A FUNCTIONAL LOGARITHMIC FORMULA FOR HYPERGEOMETRIC FUNCTIONS $\text{}_3\text{F}_2$.

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1. Introduction

For $\alpha_i, \beta_j \in \mathbb{C}$ with $\beta_j \not\in \mathbb{Z}_{\leq 0}$, the generalized hypergeometric function is defined by a power series expansion

$$\text{}_p\text{F}_{p-1} \left( \begin{array}{c} \alpha_1, \cdots, \alpha_p \\ \beta_1, \cdots, \beta_{p-1} \end{array} ; x \right) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_{p-1})_n} \frac{x^n}{n!},$$

where $(\alpha)_0 := 1$, $(\alpha)_n := \alpha(\alpha + 1) \cdots (\alpha + n - 1)$ for $n \geq 1$ denotes the Pochhammer symbol. When $p = 2$, this is called the Gauss hypergeometric function. This has the analytic continuation to $\mathbb{C}$, and then becomes the multi-valued function which is locally holomorphic on $\mathbb{C} \setminus \{0, 1\}$. A number of formulas are discovered since 19th century (e.g. [12] Chap. 15,16), and they are applied in lots of areas in mathematics. In the present, the hypergeometric function is one of the most important tools in mathematics.

In [5], we discussed the special values of $\text{}_3\text{F}_2 \left( \begin{array}{c} 1,1, q \\ a, b \end{array} ; x \right)$ at $x = 1$, and gave a sufficient conditions for that it is a $\overline{\mathbb{Q}}$-linear combination of log of algebraic numbers, namely

$$\text{}_3\text{F}_2 \left( \begin{array}{c} 1,1, q \\ a, b \end{array} ; 1 \right) \in \overline{\mathbb{Q}} + \overline{\mathbb{Q}} \log \overline{\mathbb{Q}}^\times := \left\{ a + \sum_{i=1}^{n} b_i \log c_i \mid a, b_i, c_i \in \overline{\mathbb{Q}}, c_i \neq 0, n \in \mathbb{Z}_{\geq 0} \right\}.$$

The goal of this paper is to give its functional version. To be precise, set

$$\overline{\mathbb{Q}(x)} + \overline{\mathbb{Q}(x)} \log \overline{\mathbb{Q}(x)}^\times := \left\{ f + \sum_{i=1}^{n} g_i \log h_i \mid f, g_i, h_i \in \overline{\mathbb{Q}(x)}, h_i \neq 0, n \in \mathbb{Z}_{\geq 0} \right\}$$

where $\overline{\mathbb{Q}(x)}$ denotes the algebraic closure of the field of rational functions $\mathbb{Q}(x)$. We say the logarithmic formula holds for a function $f(x)$ if it belongs to the above set. The main theorem is to give a sufficient condition on $(a, b, q)$ for that the log formula holds for the hypergeometric function $\text{}_3\text{F}_2 \left( \begin{array}{c} 1,1, q \\ a, b \end{array} ; x \right)$. Recall that two proofs are presented in [5], one uses hypergeometric fibrations and the other uses Fermat surfaces. In this paper we follow the method of hypergeometric fibrations, while a new ingredient is employed from [5]. It seems impossible to prove the functional log formula according to the method of Fermat.

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By developing the technique here, we can get explicit log formula in some cases. For example, let

\[
e_1(x) := \frac{1}{2} + x^{-\frac{1}{3}} \left( -\frac{1}{4} + \frac{x}{8} + \frac{1}{4} \sqrt{1-x} \right)^\frac{2}{3} + x^{-\frac{1}{3}} \left( -\frac{1}{4} + \frac{x}{8} - \frac{1}{4} \sqrt{1-x} \right)^\frac{2}{3}
\]

\[
e_2(x) := \frac{1}{2} + e^{-2\pi i/3} x^{-\frac{1}{3}} \left( -\frac{1}{4} + \frac{x}{8} + \frac{1}{4} \sqrt{1-x} \right)^\frac{2}{3} + e^{2\pi i/3} x^{-\frac{1}{3}} \left( -\frac{1}{4} + \frac{x}{8} - \frac{1}{4} \sqrt{1-x} \right)^\frac{2}{3}
\]

\[
e_3(x) := \frac{1}{2} + e^{2\pi i/3} x^{-\frac{1}{3}} \left( -\frac{1}{4} + \frac{x}{8} + \frac{1}{4} \sqrt{1-x} \right)^\frac{2}{3} + e^{-2\pi i/3} x^{-\frac{1}{3}} \left( -\frac{1}{4} + \frac{x}{8} - \frac{1}{4} \sqrt{1-x} \right)^\frac{2}{3}
\]

\[
p_{\pm} = p_{\pm}(x) := \left( \frac{1 \pm \sqrt{1-x}}{\sqrt{x}} \right)^\frac{2}{3}, \quad q_{j}(x) := \frac{1 - \sqrt{3x \cdot e_j(x)}}{1 + \sqrt{3x \cdot e_j(x)}}.
\]

Then

\[
_{3}F_{2} \left( \frac{1}{6}, \frac{1}{6}, \frac{1}{6}; \frac{1}{x} \right) = \frac{5\sqrt{3}}{36} x^{-\frac{1}{2}} \left[ (p_{\pm} + p_{-}) \log \left( \frac{q_{1}}{q_{2}} \right) + (e^{\frac{2\pi i}{3}} p_{+} + e^{-\frac{2\pi i}{3}} p_{-}) \log \left( \frac{q_{2}}{q_{3}} \right) \right].
\]

However there remains technical difficulties arising from algebraic cycles to obtain explicit log formula in more general cases.

2. Main Theorem

Let \( \hat{\mathbb{Z}} = \lim_{\rightarrow n} \mathbb{Z}/n\mathbb{Z} \) be the completion, and \( \hat{\mathbb{Z}}^{\times} = \lim_{\rightarrow n} (\mathbb{Z}/n\mathbb{Z})^{\times} \) the group of units. The ring \( \hat{\mathbb{Z}} \) acts on the additive group \( \mathbb{Q}/\mathbb{Z} \) in a natural way, and then it induces \( \hat{\mathbb{Z}}^{\times} \cong \text{Aut}(\mathbb{Q}/\mathbb{Z}) \). We denote by \( \{x\} := x - \lfloor x \rfloor \) the fractional part of \( x \in \mathbb{Q} \). The map \( \{-\} : \mathbb{Q} \to [0,1) \) factors through \( \mathbb{Q}/\mathbb{Z} \), which we denote by the same notation.

Theorem 2.1 (Logarithmic Formula). Let \( q, a, b \in \mathbb{Q} \) satisfy that none of \( q, a, b, q-a, q-b, q-a-b \) is an integer. Suppose

\[
1 = \{sa\} + \{sb\} + 2\{-sq\} - \{s(a-q)\} - \{s(b-q)\}
\]

\[
(\iff) \quad \min(\{sa\},\{sb\}) < \{sq\} < \max(\{sa\},\{sb\})
\]

for all \( s \in \hat{\mathbb{Z}}^{\times} \). Then

\[
_{3}F_{2} \left( \frac{n_{1}, n_{2}, q}{a, b}; x \right) \in \mathbb{Q}(x) + \mathbb{Q}(x) \log \mathbb{Q}(x)^{\times}
\]

for any integers \( n_{i} > 0 \).

As we shall see in [41] one can shift the indices \( n_{i}, q, a, b \) by arbitrary integers by applying differential operators. Thus it is enough to prove the log formula for

\[
_{3}F_{2} \left( \frac{1, 1, q}{a, b}; x \right).
\]

Recall the main theorem of [5] which asserts that if

\[
2 = \{sq\} + \{s(a-q)\} + \{s(b-q)\} + \{s(q-a-b)\}
\]

for all \( s \in \hat{\mathbb{Z}}^{\times} \), then

\[
_{3}F_{2} \left( \frac{1, 1, q}{a, b}; x \right) \in \mathbb{Q} + \mathbb{Q} \log \mathbb{Q}^{\times}
\]
as long as it converges (⇔ a + b > q + 2). It is easy to see (2.1) ⇒ (2.2) while the converse is no longer true (e.g. (a, b, q) = (1/6, 1/4, 1/2)). Theorem 2.1 does not cover all of the main theorem of [5].

**Conjecture 2.2** (cf. [3] Conjecture 5.2). The converse of Theorem 2.1 is true.

In the seminal paper [7], Beukers and Heckman gave a necessary and sufficient condition for that \( pF_{p-1} \) is an algebraic function, or equivalently its monodromy group is finite. Let \( a_i, b_j \in \mathbb{Q} \). Then their theorem tells that, under the condition that \( \{ a_i \} \neq \{ b_j \} \) and \( \{ a_i \} \neq 0 \),

\[
p_{F_{p-1}} \left( \frac{a_1, \ldots, a_p}{b_1, \ldots, b_{p-1}} ; x \right) \in \overline{\mathbb{Q}(x)}
\]

if and only if \( \{ sa_1, \ldots, sa_p \} \) and \( \{ sb_1, \ldots, sb_{p-1} \} \) interlace for all \( s \in \mathbb{Z}_x^\times \) (loc.cit. Theorem 4.8). Here we say that two sets \( \{ \alpha_1, \ldots, \alpha_p \} \) and \( \{ \beta_1, \ldots, \beta_p \} \) interlace if and only if

\[
\alpha_1 < \beta_1 < \cdots < \alpha_p < \beta_p \text{ or } \beta_1 < \alpha_1 < \cdots < \beta_p < \alpha_p
\]

when ordering \( \alpha_1 < \cdots < \alpha_p \) and \( \beta_1 < \cdots < \beta_p \). In this terminology, (2.1) is translated into that \( \{ sq \} \) and \( \{ \{ sa \}, \{ sb \} \} \) interlace. Our main theorem 2.1 has no intersection with their theorem, while they are obviously comparable.

3. **Hypergeometric Fibrations**

3.1. **Definition.** Let \( R \) be a finite-dimensional semisimple \( \mathbb{Q} \)-algebra. Let \( e : R \to E \) be a projection onto a number field \( E \). Let \( X \) be a smooth projective variety over \( k_{d\text{et}} \), and \( f : X \to \mathbb{P}^1 \) a surjective map endowed with a multiplication on \( R^1 f_* \mathcal{O}_X \) by \( R \) where \( U \subset \mathbb{P}^1 \) is the maximal Zariski open set such that \( f \) is smooth over \( U \). We say \( f \) is a hypergeometric fibration with multiplication by \( (R, e) \) (abbreviated HG fibration) if the following conditions hold. We fix an inhomogeneous coordinate \( t \in \mathbb{P}^1 \).

(a) \( f \) is smooth over \( \mathbb{P}^1 \setminus \{ 0, 1, \infty \} \),

(b) \( \dim_E (R^1 f_* \mathcal{O}_X)(e) = 2 \) where we write \( V(e) := E \otimes_{e, R} V \) the \( e \)-part,

(c) Let \( \text{Pic}^0_f \to \mathbb{P}^1 \setminus \{ 0, 1, \infty \} \) be the Picard fibration whose general fiber is the Picard variety \( \text{Pic}^0(f^{-1}(t)) \), and let \( \text{Pic}^0_f(e) \) be the component associated to the \( e \)-part \( (R^1 f_* \mathcal{O}_X)(e) \) (this is well-defined up to isogeny). Then \( \text{Pic}^0_f(e) \to \mathbb{P}^1 \setminus \{ 0, 1, \infty \} \) has a totally degenerate semistable reduction at \( t = 1 \).

The last condition (c) is equivalent to say that the local monodromy \( T \) on \( (R^1 f_* \mathcal{O}_X)(e) \) at \( t = 1 \) is unipotent and the rank of log monodromy \( N := \log(T) \) is maximal, namely \( \text{rank}(N) = \frac{1}{2} \dim_{\mathbb{Q}} (R^1 f_* \mathcal{O}_X)(e) = [E : \mathbb{Q}] \) by the condition (b)).

3.2. **HG fibration of Gauss type.** Let \( f : X \to \mathbb{P}^1 \) be a fibration over \( \overline{\mathbb{Q}} \) whose general fiber \( f^{-1}(t) \) is a nonsingular projective model of an affine curve

\[
y^N = x^a(1 - x)^b(1 - tx)^{N-b}, \quad 0 < a, b < N, \gcd(N, a, b) = 1.
\]

\[ \text{(3.1)} \]

\( f \) is smooth outside \( \{ 0, 1, \infty \} \) so that the condition (a) is satisfied. The group \( \mu_N \) of \( N \)-th roots of unity acts on \( f^{-1}(t) \) by \( (x, y, t) \to (x, \zeta y, \zeta t) \) for \( \zeta \in \mu_N \), which gives rise to a multiplication on \( R^1 f_* \mathcal{O}_X \) by the group ring \( R_0 := \mathbb{Q}[\mu_N] \).
Lemma 3.1. [4, Proposition 3.1] Let \( e_0 : R_0 := \mathbb{Q}[\mu_N] \rightarrow E_0 \) be a projection onto a number field \( E_0 \). Then \((R_0,e_0)\) satisfies the conditions (b) and (c) if and only if \( ad \neq 0 \) and \( bd \neq 0 \) modulo \( N \) where \( d := \sharp \text{Ker}[\mu_N \rightarrow R_0^\times \rightarrow E_0^\times] \).

Definition 3.2. We say that \( f \) is a HG fibration of Gauss type with multiplication by \((\mathbb{Q}[\mu_N],e)\) if \( ad \neq 0 \) and \( bd \neq 0 \) modulo \( N \).

Let \( \chi : R_0 \rightarrow \overline{\mathbb{Q}} \) be a homomorphism of \( \mathbb{Q}\)-algebra factoring through \( e \). Let \( n \) be an integer such that \( \chi(\zeta) = \zeta^{-n} \) for all \( \zeta \in \mu_N \). Note \( \gcd(n,N) = 1 \). By [4] (13), p.917, \( H^1_{dr}(X_t)(\chi) \cap H^{1,0} \) is spanned by a 1-form

\[
\omega_n := \frac{x^n(1-x)^{bn}(1-tx)^{cn}}{y^n} dx,
\]

\[a_n := \left\lfloor \frac{an}{N} \right\rfloor, \quad b_n := \left\lfloor \frac{bn}{N} \right\rfloor, \quad c_n := \left\lfloor \frac{Nn-bn}{N} \right\rfloor = n-bn-1.
\]

Let \( P_1 \) (resp. \( P_2 \)) be a point of \( X_t \) above \( x = 0 \) (resp. \( x = 1 \)). There are \( \gcd(N,a)\)-points above \( x = 0 \) (resp. \( \gcd(N,b)\)-points above \( x = 1 \)). Let \( u \) be a path from \( P_1 \) to \( P_2 \) above the real interval \( x \in [0,1] \). It defines a homology cycle \( u \in H_1(X_t,\{P_1,P_2\};\mathbb{Z}) \) with boundary. Put \( d_1 := \gcd(N,a), d_2 := \gcd(N,b) \). Since \( \sigma^{d_1}P_1 = P_1 \) and \( \sigma^{d_2}P_2 = P_2 \) for \( \sigma \in \mu_N \) an automorphism, one has a homology cycle

\[\delta := (1-\sigma^{d_1})(1-\sigma^{d_2})u \in H_1(X_t,\mathbb{Z}).\]

By an integral expression of Gauss hypergeometric functions (e.g. [6] p.4, 1.5 or [10] p.20, (1.6.6)), one has

\[
\int_{\delta} \omega_n = (1-\zeta^{-nd_1})(1-\zeta^{-nd_2}) \int_0^1 \omega_n = (1-\zeta^{-nd_1})(1-\zeta^{-nd_2})B(\alpha_n,\beta_n)2F_1(\alpha_n,\beta_n;\alpha_n+\beta_n;t),
\]

where \( \sigma(y) = \zeta y \) and

\[\alpha_n := \left\{ \frac{-an}{N} \right\}, \quad \beta_n := \left\{ \frac{-bn}{N} \right\}.
\]

This shows that the monodromy on the 2-dimensional \( H_1(X_t,\mathbb{C})(\chi) \) is isomorphic to the monodromy of the hypergeometric equation

\[D_t(D_t + \alpha_n + \beta_n - 1) - t(D_t + \alpha_n)(D_t + \beta_n), \quad D_t := t \frac{d}{dt}
\]

with the Riemann scheme

\[
\begin{vmatrix}
  t = 0 & t = 1 & t = \infty \\
  0 & 0 & \alpha_n \\
 1 - \alpha_n - \beta_n & 0 & \beta_n
\end{vmatrix}
\]

In particular, the monodromy is irreducible as \( \alpha_n,\beta_n \notin \mathbb{Z} \).

3.3. Hodge numbers. Let \( f : X \rightarrow \mathbb{P}^1 \) be a HG fibration with multiplication by \((R_0,e_0)\). Following [4] §4.1, we consider motivic sheaves \( \mathcal{M} \) and \( \mathcal{M}^\times \) which are defined in the following way. Let \( S := \mathbb{A}^1_{\mathbb{Q}} \setminus \{0,1\} \) be defined over \( \overline{\mathbb{Q}} \) with coordinate \( \lambda \). Let \( \mathbb{P}^1_S := \mathbb{P}^1 \times S \) and denote the coordinates by \((t,\lambda)\). Put \( \mathbb{P}_S \supset \mathcal{W} := (\mathbb{A}^1_{\mathbb{Q}} \setminus \{0,1\} \times S) \setminus \Delta \) where \( \Delta \) is the diagonal subscheme. Let \( l \geq 1 \) be an
Then $S$ a projection onto a number field $k/l$, eigenvalues of $T$ are at most 2, and each $\delta$ has the Hodge numbers of the determinant $\det E W_2 \mathcal{H}(e)$.

Proof. We first note that $\dim_{\mathbb{Q}} W_2 \mathcal{H}(e) = 2$ (Proposition 4.3). We employ two results from [2] and [9] respectively. First of all, it follows from [2] Theorem 4.2 that one has the Hodge numbers of the determinant $D := \det E W_2 \mathcal{H}(e) = \Lambda^2 E W_2 \mathcal{H}(e)$. The result is

$$D(p, q, r, s) = \begin{cases} (0, 0, 0, 0) & \text{if } d_\chi = 2 \\ (0, 0, 1, 0, 0) & \text{if } d_\chi = 1 \\ (0, 0, 0, 1, 0) & \text{if } d_\chi = 0 \end{cases}$$
where we put $D^{p,4-p}_\chi := \dim \Gr^p F D(\chi)$. Since $D^{p,4-p}_\chi = 1 \iff 2h^{2,0}_\chi + h^{1,1}_\chi = p$, this implies

$$(h^{2,0}_\chi, h^{1,1}_\chi, h^{0,2}_\chi) = \begin{cases} (1, 1, 0) & \text{if } d_\chi = 2 \\ (0, 2, 0) \text{ or } (1, 0, 1) & \text{if } d_\chi = 1 \\ (0, 1, 1) & \text{if } d_\chi = 0 \end{cases}$$

which completes the proof in the case $d_\chi \neq 1$. Suppose $d_\chi = 1$. We want to show that $(h^{2,0}_\chi, h^{1,1}_\chi, h^{0,2}_\chi) = (1, 0, 1)$ cannot happen. By [3] Theorem 5.8, the underlying connection of $W_2 \mathcal{H}(\chi)$ is defined by the hypergeometric differential operator as in loc.cit. One can apply the main theorem in [9] and then the possible triplets of the Hodge numbers are at most $(2, 0, 0), (0, 2, 0), (0, 0, 2)$. In particular the case $(h^{2,0}_\chi, h^{1,1}_\chi, h^{0,2}_\chi) = (1, 0, 1)$ is excluded. This completes the proof in case $d_\chi = 1$. □

**Remark 3.4.** In the latter half of the proof of Theorem 3.3 there is an alternative discussion without using the main theorem of [9]. Let $\pi_0 : \mathbb{P}^1 \to \mathbb{P}^1$ be a map given by $t \mapsto -t^4$. Let $\mathcal{M}_0 := \pi_0^* \mathcal{O} \otimes R^1 f_* \mathbb{Q}$ be a VHdR on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. Put $H_0 := H^1(\mathbb{P}^1 \setminus \{0, 1, \infty\}, \mathcal{M}_0)$. Let $\psi_{\lambda=0}$ denotes the nearby cycle functor. Then one can construct an injection

$$E \cong W_2 H_0(e) \longrightarrow \psi_{\lambda=0} W_2 \mathcal{H}(e)$$

of mixed Hodge-de Rham structures. The cohomology group $W_2 H_0(e)$ is studied in detail in [4]. In particular, if $d_\chi = 1$, then the Hodge type of $W_2 H_0(\chi)$ is $(1, 1)$. Hence $h^{1,1}_\chi > 0$ by the above injection, which excludes the case $(h^{2,0}_\chi, h^{1,1}_\chi, h^{0,2}_\chi) = (1, 0, 1)$.

**Corollary 3.5.** $W_2 \mathcal{H}(e)$ is a Tate VHdR of type $(1, 1)$ if and only if $d_\chi = 1$ for all $\chi : R \to \mathbb{Q}$, equivalently

$$2\{-sk_0/l\} + \sum_{i=1}^2 \{s\beta_{i}^{\chi_0}\} - \{s(\beta_{i}^{\chi_0} - k_0/l)\} = 1$$

for all $\chi : R \to \mathbb{Q}$ where $\chi_0$ is a fixed one and $\beta_{i}^{\chi_0}, k_0$ are the rational numbers arising from $\chi_0$.

### 3.4. Beilinson Regulator

Let $\psi_{t=1}$ be the nearby cycle functor along the function $t - 1$ on $\mathcal{W}$, and put

$$C := \Gr^W_2 \psi_{t=1} \mathcal{M} \cong \pi_* \mathcal{Q} \otimes (\Gr^W_2 \psi_{t=1} R^1 f_* \mathcal{O})$$

a VHdR on $S$. Then there is a natural embedding

$$C \otimes \mathbb{Q}(-1) \longrightarrow \mathcal{H}/W_2$$

and it gives an extension

$$0 \longrightarrow W_2 \mathcal{H}(e) \longrightarrow E \longrightarrow C(e) \otimes \mathbb{Q}(-1) \longrightarrow 0 \quad (3.7)$$

of VMHdR with multiplication by $E$ which is induced from (3.6). Note $C(e)$ is one-dimensional over $E$ and endowed with Hodge type $(1, 1)$ by (e) in (3.1).

In [3] §5 we discussed the extension data of (3.7). More precisely let $E_{zar}$ be the Zariski sheaf of polynomial functions (with coefficients in $\mathbb{Q}$) on $S = L_{\infty} \setminus \{0, 1\}$ with coordinate $\lambda$. Let $E^{an}$ be the sheaf of analytic functions on $S^{an} = \mathbb{C}^{an} \setminus \{0, 1\}$. 
Let $a : S^{an} \to S^{zar}$ be the canonical morphism from the analytic site to the Zariski site. Set

$$\mathcal{J} := \text{Coker}[a^{-1} F^2 W_2 \mathcal{H}_{dR} \oplus a^{-1} W_2 \mathcal{H}_{B} \to \mathcal{O}_{\mathcal{S}} \otimes_{a^{-1} \mathcal{O}_{\mathcal{S}}} a^{-1} W_2 \mathcal{H}_{dR}]$$

a sheaf on the analytic site $\mathbb{C}^{an} \setminus \{0, 1\}$. Let $h : S \to S$ be a generically finite and dominant map such that $\sqrt{\lambda - 1} \in \mathbb{Q}(\bar{S})$. Then $h^* C(e)$ is a constant VHdR of type $(1, 1)$. The connecting homomorphism arising from (3.7) gives a map

$$\rho : h^* C(e) \otimes \mathbb{Q}(1) \to \Gamma(\mathcal{S}^{an}, h^* \mathcal{J}(e)) \quad (3.8)$$

(see [3] §5.2 for the detail). This agrees with the Beilinson regulator map on the motivic cohomology supported on singular fibers. Let $\pi : \mathbb{P}^1_S := \mathbb{P}^1 \times_{\bar{S}} S \to \mathbb{P}^1$ be given by $(s, \lambda') \mapsto h(\lambda') - s$. Let

$$\begin{array}{c}
X_S \\
\downarrow g \\
S
\end{array} \xleftarrow{i} \begin{array}{c}
\mathbb{P}^1_S \times_{\mathbb{P}^1} X \\
\downarrow f_S \\
\mathbb{P}^1_S
\end{array} \xrightarrow{\pi} \begin{array}{c}
X \\
\downarrow f \\
\mathbb{P}^1
\end{array}$$

with $i$ desingularization and $p$ the 2nd projection. Let

$$\text{reg} : H^3_{\mathcal{M}}(X_S, \mathbb{Q}(2)) \to H^3_{\mathcal{M}}(X_S, \mathbb{Q}(2)) = \text{Ext}^3_{\text{MHM}(X_S)}(\mathbb{Q}, \mathbb{Q}(2))$$

be the Beilinson regulator map where $\text{MHM}(\bar{S})$ denotes the category of mixed Hodge modules on $\bar{S}$. There is the canonical surjective map

$$\text{Ext}^3_{\text{MHM}(X_S)}(\mathbb{Q}, \mathbb{Q}(2)) \to \text{Ext}^1_{\text{VMHdR}(\bar{S})}(\mathbb{Q}, R^2 g_* \mathbb{Q}(2)).$$

Let $U_{\bar{S}} \subset \mathbb{P}^1_{\bar{S}}$ be a Zariski open set on which $f_S$ is smooth and projective. Put

$$H^3_{\mathcal{M}}(X_S, \mathbb{Q}(2))_0 := \text{Ker}[H^3_{\mathcal{M}}(X_S, \mathbb{Q}(2)) \to H^3_{\mathcal{M}}(f^{-1}_S(U_{\bar{S}}), \mathbb{Q}(2))]$$

and $(R^2 g_* \mathbb{Q}(2))_0 := \text{Ker}[R^2 g_* \mathbb{Q}(2) \to p_* (R^2 (f_S)_* \mathbb{Q}(2)|_{U_{\bar{S}}})]$. Then there is the canonical surjective map $(R^2 g_* \mathbb{Q}(2))_0 \to h^* W_2 \mathcal{H}(2)$ and the Beilinson regulator map induces

$$\text{reg}_0 : H^3_{\mathcal{M}}(X_S, \mathbb{Q}(2))_0 \to \text{Ext}^1_{\text{VMHdR}(\bar{S})}(\mathbb{Q}, h^* W_2 \mathcal{H}(2)) \to \Gamma(\mathcal{S}^{an}, h^* \mathcal{J}).$$

The compatibility with [3] is given by the commutativity of a diagram

$$\begin{array}{c}
H^3_{\mathcal{M}}(X_S, \mathbb{Q}(2)) \\
\downarrow \rho \\
H^3_{\mathcal{M}}(X_S, \mathbb{Q}(2))_0 \xrightarrow{\text{reg}_0} \Gamma(\mathcal{S}^{an}, h^* \mathcal{J})
\end{array} \quad (3.9)$$

where $D_{\bar{S}} := X_{\bar{S}} \setminus U_{\bar{S}}$. 
3.5. **Regulator Formula for HG fibrations of Gauss type.** One of the main results in [3] (which we call regulator formula) is an explicit description of the map $\rho$ in (3.8). Here we apply [3] Theorem 5.9 (=a precise version of regulator formula) to the case that $f$ is a HG fibration of Gauss type (see Definition 3.2).

Let $f : X \to \mathbb{P}^1$ be a HG fibration of Gauss type with multiplication by $(R_0 := \mathbb{Q}[\mu_N], e_0)$ as in Definition 3.2. Let $\chi : E_0 \to \mathbb{Q}$ be a homomorphism such that $\sigma(\zeta) = \zeta^{-n}$. Recall from (3.2) that $F^1H^1_{\text{dR}}(X_t)(\chi)$ is one dimensional and spanned by a 1-form

$$\omega_n := \frac{x^a(n - x)^b(1 - tx)^c}{y^n} dx,$$

$$a_n := \left\lfloor \frac{an}{N} \right\rfloor, \ b_n := \left\lfloor \frac{bn}{N} \right\rfloor, \ c_n := \left\lfloor \frac{Nn - bn}{N} \right\rfloor = n - b_n - 1$$

where $n \in \{1, 2, \ldots, N - 1\}$ such that $\chi(\zeta) = \zeta^{-n}$ for $\forall \zeta \in \mu_N$.

**Lemma 3.6.** Let $D_0, D_1$ be the reduced singular fibers over $t = 0, 1$. We assume that $D_0 + D_1$ is a NCD. Then $\omega_n \in \Gamma(\mathbb{P}^1 \setminus \{\infty\}, f_*\Omega^1_{X/\mathbb{P}^1}(\log D_0 + D_1))$.

**Proof.** Put $S = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ and $U = f^{-1}(S)$. Let $\mathcal{H} := H^1_{\text{dR}}(U/S)$ be the bundle and $\nabla : \mathcal{H} \to \Omega^1_{S} \otimes \mathcal{H}$ the Gauss-Manin connection. Let $D_\infty$ be the reduced singular fibers over $t = \infty$ and assume that it is a NCD. Put $T := \{0, 1, \infty\}$. Recall that the sheaf $\Omega^1_{X/\mathbb{P}^1}(\log D)$ ($D := D_0 + D_1 + D_\infty$) is defined by the exact sequence

$$0 \to f_*\Omega^1_{\mathbb{P}^1}(\log T) \to \Omega^1_X(\log D) \to \Omega^1_{X/\mathbb{P}^1}(\log D) \to 0.$$ 

Let $\mathcal{H}_\xi$ be Deligne’s canonical extension over $\mathbb{P}^1$. This is characterized as a subbundle $\mathcal{H}_\xi \subset j_*\mathcal{H}$ ($j : S \hookrightarrow \mathbb{P}^1$) which satisfies

- $\nabla$ has at most log poles, $\nabla : \mathcal{H}_\xi \to \Omega^1_{\mathbb{P}^1}(\log(0 + 1 + \infty)) \otimes \mathcal{H}_\xi$,
- The eigenvalues of residue $\text{Res}(\nabla)$ at $t = 0, 1, \infty$ belong to $[0, 1)$.

Then there is an isomorphism

$$\mathcal{H}_\xi \cong R^1f_*\Omega^1_{X/\mathbb{P}^1}(\log D)$$

(11.20) and $F^1\mathcal{H}_\xi := \mathcal{H}_\xi \cap j_*F^1\mathcal{H} \cong f_*\Omega^1_{X/\mathbb{P}^1}(\log D)$ (loc.cit. 4.20 (ii)). Hence the desired assertion is equivalent to

$$t\omega_n \in \Gamma(\mathbb{P}^1 \setminus \{\infty\}, \mathcal{H}_\xi).$$

To show this, we give a local frame of $\mathcal{H}_\xi$ at $t = 0, 1$ explicitly. Let

$$\eta_n := \frac{x^a(n - x)^b(1 - tx)^c}{y^n} dx,$$

and put

$$\beta_1^\chi := \left\{\frac{-an}{N}\right\}, \ \beta_2^\chi := \left\{\frac{-bn}{N}\right\}.$$

Recall from (3.2) a homology cycle $\delta := (1 - \sigma^{d_1})(1 - \sigma^{d_2})u \in H_1(X_t, \mathbb{Z})$. Then

$$\int_{\delta} \omega_n = (1 - \zeta^{-d_1})(1 - \zeta^{-d_2})B(\beta_1^\chi, \beta_2^\chi)F(\beta_1^\chi, \beta_2^\chi, \beta_1^\chi + \beta_2^\chi; t),$$

$$\int_{\delta} \eta_n = (1 - \zeta^{-d_1})(1 - \zeta^{-d_2})B(\beta_1^\chi, \beta_2^\chi + 1)F(\beta_1^\chi, \beta_2^\chi, 1 + \beta_1^\chi + \beta_2^\chi; t).$$


This shows that $\omega_n$ and $\eta_n$ are basis of the $\chi$-part $\mathcal{H}(\chi)$ of the bundle (over a Zariski open set of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$). Denote by $\mathcal{H}(\chi)^*$ the dual connection, and by $\{\omega_n^*, \eta_n^*\}$ the dual basis. Then

$$\left( \int_{\delta} \omega_n \right) \omega_n^* + \left( \int_{\delta} \eta_n \right) \eta_n^*$$

is annihilated by the dual connection, and hence

$$d \left( \int_{\delta} \omega_n \right) \omega_n^* + d \left( \int_{\delta} \eta_n \right) \eta_n^* + \left( \int_{\delta} \omega_n \right) \nabla(\omega_n^*) + \left( \int_{\delta} \eta_n \right) \nabla(\eta_n^*) = 0. \quad (3.13)$$

Now (3.10) is immediate. □

Let $e_0 : \mu_N \to E_0^\chi$ be an injective homomorphism. Then the condition in Lemma 3.11 is satisfied. Let $e : R := \mathbb{Q}[\mu, \mu_N] \to E$ be a projection such that $\text{Ker}(e) \supset \text{Ker}(e_0)$. Let $\chi : R \to \mathbb{Q}$ be a homomorphism factoring through $e$. Fix integers $k, n$ such that

$$\chi(\zeta_1, \zeta_2) = \zeta_1^k \zeta_2^n, \quad \forall (\zeta_1, \zeta_2) \in \mu^k \times \mu^N.$$

Note $\gcd(n, N) = 1$ as $e_0 : \mu_N \to E_0^\chi$ is injective. Let

$$\beta_1 := \left\{ \frac{-na}{N} \right\}, \quad \beta_2 := \left\{ \frac{-nb}{N} \right\}, \quad \alpha_1 := 0, \quad \alpha_2 := 1 - \beta_1^N - \beta_2^N \quad (3.14)$$

which do not depend on the choice of $n$. Then $e^{2\pi i \alpha_1}$ (resp. $e^{2\pi i \beta_2}$) are eigenvalues of the local monodromy $T_0$ at $t = 0$ (resp. $T_\infty$ at $t = \infty$) on $R^1 f_* \mathcal{C}(\chi) \cong \mathbb{C}^2$ (see (3.9)). The relative 1-form $\omega := t \omega_n$ satisfies the conditions $\mathbf{P1}, \mathbf{P2}$ in 3 §4.5:

$\mathbf{P1}$ $\int_{\gamma} \omega$ (with $\gamma \in H_1(X_t)$) is spanned by $tF(\beta_1^N, \beta_2^N, 1; 1 - t)$ and $tF(\beta_1^N, \beta_2^N, \beta_1^N + \beta_2^N; t)$. (This follows from (3.11)).

$\mathbf{P2}$ $\omega \in F(\mathbb{P}^1 \setminus \{\infty\}; f_* \Omega^1_{X/P}(\log D))$. (This is Lemma 3.10).

We thus can apply the regulator formula (3 Thm.5.9). In our particular case, it is stated as follows (the notation is slightly changed for the use in below).
Theorem 3.7. Let $e_0, e, \chi$ be as above, and let $\alpha_i^\chi, \beta_j^\chi$ be as in (3.14). Assume that $k/l, k/l - \beta_1^\chi, k/l - \beta_2^\chi, k/l - \beta_3^\chi - \beta_2^\chi \notin \mathbb{Z}$. Put

$$
\mathcal{F}_1(\lambda) := (1 - \lambda)^{k/l-1} F_3 \left( \frac{1, 1, 1 - k/l}{2 - \beta_1^\chi, 2 - \beta_2^\chi}; (1 - \lambda)^{-1} \right),
$$

$$
\mathcal{F}_2(\lambda) := (1 - \lambda)^{k/l-1} F_3 \left( \frac{1, 1, 2 - k/l}{2 - \beta_1^\chi, 2 - \beta_2^\chi}; (1 - \lambda)^{-1} \right).
$$

Let $\rho(\chi)$ be the $\chi$-part of the map $\rho$ in (3.3). Let

$$
\rho(\chi) = (\phi_1(\lambda), \phi_2(\lambda)) \in (\mathcal{O}^{an})^{\otimes 2} \cong \mathcal{O}^{an} \otimes W_2 \mathcal{H}_{\text{dr}}(\chi)
$$

be a local lifting where the above isomorphism is with respect to the $\mathbb{Q}$-frame of $W_2 \mathcal{H}_{\text{dr}}(\chi)$. Define rational functions $E^{(r)}_i = E^{(r)}_i(\chi) \in \mathbb{Q}(\chi)$ for $r \in \mathbb{Z}_{\geq -1}$ in the following way. Write $a := 2 - \beta_1^\chi, b := 2 - \beta_2^\chi$. Put

$$
A(s) := \frac{s(a + b + 2s - 3 - s(1 - \lambda)^{-1})}{(a + s - 1)(b + s - 1)}, \quad B(s) := \frac{s(1 - s)(1 - (1 - \lambda)^{-1})}{(a + s - 1)(b + s - 1)}.
$$

Define $C_i(s)$ and $D_i(s)$ by

$$
\begin{pmatrix}
C_{i+1}(s) \\
D_{i+1}(s)
\end{pmatrix}
= \begin{pmatrix}
A(s) & 1 \\
B(s) & 0
\end{pmatrix}
\begin{pmatrix}
C_i(s + 1) \\
D_i(s + 1)
\end{pmatrix}, \quad \begin{pmatrix}
C_{1}(s) \\
D_{1}(s)
\end{pmatrix} := \begin{pmatrix}
0 \\
1
\end{pmatrix}.
$$

Then $E^{(r)}_i$ is given by

$$
E^{(r)}_1 = \lambda C_0(k/l) + (1 - \lambda)C_{r+1}(k/l), \quad E^{(r)}_2 = \lambda D_0(k/l) + (1 - \lambda)D_{r+1}(k/l).
$$

(3.15)

Then for infinitely many integers $r > 0$, we have

$$
\phi_1(\lambda) \equiv C_1(1 - \lambda)^{-r}[E^{(r)}_1(\chi), \mathcal{F}_1(\lambda)] + E^{(r)}_2(\chi)\mathcal{F}_2(\lambda),
$$

$$
\phi_2(\lambda) \equiv C_2(1 - \lambda)^{-r-1}[E^{(r-1)}_1(\chi), \mathcal{F}_1(\lambda)] + E^{(r-1)}_2(\chi)\mathcal{F}_2(\lambda)]
$$

modulo $\mathbb{Q}(\lambda)$ with some $C_1, C_2 \in \mathbb{Q}^\times$.

We note that $(N, l, k, n, a, b)$ in Theorem 3.7 can run over the set of all pairs of integers satisfying

- $0 < a, b < N$, $\gcd(N, a, b) = 1$ and $\gcd(n, N) = 1$,
- $k/l, k/l - \beta_1^\chi, k/l - \beta_2^\chi, k/l - \beta_3^\chi - \beta_2^\chi \notin \mathbb{Z}$ (see (3.14) for definition of $\beta_j^\chi$).

4. Proof of Main Theorem

We are now in a position to prove Theorem 2.1 (Log Formula).

There are formulas

$$
(b_1 - 1) F_2 \left( \frac{a_1, a_2, a_3}{b_1, b_2}; x \right) = \left( b_1 - 1 + x \frac{d}{dx} \right) F_2 \left( \frac{a_1, a_2, a_3}{b_1, b_2}; x \right),
$$

$$
a_1 \cdot F_2 \left( \frac{a_1 + 1, a_2, a_3}{b_1, b_2}; x \right) = \left( a_1 + x \frac{d}{dx} \right) F_2 \left( \frac{a_1, a_2, a_3}{b_1, b_2}; x \right),
$$

$$
(a_2 - b_1)(a_1 - b_1)(a_3 - b_1) F_2 \left( \frac{a_1, a_2, a_3}{b_1 + 1, b_2}; x \right) = \theta_1 \left( F_2 \left( \frac{a_1, a_2, a_3}{b_1, b_2}; x \right) \right),
$$

$$
(a_1 - b_1)(a_1 - b_2) F_2 \left( \frac{a_1 - 1, a_2, a_3}{b_1, b_2}; x \right) = \theta_2 \left( F_2 \left( \frac{a_1, a_2, a_3}{b_1, b_2}; x \right) \right),
$$

$$
(a_2 - b_1)(a_2 - b_1)(a_3 - b_1) F_2 \left( \frac{a_1, a_2, a_3}{b_1 + 1, b_2}; x \right) = \theta_1 \left( F_2 \left( \frac{a_1, a_2, a_3}{b_1, b_2}; x \right) \right),
$$

$$
(a_1 - b_1)(a_1 - b_2) F_2 \left( \frac{a_1 - 1, a_2, a_3}{b_1, b_2}; x \right) = \theta_2 \left( F_2 \left( \frac{a_1, a_2, a_3}{b_1, b_2}; x \right) \right),
$$

$$
(a_2 - b_1)(a_2 - b_1)(a_3 - b_1) F_2 \left( \frac{a_1, a_2, a_3}{b_1 + 1, b_2}; x \right) = \theta_1 \left( F_2 \left( \frac{a_1, a_2, a_3}{b_1, b_2}; x \right) \right),
$$

$$
(a_1 - b_1)(a_1 - b_2) F_2 \left( \frac{a_1 - 1, a_2, a_3}{b_1, b_2}; x \right) = \theta_2 \left( F_2 \left( \frac{a_1, a_2, a_3}{b_1, b_2}; x \right) \right),
$$

$$
(a_2 - b_1)(a_2 - b_1)(a_3 - b_1) F_2 \left( \frac{a_1, a_2, a_3}{b_1 + 1, b_2}; x \right) = \theta_1 \left( F_2 \left( \frac{a_1, a_2, a_3}{b_1, b_2}; x \right) \right),
$$

$$
(a_1 - b_1)(a_1 - b_2) F_2 \left( \frac{a_1 - 1, a_2, a_3}{b_1, b_2}; x \right) = \theta_2 \left( F_2 \left( \frac{a_1, a_2, a_3}{b_1, b_2}; x \right) \right).
where
\[
\begin{align*}
\theta_1 &:= -a_1a_2a_3 + (a_2 - b_1)(a_1 - b_1)(a_3 - b_1) \\
&\quad + b_1(b_2 + (b_1 - a_1 - a_2 - a_3 - 1)x)\frac{d}{dx} + b_1(x - x^2)\frac{d^2}{dx^2} \\
\theta_2 &:= (a_1 - b_1)(a_1 - b_2) - a_2a_3x \\
&\quad + ((b_1 + b_2 - a_1) - (a_2 + a_3 + 1)x)\frac{d}{dx} + (1 - x)x^2\frac{d^2}{dx^2}.
\end{align*}
\]
Therefore if one can show the log formula for \(3F_2\) then one immediately has the log formula for \(3F_2\) for arbitrary integers \(n_1, n_2 > 0\) and \(n_3, n_4, n_5 \in \mathbb{Z}\).

We keep the setting and the notation in §3.5. Suppose that
\[
1 = 2\{s\beta_2^3\} + \sum_{i=1}^2\{s\beta_i^3\} - \{s(\beta_i^3 - k/l)\}, \quad \forall s \in \tilde{\mathbb{Z}}^\times. \tag{4.1}
\]
Then it follows from Corollary 3.3 that \(W_2\mathcal{H}(e)\) is a Tate HdR structure of type \((1, 1)\). Let us look at the map \(\rho^{(\chi)}\) in Theorem 3.7. This turns out to be the Beilinson regulator by the diagram (3.9). Since \(W_2\mathcal{H}(e)\) is Tate, it is generated by the divisor classes of the geometric generic fiber \(X_{\mathbb{F}}\) of \(\mathcal{G}\). This implies that the image of \(\text{reg}\) in (3.9) is generated by the images of \(H^1_{\text{dR}}(D_i, \mathbb{Q}(1))\) where \(D_i\) runs over the generators of the Neron-Severi group \(\text{NS}(X_{\mathbb{F}}) \otimes \mathbb{Q}\) and \(\tilde{D}_i \rightarrow D_i\) is the desingularization. As is well-known, \(H^1_{\text{dR}}(\tilde{D}_i, \mathbb{Q}(1)) \cong \pi_i^* \otimes \mathbb{Q}\) as \(\tilde{D}_i\) is smooth projective, and the Beilinson regulator on it is given by the logarithmic function. Therefore we have
\[
\phi_1(\lambda), \phi_2(\lambda) \in \overline{\mathbb{Q}(\lambda)} + \mathbb{Q}(\lambda)\log \mathbb{Q}(\lambda)^\times. \tag{4.2}
\]
We now apply Theorem 3.7. If one can show
\[
\begin{vmatrix}
E_1^{(r)}(s) & E_2^{(r)}(s) \\
E_1^{(r-1)}(s) & E_2^{(r-1)}(s)
\end{vmatrix} \neq 0
\]
for almost all \(r > 0\), then this implies \(\mathcal{F}_i(\lambda) \in \overline{\mathbb{Q}(\lambda)} + \mathbb{Q}(\lambda)\log \mathbb{Q}(\lambda)^\times\), which finishes the proof of Theorem 2.1. To do this, recall (3.13). Letting
\[
E_1^{(r)}(s) := \lambda C_r(s) + (1 - \lambda)C_{r+1}(s), \quad E_2^{(r)}(s) := \lambda D_r(s) + (1 - \lambda)D_{r+1}(s),
\]
we want to show
\[
\begin{vmatrix}
E_1^{(r)}(s) & E_2^{(r)}(s) \\
E_1^{(r-1)}(s) & E_2^{(r-1)}(s)
\end{vmatrix} \neq 0 \tag{4.3}
\]
for almost all \(r > 0\). Since
\[
\begin{pmatrix}
E_1^{(r+1)}(s) & E_2^{(r+1)}(s) \\
E_2^{(r+1)}(s) & E_2^{(r+1)}(s)
\end{pmatrix} = \begin{pmatrix}
A(s) & 1 \\
B(s) & 0
\end{pmatrix}
\begin{pmatrix}
E_1^{(r)}(s + 1) & E_1^{(r-1)}(s + 1) \\
E_2^{(r)}(s + 1) & E_2^{(r-1)}(s + 1)
\end{pmatrix}
\]
(4.3) is reduced to show
\[
\begin{vmatrix}
E_1^{(0)}(k/l + r) & E_2^{(0)}(k/l + r) \\
E_1^{(-1)}(k/l + r) & E_2^{(-1)}(k/l + r)
\end{vmatrix} \neq 0
\]
for any integers \( r \). However this follows by
\[
\begin{vmatrix}
E_1^{(0)}(s) & E_2^{(0)}(s) \\
E_1^{(-1)}(s) & E_2^{(-1)}(s)
\end{vmatrix}
= \frac{\lambda + (1 - \lambda)A(s)}{1 - \lambda} \begin{pmatrix}
1 & (1 - \lambda)B(s) \\
\lambda & \lambda
\end{pmatrix}
\]
\[
= \lambda \frac{(a - 1)(b - 1)\lambda + s(a + b - 2)}{(s + a - 1)(s + b - 1)}, \quad (a := 2 - \beta_1^x, b := 2 - \beta_2^x)
\]
and the fact \( \beta_i^x \not\in \mathbb{Z} \) (see (3.14)) and \( k/l - \beta_i^x \not\in \mathbb{Z} \) as is assumed. This completes the proof of Theorem 2.1.

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