TWO-DIMENSIONAL ANISOTROPIC KPZ GROWTH AND LIMIT SHAPES
ALEXEI BORODIN AND FABIO TONINELLI

ABSTRACT. A series of recent works focused on two-dimensional interface growth models in the so-called Anisotropic KPZ (AKPZ) universality class, that have a large-scale behavior similar to that of the Edwards-Wilkinson equation. In agreement with the scenario conjectured by D. Wolf [35], in all known AKPZ examples the function \( v(\rho) \) giving the growth velocity as a function of the slope \( \rho \) has a Hessian with negative determinant ("AKPZ signature"). While up to now negativity was verified model by model via explicit computations, in this work we show that it actually has a simple geometric origin in the fact that the hydrodynamic PDEs associated to these non-equilibrium growth models preserves the Euler-Lagrange equations determining the macroscopic shapes of certain equilibrium two-dimensional interface models. In the case of growth processes defined via dynamics of dimer models on planar lattices, we further prove that the preservation of the Euler-Lagrange equations is equivalent to harmonicity of \( v \) with respect to a natural complex structure.

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1. INTRODUCTION

In this work we consider stochastic interface growth models, i.e., irreversible Markov dynamics of \( d \)-dimensional height functions \( \{h_x\}_{x \in \mathbb{Z}^d} \). For reasons that we explain below, we are especially interested in the case \( d = 2 \).

We refer the reader, e.g., to [1] for physical motivations: propagation of fronts in combustion, crystal growth, etc. There are various quantities of physical interest related to the large-scale evolution of such interfaces. An obvious one is the asymptotic speed of growth \( v \), i.e., the average increase of interface height in unit time, for large times. Actually, the speed of growth is in general a slope-dependent quantity \( v = v(\rho) \), where \( \rho \in \mathbb{R}^d \) is the local average interface slope. Other natural quantities are fluctuation exponents. In the long-time limit \( t \to \infty \), the law of the height gradients \( \{h_x(t) - h_y(t)\}_{x,y \in \mathbb{Z}^d} \) of an initially flat profile of slope \( \rho \) is expected to converge to a (non-reversible) stationary state \( \pi_{\rho} \). The standard deviation of the height difference \( (h_x - h_y) \) at stationarity is expected to behave like \( \text{const.} \times (1 + |x - y|^\alpha) \) for large \( |x - y| \), with \( \alpha \) the roughness exponent.
Similarly, the growth exponent \( \beta \) is defined\(^1\) so that the standard deviation of \( (h_x(t) - h_x(0)) \) grows like \( t^\beta \) for large \( t \).

The two-dimensional case \( d = 2 \) is particularly interesting: it is expected that there is a rich interplay between the convexity properties of \( v \) and the triviality or not of the critical exponents, where triviality means that they coincide with those of the linear Edwards-Wilkinson (EW) equation, recalled below. (Let us recall that \( \alpha_{EW}(d) = (2 - d)/2 \), \( \beta_{EW}(d) = (2 - d)/4 \) and that for \( d = 2 \) fluctuation growth in space of time for the EW equation is logarithmic.) The following is expected:

**Conjecture 1.1.** If the Hessian matrix \( D^2 v(\rho) \) of \( v(\rho) \) has two eigenvalues of the same sign, i.e., if \( \det(D^2 v(\rho)) > 0 \), then \( \alpha \neq \alpha_{EW}(2) = 0 \), \( \beta \neq \beta_{EW} = 0 \), while equalities hold if \( \det(D^2 v(\rho)) \leq 0 \).

In the former case the growth model is said to belong to the Isotropic KPZ class and to the Anisotropic KPZ (AKPZ) class in the latter. This situation is in contrast with that of \( d = 1 \) growth models where it is believed\(^2\), and mathematically proven in several concrete examples\(^3\), that \( \beta = 1/3 \neq \beta_{EW}(1) = 1/4 \) as soon as \( v'(\rho) \neq 0 \).

Conjecture\(^4\) originated from the work\(^5\) by D. Wolf, who studied the large-scale behavior of the two-dimensional KPZ equation

\[
\partial_t h(x, t) = \Delta h(x, t) + \lambda (\nabla h(x, t), H \nabla h(x, t)) + \dot{W}(x, t), \quad x \in \mathbb{R}^2, \quad (1.1)
\]

via (perturbative in \( \lambda \)) Renormalization Group (RG) arguments. In this equation, \( \dot{W} \) is a (suitably regularized) space-time white noise, \( \lambda \) is a real parameter tuning the strength of the non-linearity, and \( H \) is a real symmetric \( 2 \times 2 \) matrix. This equation reduces to the EW equation when \( \lambda = 0 \), and it is believed to capture the large-scale behavior of the growth model if \( H \) is chosen to be \( D^2 v(\rho) \) as above. The outcome of Wolf’s work is that for \( \det(H) > 0 \) the non-linearity is relevant (i.e., the large-scale properties of the solution of (1.1) differ from those of the EW equation as soon as \( \lambda \neq 0 \)), while if \( \det(D^2 v(\rho)) \leq 0 \) and \( \lambda \) is small enough then the non-linearity is irrelevant.

Wolf’s predicted relation between the sign of \( \det(D^2 v(\rho)) \) and the non-triviality of the exponents \( \alpha, \beta \) has been successfully tested numerically both on the KPZ equation itself\(^6\) and on microscopic growth models\(^7\). In particular, for models in the Isotropic KPZ class, exponents \( \alpha, \beta \) are known with large numerical precision and they seem to be universal\(^8\). There are also rigorous results about some specific AKPZ growth models, that we briefly review in Section \ref{sec:rigorous-results}. Apart from Wolf’s RG computations, however, we are not aware of any sound theoretical, let alone rigorous, argument supporting Conjecture\(^9\)\(^\dagger\). Even for those AKPZ growth models of Section \ref{sec:rigorous-results} for which logarithmic fluctuation growth can be proved, the presently existing proofs of \( \det(D^2 v(\rho)) \leq 0 \) are based on explicit computations that give no intuition on the underlying mechanism. The goal of the present work is to shed new light on this issue.

\(^1\)More precisely, one should look at the standard deviation of \( h_x(t) - h_x(0) \) with \( x(t) = x - Dv(\rho)t \), where \( Dv(\rho) \in \mathbb{R}^d \) is the differential of \( v(\cdot) \) computed at \( \rho \). In other words, to correctly define the growth exponent one has to choose a reference frame moving along the line \( (x(t), t) \) in space-time, that is the characteristic line of the PDE (1.1).

\(^2\)In the Isotropic KPZ class, the triviality or not of the critical exponents, where triviality means that they coincide with those of the linear Edwards-Wilkinson (EW) equation, recalled below. (Let us recall that \( \alpha_{EW}(d) = (2 - d)/2 \), \( \beta_{EW}(d) = (2 - d)/4 \) and that for \( d = 2 \) fluctuation growth in space of time for the EW equation is logarithmic.) The following is expected:

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Before coming to that, let us mention that there are various growth processes for which one can prove, via a sub-additivity argument \cite{31}, that the speed $v$ is a convex function of $\rho$. This includes the cube-stacking model of \cite{32}, that is equivalent to Glauber dynamics of interfaces of the three-dimensional Ising model at zero temperature and positive magnetic field. Convexity of $v$ makes such models natural candidates for representatives of the Isotropic KPZ universality class. However, apart from refined numerics, nothing is known rigorously on their stationary states and their fluctuation exponents. As we argue at the end of Section 1.2, we have reasons to believe that their stationary states $\pi_\rho$ are not of Gibbs type.

1.1. Previous results on AKPZ models. Here we briefly review some mathematical results on AKPZ growth models; see also \cite{34} for a recent review. The first result we are aware of in this direction is the study \cite{28} of the Gates-Westcott evolution \cite{18}, where space is continuous in one direction and discrete in the other. This growth model can be seen as a collection of mutually interacting, one-dimensional PolyNuclear Growth models. The authors of \cite{28} observed that evolution preserves a family of translation-invariant Gibbs measures that can be described in the free-fermionic language and, therefore, have determinantal correlations. This allowed them to prove that stationary fluctuations are logarithmic (whence $\alpha = 0$) and to compute the speed $v(\rho)$, which by direct inspection turns out to satisfy $\det(D^2v(\rho)) < 0$.

The other AKPZ growth models we mention below have the common feature of being defined in terms of dimer coverings of bipartite planar lattices $G$ (in particular, the honeycomb lattice and the square grid) or, equivalently, in terms of random tilings of the plane. It is a well-known fact \cite{23} that dimer coverings of $G$ are in bijection with integer-valued height functions defined on the dual graph $G^*$. See, e.g., Fig. 1 for lozenge tilings.

A general mechanism for obtaining AKPZ two-dimensional growth processes was suggested in \cite{7}, where the case of a particular continuous-time dynamics on the lozenge tilings of a half-plane was considered in detail as the
main example (see also [8]). In this case, logarithmic growth of fluctuations in time and space was proven (whence $\alpha = \beta = 0$), the speed was computed, and the AKPZ signature $\det(D^2v(\rho)) < 0$ was verified. The growth models generated by the procedure of [7] have the property of preserving families of Gibbs measures in certain unbounded regions of the plane. One can also prove that full-plane translation invariant Gibbs measures $\pi_\rho$ are stationary [33] (this requires extra non-trivial work because these dynamics involve unbounded particle jumps, which might cause the process to be ill-defined). A particular modification designed to preserve Gibbs measures inside bounded regions (hexagonal regions for lozenge tilings) was also suggested at the same time [10]. Other (discrete time) dynamics for lozenge tilings that fit into the formalism of [7] were considered in [3]; their speed of growth $v(\rho)$ was computed there, and one can check that $\det(D^2v(\rho)) < 0$. Let us mention that an exposition of the general construction of [7] in terms of the formalism of Schur processes is available in [2] and another example for dimers on a square-hexagon lattice can be found in [9].

Another two-dimensional dynamics that recently attracted attention in the context of AKPZ growth model is the domino shuffling algorithm. While this evolution fits into the general framework of [7], it was actually introduced much earlier in [16, 29], as an algorithm that perfectly samples domino tilings of special domains of the plane called “Aztec diamonds”. However, the algorithm can be defined in more general domains: in particular, in the whole plane. Thanks to the above-mentioned mapping between a domino tiling and its discrete height function, the shuffling algorithm can be viewed as a two-dimensional growth model. In the whole plane, it preserves translation-invariant Gibbs measures of domino tilings. In the recent work [11], the shuffling algorithm was studied from the point of view of interface growth, and it was proven to belong to the AKPZ class, both in terms of vanishing of the growth exponents and in the sense that $\det(D^2v(\rho)) < 0$. An interesting fact observed in [11] is that, if the underlying dimer model is given non-uniform but periodic weights on $\mathbb{Z}^2$, the speed function $v(\rho)$ can be singular (non-differentiable) at isolated values of the slope $\rho$ (the case of weights with period 2 in both space directions is worked out explicitly in [11]). At those slopes, growth exponents $\alpha, \beta$ are still zero but the fluctuation variance is bounded in time and space.

We conclude this (incomplete) list of results by mentioning two more examples. The first one is an AKPZ growth model based on domino tilings in the plane, defined in Section 3.1 of [33], that at first sight does not seem to fit into the general formalism of [7]. The proof of logarithmic growth fluctuations, implying $\alpha = \beta = 0$, as well as the computation of the speed of growth and a direct verification that $\det(D^2v(\rho)) < 0$ can be found in [12]. The second one is the so-called $q$-Whittaker process, introduced in [4] in the wider context of Macdonald processes. The $q$-Whittaker dynamics reduces to the growth process of [7,8] in the limit $q \to 0$ and to a Gaussian growth model (that has the same asymptotic large-scale behavior as the EW equation) in the $q \to 1$ limit [5,6]. For every intermediate value of $q$, it preserves certain Gibbs measures, both on the torus [14] and in certain
unbounded domains of the plane. Logarithmic growth of fluctuations is not proven but conjectured.

For some of the models mentioned in this section, a hydrodynamic limit has been proven: the height profile, rescaled as $\epsilon^{-1} h_{\epsilon^{-1} x}$, converges as $\epsilon \to 0$ to a non-random limit profile $\bar{h}(x, t)$, that solves a first-order non-linear PDE of Hamilton-Jacobi type. See for instance [7, Th. 1.2] and [27] for a lozenge tiling dynamics, and [36] for the domino shuffling algorithm. Most likely, a similar convergence holds for the other models as well.

1.2. Preservation of Gibbs measures and of equilibrium shapes. A feature common to most (and possibly all) of the models cited in the previous section is that they admit certain translation-invariant Gibbs measures as stationary, but not reversible, measures for the height gradients (the law of the height itself is not stationary, since there is a non-zero average growth). In almost all of these models, Gibbs states have a determinantal (or free-fermion) structure, but this is not a general feature of AKPZ models: the $q$-Whittaker process is not free-fermionic, for instance. The potential associated to these Gibbs measures is local and completely explicit: in several cases, for example for the lozenge dynamics of [7,8], the Gibbs property means simply that, conditionally on the interface configuration $h_{\Lambda_c}$ outside any finite domain $\Lambda$ of the lattice, the configuration inside $\Lambda$ is uniformly distributed among all possible configurations compatible with $h_{\Lambda_c}$. Actually, the general construction of [7] guarantees that these growth models preserve also non-translation invariant Gibbs measures, in the sense that if the initial condition is one such measure, then the law of the interface at a later time is another such measure (in general, a different one). See [7, Fig. 1.3] for a configuration sampled from one of these measure for lozenge tilings. Another example of preservation of non-translation-invariant Gibbs measure is provided by the domino shuffling algorithm: if at time 0 the configuration is sampled from the Gibbs distribution in an Aztec diamond of size $n \times n$, at time $T$ it is distributed according to the Gibbs distribution in an Aztec diamond of size $(n + T) \times (n + T)$.

Preservation of Gibbs measures has a direct consequence on the hydrodynamic PDE of these growth models. To explain this, let us forget dynamics for a moment. According to the general principle of statistical mechanics, on large scales (i.e., rescaling height as $\epsilon^{-1} h_{\epsilon^{-1} x}$ and letting $\epsilon \to 0$), the height function sampled from a Gibbs measure concentrates around a non-random profile $\bar{h}$, the equilibrium shape. This profile is determined by minimizing a surface tension functional $\int \sigma(\nabla \bar{h}) dx$, with suitable boundary conditions. More explicitly, the function $\bar{h}(x)$ satisfies the Euler-Lagrange equation associated to such functional; due to convexity (see, e.g., [30, Ch. 4]) of the surface tension $\sigma(\cdot)$, this is a second-order PDE of elliptic type. Back to the growth models: the fact that the above AKPZ evolutions preserve Gibbs measures translates into the fact that their hydrodynamic PDE preserves the Euler-Lagrange equation. Namely, if the initial profile satisfies the Euler-Lagrange equation, then so does it at later times $t$.

The main contribution of the present work, Theorem 2.1, is an elementary argument that shows that, roughly speaking, if the hydrodynamic PDE of a two-dimensional growth model preserves the Euler-Lagrange equation
of some equilibrium interface model, then the speed of growth function \( v \) satisfies \( \det(D^2v(\rho)) \leq 0 \). In the particular case where the growth model is defined in terms of lozenge tilings, or, more generally, in terms of a planar dimer model we show a stronger result (Theorems 3.2 and 3.3): the hydrodynamic PDE preserves the Euler-Lagrange equation of the dimer model if and only if the speed \( v \) is a harmonic function with respect to a certain natural complex-valued variable defined in terms of the interface slope [22,24].

We conclude this introduction with a couple of observations. First, we emphasize that Theorem 2.1 implies that the hydrodynamic PDE of Isotropic KPZ models, like the cube-stacking dynamics, cannot preserve the Euler-Lagrange equation of an equilibrium interface model with convex surface tension. This suggests the possibility that the stationary states of such growth processes are not of Gibbs type. The occurrence of non-Gibbs stationary states in irreversible interacting particle systems is believed to be a generic feature [26] but it has been proven only in very few examples (cf. Section 4.5.4 of [17] and references therein). Secondly, our argument might be instrumental in proving \( \det(D^2(\rho)) \leq 0 \) for some AKPZ models for which the explicit computation of \( v \) is not possible. We think, in particular, of the above-mentioned \( q \)-Whittaker process, whose invariant (Gibbs) measures are not of determinantal type and do not allow for explicit computations. A last remark is that, while in equilibrium statistical mechanics it is usually possible to guess the universality class a model belongs to from the symmetries of the Hamiltonian, for growth models it is not clear how to guess the convexity properties of \( v(\cdot) \) from the definition of the generator.

### 2. Euler-Lagrange equation and AKPZ signature

From this point on, we denote macroscopic profiles (time-dependent as well as time-independent) by \( h \) instead of \( \bar{h} \), since the microscopic height \( h_x \) will not appear in any of the subsequent arguments.

Let \( \sigma : \mathbb{R}^2 \to \mathbb{R} \) be the surface tension function of an equilibrium two-dimensional interface model, with \( \sigma(\rho) \) denoting the surface tension at slope \( \rho \). We will denote by \( \sigma_{i,j} \) the derivative of \( \sigma \) w.r.t. components \( i, j \) of \( \rho \), and by \( \Sigma(\rho) \) the \( 2 \times 2 \) matrix with

\[
\Sigma(\rho)_{i,j} := \sigma_{i,j}(\rho).
\]  

(2.1)

Since \( \sigma \) is convex, \( \Sigma \) is positive definite (possibly not strictly). In many natural examples (notably the dimer model discussed in Section 3), the interface configurations are Lipschitz, and the allowed slopes belong to some convex set \( N \) that in the dimer model setting is called “Newton Polygon”. In this case, \( \sigma \) equals \( +\infty \) outside \( N \) and \( 0 \) on the boundary of \( N \).

An “equilibrium shape” is a height profile \( h : \mathbb{R}^2 \to \mathbb{R} \) that locally minimizes the surface tension functional

\[
F(\varphi) = \int_{\mathbb{R}^2} \sigma(\nabla \varphi) dx.
\]  

(2.2)

(One can also consider the optimization problem restricted to a sub-domain \( U \) of the plane, in which case the boundary height \( h|_{\partial U} \) on the boundary of \( U \) is fixed.) More precisely, for every finite subset \( V \subset \mathbb{R}^2 \) whose boundary
is a smooth simple curve, the minimum of the functional
\[
\int_V \sigma(\nabla \varphi) \, dx
\]  
(2.3)
over functions on \(V\) with boundary height \(\varphi|_{\partial V} \equiv h|_{\partial V}\) is realized by the restriction of \(h\) itself to \(V\). At every point where \(h\) is \(C^2\), it satisfies the Euler-Lagrange equation
\[
L[h](x) := \sum_{i,j=1}^2 \sigma_{i,j}(\nabla h(x)) \partial_{x_i x_j}^2 h(x) = 0,
\]  
(2.4)
that is an elliptic PDE. While for \(d = 1\) solutions of the Euler-Lagrange equation are affine, this is not necessarily the case in dimension \(d = 2\) (and higher).

Now consider a two-dimensional growth model and assume it satisfies a hydrodynamic limit with speed function \(v\). Namely, its random height profile, once rescaled as explained at the end of Section 1.1, converges to the solution of the deterministic Hamilton-Jacobi equation
\[
\begin{cases}
\partial_t h(x,t) = v(\nabla h(x,t)) \\
h(x,0) = h_0(x)
\end{cases}
\]  
(2.5)
where \(v(\cdot)\) is some function defined on the set \(N\) of allowed slopes. Non-linear Hamilton-Jacobi equations like (2.5) are well-known to develop singularities (discontinuities of \(\nabla h\)) in finite time, and there is not a unique way of defining a weak solution after the time of appearance of singularities. However, the physically relevant solution defined for all times is the so-called viscosity solution \([15]\), that can be obtained by adding \(\nu \Delta h(x,t)\) to \(v(\nabla h(x,t))\) and then sending \(\nu \to 0^+\). We let \(h(t)\) denote the viscosity solution of (2.5) at time \(t\). We assume henceforth that \(v(\cdot) : N \to \mathbb{R}\) is smooth enough so that the viscosity solution exists and is unique.

Our first result says, roughly speaking, that if the PDE (2.5) preserves the Euler-Lagrange equation (2.4), then the determinant of the Hessian of \(v\) cannot be positive. More precisely:

**Theorem 2.1.** Let \(\rho\) be a slope in the interior of \(N\) where both \(v(\cdot)\) and \(\sigma(\cdot)\) are \(C^2\) differentiable and where \(\Sigma(\rho)\) in (2.1) is strictly positive definite. Assume that there exists an equilibrium shape \(h\) and a point \(x \in \mathbb{R}^2\) where \(h(\cdot)\) is \(C^2\) differentiable, and such that:
- \(\nabla h(x) = \rho\) and the Hessian of \(h\) at \(x\) is not zero;
- one has
\[
L[h(t)](y) = 0
\]  
(2.6)
for \(y\) in a neighborhood of \(x\) and \(t\) sufficiently small.

Then, \(\det(D^2v(\rho)) \leq 0\).

**Proof.** For sufficiently small times the PDE (2.5) can be solved by the method of characteristics, and the solution is \(C^2\) in the neighborhood of \(x\). Take \(x = x(t)\) that runs along the characteristic line started at \(x\), i.e.,
\[
\frac{dx(t)}{dt} = -Dv(\nabla h(x(t),t)), \quad x(0) = x,
\]  
(2.7)
with $Dv$ the gradient of $v$. Recall that $\nabla h$ is constant along the characteristic lines, so that $dx/dt = -Dv(\nabla h(x))$ is constant. Call
\begin{equation}
R(x) := \frac{d}{dt}C[h(t)|(x(t))]\bigg|_{t=0},
\end{equation}
that equals zero by the assumption of the theorem.

Since $\nabla h$ is constant along characteristics, Eq. (2.8) gives
\begin{equation}
R(x) = \sum_{i,j=1}^{2} \sigma_{i,j}(\nabla h(x)) \frac{d}{dt}\partial_{x_{i}x_{j}}h(x(t),t)\bigg|_{t=0}.
\end{equation}

Using the chain rule for derivatives together with the definition (2.7) of characteristic lines we get
\begin{equation}
\frac{d}{dt}\partial_{x_{i}x_{j}}h(x(t),t)\bigg|_{t=0} = \sum_{\ell,k=1}^{2} D_{\ell,k}^{2}v(\nabla h(x))\partial_{x_{i}x_{j}}h(x(t),t)\partial_{x_{k}x_{k}}h(x(t),t)\bigg|_{t=0},
\end{equation}
where
\begin{equation}
D_{\ell,k}^{2}v(\rho) := \frac{\partial^{2}}{\partial \rho_{\ell}\partial \rho_{k}}v(\rho).
\end{equation}

Altogether, we have obtained
\begin{equation}
0 = R(x) = \sum_{\ell,k=1}^{2} [D_{\ell,k}^{2}v(\nabla h(x))]A_{\ell,k}(x)
\end{equation}
with
\begin{equation}
A_{\ell,k}(x) = \sum_{i,j=1}^{2} \sigma_{i,j}(\nabla h(x)) \partial_{x_{i}x_{j}}h(x) \partial_{x_{k}x_{k}}^{2}h(x).
\end{equation}

For lightness of notation, call $M := D^{2}v(\rho)$ the Hessian matrix of $v$ computed at $\rho$ and write $\sigma_{i,j}$ instead of $\sigma_{i,j}(\nabla h(x))$. Assume by contradiction that $\det(M) > 0$ and that $M$ is positive definite (the argument works analogously if $\det(M) > 0$ and $M$ is negative definite), so that we can write it as $M = \sqrt{M}\sqrt{M}$. Also call $u_{k}, k = 1, 2$, the vector $(\partial_{x_{1}x_{k}}^{2}h(x), \partial_{x_{2}x_{k}}^{2}h(x))$. We rewrite (2.11) as
\begin{equation}
0 = \sigma_{1,1}\|\sqrt{M}u_{1}\|^{2} + \sigma_{2,2}\|\sqrt{M}u_{2}\|^{2} + 2\sigma_{1,2}(\sqrt{M}u_{1}, \sqrt{M}u_{2}).
\end{equation}

Since the matrix $\Sigma$ is strictly positive definite, 
\begin{equation}
|\sigma_{1,2}| < \sqrt{\sigma_{1,1}\sigma_{2,2}} \quad \text{and} \quad \sigma_{1,1}, \sigma_{2,2} > 0.
\end{equation}

By the assumption that the Hessian of $h$ at $x$ is non-zero, either $u_{1}$ or $u_{2}$ is non-zero. If one of them, say $u_{2}$, is zero, then one gets
\begin{equation}
\sigma_{1,1}\|\sqrt{M}u_{1}\|^{2} = 0
\end{equation}
so that $M$ is not strictly positive definite, which is a contradiction. Also, if either $\sqrt{M}u_{1}$ or $\sqrt{M}u_{2}$ is zero, then again $\det(M) = 0$ which contradicts the hypothesis $\det(M) > 0$. From now on we can therefore assume that $\sqrt{M}u_{1}, \sqrt{M}u_{2} \neq 0$. Also, $\sqrt{M}u_{1}$ cannot be orthogonal to $\sqrt{M}u_{2}$; otherwise
\begin{equation}
0 = \sigma_{1,1}\|\sqrt{M}u_{1}\|^{2} + \sigma_{2,2}\|\sqrt{M}u_{2}\|^{2},
\end{equation}

which is not possible since \( \sigma_{1,1} > 0, \sigma_{2,2} > 0 \). In the remaining case where 
\((\sqrt{M}u_1, \sqrt{M}u_2) \neq 0, \)
\[
0 \geq \sigma_{1,1}\|\sqrt{M}u_1\|^2 + \sigma_{2,2}\|\sqrt{M}u_2\|^2 - 2|\sigma_{1,2}|(\sqrt{M}u_1, \sqrt{M}u_2)| \\
> \sigma_{1,1}\|\sqrt{M}u_1\|^2 + \sigma_{2,2}\|\sqrt{M}u_2\|^2 \\
- 2\sqrt{\sigma_{1,1}\|\sqrt{M}u_1\|^2}\sqrt{\sigma_{2,2}\|\sqrt{M}u_2\|^2} \geq 0
\]
that is a contradiction. Altogether, it is not possible that \( \det(M) > 0 \). \( \square \)

3. Dimer model: Burgers equation and harmonicity of \( v \)

In this section, we assume that \( \sigma \) is the surface tension of a dimer model on a bipartite, planar, periodic graph. We refer, e.g., to \( [23] \) for an introduction to dimer models and to \( [25] \) for the definition of their ergodic Gibbs measures; we recall here a minimum of basic facts. Given a planar, bipartite, periodic weighted graph \( G = (V,E) \) (positive weights are associated to edges), dimer configurations are perfect matchings of \( G \), and a canonical construction allows to associate to a dimer configuration an integer-valued height function on faces of \( G \) (the map is bijective up to a global additive constant for the height). The possible slopes of the height function belong to a convex polygon \( N \), called “Newton polygon”, whose vertices have integer coordinates. For every slope \( \rho \) in the interior of \( N \) there exists a unique translation invariant, ergodic Gibbs measure \( \pi_\rho \) on dimer coverings, with average slope \( \rho \). Conditionally on the dimer configuration outside any finite sub-graph \( \tilde{G} \), \( \pi_\rho \) assigns to any admissible dimer configuration in \( \tilde{G} \) a probability proportional to the product of weights of edges occupied by dimers. Slopes \( \rho \) that are in the interior of \( N \) and whose coordinates are not both integer are called “liquid slopes”. In this case, \( \pi_\rho \) is called a “liquid phase” and it is known that dimer correlations decay polynomially and the height field behaves like a log-correlated massless Gaussian field on large scales. If \( \rho \) is in the interior of \( N \) but has integer coordinates, then for generic choice of the edge weights the measure \( \pi_\rho \) is a “gaseous” or “smooth” phase, with exponentially decaying correlations and \( O(1) \) height fluctuations\(^2\). A crucial object for the dimer model is the so-called characteristic polynomial \( P(z,w) \): this is a Laurent polynomial in \( z, w \in \mathbb{C} \), that is associated to the weighted graph \( G \). For instance, the surface tension \( \sigma(\cdot) \) is obtained as the Legendre transform with respect of \( B = (B_1, B_2) \in \mathbb{R}^2 \) of

\[
\frac{1}{(2\pi i)^2} \int_{|z| = e^{B_1}} \frac{dz}{z} \int_{|w| = e^{B_2}} \frac{dw}{w} \log P(z, w).
\]

For the dimer model, it is known that the Euler-Lagrange equations \( (2.4) \) for equilibrium shapes \( h \) can be expressed via a first-order PDE, that was called complex Burgers equation in \( [24] \). Namely, for \( x = (x_1, x_2) \), define

\(^2\)For special edge weights, the measure \( \pi_\rho \) can be liquid (with polynomially decaying correlations) even for integer-valued slopes \([25]\).
non-zero complex numbers $z = z(x), w = w(x)$ via the relation

$$P(z, w) = 0, \quad \nabla h(x) = \frac{1}{\pi i}(-\arg w, \arg z).$$

(3.2)

According to [24, Theorem 1], at every point $x$ in the liquid region (i.e., such that in the neighborhood of $x$, $h$ is $C^1$ with $\nabla h$ a liquid slope), the Euler-Lagrange equation (2.4) is equivalent to the “complex Burgers equation”

$$\frac{z x_1}{z} + \frac{w x_2}{w} = 0,$$

(3.3)

where $f_{x_i}$ denotes the derivative of $f$ w.r.t. $x_i$. Note that relation (3.2) does not really fix $z, w$ because the argument is a multi-valued function; as explained in [24], the statement is that there exists a branch of the argument for which equality (3.2) is satisfied.

Take a point $x_0$ in the liquid region, where the gradient of $h$ is

$$\rho = \nabla h(x_0) = \frac{1}{\pi}(-\arg w_0, \arg z_0)$$

(3.4)

for some $z_0, w_0$ such that $P(z_0, w_0) = 0$. Locally around $(z_0, w_0)$ we can expand $P(z, w)$ as

$$P(z, w) = p_1(z - z_0) + p_2(w - w_0) + O(\| (z, w) - (z_0, w_0) \|^2)$$

(3.5)

with $p_1 = \partial_z P(z_0, w_0), p_2 = \partial_w P(z_0, w_0)$. As discussed in [24, Sec. 2.3], the ratio $p_1/p_2$ is neither zero nor infinite. Then, by the implicit function theorem we can locally write $z$ as a differentiable function $z(w)$ or, conversely, write $w$ as a differentiable function $w(z)$.

An identity that will be important in the following is that

$$\frac{\partial \log |z|}{\partial \nabla_{x_2} h} = \frac{\partial \log |w|}{\partial \nabla_{x_1} h},$$

(3.6)

that follows from [24, Eq. (12)]. We have also

$$\frac{\partial \log |z|}{\partial \nabla_{x_1} h} = \frac{\partial \log z}{\partial \nabla_{x_1} h}, \quad \frac{\partial \log |z|}{\partial \nabla_{x_2} h} = \frac{\partial \log z}{\partial \nabla_{x_2} h} - i\pi,$$

$$\frac{\partial \log |w|}{\partial \nabla_{x_1} h} = \frac{\partial \log w}{\partial \nabla_{x_1} h} + i\pi, \quad \frac{\partial \log |w|}{\partial \nabla_{x_2} h} = \frac{\partial \log w}{\partial \nabla_{x_2} h},$$

(3.7)

that follow from (3.2).

Relations $w = w(z)$ and (3.2) give a bijection between $z$ in a neighborhood of $z_0$ and the interface slope $\nabla h = (\nabla_{x_1} h, \nabla_{x_2} h)$ in a neighborhood of $\nabla h(x_0)$. Therefore, we can write locally (i.e., for $\nabla h$ close to $\nabla h(x_0)$) the speed $v(\nabla h)$ as some function $f(z(\nabla h))$.

Remark 3.1. Globally, the mapping from $z$ to the slope is multi-valued, since for a given $z$ there may be many $w$ satisfying $P(z, w) = 0$. Notable exceptions are the dimer model on the honeycomb lattice and the square grid with translation-invariant weights. We will briefly come back to these special cases in Section 3.4. See instead [17] for a growth model in a case where the solution of $P(z, w) = 0$ has several branches.

---

3Our $z, w$ correspond to the complex conjugates of the similarly denoted complex numbers in [24].
Let $h = h(x,t)$ be the viscosity solution of the PDE (2.5). For $t$ small and $x$ around $x_0$ the solution is smooth, and we let $z = z(x,t)$ be defined by (3.2) with $h$ replaced by $h(t)$, with a choice of the branch of the argument so that $z(x_0,0) = 0$. Define also

$$\Delta = zw_{x_2} + wz_{x_1},$$

so that $\Delta = 0$ iff the complex Burgers equation (3.3) holds.

**Theorem 3.2.** Under the same assumptions as for Theorem 2.1 assume in addition that $\sigma$ is the surface tension function of a dimer model and $\rho$ is a slope of the liquid phase. Then, $f(\cdot)$ is a harmonic function at $z_0$, i.e.,

$$\frac{\partial^2 f(z)}{\partial \Re z^2} + \frac{\partial^2 f(z)}{\partial \Im z^2} = 0$$

for $z = z_0$, where $z_0$ is related to $\rho$ as in (3.4).

Conversely, we have:

**Theorem 3.3.** Let $h$ be an equilibrium shape and $h(t)$ be the solution of (2.5) with initial condition $h$ and $v(\cdot) = f(z(\cdot))$. Assume that $\Delta$ defined in (3.8) is differentiable in time and space for $x \in A, t \in [0,T]$, for some neighborhood $A$ of $x_0$ and some $T > 0$. If $f(z)$ is a harmonic function of $z$, then $h(t)$ satisfies the complex Burgers equation for $(x,t) \in A \times [0,T]$. The restriction to small times is simply due to the fact that at large times the solution $h(t)$ might not be pointwise differentiable in space and time, in which case $\Delta$ is not well defined.

**Remark 3.4.** Together with Theorem 2.1 Theorem 3.2 implies the following: if $v(\nabla h) = f(z(\nabla h))$ with $f$ harmonic at every $z$ corresponding to a liquid slope, then $\det(D^2 v) \leq 0$. It should be possible to obtain this also directly by calculus, but we find that the path going through Theorem 2.1 is more illuminating.

**Proof of Theorems 3.2 and 3.3** We start by writing

$$\partial_t \log |z| = \partial_t \log |z| + i \partial_t \arg(z) = \partial_t \log |z| + i \pi \partial_{x_2} f,$$

where we used $\arg(z) = \nabla_{x_2} h$ and $\partial_t h = f(\nabla h)$. Also,

$$\partial_t \log |z| = \frac{\partial \log |z|}{\partial \nabla_{x_1} h} \partial_{x_1} f + \frac{\partial \log |z|}{\partial \nabla_{x_2} h} \partial_{x_2} f$$

$$= \frac{\partial \log |z|}{\partial \nabla_{x_1} h} \frac{\partial |z|}{\partial h} + \frac{\partial \log |w|}{\partial \nabla_{x_2} h} \partial_{x_2} f$$

$$= \frac{\partial f}{\partial \Re z} \Re \left[ z_{x_1} \frac{\partial \log |z|}{\partial \nabla_{x_1} h} + z_{x_2} \frac{\partial \log |w|}{\partial \nabla_{x_2} h} \right] + \frac{\partial f}{\partial \Im z} \Im \left[ z_{x_1} \frac{\partial \log |z|}{\partial \nabla_{x_1} h} + z_{x_2} \frac{\partial \log |w|}{\partial \nabla_{x_2} h} \right]$$

$$= \frac{\partial f}{\partial \Re z} \Re \left[ z_{x_1} \frac{\partial \log z}{\partial \nabla_{x_1} h} + z_{x_2} \left( \pi + \frac{\partial \log w}{\partial \nabla_{x_1} h} \right) \right]$$

$$+ \frac{\partial f}{\partial \Im z} \Im \left[ z_{x_1} \frac{\partial \log z}{\partial \nabla_{x_1} h} + z_{x_2} \left( \pi + \frac{\partial \log w}{\partial \nabla_{x_1} h} \right) \right].$$
In the second step we used (3.6), in the third the identities
\[ \partial_{x_1} f(z) = \Re z \frac{\partial f(z)}{\partial \Re z} + \Im z \frac{\partial f(z)}{\partial \Im z}, \]
\[ \partial_{x_2} f(z) = \Re z \frac{\partial f(z)}{\partial \Re z} + \Im z \frac{\partial f(z)}{\partial \Im z}, \]
and in the last (3.7).

Note that, writing \( z = z(w) \) and recalling the definition (3.8) of \( \Delta \), one has
\[ z_{x_2} = z'(w)w_{x_2} = z'(w) \left[ \frac{\Delta}{z} - \frac{w}{z} \right], \]
so that
\[ z \frac{\partial \log z}{\partial \nabla_{x_1} h} + z_{x_2} \left( i\pi + \frac{\partial \log w}{\partial \nabla_{x_1} h} \right) = i\pi z_{x_2} + \frac{\Delta}{z} z'(w) \frac{\partial \log w}{\partial \nabla_{x_1} h}, \]
because
\[ z \partial_{x_1,1} \log z = z'(w)w \partial_{x_1,1} \log w = \partial_{x_1,1} \log w = 0. \]

Together with (3.10), (3.11), (3.12), we obtain
\[ \partial_t \log z = \pi \left[ i\partial_{x_2} f - \Im z \frac{\partial f}{\partial \Re z} + \Re z \frac{\partial f}{\partial \Im z} \right] + a = 2i\pi z_{x_2} \frac{\partial f}{\partial z} + a \]
with \( \partial_z f = (1/2)(\partial_{\Re z} - i\partial_{\Im z})f \) and
\[ a = \frac{\partial f}{\partial \Re z} \Re \left[ \frac{\Delta}{z} z'(w) \frac{\partial \log w}{\partial \nabla_{x_1} h} \right] + \frac{\partial f}{\partial \Im z} \Im \left[ \frac{\Delta}{z} z'(w) \frac{\partial \log w}{\partial \nabla_{x_1} h} \right]. \]

Similarly, one gets
\[ \partial_t \log w = \pi \left[ -i\partial_{x_1} f + \Im z \frac{\partial f}{\partial \Re z} - \Re z \frac{\partial f}{\partial \Im z} \right] + b = -2i\pi z_{x_1} \frac{\partial f}{\partial z} + b \]
with
\[ b = \frac{\partial f}{\partial \Re z} \Re \left[ \frac{\Delta}{z} z'(w) \frac{\partial \log w}{\partial \nabla_{x_1} h} \right] + \frac{\partial f}{\partial \Im z} \Im \left[ \frac{\Delta}{z} z'(w) \frac{\partial \log w}{\partial \nabla_{x_1} h} \right]. \]

From (3.8) we see that
\[ \partial_t \Delta = zw_{x_2} \left[ 2i\pi z_{x_2} \frac{\partial f}{\partial z} + a \right] + z \partial_{x_2} \left[ w \left( -2i\pi z_{x_1} \frac{\partial f}{\partial z} + b \right) \right] \]
\[ + w z_{x_1} \left[ -2i\pi z_{x_1} \frac{\partial f}{\partial z} + b \right] + w \partial_{x_1} \left[ z \left( 2i\pi z_{x_2} \frac{\partial f}{\partial z} + a \right) \right]. \]
Computing the derivatives in (3.18) and simplifying, one is left in the end with
\[
\partial_t \Delta = (a + b) \Delta + wz(a_x + b_x) + 2i\pi(z_{x_2} - z_{x_1}) \Delta \frac{\partial f}{\partial z} \\
+ 2i\pi wz \left[ z_{x_2} \partial_{x_1} f - z_{x_1} \partial_{x_2} f \right] \\
= (a + b) \Delta + wz(a_x + b_x) + 2i\pi(z_{x_2} - z_{x_1}) \Delta \frac{\partial f}{\partial z} \\
- \pi zw (\Im z_{x_2} \Re z_{x_1} - \Re z_{x_2} \Im z_{x_1}) \left( \frac{\partial^2 f(z)}{\partial \Re z^2} + \frac{\partial^2 f(z)}{\partial \Im z^2} \right),
\]
that is the main achievement of the computation.

Let us prove Theorem 3.2. By assumption, \( \Delta \) is zero at time zero and \( \partial_t \Delta \) is also zero, so that
\[
0 = zw (\Im z_{x_2} \Re z_{x_1} - \Re z_{x_2} \Im z_{x_1}) \times \left( \frac{\partial^2 f(z)}{\partial \Re z^2} + \frac{\partial^2 f(z)}{\partial \Im z^2} \right).
\]
Note that \( (\Im z_{x_2} \Re z_{x_1} - \Re z_{x_2} \Im z_{x_1}) \) vanishes only if \( z_{x_2}/z_{x_1} \) is real. On the other hand, (3.13) implies that if \( \Delta = 0 \) then
\[
\frac{z_{x_2}}{z_{x_1}} = -z'(w) \frac{w}{z} = -\frac{wP_w(z, w)}{zP_z(z, w)}
\]
and, as shown in [24] Sec. 2.3], the r.h.s. belongs to \( \mathbb{C} \setminus \mathbb{R} \). The claim of the theorem then follows.

As for Theorem 3.3, recall from definitions (3.16) and (3.17) that \( a, b \) are linear in \( \Delta \) and that \( f \) is harmonic by assumption. If we consider \( z, w \) as known functions of space and time, then one sees that the r.h.s. of (3.19) is Lipschitz in \( \Im \Delta, \Re \Delta \) and its derivatives w.r.t. \( x_1, x_2 \). As long as \( z \) is sufficiently regular in space and time so that all derivatives exist, we see that \( \Delta \) remains zero if it is zero initially, as it is the solution of an initial value problem with Lipschitz right-hand side.

\[ \square \]

3.1. A couple of concrete examples. For the dimer model on the honeycomb lattice with uniform weights, the characteristic polynomial is [23] \( P(z, w) = z + w - 1 \) and (with a conventional choice of coordinates on the honeycomb lattice) the Newton polygon \( N \) is the triangle with vertices \((0, 0), (1, 0), (0, 1)\). The condition \( P(z, w) = 0 \) implies \( w = 1 - z \), and the mapping from \( z \) to \( \nabla h \) induced by (3.2) is a bijection from the upper half complex plane \( \mathbb{H} \) to \( N \). The \( z \)-to-slope mapping is illustrated in Figure 2. Some of the two-dimensional growth models mentioned in the introduction, notably those defined in [7] and [3], are known to preserve the Gibbs measures of the honeycomb dimer model. For these models, the speed of growth turns out to be a harmonic function of \( z \) that, once expressed in terms of \( \nabla h \), can be checked by direct computation to have AKPZ signature: \( \det(D^2 v) \leq 0 \). Thanks to our Theorems 3.3 and 2.1 AKPZ signature follows simply by harmonicity.
Another natural example is when $G$ is the square grid $\mathbb{Z}^2$ with unit weights. With a natural choice of Kasteleyn matrix one finds the characteristic polynomial

$$P(z, w) = -1 + \frac{1}{z} + \frac{1}{w} + \frac{1}{zw}.$$  

Then, $P(z, w) = 0$ gives $w = (z + 1)/(z - 1)$, and one easily checks that relations (3.2) give again a bijection between the upper half-plane and the Newton polygon, that in this case is the square $N$ with vertices $(0, 0), (1, 0), (0, 1), (1, 1)$. See Fig. 3 and also the discussion in [12, Sec. 2.4 and Fig. 3]. As we mentioned in the introduction, the two-dimensional domino-tiling growth model defined in Section 3.1 of [33] admits the Gibbs measures $\pi_\rho$ of the dimer model on $\mathbb{Z}^2$ as stationary states. The speed of growth $v(\cdot)$ was computed in [12], while it has quite a complicated expression in terms of $\nabla h$, see [12, Eq. (2.6)], once expressed in terms of $z$ it equals just $\pi^{-1} \Im z$. In [12, App. B], a lengthy but direct computation shows that $\text{det}(D^2v) \leq 0$; again, this can be obtained as an immediate consequence of our present results.

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