A THEOREM ON ANALYTIC STRONG MULTIPLICITY ONE

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Abstract

Let $K$ be an algebraic number field, and $\pi = \otimes \pi_v$ an irreducible, automorphic, cuspidal representation of $\text{GL}_m(\mathbb{A}_K)$ with analytic conductor $C(\pi)$. The theorem on analytic strong multiplicity one established in this note states, essentially, that there exists a positive constant $c$ depending on $\varepsilon > 0$, $m$, and $K$ only, such that $\pi$ can be decided completely by its local components $\pi_v$ with norm $N(v) < c \cdot C(\pi)^{2m+\varepsilon}$.

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1. Introduction

Let $K$ be an algebraic number field, let $\pi$ and $\pi'$ be two cuspidal automorphic representations of $\text{GL}_m(\mathbb{A}_K)$ with restricted tensor product decompositions $\pi = \otimes \pi_v$ and $\pi' = \otimes \pi'_v$. The strong multiplicity one theorem states that if $\pi_v \cong \pi'_v$ for all but finitely many places $v$, then $\pi = \pi'$.

The reader is referred to [1] for history and references in this direction.

The analytic version of the above theorem gives, in terms of the analytic conductor $C(\pi)$ of $\pi$ defined in (2.14), more precise information on the number of places needed to decide a cuspidal automorphic representation $\pi$ of $\text{GL}_m(\mathbb{A}_K)$. Such an analytic result was first established by Moreno [10], by applying zero-free regions of the Rankin-Selberg $L$-function of two automorphic representations. To state the result, let $\mathcal{B}_m(Q)$ denote the set of all cuspidal automorphic representations $\pi$ on $\text{GL}_m(\mathbb{A}_K)$ with analytic conductors $C(\pi)$ less than a large real number $Q$. Suppose that $\pi = \otimes \pi_v$ and $\pi' = \otimes \pi'_v$ are in $\mathcal{B}_m(Q)$ with $m \geq 2$. Then, according to [10], there exist positive constants $c$ and $d$ such that, if $\pi_v \cong \pi'_v$ for all finite places $v$ with norm

$$N(v) \leq \begin{cases} cQ^d, & \text{for } m = 2, \\ c \exp(d \log^2 Q), & \text{for } m \geq 3, \end{cases}$$

(1.1)

then $\pi = \pi'$.

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Using a different method, Brumley [1] strengthened Moreno’s result in (1.1) to

\[ N(v) \leq cQ^d, \quad \text{for } m \geq 1, \]  

(1.2)

where \( c \) and \( d \) are positive constants depending on \( m \). Brumley’s method [1] actually proves that

\[ d = \frac{17}{2}m - 4 + \varepsilon \]  

(1.3)

is acceptable in (1.2). By applying a still different method, the second named author [16] showed essentially that, for all \( K \), the \( d \) in (1.2) can be reduced to

\[ d = 4m + \varepsilon. \]  

(1.4)

The purpose of this note is to show that a suitable modification of the argument in [16] actually gives the acceptable value

\[ d = 2m + \varepsilon \]  

(1.5)

in (1.2). To state the result, let \( A_M(Q) \) denote the set of all cuspidal automorphic representations \( \pi \) on \( GL_m(\mathbb{A}_K) \), with \( 1 \leq m \leq M \), whose analytic conductors \( C(\pi) \) are less than a large real number \( Q \). Thus,

\[ A_M(Q) = \bigcup_{m \leq M} B_m(Q), \]

and the main result of this note can be stated as follows.

**Theorem.** Let \( \pi = \otimes \pi_v \) and \( \pi' = \otimes \pi'_v \) be in \( A_M(Q) \). Then there exists a constant \( c = c(\varepsilon, M, K) \) depending on \( \varepsilon > 0, M, \) and \( K \) only, such that if \( \pi_v \cong \pi'_v \) for all finite places with norm

\[ N(v) < cQ^{2M+\varepsilon} \]

then \( \pi = \pi' \).

To prove the Theorem, we will exploit, among other things, Landau’s classical idea in [7].

2. PRELIMINARIES ON AUTOMORPHIC \( L \)-FUNCTIONS

In this section, we summarize some basic properties of automorphic \( L \)-functions, which will be used in the proof of the Theorem in §3.

Let \( K \) be an algebraic number field of degree \( [K : \mathbb{Q}] = l \), with adèlle ring \( \mathbb{A}_K = K_\infty \times \mathbb{A}_{K,f} \), where \( K_\infty \) is the product of the Archimedean completions of \( K \), and the ring \( \mathbb{A}_{K,f} \) of finite adèles is a restricted direct product of the completions \( K_v \) over non-Archimedean places \( v \). Suppose
that $\pi$ is an automorphic irreducible cuspidal representation of $\text{GL}_m(\mathbb{A}_K)$. Then $\pi$ is a restricted tensor product
\[
\pi = \otimes_v \pi_v = \pi_{\infty} \otimes \pi_f,
\] (2.1)
where $v$ runs over all places of $K$, and $\pi_v$ is unramified for almost all finite places $v$. At every finite place $v$ where $\pi_v$ is unramified we associate a semisimple conjugacy class
\[
A_{\pi,v} = \begin{pmatrix}
\alpha_{\pi,v}(1) \\
\vdots \\
\alpha_{\pi,v}(m)
\end{pmatrix},
\]
and define the local $L$-function for the finite place $v$ as
\[
L(s, \pi_v) = \det \left( I - A_{\pi,v} \frac{q_v^s}{q_v^m} \right)^{-1} = \prod_{j=1}^{m} \left( 1 - \frac{\alpha_{\pi,v}(j)}{q_v^j} \right)^{-1}
\] (2.2)
where $q_v = N(p_v) = N(v)$ is the norm of $K_v$. It is possible to write the local factors at ramified places $v$ in the form of (2.2) with the convention that some of the $\alpha_{\pi,v}(j)$’s may be zero. The finite part $L$-function $L(s, \pi_f)$ is defined as
\[
L(s, \pi_f) = \prod_{v<\infty} L(s, \pi_v).
\] (2.3)
And this Euler product is proved to be absolutely convergent for $\sigma = \Re s > 1$. Also, the Archimedean $L$-function is defined as
\[
L(s, \pi_{\infty}) = \pi^{-ms/2} \prod_{j=1}^{m} \Gamma \left( \frac{s + b_\pi(j)}{2} \right)
\] (2.4)
The coefficients $\{\alpha_{\pi,v}(j)\}_{1 \leq j \leq m}$ and $\{b_\pi(j)\}_{1 \leq j \leq m}$ are called local parameters of $\pi$, respectively, at finite places and at infinite places. For them, a trivial bound states that
\[
|\alpha_{\pi,v}(j)| \leq \sqrt{q}, \quad |\Re b_\pi(j)| \leq \frac{1}{2}.
\]
In connection with (2.1), the complete $L$-function associated to $\pi$ is defined by
\[
L(s, \pi) = L(s, \pi_{\infty})L(s, \pi_f).\] (2.5)
This complete $L$-function has an analytic continuation and entire, and satisfies the functional equation
\[
L(s, \pi) = W_{\pi} q_{\pi}^{-\frac{1}{3}} L(1 - s, \tilde{\pi})
\]
where $\tilde{\pi}$ is the contragredient of $\pi$, $W_\pi$ a complex number of modulus 1, and $q_\pi$ a positive integer called the arithmetic conductor of $\pi$ [3].

Let $\pi = \otimes_{\nu} \pi_{\nu}$ and $\pi' = \otimes_{\nu'} \pi'_{\nu'}$ be automorphic irreducible cuspidal representations of $GL_m(\mathbb{A}_K)$ and $GL_{m'}(\mathbb{A}_K)$, respectively. The finite part Rankin-Selberg $L$-function associated to $\pi$ and $\pi'$ is defined by

$$L(s, \pi \times \pi') = \prod_{\nu<\infty} L(s, \pi_{\nu} \times \pi'_{\nu}),$$

(2.6)

where

$$L(s, \pi_{\nu} \times \pi'_{\nu}) = \prod_{j=1}^{m} \prod_{j'=1}^{m'} \left( 1 - \frac{\alpha_{\pi_{\nu},v}(j)\alpha_{\pi'_{\nu},v}(j')}{q_v^s} \right)^{-1}$$

(2.7)

are finite local $L$-functions for unramified finite places $\nu$, i.e. where $\pi_{\nu}$ and $\pi'_{\nu}$ are both unramified. It can be defined similarly at places $\nu$ where $\pi_{\nu}$ or $\pi'_{\nu}$ are ramified. This Euler product is proved to be absolutely convergent for $\sigma > 1$, where $L(s, \pi \times \pi')$ has a Dirichlet series expression of the form

$$L(s, \pi \times \pi') = \sum_{n=1}^{\infty} \frac{a_{\pi \times \pi'}(n)}{n^s}. \quad (2.8)$$

The complete Rankin-Selberg $L$-function is defined by

$$L(s, \pi \times \pi') = L(s, \pi_\infty \times \pi'_\infty) L(s, \pi_f \times \pi'_f)$$

(2.9)

with

$$L(s, \pi_\infty \times \pi'_\infty) = \pi^{-mm'l's/2} \prod_{j=1}^{mm'l} \Gamma \left( \frac{s + b_{\pi \times \pi'}(j)}{2} \right). \quad (2.10)$$

When the infinite place $\nu$ is unramified for both $\pi$ and $\pi'$, we have

$$\{b_{\pi \times \pi'}(j)\}_{1 \leq j \leq mm'} = \{b_{\pi}(j) + b_{\pi'}(j')\}_{1 \leq j \leq m, 1 \leq j' \leq m'}. \quad \{b_{\pi \times \pi'}(j)\}_{1 \leq j \leq mm'}$$

By Shahidi [12, 13, 14, 15], the complete $L$-function $L(s, \pi \times \pi')$ has an analytic continuation to the entire complex plane, and satisfies the functional equation

$$L(s, \pi \times \pi') = W_{\pi \times \pi'} q_{\pi \times \pi'}^\frac{1}{2} L(1 - s, \tilde{\pi} \times \pi')$$

(2.11)

where $W_{\pi \times \pi'}$ is a complex constant of modulus 1, and $q_{\pi \times \pi'}$ is a positive integer. By Jacquet-Shalika [3] and Moeglin-Waldspurger [9], we know that $L(s, \pi \times \pi')$ is holomorphic when $\pi' \neq \pi \otimes |\det|^\tau$ for any $\tau \in \mathbb{R}$, and the only poles of $L(s, \pi \times \pi')$ are simple poles at $s = i\tau_0$ and
$1 + i\tau_0$, when $m = m'$ and $\pi' = \pi \otimes |\det |^{i\tau_0}$ for some $\tau_0 \in \mathbb{R}$. Finally, by Gelbart-Shahidi [4], $L(s, \pi \times \tilde{\pi}')$ is meromorphic of order one away from its poles, and bounded in the vertical strips.

Following Iwaniec-Sarnak [5], we define the analytic conductors of $L(s, \pi)$ and $L(s, \pi \times \tilde{\pi}')$, respectively, as

$$C(\pi; t) = q_{\pi} \prod_{j=1}^{m'l}(1 + |it + b_\pi(j)|), \quad (2.12)$$

and

$$C(\pi, \tilde{\pi}'; t) = q_{\pi \times \tilde{\pi}'} \prod_{j=1}^{mm'l}(1 + |it + b_{\pi \times \tilde{\pi}'}(j)|). \quad (2.13)$$

Setting $t = 0$ in the above definitions, we write

$$C(\pi) = C(\pi; 0), \quad C(\pi, \tilde{\pi}') = C(\pi, \tilde{\pi}'; 0), \quad (2.14)$$

which are called, respectively, the analytic conductors of $\pi$ and of $\pi \times \tilde{\pi}'$. They satisfy the inequality

$$C(\pi, \tilde{\pi}') \leq C(\pi)^{m'}C(\pi')^m, \quad (2.15)$$

which follows from Bushnell-Henniart [2] or Ramakrishnan-Wang [11].

### 3. Proof of the Theorem

Let $\pi = \otimes \pi_v$ and $\pi' = \otimes \pi'_v$ be in $A_M(Q)$. If they are twisted equivalent, i.e. $\pi = \pi' \otimes |\det |^{i\tau}$ for some $\tau \in \mathbb{R}^\times$, then $\pi_v \not\sim \pi'_v$ for all finite places $v$ with at most one exception. We may therefore suppose that, for all $\tau \in \mathbb{R}$,

$$\pi \not\sim \pi' \otimes |\det |^{i\tau}. \quad (3.1)$$

To compare $\pi$ with $\pi'$, we form the Rankin-Selberg $L$-function $L(s, \pi \times \tilde{\pi}')$, and exploit its Dirichlet series expansion (2.8), which holds for $\sigma > 1$. By (3.1), the functions $L(s, \pi \times \tilde{\pi}')$ and $L(s, \pi_f \times \tilde{\pi}'_f)$ are entire functions in the whole complex plane. Define

$$S(x; \pi, \tilde{\pi}') = \sum_{n=1}^{\infty} a_{\pi \times \tilde{\pi}'}(n)w\left(\frac{n}{x}\right), \quad (3.2)$$
where \( w(x) \) is a non-negative real valued function of \( C_c^\infty \) with compact support in \([0, 3]\), and we may specify

\[
  w(x) = \begin{cases} 
  0, & \text{for } x \not\in [0, 3], \\
  e^{-\frac{x}{2}}, & \text{for } x \in (0, 1], \\
  e^{-\frac{3-x}{2}}, & \text{for } x \in [2, 3) 
  \end{cases}
\]
as in [16, §3]. Thus, for any positive integer \( k \), the derivative \( w^{(k)}(x) \) has exponential decay as \( x \to 0 \) or 3. Consequently, the Mellin transform

\[
  W(s) = \int_0^\infty w(x)x^{s-1}dx
\]
is an analytic function of \( s \); if \( \sigma < -1 \) then

\[
  W(s) \ll A,\sigma \frac{1}{(1 + |t|)^A}
\]
for arbitrary \( A > 0 \), by repeated partial integration. By Mellin inversion,

\[
  w(x) = \frac{1}{2\pi i} \int_{(2)} W(s)x^{-s}ds,
\]

where \((c)\) means the vertical line \( \sigma = c \). Inserting this back to (3.2), and then using Dirichlet series expansion (2.8), we have

\[
  S(x; \pi, \tilde{\pi}') = \frac{1}{2\pi i} \sum_{n=1}^\infty a_{\pi \times \tilde{\pi}'}(n) \int_{(2)} W(s) \left( \frac{n}{x} \right)^{-s} ds
  = \frac{1}{2\pi i} \int_{(2)} x^sW(s)L(s, \pi \times \tilde{\pi}') ds,
\]

where the interchange of summation and integral is guaranteed by the absolute convergence of (2.8) on the line \( \sigma = 2 \). A pre-convexity bound like

\[
  L(s, \pi \times \tilde{\pi}') \ll C(\pi, \tilde{\pi}'; t)^B,
\]
where \( B > 0 \) is some constant, can be obtained by standard method, as pointed out in [11, §1]. Since \( W(s) \) is rapid decay and \( L(s, \pi \times \tilde{\pi}') \) is entire, we may shift the contour above to the vertical line \( \sigma = -H \), getting

\[
  S(x; \pi, \tilde{\pi}') = \frac{1}{2\pi i} \int_{(-H)} x^sW(s)L(s, \pi \times \tilde{\pi}') ds, \tag{3.3}
\]
where \( H > 1 \) is a large real number to be specified later.
We are going to apply the functional equation (2.11) to (3.3). To this end, we rewrite (2.11) as

\[ L(s, \pi \times \tilde{\pi}' f) = W(\pi \times \tilde{\pi}) q^{\frac{1}{2} - \frac{s}{2}} G(s) L(1 - s, \tilde{\pi} \times \pi' f), \]  

(3.4)

where

\[ G(s) = \frac{L(1 - s, \tilde{\pi}_\infty \times \pi'_\infty)}{L(s, \pi_\infty \times \tilde{\pi}'_\infty)}. \]  

(3.5)

We need to estimate \( G(s) \) on the line \( \sigma = -H \), avoiding the poles of the nominator of \( G(s) \). This will be done in the Lemma in §4, which asserts that, for every large positive integer \( n \), there is an \( H \in [n, n + 1] \) such that, on the vertical line \( \sigma = -H \),

\[ G(-H + it) \ll_{H,M,K} (1 + |t|)^{mm'/l} \prod_{j=1}^{mm'/l} (1 + |b_{\pi \times \tilde{\pi}'}(j)|)^{\frac{1}{2} + H}. \]  

(3.6)

Now we apply the functional equation (3.4),

\[ S(x; \pi, \tilde{\pi}') = \frac{1}{2\pi i} \int_{(-H)} x^s W(s) W_{\pi \times \tilde{\pi}} q^{\frac{1}{2} - \frac{s}{2}} G(s) L(1 - s, \tilde{\pi} \times \pi'_f) ds \]

\[ = \frac{1}{2\pi i} \int_{(-H)} x^s W(s) W_{\pi \times \tilde{\pi}} q^{\frac{1}{2} - \frac{s}{2}} G(s) \left( \sum_{n=1}^{\infty} \frac{a_{\pi \times \tilde{\pi}'}(n)}{n^{1-s}} \right) ds \]

\[ = \frac{1}{2\pi i} \sum_{n=1}^{\infty} \frac{a_{\pi \times \tilde{\pi}'}(n)}{n^{1+H}} \int_{(-H)} x^s W(s) W_{\pi \times \tilde{\pi}} q^{\frac{1}{2} - \frac{s}{2}} G(s) n^s ds. \]

Here the interchange of summation and integral is guaranteed by the absolute convergence of the Dirichlet series as well as the rapid decay of \( W(s) \). Using these facts again, and inserting (3.6) into the last integral, we get

\[ S(x; \pi, \tilde{\pi}') \ll_{H,M,K} \int_{(-H)} \left| x^s W(s) W_{\pi \times \tilde{\pi}} q^{\frac{1}{2} - \frac{s}{2}} G(s) \right| ds \]

\[ \ll_{H,M,K} x^{-H} q^{\frac{1}{2} + H} \prod_{j=1}^{mm'/l} (1 + |b_{\pi \times \tilde{\pi}'}(j)|)^{\frac{1}{2} + H} \]

\[ = x^{-H} C(\pi, \tilde{\pi}')^{\frac{1}{2} + H}. \]  

(3.7)

This upper bound corresponds to [16 (10)].
To establish the Theorem, we need a lower bound for $S(x; \pi, \tilde{\pi}')$. Under the assumption of (3.1), we further suppose that $\pi_v \cong \pi'_v$ for all finite places with norm $N(v) < x$. Then

$$S(x; \pi, \tilde{\pi}') = S(x; \pi, \tilde{\pi}).$$  
(3.8)

A lower bound for $S(x; \pi, \tilde{\pi})$ is obtained in [1, Lemma 2]; see also [16, Proposition 4]. Thus, we have

$$S(x; \pi, \tilde{\pi}) \gg \frac{x^m}{\log x}. \quad (3.9)$$

Combining (3.7), (3.8), and (3.9), we get

$$\frac{x^m}{\log x} \ll S(x; \pi, \tilde{\pi}') \ll_{H,M,K} x^{-H} C(\pi, \tilde{\pi}')^{1+\delta/2}. \quad (4.1)$$

One therefore has

$$x < c \cdot C(\pi, \tilde{\pi}')^{H+1/2} x^{-H} C(\pi, \tilde{\pi}')^{1+\delta/2},$$

where $c$ is a constant depending on $H, M, K$. Taking $H$ sufficiently large, this becomes

$$x < c \cdot C(\pi, \tilde{\pi}')^{1+\epsilon},$$

and now the constant $c$ depends on $\epsilon, M, K$. The assertion of the theorem finally follows from this and (2.15).

4. An Estimate for $G(s)$

In this section, we give an estimate for $G(s)$ defined as in (3.5) on a vertical line $s = -H + it$, where $H$ is a large real number to be decided suitably. Recall Stirling’s formula that

$$|\Gamma(\sigma + it)| = \sqrt{2\pi e^{-\frac{1}{2}t^2}} |t|^{\sigma - \frac{1}{2}} \left(1 + O_{\sigma, \delta} \left(\frac{1}{1+|t|}\right)\right),$$

which holds for $s = \sigma + it$ away from all poles of $\Gamma(s)$ by at least $\delta > 0$; note that the implied constant depends on $\sigma$ and $\delta$.

To do this, we should first locate the poles of the nominator of $G(s)$, i.e. poles of

$$L(1-s, \tilde{\pi}_\infty \times \pi'_\infty) = \pi^{-mm'l/s/2} \prod_{j=1}^{mm'l} \Gamma \left(\frac{1-s + b_{\tilde{\pi} \times \pi'}(j)}{2}\right). \quad (4.1)$$

according to (2.13). These poles are easily to be seen as

$$P_{n,j} = 2n + 1 + b_{\tilde{\pi} \times \pi'}(j), \quad n = 0, 1, 2, \cdots, \quad j = 1, \cdots, mm'l.$$
As in [8], we let \( \mathbb{C}(m, m') \) be the complex plane with the discs

\[
|s - P_{n,j}| < \frac{1}{8mm'l}, \quad n = 0, 1, 2, \cdots, \quad j = 1, \cdots, mm'l
\]

excluded. Thus, for any \( s \in \mathbb{C}(m, m') \), the quantity

\[
\frac{1 - s + b_{\pi x \pi'}(j)}{2}
\]

is away from all poles of \( \Gamma(s) \) by at least \( 1/(16mm'l) \), and therefore Stirling’s formula applies to (4.1). Of course, Stirling’s formula also applies to the denominator of \( G(s) \). Writing \( b_{\pi x \pi'}(j) = u(j) + iv(j) \) and \( s = \sigma + it \in \mathbb{C}(m, m') \), we have

\[
G(s) = \pi^{-\frac{mm'l}{2} + mm'l}s \prod_{j=1}^{mm'l} \frac{\Gamma\left(\frac{1-s+b_{\pi x \pi'}(j)}{2}\right)}{\Gamma\left(\frac{s+b_{\pi x \pi'}(j)}{2}\right)} \ll_{\sigma,M,K} \prod_{j=1}^{mm'l} \frac{|t + v(j)|^{-\frac{1-\sigma + u(j)}{2} + \frac{1}{2}}}{|t + v(j)|^{-\frac{1}{2} - \sigma + \frac{1}{2}}}
\]

where we have used \( \{b_{\pi x \pi'}(j)\}_{1 \leq j \leq mm'l} = \{b_{\pi x \pi'}(j)\}_{1 \leq j \leq mm'l}. \) It follows that, for \( \sigma < 1/2 \),

\[
G(s) \ll_{\sigma,M,K} (1 + |t|)^{mm'l(\frac{1}{2} - \sigma)} \prod_{j=1}^{mm'l} (1 + |v(j)|)^{\frac{1}{2} - \sigma}, \quad (4.2)
\]

which can be written as

\[
G(s) \ll_{\sigma,M,K} (1 + |t|)^{mm'l(\frac{1}{2} - \sigma)} \prod_{j=1}^{mm'l} (1 + |b_{\pi x \pi'}(j)|)^{\frac{1}{2} - \sigma}. \quad (4.3)
\]

Now we give a remark about the structure of \( \mathbb{C}(m, m') \). For \( j = 1, \cdots, mm'l \), denote by \( \beta(j) \) the fractional part of \( v(j) \). In addition we let \( \beta(0) = 0 \) and \( \beta(mm'l + 1) = 1 \). Then all \( \beta(j) \in [0, 1] \), and hence there exist \( \beta(j_1), \beta(j_2) \) such that \( \beta(j_2) - \beta(j_1) \geq 1/(3mm'l) \) and there is no \( \beta(j) \) lying between \( \beta(j_1) \) and \( \beta(j_2) \). It follows that the strip \( S_0 = \{s : \beta(j_1) + 1/(8mm'l) \leq \Re s \leq \beta(j_2) - 1/(8mm'l)\} \) is contained in \( \mathbb{C}(m, m') \). Consequently, for all \( n = 0, 1, 2, \cdots \), the strips

\[
S_n = \left\{s : -n + \beta(j_1) + \frac{1}{8mm'l} \leq \Re s \leq -n + \beta(j_2) - \frac{1}{8mm'l}\right\}
\]
are subsets of $C(m,m')$. Therefore, for each $n \geq 1$, one can choose a vertical line $\sigma = -H$ lying in the strip $S_n$, and therefore (4.3) holds on the line $\sigma = -H$. This proves the following result.

**Lemma.** Let $G(s)$ be as in (2.5). Then for each $n \geq 1$, there is an $H \in [n, n+1]$, such that on the line $\sigma = -H$ we have

$$G(-H + it) \ll_{H,M,K} (1 + |t|)^{mn'l(\frac{1}{2}+H)} \prod_{j=1}^{mn'l} \left(1 + |b_{\pi \times \tilde{\pi}'}(j)|\right)^{\frac{1}{2}+H}.$$  

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