Integrability from 2d $\mathcal{N} = (2, 2)$ dualities

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Received 24 April 2015, revised 4 July 2015
Accepted for publication 27 July 2015
Published 1 September 2015

Abstract

We study integrable models in the context of the recently discovered Gauge/YBE correspondence, where the Yang–Baxter equation (YBE) is promoted to a duality between two supersymmetric gauge theories. We study flavored elliptic genus of 2d $\mathcal{N} = (2, 2)$ quiver gauge theories, which are defined from statistical lattices regarded as quiver diagrams. Our $R$-matrices are written in terms of theta functions and simplify considerably when the gauge groups at the quiver nodes are Abelian. We also discuss the modularity properties of the $R$-matrix, reduction of 2d index to 1d Witten index, and string theory realizations of our theories.

Keywords: 2d supersymmetry, susy field theory, Yang–Baxter equation

1. Introduction

In this paper we discuss the celebrated Yang–Baxter equation (YBE), which is one of the most fundamental characterizations of integrable models (see, e.g., [1] and references therein). YBE has several different expressions. For concreteness let us here use the following version, formulated in the language of the Interaction-Round-a-Face (IRF) model (we will comment on the formulation as a vertex model later):

$$
\sum_g \mathcal{R}(u) \begin{bmatrix} f & g \\ a & b \end{bmatrix} \mathcal{R}(u + v) \begin{bmatrix} d & c \\ g & b \end{bmatrix} \mathcal{R}(v) \begin{bmatrix} e & d \\ f & g \end{bmatrix}
= \sum_g \mathcal{R}(v) \begin{bmatrix} g & c \\ a & b \end{bmatrix} \mathcal{R}(u + v) \begin{bmatrix} e & g \\ f & a \end{bmatrix} \mathcal{R}(u) \begin{bmatrix} e & d \\ g & c \end{bmatrix}.
$$

Here $\mathcal{R}(u)$ is known as the $R$-matrix, where the parameters $u, v$ are continuous parameters called spectral parameters. Each
The indices $a, b, c, d$ run over the possible values the spins could take in a given integrable model; for example, in the Ising model we have $a, b, ... = \pm 1 \in \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$.

A graphical representation of YBE is given in figure 1. Here an $R$-matrix is represented by a parallelogram, whose four vertices are associated with the four indices $a, b, c, d$ of the $R$-matrix $[d \ a] \ [c \ b]$.

Given a solution to YBE (1.1), we can define an integrable model by associating a Boltzmann weight $R(F)$ to a face $F$ and by summing over all the possible spin configurations:

$$Z_{2d \text{ integrable spin}} = \sum_{\text{spins}} \prod_{F \text{-face}} R(F).$$

The YBE ensures that the transfer matrices commute (under appropriate boundary conditions$^4$), and when transfer matrices are expanded with respect to spectral parameters, we obtain an infinite set of conserved charges.

The YBE is a highly over constrained equation. For example, suppose that the spins take values in $\mathbb{Z}_N := \mathbb{Z}/N\mathbb{Z}$, with $N$ large. Then the $R$-matrix has $O(N^2)$ variables, whereas the constraints from the YBE grow as $O(N^3)$. This problem becomes more severe for integrable models with continuous spins (our following model fits this category), which can be formally thought of as the limit $N \to \infty$, and it looks almost impossible to find any solution at all. Despite these naive expectations, people have found a variety of solutions of YBE. The natural question is then why integrable models exist.

Recently there is a new look on this long-standing problem [2] (see also [3, 4]; this correspondence is called the Gauge/YBE correspondence). Namely the YBE is promoted ("categorified") to a duality (Yang–Baxter duality) between two supersymmetric quiver gauge theories, which duality in turn follows from a sequence of $4d$ $\mathcal{N} = 1$ Seiberg dualities. The basic logic is that this duality automatically generates a mathematical equality of the partition functions of the two theories, and the YBE follows directly from the Yang–Baxter duality.

The lift of the YBE to a duality is rather powerful, since duality is not about a single identity but rather a set of such identities—namely, we can compute various partition functions and observables (satisfying certain constraints to be discussed later), and each of these

$^4$ For example, we can choose fixed or periodic boundary conditions.
gives rise to (in general) different solutions for the YBE. For example, in [2] the 4d lens index [5], the twisted partition function on $S^4 \times S^1/\mathbb{Z}_r$, gave rise to a large class of integrable models which are previously unknown in the literature. We can also discuss various degenerations of the model. For $r = 1$ the model reduces to the ‘master solution’ of [6] (see also [7–9]), where the name ‘master’ originates from the fact that it successfully reproduced all the known solutions of the star–triangle relations with positive Boltzmann weights. The degeneration of this model gives rise to a variety of integrable models [6, 7, 10–13], eventually all the way down to the Ising model, which is at the bottom of this hierarchy of integrable models.

The natural question is whether we can adopt the same logic to supersymmetric field theories in dimensions other than four. In [2] it has already been pointed out that a similar story works for (for example, supersymmetric $S^4 \times S^2$ partition function of) 3d $\mathcal{N} = 2$ version of Seiberg duality (Aharony duality [14]); as far as the underlying combinatorial structure of the quiver is the same, we should obtain a solution to the YBE.

The goal of this paper is to generalize the logic of [2] to 2d $\mathcal{N} = (2, 2)$ quiver gauge theories. With the help of the Seiberg-like duality in 2d, we propose a 2d version of the Yang–Baxter duality, and compute supersymmetric partition function on $T^2$ (elliptic genus) [16–18]. This automatically gives rise to 2d classical integrable models:

$$\mathcal{I}_{2d\mathcal{N} = (2, 2) \text{ theory}}[T^2] = Z_{2d \text{ integrable spin-}}$$

The resulting 2d spin system obeys either the periodic or fixed boundary condition, depending on whether we have a quiver diagram on a torus or on a plane. Correspondingly, the quiver diagram is drawn either on a torus [19] or on a disc [20, 21].

While this is to some extent a simple adoption of existing 4d techniques to 2d, we encounter some new features. Along the way we will also clarify some aspects of the Gauge/YBE correspondence itself.

The first subtlety is that the $T^2$ partition function has a subtlety in the choice of the contour of integration, which should be taken into account in the discussion of integrable models. Second, the $R$-matrices are written in terms of the well-known theta functions, and we can directly prove the YBE by evaluating the integrals. This would be helpful to those mathematical physicists who do not wish to go through the derivation from non-perturbative dualities in gauge theories. This contrasts with the case of the 4d, where the invariance of the $S^3 \times S^1/\mathbb{Z}_r$ index for a Seiberg-dual pair is not proven mathematically. Third, for the case of the $T^2$ partition function, our $R$-matrix (and hence the resulting integrable model) depends on a modular parameter $\tau$ of the torus and has a nice modular property under the $SL(2, \mathbb{Z})$ action on $\tau$.

The rest of this paper is organized as follows. In section 2 we review the basic logic of the Gauge/YBE correspondence, and apply it to the case of 2d $\mathcal{N} = (2, 2)$ Seiberg-like duality. Along the way we highlight some of the key ingredients which are crucial for the discussion of this paper. In section 3 we explicitly construct the integrable model corresponding to $T^2$. We also include comments on brane realizations in section 4. The final

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5 Towards the completion of this paper, we received [15], which also discusses integrable models associated with 2d $\mathcal{N} = (2, 2)$ theories.

6 For the case of the $S^3 \times S^1$ partition function ($r = 1$ in our previous notation), this follows from the identity of [22], as pointed out in [23].

7 See, however, discussion of the $SL(3, \mathbb{Z})$ modularity of the 4d $S^3 \times S^1$ index [24].
2. Yang–Baxter equation from duality

In this section we summarize the basic logic of how to associate an integrable model to a 2d Seiberg-like duality. The explanation of the Gauge/YBE correspondence here closely follows [2]; however, the presentation here is improved in a number of technical points. We also emphasize several key aspects crucial for the 2d duality, which are implicit in the 4d discussion of [2].

2.1. Seiberg-like duality

In the construction of [2], the crucial input for the 4d Yang–Baxter duality was the 4d Seiberg duality [25]. We will therefore look for a Seiberg-like duality for 2d theories. There are several different versions of Seiberg-like dualities for 2d theories in the literature, both for Abelian and non-Abelian gauge groups, see, e.g., [26–28]. For our purposes we need a version [16, 29] whose matter contents are essentially the same as that of the 4d $\mathcal{N}=1$ Seiberg duality. Let us quickly summarize this duality.

The 2d $\mathcal{N}=2,2$ theory is a gauged linear sigma model (GLSM), and its Lagrangian can be obtained from the dimensional reduction of the parent 4d $\mathcal{N}=1$ theory. The definition of the chiral and vector multiplets, for example, works in a similar manner. There are several differences, however. One difference is that we can define a twisted superfield $\Sigma := T_v D V$, satisfying $D_v \Sigma = D_v \Sigma = 0$. We can furthermore consider the twisted superpotential

$$L_{\text{FI}} = \frac{1}{2} \left( -t \int d^2 \theta \, \Sigma + (\text{c.c.}) \right) = -r D + \theta F_0. \tag{2.1}$$

where we defined the complexified FI parameter $t$ by

$$t := r + i\theta. \tag{2.2}$$

Here $\theta$ angle is periodic, $\theta \sim \theta + 2\pi$, and it hence it is natural to consider the exponentiated single-valued variable

$$z := (-1)^N e^i. \tag{2.3}$$

Here we take the gauge group to be $U(N_f \equiv N)$. Note that we include the diagonal $U(1)$ factor in the gauge group, which plays a crucial role in the dynamics of 2d $\mathcal{N}=(2,2)$ theories; this sharply contrasts with the case of 4d $\mathcal{N}=1$ theory, where the $U(1)$ factor decouples.

On one side of the duality (electric theory) we have $U(N)$ gauge theory with $N_f$ flavors, i.e., $N_f$ fundamentals $q_i$ and $N_f$ antifundamentals $\bar{q}_{\bar{i}}$. Here $i = 1, \ldots, N_f$ is the index for the flavor symmetry. We have no superpotential: $W_{\text{electric}} = 0$. We also turn on the twisted superpotential (2.1) with complexified FI parameter $z_{\text{electric}}$.

On the other side (magnetic theory) we have $U(N_f - N)$ gauge theory with $N_f$ flavors, i.e., $N_f$ fundamentals $Q_i$ and $N_f$ antifundamentals $\bar{Q}_{\bar{i}}$. We also have a meson field $M_{ij}$ with superpotential coupling

$$W_{\text{magnetic}} = \text{Tr} \left( \bar{Q}_{\bar{i}} M_{ij} Q_i \right). \tag{2.4}$$

We also turn on the twisted superpotential (2.1) with complexified FI parameter $z_{\text{magnetic}}$. 

section (section 5) contains concluding remarks, with technical material summarized in the two appendices.
The statement for the duality is that the electric and magnetic theories flow in the IR to the same fixed point, if we identify $\varepsilon_{\text{electric}} = \varepsilon_{\text{magnetic}}^{-1}$. (2.5)

It turns out that this is part of the transformation properties of the cluster $y$-variable [30].

In the literature there are several consistency checks for this duality [16–18, 31]. It is shown that they have the same chiral ring as well as the twisted chiral ring. They also have the same $S^2$ partition function and the $T^2$ partition function, and finally the duality has a geometrical counterpart in the geometry of the Grassmannian $\text{Gr}(N, N_f)$. This duality can be checked by explicitly computing the $T^2$ partition functions, which in turn ensures the YBE. Note that our construction of integrable models in itself does not rely on the full non-perturbative duality, but rather on the equality of the specific partition function (namely, the $T^2$ partition function). However, the underlying gauge theory duality is the ultimate reason why these integrable models should exist.

For our purposes of constructing integrable models, what is important here is that the matter content for the 2d Seiberg-like duality is essentially the same as the 4d Seiberg duality. This means that we can immediately borrow most of the results from [2]. For example, this Seiberg-like duality for a quiver gauge theory is represented graphically as in figure 2. As explained in [2], this is the gauge theory counterpart of the relation known as the star–star relation [32, 33] in integrable models. The star–star relation ensures the YBE; correspondingly, 2d Seiberg duality implies the 2d version of the Yang–Baxter duality.

For simplicity of the presentation in the following we will concentrate on the case $N_f = 2N$, with all the quiver nodes having gauge group $U(N)$ with the same rank $N$. One advantage of this choice is that the rank of the gauge group, and hence the number of the

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Figure 2. The change of the quiver diagram for the 2d Seiberg-like duality. A circle (box) represents a gauge (global) symmetry. In the following we specialize to the case $N_c = N_b = N_f = N_g = N$ (and hence $N_f = 2N$ for the gauge group in the middle). The parameters $\alpha, \beta, \cdots$ represent the $R$-charge of the bifundamental fields. For a closed loop (triangle), their $R$-charges sum up to 2.

Figure 3. The $R$-matrix is obtained from the partition function of the quiver on the right.

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8 However, only a limited subset of the solutions to the YBE originates from the star–star relation.
components of the spin of the integrable model at a lattice site, is preserved by the Seiberg-like duality.

2.2. Constraints on partition functions

Let us here be more precise as to which partition functions we could consider in (1.4). Closer inspection reveals that the logic of [2] relies crucially on the following four general properties of the partition function.

**Invariance under the IR duality.** The first requirement is that the partition function coincides for two UV dual theories, both of which flow to the same IR fixed point. This ensures the invariance of the partition function under the Yang–Baxter duality. Note that this is the case if the partition function is independent of the gauge coupling constant, which is indeed the case for $T^2$ partition function discussed in this paper$^9$.

The invariance sometimes holds up to overall constant factors or up to the appropriate change of parameters. In these cases, care is needed for the identification of a precise mathematical identity.

**Factorization.** Let us consider the partition function for a quiver gauge theory, with the gauge field (matter) associated with the vertex (edge) of the quiver diagram. Then the second requirement is that the classical as well as 1-loop contribution to the integrand of the integral/sum expression for the partition function for this quiver gauge theory factors into contributions from gauge field and matters. Schematically,

$$ Z_{\text{quiver}} = \int Z_{\text{vertex}} Z_{\text{edge}}, $$

(2.6)

where the integral here could represent either a sum or an integral or their hybrids. This ensures that the resulting expression has an interpretation as a statistical mechanical model with the nearest neighbor as well as self-interactions among spins at different sites.

**Gauging/gluing.** Suppose that we have a theory with a flavor symmetry $G$ and write its partition function as $Z_{\text{before gauging}}[a]$, where $a$ is (a set of) parameter(s) corresponding to the background gauge field for the symmetry $G$. Let us next gauge the symmetry $G$ to promote it to a gauge symmetry. The third requirement is that the partition function for the resulting theory takes the form

$$ Z_{\text{after gauging}} = \int [da] Z_{\text{gauge field}}[a] Z_{\text{global}}[a]. $$

(2.7)

where $Z_{\text{gauge field}}[a]$ is the contribution from the gauge field as well as its superpartner(s), and the integral is over $a$ with appropriate measure $[da]$. In other words, $Z_{\text{after gauging}}$ with a global symmetry $G$ can be computed in two steps, first keeping the symmetry $G$ as a flavor symmetry and then gauging $G$.

This condition should hold for any consistent localized partition function (it is a supersymmetric counterpart of Fubini’s theorem for the path integral), and in many cases (including the 4d $\mathcal{N} = 1$ lens index discussed in [2]) is satisfied trivially. However, there are subtleties for the 2d index, on which we will comment later in this paper.

**$R$-charge.** For the construction of infinite-many conserved charges it is crucial to have spectral parameters. In the Gauge/YBE correspondence, $R$-charge is identified with the $R$-charges of the bifundamental chiral multiplets. It is therefore important that the partition function depends non-trivially on the $R$-charges of the fields.

$^9$ Even if the partition function depends non-trivially on the gauge coupling constant, we should be able to extract a precise mathematical identity from a gauge theory duality, as long as we can keep track of the dependence of the partition function on the gauge coupling constant.
2.3. R-matrix

Once the conditions noted earlier are satisfied, we can write down the integrable model and the R-matrix. Let us briefly summarize the minimal material; for details, readers are referred to [2].

The basic idea is to identify the quiver diagram with the lattice of the statistical mechanical model.

An edge $e \in E$ of the quiver diagram, starting from a vertex $t(e)$ and ending on another $h(e)$, represents the nearest-neighbor interaction between the spins at $t(e)$ and $h(e)$. In gauge theory, this represents a 2d $\mathcal{N} = (2, 2)$ chiral multiplet$^{10}$ with R-charge $r$, whose partition function we denote by $\mathbb{W}_r^e = \mathbb{W}_r(t(e), h(e))$ ($\mathbb{W}$ stands for ‘weight’).

We have the relation

$$\mathbb{W}_r^e(a, b)\mathbb{W}_r^{2-1}(b, a) = 1.$$  (2.8)

This reflects that fact that the sum of the R-charges of the corresponding chiral multiplets is two, and hence we can write down a mass term. This means that we can integrate out the fields in the IR, leading to the trivial partition function, as in the right hand side of (2.8).

We also have a 2d $\mathcal{N} = (2, 2)$ vector multiplet at a vertex $v$. This represents the self-interactions among the spin $s_v$. We denote the corresponding partition function as $\mathbb{S}^v$. (Note that we here used the fact that the superpotential in (2.4) has R-charge 2; it then follows, for example, that the R-charges for the electric quarks and magnetic quarks sum up to 1.

The R-matrix, the Boltzmann weight for a face $F$ in the IRF model, is defined to be a modification of (2.10) and (2.11):$^{12}$

10 2d $\mathcal{N} = (2, 2)$ chiral and vector multiplets are dimensional reductions of their 4d $\mathcal{N} = 1$ counterparts.

11 Here we denoted the sum over spin by the symbol $\sum$; however, in our practical applications later (associated with $T^7$ partition function), the ‘spin’ will be a set of continuous parameters taking values in the Cartan of the gauge group, and the ‘sum’ will actually be integrals over these continuous parameters.

12 Equation (2.9) in version 1 of [2] contained a typo, which is corrected here.
where in the last line we used the identity (2.10).

For a face $F$ with external vertices $a, b, c$ as in figure 1, we denote the rapidity parameters of four edges by $\alpha, \beta, \gamma, \delta$. The vanishing of the beta function for the gauge coupling in the IR imply that they satisfy the relation

$$\alpha + \beta + \gamma + \delta = 2. \quad (2.14)$$

The square root factor of $\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{\square
This subtlety, which was known in the integrable model literature even before the discovery of Seiberg duality, is the manifestation of the presence of the mesons in Seiberg duality. Indeed, the $R$-matrix as defined in (2.12) has a nice symmetry, thanks to the Seiberg-like duality:

$$
\begin{vmatrix}
\delta \\
\alpha \\
\beta \\
\end{vmatrix}
\begin{vmatrix}
d \\
a \\
\end{vmatrix}
= \frac{S_{b b} S_{d d}}{S_{S_{S} c}}
\begin{vmatrix}
1 - \delta & 1 - \gamma \\
1 - \alpha & 1 - \beta \\
\end{vmatrix}
\begin{vmatrix}
a \\
b \\
\end{vmatrix}.
$$

(2.16)

To discuss YBE, we need to glue the $R$-matrices. At the level of the quiver gauge theories, this corresponds to concatenating the quiver diagram (see figure 5).

When we combine two quivers, it sometimes happens that the half-chiral multiplets are combined together. If they are of the opposite orientations, we cancel them; if of the same orientation, we replace them by a full chiral multiplet.

To be more precise, we have to take into account the constraint from $R$-charges. For example, in figure 5 the two arrows with $R$-charges $2 - \epsilon - \zeta$ and $2 - \gamma - \delta$ cancel in the end, and in order for this to happen we need to be able to write down a mass term for them. This leads to the constraint

$$
(2 - \epsilon - \zeta) + (2 - \gamma - \delta) = 2.
$$

(2.17)

This constraint remains even after the chiral multiplet is integrated out, but then now the equation should lead to $\epsilon + \zeta + \gamma + \delta = 2$; namely, we generate a new superpotential term corresponding to a face in the middle, with its superpotential normalized to be 2.
Following the similar logic, we can verify that the definitions of the partition function as an IRF model (1.3) coincides with that as a vertex model (2.9).

We can repeat this exercise and generate a quiver for the left hand side of figure (1.1). A new feature here is that we do not have a new internal vertex, which we interpret as a gauge node (see figure 6). The YBE (1.1) is now replaced with the Yang–Baxter duality, a duality between two different quivers as shown in figure 7.

Figure 7. For the YBE, we can glue three $R$-matrices, following the rule of figure 6. After canceling/combining the half-chiral multiplets in the equality on both sides of the equation, we obtain the duality of the last line, which was precisely the Yang–Baxter duality of [2].

Following the similar logic, we can verify that the definitions of the partition function as an IRF model (1.3) coincides with that as a vertex model (2.9).

We can repeat this exercise and generate a quiver for the left hand side of figure (1.1). A new feature here is that we do not have a new internal vertex, which we interpret as a gauge node (see figure 6). The YBE (1.1) is now replaced with the Yang–Baxter duality, a duality between two different quivers as shown in figure 7.
The identity coming from the Yang–Baxter duality then takes the form
\[
\sum_{g} R^{\delta \gamma}_{\alpha \beta} \left[ f^{g}_{a} h^{g}_{b} \right] R^{\mu \lambda}_{\iota \kappa} \left[ d^{c}_{g} h^{c}_{b} \right] R^{\theta \eta}_{\epsilon \zeta} \left[ e^{d}_{g} h^{d}_{b} \right] \bigg|_{\kappa = \gamma + \zeta}
= \sum_{g} R^{\theta \eta}_{\epsilon \zeta} \left[ e^{d}_{g} h^{d}_{b} \right] R^{\mu \lambda}_{\iota \kappa} \left[ d^{c}_{g} h^{c}_{b} \right] R^{\delta \gamma}_{\alpha \beta} \left[ e^{d}_{g} h^{d}_{b} \right] \bigg|_{\kappa = \gamma + \zeta},
\]
where we have
\[
\alpha + \beta + \gamma + \delta = \epsilon + \zeta + \eta + \theta = \iota + \kappa + \lambda + \mu = 2,
\]
as well as the new relation coming from the vanishing of the beta function for the gauge coupling constant at vertex \( g \):
\[
\kappa = \gamma + \zeta.
\]
Since we have total of four constraints for the \( 4 \times 3 = 12 \) parameters, we have \( 8 \) independent parameters, and these play the role of spectral parameters.

The equation (2.18) is already essentially the YBE (1.1). However, the difference is that our \( R \)-matrix \( R^{\delta \gamma}_{\alpha \beta} \left[ d^{c}_{g} h^{c}_{b} \right] \) has three independent spectral parameters \( \alpha, \beta, \gamma, \delta \) with the constraint (2.14), while the \( R \)-matrices \( R(u) \left[ d^{c}_{a} h^{c}_{b} \right] \) in most of the models in the literature have only one spectral parameter; the former is more general than the latter. To specialize to the one-variable \( R \)-matrix, we choose
\[
R(u) \left[ d^{c}_{a} h^{c}_{b} \right] = R^{\delta \gamma}_{\alpha \beta} \left[ d^{c}_{a} h^{c}_{b} \right].
\]
Physically this is a situation where \( SU(N)^{4} \) global symmetry (represented by the four \( SU(N) \) nodes) enhances to \( SU(2N)^{2} \). We can then check that (2.18) reduces to (1.1); the constraint \( \kappa = \gamma + \zeta \) accounts for the arguments \( u, v, u + v \) in (1.1).

In the following we sometimes suppress the spectral parameters and use the shorthand notation \( R \left[ d^{c}_{a} h^{c}_{b} \right] = R^{\delta \gamma}_{\alpha \beta} \left[ d^{c}_{a} h^{c}_{b} \right] \).

3. Integrability from 2d index

3.1. 2d index

Let us now apply the logic of our previous section to the 2d index (also known as flavored elliptic genus), namely a supersymmetric partition function on \( \mathbb{T}^{2} \). For a given 2d \( \mathcal{N} = (2, 2) \) theory, the 2d index is defined by
\[
I(a; q, y) = \text{Tr}_{\text{RR}} \left[ (-1)^{F} q^{H_{e}} q^{H_{s}} y^{J_{L}} \prod_{j} a_{j}^{f_{j}} \right],
\]
where the trace is over the RR ground state and the fermions have periodic boundary conditions. Also, \( F \) is the fermion number, \( H_{e} \) is the left-moving Hamiltonian, \( J_{L} \) is the left-moving \( U(1) \) current and \( f_{j}^{-} \) is the flavor symmetry charges.
The 2d RR index takes the factorized form [17, 18]:

\[ \mathcal{I}_{\text{total}} = \int_{\text{Cartan}} \mathcal{I}_{\text{chiral}} \mathcal{I}_{\text{vector}}, \quad (3.2) \]

where the integral is over the Cartan of the gauge group. The index of a single chiral bifundamental multiplet with \( R \)-charge \( r \) is

\[ \mathcal{I}_{\text{chiral}}(a_i, b_j; q, y) = \prod_{i,j=1}^{N} \Delta(y^{a_i}b_j^{-1}; q, y). \quad (3.3) \]

where \( a_i, b_j \) denote the Cartan variables for the \( SU(N) \times SU(N) \) symmetry of the bifundamental. The index of a \( U(N) \) vector multiplet is

\[ \mathcal{I}_{\text{vector}}^{U(N)}(a; q, y) = \frac{1}{N!} \left( \frac{\eta(q)^3}{i\theta_1(1/y; q)} \right)^N \prod_{1 \leq i < j \leq N} \left( 1 - \frac{a_i}{a_j} \right)^{-1} (a_i; q, y)^{-1}. \quad (3.4) \]

Here and in the following we use

\[ \Delta(a; q, y) := \frac{\theta_1(1/y; a; q)}{\theta_1(a; q)}, \quad (3.5) \]

as well as the Dedekind eta function and the Jacobi theta function:

\[ \eta(q) := q^{1/24} \prod_{i=1}^{\infty} (1 - q^i), \quad (3.6) \]

\[ \theta_1(y; q) := -iq^{1/4} \prod_{i=1}^{\infty} (1 - q^i)(1 - yq^i)(1 - q^{1-i}/y). \quad (3.7) \]

For later convenience, we also define a shorthand notation of the index of a chiral multiplet with \( R \)-charge \( r \),

\[ \Delta_x(x; q, y) := \Delta(y^{x_i}; q, y). \quad (3.8) \]

In some cases we omit the arguments \( q, y \) to write \( \Delta_x(x) \).

### 3.2. R-matrix from 2d index

The integrable model has spins taking values in \( z = \{z_i\} = U(1)^N \). This means that the ‘sum’ over the spins is actually an integral:

\[ \sum_z \rightarrow \frac{1}{N!} \int_{|z_i|=1} \prod_{i=1}^{N} \frac{dz_{z_i}}{2\pi i z_{z_i}}. \quad (3.9) \]

The weights are given by

\[ \mathcal{W}_v(t(e), h(e)) = \Delta_{2r_{\nu_0}-r_{\nu_0}}(z_{t(e)}, z_{h(e)}^{-1}; q, y) \quad (3.10) \]

and

\[ \mathcal{S}_v = \frac{1}{N!} \left( \frac{\eta(q)^3}{i\theta_1(1/y; q)} \right)^N \prod_{i \neq j} \frac{1}{\Delta(z_{v_i}z_{v_j}^{-1}, q, y)}. \quad (3.11) \]
The $R$-matrix is defined by (2.12):

$$
R \left( \begin{array}{l}
\frac{\delta}{\alpha} \\
\frac{\gamma}{\beta}
\end{array} \right) \begin{bmatrix} d & c \\ a & b \end{bmatrix}(q, y) = \prod_{i,j=1}^{N} \frac{\Delta_{2-\delta-\alpha}(d_{i}a_{j}^{-1})}{\Delta_{2-\beta-\gamma}(b_{j}c_{i}^{-1})} \frac{1}{N!} \left( \frac{\eta(q)^{i}}{i\theta_{i}(r^{-1}; q)} \right)^{y_{i}} \prod_{i\neq j} \frac{1}{\Delta(a_{i}a_{j}^{-1}; q, y)} \prod_{i} \frac{1}{\Delta(c_{i}c_{j}^{-1}; q, y)} \prod_{i} \frac{d_{i}}{2\pi iz_{i}} \prod_{i\neq j} \frac{1}{\Delta(z_{i}z_{j}^{-1}; q, y)}
\right) (3.12)
$$

This $R$-matrix satisfies (2.16) as well as (2.18).

There is one subtlety in the present discussion. To define the answer unambiguously in (3.2) it is important to specify the integration cycle. This is a rather non-trivial problem, since naively there are poles right on the integration contour. The general prescription in terms of Jeffrey–Kirwan residue [34, 35] was given in [18], which unfortunately obscures the factorization (2.7) (see, however, the discussion in appendix A).

3.3. Abelian case

In the Abelian ($U(1)$) case, the expression for $R$-matrix simplifies dramatically:

$$
R^{U(1)} \left( \begin{array}{l}
\frac{\delta}{\alpha} \\
\frac{\gamma}{\beta}
\end{array} \right) \begin{bmatrix} d & c \\ a & b \end{bmatrix}(q, y) = \frac{\Delta_{2-\delta-\alpha}(d_{a}^{-1})}{\Delta_{2-\beta-\gamma}(b_{c}^{-1})} \prod_{i\neq j} \frac{d_{i}}{2\pi iz_{i}} \frac{\Delta(x_{a}z_{i}^{-1})}{\Delta(x_{b}z_{j}^{-1})} \frac{\Delta(x_{c}z_{i}^{-1})}{\Delta(x_{d}z_{j}^{-1})}.
\right) (3.13)
$$

In this case, the Jeffrey–Kirwan residue prescription simplifies, and we can appeal to a simpler prescription (see [16, 17] and appendix A for more on this).

To explain this, let us first note that the position of the poles can be read from the identity

$$
\Delta(z; q, r) = \frac{\theta(t; q)}{(q; q)^{2}} \sum_{t \in \mathbb{Z}} \frac{t^{i}}{1 - zq^{i}},
\right) (3.14)
$$

with poles at $z = q^{-1}$. Note also that thanks to the relation

$$
\Delta(qx) = \frac{1}{y} \Delta(x),
\right) (3.15)
$$

the integrand of (3.13) is invariant under the shift $z \rightarrow qz$, and the integrand naturally is a function on a torus. Due to the residue theorem, we obtain a trivial answer if we combine all the residues inside the torus. We should rather pick up a subset of residues, and the correct choice is to pick up those residues with positive (or negative, up to the overall minus sign of the partition function) charges [16, 17]. For the case at hand, this amounts to taking the residues at $z = a$ and $z = c$, and hence the integral can be worked out explicitly to obtain
\[ R^{U(1)} \left( \frac{\delta}{\alpha}, \frac{\gamma}{\beta} \right) \left[ \frac{d}{a}, \frac{c}{b} \right] (q, y)(a, b, c, d; q, y) \]
\[ = \sqrt{\frac{\Delta_{a-b-\alpha}(da^{-1})\Delta_{b-a-\gamma}(bc^{-1})}{\Delta_{a+b}(ab^{-1})\Delta_{b+a}(cd^{-1})}} \times \left[ \Delta_{a+b}(ab^{-1})\Delta_{b+a}(cd^{-1}) \Delta_{\gamma+\delta}(ca^{-1}) \right. \]
\[ \left. + \Delta_{\gamma+\delta}(ca^{-1})\Delta_{a+b}(ab^{-1})\Delta_{b+a}(cd^{-1}) \Delta_{\gamma-\alpha}(ac^{-1}) \right] \]
\[ = \sqrt{\frac{\Delta_{a+b}(ab^{-1})\Delta_{b+a}(cd^{-1})}{\Delta_{a+b}(ab^{-1})\Delta_{b+a}(cd^{-1})}} \Delta_{\gamma-\alpha}(ac^{-1}). \quad (3.16) \]

### 3.4. Modularity

The RR index on $\mathbb{T}^2$ has a natural modular property. Under the modular transformation, the Dedekind eta function and Jacobi theta function transform as follows:

\[ \eta(e^{2\pi i(-\frac{1}{j})}) = \sqrt{-17} \eta(e^{2\pi i j}), \quad (3.17) \]

\[ \theta_1(e^{2\pi i(-\frac{1}{j})}; e^{2\pi i(-\frac{1}{j})}) = i\sqrt{-17} e^{\pi i \frac{\tau}{j}} \theta_1(e^{2\pi i j}; e^{2\pi i j}), \quad (3.18) \]

where $q := e^{2\pi i}$ and $y := e^{2\pi i \tau}$ are the original parameters, while $\bar{q} := e^{2\pi i(-\frac{1}{j})}$ and $\bar{y} := e^{2\pi i(-\frac{1}{j})}$ are the parameters after modular transformation.

The modular property of a chiral multiplet with $R$-charge $r$ and some flavor fugacity $a$ is then

\[ \Delta(\bar{y} + \bar{a}; \bar{q}, \bar{y}) = \theta_1(e^{2\pi i(-\frac{1}{j})}; e^{2\pi i(-\frac{1}{j})}) \theta_1(e^{2\pi i(-\frac{1}{j})}; e^{2\pi i(-\frac{1}{j})}) \]
\[ = e^{\frac{\pi}{12}(1 - r)\zeta^2 - 2u_\alpha} \frac{\theta_1(e^{2\pi i(-\frac{1}{j})}; e^{2\pi i(-\frac{1}{j})})}{\theta_1(e^{2\pi i j}; e^{2\pi i j})} \]
\[ = e^{\frac{\pi}{12}(1 - r)\zeta^2 - 2u_\alpha} \Delta(\bar{y} + \bar{a}; q, t), \quad (3.19) \]

where we defined $u_\alpha$ by $a = e^{2\pi i u_\alpha}$. Notice that the coefficient of the $\zeta^2$ term in the modular weight is $1/3$ of the central charge of the chiral multiplet,

\[ c = 3 \text{tr} \gamma_5 J_5^2 = 3 \left( \frac{r}{2} - 1 \right)^2 - \left( \frac{r}{2} \right) = 3(1 - r), \quad (3.20) \]

and the coefficient of the linear term is $-A^a_{1/3}$, where $A^a$ is the anomaly of flavor symmetry $F_a$. 

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\[ A^a = \text{tr} \gamma_y J_L F_a = \left( \left( \frac{r}{2} - 1 \right) - \left( \frac{r}{2} \right) \right) E_a. \]  

(3.21)

Similarly, under modular transformation, the vector multiplet \( T^{\text{vec}(z_i; q, t)} \) behaves as

\[
\left( \frac{\eta(q)^3}{\eta(1)} \right)^N \prod_{i,j} \Delta\left( \frac{z_i}{z_j}; q, \bar{q} \right)^{-1} = (-\tau)^N e^{-2\pi N^2 \zeta^2} \left( \frac{\eta(q)^3}{\eta(1)} \right)^N \prod_{i,j} \Delta\left( \frac{z_i}{z_j}; q, \bar{q} \right)^{-1}.
\]

(3.22)

Again the coefficient of \( \zeta^2 \) gives the correct central charge for vector multiplets. In general, one can read the central charge and anomalies for a theory from the modular weight of its index,

\[
\mathcal{I}(e^{2\pi i(-z_0)}, e^{2\pi i(-t)}; e^{2\pi i(-i)}) = e^{2\pi i(\xi^2 - 2\zeta \mu \nu \zeta)} \mathcal{I}(e^{2\pi i u_i}, e^{2\pi i r}, e^{2\pi i c}).
\]

(3.23)

One can then derive the modular property of the R-matrix (3.12),

\[
\mathcal{R} \left( \begin{array}{c} \delta \\ \alpha \\ \beta \end{array} \right) \left[ \begin{array}{c} \delta & \gamma \\ a & b \end{array} \right] = (-\tau)^N e^{-2\pi \zeta^2} \mathcal{R} \left( \begin{array}{c} \delta \\ \alpha \\ \beta \end{array} \right) \left[ \begin{array}{c} d & e \\ a & b \end{array} \right].
\]

(3.24)

The number \( 3N^2 \) represents the central charge \( 3N^2 \), which can be directly computed from the spectrum of the theory\(^{14}\).

### 3.5. Dimensional reduction

We can consider the dimensional reduction of our 2d \( \mathcal{N} = (2, 2) \) theory on \( S^1 \). We expect that our 2d \( \mathcal{N} = (2, 2) \) theory will reduce to 1d \( \mathcal{N} = 4 \) supersymmetric quantum mechanics, and the 2d Seiberg-like duality to a duality in 1d.

In our setup, we can choose the \( S^1 \) of the dimensional reduction to be one of the cycles of \( \mathcal{T} \). This is the reduction to the 1d index derived recently in [37, 38]. In appendix B we also derive the same result by the reduction procedure.

The quantity relevant for us is the Witten index for the \( \mathcal{N} = 4 \) supersymmetric quantum mechanics, twisted by a subgroup of the R-symmetry commuting with the supercharge:

\[
\text{Tr} \left[ (-1)^F e^{2\pi i \epsilon / \mathcal{W}} e^{-\beta H} \right].
\]

(3.25)

When we regard \( \mathcal{N} = 4 \) quantum mechanics as \( \mathcal{N} = 2 \) quantum mechanics, part of the \( \mathcal{N} = 4 \) R-symmetry looks like a flavor symmetry for the \( \mathcal{N} = 2 \) theory, and \( J_\epsilon \) generates the flavor symmetry there.

The Boltzmann weight for an edge, i.e., the 1-loop determinant for a bifundamental chiral multiplet is\(^{15}\)

\[
\mathcal{W}^{(\alpha \beta)}(a_{\alpha \epsilon(\ell)}, a_{\beta \epsilon(\ell)}) = \prod_{\ell,j=1}^{N} \frac{\sinh(\alpha_{\ell \epsilon(\ell)} - \beta_{\ell \epsilon(\ell)} + \zeta)}{\sinh(\alpha_{\ell \epsilon(\ell)} - \beta_{\ell \epsilon(\ell)})},
\]

(3.26)

where the 2d integrable variables \( a_\ell \) are here replaced by their 1d counterparts \( a_\ell \). The Boltzmann weight for a vertex, i.e., the 1-loop determinant for a vector multiplet is

\(^{14}\) There is an analogous statement for the 4d \( \mathcal{N} = 1 \) quiver gauge theories: the high temperature limit of the \( S^1 \times S^3 \) reproduces a linear combination of central charges \( a_\ell \), \( c_\ell \) [36].

\(^{15}\) In the literature, the function \( \sinh \) is replaced by \( \sin \) in (3.26) and (3.27). This amounts to the rotation of the contour, which does not affect the answer as long as we sum over the same set of residues. The expression with \( \sinh \) is the one which naturally arises from the dimensional reduction; see appendix B.
The $R$-matrix is given by

$$\mathcal{R} \left( \frac{\delta}{\alpha} \frac{\gamma}{\beta} \right) \left[ \begin{array}{cc} d & c \\ a & b \end{array} \right] (z)$$

\[= \sqrt{\prod_{i,j=1}^{N} \Delta_{2-\delta-\alpha} (d_i - a_i) \Delta_{2-\beta-\gamma} (b_i - c_i)} \times \frac{\prod_{i,j=1}^{N} \Delta (a_i - a_j, z) \prod_{i,j=1}^{N} \Delta (c_i - c_j, z)}{N!} \prod_{i,j=1}^{N} \Delta (z_i - z_j, z) \times \prod_{i,j=1}^{N} \Delta_{\alpha} (a_j - z_i) \Delta_{\beta} (z_i - b_i) \Delta_{\gamma} (c_i - z_i) \Delta_{\delta} (z_i - d_i), \]

where we defined

$$\Delta(x, z) := \frac{\sinh (x - z)}{\sinh (x)},$$

$$\Delta_{\alpha}(x, z) := \frac{\sinh \left( x + \left( \frac{z}{z} - 1 \right) z \right)}{\sinh \left( x + \frac{z}{z} \right)}. \quad (3.29, 3.30)$$

The integral is again understood to be defined in terms of the Jeffrey–Kirwan residue. For $U(1)$ theories, the contour prescription gives

$$\mathcal{R}^{U(1)} \left( \frac{\delta}{\alpha} \frac{\gamma}{\beta} \right) \left[ \begin{array}{cc} d & c \\ a & b \end{array} \right] (z)$$

\[= \sqrt{\Delta_{\alpha+\beta} (a - d, z) \Delta_{\alpha+\beta} (a - b, z)} \times \frac{\Delta_{\beta+\delta} (e - b, z) \Delta_{\beta+\delta} (e - d, z)}{\Delta_{\gamma-a} (c - a, z)} \times \frac{\Delta_{\gamma-a} (c - b, z) \Delta_{\gamma-a} (c - d, z)}{\Delta_{\alpha+\beta} (a - b, z) \Delta_{\alpha+\beta} (a - d, z)} \Delta_{\gamma-a} (c - a, z). \quad (3.31)\]

4. **Comments on brane realizations**

One novel aspect of the Gauge/YBE correspondence is that the integrability resides not in each individual quiver gauge theory, but in a class of gauge theories. In other words, integrability is in the ‘theory space,’ and to properly understand, for example, the meaning of conserved charged in integrable models we are required to go beyond the familiar territory of conventional quantum field theories and discuss the theory space inside a new framework.

One candidate for such a framework is the string theory—it is expected that different gauge theories are realized as different configurations of branes, and branes themselves should be regarded as dynamical degrees of freedom in the string theory. It is therefore natural

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16 The exploration of any structure of the theory space is a fascinating topic. For recent attempts, see, e.g., entanglement [39] and cluster algebras [30, 40].
to discuss the string theory realizations of the quiver gauge theories as a hint for the existence of the integrable structure therein.

Let us here study the case of 4d $\mathcal{N} = 1$ quiver gauge theories discussed in [2–4]; we can realize 2d $\mathcal{N} = (2,2)$ theories by dimensional reduction (or T-duality) of these theories. For this case, the relevant brane configurations are known, both for torus quivers and planar quivers.

Let us for concreteness consider case of the torus quivers. The relevant type IIB brane configuration [41–43] (mirror to type IIA description of [44]) is shown in Table 1.

|       | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|-------|---|---|---|---|---|---|---|---|---|---|
| D5    | o | o | o | o | o | o |   |   |   |   |
| NS5   | o | o | o | o |   |   | Σ (2-dim surface) |

Table 1. The five-brane configuration realizing 4d $\mathcal{N} = 1$ quiver gauge theories.

Now the crucial point here is the Seiberg duality can be understood as a rearrangement of these 1-cycles; in fact, as pointed out in [3, 45] the rapidity lines of integrable models match precisely with the zig-zag paths, which in the language of [41–43] are the intersection of D5-branes with NS5-branes, and YBE is to exchange the relative position of these 1-cycles.\footnote{More precisely, YBE is really a double Yang–Baxter move, [3], namely YBE applied twice.}

What do we gain from this? The basic idea is that we should then be able to describe integrability in terms of the effective theory on the 5-branes (say N D5-branes). The intersection with NS5-brane in this viewpoint appears as Bogomol’nyi–Prasad–Sommerfield (BPS) defects inside that theory, and consequently the integrability is translated into the statements about the rearrangement of BPS defects. This way, the problem of ‘theory space’ is turned into a more tractable problem of the discussion of BPS defects inside a supersymmetric gauge theory. We still obtain a class of theories in the sense that the insertion of BPS defects (disorder operators) changes the definition of the path integral; however, at least the starting point is always a single gauge theory.

The idea that integrability follows from the rearrangement of defects, especially line defects, goes back to the discussion of line defects in pure Chern–Simons theory [46], and more recently in the work of [47]. The latter reference in particular realizes the spectral parameters, which are absent in the descriptions of [46].

These considerations naturally lead us to the question if there are any relations between the present work and the work of [47]. While there are many similarities, one cautionary remark is that the actual integrable models studied in [47] is the Heisenberg XXX spin chain model, whereas the integrable models discussed in [2] and here tend to be more complicated models (chiral Potts models and six-vertex models, and their sophisticated generalizations), and while there are some connections [48], the connection is at best not direct.

Despite this cautionary remark, the similarity is fascinating, and it would be an interesting problem to pursue this type of reasoning further to elucidate the integrable structures in...
the Gauge/YBE correspondence. One possible clue is that, after the compactification of 3-direction, the 5-brane systems of table 1, are dual to the description of codimension 2 defects inside the M5-brane theory.

5. Conclusion

In this paper we constructed integrable models (solutions to the YBE) from the $T^2$ partition function of the 2d $\mathcal{N} = (2, 2)$ quiver gauge theories and the dualities among them (namely Seiberg-like duality and Yang–Baxter duality).

The resulting integrable model has an $R$-matrix written in term of theta functions, and the former has a nice modularity behavior. As an example of the reduction, we worked out the reduction of the 2d index to the 1d index of the dimensionally reduced $\mathcal{N} = 4$ theory.

Along the way we clarified some technical aspects of the Gauge/YBE correspondence, and also encountered several new ingredients, which are not present in their 4d counterparts [2].

Here are some open questions:

- It would be interesting to compare our solutions to the known solutions in the literature, and also to identify the quantum-group-like structure underlying our solutions. There are well-known solutions of YBE in terms of theta function; however, our models are atypical in that we have continuous spin variables.
- Given a solution of the integrable model, we can study its degeneration; the integral model of [2] reproduces in this way many known integrable models, including the Ising models and their generalizations. In particular, the root of unity degeneration of the model gave rise to integrable models with discrete spins. It would hence be interesting to study the root-of-unity degeneration of our models.
- We can try to replace the $T^2$ partition function by the $S^2$ partition function [29, 49]. One subtlety in this case is that the $S^2$ partition function depends on the complexified Fayat–Iliopoulos (FI) parameter, which transforms non-trivially under the Seiberg-like duality. In fact, it was observed in [30] that the FI parameter transforms as a cluster variable in the theory of cluster algebras [50]. This means that the statistical model coming from the $S^2$ partition function does not solve the standard YBE, but rather a generalization of it, where integrability is combined with the cluster algebra.
- The vacua of 2d $\mathcal{N} = (2, 2)$ theories are described by Bethe Ansatz equation of the integrable model (Gauge/Bethe correspondence [51]). It would be interesting to understand the relation between Gauge/Bethe correspondence and Gauge/YBE correspondence. Let us here point out that the integrable structure there is of a rather different nature from the integrable structured discussed in this paper. First, the precise integrable models there are XXX models and their generalizations, while here we have the six-vertex models and their cousins. Second, in [51] the Bethe Ansatz equations play a role, whereas here the Boltzmann weights and the $R$-matrix of the integrable models play direct roles. Third, [51] is about the vacuum structure of 2d $\mathcal{N} = (2, 2)$ theories, whereas our story here is about the supersymmetric partition functions of the 2d $\mathcal{N} = (2, 2)$ theories.
- It is natural to consider the Gauge/YBE correspondence for the 4d $\mathcal{N} = 1$ partition function on $S^2 \times T^2$ [52–54]. Unfortunately, the $R$-charge on $S^2 \times T^2$ should satisfy an integrability constraint, and consequently continuous, spectral parameter seems to be lost in this setup.
• The 4d $\mathcal{N} = 1$ theories discussed in [2] are known to have explicit brane realizations (see, e.g., [43, 55], as well as section 4). Naively one might imagine that we can dimensionally reduce these 4d $\mathcal{N} = 1$ theories on $T^2$ to obtain the 2d $\mathcal{N} = 2$ theories discussed here, leading to the $T^2$ compactification of relevant brane configurations. However, the dimensional reduction of the 4d Seiberg duality requires a careful analysis [56], in particular for quiver gauge theories, and it remains to carry out these reductions in detail on the field theory side. This type of analysis is also needed for the a proper understanding of the 3d $\mathcal{N} = 2$ theories discussed in [3, 4].

• The YBE has a higher dimensional generalization; for example in three dimensions, the relevant equation is the tetrahedron equation. The question is if these equations have their supersymmetric counterparts (see [57] for a recent result).

Acknowledgments

We would like to thank the Simons Center for geometry and physics and the 2014 Simons workshop for hospitality, where this work was initiated. We would like to thank Ibou Bah, Jacque Distler, Abhijit Gadde, Kentaro Hori, Ken Intriligator, Bei Jia, Andy Neitzke, Jaewon Song and Cumrun Vafa for stimulating discussion. The contents of this talk were presented by WY at USC (Oct. 2014), UT Austin (Nov. 2014), Caltech (Dec. 2014), and UCSD (Feb. 2015), and by MY at IPMU (Nov. 2014) and KIAS (Dec. 2014). The work of WY is supported in part by the S Fairchild scholarship, by DOE grant DE-FG02-92-ER40701, and by the W Burke Institute for theoretical physics. The research of MY is supported in part by the World Premier International Research Center Initiative (MEXT, Japan), by the JSPS Program for Advancing Strategic International Networks to Accelerate the Circulation of Talented Researchers, by JSPS KAKENHI Grant Number 15K17634, and by the Institute for Advanced Study.

Appendix A. Proof of YBE for $T^2$ partition function

In this appendix, we explicitly prove the YBE for the $T^2$ partition function. We hope this is useful for the mathematical-oriented readers who wish to skip the gauge theories dualities and illustrate our contour prescription, which is crucial for the YBE.

A.1. Proof of 2d Seiberg-like duality for 2d index

Let us write down the 2d index for the right quiver in figure 2:

$$\mathcal{I}_{\{e,e,e,e,j\}}(a, b, c, d, q, y) = \frac{1}{N!} \left( \frac{\eta(q)^3}{i \theta_1(y^{-1}; q)} \right)^N \prod_{i,j=1}^{N} \Delta_{2-j-e}(a_i b_j^{-1}) \Delta_{2-j-e}(c_i d_j^{-1}) \times \oint \frac{dz_j}{2 \pi i z_j} \prod_{i,j=1}^{N} \frac{1}{\Delta(z_i z_j^{-1}; q, y)} \times \prod_{i,j=1}^{N} \Delta_{a_i}(z_j a_i^{-1}) \Delta_{b_j}(b_i z_j^{-1}) \Delta_{c_i}(z_i c_i^{-1}) \Delta_{d_j}(d_i z_j^{-1}). \tag{A.1}$$
where the functions \( \Delta(a; q, y) \) and \( \overline{\Delta}(x) = \overline{\Delta}_z(x; q, y) \) are defined in (3.5) and (3.8). The fundamental identity representing the 2d Seiberg-like duality is

\[
\mathcal{I}_{\{r_a, r_b, r_c, r_d\}}(a, b, c, d; q, y) = \mathcal{I}_{\{1-r_a, 1-r_b, 1-r_c, 1-r_d\}}(b, c, d, a; q, y). \quad (A.2)
\]

We first prove (A.2).

Let us define the following index,

\[
\mathcal{I}_{\text{SD}}(\mathbf{A}, \mathbf{B}; q, y) = \frac{1}{\mathcal{N}!} \left( \frac{\eta(q)^3}{i\theta_i(y^{-1}; q)} \right)^N \oint \prod_{a=1}^{\mathcal{N}} \Delta(z_a A^{-1}_a; q, y) \Delta(B^{-1}_{i_a}; q, y) \prod_{\alpha, \beta} \Delta(z_\alpha z_\beta^{-1}; q, y),
\]

which can be viewed as the index of a 2d \((2, 2)\) theory with \(U(N)\) gauge group and \(2N\) fundamental and anti-fundamental chirals. \(\mathbf{A}\) and \(\mathbf{B}\) are shorthand notation for \(\{A_i\}\) and \(\{B_i\}\), and we set

\[
|A_i| > 1, \quad |q A_i| < 1, \quad i = 1, \ldots, 2N,
\]

\[
|B_i| < 1, \quad |q^{-1} B_i| > 1, \quad i = 1, \ldots, 2N. \quad (A.4)
\]

Now let us prove the following identity (this is already in version 2 in [16]):

\[
\mathcal{I}_{\text{SD}}(\mathbf{A}, \mathbf{B}; q, y) = \prod_{i, j=1}^{2N} \Delta\left(\frac{B_i}{A_j}; q, y\right) \mathcal{I}_{\text{SD}}\left(y^{-\frac{1}{2}} A^{-1}, y^{\frac{1}{2}} B^{-1}; q, y\right), \quad (A.5)
\]

where \(A^{-1}\) and \(B^{-1}\) are shorthand notation for \(\{A_i^{-1}\}\) and \(\{B_i^{-1}\}\). The left hand side of (A.5) can be computed by the residue prescription in [16]. The poles can be picked as

\[
z_\alpha = B_{i_\alpha}, \quad (A.6)
\]

where all the other poles in side the unit circle like \(q^k A_i\)’s and \(q^{k} B_i\)’s have opposite residues and will not contribute to the integral. The result is

\[
\mathcal{I}_{\text{SD}}(\mathbf{A}, \mathbf{B}; q, y)
\]

\[
= \sum_{\{i\}_a, \{i\}_b} \prod_{j=1}^{2N} \Delta\left(B_{i_j} A_{j}^{-1}; q, y\right) \prod_{a=1}^{\mathcal{N}} \prod_{j \neq i_a}^{2N} \Delta\left(B_{i_a} B_{i_j}^{-1}; q, y\right)
\]

\[
= \sum_{\{i\}_a, \{i\}_b} \prod_{j=1}^{2N} \Delta\left(B_{i_j} A_{j}^{-1}; q, y\right) \prod_{s \in \{i\}_a} \prod_{r \in \{i\}_b} \Delta\left(B_{i_s} B_{i_r}^{-1}; q, y\right). \quad (A.7)
\]

The right hand side of equation (A.5) can be computed similarly. We pick up the residues for the poles at

\[
z_\alpha = y^{\frac{1}{2}} B_{i_\alpha}^{-1},
\]

and the result is

\[
\mathcal{I}_{\text{SD}}\left(y^{-\frac{1}{2}} A^{-1}, y^{\frac{1}{2}} B^{-1}; q, t\right)
\]

\[
= \sum_{\{i\}_a, \{i\}_b} \prod_{j=1}^{2N} \Delta\left(y A_{j} B_{i_\alpha}^{-1}; q, y\right) \prod_{a=1}^{\mathcal{N}} \prod_{j \neq i_a}^{2N} \Delta\left(B_{i_a} B_{i_j}^{-1}; q, y\right)
\]

\[
= \sum_{\{i\}_a, \{i\}_b} \prod_{j=1}^{2N} \Delta\left(y A_{j} B_{i_\alpha}^{-1}; q, y\right) \prod_{a=1}^{\mathcal{N}} \prod_{j \neq i_a}^{2N} \Delta\left(B_{i_a} B_{i_j}^{-1}; q, y\right). \quad (A.8)
\]
\[
= \sum \prod_{\{i\}_a \in \{\tau\}_a} 2N \Delta(yA_jB_x^{-1}; q, y) \prod_{s \in \{\tau\}_a} \prod_{r \in \{\tau\}_a} \Delta(B_jB_r^{-1}; q, y) \\
= \sum \prod_{\{i\}_a \in \{\tau\}_a} 2N \Delta(yA_jB_x^{-1}; q, y) \prod_{s \in \{\tau\}_a} \prod_{r \in \{\tau\}_a} \Delta(B_jB_r^{-1}; q, y). 
\] (A.9)

Since
\[
\prod_{s \in \{\tau\}_a} 2N \Delta(B_jA_j^{-1}; q, y) = \frac{\prod_{i,j=1}^{2N} \Delta(B_iA_i^{-1}; q, y)}{\prod_{i,j=1}^{2N} \Delta(B_iA_i^{-1}; q, y)} \\
= \prod_{i,j=1}^{2N} \Delta(B_iA_i^{-1}; q, y) \prod_{s \in \{\tau\}_a} \Delta(yA_jB_x^{-1}; q, y), 
\] (A.10)

one can immediately verify the equality of the left hand side and right hand side of (A.5), and hence we have proven (A.5).

Now, the integral (A.2) can be written as
\[
\mathcal{I}(a, b, c, d; q, y) = \mathcal{D}_{2 - e - r_0}(a_i b_j^{-1}) \mathcal{D}_{2 - e - r_0}(c_i d_j^{-1}) \\
\times \mathcal{I}_{SD}(\{y^2a_i, y^2c_i\}, \{y^2b_i, y^2d_i\}; q, y). 
\] (A.11)

The requirement of pole position (A.4) is satisfied if
\[
|a_i| = |b_i| = |c_i| = |d_i| = 1, \quad i = 1, \ldots, N, \\
|q| < 1, \quad |y| < 1, \quad 0 \leq |r| \leq 1. 
\] (A.12)

Using (A.5) we see
\[
\mathcal{I}_{SD}(\{y^2a_i, y^2c_i\}, \{y^2b_i, y^2d_i\}; q, y) \\
= \prod_{i,j=1}^{N} \Delta(y^2a_i b_i^{-1}; q, y) \Delta(y^2b_i d_i^{-1}; q, y) \Delta(y^2b_i c_i^{-1}; q, y) \\
\times \mathcal{I}_{SD}(\{y^2a_i, y^2c_i\}, \{y^2b_i, y^2d_i\}; q, y) \\
= \mathcal{D}_{2 - e - r_0}(a_i b_i^{-1}) \mathcal{D}_{2 - e - r_0}(d_i c_i^{-1}) \\
\times \mathcal{I}_{SD}(\{y^2a_i, y^2c_i\}, \{y^2b_i, y^2d_i\}; q, y). 
\] (A.13)

Using the identities
\[
\mathcal{D}_{2 - e - r_0}(a_i b_i^{-1}) \mathcal{D}_{2 - e - r_0}(b_i a_i^{-1}) = 1, 
\] (A.14)

we arrive at (A.2).

A.2. Consistency of residue prescription with gluing

Since the Yang–Baxter duality is a sequence of the 2d Seiberg-like duality (applied four times), the fundamental identity proven earlier should automatically imply the YBE. The only
subtlety here is that (as already commented in section 3.2) the contour prescription obscures (2.7). While (2.7) is expected to hold on physical grounds, it would still be desirable to check explicitly that the resulting integral is independent of the order of integration of the two sides of the YBE. For simplicity of the presentation, we specialize to the case $N = 1$, where the contour prescription simplifies.

Let us consider the quiver in figure A1. This is a quiver with three fundamental quivers (figure 2) glued together. Part of its index can be written as

$$\int \cdots \int [de] \prod_{\alpha, \beta} \mathcal{D}_{\alpha_\beta} \left( \frac{c}{a}, \frac{d}{e}, \frac{e}{c}, \frac{f}{j}, \frac{j}{f}, \frac{h}{b}, \frac{b}{h}, \frac{a}{c} \right) \cdots,$$

where $[dc]$ and $[de]$ are shorthand notation for combination of measure and vector multiplet contribution, and $\mathcal{D}_{\alpha_\beta}(x)$ is the shorthand notation for

$$\prod_{\alpha, \beta = 1}^{N} \left( y^{\frac{c_\alpha}{a_\beta}} \frac{y^{\alpha}}{a_\beta} ; q, y \right).$$

We keep only the part related to node $c$ and node $e$ in the index.

We would like to set

$$q < y^\frac{1}{2} < 1.$$  

Hence $y^z a_n$ and $y^{-z} a_n q^n$ with positive integer $n$ are always in the unit circle and $y^z a_n q^{-n}$ is always outside the unit circle. On the other hand, $y^{-z} a_n^{-1}$ and $y^{-z} a_n^{-1} q^{-n}$ are outside the unit circle and $y^{-z} a_n^{-1} q^n$ is inside the unit circle. Remember we always put the flavor fugacity $a_n$ on the unit circle.

Now let us look at the integral (A.15). For simplicity, we look at the $U(1)$ case. If we integrate over $c$ first, the only contributions are from the poles at

$$y^{\frac{c_\alpha}{a_\beta}} b, \quad y^{\frac{c_\alpha}{a_\beta}} d,$$

and the integral becomes

$$\int \cdots \int [de] \prod_{\alpha, \beta} \mathcal{D}_{\alpha_\beta} \left( \frac{e}{f}, \frac{c}{h}, \frac{f}{j}, \frac{j}{f}, \frac{h}{b}, \frac{b}{h}, \frac{a}{c} \right) \times \left[ \mathcal{D}_{\alpha_\beta + \tau_\alpha} \left( \frac{\tau_\alpha + \tau_{\beta}}{a} \right) \mathcal{D}_{\alpha_\beta + \tau_\alpha} \left( \frac{\tau_{\beta} + \tau_{\alpha}}{a} \right) \mathcal{D}_{\alpha_\beta - \tau_\beta} \left( \frac{d}{b} \right) \right]$$

$$+ \mathcal{D}_{\alpha_\beta + \tau_\alpha} \left( \frac{d}{a} \right) \mathcal{D}_{\alpha_\beta + \tau_\alpha} \left( \frac{d}{e} \right) \mathcal{D}_{\alpha_\beta - \tau_\beta} \left( \frac{b}{d} \right).$$

We then integrate over $e$ and pick up the residue at

$$y^{\frac{c_\alpha}{a_\beta}} f, \quad y^{\frac{c_\alpha+\tau_{\beta}}{a_\beta}} b, \quad y^{\frac{c_\alpha+\tau_{\beta}}{a_\beta}} d,$$

Figure A1. The quiver for the YBE; compare with figure 7.
we get
\[
\int \ldots \left[ \Delta_{x_k+e_x} \left( \frac{b}{a} \right) \Delta_{x_k-r_k} \left( \frac{d}{b} \right) \Delta_{x_k+e_x+r_k} \left( \frac{b}{f} \right) \Delta_{x_k+r_k+e_x+r_k} \left( \frac{b}{h} \right) \Delta_{x_k-r_k-e_x} \left( \frac{j}{b} \right) \right] \\
+ \Delta_{x_k+e_x} \left( \frac{b}{a} \right) \Delta_{x_k-r_k} \left( \frac{d}{b} \right) \Delta_{x_k-e_x-r_k} \left( \frac{b}{f} \right) \Delta_{x_k+r_k-e_x} \left( \frac{j}{h} \right) \\
+ \Delta_{x_k+e_x} \left( \frac{d}{a} \right) \Delta_{x_k-r_k} \left( \frac{d}{b} \right) \Delta_{x_k+e_x+r_k} \left( \frac{d}{f} \right) \Delta_{x_k+r_k+e_x+r_k} \left( \frac{d}{h} \right) \Delta_{x_k-r_k-e_x} \left( \frac{j}{d} \right) \\
+ \Delta_{x_k+e_x} \left( \frac{d}{a} \right) \Delta_{x_k-r_k} \left( \frac{b}{a} \right) \Delta_{x_k+e_x+r_k} \left( \frac{d}{f} \right) \Delta_{x_k+r_k+e_x+r_k} \left( \frac{j}{h} \right) \Delta_{x_k-r_k-e_x} \left( \frac{c}{j} \right) \right].
\] (A.21)

Now let us integrate over e first; we have poles at
\[
y^{\pm} c, \quad y^{\pm} d,
\] (A.22)
and the result is
\[
\int \ldots \int \mathrm{d}c \Delta_{x_k} \left( \frac{c}{a} \right) \Delta_{x_k} \left( \frac{d}{c} \right) \\
\times \left[ \Delta_{x_k+e_x} \left( \frac{c}{d} \right) \Delta_{x_k-r_k} \left( \frac{c}{j} \right) \Delta_{x_k+e_x+r_k} \left( \frac{j}{c} \right) + \Delta_{x_k+e_x} \left( \frac{j}{c} \right) \Delta_{x_k-r_k+e_x-r_k} \left( \frac{j}{c} \right) \Delta_{x_k-r_k-e_x} \left( \frac{c}{j} \right) \right].
\] (A.23)

then we integrate over c; It might seem that we have an extra pole at
\[
y^{\pm} c, \quad y^{\pm} d;
\] (A.24)
however, the residue is zero because \(\Delta_{x_k-r_k-e_x} \left( \frac{j}{c} \right)\) and \(\Delta_{x_k-r_k-e_x} \left( \frac{c}{j} \right)\) terms have exactly opposite contribution and cancel out; hence, we consider only the poles at
\[
y^{\pm} b, \quad y^{\pm} d.
\] (A.25)
The result is
\[
\int \ldots \left[ \Delta_{x_k+e_x} \left( \frac{b}{a} \right) \Delta_{x_k-r_k} \left( \frac{d}{b} \right) \Delta_{x_k+e_x+r_k} \left( \frac{b}{f} \right) \Delta_{x_k+r_k+e_x+r_k} \left( \frac{b}{h} \right) \Delta_{x_k-r_k-e_x} \left( \frac{j}{b} \right) \right] \\
+ \Delta_{x_k+e_x} \left( \frac{b}{a} \right) \Delta_{x_k-r_k} \left( \frac{d}{b} \right) \Delta_{x_k-e_x-r_k} \left( \frac{b}{f} \right) \Delta_{x_k+r_k-e_x} \left( \frac{j}{h} \right) \\
+ \Delta_{x_k+e_x} \left( \frac{d}{a} \right) \Delta_{x_k-r_k} \left( \frac{d}{b} \right) \Delta_{x_k+e_x+r_k} \left( \frac{d}{f} \right) \Delta_{x_k+r_k+e_x+r_k} \left( \frac{d}{h} \right) \Delta_{x_k-r_k-e_x} \left( \frac{j}{d} \right) \\
+ \Delta_{x_k+e_x} \left( \frac{d}{a} \right) \Delta_{x_k-r_k} \left( \frac{b}{a} \right) \Delta_{x_k+e_x+r_k} \left( \frac{d}{f} \right) \Delta_{x_k+r_k+e_x+r_k} \left( \frac{j}{h} \right) \Delta_{x_k-r_k-e_x} \left( \frac{d}{j} \right) \right].
\] (A.26)
The result is exactly the same as the previous result. We have proved explicitly the order of integration does not affect the result under residue prescription in \(U(1)\) case.

**Appendix B. Dimensional reduction of \(\Delta(a; q, t)\)**

Let us discuss the dimensional reduction of the 2d index to the 1d index. The discussion here is similar to the reduction of the \(S^3 \times S^1\) partition function to \(S^1\) [58–60].
When the radius of the thermal cycle shrinks to zero, we expect all the fugacities to approach an identity. They can nevertheless approach to an identity with a different scaling limit; first let us rewrite the fugacities as

\[ q = e^{2\pi i \tau}, \quad y = e^{2\pi i \zeta}, \quad a = e^{2\pi i \nu}. \]  

Dimension reduction means we set \( \tau = i\beta \) then take the limit \( \beta \to 0 \), and also scale \( \zeta \) and \( \nu \) with \( \beta \),

\[ \zeta = \beta \zeta, \quad \nu = \beta \nu, \]

hence

\[ \Delta(a; q, y) = \prod_{n=0}^{\infty} \frac{(1 - yaq^{n+1})(1 - y^{-1}a^{-1}q^{n})}{(1 - aq^{n+1})(1 - a^{-1}q^{n})} \]

\[ = \prod_{n=0}^{\infty} \frac{(1 - e^{-2\pi i ((a-\zeta)+n)})(1 - e^{-2\pi i (\zeta-n)})}{(1 - e^{-2\pi i (a-\nu+n)})(1 - e^{-2\pi i n})}. \]

In the limit \( \beta \to 0 \),

\[ \lim_{\beta \to 0} \Delta(a; q, y) = \frac{\pi (a - z)}{\pi a} \prod_{n=1}^{\infty} \frac{(-i(a - z) + n)(-ia + n)(ia + n)}{(-ia + n)(ia + n)} \]

\[ = \frac{\pi (a - z)}{\pi a} \prod_{n=1}^{\infty} \frac{1 + \frac{a - z}{a^2}}{1 + \frac{a^2}{a^2}} = \frac{\sinh \pi (a - z)}{\sinh \pi a}. \]

Similarly, one can show that under the limit \( \beta \to 0 \),

\[ \lim_{\beta \to 0} \frac{\eta(q)^3}{i\theta_1(y^{-1}; q)} = -\frac{1}{2i\beta \sinh \pi z}. \]

This means that under the dimension reduction, the 2d index of a theory with \( U(N) \) gauge group and fundamental chirals becomes

\[ \left( -\frac{1}{2i\beta \sinh \pi z} \right)^N \int da \prod_{i \neq j} \frac{\sinh \pi (a_i - a_j)}{\sinh \pi ((a_i - a_j) - z)} \prod_i \frac{\sinh \pi \left( a_i + \left( \frac{z}{2} - 1 \right) \right)}{\sinh \pi \left( a_i + \frac{z}{2} \right)}. \]

This result, up to a divergent factor proportional to \( 1/\beta \), matches with the 1d index obtained in [37, 38].

References

[1] Baxter R 2007 Exactly Solved Models in Statistical Mechanics (New York: Dover)
[2] Yamazaki M 2014 New integrable models from the gauge/YBE correspondence J. Statist. Phys. 154 895
[3] Yamazaki M 2012 Quivers, YBE and 3-manifolds J. High Energy Phys. JHEP05(2012)147
[4] Terashima Y and Yamazaki M 2012 Emergent 3-manifolds from 4d superconformal indices Phys. Rev. Lett. 109 091602
[5] Benini F, Nishioka T and Yamazaki M 2012 4d Index to 3d Index and 2d TQFT Phys. Rev. D 86 065015
[6] Bazhanov V V and Sergeev S M 2012 Elliptic gamma-function and multi-spin solutions of the Yang–Baxter equation Nucl. Phys. B 856 475–96
[7] Bazhanov V V and Sergeev S M 2010 A Master solution of the quantum Yang–Baxter equation and classical discrete integrable equations Adv. Theor. Math. Phys. 16 65–95
[8] Spiridonov V 2012 Elliptic beta integrals and solvable models of statistical mechanics Contemp. Math. 563 181–211
[9] Bazhanov V V, Kels A P and Sergeev S M 2013 Comment on star–star relations in statistical mechanics and elliptic gamma-function identities J. Phys. A: Math. Theor. 46 152001
[10] Bazhanov V V, Mangazeev V V and Sergeev S M 2007 Faddeev–Volkov solution of the Yang–Baxter equation and discrete conformal symmetry Nucl. Phys. B 784 234–58
[11] Volkov A 1992 Quantum Volterra model Phys. Lett. A 167 345–55
[12] Faddeev L and Volkov A Y 1993 Abelian current algebra and the Virasoro algebra on the lattice Phys. Lett. B 315 311–8
[13] Fioravanti D and Rossi M 2002 A Braided Yang–Baxter algebra in a theory of two coupled lattice quantum KdV: algebraic properties and ABA representations J. Phys. A: Math. Gen. 35 3647–82
[14] Aharony O 1997 IR duality in $d = 3 \ N = 2$ supersymmetric USp(2N(c)) and U(N(c)) gauge theories Phys. Lett. B 404 71–76
[15] Yagi J 2015 Quiver gauge theories and integrable lattice models arXiv:1504.0405
[16] Gadde A and Gukov S 2014 2d Index and Surface operators J. High Energy Phys. JHEP03 (2014)080
[17] Benini F, Eager R, Hori K and Tachikawa Y 2014 Elliptic genera of two-dimensional $N = 2$ gauge theories with rank-one gauge groups Lett. Math. Phys. 104 465–93
[18] Benini F, Eager R, Hori K and Tachikawa Y 2014 Elliptic genera of 2d $N = 2$ gauge theories Lett. Math. Phys. 104 465–93
[19] Franco S, Hanany A, Kennaway K D, Vegh D and Wecht B 2006 Brane dimers and quiver gauge theories J. High Energy Phys. JHEP01(2006)096
[20] Franco S 2012 Bipartite field theories: from D-Brane probes to scattering amplitudes J. High Energy Phys. JHEP11(2012)465
[21] Xie D and Yamazaki M 2012 Network and seiberg duality J. High Energy Phys. JHEP09(2012)036
[22] Rains E M 2010 Transformations of elliptic hypergeometric integrals Ann. Math. (2) 171 169–243
[23] Dolan F and Osborn H 2009 Applications of the superconformal index for protected operators and q-Hypergeometric identities to $N = 1$ dual theories Nucl. Phys. B 818 137–78
[24] Spiridonov V and Vartanov G 2012 Elliptic hypergeometric integrals and ’t Hooft anomaly matching conditions J. High Energy Phys. JHEP06(2012)016
[25] Seiberg N 1995 Electric–magnetic duality in supersymmetric nonAbelian gauge theories Nucl. Phys. B 435 129–46
[26] Hori K and Tong D 2007 Aspects of non-abelian gauge dynamics in two-dimensional $N = (2, 2)$ theories J. High Energy Phys. JHEP05(2007)079
[27] Caldararu A, Distler J, Hellerman S, Panetev T and Sharpe E 2010 Non-birational twisted derived equivalences in abelian GLSMs Commun. Math. Phys. 294 605–45
[28] Hori K 2013 Duality in two-dimensional (2, 2) supersymmetric non-abelian gauge theories J. High Energy Phys. JHEP10(2013)121
[29] Benini F and Cremonesi S 2015 Partition functions of $\mathcal{N} = (2, 2)$ gauge theories on $S^2$ and vortices Commun. Math. Phys. 334 1483–527
[30] Benini F, Park D S and Zhao P 2014 Cluster algebras from dualities of 2d $N = (2, 2)$ quiver gauge theories arXiv:1406.2699
[31] Jia B, Sharpe E and Wu R 2014 Notes on nonabelian (0, 2) theories and dualities J. High Energy Phys. JHEP08(2014)017
[32] Baxter R 1986 The Yang–Baxter equations and the zamolodchikov model Physica D 18 321–47
[33] Bazhanov V and Baxter R 1992 New solvable lattice models in three-dimensions J. Statist. Phys. 69 453–585
[34] Jeffrey L C and Kirwan F C 1995 Localization for nonabelian group actions Topology 34 291–327
[35] Szenes A and Vergne M 2004 Toric reduction and a conjecture of Batyrev and Materov Invent. Math. 158 453–95
[36] di Pietro L and Komargodski Z 2014 Cardy formulae for SUSY theories in $d = 4$ and $d = 6$ J. High Energy Phys. JHEP12014031
[37] Hori K, Kim H and Yi P 2014 Witten Index and Wall Crossing J. High Energy Phys. JHEP12014031
38. Cordova C and Shao S-H 2014 An index formula for supersymmetric quantum mechanics arXiv:1406.7853
39. Yamazaki M 2013 Entanglement in theory space Europhys. Lett. 103 21002
40. Terashima Y and Yamazaki M 2014 3d $N=2$ theories from cluster algebras PTEP 023 B01
41. Imamura Y 2006 Global symmetries and ’t Hooft anomalies in brane tilings J. High Energy Phys. JHEP12(2006)041
42. Imamura Y, Isono H, Kimura K and Yamazaki M 2007 Exactly marginal deformations of quiver gauge theories as seen from brane tilings Prog. Theor. Phys. 117 923–55
43. Yamazaki M 2008 Brane Tilings and their applications Fortsch. Phys. 56 555–686
44. Feng B, He Y-H, Kennaway K D and Vafa C 2008 Dimer models from mirror symmetry and quivering amoebae Adv. Theor. Math. Phys. 12 3
45. Hanany A and Vegh D 2007 Quivers, tilings, branes and rhombi J. High Energy Phys. JHEP10(2007)029
46. Witten E 1989 Gauge theories and integrable lattice models Nucl. Phys. B 322 629
47. Costello K 2013 Supersymmetric gauge theory and the Yangian arXiv:1303.2632
48. Bazhanov V and Stroganov Y 1990 Chiral Potts model as a descendant of the six vertex model J. Statist. Phys. 59 799–817
49. Doroud N, Gomis J, le Floch B and Lee S 2013 Exact results in $D = 2$ supersymmetric gauge theories J. High Energy Phys. 1305 JHEP05(2013)093
50. Fomin S and Zelevinsky A 2007 Cluster algebras: IV. Coefficients Compos. Math. 143 112–64
51. Nekrasov N A and Shatashvili S L 2009 Supersymmetric vacua and Bethe ansatz Nucl. Phys. B 192-193 91–112
52. Closset C and Shamir I 2014 The $\mathcal{N}=1$ chiral multiplet on $T^2 \times S^2$ and supersymmetric localization J. High Energy Phys. JHEP03(2014)040
53. Benini F and Zaffaroni A 2015 A topologically twisted index for three-dimensional supersymmetric theories J. High Energy Phys. JHEP07(2015)127
54. Honda M and Yoshida Y 2015 Supersymmetric index on $T^2 \times S^2$ and elliptic genus arXiv:1504.0435
55. Heckman J J, Vafa C, Xie D and Yamazaki M 2013 String theory origin of bipartite SCFTs J. High Energy Phys. JHEP05(2013)148
56. Aharony O, Razamat S S, Seiberg N and Willett B 2013 3d dualities from 4d dualities J. High Energy Phys. 1307 JHEP07(2013)149
57. Gadde A and Yamazaki M (0, 2) trialitys and the tetrahedron equation, to appear
58. Dolan F A H, Spiridonov V P and Vartanov G S 2011 From 4D superconformal indices to 3D partition functions Phys. Lett. B 704 234
59. Gadde A and Yan W 2012 Reducing the 4d index to the $S^3$ partition function J. High Energy Phys. JHEP12(2012)003
60. Imamura Y 2011 Relation between the 4d superconformal index and the $S^3$ partition function J. High Energy Phys. JHEP092011133