ON HIGHER-ORDER ANISOTROPIC PERTURBED CAGINALP PHASE FIELD SYSTEMS

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ABSTRACT. Our aim in this paper is to study the existence and uniqueness of solution for hyperbolic relaxations of higher-order anisotropic Caginalp phase field systems with homogeneous Dirichlet boundary conditions with regular potentials.

1. INTRODUCTION

Caginalp proposed in [6,8,9] two phase-field systems, namely,

\[
\frac{\partial u}{\partial t} - \Delta u + f(u) = \frac{\partial T}{\partial t} \tag{1}
\]

\[
\frac{\partial T}{\partial t} - \Delta T = -\frac{\partial u}{\partial t} \tag{2}
\]

called nonconserved system, and

\[
\frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) = -\Delta T \tag{3}
\]

\[
\frac{\partial T}{\partial t} - \Delta T = -\frac{\partial u}{\partial t} \tag{4}
\]

called conserved system (in the sense that, when endowed with Neumann boundary conditions, the spatial average of \(u\) is conserved). In this context, \(u\) is the order parameter, \(T\) is the relative temperature (defined as \(T = \tilde{T} - T_E\), where \(\tilde{T}\) is the absolute temperature and \(T_E\) is the equilibrium melting temperature) and \(f\) is the derivative of a double-well potential \(F\) (a typical choice is \(F(s) = \frac{1}{4}(s^2 - 1)^2\), hence the usual cubic nonlinear term \(f(s) = s^3 - s\)). Furthermore, we have set all physical parameters equal to one. These systems have been introduced to model phase transition phenomena, such as melting-solidification phenomena, and have been much studied from a mathematical point of view. We refer the reader to, e.g., [3,4,13]. Both systems are based on the (total Ginzburg-Landau) free energy

\[
\Psi_{GL} = \int_\Omega \left( \frac{1}{2} |\nabla u|^2 + F(u) - uT - \frac{1}{2} T^2 \right) dx, \tag{5}
\]
where $\Omega$ is the domain occupied by the system (we assume here that it is a bounded and regular domain of $\mathbb{R}^3$, with boundary $\Gamma$), and the enthalpy
\begin{equation}
H = u + T.
\end{equation}

As far as the evolution for the order parameter is concerned, one postulates the relaxation dynamics (with relaxation parameter set equal to one)
\begin{equation}
\frac{\partial u}{\partial t} = -\frac{D\Psi_{GL}}{Du},
\end{equation}
for the nonconserved model, and
\begin{equation}
\frac{\partial u}{\partial t} = \Delta \frac{D\Psi_{GL}}{Du},
\end{equation}
for the conserved one, where $\frac{D}{Du}$ denotes a variational derivative with respect to $u$, which yields (1) and (2). Then, we have the energy equation
\begin{equation}
\frac{\partial H}{\partial t} = -\text{div} q,
\end{equation}
where $q$ is the heat flux. Assuming finally the usual Fourier law for heat conduction,
\begin{equation}
q = -\nabla T;
\end{equation}
we obtain (3) or (4).

In (5), the term $|\nabla u|^2$ models short-ranged interactions. It is, however, interesting to note that such a term is obtained by truncation of higher-order ones (see [10]); it can also be seen as a first-order approximation of a nonlocal term accounting for long-ranged interactions (see [14, 15]).

In the late 1960’s, several authors proposed a heat conduction theory based on two temperatures (see [11, 24]). More precisely, one now considers the conductive temperature $T$ and the thermodynamic temperature $\alpha$. In particular, for simple materials, these two temperatures are shown to coincide. However, for non-simple materials, they differ and are related as follows:
\begin{equation}
\alpha = T - \Delta T.
\end{equation}
In that case, the free energy reads, in terms of the (relative) thermodynamic temperature $\alpha$,
\begin{equation}
\Psi_{GL} = \int_{\Omega} \left( \frac{1}{2} |\nabla u|^2 + F(u) - u\alpha - \frac{1}{2} \alpha^2 \right) dx,
\end{equation}
and (8) yields, in view of (12), the following evolution equation for the order parameter
\begin{equation}
\frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) = -\Delta(T - \Delta T).
\end{equation}
Furthermore, the enthalpy now reads
\begin{equation}
H = u + \alpha = u + T - \Delta T,
\end{equation}
which yields, owing to (9), the energy equation
\begin{equation}
\frac{\partial T}{\partial t} - \Delta \frac{\partial T}{\partial t} + \text{div} q = -\frac{\partial u}{\partial t}.
\end{equation}
Finally, the heat flux is given, in the type III theory with two temperatures, by (see [16, 22])
\begin{equation}
q = -\nabla \theta - \nabla T,
\end{equation}
where

\[(17) \quad \theta(t, x) = \int_0^t T(t, x) dx + \theta_0(x), \]

is the conductive thermal displacement. Noting that \( T = \frac{\partial \theta}{\partial t} \), we finally deduce from (13), the following variant of the Caginalp phase-field system

\[(18) \quad \frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) = -\Delta \left( \frac{\partial \theta}{\partial t} - \Delta \frac{\partial \theta}{\partial t} \right), \]

\[(19) \quad \frac{\partial^2 \theta}{\partial t^2} - \Delta \frac{\partial^2 \theta}{\partial t^2} - \Delta \frac{\partial \theta}{\partial t} - \Delta \theta = -\frac{\partial u}{\partial t}. \]

G. Caginalp and E. Esenturk recently proposed in [7] (see also [12]) higher-order phase field models in order to account for anisotropic interfaces (see also [17] for other approaches which, however, do not provide an explicit way to compute the anisotropy). More precisely, these authors proposed the following modified (total) free energy:

\[(20) \quad \Psi_{HOG} = \int_\Omega \left( \frac{1}{2} \sum_{i=1}^{k} \sum_{|\beta|=i} a_\beta |D^{|\beta|} u|^2 + F(u) - u \alpha - \frac{1}{2} \alpha^2 \right) dx, \quad k \in \mathbb{N}, \]

where, for \( \beta = (k_1, k_2, k_3) \in (\mathbb{N} \cup \{0\})^3 \),

\[|\beta| = k_1 + k_2 + k_3 \]

and, for \( \beta \neq (0, 0, 0) \),

\[D^\beta = \frac{\partial^{|\beta|}}{\partial x_1^{k_1} \partial x_2^{k_2} \partial x_3^{k_3}}\]

(we agree that \( D^{(0,0,0)} w = w \)).

A. Miranville studied in [20] the corresponding nonconserved higher-order phase-field system.

As far as the conserved case is concerned, the above generalized free energy yields, proceeding as above, the following evolution equation for the order parameter \( u \):

\[(21) \quad \frac{\partial u}{\partial t} - \Delta \sum_{i=1}^{k} (-1)^i \sum_{|\beta|=i} a_\beta D^{2|\beta|} u - \Delta f(u) = -\Delta \left( \frac{\partial \theta}{\partial t} - \Delta \frac{\partial \theta}{\partial t} \right). \]

In particular, for \( k = 1 \) (anisotropic conserved Caginalp phase-field system), we have an equation of the form

\[(22) \quad \frac{\partial u}{\partial t} + \Delta \sum_{i=1}^{3} a_i \frac{\partial^2 u}{\partial x_i^2} - \Delta f(u) = -\Delta \left( \frac{\partial \theta}{\partial t} - \Delta \frac{\partial \theta}{\partial t} \right), \]

and, for \( k = 2 \) (fourth-order anisotropic conserved Caginalp phase-field system), we have an equation of the form

\[(23) \quad \frac{\partial u}{\partial t} - \Delta \sum_{i,j=1}^{k} a_{ij} \frac{\partial^4 u}{\partial x_i^2 \partial x_j^2} + \Delta \sum_{i=1}^{3} a_i \frac{\partial^2 u}{\partial x_i^2} - \Delta f(u) = -\Delta \left( \frac{\partial \theta}{\partial t} - \Delta \frac{\partial \theta}{\partial t} \right). \]

L. Cherfils, A. Miranville and S. Peng studied in [18] the corresponding higher-order isotropic equation (without the coupling with the temperature), namely, the equation

\[\frac{\partial u}{\partial t} - \Delta P(-\Delta) u - \Delta u = 0,\]
where
\[ P(s) = \sum_{i=1}^{n} a_i s^i, \quad a_k \geq 0, \quad k \geq 1, \]
endowed with the Dirichlet/Navier boundary conditions
\[ u = \Delta u = \cdots = \Delta^k u = 0 \quad \text{on} \quad \Gamma. \]
Our aim in this paper is to study the perturbed model of the higher-order anisotropic equation (21), coupled with the temperature equation (19). The perturbation is expressed by the presence of the term \( \epsilon (-\Delta) \left( \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} \right) \) in the equation (21). Then, we have an hyperbolic relaxation of the viscous Cahn-Hilliard equation. When \( \epsilon = 0 \), the system has already been studied without the effects of anisotropy (see [22]), and with effects of anisotropy (see [23]). In particular, we obtain the existence and uniqueness of solutions.

2. Setting of the problem

We consider the following initial and boundary value problem, for \( k \in \mathbb{N}^+ (k \geq 2) \):
\[ \epsilon (-\Delta) \left( \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} \right) + \Delta \sum_{i=1}^{k} (-1)^i \sum_{|\beta|=i} a_{\beta} D^\beta u - \Delta f(u) = -\Delta \left( \frac{\partial \theta}{\partial t} - \Delta \frac{\partial \theta}{\partial t} \right), \]
\[ \frac{\partial^2 \theta}{\partial t^2} - \Delta \frac{\partial^2 \theta}{\partial t^2} - \Delta \frac{\partial \theta}{\partial t} - \Delta \theta = -\frac{\partial u}{\partial t}, \]
\[ u|_{t=0} = u_0, \quad \frac{\partial u}{\partial t}|_{t=0} = u_1, \quad \theta|_{t=0} = \theta_0, \quad \frac{\partial \theta}{\partial t}|_{t=0} = \theta_1. \]
We assume that
\[ a_{\beta} > 0, \quad |\beta| = k, \]
and \( \epsilon > 0 (\epsilon \leq \epsilon_0, \epsilon_0 > 0) \) is a relaxation parameter or the perturbation parameter.

We consider the regular potential \( f(s) = \sum_{i=1}^{2p-1} b_i s^i, \quad b_{2p-1} > 0, \quad p \geq 2 (p \in \mathbb{N}) \)
which satisfies the following properties:
\[ f \in C^2(\mathbb{R}), \quad f(0) = 0, \]
\[ f'(s) \geq -c_0, \quad c_0 \geq 0, \]
\[ f(s) s \geq c_1 F(s) - c_2, \quad c_1 > 0, \quad c_2 \geq 0, \quad s \in \mathbb{R}, \]
where \( F(s) = \int_{0}^{s} f(r) dr \).

**Remark 2.1.** (see [19]) We can more generally consider a polynomial potential of the form
\[ f(r) = \sum_{i=0}^{2p+2} a_i r^i, \quad a_{2p+2} > 0, \quad p \geq 1, \]
or even a regular potential \( f \) having a polynomial growth of the form
\[ a_{2p+2} r^{2p+2}, \quad a_{2p+2} > 0, \quad p \geq 1, \]
at infinity.

We introduce the elliptic operator $A_k$ defined by

\[(32) \quad \langle A_k v, w \rangle_{H^{-k}(\Omega), H_0^k(\Omega)} = \sum_{|\beta|=k} (D^\beta v, D^\beta w),\]

where $H^{-k}(\Omega)$ is the topological dual of $H_0^k(\Omega)$. Furthermore, $\langle \cdot, \cdot \rangle$ denotes the usual $L_2$-scalar product, with associated norm $\|\cdot\|$; more generally, we denote by $\|\cdot\|_X$ the norm on the Banach space $X$. We can note that

\[(v, w) \in (H_0^k(\Omega))^2 \mapsto \sum_{|\beta|=k} a_\beta ((D^\beta v, D^\beta w)),\]

is bilinear, symmetric, continuous and coercive, so that $A_k : H_0^k(\Omega) \mapsto H^{-k}(\Omega)$ is indeed well defined. It then follows from elliptic regularity results for linear elliptic operators of order $2k$ (see [1] and [2]) that $A_k$ is a strictly positive, selfadjoint and unbounded linear operator with compact inverse, with domain

\[D(A_k) = H^{2k}(\Omega) \cap H_0^k(\Omega),\]

where, for $v \in D(A_k)$,

\[A_k v = (-1)^k \sum_{|\beta|=k} a_\beta D^{2\beta} v.\]

We further note that $D(A_k^{\frac{1}{2}}) = H_0^k(\Omega)$ and, for $(v, w) \in D(A_k^{\frac{1}{2}})^2$

\[\langle A_k^{\frac{1}{2}} v, A_k^{\frac{1}{2}} w \rangle = \sum_{|\beta|=k} a_\beta ((D^\beta v, D^\beta w)).\]

We finally note that (see, e.g., [5]) $\|A_k\|$ (resp., $\|A_k^{\frac{1}{2}}\|$) is equivalent to the usual $H^{2k}$-norm (resp. $H^k$-norm) on $D(A_k)$ (resp., $D(A_k^{\frac{1}{2}})$).

Similarly, we can define the linear operator $\overline{A}_k = -\Delta A_k$, where

\[\overline{A}_k : H_0^{k+1}(\Omega) \mapsto H^{-k-1}(\Omega),\]

which is a strictly positive, selfadjoint and unbounded linear operator with compact inverse, with domain

\[D(\overline{A}_k) = H^{2k+2}(\Omega) \cap H_0^{k+1}(\Omega),\]

where, for $v \in D(\overline{A}_k)$,

\[\overline{A}_k v = (-1)^{k+1} \Delta \sum_{|\beta|=k} a_\beta D^{2\beta} v.\]

Furthermore, $D(\overline{A}_k^{\frac{1}{2}}) = H_0^{k+1}(\Omega)$ and for $(v, w) \in (D(\overline{A}_k^{\frac{1}{2}}))^2$

\[\langle \overline{A}_k^{\frac{1}{2}} v, \overline{A}_k^{\frac{1}{2}} w \rangle = \sum_{|\beta|=k} a_\beta ((\nabla D^\beta v, \nabla D^\beta w)).\]

Besides, $\|\overline{A}_k\|$ (resp., $\|\overline{A}_k^{\frac{1}{2}}\|$) is equivalent to the usual $H^{2k+2}$-norm (resp., $H^{k+1}$-norm) on $D(\overline{A}_k)$ (resp., on $D(\overline{A}_k^{\frac{1}{2}})$).

We finally consider the operator $\hat{A}_k = (-\Delta)^{-1} A_k$, where

\[(33) \quad \hat{A}_k : H_0^{k-1}(\Omega) \mapsto H^{-k}(\Omega);\]
Having this, we rewrite (24) as

\[\tilde{A}_k = (\Delta)^{-1}A_k.\]

We have the (see [20,21])

**Lemma 2.1.** The operator \(\tilde{A}_k\) is a strictly positive, selfadjoint and unbounded linear operator with compact inverse, with domain

\[D(\tilde{A}_k) = H^{2k-2}(\Omega) \cap H_0^{k-1}(\Omega),\]

where, for \(v \in D(\tilde{A}_k)\),

\[\tilde{A}_k v = (-1)^k \sum_{|\beta|=k} a_{\beta} D^{2\beta}(-\Delta)^{-1}v.\]

Furthermore, \(D(\tilde{A}_k^2) = H_0^{k-1}(\Omega)\) and, for \((v, w) \in (D(\tilde{A}_k^2))^2\),

\[\langle (\tilde{A}_k^2 v, \tilde{A}_k^2 w) \rangle = \sum_{|\beta|=k} a_{\beta} \langle (D^{\beta}(-\Delta)^{-\frac{1}{2}}v, D^{\beta}(-\Delta)^{-\frac{1}{2}}w) \rangle.\]

Besides, \(\|\tilde{A}_k\|\) (resp., \(\|\tilde{A}_k^2\|\)) is equivalent to the usual \(H^{2k-2}\)-norm (resp., \(H^{k-1}\)-norm) on \(D(\tilde{A}_k^2)\) (resp., \(D(\tilde{A}_k^4)\)).

**Proof.** (see [20,21]) We first note that \(\tilde{A}_k\) clearly is linear and unbounded. Then, since \((-\Delta)^{-1}\) and \(A_k\) commute, it easily follows that \(\tilde{A}_k\) is selfadjoint. Next, the domain of \(\tilde{A}_k\) is defined by

\[D(\tilde{A}_k) = \{v \in H_0^k(\Omega), \tilde{A}_k v \in L^2(\Omega)\}.\]

Noting that \(\tilde{A}_k v = f, f \in L^2(\Omega), v \in D(\tilde{A}_k)\), is equivalent to \(A_k v = -\Delta v\), where \(-\Delta f \in H^2(\Omega)\), it follows from the elliptic regularity results of [1,2] that \(v \in H^{2k-2}(\Omega)\), so that \(D(\tilde{A}_k) = H^{2k-2}(\Omega) \cap H_0^{k-1}(\Omega)\).

Noting then that \(\tilde{A}_k^{-1}\) maps \(L^2(\Omega)\) onto \(H^{2k-2}(\Omega)\) and recalling that \(k \geq 2\), we deduce that \(\tilde{A}_k\) has compact inverse.

We now note that, considering the spectral properties of \(-\Delta\) and \(A_k\) (see, e.g., [25]) and recalling that these two operators commute, \(-\Delta\) and \(A_k\) have a spectral basis formed of common eigenvectors. This yields that, \(\forall s_1, s_2 \in \mathbb{R}, (-\Delta)^{s_1}\) and \(A_k^{s_2}\) commute.

Having this, we see that \(\tilde{A}_k^2 = (-\Delta)^{-\frac{1}{2}}A_k^2\), so that \(D(\tilde{A}_k^2) = H_0^{k-1}(\Omega)\), and, for \((v, w) \in (D(\tilde{A}_k^2))^2\),

\[\langle (\tilde{A}_k^2 v, \tilde{A}_k^2 w) \rangle = \sum_{|\beta|=k} a_{\beta} \langle (D^{\beta}(-\Delta)^{-\frac{1}{2}}v, D^{\beta}(-\Delta)^{-\frac{1}{2}}w) \rangle.\]

Finally, as far as the equivalences of norms are concerned, we can note that for, instance, the norm \(\|\tilde{A}_k\|\) is equivalent to the norm \(\|(-\Delta)^{-\frac{1}{2}}\|_{H^s(\Omega)}\) and, thus, to the norm \(\|(-\Delta)^{\frac{k-1}{2}}\|\). \(\Box\)

Having this, we rewrite (24) as

\[(34) \quad (-\Delta) \left(\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t}\right) + \frac{\partial u}{\partial t} - \Delta A_k u - \Delta B_k u - \Delta f(u) = -\Delta \left(\frac{\partial \theta}{\partial t} - \Delta \frac{\partial \theta}{\partial t}\right),\]

where

\[B_k u = \sum_{i=1}^{k-1} (-1)^i \sum_{|\beta|=i} a_{\beta} D^{2\beta} u.\]
Throughout the paper, the same letters $c, c'$ and $c''$ denote (generally positive and independent of $\epsilon$) constants which may vary from line to line. Similarly, the same letter $Q$ denotes (positive and independent of $\epsilon$) monéte increasing (with respect to each argument) and continuous functions which may vary from line to line.

3. A priori estimates

We multiply (34) by $(-\Delta)^{-1} \frac{\partial u}{\partial t}$ and (25) by $\frac{\partial \theta}{\partial t} - \Delta \frac{\partial \theta}{\partial t}$, sum the two resulting equalities and integrate over $\Omega$ and by parts. This gives

$$\frac{d}{dt} \left( \epsilon \| \frac{\partial u}{\partial t} \|^2 + \| A_k^\frac{1}{2} u \|^2 + B_k^\frac{1}{2} [u] + 2 \int_{\Omega} F(u)dx + \| \frac{\partial \theta}{\partial t} - \Delta \frac{\partial \theta}{\partial t} \|^2 + \| \nabla \theta \|^2 + \| \Delta \theta \|^2 \right)$$

(35)

$$+ 2 \epsilon \| \frac{\partial u}{\partial t} \|^2 + 2\| \nabla \frac{\partial \theta}{\partial t} \|^2 + 2\| \Delta \frac{\partial \theta}{\partial t} \|^2 + 2\| \frac{\partial u}{\partial t} \|^2 -_1 = 0,$$

where

$$B_k^\frac{1}{2} [u] = \sum_{i=1}^{k-1} \sum_{|\beta|=i} a_\beta \| D^\beta u \|^2$$

(note that $B_k^\frac{1}{2} [u]$ is not necessarily nonnegative). We can note that, owing to the interpolation inequality

$$\| v \|_{H^i} \leq c(i) \| v \|_{H^m} \| v \|_{1}^{1 - \frac{i}{m}}$$

$v \in H^m(\Omega), \ i \in \{1, 2, \ldots, m - 1\}, \ m \geq 2,$

there holds (see [20, 21])

$$\| B_k^\frac{1}{2} [u] \| \leq \frac{1}{2} \| A_k^\frac{1}{2} u \|^2 + c\| u \|^2.$$

This yields, employing (31),

$$\| A_k^\frac{1}{2} u \|^2 + B_k^\frac{1}{2} [u] + 2 \int_{\Omega} F(u)dx \geq \frac{1}{2} \| A_k^\frac{1}{2} u \|^2 + \int_{\Omega} F(u)dx + c\| u \|_{L^p}^2 - c'\| u \|^2 - c'$$

whence

$$\| A_k^\frac{1}{2} u \|^2 + B_k^\frac{1}{2} [u] + 2 \int_{\Omega} F(u)dx \geq c\left( \| u \|_{H^k}^2 + \int_{\Omega} F(u)dx \right) - c', \ c > 0,$$

noting that, owing to Youngs inequality,

$$\| u \|^2 \leq \epsilon \| u \|_{L^p}^2 + c(\epsilon), \ \forall \epsilon > 0.$$

We obtain a differential equality of the form

$$\frac{dE_1}{dt} + 2\epsilon \| \frac{\partial u}{\partial t} \|^2 + 2\| \nabla \frac{\partial \theta}{\partial t} \|^2 + 2\| \Delta \frac{\partial \theta}{\partial t} \|^2 + 2\| \frac{\partial u}{\partial t} \|^2 -_1 = 0,$$

where

$$E_1 = \epsilon \left( \| \frac{\partial u}{\partial t} \|^2 + \| A_k^\frac{1}{2} u \|^2 + B_k^\frac{1}{2} [u] + 2 \int_{\Omega} F(u)dx + \| \frac{\partial \theta}{\partial t} - \Delta \frac{\partial \theta}{\partial t} \|^2 + \| \nabla \theta \|^2 + \| \Delta \theta \|^2 \right),$$

(note that $\| \frac{\partial \theta}{\partial t} - \Delta \frac{\partial \theta}{\partial t} \|^2 = \| \frac{\partial \theta}{\partial t} \|^2 + 2\| \nabla \frac{\partial \theta}{\partial t} \|^2 + \| \Delta \frac{\partial \theta}{\partial t} \|^2$) satisfies, owing to (39),

$$E_1 \geq c \left( \epsilon \left( \| \frac{\partial u}{\partial t} \|^2 + \| u \|_{H^k}^2 + \int_{\Omega} F(u)dx + \| \frac{\partial \theta}{\partial t} \|_{H^2}^2 + \| \theta \|_{H^2}^2 \right) - c', \ c > 0.\right.$$

Note indeed that
\[ \| A_k^{\frac{3}{2}} u \|^2 + B_k^{\frac{3}{2}} [u] \leq c \| u \|^2_{H^k}, \quad c > 0, \]

which yields
\[ E_1 \leq c \left( \epsilon \| \frac{\partial u}{\partial t} \|^2 + \| u \|^2_{H^k} + \| u \|^2_{L^2(\Omega)} + \| \frac{\partial \theta}{\partial t} \|^2_{H^2} + \| \theta \|^2_{H^2} \right) + c', \quad c > 0. \]

We then multiply (34) by \((-\Delta)^{-1} u\) and obtain,
\[ \frac{d}{dt} \left( 2\epsilon \left( \left( \frac{\partial u}{\partial t}, u \right) \right) + \epsilon \| u \|^2 + \| u \|^2_{-1} \right) + 2\| A_k^{\frac{3}{2}} u \|^2 + 2B_k^{\frac{3}{2}} [u] + 2 \left( f(u), u \right) \]
\[ = 2 \left( \left( \frac{\partial \theta}{\partial t} - \Delta \frac{\partial \theta}{\partial t}, u \right) \right) + 2\epsilon \| \frac{\partial u}{\partial t} \|^2. \]

Since, owing to (31), we have
\[ \left( \left( f(u), u \right) \right) \geq c_1 \int_\Omega F(u) dx - c_2 |\Omega|, \]

then, we find
\[ \frac{d}{dt} \left( 2\epsilon \left( \left( \frac{\partial u}{\partial t}, u \right) \right) + \epsilon \| u \|^2 + \| u \|^2_{-1} \right) \]
\[ + c \left( \| A_k^{\frac{3}{2}} u \|^2 + 2B_k^{\frac{3}{2}} [u] + 2 \int_\Omega F(u) dx + 2 \epsilon \left( \left( \frac{\partial u}{\partial t}, u \right) \right) \right) \]
\[ \leq c_1' \| \Delta \frac{\partial \theta}{\partial t} \|^2 + c_2' c \epsilon \| \frac{\partial u}{\partial t} \|^2 + c_2''. \]

Multiplying (25) by \(-\Delta \theta\) and integrating over \(\Omega\), we have
\[ \frac{d}{dt} \left( \| \Delta \theta \|^2 - 2 \left( \left( \frac{\partial \theta}{\partial t}, \Delta \theta \right) \right) \right) + 2 \left( \left( \Delta \frac{\partial \theta}{\partial t}, \Delta \theta \right) \right) + c_2 \| \Delta \theta \|^2 \]
\[ \leq c_2' \left( \| \Delta \frac{\partial \theta}{\partial t} \|^2 + \| \nabla \Delta \theta \|^2 \right) + c_2'' c \epsilon \| \frac{\partial u}{\partial t} \|^2. \]

We sum (41), \(\delta_1(44)\) and \(\delta_2(45)\), where \(\delta_i > 0, i = 1, 2\), is chosen small enough, so that
\[ \epsilon \| \frac{\partial u}{\partial t} \|^2 + \delta_1 \left( 2\epsilon \left( \left( \frac{\partial u}{\partial t}, u \right) \right) + \epsilon \| u \|^2 \right) \geq c \left( \epsilon \| \frac{\partial u}{\partial t} \|^2 + \epsilon \| u \|^2 \right), \quad c > 0, \]
\[ \| \frac{\partial \theta}{\partial t} - \Delta \frac{\partial \theta}{\partial t} \|^2 + \| \Delta \theta \|^2 + \delta_2 \left( \| \Delta \theta \|^2 - 2 \left( \left( \frac{\partial \theta}{\partial t}, \Delta \theta \right) \right) \right) \]
\[ \geq c' \left( \| \frac{\partial \theta}{\partial t} \|^2_{H^2} + \| \Delta \theta \|^2 \right), \]
\[ 2 - \delta_1 c'' - \delta_2 c'' > 0, \]
\[ 2 - \delta_1 c'' > 0, \]

and have an inequality of the form
\[ \frac{dE_2}{dt} + cE_2 + c' \| \frac{\partial \theta}{\partial t} \|^2_{H^2} + 2 \| \frac{\partial u}{\partial t} \|^2_{-1} \leq c'', \quad c > 0, \]

where
\[ E_2 = E_1 + \delta_1 \left( 2\epsilon \left( \left( \frac{\partial u}{\partial t}, u \right) \right) + \epsilon \| u \|^2 + \| u \|^2_{-1} \right) \]
\[ + \delta_2 \left( \| \Delta \theta \|^2 - 2 \left( \left( \frac{\partial \theta}{\partial t}, \Delta \theta \right) \right) + 2 \left( \left( \Delta \frac{\partial \theta}{\partial t}, \Delta \theta \right) \right) \right) \]
satisfies, according to (43),

\[ E_2 \geq c \left( \epsilon \| \frac{\partial u}{\partial t} \|^2 + \epsilon \| u \|^2 + \| u \|^2 + \| \theta \|^2 \right) - c', \]

and

\[ E_2 \leq c \left( \epsilon \| \frac{\partial u}{\partial t} \|^2 + \epsilon \| u \|^2 + \| u \|^2 + \| \theta \|^2 \right). \]

It follows from (47) – (48) and Gronwall’s lemma that

\[ \epsilon \| \frac{\partial u}{\partial t} \|^2 + \epsilon \| u \|^2 + \| u \|^2 + \| \theta \|^2 \]

\[ \leq ce^{-c't} \left( \| u_0 \|^2 + \epsilon \| u_0 \|^2 + \| u_0 \|^2 + \| u_1 \|^2 + \| \theta_0 \|^2 \right) + c'', \]

where \( c, c' \) and \( c'' \) are independent of \( \epsilon \) and \( r > 0 \) given.

We multiply (34) by \((-\Delta)^{-1} \frac{\partial^2 u}{\partial t^2}\) and integrate over \( \Omega \) to obtain

\[ \epsilon \| \frac{\partial^2 u}{\partial t^2} \|^2 + \frac{1}{2} \frac{d}{dt} \left( \| \frac{\partial u}{\partial t} \|^2 - \epsilon \| \frac{\partial u}{\partial t} \|^2 \right) + ((A_1 u, \frac{\partial^2 u}{\partial t^2})) \]

\[ + ((B_1 u, \frac{\partial^2 u}{\partial t^2})) + ((f(u), \frac{\partial^2 u}{\partial t^2})) \]

\[ = \left( \frac{\partial}{\partial t} - \Delta \frac{\partial}{\partial t}, \frac{\partial^2 u}{\partial t^2} \right). \]

Owing to the interpolation inequality (37), we find

\[ \frac{d}{dt} \left( \| \frac{\partial u}{\partial t} \|^2 - \epsilon \| \frac{\partial u}{\partial t} \|^2 \right) + \epsilon \| \frac{\partial^2 u}{\partial t^2} \|^2 \leq c \left( \| f(u) \|^2 + \| \frac{\partial \theta}{\partial t} \|^2 \right), \]

where \( c \) is independent of \( \epsilon \).

It follows from the continuity of \( f \) and the continuous embedding \( H^k(\Omega) \subset C(\overline{\Omega}) \) for \( k \geq 2 \) that

\[ \| f(u) \| \leq Q(\| u \|_{H^k}). \]

Thanks to (49), we have

\[ \frac{d}{dt} \left( \| \frac{\partial u}{\partial t} \|^2 - \epsilon \| \frac{\partial u}{\partial t} \|^2 \right) + \epsilon \| \frac{\partial^2 u}{\partial t^2} \|^2 \]

\[ \leq Q \left( \| u_0 \|_{H^k \cap L^2}, \| \theta_0 \|_{H^2}, \| u_1 \|, \| \theta_1 \|_{H^2} \right) e^{-ct} + c' + c'' \| A_1 u \|^2, \]

where \( c, c' \) and \( Q \) are independent of \( \epsilon \).
We further assume that $f$ is of class $C^k$. Multiplying (34) by $\dot{A}_k \frac{\partial u}{\partial t}$, we obtain

$$
\begin{align*}
\frac{d}{dt} \left( \epsilon \| A^k \frac{\partial u}{\partial t} \|^2 + \| A_k u \|^2 + ((A_k u, B_k u)) \right) &+ 2\epsilon \| A^k \frac{\partial u}{\partial t} \|^2 \\
+ 2\| A^k \frac{\partial u}{\partial t} \|^2 &+ 2 \left( (A^k f(u), A^k \frac{\partial u}{\partial t}) \right) \\
= &\ 2 \left( (\frac{\partial \theta}{\partial t} - \Delta \frac{\partial \theta}{\partial t}, A_k \frac{\partial u}{\partial t}) \right).
\end{align*}
$$

We rewrite the right-hand side of the above estimate as follows

$$
\left( (\frac{\partial \theta}{\partial t} - \Delta \frac{\partial \theta}{\partial t}, A_k \frac{\partial u}{\partial t}) \right) = \frac{d}{dt} \left( (\frac{\partial \theta}{\partial t} - \Delta \frac{\partial \theta}{\partial t}, A_k u) \right) - \left( (\frac{\partial^2 \theta}{\partial t^2} - \Delta \frac{\partial^2 \theta}{\partial t^2}, A_k u) \right).
$$

We then have

$$
\begin{align*}
\frac{d}{dt} \left( \epsilon \| A^k \frac{\partial u}{\partial t} \|^2 + \| A_k u \|^2 + ((A_k u, B_k u)) \right) &+ 2\epsilon \| A^k \frac{\partial u}{\partial t} \|^2 \\
+ 2\| A^k \frac{\partial u}{\partial t} \|^2 &+ 2 \left( (A^k f(u), A^k \frac{\partial u}{\partial t}) \right) \\
= &\ -2 \left( (\frac{\partial^2 \theta}{\partial t^2} - \Delta \frac{\partial^2 \theta}{\partial t^2}, A_k u) \right) - 2 \left( (A^k f(u), A^k \frac{\partial u}{\partial t}) \right),
\end{align*}
$$

which yields, noting that $\| A^k f(u) \| \leq Q\|u\|_{H^k}$,

$$
\begin{align*}
\frac{d}{dt} \left( \epsilon \| A^k \frac{\partial u}{\partial t} \|^2 + \| A_k u \|^2 + ((A_k u, B_k u)) \right) &+ 2\epsilon \| A^k \frac{\partial u}{\partial t} \|^2 \\
+ \epsilon \| A^k \frac{\partial u}{\partial t} \|^2 &+ 2 \left( \dot{A}_k \frac{\partial u}{\partial t} \right) \\
\leq &\ \frac{1}{2} \| \frac{\partial^2 \theta}{\partial t^2} \|^2_{H^2} + Q\|u\|_{H^k} + \epsilon \|A_k u\|^2.
\end{align*}
$$

We next multiply (25) by $-\Delta \frac{\partial \theta}{\partial t}$ and integrate over $\Omega$, we find

$$
\begin{align*}
\frac{d}{dt} \| \Delta \frac{\partial \theta}{\partial t} \|_{H^2} &+ \| \frac{\partial^2 \theta}{\partial t^2} \|_{H^2} \leq 2(\|\Delta \theta\|^2 + \| \frac{\partial u}{\partial t} \|^2).
\end{align*}
$$

Summing (54), $\delta_3$ times (53) and (46), where $\delta_3 > 0$ is small enough. Applying the Gronwall’s lemma, owing to (47) and the interpolation inequality (37), we obtain

$$
\|u\|_{H^{2k}} \leq e^{ct} Q(\|u_0\|_{H^{2k}}, \|u_1\|_{H^k}, \|\theta_0\|_{H^2}, \|\theta_1\|_{H^2}) + c',
$$

so that

$$
\epsilon \int_t^{t+\tau} \| \frac{\partial^2 u}{\partial t^2} (s) \|^2 ds \leq e^{ct} Q(\|u_0\|_{H^{2k}}, \|u_1\|_{H^k}, \|\theta_0\|_{H^2}, \|\theta_1\|_{H^2}) + c'.
$$

Multiplying (25) by $(-\Delta)^{-1} \frac{\partial^2 \theta}{\partial t^2}$ and integrating over $\Omega$, we obtain

$$
\begin{align*}
\frac{d}{dt} \| \frac{\partial \theta}{\partial t} \|^2 &+ 2 \left( \| \frac{\partial^2 \theta}{\partial t^2} \|^2_{-1} + \| \frac{\partial^2 \theta}{\partial t^2} \|^2 \right) \\
= &\ -2 \left( (\frac{\partial^2 \theta}{\partial t^2}, (-\Delta)^{-1} \frac{\partial^2 \theta}{\partial t^2}) \right) + 2 \left( \theta, \frac{\partial^2 \theta}{\partial t^2} \right).
\end{align*}
$$

Employing Hlder’s inequality, we find

$$
\begin{align*}
\frac{d}{dt} \| \frac{\partial \theta}{\partial t} \|^2 &+ 2 \left( \| \frac{\partial^2 \theta}{\partial t^2} \|^2_{-1} + \| \frac{\partial^2 \theta}{\partial t^2} \|^2 \right) \leq 2 \| \frac{\partial u}{\partial t} \|_{-1} \| \frac{\partial^2 \theta}{\partial t^2} \|^2_{-1} + 2 \|\theta\| \| \frac{\partial^2 \theta}{\partial t^2} \|,
\end{align*}
$$
which implies
\[ \frac{d}{dt} \| \frac{\partial \theta}{\partial t} \|^2 + c \left( \| \frac{\partial^2 \theta}{\partial t^2} \|^2 + 1 \right) \leq c' \left( \| \frac{\partial u}{\partial t} \|^2 + \| \theta \|^2 \right), \]
thanks to continuous embedding
\[ \| \frac{\partial u}{\partial t} \|_1 \leq c \| \frac{\partial u}{\partial t} \|. \]
Hence
\[ \frac{d}{dt} \| \frac{\partial \theta}{\partial t} \|^2 + c \left( \| \frac{\partial^2 \theta}{\partial t^2} \|^2 + 1 \right) \leq c' \left( \| \frac{\partial u}{\partial t} \|^2 + \| \theta \|^2 \right). \]
In particular, owing to (49), we obtain
\[ \int_0^r \left( \| \frac{\partial^2 \theta}{\partial t^2} \|^2 + c \left( \| \frac{\partial^2 \theta}{\partial t^2} \|^2 + 1 \right) \right) ds \leq Q \left( \| u_0 \|_{H^\lambda \cap L^2}, \| \theta_0 \|_{H^2}, \| u_1 \|, \| \theta_1 \|_{H^2} \right) e^{-ct} + c' t, \]
where \( r > 0 \) given.

We now multiply (34) by \( \frac{\partial u}{\partial t} \) and (25) by \( -\Delta (\frac{\partial u}{\partial t} - \Delta \frac{\partial u}{\partial t}) \) and integrate over \( \Omega \). Summing these two results, we obtain
\[ \frac{d}{dt} \left( \epsilon \| \frac{\partial u}{\partial t} \|^2 + \| D^k \theta \|_{L^2}^2 + \| D^k u \|_{L^2}^2 + \| \Delta \theta \|_{H^1}^2 + \| \frac{\partial u}{\partial t} \|_{L^2}^2 + \| \Delta \frac{\partial u}{\partial t} \|_{L^2}^2 \right) + c \left( \epsilon \| \frac{\partial u}{\partial t} \|^2 + \| \Delta \frac{\partial u}{\partial t} \|^2 + \| \frac{\partial u}{\partial t} \|^2 \right) \leq c' \| \Delta f(u) \|^2, \]
where
\[ D^k \theta = \sum_{i=1}^{k-1} \sum_{|\beta|=1} a_\beta \| \nabla D^\beta u \|^2. \]
Noting that \( f \in C^2(\mathbb{R}) \) then
\[ \| \Delta f(u) \| \leq Q(\| u \|_{H^2}). \]
We obtain
\[ \frac{d}{dt} E_3 + c \left( \epsilon \| \frac{\partial u}{\partial t} \|^2 + \| \nabla \frac{\partial u}{\partial t} \|^2 + \| \Delta \frac{\partial u}{\partial t} \|^2 + \| \frac{\partial u}{\partial t} \|^2 \right) \leq Q \left( \| u_0 \|_{H^s \cap L^2}, \| \theta_0 \|_{H^2}, \| u_1 \|, \| \theta_1 \|_{H^2} \right) e^{-ct} + c'', \]
where
\[ E_3 = \epsilon \| \frac{\partial u}{\partial t} \|^2 + \| D^k u \|_{L^2}^2 + \| D^k \theta \|_{L^2}^2 + \| \Delta \theta \|_{H^1}^2 + \| \nabla \Delta \theta \|_{H^1}^2 + \| \nabla \partial \theta \|^2 + \| \Delta \partial \theta \|^2, \]

(note that \( \| \nabla \frac{\partial u}{\partial t} - \nabla \Delta \frac{\partial u}{\partial t} \|^2 = \| \nabla \frac{\partial u}{\partial t} \|^2 + 2 \| \Delta \frac{\partial u}{\partial t} \|^2 + \| \nabla \Delta \frac{\partial u}{\partial t} \|^2 \|).)

Multiplying next (34) by \( u \) and integrating over \( \Omega \), we obtain
\[ \frac{d}{dt} \left( \| u \|^2 + 2 \epsilon \left( \| \frac{\partial u}{\partial t} \|_{L^2}^2 \| u \|^2 \right) + c_3 \left( \| \nabla^2 u \|_{L^2}^2 + B_k^h \| u \| \right) \right) + 2 \epsilon \left( \| \nabla \frac{\partial u}{\partial t} \|_{L^2}^2 \| u \|^2 \right) \leq c_3' \left( \| \frac{\partial \theta}{\partial t} \|_{H^2}^2 + \| \nabla \frac{\partial u}{\partial t} \|^2 \right). \]
Summing (46), (60) and $\delta_4$ times (61), where $\delta_4 > 0$ is chosen small enough, we find a differential inequality of the form

$$
\frac{dE_4}{dt} + \epsilon \left( E_3 + \||u_t\||^2_{L^2} + \||u\||^2 + \||\Delta \theta_t\||^2 \right)
\leq Q \left( \|u_0\|_{H^{k+1}\cap L^{2p}}, \|\theta_0\|_{H^2}, \|u_1\|, \|\theta_1\|_{H^3} \right) e^{-c't} + c'',
$$

where

$$
E_4 = E_2 + \epsilon \||\nabla \frac{\partial u}{\partial t}||^2 + \||u_t||^2 + \||u||^2 + \||\Delta \theta_t||^2 + \||\nabla \Delta \theta||^2
+ \||\nabla \frac{\partial \theta}{\partial t} - \nabla \Delta \frac{\partial \theta}{\partial t}||^2 + \delta_4 \left( \|u\|^2 + 2\epsilon \left( \|\nabla \frac{\partial u}{\partial t}, \nabla u\| + \epsilon \|\nabla u||^2 \right) \right),
$$
satisfies, owing to (47) and the interpolation inequality (37),

$$
E_4 \geq c \left( \|u\|^2_{H^{k+1}} + \epsilon \||\nabla \frac{\partial u}{\partial t}||^2 + \epsilon \|\nabla u||^2
+ \int_{\Omega} F(u)dx + \|\theta\|^2_{H^3(\Omega)} + \||\frac{\partial \theta}{\partial t}||^2_{H^3(\Omega)} \right) - c',
$$

and, owing to (48),

$$
E_4 \leq c \left( \|u\|^2_{H^{k+1}} + \epsilon \||\nabla \frac{\partial u}{\partial t}||^2 + \epsilon \|\nabla u||^2 + 2 \int_{\Omega} F(u)dx + \|\theta\|^2_{H^3(\Omega)} + \||\frac{\partial \theta}{\partial t}||^2_{H^3(\Omega)} \right).
$$

In particular, it follows from (62) and (63) – (64) that

$$
\|u\|^2_{H^{k+1}} + \epsilon \||\nabla \frac{\partial u}{\partial t}||^2 + \epsilon \|\nabla u||^2 + \||\frac{\partial \theta}{\partial t}||^2_{H^3} + \|u||^2_{L^{2p}} + \|\theta\|^2_{H^3}
\leq Q \left( \|u_0\|_{H^{k+1}\cap L^{2p}}, \|\theta_0\|_{H^3}, \|u_1\|_{H^1}, \|\theta_1\|_{H^3} \right) e^{-c't} + c',
$$

and

$$
\int_{t}^{t+r} \left( \||\frac{\partial u}{\partial t}(s)||^2_{L^1} + \||\frac{\partial u}{\partial t}(s)||^2_{L^2} + \||\frac{\partial \theta}{\partial t}(s)||^2_{H^1} \right) ds
\leq Q \left( \|u_0\|_{H^{k+1}\cap L^{2p}}, \|\theta_0\|_{H^3}, \|u_1\|_{H^1}, \|\theta_1\|_{H^3} \right) e^{-c't} + c',
$$

where $c, c'$ are independent of $\epsilon$ and $r > 0$ given.

We multiply (25) by $\frac{\partial^2 \theta}{\partial t^2}$ and integrate over $\Omega$. We obtain

$$
\frac{d}{dt} \||\nabla \frac{\partial \theta}{\partial t}||^2 + 2\||\nabla \frac{\partial \theta}{\partial t}||^2 + 2\||\nabla \frac{\partial^2 \theta}{\partial t^2}||^2 + 2\left( (-\Delta \theta, \frac{\partial^2 \theta}{\partial t^2}) \right) = 2\left( \frac{\partial u}{\partial t}, \frac{\partial^2 \theta}{\partial t^2} \right).
$$

Applying Hlder’s and Young’s inequalities, we obtain

$$
\frac{d}{dt} \||\nabla \frac{\partial \theta}{\partial t}||^2 + c \||\frac{\partial^2 \theta}{\partial t^2}||^2_{H^3} \leq c' \left( \||\frac{\partial u}{\partial t}||^2 + \||\nabla \theta||^2 \right).
$$

Thanks to (49), we find

$$
\frac{d}{dt} \||\nabla \frac{\partial \theta}{\partial t}||^2 + c \||\frac{\partial^2 \theta}{\partial t^2}||^2_{H^3} \leq Q \left( \|u_0\|_{H^{k+1}\cap L^{2p}}, \|\theta_0\|_{H^3}, \|u_1\|, \|\theta_1\|_{H^3} \right) e^{-c't} + c'', \quad c' > 0.
$$

In particular, we have

$$
\int_{t}^{t+r} \||\frac{\partial^2 \theta}{\partial t^2}(s)||^2_{H^3} ds \leq Q \left( \|u_0\|_{H^{k+1}\cap L^{2p}}, \|\theta_0\|_{H^3}, \|u_1\|, \|\theta_1\|_{H^3} \right) e^{-c't} + c'', \quad t \geq 0, \quad r > 0 \text{ given.}$$
Multiplying (34) by $\frac{\partial^2 u}{\partial t^2}$, we have
\[
2\epsilon \| \nabla^2 u \| ^2 + \frac{d}{dt} \left( \| \frac{\partial u}{\partial t} \| ^2 + \epsilon \| \nabla \frac{\partial u}{\partial t} \| ^2 \right) + 2 \left( \langle \nabla A_k u, \nabla^2 u \rangle \right) \\
+ 2 \left( \langle \nabla B_k u, \nabla^2 u \rangle \right)
\]
(69)
\[
= 2 \left( \langle \nabla \frac{\partial \theta}{\partial t} - \nabla \Delta \frac{\partial \theta}{\partial t}, \nabla^2 u \rangle \right) - 2 \left( \langle \nabla u f'(u), \nabla^2 u \rangle \right).
\]

It follows from the continuity of $f'$ and the continuous embedding $H^k(\Omega) \subset C(\overline{\Omega})$ (recall that $k \geq 2$), that
\[
\left| \left( \langle \nabla u f'(u), \nabla^2 u \rangle \right) \right| \leq Q(\|u\|_{H^3}) \| \nabla u \| \| \nabla^2 u \|, \\
\leq Q(\|u\|_{H^5}) \| \nabla^2 u \|,
\]
where $Q$ is independent of $\epsilon$.

Employing Holder’s and Young’s inequalities and the interpolation inequality (37), (69) becomes
\[
\epsilon \| \nabla^2 u \| ^2 + \frac{d}{dt} \left( \| \frac{\partial u}{\partial t} \| ^2 + \epsilon \| \nabla \frac{\partial u}{\partial t} \| ^2 \right) \\
\leq \frac{c}{\epsilon} \left( \| \nabla A_k u \| ^2 + \| \nabla B_k u \| ^2 + \| \nabla u \| ^2 + \| \frac{\partial \theta}{\partial t} \| _{H^3} + Q(\|u\|_{H^k}) \right),
\]
which can be written in the form
\[
\epsilon \| \nabla^2 u \| ^2 + \frac{d}{dt} \left( \| \frac{\partial u}{\partial t} \| ^2 + \epsilon \| \nabla \frac{\partial u}{\partial t} \| ^2 \right) \leq \frac{c}{\epsilon} \left( \| u \| _{H^{2k+1}} ^2 + \| \frac{\partial \theta}{\partial t} \| _{H^3} + Q(\|u\|_{H^k}) \right),
\]
where $c$ is independent of $\epsilon$.

Owing to (65), we find
\[
\frac{d}{dt} \left( \| \frac{\partial u}{\partial t} \| ^2 + \epsilon \| \nabla \frac{\partial u}{\partial t} \| ^2 \right) + \epsilon \| \nabla u \| ^2 \leq \frac{1}{\epsilon} (Qe^{-ct} + c' + c'' \| u \| _{H^{2k+1}} ^2),
\]
where $Q = Q\left( \| u_0 \| _{H^{k+1} \cap L^2}, \| \theta_0 \| _{H^3}, \| u_1 \| _{H^1}, \| \theta_1 \| _{H^2} \right)$ is independent of $\epsilon$.

We multiply (34) by $A_k \frac{\partial u}{\partial t}$ and integrate over $\Omega$, we obtain
\[
\frac{d}{dt} \left( \epsilon \| A_k \frac{\partial u}{\partial t} \| ^2 + \| \nabla A_k u \| ^2 + (\langle \nabla A_k u, \nabla B_k u \rangle) \right) + 2\epsilon \| \nabla A_k \frac{\partial u}{\partial t} \| ^2 + 2\| A_k \frac{\partial u}{\partial t} \| ^2 \\
= 2(\langle \nabla A_k f'(u), \nabla A_k \frac{\partial u}{\partial t} \rangle) + 2(\langle \nabla \frac{\partial \theta}{\partial t} - \nabla \Delta \frac{\partial \theta}{\partial t}, \nabla A_k \frac{\partial u}{\partial t} \rangle).
\]

Noting that
\[
(\langle \nabla \frac{\partial \theta}{\partial t} - \nabla \Delta \frac{\partial \theta}{\partial t}, \nabla A_k \frac{\partial u}{\partial t} \rangle) = \frac{d}{dt} \left( \langle \nabla \frac{\partial \theta}{\partial t} - \nabla \Delta \frac{\partial \theta}{\partial t}, \nabla \frac{\partial u}{\partial t} \rangle \right) - \langle \nabla \frac{\partial^2 \theta}{\partial t^2} - \nabla \Delta \frac{\partial^2 \theta}{\partial t^2}, \nabla A_k u \rangle.
\]
We then have
\[
\frac{d}{dt} \left( \epsilon \| A_k \frac{\partial u}{\partial t} \| ^2 + \| \nabla A_k u \| ^2 + (\langle \nabla A_k u, \nabla B_k u \rangle) \right) - 2(\langle \nabla \frac{\partial \theta}{\partial t} - \nabla \Delta \frac{\partial \theta}{\partial t}, \nabla A_k u \rangle) \\
+ 2\epsilon \| A_k \frac{\partial u}{\partial t} \| ^2 + 2\| A_k \frac{\partial u}{\partial t} \| ^2
\]
\[ = 2 \left( (\nabla A_k^2 f(u), \nabla A_k^2 \frac{\partial u}{\partial t}) - 2 \left( (\nabla \frac{\partial^2 \theta}{\partial t^2} - \nabla \Delta \frac{\partial^2 \theta}{\partial t^2}, \nabla A_k u) \right) \right), \]

which can be written
\[
\frac{d}{dt} \left( \epsilon ||\nabla A_k^2 \frac{\partial u}{\partial t}||^2 + ||\nabla A_k u||^2 + ((\nabla A_k u, \nabla B_k u)) \right) = \frac{1}{2} \frac{\partial^2 \theta}{\partial t^2} \|\nabla A_k u\|^2 + c\|\nabla A_k u\|^2 + \frac{1}{\epsilon} ||A_k^2 f(u)||^2,
\]

which yields, noting that \( ||\nabla A_k^2 f(u)|| \leq Q(||u||_{H^{k+1}}) \),
\[
\frac{d}{dt} \left( \epsilon ||\nabla A_k^2 \frac{\partial u}{\partial t}||^2 + ||\nabla A_k u||^2 + ((\nabla A_k u, \nabla B_k u)) \right) + \epsilon \|\nabla A_k^2 \frac{\partial u}{\partial t}||^2 + 2 \frac{\partial u}{\partial t} \leq \frac{1}{2} \frac{\partial^2 \theta}{\partial t^2} ||\nabla A_k u||^2 + c\|\nabla A_k u\|^2 + \frac{1}{\epsilon} \|A_k^2 f(u)||^2,
\]

Multiplying (25) by \( \Delta^2 \frac{\partial \theta}{\partial t} \) and integrating over \( \Omega \), we obtain
\[
\frac{d}{dt} ||\nabla \Delta \frac{\partial \theta}{\partial t}||^2 + \frac{\partial^2 \theta}{\partial t^2} \|\nabla u\|^2 \leq c(||\theta||_{H^3}^2 + ||\nabla \frac{\partial u}{\partial t}||^2).
\]

Summing (72), \( \delta_5 \) times (71) and (62), where \( \delta_5 > 0 \) is small enough. Applying Gronwall’s lemma and thanks to (63) and the interpolation inequality (37), we find
\[
||u||_{H^{2k+1}} \leq c^{\epsilon}Q(||u_0||_{H^{2k+1}}, ||u_1||_{H^{k+1}}, ||\theta_0||_{H^3}, ||\theta_1||_{H^3}) + c'.
\]

We finally deduce that
\[
\epsilon \int_t^{t+r} ||\frac{\partial^2 u}{\partial t^2}(s)||_{H^2}^2 ds \leq c^{\epsilon}Q(||u_0||_{H^{2k+1}}, ||u_1||_{H^{k+1}}, ||\theta_0||_{H^3}, ||\theta_1||_{H^3}) + c',
\]

where \( c, c' \) and \( Q \) are independent of \( \epsilon \).

4. Existence and Uniqueness of Solutions

We have the following theorem

**Theorem 4.1. a):** We assume that \((u_0, u_1, \theta_0, \theta_1) \in (H^0_0(\Omega) \cap L^{2p}(\Omega)) \times L^2(\Omega) \times (H^2(\Omega) \cap H^1_0(\Omega))^2\) then, (24) – (27) possesses a unique weak solution \((u, \theta)\) such that, for all \( T > 0 \),
\[
\begin{align*}
{u} & \in L^\infty(\mathbb{R}^+; H^0_0(\Omega) \cap L^{2p}(\Omega)), \\
\theta & \in L^\infty(\mathbb{R}^+; H^2(\Omega) \cap H^1_0(\Omega)), \\
\frac{\partial u}{\partial t} & \in L^\infty(\mathbb{R}^+; L^2(\Omega)) \cap L^2(0, T; H^{-1}(\Omega)), \\
\frac{\partial \theta}{\partial t} & \in L^\infty(\mathbb{R}^+; L^2(\Omega)) \cap L^2(0, T; H^{-1}(\Omega)), \\
\frac{\partial^2 \theta}{\partial t^2} & \in L^2(0, T; L^2(\Omega))
\end{align*}
\]

and
\[
\frac{d}{dt} ((\epsilon \frac{\partial u}{\partial t}, v) + ((\epsilon \frac{\partial u}{\partial t}, v) + ((\epsilon \frac{\partial u}{\partial t}, v))) + \sum_{i=1}^{k} \sum_{|\beta|=i} a_i((D^\beta u, D^\beta v) + ((f(u), v)))
\]
\[ \frac{d}{dt}(\theta, v) + (\nabla \frac{\partial \theta}{\partial t}, \nabla v), \quad \forall \ v \in H_0^1(\Omega), \]
\[ \frac{d}{dt}(\frac{\partial u}{\partial t}, w) + \frac{d}{dt}(\nabla \frac{\partial \theta}{\partial t}, \nabla w) + (\nabla \frac{\partial \theta}{\partial t}, \nabla w) + ((\nabla \theta, \nabla \omega)) = -((\frac{\partial u}{\partial t}, w)), \quad \forall \ w \in C_0^\infty(\Omega), \]
in the sense of distributions.

b): If we further assume that \((u_0, u_1, \theta_0, \theta_1) \in (H^{k+1}(\Omega) \cap H_0^k(\Omega) \cap L^{2p}(\Omega)) \times H_0^1(\Omega) \times (H^3(\Omega) \cap H_0^1(\Omega))^2\), then for all \(T > 0\),
\[ u \in L^\infty(0, T; (H^{k+1}(\Omega) \cap H_0^k(\Omega)) \cap L^{2p}(\Omega)), \]
\[ \theta \in L^\infty(0, T; H^3(\Omega) \cap H_0^1(\Omega)), \]
\[ \frac{\partial u}{\partial t} \in L^\infty(0, T; H_0^1(\Omega)), \]
\[ \frac{\partial \theta}{\partial t} \in L^\infty(0, T; H_0^1(\Omega)) \]
and
\[ \frac{\partial^2 \theta}{\partial t^2} \in L^2(0, T; H_0^1(\Omega)). \]

c): If we further assume that \(f\) is of class \(C^k\) and \((u_0, u_1) \in (H^{2k}(\Omega) \cap H_0^k(\Omega))^2\) then
\[ u \in L^\infty(0, T; H^{2k}(\Omega) \cap H_0^k(\Omega)), \]
\[ \frac{\partial^2 u}{\partial t^2} \in L^2(0, T; L^2(\Omega)). \]

d): If we further assume that \(f\) is of class \(C^{k+1}\) and \((u_0, u_1) \in (H^{2k+1}(\Omega) \cap H_0^k(\Omega))^2\) then
\[ u \in L^\infty(0, T; H^{2k+1}(\Omega) \cap H_0^k(\Omega)), \]
\[ \frac{\partial^2 u}{\partial t^2} \in L^2(0, T; H_0^1(\Omega)). \]

The proofs of existence and regularity in a), b), c) and d) follow from the a priori estimates derived in the previous section and, e.g., a standard Galerkin scheme.

We then have the following theorem

**Theorem 4.2.** The system (24) – (27) possesses a unique solution with the above regularity.

*Proof.* Let now \((u^{(1)}, \frac{\partial u^{(1)}}{\partial t}, \theta^{(1)}, \frac{\partial \theta^{(1)}}{\partial t})\) and \((u^{(2)}, \frac{\partial u^{(2)}}{\partial t}, \theta^{(2)}, \frac{\partial \theta^{(2)}}{\partial t})\) be two solutions to (24)-(27) with initial data \((u_0^{(1)}, \theta_0^{(1)}, u_1^{(1)}, \theta_1^{(1)})\) and \((u_0^{(2)}, \theta_0^{(2)}, u_1^{(2)}, \theta_1^{(2)})\), respectively.

We set
\[ (u, \frac{\partial u}{\partial t}, \theta, \frac{\partial \theta}{\partial t}) = (u^{(1)} - u^{(2)}, \frac{\partial u^{(1)}}{\partial t} - \frac{\partial u^{(2)}}{\partial t}, \theta^{(1)} - \theta^{(2)}, \frac{\partial \theta^{(1)}}{\partial t} - \frac{\partial \theta^{(2)}}{\partial t}), \]
and
\[ (u_0, u_1, \theta_0, \theta_1) = \left(u_0^{(1)} - u_0^{(2)}, u_1^{(1)} - u_1^{(2)}, \theta_0^{(1)} - \theta_0^{(2)}, \theta_1^{(1)} - \theta_1^{(2)}\right), \]
then \((u, \theta)\) satisfies the following system

\[
\begin{align*}
(75) & \quad \epsilon \frac{\partial^2 u}{\partial t^2} + A_\epsilon \frac{\partial u}{\partial t} + A_k u + B_k u + (f(u(1)) - f(u(2))) = \frac{\partial \theta}{\partial t} - \Delta \frac{\partial \theta}{\partial t}, \\
(76) & \quad \frac{\partial \theta}{\partial t^2} - \Delta \frac{\partial^2 \theta}{\partial t^2} - \Delta \frac{\partial \theta}{\partial t} - \Delta \theta = -\frac{\partial u}{\partial t},
\end{align*}
\]

\[
\begin{align*}
\epsilon \frac{\partial u}{\partial t} + f(u(1)) - f(u(2)) = 0,
\end{align*}
\]

where \(A_\epsilon = \epsilon I + (-\Delta)^{-1}\).

Multiplying \((75)\) by \(\frac{\partial u}{\partial t}\) and integrating over \(\Omega\), we obtain

\[
\begin{align*}
\frac{d}{dt} \left( \epsilon \frac{\partial u}{\partial t} \right)^2 + \| A_k^\frac{1}{2} u \|^2 + B_k^\frac{1}{2} |u| + 2 \| \frac{\partial u}{\partial t} \|^2 + 2 \| \frac{\partial u}{\partial t} \|_{-1}^2 \\
+ 2 \left( \left( f(u(1)) - f(u(2)) \right), \frac{\partial u}{\partial t} \right) \\
= 2 \left( \left( \frac{\partial \theta}{\partial t} - \Delta \frac{\partial \theta}{\partial t}, \frac{\partial u}{\partial t} \right) \right),
\end{align*}
\]

which can be written in the form

\[
\begin{align*}
(77) & \quad \frac{d}{dt} \left( \epsilon \| \frac{\partial u}{\partial t} \|^2 + \| A_k^\frac{1}{2} u \|^2 + B_k^\frac{1}{2} |u| \right) + 2 \| \frac{\partial u}{\partial t} \|_{-1}^2 \\
& \leq 2 \left( \left( \frac{\partial \theta}{\partial t} - \Delta \frac{\partial \theta}{\partial t}, \frac{\partial u}{\partial t} \right) \right) + 2 \left| \left( f(u(1)) - f(u(2)) \right), \frac{\partial u}{\partial t} \right|.
\end{align*}
\]

Noting that

\[
\begin{align*}
(78) & \quad f(u(1)) - f(u(2)) = u \int_0^1 f'(su(1)) + (1-s)u(2) ds.
\end{align*}
\]

Since

\[
\| f'(su(1)) + (1-s)u(2) \|_{L^\infty} \leq Q(\| u(1) \|_{L^\infty} + \| u(2) \|_{L^\infty}), \quad s \in [0, 1],
\]

and the fact that \(u(1), u(2) \in H^k(\Omega)\), owing to the continuous embedding \(H^k(\Omega) \subset C(\overline{\Omega})\) for \(k \geq 2\), then, we have

\[
\begin{align*}
(79) & \quad \left| \left( f(u(1)) - f(u(2)) \right), \frac{\partial u}{\partial t} \right| \leq c \| u \|_{H^k} \| \frac{\partial u}{\partial t} \|.
\end{align*}
\]

Multiplying next \((76)\) by \(\frac{\partial \theta}{\partial t} - \Delta \frac{\partial \theta}{\partial t}\) and integrating over \(\Omega\), we obtain the following differential equality

\[
\begin{align*}
(80) & \quad \frac{d}{dt} \left( \| \frac{\partial \theta}{\partial t} \|^2 + \| \theta \|^2_{H^2(\Omega)} \right) + 2 \| \frac{\partial \theta}{\partial t} \|_{H^2(\Omega)}^2 = -2 \left( \left( \frac{\partial \theta}{\partial t} - \Delta \frac{\partial \theta}{\partial t}, \frac{\partial u}{\partial t} \right) \right).
\end{align*}
\]

We sum \((77)\) and \((80)\), and obtain, owing to \((79)\)

\[
\begin{align*}
\frac{d}{dt} \left( \| \frac{\partial u}{\partial t} \|^2 + \| A_k^\frac{1}{2} u \|^2 + B_k^\frac{1}{2} |u| + \| \frac{\partial \theta}{\partial t} - \Delta \frac{\partial \theta}{\partial t} \|^2 + \| \theta \|^2_{H^2(\Omega)} \right) \\
+ 2 \left( \left( \| \frac{\partial \theta}{\partial t} \|^2_{H^2(\Omega)} + \epsilon \| \frac{\partial u}{\partial t} \|^2 + \| \frac{\partial \theta}{\partial t} \|_{-1}^2 \right) \right) \\
\leq c \left( \epsilon \| \frac{\partial u}{\partial t} \|^2 + \| u \|^2_{H^k} \right),
\end{align*}
\]
hence, we have a differential inequality of the form
\[ \frac{dE_5}{dt} + 2(\|\frac{\partial \theta}{\partial t}\|_{H^2(\Omega)}^2 + \epsilon \|\frac{\partial u}{\partial t}\|^2 + \|\frac{\partial u}{\partial t}\|_{L^2}^2) \leq cE_5. \]

In particular,
\[ (81) \quad \frac{dE_5}{dt} \leq cE_5, \]
where
\[ E_5 = \epsilon\|\frac{\partial u}{\partial t}\|^2 + \|A_k^2 u\|^2 + B_k^2[u] + \|\frac{\partial \theta}{\partial t} - \Delta \frac{\partial \theta}{\partial t}\|^2 + \|\theta\|_{H^2(\Omega)}^2. \]

Note indeed that, (see \[20, 21\])
\[ c'\|u\|_{L^2}^2 \leq \|A_k^2 u\|^2 + B_k^1[u] \leq \epsilon \|u\|_{L^2}^2 \quad \text{where} \quad c, c' > 0, \]
which implies
\[ (82) \quad E_5 \geq c (\epsilon\|\frac{\partial u}{\partial t}\|^2 + \|u\|_{L^2}^2 + \|\frac{\partial \theta}{\partial t}\|_{H^2(\Omega)}^2 + \|\theta\|_{H^2(\Omega)}^2). \]

It follows from (81) – (82) and Gronwals lemma that
\[ \|u\|_{L^2}^2 + \epsilon\|\frac{\partial u}{\partial t}\|^2 + \|\frac{\partial \theta}{\partial t}\|_{H^2}^2 + \|\theta\|_{H^2}^2 \leq c (\|u_0\|_{H^k}^2 + c\|u_1\|_{H^k}^2 + \|\theta_0\|_{H^2}^2 + \|\theta_1\|_{H^2}^2), \]

hence the uniqueness, as well as the continuous dependence with respect to the initial data in the $H^k(\Omega) \times L^2(\Omega) \times H^2(\Omega) \times H^2(\Omega)$-norm. \[\square\]

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