We consider a generalization of the fundamental theorem of finitely generated abelian groups for some non-abelian groups, which is called OGS. First, we consider the dihedral group $\text{Dih}(A)$, which is a non-abelian extension of an abelian group $A$ by an involution. Then, we focus on a special case of $\text{Dih}(A)$, where $A$ is cyclic, which is the two-generated Coxeter group $I_2(m)$. We mention interesting connections between the reduced Coxeter presentation and a particular OGS canonical presentation, which we call the standard OGS canonical presentation. These connections motivate us to offer a generalization of the standard OGS to the $A$-type Coxeter group, which can be considered as the dual family to the $I$-type Coxeter groups. The $A$-type Coxeter graph is connected, with edges limited to simply laced, but the number of vertices is not limited, and depends on $n$. The $(n-1)$-generated $A$-type Coxeter groups can be considered as the symmetric group $S_n$ for an arbitrary $n$. We mention the standard and the dual-standard OGS of $S_n$, where, The standard OGS canonical form of $S_n$ has a special interest in combinatorics, since in 2001, R. M. Adin, and Y. Roichman has proved that sum of the exponents in the canonical form is coincide with the major-index of the permutation, which is equi-distributed with the Coxeter length. In this paper we extend the results of Adin and Roichman very significantly, where we show interesting properties of the exchange laws, we define standard OGS elementary factorization, which connects between the standard OGS and the descent set of a permutation. Then, by using the standard OGS elementary factorization, we find a new explicit formula for the Coxeter length of an element of $S_n$, and we give a new algorithm for the standard OGS canonical form and the descent set of the inverse element of an arbitrary element of $S_n$. 

**Keywords:** Basis, Ordered Generating Systems, Exchange laws, Standard OGS elementary factorization, Dihedral groups, Symmetric groups, Permutation combinatorics.

**MSC 2010 classification:** 05E15, 06F15, 20B05, 20B20, 20B30, 20F60, 20K01.
1 Introduction

One of the most important theorems of Linear Algebra is that every vector-space \( V \) over a field \( \mathbb{F} \) has a basis, i.e. there are elements \( v_1, v_2, \ldots, v_n \) in \( V \), such that every vector \( v \) in \( V \) has a unique presentation of a form:

\[
v = \alpha_1 \cdot v_1 + \alpha_2 \cdot v_2 + \cdots + \alpha_n \cdot v_n, \quad \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{F}
\]

Thus, the vector \( v \) in \( V \) can be expressed by its \( n \) coordinates \( (\alpha_1, \alpha_2, \ldots, \alpha_n) \). There is a generalization of the basis for finitely generated abelian groups. Let \( A \) be a finitely generated abelian group, then by the fundamental theorem of finitely generated abelian groups there exist generators \( a_1, a_2, \ldots, a_n \), such that every element \( a \) in \( A \) has a unique presentation of a form:

\[
g = a_1^{i_1} \cdot a_2^{i_2} \cdots a_n^{i_n}.
\]

where, \( i_1, i_2, \ldots, i_n \) are \( n \) integers such that for \( 1 \leq k \leq n, 0 \leq i_k < |g_k| \), where \( a_k \) has a finite order of \( |a_k| \) in \( A \), and \( i_k \in \mathbb{Z} \), where \( a_k \) has infinite order in \( A \). The mentioned presentation is the canonical presentation of \( a \in A \) according to the basis \( (a_1, a_2, \ldots, a_n) \).

Since \( A \) is an abelian group, the following exchange laws hold: \( a_k \cdot a_j = a_j \cdot a_k \), for each \( 1 \leq j, k \leq n \). The exchange law allows us to put each product of two elements of the group \( A \) onto the mentioned canonical form. Therefore, we can write every element \( g \) of an abelian group \( A \) as a \( n \)-tuple of integers \( (i_1, i_2, \ldots, i_n) \), where \( i_k \in \mathbb{Z}_{|a_k|} \) for \( 1 \leq k \leq n \) (we denote by \( \mathbb{Z}_r \) the set of integers modulo \( r \), and we denote by \( \mathbb{Z}_\infty \) the set of the integers).

Thus,

\[
(i_1, i_2, \ldots, i_n) + (j_1, j_2, \ldots, j_n) = (i_1 + j_1, i_2 + j_2, \ldots, i_n + j_n),
\]

where, \( i_k + j_k \) is the group operation in the additive group of \( \mathbb{Z}_{|a_k|} \). In this paper we consider a generalization of the canonical form by a given basis as it arises from the fundamental theorem of Abelian groups to the non-abelian case in the following way:

**Definition 1.1** Let \( G \) be a non-abelian group. The ordered sequence of \( n \) elements \( \langle g_1, g_2, \ldots, g_n \rangle \) is called an Ordered Generating System of the group \( G \) or by shortened notation, OGS(G), if every element \( g \in G \) has a unique presentation in the form

\[
g = g_1^{i_1} \cdot g_2^{i_2} \cdots g_n^{i_n},
\]

where, \( i_1, i_2, \ldots, i_n \) are \( n \) integers such that for \( 1 \leq k \leq n, 0 \leq i_k < r_k \), where \( r_k \mid |g_k| \) in case the order of \( g_k \) is finite in \( G \), or \( i_k \in \mathbb{Z} \), in case \( g_k \) has infinite order in \( G \). The mentioned canonical form is called OGS canonical form. For every \( q > p, 1 \leq x_q < r_q \), and \( 1 \leq x_p < r_p \) the relation

\[
g_q^{x_q} \cdot g_p^{x_p} = g_1^{x_1} \cdot g_2^{x_2} \cdots g_n^{x_n},
\]

is called exchange law.

In contrast to finitely generated abelian groups, the existence of an OGS is generally not true for every finitely generated non-abelian group. Even in case of two-generated infinite non-abelian groups it is not too hard to find counter examples. For example,
the Baumslag-Solitar groups $BS(m,n)$ [5], where $m \neq \pm 1$ or $n \neq \pm 1$, or most of the cases of the one-relator free product of a finite cyclic group generated by $a$, with a finite two-generated group generated by $b,c$ with the relation $a^2 \cdot b \cdot a \cdot c = 1$ [13], do not have an OGS. Even the question of the existence of an OGS for a general finite non-abelian group is still open. Moreover, contrary to the abelian case where the exchange law is just $g_q \cdot g_p = g_p \cdot g_q$, in most of the cases of non-abelian groups with the existence of an OGS, the exchange laws are very complicated. Although there are some specific non-abelian groups where the exchange laws are very convenient and have very interesting properties (For example, in the case of $PSL_2(q)$ there is an OGS which is closely connected to the $BN$ - pair decomposition, where the exchange laws yield some interesting recursive sequences over finite fields [15]). In this paper we deal with the two most significant classes of Coxeter groups (namely the $I$-type and the $A$-type), which have an OGS canonical presentation strongly connected to the presentation in Coxeter generators, and with very interesting and surprising exchange laws. The paper is divided as follows: In section 2 we describe an OGS canonical form and the related exchange laws for the family of the dihedral groups $Dih(A)$, which are non-abelian extensions of an abelian group $A$ by a cyclic group of order 2. Then, we focus in the case of $Dih(A)$, where the abelian group $A$ is cyclic, since then the group is a Coxeter group with two generators. We show a connection between the canonical form according to an OGS, which we call standard OGS, and the reduced Coxeter presentation of it. In section 3 we show a generalization of the standard OGS canonical form to the $A$-type Coxeter groups, which can be considered as a dual family of the $I$-type Coxeter groups, where instead of limiting the number of vertices to two in the Coxeter graph, we limit the lace of the edges to be simply laced. The $(n-1)$-generated $A$-type Coxeter group is the symmetric group $S_n$, which can be considered as the permutation group on $n$ elements. Therefore, a lot of work has been accomplished concerning the connections between permutation invariants and the Coxeter length of the elements by Brenti, Bjorner [6], Foata, Schützenberger [7], Grasia, Gessel [8], Reiner [10], Stanley [16], Steingrimsson [17], Bagno, Garber, Mansour, Shwartz [4], and many others. In the same aspect, Adin and Roichman [1] introduced a presentation of an OGS canonical form for the symmetric group $S_n$, for the hyperoctahedral group $B_n$, and for the wreath product $Z_m \wr S_n$. In the case of $S_n$, the OGS which they used, coincides with the standard OGS by our definition. Adin and Roichman proved that for every element of $S_n$ presented in the standard OGS canonical form, the sum of the exponents of the OGS equals the major-index of the permutation. Moreover, by using an OGS canonical form, Adin and Roichman generalized the theorem of MacMahon [9] to the $B$-type Coxeter group, and to the wreath product $Z_m \wr S_n$. A few years later, that OGS canonical form was generalized for complex reflection groups by Shwartz, Adin and Roichman [13]. Although an OGS canonical form for the symmetric groups $S_n$ has been already introduced, and a lot of work has been done concerning permutation statistics and Coxeter length of elements in the symmetric groups, nothing has been carried out yet concerning very important and very interesting aspects of the OGS canonical forms, like exchange laws, an algorithm for a canonical form of the inverse element of a given element of the symmetric group, or an explicit formula for the Coxeter length of a given element of $S_n$. In this paper, we find the related exchange laws for the standard and for the dual-standard OGS canonical forms of the symmetric group $S_n$, with very interesting and surprising properties. By using the standard OGS, we
define standard OGS elementary elements, which coincide with the elements of $S_n$ with a single descent. Then, we define standard OGS elementary factorization onto standard OGS elementary factors, such that the number of the elementary factors of an element $\pi \in S_n$ equals to the size of the descent set of $\pi$. We apply the standard OGS elementary factorization of $\pi$ to introduce an algorithm for the standard OGS canonical form and the descent set of the inverse element $\pi^{-1}$. We also give a new explicit formula for the Coxeter length of a permutation in the symmetric group $S_n$ by using the standard OGS canonical form, and the standard OGS elementary factorization.

2 OGS Canonical forms and exchange laws for $Dih(A)$

In this section we show an OGS canonical form, with very simple exchange laws, for the dihedral groups, a very important family of non-abelian groups. Then, we show connections between the mentioned OGS canonical form and the presentation in Coxeter generators of the two-generated Coxeter groups, which are dihedral groups.

**Definition 2.1** Let $A$ be an abelian group. The map $\phi : A \to A$, such that $\phi(a) = a^{-1}$ is an automorphism of $A$. Then, we define $Dih(A)$ to be the dihedral group of order $2|A|$, as an extension of the group $A$ by an involution $b$, where $b(a) = a^{-1}$, for every $a \in A$.

Obviously, the relations of $Dih(A)$ are the relations of $A$ and conjugation of the elements of $A$ to their inverse by an involution $b$, i.e., $b^{-1} \cdot a \cdot b = a^{-1}$ for every $a \in A$, which is equivalent to: $b \cdot a = a^{-1} \cdot b$, for every $a \in A$. Thus, every element of $Dih(A)$ has a unique presentation in the following canonical form

$$g = a_1^{i_1} \cdot a_2^{i_2} \cdots a_n^{i_n} \cdot b^j$$

where, $i_1, i_2, \ldots, i_n$ are $n$ integers such that for $1 \leq k \leq n$, $0 \leq i_k < |a_k|$, where $a_k$ has a finite order of $|a_k|$ in $A$, and $i_k \in \mathbb{Z}$, where $a_k$ has infinite order in $A$, and $0 \leq j < 2$. By Definition 2.1, the ordered sequence $(a_1, a_2, \ldots, a_n, b)$ is an OGS for $G = Dih(A)$. The relation $b \cdot a = a^{-1} \cdot b$ for every $a \in A$ implies exchange laws of the form $b \cdot a_{k}^{i_k} = a_{k}^{-i_k} \cdot b$.

Therefore,

$$(i_1, i_2, \ldots, i_n, j) + (p_1, p_2, \ldots, p_n, q) = (i_1 + (-1)^j \cdot p_1, i_2 + (-1)^j \cdot p_2, \ldots, i_n + (-1)^j \cdot p_n, j + q),$$

where, the operation "+" in $i_k + (-1)^j \cdot p_k$ is the group operation in the additive group of $\mathbb{Z}_n$, and $j + q$ is the group operation in the additive group of $\mathbb{Z}_2$.

**Proposition 2.2** Every ordered sequence of a form

$$\langle a_{\pi(1)}, \ldots, a_{\pi(w)}, b, a_{\pi(w+1)}, \ldots, a_{\pi(n)} \rangle,$$

where, $w$ is an arbitrary integer such that $1 \leq w \leq n$, and $\pi$ is a permutation of the elements in the set of the $n$ integers $\{1, 2, \ldots, n\}$, forms an OGS for $G = Dih(A)$, where the exchange laws of the given OGS canonical form is the following:

- $a_{\pi(k)} \cdot a_{\pi(j)} = a_{\pi(j)} \cdot a_{\pi(k)}$, for $1 \leq j, k \leq n$ (i.e., commutative exchange laws);
\[ b \cdot a_{\pi(k)}^{i_{\pi(k)}} = a_{\pi(k)}^{-i_{\pi(k)}} \cdot b, \text{ for } 1 \leq \pi(k) \leq w; \]

\[ a_{\pi(k)}^{i_{\pi(k)}} \cdot b = b \cdot a_{\pi(k)}^{-i_{\pi(k)}}, \text{ for } w + 1 \leq \pi(k) \leq n. \]

**Proof.** The proof comes directly by the definition of \( G = \text{Dih}(A) \) as an extension of the abelian group \( A \) by an involution \( b \), according to the automorphism \( b(a) = a^{-1} \), for every \( a \in A \).

Now, we show the \( OGS \) canonical form of the inverse element \( g^{-1} \) for a given element \( g \in \text{Dih}(A) \).

**Proposition 2.3** Let \( G = \text{Dih}(A) \) for an abelian group \( A \), where \( A \) has a basis \( \langle a_1, a_2, \ldots, a_n \rangle \) by the fundamental theorem of abelian groups. Consider the \( OGS \) canonical form with \( OGS(G) = \langle a_{\pi(1)}, \ldots, a_{\pi(w)}, b, a_{\pi(w+1)}, \ldots, a_{\pi(n)} \rangle \), where \( w \) is an arbitrary integer such that \( 1 \leq w \leq n \), and \( \pi \) is a permutation of the elements in the set of the \( n \) integers \( \{n\} = \{1, 2, \ldots, n\} \).

- If \( g \in A \), i.e.,
  \[ g = a_{\pi(1)}^{i_{\pi(1)}} \cdot a_{\pi(2)}^{i_{\pi(2)}} \cdot \ldots \cdot a_{\pi(w)}^{i_{\pi(w)}} \cdot a_{\pi(w+1)}^{i_{\pi(w+1)}} \cdot \ldots \cdot a_{\pi(n)}^{i_{\pi(n)}} \]
  then
  \[ g^{-1} = a_{\pi(1)}^{[a_{\pi(1)}]^{-i_{\pi(1)}}} \cdot a_{\pi(2)}^{[a_{\pi(2)}]^{-i_{\pi(2)}}} \cdot \ldots \cdot a_{\pi(w)}^{[a_{\pi(w)}]^{-i_{\pi(w)}}} \cdot a_{\pi(w+1)}^{[a_{\pi(w+1)}]^{-i_{\pi(w+1)}}} \cdot \ldots \cdot a_{\pi(n)}^{[a_{\pi(n)}]^{-i_{\pi(n)}}}; \]

- If \( g \notin A \), i.e.,
  \[ g = a_{\pi(1)}^{i_{\pi(1)}} \cdot a_{\pi(2)}^{i_{\pi(2)}} \cdot \ldots \cdot a_{\pi(w)}^{i_{\pi(w)}} \cdot b \cdot a_{\pi(w+1)}^{i_{\pi(w+1)}} \cdot \ldots \cdot a_{\pi(n)}^{i_{\pi(n)}} \]
  then \( g^{-1} = g \) (i.e., \( g \) is an involution).

**Proof.** If \( g \in A \), then we conclude \( g^{-1} \) by the fundamental theorem of abelian groups. Therefore, assume \( g \notin A \), which means the \( OGS \) canonical form of \( g \) involves \( b \). Notice, the relation \( b \cdot a = a^{-1} \cdot b \), for every \( a \in A \), applies:

\[ (a' \cdot b \cdot a'')^2 = a' \cdot (b \cdot a'') \cdot b \cdot a'' \]
\[ = a' \cdot (a'')^{-1} \cdot (a')^{-1} \cdot b \cdot (a'') \]
\[ = a' \cdot (a'')^{-1} \cdot (a')^{-1} \cdot a'' \]
\[ = a' \cdot (a')^{-1} \cdot (a'')^{-1} \cdot a'' \]
\[ = 1, \]

for every \( a', a'' \in A \). Thus,

\[ (a' \cdot b \cdot a'')^{-1} = a' \cdot b \cdot a'' \]

for every \( a', a'' \in A \).

The following example shows how we can multiply two arbitrary elements in \( \text{Dih}(A) \), which are presented in \( OGS \) canonical form. 

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Example 2.4 Let $A$ be $\mathbb{Z}_9 \oplus \mathbb{Z}_3$, where the elements $a_1$ and $a_2$ generates $A$, such that $|a_1| = 9$, $|a_2| = 3$, and every element in $A$ has a unique presentation in the canonical form $a_1^{i_1} \cdot a_2^{i_2}$, where $0 \leq i_1 < 9$, and $0 \leq i_2 < 3$. Now, consider the group $Dih(\mathbb{Z}_9 \oplus \mathbb{Z}_3)$, which is the extension of $\mathbb{Z}_9 \oplus \mathbb{Z}_3$ by an involution $b$ such that $b \cdot a = a^{-1} \cdot b$, for every $a \in A$. Then, every element of $Dih(\mathbb{Z}_9 \oplus \mathbb{Z}_3)$ has a unique presentation in a canonical form $a_1^{i_1} \cdot a_2^{i_2} \cdot b^j$, where $0 \leq i_1 < 9$, $0 \leq i_2 < 3$, and $0 \leq j < 2$, with the exchange laws:

- $a_2^j \cdot a_1^{i_1} = a_1^{i_1} \cdot a_2^j$;
- $b \cdot a_1^{i_1} = a_1^{9-i_1} \cdot b$;
- $b \cdot a_2^j = a_2^{3-i_2} \cdot b$.

Thus, for example let $x$ and $y$ be the following elements: $x = a_1^4 \cdot a_2^2 \cdot b$, and $y = a_1^7 \cdot a_2 \cdot b$, then by the exchange laws the following holds:

$$x \cdot y = a_1^4 \cdot a_2^2 \cdot b \cdot a_1^7 \cdot a_2 \cdot b = a_1^1 \cdot a_2^2 \cdot a_1^{9-7} \cdot a_2^{3-2} \cdot b \cdot b = a_1^6 \cdot a_2^3 \cdot b^2 = a_1^4.$$

2.1 The Coxeter group $I_2(m)$

There is a special interest in the family dihedral groups $Dih(A)$, where $A$ is a cyclic group. Let $A$ be a cyclic group of order $m$ ($m$ might be $\infty$), then $Dih(A)$ is a two-generated Coxeter group $I_2(m)$, of order $2m$, where $m$ is finite, or order $\infty$, in case $m = \infty$. We recall the Coxeter presentation of $I_2(m)$, and some basic properties of it, as described in [3]:

- $I_2(m) = \langle s_1, s_2 \mid s_1^2 = s_2^2 = 1, (s_1 \cdot s_2)^m = 1 \rangle$ in case of finite $m$;
- $I_2(\infty) = \langle s_1, s_2 \mid s_1^2 = s_2^2 = 1 \rangle$, i.e., $I_2(\infty) = \mathbb{Z}_2 \ast \mathbb{Z}_2$.

Now, define $b$ to be $s_1$, and define $a$ to be $s_1 \cdot s_2$. Then, $\langle b, a \rangle$ is an OGS for $I_2(m)$ with the exchange law $a \cdot b = b \cdot a^{m-1}$, in case of finite $m$, or $a \cdot b = b \cdot a^{-1}$ in case of $m = \infty$. Now, Consider the presentation of the elements of $I_2(m)$ in Coxeter generators.

Proposition 2.5 Let $G = I_2(m)$, with the Coxeter generators $s_1, s_2$, and let $b = s_1$, $a = s_1 \cdot s_2$, then the following holds:

- $b = s_1$;
- $b \cdot a = s_2$;
- $b \cdot a^i = s_2 \cdot s_1 \cdot s_2 \cdots s_1 \cdot s_2 = (s_2 \cdot s_1)^{i-1} \cdot s_2$, for every $1 < i \leq \frac{m+1}{2}$ in case of finite $m$, and for every $i > 1$ in case of infinite $m$;
- $b \cdot a^i = s_1 \cdot s_2 \cdot s_1 \cdots s_1 \cdot s_2 = (s_1 \cdot s_2)^{m-1} \cdot s_1$, for every $\frac{m+1}{2} \leq i < m$ in case of finite $m$, and for every $i < 0$ in case of infinite $m$;
- $a^i = s_1 \cdot s_2 \cdot s_1 \cdots s_1 \cdot s_2 = (s_1 \cdot s_2)^i$, for every $0 < i \leq \frac{m}{2}$ in case of finite $m$, and for every $i > 0$ in case of infinite $m$;
- $a^i = s_2 \cdot s_1 \cdot s_2 \cdots s_2 \cdot s_1 = (s_2 \cdot s_1)^{m-1}$, for every $\frac{m}{2} \leq i < m$ in case of finite $m$, and for every $i < 0$ in case of infinite $m$.  


Proposition 2.6 Let $G$ be $I_2(m)$ for a finite $m$, then the Coxeter length is equidistributed with the length (sum of the exponents) in the OGS canonical form by $\text{OGS}(I_2(m)) = \langle b, a \rangle$.

Proof. Let $G = I_2(m)$, then the following hold:

- By [6], there are exactly two elements with Coxeter length $i$ for every $1 \leq i \leq m - 1$. Namely $(s_1 \cdot s_2)^i$ for an even $i$, $(s_1 \cdot s_2)^{i-1} \cdot s_1$ and $(s_2 \cdot s_1)^{i-1} \cdot s_2$ for an odd $i$;

- By [6], there is exactly one element with Coxeter length 0 and Coxeter length $m$. The element with Coxeter length $m$ is $(s_1 \cdot s_2)^m = (s_2 \cdot s_1)^m$ in case of even $m$, and $(s_1 \cdot s_2)^{m-1} \cdot s_1 = (s_2 \cdot s_1)^{m-1} \cdot s_2$ in case of odd $m$;

- The presentation of the elements in $I_2(m)$ in the canonical form according to the sequence $\{b, a\}$ is $b^j \cdot a^i$, such that $0 \leq j \leq 1$, $0 \leq i \leq m - 1$. Therefore, there are two elements of length $i$ for every $1 \leq i \leq m - 1$, namely $a^i$, and $b \cdot a^{i-1}$. The identity is the only element of length 0, and $b \cdot a^{m-1}$ is the only one element of length $m$.

In Propositions 2.5 and 2.6, we show interesting connections between two presentations of the two-generated Coxeter group, $I_2(m)$:

- The presentation in Coxeter generators;

- The OGS canonical presentation for $\text{OGS}(I_2(m)) = \langle b, a \rangle$, where $b = s_1$ and $a = s_1 \cdot s_2$.

Therefore, we call $\text{OGS}(I_2(m)) = \langle b, a \rangle$, the standard OGS of $I_2(m)$.

Remark 2.7 The geometric meaning of the group $I_2(m) = \text{Dih}(\mathbb{Z}_m)$ is the symmetry group of a regular $m$-sided polygon, where the $m$ elements of $\mathbb{Z}_m$ present the rotations of the polygon, and the $m$ elements of the form $b \cdot a^i$ (where, $0 \leq i < m$) present the $m$ reflections of the polygon. The case of $m = 3$ is the non-abelian group which has the smallest order, and in this case $I_2(3) = \text{Dih}(\mathbb{Z}_3)$ is the symmetric group on 3 elements, which is denoted by $S_3$.

3 OGS canonical forms and exchange laws for the symmetric group $S_n$

In the last section we have shown interesting connections between the presentation in the standard OGS canonical form, and the presentation in Coxeter generators for every two-generated Coxeter group $I_2(m)$. These connections motivate us to look at a generalization for Coxeter groups with more than two Coxeter generators. The family $I_2(m)$ is the natural generalization of $I_2(3)$, changing the lace of the edge connecting the two vertices on the Coxeter graph to arbitrary lace, but fixing the number of vertices to two.
The natural analogous generalization of $I_2(3)$ is the $A$-type Coxeter group with arbitrary $n$ generators, which we get by changing the number of vertices on the Coxeter graph, but keeping it connected and simply laced by connecting every two adjacent vertices in the Coxeter graph by a simply laced edge. Therefore, in this section we consider the $A$-type Coxeter groups, where the Coxeter graph is a line with arbitrary $n$ vertices, but the lace of the connecting edges is fixed to be simply laced for every two adjacent vertices in the Coxeter graph. Notice, the smallest non-abelian $I$-type Coxeter group $I_2(3)$ is the same to the smallest non-abelian $A$-type Coxeter group $A_2$, where the Coxeter graph is just two vertices, which are connected by a simply-laced edge. Therefore, in this section we consider the $A$-type Coxeter group by a simply laced edge. Thus, in this section we consider the $(n-1)$-generated $A$-type Coxeter group can considered as the symmetric group on $n$ elements, $S_n$. Adin and Roichman [1] have introduced an OGS canonical form for the family of Coxeter groups $S_n$ and $B_n$ s.t.; the length according to that form is equi-distributed with the Coxeter length (as has been shown for $I_2(m)$). They also give a combinatorial interpretation for the length of an element in $S_n$ or in $B_n$ according to that $OGS$ canonical form, which is called flag-major-index and motivated by MacMahon’s theorem for the symmetric groups [9]. In this section we find the exchange laws for that $OGS$ canonical form of $S_n$, and for the dual $OGS$ canonical form. We also show interesting properties of the exchange laws, which allows for an efficient multiplication of elements in $S_n$. We give a new explicit formula for the Coxeter length of an element in $S_n$ using the $OGS$ canonical form. Finally, we find a formula for the $OGS$ canonical form of the inverse element of an arbitrary element of $S_n$.

We start with some basic definitions concerning the symmetric group $S_n$.

**Definition 3.1** Let $S_n$ be the symmetric group on $n$ elements, then:

- The symmetric group $S_n$ is an $(n-1)$-generated simply-laced Coxeter group which has the presentation of:

  $$\langle s_1, s_2, \ldots, s_{n-1} | s_i^2 = 1, (s_i \cdot s_{i+1})^3 = 1, (s_i \cdot s_j)^2 = 1 \text{ for } |i - j| \geq 2 \rangle$$

- The group $S_n$ can be considered as the permutation group on $n$ elements. A permutation $\pi \in S_n$ is denoted by $[\pi(1) ; \pi(2) ; \ldots ; \pi(n)]$ (i.e., $\pi = [2; 4; 1; 3]$ is a permutation in $S_4$ which satisfies $\pi(1) = 2$, $\pi(2) = 4$, $\pi(3) = 1$, and $\pi(4) = 3$);

- Every permutation $\pi \in S_n$ can be presented in a cyclic notation, as a product of disjoint cycles of the form $(i_1, i_2, \ldots, i_m)$, which means $\pi(i_k) = i_{k+1}$, for $1 \leq k \leq m - 1$, and $\pi(i_m) = i_1$ (i.e., The cyclic notation of $\pi = [3; 4; 1; 5; 2]$ in $S_5$, is $(1, 3)(2, 4, 5)$);

- The Coxeter generator $s_i$ can be considered the permutation which exchanges the element $i$ with the element $i + 1$, i.e., the transposition $(i, i + 1)$;

- We consider multiplication of permutations in left to right order; i.e., for every $\pi_1, \pi_2 \in S_n$, $\pi_1 \cdot \pi_2(i) = \pi_2(j)$, where $\pi_1(i) = j$ (in contrary to the notation in [4], [11] where, Brenti, Bjorner, Adin, Roichman and other people have considered right to left multiplication of permutations);

- For every permutation $\pi \in S_n$, the Coxeter length $\ell(\pi)$ is the number of inversions in $\pi$, i.e., the number of different pairs $i, j$, s.t. $i < j$ and $\pi(i) > \pi(j)$;
• For every permutation $\pi \in S_n$, the set of the locations of the descents is defined to be
\[
\text{des}(\pi) = \{1 \leq i \leq n-1 | \pi(i) > \pi(i+1)\},
\]
and
\[
i \in \text{des}(\pi) \text{ if and only if } \ell(s_i \cdot \pi) < \ell(\pi)
\]
(i.e., $i$ is a descent of $\pi$ if and only if multiplying $\pi$ by $s_i$ in the left side shortens the Coxeter length of the element.);

• For every permutation $\pi \in S_n$, the major-index is defined to be
\[
\text{maj}(\pi) = \sum_{\pi(i) > \pi(i+1)} i
\]
(i.e., major-index is the sum of the locations of the descents of $\pi$).

3.1 The standard and the dual-standard $OGS$ canonical forms and the exchange laws

Theorem 3.2 Let $S_n$ be the symmetric group on $n$ elements. For every $2 \leq m \leq n$, define $t_m$ to be the product $\prod_{j=1}^{m-1} s_j$. The element $t_m$ is the permutation $[m; 1; \ldots; m-1]$, which is the $m$-cycle $(m, m-1, \ldots, 1)$ in the cyclic notation of the permutation. Then, the elements $t_n, t_{n-1}, \ldots, t_2$ generates $S_n$, and every element of $S_n$ has a unique presentation in each one of the following $OGS$ canonical forms:

1. $t_2^0 \cdot t_3^1 \cdots t_n^{m}$, where $0 \leq i_k < k$ for $2 \leq k \leq n$;
2. $t_n^{m} \cdot t_{n-1}^{m-1} \cdots t_2^1$, where $0 \leq i_k < k$ for $2 \leq k \leq n$.

Proof. The theorem has been stated, and has been used in $\prod$. We give a detailed proof of it. The proof is by induction on $n$. For $n = 2$, we have $S_2$ a cyclic group of order 2, which is generated by $t_2$. Therefore, the theorem holds trivially. Now, assume by induction that the theorem holds for every $m$ such that $2 \leq m \leq n-1$. Thus, for each $1 \leq m \leq n-1$, every element in $S_m$ has a unique presentation in a form $t_m^i \cdot t_{m-1}^j \cdots t_2^k$, where $0 \leq i_k < k$ for $2 \leq k \leq m$, and every element in $S_m$ has a unique presentation also in a form $t_2^k \cdot t_3^j \cdots t_m^i$, where $0 \leq i_k < k$ for $2 \leq k \leq m$. Now, let us consider the group $S_n$. There is a subgroup $\hat{S}_{n-1}$ of $S_n$ which contains all the elements of $S_n$ such that $n$ is a fixed point. It is easy to show that $\hat{S}_{n-1}$ is isomorphic to $S_{n-1}$. Then, by the induction hypothesis, every element in $\hat{S}_{n-1}$ has a unique presentation in a form $t_{n-1}^i \cdot t_{n-2}^j \cdots t_2^k$, where $0 \leq i_k < k$ for $2 \leq k \leq n-1$, and also in a form $t_2^k \cdot t_3^j \cdots t_{n-1}^i$, where $0 \leq i_k < k$ for $2 \leq k \leq n-1$. The index $[S_n : \hat{S}_{n-1}] = n$, and it is easy to show that the $n$ elements $t_n^i$, where $0 \leq i < n$ are both left and right coset representatives of $\hat{S}_{n-1}$ in $S_n$. Thus, the theorem holds for $S_n$. ■

Remark 3.3 Contrary to the abelian groups and to the dihedral groups, in the case of $S_n$ for $n \geq 4$, the order in which the $n-1$ generators $t_2, t_3, \ldots, t_n$ can appear in the $OGS$ canonical form is important, and there is no every ordered sequence $\langle t_{\pi(2)}, t_{\pi(3)}, \ldots, t_{\pi(n)} \rangle$
forms an OGS for $S_n$, where $\pi$ is an arbitrary permutation of the elements in the set of the $n-1$ integers $\{2,3,\ldots,n\}$. For example, consider the group $S_4$, which is generated by the elements $t_2, t_3, t_4$. Then, it is satisfied that $t_2^3 \cdot t_2 = t_3 \cdot t_4$. Thus, there is no unique presentations of the elements of $S_4$ in the form $t_2^3 \cdot t_4^4 \cdot t_2^3$, where $0 \leq i_k < k$. Therefore, the ordered sequence $\langle t_3, t_4, t_2 \rangle$ does not form an OGS for $S_n$.

As a conclusion we consider the following two OGS for $S_n$.

- The standard OGS for $S_n$: $OGS(S_n) = \langle t_2, t_3, \ldots, t_n \rangle$;
- The dual-standard OGS for $S_n$: $OGS(S_n) = \langle t_n, t_{n-1}, \ldots, t_2 \rangle$.

The standard OGS for $S_n$ has been considered in [1] for combinatorial interest too. By [1], the sum of the exponents of the generators according to the sequence, $\sum_{j=2}^{n} i_j$, (i.e., the length of the element $\pi$ in the canonical form according to the standard OGS) is the major-index of the permutation $\pi$.

Both the standard and the dual-standard OGS give exchange laws with very interesting properties which we describe now.

**Proposition 3.4** The following holds:

1. For transforming the element $t_p^q \cdot t_p^{i_p}$ ($p < q$) onto the OGS canonical form $t_2^q \cdot t_3^{i_2} \cdots t_n^{i_n}$, i.e., according to the standard OGS, one needs to use the following exchange laws:

$$t_q^i \cdot t_p^j = \begin{cases} t_{q+i_p}^i \cdot t_{p+i_q}^j \cdot t_q^{i_p} & q - i_q \geq p \\ t_{q+i_p}^{p+i_q} \cdot t_{p+i_q}^{p-i_q} \cdot t_q^{i_p} & i_p \leq q - i_q \leq p \\ t_{q+i_p}^{p+i_q} \cdot t_{p+i_q}^{q-i_q} \cdot t_q^{i_p} & q - i_q \leq i_p \end{cases}$$

2. Similarly, for transforming the element $t_p^{i_p} \cdot t_q^{i_q}$ ($p < q$) onto the OGS canonical form $t_n^1 \cdot t_{n-1}^1 \cdots t_2^1$, i.e., according to dual-standard, one needs to use the following exchange laws:

$$t_p^{i_p} \cdot t_q^{i_q} = \begin{cases} t_{p+i_q}^{i_p} \cdot t_{q+i_p}^{i_q} \cdot t_{q-i_q}^{p-i_p} \cdot t_{p-i_q}^{p-i_p} & i_q \geq p \\ t_{p+i_q}^{q-i_q} \cdot t_{q-i_q}^{p-i_p} \cdot t_{p-i_q}^{q-i_q} \cdot t_{q+i_p}^{p-i_p} & p - i_q \leq i_q \leq p \\ t_{p+i_q}^{q-i_q} \cdot t_{q-i_q}^{p-i_p} \cdot t_{p-i_q}^{q-i_q} & i_q \leq p - i_q \end{cases}$$

**Proof.** First, consider the standard OGS of $S_n$. Then, we look at the exchange laws for $t_q^i \cdot t_p^{i_p}$, where $q > p$. Since all $j \in \{1,2,\ldots,n\}$ such that $j > q$ are fixed by $t_q^i \cdot t_p^{i_p}$, we may consider the element $t_q^i \cdot t_p^{i_p}$ in $S_q$ instead of considering them in $S_n$. Therefore, the operation " + "
is considered addition modulo $q$. Now, look at the elements $t_q^{i_q}$ and $t_p^{i_p}$ as permutations in $S_q$, then the following is satisfied:

- $t_q^{i_q}(j) = j - i_q$ for every $1 \leq j \leq q$;
- $t_q^{i_q}(j) = j - i_p + p = j - (i_p - p + q)$ for every $1 \leq j \leq i_p$;
- $t_q^{i_q}(j) = j - i_p$ for every $i_p + 1 \leq j \leq p$;
- $t_q^{i_q}(j) = j$ for every $p + 1 \leq j \leq q$.

Let $x$ be the conjugate of $t_q^{i_q}$ by the element $t_q^{i_q}$, i.e., $x := t_q^{i_q} \cdot t_p^{i_p} \cdot t_q^{-i_q}$. Then, by conjugating laws of permutations, the permutation $x$ satisfies the following by considering the inequalities modulo $q$ (i.e., if $q = 10$, then $8 \leq j \leq 2$ means the set of integers $\{8, 9, 10, 1, 2\}$):

- $x(j) = j - (i_p - p + q)$ for $i_q + 1 \leq j \leq i_q + i_p$;
- $x(j) = j - i_p$ for $i_q + i_p + 1 \leq j \leq i_q + p$;
- $x(j) = j$ for $i_q + p + 1 \leq j \leq i_q$.

Thus, by using $x(j)$ we get the following canonical form for $x$:

- $x = t_q^{i_q+i_p-1} \cdot t_p^{i_p}$ in case $q \geq i_q + p$ (i.e., $q - i_q \geq p$);
- $x = t_q^{i_q-i_p} \cdot t_p^{i_q} \cdot t_q^{i_q}$ in case $i_q + i_p \leq q \leq i_q + p$ (i.e., $i_p \leq q - i_q \leq p$);
- $x = t_q^{i_q+i_p-1} \cdot t_p^{i_q} \cdot t_q^{i_q}$ in case $q \leq i_q + i_p$ (i.e., $q - i_q \leq i_p$).

Now, right multiplications of each one of the three equalities by $t_q^{i_q}$ gives the desired exchange laws for $t_q^{i_q} \cdot t_p^{i_p}$.

Now, consider the dual-standard OGS of $S_n$. Since $(t_q^{i_q} \cdot t_p^{i_p})^{-1} = t_p^{i_p-1} \cdot t_q^{i_q}$, we get the exchange laws for $t_q^{i_q} \cdot t_p^{i_p}$ easily by taking the inverse of the the exchange laws for $t_q^{i_q} \cdot t_p^{i_p}$, and substituting $i_p = p - i_p$, $i_q = q - i_q$. □

From the described exchange laws for $S_n$ we conclude the following properties:

**Proposition 3.5** The standard OGS canonical form of $t_q^{i_q} \cdot t_p^{i_p}$, and the dual-standard canonical form of $t_p^{i_q} \cdot t_q^{i_q}$ (where, $p < q$) are products of non-zero powers of at most three different canonical generators, where the following holds:

1. The standard OGS canonical form of $t_q^{i_q} \cdot t_p^{i_p}$ ($p < q$, $i_p > 0, i_q > 0$) is a product of non-zero powers of two different canonical generators if and only if $q - i_q = p$ or $q - i_q = i_p$, and then the following hold:

$$t_q^{i_q} \cdot t_p^{i_p} = \begin{cases} t_q^{i_q+i_p} \cdot t_q^{i_q} & q - i_q = p \\ t_p^{i_p-1} \cdot t_q^{i_q} & q - i_q = i_p. \end{cases}$$
2. The dual-standard OGS canonical form of $t'^p q' \cdot t'^q p'$ ($p < q$, $i_p' > 0, i_q' > 0$) is a product of non-zero powers of two different canonical generators if and only if $i_q' = p$, or $i_q' = p - i_p'$, and then the following holds:

$$t'^p q' \cdot t'^q p' = \begin{cases} t'^{i_p'} q' \cdot i_{q-2}^{-i_p'} & i_q' = p \\ t'^q p' \cdot t'^{q-p} q' - i_p' & i_q' = p - i_p'. \end{cases}$$

Proof. The results come directly from Propositions 3.4 and 3.5.

Proposition 3.6 The exchange laws for $S_n$ which are described in Propositions 3.4 and 3.5 satisfy the following properties:

1. The standard OGS canonical form of the element $t'^i q' \cdot t'^j p'$ for $q > p$ is either $t'^a \cdot t'^b \cdot t'^c$, or $t'^a \cdot t'^b$ for $a < b < c$, where $c = q$ and all of the numbers: $a$, $b$, $i_a$, $i_b$, and $i_c$, are linear combinations of at most three different elements from $\{p, q, i_p, i_q\}$ with co-efficients 1 or -1;

2. The dual-standard canonical form of the element $t'^i q' \cdot t'^j p'$ for $q > p$ is either $t'^a \cdot t'^b', t'^a \cdot t'^b$, or $t'^a \cdot t'^b$ for $a > b > c$, where $a = q$ and all of the numbers: $b$, $c$, $i_a$, $i_b$, and $i_c$, are linear combination of at most four elements from $\{p, q, i_p', i_q\}$ with co-efficients 1 or -1 or 2 (The co-efficient 2 appears just in one of the three cases for $i_b'$, as a co-efficient of $p$).

Proof. The results come directly from Propositions 3.4 and 3.5.

Example 3.7 Consider the following elements $x, y \in S_5$. Let $x = t_2 \cdot t_3^2 \cdot t_4 \cdot t_5^3$, which is the permutation $[3; 2; 5; 4; 1]$ which is the 3-cycle $(1, 3, 5)$. Let $y = t_2 \cdot t_3 \cdot t_4^2 \cdot t_5^3$, which is the permutation $[1; 4; 2; 5; 3]$, which is the 4-cycle $(2, 4, 5, 3)$. Now, we find the standard OGS canonical form of the product $\pi = x \cdot y$ by using the exchange laws described in
Propositions 3.4 and 3.5:

Assume in induction that the proposition holds for every $k$

Proposition 3.5, is satisfied:

Let $\pi = x \cdot y$ is the permutation $[2; 4; 3; 5; 1]$, which is the 4-cycle $(1, 2, 4, 5)$.

The next proposition shows the standard OGS canonical form of some conjugates of $t_k$ and $t_k^{-1}$.

**Proposition 3.8** Let $G = S_n$ and consider the standard OGS canonical form. Then, for every $2 \leq k \leq n - 1$ the following holds:

$$
\left( \prod_{j=0}^{n-k-1} t_{n-j} \right) \cdot t_k \cdot \left( \prod_{j=0}^{n-k-1} t_{n-j} \right)^{-1} = t_{n-k+1}^{-1} \cdot t_n.
$$

$$
\left( \prod_{j=0}^{n-k-1} t_{n-j} \right) \cdot t_k^{-1} \cdot \left( \prod_{j=0}^{n-k-1} t_{n-j} \right)^{-1} = t_{n-k}^{-1} \cdot t_{n-1}^{-1}.
$$

**Proof.** The proof is by induction on $n - k$. If $n - k = 1$ (i.e., $k = n - 1$, then by Proposition 3.3)

$$
t_n \cdot t_{n-1} \cdot t_{n}^{-1} = t_2 \cdot t_n^2 \cdot t_n^{-1} = t_2 \cdot t_n.
$$

Assume in induction that the proposition holds for every $k$ such that $n - k < k'$ for some $k' \geq 1$ and we prove it for $n - k = k'$. Then, by the induction hypothesis, the following is satisfied:

$$
\left( \prod_{j=0}^{n-k'-1} t_{n-j} \right) \cdot t_{k'} \cdot \left( \prod_{j=0}^{n-k-1} t_{n-k} \right)^{-1} = t_n \cdot t_{n-k'}^{-1} \cdot t_{n-1} \cdot t_{n}^{-1}.
$$
Then, by Propositions 3.4 and 3.5 the following holds:

\[ t_n \cdot t_{n-k}^{n-k-1} \cdot t_{n-1} \cdot t_{n-1}^{-1} = t_{n-k} \cdot t_{n-k+1}^{n-k-1} \cdot t_n \cdot t_{n-1} \cdot t_{n-1}^{-1} \]

\[ = t_{n-k} \cdot t_{n-k+1}^{n-k-1} \cdot t_{2} \cdot t_{2}^{-1} \cdot t_{n-1}^{-1} \]

\[ = t_{n-k} \cdot t_{n-k+1}^{n-k-1} \cdot t_{n-k+1} \cdot t_n \]

\[ = t_{n-k+1}^{-1} \cdot t_n. \]

Thus, the first part of the proposition holds for every \( k \). Now,

\[
\left( \prod_{j=0}^{n-k-1} t_{n-j} \right) \cdot t_k^{-1} \cdot \left( \prod_{j=0}^{n-k-1} t_{n-j} \right)^{-1} = \left( \prod_{j=0}^{n-k-1} t_{n-j} \right) \cdot t_k \cdot \left( \prod_{j=0}^{n-k-1} t_{n-j} \right)^{-1} \]

\[ = (t_{n-k+1}^{-1} \cdot t_n)^{-1} = t_{n-k+1}^{-1} \cdot t_n. \]

By Proposition 3.5

\[ t_{n-1} \cdot t_{n-k+1} = t_{n-1}^{-1} \cdot t_{n-k+1} = t_{n-1}^{-1} \cdot t_{n}^{-k+1} \cdot t_{n}^{-k+1}. \]

3.2 Normal form, and its connection to the standard OGS canonical form

From now on, we consider just the standard OGS canonical form.

In this subsection, we recall the definition of a normal form of elements of \( S_n \) in Coxeter generators, which is described in detail in Brenti and Bjorner’s book, “Combinatorics of Coxeter groups” [6]. Then, we find a connection of the normal form to the standard OGS canonical form.

**Definition 3.9** By [6], every element \( \pi \) of \( S_n \) can be presented uniquely in the following normal reduced form, which we denote by \( \text{norm}(\pi) \):

\[ \text{norm}(\pi) = \prod_{u=1}^{n-1} \prod_{r=0}^{y_u-1} s_{u-r}. \]

such that \( y_u \) is a non-negative integer where, \( 0 \leq y_u \leq u \) for every \( 1 \leq u \leq n - 1 \).

We denote by \( \ell(\pi) \), the Coxeter length of an element \( \pi \in S_n \), which is the number of Coxeter generators \( s_j \) which are used in the reduced presentation of \( \pi \). By our notation of \( \text{norm}(\pi) \),

\[ \ell(\pi) = \sum_{u=1}^{n-1} y_u. \]
Example 3.10 Let \( m = 8, \ y_2 = 2, \ y_4 = 3, \ y_5 = 1, \ y_8 = 4, \) and \( y_1 = y_3 = y_6 = y_7 = 0, \) then

\[
\text{norm}(\pi) = (s_2 \cdot s_1) \cdot (s_4 \cdot s_3 \cdot s_2) \cdot s_5 \cdot (s_8 \cdot s_7 \cdot s_6 \cdot s_5).
\]

\[
\ell(\pi) = 2 + 3 + 1 + 4 = 10.
\]

Notice, \( t_k = \prod_{u=1}^{k-1} s_u \) is already presented in the mentioned normal form according to [6], with \( y_u = 1 \) for every \( 1 \leq u \leq k - 1. \)

**Proposition 3.11** Consider the symmetric group \( S_n. \) then for every \( 2 \leq k \leq n, \) and \( 1 \leq i_k \leq k - 1 \) the following holds:

\[
\text{norm}(t_k^{i_k}) = \prod_{u=i_k}^{k-1} \prod_{r=0}^{i_k-1} s_{u-r}.
\]

\[
\ell(t_k^{i_k}) = k \cdot i_k - i_k^2.
\]

**Proof.** First, notice, by the definition of \( t_{i_k+1}, \) we conclude: \( \prod_{r=0}^{i_k-1} s_{i_k-r} = t_{i_k+1}^{-1} = t_{i_k+1}. \)

Notice also, \( \prod_{r=0}^{i_k-1} s_{u-r}, \) for \( i_k < u \leq k - 1, \) is a conjugate of \( \prod_{r=0}^{i_k-1} s_{i_k-r} \) by \( \prod_{u'=i_k+2}^{u} t_{u'}^{-1}. \)

Hence,

\[
\prod_{u=i_k}^{k-1} \prod_{r=0}^{i_k-1} s_{u-r} = t_{i_k+1}^{-1} \cdot \prod_{u=i_k+2}^{k} \left( \prod_{r=0}^{u-i_k-2} t_{u-r} \cdot t_{i_k+1}^{-1} \cdot \left( \prod_{r=0}^{u-i_k-2} t_{u-r}^{-1} \right)^{-1} \right).
\]

Now, assume in induction that

\[
t_{i_k+1}^{-1} \cdot \prod_{u=i_k+2}^{k-1} \left( \prod_{r=0}^{u-i_k-2} t_{u-r} \cdot t_{i_k+1}^{-1} \cdot \left( \prod_{r=0}^{u-i_k-2} t_{u-r}^{-1} \right)^{-1} \right) = t_{k-1}^{i_k}.
\]

Then,

\[
t_{i_k+1}^{-1} \cdot \prod_{u=i_k+2}^{k-1} \left( \prod_{r=0}^{u-i_k-2} t_{u-r} \cdot t_{i_k+1}^{-1} \cdot \left( \prod_{r=0}^{u-i_k-2} t_{u-r}^{-1} \right)^{-1} \right)
\]

\[= t_{k-1}^{i_k} \cdot \left( \prod_{r=0}^{k-i_k-2} t_{k-r} \cdot t_{i_k+1}^{-1} \cdot \left( \prod_{r=0}^{k-i_k-2} t_{k-r}^{-1} \right)^{-1} \right).\]

Then, by Proposition 3.8

\[
\left( \prod_{r=0}^{k-i_k-2} t_{k-r} \cdot t_{i_k+1}^{-1} \cdot \left( \prod_{r=0}^{k-i_k-2} t_{k-r}^{-1} \right)^{-1} \right) = t_{k-1}^{k-(i_k+1)} \cdot t_{k-1}^{i_k}.
\]

Hence,

\[
\prod_{u=i_k}^{k-1} \prod_{r=0}^{i_k-1} s_{u-r} = t_{k-1}^{i_k} \cdot t_{k-1}^{k-(i_k+1)} \cdot t_{k-1}^{i_k} = t_{k}^{i_k}.
\]

Then, by using \( \text{norm}(t_k^{i_k}) \), we get \( \ell(t_k^{i_k}) = (k - i_k) \cdot i_k = k \cdot i_k - i_k^2. \) ■
Example 3.12

\[ t_7^3 = (s_3 \cdot s_2 \cdot s_1) \cdot (s_4 \cdot s_3 \cdot s_2) \cdot (s_5 \cdot s_4 \cdot s_3) \cdot (s_6 \cdot s_5 \cdot s_4). \]

\[ \ell(t_7^3) = 7 \cdot 3 - 3^2 = 12. \]

Proposition 3.13 Assume \( \text{norm}(\pi) = \prod_{r=0}^{v-1} s_{k-r} \), where \( k, v \) are positive integers, such that \( v \leq k \), then the standard OGS canonical form of \( \pi \) is the following:

- \( \pi = t_{k+1}^k \cdot t_{k+1}^v \) in case \( v < k \)
  (i.e., \( \text{norm}(\pi) = s_k \cdot s_{k-1} \cdots s_{k-v+1} \), where \( k - v + 1 \geq 2 \)).

- \( \pi = t_{k+1}^k \) in case \( v = k \)
  (i.e., \( \text{norm}(\pi) = s_k \cdot s_{k-1} \cdots s_1 \)).

Proof. Assume \( \pi = t_{k+1}^k \). By Proposition 3.11, \( t_{k+1}^k = \prod_{r=0}^{k-1} s_{k-r} \).

Now, assume \( \pi = t_{k+1}^k \cdot t_{k+1}^v \). Then, \( \pi = (t_{k+1}^k)^{-1} \cdot t_{k+1}^v \). By using Proposition 3.11,

\[ \text{norm}(t_{k+1}^k) = (s_v \cdot s_{v+1} \cdots s_1) \cdot (s_{v+1} \cdot s_{v+2} \cdots s_2) \cdots (s_{k-1} \cdot s_k \cdots s_{k-v}) \]

and

\[ \text{norm}(t_{k+1}^v) = (s_v \cdot s_{v+1} \cdots s_1) \cdot (s_{v+1} \cdot s_{v+2} \cdots s_2) \cdots (s_{k-1} \cdot s_k \cdots s_{k-v+1}). \]

Thus,

\[ \text{norm}(t_{k+1}^{v+1}) = \text{norm}(t_{k+1}^v) \cdot (s_k \cdot s_{k-1} \cdots s_{k-v}). \]

Hence,

\[ \text{norm}(\pi) = \text{norm}(t_{k+1}^{v+1}) = \prod_{r=0}^{v-1} s_{k-r}. \]

Example 3.14 Assume, \( \text{norm}(\pi) = s_5 \cdot s_4 \cdot s_3 \), then \( \pi = t_5^2 \cdot t_6^3 \) in the standard OGS canonical form.

Proposition 3.15 Let \( \pi = t_{k_1}^{k_1-v} \cdot t_{k_2}^v \) be an element of \( S_n \), which is presented in the standard OGS canonical form, where \( v \) is a positive integer such that \( 1 \leq v \leq k_1 - 1 \).

Then,

\[ \text{norm}(\pi) = \text{norm}(t_{k_1}^{k_1-v} \cdot t_{k_2}^v) = \prod_{u=k_1}^{k_2-1} \prod_{r=0}^{v-1} s_{u-r}. \]

\[ \ell(\pi) = (k_2 - k_1) \cdot v. \]

Proof. Let \( \pi = t_{k_1}^{k_1-v} \cdot t_{k_2}^v \) be an element of \( S_n \), which is presented in the standard OGS canonical form. Then, \( \pi = \prod_{u=k_1}^{k_2-1} t_{u-v}^u \cdot t_{u+1}^v \). By proposition 3.13,

\[ \text{norm}(t_{u}^{-v} \cdot t_{u+1}^{v}) = \prod_{r=0}^{v-1} s_{u-r}. \]

Therefore, \( \text{norm}(\prod_{u=k_1}^{k_2-1} t_{u-v}^u \cdot t_{u+1}^v) = \prod_{u=k_1}^{k_2-1} \prod_{r=0}^{v-1} s_{u-r}. \)

Therefore, obviously, \( \ell(\pi) = (k_2 - k_1) \cdot v. \) ■

The next Proposition describes the connection between the normal form of \( S_n \) in Coxeter generators, and the standard OGS canonical form of \( S_n \).
Proposition 3.16 Let $\pi \in S_n$, such that $\text{norm}(\pi) = \prod_{u=1}^{n-1} \prod_{r=0}^{y_u-1} s_{u-r}$, where $0 \leq y_u \leq u$ is a non-negative integer ($y_u = 0$ means $\text{norm}(\pi)$ does not contain any segment of decreasing indices starting with $s_u$). Then, the standard OGS canonical form of $\pi$ is:

$$
\prod_{j=2}^{n} t_j^{i_j},
$$

such that:

- If $y_j \leq y_{j-1}$, then $i_j = y_{j-1} - y_j$;
- If $y_j > y_{j-1}$, then $i_j = j + y_{j-1} - y_j$;
- $i_n = y_{n-1}$ (We may assume $y_n = 0$, and using $i_n = y_{n-1} - y_n$).

Proof. Let $\pi \in S_n$, such that $\text{norm}(\pi) = \prod_{u=1}^{n-1} \prod_{r=0}^{y_u-1} s_{u-r}$, where $0 \leq y_u \leq u$ is a non-negative integer. Then, by Proposition 3.16, the following holds:

- If $0 < y_u < u$, then $\prod_{r=0}^{y_u-1} s_{u-r} = t_u^{y_u - u} \cdot t_{u+1}^{y_u}$;
- If $y_u = u$, then $\prod_{r=0}^{y_u-1} s_{u-r} = \prod_{r=0}^{y_u-1} s_{u-r} = t_u^n$;

Thus, by substituting instead of $\prod_{r=0}^{y_u-1} s_{u-r}$, the standard presentation in canonical form, and by using $t_u^n = 1$ for $2 \leq u \leq n$, we get the desired result. \( \blacksquare \)

Example 3.17 Let $\pi \in S_9$ with the following normal form in Coxeter generators:

$$
\text{norm}(\pi) = s_1 \cdot (s_3 \cdot s_2) \cdot (s_4 \cdot s_3 \cdot s_2) \cdot (s_6 \cdot s_5 \cdot s_4 \cdot s_3) \cdot (s_7 \cdot s_6) \cdot (s_8 \cdot s_7).
$$

Then, $y_1 = 1, y_2 = 0, y_3 = 2, y_4 = 3, y_5 = 0, y_6 = 4, y_7 = 2, y_8 = 2$. Thus, the standard OGS canonical form of $\pi$ is the following:

$$
\pi = t_2^1 \cdot t_3^{3-2} \cdot t_4^{4-1} \cdot t_5^3 \cdot t_6^{6-4} \cdot t_7^2 \cdot t_8^0 \cdot t_9^2 = t_2 \cdot t_3^1 \cdot t_4^{3} \cdot t_5^{2} \cdot t_6^{6} \cdot t_7^{2} \cdot t_8^{1}.
$$

Conclusion 3.18 Let $\omega_n$ be the longest element of $S_n$ according to the Coxeter presentation (i.e. $\text{norm}(\omega_n) = \prod_{u=1}^{n-1} \prod_{r=0}^{y_u-1} s_{u-r}$). Then, the following hold:

- $\omega_n = \prod_{j=2}^{n} t_j^{j-1}$ in the presentation by the standard OGS canonical form;
- The subgroup of $S_n$ which is generated by $\omega_n$ and $\omega_{n-1}$ is the Coxeter group $I_2(n)$, where the following holds:

- $b = \prod_{j=2}^{n-1} t_j^{j-1}, \ a = t_n$,

form a standard OGS for $I_2(n)$, such that the presentation of $a^i$ and of $b \cdot a^i$ for $0 \leq i < n$, in the standard OGS canonical form of $S_n$ is as follows:

$$
a^i = t_i^n, \ b \cdot a^i = \omega_{n-1} = \prod_{j=2}^{n-1} t_j^{j-1} t_i^n,
$$

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\[ b' = \prod_{j=2}^{n} t_j^{j-1}, \quad a' = t_n^{-1}, \]

form a standard OGS for \( I_2(n) \), such that the presentation of \( a'^i \) and of \( b' \cdot a'^i \) for \( 0 \leq i < n \), in the standard OGS canonical form of \( S_n \) is as follows:

\[ a'^i = t_n^{n-i}, \quad b' \cdot a'^i = \omega_n = \prod_{j=2}^{n-1} t_j^{j-1} t_n^{n-1-i}. \]

**Proof.** Since \( \text{norm}(\omega_n) = \prod_{u=1}^{n-1} \prod_{v=0}^{k-1} s_u \cdot v \), the standard OGS canonical form of \( \omega_n \) comes directly from Proposition 3.16. By considering the permutation presentation of \( \omega_n \) and \( \omega_{n-1} \), we get by Remark 2.7, \( \omega_{n-1} \) and \( \omega_n \) generates the Coxeter group \( I_2(n) \). Then, by using Propositions 2.5, 2.2 we get the desired standard OGS canonical forms for \( I_2(n) \). 

### 3.3 Standard OGS elementar factorization, and the Coxeter length

In this subsection, we define standard OGS elementary elements, and the standard OGS elementary factorization of elements of \( S_n \) onto a product of standard OGS elementary factors, which we need to describe the Coxeter length of an arbitrary \( \pi \in S_n \), which is presented in the standard OGS canonical form. The standard OGS elementary factorization is used for the algorithm of the standard OGS canonical form of the inverse element \( \pi^{-1} \) for an arbitrary \( \pi \in S_n \) as well, which we will show in the next subsection.

**Definition 3.19** Let \( \pi \in S_n \), where \( \pi = \prod_{j=1}^{m} t_{k_j}^{i_{k_j}} \) is presented in the standard OGS canonical form, with \( i_{k_j} > 0 \) for every \( 1 \leq j \leq m \). Then, \( \pi \) is called standard OGS elementary element of \( S_n \), if

\[ \sum_{j=1}^{m} i_{k_j} \leq k_1. \]

**Definition 3.20** Let \( \pi = \prod_{j=1}^{m} t_{k_j}^{i_{k_j}} \) a standard OGS elementary element of \( S_n \) which is presented in the standard OGS canonical form, with \( i_{k_j} > 0 \) for every \( 1 \leq k \leq m \). Then, for every \( 1 \leq j \leq m \), \( \rho_j \), \( \varphi_j \), and \( \vartheta_j \) are defined to be as follows:

- \( \rho_j = \sum_{x=j}^{m} i_{k_x} \);
- \( \varphi_j = \sum_{x=1}^{j} i_{k_x} \);
- \( \vartheta_j = k_j - \rho_1 = k_j - \varphi_m. \)

**Remark 3.21** Let \( \pi = \prod_{j=1}^{m} t_{k_j}^{i_{k_j}} \) a standard OGS elementary element of \( S_n \) which is presented in the standard OGS canonical form, with \( i_{k_j} > 0 \) for every \( 1 \leq j \leq m \), let \( \rho_j \)
and \( q_j \) be integers as defined in Definition \textbf{3.20}. Then, by \( \prod \), the parameters \( \rho \) and \( \vartheta \) are connected to the major-index of \( \pi \), such that:

\[
\rho_1 = \vartheta_m = \text{maj} (\pi).
\]

Thus, we often write \( \text{maj} (\pi) \) instead of \( \rho_1 \) or instead of \( \vartheta_m \), which appears several times in the paper, especially in the algorithm of the presentation of the inverse element in the standard \( OGS \) canonical form. In particular, an element \( \pi = \prod_{j=1}^{m} i_{k_j} \), which is presented in the standard \( OGS \) canonical form, with \( i_{k_j} > 0 \) for \( 1 \leq j \leq m \), is a standard \( OGS \) elementary element if and only if

\[
\text{maj} (\pi) \leq k_1.
\]

**Proposition 3.22** Let \( \pi = \prod_{j=1}^{m} i_{k_j} \) a standard \( OGS \) elementary element of \( S_n \) which is presented in the standard \( OGS \) canonical form, with \( i_{k_j} > 0 \) for every \( 1 \leq j \leq m \), let \( \rho_j, \vartheta_j \) and \( \vartheta_j \) be integers as defined in Definition \textbf{3.20}. Then, \( 1 \leq j_1 < j_2 \leq m \) if and only if the following holds:

- \( \rho_{j_1} > \rho_{j_2} \);
- \( \vartheta_{j_1} < \vartheta_{j_2} \);
- \( \vartheta_{j_1} < \vartheta_{j_2} \).

Moreover, for every \( 1 \leq j \leq m \), we have:

- \( \rho_j > 0 \);
- \( \vartheta_j > 0 \);
- \( \vartheta \geq 0 \);
- \( \vartheta_j = 0 \) if and only if \( j = 1 \) and \( k_1 = \text{maj} (\pi) \);
- If \( \vartheta_1 = 0 \), then \( m > 1 \).

**Proof.** The proof comes directly from the definition of \( \rho_j, \vartheta_j \), and \( \vartheta_j \). □

**Theorem 3.23** Let \( \pi = \prod_{j=1}^{m} i_{k_j} \) be a standard \( OGS \) elementary element of \( S_n \), presented in the standard \( OGS \) canonical form, with \( i_{k_j} > 0 \) for every \( 1 \leq j \leq m \). Then, the following are satisfied:

\[
\pi = \begin{cases} 
(t_{k_1}^{\rho_1}, t_{k_2}^{\rho_2}, t_{k_3}^{\rho_3}, \ldots, t_{k_{m-1}}^{\rho_{m-1}} t_{k_m}^{\rho_m}) & k_1 > \rho_1 \\
(t_{k_1}^{\rho_2}, t_{k_2}^{\rho_2}, t_{k_3}^{\rho_3}, \ldots, t_{k_{m-1}}^{\rho_{m-1}} t_{k_m}^{\rho_m}) & k_1 = \rho_1
\end{cases}
\]

\[
\text{norm}(\pi) = \prod_{u=\rho_1}^{k_1-1} \prod_{r=0}^{\rho_1-1} s_{u-r} \cdot \prod_{u=k_1}^{k_2-1} \prod_{r=0}^{\rho_2-1} s_{u-r} \cdot \prod_{u=k_2}^{k_3-1} \prod_{r=0}^{\rho_3-1} s_{u-r} \cdots \prod_{u=k_{m-1}}^{k_m-1} \prod_{r=0}^{\rho_{m-1}} s_{u-r} \cdot \prod_{u=k_m}^{\rho_m} s_{u-r},
\]

for \( 1 \leq x \leq m \);
\[ \ell(\pi) = \sum_{j=1}^{m} k_j \cdot i_{k_j} - (i_{k_1} + i_{k_2} + \cdots + i_{k_m})^2 = \sum_{j=1}^{m} k_j \cdot i_{k_j} - (maj(\pi))^2; \]

- Every sub-word of \( \pi \) is a standard OGS elementary element too. In particular, for every two sub-words \( \pi_1 \) and \( \pi_2 \) of \( \pi \), such that \( \pi = \pi_1 \cdot \pi_2 \), it is satisfied:
\[
\ell(\pi) = \ell(\pi_1 \cdot \pi_2) < \ell(\pi_1) + \ell(\pi_2);
\]

- \( \ell(s_t \cdot \pi) = \begin{cases} 
\ell(\pi) - 1 & r = \sum_{j=1}^{m} k_{i_j}, \\
\ell(\pi) + 1 & r \neq \sum_{j=1}^{m} k_{i_j}.
\end{cases} \]

i.e., \( des(\pi) \) contains just one element, which means \( des(\pi) = \{maj(\pi)\} \).

**Proof.** Let \( \pi = \prod_{j=1}^{m} t^{i_{k_j}} \in S_n \), s.t. \( k_1 < k_2 < \ldots < k_m \), and \( i_{k_j} > 0 \) for \( 1 \leq j \leq m \), be a standard OGS elementary element. For \( 1 \leq x \leq m \), let \( \rho_x = \sum_{j=x}^{m} i_{k_j} \). Then, we get \( i_{k_x} = \rho_x - \rho_{x+1} \) for every \( 1 \leq x \leq m - 1 \) and \( \rho_m = \sum_{j=1}^{m} i_{k_j} \). Thus,

\[
\pi = \begin{cases} 
(t_{k_1}^{\rho_1} \cdot t_{k_2}^{\rho_2} \cdot t_{k_3}^{\rho_3} \cdots t_{k_{m-1}}^{\rho_{m-1}} \cdot t_{k_m}^{\rho_m}) & k_1 > \rho_1, \\
(t_{k_1}^{\rho_1} \cdot t_{k_2}^{\rho_2} \cdot t_{k_3}^{\rho_3} \cdots t_{k_{m-1}}^{\rho_{m-1}} \cdot t_{k_m}^{\rho_m}) & k_1 = \rho_1.
\end{cases}
\]

Now we turn to the proof of the second part of the proposition. By applying Proposition 3.11 for the normal form of the sub-word \( t_{k_1}^{\rho_1} \) and applying Proposition 3.15 for the normal form of every sub-word \( t_{k_x}^{-\rho_x+1} \cdot t_{k_{x+1}}^{\rho_{x+1}} \), we get the desired normal form for \( \pi \) according to 6. Now, we turn to the proof of the third part of the proposition. By the formula of \( norm(\pi) \):

\[
\ell(\pi) = (k_1 - \rho_1) \cdot \rho_1 + \sum_{j=2}^{m} (k_j - k_{j-1}) \cdot \rho_j \\
= \sum_{j=1}^{m-1} k_j \cdot (\rho_j - \rho_{j+1}) + k_m \cdot \rho_m - \rho_1^2 \\
= \sum_{j=1}^{m} k_j \cdot i_{k_j} - (maj(\pi))^2.
\]

Now, we turn to the proof of the forth part of the proposition. Assume \( \pi_1 \) and \( \pi_2 \) are two sub-words of \( \pi \), such that \( \pi = \pi_1 \cdot \pi_2 \). Then, the standard OGS presentation of \( \pi_1 \) and \( \pi_2 \) as follows:

\[
\pi_1 = \prod_{j=1}^{w-1} t_{k_j}^{i_{k_j}} \cdot t_{k_w}^{i_{k_w}} \quad \pi_2 = \prod_{j=w+1}^{m} t_{k_j}^{i_{k_j}}.
\]
where, $1 \leq w \leq m$, and $i'_{k_w} + i''_{k_w} = i_{k_w}$. Obviously,

$$maj(\pi_1) = \sum_{j=1}^{w-1} i_{k_j} + i'_{k_w} \leq maj(\pi) \leq k_1,$$

$$maj(\pi_2) = \sum_{j=w+1}^{m} i_{k_j} + i''_{k_w} \leq maj(\pi) \leq k_1 \leq k_w.$$ 

Thus, $\pi_1$ and $\pi_2$ are standard OGS elementary elements too. Since $\pi = \pi_1 \cdot \pi_2$, obviously, $maj(\pi) = maj(\pi_1) + maj(\pi_2)$. Thus,

$$\ell(\pi_1) + \ell(\pi_2) = \sum_{j=1}^{m} k_j \cdot i_{k_j} - (maj(\pi_1))^2 - (maj(\pi_2))^2 > \sum_{j=1}^{m} k_j \cdot i_{k_j} - (maj(\pi))^2 = \ell(\pi).$$

Now, we turn to the proof of the last part of the proposition. Recall, $t_j = s_1 \cdot s_2 \cdots s_{j-1}$, therefore

$$t_j(p) = \begin{cases} j & p = 1 \\ p - 1 & 2 \leq p \leq j \\ p & p \geq j + 1 \end{cases}$$

Hence, by using $\rho_1 \leq k_1$, the following holds:

- $\pi(p) = k_1 - \rho_1 + p$, for $1 \leq p \leq i_{k_1}$;
- $\pi(p) = k_q - \rho_1 + p$, for $2 \leq q \leq m$ and $\varrho_{q-1} + 1 \leq p \leq \varrho_q$;
- In particular, $\pi(\rho_1) = \pi(\varrho_m) = k_m$;
- $\pi(p) = p - \rho_1$, for $\rho_1 + 1 \leq p \leq k_1$;
- $\pi(p) = p - \rho_1$, for $2 \leq q \leq m$ and $k_{q-1} \leq p \leq k_q$;
- $\pi(p) = p$, for $k_m + 1 \leq p \leq n$.

We have $s_j$ shorten the length of $\pi$ if and only if $j \in des(\pi)$. By the observation of $\pi(p)$ for $r + 1 \leq p \leq n$ and by Proposition 3.22, $\pi(\rho_1 + 1) < \pi(\rho_1 + 2) < \cdots < \pi(n)$. Since $k_1 < k_2 < \cdots < k_m$, it follows $\pi(1) < \pi(2) < \cdots < \pi(\rho_1)$. Therefore, $des(\pi)$ contains $\rho_1$ only, and thus, $\ell(s_{\rho_1} \cdot \pi) = \ell(\pi) - 1$, and $\ell(s_j \cdot \pi) = \ell(\pi) + 1$, for $j \neq \rho_1$.

**Proposition 3.24** Let $\pi = \prod_{j=1}^{m} t_{k_j}$ be a standard OGS elementary element of $S_n$, presented in the standard OGS canonical form, with $i_{k_j} > 0$ for every $1 \leq j \leq m$. Let $t_x^x \in S_n$ such that $x \leq k_1 - maj(\pi)$. Then, the presentation of $\pi \cdot t_x^x$ in the standard OGS canonical form is as follows:

$$\pi \cdot t_x^x = t_{\rho_1 + x}^\rho \cdot t_{\rho_1 + x}^\rho \cdot \pi.$$
**Proof.** The proof is in induction on \( m \). For the case \( m = 1 \), the proposition holds by using Proposition 3.4 where considering \( t_{ij} q \cdot t_{ij} p \) in case \( q - i_q \geq p \). Assume in induction that the proposition holds for \( m' < m \), and we prove it for \( m' = m \). Thus, by the induction hypothesis:

\[
\prod_{j=2}^{m} t_{kj}^{i_{kj}} \cdot i_{x}^{i_{x}} = \prod_{j=2}^{m} t_{kj}^{i_{kj}} \cdot t_{p1+x}^{i_{p1+x}} \cdot \prod_{j=2}^{m} t_{kj}^{i_{kj}}.
\]

By our assumption, \( k_1 - p_1 \geq x \) and \( x > i_x \). Therefore, \( k_1 - i_{k1} > p_2 + i_x \). Hence, by using Proposition 3.4 for the case \( q - i_q \geq p \), where \( q = k_1, i_q = i_{k1}, p = p_2 + i_x \), and \( i_p = p_2 \), we have:

\[
t_{k1}^{i_{k1}} \cdot t_{p2+x}^{i_{p2+x}} = t_{p1+i_x}^{i_{p1+i_x}} \cdot t_{k1}^{i_{k1}}.
\]

The assumption \( k_1 - p_1 \geq x \) implies also \( k_1 - i_{k1} \geq p_2 + x \). Hence, by using again Proposition 3.4 for the case \( q - i_q \geq p \), where \( q = k_1, i_q = i_{k1}, p = p_2 + x \), and \( i_p = p_2 \), we have:

\[
t_{k1}^{i_{k1}} \cdot t_{p2+x}^{i_{p2+x}} = t_{p1+i_x}^{i_{p1+i_x}} \cdot t_{k1}^{i_{k1}}.
\]

Thus,

\[
\prod_{j=1}^{m} t_{kj}^{i_{kj}} \cdot i_{x}^{i_{x}} = \prod_{j=1}^{m} t_{kj}^{i_{kj}} \cdot t_{p1+i_x}^{i_{p1+i_x}} \cdot \prod_{j=1}^{m} t_{kj}^{i_{kj}}.
\]

Now, look at the sub-word

\[
t_{p1+i_x}^{i_{p1+i_x}} \cdot t_{k1}^{i_{k1}}.
\]

By using Proposition 3.5 for the case \( q - i_q = p \), where \( q = p_1 + i_x, i_q = p_2, p = i_{k1} + i_x \), and \( i_p = i_{k1} \), we have:

\[
t_{p1+i_x}^{i_{p1+i_x}} \cdot t_{k1}^{i_{k1}} = t_{p1+i_x}^{i_{p1+i_x}} \cdot t_{p1+i_x}^{i_{p1+i_x}};
\]

Thus,

\[
\prod_{j=1}^{m} t_{kj}^{i_{kj}} \cdot i_{x}^{i_{x}} = \prod_{j=1}^{m} t_{kj}^{i_{kj}} \cdot t_{p1+i_x}^{i_{p1+i_x}} \cdot \prod_{j=1}^{m} t_{kj}^{i_{kj}}.
\]

Therefore, the proposition holds for every \( m' \). ■

**Proposition 3.25** Let \( \pi \in S_n \), where

\[
\text{norm}(\pi) = \prod_{u=p}^{q} \prod_{r=0}^{y_{u}-1} s_{u-r},
\]

such that \( y_{u+1} \leq y_u \), for \( p \leq u \leq q - 1 \) (For a convenience, we omit from the formula in Definition 3.7 \( u < p \), and \( u > q \), where \( y_u = 0 \)), then \( \pi \) is a standard OGS elementary element, with the following standard OGS canonical form:

\[
\prod_{j=1}^{m} t_{kj}^{i_{kj}}
\]

with \( i_{kj} > 0 \) for every \( 1 \leq j \leq m \), satisfies the following conditions:
If \( yp = p \), then:
- \( p = \text{maj}(\pi) < k_1 \);
- \( k_j = u \), for \( 1 \leq j \leq m-1 \), if and only if \( y_u < y_{u-1} \), and \( i_{k_j} = y_{u-1} - y_u \);
- \( k_m = q + 1 \), and \( i_{k_m} = y_q \).

If \( yp < p \), then:
- \( k_1 = \text{maj}(\pi) = p \), and \( i_{k_1} = p - y_p \);
- \( k_j = u \), for \( 2 \leq j \leq m-1 \), if and only if \( y_u < y_{u-1} \), and \( i_{k_j} = y_{u-1} - y_u \);
- \( k_m = q + 1 \), and \( i_{k_m} = y_q \).

**Proof.** Consider \( \text{norm}(\pi) = \prod_{u=p}^{q} \prod_{r=0}^{p-1} s_{u-r} \). By our assumption, \( y_u \leq y_{u-1} \) for every \( u \) except \( u = p \). Thus, by Proposition 3.16, the standard OGS canonical form of \( \pi \) is \( \pi = \prod_{j=1}^{m} t_{k_j}^{i_j} \), where \( i_{k_j} > 0 \) for every \( 1 \leq j \leq m \), with the following conditions:
- Since \( y_u = 0 \) for \( u < p \), we have \( k_1 \geq p \). Therefore, in case \( y_p < p \), \( k_1 = p \) and \( i_{k_1} = y_p \). In case \( y_p = p \), \( k_1 \) is the smallest \( u > p \), such that \( y_u < y_{u-1} = p \), and \( i_{k_1} = p - y_u \);
- For \( 2 \leq j < m \), \( k_j = u \), such that \( y_u < y_{u-1} \) and \( i_{k_j} = y_{u-1} - y_u \);
- Since \( y_u = 0 \) for \( u > q \), we have \( k_m = y_{q+1} \) and \( i_{k_m} = y_q \);
- \( \text{maj}(\pi) = \sum_{j=1}^{m} i_{k_j} = p - y_p + \sum_{u=p}^{q} (y_u - y_{u-1}) + y_q = p \).

Now, we define the main definition of the paper, **Standard OGS elementary factorization**, which allows us to present every \( \pi \in S_n \) as a product of standard OGS elementary elements, by using the standard OGS of \( \pi \).

**Definition 3.26** Let \( \pi \in S_n \). Let \( z(\pi) \) be the minimal number, such that \( \pi \) can be presented as a product of standard OGS elementary elements, with the following conditions:

\[
\pi = \prod_{v=1}^{z(\pi)} \pi^{(v)}, \quad \text{where} \quad \pi^{(v)} = \prod_{j=1}^{m(v)} t_{h_j}^{i_j(v)},
\]

by the presentation in the standard OGS canonical form for every \( 1 \leq v \leq z(\pi) \) and \( 1 \leq j \leq m^{(v)} \) such that:
- \( i_j^{(v)} > 0 \);
- \( \sum_{j=1}^{m^{(v)}} i_j^{(v)} \leq h_1^{(1)} \), i.e., \( \text{maj}(\pi^{(1)}) \leq h_1^{(1)} \).

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- \( h^{(v-1)}_{m(v-1)} \leq \sum_{j=1}^{n} i^{(v)}_j \leq h^{(v)}_1 \) for \( 2 \leq v \leq z \)

i.e., \( h^{(v-1)}_{m(v-1)} \leq \text{maj} (\pi^{(v)}) \leq h^{(v)}_1 \) for \( 2 \leq v \leq z \).

Then, the mentioned presentation is called **Standard OGS elementary factorization** of \( \pi \). Since the factors \( \pi^{(v)} \) are standard OGS elementary elements, they are called standard OGS elementary factors of \( \pi \).

The next theorem shows some very important properties of the standard OGS elementary factorization, which is connected to the descents of \( \pi \), and we give an explicit formula for the Coxeter length of an arbitrary \( \pi \in S_n \) by using the standard OGS.

**Theorem 3.27** Let \( \pi = \prod_{j=1}^{m} t^{i_{kj}}_j \) be an element of \( S_n \) presented in the standard OGS canonical form, with \( i_{kj} > 0 \) for every \( 1 \leq j \leq m \). Consider the standard OGS elementary factorization of \( \pi \) with all the notations used in Definition 3.26. Then, the following properties hold:

- The standard OGS elementary factorization of \( \pi \) is unique, i.e., the parameters \( z(\pi), m^{(v)} \) for \( 1 \leq v \leq z(\pi) \), \( h^{(v)}_j \), and \( i^{(v)}_j \) for \( 1 \leq j \leq m^{(v)} \), are uniquely determined by the standard OGS canonical form of \( \pi \), such that:
  - For every \( h^{(v)}_j \) there exists exactly one \( k^{(v)}_j \) (where, \( 1 \leq j^{(v)} \leq m^{(v)} \)), such that \( h^{(v)}_j = k^{(v)}_j \);
  - If \( h^{(v)}_j = k^{(v)}_j \), for some \( 1 \leq v \leq z(\pi) \), \( 1 < j < m^{(v)} \), and \( 1 \leq j^{(v)} \leq m^{(v)} \), then \( i^{(v)}_j = i^{(v)}_j \);
  - If \( h^{(v_1)}_j = h^{(v_2)}_j \), where \( 1 \leq v_1 < v_2 \leq z(\pi) \), \( 1 \leq j_1 \leq m^{(v_1)} \), and \( 1 \leq j_2 \leq m^{(v_2)} \), then necessarily \( v_1 = v_2 - 1 \), \( j_1 = m^{(v_1)} \), \( j_2 = 1 \), and
    \[ h^{(v_1-1)}_{m^{(v_2-1)}} = h^{(v_2)}_1 = \text{maj} (\pi^{(v_2)}) = k^{(v_2)}_j, \]
    for some \( j^{(v_2)} \), such that \( i^{(v_2-1)}_{m^{(v_2-1)}} + i^{(v_2)}_1 = i^{(v_2)}_j \);

- \( \text{norm}(\pi) = \prod_{v=1}^{z(\pi)} \text{norm}(\pi^{(v)}) \);

- \( \ell(s_r \cdot \pi) = \begin{cases} \ell(\pi) - 1 & r = \sum_{j=1}^{m^{(v)}} i^{(v)}_j \quad \text{for} \quad 1 \leq v \leq z(\pi) \\ \ell(\pi) + 1 & \text{otherwise} \end{cases} \)

i.e.,

\[ \text{des}(\pi) = \bigcup_{v=1}^{z(\pi)} \text{des}(\pi^{(v)}) = \{ \text{maj}(\pi^{(v)}) \mid 1 \leq v \leq z(\pi) \}; \]
\[ \ell(\pi) = \sum_{v=1}^{z(\pi)} \ell(\pi^{(v)}) \]
\[ = \sum_{v=1}^{z(\pi)} \sum_{j=1}^{m(\pi)} h_j^{(v)} \cdot \iota_j^{(v)} - \sum_{v=1}^{z(\pi)} \left( \text{maj} \left( \pi^{(v)} \right) \right)^2 \]
\[ = \sum_{x=1}^{m} k_x \cdot i_{k_x} - \sum_{v=1}^{z(\pi)} \left( \text{maj} \left( \pi^{(v)} \right) \right)^2 \]
\[ = \sum_{x=1}^{m} k_x \cdot i_{k_x} - \sum_{v=1}^{z(\pi)} (c^{(v)})^2, \quad \text{where } c^{(v)} \in \text{des}(\pi). \]

**Proof.** Let \( \pi = \prod_{i=1}^{m} t_{i_{k_i}} \), such that, \( 2 \leq k_1 < k_2 < \ldots < k_m \leq n \), and \( i_{k_i} > 0 \) for every \( 1 \leq j \leq m \). We build the standard OGS elementary factorization of \( \pi \) in the following way. Let us start with the structure of \( \pi^{(z(\pi))} \). Consider the smallest integer \( r \), such that \( \sum_{x=m-r+1}^{m} i_{k_x} \geq k_{m-r} \), and fit \( m^{(z(\pi))} \) to be \( r \), and \( h_y^{(z(\pi))} \) to be \( k_{m-r+y} \) for every integer \( 1 \leq y \leq r \). We set \( t^{(z(\pi))}_y \) to be \( t_{i_{k_{m-r+y}}} \) for \( 2 \leq y \leq r \), and \( t^{(z(\pi))}_1 \) as follows: Let \( i_{k_{m-r+1}} \) be \( i_{k_{m-r+1}} \) in case \( \sum_{x=m-r+1}^{m} i_{k_x} \leq k_{m-r+1} \), and let \( t^{(z(\pi))}_1 \) be \( k_{m-r+1} - \sum_{x=m-r+1}^{m} i_{k_x} \) in case \( \sum_{x=m-r+1}^{m} i_{k_x} \geq k_{m-r+1} \). Now, we have \( \pi = \pi' \cdot \pi^{(z(\pi))} \), where

- in case \( \sum_{x=m-r+1}^{m} i_{k_x} \leq k_{m-r+1} \):

\[ \pi' = i_{k_1}^{i_{k_1}} \cdot i_{k_2}^{i_{k_2}} \cdots i_{k_{m-r}}^{i_{k_{m-r}}} \]
\[ \pi^{(z(\pi))} = t_{k_{m-r+1}}^{i_{k_{m-r+1}}} \cdots t_{k_{m-1}}^{i_{k_{m-1}}} \cdots t_{k_m}^{i_{k_m}}. \]

Thus,
\[ k_{m-r} < \text{maj} \left( \pi^{(z(\pi))} \right) = \sum_{x=m-r+1}^{m} i_{k_x} \leq k_{m-r+1}; \]

- in case \( \sum_{x=m-r+1}^{m} i_{k_x} \geq k_{m-r+1} \):

\[ \pi' = i_{k_1}^{i_{k_1}} \cdot i_{k_2}^{i_{k_2}} \cdots i_{k_{m-r}}^{i_{k_{m-r}}} \cdot t_{k_{m-r+1}}^{i_{k_{m-r+1}}-k_{m-r+1}+\sum_{x=m-r+1}^{m} i_{k_x}} \cdot t_{k_{m-r+1}}^{i_{k_{m-r+1}}} \cdots t_{k_{m-1}}^{i_{k_{m-1}}} \cdot t_{k_m}^{i_{k_m}}. \]

Thus,
\[ \text{maj} \left( \pi^{(z(\pi))} \right) = k_{m-r+1}. \]

Now, we look at \( \pi' \) and we construct \( \pi^{(z(\pi)-1)} \) from the terminal segment of \( \pi' \) in the same way as we constructed \( \pi^{(z(\pi))} \) from the terminal segment of \( \pi \), and we get \( \pi' = \pi'' \cdot \pi^{(z(\pi)-1)} \). Then, \( \pi = \pi'' \cdot \pi^{(z(\pi)-1)} \cdot \pi^{(z(\pi))} \). We continue in the same way, by defining \( \pi^{(x)} \) for every \( 1 \leq x \). Finally, we get \( \pi = \prod_{x=1}^{n} \pi^{(x)} \).
Since $\pi^{(v)}$ is a standard OGS elementary element for all $1 \leq v \leq z(\pi)$, such that $h_1^{(v)} \geq h_{m(v-1)}^{(v-1)}$ for every $2 \leq v \leq z(\pi)$, by using Theorem 3.23 we have

$$\text{norm}(\pi) = \prod_{v=1}^{z(\pi)} \text{norm}(\pi^{(v)}).$$

Now, we prove the next part of the theorem. The proof is in induction on $z(\pi)$. By the last part of Theorem 3.23, $\ell(s_r \cdot \pi^{(1)}) = \ell(\pi^{(1)}) - 1$ if and only if $r = \sum_{j=1}^{m^{(1)}} l_j^{(1)}$. Therefore, this part of the theorem holds for $v = 1$. Now, assume in induction, for every $v \leq z(\pi) - 1$:

$$\ell(s_r \cdot \pi) = \ell(s_r \cdot \prod_{v=1}^{z(\pi)-1} \pi^{(v)} \cdot \pi(z(\pi))) \leq \ell(s_r \cdot \prod_{v=1}^{z(\pi)-1} \pi^{(v)}) + \ell(\pi(z(\pi))),$$

for some $w \leq v$. Now, consider $v = z(\pi)$. Let $r \in maj \left(\pi^{(v^{'})}\right)$ for some $1 \leq v^{'} \leq z(\pi) - 1$. Then,

$$\ell(s_r \cdot \pi) = \ell(s_r \cdot \prod_{v=1}^{z(\pi)-1} \pi^{(v)} \cdot \pi(z(\pi))) \leq \ell(s_r \cdot \prod_{v=1}^{z(\pi)-1} \pi^{(v)}) + \ell(\pi(z(\pi))).$$

Since $r \in maj \left(\pi^{(v^{'})}\right)$ for some $1 \leq v^{'} \leq z(\pi) - 1$, by our induction hypothesis,

$$\ell(s_r \cdot \prod_{v=1}^{z(\pi)-1} \pi^{(v)}) = \ell(\prod_{v=1}^{z(\pi)-1} \pi^{(v)}) - 1.$$

Thus,

$$\ell(s_r \cdot \pi) \leq \ell(\prod_{v=1}^{z(\pi)-1} \pi^{(v)}) - 1 + \ell(\pi(z(\pi))) = \ell(\pi) - 1.$$

By using $\text{norm}(\pi) = \sum_{v=1}^{z(\pi)} \text{norm}(\pi^{(v)})$, and by the property that a product of an element by a Coxeter generator $s_r$ either shortens or lengthens the length of the element by 1, we conclude

$$\ell(s_r \cdot \pi) = \ell(\pi) - 1,$$

for every $r \in maj \left(\pi^{(v^{'})}\right)$ for some $1 \leq v^{'} \leq z(\pi) - 1$. Notice also, that the sum of all $r$ such that $\ell(s_r \cdot \pi) = \ell(\pi) - 1$ is the sum of the locations of all the descents of $\pi$, which is $maj(\pi)$. By [1],

$$maj(\pi) = \sum_{v=1}^{z(\pi)} \sum_{j=1}^{m^{(v)}} l_j^{(v)}$$

and

$$maj \left(\prod_{v=1}^{z(\pi)-1} \pi^{(v)}\right) = \sum_{v=1}^{z(\pi)-1} \sum_{j=1}^{m^{(v)}} l_j^{(v)}.$$
Let \( q \) be the number of descents of \( \pi \), which are not descents of \( \pi^{(v)} \) for any \( v < z(\pi) \), and denote by \( r_x \) (where, \( 1 \leq x \leq q \)) the descents such that \( r_x \in \text{des}(\pi) \) and \( r_x \notin \text{des}(\pi^{(v)}) \) for \( v < z(\pi) \). Then, the following holds:

\[
\sum_{x=1}^{q} r_x = \text{maj}(\pi) - \text{maj}\left(\prod_{v=1}^{z(\pi)-1} \pi^{(v)}\right) = \sum_{v=1}^{z(\pi)-1} i_j^{(v)} - \sum_{j=1}^{m(\pi^{(v)})} i_j^{(v)} = \sum_{j=1}^{m(\pi^{(v)})} \text{i}_j^{(z(\pi))} = \text{maj}\left(\pi^{(z(\pi))}\right),
\]

and by \( [6] \)

\[
s_{r_x} \cdot \left(\prod_{v=1}^{z(\pi)-1} \pi^{(v)}\right) \cdot \pi^{(z(\pi))} = \left(\prod_{v=1}^{z(\pi)-1} \pi^{(v)}\right) \cdot \tilde{\pi}_z(\pi)
\]

where, we get \( \tilde{\pi}_z(\pi) \) from \( \pi^{(z(\pi))} \) by omitting one Coxeter generator from a reduced presentation of it (i.e., \( s_{r_x} \) shortens by 1 the length of the segment \( \pi^{(z(\pi))} \) of \( \pi \)).

Notice, by Theorem 3.23, the first letter from left to right of \( \text{norm}(\pi^{(z(\pi))}) \) is \( s_{\text{maj}}(\pi^{(z(\pi))}) \). We already proved \( \text{norm}(\pi) = \text{norm}(\prod_{v=1}^{z(\pi)-1} \pi^{(v)}) \cdot \text{norm}(\pi^{(z(\pi))}) \). Therefore, the first letter from left to right of the segment \( \text{norm}(\pi^{(z(\pi))}) \) in \( \text{norm}(\pi) \) is \( s_{\text{maj}}(\pi^{(z(\pi))}) \) too. Thus, any \( r_x < \text{maj}(\pi^{(z(\pi))}) \) cannot shorten the length of the segment \( \text{norm}(\pi^{(z(\pi))}) \) in \( \text{norm}(\pi) \). Thus, \( q = 1 \), and \( r_1 = \text{maj}(\pi^{(z(\pi))}) \) is the only element in \( \text{des}(\pi) \) which is not in \( \text{des}(\pi^{(v)}) \), for \( 1 \leq v \leq z(\pi) - 1 \). That proves

\[
\text{des}(\pi) = \bigcup_{v=1}^{z(\pi)} \text{des}(\pi^{(v)}) = \{\text{maj}(\pi^{(v)}) \mid 1 \leq v \leq z(\pi)\}.
\]

Now, we prove the last part of the theorem, the explicit formula for length \( \pi \in S_n \). Since \( \text{norm}(\pi) = \prod_{v=1}^{z(\pi)} \text{norm}(\pi^{(v)}) \) by a former part of the proposition, and \( \text{norm}(\pi) \) is a reduced Coxeter presentation of \( \pi \), we get

\[
\ell(\pi) = \sum_{v=1}^{z(\pi)} \ell(\pi^{(v)}).
\]

Since \( \pi^{(v)} \) is a standard \( OGS \) elementary factor of \( \pi \) for every \( 1 \leq v \leq z(\pi) \), by Theorem 3.23

\[
\ell(\pi^{(v)}) = \sum_{j=1}^{m(v)} h_j^{(v)} \cdot i_j^{(v)} - \left(\text{maj}(\pi^{(v)})\right)^2.
\]

By the first part of the theorem,

\[
\sum_{v=1}^{z(\pi)} m^{(v)} \sum_{j=1}^{m(v)} h_j^{(v)} \cdot i_j^{(v)} = \sum_{x=1}^{m} k_x \cdot i_{k_x}.
\]
By a former part of the theorem, $c^{(v)} \in \text{des} \left( \pi \right)$ if and only if $c^{(v)} = \text{maj} \left( \pi^{(v)} \right)$ for some $1 \leq v \leq z(\pi)$. Therefore, we get

$$
\ell(\pi) = \sum_{x=1}^{m} k_x \cdot i_{k_x} - \sum_{v=1}^{z(\pi)} \left( c^{(v)} \right)^2, \quad \text{where } c^{(v)} \in \text{des} \left( \pi \right).
$$

**Example 3.28** Let $\pi = t_3 \cdot t_4^2 \cdot t_6^3 \cdot t_7^4 \cdot t_9^3 \cdot t_{10}^2$. Then, the standard OGS elementary factors of $\pi$ are as follows:

$$
\begin{align*}
\pi^{(1)} &= t_3 \cdot t_4^2, \\
\pi^{(2)} &= t_6^3 \cdot t_7,
\end{align*}
$$

and

$$
\pi^{(3)} = t_7^2 \cdot t_9^3 \cdot t_{10}^2.
$$

Now we compute $\text{norm}(\pi^{(1)})$, $\text{norm}(\pi^{(2)})$, and $\text{norm}(\pi^{(3)})$ by using Theorem 3.23:

$$
\begin{align*}
\text{norm}(\pi^{(1)}) &= s_3 \cdot s_2, \\
\text{norm}(\pi^{(2)}) &= (s_5 \cdot s_4 \cdot s_3 \cdot s_2 \cdot s_1) \cdot s_6, \\
\text{norm}(\pi^{(3)}) &= (s_7 \cdot s_6 \cdot s_5 \cdot s_4 \cdot s_3) \cdot (s_8 \cdot s_7 \cdot s_6 \cdot s_5 \cdot s_4) \cdot (s_9 \cdot s_8).
\end{align*}
$$

Therefore,

$$
\text{norm}(\pi) = \text{norm}(\pi^{(1)}) \cdot \text{norm}(\pi^{(2)}) \cdot \text{norm}(\pi^{(3)})
$$

$$
= [s_3 \cdot s_2] \cdot [(s_5 \cdot s_4 \cdot s_3 \cdot s_2 \cdot s_1) \cdot s_6] \cdot [(s_7 \cdot s_6 \cdot s_5 \cdot s_4 \cdot s_3) \cdot (s_8 \cdot s_7 \cdot s_6 \cdot s_5 \cdot s_4) \cdot (s_9 \cdot s_8)].
$$

$$
\pi^{(1)} = [1; 3; 4; 2], \quad \pi^{(2)} = [2; 3; 4; 5; 7; 1; 6], \quad \pi^{(3)} = [1; 2; 5; 6; 7; 9; 10; 3; 4; 8], \\
\pi = \pi^{(1)} \cdot \pi^{(2)} \cdot \pi^{(3)} = [2; 6; 7; 5; 10; 1; 9; 3; 4; 8].
$$

$$
\begin{align*}
\text{des} \left( \pi^{(1)} \right) &= \left\{ \text{maj} \left( \pi^{(1)} \right) \right\} = \{3\}, \quad \text{des} \left( \pi^{(2)} \right) &= \left\{ \text{maj} \left( \pi^{(2)} \right) \right\} = \{5\}, \\
\text{des} \left( \pi^{(3)} \right) &= \left\{ \text{maj} \left( \pi^{(3)} \right) \right\} = \{7\}, \\
\text{des} \left( \pi \right) &= \text{des} \left( \pi^{(1)} \right) \cup \text{des} \left( \pi^{(2)} \right) \cup \text{des} \left( \pi^{(3)} \right) = \{3, 5, 7\}.
\end{align*}
$$

$$
\ell(\pi) = \ell(\pi^{(1)}) + \ell(\pi^{(2)}) + \ell(\pi^{(3)})
$$

$$
= (3 \cdot 1 + 4 \cdot 2 - 3^2) + (6 \cdot 4 + 7 \cdot 1 - 5^2) + (7 \cdot 2 + 9 \cdot 3 + 10 \cdot 2 - 7^2)
$$

$$
= (3 \cdot 1 + 4 \cdot 2 + 6 \cdot 4 + 7 \cdot 3 + 9 \cdot 3 + 10 \cdot 2) - (3^2 + 5^2 + 7^2)
$$

$$
= 20.
$$
In Theorem 3.27 we have shown a very strong connection between the standard OGS elementary factorization, the Coxeter length function, and the descent set of an element $\pi \in S_n$. Therefore, in the next proposition, we show how we conclude the normal form of the elementary factors in the standard OGS elementary factorization, from a normal form of a given $\pi \in S_n$.

**Proposition 3.29** Let $\pi \in S_n$, where

$$\text{norm}(\pi) = \prod_{u=p}^{q} \prod_{r=0}^{u-1} s_{u-r}.$$  

Denote by $u_1, u_2, \ldots, u_z$ the values of $u$ such that

- $u_1 = p$;
- $y_{u_v} > y_{u_v-1}$, for $1 \leq v \leq z$;
- $u_1 < u_2 < \cdots < u_z$.

Moreover, let $u_{z+1} = q + 1$. Then, $\pi = \prod_{v=1}^{z} \pi^{(v)}$ is a standard OGS elementary factorization of $\pi$, such that

$$\text{norm}(\pi^{(v)}) = \prod_{u=u_v}^{u_{v+1}-1} \prod_{r=0}^{u-1} s_{u-r}.$$  

**Proof.** Consider

$$\text{norm}(\pi^{(v)}) = \prod_{u=u_v}^{u_{v+1}-1} \prod_{r=0}^{u-1} s_{u-r},$$

for $1 \leq v \leq z(\pi)$. Since $y_{u_j+1} \leq y_{u_j}$, for every $u_j \leq j < u_{v+1} - 1$, by Proposition 3.25 we have that $\pi^{(v)}$ is a standard OGS elementary element, with $\text{maj}(\pi^{(v)}) = u_v$. Assume,

$$\text{norm}(\pi^{(v-1)}) = \prod_{u=u_v}^{u_{v-1}-1} \prod_{r=0}^{u-1} s_{u-r},$$

where, the presentation of $\pi^{(v-1)}$ in standard OGS canonical form is

$$\pi^{(v-1)} = \prod_{j=1}^{m^{(v-1)}} t_{h^{(v-1)}_j}^{j^{(v-1)}},$$

for some $m^{(v-1)}$, $1 \leq j \leq m^{(v-1)}$, $u_{v-1} \leq h^{(v-1)}_j \leq u_v$, and $0 < j^{(v-1)}_j < h^{(v-1)}_j$. Since by Proposition 3.25

$$\text{maj}(\pi^{(v)}) = u_v \geq h^{(v-1)}_j$$

for every $1 \leq j \leq m^{(v-1)}$, we have

$$h^{(v-1)}_{m^{(v-1)}} \leq \text{maj}(\pi^{(v)}) \leq h^{(v)}_1.$$
Therefore, by Definition 3.26, $\pi = \prod_{v=1}^{z(\pi)} \pi^{(v)}$ is a standard OGS elementary factorization of $\pi$. 

The next three propositions, show some other interesting properties of the standard OGS elementary factorization, concerning exchange properties, parabolic subgroups, and elementary factors, which commute.

**Proposition 3.30** Let $\sigma \in S_n$, presented in standard OGS elementary factorization, with all the notations used in Definition 3.26. Let $t_q^{i_q} \in S_n$ such that $q - i_q \geq t_{m(z(\pi))}$. Then, the following holds:

$$t_q^{i_q} \cdot \pi = \prod_{v=1}^{z(\pi)} \left( \prod_{j=1}^{m^{(v)}} t_j^{i_{(v)}+i_q} \cdot \prod_{j=1}^{m^{(v)}} t_j^{i_{(v)}+i_q} \right) \cdot t_q^{i_q} \cdot \pi. $$

**Proof.** The proof is in double induction on $z(\pi)$ and on $m^{(v)}$, for $1 \leq v \leq z(\pi)$. First, consider the case $z(\pi) = 1$ (i.e., $\pi$ is a standard OGS elementary element). For the case of $z(\pi) = 1$ and $m^{(1)} = 1$, the proposition holds by using Proposition 3.4, where considering $t_q^{i_q} \cdot t_p^{i_p}$ in case $q - i_q \geq p$. Assume in induction that the proposition holds for $m^{(1)} < r$ for some $r$, and we prove it for $m^{(1)} = r$. Therefore, by the induction hypothesis we have:

$$t_q^{i_q} \cdot \pi = \prod_{j=1}^{r-1} t_j^{i_{(1)}+i_q} \cdot \prod_{j=1}^{r-1} t_j^{i_{(1)}+i_q} \cdot t_q^{i_q} \cdot \pi. $$

By Proposition 3.4, where considering $t_q^{i_q} \cdot t_p^{i_p}$ in case $q - i_q \geq p$, 

$$t_q^{i_q} \cdot t_p^{i_{(1)}} = t_q^{i_q} \cdot t_p^{i_{(1)}} \cdot t_q^{i_q} \cdot t_p^{i_{(1)}}. $$

Notice, $\prod_{j=1}^{r-1} t_j^{i_{(1)}}$ is a sub-word of $\pi$, which is a standard OGS elementary element.

Therefore, by Theorem 3.23, $\prod_{j=1}^{r-1} t_j^{i_{(1)}}$ is a standard OGS elementary element too.

Then, by using Proposition 3.24 for $\pi = \prod_{j=1}^{r-1} t_j^{i_{(1)}+i_q}$, $x = i_q + i_{(1)}$, and $i_x = i_q$, we get:

$$\prod_{j=1}^{r-1} t_j^{i_{(1)}+i_q} \cdot t_{(1)}^{i_{(1)}+i_q} = \prod_{j=1}^{r-1} t_j^{i_{(1)}+i_q} \cdot \prod_{j=1}^{r-1} t_j^{i_{(1)}+i_q} \cdot \prod_{j=1}^{r-1} t_j^{i_{(1)}+i_q} \cdot \prod_{j=1}^{r-1} t_j^{i_{(1)}+i_q} \cdot t_q^{i_q} \cdot \pi. $$

Thus, we get

$$t_q^{i_q} \cdot \pi = t_q^{i_q} \cdot \prod_{j=1}^{r-1} t_j^{i_{(1)}+i_q} \cdot t_q^{i_q} \cdot \pi.$$
in case \( \pi \) is a standard OGS elementary element. Now, assume in induction the proposition holds for every \( z < z(\pi) \) and we prove it for \( z = z(\pi) \). Then, 

\[
t^q \cdot \pi = \prod_{v=1}^{z(\pi)-1} t^{i q}_{v \rho_1^{(v)} + i_q} \cdot \prod_{j=1}^{m^{(v)}} t^{j}_{h_j^{(v)} + i_q} \cdot t^q \cdot \pi^{(z(\pi))}.
\]

by the induction hypothesis. Since \( \pi^{(z(\pi))} \) is a standard OGS elementary element, we have 

\[
t^q \cdot \pi^{(z(\pi))} = t^q \cdot \prod_{j=1}^{m^{(z(\pi))}} t^{j}_{h_j^{(z(\pi))} + i_q} \cdot t^q.
\]

Therefore, we get 

\[
t^q \cdot \pi = \prod_{v=1}^{z(\pi)} t^{i q}_{v \rho_1^{(v)} + i_q} \cdot \prod_{j=1}^{m^{(v)}} t^{j}_{h_j^{(v)} + i_q} \cdot t^q.
\]

\[\square\]

**Proposition 3.31** Let \( \pi \in S_n \), presented in standard OGS elementary factorization, with all the notations used in Definition 3.26. Then, the following properties are satisfied:

- The set of elements \( \pi \) such that \( \text{maj} (\pi^{(v)}) = h_1^{(v)} \) for all \( 1 \leq v \leq z(\pi) \), is the parabolic subgroup generated by \( s_2, \ldots, s_{n-1} \) (i.e., the permutations of \( S_n \), with 1 as a fixed point);

- The set of elements \( \pi \) such that \( \text{maj} (\pi^{(v)}) = h_1^{(v)} \) and \( i_1^{(v)} \geq d \) for all \( 1 \leq v \leq z(\pi) \), is the parabolic subgroup generated by \( s_d+1, \ldots, s_{n-1} \) for some \( 1 \leq d \leq n-2 \) (i.e., the permutations of \( S_n \), with \( j \) as fixed point for every \( 1 \leq j \leq d \)).

**Proof.** Let \( \pi = \prod_{v=1}^{z(\pi)} \pi^{(v)} \), where \( \pi^{(v)} \) satisfies the conditions of Definition 3.26, \( h_1^{(v)} = \text{maj} (\pi^{(v)}) \), and \( i_1^{(v)} \geq d \) for some \( d \geq 1 \) and for every \( 1 \leq v \leq z(\pi) \). Let 

\[
\pi' = \prod_{v=1}^{z(\pi)} \pi^{(v)}.
\]

such that:

\[
\pi' = \prod_{v=1}^{z(\pi)} \left( t^{i_1^{(v)} - d}_{h_1^{(v)} - d} \cdot \prod_{j=1}^{m^{(v)}} t^{j}_{h_j^{(v)} - d} \right).
\]

Obviously, for every \( 1 \leq v \leq z(\pi) \):

\[
\text{maj} (\pi^{(v)}) = \sum_{j=1}^{m^{(v)}} i_j^{(v)} - d = \text{maj} (\pi^{(v)}) - d = h_1^{(v)} - d < h_2^{(v)} - d
\]

and for every \( 2 \leq v \leq z(\pi) \):

\[
\text{maj} (\pi^{(v)}) = \sum_{j=1}^{m^{(v)}} i_j^{(v)} - d \geq h_1^{(v-1)} - d.
\]
Since for every $1 \leq v \leq z(\pi)$, $i^{(v)}_1 \geq d$, we have \[ \begin{cases} i^{(v)}_1 - d = 0 \quad & i^{(v)}_1 = d \\ i^{(v)}_1 - d > 0 \quad & i^{(v)}_1 > d \end{cases} \] Thus, by defining $m^{(v)}$, $h'_j^{(v)}$, and $t'_j^{(v)}$ for $1 \leq v \leq z(\pi)$ and $1 \leq j \leq m^{(v)}$ in the following way:

- In case $i^{(v)}_1 > d$: $m^{(v)} = m^{(v)}$, $h'_j^{(v)} = h_j^{(v)} - d$, $i^{(v)}_1 = i^{(v)} - d$, $t'_j^{(v)} = t_j^{(v)}$ for $2 \leq j \leq m^{(v)}$;
- In case $i^{(v)}_1 = d$: $m^{(v)} = m^{(v)} - 1$, $h'_j^{(v)} = h_j^{(v)} - d$, $t'_j^{(v)} = t_{j+1}^{(v)}$

notice that $\pi' = \prod_{v=1}^{z(\pi)} \pi^{(v)}$ is a standard OGS elementary factorization of $\pi'$ with the elementary factors $\pi_m^{(v)} = \prod_{j=1}^{r'_v} t_{j}^{(v)}$. Thus, by Proposition 3.31, $\pi = t_q \cdot \pi' \cdot t_q^{-d}$ for some $q$ such that $q - d \geq h'(z(\pi)) = h(z(\pi)) - d$ (i.e., $q \geq h(z(\pi))$). Therefore, $\pi$ is a conjugate of $\pi'$ by $t_q^{-d}$. Hence, by choosing $q = h'_m(z(\pi)) + d = h_m(z(\pi))$, we have $t_q \cdot \pi' \cdot t_q^{-d} = \pi \in \langle s_{d+1}, \ldots, s_{n-1} \rangle$. □

**Proposition 3.32** Let $\pi_1 = \prod_{j=1}^{t_{k_j}^{(v)}} t_k^{(v)}$ and $\pi_2 = \prod_{j=1}^{w} t_k^{(v)}$ be two elements of $S_n$ which are presented in the standard OGS canonical form, with the following properties:

- $maj(\pi_2) = \sum_{j=1}^{w} i_{r_j} = r_1$
- $i_{r_1} \geq k_m$

Then, $\pi_1 \cdot \pi_2 = \pi_2 \cdot \pi_1$ (i.e., $\pi_1$ and $\pi_2$ are commute).

**Proof.** Since $maj(\pi_2) = r_1$, $\pi_2$ is a standard OGS elementary element. Therefore, by Proposition 3.31, $\pi_2$ is in the parabolic subgroup $\langle s_{r_1+1}, \ldots, s_{n-1} \rangle$. Since $\pi_1 = \prod_{j=1}^{t_{k_j}^{(v)}}$, by the presentation in standard OGS canonical form, we get by the definition of $t_k$, for $1 \leq j \leq m$, that $\pi_1$ is in the parabolic subgroup $\langle s_1, \ldots, s_{k_m-1} \rangle$. Since $r_1 \geq k_m$, we have $(i_{r_1} + 1) - (k_m - 1) \geq 2$. Therefore, every element in $\langle s_{r_1+1}, \ldots, s_{n-1} \rangle$ commutes with every element in $\langle s_1, \ldots, s_{k_m-1} \rangle$. □

### 3.4 The standard OGS canonical form of inverse elements in $S_n$

In this subsection, we give a formula for the standard OGS canonical form of $\pi^{-1}$, the inverse element of $\pi$. Obviously, $(t_k^{(v)})^{-1} = t_k^{-i_k}$ for every $2 \leq k \leq n$. Therefore, we first consider $\pi^{-1}$ for elements $\pi \in S_n$, which have the form $t_k^{(v_1)} \cdot t_{k_2}^{(v_2)}$, where $k_1 < k_2$. We get the following proposition.
Proposition 3.33 \[ t_{i_{k_1}} \cdot t_{i_{k_2}} \], where \( k_1 < k_2 \). Then, the following holds:

\[
\pi^{-1} = \begin{cases} 
  t_{k_1-k_2} \cdot t_{k_2} & i_{k_2} > k_1 \\
  t_{k_2-k_1} \cdot t_{k_2} & i_{k_2} = k_1 \\
  t_{k_1-k_2} \cdot t_{k_2} & i_{k_2} = k_1 - i_{k_1} < i_{k_2} < k_1 \\
  t_{i_{k_1}} \cdot t_{k_2} & i_{k_2} = k_1 - i_{k_1} = i_{k_2} \\
  t_{i_{k_1}} \cdot t_{k_2} & i_{k_2} = k_1 - i_{k_1} < i_{k_2} = k_1 \\
  t_{i_{k_1}} \cdot t_{k_2} & i_{k_2} = k_1 - i_{k_1} = i_{k_2} \\
  t_{i_{k_1}} \cdot t_{k_2} & i_{k_2} = k_1 - i_{k_1} < i_{k_2} = k_1 \\
\end{cases}
\]

Proof. Let \( \pi = t_{i_{k_1}} \cdot t_{i_{k_2}} \). Then, by Proposition \[ \text{Proposition 3.33} \], \( \pi \) can be presented in the dual-standard OGS canonical form. Now, by considering the inverse, we get the presentation of \( \pi^{-1} \) in the standard OGS canonical form.

Now, we introduce an algorithm for the presentation of \( \pi^{-1} \) in the standard OGS canonical form for an arbitrary \( \pi \in S_n \), which is also presented in the standard OGS canonical form. In the first step we consider the case where \( \pi \) is a standard OGS elementary element of \( S_n \).

Theorem 3.34 Let \( \pi = \prod_{j=1}^{m} t_{i_{k_j}} \), a standard OGS elementary element of \( S_n \), presented in the standard OGS canonical form (i.e., \( \text{maj} (\pi) \leq k_1 < k_2 < \cdots < k_m \)), with \( i_{k_j} > 0 \) for every \( 1 \leq j \leq m \). For every \( 1 \leq j \leq m \), let \( \rho_j, \vartheta_j \), and \( \vartheta_j \) integers as defined in Definition \[ \text{Definition 3.30} \] and denote \( \kappa_{-j} = \vartheta_{j-1} + \vartheta_j = k_j - \rho_j \), \( \kappa_{+j} = \vartheta_j + \vartheta_j = k_j - \rho_{j+1} \). (We consider \( \vartheta_0 = \rho_{m+1} = 0 \). Then,

\[
\pi^{-1} = t_{\kappa_{-1}} \cdot (t_{\kappa_{-2}} \cdot t_{\kappa_{-3}}) \cdots (t_{\kappa_{-m+1}} \cdot t_{\kappa_{-m+2}}) \cdot (t_{\kappa_{+1}} \cdot t_{\kappa_{+2}}) \cdots (t_{\kappa_{+m-1}} \cdot t_{\kappa_{+m+1}}).
\]

In the specific case, where \( k_1 = \rho_1 = \text{maj} (\pi) \), we have \( \kappa_{-1} = \vartheta_1 = 0 \), thus:

\[
\pi^{-1} = (t_{\kappa_{-1}} \cdot t_{\kappa_{+1}}) \cdot (t_{\kappa_{-2}} \cdot t_{\kappa_{+2}}) \cdots (t_{\kappa_{-m+1}} \cdot t_{\kappa_{-m+2}}) \cdot (t_{\kappa_{+1}} \cdot t_{\kappa_{+m+1}}).
\]

Therefore,

\[
des (\pi^{-1}) = \begin{cases} 
  \{ \kappa_{-j} \mid 1 \leq j \leq m \} & k_1 > \text{maj} (\pi) \\
  \{ \kappa_{-j} \mid 2 \leq j \leq m \} & k_1 = \text{maj} (\pi). 
\end{cases}
\]

Proof. Let \( \pi = \prod_{j=1}^{m} t_{i_{k_j}} \), a standard OGS elementary element of \( S_n \) is presented in the standard OGS canonical form, with \( i_{k_j} > 0 \) for every \( 1 \leq j \leq m \). Then, by the first part of Theorem \[ \text{Theorem 3.23} \],

\[
\pi = \begin{cases} 
  t_{i_{k_1}} \cdot (t_{k_1}^{\rho_2} \cdot t_{k_2}^{\rho_3}) \cdots (t_{k_{m-1}}^{\rho_m} \cdot t_{k_m}) & k_1 > \rho_1 \\
  (t_{k_1}^{\kappa_{+1}} \cdot t_{k_2}^{\kappa_{+2}}) \cdots (t_{k_{m-1}}^{\kappa_{+m-1}} \cdot t_{k_m}^{\rho_m}) & k_1 = \rho_1 
\end{cases}
\]

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Now, we prove the formula for $\pi^{-1}$ in induction on $m$. If $m = 1$ and $k_1 > \rho_1$, then $\pi = t_{k_1}^{\rho_1}$. Therefore, $\kappa_{+1} = k_1$, $\rho_1 = i_{k_1}$, $\vartheta_1 = k_1 - i_{k_1}$ and obviously,$\kappa_- = k_1 - \vartheta_1 = k_1 - k_1$, $\kappa_- = k_1 - \vartheta_1$, $\kappa_{+1} = k_1$, $\rho_1 = i_{k_1}$, $\vartheta_1 = k_1 - i_{k_1}$ and obviously,$\kappa_{-1} = k_1 - \vartheta_1 = k_1 - k_1$, $\kappa_{+1} = k_1$, $\rho_1 = i_{k_1}$, $\vartheta_1 = k_1 - i_{k_1}$ and obviously,$\pi^{-1} = t_{k_1}^{\vartheta_1} = t_{k_1}^{\vartheta_1}$ in this case. If $m = 2$ and $k_1 = \rho_1$, then $\pi = t_{k_1}^{\vartheta_1} t_{k_2}^{\vartheta_2}$.

Therefore, $i_{k_1} = k_1 - i_{k_2} = \kappa_{+1} - \kappa_{-1} = k_1 - k_2$, and $\kappa_{+2} = k_2$. By Proposition 3.33, $\pi^{-1} = t_{k_1}^{\vartheta_1} t_{k_2}^{\vartheta_2}$.

Thus, the theorem holds in this case. Now, assume the theorem holds for $m' < m$, and we prove it for $m' = m$. Consider $\pi = \pi' \cdot \pi_m$

where,

$$\pi' = \begin{cases} t_{k_1}^{\rho_1} \cdot (t_{k_1}^{\rho_2} \cdot (t_{k_2}^{\rho_2} \cdot (t_{k_3}^{\rho_2} \cdot \cdots (t_{k_{m-2}}^{\rho_2} \cdot t_{k_{m-1}}^{\rho_2})))) & \text{if } k_1 > \rho_1 \\ (t_{k_1}^{\rho_2} \cdot (t_{k_2}^{\rho_2} \cdot (t_{k_3}^{\rho_2} \cdot \cdots (t_{k_{m-2}}^{\rho_2} \cdot t_{k_{m-1}}^{\rho_2})))) & \text{if } k_1 = \rho_1 \end{cases}$$

and

$$\pi_m = t_{m-1}^{\rho_1} \cdot \pi_m \cdot t_{m-1}^{\rho_1} \cdot \pi_m.$$

Then,

$$\pi^{-1} = \pi_m^{-1} \cdot \pi_m^{-1} \cdot \pi_m^{-1}.$$

By our induction hypothesis,

$$\pi'_{m-1} = t_{k_{m-1}}^{\rho_1} \cdot (t_{k_{m-2}}^{\rho_2} \cdot (t_{k_{m-3}}^{\rho_2} \cdot \cdots (t_{k_{m-2}}^{\rho_2} \cdot t_{k_{m-1}}^{\rho_2}))))$$

in case $k_1 > \rho_1$.

$$\pi^{-1}_{m-1} = t_{k_{m-1}}^{\rho_1} \cdot (t_{k_{m-2}}^{\rho_2} \cdot (t_{k_{m-3}}^{\rho_2} \cdot \cdots (t_{k_{m-2}}^{\rho_2} \cdot t_{k_{m-1}}^{\rho_2}))))$$

in case $k_1 = \rho_1$.

By Proposition 3.33, $\pi_{m-1}^{-1} = t_{k_{m-1}}^{\rho_1} \cdot t_{k_{m-1}}^{\rho_1} \cdot \pi_{m-1}^{\rho_1}.$

Since $\kappa_{-1} = \vartheta_{j-1} + \vartheta_j$ and $\kappa_{+1} = \vartheta_{j} + \vartheta_{j}$, the following are satisfied:

- $\max_{j} (\pi_m^{-1}) = \kappa_{+1} = \kappa_{m-1} + k_m - k_{m-1} = \kappa_{-1};$
- $\kappa_{-1} > \kappa_{+1}$ for $1 \leq x \leq m - 2;$
- $\kappa_{-1} > \kappa_{m-1};$
- $\kappa_{+1} > \kappa_{m-1}$ for $1 \leq x \leq m - 2;$
- $\kappa_{+1} > \kappa_{m-1};$

Therefore, by Proposition 3.32, $\pi_{m-1}^{-1}$ commutes with the following segments of $\pi'_{m-1}$:

- $t_{k_{m-1}}^{\rho_1}$;
- $t_{k_{m-2}}^{\rho_2} \cdot t_{k_{m-1}}^{\rho_2}$, for $2 \leq x \leq m - 2;$

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Thus,
\[
\pi^{-1} = \pi_{m}^{-1} \cdot \pi_{m}^{-1} = \theta_{\kappa_{+}, 1} \cdot \left( t_{\kappa_{+}, 1} \cdot \theta_{\kappa_{+}, 2} \right) \cdot \cdots \cdot \left( t_{\kappa_{+}, m-3} \cdot \theta_{\kappa_{+}, m-2} \right) \cdot t_{\kappa_{+}, m-1} \cdot \left( \pi_{m}^{-1} \cdot t_{\kappa_{m-1}}^{\varphi_{m}} \right),
\]
in case \( k_{1} > \rho_{1} \), and
\[
\pi^{-1} = \pi_{m}^{-1} \cdot \pi_{m}^{-1} = \left( t_{\kappa_{-}, 1} \cdot \theta_{\kappa_{-}, 2} \right) \cdot \cdots \cdot \left( t_{\kappa_{-}, m-2} \cdot \theta_{\kappa_{-}, m-1} \right) \cdot \left( \pi_{m}^{-1} \cdot t_{\kappa_{m-1}}^{\varphi_{m}} \right),
\]
in case \( k_{1} = \rho_{1} \). Now, notice,
\[
\pi_{m}^{-1} \cdot t_{\kappa_{m-1}}^{\varphi_{m-1}} = t_{\kappa_{m-1}}^{\varphi_{m-1}} \cdot \varphi_{m-1} \cdot t_{\kappa_{m-1}}^{\varphi_{m-1}}.
\]
Therefore, by Proposition 3.35 using the exchange law \( t_{q}^{\varphi} \cdot t_{p}^{\varphi} \), for the case \( q - i_{q} = p \), where \( q = k_{m}, p = k_{m-1}, i_{q} = k_{m} - k_{m-1} \), and \( i_{p} = \theta_{m-1} \):
\[
t_{k_{m}}^{m-1} \cdot t_{\kappa_{m-1}}^{\varphi_{m-1}} = t_{k_{m}}^{m-1} \cdot \varphi_{m-1} \cdot t_{k_{m}}^{m-1}.
\]
Again by Proposition 3.35 now using the exchange law \( t_{q}^{\varphi} \cdot t_{p}^{\varphi} \), for the case \( q - i_{q} = i_{p} \), where \( q = \kappa_{-m}, p = \theta_{m}, i_{q} = \kappa_{-m} + 1, \) and \( i_{p} = k_{m} - k_{1} \):
\[
t_{\kappa_{m-1}}^{m} \cdot t_{\kappa_{m-1}}^{m-1} \cdot t_{\kappa_{m-1}}^{m-1} \cdot \varphi_{m-1} \cdot t_{\kappa_{m-1}}^{m-1} = t_{\kappa_{m-1}}^{m} \cdot \varphi_{m-1} \cdot t_{\kappa_{m-1}}^{m-1} \cdot \varphi_{m-1} \cdot t_{\kappa_{m-1}}^{m-1},
\]
since \( \varphi_{m-1} = \theta_{m} - (k_{m} - k_{m-1}) \) and \( \kappa_{-m} - \theta_{m} = \varphi_{m-1} \).

Thus, the standard \( OGS \) canonical form of \( \pi^{-1} \) as follows:
\[
\pi^{-1} = \theta_{\kappa_{+}, 1} \cdot \left( t_{\kappa_{+}, 1} \cdot \theta_{\kappa_{+}, 2} \right) \cdot \left( t_{\kappa_{+}, 2} \cdot \theta_{\kappa_{+}, 3} \right) \cdots \cdot \left( t_{\kappa_{+}, m-1} \cdot \theta_{\kappa_{+}, m} \right).
\]
(1)

In the specific case, where \( k_{1} = maj(\pi) \):
\[
\pi^{-1} = \left( t_{\kappa_{+}, 1} \cdot \theta_{\kappa_{+}, 2} \right) \cdot \left( t_{\kappa_{+}, 2} \cdot \theta_{\kappa_{+}, 3} \right) \cdots \cdot \left( t_{\kappa_{+}, m-1} \cdot \theta_{\kappa_{+}, m} \right).
\]
(2)

Since for every \( 1 \leq j \leq m, \theta_{j-1} + \theta_{j} = \kappa_{-j} \), obviously, Equations (1) and (2) are standard \( OGS \) elementary factorizations of \( \pi^{-1} \) with the following properties:

- If \( k_{1} > maj(\pi) \), then \( \pi^{-1} \) has \( m \) elementary factors of the form \( t_{\kappa_{-j}}^{\varphi_{j}} \cdot t_{\kappa_{+}, j}^{\varphi_{j}} \) (we consider \( \varphi_{0} = 0 \)), for \( 1 \leq j \leq m \);
- If \( k_{1} = maj(\pi) \), then \( \pi^{-1} \) has \( m - 1 \) elementary factors of the form \( t_{\kappa_{-j}}^{\varphi_{j}} \cdot t_{\kappa_{+}, j}^{\varphi_{j}} \), for \( 2 \leq j \leq m \);

Hence, we get the desired result of the theorem about the descent set of \( \pi^{-1} \). ■

**Example 3.35** Let \( \pi = t_{17}^{1} \cdot t_{19}^{1} \cdot t_{22}^{1} \cdot t_{24}^{1} \). Since \( k_{1} = maj(\pi) = 17 \), \( \pi \) is a standard \( OGS \) elementary element of \( S_{24} \). The permutation presentation of \( \pi \):
\[
\pi = [1; 2; 3; 4; 5; 6; 7; 8; 9; 12; 13; 17; 18; 19; 22; 23; 24; 10; 11; 14; 15; 16; 20; 21].
\]
Notice, \( \pi(j) = j \) for \( 1 \leq j \leq 9 \), which verifies the result of Proposition 3.37, since \( k_1 = \text{maj}(\pi) = 17 \), and \( i_{k_1} = 9 \). By Theorem 3.24 we have:

\[
\text{norm}(\pi) = s_{17} \cdot s_{16} \cdot s_{15} \cdot s_{14} \cdot s_{13} \cdot s_{12} \cdot s_{11} \cdot s_{10} \cdot s_{18} \cdot s_{17} \cdot s_{16} \cdot s_{15} \cdot s_{14} \cdot s_{13} \cdot s_{12} \cdot s_{11} \cdot s_{20} \cdot s_{19} \cdot s_{18} \cdot s_{17} \cdot s_{16} \cdot s_{15} \cdot s_{21} \cdot s_{20} \cdot s_{19} \cdot s_{18} \cdot s_{17} \cdot s_{16} \cdot s_{22} \cdot s_{21} \cdot s_{20} \cdot s_{19} \cdot s_{18} \cdot s_{17} \cdot s_{23} \cdot s_{22} \cdot s_{21} \cdot s_{20} \cdot s_{19} \cdot s_{18} \cdot s_{17}
\]

We derive \( \text{norm}(\pi^{-1}) \), which we get just by reading the letters of \( \pi \) in the direction of right to left, and down to up. Thus,

\[
\text{norm}(\pi^{-1}) = s_{11} \cdot s_{10} \cdot s_{12} \cdot s_{11} \cdot s_{16} \cdot s_{15} \cdot s_{14} \cdot s_{13} \cdot s_{12} \cdot s_{17} \cdot s_{16} \cdot s_{15} \cdot s_{14} \cdot s_{13} \cdot s_{12} \cdot s_{18} \cdot s_{17} \cdot s_{16} \cdot s_{15} \cdot s_{14} \cdot s_{13} \cdot s_{12} \cdot s_{21} \cdot s_{20} \cdot s_{19} \cdot s_{18} \cdot s_{17} \cdot s_{16} \cdot s_{22} \cdot s_{21} \cdot s_{20} \cdot s_{19} \cdot s_{18} \cdot s_{17} \cdot s_{23} \cdot s_{22} \cdot s_{21} \cdot s_{20} \cdot s_{19} \cdot s_{18} \cdot s_{17}
\]

Now, we compute the standard OGS canonical form of \( \pi^{-1} \) by the formula in Theorem 3.34.

\[
k_1 = 17, \quad k_2 = 19, \quad k_3 = 22, \quad k_4 = 24;
\]

\[
i_{k_1} = 9, \quad i_{k_2} = 2, \quad i_{k_3} = 3, \quad i_{k_4} = 3.
\]

Thus,

\[
\varrho_1 = 9, \quad \varrho_2 = 9 + 2 = 11, \quad \varrho_3 = 9 + 2 + 3 = 14, \quad \varrho_4 = \text{maj}(\pi) = 9 + 2 + 3 + 3 = 17;
\]

\[
\varrho_1 = 17 - 17 = 0, \quad \varrho_2 = 19 - 17 = 2, \quad \varrho_3 = 22 - 17 = 5, \quad \varrho_4 = 24 - 17 = 7.
\]

and

\[
\kappa_{-2} = \varrho_1 + \varrho_2 = 9 + 2 = 11, \quad \kappa_{+2} = \varrho_2 + \varrho_3 = 11 + 5 = 16, \quad \kappa_{-3} = \varrho_2 + \varrho_3 = 11 + 5 = 16, \quad \kappa_{+3} = \varrho_3 + \varrho_4 = 14 + 5 = 19, \quad \kappa_{-4} = \varrho_3 + \varrho_4 = 14 + 7 = 21, \quad \kappa_{+4} = \varrho_4 + \varrho_4 = 17 + 7 = 24.
\]

Thus, by using Theorem 3.34 for the case of \( k_1 = \rho_1 \),

\[
\pi^{-1} = t_{11}^9 \cdot t_{13}^2 \cdot t_{16}^{11} \cdot t_{19}^5 \cdot t_{21}^{14} \cdot t_{24}^7.
\]
Recall from Definition 3.20:

Let \( \pi \) be as defined in Definition 3.20. Assume for every \( 1 \leq v \leq z(\pi) \) holds:

- \( \kappa_{-2} = 11 \) and \( \pi^{-1}(11) = 19 > 10 = \pi^{-1}(12) \);
- \( \kappa_{-3} = 16 \) and \( \pi^{-1}(16) = 22 > 12 = \pi^{-1}(17) \);
- \( \kappa_{-4} = 21 \) and \( \pi^{-1}(21) = 24 > 15 = \pi^{-1}(22) \).

Thus,

\[
\text{des} (\pi^{-1}) = \{ \kappa_{-2}, \kappa_{-3}, \kappa_{-4} \} = \{ 11, 16, 21 \}.
\]

Notice:

- \( \kappa_{-1} = v_{1} \)
- \( \kappa_{-2} = v_{2} \)\( \vdots \)
- \( \kappa_{-n} = v_{n} \)

Indeed, the permutation presentation of \( \pi^{-1} \):

\[\pi^{-1} = [1; 2; 3; 4; 5; 6; 7; 8; 9; 18; 19; 10; 11; 20; 21; 22; 12; 13; 14; 23; 24; 15; 16; 17].\]

Now, we generalize Theorem 3.34 for general \( \pi \in S_{n} \). We consider the standard OGS elementary factorization \( \pi = \prod_{v=1}^{z(\pi)} \pi^{(v)} \) with the standard OGS elementary factor \( \pi^{(v)} = \prod_{j=1}^{m(v)} t_{h_{j}}^{i_{j}(v)} \) for \( 1 \leq v \leq z(\pi) \). First, we show the standard OGS canonical form for \( \pi^{-1} \) in a special case, where \( \pi^{(v)} \) and \( \pi^{(w)} \) are commute for all \( 1 \leq v, w \leq z(\pi) \).

Recall from Definition 3.20:

- \( \rho_{j}^{(v)} \) is defined to be \( \sum_{z=j}^{m(v)} t_{z}^{(v)} \);
- \( \varphi_{j}^{(v)} \) is defined to be \( \sum_{z=1}^{j} t_{z}^{(v)} \);
- \( \tilde{\varphi}_{j}^{(v)} \) is defined to be \( h_{j}^{(v)} - \rho_{1}^{(v)} \).

Theorem 3.36 Let \( \pi \in S_{n} \), presented in standard OGS elementary factorization, with all the notations used in Definition 3.20. Assume \( \pi^{(v)} \) satisfies the following conditions:

- \( \text{maj} (\pi^{(v)}) = h_{1}^{(v)} \) for \( 2 \leq v \leq z(\pi) \);
- \( i_{1}^{(v)} \geq h_{m(v-1)}^{(v)} \) for \( 2 \leq v \leq z(\pi) \).

For every \( 1 \leq v \leq z(\pi) \), \( 1 \leq j \leq m(v) \), and \( v \leq r \leq z(\pi) \), let \( \varphi_{j}^{(v)} \) and \( \varphi_{j}^{(r)} \) be integers as defined in Definition 3.27. Denote \( \kappa_{v}^{(v)} = \varphi_{j-1}^{(v)} + \varphi_{j}^{(v)} = h_{j}^{(v)} - \rho_{j}^{(v)} \) and \( \kappa_{v}^{(v)} = \varphi_{j}^{(v)} + \tilde{\varphi}_{j}^{(v)} = h_{j}^{(v)} - \rho_{j+1}^{(v)} \) (We consider \( \varphi_{0}^{(v)} = \rho_{m(v)+1}^{(v)} = 0 \)). Then, the following holds:

- \( \pi^{(v)} \) and \( \pi^{(w)} \) are commute for all \( 1 \leq v, w \leq z(\pi) \);

\[
\pi^{-1} = \prod_{v=1}^{z(\pi)} \prod_{j=1}^{m(v)} (t_{\varphi_{j-1}^{(v)}}^{(\kappa_{v}^{(v)})} \cdot t_{\tilde{\varphi}_{j}^{(v)}}^{(\kappa_{v}^{(v)})}),
\]

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Proof.
Let \( \pi^{(v)} = \prod_{j=1}^{m^{(v)}} t_{j}^{(v)} \), for \( 1 \leq v \leq z(\pi) \). Since for every \( w > v \), \( \text{maj} (\pi^{(w)}) = h_{1}^{(w)} \), and \( i_{1}^{(w)} \geq h_{m^{(w-1)}}^{(w)} \geq h_{m^{(v)}}^{(v)} \), by Proposition 3.32, \( \pi^{(v)} \) commutes with \( \pi^{(w)} \), for every \( 1 \leq v < w \leq z(\pi) \). Therefore, \( \pi^{-1} = \prod_{v=1}^{z(\pi)} (\pi^{(v)})^{-1} \). Then, by using Theorem 3.34 we get the desired result.

Now, we turn to the general case, where considering \( \pi \in S_{\mathbb{N}} \) such that the standard OGS elementary factors do not necessarily commute. In the first step, we generalize the definitions of \( \kappa_{-j}^{(v)} \) and \( \kappa_{+j}^{(v)} \) for an arbitrary \( \pi \in S_{\mathbb{N}} \), which were defined in Theorems 3.34 and 3.36, for the special cases of \( \pi \).

Definition 3.37 For every \( 1 \leq v \leq z(\pi) \), \( 1 \leq j \leq m^{(v)} \), and \( v \leq r \leq z(\pi) \), define \( \kappa_{-j}^{(v-r)} \) and \( \kappa_{+j}^{(v-r)} \) in the following recursive way:

- \( \kappa_{-j}^{(v-r)} = \vartheta_{j}^{(v)} - \vartheta_{j-1}^{(v)} = h_{j}^{(v)} - \rho_{j}^{(v)} \),
- \( \kappa_{+j}^{(v-r)} = \vartheta_{j}^{(v)} - \vartheta_{j}^{(v)} = h_{j}^{(v)} - \rho_{j+1}^{(v)} \),

for every \( 1 \leq v \leq z(\pi) \) and for every \( 1 \leq j \leq m^{(v)} \) (where in the formula for \( j = 1 \), and for \( j = m^{(v)} \) we consider \( \vartheta_{0}^{(v)} = \rho_{m^{(v)}+1}^{(v)} = 0 \));
- If \( \vartheta_{j-1}^{(r+1)} \leq \kappa_{-j}^{(v-r)} < \vartheta_{j'}^{(r+1)} \), for some \( 1 \leq j' \leq m^{(r+1)} \), then \( \kappa_{-j}^{(v-r+1)} = \kappa_{-j}^{(v-r)} + \vartheta_{j'}^{(r+1)} \);
- If \( \vartheta_{j-1}^{(r+1)} < \kappa_{+j}^{(v-r)} \leq \vartheta_{j'}^{(r+1)} \), for some \( 1 \leq j' \leq m^{(r+1)} \), then \( \kappa_{+j}^{(v-r+1)} = \kappa_{+j}^{(v-r)} + \vartheta_{j'}^{(r+1)} \);

In case of \( r = z(\pi) \):
- Denote \( \kappa_{-j}^{(v-z(\pi))} \) by just \( \kappa_{-j}^{(v)} \);  
- Denote \( \kappa_{+j}^{(v-z(\pi))} \) by just \( \kappa_{+j}^{(v)} \).

The next proposition shows that Definition 3.37 just generalizes the definitions of \( \kappa_{-j}^{(v)} \) and \( \kappa_{+j}^{(v)} \) in the special case of Theorem 3.36.

Proposition 3.38 If \( \kappa_{+j}^{(v-m^{(v)})} = \kappa_{+j}^{(v-m^{(v)})} \), for every \( 1 \leq v \leq z(\pi) \), \( v \leq r \leq z(\pi) \), then the following holds:
- \( \kappa_{-j}^{(v-r)} = \kappa_{-j}^{(v-r)} \) and \( \kappa_{+j}^{(v-r)} = \kappa_{+j}^{(v-r)} \), for every \( 1 \leq v \leq z(\pi) \), \( v \leq r \leq z(\pi) \), and \( 1 \leq j \leq m^{(v)} \).
For every $\pi(v)$ commutes with $\pi(r)$, for every $1 \leq v \leq z(\pi)$ and $v \leq r \leq z(\pi)$.

Proof. Assume $\kappa_{+,m(v)}^{(v \rightarrow r)} = \kappa_{+,m(v)}^{(v \rightarrow r)}$, for every $1 \leq v \leq z(\pi)$ and $v \leq r \leq z(\pi)$. Then, in particular,

$$\kappa_{+,m(v)}^{(v \rightarrow v)} = \kappa_{+,m(v)}^{(v \rightarrow v+1)}$$

for every $1 \leq v < z(\pi)$. Thus, by Definition 3.37,

$$\kappa_{+,m(v)}^{(v \rightarrow v+1)} = \kappa_{+,m(v)}^{(v \rightarrow v)} \leq \vartheta_j^{(v+1)}$$

such that $\vartheta_j^{(v+1)} = 0$. Then, by Proposition 3.22

$$j' = 1 \maj \left( \pi^{(v+1)} \right) = h_1^{(v+1)}.$$  

Notice, by Definition 3.37, $\kappa_{+,m(v)}^{(v \rightarrow v)} \leq \kappa_{+,m(v)}^{(v \rightarrow v)}$ and $\kappa_{+,m(v)}^{(v \rightarrow v)} \leq \kappa_{+,m(v)}^{(v \rightarrow v)}$ for every $1 \leq j \leq m(v)$. Thus, we get

$$\kappa_{+,m(v)}^{(v \rightarrow v)} \leq \kappa_{+,m(v)}^{(v \rightarrow v+1)} \leq \kappa_{+,m(v)}^{(v \rightarrow v)}$$

as well, for every $1 \leq j \leq m(v)$. Hence,

$$\kappa_{+,m(v)}^{(v \rightarrow v+1)} = \kappa_{+,m(v)}^{(v \rightarrow v)} + \vartheta_1^{(v+1)} = \kappa_{+,m(v)}^{(v \rightarrow v)} + \vartheta_1^{(v+1)} = \kappa_{+,m(v)}^{(v \rightarrow v)},$$

for every $1 \leq v < z(\pi)$ and $1 \leq j \leq m(v)$. Hence, $\kappa_{+,m(v)}^{(v \rightarrow v)} = \kappa_{+,m(v)}^{(v \rightarrow v)}$ and $\kappa_{+,m(v)}^{(v \rightarrow v)} = \kappa_{+,m(v)}^{(v \rightarrow v)}$, for every $1 \leq v \leq z(\pi)$, $v \leq r < z(\pi)$, and $1 \leq j \leq m(v)$. Now, since

$$\kappa_{+,m(v)}^{(v \rightarrow v+1)} = \kappa_{+,m(v)}^{(v \rightarrow v)} + \vartheta_1^{(v+1)} = \kappa_{+,m(v)}^{(v \rightarrow v)} + \vartheta_1^{(v+1)} = h_1^{(v+1)},$$

by Theorem 3.39, $\pi(v)$ commutes with $\pi(r)$ for every $1 \leq v < z(\pi)$, and $v < r \leq z(\pi)$. ■

The next six propositions describe some important properties of $\kappa_{-,j}^{(v \rightarrow r)}$ and $\kappa_{+,j}^{(v \rightarrow r)}$, which are essential for understanding the formula of the standard OGS canonical form of $\pi^{-1}$.

Proposition 3.39 For every $1 \leq v \leq z(\pi)$, $1 \leq j \leq m(v)$, and $v \leq r \leq z(\pi)$, $\kappa_{-,j}^{(v \rightarrow r)}$ and $\kappa_{+,j}^{(v \rightarrow r)}$ satisfy the following properties:

- $\kappa_{-,j}^{(v \rightarrow r)} < \kappa_{+,j}^{(v \rightarrow r)}$, and in particular, $\kappa_{-,j}^{(v \rightarrow v)} < \kappa_{+,j}^{(v \rightarrow v)}$, for every $1 \leq v \leq z(\pi), v \leq r \leq z(\pi)$, and $1 \leq j \leq m(v)$;

- $\kappa_{+,j}^{(v \rightarrow r)} < \kappa_{-,j+1}^{(v \rightarrow r)}$, and in particular, $\kappa_{+,j}^{(v \rightarrow v)} < \kappa_{-,j+1}^{(v \rightarrow v)}$, for every $1 \leq v \leq z(\pi), v \leq r \leq z(\pi)$, and $2 \leq j \leq m(v) - 1$.  

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Therefore, it for \( \leq \) for every 1 \( \kappa \) \( j \) \( \leq \) for every 1 \( \kappa \). Assume by induction the correctness of the proposition for \( r' \leq r \), and we prove it for \( r' = r + 1 \). By Definition 3.37:

\[
\kappa_{v \rightarrow r+1} = \kappa_{v \rightarrow r} + \vartheta_{j'}^{(r+1)}, \quad \text{where} \quad \vartheta_{j' \rightarrow 1}^{(r+1)} \leq \kappa_{v \rightarrow r} < \vartheta_{j'}^{(r+1)};
\]

\[
\kappa_{j'' \rightarrow r+1} = \kappa_{v \rightarrow r} + \vartheta_{j''}^{(r+1)}, \quad \text{where} \quad \vartheta_{j'' \rightarrow 1}^{(r+1)} \leq \kappa_{v \rightarrow r} < \vartheta_{j''}^{(r+1)}.
\]

By the induction hypothesis,

\[
\kappa_{v \rightarrow r} < \kappa_{v \rightarrow r}, \quad \kappa_{v \rightarrow r} < \kappa_{v \rightarrow r+1}.
\]

Therefore,

\[
\vartheta_{j' \rightarrow 1}^{(r+1)} \leq \kappa_{v \rightarrow r} < \kappa_{v \rightarrow r} \leq \vartheta_{j'}^{(r+1)}.
\]

Thus, by Proposition 3.32 \( j' \leq j'' \), and \( \vartheta_{j'}^{(r+1)} \leq \vartheta_{j''}^{(r+1)} \). Hence,

\[
\kappa_{v \rightarrow r+1} = \kappa_{v \rightarrow r} + \vartheta_{j'}^{(r+1)} < \kappa_{v \rightarrow r} + \vartheta_{j''}^{(r+1)} = \kappa_{v \rightarrow r+1},
\]

for every \( 1 \leq v \leq z(\pi), 1 \leq j \leq m(v) \). By a similar argument, we have

\[
\kappa_{v \rightarrow r+1} < \kappa_{v \rightarrow r+1}.
\]

for every \( 1 \leq v \leq z(\pi), 1 \leq j \leq m(v) - 1 \). ~

**Proposition 3.40** For every \( 1 \leq v < z(\pi) \) and \( w > v \), let \( j'_{w,v} \) and \( j''_{w,v} \) be integers, such that

\[
\vartheta_{j'_{w,v}}^{(w)} \leq \kappa_{v \rightarrow w-1} < \vartheta_{j''_{w,v}}^{(w)} < \kappa_{v \rightarrow w-1} \leq \vartheta_{j''_{w,v}}^{(w)}.
\]

Then, the following holds:

\[
\kappa_{v \rightarrow w} \leq \kappa_{v \rightarrow w} < \kappa_{v \rightarrow w} < \kappa_{v \rightarrow w} \leq \kappa_{v \rightarrow w},
\]

for every \( r \) such that \( w \leq r \leq z(\pi) \). In particular,

\[
\kappa_{v \rightarrow w} \leq \kappa_{v \rightarrow w} < \kappa_{v \rightarrow w} < \kappa_{v \rightarrow w} \leq \kappa_{v \rightarrow w}.
\]

**Proof.** The proof of the result is in induction on the value of \( r \). By Definition 3.37 we have:

\[
\kappa_{v \rightarrow w} = \kappa_{v \rightarrow w} + \vartheta_{j'_{w,v}}^{(w)}, \quad \text{in case} \quad \vartheta_{j'_{w,v}}^{(w)} \leq \kappa_{v \rightarrow w} < \vartheta_{j'_{w,v}}^{(w)}.
\]

Where,

\[
\kappa_{v \rightarrow w} = \vartheta_{j''_{w,v}}^{(w)} + \vartheta_{j''_{w,v}}^{(w)} = \vartheta_{j''_{w,v}}^{(w)} + \vartheta_{j''_{w,v}}^{(w)}.
\]

Therefore, by again using Definition 3.37 we have \( \kappa_{v \rightarrow w} \leq \kappa_{v \rightarrow w} < \kappa_{v \rightarrow w} \), for every \( w > v \).

Now assume by induction \( \kappa_{v \rightarrow r'} \leq \kappa_{v \rightarrow r'} < \kappa_{v \rightarrow r'} \) for every \( r' \) such that \( w \leq r' \leq r \). Now, by Definition 3.37 notice the following,
Proposition 3.42

Assume one of the following conditions holds for some $\kappa$

Proof.

Consider $v$, $v'$, then by Proposition 3.22, $v > v'$ cannot happen for $\kappa$

Therefore, $\kappa < \kappa < \kappa$.

Then, by Proposition 3.22 $\kappa < \kappa < \kappa$.

Thus, we get $\kappa < \kappa < \kappa < \kappa$.

By the same argument, we get

\[
\kappa < \kappa < \kappa < \kappa,
\]

which implies $\kappa < \kappa < \kappa$, for every $1 \leq v \leq z(\pi) - 1$. Thus, the proposition holds for every $1 \leq v \leq z(\pi)$, $w > v$, and $w \leq r \leq z(\pi)$.

Proposition 3.41 Assume one of the following conditions holds for some $1 \leq v, v' \leq z(\pi)$, $1 \leq j \leq m(v) - 1$, $1 \leq j' \leq m(v')$, $v, v' \leq r \leq z(\pi)$, and $c \in \{+,-\}$:

- $\kappa < \kappa < \kappa$
- $\kappa < \kappa$
- $\kappa < \kappa$
- $\kappa < \kappa$

then necessarily $v' > v$ (i.e., the interval $(\kappa, \kappa)$ does not contain $\kappa$ such that, $v' \leq v$).

Proof. Consider $\kappa < \kappa$. Then, by Proposition 3.40, for every $v > v'$ there exists $1 \leq j \leq m(v)$ such that $\kappa < \kappa < \kappa < \kappa$. Therefore, the case where,

\[
\kappa < \kappa < \kappa < \kappa < \kappa < \kappa,
\]

cannot happen for $v > v'$. By Proposition 3.39 the mentioned cases cannot happen for $v = v'$ as well. Therefore, necessarily, $v' > v$ under the conditions of the proposition.

Proposition 3.42 $\kappa < \kappa$, for any $1 \leq v', v'' \leq z(\pi)$, $1 \leq j' \leq m(v')$, and $1 \leq j'' \leq m(v'')$. 

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Proof. If \( v' = v'' \) for some \( 1 \leq v' \leq z(\pi) \), then by Proposition 3.39 we have \( \kappa_{-j}^{(v')} \neq \kappa_{+j}^{(v'')} \), for all \( 1 \leq v' \leq z(\pi) \), \( 1 \leq j', j'' \leq m(v) \). Now, assume \( v' < v'' \). Then, by Proposition 3.40 there exists \( 1 \leq j \leq m(v'') \), such that \( \kappa_{-j}^{(v')} \leq \kappa_{-j}^{(v'')} < \kappa_{+j}^{(v'')} \).

By our assumption, \( \kappa_{-j}^{(v')} = \kappa_{+j}^{(v'')} \).

Therefore, \( \kappa_{-j}^{(v'')} \leq \kappa_{-j}^{(v'')} < \kappa_{+j}^{(v'')} \).

By Proposition 3.39
\[
\kappa_{+j}^{(v')} > \kappa_{+j}^{(v'')} \text{ if and only if } j > j'', \quad \kappa_{-j}^{(v'')} < \kappa_{+j}^{(v'')} \text{ if and only if } j \leq j''.
\]

Therefore, there is no \( j \) such that \( \kappa_{-j}^{(v'')} \leq \kappa_{+j}^{(v'')} < \kappa_{+j}^{(v'')} \). Thus, \( \kappa_{-j}^{(v')} \neq \kappa_{+j}^{(v'')} \), for all \( 1 \leq j' \leq m(v') \), and all \( 1 \leq j'' \leq m(v'') \), in case \( v' < v'' \).

By a similar argument we have \( \kappa_{-j}^{(v')} \neq \kappa_{+j}^{(v'')} \), for any \( 1 \leq j' \leq m(v') \) and any \( 1 \leq j'' \leq m(v'') \), in case \( v' > v'' \). \( \blacksquare \)

Proposition 3.43 For every integer \( v_1, v_2 \), and \( j \), such that \( 1 \leq v_1 < v_2 \leq z(\pi) \), and \( 1 \leq j \leq m(v_1) \), the parameters \( \kappa_{-j}^{(v_1)}, \kappa_{+j}^{(v_1)}, \kappa_{+j+m(v_2)}, \kappa_{-j}^{(v_2)} \) satisfy the following properties:

- If \( 1 \leq v_1 < v_2 \leq z(\pi) \), then \( \kappa_{+j}^{(v_1)} \leq \kappa_{+j+m(v_2)} \) and \( \kappa_{-j}^{(v_1)} < \kappa_{+j+m(v_2)} \), for every \( 1 \leq j \leq m(v_1) \). In particular, \( \kappa_{+j+m(z(\pi))}^{(v_1)} \geq \kappa_{+j}^{(v_1)} \), for every \( 1 \leq v \leq z(\pi), 1 \leq j \leq m(v_1), c \in \{+, -, \} \).

- If \( 1 \leq v_1 < v_2 \leq z(\pi) \), then \( \kappa_{+j}^{(v_2)} \leq \kappa_{-j}^{(v_2)} \) and \( \kappa_{+j}^{(v_1)} < \kappa_{+j}^{(v_2)} \), for every \( 1 \leq j \leq m(v_2) \). In particular, \( \kappa_{+j}^{(z(\pi))} \leq \kappa_{+j}^{(v_2)} \), for every \( 1 \leq v \leq z(\pi), 1 \leq j \leq m(v_2), c \in \{+, -, \} \).

Proof. Let \( v_1, v_2 \) be integers such that \( 1 \leq v_1 < v_2 \leq z(\pi) \). Let \( 1 \leq j \leq m(v_1) \). Then, by Proposition 3.40 there exists \( 1 \leq j', j'' \leq m(v_1) \) such that \( \kappa_{-j}^{(v_1)} < \kappa_{+j}^{(v_1)} \leq \kappa_{+j}^{(v_1)} \) and \( \kappa_{-j}^{(v_2)} \leq \kappa_{+j}^{(v_2)} \). By Proposition 3.39 \( \kappa_{+j}^{(v_1)} < \kappa_{+j}^{(v_2)} \) and \( \kappa_{-j}^{(v_1)} \leq \kappa_{-j}^{(v_2)} \). Hence, we get the first part of the proposition. By a similar argument we prove the second part of the proposition. \( \blacksquare \)

Proposition 3.44 Assume \( \kappa_{+j}^{(v)} = 0 \) for some \( 1 \leq v \leq z(\pi), 1 \leq j \leq m(v) \), and \( c \in \{+, -, \} \). Then, necessarily \( j = 1, c = (-) \), and for every \( v' \geq v \), \( m(v') > 1 \), and both \( \kappa_{-j}^{(v')} = 0 \) and \( \vartheta_1^{(v')} = 0 \). Moreover, if \( \vartheta_1^{(v')} = 0 \) for every \( v' \geq v \), for some \( 1 \leq v \leq z(\pi) \), then necessarily \( \kappa_{-j}^{(v')} = 0 \) for every \( v' \geq v \), as well.

Proof. Since \( \kappa_{+j}^{(v)} \) is the sum of \( \vartheta_j^{(v)} \) with elements of the form \( \vartheta_j^{(v')} \), where by Proposition 3.52 \( \vartheta_j^{(v')} > 0 \) and \( \vartheta_j^{(v')} \geq 0 \) for every \( 1 \leq v \leq z(\pi) \) and \( 1 \leq j \leq m(v) \), obviously, \( \kappa_{+j}^{(v)} > 0 \). By the same argument, \( \kappa_{+j}^{(v)} > 0 \) for \( j \geq 2 \) as well. Thus, if \( \kappa_{+j}^{(v)} = 0 \) for some
If there exists integers η and 1 such that

\[ \eta(v) \leq z(\pi) \text{ and } 1 \leq j < m(v), \] then necessarily \( j = 1 \). Now, by using Proposition 3.43, \( \kappa(v) \leq \kappa(v') \) for every \( v' > v \). Thus, we get \( \kappa(v) = 0 \) for every \( v' \geq v \). Now, since \( \kappa(v) \geq \kappa(v') = \eta(v') \), and by Proposition 3.22, \( \theta_1(v') \geq 0 \), we conclude \( \theta_1(v') = 0 \) for every \( v' \geq v \) as well, and therefore, \( m(v) > 1 \) by using the same proposition. Now, assume \( \theta_1(v') = 0 \) for every \( v' \geq v \), for some \( 1 \leq v \leq z(\pi) \), then by Definition 3.37 we get \( \kappa(v) = 0 \) for every \( v' \geq v \), as well.

Now, we introduce two new parameters \( \chi \) and \( \eta \), which are defined by the different values of \( \kappa(v) \) and \( \kappa(v') \), for \( 1 \leq v \leq z(\pi) \) and \( 1 \leq j < m(v) \), and which are essential for the formula of the standard OGS canonical form of \( \pi^{-1} \).

**Definition 3.45** For every \( 1 \leq v \leq z(\pi) \) and \( 1 \leq j < m(v) \), \( \chi(v) \) is defined in the following way:

If there exists integers \( v' < v \) and \( 1 \leq j' < m(v') \), such that

\[ \kappa(v') \leq \kappa(v) \leq \kappa(v') \text{ or } \kappa(v') < \kappa(v) \leq \kappa(v') \text{ or } \kappa(v') \leq \kappa(v) < \kappa(v'), \]

then define \( \chi(v) \) to be the largest \( v' \) such that \( v' < v \), otherwise \( \chi(v) \) is defined to be 0.

**Definition 3.46** For every \( 1 \leq v \leq z(\pi) \) and \( 1 \leq j < m(v) \), \( \eta(v) \) is defined in the following way:

\[ \eta(v) = \sum_{v'=\chi(v)+1}^{v-1} \theta_1(v'), \]

such that

\[ \kappa(v') \leq \kappa(v) \leq \kappa(v') \text{ or } \kappa(v') < \kappa(v) \leq \kappa(v') \text{ or } \kappa(v') \leq \kappa(v) < \kappa(v'), \]

for some \( 1 \leq j \leq v \leq m(v) \).

In the next three propositions we find important connections between the parameters \( \chi(v) \), \( \eta(v) \), and the parameters \( \theta_1(v) \), \( \theta_1(v') \), \( \kappa(v) \), and \( \kappa(v') \), where the third proposition (Proposition 3.49), leads to the standard OGS elementary factorization of \( \pi^{-1} \).

**Proposition 3.47** For every \( 1 \leq v \leq z(\pi) \) and \( 1 \leq j \leq m(v) \),

\[ \eta(v) < \phi(v). \]

**Proof.** The proof is in induction on the value of \( v-\chi(v) \). If \( v-\chi(v) = 1 \), then \( \chi(v) = v-1 \), and \( \eta(v) = 0 \). Therefore, obviously, \( \eta(v) < \phi(v) \) in case \( \eta(v) = 0 \). Now, assume in induction that \( \eta(v) < \phi(v) \), in case \( v-\chi(v) < v \), for some \( v \geq 1 \), and we prove the proposition for the case \( v-\chi(v) = v \). By Definition 3.46, \( \eta(v) = \sum_{v'=\chi(v)+1}^{v-1} \theta_1(v') \), such that
Proposition 3.48

\( \kappa^{(v')}_{-j; v'} < \kappa^{(v')}_{+j; v'} \). Since \( v - \chi_j^{(v)} = \hat{v} + 1 \), we get \( \eta_j^{(v)} = \sum_{v' = \hat{v} - 1}^{v - 1} \theta_{j,v'}^{(v')} \), such that
\( \kappa^{(v')}_{-j; v'} < \kappa^{(v')}_{+j; v'} \). Thus, consider the values of \( \kappa^{(v')}_{-j; v'} \) for \( v - \hat{v} \leq v' \leq v - 1 \).

Denote by \( \hat{v} \) the smallest integer such that \( v - \hat{v} \leq \hat{v} \leq v - 1 \) and for every \( v - \hat{v} \leq v' \leq v - 1 \), \( \kappa^{(v')}_{-j; v'} \leq \kappa^{(\hat{v})}_{-j; v'} \). Then, the following holds:

- \( \eta_j^{(\hat{v})} = \sum_{v' = \hat{v} - 1}^{\hat{v} - 1} \theta_{j,v'}^{(v')} \);
- \( \kappa^{(\hat{v}-v-1)}_{-j; v'} = \theta_{j,v-1}^{(v')} + \sum_{v' = \hat{v}}^{v-1} \theta_{j,v'}^{(v')} \).

Since \( \hat{v} \leq v \), by the induction hypothesis, \( \eta_j^{(\hat{v})} = \theta_{j,v-1}^{(v)} \), and since \( \kappa^{(\hat{v})}_{-j; v'} < \kappa^{(v)}_{+j; v} \), by Definition 3.37, we conclude, \( \kappa^{(\hat{v}-v-1)}_{-j; v'} < \theta_{j}^{(v)} \). Hence,

\[ \eta_j^{(v)} = \eta_j^{(\hat{v})} + \kappa^{(\hat{v}-v-1)}_{-j; v'} - \theta_{j,v-1}^{(v)} < \theta_{j,v-1}^{(v)} + \theta_j^{(v)} - \theta_{j,v-1}^{(v)} = \theta_j^{(v)}. \]

\[ \blacksquare \]

**Proposition 3.48** Consider a non-decreasing monotonic sequence \( \{a_x\}_{x=1}^r \) of all the integers of the form \( \kappa^{(v)}_{-i} \) and \( \kappa^{(v)}_{+i} \), where \( 1 \leq v \leq z(\pi) \) and \( 1 \leq i \leq m^{(v)} \), with the following condition:

- If \( \kappa^{(v)}_{-j; i} = \kappa^{(v)}_{+j; i} \) for some \( 1 \leq v_1 < v_2 \leq z(\pi) \), and \( 1 \leq j_x \leq m^{(v_x)} \) then \( \kappa^{(v)}_{-j; x} \) is prior to \( \kappa^{(v)}_{+j; x} \) in the sequence \( \{a_x\}_{x=1}^r \).

- If \( \kappa^{(v)}_{+j; i} = \kappa^{(v)}_{+j; i} \) for some \( 1 \leq v_1 < v_2 \leq z(\pi) \), and \( 1 \leq j_x \leq m^{(v_x)} \) then \( \kappa^{(v)}_{+j; x} \) is prior to \( \kappa^{(v)}_{+j; x} \) in the sequence \( \{a_x\}_{x=1}^r \).

Then, the following holds:

- \( a_1 = \kappa^{(z(\pi))}_{-1} \), \( a_r = \kappa^{(z(\pi))}_{+m^{(z(\pi))}} \);

- If \( a_x = \kappa^{(v)}_{+j; x} \) and \( a_{x+1} = \kappa^{(v+1)}_{+j; x+1} \), for some \( 1 \leq x \leq r - 1 \), \( 1 \leq j_x \leq m^{(v_x)} \), \( 1 \leq j_{x+1} \leq m^{(v_{x+1})} \), and \( c_{x+1} \in \{-, +\} \), then \( v_{x+1} \geq v_x \), where the following are satisfied:
  - In case \( c_{x+1} = (-) \) then necessarily, \( v_{x+1} = v_x \) and \( j_{x+1} = j_x + 1 \);
  - In case \( c_{x+1} = (+) \) where, \( a_x < a_{x+1} \), then necessarily, \( v_{x+1} > v_x \) and \( \chi^{(v_{x+1})}_{j_{x+1}} = v_x \).

- If \( a_x = \kappa^{(v)}_{-j; x} \) and \( a_{x-1} = \kappa^{(v-1)}_{-j; x-1} \), for some \( 2 \leq x \leq r \), \( 1 \leq j_x \leq m^{(v_x)} \), \( 1 \leq j_{x-1} \leq m^{(v_{x-1})} \), and \( c_x \in \{-, +\} \), then \( v_{x-1} \geq v_x \), where the following are satisfied:
  - In case \( c_{x-1} = (-) \) where, \( a_x > a_{x-1} \), necessarily, \( v_{x-1} > v_x \) and \( \chi^{(v_{x-1})}_{j_{x-1}} = v_x \). 

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In case $c_{x-1} = (+)$, necessarily, $v_{x-1} = v_x$ and $j_{x-1} = j_x - 1$.

- If $a_x = \kappa_{-j_x}^{(v_x)}$ and $a_{x+1} = \kappa_{+j_{x+1}}^{(v_{x+1})}$, for some $1 \leq x \leq r - 1$, $1 \leq j_x \leq m(v_x)$, $1 \leq j_{x+1} \leq m(v_{x+1})$, then either $v_{x+1} = v_x$ and then $j_{x+1} = j_x$ as well, or necessarily, $\chi_{v_x}^{(v_x)} = \chi_{(v_x+1)}^{(v_x+1)}$.

**Proof.** The first part of the proposition is a direct consequence of Proposition 3.43.

Thus, we turn to the second part of the proposition. Assume, $a_x = \kappa_{v_x}^{(v_x)}$ and $a_{x+1} = \kappa_{v_{x+1}}^{(v_{x+1})}$, for some $1 \leq x \leq r - 1$, $1 \leq v_x, v_{x+1} \leq z(\pi)$, $1 \leq j_x \leq m(v_x)$, $1 \leq j_{x+1} \leq m(v_{x+1})$, and $c_{x+1} \in \{+, -\}$. Since $a_{x+1}$ is a successive element to $a_x = \kappa_{v_x}^{(v_x)}$, necessarily $\kappa_{+j_x}^{(v_x)} \leq a_{x+1} \leq \kappa_{-j_{x+1}}^{(v_{x+1})}$. Therefore, in case $a_{x+1} \nless a_x$, by Proposition 3.41 we have $v_{x+1} > v_x$, unless $a_{x+1} = \kappa_{v_x}^{(v_x)}$. If $a_{x+1} = a_x$, then by the assumption of the proposition $v_{x+1} > v_x$. Now, assume $c_{x+1} = (-)$. Then, $a_{x+1} = \kappa_{v_x}^{(v_x)}$. Therefore, by Proposition 3.41 we have $v_x > v_{x+1}$ unless, $a_{x+1} = \kappa_{v_x}^{(v_x)}$. Hence, in case $c_{x+1} = (-)$ and $c_x = (+)$, the only possibly case is

$$a_x = \kappa_{v_x}^{(v_x)} \quad a_{x+1} = \kappa_{v_{x+1}}^{(v_{x+1})}.$$  

Now, assume $c_{x+1} = (+)$. Then, $a_x = \kappa_{v_x}^{(v_x)}$ and $a_{x+1} = \kappa_{v_{x+1}}^{(v_{x+1})}$, where we have already shown $v_{x+1} > v_x$. By Proposition 3.40 for every $\nu' > v_x$, there exists $j'_{\nu':v_x}$ such that $\kappa_{-j'_{\nu':v_x}}^{(\nu')} < a_x = \kappa_{v_x}^{(v_x)} \leq \kappa_{+j_{x+1}}^{(v_{x+1})}$. Since $a_{x+1} = \kappa_{v_{x+1}}^{(v_{x+1})}$ is a successive element to $a_x$, we have $\kappa_{-j'_{\nu':v_x}}^{(\nu')} < a_{x+1} = \kappa_{v_{x+1}}^{(v_{x+1})} \leq \kappa_{+j_{x+1}}^{(v_{x+1})}$, for every $v_x \leq \nu' \leq v_{x+1} - 1$. Thus, if $a_{x+1} \nless a_x$, then $\kappa_{-j'_{\nu':v_x}}^{(\nu')} < a_{x+1} = \kappa_{v_{x+1}}^{(v_{x+1})} < \kappa_{j_{x+1}}^{(v_{x+1})}$, for every $v_x \leq \nu' \leq v_{x+1} - 1$.

Hence, by Definition 3.45 $\chi_{j_{x+1}}^{(v_{x+1})} \leq v_x$. On the other hand, since $a_{x+1}$ is successive to $a_x = \kappa_{v_{x+1}}^{(v_{x+1})}$, we have $\chi_{j_{x+1}}^{(v_{x+1})} \leq v_x$. Thus, we conclude $\chi_{j_{x+1}}^{(v_{x+1})} = v_x$ in case $a_{x+1} > a_x$ and $c_{x+1} = (+)$. By a similar argument the third part of the proposition can be proved as well. Hence, we turn to the last part of the proposition.

Assume, $a_x = \kappa_{v_x}^{(v_x)}$ and $a_{x+1} = \kappa_{v_{x+1}}^{(v_{x+1})}$, for some $1 \leq x \leq r - 1$, $1 \leq j_x \leq m(v_x)$, $1 \leq j_{x+1} \leq m(v_{x+1})$. If $v_{x+1} > v_x$, then by Proposition 3.40 for every $v_x \leq \nu' \leq v_{x+1} - 1$, there exists $1 \leq j'_{\nu':v_x} \leq m(v')$ such that

$$\kappa_{-j'_{\nu':v_x}}^{(\nu')} \leq a_x = \kappa_{v_x}^{(v_x)} \leq \kappa_{+j_{x+1}}^{(v_{x+1})}.$$  

Since $a_{x+1} = \kappa_{+j_{x+1}}^{(v_{x+1})}$ is a successive to $a_x = \kappa_{v_x}^{(v_x)}$, we get

$$\kappa_{-j'_{\nu':v_x}}^{(\nu')} \leq a_{x+1} = \kappa_{v_{x+1}}^{(v_{x+1})} \leq \kappa_{+j_{x+1}}^{(v_{x+1})}.$$  

as well. Thus, by Definition 3.45 $\chi_{j_{x+1}}^{(v_{x+1})} \leq v_x$. On the other hand, since
For every \( \kappa \), the standard Proposition 3.49 Let \( \tilde{\kappa} \). Moreover, since \( a_{x+1} = \kappa^{(v_{x+1})}_{+} \) is a successive to \( a_x = \kappa^{(v_x)}_{-} \), we have for every \( v' < v_x \), one of the following holds:

\begin{itemize}
  \item \( \kappa^{(v')}_{-} \leq \kappa^{(v_x)}_{-} < \kappa^{(v_{x+1})}_{+} \leq \kappa^{(v')}_{+} \);
  \item \( \kappa^{(v')}_{+} < \kappa^{(v_x)}_{-} < \kappa^{(v_{x+1})}_{+} < \kappa^{(v')}_{-} + 1 \);
  \item \( \kappa^{(v_x)}_{-} < \kappa^{(v_{x+1})}_{+} < \kappa^{(v')}_{-} \);
  \item \( \kappa^{(v')}_{+} + m(v') < \kappa^{(v_x)}_{-} < \kappa^{(v_{x+1})}_{+} \).
\end{itemize}

for some \( 1 \leq j' \leq m(v') \). Therefore, by Definition 3.45 we conclude \( \chi^{(v_{x+1})}_{j+x+1} = \chi^{(v_x)}_{j+x} \) in case \( v_{x+1} > v_x \). By a similar argument it can be shown that \( \chi^{(v_{x+1})}_{j+x+1} = \chi^{(v_x)}_{j+x-1} \) in case \( v_{x+1} < v_x \) as well. If \( v_{x+1} = v_x \), then by Proposition 3.39 \( a_x = \kappa^{(v_x)}_{-} \) and \( a_{x+1} = \kappa^{(v_x)}_{+} \).

**Proposition 3.49** Let \( \{a_x\}_{x=1}^{\infty} \) be a non-decreasing monotonic sequence of \( \kappa^{(v)}_{+} \) and \( \kappa^{(v)}_{-} \) with the same conditions as in Proposition 3.48. Let \( \tilde{\pi} = \prod_{x=1}^{\infty} t_x^{a_x} \), where,

\[
\hat{l}_x = \begin{cases}
    \vartheta_j^{(v)} + \eta_j^{(v)} & a_x = \kappa^{(v)}_{+} \\
    \kappa^{(v)}_{-} - (\vartheta_j^{(v)} + \eta_j^{(v)}) & a_x = \kappa^{(v)}_{-}
\end{cases}
\]

For every \( 1 \leq w \leq z(\tilde{\pi}) \), consider

\[
\tilde{\pi} = \prod_{w=1}^{z(\tilde{\pi})} \tilde{\pi}(w),
\]

the standard OGS elementary factorization of \( \tilde{\pi} \), with the elementary factors

\[
\tilde{\pi}(w) = \prod_{j=1}^{m(w)} t_j^{z_j(w)} h_j^{t_j(w)},
\]

Define \( \tilde{\pi}_j(w) \) to be the following terminal segment of \( \tilde{\pi}(w) \):

\[
\tilde{\pi}_j(w) = \prod_{x=j}^{m(w)} t_x^{z_x(w)} h_x^{t_x(w)}.
\]

Then, the following holds:
• $z(\pi)$, the number of standard OGS elementary factors of $\pi$ equals to the number of different non-zero values $\kappa_{-j}^{(v)}$, where $1 \leq v \leq z(\pi)$, and $1 \leq j \leq m^{(v)}$;

• For every $1 \leq w \leq z(\pi)$, $\text{maj}\left(\tilde{\pi}(w)\right) = \kappa_{-j}^{(v)}$ for some $1 \leq v \leq z(\pi)$, and for every given non-zero value of $\kappa_{-j}^{(v)}$, there exists exactly one $w$ such that

$$\text{maj}\left(\tilde{\pi}(w)\right) = \kappa_{-j}^{(v)}$$

(i.e., There is a one-to-one correspondence between the non-zero different values of $\kappa_{-j}^{(v)}$ and $\text{maj}\left(\tilde{\pi}(w)\right)$);

• If $j \neq 1$, or $j \neq m^{(v)}$ for some $1 \leq w \leq z(\pi)$, then $\tilde{h}_{j}^{(w)} = \kappa_{+j}^{(v)}$ for some $1 \leq v \leq z(\pi)$, and $1 \leq j \leq m^{(v)}$. (i.e., If $\tilde{h}_{j}^{(w)} = \kappa_{-j}^{(v)}$ for some $1 \leq v \leq z(\pi)$, and $1 \leq j \leq m^{(v)}$, then necessarily $j = 1$ or $j = m^{(v)}$);

• If $\tilde{h}_{m^{(w)}}^{(w)} = \kappa_{-j}^{(v)}$ for some $1 \leq v \leq z(\pi)$ and $1 \leq j \leq m^{(v)}$, then

$$\text{maj}\left(\tilde{\pi}(w)\right) = \tilde{z}_{m^{(w)}}^{(w)} = \sum_{v' = \tilde{v}^{(w)}_{j} + 1}^{z(\pi)} \tilde{\vartheta}_{j_{v'}^{(w)}}^{(v')},$$

where, $\kappa_{-j_{v'}^{(w)} - v}^{(v')} \leq \kappa_{-j}^{(v)} < \kappa_{+j_{v'}^{(w)} - v}^{(v')}$ and $\tilde{v}$ is the maximal value of $1 \leq v \leq z(\pi)$ such that $\kappa_{-j}^{(v)} = \kappa_{-j}^{(v)}$ for some $1 \leq j \leq m^{(v)}$;

• If $\tilde{h}_{j}^{(w)} = \kappa_{+j}^{(v)}$ for some $1 \leq v \leq z(\pi)$ and $1 \leq j \leq m^{(v)}$, then

$$\text{maj}\left(\tilde{\pi}_{j}^{(w)}\right) = \sum_{v' = \tilde{v}_{j}^{(w)} + 1}^{z(\pi)} \tilde{\vartheta}_{j_{v'}^{(w)}}^{(v')},$$

where, $\kappa_{-j_{v'}^{(w)} - v}^{(v')} < \kappa_{+j}^{(v)} \leq \kappa_{+j_{v'}^{(w)} - v}^{(v')}$ and $\tilde{v}$ is the minimal value of $1 \leq v \leq z(\pi)$ such that $\kappa_{+j}^{(v)} = \kappa_{+j}^{(v)}$ for some $1 \leq j \leq m^{(v)}$;

• If $\tilde{h}_{m^{(w)}}^{(w)} = \kappa_{+j}^{(v)}$ for some $1 \leq v \leq z(\pi)$, and $1 \leq j \leq m^{(v)}$ then one of the following holds:

  - $v = z(\pi)$;
  - For every $v' > v$, $\kappa_{+1}^{(v')} = \vartheta_{1}^{(v')} = 0$, and $\kappa_{+1}^{(v)} < \kappa_{+1}^{(v')}$;

• If $\tilde{h}_{1}^{(w)} = \kappa_{-j}^{(v)}$ for some $1 \leq v \leq z(\pi)$, and $1 \leq j \leq m^{(v)}$, then $\tilde{\nu}_{1}^{(w)} = \vartheta_{1}^{(w)} = \nu_{j-1}^{(w)} - \nu_{j-1}^{(v)}$, where $\tilde{v}$ is the minimal value of $1 \leq v \leq z(\pi)$ such that $\kappa_{-j}^{(v)} = \kappa_{-j}^{(v)}$ for some $2 \leq j \leq m^{(v)}$;

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• If $\tilde{h}_{1}^{(w)} = \kappa_{+;j}^{(v)}$ for some $1 \leq v \leq z(\pi)$, and $1 \leq j \leq m^{(v)}$ then
  \[
  \text{maj} \left( \tilde{\pi}^{(u)} \right) = \kappa_{-;1}^{(v)} < \tilde{h}_{1}^{(u)} = \kappa_{+;j}^{(v)};
  \]
  for some $1 \leq \tilde{v} \leq z(\pi)$;

• If $\text{maj} \left( \tilde{\pi}^{(u)} \right) = \kappa_{-;1}^{(v)}$, for some $1 \leq v \leq z(\pi)$, then
  \[
  \text{maj} \left( \tilde{\pi}^{(u)} \right) < \tilde{h}_{1}^{(u)};
  \]

• $\tilde{h}_{m^{(w-1)}}^{(w-1)} = \kappa_{+;j}^{(v)}$ and $\tilde{h}_{1}^{(w)} = \kappa_{+;j}^{(v)}$ for some $1 \leq v, \tilde{v} \leq z(\pi)$, $1 \leq j \leq m^{(v)}$, and $1 \leq \tilde{j} \leq m^{(\tilde{v})}$ if and only if
  \[
  \tilde{h}_{m^{(w-1)}}^{(w-1)} < \text{maj} \left( \tilde{\pi}^{(u)} \right) = \kappa_{-;1}^{(v)} = \kappa_{-;j+1}^{(v)} < \tilde{h}_{1}^{(w)};
  \]
  for some $1 \leq \tilde{v} < m^{(w)}$, and $v$ satisfied one of the following conditions:
  
  - $v = z(\pi)$;
  - For every $v' > v$, $\kappa_{-;1}^{(v')} = \vartheta_{1}^{(v')} = 0$, and $\kappa_{-;j+1}^{(v')} < \kappa_{+;1}^{(v')}$.

**Proof.** Consider the standard OGS elementary factorization of $\tilde{\pi}$, with the elementary factors
  \[
  \tilde{\pi}^{(w)} = \prod_{j=1}^{m^{(w)}} \tilde{t}_{j}^{(w)}; 
  \]
  First, look at $\tilde{\pi}^{(w)}$ for $w = z(\tilde{\pi}^{(w)})$ (i.e., the last elementary factor of $\tilde{\pi}^{(w)})$. Then, by Proposition 3.48
  \[
  \tilde{h}_{m^{(w)}}^{(w)} = \kappa_{+;m^{(z(\pi))}}^{(z(\pi))}.
  \]

  Let $\beta$ be the largest $1 \leq x \leq r$ such that $a_{x} = \kappa_{-;j_{x}}^{(v_{x})}$ for some $1 \leq v_{x} \leq z(\pi)$, and $1 \leq j_{x} \leq m^{(v_{x})}$. Then, for every $x > \beta$, $a_{x} = \kappa_{+;j_{x}}^{(v_{x})}$. Therefore, by Proposition 3.39
  $j_{\beta} = m^{(v_{\beta})}$ and $j_{\beta} = m^{(v_{\beta})}$, for all $x > \beta$. Moreover, since $a_{x} = \kappa_{+;j_{x}}^{(v_{x})}$, for $x > \beta$, by Proposition 3.48 we conclude, $v_{x_{1}} < v_{x_{2}}$ for every $\beta < x_{1} < x_{2} \leq r$. Thus, by considering
  \[
  \prod_{x=\beta}^{r} \eta_{a_{x}}^{i_{x}}
  \]
  The following holds:
  \[
  a_{\beta} = \kappa_{-;m^{(v_{\beta})}}^{(v_{\beta})}, \quad i_{\beta} = \kappa_{-;m^{(v_{\beta})}}^{(v_{\beta})} - \vartheta_{m^{(v_{\beta})}}^{(v_{\beta})} - \eta_{m^{(v_{\beta})}}^{(v_{\beta})} - 1;
  \]
  where,
By Definition 3.37:
\[
\kappa(v_{\beta}) = \varrho_m(v_{\beta}) + \sum_{v' = v_{\beta} + 1}^{z(\pi)} \varphi(v') m(v'),
\]
such that:
\[
\kappa(v_{\beta}) - m(v_{\beta}) \leq \kappa(v_{\beta}) < \kappa(v_{\beta} + 1).
\]

By Definition 3.46:
\[
\eta(v_{\beta}) = \sum_{v' = v_{\beta} - 1}^{z(\pi)} \varphi(v') m(v'),
\]
such that:
\[
\kappa(v_{\beta}) - m(v_{\beta}) \leq \kappa(v_{\beta}) < \kappa(v_{\beta} + 1).
\]

For \(\beta + 1 \leq x \leq r\):
\[
a_x = \kappa_{m(v_{\beta})}(v_{\beta}), \quad i_x = \varphi_{m(v_{\beta})},
\]
such that by Definition 3.43 and by Propositions 3.40, 3.48 the following properties hold:
\[
- \quad v_{\beta + 1} = \chi_{m(v_{\beta})} - 1 + 1;
- \quad v_{x + 1} = v_x + 1, \text{ for } \beta < x < r;
- \quad v_r = z(\pi).
\]

Thus, by using Definition 3.37 for every \(x\) such that \(\beta + 1 \leq x \leq r\) the following are satisfied:
\[
a_x = \kappa_{m(v_{\beta})}(v_{\beta}) = \varrho_{m(v_{\beta})} + \sum_{v' = v_x}^{z(\pi)} \varphi(v') m(v') > \sum_{v' = v_x}^{z(\pi)} \varphi(v') m(v') = \sum_{x' = x}^{r} \varphi_{m(v_{\beta})},
\]
Hence, by Definition 3.26
\[
\prod_{x = \beta + 1}^{r} \frac{t_{i_x}}{a_x}
\]
is a terminal segment of \(\tilde{\pi}(w)\), for \(w = z(\pi)\).

Since \(\eta_{m(v_{\beta}) - 1} = \sum_{v' = \chi_{m(v_{\beta})} - 1}^{v_{\beta} - 1} \varphi(v') m(v')\), by using Proposition 3.47.
\[a_\beta = \kappa_{-\cdot m(v_\beta)}^{(v_\beta)} = \varrho_{m(v_\beta)}^{(v_\beta)} - 1 + \sum_{v' = v_\beta} z(\pi) \varrho_{m(v')}^{(v')} m(v')\]

\[\geq \kappa_{-\cdot m(v_\beta)}^{(v_\beta)} = a_\beta.\]

We have also

\[\sum_{x = \beta + 1}^r i_x = i_\beta + \sum_{x = \beta + 1}^r i_x\]

\[= \kappa_{-\cdot m(v_\beta)}^{(v_\beta)} - \sum_{v' = \chi^{(v_\beta)}_{m(v_\beta)} + 1}^{v_\beta - 1} \varrho_{m(v')}^{(v')} + \sum_{v' = \chi^{(v_\beta)}_{m(v_\beta)} + 1} z(\pi) \varrho_{m(v')}^{(v')} m(v')\]

\[= \kappa_{-\cdot m(v_\beta)}^{(v_\beta)} + z(\pi) \varrho_{m(v')}^{(v')}\]

\[\geq \kappa_{-\cdot m(v_\beta)}^{(v_\beta)} = a_\beta.\]

Therefore, by considering

\[\tilde{\pi}(w) = \prod_{j=1}^{m(w)} t_{j_1}^{(w)} h_j^{(w)};\]

for \(w = z(\tilde{\pi})\), the following holds:

\[\text{• In case } m_l^{(v_\beta)} > 1 \text{ (i.e., } \varrho_{m_l^{(v_\beta)} - 1}^{(v_\beta)} > 0):\]

\[\tilde{h}_1^{(w)} = \kappa_{-\cdot m_l^{(v_\beta)}}^{(v_\beta)}; \quad \tilde{t}_1^{(w)} = \varrho_{m_l^{(v_\beta)} - 1}^{(v_\beta)} - \eta_{m_l^{(v_\beta)} - 1}^{(v_\beta)}.\]
If $2 \leq j \leq m^{z(\pi)}$, then
\[ \tilde{h}_j(w) = \kappa_{+;m^{(v_x)}}^{(v_x)}, \]
for some $x > \beta$. Moreover, by Definition 3.45 and Proposition 3.40 for every $\chi_{m^{(v)}} - 1 < v \leq z(\pi)$, there exists $2 \leq j \leq m^{z(\pi)}$, such that
\[ \tilde{h}_j(w) = \kappa_{+;m^{(v)}}^{(v)}, \]
Since there is a possibility for $\kappa_{+;m^{(v)}}^{(v)} = \kappa_{+;m^{(v)}}^{(v')}$, where $v' \neq v$,
\[ \tilde{\pi}_j(w) = \sum_{x=x_0}^{x_0} \theta_{m^{(v_x)}}^{(v_x)}, \]
such that $\kappa_{+;m^{(v_x)}}^{(v_x)} = \kappa_{+;m^{(v_x)}}^{(v_x)}$ for every $x$ where, $\hat{x} \leq x \leq x_0$. Therefore, for every $2 \leq j \leq m^{(w)}$,
\[ \text{maj} (\tilde{\pi}_j) = \sum_{x=x_0}^{z(\pi)} \theta_{m^{(v_x)}}^{(v_x)}, \]
such that $\hat{x}$ is the minimal $x > \beta$, such that
\[ \tilde{h}_j(w) = \kappa_{+;m^{(v_x)}}^{(v_x)}, \]
By former observation:
\[ \text{maj} (\tilde{\pi}_j(w)) = \sum_{j=1}^{m^{(w)}} \tilde{h}_j(w) = \kappa_{-;m^{(v_\beta)}}^{(v_\beta)}; \]

- In case, where $m^{(v_\beta)} = 1$, we have, $\theta_{m^{(v_\beta)}}^{(v_\beta)}_{m^{(v_\beta)}-1} = 0$. Thus, by Propositions 3.47 and 3.48
\[ \chi_{m^{(v_\beta)} - 1}^{(v_\beta)} = 0, \quad \chi_{m^{(v_\beta)} - 1}^{(v_\beta)} = v_\beta - 1. \]
Thus,
\[ \tilde{h}_j(w) \neq \kappa_{-;m^{(v_\beta)}}^{(v_\beta)}, \]
and therefore, we have for every $1 \leq j \leq m^{(w)}$:
\[ \tilde{h}_j(w) = \kappa_{+;m^{(v_x)}}^{(v_x)} \]
for some $x > \beta$, where the other observations are the same as in the case of $m^{(v_\beta)} > 1.$
In particular,
\[ \text{maj} \left( \pi^{(w)}(w) \right) = \sum_{j=1}^{m^{(w)}} \tilde{i}^{(w)}_j = \kappa^{(v_{\beta})}_{m^{(v_{\beta})}}; \]
even in case where, \( \tilde{h}^{(w)}_1 \neq \kappa^{(v_{\beta})}_{-m^{(v_{\beta})}}. \)

Now, look at \( \tilde{h}^{(w-1)}_m \) for \( w = z(\pi). \) Let \( \zeta \) be the number of different values of \( x, \) such that \( a_x = a_{\beta}, \) i.e.,
\[ a_x = \kappa^{(v_x)}_{m^{(v_{\beta})}} = \kappa^{(v_{\beta})}_{-m^{(v_{\beta})}} = a_{\beta}, \]
for \( \beta - \zeta + 1 \leq x \leq \beta. \) Notice, by the assumption of the proposition, \( v_{x_1} > v_{x_2} \) for \( \beta - \zeta + 1 \leq x_1 < x_2 \leq \beta. \) Thus, the maximal value of \( v_x \) such that \( \kappa^{(v_x)}_{m^{(v_x)}} = \kappa^{(v_{\beta})}_{-m^{(v_{\beta})}} \)
is for \( x = \beta - \zeta + 1. \) Notice, \( \tilde{i}^{(w)}_1 = \varrho^{(v_{\beta})}_{m^{(v_{\beta})}} - \eta^{(v_{\beta})}_{m^{(v_{\beta})}} \)
in case \( \tilde{h}^{(w)}_1 = \kappa^{(v_{\beta})}_{-m^{(v_{\beta})}}. \) Therefore,
\[ \sum_{x = \beta - \zeta + 1}^{\beta} i_x - \left( \varrho^{(v_{\beta})}_{m^{(v_{\beta})}} - \eta^{(v_{\beta})}_{m^{(v_{\beta})}} \right), \]
the possibly exponent of \( \tilde{h}^{(w-1)}_m \in \tilde{h}^{(w-1)}_m = \kappa^{(v_{\beta})}_{m^{(v_{\beta})}}. \) Then, by using the definitions of \( i_x, \ k^{(v_x)}_{m^{(v_{\beta})}}, \) and \( \eta^{(v_x)}_{m^{(v_{\beta})}}, \) for \( \beta - \zeta + 1 \leq x \leq \beta, \) we conclude the following property:
\[ \sum_{x = \beta - \zeta + 1}^{\beta - 1} i_x - \left( \varrho^{(v_{\beta})}_{m^{(v_{\beta})}} - \eta^{(v_{\beta})}_{m^{(v_{\beta})}} \right) \]
\[ = \sum_{x = \beta - \zeta + 1}^{\beta - 1} \left( k^{(v_x)}_{m^{(v_{\beta})}} - \varrho^{(v_x)}_{m^{(v_{\beta})}} - \eta^{(v_x)}_{m^{(v_{\beta})}} \right) + \]
\[ + \sum_{v' = v_{\beta} + 1}^{m^{(v_{\beta})}} \varrho^{(v')}_{m^{(v')}} - \left( \varrho^{(v_{\beta})}_{m^{(v_{\beta})}} - \eta^{(v_{\beta})}_{m^{(v_{\beta})}} \right) \]
\[ = (\zeta - 1) \cdot k^{(v_{\beta})}_{-m^{(v_{\beta})}} + \sum_{v' = v_{\beta} - \zeta + 1 + 1}^{m^{(v_{\beta})}} \varrho^{(v')}_{m^{(v')}}. \]
Therefore, the following holds:
- In case, where there exists at least one \( v' > v_{\beta - \zeta + 1}, \) such that \( \varrho^{(v')}_{m^{(v')}} > 0, \)
\[ \tilde{h}^{(w-1)}_{m^{(w-1)}} = k^{(v_{\beta})}_{-m^{(v_{\beta})}}; \]
\[ \tilde{i}^{(w-1)}_{m^{(w-1)}} = \sum_{v' = v_{\beta - \zeta + 1} + 1}^{w} \varrho^{(v')}_{m^{(v')}}; \]
In case, where $\vartheta^{(v')}_{m^{(v')}} = 0$, for every $v' > v_{\beta - \zeta + 1}$, which means necessarily $v_{\beta - \zeta + 1} = z(\pi)$ (The case where for every $v' > v_{\beta - \zeta + 1}$, we have $\kappa^{(v')}_{-1:1} = \vartheta^{(v')}_{1} = 0$, $\kappa^{(v')}_{-m^{(v')}_{-1:1}} < \kappa^{(v')}_{+1:1}$ and $m^{(v')}_{1} = 1$ cannot happen by Proposition 3.44):

$$\sum_{v' = v_{\beta - \zeta + 1} + 1}^{z(\pi)} \vartheta^{(v')}_{m^{(v')}} = 0.$$ 

Therefore, $\tilde{h}^{(w-1)}_{m^{(w-1)}} \neq \kappa^{(v_{\beta})}_{-m^{(v_{\beta})}_{-1:1}}$.

Notice the following observation:

In case, where both $m^{(v_{\beta})} = 1$ and $v_{\beta - \zeta + 1} = z(\pi)$:

$$a_{\beta} = \kappa^{(v_{\beta})}_{-1:1} = \vartheta^{(v_{\beta})}_{1}, \quad i_{\beta} = 0.$$ 

Therefore, there is no $1 \leq w \leq z(\tilde{\pi})$, and $1 \leq j \leq m^{(w)}$, such that $\tilde{h}^{(j)}_{j} = \kappa^{(v_{\beta})}_{-1:1}$.

Although, $maj\left(\tilde{\pi}(z(\tilde{\pi}))\right) = \kappa^{(v_{\beta})}_{-1:1}$, i.e.,

$$\tilde{h}^{(z(\tilde{\pi}) - 1)}_{m^{(z(\tilde{\pi}) - 1)}} < \kappa^{(v_{\beta})}_{-m^{(v_{\beta})}_{-1:1}} < \tilde{h}^{(z(\tilde{\pi}))}_{1}.$$ 

Thus, all the parts of the proposition hold for the terminal elementary factor $\tilde{\pi}(z(\tilde{\pi}))$ of $\tilde{\pi}$. Now, assume in induction, the proposition holds for $\tilde{\pi}^{(w')}$, where $w' > w$ for some $1 \leq w < z(\tilde{\pi})$, and we prove it for $w' = w$. Consider

$$\tilde{\pi}^{(w)} = \prod_{j=1}^{m^{(w)}} \tilde{h}^{(j)}_{j}.$$ 

By the induction hypothesis,

$$a_{\alpha} = maj\left(\tilde{\pi}^{(w+1)}\right) = \kappa^{(v_{\alpha})}_{-1:1},$$

for some $1 \leq v_{\alpha} \leq z(\pi)$ and $1 \leq j_{\alpha} \leq m^{(v_{\alpha})}$ ($\beta$ is just $\alpha$ for $w' = z(\tilde{\pi})$). For every $v' > \chi^{(v_{\alpha})}_{1:1}$, let $j_{v':\alpha}$ be a positive integer such that

$$\kappa^{(v')}_{-j_{v':\alpha}} \leq \kappa^{(v_{\alpha})}_{-1:1} < \kappa^{(v')}_{+j_{v':\alpha}}.$$ 

Let $\tilde{v}_{\alpha}$ be the maximal value of $v_{\alpha} \leq v \leq z(\pi)$, such that $\kappa^{(v_{\alpha})}_{-j_{\alpha}} = \kappa^{(v_{\alpha})}_{-1:1}$ for some $1 \leq j_{\alpha} \leq m^{(v_{\alpha})}$. Then, by the induction hypothesis the following holds:
• If there exists at least one \( v' > \bar{v}_\alpha \), such that \( \vartheta_j^{(v')} > 0 \),

\[
\tilde{h}_m^{(w)} = \kappa_{j\alpha}^{(v_\alpha)} , \quad \tilde{r}_m^{(w)} = \sum_{\bar{v}_\alpha + 1}^{z(\pi)} \vartheta_j^{(v')} .
\]

• If \( \vartheta_j^{(v')} = 0 \), for every \( v' > \bar{v}_\alpha \) (i.e., either \( \bar{v}_\alpha = z(\pi) \) or for every \( v' > \bar{v}_\alpha \) we have \( \vartheta_1^{(v')} = 0 \) and \( \kappa_{j\alpha}^{(v_\alpha)} < \kappa_{\bar{v}_\alpha + 1}^{(v_\alpha)} \), then:

\[
\sum_{v'=\bar{v}_\alpha + 1}^{z(\pi)} \vartheta_j^{(v')} = 0 .
\]

Therefore,

\[
\tilde{h}_m^{(w)} = \kappa_{j\alpha}^{(v_\alpha)} .
\]

Since by the induction hypothesis,

\[
a_\alpha = \max \left( \tilde{\pi}^{(w+1)} \right) = \kappa_{-j\alpha}^{(v_\alpha)},
\]

by Proposition 3.48 either

\[
\tilde{h}_m^{(w)} = \kappa_{-\bar{v}}
\]

such that \( \bar{v} > v_\alpha \), or

\[
\tilde{h}_m^{(w)} = \kappa_{\bar{v} + j\alpha - 1}^{(v_\alpha)} .
\]

Let \( \alpha' \) be the largest integer such that \( \alpha' < \alpha \), and \( a_{\alpha'} = \kappa_{-j\alpha'}^{(v_\alpha)} < a_\alpha \), for some \( 1 \leq v_\alpha' \leq z(\pi) \) and \( 1 \leq j_{\alpha'} \leq m(v_\alpha') \). Therefore, \( a_x = \kappa_{+j_x}^{(v_x)} \) for every \( x \) such that \( x > \alpha' \) and \( a_x < \kappa_{-j_x}^{(v_x)} \). If \( x_1 < x_2 \) and \( \kappa_{-j\alpha'}^{(v_x)} < a_x \leq a_{x_2} < \kappa_{-j\alpha}^{(v_x)} \), then by Proposition 3.48 \( v_{x_1} < v_{x_2} \). Notice, the maximal possible value of \( v_x \) such that \( a_x = \kappa_{+j_x}^{(v_x)} \) is \( v_x = \bar{v}_\alpha \) and then \( a_x = \kappa_{+j_x - 1}^{(v_x)} \), where \( a_{x+1} = \kappa_{-j_x}^{(v_x)} = \kappa_{-j_x}^{(v_\alpha)} \). Therefore, if

\[
a_x = \kappa_{+j_x}^{(v_x)} = \vartheta_j^{(v_x)} + \sum_{v'>v_x+1}^{z(\pi)} \vartheta_j^{(v')},
\]

where, \( j_{v',x} \) is a positive integer such that

\[
\kappa_{-j_{v',x}}^{(v_x)} < \kappa_{+j_x}^{(v_x)} \leq \kappa_{+j_{v',x}}^{(v_x)} ;
\]

then, by Proposition 3.48 \( \chi_j^{(v_x)} = v_{x-1} \), in case \( x > \alpha' + 1 \). Therefore,

\[
i_x = \vartheta_j^{(v_x)} + \eta_j^{(v_x)} = \sum_{v'=v_x+1}^{v_x-1} \vartheta_j^{(v')} + \vartheta_j^{(v_x)},
\]

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and for \( x = \alpha' + 1 \):

\[
i_x = \vartheta_{j_x}^{(v_x)} + \eta_{j_x}^{(v_x)} = \sum_{v' = \chi^{(v_x)}_x + 1}^{v_x - 1} \vartheta_{j_x}^{(v')} + \vartheta_{j_x}^{(v_x)}.
\]

Therefore, by using Proposition 3.47

\[
a_x = \kappa_{+j_x}^{(v_x)} = \vartheta_{j_x}^{(v_x)} + \vartheta_{j_x}^{(v_x)} + \sum_{v' = v_x + 1}^{z(\pi)} \vartheta_{j_x}^{(v')} > \eta_{j_x}^{(v_x)} + \vartheta_{j_x}^{(v_x)} + \sum_{v' = v_x + 1}^{z(\pi)} \vartheta_{j_x}^{(v')} = \sum_{x' = x}^{\alpha - 1} i_{x'} = maj \left( \hat{\pi}^{(w)}_x \right).
\]

Hence, by Definition 3.26

\[
\prod_{x = \alpha' + 1}^{\alpha - 1} t_{a_x}^{i_x}
\]

is a terminal segment of \( \hat{\pi}^{(w)} \).

Notice, \( x = \alpha' \) is the largest \( x \), such that \( a_{\alpha'} = \kappa_{-j_x}^{(v_x)} < \kappa_{-j_x}^{(v_0)} \). Therefore, \( j_{v':x} = j_{v':\alpha'} \), for every \( \chi_j^{(v_x)} < v' < v_x \), and \( j_{v'} = j_{v':\alpha'} \) for \( v' = v_x \), in case \( a_x = \kappa_{+j_x}^{(v_x)} \) for \( x > \alpha' \) and \( a_x < \kappa_{-j_x}^{(v_0)} \). By Proposition 3.48 \( \chi_{j_x}^{(v_x)} = \chi_{j_x}^{(v_x+1)} \), in case \( a_{\alpha'+1} = \kappa_{+j_x}^{(v_x+1)} \). Thus,

\[
a_{\alpha'} = \kappa_{-j_x}^{(v_{\alpha'})} = \vartheta_{j_x}^{(v_{\alpha'})} + \vartheta_{j_x}^{(v_{\alpha'})} + \sum_{v' = v_{\alpha'} + 1}^{z(\pi)} \vartheta_{j_x}^{(v')} > \eta_{j_x}^{(v_{\alpha'})} + \vartheta_{j_x}^{(v_{\alpha'})} + \sum_{v' = v_{\alpha'} + 1}^{z(\pi)} \vartheta_{j_x}^{(v')} = \sum_{x' = x}^{\alpha - 1} i_{x'} = maj \left( \hat{\pi}^{(w)}_{\alpha'+1} \right).
\]

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We have also
\[
\sum_{x=\alpha'}^{\alpha-1} i_x = i_{\alpha'} + \sum_{x=\alpha'+1}^{\alpha-1} i_x = \kappa_{-j\alpha'}(v_{\alpha'}) - \vartheta_{j\alpha'}(v_{\alpha'}) - \sum_{v' = \chi_{j\alpha'}^{-1}}^{v_{\alpha'}-1} \vartheta_{j'\alpha'}(v') + \sum_{v' = \chi_{j\alpha'}^{-1}+1}^{z(\pi)} \vartheta_{j'\alpha'}(v') = \kappa_{-j\alpha'}(v_{\alpha'}) = a_{\alpha'}.
\]

Hence, by the same argument as in the case of \(\tilde{\pi}(z(\pi))\) the following holds for
\[
\tilde{\pi}(w) = \prod_{j=1}^{m(w)} \tilde{t}_j(w):\]

**In case** \(j_{\alpha'} > 1\) (i.e., \(\vartheta_{j\alpha'}(v_{\alpha'}) > 0\)):
\[
\tilde{h}_1(w) = \kappa_{-j\alpha'}(v_{\alpha'}), \quad \tilde{t}_1(w) = \tilde{\vartheta}_{j\alpha'}(v_{\alpha'}) = \vartheta_{j\alpha'}(v_{\alpha'}) - \eta_{j\alpha'-1}(v_{\alpha'}).
\]

If \(j \neq 1\) or \(j \neq m(w)\) (i.e., \(\tilde{h}_1(w)\) is not the first and is not the last letter of \(\tilde{\pi}(w)\)) then,
\[
\tilde{h}_j(w) = \kappa_{+j\alpha'}(v_x),
\]
for some \(x > \alpha'\), such that \(v_x \leq \tilde{v}_x\). Now we consider \(\tilde{t}_1(w)\). Let \(\zeta'\) be the number of \(x > \alpha'\) such that \(a_x = \kappa_{+j\alpha'}(v_x)\), for a specific \(1 \leq v \leq z(\pi)\) and \(1 \leq j \leq m(v)\) (i.e., \(\kappa_{+j\alpha'}(v_x) = \kappa_{+j\alpha'}(v_x)\) for every \(\alpha' < x' < x < x' + \zeta' - 1\)). Then,
\[
\tilde{t}_j(w) = \sum_{v' = \chi_{j\alpha'}^{-1}}^{v_{x'+\zeta'-1}} \vartheta_{j'\alpha'}(v'),
\]
where,
- \(\chi_{j\alpha'}(v_{x'}) = v_{x'-1} + 1\) for \(x' > \alpha' + 1\);
- \(\chi_{j\alpha'}(v_{x'}) = \chi_{j\alpha'-1}(v_{x'})\) for \(x' = \alpha' + 1\);
- \(x = x'\) is the minimal value of \(v_x\) such that \(\kappa_{+j\alpha'}(v_x) = \kappa_{+j\alpha'}(v_{x'})\);
- \(x = x' + \zeta' - 1\) is the maximal value of \(v_x\) such that \(\kappa_{+j\alpha'}(v_{x'+\zeta'-1}) = \kappa_{+j\alpha'}(v_{x' + \zeta' - 1})\) for some \(1 \leq v \leq z(\pi)\) and \(1 \leq j \leq m(z(\pi))\).
Now, we consider $h_{m(w)}$. By the induction hypothesis: In case of $h_{m(w)} = \kappa_{v:j}$ for some $1 \leq v \leq z(\pi)$, and $1 \leq j \leq m(v)$, one of the following holds:

- either $v = z(\pi)$;
- or for every $v' > v$, $\kappa(v') = 0$, and $\kappa_{v:j-1} < \kappa_{v:j}$.

Otherwise,

$$h_{m(w)} = \kappa_{v:j}, \quad z_{m(w)} = \sum_{v' = v+1}^{\pi} \theta(v').$$

Therefore,

$$\text{maj}(\tilde{\pi}(w)) = \sum_{j=1}^{m(w)} \tilde{z}(w) = \kappa(v'_{\alpha} + 1).$$

- In case, where $j_{\alpha'} = 1$, we have, $\theta_{j_{\alpha'-1}} = 0$. Thus, by Propositions 3.47, 3.48

$$\eta(v_{\alpha'}) = 0, \quad \chi_{j_{\alpha'-1}} = v_{\alpha'} - 1.$$ 

Therefore,

$$\tilde{h}_1(w) \neq \kappa(v_{\alpha'}).$$

The further $\tilde{h}_j(w)$ is the same like in the case of $j_{\alpha'} > 1$. Also,

$$\text{maj}(\tilde{\pi}(w)) = \sum_{j=1}^{m(w)} \tilde{z}(w) = \kappa(v_{\alpha'})$$

even in case where, $\tilde{h}_1(w) \neq \kappa(v_{\alpha'}).$

Now, look at $h_{m(w-1)}$. Let $\zeta$ be the number of different values of $x$, such that $a_x = a_{\alpha'}$, i.e.,

$$a_x = \kappa_{j_x} = \kappa_{j_{\alpha'}},$$

for $\alpha' - \zeta + 1 \leq x \leq \alpha'$. Then, by the same argument as in the case of $w = z(\pi)$ the following holds:

- In case, where there exists at least one $v' > v_{\alpha'-\zeta+1}$, such that $\theta_{j_{\alpha',v'}} > 0$,

$$h_{m(w-1)} = \kappa(v_{\alpha'}), \quad z_{m(w-1)} = \sum_{v' = v_{\alpha'-\zeta+1}+1}^{w} \theta(v').$$
• In case, where \( \vartheta_{j,v',\alpha'} = 0 \), for every \( v' > v_{\alpha'-\xi+1} \), by Proposition 3.34 either 
\[ v_{\alpha'-\xi+1} = z(\pi) \text{ or for every } v' > v_{\alpha'-\xi+1}, \text{ we have } \kappa_{\alpha'-\xi+1}(v') = 0 \text{ and } \kappa_{\alpha'-\xi+1}(v') < \kappa_{\alpha'-\xi+1}(v') \]. Then, the following holds:

\[ \sum_{v'=v_{\alpha'-\xi+1}}^{z(\pi)} \vartheta_{j,v',\alpha'} = 0. \]

Therefore,

\[ \tilde{H}_{m(w-1)}(w-1) = \kappa_{\alpha'-\xi+1}(v'). \]

Thus, the proposition has been proved for \( w' = w \). Hence, the proposition holds for \( \tilde{w} \).

The next theorem proves the formula for the standard OGS canonical form of \( \pi^{-1} \) for an arbitrary \( \pi \in S_n \), which is presented in the standard OGS canonical form as well.

**Theorem 3.50** Let \( \pi \in S_n \), presented in standard OGS elementary factorization, with all the notations used in Definition 3.39. For every \( 1 \leq v \leq z(\pi) \) and \( 1 \leq j \leq m(v) \), let \( \vartheta_j^{(v)}, \kappa_{\alpha+1}^{(v)}, \kappa_{\alpha-j}^{(v)} \), and \( \eta_j^{(v)} \) be non-negative integers as defined in Definitions 3.37 and 3.40. Then, the standard OGS canonical form of \( \pi^{-1} \) is \( \prod_{j=1}^{\tilde{m}} \tilde{k}_j \), where \( \tilde{k}_j \), \( \tilde{k}_j \), satisfy the following conditions for every \( 1 \leq j \leq \tilde{m} \):

1. \( \tilde{k}_j = \kappa_{\alpha-j}^{(v)} \) or \( \tilde{k}_j = \kappa_{\alpha+1}^{(v)} \), for \( 1 \leq v \leq z(\pi) \), and \( 1 \leq j \leq m(v) \);
2. If \( \tilde{k}_j = \kappa_{\alpha+1}^{(v)} \), then \( \tilde{k}_j = \sum_{\gamma} (\vartheta_{j,\gamma}^{(v)} + \eta_{j,\gamma}^{(v)}) \), such that \( \kappa_{\alpha-j+1}^{(v)} = \kappa_{\alpha-j+1}^{(v)} \);
3. If \( \tilde{k}_j = \kappa_{\alpha-j}^{(v)} \), then \( \tilde{k}_j = \kappa_{\alpha-j}^{(v)} - \sum_{\gamma} (\vartheta_{j,\gamma}^{(v)} + \eta_{j,\gamma}^{(v)}) \), such that \( \kappa_{\alpha-j}^{(v)} = \kappa_{\alpha-j}^{(v)} \);
4. \( z(\pi^{-1}) \), the number of standard OGS elementary factors of \( \pi^{-1} \) equals to the number of different non-zero values \( \kappa_{\alpha-j}^{(v)} \), where \( 1 \leq v \leq z(\pi) \), and \( 1 \leq j \leq m(v) \);
5. For every \( 1 \leq w \leq z(\pi^{-1}) \), \( \text{maj} \left( \pi^{-1}(w) \right) = \kappa_{\alpha-j}^{(v)} \) for some \( 1 \leq v \leq z(\pi) \), and for every given non-zero value of \( \kappa_{\alpha-j}^{(v)} \), there exists exactly one \( w \) such that \( \text{maj} \left( \pi^{-1}(w) \right) = \kappa_{\alpha-j}^{(v)} \) (i.e., There is a one-to-one correspondence between the non-zero different values of \( \kappa_{\alpha-j}^{(v)} \) and the descent set of \( \pi^{-1} \)).

**Proof.** The proof is in induction on \( z(\pi) \). If \( z(\pi) = 1 \), then \( \pi \) is a standard OGS elementary element. Then, by Definition 3.40 \( \pi_1^{(1)} = 0 \), for every \( 1 \leq j \leq m \). Therefore, by Definition 3.40 \( \eta_j^{(1)} = 0 \). By Definition 3.37 \( \kappa_{\alpha-j}^{(1)} - \vartheta_j^{(1)} = \vartheta_{j-1}^{(1)} \). Therefore, the result of the theorem for the case of \( \pi \) a standard OGS elementary element holds by Theorem 3.34. Now, assume in induction that the theorem holds for \( z' < z(\pi) \), and we prove...
the correctness of it for \( z' = z(\pi) \). Denote by \( \pi' \) the element \( \prod_{\nu=1}^{\pi(z(\pi))} \pi(\nu) \), and consider \( \pi = \pi' \cdot \pi(z(\pi)) \). For a convenience we write \( \varrho_j, \vartheta_j, \kappa_{+, j}, \kappa_{- j}, \chi_j \), and \( \eta_j \) instead of \( \varrho_j(z(\pi)), \vartheta_j(z(\pi)), \kappa_{+, j}(z(\pi)), \kappa_{- j}(z(\pi)), \chi_j(z(\pi)), \), and \( \eta_j(z(\pi)) \). Obviously,

\[
\pi^{-1} = (\prod_{\nu=1}^{\pi(z(\pi))} \pi(\nu))^{-1} = (\pi(z(\pi)))^{-1} \cdot \pi'^{-1}.
\]

Thus, by using Theorem 3.34

\[
\pi^{-1} = \prod_{j=1}^{m(\pi(z(\pi)))} \left( t_{\varrho_{- j}} \cdot t_{\vartheta_{j}} \right) \cdot \pi'^{-1},
\]

where, we consider \( \varrho_0 = 0 \). By the induction hypothesis, the standard OGS canonical form of \( \pi'^{-1} \) is as follows:

\[
\pi'^{-1} = \prod_{d=1}^{m'} t_{k'_d}
\]

where

\[
k'_d \in \{ \kappa_{-, j}^{(v \rightarrow z(\pi) - 1)}, \kappa_{+, j}^{(v \rightarrow z(\pi) - 1)} \mid 1 \leq v \leq z(\pi) - 1, 1 \leq j \leq m(\nu) \},
\]

- If \( k'_d = \kappa_{-, j}^{(v \rightarrow z(\pi) - 1)} \), then
  \[
i_{k'_d} = \kappa_{-, j}^{(v \rightarrow z(\pi) - 1)} - \sum_{\gamma} \left( \varrho_{\gamma j}^{(v \gamma)} + \eta_{\gamma j}^{(v \gamma)} \right), \]

  where, \( \kappa_{-, j}^{(v \rightarrow z(\pi) - 1)} = \kappa_{-, j}^{(v \rightarrow z(\pi) - 1)} \).

- If \( k'_d = \kappa_{+, j}^{(v \rightarrow z(\pi) - 1)} \), then
  \[
i_{k'_d} = \sum_{\gamma} \left( \vartheta_{\gamma j}^{(v \gamma)} + \eta_{\gamma j}^{(v \gamma)} \right)
\]

  where, \( \kappa_{+, j}^{(v \rightarrow z(\pi) - 1)} = \kappa_{+, j}^{(v \rightarrow z(\pi) - 1)} \).

Now, consider \( t_{k'_d}^{i_{k'_d}} \) for \( 1 \leq d \leq m' \). By Proposition 3.32 \( t_{k'_d}^{i_{k'_d}} \) commutes with every sub-word of \( \left( \pi(z(\pi)) \right)^{-1} \) of the form \( t_{\varrho_{- j}}^{\varrho_{- j}} \cdot t_{\vartheta_{j}}^{\vartheta_{j}} \), such that \( k'_d \leq \varrho_{j-1} \). Therefore, the following holds: If \( k'_d \) for some \( 1 \leq d \leq m' \) satisfies that \( \varrho_{g-1} < k'_d \leq \varrho_{g} \), for some \( 1 \leq g \leq m(\pi) \), then \( t_{k'_d}^{i_{k'_d}} \) commutes with the sub-word

\[
\prod_{j=g+1}^{m(\pi)} \left( t_{\varrho_{- j}}^{\varrho_{- j}} \cdot t_{\vartheta_{j}}^{\vartheta_{j}} \right),
\]

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but does not commute with the sub-word
\[ t_{\kappa_{-i}^{\varphi}}^{\theta_{-i}^{\varphi}} \cdot t_{\kappa_{i}^{\varphi}}^{\theta_{i}^{\varphi}} \]

of \( \pi^{(\varphi)} \). For every \( 1 \leq j \leq m^{(\varphi)} \), denote by \( \mu_j \) the number of integers \( k'_d \) such that \( \varphi_{j-1} < k'_d \leq \varphi_j \), where \( 1 \leq d \leq m' \), denote by \( (\pi')^{-1}_j \) the sub-word \( \prod_{x=1}^{\mu_j} t_{k'^{j}_x} \) of \( \pi^{-1} \), where \( \varphi_{j-1} < k'^{j}_x \leq \varphi_j \). (It might happen that \( \mu_j = 0 \) for some \( j \), if there is no \( k'_d \) such that \( \varphi_{j-1} < k'_d \leq \varphi_j \). Then, \( (\pi')^{-1}_j \) is the empty word). Then, we have

\[
\pi^{-1} = \prod_{j=1}^{m^{(\varphi)}} \left( t_{\kappa_{-i}^{\varphi}}^{\theta_{-i}^{\varphi}} \cdot t_{\kappa_{i}^{\varphi}}^{\theta_{i}^{\varphi}} \cdot (\pi')^{-1}_j \right) = \prod_{j=1}^{m^{(\varphi)}} \left( t_{\kappa_{-i}^{\varphi}}^{\theta_{-i}^{\varphi}} \cdot t_{\kappa_{i}^{\varphi}}^{\theta_{i}^{\varphi}} \cdot \prod_{x=1}^{\mu_j} t_{k'^{j}_x} \right). \tag{3}
\]

If \( k'_m \leq \varphi_{m^{(\varphi)}}, \) then there is a terminal segment of \( \pi^{-1} \), which is a terminal segment of \( (\pi^{(\varphi)})^{-1} \). Then, denote by \( \varphi \) the value of \( j \) such that \( \varphi_{j-1} < k'_m \leq \varphi_j \), and denote by \( \pi_0^{-1} \) the terminal segment of \( \pi^{-1} \) of the form:

\[
\pi_0^{-1} = \prod_{j=\varphi+1}^{m^{(\varphi)}} \left( t_{\kappa_{-i}^{\varphi}}^{\theta_{-i}^{\varphi}} \cdot t_{\kappa_{i}^{\varphi}}^{\theta_{i}^{\varphi}} \right). \tag{4}
\]

Since \( k'_m \leq \varphi \), and \( k'_d < k'_m \) for every \( d < m' \), we have \( k'_d \leq \varphi \) for every \( j \geq \varphi \). Moreover, by the induction hypothesis, every \( \kappa^{(v+\pi(z)-1)} \) equals to \( k'_d \) for some \( 1 \leq d \leq m' \). Therefore, \( \kappa^{(v+\pi(z)-1)} \leq \varphi \) for every \( j \geq \varphi \). Then, by Proposition \ref{prop:3.39}, \( \kappa^{(v)}_{c+;m^{(v)}} \leq \kappa^{(v)}_{c;\varphi} \). Since by Proposition \ref{prop:3.39}, \( \kappa^{(v)}_{c+;\varphi} < \kappa^{(v)}_{c;j} \), \( \kappa^{(v)}_{c;j'} < \kappa^{(v)}_{c+;m^{(v)}} \), and \( \kappa^{(v)}_{c;\varphi} < \kappa^{(v)}_{c;j} \) for every \( j > \varphi \), \( 1 \leq v \leq \zeta^{(v)}(\varphi) - 1 \), \( 1 \leq j' \leq m^{(v)} - 1 \), and \( c \in \{+, -, \} \), we conclude \( \kappa^{(v)}_{c;j'} < \kappa^{(v)}_{c;j} \). Thus, by Definition \ref{def:3.46}, \( \eta_j = 0 \) for every \( j \geq \varphi \). Therefore, the Theorem holds for the terminal segment \( \pi_0^{-1} \) of \( \pi^{-1} \).

Now, we turn to the sub-word

\[
t_{\kappa_{-i}^{\varphi}}^{\theta_{-i}^{\varphi}} \cdot t_{\kappa_{i}^{\varphi}}^{\theta_{i}^{\varphi}} \cdot (\pi')^{-1}_\varphi = t_{\kappa_{-i}^{\varphi}}^{\theta_{-i}^{\varphi}} \cdot t_{\kappa_{i}^{\varphi}}^{\theta_{i}^{\varphi}} \cdot \prod_{x=1}^{\mu_\varphi} t_{k'^{\varphi}_x} \tag{5}
\]

of \( \pi^{-1} \).

Notice, by the induction hypothesis, all the conditions of Proposition \ref{prop:3.39} hold for \( \tilde{\pi} = (\pi')^{-1} \). Consider the standard OGS elementary factorization of \( (\pi')^{-1} \):

\[
(\pi')^{-1} = \prod_{w' = 1}^{\zeta(\pi') - 1} m^{(w')} \prod_{x=1}^{\mu_{w'}} t_{k'^{\varphi}_{w'}} \tag{6}
\]
Define \((\pi')^{-1}_\varphi\) to be
\[
(\pi')^{-1}_\varphi = \prod_{x=1}^{\mu(\varphi)} t_{k^x_{\varphi x}}^{{i^x_{\varphi}}}.
\]

Since \((\pi')^{-1}_\varphi\) is a terminal segment of \((\pi')^{-1}\), its standard OGS elementary factorization is as follows:
\[
(\pi')^{-1}_\varphi = \prod_{w'=w_\varphi+1}^z (\pi')^{-1} \prod_{x=1}^{m(w')} t_{h_x^{(w')}}^{{i^x_{\varphi}}},
\]
for some \(1 \leq w \leq z(\pi'^{-1})\) and \(1 \leq j_\varphi \leq m(w_\varphi)\), where
\[
\tilde{\pi}_{j_\varphi}(w_\varphi) = \prod_{x=j_\varphi}^{m(w_\varphi)} t_{h_x^{(w_\varphi)}}^{{i^x_{\varphi}}}.
\]

Notice, all the elementary factors of \((\pi')^{-1}_\varphi\) apart from the first factor \(\tilde{\pi}_{j_\varphi}(w_\varphi)\) coincide with the terminal elementary factors \((\pi')^{-1}(w')\) of \((\pi')^{-1}\), for \(w' \geq w_\varphi + 1\).

Now, look at
\[
t^{\varphi_1}_{\kappa-\varphi} \cdot t^{\varphi_2}_{\kappa+\varphi} \cdot (\pi')^{-1}_\varphi.
\]
Notice, by Definition 3.37 \(\kappa_{+;-\varphi} - \varphi = \varphi\). By our assumption, \(\varphi \geq \tilde{h}_x^{(w_\varphi)}\) for \(j_\varphi \leq x \leq m(w_\varphi)\) and also \(\varphi \geq \tilde{h}_x^{(w')}\) for every \(w' > w_\varphi\) and \(1 \leq x \leq m(w')\). Thus, by Proposition 3.30 for \(q = \kappa_{+;\varphi}\) and \(i_q = \varphi\), we have:
\[
t^{\varphi_1}_{\kappa_{+;\varphi}} \cdot (\pi')^{-1}_\varphi = t^{\varphi_1}_{\rho_\varphi + \varphi} \cdot (\tilde{\pi})^{-1}_\varphi \cdot t^{\varphi_2}_{\kappa_{+;\varphi}}.
\]
where,
\[
(\tilde{\pi})^{-1}_\varphi = \prod_{x=j_\varphi} \left( t_{h_x^{(w_\varphi)}}^{{i^x_{\varphi}}} \prod_{w'=w_\varphi+1}^z \left( t_{h_x^{(w')}}^{{i^x_{\varphi}}} \prod_{x=1}^{m(w')} t_{h_x^{(w')}}^{{i^x_{\varphi}}} \right) \right).
\]

Thus, by Proposition 3.30 the standard OGS of \((\tilde{\pi'})^{-1}_\varphi\) is the following:

- \(t_{k^x_{\varphi x} + \varphi}\), for \(1 \leq j \leq \mu(\varphi)\);
\[ t_{maj}((\pi')^{-1}(w')) + \vartheta, \] for some \( w' > w_\varphi \), such that
\[ \tilde{h}_{m(w'-1)}^{(w'-1)} < \maj \left( (\pi')^{-1}(w') \right) < \tilde{h}_{1}^{(w')}. \]

Notice also, by Definition 3.37 $\kappa_{-;\varphi} = \varrho_{\varphi-1} + \vartheta_{\varphi}, \kappa_{+;\varphi} = \varrho_{\varphi} + \vartheta_{\varphi}$. By our assumption $\varrho_{\varphi-1} < k_{\varphi j}^t \leq \varrho_{\varphi}$ for every $1 \leq j \leq \mu(\varphi)$. Therefore,
\[ \kappa_{-;\varphi} < k_{\varphi j}^t + \vartheta \leq \kappa_{+;\varphi}. \]

We have also, $\tilde{h}_{m(w'-1)}^{(w'-1)} \leq \maj \left( (\pi')^{-1}(w') \right) \leq \tilde{h}_{1}^{(w')}$, for every $w' > w_\varphi$. Therefore,
\[ \kappa_{-;\varphi} < \maj \left( (\pi')^{-1}(w') \right) + \vartheta \leq \kappa_{+;\varphi} \]
also for every $w' > w_\varphi$. Thus, the non-zero exponent standard $OGS$ of $(\pi')^{-1}$ included in the non-zero exponent standard $OGS$ of $\pi^{-1}$.

Now, notice, If $w' > w_\varphi$ and $1 \leq x \leq m(w_\varphi)$, or $w' = w_\varphi$ and $j_\varphi \leq x \leq m(w_\varphi)$, then by Proposition 3.49 the following holds:

- $\rho_{1}^{(w')} = \maj \left( (\pi')_{\varphi}^{-1}(w') \right) = \kappa_{-;j}^{(v \rightarrow z(\pi)-1)}$, for some $1 \leq v \leq z(\pi)$ and $1 \leq j \leq m(v)$;

- $\tilde{h}_{x}^{(w')} = \kappa_{c;j}^{(v \rightarrow z(\pi)-1)}$, for some $1 \leq v \leq z(\pi)$ and $1 \leq j \leq m(v)$, and $c \in \{+, -\}$, such that in case
  \[ \tilde{h}_{x}^{(w')} = \kappa_{-;j}^{(v \rightarrow z(\pi)-1)}, \]
  necessarily, $x = 1$ or $x = m(w')$ (i.e., $t_{\kappa_{-;j}^{(v \rightarrow z(\pi)-1)}}$ can appear just as a first or the last letter of an elementary factor $(\pi')_{\varphi}^{-1}(w')$).

Since for every $w' \geq w_\varphi$ and $1 \leq x \leq m(w_\varphi)$,
\[ \varrho_{\varphi-1} < \tilde{h}_{x}^{(w')} = \kappa_{c;j}^{(v \rightarrow z(\pi)-1)} \leq \varrho_{\varphi} \]
and for $w' > w_\varphi$
\[ \varrho_{\varphi-1} < \maj \left( (\pi')_{\varphi}^{-1}(w') \right) = \kappa_{-;j}^{(v \rightarrow z(\pi)-1)} \leq \varrho_{\varphi}, \]
by Definition 3.37 $\tilde{h}_{x}^{(w')} + \vartheta = \kappa_{c;j}^{(v \rightarrow z(\pi)-1)} + \vartheta = \kappa_{c;j}^{(v)}$. 

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for some 1 ≤ v ≤ z(π) and 1 ≤ j ≤ m(\(v\)). Therefore, the standard OGS canonical form of \((\pi')^{-1}\) is a product of exponents of \(t_{\kappa^{(v)}_{-j}}\) and \(t_{\kappa^{(v)}_{+j}}\), such that the following holds:

- If \(k'_{\varphi,j'} + \vartheta_{\varphi} = \kappa^{(v)}_{+j}\) for some 1 ≤ v ≤ z(π), 1 ≤ j ≤ m(\(v\)), and 1 ≤ j' ≤ \(\mu(\varphi)\), then by Proposition 3.32 \(k'_{\varphi,j'} + \vartheta_{\varphi} \neq \kappa^{(v)}_{-j}\), for any 1 ≤ \(\tilde{v}\) ≤ z(π) and 1 ≤ \(\tilde{j}\) ≤ m(\(\tilde{v}\)). Thus, \(k'_{\varphi,j'} + \vartheta_{\varphi} \neq \text{maj} \left( (\pi')^{-1}(w') \right) + \vartheta_{\varphi} \) for any \(w' > w_{\varphi}\).

Therefore, by Proposition 3.30

\[
\tilde{\text{i}}^{\kappa^{(v)}_{+j}}_{\kappa^{(v)}_{-j}} = \sum_{\gamma} \left( \vartheta^{(v_{\gamma})}_{\text{j}_{\gamma}} + \vartheta^{(v_{\gamma})}_{\bar{\text{j}}_{\gamma}} \right), \tag{11}
\]

such that \(\kappa^{(v)}_{+j} = \kappa^{(v)}_{+j}\), where

\[
\kappa^{(v)}_{-\varphi} = \vartheta_{\varphi-1} + \vartheta_{\varphi} \leq \kappa^{(v)}_{-\varphi} \leq \vartheta_{\varphi} + \vartheta_{\varphi} = \kappa^{(v)}_{+\varphi};
\]

- If \(k'_{\varphi,j'} + \vartheta_{\varphi} = \kappa^{(v)}_{-j}\) for some 1 ≤ v ≤ z(π), 1 ≤ j ≤ m(\(v\)), and 1 ≤ j' ≤ \(\mu_{\varphi}\), then by Proposition 3.42 \(k'_{\varphi,j'} + \vartheta_{\varphi} = \text{maj} \left( (\pi')^{-1}(w') \right) + \vartheta_{\varphi} \) for some \(w' > w_{\varphi}\).

Therefore, by Proposition 3.30

\[
\tilde{\text{i}}^{\kappa^{(v)}_{-j}}_{\kappa^{(v)}_{-j}} = \sum_{\gamma} \left( \vartheta^{(v_{\gamma})}_{\text{j}_{\gamma}} + \vartheta^{(v_{\gamma})}_{\bar{\text{j}}_{\gamma}} \right),
\]

such that \(\kappa^{(v)}_{-\varphi} = \kappa^{(v)}_{-\varphi}\), where

\[
\kappa^{(v)}_{-\varphi} = \vartheta_{\varphi-1} + \vartheta_{\varphi} \leq \kappa^{(v)}_{-\varphi} \leq \vartheta_{\varphi} + \vartheta_{\varphi} = \kappa^{(v)}_{+\varphi};
\]

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Now, look at the sub-word \( \pi \) of \( h \).

Therefore, by Proposition \( \ref{3.49} \), the exponent of \( t_{maj}(\pi')^{-1(w')} \) is \( i_{maj}(\pi')^{-1(w')} = 0 \) in the standard OGS canonical form. By Proposition \( \ref{3.49} \), \( maj(\pi')^{-1(w')} = \kappa_{v \rightarrow z(\pi)'-1} \) for some \( 1 \leq v \leq z(\pi) \) and \( 1 \leq j \leq m(v) \). Therefore, by the induction hypothesis,

\[
i_{maj}(\pi')^{-1(w')} = 0 = \kappa_{v \rightarrow z(\pi)'-1} - \sum_\gamma (\vartheta_j^{(v)} + \eta_j^{(v)}) \]

such that \( \kappa_{v \rightarrow z(\pi)'-1} = \kappa_{v \rightarrow z(\pi)'-1} \). Thus, by Proposition \( \ref{3.49} \)

\[
\tilde{i}_{maj}(\pi')^{-1(w')} = \vartheta_\varphi = \kappa_{v \rightarrow z(\pi)'-1} + \vartheta_\varphi - \sum_\gamma (\vartheta_j^{(v)} + \eta_j^{(v)})
\]

\[
= \kappa_{v \rightarrow z(\pi)'-1} - \sum_\gamma (\vartheta_j^{(v)} + \eta_j^{(v)}) \]

such that \( \kappa_{v \rightarrow z(\pi)'-1} = \kappa_{v \rightarrow z(\pi)'-1} \), where

\[
\kappa_{v \rightarrow z(\pi)'-1} = \vartheta_\varphi - 1 + \vartheta_\varphi < \kappa_{v \rightarrow z(\pi)'-1} \vartheta_\varphi + \vartheta_\varphi = \kappa_{v \rightarrow z(\pi)'-1} \varphi \; \varphi \; \varphi.
\]

Now, look at the sub-word

\[
t_\varphi^{\vartheta_\varphi-1} \cdot t_\varphi^{\vartheta_\varphi} \rho_{\varphi}^{(w_\varphi)} + \vartheta_\varphi
\]

of \( \pi^{-1} \). If \( j_\varphi > 1 \), by Proposition \( \ref{3.49} \), \( \rho_{\varphi}^{(w_\varphi)} < \tilde{h}_{j_\varphi-1}^{(w_\varphi)} \). By our assumption, \( \tilde{h}_{j_\varphi-1}^{(w_\varphi)} < k_{j_\varphi}' \leq \vartheta_\varphi - 1 \). Thus,

\[
\rho_{\varphi}^{(w_\varphi)} + \vartheta_\varphi < \vartheta_\varphi - 1 + \vartheta_\varphi = \kappa_{v \rightarrow z(\pi)'-1} \varphi \; \varphi \; \varphi.
\]

Therefore, by using Proposition \( \ref{3.49} \) for the case \( q - i_q = i_p \), where \( q = \kappa_{v \rightarrow z(\pi)'-1} \), \( i_q = \vartheta_\varphi - 1 \), \( p = \rho_{\varphi}^{(w_\varphi)} + \vartheta_\varphi \), and \( i_p = \vartheta_\varphi \), we have:

\[
t_\varphi^{\vartheta_\varphi-1} \cdot t_\varphi^{\vartheta_\varphi} \rho_{\varphi}^{(w_\varphi)} + \vartheta_\varphi = t_{\varphi}^{\rho_{\varphi}^{(w_\varphi)} - \vartheta_\varphi - 1} \cdot t_\varphi^{\vartheta_\varphi - 1} \rho_{\varphi}^{(w_\varphi)}.
\]

(12)

By Equation \( \ref{8} \),

\[
\rho_{\varphi}^{(w_\varphi)} = maj(\pi_{j_\varphi}^{(w_\varphi)}).
\]

Therefore, by Proposition \( \ref{3.49} \) the following holds:
• If \( \tilde{h}_{\varphi}^{(w)} = \kappa_{+j}^{(v \to z(\pi) - 1)} \), then

\[
\rho_{\varphi}^{(w)} = \sum_{v' = \chi_{\varphi}^{(v)}}^{z(\pi) - 1} \vartheta_{j_{\varphi}^{(v')}}^{(v')} \]

where,

\[
\kappa_{+j_{\varphi}^{(v')}}^{(v' \to z(\pi) - 1)} < \kappa_{+j}^{(v \to z(\pi) - 1)} \leq \kappa_{+j_{\varphi}^{(v')}}^{(v' \to z(\pi) - 1)}.
\]

• If \( \tilde{h}_{\varphi}^{(w)} = \kappa_{-j}^{(v \to z(\pi) - 1)} \) and \( j_{\varphi} = m^{(w)} \), then

\[
\rho_{\varphi}^{(w)} = \sum_{v' = v + 1}^{z(\pi) - 1} \vartheta_{j_{\varphi}^{(v')}}^{(v')} \]

where,

\[
\kappa_{-j_{\varphi}^{(v')}}^{(v' \to z(\pi) - 1)} \leq \kappa_{-j}^{(v \to z(\pi) - 1)} < \kappa_{+j_{\varphi}^{(v')}}^{(v' \to z(\pi) - 1)}.
\]

Notice, \( \tilde{h}_{\varphi}^{(w)} + \vartheta_{\pi} = \kappa_{c_{ij}}^{(v)} \) in case \( \tilde{h}_{\varphi}^{(w)} = \kappa_{c_{ij}}^{(v \to z(\pi) - 1)} \) for \( c \in \{+,-\} \). Notice also, \( \tilde{h}_{\varphi}^{(w)} + \vartheta_{\pi} \) is the smallest value of \( \kappa_{c_{ij}}^{(v)} \), which is greater than \( \kappa_{-j_{\varphi}} \), unless \( j_{\varphi} = 1 \) and \( \tilde{h}_{\varphi}^{(w)} = \text{maj} \ (\tilde{\pi}^{(w)}_{\varphi}) \). Therefore, by Proposition 3.48 the following holds:

• If \( \tilde{h}_{\varphi}^{(w)} + \vartheta_{\pi} = \kappa_{-j}^{(v)} \) (i.e., \( \tilde{h}_{\varphi}^{(w)} = \kappa_{-j}^{(v \to z(\pi) - 1)} \)), for some \( 1 \leq v \leq z(\pi) \) and \( 1 \leq j \leq m^{(v)} \), then \( \chi_{\varphi}^{(v)} = v \);

• If \( \tilde{h}_{\varphi}^{(w)} + \vartheta_{\pi} = \kappa_{+j}^{(v)} \) (i.e., \( \tilde{h}_{\varphi}^{(w)} = \kappa_{+j}^{(v \to z(\pi) - 1)} \)), for some \( 1 \leq v \leq z(\pi) \) and \( 1 \leq j \leq m^{(v)} \), then \( \chi_{\varphi}^{(v)} = \chi_{j}^{(v)} \);

Therefore, unless \( j_{\varphi} = 1 \) and \( \tilde{h}_{\varphi}^{(w)} = \text{maj} \ (\tilde{\pi}^{(w)}_{\varphi}) \),

\[
\rho_{\varphi}^{(w)} = \sum_{v' = \chi_{\varphi}^{(v)}}^{z(\pi) - 1} \vartheta_{j_{\varphi}^{(v')}}^{(v')} \]

where,

\[
\kappa_{-j_{\varphi}^{(v')}}^{(v' \to z(\pi) - 1)} < \kappa_{-j_{\varphi}}^{(v)} < \kappa_{+j_{\varphi}^{(v')}}^{(v' \to z(\pi) - 1)}.
\]

Thus, by Definition 3.49,

\[
\rho_{\varphi}^{(w)} = \eta_{\varphi}^{(v)} - 1.
\]

Hence, Equation 12 can be written in the form

\[
t_{\kappa_{-j_{\varphi}}^{(v)}} \cdot t_{\rho_{\varphi}^{(w)}}^{\varphi} \cdot \vartheta_{\varphi}^{(w)} \cdot \vartheta_{\varphi}^{(w)} = t_{\eta_{\varphi}^{(v)}}^{\varphi} \cdot \eta_{\varphi}^{(v)} \cdot t_{\kappa_{-j_{\varphi}}^{(v)}} \cdot \vartheta_{\varphi}^{(w)}.
\]

(14)
Therefore, the following holds unless \( \varphi = 1 \) and \( \text{maj} \left( \tilde{\pi}^{(w_\varphi)} \right) \geq g_{j-1} \):
\[
 t_{\varphi}^{\theta_{\varphi-1}} \cdot t_{\varphi+1}^{\theta_{\varphi}} \cdot (\pi')^{-1}_{\varphi} = t_{\varphi}^{\theta_{\varphi-1}} \cdot t_{\varphi+1}^{\theta_{\varphi}-1} \cdot (\pi')^{-1}_{\varphi} \cdot t_{\varphi+1}^{\theta_{\varphi}}.
\]  
(15)

Now, assume \( \varphi = 1 \) and \( \text{maj} \left( \tilde{\pi}^{(w_\varphi)} \right) \geq g_{j-1} \). By Proposition 3.49, \( \text{maj} \left( \tilde{\pi}^{(w_\varphi)} \right) = \kappa_{(v_0 \rightarrow z(\pi)-1)}^{(v_0 \rightarrow z(\pi)-1)} \), for some \( 1 \leq v_0 \leq z(\pi) - 1 \) and \( 1 \leq j_0 \leq m^{(v)} \). Notice, also \( \tilde{h}_{m^{(w_\varphi)-1}} < \text{maj} \left( \tilde{\pi}^{(w_\varphi)} \right) \), since otherwise \( w_\varphi \) would be smaller by one, and \( \varphi \) would be \( m^{(w_\varphi)} \). Thus, by Proposition 3.49 either \( v_0 = z(\pi) - 1 \), or for every \( v_0 < v' \leq z(\pi) - 1 \), we have \( \kappa_{(v')}^{(v')} = \theta_{(v')}^{1} = 0 \) and \( \kappa_{(v_0 \rightarrow z(\pi)-1)}^{(v_0 \rightarrow z(\pi)-1)} < \kappa_{(v')_0}^{(v')} \). By Proposition 3.40, the same holds for \( \kappa_{(v_0)}^{(v_0)} = \text{maj} \left( \tilde{\pi}^{(w_\varphi)} \right) \) too (i.e., either \( v_0 = z(\pi) - 1 \), or for every \( v_0 < v' \leq z(\pi) - 1 \), we have \( \kappa_{(v')}^{(v')} = \theta_{(v')}^{1} = 0 \) and \( \kappa_{(v_0)}^{(v_0)} < \kappa_{(v')}^{(v')} \). Notice also, \( \kappa_{(v_0)}^{(v_0)} = \text{maj} \left( \tilde{\pi}^{(w_\varphi)} \right) + \partial_\varphi \), is the smallest \( \kappa_{(v_0)}^{(v_0)} \) which is larger than \( \kappa_{-1}^{(v_0)} \). Thus, by Proposition 3.48 \( \chi_{-1}^{(v_0)} = v_0 \). Then, by Definition 3.46 \( \eta_{-1}^{(v_0)} = 0 \). Thus, in case \( \eta_{-1}^{(v_0)} = 0 \), we have
\[
 t_{\varphi}^{\theta_{\varphi-1}} \cdot t_{\varphi+1}^{\theta_{\varphi}} \cdot (\pi')^{-1}_{\varphi} = t_{\varphi}^{\theta_{\varphi-1}} \cdot \left( t_{\varphi+1}^{\theta_{\varphi}} \cdot (\pi')^{-1}_{\varphi} \right) \cdot t_{\varphi+1}^{\theta_{\varphi}}.
\]  
(16)

where, \( \varphi = 1 \) and \( \rho_{1}^{(w_\varphi)} = \text{maj} \left( \tilde{\pi}^{(w_\varphi)} \right) = \tilde{h}_{1}^{(w_\varphi)} \geq g_{\varphi-1} \).

Thus, by defining
\[
 (\pi')^{-1}_{\varphi} = \begin{cases} 
 \eta_{\varphi-1} < 0 \\
 \eta_{\varphi-1} = 0 
\end{cases}
\]
we have:
\[
 t_{\varphi}^{\theta_{\varphi-1}} \cdot t_{\varphi+1}^{\theta_{\varphi}} \cdot (\pi')^{-1}_{\varphi} = t_{\varphi}^{\theta_{\varphi-1}} \cdot (\pi')^{-1}_{\varphi} \cdot t_{\varphi+1}^{\theta_{\varphi}}.
\]  
(17)

If \( \varphi = 1 \), then \( \eta_{\varphi-1} = 0 \). Thus, the Theorem has been proved for case \( \varphi = 1 \). If \( \varphi > 1 \), then consider a segment of \( \pi^{-1} \) by combining Equations 3 and 17
\[
\pi^{-1} = \prod_{j=1}^{\varphi-2} \left( t_{\varphi}^{\theta_{\varphi-1}} \cdot t_{\varphi+1}^{\theta_{\varphi}} \cdot (\pi')^{-1}_{\varphi} \right) 
\]
\[
\quad\cdot \left( t_{\varphi}^{\theta_{\varphi}} \cdot t_{\varphi+1}^{\theta_{\varphi}} \cdot (\pi')^{-1}_{\varphi} \cdot t_{\varphi+1}^{\theta_{\varphi}} \right) 
\]
\[
\quad\cdot \left( t_{\varphi}^{\theta_{\varphi}} \cdot t_{\varphi+1}^{\theta_{\varphi}} \cdot (\pi')^{-1}_{\varphi} \cdot t_{\varphi+1}^{\theta_{\varphi}} \right) 
\]
\[
= \prod_{x=1}^{\mu_{\varphi-1}} t_{\varphi}^{\theta_{\varphi-1}} \cdot t_{\varphi+1}^{\theta_{\varphi}} \cdot (\pi')^{-1}_{\varphi}.
\]

where, by Equation 3
\[
(\pi')^{-1}_{\varphi-1} = \prod_{x=1}^{\mu_{\varphi-1}} t_{\varphi}^{\theta_{\varphi-1}} \cdot t_{\varphi+1}^{\theta_{\varphi}} \cdot (\pi')^{-1}_{\varphi}.
\]
such that

\[ \vartheta_{\varphi - 2} < k'_{(\varphi - 1)_x} < k'_{(\varphi - 1)_{x+1}} \leq \vartheta_{\varphi - 1}, \]

for every \( 1 \leq x < \mu_{\varphi - 1} \).

Consider the segment

\[ t^\vartheta_{\varphi - 2} \cdot t^\vartheta_{\varphi - 1} \cdot \prod_{x=1}^{\mu_{\varphi - 1}} t^{k'_{(\varphi - 1)_x}} \cdot t^\vartheta_{\varphi - 1} = (18) \]

\[ (\pi')_{\varphi - 1} \cdot t^{\eta_{\varphi - 1}} = \pi_{j_{\varphi - 1}} \cdot \prod_{x=1}^{w'_{\varphi - 1} - 1} t^x_{j_{\varphi - 1}} \cdot \prod_{x=1}^{j_{\varphi - 1}} t^x_{w_{\varphi - 1}} \cdot \prod_{x=1}^{j_{\varphi - 1}} t^x_{\varphi - 1}, \quad (19) \]

for some \( 1 \leq w \leq z(\pi'^{-1}) \) and \( 1 \leq j \leq m(w_{\varphi}) \), where

\[ \tilde{\pi}_{j_{\varphi - 1}} = \prod_{x=j_{\varphi - 1}}^{w_{\varphi - 1}} t^x_{j_{\varphi - 1}} \cdot \tilde{h}_{x}(w_{\varphi - 1}). \quad (20) \]

Consider the terminal segment of \( (\pi')_{\varphi - 1} \cdot t^{\eta_{\varphi - 1}} \) in Equation (19)

\[ \prod_{x=1}^{j_{\varphi - 1}} t^x_{w_{\varphi}} \cdot \tilde{h}_{x}(w_{\varphi - 1}) \cdot \tilde{h}_{x}(w_{\varphi}) \cdot t^{j_{\varphi - 1}}_{\varphi - 1} \cdot \tilde{h}_{x}(w_{\varphi - 1}) \cdot \tilde{h}_{x}(w_{\varphi}) \quad (21) \]

By Equation (13) \( \eta_{\varphi - 1} = \rho_{\varphi - 1}(w_{\varphi}) \), and by Definition (3.20) \( \rho_{\varphi - 1}(w_{\varphi}) = \sum_{x=j_{\varphi}}^{m(w_{\varphi})} t^x_{j_{\varphi}} \). Thus,

\[ \sum_{x=1}^{j_{\varphi - 1}} t^x_{w_{\varphi}} + \eta_{\varphi - 1} = \sum_{x=1}^{j_{\varphi - 1}} t^x_{w_{\varphi}} = m a j \left( \tilde{\pi}(w_{\varphi}) \right). \quad (22) \]

Thus, Equation (19) is a standard OGS elementary factorization of \( (\pi')_{\varphi - 1} \cdot t^{\eta_{\varphi - 1}} \), with the elementary factors \( \tilde{\pi}_{j_{\varphi - 1}} \cdot \prod_{x=1}^{w_{\varphi - 1} + 1} t^x_{j_{\varphi - 1}} \cdot \tilde{h}_{x}(w_{\varphi}) \), where \( w_{\varphi - 1} + 1 \leq w' \leq w_{\varphi} - 1 \), and the terminal elementary factor \( \prod_{x=1}^{j_{\varphi - 1}} t^x_{w_{\varphi}} \cdot \tilde{h}_{x}(w_{\varphi - 1}) \cdot \tilde{h}_{x}(w_{\varphi}) \). Notice, similar to the OGS elementary factorization of \( (\pi')_{\varphi - 1} \), we have that all the elementary factors of \( (\pi')_{\varphi - 1} \) apart from the first factor \( \tilde{\pi}_{j_{\varphi - 1}} \) and the last factor, coincides with the terminal elementary factors \( (\pi')_{\varphi - 1} \) of \( (\pi')_{\varphi - 1} \), for \( w_{\varphi - 1} + 1 \leq w' \leq w_{\varphi} - 1 \).
Notice, by Definition 3.37 \( \kappa_{+;\varphi-1} - \vartheta_{\varphi-1} = \vartheta_{\varphi-1} \). By our assumption, \( \vartheta_{\varphi-1} > \tilde{h}_x(w_{\varphi-1}) \) for \( j_{\varphi-1} \leq x \leq m(w_{\varphi-1}) \), \( \vartheta_{\varphi-1} > \tilde{h}_x(w') \) for every \( w_{\varphi-1} < w' < w_{\varphi} \) and \( 1 \leq x < m(w) \), and \( \vartheta_{\varphi-1} > \tilde{h}_x(w_{\varphi}) \) for \( 1 \leq x \leq j_{\varphi} - 1 \). Thus, by Proposition 3.30 for \( q = \kappa_{+;\varphi} \) and \( i_q = \vartheta_{\varphi} \), we have:

\[
\begin{align*}
 t^\vartheta_{\kappa_{+;\varphi-1}} \cdot (\tilde{\pi}')^{-1}_{\varphi-1} \cdot t^\vartheta_{\varphi-1} \cdot t^\vartheta_{\varphi-1} = t^{(w_{\varphi-1})} = t^{(w_{\varphi-1})_{\rho_{j_{\varphi-1}} + \vartheta_{\varphi-1}}} \cdot (\tilde{\pi}')^{-1}_{\varphi-1} \cdot t^{\vartheta_{\varphi-1} + \eta_{\varphi-1}},
\end{align*}
\]

where,

\[
(\tilde{\pi}')^{-1}_{\varphi-1} = \prod_{x = j_{\varphi-1}}^{m(w_{\varphi-1})} t^{(w_{\varphi-1})}_{\tilde{h}_x(w_{\varphi-1}) + \vartheta_{\varphi-1}} \cdot \prod_{x = j_{\varphi-1}}^{m(w')} \cdot \prod_{x = 1}^{j_{\varphi-1}} t^{(w_{\varphi})}_{\tilde{h}_x(w_{\varphi}) + \vartheta_{\varphi-1}}.
\]

Thus, by Proposition 3.30 the standard OGS of \((\tilde{\pi}')^{-1}_{\varphi-1}\) is the following:

- \( t^\vartheta_{\kappa_{+;\varphi-1}} \), for \( 1 \leq j \leq \mu(\varphi - 1) \);
- \( t^\vartheta_{\varphi-1} \), for some \( w_{\varphi-1} < w' \leq w_{\varphi} \), such that

\[
\tilde{h}_{\min(w_{\varphi-1})}^{-1} < \max((\pi')^{-1}(w')) < \tilde{h}_{1}^{-1}(w'),
\]

where, by the same argument as in the case of \((\tilde{\pi}')^{-1}_{\varphi}\), the standard OGS canonical form of \((\tilde{\pi}')^{-1}_{\varphi-1}\) is a product of exponents of \( t^{\vartheta_{\kappa_{+;\varphi-1}}} \) and \( t^{\vartheta_{\kappa_{+;\varphi}}} \), such that the following holds:

- If \( \kappa'_{+;j'} + \vartheta_{\varphi-1} = \kappa_{+;\varphi} \), such that the following holds:

\[
\tilde{i}_{\kappa_{+;\varphi}}^{-1} = \tilde{i}_{\kappa_{+;\varphi}}^{-1} = \sum_{\alpha}^{\vartheta_{\kappa_{+;\varphi}} + \eta_{\kappa_{+;\varphi}}},
\]

such that \( \kappa_{+;\varphi} = \kappa_{+;\varphi} \), where

\[
\kappa_{-;\varphi-1} = \vartheta_{\varphi-1} + \vartheta_{\varphi-1} < \kappa_{+;\varphi} \leq \vartheta_{\varphi-1} + \vartheta_{\varphi-1} = \kappa_{+;\varphi-1};
\]
• If $k_{\varphi'}^v + \vartheta_{\varphi-1} = \kappa_{-;j}^{(v)}$ for some $1 \leq v \leq z(\pi)$, $1 \leq j \leq m^{(v)}$, and $1 \leq j' \leq \mu(\varphi-1)$, then

\[
\tilde{\kappa}_{-;j}^{(v)} = \tilde{\kappa}_{-;j}^{(v)} + \vartheta_{\varphi-1} = \kappa_{-;j}^{(v)} + \vartheta_{\varphi-1} - \sum_{\alpha} (\vartheta_{j\alpha}^{(v)} + \eta_{j\alpha-1}^{(v)}) = \kappa_{-;j}^{(v)} - \sum_{\alpha} (\vartheta_{j\alpha}^{(v)} + \eta_{j\alpha-1}^{(v)}),
\]

such that $\kappa_{-;j}^{(v)} = \kappa_{-;j}^{(v)}$, where

\[
\kappa_{-;j}^{(v)} = \kappa_{-;j}^{(v)} < \vartheta_{\varphi-1} < \kappa_{-;j}^{(v)} + \vartheta_{\varphi-1} = \kappa_{+;\varphi-1};
\]

• If $\tilde{\varphi}_{m^{(v')}} = \max \left( (\pi')^{-1} \right) < \tilde{\varphi}_{m^{(v')}}$, for some $w_{\varphi-1} < w' \leq w_{\varphi}$, then by Proposition 3.30, the exponent of $\tilde{\varphi}_{(\max)^{-1}(w')}$ is $\tilde{i}_{\max}(t_{(\pi')^{-1}(w')}) = 0$ in the standard OGS canonical form of $(\pi')_{\varphi-1}$. We have also $\max \left( (\pi')^{-1} \right) = \kappa_{-;j}^{(v)}$ for some $1 \leq v \leq z(\pi)$ and $1 \leq j \leq m^{(v)}$. By similar argument as in the case of $(\pi')_{\varphi-1}$, we conclude

\[
\tilde{\varphi}_{m^{(v')}} = \vartheta_{\varphi-1} = \kappa_{-;j}^{(v)} + \vartheta_{\varphi} - \sum_{\alpha} (\vartheta_{j\alpha}^{(v)} + \eta_{j\alpha-1}^{(v)}) = \kappa_{-;j}^{(v)} - \sum_{\alpha} (\vartheta_{j\alpha}^{(v)} + \eta_{j\alpha-1}^{(v)}),
\]

such that $\kappa_{-;j}^{(v)} = \kappa_{-;j}^{(v)}$, where

\[
\kappa_{-;\varphi} = \vartheta_{\varphi-1} + \vartheta_{\varphi} < \kappa_{-;j}^{(v)} < \vartheta_{\varphi} + \vartheta_{\varphi} = \kappa_{+;\varphi};
\]

Finally, analogously to Equations 15, 16 we conclude the following:

\[
t^{\frac{\vartheta_{\varphi-2}}{\kappa_{-;\varphi-1}} + \frac{\vartheta_{\varphi-1}}{\kappa_{+;\varphi-1}} \cdot (\pi')^{-1} \frac{1}{\varphi-1} t^{\frac{\vartheta_{\varphi-1}}{\kappa_{-;\varphi-1}}} = t^{\vartheta_{\varphi-1} - \frac{\vartheta_{\varphi-2}}{\theta_{\varphi-2}} \cdot \kappa_{-;\varphi-1} - \frac{\vartheta_{\varphi-1} + \vartheta_{\varphi-2}}{\kappa_{+;\varphi-1}}},
\]

where

\[
(\tilde{\pi}')^{-1} \varphi-1 = \begin{cases} (\tilde{\pi}')^{-1} \varphi-1 & \eta_{\varphi-2} > 0 \\ t^{\frac{\vartheta_{\varphi-1}}{\kappa_{-;\varphi-1}}} \cdot (\tilde{\pi}')^{-1} \varphi-1 & \eta_{\varphi-2} = 0 \end{cases}
\]

(25)
If $\varphi = 2$, then $\vartheta_{\varphi - 2} = 0$, and the theorem has been proved. If $\varphi > 2$, we continue in the same way, where finally we get

$$
\pi^{-1} = \left( (\pi')^{-1}_{1} \cdot t^{\vartheta_{1} + \eta_{1}} \right) \prod_{j=2}^{\varphi} \left( t^{\vartheta_{j-1} - \eta_{j-1}} \cdot (\pi')^{-1}_{j} \cdot t^{\vartheta_{j} + \eta_{j}} \right) \cdot \pi_{0}^{-1},
$$

(26)

where, for every $1 \leq j \leq \varphi$, $(\tilde{\pi}')^{-1}_{j}$ is defined analogously to the definition of $(\tilde{\pi}')^{-1}_{\varphi}$.

Hence, the OGS canonical form of $\pi^{-1}$ is as follows:

$$
\pi^{-1} = \prod_{j=1}^{\tilde{m}} t_{\tilde{k}_{j}}^{\pi^{(v)}}
$$

such that:

- $\tilde{k}_{j} = \kappa^{(v)}_{-j}$ or $\tilde{k}_{j} = \kappa^{(v)}_{+j}$, for $1 \leq v \leq z(\pi)$, and $1 \leq j \leq m^{(v)}$;
- If $\tilde{k}_{j} = \kappa^{(v)}_{+j}$, then $\tilde{t}_{\tilde{k}_{j}} = \sum_{\gamma} \left( \vartheta^{(v)}_{j,\gamma} + \eta^{(v)}_{j,\gamma} \right)$, such that $\kappa^{(v)}_{+j} = \kappa^{(v)}_{+j,\gamma}$;
- If $\tilde{k}_{j} = \kappa^{(v)}_{-j}$, then $\tilde{t}_{\tilde{k}_{j}} = \kappa^{(v)}_{-j} - \sum_{\gamma} \left( \vartheta^{(v)}_{j,\gamma} + \eta^{(v)}_{j,\gamma-1} \right)$, such that $\kappa^{(v)}_{-j} = \kappa^{(v)}_{-j,\gamma}$.

Thus, $\pi^{-1}$ satisfies all the conditions of Proposition 3.49. Hence,

- $z(\pi^{-1})$, the number of standard OGS elementary factors of $\pi^{-1}$ equals to the number of different non-zero values $\kappa^{(v)}_{-j}$, where $1 \leq v \leq z(\pi)$, and $1 \leq j \leq m^{(v)}$;
- For every $1 \leq w \leq z(\pi^{-1})$, $maj \left( \pi^{-1}(w) \right) = \kappa^{(v)}_{-j}$ for some $1 \leq v \leq z(\pi)$, and for every given non-zero value of $\kappa^{(v)}_{-j}$, there exists exactly one $w$ such that $maj \left( \pi^{-1}(w) \right) = \kappa^{(v)}_{-j}$ (i.e., There is a one-to-one correspondence between the non-zero different values of $\kappa^{(v)}_{-j}$ and the descent set of $\pi^{-1}$).

Example 3.51 Consider

$$
\pi = t_{3}^{3} \cdot t_{10}^{3} \cdot t_{12}^{5} \cdot t_{13}^{3} \cdot t_{15}^{5} \cdot t_{17}^{2} \cdot t_{18}^{7}.
$$

The permutation presentation of $\pi$:

$$
\pi = [12; 13; 15; 17; 2; 3; 6; 9; 10; 16; 4; 5; 16; 18; 1; 7; 8; 11].
$$

Thus,

$$
\text{des}(\pi) = \{4, 10, 14\}.
$$
Now, the standard OGS elementary factorization of $\pi$ is as follows:
\[
\pi^{(1)} = t_9^3 \cdot t_{10}, \quad \pi^{(2)} = t_{10}^2 \cdot t_{12}^5 \cdot t_{13}^3, \quad \pi^{(3)} = t_{15}^5 \cdot t_{17}^2 \cdot t_{18}.
\]

Notice, the number of elementary factors equals to the number of the descents of $\pi$, where each descent of $\pi$ corresponds to the sum of the exponents of an elementary factor.

The generator-length of the elementary factors are: $m^{(1)} = 2, \quad m^{(2)} = 3, \quad m^{(3)} = 3$.

Now, we consider the parameters which we use for the computation of $\pi^{-1}$.

\[
\begin{align*}
h_1^{(1)} &= 9, \quad h_2^{(1)} = 10, \\
h_1^{(2)} &= 10, \quad h_2^{(2)} = 12, \quad h_3^{(2)} = 13, \\
h_1^{(3)} &= 15, \quad h_2^{(3)} = 17, \quad h_3^{(3)} = 18, \\
t_1^{(1)} &= 3, \quad t_2^{(1)} = 1, \\
t_1^{(2)} &= 2, \quad t_2^{(2)} = 5, \quad t_3^{(2)} = 3, \\
t_1^{(3)} &= 5, \quad t_2^{(3)} = 7, \quad t_3^{(3)} = 14.
\end{align*}
\]

Therefore,
\[
\begin{align*}
\vartheta_1^{(1)} &= 3, \quad \vartheta_2^{(1)} = 3 + 1 = 4 = \text{maj} \left( \pi^{(1)} \right), \\
\vartheta_1^{(2)} &= 2, \quad \vartheta_2^{(2)} = 2 + 5 = 7, \quad \vartheta_3^{(2)} = 2 + 5 + 3 = 10 = \text{maj} \left( \pi^{(2)} \right), \\
\vartheta_1^{(3)} &= 5, \quad \vartheta_2^{(3)} = 5 + 2 = 7, \quad \vartheta_3^{(3)} = 5 + 2 + 7 = 14 = \text{maj} \left( \pi^{(3)} \right).
\end{align*}
\]

Thus, by the formula $\vartheta_j^{(v)} = h_j^{(v)} - \text{maj} \left( \pi^{(v)} \right)$:
\[
\begin{align*}
\vartheta_1^{(1)} &= 9 - 4 = 5, \quad \vartheta_2^{(1)} = 10 - 4 = 6, \\
\vartheta_1^{(2)} &= 10 - 10 = 0, \quad \vartheta_2^{(2)} = 12 - 10 = 2, \quad \vartheta_3^{(2)} = 13 - 10 = 3, \\
\vartheta_1^{(3)} &= 15 - 14 = 1, \quad \vartheta_2^{(3)} = 17 - 14 = 3, \quad \vartheta_3^{(3)} = 18 - 14 = 4.
\end{align*}
\]

Hence, by using the formula $\kappa_{-;j}^{(v;v)} = \vartheta_{j-1}^{(v)} + \vartheta_j^{(v)}$ and $\kappa_{+;j}^{(v;v)} = \vartheta_j^{(v)} + \vartheta_{j+1}^{(v)}$:
\[
\begin{align*}
\kappa_{-;1}^{(1;1)} &= 0 + 5 = 5, \quad \kappa_{-;2}^{(1;1)} = 3 + 6 = 9, \\
\kappa_{+;1}^{(1;1)} &= 3 + 5 = 8, \quad \kappa_{+;2}^{(1;1)} = 4 + 6 = 10. \\
\kappa_{-;1}^{(2;2)} &= 0 + 0 = 0, \quad \kappa_{-;2}^{(2;2)} = 2 + 2 = 4, \quad \kappa_{-;3}^{(2;2)} = 7 + 3 = 10.
\end{align*}
\]
\[ \kappa^{(2\to 2)}_{+1} = 2 + 0 = 2, \quad \kappa^{(2\to 2)}_{+2} = 7 + 2 = 9, \quad \kappa^{(2\to 2)}_{+3} = 10 + 3 = 13. \]

\[ \kappa^{(3\to 3)}_{-1} = 0 + 1 = 1, \quad \kappa^{(3\to 3)}_{-2} = 5 + 3 = 8, \quad \kappa^{(3\to 3)}_{-3} = 7 + 4 = 11, \]
\[ \kappa^{(3\to 3)}_{+1} = 5 + 1 = 6, \quad \kappa^{(3\to 3)}_{+2} = 7 + 3 = 10, \quad \kappa^{(3\to 3)}_{+3} = 14 + 4 = 18. \]

Now, compute \( \kappa^{(v\to r)}_{-j} \) and \( \kappa^{(v\to r)}_{+j} \), for \( 1 \leq v \leq 3 \), \( v < r \leq 3 \), and \( 1 \leq j \leq m^{(v)} \), by using the algorithm in Definition 3.37.

Since \( \varrho_{1}^{(2)} < \kappa^{(1\to 1)}_{-1} < \varrho_{2}^{(2)} : \quad \kappa^{(1\to 2)}_{-1} = \kappa^{(1\to 1)}_{-1} + \varrho_{2}^{(2)} = 5 + 2 = 7. \]

Since \( \varrho_{2}^{(3)} < \kappa^{(1\to 2)}_{-1} < \varrho_{3}^{(3)} : \quad \kappa^{(1\to 3)}_{-1} = \kappa^{(1\to 2)}_{-1} + \varrho_{3}^{(3)} = 7 + 4 = 11. \]

Since \( \varrho_{2}^{(3)} < \kappa^{(1\to 1)}_{+1} < \varrho_{2}^{(2)} : \quad \kappa^{(1\to 2)}_{+1} = \kappa^{(1\to 1)}_{+1} + \varrho_{2}^{(2)} = 8 + 3 = 11. \]

Since \( \varrho_{2}^{(3)} < \kappa^{(1\to 2)}_{+1} < \varrho_{3}^{(3)} : \quad \kappa^{(1\to 3)}_{+1} = \kappa^{(1\to 2)}_{+1} + \varrho_{3}^{(3)} = 11 + 4 = 15. \]

Since \( \varrho_{2}^{(2)} < \kappa^{(2\to 2)}_{-1} < \varrho_{3}^{(3)} : \quad \kappa^{(2\to 3)}_{-1} = \kappa^{(2\to 2)}_{-1} + \varrho_{3}^{(3)} = 10 + 4 = 14. \]

Since \( \varrho_{1}^{(2)} < \kappa^{(2\to 2)}_{+1} < \varrho_{1}^{(3)} : \quad \kappa^{(2\to 3)}_{+1} = \kappa^{(2\to 2)}_{+1} + \varrho_{1}^{(3)} = 3 + 1 = 4. \]

Since \( \varrho_{1}^{(2)} < \kappa^{(2\to 2)}_{+1} < \varrho_{1}^{(3)} : \quad \kappa^{(2\to 3)}_{+1} = \kappa^{(2\to 2)}_{+1} + \varrho_{1}^{(3)} = 2 + 1 = 3. \]

Since \( \varrho_{2}^{(3)} < \kappa^{(2\to 2)}_{-2} < \varrho_{3}^{(3)} : \quad \kappa^{(2\to 3)}_{-2} = \kappa^{(2\to 2)}_{-2} + \varrho_{3}^{(3)} = 4 + 1 = 5. \]

Since \( \varrho_{2}^{(3)} < \kappa^{(2\to 2)}_{+2} < \varrho_{3}^{(3)} : \quad \kappa^{(2\to 3)}_{+2} = \kappa^{(2\to 2)}_{+2} + \varrho_{3}^{(3)} = 9 + 4 = 13. \]

Since \( \varrho_{2}^{(3)} < \kappa^{(2\to 2)}_{-3} < \varrho_{3}^{(3)} : \quad \kappa^{(2\to 3)}_{-3} = \kappa^{(2\to 2)}_{-3} + \varrho_{3}^{(3)} = 10 + 4 = 14. \)
Since $\varrho_3^{(3)} < \kappa_{+3}^{(2\rightarrow 2)} < \varrho_3^{(3)}$: \[\kappa_{-3}^{(2\rightarrow 3)} = \kappa_{-3}^{(2\rightarrow 2)} + \varrho_3^{(3)} = 13 + 4 = 17.\]

Thus,

\[\begin{align*}
\kappa_{-1}^{(1)} &= 11, & \kappa_{-2}^{(1)} &= 16, \\
\kappa_{+1}^{(1)} &= 15, & \kappa_{+2}^{(1)} &= 17.
\end{align*}\]

\[\begin{align*}
\kappa_{-1}^{(2)} &= 1, & \kappa_{-2}^{(2)} &= 5, & \kappa_{-3}^{(2)} &= 14, \\
\kappa_{+1}^{(2)} &= 3, & \kappa_{+2}^{(2)} &= 13, & \kappa_{+3}^{(2)} &= 17.
\end{align*}\]

\[\begin{align*}
\kappa_{-1}^{(3)} &= 1, & \kappa_{-2}^{(3)} &= 8, & \kappa_{-3}^{(3)} &= 11, \\
\kappa_{+1}^{(3)} &= 6, & \kappa_{+2}^{(3)} &= 10, & \kappa_{+3}^{(3)} &= 18.
\end{align*}\]

Thus, the values of $\kappa$ in increasing order are:

\[\begin{align*}
\kappa_{-1}^{(3)} &= \kappa_{-1}^{(2)} = 1, & \kappa_{+1}^{(2)} &= 3, & \kappa_{-2}^{(2)} &= 5, & \kappa_{+2}^{(2)} &= 6, & \kappa_{-3}^{(3)} &= 8, & \kappa_{+3}^{(3)} &= 10, \\
\kappa_{+3}^{(3)} &= \kappa_{-1}^{(1)} = 11, & \kappa_{+1}^{(1)} &= 13, & \kappa_{-2}^{(2)} &= 14, & \kappa_{+2}^{(2)} &= 15, & \kappa_{-2}^{(1)} &= 16, \\
\kappa_{+2}^{(1)} &= \kappa_{+3}^{(2)} = 17, & \kappa_{-3}^{(3)} &= 18.
\end{align*}\]

Now, we find $\chi_j^{(v)}$ and $\eta_j^{(v)}$ for $1 \leq v \leq 3$, and $1 \leq j \leq m^{(v)}$, by using Definitions 3.44 and 3.46. Obviously, $\chi_j^{(1)} = 0$. Therefore, $\eta_j^{(1)} = 0$, for every $1 \leq j \leq 2$.

Now we compute $\chi_j^{(v)}$ and $\eta_j^{(v)}$, for $2 \leq v \leq 3$, and $1 \leq j \leq m^{(v)} - 1$.

$\kappa_{+2}^{(2)} < \kappa_{-1}^{(3)}$ applies $\chi_1^{(2)} = 1$. Therefore, $\eta_1^{(2)} = 0$.

$\kappa_{-1}^{(2)} < \kappa_{+2}^{(2)} < \kappa_{-1}^{(1)}$ applies $\chi_2^{(2)} = 0$. Therefore, $\eta_2^{(2)} = \varrho_1^{(1)} = 5$.

$\kappa_{-2}^{(2)} < \kappa_{+3}^{(3)} < \kappa_{-1}^{(2)}$ and $\kappa_{+1}^{(2)} < \kappa_{-1}^{(1)}$ applies $\chi_1^{(3)} = 1$. Therefore, $\eta_1^{(3)} = \varrho_2^{(2)} = 2$.

$\kappa_{-3}^{(2)} < \kappa_{+2}^{(2)} < \kappa_{-1}^{(3)}$ and $\kappa_{+2}^{(2)} < \kappa_{-1}^{(1)}$ applies $\chi_1^{(3)} = 1$. Therefore, $\eta_1^{(3)} = \varrho_2^{(2)} = 2$.

Thus, by Theorem 3.50, the standard OGS canonical form of $\pi^{-1}$ is a product of all the elements of the form

\[\frac{\kappa_j^{(v)}}{\kappa_{-j}^{(v)}} \left( \varrho_j^{(v)} + \eta_j^{(v)} \right) \]

where, $1 \leq v \leq 3$ and $1 \leq j \leq m^{(v)}$.
Therefore, by putting \( \kappa_{-j}^{(v)} \) and \( \kappa_{-j}^{(v)} \) in increasing order, and substituting the corresponding \( \vartheta_{j}^{(v)} \) or \( \eta_{j}^{(v)} \) or \( \eta_{j-1}^{(v)} \) in the exponent for every \( 1 \leq v \leq 3 \) and \( 1 \leq j \leq m^{(v)} \), we get the following standard OGS canonical form of \( \pi^{-1} \):
\[
\pi^{-1} = t_5^0 \cdot t_5^5 \cdot t_6^{1+2} \cdot t_8^{8-(3+2)} \cdot t_{10}^{3+2} \cdot t_{11}^{11-(4+2+5)} \cdot t_{13}^{2+5} \cdot t_{14}^{14-(3+5)} \cdot t_{15}^{5} \cdot t_{16}^{16-6} \cdot t_{17}^{6+3} \cdot t_{18}^{4}
\]
\[
= t_5^3 \cdot t_6^3 \cdot t_8^5 \cdot t_{10}^6 \cdot t_{13}^7 \cdot t_{14}^6 \cdot t_{15}^5 \cdot t_{16}^9 \cdot t_{17}^4 \cdot t_{18}^4.
\]

The standard OGS elementary factorization of \( \pi^{-1} \):

The 6 different values of \( \kappa_{-j}^{(v)} \), such that \( 1 \leq v \leq 3 \), and \( 1 \leq j \leq m^{(v)} \), implies 6 different OGS elementary factors as follows:

- \( \kappa_{-1}^{(3)} = \kappa_{-1}^{(2)} = 1 \) implies \( \text{maj} \left( \pi^{-1(1)} \right) = 1 \);
- \( \kappa_{-2}^{(2)} = 5 \) implies \( \text{maj} \left( \pi^{-1(2)} \right) = 5 \);
- \( \kappa_{-2}^{(3)} = 8 \) implies \( \text{maj} \left( \pi^{-1(3)} \right) = 8 \);
- \( \kappa_{-3}^{(3)} = \kappa_{-1}^{(1)} = 11 \) implies \( \text{maj} \left( \pi^{-1(4)} \right) = 11 \);
- \( \kappa_{-3}^{(2)} = 14 \) implies \( \text{maj} \left( \pi^{-1(5)} \right) = 14 \);
- \( \kappa_{-3}^{(1)} = 16 \) implies \( \text{maj} \left( \pi^{-1(6)} \right) = 16 \);

- \( \pi^{-1(1)} = t_5, \quad \pi^{-1(2)} = t_5^2 \cdot t_6^3, \quad \pi^{-1(3)} = t_8^3 \cdot t_{10}^5, \quad \pi^{-1(4)} = t_{13}^7 \cdot t_{14}^5, \quad \pi^{-1(5)} = t_{14}^7 \cdot t_{15}^3 \cdot t_{16}^7, \quad \pi^{-1(6)} = t_{16}^3 \cdot t_{17}^9 \cdot t_{18}^4. \)

Indeed, the permutation presentation of \( \pi^{-1} \):
\[
\pi^{-1} = [15; 5; 6; 11; 12; 7; 16; 17; 8; 9; 18; 1; 2; 13; 3; 10; 4; 14].
\]

Where,
\[
\text{des} \left( \pi^{-1} \right) = \{1, 5, 8, 11, 14, 16 \}.
\]

4 Conclusions and future plans

In the paper, we introduced a quite interesting generalization of the fundamental theorem for abelian groups to two important and very elementary families of non-abelian Coxeter groups, the \( I \)-type (where the number of vertices on the Coxeter graph is always two, the smallest possibly number of vertices for a non-abelian Coxeter group, although the lace of the connecting edge depends on the group, and can be any integer), and the \( A \)-type (where the number of vertices depends on the group, and can be any integer greater than one, although the lace of the connecting edges are simply laced, and with
a minimal number of connections such that the Coxeter graph is connected and non-
abelian). We showed canonical forms, with very interesting exchange laws, and quite
interesting properties concerning the Coxeter lengths of the elements. The interesting
results for the two elementary families of non-abelian Coxeter groups motivate gener-
ization for further families of Coxeter and generalized Coxeter groups, which have an
importance in the classification of Lie-Algebras and the Lie-type simple groups, and in
other fields of mathematics, such as algebraic geometry for classification of fundamental
groups of Galois covers of surfaces \cite{2}. In the first step it is interesting to generalize
the standard \textit{OGS} canonical forms and the exchange laws for the finite classical families
of \textit{B} and \textit{D}-type, which have presentations as signed permutations, then to the affine
classical families \textit{A}, \textit{B}, \textit{C}, and \textit{D}, and also to other generalizations of the mentioned
Coxeter groups, as the complex reflection groups \textit{G}(_{r,p,n}) \cite{12} or the generalized affine
classical groups, the definition of which is described in \cite{11,3}.

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