Software holography: interferometric data analysis for the challenges of next generation observatories

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ABSTRACT

Next generation radio observatories such as the Murchison Widefield Array (MWA), the Long Wavelength Array (LWA), the LOw Frequency ARray (LOFAR), the Combined Array for Research Millimeter-wave Astronomy (CARMA) and the Square Kilometer Array (SKA) provide a number of challenges for interferometric data analysis. These challenges include heterogeneous arrays, direction-dependent instrumental gain, and refractive and scintillating atmospheric conditions. From the analysis perspective, this means that calibration solutions cannot be described using a single complex gain per antenna. In this paper, we use the optimal map-making formalism developed for cosmic microwave background analyses to extend traditional interferometric radio analysis techniques – removing the assumption of a single complex gain per antenna and allowing more complete descriptions of the instrumental and atmospheric conditions. Due to the similarity with holographic mapping of radio antenna surfaces, we call this extended analysis approach software holography. The resulting analysis algorithms are computationally efficient, unbiased and optimally sensitive. We show how software holography can be used to solve some of the challenges of next generation observations, and how more familiar analysis techniques can be derived as limiting cases.

Key words: instrumentation: interferometers – techniques: interferometric – cosmology: miscellaneous.

1 INTRODUCTION

Motivated by the requirements of next generation radio observatories, we examine an alternative approach for formulating optimal radio analyses. The cosmic microwave background (CMB) optimal map-making (OMM) formalism provides an elegant way to translate from a mathematical description of the measurement to provably optimal analyses (Tegmark 1997a,b), and underlies most current CMB observations. In this paper, we use this OMM formalism to describe the interferometric data analysis problem, and show how many of the issues faced by next generation arrays can be naturally incorporated into this framework.

After briefly introducing OMM and how it can be used with interferometric data in Section 2, we use this framework to expand the mathematical descriptions to include direction-dependent antenna response (Section 3), heterogeneous arrays (Section 3.1), wide-field refractive atmospheric distortions (Section 4.1), and scintillating distortions (Section 4.2). We then conclude in Section 5 with a discussion of wide-field effects and comparing this analysis approach with faceting and other commonly used techniques for analysing interferometric data with direction-dependent calibration and distortion.

In a recent paper, Bhatnagar et al. (2008) detail a new analysis for data with direction-dependent antenna gains which is functionally identical to the analysis we develop in Section 3.1. While these papers were developed independently, we believe they should be considered as a complementary pair – Bhatnagar et al. demonstrate increased fidelity in the context of traditional radio astronomy software, while we provide a theoretical foundation for the extended software holography technique and apply it to a number of other problems facing next generation interferometric arrays. Specifically, in this paper we will:

(i) place the work of Bhatnagar et al. (2008) on an alternate theoretical foundation;
(ii) extend the ideas of software holography to refractive and scintillating atmospheric distortions;
(iii) provide a first step towards using CMB deconvolution techniques with interferometric data, enabling high-precision statistical measurements such as 21 cm Epoch of Reionization power spectrum measurements.
2 OPTIMAL MAP MAKING

In this section, we briefly introduce the OMM formalism that underpins most CMB data analyses (Tegmark 1997a,b), then as an example show how this description can be used to describe the traditional algorithms used in radio astronomy analysis software such as AIPS, MIRIAD and CASA.

There are two key steps in the OMM method: a mathematical description of the measurement and the optimal reconstruction based on this measurement description. In general, one describes the observation in the following form:

\[ d = Mx + n, \]

(1)

where \( d \) is a vector of the measurements and \( x \) is the true value one is measuring. \( M \) is then a matrix operator that describes the measurement process, including all instrumental, atmospheric and data handling effects, and \( n \) is detector noise with covariance matrix \( N = \langle nn^T \rangle \).

If we assume the measurement can be expressed as a linear equation of the form in equation (1) and the noise is Gaussian and uncorrelated with the signal (both are true for radio astronomy), it can then be proved that the minimally biased maximum likelihood estimator for \( x \) is given by

\[ \hat{x} = (M^TN^{-1}M)^{-1}M^TN^{-1}d \]

(Tegmark 1997a). Equation (2) can be viewed as consisting of two separate parts:

\[ \hat{x} = \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} (M^TN^{-1}M)^{-1}M^TN^{-1}d \\ \end{bmatrix} \]

(3)

In the first step, the measurements are weighted by their signal-to-noise ratio (including covariant noise) and translated by the conjugate transpose of the measurement description \( M \) back into the coordinates of the input sky. Effectively, this forms a ‘dirty map’ at the end of step one. The second part of equation (3) then represents a deconvolution step. While the OMM method implies a particular style of deconvolution that has been very successful for CMB analysis, one could also use CLEAN, MEM or other non-linear deconvolution algorithms.

For our case, we are only concerned with the first part of equation (3), and can thus rewrite the relationship as

\[ \hat{x} = D^{-1}M^TN^{-1}d, \]

(4)

where \( D^{-1} \) represents the deconvolution algorithm of the reader’s choice. What is powerful about the OMM framework is that the dirty map formed by the first part of equation (3) can be proved to be unbiased and efficient (Tegmark 1997a). In particular, the lossless nature guarantees that all of the information present in the individual measurements \( d \) is retained in the ‘dirty map’ formed at the end of Part 1.1 Often there is significant data compression in forming this lossless intermediate representation. For example, in CMB satellites, the huge number of time-ordered data values are reduced to an unprocessed temperature map, and for Very Large Array (VLA) interferometry the raw visibilities are reduced to a dirty map.

2.1 Standard interferometric techniques

It is instructive at this point to write down the standard interferometric analysis of AIPS,\(^2\) MIRIAD\(^3\) (Sault, Teuben & Wright 1995) and CASA\(^4\) using the OMM formalism. It is traditional to assume a small field of view when deriving the interferometric analysis equations (e.g. Clark 1999, equations 1–8), and we make the same assumption here as it simplifies the notation in the following sections and helps focus the reader on the unique characteristics of the software holography approach. However, we realize this is a bit of a straw man comparison as there are more advanced techniques in general usage. In Section 5, we will return to show how wide-field effects can be included in all of the developments presented here and discuss how this work is related to the more modern approaches of Cornwell & Perley (1992), Sault, Staveley-Smith & Brouw (1996) and Bhatnagar et al. (2008).

We start with a description of the measurement, for a standard mid-frequency observation with an array like the VLA:

\[ v_b = g_b \int \delta(u - u_b) \left[ \int e^{-i2\pi u \cdot \theta} I(\theta) d\theta \right] d^2u + n_v. \]

(5)

Equivalently, we can sample the sky \( I(\theta) \) very finely and rewrite equation (5) in linear algebra notation:

\[ m(v) = G(v, u)S(v, u)F(u, \theta)I(\theta) + n(v). \]

(6)

In other words the measurement equations (5) and (6) take the true sky \( I(\theta) \), Fourier transform to form the true \( u, v \) distribution (represented by the two-dimensional vector \( u \), see Table 1), sample the true \( u, v \) distribution with the baseline distribution of the observation \( S \) (a set of \( \delta \)-functions at each baseline location) to form the visibilities, multiply by the complex gain \( g_b \) (or \( G \)) appropriate for each visibility (can include both instrumental and atmospheric/ionospheric effects), and add the per visibility thermal noise \( n_v \). Note that in the linear algebra version, the symbols in parentheses represent the coordinates being transformed to and from and are not arguments – thus \( S(v, u) \) is transforming from a vector of \( u, v \) locations \( \{u\} \) to a vector of visibilities \( v \). In the remainder of this paper, we will use the linear algebra notation as it more elegantly expresses some of the transformations we are interested in. (Please see Table 1 for how to translate these into integral notation as needed.)

Using the description of the measurement in equation (6), we can directly write down the optimal analysis as

\[ \hat{I}(\theta) = D^{-1}F^T(\theta, u)S^T(\theta, u)G^T(v, u)N^{-1}m(v). \]

(7)

The analysis is essentially weighting by the noise, applying the steps describing the measurement in reverse order (and conjugate transposed), and deconvolving. Again describing the process: we start with a vector of measured visibilities \( m \), weight them by the inverse noise covariance matrix (high noise channels receive less weight), multiply by the transpose of the gain \( G^T \) to correct the phase, then grid the visibilities to the \( u, v \) plane with \( S^T \) and Fourier transform to form a dirty map of the sky.\(^5\)

Equation (7) is the traditional analysis as implemented in several current interferometric software packages. Often the terms in

1 Lossless here means that all of the sky information that was in the visibilities is preserved in the intermediate map. This does not mean one can obtain the true sky, as the measurement process itself removes a lot of information (incomplete \( u, v \) coverage, etc.), only that the information content of the measurements has been preserved in the analysis.

2 http://www.aips.nrao.edu/
3 http://www.atnf.csiro.au/computing/software/miriad/
4 http://casa.nrao.edu/
5 In some radio software implementations the conjugate reciprocal of the gain is used \((1/g^*_b) \) or \( G^T \) as opposed to the gain conjugate \((g^*_b \) or \( G^T \)) as depicted in equation (7). This difference is usually unimportant in a non-linear \( \text{CLEAN} \)-like algorithm (other than the units of the intermediate map), but the version in equation (7) maximizes the signal-to-noise ratio.
The problem encountered by next generation arrays is that equation (6) does not accurately describe their measurement – there are a number of assumptions about the measurement embedded in this measurement description. Describing the gain as a per-visibility complex number $G(v, θ)$ assumes that the gain and phase are uniform across the antenna field of view. Several next generation instruments have instrumental gain and phase which vary as a function of direction within the field of view, fundamentally breaking this assumption. Similarly, atmospheric distortions with length-scales smaller than the field of view cannot be expressed as a single per-baseline complex number. In addition, the flat sky assumption is incorporated in the $θ$ coordinate system, hampering the analysis of wide-field observations.

The remainder of this paper largely consists of rewriting equation (6) to accurately describe the measurements proposed with next generation arrays, and using the OMM formalism to determine the appropriate analysis methods. Throughout this paper, we assume the antenna calibration has been determined separately. In the current software, the overall antenna and atmospheric delay is incorporated in the self-cal calibration step using the results of self-cal; sometimes the Fourier transform is incorporated into the deconvolution step for computational reasons; and if a fast Fourier transform (FFT) is used, anti-aliasing filters must be added. However, the fundamental algorithm is the same. This is reassuring as it has long been known that this algorithm is optimal if the description of the measurement in equation (6) holds.

The section concentrates on instrumental calibration when the gain and phase vary as a function of direction within the field of view. This case is commonly encountered in wide-field imaging, such as polarimetric observations. Either approach to determining the calibration can be used in this formalism.

3 POSITION DEPENDENT CALIBRATION AND HETEROGENEOUS ARRAYS

This section concentrates on instrumental calibration when the gain and phase vary as a function of direction within the field of view. This case is commonly encountered in wide-field imaging, such as polarimetric observations. Either approach to determining the calibration can be used in this formalism.

| Linear algebra | Integral notation | Comments |
|----------------|------------------|----------|
| $u$            | $u$ or $\{u, v\}$ | $u, v$ coordinates. In this paper, we condense this to a single two-dimensional vector $u$ to make the notation more compact and avoid confusion with visibilities. |
| $v$            | $v$ or $v$       | ‘Visibility’ coordinates, or a vector listing the visibilities. In integral notation usually indicated as a subscript. |
| $n(v)$         | $n_b$           | Thermal noise per visibility. The matrix $N$ is formed by the outer product of two vectors of the thermal noise, and allows correlated noise to be included in the linear algebra notation (e.g. cable cross-talk). |
| $m(v)$         | $m_v$ or $v_b$  | A vector of measured visibilities. Usually expressed with subscripts in integral notation. |
| $I(θ)$         | $I(θ)$          | The true sky brightness distribution. Note that in the linear algebra notation, this is a vector of sky locations – thinking of this as a two-dimensional ‘matrix’ of values breaks the linear algebra notation (requires all operators to be diagonal). |
| $F(u, θ)$      | $\int e^{-2πιuv θ} d^2θ$ | Fourier transform. May be replaced with an FFT with the addition of an anti-aliasing filter. |
| $S(v, u)$      | $\int δ(u - u_b) d^2u$ | Sampling function which selects the locations in the $u, v$ plane which are measured by an interferometric baseline $b$ to create a visibility. In the linear algebra notation, the result is a vector of the visibilities ($v$). |
| $G(v, θ)$      | $g_b$           | A single complex gain per visibility. The matrix version has entries only along the diagonal. |
| $B(θ, θ)$      | $B(θ)$          | The power response of a pair of antennae. As both the gain and phase may change with direction for each antenna, this is a complex function. The $B$ operator is diagonal. |
| $B(u, u)$      | $B(u) \ast or \int B(u' - u) d^2u$ | Power response of a pair of antennae in $u, v$ coordinates. The Fourier transform of $B(θ)$ and includes the convolution created by translating the multiplication in $θ$ coordinates to $u$. Due to the physics of antennae the $B$ operator is sparse (of limited extent in $u$). |
| $\hat{B}(v, u)$ | $\int δ(u - u_b' B_b(u' - u) d^2u d^2u'$ | Power response of the antennae in a particular baseline. Effectively, this is a convolution over the sky in $u, v$ coordinates combined with a delta-function to select the baseline sampled by that antenna pair. The result is a vector of visibilities. |
| $W_i(u, u)$    | $W_i(u) \ast or \int W_i(u' - u) d^2u$ | The electric field response of an antenna in $u, v$ coordinates. This is the holographic antenna pattern, and is the Fourier transform of the direction-dependent gain $W_i(θ)$. |
| $H_i\{l, m, w\}$ | $e^{-2πιwl(\sqrt{1 - l^2 - m^2} - 1)]}$ | $w$-projection. For non-coplanar baselines in the narrow field limit can be interpreted as Fresnel diffraction (Cornwell et al. 2008). |
| $D^{-1}$       | $-$             | Deconvolution. There are many styles of deconvolution, many of which are non-linear (cannot be expressed as an integral or linear algebra equation). Any kind of deconvolution can be used with the results presented in this paper. |
as low-frequency observations with the VLA and all upcoming low-frequency arrays such as the MWA, LWA and LOFAR. The additional complication of atmospheric calibration will be delayed until Section 4.

We will first assume that the gain pattern is identical for all antennae. The standard measurement description in equation (6) can be modified to form

\[ m(v) = S(v, u)F(u, \theta)B(\theta, \theta)I(\theta) + n(v). \]  

(8)

We have replaced the per baseline gain \( G(v, v) \) with a direction dependent complex power pattern \( B(\theta, \theta) \). The beam pattern attenuates the signal seen by the interferometer, but the remainder of the measurement is unaffected.

Again following the OMM framework, our analysis method should be

\[ \hat{I}(\theta) = D^{-1}B^T(\theta, \theta)F^T(\theta, u)S^T(u, v)N^{-1}m(v). \]  

(9)

This is largely identical to equation (7), except that we have multiplied by the beam transpose \( B^T \). In forming this transpose, we take the complex conjugate of the gain towards each sky pixel \( \theta \) and reorder the entries. This means that a pixel gain of \( \frac{1}{e}e^{j\phi} \) becomes \( \frac{1}{e}e^{-j\phi} \): the phase is corrected but the gain amplitude is applied a second time. The amplitude of the ‘dirty map’ formed just before the deconvolution is attenuated by the beam shape squared. The signal is attenuated once in the measurement description \( B \) in equation (8), and again by the analysis procedure \( (B^T \) in equation (9)).

While puzzling at first the highest signal-to-noise ratio is achieved if signals are variance weighted, and the measured signal-to-noise ratio in a pixel is given by the beam pattern. The NRAO VLA Sky Survey (NVSS) team uses beam-squared weighting to add overlapping maps for exactly this reason (Condon et al. 1998), and it is gratifying to see this result falls out of this derivation.

3.1 Heterogeneous arrays

The development above assumed that the directional responses of all the antennae were identical. For many upcoming observations the antennae are not identical, either due to antenna-to-antenna variation such as the MWA, or due to a mix of different antenna types as in CARMA. In this section, we will remove the assumption of identical antenna responses, and follow a slightly more detailed derivation to illustrate use of the OMM method.

The easiest way to extend equation (8) to heterogeneous arrays is to subscript the beam pattern, giving each baseline a unique power pattern:

\[ m(v) = S(v, u_i)F(u_i, \theta_i)B_i(\theta_i, \theta_i)I(\theta_i) + n(v). \]  

(10)

This equation creates a different observed sky for each antenna pair \( \theta_i \), and the remainder of the measurement remains the same. Unfortunately, this is a computationally expensive description of the measurement as each of the \( b \) separate observed skies require a Fourier transform and sampling, and it leads to a computationally expensive analysis algorithm:

\[ \hat{I}(\theta) = D^{-1}F^T(\theta, u)B^T(\theta, \theta)S^T(u, v)N^{-1}m(v). \]  

(11)

Here, each baseline is gridded (single \( u, v \) \( \delta \)-function) and Fourier transformed to produce a single fringe, which is then attenuated by the power pattern appropriate for that baseline before being added to a common dirty map. Conceptually, this is correct. The fringe should only be added to portions of the image seen by that antenna pair, and because \( B \) is complex the location of fringe peaks can shift from one portion of the image to another in response to direction dependent phase response.

Fortunately, there is a more efficient way to perform the same analysis. Returning to the measurement description in equation (10), we can recast the problem:

\[ m(v) = S(v, u_b)F(u_b, \theta_b)B_b(\theta_b, \theta)I(\theta) + n(v), \]

\[ m(v) = S(v, u_b)B_b(\theta_b, \theta)F(u_b, \theta)I(\theta) + n(v), \]

\[ m(v) = B(v, u)F(u, \theta)I(\theta) + n(v). \]  

(12)

In going from line 1 to 2, we have simply pulled the power pattern \( B \) to the other side of the Fourier transform and expressed the power response in \( u, v \) coordinates (the operators commute because the Fourier transform is unitary, and \( B_b \) becomes a convolution). The beam pattern is still baseline dependent in line 2, but the sampling function \( S \) is already selecting out individual baselines to create visibility measurements, so these operators can be combined in the last line to form \( B(v, u) \).

In words, the input sky is transformed to \( u, v \) coordinates, then the appropriate region of the \( u, v \) plane is integrated to form the visibility measured by that pair of antennae using the unique power response of that antenna pair. It is interesting to note that equation (12) is the algorithm used for simulating the response of interferometric observations by the MIT Array Performance Simulator (MAPS).

Before moving to the analysis, a natural question is the size (support) of the power pattern \( B \) in \( u, v \) coordinates – if it covers a significant portion of the \( u, v \) plane, it would remain computationally expensive. Each antenna has a response to the incident electric field \( \mathbf{W}_a(\mathbf{\theta}, \phi) \), where the response is complex to capture both the electric field gain and phase delay. We can transform the antenna response to \( u, v \) coordinates \( \mathbf{W}_a(u, u) \). This transformed antenna response is exactly what is obtained during a holographic measurement of the antenna gain (Scott & Ryle 1977). Because the electric field outside the antenna physically cannot be added into the received signal, \( \mathbf{W}_a(u, u) \) has the same size as the antenna. The power pattern observed by a baseline is given by the multiplication of the constituent antenna sky responses, or the convolution of the \( u, v \) plane responses of antennae \( i \) and \( j \):

\[ B_i(u, u) = \mathbf{W}_a^T(u, u) + \mathbf{W}_a(u, u). \]  

(13)

Thus, the power pattern \( B(u, u) \) is very compact in the \( u, v \) plane, approximately twice the physical width of an antenna. This compact feature is why it forms the basis of array simulators.

Moving to the analysis, we can again use OMM to form an optimal analysis approach:

\[ \hat{I}(\theta) = D^{-1}F^T(\theta, u)B^T(\theta, \theta)S^T(u, v)N^{-1}m(v). \]  

(14)

In this analysis, we have replaced the simple \( \delta \)-function gridding of \( S^T(u, v) \) with a gridding function \( B^T(u, v) \) that spreads the visibility out on the \( u, v \) plane using the power pattern response of that particular antenna pair. Effectively, the direction and baseline dependent instrumental calibration has become part of the gridding kernel. Because this uses the holographic antenna response

6 Reflections from outside the antenna and wide-field \( w \)-projection effects can make the response slightly larger, but it remains very compact in the \( u, v \) plane.

7 When using an FFT, an anti-aliasing filter must be added to the power response kernel using an additional convolution. This has no impact on the final precision if the effects of the anti-aliasing filter are included in the deconvolution.
4 ATMOSPHERIC AND IONOSPHERIC DISTORTIONS

Lonsdale (2005) provides a review of atmospheric and ionospheric distortions. In that work, the effects of atmospheric disturbance are separated into four regimes, depending on the characteristics of the interferometer and the length-scales of atmospheric disturbances. Briefly these regimes are the following.

(i) A narrow field of view and short baselines. Distortion appears as a translation of entire field, correctable with a tip–tilt compensation.

(ii) A narrow field of view and long baselines. Independent phase delay for each antenna, but the delay applies to the entire antenna field of view. Appears as scintillation, but the same scintillation pattern for all sources in the field. Correctable with single calibrator adaptive optics and self-cal, and is typical for VLA observations.

(iii) Short baselines but a wide-field of view. Distortion is purely refractive, but varies across the field of view. Sometimes described as a rubber-sheet distortion and is typical of MWA observations.

(iv) Wide-field of view and long baselines. The worst case, as sources scintillate across the field of view with a position dependent scintillation screen. This is the challenge faced by the LWA and LOFAR at the longest baselines.

Regimes (i) and (ii) are described by the standard measurement description (equation 6), and are well handled with current analysis algorithms. Regimes (iii) and (iv) cannot be described as a single phase delay per antenna, because the delay varies across the field of view. In the following sections, we will explore how to use OMM to analyse observations in the challenging atmospheric conditions of regimes (iii) and (iv).

4.1 Wide-field refractive distortions

In regime (iii), atmospheric and ionospheric distortions appear as a rubber-sheet distortion: the apparent positions of sources are shifted but they do not appear to scintillate. Mathematically, this can be described by

$$m(v) = \hat{B}(v, u) F(u, t) A(\theta'; \theta; t) I(\theta) + n(v').$$

Here, we have added a time-dependent atmospheric distortion $A(\theta'; \theta; t)$ that moves the apparent locations of the sources seen by the array from $\theta$ to $\theta'$. The atmospheric distortion presented here can also include position dependent absorption and Faraday rotation, as long as the distortion is the same for all antennae in the array. Because the atmospheric distortion is time-dependent, the coordinate mapping from the real sky to the apparent sky changes. This means that the associated $u, v$ coordinates and visibilities are also time dependent as indicated. We have also included baseline dependent antenna calibration ($B$) in this example to illustrate how effects can be stacked.

The corresponding analysis is

$$\hat{I}(\theta) = D^{-1} A(\theta'; \theta; t) F^T(u, t) B^T(u, v) N^{-1} m(v).$$

Through the Fourier transform, this is identical to the analysis in the previous section. However, because of the time-dependent nature of the atmospheric distortion, we can only grid and Fourier transform visibilities from one atmospheric realization. Effectively, we are creating instrumentally calibrated snapshot images of the apparent sky, which we then correct with a rubber-sheet correction so sources appear in their true locations ($\theta$), and then stack the snapshot images. The time-scale of the snapshot images is set by the atmospheric...
distortions. This analysis is effectively the approach taken by the MWA (Mitchell et al. 2007), and the ionospheric distortion timescale sets the 8 s snapshot cadence of the instrument.

It is possible to Fourier transform the atmospheric operator and describe it in the $u, v$ plane, in analogy to what we did with the antenna power response in Section 3.1. We will look at this approach in the context of the next section, but the snapshot imaging appears to be a better solution for most interferometers operating in the rubber-sheet conditions considered here.

### 4.2 Wide-field scintillating distortions

Scintillating wide-field distortions of regime (iv) are the most difficult, because each antenna sees a different atmospheric phase screen. This is the analysis challenge faced by LOFAR and the LWA.

To describe the wide-field scintillation seen in regime (iv), we need to make our atmosphere model more general. The wavefront observed by an individual antenna can be described as a direction-dependent phase delay $L_i(\theta, \theta, t)$ (labelled $L$ in sympathy with the challenges faced by LOFAR and the LWA). The atmospheric distortion seen by a baseline is then given by

$$A_i(\theta, \theta) = L_i^T(\theta, \theta)L_i(\theta, \theta),$$

$$A_j(\mathbf{u}, \mathbf{u}) = L_j^T(\mathbf{u}, \mathbf{u})L_j(\mathbf{u}, \mathbf{u}).$$

Effectively, $A_i$ is describing the distortion to the fringe pattern for that baseline. Under the regime (iv) atmospheric conditions, the fringe pattern for a single visibility will not appear as a simple grating across the field, but more like the lines on a topographic map as the direction dependent ionospheric delay shifts and distorts the locations of the fringe peaks.

We can use equations (17) and (18) to describe the measurement as

$$m(v) = \mathbf{B}^T(\mathbf{v}, \mathbf{u}, \mathbf{b})F(u_{i,b}, v_{i,b}, w_{i,b})A_i(\theta_{i,b}, \mathbf{v}; t)I(\theta) + n(v'),$$

again leaving out wide-field curved sky effects to simplify the discussion (we will add these back in Section 5). Not only is the distortion time variable, it is different for every baseline. This leads to an analysis of the form

$$\hat{I}(\theta) = D^{-1} \mathbf{A}^T(\theta, \theta, t)F(\theta_{i,b}, \mathbf{u}_{i,b})\mathbf{B}^T(\mathbf{u}_{i,b}, \mathbf{v})N^{-1} m(v).$$

This analysis is very computationally expensive. In words, we must Fourier transform each calibrated visibility to create a snapshot image of that fringe, which is then corrected by the baseline dependent atmospheric operator $\mathbf{A}^T$. The individual fringes are then co-added to form a snapshot image, and successive images are stacked to form an integrated dirty map.

As an alternative, we could Fourier transform the atmospheric distortion and pull it to the left of the Fourier transform in analogy to equation (12) to obtain

$$m(v) = \mathbf{B}^T(\mathbf{v}, \mathbf{u}, \mathbf{b})A_i(\mathbf{u}_{i,b}, \mathbf{v}; t) \mathbf{F}(\mathbf{u}, \theta)I(\theta) + n(v').$$

As implied by the square brackets, we can now envision combining the position dependent atmospheric and instrumental distortions into a single $u, v$ plane operation. This would give us the analysis procedure

$$\hat{I}(\theta) = D^{-1} \mathbf{F}^T(\theta, \mathbf{u}) \left[ \mathbf{A}^T(\mathbf{u}, \mathbf{b})\mathbf{B}^T(\mathbf{u}, \mathbf{v}) \right] N^{-1} m(v).$$

At first glance, this appears to be the obvious solution: correct both the scintillating ionosphere and instrumental gain in one baseline dependent gridding step. The difficulty is that unlike the beam response, the atmospheric distortion is not well localized in the $u, v$ plane (Matejek & Morales 2009).

The difference in the locality of the instrumental and atmospheric terms is related to the physics of the two distortions. The electric field response $\mathbf{W}$ of an antenna is fundamentally the sum of the electric field collected at each location on the antenna ($\mathbf{W} = w_i + w_2 + \cdots$). These terms are complex, and if they are added out of phase, they interfere and destroy the response of the antenna.

The atmospheric delay $L$ on the other hand is multiplicative: if we decompose the atmosphere into many terms, each rotates the phase by a certain delay angle $\theta$. Mathematically, this ends up with a form of $e^{\pm i\theta_i d_i}$.

Due to the multiplicative nature, this is not compact in the $u, v$ plane and strong beating effects between atmospheric modes come into play—the spatial equivalent of intermodulation distortion. We refer the interested reader to Matejek & Morales (2009).

In conclusion, if the ionospheric phase screen is just a little more complicated than a single per-antenna delay (regime 2) but can be described with only a couple of large sinusoidal modes for each antenna [barely into regime (iv)], the $u, v$ plane atmospheric correction described in equation (22) may be useful. However, for complex direction and antenna dependent atmospheres, the $u, v$ size of the correction becomes enormous. Under these most challenging conditions, one would be best served with the conceptually simpler snapshot fringe approach of equation (20).

### 5 DISCUSSION

To fully integrate software holography into modern interferometric data analysis, there are a few loose ends we should tie up, including projection effects and a comparison with multifaceting techniques.

So far we have used a flat sky ($\theta$) and simple Fourier relationship to simplify the notation and help focus on the unique characteristics of software holography. In general, this is a poor assumption, particularly in the context of wide-field atmospheric distortions. Fortunately, wide-field/w-projection effects can be easily added.

Returning to basics and following the discussion in lecture 1 of Synthesis Imaging in Radio Astronomy (Clark 1999), the general spatial correlation relationship can be written as (their equations 1–5):

$$\mathbf{V}(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \mathbf{C} \{ \mathbf{u}, \mathbf{v}, \mathbf{w} \} s \mathbf{I}(\mathbf{s}),$$

where

$$\mathbf{C} \{ \mathbf{u}, \mathbf{v}, \mathbf{w} \} s = \int e^{-2\pi i u \mathbf{w} / c} d^2 s$$

and $\mathbf{w} = \{ u, v, w \} = r_1 - r_2$. Following Cornwell, Golap & Bhatnagar (2003, 2008), in $\{ u, v, \mathbf{w} \}$ coordinates the correlation relation $\mathbf{C}$ can be decomposed into a Fourier transform, a coordinate conversion $\mathbf{T}$ from $s \rightarrow \{ l, m \}$, and an additional term $\mathbf{H}$ (their equation 10):

$$\mathbf{C} = \mathbf{F}(\mathbf{u}, \mathbf{v}, \{ l, m \}) \mathbf{H}(\mathbf{l}, \mathbf{m}, \mathbf{w}, \{ l, m \}) \mathbf{T}(\mathbf{l}, \mathbf{m}, \mathbf{s}),$$

where $\mathbf{H} = e^{-2\pi i \mathbf{w} (\sqrt{l^2 + m^2 - 1})}$. This gives us the three standard limiting cases:

(i) If the field of view is small, $\mathbf{H}$ is negligible and can be ignored.

(ii) If the array is coplanar, $\mathbf{H}$ can be kept negligible at the cost of a time-dependent coordinate conversion $\mathbf{T}(\{ l', m' \}, s)$. For observatories that contend with a direction-dependent atmospheric
refraction (Section 4.1), this coordinate transformation can be combined with the time-dependent atmospheric distortion \( A(\theta', \theta; t) \) and corrected at no additional cost.

(iii) In the most general case, we can follow the w-projection technique developed by Cornwell et al. (2003, 2008). This is equivalent to pulling \( H \) through the Fourier transform to create a wide-field \( u, v \) correction \( H(u, u; w, t) \).

Any of these limits can be added to the analyses developed in this paper by inserting the appropriate operators.

As an aside, Cornwell et al. worked around the pole in \( H(u, u) \) by using the anti-aliasing filter in the gridding step, even though this filter is only needed for FFTs. In the software-holography context, this can be more physically interpreted as a convolution with the holographic beam pattern \( B(u, u) \). Since any real antenna has a non-zero size, the pole in \( H \) naturally goes away for any physical system.

All of the analysis issues approached in this paper have been successfully tackled in the past with mosaic imaging and multifaceting deconvolution techniques of Cornwell & Perley (1992) and Sault et al. (1996). These techniques subdivide the field of view and apply a separate complex gain calibration to each facet, allowing the atmospheric and instrumental calibration to change across the field in a stepwise fashion. Faceting has the definite advantage of being a proven technique behind many astronomy results, but for next generation arrays its requirements on data storage and computational efficiency may make software holography an attractive alternative.

The computational requirements of faceting are driven by how facetting is integrated into the deconvolution process. When a faceted dirty image is produced, the facet edges distort the apparent array beam of sources in neighbouring facets. This is typically dealt with by using the dirty map only as an intermediate step in the deconvolution process – visibilities are used to make a faceted map, sources are identified and subtracted from the raw visibility data, and a new faceted dirty map is created for the subsequent iteration of the deconvolution algorithm. The deconvolution algorithm must always work on the full visibility data set, as the dirty image contains artefacts which cannot be easily removed.

Bhatnagar et al. (2008) have recently demonstrated the use of software holography within a traditional non-linear deconvolution algorithm. Their algorithm is significantly faster than faceting and does not suffer the discontinuities and artefacts in the intermediate image. However, their algorithm still subtracts the sky model from the raw visibilities, using the dirty map only as an intermediate step in the deconvolution process.

For next generation radio arrays with hundreds of elements and wide fields of view the raw visibility data can be very large: for example, the correlated data rate for the MWA is \( \sim 19 \text{ GB s}^{-1} \) over just 31 MHz of bandwidth. The OMM technique was developed in response to these same computational problems as faced by the CMB community (Tegmark 1997b). The time series data from a satellite such as the Wilkinson Microwave Anisotropy Probe (WMAP) and Planck is analogous to interferometric visibilities and similarly voluminous, and deconvolution of the time series data quickly becomes computationally impractical. OMM allows all of the information in the raw data measurements to be preserved in the intermediate map, reducing both storage needs and the computational requirements of deconvolution. The precision of CMB measurements is a testament to the power of the OMM formalism.

In interferometric software holography, forming the intermediate map is computationally very efficient. The direction-dependent gain of each antenna can be corrected by gridding with the baseline dependent \( u, v \) power pattern. This gridding kernel is very compact, leading to an efficient imaging algorithm. The atmospheric corrections are less efficient than the instrumental calibration, but the final map has no significant artefacts. Thus, the deconvolution algorithms can work directly on the lossless ‘dirty map’ formed by software holography, without referring to the raw visibilities. Deconvolving intermediate maps should be no slower than deconvolving the raw visibilities (while requiring much less storage space), and potentially could be much faster if techniques from the CMB community can be effectively used. It should also be noted that the use of spherical harmonics (Ng 2001), wavelets (McEwen & Scaife 2008), or other linear approaches to the wide-field interferometry problem can be incorporated into the software holography approach.

It is hoped the software holography techniques presented in this paper will assist the development of analysis systems for next generation instrumentation, and enable precise interferometric measurements such as power spectrum detection of 21 cm emission from the Epoch of Reionization.

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REFERENCES

Bhatnagar S., Cornwell T. J., Golap K., Uson J. M., 2008, A&A, 487, 419
Clark B. G., 1999, in Taylor B. B., Carilli C. L., Perley R. A., eds, ASP Conf. Ser., Synthesis Imaging in Radio Astronomy II. Astron. Soc. Pac., San Francisco, p. 1
Condon J. J., Cotton W. D., Greisen E. W., Yin Q. F., Perley R. A., Taylor G. B., Broderick J. J., 1998, AJ, 115, 1693
Cornwell T. J., Perley R. A., 1992, A&A, 261, 353
Cornwell T., Golap K., Bhatnagar S., 2003, EVLA Memo Series, 67, 15
Cornwell T. J., Golap K., Bhatnagar S., 2008, IEEE J. Select. Topics Signal Process., 2, 647
Lonsdale C. J., 2005, in ASP Conf. Ser. Vol. 345, From Clark Lake to the Long Wavelength Array: Bill Erickson’s Radio Science. Astron. Soc. Pac., San Francisco, p. 399
McEwen J. D., Scaife A. M. M., 2008, MNRAS, 389, 1163
Matejek M., Morales M. F., 2009, arXiv:0911.3942
Mitchell D., Greenhill L., Wayth R. B., Ord S., Sault R., Doeleman S. S., Kasper J. C., Morales M. F., 2007, BAAS, 39, 744
Ng K.-W., 2001, Phys. Rev. D, 63, 123001
Sault R. J., Teuben P. J., Wright M. C. H., 1995, in Shaw R. A., Payne H. E., Hayes J. J. E., eds, ASP Conf. Ser. Vol. 77, Astronomical Data Analysis Software and Systems IV. A Retrospective View of MIRIAD. Astron. Soc. Pac., San Francisco, p. 433
Sault R. J., Staveley-Smith L., Brouw W. N., 1977, MNRAS, 178, 539
Scott P. F., Ryle M., 1977, MNRAS, 178, 539
Tegmark M., 1997a, ApJ, 480, L87
Tegmark M., 1997b, Phys. Rev. D, 55, 5895

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