FREQUENCY-DEPENDENT DISPERSION MEASURES AND IMPLICATIONS FOR PULSAR TIMING

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ABSTRACT

The dispersion measure (DM), the column density of free electrons to a pulsar, is shown to be frequency dependent because of multipath scattering from small-scale electron-density fluctuations. DMs vary between propagation paths whose transverse extent varies strongly with frequency, yielding arrival times that deviate from the high-frequency scaling expected for a cold, uniform, unmagnetized plasma (1/frequency\textsuperscript{3}). Scaling laws for thin phase screens are verified with simulations; extended media are also analyzed. The rms DM difference across an octave band near 1.5 GHz is \( \approx 4 \times 10^{-3} \) pc cm\textsuperscript{-3} for pulsars at \( \approx 1 \) kpc distance. The corresponding arrival-time variations are a few to hundreds of nanoseconds for DM \( \lesssim 30 \) pc cm\textsuperscript{-3} but increase rapidly to microseconds or more for larger DMs and wider frequency ranges. Chromatic DMs introduce correlated noise into timing residuals with a power spectrum of “low pass” form. The correlation time is roughly the geometric mean of the refraction times for the highest and lowest radio frequencies used, ranging from days to years, depending on the pulsar. We discuss implications for methodologies that use large frequency separations or wide bandwidth receivers for timing measurements. Chromatic DMs are partially mitigable by including an additional chromatic term in arrival time models. Without mitigation, an additional term in the noise model for pulsar timing is implied. In combination with measurement errors from radiometer noise, an arbitrarily large increase in total frequency range (or bandwidth) will yield diminishing benefits and may be detrimental to overall timing precision.

Key words: gravitational waves – ISM: structure – pulsars: general – stars: neutron

1. INTRODUCTION

Pulsar arrival times with sub-microsecond accuracy are required for the detection of nanohertz-frequency gravitational waves using pulsar timing arrays (e.g., Foster & Backer 1990) and have payoffs in related areas, such as precision tests of theories of gravity (Kramer et al. 2006), the determination of neutron star masses (Antoniadis et al. 2013), and the characterization of microstructure in the interstellar medium (ISM; Isaacman & Rankin 1977).

A pulse’s time of arrival (TOA) includes a group delay term, \( t_{DM}(\nu) = K \nu^{-2} \text{DM} \), where the dispersion measure (DM) is the integral of the electron number density along the line of sight (LOS), \( \nu \) is the frequency, \( K = c \epsilon_e / 2 \pi \), and \( r_c \) is the classical electron radius. A key step in arrival-time analysis is the removal of the dispersion term. Estimates of DM based on TOA differences between two or more frequencies show that DM is epoch dependent for most well-studied pulsars (Isaacman & Rankin 1977; Hamilton et al. 1985; Cordes et al. 1990; Phillips & Wolszczan 1991; Backer et al. 1993; Kaspi et al. 1994; Ilyasov et al. 2005; Ramachandran et al. 2006; You et al. 2007; Demorest et al. 2013; Keith et al. 2013; Petroff et al. 2013; Fonseca et al. 2014). Temporal variations include a slow, systematic trend as a pulsar moves toward or away from the solar system along with stochastic variations from pulsar motion that causes the LOS to sample electron-density fluctuations on a variety of scales. Annual variations in DM arise from the interplanetary medium and possibly from other causes (M. Lam et al. 2015, in preparation).

In this paper we analyze the frequency dependence of the DM that results from multipath scattering by electron density variations in the ISM. The DM along each ray path is slightly different and the size of the ray-path bundle increases monotonically with decreasing frequency, leading to a net difference in dispersive delays in the sum over all ray paths. The variation with frequency of DM(\( \nu, t \)) stays constant for a refraction time, which is the timescale for the ray-path bundle to move by its transverse extent and can range from hours to years (Rickett et al. 1984; Stinebring et al. 2000). Our analysis includes the general case of an arbitrary medium described by a wavenumber spectrum that varies along the LOS. We give specific results for the case of a thin screen and for a medium with homogeneous statistics for electron-density variations. We compare analytical results with simulations of scattering and dispersion. Our analysis is for the strong scattering regime where all of the flux from a pulsar is scattered. In the summary and conclusions section we briefly discuss the weak scattering regime and its role in precision timing.

Our results complement those of Lam et al. (2015), who quantified TOA errors that result from estimates for DM that make use of timing observations at different frequencies that are not made on the same day. Here we show that even simultaneous multi-frequency measurements yield TOA errors because the DM varies with frequency.

Section 2 derives the basic effect and gives specific results for a thin screen and for a medium with uniform statistics. Section 3 concerns implications for pulsar timing methodology and precision. Section 4 discusses our results and summarizes our conclusions. In Appendix A we derive scattering quantities and in Appendix B we derive the rms two-frequency variation of DM.

2. DM VARIATIONS IN TIME AND FREQUENCY

There are several underlying causes for temporal variations of DM but only electron density fluctuations in the ISM...
produce a large enough frequency dependence to be important in precision timing.\(^4\) For a spatially uniform \(n_e\), the DM is independent of frequency and any epoch dependence comes from changes in the pulsar distance \(D\). Motions of the pulsar and Earth with a combined velocity of \(100\,\text{V}_{100}\,\text{km\,s}^{-1}\) parallel to the LOS yield \(\delta\text{DM} \sim 3 \times 10^{-8}\,\text{pc\,cm}^{-3}\,\text{V}_{100}\,\text{t}_{\text{year}}\,n_{e,0.8}\) over \(t_{\text{year}}\) years for a typical average electron density of \(n_e = 0.03\,\text{cm}^{-3}\). Linear trends in DM are indeed seen (e.g., Keith et al. 2013) but this effect is likely to be frequency independent.

Chromatic DMs result from multipath propagation caused by diffraction and refraction from interstellar electron density variations. The effect we identify is due to geometrical path-length differences alone. Gravitational lensing, for example, could produce multiple ray paths through a medium with constant electron density, but the resulting DM variations would be negligible \((\lesssim 10^{-16}\,\text{pc\,cm}^{-3})\) given observational bounds on time delays between paths \(<1\,\mu\text{s}\). Moreover, they would be achromatic. We also emphasize that geometrical path-length differences from electron-density scattering also produce negligible changes in DM. For a typical scattering angle of \(\theta_s = 1\,\text{mas}\) and \(D = 1\,\text{kpc}\), the path length difference \(\Delta d/2 \sim 10^{-14}\,\text{pc}\) yields a DM difference of only \(3 \times 10^{-18}\,n_{e,0.8}\,\text{pc\,cm}^{-3}\), far too small to measure. Measurable DM variations require actual density fluctuations in the ISM on approximately AU scales that are not inconsistent with a power-law spectrum of fluctuations, which include much smaller scales \(\lesssim 10^4\,\text{km}\) that scatter radio waves.

Density variations exist over a wide range of length scales from Kiloparsecs to around \(10^3\,\text{km}\) (e.g., Armstrong et al. 1995) and the smaller ones are responsible for multipath propagation. The DM varies slightly between ray paths and the cross-sectional area of the ray-path bundle at any position along the LOS is strongly frequency dependent, \(\propto \nu^4\). Density microstructure therefore plays two roles: causing multipath propagation and providing variable path integrals of the electron density. Figure 1 shows frequency-dependent ray paths for a thin scattering screen and for a medium that fills the volume between us and a pulsar; it also defines some of our notation.

Figure 1. Geometries for scattering from a thin screen (left) and filled medium (right). Black lines show simulated ray paths at frequency \(\nu\) and red lines are for frequency \(\nu' = \nu/2\). The pulsar-observer distance is \(D\). The thin screen is at a distance \(D_s\) from the pulsar at \(p\) and \(D' = D - D_s\) from the observer at \(o\).

We model the electron density to be \(n_e(x) = n_e(x) + \delta n_e(x)\), where \(n_e(x)\) is a constant local mean and \(\delta n_e(x)\) is the zero-mean fluctuating part described by a wavenumber spectrum \(P_{\delta n_e}(q, \nu)\) that can vary slowly along the LOS (\(z\)-axis). We adopt a power-law wavenumber spectrum of the form

\[
P_{\delta n_e}(q, \nu) = C^2\nu q^{-\beta}, \quad q_0 \leq q \leq q_1.
\]

(1)

The dependence on only the magnitude of \(q\) is a simplifying assumption, discussed further in the next subsection, that conflicts with some observations that show elliptical scattered images but is consistent with others that show circular images.

The electromagnetic phase perturbation \(\phi\) from the index of refraction in a cold plasma\(^5\) is proportional to the integrated electron density along a ray path (Rickett 1990, Appendix A, Equation (A5)),

\[
\phi(x) = -\lambda_e \int_{\nu} \frac{dz}{\nu} n_e(x')(z),
\]

(2)

where \(x\) is the transverse location in the observation plane a distance \(D\) from the pulsar while \(x'\) is a transverse vector at a location \(z\) along the LOS.

For any single ray path the phase perturbation corresponds to a dispersive time delay \(t_{\text{DM}} = (1/2\pi)\lambda/(d\phi/d\nu)\) characterized by a dispersion measure, \(DM = -\phi/\lambda_e\). Converting \(\lambda_e\) to DM units at 1 GHz yields a natural value for one radian of phase, \(\delta\text{DM}_0 = 3.84 \times 10^{-8}\,\text{pc\,cm}^{-3}\). Since we know that pulsars are scattered significantly at radio frequencies, phase perturbations \(\phi_{\nu}\) on the Fresnel scale \(\nu = \sqrt{4D^2/2\pi} \sim 10^{11}\,\text{cm}\) are necessarily many radians. The Fresnel scale corresponds to less than one day of transverse motion for pulsar velocities \(\sim 100\,\text{km\,s}^{-1}\). We therefore expect DM to vary by large multiples of \(\delta\text{DM}_0\) on timescales of weeks and longer because the rms phase grows with increasing time span for media with a spectrum like that in Equation (1). By similar reasoning, we expect frequency-dependent DM differences \(\delta\text{DM}\) because ray paths at widely spaced frequencies have separations much larger than the Fresnel scale.

Figure 2 shows the manifestation of chromatic DMs in simulated arrival times versus frequency. The curve in the main part of the figure is dominated by the mean DM = 10\,\text{pc\,cm}^{-3}.

\(^4\) The effects of the solar wind can be observed in MSP observations within \(\sim 10\) solar radii of the Sun (You et al. 2007); however these observations are unsuitable for pulsar timing because of elevated system temperatures and instrumental distortions associated with the radiation from the nearby Sun.

\(^5\) We assume that the electron density and magnetic field are small and that frequencies are large enough so that only the term linear in electron density is important.
The origin of dispersive delays in the scattering-screen context implies that an estimate of DM from multiple frequency observations constitutes an average over all of the propagation paths that yield the measured wave field. Thus, while the wave field is the physical quantity that is superposed, measured dispersive delays can be considered in terms of an average DM. For a medium that is extended along the LOS (e.g., multiple screens), the situation is unaltered because geometric delays accrue according to a random walk while the total dispersive delay along any given path is related to the net column density of electrons.

Different pulsar and frequency combinations show a wide range of scattering parameters. Small-DM pulsars observed at high frequencies may show only a single scintillation maximum (a “scintle”) across the receiver band during the observation time while others can show a large number of scintles. The corresponding scattered image has a width and structure that is related to the correlation bandwidth of the scintles and the number of scintles, \( N_s \). In our analysis below, we assume the image is smooth. Speculles that appear in real images will add variance to DM estimates that would involve \( 1/\sqrt{N_s} \) factors at the various frequencies used, but the amount depends on how pulsar data are weighted in time and frequency to produce an arrival time.

The measurable DM is an average over all ray paths that reach Earth. At any location along the integration path, the narrow ray-path bundle spans a transverse area described by \( x(z) \) that also depends on frequency. We define a frequency-dependent averaging function \( A_\nu(x, z) \) that is normalized to unit area, \( \int dx A_\nu(x, z) = 1 \). The averaged DM is the LOS integral of the convolution of the averaging function with the electron density,

\[
\text{DM}(\nu, x) = \langle \text{DM}(x) \rangle + \int_0^D dz' \int_{-\infty}^{\infty} dx' A_\nu(x-x', z') \times \delta n_e(x', z'),
\]

(4)

where the frequency-independent term \( \langle \text{DM}(x) \rangle \) is the ensemble average (denoted by angular brackets) of the integrated electron density \( n_e \). An appropriate averaging function \( A_\nu(x, z) \) is the scattered image of the pulsar that generally also includes refraction, which offsets the image from the actual pulsar direction. The scattered image \( I(\theta) \) is the two-dimensional Fourier transform of the visibility function \( \Gamma(b) = \exp(-D_\phi(b)/2) \), where \( D_\phi \) is the phase structure function defined in Appendix A. The phase structure function is the mean-square phase difference of the wavefield between two points separated by distance \( b \) transverse to the LOS. In accordance with our assumptions about the spectrum \( P_{m_0} \), \( D_\phi \) depends only on the magnitude of \( b \).

In the most general case, the image can be elliptical or split into multiple subimages. Elliptical images have been seen for some scattered sources while others are nearly axisymmetric. An extreme case for a nearby pulsar was given by Brislkin et al. (2010), who infer an image that is predominantly Gaussian-like but has ~1% of the flux spread into a long, linear image with high aspect ratio. Refraction angles are evidently smaller than scattering angles based on the lack of significant angular wandering in very long baseline interferometry images (Gwinn et al. 1988a, 1988b; Fey & Mutel 1993) and signify consistency with a Kolmogorov-like spectrum having a spectral

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**Figure 2.** Trajectories of a dispersed pulse in the time-frequency plane for 100 realizations using a phase screen with \( \phi_F = 30 \) rad at 1 GHz. The dedispersed trajectories are shown aligned with \( r = 0 \), and the inset shows details of the dedispersed trajectories for the lower part of the frequency band.

The inset shows the deviations from the mean curve from 0.4 to 0.8 GHz for 100 realizations of a phase screen with \( \phi_F = 30 \) rad, demonstrating the spread in arrival times over \( \pm 1 \mu s \) at the lowest frequency.

### 2.1. Frequency-dependent Averaging Over Ray Paths

The wave field that reaches a telescope is a superposition from multiple propagation paths, each with its own propagation delay. The total delay includes a geometrical term resulting from the propagation distance and a dispersive term associated with the electron column density. For a thin screen that changes only the phase of incident radiation, the two delays can be associated with respective phase terms in the Kirchhoff diffraction integral (KDI). The measurable wave field given by the KDI is proportional to the integral across the screen of \( e^{i\Phi(x', z)} \) where \( x' \) is a vector in the phase screen transverse to the LOS and

\[
\Psi(x', \nu) = \phi_0(x') + \phi(x').
\]

(3)

The geometric phase is \( \phi_0(x') = \varphi_x |x'|^2/\lambda D_I(1 - D_I/D) \) while the phase \( \phi(x') \) is given by the expression in Equation (2) and includes contributions from a wide range of scales that diffract and refract incident radiation. Diffraction from small scales causes radiation to be seen from a finite area on the screen. Large scales that dominate \( \phi \) cause it to differ across the finite area. The group delay, given by \( d\Phi/d\nu \), thus includes two terms, one geometric that is achromatic and the other dispersive, scaling as \( \nu^{-2} \). For a screen with stochastic variations, the two terms are statistically independent because one depends on small-scale variations and the other on large scales. The geometric delay is typically referred to as the scattering delay and is observationally distinct from the dispersive term. We may therefore consider them separately while keeping in mind that both contribute. For low-DM pulsars, however, the scattering delay is quite small at ~GHz frequencies while the dispersive delay and its fluctuations are significant. For more distant pulsars with higher DMs, the scattering delays also need to be considered in any methodology used to estimate DMs. Nonetheless, fluctuations in DM across the scattering disk may be considered separately.
index $\beta < 4$ (Cordes et al. 1986; Romani et al. 1986; Rickett 1990). In addition, for media having wavenumber spectra like those that are consistent with a wide range of scattering and scintillation measurements, refraction angles are expected to be small. Since our goal is to describe the basic phenomenon of frequency-dependent DMs rather than address all possible varieties of scattering and refraction, the essential features are captured with a simplified approach that allows analytical tractability. The calculation undertaken here simply looks at the difference in electron column density that results from the frequency dependence of the ray-path bundle that samples the medium. The calculation of the arrival-time difference does not take into account additional delays that result from lateral shifts of the ray-path bundle due to refraction or changes due to focusing and defocusing by quadratic phase changes. We emphasize therefore that our results likely underestimate the total frequency-dependence of DMs.

First we make some simple estimates of DM variations. From the phase structure function $D_{\phi}(b)$ we show that the rms variation of $\text{DM}$ across the screen-averaging area scales as the square of the rms Fresnel phase $\phi_F$, and in the next section we derive a precise scaling law for the rms difference in $\text{DM}$ between two frequencies. The DM structure function is equal to $D_{\phi}(b)/\lambda k^2$. Without any ray-path averaging, it can be written in terms of the rms Fresnel phase as

$$D_{\text{DM}}(b) = \frac{\phi_F^2}{(\lambda k)^2} \left( \frac{b}{\lambda k} \right)^{-2}.$$  \hspace{1cm} (5)

Averaging over ray-paths yields an rms DM difference obtained by averaging the DM structure function over a circular area with radius $\sigma_F \sim \nu \phi_F^{2/(\nu-2)} \gg \nu \phi_F$ (which can be derived using Equations (38), (39), (42) and (50)). This prescription is applicable only for moderate to strong scattering with $\phi_F \gtrsim 1$. We discuss weak scattering in Section 3.4. The averaging radius is essentially the "refraction" scale for refractive scintillations, which is equal to the observed scattering angle projected onto the screen at a distance $D'$ from Earth (c.f. Figure 1); it is also the minimum scale for DM variations. This gives for $\nu$ in GHz,

$$\Delta \text{DM}_{\text{rms}} \sim \phi_F^2/\lambda k \gtrsim 3.84 \times 10^{-8} \text{ pc cm}^{-1} \nu \phi_F^2.$$  \hspace{1cm} (6)

The scaling of $\Delta \text{DM}_{\text{rms}}$ as the square of $\phi_F$ arises because the screen phase and the averaging radius are both approximately linear in $\phi_F$. For a Kolmogorov medium ($\beta = 11/3$), the Fresnel phase scales with frequency as $\nu^{-17/12}$ so $\Delta \text{DM}_{\text{rms}} \propto \nu^{-11/6}$.

2.2. Simulations of Phase Screens and Ray-path Averaging

Phase screens were simulated using approaches similar to those presented in Cordes et al. (1986), Coles et al. (1987, 2010), Foster & Cordes (1990), and Hu et al. (1991). An array in the wavenumber domain was filled with white, Gaussian, Hermitian noise shaped by the square root of the ensemble-average power spectrum. A two-dimensional inverse FFT yields the phase screen. To include low-frequency components excluded by the FFT, we separately added wavenumber components with periods up to three times the array size in each direction. Equation (38) shows that the phase structure function scales as $\phi_F^2$ and so too must the DM structure function in Equation (5). This means that phase fluctuations can be specified with the Fresnel phase at a fiducial frequency, which we took to be 1 GHz. We used this property to generate phase screens efficiently. We simulated a set of phase screens for small values $\phi_F = 5$ rad, which require fairly small 2D arrays. These screens can be scaled to other values of the Fresnel phase since the spatial scale is specified in units of the Fresnel scale, which is independent of $\phi_F$. We verified that this procedure was correct by explicitly simulating a few large phase screens.

Figure 3 shows an example phase screen and the averaging areas projected onto the screen for a 5:1 range of frequencies. DM differences are due solely to the change in averaging area. At different epochs the averaging areas sample different parts of the screen, producing stochastic variations in DM that are correlated over the time it takes for the averaging area to move by its diameter. The difference in DM between a pair of frequencies is constant over the correlation time for the higher of the two frequencies, which has the smaller scattering disk.

2.3. Two-frequency DM Differences

In our detailed analysis, we use averaging functions that are symmetric and concentric at different frequencies. In particular, we adopt a symmetric Gaussian function for $A_{\nu}(x, z),$

$$A_{\nu}(x, z) = \left[ -2\pi x_1^2 (z, \nu) \right]^{-1/2} e^{-x^2/2\sigma_{\nu}^2 (z, \nu)},$$  \hspace{1cm} (7)

where the one-dimensional width $\sigma_{\nu}$ is proportional to the observed scattering angle and therefore scales with frequency as $\sigma_{\nu} \propto \nu^{-\alpha}$ with $x_\theta \approx 2$. We have confirmed numerically that the Gaussian averaging function yields nearly identical results to usage of the scattering image appropriate for a Kolmogorov spectrum of fluctuations.

The difference in $\text{DM}$ between two frequencies $\nu$ and $\nu'$ (measured at the same location $x$, corresponding to the same epoch),

$$\Delta \text{DM}(\nu, \nu', x) = \text{DM}(\nu', x) - \text{DM}(\nu, x),$$  \hspace{1cm} (8)

has an rms difference,

$$\sigma_{\text{DM}}(\nu, \nu') = \left\{ \Delta \text{DM}(\nu, \nu', x) \right\}^{1/2}.$$  \hspace{1cm} (9)
Electron-density wavenumber spectra with $\beta > 2$ produce DM variations that are dominated by the largest scales. For these $\beta$, the relevant scales range from the smoothing length $D_x = x_{lo} \Delta x$ and the "outer scale" of the spectrum that is likely determined by the sizes of structures and clouds in the ISM. The largest structures correspond to time variations in DM up to $10^4$ years or longer for parsec scales and characteristic velocities $\sim 100$ km s$^{-1}$. We are concerned with much shorter timescales for which the mean-square difference is an appropriate tool because large scale variations $\Delta x$ cancel out.

In Appendix B we derive the rms DM difference using Gaussian smoothing functions and express the result in two forms, one that uses the scattering measure $\sigma_{\text{SM}}(\nu, \nu') = F_{\beta}(\nu/\nu')$ and a second that uses the Fresnel phase evaluated at the Fresnel scale $D_{\text{Fs}}$.

$$
\sigma_{\text{SM}}(\nu, \nu') = F_{\beta}(\nu/\nu')
\times \left\{ G_{\beta} Q_{\beta} \frac{c^{\beta/2} D^{(3-2)/2} \nu^{-\beta/2}}{c_k} SM \right\}
\times \left\{ g_{\beta} q_{\beta} \left( \frac{D_{\text{Fs}}(\nu)}{c_k} \right) \right\}
\times \text{Scattering Measure} \times \text{Fresnel Phase.}
$$

The dimensionless quantities $Q_{\beta}$, $q_{\beta}$ depend only on the wavenumber spectrum while the dimensionless quantities $G_{\beta}$, $g_{\beta}$ depend on the LOS distribution of $C_n^2$ (thin screen versus statistically uniform medium, etc.) as well as on the spectrum. Values for Kolmogorov media are given in Table 1. All of the

Figure 3. Phase screen for electron-density fluctuations with a Kolmogorov wavenumber spectrum that has $\phi_x = 5$ rad rms phase variation on the Fresnel scale. The $x$ and $y$ coordinates are in units of the Fresnel scale at 1 GHz and the rms phase difference grows as $\Delta^{5/6}$ between two points separated by $\Delta = [(\Delta x)^2 + (\Delta y)^2]^{1/2}$. Spatial units can be converted to time units using $\Delta x = V_x \Delta t$. Using a transverse velocity $V_x = 100$ km s$^{-1}$, the Fresnel scale for 1 GHz and 1 kpc distance corresponds to 0.14 day and the entire span of $x$ corresponds to 0.9 year. Left: phase screens without (top) and with spatial smoothing. The bottom three panels are for frequencies of 1.5, 0.75, and 0.3 GHz, as labeled in the right hand panels. Plotted circles (barely discernible in the middle two frames) are projected scattering-disk sizes and represent the region with 1/e radius on the phase screen that is averaged to produce the DM difference at any given epoch and frequency. The area scales as $\nu^{2/5}$. Right: phase variations corresponding to trajectories along the dashed lines in the left-hand panels. The 1D curves are not smoothed versions of $\phi_x$ in the top panel because the smoothing is two-dimensional.
relative frequency dependence is contained in the function \( F_\beta(r) \),

\[
F_\beta(r) = \left( 2^{(4-\beta)/3} + r^{2/(\beta-2)} \right)^{\beta-2}/2 - r^3 - 1 \right)^{1/2},
\]

where \( r \equiv \nu/\nu' \). This vanishes for \( r = 1 \), as expected and increases monotonically with \( r \) for \( 2 < \beta < 4 \).

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**Figure 4.** (Left) A single realization showing DM(t) at 8 frequencies (\( \nu' \)) from 0.3 to 1.5 GHz, as labeled. The DM values have been normalized by the quantity \( \nu_R(t) \) for \( \nu = 1 \) GHz, the reference frequency. Since \( \nu \) is expressed in GHz units, the plotted quantity is in standard units for DM. (Right) Top panel: time series of DM(t) differences between 1.5 GHz and all lower frequencies. Bottom panel: normalized autocorrelation functions of the difference time series in the top panel along with the ACF of the DM difference between the two lowest frequencies (0.4 and 0.5 GHz, dashed line). The main lobe of each ACF near zero lag represents the characteristic spatial scale of the DM difference. Spatial units can be converted to time units using \( \Delta t = \nu_1 \Delta x \). Using a transverse velocity of 100 km s\(^{-1}\) and a Fresnel scale for 1 GHz and 1 kpc distance, a Fresnel scale corresponds to 0.14 day.

**Figure 5.** Power spectra of dispersion measure variations \( \Delta \text{DM}(\nu, x) \) (heavy red and black lines) and spectra of DM differences \( \Delta \text{DM}(\nu', \nu, x) \), as designated in the legends. The results are based on 1000 realizations of screens with \( \theta_0 = 30 \) rad and a Kolmogorov spectrum (\( \beta = 11/3 \)), which yields a power spectrum for DM variations with a \(-8/3\) slope, as indicated by the red dashed line. The conversion to temporal frequency in cycles per year uses a velocity of 100 km s\(^{-1}\) and a radio frequency of 1 GHz. Power spectra roll off at high fluctuation frequencies because of spatial averaging from scattering. The spectra of DM differences decrease in amplitude as the frequency difference gets smaller and as the mean of the two frequencies gets smaller.

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**Table 1**

Relevant Factors for a Kolmogorov Medium (\( \beta = 11/3 \))

| Geometry Independent Factors: | Quantity | \( \beta = 11/3 \) |
|-------------------------------|----------|-------------------|
| \( f_\beta \) | 88.3 | |
| \( q_\beta \) | 1.15 | |
| \( F_{1/3}(x) \) | 1.056 | 6.64 |
| \( F_{1/3}(2) \) | 1.056 | |
| \( F_{1/3}(5) \) | 1.056 | |

**Geometry Dependent Factors:**

| Quantity | Thin Screen | Uniform | Plane Wave |
|----------|-------------|---------|------------|
| \( D_{\text{eff}}/D \) | \( x(1-x) \) | 1/4 | 1 |
| \( \text{SM}_{\text{eff}}/\text{SM} \) | \( x^{5/3} \) | 3/8 | 1 |
| \( h(x) \) | \( 1 - x/D \) | \((x/D)(1 - x/D)\) | 1 |
| \( H_{\beta}(x) \) | 0.056 | \((1 - x)^{5/3}\) as \( x \to 0 \) |
| \( G_{\beta} \) | 0.145 | 1 |
| \( g_{\beta} \) | 1 | 0.46 | 1 |

For a Kolmogorov spectrum,

\[
\sigma_{\text{DM}}(\nu, \nu') = F_{1/3}(r) \left\{ \begin{array}{ll}
3.76 & \times \ 10^{-5} \text{ pc cm}^{-3} \ G_{1/3} D^{5/6} \nu^{-11/6} \ \text{SM}_{-3.5} \\
4.42 & \times \ 10^{-5} \text{ pc cm}^{-3} \ g_{1/3} \frac{\nu_{R}^2(\nu)}{1000} \end{array} \right. \\
\text{Scattering Measure} \\
\text{Fresnel Phase},
\]

where \( \text{SM} \) is expressed in units of \( 10^{-3.5} \text{ kpc m}^{-20/3} \), the distance \( D \) in Kiloparsecs, and the frequency \( \nu \) in GHz; \( \phi_{\text{R}}(\nu) \) is in radians. The LOS is characterized by either SM or \( \phi_{\text{R}} \), which can be estimated from the scintillation bandwidth \( \Delta f_{\text{SS}} \) and timescale \( \Delta t_{\text{SS}} \). These are defined, respectively, as the
half-width at half-maximum of the intensity correlation function versus frequency difference and as the half-width at $1/e$ of the intensity correlation function versus time lag.\footnote{Usage of the half widths at half maximum and at $1/e$ for the two cases is natural given the mathematical forms of the correlation functions for media with a square-law phase structure function.} For a thin screen, \begin{equation}
SM_{-3.5} = 0.74 \nu^{11/3} (\Delta \nu_d D)^{-5/6} \left( \frac{D_s/D}{1-D_s/D} \right) \end{equation}
while for a statistically homogenous medium with constant $C_n^2$, \begin{equation}
SM_{-3.5} = 2.26 \nu^{11/3} (\Delta \nu_d D)^{-5/6}.
\end{equation}

To get $\phi_F$ we use the relationship between the pulse broadening time $\tau_d$ and the scintillation bandwidth for a thin screen, $\tau_d = C_1/2\pi \Delta \nu_d$, where $C_1$ is a constant of the order of unity (Cordes & Rickett 1998; Lambert & Rickett 1999), \begin{align}
\phi_F(\nu) &= \sqrt{2} \frac{C_1}{(D_s/D)(1-D_s/D)} \left( \frac{\nu}{\Delta \nu_d} \right)^{3-2/4} \\
&\approx 9.6 \text{ rad} \left( \frac{\nu/\Delta \nu_d}{100} \right)^{5/12},
\end{align}
where the quantity in square brackets $\sim \mathcal{O}(1)$ and the approximate expression is for a Kolmogorov spectrum. A ratio $\nu/\Delta \nu_d \sim 100$ is of the order of the value for nearby millisecond pulsars (MSPs) at $\nu = 1 \text{ GHz}$. A similar expression can be written in terms of the diffractive scintillation (DISs) timescale $\Delta \tau_{\text{ISS}}$ and the effective velocity $V_{\text{eff}}$ by which the LOS changes with time, \begin{equation}
\phi_F(\nu) \approx 8 \text{ rad} \left( \sqrt[V_{\text{eff},1000}]{\Delta \tau_{\text{ISS},1000}} \right)^{-5/6},
\end{equation}
with $V_{\text{eff}} = 100V_{\text{eff,1000}} \text{ km s}^{-1}$, and $\Delta \tau_{\text{ISS}} = 10^3 \Delta \tau_{\text{ISS,1000}} \text{ s}$. A relation between scattering measure (SM) and $\phi_F$ is given in Equation (39).

Figure 6 shows $\Delta DM$ plotted against frequency ratio $\nu/\nu'$ (black lines) based on simulations of phase screens with $\phi_F = 30 \text{ rad at } 1 \text{ GHz}$. We also show the predicted rms $\sigma_{\text{MS}}(\nu, \nu')$ (red lines) indicating statistical consistency between the theoretical and simulation results. Individual realizations show a tendency for $\Delta DM$ to be dominated by a linear dependence on $\nu - \nu'$, though there are counterexamples that show a significant quadratic dependence. In Section 3 we utilize this approximate linearity as a basis for possible mitigation of the effect in timing data.

3. IMPLICATIONS FOR TIMING ACCURACY

There are many complications to the estimation of arrival times. Recent work has addressed the removal of time-variable DMs (e.g., Lee et al. 2014; Liu et al. 2014; Pennucci et al. 2014) but no work to our knowledge has aimed at mitigating the frequency dependence. Other chromatic ISM effects include delays from diffraction and refraction that have been discussed elsewhere (e.g., Foster & Cordes 1990; Rickett 1990; Cordes & Shannon 2010). Also important is the evolution with frequency of the intrinsic shape of a pulsar’s pulse (e.g., Craft & Cornella 1968; Craft 1970; Ahuja et al. 2007; Hassall et al. 2012; Liu et al. 2014; Pennucci et al. 2014), which produces a systematic TOA error versus frequency that is partially covariant with DM variations but is assumed to be independent of epoch. Since these and other effects are largely independent, their variances add and therefore can be discussed separately from the frequency-dependent effect analyzed here.

We consider multiple frequency measurements at a fixed epoch, so we drop any explicit time dependence from $\Delta DM$. A simple model for the TOA at frequency $\nu$ includes dispersive delays and measurement errors as perturbations of the “true” TOA, $t_\infty$, \begin{equation}
t_{\nu} = t_\infty + \frac{K \Delta DM(\nu)}{\nu^2} + \epsilon_{\nu}.
\end{equation}

The quantity $\epsilon_{\nu}$ is an additive, frequency-dependent error that is uncorrelated between different frequencies for radio-pulsar noise but is highly correlated for intrinsic pulse jitter, at least over modest frequency separations.

### 3.1. Dual Frequency Measurements

An observing strategy exploits the lever-arm provided by widely spaced frequencies to obtain a high-precision estimate for DM, which then is used to estimate a DM-compensated arrival time, $t_\infty$. Here we derive the resulting errors in DM and $t_\infty$. Consider two arrival times $t_{\nu}$, $t_{\nu'}$ measured at frequencies $\nu$ and $\nu'$ < $\nu$ at the same epoch. The usual operational practice is to estimate DM (denoted by the caret) by inverting Equation (17) under the assumption that DM is frequency independent, \begin{equation}
\hat{\Delta DM} = \frac{t_{\nu} - t_{\nu'}}{\nu^2 - \nu'^2}.
\end{equation}
The dispersion delay is then removed from the measured TOA \( t_\nu \) to estimate the infinite-frequency TOA,
\[
\tilde{t}_\infty = t_\nu - \frac{K \text{DM}}{\nu^2} = t_\infty + \frac{K}{\nu^2} \left[ \text{DM}(\nu) - \text{DM} \right] + \epsilon_\nu, \tag{19}
\]
For a frequency ratio \( r = \nu/\nu' \), the difference between the estimated and true DM(\( \nu \)) is
\[
\text{DM}(\nu) - \text{DM} = \left( \frac{r^2}{r^2 - 1} \right) \left[ \text{DM}(\nu) - \text{DM}(\nu') \right] + \frac{\nu^2 (\epsilon_\nu - \epsilon'_\nu)}{K}, \tag{20}
\]
and the resulting error in the infinite-frequency TOA is
\[
\delta t_\infty \equiv t_\infty - \tilde{t}_\infty = \frac{K}{\nu'} \left( \frac{r^2}{r^2 - 1} \right) \Delta \text{DM}(\nu, \nu') + \left( \frac{\epsilon_\nu - r^2 \epsilon'_\nu}{r^2 - 1} \right). \tag{21}
\]
The estimator \( \tilde{t}_\infty \) is unbiased if the DM variations and the errors \( \epsilon_\nu \) have zero mean values over an ensemble. The contribution to \( \delta t_\infty \) from the measurement error \( \epsilon_\nu \) at the higher frequency is enhanced by a factor \( r^2 \), which means that the error at the lower frequency \( \epsilon'_\nu \) can be larger than the high-frequency error by a factor equal to a modest fraction of \( r^2 \) and thus reduce the precision of \( \tilde{t}_\infty \).

The combined rms timing error \( \sigma_{t_\infty} \) is the quadratic sum of the individual rms errors,
\[
\sigma_{t_\infty} = \left( \sigma_{t_\infty, \text{DM}}^2 + \sigma_{t_\infty, \epsilon}^2 \right)^{1/2}. \tag{22}
\]

### 3.1.1. TOA Error from Frequency-dependent DMs

The rms DM difference \( \sigma_{\text{DM}(\nu, \nu')} \) defined in Equations (10)–(11) implies an rms error in TOA,
\[
\sigma_{t_\infty, \text{DM}} = \left( \frac{r^2}{r^2 - 1} \right) \frac{K}{\nu^2} \sigma_{\text{DM}(\nu, \nu')} = E_\beta (r)
\times \left\{ \begin{aligned}
(2\pi)^{-1} G_\beta \left( \frac{a^2}{\nu^2} \right) D\left( \beta - 2 \right) 2 \nu^{-\beta + 4/2} \text{SM,} \\
\text{Scattering Measure} \\
\times \left( g_{\beta} \left[ \phi_{\beta}(\nu) \right] / 2 \nu \right) \text{Fresnel Phase},
\end{aligned} \right. \tag{23}
\]
where the \( r \)-dependent factors have been combined into a timing-error function valid for \( 2 < \beta < 4 \) (see Figure 14 in Appendix B),
\[
E_\beta (r) = \frac{r^2 F_\beta(r)}{r^2 - 1}. \tag{24}
\]

Figure 7 shows \( \sigma_{t_\infty, \text{DM}}(\nu, \nu') \) versus \( \nu' \) for \( \nu = 2 \) GHz for four values of average DM. We calculated these curves by estimating the Fresnel phase at the reference frequency \( \nu = 2 \) GHz using Equation (15) and then expressing the scintillation bandwidth at the highest frequency \( \nu \) in terms of the pulse broadening time using \( \Delta \omega_d = C_\nu / 2 \pi \tau_d \). We get \( \tau_d \) from the empirical relation \( \log_{10} \tau_d (\mu s) = -3.46 + 0.154 x + 1.07 x^2 - [2 \beta/(\beta - 2)] \log_{10} \nu \) with \( x = \log_{10} \text{DM} \) and for \( \nu \) in GHz (e.g., Bhat et al. 2004). The variation about this mean relation is 0.7 in \( \log_{10} \tau_d (\mu s) \). We use \( \beta = 11/3 \), although Bhat et al. find a best-fit value that is slightly greater. The curves may therefore underestimate the timing error because we use \( \nu = 2 \) GHz. The figure demonstrates the strong dependence of the TOA error on chromatic DMs because \( \tau_d \) and thus \( \phi_{\beta} \) increase nonlinearly with DM and the TOA error scales as the square of \( \phi_{\beta} \).

Figure 8 complements Figure 7 by showing the TOA error for observations that involve different frequency ranges and for

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7 Specifically, Bhat et al. found a best fit \( \alpha = 3.86 \pm 0.16 \) for the coefficient of the \( \log_{10} \nu \) term, which corresponds mathematically to \( \beta = 2 \alpha / \alpha - 2 = 4.15 \pm 0.17 \), but this expression applies only for \( \beta < 4 \). So a conservative interpretation is that Bhat et al.’s result corresponds to \( \beta > 11/3 \). Alternative interpretations involve other aspects of the scattering geometry rather than the form of the wavenumber spectrum.
two different but rather low values of DM (10 and 30 pc cm\(^{-3}\)). Each curve in the figure corresponds to two-frequency observations where the higher frequency \(\nu\) is the right-most frequency (2, 0.8, and 0.4 GHz) and the lower frequency \(\nu'\) is the horizontal axis. Obtaining TOAs with <0.1 \(\mu s\) precision clearly requires a high frequency of no less than 0.8 GHz for low-DM pulsars. The curves presented in Figures 7 and 8 demonstrate that low frequency telescopes operating at frequencies below 0.4 GHz will be useful for providing precision TOAs with errors \(\leq 0.1 \mu s\) from chromatic DMs only for low-DM pulsars (DM \(\lesssim 20\) pc cm\(^{-3}\)) and they would need to be coupled with higher-frequency telescopes operating at \(\gtrsim 2\) GHz. Note however, that the total DM-corrected timing error also involves radiometer noise and pulse jitter, which favor the wide frequency spans that are discussed next.

3.1.2. Measurement Errors from Radiometer Noise, Pulse Jitter, and DISS

To treat any kind of measurement error, we use a correlation function (normalized to unit maximum) \(\rho_{\nu',\nu} = \langle \epsilon_{\nu'} \epsilon_{\nu} \rangle / \sigma_{\nu'} \sigma_{\nu}\) between the errors at two frequencies. The rms of the second term in Equation (21) is then

\[
\sigma_{\nu',m} = \left[ r^4 \sigma_{\nu'}^2 + \sigma_{\nu}^2 - 2r^2 \sigma_{\nu'} \sigma_{\nu} \rho_{\nu',\nu} \right]^{1/2} / \left[ r^2 - 1 \right].
\]  

The TOA error due to radiometer noise (\(\epsilon = m\)) is uncorrelated between TOAs obtained using non-overlapping bandpasses, so \(\rho_{\nu',\nu} = 0\) and

\[
\sigma_{\nu',m} = \frac{r^4 \sigma_{\nu'}^2 + \sigma_{\nu}^2}{r^2 - 1}.
\]  

In our analysis, we consider a power-law scaling of the TOA error from radiometer noise,

\[
\frac{\sigma_{m,\nu'}}{\sigma_{m,\nu}} = \left( \frac{\nu'}{\nu} \right)^{x_m} \equiv r^{-x_m}.
\]  

We then rewrite Equation (27) as

\[
\sigma_{\nu',m} = \frac{\sigma_{m,\nu} \left( r^4 + r^{-x_m} \right)^{1/2}}{\left| r^2 - 1 \right|}.
\]  

The TOA error is insensitive to the scaling index, \(x_m\). In the limit \(r \to \infty\) there is no dependence on the index while for \(r \to 1^+\), the leading term is also independent of the index.

Individual pulses show phase and amplitude jitter that cause small shape changes in the averages of large numbers of pulses used to calculate TOAs. The resulting TOA error is correlated between frequencies, sometimes highly so. If perfectly correlated (\(\rho_{\nu',\nu} = 1\) and identical, jitter produces no error in DM because the TOAs move in tandem at the two frequencies (c.f. Equation (20)). The resulting TOA error from jitter is then simply \(\sigma_{j}\). However, single pulses and average profiles evolve slowly with frequency, yielding random and systematic TOA errors, respectively. Generally the jitter correlation will be less than 100%, yielding a larger contribution to the TOA error than from perfectly correlated jitter. However, the frequency separation over which single pulses decorrelate is large for the few cases that have been studied. These include the Crab pulsar (Sallmen et al. 1999) which decorrelates over about 0.7 GHz and the MSP J0437–4715, which decorrelates over a 2-GHz bandwidth (Shannon et al. 2014). For the brightest pulsars, the TOA errors from jitter and radiometer noise are comparable (Shannon & Cordes 2012; Dolch et al. 2014; Shannon et al. 2014) so when pulse jitter is not completely correlated, mis-estimation of DM is comparable to that from radiometer noise.

We adopt a power-law scaling analogous to that for radiometer noise,

\[
\frac{\sigma_{j,\nu'}}{\sigma_{j,\nu}} = \left( \frac{\nu'}{\nu} \right)^{y_j} \equiv r^{-y_j}.
\]
The resulting expression for the jitter-induced timing error for perfect correlation between frequencies \( (\rho_{\nu,\nu'} = 1) \) is
\[
\sigma_{\tau_{\nu,\nu'}} = \sigma_{\nu} \left\{ \frac{1 - \nu^{-x_j}}{\nu^2 - 1} \right\}.
\]

DISS have correlation times and bandwidths that range, respectively, from minutes to hours and \( \sim 100 \) kHz to 100 s of MHz at 1 GHz for currently timed MSPs, which tend to have low DMs \( \lesssim 30 \) pc cm\(^{-3}\). DISS causes TOA errors because the associated pulse broadening function—the scattering impulse response of the ISM that is convolved with a pulsar’s pulse—is stochastic on the DISS correlation timescale. The rms TOA error at an individual frequency is much smaller than those from radiometer noise and jitter for nearby MSPs but can be comparable for more distant ones (M. Lam et al. 2015, in preparation).

Scintillations of a low-DM MSP will yield a non-zero correlation \( \rho_{\nu,\nu'} \) for observations made nearly simultaneously (e.g., within less than one hour) and with frequencies separated by no more than a correlation bandwidth. Most current timing observations are made with large-enough bandwidths or frequency separations and many observations are made non-simultaneously, in which case \( \rho_{\nu,\nu'} = 0 \).

3.1.3. Systematic Errors from Profile Evolution

Profile evolution \( (\epsilon = \text{“pe”}) \) is known to introduce systematic errors in TOAs because the average profile changes smoothly with frequency. They can also be described using Equation (21). However, unlike the random errors in the previous section, profile evolution can be mitigated by exploiting its apparent epoch independence (e.g., Pennucci et al. 2014) and thus differs from the chromatic DM effect that varies with epoch.

3.1.4. Assessment of Two-frequency Timing

It is often assumed that wider frequency separations yield more precise DMs and arrival times. We assess this approach by considering the timing error from frequency-dependent DMs combined with measurement errors.

In Figure 9 we show timing errors plotted against lower frequency \( \nu' \) for an upper frequency \( \nu = 0.8 \) GHz. The left panel shows \( \sigma_{\nu_0} \) from Equation (22) and the individual contributions to it from frequency-dependent DMs, radiometer noise, and pulse jitter using Equations (25), (27), and (31), respectively. The specific case shown uses rms values of 50 ns for both the noise and jitter contributions (which are specified for the higher frequency \( \nu' \)) and frequency scaling indices \( x_\nu = -1 \) and \( x_j = 0 \). The DM error was calculated for a SM \( \log_{10} SM = -3.5 \), a value that is typical of pulsars within 1 kpc of Earth. The figure shows that rather than decreasing monotonically as the frequency range gets larger (i.e., for lower \( \nu' \)), as expected from radiometer noise alone (red line), the TOA error reaches a minimum at \( \nu' \sim \nu/2 \) and then rises as the frequency-dependent DM contribution (dashed line) begins to dominate. Other cases with different mixtures of radiometer noise and jitter and different indices \( x_\nu \) and \( x_j \neq 0 \) are qualitatively similar.

The right-hand panel of Figure 9 shows the total TOA error for a variety of admixtures of rms TOA contributions from radiometer noise and jitter combined with the same error from the chromatic-DM effect for the same value of SM used in the left-hand panel. All cases show a minimum in TOA error, which moves to lower frequency as the radiometer-noise error gets larger. These results show that there are diminishing if not negligible returns once the frequency ratio becomes larger than about a factor of two (depending on the admixture of timing errors) and if no mitigation procedure is used. We note that the chromatic-DM effect differs from the evolution of pulse shapes with frequency (mentioned earlier) for which there are mitigation procedures like those discussed by Pennucci et al.
(2014). The essential difference is that profile evolution appears to be deterministic in frequency and epoch independent whereas frequency-dependent DMs are stochastic.

Figure 10 shows the same cases but for an upper frequency \( \nu = 2 \, \text{GHz} \). Most of the plotted curves show a monotonic decline as \( \nu' \) gets smaller because the chromatic DM effect is small at 2 GHz for the SM used, \( \log_{10} \text{SM} = -3.5 \). Only for the case where \( \sigma_m \) and \( \sigma_f \) are both small (10 ns) does the chromatic DM contribution cause an upturn. TOA errors will trend toward the curves shown in Figure 9 for lines of sight with larger SMs, indicating that optimization of TOA errors with respect to the frequency range of two-frequency measurements needs to be done on a pulsar-by-pulsar basis.

These results on two-frequency measurements suggest that increases in bandwidth with wideband systems with continuous frequency coverage will also provide diminishing returns unless high-enough frequencies can be used or the frequency-dependent DM can be mitigated. We explicitly show that this is the case in the next section.

### 3.2. Wideband Timing Measurements

Digital backend systems developed recently for pulsar observations have much larger frequency ranges (1.8:1) than previous systems and provide the opportunity to estimate TOAs over a continuous range of frequencies rather than using two narrowband frequencies with a wide separation (e.g., Pennucci et al. 2014). Even wider bandwidth systems are contemplated with 4:1 (or larger) frequency ranges.

We analyze wideband systems with arbitrary total bandwidths by using standard least-squares fitting methods to fit data without and with frequency-dependent DMs. We let the data vector \( \mathbf{D} \) comprise a set of TOAs \( \{ t_{\nu_k} \}, k = 1, \ldots, N_\nu \) and their measurement errors \( \sigma_{t_{\nu_k}} \) for \( N_\nu \) separate frequencies measured at the same epoch. For simplicity, we consider only radiometer noise in the wideband analysis, corresponding to a diagonal covariance matrix, \( \mathbf{C} = \text{diagonal} \left\{ \sigma_{t_{\nu_k}}^2 \right\} \). For a design matrix \( \mathbf{X} \) and a linear model \( \mathbf{D} = \mathbf{X} \theta + \epsilon \), the solution vector is \( \theta = (\mathbf{X}^\prime \mathbf{C}^{-1} \mathbf{X})^{-1} \mathbf{X}^\prime \mathbf{C}^{-1} \mathbf{D} \) and the covariance matrix for the parameters is \( \mathbf{P}_\theta = (\mathbf{X}^\prime \mathbf{C}^{-1} \mathbf{X})^{-1} \).

With respect to the timing model of Equation (17), we consider four alternatives.

1. First is where the frequency-dependent DM is negligible so DM is constant in frequency and the only errors are from radiometer noise. This would be the case for a very low-DM pulsar or measurements at high frequencies \( \nu \gg 1 \, \text{GHz} \). The data are fitted with a parameter vector \( \theta = \text{col}(t_{\nu_0}, K\text{DM}) \) and corresponding design matrix \( \mathbf{X} = \text{matrix}(1 \, \nu_i^{-2}), i = 1, \ldots, N_\nu \). The solution vector is unbiased.

2. The second case is where the frequency dependence of DM is significant but the data are still fitted with a constant DM model. The results are generally biased.

3. The third case is motivated by recognizing in Figure 6 that DM differences \( \Delta\text{DM}(\nu', \nu) \) have a tendency to appear roughly (but not exactly) linear in \( \nu - 1 \); this suggests that a term in the fitting function scaling as \( \nu^{-3} \) will absorb much of the effect and improve the estimate for \( t_{\nu_0} \).

4. The fourth case is the same as the third except the fitting function includes a \( \nu^{-x} \) term, where \( x \) is also fitted for as a (nonlinear) parameter instead of fixing it at \( x = 3 \).

Example results in Figure 11 show the TOA error plotted against total bandwidth when the highest frequency is fixed at \( \nu = 2 \, \text{GHz} \). The bandwidth is varied from 0.4 to 1.6 GHz and a total of 20 separate subbands are used across the bandwidth. DM variations are for a phase screen with \( \phi_\text{E} = 30 \, \text{rad} \) at \( \nu = 1 \, \text{GHz} \). In the left-hand panel we have used an optimistic 10 ns for the radiometer noise at the fiducial frequency of 1 GHz while in the right-hand panel we have used 100 ns. We do not include pulse jitter in these examples because results shown previously indicate that it is secondary. Curves are shown for the four cases described above along with a fifth case that is an extreme version of case 2 where the noise is assumed negligible. The results are similar to those obtained in the two-frequency case. Namely, for the 10-ns noise case, increases in bandwidth yield improvements until the bandwidth is about 1 GHz and then the results degrade if there is no explicit fitting for the frequency-dependent DM. With such fitting (cases 3 and

![Figure 10](Image)

Figure 10. (Left) RMS error of \( t_{\nu_0} \) for two-frequency measurements as a function of the lower frequency \( \nu' \) for a fiducial (highest) frequency \( \nu = 2 \, \text{GHz} \). See caption for Figure 9 for further explanation.
Figure 11. rms TOA error from continuous wideband fitting for DM and $t_{\infty}$ (and other parameters) over a continuous frequency range with a fixed upper frequency of 2 GHz and a bandwidth as designated by the abscissa. A bandwidth of 0.5 GHz, for example, designates fitting from 1.5 to 2 GHz while a bandwidth of 1.6 GHz is for fitting from 0.4 to 2 GHz. The phase screen has a Fresnel rms phase of 30 rad and the rms radiometer noise is a optimistic 10 ns (left) and more realistic 100 ns (right) that is assumed to be constant in frequency. A total of 2000 phase-screen realizations was used. The cases shown include (1) fixed true DM (i.e., no frequency dependent DM) and a timing model that includes only a $t_{\infty}$ and a $\nu^{-2}$ term; (2) variable DM without noise added and a fit for fixed DM; (3) variable DM with noise added and a fit for fixed DM; (4) as with (3) but with a fit that also includes a $\nu^{-X}$ term with fixed $X = 3$; and (5) as with (4) but where the exponent $X$ is also fitted for.

Figure 12. Results from wideband fitting for DM and $t_{\infty}$ over a continuous frequency range from 0.4 to 2 GHz. The phase screen has a Fresnel rms phase of 30 rad and the rms radiometer noise is constant in frequency: 10 ns (left set of panels) and 100 ns (right). Left column: histograms of the error in $t_{\infty}$ based on 2000 realizations of a phase screen. rms values for $t_{\infty}$ are given in each panel. Right column: histograms of the difference in DM from the value at the highest frequency. rms values for $D_M$ are given in units of $10^{-4}$ pc cm$^{-3}$. First row: the true DM is fixed and the timing model includes only a $\nu^{-2}$ term; Second row: same as the first case except that DM varies with frequency; Third row: DM varies with frequency and the timing model includes both a $\nu^{-2}$ and a $\nu^{-X}$ term with $X = 3$ (fixed); and Fourth row: same as previous except that the exponent is also fitted for.

4 above), the bandwidth can be increased another 20%–40% before the timing errors degrade. For the larger 100-ns noise in the right-hand panel, TOA errors are dominated by radiometer noise except for the largest bandwidths for which there is an increase in timing error.

Figure 12 shows additional information on fitting results using histograms of the errors in $t_{\infty}$ and DM. The two cases shown are for the same fits and data model used to produce Figure 11. The top rows show fitting results when the DM is constant in frequency and the sole source of fitting error is the additive noise. Other rows show the results for the different methods outlined above to deal with the frequency dependence of DM. Overall, Figures 11 and 12 show that allowance for the frequency dependence can reduce the timing error but cannot achieve the same results as for a constant DM.

3.3. Other Mitigation Methods

So far we have discussed estimation and removal of dispersion using data obtained at a single epoch. To date, most methods used by groups aiming to detect gravitational waves have used multiple epochs to estimate DM at the epoch of any particular arrival time. While using multiple epochs can
be deleterious (e.g., Lam et al. 2015), they can be implemented with algorithms that take into account the correlation times of DM variations that result from ray-path averaging. These times can be days to months or longer depending on the frequency and ISM along the LOS.

It is beyond the scope of this paper to develop a multi-epoch approach. However, we can illustrate a two-frequency approach that smooths the DM time series at a high frequency when large DM variations are present. Figure 13, shows DM variations at 0.2 and 1.5 GHz from a simulation for a phase screen with \( f_p = 30 \) rad for 1 GHz. In the top panel, the high-frequency time series has been optimally smoothed using a Gaussian smoothing function that minimizes the mean-square difference with the low-frequency time series. The bottom panel of the figures shows the smoothed and unsmoothed DM difference between the two frequencies. While the differences have been reduced by smoothing the high-frequency DM, the unsmoothed difference between the two frequencies is still noticeable because one-dimensional smoothing cannot model the two-dimensional smoothing that occurs in the ISM from scattering. Nonetheless, this approach can be used as a mitigation procedure.

3.4. Weak Scattering

Our analysis applies to the strong scattering regime where \( f_p > 1 \). Nearby pulsars observed at higher frequencies may be in weak scattering with \( f_p \ll 1 \). In this regime, the scattered pulsar image consists of a very compact image of the unscattered pulsar with a fraction \( \sim 1 - f_p^2 \) of the total flux combined with a scattered image with the remaining fraction. For small \( f_p \), the cross-sectional area of the total image that averages the phase screen is very small and the DM will be nearly achromatic. This would be another advantage of using high frequencies—defined as those above the transition frequency to weak scattering. The transition frequency is derived by requiring \( f_p < 1 \) rad and using Equation (39) to get \( f_{\text{trans}} \approx 8 \, \text{GHz} \, (S_{2134} / D^3)^{1/3} \). For some pulsars, the transition frequency is \( \lesssim 1 \) GHz (e.g., Rickett et al. 2000). However, precise DM estimates require multiple-frequency (or wideband) observations that almost always will use lower frequencies that are in the strong-scattering regime. Note that these lower frequencies may not be low frequencies. We therefore see no reason to avoid the chromatic aspect of DMs in precision timing.

Further work is needed to explore the role of high-frequency observations in pulsar-timing campaigns that require the highest precision. We defer a detailed analysis to a future publication.

4. SUMMARY AND CONCLUSIONS

We have shown that DMs are chromatic because microstructure in the interstellar electron density causes multipath propagation that is strongly frequency dependent. We have characterized the effect in terms of the average over ray paths, \( D_M \), using an averaging kernel that is frequency dependent. Results were given for media having a power-law electron-density wavenumber spectrum and for arbitrary variations in amplitude of the spectrum along the LOS. We verified our analytical results with simulations of phase screens with a Kolmogorov spectrum. Differences in \( D_M \) between two frequencies scale as \( f_p^2 / \lambda_k \propto \text{SM} \), where \( f_p \) is the r.m.s. phase across a Fresnel scale at the highest frequency used and SM is the SM. Observations at frequencies \( \lesssim 1 \) GHz will have \( f_p \gg 1 \) for most pulsars and \( f_p \propto \nu^{-1/2} \). The specific variation of \( D_M \) with frequency will remain constant in time over a fraction timescale of hours to months (or longer) that is generally larger for larger-DM pulsars. The joint variation of \( D_M \) in time and frequency thus differs from that of intrinsic profile evolution, which appears to be epoch independent for MSPs. Longer period pulsars show state switching (nulling, mode changes, etc.) on a wide range of timescales, but the switching statistics appear to have stationary statistics.

As yet, there has been no definitive direct detection of chromatic DMs. Most DM measurements reported in the literature are derived from two-frequency measurements that cannot probe any frequency dependence. However, multi-frequency DM measurements of the MSP B1937+21 (J1939 +2134) have shown apparent chromatic DMs (Cordes et al. 1990; Ramachandran et al. 2006). While scattering-induced chromatic DMs discussed were advanced as a possible explanation in these papers, other effects were also suggested by (Cordes et al. 1990), including profile evolution and differential emission from the pulsar. (Ramachandran et al. 2006) interpreted timing residuals at separate frequencies in terms of variations of DM in both time and frequency, with an implied difference of the order of \( \sim 10^{-4} \) pc cm\(^{-3} \) between 327 and 610 MHz. They were unable to establish whether scattering-induced smoothing of the DM was present and explained their non-detection as the result of inadequately sensitive observations.

Another approach has been to search for departures from the standard \( \nu^{-2} \) dispersion delay expected for a tenuous, cold, unmagnetized plasma used to put limits on chromatic timing effects (e.g., Hassall et al. 2012, and references therein). Such departures are complicated by profile evolution with frequency that is mitigated by identifying fiducial pulse phases that yield consistency with the cold-plasma law. Most of the objects analyzed this way (pulsars and fast radio bursts of apparent extragalactic origin) have much coarser timing precision than the MSPs used in pulsar timing arrays. The best prospects for detection are from a bright, high-DM pulsar that has minimal

![Figure 13. Test of one-dimensional smoothing to estimate the two-dimensional averaging from scattering using DM variations for a scattering screen with \( f_p = 30 \) rad. The left-hand axis label is DM units and the right-hand axis is in time-delay units. (Top): \( D_M(\nu) \) vs. \( x \) for two widely spaced frequencies, 0.2 and 1.5 GHz. Also shown (red line) is the smoothed 1.5 GHz variation that best matches the 0.2 GHz curve using a Gaussian smoothing function. (Bottom): DM differences without and with smoothing of the high frequency variation.](image-url)
profile evolution over the frequency range needed to probe DM variations. Profile evolution may be disentangled from chromatic DMs by exploiting the lack of (or minimal) epoch dependence of profile evolution in comparison with variations in DM that have a characteristic correlation time for each frequency. It is conceivable that some of the frequency-dependent timing variations observed from the MSP J1909–3744 (e.g., Figure 7 in Manchester et al. 2013) include the chromatic DM effect.

Chromatic DMs have a significant impact on pulsar timing applications where sub-microsecond timing accuracy is needed, such as detection of gravitational waves with pulsar-timing arrays and high-order general relativistic effects in binary pulsars. All pulsar timing applications depend on removing dispersion delays with high accuracy to estimate TOAs unaffected by propagation through the ISM, which we have called \(t_\infty\). We have analyzed errors in \(t_\infty\) resulting from methodologies that assume DMs are achromatic for two cases, one where TOAs are measured at pairs of widely separated frequencies and a second that uses continuous, wideband systems. Chromatic DMs introduce errors in \(t_\infty\) that depend strongly on the pulsar’s mean DM and on the particular range of frequencies used. For nearby pulsars with \(DM \lesssim 30 \text{ pc cm}^{-3}\) and an octave frequency range with 1.5 GHz as the highest frequency, TOA errors solely from the chromatic part of DM are of the order of a few to hundreds of nanoseconds for observations extending down to 1 GHz or 0.2 GHz. Timing errors increase rapidly with increasing mean DM and timing residuals will be correlated on timescales related to those of refractive interstellar scintillations.

Timing errors from radiometer noise and pulse jitter combined with chromatic DMs show a broad minimum as a function of total frequency range (or bandwidth) that is of the order of an octave in frequency. This arises because TOAs improve monotonically with bandwidth as far as radiometer noise is concerned, but the opposite is true for chromatic DMs. The simplest prescription for optimizing TOA precision is to use an upper frequency that is as high as possible. The choice of upper frequency is strongly pulsar and telescope dependent.

Chromatic DMs also need to be considered in combination with other chromatic effects, including intrinsic pulse profile evolution with frequency and additional interstellar delays that result directly from scattering and refraction that have different frequency dependences than any of the DM effects. A comprehensive assessment of effects like that in Cordes & Shannon (2010) that includes the chromatic DM effect is deferred to a separate paper. Even though the timing errors from chromatic DMs are smaller than other effects, they nonetheless may inhibit improvements in timing accuracy that otherwise might be obtainable. It is therefore important to confirm that chromatic DMs are present in timing data at predicted levels and develop ways to mitigate them, if possible. If not, they need to be part of the noise model for timing analysis and incorporated into the covariance matrix used in model fitting.

In a separate article we will assess the role of chromatic DMs for all objects currently being observed in pulsar timing array campaigns to detect gravitational waves and we will also assess different methodologies for using existing and future telescopes. We will also identify particular pulsars that are good candidates for direct detection of DM chromaticity.

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**APPENDIX A**

**PHASE STRUCTURE FUNCTION AND SCATTERING ANGLE**

Here we summarize standard quantities use for characterizing scattering media relevant to interstellar scattering (e.g., Rickett 1990), including the phase structure function, the wavenumber spectrum for electron density variations, and the SM. We also derive quantities needed for our analysis of DMs,
including the scattering diameter and diffraction scale using scaling laws involving the Fresnel radius, Fresnel phase, and SM.

The phase structure function is the integral from a point source to an observer at distance $D$,

$$D_{\phi}(b) = \left\{ \phi(x) - \phi(x + b) \right\}^2 = 4\pi (r_c \lambda)^2 \times \int_0^D dz \int dq_{\perp} P_{\text{sm}}(q_{\perp}, z) \left( 1 - e^{iq_{\perp} b z / D} \right), \quad (32)$$

where $b$ is a spatial offset transverse to the LOS in the observer’s plane and $q_{\perp}$ a vector wavenumber transverse to the LOS. Angular brackets denote an ensemble average. This form applies to a wavenumber spectrum $P_{\text{sm}}$ whose extent in wavenumber is much narrower than $D^{-1}$, has a shape independent of $z$, and an amplitude that varies slowly with $z$. Normalization is so that the mean-square electron density is $\langle \rho^2 \rangle = \phi_F^2 / f_\beta r_F^{-2}$.

SMs indicate various degrees of anisotropy of density fluctuations, but the isotropic case is easier to analyze. To treat the anisotropic case is tedious and does not add any further insights to the results obtained for the isotropic case. For isotropic irregularities only the magnitudes of $b$ and $q_{\perp}$ matter, yielding

$$D_{\phi}(b) = 8\pi^2 (r_c \lambda)^2 \int_0^D dz \int dq_{\perp} q_{\perp} P_{\text{sm}} \times (q_{\perp}, z) \left[ 1 - J_0(q_{\perp} b z / D) \right], \quad (33)$$

where $J_0$ is the Bessel function of the first kind. We adopt a power-law wavenumber spectrum,

$$P_{\text{sm}}(q_{\perp}, z) = C_n^2(z) q^{-\beta}, \quad q_0 \leq q \leq q_1, \quad (34)$$

where $C_n^2(z)$ varies slowly with $z$ on length scales much larger than the outer scale, $2 \pi / q_0$. For $b \ll q_1^{-1}$ (i.e., $b$ smaller than the inner scale), the phase structure function is quadratic in $b$ while for $b \gg q_0^{-1}$ it asymptotes to twice the total variance of the phase. In the intermediate regime where $q_1^{-1} \ll b \ll q_0^{-1}$, the phase structure function is (Cordes & Rickett 1998, Equation (B6))

$$D_{\phi}(b) \approx f_\beta \left( \lambda r_c \right)^2 SM_{\text{eff}} b^{\beta-2}, \quad f_\beta = \frac{8\pi^2}{(\beta - 2) 2^{\beta-2}}, \quad (35)$$

where the effective SM is the LOS, weighted integral of $C_n^2$,

$$SM_{\text{eff}} = \int_0^D dz C_n^2(z) \left( \frac{z}{D} \right)^{\beta-2} \equiv SM \times \int dz C_n^2(z) (z / D)^{\beta-2} \int dz C_n^2(z) \equiv SM \left( \frac{z}{D} \right)^{\beta-2} C_n^2(z). \quad (36)$$

Angular brackets denote an average over the LOS using $C_n^2(z)$ as a weighting function and the SM is

$$SM = \int_0^D dz C_n^2(z). \quad (37)$$

For a screen, $SM_{\text{eff}} = (D_s / D)^{\beta-2} SM$ and for a uniform medium with $C_n^2 = \text{constant}$, $SM_{\text{eff}} = SM / (\beta - 1)$. A plane wave incident on a foreground scattering medium (e.g., an extragalactic pulse incident on the Milky Way or a distant pulsar viewed through a nearby H II region) so that $D_s / D \rightarrow 1$ gives $SM_{\text{eff}} = SM$. Values for $SM_{\text{eff}} / SM$ are given in Table 1 for a Kolmogorov spectrum along with other parameters.

Alternatively we can express the phase structure function in terms of the rms phase $\phi_F$ across a Fresnel radius $r_F$ in the screen. We define the Fresnel scale using $r_F^2 = [(\lambda D) / 2\pi] D_{\text{eff}} / D$, where for a thin screen $D_{\text{eff}} = D_s D_{\text{eff}} / D = D (D_s / D) (1 - D_s / D)$. For a statistically uniform medium we assume $D_{\text{eff}} = D / 4$. Taking into account that a transverse scaleb at an observer’s position corresponds to a scale $zb / D$ at a position $z$ along the LOS, we have for a thin screen at $z = D_s$,

$$D_{\phi}(b) = \phi_F^2 \left( \frac{D_s}{D} \right)^{\beta-2} \left( \frac{b}{r_F} \right)^{\beta-2}. \quad (38)$$

From this and Equation (37) we solve for SM in terms of $\phi_F$,

$$SM = \frac{1}{f_\beta r_F^{-2} (\lambda r_c)^2} \phi_F^2. \quad (39)$$

In the following we assume the same relation holds generally though we have derived it from the thin-screen case.

The scattered image of a point source has longer tails than a Gaussian function for $\beta < 4$ and an inner scale $2\pi / q_1$ much smaller than the Fresnel scale (e.g., Rickett 1990). However it is convenient to characterize the main part of the image with an equivalent Gaussian whose visibility function has the same $1/e$ width. This defines the spatial scale $b_c$ using $D_{\phi}(b_c) = 2$,

$$b_c = \left[ \frac{2}{f_\beta (\lambda r_c)^2 SM_{\text{eff}}} \right]^{1/(\beta-2)}, \quad (40)$$

from which the rms angular size $\sigma_0$, the $1/e$ half width, and the FWHM are calculated as

$$\sigma_0 = \frac{\theta_c}{\sqrt{2}} = \frac{\theta_{\text{FWHM}}}{2\sqrt{2 \ln 2}} = \frac{\lambda}{\sqrt{2} \pi b_c}. \quad (41)$$

The power-law spectrum yields an rms angular size that we factor into a scattering size $\sigma_0$, and a geometry-dependent factor,

$$\sigma_0 \equiv \sigma_0 \left( \frac{SM_{\text{eff}}}{SM} \right)^{1/(\beta-2)}, \quad \sigma_0 = \frac{1}{\pi} \left[ \frac{\lambda^2 f_\beta^2 r_c^2 SM_{\text{eff}}}{2^{\beta/2}} \right]^{1/(\beta-2)} \quad (42)$$
APPENDIX B
TWO-FREQUENCY CROSS CORRELATION
AND STRUCTURE FUNCTION

We calculate the mean-square of the difference \(\Delta \text{DM}(\nu', x) = \text{DM}(\nu', x) - \text{DM}(\nu, x)\) defined in the main text to get the two-frequency structure function,

\[
\sigma_{\text{DM}}^2(\nu, \nu') = \left\langle \left[ \Delta \text{DM}(\nu, \nu') \right]^2 \right\rangle = C_{\text{DM}}(\nu, \nu') + C_{\text{DM}}(\nu', \nu') - 2C_{\text{DM}}(\nu', \nu),
\]

where the cross correlation of \(\text{DM} = \text{DM} - \langle \text{DM} \rangle\) between two frequencies is

\[
C_{\text{DM}}(\nu, \nu') = \langle \delta \text{DM}(\nu, x) \delta \text{DM}(\nu', x) \rangle = \int dx' x'' \int dz dz'' A_\nu(x') A_{\nu'}(x'', z’') \times \langle \delta n_e(x', z') \delta n_e(x'', z'') \rangle.
\]

The \(z\) integrals are from 0 to \(D\) and the \(x\) integrals are over an infinite plane. We define the cross-correlation function of the averaging function,

\[
C_A(\delta x, z, \nu, \nu') = \int dx A_{\nu'}(\delta x) \times A_\nu(\delta x),
\]

and assume that it changes slowly in \(z\). By changing variables from \(x', x''\) to \(x = (x' + x'')/2\) and \(\delta x = x' - x''\) and from \(z', z''\) to \(z = (z' + z'')/2\) and \(\delta z = z' - z''\), and using the hierarchy of scales assumed in Appendix A, the integration over \(\delta z\) gives \(2\pi \delta z\) and we obtain

\[
\sigma_{\text{DM}}^2(\nu, \nu') = \frac{1}{2} \int dz \int d\delta x \int d\delta z D_{\text{DM}}(\delta x, \delta z; z) \times \left[ 2C_A(\delta x, z, \nu, \nu') - C_A(\delta x, z, \nu, \nu') - C_A(\delta x, z, \nu', \nu') \right].
\]

Using Equation (42) to evaluate \(\delta n_e(\nu)\), the rms DM difference becomes

\[
\sigma_{\text{DM}}(\nu, \nu') = G_3 Q_3 \left[ D(\beta - 2) SM \nu^2 \Delta^2 F_3(\nu/\nu') \right],
\]

where the two dimensionless quantities are

\[
Q_3 = \left[ \left( \frac{2\pi}{\sqrt{g}} \right)^{4\beta} \frac{\Gamma(2 - \beta/2)}{f_3} \right]^{1/2} \quad \text{and} \quad G_3 = \left[ H_3 \left( \frac{\text{SM}_{\text{eff}}}{\text{SM}} \right) \right]^{1/2}.
\]

All of the geometry-dependent factors are consolidated into \(G_3\).

We also express \(\sigma_{\text{DM}}(\nu, \nu')\) in terms of the Fresnel phase by substituting for \(\text{SM}\) from Equation (39),

\[
\sigma_{\text{DM}}(\nu, \nu') = g_3 q_3 \nu F_3(\nu/\nu') \left( \frac{\sigma_0^2}{\nu^2} \right)^{1/T_3},
\]

where

\[
q_3 = \left[ 2^{3/2} \pi^2 \frac{\Gamma(2 - \beta/2)}{f_3} \right]^{1/2}, \quad g_3 = \left[ H_3 \left( \frac{D}{D_{\text{eff}}} \right)^{\beta/2} \left( \frac{\text{SM}_{\text{eff}}}{\text{SM}} \right) \right]^{1/2}.
\]

Values of \(Q_3, G_3, q_3,\) and \(g_3\) are given in Table 1 for a Kolmogorov spectrum.

APPENDIX C
DM POWER SPECTRA

The temporal power spectrum of a DM time series corresponds to a slice through the spatial variation of DM under the assumption that spatial variations are frozen in time. This appears to be a good assumption because pulsar velocities and the Earth’s orbital velocity are larger than typical ISM velocities (though special lines of sight may traverse fast
supernova shocks). Using Equation (10) of Cronyn (1970) expressed in the notation of Appendix A, the fluctuation spectrum for motion along the x axis with velocity \( v_x \) is

\[
S_{DM}(f) = \left( \frac{2\pi}{v_x} \right)^2 \int_0^D dz \int dq_x P_{n\omega}(2\pi f/v_x, q_x, 0, z) \quad \text{(56)}
\]

For the power-law wavenumber spectrum of Appendix A, this becomes

\[
S_{DM}(f) = \frac{\pi^{1/2}}{2(2\pi)^{3/2}} \Gamma((\beta - 1)/2) \left( \frac{f}{v_x} \right)^{1-\beta} \left( \frac{v_x}{v} \right)^{8/3} \times \frac{1}{2^{5/3} \pi^{1/6}} \Gamma(11/6) \left( \frac{v}{v_x} \right)^{8/3}
\]

where the second equality is for a Kolmogorov spectrum, \( \beta = 8/3 \). The spectrum of the scattering-averaged \( DM \) includes low-pass filtering due to angular scattering,

\[
S_{DM}(f) = \left( \frac{2\pi}{v_x} \right)^2 \int_0^D dz \int dq_x P_{n\omega}(2\pi f/v_x, q_x, 0, z) \times \left| \tilde{A}_0 \left( 2\pi f/v_x, q_x, 0, z \right) \right|^2
\]

where \( \tilde{A}_0(q, z) \) is the Fourier transform of the smoothing function \( A_0(x, z) \) (c.f. Equation (7)).

The DM difference \( \Delta DM(\nu, \nu', x) \) will have a fluctuation spectrum of the form

\[
S_{\Delta DM}(\nu, \nu', f) = \left( \frac{2\pi}{v_x} \right)^2 \int_0^D dz \int dq_x P_{n\omega}(2\pi f/v_x, q_x, 0, z) \times \left[ e^{-f^2 q_x^2 + q_z^2} \sigma_0^2 + e^{-f^2 q_x^2 + q_z^2} \sigma_0^2 \right] - 2e^{-f^2 q_x^2 + q_z^2} \sigma_0^2 (\nu' + \sigma_0^2)
\]

(59)

This spectrum includes low-pass filtering but it also has the property, as with the two-frequency variance of Equation (48), that it decreases to zero as \( \nu' \rightarrow \nu \).

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