Graded Lie Algebra Generating of Parastatistical Algebraic Structure *

Weimin Yang and Sicong Jing
Department of Modern Physics, University of Science and Technology of China, Hefei 230026, P.R.China

January 9, 2022

Abstract

A new kind of graded Lie algebra (we call it $Z_{2,2}$ graded Lie algebra) is introduced as a framework for formulating parasupersymmetric theories. By choosing suitable bose subspace of the $Z_{2,2}$ graded Lie algebra and using relevant generalized Jacobi identities, we generate the whole algebraic structure of parastatistics.

1 Introduction

Generalized statistics and supersymmetry are two main interests of theoretical physics in recent years. Generalized statistics first was introduced in the form of parastatistics as an exotic possibility extending the Bose and Fermi statistics [1]. A basic idea of supersymmetry is the mixture of particles of different statistics, normally taken to be bosons and fermions [2]. Though supersymmetry and parastatistics may be unified in the form of parasupersymmetry [3], nevertheless, by the algebraic construction, the two concepts seem to be independent.

It is well known that mathematical basis of supersymmetry is $Z_2$ graded Lie algebra. In this kind of Lie algebra, although not only commutators but also anticommutators are involved, the basic algebraic structure is still bilinear. The characteristic of algebraic relations of parastatistics, however, is trilinear commutation relations, or double commutators and anticommutators. This will cause some intrinsic difficulties to formulating the parasupersymmetric theories within a framework of the usual $Z_2$ graded Lie algebra.

In order to provide a suitable framework to describe parasupersymmetric theories, we introduce a new kind of graded Lie algebra (maybe call it $Z_{2,2}$ graded Lie algebra) in this letter, which can be considered as a generalization or extension of the ordinary $Z_2$ graded Lie algebra. In contrast with the latter in which there are only two subspaces

*This project supported by National Natural Science Foundation of China and IWTZ 1298.
(bose and fermi or even and odd subspaces), the former has four subspaces which can be called bose, fermi, parabose and parafermi subspaces respectively (see the last section for the reason). We would like to point out that by choosing appropriate bose subspace of the $Z_{2,2}$ graded Lie algebra and using relevant Jacobi identities, one can derive all the algebraic relations of a system consisting of parabosons and parafermions. Therefore, one may analyse supersymmetric properties of such a parasystem more effectively and more conveniently on the basis of $Z_{2,2}$ graded Lie algebra.

This letter is arranged as follows. In section 2 we introduce a formal definition of the $Z_{2,2}$ graded Lie algebra and discuss main differences between $Z_2$ and $Z_{2,2}$ graded Lie algebras. As an example, in section 3 we choose an algebra $u(1,1)$ as the bose subspace of $Z_{2,2}$ and construct a whole $Z_{2,2}$ grading of $u(1,1)$ by virtue of the generalized Jacobi identities. Then in section 4 we demonstrate the derived $Z_{2,2}$ graded Lie algebra include all the algebraic relations of parastatistics for a system with one parabose and one parafermi degree of freedom. Some remarks are also in the last section.

2 $Z_{2,2}$ graded Lie algebra

Let $L$ be a vector space over a field $K$, which is a direct sum of four subspaces $L_{ij}$ (i,j=0,1), i.e.,

$$L = L_{00} \oplus L_{01} \oplus L_{10} \oplus L_{11}. \quad (1)$$

For any two generators (vectors) in $L$, we define a composition (or product) rule, written $\circ$, with the following properties

(i) Closure: For $\forall u, v \in L$, we have $u \circ v \in L$, i.e.,

$$L \times L \to L. \quad (2)$$

(ii) Bilinearity: For $\forall u, v, w \in L$, $c_1, c_2 \in K$, we have

$$(c_1 u + c_2 v) \circ w = c_1 u \circ w + c_2 v \circ w,$$

$$w \circ (c_1 u + c_2 v) = c_1 w \circ u + c_2 w \circ v. \quad (3)$$

(iii) Grading: For $\forall u \in L_{ij}, v \in L_{mn}, (i, j, m, n = 0, 1)$, we have

$$u \circ v = w \in L_{(i+m)mod2,(j+n)mod2}. \quad (4)$$

For instance, $L_{00} \times L_{00} \to L_{00}, L_{01} \times L_{10} \to L_{11}, L_{10} \times L_{11} \to L_{01}, \cdots$.

(iv) Supersymmetrization: For $\forall u \in L_{ij}, v \in L_{mn}$, we have

$$u \circ v = -(-1)^{g(u) \cdot g(v)} v \circ u, \quad (5)$$

here we assign to any $u \in L_{ij}$ a degree $g(u) = (i, j)$ which satisfies

$$g(u) \cdot g(v) = (i, j) \cdot (m, n) = im + jn, \quad (6)$$

2
and
\[ g(u) + g(v) = (i, j) + (m, n) = (i + m, j + n), \]
where \( v \in L_{mn} \). Obviously, the \( g(u) \) looks like a two-dimensional vector and the expressions in Eqs.(6)-(7) are exactly dot product and additive operations of the two-dimensional vectors.

(v) Generalized Jacobi identities: For \( \forall u \in L_{ij}, v \in L_{kl}, w \in L_{mn}, (i, j, k, l, m, n = 0, 1) \), we have
\[ u \circ (v \circ w) - (w \circ u) \circ v - (u \circ v) \circ w = 0. \]

It is easily to know that there are totally 20 different possibilities for constructing the generalized Jacobi identities from 4 subspaces of \( L \) \( (C_4^1 + C_4^3 + C_4^5 = 20) \).

This completes the definition of \( Z_{2,2} \) graded Lie algebra. We now discuss it in more detail. Firstly, we define the product on \( L \) \( (\circ: L \times L \to L) \) as
\[ \circ : u \circ v = uv - (-1)^{g(u)\cdot g(v)} vu, \]
for \( \forall u, v \in L \). One can convince oneself that the composition law (8) is a product which satisfies all conditions of the product of a \( Z_{2,2} \) graded Lie algebra as defined by Eqs.(2-7). Then we consider this product separately on the subspaces \( L_{00}, L_{01}, L_{10}, L_{11}, \) and between them. Let generators \( X_{ij} \) and \( X'_{ij} \) belong to the subspace \( L_{ij} \). According to Eqs.(4) and (9), we can write out the following ten products for ten different generator combinations
\[
\begin{align*}
\circ & : L_{00} \times L_{00} \to L_{00}, \quad X_{00} \circ X'_{00} = [X_{00}, X'_{00}]; \\
\circ & : L_{01} \times L_{01} \to L_{00}, \quad X_{01} \circ X'_{01} = \{X_{01}, X'_{01}\}; \\
\circ & : L_{10} \times L_{10} \to L_{00}, \quad X_{10} \circ X'_{10} = \{X_{10}, X'_{10}\}; \\
\circ & : L_{11} \times L_{11} \to L_{00}, \quad X_{11} \circ X'_{11} = [X_{11}, X'_{11}]; \\
\circ & : L_{00} \times L_{01} \to L_{01}, \quad X_{00} \circ X_{01} = [X_{00}, X_{01}]; \\
\circ & : L_{00} \times L_{10} \to L_{10}, \quad X_{00} \circ X_{10} = [X_{00}, X_{10}]; \\
\circ & : L_{00} \times L_{11} \to L_{11}, \quad X_{00} \circ X_{11} = [X_{00}, X_{11}]; \\
\circ & : L_{01} \times L_{10} \to L_{11}, \quad X_{01} \circ X_{10} = [X_{01}, X_{10}]; \\
\circ & : L_{01} \times L_{11} \to L_{10}, \quad X_{01} \circ X_{11} = \{X_{01}, X_{11}\}; \\
\circ & : L_{10} \times L_{11} \to L_{01}, \quad X_{10} \circ X_{11} = \{X_{10}, X_{11}\}.
\end{align*}
\]
Thus we see that the \( Z_{2,2} \) graded Lie algebra indeed generalizes and extends the notion of \( Z_2 \) graded Lie algebra in two aspects. One is about the grading. In \( Z_2 \) case there are only two subspaces, and in \( Z_{2,2} \) case four ones, which require the grading of \( Z_{2,2} \) must be some kind of double grading, or twofold grading. Another is the definition of the degree. In \( Z_2 \) case the degree is a number (taken value 0 or 1), and in \( Z_{2,2} \) case the degree is a two-dimensional vector (taken value from the set of \((0,0), (0,1), (1,0)\) and \((1,1)\)). Here we want to point out that there exists also a significant difference between these two graded Lie algebras concerning the generalized Jacobi identities.
Among these matrices, $H$ are symmetric and $L$ are antisymmetric matrices. From the third line of Eq.(13), we observe that the matrices must satisfy the following constraint relations

$$
\begin{align*}
K\mu H\nu + (K\mu H\nu)^T &= C_{\mu \nu \lambda} H\lambda, \\
v\mu v\nu + (v\mu v\nu)^T &= C_{\mu \nu \lambda} s\lambda, \\
& = (K\mu)_{\alpha \beta} n_{\beta} + n_{\alpha} (v\mu v\nu)^T, \\
& = (K\mu)_{\alpha \beta} n_{\beta} + n_{\alpha} (v\mu v\nu)^T = (H\mu)_{\alpha \beta} n_{\beta} + (m\alpha) n_{\beta} + l\alpha n_{\beta},
\end{align*}
$$

$$
\begin{align*}
l\alpha n_{\beta} + n_{\alpha} (v\mu v\nu)^T &= (K\mu)_{\alpha \beta} n_{\beta}, \\
l\alpha m_{\beta} + (m\alpha) n_{\beta} &= (H\mu)_{\alpha \beta} n_{\beta},
\end{align*}
$$

Notice that only the first three kinds of identities in Eq.(11) appear in the generalized Jacobi identities of $Z_2$ graded Lie algebra, however, all the four kinds of identities can be found in the structure of the generalized Jacobi identities of $Z_{2,2}$ graded Lie algebra.

3 \textbf{Z}_{2,2} \textbf{ grading of u(1,1) algebra}

As an example, we discuss the $Z_{2,2}$ grading of u(1,1) algebra, which has four generators $X_\mu, (\mu = 1, 2, 3, 4)$. To do this, we take the u(1,1) algebra as the subspace $L_{00}$, in which the product is

$$
o : L_{00} \times L_{00} \to L_{00}, \quad [X_\mu, X_\nu] = C_{\mu \nu \lambda} X_\lambda. \quad (12)
$$

$C_{\mu \nu \lambda}$ are the structure constants of the u(1,1) algebra, whose nonzero components are $C_{122} = -C_{212} = 1, C_{133} = -C_{313} = 1, C_{231} = -C_{321} = -2$. We denote generators of $L_{01}, L_{10}$ and $L_{11}$ by $Q_\alpha (\alpha = 1, \ldots, \text{dim} L_{01}), Y_\nu (\nu = 1, \ldots, \text{dim} L_{10})$ and $Z_\beta (\beta = 1, \ldots, \text{dim} L_{11})$, respectively. According to the product rules in Eq.(10), we may introduce the following coefficient matrices $K_\mu, H_\mu, s_\mu, t_\mu, u_\mu, v_\mu, l_\alpha, m\alpha and n\alpha$, to write explicitly out these products

$$
\begin{align*}
&= (H_\mu)_{\alpha \beta} X_\mu, \\
&= (K_\mu)_{\alpha \beta} Y_\nu, \\
&= (K_\mu)_{\alpha \beta} Q_\alpha, \\
&= (K_\mu)_{\alpha \beta} L_\beta, \\
&= (K_\mu)_{\alpha \beta} n_{\beta} + n_{\alpha} (v\mu v\nu)^T, \\
&= (K_\mu)_{\alpha \beta} n_{\beta} + n_{\alpha} (v\mu v\nu)^T = (H_\mu)_{\alpha \beta} n_{\beta} + (m\alpha) n_{\beta} + l\alpha n_{\beta},
\end{align*}
$$

Among these matrices, $H_\mu (\mu = 1, 2, 3, 4)$ are symmetric $\text{dim} L_{01} \times \text{dim} L_{01}$ matrices, $s_\mu$ are symmetric $\text{dim} L_{10} \times \text{dim} L_{10}$ ones, and $t_\mu$ are antisymmetric $\text{dim} L_{11} \times \text{dim} L_{11}$ ones. From the third line of Eq.(13), we observe that the matrices $l_\alpha, m\alpha and n\alpha$ are square matrices only if $\text{dim} L_{10} = \text{dim} L_{11}$, so we consider just this case in what follows. Of course, these matrices are restricted by the generalized Jacobi identities. Considering all the possible generalized Jacobi identities, we find that these matrices must satisfy the following constraint relations

$$
\begin{align*}
&= (K_\mu)_{\alpha \beta} n_{\beta} + n_{\alpha} (v\mu v\nu)^T, \\
&= (K_\mu)_{\alpha \beta} n_{\beta} + n_{\alpha} (v\mu v\nu)^T = (H_\mu)_{\alpha \beta} n_{\beta} + (m\alpha) n_{\beta} + l\alpha n_{\beta},
\end{align*}
$$

$$
\begin{align*}
l\alpha m_{\beta} + (m\alpha) n_{\beta} &= (H_\mu)_{\alpha \beta} n_{\beta},
\end{align*}
$$

$$
\begin{align*}
l\alpha m_{\beta} + (m\alpha) n_{\beta} &= (H_\mu)_{\alpha \beta} n_{\beta},
\end{align*}
$$

$$
\begin{align*}
l\alpha m_{\beta} + (m\alpha) n_{\beta} &= (H_\mu)_{\alpha \beta} n_{\beta},
\end{align*}
$$

4
\[(n_\alpha)_{ji}(l_\alpha)_{kl} + (n_\alpha)_{ki}(l_\alpha)_{jl} = -(s_\mu)_{jk}(v_\mu)_{li},\]
\[(n_\alpha)_{ji}(m_\alpha)_{kl} - (n_\alpha)_{jk}(m_\alpha)_{li} = (t_\mu)_{ki}(u_\mu)_{jl},\]
\[(H_\mu)_{\alpha\beta}(K_\mu)_{\gamma\delta} + (H_\mu)_{\beta\gamma}(K_\mu)_{\alpha\delta} + (H_\mu)_{\gamma\alpha}(K_\mu)_{\beta\delta} = 0,\]
\[(s_\mu)_{ij}(u_\mu)_{kl} + (s_\mu)_{jk}(u_\mu)_{li} + (s_\mu)_{ki}(u_\mu)_{jl} = 0,\]
\[(t_\mu)_{ij}(v_\mu)_{kl} + (t_\mu)_{jk}(v_\mu)_{li} + (t_\mu)_{ki}(v_\mu)_{jl} = 0,\]

where the notation "T" means transpose of a matrix, and the ranges of the indices are \(\mu, \nu, \lambda = 1, 2, 3, 4; \alpha, \beta, \gamma, \delta = 1, \ldots, \dim L_{01}; i, j, k, l = 1, \ldots, \dim L_{10} = \dim L_{11}\), respectively. Carefully observing these relations, we realize that \(K_\mu\) and \(H_\mu\) have to be \(\dim L_{01} \times \dim L_{01}\) matrices, and all the other ones are \(\dim L_{10} \times L_{10}\). For the simplest nontrivial case, i.e., \(\dim L_{01} = 4\) and \(\dim L_{10} = \dim L_{11} = 2\), by using the trial-and-error method, we find out the following solutions of these matrices

\[
K_1 = \frac{1}{2} \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix},
K_2 = \begin{pmatrix}
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{pmatrix},
K_3 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
K_4 = \frac{1}{2} \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix};
\]

\[
H_1 = 2 \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix},
H_2 = 2 \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
H_3 = 2 \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix},
H_4 = 2 \begin{pmatrix}
0 & -1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1
\end{pmatrix};
\]

\[
s_1 = 4 \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix},
\quad s_2 = 4 \begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix},
\quad s_3 = 4 \begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix},
\quad s_4 = 4 \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix};
\]

\[
t_1 = \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix},
\quad t_2 = \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix},
\quad t_3 = \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix},
\quad t_4 = 4 \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix};
\]

\[
u_1 = \frac{1}{2} \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix},
\quad u_2 = \begin{pmatrix}
0 & 0 \\
0 & -1
\end{pmatrix},
\quad u_3 = \begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix},
\quad u_4 = \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix};
\]

\[
v_1 = \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix},
\quad v_2 = \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix},
\quad v_3 = \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix},
\quad v_4 = \frac{1}{2} \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix};
\]

\[5\]
\[
\begin{align*}
l_1 &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \\
l_2 &= \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \\
l_3 &= \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \\
l_4 &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}; \quad (21) \\
m_1 &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \\
m_2 &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \\
m_3 &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \\
m_4 &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}; \quad (22) \\
n_1 &= 2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \\
n_2 &= 2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \\
n_3 &= 2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \\
n_4 &= 2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \quad (23)
\end{align*}
\]

By checking straightforwardly one will believe that these matrices satisfy all of the constraint relations (14). Thus we obtain the \(Z_{2,2}\) grading of \(u(1,1)\) algebra. It has 12 generators: \(X_\mu\) and \(Q_\alpha\) for \(\mu, \alpha = 1, 2, 3, 4\), and \(Y_i\) and \(Z_j\) for \(i, j = 1, 2\), which satisfy the following nonzero commutation and anticommutation relations

\[
\begin{align*}
[X_1, X_2] &= X_2, \quad [X_1, X_3] = -X_3, \quad [X_2, X_3] = -2X_1; \quad (24) \\
\{Q_1, Q_2\} &= 2X_1 - 2X_4, \quad \{Q_1, Q_4\} = 2X_3, \\
\{Q_3, Q_4\} &= 2X_1 + 2X_4, \quad \{Q_2, Q_3\} = 2X_2; \quad (25) \\
\{Y_1, Y_2\} &= 4X_2, \quad \{Y_1, Y_2\} = 4X_1, \quad \{Y_2, Y_2\} = 4X_3; \quad (26) \\
\{Z_1, Z_2\} &= 4X_4; \quad (27)
\end{align*}
\]

\[
\begin{align*}
[X_1, Q_1] &= -\frac{1}{2}Q_1, \quad [X_1, Q_2] = \frac{1}{2}Q_2, \quad [X_1, Q_3] = \frac{1}{2}Q_3, \\
[X_1, Q_4] &= -\frac{1}{2}Q_4, \quad [X_2, Q_1] = -Q_3, \quad [X_2, Q_4] = -Q_2, \\
[X_3, Q_2] &= Q_4, \quad [X_3, Q_3] = Q_1, \quad [X_4, Q_1] = -\frac{1}{2}Q_1, \\
[X_4, Q_2] &= \frac{1}{2}Q_2, \quad [X_4, Q_3] = -\frac{1}{2}Q_3, \quad [X_4, Q_4] = \frac{1}{2}Q_4; \quad (28)
\end{align*}
\]

\[
\begin{align*}
[X_1, Y_1] &= \frac{1}{2}Y_1, \quad [X_1, Y_2] = -\frac{1}{2}Y_2, \\
[X_2, Y_2] &= -Y_1, \quad [X_3, Y_1] = Y_2; \quad (29) \\
[X_4, Z_1] &= \frac{1}{2}Z_1, \quad [X_4, Z_2] = -\frac{1}{2}Z_2; \quad (30) \\
\{Q_1, Y_1\} &= Z_2, \quad \{Q_2, Y_2\} = -Z_1, \\
\{Q_3, Y_2\} &= -Z_2, \quad \{Q_4, Y_1\} = Z_1; \quad (31) \\
\{Q_1, Z_1\} &= Y_2, \quad \{Q_2, Z_2\} = Y_1, \\
\{Q_3, Z_1\} &= Y_1, \quad \{Q_4, Z_2\} = Y_2; \quad (32) \\
\{Y_1, Z_1\} &= 2Q_2, \quad \{Y_1, Z_2\} = 2Q_3, \\
\{Y_2, Z_1\} &= 2Q_4, \quad \{Y_2, Z_2\} = 2Q_1. \quad (33)
\end{align*}
\]
Algebraic structure of parastatistics

4 Algebraic structure of parastatistics

If we identify the generators \( Y_1 \) and \( Y_2 \) in the subspace \( L_{10} \) with parabose creation and annihilation operators \( a^\dagger \) and \( a \) respectively, meanwhile, identify the generators \( Z_1 \) and \( Z_2 \) in the subspace \( L_{11} \) with parafermi creation and annihilation operators \( f^\dagger \) and \( f \) respectively, we will obtain a whole algebraic structure of a system with one paraboson and one parafermion from the \( Z_2 \), \( Z_2 \) graded Lie algebra of \( u(1, 1) \) (Eqs.(24)-(33)). In fact, in this case, according to Eqs.(26) and (27), the four generators \( X_\mu \) in the subspace \( L_{00} \) can be written as

\[
X_1 = \frac{1}{4} \{ a^\dagger, a \}, \quad X_2 = \frac{1}{2} a^2, \quad X_3 = \frac{1}{2} a^\dagger, \quad X_4 = \frac{1}{4} [f^\dagger, f],
\]

and the four generators \( Q_\alpha \) in the \( L_{01} \) subspace will have the forms

\[
Q_1 = \frac{1}{2} \{ a, f \}, \quad Q_2 = \frac{1}{2} \{ a^\dagger, f^\dagger \}, \quad Q_3 = \frac{1}{2} \{ a^\dagger, f \}, \quad Q_4 = \frac{1}{2} \{ a, f^\dagger \},
\]

from Eq.(33). Obviously, here \( X_1 \) and \( X_4 \) are hermite operators, \( X_2, Q_1 \) and \( Q_3 \) are hermitian conjugate to \( X_3, Q_2 \) and \( Q_4 \) respectively. Then the Eq.(29) will give standard trilinear algebraic relations for paraboson

\[
[\{ a^\dagger, a \}, a^\dagger] = 2a^\dagger, \quad [\{ a^\dagger, a \}, a] = -2a, \quad [a^\dagger, a] = -2a^\dagger, \quad [a^2, a^\dagger] = 2a,
\]

and the Eq.(30) will give standard trilinear algebraic relations for parafermion

\[
[\{ f^\dagger, f \}, f^\dagger] = 2f^\dagger, \quad [\{ f^\dagger, f \}, f] = 2f.
\]

It is worth pointing out that the Eqs.(31) and (32) will lead to

\[
[\{ a, f \}, a^\dagger] = 2f, \quad [\{ a^\dagger, f \}, a] = -2f, \quad [\{ a, f \}, f^\dagger] = 2a, \quad [\{ a, f^\dagger \}, f] = -2a,
\]

together with the adjoint ones, which are exactly algebraic relations between paraboson and parafermion with same parastatistical order \( p \). The remaining relations of the \( Z_{2,2} \) graded Lie algebra (i.e., Eqs.(24), (25) and (28)) just mean that the generators \( X_\mu \) and \( Q_\alpha \) form a ordinary \( Z_2 \) graded Lie algebra. Usually the generators \( X_\mu \) and \( Q_\alpha \) are called bose and fermi (or even and odd) generators, and the subspaces \( L_{00} \) and \( L_{01} \) the bose and fermi subspaces, respectively. Therefore, it is reasonable to call the subspaces \( L_{10} \) and \( L_{11} \) the parabose and parafermi subspaces respectively.

In summary, we generalize the \( Z_2 \) graded Lie algebra to a more complicated grading scheme, \( Z_{2,2} \) grading, in this letter. By choosing appropriately the bose subspace of the \( Z_{2,2} \) graded Lie algebra, and using the generalized Jacobi identities, one may derive all parastatistical algebraic relations. It is worth mentioning that Biswas and Soni also pointed out that the even and odd generators made up of parabosons and parafermions may give operator realizations of graded Lie algebras. The key difference between Ref.(5) and this letter is we generalize the ordinary \( Z_2 \) grading to a new \( Z_{2,2} \) grading,
so that all of the parastatistical algebraic relations can automatically appear in our structure. The $Z_{2,2}$ graded Lie algebra also provides a more potential framework to investigate various possible supersymmetric problem, such as supersymmetry between boson and parafermion $\tilde{\mathbb{B}}$, boson and paraboson $\tilde{\mathbb{P}}$, and so on. Work in this direction is on progress.

References

[1] Green H S 1953 *Phys. Rev.* **90** 270
[2] Sohnius M F 1985 *Phys. Rep.* **128** 39
[3] Rubakov V A and Spiridonov V P 1988 *Mod. Phys. Lett.* **A 3** 1337
[4] Greenberg O W and Messiah A M L 1965 *Phys. Rev.* **B 138** 1155
[5] Biswas S N and Soni S K 1988 *J. Math. Phys.* **29** 16
[6] Beckers J and Debergh N 1990 *Nucl. Phys.* **B340** 767
[7] Plyushchay M 1999 [hep-th/9903130](https://arxiv.org/abs/9903130) v2