A non-conserving coagulation model with extremal dynamics

Róbert Juhász

Research Institute for Solid State Physics and Optics, H-1525 Budapest,
PO Box 49, Hungary
E-mail: juhasz@szfki.hu

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Abstract. A coagulation process in a set of random masses, in which two randomly chosen masses and the smallest mass of the set multiplied by some fixed parameter $\omega \in [-1, 1]$ are iteratively added, is studied. Besides masses (or primary variables), secondary variables are also considered that are correlated with primary variables and coagulate according to the above rule with $\omega = 0$. This process interpolates between known statistical physical models: the case $\omega = -1$ corresponds to the strong disorder renormalization group transformation of certain disordered quantum spin chains whereas $\omega = 1$ describes coarsening in the one-dimensional Glauber–Ising model. The case $\omega = 0$ is related to the renormalization group transformation of a recently introduced graph with a fat-tail edge-length distribution. In the intermediate range $-1 < \omega < 1$, the exponents $\alpha_\omega$ and $\beta_\omega$ that characterize the growth of the primary and the secondary variable, respectively, are accurately estimated by analysing the differential equations describing the process in the continuum formulation. According to the results, the exponent $\alpha_\omega$ varies monotonically with $\omega$ while $\beta_\omega$ has a maximum at $\omega = 0$.

Keywords: disordered systems (theory), spin chains, ladders and planes (theory), renormalization group, irreversible aggregation phenomena (theory)
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### 1. Introduction

Coagulation processes arise in various areas of physics; one may think of polymerization, growth of ordered domains in non-equilibrium magnetic systems [1], dynamics of droplets when water condenses on non-wetting surfaces [2], etc. The substance, or ‘mass’, that aggregates is very frequently not conserved during the process: for example, agglomerating insoluble inclusions in molten metal may be lost from the melt by attachment to the wall of the vessel [3]. Therefore the theoretical investigation of the kinetics of such non-conserving coagulation processes is of great importance. Moreover, the models developed for the description of such systems may show interesting behaviour: the Smoluchowsky equation with certain coagulation kernels exhibits a gelation transition and, in general, even the simplest models with conserved mass may have non-trivial solutions; see e.g. [3] and references therein. Beside quite realistic ones, there is a special class of (possibly non-conserving) coagulation models where only the actually smallest one among the masses is active while the other masses are temporarily inert. This type of extremal dynamics can be regarded as a rough approximation for models where the reaction rates are decreasing functions of the mass of particles. In what follows, we shall survey three processes with extremal dynamics in detail. We mention, however, that models of this type have also been introduced in the context of dynamics of growing and coalescing droplets [2] or multispecies pair annihilation reactions [4].

In the one-dimensional Glauber–Ising model started from a random initial state at zero temperature, the domain walls move as independent random walkers and annihilate upon meeting. While the closest pairs of walls come together and annihilate, the other domain walls hardly move. A simplified model of evolution of distances $X_i$ between adjacent walls can be formulated as follows [5,6]. The shortest interval $X_m$ is eliminated together with the two adjacent intervals $X_1$ and $X_2$ and replaced by $\tilde{X} = X_1 + X_2 + X_m$. As the density of walls tends to zero, the distributions of intervals at different times become

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self-similar, depending on a single time-dependent length scale, and the corresponding scaling function can be calculated exactly \[5,7,8\]. Another quantity of interest is the fraction of space which has never been traversed by a domain wall. The length \(Y_i\) of such parts of intervals transforms in the way given by \(\tilde{Y} = Y_1 + Y_2\) when the shortest interval is eliminated. The characteristic value of \(X\) depends on the fraction \(c\) of the initial intervals that have not yet been eliminated as \(X \sim c^{-\alpha}\), obviously, with \(\alpha = 1\), while it has been found that \(Y \sim c^{-\beta}\), where the persistence exponent \(\beta = 0.82492412\ldots\) is the zero of a transcendental equation \(7\). In addition to this, the autocorrelation exponent has also been exactly calculated in this model \(9\). To obtain this quantity, the overlap \(Z_i\) of an interval with its initial state that transforms as \(\tilde{Z} = Z_1 + Z_2 - Z_m\) had to be considered. Later, a generalization of persistence was studied in the same model, which required one to introduce an auxiliary variable transforming as \(\tilde{Y} = Y_1 + Y_2 + pY_m\) \(10\). Here, the generalized persistence exponent has been found to vary monotonically with the partial survival factor \(p\) in the range \(-1 \leq p \leq 1\).

The next example is the strong disorder renormalization group transformation of inhomogeneous quantum spin chains \(11\). Here, the degrees of freedom related to the largest coupling (a bond between neighbouring spins or a local external field) are eliminated one after another. In terms of logarithmic couplings, \(X_i\), the renormalization rule generally reads as \(\tilde{X} = X_1 + X_2 - X_m\), where \(\tilde{X}\) is a newly formed effective variable and \(X_1, X_2\) are variables adjacent to the smallest one, \(X_m\). For the relation between these variables and the couplings in the particular Hamiltonians we refer the reader to \(12\). A variable \(Y_i\) that transforms according to the rule \(\tilde{Y} = Y_1 + Y_2\) under such a renormalization step can be interpreted in the case of a particular model, the transverse field Ising chain, as the magnetic moment of a spin. For this process with i.i.d. random initial variables \(X_i\), which corresponds to critical spin chains, the distribution of \(X\) flows again to a fixed point where it shows scaling behaviour. The characteristic value of \(X\) increases in the course of the process as \(X \sim c^{-\alpha}\) with \(\alpha = 1/2\), while the variable \(Y\) grows as \(Y \sim c^{-\beta}\) with \(\beta = (1 + \sqrt{5})/4 = 0.809016\ldots\) \(12\). Note that the coagulation rules in the above two models differ only in the sign of \(X_m\), which leads to different exponents \(\alpha\) and \(\beta\).

Our third example is a random graph where three edges emanate from each node, and which is built on a regular one-dimensional lattice by adding long edges in the following way. To each edge of the one-dimensional lattice that we call short edges, a random weight \(X_i\) is assigned. Defining the length of a path as the sum of weights of the edges it contains, the closest pair of nodes of degree 2 with respect to this metric is chosen and connected by an edge of unit weight. This step is then repeated until all nodes become of degree 3 \(13\). For this graph, a renormalization procedure can be formulated where loops are eliminated step by step in reverse order compared to the construction procedure. Formally, the shortest edge with the minimal weight \(X_m\) is eliminated together with the nodes that it connects, as well as with the neighbouring short edges with weights \(X_1, X_2\), and a new effective short edge is formed with a weight calculated asymptotically as \(\tilde{X} = X_1 + X_2\). According to numerical results, the characteristic value of the effective weights grows as \(X \sim c^{-\alpha}\) with \(\alpha = 0.826(1)\) \(13\). This exponent characterizes at the same time the diameter of finite graphs with \(N\) nodes with respect to the above metric via \(D(N) \sim N^\alpha\).

As can be seen, these seemingly different problems can be treated in a common framework and can be interpreted as coagulation processes with extremal dynamics. In the first example, the total sum of the variables \(X_i\) is conserved while in the latter two cases...
it is not. We will study in this work a coagulation model controlled by a parameter $\omega$ that interpolates continuously between the first two models and incorporates the third one as a special case, as well. We are interested in the exponents $\alpha_\omega$ and $\beta_\omega$ for intermediate values of the parameter $\omega$ and shall provide accurate estimates for $\alpha_\omega$ that is obtained as the root of a transcendental equation while $\beta_\omega$ is accurately determined by the numerical analysis of a system of non-linear differential equations. We shall see that $\alpha_\omega$ varies monotonically between the corresponding values of the two marginal models, while, unlike the generalized persistence exponent of the model with partial survival mentioned above \cite{10}, the exponent $\beta_\omega$ shows a maximum when $\omega$ is varied. As can be seen, the transformation rule of the variable $Y$ does not depend directly on the parameter $\omega$ but it is influenced indirectly via the correlations emerging between $X$ and $Y$, the strength of which is controlled by $\omega$. Therefore our results may contribute to the understanding of the role of correlations in such models. Moreover, these investigations provide an accurate estimate for the diameter exponent of the graph quoted above, for which we obtain $\alpha = 0.82617561$ in agreement with the previous numerical result.

The rest of the paper is organized as follows. In section 2, the model and its continuum description are introduced. In sections 3 and 4, the method of approximative determination of the exponents $\alpha_\omega$ and $\beta_\omega$ is presented. (Some calculations are given in the appendix.) Finally, results are discussed in section 5.

2. The model and its continuum formulation

2.1. Definition of the model

Let us consider a finite set of positive vectors $V_i = (X_i, Y_i)$ indexed by the integers $i = 1, 2, \ldots, N$. We assume, moreover, that $N$ is odd. The vectors are independent, identically distributed random variables drawn from a continuous distribution $\rho(X,Y)dXdY$, for which we require that all moments exist. The first components $X_i$ and the second components $Y_i$ are called primary and secondary variables, respectively. Assume, furthermore, that $\omega \in [-1,1]$ is a fixed real number. Now, the following procedure is considered on this set. The vector $V_m$ with the smallest primary variable is chosen and, at the same time, two further vectors $V_i$ and $V_j$ are chosen at random from the set. These three vectors are removed and a new vector $\tilde{V}$ with components

$$\tilde{X} = X_i + X_j + \omega X_m, \quad \tilde{Y} = Y_i + Y_j$$

(1)

is added to the set. Thereby the number of vectors in the set is reduced by two. Note that the vectors remain independent after such an operation and that

$$\tilde{X} \geq X_i, X_j, X_m$$

(2)

even for $\omega = -1$. This step is then iterated until a single vector $V_N = (X_N, Y_N)$ is left in the set. In this general formulation, the cases $\omega = 1, -1, 0$ correspond to the three models in the order in which they were quoted in section 1. On the basis of the known asymptotical behaviour of $X_N$ and $Y_N$ for large $N$ in the marginal cases $\omega = -1, 1$, we expect

$$X_N \sim N^{\alpha_\omega} \quad \text{and} \quad Y_N \sim N^{\beta_\omega}$$

(3)

to hold also for intermediate parameter values $-1 < \omega < 1$ with some exponents $\alpha_\omega$ and $\beta_\omega$ that may depend on $\omega$. 

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2.2. Continuum formulation

Now, we consider the continuum limit \( N \to \infty \) and introduce the probability density \( P_\Gamma(X) \) of the primary variable that has the support \( \Gamma \leq X < \infty \) and that depends on the lower boundary \( \Gamma \) as a parameter. The function \( P_\Gamma(X) \) is normalized as \( \int_{\Gamma}^{\infty} P_\Gamma(X) \, dX = 1 \) for any \( \Gamma \). Following [7], we consider, furthermore, the expected value \( \bar{Y}_\Gamma(X) \) of the secondary variable under the condition that the primary variable is \( X \). In the continuum limit, the system is described by these two functions of \( X \), which depend on the lower boundary of the support \( \Gamma \) as a parameter. The inequality (2) implies that, as the fraction of vectors \( c_\Gamma \) that have not yet been eliminated decreases in the course of the coagulation process, the lower edge \( \Gamma \) of the distribution continuously increases. As is shown in the appendix, one may write the following differential equation for \( P_\Gamma(X) \):

\[
\frac{\partial P_\Gamma(X)}{\partial \Gamma} = P_\Gamma(\Gamma) \Theta[X - (2 + \omega)\Gamma] \int_{\Gamma}^{X-(1+\omega)\Gamma} P_\Gamma(X') P_\Gamma(X - X' - \omega \Gamma) \, dX',
\]

(4)

where \( \Theta(X) \) is the Heaviside step function. The fraction \( c_\Gamma \) is related to \( \Gamma \) as \( \frac{dc_\Gamma}{c_\Gamma} = -2P_\Gamma(\Gamma) \, d\Gamma \) or, equivalently,

\[
\frac{dc_\Gamma}{d\Gamma} = -2P_\Gamma(\Gamma)c_\Gamma.
\]

(5)

The function \( Q_\Gamma(X) \), defined as

\[
Q_\Gamma(X) \equiv P_\Gamma(X) \bar{Y}_\Gamma(X),
\]

(6)

can be shown to obey the differential equation

\[
\frac{\partial Q_\Gamma(X)}{\partial \Gamma} = 2P_\Gamma(\Gamma) \Theta[X - (2 + \omega)\Gamma] \int_{\Gamma}^{X-(1+\omega)\Gamma} Q_\Gamma(X') P_\Gamma(X - X' - \omega \Gamma) \, dX'.
\]

(7)

The derivation of this equation is given again in the appendix.

2.3. Fixed point solution

For the marginal cases \( \omega = -1, 1 \), it is known that, for any well-behaving initial distributions \( \rho(X,Y) \) with finite moments, the solutions of equations (4) and (7) tend to a universal fixed point solution \( P^*_\Gamma(X), Q^*_\Gamma(Y) \) in the limit \( \Gamma \to \infty \) that has the scaling property

\[
P^*_\Gamma(X) = \Gamma^{\delta_\omega-1} f(X/\Gamma), \quad Q^*_\Gamma(X) = \Gamma^{\delta_\omega-1} g(X/\Gamma),
\]

(8)

with some number \( \delta_\omega \) that is related to the growth exponents as

\[
\delta_\omega = \beta_\omega / \alpha_\omega.
\]

(9)

Therefore we expect this to hold also for intermediate parameter values \(-1 < \omega < 1\) with some (a priori unknown) exponent \( \delta_\omega \) that may depend on \( \omega \). Indeed, the functions in equations (8) solve equations (4) and (7) provided that the universal scaling functions

1 This can be seen from the equation \( \bar{Y}_\Gamma(\Gamma) \equiv Q^*_\Gamma(\Gamma)/P^*_\Gamma(\Gamma) = \Gamma^{\delta_\omega} g(1)/f(1) \) that indicates the asymptotical relation \( Y \sim X^{\delta_\omega} \) between the typical values of primary and secondary variables.
f(x) and g(x) satisfy the following differential equations:
\[
\frac{d[x f(x)]}{dx} = -f_1 \Theta(x - 2 - \omega) \int_1^{x-1-\omega} f(x') f(x - x' - \omega) \, dx',
\]
(10)
\[
\frac{d[x^{1-\delta} g(x)]}{dx} = -2 f_1 \Theta(x - 2 - \omega) \int_1^{x-1-\omega} g(x') f(x - x' - \omega) \, dx',
\]
(11)
where the notation \( f_1 \equiv f(1) \) has been used. For an alternative derivation of these equations in the case \( \omega = 1 \), see [7]. Using the fixed point solution, equation (5) can be integrated yielding an asymptotic relation in the large \( \Gamma \) limit:
\[
\Gamma \sim C^{1/2 f_1}. \tag{12}
\]
Comparing this with equation (3), we obtain the relation
\[
\alpha_\omega = \frac{1}{2 f_1}. \tag{13}
\]

3. Approximative determination of \( \alpha_\omega \)

As can be seen, equation (10) does not contain \( g(x) \) and together with equation (13) it constitutes an autonomous problem for the calculation of the exponent \( \alpha_\omega \). For the special case \( \omega = -1 \), the solution of equation (10) is of simple form: \( f(x) = e^{-x+1} \), this yields \( \alpha_{-1} = \frac{1}{2} \). In the other marginal case, \( \omega = 1 \), the Laplace transform of the solution is known [5, 8] and \( \alpha_1 = 1 \). For the case \(-1 < \omega < 1\), where equation (10) is not soluble, we shall construct an approximative solution that enables us to give an accurate estimate of \( \alpha_\omega \). An alternative approach related to the numerical analysis of the Laplace transforms is presented in section 4.

Some properties of the scaling function \( f(x) \) can be easily established by investigating equation (10) without knowing the exact solution. Apparently, the rhs of equation (10) and, as a consequence, \( f(x) \) is non-analytical at \( x = x_1 \equiv 2 + \omega \). But, as \( f(x) \) itself appears on the rhs as a convolution with a shifted argument \( x - 1 - \omega \), the rhs as well as \( f(x) \) must be non-analytical also at \( x = x_2 \equiv x_1 + 1 + \omega \). Iterating this argument, it turns out that there are infinitely many points where \( f(x) \) is non-analytical. To be precise, one can show by recursion that the 2nth derivative of \( f(x) \) is discontinuous at\(^2\)
\[
x_n = 1 + (1 + \omega)n, \quad n = 0, 1, 2, \ldots. \tag{14}
\]
Furthermore, the function value of \( f(x) \) at some \( x' \) is determined by \( f(x) \) in the restricted domain \( (1, x' - 1 - \omega) \). Due to this property, \( f(x) \) can be constructed in the intervals \([x_n, x_{n+1}]\) step by step starting with \( n = 0 \). However, the solution becomes more and more complicated for increasing \( n \) as it contains multiple integrals that cannot be evaluated analytically. In the domains \([x_n, x_{n+1}], n = 1, 2, \ldots\), the function \( f(x) \) can be written in the following form:
\[
f(x) = \frac{1}{x} \sum_{i=0}^{n} f_1^{2i+1} C_{\omega}^{(2i+1)}(x), \quad x_n \leq x \leq x_{n+1}, \tag{15}
\]
\(^2\) For a more direct route to this result for the case \( \omega = 1 \), where the explicit form of the scaling function \( f(x) \) is available, see [8].

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Figure 1. Graphs of the functions $f(x)$, $g(x)$ and $f_\infty(x)$ for $\omega = 0$. The first two are obtained by numerical integration of equations (10) and (11), whereas the last one is given in equation (17).

whereas $f(x) = 0$ if $x < x_0$. The functions $C^{(2i+1)}(x)$ are independent of $f_1$ and the first three of them read as

$$
C^{(1)}(x) = 1, \quad C^{(3)}(x) = -2 \int_{x_1}^{x} \frac{\ln(x' - \omega - 1)}{x' - \omega} \, dx',
$$

$$
C^{(5)}(x) = -2 \int_{x_2}^{x} \int_{x_0}^{x' - x_2} \frac{C^{(3)}(x' - x'' - \omega)}{x''(x' - x'' - \omega)} \, dx'' \, dx'.
$$

Substituting an exponential trial function in equation (10), we obtain that the asymptotical solution $f_\infty(x)$ in the limit $x \to \infty$ is of the form

$$
f_\infty(x) = a f_1 e^{-a(x + \omega)},
$$

with the some number $a$ that is not fixed by this substitution. The graph of $f(x)$ for $\omega = 0$ is shown in figure 1. As can be seen, $f(x)$ tends rapidly to $f_\infty(x)$ for increasing $x$. This suggests an approximation for $f(x)$ in which $f(x)$ is replaced by the simple asymptotical function $f_\infty(x)$ for large $x$. To be precise, the nth $(n = 0, 1, 2, \ldots)$ approximant $f^{(n)}(x)$ is defined as

$$
f^{(n)}(x) = \begin{cases} 
  f(x), & \text{if } x \leq x_n, \\
  f_\infty(x), & \text{otherwise.}
\end{cases}
$$

The two unknown parameters $f_1$ and $a$ are determined by the requirements that $f^{(n)}(x)$ is continuous at $x = x_n$, i.e.

$$
f(x_n) = f_\infty(x_n),
$$

where $f(x) = 0$ if $x < x_0$. The functions $C^{(2i+1)}(x)$ are independent of $f_1$ and the first three of them read as

$$
C^{(1)}(x) = 1, \quad C^{(3)}(x) = -2 \int_{x_1}^{x} \frac{\ln(x' - \omega - 1)}{x' - \omega} \, dx',
$$

$$
C^{(5)}(x) = -2 \int_{x_2}^{x} \int_{x_0}^{x' - x_2} \frac{C^{(3)}(x' - x'' - \omega)}{x''(x' - x'' - \omega)} \, dx'' \, dx'.
$$

Substituting an exponential trial function in equation (10), we obtain that the asymptotical solution $f_\infty(x)$ in the limit $x \to \infty$ is of the form

$$
f_\infty(x) = a f_1 e^{-a(x + \omega)},
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f^{(n)}(x) = \begin{cases} 
  f(x), & \text{if } x \leq x_n, \\
  f_\infty(x), & \text{otherwise.}
\end{cases}
$$

The two unknown parameters $f_1$ and $a$ are determined by the requirements that $f^{(n)}(x)$ is continuous at $x = x_n$, i.e.

$$
f(x_n) = f_\infty(x_n),
$$

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and that it is normalized as
\[ \int_{x_0}^{x_n} f(x) \, dx + \int_{x_n}^{\infty} f_\infty(x) \, dx = 1. \]  
(20)

Using the expression in equation (15), straightforward calculations yield that the nth approximant \( f_1^{(n)} \) \((n > 0)\) is the root of the following transcendental equation:
\[
\sum_{i=0}^{n-1} [f_1^{(n)}]^{2i+1} C_{\omega}^{(2i+1)}(x_n) + \frac{x_n}{x_n + \omega} \left[ 1 - \sum_{i=0}^{n-1} [f_1^{(n)}]^{2i+1} N_{\omega}^{(2i+1)}(x_n) \right] \\
\times \ln \left[ f_1^{(n)} - \sum_{i=0}^{n-1} [f_1^{(n)}]^{2i+2} N_{\omega}^{(2i+1)}(x_n) \right] = 0, 
\]  
(21)

where the function \( N_{\omega}^{(2i+1)}(x) \) has been introduced as
\[
N_{\omega}^{(2i+1)}(x) \equiv \int_{x_i}^{x} \frac{C_{\omega}^{(2i+1)}(x')}{x'} \, dx'. \]  
(22)

We have numerically calculated the root of equation (21) and the nth approximant \( \alpha_{\omega}^{(n)} \) of \( \alpha_{\omega} \) by using equation (13) for \( n = 1, 2, 3 \) and for several values of \( \omega \). This has necessitated the numerical evaluation of the integrals in equation (22) for \( n > 1 \). Results are shown in figure 2 and some numerical values are given in table 1. As can be seen, the approximants \( \alpha_{\omega}^{(n)} \) converge rapidly with increasing \( n \) and they increase monotonically with \( \omega \). The best estimate for the diameter exponent of the graph cited in section 1 is \( \alpha_{\omega}^{(3)} = 0.82617561 \).

4. Numerical determination of \( \beta_{\omega} \)

Next, we turn to the determination of the exponent \( \delta_{\omega} \) (and, at the same time, \( \beta_{\omega} \) through equation (9)), which requires the analysis of the full problem, i.e. the system of differential
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Table 1. Approximants of the exponents $\alpha_\omega$, $\delta_\omega$ and $\beta_\omega$ for different values of $\omega$.

| $\omega$ | $\alpha^{(1)}_\omega$ | $\alpha^{(2)}_\omega$ | $\alpha^{(3)}_\omega$ | $\delta_\omega$ | $\beta_\omega = \delta_\omega \alpha^{(3)}_\omega$ |
|----------|-----------------|-----------------|-----------------|-----------------|-----------------|
| -0.9     | 0.54752760      | 0.54752815      | 0.54752815      | 1.48973578      | 0.81567227      |
| -0.8     | 0.59036797      | 0.59037862      | 0.59037860      | 1.38841226      | 0.81968889      |
| -0.7     | 0.66421085      | 0.66434418      | 0.66434376      | 1.23995279      | 0.82375490      |
| -0.6     | 0.72630756      | 0.72630667      | 0.7263067        | 1.13643762      | 0.82540221      |
| -0.5     | 0.77952410      | 0.77952421      | 0.77952421      | 1.05965756      | 0.82608118      |
| -0.4     | 0.82617193      | 0.82617561      | 0.82617561      | 1.02809664      | 0.82614953      |
| -0.3     | 0.87838834      | 0.87839245      | 0.87839245      | 1.00000000      | 0.82614953      |
| -0.2     | 0.93054355      | 0.93054856      | 0.93054856      | 0.99999999      | 0.82614953      |
| -0.1     | 0.98270074      | 0.98270615      | 0.98270615      | 0.99999999      | 0.82614953      |
| 0.0      | 1.03485830      | 1.03486378      | 1.03486378      | 0.99999999      | 0.82614953      |
| 0.1      | 1.08701708      | 1.08702369      | 1.08702369      | 0.99999999      | 0.82614953      |
| 0.2      | 1.13917739      | 1.13923490      | 1.13923490      | 0.99999999      | 0.82614953      |
| 0.3      | 1.19133770      | 1.19140511      | 1.19140511      | 0.99999999      | 0.82614953      |
| 0.4      | 1.24349811      | 1.24356713      | 1.24356713      | 0.99999999      | 0.82614953      |
| 0.5      | 1.29565852      | 1.29573063      | 1.29573063      | 0.99999999      | 0.82614953      |
| 0.6      | 1.34781912      | 1.34789474      | 1.34789474      | 0.99999999      | 0.82614953      |
| 0.7      | 1.39997973      | 1.39995415      | 1.39995415      | 0.99999999      | 0.82614953      |
| 0.8      | 1.45214034      | 1.45211715      | 1.45211715      | 0.99999999      | 0.82614953      |
| 0.9      | 1.50430095      | 1.50427636      | 1.50427636      | 0.99999999      | 0.82614953      |
| 1.0      | 1.55646156      | 1.55643798      | 1.55643798      | 0.99999999      | 0.82614953      |

equations (10) and (11). Prior to this, a few remarks concerning the scaling function $g(x)$ are in order. First, as a consequence of the definition in equation (6), $g(x)$ apparently inherits the singularity properties of $f(x)$ discussed in section 3. Furthermore, it can be written in a form analogous to equation (15). In the domain $[x_0, x_1]$, it has a simple form:

$$g(x) = g(1)x^{\delta_\omega - 1}, \quad x_0 \leq x \leq x_1. \quad (23)$$

Second, the differential equation (11) gives the scaling function $g(x)$ only up to a multiplicative constant. This non-universal constant depends on the initial distribution $\rho(X, Y)$ and it is fixed in a non-trivial way by the original equations (4) and (7) that are valid for any $\Gamma$. Third, equation (11) contains the a priori unknown parameter $\delta_\omega$ that must be fixed by physical considerations about the solution that depends on $\delta_\omega$. Namely, the physically acceptable solution must be non-negative and must have the only reasonable asymptotics allowed by equation (11):

$$g_\infty(x) \simeq \text{const} \cdot x e^{-ax}, \quad (24)$$

where the number $a$ is the same as that appearing in equation (17). Numerical analysis of equation (11) shows that these requirements are fulfilled only for a single value of the parameter $\delta_\omega$.

Following [7], it is, however, simpler to analyse the Laplace transform of the equations (10) and (11). Introducing the functions

$$\phi(p) = \int_1^\infty e^{-px} f(x) \, dx, \quad \psi(p) = \int_1^\infty e^{-px} g(x) \, dx, \quad (25)$$

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equations (10) and (11) transform to
\[ p\phi'(p) = f_1 [e^{-\omega p} \phi^2(p) - e^{-p}], \]  
\[ p\psi'(p) = -\delta_\omega \psi(p) - g_1 e^{-p} + 2f_1 e^{-\omega p} \psi(p) \phi(p), \]
where the prime denotes differentiation with respect to \( p \) and \( g_1 \equiv g(1) \). These equations are not soluble in the parameter range \(-1 < \omega < 1\) but asymptotical expressions of the solution can be established. The functions \( \phi(p) \) and \( \psi(p) \) have the small-\( p \) expansions
\[ \phi(p) = \sum_{n=0}^{\infty} a_n p^n, \quad \psi(p) = g_1 \sum_{n=0}^{\infty} b_n p^n. \]  
Substituting these into equations (26) and (27), we obtain that the expansion coefficients for \(-1 < \omega < 1\) are given by \( a_0 = 1 \), \( b_0 = 1/(2f_1 - \delta_\omega) \) and by the following recursion relations for \( n > 0 \):
\[ a_n = \frac{((-1)^n/n!)(\omega^n - 1) + \sum_{0 \leq i,j,k < n: i+j+k=n} ((-\omega)^i/i!)a_ja_k}{(n/f_1) - 2}, \]  
\[ b_n = \frac{((-1)^n/n!)(\omega^n/(2f_1-\delta_\omega) - 1/(2f_1)) + (1/(2f_1 - \delta_\omega))a_n + \sum_{0 \leq i,j,k < n: i+j+k=n} (e^{-\omega} - 1/id!a_jb_k}{(n + \delta_\omega)/(2f_1) - 1}. \]

Next, we discuss the large-\( p \) behaviour of \( \psi(p) \). The differential equation (27) can have two kinds of asymptotical solutions depending on the parameter \( \delta_\omega \). If the second term on the rhs dominates, we obtain
\[ \psi'(p) \simeq -g_1 \frac{e^{-p}}{p}, \]  
while, if the first term dominates, we obtain
\[ \psi(p) \simeq \text{const} \cdot p^{-\delta_\omega}. \]  
On the other hand, it follows from equations (25) that \( \psi(p) \) must have the large-\( p \) asymptotics \( \psi(p) \simeq g_1 e^{-p}/p[1 + O(1/p)] \). Using that the function \( g(x) \) is explicitly known in the domain \( 1 \leq x \leq 2 + \omega \) (see equation (23)), we can obtain a more accurate asymptotical form for \( 1/p \ll 1 + \omega \). Replacing \( g(x) \) by the function in equation (23) over the entire domain \( x \geq 1 \), the integral in equations (25) can be evaluated, yielding
\[ \psi_\infty(p) = g_1 e^{-p} \sum_{k=0}^{\infty} \left( \frac{\delta_\omega - 1}{k} \right) \frac{k!}{p^{k+1}}, \quad p \gg \frac{1}{1 + \omega}. \]  
This shows that the physically acceptable asymptotics is that given in equation (31) whereas that in equation (32) is non-physical. Numerical analysis of the differential equations (26) and (27) with the correct value of \( f_1 \) shows the following behaviour of the

\[^3\] These series expansions are also valid for \( \omega = -1 \) with \( a_3 = 5/2 \), and for \( \omega = 1 \) with \( a_1 = -2e^\gamma \) [7], where \( \gamma \) is Euler’s constant, given by \( \gamma = -\int_0^\infty \ln te^{-t} dt = 0.577215 \ldots \)
solution when the parameter $\delta_\omega$ is varied: for small enough $\delta_\omega$, the function $\psi(p)$ is non-monotonic and tends to zero from below for increasing $p$ as given in equation (32); for large enough $\delta_\omega$, the function $\psi(p)$ decays monotonically to zero again with the asymptotics given in equation (32). These parameter regimes are separated by a ‘critical’ value of $\delta_\omega$. At this value, the solution decays monotonically to zero with the physically acceptable asymptotics given in equation (31).

The numerical estimation of $\delta_\omega$ is based on this scenario: the differential equations (26) and (27) are integrated from $p = 0$ to some large $p$ and the true value of $\delta_\omega$ is selected using the condition that $\psi(p)$ has the correct asymptotics. We have assumed here that $f_1$ is already at our disposal. This can be obtained either by the approximative procedure described in section 3 or, like in the above method, from the condition that $\phi(p)$ has the correct asymptotics given by equation (33) with $\delta_\omega = 0$.

Before presenting numerical results on $\delta_\omega$, we show that, for the case $\omega = 0$, the assumption of the uniqueness of the value $\delta_\omega$ that corresponds to the correct asymptotics implies that $\delta_0 = 1$. For $\omega = 0$, the primary and the secondary variables coagulate according to the same rules; see equation (1). If these variables are initially perfectly correlated, i.e. $X_i = bY_i$ with some common constant $b$ for all $i$, it is obvious that $\alpha_0 = \beta_0$. Nevertheless, this equality holds for general initial distributions, as well. Indeed, it is easy to check that for $\omega = 0$, the function

$$\psi(p) = \frac{g_1}{f_1} \phi'(p)$$

solves equation (27) provided that $\delta_0 = 1$ and $\phi(p)$ is the solution of equation (26).

In terms of the scaling functions, equation (34) reads as $g(x) = x f(x)(g_1/f_1)$. As the asymptotics of the solution in equation (34) is physically acceptable, we conclude that

$$\delta_0 = 1.$$  \hfill (35)

The details of the numerical determination of $\delta_\omega$ are the following. The best approximant that we have, $f_1^{(3)}$, has been substituted in equations (26) and (27) and a trial value for $\delta_\omega$ has been chosen. As the derivatives $\phi'(p)$ and $\psi'(p)$ calculated from these equations are of the form $0/0$ at $p = 0$, the functions $\phi(p)$ and $\psi(p)$ were first calculated at $p = 0.05$ by using the small-$p$ expansion in equation (28) in order to avoid numerical uncertainties of the integration in the vicinity of $p = 0$. Then, starting from $p = 0.05$, the differential equations were integrated by the Bulirsch–Stoer method [14] to some $p_f$, where $\psi(p_f)$ is compared to the asymptotical form in equation (33). In practice, we have monitored the derivative $\psi'(p_f)$ rather than $\psi(p_f)$ and $p_f = 12$ was sufficiently large that the asymptotical value (at the critical $\delta_\omega$) is reached within the numerical accuracy of the integration. The true $\delta_\omega$ was then selected using the condition $\psi'(p_f) = \psi'_\omega(p_f)$. The exponent $\delta_\omega$ determined in this way is plotted against $\omega$ in figure 3 whereas $\beta_\omega = \delta_\omega \alpha_\omega^{(3)}$ is plotted against $\omega$ in figure 4. Some numerical values can be found in table 1. As can be seen, $\delta_\omega$ decreases monotonically with $\omega$ but $\beta_\omega$ has a maximum at $\omega = 0$. The latter observation will be explained in section 5.

5. Discussion

We have shown that two problems, the renormalization group procedure for certain disordered quantum spin chains and a simple model describing coarsening in the Glauber–
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Figure 3. Numerically calculated exponent $\delta_\omega$ plotted against $\omega$.

Figure 4. Numerically calculated exponent $\beta_\omega = \delta_\omega \alpha_\omega^{(3)}$ plotted against $\omega$. The horizontal lines indicate known values in the marginal cases $\omega = -1, 1$. Ising model, can be deformed into each other by varying a single parameter. The interpolating model is a special type of non-conserving coagulation process where only the actually smallest mass is active. In the range $-1 \leq \omega \leq 1$, the exponents $\alpha_\omega$ and $\beta_\omega$ that characterize the growth of the primary and the secondary variables, respectively, vary continuously with $\omega$. The latter exponent exhibits a maximum, which contrasts with the monotonic dependence of the generalized persistence exponent on the partial survival factor that appears directly in the transformation rule of the secondary variable [10]. Although, we have focused on the range $-1 \leq \omega \leq 1$, the equations written down in this work are valid also for $\omega > 1$. In that case, the growth of the primary variable becomes super-linear, meaning that $\alpha_\omega > 1$.

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An intriguing feature of the process studied in this work is the universality with respect to the initial distribution of the variables: for a fixed $\omega$, any sufficiently rapidly decaying initial distribution tends at late times to a universal distribution that displays scaling. Although the process is universal in this sense, we have pointed out that it is sensitive to the variations of the reaction rules parametrized by $\omega$. The dependence of $\alpha_\omega$ on $\omega$ is obvious since the transformation rule of the primary variable contains $\omega$ explicitly. The growth of the secondary variable is, however, affected by $\omega$ in a more subtle way. Focusing on the secondary variables, the difference from the process of primary variables with $\omega = 0$ is that, here, it is not exactly the smallest variable that is removed from the set. This is the reason for $\beta_\omega$ being unequal to $\alpha_0$ for $\omega \neq 0$. Nevertheless, for any $\omega$, the removed secondary variable is typically relatively small since $X_i$ and $Y_i$ become positively correlated in the course of the process. Due to these correlations, the strength of which is controlled by $\omega$, the variation of $\beta_\omega$ is relatively slight. Indeed, it is smaller by an order of magnitude than that of $\alpha_\omega$.

For $\omega = 0$, we have shown that $\alpha_0 = \beta_0$ even if the primary and the secondary variables are initially not perfectly correlated. This can be understood also on a microscopic level since, in this case, the vectors in the set are sums of an increasing number of initial vectors. Thus, the ratios $\tilde{X}_i/\tilde{Y}_i$ tend stochastically to a common constant in the limit $\Gamma \to \infty$ for all $i$. In words, the two kinds of variables become asymptotically perfectly correlated for $\omega = 0$. Now, we are in a position to understand why the exponent $\beta_\omega$ is maximal at $\omega = 0$. At that point, the correlations are (at least asymptotically) perfect and almost always the smallest one among the secondary variables is removed. For $\omega \neq 0$, however, the correlations are no longer perfect and, as a consequence, it is not strictly the smallest secondary variables that are eliminated. Therefore the fastest growth of $Y$ is realized at $\omega = 0$.

In a general respect, the benefit of the analysis carried out in this work is that the numerical technique developed here for obtaining accurate estimates of the growth exponents may also apply to other non-soluble coagulation processes with extremal dynamics.

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**Appendix**

When the primary variables in the infinitesimal interval $(\Gamma, \Gamma + d\Gamma)$ are eliminated in the course of the process, we may write for the change of the probability density $P_\Gamma(X)$

$$
P_{\Gamma+d\Gamma}(X) = \left\{ P_\Gamma(X) + P_\Gamma(\Gamma)d\Gamma \int dX_1 \int dX_2 P_\Gamma(X_1)P_\Gamma(X_2)[\delta(X - X_1 - X_2 - \omega\Gamma) \\
- \delta(X - X_1) - \delta(X - X_2)] \right\} \frac{1}{1 - 2P_\Gamma(\Gamma)d\Gamma}.
$$

(A.1)

Here, the first term of the integrand is related to the newly generated primary variable while the other two terms are related to the removed ones. The factor $1/[1 - 2P_\Gamma(\Gamma)d\Gamma]$
ensures that the distribution remains normalized. Expanding the lhs of equation (A.1) and retaining only terms of order \(d\Gamma\), we arrive at the differential equation (4).

The expected value \(\bar{Y}_\Gamma(X)\) of the newly generated secondary variable under the condition that the generated primary variable is \(X\) is given as

\[
\bar{Y}_\Gamma(X) = \left\{ \int \int dX_1 dX_2 [\bar{Y}_\Gamma(X_1) + \bar{Y}_\Gamma(X_2)] P_\Gamma(X_1) P_\Gamma(X_2) \delta(X - X_1 - X_2 - \omega \Gamma) \right\} / I_\omega(X),
\]

and

\[
\bar{Y}_{\Gamma+d\Gamma}(X) = \left[ P_\Gamma(X) - 2d\Gamma P_\Gamma(\Gamma) P_\Gamma(X) \right] \bar{Y}_\Gamma(X) + d\Gamma P_\Gamma(\Gamma) I_\omega(X) \bar{Y}_\Gamma(X).
\]

This leads to the differential equation

\[
\frac{\partial P_\Gamma(X)\bar{Y}_\Gamma(X)}{\partial \Gamma} = P_\Gamma(\Gamma) \Theta[X - (2 + \omega)\Gamma] I_\omega(X) \bar{Y}_\Gamma(X),
\]

where we have made use of equation (4). Rewriting this equation in terms of \(Q_\Gamma(X)\) given in equation (6), one arrives at equation (7).

References

[1] Bray A J, 1994 Adv. Phys. 43 357
[2] Derrida B, Godrèche C and Yekutieli I, 1990 Europhys. Lett. 12 385
[3] Wattis J A D, McCartney D G and Gudmundsson T, 2004 J. Eng. Math. 49 113
[4] Deloubrière O, Hilhorst H J and Täuber U C, 2002 Phys. Rev. Lett. 89 250601
[5] Nagai T and Kawasaki K, 1986 Physica A 134 483
[6] Kawasaki K, Ogawa A and Nagai T, 1988 Physica B 149 97
[7] Bray A J, Derrida B and Godrèche C, 1994 Europhys. Lett. 27 175
[8] Rutemberg A D and Bray A J, 1994 Phys. Rev. E 50 1900
[9] Bray A J and Derrida B, 1995 Phys. Rev. E 51 R1633
[10] Majumdar S N and Bray A J, 1998 Phys. Rev. Lett. 81 2626
[11] Ma S-K, Dasgupta C and Hu C-K, 1979 Phys. Rev. Lett. 43 1434
[12] Fisher D S, 1992 Phys. Rev. Lett. 69 534
[13] Juhász R, 2008 Phys. Rev. E 78 066106
[14] Press W H, Teukolsky S A, Wetterling W T and Flannery B P, 1992 Numerical Recipes in C (Cambridge: Cambridge University Press)