Scalar Quadratic Maximum-likelihood Estimators for the CMB Cross-power Spectrum

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Abstract

Estimating the cross-correlation power spectra of the cosmic microwave background, in particular, the \(TB\) and \(EB\) spectra, is important for testing parity symmetry in cosmology and diagnosing insidious instrumental systematics. The quadratic maximum-likelihood (QML) estimator provides optimal estimates of the power spectra, but it is computationally very expensive. The hybrid pseudo-\(C\) estimator is computationally fast but performs poorly on large scales. As a natural extension of previous work, in this article, we present a new unbiased estimator based on the Smith–Zaldarriaga (SZ) approach of \(E-B\) separation and the scalar QML approach to reconstruct the cross-correlation power spectrum, called the QML-SZ estimator. Our new estimator relies on the ability to construct scalar maps, which allows us to use a scalar QML estimator to obtain the cross-correlation power spectrum. By reducing the pixel number and algorithm complexity, the computational cost is nearly one order of magnitude smaller and the running time is nearly two orders of magnitude faster in the test situations.

Unified Astronomy Thesaurus concepts: Cosmic microwave background radiation (322)

1. Introduction

In the past two decades, a series of cosmic microwave background (CMB) experiments, e.g., DASI (Kovac et al. 2002), WMAP (Benabed et al. 2001; Hinshaw et al. 2007; Komatsu et al. 2011), BOOMERAnG (Montroy et al. 2006), QUAD (Brown et al. 2009), BICEP (Chiang et al. 2010), QUIET (QUIET Collaboration et al. 2012), ACT (Naess et al. 2014), Planck (Planck Collaboration et al. 2014), and SPTpol (Henning et al. 2018), have already provided large amounts of high-quality data, responsible for obtaining tight constraints on the cosmological parameters. As the study of cosmology enters the age of precision, research on CMB cross-power spectra becomes possible and plays a more important role, both for data characterization and their scientific interpretation.

In the early cosmological scenario, quantum fluctuations produce primordial density and primordial gravitational waves. Both contribute to the CMB temperature anisotropy, and the latter also produces a distinguishable feature of CMB polarization. According to the standard cosmological model, primordial gravitational waves can generate not only the autocorrelation \(TT\), \(EE\), and \(BB\) power spectrum of the CMB, but also the \(TE\) power spectrum. The classic cosmological model believes that the physical mechanism in the process of photon propagation is parity invariant, which results in the spectra of \(EB\) and \(TB\) vanishing (Seljak & Zaldarriaga 1997; Kamionkowski et al. 1997; Krauss et al. 2010; Garcia-Bellido 2011). The information of these power spectra can be used to probe the primordial fluctuations. The \(TB\) and \(EB\) power spectra are good null tests and can be used to detect the presence of an instrument and/or astrophysical system effects (Hu et al. 2003; Yadav et al. 2010). In addition, some nonstandard cosmological mechanisms that could produce nonvanishing cross-spectra and reconstruct the cross-power spectrum are also of great significance in checking some parity-violating interactions and go beyond the standard models (Lue et al. 1999; Feng et al. 2006; Li & Zhang 2008; Wang et al. 2013; Zhao & Li 2014a, 2014b; Zhu et al. 2013; Qiao et al. 2020). These tests may have far-reaching consequences for our understanding of the universe.

Associated with CMB experimental developments, many investigations of the techniques have been undertaken to reconstruct the cross-correlation power spectrum from maps of the partial CMB sky. A standard approach presented in Hivon et al. (2002) and Tristram et al. (2005) is the most straightforward way to construct the power spectrum for a partial-sky situation, which usually is used to estimate temperature multipoles and \(E\)-mode multipoles. Because in the standard approach there is no explicit correction for the \(E\)-to-\(B\) leakage, the performance of this approach to estimate \(B\)-mode multipoles is poor. To solve the problem that exists in the standard approach, several extensions of the standard pseudo-\(C\) (PCL) methods (Hansen & Gorski 2003; Smith 2006; Smith & Zaldarriaga 2007; Zhao & Baskaran 2010; Kim & Naselsky 2010; Kim 2011; Grain et al. 2012; Liu et al. 2019; Ghosh et al. 2021) have been proposed. These methods reconstruct the power spectra by inverting the linear system relating the full-sky power to the power from the incomplete sky. These methods are based on fast spherical harmonic transforms, with the advantage of speeding up their computation. The method proposed in Smith (2006) and Smith & Zaldarriaga (2007; hereafter the SZ method) was shown to be the PCL estimator (Ferté et al. 2013) with the smallest errors. In our paper, we adopt a hybrid approach (Grain et al. 2012) to reconstruct \(TB\) and \(EB\), where the \(T\)-mode and \(E\)-mode multipoles are obtained from the standard estimator and the \(B\)-mode multipoles are obtained from the SZ estimator.

The quadratic maximum-likelihood (QML) method (Tegmark & de Oliveira-Costa 2001), which is a pixel-based estimator, provides another way to solve the \(E-B\) mixing problem. It has the advantage of minimizing spectra uncertainties. However, it involves matrix inversions and multiplications, which significantly...
increases the calculation time and the demand for computational memory. Combining the advantages of the above two methods, we propose another new method to reconstruct the large-scale cross-correlation power spectrum: the QML-SZ method. The QML-SZ method uses the SZ method to derive the pure E-mode map $\hat{E}(\hat{n})$ and the pure B-mode map $\hat{B}(\hat{n})$ from the Stokes $Q$ and $U$ maps, which can be ultimately treated as scalar fields, as the T-mode map $T(\hat{n})$ is a scalar map. These scalar maps enable us to use the QML method developed for CMB temperature maps to calculate $TE$, $TB$, and $EB$ power spectrum. Because we adopt the scalar-mode QML method to reconstruct the cross-correlation power spectrum, the number of pixels drops to one-third of the standard QML method. This means that the computational running time will be much shorter than the standard QML estimator and drastically reduce our computational requirements.

This paper is organized as follows. In Section 2, we present our conventions and notation for describing CMB fields on the sphere. In Section 3, we review the three PCL estimators and SZ estimator first, and then introduce the scalar-mode QML method and combine it with the hybrid PCL method to construct the QML-SZ estimator. The simulation setup and details can be found in Section 4. In Section 5, we apply these methods to realistic situations and make a comprehensive comparison of their performance. Conclusions and discussions are given in Section 6. The Appendix shows the results of the TE power spectrum.

2. Notation and Conventions

In this section, we will briefly summarize the notations and definitions used in this paper. The CMB temperature fluctuation on the sphere is a scalar field, with fluctuations $\Delta T(\hat{n})$ at the level of $10^{-5}$ of the average value $T = 2.725$ K. For full-sky observations, the CMB temperature fluctuations can be expanded in spherical harmonics as

$$\Delta T(\hat{n}) = \sum_{\ell m} T_{\ell m} Y_{\ell m}(\hat{n}),$$

where $\hat{n}$ denotes the line of sight, $Y_{\ell m}(\hat{n})$ are the spherical harmonics, and $T_{\ell m}$ are the corresponding coefficients. The linearly polarized CMB polarization field does not contain a circular polarization component. Therefore, it can be characterized by Stokes parameters $Q$ and $U$. We can define $\pm P(\hat{n})$ as follows:

$$\pm P(\hat{n}) = Q(\hat{n}) \pm iU(\hat{n}).$$

The relationship between these partial-sky coefficients and the full-sky coefficients can be expressed as

$$\pm P(\hat{n}) = \sum_{\ell m} a_{\pm, \ell m} Y_{\ell m}(\hat{n}).$$

The polarization field can be decomposed into $E$- and $B$-mode parts, which can be expressed as linear combinations of $a_{\pm, \ell m}$:

$$E_{\ell m} = -\frac{1}{2} [a_{+, \ell m} + a_{-, \ell m}],$$

$$B_{\ell m} = -\frac{1}{2} [a_{+, \ell m} - a_{-, \ell m}].$$

One can construct the scalar $E$-mode and pseudoscalar $B$-mode fields as

$$E(\hat{n}) = \sum_{\ell m} E_{\ell m} Y_{\ell m}(\hat{n}), \quad B(\hat{n}) = \sum_{\ell m} B_{\ell m} Y_{\ell m}(\hat{n}).$$

Finally, the power spectrum estimate can be obtained as follows (Grishchuk & Martin 1997; Zhao et al. 2009):

$$C_{\ell}^{XZ} = \frac{1}{2\ell + 1} \sum_{m} X_{\ell m} Z_{\ell m}^*,$$

where $X, Z \in \{T, E, B\}$.

3. Power Spectrum Estimators

3.1. Pseudo-C$\ell$ Estimator

In this subsection, we will introduce standard and pure harmonic coefficients for incomplete sky coverage. The $E$- and $B$-mode decomposition is not unique for an incomplete sky, which leads to leakage from $E$ to $B$ and $B$ to $E$. The standard harmonic coefficient relations have no explicit correction for this leakage while the pure method corrects for the leakage problem. Because the $E$ modes have much larger power than the $B$ modes, the pure method is only used in the $B$-mode case. So, the PCL estimator in this work uses a standard simple harmonic calculation for the $T, E$ modes and the pure method for $B$ modes. We will first summarize these definitions before defining the PCL estimator relations.

3.1.1. Standard Harmonic Coefficient Definitions

For an incomplete-sky observation, defined by a binary mask $W$, we can intuitively define the partial sky $T'$, $E'$, and $B'$-mode harmonic coefficients (indicated by overhead tilde) as

$$\tilde{T}_{\ell m} = \int \Delta T W Y_{\ell m}^* \, d\hat{n},$$

$$\tilde{E}_{\ell m} = -\frac{1}{2} \int W [+P \, +2Y_{\ell m}^* + -P \, -2Y_{\ell m}^*] \, d\hat{n},$$

$$\tilde{B}_{\ell m} = -\frac{1}{2} \int W [+P \, +2Y_{\ell m}^* - -P \, -2Y_{\ell m}^*] \, d\hat{n}.$$

The coupling matrices, $K_{\ell m'}^{XY}$, can be derived from the definitions, and the full form of these matrices can be found in Ferté et al. (2013).
3.1.2. Pure Harmonic Coefficient Definitions

In any CMB experiment, we perform E–B decomposition on an incomplete sky due to foreground masking or survey footprint. As we stated before, this leads to leakage from one type of polarization mode to another. Because the $B$ modes are orders of magnitude smaller than the $E$-mode signal, the $E$-to-$B$ leakage is a critical problem for CMB polarization experiments. Various methods have been proposed in the literature to avoid the $E$–$B$ leakage problem, such as Bunn et al. (2003), Bunn (2011), Lewis (2003), Cao & Fang (2009), Louis et al. (2013), Grahn et al. (2009), Smith (2006), Smith & Zaldarriaga (2007), Zhao & Baskaran (2010), Kim & Naselsky (2010), Santos et al. (2016, 2017), and Ghosh et al. (2021). The pure method (also called the SZ method), proposed in Smith (2006) and Smith & Zaldarriaga (2007), has been shown (Ferté et al. 2013) to have the best performance in reducing error bars. The key to this approach is to apply the spin-raising and spin-lowering operators (Newman & Penrose 1966), $\sigma$ and $\bar{\sigma}$, on $P$ to construct two scalar (pseudoscalar) fields $E$ and $B$. The expression for pure $E$ and $B$ fields as defined in Smith & Zaldarriaga (2007) and Zhao & Baskaran (2010) is

$$E(\hat{n}) = -\frac{1}{2} \{ \bar{\sigma} \sigma P_{\ell m}(\hat{n}) + \sigma \bar{\sigma} P_{\ell m}(\hat{n}) \},$$

(11)

$$B(\hat{n}) = -\frac{1}{2i} \{ \bar{\sigma} \sigma P_{\ell m}(\hat{n}) - \sigma \bar{\sigma} P_{\ell m}(\hat{n}) \}. $$

(12)

The pure $E$ and pure $B$ fields defined here are two mutually orthogonal scalar and pseudoscalar fields. The $E(\hat{n})$ and $B(\hat{n})$ fields can be decomposed in terms of spherical harmonics as usual, with spherical harmonic coefficients $E_{\ell m}$ and $B_{\ell m}$. They can be computed from the pure fields as

$$E_{\ell m} = \int E(\hat{n}) Y_{\ell m}^*(\hat{n}) d\Omega,$$

(13)

$$B_{\ell m} = \int B(\hat{n}) Y_{\ell m}^*(\hat{n}) d\Omega.$$  

(14)

This new pure $E$- and $B$-mode spherical harmonic coefficients are related to the full-sky $E$- and $B$-mode coefficients as (Seljak & Zaldarriaga 1997)

$$E_{\ell m} = N_{\ell,2} E_{\ell m},$$

(15)

$$B_{\ell m} = N_{\ell,2} B_{\ell m}. $$

(16)

For an incomplete-sky observation defined by the window function $W(\hat{n})$, the partial-sky harmonic coefficients of the $E$ and $B$ fields (with an overhead tilde) are defined as (Efstathiou 2004)

$$\tilde{E}_{\ell m} = -\frac{1}{2} \int d\Omega \{ P_{\ell m}(\hat{n}) [\bar{\sigma} \sigma W(\hat{n}) Y_{\ell m}(\hat{n})] \}^*, $$

(17)

$$\tilde{B}_{\ell m} = -\frac{1}{2i} \int d\Omega \{ P_{\ell m}(\hat{n}) [\bar{\sigma} \sigma W(\hat{n}) Y_{\ell m}(\hat{n})] \}^* - \bar{P}_{\ell m}(\hat{n}) \{ \bar{\sigma} \sigma W(\hat{n}) Y_{\ell m}(\hat{n}) \}^*. $$

(18)

These expressions can be expanded and simplified further for implementation and full expressions can be found in Wang et al. (2016). Once the coefficients $\tilde{E}_{\ell m}$ and $\tilde{B}_{\ell m}$ are derived, the scalar fields in our observation window $W(\hat{n})E(\hat{n})$ and $W(\hat{n})B(\hat{n})$ can be directly obtained by inverse harmonic transforms.

The partial-sky pure harmonic coefficients $\tilde{E}_{\ell m}$ and $\tilde{B}_{\ell m}$ are related to the full-sky harmonic coefficients $E_{\ell m}$ and $B_{\ell m}$ as follows:

$$\tilde{E}_{\ell m} = \sum_{\ell' m'} [K_{\ell m,\ell'm}^E E_{\ell' m'} + i K_{\ell m,\ell'm}^B B_{\ell' m'}],$$

(19)

$$\tilde{B}_{\ell m} = \sum_{\ell' m'} [-i K_{\ell m,\ell'm}^E E_{\ell' m'} + K_{\ell m,\ell'm}^{BB} B_{\ell' m'}],$$

(20)

where the pure field mixing kernels are denoted by $K_{\ell m,\ell'm}^{XY}$. Their detailed expressions are given in Grain et al. (2009) and Ferté et al. (2013). The cross-mixing matrix for $EB$ or $BE$ for the pure method is orders of magnitude smaller than that for the standard definition of Equation (10). This implies that the pure fields are nearly orthogonal with very small mixing between the two polarization modes.

3.1.3. PCL Estimator Definition

In this work, the PCL estimator is constructed with standard $E$-mode and pure-$B$-mode definitions. The $T$-mode definition is unchanged. The cross-spectra estimators are defined as follows:

$$\langle \hat{C}_{\ell}^{TE} \rangle = \frac{1}{2\ell + 1} \sum_m (T_{\ell m}^{(std)} E_{\ell m}^{(std)*}),$$

(21)

$$\langle \hat{C}_{\ell}^{TB} \rangle = \frac{1}{2\ell + 1} \sum_m (T_{\ell m}^{(std)} B_{\ell m}^{*}),$$

(22)

$$\langle \hat{C}_{\ell}^{EB} \rangle = \frac{1}{2\ell + 1} \sum_m (E_{\ell m}^{(std)} B_{\ell m}^{*}),$$

(23)

where $N_{\ell,s} = (\ell + s)!/\ell! (s - \ell)!$ and $\langle \cdots \rangle$ implies averaging over realizations.

To reconstruct the actual CMB cross-power spectra, the relationship between the partial-sky cross-spectra (denoted by an overhead tilde) and the full-sky spectra is necessary. For the $TE$ and $TB$ parts,

$$\langle \hat{C}_{\ell}^{TE} \rangle = \sum_{\ell'} \{ M_{\ell \ell'}^{TE,TE} C_{\ell'}^{TE} + M_{\ell \ell'}^{TE,EB} C_{\ell'}^{EB} \},$$

(24)

$$\langle \hat{C}_{\ell}^{TB} \rangle = \sum_{\ell'} \{ M_{\ell \ell'}^{TB,TE} C_{\ell'}^{TE} + M_{\ell \ell'}^{TB,EB} C_{\ell'}^{EB} \}. $$

(25)

The $EB$ part is more complicated, and their relationship is given by the following expression:

$$\langle \hat{C}_{\ell}^{EE} \rangle = \sum_{\ell'} \{ M_{\ell \ell'}^{EE,EE} C_{\ell'}^{EE} + M_{\ell \ell'}^{EE,EB} C_{\ell'}^{EB} \} + \sum_{\ell'} \{ M_{\ell \ell'}^{EB,EE} C_{\ell'}^{EE} + M_{\ell \ell'}^{EB,EB} C_{\ell'}^{EB} \},$$

(26)

$$\langle \hat{C}_{\ell}^{BB} \rangle = \sum_{\ell'} \{ M_{\ell \ell'}^{BB,EE} C_{\ell'}^{EE} + M_{\ell \ell'}^{BB,EB} C_{\ell'}^{EB} \} + \sum_{\ell'} \{ M_{\ell \ell'}^{EB,EE} C_{\ell'}^{EE} + M_{\ell \ell'}^{EB,EB} C_{\ell'}^{EB} \}. $$

(27)

The detailed expression of the mixing matrices, $M_{\ell \ell'}^{XY}$, can be derived from the definition and are listed in the appendix of Ferté et al. (2013). The PCL estimator for this work has been implemented with the Python package of NaMaster (Alonso et al. 2019).
3.2. Standard QML Estimator

For CMB observations with any sky coverage, Tegmark & de Oliveira-Costa (2001) defined the optimal QML estimator for the temperature and polarization. In this section, we will briefly review the QML estimator. We define the input data vector, $\mathbf{x}$, consisting of the temperature and the Stokes $Q$ and $U$ fields (with respect to a fixed coordinate system) at the $i$th pixel as

$$\mathbf{x}_i = \begin{pmatrix} \Delta T_i \\ Q_i \\ U_i \end{pmatrix} + \begin{pmatrix} n_{i1}^T \\ n_{i2}^Q \\ n_{i3}^U \end{pmatrix},$$

(29)

where $n_{i1}^T$ denotes the noise. The optimal quadratic estimate of the power spectrum, $\gamma_{X}^{XY}$, is defined as (Tegmark & de Oliveira-Costa 2001)

$$\gamma_{X}^{XY} = \mathbf{x}_i^T \mathbf{E}_{\ell}^{XY} \mathbf{x}_j - b_{X}^{XY}, \quad X, Y \in [T, E, B],$$

(30)

where $t$ indicates the matrix transpose operation. Here $i, j$ are the indices over pixels, and $\mathbf{E}_{\ell}^{XY}$ is a $3 \times 3$ matrix. The bias term, $b_{X}^{XY}$, corrects for the noise bias and is computed as $\text{Tr} \{ \mathbf{E}_{\ell}^{XY} \}$, assuming the noise to be uncorrelated between pixels. In these relations, we have assumed the summation convention. The $\mathbf{E}_{\ell}^{XY}$ matrices are computed as

$$\mathbf{E}_{\ell}^{XY} = \frac{1}{2} C^{-1} \frac{\partial C}{\partial \ell_{XY}} C^{-1},$$

(31)

with the covariance matrix of $\mathbf{x}$ denoted by $C$. The detailed expressions for the covariance matrix can be found in Tegmark & de Oliveira-Costa (2001).

The $\gamma_{X}^{XY}$ gives an unbiased estimate of the actual power spectra $C_{XY}$. Using Equations (30) and (31) we can get the expectation value of $\gamma_{X}^{XY}$ as

$$\langle \gamma_{X}^{XY} \rangle = F_{\ell}^{XYPO} C_{\ell}^{PO},$$

(32)

where $X, Y, P, Q \in [T, E, B]$, and we have used the Fisher matrix defined as

$$F_{\ell}^{XYPO} = \frac{1}{2} \text{Tr} \left[ \frac{\partial C}{\partial \ell_{XY}} C^{-1} \frac{\partial C}{\partial \ell_{PO}} C^{-1} \right].$$

(33)

When $F$ is invertible, one can define unbiased estimates of the true power spectra via

$$C_{\ell}^{XY} = F_{\ell}^{-1} \gamma_{\ell}^{XY}.$$

(34)

The covariance matrix of $\gamma_{X}^{XY}$ is then given by

$$\langle \gamma_{X}^{XY} \gamma_{Y}^{PO} \rangle - \langle \gamma_{X}^{XY} \rangle \langle \gamma_{Y}^{PO} \rangle = F_{\ell}^{XYPO} \equiv 2 \text{Tr} \{ \mathbf{C} \mathbf{E}_{\ell}^{XY} \mathbf{C} \mathbf{E}_{\ell}^{PO} \},$$

(35)

with $F_{\ell}^{XYPO}$ being the Fisher matrix. Hence, the covariance matrix of the actual power spectrum estimates of Equation (34) is

$$\langle \Delta \hat{C}_\ell \Delta \hat{C}_\ell \rangle = F^{-1}.$$

(36)

The QML estimators for this work have been implemented with the xQML Python package (Vanneste et al. 2018).

3.3. QML-SZ Estimator

The standard QML estimator, for CMB $TQU$ maps, described before is optimal but it is computationally prohibitive at high resolutions. We attempt to reduce the size of the computation problem by essentially reducing the estimation of each cross-spectra as its own scalar problem. We have previously demonstrated this method for the $B$-mode auto spectrum in Chen et al. (2021).

We use the pure method definitions of simple harmonic coefficients to compute pure-$E$- and pure-$B$-mode fields. With this definition, any correlations due to $E$-to-$B$ or $B$-to-$E$ leakages should be suppressed by a few orders of magnitude. We would then treat the cross-spectra computation as a scalar problem with the scalar-temperature-only QML method. We will outline this “scalar” treatment of the cross-spectra below.

Let $d_{\ell}^{XY}$ denote the $i$th pixel value in the scalar map $s_{\ell}^{X}$, where $s_{\ell}^{X}$ is the signal and $n_{\ell}^{X}$ is the noise in the individual pixel. The covariance matrix $C_{XY}$ of observations $d_{\ell}^{XY}$ and $d_{\ell}^{XY}$, with $X, Y \in \left[ T, E, B \right]$, and $X \neq Y$, can be written as

$$C_{XY}^{\ell} = \left( d_{\ell}^{X} d_{\ell}^{Y} \right) = 2 \ell + 1 \frac{4}{4 \pi} C_{XY}^{\ell} P_{\ell}(z) + N_{XY}^{\ell},$$

(37)

where $C_{XY}^{\ell}$ is the cross-spectrum corresponding to the scalar signal maps $s_{\ell}^{X}(\hat{n})$ and $s_{\ell}^{Y}(\hat{n})$. $P_{\ell}$ denotes a Legendre polynomial, and $z$ is the cosine of the angle between the two pixels under consideration. Here, $N_{XY}^{\ell}$ is the noise covariance matrix.

We can define the quadratic estimator with the “scalar” approximation as

$$\gamma_{XY}^{\ell} = \left( d_{\ell}^{X} d_{\ell}^{Y} \right) = F_{\ell}^{XY} d_{\ell}^{X} d_{\ell}^{Y} - b_{XY}^{\ell}.$$

(38)

The matrices $F_{\ell}^{XY}$ have a similar form to the equation in Section B of Vanneste et al. (2018):

$$F_{\ell}^{XY} = \frac{1}{2} \left( C_{\ell}^{XY} - \frac{\partial C_{\ell}^{XY}}{\partial \ell_{XY}} \right)^{-1},$$

(39)

where $\frac{\partial C_{\ell}^{XY}}{\partial \ell_{XY}} = C_{\ell}^{XX} F_{\ell}^{XY} + C_{\ell}^{XY} F_{\ell}^{XY}$. Likewise, the mode-mixing matrix, $F_{\ell}^{XY}$, expression becomes

$$F_{\ell}^{XY} = \frac{1}{2} \text{Tr} \left[ \left( C_{\ell}^{XX} \right)^{-1} \frac{\partial C_{\ell}^{XY}}{\partial \ell_{XY}} \left( C_{\ell}^{XY} \right)^{-1} \frac{\partial C_{\ell}^{XY}}{\partial \ell_{XY}} \right].$$

(40)

Finally, the QML estimator for the cross-power spectrum $\hat{C}_{XY}$ is given by

$$\hat{C}_{XY} = \left( F_{\ell}^{XY} \right)^{-1} \gamma_{XY}^{\ell}.$$

(41)

When computing the $\mathcal{E}$ or $\mathcal{B}$ fields, we must use a proper sky apodization to avoid numerical divergences in the calculation of the window function derivatives. Wang et al. (2016) and Kim (2011) have shown that a Gaussian smoothing kernel induces very small leakage in the final map. In this work, we will use Gaussian apodization to obtain the $\mathcal{E}$ and $\mathcal{B}$ maps for the QML-SZ method. For the $i$th pixel in the region allowed by the binary mask, the apodized window is defined as

$$W_i = \begin{cases} \frac{1}{2} + \frac{1}{2} \text{erf} \left( \frac{\delta_i - \delta_c}{\sqrt{2} \sigma} \right), & \delta_i < \delta_c, \\ 1, & \delta_i > \delta_c \end{cases},$$

(42)
where $\delta_i$ is the closest distance between the $i$th observed pixel from the boundary of the allowed region, $\sigma = \text{FWHM}/\sqrt{8 \ln 2}$ with FWHM denoting the full width at half maximum of the Gaussian kernel, and $\delta_u$ is the apodization length that acts as an additional adjustable parameter.

The steps to estimate the cross-spectrum with the QML-SZ estimator can be summarized as follows: First, for the given observed $Q$ and $U$ polarization maps, we construct a partial-sky pure-$E$-map, $\mathcal{E}(\theta)$, and pure-$B$-map, $\mathcal{B}(\theta)$, using a Gaussian apodized window function. For the $T$ map, we just use the observed $T$ map directly. Then, we use the two maps and the corresponding fiducial cross-spectrum as input to estimate the cross-spectrum $C_{TT}^{XX}$. In comparison to the standard QML estimator, our goal with the QML-SZ method is to simplify the calculation without compromising significantly on the accuracy or error bars. We implement the QML-SZ estimator with the modified xQML Python package.

4. Simulation Setup

In this section, we will outline the simulation pipeline used in this work. Here, we consider two cases of future CMB polarization experiments: a space-based and a ground-based CMB polarization experiment. For the space-based experiment case, we consider the 2018 Planck common polarization mask, which masks the Galactic foregrounds and the point sources resolved in Planck maps. We fill in all point sources smaller than $5'$, as well as the extended source masking at high Galactic latitudes ($|b| > 45'$) by using the HEALPix process_mask subroutine. So, we obtain a $\approx 78\%$ sky coverage patch for the space-based CMB experiment shown in the upper panel of Figure 1. Second, we consider the AliCPT-1 experiment (Li et al. 2017; Salatino et al. 2020), a CMB experiment in the Northern Hemisphere, to represent a ground-based CMB experiment. The observed area covers $\approx 15.1\%$ of the full sky and its binary mask is shown in the lower panel of Figure 1.

The observed data are assumed to consist of two parts: the CMB signal and the instrumental noise. The input power spectra for CMB maps are computed with CAMB\(^6\) (Lewis et al. 2000), using the 2018 Planck cosmological parameters as given by Planck Collaboration et al. (2020), with lensing and the tensor-to-scalar ratio $r = 0.05$. The CMB maps are produced using the synfast subroutine of HEALPix\(^7\) at NSIDE = 512 with $\ell_{\text{max}} = 1024$. For the noise map, we assumed a Gaussian homogeneous noise model with the noise rms set to 3$\mu$K-arcmin, which is a typical value for the next generation of space-based experiments (Hazumi et al. 2020; Hanany et al. 2019; Finelli et al. 2018; Zhao 2011; Huang et al. 2015). This equates to a white-noise level of 0.44$\mu$K-pixel at at NSIDE = 512. Finally, adding the CMB signal map and noise map together, we obtain the simulated observation. Depending on the method, we further preprocess these maps, as detailed below.

4.1. Space-based Experiment

After obtaining the observed maps, the subsequent preprocessing steps are different for the three estimators:

- \textbf{PCL estimator:} We smooth the masked TQU map with a Gaussian smoothing with FWHM = 20' first. We apodize our observation window with a “C2” (cosine) apodization function for the PCL estimator to reduce mode mixing and leakage in this work. The weight in the $i$th pixel is given as (Alonso et al. 2019)

$$W_i = \begin{cases} \frac{1}{2} [1 - \cos(\pi \delta_i^u)] & \delta_i^u < 1 \\ 1 & \text{otherwise} \end{cases}$$

where $\delta_i^u = \sqrt{(1 - \cos \delta_i)/(1 - \cos \delta_u)}$. For the space-based experiment, case $\delta_i^u$ is set to $6^\circ$.

- \textbf{Standard QML estimator:} Downgrading high-resolution maps to low-resolution maps is the key to the standard QML estimator. To extract the large-scale information, we degrade the resolution to HEALPix NSIDE = 16 with the following smoothing:

$$f(\ell) = \begin{cases} 1, & \ell \leq N_{\text{side}} \\ \frac{1}{2} \left(1 + \cos \left(\frac{(\ell - N_{\text{side}})\pi}{2N_{\text{side}}}\right)\right), & N_{\text{side}} < \ell \leq 3N_{\text{side}} \\ 0, & \ell > 3N_{\text{side}} \end{cases}$$

We smooth the $a_{\ell m}$ obtained from the input maps at NSIDE = 512 with the smoothing function of Equation (44). We then obtain the smoothed map by performing inverse spherical harmonic transforms.

- \textbf{QML-SZ estimator:} In the QML-SZ estimator, the power spectrum of the pure-$E$ map $\mathcal{E}(\theta)$ and pure-$B$ map $\mathcal{B}(\theta)$ is blue. We smooth the observed maps with FWHM = 8° to suppress higher multipoles and then mask them. We then use the SZ method to derive the pure-$E$ map $\mathcal{E}(\theta)$ and pure-$B$ map $\mathcal{B}(\theta)$ using a Gaussian apodized mask with $\sigma = 10^{-6}$, $\delta_u = 1^\circ$. Finally, we use the ud_grade subroutine of HEALPix to downgrade the smoothed mask, $T$ map, pure-$E$-mode map, and pure-$B$-mode map to NSIDE = 16. In the smoothed mask, we set all pixels with values $< 0.99$ to zero.

4.2. Ground-based Experiment

For the ground-based experiment, the coadded maps are multiplied by the AliCPT binary mask (see Figure 1) to keep only the fraction of sky observed in the considered experiment. Subsequent processing for the hybrid PCL estimator and standard QML estimator is similar to that for the space-based experiment case, we just modify the values of some parameters. For the hybrid PCL estimator, $\delta_i^u$ in Equation (43) is set to $10^\circ$. For the standard QML estimator, we set the target resolution NSIDE = 32. But for the QML-SZ estimator, we use a different way to deal with the “observed” CMB maps and noise maps. First, we use the SZ method to obtain the pure $E$-mode and $B$-mode maps from the smoothed $Q$ and $U$ maps using a Gaussian apodized mask with $\sigma = 10^{-4}$, $\delta_u = 0.5^\circ$. We obtain the spherical harmonic coefficients of the maps at NSIDE = 512 and set all harmonic coefficients to zero above $\ell_{\text{max}}$. We use these $a_{\ell m}$ with this cutoff to reconstruct the map at NSIDE = 512 but without the information above $\ell_{\text{max}}$. This methodology is applied to the $T$ map and $\mathcal{E}$ and $\mathcal{B}$ maps at NSIDE = 512 with $\ell_{\text{max}} = 192$. Then, we downgrade these maps to the targeted NSIDE = 32.
The results of the space-based experiment are shown in Figure 2. Considering that there is a significant difference in the order of magnitude on the error bars for $\ell < 10$ and for $\ell \geq 10$, the total range is split into two figures. We find that for both cross-spectra, all three estimators can get unbiased estimates of the input power spectrum but all the QML estimators have smaller error bars in the entire multipole range. We also notice that while the standard QML method has nearly optimal error bars throughout the entire multipole range, the QML-SZ method has suboptimal error bars for the lowest multipoles. The error bars for $\ell \leq 5$ show a significant increase for the QML-SZ estimator. This behavior is caused by the blue input power spectra for the QML-SZ method due to the $N_{\ell,2}$ or $N_{\ell,2}^2$ factors in the cross-spectra. Downgrading the map using ud_grade, there is some power leakage from high multipoles to low multipoles, leading to an increase in uncertainty at low multipoles, as shown in Chen et al. (2021).

We note that the QML estimator uses a binary mask, while the QML-SZ method uses the apodized mask. This apodization reduces the effective $f_{\text{sky}}$ for the QML-SZ method. While the performance of the QML-SZ estimator is not as good as that of the standard QML method, it is still a fast and reliable solution for power-spectrum estimation, except for the lowest few multipoles.

As we are working on a partial sky, the different multipoles are coupled. We can use the normalized covariance matrices to quantify the coupling between different multipoles; it is defined as

$$C_{\ell\ell'} = \frac{\text{cov}(\hat{C}_\ell, \hat{C}_{\ell'})}{\sqrt{\text{var}(\hat{C}_\ell)}\text{var}(\hat{C}_{\ell'})},$$

and the results of the space-based experiment case are shown in Figure 3. For the case considered here, all covariance matrices are approximately diagonal, which means the power spectra estimate only those weakly coupled different multipoles.

### 5.2. Ground-based Experiment

In ground-based experiments, we observe only a small fraction of the sky but with high sensitivity. This means we can choose a larger NSIDE and extend the $\ell_{\text{max}}$ to higher multipoles. Here, we use the AliCPT binary mask with $f_{\text{sky}} \sim 15.1\%$ to calculate the final results, and the estimated power spectra are binned with a bandwidth of $n_{\text{bin}} = 11$.

The results of the reconstructed cross-correlation power spectra are shown in Figure 4. We find that all the methods discussed here can give unbiased estimates of both the $TB$ and $EB$ band powers. When we focus on the error bars, we find that all three methods have near-optimal error bars in the entire multipole range. As the performance of the PCL estimator is as good as that of the QML methods, considering it is faster than QML-based estimators, it seems that we do not need to apply QML-based estimators to reconstruct the cross-correlation power spectrum for the ground-based experiment case.

In Figure 5, we have shown the normalized covariance matrices for the power-spectrum estimates with two QML methods; the first two pictures are the covariance matrix of $TB$ and the last two pictures are the covariance matrix of $EB$. For the ground-based experiment, we also find our covariance matrices to be approximately diagonal, showing that the band power leakages have been suitably removed.

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**Figure 1.** Binary masks showing the observed sky patch for a space-based experiment (top) and for the ground-based experiment (bottom) as considered in this work. The red area is the observed area, and the gray area is the masked area. The plots are in Galactic coordinate systems. The sky fractions are 78.8% and 15.1%, respectively.

**Figure 2.** Considering that there is a significant difference in the order of magnitude on the error bars for $\ell < 10$ and for $\ell \geq 10$, the total range is split into two figures. We find that for both cross-spectra, all three estimators can get unbiased estimates of the input power spectrum but all the QML estimators have smaller error bars in the entire multipole range. We also notice that while the standard QML method has nearly optimal error bars throughout the entire multipole range, the QML-SZ method has suboptimal error bars for the lowest multipoles. The error bars for $\ell \leq 5$ show a significant increase for the QML-SZ estimator. This behavior is caused by the blue input power spectra for the QML-SZ method due to the $N_{\ell,2}$ or $N_{\ell,2}^2$ factors in the cross-spectra. Downgrading the map using ud_grade, there is some power leakage from high multipoles to low multipoles, leading to an increase in uncertainty at low multipoles, as shown in Chen et al. (2021).

**Figure 3.** For the case considered here, all covariance matrices are approximately diagonal, which means the power spectra estimate only those weakly coupled different multipoles.

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**5. Realistic Examples**

In this section, we discuss the results of the above estimators applied to estimate the cross-correlation power spectra at the space-based experiment case in Section 5.1 and the ground-based experiment case in Section 5.2. We compare the computational requirements for these three methods in Section 5.3. The QML-SZ estimator can be used to estimate the $TE$ power spectrum. However, because the $TE$ power spectrum does not involve the $E-B$ leakage problem and it is signal dominated, the hybrid PCL estimator is adequate, and we do not need to reconstruct the $TE$ power spectrum using the QML-SZ estimator. We only show our results of the $TB$ and $EB$ spectrum estimates here. The $TE$ spectrum result is shown and discussed in the Appendix. In this work, our power-spectrum estimates for any estimator is a mean of 1000 random simulations, and the errors are computed as the standard deviation of the samples.

**5.1. Space-based Experiment**

One major advantage of space-based experiments is the ability to observe the full sky and therefore making measurements of the lowest multipoles of the power spectra. We simply use our estimators on the preprocessed maps to obtain the $TB$ and $EB$ cross-spectra.
5.3. Computational Performance

The computational complexity of the QML estimator is $O(N_d^3)$, where $N_d$ is the length of the data vector. Chen et al. (2021) has shown that $N_d$ for the QML-SZ method is one-third of the $N_d$ for the standard QML estimator. This significantly reduces the computational requirements for the QML-SZ method. In Tables 1 and 2, we summarize the computational parameters for these two QML estimators for the space-based and ground-based experiment cases, respectively. We run our computations on an Intel Xeon E2620 2.10 GHz workstation and list the $\text{NSIDE}$, $N_d$, $\ell_{\text{max}}$, RAM (in gigabytes), and computation time for a single computation.

As shown in Table 1, for the QML-SZ estimators, the computation time is only about $1/280$ of that for a classic QML estimator. This happens because the data vector size, $N_d$, for the scalar QML method is about one-third that for the classic QML method. Additionally, the QML-SZ method is based on the scalar-mode QML method, and its algorithm complexity is lower than that of the full-form standard QML method, thereby the computation is faster.

In Table 2, we show the same set of parameters for the ground-based experiment case. In this example, we compute the QML methods at $\text{NSIDE} = 32$. In Table 2, we show that the QML-SZ method saves on both computation time and memory.
requirements. Comparing with the results listed in the Table 1, we find that with the increase in $N_{\ell}$, the advantages of the QML-SZ method in computing time becomes obvious. For the ground-based case, the QML-SZ method is near optimal in the entire multipole range of interest, and their computational requirements imply that they can be applied to higher-resolution maps to compute the power spectrum at higher multipoles.

6. Discussions and Conclusions

In this work, we present one novel unbiased alternative estimator to estimate the cross-correlation power spectrum, called the QML-SZ estimator. We used the SZ method to solve the $E$-to-$B$ leakage problem existing in CMB polarized fields and obtain scalar $E$-mode map, $\mathcal{E}(\hat{n})$, and $B$-mode map, $B(\hat{n})$. By using the SZ method, we eliminate the coupling relationship between the polarization fields and decompose the polarization $QU$ maps into an independent pure $E$-mode map, $\mathcal{E}(\hat{n})$, and pure $B$-mode map, $B(\hat{n})$. After that, all CMB information is contained in three scalar maps, $T(\hat{n})$, $\mathcal{E}(\hat{n})$, and $B(\hat{n})$. Through the scalar QML estimator, we can reconstruct the cross-correlation power spectrum from these scalar maps.

We apply three different estimators to both space-based CMB experiment and ground-based CMB experiment; the details of the simulation steps can be found in Section 4. The final results of the $TE$ power spectrum are shown in the Appendix. For the Planck case, we find that the errors of the QML-SZ method are significantly larger than the hybrid PCL estimator and standard QML estimator for $\ell \lesssim 5$, while it performs near optimally for the rest of the multipole range. As to the ground-based CMB experiment, the performance of hybrid PCL as well as the other two QML methods. For the space-based CMB experiment scenario, all $TB$ and $EB$ power spectra reconstructed by QML estimators have better performance in reducing the error bars than the hybrid PCL estimator and standard QML estimator for $\ell \lesssim 5$, which leads to the low performances.

we can reconstruct the cross-correlation power spectrum from these scalar maps.
experiment, the performance of the hybrid PCL to estimate the $TB$ and $EB$ power spectra as well as the other two QML estimators too. That means the hybrid PCL is the best choice in most realistic ground-based CMB experiment scenarios.

Compared with the traditional QML estimator, the pixel number of the QML-SZ estimator is just $\sim 1/3$ of that of the traditional QML estimator, which greatly reduces the computational requirements (both the memory requirement and computation time).

In conclusion, we present a new estimator for estimating the cross-correlation power spectrum and compare it with the other two classic methods, the hybrid PCL estimator and standard QML estimator, in this article. We found that for the ground-based CMB experiment, the performance of a PCL estimator is good enough, the error bars of which are very close to the optimal errors and also have obvious advantages in calculation speed, we do not need to develop a QML-based estimator. The biggest advantage of our method is used to estimate the $TB$ and $EB$ power spectra in space-based cases. In this situation, the QML-SZ estimator has better performance than the hybrid PCL estimator in the entire multipole range of analysis. While the errors of the QML-SZ results are not as small as those of the standard QML method for the lowest few multipoles, considering the QML-SZ estimator saves a lot of running time and memory, it is still worth continuing research to improve its performance. We can study new filtering algorithms or try more degraded schemes to suppress the influence of high multipoles in future work.

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**Appendix**

**TE Power Spectrum**

The $TE$ power spectrum does not involve the $E-B$ leakage problem, so the PCL estimator is adequate in most situations. For the space-based experiment case, we show the final results of three estimators in Figure 6.

We find that all three methods give unbiased estimates for the TE band powers. The standard QML method has nearly optimal error bars throughout the entire multipole range, while the PCL method has suboptimal error bars. For the QML-SZ estimator, the error bars for $\ell \leq 5$ show a significant increase. The power spectrum of the scalar $E$ map $\hat{E}(\hat{n})$ is $C_{\ell}^{EE} = N_{\ell}^2 C_{\ell}^{EE}$, due to the existence of coefficient factors $N_{\ell}^2$, the power from high multipoles leaks to lower multipoles and increases the uncertainty on large angular scales. Based on the current framework, in order to solve this problem, we need to present a better downgrade method, and we will try to do this in our follow-up work. Here we just show a new possible method for estimating the $TE$ power spectrum.

The results of the ground-based experiment case are shown in Figure 7. We find that all three methods give quite similar results. All the estimators are unbiased, and the error bars are close to each other for every multipole. As well known, the PCL method is much faster than the QML-based methods, so we do not need to develop a new calculation estimator for the small-scale case to reconstruct $TE$ band powers.

Finally, the normalized covariance matrices for the power-spectrum estimates with the QML methods shown in Figure 8 and in Figure 9 and as you can see that the band power leakages have been suitably removed.

![Figure 6. Plot of the results for TE-mode power-spectrum estimates for the realistic space-based CMB experiment. The observed sky is simulated at NSIDE = 512 with $\ell_{\text{max}} = 1024$ and the noise level is set to $3\mu$K-arcmin. The input $TE$-mode power spectrum is shown with the black curve. The classic QML method results are shown by the orange line and the QML-SZ method results with the green line. These results are computed at NSIDE = 16. We also show the PCL estimator results, obtained with NaMaster, with $\xi = 6^\circ C2$ apodization, with the blue line. The gray region denotes the analytical approximation of the error bounds. The data points are the mean of 1000 estimates, and the error bars are given by the standard deviation of the estimates.](image-url)
Figure 7. Plot of the results for $TE$-mode power-spectrum estimates for the realistic ground-based CMB experiment. The observed sky is simulated at $\text{NSIDE} = 512$ with $\ell_{\text{max}} = 1024$ and the noise level is set to $3 \mu K$-arcmin. The input $TE$-mode power spectrum is shown with the black curve. The classic QML method results are shown with orange line, and QML-SZ method results are shown with green curve. These results are computed at $\text{NSIDE} = 32$. We also show PCL estimator results, obtained with NaMaster, with $\delta_C = 10^\circ$ C2 apodization, with blue line. The gray region denotes the analytical approximation of the error bounds. The data points are the mean of 1000 estimates, and the error bars are given by the standard deviation of the estimates.

Figure 8. Normalized covariance matrices $\hat{C}_{\ell_1\ell_2} = \text{cov}(\hat{C}_\ell, \hat{C}_\ell')/\sqrt{\text{var}(\hat{C}_\ell)\text{var}(\hat{C}_\ell')}$ of the QML methods for the space-based experiment. The matrices are obtained from estimates of 1000 simulations for the classic QML estimator (left) and for QML-SZ estimator (right).

Figure 9. Normalized covariance matrices $\hat{C}_{\ell_1\ell_2} = \text{cov}(\hat{C}_\ell, \hat{C}_\ell')/\sqrt{\text{var}(\hat{C}_\ell)\text{var}(\hat{C}_\ell')}$ of the QML methods for the ground-based experiment. The matrices are obtained from estimates of 1000 simulations for classic QML estimator (left) and for QML-SZ estimator (right).
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References

Alonso, D., Sanchez, J., Slosar, A. & LSST Dark Energy Science Collaboration 2019, MNRAS, 484, 4127
Benabed, K., Bernardeau, F., & van Waerbeke, L. 2001, PhRvD, 63, 043501
Brown, M. L., Ade, P., Bock, J., et al. 2009, ApJ, 705, 978
Bunn, E. F. 2011, PhRvD, 83, 083003
Bunn, E. F., Zaldarriaga, M., Tegmark, M., & de Oliveira-Costa, A. 2003, PhRvD, 67, 023501
Cao, L., & Fang, L.-Z. 2009, ApJ, 706, 1545
Chen, J., Ghosh, S., Liu, H., et al. 2021, ApJS, 257, 27
Chiang, H. C., Ade, P. A. R., Barkats, D., et al. 2010, ApJ, 711, 1123
Efstathiou, G. 2004, MNRAS, 349, 603
Feng, B., Li, M., Xia, J.-Q., Chen, X., & Zhang, X. 2006, PhRvL, 96, 221302
Grain, J., Tristram, M., & Stompor, R. 2009, PhRvD, 79, 123515
Grain, J., Tristram, M., & Stompor, R. 2012, PhRvD, 86, 076005
Grishchuk, L. P., & Martin, J. 1997, PhRvD, 56, 1924
Hanany, S., Alvarez, M., Artis, E., et al. 2019, BAAS, 51, 194
Hansen, F. K., & Görski, K. M. 2003, MNRAS, 343, 559
Hazumi, M., Ade, P. A. R., Adler, A., et al. 2020, Proc. SPIE, 11443, 114432F
Hinshaw, G., Sailer, J. T., Reichardt, C. L., et al. 2018, ApJ, 852, 97
Hinshaw, G., Nolta, M. R., Bennett, C. L., et al. 2007, ApJS, 170, 288
Hivon, E., Görski, K. M., Netterfield, C. B., et al. 2002, ApJ, 567, 2
Hu, W., Hedman, M. M., & Zaldarriaga, M. 2003, PhRvD, 67, 043004
Huang, Q.-G., Wang, S., & Zhao, W. 2015, JCAP, 2015, 035
Kamionkowski, M., Kosowsky, A., & Stebbins, A. 1997, PhRvL, 78, 2058
Kim, J. 2011, A&A, 531, A32
Kim, J., & Naselsky, P. 2010, A&A, 519, A104
Komatsu, E., Smith, K. M., Dunkley, J., et al. 2011, ApJS, 192, 18
Kovac, J. M., Leitch, E., Pryke, C., et al. 2002, Nat, 420, 772
Krauss, L. M., Dodelson, S., & Meyer, S. 2010, Sci, 328, 989
Lewis, A. 2003, PhRvD, 68, 083509
Lewis, A., Challinor, A., & Lasenby, A. 2000, ApJ, 538, 473
Li, H., Li, S.-Y., Liu, Y., et al. 2017, arXiv:1710.03047
Li, M., & Zhang, X. 2008, PhRvD, 78, 103516
Liu, H., Creswell, J., von Hausegger, S., & Naselsky, P. 2019, PhRvD, 100, 023538
Louis, T., Niess, S., Das, S., Dunkley, J., & Sherwin, B. 2013, MNRAS, 435, 2040
Lue, A., Wang, L., & Kamionkowski, M. 1999, PhRvL, 83, 1506
Monttroy, T. E., Ade, P. A. R., Bock, J. J., et al. 2006, ApJ, 647, 813
Naess, S., Hasselfield, M., McMahon, J., et al. 2014, JCAP, 2014, 007
Newman, E. T., & Penrose, R. 1996, JMP, 7, 863
Planck Collaboration, Ade, P. A. R., Aghanim, N., et al. 2014, A&A, 571, A16
Planck Collaboration, Aghanim, N., Akrami, Y., et al. 2020, A&A, 641, A6
Qiao, J., Zhu, T., Zhao, W., & Wang, A. 2020, PhRvD, 101, 043528
QUIET Collaboration, Araujo, D., Bischoff, C., et al. 2012, ApJ, 760, 145
Salatino, M., Auffermann, J., Thompson, K. L., et al. 2020, Proc. SPIE, 11453, 114532A
Santos, L., Wang, K., Hu, Y., Fang, W., & Zhao, W. 2017, JCAP, 2017, 043
Santos, L., Wang, K., & Zhao, W. 2016, JCAP, 2016, 029
Seljak, U., & Zaldarriaga, M. 1996, ApJ, 469, 437
Seljak, U., & Zaldarriaga, M. 1997, PhRvL, 78, 2054
Smith, K. M. 2006, PhRvD, 74, 083002
Smith, K. M., & Zaldarriaga, M. 2007, PhRvD, 76, 043001
Tegmark, M., & de Oliveira-Costa, A. 2001, PhRvD, 64, 063001
Tristram, M., Macías-Pérez, J. F., Renault, C., & Santos, D. 2005, MNRAS, 358, 833
Vanneste, S., Henrot-Versillé, S., Louis, T., & Tristram, M. 2018, PhRvD, 98, 103526
Wang, A., Wu, Q., Zhao, W., & Zhu, T. 2013, PhRvD, 87, 103512
Wang, Y.-F., Wang, K., & Zhao, W. 2016, RAA, 16, 59
Yadav, A. P. S., Su, M., & Zaldarriaga, M. 2010, PhRvD, 81, 063512
Zhao, W. 2011, JCAP, 2011, 007
Zhao, W., & Baskaran, D. 2010, PhRvD, 82, 023001
Zhao, W., Baskaran, D., & Grishchuk, L. P. 2009, PhRvD, 79, 023002
Zhao, W., & Li, M. 2014a, PhRvD, 89, 103518
Zhao, W., & Li, M. 2014b, PhRvD, 90, 043529
Zhu, T., Zhao, W., Huang, Y., Wang, A., & Wu, Q. 2013, PhRvD, 88, 063508