Functionally graded prismatic triangular rod under torsion

E. T. Akinlabi1*, M. N. Mikhin2,3**, E. V. Murashkin4***

1University of Johannesburg, Johannesburg, South Africa;
2Branch of the Russian State University for the Humanities, Domodedovo, Russia;
3Moscow Aviation Institute, National Research University, Stupino;
4Ishlinsky Institute for Problems in Mechanics of RAS, Moscow, Russia.
E-mail: *etakinlabi@uj.ac.za, **mikhin@inbox.ru, ***murashkin@ipmnet.ru

Abstract. The paper considers the problem of torsion of a growing viscoelastic prismatic rod with integral boundary conditions at the ends. The process of continuous growth under the influence of torque is studied. The distributions of the intensity of tangential stresses at various stages of the building process are investigated. The calculations of the torsion problem of a prismatic rod with a section in the form of a regular triangle are presented.

1. Introduction
The problem of computation and simulation of residual stresses occurring in objects manufactured by additive technologies is actual one for Mechanics of Solids. Such objects are usually demonstrate the properties inherent for the functionally graded materials [1, 2, 3].

Additive manufacturing technologies are a particular case of growth processes. Mathematical modeling of additive manufacturing technologies is aimed at improving the performance of device, machine, and mechanism parts. The fundamentally new mathematical models considered in the paper describe the evolution of the end product stress-strain state in additive manufacturing and are of general interest for modern technologies in engineering, medicine, electronics industry, aerospace industry, and other fields (see, e.g., [4, 5, 6]).

A solution of applied problem of growing solids mechanics is a sophisticated and time-consuming procedure [7, 8, 9, 10, 11, 13, 12]. An substantial feature of the formulation of boundary value problems of the mechanics of growing solids is the formulation of boundary conditions on the interface between the source material and the added part [7, 16, 14, 15].

2. Growth Problem
Consider a homogeneous viscoelastic aging material manufactured at initial time, occupying a certain prismatic region Π1 with a cross section Ω1, having a boundary L1. The load τ0 is applied to the ends of the rod. These forces are statically equivalent to the pair with the moment M(t).

The lateral surface of the body Π1 is assumed free of stress.

At the moment τ1 ≥ τ0, a continuous growing process of the solid begins by attaching to it elements manufactured simultaneously with it. At the same time, the new incremental elements are not strained. Denote by L(t) the boundary of the cross section Ω(t) changing over time, while L(τ1) = L1 and Ω(τ1) = Ω1. The boundary L(t) of the section Ω(t) consists of two subsections
\( L(t) = L^*(t) \cup L_\sigma(t) \), where is the growing boundary to which the material is attached at the actual moment, and \( L^*(t) = L^* \) at \( \tau \leq \tau_1 \), \( L_\sigma(t) \) is the stress free boundary.

We assume that the moment of the load application to incremental elements \( \tau_0 = \tau_0(x_1, x_2) \) coincides with the moment of their attachment to the growing solid \( \tau^* = \tau^*(x_1, x_2) \).

At the moment \( \tau_2 \geq \tau_1 \) the growing solid stops, and from that moment it occupies the region \( \Pi_2 = (\tau_2) \) with the transverse section \( \Omega_2 = \Omega(\tau_2) \), having the boundary \( L_2 = L(\tau_2) \). Note that everywhere below fairly slow processes are considered, such that inertial terms can be neglected in the equations of motion.

### 3. Boundary Value Problem

The boundary value problem for the main (fixed) viscoelastic aging solid on the time interval \([\tau_0, \tau_1]\) is a traditional torsion problem like considered in [7].

The initial-boundary value problem for a continuously growing solid on the time interval \( t \in [\tau_1, \tau_2] \) is determined of equilibrium equations in Cartesian coordinate

\[
\frac{\partial \sigma_{13}}{\partial x_3} = 0, \quad \frac{\partial \sigma_{23}}{\partial x_1} = 0, \quad \frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} = 0. \tag{1}
\]

Cauchy relations between strain rates and displacement rates are furnished by

\[
D_{11} = \frac{\partial v_1}{\partial x_1} = 0, \quad D_{22} = \frac{\partial v_2}{\partial x_2} = 0, \quad D_{33} = \frac{\partial v_3}{\partial x_3} = 0, \quad D_{12} = \frac{1}{2} \left( \frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2} \right) = 0, \quad D_{13} = \frac{1}{2} \left( \frac{\partial v_1}{\partial x_3} + \frac{\partial v_3}{\partial x_1} \right), \quad D_{23} = \frac{1}{2} \left( \frac{\partial v_2}{\partial x_3} + \frac{\partial v_3}{\partial x_2} \right). \tag{2}
\]

Constitutive equations can be assumed as follows

\[ \sigma_{13} = 2G(I + N_{\tau_0})\varepsilon_{13}, \quad \sigma_{23} = 2G(I + N_{\tau_0})\varepsilon_{23}, \]

\[ \tau_0(x_1, x_2) = \begin{cases} \tau_0, & (x_1, x_2) \in \Omega_1, \\ \tau^*, & (x_1, x_2) \in \Omega^*(t), \end{cases} \]

\[ (I + N_{\tau_0})^{-1} = I - L_{\tau_0}, \quad 2G = \frac{E}{1 + v}, \quad K = \frac{1}{1 - 2v}, \]

\[ L_{sf}(t) = \int_s^t f(\tau)K_1(t, \tau)d\tau, \quad K_1(t, \tau) = G(\tau) \frac{\partial}{\partial \tau} \left[ G^{-1}(\tau) + \omega(t, \tau) \right]. \]

Boundary condition on the fixed part of the boundary is stated as

\( (x_1, x_2) \in L_\sigma(t) : \quad n_1\sigma_{13} + n_2\sigma_{23} = 0; \tag{4} \)

Boundary condition on growing surface \( L^*(t) \)

\( (x_1, x_2) \in L^*(t) : \quad n_1 \frac{\partial \sigma_{13}}{\partial t} + n_2 \frac{\partial \sigma_{23}}{\partial t} = 0; \tag{5} \)

Equilibrium conditions of end sections \( \Omega(t) \)

\[
M(t) = \int_{\Omega(t)} (x_1\sigma_{23} - x_2\sigma_{13})dx_1dx_2,
\]

\[
\int_{\Omega(t)} \sigma_{13}dx_1dx_2 = \int_{\Omega(t)} \sigma_{23}dx_1dx_2 = 0. \tag{6}
\]
In equations (1)–(6) the following notation is used: $n = \{n_1, n_2\}$ is the unit vector of the external normal of the side surface; $p = \{p_1, p_2\}$ is the traction vector; $\Omega^*(t) = \Omega(t) \setminus \Omega_1$ — the part of the solid formed during the growth (additional body); $E = E(t)$ and $G = G(t)$ are the tensile and shear elastic moduli; $\omega(t, \tau)$ — shear measure; $K_1(t, \tau)$ is the creep kernel; Poisson’s ratios of elastic strain and creep strain coincide and are equal to $\nu$, $I$ is the identity operator. The values of all functions at time $t_0 \leq t \leq t_1$ are known from the solution of the problem for the main solid.

Distinctive features of the initial boundary value problem for a growing solid (1)–(6), which take it beyond the framework of the classical problems of mechanics of solids, are: violation of the compatibility condition for deformations in the region occupied by the additional body, and only it analogue and analogue of the Cauchy relations in the velocities of the corresponding quantities (this circumstance allows us to take into account this fact that the incremented elements can be subjected to deforming influences until they attaching in the main solid depending on the processes occurring in the solid); dependence of the constitutive equations on the function $\tau_0 = \tau_0(x_1, x_2)$, which may have discontinuities of the first kind.

Taking account of the notation $\sigma_0^{ij} = (I - L_0) \sigma_{ij} G^{-1}$, we can transform the problem of a growing viscoelastic solid with the constitutive equations (1)–(6) to the problem of growing elastic solid described by Hooke’s law.

4. Torsion of a growing triangular prismatic rod

As an example, consider the torsion problem of a growing prismatic rod with a cross section in the form of a regular triangle under the action of a torque $M(t)$. The boundary of the cross section $L(t)$ is the growth boundary, i.e. $L(t) = L^*(t)$. The material of the medium is viscoelastic and aging, i.e. its properties are time-dependent. Consider the core extension according to the similarity law, in which the side of an equilateral triangle doubles during the extension $a_2 = 2a_1$.

We assume that the new incremental elements are free of stress. In virtue of the mathematical equivalence of the obtained problems at each stage under consideration, it is sufficient to directly consider the growing stage. Then the initial-boundary-value problem in terms of velocities takes the form

$$\frac{\partial S_{13}}{\partial x_1} + \frac{\partial S_{23}}{\partial x_2} = 0, \quad D_{13} = \frac{1}{2} \left( \frac{\partial v_3}{\partial x_1} + \frac{\partial v_1}{\partial x_3} \right), \quad D_{23} = \frac{1}{2} \left( \frac{\partial v_3}{\partial x_2} + \frac{\partial v_2}{\partial x_3} \right),$$

$$S_{13} = 2D_{13}, \quad S_{23} = 2D_{23}, \quad (x_1, x_2) \in L(t) : \quad n_1 S_{13} + n_2 S_{23} = 0,$$

$$\frac{dM^0(t)}{dt} = \int_{\Omega(t)} (x_1 S_{23} - x_2 S_{13}) dx_1 dx_2. \tag{7}$$

Here $S_{ij}$ are the stress rates, $D_{ij}$ are the strain rates.

For the given torsion $\theta(t)$, $\theta'_t(t)$, we can find values $v_1$, $S_{13}$ and $S_{23}$:

$$v_1 = -\theta'_t(t) x_2 x_3, \quad v_2 = \theta'_t(t) x_1 x_3, \quad v_3 = \theta'_t(t) \left( 3x_1^2 x_2 - x_3^2 \right) / (6a(t)),$$

$$S_{13} = \theta'_t(t) \left( \frac{\partial \varphi_1}{\partial x_1} - x_2 \right) = \theta'_t(t) \left( x_1 - a(t) \right) x_2,$$

$$S_{23} = \theta'_t(t) \left( \frac{\partial \varphi_1}{\partial x_2} + x_1 \right) = \theta'_t(t) \left( 2a(t) x_1 x_2 \right). \tag{8}$$
Actual stresses and displacements are restored using the formulas

\[
\sigma_{ij}(x_1, x_2, t) = G(t) \left\{ \sigma_{ij}(x_1, x_2, \tau_0) \left[ 1 + \int_{\tau_0}^{t} R(t, \tau)d\tau \right] + \right.
\]
\[
\left. + \int_{\tau_0}^{t} \left[ S_{ij}(x_1, x_2, \tau) + \int_{\tau}^{\tau_0} S_{ij}(x_1, x_2, \varsigma) d\varsigma R_{ij}(t, \tau) \right] d\tau \right\}
\]
\[
u(x_1, x_2, t) = u(x_1, x_2, \tau_0) + \int_{\tau_0}^{t} \nu(x_1, x_2, \tau)d\tau,
\theta(t) = \theta(\tau_0) + \int_{\tau_0}^{t} \theta'(\tau)d\tau.
\]

Finally, one can compute \( M(t) \) based on the first formula from (6).

For the given momentum \( M(t) \) one can find derivative \( \frac{dM^0(t)}{dt} \)

\[
\frac{dM^0(t)}{dt} = M'(t) + \int_{\tau_0}^{t} \frac{\partial M(t)}{\partial \tau} \frac{\partial \omega(t, \tau)}{\partial t} d\tau + M(\tau_0(x_1, x_2)) \frac{\partial \omega(t, \tau_0(x_1, x_2))}{\partial t}.
\]

Thus the torsion rate \( \theta'(t) \) is calculated by

\[
\theta'(t) = \frac{5}{9\sqrt{3}a^4(t)} \frac{dM^0(t)}{dt}
\]

Velocities \( v_i \) can be found by (8), and stress rates \( S_{13} \) and \( S_{23} \) are reads by

\[
S_{13} = \frac{5(x_1 - a(t))x_2}{9\sqrt{3}a^5(t)} \frac{dM^0(t)}{dt},
\]
\[
S_{23} = \frac{5(x_1^2 + 2a(t)x_1 - x_2^2)}{18\sqrt{3}a^5(t)} \frac{dM^0(t)}{dt}.
\]

Actual stresses, displacements and torsion are restored by the formulas (9).

To build solutions at the stages before and after growing, it is enough to take \( t = \tau_1 \) and \( t = \tau_2 \) respectively.

5. Conclusion

The paper developed the theory of surface growth to study the torsion problem of growing solids in the case when the rate of deformation of the solid surface due to the loads and interference of incremented elements can be neglected compared to the velocity of the boundary due to the influx of new material to this surface.

The formulation of the arising classical and nonclassical initial boundary value problems is given. Methods are proposed for solving such problems, based on the reduction of nonclassical problems of growing viscoelastic aging solids to problems of the theory of elasticity with a certain parameter, using the theory of analytical functions to solve the latter, and restoring the true characteristics of the stress strain state using the obtained decryption formulas.

It is established that during torsion in the finished body without taking into account the process of growth, the maximum intensity of tangential stresses is reached at the boundary of the body. When building, the maximum intensity of tangential stresses can be achieved at the interface between the main and additional bodies, at the border of the finished body and at an arbitrary point of the additional body.

The results obtained can serve as the basis for solving important applied problems for parts and structural elements manufactured using modern additive technologies.
Acknowledgments
The paper is financially supported by joint science and technology collaboration South Africa (National Research Foundation) and Russia (Russian Foundation for Basic Research) (project No. 19-51-60001 / RUSA180527335500) and by the Ministry of Science and Higher Education of the Russian Federation.

References
[1] Akinlabi E T and Akinlabi S A 2012 Effect of heat input on the properties of dissimilar friction stir welds of aluminium and copper American Journal of Materials Science 2 5 147–152
[2] Mahamood R M, Akinlabi E T, Shukla M and Pityana S 2013 Scanning velocity influence on microstructure, microhardness and wear resistance performance of laser deposited Ti6Al4V/TiC composite Materials & design, 50 656–666
[3] Mahamood R M and Akinlabi E T 2017 Functionally graded materials (Springer)
[4] Manzhirov A V 1995 The general non-inertial initial-boundary value problem for a viscoelastic ageing solid with piecewise-continuous accretion J. Appl. Math. Mech. 59 (5) 805–16
[5] Manzhirov A V 2016 Fundamentals of mechanical design and analysis for AM fabricated parts Proc. Manufact. 7 59–65
[6] Manzhirov A V 2017 Advances in the theory of surface growth with applications to additive manufacturing technologies Proc. Engng 173 11-6
[7] Arutyunyan N Kh, Drozdov A D, and Naumov V E 1987 Mechanics of growing viscoelastic-plastic solids (Moscow, Nauka) [In Russian]
[8] Arutyunyan N Kh, Naumov V E, and Radayev Yu N 1992 Dynamic expansion of an elastic layer. Part 1. Motion of a flow of precipitated particles at a variable rate Izv. Akad. Nauk. Mekh. Tverd. Tela No. 5 6–24 [In Russian]
[9] Arutyunyan N Kh, Naumov V E, and Radayev Yu N 1992 Dynamical expansion of an elastic layer. Part 2. The case of drop of accreted particles at a constant rate Izv. Akad. Nauk. Mekh. Tverd. Tela No. 6 99–112 [In Russian]
[10] Stadnik N E and Dats E P 2018 Continuum mathematical modelling of pathological growth of blood vessels Journal of Physics: Conference Series 991 012075
[11] Stadnik N E, Murashkin E V, and Dats E P 2019 Residual stresses in blood vessel wall during atherosclerosis AIP Conference Proceedings 2116 1 380013
[12] Murashkin E and Dats E 2017 Thermoelastoplastic Deformation of a Multilayer Ball Mechanics of Solids 52 5 30–36
[13] Burenin A, Murashkin E and Dats E 2018 Residual stresses in AM fabricated ball during a heating process AIP Conference Proceedings 1959 070008
[14] Murashkin E V and Radayev Yu N 2019 On a differential constraint in asymmetric theories of mechanics of growing solids Izv. Ros. Akad. Nauk Mekh. Te. Tela No. 6 38–46
[15] Murashkin E V and Radayev Yu N 2019 On a class of constitutive equations on a growing surface Vestn. ChGPU I.Ya. Yakovleva. Ser. Lim. St. Mech. No. 3(41) 11–29
[16] Murashkin E V and Radayev Yu N 2019 On a differential constraint in the continuum theory of growing solids J. Samara State Tech. Univ., Ser. Phys. Math. Sci. 23 4 1–11