On Block Representations and Spectral Properties of Semimagic Square Matrices

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Abstract

Using the decomposition of semimagic squares into the associated and balanced symmetry types as a motivation, we introduce an equivalent representation in terms of block-structured matrices. This block representation provides a way of constructing such matrices with further symmetries and of studying their algebraic behaviour, significantly advancing and contributing to the understanding of these symmetry properties. In addition to studying classical attributes, such as dihedral equivalence and the spectral properties of these matrices, we show that the inherent structure of the block representation facilitates the definition of low-rank semimagic square matrices. This is achieved by means of tensor product blocks. Furthermore, we study the rank and eigenvector decomposition of these matrices, enabling the construction of a corresponding two-sided eigenvector matrix in rational terms of their entries. The paper concludes with the derivation of a correspondence between the tensor product block representations and quadratic form expressions of Gaussian type.

1 Introduction

The magic square of order 3 or “Loh Shu square” \( \begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix} \) was known in China as early as the Warring States period (481 BC – 221 BC). Up to rotations and reflections, it gives the unique arrangement of the numbers 1, \ldots, 9 in a 3 x 3 square such that all the rows and columns and both diagonals add up to the same number (in this case, 15). More generally, an arrangement of any \( n^2 \) numbers in an \( n \times n \) square such that all rows, columns and both diagonals add up to the same number is called a magic square; a semimagic square if the requirement on the diagonals is dropped \([1], [4]\).

A. C. Thompson \([11]\) was one of the first to publish on the subject of matrix multiplication of magic squares, observing that the above 3 x 3 magic square, considered as a square matrix, retains its symmetry for all odd integer powers. We know now that this property is due to the fact that this is a so-called associated semimagic square (or type A matrix) and that the type A matrices form a \( \mathbb{Z}_2 \)-graded algebra with the so-called balanced semimagic squares (or type B matrices) of the same dimension; such that a product of any two type A matrices gives a type B matrix, while the product of a type A and a type B matrix is of type A, and the type B matrices form a matrix subalgebra by themselves (cf. \([8], [2]\) Lemma 1.1). Any semimagic square matrix can be written as a sum of an associated and a balanced semimagic square matrix, so the above graded algebra is in fact the matrix algebra of semimagic squares \([2]\) Lemma 2.1).
In this paper we introduce an equivalent block representation of $n \times n$ semimagic squares by means of conjugation with a symmetric involution matrix $X_n$; in the transformed representation, split into a $2 \times 2$ array of block matrix components, the types A and B correspond to the off-diagonal and on-diagonal blocks, respectively, so the symmetry type decomposition of the semimagic square matrix can easily be read off. Moreover, it turns out that the block representation reveals other traits of the semimagic square and makes them more accessible to analysis than the original form. For example, based on the groundwork presented here, the authors have studied a wider range of matrix symmetries, including the type of ‘most perfect pandiagonal magic squares’ [10], using the block representation, identifying further structured algebras of semimagic square (and related) matrices [6].

The present work is structured as follows. In Section 2, we introduce the block representation and derive its properties equivalent to the symmetry types A and B; there is a curious fundamental difference between the cases of even-dimensional and odd-dimensional matrices. As the block representation is an algebra isomorphism which is covariant under transposition, it is possible to study algebraic properties by looking at the transformed matrices instead of the original semimagic squares. Moreover, the dihedral operations of rotation and reflection, typically resulting in different matrices associated with essentially the same semimagic square, are shown to be expressed by a simple sign change in different component blocks.

In Section 3, we address the question of how magic squares can be recognised in terms of their block representation; as the type A part is always magic, this is a question for type B matrices only. As a result, we show that for any balanced magic square matrix, some power of it will be either trivial (i.e. with all matrix entries the same) or non-magic.

In Section 4, we consider the existence of inverses or quasi-inverses of semimagic square matrices, and more specifically consider the ranks and eigenvector decomposition of associated magic square matrices and their (balanced) matrix algebra squares. Here the concept of parasymmetry, i.e. the property of a matrix to have a symmetric square, plays a crucial role. The case of matrices whose block representation is composed of rank 1 blocks allows a particularly detailed analysis, including the explicit construction of a two-sided (both right and left) eigenvector matrix.

In the final section, we use the latter results to establish a connection to quadratic forms of the type studied in number theory [5].

2 Block Representations and Dihedral Symmetries

Definition. An $n \times n$ matrix $M \in \mathbb{R}^{n \times n}$ is called semimagic square of weight $w$ if each of its rows and columns sums to $nw$. If, in addition, each of its two diagonals also adds up to $nw$, it is called a magic square.

The following two centre-point symmetry types of semimagic square matrices are of interest.

Definition. Let $M$ be an $n \times n$ semimagic square matrix of weight $w$.

(a) The matrix $M$ is called associated if each entry and its mirror entry w.r.t. the centre of the matrix add to $2w$, i.e. if

$$M_{ij} + M_{n+1-i,n+1-j} = 2w \quad (i, j \in \{1, \ldots, n\}).$$

(b) The matrix $M$ is called balanced if each entry is equal to its mirror entry w.r.t. the centre of the matrix, i.e.

$$M_{ij} - M_{n+1-i,n+1-j} = 0 \quad (i, j \in \{1, \ldots, n\}).$$
These properties can be equivalently expressed in a more convenient way using the following conventions. Let \( 1_n \in \mathbb{R}^n \) be the vector which has all entries equal to 1. Let \( \mathcal{E}_n = (1)_{i,j=1}^n \in \mathbb{R}^{n \times n} \) be the \( n \times n \) matrix which has all entries equal to 1. Moreover, let \( \mathcal{J}_n = (\delta_{i,n+1-j})_{i,j=1}^n \in \mathbb{R}^{n \times n} \), where \( \delta \) is the Kronecker symbol, be the \( n \times n \) matrix which has entries 1 on the antidiagonal and 0 otherwise. Later, we shall also use the notations \( 0_n \) for the null vector in \( \mathbb{R}^n \), \( \mathcal{O}_n = (0)_{i,j=1}^n \) for the \( n \times n \) null matrix, and \( \mathcal{T}_n = (\delta_{ij})_{i,j=1}^n \) for the \( n \times n \) unit matrix.

Note that we consider the elements of \( \mathbb{R}^n \) as column vectors, i.e. matrices with \( n \) rows and a single column, throughout; row vectors are represented in the form \( v^T \), where \( v \in \mathbb{R}^n \).

Lemma 1. Let \( M \in \mathbb{R}^{n \times n} \).

(a) The following statements are equivalent.

(i) \( M \) is semimagic square of weight \( w \);
(ii) \( M \mathcal{E}_n = nw \mathcal{E}_n = M^T \mathcal{E}_n \);
(iii) \( 1_n \) is an eigenvector, with eigenvalue \( nw \), for both \( M \) and its transpose \( M^T \).

(b) If \( M \) is semimagic square, then it is associated if and only if \( M + \mathcal{J}_n M \mathcal{J}_n = 2w \mathcal{E}_n \).

(c) If \( M \) is semimagic square, then it is balanced if and only if \( M = \mathcal{J}_n M \mathcal{J}_n \).

Let \( S_n \) be the set of all \( n \times n \) semimagic square matrices, and \( A_n, B_n \) the subsets of associated and balanced semimagic square matrices, respectively. Then \( S_n \) is a subalgebra of the standard \( n \times n \) matrix algebra, and \( A_n \) and \( B_n \) are vector subspaces of \( S_n \). Moreover, \( B_n \) is a subalgebra of \( S_n \), and
\[
A_n A_n \subset B_n, \quad A_n B_n \subset A_n, \quad B_n A_n \subset A_n. \tag{2.1}
\]
Furthermore, \( A_n \) and \( B_n \) generate the whole algebra \( S_n \) in the following way. Denoting by \( S_n^0 \) the subset (in fact, subalgebra) of weight 0 semimagic square matrices, and setting \( A_n^0 = A_n \cap S_n^0 \), \( B_n^0 = B_n \cap S_n^0 \), we have
\[
A_n \cap B_n = \mathbb{R} \mathcal{E}_n, \\
A_n^0 \cap B_n^0 = \{ \mathcal{O}_n \}, \\
\text{and} \quad S_n = A_n^0 + B_n^0 + \mathbb{R} \mathcal{E}_n,
\]
where every \( n \times n \) semimagic square matrix of weight \( w \) can be written as a sum of unique elements of \( A_n^0 \) and \( B_n^0 \), and \( w \mathcal{E}_n \) (cf. [2] Lemma 2.3).

Even Dimensional Case

We first consider \( 2n \times 2n \) matrices, \( n \in \mathbb{N} \). Let
\[
\mathcal{X}_{2n} = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathcal{I}_n & \mathcal{J}_n \\ \mathcal{J}_n & -\mathcal{I}_n \end{pmatrix} \in \mathbb{R}^{2n \times 2n}.
\]
Clearly \( \mathcal{X}_{2n}^T = \mathcal{I}_{2n} \) and \( \mathcal{X}_{2n}^T = \mathcal{X}_{2n} \), so \( \mathcal{X}_{2n} \) is an orthogonal symmetric involution. Conjugation with the matrix \( \mathcal{X}_{2n} \) gives rise to a block representation of matrices in \( S_{2n} \) in which the symmetry type can easily be read off. This also provides a convenient and systematic way of constructing semimagic square matrices with (or without) centre-point symmetries.

Theorem 1. Let \( M \in \mathbb{R}^{2n \times 2n} \).
(a) The matrix $M \in A_{2n}^o$ if and only if

$$M = \mathcal{X}_{2n} \left( \begin{array}{cc} O_n & V^T \\ W & O_n \end{array} \right) \mathcal{X}_{2n}$$

where $V, W \in \mathbb{R}^{n \times n}$ have row sum 0, i.e.

$$V1_n = 0_n, \quad W1_n = 0_n.$$

(b) The matrix $M \in B_{2n}^o$ if and only if

$$M = \mathcal{X}_{2n} \left( \begin{array}{cc} Y & O_n \\ O_n & Z \end{array} \right) \mathcal{X}_{2n}$$

where $Y \in S_n^o$ and $Z \in \mathbb{R}^{n \times n}$.

(c) The weight matrix $\mathcal{E}_{2n}$ satisfies

$$\mathcal{E}_{2n} = \mathcal{X}_{2n} \left( \begin{array}{cc} 2\mathcal{E}_n & O_n \\ O_n & O_n \end{array} \right) \mathcal{X}_{2n}.$$

In view of the decomposition of general semimagic square matrices mentioned above, we see that, as a consequence of Theorem 1, any even-dimensional, weight $w$ semimagic square matrix has, after conjugation with $\mathcal{X}_{2n}$, the block representation

$$\left( \begin{array}{cc} Y + 2w\mathcal{E}_n & V^T \\ W & Z \end{array} \right),$$

with a weight 0 semimagic square matrix $Y$, matrices $V, W$ with row sum 0 and a matrix $Z$ which can be any $n \times n$ matrix. Evidently, this block representation clearly shows the decomposition into an associated and a balanced matrix, corresponding to setting the two diagonal or the two off-diagonal blocks equal to 0, respectively.

From the block representation, it is very straightforward to generate all matrices in $A_{2n}^o$ (and hence, by adding a multiple of $\mathcal{E}_{2n}$, in $A_{2n}$); indeed, the conditions on the matrices $V, W$ can very easily be satisfied, as $n - 1$ columns can be arbitrary when the last column is chosen so that the rows add to 0. The construction of a general matrix in $B_{2n}$ is a bit more complicated, as $Z$ can be chosen arbitrarily, but $Y$ must be a semimagic square matrix. At least this reduces the dimension of the problem from $2n$ to $n$.

As the matrix $\mathcal{X}_{2n}$ is symmetric (i.e., equal to its transpose) and involutory (i.e., its own inverse matrix), the following observation is straightforward.

**Lemma 2.** (a) The block representation is an algebra isomorphism; indeed,

$$\mathcal{X}_{2n}(\alpha M + N)\mathcal{X}_{2n} = \alpha \mathcal{X}_{2n}M\mathcal{X}_{2n} + \mathcal{X}_{2n}N\mathcal{X}_{2n} \quad (\alpha \in \mathbb{R}; M, N \in \mathbb{R}^{2n \times 2n})$$

and

$$\mathcal{X}_{2n}(MN)\mathcal{X}_{2n} = \mathcal{X}_{2n}M\mathcal{X}_{2n}\mathcal{X}_{2n}N\mathcal{X}_{2n} \quad (M, N \in \mathbb{R}^{2n \times 2n}).$$

(b) The block representation of the transposed matrix is the transpose of the block representation of the original matrix; indeed,

$$\mathcal{X}_{2n}M^T\mathcal{X}_{2n} = (\mathcal{X}_{2n}M\mathcal{X}_{2n})^T \quad (M \in \mathbb{R}^{2n \times 2n}).$$
Proof of Theorem 1. We begin by writing the matrix $M$ in the form

$$M = \begin{pmatrix} A & C \\ B & D \end{pmatrix},$$

where $A, B, C, D \in \mathbb{R}^{n \times n}$; then

$$\mathcal{J}_{2n} M \mathcal{J}_{2n} = \begin{pmatrix} \mathcal{J}_{n} D \mathcal{J}_{n} & \mathcal{J}_{n} B \mathcal{J}_{n} \\ \mathcal{J}_{n} C \mathcal{J}_{n} & \mathcal{J}_{n} A \mathcal{J}_{n} \end{pmatrix}.$$  

(a) Assume that $M \in A^{o}_{2n}$. Then, by Lemma 1 (b),

$$\mathcal{O}_{2n} = M + \mathcal{J}_{2n} M \mathcal{J}_{2n} = \begin{pmatrix} \mathcal{J}_{n} (D + \mathcal{J}_{n} A \mathcal{J}_{n}) \mathcal{J}_{n} & \mathcal{J}_{n} (B + \mathcal{J}_{n} C \mathcal{J}_{n}) \mathcal{J}_{n} \\ B + \mathcal{J}_{n} C \mathcal{J}_{n} & D + \mathcal{J}_{n} A \mathcal{J}_{n} \end{pmatrix}$$

(where we used $\mathcal{J}_{n}^{2} = I_{n}$). This is equivalent to $B = -\mathcal{J}_{n} C \mathcal{J}_{n}$, $D = -\mathcal{J}_{n} A \mathcal{J}_{n}$. Hence,

$$\mathcal{X}_{2n} M \mathcal{X}_{2n} = \frac{1}{2} \begin{pmatrix} I_{n} & \mathcal{J}_{n} \\ \mathcal{J}_{n} & -I_{n} \end{pmatrix} \begin{pmatrix} A & C \\ -\mathcal{J}_{n} C \mathcal{J}_{n} & -\mathcal{J}_{n} A \mathcal{J}_{n} \end{pmatrix} \begin{pmatrix} I_{n} & \mathcal{J}_{n} \\ \mathcal{J}_{n} & -I_{n} \end{pmatrix}$$

$$= \begin{pmatrix} \mathcal{O}_{n} & A \mathcal{J}_{n} - C \\ \mathcal{J}_{n} A + \mathcal{J}_{n} C \mathcal{J}_{n} & \mathcal{O}_{n} \end{pmatrix}.$$  

(2.2)

By Lemma 1 (a), $M$ is a weight 0 semimagic square matrix if and only if $1_{2n}$ is an eigenvector of both $M$ and $M^{T}$ for eigenvalue 0. In view of Lemma 2, this is equivalent to

$$\mathcal{X}_{2n} 1_{2n} = \sqrt{2} \begin{pmatrix} 1_{n} \\ 0_{n} \end{pmatrix}$$

(2.3)

being an eigenvector, for eigenvalue 0, of both $\mathcal{X}_{2n} M \mathcal{X}_{2n}$ and $(\mathcal{X}_{2n} M \mathcal{X}_{2n})^{T}$. From (2.2), this corresponds to the conditions

$$\mathcal{J}_{n} (A + C \mathcal{J}_{n}) 1_{n} = 0_{n}, \quad (A \mathcal{J}_{n} - C)^{T} 1_{n} = 0_{n}.$$  

Thus $V = (A \mathcal{J}_{n} - C)^{T}$ and $W = \mathcal{J}_{n} A + \mathcal{J}_{n} C \mathcal{J}_{n}$ will have row sum 0.

Conversely, given $V, W \in \mathbb{R}^{n \times n}$ with row sum 0, we take $A = \frac{1}{2}(V^{T} \mathcal{J}_{n} + \mathcal{J}_{n} W)$, $C = \frac{1}{2}(\mathcal{J}_{n} W \mathcal{J}_{n} - V^{T})$ and further $B = -\mathcal{J}_{n} C \mathcal{J}_{n}$, $D = -\mathcal{J}_{n} A \mathcal{J}_{n}$, to construct a weight 0 semimagic square matrix.

(b) Assume $M \in B^{o}_{n}$. Then, by Lemma 1 (c),

$$\mathcal{O}_{2n} = M - \mathcal{J}_{2n} M \mathcal{J}_{2n} = \begin{pmatrix} -\mathcal{J}_{n} (D - \mathcal{J}_{n} A \mathcal{J}_{n}) \mathcal{J}_{n} & -\mathcal{J}_{n} (B - \mathcal{J}_{n} C \mathcal{J}_{n}) \mathcal{J}_{n} \\ B - \mathcal{J}_{n} C \mathcal{J}_{n} & D - \mathcal{J}_{n} A \mathcal{J}_{n} \end{pmatrix},$$

which is equivalent to $B = \mathcal{J}_{n} C \mathcal{J}_{n}$, $D = \mathcal{J}_{n} A \mathcal{J}_{n}$. Hence

$$\mathcal{X}_{2n} M \mathcal{X}_{2n} = \frac{1}{2} \begin{pmatrix} I_{n} & \mathcal{J}_{n} \\ \mathcal{J}_{n} & -I_{n} \end{pmatrix} \begin{pmatrix} A & C \\ \mathcal{J}_{n} C \mathcal{J}_{n} & \mathcal{J}_{n} A \mathcal{J}_{n} \end{pmatrix} \begin{pmatrix} I_{n} & \mathcal{J}_{n} \\ \mathcal{J}_{n} & -I_{n} \end{pmatrix}$$

$$= \begin{pmatrix} \mathcal{O}_{n} + C \mathcal{J}_{n} \\ \mathcal{J}_{n} A + \mathcal{J}_{n} C \mathcal{J}_{n} - \mathcal{J} C \mathcal{J}_{n} \end{pmatrix}. $$

(2.4)

As before, the condition that $M$ is a semimagic square matrix of weight 0 means that is an eigenvector with eigenvalue 0 of both $\mathcal{X}_{2n} M \mathcal{X}_{2n}$ and $(\mathcal{X}_{2n} M \mathcal{X}_{2n})^{T}$; by (2.4) this is equivalent to

$$(A + C \mathcal{J})^{T} 1_{n} = 0_{n}.$$  

By Lemma 1 (a), $Y = A + C \mathcal{J} \in S^{o}_{n}$.  

Conversely, given $Y \in S^{o}_{n}$ and $Z \in \mathbb{R}^{n \times n}$, we take $A = \frac{1}{2}(Y + \mathcal{J}_{n} Z \mathcal{J}_{n})$, $C = \frac{1}{2}(Y \mathcal{J}_{n} - \mathcal{J}_{n} Z)$, and further $B = \mathcal{J}_{n} C \mathcal{J}_{n}$, $D = \mathcal{J}_{n} A \mathcal{J}_{n}$, to construct a weight 0 balanced semimagic square matrix.
(c) By a straightforward calculation,
\[ X_{2n}E_{2n}X_{2n} = \frac{1}{2} \begin{pmatrix} I_n & J_n \\ J_n & -I_n \end{pmatrix} \begin{pmatrix} E_n & E_n \\ E_n & -E_n \end{pmatrix} \begin{pmatrix} I_n & J_n \\ J_n & -I_n \end{pmatrix} = \begin{pmatrix} 2E_n & O_n \\ O_n & O_n \end{pmatrix}. \]

**Odd Dimensional Case**

We now consider \((2n + 1) \times (2n + 1)\) matrices, \(n \in \mathbb{N}\). Let
\[ X_{2n+1} = \begin{pmatrix} \frac{1}{\sqrt{2}}I_n & 0_n \\ 0_n & \frac{1}{\sqrt{2}}J_n \end{pmatrix} \in \mathbb{R}^{(2n+1) \times (2n+1)}. \]
(The matrix \(X_{2n+1}\) turns into \(X_{2n}\) when its central row and column are deleted.) The matrix \(X_{2n+1}\) is a symmetric involution, i.e. \(X_{2n+1}^T = X_{2n+1}\) and \(X_{2n+1}^2 = I_{2n+1}\). Conjugation with \(X_{2n+1}\) gives rise to the following block representation of odd-dimensional semimagic square matrices.

**Theorem 2.** Let \(M \in \mathbb{R}^{(2n+1) \times (2n+1)}\).

(a) The matrix \(M \in A_{2n+1}^0\) if and only if
\[ M = X_{2n+1} \begin{pmatrix} O_n & 0_n & V^T \\ 0_n^T & 0 & -\sqrt{2}(V1_n)^T \\ W & -\sqrt{2}W1_n & O_n \end{pmatrix} \]
with matrices \(V, W \in \mathbb{R}^{n \times n}\). Moreover, \(M\) will have rational entries if and only if \(V\) and \(W\) have rational entries.

(b) The matrix \(M \in B_{2n+1}^0\) if and only if
\[ M = X_{2n+1} \begin{pmatrix} Y & -\sqrt{2}Y1_n & O_n \\ -\sqrt{2}(Y1_n)^T & 21_n^TY1_n & 0_n^T \\ O_n & 0_n & Z \end{pmatrix} \]
with matrices \(Y, Z \in \mathbb{R}^{n \times n}\). Moreover, \(M\) will have rational entries if and only if \(Y\) and \(Z\) have rational entries.

(c) The matrix \(E_{2n+1}\) satisfies
\[ E_{2n+1} = X_{2n+1} \begin{pmatrix} 2E_n & \sqrt{2}1_n & O_n \\ \sqrt{2}1_n^T & 1 & 0_n^T \\ O_n & 0_n & O_n \end{pmatrix} \]

Note that there are no conditions on the matrices \(V, W, Y\) and \(Z\) in Theorem 2 so the block representation gives a very simple way of constructing all odd-dimensional semimagic square matrices (with or without centre-point symmetries). Indeed, it is evident from the theorem that the general element of \(S_{2n+1}\) will be of the form
\[ M = X_{2n+1} \begin{pmatrix} Y + 2wE_n & \sqrt{2}(wI_n - Y)1_n \\ \sqrt{2}(wI_n - Y)1_n^T & w + 21_n^TY1_n & -\sqrt{2}(V1_n)^T \\ W & -\sqrt{2}W1_n & Z \end{pmatrix} \]
(2.5)
with arbitrary $V, W, Y, Z \in \mathbb{R}^{n \times n}$. Note that, in contrast to the even-dimensional case, adding the weight $w$ to $M$ is not equivalent to adding a weight to the matrix $Y$. Incidentally, the choices $n = 1$, $w = 5$, $V = (2)$, $W = (4)$ and $Y = Z = (0)$ give the Loh Shu square.

Proof of Theorem 2. Writing $M$ in the form

$$ M = \begin{pmatrix} A & v & C \\ w^T & x & y^T \\ B & z & D \end{pmatrix} $$

with $x \in \mathbb{R}$, $v, w, y, z \in \mathbb{R}^n$ and $A, B, C, D \in \mathbb{R}^{n \times n}$, we find

$$ J_{2n+1} M J_{2n+1} = \begin{pmatrix} J_n D J_n & J_n z & J_n B J_n \\ y^T J_n & x & w^T J_n \\ J_n C J_n & J_n v & J_n A J_n \end{pmatrix} $$

(a) The weight 0 association condition of Lemma 1 (b) now takes the form

$$ O_{2n+1} = M + J_{2n+1} M J_{2n+1} = \begin{pmatrix} J_n (J_n A J_n + D) J_n & v + J_n z & J_n (J_n C J_n + B) J_n \\ w^T + y^T J_n & 2x & (w^T + y^T J_n) J_n \\ B + J_n C J_n & J_n (v + J_n z) & D + J_n A J_n \end{pmatrix}, $$

so $x = 0$, $y = -J_n w$, $z = -J_n v$, $B = -J_n C J_n$ and $D = -J_n A J_n$. Thus, if $M$ is a weight 0 associated semimagic square matrix, then

$$ X_{2n+1} M X_{2n+1} = X_{2n+1} \begin{pmatrix} A & v & C \\ w^T & 0 & -w^T J_n \\ -J_n C J_n & -J_n v & -J_n A J_n \end{pmatrix} X_{2n+1} $$

$$ = \begin{pmatrix} O_n & 0 & A J_n - C \\ 0 & J_n A + J_n C J_n & \sqrt{2} J_n v \\ 0 & \sqrt{2} w^T J_n & O_n \end{pmatrix}. $$

By Lemma 1 (a), $M \in S^3_{2n+1}$ if and only if $1_{2n+1}$ is an eigenvector with eigenvalue 0 of $M$ and of $M^T$. Since

$$ X_{2n+1} 1_{2n+1} = \begin{pmatrix} \sqrt{2} 1_n \\ 1 \\ 0_n \end{pmatrix}, \quad (2.6) $$

we see that

$$ 0_{2n+1} = M 1_{2n+1} = M X_{2n+1} X_{2n+1} 1_{2n+1} $$

if and only if

$$ 0_n = W \sqrt{2} 1_n + \sqrt{2} J_n v, $$

where we set $W = J_n A + J_n C J_n$; this gives $J_n v = -W 1_n$. Similarly, setting $V = (A J_n - C)^T,$

$$ 0_{2n+1} = M^T 1_{2n+1} = M^T X_{2n+1} X_{2n+1} 1_{2n+1} $$

if and only if

$$ 0_n = V \sqrt{2} 1_n + \sqrt{2} J_n w, $$
and hence \( w^T J_n = -(V1_n)^T \).

Conversely, given matrices \( V, W \in \mathbb{R}^{n \times n} \), we take \( A = \frac{1}{2}(J_n W + V^T J_n) \), \( C = \frac{1}{2}(J_n W J_n - V^T) \) and further \( B = -J_n C J_n \), \( D = -J_n A J_n \), \( x = 0 \), \( v = -J_n W 1_n \), \( w = -J_n V 1_n \), \( y = V1_n \) and \( z = W1_n \) to construct a weight 0 associated semimagic square matrix.

From these formulae, it is evident that \( M \) has rational entries if and only if \( V, W \) have rational entries.

(b) The balance condition of Lemma 1 (c) takes the form

\[
\mathcal{O}_{2n+1} = M - J_n M J_n
\]

\[
= \begin{pmatrix}
J_n (J_n A J_n - D) J_n & v - J_n z & J_n (J_n C J_n - B) J_n \\
J_n (J_n C J_n - v) & y^T J_n - w^T & (y^T J_n - w^T) J_n \\
B - J_n C J_n & J_n (J_n z - v) & D - J_n A J_n
\end{pmatrix},
\]

so \( z = J_n v \), \( y = J_n w \), \( B = J_n C J_n \) and \( D = J_n A J_n \); there is no condition on \( x \). Then

\[
\mathcal{X}_{2n+1} M \mathcal{X}_{2n+1} = \begin{pmatrix}
A + C J_n & \sqrt{2} v & \mathcal{O}_n \\
\sqrt{2} w^T & x & 0^T \\
\mathcal{O}_n & 0_n & J_n A J_n - J_n C
\end{pmatrix},
\]

and hence, using Lemma 1 (a) and (2.6), \( 0_{2n+1} = M1_{2n+1} \) if and only if

\[
0_{n+1} = \left( \frac{\sqrt{2}(A + C J_n) 1_n + v}{2w^T 1_n + x} \right),
\]

giving \( x = -2w^T 1_n \) and \( v = -Y 1_n \), where we set \( Y = A + C J_n \). Similarly, \( 0_{2n+1} = M^T 1_{2n+1} \) if and only if

\[
0_{n+1} = \left( \frac{\sqrt{2}(Y^T 1_n + w)}{2v^T 1_n + x} \right),
\]

so \( x = -2v^T 1_n \) and \( w = -Y^T 1_n \). This determines \( v \) and \( w \) and gives two conditions on \( x \), which turn out to be each equivalent to \( x = 21_n^T Y 1_n \).

For the converse, we take \( A = \frac{1}{2}(Y + J_n Z J_n) \), \( C = \frac{1}{2}(Y J_n - J_n Z) \), and further \( B = J_n C J_n \), \( D = J_n A J_n \), \( v = -Y 1_n \), \( w = -Y^T 1_n \), \( x = 21_n^T Y 1_n \), and \( y = -J_n Y^T 1_n \) and \( z = -J_n Y 1_n \) to construct a weight 0 balanced semimagic square matrix.

From these formulae it is evident that \( M \in \mathbb{Q}^{(2n+1)\times(2n+1)} \) if and only if \( Y, Z \in \mathbb{Q}^{n \times n} \).

(c) This is a straightforward calculation.

The unique decomposition of a semimagic square matrix into a weightless associated, a weightless balanced semimagic square matrix and a multiple of \( \mathcal{E}_n \) (where \( n \) is the dimension of the matrix) can easily be read off its block representation. Associated and balanced semimagic square matrices are clearly identified by the presence and position of null blocks in their block representation. Moreover, the block representation can be used to characterise other matrix symmetries in a convenient manner. For example, in the study of semimagic or magic squares, it is usually the relative arrangement of numbers in the square which is the object of interest, not their fixed positioning in a matrix; therefore matrices which differ only by rotation or reflection will be identified with one another. In other words, the equivalence classes in \( S_n \) with respect to the dihedral group for the \( n \times n \) square will be considered. In the block representation, the action of the dihedral group translates into transposition and sign inversion of the right or bottom half of the block matrix. Thus we have the following result.
Theorem 3. Let $M \in S_n$ and
\[
M = X_n \begin{pmatrix} \tilde{Y} & \tilde{V}^T \\ \tilde{W} & Z \end{pmatrix} X_n
\] (2.7)

its block representation. Then the dihedral equivalence class of $M$ is
\[
\left\{ X_n \begin{pmatrix} \tilde{Y} & s\tilde{V}^T \\ t\tilde{W} & stZ \end{pmatrix} X_n, X_n \begin{pmatrix} \tilde{Y}^T & s\tilde{W}^T \\ t\tilde{V} & stZ^T \end{pmatrix} X_n \mid s, t \in \{1, -1\} \right\}.
\]

Remark. If $n$ is even, then $\tilde{Y}, \tilde{V}, \tilde{W}$ are the matrices $Y, V, W$ of Theorem 1; if $n$ is odd, then they include parts of the centre row and column in the block representation of Theorem 2. (In particular, $\tilde{V}$ and $\tilde{W}$ will then no longer be square matrices.) In either case, $Z$ is the matrix denoted by the same letter in Theorems 1, 2.

Proof. The dihedral group is generated by the two operations of reflection along the diagonal and reflection along the horizontal centreline. In matrix terms, these correspond to taking the transpose of the matrix and to left multiplication with the matrix $J_n$, respectively. Transposition carries over directly to the block representation, as seen in Lemma 2 (b). Since $X_n J_n M X_n = X_n J_n X_n M X_n$, left multiplication of $M$ with $J_n$ translates into left multiplication of its block representation with the matrix
\[
X_n J_n X_n = \begin{pmatrix} I_k & O_k \\ O_k & -I_k \end{pmatrix}
\]
in the even-dimensional case $n = 2k$, and
\[
X_n J_n X_n = \begin{pmatrix} I_{k+1} & O_{k+1,k} \\ O_{k+1,k} & -I_k \end{pmatrix}
\]
in the odd-dimensional case $n = 2k + 1$; here $O_{k,l}$ denotes the null matrix with $k$ rows and $l$ columns. Hence
\[
J_n M = X_n \begin{pmatrix} \tilde{Y} & \tilde{V}^T \\ -\tilde{W} & -Z \end{pmatrix} X_n. \tag{2.8}
\]

Similarly, reflection of $M$ along the vertical centreline corresponds to multiplication with $J_n$ on the right, and hence to inverting the sign of the rightmost $k$ columns in the block representation. (This operation is of course derived from the group generators as $(J_n M^T)^T = M J_n$.)

\[\square\]

3 Magic Square Matrices

For a semimagic square matrix to be magic, its two diagonals also need to add up to the row and column sum. For associated semimagic square matrices, this is always the case. However, for balanced semimagic squares, which, since $E_n$ is magic, we can assume without loss of generality to have weight 0, this gives an additional condition.

The sum of the diagonal entries of the matrix $M$ is equal to its trace, $\text{tr} M$; the sum of the entries on the second diagonal is the trace of the matrix after reflection along the horizontal (or vertical) centreline, i.e. equal to $\text{tr}(J_n M)$. As the trace of a product of matrices is invariant under cyclic permutations of its factors and hence the trace of a semimagic square matrix is equal to the trace of its block representation, we see that for a weight 0 balanced semimagic square matrix $M$, using the representation (2.7),
\[
\text{tr} M = \text{tr}(X_n \begin{pmatrix} \tilde{Y} & O \\ O & Z \end{pmatrix} X_n) = \text{tr} \tilde{Y} + \text{tr} Z,
\]
and by (2.8)

$$\text{tr}(\mathcal{J}_n M) = \text{tr}(\mathcal{X}_n \begin{pmatrix} \tilde{Y} & O_n \\ O_n & -Z \end{pmatrix} \mathcal{X}_n) = \text{tr} \tilde{Y} - \text{tr} Z.$$ 

As both must vanish for the matrix to be magic, this means that the traces of $\tilde{Y}$ and of $Z$ must separately be 0. (As all diagonal entries of the block representation of a weight 0 associated semimagic square matrix are 0, it is evident that such matrices are always magic.) This gives rise to the following statement.

**Theorem 4.** (a) A $2n \times 2n$ matrix $M$ is a balanced magic square matrix of weight $w$ if and only if

$$M = \mathcal{X}_{2n} \begin{pmatrix} Y & O_n \\ O_n & Z \end{pmatrix} \mathcal{X}_{2n},$$

where $Y$ is an $n \times n$ semimagic square matrix of weight $2w$ and $\text{tr} Y = 2nw$, and $Z$ is any $n \times n$ matrix with $\text{tr} Z = 0$.

(b) A $(2n + 1) \times (2n + 1)$ matrix $M$ is a balanced magic square matrix of weight $w$ if and only if

$$M = \mathcal{X}_{2n+1} \begin{pmatrix} \tilde{Y} & O_{n+1,n} \\ O_{n,n+1} & Z \end{pmatrix} \mathcal{X}_{2n+1},$$

where

$$\tilde{Y} = \begin{pmatrix} Y + 2w \mathcal{E}_n & \sqrt{2} (w \mathcal{I}_n - Y)_{1n} \\ \sqrt{2} (w \mathcal{I}_n - Y)_{1n}^T & w + 21_n^T Y 1_n \end{pmatrix},$$

$Y, Z$ are any $n \times n$ matrices such that $\text{tr} Y = -21_n^T Y 1_n$ (corresponding to $\text{tr} \tilde{Y} = (2n+1)w$) and $\text{tr} Z = 0$.

**Remarks.**

1. The condition on the matrix $Y$ in Theorem 4 (a) almost makes it a magic square matrix, but not quite. For example, the matrix

$$Y = \begin{pmatrix} 1 & 2 & -3 \\ -6 & 1 & 5 \\ 5 & -3 & -2 \end{pmatrix}$$

is a weight 0 semimagic square matrix with vanishing trace, but its second diagonal does not add up to 0. Nevertheless, this matrix is a suitable upper block $Y$ for the block representation of a $6 \times 6$ balanced magic square matrix, choosing any $3 \times 3$ matrix $Z$ of vanishing trace for the lower block.

2. Bearing in mind that $1_n^T Y 1_n$ is just the sum of all entries of $Y$, the condition on the matrix $Y$ in Theorem 4 (b) means that the off-diagonal entries of $Y$ sum to $-\frac{3}{2}$ times its trace. For example,

$$Y = \begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix}$$

has this property, and picking a matrix $Z$ with vanishing trace, we obtain a balanced magic square matrix by the calculation

$$\mathcal{X}_5 \begin{pmatrix} 1 & -2 & \sqrt{2} & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & \sqrt{2} & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 2 & -1 \end{pmatrix} \mathcal{X}_5 \begin{pmatrix} 0 & 0 & 1 & -2 & 1 \\ 1 & 1 & 0 & 0 & -2 \\ 0 & 1 & -2 & 1 & 0 \\ -2 & 0 & 0 & 1 & 1 \\ 1 & -2 & 1 & 0 & 0 \end{pmatrix}.$$
3. Theorem 4 gives a simple method of constructing balanced magic square matrices. General magic square matrices can be obtained by adding any associated semimagic (and hence magic) square matrix, or equivalently by filling in the off-diagonal blocks in the block representation with matrices satisfying the conditions in Theorem 1 (a) or 2 (a).

**Definition.** We call an \( n \times n \) magic square matrix *trivial* if it is a multiple of the matrix \( E_n \).

Any associated semimagic square matrix is magic; by (2.1), its square will be a balanced semimagic matrix. One may wonder whether it is again magic. For the case of even-dimensional associated magic square matrices with rank 1 blocks (and, w.l.o.g., weight 0), the following alternative holds.

**Theorem 5.** Consider the (non-trivial, weight 0) \( 2n \times 2n \) associated magic square

\[
M = X_{2n} \begin{pmatrix} O_n & V^T \\ W & O_n \end{pmatrix} X_{2n}
\]

with rank 1 blocks \( V,W \). Then exactly one of the following statements is true.

(a) \( M^2 \) has rank 2 and is not magic, and none of its powers are magic;

(b) \( M^2 \) has rank 0 or 1 and is magic and nilpotent, in fact \( M^4 = O_{2n} \).

**Proof.** There are vectors \( u,v,x,y \in \mathbb{R}^n \setminus \{0_n\} \) such that \( V = uv^T \), \( W = xy^T \). The block representation of \( M^2 \) is

\[
M^2 = X_{2n} \begin{pmatrix} V^T W & O_n \\ O_n & WV^T \end{pmatrix} X_{2n}
\]

with blocks

\[
V^T W = vu^T xy^T, \quad WV^T = xy^T vu^T.
\]

If \( u^T x = 0 \) and \( y^T v \neq 0 \), then \( V^T W = O_n \) and \( WV^T \) is a rank 1 matrix; it only has eigenvalue 0. Indeed, any eigenvector must be in the range of the matrix, i.e. a multiple of \( x \), which is in the null space of the matrix by hypothesis. Since the trace of a matrix is equal to the sum of its eigenvalues, repeated according to algebraic multiplicity, it follows that both \( V^T W \) and \( WV^T \) have vanishing trace, and by Theorem 4 this implies that \( M^2 \) is magic. Furthermore,

\[
WV^T WV^T = x(y^T v)(u^T x)(y^T v)u^T = O_n,
\]

so the matrix is nilpotent.

The case where \( y^T v = 0 \) and \( u^T x \neq 0 \) is analogous. If both \( u^T x = 0 \) and \( y^T v = 0 \), then \( M^2 = O_{2n} \).

If both \( u^T x \neq 0 \) and \( y^T v \neq 0 \), then both \( V^T W \) and \( WV^T \) have a non-zero eigenvalue with eigenvector \( v \neq 0_n, x \neq 0_n \), respectively, as

\[
V^T W v = v(u^T x)(y^T v), \quad WV^T x = x(y^T v)(u^T x);
\]

in fact they both have the *same* eigenvalue \((u^T x)(y^T v) \neq 0\). Hence each block in (3.1) has rank 1 and a non-zero trace, so \( M^2 \) has rank 2 and, by Theorem 4, is not magic. For any \( N \in \mathbb{N} \), its \( N \)th power has block representation

\[
M^{2N} = X_{2n} \begin{pmatrix} (V^T W)^N & O_n \\ O_n & (WV^T)^N \end{pmatrix} X_{2n},
\]
and as both blocks have the non-zero eigenvalue \(((u^T x)(v^T y))^N\), \(M^{2N}\) is not magic.

In case (b) of Theorem 5, the matrix may have rank 1 but only eigenvalue 0. This means that its Jordan normal form will have a single off-diagonal 1 and zeros otherwise, so the geometric multiplicity of the eigenvalue 0 of \(M\) will be \(2n - 1\).

It is an open question whether Theorem 5 generalises to higher-rank matrices. It is generally true that if \(W V^T = \mathcal{O}_n\), then \(V^T W\) has no non-zero eigenvalues, so \(M^2\) will then be magic and nilpotent; and similarly if \(V^T W = \mathcal{O}_n\).

We have, however, the following general result, which shows that non-trivial magic has limited power.

**Theorem 6.** Let \(M\) be a balanced magic square matrix of size \(m \times m\), where \(m = 2n\) or \(m = 2n - 1\). Then there is \(N \in \{1, \ldots, n\}\) such that \(M^N\) is either trivial or not magic.

**Remark.** The loss of magic may be temporary, as the magic property may recur when higher powers are involved. For example, the balanced magic square matrix

\[
\begin{pmatrix}
0 & -1 & 1 & 0 \\
1 & 0 & 0 & -1 \\
-1 & 0 & 0 & 1 \\
0 & 1 & -1 & 0
\end{pmatrix}
= X_4 \begin{pmatrix} 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 \\
0 & 0 & -2 & 0 \end{pmatrix} X_4
\]

has non-magic even and magic odd powers. On the other hand, triviality, i.e. the property of being a multiple of \(E\), obviously persists to all higher powers.

For the proof of Theorem 6 we require the following Vandermonde-type lemma.

**Lemma 3.** If the numbers \(\lambda_1, \ldots, \lambda_n \in \mathbb{C}\) satisfy

\[
\sum_{j=1}^{n} \lambda_j^k = 0 \quad (k \in \{1, \ldots, n\}),
\]

then \(\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0\).

**Proof** by induction. For \(n = 1\) the statement is trivial. Assume \(n \in \mathbb{N}\) is such that the statement is true for \(n - 1\). Then, considering the polynomial

\[
p(x) = \prod_{j=1}^{n} (x - \lambda_j) = \sum_{k=0}^{n} \alpha_k x^k,
\]

we see that

\[
0 = \sum_{j=1}^{n} p(\lambda_j) = \sum_{j=1}^{n} \sum_{k=0}^{n} \alpha_k \lambda_j^k = \sum_{k=0}^{n} \alpha_k \sum_{j=1}^{n} \lambda_j^k = n\alpha_0
\]

by (3.2), so \(\alpha_0 = 0\). Therefore 0 is a root of \(p\), so one of the \(\lambda_j\) vanishes. By suitable renumbering we can assume without loss of generality that \(\lambda_n = 0\); then \(\lambda_1, \ldots, \lambda_{n-1}\) satisfy the conditions (3.2) for \(n - 1\). By induction, they all vanish. \(\square\)

**Remark.** The lemma and its proof generalise to the case where (3.2) is replaced with

\[
\sum_{j=1}^{n} c_j \lambda_j^k = 0 \quad (k \in \{1, \ldots, n\})
\]
with non-zero coefficients $c_j \in \mathbb{C} \setminus \{0\}$.

**Proof** of Theorem 6. We can assume without loss of generality that $M$ has weight 0. By Theorem 1 (b) or Theorem 2 (b),

$$M = \mathcal{X}_m \begin{pmatrix} \tilde{Y} & \mathcal{O} \\ \mathcal{O} & Z \end{pmatrix} \mathcal{X}_m,$$

where $\tilde{Y}$ is an $n \times n$ matrix and $Z$ is an $n \times n$ or an $(n - 1) \times (n - 1)$ matrix, depending on whether $m$ is even or odd. If we assume that the powers $M, M^2, \ldots, M^n$ are all magic, then $\tilde{Y}, \tilde{Y}^2, \ldots, \tilde{Y}^n$ and $Z, Z^2, \ldots, Z^n$ all have trace 0, by Theorem 4. As the eigenvalues of $\tilde{Y}^k$ are just the $k$th powers of the eigenvalues of $\tilde{Y}$, this means that the eigenvalues of $\tilde{Y}$ satisfy the hypothesis of Lemma 3 and thus are all zero, and likewise the eigenvalues of $Z$ all vanish. Consequently, the Jordan normal forms of $\tilde{Y}$ and of $Z$ have all zero entries except possibly some entries 1 above the diagonal, and hence are nilpotent; in particular, $\tilde{Y}^n = \mathcal{O}$, $Z^n = \mathcal{O}$, and hence $M^n = \mathcal{O}_m$. \hfill \qed

Since the square of an associated magic square is a balanced semimagic square, Theorem 6 has the following immediate consequence.

**Corollary 1.** Let $M$ be an associated magic square matrix of size $2n \times 2n$ or $(2n - 1) \times (2n - 1)$. Then there is $N \in \{1, \ldots, n\}$ such that $M^{2N}$ is either trivial or not magic.

Of course, all odd powers of an associated magic square matrix are again associated (semi)magic square matrices and hence, in particular, magic.

This settles the matter for associated and for balanced magic square matrices; however, it does not solve the general case. For example, taking the block form of even-dimensional matrices,

$$\begin{pmatrix} Y & V^T \\ W & Z \end{pmatrix}^2 = \begin{pmatrix} Y^2 + V^T W & Y V^T + V^T Z \\ W Y + Z W & W V^T + Z^2 \end{pmatrix},$$

and the conditions $0 = \text{tr} Y^2 + \text{tr} V^T W$, $0 = \text{tr} Z^2 + \text{tr} W V^T$ do not easily relate to the properties of these matrices separately. However, we can apply the idea of the proof of Theorem 6 directly to the whole matrix, using the fact that for a weight 0 magic square matrix it is necessary, though not sufficient, that its trace vanish. This gives the following general statement, with a less tight upper bound on the number $N$.

**Theorem 7.** Let $M$ be a magic square. Then there is some positive integer $N$, not greater than the dimension of $M$, such that $M^N$ is either trivial or not magic.

### 4 Ranks, Quasi-Inverses and Spectral Properties

In Theorem 10, we have seen that adding a non-zero weight increases the rank of a weightless even-dimensional associated magic square matrix by 1. However, this can never lead to full rank. Indeed, since the block component matrices $V, W$ in the block representation

$$\begin{pmatrix} 2w \varepsilon_n & V^T \\ W & \mathcal{O}_n \end{pmatrix}$$

have row sum 0 and therefore rank $\leq n - 1$, the total rank of the matrix is at most $2n - 2$ if the weight is 0, and $2n - 1$ otherwise. In particular, an even-dimensional associated magic square matrix is never regular and does not have an inverse.
An $n \times n$ semimagic square matrix with weight zero will always have the vector $1_n$ in its null space and therefore cannot be regular; however we can define a (left or right) quasi-inverse to be a matrix which multiplies the square (to the left or right) to give the weightless part of the (semimagic) unit matrix,

$$U_n := I_n - \frac{1}{n} \mathcal{E}_n.$$ 

An even-dimensional associated magic square matrix, weighted or not, will not have a left nor a right quasi-inverse, as can be seen from the block representation

$$U_{2n} = X_{2n} \begin{pmatrix} U_n & \mathcal{O}_n \\ \mathcal{O}_n & I_n \end{pmatrix} X_{2n};$$ \hspace{1cm} (4.1)

the upper right-hand block of a right quasi-inverse of the block representation would have to be a right inverse of $W$, while the lower left-hand block of a left quasi-inverse would have to be a left inverse of $V$, and clearly neither is possible.

However, in the odd-dimensional case, the block representation is (see Theorem 2)

$$\begin{pmatrix} 2w\mathcal{E}_n & w\sqrt{2}1_n \\ w\sqrt{2}1_n^T & w \\ W & -\sqrt{2}W1_n \\ \mathcal{O}_n & \mathcal{O}_n \end{pmatrix},$$

where the matrices $V, W$ may have full rank $n$. Hence the maximal rank is $2n$ if the weight is 0 and $2n + 1$ otherwise; in the latter case, the matrix has full rank and therefore an inverse.

For a rank $2n$, $(2n + 1) \times (2n + 1)$ associated magic square matrix with weight 0, a right quasi-inverse can always be constructed, bearing in mind that $V, W$ have full rank $n$ and the matrix $I_n + 2\mathcal{E}_n$ is regular, as $-\frac{1}{2}$ is not an eigenvalue of $\mathcal{E}_n$. Indeed,

$$\begin{pmatrix} 0 & 0 & V^T \\ 0 & 0 & -\sqrt{2}(V1_n)^T \\ W & -\sqrt{2}W1_n & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & V'^T \\ 0 & 0 & -\sqrt{2}(V'1_n)^T \\ W' & -\sqrt{2}W'1_n & 0 \end{pmatrix} = X_{2n+1}U_{2n+1}X_{2n+1},$$

taking $W' := (V^T)^{-1}(1 - \frac{2\mathcal{E}_n}{2n+1})$, $V'^T := (1 + 2\mathcal{E}_n)^{-1}W^{-1}$. Note that the quasi-inverse is again the block structure matrix of a weight 0 associated magic square matrix. Here

$$X_{2n+1}U_{2n+1}X_{2n+1} = \begin{pmatrix} 1 - \frac{2\mathcal{E}_n}{2n+1} & -\sqrt{2}\frac{1}{2n+1} \\ -\sqrt{2}\frac{1}{2n+1} & 0_n \\ \mathcal{O}_n & \mathcal{O}_n \end{pmatrix}.$$

In summary, we have the following statement.

**Theorem 8.** (a) Even-dimensional associated magic square matrices, with or without weight, never have full rank, nor a quasi-inverse.

(b) Odd-dimensional associated magic square matrices may have full rank if weighted; if the weight is 0, then the maximal rank is 1 less than the dimension, and in this case left and right quasi-inverses exist.

Turning to the case of (weightless) balanced semimagic square matrices, we note that in the block representation for the odd-dimensional case,

$$\begin{pmatrix} Y & -\sqrt{2}Y1_n & 0 \\ -\sqrt{2}1_n^TY & 21_n^TY1_n & 0 \\ 0 & 0 & Z \end{pmatrix},$$
the top left \((n + 1) \times (n + 1)\) matrix \(Y\) has maximal rank \(n\) (when \(Y\) has rank \(n\)), because the \(n + 1\)st row is a linear combination of the first \(n\) rows. Therefore the maximal rank of the balanced semimagic square, achieved if both \(Y, Z\) have full rank \(n\), is \(2n\), and there is no inverse. However, there is always a quasi-inverse in this case; with \(Y' := (1 + 2E_n)^{-1}Y^{-1}(I_n - \frac{2}{2n+1}E_n)\),

\[
\begin{pmatrix}
Y & -\sqrt{2}Y1_n^T \\
-\sqrt{2}1_n^TY & 21_n^TY1_n
\end{pmatrix}
\begin{pmatrix}
Y' & -\sqrt{2}Y'1_n^T \\
-\sqrt{2}1_n^TY' & 21_n^TY'1_n
\end{pmatrix}
= \begin{pmatrix}
I_n - \frac{2}{2n+1}E_n & -\sqrt{2}1_n^T \\
-\sqrt{2}1_n & 1 - \frac{1}{2n+1}
\end{pmatrix},
\]
and completing a \((2n + 1) \times (2n + 1)\) matrix with \(Z^{-1}\) in the lower right corner, we obtain the block representation of a right quasi-inverse which is again a weightless balanced semimagic square matrix.

The block representation of a \(2n\)-dimensional weightless balanced semimagic square matrix is

\[
\begin{pmatrix}
Y & 0 \\
0 & Z
\end{pmatrix},
\]
where \(Y\) is a weightless semimagic square matrix. Hence the maximal possible rank is \(2n - 1\), and there will not be an inverse. Regarding a quasi-inverse, we see from (4.11) that \(Z\) must be invertible and \(Y\) must have a quasi-inverse. If \(n\) is odd and \(Y\) has maximal rank \(n - 1\) and is either associated or balanced, then there exists a quasi-inverse by the above considerations. If \(n\) is even and \(Y\) is associated, then no quasi-inverse exists. If \(n\) is even and \(Y\) is balanced, then we can apply these considerations recursively to the block structure of \(Y\).

Unfortunately, the case of mixed type, i.e. of a general weightless semimagic square matrix, seems rather more difficult to analyse; this case will generally occur when applying the above reduction to an even-dimensional balanced semimagic square matrix. Also, the case of a balanced semimagic square matrix with weight seems rather difficult, as it is not obvious whether the addition of a weight always raises the rank by 1.

**Theorem 9.** Let

\[ M_0 = \mathcal{X}_{2n} \begin{pmatrix} O_n & V^T \\ W & O_n \end{pmatrix} \mathcal{X}_{2n} \]

be a weightless associated magic square matrix, and for \(w \in \mathbb{R} \setminus \{0\}\), let \(M_w := M_0 + w \mathcal{E}_{2n}\). Then \(M_w^2\) has the same non-zero eigenvalues (including multiplicities) as \(M_0^2\), with \(w\)-independent eigenvectors, and the additional eigenvalue \(2n^2w^2\). Moreover, \(\text{rk} M_w^2 = \text{rk} M_0^2 + 1\).

**Proof.** Since \(M_0 \mathcal{E}_{2n} = O_{2n} = \mathcal{E}_{2n}M_0\),

\[ M_w^2 = M_0^2 + w (M_0 \mathcal{E}_{2n} + \mathcal{E}_{2n}M_0) + w^2 \mathcal{E}_{2n}^2 = M_0^2 + 2nw^2 \mathcal{E}_{2n}. \]

Now assume that \(\lambda \neq 0\) is an eigenvalue of the block representation of \(M_0^2\) (which, as \(\mathcal{X}_{2n}\) is unitary, has the same eigenvalues as \(M_0^2\) ), so there is a vector

\[ w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in \mathbb{C}^{2n} \setminus \{0\} \]

such that

\[ \begin{pmatrix} V^TW & O_n \\ O_n & WV^T \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \lambda \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}. \]
Then $V^TWw_1 = \lambda w_1$, so $w_1 \in \text{ran } V^T$, i.e. $w_1$ is a linear combination of columns of $V^T$, and hence $\mathbf{1}_n^T w_1 = 0$. Consequently, $\mathcal{E}_n w_1 = 0_n$. Therefore, considering the block representation of $M_w^2$,

$$
\begin{pmatrix}
V^TW + 4n w^2 \mathcal{E}_n & \mathcal{O}_n \\
\mathcal{O}_n & WV^T
\end{pmatrix}
\begin{pmatrix}
w_1 \\
w_2
\end{pmatrix}
= \begin{pmatrix}
\lambda w_1 + 0_n \\
\lambda w_2
\end{pmatrix}
= \lambda
\begin{pmatrix}
w_1 \\
w_2
\end{pmatrix},
$$

which shows that $\lambda$ is still an eigenvalue of $M_w^2$. Moreover, using the fact that $W \mathbf{1}_n = 0_n$, we see that

$$
\begin{pmatrix}
V^TW + 4n w^2 \mathcal{E}_n & \mathcal{O}_n \\
\mathcal{O}_n & WV^T
\end{pmatrix}
\begin{pmatrix}
1_n \\
0_n
\end{pmatrix}
= 4n^2 w^2
\begin{pmatrix}
1_n \\
0_n
\end{pmatrix},
$$

so $4n^2 w^2$ is a further eigenvalue of $M_w^2$.

Conversely, if $\lambda$ is an eigenvalue of $M_w^2$, so

$$
\begin{pmatrix}
V^TW + 4n w^2 \mathcal{E}_n & \mathcal{O}_n \\
\mathcal{O}_n & WV^T
\end{pmatrix}
\begin{pmatrix}
w_1 \\
w_2
\end{pmatrix}
= \lambda
\begin{pmatrix}
w_1 \\
w_2
\end{pmatrix},
$$

then we can write $w_1 = \alpha \mathbf{1}_n + z$, where $z \in \mathbb{R}^n$ with $z^T \mathbf{1}_n = 0$, and find

$$(V^TW + 4n w^2 \mathcal{E}_n)w_1 = V^TWz + 4n^2 w^2 \alpha \mathbf{1}_n = \lambda z + \lambda \alpha \mathbf{1}_n.$$ 

Since $V^TWz$ is orthogonal to $\mathbf{1}_n$, this implies

$$V^TWz = \lambda z, \quad \lambda \alpha = 4n^2 w^2 \alpha,$$

so $\lambda = 4n^2 w^2$ unless $\alpha = 0$, and $\lambda$ is an eigenvalue of $M_0^2$ unless $z = w_2 = \mathbf{1}_n$.

For the statement about ranks, note that $V^TW$ maps the orthogonal complement $\{ \mathbf{1}_n \}^\perp \subset \mathbb{R}^n$ into itself, so adding a non-zero multiple of $\mathcal{E}_n$ adds one dimension, parallel to $\mathbf{1}_n$, to the range of the matrix. \hfill $\square$

We remark that the following analogous statement holds for the eigenvalues of the associated magic square matrix itself.

**Theorem 10.** Let $M_0$ and $M_w$ be as in Theorem 9. Then $M_w$ has the same non-zero eigenvalues as $M_0$ and an additional simple eigenvalue $2nw$. The eigenvectors are independent of $w$. Moreover, $\text{rk } M_w = \text{rk } M_0 + 1$.

**Proof.** Assume

$$
\begin{pmatrix}
0 & V^T \\
W & 0
\end{pmatrix}
\begin{pmatrix}
w_1 \\
w_2
\end{pmatrix}
= \lambda
\begin{pmatrix}
w_1 \\
w_2
\end{pmatrix},
$$

with non-zero $\lambda$; then $V^Tw_2 = \lambda w_1$ implies that $\mathbf{1}_n^T w_1 = 0$. The remainder of the proof is completely analogous to that of Theorem 9. \hfill $\square$

Hence we can omit the weight of the associated magic square w.l.o.g. in the following. The weightless squared block matrix

$$
\begin{pmatrix}
V^TW & \mathcal{O}_n \\
\mathcal{O}_n & WV^T
\end{pmatrix},
$$

is obviously the direct sum of the matrices $V^TW$ and $WV^T$, with separate quadratic forms — the $\xi$ part and the $\eta$ part of the quadratic form (5.1) — and correspondingly the quadratic
form generated by $M^2$ splits in a natural way into two quadratic forms which are generated by $BA$ and $AB$, respectively. These forms will be expressible as a sum of squares (possibly with irrational coefficients) if both $V^TW$ and $WV^T$ are diagonalisable, in particular if they are symmetric. This motivates the following definition.

**Definition.** We call the associated magic square matrix $M$ *parasymmetric* if its square $M^2$ is a symmetric matrix.

In terms of the block representation, parasymmetry can be characterised as follows if the two constituent blocks of $M$ have rank 1.

**Lemma 4.** Let $V, W \in \mathbb{R}^{n \times n}$ have row sum 0 and rank 1, and consider the associated magic square matrix

$$M = X_{2n} \begin{pmatrix} O_n & V^T \\ W & O_n \end{pmatrix} X_{2n}.$$

If $M^2 \neq O_{2n}$, then $M$ is parasymmetric if and only if $W$ is a multiple of $V$.

**Proof.** We can write $V = uv^T$, $W = xy^T$ with non-null vectors $u, v, x, y \in \mathbb{R}^n \setminus \{0_n\}$; then $V^T = vu^T$ and $W^T = yx^T$. Now if $V^TW$ is symmetric, then

$$(u^T x) vy^T = V^TW = W^TV = (x^T u) yv^T,$$

so either $u^T x = 0$ (and note that $u^T x = x^T u$) or $vy^T = yv^T$; but in the latter case we see, multiplying by $y$ on the right, that

$$v(y^T y) = y(v^T y),$$

so $v$ and $y$ are linearly dependent. Similarly, if $WV^T$ is symmetric, then either $v^T y = 0$ or $u$ and $x$ are linearly dependent.

Now, if it should happen that $u^T x = 0$, then we must also have $v^T y = 0$ (since otherwise, by the above, $u, x$ will be simultaneously orthogonal and linearly dependent, which is impossible as both are non-null vectors), and vice versa; and in this situation $V^TW = O_n = WV^T$, which would imply $M^2 = O_{2n}$.

Hence, we find that $x, y$ are multiples of $u, v$, respectively, so $W$ is a multiple of $V$.

The converse statement is obvious. \[\square\]

More generally, a matrix is diagonalisable by conjugation with an orthogonal matrix if it commutes with its transpose, i.e. if it is *normal*. Obviously, any symmetric matrix is normal. If $M$ is normal, then it is easy to see that $M^2$ is normal, too; the converse is not so clear. In analogy to parasymmetry, we call the matrix $M$ *paranormal* if $M^2$ is normal. However, this turns out to be no more general than parasymmetry as far as associated magic square matrices with rank 1 blocks are concerned, as the following result shows.

**Lemma 5.** Let $M \in \mathbb{R}^{2n \times 2n}$ be a weightless associated magic square matrix with rank 1 blocks $V, W$ such that $M^2 \neq O_{2n}$. If $M$ is paranormal, then it is parasymmetric.

**Proof.** Let $V, W$ and $u, v, x, y$ be as in the proof of Lemma 4. Then the paranormality means that

$$V^TWV^T = W^TVV^T,$$
$$WV^TWV^T = VWTV^T;$$

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in terms of the generating vectors, this gives the two identities
\[ v(u^T x)(y^T y)(x^T u)v^T = y(x^T u)(v^T v)(u^T x)y^T, \]
\[ x(y^T v)(u^T u)(v^T y)x^T = u(yv^T)(x^T x)(y^T v)y^T, \]
and by the same reasoning as above, this implies that \( u^T x = 0 \) or \( v, y \) are linearly dependent, and that \( v^T y = 0 \) or \( x, u \) are linearly dependent. As before, the cases of orthogonality can only occur together and then give a trivial \( M^2 \); it follows that \( V, W \) are linearly dependent and hence that the matrix is parasymmetric. □

The following observation on the eigenvalues of a squared weightless associated magic square matrix with rank 1 blocks follows from the proof of Theorem 5.

**Theorem 11.** If \( M \) is the weightless associated magic square matrix
\[
M = X_{2n} \begin{pmatrix} O_n & vu^T \\ xy^T & O_n \end{pmatrix} X_{2n} \tag{4.2}
\]
with rank 1 block components \( V = uv^T \), \( W = xy^T \), then \( M^2 \) has eigenvalues 0 and \( (u^T x)(y^T v) = \text{Tr} V^T W \). In particular, if \( V, W \) have integer entries, then the eigenvalues of \( M^2 \) are integers.

**Remarks.** 1. If the eigenvalue \( (u^T x)(y^T v) \) is non-zero, then its (algebraic and geometric) multiplicity will be 2, as it will be an eigenvalue of both the upper left and the lower right blocks in the block representation of \( M^2 \), with eigenvectors \( \begin{pmatrix} v \\ 0_n \end{pmatrix} \) and \( \begin{pmatrix} 0_n \\ x \end{pmatrix} \), respectively. However, when exactly one of the products \( V^T W, W^T V \) vanishes, then its geometric multiplicity will only be 1. This can happen in the non-parasymmetric case, as the non-orthogonality of generating vectors for non-trivial magic squares, found in the proof of Lemma 4, need not hold in this case. Note that in this situation, this eigenvalue will indeed be 0, so the matrix \( M^2 \) will have eigenvalue 0 only with algebraic multiplicity \( 2n \) and geometric multiplicity \( 2n - 1 \). For example, consider
\[
V = uv^T = \begin{pmatrix} 1 \\ -3 \\ -5 \\ 7 \end{pmatrix} (1, -1, -1, 1),
\]
\[
W = xy^T = 8 \begin{pmatrix} 1 \\ -3 \\ -5 \\ 7 \end{pmatrix} (1, -1, 1, -1);
\]
here \( x \) is a multiple of \( u \), but \( y^T v = 0 \), so \( W^T V = 0 \). The resulting weightless associated magic square matrix
\[
M = \frac{1}{2} \begin{pmatrix}
63 & -61 & 53 & -55 & -57 & 59 & -51 & 49 \\
-47 & 45 & -37 & 39 & 41 & -43 & 35 & -33 \\
-31 & 29 & -21 & 23 & 25 & -27 & 19 & -17 \\
15 & -13 & 5 & -7 & -9 & 11 & -3 & 1 \\
-1 & 3 & -11 & 9 & 7 & -5 & 13 & -15 \\
17 & -19 & 27 & -25 & -23 & 21 & -29 & 31 \\
33 & -35 & 43 & -41 & -39 & 37 & -45 & 47 \\
-49 & 51 & -59 & 57 & 55 & -53 & 61 & -63
\end{pmatrix}
\]
has rank 2, but its square \( M^2 \) only has rank 1.
2. If \((u^T x)(y^T v) \neq 0\), then corresponding linearly independent (right) eigenvectors of \(M^2\) will be
\[
\sqrt{2} \lambda_{2n} \begin{pmatrix} v \\ 0 \end{pmatrix} = \begin{pmatrix} v \\ J_n v \end{pmatrix}, \quad -\sqrt{2} \lambda_{2n} \begin{pmatrix} 0_n \\ x \end{pmatrix} = \begin{pmatrix} -J_n x \\ x \end{pmatrix};
\]
the first of these is even, the other odd under reflection (i.e. multiplication with \(J_{2n}\)). These eigenvectors are orthogonal for structural reasons, reflecting the fact that they belong to different direct summands in the block representation of \(M^2\). Clearly, if \(v, x\) have integer entries, then so do these eigenvectors.

We note that there are also the left eigenvectors (i.e. eigenvectors of \((M^2)^T\)) given by
\[
\begin{pmatrix} y \\ J_n y \end{pmatrix}, \quad \begin{pmatrix} -J_n u \\ u \end{pmatrix};
\]
again, these eigenvectors are structurally orthogonal.

3. In the parasymmetric case \(y = v, x = ku\), where \(u, v \in \mathbb{R}^n \setminus \{0\}\), the matrix \(M^2\) always has a non-zero eigenvalue \(k(u^T u)(y^T v)\).

4. In the situation of Theorem 11 with \(u^T x \neq 0, y^T v \neq 0\), the matrix \(M\) has the two simple non-zero eigenvalues \(\sqrt{(u^T x)(y^T v)}, -\sqrt{(u^T x)(y^T v)}\). Indeed, any eigenvector of the block representation
\[
\begin{pmatrix} \mathcal{O}_n & vu^T \\ xy^T & \mathcal{O}_n \end{pmatrix}
\]
for non-zero eigenvalue \(\lambda\) can easily be seen to be of the form \(\begin{pmatrix} \alpha v \\ \beta x \end{pmatrix}\) with \(\alpha, \beta \neq 0\), and hence \(\lambda^2 = (u^T x)(y^T v)\). As the trace of the matrix vanishes and 0 is the only other eigenvalue, both signs of the square root occur. The coefficients for the eigenvectors can be chosen as \(\alpha = \lambda, \beta = y^T v\), so if \(u, v, x, y\) are integer vectors and \(\lambda\) is an integer, then there are integer eigenvectors as well. In the parasymmetric case with integer vectors \(u, v\) and integer parasymmetry factor \(k\), the eigenvalues will be integers if and only if the square-free part of \(k\) is equal to the square-free part of the product \((v^T v)(u^T u)\).

**Lemma 6.** Let \(n \geq 2\), and let \(M \in \mathbb{R}^{2n \times 2n}\) be a weightless associated magic square matrix with \(n \times n\) rank 1 block components
\[
M = \mathcal{X}_{2n} \begin{pmatrix} \mathcal{O}_n & vu^T \\ xy^T & \mathcal{O}_n \end{pmatrix} \mathcal{X}_{2n},
\]
where \(u, v, x, y \in \mathbb{R}^n\) such that \(\lambda := (u^T x)(y^T v) \neq 0\). Then the minimal polynomial of \(M\) is
\[
x^3 - \lambda x.
\]

**Remark.** In the case \(n = 1\), the minimal polynomial is \(x^2 - \lambda\).

**Proof.** By straightforward calculation,
\[
M^3 = \mathcal{X}_{2n} \begin{pmatrix} \mathcal{O}_n & vu^T \\ xy^T & \mathcal{O}_n \end{pmatrix} \begin{pmatrix} \mathcal{O}_n & vu^T \\ xy^T & \mathcal{O}_n \end{pmatrix} \begin{pmatrix} \mathcal{O}_n & vu^T \\ xy^T & \mathcal{O}_n \end{pmatrix} \mathcal{X}_{2n} = \mathcal{X}_{2n} \begin{pmatrix} \\ & v(u^T x)(y^T v)u^T \\ & x(y^T v)(u^T x) \\ & \mathcal{O}_n \end{pmatrix} \mathcal{X}_{2n},
\]
\[
= (u^T x)(y^T v) \mathcal{X}_{2n} \begin{pmatrix} \mathcal{O}_n & vu^T \\ xy^T & \mathcal{O}_n \end{pmatrix} \mathcal{X}_{2n} = \lambda M.
\]
To see that this is indeed the minimal polynomial for \( M \), we note that the block representation of \( M \) is not a multiple of the unit matrix, which rules out a linear polynomial, and that when inserted in a quadratic polynomial \( x^2 + ax + b \),

\[
\begin{pmatrix}
v(u^T x) y^T \\
o_n \\
x(y^T v) u^T
\end{pmatrix} + a \begin{pmatrix}
o_n \\
v u^T \\
o_n
\end{pmatrix} + b \begin{pmatrix}
o_n \\
o_n \\
o_n \ I_n
\end{pmatrix} = O_{2n}
\]

implies \( v(u^T x) y^T = -b I_n = x(y^T v) u^T \), which is impossible, as \( I_n \) has rank \( n > 1 \).

To conclude this section, we show a construction yielding a two-sided, regular eigenvector matrix for \( M^2 \), where \( M \) is the rank \( 1 + 1 \) associated magic square matrix as defined in (4.2). Here ‘two-sided’ means that the columns of the matrix are right eigenvectors of \( M^2 \) while its rows are left eigenvectors of \( M^2 \). We begin by considering the two right eigenvectors (4.3) of \( M^2 \) corresponding to the non-zero eigenvalue \( \lambda = (u^T x)(y^T v) \). These eigenvectors are placed side by side to form a \( 2n \times 2 \) matrix

\[
P_1 = \begin{pmatrix} B_1 & A_1 \\ A_1 & C_1 \end{pmatrix} = \begin{pmatrix} v & -J_n x \\ J_n v & x \end{pmatrix},
\]

where \( A_1 = \begin{pmatrix} v_n & -x_1 \\ x_1 & v_n \end{pmatrix} \) and \( B_1 \) and \( C_1 \) are \( (n - 1) \times 2 \) matrices such that \( C_1 = J_{n-1} B_1 \sigma_3 \) (with \( \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \)). We make the assumption that \( v_n, x_1 \neq 0 \), so that the matrix \( A_1 \) is regular. The construction below can be generalised to the case where \( A_1 \) is any regular matrix composed of two rows of \( P_1 \) (and indeed one can always find two linearly independent rows of \( P_1 \) because its columns are linearly independent), but we take the centre rows in the following for simplicity.

Now define

\[
\hat{P}_1 = -P_1 A_1^{-1} = \begin{pmatrix} -B_1 A_1^{-1} \\ -J_2 \\ -C_1 A_1^{-1} \end{pmatrix}.
\]

Similarly, starting from the matrix of left eigenvectors (4.4), we set

\[
P_2 = \begin{pmatrix} B_2 & A_2 \\ A_2 & C_2 \end{pmatrix} = \begin{pmatrix} y & -J_n u \\ J_n y & u \end{pmatrix},
\]

and assuming \( y_n, u_1 \neq 0 \), so that \( A_2 = \begin{pmatrix} y_n & -u_1 \\ y_n & u_1 \end{pmatrix} \) is regular, we define

\[
\hat{P}_2 = -P_2 A_2^{-1} = \begin{pmatrix} -B_2 A_2^{-1} \\ -J_2 \\ -C_2 A_2^{-1} \end{pmatrix}.
\]

The columns of \( \hat{P}_1 \) and \( \hat{P}_2 \) will still be linearly independent eigenvectors, for eigenvalue \( \lambda \), of \( M^2 \) and of \( (M^2)^T \), respectively, but in general they will no longer be orthogonal unless \( v_n^2 = x_1^2 \) and \( y_n^2 = u_1^2 \). However, due to the special structure of our chosen matrices \( A_j \), they will have the symmetry that the second column is the reversal of the first. Indeed, \( J_2 A_j \sigma_3 = A_j \), so \( \sigma_3 A_j^{-1} J_2 = A_j^{-1} (j \in \{1, 2\}) \), and it follows that

\[
J_{2n} \hat{P}_j J_2 = \begin{pmatrix} -J_{n-1} C_j A_j^{-1} \\ -J_2 \\ -J_{n-1} B_j A_j^{-1} \end{pmatrix} J_2 = \begin{pmatrix} -B_j \sigma_3 A_j^{-1} J_2 \\ -J_2 \\ -C_j \sigma_3 A_j^{-1} J_2 \end{pmatrix} = \hat{P}_j \quad (j \in \{1, 2\}),
\]

where \( \hat{P}_j \) is the two-sided regular eigenvector matrix of \( M^2 \), satisfying the condition that its columns are right eigenvectors of \( M^2 \) while its rows are left eigenvectors of \( M^2 \).
which means that swapping the columns of \( \tilde{P}_j \) is tantamount to turning them upside down.

The matrices \( \tilde{P}_1 \) and \( \tilde{P}_2 \) have the following remarkable connection with the matrix \( M \). Using the notation \( (\cdot ; \cdot ; \cdot) \) to express that the three matrices are juxtaposed to form one \( 2n \times 2n \) matrix, we calculate

\[
(O_{2n,n-1} | \tilde{P}_1 | O_{2n,n-1})M = -P_1(O_{2n,n-1} | A_1^{-1} | O_{2n,n-1})A_2n \begin{pmatrix} O_n & vu^T \\ xy^T & O_n \end{pmatrix} X_{2n} = -M,
\]

and similarly \( (O_{2n,n-1} | \tilde{P}_2 | O_{2n,n-1})M^T = -M^T \).

**Theorem 12.** Let \( u, v, x, y \in \mathbb{R}^n \) such that \( \lambda := (u^T x)(y^T v) \neq 0 \) and \( u_1, v_1, x, y \neq 0 \). Let \( M \) be the matrix \( \begin{pmatrix} 4.2 \end{pmatrix} \), and let \( \tilde{P}_1 \) and \( \tilde{P}_2 \) be defined as in \( \begin{pmatrix} 4.3 \end{pmatrix}, \begin{pmatrix} 4.6 \end{pmatrix} \). Then

\[
P = I_{2n} + (O_{2n,n-1} | \tilde{P}_1 | O_{2n,n-1}) + \begin{pmatrix} O_{n-1,2n} \\ \tilde{P}_2^T \\ O_{n-1,2n} \end{pmatrix}
\]

is a two-sided eigenvector matrix for \( M^2 \), so that \( M^2 P = P \text{diag}(0_{n-1}, \lambda_1, 0_{n-1}) \) and \( PM^2 = \text{diag}(0_{n-1}, \lambda_1, 0_{n-1}) P \). Moreover, \( P \) has the inverse

\[
P^{-1} = \text{diag}(1_{n-1}, 0_2, 1_{n-1}) - \frac{M^2}{\lambda}.
\]

**Proof.** Using the second of the above identities and the fact that the columns of \( \tilde{P}_1 \) are eigenvectors of \( M^2 \), we find

\[
M^2 P = M^2 + M^2 (O_{2n,n-1} | \tilde{P}_1 | O_{2n,n-1}) + MM \begin{pmatrix} O_{n-1,2n} \\ \tilde{P}_2^T \\ O_{n-1,2n} \end{pmatrix}
\]

on the other hand, since the central \( 2 \times 2 \) part of \( \tilde{P}_2 \) is equal to \( -I_2 \), we have

\[
P \text{diag}(0_{n-1}, \lambda_1, 0_{n-1}) = \text{diag}(0_{n-1}, \lambda_1, 0_{n-1}) + (O_{2n,n-1} | \tilde{P}_1 | O_{2n,n-1}) \text{diag}(0_{n-1}, \lambda_1, 0_{n-1})
\]

\[
+ \begin{pmatrix} O_{n-1,2n} \\ \tilde{P}_2^T \\ O_{n-1,2n} \end{pmatrix} \text{diag}(0_{n-1}, \lambda_1, 0_{n-1})
\]

\[
= \text{diag}(0_{n-1}, \lambda_1, 0_{n-1}) + \lambda (O_{2n,n-1} | \tilde{P}_1 | O_{2n,n-1}) - \text{diag}(0_{n-1}, \lambda_1, 0_{n-1}).
\]

The relation for \( PM^2 \) follows by a pair of completely analogous calculations.

To verify the inverse relation

\[
P^{-1} = \text{diag}(1_{n-1}, 0_2, 1_{n-1}) - \frac{M^2}{\lambda},
\]
we note that
\[
\text{diag}(1_{n-1}, 0_2, 1_{n-1}) - \frac{M^2}{\lambda} P = \text{diag}(1_{n-1}, 0_2, 1_{n-1}) P - P \text{diag}(0_{n-1}, 1_2, 0_{n-1})
\]
\[
= \text{diag}(1_{n-1}, 0_2, 1_{n-1}) + \text{diag}(1_{n-1}, 0_2, 1_{n-1}) \left( O_{2n,n-1} \mid \tilde{P}_1 \mid O_{2n,n-1} \right) \\
+ \text{diag}(1_{n-1}, 0_2, 1_{n-1}) \left( O_{n-1,2n} \tilde{P}_2^T \right) - \text{diag}(0_{n-1}, 1_2, 0_{n-1}) \\
- \left( O_{2n,n-1} \mid \tilde{P}_1 \mid O_{2n,n-1} \right) \text{diag}(0_{n-1}, 1_2, 0_{n-1}) - \left( O_{n-1,2n} \tilde{P}_2^T \right) \text{diag}(0_{n-1}, 1_2, 0_{n-1}) \\
= \text{diag}(1_{n-1}, 0_2, 1_{n-1}) + \left( O_{2n,n-1} \mid \tilde{P}_1 \mid O_{2n,n-1} \right) + \text{diag}(0_{n-1}, 1_2, 0_{n-1}) + O_{2n} \\
- \text{diag}(0_{n-1}, 1_2, 0_{n-1}) - \left( O_{2n,n-1} \mid \tilde{P}_1 \mid O_{2n,n-1} \right) + \text{diag}(0_{n-1}, 1_2, 0_{n-1}) \\
= \text{diag}(1_{2n}) = I_{2n}.
\]
The opposite product follows similarly. 

\[\square\]

\textbf{Remark.} In the parasymmetric case, we have \( y = v, x = ku \), which, following the above construction, gives rise to the eigenvector matrices
\[
\begin{pmatrix} y \\
\mathcal{J}_ny \\
ku 
\end{pmatrix} = P_2 \begin{pmatrix} 1 & 0 \\
0 & k \end{pmatrix}.
\]
However, in this situation the vector \( \begin{pmatrix} -\mathcal{J}_nu \\
u \end{pmatrix} \) will be an eigenvector just as well as \( \begin{pmatrix} -\mathcal{J}_nu \\
ku \end{pmatrix} \), so we can begin with this vector and take \( P_1 = P_2 \) instead of the above. Hence, in this instance, \( P = I_{2n} + \left( O_{n-1} \mid \tilde{P}_1 \mid O_{n-1} \right) + \left( O_{n-1} \mid \tilde{P}_2 \mid O_{n-1} \right)^T \) will be a symmetric matrix.

\section{Quadratic Forms from Squares of Associated Magic Squares}

In this section we focus on the case of \( 2n \times 2n \) associated magic squares, establishing a connection between their block representation vectors and certain types of quadratic forms. Here, our aim is just to establish a link between the associated magic square matrices with rank \( 1 + 1 \) block representations and quadratic forms. A deeper examination of how the vector structures and quadratic forms interrelate over both the field of rationals and the ring of integers will be left to later further investigation.

We start from the observation that the block representation of the associated magic square
\[
M = X_{2n} \begin{pmatrix} 2w\mathcal{E}_n & V^T \\
W & O_n \end{pmatrix} X_{2n}
\]
has a natural decomposition into 3 parts, as
\[
\begin{pmatrix} 2w\mathcal{E}_n & V^T \\
W & O_n \end{pmatrix} = 2w \begin{pmatrix} \mathcal{E}_n & O_n \\
O_n & O_n \end{pmatrix} + \begin{pmatrix} O_n & V^T \\
O_n & O_n \end{pmatrix} + \begin{pmatrix} O_n & O_n \\
W & O_n \end{pmatrix} =: 2we + a + b,
\]
where \( a^2 = O_n, b^2 = O_n, e^2 = ne, ae = O_n, eb = O_n, ea = O_n, be = O_n \) (the last two identities following from the fact that the rows of \( V \) and \( W \) sum to 0), and
\[
ab = \begin{pmatrix} V^TW & O_n \\
O_n & O_n \end{pmatrix}, \quad ba = \begin{pmatrix} O_n & O_n \\
O_n & WV^T \end{pmatrix}.
\]
This corresponds to a splitting of the magic square

\[ M = w\mathcal{E}_{2n} + A + B, \]

where

\[ A = \mathcal{X}_{2n}a\mathcal{X}_{2n} = \frac{1}{2} \begin{pmatrix} V^TJ & -V^T \\ JW^T & JW \end{pmatrix}, \quad B = \mathcal{X}_{2n}b\mathcal{X}_{2n} = \frac{1}{2} \begin{pmatrix} JW & JW^T \\ -W & -W^T \end{pmatrix}. \]

Now if we consider the (balanced semimagic square) matrix

\[ M^2 = \mathcal{X}_{2n} \begin{pmatrix} 4nw^2\mathcal{E}_n + V^TW & \mathcal{O}_n \\ \mathcal{O}_n & WVT \end{pmatrix} \mathcal{X}_{2n}, \]

its block representation generates the quadratic form on \( \mathbb{R}^{2n} \)

\[ \begin{pmatrix} \xi \\ \eta \end{pmatrix}^T \mathcal{X}_{2n} M^2 \mathcal{X}_{2n} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = 4nw^2\xi^T\mathcal{E}_n\xi + \xi^TV^TW\xi + \eta^TWVT\eta \quad (\xi, \eta \in \mathbb{R}^n). \quad (5.1) \]

Splitting the vector \( \xi \) into a part parallel to \( 1_n \) and a part orthogonal to \( 1_n \),

\[ \xi = \frac{1}{n}(1_n^T\xi)1_n + (\xi - \frac{1}{n}(1_n^T\xi)1_n), \]

we find

\[ \mathcal{E}_n\xi = \frac{1}{n}(1_n^T\xi)n1_n + 0_n \]

and

\[ W\xi = 0_n + W(\xi - \frac{1}{n}(1_n^T\xi)1_n); \]

this means that the first term in the quadratic form (5.1) only acts non-trivially in the subspace of \( \mathbb{R}^n \) spanned by \( 1_n \), the second term only in the orthogonal subspace. Hence, by effectively restricting the quadratic form to the subspace \( 1_n^T \oplus \mathbb{R}^n \), we can fully separate off the weight and essentially reduce the study to the weightless matrix. (This is related to the spectral separation property of the weight matrix observed in Theorem 9.)

Considering more specifically the quadratic form (5.1) with weight \( w = 0 \) and taking for \( M \) a parasmatic associated magic square matrix with rank 1 blocks (4.2) with \( y = v, x = ku \), we find

\[ \xi^TV^TW\xi + \eta^TWVT\eta = k(u^T)(v^T\xi)^2 + k(v^T)(u^T\eta)^2, \]

so setting

\[ \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \mathcal{X}_{2n}X = \frac{1}{\sqrt{2}} \begin{pmatrix} X_1 + \mathcal{J}_nX_2 \\ \mathcal{J}_nX_1 - X_2 \end{pmatrix} \]

for \( X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \in \mathbb{R}^{2n} = \mathbb{R}^n \oplus \mathbb{R}^n \), we obtain the quadratic form

\[ X^TM^2X^T = \frac{k}{2}(u^Tu)^2 + \frac{k}{2}(v^Tv)^2 \quad (X \in \mathbb{R}^{2n}). \]
Introducing the reduced variables
\[ z_1 = \begin{pmatrix} v \\ \mathcal{J}_n v \end{pmatrix}^T w, \quad z_2 = \begin{pmatrix} -\mathcal{J}_n u \\ u \end{pmatrix}^T X, \]
this gives the reduced quadratic form
\[ q_1(z_1, z_2) = \frac{k}{2} \left( (u^T u) z_1^2 + (v^T v) z_2^2 \right). \]

Alternatively we can represent the quadratic form in terms of the spectral decomposition of \( M^2 \), still focussing on the case of a rank 1+1 matrix \( M \), but not necessarily parasymmetric. We begin with any regular (right) eigenvector matrix \( \lambda \), (not necessarily the first two columns) which are eigenvectors of \( M^2 \) for the non-zero eigenvalue \( \lambda \), while the other columns \( b_3, \ldots, b_{2n} \) are eigenvectors for eigenvalue 0.

Then, with the transformation \( X = P\alpha = \sum_{j=1}^{2n} \alpha_j b_j \) (where we sort the indices of the vector \( \alpha \in \mathbb{R}^{2n} \) in the manner corresponding to the numbering of the columns of \( P \)), we find the quadratic form
\[ q(\alpha) = X^T M^2 X = (\alpha_1 b_1 + \alpha_2 b_2 + \sum_{j=3}^{2n} \alpha_j b_j)^T \lambda (\alpha_1 b_1 + \alpha_2 b_2) \]
\[ = \lambda \left( b_1^T b_1 \alpha_1^2 + 2 b_1^T b_2 \alpha_1 \alpha_2 + b_2^T b_2 \alpha_2^2 + \sum_{j=3}^{2n} (b_j^T b_1 \alpha_j \alpha_1 + b_j^T b_2 \alpha_j \alpha_2) \right). \]

In the parasymmetric case where \( M^2 \) is symmetric, \( b_3, \ldots, b_{2n} \) are orthogonal on \( b_1, b_2 \), so the quadratic form simplifies to the binary form
\[ q_2(\alpha_1, \alpha_2) = X^T M^2 X = \lambda (b_1^T b_1 \alpha_1^2 + 2 b_1^T b_2 \alpha_1 \alpha_2 + b_2^T b_2 \alpha_2^2); \]
if also \( b_1 \) and \( b_2 \) are mutually orthogonal (as is the case for the ‘natural’ eigenvectors \([4,3]\), but not necessarily for the transformed column eigenvectors of the matrix \( P \) constructed in Theorem \([12]\), then we get the simple form
\[ q_3(\alpha_1, \alpha_2) = X^T M^2 X = \lambda (b_1^T b_1 \alpha_1^2 + b_2^T b_2 \alpha_2^2). \]

We now illustrate these results with a parasymmetric and a non-parasymmetric example.

**Example 1.** Let \( u^T = (11, -13, -19, 21) \), \( x = 2u \) and \( v^T = y^T = (-1, 1, 1, -1) \) be our vectors in \( \mathbb{R}^4 \), and
\[ M = \mathcal{X}_8 \left( \begin{array}{cc} \mathcal{O}_4 & vu^T \\ xy^T & \mathcal{O}_4 \end{array} \right) \mathcal{X}_8 \frac{1}{2} \]
\[ = \begin{pmatrix} -63 & 61 & 55 & -53 & -31 & 29 & 23 & -21 \\ 59 & -57 & -51 & 49 & 27 & -25 & -19 & 17 \\ 47 & -45 & -39 & 37 & 15 & -13 & -7 & 5 \\ -43 & 41 & 35 & -33 & -11 & 9 & 3 & -1 \\ 1 & -3 & -9 & 11 & 33 & -35 & -41 & 43 \\ -5 & 7 & 13 & -15 & -37 & 39 & 45 & -47 \\ -17 & 19 & 25 & -27 & -49 & 51 & 57 & -59 \\ 21 & -23 & -29 & 31 & 53 & -55 & -61 & 63 \end{pmatrix}. \]

Then \( M^2 \) has the block representation
\[ M^2 = 8 \mathcal{X}_8 \left( \begin{array}{cccc} 273 & -273 & -273 & 273 \\ -273 & 273 & 273 & -273 \\ -273 & 273 & 273 & -273 \\ 273 & -273 & -273 & 273 \end{array} \right) \mathcal{X}_8. \]
Formula (5.2) immediately gives the quadratic form
\[ q_1(z_1, z_2) = 1092z_1^2 + 4z_2^2. \]
The matrix \( M^2 \) has the eigenvectors (4.3)
\[
\begin{align*}
b_1 &= \begin{pmatrix} v \\ J_nv \end{pmatrix}, & b_2 &= \begin{pmatrix} -J_n u \\ u \end{pmatrix}
\end{align*}
\]
(where we divided the vector \( b_2 \) by \( k \), as suggested in the Remark after Theorem [12] for eigenvalue \( \lambda = k(u^Tu)(v^Tv) = 8736 \); so \( b_1^Tb_1 = 2v^Tv = 8, b_2^Tb_2 = 2u^Tu = 2184 \), and these vectors are orthogonal. This gives rise to the quadratic form (5.5)
\[ q_3(\alpha_1, \alpha_2) = 8736(8\alpha_1^2 + 2184\alpha_2^2) = 17472(4\alpha_1^2 + 1092\alpha_2^2). \]

Applying the construction of Theorem [12], we obtain the rational symmetric (left and right) eigenvector matrix
\[
P = \frac{1}{11} \begin{pmatrix}
11 & 0 & 0 & -16 & 5 & 0 & 0 & 0 \\
0 & 11 & 0 & 15 & -4 & 0 & 0 & 0 \\
0 & 0 & 11 & 12 & -1 & 0 & 0 & 0 \\
-16 & 15 & 12 & -11 & 0 & -1 & -4 & 5 \\
5 & -4 & -1 & 0 & -11 & 12 & 15 & -16 \\
0 & 0 & 0 & -4 & 15 & 0 & 11 & 0 \\
0 & 0 & 0 & 5 & -16 & 0 & 0 & 11
\end{pmatrix}.
\]

Taking the middle columns of this matrix for the (non-orthogonal) eigenvectors \( b_1, b_2 (= J_b b_1) \), we now have \( b_1^Tb_1 = b_2^Tb_2 = \frac{788}{121}, b_1^Tb_2 = -\frac{304}{121} \), and formula (5.4) gives the quadratic form
\[ q_2(\alpha_1, \alpha_2) = 8736 \left( \frac{788}{121} \alpha_1^2 - \frac{608}{121} \alpha_1 \alpha_2 + \frac{788}{121} \alpha_2^2 \right) = \frac{34944}{121}(197\alpha_1^2 - 152\alpha_1 \alpha_2 + 197\alpha_2^2). \]

We note that Theorem [12] also gives the inverse for \( P \),
\[
P^{-1} = \text{diag}(13, 0_2, 1_3) - \frac{M^2}{\lambda} = \frac{8}{\lambda} \begin{pmatrix}
735 & 336 & 273 & -252 & -21 & 0 & -63 & 84 \\
336 & 775 & -260 & 241 & 32 & -13 & 44 & -63 \\
273 & -260 & 871 & 208 & 65 & -52 & -13 & 0 \\
-252 & 241 & 208 & -197 & -76 & 65 & 32 & -21 \\
-21 & 32 & 65 & -76 & -197 & 208 & 241 & -252 \\
0 & -13 & -52 & 65 & 208 & 871 & -260 & 273 \\
-63 & 44 & -13 & 32 & 241 & -260 & 775 & 336 \\
84 & -63 & 0 & -21 & -252 & 273 & 336 & 735
\end{pmatrix},
\]
and \( P^{-1}M^2P = PM^2P^{-1} = \text{diag}(0_3, \lambda, \lambda, 0_3). \)

Note that \( q_1, q_2 \) and \( q_3 \), as constructed above, are three different quadratic forms. They are of course all equivalent (and indeed equivalent to the simple circular binary \( x_1^2 + x_2^2 \)) on \( \mathbb{R}^2 \). The forms \( q_1 \) and \( q_3 \) are still equivalent on \( \mathbb{Q}^2 \), via the variable transformation \( z_1 = 8\alpha_1, z_2 = 2184\alpha_2 \), however, they are not equivalent on \( \mathbb{Z}^2 \), as their discriminants differ (cf. [5 §157, [3 §56].

In our second example, we consider the block decompositions and non-positive-definite quadratic form that results from our rational eigenvector transformation applied to a non-parasymmetric type A matrix.

**Example 2.** Let \( u^T = (10, -14, -18, 22), x^T = (23, -25, -39, 41) \) and \( v^T = y^T = (-1, 1, 1, -1) \in \mathbb{R}^4 \), and
\[
M = X_8 \begin{pmatrix}
\mathcal{O}_4 & v u^T \\
x y^T & \mathcal{O}_4
\end{pmatrix} X_8 = \begin{pmatrix}
-63 & 59 & 55 & -51 & -31 & 27 & 23 & -19 \\
-61 & -57 & -53 & 49 & 29 & -25 & -21 & 17 \\
47 & -43 & -39 & 35 & 15 & -11 & -7 & 3 \\
-45 & 41 & 37 & -33 & -13 & 9 & 5 & -1 \\
1 & -5 & -9 & 13 & 33 & -37 & -41 & 45 \\
-3 & 7 & 11 & -15 & -35 & 39 & 43 & -47 \\
-17 & 21 & 25 & -29 & -49 & 53 & 57 & -61 \\
19 & -23 & -27 & 31 & 51 & -55 & -59 & 63
\end{pmatrix}.
\]
Then $M^2$ has the block representation

$$M^2 = 8X_8 \begin{pmatrix} 273 & -273 & -273 & 273 & 0 & 0 & 0 & 0 \\ -273 & 273 & 273 & -273 & 0 & 0 & 0 & 0 \\ -273 & 273 & 273 & -273 & 0 & 0 & 0 & 0 \\ 273 & -273 & -273 & 273 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 115 & -161 & -207 & 253 \\ 0 & 0 & 0 & 0 & -125 & 175 & 225 & -275 \\ 0 & 0 & 0 & 0 & -195 & 273 & 351 & -429 \\ 0 & 0 & 0 & 0 & 205 & -287 & -369 & 451 \end{pmatrix} X_8,$$

which is clearly non-symmetric. Here we cannot use the formulae obtained above for the parasyymmetric case, but we can construct the quadratic form from the general formula (5.3). The non-zero eigenvalue is $\lambda = (u^T x)(y^T v) = 8736$, and the (right and left) eigenvector matrix constructed as in Theorem 12 has the form

$$P = \frac{1}{115} \begin{pmatrix} 115 & 0 & 0 & -160 & 45 & 0 & 0 & 0 \\ 0 & 115 & 0 & 155 & -40 & 0 & 0 & 0 \\ -184 & 161 & 138 & -115 & 0 & -23 & -46 & 69 \\ 69 & -46 & -23 & 0 & -115 & 138 & 161 & -184 \\ 0 & 0 & 0 & -5 & 120 & 115 & 0 & 0 \\ 0 & 0 & 0 & -40 & 155 & 0 & 115 & 0 \\ 0 & 0 & 0 & 45 & -160 & 0 & 0 & 115 \end{pmatrix}.$$

Now ordering the columns of $P$ as $P = (b_3, b_4, b_5, b_1, b_2, b_8, b_7, b_6)$ to define eigenvectors $b_1, \ldots, b_8$, we see that $b_2 = J_8 b_1$ and $b_{j+3} = J_8 b_j$ ($j \in \{3, 4, 5\}$), and calculate further $b_1^T b_1 = b_2^T b_2 = \frac{80900}{115^2}$, $b_3^T b_2 = -\frac{28000}{115^2}$ as well as $b_3^T b_1 = -b_3^T b_2 = -b_6^T b_2 = -b_6^T b_1 = b_4^T b_2 = -b_4^T b_1 = b_5^T b_2 = -\frac{6900}{115^2}$ and $b_5^T b_1 = -b_5^T b_2 = -b_8^T b_1 = b_8^T b_2 = -\frac{2070}{115^2}$. Hence

$$\sum_{j=3}^{8} (b_j^T b_1 \alpha_j \alpha_1 + b_j^T b_2 \alpha_j \alpha_2) = \sum_{j=3}^{5} (b_j^T b_1 (\alpha_j - \alpha_{j+3}) \alpha_1 + b_j^T b_2 (\alpha_j - \alpha_{j+3}) \alpha_2) = \sum_{j=3}^{5} (b_j^T b_1) (\alpha_j - \alpha_{j+3}) (\alpha_1 - \alpha_2);$$

in our example,

$$\sum_{j=3}^{5} (b_j^T b_1) (\alpha_j - \alpha_{j+3}) = \frac{6}{115} (4 (\alpha_3 - \alpha_6) - (\alpha_4 - \alpha_7) - 3 (\alpha_5 - \alpha_8));$$

and in view of the single variable $\alpha_7$ we can see that, irrespective of the number field or ring from which the variables $\alpha_3, \ldots, \alpha_8$ are taken, their combined effect is that of a single variable, say $\alpha_0$, from that field or ring.

In summary, we obtain the ternary quadratic form (5.3)

$$q(\alpha_1, \alpha_2, \alpha_0) = 8736 \left( \frac{4}{529} (809\alpha_1^2 - 560\alpha_1\alpha_2 + 809\alpha_2^2) + \frac{6}{115} \alpha_0 (\alpha_1 - \alpha_2) \right).$$

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