Seiberg-Witten equation on a manifold with rank 2-foliation

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Abstract  Let $M$ be a closed oriented 4-manifold admitting a rank-2 oriented foliation with a metric of leafwise positive scalar curvature. If $b^+ > 1$, then we will show that the Seiberg-Witten invariant vanishes for all spin structures.

Keywords  Foliation, Bundle-like metric, Seiberg-Witten invariant

AMS Subject Classifications  53C12, 58H10, 57R57

1 Introduction

A natural question in Riemannian geometry is: When does a closed manifold $M$ admit a Riemannian metric with positive scalar curvature? From the Lichnerowicz formula [5], we know that $\hat{A}(M)$ is a topological obstruction for a manifold to admit a metric of positive scalar curvature. A manifold equipped with a foliation is called a foliated manifold. We can define the leafwise scalar curvature as follows: For any $x \in M$, the integrable subbundle $F_x$ determines a leaf $F_x$, such that $TF_x = F_x$. Hence the metric on $M$ determines a Riemannian metric on $F_x$. Using the Levi-Civita connection, one has the scalar curvature of this metric, which is denoted by $\text{scal}^{F_x}$.

By the results given by A. Connes [2] and W. Zhang [13] respectively, it is known that $\hat{A}(M)$ is also a topological obstruction for the foliation admitting metric of positive leafwise scalar curvature.

Theorem 1.1 (A. Connes 1986) For a closed oriented manifold, let $\mathcal{F}$ be a spin foliation. If $\hat{A}(M) \neq 0$, then $\mathcal{F}$ does not admit any metric with positive scalar curvature.

Theorem 1.2 (W. Zhang 2017) For a closed oriented spin manifold, let $\mathcal{F}$ be a foliation. If $\hat{A}(M) \neq 0$, then $\mathcal{F}$ does not admit any metric with positive scalar curvature.

Later Zhang [14] posed the following question.

Question: Given a closed oriented 4-manifold with a foliation with positive leafwise scalar curvature, does the Seiberg-Witten invariant vanish?

On the other hand, for oriented smooth 4-manifolds, Seiberg-Witten invariant plays an important role to study the obstruction for manifold admitting some geometric and topological structures. One well-known result is that a closed oriented 4-manifold with non-trivial Seiberg-Witten invariant can not admit any metric of positive scalar curvature.

In this paper we will give a partial answer to the above question. Let $M$ denote a closed oriented smooth 4-manifold with $b^+ > 1$ and admit a rank-2 foliation $\mathcal{F}$, where $b^+$ denotes the...
dimension of the self-dual part of the second cohomology groups, see \([4, \text{Section 1}]\). Suppose that \(\mathcal{F}\) admits a positive scalar curvature metric. Then, we show that each leaf is compact and the following result holds.

**Theorem 1.3** Let \(M\) denote a closed oriented smooth 4-manifold with a rank-2 foliation admitting a leafwise positive scalar curvature metric. Suppose that the positive part of the second Betti number is greater than 1, i.e. \(b^+ > 1\). Then the Seiberg-Witten invariant vanishes for all spin\(^c\) structures.

The structure of this paper is as follows: in Section 2, we will show that each leaf is compact and the manifold \(M\) admits a bundle-like metric under the condition of Proposition 2.2; in Section 3, we review the classical theory of the Seiberg-Witten equation and give a proof to Theorem 1.3 via the adiabatic limit method.

**Acknowledgement** The author would like to express the special thanks to Huitao Feng for the introduction of such a problem and Mikio Furuta for the invaluable discussion. The author wants to thank Clifford Taubes for sharing the idea to proving the compactness of the leaf. The research is supported by the FMSP.

## 2 Compact leaf and bundle-like metric

### 2.1 Compact leaf

In this subsection, we show that each leaf in the manifold satisfying the condition of Theorem 1.3 must be compact. In general, we can establish the following proposition.

**Proposition 2.1** Let \(M\) denote a closed smooth \(n\)-manifold with a rank-2 foliation. Suppose that the foliation admits a metric with leafwise positive scalar curvature. Then, each leaf is compact. Moreover, each leaf is diffeomorphic to either \(S^2\) or \(\mathbb{R}P^2\).

**Proof** From the compactness of \(M\), we know that each leaf is a complete Riemannian submanifold with the induced metric. It suffices to show that for a given geodesic in a leaf, there always exists a conjugate point in this geodesic. Since the dimension of each leaf is 2, the leafwise sectional curvature equals half of the leafwise scalar curvature, which implies that the sectional curvature of each leaf stays away from zero, i.e. there exists some number \(c_0 > 0\) such that over each leaf the leafwise sectional curvature is strictly greater than \(c_0\). Let \(\mathcal{F}_x\) denote the leaf along a point \(x \in M\); we need to show that it is compact. Choose a geodesic \(\gamma\) starting at \(x\) in \(\mathcal{F}_x\) and consider the Jacobi field along this geodesic

\[
\frac{d^2 J}{dt^2} + \kappa J = 0, \quad J(0) = 0,
\]

where \(\kappa\) denotes the sectional curvature. By [1, Chapter 5], we can write the Jacobi field as follows:

\[
J(t) = f(t)e(t),
\]

where \(e(t)\) is the parallel normal vector field along the geodesic, i.e. \(\gamma'(t) \perp e(t)\) and \(\frac{de(t)}{dt} = 0\) and \(f\) is a function along the geodesic such that \(f(0) = 0\). We rewrite the Jacobi equation as

\[
\frac{d^2 f(t)}{dt^2} + \kappa(t)f(t) = 0.
\]
We claim that \( f(t) \) must vanish at some point of the geodesic. Suppose it is not true, without loss of generality we assume that \( f(t) > 0 \). Consider the derivative of \( f \), we can categorize it into two cases:

1. There exists \( t_0 > 0 \) such that \( f'(t_0) < 0 \). By the equation \( f''(t) = (f'(t))' = -\kappa(t)f(t) < 0 \), we have \( f'(t) < f'(t_0) < 0 \) for any \( t \geq t_0 \). Thus, there exists some point \( t_1 \) such that \( f(t_1) < 0 \), which contradicts to the hypothesis.

2. For all \( t \), we have \( f'(t) \geq 0 \). By choosing a small enough \( t'_0 > 0 \), we have \( f''(t) = -\kappa(t)f(t) \leq -cf(t'_0) \) for \( t \geq t'_0 \). Apply the similar argument, we can deduce that there exists a point \( t_2 \) such that \( f'(t_2) < 0 \), which contradicts to the hypothesis.

By the above argument, we can always find a point such that \( f \) vanishes. Hence, each leaf is compact. By the classification of closed 2-manifold with positive sectional curvature, one has that each leaf is diffeomorphic to either \( S^2 \) for the orientable leaf or \( \mathbb{R}P^2 \) for the non-orientable leaf.

Under the condition of Theorem 1.3, we will show that for each leaf there is a saturated neighborhood tubular neighborhood as defined in [11]. In particular, for the oriented foliation case, we can show that the manifold becomes a fibration if the foliation admits a metric with positive scalar curvature.

**Proposition 2.2** Let \( M \) denote a closed smooth \( n \)-manifold with a rank-2 oriented foliation \( \mathcal{F} \). Suppose that the foliation admits a metric with leafwise positive scalar curvature. Then, \( M/\mathcal{F} \) is a fibration over some closed manifold \( B \) whose fiber is diffeomorphic to \( S^2 \) i.e. there exists a submersion

\[
\pi : M \to B
\]

and for each point \( p \in B \) we have a diffeomorphism \( \pi^{-1}(p) \cong S^2 \).

**Proof** It suffices to show that \( M/\mathcal{F} \) is a smooth manifold. Proposition 2.1 tells that each leaf is \( S^2 \). Fixing a leaf \( \mathcal{L} \) and local transversal \( T \), we have a leaf holonomy [7, Chapter 1.7]

\[
Hol : \pi_1(\mathcal{L}, x_0) \to Diff_{x_0}(T),
\]

from the fundamental group of this leaf for the fixed point \( x_0 \in \mathcal{L} \) to the germs of the local diffeomorphism of the transverse manifold at \( x_0 \). Since \( \pi_1(S^2, x_0) \) is trivial, this holonomy is trivial. Therefore, on each leaf \( \mathcal{L} \), there is a neighborhood \( N(\mathcal{L}) \) which is diffeomorphic to the standard product \( S^2 \times D^2 \). It implies that the quotient \( M/\mathcal{F} \) has a smooth manifold structure.

In general, for the non-orientable foliation case, we have the following proposition.

**Proposition 2.3** Let \( M \) denote a closed oriented smooth 4-manifold with a rank-2 foliation \( \mathcal{F} \). Suppose that the foliation admits a metric with leafwise positive scalar curvature. Then \( M/\mathcal{F} \) is an orbifold.

**Proof** We give a sketch of the proof. If each leaf is orientable, then it is known that each leaf is diffeomorphic to \( S^2 \), and the above arguments of Proposition 2.2 assert that \( M/\mathcal{F} \) is a
manifold. In general, we consider the double-covering $\tilde{M}$ of $M$ with the canonical lift foliation $\tilde{F}$. We have that $\tilde{M}/\tilde{F}$ is a manifold, and the double covering $\tilde{M}/\tilde{F} \to M/\mathcal{F}$ equips an orbifold structure for $M/\mathcal{F}$.

Combining Proposition 2.2 and Proposition 2.3, we have the following proposition.

**Proposition 2.4** Let $M$ denote a closed oriented smooth 4-manifold with a rank-2 foliation $\mathcal{F}$. Suppose that the foliation admits a metric with leafwise positive scalar curvature. Then $M$ admits a bundle-like metric such that the restriction to the foliation coincides with $g_{\mathcal{F}}$.

**Proof** By Proposition 2.3, we have that $M/\mathcal{F}$ is an orbifold. It is well-known that any orbifold admits a Riemannian metric. We can pull back a metric of $M/\mathcal{F}$ to an $\mathcal{F}$-transverse metric on $M$ and complete to a bundle-like metric.

**Remark:**
1. Following the same idea, one can show that: If a closed manifold $M$ with foliation $\mathcal{F}$ has each compact leaf and has finite leafwise holonomy, then $M/\mathcal{F}$ is an orbifold.
2. Conversely, if each leaf is compact and $M$ admits a bundle-like metric, then $M/\mathcal{F}$ is an orbifold (c.f. [7, Proposition 3.7]).
3. If the holonomy is not finite, there is an example (c.f. [10]) such that even though the leaves are compact, the quotient space $M/\mathcal{F}$ fails to be an orbifold.

### 2.2 Bundle-like metrics

For a foliated manifold, a notion of bundle-like metric was firstly posted by Reinhart [9]. Let $\mathcal{F}$ be an integrable subbundle of the tangent vector bundle $TM$ of a smooth Riemannian manifold $(M, g)$. Then, we have the associated foliation $\mathcal{F}$. There is a splitting for this metric $g = g_{\mathcal{F}} \oplus g_{\mathcal{F}^\perp}$, and an isomorphism $\mathcal{F}^\perp \cong Q$, where $Q$ denotes the quotient $TM/\mathcal{F}$. $Q$ inherits a metric $g_Q = g_{\mathcal{F}^\perp}$. We say $g_Q$ is bundle-like, if $L_v g_Q \equiv 0$, for all $v \in \Gamma(\mathcal{F})$, here $L_v$ denotes the Lie-derivative associated with $v$. Given a bundle-like metric, we define $g_{\beta} = (\beta^2 g_{\mathcal{F}}) \oplus g_Q$.

Denote by $\nabla^g$ the associated Levi-Civita connection and $\langle \rangle$ the metric of $g_0$.

**Lemma 2.5** By the straightforward calculation, we have that

\begin{equation}
\langle \nabla^g_{e_i} e_j, e_k \rangle = O(1), \quad \langle \nabla^g_{e_i} e_j, f_k \rangle = O(\beta^2).
\end{equation}
where \( e_i \in \Gamma(F) \), \( f_i \in \Gamma(Q) \).

**Theorem 2.6** The scalar curvature \( \text{Scal}^\beta \) associated with the metric \( g^\beta \) can be expressed as follows:

\[
\text{Scal}^\beta = \frac{\text{Scal}^F}{\beta^2} + O(1).
\]

**Proof** Let \( p : TM \to F \) and \( p^\perp : TM \to F^\perp \) be the orthogonal projection maps. For \( e_i, e_j \in \Gamma(F) \), we get

\[
\langle R^\beta(e_i, e_j)e_i, e_j \rangle = \langle \nabla^\beta_{e_i} (p + p^\perp) \nabla^\beta_{e_i} e_i, e_j \rangle - \langle \nabla^\beta_{e_i} (p + p^\perp) \nabla^\beta_{e_i} e_i, e_j \rangle - \langle \nabla^\beta_{[e_i, e_j]} e_i, e_j \rangle
\]

\[
= \langle R^F(e_i, e_j)e_i, e_j \rangle - \beta^2 \langle p^\perp \nabla_{e_i} e_i, \nabla_{e_j} e_j \rangle + \beta^2 \langle (p^\perp \nabla_{e_i} e_i, \nabla_{e_j} e_j) \rangle
\]

\[
= \langle R^F(e_i, e_j)e_i, e_j \rangle + O(\beta^2).
\]

For \( e_i \in \Gamma(F) \), \( f_j \in \Gamma(Q) \), we have

\[
\langle R^\beta(e_i, f_j)e_i, f_j \rangle = \beta^2 \langle \nabla^\beta_{e_i} p \nabla_{f_j} e_i, f_j \rangle + \beta^2 \langle \nabla_{[e_i, f_j]} p \nabla_{e_i, f_i} \rangle - \beta^2 \langle \nabla_{e_i} p \nabla_{e_i, f_i} \rangle
\]

\[
- \beta^2 \langle \nabla_{f_j} p \nabla_{e_i} e_i, f_j \rangle - \beta^2 \langle \nabla_{[e_i, f_j]} e_i, f_j \rangle
\]

\[
= O(\beta^2).
\]

Similarly, for \( f_i, f_j \in \Gamma(Q) \), we have

\[
\langle R^\beta(f_i, f_j)f_i, f_j \rangle = \beta^2 \langle \nabla_{f_i} p \nabla^\beta_{f_j} f_i, f_j \rangle + \beta^2 \langle \nabla_{f_i} p \nabla^\beta_{f_j} f_i, f_j \rangle - \beta^2 \langle \nabla_{f_i} p \nabla^\beta_{f_i} f_i, f_j \rangle
\]

\[
- \langle \nabla_{f_i} p \nabla_{f_j} f_i, f_j \rangle - \langle \nabla_{[f_i, f_j]} f_i, f_j \rangle
\]

\[
= O(1).
\]

Combining the above three formulas, one gets the desired result. \( \square \)

### 3 Seiberg-Witten invariant and vanishing theorem

In this section, we will give a proof of Theorem 1.3. The idea to show the theorem is as follows: By Proposition 2.4, if the hypothesis of Theorem 1.3 is satisfied, then in Theorem 2.6 we may choose \( \beta \) small enough so that the scalar curvature of the manifold is positive. Thus, the conclusion of Theorem 1.3 follows. Before proceeding, we review the Seiberg-Witten invariant and the result that if a closed oriented 4-manifold has positive scalar curvature and \( b^+ > 1 \), then the Seiberg-Witten invariant vanishes.

Let \((M, g)\) be a closed oriented Riemannian 4-manifold with a spin$^c$-structure $s$. Let $S^\pm$ denote the spinor bundles associate to $s$, there is a well-defined Dirac operator

\[
\mathcal{D}_A : \Gamma(S^+) \to \Gamma(S^-),
\]
where $A$ is a connection on the determinant line bundle of this spin$^c$ structure $s$. We give a brief introduction to the classical Seiberg-Witten theory, see [8, Chapter 3.4] for more details. Let $\mathcal{A}$ denote all the connections on the determinant line bundle, for $(A, \Phi) \in \mathcal{A} \times \Gamma(S^+)$, we define Seiberg-Witten equation as follows:

\[
\begin{cases}
F_A^+ = q(\Phi) \\
\bar{D}_A \Phi = 0
\end{cases}
\]

where $q(\Phi) = \Phi \otimes \Phi^* - \frac{|\Phi|^2}{2}$ and we used the identification

\[ cl_+ : \Lambda^{2,+} \otimes \mathbb{C} \to \text{End}_0(S^+) \]

between the self-adjoint two forms and traceless endomorphism of $S^+$. The moduli space $\mathcal{M}(s)$ is the space consisting of all solutions $(A, \Phi)$ mod the gauge group $\mathcal{G} = C^\infty(M, S^1)$, where the gauge action is defined by the following: for each $g \in \mathcal{G}$,

\[ g(A, \Phi) = (A - 2g^{-1}dg, g \cdot \Phi), \]

where $\cdot$ denotes the Clifford multiplication. We can also perturb the equations, by adding a self-dual two-form $\eta$, namely to solve the equations

\[
\begin{cases}
F_A^+ = q(\Phi) + \eta \\
\bar{D}_A \Phi = 0
\end{cases}
\]

we write $\mathcal{M}_\eta(s)$ for the perturbed moduli space. The formal dimension of the moduli space $\mathcal{M}_\eta(s)$ is

\[ d(s) = \frac{1}{4}(c_1(\det(s)) \cdot c_1(\det(s)) - 2e(M) - 3\sigma(M)), \]

where $e(M)$ denotes the Euler number and $\sigma(M)$ denotes the signature.

We have the following well-known results about the moduli space:

1. The moduli space $\mathcal{M}_\eta(s)$ is compact.
2. The orientation of $H^0(M) \otimes H^1(M) \otimes H^+(M)$ induces the orientation of the moduli space $\mathcal{M}_\eta(s)$
3. For an open dense set of the perturbation $\eta$, the moduli space is a smooth manifold consisting of irreducible solution(i.e. $\Phi \neq 0$).

By fixing an orientation of the moduli space, we define the Seiberg-Witten invariant to be zero, if the formal dimension is odd or less than zero, otherwise the Seiberg-Witten invariant $s$ is to be the following:

\[ SW_{\eta}(s) = \int_{\mathcal{M}_\eta(s)} \mu^{1/2}, \]

where $\mu$ denotes the first Chern class of the canonically associated principal $S^1$-bundle, i.e. the solutions mod the reference gauge group $\mathcal{G}_0 = \{ u \in \mathcal{G} | u(x_0) \equiv 1 \}$, where $x_0 \in M$ is a fixed point. If $b^+(M) > 1$, then it is known that the moduli space is generically independent of the choice of the perturbation and the metric. In this case, we often omit the subscript of the perturbation.
We review the classical result of the local estimate for Seiberg-Witten equation [8, Chapter 4]. Let \((A, \Psi)\) be solution of the Seiberg-Witten equation, by Weitzenböck formula one deduces that
\[
0 = \frac{1}{2} \Delta |\Psi|^2 + |\nabla A \Psi|^2 + \frac{1}{4} \text{Scal} |\Psi|^2 + \frac{1}{4} |\Psi|^4,
\]
hence
\[
\Delta |\Psi|^2 + \frac{1}{2} \text{Scal} |\Psi|^2 \leq 0.
\]
At the maximal point of \(|\Psi|^2\), one gets that \(\Psi = 0\) in order that the scalar curvature is positive, so that the Seiberg-Witten invariant vanishes. Together with Proposition 2.4, Theorem 2.6, and letting \(\beta \to 0\), we proved Theorem 1.3. Therefore, the Seiberg-Witten invariant is an obstruction for the closed oriented 4 manifold admitting a rank 2-foliation with positive scalar curvature.

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