Logical locality entails
frugal distributed computation over graphs

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Abstract

First-order logic is known to have limited expressive power over finite structures. It enjoys in particular the locality property, which states that first-order formulae cannot have a global view of a structure. This limitation ensures on their low sequential computational complexity. We show that the locality impacts as well on their distributed computational complexity. We use first-order formulae to describe the properties of finite connected graphs, which are the topology of communication networks, on which the first-order formulae are also evaluated. We show that over bounded degree networks and planar networks, first-order properties can be frugally evaluated, that is, with only a bounded number of messages, of size logarithmic in the number of nodes, sent over each link. Moreover, we show that the result carries over for the extension of first-order logic with unary counting.

1 Introduction

Logical formalisms have been widely used in many areas of computer science to provide high levels of abstraction, thus offering user-friendliness while increasing the ability to perform verification. In the field of databases, first-order logic constitutes the basis of relational query languages, which allow to write queries in a declarative manner, independently of the physical implementation. In this paper, we propose to use logical formalisms to express properties of the topology of communication networks, that can be verified in a distributed fashion over the networks themselves.

We focus on first-order logic over graphs. First-order logic has been shown to have limited expressive power over finite structures. In particular, it enjoys the locality property, which states that all first-order formulae are local [Gai82], in the sense that local areas of the graphs are sufficient to evaluate them.

First-order properties have been shown to be computable with very low complexity in both sequential and parallel models of computation. It was shown that first-order properties can be evaluated in linear time over classes of bounded degree graphs [See95] and over classes of locally tree-decomposable graphs [FG01]. These results follow from the locality of the logic. It was also

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1 Locally tree-decomposable graphs generalize bounded degree graphs, planar graphs, and graphs of bounded genus.
shown that they can be evaluated in constant time over Boolean circuits with unbounded fan-in \( \text{AC}^0 \) \cite{Imm89}. These bounds lead us to be optimistic on the complexity of the distributed evaluation of first-order properties.

We consider communication networks based on the message passing model \cite{AW04}, where nodes exchange messages with their neighbors. The properties to be evaluated concern the graph which forms the topology of the network, and whose knowledge is distributed over the nodes, who are only aware of their 1-hop neighbors. We thus focus on connected graphs.

In distributed computing, the ability to solve problems locally has attracted a strong interest since the seminal paper of Linial \cite{Lin92}. The ability to solve global problems in distributed systems, while performing as much as possible local computations, is of great interest in particular to ensure scalability. Moreover relying as much as possible on local information improves fault-tolerance. Finally, restricting the computation to local areas allows to optimize time and communication complexity.

Naor and Stockmeyer \cite{NS95} showed that there were non-trivial locally checkable labelings that are locally computable, while on the other hand lower-bounds have been exhibited, thus resulting in non-local computability results \cite{KMW04,KMW06}.

Different notions of local computation have been considered. The most widely accepted restricts the time of the computation to be constant, that is independent of the size of the network \cite{NS95}, while allowing messages of size \( O(\log n) \), where \( n \) is the size of the network. This condition is rather stringent. Naor and Stockmeyer \cite{NS95} show their result for a restricted class of graphs (eg bounded odd degree). Godard et al. used graph relabeling systems as the distributed computational model, defined local computations as graph relabeling systems with locally-generated local relabeling rules, and characterized the classes of graphs that are locally computable \cite{GMM04}.

Our initial motivation is to understand the impact of the logical locality on the distributed computation, and its relationship with local distributed computation. It is easy to verify though that there are simple properties (expressible in first-order logic) that cannot be computed locally. Consider for instance the property “There exist at least two distinct triangles”, which requires non-local communication to check the distinctness of the two triangles which may be far away from each other. Nevertheless, first-order properties do admit simple distributed computations.

We thus introduce frugal distributed computations. A distributed algorithm is frugal if during its computation only a bounded number of messages of size \( O(\log n) \) are sent over each link. If we restrict our attention to bounded degree networks, this implies that each node is only receiving a bounded number of messages. Frugal computations resemble local computations over bounded degree networks, since the nodes are receiving only a bounded number of messages, although these messages can come from remote nodes through multi-hop paths.

We prove that first-order properties can be frugally evaluated over bounded degree networks and planar networks (Theorem 2 and Theorem 4). The proofs are obtained by transforming the centralized linear time evaluation algorithms \cite{See95,FG01} into distributed ones satisfying the restriction that only a bounded number of messages are sent over each link. Moreover, we show that the results carry over to the extension of first-order logic with unary counting. While the transformation of the centralized linear time algorithm is simple for first-order properties over bounded degree networks, it is quite intricate for first-order properties over planar networks. The most intricate part is the distributed construction of an ordered tree decomposition for some subgraphs of the planar network, inspired by the distributed algorithm to construct an ordered tree decomposition for planar networks with bounded diameter in \cite{GW09}. 

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Intuitively, since in the centralized linear time computation each object is involved only a bounded number of times, in the distributed computation, a bounded number of messages sent over each link could be sufficient to evaluate first-order properties. So it might seem trivial to design frugal distributed algorithms for first-order properties over bounded degree networks and planar networks. Nevertheless, this is not the case, because in the centralized computation, after visiting one object, any other object can be visited, but in the distributed computation, only the adjacent objects (nodes, links) can be visited.

The paper is organized as follows. In the next section, we recall classical graph theory concepts, as well as Gaifman’s locality theorem. In Section 3 we consider the distributed evaluation of first-order properties over respectively bounded degree and planar networks. Finally, in Section 4, we consider the distributed evaluation of first-order logic with unary counting. Proofs can be found in the appendix.

2 Graphs, first-order logic and locality

In this paper, our interest is focused to a restricted class of structures, namely finite graphs. Let \( G = (V,E) \), be a finite graph. We use the following notations. If \( v \in V \), then \( \deg(v) \) denotes the degree of \( v \). For two nodes \( u,v \in V \), the distance between \( u \) and \( v \), denoted \( \text{dist}_G(u,v) \), is the length of the shortest path between \( u \) and \( v \). For \( k \in \mathbb{N} \), the \( k \)-neighborhood of a node \( v \), denoted \( N_k(v) \), is defined as \( \{ w \in V | \text{dist}_G(v,w) \leq k \} \). If \( \bar{v} = v_1 \ldots v_p \) is a collection of nodes in \( V \), then the \( k \)-neighborhood of \( \bar{v} \), denoted \( N_k(\bar{v}) \), is defined by \( \bigcup_{1 \leq i \leq p} N_k(v_i) \). For \( X \subseteq V \), let \( \langle X \rangle^G \) denote the subgraph induced by \( X \).

Let \( G = (V,E) \) be a connected graph, a tree decomposition of \( G \) is a rooted labeled tree \( T = (T,F,r,B) \), where \( T \) is the set of vertices of the tree, \( F \subseteq T \times T \) is the child-parent relation of the tree, \( r \in T \) is the root of the tree, and \( B \) is a labeling function from \( T \) to \( 2^V \), mapping vertices \( t \) of \( T \) to sets \( B(t) \subseteq V \), called bags, such that

1. For each edge \( (v,w) \in E \), there is a \( t \in T \), such that \( \{v,w\} \subseteq B(t) \).
2. For each \( v \in V \), \( B^{-1}(v) = \{ t \in T | v \in B(t) \} \) is connected in \( T \).

The width of \( T \), \( \text{width}(T) \), is defined as \( \max \{ |B(t)| - 1 | t \in T \} \). The tree-width of \( G \), denoted \( \text{tw}(G) \), is the minimum width over all tree decompositions of \( G \). An ordered tree decomposition of width \( k \) of a graph \( G \) is a rooted labeled tree \( T = (T,F,r,L) \) such that:

- \( (T,F,r) \) is defined as above,
- \( L \) assigns each vertex \( t \in T \) to a \( (k+1) \)-tuple \( \overline{b} = (b_1, \ldots, b_{k+1}) \) of vertices of \( G \) (note that in the tuple \( \overline{b} \), vertices of \( G \) may occur repeatedly),
- If \( L'(t) := \{ b_j \in L(t) = (b_1, \ldots, b_{k+1}), 1 \leq j \leq k+1 \} \), then \( (T,F,r,L') \) is a tree decomposition.

The rank of an (ordered) tree decomposition is the rank of the rooted tree, i.e. the maximal number of children of its vertices.

We consider first-order logic (FO) over the signature \( E \), where \( E \) is a binary relation symbol. The syntax and semantics of first-order formulae are defined as usual [EF99]. The quantifier rank of a formula \( \varphi \) is the maximal number of nestings of existential and universal quantifiers in \( \varphi \).

A graph property is a class of graphs closed under isomorphisms. Let \( \varphi \) be a first-order sentence, the graph property defined by \( \varphi \), denoted \( \mathcal{P}_\varphi \), is the class of graphs satisfying \( \varphi \).

The distance between nodes can be defined by first-order formulae \( \text{dist}(x,y) \leq k \) stating that the distance between \( x \) and \( y \) is no larger than \( k \), and \( \text{dist}(x,y) > k \) is an abbreviation

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of \(-\text{dist}(x, y) \leq k\). In addition, let \(\bar{x} = x_1 \ldots x_p\) be a list of variables, then \(\text{dist}(\bar{x}, y) \leq k\) is used to denote \(\bigvee_{1 \leq i \leq p} \text{dist}(x_i, y) \leq k\).

Let \(\varphi\) be a first-order formula, \(k \in \mathbb{N}\), and \(\bar{x}\) a list of variables not occurring in \(\varphi\), then the formula bounding the quantifiers of \(\varphi\) to the \(k\)-neighborhood of \(\bar{x}\), denoted \(\varphi^{(k)}(\bar{x})\), can be defined easily in first-order logic by using formulae \(\text{dist}(\bar{x}, y) \leq k\). For instance, if \(\varphi := \exists y \psi(y)\), then \(\varphi^{(k)}(\bar{x}) := \exists y \left(\text{dist}(\bar{x}, y) \leq k \land (\psi(y))^{(k)}(\bar{x})\right)\).

We can now recall the notion of logical locality introduced by Gaifman \cite{Gai82, EF99}.

**Theorem 1.** \cite{Gai82} Let \(\varphi\) be a first-order formula with free variables \(u_1, \ldots, u_p\), then \(\varphi\) can be written in Gaifman Normal Form, that is into a Boolean combination of (i) sentences of the form:

\[
\exists x_1 \ldots \exists x_s \left( \bigwedge_{1 \leq i < j \leq s} d(x_i, x_j) > 2r \land \bigwedge_i \psi^{(r)}(x_i) \right) \quad (1)
\]

and (ii) formulae of the form \(\psi^{(t)}(\bar{y})\), where \(\bar{y} = y_1 \ldots y_q\) such that \(y_i \in \{u_1, \ldots, u_p\}\) for all \(1 \leq i \leq q\), \(r \leq 7^{k-1}\), \(s \leq p + k\), \(t \leq (7^k - 1)/2\) (\(k\) is the quantifier rank of \(\varphi\)).

Moreover, if \(\varphi\) is a sentence, then the Boolean combination contains only sentences of the form \(1\).

The locality of first-order logic is a powerful tool to demonstrate non-definability results \cite{Lib97}. It can be used in particular to prove that counting properties, such as the parity of the number of vertices, or recursive properties, such as the connectivity of a graph, are not first-order.

### 3 Distributed evaluation of FO

We consider a message passing model of distributed computation \cite{AW04}, based on a communication network whose topology is given by a graph \(G = (V, E)\) of diameter \(\Delta\), where \(E\) denotes the set of bidirectional communication links between nodes. From now on, we restrict our attention to finite connected graphs.

We assume that the distributed system is asynchronous and has no failure. The nodes have a unique identifier taken from \(1, 2, \ldots, n\), where \(n\) is the number of nodes. Each node has distinct local ports for distinct links incident to it. The nodes have states, including final accepting or rejecting states.

For simplicity, we assume that there is only one query fired in the network by a requesting node. We assume also that a breadth-first-search (BFS) tree rooted on the requesting node has been pre-computed in the network, such that each node stores locally the identifier of its parent in the BFS-tree, and the states of the ports with respect to the BFS-tree, which are either “parent” or “child”, denoting the ports corresponding to the tree edges, or “horizon”, “upward”, “downward”, denoting the ports corresponding to the non-tree edges to some node with the same, smaller, or larger depth in the BFS-tree. The computation terminates, when the requesting node reaches a final state.

Let \(\mathcal{C}\) be a class of graphs. A distributed algorithm is said to be frugal over \(\mathcal{C}\) if there is a \(k \in \mathbb{N}\) such that for any network \(G \in \mathcal{C}\) of \(n\) nodes and any requesting node in \(G\), the distributed computation terminates, with only at most \(k\) messages of size \(O(\log n)\) sent over each link. If we

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\(^2\)The bound on \(r\) has been improved to \(4^k - 1\) in \cite{KL04}.
restrict our attention to bounded degree networks, frugal distributed algorithms implies that each node only receives a bounded number of messages. Frugal computations resemble local computations over bounded degree networks, since the nodes receive only a bounded number of messages, although these messages can come from remote nodes through multi-hop paths.

Let $C$ be a class of graphs, and $\phi$ an FO sentence, we say that $\phi$ can be distributively evaluated over $C$ if there exists a distributed algorithm such that for any network $G \in C$ and any requesting node in $G$, the computation of the distributed algorithm on $G$ terminates with the requesting node in the accepting state if and only if $G \models \phi$. Moreover, if there is a frugal distributed algorithm to do this, then we say that $\phi$ can be frugally evaluated over $C$.

For centralized computations, it has been shown that Gaifman’s locality of FO entails linear time evaluation of FO properties over classes of bounded degree graphs and classes of locally tree-decomposable graphs [See95, FG01]. In the following, we show that it is possible to design frugal distributed evaluation algorithms for FO properties over bounded degree and planar networks, by carefully transforming the centralized linear time evaluation algorithms into distributed ones with computations on each node well balanced.

### 3.1 Bounded degree networks

We first consider the evaluation of FO properties over bounded degree networks. We assume that each node stores the degree bound $k$ locally.

**Theorem 2.** FO properties can be frugally evaluated over bounded degree networks.

Theorem 2 can be shown by using Hanf’s technique [FSV95], in a way similar to the proof of Seese’s seminal result [See95].

Let $r \in \mathbb{N}$, $G = (V, E)$, and $v \in V$, then the $r$-type of $v$ in $G$ is the isomorphism type of $(\langle N_r(v) \rangle^G, v)$. Let $r, m \in \mathbb{N}$, $G_1$ and $G_2$ be two graphs, then $G_1$ and $G_2$ are said to be $(r, m)$-equivalent if and only if for every $r$-type $\tau$, either $G_1$ and $G_2$ have the same number of vertices with $r$-type $\tau$ or else both have at least $m$ vertices with $r$-type $\tau$. $G_1$ and $G_2$ are said to be $k$-equivalent, denoted $G_1 \equiv_k G_2$, if $G_1$ and $G_2$ satisfy the same FO sentences of quantifier rank at most $k$. It has been shown that:

**Theorem 3.** [FSV95] Let $k, d \in \mathbb{N}$. There exist $r, m \in \mathbb{N}$ such that $r$ (resp. $m$) depends on $k$ (resp. both $k$ and $d$), and for any graphs $G_1$ and $G_2$ with maximal degree no more than $d$, if $G_1$ and $G_2$ are $(r, m)$-equivalent, then $G_1 \equiv_k G_2$.

Let us now sketch the proof of Theorem 2 which relies on a distributed algorithm consisting of three phases. Suppose the requesting node requests the evaluation of some FO sentence with quantifier rank $k$. Let $r, m$ be the natural numbers depending on $k, d$ specified in Theorem 3.

**Phase I** The requesting node broadcasts messages along the BFS-tree to ask each node to collect the topology information in its $r$-neighborhood;

**Phase II** Each node collects the topology information in its $r$-neighborhood;

**Phase III** The $r$-types of the nodes in the network are aggregated through the BFS-tree to the requesting node up to the threshold $m$ for each $r$-type. Finally the requesting node decides whether the network satisfies the FO sentence or not by using the information about the $r$-types.

It is easy to see that only a bounded number of messages are sent over each link in Phase I and II. Since the total number of distinct $r$-types with degree bound $d$ depends only upon $r$ and $d$ and
each $r$-type is only counted up to a threshold $m$, it turns out that over each link, only a bounded number of messages are sent in Phase III as well. So the above distributed evaluation algorithm is frugal over bounded degree networks.

### 3.2 Planar networks

We now consider the distributed evaluation of FO properties over planar networks.

A *combinatorial embedding* of a planar graph $G = (V,E)$ is an assignment of a cyclic ordering of the set of incident edges to each vertex $v$ such that two edges $(u,v)$ and $(v,w)$ are in the same face iff $(v,u)$ is immediately before $(v,w)$ in the cyclic ordering of $v$. Combinatorial embeddings, which encode the information about boundaries of the faces in usual embeddings of planar graphs into the planes, are useful for computing on planar graphs. Given a combinatorial embedding, the boundaries of all the faces can be discovered by traversing the edges according to the above condition.

We assume in this subsection that a combinatorial embedding of the planar network is distributively stored in the network, i.e. a cyclic ordering of the set of the incident links is stored in each node of the network.

**Theorem 4.** FO properties can be frugally evaluated over planar networks.

For the proof of Theorem 4, we first recall the centralized linear time algorithm to evaluate FO properties over planar graphs in [FG01]³.

Let $G = (V,E)$ be a planar graph and $\varphi$ be an FO sentence. From Theorem 4 we know that $\varphi$ can be written into Boolean combinations of sentences of the form (1),

$$
\exists x_1...\exists x_s \left( \bigwedge_{1 \leq i < j \leq s} d(x_i, x_j) > 2r \land \bigwedge_i \psi^{(r)}(x_i) \right).
$$

It is sufficient to show that sentences of the form (1) are linear-time computable over $G$. The centralized algorithm to evaluate FO sentences of the form (1) over planar graphs consists of the following four phases:

1. Select some $v_0 \in V$, let $\mathcal{H} = \{G[i, i + 2r] | i \geq 0\}$, where $G[i, j] = \{v \in V | i \leq \text{dist}_G(v_0, v) \leq j\}$;
2. For each $H \in \mathcal{H}$, compute $K_r(H)$, where $K_r(H) := \{v \in H | N_r(v) \subseteq H\}$;
3. For each $H \in \mathcal{H}$, compute $P_H := \{v \in K_r(H) | H \models \psi^{(r)}(v)\}$;
4. Let $P := \cup H P_H$, determine whether there are $s$ distinct nodes in $P$ such that their pairwise distance is greater than $2r$.

In the computation of the 3rd and 4th phase above, an automata-theoretical technique to evaluate Monadic-Second-Order (MSO) formulae in linear time over classes of graphs with bounded tree-width [Cou90, FG06, FFG02] is used. In the following, we recall this centralized evaluation algorithm.

MSO is obtained by adding set variables and set quantifiers into FO, such as $\exists X \varphi(X)$ (where $X$ is a set variable). MSO has been widely studied in the context of graphs for its expressive power. For instance, 3-colorability, transitive closure or connectivity can be defined in MSO [Cou08].

³In fact, in [FG01], it was shown that FO is linear-time computable over classes of locally tree-decomposable graphs.
The centralized linear time evaluation of MSO formulae over classes of bounded tree-width graphs goes as follows. First an ordered tree decomposition $T$ of the given graph is constructed. Then from the given MSO formula, a tree automaton $A$ is obtained. Afterwards, $T$ is transformed into a labeled tree $T'$, finally $A$ is ran over $T'$ (maybe several times for formulae containing free variables) to get the evaluation result.

In the rest of this section, we design a frugal distributed algorithm to evaluate FO sentences over planar networks by adapting the above centralized algorithm. The main difficulty is to distribute the computation among the nodes such that only a bounded number of messages are sent over each link during the computation.

**Phase I** The requesting node broadcasts the FO sentence of the form \([i \geq 0 \mid v \in G[i, i + 2r]]\) to all the nodes in the network through the BFS tree;

**Phase II** For each $v \in V$, compute $C(v) := \{i \geq 0 \mid v \in G[i, i + 2r]\}$;

**Phase III** For each $v \in V$, compute $D(v) := \{i \geq 0 \mid N_r(v) \subseteq G[i, i + 2r]\}$;

**Phase IV** For each $i \geq 0$, compute $P_i := \{v \in V \mid i \in D(v), (G[i, i + 2r])G = \psi^{(r)}(v)\}$;

**Phase V** Let $P := \bigcup_i P_i$, determine whether there are $s$ distinct nodes labeled by $P$ such that their pairwise distance is greater than $2r$.

Phase I is trivial. Phase II is easy. In the following, we illustrate the computation of Phase III, IV, and V one by one.

We first introduce a lemma for the computation of Phase III.

For $W \subseteq V$, let $K_i(W) := \{v \in W \mid N_i(v) \subseteq W\}$. Let $D_i(v) := \{j \geq 0 \mid v \in K_i(G[j, j + 2r])\}$.

**Lemma 5.** For each $v \in V$ and $i > 0$, $D_i(v) = C(v) \cap \bigcap_{w: (v, w) \in E} D_{i-1}(w)$.

With Lemma 5, $D(v) = D_i(v)$ can be computed in an inductive way to finish Phase III: Each node $v$ obtains the information $D_{i-1}(w)$ from all its neighbors $w$, and performs the in-node computation to compute $D_i(v)$.

Now we consider Phase IV.

Because $\psi^{(r)}(x)$ is a local formula, $\psi^{(r)}(x)$ can be evaluated separately over each connected component of $G[i, i + 2r]$ and the results are stored distributively.

Let $C_i$ be a connected component of $G[i, i + 2r]$, and $w_1^i, \ldots, w_j^i$ be all the nodes contained in $C_i$ with distance $i$ from the requesting node. Now we consider the evaluation of $\psi^{(r)}(x)$ over $C_i$.

Let $C'_i$ be the graph obtained from $C_i$ by including all ancestors of $w_1^i, \ldots, w_j^i$ in the BFS-tree, and $C^*_i$ be the graph obtained from $C'_i$ by contracting all the ancestors of $w_1^i, \ldots, w_j^i$ into one vertex, i.e. $C^*_i$ has one more vertex, called the virtual vertex, than $C_i$, and this vertex is connected to $w_1^i, \ldots, w_j^i$. It is easy to see that $C^*_i$ is a planar graph with a BFS-tree rooted on $v^*$ and of depth at most $2r + 1$. So $C^*_i$ is a planar graph with bounded diameter.

An ordered tree decomposition for planar networks with bounded diameter can be distributively constructed with only a bounded number of messages sent over each link as follows [GW09]:

- Do a depth-first-search to decompose the network into blocks, i.e. biconnected components;
- Construct an ordered tree decomposition for each nontrivial block: Traverse every face of the block according to the cyclic ordering at each node, triangulate all those faces, and connect the triangles into a tree decomposition by utilizing the pre-computed BFS tree;
- Finally the tree decompositions for the blocks are connected together into a complete tree decomposition for the whole network.
By using the distributed algorithm for the tree decomposition of planar networks with bounded
diameter, we can construct distributively an ordered tree decomposition for $C_i^*$, while having the
virtual vertex in our mind, and get an ordered tree decomposition for $C_i$.

With the ordered tree decomposition for $C_i$, we can evaluate $\psi^{(r)}(x)$ over $C_i$ by using the
automata-theoretical technique, and store the result distributively in the network (each node stores
a Boolean value indicating whether it belongs to the result or not).

A distributed post-order traversal over the BFS tree can be done to find out all the connected
components of $G[i, i + 2r]'s$ and construct the tree decompositions for these connected components
one by one.

Finally we consider Phase V.

Label nodes in $\bigcup P_i$ with $P_i$. Then consider the evaluation of FO sentence $\varphi'$ over the vocabulary \{E, P\},

$$\exists x_1...\exists x_s \left( \bigwedge_{1 \leq i < j \leq s} d(x_i, x_j) > 2r \land \bigwedge_i P(x_i) \right).$$

Starting from some node $w_1$ with label $P$, mark the vertices in $N_{2r}(w_1)$ as $Q$, then select some
node $w_2$ outside $Q$, and mark those nodes in $N_{2r}(w_2)$ by $Q$ again, continue like this, until $w_l$ such
that either $l = s$ or all the nodes with label $P$ have already been labeled by $Q$.

If $l < s$, then label the nodes in $\bigcup_{1 \leq i \leq l} N_{4r}(v_i)$ as $I$. Each connected component of $\langle I \rangle^G$ has
diameter no more than $4lr < 4sr$. We can construct distributively a tree decomposition for each
connected component of $\langle I \rangle^G$, and connect these tree decompositions together to get a complete
tree-decomposition of $\langle I \rangle^G$, then evaluate the sentence $\varphi'$ by using this complete tree decomposition.

The details of the frugal distributed evaluation algorithm can be found in the appendix.

4 Beyond FO properties

We have shown that FO properties can be frugally evaluated over respectively bounded degree and
planar networks. In this section, we extend these results to FO unary queries and some counting
extension of FO.

From Theorem 1 FO formula $\varphi(x)$ containing exactly one free variable $x$ can be written into
the Boolean combinations of sentences of the form (1) and the local formulae $\psi^{(l)}(x)$. Then it is
not hard to prove the following result.

**Theorem 6.** FO formulae $\varphi(x)$ with exactly one free variable $x$ can be frugally evaluated over
respectively bounded degree and planar networks, with the results distributively stored on the nodes
of the network.

Counting is one of the ability that is lacking to first-order logic, and has been added in com-
mercial relational query languages (e.g. SQL). Its expressive power has been widely studied
in the literature. Libkin [Lib97] proved that first-order logic with counting still enjoys Gaifman locality property. We prove that Theorem 2 and Theorem 4 carry over as
well for first-order logic with unary counting.

Let FO(#) be the extension of first-order logic with unary counting. FO(#) is a two-sorted
logic, the first sort ranges over the set of nodes $V$, while the second sort ranges over the natural
numbers $\mathbb{N}$. The terms of the second sort are defined by: $t := \#x.\varphi(x) | t_1 + t_2 | t_1 \times t_2$, where $\varphi$
is a formula over the first sort with one free variable $x$. Second sort terms of the form $\#x.\varphi(x)$ are
called basic second sort terms.
The atoms of FO(#) extend standard FO atoms with the following two unary counting atoms: 
\( t_1 = t_2 \mid t_1 < t_2 \), where \( t_1, t_2 \) are second sort terms. Let \( t \) be a second sort term of FO(#), 
\( G = (V, E) \) be a graph, then the interpretation of \( t \) in \( G \), denoted \( t^G \), is defined as follows:
- \((\#x.\varphi(x))^G\) is the cardinality of \( \{ v \in V \mid G \models \varphi(v) \} \);
- \( (t_1 + t_2)^G \) is the sum of \( t_1^G \) and \( t_2^G \);
- \( (t_1 \times t_2)^G \) is the product of \( t_1^G \) and \( t_2^G \).

The interpretation of FO(#) formulae is defined in a standard way.

**Theorem 7.** FO(#) properties can be frugally evaluated over respectively bounded degree and planar networks.

The proof of the theorem relies on a normal form of FO(#) formulae. A sketch can be found in the appendix.

## 5 Conclusion

The logical locality has been shown to entail efficient computation of first-order logic over several classes of structures. We show that if the logical formulae are used to express properties of the graphs, which constitute the topology of communication networks, then these formulae can be evaluated very efficiently over these networks. Their distributed computation, although not local \cite{Lin92, NS95, Pel00}, can be done frugally, that is with a bounded number of messages of logarithmic size exchanged over each link. The frugal computation, introduced in this paper, generalizes local computation and offers a large spectrum of applications. We proved that first-order properties can be evaluated frugally over respectively bounded degree and planar networks. Moreover the results carry over to the extension of first-order logic with unary counting. The distributed time used in the frugal evaluation of FO properties over bounded degree networks is \( O(\Delta) \), while that over planar networks is \( O(n) \).

We assumed that some pre-computations had been done on the networks. If no BFS-tree has been pre-computed, the construction of a BFS-tree can be done in \( O(\Delta) \) time and with \( O(\Delta) \) messages sent over each link \cite{BDLP08}.

Beyond its interest for logical properties, the frugality of distributed algorithms, which ensures an extremely good scalability of their computation, raises fundamental questions, such as deciding what can be frugally computed. Can a Hamiltonian path for instance be computed frugally?

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A Distributed Evaluation of FO over planar networks: Phase II

The purpose of Phase II is to compute \( C(v) := \{ i \geq 0 | v \in G[i, i + 2r] \} \) for each \( v \in V \).

A pre-computed breadth-first-search (BFS) tree rooted on the requesting node is distributively stored in the network such that each node \( v \) stores the identifier of its parent in the BFS-tree \((\text{parent}(v))\), and the states of the ports with respect to the BFS-tree \((\text{state}(l))\) for each port \( l \), which are either “parent”, or “child”, or “horizon”, or “downward”, or “upward”. Moreover, we suppose that each node \( v \) stores in \( \text{depth}(v) \) its depth in the BFS tree, i.e. the distance between \( v \) and the requesting node.

The distributed algorithm is presented by describing the message processing at each node \( v \).

**Initialization**

The requesting node sets \( \text{treeDepth} := 0 \).

The requesting node sends message \text{TREEDEPTH} over all its ports with state “child”.

**Message TREEDEPTH over port \( l \)**

- \( \text{treeDepth} := \text{depth}(v) \).
- if \( v \) is not a leaf then
  - \( v \) sends message \text{TREEDEPTH} over all ports with state “child”.
- else \( v \) sends message \text{ACKTREEDEPTH}(\text{treeDepth}) over the port \( l' \) with state “parent”.

**Message ACKTREEDEPTH(\( sd \)) over port \( l \)**

- \( \text{treeDepth} := \max\{\text{treeDepth}, sd\} \).
- if \( v \) has received ACKTREEDEPTH messages over all its ports with state “child” then
  - if \( v \) is the requesting node then
    - \( v \) sends message \text{STARTCOVER}(\text{treeDepth}) over all ports with state “child”.
  - else \( v \) sends message \text{ACKTREEDEPTH}(\text{treeDepth}) over the port \( l' \) with state “parent”.
- end if

**Message STARTCOVER(\( td \)) over port \( l \)**

- \( \text{treeDepth} := td \).
- if \( \text{treeDepth} \leq 2r \) then
  - \( C(v) := \{0\} \).
- else
  - \( C(v) := \{ i \in \mathbb{N} | \max\{\text{depth}(v) - 2r, 0\} \leq i \leq \min\{\text{depth}(v), \text{treeDepth} - 2r\} \} \).
- end if
- if \( v \) is not a leaf then
  - \( v \) sends message \text{STARTCOVER}(\text{treeDepth}) over all ports with state “child”.
- else \( v \) sends message \text{ACKCOVER} over the port \( l' \) with state “parent”.

**Message ACKCOVER over port \( l \)**

- if \( v \) has received message \text{ACKCOVER} over all its ports with state “child” then
  - if \( v \) is not the requesting node then
    - \( v \) sends message \text{ACKCOVER} over the port \( l' \) with state “parent”.
- end if
- end if
B Distributed Evaluation of FO over planar networks: Phase III

The purpose of Phase III is to compute \( D(v) := \{ i \geq 0 | N_r(v) \subseteq G[i, i + 2r] \} \) for each \( v \in V \).

When the requesting node receives message ACKCOVER from all its children, it knows that the computation of Phase II is over. Then it can starts the computation of Phase III.

We first introduce a lemma.

For \( W \subseteq V \), let \( K_i(W) := \{ v \in W | N_i(v) \subseteq W \} \). Let \( D_i(v) := \{ j \geq 0 | v \in K_i[G[j, j + 2r]] \} \).

**Lemma 5** For each \( v \in V \) and \( i > 0 \), \( D_i(v) = C(v) \cap \bigcap_{w: (v, w) \in E} D_{i-1}(w) \).

**Proof.**

\[
j \in D_i(v) \iff v \in K_i(G[j, j + 2r]) \iff N_i(v) \subseteq G[j, j + 2r] \\
\iff v \in G[j, j + 2r] \text{ and } \forall w ((v, w) \in E \rightarrow N_{i-1}(w) \subseteq G[j, j + 2r]) \\
\iff j \in C(v) \text{ and } \forall w ((v, w) \in E \rightarrow w \in K_{i-1}(G[j, j + 2r])) \\
\iff j \in C(v) \text{ and } \forall w ((v, w) \in E \rightarrow j \in D_{i-1}(w)) \\
\iff j \in C(v) \cap \bigcap_{w: (v, w) \in E} D_{i-1}(w)
\]

\[\square\]

With Lemma 5, \( D(v) \)'s can be computed in an inductive way: Each node \( v \) obtains the information \( D_{i-1}(w) \) from all its the neighbors \( w \), and does the in-node computation.

The distributed algorithm is given by describing the message processing at each node \( v \).

The following proposition can be proved on the \( idx(v) \)'s in the above distributed algorithm.

**Proposition 8.** During the computation of Phase III, for each node \( v, w \) such that \( (v, w) \in E \), \( |idx(v) - idx(w)| \leq 1 \).

**Proof.** To the contrary, suppose that \( idx(v) - idx(w) > 1 \) for some \( v, w : (v, w) \in E \).

From the distributed algorithm, we know that \( v \) has completed the computation of \( D_{idx(v)-1} \), so it has received messages \( KERNEL(idx(v) - 2, DD) \) over all its ports. In particular, \( v \) has received message \( KERNEL(idx(v) - 2, DD) \) over the port \( l' \) such that \( v \) is connected to \( w \) through \( l' \). But then, we have \( idx(w) - 1 \geq idx(v) - 2 \), i.e. \( idx(v) - idx(w) \leq 1 \), a contradiction. \[\square\]

During the computation of Phase III, for each link \( (v, w) \in E \), the number of "KERNEL" messages sent over \( (v, w) \) is no more than \( 2r \). Therefore, during the distributed computation of Phase III, only \( O(1) \) messages are sent over each link.

C Distributed Evaluation of FO over planar networks: Phase IV

The purpose of Phase IV is to compute \( P_i := \{ v \in V | i \in D(v), G[i, i + 2r] \models \psi^{(r)}(v) \} \) for each \( i \geq 0 \).

Because our distributed algorithm for Phase IV is obtained by transforming the centralized evaluation algorithm for MSO formulae over classes of graphs with bounded tree-width, we first recall it in the following.
Initialization

The requesting node sends message INIT over all ports with state “child”.

Message INIT over port $l$

$idx(v) := 1, D(v) := C(v)$.

$% idx(v)$ is the index $i$ such that $D_i(v)$ is to be computed next.

$neighborKernel(v) := \emptyset$.

$bKernelOver(v) := false$.

if $v$ is not a leaf then
  $v$ sends message INIT over all ports with state “child”.
else
  $v$ sends message ACKINIT over the port $l'$ with state “parent”.
end if

Message ACKINIT over port $l$

if $v$ has received ACKINIT messages over all ports with state “child” then
  if $v$ is not the requesting node then
    $v$ sends message ACKINIT over the port $l'$ with state “parent”.
  else
    $v$ sends message STARTKERNEL over all ports with state “child”.
  end if
end if

C.1 Centralized evaluation of MSO formulae over classes of graphs with bounded tree-width

We first recall the centralized linear time evaluation of MSO sentences.

Let $\Sigma$ be some alphabet. A tree language over alphabet $\Sigma$ is a set of rooted $\Sigma$-labeled binary trees. Let $\varphi$ be an MSO sentence over the vocabulary $\{E_1, E_2\} \cup \{P_c | c \in \Sigma\}$, $(E_1, E_2$ are respectively the left and right children relations of the tree), the tree language accepted by $\varphi$, $L(\varphi)$, is the set of rooted $\Sigma$-labeled trees satisfying $\varphi$.

Tree languages can also be recognized by tree automata. A deterministic bottom-up tree automaton $A$ is a quintuple $(Q, \Sigma, \delta, f_0, F)$, where $Q$ is the set of states; $F \subseteq Q$ is the set of final states; $\Sigma$ is the alphabet; and

- $\delta : (Q \cup Q \times Q) \times \Sigma \rightarrow Q$ is the transition function; and

- $f_0 : \Sigma \rightarrow Q$ is the initial-state assignment function.

A run of tree automaton $A = (Q, \Sigma, \delta, f_0, F)$ over a rooted $\Sigma$-labeled binary tree $T = (T, F, r, L)$ produces a rooted $Q$-labeled tree $T' = (T, F, r, L')$ such that

- If $t \in T$ is a leaf, then $L'(t) = f_0(t)$;

- Otherwise, if $t \in T$ has one child $t'$, then $L'(t) = \delta(L'(t'), L(t))$;

- Otherwise, if $t \in T$ has two children $t_1, t_2$, then $L'(t) = \delta(L'(t_1), L'(t_2), L(t))$.

Note that for each deterministic bottom-up automaton $A$ and rooted $\Sigma$-labeled binary tree $T$, there is exactly one run of $A$ over $T$.

The run $T' = (T, F, r, L')$ of $A = (Q, \Sigma, \delta, f_0, F)$ over a rooted $\Sigma$-labeled binary tree $T = (T, F, r, L)$ is accepting if $L'(r) \in F$. 

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Message STARTKERNEL over port \( l \)

\( v \) sends message \( \text{KERNEL}(\text{idx}(v) - 1, D(v)) \) over all ports.

\[ \text{if } v \text{ is not the leaf then} \]
\[ \text{v sends message STARTKERNEL over all ports with state “child”.} \]
\[ \text{end if} \]

Message KERNEL(\( i, ND \)) over port \( l \)

Let \( \text{neighborKernel}(v) := \text{neighborKernel}(v) \cup \{(l, i, ND)\} \).

\[ \text{if } i = \text{idx}(v) - 1 \text{ then} \]
\[ \text{for each port } l', \text{there is a tuple } (l', \text{idx}(v) - 1, DD) \in \text{neighborKernel}(v) \text{ for some } DD \text{ then} \]
\[ \text{D}(v) := D(v) \cap DD. \]
\[ \text{neighborKernel}(v) := \text{neighborKernel}(v) \setminus \{(l', \text{idx}(v) - 1, DD)\}. \]
\[ \text{end for} \]
\[ \text{idx}(v) := \text{idx}(v) + 1. \]
\[ \text{if } \text{idx}(v) \leq r \text{ then} \]
\[ \text{v sends message KERNEL(\( \text{idx}(v) - 1, D(v) \)) to all its neighbors.} \]
\[ \text{else} \]
\[ \text{bKERNELOVER}(v) := \text{true}. \]
\[ \text{if } v \text{ is a leaf or } v \text{ has received message KERNELOVER over all ports with state “child” then} \]
\[ \text{v sends message KERNELOVER over the port } l' \text{ with state “parent”.} \]
\[ \text{end if} \]
\[ \text{end if} \]
\[ \text{end if} \]

Message KERNELOVER over port \( l \)

\[ \text{if } \text{bKERNELOVER}(v) = \text{true} \text{ and } v \text{ has received KERNELOVER over all ports with state “child” then} \]
\[ \text{if } v \text{ is not the requesting node then} \]
\[ \text{v sends message KERNELOVER over the port with state “parent”.} \]
\[ \text{end if} \]
\[ \text{end if} \]

A rooted \( \Sigma \)-labeled binary tree \( T = (T, F, r, L) \) is accepted by a tree automaton \( A = (Q, \Sigma, \delta, f_0, F) \) if the run of \( A \) over \( T \) is accepting. The tree language accepted by \( A \), \( \mathcal{L}(A) \), is the set of rooted \( \Sigma \)-labeled binary trees accepted by \( A \).

The next theorem shows that the two notions are equivalent.

**Theorem 9.** [TW68] Let \( \Sigma \) be a finite alphabet. A tree language over \( \Sigma \) is accepted by a tree automaton iff it is defined by an MSO sentence. Moreover, there are algorithms to construct an equivalent tree automaton from a given MSO sentence and to construct an equivalent MSO sentence from a given automaton.

The centralized linear time algorithm to evaluate an MSO sentence \( \varphi \) over a graph \( G = (V, E) \) with tree-width bounded by \( k \) works as follows:

**Step 1** Construct an ordered tree decomposition \( T = (T, F, r, L) \) of \( G \) of width \( k \) and rank \( \leq 2 \);

**Step 2** Transform \( T \) into a \( \Sigma_k \)-labeled binary tree \( T' = (T, F, r, \lambda) \) for some finite alphabet \( \Sigma_k \);

**Step 3** Construct an MSO sentence \( \varphi^* \) over vocabulary \( \{E_1, E_2\} \cup \{P_c | c \in \Sigma_k\} \) from \( \varphi \) (over vocabulary \( \{E\} \)) such that \( G \models \varphi \text{ iff } T' \models \varphi^* \);
Step 4 From \( \varphi^* \), construct a bottom-up binary tree automaton \( A \), and run \( A \) over \( T' \) to decide whether \( T' \) is accepted by \( A \).

For Step 1, it has been shown that a tree decomposition of graphs with bounded tree-width can be constructed in linear time \[\text{[Bod93]}\]. It follows from Theorem 9 that Step 4 is feasible. Now suppose that an ordered tree decomposition \( T = (T, F, r, L) \) of \( G = (V, E) \) of width \( k \) and rank \( \leq 2 \) has been constructed, we recall how to perform Step 2 and Step 3 in linear time.

For Step 2, a rooted \( \Sigma_k \)-labeled tree \( T' = (T, F, r, \lambda) \), where \( \Sigma_k = 2^{[k+1]^2} \times 2^{[k+1]^2} \times 2^{[k+1]^2} \) \( ([k+1] = \{1, 2, \ldots, k+1\}) \), can be obtained from \( T \) as follows: The new labeling \( \lambda \) over \( (T, F) \) is defined by \( \lambda(t) = (\lambda_1(t), \lambda_2(t), \lambda_3(t)) \), where

\[
\begin{align*}
\lambda_1(t) &:= \{(j_1, j_2) \in [k+1]^2 | (b^t_{j_1}, b^t_{j_2}) \in E\}.
\lambda_2(t) &:= \{(j_1, j_2) \in [k+1]^2 | b^t_{j_1} = b^t_{j_2}\}.
\lambda_3(t) &:= \left\{ \begin{array}{ll}
\{(j_1, j_2) \in [k+1]^2 | b^t_{j_1} = b^t_{j_2}\} & \text{for the parent } t' \text{ of } t, \text{ if } t \neq r \\
\emptyset & \text{if } t = r
\end{array} \right\}
\end{align*}
\]

For Step 3, we recall how to translate the MSO sentence \( \varphi \) over the vocabulary \{\( E \)\} into an MSO sentence \( \varphi^* \) over the vocabulary \{\( E_1, E_2 \)\} \( \cup \{P_c | c \in \Sigma_k\} \) such that \( G \models \varphi \) iff \( T' \models \varphi^* \). The translation relies on the observation that elements and subsets of \( V \) can be represented by \( (k+1) \)-tuples of subsets of \( T \). For each element \( v \in V \) and \( i \in [k+1] \), let

\[
U_i(v) := \left\{ \begin{array}{ll}
\{t(v)\} & \text{if } b^t_{i}(v) = v, \text{ and } b^t_{j}(v) \neq v \text{ for all } j : 1 \leq j < i \\
\emptyset & \text{otherwise}
\end{array} \right\}
\]

where \( t(v) \) is the minimal \( t \in T \) (with respect to the partial order \( \preceq \)) such that \( v \in \{b^t_1, \ldots, b^t_{k+1}\} \). Let \( \overline{U}(v) = (U_1(v), \ldots, U_{k+1}(v)) \).

For each \( S \subseteq V \) and \( i \in [k+1] \), let \( U_i(S) := \bigcup_{v \in S} U_i(v) \), and let \( \overline{U}(S) = (U_1(S), \ldots, U_{k+1}(S)) \).

It is not hard to see that for subsets \( U_1, \ldots, U_{k+1} \subseteq T \), there exists \( v \in V \) such that \( \overline{U} = \overline{U}(v) \) iff

- (1) \( \bigcup_{i=1}^{k+1} U_i \) is a singleton;
- (2) For all \( t \in T \), \( i < j < k+1 \), if \( t \in U_j \), then \((i, j) \notin \lambda_2(t)\);
- (3) For all \( t \in T \), \( i, j < k+1 \), if \( t \in U_i \), then \((i, j) \notin \lambda_3(t)\).

Moreover, there is a subset \( S \subseteq V \) such that \( \overline{U} = \overline{U}(S) \) iff conditions (2) and (3) are satisfied. Using the above characterizations of \( \varphi \) and \( \varphi^* \), it is easy to construct MSO formulae \( \text{Elem}(X_1, \ldots, X_{k+1}) \) and \( \text{Set}(X_1, \ldots, X_{k+1}) \) over \( \{E_1, E_2\} \cup \{P_c | c \in \Sigma_k\} \) such that \( T' \models \text{Elem}(U) \) iff there is a \( v \in V \) such that \( U = \overline{U}(v) \).

\[T' \models \text{Set}(U) \] iff there is a \( S \subseteq V \) such that \( U = \overline{U}(S) \).

Lemma 10. \[\text{[FFG02]}\] Every MSO formula \( \varphi(X_1, \ldots, X_l, y_1, \ldots, y_m) \) over vocabulary \( E \) can be effectively translated into a formula \( \varphi^*(\overline{X}_1, \ldots, \overline{X}_l, \overline{Y}_1, \ldots, \overline{Y}_m) \) over the vocabulary \( \{E_1, E_2\} \cup \{P_c | c \in \Sigma_k\} \) such that

- (1) For all \( S_1, \ldots, S_l \subseteq V \), and \( v_1, \ldots, v_m \in V \), \( G \models \varphi(S_1, \ldots, S_l, v_1, \ldots, v_m) \) iff \( T' \models \varphi^*(\overline{U}(S_1), \ldots, \overline{U}(S_l), \overline{U}(v_1), \ldots, \overline{U}(v_m)) \).
- (2) For all \( U_1, \ldots, U_l, W_1, \ldots, W_m \subseteq T \) such that \( T' \models \varphi^*(\overline{U}_1, \ldots, \overline{U}_l, \overline{W}_1, \ldots, \overline{W}_m) \), there exist \( S_1, \ldots, S_l \subseteq V \), and \( v_1, \ldots, v_m \in V \) such that \( \overline{U}_i = \overline{U}(S_i) \) for all \( 1 \leq i \leq l \) and \( \overline{W}_j = \overline{U}(v_j) \) for all \( 1 \leq j \leq m \).
Now we recall the evaluation of MSO formulae containing free variables over classes of graphs with bounded tree-width \cite{FG02}. Let $\varphi(X_1, \ldots, X_l, y_1, \ldots, y_m)$ be an MSO formula containing free set variables $X_1, \ldots, X_l$ and first-order variables $y_1, \ldots, y_m$.

Like the evaluation of MSO sentences, the evaluation algorithm also consists of four steps. The first two steps of the evaluation is the same as those of the evaluation of MSO sentences. The 3rd step is also similar, a formula $\varphi^*(\overline{X}_1, \ldots, \overline{X}_l, \overline{Y}_1, \ldots, \overline{Y}_m)$ over the vocabulary $\{E_1, E_2\} \cup \{P_c | c \in \Sigma_k\}$ is obtained from $\varphi(X_1, \ldots, X_l, y_1, \ldots, y_m)$ (over the vocabulary $\{E\}$) such that the conditions specified in Lemma \ref{thm:soundness} are satisfied. The main difference is in the 4th step.

Because $\varphi^*$ is not a sentence and Theorem \ref{thm:automatonconstruction} only applies to MSO sentences, we cannot construct directly a tree automaton from $\varphi^*$ and run the automaton over $T'$. However, we can replace the free set variables in $\varphi^*(\overline{X}_1, \ldots, \overline{X}_l, \overline{Y}_1, \ldots, \overline{Y}_m)$ by some appropriate new unary relation names and transform it into a sentence $\varphi^{**}$. Let $\Sigma_k' := \Sigma_k \times \{0, 1\}^{(k+1)(l+m)}$, then from $\varphi^{**}$, an automaton $A = (Q, \Sigma_k', \delta, f_0, F)$ can be constructed such that for each $\Sigma_k'$-labeled tree $S'$, $S' \models \varphi^{**}$ if and only if $A$ accepts $S'$.

A $\Sigma_k$-labeled tree $S = (S, H, r, M)$ together with $\overline{U}_1, \ldots, \overline{U}_l, \overline{W}_1, \ldots, \overline{W}_m \subseteq S$ leads to a $\Sigma_k'$-labeled tree $(S, H, r, M')$, denoted by $(S; \overline{U}_1, \ldots, \overline{U}_l, \overline{W}_1, \ldots, \overline{W}_m)$, in a natural way: $M'(s) = (M(s), \bar{e}, \bar{\theta})$, where

$$
\varepsilon_{(k+1)(i-1)+j} = 1 \text{ iff } s \in U_i^j \text{ for all } 1 \leq i \leq l, 1 \leq j \leq k + 1, \text{ and }
$$

$$
\theta_{(k+1)(i-1)+j} = 1 \text{ iff } s \in W_i^j \text{ for all } 1 \leq i \leq m, 1 \leq j \leq k + 1.
$$

Then given a $\Sigma_k$-labeled tree $S$, the computation of the set

$$
\varphi^*(S) := \{\overline{U}_1, \ldots, \overline{U}_l, \overline{W}_1, \ldots, \overline{W}_m \subseteq S | S \models \varphi^*(\overline{U}_1, \ldots, \overline{U}_l, \overline{W}_1, \ldots, \overline{W}_m)\}
$$

can be reduced to the computation of the set

$$
\mathcal{A}(S) := \{\overline{U}_1, \ldots, \overline{U}_l, \overline{W}_1, \ldots, \overline{W}_m \subseteq S | A \text{ accepts } (S; \overline{U}_1, \ldots, \overline{U}_l, \overline{W}_1, \ldots, \overline{W}_m)\}.
$$

Now we recall how $S = (S, H, r, M)$ can be passed by $A = (Q, \Sigma_k', \delta, f_0, F)$ for three times, first in bottom-up, then top-down, finally bottom-up again, to compute $\mathcal{A}(S)$.

1. **Bottom-up.** From leaves to the root, for each $s \in S$, the set of “potential states” of $s$, denoted $Pot_s$, is computed inductively: If $s$ is a leaf, then $Pot_s := \{f_0(M(s), \bar{e}, \bar{\theta}) | \bar{e} \in \{0, 1\}^{l(k+1)}, \bar{\theta} \in \{0, 1\}^{m(k+1)}\}$. For an inner vertex $s$ with a child $s'$,

$$
Pot_s := \{\delta(q', (M(s), \bar{e}, \bar{\theta})) | q' \in Pot_{s'}, \bar{e} \in \{0, 1\}^{l(k+1)}, \bar{\theta} \in \{0, 1\}^{m(k+1)}\}.
$$

For an inner vertex $s$ with two children $s_1$ and $s_2$,

$$
Pot_s := \{\delta(q_1, q_2, (M(s), \bar{e}, \bar{\theta})) | q_1 \in Pot_{s_1}, q_2 \in Pot_{s_2}, \bar{e} \in \{0, 1\}^{l(k+1)}, \bar{\theta} \in \{0, 1\}^{m(k+1)}\}.
$$

2. **Top-down.** Starting from the root $r$, for each $s \in S$, the set of “successful states” of $s$, denoted $Suc_s$, is computed: let $Suc_r := F \cap Pot_r$, and for $s \in S$ with parent $t$ and no sibling,

$$
Suc_s := \{q \in Pot_s | \exists \bar{e}, \bar{\theta}, \text{ such that } \delta(q, (M(t), \bar{e}, \bar{\theta})) \in Suc_t\}.
$$

For $s \in S$ with parent $t$ and a sibling $s'$,

$$
Suc_s := \{q \in Pot_s | \exists q' \in Pot_{s'}, \bar{e}, \bar{\theta}, \text{ such that } \delta(q, q', (M(t), \bar{e}, \bar{\theta})) \in Suc_t\}.
$$
(3) **Bottom-up again.** For \( s \in S \), let \( S_s \) denote the subtree of \( S \) with \( s \) as the root. Starting from the leaves, for each \( s \in S \) and \( q \in Suc_s \), compute \( Sat_{s,q} \). Intuitively, a tuple \( \overline{B},\overline{C} \subseteq S_s \) is in \( Sat_{s,q} \) if it is the restriction of a “satisfying assignment” \( \overline{B},\overline{C} \in A(S) \) to \( S_s \), and for the run of \( A \) over \( (S;\overline{B},\overline{C}) \), the state of the run at \( s \) is \( q \).

Let \( s \in S \) and \( q \in Suc_s \). Set \( B^*_1 := \{s\} \) and \( B^*_0 := \emptyset \).

If \( s \) is a leaf, then

\[
Sat_{s,q} := \{(B_{e_1}, \ldots, B_{e_{i(k+1)}}; C_{\theta_1}, \ldots, C_{\theta_{m(k+1)}}) \mid q = f_0(M(s), \bar{\varepsilon}, \bar{\theta})\}
\]

If \( s \) is an inner vertex with one child \( s' \), then

\[
Sat_{s,q} := \left\{ \begin{array}{c}
(B_1' \cup B_2' \cup B_{e_1} \cup B_{e_{i(k+1)}}; C_1' \cup C_{\theta_1} \cup \cdots \cup C_{\theta_{m(k+1)}} \cup C_{\theta_{m(k+1)}}) \\
\exists q' \in Suc_{s'} \text{ such that, } q = \delta(q', (M(s), \varepsilon, \bar{\theta})), (\overline{B'}, \overline{C'}) \in Sat_{s',q'}
\end{array} \right\}
\]

If \( s \) is an inner vertex with two children \( s_1 \) and \( s_2 \), then

\[
Sat_{s,q} := \left\{ \begin{array}{c}
\exists q_1 \in Suc_{s_1}, q_2 \in Suc_{s_2} \text{ such that, } q = \delta(q_1, q_2, (M(s), \varepsilon, \bar{\theta})), (\overline{B'}, \overline{C'}) \in Sat_{s_1, q_1}, (\overline{B'}, \overline{C'}) \in Sat_{s_2, q_2}
\end{array} \right\}
\]

Then \( A(S) = \bigcup_{q \in Suc_s} Sat_{s,q} \).

Therefore, we can run \( A \) over \( T' \) for three times to compute \( A(T') \). Finally from \( A(T') \), we can construct \( \varphi(G) = \{(S_1, \ldots, S_i, v_1, \ldots, v_m) \mid G \models \varphi(S_1, \ldots, S_i, v_1, \ldots, v_m)\} \) according to the mechanism to encode the elements and sets of \( V \) into the subsets of \( T' \).

### C.2 Distributed evaluation of \( \psi^{(r)}(x) \) over \( G[i, i + 2r] \)'s

Now we consider the distributed evaluation of \( \psi^{(r)}(x) \) over \( G[i, i + 2r] \)'s.

Because \( \psi^{(r)}(x) \) is a local formula, it is sufficient to evaluate \( \psi^{(r)}(x) \) over each connected component of \( G[i, i + 2r] \).

Let \( C_i \) be a connected component of \( G[i, i + 2r] \), and \( w_1^i, \ldots, w_k^i \) be the nodes contained in \( C_i \) with distance \( i \) from the requesting node. Now we consider the evaluation of \( \psi^{(r)}(x) \) over \( C_i \).

Let \( C_i' \) be the graph obtained from \( C_i \) by including all ancestors of \( w_1^i, \ldots, w_k^i \), and \( C_i^* \) be the graph obtained from \( C_i' \) by contracting all the ancestors of \( w_1^i, \ldots, w_k^i \) into one vertex \( v^* \), i.e. \( C_i^* \) has one more vertex \( v^* \) than \( C_i' \), and \( v^* \) is connected to \( w_1^i, \ldots, w_k^i \). It is easy to see that \( C_i^* \) is a planar graph with a BFS tree rooted on \( v^* \) with depth at most \( 2r + 1 \). Consequently \( C_i^* \) is a planar graph with bounded diameter, a graph with bounded tree-width. Because \( C_i \) is a subgraph of \( C_i^* \), \( C_i \) is a planar graph with bounded tree-width as well.

Our purpose is to construct distributively an ordered tree decomposition for \( C_i \), and evaluate \( \psi^{(r)}(x) \) by using the automata-theoretic technique.

The distributed construction of an ordered tree decomposition for a planar network with bounded diameter is as follows [GW09]:

- Do a depth-first-search to decompose the network into blocks, i.e. biconnected components;
• Construct an ordered tree decomposition for each nontrivial block: Traverse every face of the block according to the cyclic ordering at each node, triangulate all those faces, and connect the triangles into a tree decomposition by utilizing the pre-computed BFS tree;

• Finally the tree decompositions for the blocks are connected together into a complete tree decomposition for the whole network.

The blocks of \( C_i^* \) enjoy the following property.

**Lemma 11.** Let

- \( C_i \) be a connected component of \( G[i, i+2r] \),
- \( w_1^i, \ldots, w_l^i \) be all the nodes contained in \( C_i \) with distance \( i \) from the requesting node,
- \( C_i' \) be the graph obtained from \( C_i \) by including all ancestors of \( w_1^i, \ldots, w_l^i \),
- \( C_i^* \) be the graph obtained from \( C_i' \) by contracting all the ancestors of \( w_1^i, \ldots, w_l^i \) into one vertex.

Then the virtual vertex \( v^* \) and all the \( w_1^i, \ldots, w_l^i \) are contained in a unique block \( B_0 \) of \( C_i^* \), and for each block \( B \neq B_0 \), there is a \( w_j^i \) such that

\[
V(B) \subseteq \{ u \in V(C_i) | u \text{ is a descendant of } w_j^i \text{ in the BFS tree} \}.
\]

The distributed tree decomposition of \( C_i \) can be constructed as follows: Starting from some \( w_j^i \) \((1 \leq j \leq l)\), do a depth-first-search to decompose \( C_i' \) into blocks by imagining that there is a virtual node \( v^* \), then \( v^* \) and all \( w_1^i, \ldots, w_l^i \) belong to a unique biconnected component \( B_0 \). Construct an ordered tree decomposition for each block, and do some special treatments for \( B_0 \) (when the virtual node \( v^* \) is visited). Finally connect these tree decompositions together in a suitable way to get a complete tree decomposition of \( C_i \).

Moreover, a post-order traversal over the BFS tree can be done to construct the tree decompositions for connected components of all \( G[i, i+2r] \)'s one by one.

With the ordered tree decomposition for \( C_i \), \( \psi^{(r)}(x) \) can be evaluated over \( C_i \) as follows: the node \( w_j^i \) first transforms \( \psi^{(r)}(x) \) into a formula \( \psi^*(U_1, \ldots, U_{k+1}) \) over the vocabulary \( \{E_1, E_2\} \cup \{P_c | c \in \Sigma_k\} \) satisfying the condition in Lemma 10. Then from \( \psi^* \), constructs an automaton \( A \) over \( \Sigma_k \)-labeled trees, and sends \( A \) to all the nodes in \( C_i \). The ordered tree decomposition is then transformed into a \( \Sigma_k \)-labeled tree \( T' \). Finally \( A \) is ran over \( T' \) for three times to get \( A(T') \), and the evaluation result of \( \psi^{(r)}(x) \) over \( C_i \) is distributively stored on the nodes of \( C_i \).

Because the most intricate part of Phase IV is the distributed construction of an ordered tree decomposition for each connected component \( C_i \) of \( G[i, i+2r] \). In the following, we only illustrate how to do a post-order traversal of the BFS tree to decompose each connected component \( C_i \) of \( G[i, i+2r] \) into blocks and construct an ordered tree decomposition for each block of \( C_i \), and omit the other parts of Phase IV.
Initialization

The requesting node sets \( \text{traversed}(1) := \text{true} \), and sends message \text{POSTTRAVERSE} over port 1.

Message \text{POSTTRAVERSE} over port \( l \)

% Without loss of generality, suppose that \( \text{treeDepth} > 2r \).

if \( \text{depth}(v) + 2r = \text{treeDepth} \) or \( v \) is a leaf then
  if \( \text{treeDecompOver}(\text{depth}(v)) = \text{false} \) then
    \( \text{DFSDepth}(\text{depth}(v), v) := 1, \text{DFSLow}(\text{depth}(v), v) := 0. \)
    \( \text{DFSRoot}(\text{depth}(v)) := v, \text{DFSVisited}(\text{depth}(v), v) := \text{true}. \)
    \( l' := \) the minimal port with state “child” or “downward” or “horizon”.
    \( \text{DFSState}(\text{depth}(v), l') := \text{“child”}. \)
    \( v \) sends \( \text{DFSFORWARD}(\text{depth}(v), v, 1, 1) \) over \( l' \).
  else
    \( v \) sends \text{BACKTRACK} over port \( l' \) such that \( \text{state}(l') = \text{“parent”}. \)
  end if
else if \( \text{depth}(v) + 2r < \text{treeDepth} \) then
  \( l' := \) the minimal port with state “child”.
  \( \text{traversed}(l') := \text{true}, v \) sends message \text{POSTTRAVERSE} over \( l' \).
end if

Message \text{BACKTRACK} over port \( l \)

if there is at least one port \( l' \) such that \( \text{state}(l') = \text{“child”} \) and \( \text{traversed}(l') = \text{false} \) then
  \( l' := \) the minimal such port.
  \( \text{traversed}(l') := \text{true}, v \) sends \text{POSTTRAVERSE} over \( l' \).
else
  if \( \text{treeDecompOver}(\text{depth}(v)) = \text{false} \) then
    \( \text{DFSDepth}(\text{depth}(v), v) := 1, \text{DFSLow}(\text{depth}(v), v) := 0. \)
    \( \text{DFSRoot}(\text{depth}(v)) := v, \text{DFSVisited}(\text{depth}(v), v) := \text{true}. \)
    \( l' := \) minimal port with state “child” or “downward” or “horizon”.
    \( \text{DFSState}(\text{depth}(v), l') := \text{“child”}. \)
    \( v \) sends \( \text{DFSFORWARD}(\text{depth}(v), v, 1, 1) \) over \( l' \).
  else
    if \( v \) is not the requesting node then
      \( v \) sends \text{BACKTRACK} over port \( l' \) such that \( \text{state}(l') = \text{“parent”}. \)
    end if
  end if
end if
Message DFSFORWARD($rBFSDepth$, $rId$, $nextBlockId$, $parentDFSDepth$) over port $l$

if $DFSvisited(rBFSDepth, v) = \text{false}$ then
    $DFSvisited(rBFSDepth, v) := \text{true}$, $DFSState(rBFSDepth, l) := \text{“parent”}$.
    $DFSRootId(rBFSDepth) := rId$, $DFSDepth(rBFSDepth, v) := parentDFSDepth + 1$.
    $DFSlow(rBFSDepth, v) := DFSDepth(rBFSDepth, v)$.

if $\text{depth}(v) = rBFSDepth$ then
    for each port $l' \neq l$ such that $\text{state}(l') = \text{“child”}$ or $\text{“horizon”}$ or $\text{“downward”}$ do
        $DFSState(rBFSDepth, l') := \text{“unvisited”}$.
    end for
    $DFSlow(rBFSDepth, v) := 0$.

if there exist at least one port $l' \neq l$ such that $\text{state}(l') = \text{“child”}$ or $\text{“horizon”}$ or $\text{“downward”}$ then
    Let $l'$ be the minimal such port.
    $v$ sets $DFSState(rBFSDepth, l') := \text{“child”}$.
    $v$ sends DFSFORWARD($rBFSDepth$, $rId$, $nextBlockId$, $DFSDepth(rBFSDepth, v)$) over $l'$.
else
    $v$ sends DFSBACKTRACK($rBFSDepth$, $nextBlockId$, $DFSlow(rBFSDepth, v)$) over $l$.
    $v$ sends DFSBLOCKACK($rBFSDepth$) over $l$.
end if

else if $\text{depth}(v) = rBFSDepth + 2r$ then
    for each port $l' \neq l$ such that $\text{state}(l') = \text{“parent”}$ or $\text{“horizon”}$ or $\text{“upward”}$ do
        $DFSState(rBFSDepth, l') := \text{“unvisited”}$.
    end for
    if there exist at least one port $l' \neq l$ such that $\text{state}(l') = \text{“parent”}$ or $\text{“horizon”}$ or $\text{“upward”}$ then
        Let $l'$ be the minimal such port.
        $v$ sets $DFSState(rBFSDepth, l') := \text{“child”}$.
        $v$ sends DFSFORWARD($rBFSDepth$, $rId$, $nextBlockId$, $DFSDepth(rBFSDepth, v)$) over $l'$.
    else
        $v$ sends DFSBACKTRACK($rBFSDepth$, $nextBlockId$, $DFSlow(rBFSDepth, v)$) over $l$.
        $v$ sends DFSBLOCKACK($rBFSDepth$) over $l$.
    end if
else
    $DFSState(rBFSDepth, l') := \text{“unvisited”}$ for each port $l' \neq l$.
    if there exist at least one port $l' \neq l$ then
        Let $l'$ be the minimal such port.
        $v$ sets $DFSState(rBFSDepth, l') := \text{“child”}$.
        $v$ sends DFSFORWARD($rBFSDepth$, $rId$, $nextBlockId$, $DFSDepth(rBFSDepth, v)$) over $l'$.
    else
        $v$ sends DFSBACKTRACK($rBFSDepth$, $nextBlockId$, $DFSlow(rBFSDepth, v)$) over $l$.
        $v$ sends DFSBLOCKACK($rBFSDepth$) over $l$.
    end if
else
    $DFSState(rBFSDepth, l) := \text{“non-tree-forward”}$.
    $v$ sends DFSRESTART($rBFSDepth$, $nextBlockId$, $DFSDepth(rBFSDepth, v)$) over $l$.
end if
Message DFSBACKTRACK($rBFSDepth$, $nextBlockId$, $childDFSLow$) over port $l$

if $childDFSLow = DFSDepth(rBFSDepth, v)$ then
  DFSState($rBFSDepth$, $l$) := “closed”.
  blockIds($rBFSDepth$) := blockIds($rBFSDepth$) ∪ {$nextBlockId$}.
  blockPorts($rBFSDepth$, $nextBlockId$) := {$l$}.
  v sends message DFSINFORM($rBFSDepth$, $nextBlockId$) over $l$.
  $nextBlockId := nextBlockId + 1$.
else if $childDFSLow > DFSDepth(rBFSDepth, v)$ then
  DFSState($rBFSDepth$, $l$) := “childBridge”.
else
  DFSState($rBFSDepth$, $l$) := “backtracked”.
  DFSLow($rBFSDepth$, $v$) := min{$DFSLow(rBFSDepth, v), childDFSLow$}.
end if

if $depth(v) = rBFSDepth$ then
  if there exists at least one port $l'$ such that $DFSState(rBFSDepth, l') = “unvisited” and
  state($l'$) = “child” or “horizon” or “downward” then
    Let $l'$ be the minimal such port.
    DFSState($rBFSDepth$, $l'$) := “child”.
    v sends message DFSFORWARD($rBFSDepth$, DFSRootId($rBFSDepth$),
    $nextBlockId$, DFSDepth($rBFSDepth$, $v$)) over $l'$.
  else if $v = DFSRootId(rBFSDepth)$ then
    blockIds($rBFSDepth$) := blockIds($rBFSDepth$) ∪ {$nextBlockId$}.
    specialBlockId($rBFSDepth$) := $nextBlockId$.
    blockPorts($rBFSDepth$, $nextBlockId$) := {$l'|DFSState(rBFSDepth, l') = “backtracked”}.
    v sends message DFSINFORM($rBFSDepth$, $nextBlockId$) over all $l'$ such that $DFSState(rBFSDepth, l') = “backtracked”$.
  else
    v sends message DFSBACKTRACK($rBFSDepth$, $nextBlockId$, DFSLow($rBFSDepth$, $v$))
    over $l'$ such that $DFSState(rBFSDepth, l') = “parent”$.
  end if
else if $depth(v) = rBFSDepth + 2$ then
  if there exists at least one port $l'$ such that $DFSState(rBFSDepth, l') = “unvisited”
  and state($l'$) = “parent” or “horizon” or “upward” then
    Let $l'$ be the minimal such port.
    DFSState($rBFSDepth$, $l'$) := “child”.
    v sends message DFSFORWARD($rBFSDepth$, DFSRootId($rBFSDepth$),
    $nextBlockId$, DFSDepth($rBFSDepth$, $v$)) over $l'$.
  else
    v sends message DFSBACKTRACK($rBFSDepth$, $nextBlockId$, DFSLow($rBFSDepth$, $v$))
    over $l'$ such that $DFSState(rBFSDepth, l') = “parent”$.
  end if
else
  if there exists at least one port $l'$ such that $DFSState(rBFSDepth, l') := “unvisited” then
    Let $l'$ be the minimal such port.
    DFSState($rBFSDepth$, $l'$) := “child”.
    v sends message DFSFORWARD($rBFSDepth$, DFSRootId($rBFSDepth$),
    $nextBlockId$, DFSDepth($rBFSDepth$, $v$)) over $l'$.
  else
    v sends message DFSBACKTRACK($rBFSDepth$, $nextBlockId$, DFSLow($rBFSDepth$, $v$))
    over $l'$ such that $DFSState(rBFSDepth, l') = “parent”$.
  end if
end if
Message DFSINFORM(rBFSDepth, blockId) over port \( l \).

\[
\text{if } \text{blockId} \notin \text{blockIds}(rBFSDepth) \text{ then}
\]
\[
\text{blockIds}(rBFSDepth) := \text{blockIds}(rBFSDepth) \cup \{\text{blockId}\}.
\]
\[
\text{blockPorts}(rBFSDepth, \text{blockId}) :=
\{[l'] : \text{DFSState}(rBFSDepth, l') = \text{“parent”} \text{ or “non-tree-backward” or “backtracked”}\}.
\]
\[
\text{if there are ports } l' \text{ such that } \text{DFSState}(rBFSDepth, l') = \text{“non-tree-backward”} \text{ then}
\]
\[
v \text{ sends DFSBLOCKPORT}(rBFSDepth, \text{blockId})
\]
\[
\text{ over all ports } l' \text{ such that } \text{DFSState}(rBFSDepth, l') = \text{“non-tree-backward”}.
\]
\[
\text{else}
\]
\[
\text{if there exists at least one port } l' \text{ such that } \text{DFSState}(rBFSDepth, l') = \text{“backtracked”} \text{ then}
\]
\[
v \text{ sends message DFSINFORM}(rBFSDepth, \text{blockId}) \text{ over all these ports.}
\]
\[
\text{else}
\]
\[
v \text{ sends message DFSBLOCKOVER}(rBFSDepth, \text{blockId})
\]
\[
\text{ over the port } l' \text{ such that } \text{DFSState}(rBFSDepth, l') = \text{“parent”}.
\]
\[
\text{if there are no ports } l'
\]
\[
\text{ such that } \text{DFSState}(rBFSDepth, l') = \text{“closed”} \text{ or “backtracked” or “childBridge”} \text{ then}
\]
\[
v \text{ sends DFSBLOCKACK over } l' \text{ such that } \text{DFSState}(rBFSDepth, l') = \text{“parent”}.
\]
\[
\text{end if}
\]
\[
\text{end if}
\]
\[
\text{end if}
\]

Message DFSBLOCKOVER(rBFSDepth, blockId) over port \( l \).

\[
bDFSBlockOver(rBFSDepth, l) := \text{true}.
\]
\[
\text{if } v = \text{DFSRootId}(rBFSDepth) \text{ then}
\]
\[
\text{if } bDFSBlockOver(rBFSDepth, l') = \text{true} \text{ for each } l' \text{ such that}
\]
\[
l' \in \text{blockPorts}(rBFSDepth, \text{blockId}), \text{ and } bDFSBlockAck(rBFSDepth, l') = \text{true} \text{ for each port } l'
\]
\[
such that \text{DFSState}(rBFSDepth, l') = \text{“closed”} \text{ or “backtracked” or “childBridge”} \text{ then}
\]
\[
v \text{ sends messages to do a post-order traversal of the constructed DFS tree}
\]
\[
in order to do the tree decomposition for each block,
\]
\[
\text{by using the subtrees of the BFS tree } T, \text{ moreover, some special treatment should be done}
\]
\[
\text{for the block containing all vertices } v' \text{’s such that } \text{depth}(v') = \text{depth}(v).
\]
\[
v \text{ sends messages to connect all these tree decompositions of the blocks together}
\]
\[
to get a complete tree decomposition.
\]
\[
v \text{ sends message BACKTRACK over } l' \text{ such that } \text{state}(l') = \text{“parent”}.
\]
\[
\text{end if}
\]
\[
\text{end if}
\]
\[
\text{else}
\]
\[
\text{if } \text{DFSState}(rBFSDepth, l) = \text{“backtracked”} \text{ then}
\]
\[
\text{if } bDFSBlockOver(rBFSDepth, l') = \text{true} \text{ for each } l'
\]
\[
such that \text{DFSState}(rBFSDepth, l') = \text{“backtracked”} \text{ then}
\]
\[
v \text{ sends DFSBLOCKOVER}(rBFSDepth, \text{blockId}) \text{ over } l'
\]
\[
such that \text{DFSState}(rBFSDepth, l') = \text{“parent”}.
\]
\[
\text{end if}
\]
\[
\text{else}
\]
\[
\text{if } bDFSBlockOver(rBFSDepth, l') = \text{true} \text{ for each } l'
\]
\[
such that \text{DFSState}(rBFSDepth, l') = \text{“closed”},
\]
\[
\text{and } bDFSBlockAck(rBFSDepth, l') = \text{true} \text{ for each port } l'
\]
\[
such that \text{DFSState}(rBFSDepth, l') = \text{“closed”} \text{ or “backtracked” or “childBridge”} \text{ then}
\]
\[
v \text{ sends DFSBLOCKACK}(rBFSDepth) \text{ over } l' \text{ such that } \text{DFSState}(rBFSDepth, l') = \text{“parent”}.
\]
\[
\text{end if}
\]
\[
\text{end if}
\]
\[
\text{end if}
\]
\[
22
\]
Message DFSBLOCKACK\((rBFSDepth)\) over port \(l\).

\(bDFSB\text{lock}Ack(rBFS\text{Depth}, l) := true.\)

\textbf{if} \(v = DFS\text{Root}\text{Id}(rBFS\text{Depth})\) \textbf{then}

\hspace{1em} \textbf{if} \(bDFS\text{BlockOver}(rBFS\text{Depth}, l') = true\) for each \(l'\) such that \(DFS\text{State}(rBFS\text{Depth}, l') = \text{"backtracked"},\)

\hspace{2em} \text{and} \(bDFS\text{BlockAck}(rBFS\text{Depth}, l') = true\) for each port \(l'\)

\hspace{3.5em} \text{such that} \(DFS\text{State}(rBFS\text{Depth}, l') = \text{"closed" or "backtracked" or "childBridge"}\) \textbf{then}

\hspace{5em} \(v\) sends messages to construct a tree decomposition for each block.

\hspace{1em} From Lemma 11, each block, except the block containing all \(v'\)'s such that \(\text{depth}(v') = \text{depth}(v),\)

\hspace{2em} is a planar network with bounded diameter.

\hspace{1em} A tree decomposition for each block can be distributively constructed

\hspace{2em} by doing a postorder traversal of the subtree of the constructed DFS tree,

\hspace{3em} visiting all the boundaries of the faces of the block (which are cycles),

\hspace{4em} triangulating each face of the block, and using the subtrees of the pre-computed BFS tree
to get a tree decomposition (c.f. [GW09]).

\hspace{2em} Moreover, some special treatments should be done

\hspace{3em} for the block containing all \(v'\)'s such that \(\text{depth}(v') = \text{depth}(v).\)

\hspace{4em} Then \(v\) sends messages to connect all these tree decompositions of the blocks together

\hspace{5em} to get a complete tree decomposition and evaluate \(\psi(r)(x)\) by using this tree decomposition.

\hspace{1em} Finally \(v\) sends message BACKTRACK over \(l'\) such that \(\text{state}(l') = \text{"parent"}.\)

\textbf{end if}

\textbf{else}

\hspace{1em} \textbf{if} \(bDFS\text{BlockOver}(rBFS\text{Depth}, l') = true\) for each \(l'\) such that \(DFS\text{State}(rBFS\text{Depth}, l') = \text{"closed"},\)

\hspace{2em} \text{and} \(bDFS\text{BlockAck}(rBFS\text{Depth}, l') = true\) for each port \(l'\)

\hspace{3.5em} \text{such that} \(DFS\text{State}(rBFS\text{Depth}, l') = \text{"closed" or "backtracked" or "childBridge"}\) \textbf{then}

\hspace{5em} \(v\) sends DFSBLOCKACK\((rBFS\text{Depth})\) over \(l'\) such that \(DFS\text{State}(rBFS\text{Depth}, l') = \text{"parent"}.\)

\textbf{end if}

\textbf{end if}

Message DFSBLOCKPORT\((rBFS\text{Depth}, blockId)\) over port \(l\).

\(\text{blockPorts}(rBFS\text{Depth}, blockId) := \text{blockPorts}(rBFS\text{Depth}, blockId) \cup \{l\}.\)

\(v\) sends message DFSBLOCKPORTACK\((rBFS\text{Depth}, blockId)\) over port \(l\).

Message DFSBLOCKPORTACK\((rBFS\text{Depth}, blockId)\) over port \(l\).

\(\text{blockPortAck}(rBFS\text{Depth}, l) := true.\)

\textbf{if} \(\text{blockPortAck}(rBFS\text{Depth}, l') = true\) for each \(l'\)

\hspace{1em} such that \(DFS\text{State}(rBFS\text{Depth}, l') = \text{"non-tree-backward"}\) \textbf{then}

\hspace{2em} \textbf{if} there exists at least one port \(l'\) such that \(DFS\text{State}(rBFS\text{Depth}, l') = \text{"backtracked"}\) \textbf{then}

\hspace{3.5em} \(v\) sends message DFSINFORM\((rBFS\text{Depth}, blockId)\) over all these ports.

\hspace{2em} \textbf{else}

\hspace{3.5em} \(v\) sends message DFSBLOCKOVER\((rBFS\text{Depth}, blockId)\)

\hspace{5em} over the port \(l'\) such that \(DFS\text{State}(rBFS\text{Depth}, l') = \text{"parent"}.\)

\hspace{4.5em} \textbf{if} there are no ports \(l'\)

\hspace{5.5em} such that \(DFS\text{State}(rBFS\text{Depth}, l') = \text{"closed" or "backtracked" or "childBridge"}\) \textbf{then}

\hspace{7em} \(v\) sends DFSBLOCKACK over \(l'\) such that \(DFS\text{State}(rBFS\text{Depth}, l') = \text{"parent"}.\)

\hspace{4em} \textbf{end if}

\hspace{3.5em} \textbf{end if}

\hspace{2em} \textbf{end if}

\hspace{1.5em} \textbf{end if}
Message DFSRESTART\((rBFS\text{Depth}, nextBlockId, ancestorDFSDepth)\) over port \(l\).

\[
\text{DFSState}(rBFS\text{Depth}, l) := \text{“non-tree-backward”},
\]

\[
\text{DFSLow}(rBFS\text{Depth}, v) := \min\{\text{DFSLow}(rBFS\text{Depth}, v), ancestorDFSDepth\}.
\]

**if** depth\((v) = rBFS\text{Depth} \text{ then}**

**if** there exist at least one port \(l'\) such that \(\text{DFSState}(rBFS\text{Depth}, l') := \text{“unvisited”}

and state\((l') := \text{“child” or “horizon” or “downward” then}**

Let \(l'\) be the minimal such port, \(\text{DFSState}(rBFS\text{Depth}, l') := \text{“child”}.

\(v\) sends DFSFORWARD\((rBFS\text{Depth}, DFS\text{RootId}(rBFS\text{Depth}), nextBlockId, DFS\text{Depth}(rBFS\text{Depth}, v))\) over \(l'\).

**else if** \(v = DFS\text{RootId}(rBFS\text{Depth})\) **then**

\(\text{blockIds}(rBFS\text{Depth}) := \text{blockIds}(rBFS\text{Depth}) \cup \{nextBlockId\}.

\(\text{specialBlockId}(rBFS\text{Depth}, nextBlockId) := \{l'\text{DFSState}(rBFS\text{Depth}, l') = \text{“backtracked”}\}.

\(v\) sends message DFSINFORM\((rBFS\text{Depth}, nextBlockId)\) over all \(l'\) such that \(\text{DFSState}(rBFS\text{Depth}, l') = \text{“backtracked”}.

**else**

\(v\) sends DFSBACKTRACK\((rBFS\text{Depth}, nextBlockId, DFSLow(rBFS\text{Depth}, v))\)

over \(l'\) such that \(\text{DFSState}(rBFS\text{Depth}, l') = \text{“parent”}.

**if** there are no ports \(l'\) such that

\(\text{DFSState}(rBFS\text{Depth}, l') = \text{“closed” or “backtracked” or “childBridge” then}**

\(v\) sends DFSBLOCKACK\((rBFS\text{Depth})\) over \(l'\) such that \(\text{DFSState}(rBFS\text{Depth}, l') = \text{“parent”}.

**end if**

**end if**

**else if** depth\((v) = rBFS\text{Depth} + 2r \text{ then}**

**if** there exist at least one port \(l'\) such that \(\text{DFSState}(rBFS\text{Depth}, l') := \text{“unvisited”}

and state\((l') := \text{“parent” or “horizon” or “upward” then}**

Let \(l'\) be the minimal such port, \(\text{DFSState}(rBFS\text{Depth}, l') := \text{“child”}.

\(v\) sends DFSFORWARD\((rBFS\text{Depth}, DFS\text{RootId}(rBFS\text{Depth}), nextBlockId, DFS\text{Depth}(rBFS\text{Depth}, v))\) over \(l'\).

**else**

\(v\) sends DFSBACKTRACK\((rBFS\text{Depth}, nextBlockId, DFSLow(rBFS\text{Depth}, v))\)

over \(l'\) such that \(\text{DFSState}(rBFS\text{Depth}, l') = \text{“parent”}.

**if** there are no ports \(l'\) such that

\(\text{DFSState}(rBFS\text{Depth}, l') = \text{“closed” or “backtracked” or “childBridge” then}**

\(v\) sends DFSBLOCKACK\((rBFS\text{Depth})\) over \(l'\) such that \(\text{DFSState}(rBFS\text{Depth}, l') = \text{“parent”}.

**end if**

**end if**

**else**

**if** there exist at least one port \(l'\) such that \(\text{DFSState}(rBFS\text{Depth}, l') := \text{“unvisited”}

and state\((l') := \text{“child”} \text{ then}**

Let \(l'\) be the minimal such port, \(\text{DFSState}(rBFS\text{Depth}, l') := \text{“child”}.

\(v\) sends DFSFORWARD\((rBFS\text{Depth}, DFS\text{RootId}(rBFS\text{Depth}), nextBlockId, DFS\text{Depth}(rBFS\text{Depth}, v))\) over \(l'\).

**else**

\(v\) sends DFSBACKTRACK\((rBFS\text{Depth}, nextBlockId, DFSLow(rBFS\text{Depth}, v))\)

over \(l'\) such that \(\text{DFSState}(rBFS\text{Depth}, l') = \text{“parent”.}

**if** there are no ports \(l'\) such that

\(\text{DFSState}(rBFS\text{Depth}, l') = \text{“closed” or “backtracked” or “childBridge” then}**

\(v\) sends DFSBLOCKACK\((rBFS\text{Depth})\) over \(l'\) such that \(\text{DFSState}(rBFS\text{Depth}, l') = \text{“parent”}.

**end if**

**end if**
Let $C_i$ be a connected component of $G[i, i + 2r]$, and $w_i^1, \ldots, w_i^j$ be all the nodes contained in $C_i$ with distance $i$ from the requesting node. In the following, we will consider the distributed construction of an ordered tree decomposition for the special block of $C_i$ which contains all $w_i^1, \ldots, w_i^j$, by imagining that there is a virtual vertex $v_i^*$ connected to all $w_i^1, \ldots, w_i^j$, and illustrate the special treatments that should be done.

The distributed construction of an ordered tree decomposition for the special block of $C_i$, by imagining that there is a virtual vertex and doing some special treatments.

Initialization

Let $u_0$ satisfy that $\text{DFSRoot}(\text{depth}(u_0)) = u_0$.
Suppose for each node $v$ and $i \in C(v)$, $v$ stores in $\text{BFSAncestors}(i)$ a list of all its ancestors that are of depth from $i$ to $\text{depth}(v)$ in the BFS tree.
Suppose for each $v$ and port $l'$, $v$ stores in $\text{neighbor}(l')$ the neighbor of $v$ corresponding to $l'$.
Let $l$ be the minimal port $l$ such that $\text{DFSState}(\text{depth}(u_0), l) = \text{"backtracked"}$.
$\text{DFSPostTraversed}(\text{depth}(u_0), l) := \text{true}$.

Message $\text{DFSPOSTTRAVERSE}(r\text{BFSDepths}, sp\text{BlockId})$ over port $l$.

if $v$ has no ports $l'$ such that $l' \in \text{blockPorts}(r\text{BFSDepths}, sp\text{BlockId})$
and $\text{DFSState}(r\text{BFSDepths}, l') = \text{"backtracked"}$ then
if there exist $l'$ such that $l' \in \text{blockPorts}(r\text{BFSDepths}, sp\text{BlockId})$
and $\text{arcVisited}(r\text{BFSDepths}, sp\text{BlockId}, v, \text{neighbor}(l')) = \text{false}$ then
for each such $l'$ do
  $\text{arcVisited}(r\text{BFSDepths}, sp\text{BlockId}, v, \text{neighbor}(l')) := \text{true}$.
  $v$ sends $\text{DFSFACESTART}((r\text{BFSDepths}, sp\text{BlockId}, v, \text{neighbor}(l')), \text{BFSAncestor}(r\text{BFSDepths}), \text{BFSAncestor}(r\text{BFSDepths}))$ over $l'$.
end for
else
  $v$ sends $\text{DFSPOSTBACKTRACK}(r\text{BFSDepths}, sp\text{BlockId})$ over $l'$ such that $\text{DFSState}(r\text{BFSDepths}, l') = \text{"parent"}$.
end if
else
Let $l'$ be the minimal port $l'$ such that
$\text{DFSPostTraversed}(r\text{BFSDepths}, l') = \text{false}$ and $\text{DFSState}(r\text{BFSDepths}, l') = \text{"backtracked"}$.
$\text{DFSPostTraversed}(r\text{BFSDepths}, l') := \text{true}$.
$v$ sends $\text{DFSPOSTTRAVERSE}(r\text{BFSDepths}, sp\text{BlockId})$ over $l'$.
end if
Message DFSFACESTART((rBFSDepth,spBlockId, u1, u2), (w1, ..., w_r), (w'_1, ..., w'_r)) over port l
arcVisited(rBFSDepth,spBlockId,neighbor(l), v) := true.
Let l' be the port such that (v, neighbor(l')) is immediately before (v, neighbor(l)) in the cyclic ordering.
if \(\text{depth}(v) = rBFSDepth\) and state(l') = “parent” or “upward” then
\(v\) sends DFSPECIALTREAT((rBFSDepth,spBlockId,u1,u2,v,BFSAncestors(rBFSDepth))) over l.
else if \(\text{depth}(v) = rBFSDepth + 2r\) and state(l') = “child” or “downward” then
Let l'' be the port satisfying that (v, neighbor(l'')) is the first arc before (v, neighbor(l)) in the cyclic ordering such that l'' \(\in \text{blockPorts}(rBFSDepth, spBlockId)\).
if \(v = u_2\) then
\(v\) sends DFSFACESTART((rBFSDepth,spBlockId, u1,u2), (w_1, ..., w_r), BFSAncestors(rBFSDepth)) over port l''.
else
\(Bag(rBFSDepth, spBlockId, u_1, \text{neighbor}(l), v) := \{(u_1, u_2), \text{list}_{3k+1}((w_1, ..., w_r) \cdot (w'_1, ..., w'_r)) \cdot BFSAncestors(rBFSDepth))\).
list_{3k+1}(x) generates a list of length 3k + 1 by repeating the last element of x.
\(v\) sends DFSACKFACESTART((rBFSDepth,spBlockId,u1,u2), (w_1, ..., w_r), BFSAncestors(rBFSDepth)) over port l.
if \(\text{neighbor}(l'') = u_1\) then
\(v\) sends DFSFACEOVER((rBFSDepth,spBlockId,u1,u2), \(\text{neighbor}(l), (w'_1, ..., w'_r), BFSAncestors(rBFSDepth)\)) over port l''.
else
\(v\) sends DFSFACEWALK((rBFSDepth,spBlockId,u1,u2), (w_1, ..., w_r), BFSAncestors(rBFSDepth)) over port l''.
end if
end if
arcVisited(rBFSDepth,spBlockId,v, neighbor(l'')) := true.
else
if \(v = u_2\) then
\(v\) sends DFSFACESTART((rBFSDepth,spBlockId, u1,u2), (w_1, ..., w_r), BFSAncestors(rBFSDepth)) over port l'.
else
\(Bag(rBFSDepth, spBlockId, u_1, \text{neighbor}(l), v) := \{(u_1, u_2), \text{list}_{3k+1}((w_1, ..., w_r) \cdot (w'_1, ..., w'_r)) \cdot BFSAncestors(rBFSDepth))\).
\(v\) sends DFSACKFACESTART((rBFSDepth,spBlockId,u1,u2), (w_1, ..., w_r), BFSAncestors(rBFSDepth)) over port l.
if \(\text{neighbor}(l') = u_1\) then
\(v\) sends DFSFACEOVER((rBFSDepth,spBlockId,u1,u2), \(\text{neighbor}(l), (w'_1, ..., w'_r), BFSAncestors(rBFSDepth)\)) over port l'.
else
\(v\) sends DFSFACEWALK((rBFSDepth,spBlockId,u1,u2), (w_1, ..., w_r), BFSAncestors(rBFSDepth)) over port l'.
end if
end if
arcVisited(rBFSDepth,spBlockId,v, neighbor(l')) := true.
end if
Message DFSACKFACESTART((rBFSDepth, spBlockId, u_1, u_2), (w_1, \ldots, w_r), (w_1', \ldots, w_s')) over port l

Bag(rBFSDepth, spBlockId, u_1, v, neighbor(l)) :=

\langle(u_1, u_2), \text{list}_{3k+1}((w_1, \ldots, w_r) \cdot \text{BFSAncestors}(rBFSDepth) \cdot (w_1', \ldots, w_s'))\rangle.

Message DFSFACEOVER((rBFSDepth, spBlockId, u_1, u_2), v', (w_1, \ldots, w_r), (w_1', \ldots, w_s')) over port l

arcVisited(rBFSDepth, spBlockId, neighbor(l), v) := true.

Bag(rBFSDepth, spBlockId, v, v', neighbor(l)) :=

\langle(u_1, u_2), \text{list}_{3k+1}(\text{BFSAncestors}(rBFSDepth) \cdot (w_1, \ldots, w_r) \cdot (w_1', \ldots, w_s'))\rangle.

if arcVisited(rBFSDepth, spBlockId, neighbor(l), v) = true

for each l' \in \text{blockPorts}(rBFSDepth, spBlockId) then

if v \neq \text{DFSRoot}(rBFSDepth) then

v sends DFSPOSTBACKTRACK(rBFSDepth, spBlockId) over port l'

such that DFSState(rBFSDepth, l') = “parent”.

end if

end if

Message DFSFACEWALK((rBFSDepth, spBlockId, u_1, u_2), (w_1, \ldots, w_r), (w_1', \ldots, w_s')) over port l

arcVisited(rBFSDepth, spBlockId, neighbor(l), v) := true.

Bag(rBFSDepth, spBlockId, u_1, neighbor(l), v) :=

\langle(u_1, u_2), \text{list}_{3k+1}((w_1, \ldots, w_r) \cdot \text{BFSAncestors}(rBFSDepth))\rangle.

Let l' be the port such that (v, neighbor(l')) is immediately before (v, neighbor(l)) in the cyclic ordering.

if depth(v) = rBFSDepth and state(l') = “parent” or “upward” then

v removes the bag (rBFSDepth, spBlockId, u_1, neighbor(l), v).

arcVisited(rBFSDepth, spBlockId, neighbor(l), v) := false.

v sends DFSPECIALTREAT((rBFSDepth, spBlockId, u_1, u_2, v, \text{BFSAncestors}(rBFSDepth)) over l.

else if depth(v) = rBFSDepth + 2r and state(l') = “child” or “downward” then

Let l'' be the port satisfying that (v, neighbor(l'')) is the first arc before (v, neighbor(l)) in the cyclic ordering such that l'' \in \text{blockPorts}(rBFSDepth, spBlockId).

if neighbor(l'') = u_1 then

v sends DFSFACEOVER((rBFSDepth, spBlockId, u_1, u_2), neighbor(l), (w_1', \ldots, w_s'), \text{BFSAncestors}(rBFSDepth)) over port l''.

else

v sends DFSFACEWALK((rBFSDepth, spBlockId, u_1, u_2), (w_1, \ldots, w_r), \text{BFSAncestors}(rBFSDepth)) over port l''.

end if

arcVisited(rBFSDepth, spBlockId, v, neighbor(l'')) := true.

else

if neighbor(l'') = u_1 then

v sends DFSFACEOVER((rBFSDepth, spBlockId, u_1, u_2), neighbor(l), (w_1', \ldots, w_s'), \text{BFSAncestors}(rBFSDepth)) over port l'.

else

v sends DFSFACEWALK((rBFSDepth, spBlockId, u_1, u_2), (w_1, \ldots, w_r), \text{BFSAncestors}(rBFSDepth)) over port l'.

end if

arcVisited(rBFSDepth, spBlockId, v, neighbor(l'')) := true.

end if
Message DFSSPECIALTREAT(rBFSDepth, spBlockId, u1, u2, v′, (w1, · · · , wτ)) over l.

Let l′ be the port such that (v, neighbor(l)) is immediately before (v, neighbor(l′)).

if depth(v) = rBFSDepth and state(l′) = “parent” or “upward” then
  Bag(rBFSDepth, spBlockId, v, v′, u′) := ((v, neighbor(l)), list_{k+1}(BFSAncestors(rBFSDepth) · (w1, · · · , wτ) · (w1, · · · , wτ))).
  v sends DFSSPECIALFACESTART((u1, (rBFSDepth, spBlockId, v, neighbor(l))), BFSAncestors(rBFSDepth), BFSAncestors(rBFSDepth)) over l.
  arcVisited(rBFSDepth, spBlockId, v, neighbor(l)) := true.
else if depth(v) = rBFSDepth + 2r and state(l′) = “child” or “downward” then
  Let l′′ be the port satisfying that (v, neighbor(l′′)) is the first arc after (v, neighbor(l)) in the cyclic ordering such that l′′ ∈ blockPorts(rBFSDepth, spBlockId).
  if there is a bag (rBFSDepth, spBlockId, u1, neighbor(l′′), v) stored in v then
    v removes the bag (rBFSDepth, spBlockId, u1, neighbor(l′′), v).
  else if neighbor(l′′) = u1 and
    there is a bag (rBFSDepth, spBlockId, u1, v, neighbor(l)) stored in v then
    v removes the bag (rBFSDepth, spBlockId, u1, v, neighbor(l)).
  end if
  arcVisited(rBFSDepth, spBlockId, v, neighbor(l)) := false.
  arcVisited(rBFSDepth, spBlockId, neighbor(l′′), v) := false.
  v sends DFSSPECIALTREAT(rBFSDepth, spBlockId, u1, u2) over l′′.
else
  if there is a bag (rBFSDepth, spBlockId, u1, neighbor(l′), v) stored in v then
    v removes the bag (rBFSDepth, spBlockId, u1, neighbor(l′), v).
  end if
  arcVisited(rBFSDepth, spBlockId, neighbor(l′), v) := false.
  arcVisited(rBFSDepth, spBlockId, neighbor(l′), v) := false.
  v sends DFSSPECIALTREAT(rBFSDepth, spBlockId, u1, u2) over l′.
end if
Message DFSSPECIALFACESTART($u_0$, $(rBFSDepth, spBlockId, u_1, u_2), (w_1, \ldots, w_r), (w'_1, \ldots, w'_p))$ over $l$ arcVisited($rBFSDepth, spBlockId, neighbor(l), v$) := true.

Let $l'$ be the port such that ($v, neighbor(l')$) is immediately before ($v, neighbor(l)$) in the cyclic ordering.

if depth($v$) = $rBFSDepth$ and state($l'$) = “parent” or “upward” then
  if $v = u_2$ then
    Bag($rBFSDepth, spBlockId, u_1, v, v$) :=
    $\langle (u_1, u_2), list_{3k+1}((w_1, \ldots, w_r) \cdot BFSAncestors(rBFSDepth)) \cdot BFSAncestors(rBFSDepth) \rangle$.
  else
    Bag($rBFSDepth, spBlockId, u_1, neighbor(l), v$) :=
    $\langle (u_1, u_2), list_{3k+1}((w_1, \ldots, w_r) \cdot (w'_1, \ldots, w'_p) \cdot BFSAncestors(rBFSDepth)) \rangle$.
  end if

$v$ sends DFSSPECIALFACEOVER($u_0$, $(rBFSDepth, spBlockId, u_1, u_2)$) over $l$.

else if depth($v$) = $rBFSDepth + 2r$ and state($l'$) = “child” or “downward” then
  Let $l''$ be the port satisfying that ($v, neighbor(l'')$) is the first arc before ($v, neighbor(l)$) in the cyclic ordering such that $l'' \in blockPorts(rBFSDepth, spBlockId)$.

  if $v = u_2$ then
    $v$ sends message DFSSPECIALFACESTART($u_0$, $(rBFSDepth, spBlockId, u_1, u_2), (w_1, \ldots, w_r), BFSAncestors(rBFSDepth)$) over port $l''$.
  else
    Bag($rBFSDepth, spBlockId, u_1, neighbor(l), v$) :=
    $\langle (u_1, u_2), list_{3k+1}((w_1, \ldots, w_r) \cdot BFSAncestors(rBFSDepth)) \rangle$.
    $v$ sends DFSSPECIALACKFACESTART($rBFSDepth, spBlockId, u_1, u_2), (w_1, \ldots, w_r), BFSAncestors(rBFSDepth)$) over port $l$.
    $v$ sends DFSSPECIALFACEWALK($u_0$, $(rBFSDepth, spBlockId, u_1, u_2), (w_1, \ldots, w_r), BFSAncestors(rBFSDepth)$) over port $l''$.
  end if

  arcVisited($rBFSDepth, spBlockId, v, neighbor(l'')$) := true.

else
  if $v = u_2$ then
    $v$ sends DFSSPECIALFACESTART($u_0$, $(rBFSDepth, spBlockId, u_1, u_2), (w_1, \ldots, w_r), BFSAncestors(rBFSDepth)$) over port $l'$.
  else
    Bag($rBFSDepth, spBlockId, u_1, neighbor(l), v$) :=
    $\langle (u_1, u_2), list_{3k+1}((w_1, \ldots, w_r) \cdot BFSAncestors(rBFSDepth)) \rangle$.
    $v$ sends DFSSPECIALACKFACESTART($rBFSDepth, spBlockId, u_1, u_2), (w_1, \ldots, w_r), BFSAncestors(rBFSDepth)$) over port $l$.
    $v$ sends DFSSPECIALFACEWALK($u_0$, $(rBFSDepth, spBlockId, u_1, u_2), (w_1, \ldots, w_r), BFSAncestors(rBFSDepth)$) over port $l'$.
  end if

  arcVisited($rBFSDepth, spBlockId, v, neighbor(l'')$) := true.
end if
Message DFSSPECIALACKFACESTART((rBFSDepth, spBlockId, u₁, u₂),
(w₁, …, wᵣ), (w′₁, …, w′ᵣ)) over l
Bag(rBFSDepth, spBlockId, u₁, v, neighbor(l)) :=
⟨(u₁, u₂), list₃ᵏ₊₁(((w₁, …, wᵣ) · BFSAncestors(rBFSDepth) · (w′₁, …, w′ᵣ))).

Message DFSSPECIALFACEWALK(u₀, (rBFSDepth, spBlockId, u₁, u₂), (w₁, …, wᵣ), (w′₁, …, w′ᵣ)) over l
arcVisited(rBFSDepth, spBlockId, u₁, neighbor(l), v) := true.
Bag(rBFSDepth, spBlockId, u₁, neighbor(l), v) :=
⟨(u₁, u₂), list₃ᵏ₊₁(((w₁, …, wᵣ) · (w′₁, …, w′ᵣ) · BFSAncestors(rBFSDepth))).
Let l’ be the port such that (v, neighbor(l’)) is immediately before (v, neighbor(l)) in the cyclic ordering.
if depth(v) = rBFSDepth and state(l’)=“parent” or “upward” then
  v sends DFSSPECIALFACEOVER(u₀, (rBFSDepth, spBlockId, u₁, u₂)) over l.
else if depth(v) = rBFSDepth + 2r and state(l’)=“child” or “downward” then
  Let l” be the port satisfying that (v, neighbor(l”)) is the first arc before (v, neighbor(l)) in the cyclic ordering such that l” ∈ blockPorts(rBFSDepth, spBlockId).
  v sends DFSSPECIALFACEWALK(u₀, (rBFSDepth, spBlockId, u₁, u₂),
(w₁, …, wᵣ), BFSAncestors(rBFSDepth)) over port l”.
  arcVisited(rBFSDepth, spBlockId, v, neighbor(l”)) := true.
else
  v sends DFSSPECIALFACEWALK(u₀, (rBFSDepth, spBlockId, u₁, u₂),
(w₁, …, wᵣ), BFSAncestors(rBFSDepth)) over port l’.
  arcVisited(rBFSDepth, spBlockId, v, neighbor(l’)) := true.
end if
end if

Message DFSSPECIALFACEOVER(u₀, (rBFSDepth, spBlockId, u₁, u₂)) over l
if v = u₀ then
  if arcVisited(rBFSDepth, spBlockId, neighbor(l’), v) = true
    for each l ∈ blockPorts(rBFSDepth, spBlockId) then
      if v ≠ DFSRoot(rBFSDepth) then
        v sends DFSPOSTBACKTRACK(rBFSDepth, spBlockId) over port l’ such that DFSState(rBFSDepth, l’) = “parent”.
      end if
    end if
  end if
else
  Let l’ be the port such that (v, neighbor(l)) is immediately before (v, neighbor(l’)).
  if depth(v) = rBFSDepth + 2r and state(l’)=“child” or “downward” then
    Let l” be the port satisfying that (v, neighbor(l”)) is the first arc after (v, neighbor(l)) in the cyclic ordering such that l” ∈ blockPorts(rBFSDepth, spBlockId).
    v sends DFSSPECIALFACEOVER(u₀, (rBFSDepth, spBlockId, u₁, u₂)) over l”.
  else
    v sends DFSSPECIALFACEOVER(u₀, (rBFSDepth, spBlockId, u₁, u₂)) over l’.
  end if
end if
D Distributed Evaluation of FO over planar networks: Phase V

Label nodes in \( \bigcup_i P_i \) with \( P \).

Then consider the evaluation of FO sentence \( \varphi' \) over the vocabulary \( \{E, P\} \), where

\[
\varphi' := \exists x_1 \ldots \exists x_s \left( \bigwedge_{1 \leq i < j \leq s} d(x_i, x_j) > 2r \land \bigwedge_{i} P(x_i) \right).
\]

Starting from some node \( w_1 \) with label \( P \), mark the vertices in \( N_{2r}(w_1) \) as \( Q \), then select some node \( w_2 \) outside \( Q \), and mark those nodes in \( N_{2r}(w_2) \) by \( Q \) again, continue like this, until \( w_l \) such that either \( l = s \) or all the nodes with label \( P \) have already been labeled by \( Q \).

If \( l < s \), then label the nodes in \( \bigcup_{1 \leq i \leq l} N_{4r}(v_i) \) as \( I \). Then each connected component of \( \langle I \rangle_G \) has diameter no more than \( 4l r < 4sr \). We can construct distributively a tree decomposition for each connected component of \( \langle I \rangle_G \), and connect these tree decompositions together to get a complete tree-decomposition of \( \langle I \rangle_G \), then evaluate the sentence \( \varphi' \) by using this complete tree decomposition.

E The proof of Theorem 7

The proof of Theorem 7 relies on a normal form of FO(#) formulae.

Lemma 12. FO(#) formulae can be rewritten into a Boolean combinations of (i) first-order formulae and (ii) sentences of the form \( t_1 = t_2 \) or \( t_1 < t_2 \) where \( t_i \) are second sort terms, and for each second sort term \( \#x.\varphi(x) \) occurring in \( t_i \), \( \varphi(x) \) is a first-order formula.

The proof of the lemma can be done by a simple induction on the syntax of FO(#) formulae.

Proof. Theorem 7 (sketch)

From Theorem 2 and Theorem 4, we know that FO formulae can be evaluated over bounded degree and planar networks with only a bounded number of messages sent over each link. From the normal form of FO(#) formulae (Lemma 12), it is sufficient to prove that sentences of the form \( t_1 = t_2 \) or \( t_1 < t_2 \) can be frugally evaluated over the two types of networks.

By induction, we can show that for all second sort terms \( t \), \( t^G \) is bounded by \( n^{|t|} \) (where \( |t| \) is the number of symbols in \( t \), and \( n \) is the size of \( V \)). Therefore, \( t^G \) can be encoded in \( O(\log n) \) bits.

At first we consider the computation of the term \( \#x.\varphi(x) \) (\( \varphi \) is a first-order formula with only one free variable \( x \)).

The requesting node starts the frugal evaluation of \( \varphi(x) \) (Theorem 6), then each node \( v \) knows whether \( \varphi(v) \) holds or not. Now the requesting node can aggregate the result of \( \#x.\varphi(x) \) by using the pre-computed BFS-tree.

If \( t_1 \) and \( t_2 \) can be frugally computed, then \( t_1 + t_2 \), \( t_1 - t_2 \) and \( t_1 \times t_2 \) can be frugally computed as well by just computing \( t_1 \) and \( t_2 \) separately, and computing \( t_1 + t_2 \), \( t_1 - t_2 \) or \( t_1 \times t_2 \) by in-node computation. Thus all FO(#) sentences of the form \( t_1 = t_2 \) and \( t_1 < t_2 \) can be frugally computed. \( \Box \)