Classical dimer model with anisotropic interactions on the square lattice

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We discuss phase transitions and the phase diagram of a classical dimer model with anisotropic interactions defined on a square lattice. For the attractive region, the perturbation of the orientational order parameter introduced by the anisotropy causes the Berezinskii-Kosterlitz-Thouless transitions from a dimer-liquid to columnar phases. According to the discussion by Nomura and Okamoto for a quantum-spin chain system [J. Phys. A 27, 5773 (1994)], we provide criteria to determine transition points and also universal level-splitting conditions. Subsequently, we perform numerical diagonalization calculations of the nonsymmetric real transfer matrices up to linear dimension specified by \( L = 20 \) and determine the global phase diagram. For the repulsive region, we find the boundary between the dimer-liquid and the strong repulsion phases. Based on the dispersion relation of the one-string motion, which exhibits a two-fold “zero-energy flat band” in the strong repulsion limit, we give an intuitive account for the property of the strong repulsion phase.

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I. INTRODUCTION

In the early 1960s, Kasteleyn [1] and Temperley and Fisher [2, 3] studied the classical dimer model (DM) defining the statistical-mechanical problem of the covering of a lattice by dimers. They treated the DM on, for example, the square lattice in the thermodynamic limit, and obtained the partition function to give the extensive entropy of an ensemble of dimer configurations. In particular, it was shown that the close-packed model defined on planar lattices can be solved exactly by Pfaffian techniques [4]. The properties of these ensembles were studied in subsequent research. For example, those defined on bipartite (non-bipartite) lattices such as the square (triangular) lattice exhibit critical (off-critical) behavior [5, 6], which are now thought to reflect an existence (absence) of the height representations for the DMs [7, 8, 9, 10, 11]. While the relevance of dimers as diatomic molecules adsorbed on a lattice is clear and direct, it can be also related to other degrees of freedom [12]. For instance, the zero-temperature Ising-spin antiferromagnet on a triangular (Villain) lattice can be related to the DM on a hexagonal (square) lattice, where each dimer represents an unsatisfied bond of spins [5, 8, 13]. Also widely known is the string representation whereby dimer systems under a certain condition can be related to loop gases whose configurations are classified by winding numbers (see below) [7, 8, 13]. This correspondence has been utilized in discussions of polymer systems [13]. More importantly, Rokhsar and Kivelson introduced quantum dimer models (QDMs) to describe the valence-bond physics in quantum Heisenberg antiferromagnets, where the dimer represents a tightly binding singlet pair of quantum spins [14].

These are but a fraction of the examples that show DMs having importance in a wide range of research and drawing attention over the years; in particular, current interest has been mainly focused on exotic phases with topological orders observed in the QDMs [15]. More recently, Blunt et al. reported on an adsorption experiment of certain rod-like organic molecules on graphite [16] and explained its relevance to the DM on a hexagonal lattice which includes interactions between neighboring dimers [17]. This exhibits the fact that interaction effects in classical dimers are also important from both theoretical and experimental viewpoints [18].

In this paper, we investigate an interacting dimer model (IDM) defined on a square lattice: suppose the lattice constant \( a = 1 \) and let \( \Lambda \) denote the set of lattice sites. Then the following reduced Hamiltonian expresses interactions between two nearest-neighbor dimers:

\[
H = - \sum_{(k,l) \in \Lambda} \left[ K_h n(k + \frac{1}{2}, l) n(k + \frac{1}{2}, l + 1) + K_v n(k, l + \frac{1}{2}) n(k + 1, l + \frac{1}{2}) \right],
\]

where lattice sites and lattice bonds are denoted, respectively, as \((k, l)\) and \((k + \frac{1}{2}, l)\) with \( k, l \in \mathbb{Z} \). We locate the dimer occupation numbers \( n(k + \frac{1}{2}, l) = 0 \) or 1 (binary variables) on the bonds. The first (second) term on the right-hand side of Eq. (1) represents an interaction between parallel horizontal (vertical) dimers [see Fig. 1(a)]. Defining the local Boltzmann weights as

\[
h = \exp(K_h), \quad v = \exp(K_v),
\]

with subscripts on the \( K \)'s referring to the corresponding dimer pair, the partition function \( Z(h, v) \) is then expressible as a summation with respect to the dimer configurations \( C \) on \( \Lambda \),

\[
Z(h, v) = \sum_C h^{N_h(C)} v^{N_v(C)},
\]

where \( N_h(C) \) [\( N_v(C) \)] represents the number of plaquettes with parallel horizontal (vertical) dimers. For large \( h \) or \( v \), an attractive case, a twofold- or a fourfold-degenerate state with columnar order is expected to be
The effective way to determine phase transition points [27] can provide an effective way to determine phase transition points [27]. For this reason, we shall also employ the same strategy called level-spectroscopy analysis [25, 26] can provide an effective way to determine phase transition points [27].

In numerical calculations of universal quantities, e.g., Fig. 1(a), to a string configuration is performed via an XOR operation with reference configuration [Fig. 1(b)]. The XOR operation takes the exclusive OR between occupation (binary) numbers in these two configurations over each bond. Consequently, we obtain strings running in the y direction [see the two gray lines in Fig. 1(a)]. Due to the close-pack condition, they have no end points, and thus the string configurations for a $L \times L$ system ($L$ is an even number) with periodic boundary conditions can be characterized by winding numbers $(N_x, N_y)$ satisfying $0 \leq N_{x,y} \leq L$. While in the numerical calculation of the transfer matrices we shall employ $N_y$ as a conserved quantity in the row-to-row transfer of configurations, we would rather use a quantity

$$M \equiv N_y - L/2$$

For later convenience, we shall briefly explain here the string representation of the DM on $\Lambda$. As explicitly explained in Ref. [29], the transformation of a dimer configuration, e.g., Fig. 1(a), to a string configuration is performed via an XOR operation with reference configuration [Fig. 1(b)]. The XOR operation takes the exclusive OR between occupation (binary) numbers in these two configurations over each bond.

The organization of this paper is as follows: In Sec. II, we review previous research results to give an effective description of the low-energy and long-distance behavior of the IDMs [22]. In particular, the operator content of the theory, including expressions of local order parameters and defect operators, and their scaling dimensions are explained in detail. We then provide the conditions to determine the BKT-transition points and clarify some universal relations among excitation levels, which serve as a check of our calculational results. In doing this, a correspondence with a frustrated quantum-spin chain system plays a guiding role. Thus, this correspondence will be emphasized and referred to when appropriate. In Sec. III, we summarize our numerical study and results obtained by the transfer-matrix calculations based on the conformal field theory (CFT) [30]. First, we demonstrate that the theoretical predictions in Sec. II can be observed precisely via numerical analysis of the excitation spectra. Next, we provide the global phase diagram of interacting dimers, which includes the BKT, the second-order, and the first-order transition lines. Also, in a strong repulsion region, we expect a highly degenerate phase including the staggered state. We calculate the string-number dependence of the free-energy density for the isotropic case. Furthermore, we investigate the “dispersion relation” of the one-string motion, and then based on these data we shall try to give an insight into properties of the strong repulsion phase. The last section, Sec. IV, is devoted to discussion and summary. We also provide our method to evaluate the original BKT transition in the isotropic system. Finally, we compare our data with previous research results.

II. THEORY

Continuum field theories offer unified approaches to investigate phase transitions in interacting systems on lat-
tices. They are derived in the scaling limit, \( a \to 0 \) while keeping \( \mathbf{x} = (x_1, x_2) = (ak, al/\zeta) \) finite. Here \( \zeta \) is the geometric factor, taking a fixed value, e.g., \( \zeta = 1 \) \((2/\sqrt{3})\) for isotropic systems on a square (triangular) lattice. However, for anisotropic systems, renormalization of \( \zeta \) is necessary due to interactions leading to non-universal values. The renormalized \( \zeta \) can be also related to the velocity of an elementary excitation observed in Tomonaga-Luttinger liquids \[21\]. Thus, \( \zeta \) disappears from the theoretical description if we properly employ its renormalized value; but as we will see in Sec. \[\Pi\] it becomes rather important in numerical calculations.

According to PLF, the effective description of the IDM takes the form of a sine-Gordon field theory \[20\]. In the present case, its expression is given by the Lagrangian density \( \mathcal{L} = \mathcal{L}_0 + \mathcal{L}_2 + \mathcal{L}_4 \) with

\[
\mathcal{L}_0 = \frac{K}{2\pi} (\nabla \phi)^2, \\
\mathcal{L}_2 = \frac{y_2}{2\pi\alpha^2} : \cos 2\sqrt{2}\phi ;, \\
\mathcal{L}_4 = \frac{y_4}{2\pi\alpha^2} : \cos 4\sqrt{2}\phi ;.
\]

We denote the course-grained height field in the two-dimensional Euclidean space as \( \phi(x) \), which satisfies a periodicity in height space of \( \sqrt{2}\phi = \sqrt{2}\phi + 2\pi N \) \((N \in \mathbb{Z})\). Due to the close-packing of the dimers, the defect operators given in terms of the disorder field \( \theta \) dual to \( \phi \) are absent from \( \mathcal{L} \), so that it represents a roughening phase or flat phases of an interface model in three dimensions. The theoretical parameter \( K \) (the Gaussian coupling) describes the stiffness, and determines the dimensions of the operators on the Gaussian fixed line \( \mathcal{L}_0 \).

In our notation, the vertex operator with \( a \) electric and \( n \) magnetic charges is given by \( \mathcal{O}_{m,n} = e^{im\sqrt{2}\theta + in\sqrt{2}\phi} \) whose dimension is

\[
X_{m,n} = \frac{1}{2} (K^{-1}m^2 + Kn^2).
\]

Therefore, \( K = 1 \) \((K = 4)\) represents the condition that the perturbation \( \mathcal{L}_2 \) \((\mathcal{L}_4)\) becomes marginal.

To make our discussion more concrete, we introduce the average \( (a) \) and the difference \( (d) \) of the couplings as

\[
K_{a,d} = \frac{1}{2} (K_v \pm K_h)
\]

where the first (second) subscript refers to the upper (lower) sign. Then, around the non-interacting point, the parameters in \( \mathcal{L} \) are roughly given by

\[
K \simeq \frac{1}{2} + c_1 K_a, \quad y_2 \simeq c_2 K_d, \quad \text{and} \quad y_4 \simeq -c_3
\]

\((c_{1,2,3} > 0)\). We see that the attractive interaction \( K_a > 0 \) increases \( K \), and tends to stabilize the columnar states. Also, \( y_2 \) in \( \mathcal{L}_2 \) (the orientational order parameter) is proportional to the difference, \( K_d \), while \( y_4 \) in \( \mathcal{L}_4 \) which is a remnant from the discreteness of the square lattice is almost constant. For \( K > 4 \), both nonlinear terms are relevant, but they are not competing against each other, so the fourfold-degenerate columnar state stabilized by \( \mathcal{L}_4 \) is only lifted to realize twofold-degenerate columnar states by \( \mathcal{L}_2 \) (see below). Since the BKT transition by \( \mathcal{L}_4 \) was already discussed in the literature \[20, 21\], we shall focus our attention on the role of \( \mathcal{L}_2 \).

According to the standard argument \[32\], the renormalization-group (RG) flow diagram of the sine-Gordon model \( \mathcal{L}_0 + \mathcal{L}_2 \) (here \( \mathcal{L}_4 \) is irrelevant) is expressed by the BKT RG equations \[23, 24\]; we depict it by employing the coupling constants \( y_0 = 2 - 2K \) and \( y_2 \) in Fig. 2 where the separatrices \( y_2 = \mp y_0 \) separate the dimer-liquid phase from two types of twofold-degenerate columnar phases, namely, the horizontal columnar (HC) state consisting of dimers in the horizontal direction and the vertical columnar (VC) state consisting of dimers in the vertical direction. Now, we can point out that our task to treat the BKT transitions in the IDM can be related to the investigation of the spin-\( \frac{1}{2} \) XXZ chain with next-nearest-neighbor interaction because these share the same effective description \[23, 28\]. To specify the relationship, we introduce the following operators:

\[
O_0 = \sqrt{2} \cos \sqrt{2} \phi, \\
O_{1,2} = \exp(\pm i\sqrt{2}\theta), \\
O_3 = \sqrt{2} \sin \sqrt{2}\phi.
\]

Here \( O_{0,3} \) stand for the horizontal and the vertical components of the columnar local order parameter, and take expectation values \( \langle O_0 \rangle \neq 0 \) and \( \langle O_3 \rangle = 0 \) \((\langle O_0 \rangle = 0 \) and \( \langle O_3 \rangle \neq 0) \) in the HC (VC) phase; \( O_{1,2} \) are the de-
(14), Therefore, following their argument, we shall dis-
calculate the scaling dimensions of these operators

to the discussion by NO, we can find the criterion to
determining the BKT-transition point in our IDM.

Consider a system with a finite-strip geometry, viz., a
narrow band of width $L$ along the $x$ direction and infi-
nite length along the $y$ direction. The periodic bound-
ary condition is imposed across the width of the strip.
The finite-size corrections to the scaling dimensions of
the above operators are our key quantities to be evalu-
ated analytically and numerically. Here, we first consider
the system near the separatrix $y_2 = -y_0$ where a small
parameter $t$ can be introduced, so that $y_2 = -y_0(1 + t
(\|t\| \ll 1)$. Next, the conformal perturbation calculations
of the renormalized scaling dimensions were performed
using the sine-Gordon Lagrangian density, for which the
results can be summarized as follows:

\begin{align}
x_0 &\approx \frac{1}{2} - \frac{1}{4} y_0(l) (1 + 2t) , \\
x_{1,2} &\approx \frac{1}{2} - \frac{1}{4} y_0(l) , \\
x_3 &\approx \frac{1}{2} + \frac{1}{4} y_0(l) (3 + 2t) ,
\end{align}

($l = \ln L$ is the logarithmic scale length) \[33\]. According
to the discussion by NO, we can find the criterion to
determine the BKT-transition point $t = 0$ (i.e., the level-
crossing condition)

\begin{equation}
x_0 = x_{1,2}
\end{equation}

and the level-splitting condition as

\begin{equation}
\frac{3x_{0,1,2} + x_3}{4} = \frac{1}{2},
\end{equation}

The latter is one of the universal relations among the
excitation levels on the separatrix \[34\], and enables us
to check the consistency of the calculations. Second, we
investigate the system near the separatrix $y_2 = y_0$; it
proceeds in an analogous way to the above. Writing $y_2 = y_0(1 + t)$, we then obtain the dimensions as

\begin{align}
x_0 &\approx \frac{1}{2} + \frac{1}{4} y_0(l) (3 + 2t) , \\
x_{1,2} &\approx \frac{1}{2} - \frac{1}{4} y_0(l) , \\
x_3 &\approx \frac{1}{2} - \frac{1}{4} y_0(l) (1 + 2t) .
\end{align}

Thus, the level-crossing condition needed to determine
the transition point is given by

\begin{equation}
x_{1,2} = x_3 ,
\end{equation}

and the level-splitting condition is given by

\begin{equation}
\frac{x_0 + 3x_{1,2,3}}{4} = \frac{1}{2}.
\end{equation}

In analogy to the quantum-spin chain, each level crossing
represent an emergence of a SU(2) multiplet structure
consisting of the singlet and the triplet states (e.g., $x_3$
and $x_{0,1,2}$ at $y_2 = -y_0$). Since these are the low-energy
levels in the level-1 SU(2) Wess-Zumino-Witten model
\[34\], our criteria, Eqs. (17) and (22), are natural and also
convincing from this viewpoint.

At this stage, two comments are in order about the ad-
vantage in using these relations (the level-spectroscopy
approach) and the structure of the phase diagram. In
Sec. III, we will outline the numerical transfer-matrix
calculations performed to obtain the phase diagram.
Although it can treat systems with a strip geometry,
accessible sizes are strongly restricted to small values, e.g.,
$L \leq 20$ in our calculations. In the BKT transition, as
seen above, the correction terms are typically given by
the logarithmic form $y_0 \simeq 1/\ln(L/L_0)$. If we employ
the standard KT criterion such as $x_0 = \frac{1}{2}$ to determine
the transition point, its finite-size estimates include these,
and thus exhibit a slow convergence in their extrapolation
to the thermodynamical limit. Alternatively, criteria (17)
and (22) take the logarithmic corrections into account, so
they provide finite-size estimates with fast convergences
\[26\]. Consequently, we can employ the following least-
squares-fitting form in extrapolating the finite-size data
$Q(L)$ to the thermodynamic limit $L \to \infty$:

\begin{equation}
Q(L) \simeq Q(\infty) + a/L^2 + b/L^4,
\end{equation}

which includes the $1/L^2$ term stemming from the $x = 4$
irrelevant operators as the leading universal correction
\[35\].

The biases from the finite-strip geometry disappear in
the limit, and the phase diagram is symmetric with re-
spect to the isotropic line $K_h = K_v$ which is one of the
inherent properties of the model. Here, we describe how

The table below shows the discrete symmetries of the principal operators ($O_{1,5}$ are referred to in Sec. IV). The expressions $M$, $k_x$, and $P$ represent the string number, the momentum in the $x$ direction, and the parity for the reflection $\sigma_x$, respectively (see text). Also given are notations and identifications in both dimer and quantum-spin (optional) languages.

\begin{center}
\begin{tabular}{|c|c|c|c|c|}
\hline
Notations & Operators & Identifications & $M$ & $k_x$ & $P$ \\
\hline
$O_0$ & \sqrt{2}\cos\sqrt{2}\phi & HC (Néel) & 0 & $\pi$ & $-1$ \\
$O_{1,2}$ & $\exp(\pm\sqrt{2}\phi)$ & monomer (doublet) & \pm1 & 0 & $+1$ \\
$O_3$ & $\sqrt{2}\sin\sqrt{2}\phi$ & VC (dimer) & 0 & 0 & $+1$ \\
$O_4$ & \sqrt{2}\cos 2\sqrt{2}\phi & Orientational & 0 & 0 & $+1$ \\
$O_5$ & \sqrt{2}\sin 2\sqrt{2}\phi & Plaquette & 0 & $\pi$ & $-1$ \\
\hline
\end{tabular}
\end{center}
the symmetry of the lattice model is embedded in the sine-Gordon field theory. We shall consider the generators of the $C_{4v}$-point group: the $\pi/2$ rotation ($C_4$) and the reflection in the $x$ axis ($\sigma_x$) about the original site [see Fig. 1(c)]. As well as coordinate transformations, these bring about the following changes in the height field \cite{20,22}:

\begin{equation}
C_4 : \sqrt{2}\phi \rightarrow \sqrt{2}\phi - \pi/2, \quad \sigma_x : \sqrt{2}\phi \rightarrow \pi - \sqrt{2}\phi.
\end{equation}

All other elements are obtained from Eq. \cite{24}; among them, we shall investigate transformations of the Lagrangian density and the principal operators by reflection about the diagonal line, $\sigma_d (= \sigma_x \circ C_4)$ [see Fig. 1(c)], which shifts the field as

\begin{equation}
\sigma_d : \sqrt{2}\phi \rightarrow \pi/2 - \sqrt{2}\phi.
\end{equation}

Since the orientational order parameter $L_2$ is odd for $\sigma_d$, the Lagrangian density transforms as

\begin{equation}
\sigma_d : L(K,y_2,y_4) \rightarrow L(K,-y_2,y_4).
\end{equation}

This indicates a connection between the positive and the negative values of $K_d$. In addition, the above four operators transform as

\begin{equation}
\sigma_d : O_0 \rightarrow O_3, \quad O_{1,2} \rightarrow O_{1,2}, \quad O_3 \rightarrow O_0.
\end{equation}

Thus, the symmetry operation $\sigma_d$ interchanges the roles of the HC and the VC operators while leaving unchanged the doublet of the monomer excitations. Consequently, as expected, the level-crossing and the level-splitting conditions for $K_d < 0$, Eqs. \cite{17} and \cite{18}, are translated to those for $K_d > 0$, Eqs. \cite{22} and \cite{23}. In Sec. III we shall provide some numerical data to check this symmetry.

III. NUMERICAL CALCULATIONS

Now, consider a system on $\Lambda$ with the $L \times \infty$ stripe geometry and introduce the transfer matrix $T_M(L)$ connecting nearest-neighbor rows in the $y$ direction [see Fig. 2(a)] \cite{20,21}. As mentioned in Sec. I the string number in the $y$ direction, $M$, is a conserved quantity, which can thus be specified explicitly. We denote the eigenvalues as $\lambda_p(L)$ and their logarithms as $E_p(L) = -\ln|\lambda_p(L)|$ ($p$ specifies an excitation level such as those listed in the above). Then, the conformal invariance provides direct expressions of the central charge $c$ and the scaling dimension $x_p$ in the critical systems as \cite{32,37}:

\begin{equation}
E_q(L) \simeq Lf - \frac{\pi}{6\zeta}, \quad \Delta E_p(L) \simeq \frac{2\pi}{L\zeta}x_p.
\end{equation}

Here, $E_q(L)$, $\Delta E_p(L) (= E_p(L) - E_q(L))$, and $f$ correspond to the ground-state energy, an excitation gap, and a free-energy density, respectively. The ground state is found in the $M = 0$ ($N_y = L/2$) sector, and the excited levels are also in the sectors specified by the discrete symmetries given in Table 1. In addition, since, independent of the value of $K$, the scaling dimension of a level-1 descendant is equal to 1, it has been utilized to estimate a velocity of elementary excitation in the Tomonaga-Luttinger liquid (see, for example, \cite{23}). According to their treatment, the effective geometric factor (i.e., inverse velocity) can be also calculated from the descendant level, say $E_\zeta$, as \cite{27}:

\begin{equation}
\zeta^{-1} = \lim_{L \rightarrow \infty} \frac{\Delta E_\zeta(L)}{2\pi/L}.
\end{equation}

The corresponding excitation with small momentum can be found numerically. In calculating $c$ and $x$ from the excitation gaps via Eq. \cite{29}, an estimate of $\zeta$ first needs to be obtained. However, it is not necessary in determining the BKT-transition points by Eqs. \cite{17} and \cite{22}, because these are homogeneous equations of $x$, and thus the gaps—instead of dimensions—can be used. This is one of the advantages of the level-spectroscopy approach \cite{27}.

In the following, we shall provide our results from numerical calculations for systems up to size $L = 20$. The methodological aspects of transfer-matrix calculations have been well explained in the literature \cite{21}. Furthermore, due to the sparse nature of the matrices, we can output all elements to hard disk. Then, using the ARPACK library \cite{38}, we can calculate the dominant eigenvalues of the nonsymmetric real matrices.

As a demonstration we give the $K_d$ dependence of the scaling dimensions at $K_d = 0$ in Fig. 3(a), where $x_0$, $x_{1,2}$, and $x_3$ are plotted by solid, dotted, and dashed lines, respectively. While the data are for a system with $L = 20$, we can see the excitation spectra approach quite close to the exact ones at the non-interacting point $K_d = 0$ \cite{39}. Furthermore, we find that the HC and the VC excitations interchange their behaviors at $K_d = 0$, and the former (the latter) shows a level crossing with the doublet excitations at a certain negative (positive) value. According to theoretical predictions \cite{17} and \cite{22}, these can provide finite-size estimates of the BKT-transition points to the HC and the VC phases; we shall give some evidence to support our augment. In Fig. 3(b), we exhibit extrapolations of the finite-size estimates to the thermodynamic limit according to Eq. \cite{24}. The downward (upward) pointing triangles exhibit values of $-K_d$ ($K_d$) at which the crossings between $x_0$ and $x_{1,2}$ ($x_{1,2}$ and $x_3$) occur in the systems with $L = 16$, 18, and 20. Their extrapolated values strongly agree with each other (i.e., their deviation is within 0.01%), which is the obvious condition to be satisfied. In Figs. 3(c) and 3(d), we plot averaged values, i.e., the left-hand sides (LHSs) of Eqs. \cite{15} and \cite{23}, as well as the dimensions at the BKT-transition points estimated in Fig. 3(b). In both cases, the extrapolated values of the averages agree with the theoretical value of $1/2$, which exhibits universal level splittings due to logarithmic corrections expected at the transition points.

These observations show that both our strategy and numerical procedure are valid also for investigations of the IDM \cite{25}.
estimate the central charge along the line IV. Our method and compare our results with these in Sec. Results were previously available [20, 21]. We shall explain (double circle in the figure) although some numerical re-

We provide our estimation of the point by our approach but it is eventually terminated by another BKT transi-

≤

perturbation. However, as expected, their product main-

1/L

correlation length is given by \( 1/\xi \propto K \) to the unstable Gaussian fixed line; the

\( L \)

tains a value bring about second-order phase tran-

\( K=4 \)

FIG. 3: (a) An example of the \( K_d \) dependences of scaling dimensions at \( K_d = 0 \) (for \( L = 20 \) case). An inserted key identifies excitations and lines. (b) The finite-size estimates of the transition points to the HC (VC) phase denoted by downward (upward) pointing triangles are extrapolated according to Eq. (24) (see solid lines). (c) and (d) Checks of the universal level-splitting conditions [18] and [20, 21] at the BKT-transition points. An inserted key identifies excitations and marks; \( x_{av} \) in (c) [(d)] is the LHS of Eq. (18) [Eq. (20)], and the least-squares fit solid line is exhibited.

In Fig. 4 we summarize our results of numerical calculations for the global phase diagram of our model [1], in which the diagonal line (the center point) corresponds to the isotropic (noninteracting) system. Due to the fact that the phase diagram is symmetric about the line, it is sufficient to explicitly calculate one side of the whole parameter space; our calculations are thus restricted to the region \( K_d \leq 0 \) (i.e., the upper-left triangular area). The open circles with solid lines give the phase bound-

\( x \)

1/L

av

\( x \)

1,2

0

The dimer-liquid region with \( 1 \) repulsion

\( K=1 \)

liquid

\( K=2 \),

\( K=3 \)

\( K=4 \), and \( \frac{1}{2} \). In addition to the points (plus marks) associated with \( K = 1 \), 2, and 3 on the isotropic line are given in the figure. As expected, the dimer-liquid phase spreads over the area satisfying the condition \( 0 < K \leq 4 \). For \( 1 \leq K \leq 4 \), this phase only survives on the isotropic line, but it is eventually terminated by another BKT transition caused by the marginally relevant \( L_4 \) perturbation. We provide our estimation of the point by our approach (double circle in the figure) although some numerical results were previously available [20, 21]. We shall explain our method and compare our results with these in Sec. IV.

To check the criticality of the dimer-liquid phase, we estimate the central charge along the line \( K_t = 0 \) by the use of relations [20] and [30]. In Fig. 5 we give the \( K_t \) dependencies of the effective geometric factor \( \zeta \) (diamonds), the coefficient of the \( 1/L \) correction \( \gamma = c/\zeta \) (squares), and their product to estimate \( c \) (circles). With increasing anisotropy, \( \zeta \) deviates from the isotropic value of 1 and approaches a certain value around 1.5 in the limit \( K_t \to -\infty \). Simultaneously, \( \gamma \) declines in value and thus cannot itself give the universal amplitude of the finite-size correction. However, as expected, their product maintains a value \( c = 1 \) within the dimer-liquid phase, and hence the proper normalization using the effective geometric factor is necessary for anisotropic systems. In the attractive region, one finds a point at which the central charge exhibits a steep decrease. We can check that the point is almost on the phase transition boundary to the VC phase (see the vertical arrow), and thus that it is consistent with the level-crossing calculations.

The dimer-liquid region with \( 1 \leq K \leq 4 \) corresponds to the unstable Gaussian fixed line; the \( L_2 \) perturbation, except for \( K = 1 \), brings about second-order phase transitions to the columnar phases. In this case, as \( 1/\xi \propto |K_d|^\gamma \), the critical exponent characterizing the diverging correlation length is given by \( 1/\nu = 2 - X_{2,0} = 2 - 2/K \) [30]. To treat this transition, we have performed a finite-

FIG. 4: (Color online) Global phase diagram. The diagonal line (the center point) corresponds to the isotropic (noninteracting) system. The circles with solid lines separate the dimer-liquid phase from the HC and the VC phases. The squares with solid lines exhibit the condition \( K = 0 \), which is the boundary of the dimer-liquid phase. Also plotted are the contour lines of \( K = \frac{\sqrt{2}}{2}, \frac{\sqrt{5}}{3}, \) and \( \frac{\sqrt{5}}{4} \) as well as the points (plus marks) of \( K = 1, 2, \) and 3. The double circle indicates another BKT-transition point brought about by \( L_4 \).
the liquid phase is absent in the more attractive region, whence the phase transition between the HC and the VC phases becomes first order (the solid line) accompanied by a jump in the phase-locking point $\langle \sqrt{2} \phi \rangle$ from 0 or $\pi$ to $\pi/2$ or $3\pi/2$.

Finally, we discuss the transition to the strong repulsion phase (the upper-right gray-color region) at which the stiffness of the Gaussian model vanishes (see squares with solid lines in Fig. 4). In terms of the height model, this vanishing permits the interface to tilt globally without cost $[20, 40]$. To get some deeper insight, we shall see this degeneration with high probability.

The above-mentioned $O(L)$ degeneracy inferred from the instability of the Gaussian criticality seems to be consistent with this $M$ independence. However, there still exists a discrepancy in the degree of degeneracy with the staggered state; we shall discuss this issue for the rest of this section.

It is known that the degeneracy of the staggered state is subextensive, i.e., $\propto \exp(aL)$ $[21, 45]$. This is because, as depicted in Fig. 5(b), the $\pi/2$ counterclockwise simultaneous rotation of all dimers along a dotted line in the [11] direction can be performed independently of each other—the same holds also for clockwise rotations along the lines in the [11] direction (a dashed line is an example). Thus, if one chooses a certain direction, the staggered state is completely ordered in that direction and completely disordered in the other one. While the problem of how the staggered state is stabilized is quite unclear, we shall try to give an insight based on an analysis of the one-string motion. For convenience, we consider the transfer matrix connecting the next-nearest-neighbor rows, i.e., $T_{-L/2}^{2}$. Since its eigenvalues or their logarithms are squared (i.e., $\lambda^{2}$) or doubled (i.e., $2E$), respectively. We treat two sites as one unit in which four states are included. Then, we can analytically diagonalize the matrix by a Fourier transformation and obtain the $q$-dependence of the energy $2E_q$, i.e., the “dispersion relation” of the one-string motion in the $x$ direction. Here, we only show results; the details of how to construct the transfer matrix and also the calculation of eigenvalues in the one-string

FIG. 5: (Color online) The $K_v$ dependence of $\zeta$, $\gamma$, and $c$ at $K_h = 0$. An inset identifying marks and physical quantities is given. The phase boundary between the dimer-liquid and the VC phases is around $K_v \approx 0.621$ (see the arrow).

FIG. 6: (Color online) The free-energy density $f_M$ as a function of the string number $M$ and the interaction $K_a$ for the isotropic system with $L = 20$. The phase boundary between the dimer-liquid and the strong repulsion phases is estimated as $K_a \approx -1.97$. 

\[ \text{A 1-string chain is solvable} \quad [41, 42] \quad \text{and exhibits} \quad c = 1 \quad \text{criticality for the anisotropy parameter satisfying} \quad -1 \leq \Delta < 1. \quad \text{This phase is terminated at the SU(2) ferromagnetic point} \quad \Delta = 1, \quad \text{where there occurs a first-order phase transition accompanied by the vanishing of the Gaussian coupling. At this point, the ground state of the L-site chain forms a SU(2) multiplet with total spin} \quad L/2 \quad \text{and thus possesses} \quad L + 1 \quad \text{degeneracy with respect to the} \quad z \quad \text{component} \quad S^z_{\text{total}} \in [-L/2, L/2]. \quad \text{This degeneracy is also implied from the following theoretical observation: since the vertex operator corresponding to} \quad O_{0,n} \quad \text{in Sec. II expresses} \quad n \quad \text{spin flip excitation from the ground state with} \quad S^z_{\text{total}} = 0, \quad \text{the charge} \quad n \quad \text{in a L-site system is restricted to values in} \quad [-L/2, L/2]. \quad \text{In addition, since the scaling dimension} \quad X_{0,n} \quad \text{becomes zero in the limit of} \quad K \to 0, \quad \text{at least the corresponding} \quad L \quad \text{excited levels should degenerate to the ground state to realize the} \quad L + 1 \quad \text{degeneracy} \quad [43, 44]. \quad \text{Returning to our DM where the string number plays a role as a total magnetization, the ground-state energy in each topological sector,} \quad E_{M,q}(L), \quad \text{is expected to become independent of} \quad M. \quad \text{To see this degeneracy, we calculate the} \quad M \quad \text{-dependent free-energy density} \quad Lf_M(L) = E_{M,q}(L). \quad \text{In Fig. 6 we give the results for the isotropic system with} \quad L = 20. \quad \text{The average of couplings varies within} \quad -3 \leq K_a \leq 0 \quad \text{and our estimate of the transition point is} \quad K_a \approx -1.97 \quad \text{(see Sec. IV). One can see that, with decreasing} \quad K_a, \quad \text{the free energy tends to show a weaker} \quad M \quad \text{dependence and, for} \quad K_a \quad \text{smaller than roughly the transition point, it becomes almost constant and zero. The above-mentioned} \quad O(L) \quad \text{degeneracy inferred from the instability of the Gaussian criticality seems to be consistent with this} \quad M \quad \text{independence. However, there still exists a discrepancy in the degree of degeneracy with the staggered state; we shall discuss this issue for the rest of this section.} \quad \text{It is known that the degeneracy of the staggered state is subextensive, i.e.,} \quad \propto \exp(aL) \quad [21, 45]. \quad \text{This is because, as depicted in Fig. 5(b), the} \quad \pi/2 \quad \text{counterclockwise simultaneous rotation of all dimers along a dotted line in the [11] direction can be performed independently of each other—the same holds also for clockwise rotations along the lines in the [11] direction (a dashed line is an example). Thus, if one chooses a certain direction, the staggered state is completely ordered in that direction and completely disordered in the other one. While the problem of how the staggered state is stabilized is quite unclear, we shall try to give an insight based on an analysis of the one-string motion. For convenience, we consider the transfer matrix connecting the next-nearest-neighbor rows, i.e.,} \quad T_{-L/2}^{2}. \quad \text{Since its eigenvalues or their logarithms are squared (i.e.,} \quad \lambda^{2} \quad \text{or doubled (i.e.,} \quad 2E), \quad \text{respectively. We treat two sites as one unit in which four states are included. Then, we can analytically diagonalize the matrix by a Fourier transformation and obtain the} \quad q \quad \text{-dependence of the energy} \quad 2E_q, \quad \text{i.e., the “dispersion relation” of the one-string motion in the} \quad x \quad \text{direction. Here, we only show results; the details of how to construct the transfer matrix and also the calculation of eigenvalues in the one-string...} \]
sector are given in the Appendix. In Fig. 7 for several values of the interactions (isotropic cases), we draw the lower two of four bands. When the eigenvalue becomes a complex number, we take its magnitude in the plot. While the symmetric two-band structure for the non-interacting case (Fig. 7a) is deformed by interactions, there is a unique minimum at $q = 0$ for finite interaction cases (Figs. 7b and 7c). At the same time, as expressed by the dotted lines, complex-conjugate pairs of eigenvalues start to appear near the zone boundary points $q = \pm \pi$. And then, in the strong repulsion limit (Fig. 7d), we find an emergence of a two-fold degenerate zero-energy flat band. The corresponding eigenvalues are given by $\lambda_0^2 = e^{\pm i q}$, which represents the modulations with wave number $\pm q$ in the $y$ direction. Consequently, our results show dispersionless motion in the $[11]$ and the $[1\bar{1}]$ directions, which precisely reflect the above-mentioned degeneracy, and thus this flat band structure may be a signature of the subextensive degeneracy in the staggered state. If we accept this naive argument, we can conjecture that the staggered state is only realized in the limit. However, our argument is of course at a very speculative level; a full understanding should include also the degeneracy in many-string sectors.

**IV. DISCUSSIONS AND SUMMARY**

The isotropic case was discussed in detail in Refs. 20, 21, where the BKT-transition point driven by $\mathcal{L}_2$ and the first-order transition point to the strong repulsion phase were numerically obtained. We have also estimated these transition points (see the double circle and the double square in Fig. 4); in particular, for the former, the above level-spectroscopy approach has been applied. Thus, here we briefly explain our procedure and compare the results. Since the Lagrangian density $\mathcal{L}_0 + \mathcal{L}_4$ is analyzed in the region $K \approx 4$, we focus our attention not on $O_{0,3}$ but instead on the following order parameters responsible for the breaking of the $\pi/2$ rotational symmetry:

$$O_4 = \sqrt{2} \cos 2\sqrt{2} \phi,$$
$$O_5 = \sqrt{2} \sin 2\sqrt{2} \phi.$$

While the former is the orientational order parameter, the latter represents the plaquette order which has not been found in classical DMs (16, 21) (the locking points are $\langle \sqrt{2} \phi \rangle = \pi/4, 3\pi/4, 5\pi/4, and 7\pi/4$). One then finds that via the transformation $2\sqrt{2} \phi \to \sqrt{2} \phi$ and $K/4 \to K$, the Lagrangian density and the operators are reduced to

$$\mathcal{L}_0 + \mathcal{L}_4(y_1) \to \mathcal{L}_0 + \mathcal{L}_2(y_4), \quad O_{4,5} \to O_{0,3}.$$

Therefore, from the discussion in Sec. II the level-crossing and the level-splitting conditions (17) and (18) are satisfied by the scaling dimensions of these operators, say $x_{4,5}$ (here, we have taken the condition $y_4 < 0$ into account). Since the half-charge excitations $\exp(\pm i \sqrt{2} \theta)$ are absent in our system, we employ the condition

$$3x_4 + x_5 = \frac{1}{2}$$

to determine the BKT-transition point. The corresponding excitation levels can be found in the sectors specified by their symmetries given in Table I. We extrapolate the finite-size estimates up to $L = 20$ to the thermodynamic limit according to Eq. (21). We then obtain the BKT-transition point as $K_3 \approx 1.523$. As we see in Table I, the agreement with previous results is very good, which indicates that our approach is valid. Similarly, for first-order phase transition, we have estimated the point via condition $K = 0$ (see also Ref. 21) while others have determined this transition from a point of breakdown in the condition $C = 1$. Our result is closer to the estimate of Alet et al. 20 although there still exists considerable discrepancy among these estimates. Likewise for instance for phase-separation transitions observed in one-dimensional electron systems, higher-order corrections have been argued to ambiguously affect estimations 44. Thus, we think that the discrepancy in these estimates may reflect their effects.

To summarize, we investigated the anisotropically interacting dimer model on a square lattice. For the attractive case, the orientational-order-parameter perturbation introduced by the anisotropy brings about the BKT transition to the columnar phases. We pointed out the close relationship of our model to a frustrated quantum-spin chain and then found the criteria to determine the transition points. Using these, we performed
FIG. 8: (Color online) The string vacuum $|0\rangle$ and the one-string states $\{|A\rangle, |B\rangle, |C\rangle, |D\rangle\}$ are depicted in the first line. A square given by a dotted blue line indicates a unit cell which includes two sites and four bonds. Dimers and strings are given by black rectangles and gray lines, respectively. In subsequent lines, 24 microscopic processes of transfers of one-string states between two next-nearest-neighbor rows are given with weights.

TABLE II: Estimations of the BKT and the first-order transition points in the isotropic system. For the BKT transition the rotational order parameters were treated in Ref. [20]. In Refs. [20, 21], the first-order transition point were estimated from the breakdown of the condition $c \neq 1$.

| Criteria       | BKT | Criteria | First-order |
|----------------|-----|----------|-------------|
| Ref. [20] Order parameters | 1.54 | $c \neq 1$ | $-2.23$     |
| Ref. [21] c, x, etc.     | 1.5 - 1.7 | $c \neq 1$ | $-1.39$     |
| Present Equation (34)    | 1.523 | $K = 0$ | $-1.97$     |

level-spectroscopy analysis of the eigenvalue structures of the transfer matrices. Numerical results were then summarized as the global phase diagram (Fig. 4), which includes the dimer-liquid, the columnar, and the strong repulsion phases. Furthermore, we checked the level-splitting conditions and evaluated the value of the central charge, which provided solid evidence to confirm the universality of the phase transition. By contrast, for the repulsive case, although we determined the dimer-liquid phase boundary, there exist some points with unclear status within the strong repulsion phase including the staggered state. Based on the dispersion relation of the one-string motion, we gave a possible scenario for the stabilization of the staggered phase. However, although this issue still remains an open question, we now think that the nature of the nonsymmetric real matrix might have relevance to its description [46].

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APPENDIX A: A TRANSFER-MATRIX CALCULATION IN ONE-STRING SECTOR

In this appendix, we shall explain how to construct the transfer matrix in the one-string sector and an analytical calculation of eigenvalues by the use of a Fourier transformation. Since a row of the reference configuration given in Fig. 8(b) expresses a string vacuum state, we write it as $|0\rangle$ (see the top left in Fig. 8). For convenience, we treat two sites in the $x$ direction as one unit cell which includes four bonds (see squares by dotted blue lines). Then, one-
string states can be obtained via replacements of one of $L_c (= L/2)$ unit cells in $|0\rangle$ by four possible dimer configurations. From left to right of the first line in Fig. [5] (except for the vacuum), we call these as $|A_l\rangle$, $|B_l\rangle$, $|C_l\rangle$, and $|D_l\rangle$, respectively. Here, the center is supposed to be an $l$th unit cell ($l \in [1, L_c]$). Now, consider transfers of one-string states to those in the next-nearest-neighbor row in the $y$ direction. Then, one can find 24 microscopic processes, which are listed in subsequent lines in Fig. [8]. For instance, the second line shows seven microscopic processes of transfers from $|A_l\rangle$ in the first row to states in the third row, and thus exhibits an operation of the transfer matrix, i.e., $T^2_{1-L_c}|A_l\rangle$. Consequently, we can obtain the following recursion relations for the transfers in the one-string sector

$$
T^2_{1-L_c}|A_l\rangle = |C_{l-1}\rangle + v^2|A_l\rangle + h|B_l\rangle + h|C_l\rangle + |D_{l+1}\rangle + |D_{l-1}\rangle, \quad (A1)
$$

$$
T^2_{1-L_c}|B_l\rangle = h|C_{l-1}\rangle + v^2|A_l\rangle + h^2|B_l\rangle + h|C_l\rangle + |D_{l+1}\rangle + |D_{l-1}\rangle, \quad (A2)
$$

$$
T^2_{1-L_c}|C_l\rangle = |C_{l-1}\rangle + v^2|A_l\rangle + h^2|B_l\rangle + h^2|C_l\rangle + |D_{l+1}\rangle + h|B_{l+1}\rangle + h|D_{l+1}\rangle, \quad (A3)
$$

$$
T^2_{1-L_c}|D_l\rangle = v^2|C_{l-1}\rangle + v^2|B_l\rangle + v^2|D_l\rangle, \quad (A4)
$$



where coefficients represent the Boltzmann weights of interactions in the first and the second rows. Next, by the use of the Fourier transformation, we can block-diagonalize the representation of $T^2_{1-L_c}$: suppose that

$$
|X_q\rangle = \frac{1}{\sqrt{L_c}} \sum_{l=1}^{L_c} e^{-iql} |X_l\rangle \quad (X = A, B, C, D), \quad (A5)
$$

then the $q$-block representation spanned by states $\{|A_q\rangle, |B_q\rangle, |C_q\rangle, |D_q\rangle\}$ is given by a $4 \times 4$ complex nonsym-

metric matrix:

$$
[T^2_{1-L_c}]_q = \begin{pmatrix}
q^2 & h + z_q & h + \bar{z}_q & 1 + z_q \\
q^2 & h^2 + z_q & h^2 \bar{z}_q & h + z_q \\
v^2 & h + z_q & h^2 + \bar{z}_q & 1 + z_q \\
v^2 & h^2 + \bar{z}_q & v^2 & v^2
\end{pmatrix}, \quad (A6)
$$

where $z_q, \bar{z}_q \equiv e^{\pm iq}$. Hence, the characteristic equation to determine eigenvalues $\rho$ is given by

$$
\rho^4 - 2(\cos q + v^2 + h^2)\rho^3 + [v^4 - 4h(1-h)v^2 + (1-h^2)^2] \rho^2 + 2v^2(1-h)^2(1-v^2-h^2) \rho + v^4(1-h)^4 = 0. \quad (A7)
$$

Since it is invariant under a transformation $q \to -q$, a $q$ dependence of the eigenvalue structure is even with respect to the point $q = 0$. Meanwhile, in general cases we use a software to evaluate $q$ dependences of eigenvalues; in some limiting cases, Eq. (A7) becomes simple and permits us to easily manipulate: for instance, for the noninteracting case $h = v = 1$, two of the four eigenvalues are zero, and the rest is obtained from an equation $\rho^2 - 2(\cos q + 2) \rho + 1 = 0$. It then provides two real bands, as given in Fig. [7(a)]. In contrast, for the strong repulsion limit $h = v = 0$, two of the four eigenvalues are zero again, but others are complex values with a modulus of 1, i.e., $e^{\pm i q}$ [see Fig. [7(d)]. An implication of this eigenvalue structure, in particular a correspondence to the degeneracy of states in the IDM is discussed in the last part of Sec. [III].

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