Stationary analysis of certain Markov-modulated reflected random walks in the quarter plane

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Abstract

In this work, we focus on the stationary analysis of a specific class of continuous time Markov modulated reflected random walk in the quarter plane with applications in the modelling of two-node Markov-modulated queueing networks with coupled queues. The transition rates of the two-dimensional process depend on the state of a finite state Markovian background process. Such a modulation is space homogeneous in the set of inner states of the two-dimensional lattice but may be different in the set of states at its boundaries. To obtain the stationary distribution, we apply the power series approximation method, and the theory of Riemann boundary value problems. We also obtain explicit expressions for the first moments of the stationary distribution under some symmetry assumptions. Using a queueing network example, we numerically validated the theoretical findings.

Keywords: Markov modulated reflected random walks; Power series approximation; Boundary value problems; Stationary analysis; Networks with Coupled queues.

1 Introduction

Our primary aim in this work is in methodology for obtaining stationary metrics of a certain Markov-modulated two dimensional reflected random walks, as a general model describing a two-queue network with a sort of coupling, operating in a random environment. In particular, we are dealing with a Markovian process \( \{Z(t); t \geq 0\} = \{(X_1(t), X_2(t), J(t)); t \geq 0\} \) where the two-dimensional process \( \{(X_1(t), X_2(t)); t \geq 0\} \) defined on \( \mathbb{Z}_2^+ \) is called the level process. The transition rates of the two-dimensional process \( \{(X_1(t), X_2(t)); t \geq 0\} \) depend on the state of the phase process \( \{J(t); t \geq 0\} \), which is defined on a finite set \( \{0, 1, \ldots, N\} \).

Moreover, the increments of the individual processes \( \{X_1(t)\}, \{X_2(t)\} \) take values in \( \{-1, 0, 1, \ldots\} \) when \( J(t) = 0 \) and in \( \{0, 1, \ldots\} \) when \( J(t) \in \{1, \ldots, N\} \). The modulation at the state of the process is space homogeneous, except possibly at the boundaries of \( \mathbb{Z}_2^+ \).

A special example described by such a Markov modulated reflected random walk is a two-node G-queueing network (see e.g. [20] [21] [22] for an overview on G-networks) with coupled processors and service interruptions. In such a network, three classes of jobs arrive according to independent Poisson processes. Class \( P_i, i = 1, 2 \) is routed to queue \( i \), while a job of class \( P_0 \) is placed simultaneously at both queues. Each job at queue \( i \) requires exponentially distributed service time with rate \( \nu_i \). The network is operating for an exponentially distributed time with rate \( \theta_{0,j} \), and then, it switches to the

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failed mode $j$, $j = 1, \ldots, N$. In failed mode $j$, it stays for an exponentially distributed time, and then, switches either to operating mode 0 (with rate $\theta_{j,0}$), or to another failed mode $k$ (with rate $\theta_{j,k}$, $k \neq j$).

When the network is in a failed mode of either type it cannot provide service. Moreover, the arrival rates of the jobs of either type depend on the type of the mode (either operating or failed). When the network is in operational mode, and both queues are non-empty, queue 1 serves at a rate $w\nu_1$, and queue 2 at a rate $(1 - w)\nu_2$. Moreover, signals arrive at queue 1 (resp. 2) with rate $w\lambda_1$ (resp. $(1 - w)\lambda_2$) and with probability $t_{12}$ (resp. $t_{21}$) triggers the instantaneous movement of a job from queue 1 (resp. queue 2) to queue 2 (resp. queue 1), or with probability $t_{10}$ (resp. $t_{20}$) cancels a job from queue 1 (resp. queue 2). If only one queue is non-empty it serves at full capacity, i.e., with rate $\nu_i$. Similar pattern follows the signal generation. Upon receiving service at queue 1 (resp. 2), the job is either routed to queue 2 (resp. 1) with probability $r_{12}$ (resp. $r_{21}$), or leaves the network with probability $1 - r_{12}$ (resp. $1 - r_{21}$).

1.1 Related work

Markov-modulated processes are processes that are driven by an underlying Markov process, in the sense that its transition parameters are affected by the state of the underlying process. The interested reader refer to [37], [38, 39] for more information.

Such type of stochastic processes arise in many queueing applications where systems evolve in random environment (i.e., the underlying process), that may model the irregularity of the arrival process (e.g., are rush-hour phenomena), the irregularity of the service process (e.g., servers’ breakdowns, servers’ vacations, availability of resources etc.) or both. Other applications are found in biology [29], in reliability [28] etc. The vast literature on this topic shed light on the stationary behaviour of such models. A variety of approaches have been used, but the most widely applied is the matrix-analytic method. We briefly mention [33, 19, 40, 25] (not exhaustive list). However, it is known that they demand computational effort due to the large number of matrix computations. Alternatively, some authors tried to investigate special cases for which the stationary distributions adopt a simple product form, e.g., [13, 47]. Plenty of work on multidimensional level process has been devoted in deriving asymptotic formulas of the stationary distributions and studying the stability conditions [34, 35, 36, 26, 30, 31] (not exhaustive list). We also refer to [17], that dealt with a network of infinite server queues in which the joint probability generating function (pgf) is characterized in terms of a system of partial differential equations; see also [23, 4, 3] for some recent studies on Markov modulated queueing systems.

Quite recently, the authors in [24] studied the approximation of the probability of a large excursion in the busy cycle of a constrained Markov-modulated random walk in the quarter plane. Moreover, the authors in [46] recently introduced the power series approximation (PSA) method to provide approximated stationary metrics in a non-modulated random walk in the quarter plane describing a slotted time generalized processor sharing system of two queues. There, at the beginning of a slot, if both queues are nonempty, a type 1 (resp. type 2) customer is served with probability $\beta$ (resp. $1 - \beta$); see also [42, 43]. A first step towards applying PSA method to modulated queues with coupled processors was recently given in [9, 10]. Clearly, coupled processor models arise naturally when limited resources are dynamically shared among processors, e.g., [14, 15, 27, 5, 11, 12], as well as in assembly lines in manufacturing [2].
1.2 Contribution

**Fundamental contribution** By employing the generating function approach, we come up with a system of functional equations, and we perform two different methods to investigate the stationary behaviour of a class of Markov modulated birth-death process.

1. We extend the class of stochastic processes in which PSA method (see \[43, 44, 45, 46\]) can be applied, to the case of Markov-modulated reflected random walks in the quarter plane with a sort of coupling (see also \[9, 10\] for some initial work on this direction). We obtain power series expansions of the pgf of the joint stationary distribution for any state of the phase process. A recursive technique to derive their coefficients is also presented; see Section 4.

2. We also show how the theory of Riemann boundary value problems \([18, 6, 8]\) can be applied for such kind of processes; see Section 5.

3. By considering the symmetry assumption (see Section 6) at phase \(J(t) = 0\), we obtain explicit expressions for the moments of the stationary distribution of the individual processes \(\{X_j(t)\}\), \(j = 1, 2\).

**Applications** With this methodological approach we are able to analytically investigate a variety of Markov-modulated queueing networks with a sort of coupling among queues; see the example above.

The rest of the paper is organised as follows. In Section 2 the Markov-modulated reflected random walk in quarter plane is described in detail, and a system of functional equations along with some preliminary results are presented in Section 3. The main result (see Theorem 1) regarding the application of the power series approximation (PSA) method is presented in Section 4. A discussion about how the system parameters affect the system performance for a near priority system are given in Section 5. Numerical validation of the PSA method (see \[43, 44, 45, 46\]) can be applied, to the case of Markov-modulated reflected random walks in the quarter plane with a sort of coupling (see also \[9, 10\] for some initial work on this direction). We obtain power series expansions of the pgf of the joint stationary distribution for any state of the phase process. A recursive technique to derive their coefficients is also presented; see Section 4. Numerical results obtained by using the PSA, as well as some observations about how the system parameters affect the system performance for a near priority system are given in Section 6. Numerical validation of the PSA using the explicit expressions of the symmetrical case derived in Section 5 is also given. The paper concludes in Section 8.

2 Model description

Consider a two-dimensional Markovian process \(\{(X_1(t), X_2(t))\}\) and a background process \(\{J(t)\}\) on a finite state space \(S_0 = \{0, 1, \ldots, N\}\). Assume that each of \(\{X_1(t)\}, \{X_2(t)\}\) is skip-free, which means that their increments take values in \(-1, 0, 1, \ldots\) when \(J(t) = 0\) and in \(\{0, 1, \ldots\}\) when \(J(t) \neq 0\). The joint process \(\{Z(t), t \geq 0\} = \{(X_1(t), X_2(t), J(t)), t \geq 0\}\) is Markovian with state space \(E = \mathbb{Z}^+_2 \times S_0\); see Fig. 1 for the quasi birth-death (QBD) version of \(\{Y(t)\}\).

The infinitesimal generator matrix \(Q\) of \(\{Z(t), t \geq 0\}\) is represented in block form as

\[
Q = [Q(x_1, x_2)(x'_1, x'_2); (x_1, x_2), (x'_1, x'_2) \in \mathbb{Z}^+_2],
\]

where each block \(Q(x_1, x_2)(x'_1, x'_2)\) is given by \(Q(x_1, x_2)(x'_1, x'_2) = [q(x_1, x_2, j), (x'_1, x'_2, j') \in E, (x_1, x_2, j), (x'_1, x'_2, j')\) and \(q(x_1, x_2, j), (x'_1, x'_2, j')\) are the infinitesimal transition rates from state \((x_1, x_2, j)\) to \((x'_1, x'_2, j')\). Let \(\mathbb{H} = \{-1, 0, 1, \ldots\}\), \(\mathbb{H}^+ = \{0, 1, \ldots\}\). The transition rates satisfy the following rules:
1. For \( j = j' = 0 \) (i.e., phase 0, see Fig. 2 for the QBD version),
\[
q_{(x_1,x_2,0),(x'_1,x'_2,0)}(0,0) = \begin{cases}
q_{\Delta x_1,\Delta x_2}(0), & \Delta x_1, \Delta x_2 \in \mathbb{H}, \\
q_{\Delta x_1,\Delta x_2}(0), & \Delta x_1 \in \mathbb{H}, \Delta x_2 \in \mathbb{H}^+,
\end{cases}
\]
where \( \Delta x_k = x'_k - x_k, k = 1, 2 \). Moreover,
\[
q_{\Delta x_1,\Delta x_2}(0) = q_{\Delta x_1,\Delta x_2}(0), \quad q_{1,-1}(0) = 0,
\]
and for \( w \in [0,1] \),
\[
q_{1,-1}(0) = (1-w)q_{1,-1}(0), \quad i \in \mathbb{H}^+ \\
q_{-1,j}(0) = wq_{1,-1}(0), \quad j \in \mathbb{H}^+,
\]
or equivalently,
\[
\frac{q_{1,-1}(0)}{q_{-1,j}(0)} + \frac{q_{1,-1}(0)}{q_{1,-1}(0)} = 1, \quad i, j \in \mathbb{H}^+.
\]

2. For \( j = j' = 1, \ldots, N \),
\[
q_{(x_1,x_2,j),(x'_1,x'_2,j)} = q_{\Delta x_1,\Delta x_2}(j), \quad \Delta x_1, \Delta x_2 \in \mathbb{H}^+,
\]
i.e., when \( J(t) \neq 0 \), no transitions are allowed to the West, North-West, South, South-West and
South-East (see Fig. 2 for the QBD version).

3. For \( j \neq j' \),
\[
q_{(x_1,x_2,j),(x'_1,x'_2,j')} = \theta_{j,j'} a_{\Delta x_1,\Delta x_2}(j,j'), \quad \Delta x_1, \Delta x_2 \in \mathbb{H}^+,
\]
i.e., a phase change from \( j \) to \( j' \) triggers a transition of \( \{(X_1(t),X_2(t));t \geq 0\} \) from \( (x_1,x_2) \) to
\( (x'_1,x'_2) \) with probability \( a_{\Delta x_1,\Delta x_2}(j,j') \).

**Remark 1** Note that the condition displayed in [1] is not essential for the analysis that follows in
Sections 4 and 5. If it does not hold, the analysis can be easily modified. Condition [1] is only useful in
deriving easily the limiting probability of \( \{Z(t);t \geq 0\} \) at point \( (0,0,0) \); see [12].
Moreover, condition [2] is essential only for the analysis in Section 4. The analysis presented in
Section 3 is general enough to consider also the case where condition [2] does not hold. However,
some further technicalities are needed. Moreover, we can also easily modify the analysis in Section 6
to consider the case when [2] does not hold. Our main aim in this work is on the applicability of PSA
[10] in the analysis of Markov-modulated reflected random walks in the quarter plane, and this is the
reason why we employ condition [2].

Furthermore, conditions [3], [4] are essential (in the sense that \( \Delta x_1, \Delta x_2 \) should be in \( \mathbb{H}^+ \)) both for
the analysis in Section 4 and for the one in Section 5. This is because in case we allow \( \Delta x_1, \Delta x_2 \in \mathbb{H} \),
we introduce additional unknown boundary functions that make the employed analysis intractable.
Figure 1: State transition diagram of the QBD version of \( \{Z(t), t \geq 0\} \)

Figure 2: State transition diagram of the QBD version of \( \{Z(t), t \geq 0\} \) at phase 0
3 The functional equations

Assume that the system is stable, and let the equilibrium probabilities

\[ \pi_{i,j}^{(k)}(k) = \lim_{t \to \infty} P((X_1(t), X_2(t), J(t) = (i, j, k)), (i, j, k) \in E). \]

Let for \( k = 0, 1, \ldots, N \)

\[ \Pi_k(x, y) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \pi_{i,j}^{(k)} x^i y^j, |x| \leq 1, |y| \leq 1, \]

and for \( k, m = 0, 1, \ldots, N, k \neq m, \)

\[ A_{k,m}(x, y) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{i,j}^{(k,m)} x^i y^j, |x| \leq 1, |y| \leq 1. \]

By writing down the equilibrium equations we come up with the following system of functional equations

\[ R(x, y)\Pi_0(x, y) = K(x, y)[(1 - w)\Pi_0(x, 0) - w\Pi_0(0, y)] + C(x, y)\Pi_0(0, 0) + xy\sum_{k=1}^{N} \theta_{k,0}A_{k,0}(x, y)\Pi_k(x, y), \]  \hspace{1cm} (5)

\[ \Pi_k(x, y) = \frac{\sum_{m=0, m \neq k}^{N} \theta_{m,k}A_{m,k}(x, y)\Pi_m(x, y)}{D_k(x, y)}, k = 1, \ldots, N, \]  \hspace{1cm} (6)

where for \( k = 0, 1, \ldots, N, \)

\[ D_k(x, y) = S_k(x, y) + \theta_{k,}, \]

\[ S_k(x, y) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} q_{i,j}(k)(1 - x^i y^j)1\{(i,j) \neq (0,0)} \]

\[ \theta_{k,} = \sum_{m \neq k}^{N} \theta_{k,m}. \]
Moreover,

\[
R(x, y) = xyD_0(x, y) + y \sum_{j=0}^{\infty} q_{1,j}(0)(x - y^j) + x \sum_{i=0}^{\infty} q_{i,-1}(0)(y - x^i),
\]

\[
K(x, y) = x \sum_{j=0}^{\infty} q_{j,-1}(0)(y - x^j) - y \sum_{j=0}^{\infty} q_{j,1}(0)(x - y^j),
\]

\[
C(x, y) = wK(x, y) + y \sum_{j=0}^{\infty} q_{j,-1}(0)(x - y^j)
= -(1 - w)K(x, y) + x \sum_{i=0}^{\infty} q_{i,2}(0)(y - x^i).
\]

Equation [6] defines a \(N \times N\) system of linear equations

\[
L(x, y)^T P(x, y) = E(x, y) \Pi_0(x, y),
\]

where

\[
P(x, y) = (\Pi_1(x, y), \ldots, \Pi_N(x, y))^T,
\]

\[
E(x, y) = (\theta_{0,1}A_0(x, y), \ldots, \theta_{0,N}A_0(x, y))^T,
\]

\[
L(x, y) = \begin{pmatrix}
D_1(x, y) & -\theta_{1,2}A_{1,2}(x, y) & -\theta_{1,3}A_{1,3}(x, y) & \cdots & -\theta_{1,N}A_{1,N}(x, y) \\
-\theta_{2,1}A_{2,1}(x, y) & D_2(x, y) & -\theta_{2,3}A_{2,3}(x, y) & \cdots & -\theta_{2,N}A_{2,N}(x, y) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
-\theta_{N,1}A_{N,1}(x, y) & -\theta_{N,2}A_{N,2}(x, y) & -\theta_{N,3}A_{N,3}(x, y) & \cdots & D_N(x, y)
\end{pmatrix}
\]

The system of equation [7] yields

\[
\Pi_k(x, y) = F_{0,k}(x, y)\Pi_0(x, y), \quad k = 1, \ldots, N,
\]

\[
F_{0,k}(x, y) = \frac{1}{\det[L(x, y)^T]}(\text{cof}L(x, y))^T \theta(0),
\]

provided \(\det[L(x, y)^T] \neq 0\), and where \(\text{cof}L(x, y)^T\) is the cofactor matrix of \(L(x, y)\). Note that \(L(x, y)\) is a non-singular matrix for \(|x|, |y| = 1\), since it is strictly diagonally dominant. Indeed,

\[
|D_k(x, y)| \geq \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} q_{i,j}(k)(1 - |x|^j|y|^i)1_{(j,i) \neq (0,0)} + \sum_{m=0,m \neq k} A_{k,m}(x, y)
\]

\[
= \sum_{m=0,m \neq k} \theta_{k,m} > \sum_{m \neq k,0} \theta_{k,m} = \sum_{m \neq k,0} \theta_{k,m} A_{k,m}(x, y)
\]

Therefore, since \(\sum_{k=0}^{N} \Pi_k(1, 1) = 1\), the stationary probabilities of the process \(\{Z(t), t \geq 0\}\) to be in any state of the background process \(\{J(t), t \geq 0\}\) are given by,

\[
\Pi_0(1, 1) = [1 + \sum_{j=1}^{N} F_{0,j}(1, 1)]^{-1},
\]

\[
\Pi_k(1, 1) = F_{0,k}(1, 1)\Pi_0(1, 1), \quad k = 1, \ldots, N.
\]

Using [5], [8] we come up with the following functional equation:

\[
\Pi_0(x, y)[R(x, y)T(x, y) - xy] = T(x, y)\{K(x, y)\{(1 - w)\Pi_0(x, 0) \\
- w\Pi_0(0, y)\} + C(x, y)\Pi_0(0, 0)\},
\]

where \(T(x, y) = [\sum_{k=1}^{N} \theta_{k,0}A_{k,0}(x, y)F_{0,k}(x, y)]^{-1}\).
3.1 Stability condition

Using Rouché’s theorem \(^{11}\) we can show that \(K(x, y) = 0\) has a zero, say \(x = s(y), |y| = 1\), such that \(|s(y)| < 1\), and \(s(1) = 1\). Indeed, let us first write \(R(x, y) = d_1(x, y) + d_2(y)\), where

\[
\begin{align*}
    d_1(x, y) &= x[y\sum_{i=0}^{\infty} q_{i-1}^{(2)}(0)(1-x^i) - y\sum_{j=0}^{\infty} q_{-1,j}^{(1)}(0)], \\
    d_2(y) &= \sum_{j=0}^{\infty} q_{-1,j}^{(1)}(0)y^{j+1}.
\end{align*}
\]

Note that

\[
|d_1(x, y)| \geq |x||\sum_{i=0}^{\infty} q_{i-1}^{(2)}(0)(1-|x|^i) - \sum_{j=0}^{\infty} q_{-1,j}^{(1)}(0)| > \sum_{j=0}^{\infty} q_{-1,j}^{(1)}(0),
\]

and \(|d_2(y)| < \sum_{j=0}^{\infty} q_{-1,j}^{(1)}(0)\). Thus, Rouché’s theorem states that there is a zero, say \(x := s(y), |y| = 1\) such that \(|s(y)| < 1\). Thus, using \(^{10}\),

\[
\Pi_0(s(y), y) = \frac{T(s(y), y)C(s(y), y)}{R(s(y), y)T(s(y), y) - s(y)y}\Pi_0(0, 0). \tag{11}
\]

Letting \(y \to 1\), and having in mind that \( \lim_{y \to 1} s(y) = 1, \) \(^{11}\) yields (after applying once the L’Hospital rule) \(\Pi_0(0, 0)\) and the ergodicity condition of \( \{Z(t), t \geq 0\} \), i.e,

\[
\Pi_0(0, 0) = [1 + \sum_{j=1}^{N} F_{0,j}(1, 1)]^{-1} \times \lim_{y \to 1} \left( \frac{\frac{d}{dy}[R(s(y), y)T(s(y), y) - s(y)y]}{\frac{d}{dy}[T(s(y), y)C(s(y), y)]} \right) > 0. \tag{12}
\]

Remark 2 Ergodicity conditions for such type of Markov modulated two dimensional reflecting random walks are also derived in \(^{34, 35}\) using the concept of induced Markov chains \(^{16}\).

Remark 3 Note that if we consider the QBD version of \(\{Y(t)\}\), i.e., the increments of \(\{X_j(t)\}\) are in \(\{-1, 0, 1\}\) when \(J(t) = 0\), and in \(\{0, 1\}\) when \(J(t) = 1, \ldots, N\),

\[
x := s(y) = \frac{-y(q_{-1,0}^{(1)}(0)+q_{-1,1}^{(1)}(0)-q_{0,-1}^{(2)}(0)-q_{1,-1}^{(2)}(0))-\sqrt{R(y)}}{2q_{-1,0}^{(1)}(0)},
\]

\[
R(y) = \frac{[y(q_{-1,0}^{(1)}(0)+q_{-1,1}^{(1)}(0)-q_{0,-1}^{(2)}(0)-q_{1,-1}^{(2)}(0))-q_{0,-1}^{(2)}(0)]^2}{4q_{-1,0}^{(1)}(0)y(q_{-1,1}^{(1)}(0)y+q_{-1,0}^{(1)}(0))},
\]

and \(\lim_{y \to 1} s(y) = 1\); see Figure 4 for \(x = s(y), |y| = 1\).

3.2 A special case

Consider the special case \( \theta_{i,j} = 0, i, j = 1, \ldots, N, \) i.e., the process switch always from any state \(i \neq 0\), only to state 0. Then,

\[
\mathbf{L}(x, y) = \text{diag}[D_1(x, y), \ldots, D_N(x, y)], \quad \theta_{i,-} = \theta_{i,0}, i = 1, \ldots, N.
\]

In such a case,

\[
\Pi_k(x, y) = \frac{\theta_{0,k}a_{0,k}(x,y)}{d_k(x,y)}\Pi_0(x, y), k = 1, \ldots, N, \tag{13}
\]

\[
\Pi_k(1, 1) = \frac{\theta_{0,1}}{\theta_{e,0}}\Pi_0(1, 1),
\]
Figure 4: The curve $x = s(y)$, $|y| = 1$ and the unit circle.

and,

$$
\Pi_0(1, 1) = \frac{\prod_{j=1}^{N} \theta_{j,0}}{\sum_{k=0}^{N} \theta_{0,k} \prod_{j=0,j\neq k}^{N} \theta_{j,0}}.
$$

Thus, using the second in (13) we can have the rest stationary probabilities the process \{Z(t)\} to be in any phase $k$, $k = 1, \ldots, N$. Moreover, in such a case,

$$
\begin{align*}
T(x, y) &:= \frac{D(x, y)}{\sum_{j=1}^{N} \theta_{j,0} A_{j,0}(x, y) \theta_{0,j} A_{0,j}(x, y) D^{(j)}(x, y)} , \\
D^{(j)}(x, y) &:= \prod_{k=1, k\neq j}^{N} D_k(x, y), \quad j = 1, \ldots, N, \\
D(x, y) &:= \prod_{k=1}^{N} D_k(x, y).
\end{align*}
$$

\section{3.3 The trivial cases $w = 0$ and 1}

Note that when $w = 0$ (resp. $w = 1$), there are no transitions to West (resp. South) and to North-West (resp. South-East) from the interior states, when we are in phase $J(t) = 0$, i.e., $q_{-1,0}(0) = q_{-1,1}(0) = 0$ (resp. $q_{0,-1}(0) = q_{1,-1}(0) = 0$). Such cases are not of our main interest but can serve as “boundary” cases for the main subject of concern, which is the case $w \in (0, 1)$ investigated in the following sections.

In the following we only treat the case $w = 0$. Analogous results can be obtained for the case $w = 1$.

Note that for $w = 0$, the functional equation (10) reduces to

$$
\Pi_0(x, y)[R(x, y)T(x, y) - xy] = T(x, y)[K(x, y)\Pi_0(x, 0) + C(x, y)\Pi_0(0, 0)]. 
$$

(15)

It is not difficult to show by using Rouché’s theorem \[11\] (see Theorem 2 in Section 5) that the left hand side of (15) has a single zero, say $y = \hat{Y}(x)$, for $|x| = 1$, such that $|\hat{Y}(x)| < 1$, and $\hat{Y}(1) = 1$.

Then, after some algebra,

$$
\Pi_0(x, y) = \frac{T(x, y)[K(x, y)C(x, \hat{Y}(x)) - K(x, \hat{Y}(x))C(x, y)]}{K(x, \hat{Y}(x))[xy - T(x, y)R(x, y)]} \Pi_0(0, 0).
$$

The rest pgfs $\Pi_k(x, y), k = 1, \ldots, N$ can be obtained by using (8).
4 The power series approximations in $w \in (0, 1)$

Starting by (10), our aim is to construct a power series expansion of the pgf $\Pi_0(x, y)$ in $w$, and then, with this result to construct power series expansions of $\Pi_k(x, y)$ in $w$ using (13). Let

$$\Pi_k(x, y) = \sum_{m=0}^{\infty} V_m^{(k)}(x, y) w^m, \ k = 0, 1, \ldots, N.$$  

The major difficulty in solving (10) corresponds to the presence of the two unknown boundary functions $\Pi_0(x, 0), \Pi_0(0, y)$. We proceed as in [46,9,10] and observe that (10) is rewritten as

$$G(x, y)\Pi_0(x, y) - G_{10}(x, y)\Pi_0(x, 0) - G_{00}(x, y)\Pi_0(0, 0) = wG_{10}(x, y)[\Pi_0(x, y) - \Pi_0(x, 0) - \Pi_0(0, y) + \Pi_0(0, 0)],$$  

where

$$G(x, y) = T(x, y)[yD_0(x, y) + \sum_{i=0}^{\infty} q_i^{(2)}(0)(y - x^i)] - y,$$

$$G_{10}(x, y) = T(x, y)\{\sum_{i=0}^{\infty} q_i^{(2)}(0)(y - x^i) - y \sum_{j=0}^{\infty} q_j^{(1)}(0)(1 - x^{-1}y^j)\},$$

$$G_{00}(x, y) = yT(x, y)\sum_{j=0}^{\infty} q_j^{(1)}(0)(1 - x^{-1}y^j).$$  

The following theorem summarizes the main result.

**Theorem 1** Under stability condition [12],

$$V_m^{(0)}(x, y) = Q_{m-1}(x, y) \frac{G_{10}(x, y)}{G(x, y)}, \ m > 0,$$  

$$V_0^{(0)}(x, y) = \frac{G_{00}(x, y)G_{10}(x, Y(x)) - G_{00}(x, Y(x))G_{10}(x, y)}{G(x, y)G_{10}(x, Y(x))} V_0^{(0)}(0, 0),$$

$$V_m^{(k)}(x, y) = F_{0,k}(x, y)V_m^{(0)}(x, y), \ k = 1, \ldots, N, \ m \geq 0,$$

where $Y(x), |x| \leq 1$ is the only zero of $G(x, y)$ in the unit disk $|y| \leq 1$, and for $m \geq 0$,

$$Q_m(x, y) = V_m^{(0)}(x, y) - V_m^{(0)}(x, Y(x)) - V_m^{(0)}(0, y) + V_m^{(0)}(0, Y(x)),$$

with $Q_{-1}(x, y) = 0$.

**Proof 1** Starting from (16), we observe that only $\Pi_0(0, y)$ appears with a factor $w$. By expanding the pgfs as power series in $w$ and equating the coefficients of the corresponding powers of $w$, that unknown is eliminated. Indeed, we come up with

$$G(x, y)V_m^{(0)}(x, y) = G_{10}(x, y)[V_m^{(0)}(x, 0) + P_{m-1}(x, y)] + G_{00}(x, y)V_m^{(0)}(0, 0), \ m \geq 0,$$

where

$$P_m(x, y) = V_m^{(0)}(x, y) - V_m^{(0)}(x, 0) - V_m^{(0)}(0, y) + V_m^{(0)}(0, 0),$$

with $P_{-1}(x, y) = 0$. Our aim is to express $V_m^{(0)}(x, y)$ in terms of $P_{-1}(x, y)$. Note now that there exist two unknowns, i.e., $V_m^{(0)}(x, 0)$ and $V_m^{(0)}(0, 0)$. By using Rouché’s theorem [7] we show in Appendix that $G(x, y) = 0$ has a unique root, $y = Y(x)$ such that $|Y(x)| < 1, |x| = 1, x \neq 1$. Note that due to
the implicit function theorem, \( Y(x) \) is an analytic function in the unit disk. Moreover, since \( \Pi_k(x,y) \) is analytic for all \( x \) and \( y \) in the unit disk, the terms \( V^{(k)}_m(x,y) \) are also as well.

Thus, substituting in (20) yields

\[
V^{(0)}_m(x,0) = -\frac{G_{00}(x,Y(x))}{G_{10}(x,Y(x))} V^{(0)}_m(0,0) - P_{m-1}(x,Y(x)),
\]

and substituting (21) back in (20) yields after some algebra the expressions for \( V^{(0)}_m(x,y) \), \( m \geq 0 \). Then, (13) can be used to obtain the expressions for \( V^{(k)}_m(x,y) \), \( m \geq 0, k = 1,\ldots,N \) given in the second of (19).

It remains to calculate \( V^{(0)}_m(0,0) \) by using (12). Then, it follows that \( V^{(0)}_0(0,0) = \Pi_0(0,0) \) and \( V^{(0)}_m(0,0) = 0 \) for all \( m > 0 \). Hence starting from \( V^{(0)}_0 \) in (15), every \( V^{(0)}_m, m > 0 \), and \( V^{(k)}_m, m \geq 0, k = 1,\ldots,N \) can be determined iteratively via (19).

### 4.1 Performance metrics

Having obtained iteratively the coefficients \( V^{(k)}_m(x,y), m \geq 0, k = 0,1,\ldots,N \), we are able to derive the moments of the stationary distribution of \( \{Z(t); t \geq 0\} \). The most interesting are

\[
E(X_1) = \sum_{m=0}^{\infty} w^m \frac{\partial}{\partial x} \sum_{k=0}^{N} V^{(k)}_m(x,1)|_{x=1},
\]

\[
E(X_2) = \sum_{m=0}^{\infty} w^m \frac{\partial}{\partial y} \sum_{k=0}^{N} V^{(k)}_m(1,y)|_{y=1}.
\]

Let \( v_{m,1} = \frac{\partial}{\partial x} V^{(0)}_m(x,1)|_{x=1} \). Then, using (8), (9), we finally obtain

\[
E(X_1) = \frac{\sum_{k=1}^{N} \frac{\partial}{\partial x} F_0,k(x,1)|_{x=1} + [1+\sum_{k=1}^{N} F_0,k(1,1)]^2 \sum_{m=0}^{\infty} w^m v_{m,1}}{1+\sum_{k=1}^{N} F_0,k(1,1)}.
\]

Similar expression can also be derived for \( E(X_2) \). By truncating the power series in (22) we finally derive,

\[
E(X_1) = \frac{\sum_{k=1}^{N} \frac{\partial}{\partial x} F_0,k(x,1)|_{x=1} + [1+\sum_{k=1}^{N} F_0,k(1,1)]^2 \sum_{m=0}^{M} w^m v_{m,1} + O(w^{M+1})}{1+\sum_{k=1}^{N} F_0,k(1,1)}.
\]

Clearly, the larger \( M \) the better the approximation gets. Truncation yields accurate approximations for \( w \) in the neighborhood of 0. However, it is easy to note that the accurate calculation of the expressions in (22) requires the computation of the first derivatives of \( V^{(0)}_m(x,y) \) for \( m \geq 0 \). It is evident that the calculation of these coefficients is far from straightforward due to the extensive use of L’Hôpital’s rule.

Obviously, our aim is to obtain an efficient approximation for the stationary metrics in the whole domain \( w \in [0,1] \). Contrary to the truncated power series, we can use the Padé approximants (as used in [46, 42]), which are rational functions of the form

\[
[L/K](w) = \frac{\sum_{l=0}^{L} c_{1,l} w^l}{\sum_{k=0}^{K} c_{2,k} w^k},
\]

and have \( L + K + 2 \) parameters to be set. As a normalization constant we can fix \( c_{1,0} = 1 \). The rest of them can be calculated from the coefficients of the power series, by asking the derivatives of the Padé approximant in \( w = 0 \) and 1 to be equal to the derivatives of the power series of the performance metrics in \( w = 0 \) and 1 (see [46, 42]).
5 A Riemann boundary value problem

Random walks in the quarter plane, which give rise to functional equations as given in (10) are thoroughly discussed in [6, 7, 8]. Our aim is to first obtain \( \Pi_0(x, y) \) by formulating and solving a Riemann boundary value problem, and then, using that result to obtain the rest terms \( \Pi_k(x, y) \) from (8).

The analysis of the kernel equation \( U(x, y) := xy - \psi(x, y) \), where \( \psi(x, y) := R(x, y)T(x, y) \), is the crucial step to obtain \( \Pi_0(x, y) \), which is regular for \(|x| < 1\), continuous for \(|x| \leq 1\) for every fixed \(y\) with \(|y| \leq 1\); and similarly, with \(x, y\) interchanged. Obviously, this means that \( \Pi_0(x, y) \) should be finite for every zerotuple \((\bar{x}, \bar{y})\) of the kernel \(U(x, y)\) in \(|x| \leq 1, |y| \leq 1\). Let

\[
A := \{ (x, y) : \psi(x, y) = 0, |x| \leq 1, |y| \leq 1 \}.
\]

The approach is very briefly summarized as follows (see also [6, 7, 8]):

1. Restrict the analysis to a subset of \(A\), in which

\[
K(\bar{x}, \bar{y})[(1-w)\Pi_0(\bar{x}, 0) - w\Pi_0(0, \bar{y})] + C(\bar{x}, \bar{y})\Pi_0(0, 0) = 0, (\bar{x}, \bar{y}) \in A.
\]

2. Our aim is to construct through (24) the boundary function \( \Pi_0(x, 0) \) (resp. \( \Pi(0, y) \)) to be regular in \(|x| < 1\) (resp. \(|y| < 1\)) and continuous in \(|x| \leq 1\) (resp. \(|y| \leq 1\)). Then, \( \Pi_0(x, y) \), and as a consequence, all the \( \Pi_k(x, y) \), \( k = 1, \ldots, N \) can be obtained through (10). The key vehicle to accomplish this task is to find two curves \( S_1 \subset \{ x \in \mathbb{C} : |x| \leq 1 \} \), \( S_2 \subset \{ y \in \mathbb{C} : |y| \leq 1 \} \), and a one-to-one map \( x = \omega(y) \) from \( S_2 \) to \( S_1 \) such that \((\omega(y), \bar{y})\) is a zerotuple of \(U(x, y)\) for all \(\bar{y} \in S_2\). The analyticity of \(U(x, y)\) in \(x, y\) implies that by analytic continuation all zerotuples of the kernel can be constructed staring from \(S_2\).

5.1 Kernel analysis & preliminary results

Note that \( \psi(0, 0) = 0 \), and thus, kernel analysis is done following the lines in [8 Sections II.3.10-3.12]. Consider the kernel for

\[
x = gs, y = gs^{-1}, |g| \leq 1, |s| = 1.
\]

defines for \( s = e^{i\phi} \) a one-to-one mapping \( f \) such that \( x(\phi) = f(y(\phi)) \) or \( y(\phi) = f^{-1}(x(\phi)) \), \( \phi \in [0, 2\pi) \). Moreover,

\[
U(gs, gs^{-1}) = 0 \iff g^2 = \psi(gs, gs^{-1}).
\]

**Theorem 2** Under stability condition (12), the kernel \( U(gs, gs^{-1}) \) has in \(|g| \leq 1\) exactly two zeros, of which one is identically zero. Denote the other zero by \( g = g(s) \), where \( g(1) = 1 \). Moreover, for \(|s| = 1\), \( g(s) = -g(-s) \), \( g(s) = g(s) \).

**Proof 2** See Appendix [9]

Let,

\[
S_1 := \{ x : x = g(s)s, |s| = 1 \}, \quad S_2 := \{ y : y = g(s)s^{-1}, |s| = 1 \},
\]

where \( g(s) \) the positive zero of the kernel.
To proceed we have to show that both $S_1$ and $S_2$ are simple and smooth. Although it is not difficult to show that under stability conditions, $S_1$, $S_2$ are both smooth (i.e., $\frac{d}{ds}g(s)$ exists in $|s| = 1$), it is not possible to prove that, for general values of the system parameters, that the contours $S_1$ and $S_2$ are simply connected. This is due to the variety of possible locations of $x = 0$, and $y = 0$ with respect to $S_1$ and $S_2$, respectively. Numerical results have shown that for small values of $w$ both $S_1$ and $S_2$ are simply connected; see Figures 5, 6. So, we restrict to the case where $S_1$ and $S_2$ are simple curves. In our case (see [8, II.3.10]), this problem refers to the relation among $q_{-1,0}(0), q_{0,-1}(0)$. In particular

1. If $q_{-1,0}(0) < q_{0,-1}(0) \Rightarrow x = 0 \in S_1^+, y = 0 \in S_2^-$,
2. If $q_{-1,0}(0) = q_{0,-1}(0) \Rightarrow x = 0 \in S_1, y = 0 \in S_2$,
3. If $q_{-1,0}(0) > q_{0,-1}(0) \Rightarrow x = 0 \in S_1^-, y = 0 \in S_2^+$,

where $S_j^+$ (resp. $S_j^-$) denotes the interior (resp. the exterior) of $S_j$, $j = 1, 2$. Furthermore, if $s$ traverses the unit circle once, then $S_1$ is traversed twice anticlockwise, and $S_2$ is traversed twice clockwise.

For the functions

$$x = g(s)s, y = g(s)s^{-1}, \ |s| = 1,$$

which forms a zero-pair of the kernel $U(x, y) = 0$, we consider the following boundary value problem: Construct a simple smooth contour $\mathcal{L}$, a function $\lambda(z), z \in \mathcal{L}$, and two functions $x(z), y(z)$ such that

1. $z = 0 \in \mathcal{L}^+, z = 1 \in \mathcal{L}, z = \infty \in \mathcal{L}^-$,
2. $\lambda(z) : \mathcal{L} \to [0, 2\pi), \lambda(1) = 0$,
3. $x(z)$ is regular for $z \in \mathcal{L}^+$, continuous and univalent in $\mathcal{L} \cup \mathcal{L}^+$.
4. $y(z)$ is regular for $z \in \mathcal{L}^-$, continuous and univalent in $\mathcal{L} \cup \mathcal{L}^-.$
5. $x(z)$ has a simple zero at $z = 0, x(1) = 1, \lim_{z \to 0} \frac{x(z)}{z} > 0.$
the first case, where 1).

6. $y(z)$ has a simple zero at $z = \infty$, $y(1) = 1$, $\lim_{|z| \to \infty} |zy(z)| > 0$.

7. $x(z)$ (resp. $y(z)$) maps $\mathcal{L}^+$ (resp. $\mathcal{L}^-$) conformally to $S_1^+$ (resp. $S_2^+$).

8. For $z \in \mathcal{L}$, $(x^+(z), y^-(z))$ is a zero-pair of the kernel $U(x, y) = 0$, where $x^+(z) = \lim_{t \to z, t \in \mathcal{L}^+} x(t)$, $y^-(z) = \lim_{t \to z, t \in \mathcal{L}^-} y(t)$.

The solution of this boundary value problem depends on the the cases regarding the position of $x = 0$, $y = 0$ with respect to $S_1$ and $S_2$; see [8, II.3.10]. Without loss of generality, we only consider the first case, where $x = 0 \in S_1^+$, $y = 0 \in S_2^-$. The others can be treated analogously (the second case is much more complicated; see [8, II.3.12]). Following [8, II.3.10-3.11], and setting

$$x^+(z) = g(e^{\frac{i}{2} \lambda(z)})e^{\frac{i}{2} \lambda(z)}, \quad y^-(z) = g(e^{\frac{i}{2} \lambda(z)})e^{-\frac{i}{2} \lambda(z)}, \quad z \in \mathcal{L},$$

we obtain

$$x(z) = ze^{\frac{1}{2\pi}} \int_{\zeta \in \mathcal{L}} |\log\left(e^{\frac{i}{2} \lambda(z)}\right)| \frac{\zeta^{z+1} - \zeta^{-z-1}}{\sqrt{\zeta}} \, d\zeta, \quad z \in \mathcal{L}^+, \quad y(z) = e^{-\frac{1}{2\pi}} \int_{\zeta \in \mathcal{L}} |\log\left(e^{\frac{i}{2} \lambda(z)}\right)| \frac{\zeta^{z+1} - \zeta^{-z-1}}{\sqrt{\zeta}} \, d\zeta, \quad z \in \mathcal{L}^-.$$

The relation for the determination of $\mathcal{L}$ and $\lambda(z)$, $z \in \mathcal{L}$ is

$$e^{i\lambda(z)} = ze^{\frac{2}{\pi i}} \int_{\zeta \in \mathcal{L}} |\log\left(e^{\frac{i}{2} \lambda(z)}\right)| \frac{\zeta^{z+1} - \zeta^{-z-1}}{\sqrt{\zeta}} \, d\zeta, \quad z \in \mathcal{L}. \quad (27)$$

An equivalent to (27) integral equations is

$$g(e^{\frac{i}{2} \lambda(z)}) = e^{\frac{1}{4\pi}} \int_{\zeta \in \mathcal{L}} |\log\left(e^{i\lambda(z)}\right)| \frac{\zeta^{z+1} - \zeta^{-z-1}}{\sqrt{\zeta}} \, d\zeta, \quad z \in \mathcal{L}, \quad (28)$$

with $\mathcal{L} = \{ z : z = \rho(\phi)e^{i\phi}, 0 \leq \phi \leq 2\pi \}, \theta(\phi) = \lambda(\rho(\phi)e^{i\phi})$. Separating real and imaginary parts in (28) we obtain two singular integral equations in the two unknowns functions $\rho(.)$, $\theta(.)$. Note that (28) may be regarded as a generalization of the Thoerdsen’s integral equation, see [8, Section IV.2.3].
Note finally that due to the maximum modulus principle [32]:

\[ |x(z)| < 1, \ z \in \mathcal{L}^+ \cup \mathcal{L}, \ |y(z)| < 1, \ z \in \mathcal{L}^- \cup \mathcal{L}. \]

### 5.2 Solution of the functional equation

The following theorem is the main result of this section.

**Theorem 3** Under stability condition [12], for \( x \in S_1^+, \ y \in S_2^+ \),

\[
\Pi_0(x, y) = \frac{T(x, y)\Pi_0(0, 0)}{\psi(x, y) - xy} \times \left( \frac{K(x, y)}{2\pi i} \int_{\zeta \in \mathcal{L}} H(\zeta) \left[ \frac{1}{\zeta - x_{10}(z)} - \frac{1}{\zeta - y_{10}(z)} \right] d\zeta + C(x, y) \right),
\]

\[
\Pi_k(x, y) = F_{0,k}(x, y)\Pi_0(x, y), \ k = 1, \ldots, N,
\]

\[
F_{0,k}(x, y) = \frac{1}{\det[\mathbf{L}(x, y)^T]} \left( cof \mathbf{L}(x, y)^T \right)^T \theta(0),
\]

where \( x_{10}(\cdot), y_{10}(\cdot) \) conformal mappings of \( S_1, S_2 \) onto the smooth and closed contour \( \mathcal{L} \), respectively, \( \Pi_0(0, 0) \) as given in [12], and \( H(z) = \frac{C(x^+(z), y^-(z))}{K(x^+(z), y^-(z))}, \ z \in \mathcal{L} \).

**Proof 3** Since \((x^+(z), y^-(z)), \ z \in \mathcal{L}\) is a zero pair of the kernel, it should hold for \( z \in \mathcal{L} \)

\[
K(x^+(z), y^-(z))[1 - w]\Pi_0(x^+(z), 0) - w\Pi_0(y^-(z))]
+ \Pi(x^+(z), y^-(z))\Pi_0(0, 0) = 0,
\]

or equivalently

\[
\Pi_0(x^+(z), 0) = \frac{w}{1 - w}\Pi_0(0, y^-(z)) + \frac{\Pi(0, 0)}{1 - w} H(z).
\]

Note that \( \frac{w}{1 - w} \) never vanishes and thus \( \text{index}_{z \in \mathcal{L}}(\frac{w}{1 - w}) = 0 \). Moreover, \( \frac{w}{1 - w} \) satisfies (trivially) the Holder condition on \( \mathcal{L} \). The numerator and the denominator of \( H(\cdot) \) both satisfy the Holder condition on \( \mathcal{L} \), and the denominator never vanishes on \( \mathcal{L} \), except for \( z = 1 \).

Therefore, we have the hollowing non-homogeneous Riemann boundary value problem:

1. \( \Pi_0(x(z), 0) \) should be regular for \( z \in \mathcal{L}^+ \) and continuous in \( \mathcal{L}^+ \cup \mathcal{L} \), with \( \lim_{\zeta \to z, \zeta \in \mathcal{L}^+} \Pi_0(x(z), 0) = \Pi_0(x^+(z), 0), \ z \in \mathcal{L} \),
2. \( \Pi_0(0, y(z)) \) should be regular for \( z \in \mathcal{L}^- \) and continuous in \( \mathcal{L}^- \cup \mathcal{L} \), with \( \lim_{\zeta \to z, \zeta \in \mathcal{L}^-} \Pi_0(0, y(z)) = \Pi_0(0, y^-(z)), \ z \in \mathcal{L} \),
3. For \( z \in \mathcal{L} \), the boundary condition (31) is satisfied.

The solution of the above presented boundary value problem [13] is given by:

\[
(1 - w)\Pi_0(x(z), 0) = \frac{\Pi_0(0, 0)}{2\pi i} \int_{\zeta \in \mathcal{L}} H(\zeta) \frac{d\zeta}{\zeta - z} + w\Pi_0(0, 0), \ z \in \mathcal{L}^+, \]

\[
w\Pi_0(0, y(z)) = \frac{\Pi_0(0, 0)}{2\pi i} \int_{\zeta \in \mathcal{L}} H(\zeta) \frac{d\zeta}{\zeta - z} + w\Pi_0(0, 0), \ z \in \mathcal{L}^-,
\]

where \( \Pi_0(0, 0) \) is given in [12]. Denote by,

\[ z = x_{10}(x), \ x \in S_1^+, \ z = y_{10}(y), \ y \in S_2^+ \],

15
the inverse mappings, i.e., the conformal mappings of $S^+_j$ onto $\mathcal{L}$, $j = 1, 2$, respectively. Since $S_1$, $S_2$, $\mathcal{L}$ are smooth, the theorem of corresponding boundaries implies that $x_{10}(.)$ maps $S_1$ onto $\mathcal{L}$, and $y_{10}(.)$ maps $S_2$ onto $\mathcal{L}$. Thus, for $x \in S^+_1$, $y \in S^+_2$

\[(1 - w)\Pi_0(x, 0) = \frac{\Pi_{0}(0, 0)}{2\pi} \int_{\zeta \in \mathcal{L}} H(\zeta) \frac{dc}{\zeta - x_{10}(z)} + w\Pi_0(0, 0), \ z \in \mathcal{L}^+,\]

\[w\Pi_0(0, y(z)) = \frac{\Pi_{0}(0, 0)}{2\pi} \int_{\zeta \in \mathcal{L}} H(\zeta) \frac{dc}{\zeta - y_{10}(z)} + w\Pi_0(0, 0), \ z \in \mathcal{L}^-,\]

Substituting (33) in (10), it follows for $x \in S^+_1$, $y \in S^+_2$,

\[\Pi_0(x, y) = \frac{T(x, y)\Pi_{0}(0, 0)}{\psi(x, y) - xy} \times \left\{ \frac{K(x, y)}{2\pi} \int_{\zeta \in \mathcal{L}} H(\zeta) \left[ \frac{1}{\zeta - x_{10}(z)} - \frac{1}{\zeta - y_{10}(z)} \right] d\zeta + C(x, y) \right\}.
\]

By using analytic continuation arguments we can also obtain an expression for the $\Pi_0(x, y)$, for $|x| \leq 1$, $|y| \leq 1$. Using now (3), we obtain expressions for $\Pi_k(x, y)$, $k = 1, \ldots, N$.

Remark 4 Note that the analysis performed in Section 3 is general enough to be applied in the case where at least one of the following conditions are satisfied

\[\frac{q_{j-1}(0)}{q_{j-1}(0)} + \frac{q_{j+1}(0)}{q_{j+1}(0)} \neq 1, \ i, j \in \mathbb{H}^+.
\]

However, some extra technical difficulties regarding the investigation of the properties of the curves $S_1$, $S_2$ will arise. Moreover, the resulting Riemann boundary value problem will be far more complicated, where at the same time the computation of its index is a challenging task.

6 Explicit expressions of the moments for a special case

In this section we provide explicit expressions for the moments of $\{Z(t)\}$ when we consider the symmetrical assumption when the background state $J(t) = 0$. To enhance the readability we focus on the QBD version of $\{Y(t), t \geq 0\}$:

\[q_{-j, j}(0) = q_{j, -1}(0), \quad q_{j, j}(0) = q_{-j, -1}(0), \quad j \in \mathbb{H}^+,
\]

\[q_{0,j}(0) = q_{j, 0}(0), \quad j \geq 1, \quad \theta_{0,k} = \theta, \quad A_{0,k}(x, y) = A(x, y), \quad k = 1, \ldots, N.
\]

Clearly, symmetry implies $w = 1/2$, and $\Pi_0(1, 0) = \Pi_0(0, 1)$, $\Pi_0^{(1)}(1, 1) = \Pi_0^{(2)}(1, 1)$, where $\Pi_0^{(j)}(1, 1)$, $j = 1, 2$ the derivatives of $\Pi_0(x, y)$ with respect to $x$ and $y$, respectively, at $(1, 1)$. Note that under symmetry assumptions, the stability condition (12) can be written after some algebra as

\[\rho := \frac{T(1, 1)(q_{1, 0}(0) + q_{1, 1}(0))}{T(1, 1)(N\theta + q_{-1, 0}(0)) + N\theta T^{(1)}(1, 1) - 1} < 1,
\]

and

\[\Pi_0(0, 0) = \frac{2\Pi_0(1, 1)[N\theta T^{(1)}(1, 1) + T(1, 1)] + T(1, 1)(q_{-1, 0}(0) - q_{1, 0}(0) - q_{1, 1}(0) - 1)}{T(1, 1)q_{0, 0}(0)},
\]

where $T^{(1)}(1, 1)$ the derivative of $T(x, y)$ with respect to $x$ at $(1, 1)$. Let $M_m = E(X_m)$, $m = 1, 2$, the first moment of the stationary distribution of $\{X_m(t); t \geq 0\}$. The following theorem provides the main result of this section.
Theorem 4 When \( \rho < 1 \),

\[
M_1 = \Pi_0(1,1) \sum_{k=0}^{N} \frac{\partial}{\partial y} F_{0,k}(x,1)|_{x=1} + M(1 + \sum_{k=1}^{N} F_{0,k}(1,1)),
\]

\[
M_2 = \Pi_0(1,1) \sum_{k=1}^{N} \frac{\partial}{\partial y} F_{0,k}(1,y)|_{y=1} + M(1 + \sum_{k=1}^{N} F_{0,k}(1,1))
\]

where

\[
M := \frac{\Pi_0(0,0)q_{0,-1}(0)[2T^{(1)}(1,1)+T(1,1)]+(S/2)(-1)}{2[N\theta(T^{(1)}(1,1)+T(1,1)+T(1,1)(q_{1,0}(0)-q_{1,0}(0)-q_{1,1}(0))-1]},
\]

\[
S := 2 \left[ 2T^{(1)}(1,1)(N\theta + q_{1,0}(0) - q_{1,0}(0) - q_{1,1}(0)) + T(1,1) \right.
\]

\[
\times \left( N\theta + 2q_{1,0}(0) - 4q_{1,0}(0) - 5q_{1,1}(0) \right) + N\theta \frac{\partial^2}{\partial x^2} T(x,x)|_{x=1},
\]

and \( \Pi_0(1,1) \) as given in (9).

Proof 4 Note that

\[
M_1 := \sum_{k=0}^{N} \frac{\partial}{\partial y} \Pi_k(x,1)|_{x=1} = \sum_{k=0}^{N} \Pi_k^{(1)}(1,1)
\]

\[
= \Pi_0^{(1)}(1,1) + \sum_{k=1}^{N} [\Pi_0(1,1) \frac{\partial}{\partial y} F_{0,k}(x,1)|_{x=1} + F_{0,k}(1,1) \Pi_k^{(1)}(1,1)]
\]

\[
= \Pi_0^{(1)}(1,1) \left[ 1 + \sum_{k=1}^{N} F_{0,k}(1,1) \right] + \Pi_0(1,1) \sum_{k=1}^{N} \frac{\partial}{\partial y} F_{0,k}(x,1)|_{x=1}.
\]  

A similar expression can also be derived for \( E(X_2) \). Thus, we only need to find an expression for the \( \Pi_0^{(1)}(1,1) \). Setting \( x = y \), we realise that \( K(x,x) = 0 \), and thus, (10) is rewritten as

\[
\Pi_0(x,x) \frac{T(x,x)R(x,x) - x^2}{x - 1} = T(x,x)q_{0,-1}(0)\Pi_0(0,0).
\]  

By differentiating (38) with respect to \( x \) at \( x = 1 \), applying once the L'Hospital rule, and having in mind the symmetry condition at phase 0, we obtain

\[
\Pi_0^{(1)}(1,1) = \Pi_0^{(2)}(1,1) = M,
\]

as given in the first in (36). Substituting back in (37) we obtain the first in (35). Similarly we obtain the expression for \( M_2 \).

7 Numerical results

In the following, we present some numerical illustration of the theoretical results presented in Section 4. In particular, we consider a two-node queueing network with coupled processors and two types of service interruption. Two classes of jobs arrive according to independent Poisson processes. Class \( P_i \), \( i = 1,2 \) is routed to queue \( i \). Each job at queue \( i \) requires exponentially distributed service time with rate \( \nu_i \). The network is operating for an exponentially distributed time with rate \( \theta_{0,j} = \gamma_j \), and then switch to the fail mode \( j, j = 1,2 \). In failed mode \( j \) stays for an exponentially distributed time and then, switch to operating mode 0 with rate \( \theta_{j,0} = \tau_j \). When the network is in a failed mode of either type it cannot provide service.

Jobs arrival rates are \( \lambda_i^{(j)} \), \( i = 1,2, j = 0,1,2 \), i.e., depend on the type of the mode (either operating or failed). When the network is in operational mode, and both queues are non-empty, queue 1 serves at a rate \( w\nu_1 \), and queue 2 at a rate \( (1-w)\nu_2 \). If only one queue is non-empty it serves at full capacity, i.e., with rate \( \nu_i \). Upon receiving service at queue 1 (resp. 2) the job is either routed to queue 2 (resp.
1) with probability $r_{12}$ (resp. $r_{21}$), or leaves the network with probability $1 - r_{12}$ (resp. $1 - r_{21}$). For ease of computations we assume $a_{0,0}^{(k,m)} = 1$, $k \neq m$, i.e., the change in mode does not change the number of jobs in queues. Moreover, note that arrivals occur one by one, i.e., $\mathbb{H} = \{-1, 0, 1\}$, $\mathbb{H} = \{0, 1\}$ (the process $\{Y(t)\}$ is a QBD).

Note that in such a case, the stability condition \[12\] reads
\[
\begin{align*}
\frac{r_{12}}{\tau_{1} + \gamma_{1} r_{2}} & \left( \frac{\Lambda_{1}^{(0)}}{\nu_{1}^{(0)}} + \frac{\Lambda_{2}^{(0)}}{\nu_{2}^{(0)}} \right) + \frac{\gamma_{1}}{\tau_{1} + \gamma_{1} + \gamma_{2} r_{1}} \left( \frac{\Lambda_{1}^{(1)}}{\nu_{1}^{(1)}} + \frac{\Lambda_{2}^{(1)}}{\nu_{2}^{(1)}} \right) \\
& + \frac{r_{21}}{\tau_{2} + \gamma_{2} r_{1}} \left( \frac{\Lambda_{2}^{(2)}}{\nu_{2}^{(2)}} + \frac{\Lambda_{1}^{(2)}}{\nu_{1}^{(2)}} \right) < \frac{12}{\tau_{1} r_{2} + \gamma_{2} r_{1}},
\end{align*}
\]

where $\Lambda_{1}^{(k)} = \frac{\Lambda_{1}^{(k)}}{1 - r_{12} r_{21}^{(0)}}, \Lambda_{2}^{(k)} = \frac{\Lambda_{2}^{(k)}}{1 - r_{12} r_{21}^{(2)}}, k = 0, 1, 2$. Note that \[(39)\] has a clear probabilistic interpretation, since \[(39)\] equals the amount of work brought into the system per time unit, and in order the system to be stable, should be less than the amount of work departing the system per time unit.

### 7.1 Numerical validation & Influence of system parameters as $w \to 0$

Figure 7 depicts the approximations \[(23)\] as a function of $w$ for increasing values of $M$, under the set up given in Table 1. The horizontal line (i.e., $M = 0$) refers to the special case where the second queue has priority over the first one. As expected, Figure 7 confirms that the PSA approximations are accurate when $w$ is close to 0, and clearly, by adding more terms, we can have larger regions for $w$ where the accuracy is good.

| Parameters     | Values       |
|----------------|--------------|
| $(\lambda_{1}^{(1)}, \lambda_{2}^{(1)})$ | $(0.5, 0.6)$ |
| $(\lambda_{1}^{(2)}, \lambda_{2}^{(2)})$ | $(0.1, 0.2)$ |
| $(\tau_{1}, \tau_{2})$ | $(5, 6)$ |
| $(\gamma_{1}, \gamma_{2})$ | $(5, 8)$ |
| $(r_{12}, r_{21})$ | $(0.5, 0.8)$ |

We now focus on the influence of system parameters on $E(X_{2})$ for the near priority system as $w \to 0$. It is seen that as $\gamma_{2}$ increases the mean queue content in station 2 increases, and that increase becomes more apparent as $\lambda_{1}^{(0)}, \lambda_{2}^{(0)}$ increase too. This is expected since by increasing the load in system, the station 1 has always customers. Thus, sharing the server with that station, even for a small percentage of the time can have a large influence, especially when the rate of failures increases too. Similar behaviour is expected if we fix $\gamma_{2}$, and let $\gamma_{1}$ to vary.

\[\text{Note that } \frac{r_{12}}{\tau_{1} + \gamma_{1} r_{2} + \gamma_{2} r_{1}} \left( \frac{\Lambda_{1}^{(0)}}{\nu_{1}^{(0)}} + \frac{\Lambda_{2}^{(0)}}{\nu_{2}^{(0)}} \right) \text{ (resp. } \frac{\gamma_{1}}{\tau_{1} + \gamma_{1} + \gamma_{2} r_{1}} \left( \frac{\Lambda_{1}^{(1)}}{\nu_{1}^{(1)}} + \frac{\Lambda_{2}^{(1)}}{\nu_{2}^{(1)}} \right) \text{) refers to the amount of work that arrive at the system per time unit when the network is in the operating mode (resp. in the failed mode } k = 1, 2), \text{ while a job can depart from the network only when it is in the operating mode, i.e., with probability } \frac{r_{21}}{\tau_{2} + \gamma_{2} r_{1}}.\]
Figure 7: Truncation approximation for $(\lambda_1^{(0)}, \lambda_2^{(0)}) = (1, 0.8), (\gamma_1, \gamma_2) = (0.5, 0.8)$.

Figure 8: First order correction as $w \to 0, \gamma_1 = 0.5$. 
7.2 The symmetrical model

We now focus on the system analysed in Section 6, and set $\lambda^{(0)}_1 = \lambda^{(0)}_2 := \lambda$, $\gamma_1 = \gamma_2 := \gamma$ and $w = 1/2$. The rest of the parameters values are the same as those considered in subsection 7.1, i.e., we have the following set up:

\begin{table}[h]
\centering
\begin{tabular}{|c|c|}
\hline
Parameters & Values \\
\hline
$(\lambda^{(1)}_1, \lambda^{(1)}_2)$ & (0.5, 0.6) \\
$(\lambda^{(2)}_1, \lambda^{(2)}_2)$ & (0.1, 0.2) \\
$(\tau_1, \tau_2)$ & (5, 8) \\
r_{12} = r_{21} = r & 0.3 \\
\hline
\end{tabular}
\caption{Set-up.}
\end{table}

Figure 9 we shows the total number of jobs in the system, i.e., $E(X_1 + X_2)$, as a function of $\lambda$ and $\gamma$. We observe sharp increase in $E(X_1 + X_2)$ when $\gamma$ take small values for increasing $\lambda$. This is expected since by increasing $\gamma$, the system is most of the time in failed modes, and since the parameters are considered fixed in these modes, even if we increase $\lambda$ (i.e., the arrival rate when the network is in operational mode), it will not cause significant effect. However, when $\gamma$ takes small values, the network remains in operational mode more time, and thus, the effect of $\lambda$ is crucial (as well as of service rate $\nu$).

In Figure 10 we provide a comparison between the explicit expression on the expected number of jobs in queue 2 (see the second in (35)), and the one derived using the PSA for $M = 10$ and $M = 20$. We observe that when $\lambda$ takes small values, PSA provides very accurate results. As $\lambda$ increases, PSA accuracy is improved by providing further terms.
8 Conclusion

This work focused on the stationary analysis of a certain class of Markov-modulated reflected random walk with potential applications in the modelling of two-node queueing networks with coupled queues and service interruptions. We presented three methodological approaches based on the generating function approach: i) The power series approximation (PSA) method, under which we derived power series expansions of the pgfs of the stationary joint distributions for each state of the phase process, and reveal the flexibility of the PSA approach in even more complicated setting. ii) The theory of Riemann boundary value problems. iii) We also provided explicit expressions for the first moments of the stationary distribution when we assume the symmetrical assumption at the phase $J(t) = 0$. This result is obtained without solving a boundary value problem.

Our developed technique is general enough to deal with the analysis of Markov-modulated coupled queueing systems. Examples are the standard, as well as the G-queueing networks with service interruptions and coupled queues. Potential extensions of the PSA refer to the case of three (or more) queues. Unfortunately, the theory of boundary value problems has not been developed for problems with more than two queues.

A Appendix A

Theorem 5 For every $|x| = 1, x \neq 1$, $G(x, y) = 0$ has a unique zero, say $Y(x)$, in the disc $|y| < 1$.

Proof 5 In order to enhance the readability, we consider the case where $\theta_{i,j} = 0$, $i, j = 1, \ldots, N$, so that $T(x, y)$ is given in [14], and considered the QBD version of $\{Y(t); t \geq 0\}$, i.e, $\mathbb{H} = \{-1, 0, 1\}$, $\mathbb{H}^+ = \{0, 1\}$. Analogous arguments can be applied even in the general case.

The proof is based on the application of Rouche’s theorem [17, 17]. Note that $G(x, y) = 0$ is rewritten as $G_0(x, y) = h(x, y) := y \sum_{j=1}^{N} \frac{\theta_{0,j}A_{0,j}(x,y)}{B_j(x,y)}$, where $G_0(x, y) = f(x, y) - g(x, y)$, with $f(x, y) := y[\theta_{0,0} + q_{1,0}(0)(1 - x) + q_{0,1}(0) + q_{1,1}(0) + q_{0,-1}(0) + q_{1,-1}(0)]$, $g(x, y) := -y^2(q_{0,1}(0) + q_{1,0}(0)(1 - x) + q_{0,1}(0) + q_{1,1}(0) + q_{0,-1}(0) + q_{1,-1}(0))x$. We first show that $G_0(x, y) = 0$ has a unique root in $|y| < 1$, for
\(|x| = 1, \ x \neq 1\). Then, for \(|x| = 1, \ x \neq 1,\)
\[|f(x, y)| = |y|(\theta_{0, 1} + q_{1, 0}(1 - x) + q_{0, 1}(0) + q_{1, 1}(0) + q_{0, 1}(0) + q_{1, 1}(0)| \]
\[\geq |y|(\theta_{0, 1} + q_{1, 0}(1 - x) + q_{0, 1}(0) + q_{1, 1}(0) + q_{0, 1}(0) + q_{1, 1}(0)) \]
\[> |y|(\theta_{0, 1} + q_{1, 0}(1 - x) + q_{0, 1}(0) + q_{1, 1}(0)), \]
\[|g(x, y)| \leq |y|^2(q_{1, 0}(1 - x) + q_{1, 1}(0)|x|) + q_{1, 1}(0)|x| \]
\[= |y|^2(q_{1, 0}(1 - x) + q_{1, 1}(0) + q_{1, 1}(0) + q_{1, 1}(0)). \]

Then, for all \(y\) such that \(|y| = 1\), \(y \neq 1\),
\[|g(x, y)| \leq q_{0, 1}(0) + q_{1, 1}(0) + q_{1, 1}(0) + q_{1, 1}(0) < |f(x, y)|, \ |y| = 1, \ |x| = 1, \ x \neq 1,\]
which implies by Rouché’s theorem, see, e.g. [47], that \(G_0(x, y)\) has as many zeros, counted according to their multiplicity, inside \(|y| = 1\) as \(f(x, y)\). Since \(f(x, y)\) has only one zero of multiplicity 1 at \(y = 0\), yields that for every \(x\) with \(|x| = 1, \ x \neq 1\), \(G_0(x, y) = 0\) has one root inside \(|y| = 1\).

Now, note that
\[|G_0(x, y)| = |f(x, y) - g(x, y)| \geq ||f(x, y)| - |g(x, y)|| > \theta_{0, 1}.\]

Moreover,
\[| - h(x, y)| = | - y \sum_{j=1}^{N} \theta_{j, 0} \theta_{0, 1} A_{j, 0}(x, y) A_{0, j}(x, y) | \leq |y| \sum_{j=1}^{N} \theta_{j, 0} \theta_{0, 1} |A_{j, 0}(x, y)||A_{0, j}(x, y)| \]
\[< \sum_{j=1}^{N} \theta_{j, 0} \theta_{0, 1} = \theta_{0, 1}, \]
since \(|A_{0, j}(x, y)| = 1, \ |A_{j, 0}(x, y)| = 1, \ |D_j(x, y)| \geq \theta_{j, 0}, \text{ for } |x| = 1, \ |y| = 1, \ x \neq 1. \)
Thus,
\[| - h(x, y)| < \theta_{0, 1} < |G_0(x, y)|, \ |x| = 1, \ |y| = 1, \ x \neq 1.\]

Therefore, Rouché’s theorem implies that \(G_0(x, y) = y \sum_{j=1}^{N} \theta_{j, 0} \theta_{0, 1} D_j(x, y), \ i.e., \ G(x, y) = 0\) has the same number of zeros for every \(x\) with \(|x| = 1, \ x \neq 1, \text{ inside } |y| = 1, \text{ as } G_0(x, y), \text{ which we shown that has exactly one. Thus, denote this zero as } y = Y(x), \ |x| = 1, \ x \neq 1, \text{ with } |Y(x)| < 1.\]

\section*{B Appendix B}

\textbf{Proof of Theorem 2} To enhance the readability, we consider the case where \(\theta_{i, j} = 0, \ i, j = 1, \ldots, N, \text{ and considered the QBD version of } \{Y(t); \ t \geq 0\}, \ i.e., \mathbb{H} = \{-1, 0, 1\}, \mathbb{H}^+ = \{0, 1\}. \text{ Analogous arguments can be applied even in the general case. The proof is based on the application of Rouché’s theorem} [47][1]. \text{ Note that } U(g, g^{-1}) = 0 \text{ is written as } R(g, g^{-1}) = g^2 \sum_{j=1}^{N} \theta_{j, 0} \theta_{0, 1} D_j(x, y). \text{ We first show that } R(g, g^{-1}) = 0 \text{ has a single root in } |g| \leq 1, \ |s| = 1. \text{ Indeed, } R(g, g^{-1}) = 0 \text{ is rewritten as } g^2 = L(g, g^{-1}) := \frac{q_{-1, 0}(0)g + q_{0, 0}(0)g}{D_0(g, g^{-1}) + q_{-1, 0}(0) + q_{0, 0}(0) + q_{-1, 1}(0)(1-s^2) + q_{0, 1}(0)(1-s^2)}. \text{ Note that the denominator of } L(g, g^{-1}) \text{ never vanishes for } |g| \leq 1, \ |s| = 1 \text{ and in particular, } \]
\[|D_0(g, g^{-1}) + q_{-1, 0}(0) + q_{0, 0}(0) + q_{-1, 1}(0)(1-s^2) + q_{0, 1}(0)(1-s^2)| \]
\[> \theta_{0, 1} + q_{-1, 0}(0) + q_{0, 0}(0). \]
Therefore,
\[
|L(gs, gs^{-1})| \leq \frac{|q_{-1,0}(0)| + |q_{0,-1}(0)|}{|D_0(gs, gs^{-1})| + |q_{-1,0}(0)| + |q_{0,-1}(0)| + |q_{-1,1}(0)| (1-s^2) + |q_{1,-1}(0)| (1-s^2)}.
\]

Thus, by applying Rouché’s theorem \( R(gs, gs^{-1}) = 0 \) has a single zero in \(|g| < 1, |s| = 1 \). Note also that
\[
|R(gs, gs^{-1})| \geq |g|^2 (\theta_0 + q_{1,0}(1 - gs) + q_{0,1}(1 - gs^{-1} + q_{1,1}(0))(1 - |g|^2)) > \theta_{0,}.
\]

On the other hand,
\[
|g|^2 \sum_{j=1}^N \frac{\theta_{j,0} \theta_{0,j} A_{j,0}(gs, gs^{-1}) A_{j,0}(gs, gs^{-1})}{D_j(gs, gs^{-1})} \leq \sum_{j=1}^N \frac{\theta_{j,0} \theta_{0,j}}{|D_j(gs, gs^{-1})|} < \theta_{0,} < |R(gs, gs^{-1})|.
\]

Therefore, since \( R(gs, gs^{-1}) \) has a single zero in \(|g| < 1, |s| = 1 \), Rouché’s theorem states that \( R(gs, gs^{-1}) = g^2 \sum_{j=1}^N \frac{\theta_{j,0} \theta_{0,j} A_{j,0}(gs, gs^{-1}) A_{j,0}(gs, gs^{-1})}{D_j(gs, gs^{-1})} \) has a single root in \(|g| < 1, |s| = 1 \). Equivalently, \( U(gs, gs^{-1}) = 0 \) has a a single root in \(|g| < 1, |s| = 1 \).

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