Adaptive Control of Time-Varying Parameter Systems with Asymptotic Tracking

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Abstract—A continuous adaptive control design is developed for nonlinear dynamical systems with linearly parameterizable uncertainty involving time-varying uncertain parameters. The key feature of this design is a robust integral of the sign of the error (RISE)-like term in the adaptation law which compensates for potentially destabilizing terms in the closed-loop error system arising from the time-varying nature of uncertain parameters. A Lyapunov-based stability analysis ensures asymptotic tracking, and boundedness of the closed-loop signals.

I. INTRODUCTION

Adaptive control of nonlinear dynamical systems with time-varying uncertain parameters is an open and practically relevant problem. It has been well established that traditional gradient-based update laws can compensate for constant unknown parameters yielding asymptotic convergence. Moreover, the development of robust modifications of such adaptive update laws result in uniformly ultimately bounded (UUB) results for slowly varying parametric uncertainty using a Lyapunov-based analysis, under the assumption of bounded parameters and their time-derivatives (cf. [1]).

More recent results focus on tracking and parameter estimation performance improvement, though still limited to a UUB result, using various adaptive control approaches for systems with unknown time-varying parameters. One such approach involves a fast adaptation law [2], where a matrix of time-varying learning rates is utilized to improve the tracking and estimation performance under a finite excitation condition. Another approach uses a set-theoretic control architecture [3]–[5] to reject the effects of parameter variation, while restricting the system error within a prescribed performance bound. While the aforementioned approaches can potentially yield improved transient response, the results still yield UUB error systems.

Motivation exists to obtain asymptotic convergence of the tracking error to zero, despite the time-varying nature of the uncertain parameters. Robust adaptive control approaches such as [3] yield asymptotic adaptive tracking for systems with time-varying uncertain parameters; however, such approaches exploit high-gain feedback based on worst-case uncertainty, rather than an adaptive control approach that scales to compensate for the uncertainty without using worst-case gains. In [7], the iterative learning control result in [6] is extended to yield asymptotic tracking for systems with periodic time-varying parameters with known periodicity.

To the best of our knowledge, an asymptotic tracking result has not been achieved for a generalized class of nonlinear systems with unknown time-varying parameters, where the parameters are not necessarily periodic. Asymptotic tracking is difficult to achieve for the time-varying parameter case because the time-derivative of the parameter acts like an unknown exogenous disturbance in the parameter estimation dynamics, which is difficult to cancel with an adaptive update law in a Lyapunov-based stability analysis.

To illustrate this problem, consider the scalar dynamical system

\[ \dot{x}(t) = a(t)x(t) + u(t), \]

with the controller \( u(t) = -kx(t) - \dot{a}(t)x(t) \), where \( k \) is a positive constant gain, \( a(t) \) is the unknown time-varying parameter, \( \dot{a}(t) \) is the parameter estimate of \( a(t) \) and the parameter estimation error \( \ddot{a}(t) \) is defined as \( \ddot{a}(t) = a(t) - \dot{a}(t) \). The traditional stability analysis approach for such problems is to consider the Lyapunov function candidate

\[ V(x(t), \dot{a}(t)) = \frac{1}{2} \gamma x^2(t) + \frac{1}{2\gamma} \dot{a}^2(t), \]

where \( \gamma \) is a positive constant gain. The given definitions and controller yield the following time-derivative of the candidate Lyapunov function:

\[ \dot{V}(t) = -kx^2(t) + \dot{a}(t)x^2(t) + \frac{\dot{a}(t)}{\gamma}(\ddot{a}(t) - \dot{a}(t)). \]

For the constant parameter case, i.e., \( \ddot{a}(t) = 0 \), the well-known adaptive update law \( \ddot{a}(t) = \gamma x^2(t) \) will cancel the cross term \( \dot{a}(t)x^2(t) \) in \( \dot{V}(t) \). However, when the parameters are time-varying, it is unclear how to cancel or dominate \( \ddot{a}(t) \) via an update law such that \( \dot{V}(t) \) becomes at least negative semi-definite. It would be desirable to have a sliding mode-like term based on \( \dot{a}(t) \) in the adaptation law, but \( \ddot{a}(t) \) is unknown.

Note that the system [4] is considered only for illustrative purpose. This paper presents result for a general system with a vector state and a linearly parameterizable uncertainty with time-varying parameters.
Another approach could be to use a robust controller, e.g., \( u(t) = -k x(t) - \ddot{a} x(t) \), where \( \ddot{a} \) is a known constant upper bound on the norm of parameter \( |a(t)| \), or an adaptive robust controller which involves certainty equivalence in terms of the unknown bound \( \ddot{a} \). Either of these approaches would yield an asymptotic tracking result (cf., [6]), but, as stated earlier, these approaches are based on a high-gain worst case scenario, rather than an adaptive control approach that scales to compensate for the uncertainty without using worst-case gains.

A popular approach to design adaptive controllers for the time-varying parameter case is to consider robust modification of the update law and assume upper bounds of \( |a(t)| \) and \( \dot{a}(t) \) to obtain a UUB result. For instance, consider a standard gradient update law with sigma-modification [8]. \( \dot{a}(t) = \gamma x^2(t) - \gamma \sigma \dot{a}(t), \) which yields \( \dot{V}(t) = -k x^2(t) - \sigma \dot{a}^2(t) + \dot{a}(t) \left( \frac{2 \dot{a}(t)}{\sigma} + \sigma \dot{a}(t) \right) \), implying a UUB result when the parameter \( a(t) \) and its time-derivative \( \dot{a}(t) \) are bounded. Moreover, the approaches developed in results such as [2] and [4] can be used to improve the transient response of the UUB error system.

The major challenge to achieve asymptotic stability is the derivative of the time-varying parameter term in the Lyapunov analysis, which is addressed in this paper with a Lyapunov-based design approach, that is inspired by the modular adaptive control approach in [9]. This approach includes higher order dynamics which appear after taking the time-derivative of (1). Since these higher order dynamics contain the time-derivative of the parameter estimate \( \dot{a}(t) \), it is possible to design \( \dot{a}(t) \) to facilitate the subsequent stability analysis. With this motivation, a continuous adaptive control algorithm is developed for nonlinear dynamical systems with linearly parameterized uncertainty involving time-varying parameters, where a semi-global asymptotic tracking result is achieved. A key feature of the proposed method is a robust integral of the sign of the error (RISE)-like (see [9]–[12]) update law, i.e., the update law contains a signum function of the tracking error term multiplied by some desired regressor based terms. The update law also involves a projection algorithm to ensure that the parameter estimates stay within a bounded set. However, the projection algorithm introduces a potentially destabilizing term in the time-derivative of the Lyapunov function candidate, leading to an additional technical obstacle to obtain asymptotic tracking. This challenge is resolved by using an auxiliary term in the control input, which facilitates stability by providing a stabilizing term and canceling the aforementioned potentially destabilizing term in the time-derivative of the candidate Lyapunov function. With the proposed method, the closed-loop system dynamics have the same structure as previous RISE controllers [9]–[12], for which the stability analysis tools are well established, yielding asymptotic convergence of the tracking error to zero, boundedness of the parameter estimation error, and boundedness of the closed-loop signals.

## II. Dynamic Model

Consider a control affine system with the nonlinear dynamics

\[
\dot{x}(t) = h(x(t), t) + d(t) + u(t),
\]

where \( x : [0, \infty) \to \mathbb{R}^n \) denotes the state, \( h : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}^n \) denotes a continuously differentiable function, \( d : [0, \infty) \to \mathbb{R}^n \) represents an exogenous disturbance acting on the system, and \( u : [0, \infty) \to \mathbb{R}^m \) represents the control input. The function \( h(x(t), t) \) in (2) is assumed to be linearly parameterized as

\[
h(x(t), t) \triangleq Y_h(x(t), t) \theta_f(t),
\]

where \( Y_h : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}^{n \times m} \) is a known regression matrix, and \( \theta_f : [0, \infty) \to \mathbb{R}^{n+m} \) is a vector of time-varying unknown parameters.

The disturbance parameter \( d(t) \) can be appended to the \( \theta_f(t) \) vector, yielding an augmented parameter vector \( \theta : [0, \infty) \to \mathbb{R}^{n+m} \) as

\[
\theta(t) \triangleq \begin{bmatrix} \theta_f(t) \\ d(t) \end{bmatrix},
\]

and the augmented regressor \( Y : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}^{n \times (n+m)} \) can be designed as

\[
Y(x(t), t) \triangleq \begin{bmatrix} Y_h(x(t), t) & I_n \end{bmatrix}.
\]

The parameterization in (4) and (5) yields \( h(x(t), t) + d(t) = Y(x(t), t) \theta(t) \), so the dynamics in (2) can be rewritten as

\[
\dot{x}(t) = Y(x(t), t) \theta(t) + u(t).
\]

**Assumption 1.** The time-varying augmented parameter \( \theta(t) \) and its time-derivatives, i.e., \( \dot{\theta}(t) \), \( \ddot{\theta}(t) \) are bounded by known constants, i.e., \( ||\theta(t)|| \leq \bar{\theta}, \|\dot{\theta}(t)\| \leq \zeta_1, \|\ddot{\theta}(t)\| \leq \zeta_2 \), where \( \bar{\theta}, \zeta_1, \zeta_2 \in \mathbb{R}_{>0} \) are known bounding constants, and \( \|\cdot\| \) denotes the Euclidean norm.

## III. Control Design

### A. Control Objective

The objective is to design a controller such that the state tracks a smooth bounded reference trajectory, despite the time-varying nature of the uncertain parameters. The objective is quantified by defining the tracking error \( e : [0, \infty) \to \mathbb{R}^n \) as

\[
e(t) = x - x_d(t),
\]

where \( x_d : [0, \infty) \to \mathbb{R}^n \) is a reference trajectory.

\[2\] All function dependencies are suppressed equation (7) onward; assume all variables to be time dependent unless stated otherwise.
Assumption 2. The reference trajectory \( x_d(t) \) is bounded and smooth, such that \( \| x_d(t) \| \leq \bar{x}_d, \| \dot{x}_d(t) \| \leq \delta_1 \), and \( \| \ddot{x}_d(t) \| \leq \delta_2 \), where \( \bar{x}_d, \delta_1, \delta_2 \in \mathbb{R}_{>0} \) are known bounding constants.

Substituting (5) into the time-derivative of (7) yields
\[
\dot{e} = Y \theta + u - \dot{x}_d. \tag{8}
\]
To facilitate the subsequent analysis, a filtered tracking error \( r : [0, \infty) \to \mathbb{R}^n \) is defined as
\[
r \triangleq \dot{e} + \alpha e, \tag{9}
\]
where \( \alpha \in \mathbb{R}_{>0} \) is a constant control gain. Substituting (8) into (9) yields
\[
r = Y \theta + u - \dot{x}_d + \alpha e. \tag{10}
\]

**B. Control and Update Law Development**

From the subsequent stability analysis, the continuous control input is designed as
\[
u \triangleq -Y_d \dot{\theta} - \alpha e + \dot{x}_d + \mu, \tag{11}\]
where \( Y_d \triangleq Y(x_d(t), t) \) is the desired reference matrix, \( \mu : [0, \infty) \to \mathbb{R}^n \) is a subsequently defined auxiliary control term, and \( \dot{\theta} : [0, \infty) \to \mathbb{R}^{n+m} \) denotes the parameter estimate of \( \theta(t) \). Substituting the control input in (11) into the open-loop error system in (10) yields the following closed-loop system
\[
r = Y \theta - Y_d \dot{\theta} + \mu. \tag{12}\]

Adding and subtracting \( Y_d \theta \) in (12) yields
\[
r = (Y - Y_d)\theta + Y_d \dot{\theta} + \mu, \tag{13}\]
where \( \dot{\theta} : [0, \infty) \to \mathbb{R}^{n+m} \) denotes the parameter estimation error, i.e., \( \dot{\theta}(t) \triangleq \theta(t) - \dot{\theta}(t) \). Taking the time-derivative of (13) yields
\[
\dot{r} = (\dot{Y} - \dot{Y}_d)\theta + (Y - Y_d)\dot{\theta} + \dot{Y}_d \dot{\theta} + \dot{Y}\dot{\theta} - Y_d \ddot{\theta} + \ddot{\mu}. \tag{14}\]

The control variables \( \dot{\theta}(t) \) and \( \ddot{\mu}(t) \) now appear in the higher order dynamics in (14), and these control variables are designed with the use of a continuous projection algorithm \footnote{From Lemma 1 in the Appendix section, \( Y_d \Gamma Y_d^T \) is proven to be invertible, therefore it is reasonable to include \( Y_d \dot{\gamma} \) in the update law.} [Appendix E]. The projection algorithm constrains \( \dot{\theta}(t) \) to lie inside a bounded convex set \( \mathcal{B} = \{ \theta \in \mathbb{R}^{(n+m)} | \| \theta \| \leq \bar{\theta} \} \) by switching the adaptation law to its component tangential to the boundary of the set \( \mathcal{B} \) when \( \theta(t) \) reaches the boundary. A continuously differentiable convex function \( f : \mathbb{R}^{(n+m)} \to \mathbb{R} \) is used to describe the boundaries of the bounded convex set \( \mathcal{B} \) such that \( f(\theta(t)) < 0 \) \( \forall \| \theta(t) \| < \bar{\theta} \) and \( f(\theta(t)) = 0 \) \( \forall \| \theta(t) \| = \bar{\theta} \). The adaptation law is then designed as
\[
\dot{\theta} \triangleq \text{proj}(\Lambda_0(\theta(t))) = \begin{cases} 
\Lambda_0, & ||\dot{\theta}|| < \bar{\theta} \lor (\nabla f(\dot{\theta}))^T \Lambda_0 \leq 0 \\
\Lambda_1, & ||\dot{\theta}|| \geq \bar{\theta} \land (\nabla f(\dot{\theta}))^T \Lambda_0 > 0,
\end{cases} \tag{15}\]
where \( ||\dot{\theta}(0)|| < \bar{\theta}, \lor, \land \) denote the logical ‘or’, ‘and’ operators, respectively, \( \nabla \) represents the gradient operator, i.e., \( \nabla f(\dot{\theta}) = \left[ \frac{\partial f}{\partial \theta_1} \ldots \frac{\partial f}{\partial \theta_{n+m}} \right]^T \), and \( \Lambda_0 : [0, \infty) \to \mathbb{R}^{n+m} \) and \( \Lambda_1 : [0, \infty) \to \mathbb{R}^{n+m} \) are designed as
\[
\Lambda_0 \triangleq \Gamma Y_d^T (Y_d \Gamma Y_d^T)^{-1} [\beta \text{sgn}(\epsilon)], \tag{16}\]
\[
\Lambda_1 \triangleq \left( I_{m+n} - \frac{(\nabla f(\dot{\theta})) (\nabla f(\dot{\theta}))^T}{||\nabla f(\dot{\theta})||^2} \right) \Lambda_0, \tag{17}\]
respectively. In (16) and (17), \( \beta \in \mathbb{R}_{>0} \) is a constant gain, and \( \Gamma \in \mathbb{R}^{(n+m) \times (n+m)} \) is a positive-definite matrix with a block diagonal structure, i.e., \( \Gamma \triangleq \left[ \begin{array}{cc} \Gamma_1 & 0_{m \times n} \\
0_{n \times m} & \Gamma_2 \end{array} \right] \), with \( \Gamma_1 \in \mathbb{R}^{m \times m}, \Gamma_2 \in \mathbb{R}^{n \times n} \). The continuous auxiliary term \( \mu(t) \), used in the control input in (11), acts as a stabilizing term in the Lyapunov analysis to account for the side effects of the projection, and is designed as a generalized solution to
\[
\dot{\mu} \triangleq \begin{cases} 
\mu_0, & ||\dot{\theta}|| < \bar{\theta} \lor (\nabla f(\dot{\theta}))^T \Lambda_0 \leq 0 \\
\mu_1, & ||\dot{\theta}|| \geq \bar{\theta} \land (\nabla f(\dot{\theta}))^T \Lambda_0 > 0,
\end{cases} \tag{18}\]
where \( \mu(0) = 0 \) and \( \mu_0 : [0, \infty) \to \mathbb{R}^n \) and \( \mu_1 : [0, \infty) \to \mathbb{R}^n \) are defined as \( \mu_0 \triangleq -K_r \) and \( \mu_1 \triangleq \mu_0 - Y_d (\Lambda_0 - \Lambda_1) \), respectively. Substituting (13) and (18) in (14), the closed-loop dynamics can be rewritten as
\[
\dot{r} = (\dot{Y} - \dot{Y}_d)\theta + (Y - Y_d)\dot{\theta} + \dot{Y}_d \dot{\theta} + \dot{Y}\dot{\theta} - Y_d \ddot{\theta} + \ddot{\mu}. \tag{19}\]
}

for both cases, i.e., when \( ||\dot{\theta}|| < \bar{\theta} \lor (\nabla f(\dot{\theta}))^T \Lambda_0 \leq 0 \) or \( ||\dot{\theta}|| \geq \bar{\theta} \land (\nabla f(\dot{\theta}))^T \Lambda_0 > 0 \). To facilitate the subsequent analysis, (19) can be rewritten as
\[
\dot{r} = N + N_B - \beta \text{sgn}(\epsilon) - K_r - e, \tag{20}\]
where the variables \( N \triangleq [0, \infty) \to \mathbb{R}^n \) and \( N_B : [0, \infty) \to \mathbb{R}^n \) are defined as
\[
N \triangleq (\dot{Y} - \dot{Y}_d)\theta + (Y - Y_d)\dot{\theta} + e,
\]
and
\[
N_B \triangleq Y_d \dot{\theta} + \dot{Y}_d \theta - \dot{Y}_d \dot{\theta},
\]
respectively. The Mean Value Theorem (MVT) can be used to develop the following upper bound on the term \( \dot{N}(t) \).
\[ ||\hat{N}(t)|| \leq \rho(||z(t)||)||z(t)||, \]  
where \( z \triangleq [r^T \ e^T]^T \in \mathbb{R}^{2n} \) and \( \rho : \mathbb{R}^{2n} \rightarrow \mathbb{R} \) is a positive, globally invertible and non-decreasing function. By Assumption 1, Assumption 2, Corollary 1 in the Appendix, and the bounding effect of projection algorithm on \( \hat{b}(t) \), the term \( N_B(t) \) and its time-derivative \( \hat{N}_B(t) \) can be upper bounded by some constants \( \gamma_1, \gamma_2 \in \mathbb{R}_{>0} \) as

\[ ||N_B(t)|| \leq \gamma_1, \quad ||\hat{N}_B(t)|| \leq \gamma_2, \]  
respectively.

IV. STABILITY ANALYSIS

**Theorem 1.** The controller designed in \( [11] \) along with the adaptation laws designed in \( [12] \) and \( [18] \) ensure the closed-loop system is bounded and the tracking error \( ||e(t)|| \rightarrow 0 \) as \( t \rightarrow \infty \), provided that the gains \( \alpha, \beta \) are selected such that the following condition is satisfied

\[ \beta > \gamma_1 + \frac{\gamma_2}{\alpha}. \]  

**Proof:** Let \( D \subset \mathbb{R}^{2n+1} \) be an open connected set containing \( y(t) = 0 \), where \( y : [0, \infty) \rightarrow \mathbb{R}^{2n+1} \) is defined as

\[ y(t) = \begin{bmatrix} z^T(t) & \sqrt{P(t)} \end{bmatrix}^T. \]

Let \( V_L : D \times [0, \infty) \rightarrow \mathbb{R}_{>0} \) be a positive-definite candidate Lyapunov function defined as

\[ V_L(y(t), t) \triangleq \frac{1}{2} r^T r + \frac{1}{2} e^T e + P, \]

where \( P : [0, \infty) \rightarrow \mathbb{R} \) is a generalized solution to the differential equation

\[ \dot{P}(t) \triangleq -L(t), \]  
where \( P(0) \triangleq \beta \sum_{i=1}^{n} [e_i(t)] - e(0)^T N_B(0) \) and

\[ L \triangleq r^T (N_B - \beta \text{sgn}(e)). \]  

**Remark 1.** Provided that the gain condition in \( (23) \) is satisfied, \( P(t) \geq 0 \)\(^4\). Hence it is valid to use \( P(t) \) in the candidate Lyapunov function as function of the variable \( \sqrt{P(t)} \).

From \( [9], [20] \) and \( [24] \), the differential equations describing the closed-loop system are

\[ \dot{e} = r - \alpha e, \]  
\[ \dot{r} = \hat{N} + N_B - \beta \text{sgn}(e) - K r - e, \]  
\[ \dot{P} = -r^T (N_B - \beta \text{sgn}(e)). \]  

Let \( g : \mathbb{R}^{2n+1} \times [0, \infty) \rightarrow \mathbb{R}^{2n+1} \) denote the right-hand side of \( [26] - [28] \). Since \( g(y(t), t) \) is continuous almost everywhere, except in the set \( \{(y(t), t) | e = 0\} \), an absolute continuous Filippov solution \( y(t) \) exists almost everywhere (a.e.), so that \( y(t) \in K [g(y(t), t) \ a.e., except at the points in the set \( \{(y(t), t) | e = 0\} \), where the Filippov set-valued map includes unique solutions. Using a generalized Lyapunov stability theory under the framework of Filippov solutions, a generalized time-derivative of the Lyapunov function \( V_L \) exists and \( V_L(y(t), t) \in \dot{V}_L(y(t), t) \), where

\[ \dot{V}_L \triangleq \nabla V_L^T K \begin{bmatrix} \dot{e}^T & \dot{r}^T & \frac{1}{2} P^{-\frac{1}{2}} \dot{P} \end{bmatrix}^T \]

\[ \subseteq \begin{bmatrix} \dot{e}^T & \dot{r}^T & 2 P^{\frac{1}{2}} \end{bmatrix} \times K \begin{bmatrix} \dot{e}^T & \dot{r}^T & \frac{1}{2} P^{-\frac{1}{2}} \dot{P} \end{bmatrix}^T, \]  

where \( \partial V_L(y(t), t) \) denotes Clarke’s generalized gradient \( [14] \).

Substituting \( [26], [28] \) into \( [29] \) yields

\[ \dot{V}_L \triangleq r^T \dot{N} + N_B - \beta \text{sgn}(e) - K r - e \]

\[ + e^T (r - \alpha e) - r^T (N_B - \beta \text{sgn}(e)) \]

where \( K [\text{sgn}(e)] = \text{SGN}(e) \) such that

\[ \text{SGN}(e_i) = \begin{cases} \{1\}, & e_i > 0 \\ \{-1\}, & e_i < 0 \end{cases}. \]

Using \( [21] \), the expression in \( [30] \) can be upper bounded as

\[ \dot{V}_L \leq \rho(||z||)||z|| ||r|| - K ||r||^2 - \alpha e^2. \]

Using Young’s Inequality on \( \rho(||z||)||z|| ||r|| \) yields

\[ \rho(||z||)||z|| ||r|| \leq \frac{\rho^2(||z||)||z||}{2} + \frac{1}{2} ||r||^2. \]

Therefore,

\[ \dot{V}_L \leq - \frac{\rho^2(||z||)||z||^2}{2} - (K - \frac{1}{2}) ||r||^2 - \alpha e^2 \]

\[ \leq -(\lambda_3 - \frac{\rho^2(||z||)}{2}) ||z||^2, \]  

where \( \lambda_3 \triangleq \min \{\alpha, K - \frac{1}{2}\} \in \mathbb{R}_{>0} \) is a known constant. The expression in \( [31] \) can be rewritten as

\[ \dot{V}_L \leq -c ||z||^2 \forall y \in D, \]  

for some constant \( c \in \mathbb{R}_{>0} \), where

\[ D \triangleq \{y \in \mathbb{R}^{2n+1} : ||y|| \leq \rho^{-1} \left( \sqrt{2\lambda_3} \right) \}. \]

In this region, \( \lambda_3 > \frac{\rho^2(||z||)}{2} \), so a constant \( c \) satisfies \( [32] \), and larger values of \( \lambda_3 \) expand the size of \( D \). Furthermore, the relationship in \( [32] \) implies that \( V_L(y(t), t) \in L_\infty \), hence \( e(t), r(t), P(t) \in L_\infty \). These facts along with the

\footnote{See \( [10] \) for details.}
expression in (13), indicate that \( \mu(t) \in L_\infty \). The parameter estimate \( \hat{\theta}(t) \in L_\infty \) due to the projection operation. The state and its time-derivative, i.e., \( x(t), \dot{x}(t) \in L_\infty \), because \( e(t), r(t), x_d(t), \dot{x}_d(t) \in L_\infty \). Further the regression matrix \( Y(x(t), t) \in L_\infty \) since its a bounded function for a bounded argument \( x(t) \). Similarly, \( Y_d(t) \in L_\infty \), hence \( \dot{\theta} \in L_\infty \) by Corollary 1. From the expression in (11), since \( \hat{\theta}(t), e(t), \dot{x}_d(t), \mu(t) \in L_\infty, u(t) \in L_\infty \). Hence all the closed-loop signals are bounded.

Consider \( \lambda_1 \| y \|^2 \leq V_L \leq \lambda_2 \| y \|^2 \), where \( \lambda_1, \lambda_2 \in \mathbb{R}_{>0} \). To ensure \( \| z \| \leq \rho^{-1}(\sqrt{2\lambda_3}) \), it is sufficient to obtain the result from \( \| y \| \leq \rho^{-1}(\sqrt{2\lambda_3}) \). Since \( \sqrt{\lambda_2} \leq \| y \| \), then \( \sqrt{\lambda_4} \leq \rho^{-1}(\sqrt{2\lambda_3}) \), and \( V_L \) is non-increasing, so \( V_L(t) \leq V_L(0) \). Hence it sufficient to show that \( \sqrt{\lambda_4} \leq \rho^{-1}(\sqrt{2\lambda_3}) \) to ensure that \( \sqrt{\lambda_4} \leq \rho^{-1}(\sqrt{2\lambda_3}) \). Since \( \lambda_1 \| y(0) \|^2 \leq V_L(0) \) implies \( \| y(0) \| \leq \sqrt{\lambda_4} \rho^{-1}(\sqrt{2\lambda_3}) \), \( y \in S \triangleq \{ y(t) \in \mathcal{D} | y(t) < \sqrt{\lambda_4} \rho^{-1}(\sqrt{2\lambda_3}) \} \) is the region where \( y(0) \) should lie for guaranteed asymptotic stability. The gain condition \( \lambda_3 = \min\{\alpha, K - \frac{1}{\rho^2} \} \geq \frac{1}{\lambda_{\text{min}}(\Gamma)} \) needs to be satisfied according to the initial condition for asymptotic stability and the region of attraction can be made arbitrarily large to include any initial condition by increasing the gains \( \alpha \) and \( K \) accordingly. By the extension of LaSalle-Yoshizawa theorem for non-smooth systems in [14] and [15], \( e \| z(t) \|^2 \to 0 \) and hence \( e \to 0 \) as \( t \to \infty \) \( \forall \ y(0) \in S \), so the closed-loop error system is semi-globally asymptotically stable.

V. CONCLUSION

A continuous adaptive control design was presented to achieve semi-global asymptotic tracking for linearly parameterizable nonlinear systems with time-varying uncertain parameters. The key feature of this design is a RISE-like parameter update law along with a projection algorithm, which allows the system to compensate for potentially destabilizing terms in the closed-loop error system, arising due to the time-varying nature of parameters. Semi-global asymptotic tracking for the error system is guaranteed via a Lyapunov-based stability analysis. Future work will involve improvement of the parameter estimation performance of time-varying parameter systems and its extension to the system identification problem.

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APPENDIX

Lemma 1. Consider a positive-definite matrix \( \Gamma \in \mathbb{R}^{(n+m) \times (n+m)} \) such that \( \Gamma \) has the block diagonal structure as \( \Gamma_1 \triangleq \begin{bmatrix} \Gamma_1 & 0_{n \times m} \\ 0_{m \times n} & \Gamma_2 \end{bmatrix} \), where \( \Gamma_1 \in \mathbb{R}^{m \times m} \) and \( \Gamma_2 \in \mathbb{R}^{n \times n} \). The matrix \( Y(x(t), t) \Gamma Y^T(x(t), t) \) is positive-definite, and hence invertible. Furthermore, the inverse of this matrix satisfies the property \( \| (Y(x(t), t) \Gamma Y^T(x(t), t))^{-1} \|_2 \leq \frac{1}{\lambda_{\text{min}}(\Gamma)} \), where \( \| \cdot \|_2 \) denotes the spectral norm and \( \lambda_{\text{min}}(\cdot) \) denotes the minimum eigenvalue of \( \cdot \).

Proof: Substituting the definitions for \( Y(x(t), t) \) and \( \Gamma \) in
The eigenvalues of the inverse of a positive definite matrix
\( \Gamma \) for some matrix \( A \) yields
\[
\left[ \begin{array}{ccc}
Y_h(x(t), t) & 0_m \times n \\
0_n \times m & \Gamma_2
\end{array} \right]
\left[ \begin{array}{c}
Y_h(x(t), t) \\
I_n
\end{array} \right]
= Y_h(x(t), t) \Gamma_1 Y_h(x(t), t) + \Gamma_2.
\]

Since \( \Gamma \) is selected to be a positive-definite matrix, the block matrices \( \Gamma_1 \) and \( \Gamma_2 \) are both positive-definite, so the first term \( Y_h(x(t), t) \Gamma_1 Y_h(x(t), t) \) in this expression is positive semi-definite while the second term \( \Gamma_2 \) is positive-definite, hence the sum of these two terms, i.e., \( Y(x(t), t) \Gamma Y^T(x(t), t) \) is positive-definite, and therefore invertible. Furthermore, the spectral norm satisfies the property, \( \| A \|_2 = \sqrt{\lambda_{\text{max}} \{ A^T A \}} \) for some \( A \in \mathbb{R}^{p \times d} \) with \( p, q \in \mathbb{Z}_{>0} \), where \( \lambda_{\text{max}} \{ \cdot \} \) denotes the maximum eigenvalue of \( \{ \cdot \} \). Utilizing this property with \( \| (Y T)^{-1} \|_2 \) yields
\[
\| (Y T)^{-1} \|_2 = \sqrt{\lambda_{\text{max}} \left\{ (Y T)^{-1} \right\}^T (Y T)^{-1}}
= \lambda_{\text{max}} \left\{ (Y T)^{-1} \right\}.
\]

The eigenvalues of the inverse of a positive definite matrix \( B \) satisfy the property, \( \lambda_{\text{max}} \{ B^{-1} \} = \frac{1}{\lambda_{\text{min}} \{ B \}} \). Applying this property with the right-hand side of (33) yields
\[
\| (Y T)^{-1} \|_2 = \frac{1}{\lambda_{\text{min}} \{ Y T \}} \leq \frac{1}{\lambda_{\text{min}} \{ \Gamma_2 \}}.
\]

**Corollary 1.** The norm of time-derivative of the parameter estimate, \( \| \dot{\beta} \| \) can be upper bounded by a constant \( \gamma_3 \in \mathbb{R}_{>0} \). i.e., \( \| \dot{\beta} \| \leq \gamma_3 \).

**Proof:** Based on (15)
\[
\| \dot{\beta} \| = \| \text{proj}(\Lambda_0) \| \leq \| \Lambda_0 \|
= \| \Gamma Y_d^T (Y_d Y_d^T)^{-1} \beta \text{sgn}(e) \|
\leq \| \Gamma Y_d^T (Y_d Y_d^T)^{-1} \beta \|.
\]

Applying Holder's inequality to the right-hand side of (34) yields
\[
\| \dot{\beta} \| \leq \beta \| \Gamma \|_2 \| Y_d \|_2 \| (Y_d Y_d^T)^{-1} \|_2.
\]

Using Lemma 1 with the right-hand side of (35) yields
\[
\| \dot{\beta} \| \leq \beta \| \Gamma \|_2 \| Y_d \|_2 \frac{\lambda_{\text{min}} \{ \Gamma_2 \}}{\lambda_{\text{min}} \{ \Gamma_2 \}}.
\]

Given a bounded reference \( x_d(t) \), such that \( \| x_d(t) \| \leq \bar{x}_d \), the spectral norm of the desired regressor may be upper-bounded by a constant \( Y_d \in \mathbb{R}_{>0} \), i.e., \( \| Y_d \|_2 \leq \bar{Y}_d \), because \( Y_d \) is a continuously differentiable function. Therefore,
\[
\| \dot{\beta} \| \leq \beta \| \Gamma \|_2 \| \bar{Y}_d \| = \gamma_3.
\]