Concentration inequalities for locally small increments of compound empirical processes with applications to solutions of compound and risk averse stochastical programming

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Abstract

The paper deals with concentration inequalities for locally small increments of compound empirical processes. In the asymptotic theory of $m$-estimation such inequalities play an essential role in deriving convergence rates for solutions of the sample average approximation method to solve compound stochastic programs, and in particular for $m$-estimators. We develop inequalities dependent on the sample sizes with explicit terms instead of unspecified universal constants. They are applied to study the Sample Average Approximation method for compound stochastic programs. Nonasymptotic upper estimates for the deviation probabilities of the optimal solutions are derived which are dependent on the sample sizes. They allow to conclude immediately convergence rates for the optimal solutions. In the special case of classical risk neutral stochastic programs, we end up with upper estimates of deviation probabilities for $m$-estimators, and their convergence rates. Moreover, we may also demonstrate how to apply the results to sample average approximation of risk averse stochastic programs. In this respect we consider stochastic programs expressed in terms of absolute semideviations and Average Value at Risk. The investigations are based on concentration inequalities from the recent contribution [12].

Keywords: Compound stochastic program; sample average approximation; $m$-estimation; compound empirical processes; covering numbers; uniform entropy integrals; absolute semideviations, Average Value at Risk, concentration inequalities.

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1 Introduction

Consider a compound stochastic program
\[
\min_{\theta \in \Theta} \mathbb{E}[H(\theta, \mathbb{E}[G(\theta, Z_1)], Z_1)],
\]
where \( \Theta \) denotes a compact subset of \( \mathbb{R}^m \), whereas \( Z \) stands for a \( d \)-dimensional random vector with distribution \( \mathbb{P}^Z \). Compound stochastic programs extend the class of classical risk neutral stochastic programs, and they also enclose some classes of risk averse stochastic programs. Outstanding examples are stochastic programs with absolute semideviations and Average Value at Risk as objectives.

In general the parameterized distribution of the goal function \( G \) is unknown, but some information is available by i.i.d. samples. Using this information, a general device to solve approximately problem (1.1) is provided by the so-called Sample Average Approximation (SAA). For explanation, let us consider a sequence \((Z_j)_{j \in \mathbb{N}}\) of independent \( d \)-dimensional random vectors on some fixed probability space \((\Omega, \mathcal{F}, \mathbb{P})\) which are identically distributed as the \( d \)-dimensional random vector \( Z \). Then the SAA method approximates the genuine optimization problem (1.1) by the following one
\[
\min_{\theta \in \Theta} \frac{1}{n} \sum_{j=1}^{n} H\left(\theta, \frac{1}{n} \sum_{k=1}^{n} G(\theta, Z_k), Z_j\right) \quad (n \in \mathbb{N}).
\]

In the risk neutral case an important subject is to analyze asymptotic distributions of the stochastic sequence
\[
\left(\sqrt{n} \left[ \inf_{\theta \in \Theta} \frac{1}{n} \sum_{j=1}^{n} H\left(\theta, \frac{1}{n} \sum_{k=1}^{n} G(\theta, Z_k), Z_j\right) - \mathbb{E}\left[H(\theta, \mathbb{E}[G(\theta, Z_1)], Z_1)\right]\right]\right)_{n \in \mathbb{N}}.
\]
Standard results may be found in the monograph [15]. For more general compound programs only a few investigations are known, e.g. by [4], [6] and [8]. Most of the known contributions are based on analytical path properties of the process \( G \) like continuity or convexity. Very recently, in [11] the investigations have been extended in respect of allowing for more general goal functions \( G \), which make possible to apply the results to stochastic programs whose goal functions are neither continuous nor convex in the parameter.

Another aspect of the SAA is to study the behaviour of their optimal solutions \( \hat{\theta}_n \). They are known as \( m \)-estimators in the risk neutral case. The literature on \( m \)-estimation provides criteria to derive convergence rates for the solutions even in the general case (see e.g. [17], [10]). Based on the assumption that the stochastic program (1.1) has a unique solution \( \theta^* \) an essential ingredient is to require some condition on small increments \( G_n^{G,H}(\theta, \cdot) - G_n^{G,H}(\theta^*, \cdot) \) of the compound empirical process
\[
G_n^{G,H}(\theta, \cdot) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \left( H\left(\theta, \frac{1}{n} \sum_{k=1}^{n} G(\theta, Z_k), Z_j\right) - \mathbb{E}\left[H(\theta, \mathbb{E}[G(\theta, Z_1)], Z_1)\right]\right),
\]
for almost all sample sizes $n$. In this paper we are interested in the deviation probabilities

$$\mathbb{P}\left( \{ \| \hat{\theta}_n - \theta \|_m > \varepsilon \cdot n^{-\gamma} \} \right) \quad (n \in \mathbb{N}, \varepsilon > 0)$$

for suitable $\gamma \in ]0, \infty[ \) dependent on the sample size. They might be used to construct nonasymptotic confidence regions for the unique solution $\theta^*$. The construction of such nonasymptotic confidence regions was considered in \[13\] under the usual second order growth condition on the objective. It was illustrated there that the most involved part is to find proper upper estimates of the deviations $|G_{G,H}^n(\theta, \cdot) - G_{G,H}^n(\theta^*, \cdot)|$ for parameter $\theta$ near $\theta^*$. This means to look at upper estimates for the probabilities

$$\mathbb{P}\left( \sup_{\| \theta - \theta^* \|_m \leq \delta} |G_{G,H}^n(\theta, \cdot) - G_{G,H}^n(\theta^*, \cdot)| > \varepsilon \cdot n^{-\gamma} \right) \quad (\varepsilon > 0),$$

dependent on the sample size, where $\| \cdot \|_m$ stands for the Euclidean norm on $\mathbb{R}^m$. To the best of our knowledge such estimations are not investigated systematically even in the case of $m$-estimation. This is our motivation to study them as the main objects of the paper. Independently of the SAA method the compound empirical processes $G_{G,H}^n$ may be considered in their own right. Then $\theta^*$ will be any fixed element from the parameter space. Hence our aim is to find concentration inequalities for the increments $G_{G,H}^n(\theta, \cdot) - G_{G,H}^n(\theta^*, \cdot)$. Our investigations do not rely on analytical properties for the paths of $G$. Instead we restrict ourselves to processes $G$, where the associated families $\{ G(\theta, \cdot) \mid \theta \in \Theta \}$ and $\{ G(\theta, \cdot) - G(\theta^*, \cdot) \mid \theta \in \Theta, \| \theta - \theta^* \|_m \leq \delta \} \) (\delta > 0 small) of Borel measurable mappings have finite uniform entropy integrals.

The paper is organized as follows. We shall start with our main results. The first one works also for small deviations and sample sizes. However, it does not take into account tail behaviour of the parameterized random variables $G(\theta, \cdot)$ from the process $G$ underlying the compound process $G_{G,H}$. If these variables have simultaneously light tails, then the second main result improves the former one by exponential bounds. All the results provide explicit bounds instead of using unspecified universal constants. The assumptions of these results are exemplified in the case that $G$ has Hölder continuous paths. Also processes $G$ are discussed whose paths are piecewise Hölder continuous but not necessarily continuous. Value functions of two stage mixed-integer programs are typical examples for processes of such a kind.

In Section 3 we turn considerations to solutions of the SAA for compound stochastic programs with unique solutions. Upper estimates for the probabilities of the absolute errors will be derived from the concentration inequalities for the compound empirical processes and their increments. As an immediate by product these estimates give convergence rates for the solutions of the SAA. Simplifications in the case of $m$-estimation will be pointed out so that an alternative approach is provided to find convergence rates for $m$-estimators. In addition, also the application to risk averse stochastic programs in terms of absolute semideviations will be discussed in more detail. The following Section 4 is devoted to the SAA of risk averse stochastic programs under Average Value at Risk. It will be demonstrated how to obtain upper estimates for the deviation probabilities from the results from Section 3. Section 5 gathers proofs of several results from the...
different sections. The main tools are concentration inequalities from the recent paper [12] which will be recalled at the beginning of the section.

2 The main results

Let the probability space \((\Omega, \mathcal{F}, \mathbb{P})\) from the introduction be atomless and complete, and let us recall that \(\Theta\) is a compact subset of \(\mathbb{R}^m\). We shall denote the diameter of \(\Theta\) w.r.t. the Euclidean metric by \(\Delta(\Theta)\), assuming \(\Delta(\Theta) > 0\). In addition we shall focus on mappings \(G : \Theta \times \mathbb{R}^d \to \mathbb{R}\) meeting the following requirements.

(A 1) \(G(\theta, \cdot)\) is \(\mathbb{P}^Z\)-integrable for every \(\theta \in \Theta\), and \(G\) is assumed to satisfy the following continuity property

\[ G(\theta_k, \cdot) \to G(\theta, \cdot) \text{ in } \mathbb{P}^Z\text{-probability whenever } \theta_k \to \theta \text{ w.r.t. the Euclidean metric.} \]

(A 2) There is some strictly positive square \(\mathbb{P}^Z\)-integrable mapping \(\xi_G : \mathbb{R}^d \to \mathbb{R}\) such that

\[ \sup_{\theta \in \Theta} |G(\theta, z)| \leq \xi_G(z) \text{ for } z \in \mathbb{R}^d. \]

For any sample size \(n\) we want to study small local increments of the compound empirical process \(G_n^{G,H} : \Theta \times \Omega \to \mathbb{R}\), defined by

\[ G_n^{G,H}(\theta, \omega) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \left( H\left(\theta, \frac{1}{n} \sum_{k=1}^{n} G(\theta, Z_k(\omega)), Z_j(\omega)\right) - \mathbb{E}\left[H(\theta, \mathbb{E}[G(\theta, Z_1)], Z_1)\right]\right), \]

where the mapping \(H : \Theta \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}\) is measurable w.r.t. the product \(\sigma\)-algebra \(\mathcal{B}(\Theta) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d)\) of the Borel \(\sigma\)-algebra \(\mathcal{B}(\Theta)\) on \(\Theta\), the Borel \(\sigma\)-algebra \(\mathcal{B}(\mathbb{R})\) on \(\mathbb{R}\), and Borel \(\sigma\)-algebra \(\mathcal{B}(\mathbb{R}^d)\) on \(\mathbb{R}^d\), and \(H(\theta, \mathbb{E}[G(\theta, Z_1)], \cdot)\) is \(\mathbb{P}^Z\)-integrable for \(\theta \in \Theta\). More precisely, denoting the Euclidean norm on \(\mathbb{R}^m\) by \(\| \cdot \|_m\), introducing for any \(\theta \in \Theta\) the parameter subset \(U_\delta(\theta) := \{\theta \in \Theta \mid \|\theta - \theta\|_m \leq \delta\}\), we are interested in upper estimations of the following deviation probabilities

\[ \mathbb{P}\left\{ \sup_{\theta \in U_\delta(\theta^*)} \left| G_n^{G,H}(\theta, \cdot) - G_n^{G,H}(\theta^*, \cdot) \right| > \varepsilon n^{-\gamma} \right\} \quad (\varepsilon > 0) \]

for fixed \(\theta^* \in \Theta\), small \(\delta > 0\) and suitable \(\gamma > 0\). The estimations should be formulated in terms of the sample size \(n\). In order to avoid subtleties of measurability we additionally impose

(A 3) For \(\overline{\theta} \in \Theta\), \(\delta > 0\) there exist some at most countable subset \(C(\mathcal{U}_\delta(\overline{\theta}))\) of the set \(\mathcal{U}_\delta(\overline{\theta})\) and a \(\mathbb{P}^Z\)-null set \(N_{\overline{\theta},\delta}\) such that

\[ \inf_{\theta \in C(\mathcal{U}_\delta(\overline{\theta}))} \left| G(\theta, z) - G(\overline{\theta}, z) \right| = 0 \text{ if } \theta \in \mathcal{U}_\delta(\overline{\theta}) \text{ and } z \in \mathbb{R}^d \setminus N_{\overline{\theta},\delta}. \]
There exist nonnegative square $\mathbb{P}^Z$-integrable mappings $L_1, L_2$ such that
\[
|H(\theta, t, z) - H(\vartheta, s, z)|^2 
\leq L_1(z)^2 |G(\theta, z) - G(\vartheta, z)|^2 + L_2(z)^2 |t - s|^2 \quad \text{for } \theta, \vartheta \in \Theta, t, s \in \mathbb{R}, z \in \mathbb{R}^d.
\]

**Remark 2.1** We may invoke Vitalis theorem (see e.g. [1, Proposition 21.4]) to see that the mapping $\theta \mapsto \mathbb{E}[G(\theta, Z)]$ on $\Theta$ is continuous under (A 1) and (A 2). Moreover, by assumption (A 4) we may also observe that the sequence
\[
\left( H(\theta_k, \mathbb{E}[G(\theta_k, Z)]), Z_1) - H(\theta, \mathbb{E}[G(\theta, Z)], Z_1) \right)_{n \in \mathbb{N}}
\]
converges in probability to 0 for $\|\theta_k - \theta\|_m \to 0$, and together with (A 2) it is dominated by some integrable random variable. Hence by Vitalis theorem again we may show that the mapping $\theta \mapsto \mathbb{E}[H(\theta, \mathbb{E}[G(\theta, Z)], Z_1)]$ on $\Theta$ is a continuous mapping w.r.t. the Euclidean norm.

Note that under (A 1) - (A 4) together with Remark 2.1 the suprema of the increments of $G^{G,H}$ in (2.1) may be described up to $\mathbb{P}$-null sets as suprema of a most countably many random variables. In particular by completeness of $(\Omega, F, \mathbb{P})$ they are always random variables on $(\Omega, F, \mathbb{P})$.

By compactness, $\Theta = \mathcal{U}_\delta(\theta)$ holds for every $\delta$ larger than the diameter of $\Theta$ and any $\theta \in \Theta$. In particular the separability property in (A 3) is also satisfied for the entire parameter set $\Theta$. Hence assumption (A 3) will allow us to invoke results from [12] to develop upper estimations for the probabilities in (2.1). These results use general devices from empirical process theory which are based on covering numbers for classes of Borel measurable mappings from $\mathbb{R}^d$ into $\mathbb{R}$ w.r.t. $L_p$-norms. To recall these concepts adapted to our situation, let us fix any nonvoid set $\mathcal{F}$ of Borel measurable mappings from $\mathbb{R}^d$ into $\mathbb{R}$ and any probability measure $\mathbb{Q}$ on $\mathcal{B}(\mathbb{R}^d)$ with metric $d_{\mathbb{Q}, p}$ induced by the $L_p$-norm $\| \cdot \|_{\mathbb{Q}, p}$ for $p \in [1, \infty]$.

- **Covering numbers for $\mathcal{F}$**
  We use $N(\eta, \mathcal{F}, L^p(\mathbb{Q}))$ to denote the minimal number to cover $\mathcal{F}$ by closed $d_{\mathbb{Q}, p}$-balls of radius $\eta > 0$ with centers in $\mathcal{F}$. We define $N(\eta, \mathcal{F}, L^p(\mathbb{Q})) := \infty$ if no finite cover is available.

- **An envelope of $\mathcal{F}$** is defined to mean some Borel measurable mapping $C_{\mathcal{F}} : \mathbb{R}^d \to \mathbb{R}$ satisfying $\sup_{h \in \mathcal{F}} |h| \leq C_{\mathcal{F}}$. If an envelope $C_{\mathcal{F}}$ has strictly positive outcomes, we shall speak of a **positive envelope**.

- **$\mathcal{M}_{\text{fin}}$** denotes the set of all probability measures on $\mathcal{B}(\mathbb{R}^d)$ with finite support.

A central cornerstone of the results in [12] is the assumption on finiteness of uniform entropy integrals. With a slight abuse of convention, we shall call
\[
J(\mathcal{F}, C_{\mathcal{F}}, \delta) := \int_0^\delta \sup_{\mathcal{Q} \in \mathcal{M}_{\text{fin}}} \sqrt{\ln \left( 2 N\left( \varepsilon \| C_{\mathcal{F}} \|_{\mathcal{Q}, 2}, \mathcal{F}, L^2(\mathcal{Q}) \right) \right)} \, d\varepsilon \quad (2.2)
\]
the uniform entropy integral (up to $\delta$) of the function class $F$ with positive envelope $C_F$.

From now on, let us fix $\theta^* \in \Theta$, and set $U_\delta := U_\delta(\theta^*)$. For our purposes the function classes $F_\delta := \{ G(\theta, \cdot) - G(\theta^*, \cdot) \mid \theta \in U_\delta \}$ for $\delta > 0$ and $F^\Theta := \{ G(\theta, \cdot) \mid \theta \in \Theta \}$ are the relevant ones. Note that (A 2) means nothing else but the existence of some square $\mathbb{P}^\mathbb{Z}$-integrable positive envelope of $F^\Theta$. We assume:

(A 5) Under (A 2) with function $\xi^G$ and (A 4) with function $L_1$, there exist $\beta \in [0,1]$, $\delta_1, M_{\delta_1}, M^{\bar{\delta}_1} > 0$ with $\delta_1 \leq \Delta(\Theta)$, and a family $(\xi^G_{\delta})_{\delta \in [0, \delta_1]}$ of square $\mathbb{P}^\mathbb{Z}$-integrable positive envelopes $\xi^G_{\delta}$ of $F^\Theta$ satisfying

$$\|(L_1 \vee 1) : \xi^G_{\delta}\|_{L^2,2} \leq M_{\delta_1} \delta^\beta$$

and

$$J(F^\Theta, \xi^G_{\delta}, 1/8) \vee J(F^\Theta, \xi^G, 1/8) \leq M^{\bar{\delta}_1}$$

for $\delta \in [0, \delta_1]$.

Preparing the final condition, we denote by $\| \cdot \|_{m+1}$ the Euclidean norm on $\mathbb{R}^{m+1}$, and we shall use notation $V_\delta$ for the set of all $(\theta, t) \in \Theta \times \mathbb{R}$ fulfilling the inequality $\| (\theta, t) - (\theta^*, \mathbb{E}[G(\theta^*, Z_1)]) \|_{m+1} \leq \delta$ ($\delta > 0$).

(A 6) There exist $\delta_2 > 0$ and a Borel measurable mapping $m_{\Theta \times \mathbb{R}} : \Theta \times \mathbb{R} \to \mathbb{R}$ which is Lipschitz continuous on $V_{\delta_2}$ with Lipschitz-constant $K_{\delta_2}$ and satisfies

$$\mathbb{E}[H(\theta, t, Z_1)] - \mathbb{E}[H(\theta, s, Z_1)] = \int_s^t m_{\Theta \times \mathbb{R}}(\theta, u) \, du \quad \text{for } (\theta, t), (\theta, s) \in \Theta \times \mathbb{R}, t > s.$$

Recalling the constants $\delta_1, M_{\delta_1}, M^{\bar{\delta}_1}$ from (A 5) and $\delta_2 > 0, K_{\delta_2} \geq 0$ from (A 6), and setting $\bar{\psi}(\theta^*) := \mathbb{E}[G(\theta^*, Z_1)]$, the following constants will play an important role in our main results:

$$\delta^*_\beta := \delta_1 \wedge \delta_2 / (2[\delta_1^{1-\beta} + M_{\delta_1}])^{1/\beta},$$

$$M_{a,b} := K_{\delta_2} a (\delta_1^{1-\beta} + M_{\delta_1} + 2 b) + b \, |m_{\Theta \times \mathbb{R}}(\theta^*, \bar{\psi}(\theta^*))| \quad (a, b > 0),$$

$$\overline{M}_b := M_{\delta_1} (1 + \|L_2\|_{L^2,2}) + b \|L_2\|_{L^2,2} \quad (b > 0),$$

$$\mathfrak{g}(t) := \frac{t^2}{8(t+1)(5t+28)} \quad (t > 0).$$

For $a, b, \delta > 0, n \in \mathbb{N}$ we also define the following abbreviation

$$\bar{\tau}_n(a, b, \delta) := 64\sqrt{2}M^{\delta_1} + 32\sqrt{24} + \left[ \ln(a/b) - \ln(\sqrt{n} \delta^\beta) \right]^+.$$
Here we have used the usual notation from empirical process theory
\[ (\mathbb{P}_n - \mathbb{P})(f) := \frac{1}{n} \sum_{j=1}^{n} \left( f(Z_j) - \mathbb{E}[f(Z_j)] \right) \]  
(2.10)
for \( \mathbb{P}^Z \)-integrable mappings \( f : \mathbb{R}^d \to \mathbb{R} \).

Note that in view of (A 3) along with the completeness of \( (\Omega, \mathcal{F}, \mathbb{P}) \) any \( \Omega_{n,a} \) and \( \Omega_{a,b}^\delta \) belong to \( \mathcal{F} \) because in both cases these events may be described in terms of suprema of at most countably many random variables.

Assumptions (A 1) - (A 6) allow to derive the upper estimates of the probabilities in (2.11) for \( \alpha = \beta/(4 - 2\beta) \), where \( \beta \) is as in (A 5). They will be described in terms of explicit constants. We start with a rough estimation.

**Theorem 2.2** Let assumptions (A 1) - (A 6) be fulfilled with positive envelope \( \xi^G \) of \( \mathbb{P}^\Theta \) as in (A 2), \( \delta_1, M_{\delta_1}, M_{\delta_1}^\delta > 0 \) as well as \( \beta \in [0, 1] \) from (A 5), and \( \delta_2 > 0, K_{\delta_2} \geq 0 \) from (A 6). Furthermore let \( L_1, L_2 \) denote the square \( \mathbb{P}^Z \)-integrable mappings from condition (A 4), whereas \( (\xi^G_{\delta})_{\delta \in [0, \delta_1]} \) stands for the family of positive envelopes from (A 5). Finally, let \( a, b > 0 \). Then, using notations (2.3) - (2.9), the following statements hold.

1) The mapping \( \xi^{G}_{a,b,\delta} := \sqrt{(L_1(\cdot) \vee 1)^2 \cdot \xi^G(\cdot)^2 + L_2(\cdot)^2 \cdot (b + M_{\delta_1})^2 \cdot \delta^{2\beta}} \) is square \( \mathbb{P}^Z \)-integrable for every \( \delta \in [0, \delta_1] \).

2) For every \( n \in \mathbb{N} \) with \( n \geq \max\{M_{\delta_1}^2 (\delta_{\beta})^{2\beta}/2, 1 \} \cdot a \cdot \mathbb{E}[L_2(Z_1)] \cdot 9a^2/\delta_{\beta}^2 \}, \) and arbitrary \( \delta \in [0, \delta_{\beta}], \varepsilon > 0 \)
\[
\mathbb{P}\left( \sup_{\theta \in \mathcal{L}_b} \left| G^{G,H}(\theta, \cdot) - G^{G,H}(\theta^*, \cdot) \right| > \varepsilon \cdot n^{-\beta/(4-2\beta)} \right) \leq \left[ M_{\delta_1} \pi_0(a, b, \delta) + 1 \right] \cdot (\mathbb{E}[L_2(Z_1)]) \cdot M_{a,b} \delta^{\beta} n^{\beta/(4-2\beta)}/\varepsilon.
\]

3) \( \mathbb{P}(\Omega \setminus \Omega_{n,a}) \leq 6\sqrt{2} \cdot M_{\delta_1} \cdot \|\xi^G\|_{\mathbb{P}^{Z,2}} / a \) and \( \mathbb{P}(\Omega \setminus \Omega_{a,b}^\delta) \leq 6\sqrt{2} \cdot M_{\delta_1} \cdot n^{\beta/(4-2\beta)} / \varepsilon \).

The proof of Theorem 2.2 is delegated to Subsection 5.1.

We may improve the estimations in Theorem 2.2 if the involved square \( \mathbb{P}^Z \)-integrable mappings \( \xi^G, \xi^G_{\delta}, \) and \( \xi^{G}_{a,b,\delta} \) have weak tails. For preparation, we shall associate any square \( \mathbb{P}^Z \)-integrable mapping \( \xi \) with the following events
\[
B^\xi_n := \left\{ \frac{1}{n} \sum_{j=1}^{n} \xi(Z_j)^2 \leq 2\mathbb{E}[\xi(Z_1)^2] \right\} \quad (n \in \mathbb{N}).
\]  
(2.11)

The idea now is to further restrict the increments of \( G^{G,H}_n \) to \( B^G_n \), and to provide upper estimates for \( \mathbb{P}(\Omega \setminus (\Omega_{n,a} \cap B^{G}_n)) \) as well as \( \mathbb{P}(\Omega \setminus (\Omega_{a,b}^\delta \cap B^{G}_n)) \).
Theorem 2.3 Let the assumptions of Theorem 2.2 be fulfilled, in particular let \( a, b > 0 \). Furthermore \( \xi_{n,a,b} \) denotes the Borel measurable mapping defined in Theorem 2.2. Then, using notations (2.3) - (2.11), and defining \( \eta_{t,n}(a,b,\delta) := 1 + 2 \, (t + 1) \, \eta_n(a,b,\delta) \) for \( t, \delta > 0 \), the following statements hold.

1) For every \( n \in \mathbb{N} \) with \( n \geq \text{max} \{ M_b^2 \, (\delta^2)^{3/2} / 2, \, 1 \, |_\mathbb{R}_0 \, (E[L_2(Z_1)]) \cdot 9a^2 / \delta^2, \, ||\xi||^2_{P^2,2} / 2 \} \), any \( \delta \in [0, \delta^*_n] \), and arbitrary \( \varepsilon, t > 0 \)

\[
\mathbb{P}\left( \left\{ \sup_{\theta \in \Theta} | G^{G,H}(\theta, \cdot) - G^{G,H}(\theta^*, \cdot) | > \varepsilon \cdot n^{-\beta/(4-2\beta)} \right\} \cap \Omega_{n,a} \cap \Omega_{n,b}^\varepsilon \right) \\
\leq \exp \left( [\eta_{t,n}(a,b,\delta) + 1 \, |_\mathbb{R}_0 \, (E[L_2(Z_1)]) \cdot M_{a,b} \cdot g(t)] \cdot \exp \left( -\frac{\varepsilon \cdot g(t)}{n^{\beta/(4-2\beta)} \, \delta^\beta \, M_b} \right) \right) \\
+ \mathbb{P}(\Omega \setminus B_n^{G,a,b})
\]

if \( 1 \, |_\mathbb{R}_0 \, (E[L_2(Z_1)]) \cdot \delta < \varepsilon^{1/\beta} / (n^{1/(4-2\beta)} \cdot M_{n,b}^{1/\beta}) \), and

\[
\mathbb{P}{\left( \left\{ \sup_{\theta \in \Theta} | G^{G,H}(\theta, \cdot) - G^{G,H}(\theta^*, \cdot) | > \varepsilon \cdot n^{-\beta/(4-2\beta)} \right\} \right)}
\leq \exp \left( 1 \, |_\mathbb{R}_0 \, (E[L_2(Z_1)]) \cdot M_{a,b} \cdot g(t) \right) \cdot \exp \left( -\frac{\varepsilon \cdot g(t)}{n^{\beta/(4-2\beta)} \, \delta^\beta \, M_b} \right) + \mathbb{P}(\Omega \setminus B_n^{G,a,b})
\]

in case of \( \delta < \left( \varepsilon / \left( 1 \, |_\mathbb{R}_0 \, (E[L_2(Z_1)]) \cdot M_{a,b} + M_b \cdot \eta_{t,n}(a,b,\delta) \right) \right)^{1/\beta} \cdot n^{-1/(4-2\beta)} \).

2) If \( t > 0 \) such that \( a > ||\xi||_{P^2} \cdot \left( 1 + 64 \sqrt{2} \, (t + 1) \, M_{b} \right) \), and if \( n \in \mathbb{N} \) satisfies \( n \geq ||\xi||_{P^2,2}^2 / 2 \), then

\[
\mathbb{P}(\Omega \setminus \Omega_{n,a}) \leq \exp \left( -\frac{a \cdot g(t)}{||\xi||_{P^2,2}^2} \right) + \mathbb{P}(\Omega \setminus B_n^G).
\]

3) For \( t > 0, \delta \in [0, \delta_1] \) and \( n \in \mathbb{N} \) with \( n \geq M_{\delta_1}^2 \, \delta^{2\beta} / 2 \)

\[
\mathbb{P}(\Omega \setminus \Omega_{n,b}) \leq \exp \left( -\frac{b \cdot g(t)}{M_{\delta_1}} \right) + \mathbb{P}(\Omega \setminus B_n^G)
\]

is valid whenever \( b > M_{\delta_1} \cdot \left( 1 + 64 \sqrt{2} \, (t + 1) \, M_{b} \right) \).

The proof of Theorem 2.3 may be found in the Subsection 5.1.

In the following we want to discuss possible further upper estimations for the probabilities \( \mathbb{P}(\Omega \setminus B_n^G), \mathbb{P}(\Omega \setminus B_n^{G,a}), \mathbb{P}(\Omega \setminus B_n^{G,a,b}) \) in Theorem 2.3.

Remark 2.4 Let \( \xi \) denote any square \( P^2 \)-integrable mapping with event \( B_n^G \) as defined in (2.11). The following upper estimates of \( \mathbb{P}(\Omega \setminus B_n^G) \) might be used to make the upper estimations in Theorem 2.3 more explicit.
1) If the function $\xi$ is bounded by some positive constant $L$, then $\Omega \setminus B_n^\xi = \emptyset$.

2) If $\xi$ is $\mathbb{P}^Z$-integrable of order 4, we may apply Cantelli’s inequality to conclude
\[
\mathbb{P}(\Omega \setminus B_n^\xi) \leq \frac{\text{Var}[\xi(Z_1)^2]}{n \mathbb{E}[\xi(Z_1)^2]^2 + \text{Var}[\xi(Z_1)^2]} \quad \text{for } n \in \mathbb{N}.
\]

3) The upper estimate of the probability $\mathbb{P}(\Omega \setminus B_n^\xi)$ may be further improved if the random variable $\exp(\lambda \cdot \xi^2)$ is $\mathbb{P}^Z$-integrable for some $\lambda > 0$. In this case
\[
M(\xi^2) := \sup_{k \in \mathbb{N}, k \geq 2} \left( \left| \mathbb{E} \left[ (\xi(Z_1)^2 - \mathbb{E}[\xi(Z_1)^2])^k \right] \right| / k! \right)^{1/k} < \infty,
\]
and
\[
\mathbb{P}(\Omega \setminus B_n^\xi) \leq \exp \left( -n \mathbb{E}[\xi(Z_1)^2]/(8\eta^2) \right) \vee \exp \left( -n \mathbb{E}[\xi(Z_1)^2]/(4\eta) \right)
\]
for $n \in \mathbb{N}$, and any $\eta \geq M(\xi^2)$ (see [12, Remark 2.3]).

4) Let $\xi := (t_1 \xi_1^2 + t_2 \xi_2^2)^{1/2}$ for some $t_1, t_2 > 0$ and square $\mathbb{P}^Z$-integrable mappings $\xi_1, \xi_2$. Examples of such $\xi$ are provided by the mappings $\xi_{a,b}^\Theta$ used in Theorems 2.2, 2.3. Then
\[
\mathbb{P}(\Omega \setminus B_n^\xi) \leq \sum_{i=1}^2 \mathbb{P}(\Omega \setminus B_n^{\xi_i}) \quad \text{for } n \in \mathbb{N}.
\]

In particular, we might apply the results from 1) - 3) separately to $\xi_1$ and $\xi_2$. E.g. if the Borel measurable mappings $\exp(\lambda_i \cdot \xi_i^2)$ ($i = 1, 2$) are $\mathbb{P}^Z$-integrable for some $\lambda_1, \lambda_2 > 0$, then by 3)
\[
\mathbb{P}(\Omega \setminus B_n^\xi) \leq \sum_{i=1}^2 \exp \left( -n \mathbb{E}[\xi_i(Z_1)^2]/(8M(\xi_i^2)^2) \right) \vee \exp \left( -n \mathbb{E}[\xi_i(Z_1)^2]/(4M(\xi_i^2)) \right)
\]
for $n \in \mathbb{N}$.

The finiteness of uniform entropy integrals, as required implicitly in Theorems 2.2, 2.3 is always guaranteed if the involved covering numbers have polynomial rates. Indeed this relies on the observation, that by using change of variable formula several times along with integration by parts, we obtain
\[
\int_0^1 \sqrt{v \ln(K/\varepsilon)} \, d\varepsilon \leq 2\sqrt{v \ln(K)} \quad \text{for } v \geq 1, K \geq e. \quad (2.12)
\]
This observation is the starting point to provide explicit upper estimates of the uniform entropy integrals involved in the result of Theorems, 2.2, 2.3 if the objective $G$ satisfies specific analytical properties.

We start with the following condition
There exist some $\beta \in [0,1]$ and a square $\mathbb{P}^Z$-integrable strictly positive mappings $C : \mathbb{R}^d \to [0, \infty]$ such that
\[ |G(\theta, z) - G(\vartheta, z)| \leq C(z) \|\theta - \vartheta\|_m^{\beta} \text{ for } z \in \mathbb{R}^d, \theta, \vartheta \in \Theta. \]

Under (H) we may construct on the one hand explicitly square $\mathbb{P}^Z$-integrable envelopes $\xi^G$ and $\xi^G_\delta$ of $F^\Theta$ and $F^\Theta_\delta$ respectively. On the other hand we may also provide explicit upper estimates for the associated uniform entropy integrals $J(F^\Theta, \xi^G, \varepsilon)$ and $J(F^\Theta_\delta, \xi^G_\delta, \varepsilon)$. The results are gathered in the following proposition.

**Proposition 2.5** Let condition (H) be fulfilled with $\beta \in [0,1]$ and square $\mathbb{P}^Z$-integrable strictly positive mapping $C$. If $G(\theta, \cdot)$ be Borel measurable for every $\theta \in \Theta$, then (A 3) is met, and the following statements are valid

1) If $G(\theta, \cdot)$ is square $\mathbb{P}^Z$-integrable for some $\theta \in \Theta$, then $\xi^G := C(\cdot) \Delta(\Theta)^\beta + |G(\theta, \cdot)|$ is a positive square $\mathbb{P}^Z$-integrable envelope of $F^\Theta$ with
\[ J(F^\Theta, \xi^G, \varepsilon) \leq 2\varepsilon \sqrt{(3m + 1) \ln(2) + \frac{m}{\beta} \ln(2/\varepsilon)} \text{ for } \varepsilon \in [0, 1/2]. \]

2) For $\delta > 0$, the mapping $\xi^G_\delta := C(\cdot) 2^\beta \delta^\beta$ defines a positive square $\mathbb{P}^Z$-integrable envelope of $F^\Theta_\delta$ satisfying
\[ J(F^\Theta_\delta, \xi^G_\delta, \varepsilon) \leq 2\varepsilon \sqrt{(3m + 1) \ln(2) + \frac{m}{\beta} \ln(2/\varepsilon)} \text{ for } \varepsilon \in [0, 1/2]. \]

**Proof** Note that $|[G(\theta, z) - G(\theta^*, z)] - [G(\vartheta, z) - G(\theta^*, z)]| \leq C(z) \|\theta - \vartheta\|_m^{\beta}$ holds for $\theta, \vartheta \in U_\delta$, and that $0 \in F^\Theta_\delta$ is square $\mathbb{P}^Z$-integrable for $\delta > 0$. Then the statements of Proposition 2.5 are immediate consequences of separate applications of Proposition 2.6 from [12] to the function classes $F^\Theta$ and $F^\Theta_\delta$ ($\delta > 0$).

**Remark 2.6** Proposition 2.5 tells us that under (H) the results of Theorems 2.2, 2.3 may be concluded from assumptions (A 1), (A 4) and (A 6) together with the conditions

- $G(\theta, \cdot)$ is square $\mathbb{P}^Z$-integrable for some $\theta \in \Theta$.
- $(L_1 \lor 1) C(\cdot)$ is square $\mathbb{P}^Z$-integrable.

In particular we may choose in (A 5) the positive number $\delta_1 := \Delta(\Theta)$ with constants $M_1 := 2^\beta \|L_1 \lor 1\|_{pZ, 2} C(\cdot)$ and $M_1 := \sqrt{(3m + 1) \ln(2) + m \ln(16)/\beta/4}$.

Next, let us consider objective $G$ having the following kind of structure of piecewise Hölder continuity.

**PH** $G(\theta, z) = \sum_{i=1}^r \sum_{a' \in I_{\Delta(\theta, \cdot)}(z)} \mathbb{I}_{\Delta_i(\theta, \cdot) + a' \in I_{\Delta(\theta, \cdot)}}(z) \cdot G^i(\theta, z)$, where
• $r, s_1, \ldots, s_r \in \mathbb{N}$,
• $G^i$ satisfies (A 1), and (H) with $\beta_i \in [0, 1]$ as well as strictly positive square $\mathbb{P}^Z$-integrable $C^i : \mathbb{R}^d \to \mathbb{R}$ for $i \in \{1, \ldots, r\}$,
• $\Lambda_{i \delta} : \mathbb{R}^m \times \mathbb{R}^d \to \mathbb{R}$ Borel measurable with $\Lambda_{i \delta}(\cdot, z)$ affine linear for $z \in \mathbb{R}^d$
  $(i \in \{1, \ldots, r\}, \delta \in \{1, \ldots, s_i\})$,
• $a_i^l \in \mathbb{R}$ for $i \in \{1, \ldots, r\}, l \in \{1, \ldots, s_i\}$,
• $I_{i \delta} = [0, \infty[$ or $I_{i \delta} = [0, \infty[$ for $i \in \{1, \ldots, r\}$ and $l \in \{1, \ldots, s_i\}$.
• The set
  \[
  \left\{ \bigcap_{l=1}^{s_i} \left\{ \Lambda_{i \delta}(\cdot, \cdot) + a_i^l \in I_{i \delta} \right\} \mid i \in \{1, \ldots, r\}, l \in \{1, \ldots, s_i\} \right\}
  \]
  is a partition of $\mathbb{R}^d$.

In two stage mixed-integer programs the goal functions typically may be represented in this way if the random vector $\tilde{Z}$ has compact support (see [3, p. 121] along with [12]).

Note that if $G$ satisfies condition (PH), it does not have continuity in $\theta$ in advance.

For abbreviation we set $B_i(\theta) := \bigcap_{l=1}^{s_i} \{ z \in \mathbb{R}^d \mid \Lambda_{i \delta}(\theta, z) + a_i^l \in I_{i \delta} \}$ for arbitrary $i$ from $\{1, \ldots, r\}$, and we introduce the associated function classes

\[
F_{p_{ih}} := \{ 1_{B_i(\theta)} \mid \theta \in \Theta \} \quad \text{and} \quad F_{rh} := \{ G^i(\theta, \cdot) \mid \theta \in \Theta \} \quad i \in \{1, \ldots, r\}.
\]

Note that the classes $F_{p_{ih}}$ are uniformly bounded by 1. We borrow from [12] (Proposition 2.8 there) the following result concerning construction of envelope $\xi^G$ of $F^\Theta$ and upper estimation of the associated uniform entropy integrals $J(F^\Theta, \xi^G, \varepsilon)$.

**Proposition 2.7** The set $B_i(\theta)$ is a Borel subset of $\mathbb{R}^d$, and the mapping $G^i(\theta, \cdot)$ is Borel measurable for $\theta \in \Theta$ and $i \in \{1, \ldots, r\}$. In particular $G(\theta, \cdot)$ is Borel measurable for every $\theta \in \Theta$. Moreover, if $G^1(\theta, \cdot), \ldots, G^r(\theta, \cdot)$ are square $\mathbb{P}^Z$-integrable for some $\theta \in \Theta$, and if $\xi_1, \ldots, \xi_r$ denote bounded positive envelopes of the classes $F^1_{p_{ih}}, \ldots, F^r_{p_{ih}}$ respectively, then the mapping $\xi^G := \sum_{i=1}^{r} \xi_i \cdot (\Delta(\Theta)^{\beta_i} C_i(\cdot) + |G^i(\theta, \cdot)|)$ is a positive square $\mathbb{P}^Z$-integrable envelope of $F^\Theta$ satisfying

\[
J(F^\Theta, \xi^G, \varepsilon) \\
\leq 2\varepsilon \sqrt{r + 2r m \ln(3) + m \ln(4r/\varepsilon) \sum_{i=1}^{r} 1/\beta_i + \ln(2) + [5 + 2 \ln(4r/\varepsilon)] (m + 2) \sum_{i=1}^{r} s_i}
\]

for $\varepsilon \in [0, 1]$.

Let us turn over to consider the function classes $F^\Theta_{\delta}$ under representation (PH). We set $A \Delta B := A \setminus B \cup B \setminus A$ for sets $A, B$, and we shall use the following notation

\[
\overline{B}_{i \delta} = \bigcup_{\theta \in \mathcal{U}_h} B_i(\theta) \Delta B_i(\theta^*) \quad \text{for } \delta > 0, i \in \{1, \ldots, r\}. 
\]

(2.13)
Proposition 2.8 Let $G$ fulfill representation (PH) with mappings $G^1(\theta, \cdot), \ldots, G^r(\theta, \cdot)$ being $B(\mathbb{R}^d)$-measurable for $\theta \in \Theta$. Furthermore, let $C^1, \ldots, C^r : \mathbb{R}^d \to \mathbb{R}$ as well as $\beta_1, \ldots, \beta_r \in [0, 1]$ be from representation (PH), and let for $\delta > 0$ denote by $\overline{B}_{1\delta}, \ldots, \overline{B}_{r\delta}$ the sets as in (2.13). Then, for $\delta > 0$, fixing Borel subsets $\overline{B}_{1\delta}, \ldots, \overline{B}_{r\delta}$ of $\mathbb{R}^d$ with $\overline{B}_{i\delta} \supseteq \overline{B}_{j\delta} (i \in \{1, \ldots, r\})$, a positive envelope $\xi^G_{\delta}$ of the function class $\mathbb{F}^\Theta$ is defined by

$$\xi^G_{\delta}(z) := \sum_{i=1}^{r}[\delta^\beta_i C^i(z) + |G(\theta^*, z)|] 1_{\overline{B}_i} + (\delta \wedge 1)^2],$$

and

$$J(\mathbb{F}^\Theta, \xi^G_{\delta}, \varepsilon) \leq 2\varepsilon \sqrt{r + \ln(2) + c_{mr\varepsilon} + [5 + 2\ln(8(r + 1)/\varepsilon)]} \ d_{mr}
+ 2\varepsilon \sqrt{r + 4[1 + \ln(8(r + 1)/\varepsilon)]} \ d_{mr}$$

holds for every $\varepsilon \in [0, 1]$, where $c_{mr\varepsilon} := 2r m \ln(3) + m \ln(8(r + 1)/\varepsilon) \sum_{i=1}^{r} 1/\beta_i$ and $d_{mr} := (m + 2) \sum_{i=1}^{r} s_i$.

The proof of Proposition 2.8 is subject of Subsection 5.4.

Remark 2.9 Let us discuss how to apply Theorem 2.2, 2.3 if $G$ has representation (PH), and the sets $B_i(\theta)$ satisfy the following property

(*) The set \[ \{ z \in \mathbb{R}^d \mid \Lambda_{il}(\theta, z) = -a_i^l \text{ for some } \theta \in \Theta \} \] is a $\mathbb{P}^Z$-null set for all indices $i \in \{1, \ldots, r\}$ and $l \in \{1, \ldots, s_i\}$ with $I_d = [0, \infty].$

1) Under condition (*) the continuity property required in (A 1) and also condition (A 3) may be verified by routine procedures. Moreover by Proposition 2.7 we also know how to find a square $\mathbb{P}^Z$-integrable positive envelope of $\mathbb{F}^\Theta$ for (A 2).

2) Since $\Theta$ is separable, property (*) implies that any set $\overline{B}_{i\delta}$ may be described up to some $\mathbb{P}^Z$-null set as an at most countable union of Borel subsets of $\mathbb{R}^d$. Hence for $\delta > 0, i \in \{1, \ldots, r\}$, the set $\overline{B}_{i\delta}$ is $\mathbb{P}^Z$-measurable, and thus $1_{\overline{B}_{i\delta}}(Z_1)$ is a random variable on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

3) Under (A 4) with square $\mathbb{P}^Z$-integrable mapping $L_1$ we may replace (A 5) with the following condition:

(**) The mappings $(L_1 \vee 1) C^1, \ldots, (L_1 \vee 1) C^r$ are square $\mathbb{P}^Z$-integrable, and there exist $\beta \in (0, 1]$ and $\delta_1, \widehat{M}_{i\delta} > 0$ such that for every $i \in \{1, \ldots, r\}$

$$\beta \leq \beta_i \text{ and } \sup_{\delta \in [0, \delta_1]} \mathbb{E}[|L_1(Z_1) \vee 1|^2 G^i(\theta^*, Z_1)^2 1_{\overline{B}_{i\delta}}(Z_1)]/\delta^{2\beta} \leq \widehat{M}_{i\delta}.
$$

In view of 2) we may find for any $\overline{B}_{i\delta}$ some Borel subset $\widehat{B}_{i\delta}$ of $\mathbb{R}^d$ enclosing $\overline{B}_{i\delta}$ such that $1_{\widehat{B}_{i\delta}}(Z_1) = 1_{\overline{B}_{i\delta}}(Z_1) \mathbb{P}$-a.s.. Using the envelopes from Propositions 2.7, 2.8 with the sets $\widehat{B}_{i\delta}$, assumption (**) implies (A 5) with $\beta, \delta_1,$ and

$$M^{\delta_1} := (\delta_1 \vee 1) \sum_{i=1}^{r} (|L_1(1) \vee 1| C^i)_{p^2} + r \left( \widehat{M}_{i\delta_1}^{1/2} + (\delta_1 \wedge 1)^{2-\beta} \|L_1 \vee 1\|_{p^2} \right)
M^{\delta_1} := \sqrt{r + c_{mr} + \ln(2) + [5 + 2\ln(8)]} \ d_{mr}/4 + \sqrt{r + 4[1 + \ln(7)]} \ d_{mr}/4,$$
where \( r := 64(r + 1), \) and \( c_{mr} := 2r m \ln(3) + m \ln(r) \sum_{i=1}^{r} 1/\beta_i \) as well as \( d_{mr} := (m + 2) \sum_{i=1}^{r} s_i. \)

Taking the remarks 1), 2) into account the results of Theorems 2.2, 2.3 carry over under (A 4) and (A 6), where we have also more explicit descriptions for the involved constants.

3 SAA of compound stochastic programs

Throughout this section we shall investigate the following compound form of stochastic program

\[
\min_{\theta \in \Theta} \mathbb{E}[H(\theta, \mathbb{E}[G(\theta, Z_1)], Z_1)] := \min_{\theta \in \Theta} \psi_{H, G}(\theta),
\]

where the mappings \( G, H \) are the same as in Section 2. A general device to solve approximately problem (3.1) is to use the sample average approximation (SAA). For explanation let \( (Z_j)_{j \in \mathbb{N}} \) be a sequence of independent \( d \)-dimensional random vectors on the complete atomless probability \((\Omega, \mathcal{F}, \mathbb{P})\) which are identically distributed as a random vector \( Z \) with distribution \( \mathbb{P}^Z \). Then the SAA optimization problem based on the i.i.d. sample \((Z_1, \ldots, Z_n)\) associated with (3.1) reads as follows

\[
\min_{\theta \in \Theta} \frac{1}{n} \sum_{j=1}^{n} H\left(\theta, \frac{1}{n} \sum_{k=1}^{n} G(\theta, Z_k), Z_j\right) \quad (n \in \mathbb{N}).
\]

Again \( \Theta \) is a compact subset of \( \mathbb{R}^m \), and its diameter will be denoted by \( \Delta(\Theta) \).

In this section we want to derive convergence rates for sequences \( (\hat{\theta}_n)_{n \in \mathbb{N}} \) of random vectors \( \hat{\theta}_n \) which minimize the SAA problem w.r.t. the sample size \( n \). We always assume that the genuine optimization problem (3.1) has a unique solution \( \theta^* \).

Optimization problem (3.1) has a unique solution \( \theta^* \in \Theta \). (3.3)

From now on the unique solution \( \theta^* \) plays the role of the fixed \( \theta^* \in \Theta \) in Section 2. Notation \( \mathcal{U}_\delta \) and the assumptions (A 5) and (A 6) from this section have to be read w.r.t. the unique solution \( \theta^* \).

Our main purpose is to investigate the probabilities

\[
\mathbb{P}\left\{ \left\| \hat{\theta}_n - \theta^* \right\|_m > \varepsilon \cdot n^{-\gamma} \right\} \quad (n \in \mathbb{N}, \varepsilon > 0)
\]

for suitable \( \gamma \in ]0, \infty[ \). The aim is to find explicit bounds in terms of the sample sizes \( n \). Such bounds will enable us to construction nonasymptotic confidence bands for the solution \( \theta^* \).

We shall pick up an idea from [14]. There the author illustrates how to derive upper estimations for the probabilities (3.4) via variational inequalities. We shall adapt this idea to our situation. As a starting point the unique solution \( \theta^* \) will be required to fulfill the second order growth condition.
There exists $M_3 > 0$ such that the goal function $\psi_{H,\Theta}$ of optimization (3.1) satisfies
\[ \psi_{H,\Theta}(\theta) - \psi_{H,\Theta}(\theta^*) \geq M_3 \|\theta - \theta^*\|_m^2 \quad \text{for } \theta \in \Theta. \]

In the first view this condition seems to be more restrictive than the more usual local second order growth condition.

For some $\delta_3, M_{\delta_3} > 0$ the goal function $\psi_{H,\Theta}$ of optimization (3.1) satisfies
\[ \psi_{H,\Theta}(\theta) - \psi_{H,\Theta}(\theta^*) \geq M_{\delta_3} \|\theta - \theta^*\|_m^2 \quad \text{for } \theta \in \Theta \text{ with } \|\theta - \theta^*\|_m \leq \delta_3. \]

Of course the second order growth condition implies the local second order growth condition. Actually, it will turn out that both conditions are even equivalent within our setting. We define for every $\delta > 0$ the nonnegative number
\[ M^H(\delta) := \inf \{ \psi_{H,\Theta}(\theta) - \psi_{H,\Theta}(\theta^*) \mid \theta \in \Theta \setminus U_\delta \}. \]

**Lemma 3.1** Let (3.3) and the assumptions (A 1), (A 2) and (A 4), (A 7') be fulfilled. Then $\psi_{H,\Theta}$ is a continuous mapping w.r.t. the Euclidean norm. Moreover, if (A 7') holds with $\delta_3, M_{\delta_3} > 0$, then the number $M^H(\delta_3)$ is strictly positive and (A 7) is satisfied with $M_3 = M_{\delta_3} \wedge [M(\delta_3)/\Delta(\Theta)^2]$.

Lemma 3.1 will be shown in Subsection 5.5.

**Remark 3.2** Let (3.3) and (A 1), (A 2) as well as (A 4) be fulfilled. We already know from Lemma 3.1 that $\psi_{H,\Theta}$ is a continuous mapping w.r.t. the Euclidean norm. Then, if in addition the unique solution $\theta^*$ belongs to the topological interior $\text{int}(\Theta)$ of $\Theta$ w.r.t. the standard topology of $\mathbb{R}^m$, the continuity of $\psi_{H,\Theta}$ together with Lemma 3.1 imply that the growth conditions (A 7) and (A 7') are both equivalent with the following property
\[ \lim\inf_{t \to 0, y \to 0} \frac{\psi_{H,\Theta}(\theta^* + t y') - \psi_{H,\Theta}(\theta^*)}{t^2/2} > 0 \quad \text{for all } y \in \mathbb{R}^m \setminus \{0\} \]
(see [2, Proposition 3.100]). This condition in turn is valid if $\psi_{H,\Theta}$ is twice continuously differentiable at $\theta^*$ with positive definite Hessian matrix.

For abbreviation we introduce via $G^H(\theta, z) := H(\theta, E[G(\theta, Z_1)], z)$ the real-valued mapping $G^H$ on $\Theta \times \mathbb{R}^d$. Let us also recall the definition of the compound empirical process $G_n^{\theta_0}$ from Section 2.

Now, the adaption of the idea from [14] leads to the following result which will be the basic step of our investigations. It is in parts known from textbook proofs of results on convergences rates for $m$-estimators (see [10], [17]). We denote by $[x]$ the largest integer that does not exceed the real number $x$. 

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Proposition 3.3 Let (3.3) and the assumptions (A 1), (A 2) as well as (A 4), (A 7) be fulfilled. Then with $M_3 > 0$ from (A 7), and with the mapping $L_2$ from (A 4)

\[
\mathbb{P}\{\|\hat{\theta}_n - \theta^*\|_m > \varepsilon \cdot n^{-\gamma}\} \cap \Omega
\]

\[
\leq \sum_{k=K+1}^{\infty} \mathbb{P}^* \left\{ \sup_{\theta \in U_{(2^k/n^\gamma)}} \left| G_n^{G,H}(\theta, \cdot) - G_n^{G,H}(\theta^*, \cdot) \right| > M_3 2^{2(k-1)} n^{-2\gamma + 1/2} \right\} \cap \Omega
\]

\[
+ \mathbb{P}^* \left\{ \sup_{\theta \in \Theta} \left| G^H(\theta, Z_j) - \psi_{H,\Theta}(\theta) \right| > (1 + 1_{\{0\}}(\mathbb{E}[L_2(Z_1)]) \frac{\delta^2 M_3}{4} \right\} \cap \Omega
\]

\[
+ \mathbb{P}^* \left\{ \sup_{\theta \in \Theta} \left| G(\theta, Z_j) - \mathbb{E}[G(\theta, Z_1)] \frac{1}{n} \sum_{j=1}^{n} L_2(Z_j) > \frac{\delta^2 M_3}{4} \right\} \cap \Omega
\]

holds for $\Omega \in \mathcal{F}$, $n \in \mathbb{N}$, $\delta \in [0, \Delta(\Theta)]$ and $\varepsilon, \gamma \in [0, \infty]$. Here $K_\varepsilon := [\ln(\varepsilon)/\ln(2)]$, and $\mathbb{P}^*$ denotes the outer probability w.r.t. $\mathbb{P}$. If in addition $\delta = \Delta(\Theta)$, then the second and third summand on the right hand side of the inequality may be dropped.

The proof of Proposition 3.3 is delegated to Subsection 5.6.

In view of Proposition 3.3 we may utilize our main results Theorems 2.2, 2.3 to find upper estimations for the probabilities (2.1) as they provide upper estimates for the first summand on the right hand side of the inequality in the statement of Proposition 3.3 We choose $\gamma = 1/(4 - 2\beta)$, and use $\delta_{nk} := 2^k n^{-1/(4 - 2\beta)}$ for $k, n \in \mathbb{N}$. Recall also from Section 2 that under (A 3) the involved suprema of the increments of the empirical process $G^{G,H}$ are random variables of $(\Omega, \mathcal{F}, \mathbb{P})$.

Proposition 3.4 Let the assumptions of Theorem 2.2 be fulfilled. Furthermore let $a, b, \varepsilon > 0$ and $K_\varepsilon := [\ln(\varepsilon)/\ln(2)]$. Then with notations (2.3) - (2.9), (2.11), and setting

\[
\hat{\eta}(a, b, \varepsilon) := 64 \sqrt{2} M_3^{\delta_1} + 32 \sqrt{\ln(24) + [\ln(a/b) + (\beta + 1/2) \ln(2/\varepsilon)]^+},
\]

\[
n(a, b, \beta) := \frac{\sqrt{M_b} (\delta_n^{3/2})^{2\beta}}{2} \vee \frac{1}{\delta_n} \mathbb{E}[L_2(Z_1)] \frac{9 b^2}{\delta_n^2} \vee \left[ \frac{M_b}{\delta_n} (\delta_n^{2\beta+1})^{\frac{4-2\beta}{4+2\beta}} \right],
\]

the following statements are valid.

1) For $n \in \mathbb{N}$ with $n \geq n(a, b, \beta)$ and arbitrary $\delta \in [0, \delta_n^\beta]$

\[
\sum_{k=K_\varepsilon+1}^{\infty} \mathbb{P}\left\{ \sup_{\theta \in U_{(2^k/n^\gamma)}} \left| G_n^{G,H}(\theta, \cdot) - G_n^{G,H}(\theta^*, \cdot) \right| > M_3 2^{2(k-1)} n^{-\beta/(4-2\beta)} \right\} \cap \Omega_{n,a}
\]

\[
\leq \frac{10 M_b \hat{\eta}(a, b, \varepsilon) + \frac{1}{\delta_n} \mathbb{E}[L_2(Z_1)] M_{a,b}}{M_3} \left( \frac{2}{\varepsilon} \right)^{3/2-\beta} \frac{2}{\sqrt{\varepsilon} \wedge 2} + \sum_{k=K_\varepsilon+1}^{\infty} \mathbb{P}(\Omega \setminus \Omega_{n,a}^{\delta_n \wedge \sqrt{T}^2}) .
\]
2) If \( n \in \mathbb{N} \) with \( n \geq n(a, b, \beta) \lor (\|\xi_G\|^2_{P^2} / 2) \), and if for \( t > 0 \) the inequality 
\[
2^{K_2(2-\beta) - K_4 / 2} > 4 \left[ M_{a, b} \mathbb{1}_{[0, \infty]}(L_2(Z_1)) + M_b \left( 1 + 2(t + 1) \hat{n}(a, b, \tilde{\beta}) \right) \right] / M_3 \text{ holds, then}
\]
\[
\sum_{k=K_2+1}^{\infty} \mathbb{P}\left( \left\{ \sup_{\theta \in U_{\delta_n, \delta}^a} |G_n^\theta, H(\theta, \cdot) - G_n^\theta, H(\theta^*, \cdot)| > M_3 \ 2^{(k-1) n - \beta/(1 - 2\beta)} \right\} \cap \Omega_{n,a} \right)
\]
\[
\leq \exp \left( \mathbb{1}_{[0, \infty]}(\mathbb{E}[L_2(Z_1)]) \ M_{a, b} \ g(t) \right) \cdot \frac{2^{4-\beta} M_b \cdot \exp \left( - \varepsilon^{3/2 - \beta} \sqrt{\varepsilon \wedge 2} \ M_3 \ g(t) \right)}{(3 - 2\beta) \ln(2) \ M_3 \ v^{3/2 - \beta} \sqrt{\varepsilon \wedge 2}}
\]
\[
+ \sum_{k=K_2+1}^{\infty} \left[ \mathbb{P}(\Omega \setminus \Omega_{n}^a \delta_n, \delta, \sqrt{k}) + \mathbb{P}(\Omega \setminus B_{n}^{\xi_n, \sqrt{k}, \delta_n, \delta, \sqrt{k}}) \right].
\]

The proof of Proposition 3.4 is subject of Subsection 5.7.

The first summand on the right hand side of the inequality in Proposition 3.3 is already an upper estimate for the deviation probabilities (2.1) under the following specializations of (A 5) and (A 6).

(A 5') Under (A 4) with function \( L_1 \), there exist \( \beta \in [0, 1], \ M_1, M_1 > 0 \) and a family \((\xi^G_{\theta})_{\theta \in [0, \Delta(\Theta)]}\) of positive square \( P^2 \)-integrable envelopes \( \xi^G_{\theta} \) of \( P^{\Theta}_{\delta} \) satisfying
\[
\|(L_1 \lor 1) \cdot \xi^G_{\theta} \|_{P^2} \leq M_1 \delta^\beta \quad \text{and} \quad J(P^{\Theta}_{\delta}, \xi^G_{\theta \Delta(\Theta)}, 1/8) \leq M_1 \text{ for } \delta \in [0, \Delta(\Theta)].
\]

(A 6') With \( \beta \in [0, 1], \ M_1 > 0 \) from (A 5') there exist \( \delta_2 \geq 2\Delta(\Theta)^{\beta} \ [\Delta(\Theta)^{1-\beta} + M_1] \) and a Borel measurable mapping \( m_{\Theta \times \mathbb{R}} : \Theta \times \mathbb{R} \to \mathbb{R} \) which is Lipschitz continuous on \( \mathbb{V}_{\delta_2} \) with Lipschitz-constant \( K_{\delta_2} \) and satisfies
\[
\mathbb{E}[H(\theta, t, Z_1)] - \mathbb{E}[H(\theta, s, Z_1)] = \int_s^t m_{\Theta \times \mathbb{R}}(\theta, u) \ du \quad \text{for } (\theta, t), (\theta, s) \in \Theta \times \mathbb{R}, t > s.
\]

Indeed, if (A 5') and (A 6') hold, then we may drop the second and third summand on the right hand side of the inequality in Proposition 3.3. In particular the results in Proposition 3.3 already provide upper estimations of the deviation probabilities (2.1).

**Theorem 3.5** Let assumptions (A 1) - (A 3), (A 4), (A 5'), (A 6'), (A 7) be fulfilled with \( M_1, M_1 > 0 \) as well as \( \beta \in [0, 1] \) from (A 5'), \( \delta_2 \geq 2\Delta(\Theta)^{\beta} \ [\Delta(\Theta)^{1-\beta} + M_1], K_{\delta_2} \geq 0 \) from (A 6'), and \( M_3 > 0 \) as in (A 7). Furthermore let \( L_1, L_2 \) denote the square \( P^2 \)-integrable mappings from (A 4), whereas \((\xi^G_{\theta})_{\theta \in [0, \Delta(\Theta)]}\) stands for the family of positive envelopes from (A 5'). Finally, let \( a > 0 \), and let \((\hat{\theta}_n)_{n \in \mathbb{N}}\) be a sequence of minimizers...
\( \hat{\theta}_n \) for the SAA problem (3.2) w.r.t. \( n \in \mathbb{N} \). Then, using notations (2.5) and (2.8),
\[
\mathbb{P}\left\{ n^{1/(4-2\beta)} \left\| \hat{\theta}_n - \theta^* \right\|_m > \varepsilon \right\} \cap \Omega_{n,a}
\leq \sum_{k=K_1+1}^{\infty} \mathbb{P}\left\{ \sup_{\theta \in U_{n,k}} \left\| G_n^{G,H}(\theta, \cdot) - G_n^{G,H}(\theta^*, \cdot) \right\| > \frac{M_4 2^{2(k-1)}}{n^{\beta/(4-2\beta)}} \right\}
\]
holds for \( \varepsilon > 0 \) and \( n \in \mathbb{N} \). Here \( K_1 := \left\lfloor \ln(\varepsilon)/\ln(2) \right\rfloor \). In particular, if the sample size \( n \in \mathbb{N} \) satisfies the inequalities \( n \geq \left\lceil \frac{M_4^2 \Delta(\Theta)^{2\beta+1}}{1 \Delta(\Theta)} \right\rceil \), \( n \geq \left\lceil \frac{M_4^2 \Delta(\Theta) 2\beta/2}{1 \Delta(\Theta)+2} \right\rceil \), then the results from Proposition 3.4 are fulfilled. This completes the proof.

Proof Note that by (A 2) and (A 5') the mapping \( \xi^G := \xi^G_{\Delta(\Theta)} + |G(\theta^*, z)| \) is a square \( \mathbb{P}^Z \)-integrable positive envelope of \( \mathbb{P}^\Theta \) satisfying
\[
N(\eta \| \xi^G \|_{Q, Z}, \mathbb{F}^\Theta, L_2(\mathbb{Q})) \leq N(\eta \| \xi^G_{\Delta(\Theta)} \|_{Q, Z}, \mathbb{F}^\Theta_{\Delta(\Theta)}, L_2(\mathbb{Q})) \quad \text{for } Q \in \mathcal{M}_m, \eta > 0.
\]
Thus under (A 5') condition (A 5) is fulfilled with \( \delta_1 = \Delta(\Theta) \), \( M_4 = \overline{M}_1 \) and \( M_5 = \overline{M}_1^2 \). Furthermore (A 6) holds with \( \delta_2, K_2 \) from (A 6'). In particular \( \delta_5 = \Delta(\Theta) \) in (2.3). Hence the first statement follows from Proposition 3.4. Finally note that the assumptions of Proposition 3.4 are fulfilled. This completes the proof.

Example 3.6 Let (A 4) be fulfilled with Borel measurable mappings \( L_1, L_2 \), where in addition \( \mathbb{E}[L_2(Z_1)] = 0 \). In particular \( L_2(Z_1) = 0 \) \( \mathbb{P} \)-a.s. so that (A 6') is satisfied with \( m_{\Theta, x, R} \equiv 0 \). If in addition \( G \) has representation (H) with Hölder exponent \( \beta \in [0, 1] \) and stochastic Hölder constant \( C \), then under (A 1) by Remark 2.6 condition (A 5') holds with \( \overline{M}_1 = 2^\beta \|[L_1(\cdot) \vee 1] C(\cdot)\|_{2, \alpha} \) and \( \overline{M}_1 = \sqrt{(3 m+1)\beta + 4m} \ln(2) / \beta / 4 \) provided \( [L_1(\cdot) \vee 1] C(\cdot) \) as well as some \( G(\theta, \cdot, \cdot) \) are square \( \mathbb{P}^Z \)-integrable. Thus Theorem 3.3 may be applied in this case.

Concerning the second and third summands on the right hand side of the inequality in the statement of Proposition 3.3 we may apply in a quite straightforward manner Theorems 2.1, 2.2 from [12] (see also Theorem 5.1 below). It should be emphasized that by these results the involved events in the summands belong to \( \mathcal{F} \). We start with the estimation of the second summand.

Proposition 3.7 Let (3.3), (A 1) - (A 7) be fulfilled with \( L_1, L_2 \) from (A 4), \( \xi^G \) from (A 2), \( M_3 > 0 \) from (A 7). Furthermore let \( G^H(\theta, \cdot, \cdot) \) be square \( \mathbb{P}^Z \)-integrable for some \( \theta \in \Theta \). Then the real-valued mapping \( \xi_{\#}^{G,H} \) on \( \mathbb{R}^d \), defined by \( \xi_{\#}^{G,H}(z) := 4 \sqrt{[L_1(z) \vee 1]^2 \cdot \xi^G(z)^2 + \|\xi^G\|_{2, \alpha}^2 L_2(z)^2 + |G^H(\theta, z)|^2/4} \), is square \( \mathbb{P}^Z \)-integrable, and,
using notations (2.6), (2.11),

\[ \sum_{j=1}^{n} G^H(\theta, Z_j) - \psi_{H, \theta}(\theta) > 1 + \mathbb{1}_{\{0\}}(\mathbb{E}[L_2(Z_1)]) \frac{\delta^2 M_3}{4} \]  
\[ \leq \frac{128 \sqrt{2} \|C_{\theta}^H\|_{p^2, 2} [2 M^{\delta_1} + 1]}{\sqrt{n} \delta^2 M_3} \cdot (1 + \mathbb{1}_{\{0\}}(\mathbb{E}[L_2(Z_1)])) \]

is valid for \( \delta \in [0, \Delta(\Theta)] \) and \( n \in \mathbb{N} \), whereas

\[ \mathbb{P} \left( \left\{ \sup_{\theta \in \Theta} \left| \frac{1}{n} \sum_{j=1}^{n} G^H(\theta, Z_j) - \psi_{H, \theta}(\theta) \right| > (1 + \mathbb{1}_{\{0\}}(\mathbb{E}[L_2(Z_1)])) \frac{\delta^2 M_3}{4} \right\} \right) \]
\[ \leq \exp \left( -\frac{\delta^2 M_3 \cdot g(t)}{4 \|C_{\theta}^H\|_{p^2, 2}} (1 + \mathbb{1}_{\{0\}}(\mathbb{E}[L_2(Z_1)])) \right) \]
\[ + \mathbb{P}(\Omega \setminus B_n^{C_{\theta}^H}) \]

holds for \( \delta \in [0, \Delta(\Theta)] \), \( t > 0 \) and arbitrary \( n \in \mathbb{N} \) with

\[ n > \|C_{\theta}^H\|_{p^2, 2}^2 \left[ 4 + \frac{128 \sqrt{2} (t + 1)(2 M^{\delta_1} + 1)}{(1 + \mathbb{1}_{\{0\}}(\mathbb{E}[L_2(Z_1)])) \delta^2 M_3} \frac{1}{\sqrt{2}} \right]^2. \]

The proof of Proposition 3.7 is delegated to Subsection 5.8.

Let us turn over to third summand in the result of Proposition 3.3. We restrict the events to \( B_n^{\sqrt{L_2}} \), having in mind the devices from Remark 2.4 to control the probability \( \mathbb{P}(\Omega \setminus B_n^{\sqrt{L_2}}) \).

**Proposition 3.8** Let (3.3), (A 1) - (A 7) be fulfilled with \( L_2 \) from (A 4), \( C_{\theta}^G \) from assumption (A 2), and \( M^{\delta_1}, M_3 > 0 \) as in (A 5) and (A 7) respectively. Then, using notations (2.6), (2.11),

\[ \mathbb{P} \left( \left\{ \sup_{\theta \in \Theta} \left| \frac{1}{n} \sum_{j=1}^{n} G(\theta, Z_j) - \mathbb{E}[G(\theta, Z_1)] \right| > \frac{\delta^2 M_3}{4} \right\} \cap B_n^{\sqrt{L_2}} \right) \]
\[ \leq \frac{512 \sqrt{2} \delta^{\delta_1} \|C_{\theta}^G\|_{p^2, 2} \mathbb{E}[L_2(Z_1)]}{\sqrt{n} \delta^2 M_3} \]

for \( \delta \in [0, \Delta(\Theta)] \) and \( n \in \mathbb{N} \), moreover

\[ \mathbb{P} \left( \left\{ \sup_{\theta \in \Theta} \left| \frac{1}{n} \sum_{j=1}^{n} G(\theta, Z_j) - \mathbb{E}[G(\theta, Z_1)] \right| > \frac{\delta^2 M_3}{4} \right\} \cap B_n^{\sqrt{L_2}} \right) \]
\[ \leq \mathbb{1}_{\{0, \infty\}}(\mathbb{E}[L_2(Z_1)]) \exp \left( -\frac{\delta^2 M_3 \cdot g(t)}{8 \|C_{\theta}^G\|_{p^2, 2} \mathbb{E}[L_2(Z_1)]} \right) \]
\[ + \mathbb{P}(\Omega \setminus B_n^{C_{\theta}^G}) \]

holds for \( \delta \in [0, \Delta(\Theta)] \), \( t > 0 \) and arbitrary \( n \in \mathbb{N} \) with

\[ n > \frac{\|C_{\theta}^G\|_{p^2, 2}^2 \mathbb{E}[L_2(Z_1)]^2}{[\delta^2 M_3]^2} \left[ 8 + \frac{512 \sqrt{2} (t + 1) M^{\delta_1} \|C_{\theta}^G\|_{p^2, 2}^2}{\delta^2 M_3} \right]^2. \]
The proof of Proposition 3.8 is subject of Subsection 5.9.

Now, we may present the desired upper estimates of the probabilities (2.1). The most involved part is to estimate the probabilities of the intersections with the auxiliary events $\Omega_{n,a}$, $\Omega_{\hat{b},b}$ and $B_n^T$. Possible estimations of their complements are provided by Theorems 2.2, 2.3 and Remark 2.4. Actually, in view of Proposition 3.3 we only need to reassemble results from Propositions 3.4, 3.7, 3.8 to obtain the estimates of the probabilities (2.1). There exist different combinations to do so. We shall point only two versions.

**Theorem 3.9** Let (3.3) and assumptions (A 1) - (A 7) be fulfilled with mappings $\xi^G$ and $L_1,L_2$ from (A 2) and (A 4) respectively. In addition let $\delta_1,M_{\delta_1},M_0>0$ as well as $\beta \in [0,1]$ be as in (A 5), whereas $(\xi^G)_{\delta \in [0,\delta_1]}$ denotes the family of positive envelopes from (A 5). Furthermore let $\delta_2>0$, $K_{\delta_2} \geq 0$ be from (A 6), and let $M_3>0$ be as in (A 7). Fix any $\theta \in \Theta$ such that $G^H(\theta,\cdot)$ is square $P^Z$-integrable, inducing the square $P^Z$-integrable mapping $\xi^G_{\theta}$ from Proposition 3.7. Finally, let $a,b,\varepsilon > 0$, $K_{\varepsilon} := [\ln(\varepsilon)/\ln(2)]$, and let $(\hat{\theta}_n)_{n \in \mathbb{N}}$ be a sequence of minimizers $\hat{\theta}_n$ for the SAA problem (3.2) w.r.t. $n \in \mathbb{N}$. Then, using notations (2.3) - (2.9), (2.11), (3.5), (3.6), and defining

$$n_1(\theta,t,\delta) := \frac{\|\xi^G_{\theta}\|_{P^Z,2}^2}{\varepsilon^2 M_3^2} \left[ \frac{4 + 128\sqrt{2}(t+1)(2M_0^2 + 1)}{1 + 1(0)}(E[L_2(Z_1)]) \right] \frac{1/ \sqrt{2}}{\delta^2 M_3} (t, \delta > 0, \theta \in \Theta),$$

$$n_2(\theta) := \frac{\|\xi^G\|_{P^Z,2}^2 E[L_2(Z_1)]^2}{\varepsilon^2 M_3^2} \left[ 8 + 512\sqrt{2}(t+1)M_0^2 \right] (t, \delta > 0),$$

the following statements hold.

1) For every $n \in \mathbb{N}$ with $n \geq n(a,b,\beta)$, and any $\delta \in [0,\delta_\beta^n]$

$$\mathbb{P}\left\{ n^{1/(4-2\beta)} \left\| \hat{\theta}_n - \theta^* \right\|_m > \varepsilon \right\} \cap \Omega_{n,a} \cap B_n^T \leq \frac{128\sqrt{n}}{\sqrt{2}^n} \frac{\|\xi^G_{\theta}\|_{P^Z,2}}{\varepsilon M_3^2} \left[ 2M_0^2 + 1 \right]^{1+1(0)}(E[L_2(Z_1)]) \frac{1}{\delta^2 M_3} + \sum_{k=K_{\varepsilon}+1}^{K_{\varepsilon}+\delta} \mathbb{P}(\Omega \setminus \Omega_{n,a} \setminus B_n^T).$$

2) For every $\delta \in [0,\delta_\beta^n]$ and any $t > 0$, if the sample size $n \in \mathbb{N}$ satisfies the inequality $n > n_1(\theta,t,\delta) \lor n_2(t,\delta) \lor n(a,b,\beta) \lor (\|\xi^G\|_{P^Z,2}^2/2)$, and if

$$\varepsilon^{3/2-\beta} \sqrt{\varepsilon + 2} > 2^{4-\beta} \left[ M_{a,b} \mathbb{1}_{[0,\infty]}(E[L_2(Z_1)]) + M_b (1+2(t+1) \hat{\eta}(a,b)) \right]/M_3,$$

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then

\[ \mathbb{P}\left\{ n^{1/(4-2\beta)} \left\| \tilde{\theta}_n^* - \theta^* \right\|_m > \varepsilon \right\} \cap \Omega_{n,a} \cap B_n^G \]

\[ \leq \exp \left( -\frac{\sqrt{n} \delta^2 M_3 \cdot g(t)}{4 \left\| \xi G^t \right\|_{p_2,2}} (1 + 1_{\{0\}} (\mathbb{E}[L_2(Z_1)]) \right) + \mathbb{P}(\Omega \setminus B_n^G) \]

\[ + \mathbb{P}(\Omega \setminus B_n^G) \]

\[ + \exp \left( \frac{1_{[0,\infty)}(\mathbb{E}[L_2(Z_1)]) M_{a,b} g(t)}{\left( \sqrt{2} \land 1 \right) b \left\| L_2 \right\|_{p_2,2} } \mathbb{E}[L_2(Z_1)] \right) \]

\[ + \sum_{k=K_k+1}^{\infty} \mathbb{P}(\Omega \setminus \Omega_{n,\delta_n k} \cap \delta_n \sqrt{2} b) \]

\[ \leq M_{128} M_{51} M_{51} \mathbb{E}[L_2(Z_1)] \frac{1 + 64\sqrt{2} (t + 1) M_{51}}{\sqrt{2} b} \mathbb{E}[L_2(Z_1)] \]

\[ + \sum_{k=K_k+1}^{\infty} \mathbb{P}(\Omega \setminus B_n^G) \]

3) \[ \sum_{k=K_k+1}^{\infty} \sum_{\delta_n(k-1) \leq \delta} \mathbb{P}(\Omega \setminus \Omega_{n,\delta_n k} \cap \delta_n \sqrt{2} b) \]

\[ \leq M_{128} M_{51} M_{51} \mathbb{E}[L_2(Z_1)] \frac{1 + 64\sqrt{2} (t + 1) M_{51}}{\sqrt{2} b} \mathbb{E}[L_2(Z_1)] \]

\[ + \sum_{k=K_k+1}^{\infty} \mathbb{P}(\Omega \setminus B_n^G) \]

Proof The statements 1), 2) are direct consequences of Proposition 3.3 combined with Propositions 3.4, 3.7, 3.8. Using the first inequalities in each of the Propositions 3.4, 3.7, 3.8 we end up with statement 1). In view of \( 2K_c(2-\beta) - K_c^2 / 2 \geq \varepsilon^3 / \varepsilon^2 \), statement 2) may be derived by invoking the second inequalities in each of the Propositions 3.4, 3.7, 3.8. Concerning the statement 3) we obtain by statement 3) in Theorem 2.2

\[ \sum_{k=K_k+1}^{\infty} \mathbb{P}(\Omega \setminus \Omega_{n,\delta_n k} \cap \delta_n \sqrt{2} b) \leq \sum_{k=K_k+1}^{\infty} \frac{4 \sqrt{2} M_{51} M_{51} / \sqrt{2} b}{\sqrt{2} b} = \frac{1}{2 K_c} \frac{64 \sqrt{2} M_{51} M_{51}}{(\sqrt{2} - 1) b}. \]

Then the first part follows immediately from \( K_c \geq \ln(\varepsilon) / \ln(2) - 1 \). For the remaining part let \( n \in \mathbb{N} \) with \( n \geq M_{51}^2 (\delta_n^2)^{2/3} \). Then statement 3) in Theorem 2.3 implies

\[ \sum_{k=K_k+1}^{\infty} \mathbb{P}(\Omega \setminus \Omega_{n,\delta_n k} \cap \delta_n \sqrt{2} b) \leq \sum_{k=K_k+1}^{\infty} \exp \left( -\frac{\sqrt{2} b M_{51}}{\sqrt{2} b} \right) + \sum_{k=K_k+1}^{\infty} \mathbb{P}(\Omega \setminus B_n^G) \]

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for $\delta \in [0, \delta^*]$ and $t > 0$ such that $b > M_\delta \left[1 + 64\sqrt{2} (t+1) M^{\delta_1}\right]/\sqrt{\varepsilon}$. Moreover, invoking the change of variable formula

$$
\sum_{k=K_\varepsilon+1}^\infty \exp \left( -\frac{\sqrt{2^k} b(t)}{M_\delta} \right) \leq \sum_{k=K_\varepsilon+1}^\infty \int_{k-1}^k \exp \left( -\frac{\sqrt{2^y} b(t)}{M_\delta} \right) \frac{2}{y \ln(2)} dy
$$

Now, in view of $\sqrt{2^{K_\varepsilon}} \geq \sqrt{\varepsilon/2}$ the second part of statements may be derived easily. This completes the proof.

**Remark 3.10** In order to control in Theorem 3.9 the probabilities

$$
\mathbb{P}(\Omega \setminus B^G_n), \mathbb{P}(\Omega \setminus B^{G^H}_n), \sum_{k=K_\varepsilon+1}^\infty \mathbb{P}(\Omega \setminus B^{G^H}_{n,\sqrt{2^k}b,(\delta_{n,k})^\delta}), \sum_{k=K_\varepsilon+1}^\infty \mathbb{P}(\Omega \setminus B^{G^H}_{n,\sqrt{2^k}b,(\delta_{n,k})^\delta})
$$

devices from Remark 2.4 may be utilized. In particular the observation

$$
\sum_{k=K_\varepsilon+1}^\infty \mathbb{P}(\Omega \setminus B^{G^H}_{n,\sqrt{2^k}b,(\delta_{n,k})^\delta}) \leq \sum_{k=K_\varepsilon+1}^\infty \mathbb{P}(\Omega \setminus B^{L_1\lor 1}_{n,\sqrt{2^k}b,(\delta_{n,k})^\delta}) + \frac{(4 - 2\beta) \ln(n)}{(4 - 2\beta) \ln(2)} \mathbb{P}(\Omega \setminus B^{L_2}_{n})
$$

might be useful.

In principle we may use the first inequality in statement 1) from Theorem 3.9 to construct nonasymptotic confidence regions for fixed sample size. For larger sample sizes statement 2) in Theorem 3.9 together with Remark 2.4 might be utilized to refine the construction of confidence regions.

As another application of Theorem 3.9 we may draw on statement 3) there to derive the following convergence rates for the minimizers of the SAA problems.

**Corollary 3.11** Let (3.3) and assumptions (A 1) - (A 7) be fulfilled, and let $(\tilde{\theta}_n)_{n \in \mathbb{N}}$ be a sequence of minimizers of the SAA problems (3.2). Furthermore $G^H(\overline{\theta}, \cdot)$ is square $\mathbb{P}^\overline{\theta}$-integrable for some $\overline{\theta} \in \Theta$. Then

$$
\lim_{\varepsilon \to 0} \sup_{n \in \mathbb{N}} \mathbb{P} \left( \left\{ n^{1/(4-2\beta)} \| \tilde{\theta}_n - \theta^* \|_m > \varepsilon \right\} \right) = 0.
$$
Proof Let \( \eta > 0 \) and let us fix \( \delta \in [0, \delta^*_3] \). We shall use notations (2.4), (2.5), (2.8), (2.9) and (2.11).

By Theorem 2.2 we may find some \( a = b > 0 \) such that for any \( n \in \mathbb{N} \) the inequality 
\[
P(\Omega \setminus \Omega_{n,a}) + 128M_\delta/(\sqrt{2} - 1) b \leq \eta/2
\]
holds. Furthermore \( P(\Omega \setminus B_n^{(L_2)}) \rightarrow 0 \) due to the law of large numbers. Then with statements 1), 3) from Theorem 3.9 we obtain
\[
\limsup_{n \rightarrow \infty} P\left\{ n^{1/(4-2\beta)} \| \hat{\theta}_n - \theta^* \|_m > 2^K \right\} \\
\leq \frac{320 M_a |2\sqrt{2} M^\delta_1 + \sqrt{\ln(24)}| + 10 \cdot \eta_{[0, \infty]}(\mathbb{E}[L_2(Z_1)]) \cdot M_{a,a}}{M_3} 2^{(K-1) (\beta-3/2)} + \frac{\eta}{2}
\]
for \( K \in \mathbb{N} \). Since \( \beta < 3/2 \), we may find some \( K_0 \in \mathbb{N} \) such that
\[
\limsup_{n \rightarrow \infty} P\left\{ n^{1/(4-2\beta)} \| \hat{\theta}_n - \theta^* \|_m > 2^{K_0} \right\} < \eta.
\]
Hence
\[
\lim \limsup_{\varepsilon \rightarrow 0} P\left\{ n^{1/(4-2\beta)} \| \hat{\theta}_n - \theta^* \|_m > \varepsilon \right\} = 0
\]
which completes the proof. \( \square \)

Remark 3.12 In Remarks 2.2, 2.9 we have already discussed how to meet the requirements (A 1), (A 2), (A 3), (A 5) from Theorem 3.9 in the case that \( G \) has representation (H) or (PH).

Now, let us point out the special case of mapping \( H \) fulfilling \( H(\theta, t, z) = G(\theta, z) \) for \( \theta \in \Theta, t \in \mathbb{R} \), and \( z \in \mathbb{R}^d \). In this situation the minimizers \( \hat{\theta}_n \) may be viewed as so called \( m \)-estimators.

Example 3.13 Let \( (\hat{\theta}_n)_{n \in \mathbb{N}} \) be any sequence of \( m \)-estimators according to the compound SAA problems (3.2) with \( H \) defined by \( H(\theta, t, z) = G(\theta, z) \). In this situation we have the following specializations of particular assumptions of Theorem 3.9 and the upper estimations of the probabilities (3.4).

1) Assumption (A 4) is satisfied with \( L_1 := 1 \) as well as \( L_2 := 0 \), and in (A 6) we may use \( \delta_2 > 0 \) arbitrarily together with \( K_{\delta_2} := 0 \) as well as \( m_{\Theta \times \mathbb{R}} := 0 \). Moreover, \( B_n^{(L_2)} = \Omega \), using notation (2.11).

2) The mapping \( \xi_{a,b,\delta}^G \) from the display of Theorem 2.2 boils down to the mapping \( \xi_{a,b,\delta}^G = \xi^G \) from (A 5) for \( a, b > 0 \) and \( \delta \in [0, \delta_1] \) with \( \delta_1 \) as in (A 5). This implies \( \| \xi_{a,b,\delta}^G \|_{\mathbb{P}_2} \leq M_{\delta_1} \delta \) due to (A 5).

3) In view of (A 2) every \( G^H(\overline{\theta}, \cdot) = G(\overline{\theta}, \cdot) \) is square \( \mathbb{P}_2 \)-integrable. Furthermore the mapping \( \xi_{\overline{\theta}}^{G,H} \), defined in Proposition 3.4, satisfies \( \| \xi_{\overline{\theta}}^{G,H} \|_{\mathbb{P}_2} \leq 2\sqrt{5}\| G^G \|_{\mathbb{P}_2} \) for \( \overline{\theta} \in \Theta \). Here \( \xi^{G} \) stands for the positive envelope of \( F^{\Theta} \) from (A 2). Furthermore, using notation (2.11), we may conclude \( P(\Omega \setminus B_n^{(G^H)}) \leq P(\Omega \setminus B_n^{(G^G)}) + P(\Omega \setminus B_n^{(|G^H - G^G|)}) \) due to Remark 2.4.
4) The terms, introduced in (2.3), (2.4), (2.5), (3.5), (3.7) and (3.8), satisfy \( \delta^* = \delta_1 \), \( M_b = M_{a_1} \), \( M_{a,b} = n_2(t, \delta) \), \( \delta > 0 \), and

\[
n_1(\bar{\theta}, t, \delta) \leq 10 \| \xi^2 \|_{p \pi, 2} \max \{ [2\sqrt{2} + 128(t + 1)(2M^\delta_1 + 1)]^2 / [\delta^2 M_3]^2, 1 \}
\]

for \( \delta > 0, \bar{\theta} \in \Theta \).

5) Remarks 1) - 4) lead to the following simplified requirements in Theorem 3.9 that the sample size should meet:

a) \( n \geq \lfloor M^2_3 \delta_1^{2/3}/2 \rfloor \vee \lfloor M^2_3 \delta_1^{2+\beta}/12-\beta \rfloor =: \delta_{n_1} \) in statement 1).

b) In statement 2)

\[
n \geq \frac{80 \| \xi^2 \|_{p \pi, 2} [1 + 32\sqrt{2}(t + 1)(2M^\delta_1 + 1)]^2}{\delta^2 M_3} \vee (10 \| \xi^2 \|_{p \pi, 2} \vee \delta_{n_1}).
\]

6) In Theorem 3.9 the upper estimations for the deviation probabilities read as follows:

a) Statement 1)

\[
\frac{128\sqrt{10} \| \xi^2 \|_{p \pi, 2} [2M^\delta_1 + 1]}{\delta^2 M_3} + \frac{10 M_{s_a} \hat{n}(a, b, \varepsilon)}{M_3} \frac{2^{2-\beta} \beta}{\varepsilon \wedge 2} + \sum_{k = k(\varepsilon, \beta) + 1}^{\infty} \mathbb{P}(\Omega \setminus \Omega^n_{\delta, n, \delta, \sqrt{2} \varepsilon}^{\beta})
\]

for \( \delta \in [0, \delta_1] \).

b) Statement 2)

\[
\exp \left( -\frac{\sqrt{n} \delta^2 M_3 g(t)}{4\sqrt{5} \| \xi^2 \|_{p \pi, 2}} \right) \frac{M_{s_a} \hat{n}(a, b, \varepsilon)}{2^{2-\beta} \beta} \frac{\varepsilon^{3/2-\beta}}{M_3} + \frac{\varepsilon^{3/2-\beta}}{M_3} \frac{2^{2-\beta} \beta}{\varepsilon \wedge 2} \frac{\varepsilon^{3/2-\beta}}{M_3} \frac{2^{2-\beta} \beta}{\varepsilon \wedge 2}
\]

\[
\sum_{k = k(\varepsilon, \beta) + 1}^{\infty} \mathbb{P}(\Omega \setminus \Omega^n_{\delta, n, \delta, \sqrt{2} \varepsilon}^{\beta}) + \mathbb{P}(\Omega \setminus B^{\xi^2 \|_{p \pi, 2}^{\beta}}_{\delta, n, \delta, \sqrt{2} \varepsilon}^{\beta}) + \mathbb{P}(\Omega \setminus B^{\xi^2 \|_{p \pi, 2}^{\beta}}_{\delta, n, \delta, \sqrt{2} \varepsilon}^{\beta})
\]

for \( \delta \in [0, \delta_1] \), and \( t > 0 \) with \( \varepsilon^{3/2-\beta} \vee \varepsilon \wedge 2 > 2^{2-\beta} \delta_{n_1} [1 + 2(t + 1) \hat{n}(a, b, \varepsilon)] / M_3 \).

7) Considering the convergence rates of the \( m \)-estimators \( \hat{\theta}_n \), uniform tightness of the sequence \( \{ n_1^{1/(4-2\beta)} [\hat{\theta}_n - \theta^*] \} \) may be concluded from Corollary 3.11.

The framework of Theorem 3.9 fits also to study risk averse stochastic programs

\[
\inf_{\hat{\theta} \in \Theta} \rho(G(\hat{\theta}, Z_\lambda)),
\]

where in the objective the functional \( \rho \) is an absolute semideviation. This means that for \( \lambda \in [0, 1] \) the functional \( \rho = \rho_{\lambda, \lambda} \) is defined as follows

\[
\rho_{\lambda, \lambda} : L^4(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}, \; X \mapsto \mathbb{E}[X] + \lambda \mathbb{E}[\{X - \mathbb{E}[X]\}^+],
\]

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where \( L^1(\Omega, \mathcal{F}, \mathbb{P}) \) stands for the ordinary \( L^1 \)-space on \((\Omega, \mathcal{F}, \mathbb{P})\), tacitly identifying random variables which differ on \( \mathbb{P} \)-null sets only. It is well-known that absolute semideviations are increasing w.r.t. the increasing convex order (cf. e.g. [15], Theorem 6.51 along with Example 6.23 and Proposition 6.8]). Introducing the notation

\[
H_\lambda : \Theta \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}, (\theta, t, z) \mapsto G(\theta, z) + \lambda(G(\theta, z) - t)^+ \quad (\lambda \in [0, 1]),
\]

the stochastic program under absolute semideviation may also be viewed a compound stochastic program \( (3.1) \) with \( H = H_\lambda \).

**Example 3.14** Let for \( \lambda \in [0, 1] \) the mapping \( H = H_\lambda \) be defined by \( (3.9) \).

1) Condition \((A 4)\) is already fulfilled with \( L_1 = L^2 \): \( L^2 := 2(1 + \lambda) \) and \( L_2 = L^2 := 2\lambda\).
In particular \( B_n^{\sqrt{L_2}} = \Omega \), drawing on notation \((2.11)\).

2) In \((A 6)\) the mapping \( m_{\Theta \times \mathbb{R}} \) may be defined by \( m_{\Theta \times \mathbb{R}}(\theta, x) := -\lambda[1 - F_\theta(x)] \).

3) Under \((A 5)\) we have

\[
\xi_{G_{\lambda,b,\delta}} = 2\sqrt{(1 + \lambda)^2 \xi_{G}^2(\cdot)^2 + \lambda^2 (b + M_{\delta_1})^2 \delta^2 b},
\]

where \( \xi_{G_{\lambda,b,\delta}} \) is the mapping defined in Theorem 2.2. Drawing on notation \((2.11)\) this means \( B_{n}^{G_{\lambda,b,\delta}} \subseteq B_{n}^{G_{\lambda,b,\delta}} \). Moreover, the inequality \( ||\xi_{G_{\lambda,b,\delta}}||_{P^z,2} \leq 2[M_{\delta_1} (1 + 2\lambda) + b\lambda] \delta^3 \) holds due to \((A 5)\).

4) The mapping \( \xi_{G}^{H} \) from the display of Proposition 3.7 satisfies

\[
||\xi_{G}^{H}||_{P^z,2} = \tilde{M}(\lambda) := 2\sqrt{16 \left[ 1 + 2\lambda + 2\lambda^2 \right] ||\xi_{G}||_{P^z,2}^2 + \mathbb{E}[G^{H_\lambda}(\theta, Z_1)^2] \geq 8||\xi_{G}||_{P^z,2}^2}
\]

and \( B_{n}^{G_{\lambda}} \cap B_{n}^{G_{H_\lambda}(\theta, \cdot)} \subseteq B_{n}^{G_{\lambda}} \), using notation \((2.11)\).

5) The terms in \((2.5)\) and \((2.4)\) may be rewritten by \( M_b = M_\lambda := M_{\delta_1} (1 + 2\lambda) + b\lambda\) and \( M_{a,b} = M_{a,b}^\lambda := K_{\delta_1} a (\delta_1^{1-\beta} + M_{\delta_1} + 2b) + b\lambda [1 - F_{\theta^*}(\psi(\theta^*))] \).

6) In Theorem 3.9 the following conditions are imposed on the sample size \( n \):

- \( n \geq \max \left\{ (\bar{M}_b)^2 (\delta_1^{2b})^2/2, 9 a^2/\delta_1^2, [M_{a,b}^\lambda]^2 (\delta_1^{2(b+1)})^{1-\beta} \right\} =: \widehat{n}_\lambda \) in statement 1).

- \( n \geq \frac{\bar{M}(\lambda)^2 [4 + 128\sqrt{2} (t + 1)(2M_{\delta_1} + 1)]^2}{\delta_1^2 M_{\delta_1}^2} \lor \frac{\bar{M}(\lambda)^2}{2} \lor \widehat{n}_\lambda \).

7) The upper estimations of the deviation probabilities in Theorem 2.2 carry over, inserting in the results the specific constants \( \bar{M}_b = \bar{M}^\lambda_b, M_{a,b} = M_{a,b}^\lambda, \) and also \( ||\xi_{G}^{H}||_{P^z,2} = \tilde{M}(\lambda), ||L_2||_{P^z,2} = \mathbb{E}[L_2(Z_1)] = 2\lambda \). Moreover, from Corollary 2.11 uniform tightness of the sequence \((n^{1/(4-2\beta)}[\theta_n - \theta^*])_{n \in \mathbb{N}}\) may be concluded.
At the end of this section we want to direct the interest to the contribution \cite{6}. There
the authors investigate the deviation probabilities for Hausdorff distances between ap-
proximate solution sets of the genuine problem (3.1) and the SAA problems (3.2). They
achieve to find upper estimates in terms of the sample sizes. Their line of reasoning does
not rely on estimations for the deviation probabilities (2.1) of local increments of the
compound empirical process. Also the second order growth condition is avoided. How-
ever, continuity in the parameter is imposed on the objective $G$ which is also assumed
to be uniformly bounded. The difference to our paper is that our condition (A 5) is
replaced with direct requirements for the empirical Rademacher averages of $G$.

The results from \cite{6} may not be utilized directly for the deviation proba-
bilities (3.4). Moreover, the used argumentation is based on the deviation proba-
bilities for the compound empirical process $G^{G,H}$ only (see \cite{6, proof of Theorem 4.1}). By a suitable adap-
tion of the proof for Proposition 3.3 this may also be done for deviation probabilities (3.4)
(see e.g. \cite{14, proof of Lemma 1}). However, it was pointed out in \cite{14} that convergence
rates may be improved if the deviation probabilities (2.1) are taken into account.

4 Error estimates under Average Value at Risk

As before $\Theta$ is a compact subset of $\mathbb{R}^m$ with strictly positive diameter $\Delta(\Theta)$, and
$L^1(\Omega, \mathcal{F}, P)$ denotes the usual $L^1$-space on $(\Omega, \mathcal{F}, P)$, identifying the random variables
which differ on $P$-null sets only. We consider the risk averse stochastic program

$$\sup_{\theta \in \Theta} \rho_{\alpha}(G(\theta, Z_1)), \quad (4.1)$$

where the objective $\rho_{\alpha}$ is an Average Value at Risk. More precisely, for some fixed
$\alpha \in [0, 1]$ the functional $\rho_{\alpha}$ is defined by

$$\rho_{\alpha} : L^1(\Omega, \mathcal{F}, P) \to \mathbb{R}, \quad X \mapsto \int_0^1 \mathbb{I}_{[\alpha, 1]}(u) \left( F_X^+(u) \right) du,$$

where $F_X^+$ stands for the left-continuous quantile function of the distribution function $F_X$ of $X$. It is well-known that the Average Value at Risk may also be represented by

$$\rho_{\alpha}(X) = \inf_{x \in \mathbb{R}} \mathbb{E}\left[ (X + x)^+ / (1 - \alpha) - x \right],$$

where the set of minimizers consists of all numbers $-x_\alpha$, denoting by $x_\alpha$ any $\alpha$-quantile
of $X$ (see \cite{9}). In particular the Average Value at Risk is nondecreasing w.r.t. the
increasing convex order, and the empirical counterpart of (4.1) reads as follows:

$$\inf_{\theta \in \Theta} \mathcal{R}_{\rho_{\alpha}}(\hat{G}_{n, \theta}) = \inf_{\theta \in \Theta} \inf_{x \in \mathbb{R}} \frac{1}{n} \sum_{j=1}^n [ (G(\theta, Z_j) + x)^+ / (1 - \alpha) - x ] \quad (n \in \mathbb{N}). \quad (4.2)$$

Hence (4.1) may be reformulated by the auxiliary risk neutral stochastic program

$$\min_{(\theta, x) \in \Theta \times \mathbb{R}} \mathbb{E}\left[ (G(\theta, Z_1) + x)^+ / (1 - \alpha) - x \right], \quad (4.3)$$

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with the associated SAA problems

$$\inf_{(\theta,x) \in \Theta \times \mathbb{R}} \frac{1}{n} \sum_{j=1}^{n} \left[ \frac{(G(\theta, Z_j) + x)^+}{1 - \alpha} - x \right] \quad (n \in \mathbb{N}). \quad (4.4)$$

In accordance with the previous sections we assume that the solution of (4.1) is unique, and we strengthen this condition by

(4.1) has a unique solution $\theta^{\alpha,*}$ and $F_{\theta^{\alpha,*}}$ has a unique $\alpha-$quantile. \quad (4.5)

This requirement implies that the auxiliary optimization (4.3) has $\theta^{\alpha,*}, \hat{G}_{\alpha, \theta, x}$ as its unique solution. For simplification we set $x^{\alpha,*} := -F_{\theta^{\alpha,*}}(\alpha)$, and we introduce the mapping

$$\hat{G}_{\alpha} : (\Theta \times \mathbb{R}) \times \mathbb{R}^d \rightarrow \mathbb{R}, \quad ((\theta, x), z) \mapsto [G(\theta, z) + x]^+/ (1 - \alpha) - x. \quad (4.6)$$

Our intention is to study for suitable $\gamma > 0$ the probabilities

$$\mathbb{P}\left( \left\{ \| \theta^{\alpha,*}_n - \hat{\theta}^\alpha_n, \hat{x}^\alpha_n - x^{\alpha,*} \|_{m+1} > \varepsilon \cdot n^{-\gamma} \right\} \right) \quad (n \in \mathbb{N}, \varepsilon > 0) \quad (4.7)$$

of the stochastic minimizers $(\hat{\theta}^\alpha_n, \hat{x}^\alpha_n)$ according to the problems (4.4) instead of the deviation probabilities

$$\mathbb{P}\left( \left\{ \| \hat{\theta}^\alpha_n - \theta^{\alpha,*} \|_m > \varepsilon \cdot n^{-\gamma} \right\} \right) \quad (n \in \mathbb{N}, \varepsilon > 0) \quad (4.8)$$

of stochastic minimizers $\hat{\theta}^\alpha_n$ for the problems (4.2). In view of the following auxiliary result we may restrict ourselves to the probabilities (4.7) if the objective $G$ is Borel measurable.

**Lemma 4.1** Let $n \in \mathbb{N}$, and let assumption (A 2), (A 3) be fulfilled, where in addition $G$ is assumed to be measurable w.r.t. the product $\mathcal{B}(\Theta) \otimes \mathcal{B}(\mathbb{R}^d)$ of the Borel $\sigma$-algebra $\mathcal{B}(\Theta)$ on $\Theta$ and the Borel $\sigma$-algebra $\mathcal{B}(\mathbb{R}^d)$ on $\mathbb{R}^d$. Then for any random vector $\hat{\theta}^\alpha_n$ on $(\Omega, \mathcal{F}, \mathbb{P})$ which is a stochastic minimizer of problem (4.2) w.r.t. the sample size $n$, there is some random variable $\hat{x}^\alpha_n$ on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $(\hat{\theta}^\alpha_n, \hat{x}^\alpha_n)$ a stochastic minimizer of the auxiliary problem (4.4) w.r.t. the sample size $n$.

**Proof** Since $G$ is measurable w.r.t. $\mathcal{B}(\Theta) \otimes \mathcal{B}(\mathbb{R}^d)$, the process

$$\overline{X}^\alpha_n : \mathbb{R} \times \Omega \rightarrow \mathbb{R}, \quad (x, \omega) \mapsto \frac{1}{n} \sum_{j=1}^{n} \left[ \frac{G(\hat{\theta}^\alpha_n(\omega), Z_j(\omega)) + x}{1 - \alpha} - x \right]$$

is continuous in $x$ and measurable in $\omega$. Furthermore

$$\lim_{x \to -\infty} \overline{X}^\alpha_n(x, \omega) = \lim_{x \to \infty} \overline{X}^\alpha_n(x, \omega) = \infty \quad \text{for} \quad \omega \in \Omega.$$
Hence the paths of $\overline{X}_n^\alpha$ always attain their minimum. Putting all together, and noting that $\mathbb{R}$ with the usual topology is a Polish space, we may find by a version of the measurable selection theorem (e.g. [13, Theorem 6.7.22]) a random variable $\hat{\alpha}_n$ on $(\Omega, \mathcal{F}, P)$ whose realizations minimize the paths of $\overline{X}_n^\alpha$. Then obviously, $(\hat{\theta}_n, \hat{\alpha}_n)$ is a stochastic minimizer of the auxiliary problems (4.4) w.r.t. the sample size $n$. This completes the proof. $\square$

The stochastic minimizers of the auxiliary problems (4.4) may be viewed as $m$-estimators with objective $\hat{G}_n$. This suggests to utilize Theorem 3.9 along with Example 3.13 to derive upper estimates for the probabilities (4.7). Unfortunately the parameter space $\Theta \times \mathbb{R}$ in the optimization problems (4.3), (4.4) is not totally bounded so that we may not apply this result directly. We already know that under (A 1), (A 2) together with lower semicontinuity of $G$ in the parameter $\theta$, the optimization problems (4.4) have bounded solution sets (see [12]). However, they may depend on the realizations of the samples. In [12] a kind of compactification was suggested which allows to restrict the parameter set in the optimizations to suitable compact subsets. The idea is to show that with arbitrarily high probability we may find for large sample sizes events from $\mathcal{F}$ on which all solution sets of the SAA problems (4.4) are contained in a common compact superset. The following result from [12] gives a precise formulation of this idea. For preparation let for $\omega \in \Omega$ denote by $S_n(\omega)$ the set of solutions of the optimization problem

$$\inf_{(\theta, x) \in \Theta \times \mathbb{R}} \left\{ \frac{1}{n} \sum_{j=1}^n \left( G(\theta, Z_j(\omega)) + x \right)^+ / (1 - \alpha) - x \right\}.$$ 

Moreover, we introduce for any mapping $\xi^G$ as in (A 2) and arbitrary $\alpha \in [0, 1]$ as well as $\eta > 0$ the compact interval

$$I_{\xi^G, \eta}^\alpha := \left[ -\eta - \frac{E[\xi^G(Z_1)]}{1 - \alpha}, \frac{(2 - \alpha)\eta}{\alpha} + \frac{(2 - \alpha + |2 - 3\alpha|)E[\xi^G(Z_1)]}{2\alpha(1 - \alpha)} \right].$$ (4.9)

The following result has been shown in [12] (Theorem 5.7 and Lemma 5.8 along with Example 4.1).

**Proposition 4.2** Let (A 1), (A 2) be fulfilled, and let $\alpha \in [0, 1]$. Furthermore with $\xi^G$ from (A 2) let

$$A_{n, \eta}^{\alpha, \xi^G} := \left\{ \frac{1}{n} \sum_{j=1}^n \xi^G(Z_j) \leq E[\xi^G(Z_1)] + (1 - \alpha) \eta \right\} \quad (n \in \mathbb{N}, \eta > 0).$$

If $G(\cdot, z)$ is lower semicontinuous for $z \in \mathbb{R}^d$, then $S_n(\omega)$ is nonvoid for $n \in \mathbb{N}, \omega \in \Omega$, and, using notation (4.9)

$$S_n(\omega) \subseteq \Theta \times I_{\xi^G, \eta}^\alpha \quad \text{for } n \in \mathbb{N}, \eta > 0, \omega \in A_{n, \eta}^{\alpha, \xi^G}.$$

Moreover, under assumption (4.5) the unique solution $(\theta^{\alpha,*}, -F^{\alpha,*}(\alpha))$ always belongs to every parameter set $\Theta \times I_{\xi^G, \eta}^\alpha (\eta > 0)$. 

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Proposition 4.2 tells us that up to probabilities \( P(\Omega \setminus A_{\alpha,\eta}^c) \) we may focus on m-estimators according to the problems

\[
\min_{(\theta, x) \in \Theta \times I^{n}} \frac{1}{n} \sum_{j=1}^{n} \hat{G}_{\alpha}((\theta, x), Z_{j}) \quad (n \in \mathbb{N}).
\]  

(4.10)

However, in applying Example 3.13 we want to impose most of the conditions directly on the genuine objective \( G \) instead of the auxiliary ones \( \hat{G}_{\alpha}((\theta, x), Z_{j}) \).

We start with introducing for any nonvoid interval \( I \subseteq \mathbb{R} \) the function classes

\[
F_{\alpha,I}^\theta := \{ \hat{G}_{\alpha}((\theta, x), \cdot) \mid (\theta, x) \in \Theta \times I \},
\]

and, if \( x^{\alpha,*} \in I \),

\[
F_{\alpha,I,\delta} := \{ \hat{G}_{\alpha}((\theta, x), \cdot) - \hat{G}_{\alpha}((\theta^{\alpha,*}, x^{\alpha,*}), \cdot) \mid (\theta, x) \in \bigvee_{I,\delta}(\theta^{\alpha,*}, x^{\alpha,*}) \} \quad (\delta > 0),
\]

(4.12)

where \( \bigvee_{I,\delta}(\theta, \varphi) := \{(\theta, x) \in \Theta \times I \mid \| (\theta - \bar{\theta}, x - \varphi) \|_{m+1} \leq \delta \} \) for \( (\bar{\theta}, \varphi) \in \Theta \times I \).

First of all, we want to verify an analogue of (A 3) for the function classes \( F_{\alpha,I}^\theta \) in case that the genuine class \( F_{\alpha}^\theta \) fulfills this separability condition. We may observe the following basic inequalities for the functions from the classes \( F_{\alpha,I}^\theta \)

\[
| \hat{G}_{\alpha}((\theta, x), z) - \hat{G}_{\alpha}((\theta, y), z) | \leq \| G(\theta, z) - G(\theta, y) \| + (2 - \alpha) | x - y | / (1 - \alpha)
\]

(4.13)

for \( (\theta, x), (\theta, y) \in \Theta \times I \) and \( z \in \mathbb{R}^d \). Then in view of (4.13) the analogue of (A 3) for \( F_{\alpha,I}^\theta \) is already implied by (A 3) for \( F_{\alpha}^\theta \).

**Lemma 4.3** Let (A 1) - (A 3) be fulfilled, and let \( I \subseteq \mathbb{R} \) denote a nondegenerated interval. Then the restriction of \( \hat{G}_{\alpha} \) to \( (\Theta \times I) \times \mathbb{R}^d \) satisfies the separability property analogously to (A 3).

**Proof** Let \( (\bar{\theta}, \varphi) \in \Theta \times I, \varepsilon > 0 \) and \( J_\varepsilon := ]0, \varepsilon] \cap (\mathbb{Q} \cup \{ \varepsilon \}) \). According to (A 3) we may find for any \( \varphi \in J_\varepsilon \) some at most countable subset \( C(U_{\varphi}(\bar{\theta})) \subseteq U_{\varphi}(\bar{\theta}) \) and a \( \mathbb{P}^2 \)-null set \( N_{\bar{\theta},\varphi} \) such that \( \bar{\theta} \in C(U_{\varphi}(\bar{\theta})) \), and

\[
\inf_{\varphi \in C(U_{\varphi}(\bar{\theta}))} | G(\theta, z) - G(\theta, z) | = 0 \quad \text{if} \quad \theta \in U_{\varphi}(\bar{\theta}), \quad z \in \mathbb{R}^d \setminus N_{\bar{\theta},\varphi}.
\]

(4.14)

Now set

\[
K_{\varepsilon}(\bar{\theta}, \varphi) := \bigvee_{\mathbb{Q} \cup \{ \varepsilon \}}(\bar{\theta}, \varphi) \cap \bigcup_{\varphi \in J_\varepsilon} C(U_{\varphi}(\bar{\theta})) \times [I \cap (\mathbb{Q} \cup \{ \varphi \})] \quad \text{and} \quad N_{\varepsilon}(\bar{\theta}, \varphi) := \bigcup_{\varphi \in J_\varepsilon} N_{\bar{\theta},\varphi}.
\]

Note that \( K_{\varepsilon}(\bar{\theta}, \varphi) \) is an at most countable subset of \( \bigvee_{I,\varepsilon}(\bar{\theta}, \varphi) \), whereas \( N_{\varepsilon}(\bar{\theta}, \varphi) \) is a \( \mathbb{P}^2 \)-null set. Now, invoking (4.13) along with (4.14), we may show by routine procedures

\[
\inf_{(\theta, y) \in K_{\varepsilon}(\theta, y)} | \hat{G}_{\alpha}((\theta, x), z) - \hat{G}_{\alpha}((\theta, y), z) | = 0 \quad \text{if} \quad (\theta, x) \in \bigvee_{I,\varepsilon}(\theta, y), \quad z \in \mathbb{R}^d \setminus N_{\varepsilon}(\bar{\theta}, \varphi).
\]
Next, we borrow Lemma 4.3 from [12] which describes the relationship of the uniform entropy integrals $J(F^\Theta_{\alpha,I}, C_{F^\Theta_{\alpha,I}}, \varepsilon)$ with uniform entropy integrals $J(F^\Theta, C_{F^\Theta}, \varepsilon)$.

**Lemma 4.4** Let $I \subseteq \mathbb{R}$ be a nondegenerated compact interval fulfilling the property $\sup I = |\inf I| \vee |\sup I| > 0$. If $\xi$ is a square $\mathcal{P}$-integrable positive envelope of $F^\Theta$, then $C_{F^\Theta_{\alpha,I}} := 2(2-\alpha)\sqrt{\xi^2 + (\sup I)^{1/2}}/(1-\alpha)$ is a positive envelope of $F^\Theta_{\alpha,I}$ satisfying

$$J(F^\Theta_{\alpha,I}, C_{F^\Theta_{\alpha,I}}, \varepsilon) \leq \sqrt{2} J(F^\Theta, \xi, \varepsilon) + 4\varepsilon \sqrt{\ln(1/\varepsilon)} + \sqrt{2\ln(2)} \varepsilon \quad \text{for } \varepsilon \in [0, \exp(-1)]$$

We also want to find an explicit relationship between the uniform entropy integrals $J(F^\Theta_{\alpha,I,\delta}, C_{F^\Theta_{\alpha,I,\delta}}, \varepsilon)$ and $J(F^\Theta_{\delta}, C_{F^\Theta_{\delta}}, \varepsilon)$. This will be the subject of the following result.

**Lemma 4.5** Let $I \subseteq \mathbb{R}$ be a nondegenerated compact interval containing $x^{\alpha,*}$ and satisfying $\sup I = |\inf I| \vee |\sup I| > 0$. Furthermore let $\delta > 0$. If $\xi$ is a square $\mathcal{P}$-integrable positive envelope of $F^\Theta_{\delta}$, then $C_{F^\Theta_{\alpha,I,\delta}} := [\xi + (2-\alpha)\delta]/(1-\alpha)$ is a square $\mathcal{P}$-integrable positive envelope of $F^\Theta_{\alpha,I,\delta}$, and

$$J(F^\Theta_{\alpha,I,\delta}, C_{F^\Theta_{\alpha,I,\delta}}, \varepsilon) \leq 4 J(F^\Theta_{\delta}, \xi, \varepsilon/4) + 2\varepsilon \sqrt{\ln(8/\varepsilon)} \quad \text{for } \varepsilon \in [0, 1].$$

The proof of Lemma 4.5 may be found in Subsection 5.10.

Finally, we need a suitable second growth condition at the unique solution $(\theta^{\alpha,*}, x^{\alpha,*})$.

(A7) There exists $M_{3,\alpha} > 0$ such that

$$\mathbb{E}[\widehat{G}_\alpha((\theta, x), Z_1)] - \mathbb{E}[\widehat{G}_\alpha((\theta^{\alpha,*}, x^{\alpha,*}), Z_1)] \geq M_{3,\alpha} \|((\theta - \theta^*, x - x^{\alpha,*})\|_{m+1}$$

for $(\theta, x) \in \Theta \times \mathbb{R}$.

**Remark 4.6** Let $(\theta^{\alpha,*}, x^{\alpha,*})$ be the unique minimizer of the problem (4.3) and let $\xi^G$ the positive envelope of $F^\Theta$ from (A 2). Then by definition of $\widehat{G}_\alpha$ along with Jensen’s inequality, we may observe

$$\inf_{\theta \in \Theta} \mathbb{E}[\widehat{G}_\alpha((\theta, x), Z_1)] \geq \max \left\{ -x, (x - \mathbb{E}[\xi^G(Z_1)])^+/2/(1-\alpha) - x \right\} \quad \text{for } x \in \mathbb{R}.$$

This implies that the mapping $(\theta, x) \mapsto \mathbb{E}[\widehat{G}_\alpha((\theta, x), Z_1)]$ on $\Theta \times \mathbb{R}$ has bounded weak lower level sets. Furthermore by Vitalis theorem (see [1 Prop. 21.4]) this mapping may be verified to be continuous w.r.t. the Euclidean metric under (A 1) and (A 2). Thus in this case their weak lower level sets are even compact, in particular every minimizing sequence converges to its unique minimizer $(\theta^{\alpha,*}, x^{\alpha,*})$. Hence for every $\delta > 0$

$$M^\alpha(\delta) := \inf_{\|((\theta, x) \in \Theta \times \mathbb{R})/(\theta - \theta^{\alpha,*}, x - x^{\alpha,*})\|_{m+1} > \delta} \left( \mathbb{E}[\widehat{G}_\alpha((\theta, x), Z_1)] - \mathbb{E}[\widehat{G}_\alpha((\theta^{\alpha,*}, x^{\alpha,*}), Z_1)] \right) > 0.$$
This implies, following the argumentation in the proof of Lemma 3.1, that (A7) may be derived by any local version of it at (θα, xα, ε), in the same way as (A7) follows from (A7').

If θα belongs to the topological interior of Θ then we may formulate exactly as is in Remark 3.2 criteria to provide property (A7').

Now, we are prepared to derive upper estimates for the errors of the stochastic minimizers of (4.2). Actually, it is an application of Proposition 4.2 along with Example 3.13. The relevant part consists of upper estimations of the deviation probabilities (4.7) being Borel measurable and lower semicontinuous in the parameter θ, and let (4.5) and assumptions (A 1) - (A 3), (A 5), (A 7) be as in (A 7) follow from (A 7). Lemma 4.3 ensures that every ε δ, Mδ be as in (A 5), let M3,α > 0 be from (A 7), and let ξG denote the square P2-integrable mapping from (A 2). Fix any a,b,ε,η > 0. Then, using notations (2.5), (2.6), (3.3), (4.15), (4.16), and

\[
\delta_{nk} := 2^k/n^{1/(4-2\beta)} \quad (k, n \in \mathbb{N}),
\]

\[
\tilde{\alpha}, \delta_1 := \frac{[M_\delta + (2 - \alpha) \delta_{1-\beta}^2 \cdot \delta_{1-\beta}^2]}{2 (1 - \alpha)^2} \sqrt{\left[ \frac{[M_\delta + (2 - \alpha) \delta_{1-\beta}^2 \cdot \delta_{1-\beta}^2]}{(1 - \alpha)^2} \right]^{\frac{4-2\beta}{4-2\beta}}},
\]

the following statements are valid.

1) For n ∈ N with n ≥ \tilde{n}, and every δ ∈ [0, 1]

\[
\mathbb{P}\left\{ n^{1/(4-2\beta)} \| \theta_n - \theta_{\alpha, \epsilon} \|_m > \epsilon \right\} \cap \Omega_{n, n}^{\alpha, \epsilon} \leq \frac{256\sqrt{10} (2 - \alpha) \sqrt{\| \xi_G \|_{P_2}^2 + (sup T_{\xi_G, \eta})^2} \left[ 8M_\delta + 3 \right]}{(1 - \alpha) \delta^2 M_3,\alpha} + \frac{2^{1-\beta} \left[ M_\delta + (2 - \alpha) \delta_{1-\beta} \right]}{(1 - \alpha) \delta M_3,\alpha \epsilon^{3/2-\beta} \sqrt{\epsilon} \wedge 2} + \sum_{k=K_\epsilon + 1}^{\infty} \mathbb{P}\left( \Omega \setminus \Omega_{n, n}^{\alpha, \epsilon, \eta} \right) + \mathbb{P}\left( \Omega \setminus \Omega_{n, n}^{\alpha, \epsilon, \eta} \right).
\]
In particular the sequence \((n^1\theta^n)\), \(\hat{\theta}_n \in \Theta\), if \(n \in \mathbb{N}\) and \(t > 0\) such that \(n \geq \hat{n}_{\alpha,\delta}\), and

\[
\begin{align*}
\hat{n} &\geq \frac{40 (2 - \alpha)^2 \left[ \left\| \xi^G \right\|_{\mathbb{P}, 2}^2 + (\sup \mathcal{T}_{\xi^G, \eta})^2 \right]}{\alpha (1 - \alpha)^2} \left( 8 \left[ 1 + 32 \sqrt{2} (t + 1) (8M^3 + 3)^2 \right] \right), \\
\epsilon^{2\beta} \frac{\sqrt{\epsilon} \wedge 2}{2^{4\beta}} &\geq \frac{2^{4\beta} \left( M_{\delta_1} + (2 - \alpha)^{\delta_1} \right) \exp \left( -\frac{\epsilon^{3/2} \sqrt{\epsilon} \wedge 2}{2^{4\beta} \left( M_{\delta_1} + (2 - \alpha)^{\delta_1} \right) \text{ln}(2)} \right)}{(1 - \alpha) (2 - \beta) M_{3,\alpha} \text{ln}(2) \epsilon^{3/2 - \beta} \sqrt{\epsilon} \wedge 2} \left\{ \mathbb{P}(\Omega \setminus \bigcup_{\delta = \delta_{k+1}^{\alpha,\eta}} \mathcal{A}_{\delta}^{\alpha,\eta}) \right\} \\
&\quad + \sum_{k = K_{k+1}+1}^{\infty} \left[ \mathbb{P}(\Omega \setminus \bigcup_{\delta_{k+1}^{\eta}} \mathcal{A}_{\delta_i}^{\eta}) \right] \\
&\quad \quad + \mathbb{P}(\Omega \setminus \mathcal{B}_{\alpha,\eta}^{G}) \\
&\quad \quad + \mathbb{P}(\Omega \setminus \mathcal{B}_{\alpha,\eta}^{G}) \\
&\quad \quad \quad + \mathbb{P}(\Omega \setminus \mathcal{B}_{\alpha,\eta}^{G})
\end{align*}
\]

3) \(\mathbb{P}(\Omega \setminus \mathcal{A}_{\alpha,\eta}^{\alpha,\eta}) \leq 128 \sqrt{2} (2 - \alpha) (4M^3 + 1) \frac{\left\| \xi^G \right\|_{\mathbb{P}, 2}^2 + (\sup \mathcal{T}_{\xi^G, \eta})^2}{\alpha (1 - \alpha)^2}\) and \(\mathbb{P}(\Omega \setminus \mathcal{A}_{\alpha,\eta}^{\alpha,\eta}) \leq 128 \sqrt{2} (4M^3 + 1) \frac{\left\| \xi^G \right\|_{\mathbb{P}, 2}^2 + (\sup \mathcal{T}_{\xi^G, \eta})^2}{\alpha (1 - \alpha)^2}\) for \(\delta > 0\) and \(k, n \in \mathbb{N}\).

4) If \(n \in \mathbb{N}\) with \(n \geq 4 (2 - \alpha)^2 \left[ \left\| \xi^G \right\|_{\mathbb{P}, 2}^2 + (\sup \mathcal{T}_{\xi^G, \eta})^2 \right]/(\alpha (1 - \alpha)^2)\) and \(t > 0\) such that \(a > 2 \sqrt{2} (2 - \alpha) \frac{\left\| \xi^G \right\|_{\mathbb{P}, 2}^2 + (\sup \mathcal{T}_{\xi^G, \eta})^2}{\alpha (1 - \alpha)^2}\) then

\[
\mathbb{P}(\Omega \setminus \mathcal{A}_{\alpha,\eta}^{\alpha,\eta}) \leq \exp \left( -\frac{a (1 - \alpha) \cdot \mathcal{G}(t)}{2 \left\| \xi^G \right\|_{\mathbb{P}, 2}^2 + (\sup \mathcal{T}_{\xi^G, \eta})^2} \right) + \mathbb{P}(\Omega \setminus \mathcal{B}_{\alpha,\eta}^{G}).
\]

5) For \(\delta, t > 0\) and \(k, n \in \mathbb{N}\) with \(n \geq (M_{\delta_1} + (2 - \alpha)^{\delta_1} \right)^2 \delta_1^{2\beta}/(2 (1 - \alpha)^2)\)

\[
\mathbb{P}(\Omega \setminus \mathcal{A}_{\delta,\eta}^{\alpha,\eta}) \leq \exp \left( -\frac{\sqrt{2} b (1 - \alpha) \cdot \mathcal{G}(t)}{M_{\delta_1} + (2 - \alpha)^{\delta_1}} \right) + \mathbb{P}(\Omega \setminus \mathcal{B}_{\alpha,\eta}^{G})
\]

is valid whenever \(b > (M_{\delta_1} + (2 - \alpha)^{\delta_1} \left[ 1 + 4 \sqrt{2} (t + 1) (4M^3 + 1) \right]/\left( (1 - \alpha)^2 \right) \right)^{\delta_1} \right].

In particular the sequence \(\left( n^{1/(4 - 2\beta)} \hat{\theta}_n \right) \in \mathbb{N}\) is uniformly tight.
We want to apply Example 3.13 to the function class $F$ of minimizer of the stochastic program. In Lemma 4.4 it has been already pointed out how to construct a square 4.2, and satisfying $\sup_{\theta, x} G_{\alpha}(\theta, x)$ the parameter $C$ positive envelope.

Proof Since $G$ is assumed to be $\mathcal{B}(\Theta) \otimes \mathcal{B}(\mathbb{R})$-measurable and lower semicontinuous in the parameter $\theta$, the mappings

$$(\Theta \times \mathcal{I}_{\xi^G, \eta}) \times \mathbb{R}^d \to \mathbb{R}, \left((\theta, x), (z_1, \ldots, z_n)\right) \mapsto \frac{1}{n} \sum_{j=1}^{n} \hat{G}_{\alpha}(\theta, x, z_j) \quad (n \in \mathbb{N})$$

are Borel measurable and lower semicontinuous in the parameter $(\theta, x)$. Furthermore $\Theta \times \mathcal{I}_{\xi^G, \eta}$, $\mathbb{R}^d$ endowed with the topologies induced by the respective Euclidean norms are Polish spaces. Hence by compactness of $\Theta \times \mathcal{I}_{\xi^G, \eta}$ together with some specific measurable selection theorem (e.g. Corollary 1 in [3]) we may find for $n \in \mathbb{N}$ some Borel measurable mapping $(\hat{\theta}^n, \hat{x}^n) : \mathbb{R}^d \to \Theta \times \mathcal{I}_{\xi^G, \eta}$ such that $(\hat{\theta}^n, \hat{x}^n)(Z_1, \ldots, Z_n)$ is a stochastic minimizer of the SAA problem (4.10) w.r.t. the sample size $n$. Moreover, Lemma 4.3 ensures that the analogue of (A 3) is fulfilled under (A 3). Consequently, we may conclude from Proposition 4.2 that

$$(\hat{\theta}, \hat{x}) := 1_{A_{\alpha, \xi^G}}(\hat{\theta}^n, \hat{x}^n) + 1_{\Omega\setminus A_{\alpha, \xi^G}}(\hat{\theta}^n, \hat{x}^n)(Z_1, \ldots, Z_n)$$

defines another stochastic solution of (4.10) w.r.t. the sample size $n$. We may observe

$$
P\left\{\left\{ n^{1/(4-2\beta)} \left\| \hat{\theta}^n - \theta^{\alpha,*} \right\|_m > \varepsilon \right\} \cap \Omega_{n,a}^{\eta}\right\} \leq P\left\{ \left\{ n^{1/(4-2\beta)} \left\| \hat{\theta}^n - \theta^{\alpha,*}, \hat{x}^n - x^{\alpha,*}\right\|_{m+1} > \varepsilon \right\} \cap \Omega_{n,a}\right\} 
$$

For abbreviation we set $\mathcal{I} := \mathcal{I}_{\xi^G, \eta}$. It is a compact interval containing $\alpha, x^{\alpha,*}$ by Proposition 4.2 and satisfying $\sup_{\mathcal{I}} = \left[ \inf_{\mathcal{I}} \mathcal{I} \right] \vee \left[ \sup_{\mathcal{I}} \mathcal{I} \right]$. Note that $(\theta^{\alpha,*}, x^{\alpha,*})$ is the unique minimizer of the stochastic program

$$\inf_{(\theta, x) \in \Theta \times \mathcal{I}} \mathbb{E}\left[ \hat{G}_{\alpha}(\theta, x, Z_1) \right].$$

We want to apply Example 3.13 to the function class $\mathbb{F}_{\alpha, \mathcal{I}}^{\Theta}$. First of all, the restriction of $\hat{G}_{\alpha}$ to $\Theta \times \mathcal{I}$ satisfies the analogue of (A 1) due to (A 1) for $G$ along with (4.13). In Lemma 4.3 it has been already pointed out how to construct a square $\mathbb{P}$-integrable positive envelope $C_{\mathcal{I}, \delta}^{\xi^G}$ of $\mathbb{F}_{\alpha, \mathcal{I}}^{\Theta}$ out of the positive envelope $\xi^G$ from (A 2). In addition we have

$$\|C_{\mathcal{I}, \delta}^{\xi^G}\|_{\mathbb{P}^{\mathcal{I}}} = 2(2 - \alpha) \sqrt{\mathbb{E}[\hat{\xi}^G(Z_1)^2] + (\sup_{\mathcal{I}} \mathcal{I})^2}/(1 - \alpha).$$

Moreover, Lemma 4.3 ensures that the analogue of (A 3) is fulfilled under (A 3). Condition (A 7) provides the second order growth condition analogously to (A 7). So it is left to look at the analogue of property (A 5).

For this purpose let $C_{\mathcal{I}, \delta}^{\xi^G}$ denote the square $\mathbb{P}^{\mathcal{I}}$-integrable positive envelope of $\mathbb{F}_{\alpha, \mathcal{I}, \delta}^{\Theta}$ ($\delta > 0$) according to Lemma 4.3. By Lemma 4.4 we may observe

$$J\left(\mathbb{F}_{\alpha, \mathcal{I}}^{\Theta}, C_{\mathcal{I}, \delta}^{\xi^G}, 1/8\right) \leq \sqrt{2} J\left(\mathbb{F}_{\alpha, \mathcal{I}}^{\Theta}, \xi^G, 1/8\right) + \sqrt{\ln(8)}/2 + \sqrt{2 \ln(2)}/8 \leq \sqrt{2} M^{\delta_1} + 1,$$
and by (A 5) along with Lemma 4.5
\[ \| C_{\alpha I, \delta} \|_{p, 2} \leq [M_{\delta_i} + (2 - \alpha) \delta_i^{1-\beta}] \delta_i / (1 - \alpha) \] (4.19)
\[ J(F_{\alpha I, \delta}, C_{\alpha I, \delta}, 1/8) \leq 4 J(F_{\delta}, C_{\delta}, 1/32) + \sqrt{\ln(64)/4} \leq 4 M_{\delta_i} + \sqrt{\ln(64)/4} \]
for \( \delta \in [0, \delta_1] \). In particular
\[ J(F_{\alpha I, \delta}, C_{\alpha I, \delta}, 1/8) \lor J(F_{\alpha I, \delta}, C_{\alpha I, \delta}, 1/8) \leq 4M_{\delta_i} + 1 \quad \text{for} \quad \delta \in [0, \delta_1]. \] (4.20)
Hence the analogue of (A 5) is fulfilled with \( \delta_1, \beta \) from (A 5) and the family of positive envelopes \( (C_{\alpha I, \delta})_{\delta \in [0, \delta_1]} \) associated with the family \( (F_{\alpha I, \delta})_{\delta \in [0, \delta_1]} \) of function classes \( F_{\alpha I, \delta} \), where \( M_{1, \alpha} := [M_{\delta_i} + (2 - \alpha) \delta_i^{1-\beta}] / (1 - \alpha) \) and \( M_{1, \alpha} \) take the role of \( M_{\delta_i} \) and \( M_{\delta_1} \) respectively. In particular notation (3.3) reads as follows
\[ 64\sqrt{2} M_{1, \alpha} + 32\sqrt{\ln(24)} + (\ln(a/b) + (\beta + 1/2 \ln(2/\varepsilon))^{+} \leq 64\sqrt{2} (3M_{\delta_i} + 1) + \tilde{\eta}(a, b, \varepsilon). \] (4.21)

All in all we are in the position to apply Example 3.13 and Theorem 3.9 to the function classes \( F_{\alpha I} \) and \( F_{\alpha I, \delta} \). Then statement 1) follows from statements 5) a), 6) in Example 3.13 along with (4.17), (4.18) and (4.21). Concerning statement 2) note that \( \Omega \setminus B_n^{\alpha I, \delta} \subset \Omega \setminus B_n^{\alpha I, \delta} \) and \( \Omega \setminus B_n^{\alpha I, \delta} \) hold. Hence in view of (4.17), (4.18) and (4.21) the statement 2) may be obtained easily from statements 5) b), 6) in Example 3.13 along with Theorem 3.9.

Next, concerning statements 3) - 5) we apply Theorems 2.2, 2.3 to \( F_{\alpha I} \) and \( F_{\alpha I, \delta} \). Statement 3) may be derived from statement 3) in Theorem 2.2. Furthermore, drawing on \( \Omega \setminus B_n^{\alpha I, \delta} \subset \Omega \setminus B_n^{\alpha I, \delta} \) and (4.18) again, statement 4) comes from statement 2) in Theorem 2.3. Finally, statement 5) may be concluded from statement 3) in Theorem 2.3.

Implicitly we have already shown that the function class \( F_{\alpha I} \) also meets the requirements of Corollary 3.11. Hence the sequence \( \left( n^{1/(4-2\beta)} \right) \left( \hat{\theta}_n - \theta^{\alpha, x}, \hat{x}_n - x^{\alpha, x} \right) \) is uniformly tight. Then \( \left( n^{1/(4-2\beta)} \right) \left( \hat{\theta}_n - \theta^{\alpha, x} \right) \) is uniformly tight because the inclusion
\[ \left\{ n^{1/(4-2\beta)} \left( \hat{\theta}_n - \theta^{\alpha, x} \right) \right\} \subseteq \left\{ n^{1/(4-2\beta)} \left( \hat{\theta}_n - \theta^{\alpha, x} \right) \right\} \] holds for every \( \varepsilon > 0 \). This completes the proof.

Remark 4.8 Analogously to Remark 2.4 we may invoke classical concentration inequalities to provide in Theorem 4.7 upper estimations for the probabilities \( P(A_{\alpha, \eta, n}^{\xi}) \).

We want to discuss the assumptions of Theorem 4.7 if \( G \) has some specific representation. Firstly, we look at representation (H).
Example 4.9 Let $G$ have representation $(H)$ with $G(\theta, \cdot)$ being Borel measurable. Then by continuity of the paths it is known to be $\mathcal{B}(\Theta) \otimes \mathcal{B}(\mathbb{R}^d)$-measurable (e.g. [13] Lemma 6.7.3), and (A 2) is fulfilled due to Proposition 2.3. Furthermore, Remark 2.6 shows how to verify (A 5). Finally, (A 3) is an easy consequence of (H) because $\Theta$ encloses some at most countable dense subset.

Next, we want to consider the case when $G$ has representation (PH).

Example 4.10 Let $G$ be representable by (PH), being also lower semicontinuous in the parameter $\theta$ and satisfying condition (*) just before Proposition 2.7. Then Proposition 2.7 ensures that $G$ is $\mathcal{B}(\Theta) \otimes \mathcal{B}(\mathbb{R}^d)$-measurable. In Remark 2.9 it is discussed how to meet the requirements (A 1), (A 2), (A 5). The condition (A 3) follows easily from (*).

5 Proofs

Recall the notions and notations from empirical process theory introduced in the Section 2. Furthermore we set $\psi(\theta) := \mathbb{E}[G(\theta, Z_1)]$ and $\psi_{H, \Theta}(\theta) := \mathbb{E}[H(\theta, \psi(\theta), Z_1)]$ for $\theta \in \Theta$. For $n \in \mathbb{N}$ we introduce the random function $M_n : \Theta \times \Omega \rightarrow \mathbb{R}$ via $M_n(\theta, \omega) = \frac{1}{n} \sum_{j=1}^{n} H\left(\theta, \frac{1}{n} \sum_{k=1}^{n} G(\theta, Z_k(\omega)), Z_j(\omega)\right).$ (5.1)

The proof of the main result is a repeated application of Theorems 2.1, 2.2 from [12]. For ease of reference we shall recall these results now.

Theorem 5.1 Let $\Gamma \subseteq \mathbb{R}^k$ be compact, and let $G^\Gamma : \Gamma \times \mathbb{R}^d \rightarrow \mathbb{R}$ satisfy the following properties

1) $G^\Gamma(\gamma, \cdot)$ is Borel measurable for every $\gamma \in \Gamma$.

2) The associated function class $\mathcal{F}^\Gamma := \{G^\Gamma(\gamma, \cdot) \mid \gamma \in \Gamma\}$ has a square $\mathbb{P}^\mathcal{Z}$-integrable positive envelope $\xi$ with finite uniform entropy integral $J(\mathcal{F}^\Gamma, \xi, 1/2)$.

3) There exist some at most countable subset $\Gamma \subseteq \Gamma$ and $(\mathbb{P}^\mathcal{Z})^n$-null sets $N_n (n \in \mathbb{N})$ such that

$$\inf_{\gamma \in \Gamma} \mathbb{E}|G^\Gamma(\gamma, Z_1)| - \mathbb{E}|G^\Gamma(\gamma, Z_1)| = \inf_{\gamma \in \Gamma} \max_{j \in \{1, \ldots, n\}} |G^\Gamma(\gamma, z_j) - G^\Gamma(\gamma, z_j)| = 0$$

for $n \in \mathbb{N}, \gamma \in \Gamma$ and $(z_1, \ldots, z_n) \in \mathbb{R}^{dn} \setminus N_n$.

Furthermore, set $B_n^\xi := \left\{\frac{1}{n} \sum_{j=1}^{n} \xi(Z_j)^2 \leq 2\mathbb{E}[\xi(Z_1)^2]\right\}$, and let $g(t) \in \mathbb{R}$ be defined as in (2.6) for $t > 0$. Then $\sup_{\gamma \in \Gamma} \mathbb{P}(B_n^\xi(\gamma))$ is a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ such that

$$\mathbb{E}\left[\sup_{\gamma \in \Gamma} \mathbb{P}(B_n^\xi(\gamma))\right] \leq 16\sqrt{2} \frac{\|\xi\|_{\mathbb{P}^\mathcal{Z}, 2} J(\mathcal{F}^\Gamma, \xi, 1/2)}{\sqrt{n}},$$
is valid for \( n \in \mathbb{N} \), and
\[
P\left( \left\{ \sup_{\gamma \in \Gamma} \left| (P_n - P)(G^\Gamma (\gamma, \cdot)) \right| > \varepsilon \right\} \right) \leq \exp \left( \frac{-\sqrt{n}\varepsilon}{\|\xi\|_{\mathbb{P}_{2,2}} \cdot g(t)} \right) + P(\Omega \setminus B_\varepsilon^c)
\]
holds for \( t > 0 \) and arbitrary \( n \in \mathbb{N} \) with \( \varepsilon > \eta_{t,n} \) as well as \( n \geq \|\xi\|_{\mathbb{P}_{2,2}}^2 / 2 \), where
\[
\eta_{t,n} := \|\xi\|_{\mathbb{P}_{2,2}} \left[ 1 + 32\sqrt{2}(1 + t) J(F_{\Gamma}, \xi, 1/4) \right] / \sqrt{n}.
\]
In particular,
\[
P\left( \left\{ \sup_{\gamma \in \Gamma} \left| (P_n - P)(G^\Gamma (\gamma, \cdot)) \right| > \varepsilon \right\} \right) \leq \exp \left( \left[ 1 + 32\sqrt{2}(1 + t) J(F_{\Gamma}, \xi, 1/4) \right] g(t) \right) \exp \left( \frac{-\sqrt{n}\varepsilon}{\|\xi\|_{\mathbb{P}_{2,2}} \cdot g(t)} \right) + P(\Omega \setminus B_\varepsilon^c)
\]
for \( t, \varepsilon > 0 \) and \( n \geq \|\xi\|_{\mathbb{P}_{2,2}}^2 / 2 \).

The separability condition 3) in Theorem 5.1 may be simplified.

**Lemma 5.2** Let \( \Gamma \subseteq \mathbb{R}^k \) be compact, and let \( G^\Gamma : \Gamma \times \mathbb{R}^d \to \mathbb{R} \) meet requirements 1), 2) from Theorem 5.1. In addition \( G^\Gamma \) is assumed to have the separability property

1) There exist an at most countable subset \( \Gamma \) of \( \Gamma \) and some \( \mathbb{P}^\Gamma \)-null set \( N \) satisfying
\[
\inf_{\gamma \in \Gamma} |G^\Gamma (\gamma, z) - G^\Gamma (\gamma, \bar{z})| = 0 \quad \text{for } \bar{z} \in \Gamma \text{ and } z \in \mathbb{R}^d \setminus N.
\]

Then \( G^\Gamma \) satisfies the following separability condition:

(\text{SC}) There exist an at most countable subset \( \Gamma \) of \( \Gamma \) and \( (\mathbb{P}^\Gamma)^n \)-null sets \( N_n \) \( (n \in \mathbb{N}) \) such that
\[
\inf_{\gamma \in \Gamma} \left\{ \mathbb{E} \left[ \left. |G^\Gamma (\gamma, Z_1) - G^\Gamma (\gamma, \bar{Z}_1)| \right| \gamma \right] + \max_{j \in \{1, \ldots, n\}} \left| G^\Gamma (\gamma, z_j) - G^\Gamma (\gamma, \bar{z}_j) \right| \right\} = 0
\]
for \( n \in \mathbb{N} \), \( \gamma \in \Gamma \) and \( (z_1, \ldots, z_n) \in \mathbb{R}^{dn} \setminus N_n \).

**Proof** Let the at most countable subset \( \Gamma \) of \( \Gamma \) and the \( \mathbb{P}^\Gamma \)-null set \( N \) be from condition 1). Define for \( n \in \mathbb{N} \)
\[
B^n_{jp} := \left\{ \frac{N, j = p}{\mathbb{R}^d, j \neq p} \ (j, p \in \{1, \ldots, n\}) \right\} \quad \text{and} \quad N_n := \bigcup_{j=1}^n \bigcap_{p=1}^n B^n_{jp}.
\]

Then \( N_n \) is obviously a \( (\mathbb{P}^\Gamma)^n \)-null set. Now fix \( \gamma \in \Gamma \). By 1) we may find some sequence \( (\gamma_t)_{t \in \mathbb{N}} \) in \( \Gamma \) such that
\[
\lim_{t \to \infty} \left| G^\Gamma (\gamma_t, z) - G^\Gamma (\gamma, z) \right| = 0 \quad \text{for } z \in \mathbb{R}^d \setminus N.
\]
The sequence \( (\overline{G}^l(\gamma_l, Z_1))_{l \in \mathbb{N}} \) is dominated by some integrable random variable on \((\Omega, \mathcal{F}, P)\) due to condition 2) from Theorem 5.1. Hence (5.2) implies by dominated convergence theorem
\[
\lim_{l \to \infty} \mathbb{E} \left[ |\overline{G}^l(\gamma_l, Z_1) - \overline{G}^l(\overline{\gamma}, Z_1)| \right] = 0. \tag{5.3}
\]
Let \( n \in \mathbb{N} \) and \((z_1, \ldots, z_n) \in \mathbb{R}^d \setminus N_n\). By definition of the set \( N_n \) we may observe \( z_j \in \mathbb{R}^d \setminus N \) for \( j \in \{1, \ldots, n\} \). Therefore by (5.2)
\[
\lim_{l \to \infty} \max_{j \in \{1, \ldots, n\}} |\overline{G}^l(\gamma_l, z_j) - \overline{G}^l(\overline{\gamma}, z_j)| = 0,
\]
and thus in view of (5.3)
\[
\lim_{l \to \infty} \left\{ \mathbb{E} \left[ |\overline{G}^l(\gamma_l, Z_1) - \overline{G}^l(\overline{\gamma}, Z_1)| \right] + \max_{j \in \{1, \ldots, n\}} |\overline{G}^l(\gamma_l, z_j) - \overline{G}^l(\overline{\gamma}, z_j)| \right\} = 0.
\]
This completes the proof. \( \square \)

### 5.1 Proof of Theorem 2.2 and Theorem 2.3

Let (A 1) - (A 6) be fulfilled with \( \delta_1, M_{\delta_1}, M^\delta_1 > 0 \) and \( \beta \in [0, 1] \) from (A 5), whereas \( \delta_2 > 0, K_{\delta_2} \geq 0 \) are from (A 6). In addition \( M_n \) denotes the random function defined by (5.1).

As a first observation, recalling the events \( \Omega_{n,a}, \Omega_{n,b}^n \) from (2.8) and (2.9),
\[
(\theta, \frac{1}{n} \sum_{j=1}^n G(\theta, Z_j(\omega)), \frac{1}{n} \sum_{j=1}^n G(\theta^*, Z_j(\omega))) \in \mathcal{W}_{a,b}^{n, \delta} \quad \text{for} \quad \theta \in U_{\delta}, \omega \in \Omega_{n,a} \cap \Omega_{n,b}^{n, a,b},
\]
where \( \mathcal{W}_{a,b}^{n, \delta} \) denotes the set of all vectors \((\theta, t, s) \in U_{\delta} \times \mathbb{R}^2\) satisfying
\[
|t - \overline{\psi}(\theta)| \vee |s - \overline{\psi}(\theta^*)| \leq a/\sqrt{n} \quad \text{and} \quad |t - s| \leq b \delta^2/\sqrt{n}.
\]
Then we may conclude
\[
\sup_{\theta \in U_{\delta}} |M_n(\theta, \omega) - M_n(\theta^*, \omega) - \psi_{H,\Theta}(\theta) + \psi_{H,\Theta}(\theta^*)| \\
\leq \sup_{(\theta, t, s) \in \mathcal{W}_{a,b}^{n, \delta}} \left[ (P_n - P)(H(\theta, t, \cdot) - H(\theta^*, s, \cdot)|_\omega) \right] \\
+ \sup_{(\theta, t, s) \in \mathcal{W}_{a,b}^{n, \delta}} \left| \mathbb{E}[H(\theta, t, Z_1) - H(\theta^*, s, Z_1)] - \psi_{H,\Theta}(\theta) + \psi_{H,\Theta}(\theta^*) \right| \tag{5.4}
\]
for \( \omega \in \Omega_{n,a} \cap \Omega_{n,b}^{n, a,b}. \) We continue by providing an upper estimation of the second summand in (5.4).
Lemma 5.3 Let (A 1), (A 4) - (A 6) be fulfilled with \( \delta_1, M_{\delta_1} > 0 \) as well as \( \beta \in [0,1] \) from (A 5), and \( \delta_2 > 0, K_{\delta_2} \geq 0 \) being as in (A 6). Then for \( a, b > 0 \)

\[
\sup_{(\theta,t,s) \in \mathcal{W}_{a,b}^{n,\delta}} \left| \mathbb{E}[H(\theta, t, Z_1) - H(\theta^*, s, Z_1)] - \psi_{H,\theta}(\theta) + \psi_{H,\theta}(\theta^*) \right| \leq 1_{[0,\infty]}(\mathbb{E}[L_2(Z_1)]) \frac{M_{a,b} \delta^3}{\sqrt{n}}
\]

holds for \( n \in \mathbb{N} \) with \( 1_{[0,\infty]}(\mathbb{E}[L_2(Z_1)]) \cdot 2a \leq \sqrt{n} \delta_2 \), and \( \delta \leq \delta_*^\delta \), where \( M_{a,b} \) and \( \delta_*^\delta \) denote the constants defined in (2.4) and (2.3) respectively.

The proof of Lemma 5.3 is provided in Subsection 5.2.

Let us turn over to the first summand on the right hand side of (5.4). The aim is to apply Theorem 5.1 to the function class \( \mathbb{F}_{a,b}^{n,\delta} := \{ H(\theta, t, \cdot) - H(\theta^*, s, \cdot) \mid (\theta, t, s) \in \mathcal{W}_{a,b}^{n,\delta} \} \) for \( a, b, \delta > 0 \) and \( n \in \mathbb{N} \). For this purpose we have to study the covering numbers which are involved in the associated uniform entropy integrals.

Lemma 5.4 Let (A 1), (A 4) and (A 5) be satisfied with \( L_1, L_2 \) denoting the square \( \mathbb{P}^Z \)-integrable mappings from (A 4), and \( \delta_1, M_{\delta_1} > 0 \) as well as \( \beta \in [0,1] \) being as in assumption (A 5). Furthermore, let \( a, b > 0, n \in \mathbb{N}, \delta \in [0, \delta_1] \), and let \( \xi^G_{a,b} \) stand for the positive envelope of \( \mathbb{F}_{a,b}^{n,\delta} \) as in (A 5). Then a positive envelope of \( \mathbb{F}_{a,b}^{n,\delta} \) is defined by \( \xi^G_{a,b,\delta}(z) := \sqrt{L_1(z) \vee 1} \xi_{a,b}^G(z) + L_2(z) |b + M_{\delta_1}|^2 \delta^2 \), and

\[
N(\eta \| \xi^G_{a,b,\delta} \|_{\mathcal{M}_2}, \mathbb{F}_{a,b}^{n,\delta}, L^2(\mathcal{Q})) \leq \frac{144 (a/\sqrt{n} + M_{\delta_1} \delta^3)^2}{(b + M_{\delta_1}) \cdot \delta^2 \cdot \eta^2} \sup_{Q \in \mathcal{M}_f_{n,\delta}} N(\eta \| \xi^G_{a,b} \|_{\mathcal{M}_2}/4, \mathbb{F}^\Theta_{a,b}, L^2(\mathcal{Q}))
\]

for \( Q \in \mathcal{M}_f_{a,b} \) and \( \eta \in [0,1] \). In particular

\[
J(\mathbb{F}_{a,b}^{n,\delta}, \xi^G_{a,b,\delta}, \varepsilon) \leq 4 J(\mathbb{F}^\Theta_{a,b}, \xi^G_{a,b}, \varepsilon/4) + 2 \sqrt{2} \varepsilon \sqrt{\ln \left( \frac{12}{\varepsilon} \left[ \frac{a}{b \sqrt{n} \delta^3} \vee 1 \right] \right)}
\]

holds for \( \varepsilon \in [0,1] \).

The proof of Lemma 5.4 may be found in Subsection 5.3.

As a last providing step we want to verify the conditions 1) - 3) in Theorem 5.1 for the function classes \( \mathbb{F}_{a,b}^{n,\delta} \). So let \( a, b > 0, \delta \in [0, \delta_1] \) and \( \pi \in \mathbb{N} \).

The parameter set \( \mathcal{W}_{a,b}^{\pi,\delta} \) is a bounded subset of \( \mathbb{R}^{m+2} \). By Remark 2.1 the mapping \( \theta \mapsto \mathbb{E}[G(\theta, Z_1)] \) on \( \Theta \) is continuous under (A 1) and (A 2). Then it may be verified easily that \( \mathcal{W}_{a,b}^{\pi,\delta} \) is closed w.r.t. the Euclidean metric, and thus compact.

Next, the mapping \( H(\theta, t, \cdot) - H(\theta^*, s, \cdot) \) is Borel measurable for \( (\theta, t, s) \in \mathcal{W}_{a,b}^{\pi,\delta} \) due to Borel measurability of \( H \). This means that \( \mathbb{F}_{a,b}^{\pi,\delta} \) meets condition 1) in Theorem 5.1.

Condition 2) in Theorem 5.1 is already known to be fulfilled with the square \( \mathbb{P}^Z \)-integrable mapping \( \xi^G_{a,b,\delta} \) due to Lemma 5.4 along with (A 4) and (A 5).

Concerning condition 3) it suffices to verify the separability condition 1) from Lemma 5.2. According to property (A 3) we may find an at most countable subset \( C(\mathcal{U}_\delta) \subseteq \mathcal{U}_\delta \) and some \( \mathbb{P}^Z \)-null set \( \mathcal{N}^{\mathcal{U}_\delta} \) such that

\[
\inf_{\varphi \in \mathcal{C}(\mathcal{U}_\delta)} \left| G(\theta, z) - G(\varphi, z) \right| = 0 \quad \text{for} \quad \theta \in \mathcal{U}_\delta, z \in \mathbb{R}^d \setminus \mathcal{N}^{\mathcal{U}_\delta}
\]
Every subset of $\mathbb{R}^2$ is separable w.r.t. the Euclidean metric so that for each $\vartheta \in C(U_b)$ there is some at most countable subset $M(\vartheta)$ of $\{(t, s) \in \mathbb{R}^2 \mid (t, s) \in W_{a,b}\}$ which is dense w.r.t. the Euclidean metric. Then $\overline{W}_{a,b} := \{(t, s) \mid (t, s) \in M(\vartheta)\}$ is an at most countable subset of $W_{a,b}$.

Next, fix $(\overline{\vartheta}, \overline{t}, \overline{s}) \in W_{a,b}$. Then for $z \in \mathbb{R}^d \setminus N^{U_b}$ there exists some sequence $((\vartheta_l), l \in \mathbb{N}$ in $C(U_b)$ such that $\|G(\vartheta_l, z) - G(\overline{\vartheta}, \overline{z})\| \to 0$ holds. Choose $t_l := \overline{\vartheta}(t_l) - \overline{\vartheta}(\overline{t}) + \overline{t}$ for $l \in \mathbb{N}$. Hence $(\vartheta_l, t_l, s_l) \in W_{a,b}$ for $l \in \mathbb{N}$, and thus we may select for any $l \in \mathbb{N}$ some sequence $((t_l, s_l))_{q \in \mathbb{N}}$ in $M(\vartheta_l)$ which converges to $(t_l, s_l)$ w.r.t. the Euclidean metric.

Note that $\{(\vartheta_l, t_l, s_l) \mid l, q \in \mathbb{N}\}$ is a subset of $W_{a,b}$.

Now, by (A 4) along with the choices of the numbers $t_l, t_{lq}, s_{lq}$ we end up with
\[
\|\{H(\vartheta_l, t_{lq}, z) - H(\vartheta^*, s_{lq}, z)\} - [H(\overline{\vartheta}, \overline{t}, z) - H(\vartheta^*, \overline{s}, z)\]|^2 \leq 4[L_1(z)^2 + L_2(z)^2] \|\{G(\vartheta_l, z) - G(\overline{\vartheta}, z)\}|^2 + |t_{lq} - \overline{t}|^2 + |s_{lq} - \overline{s}|^2,
\]
and thus
\[
\lim_{l \to \infty} \lim_{q \to \infty} \sup \sup_{l \to \infty} \left( \|\{H(\vartheta_l, t_{lq}, z) - H(\vartheta^*, s_{lq}, z)\} - [H(\overline{\vartheta}, \overline{t}, z) - H(\vartheta^*, \overline{s}, z)\]|^2 \leq \lim_{l \to \infty} 4[L_1(z)^2 + L_2(z)^2] \|\{G(\vartheta_l, z) - G(\overline{\vartheta}, z)\}|^2 + |t_{lq} - \overline{t}|^2 = 0.
\]

Thus we have verified condition 1) in Lemma 5.2 with at most countable subset $\overline{W}_{a,b}$ and $\mathbb{P}^{\bar{Z}}$-null set $N^{U_b}$.

Summarizing, $\overline{W}_{a,b}$ meets all the requirements of Theorem 5.1.

Now, we are in the position to show Theorem 2.2 and Theorem 2.3. We start with Theorem 2.2.

**Proof of Theorem 2.2**: Statement 1) follows immediately from Lemma 5.1 along with (A 4) and (A 5). So let us turn over to statement 2), and fix $\delta \in [0, \delta^*] \cap \mathbb{N}$ satisfying the inequalities $n \geq \overline{M}_b(\delta^*)^{2\beta}/2$ and $n \geq 4[\mathbb{E}[L_2(Z_1)]] \cdot 9 a^2/\delta^2_2$. Drawing on (A 5), the square $\mathbb{P}^Z$-integrable positive envelope $e_{a,b,\delta}^{G} \in W_{a,b}$ satisfies
\[
\|e_{a,b,\delta}^{G}\|^2_{\mathbb{P}^Z, 2} \leq \frac{\overline{M}_b \cdot \delta^{2\beta}}{2} \leq \frac{\overline{M}_b \cdot (\delta^*)^{2\beta}}{2} \leq n.
\]

Thus by Lemma 5.1 along with (A 5), recalling notation (2.7),
\[
16\sqrt{2} J(P_{a,b}, e_{a,b,\delta}^{G}, 1/2) \leq \overline{P}_n(a, b, \delta).
\]

Putting together (5.5) and (5.6), the application of Theorem 5.1 to $\overline{W}_{a,b}$ yields
\[
\mathbb{E} \left( \sup_{f \in \mathbb{P}^{a,b}} \left| (\mathbb{P} - \mathbb{P})(f) \right| \right) \leq \overline{M}_b \overline{P}_n(a, b, \delta) \delta^3/\sqrt{n}.
\]
Note that $G_n^G,H(\theta, \cdot) = \sqrt{n} [M_n(\theta, \cdot) + \psi_{H, \Theta}(\theta)]$ holds for $\theta \in \mathcal{U}_\delta$ so that in view of (5.4) the inequality in statement 2) may be derived via Markov's inequality along with Lemma 5.3.

Concerning statement 3) we firstly observe that the function classes $F^\Theta$ and $F^\Theta$ meet the requirements of Theorem 5.1 due to (A 1) - (A 3) and (A 5) along with Lemma 5.2. Secondly, $J(F, C_F, 1/2) \leq 4J(F, C_F, 1/8)$ holds for any function class $F$ with positive envelope $C_F$ (see [1, Lemma 3.5.3]). Then by Theorem 5.1 statement 3) follows immediately invoking Markov's inequality along with assumption (A 5). This completes the proof. \[\square\]

Let us now turnover to Theorem 2.3.

Proof of Theorem 2.3:

We have $J(F^\Theta, \xi^G, 1/4) \leq 2 J(F^\Theta, \xi^G, 1/8)$ (see [7, Lemma 3.5.3]), and thus the inequality $J(F^\Theta, \xi^G, 1/4) \leq 2 M^{\delta_1}$ holds due to (A 5). In addition $n \geq \|\xi^G\|_{\ell^{2,a}}^2/2$ is assumed in statement 2). Then in view of conditions (A 1) - (A 3) and (A 5) along with Lemma 5.2 we apply Theorem 5.1 to the function class $F^\Theta$ to conclude statement 2).

Next, we may observe $J(F^\Theta, \xi^G, 1/4) \leq 2 J(F^\Theta, \xi^G, 1/8) \leq 2 M^{\delta_1}$ by Lemma 5.3 along with (A 5). Furthermore by assumption in statement 3) together with (A 5), the inequalities $\|\xi^G\|_{\ell^{2,a}}/2 \leq M^{\delta_1}_1 \delta^{2\beta}/2 \leq n$ hold. Therefore, Lemma 5.2 along with assumptions (A 1), (A 3) and (A 5) allow to use Theorem 5.1 for the function class $F^\Theta$ which yields statement 3).

Concerning statement 1) we set $r_{\beta,n} := n^{1/(4-2\beta)}$ and $t_2 := 1_{[0,\infty)}(L_2(Z_1))$. Combining (5.4) with Lemma 5.3 we obtain

$$\mathbb{P} \left\{ \left\{ \sup_{\theta \in \mathcal{U}_\delta} |G_n^G,H(\theta, \cdot) - G_n^G,H(\theta^*, \cdot)| > \varepsilon \cdot r_{\beta,n}^{-\beta} \right\} \cap \Omega_{n,a} \cap \Omega_{n,b} \right\} \leq \mathbb{P}^\ast \left\{ \left\{ \sup_{f \in F_{a,b}^n} |(\mathbb{P} - \mathbb{P})(f)| > \varepsilon \frac{M_{a,b} \cdot \delta^\beta}{t_2} \right\} \right\}$$

(5.7)

for $n \in \mathbb{N}$ and $a, b, \delta, \varepsilon > 0$, where $\mathbb{P}^\ast$ denotes the outer probability of $\mathbb{P}$ and $M_{a,b}$ is as in (2.31).

In order to find suitable upper estimations of the right hand side of (5.7) we want to apply Theorem 5.1 to the function class $F_{a,b}^\epsilon$. We have already verified that it meets all the requirements of Theorem 5.1. In particular, $\sup_{f \in F_{a,b}^\epsilon} |(\mathbb{P} - \mathbb{P})(f)|$ is a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$. Furthermore by Lemma 5.3 recalling notation $\eta_{t,n}(a, b, \delta)$ from the display,

$$1 + 32\sqrt{2}(t + 1) J(F_{a,b}^\epsilon, \xi^G_{a,b,\delta}, 1/4) \leq \eta_{t,n}(a, b, \delta).$$

(5.8)

Thus, together with (5.8), in case of $t_2 \cdot \delta < \varepsilon^{1/\beta} / [n^{1/(4-2\beta)} \cdot M_{a,b}^{1/\beta}]$ we may conclude from
Theorem 5.1

\[ \mathbb{P}\left( \left\{ \sup_{f \in \mathbb{F}_{n,\delta}^{a,b}} \left| (\mathbb{P}_n - \mathbb{P}) (f) \right| > \frac{\varepsilon}{r_{\beta,n}^2} - t_2 \cdot \frac{M_{a,b} \cdot \delta^\beta}{\sqrt{n}} \right\} \right) \]

\[ \leq \exp \left( \eta_n(a, b, \delta) \cdot g(t) \cdot \exp \left( -\frac{(\varepsilon - n^{\beta/(4-2\beta)} \cdot t_2 \cdot M_{a,b} \cdot \delta^\beta) \cdot g(t)}{n^{\beta/(4-2\beta)} \left\| \xi_{a,b,\delta}^G \right\|_2} \right) + \mathbb{P}(\Omega \setminus B_{n}^{G_{a,b,\delta}}) \right) \]

\leq \exp \left( \left| \eta_n(a, b, \delta) + t_2 \cdot \frac{M_{a,b}}{M_b} \cdot g(t) \right| \cdot \exp \left( -\frac{\varepsilon \cdot g(t)}{n^{\beta/(4-2\beta)} \cdot M_b \cdot \delta^\beta} \right) + \mathbb{P}(\Omega \setminus B_{n}^{G_{a,b,\delta}}) \right) \]

for \( t, \varepsilon > 0 \). Moreover, combining (5.5) and (5.8), we obtain

\[ \frac{\varepsilon}{r_{\beta,n}^2} - t_2 \cdot M_{a,b} \cdot \delta^\beta / \sqrt{n} > \left\| \xi_{a,b,\delta}^G \right\|_{2} \cdot \left[ 1 + 32 \sqrt{n} (t + 1) J(\mathbb{P}_{n,\delta}^{a,b}, s_{a,b,\delta}, 1/4) \right] / \sqrt{n} \]

for \( t, \varepsilon > 0 \) if \( \delta < \left( \varepsilon / (t \cdot M_{a,b} + M_b \cdot \eta_n(a, b, \delta)) \right)^{1/\beta} \cdot n^{-1/(4-2\beta)} \). Then a further application of Theorem 5.1 together with (5.5) yields

\[ \mathbb{P}\left( \left\{ \sup_{f \in \mathbb{F}_{n,\delta}^{a,b}} \left| (\mathbb{P}_n - \mathbb{P}) (f) \right| > \frac{\varepsilon}{r_{\beta,n}^2} - t_2 \cdot M_{a,b} \delta^\beta / \sqrt{n} \right\} \right) \]

\[ \leq \exp \left( -\frac{(\varepsilon - n^{\beta/(4-2\beta)} \cdot t_2 \cdot M_{a,b} \cdot \delta^\beta) \cdot g(t)}{n^{\beta/(4-2\beta)} \left\| \xi_{a,b,\delta}^G \right\|_2} \right) + \mathbb{P}(\Omega \setminus B_{n}^{G_{a,b,\delta}}) \]

\[ \leq \exp \left( (t_2 \cdot M_{a,b} \cdot g(t) / M_b) \cdot \exp \left( -\frac{\varepsilon \cdot g(t)}{n^{\beta/(4-2\beta)} \cdot M_b \cdot \delta^\beta} \right) + \mathbb{P}(\Omega \setminus B_{n}^{G_{a,b,\delta}}) \right) \]

for \( t, \varepsilon > 0 \) if \( \delta < \left( \varepsilon / (t_2 \cdot M_{a,b} + M_b \cdot \eta_n(a, b, \delta)) \right)^{1/\beta} \cdot n^{-1/(4-2\beta)} \). This shows statement 1) due to (5.7) and completes the proof.

\[ \square \]

5.2 Proof of Lemma 5.3

If \( \mathbb{E}[L_2(Z_1)] = 0 \), then \( H(\theta, t, Z_1) - H(\theta, \psi(\theta), Z_1) = 0 \) \( \mathbb{P} \)-a.s. for every \( \theta \in \Theta \). Thus the statement follows immediately. So let us assume that \( \mathbb{E}[L_2(Z_1)] \neq 0 \) which means that \( \mathbb{E}[L_2(Z_1)] \) is strictly positive.

Firstly, applying change of variable formula several times we may observe by (A 6)

\[ \mathbb{E}\left[ H(\theta, t, Z_1) - H(\theta, \psi(\theta), Z_1) \right] \]

\[ = - \int_0^{[t-\psi(\theta)]^-} m_{\Theta \times \mathbb{R}} (\theta, \psi(\theta) - u) \, du + \int_0^{[t-\psi(\theta)]^+} m_{\Theta \times \mathbb{R}} (\theta, \psi(\theta) + u) \, du \]
for \((\theta, t) \in \Theta \times \mathbb{R}\). Then routine calculations yield

\[
\left| \mathbb{E}[H(\theta, t, Z_1) - H(\theta, \bar{\psi}(\theta), Z_1) - H(\theta^*, s, Z_1) + H(\theta^*, \bar{\psi}(\theta^*), Z_1)] \right|
\leq \int_0^{[t, \bar{\psi}(\theta)]^-} |m_{\Theta \times \mathbb{R}}(\theta, \bar{\psi}(\theta) - u) - m_{\Theta \times \mathbb{R}}(\theta^*, \bar{\psi}(\theta^*) - u)| \, du
\]

\[
+ \int_0^{[t, \bar{\psi}(\theta)]^+} |m_{\Theta \times \mathbb{R}}(\theta, \bar{\psi}(\theta) + u) - m_{\Theta \times \mathbb{R}}(\theta^*, \bar{\psi}(\theta^*) + u)| \, du
\]

\[
+ \int_{[t, \bar{\psi}(\theta)]^- \cap [s - \bar{\psi}(\theta^*)]^-} |m_{\Theta \times \mathbb{R}}(\theta^*, \bar{\psi}(\theta^*) - u) - m_{\Theta \times \mathbb{R}}(\theta^*, \bar{\psi}(\theta^*))| \, du
\]

\[
+ \int_{[t, \bar{\psi}(\theta)]^+ \cap [s - \bar{\psi}(\theta^*)]^-} |m_{\Theta \times \mathbb{R}}(\theta^*, \bar{\psi}(\theta^*) + u) - m_{\Theta \times \mathbb{R}}(\theta^*, \bar{\psi}(\theta^*))| \, du
\]

\[
+ |t - s - \bar{\psi}(\theta) + \bar{\psi}(\theta^*)| \cdot |m_{\Theta \times \mathbb{R}}(\theta^*, \bar{\psi}(\theta^*))|
\]  

(5.9)

for \((\theta, t), (\theta^*, s) \in \Theta \times \mathbb{R}\).

Recall that by (A.6)

\[
|m_{\Theta \times \mathbb{R}}(\theta, t) - m_{\Theta \times \mathbb{R}}(\theta, s)| \leq K_{\delta_2} \|(\theta - \psi, t - s)\|_{m+1} \quad \text{for } (\theta, t), (\theta, s) \in \mathcal{V}_{\delta_2}.
\]

Now, select any \(\delta \in [0, \delta_1]\) such that \((\delta_1^{1-\beta} + M_{\delta_1}) \delta^2 \leq \delta_2/2\), and choose \(n \in \mathbb{N}\) with \(a/\sqrt{n} \leq \delta_2/2\). By (A.5) along with \(\delta \leq \delta_1\) we know that the chosen positive envelope \(\xi^G_\delta\) of \(F^G_\delta\) satisfies the inequality \(\|\xi^G_\delta\|_{p,2} \leq M_{\delta_1} \cdot \delta^\beta\). This implies

\[
\|\mathbb{E}[G(\theta, Z_1)] - \mathbb{E}[G(\theta^*, Z_1)]\| \leq \|\xi^G_\delta\|_{p,2} \leq M_{\delta_1} \cdot \delta^\beta \quad \text{for } \theta \in \mathcal{U}_\delta.
\]

Then by \(\delta \leq \delta_1\) along with \((\delta_1^{1-\beta} + M_{\delta_1}) \delta^2 \leq \delta_2/2\),

\[
\|(\theta - \theta^*, \bar{\psi}(\theta) - u - \bar{\psi}(\theta^*))\|_{m+1}
\leq \|\theta - \theta^*\|_m + |\bar{\psi}(\theta) - \bar{\psi}(\theta^*)| + |\bar{\psi}(\theta) - t| \leq (\delta_1^{1-\beta} + M_{\delta_1}) \delta^\beta + a/\sqrt{n} \leq \delta_2
\]

for \((\theta, t, u) \in \mathcal{U}_\delta \times \mathbb{R}^2\) with \(\bar{\psi}(\theta) - t \leq a/\sqrt{n}\) and \(|u| \leq |\bar{\psi}(\theta) - t|\). In particular, since \(a/\sqrt{n} < \delta_2\), the vectors \((\theta, \bar{\psi}(\theta) - u)\) and \((\theta^*, \bar{\psi}(\theta^*) - u)\) belong to \(\mathcal{V}_{\delta_2}\) in this case. Thus

\[
\int_0^{[t, \bar{\psi}(\theta)]^-} |m_{\Theta \times \mathbb{R}}(\theta, \bar{\psi}(\theta) - u) - m_{\Theta \times \mathbb{R}}(\theta^*, \bar{\psi}(\theta^*) - u)| \, du
\]

\[
+ \int_0^{[t, \bar{\psi}(\theta)]^+} |m_{\Theta \times \mathbb{R}}(\theta, \bar{\psi}(\theta) + u) - m_{\Theta \times \mathbb{R}}(\theta^*, \bar{\psi}(\theta^*) + u)| \, du
\]

\[
\leq K_{\delta_2} |\bar{\psi}(\theta) - t| \left\| (\theta - \theta^*, \bar{\psi}(\theta) - \bar{\psi}(\theta^*)) \right\|_{m+1} \leq K_{\delta_2} a (\delta_1^{1-\beta} + M_{\delta_1}) \delta^\beta/\sqrt{n} \quad (5.10)
\]

for \((\theta, t) \in \mathcal{U}_\delta \times \mathbb{R}\) with \(\bar{\psi}(\theta) - t \leq a/\sqrt{n}\). Moreover, since \((\theta^*, \bar{\psi}(\theta^*) - u) \in \mathcal{V}_{\delta_2}\) for
\[ |u| \leq \frac{a}{\sqrt{n}}, \] we may also conclude
\[
\int_{[t-s]} \frac{1}{|s-n(t^*)|} \left| m_{\Theta \times \mathbb{R}}(\theta, \psi(\theta^*) - u) - m_{\Theta \times \mathbb{R}}(\theta, \psi(\theta^*)) \right| \, du \\
+ \int_{[t-s]} \frac{1}{|s-n(t^*)|} \left| m_{\Theta \times \mathbb{R}}(\theta, \psi(\theta^*) + u) - m_{\Theta \times \mathbb{R}}(\theta, \psi(\theta^*)) \right| \, du \\
\leq K_{\delta_2} \int_{[t-s]} \frac{|u|}{|s-n(t^*)|} \, du + K_{\delta_2} \int_{[t-s]} \frac{|u|}{|s-n(t^*)|} \, du \\
\leq 2 a K_{\delta_2} |t - s - \psi(\theta^*)| \leq 2 a b K_{\delta_2} \delta^3 / \sqrt{n} \quad \text{for } (\theta, t, s) \in \mathcal{W}_{a,b}^{n,\delta}. \quad (5.11)
\]

Putting (5.9) and (5.10), (5.11) together we end up with
\[
\mathbb{E}[H(\theta, t, Z_1) - H(\theta, \psi(\theta), Z_1) - H(\theta^*, s, Z_1) + H(\theta^*, \psi(\theta^*), Z_1)] \\
\leq [K_{\delta_2} a (\delta^3 + M_{\delta_1}) + 2 a b K_{\delta_2} + b |m_{\Theta \times \mathbb{R}}(\theta, \psi(\theta^*))|] \delta^3 / \sqrt{n}
\]
for \((\theta, t, s) \in \mathcal{W}_{a,b}^{n,\delta}\). This completes the proof.

\[ \square \]

### 5.3 Proof of Lemma 5.4

For \((\theta, t, s) \in \mathcal{W}_{a,b}^{n,\delta}\) we may observe by (A 5)
\[
|t - s| \leq |t - s - \psi(\theta^*) + \psi(\theta^*)| + |\psi(\theta) - \psi(\theta^*)| \leq \frac{b \delta^3}{\sqrt{n}} + \mathbb{E}[\xi_G^G(Z_1)] \leq (b + M_{\delta_1}) \delta^3.
\]

Hence by (A 4) it is easy to verify \(\xi_{a,b,\delta}^G\) as a positive envelope of \(\mathbb{P}_{n,\delta}^{a,b}\).

Furthermore by (A 5) along with definition of \(\mathcal{W}_{a,b}^{n,\delta}\) we obtain for \((\theta, t, s) \in \mathcal{W}_{a,b}^{n,\delta}\)
\[
|t - \psi(\theta^*)| \leq |t - \psi(\theta)| + |\psi(\theta) - \psi(\theta^*)| \leq a / \sqrt{n} + \mathbb{E}[\xi_G^G(Z_1)] \leq a / \sqrt{n} + M_{\delta_1} \delta^3,
\]
\[
|s - \psi(\theta^*)| \leq a / \sqrt{n}.
\]

In particular
\[
t, s \in \mathcal{T}_{a,b}^{n,\delta} := [\psi(\theta^*) - a / \sqrt{n} - M_{\delta_1} \delta^3, \psi(\theta^*) + a / \sqrt{n} + M_{\delta_1} \delta^3],
\]
and thus \(\mathbb{P}_{a,b}^{n,\delta} \subseteq \mathbb{P}_{a,b}^{n,\delta} = \{H(\theta, t, \cdot) - H(\theta^*, s, \cdot) \mid \theta \in \mathcal{U}_a, t, s \in \mathcal{T}_{a,b}^{n,\delta}\}\).

Fix \(\eta \in [0, 1]\) and \(\mathbb{Q} \in \mathcal{M}_a\) with support \(\text{supp}(\mathbb{Q})\). We define a new Borel probability measure \(\overline{\mathbb{Q}}\) on \(\mathbb{R}^d\) by \(\overline{\mathbb{Q}}(A) := \| (L_1 \vee 1) \cdot 1_A \|^2_{\mathbb{Q},2} / \| L_1 \vee 1 \|^2_{\mathbb{Q},2}\). It is absolutely continuous w.r.t. \(\mathbb{Q}\), and thus belongs also to \(\mathcal{M}_a\). Next, let \(\theta, \vartheta \in \mathcal{U}_a\) such that the inequality \(\| G(\theta, \cdot) - G(\vartheta, \cdot) \|^2_{\mathbb{Q},2} \leq \eta \| \xi_G^G \|^2_{\mathbb{Q},2} / 4\) holds, and let \(t_1, t_2, s_1, s_2 \in \mathcal{T}_{a,b}^{n,\delta}\) satisfy

\[ \square \]
the inequalities $|t_1 - t_2|, |s_1 - s_2| \leq \eta \left[ b + M_{b_1} \right] \delta^3/6$. Then by (A 4) we may observe

$$\| [H(\theta, t_1, \cdot) - H(\theta, t_2, \cdot)] - [H(\theta, t_2, \cdot) - H(\theta, s_1, \cdot)] \|_{Q_2}^2$$

$$\leq 4 \| H(\theta, t_1, \cdot) - H(\theta, t_2, \cdot) \|_{Q_2}^2 + 4 \| H(\theta, s_1, \cdot) - H(\theta, s_2, \cdot) \|_{Q_2}^2$$

$$= 4 \sum_{z \in \text{supp}(Q)} \left\{ L_1(z)^2 |G(\theta, z) - G(\theta, \bar{z})|^2 + L_2(z)^2 (|t_1 - t_2|^2 + |s_1 - s_2|^2) \right\} Q(\{z\})$$

$$\leq 4 \| L_1 \lor 1 \|_{Q_2}^2 \| G(\theta, \cdot) - G(\theta, \cdot) \|_{Q_2}^2 + 4 \| L_2 \|_{Q_2}^2 (|t_1 - t_2|^2 + |s_1 - s_2|^2)$$

$$\leq \delta^3/36 \| (L_1 \lor 1) \cdot \xi^G \|_{Q_2}^2 / 4 + 8 \eta^2 \| L_2 \|_{Q_2}^2 (b + M_{b_1})^2 \delta^3/36 \leq \eta^2 \| \xi^G \|_{Q_2}^2 .$$

Hence, putting all together, we may conclude

$$N(\eta \| \xi^G \|_{Q_2}^2, \mathbb{F}_{a,b}^\delta, L^2(Q)) \leq N(\eta \| \xi^G \|_{Q_2}^2, \mathbb{F}_{a,b}^\delta, L^2(Q))$$

$$\leq \sup_{Q \in \mathcal{M}_{\text{fin}}} N(\eta \| \xi^G \|_{Q_2}^2, \mathbb{F}_{a,b}^\delta, L^2(Q)) : N(\eta \| \xi^G \|_{Q_2}^2, \mathbb{F}_{a,b}^\delta, L^2(Q))$$

where $N(\eta \| \xi^G \|_{Q_2}^2, \mathbb{F}_{a,b}^\delta, L^2(Q))$ stands for the minimal number to cover $\mathcal{I}_{a,b}^\delta$ by intervals of the form $[t - \eta \| \xi^G \|_{Q_2}^2, t + \eta \| \xi^G \|_{Q_2}^2]$ with $t \in \mathcal{I}_{a,b}^\delta$. It may be easily checked that

$$N(\eta \| \xi^G \|_{Q_2}^2, \mathbb{F}_{a,b}^\delta, L^2(Q)) \leq 6 (\sup_{\mathcal{I}_{a,b}^\delta} - \inf_{\mathcal{I}_{a,b}^\delta}) =\frac{12 (a/\sqrt{n} + M_{b_1} \delta^3)}{\eta \| \xi^G \|_{Q_2}^2} \cdot \frac{1}{\delta^3} \leq \frac{1}{\eta \| \xi^G \|_{Q_2}^2} \cdot \frac{1}{\delta^3}$$

holds. Now, the upper estimate of the covering number $N(\eta \| \xi^G \|_{Q_2}^2, \mathbb{F}_{a,b}^\delta, L^2(Q))$ that is claimed in the statement may be derived immediately.

For the remaining part of the proof fix $\varepsilon \in ]0, 1]$. Note that $(x + s)/(y + s) \leq (x/y) \lor 1$ holds for $s, x, y > 0$. Hence

$$\frac{a/\sqrt{n} + M_{b_1} \delta^3}{(b + M_{b_1}) \delta^3} = \frac{a/\sqrt{n} \cdot \delta^3 + M_{b_1}}{b + M_{b_1}} \leq \frac{a}{b \sqrt{n} \delta^3} \lor 1.$$

Using the change of variable formula, and invoking (2.12), we end up with

$$J(\mathbb{F}_{a,b}^\delta, \xi^G, \varepsilon)$$

$$\leq \int_0^\varepsilon \sup_{Q \in \mathcal{M}_{\text{fin}}} \sqrt{\ln \left( 2 N(\eta \| \xi^G \|_{Q_2}^2, \mathbb{F}_{a,b}^\delta, L^2(Q)) \right)} \, d\eta$$

$$+ \int_0^\varepsilon \sqrt{2 \ln \left( 12 ([a/(b \sqrt{n} \delta^3)] \lor 1)/\eta \right)} \, d\eta$$

$$\leq 4 J(\mathbb{F}_{a,b}^\delta, \xi^G, \varepsilon/4) + \varepsilon \int_0^1 \sqrt{2 \ln \left( 12 ([a/(b \sqrt{n} \delta^3)] \lor 1)/\varepsilon \right)} \, du$$

$$\leq 4 J(\mathbb{F}_{a,b}^\delta, \xi^G, \varepsilon/4) + 2 \sqrt{2} \varepsilon \sqrt{\ln \left( 12 ([a/(b \sqrt{n} \delta^3)] \lor 1)/\varepsilon \right)}.$$
5.4 Proof of Proposition 2.8

By representation (PH) we have with
\[ G(\theta, z) - G(\theta^*, z) = \sum_{i=1}^{r} \left( [G^i(\theta, z) - G^i(\theta^*, z)] \cdot 1_{B_i(\theta)}(z) + G^i(\theta^*, z) \cdot [1_{B_i(\theta)}(z) - 1_{B_i(\theta^*)}(z)] \right) \]  \hspace{1cm} (5.12)

for \( \theta \in \Theta \) and \( z \in \mathbb{R}^d \). The processes \( \{G^i(\theta, \cdot)\}_{\theta \in \Theta} \) have Hölder continuous paths according to representation (PH). Hence by triangle inequality
\[ |G(\theta, z) - G(\theta^*, z)| \leq \sum_{i=1}^{r} \left( C^i(z) \|\theta - \theta^*\|_m^\beta_i + |G^i(\theta^*, z)| \cdot 1_{B_i(\theta)}\Delta_{B_i(\theta^*)}(z) \right) \]

for \( \theta \in \Theta \) and \( z \in \mathbb{R}^d \). Then it may be concluded easily that \( \xi^G_\delta \) is a positive envelope of \( \mathbb{P}^\Theta_{\delta} \) for \( \delta > 0 \).

In the next step we want to derive for \( Q \in \mathcal{M}_{fin} \) upper estimates of the covering numbers \( N(\eta, \|\xi^G_\delta\|_{Q, 2, \mathbb{P}^\Theta_{\delta}, L^2(Q)}) \) based on representation (5.12). For this purpose let us introduce the functions \( \overline{G}^i \), \( i = 1, \ldots, r \), \( \overline{G} : \Theta \times \mathbb{R}^d \to \mathbb{R} \) via
\[ \overline{G}^i(\theta, z) := G^i(\theta, z) - G^i(\theta^*) \quad \text{for} \quad i \in \{1, \ldots, r\} \quad \text{and} \quad \overline{G}(\theta, z) = \sum_{i=1}^{r} \overline{G}^i(\theta, z) \cdot 1_{B_i(\theta)}(z). \]

Note that \( \overline{G}^i(\theta^*, \cdot) \equiv 0 \) is square \( \mathbb{P}^Z \)-integrable and \( |\overline{G}^i(\theta, z) - \overline{G}^i(\vartheta, z)| \leq C^i(z)\|\theta - \vartheta\|_m \) for \( \theta, \vartheta \in \Theta \) as well as \( i \in \{1, \ldots, r\} \). Then we may apply directly from [12] Proposition 2.8 together with formulas (5.12) and (5.13) in its proof to the function classes \( \overline{C}^\Theta_{\delta} := \{\overline{G}(\theta, \cdot) \mid \theta \in U_\delta\} \) \( (\delta > 0) \). Hence for any \( \delta > 0 \) a positive square \( \mathbb{P}^Z \)-integrable envelope \( C^{\overline{C}^\Theta_{\delta}} \) of \( \overline{C}^\Theta_{\delta} \) is defined by \( C^{\overline{C}^\Theta_{\delta}} = 2 \sum_{i=1}^{r} \delta^\beta_i C^i(z) \), and
\[ \sup_{Q \in \mathcal{M}_{fin}} N(\eta, \|C^{\overline{C}^\Theta_{\delta}}\|_{Q, 2, \overline{C}^\Theta_{\delta}, L^2(Q)}) \leq \prod_{i=1}^{r} 9^{m_{(i)}} 16^{(m+2)s_i} e^{(1+(m+2)) s_i (\lceil mn + 2 \rceil s_i + 1)} (4/\eta)^{2(m+2)s_i + m/\beta_i} \]  \hspace{1cm} (5.13)

for \( \eta \in ]0, r[ \).

Next let us introduce the auxiliary function classes
\[ \mathbb{P}^{\overline{C}^\Theta_{\delta}, i} := \{G^i(\theta^*, \cdot) \cdot [1_{B_i(\theta)} - 1_{B_i(\theta^*)}] \mid \theta \in U_\delta\} \quad (i \in \{1, \ldots, r\}, \delta > 0). \]

Concerning upper estimations for the covering numbers of these classes we need some further preparation from the theory of empirical process theory. To recall, define for a collection \( \mathcal{B} \) of subsets of \( \mathbb{R}^d \), and \( z_1, \ldots, z_n \in \mathbb{R}^d \)
\[ \Delta_n(\mathcal{B}, z_1, \ldots, z_n) := \text{cardinality of} \ \{B \cap \{z_1, \ldots, z_n\} \mid B \in \mathcal{B}\} \].
Then
\[ V(\mathcal{B}) := \inf \left\{ n \in \mathbb{N} \mid \max_{z_1, \ldots, z_n \in \mathbb{R}^d} \Delta_n(\mathcal{B}, z_1, \ldots, z_n) < 2^n \right\} \quad (\inf \emptyset := \infty) \]
is known as the index of \( \mathcal{B} \) (see [17], p. 135). In case of finite index, \( \mathcal{B} \) is known as a so-called VC-class (see [17], p. 135). The concept of VC-classes may be carried over from sets to functions in the following way. A set \( \mathcal{F} \) of Borel measurable real-valued functions on \( \mathbb{R}^d \) is defined to be a VC-subgraph class or a VC-class if the corresponding collection \( \{(z, t) \in \mathbb{R}^d \times \mathbb{R} \mid h(z) > t \mid h \in \mathcal{F}\} \) of subgraphs is a VC-class ([17], p. 141). Its VC-index \( V(\mathcal{F}) \) coincides with the index of the subgraphs. The significance of VC-subgraph classes stems from the fact that for every VC-subgraph class \( \mathcal{F} \) and any \( \mathbb{P}^Z \)-integrable positive envelope \( C_{\mathcal{F}} \) of \( \mathcal{F} \)

\[
\sup_{Q \in M_{\text{fin}}} N(\eta \|C_{\mathcal{F}}\|_{Q, 2}, \mathcal{F}, L^2(Q)) \leq e V(\mathcal{F}) \left(4e^{1/2}/\eta\right)^{2V(\mathcal{F})+1} \quad \text{if } \eta \in ]0, 1[ \tag{5.14}
\]
(see e.g. [12], Corollary 5.3). Now, it is already known that the class \( \mathcal{F}_{\delta,i} := \{1_{B_i(\theta)} \mid \theta \in \mathcal{U}_\delta\} \) is a VC-subgraph class with \( V(\mathcal{F}_{\delta,i}) \leq (m + 2)s_i + 1 \) for \( i \in \{1, \ldots, r\}, \delta > 0 \) (see [12], Lemma 5.4). Then by permanence properties of VC-subgraph classes, namely [10] Lemma 9.9, (v) along with (vi), we obtain \( \mathcal{F}_{\delta,i}^\theta \) as a VC-subgraph class with index with index

\[
V(\mathcal{F}_{\delta,i}^\theta) \leq 2V(\mathcal{F}_{\delta,i}) - 1 \leq 2(m + 2)s_i + 1
\]
for \( i \in \{1, \ldots, r\} \) and \( \delta > 0 \). Since \( C_{\mathcal{F}_{\delta,i}^\theta} := |G^\theta(\cdot, t)| 1_{B_i} + (\delta \land 1)^2 \) defines a positive envelope of \( \mathcal{F}_{\delta,i}^\theta \) we end up with

\[
\sup_{Q \in M_{\text{fin}}} N(\eta \|C_{\mathcal{F}_{\delta,i}^\theta}\|_{Q, 2}, \mathcal{F}_{\delta,i}^\theta, L^2(Q)) \leq e \left[2(m + 2)s_i + 1 \right] \left(4e^{1/2}/\eta\right)^{4(m+2)s_i} \tag{5.15}
\]
for \( i \in \{1, \ldots, r\}, \delta > 0 \) and \( \eta \in ]0, 1[ \). Next, fix \( Q \in M_{\text{fin}}, \delta, \eta > 0 \). Let \( h^0, \overline{h}_i, \underline{h}_i \in \mathcal{F}_{\delta,i}^\theta \) such that \( \|h^i - \overline{h}_i\|_{Q, 2} \leq \eta\|C_{\mathcal{F}_{\delta,i}^\theta}\|_{Q, 2}/(2r + 2) \) for \( i = 1, \ldots, r \), and \( \|h^0 - \overline{h}\|_{Q, 2} \leq \eta\|C_{\mathcal{F}_{\delta,i}^\theta}\|_{Q, 2}/(2r + 2) \). Then by \( \sqrt{\sum_{i=0}^r t_i} \geq \sum_{i=0}^r \sqrt{t_i}/(r + 1) \) for \( t_0, \ldots, t_r \geq 0 \)

\[
\| \sum_{i=0}^r h^i - \sum_{i=0}^r \overline{h}_i \|_{Q, 2} \leq \sum_{i=0}^r \|h^i - \overline{h}_i\|_{Q, 2} \leq \frac{\eta}{2(r + 1)} \left[\|C_{\mathcal{F}_{\delta,i}^\theta}\|_{Q, 2} + \sum_{i=1}^r \|C_{\mathcal{F}_{\delta,i}^\theta}\|_{Q, 2}\right]
\]

\[
\leq \frac{\eta}{2} \|C_{\mathcal{F}_{\delta,i}^\theta}\|_{Q, 2} + \sum_{i=1}^r \|C_{\mathcal{F}_{\delta,i}^\theta}\|_{Q, 2} \leq \eta\|\xi_G\|_{Q, 2}.
\]

Thus in view of representation (5.12)

\[
N(\eta\|\xi_G\|_{Q, 2}, \mathcal{F}_{\delta,i}^\theta, L^2(Q)) \leq N(\eta\|C_{\mathcal{F}_{\delta,i}^\theta}\|_{Q, 2}/(2r + 2), \mathcal{F}_{\delta,i}^\theta, L^2(Q)) \cdot \prod_{i=1}^r N(\eta\|C_{\mathcal{F}_{\delta,i}^\theta}\|_{Q, 2}/(2r + 2), \mathcal{F}_{\delta,i}^\theta, L^2(Q)) \tag{5.16}
\]
for $\Omega \in \mathcal{M}_m$ and $\delta, \eta > 0$.

Combining (5.13) and (5.15) with (5.16), we obtain for $\delta > 0$ and $\varepsilon \in [0,1]$ by change of variable formula

$$J(\Theta^G, \delta, \varepsilon) = \varepsilon \int_0^1 \sup_{\Omega \in \mathcal{M}_m} \sqrt{\ln \left( 2N(\varepsilon \eta ||\xi^G||_{Q,2}, \Theta^G, L^2(\Omega)) \right)} \, d\eta$$

$$\leq \varepsilon \sum_{i=1}^2 \int_0^1 \sqrt{v_i \ln(K_i/\eta)} \, d\eta,$$

where

$$v_1 := 2(m+2) \sum_{i=1}^r s_i + m \sum_{i=1}^r 1/\beta_i \quad \text{and} \quad v_2 := 4(m+2) \sum_{i=1}^r s_i,$$

and

$$K_{1,\varepsilon} := \frac{8(r+1)}{\varepsilon} \left\{ 2 \cdot 9^{-m} 16^{(m+2) \sum_{i=1}^r s_i} e^{r+2(m+2) \sum_{i=1}^r s_i} \prod_{i=1}^r (2^m s_i) \right\}^{1/v_1},$$

$$K_{2,\varepsilon} := \frac{8(r+1)}{\varepsilon} \left\{ e^{r+2(m+2) \sum_{i=1}^r s_i} \prod_{i=1}^r (2^m s_i) \right\}^{1/v_2}.$$

Now, we may finish the proof of Proposition 2.8 via (2.12) by routine calculations. \(\square\)

5.5 Proof of Lemma 3.1

First of all, $\psi_{H,\Theta}$ is a continuous mapping w.r.t. the Euclidean norm due to Remark 2.1.

Now, let us assume that (A 7') holds with $\delta_3, M_{\delta_3} > 0$. By continuity of the goal function $\psi_{H,\Theta}$ along with compactness of $\Theta$, and since $\theta^*$ is the unique minimizer of $\psi_{H,\Theta}$, any minimizing sequence $(\theta_k)_{k \in \mathbb{N}}$ converges to $\theta^*$. This implies $M^H(\delta_3) > 0$.

For $\theta \in \Theta \setminus \mathcal{U}_{\delta_3}$ the definition of $M^H(\delta_3)$ implies

$$\psi_{H,\Theta}(\theta) - \psi_{H,\Theta}(\theta^*) \geq [M(\delta_3)/\|\theta - \theta^*\|_m^2] \|\theta - \theta^*\|_m \geq [M(\delta_3)/\Delta(\Theta)^2] \|\theta - \theta^*\|_m^2.$$

Now, we may conclude immediately from (A 7') the second order growth condition with $M_3 = M_{\delta_3} \wedge [M(\delta_3)/\Delta(\Theta)^2]$. This completes the proof. \(\square\)

5.6 Proof of Proposition 3.3

Let $n \in \mathbb{N}$, $\delta \in [0, \Delta(\Theta)]$, and $\varepsilon, \gamma \in [0, \infty[$. Fix any $\overline{\Omega} \in \mathcal{F}$. Firstly,

$$\left\{ n^\gamma \|\hat{\theta}_n - \theta^*\|_m > \varepsilon \right\} \subseteq \left\{ \|\hat{\theta}_n - \theta^*\|_m > \delta \right\} \cup \left\{ \{n^\gamma \|\hat{\theta}_n - \theta^*\|_m > \varepsilon \} \cap \{\|\hat{\theta}_n - \theta^*\|_m \leq \delta \} \right\} \quad (5.17)$$
By definition the realizations of $\hat{\theta}_n$ minimize the paths of the random function $M_n$ defined by (5.1). Then for any $\omega \in \Omega$

$$\psi_{H,\omega}(\hat{\theta}_n(\omega)) - \psi_{H,\omega}(\theta^*)$$

$$= M_n(\theta^*, \omega) - \psi_{H,\omega}(\theta^*)) - M_n(\hat{\theta}_n(\omega), \omega) + \psi_{H,\omega}(\hat{\theta}_n(\omega)) + M_n(\hat{\theta}_n(\omega), \omega) - M_n(\theta^*, \omega)$$

$$\leq 2 \sup_{\theta \in \Theta} |M_n(\theta, \omega) - \psi_{H,\omega}(\theta)|$$

$$\leq 2 \sup_{\theta \in \Theta} |(\mathbb{P}_n - \mathbb{P})(G^H(\theta, \cdot))|_{\mathbb{P}}$$

$$+ 2 \sup_{\theta \in \Theta} \frac{1}{n} \sum_{j=1}^{n} \left| H\left(\theta, \frac{1}{n} \sum_{k=1}^{n} G(\theta, Z_k(\omega)), Z_j(\omega)\right) - H(\theta, \mathbb{E}[G(\theta, Z_1)], Z_j(\omega)) \right|$$

$$\leq 2 \sup_{\theta \in \Theta} |(\mathbb{P}_n - \mathbb{P})(G^H(\theta, \cdot))|_{\mathbb{P}} + 2 \sup_{\theta \in \Theta} |(\mathbb{P}_n - \mathbb{P})(G(\theta, \cdot))|_{\mathbb{P}} \frac{1}{n} \sum_{j=1}^{n} L_2(Z_j(\omega)),$$

where in the last step we have invoked (A.4).

In case of $\|\hat{\theta}_n(\omega) - \theta^*\|_m \in [\delta, \Delta(\Theta)]$ the application of (A.7) leads to chain of inequalities

$$\delta^2 M_3 < M_3 \|\hat{\theta}_n(\omega) - \theta^*\|_m \leq \psi_{H,\omega}(\hat{\theta}_n(\omega)) - \psi_{H,\omega}(\theta^*)$$. Furthermore, if $\delta = \Delta(\Theta)$, then

$$\mathbb{P}(\{\|\hat{\theta}_n(\omega) - \theta^*\|_m > \delta\}) = 0$$. Hence, noting $\sum_{j=1}^{n} L_2(Z_j)/n = 0$ $\mathbb{P}$-a.s. if $\mathbb{E}[L_2(Z_1)] = 0$, we end up with

$$\mathbb{P}(\{\|\hat{\theta}_n(\omega) - \theta^*\|_m > \delta\} \cap \Omega)$$

$$\leq 1_{[0,\Delta(\Theta)]}(\delta) \cdot \mathbb{P}^*\left(\left\{ \sup_{\theta \in \Theta} \left| (\mathbb{P}_n - \mathbb{P})(G^H(\theta, \cdot)) \right| > \left(1 + 1_{(0)}(\mathbb{E}[L_2(Z_1)]) \right) \delta^2 M_3/4 \right\} \cap \Omega \right)$$

$$+ 1_{[0,\Delta(\Theta)]}(\delta) \cdot \mathbb{P}^*\left(\left\{ \sup_{\theta \in \Theta} \left| (\mathbb{P}_n - \mathbb{P})(G(\theta, \cdot)) \right| \sum_{j=1}^{n} L_2(Z_j)/n > \delta^2 M_3/4 \right\} \cap \Omega \right).$$

(5.18)

Now, let $\varepsilon/n^\gamma < \|\hat{\theta}_n(\omega) - \theta^*\|_m \leq \delta$. Then $2^{k-1}/n^\gamma < \|\hat{\theta}_n(\omega) - \theta^*\|_m \leq \delta \wedge (2^k/n^\gamma)$ for some $k \in \mathbb{Z}$ satisfying $k \geq K_\varepsilon + 1$ and $2^{k-1} \leq \delta n^\gamma$. Hence by (A.7)

$$\sup_{\theta \in \Theta_{m, (2^k/n^\gamma)}} \left| M_n(\theta, \omega) - M_n(\theta^*, \omega) - \psi_{H,\omega}(\theta) + \psi_{H,\omega}(\theta^*) \right|$$

$$\geq M_n(\theta^*, \omega) - M_n(\hat{\theta}_n(\omega), \omega) - \psi_{H,\omega}(\theta^*) + \psi_{H,\omega}(\hat{\theta}_n(\omega))$$

$$\geq \psi_{H,\omega}(\hat{\theta}_n(\omega)) - \psi_{H,\omega}(\theta^*) > M_3 2^{(k-1)/n^\gamma}. \quad (5.19)$$

Finally, note that $G_n^H(\theta, \cdot) = \sqrt{n}[M_n(\theta, \cdot) - \psi_{H,\omega}(\theta)]$ is valid for $\theta \in \Theta$. Thus, combining (5.17) with (5.18) and (5.19), we may derive the statement of Proposition 3.4 easily. \qed

5.7 Proof of Proposition 3.4

Let $n \in \mathbb{N}$ with $n \geq n(a, b, \beta)$, $\delta \in [0, \delta^*_n]$, and consider $k \in \mathbb{Z}$ such that $k \geq K_\varepsilon + 1$ as well as $\delta_{n(k-1)} \leq \delta$. 47
First of all
\[
\begin{align*}
\overline{M}_{\sqrt{T_b}} & \leq \sqrt{2}^k \overline{M}_b \quad \text{and} \quad M_{a,\sqrt{T_b}} \leq \sqrt{2}^k M_{a,b} \\
\overline{\eta}_n(a, \sqrt{2}^k b, \delta_{nk} \wedge \delta) & \leq \overline{\eta}(a, b, \varepsilon) 
\end{align*}
\tag{5.20}
\]
\[
\overline{M}^2_{\sqrt{T_b}} (\delta^*)^{2\beta} \leq \overline{M}_b^2 (\delta^*)^{2\beta} + \varepsilon n^{-1/2} \leq n(a, b, \beta)^{\frac{3}{2}-\frac{2\beta}{1}} n^{-\frac{1}{2}} \leq n. \tag{5.22}
\]

In particular by \( n \geq n(a, b, \beta) \)
\[
\overline{M}^2_{\sqrt{T_b}} (\delta^*)^{2\beta} \leq \overline{M}_b^2 (\delta^*)^{2\beta} \leq n(a, b, \beta)^{\frac{3}{2}-\frac{2\beta}{1}} n^{-\frac{1}{2}} \leq n. \tag{5.21}
\]

Then by definition of \( n(a, b, \beta) \) we may apply statement 1) in Theorem 2.2 which together with (5.20) and (5.21) yields
\[
\begin{align*}
\mathbb{P} \left\{ \sup_{\theta \in U_{\beta,a}} \left| G_n^H(\theta, \cdot) - G_n^H(\theta^*, \cdot) \right| > \frac{M_3}{\sqrt{n}} \right\} \leq \frac{4}{M_b} \overline{\eta}(a, \sqrt{2}^k b, \delta_{nk} \wedge \delta) + \mathbb{I}_{\|L_2(Z_1)\|} \cdot M_{a,\sqrt{T_b}} \\
\leq \frac{4}{M_b} \overline{\eta}(a, b, \varepsilon) + \mathbb{I}_{\|L_2(Z_1)\|} \cdot M_{a,b} \frac{M_3}{2^{(2-\beta)}} \frac{2^{(2-\beta)}-1}{2^{(2-\beta)}-1} \leq \frac{5}{2} \left( \frac{2}{\varepsilon} \right)^{3/2-\beta} \sqrt{\frac{2}{\varepsilon \wedge \frac{2}{2}}}. \tag{5.23}
\end{align*}
\]

Moreover, invoking well known formulas of geometric series, we obtain
\[
\sum_{k=K_1}^{\infty} \frac{1}{2^{k(2-\beta) - K_1/2}} \leq \frac{1}{2^{K_1(2-\beta) - K_1/2}} \left( \frac{2}{2^{2-\beta} - 1} \right) \leq \left( \frac{2}{\varepsilon} \right)^{3/2-\beta} \sqrt{\frac{2}{\varepsilon \wedge \frac{2}{2}}}. \tag{5.24}
\]

Combining (5.23) with (5.24), we may derive statement 1) easily.

Concerning statement 2) let \( n \in \mathbb{N} \) with \( n \geq n(a, b, \beta) \), and let \( t > 0 \) such that
\[
2^{K_1(2-\beta) - K_1/2} > 4 \left[ M_{a,b} \mathbb{I}_{\|L_2(Z_1)\|} \cdot \overline{\eta}_n(a, \sqrt{2}^k b, \delta_{nk} \wedge \delta) \right] / M_3.
\]

Furthermore let \( k \in \mathbb{Z} \) with \( k \geq K_1 + 1 \) and \( \delta_{n(k-1)} \leq \delta \). Then firstly by (5.22) the sample size \( n \) satisfies \( n \geq \max \left\{ \overline{M}^2_{\sqrt{T_b}} (\delta^*)^{2\beta} / 2, \mathbb{I}_{\|L_2(Z_1)\|} \cdot 9 a^2 / 2, \|\xi^G\|^2_{\mathbb{P}^2_{\mathbb{P}^2}} / 2 \right\} \), and in view of (5.20) along with (5.21)
\[
\frac{M_3}{2^{2(k-1)}} \left[ M_{a,\sqrt{T_b}} \mathbb{I}_{\|L_2(Z_1)\|} + \overline{\eta}_n(a, \sqrt{2}^k b, \delta_{nk} \wedge \delta) \right] \geq \frac{2^{3\beta}}{4} \left[ M_{a,b} \mathbb{I}_{\|L_2(Z_1)\|} + \overline{\eta}(a, b, \varepsilon) \right] > 2^{k\beta}.
\]

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Hence by the second part of statement 1) in Theorem 2.3 along with (5.20)

\[ \mathbb{P}\left\{ \sup_{\theta \in U_{\eta_{nk}^*}} |G_n^{G,H}(\theta, \cdot) - G_n^{G,H}(\theta^*, \cdot)| > \frac{M_3 2^{(k-1)}}{\eta^{(2-2\beta)}} \right\} \cap \Omega_{n,a} \cap \Omega'_{\delta_{nk}^*b, b} \]

\[ \leq \exp \left( \mathbb{I}_{[0, \infty]} \left( \mathbb{E}[L_2(Z_1)] \right) \right) \frac{M_{a, \sqrt{\mathcal{T}_n}} g(t)}{M_{\sqrt{\mathcal{T}_n}}} \exp \left( \frac{-2^{(2-2\beta)} M_3 g(t)}{4 M_{\sqrt{\mathcal{T}_n}}} \right) \]

\[ + \mathbb{P} \left( \Omega \setminus B_{n, \sqrt{\mathcal{T}_n}} \right) \]

\[ \leq \exp \left( \mathbb{I}_{[0, \infty]} \left( \mathbb{E}[L_2(Z_1)] \right) \right) \frac{\sqrt{2^{(K_{c+1})^*} M_{a,b} g(t)}}{b \|L_2\|_{L_2, 2}} \exp \left( \frac{-2^{(2-2\beta) - K^{+}/2} M_3 g(t)}{4 M_b} \right) \]

\[ + \mathbb{P} \left( \Omega \setminus B_{n, \sqrt{\mathcal{T}_n}} \right) \]

Invoking the change of variable formula, we further obtain

\[ \exp \left( \frac{-2^{(2-2\beta) - k^{+}/2} M_3 g(t)}{4 M_b} \right) \]

\[ \leq \int_{k-1}^{k} \exp \left( \frac{-2^{(2-2\beta) - k^{+}/2} M_3 g(t)}{4 M_b} \right) \, du \]

\[ = \begin{cases} \int_{2^{(2-2\beta) - k^{+}/2}}^{2^{(2-2\beta) - (2-\beta)}} \exp \left( \frac{-y M_3 g(t)}{4 M_b} \right) \frac{1}{(2-\beta) \ln(2)} \, dy , & k \leq 0 \\ \int_{2^{(2-2\beta) - (2-\beta)}}^{2^{(2-2\beta) - (3/2-\beta)}} \exp \left( \frac{-y M_3 g(t)}{4 M_b} \right) \frac{1}{(3/2-\beta) \ln(2)} \, dy , & k > 0 \end{cases} \]

\[ \leq \begin{cases} \frac{1}{(2-\beta) \ln(2)} 2^{K_s(2,2-\beta)} \int_{2^{(2-2\beta) - k^{+}/2}}^{2^{(2-2\beta) - (2-\beta)}} \exp \left( \frac{-y M_3 g(t)}{4 M_b} \right) \, dy , & k \leq 0 \\ \frac{2}{(3/2-\beta) \ln(2)} 2^{K_s(3/2-\beta)} \int_{2^{(2-2\beta) - (3/2-\beta)}}^{2^{(2-2\beta) - (3/2-\beta)}} \exp \left( \frac{-y M_3 g(t)}{4 M_b} \right) \, dy , & k > 0 \end{cases} \]

Note further that

\[ \frac{1}{(2-\beta) \ln(2)} 2^{K_s(2,2-\beta)} \sqrt{(3-2\beta) \ln(2)} 2^{K_s(3/2-\beta)} \leq \frac{2}{(3-2\beta) \ln(2)} 2^{K_s(2,2-\beta) - K^{+}/2} \]

holds. Hence

\[ \sum_{k=K_{c+1}}^{\infty} \exp \left( \frac{-2^{(2-2\beta) - k^{+}/2} M_3 g(t)}{4 M_b} \right) \]

\[ \leq \frac{2}{(3-2\beta) \ln(2)} 2^{K_s(2,2-\beta) - K^{+}/2} \int_{2^{K_s(2,2-\beta) - K^{+}/2}}^{\infty} \exp \left( \frac{-y M_3 g(t)}{4 M_b} \right) \, dy \]

\[ = \frac{4 M_b}{(3/2-\beta) \ln(2)} 2^{K_s(2,2-\beta) - K^{+}/2} \exp \left( \frac{-2^{K_s(2,2-\beta) - K^{+}/2} M_3 g(t)}{4 M_b} \right) \]

Putting (5.25) and (5.26) together we may conclude statement 2) easily. This completes the proof. \[ \square \]
5.8 Proof of Proposition 3.7

We want to apply Theorem 5.1 to the function class $\mathbb{F}^{\Theta, H} := \{G^H(\theta, \cdot) \mid \theta \in \Theta\}$, and we assume that $G^H(\theta, \cdot)$ is square $\mathbb{P}^Z$-integrable for some $\theta \in \Theta$. Note that all members of $\mathbb{F}^{\Theta, H}$ are Borel measurable due to Borel measurability of $H$. In the first step we shall investigate the uniform entropy integrals and the involved covering numbers.

Lemma 5.5 Let (A 1), (A 2) and (A 4) be fulfilled with $L_1, L_2$ from (A 4) and $\xi^G$ from (A 2). Then the mapping $\xi^{G^H}$ from the display of Proposition 3.7 is a square $\mathbb{P}^Z$-integrable positive envelope of $\mathbb{F}^{\Theta, H}$ with

$$\sup_{Q \in \mathcal{M}_{fin}} N(\eta \|\xi^{G^H}\|_{Q, 2}, \mathbb{F}^{\Theta, H}, L^2(Q)) \leq 2 \sup_{Q \in \mathcal{M}_{fin}} N(2 \eta \|\xi^G\|_{Q, 2}, \mathbb{F}^\Theta, L^2(Q))/\eta$$

for $\eta > 0$. In particular $J(\mathbb{F}^{\Theta, H}, \xi^{G^H}, 1/4) \leq J(\mathbb{F}^\Theta, \xi^G, 1/2)/2 + 1$.

Proof By (A 4)

$$|G^H(\theta, z)| \leq \sqrt{L_1(z)^2 |G(\theta, z) - G(\bar{\theta}, z)|^2 + L_2(z)^2 |\bar{\psi}(\theta) - \bar{\psi}(\theta)|^2 + |G^H(\bar{\theta}, z)|}$$

$$\leq \sqrt{4L_1(z)^2 \xi^G(z)^2 + 4L_2(z)^2 \mathbb{E}[\xi^G(\bar{z})^2] + |G^H(\bar{\theta}, z)|} \leq \xi^{G^H}(z)$$

for $\theta \in \Theta, z \in \mathbb{R}^d$. Hence $\xi^{G^H}$ is a positive envelope of $\mathbb{F}^{\Theta, H}$, and it is square $\mathbb{P}^Z$-integrable because $(L_1 \lor 1) \xi^G$ as well as $L_2$ and $G^H(\bar{\theta}, \cdot)$ fulfill this property.

Fix $\eta > 0$ and $Q \in \mathcal{M}_{fin}$ with support $\text{supp}(Q)$. We may define a Borel probability measure $\Theta$ on $\mathbb{R}^d$ by $\Theta(\bar{A}) := \|(L_1 \lor 1) \cdot 1_A\|_{Q, 2}/\|(L_1 \lor 1)\|_{Q, 2}$. It is absolutely continuous w.r.t. $Q$, and thus belongs also to $\mathcal{M}_{fin}$. Next, let $\theta, \vartheta \in \Theta$ and $t, s \in \{\bar{\psi}(\theta) \mid \theta \in \Theta\}$ such that the inequalities $\|G(\theta, \cdot) - G(\vartheta, \cdot)\|_{\mathcal{F}^\Theta, 2} \leq 2\eta \|\xi^G\|_{\mathcal{F}^\Theta, 2}$ and $|t - s| \leq 2\eta \|\xi^G\|_{\mathcal{F}^\Theta, 2}$ hold. Then by property (A 4) we may observe

$$\|H(\theta, t, \cdot) - H(\theta, s, \cdot)\|_{Q, 2}^2 \leq \sum_{z \in \text{supp}(Q)} \{L_1(z)^2 |G(\theta, z) - G(\vartheta, z)|^2 + L_2(z)^2 |t - s|^2 \} Q(\{z\})$$

$$\leq \|L_1 \lor 1\|_{Q, 2}^2 \|G(\theta, \cdot) - G(\vartheta, \cdot)\|_{\mathcal{F}^\Theta, 2}^2 + \|L_2\|_{Q, 2}^2 |t - s|^2$$

$$\leq 4\eta^2 \|L_1 \lor 1\|_{Q, 2}^2 \|\xi^G\|_{\mathcal{F}^\Theta, 2}^2 + 4\eta^2 \|L_2\|_{Q, 2}^2 \|\xi^G\|_{\mathcal{F}^\Theta, 2}^2$$

$$= 4\eta^2 \|(L_1 \lor 1) \cdot \xi^G\|_{Q, 2}^2 + 4\eta^2 \|L_2\|_{Q, 2}^2 \|\xi^G\|_{\mathcal{F}^\Theta, 2}^2 \leq \eta^2 \|\xi^{G^H}\|_{Q, 2}^2/4.$$

Hence

$$N(\eta \|\xi^{G^H}\|_{Q, 2}, \mathbb{F}^{\Theta, H}, L^2(Q)) \leq \sup_{Q \in \mathcal{M}_{fin}} N(2 \eta \|\xi^G\|_{Q, 2}, \mathbb{F}^\Theta, L^2(Q)) N(2\eta \|\xi^G\|_{\mathcal{F}^\Theta, 2}, \{\bar{\psi}(\theta) \mid \theta \in \Theta\}, \cdot)$$

$$\leq \sup_{Q \in \mathcal{M}_{fin}} N(2 \eta \|\xi^G\|_{Q, 2}, \mathbb{F}^\Theta, L^2(Q)) N(\eta \|\xi^G\|_{\mathcal{F}^\Theta, 2}, [-\|\xi^G\|_{\mathcal{F}^\Theta, 2}, \|\xi^G\|_{\mathcal{F}^\Theta, 2}], \cdot)$$

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where for a compact subset \( I \subseteq \mathbb{R} \) and a positive number \( \varepsilon \) the symbol \( N(\varepsilon, I, | \cdot |) \) stands for the minimal number to cover \( I \) by intervals of the form \([t - \varepsilon, t + \varepsilon] \) with \( t \in I \). It may be easily checked that \( N(\| \xi^G \|_{| \cdot |_2}, [\| \xi^G \|_{| \cdot |_2}, \| \xi^G \|_{| \cdot |_2}], | \cdot |) \leq 2/\eta \) holds. This shows the statement concerning the covering numbers of \( \mathcal{F}^G_H \). For the remaining part of the proof we may assume without loss of generality that \( J(\mathcal{F}^G_H, \xi^G, 1) \) is finite. Then invoking the change of variable formula along with (2.12), we may conclude

\[
J(\mathcal{F}^G_H, \xi^G_H, 1/4) \leq \frac{1}{2} \int_{0}^{1/2} \sup_{Q \in \mathcal{M}_{\text{fin}}} \sqrt{\ln \left( 2N(\varepsilon, | \xi^G \|_{| \cdot |_2}, \mathcal{F}^G_H, L^2(\mathbb{Q})) \right)} + \ln(4/|2\eta|) \, d\eta
\]

This completes the proof. \( \square \)

Now, we are ready to show Proposition 3.7.

**Proof of Proposition 3.7**

Firstly, drawing on Lemma 3.5.3 from [7] together with condition (A 5), the inequalities \( J(\mathcal{F}^G, \xi^G, 1/2) \leq 4 \, J(\mathcal{F}^G, \xi^G, 1/8) \leq 4 \, M^G \) and \( J(\mathcal{F}^G_H, \xi^G_H, 1/2) \leq 2 \, J(\mathcal{F}^G_H, \xi^G_H, 1/4) \) hold. In particular by Lemma 5.5

\[
16\sqrt{2} \, \| \xi^G_H \|_{| \cdot |_2} \, J(\mathcal{F}^G_H, \xi^G_H, 1/2) \leq 32\sqrt{2} \, \| \xi^G_H \|_{| \cdot |_2} \, (2M^G + 1),
\]

and

\[
\| \xi^G \|_{| \cdot |_2} \left[ 1 + 32\sqrt{2} \, (t + 1) \, J(\mathcal{F}^G_H, \xi^G_H, 1/4) \right] \leq \| \xi^G \|_{| \cdot |_2} \left[ 1 + 32\sqrt{2} \, (t + 1) \, (2M^G + 1) \right] \quad \text{for } t > 0.
\]

If we may achieve to verify conditions 1) - 3) in the display of Theorem 5.1 for the function class \( \mathcal{F}^G_H \), then the first inequality in Proposition 3.7 follows from Theorem 5.1 along with Markov’s inequality. Moreover, the second inequality in Proposition 3.7 may be concluded directly from Theorem 5.1.

Condition 1) follows from Borel measurability of \( H \) and condition 2) may be derived by assumptions (A 1), (A 2) together with Lemma 5.5. Finally, condition 3) is an easy consequence of assumption (A 3) along with (A 4), Remark 2.1 and Lemma 5.2. The proof is complete. \( \square \)

**5.9 Proof of Proposition 3.8**

The statement of Proposition 3.8 is obvious if \( \mathbb{E}[L_2(Z_1)] = 0 \) because in this case \( \sum_{j=1}^{n} L_2(Z_j)/n = 0 \) \( \mathbb{P} \)-a.s.. So let \( \mathbb{E}[L_2(Z_1)] \neq 0 \) which means \( \mathbb{E}[L_2(Z_1)] > 0 \).

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We have
\[ P \left( \left\{ \sup_{f \in \mathbb{P}^a} \left| \left( \mathbb{P}_n - \mathbb{P} \right) (f) \right| \leq \frac{1}{n} \sum_{j=1}^{n} L_2(Z_j) > \left[ \delta^2 M_3 / 4 \right] \cap B_n^{\mathbb{T}_2} \right\} \right) \leq P \left( \left\{ \sup_{f \in \mathbb{P}^a} \left| \left( \mathbb{P}_n - \mathbb{P} \right) (f) \right| > \left[ \delta^2 M_3 / \mathbb{E}[L_2(Z_1)] \right] \right\} \right). \]

Moreover, the mapping \( G \) meets the requirements of Theorem 5.1 due to assumptions (A 1) - (A 3) together with Lemma 5.2. In addition by Lemma 3.5.3 from \[ ] along with (A 5) we may observe \( J(\mathbb{P}^G, \xi^G, 1/4) \leq 2 J(\mathbb{P}^G, \xi^G, 1/8) \leq 2 M_3^G, \) and, as a further consequence, also \( J(\mathbb{P}^G, \xi^G, 1/2) \leq 2 J(\mathbb{P}^G, \xi^G, 1/4) \leq 4 M_3^G. \) Now, the proof may be finished immediately by applying Theorem 3.1 together with Markov’s inequality. \( \square \)

### 5.10 Proof of Lemma 4.5

In view of (4.13) the Borel measurable mapping \( C_{a,\mathbb{T},\delta}^\mathbb{P} \) may be verified easily as a positive envelope of \( \mathbb{P}_a^\mathbb{I}, \mathbb{I}, \delta. \) By construction it is square \( \mathbb{P}^Z \)-integrable because \( \xi_\delta \) fulfills this property.

Next, let \( Q \in \mathcal{M}_{\eta, \mathbb{I}}, \eta > 0 \) and \( \theta, \vartheta \in \mathcal{U}_\delta \) as well as \( x, y \in \mathcal{I}_\delta := \mathcal{I} \cap [x^{\alpha,*} - \delta, x^{\alpha,*} + \delta] \) with
\[
\|G(\theta, \cdot) - G(\vartheta, \cdot)\|_{Q,2} \leq \eta \|\xi_\delta\|_{Q,2}/4 \quad \text{and} \quad |x - y| \leq \eta \delta/4.
\]

Then by (4.13) again, we may observe
\[
\| \hat{G}_\alpha((\theta, x), \cdot) - \hat{G}_\alpha((\theta^{\alpha,*}, x^{\alpha,*}), \cdot) \|_{Q,2}^2 \leq 4 \|G(\theta, \cdot) - G(\vartheta, \cdot)\|_{Q,2}^2 + (2 - \alpha)^2 |x - y|^2 / (1 - \alpha)^2 \leq \eta^2 \left( \|\xi_\delta\|_{Q,2}^2 + (2 - \alpha)^2 \delta^2 \right) / [4 (1 - \alpha)^2] \leq \eta^2 \|C_{a,\mathbb{T},\delta}^\mathbb{P}\|_{Q,2,2}/4.
\]

Hence, denoting by \( N(\eta \delta/2, \mathcal{I}_\delta, \cdot, \cdot) \) the minimal number to cover the set \( \mathcal{I}_\delta \) by intervals \([x - \eta \delta/2, x + \eta \delta/2] \) with \( x \in \mathcal{I}_\delta \), we may conclude from the observation that \( \mathbb{P}^\mathbb{I}_a, \mathbb{I}, \delta \) is a subset of \( \left\{ \hat{G}_\alpha((\theta, x), \cdot) - \hat{G}_\alpha((\theta^{\alpha,*}, x^{\alpha,*}), \cdot) \mid (\theta, x) \in \mathcal{U}_\delta \times \mathcal{I}_\delta \right\} \)
\[
N(\eta \|C_{a,\mathbb{T},\delta}^\mathbb{P}\|_{Q,2,2}, \mathbb{P}_a^\mathbb{I}, L^2(\mathbb{Q})) \leq N(\eta \|\xi_\delta\|_{Q,2}/4, \mathbb{P}_\delta^\mathbb{I}, L^2(\mathbb{Q})) \cdot N(\eta \delta/4, \mathcal{I}_\delta, \cdot, \cdot) \leq N(\eta \|\xi_\delta\|_{Q,2}/4, \mathbb{P}_\delta^\mathbb{I}, L^2(\mathbb{Q})) \cdot 4 \left( \sup \mathcal{I}_\delta - \inf \mathcal{I}_\delta \right) / (\eta \delta) \leq N(\eta \|\xi_\delta\|_{Q,2}/4, \mathbb{P}_\delta^\mathbb{I}, L^2(\mathbb{Q})) \cdot 8 / \eta.
\]

Now, invoking subadditivity of \( \sqrt{\cdot} \) together with the change of variable formula, we end up with
\[
J(\mathbb{P}_\delta^\mathbb{I}, C_{a,\mathbb{T},\delta}^\mathbb{P}, \varepsilon) \leq 4 J(\mathbb{P}_\delta^\mathbb{I}, \xi_\delta, \varepsilon/4) + \varepsilon \int_0^1 \sqrt{\ln \left( 8 / (\varepsilon \eta) \right)} \, d\eta \quad \text{for} \ \varepsilon \in [0,1].
\]

The proof may be finished immediately by applying inequality (2.12). \( \square \)

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Declarations

Conflicts of interest: The author has no competing interests to declare that are relevant to the content of this article.

Data Availability Statement: No datasets were generated or analysed during the study.

Funding: The authors did not receive support from any organization for the submitted work.

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