On the Rank of Quadratic Equations for Curves of High Degree

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Abstract. Let \( C \subset \mathbb{P}^r \) be a linearly normal curve of arithmetic genus \( g \) and degree \( d \). In Saint-Donat (CR Acad Sci Paris Ser A 274: 324–327, 1972), B. Saint-Donat proved that the homogeneous ideal \( I(C) \) of \( C \) is generated by quadratic equations of rank at most 4 whenever \( d \geq 2g+2 \). Also, in Eisenbud et al. (Amer J Math 110: 513–539, 1988) Eisenbud, Koh and Stillman proved that \( I(C) \) admits a determinantal presentation if \( d \geq 4g+2 \). In this paper, we will show that \( I(C) \) can be generated by quadratic equations of rank 3 if either \( g = 0 \) and \( d \geq 2g+2 \) or else \( g \geq 2 \) and \( d \geq 4g+4 \).

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1. Introduction

Throughout this paper, we work over an algebraically closed field \( \mathbb{K} \) of arbitrary characteristic. We denote by \( \mathbb{P}^r \) the projective \( r \)-space over \( \mathbb{K} \).

Let \( C \) be a projective integral curve of arithmetic genus \( g \) and \( \mathcal{L} \) a base point free line bundle on \( C \) of degree \( d \), defining a map

\[
\phi_{|\mathcal{L}|} : C \to \mathbb{P}^r
\]

where \( r = h^0(C, \mathcal{L}) - 1 \). We denote by \( I(C) \) the homogeneous ideal of \( \phi_{|\mathcal{L}|}(C) \) in \( \mathbb{P}^r \). The generating structure of \( I(C) \) is relatively well-understood for the cases where either

(i) \( C \) is smooth and \( \mathcal{L} = \omega_C \) is the canonical line bundle or else

(ii) \( d \geq 2g+1 \).

In case (i), Max Noether-Enriques-Petri Theorem in [17] says that if \( C \) is not hyperelliptic, then the above map is a projectively normal embedding, and \( I(C) \) is generated by quadrics except \( C \) is a plane quintic (\( g = 6 \)) or trigonal (cf. [4,15]). In [6], M. Green reproved the classical Torelli’s Theorem
by showing that $I(C)$ is generated by quadrics of rank 3 and 4. See also [1]. In case (ii), the above map is always a projectively normal embedding (cf. [2,5,14]). In [16], B. Saint-Donat proved that if $d \geq 2g + 2$, then $I(C)$ is generated by quadratic equations of rank at most 4. Also Eisenbud, Koh and Stillman in [3] proved that if $d \geq 4g + 2$ then $(C,L)$ is determinantly presented in the sense that $I(C)$ is generated by 2-minors of a 1-generic matrix of linear forms on $\mathbb{P}^r$. Concerned with these works, due to [10], we say that $(C,L)$ satisfies property QR($k$) for a positive integer $k$ if $L$ is very ample and $I(C)$ can be generated by quadrics of rank at most $k$.

For example, $(C,L)$ satisfies property QR(4) if either $d \geq 2g + 2$ (cf. [16]) or else $L = K_C$ and $I(C)$ is generated by quadrics (cf. [6]).

In this paper, we obtain the following new result about the quadratic generators of the homogeneous ideal $I(C)$.

**Theorem 1.1.** Let $C$ be a projective integral curve of arithmetic genus $g$ and $L$ a line bundle on $C$ of degree $d \geq 2g + 2$. Then, $(C,L)$ satisfies property QR(3) whenever either

(i) $\text{char } \mathbb{K} \neq 2$, $g = 0$ and $d \geq 2$, or else
(ii) $\text{char } \mathbb{K} \neq 2$, $g = 1$ and $d \geq 4$, or else
(iii) $\text{char } \mathbb{K} \neq 2,3$, $g \geq 2$ and $d \geq 4g + 4$, or else
(iv) $\text{char } \mathbb{K} = 0$, $g \geq 7$, $C$ is smooth and general in the moduli $\mathcal{M}_g$ and $d \geq 4g + 2$.

For the proofs of these statements, see Corollary 2.5, Corollary 2.3, Theorems 3.1 and 3.2.

Roughly speaking, this result says that all sufficiently high degree embedding of projective integral curves share the property QR(3). Also note that $I(C)$ contains no quadrics of rank $< 3$ since $\phi_{|L|}(C)$ is nondegenerate in $\mathbb{P}^r$.

**Theorem 1.1** is a consequence of the following interesting result on the property QR(3) of projective integral curves.

**Theorem 1.2.** Let $C$ be a projective integral curve of arithmetic genus $g$ over an algebraically closed field $\mathbb{K}$ with $\text{char } \mathbb{K} \neq 2$. Suppose that there exists an integer $\tau \geq 2g + 2$ satisfying the condition that

(†) every line bundle $L$ on $C$ of degree $\tau$ satisfies property QR(3).

Then, every line bundle on $C$ of degree $\geq \tau$ satisfies property QR(3).

For example, it is obviously true that $(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2))$ satisfies property QR(3). Then, we can conclude by Theorem 1.2 that $(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\ell))$ satisfies property QR(3) for all $\ell \geq 2$. In this way, we were able to prove the statements of Theorem 1.1 one by one using Theorem 1.2.

**Remark 1.3.** (1) Theorem 1.2 leads us to define a new invariant of projective integral curves. To be precise, let $\tau(C)$ be the smallest integer $\tau \geq 2g + 2$ such that every line bundle $L$ on $C$ of degree $\tau$ satisfies property QR(3). By Theorems 1.1 and 1.2, we know the followings:

(i) When $g = 0, 1$, it holds that $\tau(C) = 2g + 2$. 

(ii) When \( g \geq 2 \), it holds that \( 2g + 2 \leq \tau(C) \leq 4g + 4 \). In particular, \( \tau(C) \)

does exist.

(iii) Every line bundle on \( C \) of degree \( \geq \tau(C) \) satisfies property \( QR(3) \).

It will be a natural problem to find an upper bound of \( \tau(C) \) which is better than the number \( 4g + 4 \) obtained in Theorem 1.1.

(2) Let \( C \) be a projective complex smooth curve of genus \( g \geq 3 \) which is not a hyperelliptic, a plane quintic or a trigonal curve. Then, its canonical embedding \( C \subset \mathbb{P}^{g-1} \) satisfies property \( QR(4) \) (cf. [6]). By [10, Example 5.4 and 5.8], infinitely many canonical curves satisfy property \( QR(3) \). On the other hand, there is a canonical curve \( C \subset \mathbb{P}^5 \) of genus 6 which fails to satisfy property \( QR(3) \). For details, see [10, Example 6.2].

2. Proof of Theorem 1.2

This section is devoted to giving a proof of Theorem 1.2. We begin with a few elementary facts.

Notation and Remarks 2.1. Let \( C \) be a projective integral curve of arithmetic genus \( g \) and let \( L \) be a line bundle on \( C \) of degree \( d \geq 2g + 3 \), defining an embedding

\[
C_0 := \phi|_L(C) \to \mathbb{P}H^0(C, L) = \mathbb{P}^{d-g}.
\]

Also let \( p_1 \) and \( p_2 \) be two distinct smooth points of \( C_0 \). We denote by \( \mathbb{L} \) the line through \( p_1 \) and \( p_2 \) in \( \mathbb{P}^{d-g} \).

(1) Note that the line bundles \( L, L(-p_1), L(-p_2) \) and \( L(-p_1 - p_2) \) are very ample and define projectively normal embedding (cf. [2,5,14]). Furthermore, \( L, L(-p_1) \) and \( L(-p_2) \) satisfy property \( QR(4) \) (cf. [16]).

(2) Consider the cones

\[
S_1 = \text{Join}(p_1, C_0), \quad S_2 = \text{Join}(p_2, C_0), \quad T = \text{Join}(\mathbb{L}, C_0) \subset \mathbb{P}^{d-g}.
\]

The bases of the cones \( S_1, S_2 \) and \( T \) can be regarded, respectively, as the linearly normal curves

\[
C_1 := \phi|_{L(-p_1)}(C) \subset \mathbb{P}H^0(L(-p_1)) = \mathbb{P}^{d-g-1},
\]

\[
C_2 := \phi|_{L(-p_2)}(C) \subset \mathbb{P}H^0(L(-p_2)) = \mathbb{P}^{d-g-1},
\]

and

\[
C_{12} := \phi|_{L(-p_1 - p_2)}(C) \subset \mathbb{P}H^0(L(-p_1 - p_2)) = \mathbb{P}^{d-g-2}.
\]

(3) Let the homogeneous ideals of \( C_0, S_1, S_2 \) and \( T \) in \( \mathbb{P}^{d-g} \) be, respectively, \( I(C_0), I(S_1), I(S_2) \) and \( I(T) \). There are the following inclusions:

\[
I(C_0)_2 \supseteq V := I(S_1)_2 + I(S_2)_2 \supseteq I(S_1)_2 \cap I(S_2)_2 \supseteq I(T)_2
\]

Lemma 2.2. Keep the notations in Notation and Remarks 2.1. Then,

\[
I(S_1)_2 \cap I(S_2)_2 = I(T)_2.
\]

and

\[
\dim_k V = \dim_k I(C_0)_2 - 1.
\]
Proof. To show the first equality, let us suppose that there is a quadric $Q \in I(S_1)_2 \cap I(S_2)_2 \setminus I(T)_2$. Then $T \cap Q$ contains $S_1$ and $S_2$ while

$$2 \times \deg(T) = 2(d - 2) < 2(d - 1) = \deg(S_1) + \deg(S_2).$$

This is a contradiction (cf. [9, Chapter 1, Theorem 7.7]). In consequence, the desired equality $I(S_1)_2 \cap I(S_2)_2 = I(T)_2$ holds.

To show the second equality, consider the elementary formula

$$\dim_K V = \dim_K I(S_1)_2 + \dim_K I(S_2)_2 - \dim_K I(S_1)_2 \cap I(S_2)_2.$$

By Notation and Remarks 2.1.(2), we have

$$\dim_K I(C_0)_2 = \dim_K I(C_1)_2 = \dim_K I(C_2)_2 = \dim_K I(C_{12})_2.$$

Also, $C_0 \subset \mathbb{P}^{d - g}$, $C_1, C_2 \subset \mathbb{P}^{d - g - 1}$ and $C_{12} \subset \mathbb{P}^{d - g - 2}$ are projectively normal curves, and hence it holds that

$$\dim_K I(C_0)_2 = \binom{d - g}{2} - g,$$

$$\dim_K I(C_1)_2 = \dim_K I(C_2)_2 = \binom{d - g - 1}{2} - g$$

and

$$\dim_K I(C_{12})_2 = \binom{d - g - 2}{2} - g.$$

Therefore, it follows that $\dim_K V = \binom{d - g}{2} - g - 1 = \dim_K I(C_0) - 1$.  

Now, we are ready to prove Theorem 1.2.

Proof of 1.2. Firstly, note that it suffices to prove the following statement (*).

(*) $\mathcal{L}$ satisfies property QR(3) for every line bundle $\mathcal{L}$ of degree $\tau + 1$.

Let $\mathcal{L}$ be a line bundle on $\mathcal{C}$ of degree $\tau + 1$, defining a linearly normal embedding

$$\mathcal{C}_0 \subset \mathbb{P} H^0(\mathcal{L}) = \mathbb{P}^{\tau + 1 - g}.$$

Choose three distinct smooth points $p_1, p_2$ and $p_3$ of $\mathcal{C}_0$. For $i = 1, 2, 3$, let $S_i := \text{Join}(p_i, \mathcal{C}_0) \subset \mathbb{P}^{\tau + 1 - g}$.

So, a general hyperplane section of $S_i$ is isomorphic to the linearly normal curve

$$\mathcal{C}_i \subset \mathbb{P} H^0(\mathcal{L}(-p_i)) = \mathbb{P}^{\tau - g}$$

of degree $\tau$ for each $i = 1, 2, 3$. By our assumption, $I(\mathcal{C}_i)$ can be generated by quadrics of rank 3. Therefore, $I(S_i)$ can be also generated by quadrics of rank 3 since $S_i$ is a cone having $\mathcal{C}_i$ as a base. Now, we will show that

$$I(\mathcal{C}_0)_2 = I(S_1)_2 + I(S_2)_2 + I(S_3)_2. \quad (1)$$
Obviously, this completes the proof of (\(\ast\)). To prove (1), first recall that 
\[ \dim_k I(S_1)_2 + I(S_2)_2 = \dim_k I(C_0)_2 - 1 \]
by Lemma 2.2. Thus, it suffices to check that \(I(S_3)_2\) is not contained in \(I(S_1)_2 + I(S_2)_2\). Indeed, if \(I(S_3)_2\) is a subspace of \(I(S_1)_2 + I(S_2)_2\), then it holds that
\[ S_1 \cap S_2 \subset S_3. \]
In particular, the line \(\langle p_1, p_2 \rangle\) is contained in \(S_1 \cap S_2\) and hence in \(S_3\). Then, \(\langle p_1, p_2, p_3 \rangle\) is also contained in \(S_3\). Note that \(\langle p_1, p_2, p_3 \rangle\) cannot be a line since \(C_0\) is cut out by quadrics and so it does not admit a tri-secant line. Therefore, the irreducible surface \(S_3\) contains the plane \(\langle p_1, p_2, p_3 \rangle\), which is a contradiction. In consequence, \(I(S_3)_2\) is not contained in \(I(S_1)_2 + I(S_2)_2\), and hence we have
\[ \dim_k I(S_1)_2 + I(S_2)_2 + I(S_3)_2 \geq 1 + \dim_k I(S_1)_2 + I(S_2)_2 = \dim_k I(C_0)_2. \]
It follows that the desired equality in (1) holds. This completes the proof of Theorem 1.2. \(\square\)

We finish this section by applying Theorem 1.2 to the rational normal curves and elliptic normal curves.

**Corollary 2.3.** \((\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d))\) satisfies property QR(3) for all \(d \geq 2\).

**Proof.** When \(g = 0\), it is obviously true that \((\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2))\) satisfies property QR(3). Then, Theorem 1.2 implies that \((\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d))\) satisfies property QR(3) for all \(d \geq 2\). \(\square\)

**Remark 2.4.** Corollary 2.3 is firstly proved in [10, Corollary 2.4] where a rank 3 generators are explicitly given. Here, we provide another proof, which shows that every rational normal curve satisfies property QR(3) since \((\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2))\) corresponds to a plane conic curve defined by a quadric of rank 3.

**Corollary 2.5.** Let \(\mathcal{C}\) be a projective integral curve of arithmetic genus 1 and \(\mathcal{L}\) a line bundle on \(\mathcal{C}\). If \(d \geq 4\), then \((\mathcal{C}, \mathcal{L})\) satisfies property QR(3).

**Proof.** By Theorem 1.2, it suffices to show that \((\mathcal{C}, \mathcal{L})\) satisfies property QR(3) for every line bundle \(\mathcal{L}\) on \(\mathcal{C}\) of degree 4. Now, let \(\mathcal{L}\) be a line bundle of degree 4 on \(\mathcal{C}\). Then,
\[ \mathcal{C} \subset \mathbb{P}H^0(\mathcal{L}) = \mathbb{P}^3 \]
is a complete intersection of two quadrics, say \(Q_1\) and \(Q_2\). Let \(\mathbb{P}^5\) denote the projective space of quadrics in \(\mathbb{P}^3\). Also let \(W \subset \mathbb{P}^9\) be the locus of quadrics of rank \(\leq 3\). Then, by determinant computation, \(W\) is a hypersurface of degree 4.

When \(\mathcal{C}\) is smooth, the line \(\overline{Q_1, Q_2}\) in \(\mathbb{P}^9\) intersects with \(W\) exactly at four points (cf. Proposition 22.34 in [8]). Those four intersection points correspond to quadratic equations of rank 3 in \(I(\mathcal{C})\). This shows that \((\mathcal{C}, \mathcal{L})\) satisfies property QR(3).
Now, suppose that \( C \) is singular. By [13, Theorem 1.1], we may assume that
\[
Q_1 = x_0^2 + x_1 x_2 \quad \text{and} \quad Q_2 = x_3^2 + x_0 x_2
\]
if \( C \) has the cusp singularity, and
\[
Q_1 = x_0^2 + x_1 x_2 \quad \text{and} \quad Q_2 = x_3^2 + x_0 x_2 + x_0 x_3
\]
if \( C \) has the nodal singularity. This shows that \((C, L)\) satisfies property \( QR(3) \) when \( C \) is singular. As was mentioned at the beginning, this completes the proof. \( \square \)

**Remark 2.6.** In [11, Theorem IV.1.3], the author proved that every elliptic normal curve of degree \( \geq 4 \) is the scheme-theoretic intersection of the quadrics of rank 3 which contains it. Corollary 2.5 shows that an ideal-theoretic version of the statement of [11, Theorem IV.1.3] holds also.

### 3. Proof of Theorem 1.1

Here, we will complete the proof of Theorem 1.1 by verifying the following two statements.

**Theorem 3.1.** Suppose that \( \text{char} \ K \neq 2, 3 \). Let \( C \) be a projective integral curve of arithmetic genus \( g \geq 2 \) and \( L \) a line bundle on \( C \) of degree \( d \). If \( d \geq 4g + 4 \), then \((C, L)\) satisfies property \( QR(3) \).

**Theorem 3.2.** Suppose that \( \text{char} \ K = 0 \) and \( C \) is a smooth projective curve of genus \( g \geq 7 \) and general in the moduli \( M_g \). If \( L \) is a line bundle on \( C \) of degree \( d \geq 4g + 2 \), then \((C, L)\) satisfies property \( QR(3) \).

We begin with a few remarks.

**Remark 3.3.** Let \( C \) be a projective integral curve of arithmetic genus \( g \geq 1 \).

1. The abelian group \( \text{Pic}^0(C) \) is known to be divisible in the sense that every element is an \( n \)th multiple of another element for each positive integer \( n \). For details, we refer the reader to see [18, V.§ 3]. Indeed, let \( \pi : \tilde{C} \to C \) be the normalization of \( C \) and let \( \tilde{g} \) denote the genus of \( \tilde{C} \). Then, the following sequence is exact:
\[
0 \to \pi_* \left( \mathcal{O}_{\tilde{C}}^*/\mathcal{O}_{\tilde{C}}^* \right) \to \text{Pic}^0(C) \to \text{Pic}^0(\tilde{C}) \to 0
\]
The third term \( \text{Pic}^0(\tilde{C}) \) is an Abelian variety of dimension \( \tilde{g} \). In particular, it is divisible. Also the first term can be written as
\[
\pi_* \left( \mathcal{O}_{\tilde{C}}^*/\mathcal{O}_{\tilde{C}}^* \right) = \bigoplus_{P \in \tilde{C}} \tilde{O}_P^*/O_P^*
\]
where \( \tilde{O}_P \) is the integral closure of \( O_P \) in the function field of \( C \). It is known that \( \tilde{O}_P^*/O_P^* \) is always an extension of several \( \mathbb{G}_a \)'s, \( \mathbb{G}_m \)'s and groups of Witt vectors. In particular, it is a divisible group. In consequence, \( \text{Pic}^0(C) \) is a divisible group.
(2) Let $\mathcal{M}$ be a line bundle on $\mathcal{C}$ of degree $\geq 2g + 2$. Then, $(\mathcal{C}, \mathcal{M})$ satisfies property QR(4) (cf. [16]). Now, by applying [10, Corollary 5.2] to $\mathcal{M}$, it holds that if char $\mathbb{K} \neq 2, 3$, then $(\mathcal{C}, \mathcal{M}^2)$ satisfies property QR(3).

(3) Suppose that char $\mathbb{K} = 0$ and $\mathcal{C}$ is smooth. Let $\mathcal{M}$ be a line bundle on $\mathcal{C}$ of degree $2g + 1$, which defines a linearly normal embedding $\mathcal{C} \subset \mathbb{P}^{g+1}$.

By [7, Theorem 2], the following two statements are equivalent:

(i) $I(\mathcal{C})$ fails to be generated by quadrics;
(ii) there is an effective divisor $D$ of degree 3 such that $\mathcal{M} = \omega_{\mathcal{C}} \otimes \mathcal{O}_{\mathcal{C}}(D)$.

Now, we give the proofs of Theorems 3.1 and 3.2.

**Proof of Theorem 3.1.** By Theorem 1.2, we need to prove that $(\mathcal{C}, \mathcal{L})$ satisfies property QR(3) when $\mathcal{L}$ is a line bundle of degree $4g + 4$. To this aim, choose any line bundle $\mathcal{M}_0$ of degree $2g + 2$. Then, $\mathcal{L} \otimes \mathcal{M}_0^{-2}$ is a member of $\text{Pic}^0(\mathcal{C})$. In particular, there exists a line bundle $\mathcal{N} \in \text{Pic}^0(\mathcal{C})$ such that

$$\mathcal{L} \otimes \mathcal{M}_0^{-2} = \mathcal{N}^2,$$

since $\text{Pic}^0(\mathcal{C})$ is a 2-divisible group (cf. Remark 3.3.(1)). In consequence,

$$\mathcal{L} = \mathcal{M}^2,$$

where $\mathcal{M} := \mathcal{M}_0 \otimes \mathcal{N}$ is a line bundle of degree $2g + 2$. Namely, $\mathcal{L}$ is the square of a line bundle of degree $2g + 2$. Now, it follows by Remark 3.3.(2) that $(\mathcal{C}, \mathcal{L})$ satisfies property QR(3). $\square$

**Proof of Theorem 3.2.** We may assume that $\mathcal{C}$ is not hyperelliptic. By Theorem 1.2, we need to show that $(\mathcal{C}, \mathcal{L})$ satisfies property QR(3) when $\mathcal{L}$ is of degree $4g + 2$. For such a line bundle $\mathcal{L}$, one can show that there exists a line bundle $\mathcal{M}$ of degree $2g + 1$ such that

$$\mathcal{L} = \mathcal{M}^2.$$

For details, see the proof of Theorem 3.1. Now, choose a non-trivial 2-torsion line bundle $\mathcal{E}$ on $\mathcal{C}$. Let $\mathcal{C}_1 \subset \mathbb{P}^{g+1}$ and $\mathcal{C}_2 \subset \mathbb{P}^{g+1}$ be respectively the linearly normal embedding of $\mathcal{C}$ by $\mathcal{M}$ and $\mathcal{M} \otimes \mathcal{E}$. We claim that at least one of $I(\mathcal{C}_1)$ and $I(\mathcal{C}_2)$ is generated by quadrics. Indeed, suppose not. Then, by Remark 3.3.(3), we can write

$$(\mathcal{M} = \omega_{\mathcal{C}} \otimes \mathcal{O}_{\mathcal{C}}(D) \quad \text{and} \quad \mathcal{M} \otimes \mathcal{E} = \omega_{\mathcal{C}} \otimes \mathcal{O}_{\mathcal{C}}(D'))$$

for some effective divisors $D$ and $D'$ of degree 3. Then, we get

$$\mathcal{M} = \omega_{\mathcal{C}} \otimes \mathcal{O}_{\mathcal{C}}(D) = \omega_{\mathcal{C}} \otimes \mathcal{O}_{\mathcal{C}}(D') \otimes \mathcal{E}^{-1}$$

and hence $\mathcal{O}_{\mathcal{C}}(D' - D) = \mathcal{E}$ is a torsion line bundle. Now, by the main theorem in [12], the line bundle $\mathcal{O}_{\mathcal{C}}(D + D')$ must be non-special.
Theorem (G. R. Kempf, [12]). Assume that $C$ is a smooth complete algebraic curve with general moduli in characteristic zero. Let $D_0$ and $D_\infty$ be distinct effective divisors of $C$ such that $\mathcal{O}_C(D_0 - D_\infty)$ is a torsion line bundle. Then, the cohomology $H^1(C, \mathcal{O}_C(D_0 + D_\infty))$ must be zero.

Thus, we have

$$h^0(C, \mathcal{O}_C(D + D')) = 6 + 1 - g \geq 1$$

and hence $g \leq 6$, which contradicts to our assumption that $g \geq 7$. Therefore, it is shown that either $I(C_1)$ or else $I(C_2)$ must be generated by quadrics. Since

$$\mathcal{L} = \mathcal{M}^2 = (\mathcal{M} \otimes \mathcal{E})^2,$$

it follows by Remark 3.3.(2) that $(\mathcal{L}, \mathcal{L})$ satisfies property QR(3).

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