A NOTE ON LIOUVILLE TYPE EQUATIONS ON GRAPHS

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Abstract. In this note, we study the Liouville equation $\Delta u = -e^u$ on a graph $G$ satisfying certain isoperimetric inequality. Following the idea of W. Ding, we prove that there exists a uniform lower bound for the energy, $\sum_G e^u$, of any solution $u$ to the equation. In particular, for the 2-dimensional lattice graph $\mathbb{Z}^2$, the lower bound is given by 4.

1. Introduction

The Liouville equation

\begin{equation}
\Delta u + e^u = 0 \tag{1.1}
\end{equation}

on 2-dimensional manifolds has been extensively studied in the literature. From the point of view of the theory of partial differential equations, it is critical, i.e. on the borderline of Sobolev embedding theorems in 2-dimensional case, which it is closely related to so-called Moser-Trudinger inequalities, see e.g. [12], [1], [14] for references.

Let $u$ be a solution to the Liouville equation on the plane with finite energy, i.e.

\begin{equation}
\begin{cases}
\Delta u + e^u = 0 \\
\int_{\mathbb{R}^2} e^u < \infty.
\end{cases} \tag{1.2}
\end{equation}

An interesting argument initiated by Weiyue Ding, see [2], shows that

\[ \int_{\mathbb{R}^2} e^u \geq 8\pi. \]

The key ingredient of the proof is the following isoperimetric inequality: for any bounded domain $\Omega$ of finite perimeter in $\mathbb{R}^2$,

\begin{equation}
\text{Length}(\partial \Omega)^2 \geq 4\pi \cdot \text{Area}(\Omega), \tag{1.3}
\end{equation}

where $\text{Length}(\partial \Omega)$ (Area($\Omega$) resp.) denotes the length of the boundary of $\Omega$ (the area of $\Omega$ resp.). The estimate is sharp since one can construct a family of explicit solutions,

\begin{equation}
F_{x_0,\lambda}(x) := \ln \left[ \frac{32\lambda^2}{(4 + \lambda^2|x - x_0|^2)^2} \right], \quad \lambda > 0, x_0 \in \mathbb{R}^2, \tag{1.4}
\end{equation}
whose energy attain the above lower bound. Based on a delicate argument using moving
plane methods, Chen and Li [2] further proved that all solutions to (1.2) are exactly given
by (1.4).

As is well-known, one of the difficulties for the analysis on graphs lies in the lack of chain
rules for discrete Laplace operators. While linear equations have been studied extensively
on graphs, people began to consider nonlinear problems on graphs such as semilinear
equations recently. For semilinear equations with the nonlinearity of power type, one refers
to e.g. [6, 7, 9, 10]. A class of semilinear equations with the exponential nonlinearity,
so-called Kazdan-Warner equations, have been studied by [5, 3, 4, 8] on graphs. The
exponential nonlinearity usually causes additional difficulties for the analysis in the discrete
setting. In this paper, we study the Liouville type equations on graphs, analogous to (1.1),
which are special cases of Kazdan-Warner equations. Following W. Ding’s idea, we prove
a uniform lower bound of the energy for the solutions to the Liouville equations on graphs
satisfying isoperimetric inequalities analogous to (1.3), see Theorem 2.1. As a corollary,
for the 2-dimensional lattice graph which is a discrete analog of $\mathbb{R}^2$, we obtain an explicit
lower bound for the energy of solutions to the Liouville equation, see Corollary 2.2. This
could be regarded as a preliminary step to understand the Liouville type equations on
infinite graphs.

The paper is organized as follows: In the next section, we introduce some basic setting
and state our main results. Section 3 is devoted to the proof of Theorem 2.1.

2. Basic setting and main results

Let $(V,E)$ be a simple, undirected and locally finite graph, where $V$ denotes the set
of vertices and $E$ denotes the set of edges. Two vertices $x$ and $y$ are called neighbors,
denoted by $x \sim y$, if there is an edge connecting them, i.e. $\{x, y\} \in E$. We assign weights
on vertices and edges as follows:

$$\mu : V \to (0, \infty), \quad V \ni x \mapsto \mu_x$$

and

$$w : E \to (0, \infty), \quad E \ni \{x, y\} \mapsto w_{xy} = w_{yx}$$

and call the quadruple $G = (V, E, \mu, w)$ a weighted graph. For discrete measure spaces
$(V, \mu)$ and $(E, w)$, we write $\mu(A) := \sum_{x \in A} \mu_x$ and $w(B) := \sum_{e \in B} w_e$ for any subsets
$A \subset V, B \subset E$. For simplicity, for a function $u$ on $V$ we write

$$\int_V u = \sum_{x \in V} u(x) \mu_x,$$

whenever it makes sense.

The Laplacian on $G = (V, E, \mu, w)$ is defined as, for any function $u$ on $V$ and $x \in V$,

$$\Delta u(x) = \frac{1}{\mu_x} \sum_{y \in V : y \sim x} w_{xy}(u(y) - u(x)).$$
For any vertex $x$, its weighted degree is given by

$$\text{Deg}(x) := \sum_{y : y \sim x} \frac{w_{xy}}{\mu_x}.$$ 

The Laplacian is a bounded operator on $\ell^2(V, \mu)$, i.e. the Hilbert space of $\ell^2$ summable functions on $V$ w.r.t. the measure $\mu$, if and only if

$$(BLap) \quad \text{Deg}(G) := \sup_{x \in V} \text{Deg}(x) < \infty.$$ 

In this paper, we always assume $(BLap)$ holds.

For any finite subset $\Omega$ in $V$, we denote by

$$\partial \Omega := \{ \{x, y\} \in E : x \in \Omega, y \in V \setminus \Omega, \text{ or vice versa} \}$$

the (edge) boundary of $\Omega$. We say that a weighted graph $G = (V, E, \mu, w)$ satisfies 2-dimensional isoperimetric inequality, denoted by $IS_2$, if

$$(IS_2) \quad C_{IS} := \inf \frac{(w(\partial \Omega))^2}{\mu(\Omega)} > 0,$$

where the infimum is taken over all finite $\Omega \subset V$, see [13].

In this note, we study the discrete Liouville equation

$$(2.1) \quad \Delta u + e^u = 0.$$ 

on a weighted graph $G$. Following W. Ding, see Lemma 1.1 in [2], we obtain our main result, a discrete analog of energy estimate for the solutions to Liouville equation under the assumption of the isoperimetric inequality.

**Theorem 2.1.** Let $G$ be a weighted graph satisfying $(BLap)$ and $\inf_{x \in V} \mu_x > 0$. Suppose that $(IS_2)$ holds, then for any solutions $u$ of (2.1),

$$\int_V e^u \geq \frac{C_{IS}}{\text{Deg}(G)}.$$ 

**Remark 1.** One may generalize the result to the following equation

$$\Delta u + F(u) = 0,$$

for some nonnegative function $F$ on $\mathbb{R}$ satisfying $F' \geq 0$ and $F'' \geq 0$.

We denote by $\mathbb{Z}^2$ the standard lattice graph with the set of vertices $\{(x, y) \in \mathbb{R}^2 : x, y \in \mathbb{Z}\}$ and the set of edges

$$\{(x_1, y_1), (x_2, y_2) \} : |x_1 - x_2| + |y_1 - y_2| = 1 \}$$

and with weights $\mu \equiv 4$ and $w \equiv 1$. It is known that it satisfies $(IS_2)$ with $C_{IS} = 4$, see Theorem 6.30 in [11]. Then by the above theorem we have the following corollary.
Corollary 2.2. For any solution $u$ of (2.1) on the lattice $\mathbb{Z}^2$, we have

$$\int_{\mathbb{Z}^2} e^u \geq 4.$$ 

This suggests the following interesting problems for further investigation.

**Problem 1.** What is the sharp constant in Corollary 2.2, i.e.

$$C := \inf \int_{\mathbb{Z}^2} e^u,$$

where the infimum is taken over all solutions to (2.1) on $\mathbb{Z}^2$?

**Problem 2.** Is there any solution $u$ to (2.1) on $\mathbb{Z}^2$ with finite energy, i.e. $\int_{\mathbb{Z}^2} e^u < \infty$?

### 3. Proof of Theorem 2.1

For any $\sigma \in \mathbb{R}$, set

$$\Omega_\sigma = \{x \in V | u(x) \geq \sigma\}.$$ 

It is no restriction to assume that $\Omega_\sigma$ is finite for any $\sigma$, otherwise by $\inf_{x \in V} \mu_x > 0$, 

$$\int_{V} e^u = +\infty.$$

By (2.1),

$$\int_{\Omega_\sigma} e^u = \int_{\Omega_\sigma} -\Delta u = \sum_{x \in \Omega_\sigma} \sum_{y \in V : y \sim x} w_{xy}(u(x) - u(y))$$

$$= \sum_{x \in \Omega_\sigma} \sum_{y \in \Omega_\sigma : y \sim x} w_{xy}(u(x) - u(y)) + \sum_{x \in \Omega_\sigma} \sum_{y \in \Omega_\sigma : y \sim x} w_{xy}(u(x) - u(y)).$$

We denote the first summand by $A$. Then

$$A = \sum_{x,y \in \Omega_\sigma : x \sim y} w_{xy}(u(x) - u(y))$$

$$= -\sum_{x,y \in \Omega_\sigma : x \sim y} w_{xy}(u(y) - u(x))$$

$$= -\sum_{y \in \Omega_\sigma} \sum_{x \in \Omega_\sigma : x \sim y} w_{xy}(u(y) - u(x)) = -A.$$ 

This yields that $A = 0$ and we get

$$\int_{\Omega_\sigma} e^u = \sum_{e = \{x,y\} \in E, u(x) < \sigma \leq u(y)} w_{xy}(u(y) - u(x)).$$

For any $\sigma \in \mathbb{R}$, let

$$G(\sigma) = \sum_{e = \{x,y\} \in E, u(x) < \sigma \leq u(y)} \frac{w_{xy}}{u(y) - u(x)}.$$
For any subset $K \subset \mathbb{R}$, we denote by $1_K$ the characteristic function on $K$, i.e. $1_K(\sigma) = 1$ if $\sigma \in K$, and $1_K(\sigma) = 0$ otherwise. We have

$$
\int_{-\infty}^{+\infty} e^\sigma G(\sigma) d\sigma = \int_{-\infty}^{+\infty} e^\sigma \sum_{e=\{x,y\} \in E, u(y) > u(x)} w_{xy} (u(y) - u(x))^{-1} 1_{(u(x), u(y))} (\sigma) d\sigma
$$

$$
= \sum_{e=\{x,y\} \in E, u(y) > u(x)} w_{xy} (u(y) - u(x))^{-1} \int_{-\infty}^{+\infty} e^\sigma 1_{(u(x), u(y))} (\sigma) d\sigma
$$

$$
= \sum_{e=\{x,y\} \in E, u(y) > u(x)} w_{xy} \frac{e^{u(y)} - e^{u(x)}}{u(y) - u(x)}
$$

$$
\leq \sum_{e=\{x,y\} \in E, u(y) > u(x)} w_{xy} e^{u(y)}
$$

$$
\leq \text{Deg}(G) \sum_{y \in V} e^{u(y)} \mu_y,
$$

where we have used the elementary inequality $e^{b-a} - e^{a-b} \leq e^{b-a}$ for any $a < b$ and the definition of $\text{Deg}(G)$ in (BLap). Hence by the above inequality,

$$
(3.2) \quad \int_{-\infty}^{+\infty} e^\sigma G(\sigma) \int_{\Omega_\sigma} e^u d\sigma \leq \int_V e^u \int_{-\infty}^{+\infty} e^\sigma G(\sigma) d\sigma \leq \text{Deg}(G) \left( \int_V e^u \right)^2.
$$

On the other hand, by (3.1) and the Cauchy-Schwarz inequality,

$$
G(\sigma) \int_{\Omega_\sigma} e^u
$$

$$
= \left( \sum_{e=\{x,y\} \in E, u(x) < \sigma \leq u(y)} \frac{w_{xy}}{u(y) - u(x)} \right) \left( \sum_{e=\{x,y\} \in E, u(x) < \sigma \leq u(y)} w_{xy} (u(y) - u(x)) \right)
$$

$$
\geq \left( \sum_{e=\{x,y\} \in E, u(x) < \sigma \leq u(y)} w_{xy} \right)^2 = (w(\partial \Omega_\sigma))^2
$$

$$
\geq C_{IS} \cdot \mu(\Omega_\sigma),
$$

where the last inequality follows from the isoperimetric inequality. This yields that

$$
\int_{-\infty}^{+\infty} e^\sigma G(\sigma) \int_{\Omega_\sigma} e^u \geq C_{IS} \int_{-\infty}^{+\infty} \mu(\Omega_\sigma) e^\sigma = C_{IS} \int_V e^u.
$$

We prove the theorem by combining the above inequality with (3.2).

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