On compactly supported dual windows of Gabor frames

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Abstract

The main purpose of the paper is to give a characterization of all compactly supported
dual windows of a Gabor frame. As an application, we consider an iterative procedure for
approximation of the canonical dual window via compactly supported dual windows on every
step. In particular, the procedure allows to have approximation of the canonical dual window
via dual windows from certain modulation spaces or from the Schwartz space.

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proximation of the canonical dual

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1 Introduction and main results

Throughout the paper, \( \mathcal{H} \) denotes a separable Hilbert space. A sequence \( G = \{g_k\}_{k=1}^\infty \) with
elements from \( \mathcal{H} \) is a frame for \( \mathcal{H} \) [15], if there exist positive constants \( A_G \) and \( B_G \) such that

\[
A_G \|h\|^2 \leq \sum_{k=1}^\infty |\langle h, g_k \rangle|^2 \leq B_G \|h\|^2, \quad \forall h \in \mathcal{H}.
\]

Frames extend orthonormal bases, allowing redundancy, and still guarantee perfect and stable
reconstruction. Given a frame \( G \) for \( \mathcal{H} \), there always exists a frame \( F = \{f_k\}_{k=1}^\infty \) for \( \mathcal{H} \) so that

\[
h = \sum_{k=1}^\infty \langle h, g_k \rangle f_k = \sum_{k=1}^\infty \langle h, f_k \rangle g_k, \quad \forall h \in \mathcal{H};
\]

such a frame \( F \) is called a dual frame of \( G \). The so called frame operator of \( G \) is given by
\( S_G h = \sum_{k=1}^\infty \langle h, g_k \rangle g_k \); it is a bounded bijective operator on \( \mathcal{H} \) and the sequence \( \{S_G^{-1}g_k\}_{k=1}^\infty \) is
a dual frame of the frame \( G \), called the canonical dual of \( G \). When \( G \) is a frame which is not
a Schauder basis (so called overcomplete frame), it has other dual frames (infinitely many) in
addition to the canonical one.
For certain engineering fields, e.g. in signal processing, frames of Gabor structure play an essential role. Given \( g \in L^2(\mathbb{R}) \) and positive constants \( a \) and \( b \), a Gabor system is a system of the form \( \{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}} \); \( g \) is called the window of the system \( \{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}} \). Here \( E_v \) and \( T_t \) denote the standard modulation and translation operator, respectively. A Gabor frame is a Gabor system which is a frame for \( L^2(\mathbb{R}) \). If a Gabor system \( \{E_{mb}T_{na}\phi\}_{m,n\in\mathbb{Z}} \) is a dual frame of a Gabor frame \( \{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}} \), \( \phi \) is called a dual window of \( \{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}} \). For more on general frame theory, as well as on Gabor frames and other structured frames, we refer e.g. to the books [4, 8, 19, 20, 21, 24].

While in general the structure of a given frame is not inherited by the canonical dual (e.g. this is the case with wavelet frames), the canonical dual of a Gabor frame \( G = \{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}} \) is also with Gabor structure - the function \( S_G^{-1}g \) is a dual window of \( G \) (see, e.g., [13] or [8, Theorem 12.3.2]) called the canonical dual window. However, the canonical dual of a Gabor frame may fail some other nice properties desired for applications (e.g. compact support, smoothness, good time-frequency localization, and other). In particular, a Gabor frame, which is also a Schauder basis, and its only dual (the canonical one) can never be well localized in both time and frequency, cf. the Balian-Low Theorem [13, 1]. In contrast to the above, in the overcomplete case there exist Gabor frames with a nice generating window and nice dual windows with respect to certain desired properties (see, e.g., [7, 9, 11, 12, 30, 31]). Although not all the dual frames of an overcomplete Gabor frame to have to be with a Gabor structure [29], there are infinitely many dual Gabor frames. There exist characterizations of all the dual windows of a Gabor frame via the Wexler-Raz biorthogonality relation [33, 25, 14, 18] and via more constructive approaches [29, 6, 23] involving the inverse of the frame operator. For computational purposes, those of the dual Gabor frames which have compactly supported windows are of significant importance. The main purpose of this paper is to give a characterization of all the dual windows which have compact support and to avoid operator inversions:

**Theorem 1.1** Let \( g \in L^2(\mathbb{R}) \) be compactly supported and such that \( G = \{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}} \) is a frame for \( L^2(\mathbb{R}) \) for some \( a, b > 0 \). Assume that there is a compactly supported function \( g^d \in L^2(\mathbb{R}) \) so that \( G^d = \{E_{mb}T_{na}g^d\}_{m,n\in\mathbb{Z}} \) is a dual frame of \( G \). Then there is a finite set \( K \) (dependent only on \( g, g^d \), and \( a \)) so that all the compactly supported dual windows of \( G \) are in the form

\[
\phi = g^d + w - \sum_{k \in K} \sum_{j \in \mathbb{Z}} \langle g^d, E_{jb}T_{ka}g \rangle E_{jb}T_{ka}w
\]

where \( w \in L^2(\mathbb{R}) \) is compactly supported and such that \( \{E_{mb}T_{na}w\}_{m,n\in\mathbb{Z}} \) is a Bessel sequence in \( L^2(\mathbb{R}) \).

For classes of Gabor frames for which the assumptions of the above theorem are fulfilled, see, e.g., [7, 9, 10, 11, 12].

As an application of Theorem 1.1 we can give a procedure for approximation of the canonical dual window via compactly supported dual windows (Proposition 1.2). Recall that the canonical dual window of a Gabor frame \( \{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}} \) has some nice properties, e.g. it has minimal \( L^2 \)-norm among the dual windows [27, Prop. 3.2]; it belongs to the modulation space \( M^r_{1,1} \) when \( g \in M^r_{1,1} \), \( ab \in \mathbb{Q} \), and \( v \) is a polynomial weight [17, Cor. 3.5]; it belongs to the Schwartz space

\[1\] For \( \nu \in \mathbb{R} \), the modulation operator \( E_{\nu} : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \) is determined by \( E_{\nu}g(x) = e^{2\pi i \nu x}g(x), x \in \mathbb{R} \).

\[2\] For \( t \in \mathbb{R} \), the translation operator \( T_t : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \) is determined by \( T_t g(x) = g(x - t), x \in \mathbb{R} \).
\( S \) when \( g \in S \) \([27, \text{Prop. 5.5}] \) \([17, \text{Cor. 3.6}]\). However, often it is difficult to find the canonical dual in explicit form and for such cases it is of interest to consider approximations via dual frames which are explicitly determined. Furthermore, the canonical dual window of a Gabor frame with a compactly supported window is not necessarily with a compact support (though there are Gabor frames for which the canonical dual is compactly supported, see e.g. \([2, 3], [8, \text{Ex. 12.3.3}]\)). For such Gabor frames it is of interest to have approximations of the canonical dual via compactly supported dual windows.

Note that the known frame algorithm and its accelerations based on the Chebyshev method and the conjugate gradient method \([22]\) for approximation of the canonical dual of a general frame, as well as the variant of the Schulz iterative algorithm \([26, \text{Alg. IV}]\) for approximation of the canonical dual window of a Gabor frame, and various finite section methods \([32, 5]\), use iteration steps which are not necessarily dual frames. As far as we are aware, the only paper which concerns an approximation of the canonical dual window via exact dual windows is \([28]\). The algorithm in \([28]\) is designed for Gabor frames with very specific windows (totally positive functions and exponential B-splines) and it provides compactly supported dual windows which converge exponentially to the canonical dual window. Here we consider an algorithm (variation of the frame algorithm), which provides compactly supported dual windows on every iteration and applies to a quite general class of Gabor frames.

**Proposition 1.2** Let \( g, g^d \in L^2(\mathbb{R}) \) and \( a, b > 0 \) be such that \( G = \{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}} \) is a frame for \( L^2(\mathbb{R}) \) and \( G^d = \{E_{mb}T_{na}g^d\}_{m,n \in \mathbb{Z}} \) is a dual frame of \( G \). Let \( \lambda \) be a positive constant such that \( \|I - \lambda S_G\| < 1 \). Consider the sequence \( \{g^p\}_{p=0}^{\infty} \) determined as follows:

\[
g^0 := g^d, \\
g^{p+1} := \lambda g + (I - \lambda S_G)g^p, \quad p \in \mathbb{N}_0.
\]

Then \( \{g^p\}_{p=0}^{\infty} \) is a sequence of dual windows of \( G \) which converge to the canonical dual window \( S_G^{-1}g \). If \( g \) and \( g^d \) are furthermore compactly supported, then \( g^p, p \in \mathbb{N} \), are also compactly supported.

**Remark.** Note that under the assumptions of the above proposition, if \( ab \in \mathbb{Q} \) and \( g \) and \( g^d \) belong to the modulation space \( M_{1,1}^{w}(\mathbb{R}) \) \([16]\) for a submultiplicative weight \( w \) of polynomial growth (resp. to the Schwartz space \( \mathcal{S}(\mathbb{R}) \)), then applying \([17, \text{Theorem 3.4 and (S6)}]\) and using induction on \( p \) we get that the functions \( g^p, p \in \mathbb{N} \), given by \((2)\), belong to \( M_{1,1}^{w}(\mathbb{R}) \) (resp. to \( \mathcal{S}(\mathbb{R}) \)).

The paper is organized as follows. In Section 2 we provide the necessary background results needed to prove Theorem \([1,1]\). These are characterizations of all the dual frames based on any given dual frame and they can be of independent interest serving as a tool for construction of variety of dual frames. This can be very useful e.g. in cases when one searches for dual frames satisfying additional constraints motivated by applications. Section 3 is devoted to application of Section 2 for approximation of the canonical dual frame. A proof of Proposition \([1,2]\) is given and a procedure applying to general frames is also considered. In both Sections 2 and 3 we give illustrative examples and provide implementations under the Matlab environment.

## 2 Characterizations of dual frames

The main focus of this paper is Theorem \([1,1]\) - a characterization of all the compactly supported dual windows of a Gabor frame. In order to prove this theorem, we will need the following results
(Propositions 2.1 and 2.2), which are of independent interest as well. As a first step, observe that the known characterization of all the dual frames via the canonical dual [29] [6] [23] can be extended to the use of any dual frame instead of the canonical dual.

**Proposition 2.1** Let \( G = \{g_k\}_{k=1}^\infty \) be a frame for \( \mathcal{H} \) and let \( G^d = \{g^d_k\}_{k=1}^\infty \) be a dual frame of \( G \). Then all the dual frames of \( G \) are precisely the sequences

\[
\{g^d_k + w_k - \sum_{j=1}^\infty \langle g^d_k, g_j \rangle w_j\}_{k=1}^\infty,
\]

where \( W = \{w_k\}_{k=1}^\infty \) is a Bessel sequence in \( \mathcal{H} \).

**Proof.** A proof can be done in a similar way as the proof of characterizations based on the canonical dual (see, e.g., [8] Theor. 6.3.7]). For convenience of the readers, let us give a sketch of the proof. First take a sequence in the form (3) with \( W = \{w_k\}_{k=1}^\infty \) being a Bessel sequence in \( \mathcal{H} \) and write it in the form

\[
\{(T_G^d + T_W - T_W U_G T_G^d) \delta_k\}_{k=1}^\infty,
\]

where \( \delta_k \) means the \( k \)-th canonical vector, \( k \in \mathbb{N} \). Denote \( V := T_G^d + T_W - T_W U_G T_G^d \). Clearly, \( V \) is a bounded operator from \( \ell^2 \) into \( \mathcal{H} \) and \( V U_G = Id_\mathcal{H} \). Now use the known result that the dual frames of \( G \) are precisely the sequences of the form \( \{L \delta_k\}_{k=1}^\infty \), where \( L : \ell^2 \to \mathcal{H} \) is a bounded left inverse of \( U_G \) (see, e.g., [8] Lemma 6.3.5]) and conclude that \( \{V \delta_k\}_{k=1}^\infty \) is a dual frame of \( G \).

Conversely, take a dual frame \( F = \{f_k\}_{k=1}^\infty \) of \( G \). The sequence \( F \) can be written in the form (3) using for example \( W = F \). \( \square \)

In the case of Gabor frames, we can also use the known characterization of the dual Gabor frames based on the canonical dual [29] [6] [23] and extend it to the use of any other dual Gabor frame instead of the canonical one. This can be useful for frames for which an explicit formula for the canonical dual window is hard to get, but another dual window is known in explicit form (for such cases see, e.g., [12], [8], Sec. 12.5,12.6)). It is also useful for the characterization of all compactly supported dual windows (Theorem 1.1.), which could not have been done using only the known characterization based on the canonical dual, because the canonical dual window of a Gabor frame with a compactly supported generating window is not necessarily compactly supported.

**Proposition 2.2** Let \( g, g^d \in L^2(\mathbb{R}) \) and \( a, b > 0 \) be such that \( G = \{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}} \) is a frame for \( L^2(\mathbb{R}) \) and \( G^d = \{E_{mb}T_{na}g^d\}_{m,n \in \mathbb{Z}} \) is a dual frame of \( G \). Then all the dual windows of \( G \) are the functions of the form

\[
\phi = g^d + w - \sum_{j,k \in \mathbb{Z}} \langle g^d, E_{jb}T_{ka}g \rangle E_{jb}T_{ka}w,
\]

where \( w \in L^2(\mathbb{R}) \) is such that \( \{E_{mb}T_{na}w\}_{m,n \in \mathbb{Z}} \) is a Bessel sequence in \( L^2(\mathbb{R}) \).

**Proof.** A proof can be done in a similar way as the proof of [8] Prop. 12.3.6], using Proposition 2.1 instead of [8] Theorem 6.3.7]. We give a sketch of the proof for convenience of the readers.

For one of the directions, take \( w \in L^2(\mathbb{R}) \) such that \( \{E_{mb}T_{na}w\}_{m,n \in \mathbb{Z}} \) is a Bessel sequence in \( L^2(\mathbb{R}) \). Then the function \( \phi \) determined by (4) is well defined and belongs to \( L^2(\mathbb{R}) \). Using similar calculations as in [8] Lemma 12.3.1], the Gabor system \( \{E_{mb}T_{na}\phi\}_{m,n \in \mathbb{Z}} \) can be written as

\[
\{E_{mb}T_{na}g^d + E_{mb}T_{na}w - \sum_{j,k \in \mathbb{Z}} \langle E_{mb}T_{na}g^d, E_{jb}T_{ka}g \rangle E_{jb}T_{ka}w\}_{m,n \in \mathbb{Z}},
\]
which is a dual frame of $G$ by Proposition \ref{prop:2.1}.

Conversely, if $\phi$ is a dual window of $G$, then it can be written in the form \eqref{eq:1} taking $\phi = \phi$.

We can now prove the characterization of all compactly supported dual windows of a Gabor frame (in the cases when such ones exist):

**Proof of Theorem \ref{thm:1.1}** First observe that since $g$ and $g^d$ are compactly supported, one can determine a finite set $K$ (dependent only on $g$, $g^d$, and $a$) so that $g^d, E_{jb}T_{ka}g \neq 0$ if and only if $k \in K$.

Take a compactly supported $w \in L^2(\mathbb{R})$ such that $\{E_{mb}T_{na}w\}_{m,n \in \mathbb{Z}}$ is a Bessel sequence in $L^2(\mathbb{R})$. Consider the $L^2$-function $\phi$ given by \eqref{eq:1}. Clearly, $\phi$ is also with a compact support. Since the sum on $k$ in \eqref{eq:1} is actually a sum on $k \in K$, $\phi$ can be written as in \eqref{eq:1}. Thus, by Proposition \ref{prop:2.2} $\{E_{mb}T_{na}\phi\}_{m,n \in \mathbb{Z}}$ is a dual frame of $G$.

Conversely, if $\phi$ is a compactly supported dual window $G$, then $\phi$ can be written in the form \eqref{eq:1} using for example $w = \phi$. \hfill $\Box$

For implementation purposes, it is convenient to use real-valued windows. The following statement gives a characterization of all the real-valued compactly supported dual windows.

**Proposition 2.3** Under the assumptions of Theorem \ref{thm:1.1}, let $g$ and $g^d$ be furthermore real-valued. For $j,k \in \mathbb{Z}$, denote $P_{j,k}(f)(x) := \left( \int g^d(x)f(2\pi jbx)T_{ka}g(x)dx \right)f(2\pi jbx)$. Then there is a finite set $K$ (dependent only on $g$, $g^d$, and $a$) so that all real-valued compactly supported dual windows of $G$ are in the form

$$\phi = g^d + w - \sum_{k \in K}(g^d, T_{ka}g) + 2\sum_{j \in \mathbb{N}}(P_{j,k}(\cos) + P_{j,k}(\sin))T_{ka}w$$

\hspace{1cm} (5)

where $w \in L^2(\mathbb{R})$ is real-valued compactly supported and such that $\{E_{mb}T_{na}w\}_{m,n \in \mathbb{Z}}$ is a Bessel sequence in $L^2(\mathbb{R})$.

**Proof.** As in Theorem \ref{thm:1.1} there is a finite set $K$ (dependent on $g$, $g^d$, and $a$) so that $(g^d, E_{jb}T_{ka}g) \neq 0$ if and only if $k \in K$. Take a real-valued compactly supported $w \in L^2(\mathbb{R})$ such that $\{E_{mb}T_{na}w\}_{m,n \in \mathbb{Z}}$ is a Bessel sequence in $L^2(\mathbb{R})$. Using the unconditional convergence of $\sum_{j,k \in \mathbb{Z}}(g^d, E_{jb}T_{ka}g)E_{jb}T_{ka}w$, one can proceed to obtain representation with real-valued functions:

$$\sum_{k \in K} \sum_{j \in \mathbb{Z}}(g^d, E_{jb}T_{ka}g)E_{jb}T_{ka}w = \sum_{k \in K} \left((g^d, T_{ka}g) + \sum_{j \in \mathbb{N}}(g^d, E_{jb}T_{ka}g)E_{jb} + (g^d, E_{-jb}T_{ka}g)E_{-jb}\right)T_{ka}w.$$

$$= \sum_{k \in K} \left((g^d, T_{ka}g) + 2\sum_{j \in \mathbb{N}}(P_{j,k}(\cos) + P_{j,k}(\sin))\right)T_{ka}w.$$

Now let $\phi$ be determined by \eqref{eq:5} and thus being real-valued. By the above, it follows that $\phi$ satisfies \eqref{eq:1} and hence, by Theorem \ref{thm:1.1} $\phi$ is a compactly supported dual window of $G$.

For the converse part, if $\phi$ is a real-valued compactly supported dual window of $G$, then taking $w = \phi$ one can write $\phi$ as in \eqref{eq:1} and hence as in \eqref{eq:5}. \hfill $\Box$
As an illustration of Theorem 1.1 and Proposition 2.3 consider Example 2.4 and Fig. 1. The Matlab script which was used to produce Fig. 1 can be found at https://www.oeaw.ac.at/isf/ondualframes.

Example 2.4 Let $a = 1$ and $b = 1/3$. Consider the Gabor frame $G = \{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$, where $g$ is the B-spline

$$B_2(x) = \begin{cases} 
  x, & x \in [0,1), \\
  2 - x, & x \in [1,2], \\
  0, & x \not\in [0,2],
\end{cases}$$

and the function

$$h_2(x) = \begin{cases} 
  \frac{1}{3}(x + 1), & x \in [-1,0), \\
  \frac{2}{3}, & x \in [0,2), \\
  1 - \frac{1}{3}x, & x \in [2,3], \\
  0, & x \not\in [-1,3],
\end{cases}$$

which is a dual window of $G$. Consider the function $\phi$ given by (1) with $g^d = h_2$ and $\omega = \lambda B_2$, where $\lambda$ is a positive constant. Observe that the set $K$ in (1) is $\{-2, -1, 0, 1, 2\}$. By Theorem 1.1 $\phi$ is a dual window of $G$. In Fig. 1 we visualize $\phi$ for two values of $\lambda$.

Concerning the implementation of Example 2.4 note that the visualized $\phi$ on Fig. 1 (calculated using truncated sum on $j$ in (3)) is symmetric with respect to the line $x = 1$, like the given $g$ and $g^d$. Actually, in the setting of Prop. 2.3 one can show that if $g$, $g^d$, and $w$ are furthermore symmetric with respect to some line $x = x_0$ ($x_0 \in \mathbb{R}$), then any truncated representation of $\phi$ by (3) with a finite sum on $j$ gives a symmetric function with respect to $x = x_0$.

Clearly, one can consider the case of frames of translates and to write respective characterizations of the dual frames of translates in the spirit of Proposition 2.2 and Theorem 1.1.

3 Approximation of the canonical dual frame with dual frames on every step

As motivated in the introduction, our interest in this section is in approximation of the canonical dual frame via dual frames. Let us begin with the Gabor case and the approximation of the canonical dual window via compactly supported dual windows.

Proof of Proposition 1.2 Consider the sequence $\{g^p\}_{p=0}^{\infty}$ given by (2). For a given $w \in L^2(\mathbb{R})$ such that $\{E_{mb}T_{na}w\}_{m,n \in \mathbb{Z}}$ is a Bessel sequence in $L^2(\mathbb{R})$, consider the sequence $q^p$, $p \in \mathbb{N}_0$, determined as follows: $q^0 := g^d$ and $q^{p+1} := q^p + w - \sum_{j,k} \langle g^p, E_{ja}T_{kb}g \rangle E_{ja}T_{kb}w$, $p \in \mathbb{N}_0$. Using Proposition 2.2 it follows by induction that $q^p$ is a dual window of $G$ for every $p \in \mathbb{N}_0$. Choosing $w = \lambda g$, the sequence $\{q^p\}_{p \in \mathbb{N}_0}$ becomes the same as $\{q^p\}_{p \in \mathbb{N}_0}$. Now the representation

$$g^p - S_G^{-1}g = (I - \lambda S_G)^p (g^d - S_G^{-1}g), \quad p \in \mathbb{N}_0,$$

leads to $\|g^p - S_G^{-1}g\| \leq \|I - \lambda S_G\|^p (\|B_{Gd}\| + \frac{1}{\sqrt{\lambda S_G}}) \to 0$ as $p \to \infty$.

When $g$ and $g^d$ are furthermore compactly supported, using Theorem 1.1 one can conclude that every dual window $g^p$, $p \in \mathbb{N}$, is compactly supported. □
Figure 1: Visualization of $g$, $g^{d}$, and $\phi$ from Example 2.4. For (a) and (c): comparison of $\phi$ calculated with sum on $j$ truncated to 6 terms (in blue with stars) and to 7 terms (in red). For (b) and (d): the function $g$ (in green), $g^{d}$ (in blue), and $\phi$ calculated with sum on $j$ truncated to 7 terms (in red).

For general frames, the following procedure holds for approximation of the canonical dual via dual frames:

**Proposition 3.1** Let $G = \{g_k\}_{k=1}^{\infty}$ be a frame for $\mathcal{H}$ and let $G^{d} = \{g^{d}_k\}_{k=1}^{\infty}$ be a dual frame of $G$. Let $\lambda$ be a positive constant such that $\|I - \lambda S_G\| < 1$. Consider the sequence $F^p = \{f^p_k\}_{k=1}^{\infty}, p \in \mathbb{N}_0$, determined as follows:

$$F^0 := G^d,$$

$$F^{p+1} := \{\lambda g_k + f^p_k - \lambda S_G f^p_k\}_{k=1}^{\infty}, \quad p \in \mathbb{N}_0. \quad (6) \quad (7)$$

Then $F^p$ is a dual frame of $G$ for every $p \in \mathbb{N}_0$ and $\lim_{p \to \infty} F^p = \tilde{G}$ uniformly on $k$, i.e., for every $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ so that $\|f^p_k - \tilde{g}_k\| < \varepsilon$ for every $p > N_\varepsilon$ and every $k \in \mathbb{N}$.

**Proof.** For a given Bessel sequence $W = \{w_k\}_{k=1}^{\infty}$ in $\mathcal{H}$, consider the sequence $Q^p = \{q^p_k\}_{k=1}^{\infty}, n \in \mathbb{N}_0$, determined as follows: $Q^0 := G^d$ and $Q^{p+1} := \{g^p_k + w_k - \sum_{j=1}^{\infty} \langle g^p_k, g_j \rangle w_j\}_{k=1}^{\infty}, n \in \mathbb{N}_0$. Using
Proposition 2.1 it follows by induction that $Q^p$ is a dual frame of $G$ for every $p \in \mathbb{N}_0$. Choosing $W = \lambda G$, the sequence $\{Q^p\}_{p \in \mathbb{N}_0}$ becomes $\{F^p\}_{p \in \mathbb{N}_0}$.

As in the frame algorithm, for every $p, k \in \mathbb{N}$ one can write $f_k^p - S^{-1}_G g_k = (I - \lambda S)^p(f_k^0 - S^{-1} g_k)$. Therefore,

$$\|f_k^p - S^{-1}_G g_k\| \leq \|I - \lambda S\|^p(\sqrt{B_G} + \frac{1}{\sqrt{A_G}}), \quad \forall k, p \in \mathbb{N},$$

which leads to the desired conclusion. \hfill \Box

Note that the consideration of approximation of the canonical dual via dual frames in this section was naturally motivated by the results in Section 2, but it turned out to be simply related to the classical frame algorithm with difference just in the initial step. The initialization with a null vector in the classical frame algorithm (see, e.g., [21, Alg. 5.1.1]) actually prevents the sequences from the next steps of the algorithm to be dual frames - the first step leads to the sequence $\lambda G$, which can be a dual frame of $G$ only if $G$ is a $\frac{1}{\lambda}$-tight frame. By Proposition 3.1, using a dual frame of $G$ in the initialization step of this algorithm is the key to guarantee dual frames on all steps.

As a simple illustration of Proposition 3.1, consider Example 3.2 and Fig. 2. The Matlab scripts for the implementation of Proposition 3.1 and for producing Fig. 2 are available at https://www.oeaw.ac.at/isf/ondualframes.

**Example 3.2** Consider the two-dimensional Euclidean space $\mathbb{R}^2$, a frame $G$ for $\mathbb{R}^2$ and its dual frame $G^d$ as given below. Based on Proposition 3.1 we consider an approximation of the canonical dual frame with precision of the first 4 digits after the decimal dot. The letter $p$ indicates the number of iterations the algorithm took to finish with the desired precision.

(i) Consider $G = (e_1, e_2, e_2 - e_1)$ and $G^d = (e_2, e_1, e_2 - e_1)$. Running the algorithm with $\lambda$ being $1/3$, $1/4$, and $1/2(= 2/(A_G^{opt} + B_G^{opt}))$, the respective values of $p$ are 1, 7, and 14.

(ii) Consider $G$ from (i) and $G^d = (2e_1 - e_2, 2e_2 - e_1, e_1 - e_2)$. Running the algorithm with $\lambda = 1/3, 1/4, 1/2$, we get $p = 1, 8, 15$, respectively.

(iii) Consider $G$ from (i) and $G^d = (2e_1, e_2 - e_1, e_1)$. Running the algorithm with $\lambda = 1/3, 1/4, 1/2$, we get $p = 21, 29, 15$, respectively.

(iv) Consider $G = (e_1, e_2, e_2 - 2e_1)$ and $G^d = (2e_2 - e_1, e_1, e_2 - e_1)$. Running the algorithm with $\lambda$ being $1/4, 2/9$, and $2/7(= 2/(A_G^{opt} + B_G^{opt}))$, the respective values of $p$ are 28, 32, and 31. \hfill \Box

As the above examples show and as it is natural to expect, the efficiency of the algorithms in Propositions 1.2 and 3.1 depends much on the initial dual frame $G^d$ and $\lambda$. The value $2/(A_G^{opt} + B_G^{opt})$, which is an optimal one for $\lambda$ for some algorithms, might be much less efficient in the present algorithm compare to other values of $\lambda$, see Example 3.2(i)(ii). As a brief comparison to the classical frame algorithm, if we run (7) with initialization $F^0 = 0$ for the frame $G$ from Example 3.2(ii) with $\lambda = 1/3, 1/4, 1/2$, then the respective values of $p$ are 21, 29, 14 - compare to the values of $p$ in Example 3.2(i)(ii). It will be the purpose of further work to investigate deeper the efficiency of the present algorithm in dependence of $G^d$ and $\lambda$, especially in the case of Gabor frames.
(a) Ex. 3.2(i) with $\lambda = 1/3$

(b) Ex. 3.2(i) with $\lambda = 1/4$

(c) Ex. 3.2(ii) with $\lambda = 1/3$

(d) Ex. 3.2(iii) with $\lambda = 1/3$

(e) Ex. 3.2(iv) with $\lambda = 2/7$

(f) Ex. 3.2(iv) with $\lambda = 1/4$

Figure 2: The frames from Example 3.2. In green - the given frame $G$, in blue - the given dual frame $G^d$, in red - the new dual frame from the first iteration step, in magenta - the approximation of the canonical dual $\tilde{G}$ up to 4 digits after the dot (the number $p$ of the iterations is noted as an upper index of $f$).
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