Recovering an Algebraic Curve Using its Projections From Different Points
Applications to Static and Dynamic Computational Vision

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Abstract

We study how an irreducible closed algebraic curve $X$ embedded in $\mathbb{CP}^3$, which degree is $d$ and genus $g$, can be recovered using its projections from points onto embedded projective planes. The different embeddings are unknown. The only input is the defining equation of each projected curve. We show how both the embeddings and the curve in $\mathbb{CP}^3$ can be recovered modulo some actions of the group of projective transformations of $\mathbb{CP}^3$.

In particular in the case of two projections, we show how in a generic situation, a characteristic matrix of the two embeddings can be recovered. In the process we address dimensional issues and as a result establish the minimal number of irreducible algebraic curves required to compute this characteristic matrix up to a finite-fold ambiguity, as a function of their degree and genus. Then we use this matrix to recover the class of the couple of maps and as a consequence to recover the curve. For a generic situation, two projections define a curve with two irreducible components. One component has degree $d(d-1)$ and the other has degree $d$, being the original curve.

Then we consider another problem. $N$ projections, with known projections operators and $N >> 1$, are considered as an input and we want to recover the curve. The recovery can be done by linear computations in

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the dual space and in the Grassmannian of lines in $\mathbb{C}P^3$, that we denote by $G(1,3)$. Those computations are respectively based on the dual variety and on the variety of intersecting lines. In both cases a simple lower bound for the number of necessary projections is given as a function of degree and genus. A closely related question is also considered. Each point of a finite closed subset of an irreducible algebraic curve, is projected onto a plane from a point. For each point the center of projection is different. The projections operators are known. We show when and how the recovery of the algebraic curve is possible in function of the degree of the curve of minimal degree generated by the centers of projection.

These questions were motivated by applications to static and dynamic computational vision. Therefore a second part of this work is devoted to applications to this field. The results in this paper solve a long standing problem in computer vision that could not have been solved without algebraic-geometric methods.

1 Introduction

Consider an irreducible closed algebraic curve $X \in \mathbb{C}P^3$ (in the sequel we simply write $\mathbb{P}^n$ for $\mathbb{C}P^n$). This curve is projected onto several projective planes embedded in $\mathbb{P}^3$ through several center of projections, say $\{O_i\}_{i=1}^n$. Each projection mapping, denoted by $\pi_i : \mathbb{P}^3 \setminus \{O_i\} \rightarrow \mathbb{P}^2$ is presented as a $3 \times 4$ matrix, $M_i$ defined modulo multiplication by non-zero scalar. Then each point $P$ different from $O_i$ is mapped by $\pi_i$ to $M_iP$. Each projection operator $\pi_i$, via its matrix, can be regarded as a point in $\mathbb{P}^{11}$. Let $Y_i = \pi_i(X)$ be the different projections of the curve $X$. In the sequel we always deal with generic configurations, even when not mentionned explicitly.

When we consider the problem of recovering the projections maps from the projected curves, we will show that the recovery is possible only modulo some action of the group of projective transformations of $\mathbb{P}^3$ on the set of projection maps. To define this action we refer to a projection map as a point in $\mathbb{P}^{11}$. Assume that we have $n$ projections, consider the following projective variety $V = \mathbb{P}^{11} \times \ldots \times \mathbb{P}^{11}$. Let $Pr_3$ be the group of projective transformations of $\mathbb{P}^3$. We define an action of $Pr_3$ on $V$ as follows: $\theta_n : Pr_3 \rightarrow Mor(V,V), \ A \mapsto ((Q_1,\ldots,Q_n) \mapsto (M_1A^{-1},\ldots,M_nA^{-1}))$, where each matrix $M_i$ is built from the coordinates of $Q_i = [Q_{i,1},\ldots,Q_{i,12}]^T$ as follows:

$$M_i = \begin{bmatrix} Q_{i,1} & Q_{i,2} & Q_{i,3} & Q_{i,4} \\ Q_{i,5} & Q_{i,6} & Q_{i,7} & Q_{i,8} \\ Q_{i,9} & Q_{i,10} & Q_{i,11} & Q_{i,12} \end{bmatrix}$$

The geometric meaning of this action is that if we change the projective basis in $\mathbb{P}^3$, by the transformation $A$, we need to change the projection maps in accordance for the projected curves to be invariant.

We first investigate the case of two projections. Given the projected curves $Y_1$ and $Y_2$ as the only data, our first problem is to compute the characteristic
matrix (to be defined below) of two projections maps, \( \pi_1 \) and \( \pi_2 \), up to a finite-fold ambiguity. It is shown that this is equivalent to find a necessary and sufficient conditions on \( X \) for the action of \( \theta_2 \) to have a finite number of orbits. Then we show that for each orbit we can recover the curve \( X \) modulo \( \mathbb{P}_3 \). More precisely each orbit induces a curve embedded in \( \mathbb{P}^3 \) containing two irreducible components, one of degree \( d(d - 1) \) and the other of degree \( d \). The latter is the curve we are looking for.

Then we turn to another problem. The projections maps \( \pi_i, i = 1, \ldots, n \) are now assumed to be known, in addition to the projected curves \( Y_i \). We want to recover the curve \( X \). This can be performed by linear computations using either the dual varieties or the variety of lines intersecting \( X \). In both case a simple lower bound of the minimal number of projections is simply deduced.

With the variety of lines intersecting \( X \) another problem is also handled. Consider a set of \( N \) points in \( \mathbb{P}^3 \) and let \( X \) be the curve generated by these points. Each of these points is projected on a different plane by a different projection operator \( \pi_i \). Those projection operators are known. We want to recover \( X \) by linear computations. Let \( Z \) be the curve, of minimal degree, generated by the centers of projections. We give a formula of the number of constraints obtained on \( X \) as a function of the degree of \( Z \).

Finally we show how those questions were motivated by some problems related to static and dynamic computational vision. Therefore we conclude by showing how our results can be applied in that context.

Since our computations will occur in \( \mathbb{P}^3 \), we fix \([X, Y, Z, T]^T\), as homogeneous coordinates, and \( T = 0 \) as the plane at infinity.

## 2 Projection operators

Let \( \pi \) be a projection operator from \( \mathbb{P}^3 \) to an embedded projective plane \( i(\mathbb{P}^2) \) through a point \( O \). This projection can be presented by a \( 3 \times 4 \) matrix \( M \). There exists a set of simple, but very useful, properties. The kernel of \( M \) is exactly the center of projection. The transpose of \( M \) maps a line in \( i(\mathbb{P}^2) \) to the plane it defines with the center of projections, given as point of the dual space \( \mathbb{P}^3^* \). This can easily be deduced by a duality argument and a simple computation.

There exists a matrix \( \tilde{M} \), being a polynomial function of \( M \), which maps a point in \( i(\mathbb{P}^2) \) to the Plücker coordinates of the line it generates with the center of projections. If the matrix \( M \) is decomposed as follows:

\[
M = \begin{bmatrix}
\Gamma^T \\
\Lambda^T \\
\Theta^T
\end{bmatrix},
\]

then for \( p = [x, y, z]^T \), the line \( L_p = \tilde{M}p \) is given by the extensor: \( L_p = x\Delta \land \Theta + y\Theta \land \Gamma + z\Gamma \land \Lambda \), where \( \land \) denotes the meet operator in the Grassman-Cayley algebra (see \( \mathbb{P}^3^* \)). By duality, the matrix \( \tilde{M} = \tilde{M}^T \) maps lines in \( \mathbb{P}^3 \) to lines in \( i(\mathbb{P}^2) \).
Consider now two projection operators \( \pi_1 \) and \( \pi_2 \). Let \( O_1 \) and \( O_2 \) be the center of projections and \( i_1(P^2) \) and \( i_2(P^2) \) the plane of projections. Let \( e_j \) be the point of intersection of \( i_j(P^2) \) with the line \( O_j O_2 \). Let \( \sigma(e_2) \) be the pencil of lines in \( i_2(P^2) \) through \( e_2 \). It is easy to define a map from \( i_1(P^2) \setminus \{e_1\} \) to \( \sigma(e_2) \) as follows. Each point \( p \) is sent to the line given by \( \pi_2(\pi_1^{-1}(p)) \). This map is linear and its matrix is \( F = \tilde{M}_2 \tilde{M}_1 \). Following the standard terminology used in computational vision, we will call the matrix \( F \) the fundamental matrix of the pair of projections \( \pi_1 \) and \( \pi_2 \) and the points \( e_1 \) and \( e_2 \) will be respectively called the first and the second epipole. The line in the first (second) projection plane passing the first (second) epipole atre called the epipolar lines. Clearly \( \overline{WE} = e_2^T F = 0 \).

**Proposition 1** The knowledge of the fundamental matrix \( F \) of a couple of projections operators allows the recovery of the matrices \( M_1 \) and \( M_2 \) of the projections modulo the action \( \theta_2 \). More precisely the couple \( (M_1, M_2) \) is equivalent to \( ([I, O], [H, e_2]) \), where \( H = -\frac{e_2}{\|e_2\|^2} F \). The matrix \( [e_2] \) is defined as being \( \tau(e_2) \), where \( \tau \) maps any vector \( x \) of \( \mathbb{C}^3 \) to the matrix that represents the cross-product by \( x \). We have:

\[
\tau(x) = [x] = \begin{bmatrix}
0 & -x_3 & x_2 \\
x_3 & 0 & -x_1 \\
-x_2 & x_1 & 0
\end{bmatrix},
\]

and \( x = [x_1, x_2, x_3] \).

**Proof:** We are looking for a matrix \( A \in \text{Pr}_3 \) such that:

\[
M_1 = [I, O] A^{-1} \\
M_2 = [H, e_2] A^{-1}
\]

Let us write \( B = A^{-1} \) as follows:

\[
B = \begin{bmatrix}
\Omega & : & u \\
\ldots & & \ldots \\
v^T & : & 0
\end{bmatrix}
\]

Let us write \( M_i = [\overline{M}_i, m_i] \). Then it follows immediately from the definition of \( F \) that \( F = [e_2] \overline{M}_2 \overline{M}_3^{-1} \). Using the following algebraic identity for any two vectors \( x \) and \( y \) in \( \mathbb{C}^3 \): \( (x^T y) I = x y^T - [x][y] \), it is easy to prove that it is sufficient to take: \( \Omega = \overline{M}_1, u = m_1, v = \frac{1}{\|e_2\|^2} \overline{M}_2 e_2 \) and \( \lambda = \frac{1}{\|e_2\|^2} (e_2^T m_2 - e_2^T H m_1) \).

This shows that in order to characterize a couple of projections modulo the action \( \theta_2 \), we only need to compute the fundamental matrix. In the sequel we show how to recover \( F \) from two projections of an algebraic curve.
3 Two projections with unknown projection operators

In this section we deal with the first problem. A smooth and irreducible curve \( X \) embedded in \( \mathbb{P}^3 \) is projected onto two generic planes through two generic points. The projection operators \( \pi_1 \) and \( \pi_2 \) are unknown. First we want to recover the fundamental matrix of the couple \((\pi_1, \pi_2)\) from the projected curve \( Y_1 = \pi_1(X) \) and \( Y_2 = \pi_2(X) \).

3.1 Single projection

We shall mention of set of well known facts about generic projection. Let \( X \) be a smooth irreducible algebraic embedded in \( \mathbb{P}^3 \) and \( Y \) its projection on a generic plane through a center of projections, \( O \).

1. The curve \( Y \) will always contain singularities. Furthermore for a generic position of the center of projection, the only singularities of \( Y \) will be nodes.

2. The class of a planar curve is defined to be the degree of its dual curve. Let \( m \) be the class of \( Y \). Then \( m \) is constant for a generic position of the center of projections.

3. If \( d \) and \( g \) are the degree and the genus of \( X \), they are respectively, for a generic position of \( O \), the degree and the genus of \( Y \), and the Plücker formula yields:

\[
m = d(d - 1) - 2(\sharp \text{nodes}),
\]

\[
g = \frac{(d - 1)(d - 2)}{2} - (\sharp \text{nodes}),
\]

where \( \sharp \text{nodes} \) denotes the number of nodes of \( Y \). Hence the genus, the degree and the class are related by

\[
m = 2d + 2g - 2.
\]

3.2 Fundamental matrix construction

We are ready now to investigate the recovery of the fundamental matrix of a couple of projections \((\pi_1, \pi_2)\) when the only knowledge is made of the projections of a smooth irreducible curve.

As before, let \( X \) be a smooth irreducible curve embedded in \( \mathbb{P}^3 \), which cannot be embedded in a plane. The degree of \( X \) is \( d \geq 3 \). Let \( M_i, i = 1, 2 \), be the projection matrices. Let \( F, e_1 \) and \( e_2 \) be defined as before. We will need to consider the two following mappings: \( p \mapsto e_1 \vee p \) and \( p \mapsto Fp \), where \( \vee \) is the join operator \( \mathfrak{B} \), which is equivalent to the cross-product in that case. Both maps are defined on the first projection plane.
Let $Y_1$ and $Y_2$ be the projected curves. Assume that they are defined by the polynomials $f_1$ and $f_2$. Let $Y_1^*$ and $Y_2^*$ denote the dual curves, whose polynomials are respectively $\phi_1$ and $\phi_2$.

**Theorem 1** For a generic position of the centers of projections with respect to the curve $X$, there exists a non-zero scalar $\lambda$, such that for all points $p$ in the projection plane, the following equality holds:

$$\phi_2(\xi(p)) = \lambda \phi_1(\gamma(p)) \quad (1)$$

For reasons that will be clear later, we shall call this equation, the generalized Kruppa equation.

**Proof:** Let $\epsilon_i$ be the set of epipolar lines tangent to curve in image $i$. We start with the following lemma:

**Lemma 1** The two sets $\epsilon_1$ and $\epsilon_2$ are projectively equivalent. Moreover for each corresponding pair of epipolar lines $(l, l') \in \epsilon_1 \times \epsilon_2$, the multiplicities of $l$ and $l'$ as points of $Y_1^*$ and $Y_2^*$ are the same.

**Proof:** Consider the following three pencils:

- $\sigma(L) \approx \mathbb{P}^1$, the pencil of epipolar planes, that is planes containing the baseline joining the two centers of projection,
- $\sigma(e_1) \approx \mathbb{P}^1$, the pencil of epipolar lines in the first projection plane,
- $\sigma(e_2) \approx \mathbb{P}^1$, the pencil of epipolar lines in the second projection plane.

Thus we have $\epsilon_i \subset \sigma(e_i)$. Moreover if $E$ is the set of planes in $\sigma(L)$ tangent to the curve in space, then there exists a one-to-one mapping between $E$ and each $\epsilon_i$ which leaves the multiplicities unchanged. This completes the proof.

This lemma implies that both sides of equation (1) define the same algebraic set, that is the union of epipolar lines tangent to $Y_1$. Since $\phi_1$ and $\phi_2$, in the generic case, have the same degree (as stated in [1, 4]), each side can be factorized as follows:

$$\phi_1(\gamma(x, y, z)) = \prod_i (a_{1i} x + a_{2i} y + a_{3i} z)^{a_i}$$

$$\phi_2(\xi(x, y, z)) = \prod_i \lambda_i (a_{1i} x + a_{2i} y + a_{3i} z)^{b_i},$$

where $\sum_i a_i = \sum_j b_j = m$. By the previous lemma we also have: $a_i = b_i$ for all $i$.

By eliminating the scalar $\lambda$ from the generalized Kruppa equation (1) we obtain a set of bi-homogeneous equations in $F$ and $e_1$. Hence they define a variety in $\mathbb{P}^2 \times \mathbb{P}^8$. We turn our attention to the dimensional analysis of this variety. Our concern is to exhibit the conditions for which this variety is discrete. From a practical point of view, this is a step toward the recovery of the original curve.
3.3 Dimension analysis

Let \( \{ E_i(F, e_1) \}_i \) be the set of bi-homogeneous equations on \( F \) and \( e_1 \), extracted from the generalized Kruppa equation \( \mathcal{E} \). Our first concern is to determine whether all solutions of equation \( \mathcal{E} \) are admissible, that is whether they satisfy the usual constraint \( Fe_1 = 0 \). Indeed we prove the following statement:

**Proposition 2** As long as there are at least 2 distinct lines through \( e_1 \) tangent to \( Y_1 \), equation \( \mathcal{E} \) implies that \( \text{rank} F = 2 \) and \( Fe_1 = 0 \).

*Proof:* The variety defined by \( \phi_1(\gamma(p)) \) is then a union of at least 2 distinct lines through \( e_1 \). If equation \( \mathcal{E} \) holds, \( \phi_2(\xi(p)) \) must define the same variety.

There are 2 cases to exclude: If \( \text{rank} F = 3 \), then the curve defined by \( \phi_2(\xi(p)) \) is projectively equivalent to the curve defined by \( \phi_2 \), which is \( Y_2^+ \). In particular, it is irreducible.

If \( \text{rank} F < 2 \) or \( \text{rank} F = 2 \) and \( Fe_1 \neq 0 \), then there is some \( a \), not a multiple of \( e_1 \), such that \( Fa = 0 \). Then the variety defined by \( \phi_2(\xi(p)) \) is a union of lines through \( a \). In neither case can this variety contain two distinct lines through \( e_1 \), so we must have \( \text{rank} F = 2 \) and \( Fe_1 = 0 \).

As a result, in a generic situation every solution of \( \{ E_i(F, e_1) \}_i \) is admissible. Let \( V \) be the subvariety of \( \mathbb{P}^2 \times \mathbb{P}^8 \times \mathbb{P}^2 \) defined by the equations \( \{ E_i(F, e_1) \}_i \) together with \( Fe_1 = 0 \) and \( e_2^T F = 0^T \), where \( e_2 \) is the second epipole. We next compute a lower bound on the dimension of \( V \), after which we will be ready for the calculation itself.

**Proposition 3** If \( V \) is non-empty, the dimension of \( V \) is at least \( 7 - m \).

*Proof:* Choose any line \( l \) in \( \mathbb{P}^2 \) and restrict \( e_1 \) to the affine piece \( \mathbb{P}^2 \setminus l \). Let \( (x, y) \) be homogeneous coordinates on \( l \). If \( Fe_1 = 0 \), the two sides of equation \( \mathcal{E} \) are both unchanged by replacing \( p \) by \( p + \alpha e_1 \). So equation \( \mathcal{E} \) will hold for all \( p \) if it holds for all \( p \in l \). Therefore equation \( \mathcal{E} \) is equivalent to the equality of 2 homogeneous polynomials of degree \( m \) in \( x \) and \( y \), which in turn is equivalent to the equality of \((m + 1)\) coefficients. After eliminating \( \lambda \), we have \( m \) algebraic conditions on \((e_1, F, e_2)\) in addition to \( Fe_1 = 0, e_2^T F = 0^T \).

The space of all epipolar geometries, that is, solutions to \( Fe_1 = 0, e_2^T F = 0^T \), is irreducible of dimension 7. Therefore, \( V \) is at least \((7 - m)\)-dimensional.

For the calculation of the dimension of \( V \) we introduce some additional notations. Given a triplet \((e_1, F, e_2) \in \mathbb{P}^2 \times \mathbb{P}^8 \times \mathbb{P}^2 \), let \( \{ q_{1\alpha}(e_1) \} \) (respectively \( \{ q_{2\alpha}(e_2) \} \)) be the tangency points of the epipolar lines through \( e_1 \) (respectively \( e_2 \)) to the first (respectively second) projected curve. Let \( Q_\alpha(e_1, e_2) \) be the 3D points projected onto \( \{ q_{1\alpha}(e_1) \} \) and \( \{ q_{2\alpha}(e_2) \} \). Let \( L \) be the baseline joining the two centers of projections. We next provide a sufficient condition for \( V \) to be discrete.

**Proposition 4** For a generic position of the centers of projection, the variety \( V \) will be discrete if, for any point \((e_1, F, e_2) \in V \), the union of \( L \) and the points \( Q_\alpha(e_1, e_2) \) is not contained in any quadric surface.
Proof: For generic projections, there will be $m$ distinct points $\{q_{1\alpha}(e_1)\}$ and $\{q_{2\alpha}(e_2)\}$, and we can regard $q_{1\alpha}$, $q_{2\alpha}$ locally as smooth functions of $e_1$, $e_2$.

We let $W$ be the affine variety in $\mathbb{C}^3 \times \mathbb{C}^9 \times \mathbb{C}^3$ defined by the same equations as $V$. Let $\Theta = (e_1, F, e_2)$ be a point of $W$ corresponding to a non-isolated point of $V$. Then there is a tangent vector $\theta = (v, \Phi, \nu')$ to $W$ at $\Theta$ with $\Phi$ not a multiple of $F$.

If $\chi$ is a function on $W$, $\nabla_{\theta,\nu}(\chi)$ will denote the derivative of $\chi$ in the direction defined by $\theta$ at $\Theta$. For

$$\chi_{\alpha}(e_1, F, e_2) = q_{2\alpha}(e_2)^TFq_{1\alpha}(e_1),$$

the generalized Kruppa equation implies that $\chi_{\alpha}$ vanishes identically on $W$, so its derivative must also vanish. This yields

$$\nabla_{\theta,\nu}(\chi_{\alpha}) = (\nabla_{\theta,\nu}(q_{2\alpha}))^TFq_{1\alpha}$$

$$+ q_{2\alpha}^T\Phi q_{1\alpha} + q_{2\alpha}^T F(\nabla_{\theta,\nu}(q_{1\alpha})) = 0. \quad (2)$$

We shall prove that $\nabla_{\theta,\nu}(q_{1\alpha})$ is in the linear span of $q_{1\alpha}$ and $e_1$. (This means that when the epipole moves slightly, $q_{1\alpha}$ moves along the epipolar line.) Consider $\kappa(t) = f(q_{1\alpha}(e_1 + tv))$, where $f$ is the polynomial defining the image curve $Y_1$. Since $q_{1\alpha}(e_1 + tv) \in Y_1$, $\kappa \equiv 0$, so the derivative $\kappa'(0) = 0$. On the other hand, $\kappa'(0) = \nabla_{\theta,\nu}(f(q_{1\alpha})) = \text{grad}_{q_{1\alpha}}(f)^T \nabla_{\theta,\nu}(q_{1\alpha}).$

Thus we have $\text{grad}_{q_{1\alpha}}(f)^T \nabla_{\theta,\nu}(q_{1\alpha}) = 0$. But also $\text{grad}_{q_{1\alpha}}(f)^T q_{1\alpha} = 0$ and $\text{grad}_{q_{1\alpha}}(f)^T e_1 = 0$. Since $\text{grad}_{q_{1\alpha}}(f) \neq 0$ ($q_{1\alpha}$ is not a singular point of the curve), this shows that $\nabla_{\theta,\nu}(q_{1\alpha})$, $q_{1\alpha}$, and $e_1$ are linearly dependent. $q_{1\alpha}$ and $e_1$ are linearly independent, so $\nabla_{\theta,\nu}(q_{1\alpha})$ must be in their linear span.

We have that $q_{2\alpha}^T F e_1 = q_{2\alpha}^T F q_{1\alpha} = 0$, so $q_{2\alpha}^T F \nabla_{\theta,\nu}(q_{1\alpha}) = 0$: the third term of equation (2) vanishes. In a similar way, the first term of equation (2) vanishes, leaving

$$q_{2\alpha}^T \Phi q_{1\alpha} = 0. \quad (3)$$

The derivative of $\chi(e_1, F, e_2) = Fe_1$ must also vanish, which yields

$$e_2^T \Phi e_1 = 0. \quad (4)$$

From equality (4), we deduce that for every $Q_{\alpha}$, we have

$$Q_{\alpha}^T M_2^T \Phi M_1 Q_{\alpha} = 0.$$

From equality (4), we deduce that every point $P$ lying on the baseline must satisfy

$$P^T M_2^T \Phi M_1 P = 0.$$
Observe that this result is consistent with the previous proposition, since there always exist a quadric surface containing a given line and six given points. However in general there is no quadric containing a given line and seven given points. Therefore we can conclude with the following theorem.

**Theorem 2** For a generic position of the centers of projection, the generalized Kruppa equation defines the epipolar geometry up to a finite-fold ambiguity if and only if \( m \geq 7 \).

Since different curves in generic position give rise to independent equations, this result means that the sum of the classes of the projected curves must be at least 7 for \( V \) to be a finite set.

### 3.4 Recovering the curve

Let the projection matrices be \( M_1 \) and \( M_2 \). Hence the two cones defined by the projected curves and the centers of projections are given by: \( \Delta_1(P) = f_1(M_1P) \) and \( \Delta_2(P) = f_2(M_2P) \). The reconstruction is defined as the curve whose equations are \( \Delta_1 = 0 \) and \( \Delta_2 = 0 \). This curve has two irreducible components as the following theorem states.

**Theorem 3** For a generic position of the centers of projection, namely when no epipolar plane is tangent twice to the curve \( X \), the curve defined by \( \{ \Delta_1 = 0, \Delta_2 = 0 \} \) has two irreducible components. One has degree \( d \) and is the actual solution of the reconstruction. The other one has degree \( d(d-1) \).

**Proof:** For a line \( L \subset \mathbb{P}^3 \), we write \( \sigma(L) \) for the pencil of planes containing \( L \). For a point \( p \in \mathbb{P}^2 \), we write \( \sigma(p) \) for the pencil of lines through \( p \). There is a natural isomorphism between \( \sigma(e_i) \), the epipolar lines in image \( i \), and \( \sigma(L) \), the planes containing both centers of projections. Consider the following covers of \( \mathbb{P}^1 \):

1. \( X \xrightarrow{\eta_1} \sigma(L) \cong \mathbb{P}^1 \), taking a point \( x \in X \) to the epipolar plane that it defines with the centers of projection.

2. \( Y_1 \xrightarrow{\eta_2} \sigma(e_1) \cong \sigma(L) \cong \mathbb{P}^1 \), taking a point \( y \in Y_1 \) to its epipolar line in the first projection plane.

3. \( Y_2 \xrightarrow{\eta_3} \sigma(e_2) \cong \sigma(L) \cong \mathbb{P}^1 \), taking a point \( y \in Y_2 \) to its epipolar line in the second projection plane.

If \( \rho_i \) is the projection \( X \to Y_i \), then \( \eta = \eta_i \rho_i \). Let \( \mathcal{B} \) the union set of branch points of \( \eta_1 \) and \( \eta_2 \). It is clear that the branch points of \( \eta \) are included in \( \mathcal{B} \). Let \( S = \mathbb{P}^1 \setminus \mathcal{B} \), pick \( t \in S \), and write \( X_S = \eta^{-1}(S) \), \( X_t = \eta^{-1}(t) \). Let \( \mu_{X_S} \) be the monodromy: \( \pi_1(S, t) \to \text{Perm}(X_t) \), where \( \text{Perm}(Z) \) is the group of permutation of a finite set \( Z \), see [8]. It is well known that the path-connected components of \( X \) are in one-to-one correspondence with the orbits of the action of \( \text{im}(\mu_{X_S}) \) on \( X_t \). Since \( X \) is assumed to be irreducible, it has
only one component and im(µ_X) acts transitively on X_t. Then if im(µ_X) is
generated by transpositions, this will imply that im(µ_X) = Perm(X_t). In order
to show that im(µ_X) is actually generated by transpositions, consider a loop
in P^1 based at t, say l_t. If l_t does not go round any branch point, then l_t is
homotopic to the constant path in S and then µ_X([l_t]) = 1. Now in B, there
are three types of branch points:

1. branch points that come from nodes of Y_1: these are not branch points of η,
2. branch points that come from nodes of Y_2: these are not branch points of η,
3. branch points that come from epipolar lines tangent either to Y_1 or to Y_2:
   these are genuine branch points of η.

If the loop l_t goes round a point of the first two types, then it is still true
that µ_X([l_t]) = 1. Now suppose that l_t goes round a genuine branch point of
η, say b (and goes round no other points in B). By genericity, b is a simple two-fold
branch point, hence µ_X([l_t]) is a transposition. This shows that im(µ_X) is
actually generated by transpositions and so im(µ_X) = Perm(X_t).

Now consider ˜X, the curve defined by \{ ∆_1 = 0, ∆_2 = 0 \}. By Bezout’s
Theorem ˜X has degree d^2. Let ˜x ∈ ˜X. It is projected onto a point y_i in Y_i, such
that η_1(y_1) = η_2(y_2). Hence ˜X ≅ Y_1 ×_Y_2; restricting to the inverse image of
the set S, we have ˜X_S ≅ X_S × S X_S. We can therefore identify ˜X_t with X_t × X_t.
The monodromy µ_˜X can then be given by µ_˜X(x, y) = (µ_X(x), µ_X(y)). Since
im(µ_X) = Perm(X_t), the action of im(µ_X) on X_t × X_t has two orbits, namely
\{(x, x)\} ≅ X_t and \{(x, y)|x ≠ y\}. Hence ˜X has two irreducible components.
One has degree d and is X_t, the other has degree d^2 − d = d(d − 1).

This result provides a way to find the right solution for the recovery in a
generic configuration, except in the case of conics, where the two components
of the reconstruction are both admissible.

4 The N >> 1 projections problem with known
projection operators

Now we turn our attention to the second problem. N >> 1 projections maps
{π_i}_{i=1,...,N} given by N matrices \{M_i\}_{i=1,...,N} are known. Therefore N projec-
tions of an irreducible smooth algebraic curve are also provided. The problem
is to recover the original curve by linear computations as much as possible.

4.1 Curve presentation in the dual space

Let X^* be the dual variety of X. Since X is supposed not to be a line, the
dual variety X^* must be a hypersurface of the dual space [11]. Hence let Υ
be a minimal degree polynomial that represents X^*. Our first concern is to
determine the degree of Υ.
Proposition 5 The degree of \( \mathcal{Y} \) is \( m \), that is, the common degree of the dual projected curves.

Proof: Since \( X^* \) is a hypersurface of \( \mathbb{P}^3 \), its degree is the number of points where a generic line in \( \mathbb{P}^3 \) meets \( X^* \). By duality it is the number of planes in a generic pencil that are tangent to \( X \). Hence it is the degree of the dual projected curve. Another way to express the same fact is the observation that the dual projected curve is the intersection of \( X^* \) with a generic plane in \( \mathbb{P}^3 \).

Note that this provides a new proof that the degree of the dual projected curve is constant for a generic position of the center of projection.

For the recovery of \( X^* \) from multiple projection, we will need to consider the mapping from a line \( l \) of the projection plane to the plane that it defines with the center of projection. Let \( \mu : l \mapsto \hat{M}^l \) denote this mapping. There exists a link involving \( \mathcal{Y} \), \( \mu \) and \( \phi \), the polynomial of the dual projected curve: 
\[
\mathcal{Y}(\mu(l)) = 0 \quad \text{whenever} \quad \phi(l) = 0.
\]
Since these two polynomials have the same degree (because \( \mu \) is linear) and \( \phi \) is irreducible, there exists a scalar \( \lambda \) such that 
\[
\mathcal{Y}(\mu(l)) = \lambda \phi(l)
\]
for all lines \( l \in \mathbb{P}^2 \). Eliminating \( \lambda \), we get \( \binom{m+2}{m} - 1 \) linear equations on \( \mathcal{Y} \).

Since the number of coefficients in \( \mathcal{Y} \) is \( \binom{m+3}{m} \), we can state the following result:

Proposition 6 The recovery in the dual space can be done linearly using at least 
\[
k \geq m^2 + 6m + 11 \quad \text{projections.}
\]

4.2 Curve presentation in the Grassmannian of lines of \( \mathbb{P}^3 \)

Let \( G(1, 3) \) be the Grassmanian of lines of \( \mathbb{P}^3 \). Consider the set of lines in \( \mathbb{P}^3 \) intersecting the curve \( X \) of degree \( d \). This defines a subvariety of \( G(1, 3) \) which is the intersection of \( G(1, 3) \) with a hypersurface of degree \( d \) in \( \mathbb{P}^5 \), given by a homogeneous polynomial \( \Gamma \), defined modulo the \( d \)th graded piece \( I(G(1, 3))_d \) of the ideal of \( G(1, 3) \) and modulo scalars. However picking one representative of this equivalence class is sufficient to recover entirely without any ambiguity the curve \( X \). In our context, we shall call any representative of this class the Chow polynomial of the curve. We need to compute the class of \( \Gamma \) in the homogeneous coordinate ring of \( G(1, 3) \), or more precisely in its \( d \)th graded piece, \( S(G(1, 3))_d \), which dimension is \( Nd = \binom{d+5}{d+2} - \binom{d-2+5}{d-2} \).

Let \( f \) be the polynomial defining the projected curve, \( Y \). Consider the mapping that associates to a point in the projection plane the line it generates with the center of projection: \( \nu : p \mapsto \hat{M}p \). The polynomial \( \Gamma(\nu(p)) \) vanishes whenever \( f(p) \) does. Since they have same degree and \( f \) is irreducible, there exists a scalar \( \lambda \) such that for every point \( p \in \mathbb{P}^2 \), we have:
\[
\Gamma(\nu(p)) = \lambda f(p).
\]
This yields \( \binom{d+2}{d} - 1 \) linear equations on \( \Gamma \).

Hence a similar statement to that in Proposition 6 can be made:
Proposition 7 The recovery in $\mathbb{G}(1,3)$ can be done linearly using at least $k \geq \frac{1}{6}d^2 + 5d^2 + 8d + 4$ projections.

4.3 Family of projections operators and finite closed subset of points

Consider now a finite collection of points $P_i$ in $\mathbb{P}^3$. Each point is projected by a different projection map. The $N$ maps are known and so the projected points.

Let $X$ be the smooth irreducible curve generated by the points $P_i$ and $Y$ be the smooth irreducible curve, of minimal degree, generated by the center of projections.

Each projected point $p_i$ yields one linear equation on the variety of intersecting lines of $X$, namely $\Gamma(\pi^{-1}(p_i)) = 0$, where $\Gamma$ is the Chow polynomial of $X$ as before.

Let $d$ and $d'$ be respectively the degree of $X$ and $Y$. We compute the number of constraints obtained on $\Gamma$ from the projected points as a function of $d$ and $d'$.

Proposition 8 The maximal number of constraints is $N_d - (h^0(\mathcal{O}_{\mathbb{P}^5}(d - d')) - h^0(\mathcal{O}_{\mathbb{P}^5}(d - d' - 2)) + 1)$, where $N_d = \text{dim}(S(G(1,3))_d)$ is the dimension of the $d$-th graded piece of homogeneous coordinate ring of $G(1,3)$.

Proof: Each projected point generates a line with the center of projection. Let $L_1, ..., L_n$ be these $n$ lines joining $X$ and $Y$. Let $\Gamma_X$ and $\Gamma_Y$ be the Chow polynomial of $X$ and $Y$ respectively. We shall denote by $Z(\Gamma_X)$ and $Z(\Gamma_Y)$ the sets where they vanish. Let $V = Z(\Gamma_X) \cap Z(\Gamma_Y) \cap \mathbb{G}(1,3)$. For $n \gg 1$, we have

$$\{\Gamma \in H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(d)) : \Gamma(L_i) = 0, i = 1, \ldots, n\} =$$

$$\{\Gamma \in H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(d)) : \Gamma_V \equiv 0\} = I_{V, \mathbb{P}^5}(d).$$

So, we want to compute $\text{dim}(I_{V, \mathbb{P}^5}(d))$, or equivalently, $h^0(V, \mathcal{O}_V(d)) = h^0(\mathcal{O}_{\mathbb{P}^5}(d)) - \text{dim}(I_{V, \mathbb{P}^5}(d))$. Since $V$ is a complete intersection of degree $(d, d', 2)$ in $\mathbb{P}^5$, the dimension of $I_{V, \mathbb{P}^5}(d)$ should be equal to

$$h^0(\mathcal{O}_{\mathbb{P}^5}(d - 2)) + h^0(\mathcal{O}_{\mathbb{P}^5}(d - d')) - h^0(\mathcal{O}_{\mathbb{P}^5}(d - d' - 2)) + 1.$$

As a consequence

$$h^0(V, \mathcal{O}_V(d)) = N_d - (h^0(\mathcal{O}_{\mathbb{P}^5}(d - d')) - h^0(\mathcal{O}_{\mathbb{P}^5}(d - d' - 2)) + 1).$$

\[\]
5 Applications to static and dynamic computational vision

The results obtained above were motivated by some applications to computational vision. We now proceed to show how these results can be applied to this field. We start by a quick survey on linear computational vision. More details can be found in [5, 12, 6]. Some of the terminology was introduced before in section 2.

5.1 Foundations of linear computational vision

Projective algebraic geometry provides a natural framework to geometric computer vision. However one has to keep in mind that the geometric entities to be considered are in fact embedded in the physical three-dimensional Euclidean space. Euclidean space is provided with three structures defined by three groups of transformations: the orthogonal group $\text{Euc}_3$ (which defines the Euclidean structure and which is included into the affine group), $\text{Aff}_3$ (defining the affine structure and itself included into the projective group), and $\text{Pr}_3$ (defining the projective structure). We fix $[X, Y, Z, T]^T$, as homogeneous coordinates, and $T = 0$ as the plane at infinity.

5.2 A single camera system

Computational vision starts with images captured by cameras. The camera components are the following:

- a plane $\mathcal{R}$, called the retinal plane or image plane;
- a point $O$, called the optical centre or camera centre, which does not lie on the plane $\mathcal{R}$.

The plane $\mathcal{R}$ is regarded as a two dimension projective space embedded into $\mathbb{P}^3$. Hence it is also denoted by $i(\mathbb{P}^2)$. The camera is a projection machine: $\pi : \mathbb{P}^3 \setminus \{O\} \to i(\mathbb{P}^2), P \mapsto \overline{OP} \cap i(\mathbb{P}^2)$. The projection $\pi$ is determined (up to a scalar) by a $3 \times 4$ matrix $M$ (the image of $P$ being $\lambda P$).

The physical properties of a camera imply that $M$ can be decomposed as follows:

$$M = \begin{bmatrix} f & s & u_0 \\ 0 & 0 & 1 \end{bmatrix} [R; t],$$

where $(f, \alpha, s, u_0, v_0)$ are the so-called internal parameters of the camera, whereas the rotation $R$ and the translation $t$ are the external parameters.

It is easy to see that:

- The camera centre $O$ is given by $MO = 0$.
- The matrix $M^T$ maps a line in $i(\mathbb{P}^2)$ to the only plane containing both the line and $O$. 

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• There exists a matrix $\hat{M} \in \mathcal{M}_{6 \times 3}(\mathbb{R})$, which is a polynomial function of $M$, that maps a point $p \in \mathbb{P}^2$ to the line $\overline{Op}$ (optical ray), represented by its Plücker coordinates in $\mathbb{P}^5$. If the camera matrix is decomposed as follows:

$$M = \begin{bmatrix}
\Gamma^T \\
\Lambda^T \\
\Theta^T
\end{bmatrix},$$

then for $p = [x, y, z]^T$, the optical ray $L_p = \hat{M}p$ is given by the extensor:

$$L_p = x\Lambda \wedge \Theta + y\Theta \wedge \Gamma + z\Gamma \wedge \Lambda,$$

where $\wedge$ denotes the meet operator in the Grassman-Cayley algebra (see [2]).

• The matrix $\tilde{M} = \hat{M}^T$ maps lines in $\mathbb{P}^3$ to lines in $i(\mathbb{P}^2)$.

Moreover we will need in the sequel to consider the projection of the absolute conic onto the image plane. The absolute conic is simply defined by the following equations:

$$\begin{align*}
X^2 + Y^2 + Z^2 &= 0 \\
T &= 0
\end{align*}$$

By definition, the absolute conic is left invariant by Euclidean transformations. Therefore its projection onto the image plane, defined by the matrix $\omega$, is a function of the internal parameters only. By Cholesky decomposition $\omega = LU$, where $L$ (respectively $U$) is lower (respectively upper) triangular matrix. Hence it is easy to see that $U = M^{-1}$, where $M$ is the $3 \times 3$ matrix of the internal parameters of $M$.

### 5.3 A system of two cameras

Given two cameras, $(O_j, i_j(\mathbb{P}^2))_{j=1,2}$ are their components where $i_1(\mathbb{P}^2)$ and $i_2(\mathbb{P}^2)$ are two generic projective planes embedded into $\mathbb{P}^3$, and $O_1$ and $O_2$ are two generic points in $\mathbb{P}^3$ not lying on the above planes. As in 5.2, let $\pi_j : \mathbb{P}^3 \setminus \{O_j\} \to i_j(\mathbb{P}^2), P \mapsto \overline{O_jP} \cap i_j(\mathbb{P}^2)$ be the respective projections. The camera matrices are $M_i, i = 1, 2$.

#### 5.3.1 Homography between two images of the same plane

Consider the case where the two cameras are looking at the same plane in space, denoted by $\Delta$. Let

$$M_i = \begin{bmatrix}
\Gamma_i^T \\
\Lambda_i^T \\
\Theta_i^T
\end{bmatrix},$$

be the camera matrices, decomposed as above. Let $P$ be a point lying on $\Delta$. We shall denote the projections of $P$ by $p_i = [x_i, y_i, z_i]^T \cong M_iP$, where $\cong$ means equality modulo multiplication by a non-zero scalar.
The optical ray generated by $p_1$ is given by $L_{p_1} = x_1\Lambda_1 \times \Theta_1 + y_1\Theta_1 + \Gamma_1 + z_1\Gamma_1 \times \Lambda_1$. Hence $P = L_{p_1} \times \Delta = x_1\Lambda_1 \times \Theta_1 + y_1\Theta_1 + \Gamma_1 + \Delta + z_1\Gamma_1 \times \Lambda_1 \times \Delta$. Hence $p_2 \cong M_2P$ is given by the following expression: $p_2 \cong H_{\Delta}p_1$ where:

$$H_{\Delta} = \begin{bmatrix}
\Gamma^T_1(\Lambda_1 \times \Theta_1) & \Gamma^T_2(\Theta_1 \times \Gamma_1) & \Gamma^T_2(\Gamma_1 \times \Lambda_1) \\
\Lambda^T_2(\Lambda_1 \times \Theta_1) & \Lambda^T_2(\Theta_1 \times \Gamma_1) & \Lambda^T_2(\Gamma_1 \times \Lambda_1) \\
\Theta^T_2(\Lambda_1 \times \Theta_1) & \Theta^T_2(\Theta_1 \times \Gamma_1) & \Theta^T_2(\Gamma_1 \times \Lambda_1)
\end{bmatrix}.$$ 

This yields the expression of the collineation $H_{\Delta}$ between two images of the same plane.

**Definition 1** The previous collineation is called the homography between the two images, through the plane $\Delta$.

### 5.3.2 Epipolar geometry

**Definition 2** Let $(O_j, i_j(P^2), M_j)_{j=1,2}$ be defined as before. Given a pair $(p_1, p_2) \in i_1(P^1) \times i_2(P^2)$, we say that it is a pair of corresponding or matching points if there exists $P \in P^3$ such that $p_j = \pi_j(P)$ for $j = 1, 2$.

Consider a point $p \in i_2(P^2)$. Then $p$ can be the image of any point lying on the fiber $\pi_1^{-1}(p)$. The matching point in the second image must lie on $\pi_2(\pi_1^{-1}(p))$, which is, for a generic point $p$, a line on the second image. Since $\pi_1$ and $\pi_2$ are both linear, there exists a matrix $F \in M_{3 \times 3}(\mathbb{R})$, such that: $\xi(p) = \pi_2(\pi_1^{-1}(p)) = Fp$ for all but one point in the first image.

**Definition 3** The matrix $F$ is called the fundamental matrix, where as the line $l_p = Fp$ is called the epipolar line of $p$.

Let $e_1 = O_1O_2 \cap i_1(P^2)$ and $e_2 = O_1O_2 \cap i_2(P^2)$. Those two points are respectively called the first and the second epipole. It is easy to see that $Fe_1 = 0$, since $\pi_1^{-1}(e_1) = O_1O_2$ and $\pi_2(O_1O_2) = e_2$. Observe that by symmetry $F^T$ is the fundamental matrix of the reverse couple of images. Hence $F^T e_2 = 0$. Since the only point in the first image that is mapped to zero by $F$ is the first epipole, $F$ has rank 2.

Now we want to deduce an expression of $F$ as a function of the camera matrices. By the previous analysis, it is clear that $F = \hat{M}_2\hat{M}_1$. Moreover we have the following properties:

**Proposition 9** For any plane $\Delta$, not passing through the camera centres, the following equalities hold:

1. $F \cong [e_2] \times H_{\Delta}$,

where $[e_2]_x$ is the matrix associated with the cross-product as follows: for any vector $p$, $e_2 \times p = [e_2]_x p$. Hence we have:

$$[e_2]_x = \begin{bmatrix}
0 & -e_{23} & e_{22} \\
e_{23} & 0 & -e_{21} \\
-e_{22} & e_{21} & 0
\end{bmatrix}.$$
In particular, we have: $F = [e_2]^\times H_\infty$, where $H_\infty$ is the homography between the two images through the plane at infinity.

2.

$$H_\Delta^T F + F^T H_\Delta = 0. \quad (5)$$

Proof: The first equality is clear according to its geometric meaning. Given a point $p$ in the first image, $Fp$ is its epipolar line in the second image. The optical ray $L_p$ passing through $p$ meets the plane $\Delta$ in a point $Q$, which projection in the second image is $H_\Delta p$. Hence the epipolar line must be $e_2 \lor H_\Delta p$. This gives the required equality. The second equality is simply deduced from the first one by a short calculation.

**Proposition 10** For a generic plane $\Delta$, the following equality hold:

$$H_\Delta e_1 \cong e_2.$$ 

Proof: The image of $e_1$ by the homography must be the projection on the second image of the point defined as being the intersection of the optical ray generated by $e_1$ and the plane $\Delta$. Hence $H_\Delta e_1 = M_2 (L_{e_1} \land \Delta)$. But $L_{e_1} = O_1 O_2$. Thus the result must be $M_2 O_1$ (except when the plane is passing through $O_2$) that is the second epipole $e_2$.

### 5.3.3 Canonical stratification of the reconstruction

Three-dimensional reconstruction can be achieved from a system of two cameras, once the camera matrices are known. However, a typical situation is that the camera matrices are unknown. Then we face a double problem: recovering the camera matrices and the actual object. There exists an inherent ambiguity. Consider a pair of camera matrices $(M_1, M_2)$. If you change the world coordinate system by a transformation $V \in P_{r3}$, the camera matrices are mapped to $(M_1 V^{-1}, M_2 V^{-1})$. Therefore, we define the following equivalence relation:

**Definition 4** Given a group of transformations $G$, two pairs of camera matrices, say $(M_1, M_2)$ and $(N_1, N_2)$, are said to be equivalent modulo $G$ if there exists $V \in G$ such that $M_1 = VN_1$ and $M_2 = VN_2$.

Any reconstruction algorithm will always yield a reconstruction modulo a certain group of transformations. More precisely, there exist three levels of reconstruction according to the information that can be extracted from the two images and from a priori knowledge of the world.

**Projective stratum.**

When the only available information is the fundamental matrix, then the reconstruction is done modulo $P_{r3}$. Indeed, from $F$, the so-called intrinsic homography $S = -\frac{e_2}{\|e_2\|} F$ is computed and the camera matrices are equivalent to
Affine stratum.
When, in addition to the epipolar geometry, the homography between the two images through the plane at infinity, denoted by \( H_\infty \), can be computed, the reconstruction can be done modulo the group of affine transformations. Then the two camera matrices are equivalent \( ([I:0], [H_\infty; e_2]) \).

Euclidean stratum.
The Euclidean stratum is obtained by the data of the projection of the absolute conic \( \Omega \) onto the image planes, which allows the recovery the internal parameters of the cameras. Once the internal parameters of the cameras are known, the relative motion between the cameras expressed by a rotation \( R \) and a translation \( t \) can be extracted from the fundamental matrix. However only the direction of \( t \), not the norm, can be recovered. Then the cameras matrices are equivalent, modulo the group of similarity transformations, to \( ([M_1[I:0], M_2[R; t]]) \), where \( M_1 \) and \( M_2 \) are the matrices of internal parameters.

Note that the projection of the absolute conic on the image can be computed using some a priori knowledge of the world. Moreover there exist famous equations linking \( \omega_1 \) and \( \omega_2 \), the two matrices defining the projection of the absolute conic onto the images, when the epipolar geometry is given. This is the so-called Kruppa equation, defined in the following proposition.

**Proposition 11** The projections of the absolute conic onto two images are related as follows. There exists a scalar \( \lambda \) such that

\[
[e_1]^T \omega_1^* [e_1] = \lambda F^T \omega_2^* F,
\]

where \( [e_1]^\times \) is the matrix representing the cross-product by \( e_1 \) and \( \omega_1^* \) is the adjoint matrix of \( \omega_1 \).

Let \( \epsilon_i \) be the tangents to \( \pi_i(\Omega) \) through \( e_i \). Kruppa equation simply states that \( \epsilon_1 \) and \( \epsilon_2 \) are projectively isomorphic.

5.4 Applications of the previous results to computational vision
The previous results (sections 3 and 4) can be applied to computational vision in different contexts:

1. The recovery of the epipolar geometry from two images of the same smooth irreducible curve. Theorem 4 generalizes Kruppa equation to algebraic curves. Section 3.3 provides with a necessary and sufficient conditions on the degree of the curve for the epipolar geometry to be defined up to finite fold ambiguity. Note that the case of conic sections was first introduced in [14, 15].
2. The 3D reconstruction of a curve from two images is possible in a generic situation as shown in theorem 3. The case of conic was also treated in [15, 19, 20]. Note that [9] presents an algorithm for curve reconstruction using a blow-up of the projected curve. This nice result, however, does not provide any information about the relative position of the curve in \( \mathbb{P}^3 \) with respect to other elements of the scene. On the other hand, our approach based on two images allows reconstructing the curve in the context of the whole scene. Furthermore the problem of curve reconstruction was also considered in [8] from the point of view of global optimization and bundle adjustment. Our approach, on the contrary, is based on looking at algebraic curves for which the representation is more compact.

3. The 3D reconstruction of a curve from \( N >> 1 \) projection is linear using the dual space or the Grassmannian of lines \( G(1, 3) \), sections 4.1 and 4.2. The formalism of the dual space in the case of conic or quadric was also used in [10, 16].

4. The trajectory recovery of a moving point viewed by a moving camera whose matrix is known over time is a linear problem when using the variety of intersecting lines of the curve generated by the motion of the point. Moreover this gives rise to the question of counting the number of constraints that can be obtained. This is done in theorem 8. Note that our algorithm for trajectory recovery or triangulation is a complete generalization of [1].

5.5 Experiments and discussion

Now we are in a position to give some experiments of the different applications mentioned above. The algorithms induced by our theoretical analysis involve either solving polynomial systems built from noisy data or estimating high-dimensional parameters that appear linearly in equations built also from noisy data.

The first of these problems is still under very active research. In our best knowledge, one of the most powerful solver is FastGb, introduced by Jean-Charles Faugere [7, 8]. Our experiments involving solving polynomial systems have been conducted using FastGb. However at this stage, we will show only synthetic experiments. Future research will be devoted to the case of real data. The second question is a typical case of heteroscedastic estimation [17] and will be discussed below.

Recovering Epipolar Geometry from a rational cubic and two conics

We proceed to a synthetic experiment, where the epipolar geometry is computed from a rational cubic and two conics. The curves are randomly chosen, as well as the camera.
Hence the cubic is defined by the following system:

\[
22656665956452626Z^2X – 1914854993236086169Z^2T – 791130248041963297YZ – 1198609868087500022Z^2 + 893468169675527814XT + 285940501848919422T^2 – 179632615056970090YT + 277960038226472656Y^2 = 0 \\
555920076452945312XZ – 656494420457765414ZX – 1755155973545148735Y^2 + 984240461094724954XT – 613095686279510YT – 1802588912007356295Z^2 + 291319745776795474T^2 = 0 \\
1111840152905890624XZ – 1749154450800074954Z^2 + 984240461094724954XT – 613095686279510YT – 2942349361064284313Z^2 + 398814386951585134T^2 = 0
\]

The first and the second conic are respectively defined by:

\[
25X + 9Y + 40Z + 61T = 0 \\
40X^2 – 78XY + 62ZX + 11XT + 88Y^2 + YZ + 30YT + 81Z^2 – 5ZT – 28T^2
\]

and

\[
4X – 11Y + 10Z + 57T = 0 \\
-82X^2 – 48XY – 11ZX + 38XT – 7Y^2 + 58YZ – 94YT – 68Z^2 + 14ZT – 35T^2 = 0
\]

The camera matrices are given by:

\[
M_1 = \begin{bmatrix}
-87 & 79 & 43 & -66 \\
-53 & -61 & -23 & -37 \\
31 & -34 & -42 & 88
\end{bmatrix} \\
M_2 = \begin{bmatrix}
-76 & -65 & 25 & 28 \\
-61 & -60 & 9 & 29 \\
-66 & -32 & 78 & 39
\end{bmatrix}
\]

Then we form the Extended Kruppa’s Equations for each curve. From a computational point of view, it is crucial to enforce the constraint that each \( \lambda \) is different from zero. Mathematically this means that the computation is done in the localization with respect to each \( \lambda \).

As expected, we get a zero-dimension variety which degree is one. Thus there is a single solution to the epipolar geometry given by the following fundamental matrix:

\[
F = \begin{bmatrix}
-511443 & 13426 & -2669337 & 13426 \\
26246 & 671384845 & 114121 & 14061396 \\
3426650 & 228242 & 114121 & 8707255 \\
1905169 & 228242 & 114121 & 1691295 \\
\end{bmatrix}
\]

**Reconstruction of a spatial quartic in \( \mathbb{P}^3 \)**

Consider the curve \( X \), drawn in figure 1, defined by the following equations:

\[
F_1(x, y, z, t) = x^2 + y^2 – t^2 \\
F_2(x, y, z, t) = xt – (z – 10t)^2
\]

The curve \( X \) is smooth and irreducible, and has degree 4 and genus 1. We define two camera matrices:

\[
M_1 = \begin{bmatrix}
1 & 0 & 0 & 5 \\
0 & 0 & 1 & -2 \\
0 & -1 & 0 & -10
\end{bmatrix} \\
M_2 = \begin{bmatrix}
1 & 0 & 0 & -10 \\
0 & 0 & -1 & 0 \\
0 & 1 & 0 & -10
\end{bmatrix}
\]
Then the curve is reconstructed from the two projections. As expected there are two irreducible components. One has degree 4 and is the original curve, while the second has degree 12.

**Reconstruction using the Grassmannian**

For the next experiment, we consider six images of an electric wire — one of the views is shown in figure 2 and the image curve after segmentation and thinning is shown in figure 3. Hence for each of the images, we extracted a set of points lying on the thread. No fitting is performed in the image space. For each image, the camera matrix is calculated using the calibration pattern. Then we proceeded to compute the Chow polynomial $\Gamma$ of the curve in space. The curve $X$ has degree 3. Once $\Gamma$ is computed, a reprojection is easily performed, as shown in figure 4.

The computation of the Chow Polynomial involves an estimation problem. Moreover as mentioned above, the Chow polynomial is not uniquely defined. In order to get a unique solution, we have to add some constraints to the estimation problem which do not distort the geometric meaning of the Chow polynomial. This is simply done by imposing to the Chow polynomial to vanish over $W_d$ additional arbitrary points of $\mathbb{P}^5$ which do not lie on $G(1, 3)$. The number of additional points necessary to get a unique solution is $W_d = \binom{d+5}{d} - N_d$, where $d$ is the degree of the Chow polynomial.

As we shall the estimation of the Chow polynomial is a typical case of heteroscedastic estimation. Every 2D measurement $p$ is corrupted by additive noise, which we consider as an isotropic Gaussian noise $\mathcal{N}(0, \sigma)$. The variance is estimated to be about 2 pixels.

For each 2D point $p$, we form the optical ray it generates $L = \hat{M}p$. Then the estimation of the Chow polynomial is made using the optical rays $L$. In order to avoid the problem of scale, the Plücker coordinates of each line are normalized such that the last coordinate is equal to one. Hence the lines are represented by vectors in a five-dimensional affine space, denoted by $L_a$. Hence
if $\theta$ is a vector containing the coefficient of the Chow polynomial $\Gamma$, $\theta$ is solution of the following problem:

$$Z(L_u)^T \theta = 0,$$

for all optical rays, with $\|\theta\| = 1$ and $Z(L_u)$ is a vector which coordinates are monomials generated by the coordinates of $L_u$. Following [4, 17], in order to obtain a reliable estimate, the solution $\theta$ is computed using a maximum likelihood estimator. This allows to take into account the fact that each $Z(L_u)$ has a different covariance matrix, or in other terms that the noise is heteroscedastic. More precisely, each $Z(L_u)$ has the following covariance matrix:

$$C_L = J_\phi J_n \hat{M} \begin{bmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & 0 \end{bmatrix} \hat{M}^T J_n^T J_\phi^T,$$

where $M$ is the camera matrix and $J_n$ and $J_\phi$ are respectively the Jacobian matrices of the normalization of $L$ and of the map sending $L_u$ to $Z(L_u)$. That is for $L(t) = [L_1, L_2, L_3, L_4, L_5, L_6]^T$, we have:

$$J_n = \begin{bmatrix} \frac{1}{L_6} & 0 & 0 & 0 & 0 & -\frac{L_2}{L_6} \\ 0 & \frac{1}{L_6} & 0 & 0 & -\frac{L_3}{L_6} & \frac{L_2}{L_6} \\ 0 & 0 & \frac{1}{L_6} & 0 & -\frac{L_4}{L_6} & \frac{L_3}{L_6} \\ 0 & 0 & 0 & \frac{1}{L_6} & -\frac{L_5}{L_6} & \frac{L_4}{L_6} \end{bmatrix},$$

and $J_\phi$ is similarly computed. Then we use the method presented in [4] to perform the estimation. It is worth to note that the estimation is reliable because the initial guess of the algorithm was well chosen and because the number of measurements is very large. It is necessary to use a very large number of measurements for two reasons. First the dimension of the parameter space is quite high and secondly the measurements are concentrated on a part of the space (over the Grassmannian $G(1,3)$).

**Synthetic Trajectory Triangulation**

Let $P \in \mathbb{P}^3$ be a point moving on a cubic, as follows:

$$P(t) = \begin{bmatrix} t^3 \\ 2t^3 + 3t^2 \\ t^3 + t^2 + t + 1 \\ t^3 + t^2 + t + 2 \end{bmatrix}$$

It is viewed by a moving camera. At each time instant a picture is made, we get a 2D point $p(t) = [x(t), y(t)]^T = \frac{m_1^T(t)P(t)}{m_1^T(t)P(t)} \frac{m_2^T(t)P(t)}{m_2^T(t)P(t)} \frac{m_3^T(t)P(t)}{m_3^T(t)P(t)}$, where $M^T(t) = [m_1(t), m_2(t), m_3(t)]^T$ is the transpose of the camera matrix at time $t$. 

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Figure 2: One the six views of an electric thread that were used to perform the reconstruction.

Figure 3: An electric thread after segmentation and thinning.
Then we build the set of optical rays generated by the sequence. The Chow polynomial is then computed and given below:

\[
\begin{align*}
\Gamma(L_1, \ldots, L_6) &= -72L_2^3L_3 + L_1^3 - 5L_1L_4L_5 - \\
& 18L_1L_3L_6 + 57L_2L_3L_5 + 48L_2L_4L_5 - 43L_1L_2L_4 - \\
& 10L_1L_3L_5 + 21L_1L_5L_6 - 30L_1L_4L_6 - 108L_2L_3L_6 + \\
& 41L_1L_2L_5 + 69L_1L_2L_6 - 26L_1L_2L_3 - 36L_2L_4^2 - \\
& 21L_2L_3^2 + 3L_3L_5^2 - 9L_3L_5 - 12L_4^2L_5 + 6L_4L_5^2 + \\
& 4L_4^2L_6 + 20L_4^2 - 13L_3^2 + 8L_3^2 - L_3^2 + 108L_2^2L_6 - \\
& 120L_2^2L_5 + 27L_2^2L_6 - 25L_2^2L_6 + 57L_2L_3^2 + \\
& 84L_2^2L_4 + 7L_2L_3^2 - L_2L_5^2 + 31L_1L_2^2 + \\
& 5L_1L_3^2 + L_1L_5^2 - 11L_1L_2 + 7L_1L_5^2
\end{align*}
\]

From the Chow polynomial, one can extract directly the locations of the moving point at each time instant an image was made. This is done by a two steps computation. The first step consists in giving a parametric representation of the optical ray generated by the 2D measurement. During the second step, the pencil of lines passing through a generic point on the optical ray is considered. For this generic point to be of the trajectory, the Chow polynomial must vanish over the pencil. This yields a polynomial system in one variable, whose root gives the location of the 3D moving point. We show in figure 5 the recovered discrete locations of the point in 3D.

**Trajectory Triangulation from real images**

A point is moving over a conic section. Four static non-synchronized cameras are looking at it. We show on figure 6 one image of one sequence.

The camera matrices are computed using the calibration pattern. Every 2D measurement \( \mathbf{p}(t) \) is corrupted by additive noise, which we consider as an isotropic Gaussian noise \( \mathcal{N}(0, \sigma) \). The variance is estimated to be about 2 pixels.

As before the estimation is done from the set of optical rays generated by the 2D moving point. The estimation is also a case of heteroscedastic estimation,
Figure 5: The 3D locations of the point

Figure 6: A moving point over a conic section
which was handled with the method presented in [4]. The result is stable where starting with a good initial guess. In order to handle more general situations we further stabilize it by incorporating some extra constraints that come from our \textit{a-priori} knowledge of the form of the solution. The final result is presented in figure 7.

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