( \chi \) is a multiplicative character of \( F_{q^2}^* \) and \( \lambda \) is a nontrivial additive character of \( F_{q^2} \). It is convenient to set \( \chi(0) = 0 \).

We use the notation that
\[ G_1(\chi, \lambda) = \sum_{\alpha \in F_{q^2}^* \at \alpha^{q+1} = 1} \chi(-\alpha)\lambda(-\alpha) \]
and that
\[ G_2(\chi, \lambda) = \sum_{\alpha \in F_{q^2}^* \at \alpha^{q+1} = 1} \chi(\alpha) \sum_{\beta \in F_{q^2} : \beta^{q-1} = \alpha^q} \lambda(-\beta). \]
(This notation differs from the notation in the first version of the paper but the results are the same).

We also use an involution which maps a polynomial
\[ \phi(z) = z^m + \alpha_{m-1}z^{m-1} + \cdots + \alpha_1 z + \alpha_0 \]
with $\alpha_0 \neq 0$ to

$$\tilde{\phi}(z) = z^m + (\alpha_1/\alpha_0)qz^{m-1} + (\alpha_2/\alpha_0)q^2z^{m-2} + \cdots + (\alpha_{m-1}/\alpha_0)q^{m-1}z + (1/\alpha_0)^q.$$  

Note that $\phi_1\tilde{\phi}_2 = \tilde{\phi}_1\phi_2$. The total number of monic degree $m$ polynomials with coefficients in $F_{q^2}$ invariant under $\tilde{\phi}$ is $q^m + q^{m-1}$ \cite{W}. To see this note that for $m$ odd the coefficients $\alpha_0, \ldots, \alpha_{(m-1)/2}$ determine an invariant polynomial and that $\alpha_0$ must satisfy $\alpha_0^{q+1} = 1$ but that $\alpha_1, \ldots, \alpha_{(m-1)/2}$ can be any elements of $F_{q^2}$. The case of $m$ even is similar. All irreducible $\phi$ invariant under this involution have odd degree (see \cite{F1} or \cite{NP} for a proof) and the number of such polynomials is computed in \cite{F1}.

The paper \cite{F1} develops analogs of the cycle index for the finite classical groups, based on Wall’s work on its conjugacy classes \cite{W}. For the case of the unitary groups, the conjugacy classes correspond to the following combinatorial data. As was the case with $GL(n, q^2)$, an element $g$ in $U(n, q)$ associates to each monic, non-constant, irreducible polynomial $\phi$ over $F_{q^2}$ a partition $\nu_\phi$ of some non-negative integer $|\nu_\phi|$ by means of the rational canonical form. This data represents a conjugacy class if and only if (1) $|\nu_z| = 0$, (2) $\nu_\phi = \nu_{\tilde{\phi}}$, and (3) $\sum_\phi |\nu_\phi| \deg(\phi) = n$.

Note that in the statement of Lemma 3 the second product is over (unordered) pairs of distinct monic irreducible polynomials which map to each other under the involution $\tilde{\phi}$.

**Lemma 3** Let $\chi$ be a multiplicative character of $F_{q^2}^*$ and let $\lambda$ be a non-trivial additive character of $F_{q^2}$. Then

1.  

$$\prod_{\phi \neq \tilde{\phi}} \left(1 - \chi(\phi)\lambda(\phi)u^{\deg(\phi)/q^{i\deg(\phi)}}\right) \prod_{\phi \neq \tilde{\phi}} \left(1 - \chi(\phi)\lambda(\phi)\lambda(\phi)u^{2\deg(\phi)/q^{2i\deg(\phi)}}\right) = 1 + uG_1(\chi, \lambda)/q^i + u^2G_2(\chi, \lambda)/q^{2i}.$$  

2.  

$$\prod_{\phi \neq \tilde{\phi}} \left(1 + \chi(\phi)\lambda(\phi)u^{\deg(\phi)/q^{i\deg(\phi)}}\right) \prod_{\phi \neq \tilde{\phi}} \left(1 - \chi(\phi)\lambda(\phi)\lambda(\phi)u^{2\deg(\phi)/q^{2i\deg(\phi)}}\right) = 1 - uG_1(\chi, \lambda)/q^i + u^2G_2(\chi, \lambda)/q^{2i}.$$  

**Proof:** Letting $P = z^m + \alpha_{m-1}z^{m-1} + \cdots + \alpha_1z + \alpha_0$ denote a monic polynomial with coefficients in $F_{q^2}$, the left hand side is equal to
\[
\sum_{P \cdot 0 = P} \frac{\chi(P) \lambda(P) u^{\deg(P)}}{q^{i \cdot \deg(P)}}.
\]

Observe that for \( d > 2 \), the expression
\[
\sum_{P \cdot 0 = P} \frac{\chi(P) \lambda(P) u^{d}}{q^{i \cdot \deg(P)}}
\]
vanishes because (from the explicit description of invariant \( P \)) \( \alpha_{d-1} \) is equidistributed over all elements of \( F_{q^2} \) given the value of \( \alpha_0 \). The computations for \( \deg(P) = 1 \) and \( \deg(P) = 2 \) are straightforward.

The second assertion follows from the first assertion by replacing \( u \) by \(-u\) and using the fact that all irreducible polynomials invariant under \( \sim \) have odd degree. \( \square \)

**Theorem 3** Let \( \chi \) be a multiplicative character of \( F_{q^2}^* \) and let \( \lambda \) be a non-trivial additive character of \( F_{q^2} \). Then

\[
1 + \sum_{n=1}^{\infty} \frac{u^n}{|U(n, q)|} \sum_{g \in U(n, q) \subseteq GL(n, q^2)} \chi(\det(g)) \lambda(tr(g)) = \prod_{i \geq 1} \left( 1 + (-1)^{i+1} u G_1(\chi, \lambda)/q^i + u^2 G_2(\chi, \lambda)/q^{2i} \right).
\]

**Proof:** Arguing as in [F1] and using the fact that Gauss sums are multiplicative on polynomials, it follows that

\[
1 + \sum_{n=1}^{\infty} \frac{u^n}{|U(n, q)|} \sum_{g \in U(n, q)} \chi(\det(g)) \lambda(tr(g))
\]
is equal to

\[
\prod_{i \geq 1} \prod_{\phi \neq \tilde{\phi}} \left( \frac{1}{1 - (-1)^{i+1} \chi(\phi) \lambda(\phi) u^{\deg(\phi)}/q^{i \cdot \deg(\phi)}} \right) \prod_{\phi \neq \tilde{\phi}} \left( \frac{1}{1 - \chi(\phi) \lambda(\phi) u^2 \deg(\phi)/q^{2i \cdot \deg(\phi)}} \right).
\]

This product can be broken down into terms according to whether \( i \) is even or odd and the result follows from Lemma 3. \( \square \)

Note that if one factors the right hand side of the generating function of Theorem 3 into linear factors in \( u \) (which is certainly possible in odd characteristic), then the right hand side can be expanded using Euler’s identity

\[
\prod_{i \geq 1} \left( 1 - \frac{u}{q^i} \right) = \sum_{n=0}^{\infty} \frac{(-u)^n}{(q^n - 1) \cdots (q - 1)}.
\]
4 Acknowledgements

This research was supported by an NSF Postdoctoral Fellowship. The author thanks Robin Champan for helpful correspondence.

References

[D] Deligne, P., Applications de la formule des traces aux sommes trigonométriques, SGA 4 1/2, Lecture Notes in Math 569, Springer-Verlag, New York, 1978.

[E] Eichler, M., Allgemeine Kongruenz-Klasseneinteilungen der Ideale einfacher Algebren über algebraischen Zahlkörpern und ihre L-Reihen. *J. Reine Angew. Math.* 179 (1937), 227-251.

[F1] Fulman, J., Cycle indices for the finite classical groups, *J. Group Theory* 2 (1999), 251-289.

[F2] Fulman, J., Random matrix theory over finite fields, *Bull. Amer. Math Soc.*, to appear.

[F3] Fulman, J., Affine shuffles, shuffles with cuts, the Whitehouse module, and patience sorting, *J. Algebra* 231 (2000), 614-639.

[F4] Fulman, J., Applications of the Brauer complex: card shuffling, permutation statistics, and dynamical systems, *J. Algebra*, to appear. Available at [http://xxx.lanl.gov/abs/math.CO/0102105](http://xxx.lanl.gov/abs/math.CO/0102105).

[He] Herstein, I., Topics in algebra. Xerox College Publishing, Lexington, Mass.-Toronto, Ont., 1975.

[Hu] Hurt, N., Exponential sums and coding theory: a review, *Acta Appl. Math.* 46 (1997), 49-91.

[Ka] Katz, N., Gauss sums, Kloosterman sums, and monodromy groups. Annals of Mathematics Studies 116. Princeton University Press, Princeton, N.J., 1988.

[KaS] Katz. N. and Sarnak. P., Random matrices, Frobenius eigenvalues, and monodromy. American Math. Society Colloquium Publications 45. American Math. Society, Providence, RI, 1999.

[Ki1] Kim, D.S., Gauss sums for general and special linear groups over a finite field, *Archiv der Math.* 69 (1997), 297-304.
[Ki2] Kim, D.S., Gauss sums for $U(2n, q^2)$, *Glasgow Math. J.* 40 (1998), 79-95.

[Ki3] Kim, D.S., Gauss sums for $U(2n + 1, q^2)$, *J. Korean Math. Soc.* 34 (1997), 871-894.

[Ki4] Kim, D.S., Gauss sums for symplectic groups over a finite field, *Monats. Math.* 126 (1998), 55-71.

[Ko] Kondo, Gaussian sums attached to the general linear groups over finite fields. *J. Math. Soc. Japan* 15 (1963), 244-255.

[La] Lamprecht, E., Struktur und Relationen allgemeiner Gaußscher Summen in endlichen Ringen I,II. *J. Reine Angew. Math.* 197 (1957), 1-48.

[LiN] Lidl, R. and Niederreiter, H., Finite fields. Encyclopedia of Mathematics and its Applications, 20. Cambridge University Press, Cambridge, 1997.

[NP] Neumann, P. and Praeger, C., Cyclic matrices in classical groups over finite fields, *J. Algebra* 234 (2000), 367-418.

[Sa] Sarnak, P., Some applications of modular forms. Cambridge Tracts in Mathematics, 99. Cambridge University Press, Cambridge, 1990.

[Se] Serre, J.P., Majorations de sommes exponentielles, *Soc. Math. France Asterisque* 41-2 (1977), 111-126.

[St] Stong, R., Some asymptotic results on finite vector spaces, *Adv. in Appl. Math.* 9 (1988), 167-199.

[T] Terras, A., Fourier analysis on finite groups and applications. London Mathematical Society Student Texts, 43. Cambridge University Press, Cambridge, 1999.

[W] Wall, G.E., On conjugacy classes in the unitary, symplectic, and orthogonal groups, *J. Austral. Math. Soc.* 3 (1963), 1-63.