REGULAR DYNAMICS FOR STOCHASTIC
FITZHUGH-NAGUMO SYSTEMS WITH ADDITIVE NOISE ON
THIN DOMAINS

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ABSTRACT. This paper is devoted to bi-spatial random attractors of the stochastic Fitzhugh-Nagumo equations with additive noise on thin domains when the terminate space is the Sobolev space. We first established the existence of random attractor on regular space and then show that the upper semi-continuity of these attractors under the Sobolev norm when a family of \((n+1)\)-dimensional thin domains degenerates onto an \(n\)-dimensional domain.

1. Introduction. In this paper, we investigate the following stochastic Fitzhugh-Nagumo equations with Neumann boundary conditions

\[
\begin{aligned}
\hat{u}^\epsilon & - \Delta \hat{u}^\epsilon dt + \lambda \hat{u}^\epsilon dt + \alpha \hat{v}^\epsilon dt = (f(t,x,\hat{u}^\epsilon) + G(t,x))dt + \phi(x)dW_1, \quad t \geq \tau, \\
\hat{v}^\epsilon & + \sigma \hat{v}^\epsilon dt - \beta \hat{u}^\epsilon dt = H(t,x)dt + \varphi(x)dW_2, \\
\partial \hat{u}^\epsilon / \partial \nu_\epsilon & = 0, \quad \partial \hat{v}^\epsilon / \partial \nu_\epsilon = 0, \quad \text{on } \partial \Omega_\epsilon, \\
\hat{u}^\epsilon(\tau, x) & = \hat{u}^\epsilon_\tau(x), \quad \hat{v}^\epsilon(\tau, x) = \hat{v}^\epsilon_\tau(x) \quad x \in \Omega_\epsilon, \tau \in \mathbb{R},
\end{aligned}
\]

where \(\alpha, \beta, \sigma, \lambda\) are all positive numbers, \(\nu_\epsilon\) is the unit outward normal vector on \(\partial \Omega_\epsilon\), and \(\Omega_\epsilon\) is given by

\[
\Omega_\epsilon = \{x = (x^*, x_{n+1}) | x^* = (x_1, \ldots, x_n) \in Q, \quad 0 < x_{n+1} < \epsilon g(x^*)\},
\]

where \(Q\) is a bounded smooth domain in \(\mathbb{R}^n\) and \(g \in C^2(\overline{\Omega}, (0, +\infty))\). Thus, there are \(\gamma_2 > \gamma_1 > 0\) such that

\[
\gamma_1 \leq g(x^*) \leq \gamma_2, \quad \text{for all } x^* \in \overline{Q}.
\]

The nonlinearity \(f\) and the body force \(G, H\) will be specified later.

The systematic research about the limiting behavior of deterministic dissipative equations defined on thin domains was introduced by J. K. Hale, G. Raugel [17, 18] and G. Raugel, G. R. Sell [40]. Later on, more thin domains models from different points of view have been extensively studied in the literature for deterministic cases, which can be found in [2, 5, 6, 7, 11, 38, 39] and the reference therein.

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Recently, L. Shi et al [41, 42] have studied the asymptotic behavior of stochastic FitzHugh-Nagumo systems on unbounded thin domains in the state space $L^2 \times L^2$. Moreover, D. Li et al [24, 25] and F. Li et al [28, 29] had investigated the random dynamics for the stochastic reaction-diffusion equations on thin domains in the state spaces $L^2$ and $L^p$, respectively. However, the existence and convergence of random attractor for stochastic FitzHugh-Nagumo systems in the Sobolev space $H^1$ are not involved yet. The regularity of random attractors was firstly studied by [32] and with considerable developments for a large number of stochastic equations, see [14, 30, 31, 50] and the literatures therein.

In this paper, our main purpose is to establish the existence of pullback random attractor $\mathcal{A}_\epsilon$ for systems (1) on state spaces $H^1 \times L^2$ and prove that $\mathcal{A}_\epsilon$ converges to the attractor $\mathcal{A}_0$ of the limiting equation on $\Omega$ under the topology of $H^1 \times L^2$. More precisely, as $\epsilon \to 0$, the $n + 1$-dimensional thin domain $\mathcal{O}_\epsilon$ collapses onto the $n$-dimensional domain $\Omega$. We consider the limiting equation defined on $\Omega$ as follows:

$$
\begin{cases}
    d\hat{u}^0 - \frac{1}{\epsilon} \sum_{i=1}^{n} (g\hat{u}^0)_{yy_i} dt + \lambda \hat{u}^0 dt + \alpha \tilde{v}^0 dt = (f_0(t, y^*, \hat{u}^0) + G_0(t, y^*)) dt \\
    + \phi_0 dW_1, \quad t \geq \tau,
    \\
    d\tilde{v}^0 + \sigma \tilde{v}^0 dt - \beta \tilde{v}^0 dt = H_0(t, y^*) dt + \varphi_0 dW_2,
    \\
    \partial \hat{u}^0 / \partial \nu_0 = 0, \quad \partial \tilde{v}^0 / \partial \nu_0 = 0, \quad \text{on } \partial \Omega,
    \\
    \hat{u}^0(\tau, \nu_0) = \hat{u}^0_\tau(\nu_0), \quad \tilde{v}^0(\tau, \nu_0) = \tilde{v}^0_\tau(\nu_0) \quad \nu_0 \in \mathcal{O}, \quad \tau \in \mathbb{R},
\end{cases}
$$

(3)

where

$$
f_0(t, y^*, \hat{u}^0) = f(t, (y^*, 0), \hat{u}^0), \quad G_0(t, y^*) = G(t, (y^*, 0)), \quad H_0(t, y^*) = H(t, (y^*, 0)),
\quad \phi_0(y^*, 0), \quad \varphi_0(y^*, 0) = \varphi(y^*, 0),
$$

and $\nu_0$ is the unit outward normal vector on $\partial \Omega$. Similarly, we can prove that systems (3) possesses a pullback random attractor $H^1(\Omega) \times L^2(\Omega)$. Then, we obtain our main result as follows (see Theorem 4.8):

$$
\lim_{\epsilon \to 0} \text{dist}_{H^1(\mathcal{O}) \times L^2(\mathcal{O})}(\mathcal{A}_\epsilon(\tau, \omega), \mathcal{A}_0(\tau, \omega)) = 0, \quad \forall \tau \in \mathbb{R}, \quad \omega \in \Omega.
$$

However, in such a regular space $H^1$, the usual method of symbolical truncation (see [8, 12]) and high-order integrability (see [9, 51, 52]) may not work. In fact, unlike the $L^p$ case mentioned above, a main difficulty lies in non-uniformity of equivalence between the usual Sobolev norm $H^1$ and the $H^1_{\nu}$ norm on thin domain, see Lemma 2.1. By Lemma 2.1, one can see that both norms $\| \cdot \|_{H^1}$ and $\| \cdot \|_{H^1_{\nu}}$ are equivalent for each $\epsilon \in (0, \epsilon_0]$. However, this equivalence is not uniform in $\epsilon \in (0, \epsilon_0]$ owing to the last term tends to infinity as $\epsilon \to 0$. As a consequence, the method of symbolical truncation cannot work when proving uniformity of $H^1$-asymptotic compactness of the random dynamical system as $\epsilon \to 0$.

Furthermore, due to the almost sure non-differentiability of sample paths of Wiener process $W$, we cannot get higher regularity of solutions by differentiating the equation with respect to $t$ as in the deterministic case. As a consequence, no higher regularity than $H^1$ is available so that we cannot obtain the strong compactness of system by compact embeddings.

In order to prove the upper semi-continuity of $\mathcal{A}_\epsilon$ as $\epsilon \to 0$ under the topology of $H^1 \times L^2$, we have to show that the convergence of eigenvalues and uniform flattening,
which yields to the uniform asymptotic compactness in the $H^1 \times L^2$. In the thin domain problem, the eigenvalue $\lambda^*_j$ may varying as $\epsilon \to 0$. In this case, the spectrum convergence theorem by Arrieta and Carvalho [4] will be well applied. 

It is worth pointing out that such thin domains problems for random dynamical systems is contrary to the expanding domains problems [26, 33] and different from time-varying domains [20, 44]. In particular, the upper semi-continuity on thin domains is also different from time-variable problem [13, 21, 35, 34, 46], delay problem [22, 23, 43, 47, 48] and Wong-Zakai approximations [16, 15, 37, 49].

This paper is arranged as follows. In the next section, we transfer the stochastic system (1) into a pathwise random system for a fixed domain $\mathcal{O}$ and define a continuous cocycle in $L^2(\mathcal{O}) \times L^2(\mathcal{O})$. In section 3, we derive uniform estimates of solutions and uniform flattening. In the last section, we prove the existence and upper semi-continuity of attractor for stochastic Fitzhugh-Nagumo equations on $H^1 \times L^2$.

2. Random dynamical system from equation.

2.1. Transformation of the thin domain. In this section, in order to define a random cocycle for the stochastic Fitzhugh-Nagumo equations on thin domains, we first give some normal assumptions.

Let $\mathcal{O} = Q \times (0, \gamma_2)$ and $\hat{\mathcal{O}} = Q \times [0, \gamma_2)$. It is easy to see that $\mathcal{O}_\epsilon \subset \mathcal{O} \subset \hat{\mathcal{O}}$ for each $\epsilon \in (0, 1]$.

**Assumption F.** The nonlinearity $f : \mathbb{R} \times \hat{\mathcal{O}} \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the following conditions: for all $x \in \hat{\mathcal{O}}$ and $t, s \in \mathbb{R}$,

\[
\begin{align*}
|f(t, x, s)| & \leq -\alpha_1 |s|^p + \psi_1(t, x), & \psi_1 & \in L^1_{\text{loc}}(\mathbb{R}, L^\infty(\hat{\mathcal{O}})) \cap L^2_{\text{loc}}(\mathbb{R}, L^\infty(\hat{\mathcal{O}})), \\
|f(t, x, s)| & \leq \alpha_2 |s|^{p-1} + \psi_2(t, x), & \psi_2 & \in L^2_{\text{loc}}(\mathbb{R}, L^\infty(\hat{\mathcal{O}})), \\
\frac{\partial f(t, x, s)}{\partial s} & \leq \beta_1, & \left| \frac{\partial f(t, x, s)}{\partial s} \right| & \leq \alpha_3 |s|^{p-2} + \psi_3(t, x), & \psi_3 & \in L^2_{\text{loc}}(\mathbb{R}, L^\infty(\hat{\mathcal{O}})), \\
\frac{\partial f(t, x, s)}{\partial x} & \leq \psi_4(t, x), & \psi_4 & \in L^2_{\text{loc}}(\mathbb{R}, L^\infty(\hat{\mathcal{O}})),
\end{align*}
\]

where $p > 2$, $\alpha_i, \beta_1 > 0$. Moreover, the restrictions $\psi_{j,0}$ of $\psi_j$ ($j = 1, 2, 3, 4$) on $Q \times \{0\}$ satisfy the same assumption in (4)-(7) with $Q$ instead of $\hat{\mathcal{O}}$.

**Assumption G.** The force $G, H : \mathbb{R} \times \hat{\mathcal{O}} \to \mathbb{R}$ is continuous such that

\[
G \in L^2_{\text{loc}}(\mathbb{R}, L^\infty(\hat{\mathcal{O}})) \text{ and } G_0 \in L^2_{\text{loc}}(\mathbb{R}, L^\infty(Q)), \\
H \in L^2_{\text{loc}}(\mathbb{R}, H^1(\hat{\mathcal{O}})) \text{ and } H_0 \in L^2_{\text{loc}}(\mathbb{R}, H^1(Q)), \\
\phi \in C^2(\overline{Q} \times [0, \gamma_2]) \text{ and } \varphi \in C^1(\overline{Q} \times [0, \gamma_2]).
\]

Let $\mathcal{O} = Q \times (0, 1)$. We consider a transformation $T_\epsilon : \mathcal{O}_\epsilon \to \mathcal{O}$ defined by

\[
(y^*, y_{n+1}) = T_\epsilon(x^*, x_{n+1}) = (x^*, \frac{x_{n+1} + g(x^*)}{\epsilon}) \text{ for all } x = (x^*, x_{n+1}) \in \mathcal{O}_\epsilon.
\]

One can show that $T_\epsilon$ is bijective with the Jacobian matrix:

\[
J = \begin{pmatrix}
\frac{\partial(y_1, \ldots, y_{n+1})}{\partial(x_1, \ldots, x_{n+1})} & 0
\end{pmatrix} = \begin{pmatrix}
I & 0 \\
-\frac{y_{n+1}}{g}(g_1, \ldots, g_n) & \frac{1}{\epsilon g(y^*)}
\end{pmatrix}
\]
where \( I \) is the \( n \)-dimensional unit matrix, and the determinant \(|J| = \frac{1}{\epsilon g(y) y^T} \). Then by \([18, 24]\), we have \( \nabla_x \tilde{u}(x) = J^* \nabla_y \tilde{u}(y) \) and
\[
\Delta_x \tilde{u}(x) = |J| \text{div}_y(|J|^{-1} J^* \nabla_y \tilde{u}(y)) = \frac{1}{g} \text{div}_y(\Upsilon \tilde{u}(y)),
\]
where we denote by \( \tilde{u}(y) = \tilde{u}(x) \ (y = Tx, x \in \Omega) \), \( J^* \) is the transport of \( J \) and \( \Upsilon \) is the operator given by
\[
\Upsilon \tilde{u}(y) = \begin{pmatrix}
g \tilde{u}_{y_1} - g_{y_1} y_{n+1} \tilde{u}_{y_{n+1}} \\
 \\
\vdots \\
g \tilde{u}_{y_n} - g_{y_n} y_{n+1} \tilde{u}_{y_{n+1}} + \frac{1}{\epsilon g} (1 + \sum_{i=1}^{n} (y_{n+1} g_{y_i})^2) \tilde{u}_{y_{n+1}}
\end{pmatrix}
\]
(11)

Then system (1) is equivalent to the following system defined on \( \Omega \):
\[
\begin{cases}
du^* - \frac{1}{g} \text{div}_y(\Upsilon \tilde{u^*}) dt + \lambda \tilde{u^*} dt + \alpha \tilde{u}^* = (f(t, y, \tilde{u}^*) + G(t, y)) dt + \phi(t, y) dW_1, \\
d\tilde{u}^* + \sigma \tilde{u}^* dt - \beta \tilde{u}^* dt = H(t, y) dt + \varphi(t, y) dW_2, \\
\Upsilon \tilde{u^*} \cdot \nu = 0, \ \Upsilon \tilde{v} \cdot \nu = 0, \text{ on } \partial \Omega, \\
\tilde{u}^*(\tau, y) = \hat{u}^* (T^{-1} \tau, y), \ \tilde{v}^*(\tau, y) = \hat{v}^* (T^{-1} \tau, y), \ y \in \Omega, \tau \in \mathbb{R}.
\end{cases}
\]
(12)

where \( \nu \) is the unit outward normal vector on \( \partial \Omega \), and
\[
G(t, y^*, y_{n+1}) = G(t, y^*, \epsilon g(y^*) y_{n+1}), \quad H(t, y^*, y_{n+1}) = H(t, y^*, \epsilon g(y^*) y_{n+1}),
\]
\[
\phi(t, y^*, y_{n+1}) = \phi(y^*, \epsilon g(y^*) y_{n+1}), \quad \varphi(t, y^*, y_{n+1}) = \varphi(y^*, \epsilon g(y^*) y_{n+1}),
\]
\[
f(t, y^*, y_{n+1}, u) = f(t, y^*, \epsilon g(y^*) y_{n+1}, u).
\]

In the sequel, we introduce the inner product \((\cdot, \cdot)_g\) and norm on \( H_g(\Omega) \) defined by
\[
(\tilde{u}, \tilde{v})_g = \int_{\Omega} g \tilde{u} \tilde{v} dy, \quad \|	ilde{u}\|_g^2 = \int_{\Omega} g \tilde{u}^2 dy, \text{ for all } \tilde{u}, \tilde{v} \in L^2(\Omega).
\]

Obviously, the original \( L^2 \)-norm denoted by \( \| \cdot \| \) is equivalent to the new norms \( \| \cdot \|_g \). Also, we take state space defined by
\[
\|	ilde{u}\|_p^p = \int_{\Omega} g |\tilde{u}|^p dy, \text{ for all } \tilde{u} \in L^p(\Omega).
\]

Furthermore, for \( 0 < \epsilon \leq 1 \), we introduce a bilinear form \( a_\epsilon(\cdot, \cdot) : H^1(\Omega) \times H^1(\Omega) \rightarrow \mathbb{R} \) and new norms defined by
\[
a_\epsilon(\tilde{u}, \tilde{v}) = (J^* \nabla_y \tilde{u}, J^* \nabla_y \tilde{v})_g, \text{ and } \|	ilde{u}\|_{H^1_\epsilon}^2 = a_\epsilon(\tilde{u}, \tilde{u}) + \|	ilde{u}\|_g^2,
\]
for \( \tilde{u}, \tilde{v} \in H^1(\Omega) \).

In this case, we have the following norm equivalence lemma:

**Lemma 2.1.** [29] There exist \( \epsilon_0 \in (0, 1) \) and \( \eta_1, \eta_2 > 0 \) such that, for all \( \epsilon \in (0, \epsilon_0] \),
\[
\eta_1 \|	ilde{u}\|_{H^1_\epsilon}^2 \leq \eta_1 (\|	ilde{u}\|_{H^1_\epsilon}^2 + \frac{\|	ilde{u}_{y_{n+1}}\|}{\epsilon^2}) \leq \|	ilde{u}\|_{H^1_\epsilon}^2 \leq \eta_2 (\|	ilde{u}\|_{H^1_\epsilon}^2 + \frac{\|	ilde{u}_{y_{n+1}}\|}{\epsilon^2})
\]
(13)

Denote by \( A_\epsilon \) the unbounded operator on \( H_g(\Omega) \) with domain \( D(A_\epsilon) = \{ \tilde{u} \in H^2(\Omega); \ \Upsilon \tilde{u} \cdot \nu = 0 \text{ on } \partial \Omega \} \), given by
\[
A_\epsilon \tilde{u} = -\frac{1}{g} \text{div}_y(\Upsilon \tilde{u}), \text{ and so } (A_\epsilon \tilde{u}, \tilde{v})_g = a_\epsilon(\tilde{u}, \tilde{v}), \text{ for } \tilde{u} \in D(A_\epsilon), \tilde{v} \in H^1(\Omega).
Hence, the system (12) can rewrite as follows:

\[
\begin{aligned}
\frac{du^\epsilon}{dt} + A_\epsilon u^\epsilon + \lambda u^\epsilon + \alpha v^\epsilon &= f_\epsilon(t, y, u^\epsilon) + G_\epsilon(t, y) + \phi_\epsilon \frac{dW_1}{dt}, \\
\frac{dv^\epsilon}{dt} + \sigma v^\epsilon - \beta u^\epsilon &= H_\epsilon(t, y) + \varphi_\epsilon \frac{dW_2}{dt}, \\
\end{aligned}
\]

\(\omega\) is the Winner measure. We denote a group \(\omega\) defined by

\(\sum\) that if \(\omega\) has a unique solution \(\omega(\cdot) := (\omega_1(\cdot), \omega_2(\cdot))\) on the probability space \((\Omega, \mathcal{F}, P)\), where

\[\Omega = \{\omega = (\omega_1, \omega_2) \in C(\mathbb{R}, \mathbb{R}^2) : \omega_1(0) = \omega_2(0) = 0\},\]

with the usual Fréchet topology, \(\mathcal{F} = \mathfrak{B}(\Omega)\) is the Borel algebra and \(P\) is the Winner measure. We denote a group \(\{\theta_t : t \in \mathbb{R}\}\) of self-mappings on \(\Omega\) by \(\theta_t \omega(\cdot) = \omega(t + \cdot) - \omega(t)\) for \((\omega, t) \in \Omega \times \mathbb{R}\).

It is well known that \(t \mapsto z(\theta_t \omega)\) is continuous, where \(z(\omega) := (z_1(\omega_1), z_2(\omega_2))\) is defined by

\[z_1(\omega_1) = -\lambda \int_0^\infty e^{\lambda s} \omega_1(s) ds, \quad z_2(\omega_2) = -\sigma \int_{-\infty}^0 e^{\sigma \tau} \omega_2(s) ds\]

Let \((\tilde{u}, \tilde{v})\) satisfy problem (14) and introduce a new variable \((u, v)\) by

\[
\begin{aligned}
u(t, \tau, \omega, u_\tau) &= \tilde{u}(t, \tau, \omega, \tilde{u}_\tau) + \phi_{\epsilon} z_1(\theta_t \omega_1), \\
v(t, \tau, \omega, v_\tau) &= \tilde{v}(t, \tau, \omega, \tilde{v}_\tau) - \phi_{\epsilon} z_2(\theta_t \omega_2)
\end{aligned}
\]

where \(u_\tau = \tilde{u}_\tau + \phi_{\epsilon} z_1(\theta_t \omega_1), v_\tau = \tilde{v}_\tau + \varphi_{\epsilon} z_2(\theta_t \omega_2)\). Then, the equation (14) can be translated into a random equation:

\[
\begin{aligned}
\frac{du^\epsilon}{dt} + A_\epsilon u^\epsilon + \lambda u^\epsilon + \alpha v^\epsilon &= f_\epsilon(t, y, u^\epsilon + \phi_{\epsilon} z_1(\theta_t \omega_1)) + G_\epsilon(t, y) - A_\epsilon \phi_{\epsilon} z_1(\theta_t \omega_1) + \alpha \varphi_{\epsilon} z_2(\theta_t \omega_2), \\
\frac{dv^\epsilon}{dt} + \sigma v^\epsilon - \beta u^\epsilon &= H_\epsilon(t, y) + \beta \phi_{\epsilon} z_1(\theta_t \omega_1), \\
u^\epsilon(t, \tau, \omega, u_\tau) &= u_\tau, \nu^\epsilon(t, \tau, \omega, v_\tau) = v_\tau, \quad y \in \mathcal{O}, t \geq \tau.
\end{aligned}
\]

By employing Galerkin method, we can show that if \(f_\epsilon\) satisfies assumptions (4)-(7) the problem (16) has a unique solution.

**Lemma 2.2.** For any \(\tau \in \mathbb{R}, \omega \in \Omega, (u_\tau, v_\tau) \in L^2(\mathcal{O}) \times L^2(\mathcal{O})\) and \(\epsilon \in (0, \epsilon_0)\), problem (16) has a unique solution

\[(u^\epsilon, v^\epsilon) \in C([\tau, \infty), L^2(\mathcal{O}) \times L^2(\mathcal{O})) \cap L^2(\mathcal{O}, t) \in L^2(\mathcal{O}) \times L^2(\mathcal{O}))\]

for every \(T > 0\). Moreover, this solution continuously depends on \((u_\tau, v_\tau) \in L^2(\mathcal{O}) \times L^2(\mathcal{O})\) and \(t \geq \tau\).

Hereafter, we will denote the norm in \(L^\infty(\mathcal{O})\) (resp. \(H^1(\mathcal{O}) \times L^2(\mathcal{O})\)) by \(\| \cdot \|_{\infty}\) (resp. \(\| \cdot \|_{\mathcal{O}}, \| \cdot \|\)) for simplicity.

Now, we can define a family of mappings \(\Phi_\epsilon : \mathbb{R}_+ \times \mathbb{R} \times \Omega \times (L^2(\mathcal{O}) \times L^2(\mathcal{O})) \rightarrow (L^2(\mathcal{O}) \times L^2(\mathcal{O}))\) such that for every \(t \in \mathbb{R}_+, \tau \in \mathbb{R}, \omega \in \Omega\) and \((\tilde{u}_\tau, \tilde{v}_\tau) \in L^2(\mathcal{O}) \times L^2(\mathcal{O})\).
\[ L^2(\mathcal{O}) \]
\[
\Phi_\epsilon(t, \tau, \omega, (\tilde{u}_\tau, \tilde{v}_\tau)) = (\tilde{u}^\epsilon(t + \tau, \tau, \theta_{-\tau} \omega, \tilde{u}_\tau), \tilde{v}^\epsilon(t + \tau, \tau, \theta_{-\tau} \omega, \tilde{v}_\tau))
\]
\[
= \left( u^\epsilon(t + \tau, \tau, \theta_{-\tau} \omega, u_\tau) + \phi_\epsilon z_1(\theta_{i} \omega_1), v^\epsilon(t + \tau, \tau, \theta_{-\tau} \omega, v_\tau) + \varphi_\epsilon z_2(\theta_{i} \omega_2) \right) \quad (18)
\]

where \( u_\tau = \tilde{u}_\tau - \phi_\epsilon z_1(\omega_1), v_\tau = \tilde{v}_\tau - \varphi_\epsilon z_2(\omega_2) \). One can check that \( \Phi_\epsilon \) given by (18) is a continuous random cocycle (see [45]) on \( L^2(\mathcal{O}) \times L^2(\mathcal{O}) \) over \( (\Omega, \mathcal{F}, P, \{\theta_i\}_{i\in \mathbb{R}}) \).

Meanwhile, we consider the limiting equation (3) on \( \mathcal{Q} \), denote an operator \( A_0 \) by \( D(A_0) = \{ u \in H^2(\mathcal{Q}), \frac{\partial u}{\partial n} = 0 \text{ on } \partial \mathcal{Q} \} \), and for \( u \in D(A_0) \)
\[
A_0 u = -\frac{1}{g} \sum_{i=1}^{n} (g u, y_i), \quad (A_0 u, v) = \int_{\mathcal{Q}} g \nabla u \cdot \nabla v dy^*.
\]

Similar to procedure in (15), we also have
\[
\begin{align*}
\frac{du^0}{dt} + A_0 u^0 + \lambda u^0 + \alpha v^0 &= f_0(t, y^*, u^0 + \phi_0 z_1(\theta_{i} \omega_1)) + G_0(t, y^*) \\
&\quad - A_0 \phi_0(y^*) z_1(\theta_{i} \omega_1) - \alpha \varphi_0(y^*) z_2(\theta_{i} \omega_2), \\
\frac{dv^0}{dt} + \sigma v^0 - \beta u^0 &= H_0(t, y^*) + \beta \phi_0(y^*) z_1(\theta_{i} \omega_1), \\
u^0(\tau, \tau, \omega, u_\tau) &= u_\tau, v^0(\tau, \tau, \omega, v_\tau) = v_\tau, \quad y \in \mathcal{Q}, t \geq \tau.
\end{align*}
\]
and the solution determines a continuous random cocycle \( \Phi_0(t, \tau, \omega, (u^0_\tau, u^0_\tau)) \) on \( L^2(\mathcal{Q}) \times L^2(\mathcal{Q}) \).

In order to prove the existence of pullback random attractors, we take some universes \( \mathcal{D}_i, i = \epsilon, 0, 1 \), which are consisted of all set-valued mappings \( \mathcal{D}_i : \mathbb{R} \times \Omega \to 2^X, \setminus \{\emptyset\} \) satisfying,
\[
\lim_{t \to +\infty} e^{-\delta t} \| \mathcal{D}_i(\tau - t, \theta_{-i} \omega) \|_{X_i}^2 = 0, \quad \tau \in \mathbb{R}, \omega \in \Omega,
\]
where \( \delta \) is a positive constant given by
\[
\delta = \min\{\lambda, \sigma\}, \quad \lambda \text{ and } \sigma \text{ are as in } (1),
\]
and \( \| \mathcal{D} \| \) denotes supremum of norms for all elements, \( X_\epsilon = L^2(\mathcal{O}_\epsilon) \times L^2(\mathcal{O}_\epsilon), X_0 = L^2(\mathcal{Q}) \times L^2(\mathcal{Q}), X_1 = L^2(\mathcal{O}) \times L^2(\mathcal{O}) \).

**Assumption T.** The above functions satisfy some tempered conditions:
\[
\int_{-\infty}^{\tau} e^{s \delta} (\|G(s)\|_{X_\epsilon}^2 + \|H(s)\|_{X_\epsilon}^2 + \|\psi_1(s)\|_{X_\epsilon}^2 + \|\psi_2(s)\|_{X_\epsilon}^2 + \|\psi_4(s)\|_{X_\epsilon}^2) ds < \infty,
\]
\[
(21)
\]
for any \( \tau \in \mathbb{R} \), where we use \( \| \cdot \|_{X_\epsilon} \) to denote the norm in \( L^\infty(\mathcal{O}) \) throughout this paper. Also, we assume that the restrictions \( G_0, H_0, \psi_{1,0}, \psi_{2,0}, \psi_{4,0} \) defined on \( \mathcal{Q} \) satisfy the same conditions as given in (21).

### 3. Random attractors in regular space.
In this section, we will prove the uniform absorption for the \( p \)-norm of \( u^\epsilon \), which yields to the existence and upper semi-continuous of attractors in \( H^1(\mathcal{O}) \times L^2(\mathcal{O}) \). Hereafter, we drop the superscript \( \epsilon \) for convenience when there is no ambiguity and always use \( c \) to denote intrinsic constant and intrinsic random variable respectively, which will be changed everywhere.
3.1. Uniform estimates in \( p \)-times Lebesgue space. Firstly, by generalizing Lemma 3.1 and Lemma 3.2 in [41] slightly, we also have the following uniform estimates. The proof is standard and so omitted.

\[
\sup_{s \in [\tau - 3, \tau]} \left( \| u' (s, \tau - t, \theta_{t-\tau} \omega, u_0) \|^2_2 + \| v' (s, \tau - t, \theta_{t-\tau} \omega, v_0) \|^2_2 \right) \leq c \rho_1 (\tau, \omega), \tag{22}
\]

\[
\int_{\tau - 9}^{\tau} \| u (r, \tau - t, \theta_{t-\tau} \omega, u_0) \|^2_{H^1} + \| u (r, \tau - t, \theta_{t-\tau} \omega, u_0) \|^2_p \\, dr + \| v (r, \tau - t, \theta_{t-\tau} \omega, v_0) \|^2_2 \, dr \leq c \rho_1 (\tau, \omega). \tag{23}
\]

where \( \rho_1 \) is given by

\[
\rho_1 (\tau, \omega) = M_0 + M_0 \int_{-\infty}^{0} e^{\delta s} (|z_1 (\theta_{s} \omega_1)|^p + |z_2 (\theta_{s} \omega_2)|^p) \, ds
\]

\[
+ M_0 \int_{-\infty}^{0} e^{\delta s} (\| G (s + \tau) \|^2_\infty + \| H (s + \tau) \|^2_\infty + \| \psi_1 (s + \tau) \|^2_\infty + \| \psi_2 (s + \tau) \|^2_\infty) \, ds.
\]

and \( M_0 \) is a positive constant depending on \( \lambda, \sigma, \alpha, \beta \) but independent of \( \tau, \omega, D_1 \).

Note that the solution \((u', v')\) of system (16) has two components, and we only establish the regularity of \( u' \) in \( H^1 (\mathcal{O}) \). Since \( v' \) has no uniform estimates in \( H^1 (\mathcal{O}) \), therefore, the compactness of Sobolev embeddings in bounded domains cannot be used directly to obtain the asymptotic compactness of the solution. To overcome this difficult, we decompose the \( v' \) into two parts such that one part pullback approaches zero and the other part has the regularity in \( H^1 (\mathcal{O}) \). Let \( v' = v'_1 + v'_2 \) and

\[
\frac{dv'_1}{dt} + \sigma v'_1 = 0, \quad v'_1 (\tau) = v' (\tau), \tag{24}
\]

and

\[
\frac{dv'_2}{dt} + \sigma v'_2 = \beta u' + H (t, y') + \beta \phi, z_1 (\theta_{\tau} \omega_1), \quad v'_2 (\tau) = 0. \tag{25}
\]

By (24), we find that for \( t \in \mathbb{R}^+, \tau \in \mathbb{R}, \) and \( \omega \in \Omega \),

\[
\| v'_1 (\tau - t, \theta_{\tau-\tau} \omega, v_{t-\tau}) \| = e^{-2\rho t} \| v_{t-\tau} \|^2_{L^2 (\mathcal{O})} \to 0, \text{ as } t \to \infty \tag{26}
\]

In order to obtain the uniform estimates of \( v'_2 \) in \( H^1 (\mathcal{O}) \), we further assumption that:

\[
\int_{-\infty}^{0} e^{\frac{1}{2} \delta s} \| H (s + \tau) \|^2_{H^1 (\mathcal{O})} \, ds < +\infty. \tag{27}
\]

The following lemma can also be found in [41]:

**Lemma 3.1.** Suppose that Assumption \( G, F, T \) and (27) hold true. For any \( D_1 \in D_1, \tau \in \mathbb{R} \) and \( \omega \in \Omega \), there exist \( 0 < \epsilon < \epsilon_0 \) and \( T = T (D_1, \tau, \omega) \geq 1 \) such that for all \( t \geq T, (u_0, v_0) \in D_1 (\tau - t, \theta - \omega) \) and the solution \( v'_2 \) of equation (25) satisfies, respectively,

\[
\| u' (\tau, \tau - t, \theta_{t-\tau} \omega, u_0) \|_{H^1 (\mathcal{O})} \leq \rho_2 (\tau, \omega), \tag{28}
\]

\[
\| v'_2 (\tau, \tau - t, \theta_{t-\tau} \omega, v_0) \|_{H^1 (\mathcal{O})} \leq \rho_3 (\tau, \omega), \tag{29}
\]
where \( \rho_2, \rho_3 \) is given by

\[
\rho_2(\tau, \omega) = c\rho_1(\tau, \omega) + M_0 \int_{-\infty}^{0} e^{\delta s} \|\psi_4(s + \tau)\|^2_{\infty} ds,
\]

\[
\rho_3(\tau, \omega) = c\rho_2(\tau, \omega) + M_0 \int_{-\infty}^{\tau} e^{\delta s} \|H(s + \tau)\|^2_{H_1(\Omega)} ds.
\]

Then, we introduce three Gronwall-type inequalities, which will be used frequently in this paper.

**Lemma 3.2.** [36] Let \( y, y_1 \) and \( y_2 \) be nonnegative, locally integrable such that \( \frac{dy}{ds} \) is locally integrable and

\[
\frac{dy}{ds} + ay(s) + y_1(s) \leq y_2(s), \quad s \in \mathbb{R},
\]

where \( a > 0 \). If \( \tau \in \mathbb{R} \) and \( \mu > 0 \), then

\[
y(\tau) \leq \frac{1}{\mu} \int_{\tau-\mu}^{\tau} e^{a(\tau-s)} y(s) ds + \int_{\tau-\mu}^{\tau} e^{a(\tau-s)} y_2(s) ds,
\]

\[
\int_{\tau-\mu}^{\tau} y_1(s) ds \leq \frac{e^{-a\mu}}{\mu} \int_{\tau-\mu}^{\tau} y(s) ds + \int_{\tau-\mu}^{\tau} y_2(s) ds,
\]

\[
\sup_{s \in [\tau-\mu, \tau]} y(s) \leq \frac{e^{-a\mu}}{\mu} \int_{\tau-\mu}^{\tau} y(s) ds + \int_{\tau-\mu}^{\tau} y_2(s) ds.
\]

**Lemma 3.3.** Suppose that Assumption \( G, F, T \) hold true. For any \( D_1 \in D_1, \tau \in \mathbb{R} \) and \( \omega \in \Omega \), there exist \( T = T(D_1, \tau, \omega) \geq 9 \) such that

\[
\sup_{s \in [\tau-3, \tau]} \|u'(s, \tau - t, \theta_{-\tau} \omega, u_0)\|_{p}^p + \int_{\tau-3}^{\tau} \|u'(r, \tau - t, \theta_{-\tau} \omega, u_0)\|_{2p-2}^{2p-2} dr \leq R_1(\tau, \omega),
\]

\[
\sup_{s \in [\tau-1, \tau]} \|u'(s, \tau - t, \theta_{-\tau} \omega, u_0)\|_{H_1} \leq cR_1(\tau, \omega),
\]

whenever \( (u_0, v_0) \in D_1(\tau - t, \theta_{-\tau} \omega) \), where \( c > 0 \) and \( R_1(\tau, \omega) \) is a tempered variable given by

\[
R_1(\tau, \omega) = c \int_{-\infty}^{0} (\|\psi_1(\tau + r)\|_{\infty} + \|\psi_1(\tau + r)\|^2_{\infty} + \|\psi_2(\tau + r)\|_{\infty}^2 + \|G(\tau + r)\|_{\infty}^2) dr
\]

\[
+ c \int_{-\infty}^{0} (|z_1(\theta_{4} \omega)| + |z_1(\theta_{4} \omega)|^2 + |z_2(\theta_{5} \omega)|^2) dr + c\rho_1(\tau, \omega).
\]

**Proof.** Taking the inner product of the first equation of (16) with \( g|u|^{p-2}u \) in \( H_g(\Omega) \), we see that

\[
\frac{1}{p} \frac{d}{ds} \|u\|_{p}^p + \lambda \|u\|_{p}^p + \int_{\Omega} gA_x u \cdot |u|^{p-2} u dy = (f_x(s, y, \hat{u}), |u|^{p-2} u)_g - \alpha(v, |u|^{p-2} u)_g
\]

\[
+ (G_x(s, y), |u|^{p-2} u)_g - (A_x \phi_x z_1(\theta_{4} \omega), |u|^{p-2} u)_g - \alpha(\phi_x z_2(\theta_{5} \omega), |u|^{p-2} u)_g.
\]
We claim that the Laplace term is non-negativity. Indeed,

\[
\int_{\mathcal{O}} g_A u \cdot |u|^{p-2} u dy = -\frac{1}{\epsilon} \int_{\mathcal{O}} \Delta_x \tilde{u} |\tilde{u}|^{p-2} \tilde{u} dx = \frac{1}{\epsilon} \int_{\mathcal{O}} \nabla_x \tilde{u} \cdot \nabla_x (|\tilde{u}|^{p-2} \tilde{u}) dx \\
= \frac{p-2}{\epsilon} \int_{\mathcal{O}} \nabla_x \tilde{u} \cdot |\tilde{u}|^{p-4} |\tilde{u}|^2 \nabla_x \tilde{u} dx + \frac{1}{\epsilon} \int_{\mathcal{O}} \nabla_x \tilde{u} \cdot |\tilde{u}|^{p-2} \nabla_x \tilde{u} dx \\
= \frac{p-1}{\epsilon} \int_{\mathcal{O}} |\tilde{u}|^{p-2} |\nabla_x \tilde{u}|^2 dx \geq 0. \tag{36}
\]

We now use condition (4) and (5) to estimates the nonlinearity term in (35) as follows,

\[
f_{\epsilon}(s, y, \tilde{u}) u = f(s, y^*, \epsilon g(y^*) y_{n+1}, \tilde{u}) - f(s, y^*, \epsilon g(y^*) y_{n+1}, \tilde{u}) \phi_\epsilon z_1(\theta_y \omega) \\
\leq -\alpha_1 |\tilde{u}|^p + \psi_1(s, y^*, \epsilon g(y^*) y_{n+1}) \\
+ (\alpha_2 |\tilde{u}|^{p-1} + |\psi_2(s, y^*, \epsilon g(y^*) y_{n+1})|) |\phi_\epsilon(y) z_1(\theta_y \omega_1)| \\
\leq -\frac{\alpha_1}{2p+1} |u|^p + |\psi_1(s, y^*, \epsilon g(y^*) y_{n+1})| + c |\phi_\epsilon(y) z_1(\theta_y \omega_1)|^p \\
+ |\psi_2(s, y^*, \epsilon g(y^*) y_{n+1})| |\phi_\epsilon(y) z_1(\theta_y \omega_1)|,
\]

which implies that

\[
\int_{\mathcal{O}} g f_{\epsilon}(s, y, \tilde{u}) u |u|^{p-2} dy \\
\leq -\frac{\alpha_1}{2p+1} \int_{\mathcal{O}} g |u|^{2p-2} dy + \int_{\mathcal{O}} g |\psi_1(s, y^*, \epsilon g(y^*) y_{n+1})| |u|^{p-2} dy \\
+ c \int_{\mathcal{O}} g (|\phi_\epsilon(y) z_1(\theta_y \omega_1)|^p + |\psi_2(s, y^*, \epsilon g(y^*) y_{n+1})| |\phi_\epsilon(y) z_1(\theta_y \omega_1)|) |u|^{p-2} dy \tag{37}
\]

By the Young inequality \(ab^{p-2} \leq \nu b^{2p-2} + C(\nu) a^{2-\frac{2}{p}}\), we have

\[
c |\psi_1(s, y^*, \epsilon g(y^*) y_{n+1})| |u|^{p-2} \leq \frac{\alpha_1}{2p+4} |u|^{2p-2} + c |\psi_1(s)|^{2-\frac{2}{p}} \\
\leq \frac{\alpha_1}{2p+4} |u|^{2p-2} + c (|\psi_1(s, y^*, \epsilon g(y^*) y_{n+1})|^1 + |\psi_1(s, y^*, \epsilon g(y^*) y_{n+1})|^2).
\]

Similarly, by \(\phi \in C^2(\overline{\mathcal{O}} \times [0, \gamma_2])\) and so \(\phi \in L^\infty(\overline{\mathcal{O}})\),

\[
c |\phi_\epsilon z_1(\theta_y \omega_1)|^p |u|^{p-2} \leq \frac{\alpha_1}{2p+4} |u|^{2p-2} + c (|z_1(\theta_y \omega_1)|^p + |z_1(\theta_y \omega_1)|^{2p}).
\]

By the generalized Young inequality: \(abc \leq \nu a^{\frac{2p-2}{p}} + C(\nu) b^{2} + C(\nu) c^{2p-2}\), we have

\[
|u|^{p-2} (|\psi_2(s, y^*, \epsilon g(y^*) y_{n+1})| |\phi_\epsilon z_1(\theta_y \omega_1)|) \\
\leq \frac{\alpha_1}{2p+4} |u|^{2p-2} + c |\psi_2(s, y^*, \epsilon g(y^*) y_{n+1})|^2 + c z_1^{2p-2}(\theta_y \omega_1).
\]

All above estimates imply that

\[
\int_{\mathcal{O}} g f_{\epsilon}(s, y, \tilde{u}) u |u|^{p-2} dy \leq -\frac{\alpha_1}{2p+2} \|u\|^{2p-2}_{2p-2} \\
+ c (\|\psi_1(s)\|_\infty + \|\psi_1(s)\|^2_{\infty} + \|\psi_2(s)\|^2_{\infty}) + c (|z_1(\theta_y \omega_1)| + |z_1(\theta_y \omega_1)|^{2p}), \tag{38}
\]
On the other hand, by Young inequality, we have
\[
\int_{\overline{\Omega}} g_{r}(x, y)|u|^p dy - \alpha \int_{\overline{\Omega}} g|u|^p dy - \int_{\overline{\Omega}} g z_1(\theta, \omega_1) A_r \phi_r \cdot |u|^2 dy
\]
\[
\leq \frac{\alpha_1}{2p+2} \|u\|_{2p-2}^2 + c(\|v\|_g^2 + \|G(s)\|_\infty^2 + |z_1(\theta, \omega_1)|^2 + |z_2(\theta, \omega_2)|^2).
\]
(39)
Then, we substitute (36)-(39) into (35) to find for all \(s \in [\tau - 9, \tau]\),
\[
\frac{d}{ds} \|u\|_p^2 + \frac{\alpha_1}{2p+2} \|u\|_{2p-2}^2 \leq c\left( \|v\|_g^2 + \|G(s)\|_\infty^2 + \|v_1(s)\|_\infty^2 + \|v_1(s)\|_\infty^2 + \|v_2(s)\|_\infty^2 \right)
\]
\[
+ c(|z_1(\theta_0, \omega_1)| + |z_1(\theta_0, \omega_1)|^{2p} + |z_2(\theta_0, \omega_2)|^{2p}).
\]
(40)
By using the Gronwall-type inequality (31) and (32) with \(\mu = 3, a = 0\) on above and replacing \(\omega\) by \(\theta_0, \omega\), we have
\[
\sup_{s \in [\tau - 3, \tau]} \|u(s, \tau - t, \theta_0, \omega, u_0)\|_p^2 + \int_{\tau - 3}^\tau \|u(r, \tau - t, \theta_0, \omega, u_0)\|_{2p-2}^2 dr
\]
\[
\leq c \int_{\tau - 9}^\tau \|u(r, \tau - t, \theta_0, \omega, u_0)\|_p^2 + \|v(r, \tau - t, \theta_0, \omega, v_0)\|_p^2 dr
\]
\[
+ c \int_{\tau - 9}^\tau \|v_1(r)\|_\infty^2 + \|v_1(r)\|_\infty^2 + \|v_2(r)\|_\infty^2 + \|G(r)\|_\infty^2 dr
\]
\[
+ c \int_{\tau - 9}^\tau \|z_1(\theta_0, \omega_1) + |z_1(\theta_0, \omega_1)|^{2p} + |z_2(\theta_0, \omega_2)|^{2p}) dr
\]
\[
\leq c \int_{\tau - 9}^\tau \|v_1(\tau + r)\|_\infty^2 + \|v_1(\tau + r)\|_\infty^2 + \|v_2(\tau + r)\|_\infty^2 + \|G(\tau + r)\|_\infty^2 dr
\]
\[
+ c \int_{\tau - 9}^\tau \|z_1(\theta_0, \omega_1) + |z_1(\theta_0, \omega_1)|^{2p} + |z_2(\theta_0, \omega_2)|^{2p}) dr + cp_1(\tau, \omega)
\]
\[
= : R_1(\tau, \omega) < +\infty,
\]
(41)
where we have used (23). This proves (33).
To prove (34), we take the inner product of the first equation of (16) with \(u_s\) in \(H_g(\Omega)\) for \(s \in [\tau - 3, \tau]\),
\[
\|u_s\|_g^2 + \frac{1}{2} \frac{d}{ds} \langle \lambda \|u\|_g^2 + a_c(u, u) + \alpha(v, u_s) \rangle
\]
\[
= \langle f_r(s, y, \tilde{u}), u_s \rangle + \langle G_r(s, y), u_s \rangle - \langle A_r \phi_r z_1(\theta_0, \omega_1), u_s \rangle - \alpha(\phi_r z_2(\theta_0, \omega_2), u_s) \rangle.
\]
(42)
For the nonlinearity, by (5), we have
\[
|\langle f_r(s, y, \tilde{u}), u_s \rangle| \leq \frac{1}{8} \|u_s\|_g^2 + c(\|\tilde{u}\|_g^2 + \|\psi_2(s)\|_g^2)
\]
\[
\leq \frac{1}{8} \|u_s\|_g^2 + c(\|u\|_g^2 + c(\|\psi_2(s)\|_g^2 + c|z_1(\theta_0, \omega_1)|^{2p-2}).
\]
By the Young inequality, the last three terms are bounded by
\[
\langle G_r(s, y), u_s \rangle - \langle A_r \phi_r z_1(\theta_0, \omega_1), u_s \rangle - \alpha(\phi_r z_2(\theta_0, \omega_2), u_s) \rangle
\]
\[
\leq \frac{1}{8} \|u_s\|_g^2 + c(\|G(s)\|_\infty^2 + c|z_1(\theta_0, \omega_1)|^2 + |z_2(\theta_0, \omega_2)|^2
\]
\[
+ c|z_1(\theta_0, \omega_1)|^{2p} + |z_2(\theta_0, \omega_2)|^{2p}).
\]
We also have $|\alpha(v, u_0)| \leq \frac{1}{4} \|u_0\|^2 + \|v\|^2$. Hence, we get the energy inequality: for all $s \in [\tau - 3, \tau]$,

$$
\frac{d}{ds} \left( \lambda \|u\|^2 + a_\epsilon(u, u) + \|u_0\|^2 \right) \leq c(\|u\|_{L^2}^{2^p-2} + \|v\|^{2p}_2 + \|\psi_2(s)\|_{L^\infty}^2 + \|G(s)\|_{L^2}^2) 
+ c(|z_1(\theta, \omega_1)|^2 + |z_1(\theta, \omega_1)|_{L^2}^{2p} + |z_2(\theta, \omega_2)|^2).
$$

By using the Gronwall-type inequality (31) and (32) with $\mu = 1$ on above and replacing $\omega$ by $\theta - \tau \omega$, we find from (23) and (41) that

$$
\sup_{s \in [\tau - 1, \tau]} \|u(s, \tau - t, \theta - \tau \omega, u_0)\|_{L^2}^2 + \int_{\tau - 1}^\tau \|u(r, \tau - t, \theta - \tau \omega, u_0)\|_{L^2}^2 dr
\leq c\left( \int_{\tau - 3}^\tau \|u(r)\|_{L^2}^2 + \|u(r)\|_{L^{2^p}}^2 + \|v(r)\|_{L^2}^2 dr \right) + \int_{\tau - 3}^0 \|\psi_2(r + \tau)\|_{L^\infty}^2 + \|G(r + \tau)\|_{L^2}^2 dr 
+ c\int_{\tau - 3}^0 (|z_1(\theta, \omega_1)| + |z_1(\theta, \omega_1)|_{L^2}^{2p} + |z_2(\theta, \omega_2)|^2) dr
\leq cR_1(\tau, \omega).
$$

This complete the whole proof. \qed

**Lemma 3.4.** Suppose that Assumption G, F, T hold true. Let $\tau \in \mathbb{R}, \omega \in \Omega$ and $D_1 \in \mathcal{D}_1$ be fixed. Then for each $\epsilon > 0$, there exist a large $M$ and a large $T = T(\tau, \omega, D_1, \epsilon)$ independent of $\epsilon \in (0, \epsilon_0]$ such that

$$
\sup_{\epsilon \in (0, \epsilon_0]} \sup_{t \geq T} \int_{\tau - \mu}^\tau \int_{O_M} (u^\epsilon - M)^{p-1} |u^\epsilon(s, \tau - t, \theta - \tau \omega, u_0)|^p dx ds \leq \epsilon, \quad (43)
$$

$$
\sup_{\epsilon \in (0, \epsilon_0]} \sup_{t \geq T} \int_{\tau - \mu}^\tau \int_{O_{-M}} (u^\epsilon + M)^{p-1} |u^\epsilon(s, \tau - t, \theta - \tau \omega, u_0)|^p dx ds \leq \epsilon, \quad (44)
$$

whenever $(u_0, v_0) \in D_1(\tau - \mu, \theta - \tau \omega)$, where $w_+, w_- $ denotes the positive and negative truncations respectively, which is defined by

$$
w_+ := \max\{w, 0\}, \quad w_- := \max\{-w, 0\},
$$

and

$$
O_M = O_M^+(s, \tau - t) = \{ y \in O; u^\epsilon(s, \tau - t, \theta - \tau \omega, u_0)(y) \geq M \},
$$

$$
O_{-M} = O_M^-(s, \tau - t) = \{ y \in O; u^\epsilon(s, \tau - t, \theta - \tau \omega, u_0)(y) \leq -M \}.
$$

**Proof.** Firstly, we show that

$$
\lim_{M \to \infty} \sup_{s \in [\tau - 3, \tau]} \sup_{\epsilon \in (0, \epsilon_0]} \sup_{t \geq T} |O_M^+(s, \tau - t, u_0)| = 0, \quad (45)
$$

where the Lebesgue measure $|O_M|$ decreases as $M$ increases. Indeed, by Lemma 3.3, we have

$$
|O_M^+(s, \tau - t, u_0)|M^p \leq \int_{O_M} |u^\epsilon(s, \tau - t)|^p dy \leq \int_{O} |u^\epsilon(s, \tau - t)|^p dy \leq R_1(\tau, \omega),
$$

Note that $R_1$ is independent of $\epsilon \in (0, \epsilon_0]$ and $t \geq T$. Letting $M \to +\infty$ in the above inequality yields (45).
In addition, by the continuity of \( s \to z_1(\theta s \omega_1) \), there exist \( M_1 = M_1(\tau, \omega) > 0 \) such that

\[
\sup_{s \in [-1,0]} |z_1(s, \theta s \omega_1)|||\phi||_{L^\infty(O \times [0,\tau_2])} = M_1 < +\infty.
\]

By the condition (4), we can take \( M_2 > 0 \) such that

\[
f(s, x, \tilde{u}) \leq -\alpha_1 \tilde{u}^{p-1} + \psi_1(s, x)\tilde{u}^{-1}, \text{ if } \tilde{u} > M_2.
\]

(46)

Suppose that \( M \) is large enough such that \( M \geq M_1 + M_2 + 1 \), and take the inner product of first equation of (16) with \( g(u - M)^{p-1}_+ \) in \( L^2(O) \). The result is

\[
\frac{1}{p} \frac{d}{ds} \|u - M\|_p^p + \lambda(u, (u - M)^{p-1}_+) g + (A_s u, (u - M)^{p-1}_+) g + \alpha(v, (u - M)^{p-1}_+) g
\]

\[
= (f(s, y, \tilde{u}), (u - M)^{p-1}_+) g + (G_s(s, y), (u - M)^{p-1}_+) g
\]

\[
- (A_s \phi \kappa z_1(\theta s \omega_1), (u - M)^{p-1}_+) g - \alpha(\phi \kappa z_2(\theta s \omega_2), (u - M)^{p-1}_+) g.
\]

(47)

for all \( s \in [\tau - 1, \tau] \). After some simple calculations, one can prove that

\[
(A_s u, (u - M)^{p-1}_+) g \geq 0, \quad \lambda \int_O gu(u - M)^{p-1}_+ dy \geq \lambda \|u - M\|_p^p.
\]

(48)

If \( u \geq M \), then \( \tilde{u} = u + \phi \kappa(y) z_1(\theta s \omega_1) \geq u - |\phi \kappa(y) z_1(\theta s \omega_1)| \geq u - M_1 \geq M_2 \). Then, it follows (46), we get

\[
f(s, x, \tilde{u}) \leq -\alpha_1 \tilde{u}^{p-1} + \psi_1(s, x)\tilde{u}^{-1}
\]

\[
\leq -\frac{\alpha_1}{2p} u^{p-1} + |\psi_1(s, x)|u^{-1} + c|\phi \kappa z_1(\theta s \omega_1)|^{p-1},
\]

which implies the nonlinear term is equal to

\[
\int_{\mathcal{O}_M} g f(s, y^*, \epsilon g(y^*) y_{n+1, \tilde{u}})(u - M)^{p-1}_+ dy
\]

\[
\leq -\frac{\alpha_1}{2p} \int_{\mathcal{O}_M} gu(u - M)^{p-1}_+ dy + \int_{\mathcal{O}_M} g|\psi_1(s)|(u - M)^{p-2}_+ dy
\]

\[
+ \int_{\mathcal{O}_M} |\phi \kappa z_1(\theta s \omega_1)|^{p-1}(u - M)^{p-1}_+ dy
\]

\[
\leq -\frac{\alpha_1}{2p+1} \int_{\mathcal{O}_M} gu(u - M)^{p-1}_+ dy + c \int_{\mathcal{O}_M} |\psi_1(s)|^{2-\frac{2}{p}} dy
\]

\[
+ c \int_{\mathcal{O}_M} |\phi \kappa z_1(\theta s \omega_1)|^{2p-2} dy
\]

\[
\leq -\frac{\alpha_1}{2p+1} \int_{\mathcal{O}_M} gu(u - M)^{p-1}_+ dy + c \int_{\mathcal{O}_M} |\psi_1(s)|^4 + |\psi_1(s)|^2 dy
\]

\[
+ c|\mathcal{O}_M| z_1(\theta s \omega_1)^{2p-2},
\]

(49)
where \( \psi_1(s) = \psi_1(s, y^*, e g(y^*)) y_{n+1} \). Similarly, we obtain
\[
(G_\tau(s, y), (u - M)^{p-1})_g - (A_\tau \phi_1(\theta_s \omega_1), (u - M)^{p-1})_g \\
- \alpha(\phi_2(z_2(\theta_s \omega_2), (u - M)^{p-1})_g \\
\leq \frac{\alpha_1}{2^{p+3}} \int_{\Omega^e} g u^{p-1}(u - M)^{p-1} \, dy + c \int_{\Omega^e} G^2(s, y) \, dy \\
+ c(|z_1(\theta_s \omega_1)|^2 + |z_1(\theta_s \omega_1)|^{2p} + |z_2(\theta_s \omega_2)|^2)|\Omega^e|, 
\]
(50)
\[
\alpha(v, (u - M)^{p-1})_g = \alpha \int_{\Omega^e} g v(u - M)^{p-1} \, dy \\
\leq \frac{\alpha_1}{2^{p+3}} \int_{\Omega^e} g u^{p-1}(u - M)^{p-1} \, dy + \int_{\Omega^e} g v^2 \, dy. 
\]
(51)

It follows from (47) to (51) that
\[
\frac{d}{ds} \|(u - M)_+\|_{p}^p + \frac{\alpha_1}{2^{p+2}} \int_{\Omega^e} u^{p-1}(u - M)^{p-1} \, dy \\
\leq c \int_{\Omega^e} g v^2 \, dy + c \left( \int_{\Omega^e} |\psi_1(s)|^1 + |\psi_1(s)|^2 \, dy + \int_{\Omega^e} G^2(s, y) \, dy \right) \\
+ c(|z_1(\theta_s \omega_1)| + |z_1(\theta_s \omega_1)|^{2p} + |z_2(\theta_s \omega_2)|^2)|\Omega^e|. 
\]
(52)

Notice that
\[
\int_{\Omega^e} u^{p-1}(u - M)^{p-1} \, dy \geq \int_{\Omega^e} u^{p-2}(u - M)^{p} \, dy \geq M^{p-2} \|(u - M)_+\|_{p}^p, 
\]
then, (52) can be rewritten as follows:
\[
\frac{d}{ds} \|(u - M)_+\|_{p}^p + c M^{p-2} \|(u - M)_+\|_{p}^p + c \int_{\Omega^e} |u|^{p-1}(u - M)^{p-1} \, dy \\
\leq c \int_{\Omega^e} g v^2 \, dy + c \left( \int_{\Omega^e} |\psi_1(s)|^1 + |\psi_1(s)|^2 \, dy + \int_{\Omega^e} G^2(s, y) \, dy \right) \\
+ c(|z_1(\theta_s \omega_1)| + |z_1(\theta_s \omega_1)|^{2p} + |z_2(\theta_s \omega_2)|^2)|\Omega^e|. 
\]
(53)

Applying Gronwall-type inequality (31) in Lemma 3.2 with \( \mu \leq \frac{1}{3}, a = c M^{p-2} \) on (53), we have
\[
c \int_{\tau - \mu}^{\tau} \int_{\Omega^e} |u|^{p-1}(u - M)^{p-1} \, dy \\
\leq \frac{1}{\mu} e^{-c M^{p-2} \mu} \int_{\tau - 3\mu}^{\tau} \|(u(r) - M)_+\|_{p}^p \, dr + C \int_{\tau - 3\mu}^{\tau} \int_{\Omega^e} v^2(r, \tau - t) \, dy \, dr \\
+ C \int_{-3\mu}^{0} \int_{\Omega^e} G^2(r + \tau, y) + |\psi_1(r + \tau)|^1 + |\psi_1(r + \tau)|^2 \, dy \, dr \\
+ C |\Omega^e| \int_{-3\mu}^{0} (|z_1(\theta_{r+\tau} \omega_1)| + |z_1(\theta_{r+\tau} \omega_1)|^{2p} + |z_2(\theta_{r+\tau} \omega_2)|^2) \, dr.
\]

By (22), for all \( s \in [\tau - 1, \tau] \) and \( t \geq T \) with some \( T > 0 \), we can choose a small \( \mu \leq \frac{1}{3} \) such that for all \( t \geq T, M \geq 0 \),
\[
C \int_{\tau - 3\mu}^{\tau} \int_{\Omega^e} v^2 \, dy \, dr \leq C \int_{\tau - 3\mu}^{\tau} \|v(r)\|^2 \, dr \leq 3\mu C p_1(\tau, \omega) < \varepsilon. 
\]
(54)
Furthermore, by (33), \( \|u'(s, \tau-t)\|^p_p \leq R_1(\tau, \omega) \) for all \( s \in [\tau-1, \tau] \) and \( t \geq T \) with some \( T > 0 \), we take the number \( \mu \) given in (54) and choose \( M \) large enough, then we have for all \( t \geq T \),

\[
\frac{1}{\mu} e^{-cM^{p-2} \mu} \int_{\tau-3\mu}^{T} \|(u(r) - M)_+\|_p^p dr \leq \frac{1}{\mu} e^{-cM^{p-2} \mu} \int_{\tau-3\mu}^{T} \|u(r)\|_p^p dr
\]

\[
\leq \frac{1}{\mu} e^{-cM^{p-2} \mu} \int_{\tau-3\mu}^{T} R_1(\tau, \omega) dr = 3R_1(\tau, \omega)e^{-cM^{p-2} \mu} < \varepsilon. \tag{55}
\]

Finally, by using the Lebesgue theorem, we have

\[
C \int_{-3\mu}^{0} \int_{\mathcal{O}_M} G^2(r + \tau, y) + |\psi_1(r + \tau)|^2 + |\psi_2(r + \tau)|^2 dy dr < C \varepsilon, \tag{56}
\]

\[
C|\mathcal{O}_M| \int_{-3\mu}^{0} (|z_1(\theta_r + \tau_\omega)| + |z_1(\theta_r + \tau + \omega)|^2 + |z_2(\theta_r + \tau + \omega)|^2) dr < C \varepsilon, \tag{57}
\]

if \( \mu \leq \frac{1}{3} \) and \( M \) large enough, which completes the proof of (43). The proof of (44) is similar.

3.2. Convergence of eigenvalues and uniform flattening. In order to prove the \( \mathcal{D}_1 \)-pullback asymptotic compactness in \( H^1(\mathcal{O}) \) is uniform when \( \varepsilon \) is small. To end this, we need to use a convergence result of the spectrum given by Arrieta and Carvalho [4].

In the sequel, we consider the eigenvalue problem for the Laplace operator \(-\Delta^\varepsilon_x\) with the Neumann boundary condition on \( \partial \mathcal{O}_c \):

\[
-\Delta^\varepsilon_x \hat{u}^\varepsilon(x) = \lambda^\varepsilon \hat{u}^\varepsilon(x), \quad x \in \mathcal{O}_c, \quad \frac{\partial \hat{u}^\varepsilon}{\partial \nu_e} = 0, \quad \text{on} \ \partial \mathcal{O}_c. \tag{58}
\]

Let \( V^\varepsilon = \{ \hat{u}^\varepsilon \in H^1(\mathcal{O}_c) : \frac{\partial \hat{u}^\varepsilon}{\partial \nu_e} = 0 \text{ on } \partial \mathcal{O}_c \} \). Then \(-\Delta^\varepsilon_x\) is an unbounded positive operator in \( L^2(\mathcal{O}_c) \) with \( D(-\Delta^\varepsilon_x) = H^2(\mathcal{O}_c) \cap V^\varepsilon \) for all \( \varepsilon \in (0, \epsilon_0) \), where \( \epsilon_0 \) is given by Lemma 2.1.

Next, we start at an auxiliary lemma which will be found in [27]:

**Lemma 3.5.** We have the following useful inclusion in the null-expansion sense.

\[
V^{\varepsilon_1} \subset V^{\varepsilon_2}, \quad D(-\Delta^\varepsilon_1) \subset D(-\Delta^\varepsilon_2), \quad \forall 0 < \varepsilon_1 < \varepsilon_2 \leq \epsilon_0. \tag{59}
\]

It is well-known that there are countable eigenvalues \( \lambda^\varepsilon_j, j = 1, 2, \cdots \) with

\[
0 < \lambda^0_1 \leq \cdots \leq \lambda^0_j \to \infty \text{ as } j \to \infty \text{ for each } \varepsilon \in (0, \epsilon_0]
\]

for the eigenvalue problem (58), which is equivalent to the following eigenvalue problem:

\[
A_\varepsilon \hat{u}^\varepsilon(y) = \lambda^\varepsilon \hat{u}^\varepsilon(y), \quad y \in \mathcal{O}, \quad \nabla \hat{u}^\varepsilon \cdot \nu = 0, \quad \text{on} \ \partial \mathcal{O}. \tag{60}
\]

Let \( \{ \lambda^0_j \}_j \) be the sequence of eigenvalues for \(-\Delta^0_\varepsilon\), with \( D(-\Delta^0_\varepsilon) = \{ u^\varepsilon \in H^2(\mathcal{O}) : \partial u^\varepsilon/\partial \nu_0 = 0 \text{ on } \partial \mathcal{Q} \} \) (or equivalently for \( A_0 \)). By the convergence theorem of spectrum given by Arrieta and Carvalho [4], \( \lambda^\varepsilon_j \) is continuous at \( \varepsilon = 0 \), more precisely,

\[
\lim_{\varepsilon \to 0} \| \lambda^\varepsilon_j - \lambda^0_j \| = 0. \tag{61}
\]

By (61), we choose a small \( \hat{\varepsilon} \in (0, \epsilon_0] \) such that

\[
\frac{\lambda^0_j}{2} < \lambda^\varepsilon_j < \frac{3\lambda^0_j}{2}, \quad \forall \varepsilon \in (0, \hat{\varepsilon}], \quad j \in \mathbb{N}. \tag{62}
\]
Let \( e_j^{\epsilon} \in D(A_{\epsilon}) \subset \{ u^\epsilon \in H^1(\Omega) : \nabla u^\epsilon \cdot \nu = 0, \text{on } \partial \Omega \} \) be the eigenvector with respect to the eigenvalue \( \lambda_j^{\epsilon} \) of the operator \( A_{\epsilon} \). Then, the sequence \( \{ e_j^{\epsilon} \}_{j=1}^{\infty} \) consists of a complete orthonormal basis of \( L^2(\Omega) \). Let \( H_j^{\epsilon} := \text{span}\{ e_1^{\epsilon}, e_2^{\epsilon}, \ldots, e_j^{\epsilon} \} \subset H^1 \) and \( P_j^{\epsilon} : L^2(\Omega) \to H_j^{\epsilon} \) be the canonical projector. Then, each solution \( u^\epsilon \) of equation (16) has a unique decomposition

\[
u^\epsilon = u_{1,j}^{\epsilon} + u_{2,j}^{\epsilon}, \quad u_{1,j}^{\epsilon} = P_j^{\epsilon}u^\epsilon \in H_j^{\epsilon}, \quad u_{2,j}^{\epsilon} = (I - P_j^{\epsilon})u^\epsilon \in H_j^{\epsilon,\perp}.
\]

So, by the expansion of the series, we have the following useful inequalities:

\[
\| A_{\epsilon}^{1/2} u_{1,j}^{\epsilon} \|_g^2 \leq \lambda_j^{\epsilon} \| u_{1,j}^{\epsilon} \|_g^2, \quad \text{and} \quad \| A_{\epsilon} u_{2,j}^{\epsilon} \|_g^2 \geq \lambda_j^{\epsilon} a_{\epsilon}(u_{2,j}^{\epsilon}, u_{2,j}^{\epsilon}). \quad (63)
\]

**Lemma 3.6.** Suppose that Assumption \( G, F, T \) hold true. Let \( \tau \in \mathbb{R}, \omega \in \Omega \) and \( D_1 \subset D_1 \) be fixed. Then, one can choose \( \tau \) and there is an \( \epsilon > 0 \) (as given in (62)) such that

\[
\lim_{j \to \infty} \sup_{\epsilon \in (0, \epsilon_o]} \sup_{t \geq t_0} \| (I - P_j^{\epsilon})u^\epsilon(\tau, t - \tau, \theta, \omega, u_0) \|_{H^1(\Omega)}^2 = 0. \quad (64)
\]

uniformly in \((u_0, v_0) \in D_1(\tau - t, \theta, \omega)\).

**Proof.** By taking the inner product of the first equation of (16) with \( A_{\epsilon} u_{2,j}^{\epsilon} \) in \((L^2(\Omega), \| \cdot \|_g)\), we have

\[
\frac{1}{2} \frac{d}{dt} a_{\epsilon}(u_{2,j}^{\epsilon}, u_{2,j}^{\epsilon}) + \| A_{\epsilon} u_{2,j}^{\epsilon} \|_g^2 + \lambda a_{\epsilon}(u_{2,j}^{\epsilon}, u_{2,j}^{\epsilon}) + \alpha(\nu^\epsilon, A_{\epsilon} u_{2,j}^{\epsilon})_g
\]

\[
= \langle f_{\epsilon}(t, y, u^\epsilon + \phi_{\epsilon}(y)z_1(\theta, \omega_1)), A_{\epsilon} u_{2,j}^{\epsilon} \rangle_g
\]

\[
+ \langle G_{\epsilon}(t, y) - A_{\epsilon} \phi_{\epsilon}(y)z_1(\theta, \omega_1) - \alpha \phi_{\epsilon}(y)z_2(\theta, \omega_2), A_{\epsilon} u_{2,j}^{\epsilon} \rangle_g.
\]

(65)

By (5), we have

\[
|\langle f_{\epsilon}(t, y, u^\epsilon + \phi_{\epsilon}(y)z_1(\theta, \omega_1)), A_{\epsilon} u_{2,j}^{\epsilon} \rangle_g| \leq \int_{\Omega} g |f_{\epsilon}(t, y, \tilde{u}^\epsilon)||A_{\epsilon} u_{2,j}^{\epsilon}|dy
\]

\[
\leq \frac{1}{6} \| A_{\epsilon} u_{2,j}^{\epsilon} \|_g^2 + c \int_{\Omega} |f_{\epsilon}(t, y, \tilde{u}^\epsilon)|^2 dy
\]

\[
\leq \frac{1}{6} \| A_{\epsilon} u_{2,j}^{\epsilon} \|_g^2 + c \| u^\epsilon + \phi_{\epsilon}(y)z_1(\theta, \omega_1) \|_{L^2_{2p-2}}^2 + \| \psi_{2}(t) \|_{L^2_{\infty}}^2
\]

\[
\leq \frac{1}{6} \| A_{\epsilon} u_{2,j}^{\epsilon} \|_g^2 + c \| u^\epsilon \|_{L^2_{2p-2}}^2 + c |z_1(\theta, \omega_1)|^2 + |z_2(\theta, \omega_2)|^2.
\]

(66)

By the Young inequality, we have

\[
|\langle G_{\epsilon}(t, y) - A_{\epsilon} \phi_{\epsilon}(y)z_1(\theta, \omega_1) - \alpha \phi_{\epsilon}(y)z_2(\theta, \omega_2), A_{\epsilon} u_{2,j}^{\epsilon} \rangle_g|
\]

\[
\leq \frac{1}{6} \| A_{\epsilon} u_{2,j}^{\epsilon} \|_g^2 + c \| G(t) \|_{L^2_{\infty}}^2 + c |z_1(\theta, \omega_1)|^2 + |z_2(\theta, \omega_2)|^2,
\]

(67)

\[
|\alpha(\nu^\epsilon, A_{\epsilon} u_{2,j}^{\epsilon})_g| \leq \frac{1}{6} \| A_{\epsilon} u_{2,j}^{\epsilon} \|_g^2 + c \| u^\epsilon \|_{L^2_{2p-2}}^2.
\]

(68)

By (65)-(68), we obtain an energy inequality of \( u_{2,j}(s, t - \tau, \theta, \omega, u_{2,0}) \) as follows.

\[
\frac{d}{ds} a_{\epsilon}(u_{2,j}^{\epsilon}, u_{2,j}^{\epsilon}) + \| A_{\epsilon} u_{2,j}^{\epsilon} \|_g^2 + 2 \lambda a_{\epsilon}(u_{2,j}^{\epsilon}, u_{2,j}^{\epsilon})
\]

\[
\leq c \| u^\epsilon \|_{L^2_{2p-2}}^2 + \| \nu^\epsilon \|_{L^2_{2p-2}}^2 + c \| G(s) \|_{L^2_{\infty}}^2 + \| \psi_{2}(s) \|_{L^2_{\infty}}^2
\]

\[
+ c |z_1(\theta, \omega_1)|^2 + |z_2(\theta, \omega_2)|^2.
\]

(69)
By (62) and the second inequality in (63), we have
\[
\|A_e u_{2,j}\|_p^p \geq \lambda_{j+1}^p a_e(u^e_{2,j}, u^e_{2,j}) \geq \frac{\lambda_{j+1}}{2} a_e(u^e_{2,j}, u^e_{2,j}), \quad \forall e \in (0, \hat{e}], \quad j \in \mathbb{N}. \quad (70)
\]
Therefore, we throw away the third term in (69) to obtain
\[
\begin{align*}
\frac{d}{ds} a_e(u^e_{2,j}, u^e_{2,j}) + \frac{\lambda_{j+1}}{2} a_e(u^e_{2,j}, u^e_{2,j})
\leq c(\|u^e\|_{2p-2}^2 + \|v^e\|_{p}^2) + c(\|G(s)\|_{\infty}^2 + \|\psi_2(s)\|_{\infty}^2) \\
+ c(|g_1(\theta_{s-\tau}\omega_1)|^2 + |g_1(\theta_{s-\tau}\omega_1)|_{2p-2}^2 + |g_2(\theta_{s-\tau}\omega_2)|^2).
\end{align*}
\quad (71)
\]
By applying the Gronwall-type inequality (30) to (71) for a small number \(\mu \leq \frac{1}{2}\), we have
\[
\begin{align*}
a_e(u^e_{2,j}(\tau, \tau - t, \theta_{-\tau}\omega_1), u^e_{2,j}(\tau, \tau - t, \theta_{-\tau}\omega, u^e_{2,0}))
\leq & \frac{1}{\mu} \int_{\tau - \mu}^{\tau} e^{\frac{\lambda_{j+1}}{2}(s-\tau)} a_e(u^e_{2,j}(s, \tau - t, \theta_{-\tau}\omega, u^e_{2,0})), u^e_{2,j}(s, \tau - t, \theta_{-\tau}\omega, u^e_{2,0}))ds \\
& + c \int_{\tau - \mu}^{\tau} e^{\frac{\lambda_{j+1}}{2}(s-\tau)} \|v^e(s, \tau - t, \theta_{-\tau}\omega, v_0)\|_{2}^2 ds \\
& + c \int_{\tau - \mu}^{\tau} e^{\frac{\lambda_{j+1}}{2}(s-\tau)} \|\psi_1(s, \theta_{s-\tau}\omega_1)\|_{\infty}^2 ds \\
& + c \int_{\tau - \mu}^{\tau} e^{\frac{\lambda_{j+1}}{2}(s-\tau)} \|G(s)\|_{\infty}^2 ds \\
\leq & \frac{\mu}{5} \sum_{i=1}^{j} I^j_i. \quad (72)
\end{align*}
\]
We first estimate the term \(I^j_1\). By Lemma 3.3, for all \(e \in (0, \hat{e}], s \in [\tau - 1, \tau], t \geq T\) and \((u_0, v_0) \in \mathcal{D}_1(\tau - t, \theta_{-\tau}\omega), \)
\[
a_e(u^e_{2,j}(s, \tau - t, \theta_{-\tau}\omega, u^e_{2,0})), u^e_{2,j}(s, \tau - t, \theta_{-\tau}\omega, u^e_{2,0}))
= \|(I - P^e_\tau)u^e(s, \tau - t, \theta_{-\tau}\omega, u^e_0)\|_{H^1(\mathcal{O})}^2 \\
\leq \|u^e(s, \tau - t, \theta_{-\tau}\omega, u^e_0)\|_{H^1(\mathcal{O})}^2 \leq c R_1(\tau, \omega), \quad \forall j \in \mathbb{N},
\]
which further implies that for \(T > 0, t \geq T, \)
\[
I^j_1 := \sup_{e \in (0, \hat{e}], \tau - \mu, s} \int_{\tau - \mu}^{\tau} e^{\frac{\lambda_{j+1}}{2}(s-\tau)} a_e(u^e_{2,j}(s, \tau - t, \theta_{-\tau}\omega, u^e_{2,0})), u^e_{2,j}(s, \tau - t, \theta_{-\tau}\omega, u^e_{2,0}))ds \\
\leq c R_1(\tau, \omega) \int_{\tau - \mu}^{\tau} e^{\frac{\lambda_{j+1}}{2}(s-\tau)} ds \leq c R_1(\tau, \omega) \frac{2}{\lambda_{j+1}^2}, \quad (73)
\]
uniformly in \((u_0, v_0) \in \mathcal{D}_1(\tau - t, \theta_{-\tau}\omega).\) Similarly, for the above number \(\mu\) fixed, we infer from (22) that
\[
I^j_2 := \sup_{e \in (0, \hat{e}], \tau - \mu, s} \int_{\tau - \mu}^{\tau} e^{\frac{\lambda_{j+1}}{2}(s-\tau)} \|v^e(s, \tau - t, \theta_{-\tau}\omega, v_0)\|_{2}^2 ds \\
\leq c p_1(\tau, \omega) \int_{\tau - \mu}^{\tau} e^{\frac{\lambda_{j+1}}{2}(s-\tau)} ds \leq c p_1(\tau, \omega) \frac{2}{\lambda_{j+1}^2}, \quad (74)
\]
independent of \(e \in (0, \hat{e}], \tau - \mu, s\).
We now consider the term $I^3_j$ of order $2p - 2$, which is uneasy to treat (because we cannot prove the absorption in $L^{2p-2}$). We consider the decomposition of the domain:

$$
\mathcal{O} = \mathcal{O}(u^\varepsilon \geq M) \cup \mathcal{O}(u^\varepsilon \leq -M) \cup \mathcal{O}(|u^\varepsilon| < M), \quad \forall M > 0.
$$

Then, $I^3_j$ can be split into three terms:

$$
I^3_j = \begin{align*}
&\int_{\tau - \mu}^{\tau} e^{\frac{\lambda_0}{2}(s-\tau)} \int_{\mathcal{O}} |u^\varepsilon(s, \tau - t, \theta_{-\tau}\omega, u_0)(y)|^{2p-2} dy ds \\
&= \int_{\tau - \mu}^{\tau} e^{\frac{\lambda_0}{2}(s-\tau)} \left( \int_{(u^\varepsilon \geq M)} + \int_{(u^\varepsilon \leq -M)} + \int_{(|u^\varepsilon| < M)} \right) |u^\varepsilon|^{2p-2} dy ds \\
&=: I^{3,1}_j(M) + I^{3,2}_j(M) + I^{3,3}_j(M), \quad \forall M > 0, \ j \in \mathbb{N},
\end{align*}
$$

Notice that, on $\mathcal{O}(u^\varepsilon \geq M)$, we have

$$
|u^\varepsilon|^{2p-2} = |(u^\varepsilon - M + M)|^{p-1}|u^\varepsilon|^{p-1} \leq 2^p((u^\varepsilon - M)^{p-1}|u^\varepsilon|^{p-1} + M^{p-1}|u^\varepsilon|^{p-1}) \leq 2^p((u^\varepsilon - M)^{p-1}|u^\varepsilon|^{p-1} + M^{p-2}|u^\varepsilon|^p).
$$

Hence, the term $I^{3,1}_j(M)$ is bounded by

$$
I^{3,1}_j(M) : = \int_{\tau - \mu}^{\tau} e^{\frac{\lambda_0}{2}(s-\tau)} \int_{(u^\varepsilon \geq M)} |u^\varepsilon(s, \tau - t, \theta_{-\tau}\omega, u_0)(y)|^{2p-2} dy ds \\
\leq c \int_{\tau - \mu}^{\tau} \int_{(u^\varepsilon \geq M)} |u^\varepsilon(s, \tau - t, \theta_{-\tau}\omega, u_0)|^{p-1}(u^\varepsilon - M)^{p-1} dy ds \\
+ cM^{p-2} \int_{\tau - \mu}^{\tau} e^{\frac{\lambda_0}{2}(s-\tau)} \int_{\mathcal{O}} |u^\varepsilon(s, \tau - t, \theta_{-\tau}\omega, u_0)|^{p} dy ds.
$$

Similarly, on $\mathcal{O}(u^\varepsilon \leq -M)$ and fixed number $\mu$, we have

$$
|u^\varepsilon|^{2p-2} = |(u^\varepsilon + M - M)|^{p-1}|u^\varepsilon|^{p-1} \leq 2^p((u^\varepsilon + M)^{p-1}|u^\varepsilon|^{p-1} + M^{p-1}|u^\varepsilon|^{p-1}),
$$

which implies that

$$
I^{3,2}_j(M) : = \int_{\tau - \mu}^{\tau} e^{\frac{\lambda_0}{2}(s-\tau)} \int_{(u^\varepsilon \leq -M)} |u^\varepsilon(s, \tau - t, \theta_{-\tau}\omega, u_0)(y)|^{2p-2} dy ds \\
\leq c \int_{\tau - \mu}^{\tau} \int_{(u^\varepsilon \leq -M)} |u^\varepsilon(s, \tau - t, \theta_{-\tau}\omega, u_0)|^{p-1}(u^\varepsilon + M)^{p-1} dy ds \\
+ cM^{p-2} \int_{\tau - \mu}^{\tau} e^{\frac{\lambda_0}{2}(s-\tau)} \int_{\mathcal{O}} |u^\varepsilon(s, \tau - t, \theta_{-\tau}\omega, u_0)|^{p} dy ds.
$$

Let $\varepsilon > 0$. By Lemma 3.4, there is an $M_3 > 0$ such that for all $j \in \mathbb{N}$,

$$
I^{3,1}_j(M_3) + I^{3,2}_j(M_3) \leq \varepsilon + cM_3^{p-2} \int_{\tau - \mu}^{\tau} e^{\frac{\lambda_0}{2}(s-\tau)} \int_{\mathcal{O}} |u^\varepsilon(s, \tau - t, \theta_{-\tau}\omega, u_0)|^{p} dy ds,
$$

where the order $2p-2$ reduces to the order $p$. On the other hand, by (33) in Lemma 3.3, we have

$$
\sup_{\varepsilon \in (0,1]} \sup_{s \in [\tau - \mu, \tau]} \|u^\varepsilon(s, \tau - t, \theta_{-\tau}\omega, u_0)\|^p \leq cR_1(\tau, \omega),
$$

uniformly in $(u_0, v_0) \in D_1(\tau - t, \theta_{-\tau}\omega)$, which implies that

$$
I^{3,1}_j(M_3) + I^{3,2}_j(M_3) \leq \eta + cM_3^{p-2} \int_{\tau - \mu}^{\tau} e^{\frac{\lambda_0}{2}(s-\tau)} ds \leq \varepsilon + R_1(\tau, \omega) \frac{cM_3^{p-2}}{\lambda_{j+1}},
$$

$$
\sup_{\varepsilon \in \mathbb{Z}} \sup_{s \geq 0} |u^\varepsilon(s, \tau - t, \theta_{-\tau}\omega, u_0)| \leq N\varepsilon^{1/p}.
$$
The term $I^{3,3}_j(M_3)$ (on $O(|u'| < M_3)$) is bounded by
\[
I^{3,3}_j(M_3) := \int_{t-\mu}^t e^{\frac{\lambda_0}{j+1}(s-\tau)} \int_{\{(|u'| < M_3)\}} |u'(s, \tau-t, \theta_{-\tau}(\omega), u_0)(y)|^{2p-2} dy ds
\leq |O|M_3^{2p-2} \int_{t-\mu}^t e^{\frac{\lambda_0}{j+1}(s-\tau)} ds \leq \frac{cM_3^{2p-2}}{\lambda_0^{j+1}}.
\]
Since $\lambda_{j+1}^0 \rightarrow +\infty$ as $j \rightarrow \infty$, we can take $j_0 \in \mathbb{N}$ such that
\[
\lambda_{j+1}^0 \geq \varepsilon^{-1}(R_1(\tau, \omega)M_3^{2p-2} + M_3^{2p-2}).
\]
Hence, for all $j \geq j_0$,
\[
I_j^1 = I^{3,1}_j(M_3) + I^{3,2}_j(M_3) + I^{3,3}_j(M_3)
\leq \varepsilon + c(R_1(\tau, \omega)M_3^{2p-2} + M_3^{2p-2})(\lambda_{j+1}^0)^{-1} \leq c\varepsilon.
\]
Therefore, $I_j^1 \rightarrow 0$ as $j \rightarrow \infty$, uniformly in $\varepsilon \in (0, \varepsilon)$, $t \geq T$ and $(u_0, v_0) \in \mathcal{D}_1(\tau - t, \theta_{-\tau}(\omega))$.

Finally, since $G \in L^p_{p,0}(\mathbb{R}, L^\infty(\hat{\mathcal{O}})), \psi_2 \in L^p_{p,0}(\mathbb{R}, L^\infty(\hat{\mathcal{O}}))$ and $s \rightarrow z(\theta_{s-\tau}(\omega))$ is continuous, the Lebesgue controlled convergence theorem gives
\[
I_j^3 := c \int_{t-\mu}^t e^{\frac{\lambda_0}{j+1}(s-\tau)} (\|G(s)\|_\infty^2 + \|\psi_2(s)\|_\infty^2) ds \rightarrow 0,
\]
\[
I_j^5 := c \int_{t-\mu}^t e^{\frac{\lambda_0}{j+1}(s-\tau)} (|z_1(\theta_{s-\tau}(\omega_1)|^2 + |z_1(\theta_{s-\tau}(\omega_1)|^{2p-2} + |z_2(\theta_{s-\tau}(\omega_2)|^2) ds \rightarrow 0,
\]
as $j \rightarrow \infty$. The proof is complete. \hfill $\square$

4. Dynamics in $H^1(\mathcal{O}) \times L^2(\mathcal{O})$. In this section, we will prove the existence of $\mathcal{D}_1$-pullback random $(L^2(\mathcal{O}) \times L^2(\mathcal{O}), H^1(\mathcal{O}) \times L^2(\mathcal{O}))$ attractor and upper semi-continuity in $H^1(\mathcal{O}) \times L^2(\mathcal{O})$. Firstly, we borrow some well-known concept and results from the theory of non-autonomous random dynamical systems.

Definition 4.1. Let $X$ and $Y$ be Banach spaces such that $X \cap Y \neq \emptyset$, then we claim that $(X, Y)$ is limit-identical if
\[
x_n \in X \cap Y, \quad \|x_n - x_0\|_X + \|x_n - y_0\|_Y \rightarrow 0 \Rightarrow x_0 = y_0 \in X \cap Y.
\]

Definition 4.2. A bi-parametric set $\mathcal{A} = \{A(\tau, \omega)\}$ is said to be a $(X, Y)$-random attractor for a random cocycle $\Phi$ if
1. $\omega \rightarrow A(\tau, \omega)$ is $\mathcal{F}$-measurable in $X$ and $Y$, respectively;
2. $A \in \mathcal{D}$, and $A(\tau, \omega)$ is compact in $X \cap Y$;
3. $A$ is invariant, i.e. $\Phi(s, \tau, \omega)A(\tau, \omega) = A(s + \tau, \theta_{s}\omega)$ for $s \geq 0$;
4. $A$ is pullback attracting in $Y$, i.e. for every $\mathcal{D} \in \mathcal{D}$,
\[
\lim_{t \rightarrow +\infty} dist_Y(\Phi(t, \tau-t, \theta_{-t}(\omega))\mathcal{D}(\tau-t, \theta_{-t}(\omega), A(\tau, \omega)) = 0.
\]

Proposition 1. \cite{36} Let $X$ be a Banach space equipped with a universe $\mathcal{D}$ and $Y$ another Banach space such that $(X, Y)$ is limit-identical. Let $\Phi$ be a continuous cocycle on $X$ over $\mathbb{R} \times \Omega$ but take values in $Y$. Then $\Phi$ has a unique $\mathcal{D}$-pullback $(X, Y)$-attractor if
1. $\Phi$ has a closed, measurable and absorbing set $K \in \mathcal{D}$;
2. $\Phi$ is pullback $\mathcal{D}$-pullback asymptotically compact in $Y$ as well as in $X$. 

**Theorem 4.3.** Suppose that Assumption G, F, T and (27) hold true. Then there exist $\epsilon_0$ such that for each $\epsilon \in (0, \epsilon_0]$, the cocycle $\Phi_\epsilon$ generated by the problem (16) has a unique $(L^2(\Omega) \times L^2(\Omega), H^1(\Omega) \times L^2(\Omega))$-pullback random attractor $A_\epsilon = \{A_\epsilon(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$.

**Proof.** Since $H^1(\Omega) \to L^2(\Omega)$, one can show that $(L^2(\Omega) \times L^2(\Omega), H^1(\Omega) \times L^2(\Omega))$ is limit-identical. By [1], $\Phi_\epsilon$ defined (18) is a continuous cocycle on $L^2(\Omega) \times L^2(\Omega)$ and its values in $H^1(\Omega) \times L^2(\Omega)$. By [41], it is easy to see $\Phi_\epsilon$ has a closed, measurable and absorbing set $K \in \mathcal{D}_1$ and $\Phi_\epsilon$ is $\mathcal{D}_1$-pullback asymptotically compact in $L^2(\Omega) \times L^2(\Omega)$. Hence, by Proposition 1, we need only to prove that $\Phi$ is $\mathcal{D}_1$-pullback asymptotically compact in $H^1(\Omega) \times L^2(\Omega)$.

Let $D_1 \in \mathcal{D}_1$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$ be fixed and define

$$B_{\epsilon}(T) = \bigcup_{t \geq T} \Phi_\epsilon(t, \tau - t, \theta_{-t}\omega)D_1(\tau - t, \theta_{-t}\omega),$$

$$B_{\epsilon,1}(T) = \bigcup_{t \geq T} \{u^\epsilon(t, \tau - t, \theta_{-t}\omega, u_0); (u_0, v_0) \in D_1(\tau - t, \theta_{-t}\omega)\},$$

$$B_{\epsilon,2}(T) = \bigcup_{t \geq T} \{v^\epsilon(t, \tau - t, \theta_{-t}\omega, v_0); (u_0, v_0) \in D_1(\tau - t, \theta_{-t}\omega)\}.$$  

Therefore, $B_{\epsilon,1}(T), B_{\epsilon,2}(T)$ are just two components of $B_{\epsilon}(T)$. Since $\Phi_\epsilon$ is $\mathcal{D}_1$-pullback asymptotically compact in $L^2(\Omega) \times L^2(\Omega)$, for each $\epsilon > 0$, there is a $T_1$ such that $\kappa_2^2(B_{\epsilon,2}(T_1)) < \epsilon$, where the Kuratowski measure $\kappa(\cdot)$ denotes the minimal diameter of all sets constituted a finite cover.

On the other hand, by Lemma 3.6, for each $\epsilon > 0$, there is a $j = j(\epsilon) \in \mathbb{N}$ and $T_2 \geq T_1$ such that

$$||(I - P_{j}^\epsilon)B_{\epsilon,1}(T_2)||_{H^1} < \epsilon.$$  

By Lemma 3.1, we have

$$\|P_{j}^\epsilon B_{\epsilon,1}(T_2)\|_{H^1} \leq c\|B_{\epsilon,1}(T_2)\|_{H^1} < c\rho_2(\tau, \omega),$$  

which means that $P_{j}^\epsilon B_{\epsilon,1}(T_2)$ is bounded in the finitely dimensional subspace of $H^1$, and so it is pre-compact in $H^1$ with the Kuratowski measure zero. Therefore,

$$\kappa_{H^1}B_{\epsilon,1}(T_2) \leq \kappa_{H^1}P_{j}^\epsilon B_{\epsilon,1}(T_2) + \kappa_{H^1}(I - P_{j}^\epsilon)B_{\epsilon,1}(T_2) = \kappa_{H^1}(I - P_{j}^\epsilon)B_{\epsilon,1}(T_2) < \epsilon,$$

which implies that $B_{\epsilon,1}(T_2)$ is pre-compact in $H^1$. Hence, we have

$$\kappa_{H^1 \times L^2}B_{\epsilon}(T_2) \leq \kappa_{H^1}B_{\epsilon,1}(T_2) + \kappa_{L^2}B_{\epsilon,2}(T_2) < 2\epsilon,$$

and so $\Phi_\epsilon$ is $\mathcal{D}_1$-pullback asymptotically compact in $H^1(\Omega) \times L^2(\Omega)$. This complete the proof. \hfill $\Box$

**Theorem 4.4.** Suppose that Assumption G, F, T and (27) hold true. Then the cocycle $\Phi_0$, generated by system (19), has a unique $(L^2(\Omega) \times L^2(\Omega), H^1(\Omega) \times L^2(\Omega))$-pullback random attractor $A_0 \in \mathcal{D}_0$.

We finally present the upper semi-continuity of random attractors as $\epsilon \to 0$ under the topology of $H^1(\Omega) \times L^2(\Omega)$.

**Assumption A.** There exist three functions $\mu_1(\cdot), \mu_2(\cdot), \mu_3(\cdot) \in L^2_{\text{loc}}(\mathbb{R})$ such that

$$\|f_\epsilon(t, \cdot, s) - f_0(t, \cdot, s)\|_{L^2(\Omega)} \leq \mu_1(t)\epsilon, \text{ for all } t, s \in \mathbb{R},$$

$$\|G_\epsilon(t, \cdot) - G_0(t, \cdot)\|_{L^2(\Omega)} \leq \mu_2(t)\epsilon, \text{ for all } t \in \mathbb{R},$$

$$\|H_\epsilon(t, \cdot) - H_0(t, \cdot)\|_{L^2(\Omega)} \leq \mu_3(t)\epsilon, \text{ for all } t \in \mathbb{R}.$$
Given a function \( u \in L^2(O) \), we consider its average function on \( Q \) by using the average operator \( M : L^2(O) \rightarrow L^2(Q) \),
\[
(Mu)(y^*) = \int_0^1 u(y^*, y_{n+1}) dy_{n+1}.
\]

**Lemma 4.5.** \([19]\) If \( u \in H^1(O) \), then \( Mu \in H^1(Q) \) and
\[
\|u - Mu\|_{L^2(Q)} \leq c\|u\|_{H^1(O)},
\]
where \( c \) is a constant independent of \( \epsilon \).

Under the Assumption \( A \), we have the following results:

**Lemma 4.6.** \([41]\) Suppose that Assumption \( G, F, T, A \) hold true. Given a positive number \( \eta(\tau, \omega) \), if \( (\tilde{u}_\tau^\epsilon, \tilde{v}_\tau^\epsilon) \in H^1_\theta(O) \times H^1_{\Phi}(O) \) such that \( (\|\tilde{u}_\tau^\epsilon\|_{H^1_\theta(O)} + \|\tilde{v}_\tau^\epsilon\|_{H^1_{\Phi}(O)}) \leq \eta(\tau, \omega) \). Then,
\[
\lim_{\epsilon \to 0} \|\Phi_\tau(t, \tau, \omega; (\tilde{u}_\tau^\epsilon, \tilde{v}_\tau^\epsilon)) - \Phi_\tau(t, \tau, \omega; (Mu_{\tau}^\epsilon, Mu_{\tau}^\epsilon))\|_{L^2(O) \times L^2(Q)} = 0,
\]
for each \( t \geq \tau, \tau \in \mathbb{R} \) and \( \omega \in \Omega \).

**Lemma 4.7.** \([41]\) Suppose that Assumption \( G, F, T, A \) and \((27)\) hold true. Then the random attractor \( A_{\epsilon} \) is upper semi-continuous in \( L^2(O) \times L^2(O) \) at \( \epsilon = 0 \), that is,
\[
\lim_{\epsilon \to 0} dist_{L^2(O) \times L^2(O)}(A_{\epsilon}(\tau, \omega), A_0(\tau, \omega)) = 0, \quad \forall \tau \in \mathbb{R}, \quad \omega \in \Omega.
\]

We now state our main result as follows:

**Theorem 4.8.** Suppose that Assumption \( G, F, T, A \) and \((27)\) hold true. Then the random attractor \( A_{\epsilon} \) is upper semi-continuous in \( H^1(O) \times L^2(O) \) at \( \epsilon = 0 \), that is,
\[
\lim_{\epsilon \to 0} dist_{H^1(O) \times L^2(O)}(A_{\epsilon}(\tau, \omega), A_0(\tau, \omega)) = 0, \quad \forall \tau \in \mathbb{R}, \quad \omega \in \Omega.
\]

**Proof.** We split the proof into four steps.

**Step 1.** Firstly, we show the uniform boundedness of \( D_1 \)-pullback random attractor \( A_{\epsilon} \) for all \( \epsilon \in (0, \epsilon_0] \) in \( H^1(O) \times H^1(O) \). Let \( (\tilde{u}^\epsilon, \tilde{v}^\epsilon) \in A_{\epsilon}(\tau, \omega) \) for fixed \( \epsilon \in (0, \epsilon_0] \). Taking a sequence \( \{t_n\}_{n=1}^\infty \) such that \( t_n \to \infty \). Then by the invariance of \( A_{\epsilon} \), there exists \( (\tilde{u}_n^\epsilon, \tilde{v}_n^\epsilon) \in A_{\epsilon}(\tau - t_n, \theta_{-t_n} \omega) \) such that
\[
(\tilde{u}^\epsilon, \tilde{v}^\epsilon) = \Phi_\epsilon(t_n, \tau - t_n, \theta_{-t_n} \omega, (\tilde{u}_n^\epsilon, \tilde{v}_n^\epsilon)).
\]
Hence, by \((18), (24) \) and \((25)\), we have
\[
\left\{
\begin{array}{l}
\tilde{u}^\epsilon = u^\epsilon(\tau, \tau - t_n, \theta_{-t_n} \omega, u^\epsilon_{\tau - t_n}) + \phi_\epsilon z_1(\theta_{t_n} \omega_1), \\
\tilde{v}^\epsilon = v^\epsilon_1(\tau, \tau - t_n, \theta_{-t_n} \omega, v^\epsilon_1_{\tau - t_n} + \phi_\epsilon z_2(\theta_{t_n} \omega_2)) + v^\epsilon_2(\tau, \tau - t_n, \theta_{-t_n} \omega, 0),
\end{array}
\right.
\]
where \( (u^\epsilon_{\tau - t_n}, v^\epsilon_1_{\tau - t_n}) = (\tilde{u}_n^\epsilon - \phi_\epsilon z_1(\omega_1), \tilde{v}_n^\epsilon - \phi_\epsilon z_2(\omega_2)) \). Since \( A_{\epsilon} \in D_1 \), by Lemma 3.1, there exists \( N = N(\tau, \omega, \epsilon) \in \mathbb{N} \) such that for all \( n \geq N \),
\[
\begin{align*}
&\|u^\epsilon(\tau, \tau - t_n, \theta_{-t_n} \omega, u^\epsilon_{\tau - t_n}) + \phi_\epsilon z_1(\theta_{t_n} \omega_1)\|_{H^1(O)}^2 \\
&+ \|v^\epsilon_2(\tau, \tau - t_n, \theta_{-t_n} \omega, 0) + \phi_\epsilon z_2(\theta_{t_n} \omega_2)\|_{H^1(O)}^2 \leq c \rho_3(\tau, \omega),
\end{align*}
\]
where $\rho_3(\tau, \omega)$ is given by Lemma 3.1. Therefore, we can find $(u_0, v_0) \in H^1(\Omega) \times H^1(\Omega)$, such that (up to a subsequence),

$$
(u^\epsilon(\tau, \tau - t_n, \theta - \tau \omega), u^\epsilon_{-t_n}) + \phi_\epsilon z_1(\theta t_n, \omega_1), v^\epsilon_2(\tau, \tau - t_n, \theta - \tau \omega, 0) + \varphi_\epsilon z_2(\theta t_n, \omega_2)
\rightharpoonup (u_0, v_0),
$$

(81)
in $H^1(\Omega) \times H^1(\Omega)$, where $\rightharpoonup$ means the weak convergence, and

$$
\|u_0\|^2_{H^1(\Omega)} + \|v_0\|^2_{H^1(\Omega)} \leq c\rho_3(\tau, \omega).
$$

(82)

On the other hand, by (26), we have

$$
\|v^\epsilon_1(\tau, \tau - t_n, \theta - \tau \omega, v_{\tau - t_n})\| = e^{-2\sigma t_n} \|v_{\tau - t_n}\|^2_{L^2(\Omega)}
= e^{-2\sigma t_n} z(\tau - t_n, \theta - \tau \omega) \|\tilde{v}^\epsilon_n\|^2_{L^2(\Omega)}
\leq e^{-2\sigma t_n} z(\tau - t_n, \theta - \tau \omega) \|A_\epsilon(\tau - t_n, \theta - t_n, \omega)\|^2_{L^2(\Omega) \times L^2(\Omega)} \rightarrow 0, \text{ as } n \rightarrow \infty.
$$

(83)

It follows from (79) and (82) that for all $(\tilde{u}^\epsilon, \tilde{v}^\epsilon) \in A_\epsilon(\tau, \omega)$ with $\epsilon \in (0, \epsilon_0)$, we have

$$
\|\tilde{u}^\epsilon\|^2_{H^1(\Omega)} + \|\tilde{v}^\epsilon\|^2_{H^1(\Omega)} \leq c\rho_3(\tau, \omega).
$$

(84)

**Step 2.** In the sequel, we will prove that any sequence $\{(\tilde{u}_{k\epsilon}, \tilde{v}_{k\epsilon})\}_{k=1}^{\infty}$ is pre-compact in $H^1(\Omega) \times L^2(\Omega)$ as $\epsilon_k \rightarrow 0$, where $\epsilon_k \in (0, \epsilon_0]$. It suffices to prove that the corresponding subsequence $\{(\tilde{u}_{k\epsilon}, \tilde{v}_{k\epsilon})\}_{k=1}^{\infty}$ is a Cauchy sequence in $H^1(\Omega) \times L^2(\Omega)$.

By (22) and Lemma 3.1, $\Phi_{k\epsilon}$ has a random absorbing set $B \in \mathcal{D}_1$, which is defined by

$$
B(\tau, \omega) = \{(\tilde{u}, \tilde{v}) \in H^1(\Omega) \times H^1(\Omega) : \|\tilde{u}\|^2_{H^1(\Omega)} + \|\tilde{v}\|^2_{H^1(\Omega)} \leq \epsilon_1(\tau, \omega)
\text{ and } \|\tilde{u}\|^2_{H^1(\Omega)} + \|\tilde{v}\|^2_{H^1(\Omega)} \leq \frac{\epsilon}{\eta_1} \rho_3(\tau, \omega)\}.
$$

(85)

Then, by the invariance of $A_{k\epsilon}$ and the absorption of $B$, we have

$$
\bigcup_{k \in \mathbb{N}} A_{k\epsilon}(\tau, \omega) \subset B(\tau, \omega).
$$

Since $B \in \mathcal{D}_1$ is absorbed by itself, for each $k \in \mathbb{N}$ and $T > 0$, by the invariance of $A_{k\epsilon}$ again, there are $(\tilde{u}_{k\epsilon}, \tilde{v}_{k\epsilon}) \in A_{k\epsilon}(\tau - T, \theta - T \omega) \subset B(\tau - T, \theta - T \omega)$ such that

$$
(\tilde{u}_{k\epsilon}, \tilde{v}_{k\epsilon}) = \Phi_{k\epsilon}(T, \tau - T, \theta - T \omega)(\tilde{u}_{k\epsilon}, \tilde{v}_{k\epsilon})
= (\tilde{u}_{k\epsilon}(\tau, \tau - T, \theta - T \omega, \tilde{u}_{k\epsilon}), \tilde{v}_{k\epsilon}(\tau, \tau - T, \theta - T \omega, \tilde{v}_{k\epsilon})).
$$

Let now $\delta_0 > 0$ and $m \geq n$ with $m, n \in \mathbb{N}$. Since $\epsilon_k \downarrow 0$, there is an $N_1 \in \mathbb{N}$ such that $\epsilon_m \leq \epsilon_n \leq \epsilon$ for all $m \geq n \geq N_1$, where $\epsilon$ is given in Lemma 3.6.

Since $K \in \mathcal{D}$, it follows from (64) in Lemma 3.6 that there is an $j_0 \in \mathbb{N}$ such that

$$
\|(I - P_{j_0}^\epsilon)\tilde{u}_{k\epsilon}\|^2_{H^1(\Omega)} = \|(I - P_{j_0}^\epsilon)\tilde{v}_{k\epsilon}(\tau, \tau - T, \theta - T \omega, \tilde{u}_{k\epsilon})\|^2_{H^1(\Omega)}
\leq \|(I - P_{j_0}^\epsilon)\tilde{u}_{k\epsilon}(\tau, \tau - T, \theta - T \omega, B(\tau - T, \theta - T \omega))\|^2_{H^1(\Omega)} < \delta_0, \forall n \geq N_1.
$$

(86)

For all $m \geq n \geq N_1$, we have $\epsilon_m \leq \epsilon_n$. By Lemma 3.5, $D(-\Delta^\epsilon_{n}) \subset D(-\Delta^\epsilon_{m})$, and thus by (86),

$$
\|(I - P_{j_0}^\epsilon)\tilde{u}_{m}\|^2_{H^1(\Omega)} = \|(I - P_{j_0}^\epsilon)\tilde{v}_{m}(\tau, \tau - T, \theta - T \omega, \tilde{u}_{m})\|^2_{H^1(\Omega)}
= \|(I - P_{j_0}^\epsilon)\tilde{v}_{m}(\tau, \tau - T, \theta - T \omega, \tilde{u}_{m})\|^2_{H^1(\Omega)} < \delta_0.
$$

(87)
On the other hand, by Lemma 2.1, (62) and the first inequality of (63),
\[ \| P^{\tau}_{j_0} (\tilde{u}_{e_m} - \tilde{u}_{e_n}) \|_{H_1(\mathcal{O})}^2 \leq \frac{1}{\eta_1} \| A_n^{1/2} P^{\tau}_{j_0} (\tilde{u}_{e_m} - \tilde{u}_{e_n}) \|_{g}^2 \]
\[ \leq \frac{1}{\eta_1} \chi_{j_0} \| P^{\tau}_{j_0} (\tilde{u}_{e_m} - \tilde{u}_{e_n}) \|_{g}^2 \leq \frac{\gamma_2}{\eta_1} \| P^{\tau}_{j_0} (\tilde{u}_{e_m} - \tilde{u}_{e_n}) \|_{g}^2 \leq \frac{3\gamma_2}{2\eta_1} \chi_{j_0} \| \tilde{u}_{e_m} - \tilde{u}_{e_n} \|^2. \]
Since \( \{(\tilde{u}_{e_k}, \tilde{v}_{e_k})\}_k \) is convergent in \( L^2(\mathcal{O}) \times L^2(\mathcal{O}) \), it follows that there is an \( N_2 \geq N_1 \) such that for all \( m \geq n \geq N_2 \),
\[ \| \tilde{u}_{e_m} - \tilde{u}_{e_n} \|^2 < \frac{2\eta_1}{3\gamma_2 \chi_{j_0}} \delta_0 \quad \text{and so} \quad \| P^{\tau}_{j_0} (\tilde{u}_{e_m} - \tilde{u}_{e_n}) \|_{H_1(\mathcal{O})}^2 < \delta_0. \] (88)
In a word, by (86)-(88), for all \( m \geq n \geq N_2 \),
\[ \| \tilde{u}_{e_m} - \tilde{u}_{e_n} \|_{H_1(\mathcal{O})}^2 \leq \| P^{\tau}_{j_0} (\tilde{u}_{e_m} - \tilde{u}_{e_n}) \|_{H_1(\mathcal{O})}^2 + 2\| (I - P^{\tau}_{j_0}) \tilde{u}_{e_m} \|_{H_1(\mathcal{O})}^2 \]
\[ + 2\| (I - P^{\tau}_{j_0}) \tilde{u}_{e_m} \|_{H_1(\mathcal{O})}^2 \leq 5\delta_0, \]
which proves that \( \{(\tilde{u}_{e_k}, \tilde{v}_{e_k})\}_k \) is a Cauchy sequence in \( H^1(\mathcal{O}) \times L^2(\mathcal{O}) \) as desired.

**Step 3.** We construct an absorbing set \( K \subset H^1(\mathcal{O}) \times H^1(\mathcal{O}) \) such that \( K_0 = \mathcal{M}(K) \) is a closed tempered set in \( L^2(\mathcal{Q}) \times L^2(\mathcal{Q}) \) and thus \( K_0 \subset D_0 \) is attracted by the attractor \( \mathcal{A}_0 \). To this end, we define two bi-parametric sets in \( H^1(\mathcal{O}) \times H^1(\mathcal{O}) \) and \( L^2(\mathcal{Q}) \times L^2(\mathcal{Q}) \), respectively,
\[ K(\tau, \omega) \]
\[ = \{(\tilde{u}, \tilde{v}) \in H^1(\mathcal{O}) \times H^1(\mathcal{O}) : (\tilde{u}, \tilde{v}) \in B(\tau, \omega), \| \tilde{u} \|_{H_1(\mathcal{O})}^2 + \| \tilde{v} \|_{H_1(\mathcal{O})}^2 \leq \frac{c}{\eta_1} \rho_3(\tau, \omega)\}, \]
\[ K_0(\tau, \omega) = \{(\tilde{u}, \tilde{v}) \in L^2(\mathcal{O}) \times L^2(\mathcal{O}) : (\tilde{u}, \tilde{v}) \in K(\tau, \omega)\}. \]
Since \( K(\tau, \omega) \subset B(\tau, \omega) \), we have \( K \subset D_1, \) By (84) and Lemma 2.1, for any \( \epsilon \in (0, \epsilon_0), \) and \( D_1 \subset D_2, \)
\[ \| \Phi(t, \tau - t, \theta - \omega) D_2(\tau - t, \theta - \omega) \|^2_{H^1(\mathcal{O}) \times H^1(\mathcal{O})} \]
\[ \leq \eta_1 \| \Phi(t, \tau - t, \theta - \omega) D_2(\tau - t, \theta - \omega) \|^2_{H^1(\mathcal{O}) \times H^1(\mathcal{O})} \leq \frac{c}{\eta_1} \rho_3(\tau, \omega), \] (89)
if \( t \) is large enough. Therefore, \( K \subset D_1 \) is still a \( D_2 \)-pullback absorbing set. On the other hand, by Lemma 2.1 and Lemma 4.5 again, we have, for all \( (\tilde{u}, \tilde{v}), \)
\[ \| \tilde{u} - M \tilde{u} \|_{L^2(\mathcal{O})} + \| \tilde{v} - M \tilde{v} \|_{L^2(\mathcal{O})} \leq \alpha \epsilon^2 (\| \tilde{u} \|_{H^1(\mathcal{O})} + \| \tilde{v} \|_{H^1(\mathcal{O})}) \]
\[ \leq \alpha \epsilon^2 \frac{\eta_2}{\eta_1} (\| \tilde{u} \|_{H^1(\mathcal{O})} + \| \tilde{v} \|_{H^1(\mathcal{O})}) \leq \alpha \epsilon^2 \frac{\eta_2}{\eta_1} \rho_3(\tau, \omega), \]
which implies that
\[ \| M \tilde{u} \|_{L^2(\mathcal{O})} + \| M \tilde{v} \|_{L^2(\mathcal{O})} \leq 2 (\| \tilde{u} \|_{L^2(\mathcal{O})} + \| \tilde{u} - M \tilde{u} \|_{L^2(\mathcal{O})}) \]
\[ + 2 (\| \tilde{v} \|_{L^2(\mathcal{O})} + \| \tilde{v} - M \tilde{v} \|_{L^2(\mathcal{O})}) \leq \epsilon \rho_3(\tau, \omega). \]
Since \( \rho_3(\tau, \omega) \) is a tempered random variable, the above estimate yields \( K_0 \subset D_0, \)

**Step 4.** Finally, we argue the convergence of random attractors in \( H^1(\mathcal{O}) \times L^2(\mathcal{O}) \) by contradiction. Suppose (77) is not true, the there is a \( \delta_0 > 0, \epsilon_k \rightarrow 0 \) and \( (\tilde{u}_{e_k}, \tilde{v}_{e_k}) \in \mathcal{A}_e(\tau, \omega) \) such that
\[ \text{dist}_{H^1(\mathcal{O}) \times L^2(\mathcal{O})}((\tilde{u}_{e_k}, \tilde{v}_{e_k}), \mathcal{A}_0(\tau, \omega)) \geq \delta_0, \quad \forall k \in \mathbb{N}. \]
By step 2, there is a \((\tilde{u}, \tilde{v}) \in H^1(\mathcal{O}) \times L^2(\mathcal{O})\) such that (passing to a subsequence),
\[
\lim_{k \to \infty} \|(\tilde{u}_{e_k}, \tilde{v}_{e_k}) - (\tilde{u}, \tilde{v})\|_{H^1(\mathcal{O}) \times L^2(\mathcal{O})} = 0, \tag{90}
\]
and
\[
\text{dist}_{H^1(\mathcal{O}) \times L^2(\mathcal{O})}( (\tilde{u}, \tilde{v}), \mathcal{A}_0(\tau, \omega) ) \geq \delta_0. \tag{91}
\]

By Theorem 4.4 and the step 3, there is a \(T_0 > 0\) such that for all \(t \geq T_0\),
\[
\text{dist}_{H^1(\mathcal{O}) \times L^2(\mathcal{O})}( \Phi_0(t, \tau - t, \theta_{-T}\omega)K_0(\tau - t, \theta_{-T}\omega), \mathcal{A}_0(\tau, \omega) ) < \delta_0. \tag{92}
\]
Let \(T = T(B) \geq T_0\) be an absorption time when \(B \subset \mathcal{D}_1\) is absorbed by itself, where \(T\) is independent of \(\epsilon_k\). For each \(k \in \mathbb{N}\), by the invariance of \(\mathcal{A}_{e_k}\), there are \((\tilde{u}_{e_k}, \tilde{v}_{e_k}) \in \mathcal{A}_{e_k}(\tau - T, \theta_{-T}\omega) \subset B(\tau - T, \theta_{-T}\omega)\) such that
\[
(\tilde{u}_{e_k}, \tilde{v}_{e_k}) = \Phi_{e_k}(T, \tau - T, \theta_{-T}\omega)(\tilde{u}_{e_k}, \tilde{v}_{e_k})).
\]
By Lemma 3.1 and (89), there exists another absorption time \(\overline{T} = \overline{T}(K, \tau - T, \theta_{-T}\omega)\) such that for all \(t \geq \overline{T}\)
\[
\|\tilde{u}_{e_k}\|_{H^1_{k}} + \|\tilde{v}_{e_k}\|_{H^1_{k}} \leq \Phi_{e_k}(t, \tau - T - t, \theta_{-T}\omega)A_{e_k}(\tau - T - t, \theta_{-T}\omega)\|H^1_{k} \times H^1_{k}\|H^1_{k} \times H^1_{k}
\leq \Phi_{e_k}(t, \tau - T - t, \theta_{-T}\omega)K(\tau - T - t, \theta_{-T}\omega)\|H^1_{k} \times H^1_{k}\|H^1_{k} \times H^1_{k}
\leq c_1\rho_3(\tau - T, \theta_{-T}\omega), \tag{93}
\]
where we have use Lemma 2.1. This means that \(\|\tilde{u}_{e_k}\|_{H^1_{k}} + \|\tilde{v}_{e_k}\|_{H^1_{k}}\) is bounded in \(k\), and satisfies the assumption of Lemma 4.6. Hence, by Lemma 4.6, we have
\[
\|\Phi_{e_k}(T, \tau - T, \theta_{-T}\omega)(\tilde{u}_{e_k}, \tilde{v}_{e_k}) - \Phi_0(T, \tau - T, \theta_{-T}\omega)(\mathcal{M}\tilde{u}_{e_k}, \mathcal{M}\tilde{v}_{e_k})\|_{L^2 \times L^2} \to 0,
\]
as \(k \to \infty\), which implies
\[
\|\tilde{u}_{e_k} - \tilde{u}\|_{L^2(\mathcal{O})} \to 0 \text{ as } k \to \infty, \tag{94}
\]
By (90), we have \(\|\tilde{u}_{e_k} - \tilde{u}\|_{L^2(\mathcal{O})} \to 0 \text{ as } k \to \infty\), which together with (94) implies that
\[
\|\tilde{v}_{e_k} - \tilde{v}\|_{L^2(\mathcal{O})} \to 0 \text{ as } k \to \infty. \tag{95}
\]
Furthermore, we consider the sequence \((\tilde{u}_{e_k}, \tilde{v}_{e_k}) \in \mathcal{A}_{e_k}(\tau - T, \theta_{-T}\omega)\). By (93), \(\|\tilde{u}_{e_k}\|_{H^1_{k}} + \|\tilde{v}_{e_k}\|_{H^1_{k}}\) is bounded in \(k\), which along with Lemma 4.5 implies that
\[
\|\tilde{u}_{e_k} - \tilde{u}\|_{L^2(\mathcal{O})} \leq c\epsilon_k(\|\tilde{u}_{e_k}\|_{H^1_{k}} + \|\tilde{v}_{e_k}\|_{H^1_{k}}) \leq C\epsilon_k \to 0.
\]
Since \(\{(\tilde{u}_{e_k}, \tilde{v}_{e_k})\}\) has a convergent subsequence (denoted by itself) in \(H^1(\mathcal{O}) \times L^2(\mathcal{O})\) and thus in \(L^2(\mathcal{O}) \times L^2(\mathcal{O})\). Then, the above convergence shows that the corresponding subsequence \(\{(\mathcal{M}\tilde{u}_{e_k}, \mathcal{M}\tilde{v}_{e_k})\}\) is a Cauchy sequence in \(L^2(\mathcal{O}) \times L^2(\mathcal{O})\) and thus in \(L^2(\mathcal{Q}) \times L^2(\mathcal{Q})\). So, there is a \((\overline{u}_0, \overline{v}_0) \in L^2(\mathcal{Q}) \times L^2(\mathcal{Q})\) such that
\[
(\mathcal{M}\tilde{u}_{e_k}, \mathcal{M}\tilde{v}_{e_k}) \to (\overline{u}_0, \overline{v}_0) \text{ in } L^2(\mathcal{Q}) \times L^2(\mathcal{Q}) \text{ as } k \to \infty.
\]
By the continuity of the operator \(\Phi_0 : L^2(\mathcal{Q}) \times L^2(\mathcal{Q}) \to L^2(\mathcal{Q}) \times L^2(\mathcal{Q})\), we have
\[
\Phi_0(T, \tau - T, \theta_{-T}\omega)(\mathcal{M}\tilde{u}_{e_k}, \mathcal{M}\tilde{v}_{e_k}) \to \Phi_0(T, \tau - T, \theta_{-T}\omega)(\overline{u}_0, \overline{v}_0)
\]
in $L^2(\mathcal{O}) \times L^2(\mathcal{O})$, and so in $L^2(\mathcal{O}) \times L^2(\mathcal{O})$ by expending the domain. This together with (95) implies that $(\bar{u}, \bar{v}) = \Phi_0(T, \tau - T, \theta - T)\Phi(\bar{u}_0, \bar{v}_0)$ in $L^2(\mathcal{O}) \times L^2(\mathcal{O})$. So, $(\bar{u}, \bar{v}) = \Phi_0(T, \tau - T, \theta - T)\Phi(\bar{u}_0, \bar{v}_0)$ a.e. on $\mathcal{O}$, which implies

\[(\bar{u}, \bar{v}) = \Phi_0(T, \tau - T, \theta - T)\Phi(\bar{u}_0, \bar{v}_0) \text{ in } H^1(\mathcal{O}) \times L^2(\mathcal{O}).\]

Then, by the construction in step 3, it follows that $(\mathcal{M}\bar{u}_k, \mathcal{M}\bar{v}_k) \in K_0(\tau - T, \theta - T)$ for all $k \in \mathbb{N}$. Hence, the limit $(\bar{u}_0, \bar{v}_0) \in K_0(\tau - T, \theta - T)$ in view of the closedness of $K_0$.

Finally, by (92) and $T \geq T_0$, we have

\[
\text{dist}_{H^1(\mathcal{O}) \times L^2(\mathcal{O})}((\bar{u}, \bar{v}), A_0(\tau, \omega)) = \text{dist}_{H^1(\mathcal{O}) \times L^2(\mathcal{O})}(\Phi_0(T, \tau - T, \theta - T)\Phi(\bar{u}_0, \bar{v}_0), A_0(\tau, \omega)) < \delta_0.
\]

This gives a contradiction with (91). The proof is complete. \hfill \Box

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