Abstract. A complex Hadamard matrix is a square matrix $H$ with complex entries of absolute value 1 satisfying $HH^* = nI$, where $*$ stands for the Hermitian transpose and $I$ is the identity matrix of order $n$. In this paper, we first determine the image of a certain rational map from the $d$-dimensional complex projective space to $\mathbb{C}^{d(d+1)/2}$. Applying this result with $d = 3$, we give constructions of type-II matrices and complex Hadamard matrices, and more generally, type-II matrices, in the Bose–Mesner algebra of a certain 3-class symmetric association scheme. In particular, we recover the complex Hadamard matrices of order 15 found by Ada Chan. We compute the Haagerup sets to show inequivalence of resulting type-II matrices, and determine the Nomura algebras to show that the resulting matrices are not decomposable into generalized tensor products.

1. Introduction

A complex Hadamard matrix is a square matrix $H$ with complex entries of absolute value 1 satisfying $HH^* = nI$, where $*$ stands for the Hermitian transpose and $I$ is the identity matrix of order $n$. They are the natural generalization of real Hadamard matrices. Complex Hadamard matrices appear frequently in various branches of mathematics and quantum physics.

A type-II matrix, or an inverse orthogonal matrix, is a square matrix $W$ with nonzero complex entries satisfying $WW^*(-)^\top = nI$. Obviously, a complex Hadamard matrix is a type-II matrix.

A complete classification of complex Hadamard matrices, and that of type-II matrices are only available up to order $n = 5$ (see [7, 14, 10]).
Although it is shown by Craigen [7] that there are uncountably many equivalence classes of complex Hadamard matrices of order $n$ whenever $n$ is a composite number, some type-II matrices are more closely related to combinatorial objects than the others. Szollosi [16] used design theoretical methods to construct complex Hadamard matrices. Strongly regular graphs were used to construct type-II matrices in [5, 6]. See [15] for a generalization. In this paper, we construct type-II matrices and complex Hadamard matrices in the Bose–Mesner algebra of a certain 3-class symmetric association scheme. In particular, we recover the complex Hadamard matrices of order 15 found by [4].

The method of finding a complex Hadamard matrices in the Bose–Mesner algebra of a symmetric association scheme generalizes the classical work of Goethals and Seidel [9]. By assuming that the association scheme to be symmetric, the resulting complex Hadamard matrices are symmetric. It turns out that this assumption enables us to consider only the real parts of the entries of a complex Hadamard matrices, since the orthogonality can be expressed in terms of the real parts. Extending this reduction to type-II matrices, we are led to consider a rational map whose inverse is explicitly given in Section 2. In Section 3, we explain why only real parts come into play when we construct complex Hadamard matrices in the Bose–Mesner algebra of a symmetric association scheme. In Section 4 we take a particular family of a 3-class association scheme. This family was found after extensive computer experiment on the list of 3-class association schemes up to 100 vertices given in [8]. Surprisingly, most other association schemes up to 100 vertices, with the exceptions of amorphic or pseudocyclic schemes, do not admit a complex Hadamard matrix in their Bose–Mesner algebras. In Section 5 we compute the Haagerup set to show inequivalence of type-II matrices constructed in Section 4. In Section 6 we show that the Nomura algebra of each of the type-II matrices constructed in Section 4 has dimension 2. This implies that our matrices are not equivalent to a generalized tensor product by [11].

All the computer calculations in this paper are performed by magma [2].

2. The image of a rational map

We define a polynomial in three indeterminates $X, Y, Z$ as follows:

$$g(X, Y, Z) = X^2 + Y^2 + Z^2 - XYZ - 4.$$ 

Lemma 1.

$$g\left(\frac{X}{Y}, \frac{Y}{X}, \frac{X}{Z}, \frac{Z}{X}, \frac{Z}{Y}, \frac{Y}{Z}\right) = 0.$$
Proof. Straightforward. □

Lemma 2. In the rational function field with four indeterminates $X, Y, Z$ and $z$, the following identities hold:

$$w + \frac{1}{w} = Y + \frac{z^2(z^2 - 1)g + c_1 f}{(z^2 - 1)(zZ - Y)(zX - 2)},$$

(1)

$$\frac{z}{w} + \frac{w}{z} = Z + \frac{z^2(z^2 - 1)g + c_2 f}{z(z^2 - 1)(zZ - Y)(zX - 2)},$$

(2)

$$ww' = 1 + \frac{z^2 g + (2zX - zYZ + f) f}{z(zZ - Y)(zY - Z)},$$

(3)

where

$$f = z^2 - zX + 1,$$

$$g = g(X, Y, Z),$$

$$c_1 = (z^2 - 1)(zX - Z^2 + 2) - (zY - Z)^2,$$

$$c_2 = (z^2 - 1)(zX - Y^2 + 2) - (zZ - Y)^2,$$

$$w = \frac{z^2 - 1}{zZ - Y},$$

$$w' = \frac{z^{-2} - 1}{z^{-1}Z - Y}.$$
Lemma 4. Let $N = \{0, 1, \ldots, d\}$, $N_3 = \binom{N}{3}$ and $N_4 = \binom{N}{4}$. Let $a_{i,j}$ $(0 \leq i, j \leq d, i \neq j)$ be complex numbers satisfying
\begin{equation}
    a_{i,j} = a_{j,i} \quad (0 \leq i < j \leq d),
\end{equation}
\begin{equation}
    g(a_{i,j}, a_{j,k}, a_{i,k}) = 0 \quad (\{i, j, k\} \in N_3),
\end{equation}
\begin{equation}
    h(a_{i,j}, a_{i,k}, a_{i,\ell}, a_{j,k}, a_{j,\ell}, a_{k,\ell}) = 0 \quad (\{i, j, k, \ell\} \in N_4).
\end{equation}
Assume
\begin{equation}
    a_{i_0,i_1} \neq \pm 2 \quad \text{for some } i_0, i_1 \text{ with } 0 \leq i_0 < i_1 \leq d.
\end{equation}
Let $w_{i_0}$, $w_{i_1}$ be nonzero complex numbers satisfying
\begin{equation}
    \frac{w_{i_0}}{w_{i_1}} + \frac{w_{i_1}}{w_{i_0}} = a_{i_0,i_1}.
\end{equation}
Define complex numbers $w_i$ $(0 \leq i \leq d, i \neq i_0, i_1)$ by
\begin{equation}
    w_i = \frac{w_{i_1}^2 - w_{i_0}^2}{a_{i,i_1}w_{i_1} - a_{i_0,i}w_{i_0}}.
\end{equation}
Then
\begin{equation}
    \frac{w_j}{w_i} + \frac{w_i}{w_j} = a_{i,j} \quad (0 \leq i < j \leq d).
\end{equation}
Conversely, if complex numbers $\{w_i\}_{i=0}^d$ satisfy (10), then (9) holds.
Moreover, if $a_{i,j}$ $(0 \leq i < j \leq d)$ are all real and
\begin{equation}
    -2 < a_{i_0,i_1} < 2,
\end{equation}
then $|w_i| = |w_j|$ for $0 \leq i < j \leq d$.

Proof. Without loss of generality, we may assume $i_0 = 0$ and $i_1 = 1$. Thus (8) reads
\begin{equation}
    \frac{w_0}{w_1} + \frac{w_1}{w_0} = a_{0,1}.
\end{equation}
Then by (7) and (5) with $\{i, j, k\} = \{0, 1, i\}$, we find
\begin{equation}
    a_{0,i}^2 + a_{1,i}^2 - a_{0,1}a_{0,i}a_{1,i} \neq 0.
\end{equation}
First we need to check that the denominator of (9) is nonzero. Suppose $a_{1,i}w_1 - a_{0,i}w_0 = 0$. If, moreover $a_{1,i} = 0$, then we have $a_{0,i} = 0$, but this contradicts (13). Thus we have $a_{1,i} \neq 0$, and
\begin{equation}
    \frac{w_1}{w_0} = \frac{a_{0,i}}{a_{1,i}}.
\end{equation}
Then by (12), we have
\begin{equation}
    \frac{a_{1,i}^2}{a_{0,i}} + \frac{a_{0,i}}{a_{1,i}} = a_{0,1},
\end{equation}
but again this contradicts (13). Therefore, $w_i$ is well-defined. Moreover, we claim $w_i \neq 0$. Indeed, $w_i = 0$ would imply $w_i^2 = w_0^2$. Then by (12), we have $a_{0,1} = \pm 2$, which contradicts (17).

Clearly, (12) implies (10) with $(i, j) = (0, 1)$. Suppose $j \geq 2$. Observe that $x = w_j = (w_i^2 - w_0^2)/(a_{1,j}w_1 - a_{0,j}w_0)$ given in (9) is a unique common root of the equations

$$x^2 - w_0a_{0,j}x + w_0^2 = 0,$$

$$x^2 - w_1a_{1,j}x + w_1^2 = 0.$$

This can be seen by $g(a_{0,1}, a_{0,j}, a_{1,i}) = 0$. Thus (10) holds if $i = 0$ or 1. Now assume $\{i, j\} \cap \{0, 1\} = \emptyset$. By (6), we have

$$0 = w_0^2w_i^2w_jh(a_{0,1}, a_{0,i}, a_{0,j}, a_{1,i}, a_{1,j}, a_{i,j})$$

$$= \det \begin{bmatrix} 2w_0^2 & w_0w_iw_0a_{0,i} & w_0w_iw_{a_{0,1}} \\ w_0w_1a_{0,1} & 2w_1^2 & w_1w_1a_{1,i} \\ w_0w_ja_{0,j} & w_1w_ia_{1,j} & w_1w_ia_{1,i} \end{bmatrix}$$

$$= \det \begin{bmatrix} 2w_0^2 & w_0^2 + w_i^2 & w_0^2 + w_i^2 \\ w_0^2 + w_i^2 & 2w_i^2 & w_i^2 + w_i^2 \\ w_0^2 + w_i^2 & w_i^2 + w_i^2 & w_iw_ia_{1,i} \end{bmatrix}$$

$$= (w_i^2 - w_0^2)^2(w_iw_ia_{1,i} - w_i^2 - w_0^2).$$

Thus (10) holds.

Conversely, suppose that $\{w_i\}_{i=0}^d$ satisfy (10). Setting $(i, j) = (0, 1)$ in (10) gives (12). Setting $i = 1$ and replacing $j$ by $i$ in (10) gives

$$\frac{w_i}{w_1} + \frac{w_i}{w_i} = a_{1,i}.$$

By (10), we have

$$a_{1,i}w_1w_i = w_i^2 + w_i^2$$

$$= a_{0,i}w_0w_i + w_i^2 - w_0^2,$$

which gives (9).

Finally, assume that $a_{i,j}$ ($0 \leq i < j \leq d$) are all real and $-2 < a_{0,1} < 2$. Then by (8), $w_1/w_0$ is an imaginary number with absolute value 1. This means

$$(w_1/w_0)^{-1} = \overline{(w_1/w_0)}.$$  \hspace{1cm} (14)

For $j \geq 2$, set $(X, Y, Z, z) = (a_{0,1}, a_{0,j}, a_{1,j}, w_1/w_0)$ in Lemma 2. It is easy to check that all the denominators in Lemma 2 are nonzero.
Moreover,

\[ f = 0 \quad \text{(by (8))}, \quad (15) \]
\[ g = 0 \quad \text{(by (5))}, \quad (16) \]
\[ w = \frac{w_j}{w_0} \quad \text{(by (9))}, \quad (17) \]

and

\[
\bar{w}' = \frac{\left( \frac{w_1}{w_0} \right)^2 - 1}{\left( \frac{w_1}{w_0} \right)^{-1} a_{1,j} - a_{0,j}} - 1
\]
\[ = \frac{w_j}{w_0} \quad \text{(by (14))} \]
\[ = \frac{w_j^2 - w_0^2}{w_0(a_{1,j}w_1 - a_{0,j}w_0)} \]
\[ = \frac{w_j}{w_0} \quad \text{(by (9))} \]
\[ = w \quad \text{(by (17))}. \quad (18) \]

Now

\[ 1 = w w' \quad \text{(by (9), (15), (16))} \]
\[ = |w|^2 \quad \text{(by (18))} \]
\[ = \left| \frac{w_j}{w_0} \right|^2 \quad \text{(by (17))}. \]

Therefore \(|w_i| = |w_j|\) for \(0 \leq i < j \leq d\). \qed

**Theorem 1.** Let \(d, N, N_3, N_4\) be as in Lemma 4. Define \(\phi : (\mathbb{C}^\times)^{d+1} \to \mathbb{C}^{d(d+1)/2}\) by

\[
\phi(w_0, \ldots, w_d) = \left( \frac{w_i}{w_j} + \frac{w_j}{w_i} \right)_{0 \leq i < j \leq d}.
\]

Then the image of \(\phi\) coincides with the zeros of the ideal generated by the polynomials

\[ g(X_{i,j}, X_{j,k}, X_{i,k}) = 0 \quad \{i, j, k\} \in N_3, \quad (19) \]
\[ h(X_{i,j}, X_{i,k}, X_{i,\ell}, X_{j,k}, X_{j,\ell}, X_{k,\ell}) = 0 \quad \{i, j, k, \ell\} \in N_4, \quad (20) \]

where \(X_{i,j} = X_{j,i}\).
Proof. Let $I$ denote the ideal of the polynomial ring $\mathbb{C}[X_{i,j} \mid 0 \leq i < j \leq d]$ generated by (19) and (20). By Lemmas 1 and 3, the image of $\phi$ is contained in the set of zeros of $I$.

Conversely, let $a = (a_{i,j})_{0 \leq i < j \leq d}$ be a zero of $I$. If there exists $i_0, i_1$ with $0 \leq i_0 < i_1 \leq d$ and $a_{i_0,j} \neq \pm 2$, then Lemma 4 implies that $a$ is in the image of $\phi$. Next suppose that $a_{i,j} \in \{\pm 2\}$ for $0 \leq i < j \leq d$. Then

\[ a_{0,1} = \frac{a_{0,1}}{2} + \frac{2}{a_{0,1}}. \]

Also, since $g(a_{0,i}, a_{0,j}, a_{i,j}) = 0$, we have $a_{0,i}a_{0,j}a_{i,j} = 8$. Thus

\[ a_{i,j} = \frac{8}{a_{0,i}a_{0,j}} \cdot \frac{a_{0,i}^2}{4} \cdot \frac{a_{0,j}^2}{4} = \frac{1}{2}a_{0,i}a_{0,j} = \frac{a_{0,i}}{a_{0,j}} + \frac{a_{0,j}}{a_{0,i}}. \]

Therefore, $a = \phi(2, a_{0,1}, \ldots, a_{0,d})$ is in the image of $\phi$. \hfill \square

The following lemma will be used in the proof of Theorem 2.

Lemma 5. In the rational function field with three indeterminates $X_1, X_2, X_3$, set

\[ x_{i,j} = \frac{X_i}{X_j} + \frac{X_j}{X_i} \quad (0 \leq i < j \leq 3), \]

where $X_0 = 1$. Then

\[ (X_1X_2X_3 + 1)(x_{0,1}x_{0,2} + x_{0,3} - x_{1,2}) = (X_1X_2 + X_3)(x_{0,1}x_{0,2}x_{0,3} + 2 - \frac{1}{2}(x_{1,2}x_{0,3} + x_{1,3}x_{0,2} + x_{2,3}x_{0,1})). \]

Proof. Straightforward. \hfill \square

3. Type-II matrices contained in a Bose–Mesner algebra

Throughout this section, we let $\mathcal{A}$ denote a symmetric Bose–Mesner algebra with adjacency matrices $A_0 = I, A_1, \ldots, A_d$. Let $n$ be the size of the matrices $A_i$, and we denote by

\[ P = (P_{i,j})_{0 \leq i \leq d} \]

the first eigenmatrix of $\mathcal{A}$. Then the adjacency matrices are expressed as

\[ A_j = \sum_{i=0}^{d} P_{i,j}E_i \quad (j = 0, 1, \ldots, d), \]
where \( E_0 = \frac{1}{n} J, E_1, \ldots, E_d \) are the primitive idempotents of \( A \). The second eigenmatrix
\[
Q = (Q_{i,j})_{0 \leq i \leq d, 0 \leq j \leq d}
\]
is defined as \( Q = nP^{-1} \), so that
\[
E_j = \frac{1}{n} \sum_{i=0}^{d} Q_{i,j} A_i \quad (j = 0, 1, \ldots, d)
\]
holds. Since \( QP = nI \) and \( Q_{i,0} = P_{i,0} = 1 \) for \( i = 0, 1, \ldots, d \), we have
\[
\sum_{j=1}^{d} Q_{i,j} = n\delta_{i,0} - 1. \tag{21}
\]

**Lemma 6.** Let \( w_0, w_1, \ldots, w_d \) be nonzero complex numbers, and set
\[
W = \sum_{j=0}^{d} w_j A_j \in A, \tag{22}
\]
then the following are equivalent.

(i) \( W \) is a type-II matrix,

(ii) \[
\left( \sum_{j=0}^{d} w_j P_{k,j} \right) \left( \sum_{j=0}^{d} w_j^{-1} P_{k,j} \right) = n \quad (k = 1, \ldots, d). \tag{23}
\]

**Proof.** Set
\[
\beta_k = \sum_{j=0}^{d} w_j P_{k,j}, \quad \beta'_k = \sum_{j=0}^{d} w_j^{-1} P_{k,j}.
\]
Then
\[
W = \sum_{k=0}^{d} \beta_k E_k, \quad W^{(-)} = \sum_{k=0}^{d} \beta'_k E_k.
\]
Clearly, \( W \) is type-II matrix if and only if \( \beta_k \beta'_k = n \) for \( 0 \leq k \leq d \). In particular, (i) implies (ii).

To prove the converse, it suffices to show \( \beta_0 \beta'_0 = n \) provided \( \beta_k \beta'_k = n \) for \( 1 \leq k \leq d \). Since \( W \) is symmetric, the diagonal entries of \( WW^{(-)} \) are all \( n \). Thus
\[
n^2 = \text{tr} WW^{(-)} = \sum_{k=0}^{d} \beta_k \beta'_k \text{tr} E_k
\]
and hence $\beta_0\beta'_0 = n$. \hfill \Box

Lemma 7. Let $e_k$ be the polynomial in the variables $X_{i,j}$ ($0 \leq i < j \leq d$) defined by

$$
e_k = \sum_{0 \leq i < j \leq d} P_{k,i}P_{k,j}X_{i,j} + \sum_{i=0}^{d} P_{k,i}^2 - n \quad (k = 1, \ldots, d). \tag{24}$$

If the matrix $W$ given by (22) is a type-II matrix which is not equivalent to an ordinary Hadamard matrix, then the complex numbers $a_{i,j}$ defined by (10) are common zeros of the polynomials $e_k$ ($1 \leq k \leq d$) and satisfy (4)–(7).

Conversely, if $a_{i,j}$ ($1 \leq i, j \leq d$) are common zeros of the polynomials $e_k$ ($1 \leq k \leq d$) and satisfy (4)–(7), then there exist complex numbers $w_0, w_1, \ldots, w_d$ satisfying (10) such that the matrix $W$ is a type-II matrix which is not equivalent to an ordinary Hadamard matrix.

Moreover, the matrix $W$ is a scalar multiple of a complex Hadamard matrix which is not equivalent to an ordinary Hadamard matrix if and only if $a_{i,j}$ defined by (10) are common real zeros of the polynomials $e_k$ ($1 \leq k \leq d$), satisfy (4)–(7) and (11).

Proof. Observe

$$
\left( \sum_{j=0}^{d} w_j P_{k,j} \right) \left( \sum_{j=0}^{d} w_j^{-1} P_{k,j} \right) = \sum_{0 \leq i < j \leq d} P_{k,i}P_{k,j} \left( \frac{w_i}{w_j} + \frac{w_j}{w_i} \right) + \sum_{i=0}^{d} P_{k,i}^2. \tag{25}
$$

Suppose first that $W$ is a type-II matrix which is not equivalent to an ordinary Hadamard matrix. Then Lemma 6 implies that (23) holds. By (25), this implies that the complex numbers $a_{i,j}$ defined by (10) are common zeros of the polynomials $e_k$ ($1 \leq k \leq d$). Since $W$ is not equivalent to an ordinary Hadamard matrix, (7) holds. Clearly (4) holds. From Lemma 1 and 3 we see that $a_{i,j}$ satisfy (5) and (6), respectively.

Conversely, suppose that $a_{i,j}$ are common zeros of the polynomials $e_k$ ($1 \leq k \leq d$) and satisfy (4)–(7). From Lemma 4 there are complex numbers $w_0, w_1, \ldots, w_d$ satisfying (10). Then by (25), we see that (23)
holds, and hence \( W \) is a type-II matrix by Lemma 6 and \( W \) is not equivalent to an ordinary Hadamard matrix by (7).

Assume that \( W \) is a scalar multiple of a complex Hadamard matrix which is not equivalent to an ordinary Hadamard matrix. Then \( a_{i,j} \) are common real zeros of the polynomials \( e_k \) \((1 \leq k \leq d)\) and (11) holds.

Conversely, suppose that the real numbers \( a_{i,j} \) are common zeros of the polynomials \( e_k \) \((1 \leq k \leq d)\), satisfy (11)–(7) and (11). Then by Lemma 4, \( w_0, w_1, \ldots, w_d \) have the same absolute value. Therefore \( W \) is a scalar multiple of a complex Hadamard matrix. \( \square \)

4. Infinite families of complex Hadamard matrices

Let \( q \geq 4 \) be an integer, and \( n = q^2 - 1 \). We consider a three-class association scheme \( X = (X, \{R_i\}_{i=0}^3) \) with the first eigenmatrix:

\[
P = \begin{bmatrix}
1 & \frac{q^2}{2} - q & \frac{q^2}{2} & q - 2 \\
1 & \frac{q}{2} & -\frac{q}{2} & -1 \\
1 & -\frac{q}{2} + 1 & -\frac{q}{2} & q - 2 \\
1 & -\frac{q}{2} & \frac{q}{2} & -1
\end{bmatrix}.
\]

(26)

For \( q = 2^s \) with an integer \( s \geq 2 \), there exists a 3-class association scheme with the first eigenmatrix (26) (see [3, 12.1.1]).

Let \( M = \langle A_0, A_1, A_2, A_3 \rangle \) be the Bose–Mesner algebra of \( X = (X, \{R_i\}_{i=0}^3) \). Then, \( X \) has two non-trivial fusion schemes. One is an imprimitive scheme \( X_1 = (X, \{R_0, R_1 \cup R_2, R_3\}) \) with the first eigenmatrix:

\[
P_1 = \begin{bmatrix}
1 & q(q - 1) & q - 2 \\
1 & 0 & -1 \\
1 & -q + 1 & q - 2
\end{bmatrix}.
\]

(27)

Another is a primitive scheme \( X_2 = (X, \{R_0, R_1 \cup R_3, R_2\}) \) with the first eigenmatrix:

\[
P_2 = \begin{bmatrix}
1 & \frac{q^2}{2} - 2 & \frac{q^2}{2} \\
1 & \frac{q}{2} - 1 & -\frac{q}{2} \\
1 & -\frac{q}{2} - 1 & \frac{q}{2}
\end{bmatrix}.
\]

(28)

Theorem 2. Let \( w_1, w_2, w_3 \) be nonzero complex numbers. The matrix

\[
W = A_0 + w_1 A_1 + w_2 A_2 + w_3 A_3 \in M
\]

(29)

is a type-II matrix if and only if one of the following holds:

(i) \( w_1 = w_2 = w_3 \), where

\[w_3 + \frac{1}{w_3} + q^2 - 3 = 0,\]
(ii) $w_3$ is as in (i) and
\[ w_1 = w_2 = \frac{-(q - 3)w_3 + (q - 1)}{q^2 - 2q - 1}, \]

(iii)
\[ w_1 + \frac{1}{w_1} = \frac{2(q^2 - 6)}{q^2 - 4}, \quad w_2 = -1, \quad w_3 = w_1, \]

(iv)
\[ w_1 = w_3 = 1, \quad w_2 + \frac{1}{w_2} = \frac{-2(q^2 - 2)}{q^2}, \]

(v)
\[ w_1 + \frac{1}{w_1} = \frac{-2}{q}, \quad w_2 = \frac{1}{w_1}, \quad w_3 = 1, \]

(vi)
\[ w_1 + \frac{1}{w_1} = a_{0,1}, \]

and
\[ w_i = \frac{w_i^2 - 1}{a_{1,i}w_1 - a_{0,i}} \quad (i = 2, 3), \]

where
\[
\begin{align*}
a_{0,1} &= \frac{-(q - 1)(q - 2) + (q + 2)r}{2q(q + 1)}, \\
a_{0,2} &= \frac{(q + 2)(q - 1) - (q - 2)r}{2q(q - 3)}, \\
a_{0,3} &= \frac{5q^2 - 2q - 19 - (q - 1)r}{2(q + 1)(q - 3)}, \\
a_{1,2} &= \frac{2(-q^4 + 2q^3 + 4q^2 - 10q + 1 + (q - 1)r)}{q^2(q + 1)(q - 3)}, \\
a_{1,3} &= -a_{0,2}, \\
r^2 &= (17q - 1)(q - 1).
\end{align*}
\]

Note that $w_1w_2 = -w_3$ holds.

Proof. Let $K = \mathbb{Q}(r)$, and consider the polynomial ring
\[ R = K[X_{0,1}, X_{0,2}, X_{0,3}, X_{1,2}, X_{1,3}, X_{2,3}]. \]

We assume that $W$ is a type-II matrix. For $i, j \in \{0, 1, 2, 3\}$, define $a_{i,j}$ by (10), where $w_0 = 1$. We write
\[ \mathbf{a} = (a_{0,1}, a_{0,2}, a_{0,3}, a_{1,2}, a_{1,3}, a_{2,3}) \]
for brevity. Since $q^2 - 1$ is odd, $W$ is not an ordinary Hadamard matrix. Then by Lemma 7, the complex numbers $a_{i,j}$ are common zeros of the polynomials $e_k$ ($1 \leq k \leq 3$) defined in (24), and satisfy (5)–(6). This implies that $(a_{0,1}, a_{0,2}, \ldots, a_{2,3})$ is a common zero of the polynomials

$$e_k \quad (k \in \{1, 2, 3\}),$$

$$g(X_{i,j}, X_{i,k}, X_{j,k}) \quad (\{i, j, k\} \in \binom{\{0, 1, 2, 3\}}{3}),$$

$$h(X_{i,j}, X_{i,k}, X_{i,l}, X_{j,k}, X_{j,l}, X_{k,l}) \quad (\{i, j, k, l\} \in \binom{\{0, 1, 2, 3\}}{4}).$$

Let $\mathcal{I}$ be the ideal of $R$ generated by these polynomials. Then we can verify that $\mathcal{I}$ contains the polynomial $b_1b_2b_3b_4^+b_5^-$, where

$$b_1 = X_{1,2} - 2,$$

$$b_2 = (q^2 - 4)X_{1,2} + 2q^2 - 12,$$

$$b_3 = q^2X_{1,2} + 2q^2 - 4,$$

$$b_4^+ = q^2(q + 1)(q - 3)X_{1,2} - 2(-q^4 + 2q^3 + 4q^2 - 10q + 1 ± (q - 1)r).$$

First assume $a_{1,2}$ is a zero of the polynomial $b_1$. This implies $w_1 = w_2$. Let $\mathcal{I}_1$ denote the ideal generated by $\mathcal{I}$ and $b_1$. Then we can verify that $\mathcal{I}_1$ contains the polynomials $X_{0,3} + q^2 - 3$ and $c_1c_2$, where

$$c_1 = X_{0,1} + q^2 - 3,$$

$$c_2 = (q^2 - 2q - 1)X_{0,1} - (q^3 - 3q^2 - q + 7).$$

In particular,

$$a_{0,3} = w_3 + \frac{1}{w_3} = -(q^2 - 3). \quad (30)$$

If $a_{0,1}$ is a zero of the polynomial $c_1$, then let $\mathcal{I}_1'$ denote the ideal generated by $\mathcal{I}_1$ and $c_1$. We can verify that $\mathcal{I}_1'$ contains the polynomial $X_{1,3} - 2$. Hence $w_1 = w_2 = w_3$, and we have Case (1)

If $a_{0,1}$ is a zero of the polynomial $c_2$, then

$$a_{0,1} = \frac{q^3 - 3q^2 - q + 7}{q^2 - 2q - 1}. \quad (31)$$

Let $\mathcal{I}_1''$ denote the ideal generated by $\mathcal{I}_1$ and $c_2$. We can verify that

$$(q^2 - 2q - 1)X_{1,3} + (q^3 - q^2 - q - 3) \in \mathcal{I}_1''.$$ 

This implies

$$a_{1,3} = -\frac{q^3 - q^2 - q - 3}{q^2 - 2q - 1}. \quad (32)$$
Therefore, by (9), (30), (31), (32), we obtain
\[ w_1 = w_2 = \frac{w_3^2 - 1}{a_{1,3}w_3 - a_{0,1}} = \frac{-(q - 3)w_3 + (q - 1)}{q^2 - 2q - 1}. \]

This gives Case (ii).

Next assume \( a_{1,2} \) is a zero of the polynomial \( b_2 \). Let \( \mathcal{I}_2 \) denote the ideal generated by \( \mathcal{I} \) and \( b_2 \). Then we can verify that \( \mathcal{I}_2 \) contains the polynomials \( X_{0,2} + 2, X_{1,3} - 2, \) and \( (q^2 - 4)X_{0,1} - 2(q^2 - 6) \). Hence \( w_2 = -1, w_1 = w_3 \) and \( w_1 + 1/w_1 = 2(q^2 - 6)/(q^2 - 4) \). This gives Case (iii).

Next assume \( a_{1,2} \) is a zero of the polynomial \( b_3 \). Let \( \mathcal{I}_3 \) denote the ideal generated by \( \mathcal{I} \) and \( b_3 \). Then we can verify that \( \mathcal{I}_3 \) contains the polynomials \( X_{0,3} - 2 \) and \( c_4c_5 \), where
\[ c_4 = q^2X_{0,2} + 2(q^2 - 2), \quad c_5 = qX_{0,2} + 2. \]

In particular, \( w_3 = 1 \).

If \( a_{0,2} \) is a zero of the polynomial \( c_4 \), then
\[ a_{0,2} = \frac{-2(q^2 - 2)}{q^2}. \]

Let \( \mathcal{I}_3' \) denote the ideal generated by \( \mathcal{I}_3 \) and \( c_4 \). We can verify that \( \mathcal{I}_3' \) contains the polynomial \( X_{0,1} - 2 \). Hence \( w_1 = w_3 = 1, w_2 + 1/w_2 + 2(q^2 - 2)/q^2 = 0 \), and we have Case (iv).

If \( a_{0,2} \) is a zero of the polynomial \( c_5 \), then
\[ a_{0,2} = \frac{-2}{q}. \]

Let \( \mathcal{I}_3'' \) denote the ideal generated by \( \mathcal{I} \) and \( c_5 \). We can verify that \( \mathcal{I}_3'' \) contains the polynomials \( qX_{0,1} + 2 \) and \( q^2X_{1,2} + 2(q^2 - 2) \). Thus
\[ a_{0,1} = \frac{-2}{q}, \]
\[ a_{1,2} = \frac{-2(q^2 - 2)}{q^2}. \]

Since
\[ w_1w_2 + \frac{1}{w_1w_2} = \left( w_1 + \frac{1}{w_1} \right) \left( w_2 + \frac{1}{w_2} \right) - \left( \frac{w_2}{w_1} + \frac{w_1}{w_2} \right) = a_{0,1}a_{0,2} - a_{1,2}. \]
by \( (33)-(35) \), we have \( w_1w_2 = 1 \). This gives Case (v).

Next assume \( a_{1,2} \) is a zero of the polynomial \( b_1^{-} b_1^{+} \). Without loss of
generality, we may assume \( a_{1,2} \) is a zero of the polynomial \( b_1^{+} \). Let \( \mathcal{I}_4 \)
denote the ideal generated by \( \mathcal{I} \) and \( b_1^{+} \). Then we can verify that \( \mathcal{I}_4 \)
contains the following polynomials:

\[
\begin{align*}
2q(q+1)X_{0,1} + (q-1)(q-2) - (q-2)r, & \quad (36) \\
2q(q-3)X_{0,2} - (q+2)(q-1) + (q-2)r, & \quad (37) \\
2(q+1)(q-3)X_{0,3} - (5q^2 - 2q - 19) + (q-1)r, & \quad (38) \\
q^2(q+1)(q-3)X_{1,2} - 2(-q^4 + 2q^3 + 4q^2 - 10q + 1 + (q-1)r), & \quad (39) \\
2q(q-3)X_{1,3} + (q+2)(q-1) - (q-2)r. & \quad (40)
\end{align*}
\]

Therefore, we have Case (vi) by (9). Moreover, we can verify that \( \mathcal{I}_4 \)
contains the polynomial \( X_{0,1}X_{0,2} + X_{0,3} - X_{1,2} \). Thus

\[
0 = (w_1w_2w_3 + 1)(a_{0,1}a_{0,2} + a_{0,3} - a_{1,2})
= (w_1w_2 + w_3)(a_{0,1}a_{0,2}a_{0,3} + 2 - \frac{1}{2}(a_{1,2}a_{0,3} + a_{1,3}a_{0,2} + a_{2,3}a_{0,1}))
\]

by Lemma 5. Since the ideal generated by \( \mathcal{I}_4 \) and the polynomial

\[
X_{0,1}X_{0,2}X_{0,3} + 2 - \frac{1}{2}(X_{1,2}X_{0,3} + X_{1,3}X_{0,2} + X_{2,3}X_{0,1})
\]

is trivial, we conclude \( w_1w_2 + w_3 = 0 \).

Conversely, assume that \( w_1, w_2, \) and \( w_3 \) are given in Theorem 2. Then, we show that the matrix \( W \) given in (29) is a type-II matrix. To do this, form Lemma 7 we only check that \( a \) defined by (10) is a zero of the polynomials (24), (4), (5), and (6).

First assume \( w_1, w_2, \) and \( w_3 \) are as in Case (i). Then, by (10) we have

\[
a = (-q^2 + 3, -q^2 + 3, -q^2 + 3, 2, 2, 2),
\]

and this is a zero of the polynomials (24), (4), (5), and (6). Hence \( W \) is a type-II matrix.

Next assume \( w_1, w_2, \) and \( w_3 \) are as in Case (ii). This means \( w_1 = aw_3 + b \), with

\[
a = -\frac{q - 3}{q^2 - 2q - 1},
\]

\[
b = \frac{q - 1}{q^2 - 2q - 1}.
\]
We claim
\[ \frac{1}{w_1} = a \frac{1}{w_3} + b. \]
Indeed, this can be checked by showing
\[ (aw_3 + b)(a \frac{1}{w_3} + b) = 1 \]
using \( w_3 + \frac{1}{w_3} = -(q^2 - 3) \). Thus
\[ a_{0,1} = w_1 + \frac{1}{w_1} \]
\[ = \frac{q^3 - 3q^2 - q + 7}{q^2 - 2q - 1}, \]
\[ a_{1,3} = \frac{w_1}{w_3} + \frac{w_3}{w_1} \]
\[ = \frac{aw_3 + b}{w_3} + w_3 \left( a \frac{1}{w_3} + b \right) \]
\[ = 2a + b \left( w_3 + \frac{1}{w_3} \right) \]
\[ = 2a - b(q^2 - 3) \]
\[ = \frac{-q^3 + q^2 + q + 3}{q^2 - 2q - 1}. \]
Now
\[ \alpha = (a_{0,1}, a_{0,1}, -(q^2 - 3), 2, a_{1,3}, a_{1,3}) \]
is a zero of the polynomials (24), (4), (5), and (6). Hence \( W \) is a type-II matrix.

Next assume \( w_1, w_2, \) and \( w_3 \) are as in Case (iii). Then, by (10) we have
\[ \alpha = \left( \frac{2(q^2 - 6)}{q^2 - 4}, -2, \frac{2(q^2 - 6)}{q^2 - 4}, -2, \frac{2(q^2 - 6)}{q^2 - 4}, 2, \frac{-2(q^2 - 6)}{q^2 - 4} \right), \]
and this is a zero of the polynomials (24), (4), (5), and (6). Hence \( W \) is a type-II matrix.

Next assume \( w_1, w_2, \) and \( w_3 \) are as in Case (iv). Then, by (10) we have
\[ \alpha = \left( 2, \frac{-2(q^2 - 2)}{q^2}, 2, \frac{-2(q^2 - 2)}{q^2}, 2, \frac{-2(q^2 - 2)}{q^2} \right), \]
and this is a zero of the polynomials (24), (4), (5), and (6). Hence \( W \) is a type-II matrix.
Next assume $w_1, w_2, \text{ and } w_3$ are as in Case (v). Then, by (10) we have

$$a_{1,2} = a_{0,1}^2 - 2 = \frac{-2(q^2 - 2)}{q^2},$$

and

$$a = \left(-\frac{2}{q}, -\frac{2}{q}, \frac{2}{q}, -\frac{2(q^2 - 2)}{q^2}, -\frac{2}{q}, -\frac{2}{q}\right),$$

and this is a zero of the polynomials (24), (4), (5), and (6). Hence $W$ is a type-II matrix.

Lastly assume $w_1, w_2, \text{ and } w_3$ are as in Case (vi). Then $a$ is a zero of the polynomials (24), (4), (5), and (6). Hence $W$ is a type-II matrix. □

**Corollary 1.** Let $W$ be a type-II matrix in Theorem 2. Then, $W$ is a complex Hadamard matrix if and only if $W$ is given in (iii), (iv), (v), or (vi) with $r = \sqrt{(17q - 1)(q - 1)} > 0.$

**Proof.** Since $q \geq 4,$ we have $n \geq 15.$ Hence, $|w_3| \neq 1$ for Cases (i) and (ii). For Cases (iii), (iv), and (v), it is easy to see that

$$-2 < w_i + \frac{1}{w_i} < 2 \quad (i = 1, 2, 3).$$

Therefore, by Lemma 4 the Cases (iii), (iv), and (v) give complex Hadamard matrices.

Lastly, we consider the Case (vi). If $r > 0,$ then we have $0 < a_{0,1} < 2.$ In fact, since $r > 3q,$ we have $a_{0,1} > 0.$ Also,

$$a_{0,1} - 2 = \frac{-8q(q + 1)(q - 3)^2}{2q(q + 1)(5q^2 + q + 2 + (q + 2)r)} < 0.$$

Thus, we have $0 < a_{0,1} < 2.$ By Lemma 7 we have $|w_i| = 1$ for $i = 1, 2, 3.$ Therefore, $W$ is a complex Hadamard matrix.

If $r < 0,$ then

$$a_{0,1} + 2 = \frac{-8q(q + 1)(q^2 - 5)}{2q(q + 1)(3q^2 + 7q - 2 - (q + 2)r)} < 0.$$

This implies $|w_1| \neq 1.$ Therefore, $W$ is not a complex Hadamard matrix. □

Chan [4], found three complex Hadamard matrices on the line graph of the Petersen graph. This is the 3-class association scheme with the
first eigenmatrix \([26]\), where \(q = 1\), and the three matrices can be described as the matrix \(W\) in \([29]\) with \(w_1, w_2, w_3\) given as follows.

\[
\begin{align*}
w_1 &= 1, & w_2 &= -\frac{7 \pm \sqrt{15}i}{8}, & w_3 &= 1, \\
w_1 &= \frac{5 \pm \sqrt{11}i}{6}, & w_2 &= -1, & w_3 &= w_1, \\
w_1 &= -\frac{1 \pm \sqrt{15}i}{4}, & w_2 &= w_1^{-1}, & w_3 &= 1.
\end{align*}
\tag{41, 42, 43}
\]

The cases \((41)\), \((42)\) and \((43)\) are given by \((\text{vi})\), \((\text{iii})\) and \((\text{v})\), respectively, of Theorem 2. Note that \((41)\) is equivalent to the matrix \(U_{15}\) in \([16]\).

The complex Hadamard matrix of order 15 constructed in Theorem 2 \((\text{vi})\) seems to be new. This is obtained by by setting \(q = 4\) and \(r = \sqrt{201}\), and has coefficients \(w_1, w_2, w_3 = -w_1w_2\), where

\[
\begin{align*}
w_1 + \frac{1}{w_1} &= a_{0,1}, \\
a_{0,1} &= \frac{3}{20}(\sqrt{201} - 1), \\
w_2 &= \frac{a_{0,1}w_1 - 2}{a_{1,2} - a_{0,2}}, \\
a_{0,2} &= -\frac{1}{4}(\sqrt{201} - 9), \\
a_{1,2} &= \frac{3\sqrt{201} - 103}{40}.
\end{align*}
\]

We have verified using the span condition \([13\text{ Proposition 4.1}]\) that, this matrix, as well as the one given by \((41)\) are isolated, while the two matrices given by \((42)\) and \((43)\) do not satisfy the span condition.

**Remark 1.** None of the complex Hadamard matrices given in Corollary \([1]\) is a Butson–Hadamard matrix, that is, entries are roots of unity. This follows from the fact that \(a_{i,j}\) is not an algebraic integer for some \(i, j\). Indeed, for the Case \((\text{iii})\)

\[
a_{0,1} = 2 - \frac{4}{(q - 2)(q + 2)}
\]

is not an integer.

As for the Case \((\text{iv})\)

\[
a_{0,2} = -2 + \frac{4}{q^2}
\]

is not an integer.
As for the Case (v),

\[ a_{0,1} = \frac{-2}{q} \]

is not an integer.

Finally, for the Case (vi), first suppose that \( r \) is irrational. If \( a_{0,1} \) is an algebraic integer, then so is

\[ a_{0,1}' = \frac{-(q-1)(q-2)-(q+2)r}{2q(q+1)} \]

But then

\[ a_{0,1} + a_{0,1}' = -1 + \frac{4q+2}{q(q+1)} \]

is an integer, which is absurd.

Next suppose that \( r \) is rational. Then \( a_{0,1} \) is an integer, so \( a_{0,1} \in \{0, \pm 1, \pm 2\} \). But this does not occur.

**Remark 2.** Let \( q \) be an even positive integer with \( q \geq 4 \). Then \( \sqrt{(q-1)(17q-1)} \) is an integer if and only if

\[ q \in \left\{ \frac{1}{34}(\text{tr}((2177 + 528\sqrt{17})^n(433 + 105\sqrt{17}))+18) \mid n \in \mathbb{Z} \right\} \]

\[ = \{ \ldots, 41210, 10, 26, 110890, 482812730, \ldots \}. \]

5. **Equivalence**

For a type-II matrix \( W \) of order \( n \), the Haagerup set \( H(W) \) (see [10]) is defined as

\[ H(W) = \left\{ \frac{W_{i_2,j_2}W_{i_1,j_1}}{W_{i_2,j_2}W_{i_1,j_1}} \mid 1 \leq i_1, i_2, j_1, j_2 \leq n \right\}. \]

We also define

\[ K(W) = \left\{ w + \frac{1}{w} \mid w \in H(W) \setminus \{1\} \right\}. \]

Two complex Hadamard matrices \( W_1 \) and \( W_2 \) are said to be equivalent if they are type-II equivalent. It is easy to see that, if \( W_1 \) and \( W_2 \) are equivalent, then \( H(W_1) = H(W_2) \), and hence \( K(W_1) = K(W_2) \). In this section, we compute the Haagerup sets of type-II matrices constructed in Theorem 2 to conclude that some of them are inequivalent to others.

We suppose that

\[ W = \sum_{i=0}^{d} w_i A_i \]
is a complex Hadamard matrix, where $A_0, \ldots, A_d$ are the adjacency matrices of a symmetric Bose–Mesner algebra of an association scheme $(X, \{R_i\}_{i=0}^d)$, and $w_0 = 1$. Let $H(W)$ be the Haagerup set of $W$. Then

$$H(W) = \bigcup_{i=1}^4 H_i(W),$$

where

$$H_i(W) = \left\{ \frac{W_{x_1,y_1}W_{x_2,y_2}}{W_{x_2,y_1}W_{x_1,y_2}} \mid x_1, x_2, y_1, y_2 \in X, |\{x_1, x_2, y_1, y_2\}| = i \right\}$$

for $i = 1, 2, 3, 4$. Clearly,

$$H_1(W) = \{1\},$$
$$H_2(W) = \{1\} \cup \{w_i^{\pm 2} \mid i = 1, \ldots, d\}.$$  \hspace{1cm} (44)

It should be remarked that, although $H(W)$ is an invariant, none of $H_i(W)$ ($i = 2, 3, 4$) is.

**Lemma 8.** If $|X| \geq 3$, then

$$H_3(W) = \{1\} \cup \left\{ \left( \frac{w_i w_j}{w_k} \right)^{\pm 1} \mid 1 \leq i, j, k \leq d, p_{ij}^k > 0 \right\}.$$  

**Proof.** Let

$$w = \frac{W_{x_1,y_1}W_{x_2,y_2}}{W_{x_2,y_1}W_{x_1,y_2}} \in H_3(W),$$

where $|\{x_1, x_2, y_1, y_2\}| = 3$. If $x_1 = x_2$, then $w = 1$. If $x_1 = y_1$, then $w = (w_i w_j / w_k)^{-1}$, where $(x_2, x_1) \in R_i$, $(x_1, y_2) \in R_j$, $(x_2, y_2) \in R_k$, and $1 \leq i, j, k \leq d$. All other cases can be treated in a similar manner, and we conclude that $H_3(W)$ is contained in the right-hand side. Arguing in the opposite way, we obtain the reverse containment. \hspace{1cm} \Box

**Lemma 9.** Let $\Delta$ be a subset of $\{1, \ldots, d\}$. Suppose that there exists $i \in \{1, \ldots, d\}$ such that $p_{i_1,j_1}^i > 0$ for any $i_1, j_1 \in \Delta$. Then

$$H_4(W) \supset \left\{ \frac{w_{i_1} w_{i_2}}{w_{j_1} w_{j_2}} \mid i_1, i_2, j_1, j_2 \in \Delta \right\} \setminus \{1\}.$$  

In particular, if there exists $i \in \{1, \ldots, d\}$ such that $p_{i_1,j_1}^i > 0$ for any $i_1, j_1 \in \{1, \ldots, d\}$, then

$$H_4(W) \setminus \{1\} = \left\{ \frac{w_{i_1} w_{i_2}}{w_{j_1} w_{j_2}} \mid i_1, i_2, j_1, j_2 \in \{1, \ldots, d\} \right\} \setminus \{1\}.$$
Thus $X$ in Theorem 2. In what follows, let

$$
\{w_{i_1}w_{l_2} \mid i_1, i_2, j_1, j_2 \in \Delta, (i_1, j_1) \neq (j_2, i_2)\}
$$

$$
\cup \{w_{i_1}w_{l_2} \mid i_1, i_2, j_1, j_2 \in \Delta\} \setminus \{1\}.
$$

The second statement follows immediately from the first since $H_4(W) \setminus \{1\}$ is contained in the right-hand side by the definition. □

**Lemma 10.** Suppose that there exists $i, j \in \{1, \ldots, d - 1\}$ such that

$$p_{i_1, j_1}^i > 0$$

for any $i_1, j_1 \in \{1, \ldots, d - 1\}$. Moreover, suppose $p_{i, j}^d > 0$ for

$$p_{i, j}^d > 0$$

for any $j \in \{1, \ldots, d - 1\}$. Then

$$H_4(W) \setminus \{1\} = \left\{w_{i_1}w_{l_2} \mid i_1, i_2, j_1, j_2 \in \{1, \ldots, d\}\right\} \setminus \{1\}.
$$

Proof. Clearly, $H_4(W) \setminus \{1\}$ is contained in the right-hand side, so we show the reverse containment. Setting $\Delta = \{1, \ldots, d - 1\}$ in Lemma 9 we have

$$H_4(W) \setminus \{1\} \supset \left\{w_{i_1}w_{l_2} \mid i_1, i_2, j_1, j_2 \in \{1, \ldots, d - 1\}\right\} \setminus \{1\}.
$$

It remains to show that

$$\frac{w_{i_1}w_d}{w_{j_1}w_{j_2}} \in H_4(W)$$

and

$$\frac{w^2_{i_1}w_{j_2}}{w_{j_1}w_{j_2}} \in H_4(W)$$

for any $i, j \in \{1, \ldots, d - 1\}$.

Pick $(x_1, x_2) \in R_i$. Then by the assumption, there exists $y_1$ such that

$$x_1, y_1 \in R_i$$

and

$$x_1, y_1 \in R_j.$$ Similarly, there exists $y_2$ such that

$$x_1, y_2 \in R_j$$

and

$$x_1, y_2 \in R_d.$$ Now

$$\frac{w_{i_1}w_{l_2}}{w_{j_1}w_{j_2}} = \frac{W_{x_1, y_1}W_{x_2, y_2}}{W_{x_2, y_1}W_{x_1, y_2}} \in H_4(W).
$$

Replacing $i_1$ by $d$ in the above argument gives

$$\frac{w^2_{i_1}w_{j_2}}{w_{j_1}w_{j_2}} = \frac{W_{x_1, y_1}W_{x_2, y_2}}{W_{x_2, y_1}W_{x_1, y_2}} \in H_4(W).
$$

□

Below, we determine the Haagerup set of the type-II matrices given in Theorem 2. In what follows, let $\mathcal{X} = (X, \{R_i\}_{i=0}^3)$ be an association
scheme with the first eigenmatrix \((26)\), where \(q\) is an even positive integer with \(q \geq 4\). The intersection numbers of \(X\) are given by

\[
B_1 = \begin{bmatrix}
q^2 - q & 1 & 0 & 0 \\
0 & 0 & q(q-2) & 0 \\
0 & 0 & 0 & q^4 \\
0 & q^4 & q^2 & 0
\end{bmatrix},
\]

\[
B_2 = \begin{bmatrix}
\frac{q^2}{2} & 0 & \frac{q^2}{2} & 0 \\
0 & \frac{q^2}{2} & 0 & \frac{q^2}{2} \\
0 & \frac{q^2}{2} & 0 & \frac{q^2}{2} \\
q-2 & 0 & 0 & q-3
\end{bmatrix},
\]

\[
B_3 = \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & \frac{q^2}{2} & 0 & 0 \\
0 & \frac{q^2}{2} & 0 & 0 \\
q^2 & 0 & 0 & 0
\end{bmatrix},
\]

where \(B_h\) has \((i, j)\)-entry \(p_{hi}j\) (0 ≤ \(i, j\) ≤ 3).

**Lemma 11.** Let \(W = I + \sum_{i=1}^{3} w_i A_i\) be a type-II matrix belonging to the Bose-Mesner algebra of \(X\). Then

\[H(W) = \{w_i^{\pm 2} \mid i = 1, 2, 3\}
\]

\[\cup \left\{ \left( \frac{w_{i_1} w_{i_2}}{w_{i_3}} \right)^{\pm 1} \mid 1 \leq i_1, i_2, i_3 \leq 3, p_{i_2,i_3}^{i_1} > 0 \right\}
\]

\[\cup \left\{ \frac{w_{i_1} w_{i_2}}{w_{j_1} w_{j_2}} \mid i_1, i_2, j_1, j_2 \in \{1, 2, 3\} \right\}.
\]

**Proof.** Note that \(H_3(W)\) is determined by Lemma 8. We can also easily see that the assumptions of Lemma 10 are satisfied by setting \(i = 2\). This gives \(H_4(W) \setminus \{1\}\). Since \(H_1(W) = \{1\} \subset H_4(W)\), we have the desired expression for \(H(W)\). □

Using Lemma 11, we can determine the Haagerup set \(H(W)\) for each type-II matrix given in Theorem 2. Note that the description of \(H(W)\) in Table 1 is valid for all even \(q \geq 4\), even though \(p_{11}^3 = 0\) for \(q = 4\).

The elements of \(H(W)\) given in Table 1 can be found as follows:

As for the Case (i), \(K(W)\) has two elements

\[w_1 + \frac{1}{w_1} = -q^2 + 3, \quad w_1^2 + \frac{1}{w_1^2} = q^2 - 6q^2 + 7.
\]

As for (ii), setting

\[A = -\frac{q-3}{q^2 - 2q - 1}, \quad B = \frac{q-1}{q^2 - 2q - 1},
\]
we have
\[
\begin{align*}
& w_1 = w_3 A + B, \\
& \frac{A^2 + B^2 - 1}{AB} = q^2 - 3.
\end{align*}
\]
This implies
\[
\frac{1}{w_1} = \frac{1}{w_3} A + B
\]
so that
\[
\begin{align*}
& w_1 + \frac{1}{w_1} = \left( w_3 + \frac{1}{w_3} \right) A + 2B \\
& = \frac{q^3 - 3q^2 - q + 7}{q^2 - 2q - 1}.
\end{align*}
\]
The Cases [iii] and [iv] are immediate. Finally, it is clear that $K(W)$ contains $-\frac{2}{q}$ and $-2$, in the Cases [v] and [vi], respectively. We do not need the remaining elements of $K(W)$ to prove the following propositions.

**Proposition 1.** Let $W_1, \ldots, W_6$ be type-II matrices given in [i]–[vi] of Theorem 2 respectively. Then $W_1, \ldots, W_6$ are pairwise inequivalent.

**Proof.** It suffices to show that the sets $K_i = K(W_i)$ $(i = 1, \ldots, 6)$ are pairwise distinct. Since $-1 \in H(W_i)$ if and only if $i \in \{3, 6\}$, we see that $K_i \neq K_j$ whenever $i \in \{1, 2, 4, 5\}$ and $j \in \{3, 6\}$. Moreover, we have $K_i \neq K_j$ whenever $i \in \{1, 2\}$ and $j \in \{4, 5\}$, since $K_4$ and $K_5$ are contained in the interval $[-2, 2]$, while neither $K_1$ nor $K_2$ is. Also, we

| $H(W) \setminus \{1\}$ | $K(W)$ |
|-------------------------|---------|
| (i) $\{w_1^{\pm 1}, w_1^{\pm 2}\}$ | $-q^2 + 3, q^4 - 6q^2 + 7$ |
| (ii) $\{w_1^{\pm 1}, w_1^{\pm 2}, w_3^{\pm 1}, w_3^{\pm 2}, w_3^{\pm 1}, (\frac{w_1}{w_3})^{\pm 1}, (w_3^{\pm 1}, (\frac{w_1}{w_3})^{\pm 2}\}$ | $-q^2 + 3, q^4 - 6q^2 + 7, q^2 - 2q, q^2 - 2q - 1, \ldots$ |
| (iii) $\{-1, \pm w_1^{\pm 1}, \pm w_1^{\pm 2}\}$ | $-2, \pm \frac{2(q^2 - 6)}{q^2 - 4}, \pm \frac{2(q^4 - 10q^2 + 56)}{(q^2 - 4)^2}$ |
| (iv) $\{w_2^{\pm 1}, w_2^{\pm 2}\}$ | $-2, \frac{2(q^4 - 2)}{q^2}, \frac{2q^2 - 8q^2 + 8}{q^2}$ |
| (v) $\{w_1^{\pm 1}, w_1^{\pm 2}, w_1^{\pm 3}, w_1^{\pm 4}\}$ | $-2, \ldots$ |
| (vi) $\{-1, \pm w_1^{\pm 1}, \pm w_2^{\pm 1}, \pm w_2^{\pm 2}, (w_1^{\pm 1}w_2^{\pm 1})^{\pm 1}, (w_1^{\pm 1}w_2^{\pm 1})^{\pm 1}, (w_1^{\pm 1}w_2^{\pm 1})^{\pm 1}, \pm (w_1^{\pm 1}w_2^{\pm 1})^{\pm 1}, \pm (w_1^{\pm 1}w_2^{\pm 1})^{\pm 1}\}$ | $-2, \ldots$ |

**Table 1. Haagerup sets**
have $K_1 \neq K_2$ since
\[ \frac{q^3 - 3q^2 - q + 7}{q^2 - 2q - 1} \in K_2 \setminus K_1, \]
and $K_4 \neq K_5$ since
\[ \frac{2}{q} \in K_5 \setminus K_4. \]
It can be checked directly that the element
\[ a_{0,1} = \frac{-(q-1)(q-2) + (q+2)r}{2q(q+1)} \in K_6 \]
does not belong to $K_3$. Thus $K_3 \neq K_6$. \hfill \Box

**Proposition 2.** Let $W_+$ and $W_-$ be type-II matrices given in Theorem 2 (vi) with $r > 0$ and $r < 0$, respectively. Then $W_+$ and $W_-$ are inequivalent.

**Proof.** By Corollary 1, $W_+$ is a complex Hadamard matrix. Thus every element of $H(W_+)$ has absolute value 1, and hence $K(W_+)$ is contained in the interval $[-2, 2]$. On the other hand, the element
\[ a_{0,1} = \frac{-(q-1)(q-2) + (q+2)r}{2q(q+1)} \in K(W_-) \]
does not belong to the interval $[-2, 2]$. Thus $K(W_+) \neq K(W_-)$. \hfill \Box

We were able to use the Haagerup set to distinguish some of the complex Hadamard matrices in Theorem 2 This is because the Haagerup set can be described by the intersection numbers of the association scheme, and is independent of the isomorphism class. In general, if $q \geq 8$ is a power of 2, there may be many non-isomorphic association schemes with the eigenmatrix (26). We do not know whether complex Hadamard matrices having the same coefficients are equivalent if they belong to Bose–Mesner algebras of non-isomorphic association schemes.

Note that there are two type-II matrices described in Theorem 2(i) since $w_1 = w_2 = w_3$ is any one of the two zeros of a quadratic equation. Similarly, there are two type-II matrices in each of (ii), (v) in Theorem 2 Moreover, there are four type-II matrices in (vi), since $a_{0,1}^2 - 4 \neq 0$. The following lemma shows that the two type-II matrices in Theorem 2(i) are inequivalent, and so are those in Theorem 2(ii).

**Lemma 12.** Let $W$ and $W'$ be type-II matrices belonging to the Bose–Mesner algebra of an association scheme $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$. Suppose $W$ and $W'$ have $d + 1$ distinct entries, the valencies of $\mathcal{X}$ are pairwise distinct, and $\min\{p_{i1}^i \mid 0 < i \leq d\} > \frac{|X|}{2}$. If $W$ and $W'$ are type-II equivalent, then $W$ is a scalar multiple of $W'$. 
Proof. Suppose that $W$ and $W'$ are type-II equivalent. Then there exist diagonal matrices $D = \text{diag}(d_1, \ldots, d_n)$, $D' = \text{diag}(d'_1, \ldots, d'_n)$ and permutation matrices $T = (\delta_{\tau(i),j})$, $T' = (\delta_{\tau'(i),j})$ such that

$$DWD' = TWT'^{-1}$$

(48)

holds. Let

$$W = \sum_{i=0}^{d} w_i A_i,$$

$$W' = \sum_{i=0}^{d} w'_i A_i.$$  

For $x \in X$, define

$$R_1(x) = \{ y \in X \mid (x, y) \in R_1 \}.$$  

Let $x_1, x_2 \in X$ be distinct. Then $|R_1(x_1) \cap R_1(x_2)| > \frac{|X|}{2}$ by the assumption. Similarly, $|R_1(\tau(x_1)) \cap R_1(\tau(x_2))| > \frac{|X|}{2}$, hence $|\tau^{-1}(R_1(\tau(x_1)) \cap R_1(\tau(x_2)))| > \frac{|X|}{2}$. Therefore, there exists

$$y \in R_1(x_1) \cap R_1(x_2) \cap \tau^{-1}(R_1(\tau(x_1)) \cap R_1(\tau(x_2))).$$

Equivalently, $y$ satisfies

$$(x_1, y), (x_2, y), (\tau(x_1), \tau'(y)), (\tau(x_2), \tau'(y)) \in R_1.$$  

Comparing $(x_1, y)$- and $(x_2, y)$-entries of (48), we find

$$d_{x_1} w_1 d'_{y} = d_{x_2} w_1 d'_{y} = w'_1.$$  

This implies $d_{x_1} = d_{x_2}$. Since $x_1, x_2 \in X$ were arbitrary, $D = dI$ for some $d$. Similarly, $D' = d'I$ for some $d'$. We now have

$$dd' W = TW'T'^{-1}.$$  

Since $W$ has $d + 1$ distinct entries, we have

$$|X|k_i = |\{(x, y) \in X \times X \mid (TW'T'^{-1})_{x,y} = dd'w_i\}|$$

$$= |\{(x, y) \in X \times X \mid W'_{\tau(x),\tau'(y)} = dd'w_i\}|$$

$$= |\{(x, y) \in X \times X \mid W'_{x,y} = dd'w_i\}|.$$  

Since $W'$ has $d + 1$ distinct entries and the valencies are pairwise distinct, we obtain $dd'w_i = w'_i$. Hence $dd' W = W'$. □

For a matrix $W$ with nonzero complex entries, we denote its entry-wise inverse by $W^{(-)}$.  

Proposition 3. Let $W$ be a type-II matrix given in (i) of Theorem 2. Then $W$ and $W^{(-)}$ are inequivalent. The same conclusion holds if $W$ is a type-II matrix given in (ii) of Theorem 2.

Proof. First, let $W$ be a type-II matrix given in (i) of Theorem 2. Then the matrices $W$ and $W^{(-)}$ belong to the Bose–Mesner algebra of the association scheme $(X, \{R_0, R_1 \cup R_2 \cup R_3\})$. Since $W$ has two distinct entries and this association scheme satisfies the hypothesis of Lemma 12, $W$ and $W^{(-)}$ are inequivalent.

If $W$ is a type-II matrix given in (ii) of Theorem 2, then $W$ and $W^{(-)}$ belong to the Bose–Mesner algebra of the association scheme $(X, \{R_0, R_1 \cup R_2, R_3\})$. Since $W$ has three distinct entries and this association scheme satisfies the hypothesis of Lemma 12, $W$ and $W^{(-)}$ are inequivalent. □

We do not know whether the two type-II matrices in each of (iii)–(v) in Theorem 2 are equivalent or not, and whether the two type-II matrices in Theorem 2(vi) with a given sign for $r$ are equivalent or not.

6. Nomura Algebras

Since $q^2 - 1$ is a composite, there are uncountably many inequivalent complex Hadamard matrices of order $q^2 - 1$ by [7]. Indeed, such matrices can be constructed using generalized tensor product [11]. We show that none of our complex Hadamard matrices is equivalent to a generalized tensor product. This is done by showing that the Nomura algebra of any of our complex Hadamard matrices has dimension 2. According to [11], the Nomura algebra of the generalized tensor product of type-II matrices is imprimitive, and this is never the case when it has dimension 2.

For a type-II matrix $W \in M_X(\mathbb{C})$ and $a, b \in X$, we define column vectors $Y_{ab}$ by setting

$$(Y_{ab})_x = \frac{W_{xa}}{W_{xb}} \quad (x \in X).$$

The Nomura algebra $N(W)$ of $W$ is the algebra of matrices in $M_n(\mathbb{C})$ such that $Y_{ab}$ is an eigenvector for all $a, b \in X$. It is shown in [12, Theorem 1] that the Nomura algebra is a Bose–Mesner algebra. We first show that the Nomura algebra consists of symmetric matrices.

Throughout this section, we let $N$ to denote the Nomura algebra of any of the type-II matrices given in Theorem 2.

Lemma 13. The algebra $N$ is symmetric.
Proof. Suppose that $N$ is not symmetric. Then by [12, Proposition 6(i)], there exists $(b, c) \in X^2$ with $b \neq c$ such that
\[
\sum_{x \in X} \frac{W^2_{x,b}}{W^2_{x,c}} = 0
\]
This is equivalent to
\[
\sum_{j,k} p_{jk}^i \frac{w^2_j}{w_k^2} = 0
\]
for some $i \in \{1, 2, 3\}$. Using the notation (10), we have
\[
\sum_{j,k} p_{jk}^i \frac{w^2_j}{w_k^2} = \sum_{j<k} p_{jk}^i \left( \frac{w_j^2}{w_k} + \frac{w_k^2}{w_j} \right) + \sum_{j=0}^3 p_{jj}^i
\]
\[
= \sum_{j<k} p_{jk}^i \left( \left( \frac{w_j}{w_k} + \frac{w_k}{w_j} \right)^2 - 2 \right) + \sum_{j=0}^3 p_{jj}^i
\]
\[
= \sum_{j<k} p_{jk}^i (a_{j,k}^2 - 2) + \sum_{j=0}^3 p_{jj}^i.
\]
(49)
It can be verified by computer that (49) is nonzero for each of the cases (i)--(vi) in Theorem 2. \hfill \Box

Since $N$ is symmetric, the adjacency matrices of $N$ are the $(0, 1)$-matrices representing the connected components of the Jones graph defined as follows (see [12, Sect. 3.3]). The Jones graph of a type-II matrix $W \in M_X(\mathbb{C})$ is the graph with vertex set $X^2$ such that two distinct vertices $(a, b)$ and $(c, d)$ are adjacent whenever $\langle Y_{ab}, Y_{cd} \rangle \neq 0$, where $\langle , \rangle \neq 0$ denotes the ordinary (not hermitian) scalar product.

Proposition 4. The algebra $N$ coincides with the linear span of $I$ and $J$. In particular, the none of type-II matrices given in Theorem 2 is equivalent to a nontrivial generalized tensor product.

Proof. We aim to show that $\{(x, y) \mid x, y \in X, x \neq y\}$ is a connected component of the Jones graph. Fix $x, y, z$ with $(x, y), (y, z), (z, x) \in R_3$. Then in the Jones graph, $(x, y)$ and $(y, x)$ belong to the same connected component, and $(x, z)$ and $(z, x)$ belong to the same connected component, by Lemma 13.

We claim that $(x, y)$ and $(x, z)$ belong to the same connected component. Indeed, if $(x, y)$ and $(x, z)$ belong to different connected components, then $(y, x)$ and $(z, x)$ belong to different connected components. In particular,
\[
\langle Y_{xy}, Y_{xz} \rangle = \langle Y_{yx}, Y_{zx} \rangle = 0.
\]
Let
\[ c_{i,j,k} = |\{ u \in X \mid (x, u) \in R_i, (y, u) \in R_j, (z, u) \in R_k \}|. \]

Then for \( j, k \in \{1, 2\} \),
\[ c_{1,j,k} + c_{2,j,k} = c_{j,1,k} + c_{j,2,k} = c_{j,k,1} + c_{j,k,2} = p_{j,k}^3, \]
and
\[ \sum_{i,j,k=0}^{3} c_{i,j,k} \frac{w_i^2}{w_j w_k} = \sum_{i,j,k=0}^{3} c_{i,j,k} \frac{w_j w_k}{w_i^2} = 0. \]

It can be verified by computer that these conditions give rise to a polynomial equation in \( q \) which has no solution in even positive integers. Therefore, we have proved the claim. This implies that, for each equivalence class \( C \) of the equivalence relation \( R_0 \cup R_3 \), \((C \times C) \cap R_3\) is a clique in the Jones graph.

For \((x, z) \in R_1\), suppose \( \langle Y_{xy}, Y_{xz} \rangle = 0 \) for all \( y \in R_3(x) \). Then
\[
0 = \sum_{y \in R_3(x)} \langle Y_{xy}, Y_{xz} \rangle \\
= \sum_{y \in R_3(x)} \sum_{u \in X} \langle Y_{xy} \rangle_u \langle Y_{xz} \rangle_u \\
= \sum_{y \in R_3(x)} \sum_{u \in X} \frac{W_{zu}^2}{W_{yu} W_{zu}} \\
= \sum_{y \in R_3(x)} \sum_{i,j=0}^{3} \sum_{u \in R_i(x) \cap R_j(z)} \sum_{k=0}^{3} \frac{w_i^2}{w_k w_j} \\
= \sum_{i,j=0}^{3} \sum_{u \in R_i(x) \cap R_j(z)} \sum_{k=0}^{3} p_{3k} \frac{w_i^2}{w_k w_j} \\
= \sum_{i,j,k=0}^{3} p_{ij}^1 p_{3k} \frac{w_i^2}{w_k w_j}.
\]

It can be verified by computer that this leads to a polynomial equation in \( q \) which has no solution in even positive integers.
Similarly, for \((x, z) \in R_2\), suppose \(\langle Y_{xy}, Y_{xz} \rangle = 0\) for all \(y \in R_3(x)\). Then
\[
\sum_{i,j,k=0}^{3} p_i^2 p_j p_k w_i^2 w_j w_k = 0,
\]
and again this leads to a contradiction.

Therefore, for any \((x, z) \in R_1 \cup R_2\), there exists \(y \in R_3(x)\) such that \(\langle Y_{xy}, Y_{xz} \rangle \neq 0\). Since \(R_1 \cup R_2\) is connected, we see that \(R_1 \cup R_2 \cup R_3\) is a connected component of the Jones graph. Therefore, \(\dim N = 2\). □

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Appendix A. Verification by Magma

Proof of Lemma 1.
\[ F < X, Y, Z > := \text{FunctionField}(\text{Rationals}(), 3); \]
\[ g := X^2 + Y^2 + Z^2 - X*Y*Z - 4; \]
\[ g \circ \text{hom} < F \rightarrow F | X/Y + Y/X, X/Z + Z/X, Z/Y + Y/Z > \text{ eq 0}; \]

Proof of Lemma 2.
\[ Q := \text{Rationals}(); \]
\[ F4 < X, Y, Z, z > := \text{FunctionField}(Q, 4); \]
\[ f := z^2 - z*X + 1; \]
\[ g := X^2 + Y^2 + Z^2 - X*Y*Z - 4; \]
\[ c1 := (z^2 - 1)*(z*X - 2) + (z^2 - 1)*(z*Y - 2); \]
\[ c2 := (z^2 - 1)*(z*X - 2) + (z^2 - 1)*(z*Z - 2); \]
\[ w := (z^2 - 1)/(z*Z - 2); \]
\[ w + 1/w \text{ eq } Y + (z^2 - 2)*g + c1*f/(z^2 - 1)*g + c2*f/(z^2 - 1)*g + c3*f/(z^2 - 1)*g; \]
\[ z/w + w/z \text{ eq } Z + (z^2 - 2)*g + c1*f/(z^2 - 1)*g + c2*f/(z^2 - 1)*g + c3*f/(z^2 - 1)*g; \]
\[ w*w \text{ eq } 1 + (z^2 - 2)*g + c1*f/(z^2 - 1)*g + c2*f/(z^2 - 1)*g + c3*f/(z^2 - 1)*g; \]

Proof of Lemma 3.
\[ F < X0, X1, X2, X3 > := \text{FunctionField}(\text{Rationals}(), 4); \]
\[ x01 := X0/X1 + X1/X0; \]
\[ x02 := X0/X2 + X2/X0; \]
\[ x03 := X0/X3 + X3/X0; \]
\[ x12 := X1/X2 + X2/X1; \]
\[ x13 := X1/X3 + X3/X1; \]
\[ x23 := X2/X3 + X3/X2; \]
\[ (x03^2 - 4)*x12 - x03*(x01*x23 + x02*x13) + 2*(x01*x2 + x02*x13) \text{ eq 0}; \]
\[ \text{Determinant}(\text{Matrix}(F, 3, 3, [[2, x01, x02, x01, 2, x12, x03, x13, x23]]) \text{ eq 0}; \]

Proof of Lemma 4.
\[ R < w0, w1, a01, a0j, a1j > := \text{PolynomialRing}(\text{Rationals}(), 5); \]
\[ F := \text{FieldOfFractions}(R); \]
\[ I := \text{ideal}(R | w0^2 + w1^2 - w0*w1*a01, a01^2 + a0j^2 + a1j^2 - a01*a0j*a1j - 4); \]
\[ x := (w1^2 - w0^2)/(a1j*w1 - a0j*w0); \]
\[ \text{num} := \text{Numerator}(x^2 - w0*a0j*x + w0^2); \]
\[ \text{num in I}; \]

Proof of Lemma 5.
\[ F < X1, X2, X3 > := \text{FunctionField}(\text{Rationals}(), 3); \]
\[ x01 := 1/X1 + X1; \]
\[ x02 := 1/X2 + X2; \]
Proof of Theorem 2.

d := 3;

d2s := &cat [[i, j] : j in [i+1..d]] : i in [0..d-1]];
d2 := [seqset(s) : s in d2s];
R := PolynomialRing(Rationals(), #d2+2);
q := R.(#d2+2);
r := R.(#d2+1);
X := func<i, j | R.Position(d2, {i, j})>;
g := func<i, j, k | X(i, j)^2 + X(i, k)^2 + X(j, k)^2 - X(i, j)*X(i, k)*X(j, k) - 4>;
h := func<i, j, k, l | (X(k, l)^2 - 4)*X(i, j) - X(k, l)*(X(k, i)*X(l, j) + X(k, j)*X(l, i)) + 2*(X(k, i)*X(k, j) + X(l, i)*X(l, j))>;

eigenP := Matrix(R, 4, 4, [[1, 1/2*q^2 - q, 1/2*q^2, q - 2],
                          [1, 1/2*q, -1/2*q, -1],
                          [1, -1/2*q + 1, -1/2*q, q - 2],
                          [1, -1/2*q, 1/2*q, -1]]);
P := func<i, j | eigenP[i+1, j+1]>

n := &+[P(0, i) : i in [0..d]];
n eq q^2 - 1;

e := func<i | -n + &+[P(i, j)^2 : j in [0..d]] + &+[P(i, j[1])*P(i, j[2]) : X(j[1], j[2]) : j in d2s]>;
s3 := [seqset(x) : x in Subsets([0..d], 3)];
eq7 := [g(i[1], [2], i[3]) : i in s3] cat
[h(0^i, 1^i, 2^i, 3^i) : i in Sym([0..d])] cat
[e(i) : i in [1..d]] cat
[r^2 - (17*q - 1)*(q - 1)];
I := ideal<R|eq7>;

b1 := X(1, 2) - 2;
b2 := (q^2 - 4)*X(1, 2) + 2*q^2 - 12;
b3 := q^2*X(1, 2) + 2*q^2 - 4;
a12D := q^2*(q + 1)*(q - 3);
\begin{verbatim}
a12Nplus := 2*(-q^4+2*q^3+4*q^2-10*q+1 + (q-1)*r); a12Nminus := 2*(-q^4+2*q^3+4*q^2-10*q+1 - (q-1)*r); b4plus := a12D*X(1,2) - a12Nplus; b4minus := a12D*X(1,2) - a12Nminus; q*(q-1)*b1*b2*b3*b4plus*b4minus in I;
I1 := ideal<R|I,b1>;
c1 := X(0,1)+n-2;
a01D := q^2-2*q-1;
a01N := q^3-3*q^2-q+7;
c2 := a01D*X(0,1)-a01N;
X(0,3)+n-2 in I1;
(q-1)^2*c1*c2 in I1;
I11 := ideal<R|I1,c1>;
(q-1)*(q+1)*(q-2)*(q^2-5)*(X(1,3)-2) in I11;
I12 := ideal<R|I1,c2>;
a13D := q^2-2*q-1;
a13N := -(q^3-q^2-q-3);
(q-1)*(q+1)*(q-2)*(q^2-5)*(a13D*X(1,3)-a13N) in I12;
a01 := a01N/a01D;
a03 := -(n-2);
a13 := a13N/a13D;
FA<w3> := PolynomialRing(FieldOfFractions(R));
FA3 := FA/ideal<FA|w3^2+1-a03*w3>;
QA3 := FieldOfFractions(FA3);
w1 := QA3!((w3^2-1)/(a13*w3-a01));
w1 eq QA3!((-q-3)*w3+(q-1))/(q^2-2*q-1);
I2 := ideal<R|I,b2>;
q*(q-1)*(q+1)*(q-3)*(n-4)*(X(0,2)+2) in I2;
q*(q-1)*(q+1)*(q-3)*(n-4)*(X(1,3)-2) in I2;
q*(q-1)*(q+1)*(q-3)*(n-4)*((n-3)*X(0,1)-2*(n-5)) in I2;
I3 := ideal<R|I,b3>;
c4 := q^2*X(0,2)+2*(n-1);
c5 := q*X(0,2)+2;
X(0,3)-2 in I3;
(q-1)^2*c4*c5 in I3;
I31 := ideal<R|I3,c4>;
(q-1)*(q+1)*(q-2)*(X(0,1)-2) in I31;
I32 := ideal<R|I3,c5>;
(q-1)*(q+1)*(q-2)*(q*X(0,1)+2) in I32;
q^2*X(1,2)+2*(n-1) in I32;
\end{verbatim}
a02:=-2/q;
a01:=-2/q;
a12:=-2*(n-1)/q^2;
a01*a02-a12 eq 2;
I4:=ideal<R|I,b4plus>;
a01D:=2*q*(q+1);
a01N:=-(q-1)*(q-2)+(q+2)*r;
a01:=a01D*X(0,1)-a01N;
a02D:=2*q*(q-3);
a02N:=(q+2)*(q-1)-(q-2)*r;
a02:=a02D*X(0,2)-a02N;
a03D:=2*(q+1)*(q-3);
a03N:=5*q^2-2*q-19-(q-1)*r;
a03:=a03D*X(0,3)-a03N;
a12D:=q^2*(q+1)*(q-3);
a12N:=-q^4+8*q^3-4*q^2-10*q+1+(q-1)*r;
a12:=a12D*X(1,2)-a12N;
a13D:=a02D;
a13N:=-a02N;
a13:=a13D*X(1,3)-a13N;
a23D:=a01D;
a23N:=-a01N;
a23:=a23D*X(2,3)-a23N;
q*(q+1)*(q-1)^2*(q-3)^2*(q^2-5)*a01 in I4;
q*(q+1)*(q-1)^2*(q-3)^2*(q^2-5)*a02 in I4;
a03 in I4;
a12 in I4;
q*(q+1)^2*(q-1)^2*(q-3)^2*(q^2-5)*a13 in I4;
q*(q+1)^2*(q-1)^2*(q-3)^2*(q^2-5)*a23 in I4;
q^2*(q+1)*(q-1)^2*(q-3)^2*(q^2-5)*(X(0,1)*X(0,2)+X(0,3)) in I4;
I41:=ideal<R|I4,X(0,1)*X(0,2)*X(0,3)*X(1,2)>
q*(q-1)^2*(q^2-5)^2 in I41;

Conversely,
Rqr<rr,qq>:=PolynomialRing(Rationals(),2);
F:=FieldOfFractions(Rqr/ideal<Rqr|rr^2-(17*qq-1)*(qq-1)>);
d:=3;
d2s:=&cat[[[i,j]:j in [i+1..d]]:i in [0..d-1]];
d2:=[Seqset(s):s in d2s];
q:=F!qq;
r:=F!rr;
R := PolynomialRing(F, #d2);
X := func<i,j|R.Position(d2, {i, j})>;
g := func<i,j,k|X(i, j)^2 + X(i, k)^2 + X(j, k)^2 - X(i, j)*X(i, k)*X(j, k) - 4>;
h := func<i,j,k,l|(X(k, l)^2 - 4)*X(i, j) - X(k, l)*(X(k, i)*X(l, j) + X(k, j)*X(l, i)) + 2*(X(k, i)*X(k, j) + X(l, i)*X(l, j))>;
eigenP := Matrix(F, 4, 4,[
1, 1/2*q^2 - q, 1/2*q^2, q - 2,
1, 1/2*q, -1/2*q, -1,
1, -1/2*q + 1, -1/2*q, q - 2,
1, -1/2*q, 1/2*q, -1]);
P := func<i,j|eigenP[i+1,j+1]>;
n := &+[P(0, i): i in [0..d]]
  n eq q^2 - 1;
e := func<i|-n + &+[P(i, j)^2:j in [0..d]]
  + &+[P(i, j[1])*P(i, j[2])*X(j[1], j[2]):j in d2s]>;
s3 := [Setseq(x): x in Subsets({0..d}, 3)];
eq7rq := [g(i[1], i[2], i[3]): i in s3] cat
  [h(0^i, 1^i, 2^i, 3^i): i in Sym({0..d})] cat
  [e(i): i in [1..d]]
  subs1 := [-n+2, -n+2, -n+2, 2, 2, 2, 2];
&and[Evaluate(f, subs1) eq 0: f in eq7rq];
a := -(q-3)/(q^2 - 2*q - 1);
b := (q-1)/(q^2 - 2*q - 1);
a^2 + a*b*(-(n-2)) + b^2 eq 1;
a01 := (q^3 - 3*q^2 - q + 7)/(q^2 - 2*q - 1);
a*(-(n-2)) + 2*b eq a01;
a13 := (-q^3 + q^2 + q + 3)/(q^2 - 2*q - 1);
2*a - b*(n-2) eq a13;
subs2 := [a01, a01, -(n-2), 2, a13, a13];
&and[Evaluate(f, subs2) eq 0: f in eq7rq];
a01_iii := 2*(n-5)/(n-3);
subs3 := [a01_iii, -2, a01_iii, -a01_iii, 2, -a01_iii];
&and[Evaluate(f, subs3) eq 0: f in eq7rq];
a02_iv := -2*(n-1)/(n+1);
subs4 := [2, a02_iv, 2, a02_iv, 2, a02_iv];
&and[ Evaluate(f, subs4) eq 0: f in eq7rq];
subs5 := [-2/q, -2/q, -2*(n-1)/(n+1), -2/q, -2/q];
&and[ Evaluate(f, subs5) eq 0: f in eq7rq];
a01D := 2*q*(q+1);
a01N := -(q-1)*(q-2)+(q+2)*r;
a01 := a01N/a01D;
a02D := 2*q*(q-3);
a02N := (q+2)*(q-1)-(q-2)*r;
a02 := a02N/a02D;
a03D := 2*(q+1)*(q-3);
a03N := 5*q^2-2*q-19-(q-1)*r;
a03 := a03N/a03D;
a12D := q^2*(q+1)*(q-3);
a12N := 2*(-q^4+2*q^3+4*q^2-10*q+1+(q-1)*r);
a12 := a12N/a12D;
a13D := a02D;
a13N := -a02N;
a13 := a13N/a13D;
a23D := a01D;
a23N := -a01N;
a23 := a23N/a23D;
subs6 := [a01, a02, a03, a12, a13, a23];
&and[Evaluate(f, subs6) eq 0: f in eq7rq];

Proof of Corollary 1.
a01-2 eq -8*q*(q+1)*(q-3)^2/(2*q*(q+1)*(5*q^2+q+2+(q+2)*r));
a01+2 eq -8*q*(q+1)*(q^2-5)/(2*q*(q+1)*(3*q^2+7*q-2-(q+2)*r));

Isolation.
n := 15;
A0 := ScalarMatrix(n, 1);
J := Parent(A0)! [1:i in [1..n^2]];
L03 := LineGraph(OddGraph(3));
A1 := AdjacencyMatrix(L03);
A2 := A1^2 - A1 - 4*A0;
A3 := J - A0 - A1 - A2;
DM := DistanceMatrix(L03);
DM eq A1 + 2*A2 + 3*A3;

hermitianConjugate :=
func < H | Parent(H)! [ComplexConjugate(x): x in Eltseq(Transpose(H))]>;

complexHadamard := function(xyz)
    AA := [ChangeRing(A, Parent(xyz[1])): A in [A1, A2, A3]];
    return A0 + xyz[1]*AA[1] + xyz[2]*AA[2] + xyz[3]*AA[3];
end function;
spanCondition:=function(H)
F:=Parent(H[1,1]);
MnF:=Parent(H);
n:=Nrows(H);
Es:=[MnF|0:i in [1..n]];
for i in [1..n] do
  Es[i][i,i]:=1;
end for;
EsF:=[MnF|e:e in Es];
Hs:=hermitianConjugate(H);
Vn:=VectorSpace(F,n^2);
bracket:=sub<Vn|[Vn|Eltseq(v*Hs*w*H-Hs*w*H*v):v,w in EsF]>;
return Dimension(bracket) eq n^2-2*n+1;
end function;

F<s>:=QuadraticField(-15);
y:=(-7+s)/8;
H:=complexHadamard([1,y,1]);
H*hermitianConjugate(H) eq n*A0;
spanCondition(H);

F<s>:=QuadraticField(-11);
x:=(5+s)/6;
H:=complexHadamard([x,-1,x]);
H*hermitianConjugate(H) eq n*A0;
not spanCondition(H);

F<s>:=QuadraticField(-15);
x:=(-1+s)/4;
H:=complexHadamard([x,x^(-1),1]);
H*hermitianConjugate(H) eq n*A0;
not spanCondition(H);

F<s>:=QuadraticField(201);
Z:=(53-3*s)/10;
R<T>:=PolynomialRing(F);
K<z>:=ext<F|T^2-Z*T+1>;
z+1/z eq Z;
x:=1/144*((-5*Z+31)*z-25*Z+155);
xb:=1/144*((-5*Z+31)*z^(-1)-25*Z+155);
y:=1/144*((25*Z-155)*z+5*Z-31);
yb:=1/144*((25*Z-155)*z^(-1)+5*Z-31);
x*xb eq 1;
y*yb eq 1;
H:=complexHadamard([x,y,z]);
H*hermitianConjugate(H) eq n*A0;
spanCondition(H);

Remark 1.

mustbe0:=func<a01v|(2*q*(q+1)*a01v+(q-1)*(q-2))^2-((q+2)*r)^2>;
mustbe0(0) eq -8*q*(q+1)*(q-1)*(2*q+7);
mustbe0(1) eq -8*q*(q+1)*(q^2+6*q-8);
mustbe0(-1) eq -8*q*(q+1)*(2*q^2+3*q-6);
mustbe0(2) eq 8*q*(q+1)*(q-3)^2;
mustbe0(-2) eq -8*q*(q+1)*(q^2-5);

Table 1.

HWminus1:=function(w)
I3:=\{1..3\};
H3q:=\{w[i1]*w[i2]/w[i3]:i1,i2,i3 in I3
|#\{i:i in \{i1,i2,i3\}|i eq 3\} ne 2\};
H3q4:=\{w[i1]*w[i2]/w[i3]:i1,i2,i3 in I3
|#\{i:i in \{i1,i2,i3\}|i eq 3\} ne 2 and \{i1,i2,i3\} ne \{1,3\}\};
plus:=\{w[i]^2:i in I3\} join H:H in [H3q,H3q4];
return \{p join \{x^(-1):x in p\} join
{w[i1]*w[i2]/(w[j1]*w[j2]):i1,i2,j1,j2 in I3}\}
diff \{1\}:p in plus};
end function;

Rw<w1,w2,w3>:=FunctionField(Rationals(),3);
HWminus1([w1,w1,w1]) eq
\{&join\{w^s,w^(s*2):s in \{1,-1\}\};
HWminus1([w1,w1,w3]) eq
\{&join\{w^s,w^(-s*2):s in \{1,-1\},w in \{w1,w3\}\} join
&join\{\{(w^2/w3)^s,(w3/w1)^s,(w3/w1)^(-s*2)\}:s in \{1,-1\}\};
HWminus1([w1,-1,w1]) eq
\{-1\} join
&join\{s1*w1^s,s1*w1^(-s*2):s1 in \{1,-1\}\};
HWminus1([1,w2,1]) eq
\{&join\{w^s,w^(-s*2):s in \{1,-1\}\};
HWminus1([w1,w1,-1]) eq
\{\{(w^2*w2)^s,\{(w^2*w2)^s,\{(w^2*w2)^s,\{(w^2*w2)^s\}:s in \{1,-1\},k in \{1..4\}\};
HWminus1([w1,-1,-1]) eq
\{-1\} join
\{s0*w^s:k in \{1,2\}\} join
\{\{s0*w1^s1*w1^s2,(w1^s1*w1^s2)^2:s0,s1,s2 in \{1,-1\}\}
join \{s0*(w1^2*w2^(-1))^s,s0*(w1^(-1)*w2^2)^s\}
\{s,s0 in \{1,-1\}\};


// (i)
Rq<q>:=FunctionField(Rationals());
(-q^2+3)^2-2 eq q^4-6*q^2+7;
// (ii)
Rw3<w3>:=FunctionField(Rq);
A:=-(q-3)/(q^2-2*q-1);
B:=(q-1)/(q^2-2*q-1);
(A^2+B^2-1)/(A*B) eq q^2-3;
w1:=A*w3+B;
(A/w3+B)-1/w1 eq 1/w1*A*B*(w3+1/w3+(q^2-3));
// (iii)
(2*(q^2-6)/(q^2-4))^2-2 eq 2*(q^4-16*q^2+56)/(q^2-4)^2;
// (iv)
(-2*(q^2-2)/q^2)^2-2 eq 2*(q^4-8*q^2+8)/q^4;

Proof of Proposition 1. (iii)̸∼(vi)
Rq<q>:=FunctionField(Rationals());
k3a:=(q^2-6)/(q^2-4);
k3b:=2*(q^4-16*q^2+56)/(q^2-4)^2;
ra1:=(k3a+(q-1)*(q-2)/(2*q*(q+1)))/(q+2);
fac:=Factorization(Numerator(ra1^2-(17*q-1)*(q-1)));
#fac eq 1 and Degree(fac[1][1]) gt 1;
ra2:=(-k3a+(q-1)*(q-2)/(2*q*(q+1)))/(q+2);
fac:=Factorization(Numerator(ra2^2-(17*q-1)*(q-1)));
#fac eq 1 and Degree(fac[1][1]) gt 1;
rb1:=(k3b+(q-1)*(q-2)/(2*q*(q+1)))/(q+2);
fac:=Factorization(Numerator(rb1^2-(17*q-1)*(q-1)));
#fac eq 1 and Degree(fac[1][1]) gt 1;
rb2:=(-k3b+(q-1)*(q-2)/(2*q*(q+1)))/(q+2);
fac:=Factorization(Numerator(rb2^2-(17*q-1)*(q-1)));
#fac eq 1 and Degree(fac[1][1]) gt 1;

(i)̸∼(ii)
n:=q^2-1;
Numerator(((2-n)-(q^3-3*q^2-q+7)/(q^2-2*q-1))
  eq -(q-2)*(q+1)*(q^2-5);
Numerator((n^2-4*n+2)-(q^3-3*q^2-q+7)/(q^2-2*q-1))
  eq (q-2)*(q+1)*2*(q^3-2*q^2-4*q+7);
(iv)̸∼(v)
Numerator(-2*(n-1)/(n+1)-(-2)/q)
  eq -2*(q-2)*(q+1);
Numerator(2*(n^2-6*n+1)/(n+1)^2-(-2)/q)
eq 2*(q-2)*(q+1)*(q^2+2*q-4);

**Proof of Proposition 2**
ra1:=(2+(q-1)*(q-2)/(2*q*(q+1)))/(q+2);
fac:=Factorization(Numerator(ra1^2-(17*q-1)*(q-1)));
# fac eq 1 and Degree(fac[1][1]) gt 1;
ra2:=(-2+(q-1)*(q-2)/(2*q*(q+1)))/(q+2);
fac:=Factorization(Numerator(ra2^2-(17*q-1)*(q-1)));
# fac eq 1 and Degree(fac[1][1]) gt 1;

**Proof of Lemma 13**
R<r,q>:=PolynomialRing(Rationals(),2);
F:=FieldOfFractions(R);
n:=q^2-1;
fr:=r^2-(17*q-1)*(q-1);
B1:=Matrix(R,4,4,[
  0,1,0,0,
  q^2/2-q,(q-2)^2/4,(q-2)^2/4,q*(q-4)/4,
  0,q*(q-2)/4,q*(q-2)/4,q^2/4,
  0,(q-4)/2,(q-2)/2,0]);
B2:=Matrix(R,4,4,[
  0,0,1,0,
  0,q*(q-2)/4,q*(q-2)/4,q^2/4,
  q^2/2,q^2/4,q^2/4,q^2/4,
  0,q/2,(q-2)/2,0]);
B3:=Matrix(R,4,4,[
  0,0,0,1,
  0,(q-4)/2,(q-2)/2,0,
  0,q/2,(q-2)/2,0,
  q-2,0,0,q-3]);
B0:=ScalarMatrix(4,R!1);
BB:=[B0,B1,B2,B3];
pijk:=func<i,j,k|BB[i+1][j+1,k+1]>;
isSymNbas:=function(ajk)
  aijs:=[[ajk[1],ajk[2],ajk[3]],
         [1,ajk[4],ajk[5]],[1,1,ajk[6]]];
  aij:=func<i,j|aijs[i+1,j]>;
  ff:=[F|&+[pijk(j,k,i)*(aij(j,k)^2-2):j,k in [0..3]|j lt k]
\[ \{ \pi_{jk}(j,j,i) : j \in [0..3], i \in [1..3] \} \]
\[ \text{II} := \{ \text{ideal} \langle R \mid \text{fr, Numerator}(ff[k]) \rangle : k \in [1..3] \} \]
\[ \text{return } \{ \text{Basis}(\text{EliminationIdeal}(\text{II}[k], \{q\}) : k \in [1..3]) \} \]
end function;

// case (i)
subs1 := \{-n+2, -n+2, -n+2, 2, 2, 2\};
bass := \text{isSymNbas}(\text{subs1});
[\text{Factorization}(\text{bass}[i][1]) : i \in [1..3]] \text{ eq }
[\langle q-2, 1, 1, q+1, 1, q+2, 1 \rangle : i \in [1..3]];

// case (ii)
a01 := (q^3-3*q^2-q+7)/(q^2-2*q-1);
a13 := (-q^3+q^2+q+3)/(q^2-2*q-1);
subs2 := \{a01, a01, -(n-2), 2, a13, a13\};
bass := \text{isSymNbas}(\text{subs2});
b1 := q^4-10*q^2+4*q+17;
fac := [\langle q-2, 1, 1, q-1, 1, q+1, 1, b1, 1 \rangle],
[\langle q-2, 1, 1, q-1, 1, q+1, 1, b1, 1 \rangle],
[\langle q-2, 1, 1, q-1, 1, q+1, 1, q+2, 1 \rangle];
[\text{Factorization}(\text{bass}[i][1]) : i \in [1..3]] \text{ eq fac};

// case (iii)
a01_iii := 2*(n-5)/(n-3);
subs3 := \{a01_iii, -2, a01_iii, -a01_iii, 2, -a01_iii\};
bass := \text{isSymNbas}(\text{subs3});
b1 := q^6-13*q^4+28*q^2+64;
b2 := q^4-9*q^2+24;
[\text{Factorization}(\text{bass}[i][1]) : i \in [1..3]] \text{ eq }
[\langle b1, 1 \rangle, \langle b2, 1 \rangle, \langle b1, 1 \rangle];

// case (iv)
a02_iv := -2*(n-1)/(n+1);
subs4 := \{2, a02_iv, 2, a02_iv, 2, a02_iv\};
bass := \text{isSymNbas}(\text{subs4});
bass \text{ eq } [\langle (q-2)*(q-1)*(q+1)*(q+2) \rangle : i \in [1..3]];

// case (v)
subs5 := \{-2/q, -2/q, -2*(n-1)/(n+1), -2/q, -2/q\};
bass := \text{isSymNbas}(\text{subs5});
b1 := (q-2)*(q-1)*(q+1)*(q^2-2*q-4);
b3 := (q-2)*(q-1)*(q+1)*(q+2);
bass eq \[\text{[[b1],[b1],[b3]]};

// case (vi)
a01D:=2*q*(q+1);
a01N:=-(q-1)*(q-2)+(q+2)*r;
a01:=a01N/a01D;
a02D:=2*q*(q-3);
a02N:=(q+2)*(q-1)-(q-2)*r;
a02:=a02N/a02D;
a03D:=2*(q+1)*(q-3);
a03N:=5*q^2-2q-19-(q-1)*r;
a03:=a03N/a03D;
a12D:=q^2*(q+1)*(q-3);
a12N:=-2*(-q^4+2q^3+4q^2-10q+1+(q-1)*r);
a12:=a12N/a12D;
a13:=-a02;
a23:=-a01;
bass:=isSymNbas([a01,a02,a03,a12,a13,a23]);
bass eq 
\[
[q^3*(q-3)^2*(q-1)*(q+1)^2*
(q^9-q^8-12*q^7+14*q^6+49*q^5
+51*q^4-q^3-464*q^2+4664*q^2-272)],
[q^3*(q-3)^2*(q-2)*(q-1)*(q+1)^2*
(q^7+3*q^6-4*q^5+2*q^4+57*q^3-q^2-86*q+92)],
[q^2*(q-3)^2*(q-1)*(q+1)^2*
(q^7-2*q^6+4*q^5+2*q^4+57*q^3-q^2-86*q+92)]
\]

Proof of Proposition 4.

R<
111, c112, c121, c122, c211, c212, c221, c222,
w1, w2case4, w3case2, r, q>
:= PolynomialRing(Rationals(), 13);
varname:=[[i, j, k]: i, j, k in [1, 2]];
c:=func<i, j, k | R.Position(varname, [i, j, k])>;
fr:=r^2-(17*q-1)*(q-1);
w0:=1;

B1:=Matrix(R, 4, 4, [
0, 1, 0, 0,
q^2/2-q, (q-2)^2/4, (q-2)^2/4, q*(q-4)/4,
0, q*(q-2)/4, q*(q-2)/4, q^2/4,
0, (q-4)/2, (q-2)/2, 0]);
B2 := Matrix(R, 4, 4, [0, 0, 1, 0, 0, q*(q-2)/4, q*(q-2)/4, q^2/4, q^2/2, q^2/4, q^2/4, q^2/4, 0, q/2, (q-2)/2, 0]);

B3 := Matrix(R, 4, 4, [0, 0, 0, 1, 0, (q-4)/2, (q-2)/2, 0, 0, q/2, (q-2)/2, 0, q-2, 0, 0, q-3]);

B0 := ScalarMatrix(4, R!1);
BB := [B0, B1, B2, B3];
pijk := func<i, j, k | BB[i+1][j+1, k+1]>

cijk := function(i, j, k)
    if 0 in {i, j, k} then
        if [i, j, k] in {[0, 3, 3], [3, 0, 3], [3, 3, 0]} then
            return 1;
        else
            return 0;
        end if;
    elseif 3 in {i, j, k} then
        if {3} eq {i, j, k} then
            return pijk(3, 3, 3)-1;
        else
            return 0;
        end if;
    else
        return c(i, j, k);
    end if;
end function;

fx := [cijk(1, j, k)+cijk(2, j, k)-pijk(j, k, 3): j, k in [1, 2]]
fy := [cijk(j, 1, k)+cijk(j, 2, k)-pijk(j, k, 3): j, k in [1, 2]]
fz := [cijk(j, k, 1)+cijk(j, k, 2)-pijk(j, k, 3): j, k in [1, 2]]
fxyz := fx cat fy cat fz;

test := function(fa, ww)
    alpha := func<i | ww[i+1]>
    ff := &+[cijk(i, j, k)*alpha(i)^2/(alpha(j)*alpha(k))
\[
\begin{align*}
    &i, j, k \in [0..3]; \\
    &gg := \sum \left[ c_{ijk}(i, j, k) \alpha(j) \alpha(k) / \alpha(i)^2 \right] \\
    &I := \text{ideal} < R | fr, fa, \text{Numerator}(ff), \text{Numerator}(gg) > \text{ cat fxyz}; \\
    &\text{return Basis(\text{EliminationIdeal}(I, \{q\}))}; \\
\end{align*}
\]

def test2 := function(fa, ww)
    \begin{align*}
    &alpha := \text{func} < i | ww[i+1] >; \\
    &t1 := \sum \left[ p_{ijk}(i, j, 1) p_{ijk}(3, k, i) \alpha(i)^2 / (\alpha(j) \alpha(k)) \right] \\
    &I1 := \text{ideal} < R | fr, fa, \text{Numerator}(t1) >; \\
    &I2 := \text{ideal} < R | fr, fa, \text{Numerator}(t2) >; \\
    &\text{return [Basis(\text{EliminationIdeal}(I1, \{q\})),} \\
        &\text{Basis(\text{EliminationIdeal}(I1, \{q\}))]};
    \end{align*}
end function;

// case (i)
fa1 := w1^2 + (q^2 - 3) * w1 + 1;
ww := [w0, w1, w1, w1];
test(fa1, ww) eq [(q+1)*(q-1)*(q^2-5)];
test2(fa1, ww) eq [(q-2)*(q+1)*(q-1)*(q^2-5)];
// case (ii)
fa3 := w3case2^2 + (q^2 - 3) * w3case2 + 1;
w2 := (-(q-3)*w3case2+q-1)/(q^2-2*q-1);
ww := [w0, w2, w2, w3case2];
test(fa3, ww) eq [(q+1)*(q-1)*(q^2-5)];
test2(fa3, ww) eq [(q-2)*(q+1)*(q-1)*(q^2-5)*(q^2-2*q-1)^2];
// case (iii)
fa1 := R!(q^2-4)*(w1^2-2*(q^2-6)/(q^2-4)*w1+1));
ww := [w0, w1, -1, w1];
test(fa1, ww) eq [(q^2+4)*(q^2-5)];
test2(fa1, ww) eq [(q-2)*(q+1)*(q^2-5)*(q^5-5*q^4+12*q^3-24*q^2+64)];
// case (iv)
fa2 := R!(q^-2*(w2case4^2+2*(q^-2-2)/q^-2*w2case4+1));
\( \text{ww} := [w_0, 1, w_2 \text{case} 4, 1] \);
\( \text{test}(fa_2, \text{ww}) \text{ eq } [q^2*(q+1)*(q-1)] \);
\( \text{test2}(fa_2, \text{ww}) \text{ eq } [[q^2*(q-2)*(q+1)*(q-1)] : i \text{ in } [1, 2]] \);

// case (v)
\( fa_1 := R!(q*(w_1^2+2/q*w_1+1)) \);
\( \text{ww} := [w_0, w_1, 1/w_1, 1] \);
\( \text{test}(fa_1, \text{ww}) \text{ eq } [q^2*(q+1)*(q-1)] \);
\( \text{test2}(fa_1, \text{ww}) \text{ eq } [[q*(q-2)*(q+1)*(q-1)] : i \text{ in } [1, 2]] \);

// case (vi)
\( a_{01D} := 2*q*(q+1) \);
\( a_{01N} := -(q-1)*(q-2)+(q+2)*r \);
\( a_{01} := a_{01N}/a_{01D} \);
\( a_{02D} := 2*q*(q-3) \);
\( a_{02N} := (q+2)*(q-1)-(q-2)*r \);
\( a_{02} := a_{02N}/a_{02D} \);
\( a_{03D} := 2*(q+1)*(q-3) \);
\( a_{03N} := 5*q^2-2*q-19-(q-1)*r \);
\( a_{03} := a_{03N}/a_{03D} \);
\( a_{12D} := q^2*(q+1)*(q-3) \);
\( a_{12N} := 2*(-q^4+2*q^3+4*q^2-10*q+1+(q-1)*r) \);
\( a_{12} := a_{12N}/a_{12D} \);
\( a_{13} := -a_{02} \);
\( a_{23} := -a_{01} \);
\( fa_1 := a_{01D}*w_1^2-a_{01N}*w_1+a_{01D} \);
\( w_2 := (a_{01}*w_1-a_{12})*w_1-a_{02}/a_{13} \);
\( w_3 := [w_0, w_1, w_2, w_3] \);
\( \text{test}(fa_1, \text{ww}) \text{ eq } [q^{12}*(q-3)^{12}*(q-1)*(q+1)^6*(q-1/9)*(q^2-5)^3] \);
\( \text{test2}(fa_1, \text{ww}) \text{ eq } [[q^9*(q-3)^{12}*(q-1)^2*(q+1)^7*(q^2-5)^4*(q^4+6/19*q^3-48/19*q^2+8/19*q+16/19)] : i \text{ in } [1, 2]] \);
Lemma 14. Let \( d \) be a square-free positive integer, and let \( u, v \) be positive integers satisfying \( u^2 - dv^2 = 1 \) and \( u > 1 \). Let \( a \) be a positive integer. If \( x \) and \( y \) are positive integers satisfying \( x^2 - dy^2 = a \) and \( x > \sqrt{a(u + 1)/2} \), then there exist integers \( x_0, y_0, n \) with \( 0 < x_0 \leq \sqrt{a(u + 1)/2} \) such that

\[
x + \sqrt{dy} = (u + \sqrt{dv})^n(x_0 + \sqrt{dy_0}).
\]

(50)

Proof. Suppose that the statement is false, and choose the minimal counterexample \( x > \sqrt{a(u + 1)/2} \) with \( x^2 - dy^2 = a \), not expressible in the form (50). Then \( 2x^2 > a(u + 1) \), so \( \frac{a}{x^2} < \frac{2}{u+1} \). Thus

\[
u^2x^2 > (u^2 - 1)(x^2 - a) = d^2v^2y^2
\]

\[
= (u - 1)^2x^2\frac{u + 1}{u - 1}(1 - \frac{a}{x^2})
\]

\[
> (u - 1)^2x^2\frac{u + 1}{u - 1}(1 - \frac{2}{u + 1})
\]

\[
= (u - 1)^2x^2.
\]

This implies \( ux > dvy > (u - 1)x \), or equivalently,

\[
x > ux - dvy > 0.
\]

Set

\[
x_1 = ux - dvy,
\]

\[
y_1 = |uy - vx|.
\]

Then \( x > x_1 > 0 \) and \( y_1 \geq 0 \).

\[
x_1 \pm \sqrt{dy_1} = ux - dvy \pm \sqrt{d(uy - vx)}
\]

\[
= (u \mp \sqrt{dv})(x \pm \sqrt{dy}),
\]

(51)

and so

\[
x_1^2 - dy_1^2 = (u^2 - dv^2)(x - dy^2)
\]

\[
= a.
\]

Note that if \( y_1 = 0 \), then \( x_1 = \sqrt{a} \), so

\[
x_1 \leq \sqrt{a(u + 1)/2}.
\]

(52)

Since \( x \) is the minimal counterexample, we have either (52), or there exist integers \( x_0, y_0, n \) with \( 0 < x_0 \leq \sqrt{a(u + 1)/2} \) such that

\[
x_1 + \sqrt{dy_1} = (u + \sqrt{dv})^n(x_0 + \sqrt{dy_0}).
\]

(53)
In the former case, (51) implies
\[ x + \sqrt{dy} = (u + \sqrt{dv})(x_1 + \sqrt{dy_1}), \]
so (50) holds by setting \((x_0, y_0, n) = (x_1, y_1, 1)\).

In the latter case, (51) and (53) imply
\[ x + \sqrt{dy} = (u + \sqrt{dv})^{n+1}(x_0 + \sqrt{dy_0}), \]
so (50) again holds.

In either case, we obtain a contradiction to the assumption that \(x\) is a counterexample.  

Example 1. Since the Pell equation
\[ u^2 - 17v^2 = 1 \]
has a solution \((u, v) = (33, 8)\), Lemma 14 implies that all solutions of the equation
\[ x^2 - 17y^2 = 64 \]  \hspace{1cm} (54)
can be expressed by those with \(x < \sqrt{64(33 + 1)/2} = 8\sqrt{17}\) and powers of \(33 + 8\sqrt{17}\). Observe that the non-negative solutions \((x, y)\) of (54) with \(x < 8\sqrt{17}\) are
\[ \{(8, 0), (9, 1), (26, 6)\}. \]

Now Lemma 14 implies
\[ \{x + \sqrt{17}y \mid x, y \in \mathbb{Z}, x, y > 0, x^2 - 17y^2 = 64\} \]
\[ \subset \{(33 + 8\sqrt{17})^n \cdot 8 \mid n \in \mathbb{Z}\} \cup \{(33 + 8\sqrt{17})^n(9 + \sqrt{17}) \mid n \in \mathbb{Z}\} \]
\[ \cup \{(33 + 8\sqrt{17})^n(26 + 6\sqrt{17}) \mid n \in \mathbb{Z}\} \]
33\(^2-17\)*8\(^2的情况下
\[ [v:v \text{ in } [1..8]|\text{IsSquare}(17*v^2+1)] \text{ eq } [8]; \]
\[ [x:x \text{ in } [1..\text{Floor}(8*\text{Sqrt}(17))]|\text{IsSquare}(x^2-64)/17)] \text{ eq } [8,9,26]; \]

Lemma 15. Let \(q\) be an even positive integer with \(q \geq 4\). Then \(\sqrt{(q-1)(17q-1)}\) is an integer if and only if
\[ q \in \left\{ \frac{1}{34}\text{tr}((2177 + 528\sqrt{17})^n(433 + 105\sqrt{17})) + 18 \mid n \in \mathbb{Z}\right\} \]  \hspace{1cm} (55)
\[ = \left\{ \ldots, 41210, 10, 26, 110890, 482812730, \ldots \right\}. \]

Proof. Setting \(x = 17q - 9\) and \(y = \sqrt{(q-1)(17q-1)}\), we have
\[ x^2 - 17y^2 = (17q - 9)^2 - 17r^2 \]
\[ = 17^2q^2 - 2 \cdot 17 \cdot 9q + 81 - 17(q-1)(17q-1) \]
\[ = 64. \]
Thus, by Example \[1\] we have
\[x + \sqrt{17}y \in \{(33 + 8\sqrt{17})^n \cdot 8 \mid n \in \mathbb{Z}\}
\cup \{(33 + 8\sqrt{17})^n(9 + \sqrt{17}) \mid n \in \mathbb{Z}\}
\cup \{(33 + 8\sqrt{17})^n(26 + 6\sqrt{17}) \mid n \in \mathbb{Z}\}\]

Since \(q\) is even, we have \(x \equiv -9 \pmod{34}\). On the other hand, we have
\[(33 + 8\sqrt{17})^n \cdot 8 = a + b\sqrt{17} \quad \Rightarrow \quad a \equiv 8(-1)^n \pmod{34},\]
\[(33 + 8\sqrt{17})^n(9 + \sqrt{17}) = a + b\sqrt{17} \quad \Rightarrow \quad a \equiv 9(-1)^n \pmod{34},\]
\[(33 + 8\sqrt{17})^n(26 + 6\sqrt{17}) = a + b\sqrt{17} \quad \Rightarrow \quad a \equiv 8(-1)^{n+1} \pmod{34},\]

Thus
\[x + \sqrt{17}y \in \{(33 + 8\sqrt{17})^{2n+1}(9 + \sqrt{17}) \mid n \in \mathbb{Z}\}
= \{(2177 + 528\sqrt{17})^n(433 + 105\sqrt{17}) \mid n \in \mathbb{Z}\} \quad (56)\]

Thus
\[x \in \{\frac{1}{2} \text{Tr}((2177 + 528\sqrt{17})^n(433 + 105\sqrt{17})) \mid n \in \mathbb{Z}\} \quad (57)\]

Since \(q = (x + 9)/17\), we obtain \((55)\).

Conversely, if \((55)\) holds, then \((57)\) holds, and hence \((56)\) holds for some integer \(y\). Moreover, \((56)\) implies \(x^2 - 17y^2 = 64\), and hence \(y^2 = (17q - 1)(q - 1)\). \(\square\)

\(\text{K}:=\text{QuadraticField}(17);\)
\((33+8*s)^2\) \(\text{eq} \ 2177+528*s;\)
\((33+8*s)*(9+s)\) \(\text{eq} \ 433+105*s;\)
\([1/34*(\text{Trace}((2177+528*s)^n*(433+105*s))+18):n \text{ in } [-2..2]]\)
\(\text{eq} \ [41210,10,26,110890,482812730];\)

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