LIMITS OF CARTIER DIVISORS

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Abstract. Consider a one-parameter family of algebraic varieties degenerating to a reducible one. Our main result is a formula for the fundamental cycle of the limit subscheme of any family of effective Cartier divisors. The formula expresses this cycle as a sum of Cartier divisors, not necessarily effective, of the components of the limit variety.

1. Introduction

Consider a local one-parameter family of Noetherian schemes. More precisely, let \( f: X \to S \) be a flat map of Noetherian schemes, where \( S \) stands for the spectrum of a discrete valuation ring. Let \( s \) and \( \eta \) denote the special and generic points of \( S \); put \( X_s := f^{-1}(s) \) and \( X_\eta := f^{-1}(\eta) \). Assume that \( X_s \) is of pure dimension and has no embedded components.

Let \( D \) be an effective Cartier divisor of \( X \). View it as a subscheme of \( X \), and let \( \lim D \) be the schematic boundary of \( D \cap X_\eta \). Then \( \lim D \subseteq D \cap X_s \). Equality does not necessarily hold, as \( D \) may contain components of \( X_s \) in its support.

This note presents a formula for the fundamental class \([\lim D]\) of \( \lim D \) in terms of Cartier divisors of the components of \( X_s \); see Theorem 4.1. The idea used in its proof is that, even though \( D \) may not restrict to an effective Cartier divisor of a given component of \( X_s \), a suitable modification of \( D \) may. Suitable modifications may not exist. They do when \( X_s \) is reduced, a consequence of Proposition 4.2. At any rate, when they exist, a formula for \([\lim D]\) is derived by keeping track of the modifications and their restrictions to the components of \( X_s \).

The idea used in the proof of the main theorem is reminiscent of that behind the definition of limit linear series, as explained in [1]. And, in fact, the main application of the theorem so far is in computing limits of ramification points of families of linear systems. The theorem is perfectly adapted for dealing with the case of plane curves, the study of which will be done in [3]. Example 5.3 is given to show, in a very simple situation, how the theorem will be applied there.

Supported by CNPq, Proc. 303797/2007-0 and 473032/2008-2, and FAPERJ, Proc. E-26.102.769/2008.
A rough layout of the paper is as follows. In Section 2 we define modifications, and present the main technical lemmas that will allow us to keep track of them later on. Section 3 is devoted to defining cycles and limit cycles, and proving a few of their fundamental properties, among them Proposition 3.4 stating that taking the fundamental class of the limit is additive for Cartier divisors. Section 4 is the heart of the notes, containing the main result, Theorem 4.1 and the auxiliary Proposition 4.2 giving conditions for when the theorem may be applied. Finally, in Section 5 we present examples to show how Theorem 4.1 can be applied.

2. Modifications

2.1. (Setup.) Throughout the paper, $S$ will stand for the spectrum of a discrete valuation ring, $s$ for its closed point and $\eta$ for its generic point. Also, $\pi$ will denote a local parameter of $S$ at $s$.

Throughout the paper, $f : X \rightarrow S$ will stand for a map from a Noetherian scheme $X$. Set $X_s := f^{-1}(s)$ and $X_\eta := f^{-1}(\eta)$. We call $X_s$ the special fiber and $X_\eta$ the generic fiber of $f$. We will always assume $X_s$ has no embedded components. Denote by $C_1, \ldots, C_n$ the subschemes of $X_s$ defined by the primary ideal sheaves of 0 in $\mathcal{O}_{X_s}$, and $\xi_1, \ldots, \xi_n$ their generic points. We say that $C_1, \ldots, C_n$ are the irreducible primary subschemes of $X_s$.

A union of irreducible primary subschemes of $X_s$, defined by the intersection of the corresponding sheaves of ideals, will be called a primary subscheme of $X_s$. If $Y$ is a primary subscheme of $X_s$, the union of all the irreducible primary subschemes not contained in $Y$ will be called the complementary primary subscheme to $Y$ and denoted $Y^c$. By definition, the empty set and $X_s$ are to be considered primary subschemes of $X_s$.

Let $\text{Div}(X)$ denote the group of Cartier divisors of $X$, and $\text{Div}^+(X)$ the submonoid of effective Cartier divisors. We will view an element of $\text{Div}^+(X)$ as a closed subscheme of $X$. Conversely, we will write $Y \in \text{Div}^+(X)$ for any closed subscheme of $X$ defined locally everywhere by a nonzero divisor.

For each closed subscheme $Y$ of $X$, let $\mathcal{I}_Y$ denote its sheaf of ideals. If $Z$ is another closed subscheme, we write $Y \leq Z$ if $Y \subseteq Z$. If $D \in \text{Div}^+(X)$, let $Y + D$ denote the closed subscheme of $X$ given by the sheaf of ideals $\mathcal{I}_D \mathcal{I}_Y$. In addition, if $D \subseteq Y$, let $Y - D$ denote the residual subscheme, given by the conductor ideal $(\mathcal{I}_Y : \mathcal{I}_D)$. Of course, $Y - D \leq Y \leq Y + D$ and $Y = (Y - D) + D = (Y + D) - D$. 
Let Twist$(f)$ denote the free Abelian group generated by $C_1, \ldots, C_n$. An element of Twist$(f)$ will be called a twister. We say that a twister $\gamma = \sum m_i C_i$ is effective if $m_i \geq 0$ for each $i = 1, \ldots, n$, and reduced if, in addition, $m_i \leq 1$ for each $i = 1, \ldots, n$. Let Twist$^+(f) \subset$ Twist$(f)$ denote the submonoid of effective twisters. We can naturally identify the set of primary subschemes of $X_s$ with the set of reduced effective twisters.

2.2. (Modifications by primary subschemes.) Let $J$ be a coherent sheaf on $X$, and $Y$ a primary subscheme of $X_s$. Let $J_Y$ denote the restriction of $J$ to $Y$ modulo torsion. In other words, $J_Y$ is the image of the natural map

$$J|_Y \longrightarrow \bigoplus_{\xi_i \in Y} (J|_Y)_{\xi_i}.$$

Let $J(-Y)$ denote the kernel of the quotient map $J \rightarrow J_Y$. We say that $J(-Y)$ is a modification by $Y$ of $J$. By definition, $J_{\emptyset} = 0$ and $J(-\emptyset) = J$.

(We will never use the above construction for a sheaf denoted $I$. So, throughout the paper, $I_Y$ will always be understood as the sheaf of ideals of a subscheme $Y$ of $X$.)

Notice that $J(-Y)$ is also the kernel of the natural map

$$J \longrightarrow \bigoplus_{\xi_i \in Y} (J|_{X_s})_{\xi_i}.$$

Thus, for each primary subscheme $Z$ of $X_s$ containing $Y$, we have that $J(-Z) \subseteq J(-Y)$. In addition, $\pi J \subseteq J(-X_s)$. Hence, there is a natural map, $J \rightarrow J(-Y)$, obtained as the composition,

$$J \rightarrow \pi J \rightarrow J(-X_s) \rightarrow J(-Y),$$

where the first map is multiplication by $\pi$.

If $L$ is an invertible sheaf on $X$, then $(J \otimes L)(-Y) = J(-Y) \otimes L$, as subsheaves of $J \otimes L$.

The sheaves $J_Y$ and $J(-Y)$ and all of the maps above are functorial on $J$. So are the maps and the statements of the proposition below.

**Proposition 2.3.** Let $J$ be a coherent sheaf on $X$. Then the following three statements hold.

(1) For all primary subschemes $Y$ and $Z$ of $X_s$,

$$J(-Y)(-Z) = J(-Z)(-Y).$$
(2) For all primary subschemes $Y$ and $Z$ of $X_s$ such that $Z \subseteq Y^c$, the inclusions $\mathcal{J}(-Y) \to \mathcal{J}$ and $\mathcal{J}(-Z) \to \mathcal{J}$ induce injections $\mathcal{J}(-Y)_Z \to \mathcal{J}_Z$ and $\mathcal{J}(-Z)_Y \to \mathcal{J}_Y$ whose cokernels are isomorphic.

(3) For all primary subschemes $Y_1$, $Y_2$ and $Y_3$ of $X_s$ with $Y_2 \subseteq Y_1^c$ and $Y_3 \subseteq Y_2^c$, the inclusion $\mathcal{J}(-Y_1 \cup Y_2) \to \mathcal{J}(-Y_1)$ induces a short exact sequence:

$$0 \to \mathcal{J}(-Y_1 \cup Y_2)_Y \to \mathcal{J}(-Y_1)_Y \to \mathcal{J}(-Y_1)_{Y_2} \to 0.$$ 

Proof. Clearly, $\mathcal{J}(-Y)|_{X-Y} = \mathcal{J}|_{X-Y}$. In particular, the natural map 

$$(\mathcal{J}(-Y)|_{X_s})_{\xi_i} \to (\mathcal{J}|_{X_s})_{\xi_i}$$

is bijective for each $\xi_i \not\in Y$. Therefore, $\mathcal{J}(-Y)(-Z) = \mathcal{J}(-Y \cup Z)$ if $Z \subseteq Y^c$. More generally, writing $Y = Y' \cup W$ and $Z = Z' \cup W$, where $Y'$, $Z'$ and $W$ are primary subschemes such that $Y' \subseteq Z^c$ and $Z' \subseteq Y^c$, we have

$$\mathcal{J}(-Y)(-Z) = \mathcal{J}(-Y')(-W)(-Z')(-W) = \mathcal{J}(-Y')(-Z)(-W) = \mathcal{J}(-Z)(-Y).$$

As for the second statement, since $\mathcal{J}(-Y)_{\xi_i} = \mathcal{J}_{\xi_i}$ for every $\xi_i \in Z$, it follows that the naturally induced map $\mathcal{J}(-Y)_Z \to \mathcal{J}_Z$ is injective. An analogous reasoning shows that $\mathcal{J}(-Z)_Y \to \mathcal{J}_Y$ is also injective. Now,

$$\frac{\mathcal{J}_Z}{\mathcal{J}(-Y)_Z} = \frac{\mathcal{J}/\mathcal{J}(-Z)}{\mathcal{J}(-Y)/\mathcal{J}(-Y)(-Z)} = \frac{\mathcal{J}}{\mathcal{J}(-Y) + \mathcal{J}(-Z)}.$$ 

By symmetry, $\mathcal{J}_Y/\mathcal{J}(-Z)_Y$ is thus isomorphic to $\mathcal{J}_Z/\mathcal{J}(-Y)_Z$.

As for the third statement, consider the natural short exact sequence:

$$0 \to \frac{\mathcal{J}(-Y_1 \cup Y_2)}{\mathcal{J}(-Y_1 \cup Y_2)(-Y_3)} \to \frac{\mathcal{J}(-Y_1)}{\mathcal{J}(-Y_1 \cup Y_2)(-Y_3)} \to \frac{\mathcal{J}(-Y_1)}{\mathcal{J}(-Y_1 \cup Y_2)} \to 0.$$ 

By definition, the first quotient is $\mathcal{J}(-Y_1 \cup Y_2)_{Y_3}$, while the last is $\mathcal{J}(-Y_1)_{Y_2}$. Now, using the first statement,

$$\mathcal{J}(-Y_1 \cup Y_2)(-Y_3) = \mathcal{J}(-Y_1)(-Y_2)(-Y_3) = \mathcal{J}(-Y_1)(-Y_2 \cup Y_3).$$

So, we may identify the middle quotient with $\mathcal{J}(-Y_1)_{Y_2 \cup Y_3}$, and thus obtain the desired short exact sequence. □

2.4. (Modification by twistors.) Let $\mathcal{J}$ be a coherent sheaf on $X$. For each $\gamma \in \text{Twist}^+(f)$ define a subsheaf $\mathcal{J}^\gamma$ of $\mathcal{J}$ recursively as follows: if $\gamma = 0$, then $\mathcal{J}^\gamma := \mathcal{J}$; if $\gamma \neq 0$, then let

$$\mathcal{J}^\gamma := \mathcal{J}^{\gamma-C_1}(-C_1)$$

where $C_1$ denotes the first Chern class of $\mathcal{J}$.
for any $C_i$ such that $\gamma - C_i$ is effective. It follows from the first statement of Proposition 2.3 that $J^\gamma$ is well-defined, and

$$J^{\gamma_1 + \gamma_2} = (J^{\gamma_1})^{\gamma_2}$$

for every two $\gamma_1, \gamma_2 \in \text{Twist}^+(f)$. We call $J^\gamma$ the $\gamma$-modification of $J$.

If $L$ is an invertible sheaf on $X$, then $J^\gamma \otimes L = (J \otimes L)^\gamma$ as subsheaves of $J \otimes L$. Also, the $\gamma$-modifications $J^\gamma$ are functorial on $J$.

2.5. (Torsion-free rank-1 sheaves.) Let $J$ be an $S$-flat coherent sheaf on $X$. We say that $J$ is torsion-free on $X/S$ if the associated components of $J|_{X_s}$ are components of $X_s$, or equivalently, if the natural map $J|_{X_s} \to J_{X_s}$ is a bijection. We say that $J$ is of rank 1 on $X/S$ if $(J|_{X_s})_{\xi_i} \cong O_{X_s, \xi_i}$ for each $i = 1, \ldots, n$.

Proposition 2.6. Let $J$ be a torsion-free sheaf on $X/S$, and $Y$ a primary subscheme of $X_s$. Set $Z := Y^c$. Then the following four statements hold.

1. $\pi J = J(-X_s)$ and the natural maps

$$J|_{X_s} \to J_{X_s} \quad \text{and} \quad J \to \pi J$$

are isomorphisms.

2. $J(-Y)$ is torsion-free on $X/S$.

3. The natural maps $J(-Y) \to J$ and $J \to J(-Y)$ are injective and induce short exact sequences:

$$0 \to J(-Y)_Z \to J|_{X_s} \to J_Y \to 0,$$

$$0 \to J_Y \to J(-Y)|_{X_s} \to J(-Y)_Z \to 0.$$

4. If $J$ is of rank 1, so is $J(-Y)$.

Proof. The first map, $J|_{X_s} \to J_{X_s}$, is an isomorphism because the associated components of $J|_{X_s}$ are components of $X_s$. Clearly, it follows that $J(-X_s) = \pi J$. In addition, since $J$ is $S$-flat, the multiplication-by-$\pi$ map $J \to \pi J$ is an isomorphism.

As for the second statement, since $S$ is the spectrum of a discrete valuation ring, and $J$ is $S$-flat, also its subsheaf $J(-Y)$ is $S$-flat. In addition, since the multiplication-by-$\pi$ bijection $J \to \pi J$ carries $J(-Y)$ onto $(\pi J)(-Y)$, by functoriality, we have

$$J(-Y)(-X_s) = J(-X_s)(-Y) = (\pi J)(-Y) = \pi J(-Y),$$

and thus the natural map $J(-Y)|_{X_s} \to J(-Y)_{X_s}$ is an isomorphism. So $J(-Y)$ is torsion-free.

Consider now the third statement. First, recall that the natural map $J(-Y) \to J$ is an inclusion, whence injective. Second, the natural
map \( \mathcal{J} \to \mathcal{J}(−Y) \) is injective if and only if the natural map \( \mathcal{J} \to \pi \mathcal{J} \)
is an isomorphism, and this is the case by the first statement.

As for the exact sequences, the first is obtained from that in Proposition 2.3 by setting \( Y_1 := \emptyset, Y_2 := Y \) and \( Y_3 := Z \), and recalling from the first statement that \( \mathcal{J}|_{X_s} = \mathcal{J}_{X_s} \).

The second is also obtained from that in Proposition 2.3, this time by setting \( Y_1 := Y, Y_2 := Z \) and \( Y_3 := Y \). However, we use the composition of isomorphisms,

\[
\mathcal{J} \to \pi \mathcal{J} \to \mathcal{J}(−X_s),
\]
to replace the leftmost sheaf \( \mathcal{J}(−X_s)_Y \) with \( \mathcal{J}_Y \), and we use that \( \mathcal{J}(−Y) \) is torsion-free, to replace \( \mathcal{J}(−Y)|_{X_s} \) with \( \mathcal{J}(−Y)|_{X_s} \).

The fourth statement follows from the two exact sequences of the third statement. Indeed, the first one yields \( (\mathcal{J}(−Y)|_{X_s})_{\xi_i} \cong (\mathcal{J}|_{X_s})_{\xi_i} \) for each \( \xi_i \in Z \), while the second one yields \( (\mathcal{J}|_{X_s})_{\xi_i} \cong (\mathcal{J}(−Y)|_{X_s})_{\xi_i} \) for each \( \xi_i \in Y \). Thus \( \mathcal{J}(−Y) \) is of rank 1 if and only if so is \( \mathcal{J} \).

\[\square\]

3. Limits of Cartier divisors

3.1. (Cycles.) Assume \( X_s \) is of pure dimension, say \( d \). Let \( \text{Cyc}(X_s) \) denote the free Abelian group generated by all integral closed subschemes of \( X_s \) of dimension \( d−1 \). We will simply say that an element of \( \text{Cyc}(X_s) \) is a cycle. A cycle is called effective if its expression as a \( \mathbb{Z} \)-linear combination of integral subschemes involves only nonnegative coefficients.

Let \( \text{Cyc}^+(X_s) \subset \text{Cyc}(X_s) \) denote the submonoid of effective cycles.

For any coherent sheaf \( \mathcal{F} \) on \( X_s \) with support of dimension at most \( d−1 \), let

\[
[\mathcal{F}] := \sum_{Y} \ell(\mathcal{F}_{\xi_Y})[Y] \in \text{Cyc}^+(X_s),
\]

where the sum runs over all irreducible components \( Y \) of dimension \( d−1 \) of the support of \( \mathcal{F} \), with \( \xi_Y \) denoting the generic point of \( Y \). Since localization is exact and length is additive, if

\[0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0\]
is a short exact sequence of coherent sheaves on \( X_s \) with support of dimension at most \( d−1 \), then \( [\mathcal{F}] = [\mathcal{G}] + [\mathcal{H}] \), a fact we refer to as the “additivity of the bracket.” Notice as well that, if \( \mathcal{L} \) is an invertible sheaf on \( X_s \), then \( [\mathcal{F} \otimes \mathcal{L}] = [\mathcal{F}] \).

If \( W \subset X_s \) is a closed subscheme of dimension at most \( d−1 \), let \( [W] := [\mathcal{O}_W] \). We call \( [W] \) the fundamental class of \( W \).

Lemma 3.2. Assume \( X_s \) is of pure dimension. Let \( \mathcal{F} \) be a coherent sheaf on \( X \) and \( \mathcal{G} \subset \mathcal{F} \) a coherent subsheaf. Let \( C_{i_1}, \ldots, C_{i_m} \) be a
collection of distinct irreducible primary subschemes of $X_s$. Suppose $F_{\xi_j} = G_{\xi_j}$ for each $j = 1, \ldots, m$. Set 

$$Z_j := C_{i_1} \cup \cdots \cup C_{i_j} \quad \text{and} \quad Z_j' := C_{i_{j+1}} \cup \cdots \cup C_{i_m}$$

for each $j = 0, \ldots, m$. Then

\begin{equation}
\frac{F_{Z_m}}{G_{Z_m}} = \sum_{j=0}^{m-1} \frac{F_{(-Z_j)C_{i_{j+1}}}}{G_{(-Z_j)C_{i_{j+1}}}}.
\end{equation}

\textbf{Proof.} For each $j = 0, \ldots, m - 1$, apply the third statement of Proposition 2.3 with $Y_1 := Z_j$, $Y_2 := C_{i_{j+1}}$ and $Y_3 := Z_j'$ to both $\mathcal{J} := \mathcal{F}$ and $\mathcal{J} := \mathcal{G}$. By functoriality, we get a natural map of short exact sequences:

\begin{align*}
0 \rightarrow \mathcal{G}(-Z_{j+1})Z_{j+1} & \longrightarrow \mathcal{G}(-Z_j)Z_j' \longrightarrow \mathcal{G}(-Z_j)C_{i_{j+1}} \rightarrow 0 \\
0 \rightarrow \mathcal{F}(-Z_{j+1})Z_{j+1} & \longrightarrow \mathcal{F}(-Z_j)Z_j' \longrightarrow \mathcal{F}(-Z_j)C_{i_{j+1}} \rightarrow 0.
\end{align*}

Always, the vertical map to the right is injective with cokernel supported in codimension 1 in $C_{i_{j+1}}$, because $G_{\xi_{i_{j+1}}} = F_{\xi_{i_{j+1}}}$. Thus, all the vertical maps are injective with cokernel supported in codimension 1 in $X_s$, and \((3.2.1)\) holds by the snake lemma. \qed

\textbf{3.3. (Limits of Cartier divisors.)} Assume that $f : X \rightarrow S$ is flat, or equivalently, that $X_s$ is a Cartier divisor of $X$. Assume as well that $X_s$ is of pure dimension. Since $X_s$ is a Cartier divisor of $X$, also $X$, and thus $X_\eta$, are of pure dimension. For each closed subscheme $D$ of $X$, let

$$\lim D := X_s \cap \overline{D \cap X_\eta^X}.$$ 

We call $\lim D$ the \textit{limit subscheme} of $D$.

Suppose $D \cap X_\eta$ is a Cartier divisor. Since $X_\eta$ is of pure dimension, $D \cap X_\eta$ is of pure codimension 1 in $X_\eta$. Thus, since $\overline{D \cap X_\eta^X}$ is $S$-flat, also $\lim D$ is of pure codimension 1 in $X_s$. Let $[\lim D]$ denote the associated cycle. We call $[\lim D]$ the \textit{limit cycle} of $D$.

The operation $D \mapsto [\lim D]$ induces a homomorphism of monoids from $\text{Div}^+(X)$ to $\text{Cyc}^+(X_s)$, as a consequence of the proposition below.

\textbf{Proposition 3.4.} Assume that $f : X \rightarrow S$ is flat and $X_s$ has pure dimension. Let $D_1$, $D_2$ and $D_3$ be $S$-flat closed subschemes of $X$ of pure codimension 1. Assume that $D_1 \cap X_\eta$ is a Cartier divisor of $X_\eta$ and

$$D_3 \cap X_\eta = (D_1 \cap X_\eta) + (D_2 \cap X_\eta).$$
Then
\[ [D_3 \cap X_s] = [D_1 \cap X_s] + [D_2 \cap X_s]. \]

(This proposition is a slight generalization of [6], Prop. 5.12, p. 49.)

**Proof.** For each \( i = 1, 2, 3 \), since \( D_i \) is \( S \)-flat of pure codimension 1, also \( D_i \cap X_s \) is of pure codimension 1 in \( X_s \). Again by flatness, \( D_i \) is the closure of \( D_i \cap X_s \). Thus, from the hypotheses, we get that \( D_3 = D_1 \cup D_2 \) set-theoretically, and hence
\[ D_3 \cap X_s = (D_1 \cap X_s) \cup (D_2 \cap X_s) \]
set-theoretically.

Let \( W \subseteq D_3 \cap X_s \) be an irreducible component. We need only show that the coefficient of \([W]\) in the expression for \([D_3 \cap X_s]\) is the sum of those for \([D_1 \cap X_s]\) and \([D_2 \cap X_s]\). Let \( \zeta \in W \) be the generic point, and set \( A := O_{X, \zeta} \). Let \( I_1, I_2 \) and \( I_3 \) be the respective ideals of \( D_1, D_2 \) and \( D_3 \) in \( A \).

Let \( Y_1, \ldots, Y_r \) be the irreducible components of \( D_3 \) containing \( W \). These correspond to the minimal prime ideals \( p_1, \ldots, p_r \) of \( A \) containing \( I_3 \). Notice that, for \( i = 1, 2 \), since \( D_i \) and \( D_3 \) have the same pure dimension, and \( D_3 \supseteq D_i \), the minimal prime ideals of \( A \) containing \( I_i \) are those \( p_j \) such that \( p_j \supseteq I_i \).

Since \( D_i \) is \( S \)-flat, \( \pi \) is a nonzero-divisor of \( A/I_i \) for each \( i = 1, 2, 3 \). In particular, \( \pi \not\in p_j \) for any \( j \). By [5], Lemme 21.10.17.7, p. 299 or [4], Lemma A.2.7, p. 410, for \( i = 1, 2, 3 \),

\[ (3.4.1) \quad \ell(A/(I_i + \pi A)) = \sum_{j=1}^r \ell(A_{p_j}/I_i A_{p_j}) \ell(A/(p_j + \pi A)). \]

The left-hand side of (3.4.1) is the coefficient of \([W]\) in the expression for \([D_i \cap X_s]\). Thus, we need only show that, for each \( j = 1, \ldots, r \),

\[ (3.4.2) \quad \ell(A_{p_j}/I_3 A_{p_j}) = \ell(A_{p_j}/I_1 A_{p_j}) + \ell(A_{p_j}/I_2 A_{p_j}). \]

Now, since \( \pi \not\in p_j \), we have \( I_i A_{p_j} = I_i A_{\pi} A_{p_j} \) for \( i = 1, 2, 3 \). By the hypotheses of the proposition, \( I_3 A_{\pi} = I_1 I_2 A_{\pi} \), and there is a nonzero-divisor \( f_j \in A_{p_j} \) such that \( I_1 A_{p_j} = f_j A_{p_j} \). Since \( f_j \) is not a zero-divisor, multiplication by \( f_j \) induces a short exact sequence:

\[
0 \rightarrow \frac{A_{p_j}}{I_2 A_{p_j}} \rightarrow \frac{A_{p_j}}{f_j I_2 A_{p_j}} \rightarrow \frac{A_{p_j}}{f_j A_{p_j}} \rightarrow 0.
\]

The additiveness of the length yields (3.4.2).  \( \square \)
4. The main theorem

**Theorem 4.1.** Assume that \( f : X \to S \) is flat and \( X_s \) has pure dimension. Let \( D \) be an effective Cartier divisor of \( X \). Suppose that, for each \( i = 1, \ldots, n \), there are effective Cartier divisors \( E_i \) and \( F_i \) of \( X \) and a nonnegative integer \( p_i \) such that \( \xi_i \not\in E_i + F_i \) and \( D + E_i = p_iX_s + F_i \).

Then
\[
\lim D = \sum_{i=1}^{n} \left( [F_i \cap C_i] - [E_i \cap C_i] \right).
\]

**Proof.** Let \( \gamma = \sum_i r_i C_i \) be an effective twister for which \( \mathcal{I}_D \subseteq \mathcal{O}_X^\gamma \). Suppose \( r := r_1 + \cdots + r_n \) is maximal for this property. (Such \( \gamma \) exists because \( \ell(\mathcal{O}_{D,\xi_i}) < \infty \) for each \( i = 1, \ldots, n \).) Then, since \( X \) is \( S \)-flat,
\[
\mathcal{I}_D(- \sum_i \sum_{j \neq i} r_i C_j) \subseteq \mathcal{O}_X(-r(C_1 + \cdots + C_n)) = \pi^r \mathcal{O}_X.
\]

Let
\[
\mathcal{J} := \left( \mathcal{I}_D \left( - \sum_{i=1}^{n} \sum_{j \neq i} r_i C_j \right) : \pi^r \mathcal{O}_X \right) \subseteq \mathcal{O}_X.
\]

Then \( \mathcal{J}^\gamma = \mathcal{I}_D \). Note that, since \( \mathcal{I}_D \) is invertible, and hence torsion-free of rank 1 on \( X/S \), also \( \mathcal{J} \) is torsion-free of rank 1 by Proposition 2.6.

In addition, \( \mathcal{J} \not\subseteq \mathcal{O}_X(-C_i) \) for every \( i = 1, \ldots, n \), by the maximality of \( r \). After reordering the components \( C_i \), we may assume that
\[
(4.1.1) \quad r_1 \geq r_2 \geq \cdots \geq r_{n-1} \geq r_n.
\]

If \( \mathcal{K} \) is a sheaf of ideals and \( G \) is a Cartier divisor of \( X \), then the multiplication map \( \mathcal{K} \otimes \mathcal{I}_G \to \mathcal{K}\mathcal{I}_G \) is an isomorphism. Thus, for each \( \mu \in \text{Twist}^+(f) \), since \( (\mathcal{K} \otimes \mathcal{I}_G)^\mu = \mathcal{K}^\mu \otimes \mathcal{I}_G \) as subsheaves of \( \mathcal{K} \otimes \mathcal{I}_G \), and since the multiplication map carries \( \mathcal{K}^\mu \otimes \mathcal{I}_G \) onto \( \mathcal{K}^\mu \mathcal{I}_G \) and \( (\mathcal{K} \otimes \mathcal{I}_G)^\mu \) onto \( (\mathcal{K}\mathcal{I}_G)^\mu \), we have
\[
(4.1.2) \quad \mathcal{K}^\mu \mathcal{I}_G = (\mathcal{K}\mathcal{I}_G)^\mu.
\]

In addition, if \( G \cap Y \) is Cartier for a certain primary subscheme \( Y \), then \( \mathcal{I}_G|_Y = \mathcal{I}_{G \cap Y}|_Y \), and it follows that
\[
(4.1.3) \quad (\mathcal{K}\mathcal{I}_G)|_Y = \mathcal{K}_Y \mathcal{I}_{G \cap Y}|_Y.
\]

By the hypothesis of the theorem, for each \( i = 1, \ldots, n \),
\[
\pi^p \mathcal{I}_{F_i} = \mathcal{I}_{p_iX_s + F_i} = \mathcal{I}_{D + E_i} = \mathcal{I}_{D}\mathcal{I}_{E_i} = \mathcal{J}^\gamma \mathcal{I}_{E_i}.
\]

Using Equation (4.1.2), we get that \( \pi^p \mathcal{I}_{F_i} = (\mathcal{J}\mathcal{I}_{E_i})^\gamma \). Now, since \( \xi_i \not\in E_i \) and \( \mathcal{J} \not\subseteq \mathcal{O}_X(-C_i) \), we get that \( r_i \) is the largest integer \( j \) such that \( (\mathcal{J}\mathcal{I}_{E_i})^\gamma \subseteq \mathcal{O}_X(-jC_i) \). On the other hand, since also \( \xi_i \not\in F_i \), we
get that \( p_i \) is the largest integer \( j \) such that \( \pi^{\mu}I_{F_i} \subseteq O_X(-jC_i) \). Since \( \pi^{\mu}I_{F_i} = (J\mathcal{I}_{E_i})^\gamma \), we have \( p_i = r_i \). Putting

\[
\alpha_i := \sum_{j<i} (r_j - r_i)C_j \quad \text{and} \quad \beta_i := \sum_{j>i} (r_i - r_j)C_j,
\]

we get

(4.1.4) \hspace{1cm} (J\mathcal{I}_{E_i})^{\alpha_i} = I_{F_i}^{\beta_i}.

Let

(4.1.5) \hspace{1cm} \gamma' := \sum_j r'_j C_j, \quad \text{where} \quad r'_j := r_1 - r_j \text{ for each } j = 1, \ldots, n,

and set

(4.1.6) \hspace{1cm} \delta_i := \sum_{j<i} r'_j C_j + \sum_{j \geq i} r'_j C_j, \quad \text{and} \quad \epsilon_i := \sum_{j<i} r'_j C_j + \sum_{j \geq i} r'_j C_j.

Then \( \alpha_i + \delta_i = r'_i (C_1 + \cdots + C_n) \) and \( \beta_i + \delta_i = \gamma' \). Since \( J\mathcal{I}_{E_i} \) is torsion-free of rank 1, it follows from (4.1.4) that

(4.1.7) \hspace{1cm} \pi^{r'_i}J\mathcal{I}_{E_i} = (J\mathcal{I}_{E_i})^{\alpha_i + \epsilon_i} = ((J\mathcal{I}_{E_i})^{\alpha_i})^{\delta_i} \quad \text{and} \quad (I_{F_i}^{\beta_i})^{\delta_i} = I_{F_i}^{\beta_i + \delta_i} = I_{F_i}'.

Notice, for later use, that \( \epsilon_i = \alpha_i + \gamma' \).

Since \( \xi_i \notin E_i + F_i \), and since \( C_i \) is not a summand of \( \alpha_i \) or \( \beta_i \), it follows from (4.1.4) that

\[
J_{\xi_i} = (J\mathcal{I}_{E_i})_{\xi_i} = (J\mathcal{I}_{E_i})^{\alpha_i}_{\xi_i} = (I_{F_i}^{\beta_i})_{\xi_i} = (I_{F_i})_{\xi_i} = O_X(\mathcal{I}_{\xi_i} - C_i).
\]

Since this holds for each \( i = 1, \ldots, n \), and since \( J \) is torsion-free on \( X/S \), it follows that the induced map \( J|_{X_{\eta}} \rightarrow O_X \) is injective. So the inclusion \( J \rightarrow O_X \) has flat cokernel. Since \( J^\gamma = I_D \), we have that \( J|_{X_{\eta}} \) is the sheaf of ideals of \( D \cap X_{\eta} \) in \( X_{\eta} \). Thus \( \lim D \) is the subscheme of \( X_{\eta} \) with ideal sheaf \( J|_{X_{\eta}} \), and hence

(4.1.8) \hspace{1cm} [\lim D] = \left[ \frac{O_{X_{\eta}}}{J|_{X_{\eta}}} \right].

Set \( Z_1 := \emptyset \) and, for each \( j = 2, \ldots, n + 1 \), put \( Z_j := C_1 \cup \cdots \cup C_{j-1} \). By Lemma 3.2

\[
\left[ \frac{O_{X_{\eta}}}{J|_{X_{\eta}}} \right] = \sum_{i=1}^{n} \left[ \frac{O_X(-Z_i)C_i}{J(-Z_i)C_i} \right],
\]

and thus, by (4.1.8) and the additiveness of the bracket,

(4.1.9) \hspace{1cm} [\lim D] = \sum_{i=1}^{n} \left( \left[ \frac{O_{C_i}}{J_{C_i}} \right] - \left[ \frac{O_{C_i}}{O_X(-Z_i)C_i} \right] + \left[ \frac{J_{C_i}}{J(-Z_i)C_i} \right] \right).
Thus (4.1.9) becomes

\[
\frac{\mathcal{J}_{C_1}}{\mathcal{J}(-Z_i)_{C_1}} = \frac{(\mathcal{J}_E)_{C_1}}{(\mathcal{J}_E)(-Z_i)_{C_1}} = \frac{(\mathcal{J}_E)_{C_1}}{(-Z_i)_{C_1}}.
\]

Using a similar reasoning, and the additiveness of the bracket,

\[
\frac{\mathcal{O}_{C_1}}{\mathcal{J}_{C_1}} = \frac{(\mathcal{I}_E)_{C_1}}{(\mathcal{J}_E)(-Z_i)_{C_1}} = \frac{\mathcal{O}_{C_1}}{(\mathcal{J}_E)(-Z_i)_{C_1}} - [E_i \cap C_i]
\]

\[
= \frac{((\pi')^*\mathcal{O}_X)_{C_1}}{(\pi')^*\mathcal{J}_E(-Z_i)_{C_1}} - [E_i \cap C_i] = \frac{((\pi')^*\mathcal{O}_X)_{C_1}}{((\pi')^*\mathcal{J}_E)_{C_1}} - [E_i \cap C_i]
\]

\[
= \frac{((\pi')^*\mathcal{O}_X)_{C_1}}{((\pi')^*\mathcal{J}_E)_{C_1}} - [E_i \cap C_i] = \frac{((\pi')^*\mathcal{O}_X)_{C_1}}{((\pi')^*\mathcal{J}_E)_{C_1}} - [E_i \cap C_i]
\]

\[
= \frac{((\pi')^*\mathcal{O}_X)_{C_1}}{((\pi')^*\mathcal{J}_E)_{C_1}} - [E_i \cap C_i] = \frac{((\pi')^*\mathcal{O}_X)_{C_1}}{((\pi')^*\mathcal{J}_E)_{C_1}} - [E_i \cap C_i]
\]

where we used that \(\epsilon_i = \alpha_i + \gamma\). Substituting in (4.1.10), we see that we need only prove the following claim.

**Claim:** Let \(\gamma := \sum_i r_i C_i\) be an effective twister such that (4.1.1) holds. Let \(\gamma'\) be as in (4.1.5). For each \(i = 1, \ldots, n\), let \(\epsilon_i \in \text{Twist}(f)\) be as in (4.1.6), and put

\[
\theta_i(\gamma) := \frac{((\pi')^*\mathcal{O}_X)_{C_1}}{((\pi')^*\mathcal{J}_E)_{C_1}} - \frac{\mathcal{O}_{C_1}}{(\mathcal{O}_X)_{C_1}} - \frac{\mathcal{O}_{C_1}}{(\mathcal{O}_X(-Z_i)_{C_1})} + \frac{((\pi')^*\mathcal{O}_X)_{C_1}}{(\mathcal{O}_X)_{C_1}}.
\]

Then \(\theta_1(\gamma) + \cdots + \theta_n(\gamma) = 0\).

We will prove the claim by induction on the sum \(r' := r'_1 + \cdots + r'_n\). If \(r' = 0\), then \(\gamma' = 0\) and \(\epsilon_i = 0\) for each \(i = 1, \ldots, n\). The claim is
trivial in this case, as the first and second summands of \( \theta_i(\gamma) \) are zero, and the third and fourth cancel each other, for each \( i = 1, \ldots, n \).

Now, suppose \( r' > 0 \). Then one of the inequalities in (4.1.1) is strict. Let \( \ell \) be an integer, between 2 and \( n \), such that \( r_{\ell-1} > r_\ell \). For each \( i = 1, \ldots, n \), let \( t_i := r_i \) if \( i \neq \ell \) and \( t_\ell := r_\ell + 1 \). Then \( t_1 \geq \cdots \geq t_n \) as well. Set \( \tau := \sum_i t_i C_i \) and \( \tau' := \sum_i t'_i C_i \), where \( t'_i := t_1 - t_i \) for each \( i = 1, \ldots, n \). Then \( t'_i = r'_i \) for every \( i \neq \ell \), but \( t'_\ell = r'_\ell - 1 \). So, by induction, \( \theta_1(\tau) + \cdots + \theta_n(\tau) = 0 \).

Set
\[
\rho_i := \sum_{j<i} t'_i C_j + \sum_{j>i} t'_i C_j \quad \text{for } i = 1, \ldots, n.
\]
Then \( \gamma' = \tau' + C_\ell \) and
\[
\epsilon_i = \begin{cases} 
\rho_i + C_\ell & \text{if } i < \ell, \\
\rho_\ell + C_1 + \cdots + C_\ell & \text{if } i = \ell, \\
\rho_i & \text{if } i > \ell.
\end{cases}
\]
Using the above formulas and the additiveness of the bracket, we get
\[
\theta_i(\gamma) - \theta_i(\tau) = \frac{\mathcal{O}_X'(\mathcal{Z}_i)C_i}{\mathcal{O}_X(-\mathcal{Z}_i)C_i} \quad \text{if } i \neq \ell,
\]
and
\[
\theta_\ell(\gamma) - \theta_\ell(\tau) = \left[ \frac{\pi'^\ell \mathcal{O}_X(\mathcal{C}_\ell)}{\mathcal{O}_X^\ell(\mathcal{C}_\ell)} \right] - \left[ \frac{\pi'^\ell \mathcal{O}_X(\mathcal{C}_\ell)}{\mathcal{O}_X^\ell(\mathcal{C}_\ell)} \right] - \left[ \frac{\mathcal{O}_X^\ell(\mathcal{C}_\ell)}{\mathcal{O}_X^\ell(\mathcal{C}_\ell)} \right] + \left[ \frac{\mathcal{O}_X^\ell(\mathcal{C}_\ell)}{\mathcal{O}_X^\ell(\mathcal{C}_\ell)} \right] + \left[ \frac{\mathcal{O}_X^\ell(\mathcal{C}_\ell)}{\mathcal{O}_X^\ell(\mathcal{C}_\ell)} \right] - \left[ \frac{\mathcal{O}_X^\ell(\mathcal{C}_\ell)}{\mathcal{O}_X^\ell(\mathcal{C}_\ell)} \right] - \left[ \frac{\mathcal{O}_X^\ell(\mathcal{C}_\ell)}{\mathcal{O}_X^\ell(\mathcal{C}_\ell)} \right] + \left[ \frac{\mathcal{O}_X^\ell(\mathcal{C}_\ell)}{\mathcal{O}_X^\ell(\mathcal{C}_\ell)} \right] - \left[ \frac{\mathcal{O}_X^\ell(\mathcal{C}_\ell)}{\mathcal{O}_X^\ell(\mathcal{C}_\ell)} \right].
\]
Thus, since \( \theta_1(\tau) + \cdots + \theta_n(\tau) = 0 \) by induction, we need only show that
\[
\sum_{i \neq \ell} \left[ \frac{\mathcal{O}_X'(\mathcal{Z}_i)C_i}{\mathcal{O}_X(-\mathcal{Z}_i)C_i} \right] = \left[ \frac{\mathcal{O}_X'(\mathcal{Z}_\ell)C_\ell}{\pi \mathcal{O}_X^\ell(\mathcal{C}_\ell)} \right] + \left[ \frac{\mathcal{O}_X^\ell(\mathcal{C}_\ell)}{\mathcal{O}_X'(\mathcal{Z}_\ell)C_\ell} \right].
\]
Now, applying Lemma 3.2 twice, we get that

$$\sum_{i \neq \ell} \frac{O_X'(-Z_i)C_i}{O_X'(-Z_i)C_i} = \sum_{i < \ell} \frac{O_X'(-Z_i)C_i}{O_X'(-Z_i)C_i} + \sum_{i > \ell} \frac{O_X'(-Z_i)C_i}{O_X'(-Z_i)C_i} = \left[ \frac{(O_X')_i}{(O_X')_i} \right] + \left[ \frac{O_X'(-Z_{i+1})z_{i+1}}{O_X'(-Z_{i+1})z_{i+1}} \right].$$

In addition, since $\gamma' = \tau' + C_\ell$, it follows from the second statement of Proposition 2.3 that

$$\left[ \frac{(O_X')_i}{(O_X')_i} \right] = \left[ \frac{O_X'(-Z_i)C_i}{O_X'(-Z_i)C_i} \right],$$

$$\left[ \frac{O_X'(-Z_{i+1})z_{i+1}}{O_X'(-Z_{i+1})z_{i+1}} \right] = \left[ \frac{O_X'(-Z_{i+1})C_i}{\pi O_X'(-Z_{i+1})C_i} \right].$$

Equation (4.1.11) follows. \qed

**Proposition 4.2.** Assume that $f: X \to S$ is projective and flat, and that $S$ is a $k$-scheme for an infinite field $k$. Then the following two statements hold:

1. For each $i = 1, \ldots, n$ there is $G_i \in \text{Div}^+(X)$ such that $\xi_j \notin G_i$ for $j \neq i$ and $G_i$ coincides with $C_i$ at $\xi_i$.
2. If $X_s$ is reduced, for each closed subscheme $D$ of $X$ such that

   $D \cap X_s$ is a Cartier divisor there are divisors $E_i \in \text{Div}^+(X)$ with $\xi_i \notin E_i$ and nonnegative integers $p_i$ such that $D + E_i \geq p_iX_s$ and $\xi_i \notin (D + E_i) - p_iX_s$ for $i = 1, \ldots, n$.

*Proof.* Let $R$ be the ring of regular functions of $S$. Since $S$ is a $k$-scheme, $R$ is a $k$-algebra. Since $f$ is projective, $f$ factors through an embedding $\iota: X \to P_R^m$, where $P_R^m := \text{Proj}(R[t_0, \ldots, t_m])$. Let $O_X(1)$ be the restriction to $X$ of the tautological ample sheaf of $P_R^m$. Then there is an integer $d > 0$ such that $H^1(P_R^m, I_X|_{P_R^m}(d)) = 0$ and the $d$-th twist $I_{C_i}(d)$ of the sheaf of ideals of $C_i$ in $X$ is globally generated for every $i = 1, \ldots, n$.

Let $\xi_{n+1}, \ldots, \xi_{n+r}$ denote the associated points of $X_s$. Then $I_{C_i}$ is invertible at $\xi_j$ for each $j = 1, \ldots, n + r$. Indeed, this is clearly so if $j \neq i$ because $\xi_j \notin C_i$. On the other hand, $(I_{C_i})_{\xi_i}$ is the ideal of $O_{X, \xi_i}$ generated by $\pi$, which is a nonzero-divisor because $f$ is flat, whence $I_{C_i}$ is also invertible at $\xi_i$.

Since $I_{C_i}(d)$ is globally generated, and since $H^1(P_R^m, I_X|_{P_R^m}(d)) = 0$, for each $j = 1, \ldots, n + r$ there exists a degree-$d$ homogeneous polynomial $P_j \in R[t_0, \ldots, t_m]$ generating $I_{C_i}(d)$ at $\xi_j$. Since $k$ is infinite, a
general linear combination \( Q_i := \sum_j c_j P_j \) with \( c_j \in k \) generates \( \mathcal{I}_{C_i}(d) \) at \( \xi_j \) for every \( j = 1, \ldots, n + r \). Let \( G_i \subseteq X \) be the subscheme cut out by \( Q_i = 0 \). Since \( G_i \) does not vanish on \( \xi_j \) for any \( j = n + 1, \ldots, n + r \), the subscheme \( G_i \) is a Cartier divisor. It is indeed the Cartier divisor required by the first statement.

As for the second statement, as \( X_s \) is reduced, for each \( j = 1, \ldots, n \) the point \( \xi_j \) lies on the nonsingular locus of \( X_s \), whence on the nonsingular locus of \( X \). So the local ring \( \mathcal{O}_{X, \xi_j} \) is a discrete valuation ring.

For each \( j = 1, \ldots, n \), consider the ideal of \( D \) at \( \xi_j \). If it were zero, then \( D \) would contain any irreducible closed subscheme of \( X \) containing \( \xi_j \). However, among those there is at least one irreducible component of \( X \), whose generic point lies over \( \eta \) by flatness. Thus \( D \) would contain an irreducible component of \( X_\eta \), contradicting the hypothesis that \( D \cap X_\eta \) is a Cartier divisor. So the ideal of \( D \) at \( \xi_j \) is nonzero. Since \( \mathcal{O}_{X, \xi_j} \) is a discrete valuation ring, this ideal is thus a power \( p_j \) of the maximal ideal. Let \( G_1, \ldots, G_n \) be the Cartier divisors of \( X \) claimed in the first statement. Set

\[
E_i := \sum_{p_j < p_i} (p_i - p_j) G_j.
\]

Then \( E_i \) is an effective Cartier divisor of \( X \) not containing \( \xi_i \). Also, \( D + E_i \supseteq p_i X_s \). Set \( F_i := D + E_i - p_i X_s \). Then \( F_i \) is a subscheme of \( X \) that does not contain \( \xi_i \).

\[\square\]

5. Examples

**Example 5.1.** (See [4], Ex. 11.3.2, p. 203, and the references listed there.) Let \( F, A_1, A_2, G_1 \) and \( G_2 \) be forms defining hypersurfaces. Assume \( FA_i \) and \( G_j \) have the same degree, for \( i = 1, 2 \) and \( j = 1, 2 \). Assume as well that

\[
gcd(FA_2, A_1) = 1 \quad \text{and} \quad gcd(A_1 G_2 - A_2 G_1, F) = 1.
\]

Consider the pencils \( FA_i + t G_j = 0 \) for \( i = 1, 2 \), and their intersection. The hypotheses (5.1.1) imply that the intersection is proper for a general \( t \). Indeed, if the intersection were not proper, then there would be a nonnegative integer \( m \) and forms \( L_0, \ldots, L_m \) of the same positive degree such that the polynomial \( L_0 + TL_1 + \cdots + T^m L_m \) divides \( FA_1 + TG_1 \) and \( FA_2 + TG_2 \). But then \( L_0 \) would divide \( FA_1, FA_2 \) and \( A_1 G_2 - A_2 G_1 \).

Let \( W \) denote the limit of the intersection of the pencils as \( t \) goes to 0. The intersection of the hypersurfaces \( FA_1 = 0 \) and \( FA_2 = 0 \) is not proper, and thus does not reflect \( W \) well. However, \( FA_2 = 0 \) cuts a
Cartier divisor on $A_1 = 0$. In addition,

$$A_1(FA_2 + tG_2) = A_2(FA_1 + tG_1) + t(A_1G_2 - A_2G_1),$$

and $A_1G_2 - A_2G_1 = 0$ cuts a Cartier divisor on $F = 0$. Thus, by Theorem 4.1

$$[W] = [FA_2, A_1 = 0] + [A_1G_2 - A_2G_1, F = 0] - [A_1, F = 0]$$

$$= [A_2, A_1 = 0] + [A_1G_2 - A_2G_1, F = 0].$$

**Example 5.2.** (See [4], Ex. 11.3.3, p. 203.) Consider the families of plane curves parameterized by $t$:

$$x^2y - tz^3 = 0 \quad \text{and} \quad (x - t^2y)(y^2 - t^2x^2) = 0.$$ 

It is easy to compute the intersection of the above curves for general $t$, as the second curve is a union of lines. Letting $\beta$ be a primitive cubic root of unity, we see that the intersection is reduced and consists of the nine points:

$$(t^2 : 1 : t\beta^j), \quad (1 : t : \beta^j) \quad \text{and} \quad (1 : -t : -\beta^j) \quad \text{for} \quad j = 0, 1, 2.$$ 

As $t$ goes to 0, the $(t^2 : 1 : t\beta^j)$ approach $(0 : 1 : 0)$, while the remainder approach the six points $(1 : 0 : \pm\beta^j)$.

To compute these limits using Theorem 4.1, we first use Proposition 5.3 to reduce the problem to that of computing the limits of the Cartier divisors cut on $x^2y - tz^3 = 0$ by $x - t^2y = 0$ and by the lines $y \pm tx$. Call $D$ the first limit and $D_\pm$ the last two limits.

We will actually use Theorem 4.1 to compute $2[D]$, and then use Proposition 5.4 to get $[D]$. First, $x^2 = 0$ cuts a Cartier divisor on $y = 0$. Also,

$$y(x - t^2y)^2 \equiv (x^2y - tz^3) + tz^3 \mod t^2,$$

and $z^3 = 0$ cuts a Cartier divisor on $x^2 = 0$. Thus, by Theorem 4.1

$$2[D] = [x^2, y = 0] + [z^3, x^2 = 0] - [y, x^2 = 0] = 6[z, x = 0].$$

So $[D] = 3[(0 : 1 : 0)]$.

As for $D_\pm$, first $y = 0$ cuts out a Cartier divisor on $x^2 = 0$. Also,

$$x^2(y \pm tx) = (x^2y - tz^3) + t(z^3 \pm x^3),$$

and $z^3 \pm x^3$ cuts a Cartier divisor on $y = 0$. Thus, by Theorem 4.1

$$[D_\pm] = [y, x^2 = 0] + [z^3 \pm x^3, y = 0] - [x^2, y = 0]$$

$$= \sum_{j=0}^{2}[(1 : 0 : \mp\beta^j)].$$
Example 5.3. If a plane curve is smooth, its flexes are cut out by the Hessian, the determinant of the symmetric matrix of second-order partial derivatives of the form defining the curve. However, if the curve has a linear or a multiple component, the Hessian vanishes completely on that component. What are the possible limits of flexes on the curve if the curve has such components?

The above question, considered in [2], p. 151, will also be considered in more detail in [3]. Here we will just consider the simple case where the curve is reduced with just two components, and just one of them is linear, and where the curve is deformed in first order along a general direction.

Let $k$ be an algebraically closed field of characteristic zero. For each $P \in k[[t]][x, y, z]$, define the derivation $D := \partial_y(P)\partial_x - \partial_x(P)\partial_y$, where $\partial_x$, $\partial_y$ and $\partial_z$ are the canonical partial $k[[t]]$-derivations of $k[[t]][x, y, z]$. Notice that $D_P(P) = 0$. For short, let $P_x := \partial_x(P)$, $P_y := \partial_y(P)$ and $P_z := \partial_z(P)$. Define the Hessian determinant $H(P)$ and the Wronskian determinant $W(P)$:

$$H(P) := \begin{vmatrix} P_{x,x} & P_{x,y} & P_{x,z} \\ P_{y,x} & P_{y,y} & P_{y,z} \\ P_{z,x} & P_{z,y} & P_{z,z} \end{vmatrix}$$

and

$$W(P) := \begin{vmatrix} x & y & z \\ D_P(x) & D_P(y) & D_P(z) \\ D_P^2(x) & D_P^2(y) & D_P^2(z) \end{vmatrix}.$$

If $P$ is homogeneous of degree $p$, it follows from applying the Euler Formula twice that

$$z^3H(P) \equiv (p - 1)^2W(P) \mod P. \tag{5.3.1}$$

Let

$$F := xG - F_1t \in k[[t]][x, y, z],$$

where $G$ and $F_1$ are nonzero forms of degrees $d - 1$ and $d$, respectively, for an integer $d \geq 3$. Assume $G$ is irreducible and gcd($xz, G$) = 1. Then $D_F \equiv xD_G \mod (G, t)$, and hence $D_F(G) \equiv 0 \mod (G, t)$. So, using the multilinearity of the determinant, the product rule for derivations, and $(5.3.1)$ for $P := G$, we get

$$d - 1)^2W(F) \equiv (xz)^3H(G) \mod (G, t). \tag{5.3.2}$$

Since $G$ is irreducible and nonlinear, $H(G)$ cuts a divisor on the curve defined by $G$. Let $R_G$ denote the 0-cycle of $\mathbb{P}^2$ associated to this divisor. Since gcd($G, xz) = 1$, it follows from $(5.3.2)$ that also $W(F)_0$ cuts a divisor on the curve given by $G$, where $W(F)_0$ is the constant term of $W(F)$.

Since $D_F(F) = 0$, using the multilinearity of the determinant and the product rule for derivations, we get

$$G^2W(F) \equiv tW' \mod F, \tag{5.3.3}$$
where $W'$ is the Wronskian determinant:

$$W' := \begin{vmatrix} F_1 & G_y & G_z \\ D_F(F_1) & D_F(G_y) & D_F(G_z) \\ D^2_F(F_1) & D^2_F(G_y) & D^2_F(G_z) \end{vmatrix}.$$  

Now, $D_F \equiv -G\partial_y \mod (x,t)$, and so $D_F(x) \equiv 0 \mod (x,t)$. Thus, using the multilinearity of the determinant and the product rule for derivations, we get

$$(5.3.4) \quad W' \equiv -g^3w \mod (x,t),$$

where $g := G(0,y,z)$ and $w$ is the Wronskian determinant:

$$w := \begin{vmatrix} f' & g' & g'' \\ f'_{y,y} & g'_{y,y} & g''_{y,y} \\ f'_{y,z} & g'_{y,z} & g''_{y,z} \end{vmatrix},$$

with $f' := F_1(0,y,z)$, $g' := g y$ and $g'' := g z$. If $F_1$ is general enough — more precisely, if $F_1 \not\in (x,G)$ — then $w \neq 0$, and hence also $W' \not\equiv 0 \mod (x,t)$.

Now, since $f'$, $g'$ and $g''$ have degree $d$, applying Euler formula three times, we get

$$(5.3.5) \quad (d-1)^2dw = z^3h,$$

where $h$ is the “Hessian” determinant:

$$h := \begin{vmatrix} f'_{z,z} & g'_{z,z} & g''_{z,z} \\ f'_{y,z} & g'_{y,z} & g''_{y,z} \\ f'_{y,y} & g'_{y,y} & g''_{y,y} \end{vmatrix}.$$
where $W'_0$ is the constant term of $W'$. In addition, by (5.3.2), (5.3.4) and (5.3.5),

$$[W(F)_0 \cdot G] = 3[x \cdot G] + 3[z \cdot G] + R_G,$$

$$[W'_0 \cdot x] = 3[G \cdot x] + 3[z \cdot x] + R_x.$$

Thus,

$$R = R_G + R_x + 3(G \cdot x).$$

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