Relation between the Density of States in Different Cases and the Corresponding Examples

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Abstract. The study of particle system is inevitable to study its statistical law, and understanding the distribution of particles is fundamental and important. In this paper, the author points out the relation between the number of states of free particles in the relativistic and non-relativistic cases, and gives formulae for each case. Furthermore, the author enumerates some popular and basic particle systems, as well as classifies them in terms of relativistic and non-relativistic cases, also their basic formulae are deduced and enumerated in this paper. These particle systems are applicable to different formulae for the density of states for relativistic or non-relativistic cases. It is concluded that the different densities of states in different cases are related to each other through the de Broglie relation and the energy formula.

1. Introduction
When researchers study a system with a large number of particles, it is not possible or necessary to write down the equations of motion for each particle. Such systems are studied by them using statistics of microscopic properties and relating them to the macroscopic properties of the system. For example, the temperature and pressure of a gas are determined by the motion of its molecules. If the macroscopic parameters of the isolated system do not change with time. Then the system is in equilibrium [1].

Some statistical laws for particle systems such as photon gas, phonon (with zero chemical potential), weakly degenerate Bose gas and Fermi gas, and strongly degenerate Fermi gas have been studied. Based on this, researchers also calculate some thermodynamic properties of their equilibrium state, such as pressure, volume, temperature, internal energy, enthalpy and entropy. In the non-equilibrium state, some phenomena such as the fluctuation of the number of particles and the fluctuation of energy in a specific system have also been derived. In most studies, these particle systems exist independently and do not interfere with each other, but some of the connections between them may have been overlooked. In this paper, different formulae are derived for the number of possible states of free particles, belonging to the relativistic and non-relativistic particle systems. The author shows the link between the different formulas, which could shed light on larger connections and hopefully be applied to unexplored research. After that, some well-known particle systems are listed, and some of their basic formulas are systematically deduced and shown.

2. The number of possible states of a free particle
In statistical physics, the number of possible states of free particles is a very important concept, which often acts as a link in various data, and is also a fundamental and extremely significant information. Here the author gives the information about how to get the number of free particle states in theory.

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Figure 1. The length of the container is $L$ and the periodic boundary condition is adopted. Integer multiple of the de Broglie wave length $\lambda$ is equal to the length of the container $L$.

As shown in the figure 1:

$$L = |n_x|\lambda, |n_x|=0,1,2...$$

(1)

Possible values of wave vectors $k_x$:

$$k_x = \frac{2\pi}{L} n_x, \quad \lambda = \frac{2\pi}{k}$$

$$n_x = 0, \pm 1, \pm 2...$$

(2)

$k_x = \frac{2\pi}{L} n_x$ can be substituted into the de Broglie relation $\hat{p} = \hbar \hat{k}$ to obtain the possible momentum value of one-dimensional free particles:

$$p_x = \frac{2\pi\hbar}{L} n_x, \quad n_x = 0, \pm 1, \pm 2...$$

(3)

For three dimensional free particles, the particle is in a cubic container with side length $L$, and the possible values of the three momentum components of the particle $p_x, p_y, p_z$ are:

$$p_x = \frac{2\pi\hbar}{L} n_x, \quad p_y = \frac{2\pi\hbar}{L} n_y, \quad p_z = \frac{2\pi\hbar}{L} n_z,$$

$$n_x, n_y, n_z = 0, \pm 1, \pm 2...$$

$n_x, n_y, n_z$ are the quantum number representing the motion state of three-dimensional free particles (freedom is 3).

If $L$ is very large, from $p_x = \frac{2\pi\hbar}{L} n_x$,

$$dp_x = \frac{2\pi\hbar}{L}dn_x$$

(4)

Hence, from $p_x$ to $p_x + dp_x$, possible number of $p_x$ is:

$$dn_x = \frac{L}{2\pi\hbar} dp_x$$

(5)

$p_y, p_z$ are in the same way.

Within volume $V$, from $p_x$ to $p_x + dp_x$, from $p_y$ to $p_y + dp_y$, from $p_z$ to $p_z + dp_z$, the number of quantum states of a free particle is:

$$dn_xdn_ydn_z \approx (\frac{L}{2\pi\hbar})^3 dp_xdp_ydp_z$$

(6)

It can also be understood through the uncertainty relation, the phase lattice of one-dimensional phase space, which is shown in figure 2:

$$\Delta q\Delta p \approx \hbar$$

(7)
If the degree of freedom of the particle is $r$, the uncertain values of coordinates and momentum of each degree of freedom, $\Delta q_r$ and $\Delta p_r$, satisfy the uncertain relation $\Delta q_r \Delta p_r \approx \hbar$.

The product of uncertainty relations in all directions is:
$$\Delta q_1 \cdots \Delta q_r \Delta p_1 \cdots \Delta p_r \approx \hbar^3 \quad (8)$$

Hence, the above equation can be understood as the total number of quantum states of three-dimensional free particles in the space phase volume $V dp_x dp_y dp_z$.

The momentum of a free particle is described in the spherical polar coordinates of the momentum space $p, \theta, \varphi$:
$$p_x = p \sin \theta \cos \varphi, \quad p_y = p \sin \theta \sin \varphi, \quad p_z = p \cos \theta \quad (9)$$

Within volume $V$, from $p$ to $p + dp$, from $\theta$ to $\theta + d\theta$, from $\varphi$ to $\varphi + d\varphi$, the number of quantum states of a free particle is $\frac{4\pi p^2 \sin \theta \sin \varphi \rho}{\hbar^3} dp dp dp$.

$\theta$ is integrated from 0 to $\pi$, and $\varphi$ is integrated from 0 to $2\pi$:
$$\int_0^{2\pi} d\varphi \int_0^\pi \sin \theta d\theta = 4\pi \quad (10)$$

In the volume $V$, the magnitude of momentum is in the range from $p$ to $p + dp$ (momentum direction is arbitrary), and the number of possible states of free particles is:
$$D(p) = \frac{4\pi p^2 \rho}{\hbar^3} dp \quad (11)$$

This is the number of possible states $D$ for a free particle of momentum, what the authors call the most fundamental $D$, and the derivation here uses de Broglie relation, for both relativistic and non-relativistic cases. And then there are the numbers of possible states of free particles with respect to energy $\varepsilon$ and the angular frequency $\omega$, which are applicable to non-relativistic and relativistic cases respectively.

For the non-relativistic case, the energy degenerates to $\varepsilon = \frac{p^2}{2m}$, so the number of possible states for a free particle in volume $V$, in the energy range from $\varepsilon$ to $\varepsilon + d\varepsilon$, is:
$$D(\varepsilon)d\varepsilon = g \frac{2n^2}{\hbar^3} (2m)^{3/2} \varepsilon^{1/2}d\varepsilon \quad (12)$$

$g$ is the spin degree of freedom of the particle.

$D(\varepsilon)$ denotes the number of possible states per unit energy interval, which is called the density of states.

For the relativistic case, from de Broglie relation:
$$E = \hbar \nu = \hbar \omega$$
$$E = cp$$
$$p = \frac{\hbar \nu}{c} = \frac{\hbar \omega}{c} \quad (13)$$

Combined with $D(p)$, then $D(\omega)$ is:
$$D(\omega) = g \frac{\nu \omega^2}{2\pi^2 c^2} d\omega \quad (14)$$

$g$ is the spin degree of freedom of the particle.

$D(\varepsilon)$ and $D(\omega)$ can both be obtained from $D(p)$, as shown above.
3. Non-relativistic and relativistic cases

Table 1. Different formulae for relativistic and nonrelativistic particles.

| items                        | Nonrelativistic particle | Relativistic particle |
|------------------------------|--------------------------|-----------------------|
| The relationship between    | $E = \frac{p^2}{2m}$    | $E = cp$              |
| energy and momentum          |                          |                       |
| Pressure versus energy       | $P = \frac{2}{3}\rho$   | $P = \frac{1}{3}\rho$ |
| density                      |                          |                       |

Current physics studies of many particles tend to take relativity into account. The differences between relativistic and non-relativistic particles are shown in the table above.

In this article, some of the basic particle types are enumerated and classified according to relativistic factors, and the density of states above applies to them. In the relativistic case there are photonic gases (Planck's formula), solid Debye spectrum and phonons (chemical potential zero). And in the non-relativistic case there are weakly degenerate Bose gases and Fermi gases, strongly degenerate Fermi gases (free electron gases in metals).

3.1 Photon gas

Photon is a boson, obeying the Bose statistical distribution, and the photon number is not conserved. Therefore, there is no conservation of particle number as a constraint condition, and there is no $\alpha$ (the $\alpha$ is a Laplace multiplier, and the following $l$ refers to energy levels), the number distribution is:

$$a_l = \frac{\omega_l}{\exp(\beta\epsilon_l) - 1}$$  \hspace{1cm} (15)

Because $\alpha = -\frac{\mu}{kT}$, $\alpha = 0$ means that the chemical potential of the photon gas at equilibrium is zero.

The photon spin is 1, so there are two polarization degrees of freedom $g = 2$. Thus, in the black body of volume $V$, in the momentum range from $A$ to $B$, the number of photon quantum states is:

$$dn = g \frac{4\pi V}{\hbar^3} p^2 dp = \frac{8\pi V}{\hbar^3} p^2 dp$$  \hspace{1cm} (16)

In the empty pit of volume $V$, in the circular frequency range from $\omega$ to $\omega + d\omega$, the number of photon quantum states is (use $\epsilon = cp = \hbar\omega$):

$$D(\omega)d\omega = \frac{8\pi V}{\hbar^3} p^2 dp = \frac{\nu}{\pi^2\epsilon^3} \omega^2 d\omega$$  \hspace{1cm} (17)

Hence, the average number of photons in the $l$ energy level is now the average number of photons in the circular frequency range from $\omega$ to $\omega + d\omega$:

$$a_l \rightarrow a(\omega)d\omega = \frac{\nu}{\pi^2\epsilon^3} \frac{\omega^7 d\omega}{\exp(\hbar\omega/kT) - 1}$$  \hspace{1cm} (18)

Radiation field energy:

$$U(\omega, T)d\omega = \frac{\nu}{\pi^2\epsilon^3} \frac{\hbar\omega^3}{\exp(\hbar\omega/kT) - 1} d\omega$$  \hspace{1cm} (19)

That is what Planck's formula is.

With $u = U/V$, and $\omega = 2\pi v$, then the equation above can be changed to:

$$u(v, T)dv = \frac{8\pi}{c^3 \nu} \frac{h\nu^3}{\exp(\hbar\nu/kT) - 1} dv$$  \hspace{1cm} (20)

Based on modern understanding, it is easy to understand that:

At low frequencies, $x = \frac{\hbar v}{kT}$, therefore, $\Delta x = \frac{\hbar v}{kT} < 1$, $x$ can be thought of as continuous, which conforms to classical thoughts;
At high frequency, $\Delta x = \frac{\hbar}{kT} > 1$, $x$ cannot be regarded as continuous, and almost all the degrees of freedom are frozen in the ground state of $n = 0$ and cannot be excited to higher energy states. This is contrary to classical thinking, therefore, the classical statistics are valid here.

Additionally, the thermodynamics of photonic gases in doubly special relativity have also been studied [3].

### 3.2 Solid Debye spectrum

Debye regarded solid as a continuous elastic medium, and $3N$ normal vibrations were the basic fluctuations of the elastic medium.

Any elastic wave on a solid can be decomposed into the superposition of $3N$ normal vibrations. Elastic waves are of compressional and transverse types.

For a certain wave vector $k$, the compressional wave is an expansion compression wave, and there is only one vibration mode, that is, the vibration in the direction of propagation. A shear wave is a torsional wave that oscillates in two mutually perpendicular directions perpendicular to the direction of propagation.

The wave vector and polarization can be used to mark $3N$ normal vibrations. $c_i$ and $c_t$ represent the propagation velocities of P-wave and S-wave respectively. According to the wave equation, the circular frequency $\omega$ and wave vector size $k$ satisfy the following relation:

$$\omega = c_i k_i \quad \omega = c_t k_t \quad (21)$$

For the number of vibrations with frequency between $\omega$ and $\omega + d\omega$, there are:

$$dn = g \frac{V \omega^2}{2\pi^2 c^3} d\omega \quad (22)$$

The number of normal vibrations in the range from $\omega$ to $\omega + d\omega$ can be:

$$D(\omega)d\omega = \frac{V}{2\pi^2 c^3} \omega^2 d\omega \quad (23)$$

The $V$ is the volume of the solid.

With $B = \frac{V}{2\pi^2} \left( \frac{1}{c_i^3} + \frac{2}{c_t^3} \right)$, the Debye spectrum can be written as:

$$D(\omega)d\omega = B\omega^2 d\omega \quad (24)$$

Since the solid has only $3N$ normal vibrations, it is necessary to assume that there is a maximum circular frequency $\omega_D$, from $\int_{0}^{\omega_D} B\omega^2 d\omega = 3N$:

$$\omega_D^3 = \frac{9N}{B} = 18\pi^2 \frac{N}{V} \left( \frac{1}{c_i^3} + \frac{2}{c_t^3} \right)^{-1} \quad (25)$$

$\omega_D$ is called the Debye frequency.

With the Debye spectrum $D(\omega)d\omega = B\omega^2 d\omega$, the internal energy of a solid is:

$$U = -\frac{\partial}{\partial \beta} \ln Z = U_0 + \sum_{i=1}^{3N} \frac{\hbar \omega_i}{\exp(\beta\hbar\omega_i) - 1} \quad (26)$$

It can be written as:

$$U = U_0 + \int_{0}^{\omega_D} D(\omega) \frac{\hbar \omega}{\exp(\hbar\omega) - 1} d\omega = U_0 + B \int_{0}^{\omega_D} \frac{\hbar \omega^3}{\exp(\hbar\omega)} d\omega \quad (27)$$

With $\gamma = \frac{\hbar \omega}{kT}, \chi = \frac{\hbar \omega_D}{kT} = \frac{\theta_D}{T}$, then:

$$U = U_0 + 3NkT D(x), D(x) = \frac{3}{x^3} \int_0^x y^3 dy \quad (28)$$

$\theta_D$ is called the Debye temperature and is the characteristic parameter of solids.

The value of $\theta_D$ can be inferred from the heat capacity or from the elastic wave velocity of the solid, and the $\theta_D$ of the common solid is shown in the table 2.

| matter | Pb | Ag | Zn | Cu | Al | C | NaCl | KCl | MgO |
|--------|----|----|----|----|----|---|------|-----|-----|
| $\theta_D/K$ | 88 | 215 | 308 | 345 | 398 | ~1850 | 308 | 233 | ~850 |
| $\theta_D/K$ | 73 | 214 | 305 | 332 | 402 | ~320 | 243 | ~950 |
They are all close to room temperature, or not too far from room temperature.

3.3 Phonons

Similar to equilibrium radiation theory, the thermal motion of atoms in solids can also be discussed from the point of view of particles, and figure 3 could help.

From formula:

\[ E = \phi_0 + \sum_{i=1}^{3N} \hbar \omega_i \left( n_i + \frac{1}{2} \right), n_i = 0, 1, 2, \ldots \]

(29)

It can be seen that with the wave vector \( k \) and a certain polarization, the energy of the normal vibration is:

\[ \varepsilon = \hbar \omega \left( n + \frac{1}{2} \right), n = 0, 1, 2, \ldots \]

(30)

![Figure 3. Two dimensional \( k \) space, each lattice point is an oscillatory mode.](image)

The energy quantum of simple vibration can be regarded as a kind of "particle", called phonon:

Momentum \( p = \hbar k \)

Energy \( \varepsilon = \hbar \omega \)

The relationship between phonon energy and momentum is:

\[ \varepsilon = c \cdot p \]

(31)

Phonons are bosons, obeying the Bose distribution of chemical potential zero, that is, the number of bosons on a quantum singlet at thermal equilibrium is

\[ n = \frac{1}{e^{\varepsilon/T} - 1}, \]

then the internal energy of a solid at temperature \( T \) is:

\[ U = U_0 + \sum_{i=1}^{3N} \frac{\hbar \omega_i}{\exp(\hbar \omega_i/kT) - 1} \]

(32)

Experimental study of phonon gas behavior is far more difficult than theoretical elaboration, because isotopic or chemically free large size single crystals are difficult to obtain in practice [4].

3.4 Weakly degenerate Bose gases and Fermi gases

With the de Broglie thermal wavelength \( \lambda_T = \frac{\hbar}{\sqrt{2\pi mkT}} \) and \( n = \frac{N}{V} \), the non-degenerate condition is \( \varepsilon^a = \frac{V}{N} \left( \frac{2\pi mkT}{\hbar^2} \right)^{3/2} > 1 \) or \( n\lambda_T^3 = \frac{N}{V} \left( \frac{\hbar^2}{2\pi m kT} \right)^{3/2} > 1 \), and the weak degeneracy conditions is \( \varepsilon^a = \frac{V}{N} \left( \frac{2\pi mkT}{h^2} \right)^{3/2} < 1 \) or \( n\lambda_T^3 = \frac{N}{V} \left( \frac{h^2}{2\pi m kT} \right)^{3/2} < 1 \).

Within volume \( V \), within the energy range from \( \varepsilon \) to \( \varepsilon + d\varepsilon \), the number of possible microstates of the molecule is equation (12).

Conversion of the sum to continuity:

\[ \sum_{l} \omega_l = \sum_{l} \omega_l dl \rightarrow \int_{0}^{\infty} D(\varepsilon) d\varepsilon \]

(33)

Total number of molecules in the system (+ is Fermi, - is Bose):
\[ N = \sum_i a_i = \sum_i \frac{\omega_i}{\exp(\alpha + \beta \varepsilon_i) \pm 1} = g \frac{2\pi V}{h^3} (2m)^{3/2} \int_0^\infty \frac{e^{1/2} e^x}{\exp(\alpha + \beta \varepsilon) \pm 1} \] (34)

It can be used to determine the Laplace multiplier \( \alpha \).

The internal energy of the system:
\[ U = \sum_i \varepsilon_i a_i = g \frac{2\pi V}{h^3} (2m)^{3/2} \int_0^\infty \frac{e^{3/2} e^x}{\exp(\alpha + \beta \varepsilon) \pm 1} \] (35)

The variable \( x = \beta \varepsilon \) is introduced, and the two expressions above could be rewritten as:
\[ N = g \left(\frac{2\pi V}{h^3} (2mkT)^{3/2} \right) \int_0^\infty \frac{x^3 dx}{\exp(x+\beta \varepsilon) \pm 1}, \quad U = g \frac{2\pi V}{h^3} (2mkT)^{3/2} \int_0^\infty \frac{x^3 dx}{\exp(x+x) \pm 1} \] (36)

The denominator of the integrand can be expressed as:
\[ \frac{1}{\exp(x+x) \pm 1} = \frac{1}{\exp(x+x) \pm \exp(-\alpha-\varepsilon)} = \exp(-\alpha - x) \left[ 1 \mp \exp(-\alpha - x) \right] \] (37)

In the above formula, because \( e^\alpha > 1, \exp(x+x) > 1 \), so \( \exp(-\alpha - x) < 1 \).
Taylor expansion is used and the first two terms are preserved. And the integral is:
\[ N = g \left(\frac{2\pi m kT}{h^2} \right)^{3/2} V e^{-\alpha} (1 \mp \frac{1}{2^{3/2}} e^{-\alpha}), \quad U = \frac{3}{2} g \left(\frac{2\pi m kT}{h^2} \right)^{3/2} V kT e^{-\alpha} (1 \mp \frac{1}{2^{3/2}} e^{-\alpha}) \] (38)

The two expressions are divided by each other, so that \( U \) is expressed in terms of \( N \), then:
\[ U = \frac{3}{2} g \left(\frac{2\pi m kT}{h^2} \right)^{3/2} V kT e^{-\alpha} (1 \mp \frac{1}{2^{3/2}} e^{-\alpha}) = \frac{3}{2} N kT (1 \mp \frac{1}{4^{3/2}} e^{-\alpha}) \] (39)

Since \( e^{-\alpha} \) is small, the second term of the above equation \( e^{-\alpha} \), can be approximated by order 0, and the result of Boltzmann distribution can be used:
\[ e^{-\alpha} = \frac{N}{V} \left(\frac{h^2}{2\pi m kT} \right)^{3/2} \frac{1}{g} \] (40)

Then:
\[ U = \frac{3}{2} N kT (1 \pm \frac{1}{4^{3/2}} \frac{1}{g} \frac{N}{V} \left(\frac{h^2}{2\pi m kT} \right)^{3/2}) = \frac{3}{2} N kT (1 \pm \frac{1}{4^{3/2}} n\lambda_{T}^3) \] (41)
\[ pV = N kT (1 \pm \frac{1}{4^{3/2}} n\lambda_{T}^3) \] (42)

In the above formula, the first term is the usual quantity (\( U, p \)), which is calculated according to the Boltzmann distribution. The second term is the equivalent interaction caused by quantum statistical correlation.

### 3.5 Strongly degenerate Fermi gases (free electron gases in metals)

If \( e^\alpha \gg 1 \), or \( n\lambda_{T}^3 \ll 1 \), both bosons and fermions obey the Boltzmann distribution. Among them:
\[ \lambda_{T} = h(2\pi m kT)^{1/2}, \quad n = \frac{N}{V} = l^{-3} \] (43)
\[ n\lambda_{T}^3 = \left(\frac{l}{l}\right)^3 \ll 1 \] (45)

If these conditions are not satisfied, it is a degenerate gas, obeying the Bose or Fermi statistics. Taking the free electron gas in metal as an example, the properties of Fermi gas under strong degeneracy A or B are discussed.

After the atoms combine into metals, the valence electrons detach from the atoms can move in the whole metal and become the common electrons; the atoms that lose their valence electrons become ions and form a lattice; the common electron can be approximated as a nearly independent particle moving freely inside a metal, which is shown in figure 4.
According to the Fermi distribution $a_l = \frac{\omega_l}{\exp(\alpha + \beta \varepsilon_l) + 1}$, the average number of electrons in a quantum singlet of energy $\varepsilon$ at temperature $T$ is:

$$f(\varepsilon_l) = \frac{a_l}{\omega_1} = \frac{1}{\exp(\frac{\varepsilon - \mu}{kT}) + 1}, \alpha = -\frac{\mu}{kT}$$  \hspace{1cm} (46)

Taking the electron spin (two internal degrees of freedom) into account, the number of quantum states of the electron in the $\varepsilon \sim \varepsilon + d\varepsilon$ energy range within the volume $V$ is:

$$D(\varepsilon) d\varepsilon = g \frac{2\pi^2}{\hbar^3} (2m)^{3/2} \varepsilon^{1/2} d\varepsilon = \frac{4\pi^2}{\hbar^3} (2m)^{3/2} \varepsilon^{1/2} d\varepsilon$$  \hspace{1cm} (47)

Within volume $V$, in the energy range from $\varepsilon$ to $\varepsilon + d\varepsilon$, with $a_l \rightarrow dN_0(\varepsilon), \omega_l \rightarrow D(\varepsilon) d\varepsilon$, the average number of electrons is:

$$dN(\varepsilon) = \frac{4\pi^2}{\hbar^3} (2m)^{3/2} \frac{\varepsilon^{1/2}}{\exp(\frac{\varepsilon - \mu}{kT}) + 1} d\varepsilon$$  \hspace{1cm} (48)

Given electron number $N$, temperature $T$, and volume $V$, the chemical potential $\mu$ is determined by the following equation:

$$N = \frac{4\pi^2}{\hbar^3} (2m)^{3/2} \int_0^\infty \frac{\varepsilon^{1/2} d\varepsilon}{\exp(\frac{\varepsilon - \mu}{kT}) + 1}$$  \hspace{1cm} (49)

It could be known from this statement that $\mu$ is a function of temperature $T$ and electron density $n = N/V$.

The chemical potential of the electron gas at 0K is denoted by $\mu(0)$, and when $T = 0K$:

$$f(\varepsilon_l) = \frac{a_l}{\omega_1} = \frac{1}{\exp(\frac{\varepsilon - \mu}{kT}) + 1}$$  \hspace{1cm} (50)

$\varepsilon < \mu(0)$: \hspace{1cm} $\frac{\varepsilon - \mu}{kT} = -\infty, f = 1$

$\varepsilon > \mu(0)$: \hspace{1cm} $\frac{\varepsilon - \mu}{kT} = +\infty, f = 0$

This is the electron distribution at $T$ equals zero Kelvin, which is shown in figure 5.

![Figure 5. The electron distribution at $T = 0K$.](image)

At $T = 0K$, the average number of electrons in each quantum state of $\varepsilon < \mu(0)$ is 1, and the average number of electrons in each quantum state of $\varepsilon > \mu(0)$ is 0. Electrons try to occupy low energy states, but because of the Pauli exclusion principle, each quantum state can only contain one electron at most, so electrons fill up from $\varepsilon = 0$ to $\varepsilon = \mu(0)$.
\( \mu \) is the maximum energy of the electron at \( T = 0K \), which is determined by the following equation:

\[
\frac{4\pi \nu}{\hbar^3} (2m)^{3/2} \int_0^{\mu(0)} e^{1/2} d\epsilon = N
\]  

(51)

It is obtained from equation (49).

The solution \( \mu(0) \) is:

\[
\mu(0) = \frac{\hbar^2}{2m} (3\pi^2 N \nu)^{2/3}
\]  

(52)

\( \mu(0) \) is also called Fermi level. If \( \mu(0) = \frac{p_F}{2m} \), then:

\[
p_F = 2m\mu(0)^{1/2} = (3\pi^2 n)^{1/3} \hbar
\]  

(53)

\( p_F \) is the maximum momentum of the electron at \( T = 0K \), which is called Fermi momentum, and the corresponding velocity, \( v_F = \frac{p_F}{m} \), is called Fermi velocity.

When \( T = 0K \), all the space inside the sphere with radius \( p_F \) in the momentum space is occupied, which is called the Fermi sphere (or Fermi sea), and the surface is called the Fermi surface, which is shown in figure 6.

![Fermi sphere](image)

Figure 6. Fermi sphere.

Fermi temperature is defined:

\[
kT_F = \mu(0) = \frac{\hbar^2}{2m} (3\pi^2 N \nu)^{2/3}
\]  

(54)

Within volume \( V \), in the energy range from \( \epsilon \) to \( \epsilon + d\epsilon \), the average electron number is equation (48).

When \( T = 0K \), the internal energy of electronic gas is:

\[
U(0) = \frac{4\pi \nu}{\hbar^3} (2m)^{3/2} \int_0^{\mu(0)} e^{3/2} d\epsilon = \frac{3N}{5} \mu(0)
\]  

(55)

The average electron energy is \( \frac{3}{5} \mu(0) \), and the electronic gas pressure is:

\[
p(0) = \frac{2}{3} \frac{U(0)}{V} = \frac{2}{5} \eta \mu(0), \mu(0) = \frac{\hbar^2}{2m} (3\pi^2 N \nu)^{2/3}
\]  

(56)

This pressure is called the degenerate pressure of the electron gas. And this huge pressure in the metal is offset by the electrostatic force between electrons and ions.

Under the \( T = 0K \), although the Fermi gas has high energy, momentum and pressure, but state number \( \Omega = 1 \). Hence, \( S = k \ln \Omega = 0 \), in accordance with the third law of thermodynamics.

When \( T > 0K \), from:

\[
f = \frac{1}{\exp(\frac{\mu - \epsilon}{kT_F}) + 1}
\]  

(57)

It could be known that:

\[
\epsilon < \mu \quad f > \frac{1}{2}
\]

\[
\epsilon = \mu \quad f = \frac{1}{2}
\]

\[
\epsilon > \mu \quad f < \frac{1}{2}
\]  

(58)
This is the electron distribution at $T > 0K$, which is shown in figure 7.

![Figure 7. The electron distribution at $T > 0K$.](image)

At $T$ equals zero Kelvin, where electrons occupy every quantum state from zero to $\mu(0)$, it is possible to jump to higher energies because of thermal excitation at higher temperatures. The state of not being occupied. It is very unlikely that an electron in a low energy state must absorb a great deal of thermal energy in order to transition to a state of not being occupied. Therefore, the occupation of most of the states does not really change, it only changes in the energy range around $\mu$.

From the electron distribution at $T > 0K$, it can be seen that only the electrons near $\mu$ and in the order of $kT$ contribute to the heat capacity. With a rough analysis, $N_{\text{effective}}$ represents the number of effective electrons in the range of $kT$ near $\mu$:

$$N_{\text{effective}} \approx \frac{kT}{\mu} N$$  \hspace{1cm} (59)

The contribution of each electron to the heat capacity is $\frac{3}{2} kT$, then the contribution of the free electron in the metal to the heat capacity is:

$$C_V = \frac{3}{2} N k \left( \frac{kT}{\mu} \right) = \frac{3}{2} N k \frac{T}{T_F}$$  \hspace{1cm} (60)

The heat capacity of electron is much smaller than that of ion vibration at room temperature. However, in the low temperature range, the heat capacity of ion vibration decreases with temperature according to $T^3$. The electron heat capacity is proportional to $T$ and decreases slowly. Therefore, at low enough temperature, the electron heat capacity will be larger than the ion vibration heat capacity and become the major contribution to the metal heat capacity.

Therefore, under normal temperature, most electrons in the metal are in the Fermi sphere, none of these particles have a low momentum, so the conductivity is good. When $T > 0K$, a few of the electrons are going to go outside the Fermi sphere, so they are contributing to the heat capacity of the metal, but are relatively little. It is only at a very high temperature, Fermi temperature $T_F$, that the contribution of electrons to the heat capacity of the metal becomes large.

Degenerate atomic Fermi gases also provide a paradigm for strong interactions in nature [5]. And an exciting feature of them is the possibility of high-temperature superfluid that are analogs of very high temperature superconductors [6]. Additionally, the 2D Fermi gas provides a basic model for understanding pairing and superconductivity in 2D [7].

### 4. Conclusion

The formula of the density of states in the case of relativity and non-relativity can be derived from each other, through the connection of de Broglie relation and energy formula. This can help to reveal the relationships between different particle systems, and can be used to derive other formulas, and to predict and reason about their different properties. Different conditions of particles apply to their own formulas for the density of states. In the relativistic case there are photonic gases (Planck's formula), solid Debye spectrum and phonons (chemical potential zero). And in the non-relativistic case there are weakly degenerate Bose gases and Fermi gases, strongly degenerate Fermi gases (free electron gases in metals). Their rich formulas and unique concepts are listed one by one in this article. It is hoped that the relationship between the density of states formulas can be applied to the derivation of other formulas to help discover new knowledge and enlighten researchers. At the same time, these typical particle systems listed here hope to provide some researchers as a good reference and help.
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