Solving Poisson's Ratio of Single-Layer Composites Acted by Normal Force with Interface Non-Slip Condition

Yanru Li, Liqiang Wang* and Haibo Jiang

Department of Haifang Engineering, Naval Logistics Academy, Tianjin 300450, China

*Corresponding author’s e-mail: yingguanchu@126.com

Abstract. In this paper, the stress of the interface between the reinforcement and the matrix of composite is analyzed. According to the basic characteristics that the interface does not slip, the strains of the reinforcement and the matrix along the interface are assumed to be equal. The Poisson's ratio calculation formula of the single-layer composite acted by normal force is derived. The research indicates that the interface non-slip of the laminate is the basic feature of the composite. Based on this, the elastic properties of the laminate acted by normal force can be directly calculated and analyzed by equal-strain equations; The Poisson's ratio is not only a function of the Poisson's ratio and volume fraction of the reinforcement and the matrix, but also a function of the ratio of the elastic modulus of the reinforcement to the elastic modulus of the matrix, but the size of Poisson's ratio is independent of the size and shape of the reinforcement and matrix.

1. Introduction
For laminates consisting of isotropic single-layer plates, the elastic modulus and Poisson's ratio calculated from the parallel model of longitudinal force are in accordance with the linear mixing rate law, and are close to the experimental data, but there is a big difference between the elastic constant calculated by the series model of normal force and the experimental value.[1]. In order to solve the problem of excessive error, many scholars have carried out a lot of exploration. These studies reached a climax in the 1950s and 1960s. Representative studies include a modified mixture rate formula proposed by Halpain-Tsai[2], and the deformation state is a series model[3]; other similar studies include the Eshelby elastic inclusion model [4, 5], and the Mori-Tanaka mean field model [6, 7], Chamis model [8,9], Hill-Hashin-Christensen-Lo model [10-13] and bridging model [14,15].

The meso-mechanical study of the elastic modulus and Poisson's ratio of composite materials depends to a large extent on the choice of the calculation model. The model currently established is basically an abstraction of the actual structure of the composite material, which fails to reflect some important features of the real material. Therefore, when calculating other constants other than the longitudinal elastic property, there is a large error, and it is difficult to reduce the error only by improving the calculation method [16].

This paper is not prepared to establish an abstract computational model. It only analyzes the stress of the interface between the reinforcement and the matrix, and directly derives the calculation formula of the transverse Poisson's ratio of the single-layer composite. During the derivation, it is assumed that the strains of the reinforcement and the matrix are equal in the direction of the interface, and this assumption is in line with the actual situation, because the interface does not slip under normal use conditions.
2. Micro volume unit and force analysis
The composite micro-volume unit composed of two tiny and glued single-layer plates is taken as an example to study the stress of the reinforcement and the matrix near the interface. The coordinate system is shown in Figure 1.

![Figure 1. Single-layer composite micro-volume unit and coordinate system](image)

It is assumed that each single-layer board is isotropic, and the two-layer board composed of them is subjected to the normal stress in the y-axis direction.

Since the Poisson's ratio and the elastic modulus of each single-layer board are different, shear stress is generated at the interface, and the shear stress causes the two single-layer boards to have the same deformation, this is the basic assumption of this article.

At each symmetrical end of each single-layer board, equal-sized, opposite-direction virtual forces $P_x$ or $P_z$ are set to replace the shear stress. The magnitude of the virtual force is determined by the equal-strain method, that is, the single-layer board strain generated by the virtual force is equal to the strain produced by the shear stress. In order to balance the virtual force at one end of the single-layer board, a virtual force of equal size and opposite direction must be provided at the corresponding end of the other single-layer board, so the composition of the virtual forces at either end of the double-layer board is zero. The magnitude of each pair of virtual forces in Figure 1 is represented by $P_x$ or $P_z$, and the direction is determined by the arrows. The virtual force is expressed in the form of concentrated force only for the convenience of illustration and calculation. In the discussion of the deformation problem of single-layer board, the form of stress should be adopted, that is, the virtual concentrated force should be regarded as the composition force of the normal stress distributed on the side of the single-layer board.

3. Poisson's ratio formula derivation
See Figure 1. First, the strain along the z-axis is considered. The strain $\varepsilon_{1z}$ of the first layer (lower layer) single-layer plate in the z-axis direction is equal to the sum of strain $\varepsilon_{1\sigma z}$ due to the load stress $\sigma$ in the y-axis direction, the strain $\varepsilon_{1Pxz}$ due to the virtual force $P_x$, and the the strain $\varepsilon_{1Pzz}$ due to the virtual force $P_z$. That is

$$\varepsilon_{1z} = \varepsilon_{1\sigma z} + \varepsilon_{1Pxz} + \varepsilon_{1Pzz} = -\mu \frac{\sigma}{E_1} - \mu \frac{P_x/A_x}{E_1} + \frac{P_z/A_z}{E_1}$$

(1)

Where $\mu$ is the Poisson's ratio, $E$ is the modulus of elasticity, $A_x$, $A_z$ are the areas of the single-layer plate perpendicular to the x-axis and the z-axis respectively, and the subscripts 1, 2 represent the first layer (lower layer) and the second layer (upper layer) single-layer board, respectively.

Similarly, the strain $\varepsilon_{2z}$ of the second layer (upper layer) single-layer plate in the z-axis direction is equal to the sum of the strain $\varepsilon_{2\sigma z}$ due to the load stress $\sigma$, the strain $\varepsilon_{2Pxz}$ due to the virtual force $P_x$, and the strain $\varepsilon_{2Pzz}$ due to the virtual force $P_z$. That is
\[ \varepsilon_{zc} = \varepsilon_{zyc} + \varepsilon_{yzc} = -\mu_2 \frac{\sigma_z}{E_2} + \mu_2 \frac{P_z}{A_{z2}} - \frac{P_z}{A_{z2}} \]  

(2)

Correspondingly, the strain in the x-axis direction of each single-layer board is

\[ \varepsilon_{xc} = \varepsilon_{xyz} + \varepsilon_{yxc} = -\mu_1 \frac{\sigma_x}{E_1} + \mu_1 \frac{P_x}{A_{x1}} - \frac{P_x}{A_{x1}} \]  

(3)

\[ \varepsilon_{x2c} = \varepsilon_{x2yc} + \varepsilon_{y2xc} = -\mu_2 \frac{\sigma_x}{E_2} - \mu_2 \frac{P_x}{A_{x2}} + \mu_2 \frac{P_x}{A_{x2}} \]  

(4)

In the above formula

\[ A_{x1} = A_1 V_1, \quad A_{x2} = A_2 V_2 \]  

(5)

Where \( V_1 \) and \( V_2 \) are the volume fraction of the first and second single-layer boards in the double-layer board respectively, and \( A_1 \) and \( A_2 \) respectively represent the area of the double-layer board perpendicular to the x-axis and the z-axis.

In this paper, an important assumption is made that the interface between the reinforcement and the matrix does not slip, or the strains of the reinforcement and the matrix along the interface are equal, which can be expressed as

\[
\begin{align*}
\varepsilon_{1y} &= \varepsilon_{2z} \\
\varepsilon_{1x} &= \varepsilon_{2x}
\end{align*}
\]  

(6)

Based on this, the transverse Poisson’s ratio formula of single-layer composites is derived. Substituting equation (1) to (5) into equation (6), the equations are obtained.

\[
\begin{cases}
-\mu_1 \frac{\sigma_z}{E_1} + \mu_1 \frac{P_z}{A_1V_1E_1} + \frac{P_z}{A_1E_1} = -\mu_2 \frac{\sigma_x}{E_2} + \mu_2 \frac{P_x}{A_2V_2E_2} - \frac{P_x}{A_2E_2} \\
-\mu_2 \frac{\sigma_x}{E_2} - \mu_2 \frac{P_x}{A_2V_2E_2} + \mu_2 \frac{P_x}{A_2E_2} = -\mu_1 \frac{\sigma_z}{E_1} - \mu_1 \frac{P_z}{A_1V_1E_1} + \frac{P_z}{A_1E_1}
\end{cases}
\]  

(7)

Solving the equations to get the virtual force

\[
P_z = \frac{V V_2 (E_2 \mu_z - E_z \mu_z)}{V_1 (1 - \mu_z) + V_2 E_z (1 - \mu_z)} A_1 \sigma
\]  

(8)

\[
P_x = \frac{V V_2 (E_2 \mu_x - E_z \mu_z)}{V_1 (1 - \mu_z) + V_2 E_z (1 - \mu_z)} A_1 \sigma
\]  

(9)

Then examine the strain in the y-axis direction. The strain of the first layer (lower layer) single-layer board in the y-axis direction is

\[
\varepsilon_{1y} = \varepsilon_{1xy} + \varepsilon_{1y1} + \varepsilon_{1y2y} = \frac{\sigma_y}{E_1} - \mu_1 \frac{P_y}{A_{x1}} - \mu_1 \frac{P_y}{A_{x1}}
\]  

(10)

The strain of the second layer (upper layer) single-layer board in the y-axis direction is

\[
\varepsilon_{2y} = \varepsilon_{2xy} + \varepsilon_{2y2} + \varepsilon_{2y3y} = \frac{\sigma_y}{E_2} + \mu_2 \frac{P_y}{A_{x2}} + \mu_2 \frac{P_y}{A_{x2}}
\]  

(11)
Assuming $H_1$, $H_2$ and $H$ are the thickness (height) of the first layer Single-Layer board, the second layer Single-Layer board and the double layer board respectively, then the total strain of the double layer board in the $y$-axis direction is

$$\varepsilon_y = \frac{H_1\varepsilon_{1y}+H_2\varepsilon_{2y}}{H} = \frac{V_1\varepsilon_{1y}}{E_1} + \frac{V_2\varepsilon_{2y}}{E_2} = \frac{2V_1V_2(E_2\mu_1-E_1\mu_2)^2\sigma}{E_1E_2[V_1E_1(1-\mu_2)+V_2E_2(1-\mu_1)]}$$  \hspace{1cm} (12)

From this, the transverse Poisson’s ratio formula can be derived:

$$\mu_{xy} = -\frac{\varepsilon_x}{\varepsilon_y} = -\frac{\varepsilon_{2x}}{\varepsilon_y} = -\frac{\varepsilon_{2x}}{\varepsilon_y} = \frac{-\mu_2\varepsilon_{2x}}{E_2} + \frac{\mu_2\varepsilon_{2x}}{E_2} = \frac{\mu_2\varepsilon_{2x}}{E_2} + \frac{V_1E_1(1-\mu_2)(E_2\mu_1-E_1\mu_2)}{E_1E_2[V_1E_1(1-\mu_2)+V_2E_2(1-\mu_1)]}$$  \hspace{1cm} (13)

Equations (13) is the formula for calculating the Poisson’s ratio of the laminate when subjected to normal force.

4. Analysis and discussion

4.1. Change trend of transverse Poisson’s ratio

As can be seen from equation (14), the transverse Poisson's ratio is a function of the four parameters $\mu_1$, $\mu_2$, $V_1$, and $E_2/E_1$. In order to observe the trend, set $\mu_1=0.25$, $\mu_2=0.35$, given $E_2/E_1=1$ and $E_2/E_1=5$. The curve of Poisson’s ratio with volume fraction $V_1$ is shown in Figure 2.

![Figure 2. Poisson's ratio as a function of volume fraction](image)

If $E_2/E_1$ is considered as a variable, the Poisson's ratio changes with $E_2/E_1$ and $V_1$ as shown in Figure 3.

![Figure 3. Trend of Poisson's ratio with elastic modulus ratio and volume fraction](image)

It can be seen from the figure that the $E_2/E_1$ ratio is too large (for example, after more than 2), the curved surface tends to be concave. This may be because the difference between the elastic modulus of the two materials is too large, it will resist each other’s deformation, thus reducing the total Poisson's ratio.
4.2. Result and enlightenment of derivation of parallel formula by using this method

For a single-layer composite, when the direction of the external force is parallel to the interface, the reinforcement is placed in parallel with the matrix in the longitudinal direction (parallel model), and the longitudinal Poisson's ratio is consistent with the mixing ratio:

\[ \mu_{x} = \frac{\varepsilon_{x}}{\varepsilon_{z}} = V_{1}\mu_{1} + V_{2}\mu_{2} = V_{1}\mu_{1} + (1-V_{1})\mu_{2} \]

(14)

Many experiments have confirmed that the calculated values of the longitudinal Poisson's ratio formula agree well with the experimental data. If the assumption in this paper is used, that is, the strain between the reinforcement and the matrix is equal in the direction of the interface, the longitudinal Poisson's ratio formula is re-derived, and the obtained result is completely consistent with the formula (14), and the formula does not change any more (the derivation process is omitted). This phenomenon indirectly indicates that the assumption that the strains of the reinforcement and the matrix are equal in the direction of the interface is in line with the actual situation. The formula of the transverse Poisson's ratio derived under this assumption is inevitably more accurate.

4.3. On the application of transverse poisson's ratio formula

Several force-bearing structures are shown in Figure 4. The shaded parts in the figure are the reinforcements, and the white parts are the matrix. They are arranged in series and subjected to transverse force (the external force \(\sigma\) is perpendicular to the interface). Figure 4 (a) is a "two-layer board" with a macroscopic scale, and Figure 4 (b) to Figure 4 (d) are obtained by connecting the composite materials in series of Figure 4 (a), in which the reinforcements of Figure 4 (c) are connected and the matrices of Figure 4 (d) are connected.

The length, width and height dimensions of the Poisson’s ratio formula (13) do not appear, indicating that the horizontal Poisson’s ratio is independent of the size and shape of the reinforcement and the matrix, and is only related to the volume fraction. From this, it is judged that several structures shown in Figure 4 are equivalent, since the volume fractions of the reinforcing bodies are the same, so their Poisson's ratios are equal. This shows that formula (13) can be directly applied to the calculation of composite materials on a macro scale.

![Figure 4](image-url)

Figure 4 Macro-scale composite material with transverse force

As can be seen from Figure 4, the formula for calculating Poisson's ratio of single-layer plates subjected to normal loads is also applicable to multi-layer plates.

5. Conclusion

(1) That the interface of laminates does not slip is the basic feature of the composite material. Based on this, the elastic properties of laminates under normal loading can be directly calculated by means of isostrain equations.
(2) The transverse Poisson’s ratio is a function of the Poisson’s ratio of reinforcement and matrix, the ratio of elastic modulus and volume fraction, but its size is independent of the size and shape of reinforcement and matrix.

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