$q$-Virasoro/W Algebra at Root of Unity and Parafermions

H. Itoyama$^{a,b,*}$, T. Oota$^{b†}$ and R. Yoshioka$^{b‡}$

$^a$ Department of Mathematics and Physics, Graduate School of Science
Osaka City University

$^b$ Osaka City University Advanced Mathematical Institute (OCAMI)
3-3-138, Sugimoto, Sumiyoshi-ku, Osaka, 558-8585, Japan

Abstract

We demonstrate that the parafermions appear in the $r$-th root of unity limit of $q$-Virasoro/$W_n$ algebra. The proper value of the central charge of the coset model $\hat{sl}(n)_{r+\hat{sl}(n)_{m-n}}/\hat{sl}(n)_{m-n+r}$ is given from the parafermion construction of the block in the limit.
1 Introduction

Ever since the AGT relation [1, 2, 3] (the correspondence between the correlators of 2d QFT and the 4d instanton sum) was introduced, the both sides of the correspondence have been intensively studied by a number of people. For example, in the 2d side, the $\beta$-deformed matrix model is used in order to control the integral representation of the conformal block [4, 5, 6, 7, 8, 9, 10]. There are also some proposals for proving the 2d-4d connection [11, 12, 13, 14, 15]. Moreover similar correspondence has been found and examined [16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26]. Among these, we pay our attention, in this paper, to the correspondence between the coset model,

\[
\widehat{\mathfrak{sl}}(n)^r \oplus \widehat{\mathfrak{sl}}(n)^p \quad (1.1)
\]

and the $\mathcal{N} = 2$ $SU(n)$ gauge theory on $\mathbb{R}^4/\mathbb{Z}_r$ [20, 23]. Here $\widehat{\mathfrak{sl}}(n)^k$ stands for the affine Lie algebra in the representation of level $k$ and $r$ and $p$ will be specified in this paper.

On the 2d CFT side, a quantum deformation ($q$-deformation) of the Virasoro algebra [27] and the $W_n$ algebra [28, 29] is known, while the 4d gauge theories can be lifted to five-dimensional theories with the fifth direction compactified on a circle. There exists a natural generalization to the connection between the 2d theory based on the $q$-deformed Virasoro/W algebra and the five-dimensional $\mathcal{N} = 2$ gauge theory [30]. For recent developments, see, for example, [31, 32, 33, 34, 35, 36, 37]. In the previous paper [32], we proposed a limiting procedure to get the Virasoro/W block in the 2d side from that in the $q$-deformed version. On the other hand, we saw that the instanton partition function on $\mathbb{R}^4/\mathbb{Z}_r$ are generated from that on $\mathbb{R}^5$ at the same limit. This result means if we assume the 2d-5d connection, it is automatically assured that the Virasoro/W blocks generated by using the limiting procedure agree with the instanton partition function on $\mathbb{R}^4/\mathbb{Z}_r$. Our limiting procedure corresponds to a root of unity limit in $q$. A root of unity limit of the $q$-Virasoro algebra was also considered in [38]. Our limit is slightly different from this and is similar to the one used in order to construct the eigenfunctions of the spin Calogero-Sutherland model from Macdonald polynomials in [39, 40].

In the present paper we will elaborate our limiting procedure and show that the $\mathbb{Z}_r$-parafermionic CFT which has the symmetry described by (1.1) appears in the 2d side. We clarify also the relation between the free parameter $p$ and the omega background parameters in the 4d side.

The paper is organized as follows: In the next section, we review the limiting procedure for $q$-Virasoro algebra [32]. In section 3, we consider the $q$-deformed screening current and charge and show that the $\mathbb{Z}_r$-parafermion currents are derived in a natural way. In section 4, we consider the generalization to $q$-$W_n$ algebra.

2 Root of Unity Limit of $q$-Virasoro Algebra

In this section, we review the root of unity limit [32] of the $q$-deformed Virasoro algebra [27] which has two parameters $q$ and $t = q^\beta$. The defining relation is

\[
f(z'/z)T(z)T(z') - f(z/z')T(z')T(z) = \frac{(1-q)(1-t^{-1})}{(1-p)} \left[ \delta(pz/z') - \delta(p^{-1}z/z') \right], \quad (2.1)
\]
where \( p = q/t \) and

\[
f(z) = \exp \left( \sum_{n=1}^{\infty} \frac{1}{n} \frac{(1-q^n)(1-t^{-n})}{(1+p^n)} z^n \right).
\]

(2.2)

The multiplicative delta function is defined by

\[
\delta(z) = \sum_{n \in \mathbb{Z}} z^n.
\]

(2.3)

Using the \( q \)-deformed Heisenberg algebra \( \mathcal{H}_{q,t} \):

\[
[\alpha_n, \alpha_m] = -\frac{1}{n} \frac{(1-q^n)(1-t^{-n})}{(1+p^n)} \delta_{n+m,0}, \quad (n \neq 0),
\]

\[
[\alpha_n, Q] = \delta_{n,0},
\]

(2.4)

the \( q \)-Virasoro operator \( T(z) \) can be realized as

\[
T(z) =: \exp \left( \sum_{n \neq 0} \alpha_n z^{-n} \right) :p^{1/2}q^{\sqrt{\beta} \alpha_0}+: \exp \left( -\sum_{n \neq 0} \alpha_n (pz)^{-n} \right) :p^{-1/2}q^{-\sqrt{\beta} \alpha_0},
\]

(2.5)

The \( q \)-deformed chiral bosons are defined in terms of the \( q \)-deformed Heisenberg algebra as

\[
\tilde{\varphi}^{(\pm)}(z) = \tilde{\varphi}^{(\pm)}_0(z) + \tilde{\varphi}^{(\pm)}_R(z),
\]

(2.6)

where

\[
\tilde{\varphi}^{(\pm)}_0(z) = \beta^{\pm1/2}Q + \frac{2}{r} \beta^{\pm1/2} \alpha_0 \log z^r + \sum_{n \neq 0} \frac{(1+p^{-nr})}{(1-\xi^{nr})} \alpha_{nr} z^{-nr},
\]

\[
\tilde{\varphi}^{(\pm)}_R(z) = \sum_{r=1}^{\infty} \sum_{n \in \mathbb{Z}} \frac{(1+p^{-nr})}{(1-\xi^{nr})} \alpha_{nr} z^{-nr}.
\]

(2.7)

Here \( \xi_+ = q, \xi_- = t \).

Let us consider the simultaneous \( r \)-th root of unity limit in \( q \) and \( t \) which is given by

\[
q = \omega e^{-\frac{1}{\sqrt{\beta}} h}, \quad t = \omega e^{\sqrt{\beta} h}, \quad p = e^{Q_E h}, \quad h \to 0,
\]

(2.8)

where \( \omega = e^{\frac{2\pi i}{r}} \) and \( Q_E = \sqrt{\beta} - \frac{1}{\sqrt{\beta}} \). Since \( t = q^\beta \), this limit is possible if the parameter \( \beta \) takes the rational number such as

\[
\beta = \frac{rm_- + 1}{rm_+ + 1},
\]

(2.9)

where \( m_\pm \) are non-negative integers. In the limit, we have two types of bosons \( \phi(w) \) and \( \varphi(w) \) [32] respectively given by

\[
\lim_{h \to 0} \tilde{\varphi}^{(\pm)}_0(z) = \sqrt{\frac{2}{r}} \beta^{\pm1/2} \phi(w),
\]

\[
\lim_{h \to 0} \tilde{\varphi}^{(\pm)}_R(z) = \sqrt{\frac{2}{r}} \varphi(w),
\]

(2.10)
where \( w = z^r \) and
\[
\phi(w) = Q_0 + a_0 \log w - \sum_{n \neq 0} \frac{a_n}{n} w^{-n},
\]
(2.11)
\[
\varphi(w) = \sum_{\ell=1}^{r-1} \varphi^{(\ell)}(w), \quad \varphi^{(\ell)}(w) = \sum_{n \in \mathbb{Z}} \frac{\tilde{a}_{n+\ell/r}}{n + \ell/r} w^{-n-\ell/r}.
\]
(2.12)

The commutation relations are
\[
[a_m, a_n] = m \delta_{m+n,0}, \quad [a_n, Q_0] = \delta_{n,0},
\]
\[
[\tilde{a}_{n+\ell/r}, \tilde{a}_{-m-\ell'/r}] = (n + \ell/r) \delta_{m,m'} \delta_{\ell,\ell'}.
\]
(2.13)

The boson \( \phi(w) \) and the twisted boson \( \varphi(w) \) play an important role for the appearance of the \( \mathbb{Z}_r \)-parafermions.

## 3 \( \mathbb{Z}_r \)-parafermionic CFT

The \( q \)-deformed screening current and the charge are defined respectively by
\[
S^{(\pm)}(z) =: e^{\varphi^{(\pm)}(z)} :, \quad Q^{(\pm)}_{[a,b]} = \int_a^b d\xi_{\pm} z S^{(\pm)}(z),
\]
(3.1)
where the Jackson integral is defined by
\[
\int_0^a d\xi_{\pm} z f(z) = a(1-q) \sum_{k=0}^{\infty} f(aq^k)q^k.
\]
(3.2)

Multiplying the regularization factor, we obtain the screening charge in the root of unity limit, up to normalization,
\[
Q^{(\pm)}_{[a',b']} \equiv \lim_{h \to 0} \frac{(1-q^r)}{(1-q)} Q^{(\pm)}_{[a,b]} = \int_{a'}^{b'} dw \psi_1(w) : e^{\sqrt{r} \phi(w)} :,
\]
(3.3)
where we have defined [41]
\[
\psi_1(w) = \frac{A_r}{w^{(r-1)/r}} \sum_{k=0}^{r-1} \omega^k : \exp \left\{ \sqrt{\frac{2}{r}} \phi^{(k)}(w) \right\} :.
\]
(3.4)

Here \( A_r \) is the normalization factor and we have introduced
\[
\phi^{(k)}(w) \equiv \varphi(e^{2\pi i k}w).
\]
(3.5)

The correlation function is given by
\[
\langle \phi^{(k)}(w) \phi^{(k')}(w') \rangle = \log \frac{(1 - \omega^{k-k'} (w'/w)^{1/r})^r}{1 - w'/w} = \log \frac{(1 - w'/w)^{r-1}}{\prod_{j=1}^{r-1} (1 - \omega^{k-k+j} (w'/w)^{1/r})^r}.
\]
(3.6)
Note that
\[
\phi^{(k+1)}(w) = \phi^{(k)}(e^{2\pi i}w), \quad \phi^{(r+k)}(w) = \phi^{(k)}(w), \quad \sum_{k=0}^{r-1} \phi^{(k)}(w) = 0. \tag{3.7}
\]

For example, we consider the \(r = 2\) case. In the limit, we obtain
\[
\lim_{q \to 1} S(z) = :e^{\sqrt{\beta} \phi(w)} e^{\varphi(w)} :, \tag{3.8}
\]
and after the appropriate normalization, we obtain the following screening charge for the superconformal block [42, 43]:
\[
Q_{[a^2,b^2]} = \int_{a^2}^{b^2} dw \psi(w) : e^{\sqrt{\beta} \phi(w)} :, \tag{3.9}
\]
where
\[
\psi(w) \equiv \frac{i}{2\sqrt{2w}} \left( :e^{\varphi(w)} : - :e^{-\varphi(w)} : \right), \quad \langle \psi(w_1)\psi(w_2) \rangle = \frac{1}{w_1 - w_2}, \tag{3.10}
\]
is the NS fermion.

From now on we will show that the \(Z_r\)-parafermions appear in the general \(r\)-th root of unity limit. In particular, \(\psi_1(w)\) will be shown to work as the first parafermion current.

The \(Z_r\)-parafermion algebra consists of \((r - 1)\) currents \(\psi_\ell(w) (\ell = 1, \cdots, r - 1)\) satisfying the following defining relations [44]:
\[
\psi_\ell(w)\psi_{\ell'}(w') = \frac{c_{\ell,\ell'}}{(w - w')^{2\ell'/r}} \left\{ \psi_{\ell + r}(w') + O(w - w') \right\}, \quad \ell + \ell' < r, \tag{3.11}
\]
\[
\psi_\ell(w)\psi^\dagger_{\ell'}(w') = c_{\ell,r - \ell}(w - w')^{-2(r - \ell')/r} \left\{ \psi_{\ell - r}(w') + O(w - w') \right\}, \quad \ell' < \ell \tag{3.12}
\]
\[
\psi_\ell(w)\psi^\dagger_{\ell'}(w') = (w - w')^{-2\Delta_\ell} \left\{ 1 + \frac{2\Delta_\ell}{c_p}(w - w')^{2} T_{PF}(w) + O((w - w')^3) \right\}, \tag{3.13}
\]
where \(\psi^\dagger_{\ell}(w) = \psi_{r-\ell}(w)\) and
\[
\Delta_\ell = \frac{\ell(r - \ell)}{r}, \quad c_p = \frac{2(r - 1)}{r + 2}. \tag{3.14}
\]

are the conformal dimension of \(\psi_\ell(w)\) and the central charge of the parafermionic stress tensor \(T_{PF}\). The explicit form of \(T_{PF}(w)\) is given in [45]. The coefficients \(c_{\ell,\ell'}\) are given by
\[
c_{\ell,\ell'} = \sqrt{\frac{(\ell + \ell')!(r - \ell)!(r - \ell')!}{\ell!\ell'!(r - \ell - \ell')!r!}}. \tag{3.15}
\]

The OPE of (3.4) is
\[
\psi_1(w)\psi_1(w') \equiv \frac{c_{1,1}}{(w - w')^{2/r}} \left\{ \psi_2(w) + O(w - w') \right\}. \tag{3.16}
\]
Here we have defined the second parafermion,
\[
\psi_2(w) = \frac{A_r}{c_{1,1}w^{2(r-2)/r}} \sum_{k,k'=0}^{r-1} \omega^{k+k'}(1 - \omega^{k'-k})^2 \ e^{\sqrt{\tau}(\phi^{(k)}(w) + \phi^{(k')})(w))} :. \tag{3.17}
\]
Similarly, the \((\ell + 1)\)-th parafermion is obtained from \(\ell\)-th parafermion by

\[
\psi_{\ell+1}(w) \equiv \lim_{w'\to w} \frac{w - w'}{c_{1,\ell}} \psi_1(w') \psi_\ell(w).
\]  

(3.18)

In particular,

\[
\psi_1^\dagger(w) \equiv \psi_{r-1}(w) = \frac{B_r}{w^{(r-1)/r}} \sum_{\ell=1}^{r-1} \omega^\ell \exp \left\{ -\sqrt{\frac{2}{r}} \phi^{(\ell)}(w) \right\},
\]  

(3.19)

where \(B_r\) is a constant which can be determined by the relation

\[
\langle \psi_1(w) \psi_1^\dagger(w') \rangle = \frac{1}{(w - w')^{2(r-1)/r}}.
\]  

(3.20)

After all, we have the chiral boson \(\phi(w)\) coupled to \(Q_E\) and the \(\mathbb{Z}_r\)-parafermion \(\psi_\ell(w)\). Therefore, the stress tensor of the whole system is

\[
T(w) = T_B(w) + T_{PF}(w),
\]  

(3.21)

where \(T_B(w)\) stands for the usual stress tensor for the chiral boson field. The central charge is

\[
c^{(r)} = 1 - \frac{6Q_E^2}{r} + \frac{2(r-1)}{r+2} = \frac{3r}{r+2} - \frac{6Q_E^2}{r}.
\]  

(3.22)

Because \(\beta\) is restricted to the rational number \((2.9)\), \((3.22)\) is written as

\[
c^{(r,m,s)} = \frac{3r}{r+2} - \frac{6rs^2}{m(m+rs)}.
\]  

(3.23)

where we have set \(m = rm_+ + 1\) and \(s = m_+ - m_-.\) Especially, when \(s = 1\),

\[
c^{(r,m,1)} = \frac{3r}{r+2} - \frac{6r}{m(m+r)},
\]  

(3.24)

is the central charge of the unitary series of the \(\mathbb{Z}_r\)-parafermionic CFT [46].

The form of the screening charge in the case of general \(r\) is the same as that of eq. \((3.9)\).

4 Root of Unity Limit of \(q\)-\(W_n\) Algebra

In this section, we consider the generalization to the \(q\)-\(W_n\) algebra [29]. We denote by \(h\) the Cartan subalgebra of \(\mathfrak{sl}(n)\) Lie algebra. The \(q\)-\(W_n\) algebra is expressed in terms of the following \(h\)-valued \(q\)-deformed boson,

\[
\{e_a, \varphi^{(\pm)}(z)\} \equiv \tilde{\varphi}_a^{(\pm)}(z) = \tilde{\varphi}_{0,a}^{(\pm)}(z) + \tilde{\varphi}_{R,a}^{(\pm)}(z),
\]  

(4.1)

where

\[
\tilde{\varphi}_{0,a}^{(\pm)}(z) = \beta^{\pm \frac{1}{2}} Q_a + \beta^{\pm \frac{1}{2}} \alpha_{0,a} \log z + \sum_{n \neq 0} \frac{1}{\xi^{(nr/2)}_{\pm}} \xi^{-nr/2}_{\pm} \alpha_{nr,a} z^{-nr},
\]  

(4.2)

\[
\tilde{\varphi}_{R,a}^{(\pm)}(z) = \sum_{\ell=1}^{r-1} \tilde{\varphi}_{\ell,a}^{(\pm)}(z) = \sum_{\ell=1}^{r-1} \sum_{n \in \mathbb{Z}} \xi^{(nr+\ell/2)}_{\pm} \frac{1}{\xi^{-(nr+\ell/2)}_{\pm}} \alpha_{nr+\ell,a} z^{-(nr+\ell)},
\]  

(4.3)
and \( e_a (a = 1, \cdots, n - 1) \) are the simple roots and \( \langle , \rangle : \mathfrak{h}^* \otimes \mathfrak{h} \rightarrow \mathbb{C} \) is the canonical pairing. The commutation relations are given by

\[
[Q_a, \alpha_{0,b}] = C_{ab},
\]

\[
[\alpha_{n,a}, \alpha_{m,b}] = \frac{1}{n} (q^{n/2} - q^{-n/2})(t^{n/2} - t^{-n/2}) C_{ab} \delta_{n+m,0},
\]

\[
[Q_a, Q_b] = 0, \quad [\alpha_{0,a}, \alpha_{0,b}] = 0,
\]

where \( C_{ab} \) is the Cartan matrix of \( A \) type and

\[
C_{ab}(p) = \left[ 2 \right] p \delta_{a,b} - p^{1/2} \delta_{a,b-1} - p^{-1/2} \delta_{a-1,b}.
\]

The \( q \)-number is defined by

\[
[q^n]_q = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}}.
\]

Similar to the \( q \)-Virasoro case, we consider the limit,

\[
q = \omega^k e^{-\frac{b}{\sqrt{r}h}}, \quad t = \omega^k e^{-\sqrt{r}h}, \quad p = q/t = e^{Q \epsilon h}, \quad \omega = e^{\frac{2\pi i}{r}}, \quad h \rightarrow +0,
\]

where \( \omega = e^{\frac{2\pi i}{r}} \) and \( k \) is a natural number mutually prime to \( r \). The condition to be able to take this limit is that \( \beta \) is a rational number,

\[
\beta = \frac{rm_- + k}{rm_+ + k},
\]

where \( m_\pm \) are non-negative integers. Taking this limit,

\[
\lim_{h \rightarrow 0} \varphi_0^a(z) = \frac{1}{\sqrt{r}} \beta^{1/2} \phi^a(w),
\]

\[
\lim_{h \rightarrow 0} \varphi_R^a(z) = \frac{1}{\sqrt{r}} \varphi^a(w),
\]

we obtain

\[
\phi^a(w) = Q^a_0 + a_0^a \log w - \sum_{n \neq 0} \frac{1}{n} a_n^a w^{-n},
\]

\[
\varphi(w) = \sum_{\ell=1}^{r-1} \varphi_\ell(w), \quad \varphi_\ell(w) = \sum_{n \in \mathbb{Z}} \sum_{\ell=1}^{r-1} \frac{1}{n + \ell/r} \tilde{a}_n^a \epsilon^r_{n+\ell/r} w^{-(n+\ell/r)},
\]

Here we have normalized as

\[
Q^a = \frac{1}{\sqrt{r}} Q_0^a, \quad \alpha_0^a = \sqrt{r} a_0^a,
\]

\[
\alpha_{nr}^a = -(-1)^{nk} \sqrt{r} \epsilon^r_{n} a_n^a,
\]

\[
\alpha_{nr+\ell}^a = \frac{e^{i \pi k(nr+\ell)/2} - e^{-i \pi k(nr+\ell)/2}}{\sqrt{r}(n+\ell/r)} \tilde{a}_n^a \epsilon^r_{n+\ell/r}.
\]
The commutation relations are
\[
[Q^a, a^b_0] = C_{ab}, \quad [Q^a, Q^b] = 0, \quad [a^a_0, a^b_0] = 0,
\]
(4.16)
\[
[a^a_n, a^b_m] = nC_{ab}\delta_{n+m,0},
\]
(4.17)
\[
[\tilde{a}^a_{n+\ell/r}, \tilde{a}^b_{-m-\ell/r}] = \left(n + \frac{\ell}{r}\right) C_{ab}\delta_{n,m}\delta_{\ell,\ell'}.
\]
(4.18)

The correlation functions are
\[
\langle \phi^a(w) \phi^b(w') \rangle = C_{ab} \log(w - w'),
\]
(4.19)
\[
\langle \varphi^{a}_{\ell}(w) \varphi^{b}_{\ell'}(w') \rangle = \delta_{\ell+\ell',r} C_{ab} \sum_{k=0}^{r-1} \omega^{-k\ell} \log \left[ 1 - \omega^k \left( \frac{w'}{w} \right)^{\frac{r}{r-1}} \right],
\]
(4.20)
\[
\langle \varphi^a(w) \varphi^b(w') \rangle = C_{ab} \log \left[ \frac{(1 - (w'/w)^{1/r})^r}{1 - (w'/w)} \right].
\]
(4.21)

For each \(e_a\), we define
\[
\psi_{e_a}(w) = \frac{A_r}{w^{(r-1)/r}} \sum_{\ell=0}^{r-1} \omega^\ell : \exp \left[ \sqrt{\frac{1}{r} \phi^{(\ell)}_a}(w) \right] :,
\]
(4.22)

where \(A_r\) is a normalization factor and
\[
\phi^{(\ell)}_a(w) \equiv \varphi_a(e^{2\pi i \ell/w}).
\]
(4.23)

Let \(\alpha = \sum_{a=1}^{n-1} n_a e_a \in Q\), where \(n_a\) are non-negative integers and \(Q\) denotes the root lattice. We obtain the corresponding parafermion, up to its normalization,
\[
\psi_\alpha \sim \prod \psi_{e_a}^{n_a}.
\]
(4.24)

The independent parafermion can be given only for the case \(\alpha \in Q/rQ\). Not of all \(\psi_\alpha\) are independent;
\[
1 \sim \psi_{e_a} \cdots \psi_{e_a}
\]
(4.25)

For example, in the the case of \(\mathfrak{sl}(3)\) algebra and \(r = 4\), the corresponding parafermions are drawn in the Fig. 1. We define the parafermion associated with negative of a simple root by
\[
\psi^{-e_a} \sim \psi_{e_a} \psi_{e_a} \cdots \psi_{e_a}
\]
(4.26)

The normalization can be determined by the correlation functions [47],
\[
\langle \psi_\alpha(w) \psi^{-\alpha}(w') \rangle = (w - w')^{-2 + \frac{\alpha^2}{r}}
\]
(4.27)

where \(\alpha^2 = (\alpha, \alpha)\). In particular,
\[
\langle \psi_{e_a}(w) \psi^{-e_a}(w') \rangle = (w - w')^{-2 - \frac{1}{r}}.
\]
(4.28)
Fig. 1: The parafermions in the case of \( \mathfrak{sl}(3) \) and \( r = 4 \).

In the case of the \( \mathfrak{sl}(2) \) algebra, we obtain the first \( \mathbb{Z}_r \)-parafermion,

\[
\psi_1(w) = \psi_{e_1}(w).
\]  (4.29)

Similar to the case of \( n = 2 \) (3.22), the central charge is given by

\[
c_n^{(r)} = \frac{n(n-1)(r-1)}{r+n} + (n-1) \left( 1 - n(n+1) \frac{Q_E^2}{r} \right)
= \frac{r(n^2-1)}{r+n} - n(n^2-1) \frac{Q_E^2}{r}.
\]  (4.30)

When we set \( m = rm_+ + k \), \( m_- = m_+ + s \) in (4.8), this central charge becomes

\[
c_n^{(r,m,s)} = \frac{r(n^2-1)}{r+n} - \frac{rs^2n(n^2-1)}{m(m+rs)}
= \frac{(n^2-1)r(mn - n)(mn + n + r)}{(r+n)(m \frac{m+n}{s} + r)},
\]  (4.31)

which is the same as that of the coset model,

\[
\mathfrak{sl}(n)_r \oplus \mathfrak{sl}(n)\frac{m}{s} - n \sim \mathfrak{sl}(n)_{\frac{m+n}{s} + r}.
\]  (4.32)

Compared with (1.1) we find

\[
p = \frac{m}{s} - n.
\]  (4.33)

In the case of \( s = 0 \) corresponding to \( Q_E = 0 \), we have the central charge of the usual Sugawara stress tensor for \( \mathfrak{sl}(n)_r \),

\[
c_n^{(r,m,0)} = \frac{r(n^2-1)}{r+n} = c_{\mathfrak{sl}(n)_r}.
\]  (4.34)
It is well-known that the affine Lie algebra \( \widehat{\mathfrak{sl}}(n)_r \) is represented by parafermions and an auxiliary boson \([47]\). In the case of \( s = 1 \), because (4.31) becomes
\[
\epsilon_n^{(r,m,1)} = \frac{(n^2 - 1)r(m-n)(m+n+r)}{(r+n)m(m+r)},
\]
the model gives us the unitary series of the coset,
\[
\frac{\widehat{\mathfrak{sl}}(n)_r \oplus \widehat{\mathfrak{sl}}(m-n)}{\widehat{\mathfrak{sl}}(m-n+r)}.
\]

We can see how the level \( p \) is related with the omega-background parameters \( \epsilon_1 \) and \( \epsilon_2 \) in the 4-d side. Since \( \beta = -\epsilon_1/\epsilon_2 \), (4.8) yields the condition to the ratio of these parameters. Therefore, when we introduce the free parameter \( \epsilon \), \( \epsilon_{1,2} \) can be written respectively as
\[
\epsilon_1 = \epsilon(p+n+r), \quad \epsilon_2 = -\epsilon(p+n).
\]
This result suggests that the Nekrasov-Shatashvili limit \( \epsilon_1 \to 0 \) (resp. \( \epsilon_2 \to 0 \)) of the \( \mathcal{N} = 2 \) gauge theory on the \( \mathbb{R}^4/\mathbb{Z}_r \) corresponds to the critical level limit \( p + r \to -n \) (resp. \( p \to -n \)) of the coset model.

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