Identities of inverse Chevalley type for graded characters of level-zero Demazure submodules over quantum affine algebras of type $C^*$

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Abstract

We provide identities of inverse Chevalley type for the graded characters of level-zero Demazure submodules of extremal weight modules over a quantum affine algebra of type $C$. These identities express the product $e^\mu \text{gch} V_{x}^{-}(\lambda)$ of the (one-dimensional) character $e^\mu$, where $\mu$ is a (not necessarily dominant) minuscule weight, with the graded character $\text{gch} V_{x}^{-}(\lambda)$ of the level-zero Demazure submodule $V_{x}^{-}(\lambda)$ over the quantum affine algebra $U_q(g_{af})$ as an explicit finite linear combination of the graded characters of level-zero Demazure submodules. These identities immediately imply the corresponding inverse Chevalley formulas in the torus-equivariant $K$-group of the semi-infinite flag manifold $Q_G$ associated to a connected, simply-connected and simple algebraic group $G$ of type $C$. Also, we derive cancellation-free identities from the identities above of inverse Chevalley type in the case that $\mu$ is a standard basis element $\varepsilon_k$ in the weight lattice $P$ of $G$.

1 Introduction

The purpose of this paper is to prove identities (of inverse Chevalley type) for the graded characters of Demazure submodules (level-zero Demazure submodules) of extremal weight modules with level-zero extremal weight over a quantum affine algebra of type $C$.

Let $U_q(g_{af})$ be the quantum affine algebra associated to the (untwisted) affine Lie algebra $g_{af}$ whose underlying simple finite-dimensional Lie algebra is $g$. Let us denote by $W_{af}$ (resp., $W$) the Weyl group, by $h_{af}$ (resp., $h$) the Cartan subalgebra, and by $P_{af}$ (resp., $P$) the weight lattice of $g_{af}$ (resp., $g$), where $P = \sum_{i \in I} \mathbb{Z}c_i$ and $P_{af} = P + \mathbb{Z}d + \mathbb{Z}a_0$. For $x \in W_{af}$ and $\lambda \in P^+$, with $P^+ \subset P$ the set of dominant weights for $g$, let $V_{x}^{-}(\lambda)$ denote the Demazure submodule (level-zero Demazure submodule) of the extremal weight module $V(\lambda)$ with extremal weight $\lambda$ over $U_q(g_{af})$, where $\lambda \in P$ is regarded as an element of $P_{af}$ in a canonical way. In recent years, the graded characters $\text{gch} V_{x}^{-}(\lambda)$ of the level-zero Demazure submodule $V_{x}^{-}(\lambda)$ for $x \in W_{af}$, $\lambda \in P$ has been studied in several works.

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Among them, Kato-Naito-Sagaki [KNS] obtained an explicit description of the expansion of the graded character \( \text{gch} V_x^- (\lambda + \mu) \) for \( \lambda, \mu \in \mathbb{P}^+ \) as an infinite linear combination with coefficients in \( \mathbb{Z}[q, q^{-1}][\mathbb{P}] \) of graded characters \( \text{gch} V_y^- (\nu) \) for \( y \in W_{af} \) and \( \nu \in \mathbb{P} \). Also, Naito-Orr-Sagaki [NOS] obtained a similar description of the graded character \( \text{gch} V_x^- (\lambda - \mu) \) for \( \lambda, \mu \in \mathbb{P}^+ \); note that in this case, the expansion is, in fact, a finite linear combination with coefficients in \( \mathbb{Z}[q, q^{-1}][\mathbb{P}] \). Recently, Kouno-Lenart-Naito [KLN] (cf. [LNS]) obtained an explicit description of the expansion, as an infinite linear combination with coefficients in \( \mathbb{Z}[q, q^{-1}][\mathbb{P}] \), of the graded character \( \text{gch} V_x^- (\lambda + \mu) \) for \( \lambda \in \mathbb{P}^+ \) and an arbitrary \( \mu \in \mathbb{P} \) such that \( \lambda + \mu \in \mathbb{P}^+ \). This identity is of the following form:

\[
gch V_x^- (\lambda + \mu) = \sum_{y \in W_{af}, \nu \in \mathbb{P}} c_{x, \mu, \nu}^{y, \nu} \text{gch} V_y^- (\lambda),
\]

where \( c_{x, \mu, \nu}^{y, \nu} \in \mathbb{Z}[q, q^{-1}] \) for \( y \in W_{af} \) and \( \nu \in \mathbb{P} \), and \( e^\nu \) for \( \nu \in \mathbb{P} \) denotes the character of \( H \) with weight \( \nu \). Here we should mention that the coefficients \( c_{x, \mu, \nu}^{y, \nu} \) are independent of the weight \( \lambda \in \mathbb{P} \); also, for each \( y \in W_{af} \), the sum \( \sum_{\nu \in \mathbb{P}} c_{x, \mu, \nu}^{y, \nu} e^\nu \) is an element of \( \mathbb{Z}[q, q^{-1}][\mathbb{P}] \). This explicit identity is called an identity of Chevalley type.

Our main interest lies in an explicit description of the expansion of the product \( e^\nu \text{gch} V_x^- (\lambda) \) as a finite linear combination of the graded characters \( \text{gch} V_y^- (\lambda + \mu) \) for \( y \in W_{af} \) and \( \mu \in \mathbb{P} \); that is, an explicit description of the coefficients \( d_{x, \mu, \nu}^{y, \nu} \) in the identity of the following form:

\[
e^\nu \text{gch} V_x^- (\lambda) = \sum_{y \in W_{af}, \nu \in \mathbb{P}} d_{x, \mu, \nu}^{y, \nu} \text{gch} V_y^- (\lambda + \mu),
\]

where the coefficients \( d_{x, \mu, \nu}^{y, \nu} \in \mathbb{Z}[q, q^{-1}] \) are independent of the weight \( \lambda \in \mathbb{P} \). In types \( A, D, E_6, E_7 \), Kouno-Naito-Orr-Sagaki [KNS] (for minuscule weights \( \nu \)) and Lenart-Naito-Orr-Sagaki [LNOS] (for arbitrary weights \( \nu \)) gave an explicit description of the coefficients \( d_{x, \mu, \nu}^{y, \nu} \) in the identity above; strictly speaking, the identities obtained in these works are ones in the equivariant \( K \)-group of the semi-infinite flag manifold \( \mathbb{Q}_e \) associated to the connected, simply-connected and simple algebraic group \( G \) over \( \mathbb{C} \) whose Lie algebra is \( \mathfrak{g} \). In particular, these identities imply the following finiteness result: (i) the right-hand side of the identity (1.2) is a finite sum, and (ii) \( d_{x, \mu, \nu}^{y, \nu} \in \mathbb{Z}[q, q^{-1}] \) for all \( y \in W_{af} \) and \( \mu \in \mathbb{P} \). Note that this finiteness result was obtained in simply-laced types by Orr [O], but the argument therein does not seem to work in non-simply-laced types. Since the identity (1.2) can be thought of as an “inverse expansion” of the identity (1.1), we call it an identity of inverse Chevalley type.

In this paper, we study identities of inverse Chevalley type in type \( C_n \). We give an explicit description of the coefficients \( d_{x, \mu, \nu}^{y, \nu} \) in the case that \( \nu = v \varepsilon_k \) for \( v \in \mathbb{W} \), where \( \varepsilon_1, \ldots, \varepsilon_n \) is the standard basis of the weight lattice \( \mathbb{P} \cong \mathbb{Z}^n \); for any \( v, w \in \mathbb{W} \), there exists \( m = 1, \ldots, n \) such that \( v \varepsilon_m = w \varepsilon_m \) or \( v \varepsilon_m = -w \varepsilon_m \).

Now we are ready to give the main results of this paper; for the notation used in the following theorems, see Section 4.1. First, we state the “first half” of the desired identities of inverse Chevalley type.

**Theorem 1** (Corollary 4.2). For \( x = wt_\xi \in W_{af} \) with \( w \in \mathbb{W} \) and \( \xi \in \mathbb{Q}^l \), \( m = 1, \ldots, n \), and \( \lambda \in \mathbb{P}^+ \) such that \( \lambda + \varepsilon_k \in \mathbb{P}^+ \) for all \( k = 1, \ldots, m \), there holds the following identity:

\[
e^{\varepsilon_m} \text{gch} V_x^- (\lambda) = q^{(\varepsilon_m, \xi)} \sum_{B \in A(w, t_m(m))} (-1)^{|B|} \text{gch} V_{\text{end}(B) \downarrow \text{down}(B) + \xi}^-(\lambda + \varepsilon_m)
\]

\[
+ \sum_{j=1}^{m-1} \sum_{(j_1, \ldots, j_r) \in S_m, \ A_1 \in A_{\varepsilon, m-1}^{j_1}} \cdots \sum_{A_r \in A_{\varepsilon, 0}^{j_{r-1}}} (-1)^{|A_1| + \cdots + |A_r|} q^{(\varepsilon_j, \downarrow \text{down}(A_1) + \cdots + \downarrow \text{down}(A_r) + \xi)}
\]

\[
\times \sum_{B \in A(\text{end}(A_r), \Gamma_j(j))} (-1)^{|B|} \text{gch} V_{\text{end}(B) \downarrow \text{down}(B) + \downarrow \text{down}(A_1) + \cdots + \downarrow \text{down}(A_r) + \xi}^-(\lambda + \varepsilon_j).
\]

2
Note that $x$ and $w$ in Theorem 1 agree in the sense that $x = wt_{\xi}$, but $m$ is arbitrary. Next, we state the "second half" of the desired identities of inverse Chevalley type.

**Theorem 2** (= Corollary 4.4). For $x = wt_{\xi} \in W_{ad}$ with $w \in W$ and $\xi \in Q^\vee$, $m = 1, \ldots, n$, and $\lambda \in P^+$ such that $\lambda + \varepsilon_k \in P^+$ for all $k = 1, \ldots, n$ and $\lambda - \varepsilon_k \in P^+$ for $k = m + 1, \ldots, n$, there holds the following identity:

$$e^{-w^m} \text{gch } V_x^-(\lambda) = q^{-\langle m, \xi \rangle} \sum_{B \in A(w, \Theta_m)} (-1)^{|B|} \text{gch } V_{\text{end}(B) + \xi}^-(\lambda - \varepsilon_m)$$

$$+ \sum_{j=m+1}^n \sum_{(j_1, \ldots, j_r) \in S_m, A_1 \in A_{w|j_1}} \cdots \sum_{A_r \in A_{\text{end}(A_{r-1})}} (-1)^{|A_1| + \cdots + |A_r|} q^{-\langle \varepsilon_j, \text{down}(A_1) + \cdots + \text{down}(A_r) + \xi \rangle}$$

$$\times \sum_{B \in A(\text{end}(A_r), \Theta_j)} (-1)^{|B|} \text{gch } V_{\text{end}(B) + \text{down}(A_1) + \cdots + \text{down}(A_r) + \xi}^-(\lambda - \varepsilon_j)$$

$$+ \sum_{j=1}^n \sum_{(j_1, \ldots, j_r) \in S_m, A_1 \in A_{w|j_1}} \cdots \sum_{A_r \in A_{\text{end}(A_{r-1})}} (-1)^{|A_1| + \cdots + |A_r|} q^{-\langle \varepsilon_j, \text{down}(A_1) + \cdots + \text{down}(A_r) + \xi \rangle}$$

$$\times \sum_{B \in A(\text{end}(A_r), \Gamma_j(j))} (-1)^{|B|} \text{gch } V_{\text{end}(B) + \text{down}(A_1) + \cdots + \text{down}(A_r) + \xi}^-(\lambda + \varepsilon_j).$$

Note that the proofs of Theorems 1 and 2 begin with auxiliary identities (Propositions 5.1 and 5.2) derived directly from a special case of the Chevalley formula given by Proposition 3.5.

Also, observe that from the description of these identities, the finiteness result (i), (ii) mentioned above immediately follows, since every weight $\lambda \in P$ can be written as a $\mathbb{Z}$-linear combination of $\varepsilon_1, \ldots, \varepsilon_n$.

Furthermore, we give cancellation-free identities of inverse Chevalley type in the "first-half" case, i.e., in the case $\nu = \nu = w \varepsilon_1, \ldots, w \varepsilon_n$. The precise statement is as follows; in the following theorem, $p_{m,j}(w)$ denotes a suitable directed path in the quantum Bruhat graph (for the definitions, see Section 1.2).

**Theorem 3** (= Corollary 4.5). For $x = wt_{\xi} \in W_{ad}$ with $w \in W$ and $\xi \in Q^\vee$, $m = 1, \ldots, n$, and $\lambda \in P^+$ such that $\lambda + \varepsilon_k \in P^+$ for all $k = 1, \ldots, m$, there holds the following cancellation-free identity:

$$e^{-w^m} \text{gch } V_x^-(\lambda) = q^{-\langle m, \xi \rangle} \sum_{B \in A(w, \Gamma_m(m))} (-1)^{|B|} \text{gch } V_{\text{end}(B) + \xi}^-(\lambda + \varepsilon_m)$$

$$+ \sum_{j=1}^{m-1} q^{-\langle \varepsilon_j, \text{wt}(p_{m,j}(w)) + \xi \rangle} \sum_{A \in A(\text{end}(p_{m,j}(w)), \Gamma_j(j))} (-1)^{|B|} \text{gch } V_{\text{end}(B) + \text{wt}(p_{m,j}(w)) + \xi}^-(\lambda + \varepsilon_j).$$

As for the "second-half" case, i.e., the case $\nu = -w \varepsilon_1 = -w \varepsilon_1, \ldots, -w \varepsilon_n$, we provide conjectural cancellation-free identities of inverse Chevalley type in Section 1.3

As an application of our identities of inverse Chevalley type, we can prove an equivariant scalar multiplication (i.e., multiplication with the one-dimensional character $e^\nu$, $\nu \in P$, of $H$) in the $(H \times \mathbb{C}^*)$-equivariant $K$-group $K_{H \times \mathbb{C}^*}(Q_G)$ of the semi-infinite flag manifold $Q_G$ associated to $G$. To be more precise, let $Q_G^{\text{rat}}$ denote the semi-infinite flag manifold associated to $G$, that is, the reduced ind-scheme of infinite type whose set of $\mathbb{C}$-valued points is $G(\mathbb{C}(z))/H(\mathbb{C}) \cdot N(\mathbb{C}(z)))$, where $H \subset G$ is a maximal torus with Lie algebra $\mathfrak{h}$ and $N$ is the unipotent radical of a Borel subgroup $B \supset H$. For $\lambda \in P$, there exists a line bundle on $Q_G^{\text{rat}}$ associated to $\lambda$; we denote by $\mathcal{O}(\lambda)$ the sheaf corresponding to this line bundle. Also, there exist semi-infinite Schubert varieties $Q_G(x)$...
for \( x \in W_{af} \), which are subvarieties of \( Q^\text{rat}_G \); note that \( Q_G = Q_G(e) \), with \( e \in W_{af} \) the identity element. The equivariant \( K \)-group \( K_{H \times \mathbb{C}^*}(Q_G) \) is defined to be the \( \mathbb{Z}[q, q^{-1}][P] \)-submodule of (the Laurent series, in \( q^{-1} \), extension of) the Iwahori-equivariant \( K \)-group \( K'_{H \times \mathbb{C}^*}(Q_G) \), introduced in \cite{KNS}, consisting of all “convergent” infinite linear combinations with coefficients in \( \mathbb{Z}[q, q^{-1}][P] \) of the semi-infinite Schubert classes \( [Q_G(x)] \), \( x \in W_{af}^\geq := \{ wt_\xi \in W_{af} \mid w \in W, \xi \in Q^{+,\geq} \} \), where “convergence” holds in the sense of \cite{KNS} Proposition 5.11; here, \( Q^{+,\geq} := \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i^+ \) denotes the positive part of the coroot lattice \( Q^+ \).

Now, following \cite{NOS} Sect. 9], we recall how the graded characters of level-zero Demazure submodules over the quantum affine algebra \( U_q(q_{af}) \) are related to the equivariant \( K \)-group \( K_{H \times \mathbb{C}^*}(Q_G) \) of the semi-infinite flag manifold \( Q_G \). Let us define \( \mathbb{C}[q, q^{-1}][P] \)-modules \( \text{Fun}_P(\mathbb{C}(q^{-1})[[P]]) \), \( \text{Fun}_P^{\text{neg}}(\mathbb{C}(q^{-1})[[P]]) \), and \( \text{Fun}_P^{\text{pos}}(\mathbb{C}(q^{-1})[[P]]) \) by:

\[
\text{Fun}_P(\mathbb{C}(q^{-1})[[P]]) := \{ f : P \to \mathbb{C}(q^{-1})[[P]] \} ,
\]

\[
\text{Fun}_P^{\text{neg}}(\mathbb{C}(q^{-1})[[P]]) := \left\{ f \in \text{Fun}_P(\mathbb{C}(q^{-1})[[P]]) \mid f(\mu) = 0 \text{ for all } \mu \in \gamma + P^+ \right\} ,
\]

\[
\text{Fun}_P^{\text{pos}}(\mathbb{C}(q^{-1})[[P]]) := \text{Fun}_P(\mathbb{C}(q^{-1})[[P]]) / \text{Fun}_P^{\text{neg}}(\mathbb{C}(q^{-1})[[P]]) .
\]

Then there exists an injective \( \mathbb{Z}[q, q^{-1}][P] \)-module homomorphism \( \Phi : K_{H \times \mathbb{C}^*}(Q_G) \to \text{Fun}_P^{\text{pos}}(\mathbb{C}(q^{-1})[[P]]) \) such that for the class \( [\mathcal{E}] \in K_{H \times \mathbb{C}^*}(Q_G) \) of a certain quasi-coherent sheaf \( \mathcal{E} \) on \( Q_G \), the element \( \Phi([\mathcal{E}]) \in \text{Fun}_P^{\text{pos}}(\mathbb{C}(q^{-1})[[P]]) \) is given as:

\[
P \to \mathbb{C}(q^{-1})[[P]], \lambda \mapsto \sum_{j=0}^{\infty} (-1)^j \text{gch } H^j(\mathcal{E} \otimes Q_G, \mathcal{O}_Q, \mathcal{O}(\lambda));
\]

here, \( \text{gch } H^j(\mathcal{E} \otimes Q_G, \mathcal{O}_Q, \mathcal{O}(\lambda)) \) for \( i \geq 0 \) is the graded character of the \( i \)-th cohomology group \( H^i(\mathcal{E} \otimes Q_G, \mathcal{O}_Q, \mathcal{O}(\lambda)) \), which is regarded as an \( (H \times \mathbb{C}^*) \)-module. Also, it is proved in \cite{KNS} that we can take \( \mathcal{E} = \mathcal{O}_{Q_G(x)} \) for \( x \in W_{af}^\geq \), and that

\[
\text{gch } H^i(\mathcal{E} \otimes Q_G(x) \otimes Q_G, \mathcal{O}(\lambda)) = \begin{cases} 
\text{gch } V_{\lambda}^{-}(-w_0 \lambda) & \text{if } \lambda \in P^+ \text{ and } i = 0, \\
0 & \text{otherwise}; 
\end{cases}
\]

where \( w_0 \) denotes the longest element of \( W \). By making use of these results, we can translate an identity for graded characters of level-zero Demazure submodules into one in the \( (H \times \mathbb{C}^*) \)-equivariant \( K \)-group \( K_{H \times \mathbb{C}^*}(Q_G) \). Namely, if we have a finite sum of the form \( \sum_{\gamma \in W_{af}^\geq} d_{\gamma, \mu}^{P, i} (\mathcal{O}_{Q_G(x)} \otimes \mathcal{O}_Q, \mathcal{O}(-w_0 \mu)) \]

In particular, our identities for graded characters of inverse Chevalley type yield explicit identities in the \( (H \times \mathbb{C}^*) \)-equivariant \( K \)-group \( K_{H \times \mathbb{C}^*}(Q_G) \), which we call inverse Chevalley formulas.

In addition, by the specialization at \( q = 1 \) (of the coefficients \( d_{\gamma, \mu}^{P, i} \)), we obtain corresponding inverse Chevalley formulas for equivariant scalar multiplication in the \( H \)-equivariant \( K \)-group \( K_H(Q_G) \) of the semi-infinite flag manifold \( Q_G \). Here we mention that in \cite{Kati}, Kato established a \( \mathbb{Z}[P] \)-module isomorphism from \( K_H(Q_G) \) onto the formal completion \( \widehat{\mathbb{Z}[H(G/B) \otimes \mathbb{Z}[P] Q^{-+,\geq}]} \mathbb{Z}[P][Q^{+,\geq}] \) of the (small) \( H \)-equivariant quantum K-theory \( \mathbb{K}_H(G/B) = K_H(G/B) \otimes \mathbb{Z}[P][Q^{+,\geq}] \) of the finite-dimensional flag manifold \( G/B \) which sends each semi-infinite Schubert class to the corresponding (opposite) Schubert class, where \( \mathbb{Z}[P][Q^{-+,\geq}] \) (resp., \( \mathbb{Z}[P][Q^{+,\geq}] \)) denotes the ring of polynomials (resp., formal power series) with coefficients in \( \mathbb{Z}[P] \) in the Novikov variables \( Q_i = Q_i^{-+}, i \in I \). Through this \( \mathbb{Z}[P] \)-module isomorphism, we obtain inverse Chevalley formulas for equivariant scalar multiplication in \( \mathbb{K}_H(G/B) \).

This paper is organized as follows. In Section 2 we fix our basic notation, and recall the definitions of the quantum Bruhat graph and quantum alcove model. In Section 3 we briefly review
level-zero Demazure submodules and identities of Chevalley type for their graded characters. In Section 4 we state identities of inverse Chevalley type. Also, we give the cancellation-free form of the first half of these identities. In Section 5 we prove our identities of inverse Chevalley type. In Section 6 we derive the cancellation-free form of our identities of inverse Chevalley type in the first-half case.

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2 Basic setting

In this section, we fix basic notation, and review the definitions of the quantum Bruhat graph and quantum alcove model.

2.1 Lie algebras and root systems Let \( \mathfrak{g} \) be a simple Lie algebra over \( \mathbb{C} \) with Cartan subalgebra \( \mathfrak{h} \). Let \( \Delta \subset \mathfrak{h}^* := \text{Hom}_\mathbb{C}(\mathfrak{h}, \mathbb{C}) \) be the root system of \( \mathfrak{g} \), \( \Delta^+ \subset \Delta \) the set of positive roots, and \( \{ \alpha_i \}_{i \in I} \subset \Delta^+ \) the simple roots. We denote by \( \langle \cdot, \cdot \rangle \) the canonical pairing \( \mathfrak{h}^* \times \mathfrak{h} \to \mathbb{C} \). For \( \alpha \in \Delta \), we define \( \text{sgn}(\alpha) \in \{ 1, -1 \} \) as

\[
\text{sgn}(\alpha) := \begin{cases} 
1 & \text{if } \alpha \in \Delta^+, \\
-1 & \text{if } \alpha \in -\Delta^+, 
\end{cases}
\]

and set \( |\alpha| := \text{sgn}(\alpha)\alpha \in \Delta^+ \).

For \( \alpha \in \Delta \), we denote by \( \alpha^\vee \in \mathfrak{h} \) the coroot corresponding to \( \alpha \), and define the fundamental weights \( \varpi_i, \ i \in I \), by \( \langle \varpi_i, \alpha_j^\vee \rangle = \delta_{ij} \) for \( i, j \in I \). Let \( P := \sum_{i \in I} \mathbb{Z}\varpi_i \) be the weight lattice, \( Q := \sum_{i \in I} \mathbb{Z}\alpha_i \) the root lattice, and \( Q^\vee := \sum_{i \in I} \mathbb{Z}\alpha_i^\vee \) the coroot lattice. Elements of \( P^+ := \sum_{i \in I} \mathbb{Z}_{\geq 0}\varpi_i (\subset P) \) are called dominant weights. We denote by \( \mathbb{Z}[P] := \sum_{\lambda \in P} \mathbb{Z}e^\lambda \) the group algebra of \( P \), where \( \{ e^\lambda | \lambda \in P \} \) is a formal basis with relations \( e^{\lambda_1}e^{\lambda_2} = e^{\lambda_1+\lambda_2} \). Note that if \( G \) is the connected, simply-connected and simple algebraic group over \( \mathbb{C} \) whose Lie algebra is \( \mathfrak{g} \), then the element \( e^\lambda \) for \( \lambda \in P \) also denotes the one-dimensional representation (character) of the maximal torus \( H \) of \( G \) of weight \( \lambda \). In particular, \( \mathbb{Z}[P] \) is isomorphic to the representation ring \( R(H) \) of the torus \( H \).

For \( \alpha \in \Delta \), we define the reflection \( s_\alpha \in GL(\mathfrak{h}^*) \) by \( s_\alpha(\lambda) := \lambda - \langle \lambda, \alpha^\vee \rangle \alpha \), \( \lambda \in \mathfrak{h}^* \). In particular, the reflection \( s_i := s_{\alpha_i} \) for \( i \in I \) is called a simple reflection. The Weyl group \( W \) is defined as the subgroup of \( GL(\mathfrak{h}^*) \) generated by \( \{ s_i \}_{i \in I} \), i.e., \( W = \langle s_i | i \in I \rangle \subset GL(\mathfrak{h}^*) \).

2.2 Type C root system We review the standard realization of the root system of type \( C \).

Let \( \{ \varepsilon_1, \ldots, \varepsilon_n \} \) be the standard basis of \( \mathbb{R}^n \). Then, the set

\[
\Delta = \{ \pm(\varepsilon_i - \varepsilon_j) | 1 \leq i < j \leq n \} \cup \{ \varepsilon_i + \varepsilon_j | 1 \leq i < j \leq n \} \cup \{ \pm 2\varepsilon_k | 1 \leq k \leq n \}
\]

forms the root system of type \( C_n \), and the set

\[
\Delta^+ = \{ \varepsilon_i - \varepsilon_j | 1 \leq i < j \leq n \} \cup \{ \varepsilon_i + \varepsilon_j | 1 \leq i < j \leq n \} \cup \{ 2\varepsilon_k | 1 \leq k \leq n \}
\]

is the set of all positive roots. In particular, \( \alpha_i, i = 1, \ldots, n \), defined by

\[
\alpha_i := \varepsilon_i - \varepsilon_{i+1} \quad (1 \leq i \leq n - 1), \quad \alpha_n := 2\varepsilon_n
\]

are the simple roots.

For \( 1 \leq i < j \leq n \), we set

\[
(i, j) := \varepsilon_i - \varepsilon_j, \quad (i, \overline{j}) := \varepsilon_i + \varepsilon_j, \quad (i, \overline{i}) := 2\varepsilon_i.
\]

The Weyl group \( W \) of type \( C_n \) is realized as a subgroup of the permutation group of the set \( \{ \overline{i} | i = 1, 2, \ldots, n \} \) by identifying simple reflections \( s_1, \ldots, s_{n-1}, s_n \) with transpositions \( (1 \ 2), \ldots, (n - 1 \ n), (n \ \overline{n}) \), respectively.
2.3 The quantum Bruhat graph  The quantum Bruhat graph is a labeled directed graph on the Weyl group $W$, introduced by Brenti-Fomin-Postnikov [BFP].

**Definition 2.1** ([BFP Definition 6.1]). The quantum Bruhat graph $\text{QBG}(W)$ is the $\Delta^+$-labeled directed graph whose vertex set is $W$, and whose edges are given as follows. For $x, y \in W$ and $\alpha \in \Delta^+$, we have a directed edge $x \overset{\alpha}{\to} y$ if $y = x_{s_\alpha}$, and either of the following holds: (B) $\ell(y) = \ell(x) + 1$, or (Q) $\ell(y) = \ell(x) - 2(\rho, \alpha^\vee) + 1$, where $\rho := (1/2)\sum_{\alpha \in \Delta^+} \alpha$. If the condition (B) (resp., (Q)) holds, then the corresponding edge $x \overset{\alpha}{\to} y$ is called a Bruhat edge (resp., quantum edge).

For a directed path $p : w_0 \overset{\gamma_1}{\to} w_1 \overset{\gamma_2}{\to} \cdots \overset{\gamma_r}{\to} w_r$ in $\text{QBG}(W)$, we define $\text{wt}(p) \in Q^\vee$ by

$$\text{wt}(p) := \sum_{1 \leq k \leq r} \gamma_k.$$

if $w_{k-1} \overset{\gamma_k}{\to} w_k$ is a quantum edge.

2.4 The quantum alcove model  We briefly review the theory of quantum alcove model, first introduced by Lenart-Lubovsky [LL], and then generalized by [LNS].

We set $h_R := P \otimes \mathbb{R}$. For $\alpha \in \Delta$ and $k \in \mathbb{Z}$, we define a hyperplane $H_{\alpha,k}$ in $h_R$ by

$$H_{\alpha,k} := \{ \xi \in h_R \mid \langle \xi, \alpha^\vee \rangle = k \};$$

we denote by $s_{\alpha,k}$ the reflection for the hyperplane $H_{\alpha,k}$. Connected components of the space $h_R \setminus \bigcup_{\alpha \in \Delta, k \in \mathbb{Z}} H_{\alpha,k}$ are called alcoves. Two alcoves $A, B$ are called adjacent if the closures of $A$ and $B$ have an intersection, called a common wall.

**Definition 2.2** ([LP Definition 5.2]). A sequence $(A_0, A_1, \ldots, A_r)$ of alcoves $A_0, \ldots, A_r$ is called an alcove path if $A_{i-1}$ and $A_i$ are adjacent for each $i = 1, \ldots, r$. An alcove path $\Gamma = (A_0, \ldots, A_r)$ is called reduced if $\Gamma$ has a minimal length $r$ among all alcove paths from $A_0$ to $A_r$.

For adjacent alcoves $A, B$, and a root $\alpha \in \Delta$, we write $A \overset{\alpha}{\to} B$ if the common wall of $A$ and $B$ is contained in the hyperplane $H_{\alpha,k}$ for some $k \in \mathbb{Z}$, and $\alpha$ points in a direction from $A$ to $B$ (as a direction vector). We take a special alcove $A_0$, called the fundamental alcove, defined by

$$A_0 := \{ \xi \in h_R^* \mid 0 < \langle \xi, \alpha^\vee \rangle < 1 \text{ for all } \alpha \in \Delta^+ \}. $$

For $\lambda \in P$, we define $A_\lambda$ by

$$A_\lambda := A_0 + \lambda = \{ \xi + \lambda \mid \xi \in A_0 \}. $$

**Definition 2.3** ([LP Definition 5.4]). Let $\lambda \in P$. A sequence $\Gamma = (\gamma_1, \ldots, \gamma_r)$ of roots $\gamma_1, \ldots, \gamma_r \in \Delta$ is called a $\lambda$-chain if there exists an alcove path $(A_0 = A_0, A_1, \ldots, A_r = A_{-\lambda})$ such that

$$A_0 = A_0 \overset{-\gamma_1}{\to} A_1 \overset{-\gamma_2}{\to} \cdots \overset{-\gamma_r}{\to} A_r = A_{-\lambda}. $$

We say that $\Gamma$ is reduced if the corresponding alcove path $(A_0, \ldots, A_r)$ is reduced.

Let $\Gamma$ be a sequence of roots, i.e., $\Gamma = (\gamma_1, \ldots, \gamma_r)$, with $\gamma_k \in \Delta$, $k = 1, \ldots, r$.

**Definition 2.4** ([LNS Definition 17]). Let $w \in W$. A subset $A = \{i_1 < \cdots < i_s\} \subset \{1, \ldots, r\}$ is called $w$-admissible if

$$w = w_0 \overset{|\gamma_{i_1}|}{\to} w_1 \overset{|\gamma_{i_2}|}{\to} \cdots \overset{|\gamma_{i_s}|}{\to} w_s$$

is a directed path in $\text{QBG}(W)$. In this case, we define $\text{end}(A)$ by $\text{end}(A) := w_s$. Also, we set

$$A^- := \{ k \in A \mid \text{the edge } w_{k-1} \overset{|\gamma_k|}{\to} w_k \text{ is a quantum edge} \}.$$
and then define $\text{down}(A)$ by

$$\text{down}(A) := \sum_{k \in A^+} |\gamma_k|^\vee.$$ 

Also, we set

$$n(A) := \#\{k \in A \mid \gamma_k \in -\Delta^+\}.$$ 

We denote by $\mathcal{A}(w, \Gamma)$ the set of all admissible subsets.

Let $\Gamma_1, \ldots, \Gamma_r$ be sequences of roots, and $w \in W$. For a tuple $(A_1, A_2, \ldots, A_r)$ of admissible subsets $A_1 \in \mathcal{A}(w, \Gamma_1), A_2 \in \mathcal{A}(\text{end}(A_1), \Gamma_2), \ldots, A_r \in \mathcal{A}(\text{end}(A_{r-1}), \Gamma_r)$, we set

$$\text{down}(A_1, A_2, \ldots, A_r) := \text{down}(A_1) + \text{down}(A_2) + \cdots + \text{down}(A_r).$$

If $\Gamma$ is a $\lambda$-chain for some $\lambda \in P$, then we can consider additional statistics denoted by $\text{wt}$, $\text{height}$. For a $\lambda$-chain $\Gamma = (\gamma_1, \ldots, \gamma_r)$ with $\lambda \in P$, let $(A_0 = A_0, \ldots, A_r = A_{-\lambda})$ be the alcove path corresponding to $\Gamma$, and take integers $l_k \in \mathbb{Z}, k = 1, \ldots, r$, such that the common wall of adjacent alcoves $A_{k-1}$ and $A_k$ is contained in the hyperplane $H_{\gamma_k - l_k}$. Then, we define $\text{wt}(A)$ and $\text{height}(A)$ for $A = \{i_1 < \cdots < i_s\}$ by

$$\text{wt}(A) := -w s_{\gamma_{i_1} - l_{i_1}} \cdots s_{\gamma_{i_s} - l_{i_s}} (-\lambda), \quad \text{height}(A) := \sum_{k \in A^+} \text{sgn}(\gamma_k)(\langle \lambda, \gamma_k^\vee \rangle - l_k).$$

### 2.5 Specific chains of roots

In this subsection, we deal with the root system of type $C_n$. We choose specific $(-\varpi_{k-1} + \varpi_k)$-chain and $(\varpi_{k-1} - \varpi_k)$-chain, which will play a crucial role in this paper; we understand that $\varpi_0 = 0$ in this paper. Note that $-\varpi_{k-1} + \varpi_k = \varepsilon_k$. We set

$$\Gamma_k(k) := (-(1, \overline{k}), \ldots, -(k - 1, \overline{k}), -(k, \overline{k} + 1), \ldots, -(k, \overline{n}), -(k, n), \ldots, (k, k + 1)), \quad \Theta_k(k) := (-(1, k), \ldots, -(k - 1, k)), \quad \Theta_k^\ast := ((k - 1, k), \ldots, (1, k)).$$

For sequences $\Gamma = (\gamma_1, \ldots, \gamma_r), \Xi = (\xi_1, \ldots, \xi_s)$ of roots, we denote by $\Gamma \ast \Xi$ the concatenation of $\Gamma$ and $\Xi$, i.e., $\Gamma \ast \Xi := (\gamma_1, \ldots, \gamma_r, \xi_1, \ldots, \xi_s)$.

**Lemma 2.5.** The concatenation $\Gamma_{k-1,k} := \Gamma_k^\ast(k) \ast \Theta_k$ is a reduced $(-\varpi_{k-1} + \varpi_k)$-chain.

**Proof.** We set $x := s_k s_{k+1} \cdots s_n s_{n-1} \cdots s_1, \ y := s_1 \cdots s_{k-1}$, and $\mu := \varpi_1$. Then, $x$ is a minimal-length representative for the coset $xW_\mu$, where $W_\mu := \{w \in W \mid w\mu = \mu\}$, $yx$ is the minimal-length representative for the coset $\{w \in W \mid w\mu = w_0\mu\}$, and $x\mu = \varpi_{k-1} - \varpi_k$. Now, following [LNOS] Lemma 4.1], we define $\Gamma$ as follows. Let us write $x = s_{j_1} \cdots s_{j_l}, \ y = s_{i_1} \cdots s_{i_b}$, and set

$$\beta_c := s_{j_a} \cdots s_{j_{c+1}} \alpha_{j_c}, \quad 1 \leq c \leq a,$$

$$\zeta_d := s_{i_{i'}} \cdots s_{i_{d+1}} \alpha_{i_{i'}}, \quad 1 \leq d \leq b.$$ 

Then we define $\Gamma$ as $\Gamma := (\beta_1, \ldots, \beta_a, -\gamma_1, \ldots, -\gamma_b)$; note that the convention for the sign of roots in alcove paths in this paper is different from that of [LNOS]. By direct calculation, we see that this $\Gamma$ is identical to $\Gamma_{k-1,k}$. Since $\mu$ is a minuscule fundamental weight, the argument in the proof
of [LNOS] Lemma 4.1] still works in our setting of the type $C$ root system, and hence we obtain the following reduced alcove path $\Pi$ from $A_0$ to $A_0 + x\mu = A_0 - (-\varpi_{k-1} + \varpi_k) = A_{-(-\varpi_{k-1} + \varpi_k)}$:

$$\Pi : A_0 = A_0 \xrightarrow{-\beta_1} A_1 \xrightarrow{-\beta_2} \cdots \xrightarrow{-\beta_b} A_a = B_0$$

Thus we have shown that $\Gamma_{k-1,k}$ is a reduced $(-\varpi_{k-1} + \varpi_k)$-chain corresponding to $\Pi$. This proves the lemma.

**Remark 2.6.** The proof of [LNOS] Lemma 4.1] also shows that for $t = 1, \ldots, a$, the common wall of the adjacent alcoves $A_{t-1}$ and $A_t$ in the above path $\Pi$ is contained in the hyperplane $H_{\beta_t,0}$, while for $t = 1, \ldots, b$, the common wall of the adjacent alcoves $B_{t-1}$ and $B_t$ is contained in the hyperplane $H_{\gamma_t,1}$.

By reversing the order of roots in $\Gamma$ and negating all roots, we obtain a specific $(-\varpi_{k-1} - \varpi_k)$-chain.

**Corollary 2.7.** The concatenation $\Gamma_{k-1,k}^\ast := \Theta_k^\ast * \Gamma_k(k)$ is a reduced $(-\varpi_{k-1} - \varpi_k)$-chain.

### 3 Level-zero Demazure submodules over quantum affine algebras

We recall the definition of level-zero Demazure submodules over quantum affine algebras and their graded characters.

#### 3.1 Notation for affine Lie algebras and quantum affine algebras

Let $\mathfrak{g}_{af} := (\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]) \oplus \mathbb{C}c \oplus \mathbb{C}d$ be the (untwisted) affine Lie algebra associated to $\mathfrak{g}$, where $c$ is the canonical central element and $d$ is the scaling element. We denote by $\mathfrak{h}_{af}$ its Cartan subalgebra. Let $\langle \cdot, \cdot \rangle$ be the canonical pairing $\mathfrak{h}_{af}^* \times \mathfrak{h}_{af} \to \mathbb{C}$, where $\mathfrak{h}_{af}^* = \text{Hom}_\mathbb{C}(\mathfrak{h}_{af}, \mathbb{C})$. We set $I_{af} := I \cup \{0\}$. Then, the simple roots $\alpha_i$, $i \in I \subseteq I_{af}$, of $\mathfrak{g}$ can be regarded as simple roots of $\mathfrak{g}_{af}$. Let $s_i, i \in I_{af}$, be the simple reflections corresponding to $\alpha_i$. Let $W_{af} := \langle s_i \mid i \in I_{af} \rangle$ denote the (affine) Weyl group of $\mathfrak{g}_{af}$. We know that $W_{af} = \{ wt_\xi \mid w \in W, \xi \in Q^\vee \} \cong W \times Q^\vee$, where $t_\xi, \xi \in Q^\vee$, is the translation element ($\mathfrak{Kac}$).

Let $U_q(\mathfrak{g}_{af})$ be the quantum affine algebra associated to $\mathfrak{g}_{af}$, and denote by $E_i, F_i, i \in I_{af} = I \cup \{0\}$, the Chevalley generators of $U_q(\mathfrak{g}_{af})$. Then, we define $U_q^-(\mathfrak{g}_{af})$ as the subalgebra of $U_q(\mathfrak{g}_{af})$ generated by $\{ F_i \mid i \in I_{af} \}$, i.e., $U_q^-(\mathfrak{g}_{af}) = \langle F_i \mid i \in I_{af} \rangle$.

#### 3.2 Extremal weight submodules and level-zero Demazure submodules

**Definition 3.1** ([Kac] Definition 8.1.1]). Let $M$ be an integrable $U_q(\mathfrak{g}_{af})$-module, and $\lambda \in P_{af}$. An element $v \in M$ is called an **extremal weight vector of weight** $\lambda$ if $v$ is a weight vector of $\lambda$, and there exists a family $\{v_x \mid x \in W_{af}\} \subset M$ of vectors such that

1. $v_e = v$,
2. for $i \in I_{af}$ and $x \in W_{af}$, if $\langle x, \alpha_i^\vee \rangle \geq 0$, then $E_i v_x = 0$ and $F_i^{(\langle x, \alpha_i^\vee \rangle)} v_x = v_{s_i x}$, and
3. for $i \in I_{af}$ and $x \in W_{af}$, if $\langle x, \alpha_i^\vee \rangle \leq 0$, then $F_i v_x = 0$ and $E_i^{(-\langle x, \alpha_i^\vee \rangle)} v_x = v_{s_i x}$,

where $F_i^{(k)}$ and $E_i^{(k)}$, $k \geq 0$, denote the divided powers.

For $\lambda \in P_{af}$, the **extremal weight module** of weight $\lambda$, denoted by $V(\lambda)$, is the integrable weight module over $U_q(\mathfrak{g}_{af})$ whose generator is a single element $v_\lambda$, and whose defining relation is that $v_\lambda$ is an extremal weight vector of weight $\lambda$; for the precise definition of extremal weight modules, see [Kas] Proposition 8.2.2.

Let $\lambda \in P^+ \subset P_{af}$, and $x \in W_{af}$. By the definition of extremal weight vectors, there exists a family $\{v_x \mid x \in W_{af}\} \subset V(\lambda)$ of vectors satisfying the conditions in Definition 3.1 with $v = v_\lambda$. 

8
The level-zero Demazure submodule $V_x^-(\lambda)$ is a $U_q(\mathfrak{g}_{af})$-submodule of $V(\lambda)$ generated by $v_x$, i.e., $V_x^-(\lambda) = U_q(\mathfrak{g}_{af})v_x$. For $\nu \in P_{af}$, we denote by $V^-_{\nu}(\lambda)_\nu$ the weight space of $V^-_{\nu}(\lambda)$ of weight $\nu \in P_{af}$. Then we have the following weight space decomposition with respect to $\mathfrak{h}_{af}$:

$$V_x^-(\lambda) = \bigoplus_{\gamma \in Q, \ k \in \mathbb{Z}} V_x^-(\lambda)_{\lambda + \gamma + k\delta},$$

where each weight space $V_x^-(\lambda)_{\lambda + \gamma + k\delta}$, $\gamma \in Q$, $k \in \mathbb{Z}$, is a finite-dimensional $\mathbb{C}(q)$-vector space; here, $\delta$ denotes the (primitive) null root of $\mathfrak{h}_{af}$. Now we define the graded character of $V_x^-(\lambda)$ by

$${\text{gch}} V_x^-(\lambda) := \sum_{\gamma \in Q, \ k \in \mathbb{Z}} \dim(V_x^-(\lambda)_{\lambda + \gamma + k\delta}) q^k e^{\lambda + \gamma} \in \mathbb{Z}[P][[q^{-1}]],$$

where $q$ is an indeterminate (not to be confused with $q$).

The following identity is useful to compute graded characters of level-zero Demazure submodules.

**Proposition 3.2** ([LNS Proposition D.1]). Let $x \in W_{af}$ and $\lambda \in P^+$. For $\xi \in Q^\vee$, we have

$${\text{gch}} V_{x\xi}^-(\lambda) = q^{-(\lambda,\xi)} {\text{gch}} V_x^-(\lambda).$$

### 3.3 Identities of Chevalley type

Let $\lambda \in P^+$ and $x \in W_{af}$. We consider the graded character $\text{gch} V_x^-(\lambda + \mu)$, where $\mu \in P$ is such that $\lambda + \mu \in P^+$. For this, we need to introduce more notation. A partition is a weakly decreasing sequence $\chi = (\chi_1 \geq \cdots \geq \chi_l)$ of positive integers $\chi_1, \ldots, \chi_l \in \mathbb{Z}_{\geq 0}$; we call $l$ the length of $\chi$. Also, we set $|\chi| := \chi_1 + \cdots + \chi_l$, the size of $\chi$. If $\chi = \emptyset$, the empty partition, then we set $\ell(\chi) := 0$, and $|\chi| := 0$. Let $\mu \in P$, and write $\mu = \sum_{i \in I} m_i \omega_i$. We define a set $\text{Par}(\mu)$ as follows:

$$\text{Par}(\mu) := \{\chi = (\chi(i))_{i \in I} \mid |\chi(i)|, \ i \in I, \text{ are partitions such that } \ell(\chi(i)) \leq \max\{m_i, 0\}\}.$$

For $\chi = (\chi(i))_{i \in I} \in \text{Par}(\mu)$, we write $\chi(i) = (\chi^{(i)}_1 \geq \cdots \geq \chi^{(i)}_{l_i})$, where $\chi^{(i)}_1, \ldots, \chi^{(i)}_{l_i} \in \mathbb{Z}_{\geq 0}$, with $l_i = \ell(\chi^{(i)})$. We set

$$|\chi| := \sum_{i \in I} |\chi^{(i)}|, \ \ i(\chi) := \sum_{i \in I} \chi^{(i)}_1 \alpha_i^\vee \in Q^\vee, +,$$

where, if $\chi^{(i)} = \emptyset$, then we set $\chi^{(i)} := 0$.

Lenart-Naito-Sagaki ([LNS] and Kouno-Lenart-Naito [KLN] proved the following identity, called the identity of Chevalley type.

**Theorem 3.3** ([LNS Theorem 33] and [KLN Theorem 5.16]). Let $\lambda \in P^+$, $\mu \in P$, and $x = w_{af} \in W_{af}$ with $w \in W$ and $\xi \in Q^\vee$. Assume that $\lambda + \mu \in P^+$. Take a reduced $\mu$-chain $\Gamma$. Then, there holds the following identity:

$$\text{gch} V_x^-(\lambda + \mu) = \sum_{A \in A(w, \Gamma)} \sum_{\chi \in \text{Par}(\mu)} (-1)^{n(A)} q^{-\text{height}(A) - (\lambda, \xi) - |\chi|} e^{\text{wt}(A)} \text{gch} V_{\text{end}(A)_{\xi + \text{down}(A) + i(\chi)}}^-(\lambda). \quad (3.1)$$

**Remark 3.4.** Strictly speaking, Lenart-Naito-Sagaki proved an identity, called the Chevalley formula, in the equivariant $K$-group of semi-infinite flag manifolds, which is equivalent to (3.1).

Now, we consider the root system of type $C_n$, and apply the identity of Chevalley type above to the case that $\mu = -\omega_{k-1} + \omega_k = \varepsilon_k$, $k = 1, \ldots, n$, to obtain the following.

**Proposition 3.5.** Let $2 \leq k \leq n$ and $\mu := \varepsilon_k = -\omega_{k-1} + \omega_k$. Take an arbitrary reduced $\mu$-chain $\Gamma$. Let $w \in W$. For $\lambda \in P^+$ such that $\lambda + \mu \in P^+$, we have

$$\text{gch} V_w^-(\lambda + \mu) = \frac{1}{1 - q^{-(\lambda + \omega_k, \alpha_k^\vee)}} \sum_{A \in A(w, \Gamma)} (-1)^{n(A)} q^{-\text{height}(A)} e^{\text{wt}(A)} \text{gch} V_{\text{end}(A)_{\xi + \text{down}(A)}}^-(\lambda).$$

9
Proof. Since \( \mu = -w_{k-1} + w_k \), we have
\[
\overline{\text{Par}}(\mu) = \{ i_k := (\emptyset, \ldots, \emptyset, (i), \emptyset, \ldots, \emptyset) \mid i \geq 0 \},
\]
where \( \emptyset \) denotes the empty partition (of length 0), which is also regarded as \( (0) \). For \( i \geq 0 \), we have \( |i_k| = i \), and \( i(i_k) = i\alpha_k^e \). Therefore, by Theorem \[ \[3.3 \] \], we compute:
\[
gch V_n^-(\lambda + \mu) = \sum_{A \in A(w,\Gamma)} \sum_{\chi \in \text{Par}(\mu)} (-1)^n(A) q^{-\text{height}(A) - |\chi|} \varepsilon^{\text{wt}(A)} \text{gch } V_{\text{end}(A) \downarrow \text{down}(A) + \chi}(\lambda)
\]
\[
= \sum_{A \in A(w,\Gamma)} \sum_{i=0}^{\infty} (-1)^n(A) q^{-\text{height}(A) - i} \varepsilon^{\text{wt}(A)} \text{gch } V_{\text{end}(A) \downarrow \text{down}(A) + i\alpha_k^e}(\lambda)
\]
\[
= \sum_{A \in A(w,\Gamma)} \sum_{i=0}^{\infty} (-1)^n(A) q^{-\text{height}(A) - i} \varepsilon^{\text{wt}(A)} q^{-\langle \lambda, i\alpha_k^e \rangle} \text{gch } V_{\text{end}(A) \downarrow \text{down}(A)}(\lambda)
\]
\[
= \sum_{i=0}^{\infty} q^{-i} \varepsilon^{-\langle \lambda, i\alpha_k^e \rangle} \sum_{A \in A(w,\Gamma)} (-1)^n(A) q^{-\text{height}(A)} \varepsilon^{\text{wt}(A)} \text{gch } V_{\text{end}(A) \downarrow \text{down}(A)}(\lambda)
\]
\[
= \sum_{i=0}^{\infty} q^{-i} \varepsilon^{\langle \lambda + w_k, \alpha_k^e \rangle} \sum_{A \in A(w,\Gamma)} (-1)^n(A) q^{-\text{height}(A)} \varepsilon^{\text{wt}(A)} \text{gch } V_{\text{end}(A) \downarrow \text{down}(A)}(\lambda)
\]
\[
= \frac{1}{1 - q^{\langle \lambda + w_k, \alpha_k^e \rangle}} \sum_{A \in A(w,\Gamma)} (-1)^n(A) q^{-\text{height}(A)} \varepsilon^{\text{wt}(A)} \text{gch } V_{\text{end}(A) \downarrow \text{down}(A)}(\lambda),
\]
as desired; for the third equality, we used Proposition \[ \[3.2 \] \]. This proves the proposition. \( \square \)

4 Main Results

In this section, we give precise statements of identities of inverse Chevalley type in type \( C_n \). First, we give identities in which some terms may cancel. Next, we describe the cancellations in the “first half” of these identities to obtain cancellation-free ones. Also, we make a conjecture for the cancellations in the “second half” of these identities. In the rest of this paper, we assume that \( g \) is of type \( C_n \).

4.1 Identities of inverse Chevalley type

To give precise statements of our main results, we prepare additional notation. Let us define a total order \( < \) on the set \( [\overline{m}] \) by \( 1 < 2 < \cdots < n < \overline{\pi} < n - 1 < \cdots < 1 \). For \( j, m \in [\overline{m}] \) with \( j < m \), we define \( S_{m,j} \) to be the set of all strictly decreasing sequences of integers starting at \( m \) and ending at \( j \), that is,
\[
S_{m,j} := \{ (j_1, \ldots, j_r) \mid r \geq 1, j_1, \ldots, j_r \in [\overline{m}], m > j_1 > \cdots > j_r = j \}.
\]
For \( w \in W \) and \( 1 \leq l < k \leq n \), we set
\[
A_w^{k,l} := \{ A \in A(w, \Theta_k) \setminus \{ \emptyset \} \mid \text{end}(A)^{-1}w\varepsilon_k = \varepsilon_l \}.
\]
Also, for \( w \in W \), \( k \in \{1, \ldots, n\} \), and \( l < k \), we set
\[
A_w^{\overline{k},l} := \{ A \in A(w, \Gamma_k(k)) \setminus \{ \emptyset \} \mid \text{end}(A)^{-1}w(-\varepsilon_k) = \varepsilon_l \},
\]
where for simplicity, we write \( \varepsilon_{\overline{m}} = -\varepsilon_m \) for \( m \in \{1, \ldots, n\} \).

The following is the “first half” of identities of inverse Chevalley type (in which cancellations occur).

10
Theorem 4.1. For \( w \in W, \lambda \in P^+, m = 1, \ldots, n, \) and \( \lambda \in P^+ \) such that \( \lambda + \varepsilon_k \in P^+ \) for all \( k = 1, \ldots, m \), there holds the following identity:

\[
e^{u \varepsilon_m} \text{gch} V_w^-(\lambda) = \sum_{B \in A(w, \Gamma_m(m))} (-1)^{|B|} \text{gch} V_{\text{end}(B)_{\text{down}(B)}}^-(\lambda + \varepsilon_m) + \sum_{j=1}^{m-1} \sum_{(j_1, \ldots, j_r) \in S_{m,j}} \sum_{A_1 \in A_{w,m+1}^{j_1}} \cdots \sum_{A_r \in A_{w,1}^{j_r-1,j_r}} (-1)^{|A_1|+\cdots+|A_r|} \text{q}^{q^m_j} \text{gch} V_{\text{end}(A_j, \Gamma_j(j))}^-(\lambda + \varepsilon_j).
\]

We give a proof of Theorem 4.1 in Section 5. By Proposition 3.2, we obtain the following identities for an arbitrary \( x \in W_{af} \) (not only for \( x \in W \)).

Corollary 4.2. For \( x = wt_\xi \in W_{af} \) with \( w \in W \) and \( \xi \in Q^\vee \), \( m = 1, \ldots, n, \) and \( \lambda \in P^+ \) such that \( \lambda + \varepsilon_k \in P^+ \) for all \( k = 1, \ldots, m \), there holds the following identity:

\[
e^{w \varepsilon_m} \text{gch} V_x^-(\lambda) = q^{(\varepsilon_m, \xi)} \sum_{B \in A(w, \Gamma_m(m))} (-1)^{|B|} \text{gch} V_{\text{end}(B)_{\text{down}(B)+\xi}}^-(\lambda + \varepsilon_m) + \sum_{j=1}^{m-1} \sum_{(j_1, \ldots, j_r) \in S_{m,j}} \sum_{A_1 \in A_{w,m+1}^{j_1}} \cdots \sum_{A_r \in A_{w,1}^{j_r-1,j_r}} (-1)^{|A_1|+\cdots+|A_r|} \text{gch} V_{\text{end}(A_j, \Gamma_j(j))}^-(\lambda + \varepsilon_j).
\]

The following theorem is the “second half” of identities of inverse Chevalley type.

Theorem 4.3. For \( w \in W, m = 1, \ldots, n, \) and \( \lambda \in P^+ \) such that \( \lambda + \varepsilon_k \in P^+ \) for all \( k = 1, \ldots, n \) and \( \lambda - \varepsilon_k \in P^+ \) for \( k = m+1, \ldots, n \), there holds the following identity:

\[
e^{-w \varepsilon_m} \text{gch} V_w^-(\lambda) = \sum_{B \in A(w, \Theta_m)} (-1)^{|B|} \text{gch} V_{\text{end}(B)_{\text{down}(B)}}^-(\lambda - \varepsilon_m) + \sum_{j=m+1}^{n} \sum_{(j_1, \ldots, j_r) \in S_{m,j}} \sum_{A_1 \in A_{w,m+1}^{j_1}} \cdots \sum_{A_r \in A_{w,1}^{j_r-1,j_r}} (-1)^{|A_1|+\cdots+|A_r|} \text{q}^{-q^m_j} \text{gch} V_{\text{end}(A_j, \Theta_j,j)}^-(\lambda - \varepsilon_j).
\]

We give a proof of Theorem 4.3 in Section 5. Again, by Proposition 3.2, we obtain the following identities for an arbitrary \( x \in W_{af} \) (not only for \( x \in W \)).
Corollary 4.4. For $x = wt_\xi \in W_{af}$ with $w \in W$ and $\xi \in Q^v$, $m = 1, \ldots, n$, and $\lambda \in P^+$ such that $\lambda + \varepsilon_k \in P^+$ for all $k = 1, \ldots, n$ and $\lambda - \varepsilon_k \in P^+$ for $k = m + 1, \ldots, n$, there holds the following identity:

$$e^{-w_{\varepsilon_m}} \text{gch } V_w^{-}(\lambda) = q^{-\langle \varepsilon, \xi \rangle} \sum_{B \in A(w, \Theta_m)} (-1)^{|B|} \text{gch } V_{\text{end}(B)}^{\downarrow (\lambda - \varepsilon_m)} + \sum_{j=m+1}^{n} \sum_{(j_1, \ldots, j_r) \in \Sigma_{m,j} A_1 \in A_m^{\varepsilon_{j_1}}} \cdots \sum_{A_r \in A_{\text{end}(A_{r-1})}} (-1)^{|A_1| + \cdots + |A_r|} q^{-\langle \varepsilon_j, \text{down}(A_1, \ldots, A_r) + \xi \rangle} \sum_{B \in A(\text{end}(A_r), \Theta_j)} (-1)^{|B|} \text{gch } V_{\text{end}(B)}^{\downarrow (\lambda - \varepsilon_j)} + \sum_{j=1}^{m} \sum_{(j_1, \ldots, j_r) \in \Sigma_{m,j} A_1 \in A_m^{\varepsilon_{j_1}}} \cdots \sum_{A_r \in A_{\text{end}(A_{r-1})}} (-1)^{|A_1| + \cdots + |A_r|} q^{-\langle \varepsilon_j, \text{down}(A_1, \ldots, A_r) + \xi \rangle} \sum_{B \in A(\text{end}(A_r), \Gamma_j(j))} (-1)^{|B|} \text{gch } V_{\text{end}(B)}^{\downarrow (\lambda + \varepsilon_j)}.

Here we should mention that all the sums on the right-hand side of Theorems 4.1 and 4.3 together with Corollaries 4.2 and 4.4 are indeed finite sums.

4.2 Cancellation-free identities of inverse Chevalley type in the first-half case

We consider cancellations of terms in the first-half identities of inverse Chevalley type. Let $w \in W$. Take $l, m \in \{1, \ldots, n\}$ such that $l > m$. We define a directed path $p_{l,m}(w)$ in QBG(W) inductively as follows:

1. if $l - m = 1$, then $p_{l,m}(w) : w \xrightarrow{(l-1,l)} ws_{l-1};$
2. if $l - m > 1$, then assume that $p_{l',m'}(v)$ is defined for $v \in W$ and $l', m' \in \{1, \ldots, n\}$ such that $0 < l' - m' < l - m$. Take minimal $k \in \{m, \ldots, l - 1\}$ such that $w \xrightarrow{(k,l)} ws_{(k,l)}$. Then, since $k - m < l - m$, $p_{k,m}(ws_{(k,l)})$ is defined. Let $p_{l,m}(w)$ be the directed path obtained as the concatenation of the edge $w \xrightarrow{(k,l)} ws_{(k,l)}$ with the directed path $p_{k,m}(ws_{(k,l)})$.

The following theorem gives the cancellation-free identities of inverse Chevalley type in the first-half case.

Theorem 4.5. For $w \in W$, $m = 1, \ldots, n$, and $\lambda \in P^+$ such that $\lambda + \varepsilon_k \in P^+$ for all $k = 1, \ldots, m$, there holds the following cancellation-free identity:

$$e^{-w_{\varepsilon_m}} \text{gch } V_w^{-}(\lambda) = \sum_{B \in A(w, \Gamma_m(m))} (-1)^{|B|} \text{gch } V_{\text{end}(B)}^{\downarrow (\lambda + \varepsilon_m)} + \sum_{j=1}^{m-1} q^{-\langle \varepsilon_j, \text{wt}(p_{m,j}(w)) \rangle} \sum_{A(\text{end}(p_{m,j}(w), \Gamma_j(j))} (-1)^{|B|} \text{gch } V_{\text{end}(B)}^{\downarrow (\lambda + \varepsilon_j)}.$$

We give a proof of this theorem in Section 6. Again, by using Proposition 5.2, we obtain the following cancellation-free identities for an arbitrary $x \in W_{af}$ (not only for $x \in W$).
Corollary 4.6. For \( x = wt_\xi \in W_\af \) with \( w \in W \) and \( \xi \in Q^v \), \( m = 1, \ldots, n \), and \( \lambda \in P^+ \) such that \( \lambda + \varepsilon_k \in P^+ \) for all \( k = 1, \ldots, m \), there holds the following cancellation-free identity:

\[
e^{wt_m} \text{gch } V^-_x(\lambda) = q^{(\varepsilon_m,\xi)} \sum_{B \in A(w,\Gamma_m(m))} (-1)^{|B|} \text{gch } V^-_{\text{end}(B)\downarrow_{\text{down}(B) + \lambda + \varepsilon_m}}(\lambda + \varepsilon_m) + \sum_{j=1}^{m-1} q^{(\varepsilon_j,\text{wt}(p_{m,j}(w)) + \xi)} \sum_{A(\text{end}(p_{m,j}(w)),\Gamma_j(j))} (-1)^{|B|} \text{gch } V^-_{\text{end}(B)\downarrow_{\text{down}(B) + \text{wt}(p_{m,j}(w)) + \varepsilon_j}}(\lambda + \varepsilon_j).
\]

4.3 Conjectural cancellation-free identities of inverse Chevalley type in the second-half We make a conjecture for the cancellations of terms in the second-half identities of inverse Chevalley type. We define a total order \( < \) on \( [\Pi] \) by: \( 1 < 2 < \cdots < n < \Pi < n - 1 < \cdots < 1 \), and a distance function \( d(\cdot, \cdot) \) on \( [\Pi] \) as follows:

1. for \( k \in [\Pi] \), set \( d(k, k) := 0 \);
2. for \( k, l \in [\Pi] \) with \( k > l \), set \( d(k, l) := \begin{cases} k - l & \text{if } 1 \leq l < k \leq n, \\ (2n + 1 - p) - l & \text{if } l = \Pi \text{ for some } 1 \leq p \leq n \text{ and } 1 \leq l \leq n, \\ q - p & \text{if } k = \Pi \text{ and } l = \overline{q} \text{ for some } 1 \leq p \leq q \leq n; \end{cases} \)
3. for \( k, l \in [\Pi] \) with \( k < l \), set \( d(k, l) := d(l, k) \).

Also, for \( 1 \leq l \leq n \) and \( 1 \leq k \leq \overline{1} \), we define \( \gamma_{\overline{l},k} \in \Delta^+ \) by

\[
\gamma_{\overline{l},k} := \begin{cases} (k, \overline{l}) & \text{if } k = 1, \ldots, l, \\ (l, \overline{k}) & \text{if } k = l + 1, \ldots, n, \\ (l, p) & \text{if } k = \Pi \text{ for some } p = l + 1, \ldots, n. \end{cases}
\]

Now, we define a directed path \( p_{\overline{l},m}(w) \) in QBG\((W)\) for \( w \in W \), \( 1 \leq l \leq n \), and \( 1 \leq m \leq \overline{l} + 1 \) by induction on \( d(\overline{l}, \overline{m}) \) as follows; we understand that \( n + 1 := n \).

1. If \( d(\overline{l}, \overline{m}) = 1 \) (in this case, \( \gamma_{\overline{l},m} \) is a simple root), then \( p_{\overline{l},m}(w) : w \xrightarrow{\gamma_{\overline{l},m}} ws_{\gamma_{\overline{l},m}}. \)
2. If \( d(\overline{l}, \overline{m}) > 1 \), then assume that \( p_{q,m'}(v) \) is defined for \( v \in W \) and \( q, m' \in [\Pi] \), with \( q > m' \), such that \( 0 < d(q, m') < d(\overline{l}, \overline{m}) \); note that for \( 1 \leq q \leq n \), \( p_{q,m'}(v) \) is already defined in Section 4.2. Take minimal \( k \in [\Pi] \), with \( m \leq k \leq \overline{l} + 1 \), such that \( w \xrightarrow{\gamma_{\overline{l},k}} ws_{\gamma_{\overline{l},k}}. \) Then, since \( 0 < d(k, m) < d(\overline{l}, \overline{m}) \), \( p_{k,m}(ws_{\gamma_{\overline{l},k}}) \) is defined. Let \( p_{\overline{l},m}(w) \) be the directed path obtained as the concatenation of the edge \( w \xrightarrow{\gamma_{\overline{l},k}} ws_{\gamma_{\overline{l},k}} \) with the directed path \( p_{k,m}(ws_{\gamma_{\overline{l},k}}) \).

Our conjectural cancellation-free identities in the second-half case are as follows.

Conjecture 4.7. For \( w \in W \), \( m = 1, \ldots, n \), and \( \lambda \in P^+ \) such that \( \lambda + \varepsilon_k \in P^+ \) for all \( k = 1, \ldots, n \) and such that \( \lambda - \varepsilon_k \in P^+ \) for \( k = m+1, \ldots, n \), there exists some \( m \leq l \leq n \) for which the following
cancellation-free identity holds:

\[
e^{-w \varepsilon_m} \text{gch } V^-(\lambda) = \sum_{B \in A(w, \Theta_m)} (-1)^{|B|} \text{gch } V^-_{\text{end}(B) \downarrow \text{down}(B)}(\lambda - \varepsilon_m) + \sum_{k=m+1}^{n} q^{-\langle \varepsilon_k, \text{down}(p_{m,k}(w)) \rangle} \times \sum_{B \in A(\text{end}(p_{m,k}(w)), \Theta_k)} (-1)^{|B|} \text{gch } V^-_{\text{end}(B) \downarrow \text{down}(B) + \text{down}(\text{down}(p_{m,k}(w)))}(\lambda - \varepsilon_k) + \sum_{k=1}^{l} q^{\langle \varepsilon_k, \text{down}(p_{m,k}(w)) \rangle + \varepsilon} \times \sum_{B \in A(\text{end}(p_{m,k}(w)), \Gamma_k(k))} (-1)^{|B|} \text{gch } V^-_{\text{end}(B) \downarrow \text{down}(B) + \text{down}(\text{down}(p_{m,k}(w))) + \varepsilon}(\lambda + \varepsilon_k).
\]

Conjecture 4.8. For \( x = \text{wt}_\xi \in W_{af} \) with \( w \in W \) and \( \xi \in Q^V \), \( m = 1, \ldots, n \), and \( \lambda \in P^+ \) such that \( \lambda + \varepsilon_k \in P^+ \) for all \( k = 1, \ldots, n \) and such that \( \lambda - \varepsilon_k \in P^+ \) for \( k = m+1, \ldots, n \), there exists some \( m \leq l \leq n \) for which the following cancellation-free identity holds:

\[
e^{-w \varepsilon_m} \text{gch } V^x_w(\lambda) = q^{-\langle \varepsilon_m, \xi \rangle} \sum_{B \in A(w, \Theta_m)} (-1)^{|B|} \text{gch } V^-_{\text{end}(B) \downarrow \text{down}(B) + \xi}(\lambda - \varepsilon_m) + \sum_{k=m+1}^{n} q^{-\langle \varepsilon_k, \text{down}(p_{m,k}(w)) + \xi \rangle} \times \sum_{B \in A(\text{end}(p_{m,k}(w)), \Theta_k)} (-1)^{|B|} \text{gch } V^-_{\text{end}(B) \downarrow \text{down}(B) + \text{down}(\text{down}(p_{m,k}(w))) + \xi}(\lambda - \varepsilon_k) + \sum_{k=1}^{l} q^{\langle \varepsilon_k, \text{down}(p_{m,k}(w)) + \xi \rangle} \times \sum_{B \in A(\text{end}(p_{m,k}(w)), \Gamma_k(k))} (-1)^{|B|} \text{gch } V^-_{\text{end}(B) \downarrow \text{down}(B) + \text{down}(\text{down}(p_{m,k}(w))) + \xi}(\lambda + \varepsilon_k).
\]

We expect that the \( l \) in the above conjectures is either \( m \) or \( n \).

4.4 Examples of identities of inverse Chevalley type We give an example of the identities given by Theorem 4.5 and also give two examples of the conjectural identities proposed by Conjecture 4.7 in these examples, we assume that \( n = 3 \). We used SageMath [S] for calculations in these examples.

Example 4.9. We consider the product \( e^{w \varepsilon_3} \text{gch } V^-_w(\lambda) \). Let \( w = s_1 s_2 s_1 \), and take \( \lambda \in P^+ \) such that \( \lambda + \varepsilon_k \in P^+ \) for \( k = 1, 2, 3 \). We see that

\[ S_{3,1} = \{(1), (2, 1)\}, \quad S_{3,2} = \{(2)\}, \]

and that the admissible subsets which appear on the right-hand side of (11) are as follows:

\[ A_{s_1 s_2 s_1}^{3,1} = \{ \{1\} \}, \quad A_{s_1 s_2 s_1}^{3,2} = \{ \{1\} \}, \quad A_{s_2 s_1}^{2,1} = \{ \{1\} \}. \]

Table I is the list of \( \text{end}(\cdot) \) and \( \text{down}(\cdot) \) for admissible subsets \( A_1, A_2, A_3, A_4 \).
We consider the product Example 4.10.

Therefore, by Theorem 4.3, we compute:

\[
e^{s_1s_2s_1\varepsilon_3} \text{gch} V^-_{s_1s_2s_1}(\lambda) = \sum_{B \in \mathcal{A}(s_1s_2s_1, \Gamma_3(3))} (-1)^{|\mathcal{B}|} \text{gch} V^-_{\text{end}(B) \downarrow_{\text{down}(B)}(\lambda + \varepsilon_3)} + q^{(2, \alpha_2^\vee)} \sum_{B \in \mathcal{A}(s_2s_1, \text{gch}(2))} (-1)^{|\mathcal{B}|} \text{gch} V^-_{\text{end}(B) \downarrow_{\text{down}(B)}(\alpha_1^\vee + \alpha_2^\vee)}(\lambda + \varepsilon_2) + q^{(\varepsilon_1, \alpha_1^\vee + \alpha_2^\vee)} \sum_{B \in \mathcal{A}(\varepsilon_1, \Gamma_1(1))} (-1)^{|\mathcal{B}|} \text{gch} V^-_{\text{end}(B) \downarrow_{\text{down}(B)}(\alpha_1^\vee + \alpha_2^\vee)}(\lambda + \varepsilon_1)
\]

\[
- q^{(\varepsilon_1, \alpha_1^\vee + \alpha_2^\vee)} \sum_{B \in \mathcal{A}(s_2s_1, \Gamma_1(1))} (-1)^{|\mathcal{B}|} \text{gch} V^-_{\text{end}(B) \downarrow_{\text{down}(B)}(\alpha_1^\vee + \alpha_2^\vee)}(\lambda + \varepsilon_1) + q^{(\varepsilon_1, \alpha_1^\vee + \alpha_2^\vee)} \sum_{B \in \mathcal{A}(\varepsilon_1, \Gamma_1(1))} (-1)^{|\mathcal{B}|} \text{gch} V^-_{\text{end}(B) \downarrow_{\text{down}(B)}(\alpha_1^\vee + \alpha_2^\vee)}(\lambda + \varepsilon_1)
\]

\[
= \sum_{B \in \mathcal{A}(s_1s_2s_1, \Gamma_3(3))} (-1)^{|\mathcal{B}|} \text{gch} V^-_{\text{end}(B) \downarrow_{\text{down}(B)}(\lambda + \varepsilon_3)} + q^{(2, \alpha_2^\vee)} \sum_{B \in \mathcal{A}(s_2s_1, \text{gch}(2))} (-1)^{|\mathcal{B}|} \text{gch} V^-_{\text{end}(B) \downarrow_{\text{down}(B)}(\alpha_1^\vee + \alpha_2^\vee)}(\lambda + \varepsilon_2) + q^{(\varepsilon_1, \alpha_1^\vee + \alpha_2^\vee)} \sum_{B \in \mathcal{A}(\varepsilon_1, \Gamma_1(1))} (-1)^{|\mathcal{B}|} \text{gch} V^-_{\text{end}(B) \downarrow_{\text{down}(B)}(\alpha_1^\vee + \alpha_2^\vee)}(\lambda + \varepsilon_1).
\]

The above result agrees with Theorem 4.3 since we have

\[
P_{3,2}(s_1s_2s_1) : s_1s_2s_1 \xrightarrow{(2,3)} s_2s_1, \quad P_{3,1}(s_1s_2s_1) : s_1s_2s_1 \xrightarrow{(1,3)} e.
\]

Example 4.10. We consider the product \(e^{-\varepsilon_2 \text{gch} V^-_w(\lambda)}\). Let \(w = s_3s_2\), and take \(\lambda \in P^+\) such that \(\lambda + \varepsilon_k \in P^+\) for \(k = 1, 2, 3\) and such that \(\lambda - \varepsilon_k \in P^+\) for \(k = 2, 3\). We see that the sets \(S_{\lambda,k}\) for \(k = 1, 2, 3, 3\) are:

\[
S_{\lambda,1} = \{(1), (2, 1), (3, 1), (3, 1), (3, 2, 1), (3, 2, 1), (3, 3, 1), (3, 3, 2, 1)\},
S_{\lambda,2} = \{(2), (3, 2), (3, 2), (3, 3, 2)\},
S_{\lambda,3} = \{(3), (3, 3)\},
S_{\lambda,3} = \{(3)\}.
\]
and that the admissible subsets which appear on the right-hand side of (4.2) are as follows:

\[
\begin{align*}
\mathcal{A}_{s_3 s_2}^2 &= \emptyset, & \mathcal{A}_{s_3 s_2}^{7.2} &= \{\{2, 4\}, \{2, 3, 4\}\}, & \mathcal{A}_{s_3 s_2}^{7.3} &= \{\{2\}, \{2, 3\}\}, & \mathcal{A}_{s_3 s_2}^{7.5} &= \{\{4\}\}, \\
\mathcal{A}_{s_3}^{3.1} &= \emptyset, & \mathcal{A}_{s_3}^{7.2} &= \{\{2\}, \{2, 3\}\}, & \mathcal{A}_{s_3}^{7.3} &= \{\{3\}\}, & \mathcal{A}_{s_3}^{7.5} &= \emptyset, \\
\mathcal{A}_{s_2 s_3}^{3.1} &= \emptyset, & \mathcal{A}_{s_2 s_3}^{7.2} &= \{\{2\}\}, & \mathcal{A}_{s_2 s_3}^{7.3} &= \{\{2\}\}, & \mathcal{A}_{s_2 s_3}^{7.5} &= \emptyset, \\
\mathcal{A}_{s_2}^{2.1} &= \emptyset, & \mathcal{A}_{s_2}^{7.2} &= \{\{1\}\}, & \mathcal{A}_{s_2}^{7.3} &= \{\{1\}\}, & \mathcal{A}_{s_2}^{7.5} &= \emptyset.
\end{align*}
\]

Table 2 is the list of end(·) and down(·) for admissible subsets given above.

| Admissible subset | end(·) | down(·) |
|-------------------|--------|---------|
| \(A_1\)          | \(s_2 s_3\) | \(\alpha_2\) |
| \(A_2\)          | \(s_2\) | \(\alpha_2^\vee + \alpha_3^\vee\) |
| \(A_3\)          | \(s_2 s_3 s_2\) | \(0\) |
| \(A_4\)          | \(e\) | \(\alpha_2\) |
| \(A_5\)          | \(s_3\) | \(\alpha_2^\vee + \alpha_3^\vee\) |
| \(A_6\)          | \(s_2 s_3\) | \(0\) |
| \(A_7\)          | \(s_2\) | \(\alpha_2^\vee\) |
| \(A_8\)          | \(e\) | \(\alpha_3^\vee\) |
| \(A_9\)          | \(s_2\) | \(0\) |
| \(A_{10}\)       | \(s_2 s_3 s_2\) | \(0\) |
| \(A_{11}\)       | \(s_2 s_3 s_1 s_2\) | \(\alpha_2^\vee\) |
| \(A_{12}\)       | \(s_2 s_3\) | \(\alpha_2^\vee\) |
| \(A_{13}\)       | \(s_2 s_3 s_1\) | \(0\) |
| \(A_{14}\)       | \(s_2 s_1\) | \(0\) |
Therefore, by Theorem 4.3 we see that
\[ e^{-s_3s_2\varepsilon_2} gch V_{s_3s_2}^-(\lambda) \]
\[ = \sum_{B \in A(s_3s_2,\Theta_2)} (-1)^{|B|} gch V_{\text{end}(B)t_{\text{down}(B)}}^-(\lambda - \varepsilon_2) \]
\[ + q^{-(\varepsilon_3,\alpha_2')} \sum_{B \in A(s_3,\Theta_3)} (-1)^{|B|} gch V_{\text{end}(B)t_{\text{down}(B)}+\alpha_2'}^-(\lambda - \varepsilon_3) \]
\[ + \sum_{B \in A(s_3s_3s_2,\Gamma_3(3))} (-1)^{|B|} gch V_{\text{end}(B)t_{\text{down}(B)}}^-(\lambda + \varepsilon_3) \]
\[ + q^{(\varepsilon_2,\alpha_2')} \sum_{B \in A(s_3s_3,\Gamma_2(2))} (-1)^{|B|} gch V_{\text{end}(B)t_{\text{down}(B)}+\alpha_2}^-(\lambda + \varepsilon_2) \]
\[ + \sum_{B \in A(s_3s_3s_1s_2,\Gamma_1(1))} (-1)^{|B|} gch V_{\text{end}(B)t_{\text{down}(B)}}^-(\lambda + \varepsilon_1); \]
\[ (A_3, A_{10}) \]

here, the other terms cancel out. The above result agrees with Conjecture 4.7 if we take \( l = 3 \), since we have
\[ P_{\mathcal{T}_2}(s_3s_2) : s_3s_2 \xrightarrow{(2,3)} s_3, \]
\[ P_{\mathcal{T}_2}(s_3s_2) : s_3s_2 \xrightarrow{(2,3)} s_3s_3s_2, \]
\[ P_{\mathcal{T}_3}(s_3s_2) : s_3s_2 \xrightarrow{(2,3)} s_3s_3s_2, \]
\[ P_{\mathcal{T}_1}(s_3s_2) : s_3s_2 \xrightarrow{(2,3)} s_3s_3s_2 \xrightarrow{(1,3)} s_3s_3s_1s_2. \]

Example 4.11. We consider the product \( e^{-w_1} gch V_w^- (\lambda) \). Let \( w = s_1s_2s_3s_2s_1 \), and take \( \lambda \in P^+ \) such that \( \lambda + \varepsilon_k \in P^+ \) for \( k = 1, 2, 3 \) and such that \( \lambda - \varepsilon_k \in P^+ \) for \( k = 1, 2, 3 \). We see that the sets \( \mathcal{S}_{\mathcal{T}, k} \) for \( k = 1, 2, 3, \mathcal{T}, \mathcal{T} \) are:
\[ \mathcal{S}_{\mathcal{T}, 1} = \{(1), (2, 1), (3, 1), (\overline{2}, 1), (3, 2, 1), (3, 3, 2), (3, 3, 1), (\overline{2}, 3, 1), (3, 3, 1), (\overline{2}, 3, 1), (\overline{2}, 3, 2, 1), (3, 3, 2), (3, 3, 1), (\overline{2}, 3, 2, 1)\}, \]
\[ \mathcal{S}_{\mathcal{T}, 2} = \{(2), (3, 2), (\overline{2}, 2), (3, 3, 2), (3, 3, 2), (\overline{2}, 3, 2), (\overline{2}, 3, 2)\} \]
\[ \mathcal{S}_{\mathcal{T}, 3} = \{(3), (\overline{3}), (\overline{2}, 3), (\overline{2}, 3)\} \]
\[ \mathcal{S}_{\mathcal{T}, \mathcal{T}} = \{\{(\overline{2})\}\} \]

and that the admissible subsets which appear on the right-hand side of (12) are as follows:
\[ \mathcal{A}_{\mathcal{T}, 1}^{\mathcal{T}, 1} = \{\{(3)\}\}, \quad \mathcal{A}_{\mathcal{T}, 2}^{\mathcal{T}, 2} = \{\{(3, 5)\}\}, \quad \mathcal{A}_{\mathcal{T}, 3}^{\mathcal{T}, 3} = \{\{(3, 5)\}\}, \quad \mathcal{A}_{s_1s_2s_3s_2s_1}^{\mathcal{T}, 3} = \emptyset, \]
\[ \mathcal{A}_{s_1s_2s_3s_2s_1}^{\mathcal{T}, 3} = \emptyset, \quad \mathcal{A}_{s_1s_2s_3s_2s_1}^{\mathcal{T}, 3} = \emptyset, \quad \mathcal{A}_{s_1s_2s_3s_2s_1}^{\mathcal{T}, 3} = \emptyset, \]
\[ \mathcal{A}_{s_1s_2s_3s_2s_1}^{\mathcal{T}, 3} = \emptyset, \quad \mathcal{A}_{s_1s_2s_3s_2s_1}^{\mathcal{T}, 3} = \emptyset, \quad \mathcal{A}_{s_1s_2s_3s_2s_1}^{\mathcal{T}, 3} = \emptyset, \]
\[ \mathcal{A}_{s_1s_2s_3s_2s_1}^{\mathcal{T}, 3} = \emptyset, \quad \mathcal{A}_{s_1s_2s_3s_2s_1}^{\mathcal{T}, 3} = \emptyset, \quad \mathcal{A}_{s_1s_2s_3s_2s_1}^{\mathcal{T}, 3} = \emptyset. \]
Table 3 is the list of $\text{end}(\cdot)$ and $\text{down}(\cdot)$ for admissible subsets given above.

Table 3: The list of $\text{end}(A)$ and $\text{down}(A)$ for $A = A_1, \ldots, A_9$

| $A$          | $\text{end}(A)$ | $\text{down}(A)$ |
|--------------|------------------|------------------|
| $A_1$        | $e$              | $\alpha_1^\vee + \alpha_2^\vee + \alpha_3^\vee$ |
| $A_2$        | $s_1$            | $\alpha_1^\vee + \alpha_2^\vee + \alpha_3^\vee$ |
| $A_3$        | $s_1s_2s_3s_2$   | $\alpha_1^\vee$  |
| $A_4$        | $s_1$            | $\alpha_2^\vee + \alpha_3^\vee$ |
| $A_5$        | $s_1s_2$         | $\alpha_2^\vee + \alpha_3^\vee$ |
| $A_6$        | $s_1s_2s_3$      | $\alpha_3^\vee$  |
| $A_7$        | $s_1s_2$         | $\alpha_3^\vee$  |
| $A_8$        | $s_1$            | $\alpha_2^\vee$  |
| $A_9$        | $e$              | $\alpha_1^\vee$  |

Therefore, by Theorem 4.3 we see that

\[
e^{-s_1s_2s_3s_2s_1\cdot e} \text{gch} V_{s_1s_2s_3s_2s_1}^{-}(\lambda) = \sum_{B \in A(s_1s_2s_3s_2s_1, \Theta_1)} (-1)^{|B|} \text{gch} V_{\text{end}(B) \cdot \text{down}(B)}^{-}(\lambda - \varepsilon_1) + q^{-\langle\varepsilon_2, \alpha_1^\vee\rangle} \sum_{B \in A(s_1s_2s_3s_2, \Theta_2)} (-1)^{|B|} \text{gch} V_{\text{end}(B) \cdot \text{down}(B) + \alpha_1^\vee}(\lambda - \varepsilon_2) + q^{-\langle\varepsilon_3, \alpha_1^\vee + \alpha_3^\vee\rangle} \sum_{B \in A(s_1s_2s_3, \Theta_3)} (-1)^{|B|} \text{gch} V_{\text{end}(B) \cdot \text{down}(B) + \alpha_1^\vee + \alpha_3^\vee}(\lambda - \varepsilon_3) + q^{\langle\varepsilon_1, \alpha_1^\vee + \alpha_2^\vee + \alpha_3^\vee\rangle} \sum_{B \in A(e, \Gamma_1(1))} (-1)^{|B|} \text{gch} V_{\text{end}(B) \cdot \text{down}(B) + \alpha_1^\vee + \alpha_2^\vee + \alpha_3^\vee}(\lambda + \varepsilon_1);
\]

here, the other terms cancel out. The above result agrees with Conjecture 4.7 if we take $l = 1$, since we have

\[
\begin{align*}
\mathcal{P}_{T, \Omega}(s_1s_2s_3s_2s_1) : & \quad s_1s_2s_3s_2s_1 \xrightarrow{(1,2)} s_1s_2s_3s_2, \\
\mathcal{P}_{T, \Omega}(s_1s_2s_3s_2s_1) : & \quad s_1s_2s_3s_2s_1 \xrightarrow{(1,2)} s_1s_2s_3s_2 \xrightarrow{(2,3)} s_1s_2s_3, \\
\mathcal{P}_{T, \Omega}(s_1s_2s_3s_2s_1) : & \quad s_1s_2s_3s_2s_1 \xrightarrow{(1,7)} e.
\end{align*}
\]

5 Proofs of Theorems 4.1 and 4.3

We give proofs of our identities of inverse Chevalley type.

5.1 First-half case First, we consider the first-half case. The following proposition is a key to the proof of the identities.
Proposition 5.1. Let $w \in W$ and $\lambda \in P^+$ such that $\lambda + \varepsilon_k \in P^+$. Then we have
\[
\sum_{B \in A(w, \Gamma_k(k))} (-1)^{|B|} gch V_{\text{end}(B)|_{\text{down}(B)}}^-(\lambda + \varepsilon_k) = \sum_{A \in A(w, \Theta_k)} (-1)^{|A|} e^{\text{wk}} gch V_{\text{end}(A)|_{\text{down}(A)}}^-(\lambda). \tag{5.1}
\]

Proof. In this proof,

1) the sequence $\Gamma_k(k)$ is of the form:
\[
\Gamma_k(k) = (\beta_{k,k}, \ldots, \beta_{k-1,k}, \beta_{k,k+1}, \ldots, \beta_{k,k+1}, \beta_{k,n}, \ldots, \beta_{k,k+1}).
\]

2) admissible subsets $B \in A(v, \Gamma_k(k))$ for $v \in W$ are subsets of the (totally ordered) index set $I_1 := \{(1, k) \prec \cdots \prec (k-1, k) \prec (k, k+1) \prec \cdots \prec (k, n) \prec \cdots \prec (k, k+1)\}$, where $\prec$ defines a total order.

Similarly,

1) the sequence $\Gamma_{k-1,k}$ is of the form:
\[
\Gamma_{k-1,k} = (\gamma_{k,k+1}, \ldots, \gamma_{k,n}, \gamma_{k,k+1}, \gamma_{k,k+1}, \gamma_{k,k+1}, \gamma_{k,k+1}, \gamma_{1,k}, \gamma_{k,k+1}, \gamma_{k,k+1}, \gamma_{k-1,k}).
\]

2) admissible subsets $A \in \Gamma(v, \Gamma_{k-1,k})$ for $v \in W$ are subsets of the (totally ordered) index set $I_2 := \{(k, k+1) \prec \cdots \prec (k, n) \prec (k, k+1) \prec \cdots \prec (k-1, k) \prec \cdots \prec (1, k) \prec \cdots \prec (k-1, k)\}$, where $\prec$ defines a total order. Note that $I_1 \subset I_2$, and that if $\beta \prec \gamma$, then $\gamma \prec \beta$ for $\beta, \gamma \in I_1$.

By Proposition 3.3 and Lemma 2.5 we deduce that

(LHS of (5.1))
\[
= \sum_{B \in A(w, \Gamma_k(k))} (-1)^{|B|} gch V_{\text{end}(B)|_{\text{down}(B)}}^-(\lambda + \varepsilon_k) = \sum_{B \in A(w, \Gamma_k(k))} (-1)^{|B|} q^{-\langle \lambda + \varepsilon_k, \text{down}(B) \rangle} gch V_{\text{end}(B)}^-(\lambda + \varepsilon_k)
\]
\[
= \sum_{B \in A(w, \Gamma_k(k))} (-1)^{|B|} q^{-\langle \lambda + \varepsilon_k, \text{down}(B) \rangle}
\]
\[
\times \frac{1}{1 - q^{-\langle \lambda + \varepsilon_k, \alpha_k \rangle}} \sum_{A \in A(\text{end}(A), \Gamma_{k-1,k})} (-1)^{n(A)} q^{-\text{height}(A) + \text{wt}(A)} gch V_{\text{end}(A)|_{\text{down}(A)}}^-(\lambda). \tag{5.2}
\]

Let us take the alcove path
\[
(A_0 = A_0, A_1, \ldots, A_a = B_0, B_1, \ldots, B_b = A_{\varepsilon_k})
\]
corresponding to $\Gamma_{k-1,k} = \Gamma^*_k(k) \ast \Theta_k$. Then, the hyperplane containing the common wall of $A_{t-1}$ and $A_t$, $t = 1, \ldots, a$, is of the form $H_{\beta,0}$ with $\beta \in \Delta^+$. Also, the hyperplane containing the common wall of $B_{t-1}$ and $B_t$ ($t = 1, \ldots, b$) is of the form $H_{\beta,1}$ with $\beta \in \Delta^+$ (see Remark 2.6). This
implies that if we divide $A \in \mathcal{A}(v, \Gamma_{k-1,k})$ into the two parts: $A^{(1)} := A \cap \{(k, k + 1), \ldots, (1, \overline{k})\} \ (\in \mathcal{A}(v, \Gamma_{k}^{*}(k)))$ and $A^{(2)} := A \cap \{(1, k), \ldots, (k - 1, k)\} \ (\in \mathcal{A}(\text{end}(A^{(1)}), \Theta_{k}))$, then we have

$$
\text{height}(A) = \sum_{a \in A \cap A^{(1)}(1)} \langle \varepsilon_{k}, \gamma_{a}^{\vee} \rangle
$$

$$
= \left\langle \varepsilon_{k}, \sum_{a \in A \cap A^{(1)}(1)} \gamma_{a}^{\vee} \right\rangle
$$

$$
= \langle \varepsilon_{k}, \text{down}(A^{(1)}) \rangle.
$$

In addition, we have $\text{wt}(A) = \text{end}(A^{(1)})\varepsilon_{k}$. Also, since all the roots in $\Gamma_{k}^{*}(k)$ are positive roots, while those in $\Theta_{k}$ are negative roots, we see that $n(A) = |A^{(2)}|$. Therefore, we see that

\begin{equation}
= \sum_{B \in \mathcal{A}(w, \Gamma_{k}(k))} (-1)^{|B|} q^{-\langle \lambda + \varepsilon_{k}, \text{down}(B) \rangle} \sum_{A \in \mathcal{A}(\text{end}(B), \Gamma_{k-1,k})} (-1)^{|A|} \langle \varepsilon_{k}, \text{down}(A^{(1)}) \rangle e^{\text{end}(A^{(1)})\varepsilon_{k}} \text{gch} \ V_{\text{end}(A)\text{down}(A)}^{-}(\lambda)
\end{equation}

\begin{equation}
= \frac{1}{1 - q^{-\langle \lambda + \varepsilon_{k}, \alpha_{k}^{\vee} \rangle}} \sum_{B \in \mathcal{A}(w, \Gamma_{k}(k))} \sum_{A \in \mathcal{A}(\text{end}(B), \Gamma_{k-1,k})} (-1)^{|B|} (-1)^{|A^{(2)}|}
\end{equation}

\begin{equation}
\times q^{-\langle \lambda + \varepsilon_{k}, \text{down}(B) \rangle} e^{\text{end}(A^{(1)})\varepsilon_{k}} q^{-\langle \lambda + \varepsilon_{k}, \text{down}(A^{(1)}) \rangle} \text{gch} \ V_{\text{end}(A)\text{down}(A^{(2)})}^{-}(\lambda)
\end{equation}

\begin{equation}
= \frac{1}{1 - q^{-\langle \lambda + \varepsilon_{k}, \alpha_{k}^{\vee} \rangle}} \sum_{B \in \mathcal{A}(w, \Gamma_{k}(k))} \sum_{A^{(2)} \in \mathcal{A}(\text{end}(A^{(1)}), \Theta_{k})} (-1)^{|A^{(2)}|} \text{gch} \ V_{\text{end}(A^{(2)})\text{down}(A^{(2)})}^{-}(\lambda).
\end{equation}

Now, we define an involution on the set

$$
\mathcal{P} := \{(B, A^{(1)}) \mid B \in \mathcal{A}(w, \Gamma_{k}(k)), A^{(1)} \in \mathcal{A}(\text{end}(B), \Gamma_{k}^{*}(k))\}.
$$

There are the following six cases:

1. $[\text{max } B < \min A^{(1)}]$ or $[B \neq \emptyset$ and $A^{(1)} = \emptyset]$, 
2. $[\text{max } B > \min A^{(1)}]$ or $[B = \emptyset$ and $A^{(1)} \neq \emptyset]$, 
3. $\text{max } B = \min A^{(1)} = (k, k + 1)$, and one of the following holds:
   - $\text{max } (B \setminus \{(k, k + 1)\}) < \min (A^{(1)} \setminus \{(k, k + 1)\})$ or
   - $B \setminus \{(k, k + 1)\} \neq \emptyset$ and $A^{(1)} = \{(k, k + 1)\}$,
4. $\text{max } B = \min A^{(1)} = (k, k + 1)$, and one of the following holds:
   - $\text{max } (B \setminus \{(k, k + 1)\}) > \min (A^{(1)} \setminus \{(k, k + 1)\})$ or
   - $B = \{(k, k + 1)\}$ and $A^{(1)} \setminus \{(k, k + 1)\} \neq \emptyset$, 
5. $B = A^{(1)} = \{(k, k + 1)\}$,
Here we remark that, if we have a directed path \( v \overset{\alpha}{\rightarrow} vs \overset{\alpha}{\rightarrow} v \) in \( \text{QBG}(W) \) for \( v \in W \) and \( \alpha \in \Delta^+ \), then \( \alpha \) must be a simple root. Conversely, for \( v \in W \) and a simple root \( \alpha \), \( v \overset{\alpha}{\rightarrow} vs \overset{\alpha}{\rightarrow} v \) is a directed path in \( \text{QBG}(W) \). For \( (B, A^{(1)}) \in \mathcal{P} \), we define \( \iota(B, A^{(1)}) = (B', A'^{(1)}) \in \mathcal{P} \) as follows:

- if \( (B, A^{(1)}) \) satisfies (1) above, then set \( B' := B \setminus \{ \text{max } B \} \), \( A'^{(1)} := A^{(1)} \cup \{ \text{max } B \} \);
- if \( (B, A^{(1)}) \) satisfies (2) above, then set \( B' := B \cup \{ \text{min } A^{(1)} \} \), \( A'^{(1)} := A^{(1)} \setminus \{ \text{min } A^{(1)} \} \);
- if \( (B, A^{(1)}) \) satisfies (3) above, then set \( B' := B \setminus \{ \text{max}(B \setminus \{(k, k + 1)\}) \} \), \( A'^{(1)} := A' \cup \{ \text{max}(B \setminus \{(k, k + 1)\}) \} \);
- if \( (B, A^{(1)}) \) satisfies (4) above, then set \( B' := B \cup \{ \min(A^{(1)} \setminus \{(k, k + 1)\}) \} \), \( A'^{(1)} := A' \setminus \{ \min(A^{(1)} \setminus \{(k, k + 1)\}) \} \);
- if \( (B, A^{(1)}) \) satisfies (5) or (6) above, then set \( B' := B, A'^{(1)} := A^{(1)} \).

It is clear that \( \iota \) defines an involution on \( \mathcal{P} \). Moreover, in cases (1) and (3) (resp., (2) and (4)), we have

- \( |B'| = |B| - 1 \) (resp., \( |B'| = |B| + 1 \)),
- \( \text{down}(B') + \text{down}(A^{(1)}) = \text{down}(B) + \text{down}(A^{(1)}) \), and
- \( \text{end}(A'^{(1)}) = \text{end}(A^{(1)}) \).

This implies that

\[
\sum_{(B, A^{(1)}) \in \mathcal{P}} (-1)^{|B|} q^{-\langle \lambda + e_k, \text{down}(B) + \text{down}(A^{(1)}) \rangle} e^{|\text{end}(A^{(1)})| e_k} \times \sum_{A^{(2)} \in A(\text{end}(A^{(1)}), \Theta_k)} (-1)^{|A^{(2)}|} gch_{\text{end}(A^{(2))}}(V)_{\text{down}(A^{(2))}}(\lambda) = 0.
\]

Therefore, we conclude that

\[
\begin{align*}
\sum \frac{1}{1 - q^{-\langle \lambda + \frac{w_k}{\Theta_k}, \alpha_k \rangle}} & \left( -q^{-\langle \lambda + e_k, \alpha_k \rangle} e^{w_k} \sum_{A^{(2)} \in A(w, \Theta_k)} (-1)^{|A^{(2)}|} gch_{\text{end}(A^{(2))}}(V)_{\text{down}(A^{(2))}}(\lambda) \right) \\
& + e^{w_k} \sum_{A^{(2)} \in A(w, \Theta_k)} (-1)^{|A^{(2)}|} gch_{\text{end}(A^{(2))}}(V)_{\text{down}(A^{(2))}}(\lambda) \\
& = \frac{1}{1 - q^{-\langle \lambda + \frac{w_k}{\Theta_k}, \alpha_k \rangle}} \left( -q^{-\langle \lambda + e_k, \alpha_k \rangle} e^{w_k} \sum_{A^{(2)} \in A(w, \Theta_k)} (-1)^{|A^{(2)}|} gch_{\text{end}(A^{(2))}}(V)_{\text{down}(A^{(2))}}(\lambda) \right) \\
& + e^{w_k} \sum_{A^{(2)} \in A(w, \Theta_k)} (-1)^{|A^{(2)}|} gch_{\text{end}(A^{(2))}}(V)_{\text{down}(A^{(2))}}(\lambda) \\
& \text{for } B = A^{(1)} = \emptyset.
\end{align*}
\]
as desired; for the third equality, we have used that \( \langle \varepsilon_k, \alpha_k^\vee \rangle = \langle \pi_k, \alpha_k^\vee \rangle = 1 \). This completes the proof of the proposition.

\[ \Box \]

**Proof of Theorem 4.1.** We prove the assertion of the theorem by induction on \( m = 1, \ldots, n \). If \( m = 1 \), then the assertion immediately follows from Proposition 5.1 since \( \mathcal{A}(w, \Theta_1) = \{ \emptyset \} \). Let \( 1 < l \leq n \), and assume the assertion for \( m = 1, \ldots, l-1 \). We will prove the assertion for \( m = l \). Note that for \( A \in \mathcal{A}(w, \Theta_l) \setminus \{ \emptyset \} \), if the index \( k \) satisfies \( \text{end}(A)^{-1} w \varepsilon_l = \varepsilon_k \), then we have \( 1 \leq k \leq l-1 \). Therefore, by Proposition 5.1 we see that

\[
e^{w_{\varepsilon_l}} \text{gch} V_{w}^-(\lambda)
= \sum_{B \in \mathcal{A}(w, \Gamma_l(l))} (-1)^{|B|} \text{gch} V_{\text{end}(B) \downarrow \text{down}(B)}^-(\lambda + \varepsilon_l)
- \sum_{A \in \mathcal{A}(w, \Theta_l) \setminus \{ \emptyset \}} (-1)^{|A|} e^{w_{\varepsilon_l}} \text{gch} V_{\text{end}(A) \downarrow \text{down}(A)}^-(\lambda)
= \sum_{B \in \mathcal{A}(w, \Gamma_l(l))} (-1)^{|B|} \text{gch} V_{\text{end}(B) \downarrow \text{down}(B)}^-(\lambda + \varepsilon_l)
- \sum_{A \in \mathcal{A}(w, \Theta_l) \setminus \{ \emptyset \}} (-1)^{|A|} q^{-\langle \lambda, \text{down}(A) \rangle} e^{w_{\varepsilon_l}} \text{gch} V_{\text{end}(A)}^-(\lambda)
= \sum_{B \in \mathcal{A}(w, \Gamma_l(l))} (-1)^{|B|} \text{gch} V_{\text{end}(B) \downarrow \text{down}(B)}^-(\lambda + \varepsilon_l)
+ \sum_{k=1}^{l-1} \sum_{A \in \mathcal{A}(w, \Theta_l) \setminus \{ \emptyset \}} (-1)^{|A|} q^{-\langle \lambda, \text{down}(A) \rangle} \text{gch} V_{\text{end}(A)}^-(\lambda)
\text{induction hypothesis}
= \sum_{B \in \mathcal{A}(w, \Gamma_l(l))} (-1)^{|B|} \text{gch} V_{\text{end}(B) \downarrow \text{down}(B)}^-(\lambda + \varepsilon_l)
+ \sum_{k=1}^{l-1} \sum_{A \in \mathcal{A}_w^{A,k}} (-1)^{|A|} q^{-\langle \lambda, \text{down}(A) \rangle}
\times \left( \sum_{B \in \mathcal{A}(\text{end}(A), \Gamma_k(k))} (-1)^{|B|} \text{gch} V_{\text{end}(B) \downarrow \text{down}(B)}^-(\lambda + \varepsilon_k)
+ \sum_{j=1}^{k-1} \sum_{(j_1, \ldots, j_r) \in S_{k-j}} \sum_{A_1 \in \mathcal{A}_{\text{end}(A)}^{k-j_1}} \cdots \sum_{A_r \in \mathcal{A}_{\text{end}(A)}^{r-1,j_r}} (-1)^{|A_1| + \cdots + |A_r|} q^{-\langle \varepsilon_j, \text{down}(A_1, \ldots, A_r) \rangle}
\times \sum_{B \in \mathcal{A}(\text{end}(A_r), \Gamma(j))} (-1)^{|B|} \text{gch} V_{\text{end}(B) \downarrow \text{down}(A_1, \ldots, A_r, B)}^-(\lambda + \varepsilon_j) \right)
as desired. Thus, the assertion also holds for $m = l$. This proves the theorem. \hfill \square

### 5.2 Second-half case

The following proposition is a key to the proof of the second half of our identities of inverse Chevalley type.

**Proposition 5.2.** Let $w \in W$, $\lambda \in P^+$ such that $\lambda - \varepsilon_k \in P^+$. Then we have

$$\sum_{B \in A(w, \Theta_k)} (-1)^{|B|} \operatorname{gch} V_{\operatorname{end}(B)\downarrow \operatorname{down}(B)}^{-} (\lambda - \varepsilon_k)$$

$$= \sum_{A \in A(w, \Gamma_k(k))} (-1)^{|A|} e^{-w \varepsilon_k} \operatorname{gch} V_{\operatorname{end}(A)\downarrow \operatorname{down}(A)}^{-} (\lambda).$$

**Proof.** By replacing $\lambda$ in equation (5.1) with $\lambda - \varepsilon_k$ and multiplying both sides of equation (5.1) by $e^{-w \varepsilon_k}$, we obtain the desired identity. \hfill \square

**Proof of Theorem 4.3.** We prove the assertion of the theorem by downward induction on $m = n, n-1, \ldots, 1$. First, assume that $m = n$. Then, by Proposition 5.2 and Corollary 4.2, we see that

$$e^{-w \varepsilon_m} \operatorname{gch} V_{w}^{-} (\lambda)$$

$$= \sum_{B \in A(w, \Theta_n)} (-1)^{|B|} \operatorname{gch} V_{\operatorname{end}(B)\downarrow \operatorname{down}(B)}^{-} (\lambda - \varepsilon_n)$$

$$- \sum_{A \in A(w, \Gamma_n(n)) \setminus \{0\}} (-1)^{|A|} e^{-w \varepsilon_n} \operatorname{gch} V_{\operatorname{end}(A)\downarrow \operatorname{down}(A)}^{-} (\lambda)$$

$$= \sum_{B \in A(w, \Theta_n)} (-1)^{|B|} \operatorname{gch} V_{\operatorname{end}(B)\downarrow \operatorname{down}(B)}^{-} (\lambda - \varepsilon_n)$$

$$+ \sum_{k=1}^{n} \sum_{A \in A^k_w} (-1)^{|A|-1} e^{-w \varepsilon_n} \operatorname{gch} V_{\operatorname{end}(A)\downarrow \operatorname{down}(A)}^{-} (\lambda)$$

$$\sum_{B \in A(w, \Gamma_n(n)) \setminus \{0\}} (-1)^{|A|} e^{-w \varepsilon_n} \operatorname{gch} V_{\operatorname{end}(A)\downarrow \operatorname{down}(A)}^{-} (\lambda)$$

as desired. This completes the proof for $m = n$. For $m < n$, we proceed by induction on $m = n - 1, \ldots, 1$. The case $m = 1$ is trivial. For $m > 1$, we use the same method as before, but with $m$ replaced by $m-1$. This completes the proof of Theorem 4.3. \hfill \square
as desired. Let $1 \leq l < n$, and assume the assertion for $m = n, n - 1, \ldots, l + 1$. We prove the assertion for $m = l$. For $A \in \mathcal{A}(w, \Gamma((l))) \setminus \{\emptyset\}$, if the index $k$ satisfies $\text{end}(A)^{-1} w(\varepsilon_l) = \varepsilon_k$, then we have $k = 1, \ldots, n - 1, n, n, n - 1, \ldots, l + 1$. Therefore, by Proposition 5.2 (and Proposition 3.2), we compute

$$e^{-w_l \varepsilon_l} \text{gch } V_{w_l}^-(\lambda)$$

$$= \sum_{B \in \mathcal{A}(w, \Theta_l)} (-1)^{|B|} \text{gch } V_{\text{end}(B)\downarrow \text{down}(B)}^-(\lambda - \varepsilon_l)$$

$$- \sum_{A \in \mathcal{A}(w, \Gamma_l((l))) \setminus \{\emptyset\}} (-1)^{|A|} e^{-w_l \varepsilon_l} \text{gch } V_{\text{end}(A)\downarrow \text{down}(A)}^-(\lambda)$$

$$= \sum_{B \in \mathcal{A}(w, \Theta_l)} (-1)^{|B|} \text{gch } V_{\text{end}(B)\downarrow \text{down}(B)}^-(\lambda - \varepsilon_l)$$

$$+ \sum_{k=l+1}^{n} \sum_{A \in \mathcal{A}_w^k} (-1)^{|A|} e^{-w_l \varepsilon_l} \text{gch } V_{\text{end}(A)\downarrow \text{down}(A)}^-(\lambda)$$

$$+ \sum_{k=1}^{l} \sum_{A \in \mathcal{A}_w^k} (-1)^{|A|} e^{-w_l \varepsilon_l} \text{gch } V_{\text{end}(A)\downarrow \text{down}(A)}^-(\lambda)$$
\[
\begin{align*}
&= \sum_{B \in \mathcal{A}(u, \Theta)} (-1)^{|B|} gch V_{end(B)} t_{down(B)} (\lambda - \varepsilon_l) \\
&+ \sum_{k=1}^{n} \sum_{A \in \mathcal{A}^k(u)} (\sum_{l=1}^{\frac{k}{2}}) (-1)^{|A|-1} q^{-\langle \lambda, t_{down(A)} \rangle} e^{-\langle \lambda, t_{down(A)} \rangle} gch V_{end(A)}^-(\lambda) \\
&+ \sum_{k=1}^{n} \sum_{A \in \mathcal{A}^k(u)} (-1)^{|A|} e^{\langle \lambda, t_{down(A)} \rangle} gch V_{end(A)}^-(\lambda) \\
&= \sum_{B \in \mathcal{A}(u, \Theta)} (-1)^{|B|} gch V_{end(B)} t_{down(B)} (\lambda - \varepsilon_l) \\
&+ \sum_{j=1}^{n} \sum_{A \in \mathcal{A}^j} \sum_{\sum_{l=1}^{\frac{j}{2}} V_{r+1} A \in \mathcal{A}^r(j, A_{\varepsilon})} (-1)^{|A_1|+\ldots+|A_r|} \sum_{j\leq j+1} q^{-\langle \varepsilon, t_{down(A_1, \ldots, A_r)} \rangle} \\
&\times \sum_{B \in \mathcal{A}(A_1, \ldots, A_r, \Theta)} (-1)^{|B|} gch V_{end(B)} t_{down(B)} (\lambda - \varepsilon_l) \\
&+ \sum_{k=1}^{n} \sum_{A \in \mathcal{A}^k} (-1)^{|A|-1} q^{-\langle \varepsilon, t_{down(A)} \rangle} \sum_{B \in \mathcal{A}(A, \Theta)} (-1)^{|B|} gch V_{end(B)} t_{down(B)} (\lambda + \varepsilon_k) \\
&+ \sum_{j=1}^{n} \sum_{A \in \mathcal{A}^j} \sum_{\sum_{l=1}^{\frac{j}{2}} A_{\varepsilon} \in \mathcal{A}^r(j, A_{\varepsilon})} (-1)^{|A_1|+\ldots+|A_r|} \sum_{j\leq j+1} q^{-\langle \varepsilon, t_{down(A_1, \ldots, A_r)} \rangle} \\
&\times \sum_{B \in \mathcal{A}(A_1, \ldots, A_r, \Theta)} (-1)^{|B|} gch V_{end(B)} t_{down(B)} (\lambda + \varepsilon_j) \\
&= \sum_{B \in \mathcal{A}(u, \Theta)} (-1)^{|B|} gch V_{end(B)} t_{down(B)} (\lambda - \varepsilon_l) \\
&+ \sum_{j=1}^{n} \sum_{A \in \mathcal{A}^j} \sum_{\sum_{l=1}^{\frac{j}{2}} \sum_{A \in \mathcal{A}^r(j, A_{\varepsilon})} (-1)^{|A_1|+\ldots+|A_r|} \sum_{j\leq j+1} q^{-\langle \varepsilon, t_{down(A_1, \ldots, A_r)} \rangle} \\
&\times \sum_{B \in \mathcal{A}(A_1, \ldots, A_r, \Theta)} (-1)^{|B|} gch V_{end(B)} t_{down(B)} (\lambda - \varepsilon_l) \\
&+ \sum_{j=1}^{n} \sum_{A \in \mathcal{A}^j} \sum_{\sum_{l=1}^{\frac{j}{2}} A_{\varepsilon} \in \mathcal{A}^r(j, A_{\varepsilon})} (-1)^{|A_1|+\ldots+|A_r|} \sum_{j\leq j+1} q^{-\langle \varepsilon, t_{down(A_1, \ldots, A_r)} \rangle} \\
&\times \sum_{B \in \mathcal{A}(A_1, \ldots, A_r, \Theta)} (-1)^{|B|} gch V_{end(B)} t_{down(B)} (\lambda + \varepsilon_j),
\end{align*}
\]

as desired. By downward induction, this completes the proof of the theorem.
6 Proof of Theorem 4.5

We will derive the cancellation-free form of the first-half identities of inverse Chevalley type (Theorem 4.5). For this purpose, we need the following lemmas on edges of the quantum Bruhat graph. We continue to assume that $\mathfrak{g}$ is of type $C_n$. Recall that a total order $< \mathfrak{n}$ on $[\mathfrak{n}]$ is defined by: $1 < 2 < \cdots < n < \mathfrak{n} < n-\mathfrak{I} < \cdots < \mathfrak{I}$; for each $1 \leq k \leq n$, we define an order $\prec_k$ (resp., $\prec_k$) on $[\mathfrak{n}]$ by: $k \prec_k k+1 \prec_k \cdots \prec_k n \prec_k n-\mathfrak{I} \prec_k \cdots \prec_k \mathfrak{I} \prec_k 1 \prec_k 2 \prec_k \cdots \prec_k k-1 \prec_k k$ (resp., $\mathfrak{k} \prec_{\mathfrak{k}} k-1 \prec_{\mathfrak{k}} \cdots \prec_{\mathfrak{k}} 1 \prec_k 2 \prec_k \cdots \prec_k n \prec_k n-\mathfrak{I} \prec_k \cdots \prec_k \mathfrak{k}+1$). For $a_1, \ldots, a_r \in [\mathfrak{n}]$ with $r \geq 2$, we write $a_1 \prec a_2 \prec \cdots \prec a_r$ if $a_1 \prec a_2 \prec \cdots \prec a_r$ (the order $<$ is different from that introduced in the proof of Proposition 5.1). Also, on the set $[\mathfrak{n}]$, we define the sign function $\text{sgn}(\cdot)$: for $a \in [\mathfrak{n}]$, we set

$$\text{sgn}(a) := \begin{cases} 1 & \text{if } a = 1, 2, \ldots, n, \\ -1 & \text{if } a = \mathfrak{n}, n-\mathfrak{I}, \ldots, \mathfrak{I}. \end{cases}$$

We know the following useful criterion.

**Lemma 6.1** ([L, Proposition 5.7]). Let $w \in W$.

(1) Let $1 \leq k < l \leq n$. Then, $w (k,l) \to w_{s(k,l)}$ is an edge in QBG(W) if and only if there does not exist $1 < k < l$ such that $w(k) < w(j) < w(l)$.

(2) Let $1 \leq k < l \leq n$. Then, $w (k,l) \to w_{s(k,l)}$ is an edge in QBG(W) if and only if the following hold:

- $w(k) < w(l)$;
- $\text{sgn}(w(k)) = \text{sgn}(w(l))$; and
- there does not exist $k < j < l$ such that $w(k) < w(j) < w(l)$.

(3) Let $1 \leq k \leq n$. Then, $w (k,\mathfrak{k}) \to w_{s(k,\mathfrak{k})}$ is an edge in QBG(W) if and only if there does not exist $k < j < \mathfrak{k}$ such that $w(k) < w(j) < w(\mathfrak{k})$.

By using this criterion, we can show the following three lemmas.

**Lemma 6.2.** Let $w \in W$, and $1 \leq k < l < m \leq n$. Then, the following are equivalent:

(1) $w (k,m) \to w_{s(k,m)}$ and $w (l,m) \to w_{s(l,m)} (k,l) \to w_{s(l,m)s(k,l)}$;

(2) $w (k,m) \to w_{s(k,m)}$ and $w (l,m) \to w_{s(l,m)}$;

(3) $w (k,m) \to w_{s(k,m)} (l,m) \to w_{s(k,m)s(l,m)}$.

**Lemma 6.3.** Let $w \in W$. Take $1 \leq k_1 < l_1 \leq n$ and $1 \leq k_2 < l_2 \leq n$ such that $\{k_1, l_1\} \cap \{k_2, l_2\} = \emptyset$. Then, the following are equivalent:

(1) we have the directed path $w (k_1,l_1) \to w_{s(k_1,l_1)} (k_2,l_2) \to w_{s(k_1,l_1)s(k_2,l_2)}$;

(2) we have the directed path $w (k_2,l_2) \to w_{s(k_2,l_2)} (k_1,l_1) \to w_{s(k_2,l_2)s(k_1,l_1)}$.

**Lemma 6.4.** Let $w \in W$, $m = 1, \ldots, n$, and take $a_1, \ldots, a_s \in \{k \in [1, m-1] \mid w (k,m) \to w_{s(k,m)} \}$ such that $a_1 < \cdots < a_s$; by Lemma 6.2, we have the directed path

$$w = y_0 (a_1,m) \to y_1 (a_2,m) \to \cdots \to y_s (a_s,m) \to y_s$$

in QBG(W). Let us take $c < a_1$ such that
• \( y_s \xrightarrow{(c,a_1)} y_s s_{(c,a_1)} \) is an edge in \( \text{QBG}(W) \), and

• \( w \xrightarrow{(c,m)} w s_{(c,m)} \) is an edge in \( \text{QBG}(W) \).

For \( p < a_1 \), if \( w \xrightarrow{(p,m)} w s_{(p,m)} \) is an edge in \( \text{QBG}(W) \), then we have \( p < c \).

**Corollary 6.5.** Let \( w \in W, m = 1, \ldots, n \), and let \( \{ a_1 < \cdots < a_s \} = \{ k \in [1, m - 1] \mid w \xrightarrow{(k,m)} w s_{(k,m)} \} \) with \( s \geq 2 \); by Lemma 6.4, for \( b_1 < \cdots < b_s \leq s \), we have the directed path

\[
w = z_0 \xrightarrow{(a_1,m)} z_1 \xrightarrow{(a_2,m)} \cdots \xrightarrow{(a_s,m)} z_a
\]

in \( \text{QBG}(W) \). Then, \( a_1 \) is equal to the minimal \( c \) such that \( 1 \leq c < a_{b_1} \) for which \( w \xrightarrow{(c,a_{b_1})} w s_{(c,a_{b_1})} \) is an edge in \( \text{QBG}(W) \).

**Proof.** Let us take the minimal \( c \) for which \( z_u \xrightarrow{(a_{b_1})} z_w s_{(c,a_{b_1})} \) is an edge in \( \text{QBG}(W) \). Note that such a \( c \) exists since \( z_u \xrightarrow{(a_{b_1})} z_w s_{(a_{b_1},a_{b_1})} \) is an edge in \( \text{QBG}(W) \) by Lemmas 6.2 and 6.3. This also implies that \( c \leq a_1 \). Assume, for a contradiction, that \( c < a_1 \). Then, since \( a_1 \) is the minimum of the set \( \{ k \in [1, m - 1] \mid w \xrightarrow{(k,m)} w s_{(k,m)} \} \), \( w \xrightarrow{(c,m)} w s_{(c,m)} \) is not an edge in \( \text{QBG}(W) \). Also, we see that \( a_1 < a_2 \leq a_{b_1} \) and that \( w \xrightarrow{(a_1,m)} w s_{(a_1,m)} \) is an edge in \( \text{QBG}(W) \). Therefore, by Lemma 6.3, we obtain \( a_1 < c \), which is a contradiction. Hence we conclude that \( c = a_1 \), as desired. This proves the corollary. \( \Box \)

Theorem 4.5 follows immediately from the following key proposition and Theorem 4.1. Let \( Z[q^{-1}][W] \) denote the group algebra of \( W \) with coefficients in \( Z[q^{-1}] \); the elements of \( Z[q^{-1}][W] \) are of the form \( \sum_{v \in W} c_v (q^{-1})^v \), with \( c_v (q^{-1}) \in Z[q^{-1}] \).

**Proposition 6.6.** Let \( w \in W, m = 1, \ldots, n \), and \( j = 1, \ldots, m - 1 \). Then, there holds the following equality in \( Z[q^{-1}][W] \):

\[
\sum_{(j_1, \ldots, j_r) \in S_{m,j}} \sum_{A_1 \in A_{w}^{m,j_1}} \cdots \sum_{A_r \in A_{w}^{(r-1,j_r)}} (-1)^{|A_1|+\cdots+|A_r|} q^{-\lambda, \text{down}(A_1, \ldots, A_r)} \text{end}(A_r)
= q^{-\lambda, \text{down}(p_{m,j}(w))} \text{end}(p_{m,j}(w)).
\]

**Proof.** We prove the assertion of the proposition by induction on \( m - j \). If \( m - j = 1 \), then the assertion is clear since \( S_{m,m-1} = \{(m-1)\} \) and \( A_w^{m,m-1} = \{\{m-1\}\} \). Assume that \( m - j > 1 \). By Lemma 6.2, we can verify that if \( \{ a_1 < \cdots < a_s \} = \{ k \in [1, m - 1] \mid w \xrightarrow{(k,m)} w s_{(k,m)} \} \) (note that \( a_s = m - 1 \)), then

\[
A_w^{m,j} = \begin{cases} \{ \{ j, a_{c_1}, \ldots, a_{c_u} \} \mid l < c_1 < \cdots < c_u \leq s \} & \text{if } j = a_l \text{ for some } l = 1, \ldots, s, \\ \emptyset & \text{if } j \neq a_1, \ldots, a_s. \end{cases}
\]

Therefore, we compute

\[
\sum_{(j_1, \ldots, j_r) \in S_{m,j}} \sum_{A_1 \in A_{w}^{m,j_1}} \cdots \sum_{A_r \in A_{w}^{(r-1,j_r)}} (-1)^{|A_1|+\cdots+|A_r|} q^{-\lambda, \text{down}(A_1, \ldots, A_r)} \text{end}(A_r)
= \sum_{k=1}^s \sum_{(j_1, \ldots, j_r) \in S_{k,j}} \sum_{A_1 \in A_{w}^{k,j_1}} \cdots \sum_{A_r \in A_{w}^{(r-1,j_r)}} \text{end}(A_r)
\times (-1)^{|A_1|+|A_2|+\cdots+|A_r|+(r+1)} q^{-\lambda, \text{down}(A_1, \ldots, A_r)} \text{end}(A_r)
\]

27
This proves the proposition.

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