Compression with wildcards:
All exact, or all minimal hitting sets

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ABSTRACT: Our objective is the compressed enumeration (based on wildcards) of all minimal hitting sets of general hypergraphs. To the author’s best knowledge the only previous attempt towards compression, due to Toda [T], is based on BDD’s and much different from our techniques. Traditional one-by-one enumeration schemes cannot compete when the number of minimal hitting sets is large and the degree of compression is high. Our method works particularly well in these two cases: Either compressing all minimum cardinality hitting sets, or compressing all exact hitting sets.

Key words: hitting set (minimal, minimum, exact), compressed enumeration, Vertical Layout

1 Introduction

Let $W$ be a finite set (such as all sets in this article) and $\mathcal{P}(W)$ its powerset. Given a hypergraph (=set-system) $\mathbb{H} \subseteq \mathcal{P}(W)$, a $\mathbb{H}$-hitting set is a set $X \subseteq W$ such that $X \cap H \neq \emptyset$ for all hyperedges $H \in \mathbb{H}$. Let $\text{HS}(\mathbb{H})$ be the set of all hitting sets, and $\text{MHS}(\mathbb{H})$ the subset of all (inclusion-)minimal hitting sets, henceforth called MHSes. The famous Minimal Hitting Set Problem is this: Given $\mathbb{H} \subseteq \mathcal{P}(W)$, is it possible to enumerate $\text{MHS}(\mathbb{H})$ in polynomial total time? i.e. polynomial in $w := |W|$, $h := |\mathbb{H}|$, and $mhs := |\text{MHS}(\mathbb{H})|$? We refer to [1] and [2] for the history and the state of the art concerning this problem.

The objective in our article is different and can be described in picturesque ways as follows. For fixed $\mathbb{H}$ identify the MHSes with diamonds and the ordinary hitting sets (i.e. the members of $\text{HS}(\mathbb{H}) \setminus \text{MHS}(\mathbb{H})$) with worthless pebbles which, however, may be hard to distinguish from diamonds. Some friendly sponsor provides $R$ many nonempty boxes which are filled with both kinds of stones. All diamonds are distributed among the boxes but usually not all pebbles (which is just as well). Our Main Quest is to retrieve all diamonds (and only them) as efficiently as possible. A box is good if it contains at least one diamond, and bad otherwise. A box 100% filled with diamonds is very-good. As will be seen, depending on the structure of $\mathbb{H}$, very-good boxes can both be numerous and heavy! Furthermore the number of diamonds in a very-good box is found at once, and the diamonds themselves are arranged in a pleasant, compressed manner.

1 An older synonym is output polynomial time.
To get a first impression of the quality of boxes the Monte-Carlo method picks (say) 20 stones at random from each box \( \rho \), and determines the number \( \alpha(\rho) \) of diamonds among them. If \( 0 < \alpha(\rho) < 20 \) then \( \rho \) is merely-good, i.e. good but not very-good. However, if \( \alpha(\rho) = 0 \) then \( \rho \) is only likely-bad, and if \( \alpha(\rho) = 20 \) then \( \rho \) is likely-very-good. If a likely-very-good row contains thousands of stones then classifying the stones one-by-one is time-consuming. Fortunately we will provide three criteria for very-goodness which settle the issue faster. Efficient criteria for badness are harder to come by but an elegant sufficient condition exists. As to merely-good boxes \( \rho \), there are two approaches, each with benefits and drawbacks. The first is to classify the stones one-by-one. The second uses subtle machinery but has the benefit that the diamonds in \( \rho \) get repackaged into brandnew very-good boxes.

Here comes the Section break-up, phrased in more mathematical terms. The preliminaries in Section 2 concern Boolean functions and three kinds of wildcards; the \( e \)-, the \( n \)-, and the \( g \)-wildcard. All of them generalize the don’t-care symbol \( * \) familiar from describing partial models of Boolean functions. Furthermore we adopt the Vertical Layout technique used in data mining. In a nutshell, it substitutes set operations (e.g finding all suitable supersets of a given set) that involve many small sets by set operations with few large sets. Section 3 discloses the above-mentioned sponsor (i.e. the transversal \( e \)-algorithm of [3]). Section 4 explains the mathematical nature of the \( R \) boxes provided by the sponsor and goes on (Theorem 1) to show that all minimum-cardinality MHSes occur in very-good boxes, which moreover can be pinpointed at once. Section 5 elaborates the first approach towards merely-good boxes by offering four algorithms for one-by-one classification. Algorithm 1 relies on the diamonds (=MHSes) retrieved so far, whereas Algorithm 2 only relies on the knowledge of \( \mathbb{H} \). Algorithms 3 and 4 exploit tricks that are fully justified only in Section 9. Section 6 elaborates the second approach towards merely-good boxes. Sections 7 and 8 propose two criteria (each of which sufficient and necessary) for very-goodness. The first is based on inclusion-exclusion, the second on matroid theory (Rado’s Theorem).

Section 9 introduces the key concept [4] of an MC-set. By definition \( X \subseteq W \) is MC if for each \( x \in X \) there is at least one hyperedge \( H \in \mathbb{H} \) that cuts \( x \) sharply in the sense that \( H \cap X = \{ x \} \). The set-system \( MC(\mathbb{H}) \) of all MC-sets is dual to \( HS(\mathbb{H}) \) in that the former is a set-ideal, the latter a set-filter, and it holds (Theorem 2) that \( MC(\mathbb{H}) \cap HS(\mathbb{H}) = MHS(\mathbb{H}) \). Those subsets of \( W \) which are not MC, yet all their proper subsets are MC, are of particular importance. They are collected in the set-system \( MinNotMC(\mathbb{H}) \). For instance it allows us to calculate the cardinality \( |MHS(\mathbb{H})| \) without knowing \( MHS(\mathbb{H}) \). Section 10 calculates \( MinNotMC(\mathbb{H}) \). It exploits the fact that minimal set-coverings are cryptomorphic to minimal hitting sets and can hence be handled with the transversal \( e \)-algorithm. Section 11 features numerical experiments with Mathematica. In a nutshell, our compression with wildcards works the better the fewer and the larger the hyperedges are. In particular very-good play a key role here. Although promising ideas of previous Sections have not yet been implemented in Mathematica, in 11.6 we attempt a preliminary comparison of our methods with the algorithms of the two winners [4] and [5] of a competition carried out in [1].

Section 12 at first seems to abandon minimal hitting sets and turn to the different topic of exact hitting sets (EHIS). Is it that different? By definition \( Y \subseteq W \) is an EHIS for \( \mathbb{H} \) if \( |Y \cap H| = 1 \) for all \( H \in \mathbb{H} \). Under the mild assumption that \( \bigcup \mathbb{H} = W \) each EHIS must be a MHS, yet the converse fails severely in that some hypergraphs have plenty MHSes and no EHISes. Nevertheless, our previously used \( g \)-wildcards can sometimes compress the set-system \( EHS(\mathbb{H}) \) of all hitting sets. As to "sometimes", any fixed hypergraph \( \mathbb{H} \subseteq \mathcal{P}(W) \) induces a natural, apparently novel
equivalence relation $\sim$ on $W$. It turns out that compressing $EHS(\mathbb{H})$ is possible iff $\sim$ is nontrivial. Furthermore Knuth’s popular Dancing-Link algorithm shows up in Section 12 and in Theorem 4 we enumerate the perfect matchings of any graph without $K_{3,3}$-minor in polynomial total time.

2 Preliminaries on Boolean functions, partial models, wildcards, and Vertical Layout

After Boolean functions (2.1) we turn to $e$-wildcards (2.2-2.3), followed by $n$-wildcards and $g$-wildcards (2.4). In 2.5 we sieve the minimal members of any set-system $S \subseteq \mathcal{P}(W)$ and 2.6 introduces Vertical Layout.

Throughout the article for any integer $w \geq 1$ we put $[w] := \{1,2,\ldots,w\}$. For convenience usually $W := [w]$. If the powerset is concerned we write $\mathcal{P}[w]$ instead of $\mathcal{P}([w])$. Further we use the shorthand "iff" for "if and only if", and write $\subset$ (as opposed to $\subseteq$) for proper inclusion.

2.1 We freely identify bitstrings of length $w$ (also called 01-rows) with subsets of $[w]$ in the usual way; thus $X = \{2,4,5\}$ (viewed, say, as subset of $[7]$) matches $x = (0,1,0,1,0,0)$. Depending on circumstances one or the other view is preferable. We now extend 01-rows to 012-rows such as

$$r = (0,2,2,1,0,2).$$

The following type of notation that refers to the positions of the various symbols will be used throughout:

(1) $\text{zeros}(r) := \{1,5\}$, $\text{ones}(r) := \{4\}$, $\text{twos}(r) := \{2,3,6\}$.

While 01-rows encode sets, 012-rows encode set-systems because '2' is viewed as don't-care symbol which can be freely replaced by 0 or 1. Thus $r = (0,2,2,1,0,2)$ above encodes, and in fact will be identified with, the set-system

$$r = \{\{4\},\{4,2\},\{4,3\},\{4,6\},\{4,2,3\},\{4,2,6\},\{4,3,6\},\{4,2,3,6\}\}$$

which, with obvious shorthand notation (that will only be applied to sets of 1-digit numbers) can also be rendered (since elements of sets can be listed in arbitrary order) as

$$\{4,42,43,46,423,426,436,4236\} \text{ or as } \{4,24,34,46,234,246,346,2346\}. $$

As to a general 012-row $r$, if it is viewed as a set-system, this set-system is $\{\text{ones}(r) \cup S : S \subseteq \text{twos}(r)\}$. While $\text{zeros}(r)$ does not come up here, the 0’s are as important as the 1’s in the sequel (ponder what would become of $r = (0,2,2,1,0,2)$ without the 0’s).

\footnote{In the literature often $*$ is used rather than 2.}

\footnote{This is a bit sloppy but it outweighs the clumsiness of introducing an extra symbol for the represented set-system. From the context it will always be clear whether $r$ is meant to be a vector with entries from 0,1,2 or whether $r$ is a set-system.}
That leads us to \( \{0,1\} \) viewed as Boolean algebra and to Boolean functions \( f : \{0,1\}^n \to \{0,1\} \) whose basic features are assumed to be familiar to the reader, so that we only need to fix notation here. Any bitstring \( x \in \{0,1\}^n \) with \( f(x) = 1 \) is a model of \( f \). Apart from other means Boolean functions can be defined by Boolean formulas. Thus by writing
\[
f(x) := x_1 \lor x_2 \lor x_3
\]
we define a Boolean function \( f : \{0,1\}^3 \to \{0,1\} \) that e.g. satisfies \( f((0,1,1)) = 0 \lor 1 \lor 1 = 1 \). It is clear that only \((0,0,0)\) fails to be a model, and so the modelset is
\[
\text{Mod}(f) = (2,2,2) \setminus \{(0,0,0)\} = (1,2,2) \cup (2,1,2) \cup (2,2,1).
\]
The union on the righthand side is not disjoint since e.g. \((1,0,1) \in (1,2,2) \cap (2,2,1) \). Fortunately, this can be cured as follows (here and henceforth \( \uplus \) signifies disjoint union):
\[
\text{Mod}(x_1 \lor x_2 \lor x_3) \\
= (1,2,2) \\
\uplus (0,1,2) \\
\uplus (0,0,1)
\]
This idea is long known and its visualization has been coined Abraham-flag in [6]. Thus a general \( n \times n \) Abraham-flag has 1's in the main diagonal, 0's below it, and 2's above it. The row-cardinalities sum up to \( 2^{n-1} + 2^{n-2} + \cdots + 1 \) which equals \( 2^n - 1 \), as is to be expected.

In connection with Boolean functions 012-rows usually describe partial models. For instance \((1,2,2)\) is a partial model of \( x_1 \lor x_2 \lor x_3 \) in the sense that replacing the 2's by 0 or 1 in any way results in a model of \( x_1 \lor x_2 \lor x_3 \).

In addition to the don’t-care symbol ”2” we will use three further wildcards. For starters, instead of using an \( s \times s \) Abraham-flag to spell out \( \text{Mod}(x_1 \lor \cdots \lor x_s) \) we can, better still, simply define
\[
(e,e,...,e) := \text{Mod}(x_1 \lor \cdots \lor x_s).
\]
Roughly speaking, \( s \) symbols \( e \) (not necessarily adjacent) demand bitstrings to have ”at least one 1 in that area”. Combining such \( e \)-wildcards (distinguished by subscripts) gives rise to \( 012e \)-rows like
\[
(2) \quad r' = (e_1,0,2,e_1,e_2,1,0,e_2,2,2),
\]
which by definition consists of those subsets \( S \subseteq \{0,1,2\} \) that satisfy

- \( 2,7 \notin S \) (because \( \text{zeros}(r') = \{2,7\} \))
- \( 6 \in S \) (because \( \text{ones}(r') = \{6\} \))
- \( \{1,4\} \cap S = \emptyset \) (because of \( e_1,e_1 \))

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\footnote{We will only be concerned with the join and meet operations, so \( 1 \lor 1 = 0 \lor 1 = 1 \lor 0 = 1 \), \( 0 \lor 0 = 0 \) and \( 0 \land 0 = 0 \land 1 = 1 \land 0 = 0 \), \( 1 \land 1 = 1 \).}

\footnote{In likewise fashion the formula defines a unique function \( f : \{0,1\}^w \to \{0,1\} \) for every \( w > 4 \). In the sequel it will always be clear which \( w \) is meant.}

\footnote{However, Abraham-flags will reappear in 2.3 in new guise.}
• \( \{5, 8\} \cap S = \emptyset \) (because of \( e_2, e_2 \))

The fact that \( \text{twos}(r') = \{3, 9, 10\} \) reflects the fact that 3, 9, 10 don’t occur in any of the conditions. By \( e \)-bubble we mean the position-set of any given \( e \)-wildcard. Thus the \( e_2 \)-bubble of the \( e_2 \)-wildcard in (2) is \( \{5, 8\} \). It is easy to see that

\[
|r'| = 2^3 \cdot (2^2 - 1) \cdot (2^2 - 1),
\]

and that \( 2^2 - 1 \) generalizes to \( 2^s - 1 \) for \( e \)-bubbles of size \( s \).

Alternatively (but clumsier) \( r' \) in (2) could be defined as

\[
(2') \quad r' = \text{Mod}(x_2 \land x_7 \land x_6 \land (x_1 \lor x_4) \land (x_5 \lor x_8)).
\]

2.2.1 Observe that the intersection \( \rho \cap \rho' \) of an 012e-row \( \rho \) with an 012-row \( \rho' \) is either empty (when 0’s and 1’s clash) or is again a 012e-row, which arises in obvious ways:

\[
\rho = (e_1, e_1, e_2, e_2, e_3, e_3, e_3)
\]
\[
\rho' = (2, 2, 0, 0, 2, 2, 1, 0)
\]
\[
\rho \cap \rho' = (e_1, e_1, 0, 0, e_2, e_2, 1, 0)
\]

2.2.2 The set of all \( \text{minimal} \) members contained in a 012e-row will play a crucial role. One checks that the set-system \( \text{Min}(r') \) of all minimal members of the set-system \( r' \) in (2) equals

\[
(3) \quad \text{Min}(r') = \{615, 618, 645, 648\}.
\]

Generally, if the 012e-row \( r \) has \( t \geq 1 \) many \( e \)-wildcards of cardinalities \( \epsilon_1, \ldots, \epsilon_t \) then each \( X \in \text{Min}(r) \) is of type \( X = \text{ones}(r) \cup T \), where \( T \) cuts each \( e \)-bubble in exactly one element. Thus \( |T| = t \). If we define the \( \text{degree} \) of \( r \) as

\[
(4) \quad \text{deg}(r) := |\text{ones}(r)| + t,
\]

then

\[
(5) \quad \text{Min}(r) = \{X \in r : |X| = \text{deg}(r)\} \quad \text{and} \quad |\text{Min}(r)| = \epsilon_1 \cdot \epsilon_2 \cdots \epsilon_t.
\]

For general set-systems \( S \) it will be more demanding (2.5) to sieve \( \text{Min}(S) \) from \( S \). Nevertheless (5) will keep coming back even in that context.

2.3 Let us introduce higher-level Abraham-flags, i.e. constituted by certain 012e-rows as opposed to the 012-rows in 2.1. Consider

\[
(6) \quad r := (e_1, e_1, e_2, e_2, e_3, e_3, e_2, e_2, e_3).
\]

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7 Provided \( \{0, 1\}^{10} \) is taken as domain of the Boolean function induced by the Boolean formula.
8 Recall that "minimal" always means minimal with respect to set inclusion.
9 For the special case of 012-rows \( r \), i.e. \( t = 0 \), we have \( |\text{Min}(r)| = 1 \) and \( \text{deg}(r) = |\text{ones}(r)| \).
Soon we need to be able to e.g. sieve those bitstrings \((x_1, \ldots, x_9)\) from \(r\) that have at least one 1 among \(\{x_1, \ldots, x_9\}\). In other words, we need to “impose” \((e, e, e, e)\) upon \(r\), i.e. the intersection \(r \cap (e, e, e, e, 2, 2, 2, 2, 2)\) of two 012e-rows must be rewritten in a handy format. The answer is 
\[
(7) \quad r_1 := (e_1, e_1, e_2, e_2, e_3, 2, e_2, e_2, e_3) \\
r_2 := (0, 0, e_2, e_2, e_3, 1, 2, 2, e_3) \\
r_3 := (0, 0, 0, 0, 1, 1, e_2, e_2, 2)
\]

The first part of the righthand side is a novel \(3 \times 3\) Abraham-flag in the sense that the boldface main diagonal entries are either 1 (as in 2.1) or full \(e\)-wildcards. Likewise the entries below the main diagonal are again 0’s. We leave it to the reader to figure out what happens above the main diagonal, and how all of this affects the last four columns in (7). See also Section 3.1.

2.4 Dually to \(e\)-wildcards we will encounter \(n\)-wildcards which demand ”at least one 0 here”. Thus for instance

\[
(n, n, n, n) := \text{Mod}(x_1 \lor x_2 \lor x_3 \lor x_4) \\
= (0, 2, 2, 2) \\
\uplus (1, 0, 2, 2) \\
\uplus (1, 1, 0, 2) \\
\uplus (1, 1, 1, 0)
\]

Mutatis mutandis as in 2.2 we define \(n\)-bubbles and 012\(n\)-rows.

2.4.1 Apart from \(e\)-wildcards and \(n\)-wildcard\(\text{[10]}\), a third type of wildcard takes care of the requirement ”exactly one 0 here”. Namely, by definition

\[
(g, g, \ldots, g) := \{(1, 0, \ldots, 0), (0, 1, \ldots, 0), \ldots, (0, 0, \ldots, 1)\}
\]

One trivial application of these \(g\)-wildcards (and coupled \(g\)-bubbles) is the compression of \(\text{MHS}(\mathbb{H})\) for hypergraphs with disjoint hyperedges. Thus if \(\mathbb{H}_1 = \{123, 45, 6789\}\) then \(\text{MHS}(\mathbb{H}_1) = (g_1, g_1, g_1, g_2, g_2, g_3, g_3, g_3)\). Slightly more subtle and important later, one checks that \(r' = (e_1, 0, 2, e_1, e_2, 1, 0, e_2, 2, 2)\) from (2) has \(\text{Min}(r') = (g_1, 0, 0, g_1, g_2, 1, 0, g_2, 0, 0)\). Expressions like this are called 012\(g\)-rows.

2.5 Let \(\mathcal{S} \subseteq \mathcal{P}([w])\) be any set system. The problem to get \(\text{[11]}\) the set-system \(\text{Min}(\mathcal{S})\) of all minimal members of \(\mathcal{S}\) occurs frequently in discrete mathematics. The naive way to proceed is to decide for each \(X \in \mathcal{S}\) whether there is another \(Y \in \mathcal{S}\) with \(Y \subset X\). Clearly \(X\) belongs to \(\text{Min}(\mathcal{S})\) iff no such \(Y\) exists. Since deciding whether or not \(Y \subset X\) costs \(O(w)\), the overall cost is \(O(|\mathcal{S}|^2w)\).

\text{[10]}\text{We mention in passing that to some extent general clauses (i.e. with positive and negative literals) can be handled by mixing the two wildcards. For instance }\text{Mod}(x_1 \lor x_2 \lor x_3 \lor x_4) = (e, e, 2, 2, 2) \uplus (0, 0, n, n, n). \text{ Also in the present article the two wildcards will appear simultaneously, if only in Section 9.}

\text{[11]}\text{All of the sequel applies mutatis mutandis to the set system Max(\mathcal{S}) of all maximal members.}
To the author’s best knowledge (readers are welcome to teach him better) the following refinement has not appeared in the literature before. Start by grouping the members of $S$ according to their cardinalities $m_1 < m_2 < \cdots < m_s$ (often $m_{i+1} = m_i + 1$). This induces the decomposition $S = S[1] \uplus S[2] \uplus \cdots \uplus S[s]$. That costs $O(|S|w)$. It suffices to show how to calculate $Min[i] := S[i] \cap \text{Min}(S)$ for all $1 \leq i \leq s$.

Clearly $Min[1] = S[1]$ since minimum-cardinality implies minimal. Set $S'[i] := S[i]$ for $2 \leq i \leq s$. Throughout the remainder we will have $Min[i] \subseteq S'[i] \subseteq S[i]$ and the set-systems $S'[i]$ keep shrinking until they reach $S'[i] = Min[i]$. To begin with, pick any $X \in Min[1]$ and remove all $Y \in S'[i] \ (i \geq 2)$ from $S'[i]$ whenever $X \subset Y$. This costs $O(|S|w)$. Doing the same for all members $X' \in Min[1]$ costs $O(|S|w \cdot |Min[1]|) = O(|S|w \cdot \min)$ where $\min := |Min(S)|$. It is clear that afterwards $S'[2] = Min[2]$. Next for each $X \in Min[2]$ and all $Y \in S'[i] \ (i \geq 3)$ remove $Y$ from $S'[i]$ whenever $X \subset Y$ (again VL can be used). Clearly afterwards $S'[3] = Min[3]$. And so it goes on until we get $S'[s] = Min[s]$. The overall cost is $O(|S|w \cdot \min \cdot s) = O(|S|w^2 \cdot \min)$.

2.6 The operations $\lor, \land$ on $\{0, 1\}$ extend to operations on $\{0, 1\}^m$ (and they match union/intersection of sets in $P(m)$). Adopting Mathematica terminology we call the extended operations $\text{BitOr}$ and $\text{BitAnd}$. For instance, referring to the columns of the $8 \times 6$ matrix $A$ with rows $Z_1$ to $Z_8$ (Table 1), it holds that $\text{BitAnd}(col_2, col_6) = (0, 0, 1, 1, 0, 0, 0)^T$ (where the $T$ means ‘transposed’).

|     | col1 | col2 | col3 | col4 | col5 | col6 |
|-----|------|------|------|------|------|------|
| $Z_1$ | 1    | 1    | 1    | 0    | 0    | 0    |
| $Z_2$ | 1    | 0    | 0    | 0    | 1    | 0    |
| $Z_3$ | 0    | 0    | 0    | 0    | 0    | 0    |
| $Z_4$ | 0    | 0    | 0    | 1    | 1    | 1    |
| $Z_5$ | 0    | 0    | 1    | 0    | 0    | 0    |
| $Z_6$ | 0    | 0    | 1    | 0    | 1    | 0    |
| $Z_7$ | 0    | 0    | 1    | 0    | 0    | 1    |
| $Z_8$ | 0    | 1    | 0    | 1    | 0    | 1    |

Table 1: Illustrating Vertical Layout.

2.6.1 What is this good for? The fact that $\text{BitAnd}(col_2, col_6)$ had a component 1 exactly on the 3rd, 4th and 8th position tells us that among the sets $Z_1, \ldots, Z_8$ the ones that contain the set $\{2, 6\}$ are exactly $Z_3, Z_4, Z_8$. This is e.g. relevant for speeding up the method of 2.5.

2.6.2 Here comes another application. Consider the set system

$$(8) \quad \mathcal{G} := \{\{1, 2, 3\}, \{1, 5\}, \{1, 2, 6\}, \{2, 5, 6\}, \{1, 3, 4\}, \{3, 4, 5\}, \{3, 4, 6\}, \{2, 4, 6\}\}.$$  

The straightforward (=’horizontal’) way to see whether $X = \{1, 2, 5\}$ is a $\mathcal{G}$-transversal checks whether any intersection $X \cap Y \ (Y \in \mathcal{G})$ is empty. In contrast, Vertical Layout (VL) demands\footnote{This can be done “in one sweep” using the method of Vertical Layout discussed in 2.6.} to build the $8 \times 6$ matrix $A(\mathcal{G})$ whose $ith$ row $Y'_i$ is the characteristic bitstring of the $ith$ set $Y_i$ listed in (8). It happens that $A(\mathcal{G})$ is rendered in Table 1. A moment’s reflection confirms the following.

\footnote{For the history of VL see e.g. arXiv:2002.09707}
The fact that $BitOr(col_1, col_2, col_5) = (1, 1, 1, 1, 1, 1, 0, 1)^T$ does not equal $(1, 1, 1, 1, 1, 1, 1, 1)^T$, is tantamount to $X$ not being a $G$-hitting set ($X \cap Y_2 = \emptyset$). Although the formal complexities of the horizontal and vertical way coincide, in practise VL is the faster the more (small) sets $G$ contains. Simply put, computer hardware prefers doing few operations with long bitstrings to doing many operations with short bitstrings.

3 Review of the transversal $e$-algorithm

We survey the transversal $e$-algorithm (3.1) and adapt it to count or generate hitting sets of fixed cardinality (3.2). In 3.3 we indicate how the transversal $e$-algorithm dualizes to the noncover $n$-algorithm.

3.1 Consider the task to enumerate the set $HS(H_2)$ of all hitting sets of the hypergraph $H_2$ whose five hyperedges $X \subseteq [6]$ are

\[(9) \ H_1 = \{1, 2, 5\}, \ H_2 = \{3, 4\}, \ H_3 = \{4, 5, 6\}, \ H_4 = \{1, 3, 5\}, \ H_5 = \{2, 6\}.\]

One idea is to first compute the hitting sets of the hypergraph $\{H_1\}$, then the ones of $\{H_1, H_2\}$, and so forth until we obtain the hitting sets of $\{H_1, ..., H_5\} = H_2$. Calculating $HS(\{H_1\})$ is easy in view of 2.2. It consists of all bitstrings (=subsets of $[6]$) that have at least 1 on the positions 1, 2, 5, and so $HS(\{H_1\}) = (e, e, 2, 2, e, 2)$. Likewise $HS(\{H_1, H_2\}) = (e_1, e_1, e_2, e_2, e_1, 2) =: r'$.

Now it gets trickier because $H_3$ intersects $H_1$ and $H_2$, i.e. the $e_3$-wildcard supposed to be modeling $H_3$ interferes with existing $e$-wildcards. In 2.3 we indicated how this is to be handled. Recall that the row in (6), which suffered the same predicament as $r'$ above, had to be replaced by three candidate sons in (7). The essence of the transversal $e$-algorithm is to keep on picking the topmost row $r'$ of a "to do" stack of 012$e$-rows and to impose some $e$-wildcard upon $r'$, which in turn can trigger up\footnote{Here $t$ is as in (4) and (5). Concerning the "to do" stack, the standard name is Last-In-First-Out (LIFO) stack. LIFO-stacks are standard data structures which match the depth-first search of trees.} to $t$ candidate sons. Each candidate son $r_i$ must be feasible in the sense that $r_i \cap HS(H) \neq \emptyset$, for otherwise further processing of $r_i$ cannot possibly yield any hitting sets. The feasible candidate sons are put on top of the LIFO stack, the others are discarded. Fortunately deciding feasibility is easy:

\[(10) \ r \ is \ feasible \ iff \ (\forall H \in \mathbb{H})(H \not\subset \text{zeros}(r)).\]

The effect of discarding infeasible candidate sons is that in each set of candidate sons at least one will be feasible. This in turn is the reason that the $e$-algorithm runs in total polynomial time, in fact in $O(Rh^2w^2)$ time. For the fine details of this transversal $e$-algorithm\footnote{Due to its use in previous publications we stick with "transversal e-algorithm". Other than that we always use "hitting set" instead of the synonym "transversal".} the reader is referred to [3]. To summarize, for any given hypergraph $H \subseteq \mathcal{P}([w])$ the transversal $e$-algorithm renders $HS(H)$ as a disjoint union of $R$ many 012$e$-rows, thus

\[(11) \ HS(H) = \bigcup_{i=1}^R \overline{r}_i.\]

3.1.1 Applied to $H_2$ the transversal $e$-algorithm yields $HS(H_2) = \overline{r}_1 \cup \ldots \cup \overline{r}_t$, where the $\overline{r}_i$'s
are defined in Table 2.

| \( \rho_1 \) | \( e \) | \( e \) | \( 1 \) | \( 0 \) | \( 0 \) | \( 1 \) |
| \( \rho_2 \) | \( 2 \) | \( e_1 \) | \( e_2 \) | \( e_2 \) | \( 1 \) | \( e_1 \) |
| \( \rho_3 \) | \( 0 \) | \( 1 \) | \( 1 \) | \( 1 \) | \( 0 \) | \( 2 \) |
| \( \rho_4 \) | \( 1 \) | \( e \) | \( 2 \) | \( 1 \) | \( 0 \) | \( e \) |

Table 2: Representing \( HS(\mathbb{H}_2) \) as disjoint union of 012e-rows

In view of 2.2 we conclude that

\[ |HS(\mathbb{H}_2)| = |\rho_1| + \cdots |\rho_4| = (2^2 - 1) + 2(2^2 - 1)^2 + 2 + 2(2^2 - 1) = 29. \]

3.2 Let \( \mu := \mu(\mathbb{H}) \) be the minimum cardinality achieved by any hitting set of the hypergraph \( \mathbb{H} \). Often \( \mu \) gets known\(^{16}\) only after (11) has been obtained. For all \( c \in \{\mu, \mu + 1, \ldots, w\} \) we put

\[ (12) \quad HS(\mathbb{H}, c) := \{ X \in HS(\mathbb{H}) : |X| = c \}. \]

Of particular interest is the set-system

\[ (13) \quad MCHS(\mathbb{H}) := HS(\mathbb{H}, \mu) \subseteq MSH(\mathbb{H}). \]

3.2.1 In some circumstances (e.g. in [7]) it is irrelevant whether the \( \mathbb{H} \)-hitting sets are minimal; just their cardinality matters. Let us hence calculate \( |HS(\mathbb{H}, c)| \) for any fixed \( c \geq \mu \). Viewing (11) for any such \( c \) let \( \mathcal{T} \) be the set of indices \( i \leq R \) such that the 012e-row \( \rho_i \) has degree \( \leq c \). (That’s because \( \rho_i \cap HS(\mathbb{H}, c) = \emptyset \) if \( i \notin \mathcal{T} \).) Putting \( S(\rho_i) := \{ X \in \rho_i : |X| = c \} \) we get

\[ |HS(\mathbb{H}, c)| \] by summing up the numbers \( |S(\rho_i)| \) \((i \in \mathcal{T})\). It is easy to calculate the numbers \( |S(\rho_i)| \) with inclusion-exclusion; for a faster way see [3, Thm.1].

3.2.2 Suppose the set \( HS(\mathbb{H}, c) \) itself needs to be calculated. By the above each fixed set-family \( HS(\mathbb{H}, c) \) is the disjoint union of all sets \( S(\rho_i) \) \((i \in \mathcal{T})\). But sieving \( S(\rho_i) \) from \( \rho_i \) is more cumbersome than calculating \( |S(\rho_i)| \). Leaving ways of compression to the future, we only note that if \( S(\rho_i) \) has \( \alpha \) elements then by [3, Thm.2] it can be enumerated one-by-one in total polynomial time \( O(\alpha w^2) \).

3.2.3 If \( HS(\mathbb{H}, c) \) is of interest cardinality-wise (or the members themselves) for all values \( \mu \leq c \leq w \), then upon running the transversal e-algorithm each \( c \) gets processed as discussed in 3.2.1 (or 3.2.2). However, if only values \( c \leq d \) for some bound \( d \) are relevant, then it pays to adjust the transversal e-algorithm as follows. In addition to (10), the arising candidate sons should also satisfy \( \text{deg}(\rho) \leq d \). That’s because \( \text{deg}(\rho) > d \) implies that all successor rows \( \rho_i \) of \( \rho \) will have \( \text{deg}(\rho_i) \geq \text{deg}(\rho) > d \), and so cannot contain any members of \( HS(\mathbb{H}, c) \). Problem is, in contrast to the remarks after (10) it can now happen that some rows loose all their candidate sons. Nevertheless, performance in practise may be good.

\(^{16}\) According to [8] the cost of finding a minimum-cardinality transversal is \( O(1.2381^n) \) where \( n \) is the sum of \( w \) and all cardinalities \( |H| \) \((H \in \mathbb{H})\).
3.3 The family $HS(\mathbb{H})$ of all $\mathbb{H}$-hitting sets is a set-filter $\mathcal{F}$ in the sense that $(X \in \mathcal{F} \text{ and } X \subseteq Y) \Rightarrow Y \in \mathcal{F}$. Now let $S \subseteq \mathcal{P}([w])$ be a set system. Call $Z \in \mathcal{P}([w])$ a $S$-noncover if $Z \nsubseteq Y$ for all $Y \in S$. Then the family $NC(S)$ of all $S$-noncovers is a set-ideal $\mathcal{J}$ in the sense that $(X \in \mathcal{J} \text{ and } Y \subseteq X) \Rightarrow Y \in \mathcal{J}$. Consider any set-filter $\mathcal{F} \subseteq \mathcal{P}([w])$. The minimal members of $\mathcal{F}$ are called its generators and they determine $\mathcal{F}$ uniquely. Likewise for any set-ideal $\mathcal{J} \subseteq \mathcal{P}([w])$ the maximal members of $\mathcal{J}$ are called its facets and they determine $\mathcal{J}$ uniquely. Furthermore, let $\mathcal{F}$ and $\mathcal{J}$ be complementary set-systems in the sense that $\mathcal{F} \cup \mathcal{J} = \mathcal{P}([w])$. It then holds that $\mathcal{F}$ is a set-filter iff $\mathcal{J}$ is a set-ideal.

Given $\mathbb{H} \subseteq \mathcal{P}([w])$ the transversal $e$-algorithm renders the set-filter $HS(\mathbb{H})$ in the convenient format (11). Since set-filter and set-ideal are dual concepts, and so are $e$-wildcards and $n$-wildcards, it comes as no surprise that some noncover $n$-algorithm (see e.g. [6]), fed with $\mathcal{S}$ renders the set-ideal $NC(S)$ as a disjoint union of $R'$ many 012n-rows:

$$(11')\quad NC(S) = \bigcup_{i=1}^{R'} R_i.$$

4 From minimum-cardinality toward inclusion-minimal

We show that (11) persists even when all 012e-rows $\overline{R}_i$ get ”shaved” and become certain 01g-rows $\rho_i \subseteq \overline{R}_i$. Thus (11) improves to (17). It turns out that $MCHS(\mathbb{H})$ in (13) is the union of some such rows $\rho_i$. In 4.2 we comment on situations where $MCHS(\mathbb{H}) = MHS(\mathbb{H})$, and in 4.4 resume the Monte Carlo of Section 1 in order to get an estimate for $|MHS(\mathbb{H})|$.

4.1 Let $\mathbb{H} \subseteq \mathcal{P}([w])$ be a hypergraph. In the remainder of the article we assume that the transversal $e$-algorithm has rendered $HS(\mathbb{H})$ as a disjoint union of $R$ many 012e-rows as in (11). Different from [3] where these rows were coined ‘final’, here the availability of them is not the end but only the beginning. That’s why we henceforth call them semifinal 012e-rows.

Suppose $X \subseteq [w]$ is any minimal $\mathbb{H}$-hitting set. Then $X$ is contained in some semifinal 012e-row $\overline{R}_i$ because of (11). Being minimal within $HS(\mathbb{H})$, a fortiori $X$ is minimal within the smaller set-system $\overline{R}_i \subseteq HS(\mathbb{H})$, i.e. $X \in Min(\overline{R}_i)$. In view of (5) it follows that for all $1 \leq i \leq R$:

$$(14)\quad \overline{R}_i \cap MHS(\mathbb{H}) \subseteq Min(\overline{R}_i) = \{X \in \overline{R}_i : |X| = deg(\overline{R}_i)\}.$$

In particular consider $Y \in MCHS(\mathbb{H}) \subseteq MHS(\mathbb{H})$. As before $Y \in Min(\overline{R}_j)$ for some $j \leq R$. But all sets in $Min(\overline{R}_j)$ have the same cardinality as $Y$, and so are themselves in $MCHS(\mathbb{H})$. Hence $\subseteq$ in (14) becomes $=$. To summarize:

**Theorem 1:** Assume that $HS(\mathbb{H})$ is represented as disjoint union of 012e-rows $\overline{R}_i$ as in (11). Then, with notation as above, $MCHS(\mathbb{H})$ is the disjoint union of those sets $Min(\overline{R}_i)$ that have $deg(\overline{R}_i) = \mu$.

To illustrate consider $HS(\mathbb{H}_3) = \overline{R}_1 \cup \cdots \cup \overline{R}_4$ in Table 2. One checks that all these rows happen to have degree 3, and so $\mu = 3$. It follows from Theorem 1 and the fact (see 2.4.1) that sets of

---

17 Set-ideals are also called (abstract) simplicial complexes.
type $\text{Min}(\overline{\rho}_i)$ can conveniently be rendered by single 01g-rows $\rho_i$ that

\begin{equation}
\text{MC}\text{HS}(\mathbb{H}_2) = \rho_1 \cup \rho_2 \cup \rho_3 \cup \rho_4,
\end{equation}

where the $\rho_i$’s are defined below:

\begin{align*}
(16) \quad \text{Min}(\overline{\rho}_1) &= \{136, 236\} = (g, g, 1, 0, 0, 1) =: \rho_1 \\
\text{Min}(\overline{\rho}_2) &= \{235, 245, 356, 456\} = (0, g_1, g_2, 1, g_1) =: \rho_2 \\
\text{Min}(\overline{\rho}_3) &= \{234\} = (0, 1, 1, 0, 0) =: \rho_3 \\
\text{Min}(\overline{\rho}_4) &= \{124, 146\} = (1, g, 0, 1, 0, g) =: \rho_4
\end{align*}

4.2 As opposed to (15) where incidently $\text{MC}\text{HS}(\mathbb{H}_2) = \text{MHS}(\mathbb{H}_2)$, for general hypergraphs $\mathbb{H}$ only few semifinal 012e-rows $\overline{\rho}_i$ will have degree $\mu$. If only $\text{MC}\text{HS}(\mathbb{H})$ is sought then all rows $\overline{\rho}_i$ with $\deg(\overline{\rho}_i) > \mu$ are superfluous. Yet to avoid them one cannot proceed as in 3.2.3 because usually $d := \mu$ is not known in advance. However, guessing and working with some slightly larger $d > \mu$ will still beat computing all $R$ rows. (If it happens that one guesses a $d < \mu$ then the proposed method will not deliver any semifinal 012e-rows. But it will improve the next guess, and with binary search one can even pin down $\mu$.)

4.2.1 Interestingly, in the following set-up $\mu$ is known\textsuperscript{18} in advance; it even happens that $\text{MHS}(\mathbb{H}) = \text{MC}\text{HS}(\mathbb{H})$. Namely, if $\mathbb{H}$ is the family of all cocircuits [9,p.653] of a matroid then $\text{MHS}(\mathbb{H})$ is the set of all matroid bases and $\mu$ is easy to come by. In arXiv:2002.09707 (submitted) this has been implemented for the scenario where the cocircuits are the minimal cutsets of a graph $G$, in which case $\text{MHS}(\mathbb{H}) = \text{MC}\text{HS}(\mathbb{H})$ is the set of all spanning trees of $G$.

4.2.2 Suppose that $\mu$ is known, be it by binary search or by theoretical reasoning as in 4.2.1. Then one still sits with the problem (mentioned in 3.2.3) that some top-rows of the LIFO stack may lose all their candidate sons. That this cannot happen in 4.2.1 is one of the (numerically well-supported) conjectures raised in arXiv:2002.09707. In another vein, if all $H \in \mathbb{H}$ have $|H| = 2$, so $\mathbb{H}$ is the edge-set of a graph, then instead of MHSes one rather speaks of minimal vertex-covers. In this scenario $\mu$ remains hard to compute, but at least ”loosing all candidate sons” can be avoided (work in progress).

4.3 Generalizing Table 2 and (16), each semifinal 012e-row $\overline{\rho} := \overline{\rho}_i$ appearing in (11) yields the

\begin{equation}
\text{semifinal 01g-row (or simply: semifinal row)}
\end{equation}

$\rho := \text{Min}(\overline{\rho})$ where all 2’s of $\overline{\rho}$ have been replaced by 0’s and each $e$-wildcard of length $\epsilon_j$ has been replaced by a $g$-wildcard of the same length $\gamma_j := \epsilon_j$. Hence, akin to (4) and (5), the semifinal 01g-row $\rho$ has $t$ many $g$-wildcards and it holds that the $\gamma_1 \gamma_2 \cdots \gamma_t$ members of $\rho$ all have cardinality $|\text{ones}(\rho)| + t$. It follows from (11) that

\begin{equation}
\text{MHS}(\mathbb{H}) \subseteq \bigcup_{i=1}^{R} \rho_i =: SF(\mathbb{H}).
\end{equation}

\textsuperscript{18}Readers aware of other scenarios of that type, please let the author know.
Accordingly we have

\[(17') \ mhs = |MHS(\mathbb{H})| \leq sf := |\bigcup_{i=1}^R \rho_i|.
\]

The following terminology will be handy as well. A semifinal 01g-row \(\rho\) is bad if \(MHS(\mathbb{H}) \cap \rho = \emptyset\), and good otherwise. Additionally call \(\rho\) very-good if \(\rho \subseteq MHS(\mathbb{H})\), and call \(\rho\) merely-good if it is good but not very-good. Each \(X \in \rho \setminus MHS(\mathbb{H})\) a dud. For 01g-rows it holds that good \(\iff\) very-good.

\[4.4\] A simple attempt to settle ”good or bad?” is the Monte-Carlo way. That is, pick uniformly and at random \(X \in \rho\) and check (in whatever way) whether or not \(X \in MHS\). If yes, then \(\rho\) is good. If no, test some more \(X\). The more often the answer persists to be no, the likelier \(\rho\) is bad. As to ”likelier”, the density \(d := |\rho \cap MHS(\mathbb{H})| / |\rho|\) can be estimated to any desired precision \(\epsilon\) as follows. Given \(\epsilon, \delta > 0\), standard statistics yields a value \(d'\) such that (with error-probability < \(\delta\)) it holds that \(d \in [(1-\epsilon)d', (1+\epsilon)d']\). Since \(|\rho| = \gamma_1 \cdots \gamma_t\) is known, \(d'\) also yields an estimate for \(|\rho \cap MHS(\mathbb{H})|\), and hence in view of (17) for \(|MHS(\mathbb{H})|\).

\section{Four ways to sieve the MHSes from the semifinal 01g-rows}

Let \(\rho\) be a fixed semifinal 01g-row. In this Section we present four methods (Algorithm 1 to Algorithm 4) to classify all \(X \in \rho\) one-by-one, i.e. to decide whether \(X\) is an MHS or a dud. Algorithm 1 relies on 2.5 and 2.6.1, whereas Algorithm 2 uses the kind of Vertical Layout in 2.6.2. Algorithms 3 and 4 rely on presently ”magic” set-systems PotKi(\(\rho\)), respectively MC(\(\mathbb{H}\)), whose capabilities and whose acquirability will be postponed to later Sections.

\[5.1\] Referring to 2.5 let \(m_1 < m_2 < \cdots < m_s\) be the numbers that occur as cardinalities of \(\mathbb{H}\)-hitting sets. Then \(m_1 = \mu\) and \(m_s = w\). Putting \(S := SF(\mathbb{H})\) we have \(Min(S) = MHS(\mathbb{H})\), and following 2.5 we get \(MHS(\mathbb{H})\) in time \(O(sf \cdot w \cdot mhs \cdot s)\). This Algorithm 1 may compare favorably to methods in spe if \(s\) gets replaced (say) by 3, i.e. if we only care for MHSes of cardinality \(\leq \mu + 2\). In this case the cost is \(O(sf \cdot w \cdot mhs)\).

\[5.2\] Let us view the hyperedges of \(\mathbb{H}\) as bitstrings and take them as the rows of an \(h \times w\) matrix \(A\). Fix a semifinal 01g-row \(\rho\) and put \(k := \deg(\rho)\). For each fixed \(Y \in \rho\) (hence a hitting set) it holds that \(Y \in MHS(\mathbb{H})\) iff no set \(X := Y \setminus \{a\} (a \in Y)\) is an hitting set. Whether or not VL based on \(A\) (see 2.6) is used, the formal cost to classify \(X\) is \(O(kh)\). Hence classifying \(Y\) costs \(O(k^2w)\). Furthermore finding \(\rho \cap MHS(\mathbb{H})\) costs \(O(|\rho|k^2h)\), and finding \(MHS(\mathbb{H})\) with the sketched Algorithm 2 costs \(O(sf \cdot w^2h)\) (since \(|Y| = k\) becomes \(|Y| \leq w\).

\[5.2.1\] Observe that the bound in 5.1 to get \(MHS(\mathbb{H})\) does not adapt smoothly to a bound for calculating just \(\rho \cap MHS(\mathbb{H})\). This contrasts with 5.2 where we obtained \(O(|\rho|k^2h)\) for the latter task, due to the fact that it costs \(O(k^2h)\) to decide whether any \(k\)-hitting set of a hypergraph with \(h\) hyperedges is minimal. While the \(O(k^2h)\) bound for this basic decision problem has probably been observed before, let us indicate a surprising improvement of it. To fix ideas, suppose that \(k = 5\) and that the minimality of an \(\mathbb{H}\)-hitting \(Y = \{1, 2, 3, 4, 5\}\) (where \(|\mathbb{H}| = h\)) needs to be decided. In 2.6.2 the VL way to handle \(Y\) demands to calculate

\[19\] Often, yet not always, \(m_{i+1} = m_i + 1\) for all \(1 \leq i \leq s - 1\).
"col_{1234} := col_1 \lor col_2 \lor col_3 \lor col_4 \text{ and } col_{1235} := col_1 \lor col_2 \lor col_3 \lor col_5 

and col_{1245} := col_1 \lor col_2 \lor col_4 \lor col_5 \text{ and } col_{1345} := col_1 \lor col_3 \lor col_4 \lor col_5 

and col_{2345} := col_2 \lor col_3 \lor col_4 \lor col_5."

This required $5 \cdot 3 = 15$ basic BitOr operations, but one can improve that to $11$:

\[
col_{12} := col_1 \lor col_2, \ col_{123} := col_{12} \lor col_3, \ col_{124} := col_{12} \lor col_4, \ col_{1234} := col_{123} \lor col_4, \\
\col_{1235} := col_{123} \lor col_5, \ col_{1245} := col_{124} \lor col_5, \ col_{45} := col_4 \lor col_5, \ col_{345} := col_3 \lor col_{45}, \\
\col_{245} := col_2 \lor col_{45}, \ col_{1345} := col_1 \lor col_{1345}, \ col_{2345} := col_3 \lor col_{245}.
\]

Driving this idea further\(^{20}\), one can improve $O(k^2w)$ to $O(k^{4/3}w)$.

5.3 Given a semifinal 01g-row $\rho$ suppose it was possible (more on that in 9.3) to get a set-system $PotK_i(\rho) \subseteq \mathcal{P}(\{w\})$ such that any given $X \in \rho$ is a dud iff it gets killed by some $Z \in PotK_i(\rho)$ in the sense that $Z \subseteq X$. So suppose the toy row $\hat{\rho} := (1,g_1,g_1,g_2,g_2)$ has $PotK_i(\hat{\rho}) = \{15,126\}$. Since 15 kills 152, 153, 154 and 126 kills 126, we have four duds and hence $\hat{\rho} \cap MHS(\mathbb{H}) = \{136,146\}$. The availability of $PotK_i(\rho)$ facilitates a lot the calculation of $Duds(\rho) := \{X \in \rho : X \text{ is a dud} \} (= \rho \setminus MHS(\mathbb{H}))$.

Namely, embarking onto VL (which makes the more sense the larger $\rho$), we view the members of $\rho$ as bitstrings and take them as the rows of a $|\rho| \times w$ matrix $A$. Starting with $Duds(\rho) := \emptyset$ we process $PotK_i(\rho)$ one by one and update $Duds(\rho)$ accordingly as follows (Algorithm 3). Say $Z = \{2,4,7\} \in PotK_i(\rho)$. If $\col_i$ is the $i$-th column of $A$ we calculate $\col := \col_2 \land \col_4 \land \col_7$. Then $ones(\col)$ is the set of row-numbers whose corresponding rows of $A$ get killed by $Y$. Thus we update $Duds(\rho) = Duds(\rho) \cup ones(\col)$.

5.3.1 Picked from the author’s random experiments, here comes a more demanding semifinal 01g-row $\rho$. It is defined by $zeros(\rho) := \emptyset$, $ones(\rho) := \{4,5,6\}$, has g-bubbles \{1,8\}, \{2,10\}, \{3,11\}, \{7,9,12\}, and has $PotK_i(\rho) = \{Z_1, Z_2, Z_3, Z_4\} := \{\{2,4,6\}, \{2,5,6\}, \{1,5,10\}, \{5,8,10\}\}$.

Here $Z_1$ kills (exactly) the twelve sets of type $\{4,5,6\} \cup \{a_2, c, d\}$, $Z_2$ kills the sets of type $\{4,5,6\} \cup \{a, 2, c, d\}$, i.e. the same as before, $Z_3$ kills the six sets of type $\{14,5,6\} \cup \{1,10, c, d\}$, and $Z_4$ the six sets of type $\{4,5,6\} \cup \{8,10, c, d\}$. Since the killed sets happen to be either identical or disjoint, it follows from $12 + 6 + 6 = |\rho|$ that $\rho$ gets killed entirely. It is an example of a ‘sophisticated-bad’ row, the exact definition following in 9.1.

5.4 Fix some hypergraph $\mathbb{H} \subseteq \mathcal{P}(\{w\})$. Following [4] we say $S \subseteq \{w\}$ is a MC-set (or: is MC) iff for each $b \in S$ at least one $H \in \mathbb{H}$ cuts $b$ sharply from $S$, i.e. $H \cap S = \{b\}$. Each other $H' \in \mathbb{H}$ either cuts out $b$ sharply as well, or has $|H' \cap S| \geq 2$, or has $H' \cap S = \emptyset$. It is evident that a subset of a MC-set is again a MC-set. Hence the family

\(^{20}\)Interested readers are welcome to help refine the author’s handwritten notes into publishable form.
Note that MC-sets need not be hitting sets. To witness take $\mathbb H_3 := \{\{1,3\}, \{2,4\}, \{3,4\}\}$. One checks that $\{1,2\}$ is a MC-set yet not an $\mathbb H$-hitting set. However, it holds (Section 9) that $\rho \cap MC(\mathbb H) = \rho \cap MHS(\mathbb H)$ for each semifinal 01g-row $\rho$.

This suggests an elegant method for classifying any $X \in \rho$. Namely, initialize a testset $T$ to $T := \emptyset$. Process all $H \in \mathbb H$ and update $T := T \cup (X \cap H)$ (programmer’s speak) whenever $|X \cap H| = 1$. As soon as $T = X$ occurs, we know that $X \in MHS(\mathbb H)$. If it never occurs then $X \notin MHS(\mathbb H)$. Since classifying $X$ that way costs $O(hw)$ we have a method, call it Algorithm 4, that calculates $MHS(\mathbb H)$ in time $O(sf \cdot hw)$.

5.4.1 Let us indicate how VL may further speed up calculating $\rho \cap MHS(\mathbb H)$ (as always, without challenging the formal bound, in this case $O(|\rho|hw)$). For starters, the $wh$ many sets

$$S(a,H) := \{Z \in \rho : a \in H \cap Z, \ |H \cap Z| \geq 2\} \quad (a \in [w], \ H \in \mathbb H)$$

need to be calculated. To do so initialize all of them as $S(a,H) := \emptyset$. Next for each fixed $Z \in \rho$ do the following. Using VL determine all $K \in \mathbb H$ with $|K \cap Z| \geq 2$. For any such $K$ add $Z$ to all sets $S(a,K)$ ($a \in K \cap Z$). For $a \in [w]$ and $Z \in \rho$ with $a \in Z$ call $Z$ an $a$-dud if there is no hyperedge that sharply cuts out $a$ from $Z$ (and so $Z \notin MHS(\mathbb H)$). It is easy to see that

$$S(a) := \bigcap \{S(a,H) : a \in H \in \mathbb H\}$$

is the set of all $a$-duds, and that VL speeds up the calculation of $S(a)$ the more the bigger $|\rho|$. Consequently

$$Duds := \bigcup \{S(a) : a \in [w]\}$$

is the set of all duds contained in $\rho$. Put another way, $\rho \cap MHS(\mathbb H) = \rho \setminus Duds$.

6 Replacing merely-good rows by very-good rows

In Section 5 we presented four algorithms to unravel the MHSes contained in a fixed semifinal 01g-row $\rho$. Any such MHS, viewed as bitstring $x \in \{0,1\}^w$ if $x$ is a very-good row, and so one could say that each semifinal row $\rho$ is either bad or can be represented as a disjoint union of very-good rows. But it would be nice to use fewer than $|\rho|$ very-good rows to exhaust $\rho$.

Suppose we possess (more on that later) criteria that allow us to quickly classify each semifinal 01g-row as bad, merely-good, and very-good. The bad ones are thrown away, the very-good ones are in optimal shape, but what about the merely-good rows $\rho$? Aren’t we back to Square 1 and need to scan $\rho$ one by one?. Not so. We start with a toy example in 6.1 and follow up with theory in 6.2.

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21 Instead of adding whole sets $Z$, code the sets as numbers $f(Z)$. Because $\rho$ is a 01g-row, there is a natural "lexicographic" encoding. To witness, take $\rho = (g_1 g_2 g_2 g_3)$ which for simplicity has $zeros(\rho) = ones(\rho) = \emptyset$. Put $f((2,4,6)) = f(246) := 1$, $f(245) := 2$, $f(236) := 3$, ..., $f(135) := 8$.

22 To be pedantic, since by definition every 012g-row is a set of bitstrings, not $x$ but $\{x\}$ is a very-good row.
6.1 Consider the hypergraph $\mathbb{H}_4 \subseteq \mathcal{P}[\{6\}]$ with hyperedges

\[(19) \quad H_1 = \{1, 5, 6\}, \quad H_2 = \{3, 4, 5\}, \quad H_3 = \{2, 3\}, \quad H_4 = \{1, 4, 6\}. \]

Feeding the transversal e-algorithm with $\mathbb{H}_4$ yields (among others) the semifinal 01g-row $r$ in Table 3. It is good since it e.g. contains the minimal $\mathbb{H}_4$-hitting set $\{1, 2, 5\}$. Yet $r$ is not very-good since $\{1, 3, 5\} \in r \setminus MHS(r)$ is a dud (viewing that $\{1, 3\} \in HS(\mathbb{H}_4)$).

|   | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $r$ | $g_1$ | $g_2$ | $g_2$ | 1 | $g_1$ | merely-good |
| $r_1$ | 1 | $g_2$ | $g_2$ | 0 | 1 | 0 | merely-good |
| $r_2$ | 0 | $g_2$ | $g_2$ | 1 | 1 | 0 | very-good |
| $r_3$ | 0 | $g_2$ | $g_2$ | 0 | 1 | 1 | merely-good |
| $r_4$ | 1 | 1 | 0 | 0 | 1 | 0 | very-good |
| $r_5$ | 0 | 1 | 0 | 0 | 1 | 1 | very-good |
| $\rho_1$ | $g_1$ | 1 | 0 | $g_1$ | 1 | $g_1$ | very-good |
| $\rho_2$ | $g_1$ | 0 | 1 | $g_1$ | 1 | $g_1$ | merely-good |
| $\rho_3$ | 0 | 0 | 1 | 1 | 1 | 0 | very-good |

*Table 3: Replacing a merely-good row by very-good rows.*

We strive to replace $r$ by disjoint rows which are very-good and jointly contain the same minimal hitting sets as $r$. It is natural to start by picking any g-wildcard of $r$, say $(g_1, g_1, g_1)$, and *expand* $r$ accordingly as $r = r_1 \uplus r_2 \uplus r_3$ (see Table 3). We call $r_1, r_2, r_3$ the *sons* of $r$. One checks that $\{2, 4, 5\}, \{3, 4, 5\} \in MHS(\mathbb{H}_4)$, and so $r_2$ is very-good. As to $r_1$, it is merely-good. Specifically, by expanding the second g-wildcard $(g_2, g_2)$ one obtains $r_1 = (1, 0, 1, 0, 1, 0) \uplus (1, 1, 0, 0, 1, 0)$, where the first son is bad since $\{1, 3\} \in HS(\mathbb{H}_4)$, and the second (call it $r_4$) is very-good. Also $r_3$ is merely-good; it decomposes as $r_3 = (0, 0, 1, 1, 1) \uplus (0, 1, 0, 0, 1, 1)$, where the first son is bad ($36 \in HS(\mathbb{H}_4)$) and the second (call it $r_5$) is very-good. To summarize, we managed to replace the semifinal merely-good row $r$ by the final very-good rows $r_2, r_4, r_5$.

Alternatively, one can start by expanding $(g_2, g_2)$. This yields the rows $\rho_1, \rho_2$ in Table 3. One checks that $\rho_1$ is very-good, but $\rho_2$ is not. Specifically, when expanding $(g_1, g_1, g_1)$ in $\rho_2$, two of the three arising 01-rows are bad. The third one (labelled $\rho_3$) is very-good. To summarize, $r$ can even be replaced by *two* very-good rows, i.e. $\rho_1, \rho_3$.

6.2 The example above suggests the following method to replace a semifinal good row $r$ by final very-good rows that jointly contain the same MHSes as $r$. There is nothing to do if $r$ is already very-good. By induction assume that a stack is filled with disjoint merely-good 012g-rows which jointly contain exactly the MHSes contained in $r$. (Initially $r$ is the only member of the stack.) Remove the top row $r'$ from the stack. Expanding any g-wildcard of $r'$ yields candidate \[\text{sons} \ r_1, r_2, \ldots \] akin to 6.1. The very-good candidate sons are final; they are removed from the stack and stored somewhere else. The bad ones are thrown away, and the merely-good ones are put on top of the stack. It is clear that the new stack maintains the induction hypothesis. When the

\[23\] There is no danger confusing the with the kind of candidate sons in 3.1.
stack is empty, the final rows are disjoint and jointly contain the same MHSes as \( r \).

Above we used the wording “expanding any \( g \)-wildcard”. Without going into details we mention that “any” needs not be random but can be chosen in ways that likely increase compression.

7 Deciding very-goodness using inclusion-exclusion

The larger our semifinal rows \( \rho \) in (17) the more desirable is it to have efficient criteria for very-goodness and badness. In particular in Sec. 6 we reduced the handling of merely-good rows, to large extent, to the existence of such tests. In this and the next Section we offer two very-goodness tests. The one in Section 7 relies on inclusion-exclusion.

7.1 Consider a fixed semifinal 01g-row \( \rho \) triggered by \( \mathbb{H} \subseteq \mathcal{P}([w]) \). We say that \( Z \subseteq [w] \) is a potential \( \rho \)-spoiler if there is a \( Y \in \rho \) with \( Z \uplus \{a\} = Y \). In Table 4 the set system of all potential \( \rho \)-spoilers of some semifinal \( \rho \) is represented as disjoint union \( d_1 \uplus \cdots \uplus d_5 \) of 01g-rows. Its cardinality is \( 24 + \cdots + 6 = 74 \). Generally the following holds:

(20) With \( \gamma_1, \ldots, \gamma_t \) being the length of the \( g \)-wildcards of \( \rho \), the number of potential \( \rho \)-spoilers of the semifinal row \( \rho \) is \( \text{Pot} = (\gamma_2 \cdots \gamma_t) + (\gamma_1 \gamma_3 \cdots \gamma_t) + (\gamma_1 \cdots \gamma_{t-1}) + |\text{ones}(\rho)| \cdot \gamma_1 \cdots \gamma_t \).

|   | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | cardinality |
|---|---|---|---|---|---|---|---|---|---|----|----|----|-------------|
| \( \rho = \) | \( g_1 \) | \( g_1 \) | 0 | \( g_2 \) | \( g_2 \) | \( g_2 \) | \( g_3 \) | \( g_3 \) | \( g_3 \) | 1 | 1 | 630 |
| \( d_1 = \) | \( g_1 \) | \( g_1 \) | 0 | \( g_2 \) | \( g_2 \) | \( g_2 \) | \( g_3 \) | \( g_3 \) | \( g_3 \) | 0 | 1 | 24 |
| \( d_2 = \) | \( g_1 \) | \( g_1 \) | 0 | \( g_2 \) | \( g_2 \) | \( g_2 \) | \( g_3 \) | \( g_3 \) | \( g_3 \) | 1 | 0 | 24 |
| \( d_3 = \) | 0 | 0 | 0 | \( g_2 \) | \( g_2 \) | \( g_2 \) | \( g_3 \) | \( g_3 \) | \( g_3 \) | 1 | 1 | 12 |
| \( d_4 = \) | \( g_1 \) | \( g_1 \) | 0 | 0 | 0 | 0 | \( g_3 \) | \( g_3 \) | \( g_3 \) | 1 | 1 | 8 |
| \( d_5 = \) | \( g_1 \) | \( g_1 \) | 0 | \( g_2 \) | \( g_2 \) | \( g_2 \) | 0 | 0 | 0 | 0 | 1 | 1 | 6 |
| \( \delta_1 = \) | \( g_1 \) | \( g_1 \) | 0 | 1 | 0 | 0 | 0 | 0 | \( g_3 \) | \( g_3 \) | 0 | 1 | 4 |
| \( \delta_2 = \) | \( g_1 \) | \( g_1 \) | 0 | 1 | 0 | 0 | 0 | 0 | \( g_3 \) | \( g_3 \) | 1 | 0 | 4 |
| \( \delta_3 = \) | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | \( g_3 \) | \( g_3 \) | 1 | 1 | 2 |
| \( \delta_4 = \) | \( g_1 \) | \( g_1 \) | 0 | 0 | 0 | 0 | 0 | 0 | \( g_3 \) | \( g_3 \) | 1 | 1 | 4 |
| \( \delta_5 = \) | \( g_1 \) | \( g_1 \) | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 2 |

Table 4: Counting \( \rho \)-spoilers by applying inclusion-exclusion

For a semifinal \( \rho \) we define an \( \rho \)-spoiler as a potential \( \rho \)-spoiler that happens to be an \( \mathbb{H} \)-hitting set. If \( Sp = Sp(\rho, \mathbb{H}) \) is the number of \( \rho \)-spoilers, then a moment’s reflection confirms:

(21) The semifinal row \( \rho \) is very-good iff \( Sp = 0 \).

If say \( H_i, H_j, H_\ell \) are hyperedges of \( \mathbb{H} \) then we define \( N(i, j, \ell) \) as the number of potential \( \rho \)-spoilers \( Z \) with \( Z \cap H_i = Z \cap H_j = Z \cap H_\ell = \emptyset \). Since a potential spoiler is a spoiler iff it cuts
all hyperedges of $\mathbb{H}$, we can compute $Sp$ with inclusion-exclusion as

$$Sp = Pot - N(1) - N(2) - \cdots - N(h) + N(1, 2) + \cdots + (-1)^h N(1, 2, \ldots, h).$$

Calculating $2^h$ terms $N(.)$ may seem inefficient but the larger $|\rho|$ and $w$, and the smaller $h$, the more inclusion-exclusion will prevail over the "naive" way in 5.2 which spends $O(hk^2)$ time per $k$-element member $X \in \rho$.

7.2 Furthermore, based on the first three Bonferroni inequalities these implications often alleviate full-blown inclusion-exclusion:

(Bf1) $Pot - N(1) - \cdots - N(h) > 0 \Rightarrow Sp > 0$ (not very-good)
(Bf2) $Pot - N(1) - \cdots - N(h) + N(1, 2) + \cdots + N(h - 1, h) = 0 \Rightarrow Sp = 0$ (very-good)
(Bf3) $Pot - N(1) - \cdots + N(1, 2) - N(1, 2, 3) - \cdots - N(h - 2, h - 1, h) > 0$
$$\Rightarrow Sp > 0$$ (not very-good)

7.3 Full-blown inclusion-exclusion can also be avoided by other means. Recall that $N(i_1, \ldots, i_t)$ is the number of potential spoilers $Z$ with $Z \cap H_{i_1} = \cdots = Z \cap H_{i_t} = \emptyset$. But this is equivalent to $Z \cap (H_{i_1} \cup \cdots \cup H_{i_t}) = \emptyset$. If the hyperedges are all very large (say of cardinality $> w/3$) then it is likely that $U := H_{i_1} \cup \cdots \cup H_{i_t} = [w]$ even for small index sets $\{i_1, \ldots, i_t\} \subseteq [h]$. But then $N(i_1, \ldots, i_t) = 0$. (More generally ")=0" happens iff $U$ contains a $g$-bubble or cuts ones($\rho$).)

This appeals to the following more general endeavour (work in progress, arXiv:1309.6927v3). In every inclusion-exclusion problem the family of relevant index sets $\{i_1, \ldots, i_t\}$, i.e. the ones that satisfy $N(i_1, \ldots, i_t) \neq 0$, constitute a set-ideal $\mathcal{N} \subseteq \mathcal{P}([h])$. If this so-called nerve $\mathcal{N}$ is small and can be obtained in clever ways (i.e. not by scanning $\mathcal{P}([h])$), then inclusion-exclusion speeds up considerably.

7.4 According to (21) it follows from $Sp > 0$ that $\rho$ is not very-good. But $\rho$ stays merely-good (as opposed to bad) unless $Sp$ sky-rockets. To make this more precise, let us generally order the sizes of the $g$-wildcards occurring in $\rho$ as $\gamma_1 \leq \cdots \leq \gamma_t$. Then each $\rho$-spoiler $Z$ can prevent at most $\gamma_t$ many $X \in \rho$ from being in $\text{MHS}(\mathbb{H})$. Since $|\rho| = (\gamma_1 \cdots \gamma_{t-1}) \gamma_t$, we conclude:

$$\gamma_1 \leq \cdots \leq \gamma_{t-1} \leq \gamma_t \text{ and } Sp(\rho) < \gamma_1 \gamma_2 \cdots \gamma_{t-1}, \text{ then } \rho \text{ is good.}$$

Although the bound $\epsilon_1 \cdots \epsilon_{t-1}$ is sharp, in practise it is likely that for much higher values of $SP(\rho)$ the row $\rho$ remains merely-good.

8 Deciding very-goodness using Rado’s Theorem

Our second method to decide the very-goodness of a semifinal 01g-row $\rho_j$ is based on certain "critical" pairs $(\rho_i, \rho_j)$. Matroids [9] will play a crucial role. Let us jump into medias res with

---

24 These inequalities are the backrock of many theorems in probability theory and statistics. Consult any good textbook please.

25 Computational experiments have been carried out in a previous version arXiv:2008.08996v2 of this article.
Rado’s Theorem [9, p. 702]:

\[(24)\] Consider any matroid \(M\) on a set \(E\) and any family \(\{Q_i : i \in I\}\) of subsets of \(E\). Then this family has a hitting set which is \(M\)-independent iff
\[
|J| \leq \text{rank}\left(\bigcup\{Q_j : j \in J\}\right) \text{ for all } J \subseteq I.
\]

8.1 Apart from inviting matroids, here comes the second ingredient:

\[(25)\] The semifinal row \(\rho_j\) in (17) is not very-good iff there is a semifinal row \(\rho_i \neq \rho_j\) such that \(X \subseteq Y\) for some \(X \in \rho_i\) and \(Y \in \rho_j\).

Proof of (25). Assume that such \(X, Y\) with \(X \subseteq Y\) exist. Since \(X = Y\) is impossible \((\rho_i \cap \rho_j = \emptyset)\), we have \(X \subset Y\). Since \(Y\) properly contains a \(\mathbb{H}\)-hitting set, we conclude \(Y \notin \text{MHS}(\mathbb{H})\), and so \(\rho_j\) is not very-good. Conversely suppose that \(\rho_j\) is not very-good. Picking any dud \(Y \in \rho_j \setminus \text{MHS}(\mathbb{H})\) there is \(X' \in \text{MHS}(\mathbb{H})\) with \(X' \subset Y\). This \(X'\) belongs to a unique semifinal row \(\rho_i\) by (17). We have \(\rho_i \neq \rho_j\) since \(\deg(\rho_i) = |X| < |Y| = \deg(\rho_j)\). \(\square\)

In view of (25) we call \((X, Y)\) a spoiling pair for \(\rho_j\) (not to be confused with the 'spoilers' in Sec. 7) if
\[
(Y \in \rho_j) \land (\exists i \neq j)(X \in \rho_i) \land X \subset Y.
\]

When \((X, Y)\) is a spoiling pair for \(\rho_j\) then necessarily there is \(i\) such that \((\rho_i, \rho_j)\) is a critical pair in the sense that \(\deg(\rho_j) < \deg(\rho_j)\) and \(\text{ones}(\rho_i) \cap \text{zeros}(\rho_j) = \emptyset\). This speeds up searching spoiling pairs \((X, Y)\) for likely-very-good rows \(\rho_j\).

8.2 To illustrate consider a hypothetical hypergraph that has triggered the two seminal rows \(\rho_1, \rho_2\) in Table 5. In fact \((\rho_2, \rho_1)\) is a critical pair since \(\deg(\rho_2) = 4 < 5 = \deg(\rho_1)\) and \(\text{ones}(\rho_2) \cap \text{zeros}(\rho_1) = \{5\} \cap \{1, 2, 3\} = \emptyset\). In order to efficiently decide the existence of a spoiling pair \((X, Y)\) for \(\rho_1\) (with \(X \in \rho_1 = \rho_2\)), notice that any such \((X, Y)\) has \(X \cap \text{zeros}(\rho_1) = \emptyset\), and so \(X \in \rho_2\) (see Table 5). But why does \(\rho_2\) also differ from \(\rho_2\) on the rightmost part? Because the \(g_1g_1\) in \(\rho_2\) was forced to become \(01\). Now \(1\) in \(\rho_2\) triggers a 1 at the same location in \(\rho_1\), which transforms \(g_3g_3g_3\) in \(\rho_1\) to 100, i.e. replaces \(\rho_1\) by \(\rho_1'\). Dropping the common 0’s of \(\rho_1', \rho_2'\) one gets two 1g-rows \(\rho_1''', \rho_2'''\) with the same index set, in our case \(E := \{4, 5, 6, 7, 8, 9, 10, 11, 12\}\).

|   | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|----|----|----|----|----|
| \(\rho_1\) = | 0 | 0 | 0 | 1 | 1 | 1 | g_1 | g_1 | g_1 | g_2 | g_2 | g_2 | g_3 | g_3 | g_3 |
| \(\rho_2\) = | g_1 | g_2 | g_3 | g_2 | 1 | g_2 | g_2 | g_3 | g_3 | g_2 | g_1 | g_2 | g_2 |
| \(\rho_1'\) = | 0 | 0 | 0 | 1 | 1 | g_1 | g_1 | g_1 | g_2 | g_2 | g_2 | 1 | 0 | 0 |
| \(\rho_2'\) = | 0 | 0 | 0 | g_2 | 1 | g_2 | g_2 | g_3 | g_3 | g_2 | 1 | 0 | 0 |
| \(\rho_1''\) = | 1 | 1 | g_1 | g_1 | g_1 | g_2 | g_2 | g_2 | 1 |
| \(\rho_2''\) = | g_2 | 1 | g_2 | g_2 | g_3 | g_3 | g_2 | 1 |
Table 5: Deciding the existence of a spoiling pair with a Theorem of Rado

That’s when the matroid takes over. Namely, the partition $E = \{4\} \cup \{5\} \cup \{6, 7, 8\} \cup \{9, 10, 11\} \cup \{12\}$ determined by the 1’s and $g$-wildcards of $\rho''_i$ defines a so-called partition matroid $M = M(E)$ where by definition $X \subseteq E$ is $M$-independent iff $X$ cuts each part of the partition in at most one element. In contrast, the analogous partition induced by $\rho''_i$ is not used for a second matroid but rather yields the set system $\{Q_i : i \in I\}$ in (24). In our case $I = \{1, 2, 3, 4\}$ and $Q_1 = \{4, 6, 7, 11\}$, $Q_2 = \{5\}$, $Q_3 = \{8, 9, 10\}$, $Q_4 = \{12\}$. Consequently, if $X$ is an $M$-independent transversal of $\{Q_i : i \in I\}$, then $X$ extends to a spoiling pair $(X, Y)$ of $\rho''_1$ (and each spoiling pair arises this way). The existence of such spoiling pairs is handled by the rank condition in statement (24). Take say $J = \{2, 3, 4\}$. Then

$$|J| = 3 \leq 4 = \text{rank}(Q_2 \cup Q_3 \cup Q_4) = \text{rank}(\{5, 8, 9, 10, 12\}).$$

One sees that generally the cardinality of $I$ in (24) equals $\text{deg}(\rho''_2)$ which, even for large hypergraphs $\mathbb{H}$, often is a modest number (and so all $J \subseteq I$ can be evaluated painlessly).

9 The benefits of having $MC(\mathbb{H})$ and $\text{MinNotMC}(\mathbb{H})$

In Section 9 we fill in gaps in 5.3 and 5.4 and deepen our understanding of the set-ideal $MC(\mathbb{H})$ of all MC-sets. The acronym MC [4] abbreviates Minimality Condition, i.e. the fact that "being MC" is a necessary condition for "being a MHS". (So MC has nothing to do with MCHS appearing in (13).) Subsection 9.1 proves the key fact $\text{MHS}(\mathbb{H}) = \text{HS}(\mathbb{H}) \cap MC(\mathbb{H})$. The set-system $\text{MinNotMC}(\mathbb{H})$ (consisting of the generators of the complementary set-filter of $MC(\mathbb{H})$) is introduced in 9.2. Our third criterion (after Sections 7 and 8) for very-goodness appears in 9.3. In 9.4-9.5 we unravel the enigmatic set-systems $\text{PotKi}(\rho_i)$ from 5.3 and trim them to set-systems $Ki(\rho_i)$ ($i \leq R$). Using Vertical Layout these $R$ set-systems can be calculated "simultaneously".

The remainder of Section 9 relies on the dual companion of the transversal $e$-algorithm, i.e. the noncover $n$-algorithm which we glimpsed in 3.3. In 9.6 the latter represents $MC(\mathbb{H})$ as a disjoint union (32) of 012n-rows. In a sense (32) dualizes (11). The dualization continues in 9.7 in that 01g-rows get accompanied by 01g'-rows. Furthermore $\text{MHS}(\mathbb{H})$ is represented as disjoint union of set-systems $\rho_i \cap \sigma_j$, where the $\rho_i$’s are 01g-rows and the $\sigma_j$’s are 01g'-rows. Using inclusion-exclusion $|\rho_i \cap \sigma_j|$ can be calculated quickly (9.8). This enables us to calculate $|\text{MHS}(\mathbb{H})|$ without knowing $\text{MHS}(\mathbb{H})$. Merely deciding whether or not $\rho_i \cap \sigma_j = \emptyset$ works faster still and it e.g. leads to the badness-criterion (37).

9.1 Given $\mathbb{H} \subseteq \mathcal{P}([w])$, in 5.4 we defined MC-sets $X \subseteq [w]$ and saw that the set-system $MC(\mathbb{H})$ of all MC-sets is a set-ideal. The intersection of this set-ideal with the set-filter $\text{HS}(\mathbb{H})$ turns out to be highly relevant:

**Theorem 2:** For any hypergraph $\mathbb{H}$ it holds that $\text{HS}(\mathbb{H}) \cap MC(\mathbb{H}) = \text{MHS}(\mathbb{H})$.

**Proof.** Take any $X \in \text{MHS}(\mathbb{H})$ and fix $b \in X$. There are $b$-hyperedges $H$, i.e. with $b \in H$, since otherwise $X \setminus \{b\}$ would remain a hitting set, in contradiction to $X$ being minimal. Suppose none of the $b$-hyperedges were to cut $b$ sharply from $X$. Then $(X \setminus \{b\}) \cap H \neq \emptyset$ for all $b$-hyperedges $H$,
and of course \((X \setminus \{b\}) \cap H' \neq \emptyset\) for all other hyperedges \(H'\). This contradicts \(X \in \text{MHS}(\mathbb{H})\), and hence shows that \(X \notin \text{MC}(\mathbb{H})\). From \(\text{MHS}(\mathbb{H}) \subseteq \text{HS}(\mathbb{H})\) follows \(\text{MHS}(\mathbb{H}) \subseteq \text{HS}(\mathbb{H}) \cap \text{MC}(\mathbb{H})\).

Conversely pick \(Y \in \text{HS}(\mathbb{H}) \cap \text{MC}(\mathbb{H})\). Since by assumption \(Y\) is a hitting set, it suffices to show that \(Y \setminus \{b\}\) is no hitting set for all \(b \in Y\). In view of \(Y \in \text{MC}(\mathbb{H})\) some \(H_0 \in \mathbb{H}\) cuts \(b\) sharply from \(Y\), hence \(H_0 \cap (Y \setminus \{b\}) = \emptyset\), hence \(Y \setminus \{b\}\) is no hitting set. \(\square\)

Consider any \(X \in \text{HS}(\mathbb{H}) \cap \text{MC}(\mathbb{H})\) and suppose \(X\) was not a facet of \(\text{MC}(\mathbb{H})\). Then there was a facet \(Y\) with \(X \subset Y\), and so \(Y \in \text{HS}(\mathbb{H}) \cap \text{MC}(\mathbb{H})\). But in view of Theorem 2 this yields the contradiction of two comparable members of \(\text{MHS}(\mathbb{H})\). We conclude that

\[(26)\quad \text{At most the facets of } \text{MC}(\mathbb{H}) \text{ can be minimal } \mathbb{H}\text{-hitting sets.}\]

In 5.4 we found that with respect to \(\mathbb{H}_3\) the set \(\{1, 2\}\) is MC but no hitting set. One checks that \(\{1, 2\}\) is a facet of \(\text{MC}(\mathbb{H}_3)\). This shows that the by (26) necessary condition of being a facet of \(\text{MC}(\mathbb{H})\) is not sufficient for being an MHS. As another consequence of Theorem 2 we find that for each semifinal 01g-row \(\rho\) from (17) we have

\[(27)\quad \rho \cap \text{MC}(\mathbb{H}) = \rho \cap \text{HS}(\mathbb{H}) \cap \text{MC}(\mathbb{H}) = \rho \cap \text{MHS}(\mathbb{H}).\]

Because \(\text{MC}(\mathbb{H}) \subseteq \mathcal{P}([w])\) is a set-ideal by (18), we can consider the complementary set-filter \(\mathcal{F} := \mathcal{P}(\{w\}) \setminus \text{MC}(\mathbb{H})\) (see 3.3). This yields a neat sufficient condition for badness:

\[(28)\quad \text{If the semifinal 01g-row } \rho \text{ is such that } \text{ones}(\rho) \notin \text{MC}, \text{ then } \rho \text{ is bad.}\]

To prove it, all \(X \in \rho\) are supersets of \(\text{ones}(\rho)\), and so \(\text{ones}(\rho) \in \mathcal{F}\) implies \(X \in \mathcal{F}\).

A semifinal 01g-row satisfying (28) will be called easy-bad. A bad row which is not easy-bad is sophisticated-bad; an example was given in 5.3.1.

9.2 By definition the set-system

\[\text{MinNotMC}(\mathbb{H}) \quad (\text{of cardinality } mn\text{MC})\]

consists of the generatos of the set-filter \(\mathcal{F}\) in 9.1. To spell it out, \(\text{MinNotMC}(\mathbb{H})\) consists of those subsets of \([w]\) which are not MC, but all their proper subsets are MC. While \(\text{MinNotMC}(\mathbb{H})\) is beneficial, it is also expensive to compute. Before we turn to the benefits, here comes a toy example.

9.2.1 It turns out (see Sec. 10) that \(\text{MinNotMC}(\mathbb{H}_2)\) is the set-system \(\mathcal{G}\) in (8). Here \(\mathbb{H}_2\) is from (9). To summarize

\[(29) \quad \mathbb{H}_2 = \{125, 34, 456, 135, 26\} \text{ has } \text{MinNotMC}(\mathbb{H}_2) = \{123, 15, 126, 256, 134, 345, 346, 246\}.\]

For instance, \(Z := \{2, 5, 6\}\) is not MC since no \(\mathbb{H}_2\)-hyperedge cuts 6 sharply: \(Z \cap \{4, 5, 6\} = \{5, 6\}\) and \(Z \cap \{2, 6\} = \{2, 6\}\). However, let us verify that all 2-subsets \(Z' \subset Z\) (and hence all subsets) are MC. For instance, take \(Z' = \{5, 6\}\). While still \(Z' \cap \{4, 5, 6\} = \{5, 6\}\), now \(Z' \cap \{2, 6\}\) works, i.e. equals \(\{6\}\). Since also \(Z' \cap \{1, 2, 5\} = \{5\}\), the set \(Z'\) is MC. Similarly one checks

\[26\text{If the 01g-row is a 01-row then of course bad }\Leftrightarrow\text{ easy-bad.}\]
that the other 2-subsets of \( Z \), i.e. \( \{2, 6\} \) and \( \{2, 5\} \), are MC-sets.

9.3 We are now fit to return to the set-systems \( PotKi(\rho) \) in 5.3. It follows from (27) that \( X \in \rho \) is a dud iff \( X \) is no MC-set, i.e. iff \( X \) contains some member of \( MinNotMC(\mathbb{H}) \). In other words, setting

\[
PotKi(\rho) := MinNotMC(\mathbb{H})
\]

fulfils the requirement of 5.3 for whatever semifinal 01g-row \( \rho \). Trouble is, the set \( PotKi(\rho) \) may be bigger than it need be. Put another way, many members of \( PotKi(\rho) \) are just potential killers, i.e. dangerous for other rows, but not harming any \( X \in \rho \). Thus if \( Z \in MinNotMC(\mathbb{H}) \) is such that \( Z \not\subseteq X \) for all \( X \in \rho \), we are led to say \( Z \) is \( \rho \)-harmless. Putting

\[
Harmless(\rho) := \{ Z \in MinNotMC(\mathbb{H}) : Z \text{ is } \rho \text{-harmless} \},
\]

and

\[
Ki(\rho) := MinNotMC(\mathbb{H}) \setminus Harmless(\rho)
\]

we hence get a third very-goodness criterion:

(30) A semifinal 01g-row \( \rho \) is very-good iff \( Ki(\rho) = \emptyset \).

Recall that \( \mathbb{H}_2 \) triggered the \( R = 4 \) semifinal 01g-rows \( \rho_1, \ldots, \rho_4 \) in (16), all of which happened to be of the same degree and hence very-good. In accordance with (30) one verifies that indeed \( Ki(\rho_1) = \cdots = Ki(\rho_4) = \emptyset \). As crisp as (30) may look, viewing that \( MinNotMC(\mathbb{H}) \) is hard to find (Section 10), the criteria for very-goodness derived in Sections 7-8 remain attractive.

9.4 The good news is, once \( MinNotMC(\mathbb{H}) \) has been conquered, VL will yield \( Ki(\rho_i) \) simultaneously for all semifinal 01g-rows \( \rho_i \) \((1 \leq i \leq R)\). Namely, we start by initializing certain auxiliary sets to \( Ha(i) := \emptyset \) for all \( 1 \leq i \leq R \). For each fixed \( Z \in MinNotMC(\mathbb{H}) \) we will calculate the set \( I(Z) \) of all \( i \leq R \) which have \( Z \in Harmless(\rho_i) \), and accordingly update \( Ha(i) := Ha(i) \cup \{ Z \} \) for all \( i \in I(Z) \). Hence, once all \( Z \in MinNotMC(\mathbb{H}) \) have been processed, all \( Ha(i) \) will have the correct content \( Ha(i) = Harmless(\rho_i) \) (and so \( Ki(\rho_i) = MinNotMC(\mathbb{H}) \setminus Ha(i) \) is obtained).

Calculating \( I(Z) \) for fixed \( Z \) works as follows. Say \( \rho_1 = (0, 1, 0, g_1, g_2, g_2, g_2, g_2, g_2, g_2, g_2, g_2) \). It will trigger the first three 01\( \infty \)-rows \( r_1, r_2, r_3 \) of the matrix \( A \) that underlies the VL application to come. Turning all existing 1’s to 0’s, setting all existing 0’s to \( \infty \) (more on that in moment), and filling exactly one \( g_i \)-wildcard with 1’s and the others with 0’s, yields

\[
\begin{align*}
\rho_1 &= (\infty, 0, \infty, 1, 1, 0, 0, 0, 0, 0, 0), \\
r_2 &= (\infty, 0, \infty, 0, 0, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 1, 1).
\end{align*}
\]

In order to remember the number \( i = 1 \) of the semifinal 01g-row \( \rho_i \) triggering \( r_1, r_2, r_3 \) we set \( nsf(1) = nsf(2) = nsf(3) := 1 \). Say \( \rho_2 \) has two \( g \)-bubbles. Then it triggers analogous 01\( \infty \)-rows \( r_4, r_5 \) (written below \( r_1 \) to \( r_3 \) ) and we record \( nsf(4) = nsf(5) := 2 \). And so it goes on with \( \rho_3 \) up to \( \rho_R \).

Having calculated \( A \) (say it has dimensions \( 41 \times 10 \)), we can begin to process all \( Z \in MinNotMC(\mathbb{H}) \).

\[\text{27} \text{What follows is only for VL-enthusiasts.}\]
If say $Z_1 = \{5, 6, 7\}$, calculate the column $col := col_5 + col_6 + col_7$. Then $col' = (1, 2, 0, ...)$, where the fact that the second component is $\geq 2$ testifies that $Z_1$ cannot be contained in any member of the semifinal 01g-row with number $nsf(2) = 1$ (since $Z_1$ cuts one $g$-bubble of that row in $\geq 2$ elements). As another example suppose that $Z_2$ is such that the corresponding length 41 column $col''$ has 20 components equal to 1, 20 equal to 0, and the 13th component is $\infty$. How does this translate to plain language? It means that $Z_2$ is harmless only for the semifinal 01g-row $\rho_j$ ($j := nsf(13)$) because $Z_2 \cap zeros(\rho_j) \neq \emptyset$, and so $Z_2$ can’t be contained in any member of $\rho_j$.

(For all other semifinal 01g-rows $Z_2$ is a killer since it doesn’t clash with their 0’s and cuts all their g-bubbles in at most one element.) For general $Z \in MinNotMC(\mathbb{H})$ with coupled column col let $J$ is the position-set of the components $\geq 2$ that occur in col. By the above it is clear that $I(Z) = \{nsf(j) : j \in J\}$ (it doesn’t matter that $nsf(j) = nsf(j')$ for $j \neq j'$ is possible).

9.5 In 5.3 two toy examples showed how PotKi($\rho$) helps to calculate Duds($\rho$). Let us propose a more systematic way (from now on we drop PotKi($\rho$) and stick to Ki($\rho$)) that in particular speeds up the detection of bad rows. So put $Ki := Ki(\rho)$. If $\rho$ has $t$ many g-bubbles then for all $1 \leq j \leq t$ let $Ki[j]$ be the (possibly empty) set of $Z \in Ki$ that intersect exactly $j$ many g-bubbles (necessarily these intersections being singletons). Hence $Ki = Ki[0] \uplus Ki[1] \uplus \cdots \uplus Ki[t]$. Since each killer $Z \in Ki$ is necessarily disjoint from zeros($\rho$), we see that $Ki[0] = \{Z \in Ki : Z \subseteq ones(\rho)\}$. Recalling the definition of “easy-bad” in 9.1 we claim:

$$Ki[0] \neq \emptyset \text{ iff } \rho \text{ is easy-bad.}$$

Proof of (31). If $Z \in Ki[0] \neq \emptyset$, then $Z$ (being a killer) is not-MC, hence the superset ones($\rho$) is not-MC, hence $\rho$ is easy-bad. If conversely $\rho$ is easy-bad, then ones($\rho$) (being not-MC) contains some $Z \in MinNotMC(\mathbb{H})$. Obviously $Z \in Ki[0]$. $\square$

If $\rho$ is not easy-bad then it either is sophisticated-bad or ‘actually-good’. To find out fast, the second most effective killers are the ones in Ki[1] which we hence exploit first to inflate our changing set Duds($\rho$). Then we turn to Ki[2], and so forth up to Ki[t]. If in the process Duds($\rho$) ever becomes $\rho$, we can stop and know that $\rho$ is bad. As seen in 5.3, VL can be used in all of that.

9.6 Recall from 3.3 that the noncover $n$-algorithm yields for each set-system $S$ the family $NC(S)$ of all $S$-noncovers as a disjoint union of 012n-rows $\overline{\sigma_j}$. If in particular $S := MinNotMC(\mathbb{H})$ then $NC(S) = MC(\mathbb{H})$. Therefore (11’) specializes to

$$MC(\mathbb{H}) = \biguplus_{j=1}^{r^*} \overline{\sigma_j}.$$

For instance, recall that applying the transversal $e$-algorithm to $\mathbb{H}_2$ yielded $HS(\mathbb{H}_2) = \overline{\sigma_1} \uplus \cdots \uplus \overline{\sigma_4}$ (Table 2). If we dually apply the noncover $n$-algorithm to $MinNotMC(\mathbb{H}_2)$ from (29) we get $MC(\mathbb{H}_2) = \overline{\sigma_1} \uplus \cdots \uplus \overline{\sigma_6}$ (Table 6).
row-maximal sets

| $\sigma_1$ | $\sigma_2$ | $\sigma_3$ | $\sigma_4$ | $\sigma_5$ | $\sigma_6$ |
|-------------|-------------|-------------|-------------|-------------|-------------|
| 0 2 n n n 0 | 0 n 1 0 n 1 | 0 0 0 2 2 1 | 0 1 0 0 0 1 | 1 0 n n 0 2 | 1 1 0 2 0 0 |

$\text{Max}(\sigma_1) = \{234, 235, 245\} = (0, 1, g^*, g^*, g^*, 0) =: \sigma_1$

$\text{Max}(\sigma_2) = \{236, 356\} = (0, g^*, 1, 0, g^*, 1) =: \sigma_2$

$\text{Max}(\sigma_3) = \{456\} = (0, 0, 0, 1, 1, 1) =: \sigma_3$

$\text{Max}(\sigma_4) = \{26\} = (0, 1, 0, 0, 1) =: \sigma_4$

$\text{Max}(\sigma_5) = \{136, 146\} = (1, 0, g^*, g^*, 0, 1) =: \sigma_5$

$\text{Max}(\sigma_6) = \{124\} = (1, 1, 0, 1, 0, 0) =: \sigma_6$

Table 6: Representing $MC(\mathbb{H}_2)$ as disjoint union of 012n-rows

9.7 Let us keep on dualizing. To begin with, for each 012n-row $\sigma$ in (32) one gets $\text{Max}(\sigma)$ by turning all 2’s to 1’s and all $n$-wildcards to $g^*$-wildcards, where by definition $(g^*, g^*, \ldots, g^*)$ means "exactly one here". For instance $\sigma_1$ in Table 6 becomes $\sigma_1$. Generally each $\sigma_j$ from (32) induces such a 01g*-row $\sigma_j$. Akin to (17) we claim that

\[(33) \quad \text{MHS}(\mathbb{H}) \subseteq \bigcup_{j=1}^{R} \sigma_j.\]

Proof of (33). From (32) and Theorem 2 follows $\text{MHS}(\mathbb{H}) \subseteq \bigcup_{j=1}^{R} \sigma_j$. Hence each $X \in \text{MHS}(\mathbb{H})$ is in a unique row $\sigma_j$. We claim that $X \in \text{Max}(\sigma_j) = \sigma_j$. Indeed, since $X$ is a maximal member of $MC(\mathbb{H})$ by (26), it is a fortiori maximal within $\sigma_j \subseteq MC(\mathbb{H})$. □

In view of (33) we can carry over the concepts good, bad, very-good, and so on to 01g*-rows. For instance, as it is forced by (17) and (33), the 9 MHSes of $\mathbb{H}_2$ appear both in (16) and Table 6, yet $R = 4 \neq 6 = R^*$. Whereas all $\rho_i$ were very-good, $\sigma_4$ is bad; its only member $\{2, 6\}$ is MC but no MHS.

9.7.1 Recall from Section 1 that our Main Quest is to retrieve the diamonds from the boxes (=semifinal 01g-rows) as efficiently as possible. As is evident from (33) one could also retrieve the diamonds from dual boxes (= semifinal 01g*-rows). In fact, this is attempted in [4], yet in a one-by-one fashion based directly on $MC(\mathbb{H})$. We will continue to retrieve the MHSes from the semifinal 01g-rows but the "dual" 01g*-rows will play an important auxiliary role. For technical reasons (see footnote in 9.8.1) the coupled 012-rows and 012n-rows will resurface as well. For starters, observe that unless all involved rows are 01-rows it holds that $\rho_i \subset \overline{\rho}_i$ and $\sigma_j \subset \overline{\sigma}_j$.

Nevertheless, this takes place:

\[(34) \quad \text{For all } \overline{\rho}_i, \rho_i \text{ in (11) and (17), and all } \overline{\sigma}_j, \sigma_j \text{ in (32) and (33), we have } \overline{\rho}_i \cap \overline{\sigma}_j = \rho_i \cap \sigma_j.\]

Proof of (34). It suffices to show $\overline{\rho}_i \cap \overline{\sigma}_j \subseteq \rho_i \cap \sigma_j$. As we long know, $\overline{\rho}_i \cap \text{MHS}(\mathbb{H}) \subseteq \rho_i$. Similarly, as shown above, $\overline{\sigma}_j \cap \text{MHS}(\mathbb{H}) \subseteq \sigma_j$. Together with Theorem 2 follows that $\overline{\rho}_i \cap \overline{\sigma}_j = (\overline{\rho}_i \cap \text{HS}(\mathbb{H})) \cap (\overline{\sigma}_j \cap \text{MC}(\mathbb{H})) = (\overline{\rho}_i \cap \text{MHS}(\mathbb{H})) \cap (\overline{\sigma}_j \cap \text{MHS}(\mathbb{H})) \subseteq \rho_i \cap \sigma_j$. □

From Theorem 2, (11), (32), the distributivity of $\cap$ over $\cup$, and (34) follows

\[(35) \quad \text{MHS}(\mathbb{H}) = \left(\bigcup_{i=1}^{R} \overline{\rho}_i\right) \cap \left(\bigcup_{j=1}^{R^*} \overline{\sigma}_j\right) = \bigcup_{i,j \geq 1} (\overline{\rho}_i \cap \overline{\sigma}_j) = \bigcup_{i,j \geq 1} (\rho_i \cap \sigma_j).\]

9.7.2 To illustrate (35), taking $\overline{\rho}_2$ from Table 2 and $\overline{\sigma}_1$ from Table 6, it holds that $\overline{\rho}_2 \cap \overline{\sigma}_1 = \ldots$
\{235, 245\}. Generally speaking, intersecting 012e-rows with 012n-rows (or 01g-rows with 01g*-
rows) is no easier than intersecting two 012e-rows (see 2.3). As one way out one can ponder to
either expand the 012e-row or the 012n-rows as a disjoint union of 012-rows. For instance $\overline{\sigma}_1$
expands as shown in Table 7:

|   | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $\overline{\sigma}_1$ | 0 | 2 | $n$ | $n$ | $n$ | 0 |
| $\overline{\sigma}_{11}$ | 0 | 2 | 0 | 2 | 0 | 0 |
| $\overline{\sigma}_{12}$ | 0 | 2 | 1 | 0 | 2 | 0 |
| $\overline{\sigma}_{13}$ | 0 | 2 | 1 | 1 | 0 | 0 |

Table 7: Expanding a 012n-row into 012-rows.

It follows that $\overline{\rho}_2 \cap \overline{\sigma}_1 = (\overline{\rho}_2 \cap \overline{\sigma}_{11}) \cup (\overline{\rho}_2 \cap \overline{\sigma}_{12}) \cup (\overline{\rho}_2 \cap \overline{\sigma}_{13})$. Each term on the right, and generally
each intersection of a 012e-row with a 012-row, is either empty or again a 012e-row (2.2.1). In
our particular case $\overline{\rho}_2 \cap \overline{\sigma}_{11} = (0, 1, 0, 1, 1, 0)$, $\overline{\rho}_2 \cap \overline{\sigma}_{12} = (0, 1, 1, 0, 1, 0)$, $\overline{\rho}_2 \cap \overline{\sigma}_{13} = \emptyset$.

Let us argue that in the present scenario such intersections are always either empty or 01-rows.
So suppose $\overline{\rho}$ is from (11) and $\overline{\sigma}$ from (32) got again expanded into 012-rows $\overline{\sigma}_\ast$. Since each MHS
$X$ contained in $\overline{\sigma}$ is maximal within $\overline{\sigma}$, it will also be maximal within the 012-row $\overline{\sigma}_\ast$ it happens
to lie. Any two MHSes being incomparable, there cannot be another MHS in $\overline{\sigma}_\ast$. Because $\overline{\rho} \cap \overline{\sigma}_\ast$
, if nonempty, is a 012e-row that by (35) consists entirely of MHSes, this 012e-row is actually a
01-row that matches $X$.

The bottom line is this. Formula (35) likely cannot be exploited to compress $MHS(\mathbb{H})$; at most
(35) can be used for one-by-one enumeration. Whether and when this competes with the four
one-by-one schemes from Section 5, remains to be seen\textsuperscript{28}. For (35) to be competitive it will be
necessary (possibly not sufficient) that all empty intersections $\rho_i \cap \sigma_j$ can be recognized fast.

9.8 The true calling of (35) is to find the cardinality $|MHS(\mathbb{H})|!$. Namely, suppose that:

(36) For each $\overline{\rho}_i$ in (11) we can obtain (preferably few) 012n-rows $\overline{\tau}_1, \ldots, \overline{\tau}_m$ such that
$\overline{\rho}_i \cap MHS(\mathbb{H}) \subseteq (\overline{\rho}_i \cap \overline{\tau}_1) \cup \cdots \cup (\overline{\rho}_i \cap \overline{\tau}_m)$ and $\overline{\tau}_1, \ldots, \overline{\tau}_m \subseteq MC(\mathbb{H})$.

In view of (35), statement (36) is plausible. A full proof of (36) that also touches on $K(\overline{\rho})$ and
on computational issues will be given in Section 11.5. Accepting (36) we first note that from
$\overline{\rho}_i \cap \overline{\tau}_j \subseteq HS(\mathbb{H}) \cap MC(\mathbb{H}) = MHS(\mathbb{H})$ follows that "$\subseteq" in (36) in fact is "$\subset". Hence
$|\overline{\rho}_i \cap MHS(\mathbb{H})| = |\overline{\rho}_i \cap \overline{\tau}_1| + \cdots + |\overline{\rho}_i \cap \overline{\tau}_m|$. Because $|MHS(\mathbb{H})|$ is the sum of $R$ terms $|\overline{\rho}_i \cap MHS(\mathbb{H})|$, calculating $|MHS(\mathbb{H})|$ boils down to calculating $|\rho \cap \sigma|$ for an arbitrary 012e-row $\overline{\rho}$ and 012n-row
$\overline{\sigma}$ (this problem occurs in other circumstances as well). Let us apply inclusion-exclusion to do
so.

9.8.1 To fix ideas take $\overline{\rho} := (e_1, e_1, e_2, e_2, e_2)$ and $\overline{\sigma} := (n_1, n_2, n_1, n_2, n_3, n_3)$. (The presence
of entries 0, 1, 2 would only cause trivial changes in the sequel.) Let $N(e_1), N(e_2), N(e_1e_2)$ be

\textsuperscript{28}In particular 5.4.1 will be stiff competition. As opposed to 9.2 to 9.8, the method in 5.4.1 does not even rely
on MinNotMC(\mathbb{H}).
the numbers of bitstrings $x \in \overline{\sigma}$ that violate respectively, the $e_1$-bubble, the $e_2$-bubble, and both $e_i$-bubbles. By inclusion-exclusion it holds that

$|\rho \cap \sigma| = |\rho| - N(e_1) - N(e_2) + N(e_1 e_2) = 27 - 12 - 4 + 1 = 12$

in view of $|\rho| = 3 \cdot 3 \cdot 3$ (see 2.2), $N(e_1) = |(0, 0, 2, 2, n_3, n_3)| = 12$, $N(e_2) = |(2, 2, 0, 0, 0, 0)| = 4$, $N(e_1 e_2) = |(0, 0, 0, 0, 0, 0)| = 1$.

Similarly (using obvious notation) one obtains 12 as

$|\rho \cap \sigma| = |\sigma| - N(n_1) - N(n_2) - N(n_3) + N(n_1 n_2) + N(n_1 n_3) + N(n_2 n_3) - N(n_1 n_2 n_3) = 45 - 16 - 16 - 12 + 4 + 4 + 1 = 12$

in view of $|\sigma| = 3 \cdot 15, ..., N(n_3) = |(e_1, e_1, 2, 2, 1, 1)| = 12, ..., N(n_1 n_2 n_3) = |(1, 1, 1, 1, 1, 1)| = 1$.

In general we launch inclusion-exclusion on the row with the fewer wildcards. Interestingly, deciding merely whether or not $\rho \cap \sigma$ is empty, works even faster than inclusion-exclusion (more on that in 11.5.2). This speed of deciding the emptiness of $\rho \cap \sigma$ prompts us to finally state a badness criterion for semifinal 012e-rows. Thus, as opposed to (28), it is a sufficient and necessary (albeit somewhat clumsy) condition:

(37) Suppose the 012e-row $\overline{\rho}_i$ is as in (36). Then $\overline{\rho}_i$ is bad iff $\overline{\rho}_i \cap \overline{\rho}_j = \emptyset$ for all $1 \leq j \leq m$.

10 How to calculate $MinNotMC(\overline{H})$ in the first place

In order to understand how $MinNotMC(\overline{H}_2)$ in (29) was computed it pays to momentarily relabel the hyperedges of $\overline{H}_2$ in obvious ways:

(38) $H_1 = \{a, b, e\}$, $H_2 = \{c, d\}$, $H_3 = \{d, e, f\}$, $H_4 = \{a, c, \epsilon\}$, $H_5 = \{b, f\}$

Let us refine the property ‘$T$ is MC’. Thus for any set $T$ and fixed $u \in T$ we say ‘$T$ is $u$-MC’ if $crit(u, T) := \{H \in \overline{H} : H \cap T = \{u\}\}$ is nonempty. Consequently it holds for all $T \subseteq W := \{a, b, c, d, e, f\}$ that:

(39) $T$ is MC $\iff$ $T$ is $u$-MC for all $u \in T$

(40) $T$ is not-MC $\iff$ $T$ is not-$u$-MC for some $u \in T$ $\iff$ $crit(u, T) = \emptyset$ for some $u \in T$

For instance $T = \{d, c, f\}$ is not-$d$-MC because from $d \in T \cap H_i$ always follows $|T \cap H_i| \geq 2$, the relevant indices being $i = 2, 3$.

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29 Notice that $ee..e$ (of length $k$) is violated by just one bitstring $00..0$, whereas $gg..g$ is violated by $2^k - k$ many bitstrings. That’s why in the context of inclusion-exclusion we prefer to deal with $|\rho \cap \sigma|$ rather than $|\rho \cap \sigma|$. Recall from (34) that $\rho \cap \sigma = \rho \cap \sigma$.

30 For various small $\overline{H}$ ($w < 20$) the author ran both Algorithm 4 from 5.4 and the sophisticated method in spe to get $MinNotMC(\overline{H})$. Both always yielded the same result. While this is no formal proof of correctness of the method in spe, it makes correctness very likely.

31 Otherwise the elements of $H_1, ..., H_6$ clash with the indices $1, 2, ..., 6$ of $H_1, ..., H_6$. 
10.1 For $u \in W$ put $S_u := \{ i \in [h]: u \in H_i \}$ Therefore

\begin{equation}
S_a = \{1, 4\}, S_b = \{1, 5\}, S_c = \{2, 4\}, S_d = \{2, 3\}, S_e = \{1, 3, 4\}, S_f = \{3, 5\}.
\end{equation}

The fact that $\{d, c, f\}$ is not-d-MC can now be seen as tantamount to $S_d \subseteq S_c \cup S_f$. Generally the not-u-MC sets bijectively match the set coverings of $S_u$ by other $S_v$'s.

Our aim is to calculate the family $\text{MinNotMC}(\mathbb{H}_2)$ of minimal not-MC sets. According to (40) they are found among the minimal not-u-MC sets, where $u$ ranges over $W$. Let us hence find for each fixed $u \in W$ all minimal set coverings of $S_u$. The systematic method follows in 10.2, but for $\mathbb{H} = \mathbb{H}_2$ we can proceed by inspecting (41):

- The minimal set coverings of $S_a$ are $\{S_b, S_c\}, \{S_e\}$
- The minimal set coverings of $S_b$ are $\{S_a, S_f\}, \{S_c, S_f\}$
- The minimal set coverings of $S_c$ are $\{S_a, S_d\}, \{S_d, S_e\}$
- The minimal set coverings of $S_d$ are $\{S_c, S_e\}, \{S_c, S_f\}$
- The minimal set coverings of $S_e$ are $\{S_a, S_d\}, \{S_a, S_f\}, \{S_b, S_c, S_d\}, \{S_b, S_c, S_f\}$
- The minimal set coverings of $S_f$ are $\{S_b, S_d\}, \{S_b, S_e\}$

Therefore the minimal not-a-MC sets are $\{a, b, c\}$ and $\{a, e\}$, and so forth until the minimal not-f-MC sets are $\{f, b, d\}$ and $\{f, b, e\}$. The inclusion-minimal sets among these sets\footnote{For instance $\{e, a, d\}$ gets killed by $\{a, e\}$, and e.g. the double occurrence of $\{b, e, f\} = \{f, b, e\}$ can be pruned. In general proceed as in 2.5 to get the minimal sets.} are (in shorthand notation) $abc, ae, ba, f, bf, cad, cde, dcf, fbd$. Relabelling back $a \rightarrow 1, ... , f \rightarrow 6$ yields $\text{MinNotMC}(\mathbb{H}_2)$ in (29).

10.2 As is well known, finding minimal set coverings is cryptomorphic to finding minimal hypergraph transversals. Let us make this cryptomorphism explicite by recalculating the set coverings of the set $S_u$ by the set system $S := \{S_a, S_b, S_c, S_d, S_f\}$. Because $S_e = \{1, 3, 4\}$, at least one member of $S$ must cover 1; only $S_a, S_b$ can do that. Similarly, only $S_d, S_f$ can contain 3, and only $S_a, S_c$ can contain 4. Thus we define the auxiliary hypergraph triggered by $\epsilon$ as

\begin{equation}
\mathbb{H}_2^{\text{aux}}(\epsilon) := \{S_a, S_b\}, \{S_d, S_f\}, \{S_a, S_c\}.
\end{equation}

It follows that the $\mathbb{H}_2^{\text{aux}}(\epsilon)$-hitting sets are exactly the minimal set-coverings of $S_\epsilon$ by other $S_u$'s. It is natural to employ again the transversal e-algorithm to calculate all minimal $\mathbb{H}_2^{\text{aux}}(\epsilon)$-hitting sets.

The transversal e-algorithm starts by imposing the hyperedge $\{S_a, S_b\}$ of $\mathbb{H}_2^{\text{aux}}(\epsilon)$, and then imposes $\{S_d, S_f\}$. Since the two happen to be disjoint, this is achieved by the single 012e-row $r_1$ in Table 7. Imposing $\{S_a, S_c\}$ upon $r_1$ yields the two final rows $r_2$ and $r_3$. It happens that both of them are very-good, i.e. $\text{Min}(r_2)$ and $\text{Min}(r_3)$ need not be pruned further.
Table 7: Calculating all the minimal set coverings of $S_c$ with the e-algorithm

| $r_1$ | $r_2$ | $r_3$ |
|-------|-------|-------|
| $S_a$ | 1     | 0     |
| $S_b$ | 2     | 1     |
| $S_c$ | 2     | 2     |
| $S_d$ | $e'$  | $e'$  |
| $S_f$ | $e'$  | $e'$  |

| $Min(r_i)$ |
|------------|
| $\{S_a, S_d\}, \{S_a, S_f\}$ |
| $\{S_b, S_c, S_d\}, \{S_b, S_c, S_f\}$ |

11 Numerical experiments

While terminology and overall structure of the article in front of you have improved a lot compared to the 2021-version (arXiv:2008.08996v2), there is a problem concerning the 2021 Mathematica experiments: The author lacked time to substitute some of the 2021 subroutines by implementations of the superior ideas discussed in previous Sections. After long deliberation I decided to pick a few of the most telling numerical experiments done in 2021, recast them in Table 8 below, and describe them thoroughly with adapted terminology (i.e. from the present article). All experiments are characterized by the signature $(w, h, k)$ that refers to a hypergraph $\mathbb{H} \subseteq P[w]$ whose $h$ hyperedges $H \in \mathbb{H}$ are random and have uniform cardinality $|H| = k$. For some signatures (in 11.1) we managed to calculate $|\text{MHS}(\mathbb{H})|$ exactly. For other signatures $|\text{MHS}(\mathbb{H})|$ could only be approximated; e.g. because $\text{MinNotMC}(\mathbb{H})$ could not be conquered (11.2), or not even the $R$ semifinal rows in (11) could (11.3). In 11.4 and 11.5 we speculate on future improvements. Finally 11.6 compares our ”wildcard-approach” with an algorithm of Toda [5] which is based on BDD’s and which therefore also offers some kind of compression..

11.1 Whenever $|\text{MHS}(\mathbb{H})|$ could be determined exactly, the procedure usually was as follows. The (transversal) e-algorithm, fed with $\mathbb{H}$, terminates and outputs $R$ many semifinal 01g-rows $\rho_i$ (see (17)). Whenever $\text{MinNotMC}(\mathbb{H})$ could be calculated, then likewise all $R$ set-systems $Ki(\rho_i)$ could be calculated (though not yet with the nifty VL way of 9.4). In this situation the potential very-goodness of $\rho_i$ (and if yes, $|\rho_i|$) is settled at once in view of (30). How to process the remaining merely-good or bad rows $\rho_j$? We mostly used 9.5 (inflating $\text{Duds}(\rho_j)$ by processing $Ki[0] \cup \cdots \cup Ki[t]$) or \(^{33}9.6 - 9.8\) (combining the $n$-algorithm with inclusion-exclusion)

Thus one hypergraph $\mathbb{H}$ of signature $(60,20,5)$ (see Table 8) triggered $R = 26701$ semifinal 01g-rows $\rho_i$ of average degree 13. The calculation took 13 seconds. Calculating $\text{MinNotMC}(\mathbb{H})$ of cardinality $mmMC = 309$ took 1 second. Using the 9.6-9.8 way we found that $\mathbb{H}$ had $51'109'682$ MHSes. Perhaps more informative than knowing $|\text{MHS}(\mathbb{H})|$ is it to know the average 1914 of the (absolute) contents $|\rho_i \cap \text{MHS}(\mathbb{H})|$, as well as the average relative content $|\rho_i \cap \text{MHS}(\mathbb{H})|/|\rho_i| = 0.77$ (77%). (Up to small rounding error one retrieves $|\text{MHS}(\mathbb{H})|$ by multiplying with $R$ the average absolute content.) As to the $(30,50,70)$-hypergraph, since its semifinal rows have little content and $\text{MinNotMC}(\mathbb{H})$ is large, the 9.5 way was faster. For some $(70,20,30)$-hypergraph the precise value of $|\text{MHS}(\mathbb{H})|$ was obtained without the aid $\text{MinNotMC}(\mathbb{H})$ because Algorithm 4 from 5.4 managed to process all semifinal rows (including the very-good-ones) one-by-one.

\(^{33}\)The author does not remember for each signature occurring in the 2021 experiments which variant was used. Notice that the 9.6 - 9.8 variant is more powerful but has an overhead, i.e. 9.5 might be faster for moderate $|\rho_i|$.
11.2 For some \((w,h,k)\)-hypergraphs \(H\) it was possible to calculate all \(R\) semifinal rows but not the exact value of \(MHS(H)\). That is because either \(\text{MinNotMC}(H)\) was too hard to calculate (see also 11.4) and Algorithm 4 not up to the task. Or, while \(\text{MinNotMC}(H)\) could be obtained, either \(R\) or the sizes \(|\rho_i|\) were too large to process, in whatever way, the not very-good rows (see also 11.5). In this situation we picked 1000 among the \(R\) semifinal rows at random\(^{34}\) and used them to approximate the average content of semifinal rows.

There is one \(H\) which doesn’t quite fit "In this situation". For this \(H\) of signature \((100,40,3)\) the 113 potential killers in \(\text{MinNotMC}(H)\) could be calculated in just 0.4 sec. Among the 10367 semifinal rows 94\% were very-good (identified via \(Ki(\rho_i) = \emptyset\)) and their cardinalities summed up to \(3190986028403520327\). The remaining semifinal rows were all merely-good and still very dense. The 9.5 variant being out of question due to the size of \(\rho_i\), the author speculates (but doesn’t remember fore sure) that attempting the 9.6 - 9.8 variant must have failed due to the inferior 2021 subroutine for inclusion-exclusion (see 11.5.2).

11.3 In some cases not all semifinal rows could be generated, i.e. the \(e\)-algorithm failed and \(R\) was unknown. Nevertheless, one can still employ the \(e\)-algorithm to generate 1000 random semifinal \(01g\)-rows. The last three lines in Table 8 arose this way. It is interesting to compare the signatures \((100,40,3)\) and \((100,80,3)\), as well as \((30,50,7)\) to \((30,5000,7)\). As usual, if \(w,k\) stay fixed while \(h\) increases, the absolute content "deteriorates".

As to the last column in Table 8, if all \(R\) semifinal \(012e\)-rows (and whence semifinal \(01g\)-rows) could be classified (whether or not \(|MHS(H)|\) was achieved) then we evidently get the exact percentages of very-good, merely-good, and bad rows. They appear (rounded) in the last column. If not all semifinal rows could be computed, then the numbers in the last column were extrapolated by applying the Monte Carlo method to the 1000 semifinal rows (be it in 11.2 or 11.3) that were computed.

11.4 As to calculating \(\text{MinNotMC}(H)\), considerably less time was spent for running the \(w\) many auxiliary transversal \(e\)-algorithms than for minimizing the resulting set system \(S\) to \(\text{Min}(S) = \text{MinNotMC}(H)\). For instance for the \((30,50,7)\)-instance it took only 61 seconds to calculate \(S\) (of cardinality 252'211), but 2503 seconds to shrink \(S\) to \(\text{MinNotMC}(H)\) (of cardinality 55538). For the \((70,20,30)\)-instance \(\text{MinNotMC}(H)\) could not be calculated in reasonable time. Problem is, the minimization method used was inferior to the ideas in 2.5 and 2.6.1.

11.4.1 Is there hope compressing \(MHS(H)\) without knowing \(\text{MinNotMC}(H)\)? Yes there is: While the inclusion-exclusion method of Section 7 has been experimented with in the 2020-version of the present article (arXiv:2008.08996v1, Section 6.3) only for small values \(h\), there is hope (recall 7.3) to trim it considerably. Also Rado’s Theorem (Section 8) should be kept in mind as basis for a very-goodness criterion. When trying to compress \(MHS(H)\), Section 6 plays an important role as well. A second look at Section 6 shows that one can handle matters with just a very-goodness criterion. Put another way, a nontrivial badness-criterion is nice-to-have but not strictly necessary to repackagemeregood rows into fresh very-good rows.

\(^{34}\)In fact we picked the first thousand \(012e\)-rows produced by the transversal \(e\)-algorithm. One may object that these rows are not representative because they match the ‘leftmost’ 1000 nodes of the computation tree. This objection can be dismissed as follows. Whenever a top row of the LIFO stack (3.1) gets removed, we switch the new top row with a random row further below. The effect is that the first 1000 semifinal rows are as random as any other sample of 1000 semifinal rows.
11.5 As to merely calculating $|MHS(\mathbb{H})|$, let us first prove (36) from 9.8. There are two approaches to obtain the required 012n-rows $\tau_i$. Both are based on possessing $MinNotMC(\mathbb{H})$. The first approach obtains the rows $\sigma_j$ ($j \leq R^*$) in (32) by feeding the whole of $MinNotMC(\mathbb{H})$ to the (noncover) $n$-algorithm. Then for all $1 \leq j \leq R^*$ we check whether or not $\rho_i \cap \sigma_j = \emptyset$ (see 11.5.2) and take as $\{\tau_1,...,\tau_m\}$ the set of all $\sigma_j$ with $\rho_i \cap \sigma_j \neq \emptyset$. The second approach only feeds $K_i(\rho_i)$ instead of $MinNotMC(\mathbb{H})$ to the $n$-algorithm and thus obtains $m'_i$ many 012n-rows $\tau'_j$ that also do the job.

11.5.1 What are the pros and cons of the two approaches to provide each semifinal 012e-row $\rho_i$ with "its" 012n-rows guaranteed by (36)? For starters, while the calculation of the $R$-setsystems $K_i(\rho_i)$ is based on $MinNotMC(\mathbb{H})$, it works smoothly with Vertical Layout (9.4). Since all $K_i(\rho_i)$ are small subsets of $MinNotMC(\mathbb{H})$, applying the $n$-algorithm to a single $K_i(\rho_i)$ takes much less time than applying it to $MinNotMC(\mathbb{H})$. Under circumstances even the sum of all $K_i(\rho_i)$-times compares well to the $MinNotMC(\mathbb{H})$-time; e.g. when $R$ is small and/or many $\rho_i$ are very-good due to $K_i(\rho_i) = \emptyset$ and hence need not undergo the $n$-algorithm. How does $m_i$ compare to $m'_i$? In lockstep with the shorter time also the number $m'_i$ of produced 012n-rows will be smaller than the corresponding number $m_i$. Finally observe that by construction all $m_i$ rows $\tau_j$ intersect $\rho_i$, whereas this need not be the case for the $m'_i$ many rows $\tau'_j$. If, while running the $n$-algorithm on $K_i(\rho_i)$, one keeps on discarding candidate sons $\tau'$ with $\rho_i \cap \tau' = \emptyset$ (see 11.5.2), then it is guaranteed that no final 012n-row will be disjoint from $\rho_i$. In this way one can further reduce $m'_i$ but perhaps that’s not worth the effort. More computational experiments need to be carried out to clarify all of 11.5.1.

11.5.2 Two more loose ends must be addressed. First, the type of inclusion-exclusion proposed in 9.8.1 for calculating $|\rho \cap \tau|$ is superior to the type of inclusion-exclusion employed in the 2021-experiments of Table 8. Namely, as detailed in [arXiv:2008.08996v1, Sec.7.2], this slower kind of inclusion-exclusion relies on a bipartite graph whose shores are the e-wildcards of $\rho$ and the n-wildcards of $\tau$ respectively. Since in 9.8.1 we only need one kind of wildcards, the 9.8.1 implementation in spe is up to $2^t$ times faster that the current implementation (where $t$ is the number of the kind of wildcards of which there are more).

Second, deciding merely whether or not $\rho \cap \tau$ is empty works faster still than 9.8.1 type inclusion-exclusion. For starters, the intersection is clearly empty when 1’s in one row clash with 0’s in the other row. However, there can be more hidden reasons for emptiness; e.g. $(1,1,e,e) \cap (n_1,n_2,n_1,n_2) = \emptyset$. The gory details of deciding the emptiness of $\rho \cap \tau$ have been tackled in [arXiv:2008.08996v1, Sec.8], yet all of that will be recast in a separate publication that also relates the matter to deciding the satisfiability of certain Boolean functions (of type $Horn \land AntiHorn$). Another issue is the Mathematica implementation of it all, and its possible overhead that slows it down for small size inputs.
In [1] nineteen methods to calculate $MHS(\mathbb{H})$ have been pitted against each other on a common platform, using a variety of real-life datasets. Our method does not post factum fit that platform. For one thing, it is implemented in high-level Mathematica code and so far only ran on the author’s laptop (Dell Latitude 7410). Furthermore, much different from [1], all hypergraphs in Table 8 have random and equicardinal hyperedges (which in view of 11.3 may be disadvantageous). Nevertheless, let us attempt a preliminary comparison with two specific algorithms investigated in [1]. First, the Murakami-Uno-algorithm [4] (like us to some extent) relies on the MC-condition but proceeds one-by-one. Second, building on ideas of Knuth, the Toda-algorithm [5], like us, uses compression, but in more implicite ways (BDD’s). These two algorithms also happen to be the champions in [1]. Since the MC-condition has received plenty attention in Sections 9 and 10, let us devote the remainder of 11.6 to the $\mathbb{H}$-algorithm. Here come four aspects where our method seems to win out (but since talk is cheap only direct confrontation can ultimately determine the pros and cons of both).

(i) As is well known (and repeated in [6]), having the BDD of a Boolean function $f$ yields at once the cardinality of the model set $\text{Mod}(f)$. With a bit more effort (but in linear total time) one gets the model set of $f$ as a disjoint union of 012-rows. Unfortunately, when $\text{Mod}(f) = MHS(\mathbb{H})$, then the models are mutually incomparable, and so all 012-rows are necessarily 01-rows, i.e. no compression is achieved. (This is akin to 9.7.2.) Matters are alleviated but not cured by Toda’s use of zero-suppressed BDD’s (=ZDD’s). Thus the ZDD provides an implicite compression of $MHS(\mathbb{H})$ which often provided $|MHS(\mathbb{H})|$ faster than the 18 competitors in [1]. But since $MHS(\mathbb{H})$ is only output one-by-one this didn’t always mean overall victory. The Toda-algorithm is probably faster than us whenever the compression-rate is low, such as for the (30,5000,7) signature. With increasing

| $(w, h, k)$ | $R$, av.deg | mnMC | content (abs/rel) | vg, mg, bad |
|------------|-------------|------|------------------|-------------|
| (60, 20, 5) | 26701, 13 (13s) | 309, (1s) | 1914, 77% (5042s) | 43, 49, 8 |
| (70, 20, 30) | 77448, 5 (39s) | — | 8.5, 20% (186s) | 13, 62, 25 |
| (60, 20, 7) | 123584, 10 (56s) | 55538 (2564s) | 1.02, 20% (66s) | 15, 26, 59 |
| (70, 20, 5) | 13577, 14 (8s) | 256, (3s) | 113116, 86% | 68, 32, 0, |
| (70, 20, 6) | 41319, 12 (21s) | 730, (3s) | 1694, 82% | 37, 62, 2, |
| (70, 20, 9) | 91737, 10 (1546s) | — | 42, 27% | 33, 60, 7 |
| (100, 40, 3) | 103673, 33 (13s) | 113, (0.4s) | $3 \cdot 10^10$, 99% | 94, 6, 0 |
| (30, 5000, 7) | 1000, 18 (103s) | — | 0.45, 22% | 23, 14, 64 |
| (100, 80, 3) | 1000, 34 (2s) | 437, (2s) | 3000, 29% | 6, 73, 21 |
| (10000, 100, 1000) | 1000, 14 (158s) | — | $10^4$, 77% | 55, 40, 5 |

Table 8. Numerical evaluation and extrapolation of the minhit algorithm
compression-rate the tables begin to turn. Also keep in mind: Our more pleasantly compressed representation of \( MHS(\mathbb{H}) \) may be desirable enough that spending extra time on it is worthwhile.

(ii) Even when the final BDD is moderate in size, intermediate BDD’s can be excessively large, thus causing memory problems. In contrast, the LIFO stack used by the transversal \( e \)-algorithm can never contain more than \( h \) rows (this is a classic result about LIFO stacks).

(iii) In [5,p.101] Toda hopes to eventually parallelize one part of his algorithm, i.e. the calculation of a BDD that captures \( HS(\mathbb{H}) \). In contrast, parallelizing our equivalent (the transversal \( e \)-algorithm) is straightforward. In fact, the evaluation of all semifinal 012e-rows can be parallelized as well.

(iv) Like our method some algorithms in [1] have the potential for cut-off (4.3), but the Toda-algorithm seems not to be among them since it does not appear in Table 9 or 10 of [1].

12 Enumerating all exact hitting sets

In our last Section all our hypergraphs \( \mathbb{H} \subseteq \mathcal{P}([w]) \) of cardinality \( h := |\mathbb{H}| \) are full in the sense that \( \bigcup \mathbb{H} = [w] \) (to avoid trivial cases). An exact hitting set (EHS) with respect to a hypergraph \( \mathbb{H} \) is a subset \( X \subseteq [w] \) such that \( |X \cap H| = 1 \) for all \( H \in \mathbb{H} \). Because of \( \bigcup \mathbb{H} = [w] \) each \( a \in X \) belongs to some hyperedge \( H \). This implies that each EHS \( X \) is \( \text{"very MC"} \), and so a minimal hitting set. The converse fails.

In the sequel we compress the set \( EHS(\mathbb{H}) \) of all exact \( \mathbb{H} \)-hitting sets by ‘imposing’ the hyperedges one after the other (12.2-12.3). In doing so the previously used 01g-cards will be applicable even more directly, yet the trivial feasibility test (10) becomes much harder. One consequence (12.4) concerns the enumeration of all perfect matchings in certain graphs. Sections 12.1 and 12.5 deal with a natural (apparently novel) equivalence relation induced on \([w]\) by every hypergraph \( \mathbb{H} \subseteq \mathcal{P}([w]) \). It prompts one to distinguish 'degenerate' and 'nondegenerate' hypergraphs.

12.1 For a hypergraph \( \mathbb{H} = \{K_1, \ldots, K_h\} \subseteq \mathcal{P}(W) \) we say that \( x, y \in W \) are (\( \mathbb{H} \)-)equivalent (written \( \sim \)) if \( \forall (1 \leq i \leq h) \ x \in K_i \iff y \in K_i \). If the equivalence relation \( \sim \) is the identity relation, then \( \mathbb{H} \) is called nondegenerate, otherwise degenerate. For instance, if \( \mathbb{H} \) is the hypergraph of all stars of a graph (see 12.4) then \( \mathbb{H} \) is nondegenerate. On the other hand, the vertices 6, 8 are \( \mathbb{H}_1 \)-equivalent (see 2.4.1), and so \( \mathbb{H}_1 \) is degenerate. For each index set \( I \subseteq [h] \) let \( \mathbb{H}(I) \) be the set of \( a \in W \) which are in all \( K_i \)’s \( (i \in I) \) and nowhere else. Formally

\[
(43) \quad \mathbb{H}(I) := \bigcap\{K_i : i \in I\} \cap \bigcap\{W \setminus K_i : i \in [h] \setminus I\}.
\]

If \( \mathbb{H}(I) \neq \emptyset \) then \( \mathbb{H}(I) \) is a \( \sim \)-class, and each \( \sim \)-class arises this way\footnote{Why parallelization (aka distributed computing) works smoothly in all LIFO-scenarios is e.g. explained in [6,Sec. 6.5].}. It follows that \( 2^h < w \) is a sufficient condition for \( \mathbb{H} \) to be degenerate.

\footnote{In the sense that for each \( a \in X \) every \( H \) containing \( a \) cuts it out sharply.}

\footnote{In fact \( \mathbb{H}_2 \) in 3.1 has no EHSes. We mention in passing that hypergraphs \( \mathbb{H}' \) with \( MHS(\mathbb{H}') = EHS(\mathbb{H}') \) can be reckognized in polynomial time [10], and that \( MHS(\mathbb{H}') \) can be output one-by-one with polynomial delay. The most obvious instance of \( MHS(\mathbb{H}') = EHS(\mathbb{H}') \) occurs when the hyperedges of \( \mathbb{H}' \) are mutually disjoint.}

\footnote{Once more VL can be used. In brief, letting \( A \) be the \( h \times w \) whose \( i \)th row is the characteristic bitstring of}
(44) Let $\mathcal{H}$ be a hypergraph and let $r$ be any 01g-row contained in $EHS(\mathcal{H})$. Then each $g$-bubble \{a, b, \ldots\} of $r$ is contained in a $\sim$-class.

Proof of (44). Let $K \in \mathcal{H}$ be arbitrary with $a \in K$. By symmetry it suffices to show that $b \in K$. By way of contradiction suppose $b \notin K$. Fix any $X \in r$ with $a \in X$ (by definition of 01g-row there is such $X$). Then $X \cap K = \{a\}$ since $X$ is an exact hitting set. If $Y$ arises from $X$ by switching $a$ with $b$ then still $Y \in r$. But $Y \cap K = \emptyset$, which contradicts the fact that $Y$ (being in $r$) is an (exact) hitting set. □

12.2 Consider the hypergraph $\mathcal{H}_5 \subseteq \mathcal{P}[9]$ consisting of the three hyperedges

(45) $K_1 = \{2, 3, 4, 6\}$, $K_2 = \{1, 2, 3, 4, 5, 7\}$, $K_3 = \{2, 8, 9\}$.

If instead of $\{K_1, K_2, K_3\}$ we just have $\{K_1\}$, then the set of $\{K_1\}$-hitting sets, i.e. $\{X \subseteq [9] : |X \cap \{2, 3, 4, 6\}| = 1\}$, can be written\(^{43}\) as the 012g-row $r_0$ below.

|    | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  |
|----|----|----|----|----|----|----|----|----|----|
| $r_0$ | 2  | $g$ | $g$ | 2  | $g$ | 2  | 2  | 2  | pending $K_2$ |
| $r'_1$ | 2  | 0  | 0  | 0  | 2  | 1  | 2  | 2  | 2  |
| $r'_2$ | 2  | $g$ | $g$ | 2  | 0  | 2  | 2  | 2  | |
| $r_1$ | $g$ | 0  | 0  | 0  | $g$ | 1  | $g$ | 2  | 2  | pending $K_3$ |
| $r_2$ | 0  | $g$ | $g$ | $g$ | 0  | 0  | 0  | 2  | 2  | pending $K_3$ |
| $r_3$ | $g_1$ | 0  | 0  | 0  | $g_1$ | 1  | $g_2$ | $g_2$ | final |
| $r_4$ | 0  | $g$ | $g$ | $g$ | 0  | 0  | 0  | 2  | 2  | pending $K_3$ |
| $r_5$ | 0  | 0  | $g_1$ | $g_1$ | 0  | 0  | 0  | $g_2$ | $g_2$ | final |

Table 9: The working stack for the g-algorithm

In order to sieve the $\{K_1, K_2\}$-EHSes $X$ from $r_0$ we observe that $K_1 \cap K_2 = \{2, 3, 4\}$ and accordingly write $r_0 = \gamma_1 \uplus \gamma_2$ (Table 9). That helps because sieving the $\{K_1, K_2\}$-EHSes from the auxiliary rows $r'_1, r'_2$ is easy. It results in $r_1, r_2$ respectively. For both rows the imposition of $K_3$ is still pending. Each row in the stack must be tagged with this kind of information. Picking the top row of the current working stack $\{r_1, r_2\}$ we focus on $r_1$. It is evident that the subset of all $X \in r_1$ with $|X \cap K_3| = 1$ can be written as the 012g-row $r_3$ in Table 9. Row $r_3$ is final in the sense that all hyperedges have been imposed on it; this amounts to $r_3 \subseteq EHS(\mathcal{H})$.

We hence remove $r_3$ from the working stack and make it the first final row. It is clear that imposing $K_3$ on the last row $r_2$ in the working stack yields the final rows $r_4, r_5$. We hence have $EHT(\mathcal{H}_1) = r_3 \uplus r_4 \uplus r_5$. In particular $\mathcal{H}_5$ has $3 \cdot 2 + 2 \cdot 2 + 1 = 11$ exact hitting sets.

12.3 In order to generally impose a hyperedge $K$ upon a 01g-row we erect a certain Abraham-the $i$th hyperedge, one checks that $1 \sim k \iff (\text{BitOr}(\text{col}[1], \text{col}[k]) = \text{col}[1] \text{ and } \text{BitAnd}(\text{col}[1], \text{col}[k]) = \text{col}[1])$. In this way the $\sim$-class $\mathcal{T}$ can be determined. Next pick any $j \in [v] \setminus \mathcal{T}$ and determine $\mathcal{J}$ likewise. And so forth.

\(^{43}\)In Section 12 our familiar 01g-rows must be slightly generalized to 012g-rows.
flag (boldface in Table 10) akin to (7). Thus imposing $K = \{1,2,...,6\}$ upon the 012g-row in Table 10 yields $\tau_7$ to $\tau_4$.

|   | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|----|----|----|
| $\tau_6$ | $g_1$ | $g_1$ | $g_2$ | $g_2$ | $g_3$ | $g_4$ | $g_1$ | $g_1$ | $g_2$ | $g_3$ | $g_3$ | $g_4$ |
| $\tau_7$ | $g_1$ | $g_1$ | 0 | 0 | 0 | 0 | 0 | 0 | 1 | $g_3$ | $g_3$ | 1 |
| $\tau_2$ | 0 | 0 | $g_2$ | $g_2$ | 0 | 0 | $g_1$ | $g_1$ | 0 | $g_3$ | $g_3$ | 1 |
| $\tau_3$ | 0 | 0 | 0 | 0 | 1 | 0 | $g_1$ | $g_1$ | 1 | 0 | 0 | 1 |
| $\tau_4$ | 0 | 0 | 0 | 0 | 0 | 1 | $g_1$ | $g_1$ | 1 | $g_3$ | $g_3$ | 0 |

Table 10: Imposing the exact hitting set $\{1,\ldots,6\}$ upon the row $\tau_0$

Adhering to the terminology of 3.1 we call $\tau_1$ to $\tau_4$ the candidate sons of $\tau_0$ (that arise upon imposing $K$ on $\tau_0$). Again we need to know which of the candidate sons $\tau_i$ are feasible in the sense that $\tau_i \cap \text{EHS}(H) \neq \emptyset$, and infeasible candidate sons (=duds) should be cancelled. The popular Dancing-Links algorithm of Knuth which decides (though not in polynomial time) whether or not a given hypergraph admits a hitting set, is easily adapted to a feasibility test for candidate sons. Again the surviving candidate sons of $\tau_0$ are called its sons. The described method will be coined the g-algorithm.

**Theorem 3:** Let $\mathbb{H} \subseteq \mathcal{P}[w]$ be a hypergraph. Then $\text{EHS}(\mathbb{H})$ can be enumerated as a disjoint union of $R$ many 01g-rows in time $O(Rhw \cdot \text{feas}(h,w))$. Here $\text{feas}(h,w)$ upper-bounds the time for any chosen subroutine (e.g. Dancing-Links) to decide whether a hypergraph with $\leq w$ vertices and $\leq h$ hyperedges has an EHS.

**Proof.** Throughout the g-algorithm the top rows in the LIFO-stack match the nodes of a computation tree (rooted at $r_0$) whose $R$ leaves are the final rows. The length of a root-to-leaf path equals the number of impositions that were required to generate that leaf (=final row), and hence that length is at most $h$. In the worst case (i.e. when all root-to-leaf paths are mutually disjoint and have maximal length) the number of non-root nodes, i.e. the number of impositions, equals $Rh$.

What is the maximum cost $\text{imp}(h,w)$ of imposing a hyperedge on a LIFO top row $r$? Building the at most $\tau = \tau(\mathbb{H}) := \max \{|H| : H \in \mathbb{H}\}$ candidate sons $r_i$ of $r$ (by way of 0g0-Abraham-Flags) costs $O(\tau w)$. Letting $\text{feas}(h,w)$ be any time bound for checking the feasibility of a 012g-row it costs $O(\tau \text{feas}(h,w))$ to discard the infeasible candidate sons. A surviving son $r_i$ satisfies a fixed hyperedge $K$ iff in $r_i$ the bits with indices in $K$ are all 0’s except for one 1. Hence it costs $O(\tau hw)$ to tag each son with its pending hyperedge. We conclude that $\text{imp}(h,w) = O(\tau w + \tau hw + \tau \text{feas}(h,w))$ and therefore:

(46) The overall cost of imposing the hyperedges of $\mathbb{H}$ in order to pack all exact hitting sets of

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44To avoid distraction we often choose $\text{twos}(r) = \text{ones}(r) = \text{zeros}(r) = \emptyset$. Only trivial modifications would occur otherwise.

45This name was previously used by the author in other circumstances involving g-wildcards. There is no danger of confusion.

46For technical reasons we postulate that $f(h,w) \geq hw$. For every non-trivial hypergraph $\mathbb{H}$ this inequality holds anyway.
\[ H \text{ into } R \text{ disjoint 01g-rows is } O(Rh \cdot \text{imp}(h, w)) = O(Rh\tau(hw + \text{feas}(h, w))). \]

Since we postulated \( f(h, w) \geq hw \) and since \( \tau \leq w \), we have \( O(Rh\tau(hw + \text{feas}(h, w))) = O(Rhw \cdot \text{feas}(h, w)). \)

12.4 An important kind of exact hitting set arises from any graph \( G \) with vertex set \( V \) and edge set \( E \). Namely, if \( \text{star}(v) \) is the set of all edges incident with vertex \( v \) and \( \mathbb{H} := \{\text{star}(v) : v \in V\} \subseteq \mathcal{P}(E) \), then the EHSes of \( \mathbb{H} \) are exactly the perfect matchings of \( G \). Recall that \( K_{3,3} \) is the complete bipartite graph both shores of which having 3 vertices. The bipartite graph with 3 vertices on each shore, such that each vertex is adjacent to every vertex on the opposite shore, is commonly denoted as \( K_{3,3} \). A graph \( G \) is \( K_{3,3} \)-minor-free if one cannot obtain \( K_{3,3} \) from \( G \) by deleting edges and vertices of \( G \), nor by contracting edges of \( G \).

**Theorem 4:** All perfect matchings of a \( K_{3,3} \)-minor-free graph \( G \) can be enumerated in polynomial total time.

*Proof.* In our context each feasibility test performed by the \( g \)-algorithm on a 01g-row \( r \) de facto decides whether a certain minor \( G(0,1) \) of \( G \) of has a perfect matching. Specifically, the 0’s in \( r \) delete edges from \( G \) which thus becomes a sparser graph \( G(0) \). The 1’s in \( r \) constitute a partial matching \( P \) in \( G(0) \) which wants to be extended to a perfect matching of \( G(0) \). This is possible iff a certain subgraph \( G(0,1) \) of \( G(0) \) has a perfect matching. Namely, \( G(0,1) \) is obtained by removing all edges of \( P \), along with all edges incident with them. The arising isolated vertices are also removed. With \( G \) also its minor \( G(0,1) \) is \( K_{3,3} \)-minor-free. By Corollary 1 in [11] one can decide in polynomial time (in fact even NC-time) whether \( G(0,1) \) has a perfect matching. Hence the function \( \text{feas}(h, w) \) in Theorem 1 is bound by a polynomial in \( h, w \), causing the overall algorithm to run in total polynomial time. □

One can dispense with \( K_{3,3} \)-minor-freeness if one allows for randomization because deciding the existence of a perfect matching is in RNC [12,p.347]. Perfect matchings in bipartite graphs have been dealt with before [13].

12.5 Let \( \mathbb{H} = \{K_1, \ldots, K_h\} \subseteq \mathcal{P}\{\omega\} \) be a hypergraph. Generally, if a \( \sim \)-class \( C \) intersects \( K_i \), then it must be contained in \( K_i \); otherwise there were \( x, y \in C \), one in \( K_i \), the other not, which is impossible. Therefore, if \( \overline{K_i} \) denotes the set of \( \sim \)-classes contained in \( K_i \), then \( K_i = \bigcup \overline{K_i} \). The reduced hypergraph \( \mathbb{H} := \{\overline{K_1}, \ldots, \overline{K_h}\} \) has \( h_0 \leq h \) hyperedges and is nondegenerate. For instance, for \( \mathbb{H}_5 \) in (45) the \( \mathbb{H}_5 \)-classes are \( \Gamma(=\overline{5} = \overline{7}) = \{1, 5, 7\} \), \( \overline{2} = \{2\} \), \( \overline{3} = \{3, 4\} \), \( \overline{6} = \{6\} \), \( \overline{8} = \{8, 9\} \). Hence \( \overline{\mathbb{H}_5} = \{\overline{K_1}, \overline{K_2}, \overline{K_3}\} \), where \( \overline{K_1} = \{\overline{2}, \overline{3}, \overline{6}\} \), \( \overline{K_2} = \{\overline{1}, \overline{2}, \overline{3}\} \), \( \overline{K_3} = \{\overline{2}, \overline{8}\} \).

Let us connect all of this with \( g \)-wildcards. The \( g \)-bubble of the \( g \)-wildcard in row \( r_9 \) of Table 9 is \( \{2, 3, 4, 6\} \). Since this is just \( K_1 \), it is a union of \( \sim \)-classes. It follows at once from induction and the design Abraham-Flags that this property gets perpetuated:

\[(47) \text{ When applying the } g \text{-algorithm to the hypergraph } \mathbb{H}, \text{ each occuring } g \text{-bubble is a union of } \sim \text{-classes.} \]

However, once the \( g \)-algorithm has terminated, all final 01g-rows are subsets of \( EHS(\mathbb{H}) \), and so by (44) all their \( g \)-bubbles are contained in single \( \sim \)-classes. This is compatible with (47) only if each \( g \)-bubble of a final row actually is an \( \mathbb{H} \)-class.
12.5.1 In particular, when applying the $g$-algorithm to a nondegenerate hypergraph, each final 01$g$-row must be a 01-row (=bitstring). For instance, applying the $g$-algorithm to the nondegenerate hypergraph $\mathbb{H}_5$ would give the final 01-rows in the left part of Table 11:

| 1 | 2 | 3 | 5 | 8 | 1 | 5 | 7 | 2 | 3 | 4 | 6 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 1 | 1 | $\Rightarrow$ | $g_1$ | $g_1$ | $g_1$ | 0 | 0 | 0 | 1 | $g_2$ | $g_2$ |
| 0 | 0 | 1 | 0 | 1 | $\Rightarrow$ | 0 | 0 | 0 | 0 | $g$ | $g$ | 0 | $g$ | $g$ |
| 0 | 1 | 0 | 0 | 0 | $\Rightarrow$ | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |

Table 11: The $g$-algorithm necessarily enumerates $EHS(\mathbb{H}_5)$ one-by-one

One retrieves the final 01$g$-rows on the right in Table 11 by inflating each 1 at position $\overline{k}$ on the left to a $g$-wildcard as large as the class $\overline{k}$ (with the understanding that 1 stays 1 if $\overline{k}$ is a singleton).

12.6 What is the bottom line in all of that? A devil’s advocate might argue: For nondegenerate hypergraphs the $g$-algorithm offers no compression, and for degenerate hypergraphs $\mathbb{H}$ the compression can also be achieved by enumerating the EHSes of $\mathbb{H}$ with any other algorithm, and then inserting $g$-wildcards in a trivial manner.

Here is the defender’s argument: As elementary as they are, the concepts 'degenerate' and 'nondegenerate' are new. Likewise for 'g-wildcards' and 'Abraham-Flags'. Concerning 'other algorithm', the author could not google any publication concerning the enumeration of all exact hitting sets of a general hypergraph. Even concerning specific hypergraphs, the algorithm in [13] seems to be the only publication.

12.6.1 What is the importance of "degenerate/or not" in the context of $MHS(\mathbb{H})$? As testified by $\mathbb{H}_2$ in (9), the MHSes of nondegenerate hypergraphs are often compressible nevertheless. For degenerate $\mathbb{H}$ one could, as we did for EHSes, run all our techniques on the reduced hypergraph $\mathbb{H}$ and later compress further. Whether that actually gives better compression than just sticking to $\mathbb{H}$ remains to be seen.

12.7 Conclusion: This article promotes the compression of $MHS(\mathbb{H})$ by the use of wildcards. This approach is very promising for sparse hypergraphs (see (100,40,3) in Table 8), but not advisable for dense ones (see (30,50,7) in Table 8). As observed already in [3], what works particularly well in the sparse case (which we henceforth assume) is the compression of $MCHS(\mathbb{H})$, i.e. of the minimum-cardinality hitting sets. As to compressing the remainder $MHS(\mathbb{H}) \setminus MCHS(\mathbb{H})$, we apologize for having overwhelmed (or not?) the reader with a plethora of topics: Three criteria for very-goodness, many uses of Vertical Layout, the fact that $MHS(\mathbb{H}) = HS(\mathbb{H}) \cap MC(\mathbb{H})$, the proposal and calculation of $MinNotMC(\mathbb{H})$, the primal-dual approach ($e$- and $n$-wildcards) for finding $|MHS(\mathbb{H})|$, and more. While often illustrated with luscious toy-examples, many of these ideas await implementation and comparison with other approaches (collaboration is welcome). The author also appreciates to be informed of further (some are given in [6]) real-life examples of hypergraphs with few but large hyperedges. As a "side show" Section 12 turned to exact (as opposed to minimal) hitting sets. The issue of when $EHS(\mathbb{H})$ can be compressed is more clear-cut (12.6) than it was for $MHS(\mathbb{H})$. Further we touched upon Knuth’s Dancing-Links and on compressing all perfect matchings of a graph.
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