Using computer algebra to construct analytical solutions for elastostatic problems

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Abstract. The article gives a brief overview of elastostatic methods. It notes successful attempts of building analytical solutions. It is necessary to "legalize" a number of parameters of the problem in the full-parameter solution (FPS): elasticity, boundary conditions (BCs), volume forces, "geometry" of the body. To provide stiffness and geometry parameters figuring, the Lagrange interpolation and Chebyshev approximation are applicable (the FPS of the first main problem for a ball is constructed). The perturbation method has claimed to be a resource-saving technology for the "legalization" of the Poisson's ratio. To construct the state of the body under the action of volume forces, two efficient algorithms from a wide class of polynomial ones are proposed. They allow strict writing out of a particular solution. There is shown the FPS of a problem of deformation of the ball layer pinched on the surface by such forces. An approach based on the construction "reference" solutions is recommended for the "legalization" of BC parameters and volume forces.

1. Full-parameter solutions
At present, software products (ANSYS, Civil Engineering, SCAD, etc.) are widely used, allowing the calculation of the stress strain behavior (SSB) of the body and visualisation of the results. All of them provide solutions to problems in the numerical form. The effective design requires multiple calculations, even with minor variations in the parameters of the problem.

Analytical solutions used to determine states of bodies during the performance of engineering and research calculations can significantly reduce the cost of computing resources and time, as well as allow a comprehensive study of the state of an object using various research methods to achieve specific targets.

Constructing an analytical solution, explicitly involving all the parameters of a problem (full-parameter analytical solution, FPS), is always an important issue. The methodology of the computer-aided FPS construction includes the following steps:

1) correct mechanical formulation of the problem;
2) non-dimensionalisation of defining relations and parameters of the problem in accordance with the Buckingham π theorem;
3) selection of a method to obtain a solution in the analytical form;
4) accounting elastostatic parameters;
5) accounting parameters of boundary conditions (BCs) (non-dimensionalised constants);
accounting parameters of volume forces;
7) accounting the "geometry" of the body;
8) reverse transition to the dimensional values.

Proper formulation of problems. According to Hadamard, a problem is called well-formulated if the solution exists, it is unique and continuously depends on BCs [1]. The task of the linear elastostatics is to solve a system of equations (tensor-index form of recording with the "summation convention") [2], [3]:

\[
\begin{align*}
\varepsilon_{ij} &= \frac{1}{2} \left( u_{i,j} + u_{j,i} \right), \\
\sigma_{ij} &= \lambda \varepsilon \delta_{ij} + 2 \mu \varepsilon_{ij}, \quad \varepsilon = \varepsilon_{kk}, \\
\sigma_{ij} + X_i &= 0,
\end{align*}
\]

where \( u_i, X_i \) are components of the vectors of displacements and volume forces, \( \sigma_{ij}, \varepsilon_{ij} \) are components of the tensors of stresses and strains, \( \delta_{ij} \) is a Kronecker delta, \( \lambda, \mu \) are Lamé parameters.

Below, \( \zeta \) is understood as a set of field characteristics of the elastic medium (state): \( \zeta = \{ u, \varepsilon_{ij}, \sigma_{ij} \} \), \( x = \{ x \} \in V \), where \( V \) is the area occupied by the body. For the basic problems of the theory of elasticity, theorems of uniqueness of solutions are proved.

Non-dimensionalisation. The procedure of non-dimensionalisation is performed in accordance with the \( \pi \) theorem [4], which allows minimizing the number of parameters taken into account in search for a solution. As the scale factors of elastostatics it is convenient to use magnitudes of displacement (characteristic geometric size) and stress (shear modulus).

Selection of method for FPS construction. The main modern computational methods (the finite element method, the boundary element method) are implemented in many forms in computing systems and based on the traditional object of common computational tools – the “number”. They are effectively used at the present time, but require recalculation of the resolving system of equations and its solution when changing any parameter of the object under study (resource consumption) ([5], [6], [7], [8]). The result of the solution also does not guarantee the truth, therefore at the level of test problems it is necessary to perform a comparison with solutions obtained by other means.

The boundary state method (BSM) [9] by definition gives a solution that identically satisfies the defining relations of the medium, and the error of the solution can be judged by the value of the BC residual with the corresponding characteristics of the constructed boundary state. The possibility of constructing a FPS eliminates the need for the resource-consuming recalculation when performing computational procedures, which is replaced by routine calculations using ready-made analytical expressions.

The Trefftz approach ([10], [11], [12], [13], [14]), when forming the basis of approximating functions in the resolving equation with further minimization of some function along the boundary of the body, can give a similar effect, but further requires restoration of the SSB through the boundary state. This implies additional actions related to the calculation of the surface integral when determining the displacement at each point within the body, etc. (see Somigliana's formula), since the isomorphism of states within and at the boundary of the body is not provided. In the BSM, the isomorphism is initially established and the SSB of the body is determined in the numerical and analytical form. Also very significant is the fact that the formulation of a boundary value problem does not require to assemble a resolving equation with respect to any resolving function (vector of functions). BSs are formed directly in terms of the nominal characteristics of the boundary state: displacements, superficial forces, their combinations.

It should be noted successful attempts to construct analytical solutions in cases of main problems of linear anisotropy [15] and heterogeneity [16]. But these technologies are obviously losing to the more general approach of the BSM.
**Figuring of medium parameters.** For a homogeneous isotropic body, it is sufficient to provide the presence of one elastic parameter in the FPR (full-parameter solution). Set the shear modulus to be fixed and enter a small \( \alpha \) parameter for the Poisson’s ratio: \( \nu = \nu_0 + \alpha \). At \( \nu_0 = 1/4 \) it holds \( |\alpha| \leq 1/4 \). The desired state of the elastic medium in the body is sought as an asymptotic series (the superscript indicates the iteration number):

\[
\xi = \xi^0 + \alpha \xi^1 + \alpha^2 \xi^2 + \ldots
\]

For each iteration \( k \), elastostatic ratios have the classical form (1.1) with the only difference that strains are understood as values different from those by an increment defined as the result of the previous iteration (at \( k = 0 \), there is no difference). There is also a correction in the equilibrium equations, for example, for a simply connected bounded body, even in the initial absence of volume forces, there are generated fictitious forces of a polynomial nature, wherein state of medium \( \xi \) is strictly restored (see Paragraph 2) in the analytical form.

There can be used another approach to take into account the Poisson's ratio \( \nu \) in an explicit form. It requires solving a series of problems for the body of a given geometric configuration with various fixed values \( \nu \) followed by interpolation by the Lagrange polynomial:

\[
\xi(\nu) = \sum_i \prod_{j \neq i} (\nu - \nu_j) \xi(\nu_i).
\]

For the same purpose, the approximation by Chebyshev polynomials is applicable. Denote the arbitrary characteristic of state \( \xi \) by \( \tilde{\xi} \in \xi \). For the known values of characteristic \( \tilde{\xi} = f(\nu) \) on a set of points \( \nu = \{\nu_0, \nu_1, \ldots, \nu_n\} \), the polynomial approximant \( Q_m(\nu) \) is as follows [17]:

\[
Q_m(\nu) = \sum_{j=0}^m c_j P_j(\nu), \quad c_j = \sum_{i=0}^n \tilde{\xi} P_j(\nu_i), \quad s_j = \sum_{i=0}^n P_j^2(\nu_i),
\]

where \( \tilde{\xi} = f(\nu_i) \), a \( P_j(\nu) \) are orthogonal polynomials on the set of points \( \nu \). The quadratic deviation of this polynomial from this function is defined as \( S = \sum_{j=0}^m \tilde{\xi}^2 - \sum_{j=0}^m c_j^2 \).

Use Chebyshev polynomials as \( P_j \) [17]:

\[
T_0(x) = 1, \quad T_{n+1}(x) = xT_n(x) - \frac{1}{4}T_{n-1}(x), \quad (n = 0, 1, 2, \ldots), \quad x \in [-1,1].
\]

The \( \nu \) and \( x \) concordance is performed by centering: \( x = 4\nu - 1 \).

The use of the approximation approach is demonstrated in the example. The calculations have been performed by means of the Mathematica system, they are concisely and clearly described in [18].

**Example.** Elastic ball (elastic constants \( \mu, \nu \)) of radius \( R \) is loaded with axial forces: \( \mathbf{p}^0 \) (\( r = R, \phi, \theta \) are spherical coordinates):

\[
\mathbf{p}^0 = p_0 \mathbf{e}_r, \quad \mathbf{e}_r = \partial_r \mathbf{e}_x, \quad \mathbf{e}_\phi = \sin \theta \mathbf{e}_y, \quad \mathbf{e}_\theta = -\cos \theta \mathbf{e}_z, \quad \mathbf{e}_x \in [0, \pi/2], \quad \mathbf{e}_y \in [-\pi, \pi], \quad \mathbf{e}_z \in [-\pi/2, \pi/2].
\]

The system of external forces is balanced, there are no volume forces. It is required to construct a FPS describing the SSB of the ball.

The problem was solved by means of BSM [19], where, in particular, the basis of the Hilbert space of the states is prescribed; \( k \) is the number of the basis element \( \xi^{(k)} \). The desired state \( \xi \) is
determined by the Fourier series \( \zeta = \sum_{k} c_k \xi^k \). The obtained non-zero values of the Fourier coefficients for the orthonormal basis are given in Table 1.

| \( \nu \) | \( c_1 \) | \( c_4 \) | \( c_6 \) | \( c_{22} \) | \( c_{25} \) | \( c_{29} \) | \( c_{32} \) | \( c_{36} \) | \( c_{39} \) |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0.205 | 0.309 | -0.785 | -0.234 | 0.069 | -0.258 | 0.146 | 0.824 | -0.772 |
| 0.25 | 0.227 | 0.341 | -0.709 | -0.252 | 0.075 | -0.277 | 0.157 | 0.735 | -0.686 |
| 0.25 | 0.256 | 0.387 | -0.623 | -0.273 | 0.081 | -0.300 | 0.170 | 0.644 | -0.601 |
| 0.375 | 0.296 | 0.463 | -0.497 | -0.298 | 0.088 | -0.329 | 0.185 | 0.545 | -0.512 |
| 0.49999 | 0.354 | 0.614 | -0.005 | -0.329 | 0.097 | -0.367 | 0.203 | 0.429 | -0.410 |

The results of using the Lagrange method are given in [20].

FIGURING THE PARAMETERS OF BOUNDARY CONDITIONS. BCs in the main problems of the elastostatics can be set in the analytical form, containing non-fixed values of the parameters: \( p = \sum_{k=0}^{n} p_k f_k \), \( x \in \partial V \) (or \( u = \sum_{k=0}^{n} u_k f_k \)), where \( f_k \) are sets of given vector functions. Is is possible to construct a series of "reference" solutions, specifying each \( f_k \) and obtaining \( \xi^k \) corresponding to the unit value of
parameter \( p^*_k (u^*_k) \). Then the resulting state is the corresponding linear combination: 
\[
\xi = \sum_{k=0}^{n} p^*_k \xi^0_k 
\]

(\( \xi = \sum_{k=0}^{n} u^*_k \xi^0_k \)).

In the case of an infinite body, the FPS must include parameters responsible for the state at infinity (forces \( X^-, Y^-, Z^- \)). They are accounted in the similar way.

**Figuring the parameters of volume forces.** To include the parameters of the volume forces of the elastic medium as well as in the case of the parameters from BCs, their linear combination is studies, i.e. \( X = \sum_{j=0}^{n} X^*_j \chi_j \). The problem of constructing a particular solution should be decomposed into a set of several problems that successively correspond to each of the “reference” states \( \chi_j \), and make up a linear combination containing parameters \( X^*_j \). At the same time, in order to provide compensation from the violation of BCs, it is necessary to add to them the terms of sum containing exactly these parameters with the compensation of strictly retained sets of conditions at the boundary at each section of the homogeneity of the BCs.

**Figuring the geometrical parameters of the body.** The inclusion of excessive geometric parameters of the body in the FPS is quite a time-consuming task. It requires to solve a series of similar problems with subsequent approximation. The resource consumption of the process is associated with the need to generate state bases for each set of geometry parameters. This approach was used in [21] and based on the constructed FPS it was possible to find the optimal values of the parameters of an inhomogeneous elastic body, optimizing the mass of the product under strict requirements to the operating conditions.

2. **State of the elastic medium under the effect of polynomial mass forces**

Defining relations (1.1) must be performed in the vicinity of each internal point of the region \( V \subset R^3 \) with the boundary \( \partial V \) occupied by the body. They are reduced to the Lamé equations

\[
\mu u_{i,j,j} + (\lambda + \mu) u_{j,j,i} + X_i = 0, \quad (2.1)
\]

the general solution of which consists of the general solution of its homogeneous part \( u^0_i \) and any particular solution \( u^*_i \).

The general Papkovich-Neuber solution for the homogeneous part

\[
u_i^0 = 4(1-\nu)B_i - (x_j B_j + B_0)_{ij}, \quad (2.2)
\]

where \( \nu \) the Poisson’s ratio based on arbitrary harmonic vectors with components \( B_i \) and scalar \( B_0 \).

For practical purposes, the Arzhanyh-Slobodyansky solutions for the interior of a bounded area and for the exterior of a cavity in an infinite space, respectively, are more convenient.

\[
u_i^0 = 4(1-\nu)B_i + x_j B_{i,j} - x_i B_{j,j}, \quad \nu_i^0 = 4(1-\nu)B_i - (x_j B_j)_{ij}, \quad (2.3)
\]

expressed through three harmonic functions \( B_i \) in their domains.

In both cases, the presence of vector bases \( B^{(i)}_i \) from harmonic polynomials (and their inversions with respect to the sphere of unit radius) makes it possible to effectively construct the bases \( \{\xi_i^{(i)}\} \in \Xi \) of the space of internal states of the body [19] and its isomorphic space of boundary states \( \{\gamma_i^{(i)}\} \in \Gamma \).

Here, the elements of spaces are understood as sets \( \xi = \{u, \epsilon, \sigma\}, \gamma = \{u_{|_{\partial V}}, p\}, \sigma = \{\sigma_i \}_{ij} \), \( n \), respectively; \( n_j \) are the components of the unit vector of the external normal to \( \partial V \).
Having the basis of the space Ξ, it is possible to construct a specific state that corresponds not only to the defining relations, but also to BCs using classical energy methods, a new direct method of boundary states [9], and other computational tools, but with one provision: The BCs should reflect the compensation from the state ξ∗ ↔ ξγ generated by the mass forces.

The construction of a particular solution ξ∗ corresponding to the given volume forces X is a separate important task.

In a general case, there is known a representation of the displacement field inside the body V through the Green strain tensor

\[ u_i(Q) = \int_U (U_i(M;Q) X_j(M) dV) \]

where \( M, Q \) are points of impact and observation, respectively. The singular character of the integral does not allow us writing out an explicit solution, only its numerical variants can be used.

In a particular case of potential mass forces, Papkovich and Neuber found a representation

\[ u_i = \chi_i, \]

where function \( \chi \) satisfies the Poisson's equation with the potential \( \Pi \) of volume forces on the right side (\( X = \Pi_i \)):

\[ \chi_i = \frac{1 - 2\nu}{2\mu(1-\nu)} \Pi_i. \]

The authors found another approach to the construction of an explicit form of the state ξ∗ from the mass forces, which are presented in the form of polynomials of finite order. This approach has a significant practical effectiveness.

"Providing algorithm". Below, "regularity" means a polynomial representation for all \( X_i \) simply connected bounded regions \( V \). The purpose of the constructive demonstrative algorithm is providing the basis of the space of regular force vectors, i.e. filling the basis, ensuring the completeness and linear independence of its elements.

Initial assumptions: assuming \( w = x^\alpha y^\beta z^\gamma, \alpha, \beta, \gamma \in N \cup \{0\} \), it can be assumed that the use of options

\[ u = \{[w,0,0], [0,w,0], [0,0,w]\} \]

will lead to the organization of basis \( \{X^0\} \) of space \( X \) of regular vectors \( X_i \) along chain \( u \rightarrow \hat{e} \rightarrow \hat{a} \rightarrow X \), arising from (1), (2), (3). The definition \( u = \{w,0,0\} \) leads to the form

\[ X = -w \begin{pmatrix} (\lambda + 2\mu) \alpha (\alpha - 1) x^2 + \mu \beta (\beta - 1) y^2 + \mu \gamma (\gamma - 1) z^2 \\ (\lambda + \mu) \alpha \beta x^1 y^1 \\ (\lambda + \mu) \alpha \gamma x^1 z^1 \end{pmatrix}. \]  

Other basic options are obtained from (6) by the circular substitution of the indices with a corresponding change of positions of the rows of vector \( X \). Expression (6) convinces that basis \( \{X^0\} \) should be formed according to the summary of monomials of the form \( x^\alpha y^\beta z^\gamma, k = a + b + c = \alpha + \beta + \gamma - 2 \). The whole ensemble of monomials can be represented by layers with number \( k \) of the triangular pyramid of monomials \( x^\alpha y^\beta z^\gamma \). The top of the pyramid corresponds to 1 \( (a = b = c = k = 0) \).
The ribs form powers $x^k, y^k, z^k$. The faces contain pairwise products of powers:

$x^s y^{s-k}, y^s z^{s-k}, x^s z^{s-k}, s \in [0..k].$

Arrange to iterate over each layer $k$ of monomials for the first variant of the structure of vector $u$ (other variants are immediately provided by the circular substitution). The process of the "providing" iteration is accompanied by the mental construction of a broken helix (three types of links $a = \text{const}, b = \text{const}, c = \text{const}$ in each turn), tightening to the center of pyramid layer $k$. The sequential fixation of the links at level $s$ appoints the iteration of possible variants, while in each node there is a single monomial involved in $X_s$, which will never be repeated in the iteration. This guarantees both completeness and linear independence of the basis segment for layer $k$. It is clear that at $k \to \infty$ there is formed a countable basis of separable space of vectors of polynomial forces $X$.

The "providing" algorithm is not only constructive in terms of forming the basis, but in addition allows organizing the elimination of linear dependence on other elements, which are intentionally constructed earlier. The algorithm is actually equivalent to the procedure of orthogonalization of the basis in the sense of the scalar product

$$(X^{(i)} X^{(j)}) = \int \chi^{(i)} \chi^{(j)} dV.$$ 

"Sorting algorithm". The iteration of monomials of layer $k$ can be carried out in any order. Associating a plane system of coordinates to the "geometric center" of the triangular layer allows the assignment of a radius-vector to each element symbolized by a point inside an equilateral triangle. The scalar product with the radius vectors of the triangle vertices is determined by the "degree of belonging" of the object to one or another vertex. Their maximum value binds the monomial to "varieties" "A" ($x^k$), "B" ($y^k$) or "C" ($z^k$). In case of equality of scalar products the choice is indifferent. Next, assume either $\alpha = a + 2, \beta = b, \gamma = c$ ("A") or $\alpha = a, \beta = b + 2, \gamma = c$ ("B") or $\alpha = a, \beta = b, \gamma = c + 2$ ("C") and use definition (2.4) followed by the circular substitution of indices.

The constructed segment of the basis is complete, linear independent, but it is not orthogonal to other elements. However, the procedure for its implementation is simple.

Basis $X$ determines the corresponding basis of the space of internal states $\Xi$, consisting of elements $\xi = \{u, v, \sigma\}$ in accordance with expressions (1.1) and recorded in parallel with the filling of the basis of the space of mass forces. Accordingly, the decomposition of a known mass force in its basis determines the Fourier coefficients, which are also those for the formation of the internal state $\xi$ corresponding to the acting force $X$.

A new approach to the construction of a particular solution of the system of Lamé equations, while in the analytical form, is much broader and more flexible than previously known. A similar approach for force classes in the exterior neighbourhood of the singular point of space has not been guaranteed yet, but computational experiments in this area have already shown satisfactory results.

3. Formulation of problems of the theory of elasticity and their solution by the method of boundary states

One of the most time-consuming processes in solving problems by the method of boundary states is the process of orthogonalization. It is based on the orthogonalization theorem [22] implemented by the Gram-Schmidt process. The algorithm is as follows: at the k-th step, a new independent element $\varphi_k$ of the orthonormal basis is added; through it and previously constructed elements $\varphi_1, \ldots, \varphi_{k-1}$ an orthonormal element $\psi_k$ is constructed. The BSM uses [9] a more convenient matrix orthogonalization algorithm for machine-added count. It uses only cross scalar products of the elements of the original basis of internal states, which are reduced to the Gram matrix:
\[ G = \left| g_{ij} \right|_{ij} \in \mathbb{C}, \quad g_{ij} = \left( g^{(i)}, g^{(j)} \right) \in \mathbb{C}. \] Since the spaces of internal and boundary states are isomorphic, the orthonormal basis of boundary states corresponds to the orthonormal basis of internal states.

The main problems of mechanics are conveniently formulated and solved in terms of BSM. In the first (or second) main problem (according to classification by N. I. Muskhelishvili) at boundary \( \partial V \) it is set

\[ p_i \big|_{\partial V} = \varphi_i(x_i) \quad \text{(or)} \quad u_i \big|_{\partial V} = \psi_i(x_i). \] \hfill (3.1)

Adjustment of BCs in accordance with the definition of state in (1.1)

\[ p_i \big|_{\partial V} = p_i \big|_{\partial V} - p_i \big|_{\partial V}, \quad p_i \big|_{\partial V} = \sigma_i \big|_{\partial V} \right), \quad \text{(or)\hspace{10pt}u_i \big|_{\partial V} = u_i \big|_{\partial V} - u_i \big|_{\partial V}} \] reduces the solution of the problem to finding Fourier coefficients through quadratures

\[ c_{\pm} = \int_{\partial V} p_i^{(\pm)} u_i^{(\pm)} dS \quad \text{(or)\hspace{10pt}c_{\pm} = \int_{\partial V} u_i^{(\pm)} p_i^{(\pm)} dS}. \] \hfill (3.2)

In the main mixed and contact problems, it is necessary to solve an infinite system of equations (ISE) [9]:

\[ \mathbf{q} = \mathbf{c}. \] \hfill (3.3)

In the main mixed problem at boundary \( S_u \) there are set displacements \( u_i \big|_{\partial V} \), at boundary \( S_p \) – forces, \( p_i \big|_{\partial V}, S_u \cup S_p = \partial V, S_u \cap S_p = \emptyset \). The elements of the right side of ISE \( \mathbf{q} \) and ISE matrix ("skeleton" of the problem) \( \mathbf{Q}_{bn} \) are calculated as follows:

\[ \mathbf{q}_b = \int_{S_u} \left( p_i u_i^{(k)} + p_i u_i^{(k)} + p_i u_i^{(k)} \right) dS + \int_{S_p} \left( u_i p_i^{(k)} + u_i p_i^{(k)} + u_i p_i^{(k)} \right) dS, \]

\[ \mathbf{Q}_{bn} = \int_{S_u} \left( u_i^{(n)} p_i^{(n)} + u_i^{(n)} p_i^{(n)} + u_i^{(n)} p_i^{(n)} \right) dS + \int_{S_p} \left( p_i^{(n)} u_i^{(n)} + p_i^{(n)} u_i^{(n)} + p_i^{(n)} u_i^{(n)} \right) dS. \]

In the primary contact problem, there is given a set of BCs \( \{u_n, p_n, \tau_n\} \) referred to the system of coordinates "normal – tangent". The right side of ISE \( \mathbf{q} \) and "skeleton" of the problem \( \mathbf{Q} \) are calculated according to the scheme:

\[ \mathbf{q}_c = \int_{\partial V} \left( u_i p_i^{(k)} + p_i u_i^{(k)} + p_i u_i^{(k)} \right) dS, \quad \mathbf{Q}_{bn} = \int_{\partial V} \left( p_i^{(n)} u_i^{(n)} + p_i^{(n)} p_i^{(n)} + p_i^{(n)} p_i^{(n)} \right) dS. \]

Fourier coefficients are calculated using the inverse matrix: \( \mathbf{c} = \mathbf{Q}^{-1} \mathbf{q} \).

Corrections to the BCs in view of mass forces are carried out in the same way as described above. The fields marked with "•" are defined by linear combinations

\[ u_i = \sum_k c_i u_i^{(k)} , \quad \varepsilon_i^{(k)} = \sum_k c_i \varepsilon_i^{(k)} , \quad \sigma_i^{(k)} = \sum_k c_i \sigma_i^{(k)}. \]

The resulting fields are defined by superposition \( \xi = \xi^0 + \xi^\pm \).

4. Stress-strain state of the pinched elastic layer under the effect of non-potential polynomial forces

An axisymmetric body (figure 1), bounded on the lateral surface by a sphere of radius \( 2R_0 \) (the lower index 0 corresponds to the dimensional values), and at the ends – by circles of radius \( \sqrt{3}R_0 \),
constitutes a layer of thickness $2R_o$ with a spherical lateral surface. The body is isotropic, mechanically homogeneous. The Poisson’s ratio $\nu = 1/4$ corresponds to the non-dimensionalised Lamé parameters $\lambda = \mu = 1$.

**Figure 1. Spherical layer.**

The boundary of the body is fixed ($\mathbf{u}_{\partial V} = 0$), the body is affected by the volume forces of polynomial nature: $X = X_o R_o / \mu_o \{ xz, yz, 2z \}$. It is required to evaluate the SSB of the body.

According to the results of solving the problem, the dimensional values of displacements were written out, but at a fixed numerical value of coefficient $\nu$ (for the sake of convenience, the subscript 0 is omitted):

$$u_x = Xxz (0.147 - 0.033 \left( x^2 + y^2 \right) / R^2 + 0.054 z^2 / R^2 - 0.001 (x^4 + y^4) / R^4 - 0.021 z^4 / R^4 + 0.029 z^2 (x^2 + y^2) / R^4 - 0.003 x^2 y^2 / R^4) / \mu;$$

$$u_y = Xyz (0.147 - 0.033 \left( x^2 + y^2 \right) / R^2 + 0.054 z^2 / R^2 - 0.001 (x^4 + y^4) / R^4 - 0.021 z^4 / R^4 + 0.029 z^2 (x^2 + y^2) / R^4 - 0.003 x^2 y^2 / R^4) / \mu;$$

$$u_z = X (0.097 R^2 - 0.024(x^2 + y^2) - 0.082 z^2 - 0.006(x^4 + y^4) / R^2 + 0.059 z^2 (x^2 + y^2) / R^2 - 0.024 z^2 / R^2 - 0.011x^2 y^2 / R^2 + 0.004 x^2 y^2 (x^2 + y^2) / R^4 - 0.017 z^4 (x^2 + y^2) / R^4 - 0.005 z^2 (x^4 + y^4) / R^4 - 0.010 x^2 y^2 z^2 / R^4 + 0.001 (x^4 + y^4) / R^4 + 0.006 z^6 / R^4) / \mu.$$

The illustration of the resulting solution by means of BSM is shown in figure 2.

The problem retains mass forces of a special nature – radially stretching at $z > 0$ and radially compressing at $z < 0$. Axial components cause more tension as they move away from the median plane. Due to the "pinched" boundary $\partial V$ of the body it is quite clearly expressed in the axial sections of the stress field. The symmetry itself in the constructed figures testifies to the high-precision approximation of the stress field (in fact, it is built with almost absolute accuracy: if there are errors, they are only due to rounding of dimensionless physical quantities). The nature of shear stresses is impressive $\sigma_{\theta z}$: near neutral plane $z = 0$, they are negative and change their sign as they approach the end surfaces of the body.

The present study allows an important conclusion about the effectiveness of new algorithms that allow the provision of a particular solution to the boundary value problems of inhomogeneous Lamé equations corresponding to the presence of mass forces of a regular nature in the sense of the polynomial representation. It would seem that the "fanciful" character of arbitrarily given forces should not be alarming: when solving various problems of the theory of elasticity by methods of linearisation (for example, the method of perturbations) fictitious mass forces of such character are "born" at each step of linearisation.
Figure 2. Stresses in the cross section \( y = 0 \): a) radial stress \( \sigma_r \), b) circumferential stress \( \sigma_\theta \) 
\( (\sigma_\theta = \sigma_\gamma |_{y=0} ) \), c) axial stress \( \sigma_z \), d) shear stress \( \sigma_{r\theta} = \sigma_{\theta\gamma} |_{y=0} \).

5. The method of perturbations for linear inhomogeneous elastostatics

In a general case of linear inhomogeneous elastostatic medium, defining relations are (1.1). The difference consists in the fact that the elastic characteristics of the body are implied to be inhomogeneous and volume forces – nonpotential. Both are considered to be represented by series in powers of small parameter \( \alpha \):

\[
\{\lambda, \mu, X_i\} = \sum_{m=0}^{\infty} \alpha^m \{\lambda_m, \mu_m, X_{m,i}\},
\]

where all functions \( \lambda_m, \mu_m, X_{m,i} \) are assumed to be polynomial or approximate polynomials from \( x, y, z \) with a high accuracy.

The asymptotic development of state

\[
\{u_i, \varepsilon_{ij}, \sigma_{ij}\} = \sum_{n=0}^{\infty} \alpha^n \left\{u_i^n, \varepsilon_{ij}^n, \sigma_{ij}^n\right\}
\] (5.1)

after substitution in (1.1), replacement of summation variables in order to align the exponents of small parameter \( \alpha \) in the record and compare the expressions to the same powers, leads to a sequence of classical constitutive relations for a homogeneous body. Namely, in step \( n \) of the decomposition, the obtained ratios are

\[
\varepsilon_{ij}^n = \frac{1}{2} \left( u_{ij}^n + u_{ji}^n \right),
\]

\[
s_{i,j}^n = \lambda_{ij}^n \vartheta \delta_{ij}^n + 2 \mu_{ij}^n \varepsilon_{ij}^n, \quad \vartheta = \varepsilon_{\varepsilon \varepsilon},
\]

\[
s_{i,j}^n + \chi_i^n = 0,
\] (5.2)
regarding “pseudostates” \( \xi = \left\{ u_i, \varepsilon_{ij}, s_{ij} \right\} \) with fictitious volume forces \( \dot{\chi} \) that take into account the results of solutions at the previous steps: \( \dot{\chi}_i = \chi_i + \dot{\sigma}_{ij} \). By constructing the corresponding field characteristics of step \( n \), the stress field is restored as follows:

\[
\dot{\sigma}_{ij} = s_{ij} + \sum_{n=0}^{\infty} \left( \lambda_{n,ij} \delta_{ij} + 2 \mu_{n,ij} \varepsilon_{ij} \right).
\]

(5.3)

No corrections are made at the initial iteration.

The stated above convincingly shows that nonpotential volume forces are regularly encountered in the research practice, and the task of restoring the internal state from regular forces in this sense is highly relevant.

Conclusions
1. The FPS provides engineers and researchers with unique opportunities to achieve their goals.
2. The numerical and analytical form of solution by means of BSM allows accounting corrections easily (including analytical differentiation) in the expressions of stresses and volume forces.
3. The interpolation and approximation are traditional methods of parameter inclusion in functional dependences, but for the problems of mathematical physics they are resource-consuming. Using the method of perturbations significantly saves computational resources and time.
4. The classical problem of elastostatics for a homogeneous isotropic body is solved at each step of the MGSV for a linear inhomogeneous medium. The basis of the spaces of state is constructed once for all iterations.
5. Since the basis of the spaces of state of a medium for a simply connected bounded body is formed on the ground of the basis of harmonic polynomials, and the parameters of the medium and the volume forces are assumed to be regular, corrective additions to them are also polynomial.
6. The approaches to the construction of a particular solution of the defining relations of the medium presented in the paper allow estimating the internal state of the body from polynomial volume forces strictly (practically with any predetermined accuracy).
7. The technology of using "reference" solutions radically resolves the issue of including BC parameters, volume forces, conditions at infinity in the FPS.

The developed approach is also a serious basis for the organization of the continuation method in order to construct solutions to geometrically and physically nonlinear problems of the theory of elasticity.

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