ON THE STABLE 4-GENUS OF KNOTS WITH INDEFINITE SEIFERT FORM

SEBASTIAN BAADER

Abstract. Under a simple assumption on Seifert surfaces, we characterise knots whose stable topological 4-genus coincides with the genus.

1. Introduction

The topological 4-genus $g_4(K)$ of a knot $K$ is the minimal genus of a topological, locally flat surface embedded in the 4-ball with boundary $K$. A well-known theorem due to Freedman asserts that knots with trivial Alexander polynomial bound a locally flat disc in the 4-ball [2]. Unlike for the classical genus $g$, there is no known algorithm that determines the topological 4-genus of a knot. The signature bound by Kauffman and Taylor [5], $|\sigma(K)| \leq 2g_4(K)$, fails to be sharp for the simplest knots, such as the figure-eight knot. As we will see, the signature bound becomes much more effective when the topological 4-genus is replaced by its stable version $\hat{g}_4$ defined by Livingston [6]:

$$\hat{g}_4(K) = \lim_{n \to \infty} \frac{1}{n} g_4(K^n).$$

Here $K^n$ denotes the $n$-times iterated connected sum of $K$. The existence of $\hat{g}_4$ follows from general principles on subadditive functions (see Theorem 1 in [6]).

**Theorem 1.** Let $\Sigma \subset \mathbb{R}^3$ be a minimal genus Seifert surface for a knot $K$. Assume that $\Sigma$ contains an embedded annulus with framing +1 or −1. Then the following are equivalent:

(i) $\hat{g}_4(K) = g(K),$

(ii) $|\sigma(K)| = 2g(K).$

**Corollary 1.** Let $\Sigma \subset \mathbb{R}^3$ be a minimal genus Seifert surface for a knot $K$. If $\Sigma$ contains two embedded annuli with framings +1 and −1, then $\hat{g}_4(K) < g(K).$
The second condition of Theorem 1 clearly implies the first one, by the following chain of (in)equalities:

\[ n2g(K) = n|\sigma(K)| = |\sigma(K^n)| \leq 2g_4(K^n) \leq 2g(K^n) = n2g(K). \]

We do not know whether the reverse implication holds without any additional assumption on Seifert surfaces.

**Question.** Does there exist a knot \( K \) with \( |\sigma(K)| < 2g(K) \) and \( \hat{g}_4(K) = g(K) \)?

We conclude the introduction with an application concerning positive braid knots, i.e. knots which are closures of a positive braids. As shown in [1], the only positive braid knots with \( |\sigma(K)| = 2g(K) \) are torus knots of type \( T(2, n) \) (\( n \in \mathbb{N} \)), \( T(3, 4) \) and \( T(3, 5) \). Moreover, positive braid knots have a canonical Seifert surface (in fact, a fibre surface), which always contains a Hopf band with framing +1.

**Corollary 2.** Let \( K \) be a positive braid knot. Then \( \hat{g}_4(K) = g(K) \), if and only if \( K \) is a torus knot of type \( T(2, n) \) (\( n \in \mathbb{N} \)), \( T(3, 4) \) or \( T(3, 5) \).

**Acknowledgements**

I would like to thank Livio Liechti for fruitful discussions, in particular for helping me get the assumption of Theorem 1 right.

**2. Constructing tori with slice boundary**

Let \( K \subset S^3 \) be a knot with minimal genus Seifert surface \( \Sigma \). The Seifert form \( V : H_1(\Sigma, \mathbb{Z}) \times H_1(\Sigma, \mathbb{Z}) \to \mathbb{Z} \) is defined by linear extension of the formula

\[ V([x], [y]) = \text{lk}(x, y^+), \]

valid for simple closed curves \( x, y \subset \Sigma \). Here lk denotes the linking number and \( y^+ \) is a push-off of the curve \( y \) in the positive direction with respect to a fixed orientation of \( \Sigma \). The number \( V([x], [x]) \in \mathbb{Z} \) is called framing of the curve \( x \). The signature \( \sigma(K) \) of \( K \) is defined as the number of positive eigenvalues minus the number of negative eigenvalues of the symmetrised Seifert form \( V + V^T \). The Alexander polynomial of \( K \) is defined as \( \Delta_K(t) = \det(\sqrt{t}V - \frac{1}{\sqrt{t}}V^T) \). Throughout this section, we will assume that

(i) the symmetrised Seifert form on \( H_1(\Sigma, \mathbb{Z}) \) is indefinite, i.e.

\[ |\sigma(K)| < 2g(K), \]
(ii) the surface $\Sigma$ contains an embedded annulus $A$ with framing $+1$ (the case of framing $-1$ can be reduced to this by taking the mirror image of $\Sigma$).

Let $\Sigma^n$ be the Seifert surface for $K^n$ obtained by $n$-times iterated boundary connected sum of $\Sigma$. We define

$$\mathcal{F}(\Sigma) = \{ m \in \mathbb{Z} | \text{there exist a number } n \in \mathbb{N} \text{ and an embedded annulus } A \subset \Sigma^n \text{ with framing } m\}.$$

**Lemma 1.** $\mathcal{F}(\Sigma) = \mathbb{Z}$.

**Proof.** We first show that $\Sigma$ contains an embedded annulus with negative framing. The symmetrised Seifert form $q = V + V^T$ being indefinite and non-degenerate (the latter is true for all Seifert surfaces with one boundary component), there exists a vector $\alpha \in H_1(\Sigma, \mathbb{R})$ with $q(\alpha) < 0$. Since negative vectors for $q$ form an open cone in $H_1(\Sigma, \mathbb{R})$, there exists a simple closed curve $c \subset \Sigma$ with negative framing, i.e. $q([c]) < 0$. Indeed, the surface $\Sigma$ can be seen as a boundary connected sum of $g(\Sigma)$ tori; a suitable connected sum of torus knots will do.

Let $n = |q([c])|$ be the absolute value of the framing of the annulus $C \subset \Sigma$ defined by the curve $c$. We claim that $\Sigma^n$ contains an embedded annulus with framing $-1$. This can be seen by taking a split union of $C$ and $n - 1$ copies of $A$ in $\Sigma^n$ (one annulus per factor), and constructing an annulus that runs through all of these once. Here we need to choose $n - 1$ disjoint intervals connecting pairs of successive annuli, along which the new annulus will run back and forth, as sketched in Figure 1. In the same way, we may construct annuli with arbitrary framings. □

**Figure 1.**

**Lemma 2.** There exists a number $N \in \mathbb{N}$ and an embedded torus $T \subset \Sigma^N$ with one boundary component whose Seifert form is

$$\begin{pmatrix} 0 & \pm 1 \\ 0 & 0 \end{pmatrix},$$

with respect to a suitable basis of $H_1(T, \mathbb{Z})$. In particular, the boundary knot $L = \partial T$ has trivial Alexander polynomial.

**Proof.** By the second assumption, $\Sigma$ contains an embedded annulus $A$ with framing $+1$. We claim that the core curve $a$ of the annulus $A$ is non-separating. Indeed, if the curve $a$ was separating, it would bound
a surface on one side (since the boundary of $\Sigma$ is connected), so the framing of $A$ would be zero. As a consequence of the non-separation property of $a$, there exists an embedded annulus $D \subset \Sigma$ which intersects $A$ in a square. The union of $A$ and $D$ is an embedded torus $T \subset \Sigma$ with one boundary component. Let $\begin{pmatrix} 1 & b \\ c & d \end{pmatrix}$ be the matrix representing the Seifert form on $H_1(T, \mathbb{Z})$ with respect to a pair of oriented core curves of $A$ and $D$. By adding a suitable number of copies of $A$ or $B$ to $D$ in a power $\Sigma^n$, far away from the initial annulus $A \subset \Sigma$, we may impose the framing of $D$ to be $-1$, without changing its linking number with the annulus $A$. Thus we obtain an embedded torus $T' \subset \Sigma$ with Seifert form $\begin{pmatrix} 1 & b \\ c & -1 \end{pmatrix}$. An elementary base change yields

\[ \begin{pmatrix} 1 & b \\ c & -1 \end{pmatrix} \begin{pmatrix} 1 & b \\ c & -1 \end{pmatrix} = \begin{pmatrix} 1 & b - c \\ 0 & -bc - 1 \end{pmatrix}. \]

In turn, if we replace the annulus $D$ by an annulus with $-c$ additional twists around $A$, we obtain an embedded torus $T'' \subset \Sigma^n$ with Seifert form $\begin{pmatrix} 1 & b - c \\ 0 & -bc - 1 \end{pmatrix}$. As before, we may change the individual framings of $A$ and $D$ to be zero in an even larger power $\Sigma^N$. The resulting torus, which we again denote $T \subset \Sigma^N$, has Seifert form $V = \begin{pmatrix} 0 & b - c \\ 0 & 0 \end{pmatrix}$. We claim that $b - c = \pm 1$. Indeed, let $L = \partial T$ be the boundary knot of $T$. The Alexander polynomial of $L$ can be computed as

\[ \Delta_L(t) = \det(\sqrt{t}V - \frac{1}{\sqrt{t}}V^T) = \begin{vmatrix} 0 & \sqrt{t}(b - c) \\ -\frac{1}{\sqrt{t}}(b - c) & 0 \end{vmatrix} = (b - c)^2. \]

Since $\Delta_L(1) = 1$, for all knots $L$, we conclude $b - c = \pm 1$ and

\[ \Delta_L(t) = 1. \]

In order to prove Theorem 1, we need to invoke Freedman’s result ([2], see also [3] and [4]): knots with trivial Alexander polynomial are topologically slice.

Proof of Theorem 1. As mentioned in the introduction, the condition $|\sigma(K)| = 2g(K)$ implies $\tilde{g}_4(K) = g(K)$, without any assumption on the Seifert surface $\Sigma$. For the reverse implication, we assume $|\sigma(K)| < 2g(K)$ and prove $\tilde{g}_4(K) < g(K)$. By Lemma 2, there exists a number $N \in \mathbb{N}$ and an embedded torus $T \subset \Sigma^N$ with one boundary component.
\( L = \partial T \) and \( \Delta_L(t) = 1 \). According to Freedman, there exists a topological, locally flat disc \( D \) embedded in the 4-ball with boundary \( L \). We may assume that the interior of \( D \) is contained in the interior of the 4-ball. Now the union of \( D \) and \( \Sigma^N \setminus T \) is a topological, locally flat surface embedded in the 4-ball with boundary \( K^N \) and genus \( Ng(K) - 1 \). Therefore,

\[
\hat{g}_4(K) \leq g(K) - \frac{1}{N} < g(K).
\]

\( \square \)

References

[1] S. Baader: Positive braids of maximal signature, Enseign. Math 59 (2013), no. 3-4, 351-358.
[2] M. Freedman: The disk theorem for four-dimensional manifolds, Proceedings of the International Congress of Mathematicians in Warsaw (1983), 647-663.
[3] M. Freedman, F. Quinn: Topology of 4-manifolds, Princeton University Press, Princeton, NJ, 1990.
[4] S. Garoufalidis, P. Teichner: On knots with trivial Alexander polynomial, J. Differential Geom. 67 (2004), no. 1, 167-193.
[5] L. Kauffman, L. Taylor: Signature of links, Trans. Amer. Math. Soc. 216 (1976), 351-365.
[6] C. Livingston: The stable 4-genus of knots, Alg. Geom. Top. 10 (2010), 2191-2202.

Universität Bern, Sidlerstrasse 5, CH-3012 Bern, Switzerland

sebastian.baader@math.unibe.ch