QUARK COUNTING RULES:  
OLD AND NEW APPROACHES

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I discuss the subject of powerlike asymptotic behavior of hadronic form factors in pre-QCD analyses of soft (Feynman/Drell-Yan) and hard (West) mechanisms, and also recent derivation of $1/Q^2$ asymptotics of meson form factors in AdS/QCD. At the end, I briefly comment on “light-front holography” ansatz.

1. Hadronic form factors

Introduction. Experimental evidence that (exclusive) form factors of hadrons consisting of $n_q$ quarks behave like $(1/Q^2)^{n_q-1}$ for large $Q^2$, provokes expectations that there is a fundamental and/or easily visible reason for such a phenomenon, scale invariance being the most natural suspect.1 Indeed, hard rescattering in a theory with spinor constituents and dimensionless coupling constant for their interaction with an intermediary boson field provides a specific dynamical mechanism2 that produces the $(1/Q^2)^{n_q-1}$ behavior. In this approach, $n_q - 1$ is just the number of hard exchanges. Another property apparently correlated with the number of quarks in the hadron is the $\sim (1-x)^{2n_q-3}$ behavior of the (inclusive) quark distributions functions in the $x \to 1$ region. This observation suggests to look for connection between these exclusive and inclusive observables. Below in this section we discuss scenarios which display two versions of exclusive-inclusive correlation. In subsequent sections, we discuss derivation of the $1/Q^2$ behavior for meson form factors in AdS/QCD.

Soft mechanism. Powerlike behavior of hadronic form factors due to

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Feynman mechanism can be derived from the Drell-Yan formula
\[
F(Q^2) = \int_0^1 dx \int d^2k_\perp \Psi^*(x, k_\perp + (1-x)q_\perp)\Psi(x, k_\perp),
\]
which represents form factor in terms of the light-front wave function \(\Psi(x, k_\perp)\) and light-front variables \(x\) and \(k_\perp\). When the wave function \(\Psi(x, k_\perp)\) rapidly (say, exponentially) decreases for \(k_\perp \gtrsim \Lambda\), it is natural to consider the region where both \(\Psi(x, k_\perp)\) and \(\Psi^M(x, k_\perp + \bar{x}q_\perp)\) are maximal: \(i)\) \(k_\perp \lesssim \Lambda\) is small and \(ii)\) \(\bar{x} \equiv 1 - x\) is close to 0, so that \(|\bar{x}q_\perp| \lesssim \Lambda\).

If
\[
|\Psi(x, k_\perp)|^2 \sim (1 - x)^{2n - 3},
\]
then
\[
F(Q^2) \sim \int \frac{\Lambda}{Q^2} \frac{1}{x^{2n-3}} d\bar{x} \sim (1/Q^2)^{n_q - 1}.
\]
The parton distribution functions in this formalism are given by the integral of \(|\Psi(x, k_\perp)|^2\) over \(k_\perp\). The latter is dominated by \(k_\perp \lesssim \Lambda\), hence \(f(x) \sim (1 - x)^{2n_q - 3}\). Thus, changing the shape of \(f(x)\), one would change the result for form factor. In other words, there is a causal relation between the \(x \to 1\) shape of the distribution function \(f(x)\) and the \(Q^2 \to 1\) behavior of the form factor \(F(Q^2)\): form of \(f(x)\) determines \(F(Q^2)\).

**Hard mechanism.** For the Feynman/DY mechanism it was important that the fraction \(\bar{x} \equiv 1 - x\) vanishes in the \(Q^2 \to 0\) limit. Consider now the regions in DY formula (1), in which the fraction \(\bar{x}\) is finite, while the transverse momentum argument of one of the wave functions is small, e.g., the region \(|k_\perp| \ll \bar{x}|q_\perp|\), where \(\Psi(x, k_\perp)\) is maximal. Then
\[
F(Q^2) \sim \int \left|\Psi^*(x, \bar{x}q_\perp)\varphi(x)\right| dx,
\]
where
\[
\varphi(x) = \int \Psi(x, k_\perp) d^2k_\perp
\]
is the relevant distribution amplitude. In this scenario, the form factor repeats large-\(k_\perp\) behavior of the hadron wave function, e.g., if \(\Psi(x, k_\perp) \sim (1/k_\perp^2)^n\), then \(F(Q^2) \sim (1/Q^2)^n\). This mechanism was proposed by G.B. West, who used, in fact, a covariant Bethe-Salpeter (BS) formalism rather than light-front variables, writing the form factor as
\[
F(Q^2) \sim \int f(p) f(p + q) d^4p,
\]
where \(f(p)\) is treated as a function of the active parton virtuality \(t \equiv p^2\) and spectator mass \(M^2\). Assuming that \(f(t, M^2) \sim t^{-n}g(M^2)\) for large \(t\),
West concludes that $F(Q^2) \sim (1/Q^2)^n$: form factor repeats the large-$Q^2$ behavior of the BS wave function $f(p+q)$. For the deep inelastic structure function, West obtains

$$\nu W^2(x) \sim \int_{t_{\text{min}}}^{t_{\text{max}}} f^2(t, M^2) dt \sim (t_{\text{min}})^{-2n+1}, \quad (6)$$

where $t_{\text{min}} = (\frac{1-x}{1-x_N}) [M^2 - (1-x)M_N^2]$, $M_N$ being the nucleon mass. As a result, $\nu W^2(x) \sim (1-x)^{2n-1}$.

**DY vs West’s model.** If $n = n_q - 1$, the power-law predictions of the two models formally coincide. However, these results were obtained from completely different assumptions. In DY picture, the active parton is “on-shell” both before and after the collision: both $|k_\perp|$ and $|k_\perp + \bar{x}q_\perp|$ are of order $\Lambda$, and form factor $F(Q^2)$ reflects the size of phase space region in which $1-x \sim \Lambda/Q$. On the other hand, in West’s model, the active parton is highly virtual either in initial or final state, and $F(Q^2)$ reflects the $t$-dependence of WF for large virtualities $t = p^2$. Still, though the two mechanisms are completely different, the connection $(1/Q^2)^n \Leftrightarrow (1-x)^{2n-1}$ (“Drell-Yan-West relation”) holds in both models! It should be also emphasized that in West’s model, $(1/Q^2)^n$ and $(1-x)^{2n-1}$ have the same cause (large-$t$ behavior of $f(p)$), but they are not “causing” each other.

**West’s hard mechanism & pQCD.** In DY model, $n$ is not necessarily integer. Integer values of $n$ naturally appear in West’s hard scenario, where they are related to the number of hard propagators. In particular, hard exchange in a theory with a dimensionless coupling constant gives $n = n_q - 1$ [2], which is a consequence of scale invariance. In quantum chromodynamics, each hard gluon exchange is accompanied by effective coupling.

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This is apparently why the two models are confused up to the point that Eq. (1) is often referred to as “Drell-Yan-West formula”, which is absolutely incorrect because its crucial feature is incorporation of light-front variables that West did not use.
constant $\alpha_s$, i.e., $F_n(q^2) \sim (\alpha_s/Q^2)^{n_q-1}$. According to explicit calculation, the asymptotic prediction for the pion form factor in pQCD is $F_{\pi}(Q^2) \sim (2\alpha_s/\pi)s_0/Q^2$, where $s_0 = 4\pi^2 f_\pi^2 \approx 0.7 \text{GeV}^2 \sim m_\rho^2$. Compared to the VMD expectation $F_{\pi}(Q^2) \sim m_\rho^2/Q^2$, pQCD prediction is suppressed by $2\alpha_s/\pi$ factor. It is well known that the factor $\alpha_s/\pi \approx 0.1$ is penalty for an extra loop, which suggests that the hard one-gluon-exchange contribution is an $O(\alpha_s)$ correction to some $O(\alpha_s^0)$ term. The only candidate is the Feynman/DY soft contribution, which should be calculated in a nonperturbative way. In particular, in holographic AdS/QCD models considered in Refs. [8,9] one has $F_{\pi}(Q^2) \sim s_0/Q^2$, without a suppression factor.

2. Vector meson form factors in AdS/QCD

Models based on AdS/CFT correspondence are often claimed to provide nonperturbative explanation of quark counting rules for form factors that is based on conformal invariance and short-distance behavior of normalizable modes $\Phi(\zeta)$ playing the role of wave functions of initial and final hadrons. Namely, in the model of Polchinski and Strassler (that involves on the AdS side scalar fields only) one has

$$F(Q^2) = \int_0^{1/\Lambda} \Phi_P(z)J(Q,z)\Phi_P(z)\,dz/z^3, \quad (7)$$

where $J(Q,z) = zQK_1(zQ) \equiv K_1(zQ)$ is nonnormalizable mode describing the probing EM current, and normalizable modes for mesons are given by $\Phi(z) = Cz^2J_{L+1}(\beta_L k z \Lambda)$, with $K_1$ and $J_{L+1}$ being standard Bessel functions. For large $Q$, one may approximate $K_1(zQ) \sim e^{-zQ}$, and it is clear that only small $z \lesssim 1/Q$ contribute. As a result, $F_{L=0}(Q^2) \to 1/Q^4$ for the ground state. But this is not the $1/Q^2$ power that one is longing to get! To bring the result of this AdS/CFT-based model in agreement with pQCD expectations, Brodsky and de Teramond proposed to modify the basic principle of AdS/CFT correspondence, requiring that the dimension of the operator on the AdS side should be equal to the twist of the corresponding current in the 4-dimensional theory rather than to its dimension. In our papers with H.R. Grigoryan we demonstrated that in more realistic AdS/QCD models of Refs.16,17 it is possible to get $F_{L=0}(Q^2) \to 1/Q^2$ for (leading) meson form factors without challenging the Maldacena correspondence principle.

**Hard-wall model** is formulated in 5-dimensional space $\{x^\mu, z\} \equiv X^M$ having AdS$_5$ metric $ds^2 = (\eta_{\mu\nu}dx^\mu dx^\nu - dz^2)/z^2$ with a hard wall: $0 \leq z \leq z_0 = 1/\Lambda$. The basic object is the 5-dimensional (5D) vector gauge
field \( A_M (X) \) \((M = \mu, z)\) which produces 4D field \( A_\mu (x) = A_\mu (x, z = 0)\). at the UV boundary of AdS space. The 5D gauge action for the vector field is given by

\[
S_{\text{AdS}} = -\frac{1}{4g_5^2} \int d^4x \, dz \, \sqrt{g} \, \text{Tr} \left( F_{MN} F^{MN} \right) ,
\]

(8)

where \( F_{MN} \) is the field-strength tensor. The coupling constant \( g_5^2 = \frac{6\pi^2}{N_c} \) is small in large-\( N_c \) limit. The free field satisfies

\[
\Box^5 A(X) = 0 \quad \text{or} \quad \Box^4 A(x, z) + z \partial_z \left( \frac{1}{z} \partial_z A(x, z) \right) = 0 .
\]

(9)

In 4D momentum representation this gives

\[
z \partial_z \left( \frac{1}{z} \partial_z \tilde{A}(p, z) \right) + p^2 \tilde{A}(p, z) = 0 .
\]

(10)

According to AdS/QCD correspondence

\[
\tilde{A}_\mu (p, z) = \frac{A_\mu (p) V(p, z)}{V(p, 0)} \equiv \tilde{A}_\mu (p) V(p, z) ,
\]

(11)

where the bulk-to-boundary propagator \( V(p, z) \) satisfies Eq.(10). The gauge-invariant boundary condition (b.c.) \( F_{\mu z}(x, z_0) = 0 \) on the infrared (IR) wall results in Neumann b.c. \( \partial_z V(p, z_0) = 0 \), with solution

\[
V(p, z) = P \left[ Y_0 (P z_0) J_1 (P z) - J_0 (P z_0) Y_1 (P z) \right] .
\]

(12)

Using Kneser-Sommerfeld formula\(^{19}\) gives bound state expansion

\[
V(p, z) = -\sum_{n=1}^{\infty} \frac{g_5 f_n}{p^2 - M_n^2} \psi_n(z)
\]

(13)

with masses: \( M_n = \gamma_{0,n}/z_0 \) determined by zeros \( J_0 (\gamma_{0,n}) = 0 \) of Bessel functions, while the “coupling constants” \( f_n \) are given by

\[
f_n = \frac{\sqrt{2} M_n}{g_5 z_0 J_1 (\gamma_{0,n})} .
\]

(14)

They are accompanied by “\( \psi \)” wave functions

\[
\psi_n(z) = \frac{\sqrt{2}}{z_0 J_1 (\gamma_{0,n})} z J_1 (M_n z)
\]

(15)

coinciding with nonnormalizable modes of Polchinski-Strassler model.\(^{10,11}\)

These “\( \psi \)” wave functions (w.f.) obey equation of motion (10) with \( p^2 = M_n^2 \), satisfy \( \psi_n(0) = 0 \) at UV boundary, and \( \partial_z \psi_n(z_0) = 0 \) at IR boundary. They are normalized according to

\[
\int_0^{z_0} |\psi_n(z)|^2 \frac{dz}{z} = 1 .
\]

(16)
However, they do not look like bound state w.f. in quantum mechanics, see Fig. 2, left. To this end, it makes sense to introduce “φ” wave functions

\[
\phi_n(z) \equiv \frac{1}{M_n z} \partial_z \psi_n(z) = \frac{\sqrt{2}}{z_0 J_1(\gamma_0 n)} J_0(M_n z) .
\]  

(17)

According to Sturm-Liouville equation (10), they are reciprocal to “ψ” w.f.:

\[
\psi_n(z) = -z \partial_z \phi_n(z) / M_n .
\]  

(18)

The φ w.f. give couplings \( g_5 f_n / M_n \) as their values at the origin, they satisfy Dirichlet b. c. \( \phi_n(z_0) = 0 \) at confinement radius, and are normalized by

\[
\int_0^{z_0} |\phi_n(z)|^2 z dz = 1 .
\]  

(19)

The “φ” w.f. (see Fig. 2, right) are thus analogous to bound state wave functions in quantum mechanics. The difference between the two types of AdS/QCD wave functions can be easily understood: ψ w.f. correspond to vector-potential \( A_M \), while φ w.f. correspond to field strength tensor \( F_{MN} \).

Three-point function should be introduced to study form factors. It has a “Mercedes-Benz” form

\[
W(p_1, p_2, q) = \int_0^{z_0} V(p_1, z) V(p_2, z) V(q, z) \frac{dz}{z} .
\]  

(20)

For spacelike \( q \) (with \( q^2 = -Q^2 \)) we have \( V(iQ, z) \equiv J(Q, z) \) The form factors for diagonal \( n \to n \) transitions may be written

\[
F_{nn}(Q^2) = \int_0^{z_0} J(Q, z) |\psi_n(z)|^2 \frac{dz}{z} .
\]  

(21)

either in terms of ψ functions or in terms of φ functions:

\[
F_{nn}(Q^2) = \frac{1}{1 + Q^2 / 2M_n^2} \int_0^{z_0} J(Q, z) |\phi_n(z)|^2 z dz .
\]  

(22)
The overlap integral here is a direct analogue of form factors in quantum mechanics, so we define
\[ \mathcal{F}_{nn}(Q^2) \equiv \int_0^\infty J(Q, z) |\phi_n(z)|^2 z \, dz. \] (23)

The hard-wall model calculation gives
\[ \langle \rho^+(p_2, \epsilon') | J_{EM}^\mu(0) | \rho^+(p_1, \epsilon) \rangle = -\epsilon'_\beta \epsilon_\alpha \left[ \eta_{\alpha\beta}(p_1 + p_2)_\mu + 2(\eta_{\alpha\mu} q_\beta - \eta_{\beta\mu} q_\alpha) \right] F_{nn}(Q^2). \] (24)

But it is well known that vector mesons have three form factors:
\[ \langle \rho^+(p_2, \epsilon') | J_{EM}^\mu(0) | \rho^+(p_1, \epsilon) \rangle = -\epsilon'_\beta \epsilon_\alpha \left[ \eta_{\alpha\beta}(p_1 + p_2)_\mu G_1(Q^2) + (\eta^{\mu\alpha} q^\beta - \eta^{\mu\beta} q^\alpha)(G_1(Q^2) + G_2(Q^2)) - \frac{1}{4M^2} \eta^{\nu\rho} q^{\alpha}(p_1^\nu + p_2^\nu) G_3(Q^2) \right], \] (25)
i.e., 
\[ G_1(Q^2) = G_2(Q^2) = F_{nn}(Q^2) \text{ and } G_3(Q^2) = 0 \text{ [20].} \]

The form factor (23) is projected by taking the “+++” component of 3-point correlator,
\[ \mathcal{F}(Q^2) = G_1(Q^2) + \frac{Q^2}{2M^2} G_2(Q^2) - \left( \frac{Q^2}{2M^2} \right)^2 G_3(Q^2). \] (26)

For \( \rho \)-meson, this combination coincides with the IMF “LL” transition having \( \sim \alpha_s/Q^2 \) behavior in pQCD.\(^{21}\) Taking the hard-wall model prediction (23) and using that \( z \sim 1/Q \) dominate in the large-\( Q \) limit because \( J(Q, z) \rightarrow zQ K_1(Q z) \sim e^{-Qz} \), we may substitute \( \phi(z) \) by \( \phi(0) \). Thus,
\[ \mathcal{F}(Q^2) = -\frac{|\phi(0)|^2}{Q^2} \int_0^\infty d\chi \chi^2 K_1(\chi) = 2 \frac{|\phi(0)|^2}{Q^2}, \] (27)
and we get the same power of \( 1/Q^2 \) as in pQCD, but without \( \alpha_s/\pi \) factor.

**Soft-wall model**\(^{17}\) corresponds to \( z^2 \) barrier, and bulk-to-boundary propagator \( \mathcal{V}(p, z) \) can be written \( (a = -p^2/4\kappa^2) \) as\(^{14}\)
\[ \mathcal{V}(p, z) = a \int_0^1 dx x^{a-1} \exp \left[ -\frac{x}{1-x} \kappa^2 z^2 \right]. \] (28)

The propagator poles are located at \( p^2 = 4(n+1)\kappa^2 \equiv M_n^2 \) [17]:
\[ \mathcal{V}(p, z) \equiv \kappa^2 z^2 \sum_{n=0}^\infty \frac{L_n(\kappa^2 z^2)}{a + n + 1} = \sum_{n=0}^\infty \frac{g_5 f_n}{M_n^2 - p^2} \psi_n(z). \] (29)

Just like in the hard-wall case, we deal with \( \psi \) wave functions and coupling constants \( g_5 f_n \) given by their derivatives at the origin
\[ g_5 f_n = \frac{1}{z} e^{-\kappa^2 z^2} \partial_z \psi_n(z) \bigg|_{z=0} = \sqrt{8(n+1)\kappa^2}. \] (30)
Again, we can introduce the (Sturm-Liouville-) conjugate wave functions:

\[ \phi_n(z) = \frac{1}{M_n} e^{-\kappa^2 z^2} \partial_z \psi_n(z) = \frac{2}{M_n} e^{-\kappa^2 z^2} L_n^0(\kappa^2 z^2). \]  \hspace{1cm} (31)

Taking the diagonal form factor for the lowest state

\[ F_{00}(Q^2) = 2 \int_0^\infty e^{-\kappa^2 z^2} f(Q, z) \, dz \]  \hspace{1cm} (32)

and using representation (28) for \( f(Q, z) \) gives

\[ F_{00}(Q^2) = \frac{1}{1 + Q^2 / M_0^2}, \]

i.e., exact vector meson dominance. Large-\( Q^2 \) behavior of \( F \) form factor is given by the same expression (27) as in hard-wall model, the only difference being in the value of w.f. at the origin. As a result, we have

\[ F_H(\rho)(Q^2) \rightarrow 2 \frac{m_\rho^2}{Q^2}, \quad F_S(\rho)(Q^2) \rightarrow \frac{m_\rho^2}{Q^2}. \]  \hspace{1cm} (33)

3. Pion Form Factors in AdS/QCD

The full action of hard-wall model is given by

\[ S_{AdS}^B = \text{Tr} \int d^4x \int_0^{\infty} dz \left[ \frac{1}{z^3} (D^M X)^\dagger (D_M X) + \frac{3}{z^3} X^\dagger X \right. \]

\[ - \frac{1}{8g_5^2 z} \left( B_{(L)}^{MN} B_{(L)MN} + B_{(R)}^{MN} B_{(R)MN} \right) \]  \hspace{1cm} (34)

where \( DX = \partial X - iB_{(L)} X + iX B_{(R)}, B_{(L,R)} = V \pm A \) and \( X(x, z) = v(z)U(x, z)/2 \) involves the chiral field: \( U(x, z) = \exp [i\sigma^a \pi^a(x, z)] \), with the pion field \( \pi^a(x, z) \). The chiral symmetry is broken by the term \( v(z) = (m_q z + \sigma z^3) \), with \( m_q \sim \text{quark mass} \) and \( \sigma \) playing the role of quark condensate. The longitudinal component of the axial field \( A^\parallel_M(x, z) = \partial_M \psi^a(x, z) \) gives another pion field \( \psi^a(x, z) \). The model satisfies Gell-Mann–Oakes–Renner relation \( m_\pi^2 \sim m_q \). In the chiral limit \( m_q = 0 \), it is possible to get the analytic result \(^8,22\) for \( \Psi(z) \equiv \psi(z) - \pi(z) \)

\[ \Psi(z) = z \Gamma(2/3) \left( \alpha^2 \right)^{1/3} \left[ I_{-1/3}(\alpha z^3) - I_{1/3}(\alpha z^3) \right], \]  \hspace{1cm} (35)

where \( \alpha = g_5 \sigma / 3 \). \( \Psi(z) \) satisfies \( \Psi(0) = 1 \), Neumann b.c. \( \Psi'(z_0) = 0 \) and

\[ f_2^i = - \frac{1}{g_5} \left( \frac{1}{z} \partial_z \Psi(z) \right)_{z = 0} \]

The conjugate wave function is given by

\[ \Phi(z) = - \frac{1}{g_5 f_2^i} \left( \frac{1}{z} \partial_z \Psi(z) \right) = - \frac{2}{s_0} \left( \frac{1}{z} \partial_z \Psi(z) \right). \]  \hspace{1cm} (36)
Fig. 3. Pion wave functions $\Psi(z) \rightarrow \psi(\zeta,a)$ and $\Phi(z) \rightarrow \phi(\zeta,a)$ as functions of $\zeta \equiv z/z_0$ and $a \equiv \alpha z_0^3$ for $a = 0, a = 1, a = 2.26, a = 5$ and $a = 10$.

where $s_0 = 4\pi f_\pi^2 \approx 0.67$ GeV$^2$ is the usual characteristic scale for pion.

The function $\Phi(z)$ satisfies $\Phi(0) = 1$ and Dirichlet b.c. $\Phi(z_0) = 0$.

Pion EM form factor written in terms of $\Psi(z)$ looks like

$$F_\pi(Q^2) = \frac{1}{g_5 f_\pi^2} \int_0^{z_0} J(Q,z) \left[ \frac{\partial \Psi}{\partial z} \right]^2 + \frac{g_5^2 f_\pi^2}{z^4} \Psi^2(z) \right] \ dz. \quad (37)$$

To analyze form factor at large $Q^2$, we write it in terms of $\Psi(z)$ and $\Phi(z)$:

$$F_\pi(Q^2) = \int_0^{z_0} J(Q,z) \left[ g_5^2 f_\pi^2 \Phi^2(z) + \frac{9 \alpha_s^2 g_5^2 f_\pi^2}{Q^2} \right] \ dz. \quad (38)$$

For large $Q$, only $z \sim 1/Q$ part of $\Phi^2(z)$ term works, which gives

$$F_\pi(Q^2) \rightarrow \frac{2 g_5^2 f_\pi^2 \Phi^2(0)}{Q^2} = \frac{4\pi f_\pi^2}{Q^2} \equiv \frac{s_0}{Q^2}. \quad (39)$$

The curve we obtained from the AdS/QCD model (see Ref.[8]) goes above existing experimental data that give $Q^2 F_\pi(Q^2) \approx 0.4$ GeV$^2$, which means that the pion in this model is too small.

We remind that pQCD result$^6,7$ has $2\alpha_s/\pi$ factor

$$F_{\pi}^{P\text{QCD}}(Q^2) \rightarrow \frac{2\alpha_s}{\pi}, \frac{s_0}{Q^2} \sim 0.2 \ F_{\pi}^{\text{AdS/QCD}}(Q^2) \quad (40)$$

due to one-gluon exchange.

Anomalous amplitude of the $\gamma^* \gamma^* \pi^0$ transition is defined by

$$\int \langle \pi, p | T \{ J_{\text{EM}}^\mu(x) J_{\text{EM}}^\nu(0) \} | 0 \rangle e^{-iq_1 x} d^4 x$$

$$= e^{i\nu \alpha \beta} q_1 \alpha q_2 \beta \frac{N_c}{12\pi^2 f_\pi} K_{\gamma^* \gamma^* \pi^0}(Q_{1,2}^2), \quad (41)$$

where $p = q_1 + q_2$ and $q_{1,2}^2 = -Q_{1,2}^2$. Its value for real photons is fixed in QCD by axial anomaly: $K_{\gamma^* \gamma^* \pi^0}(0,0) = 1$. To consider this form factor, the AdS/QCD model should be extended. We need isoscalar fields,
which is achieved by gauging $U(2)_L \otimes U(2)_R$ and introducing the field $B_\mu = t^a B^a_\mu + \frac{1}{2} \eta_\mu$, and we also need the Chern-Simons term
\[
S^{(3)}_{CS}[B] = \frac{N_c}{24\pi^2} \epsilon^{\mu
u\rho\sigma} \text{Tr} \int d^4x dz \left( \partial_\mu B_\sigma + B_\nu \partial_\rho \sigma \right).
\] (42)

The anomalous form factor conforming to QCD anomaly is given by
\[
K(Q^2_1, Q^2_2) = \Psi(z_0) J(Q_1, z_0) J(Q_2, z_0) - \int_0^{z_0} J(Q_1, z) J(Q_2, z) \partial_z \Psi(z) dz.
\] (43)

For large $Q_1$ and/or $Q_2$ we may write
\[
K(Q^2_1, Q^2_2) \approx \frac{s_0}{2} \int_0^{z_0} J(Q_1, z) J(Q_2, z) \Phi(z) z dz.
\] (44)

If one of the photons is real, we have
\[
K(0, Q^2) \rightarrow \Phi(0) \int_0^\infty d\chi \chi^2 K_1(\chi) = \frac{s_0}{Q^2}.
\] (45)

For comparison, in pQCD $\gamma^* \gamma \pi^0$ form factor is given by
\[
K^{pQCD}(0, Q^2) = \frac{s_0}{3Q^2} \int_0^1 \frac{\varphi_\pi(x)}{x} dx = \frac{s_0}{3Q^2} I^\varphi.
\] (47)

The pQCD result agrees with AdS/QCD model if $I^\varphi = 3$, e.g., for $\varphi_\pi(x) = 6x(1-x)$ (asymptotic DA). Our model is very close to Brodsky-Lepage interpolation $K_{BL}(0, Q^2) = 1/(1 + Q^2/s_0)$ which goes above CLEO data. However, next-to-leading pQCD correction is negative which allows to fit CLEO data if one takes distribution amplitudes with $I^\varphi \approx 3$.

In case of large and equal photon virtualities, the AdS/QCD result is
\[
K(Q^2, Q^2) \rightarrow \Phi(0) \int_0^\infty d\chi \chi^3 [K_1(\chi)]^2 = \frac{s_0}{3Q^2}.
\] (46)

Note that pQCD result in this kinematics does not depend on pion DA
\[
K^{pQCD}(Q^2, Q^2) = \frac{s_0}{3} \int_0^1 \frac{\varphi_\pi(x)}{xQ^2 + (1-x)Q^2} dx = \frac{s_0}{3Q^2} I^\varphi
\] (47)

and coincides with AdS/QCD model!

For non-equal large photon virtualities, we write $Q^2_1 = (1 + \omega)Q^2$ and $Q^2_2 = (1 - \omega)Q^2$. The leading-order pQCD then gives
\[
K^{pQCD}(Q^2_1, Q^2_2) = \frac{s_0}{3Q^2} \int_0^1 \frac{\varphi_\pi(x)}{1 + \omega(2x - 1)} dx = \frac{s_0}{3Q^2} I^\varphi(\omega),
\] (48)
while the AdS/QCD model result reads

\[
K(Q_1^2, Q_2^2) \to \frac{\Phi(0)s_0}{2Q^2} \sqrt{1 - \omega^2} \int_0^\infty d\chi \chi^3 K_1(\chi \sqrt{1 + \omega}) K_1(\chi \sqrt{1 - \omega})
\]

\[
= \left( \frac{s_0}{3Q^2} \right) \left\{ \frac{3}{4\omega^2} \left[ 2\omega - (1 - \omega^2) \ln \left( \frac{1 - \omega}{1 + \omega} \right) \right] \right\} .
\]

(49)

Note, that the term enclosed in curly brackets coincides with pQCD $I_\phi(\omega)$ for $\phi(x) = 6x(1 - x)$. Indeed, using representation

\[
\chi K_1(\chi) = \int_0^\infty e^{-\chi^2/4u - u} du ,
\]

and integrating over $\chi$ we get

\[
K(Q_1^2, Q_2^2) \to \frac{s_0}{Q^2} \int_0^\infty \int_0^\infty \frac{u_1u_2 e^{-u_1-u_2} du_1du_2}{u_2(1 + \omega) + u_1(1 - \omega)} .
\]

(51)

Changing $u_2 = x\lambda$, $u_1 = (1 - x)\lambda$ and integrating over $\lambda$ gives

\[
K(Q_1^2, Q_2^2) \to \frac{s_0}{3Q^2} \int_0^1 6x(1 - x) dx \frac{1 + \omega(2x - 1)}{\pi} .
\]

(52)

Comment on “Light-Front Holography”. The AdS/CFT form factor expression (7) has structure similar to that of DY light-front formula (1), especially when the latter is written in terms of the impact parameter space w.f. $\tilde{\Psi}(x, b_\perp)$. Brodsky and de Teramond\textsuperscript{12} noticed that, identifying $z$ with $|b_\perp|\sqrt{x(1 - x)}$ and taking a special form of the light-front w.f.

\[
\tilde{\Psi}(x, b_\perp) = \frac{1}{\sqrt{2\pi}} \frac{\Phi(|b_\perp|\sqrt{x(1 - x)})}{b_\perp^2 \sqrt{x(1 - x)}} ,
\]

(53)

one can convert the 3D DY formula (1) into the 1D AdS/CFT integral (7). This observation is the basis of the “Light-Front Holography” approach.\textsuperscript{24} However, it is easy to check that if one would calculate the meson couplings $f_n$ (14), (30) from the light-front w.f. fixed by this ansatz, the results would have an extra $\sqrt{6}\pi/8$ factor (see Eqs.(88),(89) of Ref.\textsuperscript{[23]}) compared to exact AdS/QCD results (14), (30). Furthermore, this ansatz gives $8\sqrt{x(1 - x)/\pi}$ for meson distribution amplitude, while we demonstrated above that AdS/QCD results for $\gamma^*\gamma^* \to \pi^0$ form factor correspond to asymptotic $6x(1 - x)$ distribution amplitude. In general, the light-front holography ansatz\textsuperscript{12} is not consistent with AdS/QCD for any observable that depends linearly on the w.f. (rather than bilinearly as in DY formula).
4. Summary

Summarizing, we established that meson form factors in AdS/QCD are given by formulas similar to those in quantum mechanics. For large $Q$, there is only one mechanism $z \sim 1/Q$. For vector mesons, the leading $(LL)$ IMF form factor $F(Q^2)$ indeed behaves like $1/Q^2$ for large $Q^2$. In soft-wall model, $F(Q^2)$ demonstrates exact $\rho$-dominance. For pion, large-$Q^2$ asymptotics is $s_0/Q^2$ vs. pQCD result $(2\alpha_s/\pi)s_0/Q^2$. We included the anomalous amplitude into the AdS/QCD analysis, extending it to $U(2)_L \otimes U(2)_R$ and adding the Chern-Simons term. Fixing normalization by conforming to QCD anomaly, we observed that large-$Q^2$ behavior coincides then with pQCD calculations for asymptotic pion DA, the result contradicting the claim of “light-front holography” approach that meson distribution amplitude is given by $8\sqrt{x(1-x)}/\pi$. In conclusion, AdS/QCD is an instructive model for what may happen with form factors in real-world QCD.

Acknowledgements

I am very grateful to Organizers for invitation to Workshop honoring 60th anniversary of M. Shifman and their hospitality. Happy birthday, Misha! I thank H.R. Grigoryan for collaboration on the studies of form factors in AdS/QCD.

This paper is authored by Jefferson Science Associates, LLC under U.S. DOE Contract No. DE-AC05-06OR23177. The United States Government retains and the publisher, by accepting the article for publication, acknowledges that the United States Government retains a non-exclusive, paid-up, irrevocable, world wide license to publish or reproduce the published form of this manuscript, or allow others to do so, for United States Government purposes.

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