Competing many-body interactions in systems of trapped ions

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We propose and theoretically analyse an experimental configuration in which lasers induce 3-spin interactions between trapped ions. By properly choosing the intensities and frequencies of the lasers, 3-spin couplings may be dominant or comparable to 2-spin terms and magnetic fields. In this way, trapped ions can be used to study exotic quantum phases which do not have a counterpart in nature. We study the conditions for the validity of the effective 3-spin Hamiltonian, and predict qualitatively the quantum phase diagram of the system.

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Most theoretical models in condensed matter physics rely on two-body interactions to describe a wide variety of phenomena, like for example, quantum phase transitions [1]. However, the presence of many-body interactions leads to exotic quantum effects, such as the existence of topologically ordered phases with anyonic excitations, which cannot be generally induced by pairwise couplings [2]. The beauty and complexity of those exotic models have motivated a good deal of recent theoretical work. Unfortunately, theory is still ahead of experimental implementations in this field, since many-body interactions are usually negligible in most systems found in nature.

In this Letter, we show that 3-body effective spin interactions may be implemented with trapped ions, thus opening a new avenue of research beyond the traditional paradigm of condensed matter physics. The application of systems of trapped ions to the quantum simulation of spin models has been theoretically analyzed, with a focus on conventional models with 2-body interactions [3]. Furthermore, a recent proof-of-principle experiment has confirmed the validity of this idea [4]. By studying the implementation of 3-spin couplings, we show here that trapped ions can also be used to engineer new quantum states of matter, with a phenomenology that does not have a counterpart in nature. Recently, 3-body interactions have been theoretically analyzed in systems of ultracold atoms in triangular lattices [5] and cold polar molecules [6]. Our proposal would benefit from the advantages of experiments with trapped ions, such as measurement and manipulation at the single particle level.

To sketch our idea, let us consider a system of spins and a spin operator \( \sigma^z = \sum_j \sigma^z_j \). The spins are coupled to a bosonic mode \( H_0 = \delta a \dagger a \) by both a linear, \( H_1 = \sigma^z (a + a \dagger) \), and a quadratic squeezing term, \( H_2 = \sigma^z (a^2 + a^2) \). After the adiabatic elimination of the lowest order energy corrections include 2-body \( \sim (F^2/\delta) \sigma^z_j \sigma^z_k \), (M²/\delta) \sigma^z_j \sigma^z_k \), and 3-body \( \sim (MF^2/\delta^3) \sigma^z_j \sigma^z_k \sigma^z_l \) interactions. In our scheme, the idea is realized in a system of ions in a linear array of microtraps [3][5]. The internal states of the ions play the role of the effective spins. Two off-resonant laser beams are tuned such that they induce linear and squeezing couplings between internal states and the collective vibrational modes of the chain. A third off-resonant beam induces an auxiliary linear coupling to partially cancel 2-body interactions, and thus, to tune the relative strength of the 2- and 3-spin couplings. In this way, experiments may reveal quantum effects beyond the usual pairwise-induced correlations, such as a quantum tricritical point, and a quantum phase with a 3-body order parameter.

We start by considering \( N \) ions of mass \( m \) and charge \( e \), confined along a one-dimensional array of microtraps, with lattice spacing \( a \) and trapping frequencies \( \omega_n \) with \( \alpha = x, y, z \) (see fig. 1). We assume that the ions have two internal hyperfine ground state levels \( | \uparrow \rangle, | \downarrow \rangle \), and their motion can be accurately described in terms of collective quantized vibrations (i.e. phonons). Therefore, the system Hamiltonian \( (h = 1) \) becomes

\[
H_0 = -\sum_{j=1}^{N} \hbar \sigma^z_j + \sum_{n=1}^{N} \sum_{\alpha=x,y,z} \Omega_{n\alpha} a_{n\alpha}^\dagger a_{n\alpha}.
\]

(1)

Here, \( h \) is an effective magnetic field induced by a laser or microwave field coupled to the internal transition, \( \sigma^\alpha_j \) are the Pauli matrices corresponding to each ion, \( a_{n\alpha}^\dagger (a_{n\alpha}) \) are the phonon creation (annihilation) operators, and \( \Omega_{n\alpha} = \omega_n (1 + c_n \beta_n V_n) \) denotes the \( n \)-th normal mode frequency along the \( \alpha \)-axis. These frequencies are obtained by diagonalizing \( V_n = \sum_{j,k} M_{jn} V_{jk} M_{kn} \), where \( V_{jk} = \frac{1}{(l-k+\delta_{jk})} \) is the Coulomb interaction in the limit of small vibrations, \( M_{jn} \) are the normal mode wavefunctions, and \( \beta_n = e^2/4\pi e_0 m \omega_n^2 a^3 \), \( e_{x,y} = 1, e_z = -2 \). Note that in the "stiff" limit \( \beta_n \ll 1 \),

![FIG. 1: Linear array of microtraps subjected to three laser beams along the axial and radial directions.](image-url)
the trapping potential is much larger than the Coulomb repulsion, and thus vibrational frequencies are restricted to a narrow band of width $\frac{1}{4}c_0\alpha_0 \omega_0$ around the trapping frequencies $\omega_0$ (see figs. 2(a) and 2(b)).

The ion chain will be subjected to three spin-dependent dipole forces $H_1 = H_{L2} + H_{L3} + H_{L4}$, such that the Hamiltonian becomes $H = H_0 + H_1$. Here, the most general definition of a dipole force along direction $\alpha$ is

$$H_{L2} = \frac{\Omega_{L2}}{2} \sum_{j=1}^{N} \left( e^{i(k_{z2} r_{z2} - \omega_{L2} t)} + h.c. \right) \sigma_j^z,$$

which is induced by a pair of lasers in a Raman configuration, such that $\omega_{L2}$ is the detuning between the Raman beams, $k_{z2}$ is the difference between the laser wavevectors $\Omega_{L2}$ and $\Omega_{L2}$ is the two-photon Rabi frequency $\Omega_{L2}$. The position of the ions in (2) can be expressed in terms of normal modes, $r_\alpha = x_\alpha + a_\alpha + a_\alpha$, where $a_\alpha$ are the ion equilibrium positions. Let us stress that in the stiff regime, $\beta_\alpha \ll 1$, it is possible to tune the dipole forces to a certain sideband of every normal mode of the ion chain, and thus couple each spin to the whole ensemble of vibrational phonons.

FIG. 2: Phonon frequencies $\Omega_{\alpha\alpha}$ of (a) radial and (b) axial modes in the stiff limit $\beta_\alpha \ll 1$, where the corrections due to the Coulomb energy $\frac{1}{2}c_0\alpha_0 \omega_0 V_\alpha$ have opposite signs for transverse $c_\alpha = 1$ and longitudinal $c_\alpha = -2$ phonons. The coupling of spin-dependent dipole forces to vibrational modes is also shown: (a) In the radial stiff limit, we can simultaneously blue-detune every mode to the first-sideband $\omega_{L2} \gg \Omega_{\alpha\alpha}$, or to the second-sideband $\omega_{L2} \gg 2\Omega_{\alpha\alpha}$. (b) In the axial stiff limit, every mode will be red-detuned to the first-sideband $\omega_{L2} \gg \Omega_{\alpha\alpha}$.

The main effect of $H_{L1}$ and $H_{L2}$ is to induce conditional linear and squeezing terms, which will produce 2- and 3-spin couplings, respectively. This is achieved by tuning the dipole forces to the first and second blue-sidebands ($\omega_{L2} \gg \Omega_{\alpha\alpha}$, $\omega_{L2} = 2\omega_{L2} \gg 2\Omega_{\alpha\alpha}$, see fig. 2(a)). Besides, the role of $H_{L3}$ is to generate 2-body couplings with an opposite sign as those induced by $H_{L1}$, and thus to partially cancel 2-spin interactions in favour of 3-body effects. As discussed below, the optimal screening is achieved by red-detuning this force to the first axial-sideband ($\omega_{L2} \ll \Omega_{\alpha\alpha}$, see fig. 2(b)). In the limit of resolved sidebands $|\delta_\alpha| = |\Omega_{\alpha\alpha} - \omega_{L2}| \ll \Omega_{\alpha\alpha}$, and weak couplings $\Omega_{L2} \ll \Omega_{\alpha\alpha}$, we get a time-dependent interaction Hamiltonian, $H_I(t) = H_J(t) + H_s(t)$, with

$$H_d(t) = \sum_{j,n} F^x_{jn}(t) a^\dagger_{jn} \sigma_j^x + \sum_{j,n} F^z_{jn}(t) a^\dagger_{jn} \sigma_j^z + h.c.,$$

$$H_s(t) = \sum_{j,n,m} M^x_{jnm}(t) a^\dagger_{jn} a_{nx} \sigma_j^x + h.c.,$$

where we have introduced the coupling strengths $F^x_{jn}(t) = i(\Omega_{L2} \eta_{jn1} \sin \omega_{L2} t + \Omega_{L2} \eta_{jn2} \sin \omega_{L2} t) \mathcal{J}_{jn} e^{i\phi_{jn}}$, and $F^z_{jn}(t) = i(\Omega_{L2} \eta_{jn3} \sin \omega_{L2} t) \mathcal{J}_{jn} e^{i\phi_{jn}}$ for the contributions arising from the first-sideband, and analogously $M^x_{jnm}(t) = \frac{1}{2}(\Omega_{L2} \cos \omega_{L2} t) \mathcal{M}_{jn} \mathcal{M}_{nm} e^{i\phi_{nm}}$ for the second-sideband. Let us note that these couplings are switched on adiabatically, and depend upon the laser intensities through $\mathcal{M}_{LM}$, the Lamb-Dicke parameters $\eta_{LM}$, the relative phases $\phi_{LM}$, and the relative phases $\phi_{LM}$, assuming for simplicity $\phi_{L1} = \phi_{L2} = \phi$.

To get the explicit form of the 2- and 3-spin couplings, we first note that $H_0 + H_d(t)$ describes a set of forced harmonic oscillators. If we consider that dipole forces are adiabatically switched on, and $t \gg 1/(\Omega_{\alpha\alpha} - \Omega_{\alpha\alpha})$, the evolution operator corresponding to $H_0 + H_d(t)$ is $U_d(t) = e^{-itH_0 -itH_d(t)}$, with

$$H_{eff}^{(2)} = \sum_{j>k} J^{(2)}_{jk} \sigma_j^z \sigma_k^z, \quad J^{(2)}_{jk} = (J^x_{jk} + J^z_{jk})$$

$$J^{(2)}_{jk} = \frac{\Omega_{L2}}{\lambda \Omega_{\alpha\alpha}} \sum_{n\lambda} \frac{\mathcal{M}_{jn} \mathcal{M}_{kn}}{(\Omega_{\alpha\alpha} - \lambda \Omega_{L2})^2}$$

where $\lambda = \pm 1$ also considers non-resonant terms, and

$$S = \sum_{n\lambda} \xi_{jn} a^\dagger_{jn} \sigma_j^z - h.c., \quad \xi_{jn} = \int_0^t d\tau F_{jn}(\tau) e^{i\Omega_{\alpha\alpha} \tau}.$$
are the residual non-resonant spin-phonon couplings [14].

The conditions under which these error terms become negligible, which are thoroughly described below, lead to a novel Hamiltonian in quantum magnetism

\[
H_{\text{eff}} = \sum_{j<k} J^{(1)}_{jk} \sigma^z_j \sigma^z_k + \sum_{j<k,l} J^{(2)}_{jkl} \sigma^z_j \sigma^z_k \sigma^z_l - h \sum_j \sigma^x_j, \tag{8}
\]

where 2- and 3-body interactions contribute. We note here that naive scalings \( J^{(2)} \sim \gamma^2, J^{(3)} \sim \gamma^4 \) (\( \gamma \sim 0.2 \) in experiments) imply that two-spin couplings shall hinder 3-spin effects. However, as shown in Eq. (4), the contribution to the 2-body terms from \( H_{L_2} \) partially cancelled by that from \( H_{L_3} \) (\( J^{(2)}_{jk} < 0 \)). This effect becomes optimal if the parameters of the axial dipole force are carefully chosen. In particular, \( k_{L_2} \) should be an integer multiple of \( 2\pi/a \), such that the ions sit on positions with the same relative phase of the dipole force. To achieve \( J^{(2)} \sim J^{(3)} \), the laser must be red-detuned, with an intensity \( \Omega_{L_2}^2 \) and detuning \( \delta_x < 0 \) fulfilling

\[
(\Omega_{L_2}^2 k_{L_2}^2 f_1(\omega_x, \omega_z, \delta_z, \beta_x)) = (\Omega_{L_3}^2 k_{L_3}^2 f_2(\omega_x, \omega_z, \delta_z, \beta_x), \tag{9}
\]

where the functions \( f_j(\omega_x, \omega_z, \delta_x, \beta_x) \) are listed in [13]. Furthermore, to obtain short-ranged spin interacting models, the radial stiffness parameter is restricted to \( \beta_x \sim 0.4 \delta_x/\omega_x \), where \( \delta_x = \omega_x - \omega_{Lo} \) is the detuning with respect to the bare trapping frequency. Under all these experimental constraints, at reach with current technology, we come to expressions for the many-body couplings

\[
J^{(2)}_{jk} = J_2 \delta_{jk} + J^{(3)}_{jkl} = \frac{1}{3} J_3 (\Lambda_{jk} \Lambda_{kl} + \Lambda_{kj} \Lambda_{jl} + \Lambda_{lj} \Lambda_{jk}), \tag{10}
\]

where we have introduced the dipolar scaling function \( \Lambda_{jk} = 1/j - k^3 \). The strength of these couplings is

\[
J_2 = \frac{\beta_x F_x^2 \delta_x}{|\delta_x|^4}, \quad J_3 = \frac{3 \delta_x^2 \omega_x^2 F_x^2 M_x \cos \phi_x}{|\delta_x|^4}, \tag{11}
\]

where \( \chi = \beta_x \delta_x F_x (\omega_x, \omega_z, \beta_x) \). We have introduced the bare linear and quadratic intensities by \( F_x = 1/\Omega_{L_2} \eta_x, M_x = 1/\Omega_{L_3} \eta_x^2 \), and the bare Lamb-Dicke parameter \( \eta_x = k_{L_2}/\sqrt{2 \omega_x} \). Hence, the access to the wavelengths, detunings, and laser intensities, leads to the controllability of 2 and 3-body couplings.

In order to check the viability of this proposal, we should carefully deal with the residual spin-phonon coupling in Eq. (7), which contains different non-resonant terms that contribute to the error with \( O(M_x^2/\delta_x^2) \) and \( O((M_x^2 F_x^2)/\delta_x^2) \). Besides, the canonical transformation in Eq. (5) leads to an additional error that scales as \( O((F_x^2/\delta_x^2)^2) \). Hence, a feasible quantum simulation of competing many-body interactions requires the parameters \( M_x, F_x \) to be small in comparison to the laser detuning \( \delta_x \). At this stage, we can check the viability of this proposal assuming the following available experimental parameters, which fulfill all the constraints above. Considering trapping frequencies \( \omega_x \sim 10\text{MHz}, \) laser detunings \( \delta_x \sim 1.25\times 10^7 \), stiffness parameters \( \beta_x \sim 0.05, \) and setting the dipole force intensities to \( F_x \approx M_x \sim 0.1 \delta_x \), we readily obtain an effective model of competing 2- and 3-spin interactions with \( |J_3| \approx J_2 \sim 0.6 \text{kHz} \) and an error on the order of \( E \sim 10^{-2} \). Furthermore, the modification of these parameters within the constraints detailed above, allows to experimentally access different regimes where \( |J_3|/J_2 \leq 1 \) and thus observe the consequences of a competition between the many-body interactions in full glory.

Let us note that one can also go beyond the usual dipolar regime in Eq. (10) by considering \( \beta_x \omega_x/\delta_x \geq 0.5 \). In this case, exotic long range interactions between distant spins arise and offer a exceptional playground where the effects of the range of interactions can be studied. Additionally, the couplings can be raised to \( |J_3| \approx J_2 \sim 1 - 10\text{kHz} \). In order to achieve this interesting regime, one may relax the trapping frequencies or design the microtrap in such a way that the ion equilibrium distance is lowered.

The ability to independently tune the couplings \( J_2, J_3, h \) offers the opportunity to study novel quantum phases of interacting spins. To get a qualitative picture, we consider the Hamiltonian where only the nearest-neighbour terms of the dipole couplings are kept

\[
H = J_2 \sum_j \sigma^z_j \sigma^z_{j+1} + J_3 \sum_j \sigma^x_j \sigma^x_{j+1} \sigma^z_{j+2} - h \sum_j \sigma^x_j. \tag{12}
\]

The ground state is determined by the competition between the different terms in (12): \( J_2(>0) \) induces antiferromagnetic (AF) order, \( J_3(>0) \) will be shown to induce a novel ferrimagnetic (F) phase, and \( h(>0) \) encourages the system to lie in a disordered paramagnetic (P) regime, where spins are aligned along the x-direction. We start our analysis by considering the following two limits:

(i) \( J_3 = 0 \) (2-spin quantum Ising model). This case is exactly solvable and shows a quantum phase transition at the critical coupling \( J_2 = h \), between the antiferromagnetic doubly degenerate ground state \( |g_{AF} \rangle = \{ \uparrow \downarrow \cdots \uparrow \downarrow \}, \downarrow \uparrow \cdots \downarrow \uparrow \} \), and the paramagnetic phase \( |g_P \rangle \propto \sum_{j=1}^{N-1} \{ \uparrow \downarrow \cdots \uparrow \downarrow \} \). To quantify the degree of anti-ferromagnetic order, we define the order parameter \( O_{AF}(g) = \frac{1}{N^2-1} \sum_{j=1}^{N-1} \langle \sigma^x_j \sigma^x_{j+1} \rangle \), which fulfills \( O_{AF}(g_P) = 0 \), and \( O_{AF}(g_{AF}) = 1 \).

(ii) \( J_2 = 0 \) (3-spin quantum Ising model). In this case, 3-spin interactions induce a novel quantum phase which can be fully characterised by the order parameter \( O_P(g) = \frac{1}{N^2-1} \sum_{j=1}^{N-1} \langle \sigma^x_j \sigma^x_{j+1} \sigma^x_{j+2} \rangle \). The 3-spin Ising model is no longer exactly solvable, but shows self-duality properties [17], something that allows us to locate its critical point at the value \( |J_3| = h \). This point separates a phase with a four-fold ferrimagnetic state, \( |g_{4f} \rangle = \{ \uparrow \uparrow \cdots \uparrow \uparrow \}, \downarrow \downarrow \cdots \downarrow \downarrow \} \) (\( J_3 > h \)), from the paramagnetic phase \( |g_{4f} \rangle \) (\( J_3 < h \)). Remarkably, at \( J_3^2 \) there is a phase transition
which belongs to the four-state Potts model universality class. Hence, 3-spin correlations induce an exotic critical behaviour different from the Ising universality class.

From these limiting regimes, one gets a notion of the complexity of the model for general \((J_2, J_3, h)\). In addition to the critical points studied above, the system should also hold a quantum phase transition between the AF and F phases, as well as a tricritical point, at which all the magnetic orders coexist (see fig. 3(a)). This qualitative picture is supported by the finite-size numerical calculations presented in figs. 3(b) and 3(c). Note that the order parameters, \(O_{AF}, O_F\), may be measured by detecting the photoluminescence from individual ions, something that amounts to a quantum measurement of \(\sigma^z\) [18]. It is precisely this ability of performing highly accurate measurements at the single particle level, which allows us to characterize the full quantum phase diagram.

![Phase diagram](image)

**FIG. 3:** (a) Quantum phase diagram with P, AF and F phases that coexist in the tricritical point (in red). Order parameters as a function of the couplings \(J_2, J_3\) for a chain with \(N = 15\) spins: (b) Anti-ferromagnetic order parameter (c) Ferrimagnetic order parameter.

Let us briefly consider the implementation of these spin models using the more conventional linear Paul traps. Although the axial stiff limit cannot be achieved (i.e. the ion chain stability imposes \(\beta_2 \gtrsim 1\)), it is still possible to devise effective two- and three-spin interactions. Focusing on the case of three ions, a similar procedure as that presented for microtraps, but considering a single mode in both radial directions \(\alpha = x, y\) would yield

\[ H = J_2(\sigma_1^z \sigma_2^z + \sigma_2^z \sigma_3^z + \sigma_3^z \sigma_1^z) + J_3 \sigma_1^z \sigma_2^z \sigma_3^z - h(\sigma_1^z + \sigma_2^z + \sigma_3^z), \]

where there is full access to the different couplings \((J_2, J_3, h)\). In this case, we could even switch off the two-body term \(J_2 = 0\) enhancing pure three-spin interactions and vice versa. In the same spirit as [3], one could perform a proof-of-principle experiment, where an initial separable paramagnetic state \(|P\rangle = |\uparrow \uparrow \uparrow \rangle\rangle\) adiabatically evolves towards an entangled state with different types of ordering. In case we tune \(J_2 < 0, J_3 = 0\), the evolution would generate GHZ states \(|\text{GHZ}\rangle \sim |\uparrow \downarrow \uparrow \rangle + |\downarrow \uparrow \downarrow \rangle\rangle\), whereas for \(J_3 < 0, J_2 \gtrsim 0\), one generates W-states \(|W\rangle \sim |\uparrow \downarrow \uparrow \rangle + |\downarrow \uparrow \downarrow \rangle + |\downarrow \downarrow \uparrow \rangle\rangle\), studying thus the two non-equivalent classes of tripartite entanglement. Note that the Hamiltonians obtained with Paul traps are restricted to mean-field models, and microtraps should be used to implement strongly correlated systems.

In conclusion, we have shown that a system of trapped ions can be used to explore the singular phenomenology of spin models with three-body interactions. We have made a realistic experimental proposal, at reach with current technology, to access the peculiar phase diagram of an effective Hamiltonian with three-body interactions, characterized by the appearance of exotic phases without a counterpart in usual condensed matter experiments.

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[14] There is also an error term proportional to the transverse magnetic field \(H_{\text{trans}} = 2i\hbar \sum_j \xi_j a_j^\dagger a_j \sigma_j^x + h.c.\) Here, it gives a negligible contribution for the range of parameters that we are interested in \(\hbar \approx J_2, J_3\), which leads to insignificant corrections of the order \(O(F_j^x, F_j^y)\).
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