Turbulence in a Bose-Einstein condensate

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Abstract. We numerically model turbulence in a trapped atomic Bose–Einstein condensate by solving the Gross–Pitaevskii nonlinear Schroedinger equation. We find that, after an initial growth, the vortex length decays approximately as \( t^{-1} \) where \( t \) is time, consistent with experiments in turbulent superfluid helium, and that the velocity components obey power-law statistics, again in agreement with observations in turbulent superfluid helium. We find the same statistics, which contrasts to the Gaussian statistics observed in ordinary classical turbulence, in a variety of quantum fluids in two and three dimensions (trapped condensates, homogeneous condensates, vortex points, vortex filaments). We argue that the non-Gaussianity arises from the singular nature of quantised vorticity.

1. Background

Quantum turbulence is a dynamic tangle of discrete, reconnecting vortices in a quantum fluid. A number of experiments with turbulent superfluid \(^4\)He have been recently performed, for example by Walmsley & Golov (2008) at Manchester, Skrbek et al. (2009) at Prague, Salort et al. (2010) at Grenoble, Zhang & Van Sciver (2005) at Florida, Paoletti et al. (2008) at Maryland, and Guo et al. (2010) at Yale; turbulence experiments with related superfluid \(^3\)He have been carried out, for example, by Bradley et al. (2008) at Lancaster, Eltsov et al. (2007) at Helsinki, and Yano et al. (2010) at Osaka. Following the turbulence experiment of Henn et al. (2009), there is now interest in studying quantum turbulence in trapped atomic Bose–Einstein condensates (BECs). The advantage of using ultra-cold gases is that the dimensionality, the strength, and the type of interaction can be engineered by the experimentalist, whereas in liquid helium the parameters are fixed by nature. Ultra-cold Bose gases also promise well-controlled methods for vortex creation and the ability to manipulate single-vortex dynamics. On the theoretical side, the interaction between atoms in ultra-cold gases is weak, which makes the Gross-Pitaevskii equation (GPE) a quantitative model (at least at sufficiently low temperature); in liquid helium, the GPE is only a qualitative model.

The defining property of quantum fluids is that the behaviour is determined by a macroscopic (complex) wavefunction \( \psi = \sqrt{\rho} \exp(i\phi) \), hence the velocity \( \mathbf{v} \) is proportional to the gradient of the phase of \( \psi \):

\[ \mathbf{v} = \frac{\hbar}{m} \nabla \phi, \]
Figure 1. Left: A single straight vortex inside an atomic BEC: The vortex core’s surface is depicted in violet; the outer surface of the condensate is shaded in pale blue. Right: The corresponding phase profile on a slice across the condensate, through the center of the vortex core.

where $\rho = |\psi|^2$ is the density, $\phi$ is the phase, $\hbar$ is Planck’s constant, $\tilde{\hbar} = \hbar/(2\pi)$, and $m$ is the mass of the relevant boson (one atom in the case of $^4\text{He}$, one Cooper pair in the case of $^3\text{He}$). Therefore, whereas in ordinary fluids (such as air or water) the rotational motion is unconstrained and eddies can be of any size, shape and strength, in a quantum fluid the vorticity is singular, and takes the form of discrete vortex lines around which the circulation is

$$\oint_C \mathbf{v} \cdot d\mathbf{r} = \frac{\hbar}{m},$$

where $C$ is a closed path of integration around the vortex core. The quantity $\kappa = \hbar/m$ is called the quantum of circulation. It follows from Eq. 2 that the magnitude of the velocity, $v$, at distance $r$ from the axis of the vortex line, is $v = \hbar/(mr)$. It is important to notice that quantum vortices have a hollow core: the superfluid density drops from its bulk value at $r \to \infty$ to zero as $r \to 0$ over a characteristic distance which is of the order of the healing length $\xi \approx 10^{-8}$ cm for $^4\text{He}$ and $10^{-6}$ cm for $^3\text{He}$. In Figure 1, the core of a single straight vortex in a harmonically trapped BEC is shown by the violet line, and the blue shading depicts the condensate edge. To the right of Figure 1 the phase winding around the vortex core for a slice through the center of the vortex line is shown; note that the phase changes from $-\pi$ to $\pi$.

The aim of this work is to model turbulence in an atomic Bose-Einstein condensate using the GPE, and to explore the relation between quantum turbulence and turbulence in ordinary fluids (classical turbulence) in terms of the statistics of the velocity components.

2. The GPE Model

The GPE is
\[ i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V_0|\psi|^2 \psi - \mu \psi + V_{\text{ext}} \psi, \] 

where \( t \) is time, \( V_0 \) is the strength of the (delta–function repulsive) potential of interaction between the boson, \( \mu \) is the chemical potential (the energy increase upon addition of one boson), and \( V_{\text{ext}} \) is the external trapping potential (usually harmonic) which confines the condensate. The relation between the GPE and the classical Euler equation is well-known, see for example Barenghi (2008). Setting \( V_{\text{ext}} = 0 \) for simplicity, it is easy to reduce the GPE to the following equations which make the fluid dynamics nature of the condensate more apparent:

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \quad \text{and} \quad \rho \left( \frac{\partial v_i}{\partial t} + v_k \frac{\partial v_i}{\partial x_k} \right) = -\frac{\partial p}{\partial x_i} + \frac{\partial \Sigma_{ik}}{\partial x_k},
\]

where \( v_i \) is the \( i \)th Cartesian component (\( i = 1, 2, 3 \)) of \( \mathbf{v} \), and \( p = (V_0/2m^2)\rho^2 \) is the pressure. The first equation is the continuity equation. The second equation differs from the classical Euler equation because of the term \( \frac{\partial \Sigma_{ik}}{\partial x_k} \), where \( \Sigma_{ik} = (\hbar/2m)^2 \rho \partial^2 \ln \rho / \partial x_i \partial x_k \) is called the quantum stress.

It can be shown that the quantum stress is important only at length scales less than \( \xi \), so it plays a crucial role during phenomena such as vortex nucleation (creation of new vortices near boundaries) and vortex reconnections. Vortex reconnections, which were first revealed by Koplick & Levine (1993), are crucial for turbulence, and are the subject of great attention, see for example Kerr (2010), Tebbs et al. (2011), Kursa et al. (2011), Bewley et al. (2008) and Alamri et al. (2008). Away from boundaries and vortices, the quantum stress is negligible, and the GPE is essentially the compressible Euler equation. If the speeds under consideration are orders of magnitude less than the speed of sound, which is the case in large systems such as superfluid helium, the GPE reduces to the incompressible Euler equation.

3. Growth and decay of turbulence

To create turbulence in a trapped BEC we choose the technique of phase imprinting used by Leanhardt et al. (2002); although this is not the only way to generate turbulence (the method used by Henn et al. (2009) consists in shaking the trap) it is easy to model numerically. After imprinting a number of staggered straight vortices, we compute the resulting evolution using the GPE with realistic experimental parameters for a spherical trap.

Our main results have been presented in White et al. (2010). We find that during the evolution the vortices interact with each other, reconnect, become curved and Kelvin waves are excited. At first the total vortex length \( L \) increases rapidly, but then it decays, as some of the kinetic energy of the vortices is turned into sound energy, as shown for example by Nore et al. (2003), Leadbeater et al. (2003) and Barenghi et al. (2005). During the process the entire condensate wobbles visibly. The evolution of the condensate with time is shown by the sequence of images in Figure 2. A snapshot of the turbulent condensate is shown at the left of Figure 4. The three–dimensional density contour reveals the outer surface of the condensate and the vortices which it contains. The decay regime is approximately of the form \( L_T \sim t^{-1} \) (where \( L_T \) is the total vortex length), consistently with experiments Walmsley & Golov (2008) in superfluid helium at very low temperatures (see Figure 3).
4. Velocity statistics

To further characterise the turbulence, we examine the statistics of the velocity components $v_i \ (i = 1, 2, 3)$. We compute the probability density function (PDF) of each component, and compare it to the corresponding Gaussian PDF (gPDF) which fits that data set, which is

$$gPDF(v_i) = \frac{1}{\sigma \sqrt{2\pi}} \exp\left(\frac{-(v_i - \bar{\mu})^2}{2\sigma^2}\right),$$

where $\sigma_i$ is the standard deviation and $\bar{\mu}_i$ the mean. We find that superfluid velocity statistics are non-Gaussian and obey power-law dependence $PDF(v_i) \sim v_i^b$ with $-3.6 < b < -3.3$ in all three
Figure 4. Left: Snapshot of the turbulent Bose-Einstein condensate. Note the wobbly outer surface of the condensate (pale blue), and the surfaces of the vortex cores which are inside the condensate (violet). Right: PDFs of the velocity components $v_i$ $i = 1, 2, 3$ where $c_i = \sigma_i$. The solid black line is the Gaussian fit to one component to illustrate the departure from Gaussianity.

directions, consistently with the high velocity tails ($b = 3$) found experimentally by Paoletti et al. (2008) in turbulent superfluid helium. This result is in stark contrast with Gaussian statistics observed in ordinary turbulence for example by Vincent & Meneguzzi (1991), Noullez et al. (1997) and Gotoh et al. (2002).

We also find that the power-law nature of the velocity statistics does not depend on the initial condition and holds during the decay. It is also valid in turbulent two-dimensional condensates, and in homogeneous three-dimensional condensates computed by solving Eq. 3 in a cube with periodic boundary conditions and $V_{\text{ext}} = 0$.

5. Interpretation

Paoletti et al. (2008) suggested that high velocities generated by vortex reconnection events are responsible for the non-Gaussianity. We think that this is only a partial explanation. Firstly, the power-law behaviour does not change during the decay, but the vortex length, hence the frequency of reconnections, decrease. Secondly, we observe non-Gaussian PDFs in two-dimensional condensates before the occurrence of any reconnection events.

The following explanation of the non-Gaussianity is more likely. While studying the properties of vortex filament methods, Min et al. (1996) discovered that velocity statistics of singular and non-singular vortices are qualitatively different. For singular vortices, $\text{PDF}(v_i) \sim v_i^{-3}$ for an isolated vortex. For $N$ vortices, provided that the contribution of each vortex to a velocity component can be considered an independent random variable, the PDF converges to Gaussian but very slowly. This result suggests that the non-Gaussian nature of velocity PDFs is a consequence of the singular nature of quantised vortices.

To test this explanation we perform numerical experiments with two-dimensional vortex points in a periodic box. We find that if the magnitude of the velocity field around a vortex point has the standard form $v \sim 1/r$ (singular vortex), the resulting PDFs are non-Gaussian. However, if the velocity has the (non-singular) Rankine form $v \sim 1/r$ for $r > a$ and $v \sim r$ for
Figure 5. Left: Snapshot of turbulent vortex tangle in $^4$He computed using the vortex filament method. Right: PDF of superfluid velocity components $v_{sx}$ (blue circles), $v_{sy}$ (red triangles) and $v_{sz}$ (green asterisks) The overlapping black dotted, dash dotted and solid lines are respectively the Gaussian fits to the same data.

$r < a$ where $a$ is a cutoff parameter, the PDFs become Gaussian, provided that $a$ is a small but nonzero fraction of the average intervortex distance.

As a final test we consider a three-dimensional tangle of vortex filaments in a periodic box, a well-known model of superfluid turbulence pioneered by Schwarz (1988). In this model the typical distance between vortices is assumed to be much larger than the vortex core radius. Vortex lines are described as three-dimensional space curves $r(s,t)$ of infinitesimal thickness where $s$ is arclength. The governing equation of motion is

$$\frac{dr}{dt} = v_s + \alpha r' \times (v_n - v_s) - \alpha' r' \times [r' \times (v_n - v_s)],$$

(6)

where $v_n$ is the normal fluid component of liquid helium, $\alpha$, $\alpha'$ are temperature dependent friction coefficients which arise from the interaction of the vortex lines with the quasi particles which make up the normal fluid (at temperatures below 1 K in $^4$He the normal fluid is negligible and $\alpha = \alpha' = 0$), and $r' = dr/ds$ is the unit tangent vector at the point $r$. The superfluid velocity field $v_s$ induced at the point $r$ by the entire vortex configuration is given by the classical Biot-Savart law:

$$v_s = -\frac{\kappa}{4\pi} \oint_L \frac{(r - z)}{|r - z|^3} \times dz.$$  

(7)

The necessary techniques to discretize the vortex filaments, de-singularize Eq. 7 and perform vortex reconnections are described by Baggaley & Barenghi (2011a).

To model turbulent tangles created by agitating liquid helium with a grid or propellers, we assume for $v_n(r,t)$ a Kinematic Simulation of classical turbulence, see Osborne et al. (2006). Starting from some seeding vortex loops, we integrate Eq. 6 in time until we obtain a turbulent vortex tangle as shown in Fig. 5 (left), in which the vortex length fluctuates about a mean value; for details see Baggaley & Barenghi (2011b). By taking the Fourier transform of the superfluid velocity field $v_s$, we find that the superfluid energy spectrum $E(k)$ obeys the Kolmogorov scaling $E(k) \sim k^{-5/3}$, where $E(k)$ is defined by
\[ E_s = \frac{1}{V} \int \frac{1}{2} v_s^2 dV = \int_0^\infty E(k) dk, \] (8)

where \( E_s \) is the total superfluid energy (per unit density), \( V \) is volume, \( k = |k| \), and \( k \) is the three-dimensional wavenumber. The PDFs of the superfluid velocity components are shown in Fig. 5(right). We find that PDF\( (v_s,i) \propto v_{bi}^b \) (\( i = 1, 2, 3 \)) with average exponent \( b = -3.1 \). We note that a similar result has been reported recently by Adachi & Tsubota (2011), who observed non-Gaussian PDFs for superfluid turbulence created by a heat current.

6. Conclusion

This numerical investigation has shown that quantum turbulence can be created in an atomic Bose-Einstein condensate by phase imprinting. We have found that, although the turbulent condensate do not contain many quantised vortices due to its limited size, the decay of the vortex length is similar to what is observed in \(^4\)He. We have also found that the PDF of the superfluid turbulent velocity components have non-Gaussian tails, as observed in \(^4\)He, in contrast to what is observed in ordinary turbulence. Finally, we have argued that the non-Gaussian nature of these PDFs arises from the singular nature of the quantised vorticity.

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