STABILITY CONDITIONS FOR A CLASS OF DELAY DIFFERENTIAL EQUATIONS IN SINGLE SPECIES POPULATION DYNAMICS

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Abstract. We consider a class of nonlinear delay differential equations, which describes single species population growth with stage structure. By constructing appropriate Lyapunov functionals, the global asymptotic stability criteria, which are independent of delay, are established. Much sharper stability conditions than known results are provided. Applications of the results to some population models show the effectiveness of the methods described in the paper.

1. Introduction. Many examples of the biological processes in the real world involve significant delays. There are various ways in which scalar delay differential equations appear in the population biology. In general, for the single species population dynamics two processes are taken into account, that is, the reproduction/maturation and death processes. If the maturation process is taken into account, then the model would include a time delay, and therefore we can consider the general equation describing the past-dependent growth of the population as follows

\[ x'(t) = f(x(t - \tau)) - g(x(t)), \]  

where \( x(t) \) represents the number of adult individuals, \( \tau > 0 \) is an approximate time at which immature individuals mature and are recruited into the adult population, \( f(x(t - \tau)) \) is the birth rate, and \( g(x(t)) \) is the death rate.

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Equation (1.1) means that the recruitment effect at the present time \( t \) depends on the population at time \( t - \tau \) in the past. Considering this maturation process in more details, the delay effect should be ranging over certain period of the past population. In the nature, it is obvious that the process depending on the past does not depend on the exact time \( t - \tau \). But the dependence is somehow distributed. We have the distributed delay differential equation

\[
ax'(t) = \int_0^h f(x(t-s))p(s)\,ds - g(x(t)),
\]

(1.2)

where \( 0 < h < \infty \) is referred to as the duration of a distributed delay. \( p(\cdot) \) is called a delay kernel, and represents a probability density function, where \( p(s) \geq 0 \) for \( 0 < s < h \) and \( \int_0^h p(s)\,ds = 1 \). Equation (1.1) can be considered as a special case of (1.2) if we replace \( p(s) \) with \( \delta(s - \tau) \), where \( \delta \) is the Dirac delta function.

Equations of the form (1.1) and (1.2) have attracted the attention and interest of many researchers because of their practical importance. Indeed, many models from medicine, ecology and environment sciences lead to equations of the form (1.1) and (1.2), and the knowledge of the evolution of the solutions has a great interest. Examples are Beddington and May [4], Blythe et al. [6], Cooke et al. [13], Cushing [15], Karakostas et al. [25], Taylor and Sokal [35], Walther [36], the Mackey-Glass equation [30], and the Nicholson blowflies equation [31].

The stability of equilibria is one of the most important issues in the study of any model of single species population. In particular, large delays would make the steady states unstable because they would lead to oscillations that are not observed when delays are very small. The well-known example is the delayed logistic equation (see Cushing [15]; Gopalsamy [18] and Kuang [27, 28]). On the other hand, Beddington and May [4] showed that the time delay is not necessary destabilizing. Arino et al. [3] considered an alternative formulation for a delayed logistic equation for which delay does not lead to oscillations. Freedman and Gopalsamy in [16] considered the global stability of a unique positive equilibrium of (1.1) by the method of Razumikhin-type function which is a result of Haddock and Terjeki [19], when the birth and death rates satisfy some conditions. The following important question arises: What characteristic for the functions \( f(x) \) and \( g(x) \) does allow for the time delays to change the stability of a single species dynamics?

Generally, two different techniques are usually used to determine the asymptotic stability of delay differential equations. One is to study the eigenvalues of the linearized equations. The other is direct Lyapunov’s method, which is most frequently and simply employed to establish the global stability [12, 20, 28]. However, the main difficulty is to find suitable Lyapunov functionals for delay differential equations. Our main goal is to establish some sufficient and necessary conditions for the global asymptotic stability of all equilibria of (1.1) and (1.2) by the method of Lyapunov functionals (see, Section 5.3 in [20]). The analysis presented in this paper improves and extends some earlier results on global stability for delay single species population dynamics (such as in Brauer et al., [9, 10], Freedman and Gopalsamy [16], Kuang [27, 28], and Lenhart and Travis [29]).

The rest of the paper is organized as follows. In Sections 2 and 3, we derive the global asymptotic stability of the steady states of Equations (1.2) and (1.1). The results are applied to several equations and models in Section 4.
2. Stability for distributed delay case. In this section, we consider a single species population model with a distributed delay (1.2). We assume that the functions \( f, g : \mathbb{R} \rightarrow \mathbb{R} \) are continuously differentiable and satisfy \( f(0) = g(0) = 0 \) and \( xf(x) > 0, xg(x) > 0 \) for \( x \neq 0 \). Moreover, we assume that the delay kernel \( p : [0, h] \rightarrow (0, +\infty) \) is piecewise continuous and \( \int_{0}^{h} p(s) ds = 1 \).

Let \( C = C([-h, 0], \mathbb{R}) \) denote the Banach space of continuous functions from \([-h, 0]\) to \( \mathbb{R} \) equipped with the norm given by \( \| \varphi \| = \max_{-h \leq t \leq 0} |\varphi(t)| \). For continuous \( x : [-h, \infty) \rightarrow \mathbb{R} \), we define \( x_{t}(\theta) = x(t + \theta), -h \leq \theta \leq 0 \), so that \( x_{t} \in C \).

Let \( G \) be a subspace of \( C \) defined by
\[
G = \{ \varphi \in C : \varphi(\theta) > 0 \quad \text{on} \quad [-h, 0] \}
\]
We assume that the initial conditions for (1.2) are of the following type:
\[
x_{0} = \varphi, \quad \varphi \in G. \quad (2.1)
\]

The local existence and uniqueness of the solutions of the initial value problem (1.2) and (2.1) are established. We describe the solution of (1.2) and (2.1) as \( x(t)\varphi \). The global continuation of solutions is a consequence of their boundedness, i.e., if the solution \( x(t) \) is bounded, then it is defined for all \( t \in [0, \infty) \) (cf. Proposition 3.10 in [34]).

We show that all solutions of the initial value problem (1.2) and (2.1) are positive. In fact, suppose that \( x(t) \) takes nonnegative values. Then there is the first time \( t_{1} > 0 \) such that \( x(t) > 0 \) for \( t \in [-h, t_{1}] \), \( x(t_{1}) = 0 \) and \( x'(t_{1}) \leq 0 \). However,
\[
x'(t_{1}) = \int_{0}^{h} f(x(t_{1} - s)) p(s) ds - g(x(t_{1})) = \int_{0}^{h} f(x(t_{1} - s)) p(s) ds > 0,
\]
which is a contradiction. Therefore, \( x(t) \) remains positive for all \( t \geq 0 \).

Here, we will give the definition of the global asymptotic stability in this paper: The steady state of (1.2) is stable in \( G \) if for any \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that \( \varphi \in G \) and \( \| \varphi \| < \delta \) implies that \( |x(t; \varphi)| < \varepsilon \) for \( t \geq 0 \). It is globally asymptotically stable, if it is stable in \( G \) and all solutions of the initial value problem (1.2) and (2.1) tend to the steady state as \( t \rightarrow +\infty \).

From the assumption \( f(0) = g(0) = 0 \), it is obvious that the zero solution \( x = 0 \) is a steady state of (1.2). The following analysis gives the conditions for the global asymptotic stability of the zero solution of (1.2).

**Theorem 2.1.** If the functions \( f(x) \) and \( g(x) \) satisfy
\[
(\text{H1}) \quad f(x) < g(x) \quad \text{for any} \quad x > 0,
\]
then the steady state \( x = 0 \) of (1.2) is globally asymptotically stable for any delay \( \tau > 0 \).

**Proof.** Define a Lyapunov functional \( V : C \rightarrow \mathbb{R} \),
\[
V(\psi) = \psi(0) + \int_{0}^{h} p(s) \int_{-s}^{0} f(\psi(\theta)) d\theta ds. \quad (2.2)
\]
It is clear that \( V \) is continuous on the closure of \( G \) (\( \text{Cl} \ G \)). For a solution \( x(t) \) of (1.2),
\[
V(x_{t}) = x(t) + \int_{0}^{h} p(s) \int_{-s}^{0} f(x(t + \theta)) d\theta ds = x(t) + \int_{0}^{h} p(s) \int_{t-s}^{t-u} f(x(\theta)) d\theta ds,
\]
Theorem 2.2. If the functions $f(x)$ and $g(x)$ satisfy

(H2) there exists a positive $k$, such that $g(x) < f(x) < f(k)$ for $x \in (0, k)$, and $f(k) = f(x) > g(x)$ for $x \in (k, \infty)$, and $\lim_{x \to +\infty} f(x) > f(k)$,

then Equation (1.2) has a unique positive steady state $x^* = k$, and it is globally asymptotically stable for any delay $\tau > 0$.

Proof. The condition (H2) ensures that $x^* = k$ is a unique positive steady state satisfying $f(k) = g(k)$.

Define a Lyapunov functional $W: \mathcal{C} \to \mathbb{R}$,

$$W(\psi) = \psi(0) - k - \int_{0}^{\psi(0)} \frac{f(u)}{f(k)} du$$

and

$$\frac{d}{dt} V(x_t) = x'(t) + \int_{0}^{h} p(s) \{f(x(t)) - f(x(t-s))\} ds$$

$$= x'(t) + \int_{0}^{h} p(s) ds \{f(x(t)) - f(x(t-s))\}$$

$$= f(x(t)) - g(x(t)). \tag{2.3}$$

Thus the derivative of $V$ along the solution of (1.2) is

$$\dot{V}(1.2)(\psi) = f(\psi(0)) - g(\psi(0)). \tag{2.4}$$

From the assumption (H1), $\dot{V}(1.2)(\psi) \leq 0$ for $\psi \in G$. Therefore $V$ is a Lyapunov function on a set $G$ in $\mathcal{C}$ relative to Equation (1.2) (see Definition 5.3.1 in [20]). Let $S = \{\psi \in \text{Cl}G: \dot{V}(1.2)(\psi) = 0\}$ and $M$ be the largest set in $S$ which is invariant with respect to (2.1). Then from the assumption (H1) and (2.4), we have

$$S = \{\psi \in \text{Cl}G: \psi(0) = 0\}.$$  

Obviously, the set $M = \{0\}$. Theorem 5.3.1 in [20] implies that if $x(t; \varphi)$ is a bounded solution of (1.2) and (2.1), then $x(t; \varphi) \to 0$ as $t \to \infty$.

To complete the proof we will show the stability in $G$ of the zero solution and the boundedness of solutions of (1.2) and (2.1). As mentioned before, for any initial function $\varphi \in G$, $x(t; \varphi) > 0$ for all $t > 0$, that is $G$ is invariant with respect to (1.2). Thus from (2.3) and the assumption (H1), for any solution $x(t) = x(t; \varphi)$ of (1.2) and (2.1), we obtain $dV(x_t)/dt < 0$ and

$$V(x_t) < V(x_0) = V(\varphi) \quad \text{for all } t > 0.$$

Moreover, it is clear that $p$ and $f(x(t+\theta))$ are positive, which implies $x(t) < V(x_t)$. From these facts we can easily obtain the stability and the boundedness. \qed
\[ + f(k) \int_0^h p(s) \int_{s}^0 \left\{ \frac{f(x(t + \theta))}{f(k)} - 1 - \ln \frac{f(x(t + \theta))}{f(k)} \right\} d\theta ds. \tag{2.5} \]

For any fixed \( \ell > 0 \), define a subset \( U_\ell \) of \( G \) as follows:
\[ U_\ell = \{ \psi \in G : W(\psi) < \ell \}. \]

Then \( W \) is continuous on the colsure of \( U_\ell \) (\( \text{Cl} U_\ell \)). The condition (H2) yields that
\[ 1 - \frac{f(k)}{f(x)} < 0, \quad x \in (0, k) \quad \text{and} \quad 1 - \frac{f(k)}{f(x)} > 0, \quad x \in (k, \infty). \]

Hence the function \( a_1(x) = x - k - \int_x^k \frac{f(u)}{f(k)} du \) is positive when \( x > 0 \) and \( x \neq k \), and has a minimal value zero at \( x = k \). The condition \( \lim_{x \to +\infty} f(x) > f(k) \) ensures that \( a_1(x) \) tends to infinity as \( x \to +\infty \). On the other hand, the function \( a_2(x) = x - 1 - \ln x \) \((x > 0)\) is nonnegative and has a global minimum zero at \( x = 1 \). These facts show that
\[ a_1(\psi(0)) \leq W(\psi) \quad \text{and} \quad \lim_{x \to -\infty} a_1(x) = +\infty. \tag{2.6} \]

For a positive solution \( x(t) \) of (1.2),
\[
W(x_t) = x(t) - k - \int_k^x \frac{f(k)}{f(u)} du
+ f(k) \int_0^h p(s) \int_{s}^0 \left\{ \frac{f(x(t + \theta))}{f(k)} - 1 - \ln \frac{f(x(t + \theta))}{f(k)} \right\} d\theta ds
= x(t) - k - \int_k^x \frac{f(k)}{f(u)} du
+ f(k) \int_0^h p(s) \int_{t-s}^t \left\{ \frac{f(x(\theta))}{f(k)} - 1 - \ln \frac{f(x(\theta))}{f(k)} \right\} d\theta ds,
\]
and
\[
\frac{d}{dt} W(x_t) = x'(t) \left\{ 1 - \frac{f(k)}{f(x(t))} \right\} + f(k) \int_0^h p(s) \left\{ \frac{f(x(t))}{f(k)} - 1 - \ln \frac{f(x(t))}{f(k)} \right\} ds
- f(k) \int_0^h p(s) \left\{ \frac{f(x(t-s))}{f(k)} - 1 - \ln \frac{f(x(t-s))}{f(k)} \right\} ds
= \left\{ 1 - \frac{f(k)}{f(x(t))} \right\} \left\{ \int_0^h f(x(t-s)) p(s) ds - g(x(t)) \right\}
+ f(k) \int_0^h p(s) \left\{ \frac{f(x(t-s))}{f(k)} - \frac{f(x(t-s))}{f(x(t))} + \ln \frac{f(x(t-s))}{f(x(t))} \right\} ds
= - \{ g(x(t)) - f(x(t)) \} \left\{ 1 - \frac{f(k)}{f(x(t))} \right\}
+ f(k) \int_0^h p(s) \left\{ 1 - \frac{f(x(t-s))}{f(x(t))} + \ln \frac{f(x(t-s))}{f(x(t))} \right\} ds
= - \{ g(x(t)) - f(x(t)) \} \left\{ 1 - \frac{f(k)}{f(x(t))} \right\} - f(k) \int_0^h p(s) a_2 \left( \frac{f(x(t-s))}{f(x(t))} \right) ds \tag{2.7}
\]
Thus the derivative of \( W \) along the solution of (1.2) is
From (H2), the following inequality always holds for all \( x > 0, x \neq k \),

\[
W_{(1.2)}(\psi) = -\{g(\psi(0)) - f(\psi(0))\} \left\{ 1 - \frac{f(k)}{f(\psi(0))} \right\} - f(k) \int_0^h p(s)a_2 \left( \frac{f(\psi(-s))}{f(\psi(0))} \right) ds. \tag{2.8}
\]

From (H2), the following inequality always holds for all \( x > 0, x \neq k \),

\[
(g(x) - f(x)) \left( \frac{f(k)}{f(x)} - 1 \right) < 0.
\]

Note that \( a_2(x) > 0 \) for any \( x > 0, x \neq 1 \), and \( a_2(x) = 0 \) if and only if \( x = 1 \). Therefore \( W_{(1.2)}(\psi) \leq 0 \) for \( \psi \in U_{\ell} \) and \( W \) is a Lyapunov function on a set \( U_{\ell} \) in \( \mathcal{C} \) relative to Equation (1.2). Let \( S = \{ \psi \in \text{Cl} \ U_{\ell} : W_{(1.2)}(\psi) = 0 \} \) and \( M \) be the largest set in \( S \) which is invariant with respect to (2.1). Then from the assumption (H2) and (2.8), we have

\[
S = \{ \psi \in \text{Cl} \ U_{\ell} : \psi(\theta) = k \text{ for } \theta \in [-h, 0] \}.
\]

Obviously, the set \( M = \{ k \} = \{ x^* \} \). Theorem 5.3.1 in [20] implies that if \( x(t; \varphi) \) is a bounded solution of (1.2) with the initial data \( \varphi \in U_{\ell} \), then \( x(t; \varphi) \to x^* \) as \( t \to \infty \). Since \( \ell \) can be taken arbitrarily large value, we can conclude that any solutions of (1.2) and (2.1) tend to the steady state \( x^* \) as \( t \to \infty \).

Now we will show the stability of the steady state \( x^* \) and the boundedness of solutions of (1.2) with the initial data \( \varphi \in U_{\ell} \). Let \( x(t) = x(t; \varphi) \) be a solution of (1.2) with the initial data \( \varphi \in U_{\ell} \). Then \( dW(x(t))/dt \leq 0 \) as long as the solution \( x_t \) remains in \( U_{\ell} \) and

\[
W(x_t) \leq W(x_0) = W(\varphi) < \ell
\]

for all \( t > 0 \) such that the solution \( x_t \) remains in \( U_{\ell} \). This implies that \( U_{\ell} \) is invariant with respect to (1.2). Moreover, from the relation (2.6) \( a_1 \), we can obtain the stability and the boundedness.

\[ \square \]

**Remark 1.** Here two Lyapunov functionals are effective to establish the global stability conditions of zero solution and the positive steady state of (1.2), respectively. It also resolves the boundedness of any solution of (1.2). We should point out that the other Lyapunov function approach, i.e., the Razumikhin method is also suitable to establish conditions for the existence of limits of solutions and for asymptotic stability (see Freedman and Gopalsamy [16] and Kuang [28]).

**Remark 2.** It should be mentioned here that the following equation including two delays was proposed in (Kuang [28]),

\[
x'(t) = \int_\sigma^\tau f(x(t-s)) \, d\mu_1(s) - \int_0^\sigma g(x(t-s)) \, d\mu_2(s), \tag{2.9}
\]

where \( 0 < \sigma < \tau \) are constants, \( \mu_1(s) \) and \( \mu_2(s) \) are continuous nondecreasing functions, and \( \int_\sigma^\tau d\mu_1(s) = 1 \) and \( \int_0^\sigma d\mu_2(s) = 1 \). Equation (2.9) allows for delays to appear in both \( f(\cdot) \) and \( g(\cdot) \). This is important in applications since, in all real systems, changes take time, no matter how small. The sufficient conditions for global asymptotic stability of the positive steady state \( x^* \) of (2.9) is left as an open problem in [28]. By using the Lyapunov functional

\[
W(\psi) = \psi(0) - x^* - \int_{x^*}^{\psi(0)} f(u) \, du
\]
we can also establish the stability conditions for the positive steady state of (2.9), which is the same as Theorem 2.2. For a special case, (2.9) is reduced to the equation with two discrete delays

\[ x'(t) = f(x(t - \sigma - \tau)) - g(x(t - \sigma)). \] (2.10)

In this case, the results can be directly applied to (2.10).

Theorems 2.1 and 2.2 yields that the global stability of the steady states in (1.2) depends mainly on the properties of \( f \) and \( g \). The condition (H1) implies biologically that stronger regulation of the death than growth rate reduces the density of species population and intends it to extinction; The condition (H2) implies that the birth rate is higher than the death rate when the population size is smaller than the positive steady state value \( k \) and the latter rate is higher than the former when the population is larger than \( k \). These two results are reasonable from biological point of view. It should be noted that these properties on \( f \) and \( g \) ensure that the discrete or distributed time delay for maturation for the species does not give any effect on the stability of the model (harmless delay).

From a mathematical point of view, several suitable Lyapunov functionals for a class of delay differential equations have been constructed in this paper. We would like to point out that this type of Lyapunov functionals have been successfully applied to two-variable and three-variable delay systems in many previous works, such as delay epidemiological models in Huang et al. [23], delay virus infection model in Huang et al. [24]. Here it is firstly applied to the first order nonlinear delay differential equations \( x'(t) = f(x(t - \tau)) - g(x(t)) \), describing single species population growth. It appears to be a sound basis to construct Lyapunov functionals for more general nonlinear delay equations, such as \( x'(t) = F(x, x_t) \) in [7, 8, 17], which is left as our future work.

3. Stability for discrete delay case. For the discrete delay equation (1.1), we can completely carry out the same argument and obtain the following results.

**Theorem 3.1.** If the functions \( f(x) \) and \( g(x) \) satisfy (H1), then the steady state \( x = 0 \) of (1.1) is globally asymptotically stable for any delay \( \tau > 0 \).

**Proof.** Since the proof is similar to that of Theorem 2.1, we only give the Lyapunov functional and its derivative along the solution (1.1):

\[
V(\psi) = \psi(0) + \int_{-\tau}^{0} f(\psi(\theta)) d\theta.
\]

\[
\dot{V}_{(1.1)}(\psi) = f(\psi(0)) - g(\psi(0)).
\]

**Theorem 3.2.** If the functions \( f(x) \) and \( g(x) \) satisfy (H2), then Equation (1.1) has a unique positive steady state \( x^* = k \), and it is s globally asymptotically stable for any delay \( \tau > 0 \).

**Proof.** The Lyapunov functional and its derivative along the solution (1.1) are given as follows:

\[
W(\psi) = \psi(0) - k - \int_{k}^{\psi(0)} \frac{f(k)}{f(u)} du + f(k) \int_{-\tau}^{0} \left\{ \frac{f(\psi(\theta))}{f(k)} - 1 - \ln \frac{f(\psi(\theta))}{f(k)} \right\} d\theta.
\]
\[
\dot{W}_{(1.1)}(\psi) = -\{g(\psi(0)) - f(\psi(0))\left\{1 - \frac{f(k)}{f(\psi(0))}\right\} - f(k)a_2 \left(\frac{f(\psi(-\tau))}{f(\psi(0))}\right)\}.
\]

**Remark 3.** Reviewing the known conditions in [16] and [28], they both restricted the functions \(f\) and \(g\) to be strictly monotonously increasing. However, by Theorems 3.1 and 3.2, the monotonicity of \(f\) and \(g\) is not necessary for the global stability of the zero solution or positive steady state (see Figure 1.(a) and (b)). In particular, in [10] and [28], the function \(g(x)\) was further assumed to satisfy \(\lim_{x \to +\infty} g(x) = +\infty\). In Figure 1.(c), we see the example with the function \(g\) having the property \(\lim_{x \to +\infty} g(x) = A < +\infty\) and the positive steady state \(k\) is still globally asymptotically stable since (H2) holds. Obviously, (H2) gives more weaker restriction for \(f\) and \(g\) than the conditions mentioned above in [10], [16] and [28].

**Figure 1.** The functions \(f\) and \(g\) satisfy (H1) in (a), and satisfy (H2) in (b) and (c). Note that \(f\) and \(g\) are allowed to be non-monotonous and bounded as \(x \to +\infty\).
Consider an equation
\[ y'(t) = y^2(t)[P(y(t)) - Q(y(t - \tau))], \tag{3.1} \]
where \( P(y), Q(y) \in C[0, \infty), (0, \infty) \), and \( P(y) \) and \( Q(y) \) are assumed to be monotonous and continuously differentiable for \( y > 0 \).

We assume that the initial value problem of Equation (3.1) with a continuous and positive initial function has a unique positive solution \( y(t) \) defined for all \( t \geq 0 \). We also assume that Equation (3.1) has a unique positive steady state \( K \), that is, \( K \) is the unique solution of the equation \( P(y) = Q(y) \).

Now we consider the stability conditions of the positive steady state \( y^* = K \). Under the transformation \( y(t) = x^{-1}(t) \), (3.1) becomes
\[ x'(t) = Q(x^{-1}(t - \tau)) - P(x^{-1}(t)). \tag{3.2} \]

Clearly, (3.2) is equivalent to Equation (3.1) and Equation (3.2) is of the form (1.1) with \( f(x) = Q(x^{-1}) \) and \( g(x) = P(x^{-1}) \). Note that (3.2) has a unique positive steady state \( x^* = 1/K \). Therefore, Theorem 3.2 can be applied to establish stability condition of steady state \( 1/K \) of (3.2). If all solutions of (3.2) approach \( 1/K \), then all solutions of (3.1) converge to the positive steady state of \( K \). We have the following corollary for (3.1).

**Corollary 3.1.** Consider (3.1). If the functions \( P \) and \( Q \) in (3.1) satisfy

(H3) there exists a positive \( K \), such that \( Q(K) < Q(y) < P(y) \) for \( y \in (0, K) \), and \( P(y) < Q(y) < Q(K) \) for \( y \in (K, \infty) \), and \( \lim_{y \to 0^+} Q(y) > Q(K) \),

then the unique positive steady state \( y^* = K \) is globally asymptotically stable for any delay \( \tau > 0 \).

As an example, consider
\[ y'(t) = y(t) - \frac{2y^2(t)}{1 + y(t - \tau)}. \tag{3.3} \]

We can find that the initial value problem of (3.3) with a positive initial function has a unique positive solution \( y(t) \) for \( t \geq 0 \). Here the global continuity of solutions is a consequence of there boundedness. Equation (3.3) is of the form (3.1) with \( P(y) = \frac{2}{y} \) and \( Q(y) = \frac{2}{1+y} \). Obviously, (3.1) has a unique positive steady state \( y^* = 1 \) and the functions \( P(y) \) and \( Q(y) \) satisfy (H3) (see Figure 2 (a)). By Corollary 3.1, \( y^* \) is global asymptotically stable for any \( \tau > 0 \). Numerical simulation in Figure 2(b) verifies that solutions approach \( y^* \) both for \( \tau = 2 \) with initial data \( \varphi = 0.2; 1.8 \) (Blue) and \( \tau = 20 \) with initial data \( \varphi = 0.2; 1.8 \) (Black).

4. **Applications.** In this section, we apply the main results obtained in Sections 2 and 3 to some simple delay differential equations. Recently Ruan [32] gave a very excellent survey of differential equations modelling single species growth. We also apply the results to several well-known population models and show the effectiveness of the methods applied here.

4.1. **Linear scalar delay differential equations.** Consider the simple scalar delay differential equation
\[ x'(t) = bx(t - \tau) - cx(t), \tag{4.1} \]
where \( b \) and \( c \) are constants. If \( 0 < b < c \) then the condition (H1) is satisfied. Therefore the condition \( 0 < b < c \) ensures the global asymptotic stability of zero
solution. We remark that the necessary and sufficient condition of the local asymptotic stability is well-known. Please refer to Hale [20], Hayes [22], and so on.

4.2. A self-excited neuron model. Consider the simple model of a self-excited neuron with delayed excitation given by

$$x'(t) = -x(t) + \tanh(kx(t-1))$$

where $x(t)$ encodes the neuron’s “activity level” and $k > 0$. The unit delay, reflecting a scaling of the time variable, represents the transmission time between output $x(t)$ and input. Note that nonnegative (nonpositive) initial data give rise to nonnegative (nonpositive) solutions.

From the previous results and the property of function $f(x) = \tanh x$, consequently, it is easy to show the following dynamical properties of (4.2).

Proposition 4.1 (cf. p. 72, [34]). If $0 < k \leq 1$, then $x = 0$ is the only steady state of (4.2) and it is globally asymptotic stable. If $k > 1$, then there is a unique positive steady state $x = x^*$ and it is globally asymptotic stable.

In [34], the initial value problem is considered in the whole space $C$. If we restrict the region to $G$, the result in [34] is identical to our result.
4.3. **Single species stage-structured model.** The following single-species stage structured model was introduced and studied by Aiello and Freedman [1] and Aiello et al. [2]:

\[
\begin{align*}
x_1'(t) &= \alpha x_2(t) - \gamma x_1(t) - \alpha e^{-\gamma \tau} x_2(t - \tau), \\
x_2'(t) &= \alpha e^{-\gamma \tau} x_2(t - \tau) - \beta x_2^2(t),
\end{align*}
\]

(4.3)

where \(x_1(t)\) and \(x_2(t)\) are the immature and mature population density, respectively. \(\alpha > 0\) represents the birth rate, \(\gamma > 0\) is the immature death rate, \(\beta > 0\) is the mature death and overcrowding rate and \(\tau\) is the time to maturity. The term \(\alpha e^{-\gamma \tau} x_2(t - \tau)\) represents the immatures who were born at time \(t - \tau\) and survive at the time \(t\), and therefore represents the transformation of immature to mature.

From Theorem 3.2, it is easy to see that the second equation of model (4.3) has always a positive steady state \(x_2^* = \alpha e^{-\gamma \tau} / \beta\), which is globally asymptotically stable. Hence, model (4.4) has a globally asymptotically stable steady state \(E^*(x_1^*, x_2^*)\), where \(x_1^* = \alpha (1 - e^{-\gamma \tau}) x_2^* / \gamma\).

4.4. **Delayed logistic equation.** Recently, an alternative formulation for a delayed logistic equation was considered by Arino et al. [3]. It was assumed that the rate of change of the population depends on three components: growth, death and intraspecific competition (crowding or direct interference). The last two components are instantaneous and the death rate is given by a linear term whereas the intraspecific competition rate is given by the quadratic term. The equation is given as follows:

\[
x'(t) = \frac{\gamma \mu x(t - \tau)}{\mu e^{\mu \tau} + \kappa (e^{\mu \tau} - 1) x(t - \tau)} - \mu x(t) - \kappa x^2(t).
\]

(4.4)

Here the parameters \(\gamma, \mu, \kappa\) are positive constants. When \(\tau = 0\), (4.4) is a logistic ordinary differential equation.

(i) When \(\kappa = 0\), (4.4) reduces to

\[
x'(t) = \gamma e^{-\mu \tau} x(t - \tau) - \mu x(t),
\]

(4.5)

which is a linear first order scalar delay differential equation of the form of (4.1), where \(b = \gamma e^{-\mu \tau}\) and \(c = \mu\). Therefore it is easy to see that the zero solution is globally asymptotic stable if \(\gamma e^{-\mu \tau} < \mu\).

(ii) When \(\kappa > 0\), the functions \(f(x)\) and \(g(x)\) in (4.4) are represented as

\[
f(x) = \frac{\gamma \mu x}{\mu e^{\mu \tau} + \kappa (e^{\mu \tau} - 1) x} \quad \text{and} \quad g(x) = \mu x + \kappa x^2.
\]

(4.6)

It is easy to know that (4.4) always has a zero solution, and a unique positive steady state \(x^*\) when \(\gamma e^{-\mu \tau} > \mu\), where

\[
x^* = \frac{\sqrt{\mu^2 + 4 \gamma \mu (e^{\mu \tau} - 1)} + \mu (1 - 2 e^{\mu \tau})}{2 \kappa (e^{\mu \tau} - 1)}.
\]

Obviously, the functions \(f(x)\) and \(g(x)\) described by (4.6) satisfy (H1) when \(\gamma e^{-\mu \tau} \leq \mu\), and also satisfy (H2) when \(\gamma e^{-\mu \tau} > \mu\). Hence, we obtain the following global stability properties for equation (4.4), which is similar with the results in [3].

**Proposition 4.2.**  
(i) If \(\gamma e^{-\mu \tau} > \mu\), the unique positive steady state \(x^*\) of (4.4) is globally asymptotically stable.  
(ii) If \(\gamma e^{-\mu \tau} \leq \mu\), the zero solution of (4.4) is globally asymptotically stable.
4.5. Mackey-Glass model. In order to describe the control of a single population of blood cells, Mackey and Glass in [30] proposed the following nonlinear differential delay equation as its model

\[ \frac{dN}{dt} = \frac{\beta \theta^n N(t-\tau)}{\theta^n + N^n(t-\tau)} - rN(t), \]  

(4.7)

where \( \beta, \theta, r, n \) are positive constants and delay \( \tau > 0 \). \( N(t) \) denotes the density of mature cells in blood circulation, and \( \tau \) is the time delay between the production of immature cells in the bone marrow and their maturation for release in the circulating bloodstream.

Letting \( N(t) = \theta x(t) \), equation (4.7) is transformed into

\[ \frac{dx}{dt} = \frac{\beta x(t-\tau)}{1 + x^n(t-\tau)} - rx(t). \]  

(4.8)

The initial condition is given by \( x_0 = \varphi \in \mathcal{C}, \varphi(s) > 0 \) for \( s \in [-\tau, 0] \).

When \( \beta \leq r \), equation (4.8) has only a zero solution. When \( \beta > r \), in addition to zero solution, there exists a unique positive steady state \( x^* \), where \( x^* = (\beta/r - 1)^{1/n} \).

Obviously, equation (4.8) is of the form (1.1) with \( f(x) = \frac{\beta x}{1 + x^n} \) and \( g(x) = rx \).

(4.9)

When \( \beta \leq r \), since

\[ f(x) - g(x) = \left( \frac{\beta}{1 + x^n} - r \right)x \leq 0, \]

and the equality holds only at the point \( x = 0 \), it follows from Theorem 3.1 that the zero solution is globally asymptotically stable.

Further, in the case \( n \leq 1 \), the functions \( f(x) \) and \( g(x) \) satisfy (H2) under the condition \( \beta > r \). It means that the unique positive steady state \( x^* \) is globally asymptotically stable. Thus, we have the following proposition.

**Proposition 4.3.** (i) If \( \beta \leq r \), then the zero solution of (4.8) is globally asymptotically stable for any delay \( \tau > 0 \).

(ii) If \( 0 < n \leq 1 \) and \( \beta > r \), then the unique positive steady state \( x^* \) of (4.8) is globally asymptotically stable.

Note that the analysis above only gives the global stability of the positive steady state of equation (4.8) in the case \( n \leq 1 \). It implies that there is no bifurcation at \( x^* \) when \( n \leq 1 \). However, when \( n > 1 \), the functions \( f(x) \) and \( g(x) \) in (4.9) do not satisfy (H2). There have been extensive works dealing with equation (4.8) in the case \( n > 1 \). Hopf bifurcation, periodic solutions and chaotic solutions and other dynamical properties are studied when the delay increases (see e.g. Kubiaczyk and Saker [26], Ruan [32], Rost and Wu [33] and the references therein).

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