LINEAR CONNECTIVITY, SCHWARZ-PICK LEMMA AND UNIVALENCE CRITERIA FOR PLANAR HARMONIC MAPPINGS

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ABSTRACT. In this paper, we first establish the Schwarz-Pick lemma of higher-order and apply it to obtain a univalency criteria for planar harmonic mappings. Then we discuss distortion theorems, Lipschitz continuity and univalence of planar harmonic mappings defined in the unit disk with linearly connected images.

1. Introduction and main results

Let $f$ be a complex-valued function defined on a simply connected subdomain $D$ of the complex plane $\mathbb{C}$ and $f = h + g$ its decomposition (unique up to an additive constant), where $h$ and $g$ are analytic in $D$. It is convenient to choose the additive constant in such a way that $g(0) = 0$. In this case, the decomposition is unique, and it is called the canonical decomposition. Since the Jacobian $J_f$ of $f$ is given by

$$J_f = |f_z|^2 - |f_{\bar{z}}|^2 = |h'|^2 - |g'|^2,$$

$f$ is locally univalent and sense-preserving in $D$ if and only if $|g'(z)| < |h'(z)|$ in $D$. The (second) complex dilatation $\omega = g'/h'$ of a sense-preserving harmonic mapping $f$ has the property that $|\omega(z)| < 1$ in $D$ (see [17]). We refer to [12, 14, 19] for basic results in the theory of planar harmonic mappings.

We first recall that the classical Schwarz Lemma for analytic functions $f$ of $\mathbb{D}$ into itself as follows:

$$|f'(z)| \leq \frac{1 - |f(z)|^2}{1 - |z|^2}. \quad (1.1)$$

In 1920, Szász [23] extended the inequality (1.1) to the following estimate involving higher order derivatives:

$$|f^{(2m+1)}(z)| \leq \frac{(2m + 1)!}{(1 - |z|^2)^{2m+1}} \sum_{k=0}^{m} \binom{m}{k}^2 |z|^{2k}, \quad (1.2)$$

2000 Mathematics Subject Classification. Primary: 30C99; Secondary: 30C62.

Key words and phrases. Harmonic mapping, linearly connected domain, $\alpha$-close-to-convex function, John constant, univalence.

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where \( m \in \{1, 2, \ldots \} \). In 1985, Ruscheweyh [2, 20] improved (1.2) to the following form:

\[
|f^{(n)}(z)| \leq \frac{n!(1 - |f(z)|^2)}{(1 - |z|)^n(1 + |z|)}.
\]

In 1989, Colonna established an analogue of the Schwarz-Pick lemma for planar harmonic mappings.

**Theorem A.** ([13, Theorem 3]) Let \( f \) be a harmonic mapping of \( \mathbb{D} \) into \( \mathbb{C} \) such that \( \sup_{z \in \mathbb{D}} |f(z)| \leq M \), where \( M \) is a positive constant. Then for \( z \in \mathbb{D} \),

\[
\Lambda_f(z) \leq \frac{4M}{\pi} \frac{1}{1 - |z|^2}.
\]

This estimate is sharp and all the extremal functions are

\[
f(z) = \frac{2M\alpha}{\pi} \arg \left( \frac{1 + \psi(z)}{1 - \psi(z)} \right),
\]

where \( |\alpha| = 1 \) and \( \psi \) is a conformal automorphism of \( \mathbb{D} \).

Analogously to the inequality (1.3), Chen, Ponnusamy and Wang [6] established the higher order derivatives of harmonic mappings as follows.

**Theorem B.** ([6, Corollary 3.1]) Let \( f \) be a harmonic mapping of \( \mathbb{D} \) into \( \mathbb{C} \) such that \( \sup_{z \in \mathbb{D}} |f(z)| \leq M \), where \( M \) is a positive constant. Then for \( n \geq 1 \) and \( z \in \mathbb{D} \),

\[
\left| \frac{\partial^n f}{\partial z^n}(z) \right| \leq \frac{n!M}{(1 - |z|)^{n+1}} \quad \text{and} \quad \left| \frac{\partial^n f}{\partial \overline{z}^n}(z) \right| \leq \frac{n!M}{(1 - |z|)^{n+1}}.
\]

The following result is a generalization of Theorem A and an improvement of Theorem B.

**Theorem 1.** Let \( f \) be a harmonic mapping of \( \mathbb{D} \) into \( \mathbb{C} \) such that \( \sup_{z \in \mathbb{D}} |f(z)| \leq M \), where \( M \) is a positive constant. Then for \( n \geq 1 \),

\[
\left| \frac{\partial^n f}{\partial z^n}(z) \right| + \left| \frac{\partial^n f}{\partial \overline{z}^n}(z) \right| \leq \frac{n!4M}{\pi} \frac{1}{(1 - |z|)^n(1 + |z|)},
\]

where \( z \in \mathbb{D} \). The estimate of (1.4) is sharp at \( z = 0 \).

From Theorem 1, we get the following result as well.

**Corollary 1.1.** Let \( f \) be a analytic function in \( \mathbb{D} \). Then

\[
|f^{(n)}(z)| \leq \frac{n!4\sup_{\zeta \in \mathbb{D}} |\text{Re} f(\zeta)|}{\pi} \frac{1}{(1 - |z|)^n(1 + |z|)}.
\]

Let \( \mathcal{H} \) denote all non-constant harmonic mappings in \( \mathbb{D} \). We use \( \mathcal{S}_{HU} \) to denote all the univalent harmonic mappings in \( \mathbb{D} \). For \( f \in \mathcal{H} \), let

\[
M_f = \sup_{z \in \mathbb{D}} \Lambda_f(z), \quad m_f = \sup_{z \in \mathbb{D}} \lambda_f(z) \quad \text{and} \quad \mu_f = \frac{M_f}{m_f}.
\]
where
\[ \Lambda_f(z) = \max_{0 \leq \theta \leq 2\pi} |f(z) + e^{-2i\theta}f(z)| = |f(z)| + |f(z)| \]
and
\[ \lambda_f(z) = \min_{0 \leq \theta \leq 2\pi} |f(z) + e^{-2i\theta}f(z)| = |f(z)| - |f(z)|. \]
Then \( J_f = \lambda_f \Lambda_f \) if \( J_f \geq 0 \). Moreover, for
\[ T = \mathcal{H} \setminus S_{HU}, \]
we define the harmonic John constant \( \gamma \) by
\[ \gamma = \inf_{f \in T} \mu_f. \]
On the studies of John constant for analytic functions, see [16, 24].

**Theorem 2.** Let \( \gamma \) be the harmonic John constant as in (1.5). Then \( e^{\frac{\pi}{2}} \leq \gamma \leq e^\pi \).

Proofs for Theorems 1 and 2 will be given in Section 2.

A domain \( \Omega \subset \mathbb{C} \) is said to be \( M \)-linearly connected if there exists a positive constant \( M \in [1, \infty) \) such that any two points \( z, w \in \Omega \) are joined by a path \( \gamma \subset \Omega \) with
\[ \ell(\gamma) \leq M|z - w|, \]
where \( \ell(\gamma) = \inf \left\{ \int_\gamma |dz| : \gamma \subset \Omega \right\} \).

It is not difficult to verify that a 1-linearly connected domain is convex. For extensive discussions on linearly connected domains, see [1, 7, 10, 11, 15, 18].

Let \( S_H \) denote the class of all sense-preserving planar harmonic univalent mappings \( f = h + \overline{g} \) defined in \( \mathbb{D} \), where \( h \) and \( g \) are analytic function in \( \mathbb{D} \) normalized in a standard form: \( h(0) = g(0) = 0 \) and \( h'(0) = 1 \), see [12, 14]. If \( g(z) \) is identically zero on the decomposition of \( f(z) \), then the class \( S_H \) in this case reduces to the classical family \( S \) of normalized analytic univalent functions \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) in \( \mathbb{D} \). Let \( S_H^0 = \{ f = h + \overline{g} \in S_H : g'(0) = 0 \} \). It is well-known that the family \( S_H^0 \) is normal and compact (see [12, 14]).

A family \( L \) of locally univalent harmonic mappings is called a linearly invariant family (LIF) if for any \( f = h + \overline{g} \in L \) its Koebe transformation
\[ K(z) := K_\phi(f)(z) := \frac{f(\phi(z)) - f(\phi(0))}{\phi'(0)h'(\phi(0))} \]
belongs to \( L \) for all analytic automorphisms \( \phi \) of the disk \( \mathbb{D} \). The family \( L \) is called an affinely and linearly invariant family (ALIF) if it is LIF and for any \( f \in L \) its affine transformation
\[ F_\mu(z) := F_\mu(f)(z) = \frac{f(z) + \mu f(z)}{1 + \mu g'(0)} \]
belongs to \( L \) for all \( \mu \in \mathbb{D} \). The classical order of the family \( L \) is defined as
\[ \text{ord} L := \sup \{|a_2(h)| : f \in L\}. \]
(see [21, 22]). These transformations are instrumental in the investigation of distortion theorem and Lipschitz continuity of harmonic mappings \( f \in S_H^0 \).
Theorem 3. Let $f = h + \bar{g} \in S^0_H$, where $h$ and $g$ have the form

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=2}^{\infty} b_n z^n.$$ 

Then we have the following:

(I) There is a positive constant $c_1 < \infty$ such that for $\xi \in \partial \mathbb{D}$ and $0 \leq \rho \leq r < 1$,

$$\Lambda_f(r\xi) \geq \frac{1}{2^{1+c_1}}\Lambda_f(\rho\xi) \left(\frac{1-r}{1-\rho}\right)^{c_1-1}.$$ 

(II) Furthermore, if $\Omega = f(\mathbb{D})$ is a $M$-linearly connected domain with $|\omega(z)| \leq c < 1$, then $|b_2| \leq c/2$. The estimate of $|b_2|$ is sharp, and the extremal function is $f(z) = z + \frac{c}{2}z^2$.

(III) Moreover, if $\Omega = f(\mathbb{D})$ is a $M$-linearly connected domain with $|\omega(z)| \leq c < \frac{1}{2M+1}$, then there is a positive constant $c_2$ and $c_3 < 2$ such that for $\zeta_1, \zeta_2 \in \partial \mathbb{D}$,

$$|f(\zeta_1) - f(\zeta_2)| \geq c_2|\zeta_1 - \zeta_2|^{c_3}$$

and for $n \geq 2$,

$$|a_n| + |b_n| \leq n,$$

where $c_2$ depends only on $M$.

We remark that Theorem 3 is a generalization of [18, Theorem 5.7]. We conjecture that $c_1$ in Theorem 3(I) could be taken as $c_1 = 5/2$. Moreover, further computations suggest the following.

Conjecture 1. Suppose that $f = h + \bar{g} \in S^0_H$ and $\Omega = f(\mathbb{D})$ is a $M$-linearly connected domain. Then there is a positive constant $c_4 < 2$ such that for $\xi \in \partial \mathbb{D}$ and $0 \leq \rho \leq r < 1$,

$$\Lambda_f(r\xi) \geq \frac{1}{8}\Lambda_f(\rho\xi) \left(\frac{1-r}{1-\rho}\right)^{c_4-1}.$$ 

Definition 1. Let $\alpha \in [0, 1)$. A univalent analytic function $f$ is called $\alpha$-close-to convex if there is a univalent and convex analytic function $\phi$ such that

$$|\arg[f'(z)/\phi'(z)]| \leq \frac{\alpha\pi}{2} \quad \text{for} \quad z \in \mathbb{D}.$$ 

It is known that [18, Proposition 5.8] the range of every $\alpha$-close-to convex function is linearly connected.

A domain $D$ is convex in the horizontal direction (CHD) if every line parallel to the real axis has a connected intersection with $D$. One of the beautiful results of Clunie and Sheil-Small [12, Theorem 5.3] states that if $f = h + \bar{g}$ is a harmonic function that is locally univalent in $\mathbb{D}$ (i.e., $|\omega(z)| < 1$ for all $z \in \mathbb{D}$), then the function $F = h - g$ is an analytic univalent mapping of $\mathbb{D}$ onto a CHD domain if and only if $f = h + \bar{g}$ is a univalent mapping of $\mathbb{D}$ onto a CHD domain. It is easy to establish a similar result for functions which are convex in the other directions.

In [11], the authors discussed the relationship between linear connectivity of the images of $\mathbb{D}$ under the planar harmonic mappings $f = h + \bar{g}$ and under their corresponding analytic counterparts $h$. The following result is a generalization of the shearing theorem of Clunie and Sheil-Small [12].
Theorem 4. Fix \( \alpha \in [0, 1) \), and let \( f = h + \overline{g} \) be a harmonic mapping, where \( h \) and \( g \) are analytic in \( \mathbb{D} \).

(I) If \( h - g \) (or \( h + g \)) is \( \alpha \)-close-to convex and \( |\omega(z)| \leq M_1 \) for \( z \in \mathbb{D} \), then \( h \) is univalent and \( h(\mathbb{D}) \) is \( M_2 \)-linearly connected domain, where
\[
M_1 = \frac{\cos \frac{\alpha \pi}{2}}{1 + \cos \frac{\alpha \pi}{2}} \quad \text{and} \quad M_2 = \frac{1}{\cos \frac{\alpha \pi}{2} - M_1(1 + \cos \frac{\alpha \pi}{2})}.
\]

(II) If \( h - g \) (or \( h + g \)) is \( \alpha \)-close-to convex and \( |\omega(z)| \leq M_3 \) for \( z \in \mathbb{D} \), then \( f_\theta = h + e^{i\theta} \overline{g} \) is a \( K \)-quasiconformal harmonic mapping and \( f_\theta(\mathbb{D}) \) is \( M_4 \)-linearly connected domain, where \( \theta \in [0, 2\pi) \),
\[
K = \frac{1 + M_3}{1 - M_3}, \quad M_3 < \frac{\cos \frac{\alpha \pi}{2}}{2 + \cos \frac{\alpha \pi}{2}} \quad \text{and} \quad M_4 = \frac{1 + M_3}{\cos \frac{\alpha \pi}{2} - M_3(2 + \cos \frac{\alpha \pi}{2})}.
\]

The proofs of Theorems 3 and 4 will be presented in Section 3.

2. The Schwarz Lemma of Higher-Order and a Univalence Criterion for Harmonic Mappings

Lemma C. ([8, Lemma 1] or [9, Theorem 1.1]) Let \( f \) be a harmonic mapping of \( \mathbb{D} \) into \( \mathbb{C} \) such that \( \sup_{z \in \mathbb{D}} |f(z)| \leq M \) and \( f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n \bar{z}^n \), where \( M \) is a positive constant. Then \( |a_0| \leq M \) and for all \( n \geq 1 \),
\[
|a_n| + |b_n| \leq \frac{4M}{\pi}.
\]
The estimate of (2.1) is sharp, all the extremal functions are \( f(z) \equiv M \) and
\[
f_n(z) = \frac{2M \alpha}{\pi} \arg \left( \frac{1 + \beta z^n}{1 - \beta z^n} \right),
\]
where \( |\alpha| = |\beta| = 1 \).

Proof of Theorem 1. Let \( f = h + \overline{g} \), where \( h \) and \( g \) are analytic in \( \mathbb{D} \). For any fixed \( z \in \mathbb{D} \), let \( \phi(\zeta) = \frac{\zeta + \bar{z}}{1 + \zeta \bar{z}} \), where \( \zeta \in \mathbb{D} \). For \( \zeta \in \mathbb{D} \), set
\[
F(\zeta) = f(\phi(\zeta)) = h(\phi(\zeta)) + \overline{g(\phi(\zeta))} = \sum_{k=0}^{\infty} a_k \zeta^k + \sum_{k=1}^{\infty} b_k \bar{\zeta}^k.
\]
Using Lemma C and the well-known formula (cf. [2])
\[
\frac{h^{(n)}(z)(1 - |z|^2)^n}{n!} = \sum_{k=1}^{n} \binom{n-1}{n-k} z^{n-k} a_k
\]
and
\[
\frac{g^{(n)}(z)(1 - |z|^2)^n}{n!} = \sum_{k=1}^{n} \binom{n-1}{n-k} z^{n-k} b_k,
\]
we get
\[
\left( |h^{(n)}(z)| + |g^{(n)}(z)| \right) (1 - |z|^2)^n \leq \frac{1}{n!} \sum_{k=1}^{n} \frac{(n-1)}{(n-k)} |z|^{n-k} (|a_k| + |b_k|) \\
\leq \frac{4M}{\pi} \sum_{k=1}^{n} \frac{(n-1)}{(n-k)} |z|^{n-k} \\
= \frac{4M}{\pi} (1 + |z|)^{n-1},
\]
which gives
\[
|h^{(n)}(z)| + |g^{(n)}(z)| \leq \frac{n!4M(1 + |z|)^{n-1}}{(1 - |z|^2)^n}.
\]
The inequality (1.4) follows. Sharpness at \( z = 0 \) is a consequence of the sharpness part of Lemma C. So we omit the details. \( \Box \)

**Proof of Corollary 1.1.** For \( z \in \mathbb{D} \), let \( u(z) = \text{Re}(f(z)) \). Without loss of generality, we assume that \( \sup_{\zeta \in \mathbb{D}} |\text{Re}(f(\zeta))| < \infty \). By Theorem 1 and elementary calculations, we get
\[
|f^{(n)}(z)| = \left| \frac{\partial^n u}{\partial z^n} \right| + \left| \frac{\partial^n u}{\partial \overline{z}^n} \right| \leq \frac{n!4 \sup_{\zeta \in \mathbb{D}} |\text{Re}(f(\zeta))| (1 + |z|)^{n-1}}{\pi (1 - |z|^2)^n}.
\]

**Lemma D.** ([4, Corollary 4.1]) Let \( f \) be a non-constant analytic function in \( \mathbb{D} \). If \( \|f\| \leq 1 \), then \( f \) is univalent in \( \mathbb{D} \), where
\[
\|f\| = \sup_{z \in \mathbb{D}} \left\{ (1 - |z|^2) \left| \frac{f^{(n)}(z)}{f'(z)} \right| \right\}.
\]

**Proof of Theorem 2.** We first prove the left part. For \( \theta \in [0, 2\pi) \), let \( F_\theta = h + e^{i\theta}g \). For \( z \in \mathbb{D} \), set
\[
H_\theta(z) = \log \frac{F_\theta'(z)}{\sqrt{M_f m_f}}.
\]
Then for \( z \in \mathbb{D} \),
\[
(2.2) \quad \text{Re}(H_\theta(z)) \leq \frac{1}{2} \log \frac{M_f}{m_f}.
\]
By Corollary 1.1, we have
\[
(2.3) \quad (1 - |z|^2)|H_\theta'(z)| \leq \frac{4}{\pi} \sup_{z \in \mathbb{D}} |\text{Re}(H_\theta(z))|.
\]
Applying (2.2) and (2.3), we obtain
\[
(1 - |z|^2)|H_\theta'(z)| \leq \frac{2}{\pi} \log \frac{M_f}{m_f}.
\]
which gives

\[(1 - |z|^2) \left| \frac{h''(z) + e^{i\theta} g''(z)}{h'(z) + e^{i\theta} g'(z)} \right| \leq \frac{2}{\pi} \log \frac{M_f}{m_f}.\]

By using Lemma D, if

\[\frac{2}{\pi} \log \frac{M_f}{m_f} \leq 1,\]

then for all \(\theta \in [0, 2\pi)\) the function \(F_\theta\) is univalent, which implies that \(f = h + \overline{g}\) is univalent. Hence

\[\frac{M_f}{m_f} > e^{\pi},\]

which yields that \(\gamma \geq e^{\pi}\).

Next we come to prove the right part. Since all the analytic functions defined in \(\mathbb{D}\) are harmonic, by [24, Theorem], we see that \(\gamma \leq e^{\pi}\). The proof of the theorem is complete. \(\Box\)

3. LINEAR CONNECTIVITY AND UNIVALENCY CRITERION FOR HARMONIC MAPPINGS

**Proof of Theorem 3.** We first prove (I). For every constant \(\mu \in \mathbb{D}\), consider the affine mappings \(F_\mu = f + \mu \overline{f}\), where \(f = h + \overline{g} \in S_H^0\). Clearly, \(F_\mu \in S_H\). For a fixed \(\zeta \in \mathbb{D}\), we next consider the Koebe transform of \(F_\mu\) given by

\[K(z) = \frac{F_\mu\left(\frac{z + \zeta}{1 + \zeta z}\right) - F_\mu(\zeta)}{(1 - |\zeta|^2)(h'(\zeta) + \mu g'(\zeta))} = H(z) + \overline{G(z)},\]

which again belongs to \(S_H\). By elementary calculations, we get

\[H(z) = z + A_2(\zeta) z^2 + A_3(\zeta) z^3 + \cdots ,\]

where

\[A_2(\zeta) = \frac{1}{2} (1 - |\zeta|^2) \frac{h''(\zeta) + \mu g''(\zeta)}{h'(\zeta) + \mu g'(\zeta)} - \overline{\zeta}.\]

By [14, p. 87 and p. 96], we know that \(|A_2(\zeta)|\) is bounded. Without loss of generality, we assume \(|A_2(\zeta)| \leq c_1 < \infty\), which implies

\[\left| \frac{\partial}{\partial \rho} \log \left[ \frac{1 - \rho^2}{1 - \rho^2} (h'(\rho \xi) + \mu g'(\rho \xi)) \right] \right| = \left| \frac{h''(\rho \xi) + \mu g''(\rho \xi)}{h'(\rho \xi) + \mu g'(\rho \xi)} - \frac{2\rho \xi}{1 - \rho^2} \right| \leq \frac{2c_1}{1 - \rho^2},\]

where \(\xi \in \partial \mathbb{D}\). Integration leads to

\[(3.1) \frac{(1 - r^2) |h'(r \xi) + \mu g'(r \xi)|}{(1 - \rho^2) |h'(\rho \xi) + \mu g'(\rho \xi)|} \geq \left( \frac{1 - r}{1 - \rho} \cdot \frac{1 + \rho}{1 + r} \right)^{c_1},\]

which gives

\[(3.2) |h'(r \xi) + \mu g'(r \xi)| \geq |h'(\rho \xi) + \mu g'(\rho \xi)| \left( \frac{1 - r}{1 - \rho} \right)^{c_1-1} \left( \frac{1 + \rho}{1 + r} \right)^{c_1+1}.\]
By (3.2), we have

\[(3.3) \quad \Lambda_f(r\xi) \geq |h'(\rho\xi) + \mu g'(\rho\xi)| \left( \frac{1-r}{1-\rho} \right)^{c_1-1} \left( \frac{1+\rho}{1+r} \right)^{c_1+1}. \]

Applying (3.3) and the arbitrariness of \(\mu\), we see that

\[(3.4) \quad \Lambda_f(r\xi) \geq \Lambda_f(\rho\xi) \left( \frac{1-r}{1-\rho} \right)^{c_1-1} \left( \frac{1+\rho}{1+r} \right)^{c_1+1} \geq \frac{1}{2^{l+c_1}} \Lambda_f(\rho\xi) \left( \frac{1-r}{1-\rho} \right)^{c_1-1}. \]

Now, we prove that \(|b_2| \leq \frac{c}{2}\). Since \(\omega(0) = 0\) and \(|\omega(z)| \leq c\) in \(D\), follows that \(|\omega'(0)| \leq c\) and hence, from

\[\omega(z) = g'(z) = \frac{2b_2 z + 3b_3 z^2 + \cdots}{h'(z)} = 2b_2 z + (3b_3 - 4a_2 b_2) z^2 + \cdots,\]

we obtain that \(|\omega'(0)| = |2b_2| \leq c\). Then

\[(3.5) \quad |b_2| \leq \frac{c}{2}.\]

Finally, we prove the sharpness part. For \(z \in D\), let \(f(z) = h(z) + g(z)\), where \(h(z) = z\) and \(g(z) = \frac{c}{2} z^2\). For \(w \in h(D) = D\), let

\[F(w) = f(h^{-1}(w)) = w + g(h^{-1}(w)) = w + \frac{c}{2} w^2.\]

Since \(h(D)\) is convex, we see that for any \(w_1, w_2 \in h(D)\),

\[(3.6) \quad l(F(\gamma)) \leq \int_{\gamma} |F_w(w) \, dw + F_{\overline{w}}(w) \, d\bar{w}| \leq \int_{\gamma} (|F_w(w)| + |F_{\overline{w}}(w)|) \, |dw| \leq (1 + c)|w_2 - w_1|,\]

where \(\gamma\) is a line segment joining \(w_1\) and \(w_2\). On the other hand,

\[(3.7) \quad |F(w_2) - F(w_1)| \geq |w_2 - w_1| - \frac{c}{2} |w_2^2 - w_1^2| \geq (1 - c)|w_2 - w_1|.\]

Equations (3.6) and (3.7) yield that

\[l(F(\gamma)) \leq \frac{1 + c}{1 - c} |F(w_2) - F(w_1)|.\]

Hence \(f(D)\) is \((\frac{1+c}{1-c})\)-linearly connected, where \(f(z) = z + \frac{c}{2} z^2\). Therefore, the extremal function \(f(z) = z + \frac{c}{2} z^2\) shows that the estimate of (3.5) is sharp.

Next we prove the first part of (III). Let \(T_\theta = h + e^{i\theta} g\), where \(\theta \in [0, 2\pi]\). First of all, we prove that \(T_\theta\) is univalent and \(T_\theta(D)\) is a \(\frac{M(1+c)}{1-c(1+2M)}\)-linearly connected domain. For \(w \in f(D)\), let

\[H(w) = T_\theta(f^{-1}(w)) = w - G(w) + e^{i\theta} G(w),\]

where
where $G(w) = g(f^{-1}(w))$. By the chain rule, we get
\[ G_w = g' \cdot (f^{-1})_w \quad \text{and} \quad G_{\overline{w}} = g' \cdot (f^{-1})_{\overline{w}}. \]
Differentiating both sides of equation $f^{-1}(f(z)) = z$ yields the relations
\[ (f^{-1})_w \cdot h' + (f^{-1})_{\overline{w}} \cdot g' = 1 \quad \text{and} \quad (f^{-1})_w \cdot \overline{g'} + (f^{-1})_{\overline{w}} \cdot \overline{h'} = 0, \]
which imply that
\[ (f^{-1})_w = \frac{h'}{Jf} \quad \text{and} \quad (f^{-1})_{\overline{w}} = -\frac{g'}{Jf}. \]
Since $|\omega| \leq c$ (by hypotheses), we see that
\[ (3.8) \quad \Lambda_G(w) = \frac{|\omega|}{1 - |\omega|} \leq \frac{c}{1 - c}. \]
Since $f(D)$ is $M$-linearly connected, we know that for any two points $w_1, w_2 \in f(D)$, there is a curve $\gamma \subset f(D)$ joining $w_1$ and $w_2$ such that $l(\gamma) \leq M|w_1 - w_2|$. Now, we set $\Gamma = H(\gamma)$. By elementary calculations, we have
\[ (3.9) \quad H_w = 1 - \overline{G_{\overline{w}}} + e^{i\theta} G_w \quad \text{and} \quad H_{\overline{w}} = -\overline{G_w} + e^{i\theta} G_{\overline{w}}. \]
By using (3.8) and (3.9), we get
\[ (3.10) \quad l(\Gamma) = \int_{\gamma} |H_w(w) \, dw + H_{\overline{w}}(w) \, d\overline{w}| \]
\[ \leq \int_{\gamma} (1 + 2\Lambda_G(w)) |dw| \]
\[ \leq \left(1 + \frac{2c}{1 - c}\right) l(\gamma) \]
\[ \leq M \left(1 + \frac{c}{1 - c}\right) |w_1 - w_2|. \]
On the other hand, we have
\[ (3.11) \quad |H(w_2) - H(w_1)| \geq |w_2 - w_1| - 2 \int_{\gamma} \Lambda_G(w) |dw| \geq |w_1 - w_2| \left(1 - \frac{2cM}{1 - c}\right), \]
which shows that for all $\theta \in [0, 2\pi)$, $T_\theta$ is univalent. Equations (3.10) and (3.11) yield that
\[ l(\Gamma) \leq \frac{M(1 + c)}{1 - c(1 + 2M)} |H(w_2) - H(w_1)|, \]
which implies that $T_\theta(D)$ is a $\frac{M(1+c)}{1-c(1+2M)}$-linearly connected domain.

By [18, Proposition 5.6], we know that $T_\theta$ is continuous in $\overline{D}$ with values in $C \cup \{\infty\}$. Applying [18, Theorem 5.7 (5)] to $T_\theta$, we see that there is a positive constant $c_2$ and $c_3 < 2$ such that for $\zeta_1, \zeta_2 \in \partial D$,
\[ (3.12) \quad |T_\theta(\zeta_1) - T_\theta(\zeta_2)| \geq c_2 |\zeta_1 - \zeta_2|^{c_3}. \]
Inequality (3.12) and the arbitrariness of $\theta \in [0, 2\pi)$ gives
\[ |f(\zeta_1) - f(\zeta_2)| \geq c_2 |\zeta_1 - \zeta_2|^{c_3}. \]
This completes the proof of the first part of (III).
Next we show the second part of (III). By the proof of the first part in (III), we know that for all $\theta \in [0, 2\pi)$, $T_\theta = h + e^{i\theta}g$ is univalent in $\mathbb{D}$. Applying the result of de Branges in [5], we see that for all $\theta \in [0, 2\pi)$,
\[
|a_n + e^{i\theta}b_n| \leq n,
\]
which implies that $|a_n| + |b_n| \leq n$, for $n \geq 2$. □

**Proof of Theorem 4.** We first prove (I). Without loss of generality, we assume that $F = h - g$ is $\alpha$-close-to convex. By [18, Proposition 5.8], we know that $\Omega = F(\mathbb{D})$ is $M^*$-linearly connected domain, where $M^* = \frac{1}{\cos \frac{\alpha \pi}{2}}$. Let
\[
H(w) = h(F^{-1}(w)) = w + g(F^{-1}(w)).
\]
For any two distinct $w_1, w_2 \in \Omega$, by hypothesis, there is a curve $\gamma \subset \Omega$ joining $w_1$ and $w_2$ such that $l(\gamma) \leq M^*|w_1 - w_2|$. Set $\Gamma = H(\gamma)$. Then we have

(3.13) \[ l(\Gamma) = \int_\gamma |dH(w)| = \int_\gamma |H'(w) dw| \]
\[
= \int_\gamma \left| 1 + \frac{g'(F^{-1}(w))}{h'(F^{-1}(w)) - g'(F^{-1}(w))} \right| |dw|
\]
\[
\leq \int_\gamma \left( 1 + \frac{|\omega(F^{-1}(w))|}{1 - |\omega(F^{-1}(w))|} \right) |dw|
\]
\[
\leq \frac{l(\gamma)}{1 - M_1}
\]
\[
\leq \frac{M^*}{1 - M_1}|w_1 - w_2|.
\]

On the other hand,

(3.14) \[ |H(w_2) - H(w_1)| \geq |w_2 - w_1| - |g(F^{-1}(w_2)) - g(F^{-1}(w_1))| \]
\[
\geq |w_2 - w_1| - \int_\gamma \left| (g(F^{-1}(w)))' \right| |dw|
\]
\[
= \frac{1 - M_1(1 + M^*)}{1 - M_1}|w_2 - w_1|.
\]

By (3.13) and (3.14), we obtain
\[
l(\Gamma) \leq \frac{M^*}{1 - M_1(1 + M^*)}|H(w_2) - H(w_1)|,
\]
which shows that $h(\mathbb{D})$ is $M_2$-linearly connected domain, where
\[
M_2 = \frac{1}{\cos \frac{\alpha \pi}{2} - M_1(1 + \cos \frac{\alpha \pi}{2})}.
\]
The univalency of $h$ follows from (3.14).

Next we prove (II). Define
\[
T(w) = f_\theta(F^{-1}(w)) = w + g(F^{-1}(w)) + e^{i\theta}g(F^{-1}(w)).
\]
Let $\Gamma_1 = T(\gamma)$. Then we find that

\begin{equation}
(3.15) \quad l(\Gamma) = \int_{\Gamma} |dT(w)| = \int_{\Gamma} |T_w(w) \, dw + T_{\bar{w}}(w) \, d\bar{w}|
\end{equation}

\begin{align*}
&\leq \int_{\gamma} \left( 1 + \frac{2|g'(F^{-1}(w))|}{|h'(F^{-1}(w)) - g'(F^{-1}(w))|} \right) |dw| \\
&\leq \int_{\gamma} \left( 1 + \frac{2|\omega(F^{-1}(w))|}{1 - |\omega(F^{-1}(w))|} \right) |dw| \\
&\leq \frac{1 + M_3}{1 - M_3} l(\gamma) \leq \frac{M^*(1 + M_3)}{1 - M_3} |w_1 - w_2|
\end{align*}

and

\begin{align*}
(3.16) \quad |T(w_2) - T(w_1)| &\geq |w_2 - w_1| - 2 |g(F^{-1}(w_2)) - g(F^{-1}(w_1))| \\
&\geq |w_2 - w_1| - 2 \int_{\gamma} |(g(F^{-1}(w)))'| |dw| \\
&= \frac{1 - M_3(1 + 2M^*)}{1 - M_3} |w_2 - w_1|.
\end{align*}

By (3.15) and (3.16), we obtain

\[ l(\Gamma_1) \leq \frac{M^*(1 + M_3)}{1 - M_3(1 + 2M^*)} |T(w_2) - T(w_1)|, \]

which implies that $f_{\theta}(D)$ is $M_4$-linearly connected domain, where $\theta \in [0, 2\pi)$ and

\[ M_4 = \frac{1 + M_3}{\cos \frac{\alpha \pi}{2} - M_3(2 + \cos \frac{\alpha \pi}{2})}. \]

The univalency of $f_{\theta}$ follows from (3.16). Since for $z \in D$,

\[ \frac{\Lambda_{f_{\theta}}(z)}{\lambda_{f_{\theta}}(z)} \leq \frac{1 + M_3}{1 - M_3}, \]

we see that $f_{\theta}$ is a $K$-quasiconformal harmonic mapping, where $K = \frac{1 + M_4}{1 - M_3}$. □

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