Classical dynamics of a moving mirror due to radiation pressure

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Abstract. We investigate the classical dynamics of a thin moving mirror with non-zero transparency interacting with a laser by means of radiation pressure. The thin moving mirror is modelled by a delta function and it is part of a one-dimensional cavity where the other mirror is perfect and fixed at a position. We use the exact modes of the system so that the transparency and the mirror-position dependent cavity resonance frequencies are built into them. We derive the radiation pressure force from a periodic potential with period half the wavelength of the field and this allows us to give direct physical interpretations of the dynamics of the system and to obtain approximate analytic solutions. We show that within the rotating-wave-approximation the movable mirror can exhibit either a bounded or unbounded motion governed by an approaching to and a withdrawing from resonant positions where the frequency of the field is similar to one of the cavity resonance frequencies.

1. INTRODUCTION

The area of optomechanics is concerned, broadly speaking, with coupling mechanical degrees of freedom with light. The mechanical degrees of freedom could be, for example, the motion of atoms or a movable mirror. The area has attracted much attention in recent years due to the potential applications and the technological advances. The possible applications are diverse and range from optical communications [1] to studies of both classical and quantum systems [2, 3]. Some of the investigations that stand out are the possibility to explore the transition from the quantum to the classical world [3], the preparation of non-classical states of the mechanical degrees of freedom [4], and investigations of classical non-linear and possibly chaotic dynamics [5]-[13]. On the other hand, one of the technological advances that stands out is that, in some optomechanical set-ups, the mechanical degrees of freedom have been cooled to the quantum ground-state [14, 15].

One of the paradigmatic models in optomechanics consists of a one-dimensional cavity with one movable mirror. The movable mirror is a mechanical oscillator and it is coupled to the electromagnetic field by two main forces: the radiation pressure force and the thermal or bolometric force. In the former the electromagnetic field pushes the mirror, while the latter consists in the absorption of light that distorts and displaces the mirror. The dynamics of this system and similar ones can be studied with both classical and quantum mechanics. In this article we are concerned with the classical dynamics. We refer the reader to [2]-[4] and [16]-[21] for studies of the quantum dynamics of similar systems.
Depending on the experimental set-up (for example, the materials composing the movable mirror) one of the two forces mentioned above can dominate and they can operate at different time scales [9, 10, 19]. Also, the radiation pressure and thermal forces can even be in opposite directions [19]. These two types of forces alone or combined give rise to complex classical non-linear dynamics that have been studied both theoretically and experimentally. For example, in the set-up analysed in [5, 6] the classical dynamics are dominated by the radiation pressure force, while in that of [7, 8] they are dominated by the thermal force. The case where both the radiation pressure and thermal forces are relevant has also been studied [9]-[12]. In particular, [7, 8] found that there is multi-stability, that is, the amplitude of the oscillations of the mirror settles into one of several attractors (see also [13] for this phenomenon), while [9, 10] found that chaotic attractors can appear.

In this article we consider a one-dimensional cavity composed of one fixed, perfect mirror and one thin, movable mirror with non-zero transparency. The movable mirror is modelled by a delta-function and interacts with the electromagnetic field of a monochromatic laser by means of radiation pressure. We start from approximate Maxwell-Newton equations valid when the velocity and acceleration of the movable mirror are small and we consider the exact modes of the complete system. The use of the exact modes allows us to derive the radiation pressure force from a periodic potential and to interpret the dynamics directly in terms of the mirror approaching or withdrawing from positions where the frequency of the field coincides with one of the cavity resonance frequencies. Moreover, we pay special attention to determining when the approximate Maxwell-Newton equations used to describe the system are valid.

The model described in the preceding paragraph but with a fixed very thin mirror is sometimes known as the Lang-Scull-Lamb or LSL model [22]-[29]. It was introduced to explain why the laser line is so narrow, to form a more satisfactory picture of the nature of a laser mode, and to account for radiation losses through one of the mirrors of the laser without the use of phenomenological models [22]-[29].

After this article was completed we learned about reference [30]. It uses a scattering approach to describe a mobile scatterer (a mirror or atom) interacting with the electromagnetic field. The scatterer has a dispersive dielectric constant and moves with constant velocity. Moreover, the friction force and the diffusion affecting the scatterer were derived with the rotating-wave-approximation.

The article is organized as follows. In Section II we establish the system and the model under study. In Section III we restrict to a single-mode electromagnetic field (associated with a laser) and derive the potential associated with the radiation pressure force. In Section IV we investigate the dynamics of the movable mirror. The conclusions are given in Section V.

2. THE MODEL

Consider a one dimensional cavity composed of two mirrors parallel to the yz-plane. One mirror is perfect and it is fixed at \( x = 0 \). The other one can move and its mid-point has position \( x = q(t) > 0 \) at time \( t \). When it is at rest, we assume that the movable mirror is a linear, isotropic, non-magnetizable, and non-conducting (it has no free charges or currents) dielectric of thickness \( \delta_{\text{thick}} \). Also, there is a classical electromagnetic field polarized along the z-axis and we use the Gaussian system of units to describe it. One can then work in the Coulomb gauge and derive both the electric and magnetic fields from a vector potential \( \mathbf{A}(x, t) \) of the form

\[
\mathbf{A}(x, t) = A_0(x, t)\mathbf{z}.
\]  

(1)

The electric and magnetic fields are given respectively by

\[
\mathbf{E}(x, t) = -\frac{1}{c} \frac{\partial A_0}{\partial t}(x, t)\mathbf{z}, \quad \mathbf{B}(x, t) = -\frac{\partial A_0}{\partial x}(x, t)\mathbf{y},
\]  

(2)
and the equation for the vector potential is [31]

\[
\frac{\partial^2 A_0}{\partial x^2}(x, t) = -\epsilon \frac{[x - q(t)]}{c^2} \frac{\partial^2 A_0}{\partial t^2}(x, t) \quad (x > 0, \; t \in \mathbb{R}).
\]  

(3)

The dielectric function associated with the movable mirror is \(\epsilon\).

The perfect and fixed mirror at \(x = 0\) imposes the condition \(E(x, t) = B(x, t) = 0\) for \(x < 0\). Using the well-known boundary conditions for the electromagnetic field and (2) one arrives at the following boundary condition for \(A_0(x, t)\):

\[
\lim_{x \to 0+} \frac{\partial A_0}{\partial x}(x, t) = 0 \quad (t \in \mathbb{R}).
\]

(4)

Reference [31] shows that (3) is obtained by starting from Maxwell’s equations in inertial reference frames in which the movable mirror is instantaneously at rest and then using instantaneous Lorentz transformations to change back to the reference frame where the perfect mirror is fixed at \(x = 0\). Moreover, terms of order \(\dot{q}(t)/c\), \(\ddot{q}(t)/(c\omega_0)\), and higher powers of them are neglected. Here \(c\) is the speed of light in vacuum and \(\omega_0\) is the characteristic frequency of the electromagnetic field. It follows that (3) is an accurate approximation to the exact equation governing the dynamics of the field if both the velocity and acceleration of the movable mirror are small or, more precisely, if

\[
\frac{|\dot{q}(t)|}{c} , \frac{|\ddot{q}(t)|}{c\omega_0} \ll 1 \quad (t \in \mathbb{R}).
\]

(5)

Note that these conditions are reasonable, since according to (3) one has that \(A_0(x, t)\) evolves as if the mirror were instantaneously fixed at \(x = q(t)\).

Reference [31] also shows that (3) and (4) are valid for general dielectric functions \(\epsilon[x - q(t)]\).

In all that follows we assume that the movable mirror is very thin, that is,

\[
\delta_{\text{thick}} \ll \lambda = \frac{2\pi c}{\omega_0}.
\]

(6)

Therefore, one can approximate \(\epsilon[x - q(t)]\) by a delta function:

\[
\epsilon[x - q(t)] = 1 + 4\pi \chi_0 \delta [x - q(t)].
\]

(7)

Here \(\chi_0\) is a quantity with units of length. We note that movable mirrors satisfying (6) have already been used experimentally [32].

From (3), (7), and the law of conservation of linear momentum [33] for the complete system (electromagnetic field + dielectric mirror at \(x = q(t)\) + fixed perfect mirror at \(x = 0\)) one obtains that the equation governing the dynamics of the movable mirror is given by [31]

\[
M_0\dot{\ddot{q}}(t) = \lim_{\eta \to 0+} - \frac{1}{8\pi} \left\{ \left[ \frac{\partial A_0}{\partial x}(q(t) + \eta, t) \right]^2 - \left[ \frac{\partial A_0}{\partial x}(q(t) - \eta, t) \right]^2 \right\} \quad (t \in \mathbb{R}).
\]

(8)

Here \(M_0\) is the mass per unit area of the movable mirror. Note that, in accordance with (3), the force on the right-hand side of (8) corresponds to the radiation pressure exerted by the field on the mirror as if the mirror were instantaneously fixed. Also, the right-hand side of (8) is the difference of the radiation pressure exerted by the field outside the cavity minus the radiation pressure exerted by the field inside the cavity.

Since we have neglected terms of order \(\dot{q}(t)/c\), \(\ddot{q}(t)/(c\omega_0)\), and higher powers of them, we are only considering the leading term in the force affecting the movable mirror. We now comment on
what processes are not taken into account because of this approximation. The first corrections to the force on the right-hand side of (8) are a friction force proportional to $\dot{q}(t)/c$ [30, 31] and an acceleration-dependent force proportional to $\ddot{q}(t)/(c\omega_0)$ [31]. The former leads to dissipation in the motion of the movable mirror, while the latter modifies the mass of the mirror. In addition, field amplitudes corresponding to different wave-numbers are mixed if terms proportional to $\dot{q}(t)/c$ are not neglected [30, 31]. In our case the aforementioned processes are small because we demand that conditions (5) are always satisfied.

We now introduce the exact modes of the system for fixed $q(t)$.

2.1. Modes for fixed $q(t)$

In this subsection we assume that the movable mirror is fixed at $q > 0$, that is, $q(t) = q > 0$ for all $t \in \mathbb{R}$. To calculate the modes associated with (3), (4), and (7) we consider the function

$$A_{0,k}(x,t) = \text{Re} \left[ V_k(x) e^{-i\omega t} \right], \quad (9)$$

with $\omega = ck$, $k > 0$, and Re the real part of a complex number.

Substituting (9) in (4) one immediately obtains the following boundary condition:

$$V_k(0^+) = 0, \quad (10)$$

with

$$f(y^+) = \lim_{x \to y^+} f(x), \quad (11)$$

for any function $f(x)$ of a real variable $x$.

If one substitutes (9) in (3) one obtains the following differential equation:

$$\frac{d^2}{dx^2} V_k(x) + k^2 \epsilon(x - q)V_k(x) = 0 \quad (x > 0). \quad (12)$$

It follows from (7) (recall that we are considering $q(t) = q$) that (12) is equivalent to the following problem:

$$\frac{d^2}{dx^2} V_k(x) + k^2 \epsilon(x - q)V_k(x) = 0 \quad (x > 0, x \neq q),$$

$$V_k(q^+) = V_k(q^-),$$

$$\frac{dV_k}{dx}(q^+) - \frac{dV_k}{dx}(q^-) = -4\pi \chi_0 k^2 V_k(q). \quad (13)$$

Reference [24] (see also [34]) solved (10) and (13) for a coordinate system where the dielectric is fixed at $x = 0$, while the perfect mirror is fixed at $x = q > 0$. One adapts the results to our coordinate system by making the translation and reflection in which $x$ is replaced by $(q - x)$. One then concludes that they are given by

$$V_k(x, q) = \begin{cases} L_k(q) \sin(kx) & \text{if } 0 \leq x \leq q, \\ \sqrt{\frac{2}{\pi}} \sin \left[ k(x - q) + \delta_k(q) \right] & \text{if } x > q. \end{cases} \quad (14)$$

for $k > 0$. Here one has

$$L_k(q) = \sqrt{\frac{2}{\pi}} \left[ 1 + (4\pi \chi_0 k)^2 \sin^2(kq) - (4\pi \chi_0 k) \sin(2kq) \right]^{-1/2}, \quad (15)$$
and
\[ \sin [\delta_k(q)] = \sqrt{\frac{\pi}{2} L_k(q) \sin (kq)} , \]
\[ \cos [\delta_k(q)] = \sqrt{\frac{\pi}{2} L_k(q) \left[ \cos (kq) - (4\pi \chi_0 k) \sin (kq) \right]} . \]  \hspace{1cm} (16)

Observe that \( V_k(x,q) \) are real valued functions.

We want to use the modes \( V_k(x,q) \) to represent the plane wave of a laser approaching the movable mirror from the right. In order to do this we must eliminate the mirror position dependence in \( V_k(x,q) \) in the second line of (14). To accomplish this we introduce a phase shift as follows:
\[ \tilde{V}_k(x,q) = e^{i(\delta_k(q)-kq)} V_k(x,q) = \begin{cases} 
  \frac{i}{2} L_k(q) e^{i[\delta_k(q)-kq]} (e^{-ikx} - e^{ikx}) & \text{if } 0 \leq x \leq q , \\
  \frac{i}{2} \sqrt{\frac{\pi}{2}} \left( e^{-ikx} - e^{ikx} e^{2i[\delta_k(q)-kq]} \right) & \text{if } x > q .
\end{cases} \]  \hspace{1cm} (17)

Notice that (17) expresses the modes in terms of incoming and outgoing waves. A plane wave \( e^{-ikx} \) (which could be associated with a laser) with amplitude 1 comes from \( +\infty \), travels to the left, and is incident on the mirror at \( x = q > 0 \). There is a reflected plane wave \( e^{ikx} \) travelling to the right and with amplitude given by the scattering matrix \( S(k) = e^{2i[\delta_k(q)-kq]} \) with \( 2[\delta_k(q)-kq] \) the phase shift. Also, there is a standing wave \( (e^{-ikx} - e^{ikx}) \) in the region \( 0 \leq x \leq q \) produced by the plane wave coming from \( +\infty \) and the perfect mirror at \( x = 0 \). Moreover, \( L_k(q) \) is a normalization factor. The factor \( i \) has been left in (17) for two reasons. First, \( \tilde{V}_k(x,q) \) reduces to \( (2/\pi)^{1/2} \sin(kx) \) when \( \chi_0 \to 0 \) (that is, there is no movable mirror) and second, (17) is \( \sqrt{2/\pi} \) times the standard physical solution in scattering in the half-line \( (0, +\infty) \) with a Dirichlet boundary condition at \( x = 0 \). The factor \( \sqrt{2/\pi} \) is necessary for the orthogonality relation below.

The modes (14) form an orthogonal set of generalized eigenfunctions [24, 34]. Adapting those relations for the \( \tilde{V}_k(x,q) \) one obtains
\[ \delta(k-k') = \int_0^{+\infty} dx \, \epsilon(x-q) \tilde{V}_k(x,q)^* \tilde{V}_{k'}(x,q) , \quad f(x) = \int_0^{+\infty} dk \, f_k(q) \tilde{V}_k(x,q) , \]  \hspace{1cm} (18)
with \( f(x) \in L^2[0, +\infty), \, k, k' > 0 \), and the kth-mode \( f_k(q) \) given by
\[ f_k(q) = \int_0^{+\infty} dx \, \epsilon(x-q) \tilde{V}_k(x,q)^* f(x) . \]  \hspace{1cm} (19)

The transmissivity \( T \) of the dielectric is also calculated in [24]. It is given by
\[ T = \left[ 1 + \left( \frac{4\pi \chi_0 k}{2} \right)^2 \right]^{-1} . \]  \hspace{1cm} (20)
for each fixed \( k > 0 \). Therefore, the transparency of the dielectric is small if \( T \ll 1 \) or, equivalently, \( 4\pi \chi_0 k \gg 1 \).

We now list some properties of the function \( L_k(q) \) for \( k > 0 \) fixed. For their proof we refer the reader to [35]. In all that follows \( \mathbb{Z}^+ \) denotes the set of non-negative integers.

(i) \( L_k(q) \) is maximized for a discrete set of values \( q_{2n} \) of \( q \) with \( n \in \mathbb{Z}^+ \). If \( 4\pi \chi_0 k \gtrsim 5 \), then one has to good approximation
\[ kq_{2n} \simeq n\pi + \frac{1}{4\pi \chi_0 k} , \quad L_k(q_{2n}) \simeq \sqrt{\frac{2}{\pi}} (4\pi \chi_0 k) . \]  \hspace{1cm} (21)
(ii) $L_k(q)$ is minimized for a discrete set of values $q_{2n+1}$ of $q$ with $n \in \mathbb{Z}^+$. If $4\pi \chi_0 k \gtrsim 5$, then one has to good approximation

$$kq_{2n+1} \simeq \left(n + \frac{1}{2}\right)\pi + \frac{1}{4\pi \chi_0 k}, \quad L_k(q_{2n+1}) \simeq \sqrt{\frac{2}{\pi}} \left(\frac{1}{4\pi \chi_0 k}\right).$$

(22)

(iii) $q_{2n}$ tends to the values $n\pi/k$ corresponding to the case where the movable mirror is perfect [36] as the transparency tends to zero:

$$kq_{2n} \to n\pi \quad \text{as} \quad 4\pi \chi_0 k \to +\infty.$$  

(23)

(iv) $L_k(q)^2$ can be approximated by a Lorentzian if the transparency of the movable mirror is small and one restricts $q$ to an interval around $q_{2n}$ whose endpoints are not near the minimizers $q_{2n\pm 1}$, that is,

$$L_k(q)^2 \simeq \left(\frac{2}{\pi \xi^2}\right) \frac{1}{k^2(q - q_{2n})^2 + \frac{1}{\xi^2}}$$

if $k|q - q_{2n}| \ll 3/2$ and $2/\xi \equiv 2/(4\pi \chi_0 k) \ll 1$.

(v) For fixed $k > 0$, the frequency $\omega = ck$ coincides with one of the cavity resonance frequencies if and only if $q \simeq q_{2n}$ for some $n \in \mathbb{Z}^+$. If $k|q - q_{2n}| \ll 3/2$ and $2/\xi \equiv 2/(4\pi \chi_0 k) \ll 1$, then it follows from (24) that $\omega = ck$ coincides with one of the cavity resonance frequencies if and only if $k|q - q_{2n}| \leq 1/\xi^2$. Therefore, the half-width-at-half-maximum (HWHM) of the resonant position $kq_{2n}$ is $\xi^{-2}$ if $k|q - q_{2n}| \ll 3/2$ and $2/\xi \equiv 2/(4\pi \chi_0 k) \ll 1$.

Figure 1a illustrates $L_k(q)$ as a function of $q$. The Lorentzian approximation to $L_k(q)^2$ given in (24) is illustrated in figure 1b for the interval $[q_{2n} - 1/(k\xi), q_{2n} + 1/(k\xi)]$ with $n = 4$ and $\xi = 4\pi \chi_0 k$. Notice that the agreement between both is good.
2.2. Expansion in terms of the instantaneous modes

We now use the instantaneous modes of the system, that is, we consider the exact modes introduced in the previous subsection with $q(t)$ replacing $q$. One can then expand $A_0(x,t)$ in terms of these using (18) and (19) as follows:

$$A_0(x,t) = \int_0^{+\infty} dk \, Q_k[t,q(t)] \tilde{V}_k[x,q(t)],$$

(25)

with

$$Q_k[t,q(t)] = \int_0^{+\infty} dx \, \epsilon[x-q(t)] \tilde{V}_k[x,q(t)]^* A_0(x,t).$$

(26)

To obtain the equations governing the evolution of the modes $Q_k[t,q(t)]$ first substitute (25) into (3). Then neglect terms proportional to $\dot{q}(t)/c$, $[\dot{q}(t)/c]^2$, and $\ddot{q}(t)/(\omega_0)$ ($\omega_0$ the characteristic frequency of the field) and use the orthonormalization relation in (18). One obtains

$$0 = \frac{d^2}{dt^2} Q_k[t,q(t)] + \omega_k^2 Q_k[t,q(t)],$$

(27)

with $\omega_k = ck$, $k > 0$, and $t \in \mathbb{R}$. These are harmonic oscillator equations and their solution is given by

$$Q_k[t,q(t)] = g(k)e^{-i\omega kt} + g(k)^* e^{i\omega kt} e^{-i2(\delta_k[q(t)]-kq(t))},$$

(28)

for $k > 0$ and $t \in \mathbb{R}$. Here we have used that $A_0(x,t)$ must be a real quantity.

Recall that we have neglected terms proportional to $\dot{q}(t)/c$, $[\dot{q}(t)/c]^2$, and $\ddot{q}(t)/(\omega_0)$ to obtain (27). We have done this because terms of these orders were neglected to obtain (3). Furthermore, only by dropping these does one arrive to (27) and recover the physical situation envisioned in (3): the field evolves as if the movable mirror were fixed.

3. A SINGLE-MODE FIELD

Consider a monochromatic laser on the far right. It is turned on and the plane wave associated with it propagates to the left. This plane wave is reflected partially by the movable mirror at $q(t)$ and it is completely reflected by the perfect mirror located at $x = 0$. After a transient, a standing wave defined by one of the modes in (17) is approximately formed. Therefore, one approximately has a single-mode field that is exerting radiation pressure on the movable mirror and at the same time it is driving the electromagnetic field inside the cavity. We describe the dynamics of the system after the aforementioned transient.

Notice that the field does not decay because the laser is always turned on. Moreover, the restriction to a single-mode is possible because we have neglected terms of order $\dot{q}(t)/c$. Recall from the discussion at the end of Sec. II that field amplitudes corresponding to different wave-numbers are mixed if terms of order $\dot{q}(t)/c$ are not neglected. In our case this mixing is small because we demand that $|\dot{q}(t)/c| \ll 1$, see (5).

In the rest of the article we consider the physical situation described in the preceding paragraphs and we take the field as being composed of a single-mode, that is,

$$g(k) = g_0 e^{i\phi_0} \delta(k-k_0^0),$$

(29)

with $k_0^0, g_0 > 0$ and $\phi_0 \in \mathbb{R}$. In the following we take the initial phase equal to zero, that is, $\phi_0 = 0$. This corresponds to redefining the instant $t = 0$ by making a time translation.
Substituting (28) in (25) with \( g(k) \) given in (29), one obtains
\[
A_0(x, t) = 2g_0 \cos \left\{ \omega_0 t - \delta_{k_N} [g(t)] + k_N^0 q(t) \right\} \ V_{k_N} [x, q(t)] ,
\]
with \( \omega_0 \equiv ck_N^0 \).

To proceed we introduce a characteristic time and length of the system and then use these to express (8) in terms of non-dimensional quantities.

Define
\[
\Delta = \sqrt{\frac{g_0^2 \omega_0^3 \pi^2}{M_0 c^3}} .
\]
(31)

Then \( \Delta \) is a quantity with units 1/s and can be thought of as 1 over the characteristic time-scale of the movable mirror. Therefore, \( \tau = \Delta t \) is the non-dimensional time. Also, we measure lengths in units of \( 1/k_N^0 \), that is, we take \( 1/k_N^0 \) to be the characteristic length of the system. We note that the form of \( \Delta \) is dictated by the differential equation in (8), by (30), and by choosing the characteristic length to be \( 1/k_N^0 \).

Now define the non-dimensional quantities
\[
\xi = 4\pi \chi_0 k_N^0 , \quad \Omega = \frac{\omega_0}{\Delta} , \quad x(\tau) = k_N^0 q \left( \frac{\tau}{\Delta} \right) .
\]
(32)

Notice that the transmissivity \( T \) of the movable mirror given in (20) can be rewritten as
\[
T = \left[ 1 + \left( \frac{\xi}{2} \right)^2 \right]^{-1} .
\]
(33)

Here we used the definition of \( \xi \) in (32). It follows that the transparency will be small if \( \xi \) is large.

One can rewrite equation (8) for the movable mirror in terms of non-dimensional quantities using the expression for \( A_0(x, t) \) in (30) and the quantities defined in (32). One obtains that
\[
x''(\tau) = \left[ 1 + \cos \left\{ \Omega \tau + 2x(\tau) - 2\delta_{k_N} \left[ \frac{x(\tau)}{k_N^0} \right] \right\} \right] f_{\text{RWA}} [x(\tau)] .
\]
(34)

Here we have introduced the non-dimensional force
\[
f_{\text{RWA}}(x) = -\frac{1}{2} \left[ 1 - \frac{1}{1 + \xi^2 \sin^2(x) - \xi \sin(2x)} \right] .
\]
(35)

Also, derivatives with respect to \( \tau \) are denoted by a prime, while derivatives with respect to \( t \) are denoted by a dot.

If one uses the Taylor series of the function \( 1/(1 + y) \) with \( y = \xi^2 \sin^2(x) - \xi \sin(2x) \) and one neglects terms of order \( \xi^3 \) and higher powers, then (35) is the same force as in [30] in the case where the transparency of the movable mirror is large and terms of order \( \dot{q}(t)/c \) are neglected and the rotating-wave-approximation (RWA) is performed. We note that one must also change the coordinate system and Gaussian units to SI units for this comparison.

We now list a set properties of \( f_{\text{RWA}}(x) \) that will be used throughout the article. We refer the reader to [35] for their proof.

(i) \( f_{\text{RWA}}(x) \) is well defined for all \( x \in \mathbb{R} \).

This follows from the following identity valid for all \( x \in \mathbb{R} \):
\[
1 + \xi^2 \sin^2(x) - \xi \sin(2x) = \sin^2(x) + [\xi \sin(x) - \cos(x)]^2 > 0 .
\]
(36)
(ii) The maximizers of \( f_{\text{RWA}}(x) \) are denoted by \( x_{2n} \) with \( n \in \mathbb{Z}^+ \). If \( \xi \gtrsim 5 \), then one has to good approximation
\[
x_{2n} \simeq n\pi + \frac{1}{\xi}, \quad f_{\text{RWA}}(x_{2n}) \simeq \frac{\xi^2}{2} \quad (n \in \mathbb{Z}^+).
\]

(iii) The minimizers of \( f_{\text{RWA}}(x) \) are denoted by \( x_{2n+1} \) with \( n \in \mathbb{Z}^+ \). If \( \xi \gtrsim 5 \), then one has to good approximation
\[
x_{2n+1} \simeq \left(n + \frac{1}{2}\right)\pi + \frac{1}{\xi}, \quad f_{\text{RWA}}(x_{2n+1}) \simeq -\frac{1}{2} \quad (n \in \mathbb{Z}^+).
\]

(iv) The zeros of \( f_{\text{RWA}}(x) \) are denoted by \( x_n^* \) and \( x_n^{**} \) with \( n \in \mathbb{Z}^+ \). One has
\[
x_n^* = n\pi, \quad x_n^{**} \simeq n\pi + \frac{2}{\xi} \quad (n \in \mathbb{Z}^+).
\]

(v) One can approximate \( f_{\text{RWA}}(x) \) by a displaced Lorentzian
\[
f_{\text{RWA}}(x) \simeq \frac{-1}{2} + \frac{1}{2\xi^2} \left(\frac{1}{u^2 + \xi^{-2}}\right),
\]
if \( (2m - 1)\pi/2 \leq x \leq (2m + 1)\pi/2, \quad x \geq 0, \quad u = x - (m\pi + 1/\xi), \quad 1 \gg 3^{-1}(u + \xi^{-1})^2 \), and \( \xi \gg 1 \) with \( m \in \mathbb{Z}^+ \).

Before proceeding we delve a bit more on the non-dimensional quantities introduced in (32) and the equation (34) governing the dynamics of the movable mirror.

Recall that (3) and (8) are the equations governing the dynamics of the field and of the movable mirror and that these are accurate approximations to the exact equations governing the dynamics of the system when (5) are satisfied. These conditions say that the movable mirror has to move very slowly and that it has to be subject to very small accelerations so that the field evolves as if the movable mirror were instantaneously fixed. Therefore, the model of Sec. II assumes the existence of two very different time-scales: one very fast time-scale for the evolution of the field and one very slow time-scale for the evolution of the movable mirror. The former is given by \( 2\pi/\omega_0 \) while the latter is given by an unknown quantity \( 2P_{\text{mirror}} \sim 1/\Delta \) that satisfies \( 2\pi/\omega_0 \ll 2P_{\text{mirror}} \). In non-dimensional units the previous condition takes the form
\[
2\pi/\Omega = \pi\Delta/\omega_0 \ll P \equiv \Delta P_{\text{mirror}}.
\]
Notice that \( P \) is the non-dimensional time scale in which \( x(\tau) \) changes appreciably. From (41) it follows that the cosine term in (34) oscillates very rapidly and averages to zero. Hence, one can perform the rotating-wave-approximation (RWA) to obtain the approximate equation
\[
x''(\tau) = f_{\text{RWA}}[x(\tau)] \cdot
\]
We shall say that the system is in the \textit{RWA regime} or \textit{limit of low field intensity} whenever (41) is satisfied and the RWA is performed. The name \textit{RWA regime} comes from the fact that condition (41) was used to perform the RWA. On the other hand, the name \textit{limit of low field intensity} comes from the fact that \( 2\pi/\Omega \) is proportional to \( g_0 \), see (32).

From the discussion above it follows that the RWA regime is a natural setting for the model presented in Sec. II, since the movable mirror evolves on a time-scale much larger than that of the field. These two very different and separate time-scales are easily illustrated if one takes frequencies in the optical regime [35]: \( 2\pi/\omega_0 \sim 10^{-14} \) and \( P_{\text{mirror}} \sim 1/\Delta \gtrsim 10^{-6} \).

It is important to note that (41) may be satisfied, but the RWA may not be applicable. For the RWA to be applicable one needs \( \omega_0 \) several orders of magnitude larger than \( 1/P_{\text{mirror}} \sim \Delta \) (say, 4 orders of magnitude larger) as in the previous example [35].

In all that follows we restrict to the RWA regime. We now deduce the non-dimensional radiation pressure force \( f_{\text{RWA}}(x) \) from a potential.
3.1. The radiation pressure potential

From (34) it follows that the non-dimensional radiation pressure force \( f_{\text{RWA}}(x) \) can be obtained from a non-dimensional radiation pressure potential \( V_{\text{RWA}}(x) \), that is,

\[
f_{\text{RWA}}(x) = -\frac{d}{dx} V_{\text{RWA}}(x). \tag{43}
\]

An analytic expression for \( V_{\text{RWA}}(x) \) can be obtained by integrating \( f_{\text{RWA}}(x) \) from \( x' = 0 \) to \( x' = x > 0 \). One obtains that

\[
V_{\text{RWA}}(x) = \frac{x}{2} - \frac{1}{2} \tan^{-1} \left( 1 + \xi^2 \right) \tan(x) - \frac{1}{2} \left[ \tan^{-1}(\xi) + m \pi \right], \tag{44}
\]

if \( (2m - 1)\pi/2 \leq x \leq (2m + 1)\pi/2 \) and \( x \geq 0 \) with \( m \in \mathbb{Z}^+ \). Here \( \tan^{-1}(\theta) \) is periodic of period \( \pi/2 \) for \( \theta \in \mathbb{R} \).

We now list properties of \( V_{\text{RWA}}(x) \) that will be used throughout the article. For their proof we refer the reader to [35].

(i) \( V_{\text{RWA}}(x) \) is a periodic function of period \( \pi \).

This follows from the piecewise definition of \( V_{\text{RWA}}(x) \) on intervals of length \( \pi \) and the fact that \( \tan(x) \) is periodic of period \( \pi \).

(ii) The maximizers of \( V_{\text{RWA}}(x) \) are given by \( x_n^* \) in (39). Moreover,

\[
V_{\text{RWA}}(x_n^*) = 0, \tag{45}
\]

and 0 is the absolute maximum value of \( V_{\text{RWA}}(x) \).

(iii) The minimizers of \( V_{\text{RWA}}(x) \) are given by \( x_n^{**} \) in (39) and \( V_{\text{RWA}}(x_n^{**}) \) is the absolute minimum value of \( V_{\text{RWA}}(x) \). Also, if \( \xi \gtrsim 5 \), then

\[
V_{\text{RWA}}(x_n^{**}) \simeq V_{\text{RWA}}(2/\xi) \simeq -\tan^{-1}(\xi). \tag{46}
\]

(iv) For all \( x \geq 0 \) one has

\[
-\pi/2 < V_{\text{RWA}}(x) \leq 0. \tag{47}
\]

Also, \( V_{\text{RWA}}(2/\xi) \to -\pi/2 \) as \( \xi \to +\infty \).

(v) All of the wells of \( V_{\text{RWA}}(x) \) have non-dimensional length \( \pi \), which corresponds to a length of half the wavelength of the field, that is, \( \lambda_0/2 = \pi/k_n^\xi \).

Figure 2 illustrates \( V_{\text{RWA}}(x) \) for \( \xi = 2, 5, \) and 50. Notice that the depth of the wells increases and that \( V_{\text{RWA}}(x) \) tends to a saw tooth wave as \( \xi \to +\infty \). Therefore, \( V_{\text{RWA}}(x) \) can be approximated for sufficiently large \( \xi \) (say, \( \xi \gtrsim 50 \)) as follows:

\[
V_{\text{RWA}}(x) \simeq V_{\text{RWA}}^{\text{approx}}(x) \equiv \begin{cases} 
0 & \text{if } n\pi \leq x \leq n\pi + \frac{1}{2}, \\
V_{\text{RWA}} \left( \frac{2}{\xi} \right) & \text{if } n\pi + \frac{1}{2} < x \leq n\pi + \frac{3}{2}, \\
V_{\text{RWA}} \left( \frac{2}{\xi} \right) + m_+ \left[ x - \left( n\pi + \frac{2}{\xi} \right) \right] & \text{if } n\pi + \frac{3}{2} \leq x \leq (n+1)\pi.
\end{cases} \tag{48}
\]

with \( n\pi \leq x \leq (n+1)\pi, \ n \in \mathbb{Z}^+ \), and

\[
m_+ = -\frac{V_{\text{RWA}}(2/\xi)}{\pi - 2/\xi}. \tag{49}
\]

Recall that \( x = (n\pi + 1/\xi) \) is (approximately) the maximizer \( x_{2n} \) of \( f_{\text{RWA}}(x) \), see (37).
Figure 2: (Color online) The figure shows $V_{\text{RWA}}(x)$ for $\xi = 2$ (blue-solid line), 5 (red-solid line), and 50 (black-dashed line).

Figure 3a shows a close-up of one of the wells of $V_{\text{RWA}}(x)$ and $V_{\text{RWA}}^{\text{approx}}(x)$ for $\xi = 5$, while figure 3b shows the relative error $|V_{\text{RWA}}(x) - V_{\text{RWA}}^{\text{approx}}(x)|/|V_{\text{RWA}}(x)|$ for $\xi = 50$. Notice that the approximation is quite good except at thin layers at the maximizers $x = 4\pi$ and $x = 5\pi$ of $V_{\text{RWA}}(x)$ and at the maximizer $x = (4\pi + 1/\xi)$ of $f_{\text{RWA}}(x)$. These layers appear because the approximation $V_{\text{RWA}}^{\text{approx}}(x)$ does not take into account the curvature of $V_{\text{RWA}}(x)$.

Using (43) and the approximation for $V_{\text{RWA}}(x)$ given in (48) one obtains for $\xi \gg 1$ that

$$f_{\text{RWA}}(x) \simeq \begin{cases} 0 & \text{if } n\pi < x < \left(n\pi + \frac{2}{\xi}\right) \text{ and } x \neq \left(n\pi + \frac{1}{\xi}\right), \\ -m_+ & \text{if } \left(n\pi + \frac{2}{\xi}\right) < x < (n+1)\pi. \end{cases} \quad (50)$$

It follows that, for sufficiently large $\xi$, $f_{\text{RWA}}(x)$ is approximately a non-dimensional piecewise constant force.

Figure 3: (Color online) Figure 3a shows a close up of one of the wells of $V_{\text{RWA}}(x)$ (red-solid line) and the approximate $V_{\text{RWA}}^{\text{approx}}(x)$ (blue-dashed line) for $\xi = 5$ and the region $[4\pi, 5\pi]$. Figure 3b illustrates the relative error $|V_{\text{RWA}}(x) - V_{\text{RWA}}^{\text{approx}}(x)|/|V_{\text{RWA}}(x)|$ for $\xi = 50$ (red-solid line). The outer figure shows the region $[4\pi + 2/\xi, 5\pi]$, while the inside figure shows a close-up of the interval $[4\pi, 4\pi + 2/\xi]$ and the vertical line $x = 4\pi + 1/\xi$ (black-dashed line) indicating the (approximate) position of the maximizer of $f_{\text{RWA}}(x)$. 
4. DYNAMICS IN THE RWA
We now analyse and solve to good approximation (42). First notice from (42) that the non-dimensional energy \( E[x(\tau), x'(\tau)] \) of the movable mirror is conserved. It is given by

\[
E[x(\tau), x'(\tau)] = \frac{1}{2} [x'(\tau)]^2 + V_{\text{RWA}}[x(\tau)],
\]

(51)

Figure 4 shows a contour plot of the energy as a function of \( x \) and \( x' \) for \( \xi = 2 \).
Since the mirror moves under the potential \( V_{\text{RWA}}(x) \) (see (45)-(47) and figure 2) and energy is conserved, one has the following facts [35]:

(i) The motion of the mirror is bounded to one of the potential wells if and only if the mirror has non-positive energy, that is, if and only if \( E(x, x') \leq 0 \).
(ii) If the motion of the mirror is bounded to one of the potential wells, then the movable mirror has a closed orbit in phase space, see figure 4. In each of these periodic trajectories the mirror takes on its maximum speed \( v_0 \) when it is located at one of the minimizers \( x_n^{**} \) of the potential \( V_{\text{RWA}}(x) \).
(iii) The maximizers \( x_n^* = n\pi \) of \( V_{\text{RWA}}(x) \) are points of unstable equilibrium.

![Figure 4](Color online) The figure shows a contour plot of the non-dimensional energy \( E(x, x') \) of the movable mirror as a function of \( x \) and \( x' \) for \( \xi = 2 \). The contours \( E(x, x') = \pm 0.6, \pm 0.4, \pm 0.2, 0 \) are shown.

The use of the exact instantaneous modes (14) of the system allows us to give a direct physical interpretation of the periodic bounded motion of the movable mirror in terms of an approaching and withdrawing from a position where the frequency of the field coincides with one of the cavity resonance frequencies. Assume that \( \xi \gtrsim 5 \) (that is, the transparency of the movable mirror is small), \( E[x(0), x'(0)] < 0 \) (that is, the movable mirror is bound to one of the potential wells), and \( q(0) = q_{2n} \) for some \( n \in \mathbb{Z}^+ \) (that is, the movable mirror is at a position where the frequency of the field \( \omega_0 \) coincides with one of the cavity resonance frequencies). Then the field inside the cavity is much larger than the field outside of it, see figures 5a and 5b. As a consequence, the radiation pressure exerted by the field inside the cavity is much larger than that exerted by the field outside of it and the mirror is pushed to the right (recall that \( f_{\text{RWA}}(x_{2n}) \approx \xi^2/2, \) see (37)). As the mirror withdraws from the resonant position \( q_{2n} \), the frequency of the field becomes very
different from any of the cavity resonance frequencies. Therefore, the field inside the cavity decreases (see figures 5c and 5d) and the radiation pressure exerted by the field outside of the cavity eventually surpasses that of the field inside. As a consequence the mirror slows down, stops, and starts moving to the left (recall that \( f_{RWA}(x_{2n+1}) \approx -1/2 \), see (38)). As the mirror now approaches the resonant position \( q_{2n} \), the frequency of the field becomes similar to one of the cavity resonance frequencies and the field inside the cavity increases. Hence, the radiation pressure exerted by the field inside the cavity eventually surpasses that of the field outside (see figures 5a and 5b and recall that \( f_{RWA}(x_{2n}) \approx \xi^2/2 \)). The mirror then slows down, stops, and starts moving to the right and the process starts over again.

Notice that the process described in the preceding paragraph can also be explained with the radiation pressure potential \( V_{RWA}(x) \), see figure (2) and recall from (37) and (39) that the resonant positions \( x_{2n} = k^\prime_n q_{2n} \) are located between consecutive maximizers \( x_n^* \) and minimizers \( x_n' \) of \( V_{RWA}(x) \). The potential also allows one to understand the unbounded motion of the mirror. Suppose that the movable mirror has positive energy so that its motion is not bounded to one of the potential wells. As the mirror starts moving (if it moves to the right (left), then see the contours with positive energy and \( x' > 0 \) \((x' < 0)\) in Fig. 4), it accelerates as it approaches a minimizer and it decelerates as it approaches a maximizer of \( V_{RWA}(x) \). The radiation pressure is never large enough so as to decelerate the mirror to a complete stop as in the case of the bounded motion.

Now assume that \( E(x,x') < 0 \) and that \( \xi \gg 1 \) (say, \( \xi \gtrsim 50 \)), that is, the mirror is confined to one of the potential wells and has small transparency. Then one can use \( V_{RWA}^{approx}(x) \) in (48) to solve approximately (42). Notice that the mirror cannot approach \( x = n\pi \) with this approximation. Therefore, \( V_{RWA}^{approx}(x) \) cannot be used to describe accurately the motion of the mirror in a small layer at \( E(x,x') = 0 \).

If the mirror is in the \( n \)-th well, then an approximate solution of (42) is given by [35]

\[
x_{RWA}(\tau) = \begin{cases} 
  \left(n\frac{\pi + \frac{2}{\xi}}{2}\right) + v_0(\tau - \tau_{2k}) - \frac{m_+}{2}(\tau - \tau_{2k})^2 & \text{if } \tau_{2k} \leq \tau \leq \tau_{2k+1}, \\
  \left(n\frac{\pi + \frac{2}{\xi}}{2}\right) - v_0(\tau - \tau_{2k+1}) & \text{if } \tau_{2k+1} \leq \tau \leq \tau_{2(k+1)}, \\
  \left(n\frac{\pi + \frac{1}{\xi}}{2}\right) + v_0(\tau - \tau_{2(k+1)}) & \text{if } \tau_{2(k+1)} \leq \tau \leq \tau_{2(k+1)}. 
\end{cases}
\]

(52)

Here \( k \in \mathbb{Z} \) and \( v_0 \) is such that \( x(\tau_0) = n\pi + 2/\xi \) and \( v_0 = (dx/d\tau)(\tau_0) > 0 \). Notice that \( v_0 \) is the maximum (non-dimensional) speed of the mirror. Given \( k \in \mathbb{Z} \) one has

\[
\tau_{2k+1} - \tau_{2k} = \frac{2v_0}{m_+}, \quad \tau_{2(k+1)} - \tau_1 = \tau_{2k+1} - \tau_{2k+1} = \frac{1}{\xi v_0}.
\]

(53)

It follows from (52) and (53) that the periodic trajectory has period \( P \) given by

\[
P = \frac{2v_0}{m_+} + \frac{2}{\xi v_0} \approx 4v_0 + \frac{2}{\xi v_0}.
\]

(54)

Notice that \( P \) depends on the maximum speed \( v_0 \) of the mirror. Moreover, its absolute minimum value \( P_{min} \) occurs for \( v_0 = \sqrt{m_+}/\xi \approx 1/\sqrt{2\xi} \) and is given by [35]

\[
P_{min} = \frac{4}{\sqrt{m_+\xi}} \approx 4\sqrt{\frac{2}{\xi}},
\]

(55)

since \( m_+ \approx 1/2 \) for \( \xi \gg 1 \), see (47) and (49).

Figure 6 illustrates the numerical solution \( x(\tau) \) (red-solid line) of (42) and the approximate solution \( x_{RWA}(\tau) \) (blue-dashed line) given in (52) for \( \xi = 20 \) and the initial conditions...
Figure 5: (Color online) Figures 5a and 5b illustrate $V_{kN}(x, q_{2n})$ as a function of $x$. Here the position $q_{2n}$ of the mirror is such that $\omega_0$ coincides with one of the cavity resonance frequencies. Figures 5c and 5d illustrate $V_{kN}(x, q_{2n+1})$ as a function of $x$. Here the position $q_{2n+1}$ of the mirror is such that $\omega_0$ is very different from all the cavity resonance frequencies. Figures 5a and 5c show the region inside the cavity, while figures 5b and 5d show the region outside of it. In all figures the vertical blue solid or dashed line shows the position of the movable mirror. Also, all figures have $k_N^0 = 100 \text{ m}^{-1}$, $n = 4$, and $\xi = 4\pi \chi_0 k_N^0 = 50$.

$x(0) = 4\pi + 1/\xi$ and $x'(0) = 0$. Therefore, the movable mirror starts from rest at a position where the frequency $\omega_0$ of the field coincides with one of the cavity resonance frequencies. Observe that the agreement between $x(\tau)$ and $x_{RWA}(\tau)$ is very good and that (54) gives a good approximation for the period, since one can show that $P \simeq 5$.

To end this section we mention that the initial velocity of the movable mirror has to be adjusted in order to apply (52): if $E \equiv E(4\pi + 1/\xi, 0)$ is the energy of the movable mirror, then according to the approximation in (48) the corresponding initial conditions for $x_{RWA}(\tau)$ are $x(0) = 4\pi + 1/\xi$ and $x'(0) = x'_0(0)$ with $x'_0(0) = \sqrt{2[E - V_{RWA}(2/\xi)]}$.

4.1. Validity of the model

We now establish the regime of validity of the model, that is, we determine sufficient conditions for (5) to be valid and we rewrite condition (41) for the RWA regime. In the following we drop the assumption that $\xi \gg 1$ and we do not restrict the motion to be bounded.

From (51) we know that energy is conserved, that is,

$$E[x(\tau), x'(\tau)] = E_0 \quad (\tau \in \mathbb{R}).$$

Using (56) to bound the velocity of the movable mirror, (42) to bound its acceleration, (37) to bound $f_{RWA}(x)$, (47) to bound $V_{RWA}(x)$, and (32) to relate $x(\tau)$ and its derivatives with $q(t)$ and
its derivatives, it is straightforward to show that

\[ 2\sqrt{2E_0 + \pi} \ll \Omega \implies \left| \frac{\dot{q}(t)}{c} \right| \ll 1 \quad (t \in \mathbb{R}), \tag{57} \]

and

\[ 4f_{\text{RWA}}(x_{2n}) \ll \Omega^2 \implies \left| \frac{\ddot{q}(t)}{c\omega_0} \right| \ll 1 \quad (t \in \mathbb{R}). \tag{58} \]

We now illustrate (57) and (58) using the typical values \( \xi = 6.4 \) (the reflectivity of the movable mirror is 0.91) and \( \Omega \geq 10^9 \) given in [35]. From (57) it is clear that \( \left| \frac{\dot{q}(t)}{c} \right| \ll 1 \) if the energy \( E_0 \) of the movable mirror is not too large (for example, \( E_0 \leq 0 \) for bounded motion). Using (37) to evaluate \( f_{\text{RWA}}(x_{2n}) \) it follows that \( 4f_{\text{RWA}}(x_{2n}) \simeq 81.92 \ll 10^{18} \leq \Omega^2 \). Then, from (58) one obtains that \( \left| \frac{\ddot{q}(t)}{c\omega_0} \right| \ll 1 \) for all \( t \in \mathbb{R} \). Observe that this last condition is valid for both bounded and unbounded motion. Therefore, the model presented in Sec.II is an accurate description of the physical system under study if these typical values are used.

If one considers \( E_0 \leq 0 \) (bounded motion) and \( \xi \gg 1 \) (small transparency of the movable mirror), then (using (37)) the results (57) and (58) can be reduced to the following:

\[ 2\xi^2 \ll \Omega^2 \implies \left| \frac{\ddot{q}(t)}{c\omega_0} \right| , \left| \frac{\dot{q}(t)}{c} \right| \ll 1 \quad (t \in \mathbb{R}). \tag{59} \]

Using the typical value \( \Omega \geq 10^9 \) given in [35] it follows that one only requires \( \xi \ll 10^8 \) for the model in Sec. II to be valid.

Finally, we rewrite condition (41) for the RWA regime and for bounded motion. In this case \( P \) in (41) can be taken to be the period of the periodic trajectory of the mirror. If in addition \( \xi \gg 1 \) (small transparency of the movable mirror), then (55) can be used to obtain a general sufficient condition for the RWA to be valid:

\[ \Omega \gg \frac{\pi}{2} \sqrt{m+\xi} \simeq \frac{\pi}{2} \sqrt{\frac{\xi}{2}}. \tag{60} \]

Comparing (60) with (59) it follows that the low intensity limit (or RWA regime) is a natural setting for the consideration of the model established in the first section in the case where the transparency of the movable mirror is small.

5. CONCLUSIONS
In this article we investigated the classical dynamics of a thin movable mirror with non-zero transparency as it interacts with a monochromatic laser by means of radiation pressure. The movable mirror is part of a one-dimensional cavity whose other mirror is perfect and fixed. The model used in the article describes the following physical situation: a single-mode laser on the far right is always turned on. The plane wave associated with the laser travels to the left, is partially reflected by the movable mirror at \( q(t) > 0 \), and it is completely reflected by the perfect, fixed mirror at \( x = 0 \). After a transient a standing wave is formed and the laser is responsible for maintaining it. We described the dynamics after the aforementioned transient.

Starting from approximate Maxwell-Newton equations describing the dynamics of the system when the movable mirror has small speed and is subject to small accelerations, we considered the exact instantaneous modes of the system and we established an effective equation governing the dynamics of the movable mirror. This allowed us to deduce the radiation pressure force from a periodic potential with period half the wavelength of the field.
The conditions under which the approximate equations describing the dynamics of the system were derived allowed us to identify two very different time-scales in the system: one very fast time-scale for the evolution of the field and another slow time-scale for the evolution of the movable mirror. This allowed us to perform the rotating-wave-approximation and, hence, to simplify the equation governing the dynamics of the mirror. Within this approximation the energy of the movable mirror is conserved. Combining this result with the radiation pressure potential we concluded that the motion of the movable mirror can be either bounded or unbounded. In the case of unbounded motion the mirror accelerates (decelerates) as it approaches a minimizer (maximizer) of the potential. In the case of bounded motion, the movable mirror has a periodic trajectory around one of the minimizers of the potential. The use of the exact instantaneous modes and of the potential also allowed us to interpret this periodic motion as follows. As the movable mirror approaches a position where the frequency of the field is similar to one of the cavity resonance frequencies, the field inside the cavity builds up and eventually exerts a radiation pressure that surpasses that exerted by the field outside the cavity. As a consequence, the movable mirror slows down, stops, and eventually starts moving to the right. As the movable mirror withdraws form the resonant position, the frequency of the field becomes very different from all cavity resonance frequencies and the field inside the cavity is depleted. As a result, the field outside the cavity eventually exerts a radiation pressure that exceeds that of the field inside the cavity. The movable mirror then slows down, stops, and eventually starts moving to the left. Then the mirror approaches the resonant position once more and the process repeats itself over and over giving rise to the periodic motion.

We note that the radiation pressure potential also allowed us to obtain accurate analytic approximations of the solutions of the equations governing the dynamics of the movable mirror in the case were its transparency is small. Moreover, we established that for some typical values both the rotating-wave-approximation and the approximate Maxwell-Newton equations are valid.

We note that our approach differs from that taken in most articles because the movable mirror can deviate far from an equilibrium position and we considered the exact instantaneous modes of the system. In other words, we did not divide the modes of the system into modes of the cavity and modes outside of it as if the cavity mirrors were perfect and then coupled them through a phenomenological interaction. This allowed us to incorporate directly into the modes the transparency of the movable mirror and the positions of the mirror where the frequency of
the field coincides with the cavity resonance frequencies. The result of this approach is that we were able to deduce the radiation pressure force from a potential and to give a direct physical interpretation of the dynamics of the system.

Reference [35] also investigates the case where the mirror is subject both to friction and to a harmonic oscillator potential.

It is important to note that we have only considered the leading order of the electromagnetic force acting on the movable mirror, that is, we neglected terms of order $\dot{q}(t)/c$, $\ddot{q}(t)/(c\omega_0)$, and higher powers of them. Here $c$ is the speed of light in vacuum and $\omega_0$ is the characteristic frequency of the field. These terms are responsible, for example, for a friction force arising from the interaction of the movable mirror with the electromagnetic field and for a mixing of field amplitudes corresponding to different wave-numbers [30, 31]. In our case these processes are small because we always demanded that $|\dot{q}(t)/c| \ll 1$ and $|\ddot{q}(t)/(c\omega_0)| \ll 1$. In future work we shall consider the terms we neglected.

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