An impurity in a Luttinger Liquid coupled to Ohmic-class environments

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(Dated: December 9, 2020)

We investigate the impact of noisy environments on transport through a one-dimensional interacting quantum wire containing a scattering impurity. The impurity is known to induce a metal-insulator quantum phase transition as a function of the interaction strength in the wire. By considering Ohmic-class environments (ranging from Ohmic, to sub-Ohmic and super-Ohmic cases), we elucidate the influence of fluctuations on this phase transition. Within a perturbative renormalization group approach, we show that Ohmic environments keep the phase transition intact, while sub- and super-Ohmic environments, modify it into a smooth crossover at a scale that depends on the interaction strength in the wire. The system, however, still undergoes a metal-to-insulator-like transition when moving from sub-Ohmic to super-Ohmic environment noise. We further consider realistic experimental conditions due to finite length and finite temperatures of the wire, and explore the signatures of the phase transition in the temperature dependence of the impurity-induced back-scattering electrical conductance.

One of the most prominent manifestations of electron-electron interaction occurs in one-dimensional (1D) systems. There, the electronic low-energy properties exhibit Tomonaga-Luttinger liquid (TLL) behaviour with well-defined plasmonic excitations, in contrast to electron-like quasiparticles in Fermi’s liquid paradigm [1–3]. Distinct signatures of TLL are separation of spin and charge degrees of freedom [4–7], fractionalization of injected electrons [8–11], power-law suppression of tunnelling current at low-bias voltages known as the zero-bias anomaly [12–14], and Kane-Fisher impurity physics. The latter engenders the quantum phase transition between a metallic and an insulating phase induced by the presence of a backscattering impurity inside the TLL [15–17]. Such TLL features have been observed in a wide variety of experiments including nanotubes [13, 14, 18], quantum Hall edges [19, 20], cold atom platforms [21–23], and circuit quantum simulations [17].

Quantum transport through a 1D wire described by a spinless TLL theory has been the subject of intense theoretical [15, 16, 24–29] and experimental studies [30, 31]. It was shown that special care needs to be taken when considering the coupling of such wires to external electronic leads [25, 26, 32, 33]. When a clean wire is adiabatically coupled to leads [25, 26], the conductance is independent of the electron-electron interactions inside the wire, and is given by the unit of conductance $e^2/h$. However, placing an impurity amidst a wire with repulsive many-body interactions, the conductance through it becomes suppressed due to back-scattering processes, and it exhibits a power-law temperature-dependence with exponents entirely determined by the many-body interactions [25, 26, 31]. This is reminiscent of the Kane-Fisher quantum phase transition between a metallic and an insulating state at the non-interacting critical point [15, 16]. Transport across an impurity in a TLL is, therefore, of particular interest, as it reveals the interaction strength within the wire, and crucially-depends on the resulting 1D strongly-correlated state.

Motivated by technological advances in cold atom experiments [34] and circuit quantum simulations [17], there is a surge of interest in the nonequilibrium dynamics of open quantum 1D systems. Correspondingly, the problem of an impurity in a TLL is being revisited with various studies on the impact of, e.g., the induced dissipation by a lossy impurity [35–37] or a non-local two-body loss [38], as well as on their attenuation of critical phenomena [39]. Similarly, the impact of out-of-equilibrium bias across the wire has been extensively explored [40–43]. Note that to simplify the theoretical treatment, the electronic leads are routinely modelled as one-dimensional non-interacting electrons (Ohmic environment) [25, 26, 43], rather than more realistic higher-dimensional systems [44, 45]. In an attempt to consider a more realistic environment, some of us introduced a low-pass filter (finite capacitance) to the leads to account for charging effects inside them [46]. This produces a characteristic charging time to the Ohmic environment that competes with the time of flight of the TLL plasmons, altering the high-energy properties of the wire.

In this work, we go beyond the Ohmic-environment assumption and reveal how the low-energy Kane-Fisher’s physics becomes dominated by the environment fluctuations. Specifically, we consider sub-Ohmic and super-Ohmic current noise fluctuations in the environment that compete with the wire’s many-body effects. We show that while super-Ohmic dissipation localizes the particles akin to a Zeno effect, the sub-Ohmic fluctuations overtake the low-energy properties of the TLL. The impurity, then, engenders a wire–environment competition with a non-monotonous renormalization group (RG) flow, leading ultimately to a metal-to-insulator-like transition in the wire as a function of the noise statistics. Furthermore, considering realistic finite wires, we predict that the non-monotonous flow implies unusual temperature-dependent scaling of the conductance coming out from
The noise-power spectrum of the electronic leads as a function of frequency, where $\omega_c$ marks the environment bandwidth. (c) The phase diagram of a two-level system ($\phi^4$-theory) coupled to an Ohmic environment for varying dissipation strength $\alpha$. For small dissipation $\alpha < 1$, the effective tunneling, $\Gamma$, between two potential wells diverges, i.e., the particle is delocalized, but for stronger dissipation $\alpha > 1$ the particle is localized $\Gamma \to 0$ [47]. (d) Phase diagram of the TLL hosting an impurity coupled to Ohmic leads [cf. action (4)], with effective scattering potential $V$ mapped to a Sine-Gordon potential as a function of the interaction strength in the wire $K_w$ [cf. Eq. (2)].

the wire–environment competition.

Setup and microscopic model — We consider a system of interacting spinless electrons confined in a single-channel 1D wire of length $L$ that contains a single impurity. The wire is additionally adiabatically connected to metallic leads, see Fig. 1(a). The Hamiltonian of the wire reads

$$H_w = \int_{-\infty}^{\infty} dx \left[ i v_F \sum_{\eta=L,R} \alpha_\eta \Psi_\eta^\dagger(x) \partial_x \Psi_\eta(x) \right]$$

$$+ \sum_{\eta} U(x) \rho_\eta(x) \rho_\eta(x) + V_0 \left[ \Psi_L^\dagger(x_0) \Psi_R(x_0) + H.c. \right],$$

where the first term represents the kinetic energy of electrons with linearized dispersion $E_\kappa = \alpha_\eta v_F k$, $v_F$ the electron velocity, and $\alpha_L = 1$ ($\alpha_R = -1$) corresponds to the left- (right-)moving electrons with fermionic field operators $\Psi_L$ ($\Psi_R$). The second term describes local electron-electron interactions inside the wire via (normal-ordered) density operators $\rho_\eta(x) = \Psi_\eta^\dagger(x) \Psi_\eta(x)$: with a constant magnitude $U(x) = U \neq 0$ only inside the wire ($x \in [-L/2, L/2]$). The third term corresponds to a backscattering impurity of strength $V_0$ at position $x_0$.

The many-body interactions in the wire modify the well-defined fermionic excitations profoundly, resulting in the emergence of plasmons, collective excitations of bosonic nature [1, 3]. Using bosonization, we can write the fermionic fields as $\Psi_\eta = \sqrt{N/(2\pi)} \tilde{F}_\eta \exp[i k_F x + i \phi_\eta(x)]$, where $\Lambda$ is an ultraviolet cutoff, $\tilde{F}_\eta$ the Klein factor, and $\phi_\eta$ represents bosonic fields with commutation relations, $[\phi_\eta(x), \phi_\eta(y)] = -i \pi \alpha_\eta \delta(x-y)$. The density operator in terms of these bosonic fields reads $\rho_\eta(x) = \partial_x \phi_\eta(x)/(2\pi)$. Thereby, the bosonized Hamiltonian of the interacting wire takes the form

$$H_w = \frac{v_F}{4\pi K_w} \int_{-L/2}^{L/2} dx \left\{ \frac{1}{K_w} (\partial_x \phi)^2 + K_w (\partial_x \theta)^2 \right\}$$

$$+ V_0 \cos[2k_F x_0 + \sqrt{4\pi} \phi(x_0)],$$

where $\phi(x,t), \theta(x,t) = (1/\sqrt{2}) [\phi_L(x,t) \pm \phi_R(x,t)]$ satisfy the commutation relation $[\phi(x), \theta(y)] = i \pi \delta(x-y)$, and $K_w = 1/\sqrt{1 + U/(\pi v_F)}$ is the so-called Luttinger parameter, with $K_w = 1$ referring to a noninteracting wire, and $K_w < 1 (K_w > 1)$ indicating repulsive (attractive) interactions.

The wire is adiabatically connected to electronic leads, which we introduce by imposing appropriate boundary conditions $\partial_x \phi_\eta(x = \pm L/2) = 2\pi n_\eta J_\eta$, where $J_\eta(\omega)$ is the current operator in the leads. The effect of the boundaries enter the correlation functions of the wire via the noise power spectrum $S(\omega) = \langle J_\eta(\omega) J_\eta(-\omega) \rangle$, where $\langle \cdots \rangle$ denotes thermal averaging with respect to the Fermi liquid state of the leads [46–49]. We consider an Ohmic-class noise power spectrum

$$S(\omega) = \omega \left| \frac{\omega}{\omega_c} \right|^{s-1} e^{-|\omega/\omega_c|} \left[ 1 + n_b(\beta \omega) \right],$$

where $\omega_c$ is the characteristic energy scale of the environment, indicating the exponential suppression of current-to-current correlations for $\omega \gg \omega_c$. The parameter $s \in (0, 2)$ distinguishes between different cases, i.e., $s = 1$ describes an Ohmic lead, whereas $s < 1 (s > 1)$ corresponds to the sub- (super-)Ohmic case. The noise power exhibits a bosonic distribution $n_b(\beta \omega) = 1/\{\exp(\beta \omega) - 1\}$ at inverse temperature $\beta$.

To realize an Ohmic environment, it suffices to consider free fermions with a well-defined Fermi-Dirac distribution. On the other hand, non-Ohmic environments with $s \neq 1$ can be realized, for example, by electron-phonon coupling in the leads ($s > 1$) [50], or by complex RC circuit architectures ($s < 1$) [51]. In Fig. 1(b), we plot the frequency dependence of the noise spectrum for these three cases. Comparing to the Ohmic case, the current-to-current fluctuations in the sub- (super-)Ohmic leads are more dominant at lower (higher) frequencies, i.e., environmental fluctuations are slower (faster). Ohmic-class environments have been extensively studied in the framework of the spin-boson model [47, 51–54], revealing the profound influence of the environment fluctuations on the nature of the ground state, as well as on the dynamics of the system. In particular, it was shown that in the Ohmic case, there exist a critical dissipation that distinguishes between a localized phase and a delocalized one, see Fig. 1(c). In contrast, in the sub- (super-)Ohmic case,
the system is argued to be localized (delocalized) independent of dissipation strength [47]. Analogously, in this work, we investigate the impact of such current fluctuations in the leads on the transport through a disordered interacting wire.

We formulate the impurity scattering in imaginary-time path-integral formalism. The action of the bosonized system at all positions \( x \neq x_0 \) is quadratic and can be therefore integrated out, resulting in the following local (extended Sine-Gordon) action

\[
A = \int_0^\beta dt \, \varphi^+(\tau) \left[ \mathcal{G}_{\varphi\varphi}^0(\tau) \right]^{-1} \varphi(\tau) + V_0 \int_0^\beta d\tau \cos(\sqrt{4\Lambda} \varphi(\tau)),
\]

where \( \mathcal{G}_{\varphi\varphi}^0(\tau) = \int_{-\infty}^\infty d\omega \mathcal{G}_{\varphi\varphi}^0(i\omega) e^{i\omega\tau} \), with \( \mathcal{G}_{\varphi\varphi}^0(i\omega) = \int \frac{d\omega'}{2\pi i} \mathcal{G}_{\varphi\varphi}^{G^0}(\omega') \mathcal{G}_{\varphi\varphi}^{G^0}(\omega') \mathcal{G}_{\varphi\varphi}^{G^0}(\omega) \) the imaginary-time (Matsubara) Green’s function of the clean wire at \( x = x_0 \). Without loss of generality, we assume \( x_0 = 0 \) [55]. The wire’s plasmonic greater and lesser Green’s functions read

\[
\mathcal{G}_{\varphi\varphi}^+(x,x',\omega) = -i\mathcal{K}_w S(\omega)/\omega^2,
\]

and
detailed-balance holds \( \mathcal{G}_{\varphi\varphi}^-(\omega) = e^{-\beta\omega} \mathcal{G}_{\varphi\varphi}^+(\omega) \) as expected for bosons in thermal equilibrium [59]. The noise spectrum of Ohmic leads at low energies scales linearly with energy, \( S(\omega) \sim \omega \), and commonly in this limit, we observe \( \mathcal{G}_{\varphi\varphi}^0(i\omega) = K_w/[2\omega] \). More generally, we can always define a similar structure \( \mathcal{G}_{\varphi\varphi}^0(i\omega) = K(\omega)/[2\omega] \) with an energy-dependent Luttinger parameter \( K(\omega) \) incorporating also the fluctuations from the leads, see Fig. 2(a). At low frequencies \( \omega \ll \omega_c \), it can be approximated as

\[
K(\omega) = \frac{K_w}{\sin[(\pi s)/2]} \left[ \frac{\omega}{\omega_c} \right]^{s-1}.
\]

Unlike the Ohmic case, the frequency-dependence of the effective Luttinger parameter with non-Ohmic leads modifies the physics of the interacting wire. In particular, for the sub-Ohmic case (slow environmental fluctuations), at sufficiently low-frequencies, the effective interaction in the wire appears attractive \( [K(\omega) > 1] \), while for the super-Ohmic case, the interaction becomes effectively repulsive \( [K(\omega) < 1] \).

We employ a perturbative RG approach (due to the presence of infrared divergences [51]), in which we integrate out high energy fields and map the system (4) to itself, but with a smaller ultraviolet cutoff \( \Lambda' = \Lambda (1 - d l) \), i.e., \( d l = d\Lambda/\Lambda [15] \). As a result, a renormalized scattering potential \( V(\Lambda) \) (up to \( dl^2 \)) obeys the flow equation

\[
\frac{dV}{dl} = V_0 (1 - K(\Lambda)).
\]

Note that due to the noisy leads, the flow involves also the renormalization of \( K(\omega) \). The numerical solution of the flow equation is shown in Fig. 2. For the Ohmic case \( s = 1 \) [Fig. 2(b)], standard Kane-Fisher physics [15] is observed, where the wire’s Luttinger-Liquid parameter plays a decisive role, i.e., for \( K_w < 1 \) \( (K_w > 1) \) the fixed point of the RG, for \( \Lambda \to 0 \), is \( V \to \infty \) \( (V \to 0) \) for the insulating (metallic) case. The quantum critical point is at \( K_w = 1 \), corresponding to non-interacting electrons, see Fig. 1(d).

In the case of a sub-ohmic noise spectrum, see Fig. 2(c), the low-frequency noise induces an effective \( K(\omega) \) that increases with \( \Lambda \). Therefore, starting from repulsive interactions in the wire, the renormalized scattering potential exhibits a non-monotonic behaviour. Specifically, defining \( \Lambda^* \) such that \( K(\Lambda^*) = 1 \), we observe that for \( \Lambda > \Lambda^* \), the back-scattering potential increases and the transport through the wire will be suppressed, while for
\( \Lambda < \Lambda^* \), it decreases and transport ignores the impurity. The transition point \( \Lambda^* \) strongly depends on the wire’s bare Luttinger Liquid parameter \( K_w \), cf. Eq. (5).

Crucially, regardless of the specific initial microscopic parameters of the wire, the fixed point of the flow is \( K \to \infty, V \to 0 \). We compare the sub-ohmic behaviour with the super-ohmic case, where the flow lines are reversed as shown in Fig. 2(d). In this case, the effective \( K(\omega) \) is reduced and the fixed point is realized at \( K \to 0, V \to \infty \). We conclude that the quantum phase transition occurs for the Ohmic environment from the insulating state to the metallic one with quantum critical point at \( K_w = 1 \), for non-Ohmic environment is replaced by a smooth cross-over with characteristic energy scale \( \Lambda^* \) that depends on interaction strength in the wire.

**Conductance of a finite wire at finite temperature** —

At finite temperatures and for a finite length of the wire, the local plasmonic Green’s function of a clean wire is \( G^0_{\varphi \varphi}(x,x,\omega) = -iS(\omega)F(x,\omega)/\omega^2 \), with

\[
F(x,\omega) = K_w^2 - \frac{1-K_w^2}{1+K_w^2} \sum_{\alpha=\pm} \cos[\tau_l \omega(\frac{2}{\omega} + \alpha)] + 2 \frac{1}{(1 + K_w^2) - \frac{(1-K_w^2)^2}{1+K_w^2} \cos^2[\tau_l \omega]}, \tag{7}
\]

the structure function of a many-body Fabry-Pérot interferometer that is formed due to the presence of the plasmons at the boundaries [46, 49]. The finite length of the wire introduces a characteristic time scale to the system, namely, the time of flight for the plasmons \( \tau_l = L/K_w/v_F \) to cross the wire. At high frequencies, \( \omega \tau_l \gg 1 \), the system acts similarly to the infinite wire [60], whereas at small frequencies, \( \omega \tau_l \ll 1 \), \( F(x,\omega) \approx 1 \) and thereby the physics of the interacting wire is washed out and the behaviour of the system is dominated by the environment.

In order to study ac-conductance through the wire, we consider an external probe in the form of an electric potential \( U(x,t) = U(x) \cos(\omega t) \), leading to the additional term in the Hamiltonian, \( \delta H = \sum_\mathcal{F} \int dx \rho_\mathcal{F}(x) U(x) \).

The ac-conductance within linear response theory [25, 26, 60, 61], reads \( G(\omega) = -(e^2/h)[G_0 - G_b] \), where \( G_0 = -2i\omega G^0_{\varphi \varphi}(\omega) \) corresponds to the conductance through a clean wire, and \( G_b = -2i\omega G^R_{\varphi \varphi}(\omega) \) represents the correction to the conductance due to the presence of the impurity. The limit \( \omega \to 0 \) \( G_b(\omega) = (\omega/\omega_c)^{\nu-1} \), which is independent of temperature and interaction strength inside the wire. For the Ohmic case, \( \lim_{\omega \to 0} G_b(\omega) = 1 \), for sub-Ohmic it diverges, and in the super-Ohmic case it vanishes.

Thereby the ac-conductance can be written as \( G(\omega) = (e^2/h)[G_0 - G_b] \), where \( G_b = -2i\omega G^R_{\varphi \varphi}(\omega) \) is the expected power-law dependence \( G_b \propto (T/\omega_c)^{2K_w-2} \), (b) the exponential dependence \( G_b \propto \exp[-\alpha_s(T/\omega_c)^{-1}] \), and (c) \( G_b \propto (T/\omega_c)^{2s-4} \) [60]. The arrows in (b) and (c) marks \( \Lambda^* \), where \( K(\Lambda^*) = 1 \).

In the following, we focus on the temperature-dependence of the correction \( G_b \), see Fig. 3. The temperature mimics the RG flow of the renormalized scattering potential (cf. Fig. 2): (i) In the Ohmic case [Fig. 3(a)], for \( 1/\tau_l \ll T \ll \omega_c \), we obtain a power-law temperature dependence of the form \( G_b \propto (T/\omega_c)^{2K_w-2} \). Such a power-law is characteristic for critical scaling close to a quantum phase transition [60]. Specifically, for repulsive interaction, \( G_b \) grows with decreasing temperature, while it gets suppressed for attractive interactions. For a non-interacting wire, \( G_b \) is independent of temperature. Interestingly, at small temperatures \( (T/\eta_l \ll 1) \), we observe a temperature-independent behaviour for all values of \( K_w \), corresponding to the cutoff of the critical scaling by the finite length of the wire; (ii) For the sub- and super-Ohmic case [Figs. 3(b) and (c), respectively], we observe a characteristic energy-scale \( \Lambda^* \) above which the system qualitatively behaves as in the Ohmic case, i.e., with a power-law decrease governed by the noise scaling with \( \omega_c \).

However, at \( T \ll \Lambda^* \), for the sub-Ohmic case, we obtain an exponential suppression with exponents depending on the interaction strength as \( G_b \propto \exp[-K_w \alpha_s(T/\omega_c)^{-1}] \).

In contrast, in the super-Ohmic case for \( T \ll \Lambda^* \), the \( G_b \) grows with decreasing temperature in a power-law fashion with an exponent that is entirely independent of the interaction strength in the wire \( (2s-4) \). The finite length

**FIG. 3.** Temperature dependence of the impurity-induced correction to conductance through a finite-length wire \( [L = 100\nu_F/(K_w\omega_c)] \) for different interaction strengths \( (K_w) \), cf. Eq. (8). (a) Ohmic, (b) sub-Ohmic \( (s = 0.8) \), and (c) super-Ohmic case \( (s = 1.2) \). The dashed lines show in (a) the expected power-law dependence \( G_b \propto (T/\omega_c)^{2K_w-2} \), (b) the exponential dependence \( G_b \propto \exp[-\alpha_s(T/\omega_c)^{-1}] \), and (c) \( G_b \propto (T/\omega_c)^{2s-4} \) [60]. The arrows in (b) and (c) marks \( \Lambda^* \), where \( K(\Lambda^*) = 1 \).
effect at low temperatures is washed away by the noisy environment.

**Conclusion** The noise statistics of the charge fluctuation at the boundaries of an interacting wire can modify the transport through a dirty Luttinger liquid beyond the Kane-Fisher description. Within a perturbative RG analysis, we showed how an adiabatic coupling to an Ohmic-class environment can turn a repulsively-interacting wire to behave as an attractively-interacting one [and vice versa] at certain energy scales. We further outline the physical implications of the wire–environment competition for transport measurements within linear response. Specifically, at low temperatures, the impurity-induced correction to conductance (i) follows the result of Kane-Fisher [15], and critically-scales with an interaction-dependent power-law up to a finite length cutoff in the Ohmic case, (ii) get washed out in the sub-Ohmic case due to the dominant role of slow (viscous) fluctuations in the environment, and (iii) is effectively amplified as the fast charge-fluctuations (super-Ohmic) at the boundary of the wire acts similarly to a Zeno effect [47]. Future work can extend our analysis beyond the linear response paradigm and employ, e.g., non-equilibrium bosonization [42], as well as explore more sophisticated impurities that allow for incoherent scattering processes [36].

We thank A. Chiocchetta for fruitful discussions. We acknowledge financial support from the Swiss National Science Foundation.

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Supplemental Material for
An impurity in a Luttinger Liquid coupled to Ohmic-class environments
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PLASMONIC GREEN’S FUNCTION OF A CLEAN WIRE

In this section, we outline the calculation details of the plasmonic Green’s function for the interacting wire. We elaborate on its crucial dependence on the environment through the boundary conditions. As mentioned in the main text, we restrict the bosonization treatment to the interacting part of the system (wire), i.e., \( x \in [-L/2, L/2] \), and account for the presence of the environment (leads) through the following boundary conditions (continuity equation):

\[
\partial_t \phi_{L/R}(x = \pm L/2, t) = 2\pi J_{L/R}(t),
\]  

(I.1)

where the boundary operators \( J_{L/R} \) are the current operator in the non-interacting leads

\[
J_{L/R}(\omega) = \int d\omega' c_{L/R,\omega+\omega}' c_{L/R,\omega}',
\]  

(I.2)

with \( c^{(1)}_{L/R,\omega} \) the fermionic (creation) annihilation operators for right/left-moving electrons.

We use the boundary condition (I.1) to solve the equations of motions

\[
\partial_t \theta(x, t) = \frac{\nu_\ell}{K_w^2} \partial_x \varphi(x, t),
\]  

(I.3)

\[
\partial_t \varphi(x, t) = \nu_\ell \partial_x \theta(x, t),
\]  

(I.4)

for the bosonic fields \( \varphi(x, t), \theta(x, t) = (1/\sqrt{2}) \left[ \phi_L(x, t) \pm \phi_R(x, t) \right] \). We obtain a solution in the form of the original right- (left-)mover fields that reads \([1, 2]\)

\[
\phi_{L,R}(x, \omega) = \frac{2\pi}{i\omega} \left\{ J_{L/R}(\omega) \left[ \frac{4}{K_w} \cos[\omega\tau_L(x/L \pm 1/2)] \pm 2i \left( 1 + \frac{1}{K_w^2} \right) \sin[\omega\tau_L(x/L \pm 1/2)] \right] \right\} \frac{1}{\sqrt{\omega^2 + \nu_\ell^2}},
\]  

(I.5)

with \( \tau_L = L K_w/\nu_\ell \) the time-of-flight required for the plasmons to cross the wire. Note that from the particle-hole symmetry of the currents at the boundaries \( J_{L/R}(-\omega) = J_{L/R}^\dagger(\omega) \), the following symmetry for the fields holds

\[ \phi^{\dagger}_{L,R}(x, \omega) = \phi_{L/R}(x, -\omega). \]

Using the current-current correlations at the boundaries, we can define the environment noise spectrum \( S(\omega) \) as

\[
\langle J_\eta(x) J_{\eta'}(x') \rangle = S(\omega) \delta_{\eta,\eta'} \delta(\omega + \omega'), \quad \eta = L, R.
\]  

(I.6)

Thus, we can fully determine the correlation functions of the plasmonic fields as

\[
G_{\varphi^0, \varphi^0}(x, x', \omega) = -i(\varphi(x, \omega)\varphi^\dagger(x', \omega')) = -i \frac{S(\omega) \delta(\omega - \omega')}{\omega^2} F(x, x', \omega),
\]  

(I.7)

with the structure function

\[
F(x, x', \omega) = \left( \frac{1}{K_w^2} - 1 \right) \sum_{\alpha = \pm} \cos[\omega\tau_L(x + x' + \alpha L)/L] + 2 \left( \frac{1}{K_w^2} + 1 \right) \cos[\omega\tau_L(x - x')/L] \left( 1 + \frac{1}{K_w^2} \right)^2 - \left( 1 - \frac{1}{K_w^2} \right)^2 \cos^2[\omega\tau_L],
\]  

(I.8)

which encodes all the information about the interacting wire, i.e., its length \( L \), and the interaction strength \( K_w \). Note that, due to the presence of the environment and the correspondingly-imposed boundary, a Fabry-Pérot cavity
Note that Ohmic-class environments hold detailed balance, and we have Green’s function $G_{\phi_{\phi}}$. In the following, we will use the retarded bosonic Green’s function for the field $\phi$ in the lower/upper-half. Furthermore, defining $y$ where the first/second term in the parenthesis is analytic in the upper/lower-half of the complex plane, i.e., its poles are $n$ with $\tau \in \mathbb{N}$. For $K_w \in (0, 1)$, the poles are shown in Fig. 1. Note that the imaginary part of the poles grows when the repulsive interaction is decreased, and as we approach the non-interacting limit $K_w \to 1$, it becomes infinitely large.

For the purpose of performing complex integration over the structure factor (I.9), it is useful to rewrite it as a sum over two function

$$F(z) = K_w \left\{ \frac{-1}{\frac{i-K_w}{1-K_w} \exp[i\tau z] - 1} + \frac{1}{\frac{i+K_w}{1-K_w} \exp[i\tau z] - 1} \right\},$$

where the first/second term in the parenthesis is analytic in the upper/lower-half of the complex plane, i.e., its poles are in the lower/upper-half. Furthermore, defining $y = \cosh^{-1}\left[\frac{1+K_w^2}{1-K_w^2}\right]$), we have $(1 \pm K_w)/(1 \mp K_w) = \exp(\pm y)$, and therefore we can rewrite the structure function in terms of the bosonic distribution function

$$F(z) = -n_b(i\tau z - y) + n_b(i\tau z + y).$$

### OHMIC-CLASS ENVIRONMENT

We consider a generic Ohmic-class environment with noise power spectrum $S(\omega)$ defined in Eq. (3) in the main text. In the following, we will use the retarded bosonic Green’s function for the field $\phi$ of the clean wire, $G_{\phi_{\phi}}^{\text{R}}(x, t; x', t') = -i\Theta(t - t')[\phi(x, t), \phi^\dagger(x', t')]$, with $\Theta$ the Heaviside function. It can be expressed in terms of lesser and greater Green’s function $G_{\phi_{\phi}}^{<}(x, t; x', t') = -i\langle \phi^\dagger(x', t')\phi(x, t) \rangle$, and $G_{\phi_{\phi}}^{>}(x, t; x', t') = -i\langle \phi^\dagger(x, t)\phi(x', t') \rangle$, as

$$G_{\phi_{\phi}}^{R}(x, x', \omega) = i \int_{-\infty}^{\infty} d\omega' \frac{G_{\phi_{\phi}}^{<}(x, x', \omega') - G_{\phi_{\phi}}^{>}(x, x', \omega')}{\omega' - \omega - \text{i}0^+}.$$

Note that Ohmic-class environments hold detailed balance, and we have

$$G_{\phi_{\phi}}^{\geq}(x, x', \omega) = \frac{S(-\omega)}{S(\omega)} G_{\phi_{\phi}}^{<}(x, x', \omega) = e^{-\beta\omega} G_{\phi_{\phi}}^{>}(x, x', \omega),$$
which follows from Eq. (1.7) [we use the fact that $F(x, x', \omega)$ is a real and even function of $\omega$, see Eq. (1.8)], as we expect for bosons in thermal equilibrium. Thereby, the retarded Green’s function Eq. (II.1) simplifies to

$$G^R_{\varphi\varphi}(x, x', \omega) = \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} F(x, x', \omega') \left| \frac{\omega'}{\omega_c} \right|^{-1} e^{-|\omega'|/\omega_c} G_{\varphi\varphi}^{\pm}(\omega) e^{\pm i\omega t}. \quad (II.3)$$

**Ohmic-environment**

In the case of an Ohmic environment, $s = 1$, we extend the integral in Eq. (II.3) to the complex plane, and employ the analytic properties of the structure function [see discussion leading to Eq. (I.12)], to obtain the local retarded Green’s function for $x = x' = 0$

$$G^R_{\varphi\varphi}(\omega) = \frac{\pm iK_w/2}{\omega \mp i\eta} \left\{ 1 + \frac{1 - i\eta}{iK_w e^{\tau_L(\omega \mp i\eta)}} \right\}. \quad (II.4)$$

Analytically continuing the obtained retarded/advanced Green’s function to the imaginary axis (Matsubara space) [3] by substitution $\omega \mp i\eta \rightarrow i\omega$ for $\omega \geq 0$, we can rewrite the Matsubara Green’s function as $G^0_{\varphi\varphi}(i\omega) = K(\omega)/(2|\omega|)$, with

$$K(\omega) = K_w \frac{1 + \frac{1 - i\omega}{iK_w e^{\tau_L|\omega|}}}{1 - \frac{1 + i\omega}{iK_w e^{\tau_L|\omega|}}}. \quad (II.5)$$

At large frequencies, $\omega\tau_L \gg 1$, $K(\omega)$ approaches the Luttinger Liquid parameter inside the wire $K_w$, whereas for low frequencies, $\omega\tau_L < 1$, $K(\omega)$ goes to 1, resembling the TLL parameter of the noninteracting (Fermi liquid) leads.

We note the the same results can be obtained by solving the equation of motion for Matsubara Green’s function

$$\frac{1}{k(x)} \left[ -\frac{\nu_F}{k(x)} \frac{\partial^2}{\partial x^2} + \frac{k(x)\omega^2}{\nu_F} \right] G^0_{\varphi\varphi}(x, x', i\omega) = \delta(x - x'), \quad (II.6)$$

where $k(x) = K_w$ for $x \in [-L/2, L/2]$, and $k(x) = 1$ elsewhere. This is accomplished by imposing the continuity of the Matsubara Green’s function and of its derivative $\nu_F/k^2(x) \partial_x G^0_{\varphi\varphi}(x, x', i\omega)$ at the wire’s ends $x = \pm L/2$, as well as at the discontinuity at $x = x'$, namely

$$\frac{\nu_F}{k^2(x)} \partial_x G^0_{\varphi\varphi}(x, x', i\omega) \bigg|_{x = x' + 0^-} = 1. \quad (II.7)$$

In conclusion, the finite length of the wire connected to non-interacting electrons introduces an infrared cutoff $1/\tau_L$, below which the wire acts as non-interacting electrons.

**Infinite wire connected to an Ohmic-class environment**

We next consider a generic Ohmic-class environment with $s \in [0, 2]$, for which the plasmonic spectral function reads

$$\rho(\omega) = \text{Im} \{G^R_{\varphi\varphi}(\omega)\} = F(\omega) \frac{\text{sgn}(\omega)}{2\omega_c} \left| \frac{\omega}{\omega_c} \right|^{-s-2} e^{-|\omega|/\omega_c}, \quad (II.8)$$

as is shown in Fig. 2. In the super-Ohmic case, the spectral function at small frequencies, $\omega\tau_L \ll 1$, diverges slower than the ohmic case [i.e., slower than $\sim K_w/(2\omega_c)]$, while the sub-Ohmic case is diverging faster. For the interacting wire at frequencies $\omega\tau_L > 1$, the oscillations are present due to the formation of a Fabry-Pérot cavity.

Having the plasmonic spectral function, we can obtain the plasmonic Matsubara Green’s function, at Matsubara frequencies $\omega_n = 2n\pi/\beta$, $n \in \mathbb{N}$,

$$G^0_{\varphi\varphi}(i\omega_n) = \int_{-\infty}^{\infty} \frac{d\omega'}{\pi} \frac{\rho(\omega')}{\omega' - i\omega_n} = \int_{0}^{\infty} \frac{d\omega'}{\pi} \frac{\omega' \rho(\omega')}{\omega'^2 + \omega_n^2} \approx \frac{K_w}{2\omega_c} \frac{\omega_n}{\omega_c} e^{-|\omega|/\omega_c} \text{Csc}[(\pi s)/2], \quad (II.9)$$

where we used the fact that in the limit of $L \rightarrow \infty$, $F(i\omega) \rightarrow K_w$, and that we are interested in $\omega_n \ll \omega_c$. We can rewrite Eq. (II.9), as $G^0_{\varphi\varphi}(i\omega) = K(\omega)/(2|\omega|)$, with $K(\omega) = K_w \text{Csc}[(\pi s)/2]|\omega|/\omega_c|^{s-1}$. In the Ohmic case, this reduces to $K(\omega) = K_w$ as expected.
Finite wire connected to an Ohmic-class environment

In the case of a finite wire, similar to the infinite length case, we can define the effective TLL parameter as

$$\tilde{K}(\omega, L) = \omega \int_0^\infty \frac{d\omega'}{\pi} \frac{F(\omega')}{\omega^2 + \omega'^2} \frac{\operatorname{sgn}(\omega')}{\omega_c} \left| \omega' \omega_c \right|^{-2} e^{-|\omega'|/\omega_c}.$$  \hspace{1cm} (II.10)

In Fig. 3, we show the numerical evaluation of the integral above, and compare it with the approximate formula

$$\tilde{K}(\omega, L) = K(\omega) \frac{1 + K_w (1 - K_w) e^{-\tau_L |\omega|}}{1 + K_w - (1 - K_w) e^{-\tau_L |\omega|}}.$$  \hspace{1cm} (II.11)

The approximation agrees well for $\omega \ll \omega_c$, irrespective of the value of $s$. Note that from Eq. (II.11) it follows that at high frequencies, $\omega \tau_L \gg 1$, $\tilde{K}(\omega, L) = K(\omega)$ with $K(\omega)$ being the effective Luttinger parameter for $L \to \infty$, cf. Eq. (5) in the main text. At small frequencies, $\omega \tau_L \ll 1$, on the other hand, the curves corresponding to different lengths merge together as is shown in Fig. 3, exhibiting that at such small frequencies, the physics of the interacting wire is washed out and the system’s behaviour is dominated by the environment.
In this section, we present the standard perturbative RG analysis for the problem of a single impurity immersed in a TLL. After integrating-out the leads, the action of the system in imaginary-time reads
\[ A_0(\varphi) = \frac{1}{4\pi} \int_0^\beta d\tau \int_0^\beta d\tau' \int_{-L/2}^{L/2} dx \int_{-L/2}^{L/2} dx' \varphi'(x, \tau) G^{0,-1}_{\varphi\varphi}(x, \tau; x', \tau') + V_0 \int_0^\beta d\tau \cos[\gamma \varphi(x_0, \tau)], \] (III.1)
with \( \gamma = \sqrt{4\pi} \), and \( \varphi(\tau) = \int_{-\infty}^{\infty} d\omega e^{i\omega\tau} \varphi(\omega) \). For all \( x \neq x_0 \), the action is quadratic, and we can integrate out all the corresponding fields to obtain the local action [Eq. (4) in the main text], which can be decomposed as \( A(\varphi) = A_0(\varphi) + A_{\text{imp}}(\varphi) \), with
\[ A_0[\varphi] = \int_0^\beta d\tau \varphi'(\tau) G^{0,-1}_{\varphi\varphi}(\tau) \varphi(\tau), \]
\[ A_{\text{imp}}[\varphi] = V_0 \int_0^\beta d\tau \cos[\gamma \varphi(\tau)], \] (III.2)
where we have assumed \( x_0 = 0 \). We define an ultraviolet cut-off \( \Lambda \) and the corresponding scale-dependent field as \( \varphi_\Lambda(\tau) = \int_{\Lambda}^{\infty} d\omega e^{i\omega\tau} \varphi(\omega) \). For any \( \Lambda' \in [0, \Lambda] \), we can decompose the bosonic fields into low- and high-frequency fields, \( \varphi_\Lambda(\tau) = \varphi_{\Lambda'}(\tau) + h(\tau) \). Now, we turn to the functional-integral formulation of the partition function, and integrate over the high-frequency field \( h(\tau) \) to obtain \( Z = \int D[\{ \varphi_{\Lambda'} \}] e^{-A_{\text{eff}}[\{ \varphi_{\Lambda'} \}]} \), with
\[ A_{\text{eff}}[\{ \varphi_{\Lambda'}(\tau) \}] = A_0[\{ \varphi_{\Lambda'}(\tau) \}] + \langle A_{\text{imp}}[\{ \varphi_{\Lambda'}(\tau) + h(\tau) \}] \rangle_{(h(\tau))} + \langle A_{\text{imp}}^2[\{ \varphi_{\Lambda'}(\tau) + h(\tau) \}] \rangle_{(h(\tau))} + \cdots, \] (III.3)
where
\[ \langle A_{\text{imp}}[\{ \varphi_{\Lambda'}(\tau) + h(\tau) \}] \rangle_{(h(\tau))} = V_0 \int_0^\beta d\tau \int D[\{ h(\tau) \}] e^{-A_0[\{ h(\tau) \}]} \cos[\gamma (\varphi_{\Lambda'}(\tau) + h(\tau))] = \frac{V_0}{2} \int_0^\beta d\tau \left\{ e^{i\gamma \varphi_{\Lambda'}(\tau)} \int D[\{ h(\tau) \}] e^{-A_0[\{ h(\tau) \}]} e^{i\gamma h(\tau)} + c.c \right\}. \] (III.4)
Performing the Gaussian integral over high-frequency fields we obtain
\[ \langle A_{\text{imp}}[\{ \varphi_{\Lambda'}(\tau) + h(\tau) \}] \rangle_{(h(\tau))} = \frac{V_0}{2} e^{-\gamma^2 f_\Lambda^{\Lambda'} f_\tau^{\Lambda'} \varphi_{\Lambda'}^{(\omega)}(\omega)} \left\{ e^{i\gamma \varphi_{\Lambda'} + c.c} \right\}. \] (III.5)
We proceed with the RG procedure and consider an infinitesimal change of the ultraviolet cutoff \( \Lambda' = \Lambda - d\Lambda = \Lambda(1 - dl) \), \( dl = d\Lambda/\Lambda \). In order to compare the physics governed by fields \( \varphi_{\Lambda'} \) with the one governed by the fields \( \varphi_\Lambda \), we re-scale the frequency \( \omega \) to \( \omega' = \omega \Lambda/\Lambda' = (1 + dl)\omega \), and the corresponding imaginary-time \( \tau \) to \( \tau' = \tau(1 - dl) \), such that \( \tau \omega = \tau' \omega' \) [4]. Thereby, we have
\[ \langle A_{\text{imp}}[\{ \varphi_{\Lambda'}(\tau) + h(\tau) \}] \rangle_{(h(\tau))} = \frac{V_0}{2} e^{-\gamma^2 f_\Lambda^{\Lambda'} f_\tau^{\Lambda'} \varphi_{\Lambda'}^{(\omega)}(\omega)} (1 + dl) \int d\tau' \cos[\gamma \varphi_{\Lambda'}(\tau')]. \] (III.6)
Now the high-frequency degrees of freedom can be integrated out while keeping the partition function invariant
\[ Z = \int D[\{ \varphi_\Lambda \}] e^{-A_0[\{ \varphi_\Lambda \}]} - A_{\text{imp}}[\{ \varphi_\Lambda(\tau), V_0 \}]
= \int D[\{ \varphi_{\Lambda(1-dl)} \}] e^{-A_0[\{ \varphi_{\Lambda(1-dl)} \}]} - A_{\text{imp}}[\{ \varphi_{\Lambda(1-dl)}, V(\Lambda) \}] \big) + O(V_0^2), \] (III.7)
provided that the scattering potential is renormalized accordingly, i.e.,
\[ V(\Lambda) = V_0 \left[ 1 + e^{-\gamma^2 f_\Lambda^{\Lambda'} f_\tau^{\Lambda'} \varphi_{\Lambda'}^{(\omega)}(\omega)} (1 + dl) \right]. \] (III.8)
Using the plasmonic Matsubara Green’s function for an infinitely long-wire coupled to the Ohmic-class environment that we obtained in Eq. (II.9), the integral in the exponent becomes
\[ \int_\Lambda^{\Lambda(1-dl)} \frac{d\omega}{2\pi} \varphi_{\varphi\varphi}(\omega) = K_w Csc \left[ \frac{s\pi}{2} \right] \int_\Lambda^{\Lambda(1-dl)} \frac{d\omega}{\omega_c} e^{-\omega/\omega_c} \left| \frac{\omega}{\omega_c} \right|^{s-2} \]
\[ = K_w \ Csc \left[ \frac{s\pi}{2} \right] \left\{ \Gamma[s - 1, \Lambda/\omega_c] - \Gamma[s - 1, (\Lambda/\omega_c)(1 - dl)] \right\} \]
\[ = K_w \ Csc \left[ \frac{s\pi}{2} \right] \left( \frac{\Lambda}{\omega_c} \right)^{s-1} e^{-\Lambda/\omega_c} dl + O((dl)^2), \] (III.10)
where \( \Gamma \) is the incomplete Gamma function. Therefore the flow equation up to the first order in \( dl \) is

\[
\frac{dV}{dl} = V_0 \left[ 1 - K_w \csc[s \pi/2] \left( \frac{\Lambda}{\omega_c} \right)^{s-1} e^{-\Lambda/\omega_c} \right].
\] (III.11)

Taking the limit \( \omega_c \to \infty \), in the Ohmic-case \( (s = 1) \), the flow equation boils down to \( dV/dl = V_0 (1 - K_w) \), which is the known result from Kane and Fisher [5].

For a finite-length wire, we can obtain the flow equation for the scattering potential in an analogous manner. Figure 4 depicts the beta function \( dV/dl = \beta(\Lambda) \) as a function of frequency cut-off for different values of \( s \) and interaction strength \( K_w \). The action of the system in imaginary time reads

\[
\beta(x) = x \neq x \quad \text{local action}
\]

\[
\text{FIG. 4. The beta function, i.e., the right-hand side of Eq. (III.11), as a function of frequency cut-off for different values of } s \text{ and interaction strength } K_w. \text{ Vertical dashed lines mark frequencies } K_w/\tau_L \text{ (green) below which the beta function becomes independent of the interaction strength in the wire, and } \omega_c \text{ (red) above which the environment noise is cut off.}
\]

**CONDUCTANCE**

In this section, we consider an external electrical potential \( U(x, t) \), and calculate the resulting conductance through the interacting wire containing an impurity. The action of the system in imaginary time reads

\[
\mathcal{A}_{\text{tot}} \{ \varphi \} = \mathcal{A}_s \{ \varphi \} + \int_0^\beta d\tau \int_{-L/2}^{L/2} dx \ E(x, \tau) \ \varphi(x, \tau),
\] (IV.1)

where \( \mathcal{A}_s \) is given in Eq. (III.1), and \( E(x, t) = -\partial_x U(x, t) \) is the external electric field. Similar to the previous section, for all \( x \neq x_0 \), the action is quadratic, and we can integrate out all the fields at \( x \neq x_0 \) [6], and obtain the following local action

\[
\mathcal{A}_{\text{eff}} = \frac{1}{\beta} \sum_{\omega_n} \varphi(-\omega_n) \mathcal{G}_{\varphi\varphi}^{-1}(x_0, x_0, i\omega_n) \varphi(\omega_n) + V_0 \int_0^\beta d\tau \ \cos[\gamma \varphi(\tau)]
\]

\[
+ \sum_{\omega_n} \mathcal{G}_{\varphi\varphi}^{-1}(x_0, x_0, i\omega_n) \varphi(\omega_n) \int dx \ \varphi(x, x_0) E(x, \omega_n) \varphi(-\omega_n)
\]

\[= -\frac{\beta}{4} \sum_{\omega_n} \int dx \int dx' \left[ \mathcal{G}_{\varphi\varphi}(x, x', 0) - \mathcal{G}_{\varphi\varphi}(x, 0, x') \right] E(x, \omega_n) E(x', -\omega_n). \] (IV.2)

In the limit \( \omega_n \to 0 \), the structure function (I.8) and hence the Matsubara Green’s function \( \mathcal{G}(i\omega_n) \) become independent of position \( (x, x') \), and hence the local action can be approximated as

\[
\mathcal{A}_{\text{eff}} = \frac{1}{\beta} \sum_{\omega_n} \varphi(-\omega_n) \mathcal{G}_{\varphi\varphi}(i\omega_n)^{-1} \varphi(\omega_n) + V_0 \int_0^\beta d\tau \ \cos[\gamma \varphi(\tau)] + \sum_{\omega_n} \varphi(-\omega_n) U \cos(\omega t),
\] (IV.3)
where we have assumed $x_0 = 0$, and $\int_{-L/2}^{L/2} dx \ E(x, \omega_n) = U \cos(\omega t)$. We are interested in calculating the resulting ac-current, which reads [7]

$$\frac{I_{ac}}{2(\varepsilon^2/\hbar)} = \omega \langle \phi(t) \rangle = \omega \int \mathcal{D}\{\varphi\} \ \langle \phi(\varphi) \rangle e^{-\mathcal{A}_{\text{eff}}} = \omega U \int dt' \ G_{\varphi\varphi}(t, t') \cos(\omega t') = \omega \ U \text{Re} \left\{ e^{i\omega t} G_{\varphi\varphi}(i\omega) \right\},$$

(IV.4)

with $G_{\varphi\varphi}$ being the plasmonic Matsubara Green’s function in the presence of the impurity. As the exact form of the $G_{\varphi\varphi}$ is unattainable, we perform a perturbative expansion [5] in terms of the scattering potential $V_0$, and obtain

$$G_{\varphi\varphi}(i\omega_n) = \int \mathcal{D}\{\varphi\} \ \langle \phi(-\omega_n) \phi(\omega_n) \rangle e^{-\mathcal{A}_{\text{eff}}} = G_{\varphi\varphi}^{0\prime}(i\omega_n) + G_{\varphi\varphi}^{0}(i\omega_n) \Sigma(i\omega_n) G_{\varphi\varphi}^{0}(i\omega_n) + \mathcal{O}(V_0^4),$$

(IV.5)

where

$$\Sigma(i\omega_n) = -V_0^2 \frac{\gamma^2}{2} \sum_{\alpha_1, \alpha_2} \int_0^\beta d\tau_1 \int_0^\beta d\tau_2 \left[ 1 + \alpha_1 \alpha_2 \cos[\omega_n(\tau_1 - \tau_2)] \right] e^{-\frac{\gamma^2}{\beta} \sum_{\omega_n \neq 0} G_{\varphi\varphi}^{0\prime}(i\omega_n)[1 + \alpha_1 \alpha_2 \cos[\omega_n(\tau_1 - \tau_2)]]}$$

(IV.6)

is the impurity-induced self energy. In the following, we try to simplify the above expression by using the analytic properties of Matsubara Green’s functions. First, we define

$$E(\tau) = -\frac{2\gamma^2}{\beta} \sum_{\omega_n} G_{\varphi\varphi}^{0\prime}(i\omega_n)[1 - \cos(\omega_n \tau)] = -\frac{\gamma^2}{\beta} \sum_{\omega_n \neq 0} G_{\varphi\varphi}^{0\prime}(i\omega_n)[1 + e^{i\omega_n \beta} - e^{i\omega_n \tau} - e^{i\omega_n (\beta - \tau)}],$$

(IV.7)

where we have used $e^{i\omega_n \beta} = 1$. Using the Bose-Einstein distribution function $n_b(z) = [\exp(\beta z) - 1]^{-1}$, we can rewrite the sum in Eq. (IV.7) as the contour integral in the complex plane. Furthermore, we deform the contour to be parallel to the real axis, and obtain

$$E(\tau) = -\frac{\gamma^2}{\beta} \int_{-\infty}^{\infty} \frac{d\omega}{\pi i} G_{\varphi\varphi}^{0\prime\prime}(\omega) \left\{ \coth(\beta \omega/2)[1 - \cosh(\omega \tau)] + \sinh(\omega \tau) \right\},$$

(IV.8)

Therefore, the impurity-induced self energy becomes

$$\Sigma(i\omega_n) = V_0^2 \int_0^\beta d\tau \left[ 1 - \cos(\omega_n \tau) \right] e^{E(\tau)} = V_0^2 \int_0^\beta d\tau \left[ 1 - \exp(i\omega_n \tau) \right] e^{E(\tau)},$$

(IV.9)

where we have used the periodic properties of $E(\tau) = E(\beta - \tau)$. Further deforming of the contour integration in the complex plane results in

$$\Sigma(i\omega_n) = iV_0^2 \left\{ \int_0^\infty dt \left[ 1 - e^{-\omega_n t} \right] e^{E(it)} - \int_{-\infty}^0 dt \left[ 1 - e^{\omega_n t} \right] e^{E(it)} \right\}.$$  

(IV.10)

Now, we perform the analytical continuation to the real axis ($i\omega_n \rightarrow \omega + i0^+$), and obtain

$$\Sigma^R(\omega) = iV_0^2 \left\{ \int_0^\infty dt \left[ 1 - e^{i(\omega + i0^+ t)} \right] e^{E(it)} - i \int_{-\infty}^0 dt \left[ 1 - e^{-i(\omega + i0^+ t)} \right] e^{E(it)} \right\} = i\omega \tilde{I} + \mathcal{O}(\omega^2),$$

(IV.11)

with

$$\tilde{I} = -iV_0^2 \int_{-\infty}^{\infty} dt \ e^{E(it)} = -V_0^2 (\beta/2) \int_{-\infty}^{\infty} dt \ e^{E(it)}.$$  

(IV.12)

Note that in Eq. (IV.12), we once again employed the periodic properties $E(\beta - \tau) = E(it)$. Finally, the retarded Green’s function up to second order in $V_0$ reads

$$G_{\varphi\varphi}^{R}(\omega) = G_{\varphi\varphi}^{R0}(\omega)[1 + i\omega \tilde{I} G_{\varphi\varphi}^{R0}(\omega)],$$

(IV.13)

which can be employed to calculate the ac current Eq. (IV.4), and hence the ac conductance

$$G(\omega) = \frac{I_{ac}(\omega)}{U \cos(\omega t)} = 2\frac{e^2}{\hbar} \left\{ i\omega G_{\varphi\varphi}^{R0}(\omega) - \omega^2 \tilde{I} [G_{\varphi\varphi}^{R0}(i\omega)]^2 \right\}.$$  

(IV.14)

The first term in Eq. (IV.14) is the conductance of a clean wire [$G_0$ in the main text], and the second term corresponds to the back-scatterings from the impurity [$G_b$ in the main text]. We conclude this section by emphasising that the calculation of the conductance (IV.14) boils down to the evaluation of the two integrals in Eqs. (IV.12) and (IV.8).
Ohmic environment

In this subsection, we outline the analysis of the temperature dependence of the conductance through the wire which is in contact with Ohmic leads with noise power spectrum \( S(\omega) = \omega [1 + n_0(\beta \omega)] \).

**Infinitely long wire**

For an infinitely long-wire, the local retarded Green’s function is \( G_{\varphi \varphi}^{0,R}(i\omega) = \frac{iK_w}{2\omega + i\beta} e^{-|\omega|}. \) Hence, Eq. (IV.8) becomes

\[
E(it) = -\frac{K_w \gamma^2}{\pi} \int_0^\infty d\omega \frac{e^{-\omega/\omega_c}}{\omega} \left\{ \left[ 1 - \cos(\omega t) \right] \text{coth}(\beta \omega) + i \sin(\omega t) \right\} 
= -\frac{K_w \gamma^2}{\pi} \ln(1 + i\omega_c t) + \sum_{m=1}^\infty \ln \left[ 1 + \left( \frac{t}{m\beta + (1/\omega_c)} \right)^2 \right]. \tag{IV.15}
\]

Concentrating on the low temperature behaviour, \( \omega_c \beta \gg 1 \), we obtain

\[
E(it) \approx -\frac{K_w \gamma^2}{\pi} \ln \left\{ 1 + i\omega_c t \left[ \frac{\beta}{\pi t} \sinh \left( \frac{\pi t}{\beta} \right) \right] \right\}, \tag{IV.16}
\]

which makes the evaluation of Eq. (IV.12) feasible, resulting in

\[
G = \frac{e^2}{\hbar} \left[ K_w - \frac{V_0^2 K_w^2}{2\omega_c^2} \frac{\sqrt{\pi} \Gamma(K_w)}{\Gamma[1/2 + K_w]} \left( \frac{\pi}{\beta \omega_c} \right)^{2K_w - 2} \right]. \tag{IV.17}
\]

However, note that this situation is unphysical, namely in order to generate a current through the interacting wire, we have to connect the system to the electronic leads, which forces us to take into account the finite-length of the wire and the resulting frequency structure of the Luttinger liquid parameter as we shall see in the following.

**Finite wire connected to Ohmic leads**

We now consider a finite-length wire, where the Luttinger Liquid parameter acquires a frequency dependence, as shown in the discussion related to Eq. (II.5). We evaluate the integrals [Eqs. (IV.12) and (IV.8)] numerically and show the results in Fig. 5. At high temperatures \( T \tau_L > 1 \), the conductance can be approximated as

\[
G \approx \frac{e^2}{\hbar} \left[ 1 - \frac{V_0^2}{2\omega_c^2} \frac{\sqrt{\pi} \Gamma(K_w)}{\Gamma[1/2 + K_w]} \left( \frac{\pi}{\beta \Lambda} \right)^{2K_w - 2} \right], \tag{IV.18}
\]

which has been obtained similar to the previous case of infinitely large wire [cf. Eq. (IV.17), albeit considering that the dc-limit of the effective Luttinger liquid parameter is 1 (i.e., \( K(\omega \to 0) = 1 \)).

**Finite wire connected to Ohmic-class leads**

We consider a generic Ohmic-class environment with the noise power spectrum defined in Eq. (3) in the main text, and obtain the temperature dependence of the self-energy, and hence impurity-induced back-scattering conductance. We can rewrite Eq. (IV.8) as \( E(it) = E_1(it) + E_2(it) \), with the first term being the temperature-independent part

\[
E_1(it) = -\gamma^2 K_w(\omega_c)^{1-s} \int_0^\infty \frac{d\omega}{\pi} \omega^{s-2} e^{-\omega/\omega_c} \left[ 1 - e^{-i\omega t} \right] 
= K_w \frac{-\gamma^2}{\pi} \Gamma[s - 1] \left[ 1 - (1 + i\omega_c t)^{1-s} \right], \tag{IV.19}
\]
and $E_2(\tau)$ being the temperature-dependent component

$$E_2(it) = - (\omega_c)^{1-s} \gamma^2 K_w \int_0^\infty \frac{d\omega}{\pi} \omega^{s-2} e^{-\omega/t} \left[ \log(\beta\omega/2) - 1 \right] \left[ 1 - \cos(\omega t) \right]$$

$$= - \gamma^2 \frac{1}{\Gamma(s-1)} (\beta\omega_c)^{1-s} \left\{ 2\zeta[s-1,1+\frac{1}{\beta\omega_c}] - \zeta[s-1,1+\frac{1}{\beta\omega_c} - it] - \zeta[s-1,1+\frac{1}{\beta\omega_c} + it] \right\},$$

with $\zeta$ the Riemann zeta-function. The linear (in frequency) component of the self-energy [see Eq. (IV.11)] then reads

$$\tilde{I} = -V_0^2 (\beta/2) \int_{-\infty}^{\infty} dt e^{E_1(it) + E_2(it)}$$

$$= - V_0^2 (\beta/2) e^{-K_w \alpha_s} \int_{-\infty}^{\infty} dt e^{-K_w \alpha_s (1+it)/\beta} \left\{ 2\zeta[s-1,1+\frac{1}{\beta\omega_c}] - \zeta[s-1,1+\frac{1}{\beta\omega_c} - it] - \zeta[s-1,1+\frac{1}{\beta\omega_c} + it] \right\},$$

with $\alpha_s = \gamma^2 \frac{1}{\pi} \Gamma(s-1)$.

**sub-Ohmic case**

In the sub-Ohmic case, the current-current fluctuations are more pronounced at smaller frequencies, and hence at low-temperatures, the environmental effects become dominant. For this case $s < 1$, the temperature dependant part, namely Eq. (IV.20) is important. By changing the integration variable $t \rightarrow t + i\beta/2$ in Eq. (IV.21) and using $\zeta[s-1,1+q] = \zeta[s-1,q] - q^{-(s-1)}$, we obtain

$$\tilde{I} = - (\beta/2) V_0^2 e^{-K_w \alpha_s} \int dt e^{K_w \alpha_s (\beta\omega_c)^{1-s}} 2\zeta[s-1,1+\frac{1}{\beta\omega_c}] - \zeta[s-1,1+\frac{1}{\beta\omega_c} - \frac{t}{\beta}] - \zeta[s-1,1+\frac{1}{\beta\omega_c} + \frac{t}{\beta}] \right\].$$

As we are interested in the low-temperature behaviour, $\omega_c \beta \gg 1$, at which the prefactor $(\beta\omega_c)^{1-s} \gg 1$, we can employ the method of steepest descend. In other words, we have

$$\sum_{\eta = \pm} \zeta[s-1,1+\frac{1}{\beta\omega_c} + \frac{1}{2} + \eta \frac{t}{\beta}] \approx 2\zeta[s-1,1+\frac{1}{\beta\omega_c} + \frac{1}{2}] - \left( \frac{t}{\beta} \right)^2 \zeta[s+1,1+\frac{1}{\beta\omega_c} + \frac{1}{2}] s(s-1),$$

which makes the integral in Eq. (IV.22) Gaussian, resulting in

$$\tilde{I} = V_0^2 A_s e^{-B_s T^{1-s}} T^{\frac{8s}{3}},$$

FIG. 5. The conductance as a function of rescaled temperature for various interaction strengths ($K_w \leq 1$), cf. Eqs. (IV.12) and (IV.8). The dashed-dotted lines correspond to Eq. (IV.18).
with
\[ A_s = e^{K_w a_s} \left( \frac{\pi}{4K_w \Gamma [s + 1] \zeta [s + 1, \frac{1}{2} + \frac{1}{\beta \omega_c}] \omega_c^{1-s}} \right)^{1/2}, \]
\[ B_s = 2\omega_c^{1-s} \zeta [s - 1, 1 + \frac{1}{\beta \omega_c}]. \] (IV.25)

Since we are in the limit \( \beta \omega_c \gg 1 \), further temperature dependence in Eqs. (IV.25) can be neglected. In conclusion, we found that the temperature dependence of the self-energy reads \( \Sigma (i\omega) = \omega A_s \exp [-B_s T^{s-1}] T^{(s+3)/2} \), cf. Fig. (3) in the main text.

**super-Ohmic case**

We now turn into the super-Ohmic case \( s > 1 \), cf. Fig. (3) in the main text. The self-energy is
\[ \tilde{I} = -(\beta/2)V_0^2 \lim_{\omega \to 0} \int dt e^{i\omega t} e^{E(t) + (\beta/2)} = -(\beta/2)V_0^2 e^{K_w a_s} e^{-B_s T^{1-s}} \int dt e^{i\omega t} e^{\tilde{E}(t)}, \] (IV.26)
where we have defined
\[ E(it + \beta/2) = \int_0^\infty d\omega \, G_{\varphi \varphi}(i\omega) \left\{ \coth (\omega t/2) - \frac{\cos (\omega t)}{\sinh (\omega t/2)} \right\} \equiv \tilde{E}_0 + \tilde{E}(t). \] (IV.27)

In contrast to the sub-ohmic case, in the low-temperature limit the pre-factor \((\beta \omega_c)^{1-s} \ll 1\), and hence we can expand the exponential in Eq. (IV.22) obtaining
\[ \tilde{I} \approx -(\beta/2)V_0^2 e^{K_w a_s} \frac{1}{2} \int dt e^{i\omega t} \tilde{E}^2(t) = -(\beta/2)V_0^2 \int_0^\infty d\omega \frac{G_{\varphi \varphi}^2(i\omega)}{\sinh^2(\omega t/2)} \propto T^{2s-4}. \] (IV.28)

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