Envelop doubly periodic wave solutions of cubic nonlinear Schrödinger equation using the variational iteration method

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Abstract. One of the advantages of the variational iteration method is the free choice of initial guess. In this paper we use the basic idea of the Jacobian-function method to construct a generalized trial function with some unknown parameters. The cubic nonlinear Schrödinger equation is used to illustrate effectiveness and convenience of the method, some new explicit exact travelling wave solutions are obtained which include envelope bell-type soliton solution, envelope solitary wave solution, and envelope doubly periodic wave solutions.

1. Introduction
In recent years, several powerful methods have been proposed to obtain exact solutions of nonlinear partial differential equations (PDEs), such as the tanh-function method [1], the homotopy perturbation method [2-4], the variational iteration method [5-7], the exp-function method [8,9], and Jacobian-function method [10,11]. In recent years, the direct search for exact solutions of PDEs has become more and more attractive partly due to the availability of computer symbolic systems like Maple or Mathematica, which allows us to perform the complicated and tedious algebraic calculations on computer. In particular, one of the effective direct methods to construct double-periodic wave solutions of PDEs is the Jacobian-function method, which was first proposed by Fu et al. [11]. The Jacobian-function method can be used to seek solitary solutions, doubly-periodic wave solutions and periodic wave solutions of nonlinear differential equations.
In this paper, we will apply the basic idea of the Jacobian-function method to obtain the explicit exact solutions using the variational iteration method. Some new envelope doubly-periodic wave solutions and envelope solitary wave solutions are obtained for the cubic nonlinear Schrödinger equation.

2. The variational iteration method
To illustrate the basic concepts of variational iteration method [6], we consider the following differential equation

\[ Lu + Nu = g(x), \]  \hspace{1cm} (1)

where \( L \) is a linear operator, \( N \) is a nonlinear operator, and \( g(x) \) is an inhomogeneous term.
According to the variational iteration method, we can construct a correct function as follows

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\[ u_{n+1}(x) = u_n(x) + \int_{0}^{x} \lambda \left[ Lu_n(\tau) + N\bar{u}_n(\tau) - g(\tau) \right] d\tau, \]

where \( \lambda \) is a general Lagrangian multiplier, which can be identified optimally via the variational theory, the subscript \( n \) denotes the \( n \)th order approximation, \( \bar{u}_n \) is considered as a restricted variation, i.e. \( \delta\bar{u}_n = 0 \).

One of the advantages of the variational iteration method is the free choice of the initial solution, \( u_0 \). Hinted by the Jacobian-function method, we assume the initial solution can be expressed in a generalized form:

\[ u_0(x,t) = a_0 + \sum_{i=1}^{n} s_n^{i-1}(\xi(x,t))[a_i sn(\xi(x,t)) + b_i cn(\xi(x,t))] \]

where \( a_0 \), \( a_i \), and \( b_i \) are unknown constants to be determined, and \( \xi(x,t) \) is a function of \( (x, t) \), and \( n \) is determined by balancing the highest-order linear term with the nonlinear term of Eq.(1).

In order to identify the constants in the initial solution, we can set

\[ u_n(x,t) = u_{n+1}(x,t) \]

and

\[ \frac{\partial^k}{\partial t^k} u_n(x,t) = \frac{\partial^k}{\partial t^k} u_{n+1}(x,t). \]

From Eqs.(4) and (5), we obtain a set of algebraic polynomials for \( sn(\xi)cn(\xi)dn(\xi) \) \((i,j,l=0,1,2...)\). Eliminating all the coefficients of the powers of \( sn(\xi)cn(\xi)dn(\xi) \) yields a series of differential equations, from which the parameters \( a_i \), \( b_i \), and \( \xi \) are explicitly determined.

Finally, substituting \( a_i \), \( b_i \), and \( \xi \) obtained in the above into (3), we can derive the exact solutions of Eq.(1).

To illustrate the effectiveness and convenience, we consider in the next section the cubic nonlinear Schrödinger equation as an example.

### 3. The explicit exact solutions of the cubic nonlinear Schrödinger equation

The cubic nonlinear Schrödinger equation (CNLS)[12] is one of the most universal models that describe many physical nonlinear system. The equation can be applied to hydrodynamics, nonlinear optics, nonlinear acoustics, quantum condensate, heat pulses in solids and various other nonlinear instability phenomena. The cubic nonlinear Schrödinger equation in one space variable has the general form

\[ iu_t + u_{xx} + q|u|^2u = 0, \quad L_0 < x < L_1, \quad t > 0, \]

where \( i = \sqrt{-1}, q \geq 0 \) is a real parameter and \( u = u(x,t) = v(x,t) + iw(x,t) \), with initial condition:

\[ u(x,t = 0) = g(x) = g_x(x) + ig_y(x), \quad L_0 \leq x \leq L_1, \]

which \( g_x \) and \( g_y \) are real-valued continuous functions of \( x \), and boundary conditions:

\[ \frac{\partial u(L_0,t)}{\partial x} = \frac{\partial u(L_1,t)}{\partial x} = 0, \quad t \geq t_0. \]
To solve Eq. (6) by means of the proposed method, Eq. (6) can be reduced to the following coupled system of equations:

$$v_t + w_{xx} + q(v^2 + w^2)w = 0,$$

subjected to the following initial conditions

$$v(x,0) = g_p(x), \quad w(x,0) = g_s(x),$$

and the

$$\frac{\partial v(L_0,t)}{\partial x} = \frac{\partial v(L_1,t)}{\partial x} = 0, \quad \frac{\partial w(L_0,t)}{\partial x} = \frac{\partial w(L_1,t)}{\partial x} = 0,$$

where $u = u(x,t) = v(x,t) + iw(x,t)$.

To apply the method we construct the following correction functions

$$v_{n+1}(x,t) = v_n(x,t) + \int_0^t \lambda_n(\tau)\left[w_{nt} + w_{xx} + q(v_n^2 + w_n^2)w_n\right]d\tau,$$

$$w_{n+1}(x,t) = w_n(x,t) + \int_0^t \lambda_n(\tau)\left[w_{nt} - \hat{v}_n - q(v_n^2 + w_n^2)\hat{v}_n\right]d\tau,$$

where $\lambda_1$ and $\lambda_2$ are general Lagrange multipliers, $\hat{v}_{nxx}$, $\hat{v}_n$, and $\hat{w}_n$ denote restricted variation which means $\delta\hat{v}_{nxx} = \delta\hat{v}_n = \delta\hat{w}_n = 0$. Its stationary conditions can be obtained as follows:

$$1 + \dot{\lambda}_1(\tau)|_{\tau=t} = 0, \quad \dot{\lambda}_1(\tau) = 0,$$

$$1 + \dot{\lambda}_2(\tau)|_{\tau=t} = 0, \quad \dot{\lambda}_2(\tau) = 0,$$

The Lagrange multiplier, therefore, can be obtained as $\lambda_1(\tau) = -1, \lambda_2(\tau) = -1$, and the following variational iteration formula can be obtained as

$$v_{n+1}(x,t) = v_n(x,t) - \int_0^t [w_{nt} + w_{xx} + q(v_n^2 + w_n^2)w_n]d\tau,$$

$$w_{n+1}(x,t) = w_n(x,t) - \int_0^t [w_{nt} - \hat{v}_n - q(v_n^2 + w_n^2)\hat{v}_n]d\tau.$$
\[ v_n(x,t) = v_{n+1}(x,t), \quad w_n(x,t) = w_{n+1}(x,t) \quad (21) \]

and

\[ \frac{\partial^k}{\partial t^k} v_n(x,t) = \frac{\partial^k}{\partial t^k} v_{n+1}(x,t), \quad \frac{\partial^k}{\partial t^k} w_n(x,t) = \frac{\partial^k}{\partial t^k} w_{n+1}(x,t). \quad (22) \]

Using Maple, we have

\[ \frac{\partial}{\partial t} v_0(x,t) = -b_0 c \sin \theta n \xi + b_0 \lambda \cos \theta c n \xi \delta n \xi - b_1 c \sin \theta k n \xi - b_1 \lambda \cos \theta k n \xi \delta n \xi, \quad (23) \]

\[ \frac{\partial}{\partial t} w_0(x,t) = a_0 c \cos \theta k n \xi + a_0 \lambda \sin \theta c n \xi \delta n \xi + a_1 c \cos \theta k n \xi - a_1 \lambda \sin \theta k n \xi \delta n \xi. \quad (24) \]

Similarly we can obtain explicitly the expression for \( \frac{\partial v_1(x,t)}{\partial t}, \frac{\partial w_1(x,t)}{\partial t} \). Setting \( \frac{\partial v_0(x,t)}{\partial t} = \frac{\partial v_1(x,t)}{\partial t} \),

\[ \frac{\partial v_0(x,t)}{\partial t} = \frac{\partial v_1(x,t)}{\partial t} = \frac{\partial v_1(x,t)}{\partial t}, \quad (25) \]

\[ \frac{\partial w_0(x,t)}{\partial t} = \frac{\partial w_1(x,t)}{\partial t}, \quad (26) \]

Solving the algebraic equations simultaneously, we obtain

**Case 1**

\[ a_0 = b_0 = a_1 = b_1 = 0, a_2 = b_2, \quad \alpha = \pm \sqrt{2q b_2} / 2m, \quad \lambda = \pm \sqrt{2q k b_2} / m, \quad c = \sqrt{(1 - 2m^2)q b_2^2 + 2k^2 m^2} / 2m, \]

where \( b_2, k \) are free parameters, \( m \) is the modulus of the Jacobian elliptic function. We, therefore, obtain the doubly-periodic wave solution of Eqs.(9) and (10) which reads

\[ v_{1,2} = b_2 \cos[k x + (1 - 2m^2)q b_2^2 + 2k^2 m^2 / 2m^2] \cdot \text{cn}[\pm \sqrt{2q b_2} / 2m \cdot (x + 2kt)], \quad (25) \]

\[ w_{1,2} = b_2 \sin[k x + (1 - 2m^2)q b_2^2 + 2k^2 m^2 / 2m^2] \cdot \text{cn}[\pm \sqrt{2q b_2} / 2m \cdot (x + 2kt)]. \quad (26) \]

The corresponding envelope doubly periodic wave solutions of Eq.(6) are

\[ u_{1,2} = b_2 e^{i(1 - 2m^2)q b_2^2 + 2k^2 m^2 / 2m^2} \cdot \text{cn}[\pm \sqrt{2q b_2} / 2m \cdot (x + 2kt)]. \quad (27) \]

If we set \( m=1 \), we can obtain the bell-type solitary wave solutions of Eqs.(9) and (10)

\[ v_{1,2} = b_2 \cos[k x + 2k^2 - q b_2^2 / 2] \cdot \text{sech}[\pm \sqrt{2q b_2} / 2 \cdot (x + 2kt)], \quad (28) \]

\[ w_{1,2} = b_2 \sin[k x + 2k^2 - q b_2^2 / 2] \cdot \text{sech}[\pm \sqrt{2q b_2} / 2 \cdot (x + 2kt)]. \quad (29) \]

The corresponding envelope bell-type solitary wave solutions of Eq.(6) are
\[ u_{1,2}' = b_2 e^{i(kx + \frac{2m^2 - 1}{2})} \sec h \left[ \pm \sqrt{2q}b_2 \right] \left( x + 2kt \right), \quad (30) \]

which are new envelope solitary wave solutions.

**Case 2**

\[ a_0 = b_0 = a_1 = b_1 = 0, a_2 = -b_2, \alpha = \pm \sqrt{2q}b_2, \lambda = \mp \sqrt{2q}kb_2, c = \frac{(m^2 - 1)q b_2^2 - 2k^2m^2}{2m^2}, \]

where \( b_2 \) and \( k \) are free parameters. The doubly periodic wave solutions of Eqs.(9) and (10) are

\[ v_{3,4} = b_2 \cos[kx + \frac{(2m^2 - 1)q b_2^2 - 2k^2m^2}{2m^2}t] \sec h \left[ \pm \sqrt{2q}b_2 \right] (x - 2kt), \quad (31) \]

\[ w_{3,4} = -b_2 \sin[kx + \frac{(2m^2 - 1)q b_2^2 - 2k^2m^2}{2m^2}t] \sec h \left[ \pm \sqrt{2q}b_2 \right] (x - 2kt). \quad (32) \]

The corresponding envelope doubly periodic wave solutions of Eq.(6) are

\[ u_{3,4} = b_2 e^{i(kx + \frac{(2m^2 - 1)q b_2^2 + 2k^2m^2}{2m^2}t)} \sec h \left[ \pm \sqrt{2q}b_2 \right] (x - 2kt). \quad (33) \]

If we set \( m = 1 \), then we can obtain another bell-type soliton solutions.

\[ v_{3,4}' = b_2 \cos[kx + \frac{q b_2^2 - 2k^2}{2}t] \sec h \left[ \pm \sqrt{2q}b_2 \right] (x - 2kt), \quad (34) \]

\[ w_{3,4}' = -b_2 \sin[kx + \frac{q b_2^2 - 2k^2}{2}t] \sec h \left[ \pm \sqrt{2q}b_2 \right] (x - 2kt). \quad (35) \]

The corresponding envelope bell-type solitary wave solutions of Eq.(6) are

\[ u_{3,4}' = b_2 e^{i(kx + \frac{q b_2^2 - 2k^2}{2})} \sec h \left[ \pm \sqrt{2q}b_2 \right] (x - 2kt). \quad (36) \]

which have been given by Ref.[12]

**Case 3**

\[ a_0 = b_0 = 0, a_1 = b_1 = \pm ib_2, a_2 = b_2, \alpha = \pm \sqrt{2q}b_2, \lambda = \pm \sqrt{2q}kb_2, c = \frac{(m^2 - 1)q b_2^2 + k^2m^2}{m^2}, \]

where \( b_2 \) and \( k \) are free parameters. The doubly periodic wave solutions of Eqs.(9) and (10) are

\[ v_{5,6} = b_2 \cos[kx + \frac{(2m^2 - 1)q b_2^2 + k^2m^2}{m^2}t] \pm \sec h \left[ \pm \sqrt{2q}b_2 \right] (x + 2kt). \quad (37) \]

\[ w_{5,6} = b_2 \sin[kx + \frac{(2m^2 - 1)q b_2^2 + k^2m^2}{m^2}t] \pm \sec h \left[ \pm \sqrt{2q}b_2 \right] (x + 2kt). \quad (38) \]

The corresponding envelope doubly periodic wave solutions of Eq.(6) are
\[ u_{5,6} = b_2 e^{\frac{\Delta_{m}(2m^2)q_b^2+k^2m^2}{m^2}} \left[ \pm isn \left( \frac{\sqrt{2}q_b}{m} (x + 2kt) \right) + cn \left( \frac{\sqrt{2}q_b}{m} (x + 2kt) \right) \right]. \]  \tag{39}

If we set \( m=1 \), then we can obtain the solitary wave solutions.

\[ v'_{5,6} = b_2 \cos[kx + (q_b^2 + k^2)t] \left[ \pm i \tanh(\pm \sqrt{2}q_b(x + 2kt)) + \sec h(\pm \sqrt{2}q_b(x + 2kt)) \right], \tag{40} \]

\[ w'_{5,6} = b_2 \sin[kx + (q_b^2 + k^2)t] \left[ \pm i \tanh(\pm \sqrt{2}q_b(x + 2kt)) + \sec h(\pm \sqrt{2}q_b(x + 2kt)) \right]. \tag{41} \]

The corresponding envelope solitary wave solutions of Eq.(6) are

\[ u'_{5,6} = b_2 e^{i(\sqrt{2}q_bk^2 + k^2)t)} \left[ \pm i \tanh(\pm \sqrt{2}q_b(x + 2kt)) + \sec h(\pm \sqrt{2}q_b(x + 2kt)) \right]. \tag{42} \]

**Case 4**

\[ a_0 = b_0 = 0, a_i = -ib_2, b_i = ib_2, a_2 = -b_2, \alpha = \pm \sqrt{2}q_b, \lambda = \pm \sqrt{2}q_b \frac{k^2m^2}{m^2}, c = \frac{(m^2 - 2)q_b^2 - k^2m^2}{m^2}, \]

where \( b_2 \) and \( k \) are free parameters. The doubly periodic wave solutions of Eqs.(9) and (10) are

\[ v_{7,8} = b_2 \cos[kx + \frac{(m^2 - 2)q_b^2 - k^2m^2}{m^2}t] \left[ \pm isn \left( \frac{\sqrt{2}q_b}{m} (x - 2kt) \right) + cn \left( \frac{\sqrt{2}q_b}{m} (x - 2kt) \right) \right], \tag{43} \]

\[ w_{7,8} = -b_2 \sin[kx + \frac{(m^2 - 2)q_b^2 - k^2m^2}{m^2}t] \left[ \pm isn \left( \frac{\sqrt{2}q_b}{m} (x - 2kt) \right) + cn \left( \frac{\sqrt{2}q_b}{m} (x - 2kt) \right) \right]. \tag{44} \]

The corresponding envelope doubly periodic wave solutions of Eq.(6) are

\[ u_{7,8} = b_2 e^{-i(\sqrt{2}q_bk^2 + k^2)t)} \left[ \pm isn \left( \frac{\sqrt{2}q_b}{m} (x - 2kt) \right) + cn \left( \frac{\sqrt{2}q_b}{m} (x - 2kt) \right) \right]. \tag{45} \]

If we set \( m=1 \), then we can obtain another solitary wave solutions

\[ v'_{7,8} = b_2 \cos[kx - (q_b^2 + k^2)t] \left[ \pm i \tanh(\pm \sqrt{2}q_b(x - 2kt)) + \sec h(\pm \sqrt{2}q_b(x - 2kt)) \right], \tag{46} \]

\[ w'_{7,8} = b_2 \sin[kx - (q_b^2 + k^2)t] \left[ \pm i \tanh(\pm \sqrt{2}q_b(x - 2kt)) + \sec h(\pm \sqrt{2}q_b(x - 2kt)) \right]. \tag{47} \]

The corresponding envelope solitary wave solutions of Eq.(6) are

\[ u'_{7,8} = b_2 e^{-(\sqrt{2}q_bk^2 + k^2)t)} \left[ \pm i \tanh(\pm \sqrt{2}q_b(x - 2kt)) + \sec h(\pm \sqrt{2}q_b(x - 2kt)) \right]. \tag{48} \]

**4. Summary**

In this paper, we have utilized the variational iteration method combined with Jacobian-function method to study the cubic nonlinear Schrödinger equation. As a result, some new explicit exact travelling wave solutions of the cubic nonlinear Schrödinger equation have been obtained which include envelope doubly periodic wave solutions, envelope solitary wave solutions, and envelope bell-type wave solutions. As far as we know, some solutions are first found. We also found that the method is of effectiveness and convenience, and the solution procedure is of utter simplicity as well.
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