CHARACTERIZATIONS OF RECTIFIABLE METRIC MEASURE SPACES

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Abstract. We characterize \( n \)-rectifiable metric measure spaces as those spaces that admit a countable Borel decomposition so that each piece has positive and finite \( n \)-densities and one of the following: is an \( n \)-dimensional Lipschitz differentiability space; has \( n \)-independent Alberti representations; satisfies David’s condition for an \( n \)-dimensional chart. The key tool is an iterative grid construction which allows us to show that the image of a ball with a high density of curves from the Alberti representations under a chart map contains a large portion of a uniformly large ball and hence satisfies David’s condition. This allows us to apply previously known “biLipschitz pieces” results [9, 11, 12, 19] on the charts.

1. Introduction

General conditions that describe when a metric measure space is rectifiable have been elusive. Classically (that is, when the measure is defined on Euclidean space), this problem was first solved by Besicovitch [2] for Hausdorff measure and more generally by Preiss [17] for an arbitrary Radon measure. In these results it was shown that the space in \( n \)-rectifiable if and only if the \( n \)-dimensional density (see below) of the measure exists and equals 1 (Besicovitch) or is positive and finite (Preiss) at almost every point. In the metric setting, only partial answers are known. In [13] Kirchheim shows that an the \( n \)-dimensional Hausdorff density must exist and equal 1 at almost every point of an \( n \)-rectifiable metric measure space of finite \( \mathcal{H}^n \) measure. Conversely, Preiss and Tiser [18] showed that, for dimension 1, a lower Hausdorff density greater than (a number slightly less than) \( 3/4 \) is sufficient to ensure 1-rectifiability.

In a different direction, but with a similar goal, the work of David and Semmes [9, 11, 19] found conditions under which a Lipschitz function defined on an \( n \) Ahlfors regular space taking values in \( \mathbb{R}^n \) is in fact biLipschitz. In particular they showed that, if the image of every ball contains a ball of comparable radius centred at the image of centre of the first ball, a condition now known as David’s condition, then the function can be decomposed into biLipschitz pieces.

More recently, initiated by the striking work of Cheeger [4], there has been much activity in generalising the classical theorem of Rademacher to metric measure spaces. Most of all, this departed from the existing generalisations of Rademacher’s theorem, for example that of Pansu [16], by not requiring a group structure in the domain to make sense of the derivative. However, it is known that some additional structure must exist - it must be possible to decompose the measure into an integral combination of 1-rectifiable measures known as Alberti representations. See [1].

There are known examples of such spaces that are not groups (cf. [8, 10, 14]) and do not admit any rectifiable behaviour beyond the existence of Alberti representations. Indeed,
Cheeger also showed that for many of these spaces to possess a biLipschitz embedding into any Euclidean space, the dimension of the chart must equal the Hausdorff dimension of the space, which is not generally true. (More generally, a theorem of Cheeger-Kleiner [5] proves the same result but for a biLipschitz embedding into an RNP Banach space.) However, there are very natural relationships between rectifiability and differentiability. For example, Rademacher’s theorem easily extends to rectifiable sets via composition of functions (this concept is at the heart of the relationship between Alberti representations and differentiability). Further, Kirchheim’s theorem fundamentally relies on a version of Rademacher’s theorem for metric space valued Lipschitz functions defined on Euclidean space.

In this paper we give precise conditions when the notions of rectifiability and differentiability agree and hence obtain several characterizations of rectifiable metric measure spaces. Specifically, we prove the following theorem (see the next section for precise definitions of the concepts used).

**Theorem 1.1.** A metric measure space \((X, d, \mu)\) is \(n\)-rectifiable (which we denote by Property \((R)\)) if and only if there exist a countable collection of Borel sets \(U_i \subset X\) with \(\mu(X \setminus \bigcup_i U_i) = 0\) and Lipschitz functions \(\varphi_i : X \to \mathbb{R}^n\) such that the upper and lower \(n\)-densities of each \(\mu \downarrow U_i\) are positive and finite a.e. and one of the following holds:

(i) Each \((U_i, d, \mu)\) is an \(n\)-dimensional Lipschitz differentiability space;

(ii) Each \(\mu \downarrow U_i\) has \(n\) \(\varphi_i\) independent Alberti representations;

(iii) Each \(U_i\) satisfies David’s condition a.e. with respect to the function \(\varphi_i\).

We note that, for each condition, a different countable decomposition is permitted. Further, in the case that the space is rectifiable, the \(U_i\) can be chosen to have finite \(\mathcal{H}^n\) measure and so the Hausdorff density exists and equals 1 at almost every point by Kirchheim’s theorem. Therefore, the \(\mu\) density must also exist.

From this theorem we see that the density estimates force the curve fragments obtained from Alberti representations to form rectifiable sets. This is not necessarily true without this condition. For example, the Heisenberg group is a Lipschitz differentiability space consisting of a single chart of dimension two and hence has two independent Alberti representations but is purely 2-unrectifiable. (The Heisenberg group is Ahlfors 4-regular.) Furthermore, in [15], Máté constructs a measure on \(\mathbb{R}^3\) with 2 independent Alberti representations that is purely 2-unrectifiable. Indeed, our results are new even for a measure defined on Euclidean space: the only existing result is a corollary of Csörnýei-Jones [8] that states that a measure on \(\mathbb{R}^n\) with \(n\)-independent Alberti representations must be absolutely continuous with respect to Lebesgue measure.

We point out the known and easy implications in our theorem. Firstly, as mentioned above, by Kirchheim’s theorem we may take a countable decomposition of any rectifiable metric measure space so that the \(n\)-dimensional density of \(\mu\) on each piece exists and is finite and positive at almost every point of such a piece. Further, it is easy to see that \(f(A)\), for \(f : A \subset \mathbb{R}^n \to f(A)\) biLipschitz and \(A\) closed, equipped with \(\mu \ll \mathcal{H}^n\) is a Lipschitz differentiability space (the chart map is simply the inverse of \(f\)), so that (R) implies (i).

Secondly, the existence of \(n\) independent Alberti representations of an \(n\) dimensional chart in a Lipschitz differentiability space is proved in [11, Theorem 9.5]. Thus, within this paper we are interested in proving (iii) implies (ii) and that (iii) implies (R).

An immediate corollary of our theorem is the following.
Corollary 1.2. Any Ahlfors $n$-regular $n$-dimensional Lipschitz differentiability space is $n$-rectifiable.

This is a generalisation of a result of G. C. David (this David is not the same person who is the namesake of David’s condition) in [10] where it is shown that tangents of almost every point of an $n$-dimensional chart in an Ahlfors $n$-regular Lipschitz differentiability spaces are $n$-rectifiable (in fact he showed even more that they are uniformly rectifiable). In addition, he showed that $k$-dimensional charts in Ahlfors $s$-regular Lipschitz differentiability spaces for $k < s$ are strongly $k$-unrectifiable. However, the rectifiability of Ahlfors $n$-regular Lipschitz differentiability spaces themselves was still unknown. Indeed, this was the starting point for our work presented in this paper.

The proof of Theorem 1.1 will follow the same outline as that in [10] although we will need to make all the arguments effective. We quickly go over the general idea. There, G. C. David took a tangent at a point $x \in X$ and got a limiting metric space $Y$ and a limiting Lipschitz function $f : Y \to \mathbb{R}^n$ from $\varphi$. To show that $Y$ is rectifiable, G. C. David proved that $f$ was a Lipschitz quotient, which mean images of balls contain uniformly large balls, i.e., $f$ satisfies David’s condition. This allowed him to use the machinery developed by David [9] to conclude that $f$ can be decomposed into biLipschitz pieces.

We will seek to use the same biLipschitz decomposition machinery and to do so, we need to show that images of balls contain large portions of uniformly large balls. G. C. David was able to show that, when taking a tangent, the curve fragments from the Alberti representations become full Lipschitz curves of $Y$ that are pushed through $f$ to straight lines of $\mathbb{R}^n$. The use of these straight lines allowed G. C. David to show that $f$ is locally surjective in a scale invariant way.

As we cannot take a tangent, we do not have access to these full straight lines, but rather fragmented biLipschitz curves. The heart of our argument is showing that if a ball $B \subseteq X$ has a high density of points that lie on dense long fragmented curves with some certain speed relative to $\varphi$, then the image of $B$ under $\varphi$ contains most of a uniformly large ball. This will be sufficient to use the biLipschitz pieces decomposition of [11, 19].

In addition, we show that the machinery of finding big biLipschitz pieces from David’s condition is applicable when we only have control over the infinitesimal density of the measure and only for a subset of the space, rather than full Ahlfors regularity.

Finally, we note that the decomposition of a rectifiable metric measure space into Lipschitz differentiability spaces, rather than into a single Lipschitz differentiability space with many charts, is necessary. This is because of the subtle point that a countable union of Lipschitz differentiability spaces need not actually be a Lipschitz differentiability space, as the pieces may interact in undesirable ways. This is relevant since, in the definition of the derivative, we require convergence over the whole space. Of course, if the convergence were only over the set defining a chart, then the union would trivially be a Lipschitz differentiability space.

As an example, consider the subset of the plane defined by

$$\{(0) \times [0, 1]\} \cup \{(x, p/2^n) : n \in \mathbb{N}, 1 \leq p < 2^n \text{ odd}, \pm x \in [2^{-n} - 4^{-n}, 2^{-n}]\},$$

equipped with $\mathcal{H}^1$. This is a compact metric measure space of finite measure that is a countable union of Lipschitz differentiability spaces and is rectifiable. However, the function $|x|$ is not differentiable at any point of the vertical segment. More generally, the functions $|x|$ and $x$ cannot both be differentiable with respect to the same chart function and so the space
is not a 1-dimensional Lipschitz differentiability space. It cannot be a higher dimensional
Lipschitz differentiability space for several reasons. For example, it does not have 2 indepen-
dent Alberti representations or the fact that the vertical segment is a porous set of positive
measure. Further, if we restrict the measure to the vertical segment, then we see that it is
also necessary to consider only the support of $\mu$, rather than the whole space, in our theorem.

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2. Preliminaries

In this section, we introduce the various properties of our metric space. Unless specified
otherwise, we will suppose that all balls in our metric space are closed. For simplicity, we
will consider a metric measure space to be a complete and separable metric space equipped
with a finite Borel regular measure. However, our main theorem immediately generalises to
any metric measure space with a Radon measure.

A metric measure space $(X, d, \mu)$ is $n$-rectifiable if there is a countable family of Lipschitz
maps $\{f_i : A_i \to X\}$ where $A_i \subseteq \mathbb{R}^n$ is Borel so that $\mu(X \setminus \bigcup_{i=1}^{\infty} f(A_i)) = 0$ and $\mu \ll \mathcal{H}^n$. Note that, by Kirchheim’s theorem, we may suppose that each $f_i$ is in fact biLipschitz.

Remark 2.1. To prove that a space $(Y, d, \mu)$ is $n$-rectifiable, it suffices to show that for any
$\varepsilon > 0$, there exists a measurable subset $Y' \subseteq Y$ so that $\mu(Y \setminus Y') < \varepsilon$ and $Y'$ is $n$-rectifiable.
If there exist measurable subsets $Y_i \subseteq Y$ so that $\mu(Y \setminus \bigcup_{i=1}^{\infty} Y_i) = 0$, then it suffices to show
that each $Y_i$ is $n$-rectifiable.

Recall that the upper and lower $n$-densities of point $x$ in a metric measure space $(X, d, \mu)$
are the quantities

$$\Theta_n^*(x; \mu) = \limsup_{r \to 0} \frac{\mu(B(x, r))}{(2r)^n}, \quad \Theta_n^*(x; \mu) = \liminf_{r \to 0} \frac{\mu(B(x, r))}{(2r)^n}.$$  

In the case the upper and lower densities agree, we can define the $n$-density of $x$ to be
$\Theta_n(x; \mu) = \Theta_n^*(x; \mu)$.

Our first goal is to show the existence of a set of dyadic cubes that we describe now.

Proposition 2.2. Let $K \subseteq X$ be a compact subset of a metric measure space $(X, d, \mu)$ that satisfies

$$\frac{1}{C} r^n \leq \mu(B(x, r)) \leq C r^n, \quad \forall x \in X, r < R,$$

for some $R > 0$ and $C > 1$. Then there exist constants $a > 0$, $\eta > 0$, $k_K \in \mathbb{Z}$ and a collection
of subsets $\Delta = \{Q^k_{\omega} \subseteq X : k \leq k_K, w \in I_k\}$ so that $16^{k_K+2} \leq R$,

1. $\mu(K \setminus \bigcup_{\omega} Q^k_{\omega}) = 0 \quad \forall k \leq k_K,$
2. If $j \geq k$, then either $Q^k_{\omega} \subseteq Q^j_{\omega}$ or $Q^k_{\omega} \cap Q^j_{\omega} = \emptyset$.
3. For each $(j, \alpha)$ and each $k \in \{j, j+1, \ldots, k_K\}$, there exists a unique $\omega$ so that $Q^j_{\omega} \subseteq Q^k_{\omega}$,
4. For each $Q^k_{\omega}$, there is some $z_{\omega} \in K$ so that

$$B(z_{\omega}, 16^{k-1}) \subseteq Q^k_{\omega} \subseteq B(z_{\omega}, 16^{k+1}).$$
(5) For each $k, \alpha$ and $t > 0$, we have
\[ \mu\{x \in Q_\alpha^k \cap K : \text{dist}(x, X \setminus Q_\alpha^k) \leq t16^k\} \leq at^n \mu(Q_\alpha^k). \]

(6) For each $k, \alpha$,
\[ \frac{1}{C}16^{(k-1)n} \leq \mu(Q_\alpha^k) \leq C16^{(k+1)n}. \]

**Proof.** We may suppose without loss of generality that $\mu(K) > 0$. The proof of this theorem is essentially that of Theorem 11 of [7]. We assume the reader is familiar with that proof and will freely use its notation. We still do the usual partially ordered tree construction (see Lemma 13 and Definition 14 of [7]). One change is that $Q_\alpha^k$ become smaller as $k$ decreases, which does not pose any real challenge. The one other change is that, for each $k$, the maximal collection of points $z_\alpha^k$ satisfying $d(z_\alpha^k, z_\beta^k) \geq 16^k$ in the proof of Theorem 11 is now taken to be in $K$ instead of all of $X$. As our space $X$ is a true metric space (i.e. $A_0 = 1$), we can fix the constants $a_0 = 1/8$, $C_1 = 2$, and $\delta = 16$ (remembering that our scale is flipped). Properties 2-4 are satisfied by the construction of $Q_\alpha^k$ as per the proof of Theorem 11.

The proof of Property 1 remains unchanged as the fact that $z_\alpha^k \in K$ allows us to use the ball volume estimates (1) for balls centered at such points. Thus, we can continue to use the Lebesgue differentiation argument of [7] to show Property 1. Property 6 follows easily from Property 4, (1), and the fact that $16^{kK+2} \leq R$.

For the proof of Property 5 (known as the small boundaries condition), one has to make some slightly more substantial changes to the proof of the analogous property of Theorem 11 of [7]. Lemma 16 of the same paper remains unchanged (now $C_3 = 1/4$), but Lemma 17 is modified to the following statement:

**Lemma 2.3 (Modified Lemma 17).** For any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for every $Q_\alpha^k$,
\[ \mu\{x \in Q_\alpha^k : \exists \sigma \in I_{k-N} \text{ so that } x \in Q_{\sigma}^{k-N}, \text{dist}(Q_{\sigma}^{k-N}, X \setminus Q_\alpha^k) < 100 \cdot 16^{k-N}\} < \varepsilon \mu(Q_\alpha^k). \]

**Proof.** The proof of the modified Lemma 17 is similar to the proof of the original. Let $x \in Q_{\sigma}^{k-N} \cap Q_\alpha^k$ as above for some $N$ to be determined. Then we have that there exists a unique chain of cubes
\[ Q_{\sigma}^{k-N} = Q_{\sigma_{k-N}}^k \subset Q_{\sigma_{k-N+1}}^k \subset \cdots \subset Q_{\alpha}^k = Q_\alpha^k. \]

We let $z_j = z_{\sigma_j}$ be the points as in Property 4. As in the proof of Lemma 17, we have some $\varepsilon_1 > 0$ so that $d(z_i, z_j) \geq \varepsilon_1 16^j$ when $k - N + 3 \leq i < j \leq k$.

Now for each $x \in \bigcup\{Q_\sigma^{k-N} \subset Q_\alpha^k : d(Q_\sigma^{k-N}, X \setminus Q_\alpha^k) < 100 \cdot 16^{k-N}\}$, we can construct a chain of cubes and $z_{\beta(x,j)}$ as above for $k - N \leq j \leq k$. Let $S_j$ be the collection of all points $z_{\beta(x,j)}$ for all such $x$. Let $G_j = \bigcup_{z \in S_j} B(z, \varepsilon_1 16^{j-1})$. Then we have that the $G_j$ are disjoint. The rest of the proof follows the proof of Lemma 17 with obvious modifications. We have to take $N \geq 4$ large enough at this step. □

The proof of Property 5 now follows the proof in Theorem 11. One defines
\[ E_j(Q_\alpha^k) = \{Q_\beta^{k-j} \subset Q_\alpha^k : \text{dist}(Q_\beta^{k-j}, X \setminus Q_\alpha^k) \leq 100 \cdot 16^{k-j}\}, \]

We also define $e_j(Q_\alpha^k) = \bigcup_{Q \in E_j(Q_\alpha^k)} Q$. As in the proof of Lemma 17, one can verify that if $x \in Q_\alpha^k \cap K$ so that $d(x, X \setminus Q_\alpha^k) < \tau 16^k$, then $x \in e_j(Q_\alpha^k)$ where $16^{k-j-1} \geq \tau 16^k$. Thus, it
suffices to prove that there is some $C' > 0$ so that

$$
\mu(e_j(Q^k \cap K)) \leq C' 16^{-jn} \mu(Q^k), \quad \forall \alpha, k, \forall j \geq 0.
$$

As in the proof of Theorem 11, one gets by iterating Lemma 2.3 that there exists some $J \geq 0$ so that

$$
\mu(e_{mJ}(Q^k)) \leq 2^{-m} \mu(Q^k) = 16^{-jm} \mu(Q^k), \quad \forall m \geq 0.
$$

This finishes the proof. □

By Properties 2 and 6 of Proposition 2.2 we see that for each $Q^k \in \Delta$, the number of cubes $Q^{k-1}$ that $Q^k$ can contain is bounded by a number depending only on $C$ and $n$. We let

$$
\Delta_k = \{Q^k \in \Delta : \omega \in I_k\}.
$$

Given a cube $Q_0$, we let $\Delta(Q_0) = \{Q \in \Delta : Q \subseteq Q_0\}$. A similar definition gives us $\Delta_k(Q_0)$. Note that if $k$ is too big, then $\Delta_k(Q_0)$ can be empty. For a cube $Q$, we let $z_Q$ denote the point guaranteed to us to satisfy (2). We let $j(Q)$ denote the largest integer such that $Q \in \Delta_j(Q)$. For convenience, we will also set $\ell(Q) = 16^{j(Q)}$.

A set of cubes $E$ is said to be a Carleson set if there exists some constant $C > 0$ so that the following Carleson estimate is satisfied:

$$
\sum_{Q \in E, Q \subseteq Q_0} \mu(Q) \leq C \mu(Q_0), \quad \forall Q_0 \in \Delta.
$$

Here, $C$ is called the Carleson constant. Given some set $A \subseteq X$, we say that $E$ is an $A$-Carleson set if

$$
\sum_{Q \in E, Q \subseteq Q_0} \mu(Q \cap A) \leq C \mu(Q_0), \quad \forall Q_0 \in \Delta.
$$

Obviously, any subset of a (A-)Carleson set is also a (A-)Carleson set and any set of disjoint cubes is a (A-)Carleson set with constant 1. Also, any Carleson set is an $A$-Carleson set for any $A \subseteq X$.

Given some $\lambda > 1$ and some cube $Q \in \Delta$, we let

$$
\lambda Q = \{x \in X : \text{dist}(x, Q) \leq (\lambda - 1) \text{diam}(Q)\}.
$$

Let $U \subset X$ be a Borel set, $n \in \mathbb{N}$ and $\varphi : X \rightarrow \mathbb{R}^n$ Lipschitz. We say that $(U, \varphi)$ form a chart of dimension $n$ and that a function $f : X \rightarrow \mathbb{R}$ is differentiable at $x_0 \in U$ with respect to $(U, \varphi)$ if there exists a unique $Df(x_0) \in \mathbb{R}^n$ (the derivative of $f$ at $x_0$) so that

$$
\lim_{x \to x_0} \frac{|f(x) - f(x_0) - Df(x_0) \cdot (\varphi(x) - \varphi(x_0))|}{d(x, x_0)} = 0.
$$

It easily follows that if $U' \subset U$, then $(U', \varphi)$ is also a chart. A metric measure space $(X, d, \mu)$ is said to be a Lipschitz differentiability space if there exists a countable set of charts $(U_i, \varphi_i)$ so that $X = \bigcup_i U_i$ and every real valued Lipschitz function defined on $X$ is differentiable at almost every point of every chart. A Lipschitz differentiability space is said to be $n$-dimensional if every chart map is $\mathbb{R}^n$ valued.

For a metric measure space $(X, d, \mu)$, we define $\Gamma$ to be the collection of biLipschitz functions $\gamma$ defined on a compact subset of $\mathbb{R}$ taking values in $X$ (known as curve fragments)
and say that $\mu$ has an Alberti representation if there exists a probability measure on $\Gamma$ and measures $\mu_\gamma \ll H^1 \Image \gamma$ such that
\[
\mu(B) = \int_{\Gamma} \mu_\gamma(B) \, d\mathbb{P}
\]
for each Borel set $B \subset X$. Further, for a Lipschitz function $\varphi : X \to \mathbb{R}^n$ and $v > 0$, an Alberti representation is in the $\varphi$-direction of a cone $C \subset \mathbb{R}^n$, respectively has $\varphi$-speed $v$, if
\[
(\varphi \circ \gamma)'(t) \in C \setminus \{0\},
\]
respectively
\[
| (\varphi \circ \gamma)'(t) | > v \text{Lip}(\varphi, \gamma(t)) \Lip(\gamma, t),
\]
for $\mathbb{P}$ a.e. $\gamma \in \Gamma$ and $\mu_\gamma$ a.e. $t \in \text{dom} \gamma$. Finally, we say that a collection of $n$ Alberti representations are $\varphi$-independent if there exist linearly independent cones in $\mathbb{R}^n$ so that each Alberti representation is in the direction of a cone and no two Alberti representations are in the direction of the same cone. See [1, Section 2] for further details. One of the main results of [1] is that any $n$-dimensional chart in a Lipschitz differentiability space has $n$ independent Alberti representations, each with positive speed with respect to the chart map.

From now on, we will assume that $(X, d, \mu)$ is a metric measure space with
\[
0 < \Theta^*_n(x; \mu) \leq \Theta^{n*}(x; \mu) < \infty, \quad \mu\text{-a.e. } x \in X,
\]
and $(U, \varphi)$ is a chart of dimension $n$ such that $\mu_\nu U$ has $n$ $\varphi$-independent Alberti representations and $\mu(U) > 0$. Note that the density conditions imply that $X$ is pointwise doubling and so satisfies the Lebesgue density theorem. By refining the Alberti representations (see [1, Definition 5.10]) we may suppose that they are in the $\varphi$-direction of cones of width $1/1000n$. Further, by applying a linear transformation to $\varphi$, we may suppose that these Alberti representations are in the $\varphi$-direction of cones centred on the standard basis vectors. (We note that the process of refining a collection of Alberti representations requires a countable Borel decomposition of the set $U$. However, this does not affect our goal of showing that $\mu_\nu U$ is rectifiable. Further, the linear transformation will increase the width of the refined cones only by an amount predetermined by the centres and widths of the original cones and so both the centres and the widths of the refined cones may be chosen in this way.) Finally, we may suppose that $\varphi$ is 1-Lipschitz.

We are now ready to define the set of dyadic cubes that we will use for the rest of the paper. Let
\[
U(j, R) = \left\{ x \in U : \frac{1}{j} r^n \leq \mu(B(x, r)) \leq j r^n, \forall r < R \right\}.
\]
As we are supposing that $X$ has positive and finite lower and upper $n$-densities almost everywhere, we have that
\[
\mu \left( U \setminus \bigcup_{j=1}^{\infty} \bigcup_{k=1}^{\infty} U(j, k) \right) = 0.
\]
By inner regularity of $\mu$ and two applications of Remark 2.1, we may reduce proving the rectifiability of $U$ to proving the rectifiability of $K$ where $K$ is any compact subset of $U$ for
which there exist some constants $C > 1$ and $R_K > 0$—which we now fix—so that
\[
\frac{1}{C} r^n \leq \mu(B(x, r)) \leq C r^n, \quad \forall x \in K, r < R_K.
\] (5)

We let $\Delta$ denote the collection of cubes covering $K$ as given by Proposition 2.2.

For $R, v > 0$ we define the set $GP(v, R)$ to be those $y \in K$ for which, for each $1 \leq j \leq n$, there exists a $\gamma \in \Gamma$ and a $t$ in the domain of $\gamma$ such that

1. $\gamma(t) = y$;
2. for every $0 < r < 4R \text{biLip} (\gamma)$, $|B(t, r) \cap \gamma^{-1}(K)| > (1 - 1/10000n)|B(t, r)|$;
3. for every $s > s' \in \text{dom} \gamma$, $|\varphi(\gamma(s)) - \varphi(\gamma(s'))| > v d(\gamma(s), \gamma(s'))$.

Since $X$ is complete and separable and $\mu$ is Borel regular, the results of [1, Section 2] show that $GP(v, R)$ is measurable for any $v, R > 0$. Moreover, since $\mu, K$ has the Alberti representations described above, [1, Proposition 2.9] shows that $GP(v, R)$ converges to a set of full measure in $K$ as $v, R \to 0$. Finally, by enlarging the domain of any such $\gamma$ by a factor of biLip($\gamma$), we may suppose that for any $y \in GP(v, R)$, the defining $\gamma$ is 1-Lipschitz.

We also define
\[DP(v, \varepsilon, R) = \{ x \in K : \mu(GP(v, R) \cap B(x, r)) \geq (1 - \varepsilon)\mu(B(x, r)), \forall r < R\}.
\]
and
\[DC(\beta, \varepsilon, R) = \{ x \in K : |B(\varphi(x), \beta r) \cap \varphi(B(x, r) \cap K)| \geq (1 - \varepsilon)|B(\varphi(x), \beta r)|, \forall r < R\}.
\]

Note that $DC(\beta, \varepsilon, R) \subset DC(\beta', \varepsilon, R)$ and $DC(\beta, \varepsilon, R) \subset DC(\beta, \varepsilon, R')$ when $\beta' \leq \beta$ and $R' \leq R$.

The points of $DC$ satisfy one of the two conditions of David’s condition. This is the more important of the two conditions that allows one to deduce biLipschitz behavior. See Section 1 and in particular equation (9) in [9] or Section 9 and in particular Condition 9.1 and Remark 9.6 of [19] for more information on David’s condition as it was originally introduced.

We can now specify what we mean in the statement of condition [11] in Theorem 1.1. We say that $U$ satisfies David’s condition a.e. with respect to the function $\varphi : U \to \mathbb{R}^n$ if for every $\varepsilon \in (0, 1),
\[
\mu\left(U \setminus \bigcup_{j=1}^{\infty} \bigcup_{k=1}^{\infty} DC(1/j, \varepsilon, 1/k)\right) = 0,
\]
where the $DC$ sets here are defined with respect to $U$ and $\varphi$.

3. David’s condition

The goal of this section is to prove the following proposition, which proves that Condition [11] implies Condition [11] of Theorem 1.1.

**Proposition 3.1.** For all $\varepsilon \in (0, 1),
\[
\mu\left(K \setminus \bigcup_{j=1}^{\infty} \bigcup_{k=1}^{\infty} DC(1/j, \varepsilon, 1/k)\right) = 0.
\]
We will need the following lemma, which is the heart of the iterative step needed for our grid construction.

**Lemma 3.2.** Let \( x \in K \), \( v > 0 \) and \( R > r > 0 \). Let \( Q \) be an axis-parallel cube in \( \mathbb{R}^n \) that has sidelength \( \frac{v}{10n}r \) and whose center \( x_Q \) satisfies \( |\varphi(x) - x_Q| < \frac{v}{100n}r \). Let \( \{p_i\}_{i=1}^{2n} \) denote the centers of the \( 2^n \) quadrant subcubes of \( Q \). Then at least one of the following must be true:

1. There exists points in \( \{q_i\}_{i=1}^{2n} \) in \( B(x, r/2) \cap K \) so that \( |\varphi(q_i) - p_i| < \frac{v}{1000n}r \).
2. There exists some \( y \in B(x, r/2) \cap K \) so that \( B(y, \frac{v}{10000n^2}r) \cap GP(v, R) = \emptyset \) and \( \text{dist}(\varphi(y), Q^c) \geq \frac{v}{100n}r \).

**Proof.** Suppose the second alternative is false. We may suppose without loss of generality that \( x_Q = 0 \). Let us first assume that \( n = 2 \). Note then that the centers of the 4 quadrant cubes are located at

\[
(\pm \frac{v}{40n}r, \pm \frac{v}{40n}r), \quad (\pm \frac{v}{40n}r, \mp \frac{v}{40n}r).
\]

Consider the set \( B\left(x, \frac{v}{10000n^2}r\right) \cap G(v, R) \). As \( |\varphi(x) - x_Q| < \frac{v}{100n}r \), the fact that the second alternative is false means that this set is not empty and so there exist \( \gamma_0 : D_0 \to K \) and \( t_0 \in \text{dom} \gamma \) that satisfy the defining properties of \( G(v, R) \) for the cone \( C(e_1, 1/1000n^2) \) such that \( \gamma(t_0) \in B\left(x, \frac{v}{10000n^2}r\right) \cap G(v, R) \). By Property 2 of \( GP(v, R) \), we have that there exist points \( t_1, t_2 \) such that

\[
|\varphi_1(\gamma_0(t_1)) + \frac{v}{40n}r| < \frac{v}{10000n^2}r, \quad |\varphi_1(\gamma_0(t_2)) - \frac{v}{40n}r| < \frac{v}{10000n^2}r. \tag{7}
\]

Now consider all points of \( B\left(\gamma_0(t_1), \frac{v}{10000n^2}r\right) \cap G(v, R) \). By (6) in the definition of \( G(v, R) \), we have

\[
d(\gamma_0(t_1), x) \leq d(\gamma_0(t_1), \gamma_0(t_0)) + d(\gamma_0(t_0), x) \leq \frac{1}{10n}r + \frac{v}{10000n^2}r \leq \frac{1}{2}r.
\]

It is easy to see that \( d(\varphi(\gamma_0(t_1)), Q^c) \geq \frac{v}{1000}r \). Thus, as the second alternative is false, we can take such a curve \( \gamma_1 : D_1 \to K \) for the cone \( C(e_2, 1/1000n^2) \) and let \( \gamma_1(t_{10}) \in B\left(\gamma_0(t_1), \frac{v}{10000n^2}r\right) \cap K \). Again, by Property 2 of \( GP(v, R) \), there exist points \( t_{11} \) and \( t_{12} \) such that

\[
|\varphi_2(\gamma_1(t_{11})) + \frac{v}{40n}r| < \frac{v}{10000n^2}r, \quad |\varphi_2(\gamma_1(t_{12})) - \frac{v}{40n}r| < \frac{v}{10000n^2}r.
\]

As \( \gamma_1 \) is traveling in the \( \varphi \)-cone \( C(e_2, \frac{1}{1000n^2}) \) and starts off at a point near \( \gamma_0(t_1) \), we get that

\[
|\varphi_1(\gamma_1(t_{11})) + \frac{v}{40n}r| \leq |\varphi_1(\gamma_1(t_{11})) - \varphi_1(\gamma_1(t_{10}))| + |\varphi_1(\gamma_1(t_{10})) - \varphi_1(\gamma_0(t_1))| + |\varphi_1(\gamma_0(t_1)) + \frac{v}{40n}r| \\
\leq \frac{v}{10000n^3}r + \frac{v}{10000n^3}r + \frac{v}{10000n^3}r \leq \frac{v}{100n^2}r.
\]

The first term on the right hand side above comes from the fact that the curve is traveling in a cone of width \( \frac{1}{10000n^2}r \) in the direction of \( e_2 \) over a length of no more than \( \frac{v}{10n}r \). The second term comes from how we chose \( \gamma_1(t_{10}) \) and the third term comes from (7). A similar bound holds for \( |\varphi_1(\gamma_1(t_{12})) + \frac{v}{40n}r| \).
Thus, we see that $\varphi(\gamma_1(t_{11}))$ and $\varphi(\gamma_1(t_{12}))$ are within $\frac{v}{1000n^3}r$ of the centers of the left two quadrant subcubes of $Q$. One may do a similar construction starting from $\gamma_0(t_2)$ to get a curve $\gamma_2$ and points $\varphi(\gamma_2(t_{21}))$ and $\varphi(\gamma_2(t_{22}))$ that are within $\frac{v}{1000n^3}r$ of the centers of the right two quadrant subcubes of $Q$.

In the case of general $n$, one continues using curves in the remaining directions in the obvious way. That the width of the aperture of the cones is $O(n^{-2})$ and the radius of the balls in which we are looking for curves is $O(n^{-3})$ allows us to guarantee that the errors accumulated in finding new curves is controlled.

Consider one of these points $p$ that we have found near the center of a quadrant subcube. We will again assume $n = 2$ for simplicity. Let $p = \varphi(\gamma_1(t_{11}))$, for instance. Remember that $\gamma_0$ and $\gamma_1$ both satisfy (5), we have that $d(\varphi(i(s)), \varphi(i(t))) < \frac{1}{2} |\varphi(i(s)) - \varphi(i(t))|$. Thus, we have that

$$d(\gamma_1(t_{11}), x) \leq d(\gamma_1(t_{11}), \gamma_1(t_{10})) + d(\gamma_1(t_{10}), \gamma_0(t_1)) + d(\gamma_0(t_1), \gamma_0(t_0)) + d(\gamma_0(t_0), x)$$

$$\leq \frac{1}{10} r + \frac{v}{1000n^3} r + \frac{1}{10} r + \frac{v}{1000n^3} r \leq \frac{1}{2} r.$$ 

A similar estimate holds for the other quadrants. The case of general $n$ follows completely analogously. Thus, we have shown in the case that the second alternative of the lemma is false that the first alternative must be true. \qed

We can now prove that points whose neighborhoods have a high density of points in $GP$ belong to some $DC$ set.

**Lemma 3.3.** There exists some constant $\alpha > 0$ depending only on $K$ so that for any $\varepsilon, v, R > 0$ such that $R < R_K$,

$$DP(v, \varepsilon, R) \subseteq DC\left(\frac{v}{20n}, \alpha \varepsilon, R\right).$$

**Proof.** Let $x \in DP(v, \varepsilon, R)$ and let $r < R$. Since $K$ is compact, so is $\varphi(B(x, r) \cap K)$. Let $Q$ be the axis-parallel square centered at $\varphi(x)$ with sidelength $\frac{1}{10n}r$. We will show for some constant depending only on $K$ that

$$|Q \setminus \varphi(B(x, r) \cap K)| \leq C \varepsilon \mu(B(x, r)).$$

By (5), this clearly will suffice.

We will define the following stopping time process. In the first stage, we start off with $(x, Q, r)$. If the first alternative of Lemma 3.2 is satisfied with this triple, we can apply it to get $\{q_i\}_{i=1}^{2^n} \subseteq K$, points which map close to the center of the quadrant subcubes $\{Q_i\}_{i=1}^{2^n}$ of $Q$. Otherwise, we terminate the process. Assuming the process continues, for our second stage, we see if the first alternative of Lemma 3.2 applies to the triples $\{(q_i, Q, r/2)\}_{i=1}^{2^n}$. Indeed, we can apply the lemma again given the conclusions of the first alternative. We stop the process at each subcube where the first alternative fails and continue in the cubes where it doesn’t, making sure to divide $r$ by a further factor of 2 at each stage. Note that all the points of $Q$ that the process discovers has a preimage in $B(x, r)$ as the points discovered at each stage are no more than $2^{-k-1}r$ away from a point from the previous stage.

The cubes where the process terminates after finite time $\{S_i\}_{i=1}^\infty$ are disjoint dyadic subcubes of $Q$. We will upper bound their collective volume. If $S_i$ is a cube where the process terminates, then by the failure of the first alternative of Lemma 3.2, the second alternative must be true. Thus, there exists a ball $B_i \subseteq X$ with center in $K$ of radius comparable to
Thus, as \( \varphi \)

Indeed, let \( B \) for some constant \( \in \varphi \)

Proof of Proposition 3.1.

First observe that \( \beta, \varepsilon, R > 1 \) \( \text{Lipschitz,} \)

Therefore, since Lebesgue measure is continuous on balls, we see that the proof.

\( \square \)

We are now ready to prove the main proposition of this section.

\( \varphi(B(x, r) \cap K) \) is compact, we get that almost every point of \( Q \cup S_i \) is contained in \( \varphi(B(x, r) \cap K) \). We then have that

\[
|Q \setminus \varphi(B(x, r) \cap K)| \leq \bigcup_{i=1}^{\infty} S_i \leq C \varepsilon \mu(B(x, r)).
\]

\[ (8) \]

For almost every \( p \in Q \setminus \bigcup_{i=1}^{\infty} S_i \),

\[
|B(p, s) \cap \left(Q \setminus \bigcup_{i=1}^{\infty} S_i\right)| > 0, \quad \forall s > 0.
\]

In particular, the process tells us that there must be some point \( y \in B(x, r) \cap K \) that maps into \( B(p, s) \). As this holds true for all \( s > 0 \), we see that \( p \) is a limit point of \( \varphi(B(x, r) \cap K) \).

Thus, as \( \varphi(B(x, r) \cap K) \) is compact, we get that almost every point of \( Q \setminus \bigcup S_i \) is contained in \( \varphi(B(x, r) \cap K) \). We then have that

\[
|Q \setminus \varphi(B(x, r) \cap K)| \leq \bigcup_{i=1}^{\infty} S_i \leq C \varepsilon \mu(B(x, r)).
\]

\[ (8) \]

Proof of Proposition 3.1. First observe that

\[
\varphi(B(x, r) \cap K) \subset B(\varphi(y, x, r) \cap K, d(y, x))
\]

and so, since \( \varphi(B(x, r) \cap K) \) is closed, \( x \mapsto |\varphi(B(x, r) \cap K)| \) is upper semicontinuous. Therefore, since Lebesgue measure is continuous on balls, we see that \( DC(\beta, \varepsilon, R) \) is closed for any \( \beta, \varepsilon, R > 0 \) and hence measurable.

Secondly, we know that \( \mu \left( K \setminus \bigcup_j \bigcap_k GP(1/j, 1/k) \right) = 0 \) and so, by the Lebesgue density theorem, for any \( \varepsilon > 0 \), \( \mu \left( K \setminus \bigcup_j \bigcap_k DP(1/j, \varepsilon, 1/k) \right) = 0. \) Therefore, Lemma 3.3 concludes the proof.

\[ \square \]

4. BiLipschitz pieces

The main result of this section is Proposition 4.3, which will be the main step in showing that Condition (3) implies Condition (R) in Theorem 1.1. A good knowledge of [11,19] will be necessary in this section. Fix a cube \( Q_0 \in \Delta \) for the remainder of the section. We recall some terminology from [19].

Given \( \delta > 0 \), the set \( \mathcal{S}(\delta, Q_0) \) are the subcubes \( Q \in \Delta(Q_0) \) for which there is some \( W \in \Delta(Q_0) \) such that \( Q \subseteq W \) and \( |\varphi(W \cap K)| < \delta \mu(W) \). We may drop the \( Q_0 \) from the parameters list if it is obvious.

Given \( A > 1 \), we say that two dyadic cubes are \( A \)-neighbors (or just neighbors) if

\[
\text{dist}(Q, Q') \leq A(\text{diam } Q + \text{diam } Q')
\]
and 
\[ \frac{1}{A} \text{diam}(Q) \leq \text{diam}(Q') \leq A \text{diam}(Q). \]

For some \( Q \in \Delta \), we let 
\[ \tilde{Q} = \left( \bigcup \{ S \in \Delta_{j(Q)} : \text{dist}(S, Q) \leq \text{diam}(Q) \} \right) \cap Q_0. \]

Given \( A > 1 \) and \( \zeta > 0 \), the set \( M_A(\zeta, Q_0) \) are the subcubes \( Q \in \Delta(Q_0) \) that satisfy the following property:

- \( |\varphi(Q \cap K)| \geq (1 + \zeta)^{-1} \delta \mu(Q) \),
- if \( R \in \Delta \) is a neighbor of \( Q \), then \( R \subseteq Q_0 \), and

\[ (1 + \zeta)^{-1} |\varphi(Q \cap K)| \leq \frac{|\varphi(R \cap K)|}{\mu(R)} \leq (1 + \zeta) \frac{|\varphi(Q \cap K)|}{\mu(Q)}, \]

- if \( R \in \Delta \) is a neighbor of \( Q \),

\[ (1 + \zeta)^{-1} |\varphi(Q \cap K)| \leq \frac{|\varphi(R \cap K)|}{\mu(R)} \leq (1 + \zeta) \frac{|\varphi(Q \cap K)|}{\mu(Q)}. \]

We will not actually use the first property of \( M_A(\zeta, Q_0) \) cubes, but they are defined this way in [19] (although without the intersection with \( K \)) so we keep it to reduce confusion.

Finally, for \( \eta > 0 \), we define the set \( \mathcal{LD}(\eta, Q_0) \) to be the subcubes \( Q \in \Delta(Q_0) \) for which there is some \( W \in \Delta(Q_0) \) such that \( Q \subseteq W \) and \( \mu(W \cap K) < \eta \mu(W) \). Again, we may drop the \( Q_0 \) from the list of parameters of both \( M \) and \( \mathcal{LD} \) if it is obvious.

We first prove that the measure of the cubes of \( \mathcal{SI} \) can be bounded by the measure of the complement of \( DC \).

**Lemma 4.1.** There exists constants \( c, C > 0 \) depending only on \( K \), so that if \( \delta > 0 \), \( \Sigma(c\delta^n) = \bigcup_{Q \in \mathcal{SI}(c\delta^n, Q_0)} Q \) and \( \varepsilon \in (0, 1/2) \), then
\[ \mu(\Sigma(c\delta^n)) \leq C \mu(Q_0 \setminus DC(\delta, \varepsilon, 16\ell(Q_0))). \]  

**Proof.** Let \( \{ Q_i \}_{i=1}^\infty \) denote the set of maximal cubes of \( \mathcal{SI}(\delta, Q_0) \), which are obviously disjoint. Then \( \Sigma(c\delta^n) = \bigcup_{i=1}^\infty Q_i. \) Let \( Q \in Q_0 \) and suppose \( x \in B(z_Q, \ell(Q)/32) \cap DC(\delta, \varepsilon, 16\ell(Q_0)) \neq \emptyset \). Then
\[ |\varphi(Q \cap K)| \geq |\varphi(B(x, \ell(Q)/32) \cap K) \cap B(\varphi(x), \delta \ell(Q)/32)| \geq \frac{1}{2} 32^{-n} \delta^n |B(0, \ell(Q))|. \]

As \( B(x, \ell(Q)/32) \subseteq Q \), we get from [2], [3], and [4] that \( Q \notin \mathcal{SI}(c\delta^n) \) for some small enough \( c > 0 \) depending only on \( K \). Thus, if \( Q \in \mathcal{SI}(c\delta^n) \), then \( B(z_Q, \ell(Q)/32) \subseteq Q \setminus DC(\delta, \varepsilon, 16\ell(Q_0)) \). As there exists some constant \( C > 0 \) depending only on \( K \) so that \( \mu(Q_i) \leq C \mu(B(z_{Q_i}, \ell(Q_i)/32)) \), we have that
\[ \mu(\Sigma(c\delta^n)) \leq \sum_{i=1}^\infty \mu(Q_i) \leq C \sum_{i=1}^\infty \mu(B(z_{Q_i}, \ell(Q_i)/32)) \leq C \mu(Q_0 \setminus DC(\delta, \varepsilon, 16\ell(Q_0))). \]

Note that, for the final inequality, we have used the fact that the balls \( B(z_{Q_i}, \ell(Q_i)/32) \) are disjoint. \( \square \)

We can also show that the space covered by \( \mathcal{LD} \) has small volume.
Lemma 4.2. Let $\eta > 0$ and $\Lambda(\eta) = K \cap \bigcup_{Q \in \mathcal{LD}(\eta, Q_0)} Q$. Then

$$\mu(\Lambda(\eta)) < \eta \mu(Q_0).$$

Proof. Let $\{Q_i\}_{i=1}^{\infty}$ denote the set of maximal cubes of $\mathcal{LD}(\eta)$, which are obviously disjoint. Then $\Lambda(\eta) = \bigcup_{i=1}^{\infty} Q_i$ and so

$$\mu(\Lambda(\eta)) \leq \sum_{i=1}^{\infty} \mu(Q_i \cap K) < \eta \sum_{i=1}^{\infty} \mu(Q) \leq \eta \mu(Q_0).$$

\[\square\]

By Lemma 8.2 of [19] (or one can easily derive from (2)), we can choose some absolute constant $b \in (0, 1)$ so that if $x, y \in K$ are distinct points and $Q$ is the smallest cube in $\Delta$ such that $x \in Q$ and $y \in 2Q$, then

$$d(x, y) \geq 10b \text{diam}(Q).$$

Lemma 4.3. There exist constants $k > 0$ depending only on $\beta$ and $\zeta_0$, $A > 0$ depending on $k$ and $K$ so that the following holds. Let $\zeta < \zeta_0$, $A > A_0$, $\varepsilon \in (0, 1/10)$, $Q \in M_A(\zeta, Q_0)$. If $x, y \in 2Q \cap Q_0 \cap DC(\beta, \varepsilon, 16\ell(Q_0))$ are such that $d(x, y) > b \text{diam}(Q)$, then

$$|\varphi(x) - \varphi(y)| \geq k^{-1}d(x, y).$$

Proof. Given a $j \in \mathbb{Z}$, we define

$$T_j(x) = \bigcup \{Q \in \Delta_j : Q \cap B(x, 16^j) \neq \emptyset\}.$$

It is clear then that $B(x, 16^j) \subseteq T_j(x) \subseteq B(x, 16^{j+2})$.

The proof follows the proof of Proposition 9.36 of [19]. Let us suppose that the conclusion does not hold, so that $|\varphi(x) - \varphi(y)| < d(x, y)/k$ for some $k > 0$ to be determined, and seek a contradiction. Let $j_1$ be the largest integer at most $j(Q)$ so that

$$T_{j_1}(x) \cap T_{j_1}(y) = \emptyset.$$

As $d(x, y) > b \text{diam}(Q)$, we get that there exists some absolute constant $C > 0$ so that

$$0 \leq j(Q) - j_1 \leq C.$$  \hspace{1cm} (10)

We will show there exists some $c > 0$ and $k > 0$ depending only on $\beta$ so that

$$|\varphi(B(x, 16^{j_1}) \cap K) \cap \varphi(B(y, 16^{j_1}) \cap K)| \geq c|\varphi(Q \cap K)|.$$  \hspace{1cm} (11)

This then proves that

$$|\varphi(T_{j_1}(x) \cap K) \cap \varphi(T_{j_1}(y) \cap K)| \geq c|\varphi(Q \cap K)|,$$

which is equation (9.50) of [19]. The rest of the proof will just continue as in the proof of Proposition 9.36 after Remark 9.56 of [19] where it is shown that taking $\zeta$ small enough and $A$ large enough leads to a contradiction of $Q \in M_A(\zeta, Q_0)$. The only change need to be made is that the function is now defined on $K$ instead of the whole space, but this does not pose any challenge. We will leave those details to the reader.
We now prove (11). By Lemma 9.10 of [19], if we choose $A$ to be large enough, then $T_{j_1}(x)$ and $T_{j_1}(y) \subseteq Q$. Thus, as $\varepsilon \in (0, 1/4)$ and $x, y \in DC(\beta, \varepsilon, 16\ell(Q_0))$, we get that
\[
|B(\varphi(x), \beta 16^{j_1}) \cap \varphi(B(x, 16^{j_1}) \cap K)| \geq \frac{3}{4} |B(0, \beta 16^{j_1})|,
\]
\[
|B(\varphi(y), \beta 16^{j_1}) \cap \varphi(B(y, 16^{j_1}) \cap K)| \geq \frac{3}{4} |B(0, \beta 16^{j_1})|.
\]
Note that
\[
d(x, y) \leq 3 \text{diam}(Q)^{2^A} \leq 96 \cdot 16^C 16^{j_1}.
\]
Thus, if $k$ is larger than some constant depending only on $\beta$, then
\[
|\varphi(B(x, 16^{j_1}) \cap K) \cap \varphi(B(y, 16^{j_1}) \cap K)| \geq \frac{1}{4} |B(\varphi(x), \beta 16^{j_1})| = (*).
\]
By (2), $Q$ has diameter no more than $32 \cdot 16^{j(Q)} \leq 32 \cdot 16^C 16^{j_1}$. Thus, $Q \subseteq B(x, 3 \text{diam}(Q)) \subseteq B(x, 96 \cdot 16^C 16^{j_1})$. As $\varphi$ is 1-Lipschitz, we have that
\[
(*) \geq \frac{\beta^n}{384^n 16^C n} |B(\varphi(x), 96 \cdot 16^C 16^{j_1})| \geq \frac{\beta^n}{384^n 16^C n} |\varphi(Q \cap K)|.
\]
This finishes the proof of (11). \[\Box\]

The following proposition says that the set of cubes not in $\Sigma(c\delta^n)$, $LD(\eta)$, and $M_A(\zeta, Q_0)$ satisfy a $K$-Carleson estimate. It is mostly proven in [19], but we will require some nontrivial changes. The proof will be given in the appendix.

**Proposition 4.4.** Let $\eta, \delta > 0$ and $A > 1$. Then there exists some $\zeta_0 > 0$ so that for each $\zeta < \zeta_0$ there exists some $C > 0$ depending only these constants and $K$ so that
\[
\sum_{Q \in \Delta(Q_0) \setminus (M_A(\zeta, Q_0) \cup \Sigma(\delta^n) \cup LD(\eta, Q_0))} \mu(Q \cap K) \leq C \mu(Q_0).
\]

We can now prove the main result of this section. We are not keeping track of the number of biLipschitz pieces or the biLipschitz constant—although they can be estimated—as they are not necessary for our purposes.

**Proposition 4.5.** Let $\eta, \delta, \varepsilon > 0$, and $Q_0 \in \Delta$ and let $\varphi : K \to \mathbb{R}^n$ be a chart. Assume $(Q_0 \cap K) \setminus (\Sigma(c\delta^n) \cup \Lambda(\eta))$ has positive measure where $c > 0$ is the constant from Lemma 4.1. Then there exists some $C > 0$ depending only on $\delta$, $\eta$, and $K$ so that the following holds. There exist a finite collection of compact sets $\{F_j\}_{j=1}^M$ in $Q_0 \cap K$ such that $\varphi|_{F_j}$ is biLipschitz and
\[
\mu \left( (Q_0 \cap K) \setminus \bigcup_{j=1}^M F_j \right) \leq \varepsilon + \eta \mu(Q_0) + C \mu(Q_0) \setminus DC(\delta, 1/10, 16\ell(Q_0)).
\]

**Proof.** The proof will follow the usual biLipschitz decomposition method of [11,12], except we now use Lemma 4.1 and Lemma 4.2 to control the measure of $\Sigma(c\delta^n)$ and $\Lambda(\eta)$ and we localize to a $DC$ set. We proceed.

We choose $A, \zeta, k$ depending only on $\delta$ and $K$ so that Lemma 4.3 applies to points of $DC(\delta, 1/4, 16\ell(Q))$ in the cubes of $M_A(\zeta, Q_0)$ and so that $\zeta$ is also small enough to apply
Proposition 5.1. For a cube $Q$, let $\tilde{Q}$ denote the union of cubes in $Q' \in \Delta_{j(Q)}(Q_0)$ so that $Q' \cap 2Q \neq \emptyset$. For $L \geq 1$, we define the set
\[
R_L = \left\{ x \in (Q_0 \cap K)\setminus(\Sigma(c\delta^n) \cup \Lambda(\eta)) : \sum_{Q \in \Delta(Q_0) \setminus M_A(\zeta)} \chi_Q(x) \geq L \right\}.
\]
As $(Q_0 \cap K)\setminus(\Sigma(c\delta^n) \cup \Lambda(\eta))$ has positive measure, by Proposition 4.3 there exists some $L' > 0$ depending only on $A, \zeta, \delta, \eta$ so that
\[
\mu(R_{L'}) < \varepsilon. \tag{12}
\]
The usual coding argument from [12] or Section 2 of [11] required to decompose the set
\[
(Q_0 \cap DC'(\delta, 1/10, 16\ell(Q)) \setminus (\Sigma(c\delta^n) \cup \Lambda(\eta) \cup R_{L'})
\]
into the pieces on which $\varphi$ is biLipschitz remains unchanged. Indeed, the fact that we take points from $DC'(\delta, 1/10, 16\ell(Q))$ allows us to get a weak $k$-biLipschitz behavior from the cubes of $M_A(\zeta)$ by Lemma 4.3. Details are left to the reader.

The only thing left is to bound the measure of $(Q_0 \cap K)\setminus \bigcup F_j$. We have that
\[
\mu((Q_0 \cap K)\setminus \bigcup F_j) \leq \mu(R_{L'}) + \mu(\Lambda(\eta)) + \mu(\Sigma(c\delta^n)) + \mu((Q_0 \cap K)\setminus DC'(\delta, 1/10, 16\ell(Q_0))) \tag{13}
\]
\[
\leq \varepsilon + \eta\mu(Q_0) + C\mu(Q_0 \setminus DC'(\delta, 1/10, 16\ell(Q_0)))
+ \mu(Q_0 \setminus DC'(\delta, 1/10, 16\ell(Q_0)))
\leq \varepsilon + \eta\mu(Q_0) + (C + 1)\mu(Q_0 \setminus DC'(\delta, 1/10, 16\ell(Q_0))).
\]
The dependencies for $C$ are $\zeta, A, \delta, \eta$, but $\zeta$ and $A$ depend on $\delta, \eta$ and $K$ so $C$ depends only on $\delta, \eta, \zeta$. \hfill \Box

5. Proof of main theorem

We will need a preliminary lemma.

Lemma 5.1. Let $A$ be any finite measurable set in $K$ and $\varepsilon, \lambda > 0$. There exists some $J \leq k_K$ so that for every $j \leq J$ there exists a finite collection of disjoint cubes $\{Q_k\}_{k=1}^M \subset \Delta_j$ so that $\mu(A \setminus \bigcup_i Q_i) < \varepsilon$ and $\mu(Q_i \setminus A) < \lambda\mu(Q_i)$.

Proof. By Lebesgue’s differentiation theorem, there exists some $R > 0$ and some subset $A' \subseteq A$ so that $\mu(A \setminus A') < \varepsilon$ and
\[
\mu(B(x, r) \cap A) > (1 - c\lambda)\mu(B(x, r)), \quad \forall r \in (0, R), x \in A'.
\]
Here, $c > 0$ is some constant depending only on $K$ so that if $Q$ is a cube of diameter no more than $R$ which intersects $A'$ nontrivially, then
\[
\mu(Q \cap A) > (1 - \lambda)\mu(Q).
\]
That such a constant exists easily follows from (2), (4), and (5).

Thus, we can choose $J \leq k_K$ so that $32 \cdot 16^J \leq R$ and for every $j \leq J$, we let $\{Q_k\}_{k=1}^M \subset \Delta_j$ be the cubes that intersect with $A'$ nontrivially. \hfill \Box

We can now prove the main theorem.
Proof of Theorem 1.1. As mentioned in the introduction, we simply need to prove that (ii) implies (iii) and (iii) implies (R) in the main theorem.

Recall that, after Section 2, we are reduced to proving that K is n-rectifiable. Proposition 3.1 and Remark 2.1 together proves the implication of (iii) from (ii). It thus suffices to show that any DC(\(\beta, 1/10, R\)) of positive measure is n-rectifiable. We now fix such a \(\beta \in (0, 1)\) and \(R > 0\).

Let \(\varepsilon > 0\). By an application of Lemma 5.1 there exists some \(j \leq k_K\) and a finite collection of cubes \(\{Q_i\}_{i=1}^M \subset \Delta_j\) so that \(16^{j+1} \leq R\),

\[
\mu(\text{DC}(\beta, 1/10, R) \setminus \bigcup Q_i) < \frac{\varepsilon}{4},
\]

\[
\mu(Q_i \setminus \text{DC}(\beta, 1/10, R)) < \frac{\varepsilon}{8C\mu(\text{DC}(\beta, 1/10, R))}\mu(Q_i), \quad \forall i,
\]

where \(C > 0\) is the constant from Proposition 4.5. Assuming that we had initially chosen \(\varepsilon < 4C\mu(\text{DC}(\beta, 1/10, R)))\), which we can and will, we then get from (14) that

\[
\sum_{i=1}^M \mu(Q_i) < 2\mu(\text{DC}(\beta, 1/10, R)).
\]

By choosing \(\eta = \varepsilon/(8\mu(\text{DC}(\beta, 1/10, R)))\) and applying Proposition 4.3 for every \(Q_i\), there exists a finite family of compact subsets \(\{F_{i,j}\}_{j=1}^{m_i}\) of \(K\) so that \(\varphi|_{F_{i,j}}\) is biLipschitz and

\[
\mu((Q_i \cap K) \setminus \bigcup_j F_{i,j}) \leq \frac{\varepsilon}{4M} + \frac{\varepsilon}{8\mu(\text{DC}(\beta, 1/10, R))}\mu(Q_i) + C\mu(Q_i \setminus \text{DC}(\beta, 1/10, 16^{j+1}))
\]

\[
\leq \frac{\varepsilon}{4M} + \frac{\varepsilon}{8\mu(\text{DC}(\beta, 1/10, R))}\mu(Q_i) + C\mu(Q_i \setminus \text{DC}(\beta, 1/10, R)).
\]

We can apply Proposition 4.3 because we can use Lemma 4.1, Lemma 4.2, and (14) to prove that \((Q_i \cap K) \setminus (\Sigma(c\beta^n) \cup \Lambda(\eta))\) has positive measure whenever \(\varepsilon\) is chosen small enough.

We let \(F_{i,j}' = F_{i,j} \cap \text{DC}(\beta, 1/10, R)\), which clearly also satisfies

\[
\mu((Q_i \cap \text{DC}(\beta, 1/10, R)) \setminus \bigcup_j F_{i,j}') \leq \frac{\varepsilon}{4M} + \frac{\varepsilon}{8\mu(\text{DC}(\beta, 1/10, R))}\mu(Q_i) + C\mu(Q_i \setminus \text{DC}(\beta, 1/10, R)).
\]

Thus, \(\{F_{i,j}'\}_{i=1,j=1}^{M,m_i}\) are a finite collection of bounded subsets of \(\text{DC}(\beta, 1/10, R)\) on each of which \(\varphi\) is biLipschitz. The last step is to bound the size of \(\text{DC}(\beta, 1/10, R) \setminus \bigcup F_{i,j}'\). We
have
\[ \mu \left( DC(\beta, 1/10, R) \setminus \bigcup_{i,j} F'_{i,j} \right) \]
\[ \leq \mu \left( DC(\beta, 1/10, R) \setminus \bigcup Q_i \right) + \sum_{i=1}^{M} \mu \left( (Q_i \cap DC(\beta, 1/10, R)) \setminus \bigcup_{j} F'_{i,j} \right) \]
\[ \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{8\mu(DC(\beta, 1/10, R))} \sum_{i=1}^{M} \mu(Q_i) + C \sum_{i=1}^{M} \mu(Q_i \setminus DC(\beta, 1/10, R)) \]
\[ \leq \frac{\varepsilon}{4} + \frac{\varepsilon}{8\mu(DC(\beta, 1/10, R))} \sum_{i=1}^{M} \mu(Q_i) \leq \varepsilon. \]

As \( \varepsilon > 0 \) was arbitrary, this finishes the proof that \( DC(\beta, 1/10, R) \) is n-rectifiable, which, as mentioned before, proves that \( K \) is n-rectifiable. Thus, we have shown that (iii) implies (R), which finishes the proof of Theorem 1.1. \( \square \)

Appendix A. Proof of Proposition 4.4

A.1. Introduction. The proof of Proposition 4.4 closely follows the proof of Proposition 7.8 in [19], which, needless to say, a good understanding of will be necessary. Most of the proof will only require superficial changes. Thus, for convenience, we will not go into much details in these parts. There are some parts where we will have to make nontrivial modifications; most notably, we have to add an extra condition to a stopping time process defined in [19]. We will go through these modifications carefully. Our lemmas and propositions will be numbered A.X.Y' where X.Y is the numbering of the corresponding lemma or proposition in [19].

Recall that we have \( K \subset X \) satisfying (5), a collection of cubes \( \Delta \) satisfying Proposition 2.2 and a 1-Lipschitz map \( \varphi : K \to \mathbb{R}^n \). We will recall further necessary concepts from [19] in the relevant subsections below. Also recall the families of cubes \( \mathcal{L}D(\eta) \) from Section 4, which was not in [19]. This family of cubes (or more precisely, the complement of this family) will play a big role in establishing our \( K \)-Carleson bound.

A.2. Section 2 of [19]: Lemmas. In this subsection, we recall some preliminary terminology and lemmas. Most of the lemmas of Section 2 of [19] establish Carleson bounds. We will convert them to establishing \( K \)-Carleson bounds.

Recall a nonempty collection cubes \( S \subseteq \Delta \) is called a stopping time region if there is a top cube \( Q(S) \in S \) so that for every other \( Q \in S \) such that \( Q \subset Q(S) \) and \( Q' \in \Delta \) such that \( Q \subseteq Q' \subseteq Q(S) \), then \( Q' \in S \). The set of bottom cubes of a stopping time region \( S \) is
\[ b(S) = \{ Q \in \Delta : Q \subseteq Q(S), Q \notin S, \text{and } Q \text{ is maximal with respect to these properties} \}. \]

Note that these are disjoint cubes contained in \( Q(S) \) if they exist (\( b(S) \) may in fact be an empty set).

Fix a cube \( Q_0 \) and let \( \mathcal{E} \) a family of cubes. For \( x \in Q_0 \), we let \( N(x) = N_{Q_0}(x) \) denote the number of cubes in \( \mathcal{E} \) that contain \( x \). Note that all such cubes are then subsets of \( Q_0 \).

The following lemma will be very important in establishing Carleson bounds.
Lemma A.2.28'. Let $E$ be a family of cubes in $\Delta$ and suppose that there are positive constants $k, \lambda$ such that
\[
\mu(\{x \in Q_0 \cap K : N_{Q_0}(x) > L\}) \leq (1 - \lambda)\mu(Q_0 \cap K)
\]
for every cube $Q_0 \in \Delta$. Then $E$ is a $K$-Carleson set with constants depending only on $L$ and $\lambda$.

The proof requires only superficial modifications of the proof of the original Lemma 2.28 in [19] and so is omitted. Note that following that proof actually gives the stronger statement that
\[
\sum_{Q' \in E, Q' \subseteq Q} \mu(K \cap Q') \leq C\mu(K \cap Q), \quad \forall Q \in \Delta,
\]
but we will not need this.

Let $\eta > 0$. If a family of cubes $E$ is a $K$-Carleson set with constant $C$ and does not contain any cubes for which $\mu(Q \cap K) < \eta\mu(Q)$, then it follows easily from the definition of $(K)$-Carleson sets that $E$ is actually a Carleson set with constant $C/\eta$.

Given a family of cubes $E$, we let $E_A$ denote the set of cubes $Q$ such that $Q$ is an $A$-neighbor of some $Q' \in E$.

We will need the original version of Lemma 2.32 of [19].

Lemma A.2.32. Let $E$ be a Carleson set with constant $C$. Then $E_A$ is also a Carleson set with some constant $C'$ increased by a factor depending on $A$ and $K$.

We will also need the following $K$-Carleson version.

Lemma A.2.32'. Let $\eta > 0$ and $E$ be a $K$-Carleson set with constant $C$ so that $\mu(Q \cap K) \geq \eta\mu(Q)$ for all $Q \in E$. Then $E_A$ is also an $K$-Carleson set with some constant $C'$ increased by a factor depending on $K, A, \text{ and } \eta$.

Proof. As $E$ does not contain cubes for which $\mu(Q \cap K) < \eta\mu(Q)$, we have that it is a Carleson set of constant depending on $\eta$ and the $K$-Carleson constant of $E$. We then apply Lemma A.2.32 to get that $E_A$ is Carleson with constant increased by a factor depending on $A, K, \text{ and } \eta$. As Carleson sets are $K$-Carleson, we are done. \qed

Given a $T \in \Delta$, we let
\[
C_A(T) = \{Q \in \Delta : \text{there is a cube } Q' \in \Delta \text{ such that } Q \text{ and } Q' \text{ are neighbors, and either } Q \subseteq T, Q' \not\subseteq T, \text{ or } Q' \subseteq T, Q \not\subseteq T\}.
\]
See (2.33) of [19].

Lemma A.2.34'. There is a constant $D$ so that
\[
\sum_{Q \in C_A(T)} \mu(Q \cap K) \leq D\mu(T),
\]
where $D$ depends only on $A$ and $K$.

The proof requires only superficial modifications of the proof of the original Lemma 2.34 in [19] and so is omitted. Note by (3) that our cubes have the small boundary property only when restricted to $K$, but this is exactly what is needed for the lemma.
We now move to the Carleson set composition lemmas of [19]. Note that $K$-Carleson conditions do not compose because the $K$-Carleson upper bound does not take into account intersections with $K$. Thus, we will need that there exists some $\eta > 0$ so that all our cubes in the following lemmas satisfy $\mu(Q \cap K) \geq \eta \mu(Q)$. We call these cubes high density cubes. They help us turn $K$-Carleson bounds into Carleson bounds, which will lead to nice composition properties.

**Lemma A.2.44'**. Let $\eta > 0$ and $\mathcal{X}$ be a collection of cubes that is a $K$-Carleson set and for which each $Q \in \mathcal{X}$ satisfies $\mu(Q \cap K) \geq \eta \mu(Q)$. Define $\hat{\mathcal{X}}_A$ by

$$\hat{\mathcal{X}}_A = \bigcup_{T \in \mathcal{X}} C_A(T).$$

Then $\hat{\mathcal{X}}_A$ is also a $K$-Carleson set, with a constant depending only on $K$, $\eta$, and the $K$-Carleson constant for $\mathcal{X}$.

**Proof.** Fix some $Q_0 \in \Delta$. As in the proof of Lemma 2.44 of [19], we get that

$$\sum_{R \in \hat{\mathcal{X}}_A, R \subseteq Q_0} \mu(R \cap K) \leq \sum_{R \in C_A(Q_0)} \mu(R \cap K) + \sum_{T \in \mathcal{X}(Q_0)} \sum_{R \in C_A(T)} \mu(R \cap K).$$

See the argument before equation (2.47) of [19]. Using Lemma A.2.34', we can convert this to

$$\sum_{R \in \hat{\mathcal{X}}_A, R \subseteq Q_0} \mu(R \cap K) \leq C \mu(Q_0) + C \sum_{T \in \mathcal{X}(Q_0)} \mu(T).$$

As $\mathcal{X}$ is $K$-Carleson and does not contain cubes for which $\mu(Q \cap K) < \eta \mu(Q)$, we get that

$$\sum_{R \in \hat{\mathcal{X}}_A, R \subseteq Q_0} \mu(R \cap K) \leq C \mu(Q_0) + \frac{C}{\eta} \sum_{T \in \mathcal{X}(Q_0)} \mu(T \cap K) \leq C' \mu(Q_0),$$

where $C'$ depends on the previous $C$ and $\eta$. \qed

**Lemma A.2.50'.** Let $\eta > 0$ and $\mathcal{F}$ be a family of stopping-time regions that are disjoint as subsets of $\Delta$. Assume that the collection of top cubes $\{Q(S) : S \in \mathcal{F}\}$ is a $K$-Carleson set with constant $C_1$ and does not contain any cubes for which $\mu(Q \cap K) < \eta \mu(Q)$. Suppose for each $S \in \mathcal{F}$ we have a collection of cubes $\mathcal{E}(S) \subseteq S$ that is a $K$-Carleson set with constant $C_2$. Then the union

$$\mathcal{E}^* = \bigcup_{S \in \mathcal{F}} \mathcal{E}(S)$$

is a $K$-Carleson set with constant depending only on $C_1$, $C_2$, and $\eta$.

The proof requires only superficial modifications of the proof of the original Lemma 2.50 in [19] once one takes into account that $\{Q(S) : S \in \mathcal{F}\}$ contains only high density cubes and so is actually also a Carleson set. Thus, the proof is omitted.

**Lemma A.2.58'.** Let $\eta > 0$ and $\mathcal{F}$ be a family of stopping time regions that are disjoint as subsets of $\Delta$. For each $S \in \mathcal{F}$ set

$$S_A = \{Q \in S : Q' \in S \text{ whenever } Q \text{ and } Q' \text{ are neighbors }\},$$

where $\cup_{S \in \mathcal{F}} S_A$ is a $K$-Carleson set with constant depending only on $\eta$.\ued
and set
\[ B_A = \bigcup_{S \in F} ((S \cap K) \setminus S_A). \]

If the collection of top cubes \( \{Q(S) : S \in F\} \) is a \( K \)-Carleson set and satisfies \( \mu(Q \cap K) \geq \eta \mu(Q) \), then \( B_A \) is also a \( K \)-Carleson set with constant depending only on the \( K \)-Carleson constant for \( \{Q(S)\}_{S \in F} \), \( \eta \), \( A \), and \( K \).

The proof requires only superficial modifications of the proof of the original Lemma 2.58 in [19] once one takes into account that \( \{Q(S) : S \in F\} \) contains no cubes of low density and so is actually also a Carleson set. Note that the bottom cubes of a stopping time region \( b(S) \) are still Carleson with constant 1. Thus, the proof is omitted.

A.3. Section 3 of [19].

**Proposition A.3.6’.** Let \( \tau, \delta, \eta > 0 \) and \( Q_0 \in \Delta \) such that \( |\varphi(Q_0 \cap K)| \geq \delta(Q_0) \) and \( \mu(Q_0 \cap K) \geq \eta \mu(Q_0) \). There exist constants \( k, \alpha > 0 \) depending only on \( \tau, \delta, \) and \( K \) so that the following is true. There exists a family \( F \) of pairwise-disjoint stopping time regions of \( \Delta \) and a measurable subset \( E \subseteq Q_0 \cap K \) with the following properties:

(a) \( \mu(E) \geq \alpha \mu(Q_0 \cap K) \),
(b) if \( Q \in \Delta \) satisfies \( Q \subseteq Q_0 \) and \( Q \cap E \neq \emptyset \), then either \( Q \) lies in \( S \) for some stopping-time region \( S \in F \) or \( Q \in LD(\eta) \),
(c) if \( Q \in S \) and \( S \in F \), then \( Q \subseteq Q_0 \), and
   \[ (1 + \tau)^{-1} \frac{|\varphi(Q(S) \cap K)|}{\mu(Q(S))} \leq \frac{|\varphi(Q \cap K)|}{\mu(Q)} \leq (1 + \tau) \frac{|\varphi(Q(S) \cap K)|}{\mu(Q(S))}, \]
(d) \( |\varphi(Q(S) \cap K)| \geq \delta \mu(Q(S)) \) for all \( S \in F \),
(e) for each \( x \in K \), there are at most \( k \) choices of \( S \in F \) such that \( x \in Q(S) \),
(f) \( \mu(Q \cap K) \geq \eta \mu(Q) \) for all \( Q \in S \) and \( S \in F \).

**Proof.** We run the stopping time process of Section 3 of [19] on the subcubes of \( Q_0 \) but with an extra stopping time condition. Specifically, we stop at a cube \( Q \subseteq Q_0 \) if any of the following conditions are satisfied:

\[ \frac{|\varphi(Q \cap K)|}{\mu(Q)} < (1 + \tau)^{-1} \frac{|\varphi(Q_0 \cap K)|}{\mu(Q_0)}, \quad (17) \]
\[ \frac{|\varphi(Q \cap K)|}{\mu(Q)} > (1 + \tau) \frac{|\varphi(Q_0 \cap K)|}{\mu(Q_0)}, \quad (18) \]
\[ \mu(K \cap Q) < \eta \mu(Q). \quad (19) \]

Note that the first two conditions may not necessarily be disjoint from the third. We start with \( Q_0 \) and only keep the children of \( Q_0 \) that do not satisfy (17), (18), or (19). We then apply the process to the kept children and iterate the process on all the kept children. This gives us one stopping time region \( S_0 \), which we put into the singleton family \( F_0 \). We then look at the bottom cubes \( b(S_0) \). For each \( Q \in b(S_0) \) that satisfies (18) but not (19), we repeat this stopping time process to get another family of stopping time regions \( F_1 \). We repeat again the process for the bottom cubes of all stopping time regions in \( F_1 \) that satisfy (18) but not (19) to get another family of stopping time regions \( F_2 \). We keep repeating this
process over and over to more families $F_3, F_4, F_5...$. Our final family of stopping time regions will be $F = \bigcup_{i=0}^{\infty} F_i$.

Properties c, d, and e can be verified in the same way as they were in [19]. Property f is also immediate from the condition of the stopping time.

To construct $E$, for $S \in F$, let $b_1(S)$ denote all the cubes of $b(S)$ that satisfy (17) but not (19). If we define

$$E = Q_0 \setminus \bigcup_{S \in F} \bigcup_{Q \in b_1(S)} Q,$$

then we see that $E$ satisfies Property b. Indeed, the stopping time process completely terminates only when either (17) or (19) is satisfied. Thus, if $Q \cap E \neq \emptyset$, then either $Q \subseteq Q'$ for some maximal $Q'$ satisfying (19), $Q \supseteq Q'$ for some maximal $Q'$ satisfying (19), or $Q$ is disjoint from all such $Q'$. The first case, $Q \in LD(\eta)$. In the second case, we have that $Q' \subseteq Q \subseteq Q_0$. From the construction, one then see that $Q \in S$ for some $S \in F$ as the fact that $Q'$ was maximal means that $Q'$ was the when the process terminated. In the third case, we see that $Q$ then contains some $S \in F$ as the process never terminated inside $Q$ and so $Q \in F$ by similar reasoning of the second case. In all three cases, we see that $Q$ satisfies Property b.

It remains to lower bound $\mu(E)$ as in Property a. We have the following lemma.

**Lemma A.3.16'.** Let $Q$ be a cube in $\Delta$ and $\{Q_i\}_i$ be a disjoint family of subcubes for which

$$\frac{|\varphi(Q_i \cap K)|}{\mu(Q_i)} \leq (1 + \tau)^{-1}\frac{|\varphi(Q \cap K)|}{\mu(Q)}, \quad \forall i.$$

Then

$$\mu\left((Q \cap K) \setminus \bigcup_i Q_i\right) \geq \frac{\tau}{1 + \tau} |\varphi(Q \cap K)|.$$

The proof requires only superficial modifications of the proof of the original Lemma 3.16 in [19] and so will be omitted.

Let $S \in F$ and set $E_0(S) = (Q(S) \cap K) \setminus \bigcup_{R \in b_1(S)} R$. We get from Lemma A.3.16 that

$$\mu(E_0(S)) \geq \frac{\tau}{1 + \tau} |\varphi(Q(S) \cap K)| \geq \frac{\tau}{1 + \tau} \delta \mu(Q(S) \cap K),$$

where we used Property d in the last inequality. This is the analogue of equation (3.26) of [19]. The rest of the proof of Property a only requires superficial modifications of the proof of the original Property a of Lemma 3.6 in [19] and so will be omitted.

As was proven in Remark 3.46 of [19], the set $G = \bigcup_{S \in F} S$ is itself a stopping time region.

**A.4. Section 4 of [19].**

**Proposition A.4.2'.** Let $Q_0 \in \Delta$ and fix $\delta, \tau, \eta > 0$. There exists a constant $k_1$ depending only on $K$, $\delta$, and $\tau$, as well as a family $F_1$ of stopping-time regions in $\Delta$ and two collections $\{Q_i\}_{i \in I}$ and $\{P_j\}_{j \in J}$ of cubes in $M$ so that the following are true:

(a) the $Q_i$’s and $P_j$’s together are pairwise disjoint subcubes of $Q_0$ and the stopping time regions $F$ are pairwise disjoint subsets of $\Delta(Q_0)$,
(b) if $R \in \Delta(Q_0)$ then either $R \subseteq Q_i$ for some $i \in I$, $R \subseteq P_j$ for some $j \in J$, or $R \in S$ for some $S \in \mathcal{F}_1$ (but not more than one),

(c) if $Q \in S$ and $S \in \mathcal{F}_1$, then

\[
(1 + \tau)^{-1}|\varphi(Q(S) \cap K)| \leq |\varphi(Q \cap K)| \leq (1 + \tau)\frac{|\varphi(Q(S) \cap K)|}{\mu(Q(S))},
\]

(d) $|\varphi(Q(S) \cap K)| \geq \delta\mu(Q(S))$ for all $S \in \mathcal{F}_1$,

(e) the family of cubes $\{Q(S) : S \in \mathcal{F}_1\}$ is a $K$-Carleson set with constant $k_1$.

(f) $|\varphi(Q_i \cap K)| < \delta\mu(Q_i)$, \quad $\forall i \in I$.

(g) $\mu(P_j \cap K) < \eta\mu(P_j)$, \quad $\forall j \in J$.

(h) $\mu(Q \cap K) \geq \eta\mu(Q)$ for all $Q \in S$ and $S \in \mathcal{F}_1$.

We may assume that $|\varphi(Q_0 \cap K)| \geq \delta\mu(Q_0)$ and $\mu(Q_0 \cap K) \geq \eta\mu(Q_0)$ as otherwise there is nothing to do. As in [19], the union of all the stopping time regions $S \in \mathcal{F}$ where $\mathcal{F}$ is the family of stopping time regions of Proposition A.3.6 is a stopping time region itself. Thus, we apply Proposition A.3.6 to $Q_0$ to get a family of stopping time regions $\mathcal{F}(Q_0)$ and let $G$ denote the stopping time region that is the union of all cubes of $\mathcal{F}$. Let $b(G)$ denote the bottom cubes of $G$. By construction, if $Q \in b(G)$, then $Q$ has to satisfy at least one of (17) or (19). All the cubes that satisfy (19) we put into $P$. For the other cubes, we check to see if

\[
|\varphi(Q \cap K)| < \delta\mu(Q).
\]

If so, we put it into $Q_i$. Any remaining bottom cube $Q'$ satisfy $\varphi(Q' \cap K) \geq \delta\mu(Q')$ and $\mu(Q' \cap K) \geq \eta\mu(Q')$ and so we apply the stopping time process of Proposition A.3.6 on each of these to get more families of stopping time regions $\mathcal{F}(Q')$. We continue this way forever or until we run out of cubes. We let $\mathcal{F}_1$ denote the union of all these $\mathcal{F}(Q')$. Note that $\mathcal{F}_1$ are composed of stopping time regions of each $\mathcal{F}$ generated by Proposition A.3.6, not the union of these stopping time regions.

By construction, all the properties besides e are satisfied. See the proof of Proposition 4.2 of [19] for more information if needed. The proof of Property e is also similar to the proof of the analogous property in [19] with only superficial modifications. For example, let $\mathcal{G}$ denote the set of cubes for which Proposition A.3.6 was applied in the above construction and $Q(G)$ denote the set of cubes that make up the stopping time process starting at $G \in \mathcal{G}$. Thus, $\mathcal{G}$ is a subset of $\{Q(S) : S \in \mathcal{F}_1\}$ and does not contain cubes for which $\mu(Q \cap K) < \eta\mu(Q)$. We can get the following claim.

Claim A.4.16'. For each cube $R \in \Delta$, there is a measurable subset $F(R)$ of $R \cap K$ such that $\mu(F(R)) \geq \alpha\mu(R \cap K)$ and so that for each $y \in F(R)$ there is at most one $Q \in \mathcal{G}$.

The proof requires only superficial modifications of the proof of the original Claim 4.16 in [19] and so will be omitted. Note that we are using the modified set $E$ of Proposition A.3.6' (by the construction above), which also has the cubes $P_j$, but this is fine as the construction above completely terminates at these cubes.

As in [19], Claim A.4.16' and Lemma A.2.28 show that $\mathcal{G}$ is $K$-Carleson. The rest of the proof of Property e follows completely analogously as in [19]. We use that each $\{Q(S) \in \mathcal{F}_1\} \cap Q(G)$ is uniformly $K$-Carleson and each $G \in \mathcal{G}$ contains only cubes of high density cubes along with Lemma A.2.50 to establish Property e. The details are left to the reader.
A.5. Section 5 of [19]. We recall Definition 5.1 of [19], which says that a stopping time region $S$ is good if for each $Q \in S$, either all of its children are in $S$ or none of them are.

We will need the following lemma for the next section.

Lemma A.5.2'. Suppose $S \subseteq \Delta$ is a good stopping time region. Let $Q \in S$ and $\{T_i\} \subseteq S$ be a finite family of pairwise-disjoint cubes so that $T_i \subseteq Q$ for all $i$. Then there exists another finite family of pairwise disjoint cubes $\{W_j\} \subseteq S$ so that $W_j \subseteq Q$ for all $j$, each $W_j$ is disjoint from all the $T_i$, and

$$Q \cap K = \left( \bigcup_i (T_i \cap K) \right) \cup \left( \bigcup_j (W_j \cap K) \right).$$

The proof requires only superficial modifications of the proof of the original Lemma 5.2 in [19] and so will be omitted.

Proposition A.5.5'. Let $Q_0 \in \Delta$ and fix $\delta, \tau, \eta > 0$. There exists a constant $k_2$ depending on $\delta$ and $\tau$, as well as a family $F_2$ of stopping time regions in $\Delta$ and two collections $\{Q_i\}_{i \in I}$ and $\{P_j\}_{j \in J}$ of cubes so that the following are true:

(a) the $Q_i$’s and $P_i$’s together form a pairwise disjoint collection of subcubes of $Q_0$ and the stopping time regions in $F_2$ are pairwise-disjoint as subsets of $\Delta(Q_0)$,

(b) if $R \in \Delta(Q_0)$, then either $R \subseteq Q_i$ for some $i \in I$, $R \subseteq P_j$ for some $j \in J$, or $R \in S$ for some $S \in F_2$ (but not more than one),

(c) if $Q, \tilde{Q} \in S$ and $S \in F_2$, then

$$(1 + \tau)^{-2} \frac{|\varphi(Q \cap K)|}{\mu(Q)} \leq \frac{|\varphi(\tilde{Q} \cap K)|}{\mu(\tilde{Q})} \leq (1 + \tau)^2 \frac{|\varphi(Q \cap K)|}{\mu(Q)},$$

(d) $|\varphi(Q \cap K)| \geq (1 + \tau)^{-1}\delta \mu(Q)$ when $Q \in S$, $S \in F_2$,

(e) the family of cubes $\{Q(S) : S \in F_2\}$ is a $K$-Carleson set with constant $k_2$,

(f) $|\varphi(Q_i \cap K)| < \delta \mu(Q_i)$ for all $i \in I$,

(g) each $S \in F_2$ is a good stopping time region,

(h) $\mu(P_j \cap K) < \eta \mu(P_j)$ for all $j \in J$.

(i) $\mu(Q \cap K) \geq \eta \mu(Q)$ for all $Q \in S$ and $S \in F_2$.

We run the same exact stopping time region decomposition of the proof of Proposition 5.5 of [19] on $F_1$ of Proposition A.4.2 to get a family of good stopping time regions $F_2$. Thus, Property g is satisfied by construction and all other properties besides e are satisfied by the properties of $F_1$, $Q_i$, and $P_j$ of Proposition A.4.2.

As in [19], we see that a top cube $Q \in \{Q(S) : S \in F_2\}$ either belongs to $\{Q(S) : S \in F_1\}$ or has a parent that belong to some $S \in F_1$, but one of the children of the parent (a sibling of $Q$) does not belong to $S$. This comes from the good stopping time decomposition of the stopping time regions in $F_1$. From the previous proposition, the cubes that are contained in the former case are $K$-Carleson, so we do not have to worry about them. For cubes from the latter case, we have that one of the siblings $Q'$ of $Q$ must either be a top cube of some other $\tilde{S} \in F_1$ or belong to one of the family $\{Q_i\}_{i \in I}$ and $\{P_j\}_{j \in J}$.

In the second case, we have that $\{Q_i\}_{i \in I}$ and $\{P_j\}_{j \in J}$ are both Carleson sets because they are composed of disjoint cubes. Thus, Lemma A.2.32 shows that this group of cubes $Q'$ are Carleson and so also $K$-Carleson. In the first case, we have from Properties e and h
of Proposition A.4.2' that \( \{Q(S) : S \in F_1 \} \) are \( K \)-Carleson and contain only high density cubes. Thus, by Lemma A.2.32', we get that this group of cubes \( Q' \) is also \( K \)-Carleson. This finishes the proof of property e, which finishes the proof of the entire proposition.

A.6. Section 6 of [19]. We keep the same notation as in the previous sections. We recall some more notation from [19]. For a cube \( Q \in \Delta \), we let

\[
\ast Q = \bigcup \{ T \in \Delta_j(Q) : \text{dist}(T, Q) \leq \text{diam}(Q) \}.
\]

Thus, \( \hat{Q} = \ast Q \cap Q_0 \). Given some \( \sigma > 0 \), we let

\[
G(\sigma) = \left\{ Q \in \Delta(Q_0) : (1 + \sigma)^{-1} \frac{|\varphi(Q \cap K)|}{\mu(Q)} \leq (1 + \sigma) \frac{|\varphi(\hat{Q} \cap K)|}{\mu(\hat{Q})} \right\}.
\]

Let \( \tau \) be a small number and \( \delta, \eta > 0 \). Then we can use Proposition A.5.5' to get a family \( F_2 \) of stopping time regions in \( Q_0 \) along with two families of mutually disjoint subcubes \( \{Q_i\}_{i \in I} \) and \( \{P_j\}_{j \in J} \). We set

\[
G_2 = \bigcup_{S \in F_2} S.
\]

Proposition A.6.13'. Let \( \sigma, \delta, \eta > 0 \). If we choose \( \tau \) small enough, depending on \( \sigma \) and \( K \), then \( G_2 \setminus G(\sigma) \) is a \( K \)-Carleson set with constant depending only on \( \tau, \delta, \eta, \) and \( K \).

The fact that \( G_2 \) contains only cubes of high density allows us to transition from \( K \)-Carleson estimates to Carleson estimates. Keeping this in mind, most of the proof then requires only superficial modifications of the proof of the original Proposition 6.13 in [19]. For example, for some \( S \in F_2 \), we set

\[
S' = \{ Q \in S : T \in S \text{ whenever } T \in \Delta_j(Q) \text{ and } \text{dist}(T, Q) \leq \text{diam}(Q) \}.
\]

Fixing some \( Q \in S \), we set

\[
B_1 = \{ R \in S' \setminus G(\sigma) : \ast R \subseteq Q \}.
\]

We have the following lemma.

Lemma A.6.27'. There is a constant \( C_2 \) which depends only on \( K \) and \( \eta \) so that

\[
\sum_{R \in B_1} \mu(R \cap K) \leq C_2 \mu(Q \cap K).
\]

The proof follows easily from the proof of the original Lemma 6.27 and the fact that \( Q \in S \) has high density.

Using Lemma A.6.27' we get as in [19] that it suffices to prove the following modification of the fourth reduction of the proof of Proposition 6.13

Reduction A.6.29'. It suffices to show for every \( Q \in S \) that if \( \tau \) is small enough depending on \( \sigma, \eta, \) and \( K \), then there is a measurable subset \( E(Q) \) of \( Q \cap K \) such that \( \mu(E(Q)) \geq \frac{1}{2} \mu(Q \cap K) \) and there are no cubes of \( B_1 \) that intersect \( E(Q) \).

The proof of this fourth reduction requires mostly superficial modifications of the proof of the original Reduction 6.29 in [19]. For instance, we start off with the Lemma 6.39 of [19] unmodified:
Lemma A.6.39. There is a family \( \{ R_j \}_{j \in J} \) of elements of \( \mathcal{B}_1 \) such that
\[ *R_i \cap *R_j = \emptyset, \quad \text{when } i \neq j, \]
and
\[ \bigcup_{R \in \mathcal{B}_1} R \subseteq \bigcup_{j \in J} \lambda R_j, \]
where \( \lambda \) depends only on \( K \).

The proof is also unchanged.

We thus get that
\[ \bigcup_{R \in \mathcal{B}_1} (R \cap K) \subseteq \bigcup_{j \in J} (\lambda R_j \cap K). \quad (20) \]

One easily sees that
\[ \mu(\lambda R \cap K) \leq \mu(\lambda R) \leq C \mu(R), \quad \forall R \in \mathcal{B}_1, \quad (21) \]
where \( C \) depends on \( \lambda > 1 \) and \( K \). Thus, we get the following equation that is analogous to (6.47):
\[ \mu \left( \bigcup_{R \in \mathcal{B}_1} (R \cap K) \right) \leq \mu \left( \bigcup_{j \in J} (\lambda R_j \cap K) \right) \leq C \sum_{j \in J} \mu(R_j). \quad (22) \]

The rest of the proof of Reduction A.6.29' now continues as in [19] with superficial modifications. Namely, define
\[ V = \bigcup_{j \in J} *R_j. \]

Then one derives (as in [19]) that
\[ \mu(V) \leq \frac{\tau}{\sigma} 3(1 + \sigma) \mu(Q). \]

One then gets
\[ \mu \left( \bigcup_{R \in \mathcal{B}_1} (R \cap K) \right) \leq C \sum_{j \in J} \mu(R_j) \leq C \sum_{j \in J} \mu(*R_j) \leq C \mu(V) \]
\[ \leq \frac{C \tau}{\sigma} 3(1 + \sigma) \mu(Q) \leq \frac{C \tau}{\eta \sigma} 3(1 + \sigma) \mu(Q \cap K). \]

In the last inequality, we used the fact that \( Q \in S \in \mathcal{F}_2 \) has high density. Taking \( \tau \) small enough finishes the proof.

A.7. Section 7 of [19]: Final proof. We keep the same notation as the previous sections. We need the following lemma.

Lemma A.7.11'. \( \Delta(Q_0 \setminus G_2) \subseteq SI(\delta) \cup LD(\eta) \).

This follows directly from Properties b, f, and h of Proposition A.5.5'.

The rest of the proof of Proposition 4.4 now continues as in the proof of Proposition 7.8 of [19] with only superficial modifications.
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