Permutations with orders coprime to a given integer

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Abstract. Erdős and Turán, in 1967, showed that the proportion of permutations in the symmetric group Sym(n) of degree n with no cycle of length divisible by a fixed prime m is \( \Gamma \left( 1 - \frac{1}{m} \right)^{-1} \left( \frac{n}{m} \right)^{\frac{1}{m}} + O(n^{-1/2}) \). Let m be a positive square-free integer and let \( \rho(n, m) \) be the proportion of permutations of Sym(n) whose order is coprime to m. We show that there exists a positive constant \( C(m) \) such that, for all \( n \geq m \),

\[
C(m) \left( \frac{n}{m} \right)^{\frac{\phi(m)}{m} - 1} \leq \rho(n, m) \leq \left( \frac{n}{m} \right)^{\frac{\phi(m)}{m} - 1} 
\]

where \( \phi \) is Euler’s totient function.

1. Introduction

In a series of papers between 1965 and 1972, Erdős and Turán initiated a systematic study of probabilistic aspects of group theory (see, for example, [7]). One topic which has been of particular interest since this time is the distribution of element orders in finite symmetric groups, and their most relevant work for us on this topic began in [8, 9] where they studied the proportion \( p_{-m}(n) \) of elements in Sym(n) with no cycle of length divisible by a fixed prime m. Erdős and Turán obtained an explicit formula for \( p_{-m}(n) \) and determined the limiting proportion, as n grows, as

\[
p_{-m}(n) = k(m) \left( \frac{n}{m} \right)^{-\frac{1}{m}} + O(n^{-1/2}), \tag{1}
\]

where \( k(m) = \Gamma \left( 1 - \frac{1}{m} \right)^{-1} \), noting that \( \pi^{-1/2} \leq k(m) < 1 \) [8, Sections 3 and 4]. Although m was assumed to be a prime in [8], the formula for \( p_{-m}(n) \) in (1) holds for an arbitrary positive integer m, see [11], and their asymptotic arguments can be extended to give explicit convergence bounds [3, Theorem 2.3(b)], again for arbitrary m. These explicit bounds, together with analogous results for alternating groups [3, Section 3], were used to analyse algorithms for constructing transpositions and 3-cycles [3, Section 6], procedures used as components of the constructive recognition algorithms for black-box alternating and symmetric groups in [4]. Many

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other authors have also considered the proportion \( p_{-m}(n) \), see for example \([5, 6, 15]\) and the discussion in \([16]\).

Let us introduce the specific topic of interest for this paper. For positive integers \( n \) and \( m \), let \( R(n, m) \) be the set of elements of \( \text{Sym}(n) \) whose order is coprime to \( m \), and write
\[
\rho(n, m) := \frac{|R(n, m)|}{n!}.
\]
The proportion \( \rho(n, m) \) is equal to the proportion \( p_{-m}(n) \) of Erdős and Turán discussed above if and only if \( m \) is a prime power. Moreover, in \([8, \text{Lemma II}]\), Erdős and Turán demonstrate that if \( n \) is sufficiently large and \( m \) is the product of two distinct primes \( p \) and \( q \) satisfying \((\log n)^{3/4} \leq p, q \leq 10 \log n / \log \log n\), then
\[
\rho(n, m) = n^{-\frac{1}{p} - \frac{1}{q}}(1 + O(\log n^{-\frac{1}{m}})).
\]
However, the problem of determining \( \rho(n, m) \) for an arbitrary \( m \) has remained open. Thus, motivated by possible algorithmic applications, we decided to study the proportion \( \rho(n, m) \) for all positive integers \( n \) and \( m \).

Let \( \phi \) be Euler’s totient function. The set \( \pi(m) \) of prime divisors of \( m \) is significant as \( \rho(n, m) = \rho(n, m') \) and \( \phi(m)/m = \phi(m')/m' \) when \( \pi(m) = \pi(m') \). Given this fact, we will henceforth assume that \( m \) is square-free. We implicitly also assume that \( \pi(m) \subseteq \pi(n) \) since \( \rho(n, m) = \rho(n, mp) \) for primes \( p > n \). With this in mind, we now present our main result.

**Theorem 1.** Let \( m \) be a positive square-free integer. There exists a positive constant \( C(m) \) such that, for all \( n \geq m \),
\[
C(m) \left( \frac{n}{m} \right)^{\frac{\phi(m)}{m} - 1} \leq \rho(n, m) \leq \left( \frac{n}{m} \right)^{\frac{\phi(m)}{m} - 1}.
\]

In Theorem 1, the exponent \( \frac{\phi(m)}{m} - 1 \) is negative, and hence \( \left[ \frac{n}{m} \right]^{\frac{\phi(m)}{m} - 1} \leq \left( \frac{n}{m} \right)^{\frac{\phi(m)}{m} - 1} \)
and \( \left[ \frac{n}{m} \right]^{\frac{\phi(m)}{m} - 1} \geq \left( \frac{n}{m} \right)^{\frac{\phi(m)}{m} - 1} \), for \( n \geq m \). Thus, in order to prove Theorem 1 it is sufficient to prove that
\[
C(m) \left[ \frac{n}{m} \right]^{\frac{\phi(m)}{m} - 1} \leq \rho(n, m) \leq \left[ \frac{n}{m} \right]^{\frac{\phi(m)}{m} - 1}.
\]
We prove these inequalities in Section 2. In fact the upper bound holds for \( n \geq 1 \).

First we make a few remarks concerning the constant \( C(m) \) and links between Theorem 1 and the results (1) and (2).

**Remark 2.**
(a) We prove Theorem 1 with the constant
\[
C(m) := \min\{\rho(n, m) \mid m \leq n \leq 2m - 1\}.
\]
In particular, if \( m \) is a prime then \( C(m) = 1 - \frac{1}{m} \).

(b) If an element of \( \text{Sym}(n) \) has order coprime to \( m \), then the length of each of its cycles is certainly not divisible by \( m \). Hence, we have the upper bound
\[
\rho(n, m) \leq p_{-m}(n) = \prod_{i=1}^{\frac{m}{2}} (1 - \frac{1}{im}) \text{ by } [11].
\]
However, this bound grows too quickly as remarked on in (c).
(c) If $m$ is prime, then the exponent is $\frac{\phi(m)}{m} - 1 = \frac{m-1}{m} - 1 = -\frac{1}{m}$, and we obtain from Theorem 1 the result (1), apart from determining the constant $k(m)$. In fact, the exponent $\frac{\phi(m)}{m} - 1$ is equal to $-\frac{1}{m}$ if and only if $m$ is a power of a prime, and in all other cases the exponent is strictly less than $-\frac{1}{m}$. In other words, if $m$ is divisible by at least two primes then $\rho(n, m)$ grows more slowly, as $n$ increases, than $p_{m,n}(n)$ does.

(d) Suppose $m = pq$ where $p < q$ are primes. Then $\frac{\phi(m)}{m} - 1 = -\frac{1}{p} - \frac{1}{q} + \frac{1}{pq}$ and Theorem 1 appears to differ from (2) by a multiplicative factor of $n^{3/pq}$. However, in our context $m$ is fixed and $n$ increases, whereas Erdős and Turán assume for (2) that

$$
\left(\log n\right)^{3/4} \leq p < q \leq \frac{100 \log n}{\log \log n}.
$$

That is, both $m$ and $n$ increase in (2). The apparent inconsistency can be resolved by showing that (5) implies

$$
n^{1/pq} = 1 + O((\log n)^{-1/2}).
$$

For a lower bound, we have

$$
n^{1/(pq)} \geq n^{(\log n)^{-3/2}} = \left(\frac{\log n}{\log \log n}\right)^{1/2} = e^{(\log n)^{-1/2}} = 1 + O((\log n)^{-1/2}).
$$

For an upper bound we show

$$
n^{1/pq} \leq n^{(\log \log n)^2/(100(\log n)^2)} \leq 1 + O((\log n)^{-1/2}).
$$

Establishing the last inequality is the same as bounding the function

$$
f(n) := (n^{(\log n)^{-1}} - 1)(\log n)^{1/2} \text{ where } x = \frac{(\log \log n)^2}{100 \log n}.
$$

Rewriting $f(n)$ using the identity $n^{(\log n)^{-1}} = e$ gives

$$
f(n) = (e^x - 1)(\log n)^{1/2}.
$$

Since $x \to 0$ as $n \to \infty$, we can choose $n$ large enough so that $x < 1/2$. However, $0 \leq e^x - 1 < 2x$ for $0 \leq x < 1/2$ so

$$
f(n) < 2x(\log n)^{1/2} = \frac{(\log n)^2}{50(\log n)^{1/2}}.
$$

Hence $f(n) \to 0$ as $n \to \infty$, so $f(n)$ is bounded as claimed.

(e) The proofs by Erdős and Turán of results such as (1) and (2) draw heavily on tools from complex analysis. In [8, Section 5], Erdős and Turán state that it would be desirable to obtain a proof of (2) using more direct means:

“A more direct (real-variable or algebraic) approach to the determination of this coefficient would be desirable.”

The proof of Theorem 1 is principally algebraic: we determine and exploit a recursive formula for $\rho$.

(f) In a different direction, restricting $m$ to a prime number and determining the proportion $\rho(G, m)$ of elements of an arbitrary finite group $G$ whose order is coprime to $m$ has been the subject of papers by many authors. For example, see [14] when $G$ is a permutation group of degree $n$ and see [1, 12, 13] when $G$ is a finite simple classical group.
2. Proof of Theorem 1

For the remainder of the paper, fix \( m \) as a square-free positive integer. Recall that \( R(n, m) \) is the set of elements in \( \text{Sym}(n) \) of order coprime to \( m \). Since \( m \) is fixed we will write \( R(n) := R(n, m) \) and similarly (except in some formal statements) we write \( \rho(n) := \rho(n, m) \). Additionally, we denote the greatest common divisor of integers \( c \) and \( d \) by \( (c, d) \), and we write

\[
\Phi = \Phi(m) := \{ 1 \leq i \leq m \mid (i, m) = 1 \},
\]

noting that \( \phi := \phi(m) = |\Phi| \).

The following lemma generalises [3, Lemma 2.1]. For convenience, we adopt the convention that \( \rho(0) = 1 \).

**Lemma 3.** The following recursive formula holds for integers \( n \geq m > 0 \),

\[
n\rho(n) = (n - m)\rho(n - m) + \sum_{k \in \Phi} \rho(n - k).
\]

**Proof.** The permutations \( x \in R(n) \) can be enumerated according to the length \( k \) of the cycle containing the point 1. The number of choices for the cycle \( (1, i_2, \ldots, i_k) \) of \( x \) is \( (n - 1)(n - 2) \cdots (n - k + 1) \). Note that \( (k, m) = 1 \) and that the permutation induced by \( x \) on the \( n - k \) points outside \( \{1, i_2, \ldots, i_k\} \) lies in \( R(n - k) \). Thus

\[
|R(n)| = \sum_{1 \leq k \leq n \atop (k, m) = 1} (n - 1)(n - 2) \cdots (n - k + 1)|R(n - k)|.
\]

Dividing this equation by \( (n - 1)! \), and noting that \( |R(a)| = a!\rho(a) \) for all \( a \in \mathbb{N} \), we obtain

\[
n\rho(n) = \sum_{1 \leq k \leq n \atop (k, m) = 1} \rho(n - k).
\]

Replacing \( n \) above with \( n - m \) and observing that \( (k + m, m) = (k, m) \) yields

\[
(n - m)\rho(n - m) = \sum_{1 \leq k \leq n - m \atop (k, m) = 1} \rho(n - m - k) = \sum_{m + 1 \leq k \leq n \atop (k, m) = 1} \rho(n - k).
\]

Subtracting these two equations gives

\[
n\rho(n) - (n - m)\rho(n - m) = \sum_{k \in \Phi} \rho(n - k). \quad \Box
\]

We now present a technical lemma which will be of use in the proof of Theorem 1.

**Lemma 4.** Let \( y \) and \( a \) be real numbers such that \(-1 < y < 0 \) and \( a \geq 2 \). Then

\[
0 < 1 - \frac{y + 1}{a} \left( 1 - \frac{y}{a} \right) \leq \left( \frac{a - 1}{a} \right)^{y + 1} < 1 - \frac{y + 1}{a}.
\]

**Proof.** Let \( x = -1/a \) and \( x_0 = -1/2 \), and note that \( x_0 \leq x < 0 \). We seek upper and lower bounds for \( f(x) := (1 + x)^{y + 1} = \left( \frac{a}{a - 1} \right)^{y + 1} \). As \( |x| < 1 \), the binomial series below converges absolutely

\[
f(x) = \sum_{i \geq 0} \binom{y + 1}{i} x^i.
\]
Since \(-1 < y < 0\), for each \(i > 0\), the binomial coefficient
\[
\binom{y+1}{i} = \frac{(y+1)(y+1)\cdots(y-(i-2))}{i!}
\]
has \(i - 1\) negative factors. Hence, the product \((y+1)x^i\) is negative for each \(i > 0\). Therefore,
\[
f(x) = \sum_{i>0} \binom{y+1}{i} x^i < 1 + (y+1)x = 1 - \frac{y+1}{a}
\]
yielding the desired upper bound.

Now we consider the lower bound. Temporarily we assume that \(i \geq 2\). Since \((y-1)\cdots(y-(i-2))\) has \(i-2\) negative factors, the product \((y-1)\cdots(y-(i-2))x^{i-2}\) is positive for each \(i \geq 2\). Hence,
\[
0 < \prod_{j=1}^{i-2} (y-j)x = \prod_{j=1}^{i-2} (j-y)(-x) \leq \prod_{j=1}^{i-2} (j+1)(-x_0) = (i-1)!(-x_0)^{i-2}.
\]
This in turn shows that
\[
0 > \binom{y+1}{i} x^i = \frac{(y+1)(y-1)\cdots(y-(i-2))x^i}{i!} \geq \frac{(y+1)y(-x_0)^{i-2}x^2}{i}.
\]
Taking the terms with \(0 \leq i < 2\), together with the above lower bound for the sum of the terms with \(i \geq 2\), gives
\[
f(x) \geq 1 + (y+1)x + \sum_{i \geq 2} \frac{(y+1)y(-x_0)^{i-2}x^2}{i}
\]
\[
= 1 + (y+1)x + \frac{(y+1)y}{x_0^2} \left( \sum_{i \geq 2} \frac{(-x_0)^i}{i} \right) x^2.
\]
Now \(\sum_{i \geq 1} \frac{(-x_0)^i}{i} = -\log(1+x_0)\), and hence, since \(x_0 = -1/2\), we have
\[
x_0^{-2} \sum_{i \geq 2} \frac{(-x_0)^i}{i} = x_0^{-2}(x_0 - \log(1+x_0)),
\]
and this lies in the open interval \((0, 1)\). Then since \((y+1)y x^2 < 0\), we obtain the desired lower bound
\[
f(x) = (1 + x)^{y+1} > 1 + (y+1)x + (y+1)y x^2 = 1 - \frac{y+1}{a} \left( 1 - \frac{y}{a} \right).
\]
Finally, since \(-1 < y < 0\) and \(a \geq 2\), this lower bound is positive. \(\square\)

We now prove our main result.

**Proof of Theorem 1.** The result is true when \(m = 1\) and \(C(1) = 1\). Suppose \(n \geq m \geq 2\). Recall the notation \(\Phi = \Phi(m)\) and \(\phi = |\Phi|\), and write
\[
y := \frac{\phi}{m} - 1.
\]
Observe that \(-1 < y < 0\). In addition, for \(0 \leq i \leq m-1\), write
\[
x_i = |\{k \in \Phi \mid k < m-i\}| \quad \text{and} \quad y_i = |\{k \in \Phi \mid k \leq i\}|.
\]
Note that \(x_i \leq m-i-1\), \(y_i \leq i\), \(x_{i+1} \geq \phi(m)\) and \(y_i + (m-i) \geq \phi(m)\). In summary
\[
\phi(m) - i \leq x_i \leq m-i-1 \quad \text{and} \quad \phi(m) - m + i \leq y_i \leq i.
\]
We begin by proving the required upper bound, namely
\[ \rho(n) \leq \left( \frac{n}{m} \right)^y \text{ for } n \geq m \geq 2. \tag{8} \]

Although we do not require it for this proof, the upper bound above holds trivially if \(1 \leq n \leq m\) as then \(\rho(n) \leq 1 = \left[ \frac{n}{m} \right]^y = 1\). We proceed by induction on \(n\). Now let \(n \geq m + 1\), so that \(a := \left[ \frac{n}{m} \right] \geq 2\). Write \(n = am - b\), and note that \(0 \leq b \leq m - 1\). Assume the upper bound in (8) holds for all positive integers strictly less than \(n\). By Lemma 3,
\[ \rho(am - b) = \frac{(a-1)m - b}{am - b} \rho((a-1)m - b) + \frac{1}{am - b} \sum_{k \in \Phi} \rho(am - b - k). \]

By the inductive hypothesis, \(\rho((a-1)m - b) \leq (a-1)^y\). Similarly, for each \(k \in \Phi\), if \(k < m\) then \(am - b - k > (a-1)m\) so by induction \(\rho(am - b - k) \leq a^y\), and if \(k \geq m\) then \(\rho(am - b - k) \leq (a-1)^y\). Therefore, using the definition of \(x_i\) in (6), we obtain
\[ \rho(am - b) \leq \frac{(a-1)m - b}{am - b} (a-1)^y + \frac{x_b a^y + (\phi - x_b)(a-1)^y}{am - b} \]
\[ = a^y \left( \frac{a-1}{a} - \frac{b/a}{am - b} \right) (a-1)^y + \frac{x_b + (\phi - x_b)(a-1)^y}{am - b} \]
\[ = a^y \left( \frac{a-1}{a} \right)^{y+1} \left( 1 - \frac{b - a\phi + ax_b}{(a-1)(am - b)} \right) + \frac{x_b}{am - b}. \]

By Lemma 4, \((\frac{a-1}{a})^{y+1} < 1 - \frac{x_b}{m}\) and as \(y + 1 = \frac{\phi}{m}\) and \(a \geq 2\), we have
\[ \rho(am - b) \leq a^y Y \text{ where } Y = \left( 1 - \frac{\phi}{am} \right) \left( 1 - \frac{b - a\phi + ax_b}{(a-1)(am - b)} \right) + \frac{x_b}{am - b}. \]

We want to show that \(Y \leq 1\), so we write \(Y = 1 - Y_0\) where \(Y_0\) is an algebraic fraction in \(a, b, x_b, m, \phi\). It suffices, therefore, to show that \(Y_0 \geq 0\) for all input values satisfying \(a \geq 2\), \(0 \leq b < m\), and \(\phi \leq \min\{b + x_b, m\}\). Using a computer to factor \(Y_0\) giving
\[ Y_0 = Y - 1 = \frac{-m - \phi(b + x_b - \phi)}{m(a - 1)(am - b)} \leq 0. \]

Thus \(Y \leq 1\) and hence \(\rho(am - b) \leq a^y\), proving the upper bound (8) for all \(n \geq 1\).

We now turn to the lower bound. Recall the definition of \(C := C(m)\) in (4), and note that \(C > 0\) since \(\rho(n) > 0\) for all \(n \geq 1\). We will prove that,
\[ \rho(n) \geq C \left[ \frac{n}{m} \right]^y \text{ for } n \geq m \geq 2. \tag{9} \]

As for the proof of the upper bound, we use induction on \(n\). Observe that if \(m \leq n \leq 2m - 1\), then \(\left[ \frac{n}{m} \right] = 1\), and hence \(\rho(n) \geq C = C \left[ \frac{n}{m} \right]^y\) holds by (4). Now suppose \(n \geq 2m\). Then \(a := \left[ \frac{n}{m} \right] \geq 2\). Write \(n = am + b\), and note that \(0 \leq b \leq m - 1\). (Be aware that the definitions of \(a\) and \(b\) differ from their definitions in the proof of the upper bound.) Assume that the lower bound (9) holds for all positive integers strictly less than \(n\). By Lemma 3,
\[ \rho(am + b) = \frac{(a - 1)m + b}{am + b} \rho((a - 1)m + b) + \frac{1}{am + b} \sum_{k \in \Phi} \rho(am + b - k). \]
By the inductive hypothesis, \( \rho((a-1)m+b) \geq C(a-1)^y \). Similarly, for each \( k \in \Phi \), if \( k \leq b \) then \( am + b - k \geq am \) so by induction, \( \rho(am + b - k) \geq Ca^y \), and if \( k > b \) then \( am > am + b - k \geq (a-1)m \) so by induction \( \rho(am + b - k) \geq C(a-1)^y \).

Therefore, using the definition of \( y_b \) in (6), we obtain

\[
\rho(am + b) \geq C \left( \frac{(a-1)m+b}{am+b}(a-1)^y + \frac{yb}{am+b} + \frac{(\phi-y_b)(a-1)^y}{am+b} \right)
\]

By Lemma 4, since \( a \geq 2 \), \( y = \frac{a-1}{a} \) and \( -1 < y < 0 \), we have \( \left(\frac{a-1}{a}\right)^{y+1} > 1 - \frac{y+1}{a} \), so

\[
\rho(am + b) \geq C a^y \left( \left(1 - \frac{y}{a}\right) \left(1 + \frac{m}{am+b}(a-1)^y\right) + \frac{yb}{am+b} \right).
\]

Write the above expression as \( Ca^y Y \) where \( Y \) is an algebraic fraction in \( a, b, y_b, m, \phi \).

We want to show that \( Y \geq 1 \), so we write \( Y = 1 + Y_0 \). It suffices, therefore, to show that \( Y_0 \geq 0 \) for all input values satisfying \( a \geq 2, 0 \leq b < m, m \geq \phi \) and \( \phi - m + b \leq y_b \leq b \) (see (7)). We use a computer to factor \( Y_0 \) giving

\[
Y_0 = Y - 1 = \frac{(m-\phi)(am(b-y_b) + \phi(y_b-b+m-\phi))}{m^2(a-1)(am+b)} \geq 0.
\]

Therefore, \( \rho(am + b) \geq Ca^y Y \geq Ca^y \) and the claim in (9) holds for all \( n \geq m \). This establishes the lower bound and completes the proof of the theorem.

\[\Box\]

3. Conjecture and computational evidence

Let \( n \geq m > 1 \) and assume that \( m \) is square-free. First suppose that \( m \) is prime. Recall that \( p_{-m}(n) \) is the proportion of elements in \( \text{Sym}(n) \) with no cycle of length divisible by \( m \), so \( p_{-m}(n) = p_{-m}(n+i) \) for \( 0 \leq i < m \). Since \( \rho(n, m) = p_{-m}(n) \), it follows that for all \( a \geq 1 \),

\[
\rho(am, m) = \rho(am + 1, m) = \cdots = \rho(am + (m-1), m).
\]

Moreover, in this case (since \( m \) is prime),

\[
\rho(n, m) = k(m) \left( \frac{n}{m} \right)^{\frac{\phi(m)}{m}-1} + O(n^{\frac{\phi(m)}{m}})^{-2},
\]

where \( k(m) = \Gamma(1 - \frac{1}{m})^{-1} \), noting that \( \pi^{-1/2} \leq k(m) < 1 \) (see [8, Sections 3 and 4] and [3, Theorem 2.3]).

In this final section we investigate the extent to which an analogue of the relationship in (11) holds for general positive integers \( m \). We do this by presenting some computational evidence which led the authors to the statement of Theorem 1 and the stronger Conjecture 5 below.

The recursive formula for \( \rho \) in Lemma 3 provides an efficient means of computing \( \rho(n, m) \) from the values \( \rho(0, m), \rho(1, m), \ldots, \rho(m-1, m) \). In Figures 1–3 we fix the
value of \( m \) as 6, 15 and 30, respectively, and we plot
\[
f(n, m) := \rho(n, m) \cdot \left( \frac{n}{m} \right)^{1 - \frac{\phi(m)}{m}}
\]
against \( n \) for many values of \( n \) greater than \( m \).

**Figure 1.** Plot of \( f(n, 6) \) versus \( n \) for \( 7 \leq n \leq 2000 \).

**Figure 2.** Plot of \( f(n, 15) \) versus \( n \) for \( 16 \leq n \leq 2000 \).
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It is evident from Figures 1–3 that the relationships in (10) and (11) do not hold if \( m \) is composite. Rather than \( f(n, m) \) being asymptotically constant, as is the case when \( m \) is prime, Figures 1–3 suggest that for fixed \( 0 \leq b < m \) and \( n \equiv b \pmod{m} \) there exists a constant \( k(m, b) \) such that \( f(n, m) \to k(m, b) \) as \( n \to \infty \). This leads us to the following conjecture.

**Conjecture 5.** Let \( m \) be a positive square-free integer. For each \( 0 \leq b < m \), there exists a positive constant \( k(m, b) \) such that if \( n \equiv b \pmod{m} \), then

\[
\rho(n, m) = k(m, b) \left( \frac{n}{m} \right)^{\frac{\phi(m)}{m} - 1} + o\left(n^{\frac{\phi(m)}{m} - 1}\right).
\]

By using streamlined MAGMA code [2, 10] we were able to compute \( \rho(n, m) \) for \( n \) up to \( 10^6 \), and it appears that the constant \( k(m, b) \) in Conjecture 5 is independent of \( b \). We stress though that the periodic nature of \( f(n, m) \) modulo \( m \) which is indicated in Figures 1–3 is a feature which motivated the proof of Theorem 1, the statement of which gives upper and lower bounds for \( \rho(n, m) \) that are independent of \( n \). We anticipate that such bounds will be useful for algorithmic applications.

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