The Nambu–Jona-Lasinio model at $O(1/N^2)$.

J.A. Gracey,
Department of Applied Mathematics and Theoretical Physics,
University of Liverpool,
P.O. Box 147,
Liverpool,
L69 3BX,
United Kingdom.

Abstract. We write down the anomalous dimensions of the fields of the Nambu–Jona-Lasinio model or chiral Gross Neveu model with a continuous global chiral symmetry for the two cases $U(1) \times U(1)$ and $SU(M) \times SU(M)$ at $O(1/N^2)$ in a $1/N$ expansion.
The Nambu–Jona-Lasinio, (NJL), model was introduced in [1] as a theory with a continuous global chiral symmetry which is broken dynamically. It has also been studied in the context of hadronic physics following the early work of [2] and lately in, for example, [3], and chiral perturbation theory [4] since it corresponds to a low energy effective theory of the strong interactions. Further, there has also been significant interest in analysing models with four fermi interactions, like the NJL model, in other contexts such as the standard model, [5, 6, 7]. For instance, it was initially pointed out in [5] and subsequently studied in more detail in [6, 7] that the Higgs boson of the standard model could be regarded as a composite field built out of fermions in much the same way that the $\sigma$ and $\pi$ fields of the two dimensional chiral Gross Neveu, (CGN), model are bound states of the fundamental fermions of the model, [8].

One of the most widely used techniques to examine the CGN or NJL models is the large $N$ expansion where the number of fundamental fields $N$ is allowed to become large, [8], and $1/N$ can therefore be used as a dimensionless coupling constant. Consequently one can show, for example, that the models are renormalizable in $2 \leq d < 4$ dimensions in this approach [8] and simultaneously deduce that they possess a rich structure, such as dynamical symmetry breaking. Whilst the leading order large $N$ analysis is well understood, it is important to go beyond the leading order to improve our knowledge of the quantum structure. Recently, new techniques to achieve this for theories with fermionic interactions were introduced in [10] for the $O(N)$ Gross Neveu model based on the critical point self consistency methods to calculate critical exponents in the bosonic $O(N)$ $\sigma$ model, [11, 12]. In this letter we present the results of the application of the same techniques to the 4-fermi models with continuous chiral symmetry by writing down the anomalous dimensions of each of the fields in arbitrary dimensions at $O(1/N^2)$. This is one order beyond any previous analysis and it is necessary to have such higher order expressions in order to improve our understanding of the areas mentioned above. A further motivation for such a computation lies in the accurate independent evaluation of these quantities in three dimensions in order to provide precise estimates to compare with recent numerical simulations of 4-fermi models, [13].

More concretely, the lagrangians of the models we have considered are, [8],

$$L = i\bar{\psi}i\slashed{D}\psi + \sigma\bar{\psi}\gamma^5\psi + i\pi\bar{\psi}\gamma^5\psi - \frac{1}{2g^2}(\sigma^2 + \pi^2)$$

(1)
for the NJL model with a global $U(1) \times U(1)$ chiral symmetry and

$$L = i\bar{\psi}^I \sigma^I \psi^I + \sigma^I \bar{\psi}^I \psi^I + i\pi^a \bar{\psi}^I \gamma^5 \lambda^a_{IJ} \psi^J - \frac{1}{2g^2}(\sigma^2 + \pi^2)$$  \hspace{1cm} (2)$$

for the same model but with an $SU(M) \times SU(M)$ chiral symmetry. The bosonic fields $\sigma$ and $\pi^a$ are auxiliary and $g$ is the perturbative coupling constant. In (2) the generalized Pauli matrices, $\lambda^a_{IJ}$, $1 \leq a \leq (M^2 - 1)$, $1 \leq I \leq M$ are normalized to $\text{Tr}(\lambda^a \lambda^b) = 4T(R)\delta^{ab}$ and we used the conventions of [15] and [16]. Both (1) and (2) involve $N$-tuplets of fermions $1 \leq i \leq N$ and this $N$ will become our expansion parameter. We also use the conventions of [13] in defining the properties of $\gamma^5$ as $\{\gamma^\mu, \gamma^5\} = 0$, $\text{tr}(\gamma^5 \gamma^\mu_1 \ldots \gamma^\mu_n) = 0$ and $(\gamma^5)^2 = 1$ with $\text{tr}1 = 2$. We remark that our $\gamma^5$ conventions in $d$-dimensions retain the anticommutativity property. This differs from the definition given in [17] which is one formulation used to perform consistent renormalization calculations using dimensional regularization where the spacetime dimension is changed to provide a way of handling infinities, [17, 18]. There, [17], one loses Lorentz invariance in the full $d$-dimensional space where $[\gamma^\mu, \gamma^5] = 0$ when $\mu$ is not an index in the physical spacetime. By contrast, the method of [11] uses propagators defined in arbitrary but fixed dimensions which are consistent with Lorentz and conformal symmetry. Therefore it seems more appropriate here to retain the anticommutativity of the $\gamma^5$ for a Lorentz invariant formulation. (Indeed an alternative to [17] for treating $\gamma^5$ in dimensionally regularized calculations retains this condition, [18].)

In [10], the model with $\pi = 0$ was solved at $O(1/N^2)$ by the self-consistency approach of obtaining critical exponents and we have used the same methods to treat (1) and (2). Briefly, this involves solving for critical exponents at the $d$-dimensional fixed point of the field theory where the model possesses an extra scaling or conformal symmetry. With this scaling property one writes down the most general form the propagators of the fields can take, consistent with Lorentz and conformal symmetry, where the powers of the scaling form are related to the dimension and therefore the anomalous dimension of the fields, [13]. To obtain analytic expressions for these critical exponents one represents the skeleton Dyson equations of various Green’s functions by the scaling forms and solves the resulting representation of the Dyson equations for the unknown exponents order by order in $1/N$. The power of the method is illustrated by the fact that the leading order results are deduced algebraically and agree with previous work, whilst the new results at the subsequent order are deduced with a minimal amount of effort, [10-12].
For (1), the asymptotic scaling forms of the respective propagators are, in coordinate space, in the critical region,

\[ \psi(x) \sim \frac{A x}{(x^2)^{\alpha}}, \quad \sigma(x) \sim \frac{B x^2}{(x^2)^{\beta}}, \quad \pi(x) \sim \frac{C x^2}{(x^2)^{\gamma}} \]

(3)

where \( A, B \) and \( C \) are amplitudes which are independent of \( x \) and the dimensions of the fields are defined as

\[ \alpha = \mu + \frac{1}{2} \eta, \quad \beta = 1 - \eta - \chi_{\sigma}, \quad \gamma = 1 - \eta - \chi_{\pi} \]

(4)

where \( d = 2 \mu \) is the spacetime dimension, \( \eta \) is the fermion anomalous dimension and \( \chi_{\sigma} \) and \( \chi_{\pi} \) are the anomalous dimensions of the respective vertices. The anomalous dimensions are expanded in powers of \( 1/N \) via, for example, \( \eta = \sum_{i=1}^{\infty} \eta_i/N^i \). By substituting the scaling forms (3) into the skeleton Dyson equations with dressed propagators, which are illustrated in figs. 1-3, we were able to determine \( \eta_2 \) using results obtained in [13]. We found

\[ \eta_1 = -\frac{2\Gamma(2\mu - 1)}{\Gamma(\mu - 1)\Gamma(1 - \mu)\Gamma(\mu + 1)\Gamma(\mu)} \]

(5)

\[ \eta_2 = \frac{\eta_1^2}{\left[ \Psi(\mu) + \frac{2}{\mu - 1} + \frac{1}{2\mu} \right]} \]

(6)

where \( \Psi(\mu) = \psi(2\mu - 1) - \psi(1) + \psi(2 - \mu) - \psi(\mu) \) and \( \psi(x) \) is the logarithmic derivative of the \( \Gamma \)-function, as well as \( \chi_{\sigma 1} = \chi_{\pi 1} = 0 \). Equation (6) represents the first \( O(1/N^2) \) quantity to be determined for (1).

The \( O(1/N^2) \) corrections to the vertex anomalous dimensions were deduced by following the analogous calculation for the \( O(N) \) Gross Neveu model given in [16]. It involves studying the scaling properties of the 3-point functions at \( O(1/N^2) \) using a method which developed the leading order work of [19, 20] for the bosonic \( \sigma \) model on \( CP(N) \). Rather than illustrate the large number of graphs which arise at \( O(1/N^2) \), we have given in figure 4 the basic structure of the distinct graphs which arise, though the graphs with vertex counterterms have not been shown. The basic integrals corresponding to each of the graphs have been given in [13] and it was therefore a straightforward exercise to manipulate the graphs which occur in (1) to be proportional to integrals whose values are already known, [16]. For (1), at \( O(1/N^2) \) we found that the degeneracy in the \((\sigma, \pi)\) sector was not lifted but unlike at leading order the fields now have a non-zero anomalous dimension, which is a new feature, ie

\[ \chi_{\sigma 2} = \chi_{\pi 2} = -\frac{\mu^2(4\mu^2 - 10\mu + 7)\eta_1^2}{2(\mu - 1)^3} \]

(7)
We have also carried out the same calculation for the non-abelian case (2) and we note the results obtained at leading order are

\[ \eta_1 = \frac{\tilde{\eta}_1}{2} \left[ \frac{1}{M} + \frac{C_2(R)}{T(R)} \right] \]  

where \( \tilde{\eta}_1 = -2\Gamma(2\mu - 1)/[\Gamma(\mu + 1)\Gamma(1 - \mu)\Gamma(\mu - 1)] \) and

\[ \chi_{\sigma 1} = \frac{\mu \tilde{\eta}_1}{2(\mu - 1)} \left[ \frac{1}{M} - \frac{C_2(R)}{T(R)} \right] \]

\[ \chi_{\pi 1} = \frac{\mu \tilde{\eta}_1}{2(\mu - 1)} \left[ \frac{C_2(R)}{T(R)} - \frac{1}{M} - \frac{C_2(G)}{2T(R)} \right] \]

where \( \lambda^a \lambda^a = 4C_2(R)I \), \( \delta^{ac}d^{bd} = C_2(G)\delta^{ab} \) and \( C_2(R) = (M^2 - 1)/2M \), \( C_2(G) = M \) for \( SU(M) \), \[14, 15\]. Several leading order exponents were calculated in \[13\] for \( SU(2) \times SU(2) \) and our results are in agreement with them which provides us with a partial check on our calculation. At next to leading order the expressions are more involved compared to (6) and (7).

We found

\[ \eta_2 = \frac{\tilde{\eta}_1^2}{4} \left[ \left( \frac{1}{M} + \frac{C_2(R)}{T(R)} \right)^2 \left( \Psi(\mu) + \frac{2}{\mu - 1} + \frac{1}{2\mu} \right) \right. \]

\[ + \frac{\mu}{(\mu - 1)} \left( \left( \frac{1}{M} - \frac{C_2(R)}{T(R)} \right)^2 + \frac{C_2(G)C_2(R)}{2T^2(R)} \right) \left( \Psi(\mu) + \frac{3}{2(\mu - 1)} \right) \]  

\[ \chi_{\sigma 2} = \frac{\mu \tilde{\eta}_1^2}{4(\mu - 1)^2} \left[ (2\mu - 1) \left( \frac{1}{M^2} - \frac{C_2^2(R)}{T^2(R)} \right) \left( \Psi(\mu) + \frac{1}{(\mu - 1)} \right) \right. \]

\[ + \frac{\mu C_2(R)C_2(G)}{2T^2(R)} \left( \Psi(\mu) + \frac{1}{(\mu - 1)} \right) + \frac{3\mu}{2(\mu - 1)} \left( \frac{1}{M} - \frac{C_2(R)}{T(R)} \right)^2 \]

\[ + \frac{5\mu C_2(R)}{(\mu - 1)T(R)} \left( \frac{1}{M} - \frac{C_2(R)}{T(R)} \right) - \frac{2\mu}{(\mu - 1)} \left( \frac{1}{M^2} - \frac{C_2^2(R)}{T^2(R)} \right) \]

\[ + \frac{\mu^2}{2(\mu - 1)} \left( \frac{1}{M} + \frac{C_2(R)}{T(R)} \right)^2 - \frac{\mu(2\mu^2 - 5\mu + 4)}{(\mu - 1)M} \left( \frac{1}{M} + \frac{3C_2(R)}{T(R)} \right) \]

\[ + \frac{\mu}{M} \left( 3(\mu - 1)\Theta(\mu) - \frac{(2\mu - 3)}{(\mu - 1)} \right) \left( \frac{1}{M} + \frac{C_2(R)}{T(R)} \right) \]  

and

\[ \chi_{\pi 2} = \frac{\mu \tilde{\eta}_1^2}{4(\mu - 1)^2} \left[ \Psi(\mu) + \frac{1}{(\mu - 1)} \right] \]
\[ \begin{align*}
&\times \left[ (2\mu - 1) \left( \frac{1}{M} + \frac{C_2(R)}{T(R)} \right) \left( \frac{C_2(R)}{T(R)} - \frac{1}{M} - \frac{C_2(G)}{2T(R)} \right) \\
&- \frac{\mu C_2(G)}{2T(R)} \left( \frac{C_2(R)}{T(R)} - \frac{2}{M} - \frac{C_2(G)}{2T(R)} \right) \right] \\
&+ \frac{3\mu^2 \eta_1^2}{4(\mu - 1)^3} \left( \frac{C_2(R)}{T(R)} - \frac{1}{M} - \frac{C_2(G)}{2T(R)} \right)^2 \\
&+ \frac{5\mu^2 \eta_1^2}{16(\mu - 1)^2 M} \left[ 4 \left( \frac{C_2(R)}{T(R)} - \frac{1}{M} \right) - \frac{C_2(G)}{T(R)} \right] \\
&+ \frac{2\mu^2 \eta_1^2}{(\mu - 1)^3} \left[ \frac{1}{M^2} - \frac{C_2^2(R)}{T^2(R)} - \frac{C_2(G)}{MT(R)} - \frac{C_2^2(G)}{8T^2(R)} + \frac{C_2(R)C_2(G)}{T^2(R)} \right] \\
&+ \frac{1}{16} \left( \frac{C_2(R)}{T(R)} + \frac{1}{M} \right)^2 - \frac{C_2(G)}{MT(R)} - \frac{3C_2(G)C_2(R)}{2T^2(R)} + \frac{C_2^2(G)}{2T^2(R)} \\
&- \frac{3}{16} \left( \frac{1}{M} - \frac{C_2(R)}{2T(R)} \right)^2 - \frac{(2\mu^2 - 5\mu + 4)}{8} \left( \frac{3}{M^2} + \frac{C_2(R)}{T(R)M} \right) \\
&- \frac{C_2(G)}{4T(R)} \left( \frac{C_2(R)}{T(R)} - \frac{3}{M} \right) + \left( \frac{C_2(R)}{T(R)} - \frac{1}{M} \right)^2 \\
&- \frac{(\mu - 1)^2}{8} \left( 3\Theta(\mu) - \frac{(2\mu - 3)}{\mu} \right) \left( \frac{1}{M^2} - \frac{C_2(R)}{MT(R)} + \frac{C_2(G)}{MT(R)} \right) \\
&- \frac{C_2(G)}{4T(R)} \left( \frac{C_2(R)}{T(R)} - \frac{1}{M} - \frac{C_2(G)}{4T(R)} \right) \left( \frac{C_2(R)}{T(R)} - \frac{1}{M} - \frac{C_2(G)}{2T(R)} \right) \right] \tag{13}
\end{align*} \]

where \( \Theta(\mu) = \psi'(\mu) - \psi'(1) \), and we have used the results of [14] in manipulating the \( f_{abc} \) and \( q_{abc} \) tensors which arise in the 3-loop graphs of fig. 4.

We have expressed our results in as general a form as possible which allows one to check that each expression does agree with the analogous results of [10, 14] and the \( O(N) \) model.

We conclude with the observation that our results will prove to be extremely useful in establishing which other models lie in the same universality class as (1) and (2) since, for example, we have provided independent analytic expressions which can now be expanded in powers of \( \epsilon \) and compared with \( \epsilon \)-expansions of critical exponents of other models deduced from the corresponding perturbative renormalization group functions.

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Figure Captions.

Fig. 1. Dressed skeleton Dyson equation for $\psi$.

Fig. 2. Dressed skeleton Dyson equation for $\sigma$.

Fig. 3. Dressed skeleton Dyson equation for $\pi$.

Fig. 4. Vertex corrections with dressed propagators.