On The Characters of Parafermionic Field Theories

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ABSTRACT

We study cosets of the type $H_l/U(1)^r$, where $H$ is any Lie algebra at level $l$ and rank $r$. These theories are parafermionic and their characters are related to the string functions, which are generating functions for the multiplicities of weights in the affine representations. An identity for the characters is described, which apply to all the algebras and all the levels. The expression is of the Rogers Ramanujan type. We verify this conjecture, for many algebras and levels, using Freudenthal Kac formula, which calculates the multiplicities in the affine representations, recursively, up to some grade. Our conjecture encapsulates all the known results about these string functions, along with giving a vast wealth of new ones.
Some years ago the author have suggested parafermionic conformal field theo-
ries related to simple Lie algebras [1]. The characters of these conformal field theo-
tories (CFT) were shown to be expressible in terms of the string functions of the
corresponding affine Lie algebra (up to factors of Dedekind’s eta function). Our
purpose here is to describe closed expressions for many of these characters,
which are of the Rogers Ramanujan type.

Several examples of characters in CFT were studied, and were shown to be
expressible as Rogers Ramanujan type sums [2, 3, 4, 5, 6, 7, 8]. The origin of these
identities is somewhat perplexing, though, they were shown to be connected with
local state probabilities in solvable lattice models [9, 10, 11], and with thermody-
namic Bethe ansatz equations (see, e.g., [6] and refs. therein).

The characters of the parafermionic field theories are \( \chi^\Lambda_\lambda(\tau) \), where \( \Lambda \) is an
integrable dominant weight of some simple Lie algebra \( H \) at level \( l \), and \( \lambda \) is the
weight. We describe here a Rogers Ramanujan type expression when the weight
\( \Lambda = f \Lambda_g \) where \( \Lambda_g \) has mark one, and \( f \) is any integer, or \( \Lambda_g \) has dual mark one
and \( f = 1 \). Our conjecture holds for any allowed weight \( \lambda \), and for all algebras
\( H \) at any level \( l \).

We verify our conjecture using Freudenthal Kac formula, which gives the
exact characters up to a certain grade (or the dimension of the fields). We use
for this purpose a computer program that we wrote.

This provides important physical information about the parafermionic theo-
ries, along with valuable mathematical insight concerning the string functions,
which are central in the mathematics of infinite dimensional Lie algebras.

Parafermionic conformal field theories associated with Lie algebras were de-
scribed in ref. [1], along with their characters and partition functions. These
theories can be thought of as the cosets of the type,

\[
\frac{H_l}{U(1)^r},
\]

(1)

where \( H \) is any Lie algebra, of the types \( A - G \), \( l \) is the level and \( r \) is the rank.
The fields in the theory are labeled by a pair of weights \((\Lambda, \lambda)\), where \(\Lambda\) is an integrable highest weight of \(H\) at level \(l\) and \(\lambda\) is an element of the weight lattice of \(H\). We have the selection rule,

\[
\Lambda - \lambda \in M, \tag{2}
\]

where \(M\) is the root lattice of \(G\). This lattice is generated by the simple roots, which we denote by \(\alpha_a, a = 1, 2, \ldots, r\). The characters of the theory are denoted by \(\chi^\Lambda_\lambda(\tau)\) and they can be expressed in terms of the string functions of the corresponding untwisted affine Lie algebra, as follows,

\[
\chi^\Lambda_\lambda(\tau) = \eta(\tau)^r c^\Lambda_\lambda(\tau), \tag{3}
\]

where \(\eta(\tau)\) is the Dedekind’s eta function. The string functions are defined as \([12, 2]\),

\[
c^\Lambda_\lambda(\tau) = e^{-\pi i \lambda^2 \tau / l} \text{Tr}_{\mathcal{H}^\Lambda_\lambda} e^{2\pi i \tau (L_0 - c/24)} =
\]

\[
e^{2\pi i r \left(\frac{1}{2}(\Lambda + 2\rho)\Lambda / (l+g) - \Lambda^2 / (2l) - c/24\right)} \sum_{n=0}^{\infty} p_n e^{2\pi i n \tau},
\]

where \(L_0\) is the dimension of the fields, \(c = lD / (l+g)\) is the central charge, where \(D\) is the dimension of the algebra, and \(\mathcal{H}^\Lambda_\lambda\) is the representations of the affine algebra with dominant highest weight \(\Lambda\) and weight \(\lambda\). Here \(\rho\) is half the sum of positive roots and \(g\) is the dual Coxeter number, of the algebra \(H\). The integer \(p_n\) is the multiplicity of the fields with the number operator \(N = n\) (or the grade), in the representation of the affine algebra \(\hat{H}\), which have the highest weight \(\Lambda\) at level \(l\), and have a weight \(\lambda\). Thus, the string functions are the generating functions for the multiplicities of a ‘string’ of weights.
The characters of the CFT (and the string functions) obey the following relations.

\[ \chi_\Lambda^\Lambda(\tau) = \chi_{\omega(\Lambda)}(\tau) = \chi_{\Lambda+\mu}(\tau), \]  

where \( \omega \) is any element of the Weyl group of the finite algebra \( H \), denoted by \( W \), and \( \mu \in lM_L \), where \( M_L \) stands for the long root lattice of \( H \), generated by the elements \( 2\alpha_a/\alpha_a^2 \), for \( a = 1, 2, \ldots, r \).

Before getting to our conjecture for the characters, we need to introduce some notation. Let \( t_a = 2/\alpha_a^2 \). Denote also \( l_a = t_a l \), where \( l \) is the level. We define

\[ G = \{(a, m) | 1 \leq a \leq r, \quad 1 \leq m \leq l_a - 1, \quad a, m \in \mathbb{Z}\}, \]  

following ref. [6].

We define

\[ K_{ab}^{mk} = \left( \min(t_b m, t_a k) - \frac{mk}{l} \right) \alpha_a \cdot \alpha_b. \]  

Let \( C_N \) be the Cartan matrix of \( SU(N) = A_{N-1} \),

\[ (C_N)_{r,s} = 2\delta_{r,s} - \delta_{r,s+1} - \delta_{r+1,s}, \]  

where \( r, s = 1, 2, \ldots, N - 1 \). We denote the inverse matrix by

\[ B_N = (C_N)^{-1}. \]  

Our Generalized Rogers Ramanujan expression (GRR) involves a summation over the vector of non negative integers,

\[ n = \left( n_{am}^{(a)} \right)_{(a, m) \in G}. \]
We also define the element of the root lattice,

$$\lambda(n) = \sum_{(a,m) \in G} mn_m^{(a)} \alpha_a.$$  \hspace{1cm} (11)

and the Pochhammer symbol,

$$(q)_n = \prod_{(a,m) \in G} (q)_{n_m^{(a)}}, \quad (q)_k = \prod_{j=1}^{k} (1 - q^j).$$ \hspace{1cm} (12)

We also define

$$\mathcal{K}(n) = \frac{1}{2} \sum_{(a,m) \in G} \sum_{(b,k) \in G} R_{ab}^{mk} n_m^{(a)} n_k^{(b)}.$$ \hspace{1cm} (13)

A conjecture for the characters of the type $\chi^0_\lambda(\tau)$ was described by Kuniba et al. [6]. Their conjecture is

$$\chi^0_\lambda(\tau) = q^{-c_{pf}/24} \sum_{\lambda(n) = \lambda \mod lM} \frac{q^{\mathcal{K}(n)}}{(q)_n},$$ \hspace{1cm} (14)

where $c_{pf}$ is the central charge of the parafermions, $c_{pf} = c - r$. There is a summation over the integers $n$ from zero to infinity, and

$$q = e^{2\pi i \tau}. \hspace{1cm} (15)$$

This conjecture holds for all the algebras $H$ and all the levels $l$ and any weight $\lambda$ on the root lattice $M$. Our aim here is to verify this conjecture, as well as establishing expressions for the characters $\chi^A_\lambda(\tau)$, such that $\Lambda \neq 0$.

Our conjectured identities holds for all the characters $\chi^A_\lambda(\tau)$, where the weight $\Lambda$ is given by

$$\Lambda = f \Lambda_g,$$ \hspace{1cm} (16)

where $f$ is a non-negative integer and $\Lambda_g$ is the $g$th fundamental weight, and for any weight $\lambda$, which obeys the admissibility condition $\Lambda - \lambda \in M$. We also
assume that $\Lambda_g$ has mark one, i.e., $\Lambda_g \cdot \theta = 1$, where $\theta$ is the highest root and $\alpha_g^2 = 2$. Our conjecture also holds for fundamental weights which have dual mark one, $\Lambda_g \cdot \theta = 1$ assuming $\alpha_g$ is a short root, provided we take, $f = 1$.

For such weights $\Lambda = f \Lambda_g$, as above, we define

$$L_{f,g}(n) = \sum_{m=1}^{l_g-1} (B_{l_g})_{l_g-f,m} n_{m}^{(g)},$$

when $1 \leq f \leq l_g - 1$ and we otherwise assume $L_{f,g}(n) = 0$. The matrix $B_N$ was defined in eq. (9).

Our main result, which is a conjecture for $\chi^\Lambda_{\lambda}(\tau)$, is then

$$\chi^\Lambda_{\lambda}(\tau) = q^{-\Delta^\Lambda_{\lambda}} \sum_{\lambda(n) = \lambda - \Lambda \mod lM_L} \frac{q^{-L_{f,g}(n) + K(n)}}{(q)_n},$$

where there is a summation of $n$ over all the non negative integers, $n \in (\mathbb{Z}_0)^{|G|}$, and $\Delta^\Lambda_{\lambda}$ is some dimension which we do not specify. The GRR formula eq. (18) holds for all the algebras, $H$, at all the levels.

This identity is, by definition, invariant under the symmetry, eq. (5), where we take $\lambda \rightarrow \lambda + \mu$, where $\mu \in lM_L$, which is a good consistency check. It is also identical when $\Lambda = 0$ to the conjecture of Kuniba et al., eq. (14). In the case of $H = SU(2) = A_1$, a GRR expression for the characters was obtained by Lepowski et al. [2], and it agrees exactly with our conjecture. Also, for the level two simply laced algebras a GRR expression was described in refs. [7, 8], and our formula here specialises precisely to these level two results. Thus our formula, eq. (18), encapsulates all the known GRR expressions for the characters of the parafermionic field theories, along with providing a wealth of new ones.

The characters are invariant under the field identifications [1],

$$\chi^\Lambda_{\lambda}(\tau) = \chi_{\sigma(\lambda)}^{\sigma(\Lambda)}(\tau),$$

where $\sigma$ is any automorphism of the affine Dynkin diagrams, and we assumed that $\lambda$ is also an integrable weight at level $l$. When $\Lambda = f \Lambda_g$, where $\Lambda_g \cdot \theta = 1$, and
\( \alpha_g^2 = 2 \), this is equivalent to taking \( f \) to \( l - f \), and eq. (19) is indeed a symmetry of the GRR expression eq. (18), (when we transform also \( \lambda \) appropriately), providing a consistency check for this identity.

The conjecture eq. (14) was shown in ref. [6] to give the correct central charges by evaluating the \( q \to 1 \) limit, making connection with thermodynamic Bethe ansatz equations. Our conjecture eq. (18) thus also gives the correct central charges since the linear terms that we added do not change the calculation of the central charges, as they can be neglected when the elements of \( n \) are all large.

The symmetry of the characters under the Weyl group, eq. (5), \( \chi^\Lambda_\lambda (\tau) = \chi^{\Lambda_{w(\lambda)}}_{\lambda} \), where \( w \) is any element of the Weyl group, is not explicit in the GRR, eq. (18), but actually gives an expression with a different summation condition, for each \( w \in W \). These GRR sums are, of course, equal, by eq. (5), and gives a wealth of nontrivial identities, which we term generalised Slater identities, as the simplest case, for the Ising model, was described by Slater [13]. In special cases, such identities were noted already in refs. [7, 8].

We get now to the problem of verifying our general GRR identity eq. (18). The characters of the parafermionic theories are closely related to the string functions, eq. (3). These string functions are, actually, multiplicities of weights in the corresponding integrable affine highest weight representation, eq. (4). We denote this representation by \( L(\bar{\Lambda}) \) where \( \bar{\Lambda} \) is the affine highest weight, whose finite weight is \( \Lambda \) and its level is \( l \). For explanation of this notions see the book by Kac [14], or for a review, the appendix of ref. [15]. Thus, the problem of calculating the characters reduces to the problem of calculating the multiplicities of weights in the appropriate representation of the affine algebra \( \hat{H} \). Luckily, a great deal of results are known about these. We find, of particular suitability, the generalisation by Kac [14], p. 211, of Freudenthal’s formula [16], which is valid for any Kac Moody algebra, and is
\[(|\bar{\Lambda} + \bar{\rho}|^2 - |\bar{\lambda} + \bar{\rho}|^2) \dim V_{\bar{\lambda}} = 2 \sum_{\alpha \in \Delta_+} \sum_{j \geq 1} (\text{mult } \alpha)(\bar{\lambda} + j\alpha|\alpha)\dim V_{\bar{\lambda} + j\alpha}. \quad (20)\]

We specialise here to the untwisted affine algebra \( \hat{H} \). Here \( \bar{\Lambda} \) and \( \bar{\lambda} \) are the affine weights, \( \bar{\rho} \) is the affine generalisation of half the sum of positive roots, \( \Delta_+ \) is the set of positive affine roots and \( \text{mult } \alpha \) is the multiplicity of the root \( \alpha \). The scalar product \((a|b)\) is the affine scalar product and we denote by \( \dim V_{\bar{\lambda}} \) the dimension of the vector space with the weight \( \bar{\lambda} \) (or, multiplicity).

The Freudenthal Kac formula, eq. (20), is an effective tool for calculating the multiplicities in the affine representation, as it can be used recursively, for each representation, starting from the highest weight, grade by grade. For a discussion and examples for simple Lie algebras see Humphrey’s book [16]. For affine algebras, since the representation is infinite, we simply have to stop at some grade (or, dimension). This sort of computation, can be done by a computer program. We implemented this algorithm in the fortran program ALGEBRA (written by A. Abouelsaood and D. Gepner), which calculates the multiplicities either for finite or untwisted affine algebras.

Let us give some examples. We find it convenient to rearrange the elements of \( K \) in a matrix,

\[ M_{r,s} = K^{G_{[r,2]},G_{[s,2]}}, \quad (21) \]

where \( G \) and \( K \) were defined in eqs. (6,7), and \( r, s = 1, 2, \ldots, |G| \) and we denote by \( G[r, k] \) the \( k \)th entry of the \( r \)th element of \( G \).

Consider the algebra \( B_2 \) at level two. The simple roots of the algebra are \( \alpha_1 = \epsilon_1 - \epsilon_2 \) and \( \alpha_2 = \epsilon_2 \), where \( \epsilon_i \) are orthogonal unit vectors. The fundamental weights are \( \Lambda_1 = \epsilon_1 \) and \( \Lambda_2 = (\epsilon_1 + \epsilon_2)/2 \). Here \( l_1 = 2, l_2 = 4, \) and

\[ M = \frac{1}{2} \begin{pmatrix} 2 & -1 & -2 & -1 \\ -1 & 3 & 2 & 1 \\ -2 & 2 & 4 & 2 \\ -1 & 1 & 2 & 3 \end{pmatrix}. \quad (22) \]
We define the vector \( \mathbf{n} \) as, eq. (10),

\[
\mathbf{n} = (n_1, n_2, n_3, n_4) = (n_1^{(1)}, n_1^{(2)}, n_2^{(2)}, n_3^{(2)}).
\] (23)

We consider the fundamental weight of the short root, \( \Lambda_2 = (\epsilon_1 + \epsilon_2)/2 \). For \( \lambda \) we consider two cases. First, for \( \mu = 0 \) we take \( \lambda = \Lambda_2 \), and for \( \mu = 1 \) we take \( \lambda = \Lambda_2 + \alpha_2 = (\epsilon_1 + 3\epsilon_2)/2 \).

Then, our general GRR identity, eq. (18) becomes,

\[
z_\mu = q^{\Delta_\mu} \chi_{\Lambda_2 + \mu \alpha_2} = \sum_{\substack{n_1 = 0 \mod 2 \\
n_2 + 2n_3 + 3n_4 = \mu \mod 4}} \frac{q^{\mathbf{n}Mn/2 - (n_2 + 2n_3 + 3n_4)/4}}{(q)\mathbf{n}},
\] (24)

where there is a sum over all the \( n_i \) from zero to infinity, and \( \Delta_\mu \) is set to make the leading term \( q^{\mu/2} \).

We can evaluate the sum, eq. (24), by a Mathematica program and we find

\[
z_0 = 1 + 3q + 9q^2 + 22q^3 + 46q^4 + 93q^5 + 176q^6 + 319q^7 + 562q^8 + 960q^9 + O(q^{10}),
\] (25)

and

\[
q^{\frac{1}{4}} z_1 = 2 + 6q + 14q^2 + 32q^3 + 66q^4 + 128q^5 + 238q^6 + 426q^7 + 736q^8 + 1242q^9 + O(q^{10}).
\] (26)

From the program ALGEBRA we can calculate the relevant string functions which are \( c_{\Lambda_2 + \mu \alpha_2}(\tau) \), by reading off the dimensions in the representation \( L(\bar{\Lambda}_2) \) at level two. We find, up to grade 8,

\[
q^{d_0} c_{\Lambda_2}^L(\tau) = 1 + 5q + 20q^2 + 65q^3 + 185q^4 + 481q^5 + 1165q^6 + 2665q^7 + 5822q^8 + O(q^9).
\] (27)

Multiplying it by \( \eta(\tau)^2 \) according to eq. (3), we find the character up to order 8, and these agree exactly with \( z_0 \), calculated from the GRR, eq. (25). For the
other string function, we find from ALGEBRA,
\[ q^{d_1} c_{\Lambda_2 + \alpha_2}^A (\tau) = 2 + 10q + 36q^2 + 110q^3 + 300q^4 + 752q^5 + 1770q^6 + 3956q^7 + O(q^8), \]
(28)
which, again, when multiplied by \( \eta(\tau)^2 \) agrees exactly with \( z_1 \). The quantities \( d_\mu \) are some dimensions, which follow from eq. (4). Many more string functions can be compared with the GRR and indeed, they all agree, verifying our general conjecture eq. (18) for the algebra \( B_2 \) at level two.

Let us give another example which is \( SU(3) = A_2 \) at level 3. The simple roots of \( A_2 \) are \( \alpha_1 = \epsilon_1 - \epsilon_2 \) and \( \alpha_2 = \epsilon_2 - \epsilon_3 \). The fundamental weights are, \( \Lambda_1 = (2\alpha_1 + \alpha_2)/3 \) and \( \Lambda_2 = (\alpha_1 + 2\alpha_2)/3 \). Both weights have mark one. Here, \( t_1 = t_2 = 1 \) and so \( l_1 = l_2 = l = 3 \). The Cartan matrix is

\[ C_3 = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \]
(29)
and

\[ G = \{(1, 1), (1, 2), (2, 1), (2, 2)\}. \]
(30)

Here,

\[ K_{m,k}^{a,b} = (\min(m, k) - mk/3)\alpha_a \cdot \alpha_b = \]
(31)

\[ (B_3)_{mk}(C_3)_{ab} = \frac{1}{3} \left( \begin{array}{cc} 2 & 1 \\ 1 & 2 \end{array} \right)_{m,k} \left( \begin{array}{cc} 2 & -1 \\ -1 & 2 \end{array} \right)_{a,b}. \]

Also,

\[ n = (n_1^{(1)}, n_2^{(1)}, n_1^{(2)}, n_2^{(2)}). \]
(32)

From eq. (13),

\[ K(n) = \frac{1}{2} \sum_{a,b,m,k=1}^{2} K_{m,k}^{a,b} n_m^{(a)} n_k^{(b)}. \]
(33)

We consider the highest weight \( \Lambda_1 = (2\alpha_1 + \alpha_2)/3 \) and the weights \( \lambda = \)
$\Lambda_1 + \mu \alpha_1$, where $\mu = 0, 1$. Our general GRR, eq. (18), then becomes

$$z_\mu = q^{d_\mu} \chi_{\Lambda_1 + \mu \alpha_1} = \sum_{\begin{array}{c} n_1^{(1)} + 2n_2^{(1)} = \mu \mod 3 \\ n_1^{(2)} + 2n_2^{(2)} = 0 \mod 3 \end{array}} \frac{q^{\mathcal{K}(n)-(n_1^{(1)}+2n_2^{(1)})/3}}{(q)_n},$$

(34)

where $d_\mu$ is a dimension used to make the leading term $q^{\mu/3}$.

Using a Mathematica program, we evaluate $z_\mu$, eq. (34), and we find

$$z_0 = 1 + 2q + 7q^2 + 16q^3 + 36q^4 + 70q^5 + 135q^6 + 243q^7 + 431q^8 + O(q^{10}).$$

(35)

and

$$q^{-\frac{1}{3}} z_1 = 1 + 4q + 9q^2 + 22q^3 + 44q^4 + 89q^5 + 163q^6 + 297q^7 + 513q^8 + 874q^9 + O(q^{10}).$$

(36)

From the program ALGEBRA we find the string functions in the affine algebra representation,

$$q^{d_0} c_{\Lambda_1} (\tau) = 1 + 4q + 16q^2 + 50q^3 + 143q^4 + 368q^5 + 892q^6 + 2035q^7 + 4448q^8 + O(q^9),$$

(37)

and

$$q^{d_1} c_{\Lambda_1 + \alpha_1} (\tau) = 1 + 6q + 22q^2 + 70q^3 + 193q^4 + 493q^5 + 1170q^6 + 2642q^7 + O(q^8),$$

(38)

where $d_\mu$ are some dimensions which follow from eq. (4). Multiplying these string functions by $\eta(\tau)^2$, to get the characters, we find exactly the expressions calculated from the GRR, eqs. (35,36).

We studied many other string functions for this algebra, confirming our GRR formula for the algebra $A_2$ at level 3.
One may check in this fashion other algebras. We verified our GRR formula, eq. (18), for many algebras at many levels. These include the non–simply laced algebras, $B_2, B_3, C_3, G_2, F_4$ at level two, and the simply laced algebras $A_2$ at levels 3, 4, $A_3, D_4$ and $E_6$ at level 3. We checked in these algebras many of the string functions and, indeed, the GRR expressions agree exactly with the string functions, as computed by ALGEBRA.

We described here a Rogers Ramanujan type expressions for the characters of parafermionic cosets of the type $H_l/U(1)^r$ for all the algebras $H$ at level $l$, where $r$ is the rank. The characters are $\chi^\Lambda_\lambda$, where $\Lambda$ is an integrable highest weight, and $\lambda$ is a weight. Our conjecture holds for all $\Lambda$ such that $\Lambda = f\Lambda_g$, where $f$ is an integer, and $\Lambda_g \cdot \theta = 1$ where $\theta$ is the highest root, i.e., $\Lambda_g$ has mark one, and $\alpha_2^g = 2$. Our conjecture holds also for short roots, which have dual mark one, $\alpha_2^g < 2$, provided $f = 1$. Our GRR expression holds for all the weights $\lambda$. We verified this conjecture using Freudenthal Kac formula, which we computerised in the fortran program AGEBRA. This program gives the exact characters, up to some grade.

It is an interesting question to generalise our GRR expressions to the weights $\Lambda$ other than the ones above. Another interesting question is how to connect these expressions with solvable lattice models of the RSOS type (rigid solid on solid). This will, also, furnish a physical proof for these identities.

The parafermionic characters are related to the string functions. The string functions are central in the theory of affine algebras. Thus, our expressions, which give the first closed formula for these new string functions, are also very important mathematically.

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