Measuring gravitational waves from binary black hole coalescences: II. The waves’ information and its extraction, with and without templates.

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We discuss the extraction of information from detected binary black hole (BBH) coalescence gravitational wave bursts, focusing in particular on the nonlinear merger phase of the coalescence, which occurs after the gradual inspiral of the bodies in the binary and before the ringdown of the system to its final Kerr black hole state.

We report four principal results: (i) If numerical relativity simulations have not successfully produced theoretical template waveforms for the merger by the time that BBH waves are first detected by LIGO/VIRGO interferometers, or if they cannot produce a set of templates that completely covers the space of merger waveforms, then observers can use simple band-pass filters to study the merger waves. For BBHs of total mass $\lesssim 40 M_{\odot}$ which are detected via their inspiral waves, we estimate that the signal-to-noise ratio from band-pass filtering will typically be of order unity for initial and advanced LIGO interferometers. Thus, the merger waves should be just visible above the noise for typical events; rare, stronger events will be more visible, and thus more interesting. (ii) We use Bayesian statistics and the maximum likelihood framework to sketch out an optimized method for extracting the merger waveform from the detector output. The method is based on a “perpendicular projection” of the observed (noisy) signal onto an appropriate function space that incorporates all our (possibly sketchy) prior knowledge of the waveforms. We argue that the best type of “basis functions” to use to specify this function space is wavelets or wavelet-like functions, and we develop the method in some detail in the language of wavelets. In an Appendix, we sketch an extension of the method which allows one to reconstruct the two independent polarization components of the merger waves from the outputs of a network of several interferometers. (iii) We propose a computational strategy for numerical relativists to pursue, if they successfully produce computer codes for generating merger waveforms, but if running the codes is too expensive to permit an extensive survey of the merger parameter space. In this case, for LIGO/VIRGO data analysis purposes, it would be advantageous to do a very coarse survey of the parameter space aimed at exploring several qualitative issues and at determining the ranges of the several key parameters which we describe. (iv) If merger templates are available for data analysis, matched filtering can be used to make quantitative tests of general relativity in a highly dynamical and nonlinear regime, and to make measurements of the binary’s parameters. These measurements and tests can be carried out with moderate accuracy by LIGO/VIRGO, and with extremely high accuracy by the proposed space-based interferometer LISA. Using information theory, we estimate the total number of bits of information obtainable from the merger waves ($\sim 10$ to 60 bits for LIGO/VIRGO, up to $\sim 200$ bits for LISA), and estimate how much information would be lost due to numerical errors in the templates or to sparseness in the template grid. We deduce an approximate rule-of-thumb for the required accuracy of merger templates and for their spacing.

I. INTRODUCTION AND SUMMARY

A. Gravitational waves from binary black hole systems

With the kilometer-scale, ground-based interferometric gravitational-wave observatories LIGO \cite{ligo}, VIRGO \cite{vigo}, and GEO600 \cite{geo} expected to be on line and taking data within the next few years, and with the space-based interferometer LISA \cite{isa} in the planning and development stage, much effort is currently going into understanding potential gravitational-wave sources and associated data analysis issues. One potentially very interesting and important class of source is the coalescence of binary black holes (BBHs) where the two black holes have comparable masses. Such binaries with total masses $M$ in the range $10 M_{\odot} \lesssim M \lesssim 10^3 M_{\odot}$ could be detected by ground-based interferometers, and with $10^2 M_{\odot} \lesssim M \lesssim 10^8 M_{\odot}$ by LISA.

The evolution of these systems, and the gravitational waves that they emit, can be roughly divided into three successive epochs: an adiabatic inspiral epoch, in which the evolution of BBH systems is driven by radiation reaction, and which terminates roughly at the last stable
circular orbit \cite{14,15}: a violent, dynamical merger epoch; and a ringdown epoch in which the emitted gravitational waves are dominated by the $l = m = 2$ quasinormal mode radiations of the final Kerr black hole. Gravitational waves from the merger phase could be rich with information about relativistic gravity in a highly nonlinear, highly dynamical regime which is poorly understood today.

Theoretical predictions of the gravitational waveforms $h_+(t)$ and $h_\times(t)$ produced in the three phases of BBH coalescences will be useful both for detecting the gravitational-wave signal, and for interpreting and making deductions from the observed waveforms, i.e., for extracting information from the waves.

For the inspiral phase, such theoretical waveforms or waveform templates have already been computed analytically to post-2.5-Newtonian order \cite{16,17}. These templates will be accurate enough for separations $r \gtrsim 12M$ that their errors will not significantly impede wave detection; for more details see, for example, Ref. \cite{17} and Sec. \S1 below. The phase evolution of the inspiral waves between $r \sim 12M$ and $r \sim 6M$, where $M$ is the total mass of system and $r$ the distance between the black holes in Schwarzschild coordinates, will not be accurately described by the post-Newtonian approximation \cite{18}. Alternative analytic and numerical approximation schemes are under development for modeling the coalescence and computing the waves in this “Intermediate binary black hole” (IBBH) regime \cite{13,14,15}. For the purpose of this paper, we consider this IBBH regime to be part of the inspiral phase of the coalescence.

Templates for the ringdown phase of the coalescence are obtained using perturbation theory on the background of the final Kerr black hole \cite{17,18}; these templates consist of exponentially damped sinusoids.

In contrast to the situation for the inspiral and ringdown phases, there is at the present time very little theoretical understanding of gravitational waves from the merger phase, and no merger templates exist at all. Detailed understanding of the merger probably will come only from numerical relativity. One rather large effort to compute the dynamics of BBH mergers is the American Grand Challenge Alliance, an NSF funded collaboration of physicists and computer scientists at eight institutions \cite{17,18}; similar efforts are underway elsewhere. Modeling BBH mergers is an extremely difficult task; the numerical relativists who are writing codes for simulating BBH mergers are beset with many technical difficulties.

Our theoretical understanding of BBH mergers could be in any one of several different states by the time the first BBH coalescences are detected: (i) No information: The supercomputer simulation codes have not yet been successfully implemented, thus no information about waves from BBH mergers is available. (ii) Information limited in principle: A small amount of information about the waves is available. This could arise if working supercomputer codes are available, but the codes cannot simulate fully general BBH mergers, but only those in some special class (e.g., vanishing initial spins, or equal mass black holes). Or, it could arise if the codes can simulate arbitrary mergers but technical difficulties prevent the extraction of accurate gravitational waveforms; in such cases one would know at least the duration of the merger waves. (iii) Information limited in practice: Fully general BBH mergers can be simulated and waveforms can be extracted, but each run of these codes to produce a template is very expensive in terms of computer time and cost, and therefore only a small number of representative template shapes can be computed and stored. (The total number of template shapes required to cover the entire range of behaviors of BBH mergers is likely to be in the range of thousands to millions or more.) (iv) Full information: A complete set of theoretical templates has been computed and is available for data analysis. This fourth possibility seems rather unlikely in the time frame of the first detections of BBH coalescences.

B. Detecting the waves

Depending on the system’s mass, some BBH coalescence events will be most easily detected by searching for the inspiral waves, others by searching for the ringdown waves, and others by searching for the merger waves themselves (depending on the system’s mass). In paper I of this series \cite{1}, we analyzed the prospects for detecting BBH events using these three different types of searches, for initial and advanced LIGO interferometers and for LISA. We briefly review here some of the relevant aspects and conclusions of that analysis.

Low-mass BBHs [$M \lesssim 30M_\odot$ for the first LIGO interferometers; $(1+z)M \lesssim 80M_\odot$ for the advanced LIGO interferometers; $(1+z)M \lesssim 3 \times 10^6M_\odot$ for LISA, where $z$ is the source’s cosmological redshift] are best searched for via their inspiral waves. Such searches will use matched filtering with post-Newtonian templates. These low-mass binaries may be the most common type of detected BBH source. Moreover, they may well be the first detected source of gravitational waves and be detected before binary neutron star inspirals, since the range of initial LIGO interferometers for BBHs with $M \lesssim 50M_\odot$ is $\sim 250$ Mpc whereas binary neutron stars can be seen out to $\sim 25$ Mpc \cite{1,2}.

Higher mass BBH systems are better searched for via their ringdown waves or merger waves. A matched filtering search for ringdown waves will be possible as soon as data are available, since ringdown templates are simple to construct.

A matched filtering search for merger waves could be performed if a complete set of merger templates were available. We estimated in Ref. \cite{1} that the resulting event detection rate would be a factor of roughly 40 higher than the event rate from inspiral and ringdown searches for a certain range of BBH masses ($30M_\odot \lesssim$
\( M \lesssim 200M_\odot \) for initial LIGO interferometers, \( 100M_\odot \lesssim M \lesssim 400M_\odot \) for advanced LIGO interferometers, and \( 3 \times 10^6M_\odot \lesssim (1+z)M \lesssim 3 \times 10^7M_\odot \) for LISA. However, as mentioned above in Sec. I \( A \) it seems very unlikely that a complete bank of numerical templates will be available. If merger templates are not available, one can still search for the merger waves using simple bandpass filtering (i.e., using filters that throw away all signal and noise except for that within some prescribed frequency band), or more effectively using techniques such as the noise-monitoring search method described in Refs. \( [11,20] \).

The gain factor in event detection rate for noise-monitoring searches for merger waves, over inspiral and ringdown searches, will be roughly 4 to 10, depending on (among other things) whether or not one has firm information from representative supercomputer simulations about the possible durations and frequency bandwidths of merger waveforms \( [21] \).

Once a BBH event has been detected, the location of the three different phases of the waves in the data stream will be known to a fair approximation. For many detected events, though, it will not be the case that all three phases will be detectable. For instance, typical low mass BBH events which are detected via their inspiral waves will have ringdown waves that are too weak to be detected; see Ref. \( [1] \) and Sec. \( I \). Likewise, very massive systems which are detected via their ringdown waves might in some cases not yield a detectable inspiral signal.

### C. Extracting the waves' information: three scenarios

In contrast to paper I \( [1] \), where we focused on expected signal strengths and search strategies for BBH events, in this paper we focus on measurements of the merger waveform itself: on reconstructing the waveform from the instrumental data stream, and on using the measured waveforms to learn about the BBH source and about the dynamics of very strong field general relativity. At present, because merger waveforms are so poorly understood, it is hard to say how much one can learn about BBH systems from their merger waves. Both how well we can reconstruct BBH waveforms and how much we can learn from such reconstructions depend on the success of efforts to numerically simulate BBH mergers.

In this subsection, as background to the discussion of the contents of this paper in Sec. \( I \) below, we describe in general terms three possible different scenarios for data analysis of the merger waves:

- The first possibility (corresponding to situation (i) in Sec. \( I \)) is that numerical computations might provide no input at all that can be used to aid gravitational-wave data analysis. In this case, with no templates to guide the interpretation of the measured waveform, it will not be possible to obtain any information about the BBH source or about strong-field general relativity from the merger waves. One’s goal will simply be to measure as accurately as possible the merger waveform’s shape. For this waveform shape measurement, observers should make use of all possible prior information obtainable from analyses of the inspiral and/or ringdown signals, if they are detectable. (For example, if the system is detected via its inspiral waves, then one will know that the merger waves lie immediately following the inspiral waves in the data stream, and must join smoothly onto the ringdown waves.)

- Second (situations (ii) and (iii) of Sec. \( I \)), if one has only a few, representative supercomputer simulations and associated waveform templates at one’s disposal, one might simply perform a qualitative comparison between the measured waveform and templates in order to deduce qualitative information about the BBH source. For instance, simulations might demonstrate a strong correlation between the duration of the merger (in units of the total mass of the system) and the spins of the black holes in the binary. One might then be able to deduce some information about the black hole spins from the duration of the reconstructed merger waveform, without having to find a template that exactly matched the measured waveform. In this second scenario, for the purpose of reconstructing a “best fit” merger waveform from the noisy data stream, one should use the prior information from the measured inspiral and/or ringdown waves, and in addition the prior information (for example the expected range of frequencies) one has about the merger waveforms’ behaviors from the representative supercomputer simulations.

The third scenario consists of performing matched filtering analyses of the data stream with merger templates in order to measure the parameters of the BBH binary and to test general relativity. This will certainly be feasible if one has a complete set of merger templates (situation (iv) of Sec. \( I \)). However, in some cases matched filtering parameter extraction may also be feasible in situation (iii) of Sec. \( I \) where one has a working computer code for simulating BBH mergers but where each run of the code is so expensive in computer time and cost that it is not possible to calculate a complete set of templates. In such a case, after the merger waves have been detected, it may be possible to perform several runs of the supercomputer code, concentrated in the appropriate small region of parameter space compatible with one’s measurements from the inspiral and ringdown waves, in an effort to match the observed waveforms. In either case (complete set of templates or templates produced as needed), a conclusive fit between a numerical waveform and the measured waveform would be a triumph for general relativity, testing the theory in an extremely strong field, fast motion regime with no approximations, and would provide an unequivocal signature of the existence of black holes.

In this paper, as we now outline, we consider the requirements for and the implications of all three of these modes of data analysis.
D. Extracting the waves’ information: our analyses, suggested tools, and results

The four principal purposes of this paper are: (i) to review and discuss the useful information carried by all three phases of the waves and the prospects for its extraction, both with and without templates; (ii) to suggest a data analysis method that can be used in the absence of templates to obtain from the noisy data stream a “best-fit” merger waveform shape; (iii) to provide input to numerical relativity simulations by highlighting the kinds of information that supercomputer simulations can provide, other than merger templates, that can aid BBH merger data analysis; and (iv) to provide input to numerical relativity simulations by deriving some requirements that numerical templates must satisfy in order to be as useful as possible for data analysis purposes. We now turn to a detailed summary of our analyses and results in these four areas.

We first consider the situation in which very little information about the merger waveform is available to aid data analysis. The data analysis method that we suggest [item (ii) in the above paragraph] reduces in this case to band-pass filtering. In this case, observers will likely resort to simple band-pass filters to study the merger waves. The first question to address in this context is whether the merger signal is likely to even be visible; that is, whether the signal will stand out above the background noise level in the band-pass filtered detector output.

The merger signal will be visible if the band-pass filtering signal-to-noise ratio (SNR) is large compared to unity. In paper I of this series, we estimated the matched filtering signal-to-noise ratio (SNR) is large compared to order unity for low mass BBH events detected by ground-based interferometers. Thus, the last few cycles of the inspiral should be (just about) individually visible above the interferometer noise.

When templates are not available, one’s goal will be to reconstruct as well as possible the merger waveform from the noisy data stream. In Sec. IV of this paper we use Bayesian statistics and the framework of maximum likelihood estimation to sketch out an optimized method for performing such a reconstruction in the absence of theoretical templates. The method is based on a “perpendicular projection” of the observed noisy signal onto an appropriate function space that encodes all our (possibly sketchy) prior knowledge about the waveforms. We argue that the best type of “basis functions” to use to specify this function space are wavelets: functions which simultaneously allow localization in time and in frequency. We develop this reconstruction technique in detail using the language of wavelets. We show that the operation of “perpendicular projection” into the function space is a special case of Wiener optimal filtering. In Sec. V and Appendix C, we present our method for a network of several gravitational wave detectors which allows one to reconstruct, from the outputs of all the detectors in the network, the two independent waveforms \( h_+ (t) \) and \( h_\times (t) \) of the merger waves. We also show that our method for a network is an extension and generalization of a method previously suggested by Gürsel and Tinto [22]. Secs. A and B of Appendix A overlap somewhat with unpublished analyses by Sam Finn [23]. Finn uses similar mathematical techniques to analyze mathematical techniques to analyze the use of multiple interferometers to measure a stochastic background of gravitational waves and to measure waves of well-understood form, applications which are rather different from the measurement of bursts of unknown form that we consider.

Our waveform reconstruction algorithm comes in two versions: a simple version incorporating the above mentioned “perpendicular projection”, described in Sec. V B, and a more general and powerful version that allows one to build in more prior information, described in Sec. V C. If one’s prior information consists only of knowledge about the signal’s bandwidth, then the best-fit reconstructed waveform is just the band-pass filtered data stream. However, one can also build in as input to the method the expected duration of the signal, the fact that it must match up smoothly to the measured inspiral waveform, etc.; in such cases the reconstructed waveform differs from the band-pass filtered data stream.

Qualitative information about BBH merger waveforms will thus be very useful as prior information for signal...
reconstruction. Such information will also be useful as a basis for qualitative comparisons with the reconstructed waveforms in order to make qualitative deductions about the BBH source, as outlined in Sec. I C above. Supercomputer simulations should be able to provide such information, in the case where these codes can successfully simulate BBH mergers and produce templates, but where running the codes is too expensive to permit an extensive survey of the merger parameter space (i.e., too expensive to produce a complete set of templates). In this situation, a small number of representative simulations could still be extremely useful. In Sec. VII, we give examples of the types of information such supercomputer simulations could provide (short of providing a complete set of merger templates): the range of numbers of cycles in the merger waveform and how this number depends on parameters such as the initial spins of the black holes; the (closely related) range of temporal durations of merger waveforms, and how duration varies with parameters of the binary; the minimum and maximum frequencies of typical merger energy spectra; characteristics of the waveform’s time/frequency behavior (whether it involves a monotonic chirp or not, and whether in some cases it can be characterized as a modulated carrier wave or not); and which quasinormal modes are typically excited, and how strongly.

We turn next to issues concerning the use of numerical templates in data analysis. In Sec. VII, we begin to examine matched filtering of merger waves with templates. As mentioned in Sec. I C above, such matched filtering may be possible even if a complete set of merger templates does not exist: runs of merger template generation codes can be performed as part of the data analysis of measured BBH signals in an effort to produce a template that matches the measured waveform; such efforts may or may not be successful. We review in Sec. VII what one should be able to achieve with matched filtering: measurements of the binary’s physical parameters (masses, vectorial spin angular momenta, etc.) which are independent of any such measurements from the inspiral and ringdown waves; and quantitative tests of general relativity in the most extreme of domains: highly nonlinear, rapidly dynamical, highly non-spherical spacetime warpage. These measurements and tests will be possible with modest accuracy with LIGO/VIRGO, and with extremely high accuracy with LISA (for which the merger matched filtering SNRs are typically \( \gtrsim 10^4 \)).

In order for such measurements and tests to be as successful as possible, the numerically generated merger templates must satisfy certain requirements. In Sec. VIII we derive a simple formula [Eq. (8.2)] that numerical relativists can use to ensure that the waveforms produced by their simulations are sufficiently accurate for data analysis. In Sec. VIII A we describe how this formula can be used to regulate the accuracy with which the numerical simulations are carried out. The formula is derived from the following requirements: first, any signal searches that use matched filtering with merger templates should suffer a fractional loss of event rate due to template inaccuracies of no more than 3%; and second, when using templates to fit for and measure the physical parameters of the BBH source (masses, spins, etc.), the systematic errors due to template inaccuracies should always be smaller than the detector-noise induced statistical errors. The derivation of the formula from these two requirements is given in Sec. VIII B.

In Sec. IX we address again the issue of template accuracy requirements, and also the issue of the required spacing of templates in parameter space in the construction of a grid of templates, by using the mathematical machinery of information theory. In information theory, a quantity called “information” (analogous to entropy) can be associated with any measurement process: it is simply the base 2 logarithm of the number of distinguishable outcomes of the measurement \([24,25]\). Equivalently, it is the number of bits required to store the knowledge gained from the measurement. We specialize the notions of information theory to gravitational wave measurements, and define two different types of information: (i) the “total” information \( I_{\text{total}} \) which is the base 2 logarithm of the total number of distinguishable waveform shapes that the measurement could have produced; and (ii) a smaller “source” information \( I_{\text{source}} \), which is the base 2 logarithm of the total number of distinguishable waveform shapes that the measurement could have produced and that are generated by BBH mergers. This second measure of information is equivalent to the base 2 logarithm of the total number of independent BBH sources that the measurement could have distinguished. We give precise definitions of these two notions of information [Eqs. (9.2) and (9.11)] in Sec. IX. In Appendix B, we derive simple analytic approximations for the quantities \( I_{\text{total}} \) and \( I_{\text{source}} \) [Eqs. (9.5) and (9.12)], expressing them in terms of the merger signal’s matched filtering signal-to-noise ratio \( \rho \), the number of independent real data points \( N_{\text{bins}} \) in the observed signal, and the number of parameters \( N_{\text{param}} \) on which merger templates have a significant dependence. We estimate that the total information gain \( I_{\text{total}} \) is typically of the order of \( ~10 \) to \( ~120 \) bits for LIGO/VIRGO, and can be up to \( ~400 \) bits for LISA; and that the source information gain \( I_{\text{source}} \) is typically of the order of \( 10 \) to \( 70 \) bits for LIGO/VIRGO, and can be up to \( ~200 \) bits for LISA.

In Sec. IX C we estimate the loss in information about the BBH source, \( \delta I_{\text{source}} \), that would result from template inaccuracies [Eq. (9.20) below]; this allows us to re-derive the criterion for the template accuracy requirements obtained in Sec. VIII. We also estimate the loss in information \( \delta I_{\text{source}} \) that would result from having insufficiently closely spaced templates in a template grid [Eq. (9.24) below], and we deduce an approximate criterion for how closely templates must be spaced.
E. Organization of this paper

The remainder of this paper is organized as follows: In Sec. II, we define the notations and conventions that will be used throughout the paper. In Sec. III, we review in moderate detail the information obtainable from the inspiral and ringdown phases of the waves for detected BBH events, which will be used as prior information when attempting to analyze the merger phase.

In Sec. IV we discuss the visibility of BBH coalescence waveforms. In Sec. IV.A we first compute the band-pass filtering SNR for the last few cycles of the inspiral; this serves as background to the merger visibility analysis, and is relevant to the merger visibility itself: if the end of the inspiral is visible, then the beginning of the merger will most likely be visible as well. In Sec. IV.B we analyze the merger visibility.

In Sec. V we present our method for optimally reconstructing the merger waveform from the interferometer output. We derive the method in Sec. V.B, and in Appendix A we present an extension of the method to a network of several gravitational-wave detectors. In Sec. V.C we describe another extension of the method that allows one to incorporate prior information in a more effective way. In Sec. V.D we quantify the fidelity of the reconstructed waveform by defining a normalized correlation coefficient that describes how well the reconstructed wave correlates with the true waveform. We show in Appendix B that this coefficient will be close to 1 (i.e., that the reconstructed waveform will be close to the true waveform) when the signal’s band-pass filtering SNR is \( \gg 1 \).

In the remainder of the paper, we consider the situation where supercomputer simulations are able to provide some input to data analysis, either in the form of useful qualitative or semi-quantitative information about the merger, or in the form of templates. Sec. VI presents a list of the kinds of information that numerical relativists may be able to provide, short of a definitive template set, that can be used to aid data analysis. Sec. VII discusses and describes the kinds of information that numerical relativists may be able to provide, short of a definitive template set, that can be used to aid data analysis. Sec. VIII presents our derivations of criteria for determining whether templates are numerically accurate enough and closely spaced enough to be used in data analysis. In Sec. IX.A we derive an accuracy criterion from the requirement that the loss in event detection rate due to template inaccuracies in a matched filtering signal search using merger templates be no more than 3%. We also obtain, in Sec. IX.B, approximately the same criterion from demanding that systematic errors in parameter extraction using merger waveforms be small compared to the detector-noise induced statistical errors. In Sec. IX.C we rederive the accuracy criterion using the mathematical machinery of information theory. In this derivation, we require that the number of bits of information lost due to template inaccuracies be less than 1. The relevant information theoretic concepts are presented in Secs. A and B, and some of the technical calculations are relegated to Appendix B.

Finally, in Sec. XI we summarize our main conclusions.

II. NOTATIONS AND CONVENTIONS

In this section we introduce some of the conventions and notations that will be used throughout the paper. We use geometrized units in which Newton’s gravitational constant \( G \) and the speed of light \( c \) are unity. For any function of time \( a(t) \), we will use a tilde to represent that function’s Fourier transform, according to the convention

\[
\tilde{a}(f) = \int_{-\infty}^{\infty} dt e^{2\pi i ft} a(t). \tag{2.1}
\]

The output strain amplitude \( s(t) \) of a gravitational wave detector can be written as

\[
s(t) = h(t) + n(t), \tag{2.2}
\]

where \( h(t) \) is the gravitational wave signal and \( n(t) \) is the detector noise. Throughout this paper we will assume, for simplicity, that the noise is stationary and Gaussian. The statistical properties of the noise determine a natural inner product \((\ldots | \ldots)\) on the vector space of waveforms \( h(t) \), given by

\[
(h_1 | h_2) = 4 \text{Re} \int_{0}^{\infty} df \frac{\hat{h}_1(f)^* \hat{h}_2(f)}{S_h(f)}; \tag{2.3}
\]

see, for example, Refs. 24,27. In Eq. (2.3), \( S_h(f) \) is the power spectral density of the strain noise \( n(t) \). The associated norm is given by

\[
||h|| \equiv \sqrt{(h | h)}. \tag{2.4}
\]

For any waveform \( h(t) \), the matched filtering signal-to-noise ratio is given by

\[
\rho^2 = (h | h) = 4 \int_{0}^{\infty} df \frac{\hat{h}(f)^2}{S_h(f)}. \tag{2.5}
\]

On several occasions we shall be interested in finite stretches of data of length \( T \) say, represented in a discrete way as a vector of numbers instead of as a continuous function. If \( \Delta t \) is the sampling time, this vector is

\[
s = (s^1, \ldots, s^{N_{\text{bins}}}) \tag{2.6}
\]

where \( N_{\text{bins}} = T/\Delta t \), \( s^j = s(t_{\text{start}} + j \Delta t), 0 \leq j \leq N_{\text{bins}} \), and \( t_{\text{start}} \) is the starting time. In this quantity \( N_{\text{bins}} \) is the number of independent real data points (number of bins) in the measured signal; it is denoted by \( N \) in Appendices A and C. The gravitational wave signal \( h(t) \) and the
noise \( n(t) \) can similarly be represented in this way, so that \( s = h + n \), as in Eq. (2.4). We adopt the geometrical viewpoint of Dhurandhar and Schutz [29], regarding \( s \) as an element of an abstract vector space \( V \) of dimension \( N_{\text{bins}} \), and the sample points \( s^j \) as the components of \( s \) on a time domain basis \( \{ e_1, \ldots, e_{N_{\text{bins}}} \} \) of \( V \):

\[
s = \sum_{j=1}^{N_{\text{bins}}} s^j e_j. \tag{2.7}
\]

Taking a finite Fourier transform of the data stream can be regarded as a change of basis of \( V \) in which \( s \) remains fixed but its components change. Thus, a frequency domain basis \( \{ d_k \} \) of \( V \) is given by the finite Fourier transform

\[
d_k = \sum_{j=1}^{N_{\text{bins}}} e_j \exp \{ 2\pi i j k / N_{\text{bins}} \}, \tag{2.8}
\]

where \(- (N_{\text{bins}} - 1)/2 \leq k \leq (N_{\text{bins}} - 1)/2\). The corresponding frequencies \( f_k = k/T \) run from \(-1/(2\Delta t)\) to \(1/(2\Delta t)\).

More generally, if we band-pass filter the data stream down to a frequency interval of length \( \Delta f \), and consider a stretch of band-pass filtered data of duration \( T \), this stretch of data will have

\[
N_{\text{bins}} = 2T\Delta f \tag{2.9}
\]

independent real data points. In this case also we regard the set of all such stretches of data as an abstract linear space \( V \) of dimension \( N_{\text{bins}} \).

On an arbitrary basis of \( V \), we define the matrices \( \Gamma_{ij} \) and \( \Sigma^{ij} \) by

\[
\langle n^i n^j \rangle = \Sigma^{ij} \tag{2.10}
\]

and

\[
\Gamma_{ij} \Sigma^{jk} = \delta^i_k; \tag{2.11}
\]

i.e., the matrices \( \Gamma \) and \( \Sigma \) are inverses of each other. In Eq. (2.10) the angle brackets mean expected value. On the time domain basis \( \{ e_1, \ldots, e_{N_{\text{bins}}} \} \), we have

\[
\Sigma^{jk} = C_n(t_j - t_k), \tag{2.12}
\]

where \( t_j = t_{\text{start}} + j\Delta t \), and \( C_n(\tau) = \langle n(t) n(t + \tau) \rangle \) is the noise correlation function given by

\[
C_n(\tau) = \int_0^\infty df \cos[2\pi f \tau] S_h(f). \tag{2.13}
\]

We define an inner product on the space \( V \) by

\[
\langle h_1 | h_2 \rangle = \Gamma_{ij} h^i_1 h^j_2 \tag{2.14}
\]

This is essentially a discrete version of the inner product (2.3) which characterizes the detector noise: the two inner products coincide in the limit of small sampling times \( \Delta t \), and for waveforms which vanish outside of the time interval of length \( T \) [31].

Throughout this paper we shall use interchangeably the notations \( h(t) \) and \( h \) for a gravitational waveform. We shall also for the most part not need to distinguish between the inner products (2.3) and (2.14). Some generalizations of these notations and definitions to a network of several detectors are used in Appendix B.

For a given detector output \( s = h + n \), we define

\[
\rho(s)^2 = ||s||^2 = \langle s | s \rangle, \tag{2.15}
\]

which is the inner product or integral of the detector output with itself. We will call \( \rho(s) \) the magnitude of the stretch of data \( s \). From Eqs. (2.10) and (2.14) it follows that

\[
\langle \rho(s)^2 \rangle = \rho^2 + N_{\text{bins}}, \tag{2.16}
\]

where \( \rho^2 \) is the matched filtering SNR squared (2.5) of the signal \( h \), and that

\[
\sqrt{\langle (\Delta \rho(s)^2)^2 \rangle} = \sqrt{4\rho^2 + 2N_{\text{bins}}}, \tag{2.17}
\]

where \( \Delta \rho(s)^2 \equiv \rho(s)^2 - \langle \rho(s)^2 \rangle \). Thus, the magnitude \( \rho(s) \) is approximately the same as the usual SNR \( \rho \) in the limit \( \rho \gg \sqrt{N_{\text{bins}}} \) (large signal-to-noise squared per frequency bin), but is much larger than \( \rho \) when \( \rho \ll \sqrt{N_{\text{bins}}} \). The quantity \( \rho(s) \) will occur in our in our information theory calculations in Sec. IX and Appendix B.

The space \( V \) equipped with the inner product (2.14) forms a Euclidean vector space. We will also be concerned with sets of gravitational waveforms \( h(\theta) \) (equivalently, \( h(t; \theta) \)) that depend on a finite number \( n_p \) of parameters \( \theta = (\theta^1, \ldots, \theta^{n_p}) \). For example, inspiral gravitational waveforms form a set of this type, where \( \theta \) are the parameters describing the binary source. We will denote by \( S \) the manifold of signals \( h(\theta) \), which is a submanifold of dimension \( n_p \) of the vector space \( V \). We will adopt the convention that Roman indices \( i, j, k, \ldots \) will run from 1 to \( N_{\text{bins}} \), and that \( v^i \) will denote some vector in the space \( V \). Greek indices \( \alpha, \beta, \gamma \) will run from 1 to \( n_p \), and a vector \( v^\alpha \) will denote a vector field on the manifold \( S \). The inner product (2.14) induces a natural Riemannian metric on the manifold \( S \) given by

\[
ds^2 = \left( \frac{\partial h}{\partial \theta^\alpha} \right) \left( \frac{\partial h}{\partial \theta^\beta} \right) d\theta^\alpha d\theta^\beta. \tag{2.18}
\]

We shall denote this metric by \( \Gamma_{\alpha\beta} \) and its inverse by \( \Sigma^{\alpha\beta} \), relying on the index alphabet to distinguish these quantities from the quantities (2.10) and (2.11). For more details on this geometric picture, see, for example, Ref. [27].

We shall use the word detector to refer to either a single interferometer or a resonant mass antenna, and the phrase detector network to refer to a collection of detectors operated in tandem. Note that this terminology differs from that adopted in, for example, Ref. [26], where a detector network is called simply a detector.
Finally, we will use bold faced vectors like \( \mathbf{a} \) to denote either vectors in three dimensional space, or vectors in the \( N_{\text{bins}} \)-dimensional space \( V \). In Appendix A, we will use arrowed vectors (\( \mathbf{e}_1, \mathbf{e}_2 \)) to denote elements of the linear space of the output of a network of gravitational wave detectors.

### III. INFORMATION FROM THE INSPIRAL AND RINGDOWN PHASES

Different types of information will be obtainable from the three different phases of the gravitational wave signal. If the inspiral and ringdown phases are strong enough to be measurable, they will be easier to analyze than the merger phase, and the information they yield will be used as “prior information” in attempting to analyze the merger phase. For instance, from the inspiral portion of the signal it will be possible to measure the masses of the binary’s black holes to some accuracy (as we discuss below). Those measured masses will then be an input to data analysis of the merger waves, since they strongly constrain the possible values of template parameters that need to be examined when fitting a theoretical waveform to the merger signal. In this section, we review the prior information that will likely be available from measurements of the inspiral and the ringdown in typical cases.

Let us focus first on solar mass coalescences \( [(1 + z)M \lesssim 50 M_\odot \text{say}] \) measured by ground based interferometers, for which most of the prior information will come from the inspiral waveforms. The analysis of the inspiral waveforms will take place in two phases. The first phase will consist of filtering the data streams of each detector separately using “search templates” in order to detect the inspiral \( \mathbf{a} \). These search templates will depend on 2 or possibly 3 parameters. Roughly \( 10^4 \) to \( 10^5 \) distinct template shapes will be required for initial LIGO interferometers, and roughly \( 10^6 \) to \( 10^7 \) template shapes for advanced LIGO interferometers \([33,36]\). (Note that these numbers assume that the search is for generic inspiralling binaries, not simply black hole binaries. If the search were restricted to BBH systems only, these numbers would be greatly reduced: assuming that the smallest BBH systems consist of a pair of 2\( M_\odot \) binaries, the number of templates for initial LIGO interferometers is roughly \( 10^3 \), and for advanced interferometers roughly \( 10^3 \).) The second phase will consist of combining the outputs of all the detectors together and using the most accurate templates available (“extraction templates”) to analyze the signal and extract the best-fit parameter values. Such extraction templates will presumably be provided by post-Newtonian calculations, perhaps improved by the judicious use of Padé approximants \([35,36]\), and perhaps supplemented by IBBH calculations in the IBBH regime \( 6 M_\odot \lesssim r \lesssim 12 M_\odot \) (cf. the discussion in Sec. A above). In this second phase there will be 15 independent parameters to fit for. These parameters are the masses \( m_1 \) and \( m_2 \) and initial spins \( S_1 \) and \( S_2 \) of the two black holes, the luminosity distance \( D \) to the binary, the direction of the orbital angular momentum \( \mathbf{L} = \mathbf{L}/|\mathbf{L}| \), the direction \( \mathbf{n} \) from the binary to the Earth, and the arrival time \( t_a \) and orbital phase \( \phi_0 \) at some fiducial frequency. (The dependence of the templates on several of these 15 parameters, such as the luminosity distance, will be trivial and will not need to be computed numerically.)

As an example, consider a binary with two non-spinning \( 10 M_\odot \) black holes at a distance of 200 Mpc. The inspiral SNR for this system is \( \sim 100 \) for advanced LIGO interferometers \([11]\). In this optimistic case, the information obtained from the inspiral waveform will be roughly as follows \([10]\): The distance to the system will be known to \( \lesssim 2\% \), the masses will be known to \( \sim 40\% \) (although the chirp mass \( M = m^{3/5} M^{2/5} \) will likely be known to an accuracy of \( \lesssim 0.1\% \)), the arrival time to \( \sim 0.1 \) ms, the position on the sky to less than one square degree, and the angles defining \( \mathbf{L} \) and \( \phi_0 \) to \( \lesssim 10^\circ \). Also, some information will be obtained about two particular combinations of the spins \( S_1 \) and \( S_2 \) (see Refs. \([23,37,41]\) for details). As a second example, consider a binary of two \( 15 M_\odot \) black holes at \( z = 1 \), for which the inspiral SNR for advanced interferometers is \( \sim 7 \) \([11]\). For such a binary the accuracies are several times worse. The luminosity distance is measured to \( \sim 20\% \), for example, and although the chirp mass is measured to \( \lesssim 1\% \), the individual masses are only constrained to lie in the ranges \( 3 M_\odot \lesssim m_2 \lesssim 15 M_\odot \) and \( 15 M_\odot \lesssim m_1 \lesssim 100 M_\odot \) \([40]\).

Turn, now, to the information obtainable from inspiral signals for the space-based LISA interferometer. Equation (A6) of Ref. \([11]\) shows that the time \( T_{\text{insp}} \) which the gravitational wave signal spends in the interferometer’s bandwidth during the inspiral before merger is approximately

\[
T_{\text{insp}} \sim 0.4 \text{ yr} \left( \frac{(1 + z)M}{10^6 M_\odot} \right)^{-5/3} \left[ 1 - \left( \frac{(1 + z)M}{4 \times 10^7 M_\odot} \right)^{8/3} \right].
\]

(3.1)

Signal-to-noise ratios from such inspirals (or from the last year of inspiral if \( T_{\text{insp}} \geq 1 \) yr) will be \( \gtrsim 100 \) for all events with cosmological redshift \( z \lesssim 10 \) and with \( 10^3 M_\odot \lesssim (1 + z)M \lesssim 5 \times 10^7 M_\odot \); see Fig. 6 of Ref. \([11]\). Thus, detailed information about the binary’s parameters should be available for analyzing merger signals detected by LISA \([11]\). (For some LISA BBH sources, most of the inspiral SNR will come from the IBBH regime \( 6 M_\odot \lesssim r \lesssim 12 M_\odot \) discussed in the Introduction. For such sources, accurate IBBH templates will likely be needed to extract all the available inspiral information.)

In some cases with LIGO/VIRGO, and in many cases with LISA, it will also be possible to analyze the ringdown waveform using optimal filtering to extract the ringdown frequency and damping time \([31,33]\). These measurements will yield the mass \( M \) and spin parameter \( a \) of the final black hole. The accuracy of such measurements will be approximately given by \([31,33]\).
\[
\frac{\Delta M}{M} \sim \frac{2(1 - a)^{9/20}}{(S/N)_{\text{ringdown}}},
\]
\[
\Delta a \sim \frac{6(1 - a)^{1.06}}{(S/N)_{\text{ringdown}}},
\]
where \((S/N)_{\text{ringdown}}\) is the matched filtering SNR for the ringdown signal. It should also be possible to measure the ratio at which the ringdown starts to within an accuracy \(\lesssim 1/f_{\text{nr}}\). For low mass coalescences \((M \lesssim 50M_\odot)\), such measurements will only be possible for the very strongest detected events: the ringdown SNR will be \(\gtrsim 1\) only for the strongest \(\sim 1\%\) of detected events for initial and advanced LIGO interferometers \[44\]. For larger mass BBH coalescences, however, the ringdown SNR will be larger, as can be seen from Figs. 4 and 5 of Ref. \[11\], and ringdown measurements will be feasible for a reasonable fraction of detected signals. For LISA, Fig. 6 of Ref. \[11\] shows that most detected merger events will be accompanied by easily detectable ringdown signals with SNR values \(\gtrsim 100\). Thus, accurate values of \(M\) and \(a\) should be available as prior information when analyzing merger signals detected by LISA.

For the strongest detected signals, it may also be possible to measure the complex amplitudes of some of the quasinormal modes in the waveform other than the dominant \(l = m = 2\) mode. These higher order quasinormal ringing (QNRM) modes will not be as long lived as the \(l = m = 2\) mode, but they may nevertheless be detectable. The amplitudes and phases of such modes will constitute very useful information if they are measurable, since their values should be predictable by the supercomputer simulations as functions of the binary’s parameters at the start of the merger phase. The supercomputer simulations will have passed an important test if the measured mode amplitude values are consistent with known information about the initial conditions.

**IV. ANALYSIS OF THE MERGER WAVES WITHOUT TEMPLATES—VISIBILITY OF MERGER SIGNAL AFTER BAND-PASS FILTERING**

Turn now to the data analysis of the merger waves, focusing on the case in which matched filtering cannot be used. This situation will arise if supercomputer simulations are unable to produce merger templates, or if they have only produced a small sampling of the total function space \(S\) of merger waveforms when BBH signals are detected. Such a sampling should provide valuable qualitative information about the merger waveforms (as we discuss in Sec. \[V\] below), but would be too sparse to be used as a bank of optimal filters. (As mentioned in Sec. \[IV\] above, it may be possible to perform matched filtering in the absence of a complete set of templates, but this is not guaranteed).

In the absence of a complete set of theoretical templates, one’s first aim will be to reconstruct from the noisy detector output a best-guess estimate of the merger waveform \(h(t)\) \[3\]. If a small number of representative supercomputer templates are available, it may then be possible to interpret the measured waveform and obtain qualitative information about the BBH source. One very simple procedure that could be used to obtain an estimate of the waveform shape is simply to band-pass filter the data stream according to our prior prejudice about the frequency band of the merger waves (based on estimates of the merger signal bandwidth \[4\], hopefully supplemented by information from representative supercomputer simulations and from inspiral/ringdown measurements \[5\]). However, after such band-pass filtering, the merger signal may be dominated by detector noise and may not even be visible. (Signals that are visible in the noise will clearly be easier to reconstruct from the noisy data stream; we demonstrate this mathematically in Sec. \[VI\] below).

In this section we explore this issue of merger waveform visibility, by which we mean whether or not the signal stands out above the noise after band-pass filtering. A signal will be visible if the band-pass filtering SNR is large compared to unity; see, for example, the discussion in Ref. \[4\]. We use the results of Ref. \[4\] to estimate band-pass filtering SNRs, first for the inspiral waves near the end of the inspiral in Sec. \[IV\] and then for the merger waves in Sec. \[V\] below. The analysis of the inspiral waves is useful as background for the merger visibility calculation, and is also indicative of the visibility of the early merger waves (if the endpoint of inspiral is visible with band-pass filters, than one would expect that by continuity the beginning of the merger should be visible as well).

**A. Visibility of inspiral waveform**

We focus on BBH events which have been detected via their inspiral waves using matched filtering. Since the event has been detected, the inspiral matched filtering SNR must be \(\gtrsim 6\) \[4\]; however, it does not follow that the inspiral signal is visible in the data stream without matched filtering. (In fact, for neutron star-neutron star binaries the reverse is usually the case: the amplitude of the signal is rather less than the noise, and so matched filtering is very necessary to detect the waves.) We now estimate the degree of visibility of the last few cycles of the inspiral waveform for BBH coalescences.

The dominant harmonic of the inspiral waveform can be written as

\[
h(t) = h_{\text{amp}}(t) \cos[\Phi(t)],
\]
where the amplitude \(h_{\text{amp}}(t)\) and instantaneous frequency \(f(t)\) [given by \(2\pi f(t) = d\Phi/dt\)] are slowly evolving. For such waveforms, the SNR squared obtained using band-pass filtering is approximately given by the SNR
squared per cycle obtained from matched filtering [cf. Eq. (2.9) of Ref. [11]]:

\[
\left( \frac{S}{N} \right)_{\text{band-pass}}^2 \approx \left( \frac{S}{N} \right)_{\text{optimal filter, per cycle}}^2 = \frac{h_{\text{amp}}[t(f)]^2}{h_n(f)^2}. \tag{4.2}
\]

In Eq. (4.2), an rms average over source orientations has been performed, \( t(f) \) denotes the time at which the instantaneous frequency has value \( f \), and \( h_n(f) \equiv \sqrt{5} S_5(f) \). Note that the band-pass filtering SNR (4.2) is evaluated at a specific frequency, whereas typically when one discusses matched filtering SNRs, an integral approximation to the inspiral waves, which can be obtained from, for example, Eq. (3.20) of Ref. [11], and approximated when one discusses matched filtering SNRs, an integral approximation to the inspiral waves, which can be obtained from, for example, Eq. (3.20) of Ref. [11], and obtain

\[
\left( \frac{S}{N} \right)_{\text{band-pass}}^2 = \frac{64 \pi^{4/3} M^{10/3} (1+z)^{10/3} f^{4/3}}{5 D(z)^2 h_n(f)^2}. \tag{4.3}
\]

Here \( M \equiv \mu^{3/5} M^{2/5} \) is the chirp mass, \( z \) is the binary’s cosmological redshift and \( D(z) \) is the binary’s luminosity distance.

In Eq. (4.1) of Ref. [11] we introduced an analytic formula for a detector’s noise spectrum \( S_n(f) \), which, by specialization of its parameters, could describe to a good approximation either an initial LIGO interferometer, an advanced LIGO interferometer, or a space-based LISA interferometer. We now insert that formula into Eq. (4.2), and specialize to the frequency

\[
f = f_{\text{merge}} = \frac{\gamma_m}{(1+z)M}. \tag{4.4}
\]

where \( \gamma_m = 0.02 \). The frequency \( f_{\text{merge}} \) is approximately the location of the transition from inspiral to merger, as estimated in Ref. [11]. We thus obtain for the band-pass filtering SNR

\[
\left( \frac{S}{N} \right)_{\text{band-pass}}^2 \approx \frac{4 \pi^{4/3} M^5 (1+z)^5 \gamma_m^{-5/3} \alpha^3 f_m^3}{5 D(z)^2 h_m^2}, \tag{4.5}
\]

where \( \alpha, h_m, \) and \( f_m \) are the parameters used in Ref. [11] to describe the interferometer noise curve. Equation (4.5) is valid only when the redshifted mass \((1+z)M\) of the binary is smaller than \( \gamma_m / \alpha f_m \).

For initial LIGO interferometers, appropriate values of the parameters \( h_m, f_m \) and \( \alpha \) are given in Eq. (4.2) of Ref. [11]. Inserting these values into Eq. (4.5) gives

\[
\left( \frac{S}{N} \right)_{\text{band-pass}}^2 \approx 1.1 \left[ \frac{200 \text{ Mpc}}{D(z)} \right] \left[ \frac{(1+z)M}{20 M_\odot} \right]^{5/2}, \tag{4.6}
\]

which is valid for \((1+z)M \lesssim 18 M_\odot\). Now, the SNR obtained by matched filtering the inspiral signal (i.e., by correlating the inspiral data with an inspiral template over the full bandwidth of the signal) is approximately

\[
\left( \frac{S}{N} \right)_{\text{optimal}} \sim 2.6 \left[ \frac{200 \text{ Mpc}}{D(z)} \right] \left[ \frac{(1+z)M}{20 M_\odot} \right]^{5/6}. \tag{4.7}
\]

Also the quantity (4.3) must be \( \gtrsim 6 \), because, by assumption, the inspiral has in fact been detected. By eliminating the luminosity distance \( D(z) \) between Eqs. (4.6) and (4.7) we find that the band-pass filtering SNR for the last inspiral cycles of detected binaries satisfies

\[
\left( \frac{S}{N} \right)_{\text{band-pass}} \gtrsim 2.5 \left[ \frac{(1+z)M}{20 M_\odot} \right]^{5/3}. \tag{4.8}
\]

Therefore, the last few cycles of the inspiral should be individually visible above the noise for BBH events with \( 5 M_\odot \lesssim M \lesssim 20 M_\odot \) detected by initial LIGO interferometers.

We now repeat the above calculation with the values of \( h_n, f_m, \) and \( \alpha \) appropriate for advanced LIGO interferometers, which are given in Eq. (4.3) of Ref. [11]. The band-pass filtering SNR for advanced interferometers is

\[
\left( \frac{S}{N} \right)_{\text{band-pass}} \sim 1.6 \left[ \frac{1 \text{ Gpc}}{D(z)} \right] \left[ \frac{(1+z)M}{20 M_\odot} \right]^{5/2}, \tag{4.9}
\]

and the SNR obtained by matched filtering the inspiral signal is

\[
\left( \frac{S}{N} \right)_{\text{optimal}} \sim 16 \left[ \frac{1 \text{ Gpc}}{D(z)} \right] \left[ \frac{(1+z)M}{20 M_\odot} \right]^{5/6}, \tag{4.10}
\]

for \((1+z)M \lesssim 37 M_\odot\). So, with the assumption that \( \left( \frac{S}{N} \right)_{\text{optimal}} \gtrsim 6 \), we find

\[
\left( \frac{S}{N} \right)_{\text{band-pass}} \gtrsim 0.6 \left[ \frac{(1+z)M}{20 M_\odot} \right]^{5/3}. \tag{4.11}
\]

for \((1+z)M \lesssim 37 M_\odot\). Therefore, for BBH inspirals with \((1+z)M \lesssim 37 M_\odot\) detected by advanced LIGO interferometers, the last few cycles of the inspiral will be just barely individually visible above the noise, depending on the binary’s total mass \( M \). The last few cycles of the inspiral will also be visible for larger mass BBH systems, as can be seen by combining Eq. (4.2) above with Figs. 4 and 5 of Ref. [11].

For LISA, Eq. (4.3) combined with Eq. (4.3) of Ref. [11] yields

\[
\left( \frac{S}{N} \right)_{\text{band-pass}} \sim 180 \left[ \frac{1 \text{ Gpc}}{D(z)} \right] \left[ \frac{(1+z)M}{10^6 M_\odot} \right]^{5/2}, \tag{4.12}
\]

for \((1+z)M \lesssim 10^5 M_\odot\), with larger values for \( 10^5 M_\odot \lesssim (1+z)M \lesssim 3 \times 10^7 M_\odot \). Therefore individual cycles of the inspiral waveform should be clearly visible for LISA.
B. Visibility of merger waveform

Consider now the merger waveform itself. This will be visible if the SNR from band-pass filtering of the merger signal is large compared to unity. In Ref. [1] we showed that

$$\left( \frac{S}{N} \right)_{\text{band-pass, merger}} \approx \frac{1}{\sqrt{N_{\text{bins}}}} \left( \frac{S}{N} \right)_{\text{optimal, merger}},$$

(4.13)

where $N_{\text{bins}} = 2T\Delta f$; $T$ and $\Delta f$ are the expected duration and bandwidth of the merger signal. We also estimated [Eq. (3.32) of Ref. [11]] that for the merger waves,

$$\sqrt{N_{\text{bins}}} \sim 5,$$

(4.14)

although there is large uncertainty in this estimate and $N_{\text{bins}}$ will vary from event to event. Combining Eqs. (5.4) of Ref. [1] for initial LIGO interferometers, Eq. (4.13), and the threshold for detection [3]

$$\left( \frac{S}{N} \right)_{\text{optimal, inspiral}} \gtrsim 6$$

(4.15)
yields

$$\left( \frac{S}{N} \right)_{\text{band-pass, merger}} \gtrsim 0.8 \left[ \frac{(1 + z)M}{20M_\odot} \right]^{5/3}$$

(4.16)

for $(1 + z)M \lesssim 18M_\odot$. For advanced LIGO interferometers, Eq. (4.14) together with Eq. (5.5) of Ref. [11] similarly yield

$$\left( \frac{S}{N} \right)_{\text{band-pass, merger}} \gtrsim 0.2 \left[ \frac{(1 + z)M}{20M_\odot} \right]^{5/3}$$

(4.17)

for $(1 + z)M \lesssim 37M_\odot$. Note that, contrary to one’s intuition, the value (4.17) for advanced interferometers is lower than the value (4.16) for initial interferometers. This is because the advanced interferometers can detect inspirals with lower band-pass filtering SNRs than the initial interferometers, due to the larger number of cycles of the inspiral signal in the advanced interferometer’s bandwidth. Matched filtering is extremely efficient at detecting inspiral signals, and it is more so for advanced interferometers than for initial interferometers. The weaker the signals that are detectable by matched filtering, the less visible the merger waveform will be after bandpass filtering.

The SNR values (4.16) and (4.17) indicate that for typical inspiral-detected BBH systems with $M \lesssim 20M_\odot$ (initial interferometers) or $M \lesssim 40M_\odot$ (advanced interferometers), the merger signal will not be easily visible in the noise, and that only the somewhat rarer, closer events will have easily visible merger signals. This conclusion is somewhat tentative because of the uncertainty in the estimates of $N_{\text{bins}}$ and of the energy spectra discussed in Ref. [11]. Also the visibility of the merger waveform will probably vary considerably from event to event.

This conclusion only applies to low mass BBH systems which are detected via their inspiral waves. For higher mass systems which are detected directly via their merger and/or ringdown waves, the merger signal should be visible above the noise after appropriate band-pass filtering. Moreover, most merger events detected by LISA will have band-pass filtering SNRs $\gg 1$, as can be seen from Fig. 6 of Ref. [11], and thus should be easily visible.

Our crude visibility argument thus suggests that the prospects for accurately recovering the merger waveform are good only for the stronger detected merger signals. This visibility analysis also illustrates the importance of theoretical template waveforms: the SNRs that can be achieved without them will often be mediocre at best. Templates for the merger will be able to boost measured SNRs by a factor $\sqrt{N_{\text{bins}}} \sim 5$. Of course, we need to go beyond this simple analysis and try to determine the optimal method of reconstructing the shape of the merger waveform from the noisy data; we propose one method in the following section.

V. ANALYSIS OF THE MERGER WAVES WITHOUT TEMPLATES—A METHOD OF EXTRACTING A BEST-GUESS MERGER WAVEFORM FROM THE NOISY DATA STREAM

A. Overview

In the absence of a complete set of theoretical templates we would like to reconstruct from the noisy detector data stream a best-guess estimate of the merger waveform $h(t)$. In this section, we suggest and describe a method, based on the technique of maximum likelihood estimation [13–19], for performing such a waveform reconstruction.

A method for estimating the merger waveform shape $h(t)$ should use all available prior knowledge about the waveform. We will hopefully know from representative supercomputer simulations and perhaps from the measured inspiral/ringdown signals the following: (i) the approximate starting time of the merger; (ii) the fact that it starts off strongly (smoothly joining on to the inspiral waveform) and eventually dies away in quasinormal ringing; and (iii) the approximate bandwidth and duration of the signal. For those signals for which both the inspiral and the ringdown are strong enough to be detectable with optimal filtering, the duration of the merger portion of the waveform will be fairly well known, as will the frequency $f_{\text{runt}}$ of the ringdown signal onto which the merger waveform must smoothly join. The technique which we describe in this section encodes such prior information and makes use of it in reconstructing the best-guess estimate of the waveform.
We shall describe this method in the context of a single detector or interferometer. However, in a few years there will be in operation a network of several detectors (both interferometers \([1,3]\) and resonant mass antennae) and from the combined outputs of these several detectors one would like to reconstruct the two independent polarization components \(h_+(t)\) and \(h_\times(t)\) of the gravitational waves from the merger. In Appendix \([4]\) we show how to extend the waveform estimation method discussed in this section to an arbitrary number of detectors, which yields a method of reconstructing the two waveforms \(h_+(t)\) and \(h_\times(t)\).

The issue of reconstructing the waveforms \(h_+(t)\) and \(h_\times(t)\) was previously addressed by G"ursel and Tinto \([2]\), in the context of a network of three interferometers and for arbitrary bursts of gravitational waves. G"ursel and Tinto suggest a method of extracting, from the outputs of all the interferometers (i) the direction to the source, and (ii) the two gravitational waveforms. For many BBH mergers, the direction to the source will have already been determined to fairly good accuracy from the inspiral waveform \([1,7]\), and so the G"ursel-Tinto filtering method is not directly applicable. However, they do suggest in passing a method for extracting the waveforms \(h_+(t)\) and \(h_\times(t)\) when the direction to the source is given. In Appendix \([5]\) we show that our filtering method (as extended to a network of interferometers) is an extension and generalization of the G"ursel-Tinto algorithm.

The filtering methods which we consider are based on the theory of maximum likelihood estimation \([18,19]\). The use of maximum likelihood estimators has been discussed extensively by many authors in the context of gravitational waves of a known functional form, depending only on a few parameters \([26,27,32,17,31]\). In this section we consider their application to gravitational wave bursts of largely unknown shape. The resulting data analysis methods which we derive are closely related mathematically to the methods discussed previously \([26,27,32,17,31]\), but are considerably different in operational terms and in implementation.

### B. Derivation of data analysis method

We now turn to our derivation of the best-guess waveform estimator using maximum likelihood estimation. Suppose that our prior information about the merger waves includes the information that they lie inside some time interval of duration \(T\), and inside some frequency interval of length \(\Delta f\). We define \(N_{\text{bins}} = 2T\Delta f\), cf. Sec. \([1]\) above. We also suppose that we have a stretch of data to analyze of duration \(T' > T\) and with sampling time \(\Delta t < 1/(2\Delta f)\). These data lie in a linear space \(V\) of dimension

\[
N'_{\text{bins}} = 2T'/\Delta t
\]

which is strictly larger than \(N_{\text{bins}}\). Thus, \(N'_{\text{bins}}\) is the number of independent real data points in the data, and \(N_{\text{bins}}\) is the number of independent real data points in that subset of the data which we expect to contain the merger signal. Note that these definitions constitute a modification/extension of the conventions introduced in Sec. \([1]\) above, where the dimension of the space \(V\) was denoted by \(N_{\text{bins}}\). We will use, unmodified, the other conventions of Sec. \([1]\); thus, the detector output \(s\) is given by \(s = h + n\), where \(h\) the gravitational-wave signal and \(n\) the detector noise, and the vectors \(s, h, \text{and } n\) are all elements of the vector space \(V\) of dimension \(N'_{\text{bins}}\).

In our analysis below, we will allow the basis of the vector space \(V\) to be arbitrary. Thus, \(n_i\) (for example) will denote the components of the noise on this arbitrary basis. However, we will occasionally specialize to the time-domain and frequency-domain bases discussed in Sec. \([1]\) above. We will also consider wavelet bases of \(V\). Wavelet bases can be regarded as any set of functions \(w_{ij}(t)\) such that \(w_{ij}(t)\) is approximately localized in time at the time \(t = t_{\text{start}} + (i/n_T)T'\), and approximately localized in frequency at the frequency \(f_j = (j/n_F)\Delta f^{-1}\). The index \(i\) runs from 1 to \(n_T\) and \(j\) from \(-(n_F - 1)/2\) to \((n_F - 1)/2\).

Clearly the number of frequency bins \(n_F\) and the number of time bins \(n_T\) must satisfy \(n_T n_F = N'_{\text{bins}}\), but otherwise they can be arbitrary; typically \(n_T \sim n_F \sim \sqrt{N'_{\text{bins}}}\). Also, the functions \(w_{ij}\) usually all have the same shape, so that

\[
w_{ij}(t) \propto \varphi [f_j(t - t_i)],
\]

for some function \(\varphi\). For our considerations here, the shape of \(\varphi\) is not of critical importance. Also, wavelet bases are often overcomplete; the bases we discuss below are to be considered simply complete. So if the full function space of some family of wavelets is \(W\), we restrict ourselves to some complete subset \(W'\) of that space. The advantage of wavelet bases is that they they simultaneously encode frequency domain and time domain information.

Let \(p^{(0)}(h)\) be the probability distribution (PDF) that summarizes our prior information about the gravitational waveform. A standard Bayesian analysis shows that the PDF of \(h\) given the measured data stream \(s\) is \([21,13]\)

\[
p(h \mid s) = \mathcal{K} p^{(0)}(h) \exp \left[ -\Gamma_{ij}(h^i - s^i)(h^j - s^j)/2 \right],
\]

where the matrix \(\Gamma_{ij}\) is defined in Eq. \([21,13]\) and \(\mathcal{K}\) is a normalization constant \([21]\). In principle this PDF gives complete information about the measurement. Maximizing the PDF will yield the maximum likelihood estimator for the merger waveform \(h\). This estimator will be some function \(h = h(s)\), which in general will be a non-linear function. The effectiveness of the resulting estimator of the waveform will depend on how much prior information concerning the waveform shape can be encoded in the choice of prior PDF \(p^{(0)}\).

One of the simplest possibilities is to take \(p^{(0)}\) to be concentrated on some linear subspace \(U\) of the space \(V\),
and to be approximately constant inside this subspace. A multivariate Gaussian with widths very small in some directions and very broad in others would accomplish this to a good approximation. For such choices of prior PDF $p(\theta)$, the resulting maximum likelihood estimator [the function $h = h(s)$ that maximizes the PDF (5.3)] is simply the perpendicular projection $P_U$ of $s$ into $U$:

$$h_{\text{best-fit}}(s) = P_U(s), \quad (5.4)$$

where

$$P_U(s) = \sum_{i=1}^{n_U} u_{ij}(u_j | s) u_i. \quad (5.5)$$

Here, $u_1, \ldots, u_{n_U}$ is an arbitrary basis of $U$, $n_U$ is the dimension of $U$, $u_{ij} = \delta_{ij}$ and $u_{jk} = (u_j | u_k)$.

We remark that the method of filtering (5.3) is a special case of Wiener optimal filtering: it is equivalent to optimal filtering with templates that are constructed by taking linear combinations of the basis functions $u_i$. [The equivalence between maximum likelihood estimation and Wiener optimal filtering in more general contexts has been shown by Echeverria [51].] To show that our filtering method is a form of Wiener optimal filtering, define a family of template waveforms that depends on parameters $a_1, \ldots, a_{n_U}$ by

$$h(t; a_j) = \sum_{j=1}^{n_U} a_j u_j(t), \quad (5.6)$$

where $u_j(t)$ are the functions of time corresponding to the basis elements $u_j$ of $U$. If $s(t)$ is the measured detector output, define for any function $h(t)$

$$\frac{S}{N}[h(t)] = \frac{\langle h | s \rangle}{\sqrt{\langle h | h \rangle}}. \quad (5.7)$$

This is the SNR for the template $h(t)$ with the data stream $s$. The best-fit signal given by the optimal filtering method is the template which maximizes the SNR (5.7), i.e., the template $h(t; \hat{a}_j)$ such that

$$\frac{S}{N}[h(t; \hat{a}_j)] = \max_{a_1, \ldots, a_{n_U}} \frac{S}{N}[h(t; a_j)]. \quad (5.8)$$

However, it is easy to show from Eqs. (5.3)–(5.7) that

$$P_U(s) = h(t; \hat{a}_j). \quad (5.9)$$

Thus, computing the perpendicular projection (5.5) of $s$ into $U$ is equivalent to Wiener optimal filtering with the family of templates (5.6). From an operational point of view, the method of filtering (5.3) is quite different to the normal implementation of optimal filtering, which is carried out by calculating the SNR (5.7) for various parameter values, but the final best-fit signals (5.9) are identical. [Of course, Wiener optimal filtering is normally only carried out when the dependence of the waveform $h(t; a_j)$ on the parameters $a_j$ is complicated and non-linear, as when searching for inspiral waves where the parameters represent astrophysical characteristics of the binary system.]

To summarize, the maximum likelihood estimator (5.4) gives a general procedure for specifying a filtering algorithm adapted to a given linear subspace $U$ of the space of signals $V$. We will suggest below a specific choice for the subspace $U$; but first, we discuss some general issues related to making such a choice.

At the very least, we would like our choice of $U$ to effect truncation of the measured data stream in both the time domain and the frequency domain, down to the intervals of time and frequency in which we expect the merger waveform to lie. (We assume that the duration of the data being analyzed, $T'$, is somewhat longer than one’s guess of the merger duration, $T$.) Because of the uncertainty principle, such a truncation cannot be done exactly. Moreover, for fixed specific intervals of time and of frequency, there are different, inequivalent ways of approximating the signal to these intervals [52]. The differences between the inequivalent methods are essentially due to aliasing effects. Such effects cannot always be neglected in the analysis of merger waveforms, because the duration $T \sim 10M - 100M$ of the waveform is probably only a few times larger than the reciprocal of the highest frequency of interest.

It turns out that the simplest method of truncating in frequency (band-pass filtering) is, to a good approximation, a projection of the type (5.4) that we are considering. Truncating in the time domain, on the other hand, is not a projection of this type.

Let us first discuss band-pass filtering. Let $d_k$ [cf. Eq. (2.5)] be a frequency domain basis of $V$. For a given frequency interval $[f_{\text{char}} - \Delta f/2, f_{\text{char}} + \Delta f/2]$, let $U$ be the subspace of $V$ spanned by the elements $d_j$ with $|f_j - f_{\text{char}}| < \Delta f/2$, i.e., the span of the subset of the frequency domain basis that corresponds to the given frequency interval. Then the projection operation $P_U$ is to a moderate approximation just the band-pass filter:

$$P_U \left[ \sum_{j=1}^{N_{\text{lines}}} s^j d_j \right] \approx \sum_{j=1}^{N_{\text{lines}}} s^j d_j, \quad (5.10)$$

where the notation $\sum'$ means that the sum is taken only over the appropriate range of frequencies. The reason for the relation (5.10) is that the basis $d_j$ is approximately orthogonal with respect to the noise inner product (2.14): different frequency components of the noise are statistically independent up to small aliasing corrections of the order of $\sim 1/(f_{\text{char}} T')$. Thus, if our a priori information is that the signal lies within a certain frequency interval, then the maximum likelihood estimate of the signal is approximately given by passing the data stream through a band-pass filter.

An analogous statement is not true in the time domain.
If our \textit{a priori} information is that the signal vanishes outside a certain interval of time, then truncating the data stream by throwing away the data outside of this interval will not give the maximum likelihood estimate of the signal. This is because of statistical correlations between sample points just inside and just outside of the time interval; the measured data stream outside the interval gives information about what the noise inside the interval is likely to be. These correlation effects become unimportant in the limit \( T_{\text{fchar}} \to \infty \), but for BBH merger signals \( T_{\text{fchar}} \) is probably \( \lesssim 20 \) \cite{1}. The correct maximum likelihood estimator of the waveform, when our prior information is that the signal vanishes outside of a certain time interval, is given by Eq. (5.5) with the basis \( \{ u_1, \ldots, u_{n_U} \} \) replaced by the appropriate subset of the time-domain basis \( \{ e_1, \ldots, e_{N_{\text{bins}}} \} \) discussed in Sec. \[\text{II}\].

Our suggested choice of subspace \( U \) and corresponding specification of a filtering method is as follows. Pick a wavelet basis \( w_{ij} \) of the type discussed above. (The filtering method will depend only weakly on which wavelet basis is chosen). Then, the subspace \( U \) is taken to be the span of a suitable subset of this wavelet basis, according to our prior prejudice regarding the bandwidth and duration of the signal. The dimension \( n_U \) of \( U \) will be given by

\[ n_U = N_{\text{bins}} = 2T \Delta f. \tag{5.11} \]

In more detail, the filtering method would work as follows. First, band-pass filter the data stream and truncate it in time, down to intervals of frequency and time that are several times larger than are ultimately required, in order to reduce the number of independent data points \( N_{\text{bins}} \) to a manageable number. Second, for the wavelet basis \( w_{ij} \) of this reduced data set, calculate the matrix \( w_{ij} w_{ij}' = (w_{ij} \sum w_{ij}') \). Recall that the index \( i \) corresponds to a time \( t_i \) and the index \( j \) to a frequency \( f_j \) [cf. the discussion preceding Eq. (5.3)]. Third, pick out the sub-block \( \bar{w}_{ij} \bar{w}_{ij}' \) of the matrix \( w_{ij} w_{ij}' \) for which the times \( t_i \) and \( t_i' \) lie in the required time interval, and for which the frequencies \( f_j \) and \( f_j' \) lie in the required frequency interval. Numerically invert this matrix to obtain \( \bar{w}_{ij} \bar{w}_{ij}' \). Finally, the best-fit waveform is given by

\[ h_{\text{best-fit}} = \sum_{ij} \sum_{ij'} \bar{w}_{ij} \bar{w}_{ij'} (s \mid w_{ij'}) w_{ij}, \tag{5.12} \]

where \( \sum' \) means the sum over the required time and frequency intervals.

Note also that the best fit signal \( (5.12) \) would also be obtained by calculating the SNR \( (5.7) \) for the family of waveforms

\[ h(t) = \sum_{ij} c_{ij} w_{ij}(t) \tag{5.14} \]

and by maximizing over the \( c_{ij} \)'s, as discussed above. This essentially corresponds to building a family of templates with the wavelet basis, and then performing matched filtering with that bank of templates.

C. Extension of method to incorporate other types of prior information

A more sophisticated filtering method can be obtained by a generalization of the above analysis. Let us suppose that the prior PDF \( p^{(0)}(h) \) is a general multivariate Gaussian in \( h \). For example, one could choose the prior PDF to be of the form

\[ p^{(0)}(h) \propto \exp \left[ -\frac{1}{2} \sum_{ij} \frac{(h_{ij} - \bar{h}_{ij})^2}{\alpha_{ij}^2} \right], \tag{5.15} \]

where \( h_{ij} \) are the expansion coefficients of the signal \( h \) on some fixed wavelet basis \( w_{ij} \), so that \( h = \sum_{ij} h_{ij} w_{ij} \). Then, by making suitable choices of the parameters \( \bar{h}_{ij} \) and \( \alpha_{ij} \), such a PDF could be chosen to encode the information that the frequency content of the signal at early times is concentrated near \( f_{\text{merge}} \), that the signal joins smoothly onto the inspiral waveform, that at the end of merger the dominant frequency component is that of quasi-normal ringing, \textit{etc}. For any such prior PDF, it is straightforward to calculate the corresponding maximum likelihood estimator. If the prior PDF has expected value \( h_0 \) and variance-covariance matrix \( \Sigma_0 \), then the estimator is

\[ h_{\text{best-fit}}(s) = \left[ \Sigma^{-1} + \Sigma_0^{-1} \right]^{-1} \cdot \left[ \Sigma^{-1} \cdot s + \Sigma_0^{-1} \cdot h_0 \right]. \tag{5.16} \]

Such a waveform estimator could be calculated numerically.

D. Fidelity of waveform recovery

In this subsection we address the question of how close, statistically, we expect our estimated waveform \( h_{\text{best-fit}}(t) \) to be to the original gravitational waveform \( h(t) \). We can quantify the closeness by means of the correlation coefficient

\[ C \equiv \frac{(h \mid h_{\text{best-fit}})}{\sqrt{(h \mid h)} \sqrt{(h_{\text{best-fit}} \mid h_{\text{best-fit}})}}, \tag{5.17} \]
which takes values between −1 and 1. In appendix C, we show that for estimators of the form (5.4), the expected value of $C$ is approximately given by

$$\langle C \rangle \approx \frac{\rho_{\text{bin}}}{\sqrt{1 + \rho_{\text{bin}}^2}},$$  

(5.18)

where $\rho_{\text{bin}}^2$ is the matched filtering SNR squared per frequency bin, given by

$$\rho_{\text{bin}}^2 = \frac{\rho^2}{N_{\text{bins}}}.$$  

(5.19)

Thus, as one would expect, the best-guess reconstructed waveform agrees closely with the original gravitational waveform ($C$ is close to 1) when there is large SNR in each frequency bin, and vice-versa [53]. This result will also be approximately valid for waveforms obtained by simple band-pass filtering when the duration $T$ of the signal satisfies $T \gg 1/\Delta f$ (where $\Delta f$ is the frequency bandwidth of the signal).

Note that the quantity $\rho_{\text{bin}}$ is to a good approximation just the SNR which one obtains from band-pass filtering, from Eq. (4.13) above. Our criterion for the signal to be visible can therefore be written as $\rho_{\text{bin}} > 1$. So our criteria for signal visibility and for reconstructed signal fidelity turn out to be essentially identical: the fidelity of signal reconstruction is good when the merger signal is easily visible above the noise, as is fairly obvious intuitively.

VI. USING INFORMATION PROVIDED BY REPRESENTATIVE SUPERCOMPUTER SIMULATIONS

In this section we propose a computational strategy for numerical relativists to pursue, if they successfully produce computer codes capable of simulating BBH mergers, but if running such codes is too expensive to permit an extensive survey of the merger parameter space. In this case, for LIGO/VIRGO data analysis purposes, it would be advantageous to do a very coarse survey of the parameter space aimed at determining the ranges of several key parameters and at answering several qualitative questions, as we now describe.

- Do the waveforms contain a strong signature of an “innermost stable circular orbit” (ISCO) [3,4]? In the extreme mass ratio limit $\mu \ll M$, there is such an orbit, and when the smaller inspiralling black hole reaches it there is a transition from a radiation-reaction-driven inspiral to a freely falling plunge [23]. Correspondingly, there is a sharp drop in the radiated energy per unit logarithmic frequency $dE/d(\ln f)$ at the frequency corresponding to this orbit. However, in the equal-mass case, there may not be a sharp feature in the $dE/d(\ln f)$ plot, if the timescale over which the orbital instability operates is comparable to the radiation reaction timescale. Or, if the spins of the individual black holes are large and parallel to the orbital angular momentum, the inspiral may smoothly join into the merger without any plunge. In the former case, the concept of ISCO would not really be meaningful; and in the latter case, there would simply be nothing resembling an ISCO in the evolution. Simulations should be able to settle this issue.

- A closely related question is: At what frequency does the adiabatic approximation break down? As seen in a coordinate system which co-rotates with the black holes, the system evolves on a radiation-reaction timescale which is initially much longer than the orbital period [3,4]. When does this separation of timescales break down? This separation of timescales underlies proposed methods of calculating templates in the so-called Intermediate Binary Black Hole (IBBH) regime after the post-Newtonian approximation fails at $r \sim 12M$ [3,4]. Therefore, fully numerical templates will have to be used after the adiabatic approximation fails. Resolving this issue will probably require exploration of both numerical relativity simulations and IBBH calculations. If the black holes’ spins are small, one might expect the transition point to coincide with estimates of the location of the last stable circular orbit [3,4] around $r \sim 6M$; our estimate (4.4) of the frequency of the transition from inspiral to merger roughly corresponds to this expectation. But with large spins, the system might evolve adiabatically all the way into the merger. (Note, however, that numerical relativity will still need to be used to simulate such evolution, whether it is adiabatic or not.)

- What is the approximate duration of the merger signal, and how does it depend on the merger parameters such as the initial spins of the black holes and the mass ratio? The range of merger signal durations will be an important input to algorithms for reconstructing the merger waveform from the noisy data stream (see Sec. I), particularly in those cases in which the ringdown and/or inspiral signals are too weak to be seen in the data stream. Moreover, the duration of the waveform (together with its bandwidth) approximately determines the amount by which the SNR from band-pass filtering is lower than the matched filtering SNR obtained with merger templates [cf. Eq. (4.13)].

- A closely related issue is: How much energy is radiated in the merger waves relative to the ringdown waves? Operationally, this question reduces to asking what proportion of the total waveform produced during the coalescence can be accurately fit by the ringdown’s decaying sinusoid. In paper I...
we argued that if the spins of the individual black holes are large and aligned with one another and with the orbital angular momentum, then the system has too much angular momentum for it to be lost solely through the ringdown, and that therefore the ringdown waves should not dominate the merger. On the other hand, if the spins of the black holes are small, most of the radiated energy might well come out in ringdown waves.

- What is the frequency bandwidth in which most of the merger waves’ power is concentrated? In Ref. [11] we assumed that when one excises in the time domain the ringdown portion of the signal, the remaining signal has no significant power at frequencies above the quasi-normal ringing frequency of the final Kerr black hole. However, this assumption may not be valid. As with the signal’s duration, the range of bandwidths of merger waveforms will be an input to algorithms for reconstructing the merger waveform from the noisy data (see Sec. V), so this is an important issue.

- To what extent does the merger waveform chirp monotonically? If we represent the merger waves on a time-frequency wavelet basis, then we know that at early times, the waves are concentrated at one frequency with additional contributions in nearby harmonics. At the end of the merger signal, most of the power is concentrated near the frequency of quasi-normal ringing of the final black hole. One could extrapolate in the time-frequency plane a line joining twice the orbital frequency at the end of inspiral to the quasinormal ringing frequency at the start of ringdown. To what extent is the merger signal concentrated near this line in the time-frequency plane?

- How much of the merger can be described as higher order QNR modes? By convention, we have been calling that phase of the coalescence which is dominated by the most slowly damped, \( l = m = 2 \) mode the ringdown phase; but, before this mode dominates, QNR modes with different values of \( l \) and/or \( m \) are likely to be present. After the merger has evolved to the point when the merged object can be accurately described as a linear perturbation about a stationary black hole background, there might or might not be any significant subsequent period of time before the higher order modes have decayed away so much as to be undetectable. If simulations predict that higher order QNR modes are strong for a significant period of time, then these higher order QNR modes should be found by the normal ringdown search of the data stream; no extra search should be needed.

- Does the merger signal have the property that we can distinguish a “carrier waveform” and a “modulation”? This separation would require that the carrier waveform have a fairly large number of cycles at a frequency well separated from that of the modulation. It would also require some mechanism to produce modulation, one possibility being the precession of the black hole spins. It is known that spin precession does modulate the inspiral waveform \([25]\), and it is possible that a similar precession might be present during at least part of the merger.

An improved understanding of these issues would be of use both in extracting [cf. Sec. V C above] and in interpreting the merger waveforms.

VII. INFORMATION OBTAINABLE FROM THE MERGER PHASE OF THE WAVES USING TEMPLATES

In the remainder of the paper we consider the optimistic scenario in which a complete set of supercomputer generated theoretical merger waveforms is available for data analysis. In this section we describe in qualitative terms the extra information that one can extract from the merger waves using templates. In Sec. VIII we estimate how accurate numerical templates need to be for data analysis purposes, and in Sec. IX we estimate the total number of bits of information obtainable from the merger waves using templates, and discuss implications for the requirements on one’s grid of templates.

If merger templates are available, it should be possible to perform Wiener optimal filtering of the data stream for the merger signal, just as will be done for the inspiral and ringdown signals. When one has no information about the BBH system, one would simply filter the data with all numerical merger templates available, potentially a very large number. However, if the inspiral and/or the ringdown signals have already been measured (as will be the case for most detected signals), some information about the black hole binary’s constituents will be available. In such cases the total number of merger templates needed will be reduced, perhaps substantially; one need consider only those numerical templates whose parameters are commensurate with the inspiral/ringdown measurements.

It may turn out that black hole mergers have such a wide variety of behaviors that it will not be feasible to produce a complete family of templates, even with a numerical code that can evolve mergers and produce waveforms. In such an eventuality, as mentioned in Sec. IX above, the interpretation of an observed merger waveform could proceed as follows: The numerical relativists, with noisy data and numerical code in hand, carry out a series of iterated numerical simulations, trying to produce a waveform that matches the observed data. (Clearly, it would be very useful for such a procedure to have as much
prior information as possible about the system’s parameters from the inspiral and/or ringdown phases, so that the numerical relativists will know where in the binary black hole parameter space to concentrate their computational efforts.) Thus, matched filtering might be possible even if the computation of a complete set of template waveforms is too difficult to perform.

In attempting to match a merger template with gravitational-wave data, one’s primary goal would be to provide a test of general relativity rather than the measurement of parameters. A good match between the measured waveform and a numerical template would constitute a strong test of general relativity and provide the oft-quoted unambiguous detection of black holes. (Such an unambiguous detection could also come from a measurement of the quasinormal ringing signal.) Although not the primary goal, matches between numerical merger templates and the data stream would also be useful in measuring some of the system’s parameters, such as the total mass $M$ or the spin parameter $a$ of the final black hole. These merger parameter measurements could provide additional information about the source, over and above that obtainable from the inspiral and ringdown signals. For instance, in the second example discussed in Sec. II (a 30 $M_\odot$ BBH at $z = 1$), the total redshifted mass $(1 + z)M$ would be essentially unconstrained by the inspiral and ringdown waveforms, but might be extractable from the measured merger waveform. In other cases, a quantitative test of general relativity could be obtained by verifying that parameters measured from the merger phase are consistent with parameter measurements from the inspiral and ringdown phases.

A close match between measured and predicted waveforms for BBH mergers might also constrain some possible theories of gravity that generalize general relativity. Clifford Will has shown that the inspiral portion of the waveform for neutron star-neutron star mergers will strongly constrain the dimensionless parameter $\omega$ of Brans-Dicke theory. Unfortunately, the most theoretically natural class of generalizations of general relativity compatible with known experiments, the so-called scalar-tensor theories, may not be strongly constrained (if at all) by measurements of BBH mergers, since black holes, unlike neutron stars, cannot have any scalar hair in such theories.

In order for the above endeavors to be successful, the numerical templates must be sufficiently accurate. In the next section, we turn to a discussion of how accurate numerical templates need to be in order to extract the information in merger signals.

VIII. ACCURACY REQUIREMENTS FOR MERGER WAVEFORM TEMPLATES

There will be unavoidable errors in the waveform templates produced by supercomputer simulations, since these simulations are numerical. Suppose that the physical waveform for some particular source is $h(t; \theta)$, where the components of the vector $\theta = (\theta_1, \ldots, \theta_\nu)$ represent the various parameters upon which the waveform depends. Then, a simulation of the evolution of that source will predict a slightly different waveform $h(t; \theta) + \delta h(t; \theta)$, where $\delta h(t; \theta)$ is the numerical error. One would like the numerical error to be small enough not to have a significant effect on signal searches, parameter extraction or any other types of data analysis that might be carried out using the template waveforms. In this section we suggest an approximate rule of thumb (Eqs. (8.1) and (8.2)) for estimating when numerical errors are sufficiently small, and discuss its meaning and derivation.

### A. Accuracy criterion and implementation

The accuracy criterion can be simply expressed in terms of the inner product introduced in Sec. II above (which is defined by Eq. (2.3) or alternatively by Eqs. (2.11)—(2.14)). For a given template $h(t)$, our rule of thumb is that the numerical error $\delta h(t)$ should be small enough that the quantity

$$\Delta \equiv \frac{1}{2} \left( \frac{\delta h(t)}{h(t)} \right)$$

satisfies

$$\Delta \lesssim 0.01.$$  

(The fractional loss in event detection rate in signal searches is $\sim 3\Delta$, so the value of 0.01 in Eq. (8.2) is chosen to correspond to a 3% loss in event rate; see Sec. VIII B below). For the purpose of evaluating the inner product numerically, note that the absolute normalization of the noise spectrum $S_h(f)$ is unimportant, and that one could use, for example, Eqs. (4.1)—(4.3) of Ref. [11] to specify the shape of the noise spectrum.

In practice, Eq. (8.2) translates to a fractional accuracy per data point $h_j = h(t_j)$ of about $0.01/\sqrt{N_{\text{points}}}$, where $N_{\text{points}}$ is the number of numerical data points used to describe the templates, if the errors at each data point are effectively uncorrelated. If, however, these errors add coherently in the integral (8.1), the requirement on fractional accuracy at each data point will be more stringent.

It should be straightforward in principle to ensure that numerical templates satisfy the criterion (8.2). Let us schematically denote a numerically generated template as $h_{\text{num}}(t, \varepsilon)$, where $\varepsilon$ represents the set of tolerances (grid size, size of time steps, etc.) that govern the accuracy of the numerical calculation. (Representing this set of parameters by a single parameter $\varepsilon$ is an oversimplification but is adequate for the purposes of our discussion.) One can then iterate one’s calculations varying the parameter $\varepsilon$ in order to obtain templates that are...
sufficiently accurate, using the following standard type of procedure: First, calculate the template \( h_{\text{num}}(t, \varepsilon) \). Second, calculate the more accurate template \( h_{\text{num}}(t, \varepsilon') \) for some choice of \( \varepsilon' < \varepsilon \), for example \( \varepsilon' = \varepsilon/2 \). Third, make the identifications

\[
h(t) \equiv h_{\text{num}}(t, \varepsilon'),
\delta h(t) \equiv h_{\text{num}}(t, \varepsilon') - h_{\text{num}}(t, \varepsilon)
\]  

(8.3)

and insert these quantities into Eq. (8.1) to calculate \( \Delta \). This allows one to assess the accuracy of the template \( h_{\text{num}}(t, \varepsilon) \). Finally, iterate this procedure until Eq. (8.2) is satisfied.

### B. Derivation and meaning of accuracy criterion

The required accuracy for the numerical templates depends on how and for what purpose those templates are used. As discussed in the Introduction, merger templates might be used in several different ways:

- They might be used as search templates for signal searches using matched filtering. Such searches will probably not be feasible, at least initially, as they would require the computation of an inordinately large number of templates.

- For BBH events that have already been detected via matched filtering of the inspiral or ringdown waves, or by the noise-monitoring detection technique \([11,20]\) applied to the merger waves, the merger templates might be used for matched filtering in order to measure the binary’s parameters and test general relativity. This use of merger templates could correspond to the third scenario that was discussed in Sec. [C] where iterated runs of the supercomputer codes are performed to produce a template that best fits a dataset known to contain BBH merger gravitational waves. This scenario would not require that a complete set of templates be computed and stored, and thus is somewhat more feasible than matched filtering signal searches using the merger waves.

- If one has only a few, representative supercomputer simulations and their associated waveform templates at one’s disposal, one might simply perform a qualitative comparison between the measured waveform and templates in order to deduce qualitative information about the BBH source. This is the second scenario described in Sec. [C].

In this section we estimate the accuracy requirements for the first two of these uses of merger templates.

Consider first signal searches using matched filtering. The expected SNR \( \rho \) obtained for a gravitational waveform \( h(t) \) when using a template waveform \( h_T(t) \) is given by [33]

\[
\rho = \frac{(h|h_T)}{\sqrt{(h_T|h_T)}}.
\]  

(8.4)

If we substitute \( h_T(t) = h(t) + \delta h(t) \) into Eq. (8.4) and expand to second order in \( \delta h \), we find that the fractional loss \( \delta \rho/\rho \) in SNR produced by the numerical error \( \delta h(t) \) is given by

\[
\frac{\delta \rho}{\rho} = \Delta_1 + O((\delta h)^2),
\]  

(8.5)

where

\[
\Delta_1 = \frac{1}{2} \left( \frac{(\delta h|\delta h)}{(h|h)} - \frac{(\delta h|h)^2}{(h|h)^2} \right).
\]  

(8.6)

Note that the quantity \( \Delta_1 \) is proportional to \( (\delta h_1|\delta h_1) \), where \( \delta h_1 \) is the component of \( \delta h \) perpendicular to \( h \) with respect to the inner product \([2,14]\). Thus, a numerical error of the form \( \delta h(t) \propto h(t) \) will not contribute to the fractional loss \( \Delta_1 \) in SNR. This is to be expected, since the quantity \( (\delta h|\delta h) \) is independent of the absolute normalization of the templates \( h_T(t) \).

Now, the event detection rate is proportional to the cube of the SNR, and hence the fractional loss in event detection rate that results from using inaccurate numerical templates is approximately \( 3\delta \rho/\rho \) [33]. If one demands that the fractional loss in event rate be less than, say, 3%, then one obtains the criterion [62]

\[
\Delta_1 \leq 0.01.
\]  

(8.7)

It is clear from Eqs. (8.1) and (8.3) that \( \Delta_1 \leq \Delta \). Hence, the condition (8.7) is less stringent than the condition (8.2) above. The justification for imposing the more stringent criterion (8.2) rather than (8.7) derives from the use of templates for parameter extraction.

Consider next using merger templates for the purpose of measuring parameters via matched filtering. In principle, one could hope to measure all of the 15 parameters on which the merger waveforms depend by combining the outputs of several detectors with a complete bank of templates (although in practice the accuracy with which some of those 15 parameters can be measured is not likely to be very good). In the next few paragraphs we derive an approximate condition on \( \Delta \) [Eq. (8.14)] which results from demanding that the systematic errors in the measured values of all the parameters be small compared to the statistical errors due to detector noise. (We note that one would also like to use matched filtering to test general relativity with these waves; the accuracy criterion that we derive for parameter measurement will also approximately apply to tests of general relativity.)

Let the gravitational waveform be \( h(t; \theta) \), where \( \theta = (\theta^1, \ldots, \theta^{n_{\theta}}) \). Let \( \hat{\theta}^\alpha \), \( 1 \leq \alpha \leq n_{\theta} \), be the best-fit values of \( \theta^\alpha \) given by the matched-filtering process. The quantities \( \hat{\theta}^\alpha \) depend on the detector noise and are thus random variables. In the high SNR limit, the variables
\( \hat{\theta} \) have a multivariate Gaussian distribution with \( \langle \delta \hat{\theta}^\alpha \delta \hat{\theta}^\beta \rangle = \Sigma^{\alpha\beta} \),

\[ \text{(8.8)} \]

where \( \delta \hat{\theta}^\alpha = \hat{\theta}^\alpha - \langle \hat{\theta}^\alpha \rangle \) and the matrix \( \Sigma^{\alpha\beta} \) is defined after Eq. (2.18). The systematic error \( \Delta \theta^\alpha \) in the inferred values of the parameters \( \theta^\alpha \) due to the template error \( \delta h \) can be shown to be approximately

\[ \Delta \theta^\alpha = \Sigma^{\alpha\beta} \left( \delta h \right) \frac{\partial h}{\partial \theta^\beta}. \]

\[ \text{(8.9)} \]

From Eqs. (8.8) and (8.9) we find that in order to guarantee that the systematic error in each of the parameters be smaller than some number \( \varepsilon \) times that parameter's statistical error, we must have

\[ ||\delta h||^2 = \langle \delta h || \delta h \rangle \leq \varepsilon^2. \]

\[ \text{(8.10)} \]

Here \( \delta h \) is the component of \( \delta h \parallel \) parallel to the tangent space of the manifold of signals \( S h(t, \theta) \) discussed in Sec. II. It is given by

\[ \delta h \parallel = \Sigma^{\alpha\beta} \left( \delta h \right) \frac{\partial h}{\partial \theta^\beta}. \]

\[ \text{(8.11)} \]

The magnitude \( ||\delta h \parallel || \) of this component of \( \delta h \) depends on details of the number of parameters, and on how the waveform \( h(t, \theta) \) varies with these parameters. However, a strict upper bound is given by

\[ ||\delta h \parallel || \leq ||\delta h||. \]

\[ \text{(8.12)} \]

If we combine Eqs. (8.1), (8.10), and (8.12) we obtain the condition

\[ \Delta \leq \frac{\varepsilon^2}{2\rho^2}. \]

\[ \text{(8.13)} \]

If we insert reasonable estimates for \( \rho \) and \( \varepsilon \)—namely, \( \rho \simeq 7, \varepsilon \simeq 1 \)—we recover the criterion (8.2). [Note that the requirement (8.13) is probably rather more stringent than need be: the left hand side of Eq. (8.12) is likely smaller than the right hand side by a factor \( \sim \sqrt{n_p/N_{\text{bins}}} \), where \( n_p \) is the number of parameters and \( N_{\text{bins}} \) is the dimension of the total space of signals \( V \).]

In Sec. IX below we give an alternative derivation of the accuracy criterion (8.13) using information theory.

The expected order of magnitude \( \rho \simeq 7 \) of the SNR that leads to the criterion (8.2) is appropriate for ground based interferometers such as LIGO and VIRGO. However, for the space-based LISA interferometer, much higher SNRs are expected; see, e.g., Ref. [11]. Correspondingly, numerical templates used for testing relativity and measuring parameters with LISA data will have to be substantially more accurate than those used with data from ground based instruments.

IX. NUMBER OF BITS OF INFORMATION OBTAINABLE FROM THE MERGER SIGNAL AND IMPLICATIONS FOR TEMPLATE CONSTRUCTION

In this section, we describe how to use information theory to quantify how much can be learned from a gravitational-wave measurement. In information theory, a quantity called “information” (analogous to entropy) can be associated with any measurement process: it is simply the base 2 logarithm of the number of distinguishable outcomes of the measurement [24,25]. Equivalently, it is the number of bits required to store the knowledge gained from the measurement. Here we specialize the notions of information theory to gravitational wave measurements, and estimate the number of bits of information which one can gain in different cases.

Let us first consider the situation in which templates are unavailable. Suppose that our prior information describing the signal is that it lies inside some frequency band of width \( \Delta f \) say, that it lies inside some time interval of length \( T \) say. We will denote by \( I_{\text{total}} \) the base 2 logarithm of the number of waveforms \( h \) that are distinguishable by the measurement, that are compatible with our prior information, and that are compatible with our measurement of the detector output’s magnitude \( \rho(s) = ||s|| \). We give a precise version of this definition in Sec. IX.A below (Eq. (9.2)). Note that the vast majority of these \( 2^{I_{\text{total}}} \) waveforms are completely irrelevant; the BBH merger signals are a small subset (the manifold \( S \)) of all distinguishable waveforms with the above characteristics. However, without prior information about which waveforms are relevant, we cannot \textit{a priori} ignore any waveform, and so we must include in our counting even the irrelevant ones. Note also that the quantity \( I_{\text{total}} \) quantifies the amount of information we gain from the measurement about the shape of the merger waveform; however, in the absence of any templates we do not learn anything about the source of waves.

In Appendix B we derive and in Sec. IX.B we discuss an approximate formula for \( I_{\text{total}} \) in terms of the matched filtering signal-to-noise ratio \( \rho \) and the number of frequency bins \( N_{\text{bins}} \) [Eq. (9.8)]. This approximate formula can be understood with a simple, intuitive argument, which we also elucidate in Sec. IX.A.

Consider now the situation in which templates are available. In Sec. IX.B below (Eq. (9.11)) we define a quantity \( I_{\text{source}} \), which is, roughly speaking, the base 2 logarithm of the number of distinguishable waveforms that could have come from BBH mergers and that are distinguishable in the detector noise. The quantity \( I_{\text{source}} \) differs from the quantity \( I_{\text{total}} \) in that it counts only the subset of waveforms relevant to BBH mergers. Note that the information which \( I_{\text{source}} \) quantifies is information about the \textit{source} of the waves: when templates are available we can relate the waveform shape to properties of the BBH system. In Appendix B we derive an approxi-
mate formula [Eq. (9.12)] for $I_{\text{source}}$.

Finally, in Sec. [IX C] we estimate how much of the information $I_{\text{source}}$ is lost due to template numerical error [Eq. (12.0)] and due to having insufficiently many templates in one’s grid [Eq. (12.3)], and deduce requirements one’s grid of templates must satisfy in order for the loss of information to be unimportant.

### A. Total information gain

A precise definition of the total information gain $I_{\text{total}}$ is the following: Let $T$ and $\Delta f$ be a priori upper bounds for the durations and bandwidths of merger signals, and let $V$ be the vector space of signals with duration $\leq T$ inside the relevant frequency band. This vector space $V$ has dimension $\mathcal{N}_{\text{bins}} = 2T\Delta f$. Let $s$, $h$, and $n$ denote the detector output, gravitational wave signal and detector noise respectively, so that $s = h + n$. The quantities $s$, $h$, and $n$ are all elements of $V$. Let $p^{(0)}(h)$ be the PDF describing our prior information about the gravitational wave signal [3], and let $p(h | s)$ denote the posterior PDF for $h$ after the measurement, i.e., the PDF for $h$ given that the detector output is $s$. A standard Bayesian analysis shows that $p(h | s)$ will be given by

$$p(h | s) = \mathcal{K}p^{(0)}(h) \exp \left[ -|s - h| |s - h| / 2 \right] \quad (9.1)$$

where $\mathcal{K}$ is a normalization constant [31]. Finally, let $p(h | \rho(s))$ be the PDF of $h$ given that the magnitude $|s|$ of the measured signal is $\rho(s)$. We define the quantity $I_{\text{total}}$ to be

$$I_{\text{total}} \equiv \int dh \frac{p(h | s)}{p(h | \rho(s))} \log_2 \left[ \frac{p(h | s)}{p(h | \rho(s))} \right]. \quad (9.2)$$

By this definition, $I_{\text{total}}$ is the relative information of the probability distributions $p(h | \rho(s))$ and $p(h | s)$ [23]. In Appendix 3 we show that the quantity (9.2) in fact represents the base 2 logarithm of the number of distinguishable wave shapes that could have been measured and that are compatible with one’s measurement of the magnitude $\rho(s)$ of the data stream [3]. Thus, one learns $I_{\text{total}}$ bits of information about the waveform $h$ when one goes from knowing only the magnitude $|s|$ of the detector output to knowing the actual detector output $s$.

We also show in Appendix 3 that in the limit of no prior information other than $T$ and $\Delta f$, an approximate formula for the quantity (9.2) is

$$I_{\text{total}} = \frac{1}{2} \mathcal{N}_{\text{bins}} \log_2 \left[ \frac{\rho(s)^2}{\mathcal{N}_{\text{bins}}} \right] + O(\ln \mathcal{N}_{\text{bins}}). \quad (9.3)$$

The formula (9.3) is valid in the limit of large $\mathcal{N}_{\text{bins}}$ for fixed $\rho(s)^2/\mathcal{N}_{\text{bins}}$, and moreover applies only when

$$\rho(s)^2 / \mathcal{N}_{\text{bins}} > 1; \quad (9.4)$$

see below for further discussion of this point.

There is a simple and intuitive way to understand the result (9.3). Let us fix the gravitational waveform, $h$, considered as a point in the $\mathcal{N}_{\text{bins}}$-dimensional Euclidean space $V$. What is measured is the detector output $h + n$, whose location in $V$ is displaced from that of $h$. The direction and magnitude of the displacement depend upon the particular instance of the noise $n$. However, if we average over an ensemble of realizations of the noise, we can see that the displacement due to the noise is in a random direction and has rms magnitude $\sqrt{\mathcal{N}_{\text{bins}}}$ (since on an appropriate basis each component of $n$ has rms value 1). Therefore, all points $h'$ lying inside a hypersphere of radius $\sqrt{\mathcal{N}_{\text{bins}}}$ centered on $h$ are effectively indistinguishable from each other. The volume of such a hypersphere is

$$C_{\mathcal{N}_{\text{bins}}} \left( \sqrt{\mathcal{N}_{\text{bins}}} \right)^{\mathcal{N}_{\text{bins}}}, \quad (9.5)$$

where $C_{\mathcal{N}_{\text{bins}}}$ is a constant whose value is unimportant. When we measure a detector output $s$ with magnitude $\rho(s)$, the set of signals $h$ that could have given rise to an identical measured $\rho(s)$ will form a hypersphere of radius $\sim \rho(s)$ and volume

$$C_{\mathcal{N}_{\text{bins}}} \rho(s)^{\mathcal{N}_{\text{bins}}}. \quad (9.6)$$

The number of distinguishable signals in this large hypersphere will be approximately the ratio of the two volumes (13.1) and (16.1); the base 2 logarithm of this ratio is the quantity (9.3).

Equation (9.3) expresses the information gain as a function of the magnitude of the measured detector output $s$. We now re-express this information gain in terms of properties of the gravitational wave signal $h$. For a given $h$, Eqs. (2.16) and (2.17) show that the detector output’s magnitude $\rho(s)$ will be approximately

$$\rho(s)^2 \approx \rho^2 + \mathcal{N}_{\text{bins}} \pm \sqrt{\mathcal{N}_{\text{bins}}}. \quad (9.7)$$

Here $\rho^2 = ||h||^2$ is the SNR squared (2.5) that would be achieved if matched filtering were possible (if templates were available). We use $\rho$ simply as a convenient measure of signal strength; in this context, it is meaningful even in situations where templates are unavailable and where matched filtering cannot be carried out. The last term in Eq. (9.7) gives the approximate size of the statistical fluctuations in $\rho(s)^2$. We now substitute Eq. (9.7) into Eq. (9.3) and obtain

$$I_{\text{total}} = \frac{1}{2} \mathcal{N}_{\text{bins}} \log_2 \left[ 1 + \frac{\rho^2}{\mathcal{N}_{\text{bins}}} \right] \times \left[ 1 + O \left( \frac{\ln \mathcal{N}_{\text{bins}}}{\mathcal{N}_{\text{bins}}} \right) + O \left( \frac{1}{\sqrt{\mathcal{N}_{\text{bins}}}} \right) \right]. \quad (9.8)$$

Also, the condition (9.4) for the applicability of Eq. (9.3), when expressed in terms of $\rho$ instead of $\rho(s)$, becomes

$$\frac{\rho^2}{\mathcal{N}_{\text{bins}}} \pm \frac{1}{\sqrt{\mathcal{N}_{\text{bins}}}} \geq 0, \quad (9.9)$$
which will be satisfied with high probability when \( \rho \gg N_{\text{bins}}^{1/4} \). In the regime \( \rho \lesssim N_{\text{bins}}^{1/4} \), the condition (9.4) is typically not satisfied and the formula (9.3) does not apply; we show in Appendix B that in this case the information gain (9.2) is usually very small, depending somewhat on the prior PDF \( p^{(0)}(h) \). [In contexts other than BBH merger waveforms, the information gain can be large in the regime \( \rho \ll N_{\text{bins}}^{1/4} \) if the prior PDF \( p^{(0)}(h) \) is very sharply peaked. For example, when one considers measurements of binary neutron star inspirals with advanced LIGO interferometers, the information gain in the measurement is large even though typically one will have \( \rho \ll N_{\text{bins}}^{1/4} \), because we have very good prior information about inspiral waveforms.]

As an example, a typical detected BBH event might have an SNR for the merger signal of \( \rho \sim 10 \), and the number of frequency bins \( N_{\text{bins}} \) might be \( \sim 30 \). Then, Eq. (9.8) tells us that \( \sim 3 \times 10^9 \approx 2^{32} \) signals of the same magnitude could have been distinguished, thus the number of bits of information gained is \( \sim 32 \). More generally, for ground based interferometers we expect \( \rho \) to lie in the range \( 5 \lesssim \rho \lesssim 100 \), and therefore we expect \( 10 \lesssim I_{\text{total}} \lesssim 120 \); and for LISA we expect \( \rho \) to typically lie in the range \( 10^3 \lesssim \rho \lesssim 10^5 \) so that \( 200 \lesssim I_{\text{total}} \lesssim 400 \).

### B. Amount of information gained about the wave’s source

Consider now the idealized situation in which a complete family of accurate theoretical template waveforms \( h(\theta) \) are available for the merger. Without templates, we gain \( I_{\text{total}} \) bits of information about the shape of the gravitational waveform in a measurement. With templates, some—but not all—of this information can be translated into information about the BBH source. For instance, suppose in the example considered above that the number of distinguishable waveforms that could have come from BBH mergers and that are distinguishable in the detector noise is \( 2^{25} \). (This number must be less that the total number \( 2^{32} \) of distinguishable waveform shapes, since waveforms from BBH mergers will clearly not fill out the entire function space \( V \) of possible gravitational waveforms.) In this example, by identifying which template best fits the detector output, we can gain \( \sim 25 \) bits of information about the BBH source (e.g. about the black holes’ masses or spins). We will call this number of bits of information \( I_{\text{source}} \); clearly \( I_{\text{source}} \leq I_{\text{total}} \) always.

What of the remaining \( I_{\text{total}} - I_{\text{source}} \) bits of information (7 bits in the above example)? If the detector output is close to one of the template shapes, then this closeness can be regarded as evidence in favor of the theory of gravity (general relativity) used to compute the templates, so the \( I_{\text{total}} - I_{\text{source}} \) extra bits of information can be viewed as information about the validity of general relativity. If one computed templates in more general theories of gravity, one could in principle translate these \( I_{\text{total}} - I_{\text{source}} \) bits of information into a quantitative form and obtain constraints on the parameters entering into the gravitational theory. However, with only general-relativistic templates at one’s disposal, the information contained in the \( I_{\text{total}} - I_{\text{source}} \) bits will simply result in a qualitative confirmation of general relativity, in the sense that one of the general relativistic templates will provide a good fit to the data.

It is possible to give a precise definition of the number of bits of information gained about the BBH source, \( I_{\text{source}} \), in the following way. Let \( p(\theta | s) \) denote the probability distribution for the source parameters \( \theta \) given the measurement \( s \). This PDF is given by a formula analogous to Eq. (9.11)

\[
p(\theta | s) = K \cdot p^{(0)}(\theta) \exp \left[ -\frac{1}{2} (s - h(\theta)) \frac{1}{\rho} \right],
\]

where \( p^{(0)}(\theta) \) is the prior PDF for \( \theta \) and \( K \) is a normalization constant. Let \( p(\theta | \rho(s)) \) be the posterior PDF for \( \theta \) given that the magnitude \( ||s|| \) of the measured signal is \( \rho(s) \). Then we define

\[
I_{\text{source}} = \int d\theta \log_2 \left[ \frac{p(\theta | s)}{p(\theta | \rho(s))} \right].
\]

The number of bits of information \( I_{\text{source}} \) gained about the BBH source will clearly depend on the details of how the gravitational waveforms depend on the source parameters, on the prior expected ranges of these parameters, etc. In Appendix B we argue that to a rather crude approximation, \( I_{\text{source}} \) should be given by the formula (9.8) with \( N_{\text{bins}} \) replaced by the number of parameters \( N_{\text{param}} \) on which the waveform has a significant dependence:

\[
I_{\text{source}} \approx \frac{1}{2} N_{\text{param}} \log_2 \left[ 1 + \rho^2 / N_{\text{param}} \right].
\]

Note that the quantity \( N_{\text{param}} \) should be bounded above by the quantity \( n_p \) discussed in Sec. I, but may be somewhat smaller than \( n_p \). This will be the case if the waveform depends only very weakly on some of the parameters \( \theta_p \). Equation (9.12) is only valid when \( N_{\text{param}} \leq N_{\text{bins}} \). For BBH mergers we expect \( N_{\text{param}} \lesssim 15 \), which from Eq. (9.12) predicts that \( I_{\text{source}} \) lies in the range \( \sim 10 \) bits to \( \sim 70 \) bits for signal-to-noise ratios \( \rho \) in the range 5 to 100 (the expected range for ground based interferometers \([11]\)), and \( \sim 100 \) bits to \( \sim 200 \) bits for \( \rho \) in the range \( 10^3 \) to \( 10^5 \) expected for LISA \([11]\).

### C. Loss of information about source due to template inaccuracies or to sparseness of the lattice of templates

As we discussed in Sec. VIII, numerical templates will contain some unavoidable error due to the calculation
Technique. In this section we analyze how that error affects the information gained in the measurement process, and use this analysis to infer the maximum allowable template error.

Let us write

$$\mathbf{h}_T(\theta) = \mathbf{h}(\theta) + \delta\mathbf{h}(\theta),$$  
(9.13)

where $\mathbf{h}(\theta)$ denotes the true waveform shape, $\mathbf{h}_T(\theta)$ the numerical template, and $\delta\mathbf{h}(\theta)$ the numerical error. It is clear that the numerical error will reduce the amount of information (9.11) one can obtain about the source. We can make a crude estimate of the amount of reduction in the following way. We model the numerical error as a random process with

$$\langle \delta h_i \delta h_j \rangle = C_{ij},$$  
(9.14)

where for simplicity we take $C_{ij} = \lambda \Gamma_{ij}$ for some constant $\lambda$. Here $\Gamma_{ij}$ is the matrix introduced in Eq. (2.11). The expected value of $\langle \delta \mathbf{h} \mid \delta \mathbf{h} \rangle$ is then given by, from Eq. (2.14),

$$\langle \delta \mathbf{h} \mid \delta \mathbf{h} \rangle = \Sigma^{ij} \langle \delta h_i \delta h_j \rangle = \Sigma^{ij} \lambda \Gamma_{ij} = \lambda N_{\text{bins}},$$  
(9.15)

where we have used Eq. (2.10). We can write $\lambda$ in terms of the quantity $\Delta$ discussed in Sec. VII by combining Eqs. (8.1) and (9.13), yielding

$$\lambda = 2\Delta \frac{\rho^2}{N_{\text{bins}}}.$$  
(9.16)

The information $I'_{\text{source}}$, which one obtains when measuring with inaccurate templates can be calculated by treating the sum of the detector noise $\mathbf{n}$ and the template numerical error $\delta\mathbf{h}$ as an effective noise $\mathbf{n}^{(\text{eff})}$. This effective noise is characterized by the covariance matrix

$$(n'_i^{(\text{eff})} n'_j^{(\text{eff})}) = \Gamma_{ij} + \lambda \Gamma_{ij}.$$  
(9.17)

Thus, in this simplified model, the effect of the numerical error is to increase the noise by a factor $1 + \lambda$. The new information gain $I'_{\text{source}}$ is therefore given by Eq. (9.12) with $\rho$ replaced by an effective SNR $\rho'$, where

$$(\rho')^2 = \frac{\rho^2}{1 + \lambda}.$$  
(9.18)

If we now combine Eqs. (9.12), (9.16), and (9.18), we find that the loss in information due to template inaccuracy

$$\delta I_{\text{source}} = I_{\text{source}} - I'_{\text{source}},$$  
(9.19)

is given by

$$\delta I_{\text{source}} \sim \rho^2 \left( \frac{\rho^2}{N_{\text{param}} + \rho^2} \right) \left( \frac{N_{\text{param}}}{N_{\text{bins}}} \right) \Delta + O(\Delta^2).$$  
(9.20)

To ensure that $\delta I_{\text{source}} \lesssim 1$ bit, we therefore must have

$$\Delta \lesssim \frac{1}{\rho^2} \left( \frac{N_{\text{param}} + \rho^2}{\rho^2} \right) \left( \frac{N_{\text{param}}}{N_{\text{bins}}} \right).$$  
(9.21)

This condition is a more accurate version of the condition (8.13) that was derived in Sec. VII. It approximately reduces to the condition (8.13) for typical BBH events (except in the unrealistic limit $\rho^2 \ll N_{\text{param}}$), since $N_{\text{param}} \sim 10$ and $10 \ll N_{\text{bins}} \ll 10^5$. [16]

Turn next to the issue of the required degree of fineness of a template lattice; i.e., the issue of how close in parameter space successive templates must be to one another. This is mostly relevant to the third scenario described in Sec. VII, in which numerical relativists are able to simulate essentially arbitrary BBH mergers, and to carry out a large number of such simulations. We can parameterize the degree of fineness by a dimensionless parameter $\varepsilon_{\text{grid}}$ in the following way: the lattice is required to have the property that for any possible true signal $\mathbf{h}(\theta)$, there exists some template $\mathbf{h}(\theta^*)$ in the lattice with

$$\frac{(\mathbf{h}(\theta) \mid \mathbf{h}(\theta^*))}{\sqrt{(\mathbf{h}(\theta) \mid \mathbf{h}(\theta))} \sqrt{(\mathbf{h}(\theta) \mid \mathbf{h}(\theta^*))}} \geq 1 - \varepsilon_{\text{grid}}.$$  
(9.22)

The quantity $1 - \varepsilon_{\text{grid}}$ is called the minimal match [3]. Suppose that one defines a metric on the space $V$ of templates using the norm (2.4). It then follows from Eq. (9.22) that the largest possible distance $D_{\text{max}}$ between an incoming signal $\mathbf{h}(\theta)$ and some rescaled template $A\mathbf{h}(\theta^*)$ with $A > 0$ is

$$D_{\text{max}} = \sqrt{2\varepsilon_{\text{grid}} \rho},$$  
(9.23)

where $\rho$ is the matched filtering SNR (2.3) of the incoming signal.

We can view the discreteness in the template lattice as roughly equivalent to an ignorance on our part about the location of the manifold $S$ of true gravitational wave signals between the lattice points. The maximum distance any correct waveform $\mathbf{h}(\theta)$ could be away from where we may think it should be (where our ignorance is for example obtained by linearly extrapolating from the nearest points on the lattice) is of order $D_{\text{max}}$. We can crudely view this ignorance as equivalent to a numerical error $\delta \mathbf{h}$ in the templates of magnitude $||\delta \mathbf{h}|| \sim \sqrt{2\varepsilon_{\text{grid}} \rho}$. Combining Eqs. (8.1) and (9.21) shows that the loss of information $\delta I_{\text{source}}$ due to the discreteness of the grid should therefore be of order

$$\delta I_{\text{source}} \sim \rho^2 \left( \frac{\rho^2}{N_{\text{param}} + \rho^2} \right) \left( \frac{\varepsilon_{\text{grid}}}{N_{\text{param}}} \right).$$  
(9.24)

The grid fineness $\varepsilon_{\text{grid}}$ should be chosen to ensure that $\delta I_{\text{source}}$ is small compared to unity, while also taking into account that the fractional loss in event detection rate for signal searches due to the coarseness of the grid will be $\lesssim 3\varepsilon_{\text{grid}}$; see Sec. VIII B above and Refs. [3,22].
X. CONCLUSIONS

Theoretical template waveforms for the merger phase of BBH coalescences from numerical relativity will be a great aid to the analysis of detected BBH coalescence events. A complete bank of templates could be used to implement a matched filtering analysis of merger data, which would allow measurements of the binary’s parameters and tests of general relativity in a strong field, highly dynamic, highly non-spherical regime. Such matched filtering may also be possible without a complete bank of templates, if iterative supercomputer simulations are carried out in tandem with the data analysis. A match of the detected waves with those produced by numerical relativity will be a triumph for the theory of general relativity and an unambiguous signature of the existence of black holes. Qualitative information from representative supercomputer simulations will also be useful, both as an input to algorithms for extracting the merger waveform’s shape from the noisy interferometer data stream, and as an aid to interpreting the observed waveforms and making deductions about the waves’ source.

We have derived, using several rather different conceptual starting points, accuracy requirements that numerical templates must satisfy in order for them to be useful as data analysis tools. We first considered matched filtering signal searches using templates; here the loss in event rate due to template inaccuracies is simply related to the degradation in SNR, and leads to a criterion on template accuracy. Approximately the same criterion is obtained when one demands that the systematic errors in parameter extraction be small compared to the detector-noise induced statistical errors. Finally, we quantified the information that is encoded in the merger waveforms using the mathematical framework of information theory, and deduced how much of the information is lost due to template inaccuracies or to having insufficiently many templates. We deduced approximate requirements that templates must satisfy (in terms of both accuracy of individual templates and of the spacing between templates) in order that all of the waveforms information can be extracted.

The theory of maximum likelihood estimation is a useful starting point for deriving algorithms for reconstructing the gravitational waveforms from the noisy interferometer output. In this paper we have discussed and derived such algorithms in the contexts of both a single detector and a network of several detectors; these algorithms can be tailored to build-in many different kinds of prior information about the waveforms.

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APPENDIX A: WAVEFORM RECONSTRUCTION USING A NETWORK OF DETECTORS

In this appendix we describe how to extend the filtering methods discussed in Sec. V above from a single detector to a network of an arbitrary number of detectors. The underlying principle is again simply to use the maximum likelihood estimator of the waveform shape. We also explain the relationship between our waveform reconstruction method and the method of Gürsel and Tinto [24]. Secs. A1 and A2 below overlap somewhat with unpublished analyses by Sam Finn [23].

We start by establishing some notations for a network of detectors; these notations and conventions follow those of Appendix A of Ref. [27]. The output of such a network can be represented as a vector \( \mathbf{s}(t) = [s_1(t), \ldots, s_{nd}(t)] \), where \( nd \) is the number of detectors, and \( s_a(t) \) is the strain amplitude read out from the \( a \)th detector [66]. There will be two contributions to the detector output \( \mathbf{s}(t) \)—the intrinsic detector network noise \( \mathbf{n}(t) \) (a vector random process), and the true gravitational wave signal \( \mathbf{h}(t) \):

\[
\mathbf{s}(t) = \mathbf{h}(t) + \mathbf{n}(t).
\]  

We will assume that the detector network noise is stationary and Gaussian. In reality the noise will be non-stationary and non-Gaussian, but understanding the optimal method of waveform reconstruction under our idealized assumptions is an important first step towards more sophisticated waveform reconstruction algorithms that incorporate more information about the nature of the noise. With this assumption, the statistical properties of the detector network noise can be described by the auto-correlation matrix

\[
C_n(\tau)_{ab} = \langle n_a(t + \tau)n_b(t) \rangle - \langle n_a(t) \rangle \langle n_b(t) \rangle ,
\]

where the angular brackets mean an ensemble average or a time average. The Fourier transform of the correlation matrix, multiplied by two, is the power spectral density matrix:

\[
S_h(f)_{ab} = 2 \int_{-\infty}^{\infty} d\tau e^{2\pi if \tau} C_n(\tau)_{ab}.
\]

The off-diagonal elements of this matrix describe the effects of correlations between the noise sources in the various detectors, while each diagonal element \( S_h(f)_{aa} \) is
just the usual power spectral density of the noise in the ath detector. We assume that the functions $S_h(f)_{ab}$ for $a \neq b$ have been measured for each pair of detectors.

The Gaussian random process $\tilde{n}(t)$ determines a natural inner product $\langle \ldots | \ldots \rangle$ on the space of functions $\tilde{h}(t)$, which generalizes the inner product (2.3) discussed in the body of the paper in the context of a single detector. The inner product is defined so that the probability that the noise takes a specific value $\tilde{n}_0(t)$ is

$$p[\tilde{n} = \tilde{n}_0] \propto e^{-(\tilde{n}_0|\tilde{n}_0)/2},$$

and it is given by

$$\langle \tilde{g} | \tilde{h} \rangle \equiv 4 \operatorname{Re} \int_0^\infty df \, \tilde{g}(f)^* [S_h(f)^{-1}]^{ab} \tilde{h}_b(f).$$

See, e.g., Appendix A of Ref. [27] for more details.

Now, turn, to the relation between the gravitational wave signal $h_a(t)$ seen in the ath detector, and the two independent polarization components $h_+ (t)$ and $h_\times (t)$ of the gravitational waves. Let $x_a$ be the position and $d_a$ the polarization tensor of the ath detector in the detector network. By polarization tensor we mean that tensor $d_a$ for which the detector’s output $h_a(t)$ is given in terms of the waves’ transverse traceless strain tensor $h(x, t)$ by

$$h_a(t) = d_a : h(x_a, t),$$

where the colon denotes a double contraction. A gravitational wave burst coming from the direction of a unit vector $\mathbf{m}$ will have the form

$$h(x, t) = \sum_{A=+\times} h_A(t + m \cdot x) e^A_m,$$

where $e^A_m$ and $e^B_m$ are a basis for the transverse traceless tensors perpendicular to $\mathbf{m}$, normalized according to $e^A_m : e^B_m = 2 \delta^{AB}$. (Note that the notation $\mathbf{n}$ is typically used to denote direction to the source; we use instead $\mathbf{m}$ because we have denoted by $\mathbf{n}$ the detector noise.) Combining Eqs. (A4) and (A7) and switching from the time domain to the frequency domain using the convention (2.3) yields

$$\tilde{h}_a(f) = F_a^A(\mathbf{m}) \tilde{h}_A(f) e^{-2\pi i \tau_a(\mathbf{m})},$$

where the quantities

$$F_a^A(\mathbf{m}) \equiv e^A_m : d_a,$$

for $A = +\times$, are detector beam-pattern functions for the ath detector [57] and $\tau_a(\mathbf{m}) \equiv \mathbf{m} \cdot x_a$ is the time delay at the ath detector relative to the origin of coordinates.

1. Derivation of posterior probability distribution

We now construct the probability distribution $\mathcal{P}[\mathbf{m}, h_+(t), h_\times(t)|\tilde{s}(t)]$ for the gravitational waves to be coming from direction $\mathbf{m}$ with waveforms $h_+(t)$ and $h_\times(t)$, given that the output of the detector network is $\tilde{s}(t)$. Let $p^{(0)}(\mathbf{m})$ and $p^{(0)}[h_A(t)]$ be the prior probability distributions for the sky position $\mathbf{m}$ (presumably a uniform distribution on the unit sphere) and waveform shapes $h_A(t)$, respectively. A standard Bayesian analysis along the lines of that given in Ref. [2] and using Eq. (A4) gives

$$\mathcal{P}[\mathbf{m}, h_A(t)|\tilde{s}(t)] = \mathcal{K} p^{(0)}(\mathbf{m}) p^{(0)}[h_A(t)]$$

$$\times \exp \left[ - \left( \tilde{s} - \tilde{h} | \tilde{s} - \tilde{h} \right) / 2 \right],$$

where $\mathcal{K}$ is a normalization constant and $\tilde{h} = (h_1, \ldots, h_d)$ is understood to be the function of $\mathbf{m}$ and $h_A(t)$ given by (the Fourier transform of) Eq. (A8).

We simplify the expression (A10) in two stages. First, we reduce the argument of the exponential from a double sum over detectors to a single sum over detector sites. In the next few paragraphs we carry out this reduction, leading to Eqs. (A18) and (A19) below.

We assume that each pair of detectors in the detector network comes in one of two categories: (i) pairs of detectors at the same detector site, which are oriented the same way, and thus share common detector beam pattern functions $F_a^A(\mathbf{m})$ (for example the 2 km and 4 km interferometers at the LIGO Hanford site); or (ii) pairs of detectors at widely separated sites, for which the detector noise is effectively uncorrelated. Under this assumption we can arrange for the matrix $S_a(\mathbf{f})$ to have a block diagonal form, with each block corresponding to a detector site, by choosing a suitable ordering of detectors in the list $(1, \ldots, n_d)$. Let us denote the detector sites by Greek indices $\alpha, \beta, \gamma \ldots$, so that $\alpha$ runs from 1 to $n_s$, where $n_s$ is the number of sites. Let $D_a$ be the subset of the list of detectors $(1, \ldots, n_d)$ containing the detectors at the $\alpha$th site, so that any sum over detectors can be rewritten

$$\sum_a = \sum_{\alpha=1}^{n_s} \sum_{a \in D_\alpha}.$$

Thus, for example, for a 3 detector network with 2 detectors at the first site and one at the second, $D_1 = \{1, 2\}$ and $D_2 = \{3\}$. Let $F^{(A)}_a(\mathbf{m})$ denote the common value of the beam pattern functions (A9) for all the detectors at site $\alpha$. Let $S(\mathbf{f})$ denote the $\alpha$th diagonal sub-block of the matrix $S_h(\mathbf{f})$. Then if we define

$$\Lambda = \left( \tilde{s} - \tilde{h} \right) \left( \tilde{s} - \tilde{h} \right)^*,$$

[the quantity which appears in the exponential in Eq. (A10)], we obtain from Eq. (A5)

$$\Lambda = \sum_{\alpha=1}^{n_s} 4 \operatorname{Re} \int_0^\infty df \sum_{a,b \in D_\alpha} \left[ \tilde{s}_a(f)^* - \tilde{h}_a(f)^* \right]$$

$$\times \left[ S_{\alpha}(f)^{-1} \right]^{ab} \left[ \tilde{s}_b(f) - \tilde{h}_b(f) \right] .$$

(13)
Next, we note from Eq. (A8) that the value of \( \tilde{h}_a \) will be the same for all detectors at a given site. If we denote this common value by \( \bar{h}_a \), then we obtain after some manipulation of Eq. (A13)

\[
\Lambda = \sum_{\alpha=1}^{n_s} 4 \text{Re} \int_0^\infty df \left\{ \frac{|\tilde{s}_\alpha(f) - \bar{h}_a(f)|^2}{S'^{\alpha}(f)} + \Delta_\alpha(f) \right\}.
\]

(A14)

The meanings of the various symbols in Eq. (A14) are as follows. The quantity \( S'^{\alpha}(f) \) is defined by

\[
\frac{1}{S'^{\alpha}(f)} \equiv \sum_{a,b \in D_\alpha} [s_\alpha(f)^{-1}]^{ab},
\]

(A15)

and can be interpreted as the effective overall noise spectrum for site \( \alpha \). The quantity \( s_\alpha \) is given by

\[
\tilde{s}_\alpha(f) \equiv S'^{\alpha}(f) \sum_{a,b \in D_\alpha} [s_\alpha(f)^{-1}]^{ab} \tilde{s}_b(f),
\]

(A16)

and is, roughly speaking, the mean output strain amplitude of site \( \alpha \). Finally,

\[
\Delta_\alpha(f) \equiv \sum_{a,b \in D_\alpha} \tilde{s}_a(f) \tilde{s}_b(f) \left\{ [s_\alpha(f)^{-1}]^{ab} - S'^{\alpha}(f) \sum_{c,d \in D_\alpha} [s_\alpha(f)^{-1}]^{ac} [s_\alpha(f)^{-1}]^{db} \right\}.
\]

(A17)

The quantity \( \Delta_\alpha \) is independent of \( \mathbf{m} \) and \( h_A(t) \), and is therefore irrelevant for our purposes; it can be absorbed into the normalization constant \( \mathcal{K} \) in Eq. (A10). This unimportance of \( \Delta_\alpha \) occurs because we are assuming that there is some signal present. However, in situations where one is trying to assess the probability that some signal (and not just noise) is present in the outputs of the detector network, the term \( \Delta_\alpha \) is very important. In effect, it encodes the discriminating power against noise bursts which is due to the presence of detectors with different noise spectra at one site (e.g., the 2km and 4km interferometers at the LIGO Hanford site). We drop the term \( \Delta_\alpha \) from now on.

The probability distribution for the waveform shapes and sky direction is now given by, from Eqs. (A10), (A12) and (A14),

\[
\mathcal{P} \{ \mathbf{m}, h_A(t) | \tilde{s}(t) \} = \mathcal{K} p^{(0)}(\mathbf{m}) p^{(0)}[h_A(t)] e^{-\Lambda'/2},
\]

(A18)

where

\[
\Lambda' = \sum_{\alpha=1}^{n_s} 4 \text{Re} \int_0^\infty df \frac{|\tilde{s}_\alpha(f) - \bar{h}_a(f)|^2}{S'^{\alpha}(f)}.
\]

(A19)

Finally, we express this probability distribution directly in terms of the waveforms \( h_+ (t) \) and \( h_\times (t) \) by substituting Eq. (A8) into Eq. (A19), which gives

\[
\mathcal{L}' = 4 \text{Re} \int_0^\infty df \left\{ \sum_{A,B=+,-} \Theta^{AB}(f, \mathbf{m}) \left[ \tilde{h}_A(f)^* - \bar{h}_A(f)^* \right] \right\} \times \left[ \tilde{h}_B(f) - \bar{h}_B(f) \right] + S(f, \mathbf{m}),
\]

(A20)

Here

\[
\Theta^{AB}(f, \mathbf{m}) \equiv \sum_{\alpha=1}^{n_s} F^A_\alpha(\mathbf{m}) F^B_\alpha(\mathbf{m}) \frac{1}{S'^{\alpha}(f)},
\]

(A21)

\[
\tilde{h}_A(f) \equiv \Theta_{AB}(f, \mathbf{m}) \sum_{\alpha=1}^{n_s} F^B_\alpha(\mathbf{m}) \tilde{s}_\alpha(f) e^{2\pi i f \tau_\alpha(\mathbf{m})},
\]

(A22)

2. Estimating the waveform shapes and the direction to the source

Equations (A18) and (A20) constitute one of the main results of this Appendix, and give the final and general probability distribution for \( \mathbf{m} \) and \( h_A(t) \). In the next few paragraphs we discuss its implications. As mentioned at the start of the appendix, we are primarily interested in situations where the direction \( \mathbf{m} \) to the source is already known. However, as an aside, we now briefly consider the more general context where the direction to the source as well as the waveform shapes are unknown.

Starting from Eq. (A18), one could use either maximum likelihood estimators or so-called Bayes estimators \( [27, 28, 29] \) to determine “best-guess” values of \( \mathbf{m} \) and \( h_A(t) \). Bayes estimators have significant advantages over maximum likelihood estimators but are typically much more difficult to compute, as explained in, for example, Appendix A of Ref. \( [7] \). The Bayes estimator for the direction to the source will be given by first integrating Eq. (A18) over all waveform shapes, which yields

\[
\mathcal{P} \{ \mathbf{m} | \tilde{s}(t) \} = \mathcal{K} p^{(0)}(\mathbf{m}) \mathcal{D}(\mathbf{m}) \exp \left[ -2 \int_0^\infty df S(f, \mathbf{m}) \right],
\]

(A24)

where \( \mathcal{D}(\mathbf{m}) \) is a determinant-type factor that is produced by integrating over the waveforms \( h_\times(t) \). This factor encodes the information that the detector network has greater sensitivity in some directions than in others, and that other things being equal, a signal is more likely to have come from a direction in which the network is more sensitive. The Bayes estimator of \( \mathbf{m} \) is now obtained simply by calculating the expected value.
of \( \mathbf{m} \) with respect to the probability distribution (A24). The simpler, maximum likelihood estimator of \( \mathbf{m} \) is given by choosing the values of \( \mathbf{m} \) [and of \( h_A(t) \)] which maximize the probability distribution (A18), or equivalently by minimizing the quantity

\[
\int_0^\infty df \, S(f, \mathbf{m}). \tag{A25}
\]

Let us denote this value of \( \mathbf{m} \) by \( \mathbf{m}_{\text{ML}}(\hat{s}) \). Note that the quantity (A23) encodes all the information about time delays between the signals detected at the various detector sites; as is well known, directional information is obtained primarily through time delay information.

In Ref. [22], Gürsel and Tinto suggest a method of estimating \( \mathbf{m} \) from \( \hat{s}(t) \) for a network of three detectors. For white noise and for the special case of one detector per detector site, the Gürsel-Tinto estimator is the same as the maximum likelihood estimator \( \mathbf{m}_{\text{ML}}(\hat{s}) \) just discussed, with one major modification: in Sec. V of Ref. [22], Gürsel and Tinto prescribe discarding those Fourier components of the data whose SNR is below a certain threshold as the first stage of calculating their estimator.

Turn now, to the issue of estimating the waveform shapes \( h_+(t) \) and \( h_\times(t) \). In general situations where both \( \mathbf{m} \) and \( h_A(t) \) are unknown, the best way to proceed in principle would be to integrate the probability distribution (A13) over all solid angles \( \mathbf{m} \) to obtain a reduced probability distribution \( P[h_A(t)|\hat{s}(t)] \) for the waveform shapes, and to use this reduced probability distribution to make estimators of \( h_A(t) \). However, such an integration cannot be performed analytically and would not be easy numerically; in practice simpler estimators will likely be used. One such simpler estimator is the maximum likelihood estimator of \( h_A(t) \) obtained from Eq. (A18). In the case of no prior information about the waveform shape when the prior distribution \( P^{(0)}[h_A(t)] \) is very broad, this maximum likelihood estimator is simply \( \hat{h}_A(t) \) evaluated at the value \( \mathbf{m}_{\text{ML}}(\hat{s}) \) of \( \mathbf{m} \) discussed above.

For BBH mergers, in many cases the direction \( \mathbf{m} \) to the source will have been measured from the inspiral portion of the waveform, and thus for the purposes of estimating the merger waveform’s shape, \( \mathbf{m} \) can be regarded as known. The probability distribution for \( h_A(t) \) given \( \mathbf{m} \) and \( \hat{s}(t) \) is, from Eq. (A18):

\[
P[h_A(t) | \mathbf{m}, \hat{s}(t)] = K' P^{(0)}[h_A(t)] e^{-\Lambda''/2}. \tag{A26}
\]

Here \( K' \) is a normalization constant, and \( \Lambda'' \) is given by Eq. (A20) with the term \( S(f, \mathbf{m}) \) omitted. The maximum likelihood estimator of \( h_A(t) \) obtained from this probability distribution in the limit of no prior information is again just \( \hat{h}_A(t) \). The formula for the estimator \( \hat{h}_A(t) \) given by Eqs. (A15), (A16), (A21) and (A22) is one of the key results of this appendix. It specifies the best-fit waveform shape as a unique function of the detector outputs \( s_a(t) \) for any network of detectors.

### 3. Incorporating prior information

In Sec. [V], we suggested a method of reconstruction of the merger waveform shape, for a single detector, which incorporated assumed prior information as to the waveform’s properties. In this appendix, our discussion so far has neglected all prior information about the shape of the waveforms \( h_+(t) \) and \( h_\times(t) \). We now discuss waveform estimation for a network of detectors, incorporating prior information, for fixed sky direction \( \mathbf{m} \).

With a few minor modifications, the entire discussion of Sec. [V] can be applied to a network of detectors. The required modifications are as follows. First, the linear space \( V \) should be taken to be the space of pairs of waveforms \( \{h_+(t), h_\times(t)\} \), suitably discretized, so that the dimension of \( V \) is \( 2T/\Delta t \). Second, the inner product (2.14) must be replaced by a discrete version of the inner product

\[
\langle \{h_+, h_\times|\{k_+, k_\times\} \rangle = 4 \text{Re} \int_0^\infty df \, \Theta^{AB}(f, \mathbf{m}) \times \hat{h}_A(f) \ast \hat{k}_B(f), \tag{A27}
\]

since the inner product (A27) plays the same role in the probability distribution (A24) as the inner product (2.14) plays in the distribution (5.3). Third, the estimated waveforms \( \{\hat{h}_+(t), \hat{h}_\times(t)\} \) given by Eq. (A22) take the place of the measured waveform \( s \) in Sec. [V] for the same reason. Fourth, the wavelet basis used to specify the prior information must be replaced by a basis of the form \( \{w_{ij}^a(t), w_{ij}^\times(t)\} \), where \( w_{ij}^a(t) \) is a wavelet basis of the type discussed in Sec. [V] for the space of waveforms \( h_+(t) \), and \( w_{ij}^\times(t) \) is a similar wavelet basis for the space of waveforms \( h_\times(t) \). The prior information about, for example, the assumed duration and bandwidths of the waveforms \( h_+(t) \) and \( h_\times(t) \) can then be represented exactly as in Sec. [V]. With these modifications, the remainder of the analyses of Sec. [V] apply directly to a network of detectors. Thus the “perpendicular projection” estimator (7.4) and the more general estimator (5.16) (corresponding to the more general algorithm described in Sec. [V.C]) can both be applied to a network of detectors.

### 4. The Gürsel-Tinto waveform estimator

As mentioned in Sec. [V] above, Gürsel and Tinto have suggested an estimator of the waveforms \( h_+(t) \) and \( h_\times(t) \) for networks of three detector sites with one detector at each site (7.4), in the case when the direction \( \mathbf{m} \) to the source is known. In our notation, the construction of that estimator can be summarized as follows. First, assume that the estimator is some linear combination of the outputs of the independent detectors corrected for time delays:

\[
\tilde{z}^{(GT)}(\mathbf{m}) = \sum_{\alpha=1}^{3} w_{ij}^\alpha(\mathbf{m}) e^{2\pi i f \tau_\alpha(\mathbf{m})} \tilde{s}_\alpha(f). \tag{A28}
\]
Here $\hat{h}_{A}^{(GT)}$ is the Gürsel-Tinto ansatz for the estimator, and $w_{A}^{\alpha}$ are some arbitrary constants that depend on $m$. [Since there is only one detector per site we can neglect the distinction between the output $s_{A}(f)$ of an individual detector and the output $s_{A}(f)$ of a detector site.] Next, demand that for a noise-free signal, the estimator reduces to the true waveforms $h_{A}(t)$. From Eqs. (A1) and (A8) above, this requirement is equivalent to the equation

$$\sum_{\alpha=1}^{3} w_{A}^{\alpha}(m) F_{\alpha}^{B}(m) = \delta_{A}^{B}. \quad \text{(A29)}$$

There is a two-dimensional linear space of tensors $w_{A}^{\alpha}$ which satisfy Eq. (A29). Finally, choose $w_{A}^{\alpha}$ above, this requirement is equivalent to the equation

$$\sum_{A=+,-,A} \int dt \left| \hat{h}_{A}^{(GT)}(t) - h_{A}(t) \right|^2, \quad \text{(A30)}$$

where $\hat{h}_{A}^{(GT)}(t)$ is given as a functional of $h_{A}(t)$ and the detector noise $n_{A}(t)$ by Eqs. (A1), (A8) and (A28).

It is straightforward to show by a calculation using Lagrange multipliers that the resulting estimator is given by

$$\hat{h}_{A}^{(GT)}(t) = \hat{h}_{A}(t). \quad \text{(A31)}$$

In other words, the Gürsel-Tinto estimator coincides with the maximum likelihood estimators of $h_{+}(t)$ and $h_{x}(t)$ discussed in this appendix in the case of little prior information. However, the estimators discussed here generalize the Gürsel-Tinto estimator by allowing an arbitrary number of detectors per site [with the effective output and effective noise spectrum of a site being given by Eqs. (A16) and (A15) above], by allowing an arbitrary number of sites, and by allowing one to incorporate prior information about the waveform shapes.

APPENDIX B: MEASURES OF INFORMATION

In this appendix we substantiate the claims concerning information theory made in Sec. IX of the body of the paper. First, we argue that the concept of the “relative information” of two PDFs introduced in Eq. (1.2) does have the interpretation we ascribed to it: it is the base 2 logarithm of the number of distinguishable measurement outcomes. Second, we derive the approximate equations (1.8) and (1.12).

Consider first the issue of ascribing to any measurement process a “number of bits of information gained” from that process, which corresponds to the base 2 logarithm of the number of distinguishable possible outcomes of the measurement. If $p^{(0)}(x)$ is the PDF for the measured quantities $x = (x^{1}, \ldots, x^{n})$ before the measurement, and $p(x)$ is the corresponding PDF after the measurement, then the relative information of these two PDFs is defined to be

$$I = \int d^{n}x \, p(x) \log_{2} \left[ \frac{p(x)}{p^{(0)}(x)} \right]. \quad \text{(B1)}$$

In simple examples, it is easy to see that the quantity (B1) reduces to the number of bits of information gained in the measurement. For instance, if $x = (x^{1})$ and the prior PDF $p^{(0)}$ constrains $x^{1}$ to lie in some range of size $X$, and if after the measurement $x^{1}$ is constrained to lie in a small interval of size $\Delta x$, then $I \approx \log_{2}(X/\Delta x)$, as one would expect. In addition, the quantity (B1) has the desirable feature that it is coordinate independent, i.e., that the same answer is obtained when one makes a non-linear coordinate transformation on the manifold parameterized by $(x^{1}, \ldots, x^{n})$ before evaluating the quantity (B1). For these reasons, in any measurement process, the quantity (B1) can be interpreted as the number of bits of information gained.

1. Explicit formula for the total information

As a foundation for deriving the approximate formula (1.8), we derive in this subsection an explicit formula [Eq. (B14)] for the total information gain (1.2) in a gravitational wave measurement. We shall use a basis of $V$ where the matrix (2.11) is unity, and for ease of notation we shall denote by $\tilde{N}$ the quantity which was denoted by $N_{\text{ins}}$ in the body of the paper.

First, we assume that the prior PDF $p^{(0)}(h)$ appearing in Eq. (B1) is a function only of $h = ||h||$. In other words, all directions in the vector space $V$ are taken to be, a priori, equally likely, when one measures distances and angles with the inner product (2.1). It would be more realistic to make such an assumption with respect to a noise-independent inner product like $(h_{1} | h_{2}) \equiv \int dh_{1}(t)h_{2}(t)$, but if the noise spectrum $S_{h}(f)$ does not vary too rapidly within the bandwidth of interest, the distinction is not too important and our assumption will be fairly realistic.

We write the prior PDF as

$$p^{(0)}(h) \, d^{N}h = \frac{2\pi^{N/2}}{\Gamma(N/2)} h^{N-1} p^{(0)}(h) dh \equiv \tilde{p}^{(0)}(h) \, dh. \quad \text{(B2)}$$

The quantity $\tilde{p}^{(0)}(h) \, dh$ is the prior probability that the signal $h$ will have an SNR $||h||$ between $h$ and $h + dh$. The exact form of the PDF $\tilde{p}^{(0)}(h)$ will not be too important for our calculations below. A moderately realistic choice is $\tilde{p}^{(0)}(h) \propto 1/h^{3}$ with a cutoff at some $h_{1} \ll 1$. Note however that the choice $p^{(0)}(h) = 1$ corresponding to $\tilde{p}^{(0)}(h) \propto h^{N-1}$ is very unrealistic. Below we shall assume that $\tilde{p}^{(0)}(h)$ is independent of $N$.

We next write Eq. (B1) in a more explicit form. Without loss of generality we can take...
Now combining Eqs. (B4), (B8), and (B10) yields
\begin{equation}
\frac{d^Nh}{\Gamma[(N-1)/2]} \sin(\theta)^{N-2} h^{N-1} d\theta dh, \tag{B4}
\end{equation}
we can write
\begin{equation}
p(h|s) d^Nh = K_1 \tilde{p}^{(0)}(h) \sin(\theta)^{N-2} \times \exp \left[ -\frac{1}{2} (s^2 + h^2 - 2sh \cos \theta) \right] dh d\theta,
\end{equation}
where $K_1$ is a constant. If we define the function $F_N(x)$ by
\begin{equation}
F_N(x) = \frac{1}{2} \int_0^\pi d\theta \sin(\theta)^{N-2} e^{x \cos \theta}, \tag{B6}
\end{equation}
then the constant $K_1$ is determined by the normalization condition
\begin{equation}
1 = 2K_1 \int_0^\infty dh \, e^{-(s^2 + h^2)/2} \, F_N(sh) \, \tilde{p}^{(0)}(h). \tag{B7}
\end{equation}

We next calculate the PDF $p(h|\rho(s))$ appearing in the denominator in Eq. (12). From Bayes’s theorem, this PDF is given by
\begin{equation}
p(h|\rho(s)) = K \tilde{p}^{(0)}(h) p(\rho(s)|h), \tag{B8}
\end{equation}
where $p(\rho(s)|h)$ is the PDF for $\rho(s)$ given that the gravitational wave signal is $h$, and $K$ is a normalization constant. Using the fact that $p(s|h) \propto \exp \left[ -(s-h)^2 \right]$, we find using Eq. (B4) that
\begin{equation}
p(s|h) d^Ns = \frac{2^{1-N/2}}{\sqrt{\pi} \Gamma[(N-1)/2]} \sin(\theta)^{N-2} s^{N-1} \times \exp \left[ -\frac{1}{2} (s^2 + h^2 - 2sh \cos \theta) \right] ds d\theta. \tag{B9}
\end{equation}
Integrating over $\theta$ now yields from Eq. (B6)
\begin{equation}
p(\rho(s)|h) = \frac{1}{2} \ln 2 \pi \Gamma[2 - N/2] - \frac{1}{2} \ln \sin^2 \theta \int_0^\pi d\theta \sin(\theta)^{N-2} \frac{d}{dh} F_N(sh) ds. \tag{B10}
\end{equation}
Now combining Eqs. (B3), (B8), and (B10) yields
\begin{equation}
p(h|\rho(s)) d^Nh = K_2 \tilde{p}^{(0)}(h) e^{-(\rho(s)^2 + h^2)/2} F_N[\rho(s)h] \times \sin(\theta)^{N-2} dh d\theta, \tag{B11}
\end{equation}
where from Eq. (B7) the normalization constant is given by
\begin{equation}
K_2 = \frac{2\Gamma(N/2)}{\sqrt{\pi} \Gamma[(N-1)/2]} K_1. \tag{B12}
\end{equation}

We can now calculate the information $I_{total}$ by combining Eqs. (B2), (B7), (B6), (B11), and (B12). The result is
\begin{equation}
I_{total}[\rho(s),N] = -\int_0^\infty dh \, p^{(1)}(h) \, G_N[\rho(s)h] \times \log_2 \left[ \frac{2\Gamma(N/2)}{\sqrt{\pi} \Gamma[(N-1)/2]} \right]. \tag{B13}
\end{equation}

Equations (B7), (B8), and (B13) – (B15) now define explicitly the total information $I_{total}$ as a function of the parameters $\rho(s)$ and $N$ and of the prior PDF $\tilde{p}^{(0)}(h)$.

2. Approximate formula for the total information

We now derive the approximate formula (9.8) for the total information. Let $\rho_0^2 = \rho(s)^2/N$; we will consider the limit of large $\rho(s)$ and $N$ but fixed $\rho_0$. Our analysis will divide into two cases, depending on whether $\rho_0 > 1$ or $\rho_0 \leq 1$. Let us first consider the case $\rho_0 > 1$. In the large $N$ limit the result for $\rho_0 > 1$ will be independent of the prior PDF $\tilde{p}^{(0)}(h)$, which we assume has no dependence on $N$.

The first term in Eq. (B13) is the expected value $\langle G_N[\rho(s)h] \rangle$ of $G_N[\rho(s)h]$ with respect to the PDF (B15). If we change the variable of integration in this term from $h$ to $u = h/\sqrt{N}$, we find
\begin{equation}
\langle G_N[\rho(s)h] \rangle \propto \int_0^\infty du \, \tilde{p}^{(0)}(\sqrt{N}u) \, e^{-N(\rho_0^2 + u^2)/2} \times F_N(N \rho_0 u) \, G_N(N \rho_0 u). \tag{B16}
\end{equation}
From Eq. (B10) it is straightforward to show that in the limit of large $N$,
\begin{equation}
F_N(z) \approx \frac{1}{2} e^{Nq_0(\theta_c)} \sqrt{\frac{2\pi}{N |q''(\theta_c)|}}, \tag{B17}
\end{equation}
for fixed $z$. Here $q(\theta)$ is the function
\begin{equation}q(\theta) = z \cos \theta + \ln \sin \theta, \tag{B18}
\end{equation}
and $\theta_c = \theta_c(z)$ is the value of $\theta$ which maximizes the function $q(\theta)$, given implicitly by
\begin{equation}
z \sin^2 \theta_c = \cos \theta_c. \tag{B19}
\end{equation}
We similarly find that
\begin{equation}F'_N(z) \approx \frac{1}{2} e^{Nq_0(\theta_c)} \sqrt{\frac{2\pi}{N |q''(\theta_c)|}} \cos \theta_c. \tag{B20}
\end{equation}
It is legitimate to use the approximations (B17) and (B20) in the integral (B14) since the value $u_{max}(N, \rho_0)$
of \( u \) at which the PDF \( p^{(1)}(\mathcal{N}|\rho bh) \) is a maximum approaches at large \( \mathcal{N} \) a constant \( u_{\text{max}}(\rho_b) \) which is independent of \( \mathcal{N} \), as we show below.

Inserting the approximation (B17) into Eq. (B16) and identifying \( z = \rho_b u \), we find that the PDF (B15) is proportional to

\[
\exp \left[ \mathcal{N} Q(u) + O(1) \right], \tag{B21}
\]

where

\[
Q(u) = -\frac{1}{2}(\rho_b^2 + u^2) + q(\theta_c) \tag{B22}
\]

and \( \theta_c = \theta_c(z) = \theta_c(\rho_b u) \). From Eqs. (B18) and (B19) it can be shown that the function (B22) has a local maximum at

\[
u_{\text{max}} = \sqrt{\rho_b^2 - 1} \tag{B23}
\]

at which point \( \theta_c \) is given by \( \sin \theta_c = 1/\rho_b \). The form of the PDF (B22) now shows that at large \( \mathcal{N} \),

\[
\left< G^\mathcal{N}(\mathcal{N}|\rho bh) \right> \approx G^\mathcal{N}(\mathcal{N}|\rho bh u_{\text{max}}). \tag{B24}
\]

Finally, if we combine Eqs. (B13), (B17)–(B21), (B23) and (B24) and use Stirling’s formula to approximate the Gamma functions, we obtain Eq. (9.3).

Turn, next, to the case \( \rho_b < 1 \). In this case the function \( Q \) does not have a local maximum, and the dominant contribution to the integral (B16) at large \( \mathcal{N} \) comes from \( h \sim O(1) \) (rather than from \( h \sim \sqrt{\mathcal{N}} \), \( u \sim O(1) \) as was the case above). From Eq. (B6) we obtain the approximations

\[
F^\mathcal{N}(\sqrt{\mathcal{N}} w) = \sqrt{\frac{\pi}{2N}} e^{w^2/2} \left[ 1 + O(1/\sqrt{\mathcal{N}}) \right] \tag{B25}
\]

and

\[
F^\mathcal{N}_2(\sqrt{\mathcal{N}} w) = \sqrt{\frac{\pi}{2N}} e^{w^2/2} \left[ 1 + O(1/\sqrt{\mathcal{N}}) \right], \tag{B26}
\]

which are valid for fixed \( w \) at large \( \mathcal{N} \). Using Eqs. (B25), (B26), and (B13)–(B15), and using Stirling’s formula again we find that

\[
I_{\text{total}} \approx \frac{1}{2} \rho_b^2 \int_0^\infty dh \, \bar{p}^{(0)}(h) \exp \left[ - \left( 1 - \rho_b^2 \right) \frac{h^2}{2} \right] \frac{h^2}{2} \int_0^\infty dh \, \bar{p}^{(0)}(h) \exp \left[ - \left( 1 - \rho_b^2 \right) \frac{h^2}{2} \right]. \tag{B27}
\]

For simplicity we now take \( \bar{p}^{(0)}(h) \) to be a Gaussian centered at zero with width \( h_{\text{prior}}^2 \); this yields

\[
I_{\text{total}} \approx \frac{1}{2} \left[ \frac{\rho_b^2 h_{\text{prior}}^2}{1 + (1 - \rho_b^2) h_{\text{prior}}^2} \right]. \tag{B28}
\]

From Eq. (B7), the parameter \( \rho_b \) is given by

\[
\rho_b^2 = 1 + \frac{\rho^2}{\mathcal{N}_{\text{bins}}} - \frac{1}{\sqrt{\mathcal{N}_{\text{bins}}}}, \tag{B29}
\]

where the last term denotes the rms magnitude of the statistical fluctuations. Since we are assuming that \( \rho_b < 1 \), it follows that \( \rho_b^2 \approx 1 - 1/\sqrt{\mathcal{N}_{\text{bins}}} \), and therefore we obtain from Eq. (B28) that

\[
I_{\text{total}} \approx \frac{1}{2} \min \left[ h_{\text{prior}}^2, \sqrt{\mathcal{N}_{\text{bins}}} \right]. \tag{B30}
\]

Thus, if \( h_{\text{prior}} \lesssim 1 \), then the total information gain is \( \lesssim 1 \) also.

3. Approximate formula for the source information

We now turn to a discussion of the approximate formula (9.12) for the information (9.11) obtained about the source of the gravitational waves. In general, the measure of information (9.11) depends on the source parameters \( \theta \). We can evaluate the information \( I_{\text{source}} \) explicitly in the simple and unrealistic model where the dependence on the source parameters \( \theta \) is linear and where there is little prior information. In this case the manifold of possible signals is a linear subspace (with dimension \( \mathcal{N}_{\text{param}} \)) of the linear space of all possible signals (which has dimension \( \mathcal{N} \)). The integral (9.11) then reduces to an integral analogous to (9.2), and we obtain the formula (9.12) in the same way as we obtained Eq. (5.8). The result (9.12) is clearly a very crude approximation, as the true manifold of merger signals is very curved and nonlinear. Nevertheless, it seems likely that the formula (9.12) will be valid for some effective number of parameters \( \mathcal{N}_{\text{param}} \) that is not too much different from the true number of parameters on which the waveform depends.

APPENDIX C: EXPECTED VALUE OF CORRELATION COEFFICIENT

In this appendix we derive the formula (5.18) for the expected value of the correlation coefficient (5.17). We start by deriving the following general result. Let \( n = (n^1, \ldots, n^N) \) be a Gaussian random variable with \( (n) = 0 \) and \( (n^i n^j) = \Sigma^{ij} \). Let \( \mathbf{h} = (h^1, \ldots, h^N) \) be a fixed vector, and define the random variable \( \mathcal{C} \) by

\[
\mathcal{C} = \frac{\mathbf{h} \cdot \Sigma^{-1} \cdot (h + n)}{\sqrt{\mathbf{h} \cdot \Sigma^{-1} \cdot \mathbf{h} \cdot (h + n) \cdot \Sigma^{-1} \cdot (h + n)}}. \tag{C1}
\]

Then, in the regime \( \mathcal{N} \gg 1 \) and \( \rho \gg 1 \), where \( \rho^2 = \mathbf{h} \cdot \Sigma^{-1} \cdot \mathbf{h} \), we have

\[
\left< \mathcal{C} \right> = \frac{1}{\sqrt{1 + \mathcal{N}/\rho^2}} \left[ 1 + O \left( \frac{1}{\rho^2}, \frac{1}{\mathcal{N}} \right) \right]. \tag{C2}
\]
Equation (5.18) can be obtained from Eq. (C2) as follows. From Eq. (5.4), the vector \( \mathbf{h}_{\text{best-fit}} \) can be written as

\[
\mathbf{h}_{\text{best-fit}} = \mathbf{h} + \mathbf{n}_\parallel
\]

where \( \mathbf{n}_\parallel \) denotes the component of the noise \( \mathbf{n} \) in the space \( U \). Thus, the vectors \( \mathbf{h} \) and \( \mathbf{h}_{\text{best-fit}} \) which appear in Eq. (5.17) both lie in the space \( U \) of dimension \( N'_{\text{bins}} \) (although both nominally lie in the larger space \( V \) of dimension \( N'_{\text{bins}} \)). Now, identifying \( N \) and \( N'_{\text{bins}} \), we see that the quantities (5.17) and (C1) coincide, and the result (6.18) follows.

We now turn to the derivation of Eq. (6.9). First, make a linear change of variables to make \( \Sigma \) fit. (The results obtained at the end can be generalized to non-unit \( \Sigma \) by inspection.) We want to evaluate

\[
\langle C \rangle = \int dn_1 \ldots dn_N p(n) C(n),
\]

where \( C(n) \) is given by Eq. (C1). The quantity \( C(n) \) depends on \( n \) only through the combinations

\[
\alpha = \mathbf{h} \cdot \mathbf{n} = \sum_i h_i n_i
\]

and

\[
\beta = \mathbf{n} \cdot \mathbf{n} = \sum_{i=1}^N (n_i)^2.
\]

Hence

\[
\langle C \rangle = \int d\alpha \int d\beta p(\alpha, \beta) C(\alpha, \beta),
\]

where

\[
C(\alpha, \beta) = \frac{\rho^2 + \alpha}{\rho \sqrt{\rho^2 + 2\alpha + \beta}}.
\]

The probability distribution \( p(\alpha, \beta) \) can be approximately evaluated in the following way. We have

\[
p(\alpha, \beta) = p(\beta|\alpha) p(\alpha).
\]

Here \( p(\beta|\alpha) \) is the distribution for \( \beta \) given a value of \( \alpha \), and \( p(\alpha) \) is from Eq. (C5) a Gaussian with zero mean and variance \( \rho^2 \):

\[
p(\alpha) = \frac{1}{\sqrt{2\pi} \rho} \exp\left\{ -\alpha^2/(2\rho^2) \right\}.
\]

We introduce the notation

\[
\langle \ldots \rangle_\alpha = \int \ldots p(\beta|\alpha) d\beta,
\]

and define \( (\Delta \beta)^2_\alpha = \langle \beta^2 \rangle_\alpha - \langle \beta \rangle^2_\alpha \). The distribution \( p(\beta|\alpha) \) can be treated as being approximately Gaussian in the regime where \( (\Delta \beta)^2_\alpha \ll \langle \beta \rangle^2_\alpha \), which we show below is the case when \( \rho^2 \gg 1 \) and \( N \gg 1 \). Hence we need only evaluate \( \langle \beta \rangle_\alpha \) and \( (\Delta \beta)^2_\alpha \).

Without loss of generality, we can write \( \mathbf{h} = (\rho, 0, \ldots, 0) \), so that from Eq. (C1), \( \alpha = \rho n_1 \). Similarly, Eq. (C1) gives

\[
\beta = \frac{\alpha^2}{\rho^2} + \sum_{i=2}^N (n_i)^2.
\]

Using the fact that the \( n_i \) are independent normally distributed random variables, it follows from Eq. (C12) that

\[
\langle \beta \rangle_\alpha = \frac{\alpha^2}{\rho^2} + N - 1,
\]

and similarly we find

\[
(\Delta \beta)^2_\alpha = 2(N - 1).
\]

Now the integral (C12) will be dominated by contributions from the regime \( \alpha \sim (\text{a few}) \times \rho \). In this regime, we have

\[
\frac{\langle \Delta \beta \rangle_\alpha}{\langle \beta \rangle_\alpha} \sim \frac{1}{\sqrt{N}} \ll 1,
\]

which justifies our treating the PDF \( p(\beta|\alpha) \) as Gaussian. Combining these results we find

\[
p(\alpha, \beta) \approx \frac{1}{2\pi \alpha (\Delta \beta)_{\alpha}} \exp\left\{ \frac{\alpha^2}{2a^2} - \frac{(\beta - \langle \beta \rangle_\alpha)^2}{2(\Delta \beta)^2_{\alpha}} \right\}.
\]

Inserting this distribution into Eq. (C7), using Eq. (C8) and expanding to second order in \( \alpha \) gives

\[
\langle C \rangle = \int d\alpha p(\alpha) C[\alpha, \langle \beta \rangle_\alpha] \left[ 1 + O(1/N) \right]
\]

\[
= \int d\alpha p(\alpha) \frac{1 + \alpha/\rho^2}{\sqrt{(1 + \alpha/\rho^2)^2 + (N - 1)/\rho^2}}
\]

\[
\times \left[ 1 + O(1/N) \right]
\]

\[
= \frac{1}{\sqrt{1 + N/\rho^2}} \left[ 1 + O\left( \frac{1}{\rho^2}, \frac{1}{N} \right) \right],
\]

as required.

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[3] J. Hough, K. Danzmann et al., GEO600, Proposal for a 600 m Laser-Interferometric Gravitational Wave Antenna, unpublished, 1994.

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[6] P. Bender, I. Ciufolini, K. Danzmann, W. Folkner, J. Hough, D. Robertson, A. Rüdiger, M. Sandfors, R. Schilling, B. Schutz, R. Stebbins, T. Summer, P. Touboul, S. Vitale, H. Ward, and W. Winkler. LISA Interferometric Space Antenna for the detection and observation of gravitational waves, Pre-Phase A Report, December 1995 (unpublished).

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[12] K. S. Thorne, presentation at Intermediate Binary Black Hole workshop, Caltech, July 1996.

[13] R. H. Price et al., in preparation.

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[17] These eight institutions are: University of Texas, Austin; NCSA/University of Illinois; University of North Carolina, Chapel Hill; Cornell University; Syracuse University; University of Pittsburgh; Northwestern University; and Penn State University. More information about the Grand Challenge Alliance and their efforts to solve the BBH problem can be found at the WWW address http://www.npac.syr.edu/projects/bh/ see also Ref. 13 for a review.

[18] L. S. Finn, A numerical approach to binary black hole coalescence, to appear in the conference proceedings of GR14, gr-qc/9603004.

[19] When we say that a source is visible to a distance \( D \), we mean that the rate of detection of events is roughly the same as the rate of occurrence of events within a sphere of radius \( D \). Some events within this sphere will be missed, and some event outside this sphere will be detected, due to beaming and orientation effects in the emission of gravitational waves. Rare, optimally oriented sources will be visible out to several times \( D \).

[20] E. E. Flanagan, in preparation.

[21] The factor of \( \sim 10 \) increase in event rate, which is quoted in Sec. I.E. of Ref. [4], comes from a combination of a factor \( \sim 40 \) increase in event rate for matched filtering (the cube of the ratio (merger SNR)/max(ringdown SNR, inspiral SNR), taken from Figs. 4, 5 and 6 of Ref. [1]), combined with the fact that noise-monitoring searches perform a factor \( \sim 4 \) worse in event rate than matched filtering searches (the estimate of \( R \) in Sec. VI.B of Ref. [1]). This estimate of \( R \) assumes that we know roughly the range of frequency bandwidths and temporal durations of merger waves from representative supercomputer simulations, in that the parameter \( N_{\text{bins}} \) is assumed to have the value 60. If, on the other hand, we are very ignorant as to the range of frequency bandwidths and durations of the merger waves, a more appropriate, conservative value of \( N_{\text{bins}} \) for the search process might be \( \sim 300 \), which would yield instead \( R \sim 10 \) and a corresponding overall gain in event rate of noise-monitoring searches for merger waves over matched filtering searches for inspiral/ringdown waves of \( \sim 4 \).

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[26] L. S. Finn and D. F. Chernoff, Phys. Rev. D 47, 2198 (1993).

[27] E. E. Flanagan and C. Cutler, Phys. Rev. D 49, 2658 (1994).

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\[
\langle n(t)^2 \rangle = \int_0^\infty S_h(f) \, df.
\]

("One-sided spectral noise density"). This determines the constant \( 4 \) appearing in Eq. [23].

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[33] B. J. Owen, Phys. Rev. D 53, 6749 (1996).

[34] C. Cutler, T. A. Apostolatos, L. Bildsten, L. S. Finn, E. E. Flanagan, D. Kennefick, D. M. Marković, A. Ori,
E. Poisson, G. J. Sussman, and K. S. Thorne, Phys. Rev. Lett. 70, 2984 (1993).

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[36] Families of template waveforms optimized for signal searches are in the early stages of development; see B. S. Sathyaprakash, Phys. Rev. D 40, R7111 (1994); B. S. Sathyaprakash and S. V. Dhurandhar, Phys. Rev. D 44 3819 (1991); 49, 1707 (1994); A. Królak, K. D. Kokkotas, and G. Schäfer, Proceedings of the 17th Texas Symposium on Relativistic Astrophysics, Ann. N.Y. Acad. Sci., 1995; S.D. Mohanty, “Hierarchical search strategy for the detection of gravitational waves from coalescing binaries: Extension to Post-Newtonian waveforms”, gr-qc/9703088; S. Droz and E. Poisson, gr-qc/9705034 and Ref. [35] above.

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[39] T. Damour, B. R. Iyer, and B. S. Sathyaprakash, Improved filters for gravitational waves from coalescing binaries, gr-qc/9708034.

[40] The accuracy with which the various waveform parameters will be measurable is currently known only to within factors of order ∼ 2. Statistical errors in maximum likelihood estimators have been calculated in the high signal-to-noise approximation [27] and [33], and also more accurately using Monte-Carlo simulations [51]. However, ultimately in the data analysis one should use estimators (for example Bayes estimators) which perform better than maximum likelihood estimators [27,33]. In addition, all analyses to date have neglected the modulation of the waveform due to the spin-induced precession of the orbital plane, an effect which will be important for BBHs with rapidly spinning constituents [51]. Thus the true parameter-measurement accuracies are still somewhat uncertain. The mass-measurement accuracies we have quoted were obtained as follows: We used the result from result from Ref. [11] (in the high SNR approximation) that ∆(η/η) = 1.5 for two 10 M⊙ black holes with S/N = 10, where η = μ/M. M is the total mass and μ the reduced mass of the system. This implies that ∆μ/μ = 0.6. Scaling to S/N ∼ 100 gives ∆μ/μ = 0.06. Finally, the 1σ confidence limits for the two masses in the equal mass case are approximately m0 ≤ m1 ≤ m0 f+ (Δμ/μ) and m0 f− (Δμ/μ) ≤ m2 ≤ m0, where m0 is the true mass and

\[ f_\pm(x) = (1 - x)^{-3/2} \left[ 1 \pm \sqrt{1 - (1 - x)^{3/2}} \right], \]

see Ref. [27]. The other accuracies quoted were taken from Refs. [27] and [51].

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[44] The fraction of events with ringdown SNR greater than some threshold ρring = ρring/ρring3, where ρring is the detection threshold for inspiral SNRs, and α is the ratio between the actual inspiral and ringdown SNRs. This implies that all events are detected via their inspiral signals and thus have inspiral SNRs larger than ρring. Equations (5.4) and (5.5) of Ref. [1] imply that α ∼ 30 for initial and advanced LIGO interferometers for M < 50 M⊙. Taking ρring = 1 and using the typically discussed inspiral SNR threshold of ρring = 6 [14], one obtains that the fraction is about 1%.

[45] When a complete set of theoretical templates is available so that matched filtering can be carried out, reconstructing the “best-fit” waveform is of course trivial: it is just the template which maximizes the signal to noise ratio.

[46] As we show in Sec. 3, in the lack of any information about the merger signal other than its expected bandwidth, this band-pass filtered detector output is in fact the maximum likelihood estimator of the merger waveform.

[47] P. Jaranowski and A. Krolak, Phys. Rev. D 49, 1723 (1994); P. Jaranowski, K. D. Kokkotas, A. Krolak, G. Tsegas, Class. Quant. Grav. 13, 1279 (1996).

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[50] R. Balasubramanian, B. S. Sathyaprakash, S. V. Dhurandhar, Phys. Rev. D 53 3033 (1996); 54 1860 (1996); R. Balasubramanian, S.V. Dhurandhar gr-qc/9708003; C. Cutler, unpublished.

[51] F. Echeverria, unpublished PhD thesis, California Institute of Technology, addendum to Ch. 2 (1993).

[52] For example, one can first truncate the signal in the time domain, then perform a finite Fourier transform, and then throw away the Fourier coefficients outside the frequency range of interest. Or, one can first band-pass filter the data stream, and then truncate in the time domain.

[53] One could define an alternative correlation coefficient by not using the noise inner product (2.14) but instead a different inner product. An example of such an alternative correlation coefficient is

\[ C' = \frac{\int_{-T/2}^{T/2} dt h(t) h_{\text{best-fit}}(t)}{\sqrt{\int_{-T/2}^{T/2} dt h(t)^2 dt} \sqrt{\int_{-T/2}^{T/2} dt h_{\text{best-fit}}(t)^2 dt}}, \]

When aliasing effects can be neglected, it can be shown using the method of appendix C that the expected value of (C’) of this correlation coefficient is approximately given
by Eq. (5.18), but with \( \rho_{\text{bin}} \) now given by
\[
\rho_{\text{bin}}^2 = \frac{2 \int_{\Delta f} df \tilde{h}(f)^2}{T \int_{\Delta f} df S_n(f)}.
\]

[54] D. Lai and A. G. Wiseman, Phys. Rev. D 54, 3958 (1996).

[55] L. E. Kidder, Phys. Rev. D 52, 821 (1995).

Although we discuss only Wiener optimal filtering, there are more sophisticated nonlinear filtering methods [49] that could be used to give better results for parameter measurements (see Ref. [40] above). The use of such methods would be more computationally intensive than linear filtering, but on the other hand, vast amounts of supercomputer time would have already been expended to generate merger templates.

[56] A. Ori and K. S. Thorne, in preparation.

[57] T. A. Apostolatos, C. Cutler, G. J. Sussman, and K. S. Thorne, Phys. Rev. D 49, 6274 (1994).

[58] Note that the regime \( \rho \gg N_{\text{bins}}^{1/4} \) coincides with the regime in which the signal is detectable using the noise-monitoring search method discussed in Refs. [12,20].

[59] J. Bekenstein, Phys. Rev. D 3436 (1993). See Eqs. (4.17) — (4.22) and Appendix C of Ref. [22].

[60] D. Nicholson and A. Vecchio, Bayesian Bounds on Parameter Estimation Accuracy fro Compact Coalescing Binary Gravitational Wave Signals, [gr-qc/9705064].

[61] K. S. Thorne, in 300 Years of Gravitation, ed. S. W. Hawking and W. Israel (Cambridge University Press, Cambridge, 1987), pp. 330-458.

[62] K. S. Thorne, in [98], ed. D. Blair, (Cambridge University Press, Cambridge, England, 1989).

[63] In fact, for detection purposes, the criterion (8.7) is a little more stringent than need be, for the following reason. When one attempts to detect the signal, one will compute the maximum overlap between the data stream and all possible templates, and thus the relevant quantity to calculate is
\[
\max_{\Delta \theta} \frac{(h(\theta) \mid h_T(\theta + \Delta \theta))}{\sqrt{(h_T(\theta + \Delta \theta) \mid h_T(\theta))}}.
\]
which is larger than
\[
\frac{(h(\theta) \mid h_T(\theta))}{\sqrt{(h_T(\theta) \mid h_T(\theta))}}.
\]
However, we neglect this effect here as it is not possible to calculate it in the absence of a model waveform \( h(\theta) \) for merger waves. For inspiral waves, relatively crude templates will suffice to detect the waves, and more accurate templates will be needed to measure parameters.

[64] In Eqs. (B3), (B4), (B5), (B6), and (B14) we omit factors of \( N - 2 \) dimensional solid angle elements \( d\Omega_{N-2} / C_{N-2} \) which integrate to one. Thus, the equals symbol in, for example, Eq. (B2) should be interpreted to mean that the PDFs on the left-hand-side and right-hand-side will produce identical expected values of functions that depend on \( \theta \) and \( h \) only.

[65] The PDF \( p^{(0)}(h) \) describes all one’s prior information about the signal \( h \), except for the information that the maximum expected duration is \( T \) and that the maximum expected bandwidth is \( \Delta f \); that information is already encoded in the choice of the space \( V \).

[66] In fact, for detection purposes, the criterion (8.7) is a little more stringent than need be, for the following reason. When one attempts to detect the signal, one will compute the maximum overlap between the data stream and all possible templates, and thus the relevant quantity to calculate is
\[
\max_{\Delta \theta} \frac{(h(\theta) \mid h_T(\theta + \Delta \theta))}{\sqrt{(h_T(\theta + \Delta \theta) \mid h_T(\theta))}}.
\]
which is larger than
\[
\frac{(h(\theta) \mid h_T(\theta))}{\sqrt{(h_T(\theta) \mid h_T(\theta))}}.
\]
However, we neglect this effect here as it is not possible to calculate it in the absence of a model waveform \( h(\theta) \) for merger waves. For inspiral waves, relatively crude templates will suffice to detect the waves, and more accurate templates will be needed to measure parameters.

[67] In fact, this result is independent of the specific form of the norm on the space of pairs of waveforms \{\( h_1(t) \), \( h_2(t) \)\} chosen in Eq. (A30).

[68] In Eqs. (B3), (B4), (B5), (B6), and (B14) we omit factors of \( N - 2 \) dimensional solid angle elements \( d\Omega_{N-2} / C_{N-2} \) which integrate to one. Thus, the equals symbol in, for example, Eq. (B2) should be interpreted to mean that the PDFs on the left-hand-side and right-hand-side will produce identical expected values of functions that depend on \( \theta \) and \( h \) only.