GLOBAL WELL-POSEDNESS IN SOBOLEV SPACE IMPLIES
GLOBAL EXISTENCE FOR WEIGHTED $L^2$ INITIAL DATA FOR
$L^2$-CRITICAL NLS.

P. BLUE, J. COLLIANDER

Abstract. The $L^2$-critical defocusing nonlinear Schrödinger initial value problem on $\mathbb{R}^d$ is known to be locally well-posed for initial data in $L^2$. Hamiltonian conservation and the pseudoconformal transformation show that global well-posedness holds for initial data $u_0$ in Sobolev $H^1$ and for data in the weighted space $(1+|x|)u_0 \in L^2$. For the $d = 2$ problem, it is known that global existence holds for data in $H^s$ and also for data in the weighted space $(1 + |x|)^\sigma u_0 \in L^2$ for certain $s, \sigma < 1$. We prove: If global well-posedness holds in $H^s$ then global existence and scattering holds for initial data in the weighted space with $\sigma = s$.

1. Introduction

Consider the initial value problem for the $L^2$ critical nonlinear Schrödinger equation for $u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$,

$$
\begin{cases}
  i\partial_t u + \Delta u = \lambda |u|^\frac{4}{d} u \\
  u(t_0, x) = u_0(x)
\end{cases}
$$

This problem is called defocusing for $\lambda > 0$ and focusing for $\lambda < 0$. In dimension $d = 2$, equation (1.1) reduces to the cubic nonlinear Schrödinger equation, $i\partial_t u + \Delta u = \lambda |u|^2 u$, which appears widely as a model equation in Physics [9].

This problem is locally well-posed in $L^2$ or in any $H^s$ with $s \geq 0$. That is, given initial data $u_0 \in H^s(\mathbb{R}^d)$ with $s \geq 0$ there is a local existence time, $T_{lwp}$, and a local in time solution $u : [t_0 - T_{lwp}, t_0 + T_{lwp}] \times \mathbb{R}^d \to \mathbb{C}$ such that $u$ solves (1.1) and the function $u \in C_t H^s$. For $s > 0$, $T_{lwp}$ is a decreasing function of the $H^s$ norm of $u_0$.

An open problem is to prove global well-posedness in $L^2$ in the defocusing case and under an appropriate smallness condition in the focusing case. For the focusing case, it is believed that solutions with $L^2$ norm smaller than the ground state mass $\|Q\|_{L^2}$ do not blowup and in fact scatter. Explicit blow-up solutions with Schwartz class initial data and with the mass of the ground state are known to exist in the focusing case. By finding the optimal constant in the Gagliardo-Nirenberg estimate,

$$
\|u\|_{L^{2d/(d+2)}}^2 \leq \left( \frac{4}{d} \|Q\|_{L^2}^\frac{4}{d} \right) \|\nabla u\|_{L^2}^2 \|u\|_{L^2}^\frac{4}{d},
$$

Weinstein [11] proved that $H^1$ initial data with $L^2$ norm less than the ground state mass evolves globally in time. In the defocusing case, $L^2$ solutions are expected to exist globally in time and scatter. Although the $L^2$ norm of $u(t)$ is constant on the local well-posedness time interval, this norm does not control the length of the local well-posedness time $T_{lwp}$ and

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1The ground state is the unique (up to translations) positive solution of $-Q + \Delta Q = Q^3$. 

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can not be used to prove global well-posedness in $L^2$. In $H^1$, this problem has an additional conserved quantity, the energy, $E[u] = \int \frac{1}{2} |\nabla u|^2 + \frac{1}{4} |u|^4 dx$. In the defocusing case, the energy is positive and dominates the $H^1$ norm. Since the local well-posedness time is a function of the $H^1$ norm, at any time the solution persists for a uniformly long local well-posedness time, and, hence, globally in time.

Bourgain was the first to prove [2], for the cubic problem in $\mathbb{R}^2$, global well-posedness below the energy threshold $H^1$ by proving global well-posedness for data in $H^s$ in the defocusing case for $s > \frac{3}{5}$. The method in [2] involved a decomposition of the data into high and low frequencies with a sharp cut-off function in the Fourier variables. Later, the “$I$-method” [6] was used to improve this to $s > \frac{47}{42}$ for the cubic problem on $\mathbb{R}^2$.

The nonlinear Schrödinger equation (1.1) has a discrete pseudoconformal symmetry,

$$C[u](\tau, y) = v(\tau, y) = \frac{1}{|\tau|^2} e^{\frac{iy^2}{4\tau^2}} u(\frac{1}{\tau}, y).$$

This is a symmetry in the sense that, if $u(t, x)$ is a solution to the nonlinear Schrödinger equation on $(t, x) \in [t_1, t_2] \times \mathbb{R}^d$, then $v(\tau, y)$ is a solution on $\tau \in [-t_1^{-1}, -t_2^{-1}], y \in \mathbb{R}^d$. Throughout this paper, we shall use $u$, $t$, and $x$ to refer to a solution, the time variable, and the spatial variable respectively, to use $v$ as the pseudoconformal transform of $u$, and to use $\tau$ and $y$ as the arguments of $v$ which will be called the transformed time and space variables. In particular, $\tau = t^{-1}$.

We introduce the space

$$H^{0,s} = \{ u \in L^2 : |x|^s u \in L^2 \}$$

with norm

$$\|u\|_{H^{0,s}} = \|\langle x \rangle^s u\|_{L^2}$$

where $\langle x \rangle^2 = 1 + x^2$.

The pseudoconformal transform satisfies

$$E[v](\tau) = \|xu_0\|_{L^2}^2 = \|u_0\|_{H^{0,1}}^2.$$ 

The energy on the left side of this equation is already known to be independent of $\tau$. This property was used to prove global well-posedness in $L^2$ for initial data $u_0 \in H^{0,s}$ with $s > \frac{3}{5}$ in the defocusing case [2]. The proof involves a spatial decomposition analogous to the Fourier decomposition used in proving the $H^s$ global existence result. As a further consequence, [2] also proves scattering, that is the existence of functions $u_\pm \in L^2$ such that

$$\lim_{t \to \pm \infty} \|u(t) - e^{\pm it\Delta} u_\pm\|_{L^2} = 0.$$

This paper establishes that global well-posedness in $H^s$ for (1.1) implies global existence and scattering in $L^2$ for initial data in $H^{0,s}$ for (1.1). Thus, the link between $H^s$ global well-posedness and the evolution properties of $H^{0,s}$ initial data found in the $\mathbb{R}^2$ case in [2] is in fact common to all pseudoconformal or $L^2$-critical nonlinear Schrödinger initial value problems (1.1).

**Proposition 1.1.** Assume that the nonlinear Schrödinger equation (1.1) is globally well-posed in $H^s$ (with the additional hypothesis that the initial data has $L^2$ norm bounded by $\|Q\|_{L^2}$ in the focusing case).
If \( u_0 \in H^{0,s} \) (and \( \|u_0\|_{L^2} < \|Q\|_{L^2} \) in the focusing case), then there is a function \( u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C} \) which solves the nonlinear Schrödinger equation (1.1) for all time. Furthermore, there are functions \( u_{\pm} \in H^{0,s} \) such that
\[
\lim_{t \to \pm \infty} \| e^{\mp it \Delta} u(t) - u_{\pm} \|_{H^{0,s}} = 0.
\]

Based on the \( H^s \) global well-posedness result in [6], we obtain as a consequence of Proposition 1.1 that for \( s > \frac{4}{7} \), initial data in \( H^{0,s} \) evolve globally in time and scatter in \( L^2 \) under the cubic nonlinear Schrödinger flow on \( \mathbb{R}^2 \).

Global existence in \( L^2 \) for initial data in \( H^{0,s} \) means initial data \( u_0 \in H^{0,s} \), which is also in \( L^2 \), evolves as a solution in \( L^2 \) and that this solution exists for all time. This is significantly weaker than global well-posedness in \( H^s \), which means that initial data in \( H^s \) continuously evolves in \( H^s \), that this solution exists for all time, and that the time evolution map \( S_{\text{NLS}}(t,0) : u_0 \mapsto u(t) \) is continuous from \( H^s \) to \( H^s \).

The asymmetry between \( H^s \) and \( H^{0,s} \) in Proposition 1.1 is a consequence of the local theory. The initial value problem (1.1) is locally well-posed in \( H^s \); whereas, in \( H^{0,s} \), (1.1) is ill-posed, so global existence in \( L^2 \) for initial data in \( H^{0,s} \) can not be extended to \( H^{0,s} \) global well-posedness.

In the remainder of the introduction, we review \( L^2 \) and \( H^s \) local-well posedness, \( H^{0,s} \) ill-posedness, and some properties of the pseudoconformal transform. In Section 2, we prove Proposition 1.1 by showing that, for a solution with initial data in \( H^{0,s} \), the pseudoconformal transform is in \( H^s \). This is done by taking regularized approximators and showing their transforms converge in \( H^s \) at a particular transformed time \( -T_{\text{lwp}}^{-1} \). In Section 3, we show that scattering is a consequence of the construction in Section 2.

We use the notation \( S_{\text{NLS}}(t_2,t_1) \) to denote the nonlinear Schrödinger evolution map from time \( t_1 \) to time \( t_2 \), \( F[\bullet] \) for the Fourier transform, and \( \Re \alpha \) and \( \Im \alpha \) for the real and imaginary part of \( \alpha \) respectively.

1.1. Local well-posedness theory. The local well-posedness theory (see [3], [8] for a review) begins with the presentation of the nonlinear Schrödinger equation as an integral equation through Duhamel’s principle
\[
(1.6) \quad u = \Phi_{u_0}[u], \\
(1.7) \quad \Phi_{u_0}[u] = e^{it \Delta} u_0 + \int_0^t e^{i(t-t') \Delta} (\lambda |u|^2 u)(t') dt'.
\]

To prove that (1.6) has a unique solution, it is sufficient to show that \( \Phi_{u_0} \) is a contraction in an appropriate space. This space will be the Strichartz space defined below.

Definition 1.2. The pair \((q,r)\) is \( L^2 \) Strichartz admissible, or simply admissible, if
\[
\frac{2}{q} + \frac{d}{r} = \frac{d}{2}, \quad 2 \leq q \leq \infty, \quad 2 \leq r \leq 2 + \frac{4}{d - 2}.
\]

In dimension \( d = 2 \), there is the additional restriction that \( 2 < q \leq \infty \) and \( 2 \leq r < \infty \). In dimension \( d = 1 \), \( 2 \leq q \leq \infty \) and \( 2 \leq r \leq \infty \).
For $s \geq 0$, a pair $(q,r)$ is $H^s$ Strichartz admissible if
\[
\frac{2}{q} + \frac{d}{r} = \frac{d}{2} - s, \quad 2 \leq q \leq \infty, \quad 2 \leq r.
\]

The $L^2$ Strichartz norm, or simply the Strichartz norm, is
\[
\|u\|_{S^0} = \sup_{(q,r) \text{ admissible}} \|u\|_{L^q_t L^r_x}.
\]

The homogeneous $H^s$ Strichartz norm is
\[
\|u\|_{S^s} = \sup_{(q,r) \text{ admissible}} \|D^s u\|_{L^q_t L^r_x},
\]
where $D$ is the Fourier multiplier defined by $\mathcal{F}[Df](\xi) = |\xi|^s \mathcal{F}[f](\xi)$.

For $s > 0$, the $H^s$ Strichartz norm is
\[
\|u\|_{S^s} = \sup_{0 \leq \sigma \leq s} \|u\|_{S^\sigma},
\]

For an interval $I$, the spaces $S^0(I)$, $S^s(I)$, and $S^s(I)$ are the spaces with the above Strichartz norm where the $t$ integration is taken over the interval $t \in I$.

With this notation, we record the Strichartz estimates: For $s > 0$,
\[
\|e^{it \Delta} u_0\|_{S^0} \leq C \|u_0\|_{L^2},
\]
\[
\|e^{it \Delta} u_0\|_{S^s} \leq C \|u_0\|_{H^s},
\]
\[
\|e^{it \Delta} u_0\|_{S^s} \leq C \|u_0\|_{H^s}.
\]

Using Duhamel's principle, estimates for solutions to the linear Schrödinger equation, and a contraction argument, it is possible to prove local well-posedness, uniqueness, and continuity of solutions in the $L^2$ Strichartz space. The balanced pair $q = r = \frac{2(4+2)}{d}$ plays an important role in our presentation of the local well-posedness theory.

**Theorem 1.3** ($L^2$ local well-posedness). For $u_0 \in L^2$ and $\delta_1 > 0$, there is a $T_{lwp} = T_{lwp}(u_0, \delta_1) > 0$ and $u : [0, T_{lwp}] \times \mathbb{R} \to \mathbb{C}$ such that:

1. $u$ solves the nonlinear Schrödinger equation (1.1) with initial data $u_0 \forall t \in [0, T_{lwp}]$;
2. $\|u\|_{S^0([0, T_{lwp}])} \leq 2 \|u_0\|_{L^2}$;
3. $\|u\|_{L^\frac{2(4+2)}{d}([0, T_{lwp}])} \leq \delta_1$;
4. for all $t \in [0, T_{lwp}]$, $\|u(t)\|_{L^2} = \|u_0\|_{L^2}$;
5. there is a $\delta_2 > 0$, such that, if $u_0' \in L^2$ with $\|u_0 - u_0'\|_{L^2} < \delta_2$, then there is a $u' \in S^0([0, T_{lwp}])$ such that $u'$ solves the nonlinear Schrödinger equation with initial data $u_0'$; and $\|u - u'\|_{S^0([0, T_{lwp}])} \leq 2 \|u_0 - u_0'\|_{L^2}$. In particular, for $t \in [0, T_{lwp}]$,
\[
\|u(t) - u'(t)\|_{L^2} \leq 2 \|u_0 - u_0'\|_{L^2}.
\]

The $T_{lwp}$ from this theorem will be referred to as the $L^2$ local well-posedness time. The $L^2$ maximal forward time of existence, $T^*$, is the time for which there is a solution $u : [0, T^*) \to L^2$ but no solution $\tilde{u} : [0, t] \to L^2$ for $t > T^*$.

The $L^2$ local well-posedness theory and $L^\frac{2(4+2)}{d}([0, T_{lwp}])$ norm play a central role in our presentation of the $H^s$ theory. The $L^2$ local well-posedness time, $T_{lwp}$, is the
same as the $H^s$ local well-posedness time. If a solution has small $L^{2(d+2)}_{tx}([0,T_{lwp}])$ norm, then its $H^s$ norm can not grow by more than a small factor, and, if another solution has initial data close in $L^2$, then the $H^s$ distance between the two solutions can not grow by more than a small factor. The $L^2$ local well-posedness Theorem 1.3 uses $T_{lwp}$ chosen to enforce the $L^{2(d+2)}_{tx}([0,T_{lwp}])$ smallness condition.

**Theorem 1.4 ($H^s$ local well-posedness).** For $u_0 \in H^s$, the solution $u$ and local well-posedness time $T_{lwp}$ given in Theorem 1.3 satisfy:

1. for $t \in [0,T_{lwp}]$, $u(t) \in H^s$;
2. there is a $\delta_3 > 0$, such that, if $\|u\|_{L^{2(d+2)}_{tx}([0,T])} \leq \delta_3$, then $\|u\|_{S^s(I)} \leq 2\|u_0\|_{H^s}$;
3. there is a $\delta_4 > 0$, such that, if $\|u\|_{L^{2(d+2)}_{tx}([0,T])} \leq \delta_3$, $u'$ is a solution to the nonlinear Schrödinger equation with initial data $u'_0 \in H^s$, and $\|u_0 - u'_0\|_{L^2} \leq \delta_4$, then $\|u - u'\|_{S^s} \leq 2\|u_0 - u'_0\|_{H^s}$. In particular,

\[
\|u(t) - u'(t)\|_{H^s} \leq 2\|u_0 - u'_0\|_{H^s}.
\]

Since $u$ can be extended to any interval on which the $L^2$ Strichartz norm is finite, the $L^2$ Strichartz norm must diverge on intervals approaching the maximal forward time of existence.

**Corollary 1.5** (Maximal time blow up theorem). If $u$ is a solution to the nonlinear Schrödinger equation (1.1) with initial data $u_0 \in L^2$ and maximal forward time of existence $T^*$ (that is $u : [0,T^*) \rightarrow \mathbb{C}$ is a solution, but for $t > T^*$, there is no solution $\tilde{u} : [0,t] \rightarrow \mathbb{C}$), then

\[
\lim_{t \uparrow T^*} \|u\|_{S^0([0,t])} = \infty.
\]

1.2. Local evolution of $H^{0,s}$ initial data. As noted earlier, Proposition 1.1 only asserts global existence in $L^2$ for initial data in $H^{0,s}$, not $H^{0,s}$ global well-posedness, because the nonlinear Schrödinger equation is ill-posed in $H^{0,s}$. This ill-posedness is a consequence of the ill-posedness of the linear Schrödinger equation in $H^{0,s}$. It has solutions which start with finite initial $H^{0,s}$ norm, but have divergent $H^{0,s}$ norm at all later times. From these, it is possible to show that the nonlinear Schrödinger time evolution $S_{NLS}(t,0) : u_0 \mapsto u(t)$ is not continuous from $H^{0,s}$ to $H^{0,s}$. This result does not have an analogue in $H^s$, $s > 0$.

Solutions to the linear Schrödinger equation which have finite initial $H^{0,s}$ norm but divergent $H^{0,s}$ norm can be constructed from sums of Gaussians. Since the Fourier variable corresponds to velocity for the linear Schrödinger equation, Gaussian initial data of width $a^{-\frac{1}{2}}$ has a Gaussian Fourier transform with a width $a^{\frac{1}{2}}$, and hence disperses linearly in time at speed $a^{\frac{1}{2}}$. A linear combination of such initial data can be constructed to disperse with unbounded velocity and instantaneously divergent weighted norm.

**Lemma 1.6.** For $s > 0$, there is a solution $\Psi$ to the linear Schrödinger equation with initial data $\Psi_0 \in H^{0,s}$ for which $\Psi(t)$ is not in $H^{0,s}$ at any future time $t > 0$. 

Proof. Let
\[
(C_{d,s})^2 = \frac{1}{\pi^2} \int_{\mathbb{R}^d} |x|^2 s e^{-x^2} \, dx,
\]
\[
(C'_{d,s})^2 = \frac{1}{\pi^2} \int_{|x|<1} |x|^2 s e^{-x^2} \, dx,
\]
and let \( r(a,t) = \left( \frac{1 + 4a^2 t^2}{a} \right)^{\frac{1}{2}} \),

and let \( \chi_{|x|<r(a,t)} \) be the characteristic function with support on \( |x| < r(a,t) \).

Given \( A > 0 \) and \( a > 0 \), let \( \psi[A,a] \) be solutions to the linear Schrödinger with initial data
\[
u_0[A,a] = A \left( \frac{a}{\pi} \right)^{\frac{d}{2}} e^{-\frac{|x|^2}{2}}.
\]

These are given by
\[
\psi[A,a] = e^{it\Delta} u_0[A,a] = A \left( \frac{a}{\pi} \right)^{\frac{d}{2}} \frac{1}{(1 + 2ait)^{\frac{d}{2}}} e^{\frac{a|x|^2}{2(1 + 2ait)}},
\]
\[
\|e^{it\Delta} u_0[A,a]\|_{L_2^2} = A
\]
\[
\|\psi[A,a]\|_{L_2^2} = C_{d,s} A(a,t)^{s} \sim Aa^{\frac{d}{2}} t^s
\]
for large \( t \).

Let \( \psi_k = e^{it\Delta} u_0[A_k,a_k] \) with \( A_j \) chosen, in terms of \( A_k \) for \( k < j \), to be sufficiently small so that
\[
(1.11) \quad \sum_{i=k+1}^{j} A_i \leq \frac{1}{8} C'_{d,s} A_k
\]
and with \( a_j \) chosen sufficiently large so that
\[
(1.12) \quad \sum_{k<j} A_k (1 + C_{d,s} r(a_k,t)^s) \leq \frac{1}{4} C'_{d,s} A_j a_j^{\frac{d}{2}} t^s
\]
and let \( \Psi = \sum \psi_k \).

Given a fixed \( t \), let \( r_k = r(a_k,t) \) and \( \chi_k = \chi_{|x|<r_k} \). The first condition, (1.11), ensures that, for sufficiently large \( k \), on a length scale of \( |x| < r_k \), the function \( \psi_k \) dominates all the later \( \psi_j \) with \( j > k \):
\[
\|\chi_k |x|^s \psi_k\|_{L^2} = C'_{d,s} A_k r_k^s > 8 \sum_{j>k} A_j r_k^s > 8r_k^s \sum_{j>k} \|\psi_j\|_{L^2} > 4 \sum_{j>k} \|\chi_k |x|^s \psi_j\|_{L^2}
\]
The second condition, (1.12), ensures that \( A_j a_j^{\frac{d}{2}} \) grows at least exponentially. It also ensures that, for \( a_j^{\frac{d}{2}} > t^{-2} \), in \( H^{0,s} \), \( \psi_j \) dominates all of the previous \( \psi_k \) with \( k < j \):
\[
\|\chi_j |x|^s \psi_j\|_{L^2} = C'_{d,s} A_j r_j^s > C'_{d,s} A_j a_j^{\frac{d}{2}} t_j^s > 4 \sum_{k<j} A_k (1 + C_{d,s} r_k^s) > 4 \sum_{k<j} \|\psi_k\|_{H^{0,s}}
\]
Since \( \Psi \) is the sum of the \( \psi_k \), and since, at a given time \( t \), for sufficiently large \( j \), on a length scale of \( r_j \), \( \psi_j \) dominates all the other \( \psi_k \), the \( H^{0,s} \) norm of \( \Psi(t) \) is bounded below by arbitrarily large numbers and must diverge:
\[
\|\Psi(t)\|_{H^{0,s}} \geq \|\chi_j |x|^s \Psi\|_{L^2} \geq \|\chi_j |x|^s \psi_j\|_{L^2} - \sum_{k\neq j} \|\chi_j |x|^s \psi_k\|_{L^2} \geq \frac{1}{2} \|\chi_j |x|^s \psi_j\|_{L^2} \geq C'_{d,s} A_j a_j^{\frac{d}{2}} t^s \rightarrow \infty.
\]
A similar sequence of Gaussian initial data shows that the nonlinear Schrödinger equation is also ill-posed in $H^{0,s}$.

**Proposition 1.7.** The nonlinear Schrödinger equation (1.1) is not well-posed in $H^{0,s}$ for any $s > 0$: the nonlinear Schrödinger evolution from time $0$ to $t > 0$, $S_{NLS}(t,0): u_0 \mapsto u(t)$, is not continuous from $H^{0,s}$ to $H^{0,s}$.

**Proof.** It is sufficient to construct a sequence of $L^2$ solutions, $u^{[k]}$, for which $\|u^{[k]}_0\|_{H^{0,s}} \to 0$, but for which, at any $t > 0$, $\|u^{[k]}(t)\|_{H^{0,s}} \to \infty$. This construction follows from the closeness of nonlinear Schrödinger and linear Schrödinger evolutions for small initial data and from the existence of linear solutions with arbitrarily fast $H^{0,s}$ growth.

When the $L^{2(d+2)}_{tx}$ norm is sufficiently small, the difference between the linear Schrödinger and nonlinear Schrödinger evolutions is small. From Duhamel’s principle and an extension of the local well-posedness theory, it is known that there is a $\delta'$ such that, if $\|u_0\|_{L^2} \leq \delta'$, then $u$ is defined for all time, and

$$
\|u - e^{-it\Delta}u_0\|_{S^0} \leq \|u\|_{L^2}^{\frac{4}{3}} + \|u_0\|_{L^2}^{\frac{2}{3}}
$$

$$
\|u\|_{L^{2(d+2)}_{tx}} \leq 2\|u_0\|_{L^2}.
$$

For the linear Schrödinger solutions, the notation from the previous lemma will be used. In addition, $u^{[k]}$ will denote the nonlinear Schrödinger evolution of $u_0[A_k,a_k]$ with $A_k$ decreasing to zero and $a_k$ increasing to infinite, but with rates to be chosen. The index $k$ will be chosen sufficiently large so that, for $i > k$,

$$
A_i < \delta'
$$

$$
A_i^{\frac{4}{3}} < C'_{d,s} a_k^{2}\frac{2}{3}
$$

As in Lemma 1.4, the $H^{0,s}$ norm can be estimated by localizing on a length scale of $r_k$.

$$
\|u^{[k]}(t)\|_{H^{0,s}} \geq \|\chi_k|x|^s u^{[k]}\|_{L^2} \geq \|\chi_k|x|^s e^{-it\Delta}u^{[k]}_0\|_{L^2} \geq \|\chi_k|x|^s (u^{[k]} - e^{-it\Delta}u^{[k]}_0)\|_{L^2} \geq C'_{d,s} A_k r_k^{\frac{4}{3}} \|u^{[k]}_0\|_{L^2}^{\frac{2}{3}} \geq C'_{d,s} A_k r_k^{\frac{4}{3}} (1 - \frac{1}{2}) \geq A_k a_k^{\frac{2}{3}} t^s.
$$

If $A_k = a_k^{\frac{2}{3}}$ and $a_k \to \infty$, then, for each $t > 0$,

$$
\|u^{[k]}_0\|_{H^{0,s}} \sim A_k \to 0,
$$

$$
\|u^{[k]}(t)\|_{H^{0,s}} \sim A_k a_k^{\frac{2}{3}} t^s \to \infty.
$$

\[\square\]

1.3. The pseudoconformal transform and Strichartz norms. The pseudoconformal transform is a symmetry of both the linear Schrödinger equation and the pseudoconformal nonlinear Schrödinger equation (1.1) and is also an isometry on $L^2_\nu$ and the Strichartz admissible $L^q L^r$ spaces.

**Lemma 1.8.** (1) If $0 \leq t_1 \leq t_2$, and $u : [t_1, t_2] \times \mathbb{R}^d \to \mathbb{C}$ satisfies the nonlinear Schrödinger equation (1.1), then so does $C[u] = v : [-t_1^{-1}, -t_2^{-1}] \times \mathbb{R}^d \to \mathbb{C}$. 

\[\square\]
Figure 1. The map $F_t : u_0 \mapsto v(-T_{lwp}^{-1})$. $u_0$ is given in $H^{0,s} \subset L^2$, and $u_0'$ and $u_0''$ in $H^{0,s} \cap H^1$ are then chosen in a $H^{0,s}$ neighborhood of $u$.

(2) If $u : \{t_1\} \times \mathbb{R}^d \to \mathbb{C}$, then $\|u(t_1)\|_{L^2} = \|v(-t_{lwp}^{-1})\|_{L^2}$.

(3) If $u : [t_1, t_2] \times \mathbb{R}^d \to \mathbb{C}$ and $(q, r)$ is admissible, then $\|u\|_{L^q_{t \in [t_1, t_2]} L^r_x} = \|v\|_{L^q_{t \in [-t_{lwp}^{-1}, -t_{lwp}^{-1}]} L^r_x}$.

(4) Up to a reflection, the pseudoconformal transform is its own inverse: $C[C[u]](t, x) = u(t, -x)$.

These facts may be validated through explicit calculations.

2. Global existence for initial data in $H^{0,s}$

The goal is to prove global existence for initial data in $H^{0,s}$ from the assumption that there is global well-posedness in $H^s$.

Heuristically, initial data $u_0 \in H^{0,s}$ at $t_0 = 0$ can be transformed to initial data $v_0 \in H^s$ at $\tau_0 = -\infty$. Under the $H^s$ global well-posedness hypothesis, $v$ can then be defined for all time, and $u$ can be defined for all time by the inverse pseudoconformal transform. To make this heuristic rigorous, $u_0$ can be evolved to $u(T_{lwp})$ and then pseudoconformally transformed to $v$. Following this, it is sufficient to show that $v(-T_{lwp}^{-1})$ is in $H^s$ to apply the $H^s$ global well-posedness hypothesis.
In terms of the nonlinear Schrödinger evolution map, \( S_{NLS}(t_2, t_1) \), which was introduced earlier, the map \( u_0 \mapsto v(-T_{lwp}^{-1}) \) is

\[
F = C \circ S_{NLS}(t_{lwp}, 0) : L^2 \to L^2.
\]

Since the pseudoconformal transform commutes with the nonlinear Schrödinger evolution, this map can also be constructed in a different way, which is illustrated in Figure 1. The solutions are first allowed to evolve under \( S_{NLS}(t, 0) \) to time \( t \) (dark in left diagram), then pseudoconformally transformed from data at time \( t \) to data at transformed time \( -t^{-1} \) (dashed arrow from left diagram to right diagram), and finally allowed to evolve under \( S_{NLS}(-T_{lwp}^{-1} - t^{-1}) \) to transformed time \( -T_{lwp}^{-1} \) (dark in right diagram). This construction is represented by

\[
F = F_t = S_{NLS}(-T_{lwp}^{-1} - t^{-1}) \circ C \circ S_{NLS}(t, 0) : L^2 \to L^2.
\]

By the \( L^2 \) local well-posedness Theorem 1.4 and the properties of the pseudoconformal transform, in a \( L^2 \) neighborhood of \( u_0 \), \( F \) is continuous with respect to the \( L^2 \) norm.

To prove Proposition 1.1, it is sufficient to show that \( F \) can be restricted to \( F : H^{0,s} \to H^s \). This is done by initially restricting to regularized data in \( H^{0,s} \cap H^1 \), showing that each of the three steps of \( F_t \) is continuous with respect to the regularized data, and then removing the regularization. \( H^{0,s} \cap H^1 \) is a useful auxiliary space because it is preserved by the nonlinear Schrödinger evolution and pseudoconformally transforms to \( H^s \).

Because the \( H^s \) local well-posedness Theorem 1.4 uses \( L^2 \) Strichartz norms to control the divergence of nearby solutions, if \( u' \) and \( u'' \) start near \( u \) in \( L^2 \), their separation in \( H^{0,s} \cap H^1 \) can not grow by more than a constant factor. Similarly, if \( v' \) and \( v'' \) start near \( v \) in \( L^2 \), then their \( H^s \) separation can not increase by more than a constant factor. Thus the divergence of the approximants, from each other, is controlled if they start in a sufficiently small \( L^2 \) neighborhood of \( u_0 \).

This \( L^2 \) neighborhood of \( u_0 \) is illustrated by the oval in the left diagram in Figure 1. It is taken to be a ball of radius \( \delta \), and this is the \( \delta \) which appears in the following subsections. The value of \( \delta \) is dictated by the \( H^s \) local well-posedness Theorem 1.4.

Since \( F = F_t \) is independent of \( t \), it is possible to take the infimum in \( t \) of the \( H^s \) norm estimates for \( v'(-T_{lwp}^{-1}) \) and \( v''(-T_{lwp}^{-1}) \). This eliminates the dependence on the \( H^1 \) regularization and corresponds to the original heuristic of transforming \( H^{0,s} \) data at time \( t_0 = 0 \) to \( H^s \) data at \( t_0 = -\infty \). As a result \( F : H^{0,s} \cap H^1 \to H^s \) is continuous with respect to the \( H^{0,s} \) norm alone and extends uniquely to \( F : H^{0,s} \to H^s \) in a neighborhood of \( u_0 \). As a result, and contrary to the image in Figure 1, \( v(-T_{lwp}^{-1}) \) is in \( H^s \) not merely \( L^2 \).

In Subsection 2.1 it will be shown that the nonlinear Schrödinger evolution is continuous in \( H^{0,s} \cap H^1 \). In Subsection 2.2 real interpolation will be used to show that the pseudoconformal transform takes \( H^{0,s} \cap H^1 \) to \( H^s \) with a \( t \) dependent coefficient on the \( H^1 \) part of the norm. In Subsection 2.3 the \( H^s \) local well-posedness theorem will be used to show that \( H^s \) data evolves continuously in \( H^s \) from transformed time \(-t^{-1}\) to transformed time \(-T_{lwp}^{-1}\). In Subsection 2.4 the infimum in \( t \) will be taken to eliminate the \( H^1 \) dependence.

2.1. \( S_{NLS}(t, 0) : H^{0,s} \cap H^1 \to H^{0,s} \cap H^1 \). By the local well-posedness theory, the nonlinear Schrödinger evolution takes \( H^1 \to H^1 \), at least up to the \( L^2 \) local well-posedness time, \( T_{lwp} \). For the linear evolution, a simple commutator calculation
shows that the $H^1$ norm controls the growth of the $H^{0,s}$ norm. The extra terms arising in the nonlinear evolution cancel, so that the same result holds.

**Lemma 2.1.** For $u$ a solution to the nonlinear Schrödinger equation (1.1) with initial data $u_0 \in H^{0,s}$ with $0 \leq s \leq 1$ and local existence time $T_{lwp}$, there is a $\delta$ coming from the local well-posedness theory such that

1. if $u'$ solves the nonlinear Schrödinger equation with initial data $u'_0 \in H^{0,s} \cap H^1$ and with $\|u_0 - u'_0\|_{L^2} < \delta$, then $\forall t \in [0, T_{lwp}]$
   \[
   \|u'(t)\|_{H^1} \leq 2\|u'_0\|_{H^1},
   \]
   \[
   \|u'(t)\|_{H^{0,s}} \leq \|u'_0\|_{H^{0,s}} + 2\|u'_0\|_{H^1}.
   \]

2. if $u'$ and $u''$ solve the nonlinear Schrödinger equation with initial data $u'_0 \in H^{0,s} \cap H^1$ and $u''_0 \in H^{0,s} \cap H^1$ respectively and with $\|u_0 - u'_0\|_{L^2} + \|u_0 - u''_0\|_{L^2} < \delta$, then $\forall t \in [0, T_{lwp}]$
   \[
   \|u'(t) - u''(t)\|_{H^1} \leq 2\|u'_0 - u''_0\|_{H^1},
   \]
   \[
   \|u'(t) - u''(t)\|_{H^{0,s}} \leq \|u'_0 - u''_0\|_{H^{0,s}} + 2\|u'_0\|_{H^1} + 2t\|u'_0\|_{H^1}.
   \]

**Remark 2.2.** The solutions $u'$ and $u''$ are thought of as perturbations from $u$. The original solution $u$ appears in the statement of this theorem because it provides both the local well-posedness interval, $[0, T_{lwp}]$, on which we wish to control $u'$ and $u''$, and the $L^2$ estimates with which we can achieve this control.

**Proof.** From the local well-posedness Theorem 1.3, $T_{lwp}$ can be chosen small enough so that $\|u\|_{H^{0.5}\cap H^1((0,T_{lwp}))} \leq 2\|u_0 - u'_0\|_{L^2} < 2\delta$ and $\|u\|_{H^{0.5}\cap H^1((0,T_{lwp}))} < 2\delta + \frac{1}{2}\delta_2$. The function $u'$ can now be taken as the solution to estimate and from which $u''$ is a perturbation. If $2\delta + \frac{1}{2}\delta_2 \leq \delta_3$, then the $H^s$ local well-posedness Theorem 1.3 provides the estimates on the growth and separation in $H^1$.

Differentiation in time and the Cauchy-Schwartz estimate gives the growth of the weighted norms.

\[
\frac{d}{dt}\|u'(t)\|_{H^{0,s}}^2 = \frac{d}{dt}\langle u', (x)^{2s}u' \rangle.
\]

\[
2\|u'(t)\|_{H^{0,s}} \frac{d}{dt}\|u'(t)\|_{H^{0,s}} = \langle i\Delta u' - i\lambda|u'|^{2s}u', (x)^{2s}u' \rangle + \langle u', (x)^{2s}i\Delta u' - i\lambda|u'|^{2s}u' \rangle
\]

\[
= -i\langle u', (x)^{2s}u' \rangle + i\lambda\langle u', (x)^{2s}\Delta u' \rangle + i\lambda\langle u', (x)^{2s}\Delta u' \rangle - i\lambda\langle u', (x)^{2s}|u'|^{2s}u' \rangle
\]

\[
= \langle u', i\Delta^2 (x)^{2s}|u'|^2u' \rangle.
\]

\[
|2\|u'(t)\|_{H^{0,s}} \frac{d}{dt}\|u'(t)\|_{H^{0,s}} | \leq | \langle u', -4is(x)^{2s-2}Du' - 2is(x)^{2s-2}((2s-2)\frac{x^2}{(x)^2} + 1)u' \rangle |
\]

\[
\leq C\|u\|_{H^{0,s}}\|Du'\|_{L^2} + C\|u\|_{H^{0,s}}\|u\|_{L^2}.
\]

\[
\frac{d}{dt}\langle x\rangle u'^2 \leq C\|Du'\|_{L^2} + C\|u\|_{L^2} \leq C\|u\|_{H^1}.
\]
This proves that
\[ \|u'(t)\|_{H^{0,s}} \leq \|u'_0\|_{H^{0,s}} + Ct\|u'_0\|_{H^1}. \]

A similar calculation shows
\[ \|u'(t) - u''(t)\|_{H^{0,s}} \leq \|u'_0 - u''_0\|_{H^{0,s}} + \|u'(t) - u'_0\|_{H^{0,s}} + \|u''(t) - u''_0\|_{H^{0,s}} \]
\[ \leq \|u'_0 - u''_0\|_{H^{0,s}} + C(t\|u'_0\|_{H^1} + \|u''_0\|_{H^1}). \]

\[ \square \]

2.2. \( C : H^{0,s} \cap H^1 \to H^s. \) In this section, it is shown that the pseudoconformal transform takes a function \( u(t) \in H^{0,s} \cap H^1 \) to \( v(-t^{-1}) \in H^s. \) This is done by interpolation between \( L^2 \) and \( H^{0,1} \cap H^1 \) using the \( K \) method of real interpolation.

To begin, the arguments for \( L^2 \) and \( H^{0,1} \cap H^1 \) are presented. The \( L^2 \) result is part of Theorem \( \text{[1.8]} \) The \( H^{0,1} \cap H^1 \) result leads to equation \( \text{[1.3]} \), which was stated in the introduction.

**Lemma 2.3.** If \( u : \{t\} \times \mathbb{R}^d \to \mathbb{C} \) and \( v = C[u] \), then

\begin{align*}
(2.1) & \quad \|v(-t^{-1})\|_{L^2} = \|u(t)\|_{L^2}, \\
(2.2) & \quad \|v(-t^{-1})\|_{H^1} \leq \|u(t)\|_{H^{0,1}} + t\|u(t)\|_{H^1},
\end{align*}

where it is understood that the norm on the right is infinite (and the inequality trivial) if \( u(t) \) does not belong to the appropriate space.

**Proof.** Equation \( (2.1) \) is statement 2 of Lemma \( \text{[1.8]} \).

Inequality \( (2.2) \) follows by direct computation with a change of variables. This computation is simplified by recalling the notation \( \tau = -t^{-1} \) and \( ty = x. \)

\[ \|v(-t^{-1})\|_{H^1}^2 = \int |\nabla_y v(-t^{-1}, y)|^2 dy \]
\[ = \int |\nabla_y (t^2 e^{-\frac{ty^2}{4}} u(t, ty))|^2 dy \]
\[ = \int t^4 |\frac{1}{2} itye^{-\frac{ty^2}{4}} u(t, ty) + e^{-\frac{ty^2}{4}} \nabla_y u(t, ty)|^2 dy \]
\[ \leq \int |tyu(t, ty)|^2 t^4 dy + \int 2|\nabla_y u(t, ty)|^2 t^4 dy \]
\[ \leq \int |xu(t, x)|^2 dx + 2 \int t^2|\nabla_x u(t, x)|^2 dx. \]

\[ \square \]

The \( K \) method of real interpolation is now summarized from \( \text{[1]} \). The \( s \) interpolation norm of \( a \in A_0 + A_1 \) is defined by the following, if this norm is finite,

\[ (2.3) \quad K(\lambda, a; A_0, A_1) = \inf_{a_0 + a_1 = a} (\|a_0\|_{A_0}^2 + \lambda^2\|a_1\|_{A_1})^2 \]
\[ (2.4) \quad \|a\|_{s,(A_0, A_1)} = \Phi_s^2(K(\lambda, a; A_0, A_1)) \]
\[ = \int \lambda^{-2s-1} K(\lambda, a; A_0, A_1)^2 d\lambda. \]

Since only the \( K \) method of interpolation will be introduced, the \( K \) index in the norm will be omitted \( \|a\|_{s,(A_0, A_1)} = \|a\|_{s,(A_0, A_1)};K. \) If \( a \in A_0 \cap A_1 \), then
\[ \|a\|_{s,(A_0, A_1)} \leq \|a\|_{A_0}^{1-s} \|a\|_{A_1}^s. \]
The interpolation space \((A_0, A_1)_s\) is defined as the set of \(a \in A_0 + A_1\) for which \(\|a\|_{s, (A_0, A_1)}\) is finite. There are some technical issues, but since only spaces \(A_0\) and \(A_1\) which are subsets of \(L^2\) will be considered, \((A_0, A_1)_s\) will be well-defined, a Banach space, and the closure of \(A_0 \cap A_1\).

The \(K\) method of real interpolation is an exact interpolation method of exponent \(s\) [1]: if

\[
T : A_0 \to B_0 \quad \|T\|_{A_0 \to B_0} \leq M_0
\]

\[
T : A_1 \to B_1 \quad \|T\|_{A_1 \to B_1} \leq M_1,
\]

then

\[
T : (A_0, A_1)_s \to (B_0, B_1)_s
\]

\[
\|T\|_{(A_0, A_1)_s \to (B_0, B_1)_s} \leq M_0^{1-s} M_1^s.
\]

(2.5)

It is known that

\[
(L^2, H^1)_s = H^s
\]

\[
(L^2, H^{0,1})_s = H^{0,s}.
\]

By interpolating the results of Lemma 2.3 it follows that

\[
C : (L^2, H^{0,1} \cap H^1)_s \to H^s.
\]

Unfortunately, because of the inf in (2.3), it is not clear that \(\|a\|_{s, (L^2, H^{0,1} \cap H^1)} = \|a\|_{s, (L^2, H^{0,1})} + \|a\|_{s, (L^2, H^1)} = \|a\|_{H^{0,s}} + \|a\|_{H^1}\); although, we expect this is true. We will instead prove the simpler result that

\[
\|a\|_{s, (L^2, H^{0,1} \cap H^1)} \lesssim \|a\|_{H^{0,s}} + \|a\|_{H^1}.
\]

To ensure that the \(t\) dependent coefficients only appear on the \(H^1\) norm, the \(t\) dependence is kept in the interpolation calculations rather than being estimated by \(2.4\).

**Lemma 2.4.** If \(u : \{t\} \times \mathbb{R}^d \to \mathbb{C}, u(t) \in H^{0,s} \cap H^1\), and \(v = C[u]\), then

\[
\|v(-t^{-1})\|_{H^s} \leq C_{1,s} \|u\|_{H^{0,s}} + C_{2,s} t \|u\|_{H^1},
\]

where the constants \(C_{1,s}\) and \(C_{2,s}\) depend only on \(s\) and \(d\).

**Proof.** Let \(u : \{t\} \times \mathbb{R}^d \to \mathbb{C}\) with \(u(t)\) in Schwartz class. Using the \(K\)-method, it will be shown that the \(H^s\) norm is dominated by the \(H^{0,s}\) and \(H^1\) norms. The \(K\)-method of interpolation involves taking an infimum over all possible decompositions of \(u\). This infimum is dominated by any particular choice of decomposition. The decomposition which is optimal for balancing \(L^2\) with \(H^{0,1}\) will be used. This will give the \(H^{0,s}\) part of the estimate. There is no reduction in the regularity required for the estimate, since this decomposition ignores the \(H^1\) term.

\[
\|v(-t^{-1})\|_{H^s} = \|C[u](-t^{-1})\|_{H^s} = \int \lambda^{-2s-1} K(\lambda, C[u]; L^2, H^1)^2 d\lambda
\]

Using Lemma 2.3 \(K(\lambda, C[u])\) can be estimated in terms of the \(L^2\), \(H^{0,1}\), and \(H^1\) norms of \(u\). (Note that in this proof, \(u_0\) refers to part of the interpolation.
decomposition in [2,3], not the initial data.)

\[ K(\lambda, C[u]) = \inf_{u=u_0+u_1} (\|C[u_0]\|_{L^2}^2 + \lambda^2\|C[u_1]\|_{H^s}^2)^{\frac{1}{2}} \]

\[ \leq \inf_{u=u_0+u_1} (\|u_0\|_{L^2}^2 + \lambda^2\|u_1\|_{H^s}^2 + \lambda^2\|u_1\|_{H^s}^2)^{\frac{1}{2}}. \]

In the proof that \( (L^2, H^{0,1})_s = H^{0,s} \) [1], it is shown that the optimal decomposition for \( \|u_0\|_{L^2} + \lambda^2\|u_1\|_{H^{0,1}} \) is

\[ u_0 = \frac{\lambda^2(1 + x^2)}{1 + \lambda^2(1 + x^2)} u \]

\[ u_1 = \frac{1}{1 + \lambda^2(1 + x^2)} u. \]

This decomposition will be used for \( \lambda < 1 \) to bound \( K(\lambda, C[u]) \) from above.

\[ K(\lambda, C[u])^2 \leq \|u_0\|_{L^2}^2 + \lambda^2\|u_1\|_{H^{0,1}}^2 + \lambda^2\|u_1\|_{H^s}^2 \]

\[ \leq \int \left( \frac{\lambda^2(1 + x^2)^2}{(1 + \lambda^2(1 + x^2))^2} + \frac{1}{(1 + \lambda^2(1 + x^2))^2} \lambda^2(1 + x^2)|u|^2d^2x \right) \]

\[ + \lambda^2 t^2 \int |\nabla\left( \frac{1}{1 + \lambda^2(1 + x^2)} u \right)|^2dx \]

\[ \leq \int \left( \frac{\lambda^2(1 + x^2)}{1 + \lambda^2(1 + x^2)} |u|^2dx \right) \]

\[ + \lambda^2 t^2 \int |\nabla\left( \frac{1}{1 + \lambda^2(1 + x^2)} u \right)|^2dx. \]

This decomposition can be used to bound the \( H^s \) norm of \( v = C[u] \) for \( \lambda < 1 \). The decomposition \( u_0 = u \) and \( u_1 = 0 \) will be used for \( \lambda \geq 1 \).

\[ \|v(t^{-1})\|_{H^s}^2 \leq \int_0^1 \lambda^{-2s-1} K(\lambda, C[u])d\lambda + \int_1^\infty \lambda^{-2s-1} K(\lambda, C[u])d\lambda \]

\[ \leq \int_0^1 \lambda^{-2s-1} \int_{\mathbb{R}^2} \frac{\lambda^2(1 + x^2)}{1 + \lambda^2(1 + x^2)} |u|^2dxd\lambda \]

\[ + \int_0^1 \lambda^{-2s-1} \int \lambda^2 t^2 |\nabla_x\left( \frac{1}{1 + \lambda^2(1 + x^2)} u \right)|^2dxd\lambda \]

\[ + \int_1^\infty \lambda^{-2s-1} ||u_0||_{L^2}^2d\lambda \]

\[ \leq \int_0^1 \lambda^{-2s-1} \int_{\mathbb{R}^2} \frac{\lambda^2(1 + x^2)}{1 + \lambda^2(1 + x^2)} |u|^2dxd\lambda \]

\[ + \int_0^1 \lambda^{-2s+1} t^2 \int \lambda |u|^2 + |\nabla_x u|^2dxd\lambda \]

\[ + C_s \|u\|_{L^2}^2. \]

At this stage, the first term is evaluated by the substitution \( \lambda^2 = (1 + x^2)\lambda^2 \) and Fubini’s theorem. The other two pieces are estimated by direct integration and
Schrödinger evolution is controlled by $L^s$. Thus, if $t < 14$ then the pseudoconformal transformation preserves Strichartz admissible norms, \[
\|v(-t^{-1})\|_{H^s}^2 \leq \int_{\mathbb{R}^2} (1 + x^2)^s \|u\|^2 \int_0^t \lambda^{-2s-1} \frac{\lambda^2}{1 + \lambda^2} d\lambda dx + C t^2 \|u\|_{H^s}^2 + C'(1 + t^2) \|u\|_{L^2}^2 \]
\[
\leq C_{1,s} \|u\|_{H^{0,s}}^2 + C_{2,s} t^2 \|u\|_{H^1}^2.
\]

2.3. $S_{NLS}(-T_{lwp}\,^1_{LWP}, -t^{-1}) : H^s \to H^s$. Since the pseudoconformal transform preserves Strichartz admissible norms, \[
\|v\|_{L^t_{tx} \frac{2(d+2)}{d} ([t^{-1}, -T_{lwp}^{-1}])} \quad \text{is controlled by} \quad \|u\|_{L^t_{tx} \frac{2(d+2)}{d} ([0, T_{lwp}])}
\]
and can be taken to be small. Since the growth of the $H^s$ norm under the nonlinear Schrödinger evolution is controlled by $L^s_{tx \frac{2(d+2)}{d}}$, $\|v(-T_{lwp}^{-1})\|_{H^s}$ can be controlled by $\|v(-t^{-1})\|_{H^s}$ for any $t \in [0, T_{lwp}]$. The divergence of nearby solutions can be similarly controlled. From this, if $u_0^0$ and $u_0''$ are $H^{0,s} \cap H^1$ approximations of $u_0 \in H^{0,s}$, then the $H^s$ distance between $v'$ and $v''$ at transformed time $-T_{lwp}^{-1}$ is controlled by the distance between them at earlier transformed times.

**Lemma 2.5.** For $u$ a solution to the nonlinear Schrödinger equation \((1.1)\) with initial data $u_0 \in L^2$, there is a local well-posedness time $T_{lwp}$ and a $\delta$ coming from the local well-posedness theory such that:

1. if $u'$ is a solution to the nonlinear Schrödinger equation on $[0, T_{lwp}]$ with initial data $u_0' \in H^{0,s} \cap H^1$, with $\|u_0 - u_0'\|_{L^2} < \delta$ and with a $t \in [0, T_{lwp}]$ such that $v'(-t^{-1}) \in H^s$, then
\[
\|v'(-T_{lwp}^{-1})\|_{H^s} < 2\|v'(-t^{-1})\|_{H^s};
\]

2. if $u'$ and $u''$ are solutions to the nonlinear Schrödinger equation on $[0, T_{lwp}]$ with initial data $u_0' \in L^2$ and $u_0'' \in L^2$ respectively, with $\|u_0 - u_0'\|_{L^2} < \delta$ and $\|u_0 - u_0''\|_{L^2} < \delta$, and with a $t \in [0, T_{lwp}]$ such that $v'(-t^{-1}) \in H^s$ and $v''(-t^{-1}) \in H^s$, then
\[
\|v'(-T_{lwp}^{-1}) - v''(-T_{lwp}^{-1})\|_{H^s} < 2\|v'(-t^{-1}) - v''(-t^{-1})\|_{H^s}.
\]

**Remark 2.6.** As in Lemma 2.1, $u$ appears in this theorem because we want to control the size of $L^s_{tx \frac{2(d+2)}{d}}$ norms for the approximators, $u'$ and $u''$. Proof. From the $L^2$ local well-posedness Theorem 1.3, $T_{lwp}$ can be chosen small enough so that $\|u\|_{L^t_{tx} \frac{2(d+2)}{d} ([0, T_{lwp}])}$ is less than half of $\delta_3$ from the $H^s$ local well-posedness Theorem 1.4. In this case, by the $L^2$ local well-posedness Theorem 1.3
\[
\|u - u''\|_{L^t_{tx} \frac{2(d+2)}{d} ([0, T_{lwp})]} \leq 2\|u_0 - u_0''\|_{L^2} < 2\delta \quad \text{and} \quad \|u''\|_{L^t_{tx} \frac{2(d+2)}{d} ([0, T_{lwp})]} < 2\delta + \frac{3}{2}\delta_3.
\]

The function $u'$ or $v'$ can now be taken as the solution to estimate. Since the pseudoconformal transformation preserves the Strichartz admissible norms, \[
\|v'\|_{L^t_{tx} \frac{2(d+2)}{d} ([t^{-1}, -T_{lwp}^{-1}])} \leq \|v'\|_{L^t_{tx} \frac{2(d+2)}{d} ([t^{-1}, -T_{lwp}^{-1}])} \leq \|u''\|_{L^t_{tx} \frac{2(d+2)}{d} ([0, T_{lwp})]} < 2\delta + \frac{3}{2}\delta_3.
\]

Thus, if $2\delta + \frac{3}{2}\delta_3 \leq \delta_3$, then, by the $H^s$ local well-posedness Theorem 1.4
\[
\|v'\|_{S((t^{-1}, -T_{lwp}^{-1}))} < 2\|v'(-t^{-1})\|_{H^s}.
\]
If \( \|u_0 - u'_0\|_{L^2} \leq \delta \) and \( \|u_0 - u''_0\|_{L^2} \leq \delta \), then \( \|u'_0 - u''_0\|_{L^2} \leq 2\delta \). Again, by the \( H^s \) local well-posedness Theorem \[2.4\] if, in addition to the previous conditions, \( 2\delta \leq \delta_4 \), then
\[
\|v'(T_{lwp}^{-1}) - v''(T_{lwp}^{-1})\|_{H^s} \leq \|v' - v''\|_{S^s([-T_{lwp}^{-1}, -T_{lwp}^{-1}])} \leq 2\|v'(t^{-1}) - v''(t^{-1})\|_{H^s}.
\]
\( \square \)

2.4. \( F : H^{0,s} \cap H^1 \rightarrow H^s \). The results from Sections \[2.1\] \[2.2\] and \[2.3\] are now combined to show that \( F = F_t = S_{NLS}(T_{lwp}^{-1}, -t^{-1}) \circ C \circ S_{NLS}(t, 0) \) takes \( H^{0,s} \cap H^1 \) to \( H^s \). Furthermore, an explicit \( t \) dependence on the \( H^s \) norm will be found in Lemma \[2.7\] and then removed in Proposition \[2.8\] to show that \( F \) takes a \( H^{0,s} \) neighborhood of \( u_0 \) into \( H^s \).

We will first show that for \( u_0 \in H^{0,s} \), if we restrict attention to initial data which is both in a \( H^{0,s} \) neighborhood, \( N \), of \( u_0 \) and in \( H^1 \), then \( F \) maps this initial data in \( N \cap H^1 \) to \( H^s \).

**Lemma 2.7.** For \( u \) a solution to the nonlinear Schrödinger equation \[1.1\] with initial data \( u_0 \in H^{0,s} \), then there is a local well-posedness time \( T_{lwp} \) and a \( \delta \) coming from the local well-posedness theory such that:

1. if \( u' \) is a solution to the nonlinear Schrödinger equation with initial data \( u'_0 \in H^{0,s} \cap H^1 \) and \( \|u_0 - u'_0\|_{L^2} < \delta \), then for all \( t \in (0, T_{lwp}] \) we have

\[
\|v'(T_{lwp}^{-1})\|_{H^s} \leq C_s\|u'_0\|_{H^{0,s}} + C'_s t\|u'_0\|_{H^1}.
\]

2. if \( u' \) and \( u'' \) are solutions to the nonlinear Schrödinger equation with initial data \( u'_0 \in H^{0,s} \cap H^1 \) and \( u''_0 \in H^{0,s} \cap H^1 \) respectively and with \( \|u_0 - u'_0\|_{L^2} < \delta \) and \( \|u_0 - u''_0\|_{L^2} < \delta \), then for all \( t \in (0, T_{lwp}] \) we have

\[
\|v'(T_{lwp}^{-1}) - v''(T_{lwp}^{-1})\|_{H^s} \leq C_s\|u'_0 - u''_0\|_{H^{0,s}} + C'_s t\|u'_0\|_{H^1} + C''_s t\|u''_0\|_{H^1}.
\]

In other words, there is an open set \( N \in H^{0,s} \) containing \( u_0 \) for which
\[
F_t = S_{NLS}(T_{lwp}^{-1}, -t^{-1}) \circ C \circ S_{NLS}(t, 0) : N \cap H^1 \rightarrow H^s
\]
and \( F_t \) is continuous with respect to the \( H^{0,s} \cap H^1 \) topology.

**Proof.** Conditions on \( \delta \) and \( T_{lwp} \) will be found. To begin, assume \( T_{lwp} < 1 \).

Since \( u_0 \in H^{0,s} \) and \( u'_0 \) and \( u''_0 \) are \( H^{0,s} \cap H^1 \) approximations, by Lemma \[2.1\] if \( \delta \) is less than the \( \delta \) in Lemma \[2.1\] then, for \( t \in [0, T_{lwp}] \),
\[
\|u'(t)\|_{H^1} \leq 2\|u'_0\|_{H^1},
\]
\[
\|u'(t)\|_{H^{0,s}} \leq \|u'_0\|_{H^{0,s}} + 2t\|u'_0\|_{H^1},
\]
\[
\|u'(t) - u''(t)\|_{H^1} \leq 2\|u'_0\|_{H^1} + 2\|u''_0\|_{H^1},
\]
\[
\|u'(t) - u''(t)\|_{H^{0,s}} \leq \|u'_0 - u''_0\|_{H^{0,s}} + 2t\|u'_0\|_{H^1} + 2t\|u''_0\|_{H^1}.
\]

By Lemma \[2.3\] the linearity of the pseudoconformal transform and the triangle inequality,
\[
\|v'(t^{-1})\|_{H^s} \leq C\|u'_0\|_{H^{0,s}} + tC'\|u'_0\|_{H^1},
\]
\[
\|v'(t^{-1}) - v''(t^{-1})\|_{H^s} \leq C\|u'_0 - u''_0\|_{H^{0,s}} + tC'\|u'_0\|_{H^1} + tC''\|u''_0\|_{H^1}.
\]
Since \( u_0 \) is a solution with initial data in \( H^{0,s} \), \( u'_0 \) and \( u''_0 \) are \( H^{0,s} \cap H^1 \) approximations, and \( v' \) and \( v'' \) are in \( H^s \) at transformed time \(-t^{-1}\), if \( T_{lwp} \) and \( \delta \) are less than the corresponding values in Lemma 2.5, then

\[
\|v'(T_{lwp}^{-1})\|_{H^s} \leq C\|u'_0\|_{H^{0,s}} + tC', \|u''_0\|_{H^1},
\]

\[
\|v'(T_{lwp}^{-1}) - v''(T_{lwp}^{-1})\|_{H^s} \leq C\|u'_0 - u''_0\|_{H^{0,s}} + tC'\|u''_0\|_{H^1} + tC'\|u'_0\|_{H^1}.
\]

Since \( F_t : u_0 \mapsto v(-T_{lwp}^{-1}) \) and the set \( \|u_0 - u'_0\|_{L^2} < \delta \) is open in \( H^{0,s} \), this set is the \( N \) given in the statement of the theorem. By (2.7), \( F_t \) is continuous from \( H^{0,s} \cap H^1 \) to \( H^s \).

The infimum in \( t \) can be taken when estimating the \( H^s \) norm of \( F_t(u_0) = v'(T_{lwp}^{-1}) \). Since \( F = F_t : u_0 \mapsto v(-T_{lwp}^{-1}) \) is independent of \( t \), this eliminates the \( H^1 \) dependence. Eliminating the \( H^1 \) dependence shows that \( F \) is continuous from \( H^{0,s} \) to \( H^s \). If \( u_0 \) is approximated in \( H^{0,s} \) by a sequence of regularized initial data \( u^{[i]}_0 \) in \( H^{0,s} \cap H^s \), then the corresponding \( v^{[i]}(-T_{lwp}^{-1}) \) must converge in \( H^s \) to \( v(-T_{lwp}^{-1}) \). This proves that \( v(-T_{lwp}^{-1}) \) is in \( H^s \). Under the assumption of \( H^s \) global existence, \( v \) and \( u \) can exist globally.

**Proposition 2.8.** Assume that the nonlinear Schrödinger equation (1.1) is globally well-posed in \( H^s \) (with the additional hypothesis that the initial data has \( L^2 \) norm bounded by \( \|Q\|_{L^2} \) in the focusing case).

If \( u_0 \in H^{0,s} \) (and \( \|u_0\|_{L^2} < \|Q\|_{L^2} \) in the focusing case), then there is a function \( u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C} \) which solves the nonlinear Schrödinger equation (1.1) for all time.

From the local well-posedness theory, this solution remains in \( L^2 \), at each time, with constant norm and is the unique solution in the Strichartz space \( S^0 \).

**Proof.** For \( u_0 \in H^{0,s} \) at \( t_0 = 0 \), by Lemma 2.7 there is a solution \( u \) with local existence time \( T_{lwp} \). If \( u'_0 \) and \( u''_0 \) are in \( H^{0,s} \cap H^1 \) and sufficiently close to \( u_0 \) in \( H^{0,s} \), then, since the map \( F : u_0 \mapsto v(-T_{lwp}^{-1}) \) is independent of \( t \), it is possible to apply the infimum in \( t \) to (2.6) and (2.7) and obtain

\[
\|v'(T_{lwp}^{-1})\|_{H^s} \leq C_s\|u'_0\|_{H^{0,s}},
\]

\[
\|v'(T_{lwp}^{-1}) - v''(T_{lwp}^{-1})\|_{H^s} \leq C_s\|u'_0 - u''_0\|_{H^{0,s}}.
\]

Since \( H^{0,s} \cap H^1 \) is dense in \( H^{0,s} \), and \( F \) is continuous with respect to the \( H^{0,s} \) norm, if a sequence \( u^{[i]}_0 \in H^{0,s} \cap H^1 \) is chosen to converge to \( u_0 \) in \( H^{0,s} \), the \( v^{[i]}(-T_{lwp}^{-1}) \) must converge to a function \( \tilde{v} \in H^s \) and to \( v(-T_{lwp}^{-1}) \) in \( L^2 \). Since \( H^s \) is dense in \( L^2 \), \( v(-T_{lwp}^{-1}) = \tilde{v} \in H^s \).

Since the nonlinear Schrödinger evolution and the pseudoconformal transform both preserve the \( L^2 \) norm, if \( \|u_0\|_{L^2} < \|Q\|_{L^2} \), then \( \|v(-T_{lwp}^{-1})\|_{L^2} < \|Q\|_{L^2} \).

Therefore, in the defocusing case, from the assumption of global well-posedness in \( H^s \), \( v \) extends to a function \( v : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C} \) with \( v : \mathbb{R} \to H^s(\mathbb{R}^d) \). For \( t > 0 \), \( u \) can be defined by \( u = C^{-1}[v] \). By Theorem 1.8 this extension of \( u \) is a solution to the nonlinear Schrödinger equation on \([0, \infty)\). For \( t < 0 \) all the arguments of the paper can be reproduced to define \( u \) on \((\infty, 0)\). Thus, \( u \) is a solution to the nonlinear Schrödinger equation, has initial data \( u_0 \), and is defined for all \( t \).

In the focusing case, since the nonlinear Schrödinger evolution and the pseudoconformal transform both preserve the \( L^2 \) norm, if \( \|u_0\|_{L^2} < \|Q\|_{L^2} \), then \( \|v(-T_{lwp}^{-1})\|_{L^2} < \|Q\|_{L^2} \).
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\[ \|Q\|_{L^2} \] and the same argument can be applied with the additional \( L^2 \) norm hypothesis.

Remark 2.9. The process of taking \( u \) on \([0, T_{\text{wp}}]\), applying the pseudoconformal transform to get \( v \) on \((-\infty, -T_{\text{wp}}]\), and then extending \( v \) globally in time provides a function \( v \) which is defined for positive time. We remark that there is no clear relation between \( v \) at positive time and \( u \) at negative time. In some sense, \( v \) at positive time corresponds to the evolution of \( u \) “beyond infinite”, and, unless it is known a priori that the scattering states \( u_+ \) and \( u_- \) satisfy

\[ \lim_{t \to \infty} \|e^{-it\Delta}u_+ - e^{-it\Delta}u_-\|_{L^2} = 0, \]

there is no reason to believe that \( v \) at positive transformed time corresponds to \( u \) at negative time.

3. A SCATTERING LEMMA

Lemma 3.1. If \( u \) is a solution of the nonlinear Schrödinger equation (1.1) which exists globally and for which \( v \) exists globally as well, then

\[ \|u\|_{S^0([0, \infty))} \leq \infty \]

and there exists a \( u_+ \) for which

\[ \lim_{t \to \infty} \|u(t) - e^{it\Delta}u_+\|_{L^2} = 0. \]

Proof. If \( u \) and \( v \) exist globally, then, in particular they exist for \( t \in [0, 1] \) and \( \tau \in [-1, 0] \) respectively. By the maximal time blow up Theorem 1.5, \( \|u\|_{S^0([1, \infty))} = \|v\|_{S^0((-1, 0])} < \infty \). Thus \( \|u\|_{S^0([0, \infty))} < \infty \). By Duhamel’s principle and the Strichartz estimates, this means that \( \lim_{t \to \infty} \|u(t) - e^{it\Delta}u_+\|_{L^2} = 0. \)

Remark 3.2. As in Remark 2.9, it is necessary to distinguish between the forward in time extension of the pseudoconformal transform and the backwards in time extension of the pseudoconformal transform. If \( \tilde{v} \) denotes the pseudoconformal transform of \( u \) at negative time, then if \( v \) and \( \tilde{v} \) both extend globally in time, by the same argument as in this lemma, \( \|u\|_{S^0((-\infty, 0])} < \infty \) and there is a \( u_- \) for which \( \lim_{t \to -\infty} \|u(t) - e^{-it\Delta}u_-\|_{L^2} = 0 \).

Corollary 3.3. Under the hypotheses of Proposition 1.1, there are functions \( u_\pm \in H^{0,s} \) such that

\[ \lim_{t \to \pm\infty} \|e^{\mp it\Delta}u(t) - u_\pm\|_{H^{0,s}} = 0. \]

Proof. The construction in the proof of Proposition 2.8 shows that both \( u \) and \( v \) exist globally, and hence \( u \) scatters forward in time by Lemma 3.1. As noted in Remark 2.9, the same occurs backwards in time. This establishes the existence of \( u_\pm \in L^2 \).

We now introduce linearly advanced and retarded versions of \( u \) and \( v \). These have two time variables, one to record the time variable associated with the nonlinear Schrödinger evolution, and one for the advancement or retardation by the linear Schrödinger evolution.

\[ \phi(t, t') = e^{i(t'-t)\Delta}u(t) \]
\[ \psi(\tau, \tau') = e^{i(\tau'-\tau)\Delta}v(\tau) \]
The function $\phi(t, \bullet)$ is a linear solution with initial data $u(t)$ at time $t' = t$. The function $\psi(\tau, \bullet)$ is the analogous solution with initial data $v(\tau)$ at time $\tau' = \tau$. Since $v(-t^{-1})$ is the pseudoconformal transform of $u(t)$ at time $t$, and the pseudoconformal transform preserves the linear Schrödinger evolution, the pseudoconformal transform of $\phi(t, \bullet)$ with respect to the spatial variable and the second time variable is $\psi(-t^{-1}, \bullet)$.

Denoting a solution to the linear Schrödinger equation by $\phi$ and its pseudoconformal transform by $\psi$, it is known that $\psi(0)$ is the Fourier transform of $\phi(0)$. If $u_0 \in H^{0,s}$, by the construction in Section 2, then $v(\tau) \in H^{s}$. Since the linear Schrödinger evolution preserves the $H^{s}$, for all $\tau'$, $\psi(\tau, \tau') \in H^{s}$, and, in particular $\psi(\tau, 0) \in H^{s}$. This proves that the linearly retarded version of $u$ evolves in $H^{0,s}$.

$$e^{-it\Delta}u(t) = \phi(t, 0) = \mathcal{F}[\psi(\tau, 0)] \in H^{0,s}$$

Since $e^{-it\Delta}u(t)$ evolves in $H^{0,s}$, it is now possible to consider scattering in $H^{0,s}$, despite the fact that $u(t)$ itself does not evolve in $H^{0,s}$. From the $L^{2}$ scattering, it follows that $u_{\pm} = \mathcal{F}[\psi(0, 0)] = \mathcal{F}[v(0)]$. It now remains to show this limit holds in $H^{0,s}$.

$$\|e^{-it\Delta}u(t) - u_{\pm}\|_{H^{0,s}} = \|\phi(t, 0) - \mathcal{F}[v(0)]\|_{H^{0,s}}$$

$$= \|\psi(\tau, 0) - v(0)\|_{H^{s}} = \|e^{-it\Delta}v(\tau) - v(0)\|_{H^{s}}$$

$$= \|v(\tau) - e^{it\Delta}v(0)\|_{H^{s}}$$

$$\lim_{t \to \infty} \|e^{-it\Delta}u(t) - u_{\pm}\|_{H^{0,s}} = \lim_{\tau \to 0^{+}} \|v(\tau) - e^{it\Delta}v(0)\|_{H^{s}} \to 0$$

The same argument holds as $t \to -\infty$, with the usual remark on the difference between the pseudoconformal transforms for positive and negative times.

□

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Errata

In this paper, we consider the initial value problem for the $L^2$-critical nonlinear Schrödinger equation for $u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$:

\begin{equation}
\begin{aligned}
    i\partial_t u + \Delta u &= \lambda |u|^4 u \\
    u(t_0, x) &= u_0(x),
\end{aligned}
\end{equation}

with initial data $u_0$ in $H^{0,s}$, the weighted $L^2$ space with norm

$$
    \|\psi\|_{H^{0,s}} = \left( \int_{\mathbb{R}^d} (1 + x^2)^s |\psi(x)|^2 \, dx \right)^{1/2}.
$$

Our main result was that if equation (3.1) is globally well-posed for initial data in the $H^s$ Sobolev space, then all initial data $u_0 \in H^{0,s}$ generates global solutions $u \in L^2_t(L^2_x)$. We also proved that such solutions scatter.

The proof presented in Section 2 approximates $u_0 \in H^{0,s}$ by a sequence of regularised approximators in $H^{0,s} \cap H^1$. It has been brought to our attention that there appears to be an error in the final line of the proof of Lemma 2.1. In particular, although the growth of the $H^{0,s}$ norms of the approximators is controlled by the $H^1$ norm, it seems that the growth of the divergence of two approximators in $H^{0,s}$, $\|u' - u''\|_{H^{0,s}}$ is not controlled by the $H^1$ norms of the approximators. Fortunately, the same argument works if we replace $H^{0,s} \cap H^1$ by $H^{0,s} \cap H^p$ with $p = (d+1)/2$.

In the proof of Lemma 2.1, the application of Theorem 1.4 applies equally well to the persistence of regularity in $H^p$ as it did in $H^1$ in the original proof. Thus, the same norm bound applies. The calculation of the growth of the $H^{0,s}$ norm of $u'$ is correct, and, since $1 \leq (d + 1)/2$, we can trivially replace the $H^1$ norm by the $H^p$ norm. In the final calculation of Lemma 2.1, where there seems to be an error, we can sill use $\langle \cdot, \cdot \rangle$ for the $L^2$ inner product and $\langle x \rangle$ for $(1 + x^2)^{1/2}$ and make the estimate:

\begin{equation}
    \frac{d}{dt}\|u'(t) - u''(t)\|_{H^{0,s}}^2 = i\langle \Delta (u' - u''), \langle x \rangle^{2s}(u' - u'') \rangle \\
    - i\langle (u' - u''), \langle x \rangle^{2s} \Delta (u' - u'') \rangle \\
    - \lambda \langle |u'|^{\frac{4}{d}u' - |u''|^{\frac{4}{d}u', \langle x \rangle^{2s}(u' - u'')} \rangle \\
    + \lambda \langle u' - u'', \langle x \rangle^{2s}(|u'|^{\frac{2}{d}u' - |u''|^{\frac{2}{d}u''}) \rangle.
\end{equation}

The first two terms on the right can be estimated as in the estimate of the norm of $u'$, so that they are dominated by

\begin{equation}
    C\|u' - u''\|_{H^{0,s}}\|u' - u''\|_{H^1}.
\end{equation}

The argument which needs correcting concerns the estimate on the remaining terms. These are controlled by

\begin{align*}
    C\|(|u'|^{\frac{2}{d}u' - |u''|^{\frac{2}{d}u'', \langle x \rangle^{2s}(u' - u'')} |)
    \leq C\|\langle x \rangle^{s}(u' - u'')\|_{L^2}\|\langle x \rangle^{s}(|u'|^{\frac{2}{d}u' - |u''|^{\frac{2}{d}u''})\|_{L^2} \\
    \leq C\|u' - u''\|_{H^{0,s}}\left(\|\langle x \rangle^{s}u'\|_{L^2}\|u'\|_{L^\infty} + \|\langle x \rangle^{s}u''\|_{L^2}\|u''\|_{L^\infty}\right) \\
    \leq C\|u' - u''\|_{H^{0,s}}\left(\|u'\|_{H^{0,s}}\|u'\|_{H^1} + \|u''\|_{H^{0,s}}\|u''\|_{H^p}\right).
\end{align*}
Here we have used that $\rho > d/2$, to control the $L^\infty$ norm. By applying the earlier estimate on the growth of the $H^\rho$ and $H^{0,s}$ norms and restricting $t \leq 1$, we have the stronger bound by

$$
C \|u' - u''\|_{H^{0,s}} \left( (\|u_0'\|_{H^{0,s}} + \|u_0''\|_{H^{\rho}}) \|u_0'\|_{H^{\rho}}^{\frac{d}{H^{\rho}}} + (\|u''\|_{H^{0,s}} + \|u_0''\|_{H^{\rho}}) \|u_0''\|_{H^{\rho}}^{\frac{d}{H^{\rho}}} \right).
$$

Combining the estimates in (3.3) and (3.4), we have that

$$
\frac{d}{dt} \|u'(t) - u''(t)\| \leq C \left( (\|u_0'\|_{H^{0,s}} + \|u_0''\|_{H^{\rho}}) \left( 1 + \|u_0'\|_{H^{\rho}}^{\frac{d}{H^{\rho}}} \right) + (\|u''\|_{H^{0,s}} + \|u_0''\|_{H^{\rho}}) \left( \|u_0''\|_{H^{\rho}}^{\frac{d}{H^{\rho}}} + 1 \right) \right).
$$

Thus, the statement of Lemma 2.1 remains valid if we (i) replace the space $H^1$ by the space $H^\rho$, (ii) replace the $H^1$ norm by the $H^\rho$ norm in estimates of the growth of Sobolev norms, and (iii) replace the $\|u'\|_{H^1}$ norm by

$$
(\|u_0'\|_{H^{0,s}} + \|u_0''\|_{H^{\rho}}) \left( 1 + \|u_0'\|_{H^{\rho}}^{\frac{d}{H^{\rho}}} \right)
$$

and similarly for $u''$ in estimates of the growth of the $H^{0,s}$ norm.

In Sections 2.2 and 2.3, it is sufficient to replace $H^1$ by $H^\rho$. In the proof of 2.3 and 2.4, it is sufficient to use $H^1$ throughout and then conclude with the embedding of $H^\rho \hookrightarrow H^1$. In Lemma 2.5, it is sufficient to replace $H^1$ by $H^\rho$, since both are sufficient to put $v'$ and $v''$ in $H^s$.

In Section 2.4, it is sufficient to make the same three changes as in Section 2.1. The replacement of the $H^1$ norm by the term in (3.5) does not prevent the technique of taking $t \to 0$ to leave dependence only on the $H^{0,s}$ norm. No further use of the regularised approximators is made in the paper.

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