Homeomorphic Changes of Variable
and Fourier Multipliers

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Abstract. We consider the algebras $M_p$ of Fourier multipliers and show that for every bounded continuous function $f$ on $\mathbb{R}^d$ there exists a self-homeomorphism $h$ of $\mathbb{R}^d$ such that the superposition $f \circ h$ is in $M_p(\mathbb{R}^d)$ for all $p, 1 < p < \infty$. Moreover, under certain assumptions on a family $K$ of continuous functions, one $h$ will suffice for all $f \in K$. This may be contrasted with the known solution of Luzin’s problem related to the Wiener algebra.

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1. Introduction

To what extent the behavior of the Fourier series of a continuous function can be improved by a change of variable? The problem originates from a theorem of Bohr and Pál (see [1, Ch. 4, Sec. 12], [9], [23]), who showed that given a continuous real-valued function $f$ on the circle $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, there exists a self-homeomorphism $h$ of $\mathbb{T}$ such that the Fourier series of the superposition $f \circ h$ converges uniformly. In addition, the proof yields a condition on the decay of the Fourier coefficients of $f \circ h$; namely,

$$\hat{f} \circ h \in l^p(\mathbb{Z})$$

for all $p > 1$.

This result inspired the following problem posed by N. Lusin (see [1, Ch. 4, Sec. 12], [8, Ch. VII, Sec. 9]): Is it possible to attain condition (1) for $p = 1$? In other words, is it true that for every continuous function on $\mathbb{T}$ there exists a change of variable which brings it into the Wiener algebra $A(\mathbb{T})$?

It is worth noting that the original proof of the Bohr–Pál theorem involves Riemann’s theorem on conformal mappings. A purely real-analytic proof was found in [25].
A solution (in the negative) of Lusin’s problem was obtained in [22], where it is shown that there exists a continuous real-valued function $f$ on $\mathbb{T}$ such that $f \circ h \notin A(\mathbb{T})$ whenever $h$ is a self-homeomorphism of $\mathbb{T}$. Some ideas of P. Cohen were used in the proof. Simultaneously, the same result for a complex-valued $f$ was obtained in [10] (this weaker result amounts to the fact that, in general, there is no single change of variable which will bring two continuous real-valued functions into $A(\mathbb{T})$).

Subsequently, for certain function spaces, naturally arising in harmonic analysis, the question of whether every continuous function can be transformed by a suitable homeomorphic change of variable into a function that belongs to a given space, was studied by various authors. Some of these studies concern the possibility of simultaneous improvement of several functions by means of a single change of variable; we note in particular the work [11] where the Bohr–Pál theorem is extended to compact families of continuous functions.

For a survey on the subject see [23], [9]. For more recent results see [24], [4, Ch. 9], [15], [16], [18], [19]. We also mention the papers [14], [13] which study the growth of the partial sums of the Fourier series of $f \circ h$ for random homeomorphisms $h$.

In the present paper we investigate the problem on changes of variable in relation with the Fourier Multiplier Algebras. We consider multipliers on $\mathbb{R}^d$ and on the torus $\mathbb{T}^d$.

Let $G$ be one of the groups $\mathbb{R}^d$ or $\mathbb{Z}^d$ and let $\Gamma$ be its dual group, i.e., $\Gamma = \mathbb{R}^d$ or $\Gamma = \mathbb{T}^d$, correspondingly. Consider a function $m \in L^\infty(\Gamma)$ and the operator $Q$ defined by

$$\hat{Q}f = m \cdot \hat{f}, \quad f \in L^p \cap L^2(G), \quad (2)$$

where $\hat{\cdot}$ stands for the Fourier transform on $G$. The function $m$ is called an $L^p$ multiplier ($1 \leq p \leq \infty$) if

$$\|Qf\|_{L^p(G)} \leq c\|f\|_{L^p(G)}, \quad f \in L^p \cap L^2(G), \quad (3)$$

where $c > 0$ does not depend on $f$. The space of all such multipliers is denoted by $M_p(\Gamma)$. The norm on $M_p(\Gamma)$ is defined by setting $\|m\|_{M_p(\Gamma)}$ to be the smallest $c$ for which (3) holds. The space $M_p(\Gamma)$ equipped with this norm is a Banach algebra with the usual multiplication of functions. Clearly, if $p < \infty$, then the operator $Q$ that corresponds to the function $m$ can be uniquely extended to a bounded operator on $L^p(G)$, and retaining
the notation $Q$ for this extension, we have $\|Q\|_{L^p(G) \to L^p(G)} = \|m\|_{M_p(\Gamma)}$. We also note that the operator $Q$ is translation-invariant. The converse also holds for $p < \infty$: every translation-invariant bounded operator on $L^p(G)$ has the form (2), where $m \in M_p(\Gamma)$. For basic properties of multipliers see [3], [5].

It is known that $M_2$ coincides with $L^\infty$. It is also known that $M_1 = M_\infty$, and, at the same time, $M_1(\mathbb{T}^d)$ coincides with the Wiener algebra $A(\mathbb{T}^d)$ and $M_1(\mathbb{R}^d)$ coincides with the algebra $B(\mathbb{R}^d)$ of the Fourier transforms of (complex) bounded regular Borel measures on $\mathbb{R}^d$. Note that the negative solution of Luzin’s problem immediately implies a similar result for functions on the real line: there exists a bounded continuous real-valued function $f$ on $\mathbb{R}$ such that $f \circ h \notin B(\mathbb{R})$ for every self-homeomorphism $h$ of $\mathbb{R}$ (for details, see Sec. 5). Thus, in general, there is no change of variable which will bring a continuous real-valued function on $\mathbb{T}$ (a bounded continuous function on $\mathbb{R}$) into $M_1 = M_\infty$.

In this paper we show that for every bounded continuous function $f$ on $\mathbb{R}^d$ (for every continuous function $f$ on $\mathbb{T}^d$) there is a homeomorphic change of variable $h$ such that $f \circ h \in \bigcap_{1 < p < \infty} M_p$. Moreover, under certain natural assumptions on a function family $K$, one change of variable will suffice for all $f \in K$. An important role in the proof is played by a result of Sjörgen and Sjölin on Littlewood–Paley partitions.

The exact statements of our results are given in Section 2 below. Section 3 contains preliminaries. The proofs are given in Section 4. The concluding Section 5 contains several remarks and open problems; in particular, we focus on the famous Beurling–Helson theorem and briefly discuss some recent results and open questions related to this theorem and to its version for Multiplier Algebras.

2. Statement of Results

Let $f$ be a function defined on a set $E$ (we assume that $E \subseteq \mathbb{R}^d$ or $E \subseteq \mathbb{T}^d$). By $\omega(f, E, \delta)$ we denote the modulus of continuity of $f$ on $E$:

$$\omega(f, E, \delta) = \sup_{t_1, t_2 \in E, |t_1 - t_2| \leq \delta} |f(t_1) - f(t_2)|, \quad \delta \geq 0$$

(|$x$| stands for the length of a vector $x \in \mathbb{R}^d$). Let $K$ be a family of functions
on $E$. We define the modulus of continuity $\omega(K, E, \delta)$ of $K$ on $E$ by

$$\omega(K, E, \delta) = \sup_{f \in K} \omega(f, E, \delta), \quad \delta \geq 0.$$  

We say that $K$ is uniformly equicontinuous on $E$ if $\omega(K, E, \delta) \to 0$ as $\delta \to +0$.  

By $C(\mathbb{R}^d)$ and $C(\mathbb{T}^d)$ we denote the classes of continuous functions on $\mathbb{R}^d$ and $\mathbb{T}^d$, respectively. The results of this paper are the following two theorems.

**Theorem 1.** Let $f \in C(\mathbb{R}^d)$ be a bounded function. Then there exists a self-homeomorphism $h$ of $\mathbb{R}^d$ such that $f \circ h \in \bigcap_{1 < p < \infty} M_p(\mathbb{R}^d)$. Moreover, if a family $K \subseteq C(\mathbb{R}^d)$ of bounded functions is uniformly equicontinuous on every ball in $\mathbb{R}^d$, then there exists a self-homeomorphism $h$ of $\mathbb{R}^d$ such that $f \circ h \in \bigcap_{1 < p < \infty} M_p(\mathbb{R}^d)$ for all $f \in K$.

**Theorem 2.** Let $f \in C(\mathbb{T}^d)$. Then there exists a self-homeomorphism $h$ of $\mathbb{T}^d$ such that $f \circ h \in \bigcap_{1 < p < \infty} M_p(\mathbb{T}^d)$. Moreover, if a family $K \subseteq C(\mathbb{T}^d)$ is uniformly equicontinuous on $\mathbb{T}^d$, then there exists a self-homeomorphism $h$ of $\mathbb{T}^d$ such that $f \circ h \in \bigcap_{1 < p < \infty} M_p(\mathbb{T}^d)$ for all $f \in K$.

### 3. Preliminaries

In this section we recall some notions and facts from the Littlewood–Paley and multiplier theory.

A. For $1/p + 1/q = 1$ we have $M_p(\Gamma) = M_q(\Gamma)$ and $\| \cdot \|_{M_p} = \| \cdot \|_{M_q}$. If $1 \leq p_1 < p_2 \leq 2$, then $M_{p_1}(\Gamma) \subset M_{p_2}(\Gamma)$ and $\| \cdot \|_{M_{p_2}} \leq \| \cdot \|_{M_{p_1}}$ (see, e.g., [3]).

B. The indicator function $1_I$ of any rectangle $I \subseteq \mathbb{R}^d$ belongs to $M_p(\mathbb{R}^d)$ for all $p, 1 < p < \infty$, see, e.g., [3]. (As usual, we set $1_I(t) = 1$ if $t \in I$ and $1_I(t) = 0$ if $t \notin I$.)

C. Let $m \in M_p(\mathbb{R}^d), 1 \leq p \leq \infty$, and let $l: \mathbb{R}^d \to \mathbb{R}^d$ be a nondegenerate affine mapping; then $m \circ l \in M_p(\mathbb{R}^d)$ and $\| m \circ l \|_{M_p(\mathbb{R}^d)} = \| m \|_{M_p(\mathbb{R}^d)}$ (see, e.g., [5, Ch. 1, Sec. 1.3]).
D. Let \( m_0 \) be a function in \( M_p(\mathbb{R}^d) \), \( 1 \leq p \leq \infty \), vanishing outside a cube \( I \) with edges of length \( 2\pi \) parallel to the coordinate axes. Consider the extension \( m \) of \( m_0 \) to \( \mathbb{R}^d \) which is \( 2\pi \)-periodic in each variable. The function \( m \) belongs to \( M_p(\mathbb{T}^d) \), and \( \| m \|_{M_p(\mathbb{T}^d)} \leq c_p \| m_0 \|_{M_p(\mathbb{R}^d)} \), where \( c_p \) is a positive constant not depending on \( m_0 \). This theorem on extension of multipliers is well known [6, Theorem 2.3] (for \( p = 1, \infty \) the result follows from the local properties of the Fourier transforms of measures, see, e.g., [8, Ch. II, Sec. 4]).

E. Let \( I \subseteq \mathbb{R}^d \) be a rectangle. By \( S_I \) we denote the operator on \( L^p(\mathbb{R}^d) \) that corresponds to multiplication by \( 1_I \), i.e., the operator defined by

\[
\hat{S_I}(f) = 1_I \cdot \hat{f}, \quad f \in L^p \cap L^2(\mathbb{R}^d).
\]

Let \( \Delta \) be a family of rectangles which form a partition of \( \mathbb{R}^d \), i.e., a family of pairwise disjoint rectangles in \( \mathbb{R}^d \) such that the complement \( \mathbb{R}^d \setminus \bigcup_{I \in \Delta} I \) has Lebesgue measure zero. Consider the corresponding Littlewood–Paley square function \( S^\Delta(f) \):

\[
S^\Delta(f) = \left( \sum_{I \in \Delta} |S_I(f)|^2 \right)^{1/2}.
\]

The partition \( \Delta \) is called an LP partition if, for all \( p, 1 < p < \infty \), we have

\[
a_p \cdot \| f \|_{L^p(\mathbb{R}^d)} \leq \| S^\Delta(f) \|_{L^p(\mathbb{R}^d)} \leq b_p \cdot \| f \|_{L^p(\mathbb{R}^d)},
\]

where \( a_p = a_p(\Delta) \) and \( b_p = b_p(\Delta) \) are positive constants independent of \( f \). A classical example of an LP partition of the real line \( \mathbb{R} \) is the dyadic partition, that is the family of the intervals \( I_k, k \in \mathbb{Z}, \) of the form \( I_k = (2^{k-1}, 2^k) \) for \( k = 1, 2, \ldots \), \( I_0 = (-1, 1), \) and \( I_k = (-2^{-k}, -2^{-k-1}) \) for \( k = -1, -2, \ldots \).

Let \( \Delta \) be a family of rectangles which form an LP partition of \( \mathbb{R}^d \). Consider an arbitrary function \( m \in L^\infty(\mathbb{R}^d) \) constant on each rectangle \( I \in \Delta \). Then \( m \) is in \( M_p(\mathbb{R}^d) \) for all \( p, 1 < p < \infty \), and

\[
\| m \|_{M_p(\mathbb{R}^d)} \leq c(p, \Delta) \cdot \| m \|_{L^\infty(\mathbb{R}^d)},
\]

where \( c(p, \Delta) > 0 \) does not depend on \( m \) (see, e.g., [3, 1.2.7, 1.2.8]).

Given a family \( \Delta \) of intervals in \( \mathbb{R} \), let \( \Delta_d \) denote the family of rectangles in \( \mathbb{R}^d \) generated by \( \Delta \), that is, the family of all rectangles \( I \) of the form
\[ I = I_1 \times I_2 \times \ldots I_d , \text{ where each factor belongs to } \Delta. \text{ If } \Delta \text{ is an LP partition of } \mathbb{R}, \text{ then } \Delta_d \text{ is an LP partition of } \mathbb{R}^d \text{ (see [3, Theorem 1.3.4]).} \]

F. By the dyadic partition of the interval \((0, 1)\) we mean the family of the intervals \(I_k, k \in \mathbb{Z}\), defined by
\[
I_k = (1 - 2^{-k-1}, 1 - 2^{-k-2}), \quad k = 1, 2, \ldots;
\]
\[
I_0 = (1/4, 3/4);
\]
\[
I_k = (2^{k-2}, 2^{k-1}), \quad k = -1, -2, \ldots.
\]
We extend this definition to any interval \((a, b)\) in the obvious way by translation and rescaling. Suppose now that \(\Delta\) is a family of intervals which form an LP partition of \(\mathbb{R}\). Then the family of all intervals, obtained by dyadic partition of each interval \(I \in \Delta\) is an LP partition of \(\mathbb{R}\) as well. This immediately follows from a result of Sjögren and Sjölin [26, Theorem 1.2].

4. Proofs of the Theorems

First, we prove a simple auxiliary lemma.

**Lemma.** Let \(f \in C(\mathbb{R}^d)\). Then there exists a self-homeomorphism \(\psi\) of \(\mathbb{R}^d\) such that \(f \circ \psi\) is uniformly continuous on \(\mathbb{R}^d\). Moreover, if a family \(K \subseteq C(\mathbb{R}^d)\) is uniformly equicontinuous on every ball in \(\mathbb{R}^d\), then there exists a self-homeomorphism \(\psi\) of \(\mathbb{R}^d\) such that the family \(\{f \circ \psi, f \in K\}\) is uniformly equicontinuous on \(\mathbb{R}^d\).

**Proof.** We give the proof in the general case of families of functions. Consider spherical shells
\[
\Theta_j = \{x \in \mathbb{R}^d : j \leq |x| \leq j + 1\}, \quad j = 0, 1, 2, \ldots
\]
(\(\Theta_0\) is a ball). Let \(\omega_j\) be the modulus of continuity of the family \(K\) on \(\Theta_j\), i.e.,
\[
\omega_j(\delta) = \sup_{f \in K} \omega(f, \Theta_j, \delta).
\]
By assumption, \(\omega_j(\delta) \to 0\) as \(\delta \to +0\) for each \(j\). So, we can find a decreasing sequence \(b_j, j = 0, 1, 2, \ldots\), such that \(0 < b_j < 1\) and
\[
\omega_j(b_j) \to 0 \quad \text{as} \quad j \to \infty. \quad (4)
\]
We set
\[ a_j = \frac{b_j}{j+2}, \quad j = 1, 2, \ldots \]
The sequence \(a_j, j = 1, 2, \ldots\), decreases, and \(0 < a_j < 1\). Define numbers \(r_j, j = 0, 1, 2, \ldots\), by
\[ r_0 = 0; \quad r_j = \sum_{s=1}^{j} \frac{1}{a_s}, \quad j = 1, 2, \ldots \] (5)
Clearly, there exists a function \(g\) on \([0, +\infty)\) with the following properties:
(i) \(g(0) = 0\);
(ii) \(g\) is continuous and strictly increasing;
(iii) for each \(j = 0, 1, 2, \ldots\) the function \(g\) is linear on the interval \([r_j, r_{j+1}]\) and maps \([r_j, r_{j+1}]\) onto the interval \([j, j+1]\);
(iv) the slope of \(g\) on \([r_j, r_{j+1}]\) equals \(a_{j+1}\) (see (5)).
For \(x \in \mathbb{R}^d\), we set
\[ \psi(x) = g(|x|) \frac{x}{|x|}, \quad x \neq 0; \quad \psi(0) = 0. \]
One can easily see that \(\psi\) is a self-homeomorphism of \(\mathbb{R}^d\).
Consider the spherical shells
\[ \Omega_j = \{x : r_j \leq |x| \leq r_{j+1}\}, \quad j = 0, 1, 2, \ldots \]
(\(\Omega_0\) is a ball). Note, that the image of \(\Omega_j\) under \(\psi\) is the shell \(\Theta_j\). We claim that
\[ |\psi(x) - \psi(y)| \leq b_j|x - y| \quad \text{for all } x, y \in \Omega_j, \quad j = 0, 1, 2, \ldots \] (6)
Since \(\psi(x) = a_1x\) on \(\Omega_0\), we see that if \(x, y \in \Omega_0\), then
\[ |\psi(x) - \psi(y)| = a_1|x - y| = (b_1/3)|x - y| \leq b_0|x - y|. \]
Let \(j \geq 1\). Observe that the mapping \(\gamma(x) = x/|x|\) has the property that
\[ |\gamma(x_1) - \gamma(x_2)| \leq (1/r)|x_1 - x_2| \quad \text{for all } x_1, x_2 \text{ that lie outside the ball } B_r = \{x : |x| < r\}, \quad r > 0. \]
So, if \(x, y \in \Omega_j\), then
\[ |\gamma(x) - \gamma(y)| \leq \frac{1}{r_j}|x - y|. \]
At the same time
\[ |g(\|x\|) - g(\|y\|)| = a_{j+1}|x - y| \leq a_{j+1}|x - y|. \]
Whence, taking into account that \( g(\|y\|) \leq j + 1 \) and \( r_j \geq 1/a_j \), we obtain
\[ |\psi(x) - \psi(y)| = |(g(\|x\|) - g(\|y\|))\gamma(x) + g(\|y\|)(\gamma(x) - \gamma(y))| \]
\[ \leq a_{j+1}|x - y| + (j + 1)\frac{1}{r_j}|x - y| \leq (j + 2)a_j|x - y| = b_j|x - y|. \]

Thus, (6) holds.

Let \( f \in K \). We shall estimate the modulus of continuity of the superposition \( f \circ \psi \) on \( \mathbb{R}^d \). Let \( 0 < \delta \leq 1 \), and let \( x, y \in \mathbb{R}^d \), \( |x - y| \leq \delta \). Since the thickness of each shell \( \Omega_j \) is greater than 1 (it equals \( 1/a_j \)), we see that either the points \( x \) and \( y \) belong to the same shell, say \( \Omega_j \), or there are two neighboring shells \( \Omega_j \) and \( \Omega_{j+1} \) such that \( x \) is in one of them and \( y \) is in the other one. Consider the first case when \( x \) and \( y \) are in \( \Omega_j \). Then the points \( \psi(x), \psi(y) \) belong to the shell \( \Theta_j \). So (see (6)),
\[ |f \circ \psi(x) - f \circ \psi(y)| \leq \omega_j(|\psi(x) - \psi(y)|) \leq \omega_j(b_j \delta). \]

Consider the second case when \( x \in \Omega_j \) and \( y \in \Omega_{j+1} \). The line segment that joins \( x \) and \( y \) contains a point \( z \) that belongs to both \( \Omega_j \) and \( \Omega_{j+1} \). Using the estimate obtained in the first case, we have
\[ |f \circ \psi(x) - f \circ \psi(y)| \leq |f \circ \psi(x) - f \circ \psi(z)| + |f \circ \psi(z) - f \circ \psi(y)| \leq \omega_j(b_j \delta) + \omega_{j+1}(b_{j+1} \delta). \]
Thus, we see that for \( 0 < \delta \leq 1 \)
\[ \omega(f \circ \psi, \mathbb{R}^d, \delta) \leq 2 \sup_{j \geq 0} \omega_j(b_j \delta). \]

To complete the proof it remains to note that (4) implies
\[ \sup_{j \geq 0} \omega_j(b_j \delta) \to 0 \quad \text{as} \quad \delta \to +0. \]

**Proof of Theorem 1.** Let \( I = (l, r) \) be a bounded or unbounded interval in \( \mathbb{R} \). We say that intervals \( I_k \subset I \), \( k \in \mathbb{Z} \), form an ordered partition of \( I \) if \( I_k = (\theta_k, \theta_{k+1}) \), where \( \theta_k < \theta_{k+1} \) for all \( k \in \mathbb{Z} \), \( \lim_{k \to +\infty} \theta_k = r \), and \( \lim_{k \to -\infty} \theta_k = l \).
Suppose that intervals $I_{s_1, s_2} \in \mathbb{Z}$, form an ordered partition of $\mathbb{R}$. For each fixed $s_1$, let $I_{s_1, s_2}$, $s_2 \in \mathbb{Z}$, be certain intervals, which form an ordered partition of $I_{s_1}$, and for each $\nu$ and integers $s_1, s_2, \ldots, s_\nu$ let $I_{s_1, s_2, \ldots, s_\nu, s_{\nu+1}}, s_{\nu+1} \in \mathbb{Z}$, be intervals, which form an ordered partition of $I_{s_1, s_2, \ldots, s_\nu}$. Proceeding, we obtain a certain family of intervals:

$$\{I_{s_1, s_2, \ldots, s_\nu} : \nu = 1, 2, \ldots, \text{ and } s_1, s_2, \ldots, s_\nu \in \mathbb{Z}\}.$$ 

We refer to any family of intervals thus obtained as a net. For each fixed $\nu$, the intervals $I_{s_1, s_2, \ldots, s_\nu}$ are called intervals of rank $\nu$ (of a given net).

Given a set $E$ and a function $f$ on $E$, by $\text{osc}_E f$ we denote the oscillation of $f$ on $E$: $\text{osc}_E f = \sup_{t_1, t_2 \in E} |f(t_1) - f(t_2)|$.

We shall construct two nets of intervals. Clearly, the dyadic partition of the real line or of an interval (see Section 3, E and F) is an ordered partition. Consider the intervals $I_{s_1} \in \mathbb{Z}$, which form the dyadic partition of $\mathbb{R}$. If intervals $I_{s_1, s_2, \ldots, s_\nu}$ of rank $\nu$ are already defined, then we define $I_{s_1, s_2, \ldots, s_\nu, s_{\nu+1}}, s_{\nu+1} \in \mathbb{Z}$, to be the intervals that form the dyadic partition of $I_{s_1, s_2, \ldots, s_\nu}$. Thereby, we have constructed the first net, which we denote by $\alpha$. By $\alpha_d(\nu)$ we denote the family of all rectangles in $\mathbb{R}^d$ obtained as the Cartesian product of any $d$ intervals of rank $\nu$ of the net $\alpha$. It follows from the properties of LP partitions listed in Section 3, E and F, that if $m$ is a function in $L^\infty(\mathbb{R}^d)$ constant on each rectangle which belongs to $\alpha_d(\nu)$, then $m \in M_p(\mathbb{R}^d)$ for all $p, 1 < p < \infty$, and

$$\|m\|_{M_p(\mathbb{R}^d)} \leq c(p, \nu) \cdot \|m\|_{L^\infty(\mathbb{R}^d)}, \quad 1 < p < \infty,$$

where $c(p, \nu) > 0$ may depend only on $p$ and $\nu$. In what follows, we assume that

$$c(p, \nu) = \sup_{m \in \mathcal{P}_\nu, m \neq 0} \frac{\|m\|_{M_p(\mathbb{R}^d)}}{\|m\|_{L^\infty(\mathbb{R}^d)}}, \quad (7)$$

where $\mathcal{P}_\nu$ is the class of all functions in $L^\infty(\mathbb{R}^d)$ constant on each rectangle from $\alpha_d(\nu)$.

We proceed to the construction of the second net. By using the Lemma, we can assume that $K$ is a family of bounded functions uniformly equicontinuous on the whole $\mathbb{R}^d$. Let $\omega$ be the modulus of continuity of $K$. We have

$$\sup_{t_1, t_2 \in \mathbb{R}^d} |f(t_1) - f(t_2)| \leq \omega(\delta), \quad \delta > 0, \quad \text{for all } f \in K,$$
where $\omega(\delta) \to 0$ as $\delta \to +0$, and $\omega$ is nondecreasing on $[0, +\infty)$. Fix a positive decreasing sequence $\delta_\nu$, $\nu = 1, 2, \ldots$, that tends to 0 so fast that

$$\sum_{\nu=2}^\infty c\left(1 + \frac{1}{\nu}, \nu \right) \omega(\delta_{\nu-1}\sqrt{d}) < \infty. \quad (8)$$

Note that (8) implies that

$$\sum_{\nu=2}^\infty c(\nu, \nu) \omega(\delta_{\nu-1}\sqrt{d}) < \infty \quad (9)$$

for all $p$, $1 < p < \infty$. Indeed, it suffices to observe that if $\nu$ is large enough, then $c(p, \nu) \leq c(1 + 1/\nu, \nu)$ (see Section 3, A).

Let $J_{s_1, s_1}$ be intervals of length at most $\delta_1$ which form an ordered partition of the line $\mathbb{R}$. For a fixed $\nu$, assuming that all the intervals $J_{s_1, s_2, \ldots, s_\nu}$ are already defined, consider an ordered partition of each interval $J_{s_1, s_2, \ldots, s_\nu}$ by intervals $J_{s_1, s_2, \ldots, s_\nu, s_{\nu+1}}$, $s_{\nu+1} \in \mathbb{Z}$, of length at most $\delta_{\nu+1}$. Thereby, we obtain the second net, which we denote by $\beta$.

Clearly, there exists a self-homeomorphism $\varphi$ of $\mathbb{R}$ such that $\varphi(I_{s_1, s_2, \ldots, s_\nu}) = J_{s_1, s_2, \ldots, s_\nu}$ for all $\nu = 1, 2, \ldots$ and $s_1, s_2, \ldots, s_\nu$ (the intervals $I_{s_1, s_2, \ldots, s_\nu}$ and $J_{s_1, s_2, \ldots, s_\nu}$ belong to the nets $\alpha$ and $\beta$, respectively). We define a homeomorphism $h$ of $\mathbb{R}^d$ onto itself by

$$h(t) = (\varphi(t_1), \varphi(t_2), \ldots, \varphi(t_d)), \quad t = (t_1, t_2, \ldots, t_d) \in \mathbb{R}^d. \quad (10)$$

Consider an arbitrary function $f \in K$ and set $g = f \circ h$. Let us verify that $g \in \bigcap_{1 < p < \infty} M_p(\mathbb{R}^d)$. Given a rectangle $I \subset \mathbb{R}^d$ denote the center of $I$ by $c_I$. For $\nu = 1, 2, \ldots$, let $g_\nu$ be the function that takes the constant value $g(c_I)$ on each rectangle $I \in \alpha_d(\nu)$. Clearly,

$$\|g - g_\nu\|_{L^\infty(\mathbb{R}^d)} \leq \sup_{I \in \alpha_d(\nu)} \text{osc}_I g. \quad (11)$$

Note that if $I \in \alpha_d(\nu)$, then the image $h(I)$ of $I$ under $h$ is a certain rectangle whose edges are of length at most $\delta_\nu$, which implies that $\text{diam} h(I) \leq \delta_\nu \sqrt{d}$. So,

$$\text{osc}_I g = \text{osc}_{h(I)} f \leq \omega(f, \delta_\nu \sqrt{d}) \leq \omega(\delta_\nu \sqrt{d}) \quad \text{for all } I \in \alpha_d(\nu), \quad \nu = 1, 2, \ldots.$$

Therefore (see (11)),

$$\|g - g_\nu\|_{L^\infty(\mathbb{R}^d)} \leq \omega(\delta_\nu \sqrt{d}). \quad (12)$$
Hence, for $\nu \geq 2$, we obtain
\[
\|g_\nu - g_{\nu-1}\|_{L^\infty(\mathbb{R}^d)} \leq \|g_\nu - g\|_{L^\infty(\mathbb{R}^d)} + \|g - g_{\nu-1}\|_{L^\infty(\mathbb{R}^d)} \leq 2\omega(\delta_{\nu-1}\sqrt{d}). \quad (13)
\]
Let $1 < p < \infty$. Since $g_\nu \in \mathcal{P}_\nu$, it follows that $g_\nu \in M_p(\mathbb{R}^d)$, $\nu = 1, 2, \ldots$. Note that $g_\nu - g_{\nu-1} \in \mathcal{P}_\nu$ for $\nu \geq 2$; therefore (see (7) and (13)), we have
\[
\|g_\nu - g_{\nu-1}\|_{M_p(\mathbb{R}^d)} \leq c(p,\nu)2\omega(\delta_{\nu-1}\sqrt{d}).
\]
Thus,
\[
\|g_{n+m} - g_n\|_{M_p(\mathbb{R}^d)} \leq \sum_{\nu=n+1}^{n+m} \|g_\nu - g_{\nu-1}\|_{M_p(\mathbb{R}^d)} \leq \sum_{\nu=n+1}^{n+m} 2c(p,\nu)\omega(\delta_{\nu-1}\sqrt{d}). \quad (14)
\]
Taking (9) into account, we see that the sequence $g_\nu$, $\nu = 1, 2, \ldots$, converges in $M_p(\mathbb{R}^d)$ (recall that $M_p$ is a Banach space). At the same time from (12) it follows that this sequence converges to $g$ in $L^\infty(\mathbb{R}^d)$. It remains to recall that $\|\cdot\|_{L^\infty} = \|\cdot\|_{M_2} \leq \|\cdot\|_{M_p}$. This completes the proof of Theorem 1.

**Proof of Theorem 2.** Let $K$ be a uniformly equicontinuous family of functions on $\mathbb{T}^d$. We identify each function on $\mathbb{T}^d$ with a $2\pi$-periodic (in each variable) function on $\mathbb{R}^d$ in the standard manner. Thus, $K$ is a family of bounded functions uniformly equicontinuous on $\mathbb{R}^d$. Following the proof of Theorem 1 (now we do not need the Lemma), we obtain a self-homeomorphism $\varphi$ of $\mathbb{R}$ such that defining self-homeomorphism $h$ of $\mathbb{R}^d$ by (10) we have $f \circ h \in \bigcap_{1 < p < \infty} M_p(\mathbb{R}^d)$ for all $f \in K$. Consider the interval $J = \varphi^{-1}([0, 2\pi])$ which is the preimage of $[0, 2\pi]$ under $\varphi$. Let $l$ be an affine self-mapping of the real line for which $l([0, 2\pi]) = J$. We set $\varphi_1 = \varphi \circ l$ and
\[
h_1(x) = (\varphi_1(x_1), \varphi_1(x_2), \ldots, \varphi_1(x_d)), \quad x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d.
\]
Using the assertion on superpositions of multipliers with affine mappings (see Section 3, C), we conclude that $f \circ h_1 \in \bigcap_{1 < p < \infty} M_p(\mathbb{R}^d)$ for all $f \in K$. Recall that the indicator function of any rectangle in $\mathbb{R}^d$ is a multiplier for all $p$, $1 < p < \infty$ (see Section 3, B); hence
\[
1_{[0, 2\pi]^d} \cdot (f \circ h_1) \in \bigcap_{1 < p < \infty} M_p(\mathbb{R}^d)
\]
for all \( f \in K \). Since the homeomorphism \( \varphi_1 \) maps the interval \([0, 2\pi]\) onto itself, we can regard it as a self-homeomorphism of the circle \( \mathbb{T} \). So, we can regard \( h_1 \) as a self-homeomorphism of the torus \( \mathbb{T}^d \). Denote this self-homeomorphism of the torus by \( h_2 \). Clearly, \( f \circ h_2 \) is a \( 2\pi \)-periodic in each variable extension of the function \( 1_{[0,2\pi]} \cdot (f \circ h_1) \). It remains to use the theorem on periodic extensions (see Section 3, D). This completes the proof of Theorem 2.

5. Remarks and Open Problems

1. As we stated in Introduction, the negative solution of Luzin’s problem implies a similar result for functions on the real line, namely: there exists a bounded continuous real-valued function \( f \) on \( \mathbb{R} \) such that \( f \circ h \notin B(\mathbb{R}) \) for every self-homeomorphism \( h \) of \( \mathbb{R} \). One can easily verify this as follows. Let \( f \) be a function on \( \mathbb{T} \) which provides the negative solution of Luzin’s problem. Regarding \( f \) as a \( 2\pi \)-periodic function on \( \mathbb{R} \), we can assume without loss of generality that \( f(0) = f(2\pi) = 0 \). Let \( f_0 \) be the function on \( \mathbb{R} \) that coincides with \( f \) on the interval \([0,2\pi]\) and vanishes outside it. Suppose that \( f_0 \circ h \in B(\mathbb{R}) \) for some self-homeomorphism \( h \) of \( \mathbb{R} \). Clearly, if a function belongs to \( B(\mathbb{R}) \), then so does every superposition of this function with an affine mapping \( l: \mathbb{R} \to \mathbb{R} \); so, replacing \( h \) by \( h \circ l \) if necessary, we can assume that \( h \) maps \([0,2\pi]\) onto itself, and hence \( f_0 \circ h \) vanishes outside \([0,2\pi]\). It follows that \( 2\pi \)-periodic extension of \( f_0 \circ h \) to \( \mathbb{R} \) is in \( A(\mathbb{T}) \) (see Section 3, D), which is impossible, since this extension has the form \( f \circ h_1 \), where \( h_1 \) is a self-homeomorphism of the circle.

2. It is natural to consider an analogue of Luzin’s problem in the multidimensional case. Is it true that, given a real-valued function \( f \in C(\mathbb{T}^d), \ d \geq 2 \), there exists a self-homeomorphism \( h \) of \( \mathbb{T}^d \) such that \( f \circ h \in A(\mathbb{T}^d) \)? Since the group of homeomorphisms of \( \mathbb{T}^d \) with \( d \geq 2 \) is more massive than that of \( \mathbb{T} \) the question may have a positive answer for \( d \geq 2 \) despite the fact that in the one-dimensional case it is answered in the negative.

3. As in the introduction, let \( G \) be either \( \mathbb{R}^d \) or \( \mathbb{Z}^d \) and \( \Gamma \) is \( \mathbb{R}^d \) or \( \mathbb{T}^d \), correspondingly. Recall that a function \( m \in L^\infty(\Gamma) \) is called a weak type \((1,1)\) multiplier if the operator \( Q \) defined by \( \widehat{Qf} = m\widehat{f}, \ f \in L^1 \cap L^2(G) \), is of weak type \((1,1)\), i.e., satisfies the condition

\[
\text{mes}_G \{ t \in G : |Qf(t)| > \lambda \} \leq c\|f\|_{L^1(G)} / \lambda, \quad \lambda > 0,
\]
where \( \text{mes}_G \) is the Haar measure on \( G \) (the Lebesgue measure of a set in the case of \( G = \mathbb{R}^d \) and the number of elements in a set in the case of \( G = \mathbb{Z}^d \)). Let \( M_1^{\text{weak}}(\Gamma) \) denote the class of all such multipliers. We have (using Marcinkiewicz’ interpolation theorem)

\[
M_1(\Gamma) \subseteq M_1^{\text{weak}}(\Gamma) \subseteq \bigcap_{1<p<\infty} M_p(\Gamma).
\]

The authors do not know whether it is possible to improve Theorems 1 and 2 so as to attain the condition \( f \circ h \in M_1^{\text{weak}} \) for all \( f \in K \). The answer is unclear even for the families that consist of one function: Given a bounded real-valued function \( f \in C(\mathbb{R}^d) \) or a real-valued function \( f \in C(\mathbb{T}^d) \) is there a homeomorphic change of variable \( h \) such that \( f \circ h \in M_1^{\text{weak}} \)? The negative answer to this question in the one-dimensional case would strengthen the result that solves Lusin’s problem.

4. Let \( d \geq 2 \). Observe that the homeomorphism \( h \) of \( \mathbb{T}^d \) in Theorem 2 can be chosen in the form

\[
h: (t_1, t_2, \ldots, t_d) \to (\varphi(t_1), \varphi(t_2), \ldots, \varphi(t_d)),
\]

where \( \varphi \) is a self-homeomorphism of \( \mathbb{T} \). As concerns Theorem 1, it is clear from its proof that if we impose stronger assumption on \( f \) or on \( K \), namely if we assume that \( f \) is bounded and uniformly continuous or, respectively, that \( K \) is a family of bounded functions which is uniformly equicontinuous on the whole \( \mathbb{R}^d \), \( d \geq 2 \), then the corresponding homeomorphism \( h \) of \( \mathbb{R}^d \) can be chosen in the form (15) with \( \varphi \) being a self-homeomorphism of \( \mathbb{R} \). We do not know if the same is true without the above stronger assumptions. Moreover, it is unclear if every bounded real-valued function \( f \in C(\mathbb{R}^d) \) can be transformed into a multiplier by a homeomorphism of the form \( h: (t_1, t_2, \ldots, t_d) \to (\varphi_1(t_1), \varphi_2(t_2), \ldots, \varphi_d(t_d)) \), where \( \varphi_j \)'s are self-homeomorphisms of \( \mathbb{R} \) allowed to be different. It seems likely that the answer is negative.

5. The well-known Beurling–Helson theorem [2] (see also [8, Ch. VI], [9]) states that if \( \varphi \) is a continuous self-mapping of the circle \( \mathbb{T} \) satisfying the condition \( \|e^{in\varphi}\|_{A(\mathbb{T})} = O(1), n \in \mathbb{Z} \), then \( \varphi \) is linear (affine), i.e., \( \varphi(t) = \nu t + \varphi(0) \) where \( \nu \in \mathbb{Z} \). The character of growth of the norms \( \|e^{in\varphi}\|_{A(\mathbb{T})} \) for nontrivial \( \varphi \)'s is unclear in many respects. Kahane conjectured that the Beurling–Helson theorem can be considerably improved; namely, he conjectured in [7] (see also [8, Ch. VI], [9]) that the conclusion of the
Beurling–Helson theorem remains valid even if the norms $\|e^{in}\|_{A(T)}$ grow to infinity but the growth is not very fast. He also conjectured ([7], [8]) that the condition $\|e^{in}\|_{A(T)} = o(\log |n|)$ already implies linearity of $\varphi$. The first result in this direction was obtained in [17]; further strengthening is obtained in [12], however the $o(\log |n|)$-conjecture remains unproved.

Certain analogs of the Beurling–Helson theorem for the algebras $M_p$ were obtained in [20] and [21]. Note that the case $1 < p < \infty$ differs from that of $p = 1$, which corresponds to the Wiener algebra; for example (see [5, Ch. I, Sec. 1.3]), if $\varphi : T \to T$ is piecewise linear, then for all $p, 1 < p < \infty$, $\|e^{in}\|_{M_p(T)} = O(1), \quad n \in \mathbb{Z}$. (16)

To some extent the converse is also true [21, Theorem 2'] if (16) holds for some $p \neq 2$, then there exists a closed set $E(\varphi) \subset T$ of Lebesgue measure zero such that $\varphi$ is linear on the intervals complementary to $E(\varphi)$, and the set of distinct slopes of $\varphi$ is finite.

It is natural to ask how slow the norms $\|e^{in}\|_{M_p}$ can grow in the case when $\varphi$ is nowhere (i.e., on no interval) linear. The answer to this question can easily be extracted from results of this paper. For $1 < p < \infty$ the growth can be arbitrarily slow, namely: given an arbitrary positive sequence $\gamma(n), n = 0, 1, 2, \ldots,$ with $\gamma(n) \to +\infty$, there exists a nowhere linear self-homeomorphism $h$ of $T$ such that $\|e^{inh}\|_{M_p(T)} = O(\gamma(|n|)), \quad n \in \mathbb{Z}$, for all $p, 1 < p < \infty$.

To prove this assertion, we firstly observe that Theorems 1 and 2 can be supplemented by the estimate $\|f \circ h\|_{M_p} \leq c(p, 1)\|f\|_{L^\infty} + c_p(K), f \in K$. In particular, if $K$ is a compact set in the space $C(T^d)$, then there exists a homeomorphism $h : T^d \to T^d$ such that $\|f \circ h\|_{M_p(T^d)} \leq c_K(p), \quad 1 < p < \infty, \quad f \in K$. (17)

Indeed (see the end of the proof of Theorem 1), since $g_\nu \xrightarrow{M_p} g$, it follows from (14) that $\|g - g_1\|_{M_p} \leq 2c_p(K)$, where $c_p(K)$ is the sum of the series in (9). At the same time $\|g_1\|_{M_p} \leq c(p, 1)\|g_1\|_{L^\infty} \leq c(p, 1)\|f\|_{L^\infty}$.

Secondly we observe that due to sufficient flexibility in the construction of the net $\beta$ in the proof of Theorem 1, the homeomorphism $h$ in the statement of Theorems 1 and 2 can be made nowhere linear.
It remains to apply the above two observations to the family

\[ K = \left\{ \frac{e^{int}}{\gamma(|n|)}, \quad n \in \mathbb{Z} \right\} \]

(see (17)).

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