Two-frequency chimera state in a ring of nonlocally coupled Brusselators

Qionglin Dai, Danna Liu, Hongyan Cheng, Haihong Li, Junzhong Yang *
School of Science, Beijing University of Posts and Telecommunications, Beijing, China
* jzyang@bupt.edu.cn

Abstract

Chimera states, which consist of coexisting domains of spatially coherent and incoherent dynamics, have been intensively investigated in the past decade. In this work, we report a special chimera state, 2-frequency chimera state, in one-dimensional ring of nonlocally coupled Brusselators. In a 2-frequency chimera state, there exist two types of coherent domains and oscillators in different types of coherent domains have different mean phase velocities. We present the stability diagram of 2-frequency chimera state and study the transition between the 2-frequency chimera state and an ordinary 2-cluster chimera state.

Introduction

Chimera state refers to a type of fascinating hybrid dynamical states in which identically coupled units spontaneously develop into coexisting synchronous and asynchronous parts. Since its discovery in nonlocally coupled phase oscillators in 2002 [1], chimera state has become a very active research field [2, 3]. It has been extensively observed that chimera states can occur in globally coupled [4, 5] and locally coupled oscillators [6], periodic and chaotic maps [7], Stuart-Landau models [8, 9], Van der Pol oscillators [10], FitzHugh-Nagumo (FHN) oscillators [11], Hindmarsh-Rose models [12], Hodgkin-Huxley models [13] and Delayed-Feedback Systems [14]. Chimera states on random networks and on multiplex networks have been investigated [15–17]. Recently, chimera states were realized experimentally in chemical [18, 19], optical [20, 21], electronic [14], mechanical and electrochemical systems [22–25].

Different types of chimera states such as breathing chimeras [2], multi-cluster chimeras [26–28], and spiral chimeras [29, 30] have been discovered and investigated in details. However, in these chimera states, coherent oscillators always have the same mean phase velocity. In this work, we will report a new type of chimera state in which coherent oscillators may have different mean phase velocities.
Materials and methods

We consider a one-dimensional ring of $N$ nonlocally coupled Brusselator [31] in which the individual unit is coupled to $R$ neighbors on each side with coupling strength $\epsilon$:

\[
\begin{align*}
\dot{X}_k &= A - (B + 1)X_k + X_k^2Y_k + \frac{\epsilon}{2R} \sum_{j=k-R}^{k+R} D_{uu}(X_j - X_k) + D_{uv}(Y_j - Y_k), \\
\dot{Y}_k &= BX_k - X_k^2Y_k + \frac{\epsilon}{2R} \sum_{j=k-R}^{k+R} D_{uu}(X_j - X_k) + D_{uv}(Y_j - Y_k),
\end{align*}
\]

(1)

The subscript $k$ refers to the unit index, which has to be taken module $N$ (or period boundary condition). Following Ref. [11], the coupling matrix is modelled as:

\[
D = \begin{pmatrix}
D_{uu} & D_{uv} \\
D_{vu} & D_{vv}
\end{pmatrix} = \begin{pmatrix}
\cos \phi & \sin \phi \\
-\sin \phi & \cos \phi
\end{pmatrix}.
\]

(2)

Brusellator is a theoretical model for a type of autocatalytic reaction. Isolated Brusellator allows for an equilibrium at $X = A$ and $Y = B/A$. When $B > 1 + A^2$, the equilibrium becomes unstable and leads to a limit cycle.

To require that the Brusellator units work in the oscillatory regime, we set $A = 1$, $B = 2.1$. It is convenient to consider the ratio $r = R/N$, the coupling radius, which ranges from 1/($N$) (nearest-neighbor coupling) to 0.5 (global coupling). In addition, we let $\phi = \pi/2 + \theta$. Throughout the paper, we numerically simulate Eq (1) by using the fourth-order Runge-Kutta method with a time step $\delta t = 0.01$. The total number of the Brusellator units is set to $N = 1000$.

Results and discussion

We report a peculiar chimera state at $r = 0.35$, $\theta = -0.1$, and $\epsilon = 0.02$ in Fig 1. The snapshot of the variable $X_k$ in (a) and the snapshot of the phase of oscillator $\Theta_k$, defined as $e^{i\Theta_k} = (X_k + iY_k)/|X_k + iY_k|$, in (b) show the coexistence of spatially coherent domains, in which oscillators distribute their variables in a continuous way, and incoherent domains, in which the variables of oscillators are scattered. There exist two large and several small coherent domains. Oscillators in the same large coherent domain are nearly in phase while those in different large coherent domains have a phase difference between them at around $\pi$. In contrast, coherent oscillators in small coherent domains may disperse their variables over a large range such as the phase in the range of $2\pi$. The snapshot of the oscillators in the $(X, Y)$ plane in Fig 1(c) shows that oscillators do not fall onto the orbit of isolated oscillators.

Coherent and incoherent domains can be identified more clearly by the mean phase velocity of oscillators which is defined as $\omega_k = \lim_{t \to -\infty} [\Theta_k(t) - \Theta_k(t')] / (t - t')$ with $t'$ the transient time. Oscillators in a same coherent domain share the same mean phase velocity while those in a same incoherent cluster have different mean phase velocities. As shown by the profile of $\omega_k$ in Fig 1(d), there exists two large coherent domains and six small coherent domains. In an ordinary view on chimera state containing multi-coherent-cluster, all coherent oscillators share the same mean phase velocity. However, Fig 1(d) shows an extraordinary feature: oscillators in the two large coherent domains share a same mean phase velocity $\Omega_1$ while those in the other six small coherent domains share another mean phase velocity $\Omega_2$. $\Omega_1 \neq \Omega_2$ suggests that
there are no fixed phase difference between oscillators in the large and the small coherent domains. From now on, we call the chimera state as 2-frequency chimera state. The profile of the mean phase velocity shows that the six small coherent domains are partitioned evenly into two groups and spatially separated by the large coherent domains. Regardless of the antiphase between the two large coherent domains, the 2-frequency chimera state is symmetric in space under the transformation, $k \rightarrow 2k_0 - k$ with $k_0$ the location of the center of the large coherent domain or the location of the center of the middle one among the three adjacent small coherent domains (For convenience, we call the middle one in the adjacent three small coherent domains as M-domain and others as S-domains). Thereby, the coherent domains are classified as the large domain, the M-domain, and the S-domain. Within the same type of coherent domain, different domains have the same domain size.

To further characterize the 2-frequency chimera state, we consider two other measures. One is the difference between adjacent oscillators, defined as $\Delta_k = \langle \Delta_k(t) \rangle_t = \langle \sqrt{(X_k - X_{k+1})^2 + (Y_k - Y_{k+1})^2} \rangle$, with $\langle \rangle_t$ the time average and the other is the variance $\sigma_k$ of $\Delta_k(t)$. The profile of $\Delta_k$ in Fig 1(e) shows that $\Delta_k$ reaches its minima in coherent domains. $\Delta_k$ is nearly zero in the two large coherent domains, which confirms that coherent oscillators in the
same large domain are almost in phase. On the other hand, $\Delta_k$ stays at nonzero values in both M-domains and S-domains, which is in agreement with the observation that oscillators in small coherent domains are off phase as shown in Fig 1(a) and 1(b). Actually, $\Delta_k$ fluctuates with the locations of oscillators in the incoherent domains and the strongest fluctuation appears at the center part of each incoherent domain. Accordingly, Eq (1) shows that the variance $\sigma_k$ stays at its highest value at the center part of the incoherent domains. Furthermore, Fig 1(f) shows $\sigma_k = 0$ in the two large coherent domains while nonzero $\sigma_k$ in the M- and S-domains. Fig 1(g) shows a typical spatiotemporal plot of the variable $X_k$ for the 2-frequency chimera state. To be mentioned, synchronous state is stable at the parameters in Fig 1. That is, the 2-frequency chimera state coexists with the synchronous state. Moreover, the attraction basin of the synchronous state is overwhelmingly larger than that of the 2-frequency chimera state. Consequently, Eq (1) always builds up the synchronous state for arbitrary initial conditions and the establishment of the 2-frequency chimera state requires deliberately prepared initial conditions. However, at certain range of $\theta$ such as $\theta \in [0.6, 1.1]$, the synchronous state might be unstable in Eq (1) and chimera states can be easily built up for random initial conditions (The results on that are beyond the scope of this work and are not presented here.). Using these chimera states as initial conditions, we find that the 2-frequency chimera states are possible to be realized. For example, the 2-frequency chimera state in Fig 1 is generated using the chimera state at $\sigma = 0.09$ and $\theta = 0.6$.

To gain an overall view of 2-frequency chimera states, we explore the $\theta - \epsilon$ plane in the range $[-0.4, 0] \times [0.005, 0.05]$. We use the chimera state in Fig 1 as initial conditions and integrate Eq (1) for $10^5$ time units. After this interval, if the final state possesses a profile of mean phase velocity with two different coherent frequencies, we classify the 2-frequency chimera state as stable. The stability diagram of the 2-frequency chimera states is presented in Fig 2. We observe a narrow stripe extending from $\theta \simeq -0.325$ and $\epsilon \simeq 0.04$ down to $\theta \simeq 0$ and $\epsilon \simeq 0.0125$. Ordinary chimera state with two coherent clusters are developed on the right side of the stable regime of the 2-frequency chimera state. In contrast, the coherent states including synchronous state and travelling wave states appear on the left side of the stability regime (To be noted, the synchronous state is always stable in Fig 2 if arbitrary initial conditions are adopted.)

Now we investigate the transition between the 2-frequency chimera state and the 2-cluster chimera state. The typical bifurcation scenario of the transition between them is presented in Fig 3, where we fix $\theta = -0.1$ and increase the coupling strength $\epsilon$ from 0.02 to 0.03. Each row in Fig 3 has been made for different coupling strengths, starting with the chimera state in Fig 1 as initial conditions. The column A presenting the snapshots of the phases of oscillators after transient time shows that increasing $\epsilon$ turns a 2-frequency chimera state to a 2-cluster one. During the process, the two large coherent domains remain while the other small coherent domains are eliminated. Oscillators in different coherent domains are in anti-phase for the 2-cluster chimera state, which provides an explanation for the anti-phase between two large coherent domains in a 2-frequency chimera state. The columns B and C, presenting the profiles of the mean phase velocity $\omega_k$ and the profiles of $\Delta_k$, respectively, suggest that the transition is a continuous one. With the coupling strength $\epsilon$ increase, the sizes of the small coherent domains vanish gradually and, interestingly, the small coherent domains in the 2-frequency chimera state locate in the center part of the incoherent domains in the 2-cluster chimera state. The phenomenon that new coherent domains emerge out of incoherent domain with parameter change has been observed in Ref. [11]. Different from the 2-frequency chimera state, there the new coherent domains share the same mean phase velocity with previous ones. The continuous transition between the 2-frequency chimera state and the 2-cluster chimera state can be
supported by using the 2-cluster chimera states at $\epsilon = 0.03$ as initial conditions. With the coupling strength $\epsilon$ decrease, Fig 3 can be reproduced.

**Conclusion**

In conclusion, we have investigated nonlocally coupled Brusselators in a ring. We reported a new type of chimera states, 2-frequency chimera state. In a 2-frequency chimera state, there exist two types of coherent domains and oscillators in different types of coherent domains have different mean phase velocities. We explored the stability diagram of the 2-frequency chimera state in the parameter $\theta - \epsilon$ plane. We studied the transition between the 2-frequency chimera state and 2-cluster chimera state and found that the transition is a continuous one. The discovery of the 2-frequency chimera state may shed light on the future studies on chimera states.
**Author Contributions**

**Conceptualization:** Qionglin Dai, Danna Liu, Haihong Li, Junzhong Yang.

**Data curation:** Qionglin Dai, Danna Liu, Hongyan Cheng, Haihong Li, Junzhong Yang.

**Formal analysis:** Qionglin Dai, Danna Liu, Hongyan Cheng, Junzhong Yang.

**Funding acquisition:** Hongyan Cheng, Junzhong Yang.

**Investigation:** Qionglin Dai, Danna Liu, Hongyan Cheng, Haihong Li, Junzhong Yang.

**Methodology:** Haihong Li, Junzhong Yang.

**Visualization:** Qionglin Dai, Danna Liu.

**Writing – original draft:** Junzhong Yang.

**Writing – review & editing:** Qionglin Dai, Junzhong Yang.

---

**Fig 3. Bifurcation scenario.** Typical bifurcation scenario for the 2-frequency chimera state with $\theta = -0.1$. For each value of the coupling strength $\epsilon$ (increasing from the top to the bottom, $\epsilon = 0.02, 0.0225, 0.024, 0.025, 0.026, 0.0275,$ and $0.03$, respectively) the snapshots of $\Theta_k$ (column A), the profile of the mean phase velocity $\omega_k$ (column B), and the profile of $\Delta_k$ (column C) are shown. Other parameters are same as in Fig 1.

[https://doi.org/10.1371/journal.pone.0187067.g003](https://doi.org/10.1371/journal.pone.0187067.g003)
References

1. Kuramoto Y, Battogtokh D. Coexistence of coherence and incoherence in nonlocally coupled phase oscillators. Nonlinear Phenomena in Complex Systems. 2002; 5(4): 380–5.

2. Abrams DM, Strogatz SH. Chimera states for coupled oscillators. Phys. Rev. Lett. 2004 Oct; 93(17): 174102. https://doi.org/10.1103/PhysRevLett.93.174102 PMID: 15525081

3. Panaggio MJ, Abrams DM. Chimera states: Coexistence of coherence and incoherence in nonlocally coupled oscillators. Nonlinearity. 2015 Feb; 28(3): R67. https://doi.org/10.1088/0951-7715/28/3/R67

4. Sethia GC, Sen A. Chimera states: The existence criteria revisited. Phys. Rev. Lett. 2014 Apr; 112(14): 144101. https://doi.org/10.1103/PhysRevLett.112.144101 PMID: 24765967

5. Schmidt L, Krischer K. Clustering as a prerequisite for chimera states in globally coupled systems. Phys. Rev. Lett. 2015 Jan; 114(3): 034101. https://doi.org/10.1103/PhysRevLett.114.034101 PMID: 25658999

6. Laing CR. Chimeras in networks with purely local coupling. Phys. Rev. E. 2015 Nov; 92(5): 050904. https://doi.org/10.1103/PhysRevE.92.050904

7. Omelchenko I, Maistrenko Y, Hövel P, Schöll E. Loss of coherence in dynamical networks: Spatial chaos and chimera states. Phys. Rev. Lett. 2011 Jun; 106(23): 234102. https://doi.org/10.1103/PhysRevLett.106.234102 PMID: 21770506

8. Laing CR. Chimeras in networks of planar oscillators. Phys. Rev. E. 2010 Jun; 81(6): 066221. https://doi.org/10.1103/PhysRevE.81.066221

9. Zakharova A, Kapeller M, Schöll E. Chimera death: Symmetry breaking in dynamical networks. Phys. Rev. Lett. 2014 Apr; 112(15): 154101. https://doi.org/10.1103/PhysRevLett.112.154101 PMID: 24785041

10. Omelchenko I, Zakharova A, Hövel P, Siebert J, Schöll E. Nonlinearity of local dynamics promotes multi-chimeras. Chaos. 2015 Aug; 25(8): 083104. https://doi.org/10.1063/1.4927829 PMID: 26328555

11. Omelchenko I, Omelchenko OE, Hövel P, Schöll E. When nonlocal coupling between oscillators becomes stronger: Patched synchrony or multichimera states. Phys. Rev. Lett. 2013 May; 110(22): 224101. https://doi.org/10.1103/PhysRevLett.110.224101 PMID: 23767727

12. Hizanidis J, Kanas V, Bezerianos A, Bountis T. Chimera states in networks of nonlocally coupled Hindmarsh-Rose neuron models. Int. J. Bifurcat. Chaos. 2014 Mar; 24(3): 1450030. https://doi.org/10.1142/S0218127414500308

13. Sakaguchi H. Instability of synchronized motion in nonlocally coupled neural oscillators. Phys. Rev. E. 2006 Mar; 73(3): 031907. https://doi.org/10.1103/PhysRevE.73.031907

14. Larger L, Penkovsky B, Maistrenko Y. Virtual chimera states for delayed-feedback systems. Phys. Rev. Lett. 2013 Aug; 111(5): 054103. https://doi.org/10.1103/PhysRevLett.111.054103 PMID: 23952404

15. Zhu Y, Zheng Z, Yang J. Chimera states on complex networks. Phys. Rev. E. 2014 Feb; 89(2): 022914. https://doi.org/10.1103/PhysRevE.89.022914

16. Ghosh S, Kumar A, Zakharova A, Jalan S. Birth and death of chimera: Interplay of delay and multiplexing. EPL. 2016 Nov; 115(6): 60005. https://doi.org/10.1209/0295-5075/115/60005

17. Maksimenko VA, Makarov VV, Bera BK, Ghosh D, Dana SK, Goremyko MV, et al. Excitation and suppression of chimera states by multiplexing. Phys. Rev. E. 2016 Nov; 94(5): 052205. https://doi.org/10.1103/PhysRevE.94.052205 PMID: 27967153

18. Tinsley MR, Nkomo S, Showalter K. Chimera and phasecluster states in populations of coupled chemical oscillators. Nat. Phys. 2012 Sep; 8(9): 662. https://doi.org/10.1038/nphys2371

19. Schmidt L, Schönleber K, Krischer K, García-Morales V. Coexistence of synchrony and incoherence in oscillatory media under nonlinear global coupling. Chaos. 2014 Mar; 24(1): 013102. https://doi.org/10.1063/1.4858996 PMID: 24967364

20. Hagerstrom AM, Murphy TE, Roy R, Hövel P, Omelchenko I, Schöll E. Experimental observation of chimeras in coupled-map lattices. Nat. Phys. 2012 Sep; 8(9): 658. https://doi.org/10.1038/nphys2372

21. Viktorov EA, Habruseva T, Hegarty SP, Huyet G, Kelleher B. Coherence and incoherence in an optical comb. Phys. Rev. Lett. 2014 Jun; 112(12): 224101. https://doi.org/10.1103/PhysRevLett.112.224101 PMID: 24949771

22. Martens EA, Thutupalli S, Fournier A, Hallatschek O. Chimera states in mechanical oscillator networks. Proc. Natl. Acad. Sci. U S A. 2013 Jun; 110(26): 10563–7. https://doi.org/10.1073/pnas.1302880110 PMID: 23759743

23. Kapitaniak T, Kuzma P, Wojewoda J, Czolczynski K, Maistrenko Y. Imperfect chimera states for coupled pendula. Sci. Rep. 2014 Sep; 4: 6379. https://doi.org/10.1038/srep06379 PMID: 25223296
24. Olmi S, Martens EA, Thutupalli S, Torcini A. Intermittent chaotic chimeras for coupled rotators. Phys. Rev. E. 2015 Sep; 92(3):030901. https://doi.org/10.1103/PhysRevE.92.030901

25. Wickramasinghe M, Kiss IZ. Spatially organized dynamical states in chemical oscillator networks: synchronization, dynamical differentiation, and chimera patterns. PLoS ONE. 2013 Nov; 8(11): e80586. https://doi.org/10.1371/journal.pone.0080586 PMID: 24260429

26. Omelchenko I, Provata A, Hizanidis J, Schöll E, Hövel P. Robustness of chimera states for coupled FitzHugh-Nagumo oscillators. Phys. Rev. E. 2015 Feb; 91(2): 022917. https://doi.org/10.1103/PhysRevE.91.022917

27. Maistrenko YL, Vasylenko A, Sudakov O, Levchenko R, Maistrenko VL. Cascades of multiheaded chimera states for coupled phase oscillators. Int. J. Bifurcat. Chaos. 2014 Aug; 24(8): 1440014. https://doi.org/10.1142/S0218127414400148

28. Zhu Y, Li Y, Zhang M, Yang J. The oscillating two-cluster chimera state in non-locally coupled phase oscillators. EPL. 2012 Jan; 97(1): 10009. https://doi.org/10.1209/0295-5075/97/10009

29. Martens EA, Laing CR, Strogatz SH. Solvable model of spiral wave chimeras. Phys. Rev. Lett. 2010 Jan; 104(4): 044101. https://doi.org/10.1103/PhysRevLett.104.044101 PMID: 20366714

30. Gu C, St-Yves G, Davidsen J. Spiral wave chimeras in complex oscillatory and chaotic systems. Phys. Rev. Lett. 2013 Sep; 111(13): 134101 https://doi.org/10.1103/PhysRevLett.111.134101 PMID: 24116782

31. Glansdorff P, Prigogine I. Thermodynamic theory of structure, stability and fluctuations. Chichester: Wiley Interscience; 1971.