Generalized Kloosterman Sums from M2-branes

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Abstract: Kloosterman sums play a special role in analytic number theory, for expressing the integer Fourier coefficients of modular forms as an infinite sum of Bessel functions, also known as Rademacher formula. The generalization to vector-valued modular forms is known as generalized Kloosterman sums. In the paper arxiv:1404.0033, a remarkable connection between these arithmetic sums and quantum black hole entropy was found. Nevertheless, the computation was particular for one-eighth BPS black holes in $\mathcal{N} = 8$ string theory, which have a simple counting formula. Here, we review this construction and extend it to the case of one-quarter BPS black holes in $\mathcal{N} = 4$ string theory, which are counted by (mock) Jacobi forms of arbitrary index. The main result is an holographic derivation of the Kloosterman sums which includes the intricate sum over phases, and depends exactly on the spectral flow sectors and the spectrum of polar states. On the microscopic side we derive an analytic formula for the Kloosterman sums valid for any index, whereas from the macroscopic side we reproduce the same formula from the M-theory path integral on $\mathbb{Z}_c$ orbifolds of $AdS_2 \times S^1$. A key aspect of the derivation is the identification of the spectral flow sectors with the contribution of M2 branes wrapping cycles on the compactification manifold. After a careful treatment of the measure, the sum over orbifolds results in the sum over Bessels, in perfect agreement with the Rademacher expansion at any order in perturbation theory.

Keywords: holography, supergravity, Localization.
1. Introduction

Kloosterman sums are arithmetic sums of the form

\[ Kl(n, m, c) = \sum_{\substack{d \in (\mathbb{Z}/c\mathbb{Z})^* \\text{ad}=1 \text{ mod}(c)}} \exp \left[ 2\pi i \frac{d}{c} n + 2\pi i \frac{a}{c} m \right], \tag{1.1} \]

for integers \( n, m, c \). These sums appeared originally in the problem of representing large numbers in quadratic forms of four variables [1]. However, they occur most notably in the Hardy-Ramanujan-Rademacher expansion [2, 3], which we review later.

Recently, Kloosterman sums were shown to be related to non-perturbative corrections to the Bekenstein-Hawking area formula of BPS black holes [4]. Following previous work on quantum black hole entropy [4] and localization techniques in supergravity [5, 6], the authors of [4] were able to identify the sums (1.1) with additional saddles in the path integral, related to global contributions on \( AdS_2 \times S^1/\mathbb{Z}_c \) orbifolds. In particular, the exponential factor and the different sums in (1.1) were shown to arise after a careful evaluation of the Chern-Simons action of the flat connections living on the orbifold. Topologically, the orbifold corresponds to a Dhen filled solid torus parametrized by the integers \( c, d \), and so the sum over those integers in (1.1) can be understood as a sum over topologies in quantum gravity.
The essential step in uncovering the Kloosterman sums, is the application of the localization technique in the string theory path integral that computes the quantum entropy \([6, 8]\). This allows for an exact computation of the black hole entropy as function of the charges. This way we have control over the non-perturbative corrections, which is where the Kloosterman sums become more relevant.

Modular forms with non-positive weight have the remarkable property that its Fourier coefficients can be written as an infinite sum of Bessel functions, each of which comes multiplied by Kloosterman sums. This is known as Rademacher expansion \([2]\), and its generalization to vector-valued modular forms is the generalized Rademacher expansion \([9, 10]\). The idea behind the generalized version consists in writing the Jacobi form as a sum over theta functions \(\theta_\mu(\tau, z)\) multiplied by vector-valued modular forms \(f_\mu(\tau)\) \([11]\). Then, the application of the circle method to the vector-valued modular forms gives the generalized Rademacher expansion. The Kloosterman coefficients \((1.1)\) are modified to account for the fact that vector-valued modular forms transform among themselves under modular transformations. The generalized sums are schematically of the form

\[
Kl(n, m, c)_{\mu\nu} = \sum_{d \in (\mathbb{Z}/c\mathbb{Z})^* \atop ad = 1 \text{mod } c} e^{2\pi i (n - \frac{\mu^2}{4}) \frac{d}{c}} M^{-1}(\gamma)_{\nu\mu} e^{2\pi i (m - \frac{\nu^2}{4}) \frac{a}{c}}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}),
\]

(1.2)

where \(\mu, \nu\) is an index in the space of vector-valued modular forms, \(M(\gamma)_{\mu\nu}\) is the matrix that maps vector-valued modular forms to themselves under the modular transformation, and \(k\) is the index of the Jacobi form. In physical terms, we can identify the Jacobi form with the elliptic genus of the underlying microscopic CFT. The index \(k\) is usually of the order of the central charge and \(\mu, \nu\) parameterize spectral flow sectors.

The main goal of this work is to provide with a bulk string theory computation of the generalized Kloosterman sums \((1.2)\), for arbitrary index \(k\). This extends the results of \([4]\) to one-quarter BPS black holes in \(\mathcal{N} = 4\) string theory. We shall have in mind though that the black hole partition function in the \(\mathcal{N} = 4\) theory is a mock Jacobi form \([12]\), and so the usual Rademacher expansion does not apply. Nevertheless for the range of charges we will be considering the exact answer is well approximated by a Jacobi form \([13]\). In any case, the Kloosterman sums depend only on general transformation properties of the (mock)-Jacobi forms, and not on particular details. This is the feature that we want to reproduce from the bulk theory.

The analysis of \([4]\) focused on the case of one-eighth BPS black holes, which have a simple counting formula \([14]\). In this case, the black hole degeneracies are the Fourier coefficients of the weak Jacobi form

\[
\phi_{-2,1}(\tau, z) = \frac{\vartheta(\tau, z)^2}{\eta(\tau)} ,
\]

(1.3)

where \(\vartheta(\tau, z)\) is a theta function and \(\eta(\tau)\) is the Dedekind function. \(\phi_{-2,1}(\tau, z)\) is a weak Jacobi form of weight minus two and index one. The generalized Kloosterman sums \((1.2)\),

\[^1\text{We thank Atish Dabholkar for clarifying this point.}\]
in particular the matrix $M(\gamma)_{\mu\nu}$ can be constructed starting with the representation of $M$ for the generating elements $S, T$ of $SL(2, \mathbb{Z})$, and then building a general expression from the decomposition $\gamma = ST^{m_1}ST^{m_2} \ldots \in SL(2, \mathbb{Z})$. This was done in [4] with the help of a result by Jeffrey [15] in the context of compact Chern-Simons theory and Witten invariants [16]. However, the computation was specific to index one Jacobi forms such as (1.3). For arbitrary index, a similar computation is possible but it is much more challenging. Part of our work is devoted to obtaining an analytic formula for the matrix $M(\gamma)$ valid for any index, which we can use to compare with the bulk computation. Our formula is based on a result developed long time ago by H. D. Kloosterman [17], which we use extensively. In the appendix we provide with an independent proof of that formula.

From the gravity point of view, the different sums and phases in (1.2), for the index one Jacobi form (1.3), can be shown to arise from the contribution of flat connections to a Chern-Simons action on the Dpen filled solid torus $\simeq AdS_2 \times S^1/\mathbb{Z}_c$ [4]. This Chern-Simons action contains both ”gravitational” and gauge Chern-Simons terms. However, the Chern-Simons level, which maps to the index of the Jacobi form, is an arbitrary charge dependent parameter, whereas the index of the microscopic counting function is one. So to obtain agreement between the bulk computation and the microscopic prediction (1.3), one has to fix the Chern-Simons level to be exactly one [4]. This is puzzling in view of the U-duality in variance of the microscopic formula. On the other hand, we need very large central charge, and thus large Chern-Simons level, for the theory to have a semiclassical description. Our work will provide with the steppingstones to tackle this puzzle completely.

To achieve the main goal of the paper, we develop on the proposal [18] for computing the exact quantum entropy of one-quarter BPS black holes. Essentially, the proposal provides a bulk physical interpretation for the non-perturbative corrections to the entropy, related to the polar coefficients of the vector-valued modular forms. It is argued that the path integral receives, besides the attractor geometry, additional saddles, which result from quantum fluctuations of the Calabi-Yau manifold. These fluctuations lead, in turn, to a renormalization of the parameters that define the effective five dimensional Lagrangian, from which we compute the path integral using localization. Furthermore, the geometry gets corrected in such a way that only a finite number of geometries contribute. The bound on this number is also the bound imposed by the stringy exclusion principle [4]. The great advantage of this construction is that we can identify each of the Bessel functions, associated with the polar terms in the Rademacher expansion, with the perturbative quantum fluctuations around each new saddle. Then it becomes natural from the path integral point of view to include also orbifolds of those geometries, which is what we do in this work.

Following [18], the fluctuations of the Calabi-Yau can be described equivalently in terms of M2 and anti-M2 branes wrapping cycles on the Calabi-Yau and sitting at the origin of $AdS_2 \times S^1$; this picture is borrowed from the chiral primary counting of [20, 21]. It is found that the difference between the number of M2 and anti-M2 ($\text{M2}$) branes generates a large gauge

\footnote{This means the usual map from three dimensional gravity and $SL(2)$ Chern-Simons theory.}
transformation on the $U(1)$ gauge fields of supergravity. However, such gauge transformations are singular on the disk $\simeq AdS_2$. As a consequence, the holonomies around the contractible cycle, that is, the around the disk, change to account for the presence of the M2 branes; when the same number of M2 and $\overline{M}2$ is present, the total charge is zero and the gauge transformation vanishes. Following closely [4], we use this description to compute the contribution of these holonomies to the Chern-Simons action. As a result, one obtains precisely the generalized Kloosterman sums for arbitrary index $k$. In particular, we identify the spectral flow sectors $\nu$ in (1.2) with the singular gauge transformations.

In addition, we provide a simple derivation of the localization measure to include the effect of the orbifold geometries. This generalizes the result of [22] to the case of $AdS_2 \times S^1/\mathbb{Z}_c$ orbifolds, and we use this to fix the dependence of the localization finite dimensional integral on the order of the orbifold $|\mathbb{Z}_c| = c$. Such dependence is crucial for the convergence of the full answer, for the following reason. Note that, in a large charge expansion of the black hole degeneracy $d(q,p)$, the orbifold saddles lead to corrections of the form

$$
\sim \exp \left[ \frac{A}{4c} + \ldots \right], \quad A/c \gg 1
$$

(1.4)

where $A$ is the area of the horizon and the $\ldots$ denote perturbative corrections around each saddle orbifold geometry. Clearly, for sufficiently large $c$ and fixed $A$, the saddle point approximation breaks down. However, using localization one can show that such contributions are of order one, and so the sum over these order one terms leads to a potential divergence, unless the measure is correctly taken into account. It is important to stress that this divergence can not be studied using perturbative methods for the reason just explained, and only a non-perturbative off-shell computation such as localization can provide such test.

Putting together the contribution coming from the localization computation, that is, the Bessel functions, the generalized Kloosterman sums and the $|\mathbb{Z}_c|$ dependent measure, we obtain precisely the Rademacher expansion.

The plan of the paper is as follows. In section §2, we review the generalized Rademacher expansion and associated generalized Kloosterman sums. The main result is an analytic formula for the multiplier matrix, which is the core of the generalized sums. Then in section §3, we describe the holographic computation using an effective three dimensional Chern-Simons theory. A crucial step in this exercise is the inclusion of the singular gauge transformations that signal the presence of the M2 and $\overline{M}2$ branes. We show that this leads precisely to the structure of the Kloosterman sums. Finally in section §3.2, we derive the dependence of the measure on the order of the $\mathbb{Z}_c$ orbifold. We show this agrees precisely with the Rademacher expansion.

2. Generalized Kloosterman sums

In this section, we review the generalized Rademacher expansion for the Fourier coefficients of vector-valued modular forms [9, 10]. Later we provide with an analytic formula for the generalized Kloosterman sums.
Recently, an extension of the Rademacher expansion to mock-Jacobi forms was considered \cite{13}. The structure of this expansion is very similar to the usual Rademacher expansion of Jacobi forms, in the sense that we have a sum over Bessel functions dressed by Kloosterman sums. However, in the mock case, the sum contains additional Bessel functions, whose index differs from the Bessels that appear in the usual Rademacher expansion (2.10). In particular, these Bessels have integral index for integer weight $\omega$. For our purpose, we will only be considering the Bessels of half-integer index, which are common to both Jacobi and mock-Jacobi forms. In both the mock and Jacobi examples, the Kloosterman sums are determined by general modular transformation properties, and so, for our purpose, it is enough to consider the Jacobi case.

2.1 Generalized Rademacher expansion

A Jacobi form $\varphi(\tau, z)$ of weight $\omega$ and index $k$ satisfies the transformation properties

$$
\varphi\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) = (c\tau + d)^{\omega} e^{2\pi i k \frac{z^2}{c\tau + d}} \varphi(\tau, z), \quad \left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \in SL(2, \mathbb{Z}),
$$

and

$$
\varphi(\tau, z + l\tau + m) = e^{-2\pi i k (l^2 \tau + 2lz)} \varphi(\tau, z), \quad l, m \in \mathbb{Z},
$$

also known as elliptic symmetry. Using the property (2.2) we can decompose the Jacobi form as a sum over theta functions \cite{11}, that is,

$$
\varphi(\tau, z) = \sum_{\mu \mod 2k} h_\mu(\tau) \theta_{\mu, k}(\tau, z),
$$

where the theta functions are defined as

$$
\theta_{\mu, k}(\tau, z) = \sum_{n \in \mathbb{Z}} q^{k(n+\mu/(2k))^2} y^{n+2kn}, \quad q = e^{2\pi i \tau}, \quad y = e^{2\pi i z},
$$

The functions $h_\mu(\tau)$ are vector-valued modular forms and have the Fourier expansion

$$
h_\mu(\tau) = q^{-\Delta_\mu} \sum_{n=0}^{\infty} H_\mu(n) q^n.
$$

The part of $h_\mu(\tau)$ with negative $n - \Delta_\mu$ is called the polar part.

Under modular transformations the theta functions transform to themselves in the following way

$$
\theta_{\mu, k}\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) = (c\tau + d)^{1/2} e^{2\pi i k \frac{z^2}{c\tau + d}} \sum_{\nu \mod 2k} M^{-1}(\gamma)_{\mu\nu} \theta_{\nu, k}(\tau, z).
$$

The matrix $M(\gamma)_{\mu\nu}$ is called the multiplier system. For the generating elements $S, T \in SL(2, \mathbb{Z})$, one has respectively

$$
\theta_{\mu, k}(-1/\tau, z/\tau) = \sqrt{\frac{\tau}{2ki}} e^{2\pi i k \frac{z^2}{2ki}} \sum_{\nu \mod 2k} e^{-\pi i \frac{\nu^2}{4k}} \theta_{\nu, k}(\tau, z),
$$

$$
\theta_{\mu, k}(\tau + 1, z) = e^{\pi i \frac{z^2}{4k}} \theta_{\mu, k}(\tau, z).
$$
Given the modular transformation property of the Jacobi form (2.1) together with (2.3) and (2.6), one finds that the functions \( h_\mu(\tau) \) transform as

\[
 h_\mu \left( \frac{a\tau + b}{c\tau + d} \right) = (c\tau + d)^{\omega - 1/2} \sum_{\nu \mod 2k} M(\gamma)_{\mu\nu} h_\nu(\tau),
\]

which justifies the name vector-valued modular form. We see that the multiplier matrix \( M(\gamma)_{\mu\nu} \) is a representation of \( SL(2, \mathbb{Z}) \) in the space of vector-valued modular forms.

The generalized Rademacher expansion is an exact formula for the Fourier coefficients of \( \varphi(\tau, z) \). Given the theta function decomposition, it is easy to show that the Fourier coefficients are given in fact by \( H_\mu(n) \) (2.5). Following [9, 10], we have

\[
 H_\mu(n) = \frac{1}{i^{\omega + 1/2}} \sum_{m - \Delta_\nu < 0} H_\nu(m) \sum_{c=1}^{\infty} \frac{1}{c} K(n, m, c)_{\mu\nu} 
\]

\[
 \times \int_{\epsilon - i\infty}^{\epsilon + i\infty} \frac{dt}{t^{5/2 - \omega}} \exp \left[ 2\pi i \left( \frac{n - \Delta_\nu}{c} - \frac{m - \Delta_\nu}{c} \right) t \right],
\]

where \( m - \Delta_\nu < 0 \) defines the polarity and \( H_\nu(m) \) is the associated polar coefficient. The function \( K(n, m, c)_{\mu\nu} \) are the generalized Kloosterman sums

\[
 K(n, m, c)_{\mu\nu} = \sum_{0 \leq -d < c; (d, c) = 1 \atop ad=1 \text{ mod}(c)} e^{2\pi i (n - \Delta_\mu) \frac{d}{c}} M^{-1}(\gamma)_{\nu\mu} e^{2\pi i (m - \Delta_\nu) \frac{d}{c}},
\]

with no implicit sum on \( \mu, \nu \).

In the case of modular forms, the generalized Kloosterman sum (2.12) reduces to the classical definition (1.1). To see this, suppose we have a modular form with Fourier expansion

\[
 f(\tau) = q^{-n_\rho} \sum_{n \geq 0} d(n) q^n,
\]

with \( n_\rho > 0 \). In this case, \( n - \Delta_\mu \) and \( m - \Delta_\nu < 0 \) in (2.12) are replaced respectively by \( n > 0 \) and \( m - n_\rho < 0 \), which is the polarity. Moreover, since we are dealing with a modular form, we do not have spectral flow sectors \( \mu, \nu \), and hence there is no multiplier matrix. Therefore, the Kloosterman sum reduces to

\[
 K(n, m, c) = \sum_{0 \leq -d < c; (d, c) = 1 \atop ad=1 \text{ mod}(c)} e^{2\pi i n \frac{d}{c} + 2\pi i (m - n_\rho) \frac{d}{c}}.
\]

### 2.2 Analytic formula for the Multiplier Matrix

To construct an analytic formula for the matrix \( M(\gamma)_{\mu\nu} \) we can build a general representation starting with the generating elements of \( SL(2, \mathbb{Z}) \). For the case with index \( k = 1 \) this was done in [4] following a result by Jeffrey [13] in the context of Chern-Simons theory. However for
general index $k$ the problem is technically more challenging and we cannot straightforwardly use the results of [15]. Fortunately, this problem was solved long time ago by H.D. Kloosterman [17], which provides an explicit representation for the matrix $M^{-1}(\gamma)_{\mu\nu}$. This is very convenient because $M^{-1}(\gamma)_{\mu\nu}$ appears explicitly in the Kloosterman sums (2.12). In the appendix §A we give an alternative derivation of that formula.

In [17], H. D. Kloosterman provides many results on the transformation of generalized theta functions under modular transformations. For our purpose, we are interested in equations 2.15, 3.5 and 3.8 of that paper. Specializing his results to the theta functions of index $k$ (2.4), one obtains the expression

$$M^{-1}(\gamma)_{\mu\nu} = \frac{1}{(2kci)^{1/2}} \sum_{m=0}^{c-1} \exp \left[ 2\pi i \left( \frac{a(\mu + 2km)^2}{c} - \frac{v(\nu + 2km)}{2kc} + \frac{d\nu^2}{4k} \right) \right],$$

(2.15)

with $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$. This is one of the main formulas of our work.

The representation (2.15) has a few important properties that will be useful later on. First one has

$$M^{-1}(\gamma)_{\mu+2kl,\nu} = M^{-1}(\gamma)_{\mu\nu}, \ l \in \mathbb{Z},$$

(2.16)

and similarly

$$M^{-1}(\gamma)_{\mu,\nu+2kl} = M^{-1}(\gamma)_{\mu\nu}, \ l \in \mathbb{Z},$$

(2.17)

which is the statement that the representation (2.15) only depends on the equivalence class of $\mu, \nu \in \mathbb{Z}/2k\mathbb{Z}$. Second we have

$$\sum_{\sigma=0}^{2k-1} M^{-1}(\gamma)_{\mu\sigma} M^{-1}(\gamma')_{\sigma\nu} = M^{-1}(\gamma \gamma')_{\mu\nu}, \ \gamma, \gamma' \in SL(2, \mathbb{Z}).$$

(2.18)

This follows from the fact that (2.15) is a representation of $SL(2, \mathbb{Z})$. In the appendix we show explicitly how the representation (2.15) obeys this property. We also give derivations of the properties (2.16) and (2.17).

3. Holographic computation

In this section we describe the holographic dual computation of the generalized Kloosterman sums using Chern-Simons theory on $AdS_2 \times S^1/\mathbb{Z}_c$ orbifolds. The discussion is very similar to [1], which we briefly review now.

The $AdS_2 \times S^1/\mathbb{Z}_c$ orbifolds were studied originally in [23] by considering a decoupling limit of the $SL(2, \mathbb{Z})$ family of extremal black hole solutions in $AdS_3$ [19]. The inclusion of the orbifold geometry in the path integral explains non-perturbative corrections to black hole entropy of the form

$$\sim \exp \left[ \frac{A}{4c} \right],$$

(3.1)
with $A$ the horizon area; the factor of $1/c$ is a direct consequence of the orbifold. The orbifold consists in identifying points on $AdS_2$ which differ by a deficit of $2\pi/c$ angle, while performing a translation along the circle $S^1$ by $2\pi d/c$, with $d, c$ coprime; the translation along the circle renders the quotient smooth. Globally one has a solid torus $D \times S^1$, with $D$ a disk, filled with a hyperbolic metric. A choice of $(c, d)$ is equivalent to choose which cycle in the boundary torus we are making contractible in the full geometry. That is, after choosing a basis of one-cycles $C_1$ and $C_2$ on the boundary torus, we then fill the solid torus by attaching a disk to a cycle $C_c$, which becomes the contractible cycle; this is a linear combination of the basis one-cycles,

$$C_c \equiv cC_1 + dC_2,$$

(3.2)

whereas the non-contractible circle $S^1$ is identified with the linear combination

$$C_{nc} \equiv aC_1 + bC_2.$$ (3.3)

To guarantee that $C_{nc}$ has unit intersection with $C_c$, that is, $C_{nc} \cap C_c = 1$ given $C_1 \cap C_2 = 1$, we must have $ad - bc = 1$, with $a, b, c, d \in \mathbb{Z}$. From now on we denote the orbifold geometry by $M_{(c,d)}$.

In [4], it is shown that the quantum entropy path integral receives the contribution of flat connections on $M_{(c,d)}$ via their Chern-Simons action. The non-trivial feature of the computation is that, while the local contributions to the path integral give rise to contributions to the entropy that are real and of the form (3.1), the flat connections, on the other hand, give rise to phases, essentially because the Chern-Simons action is not parity invariant. In particular, the Chern-Simons action of the flat connections in the $M_{(c,d)}$ geometry gives rise to the phases that one finds in the Kloosterman sums of (1.3) [4]. For each $c$, we have to sum over $d$ and $a$, which is the element inverse of $d$ in $\mathbb{Z}/c\mathbb{Z}$. This explains the various sums in the Kloosterman formula.

Since the contribution of the flat connections is topological in nature, it is enough to consider the effective Chern-Simons action living on the solid torus defined by the geometry $M_{(c,d)}$. Moreover, we can show that the Chern-Simons action depends only on the holonomies of the flat connection along the contractible and non-contractible cycles, which simplifies greatly the discussion.

Following [4], we consider the $SL(2,\mathbb{R})_L \times SL(2,\mathbb{R})_R \times SU(2)_L \times SU(2)_R$ effective Chern-Simons action living on the geometry $M_{(c,d)}$. The theory contains in addition multiple $U(1)$ Chern-Simons terms but they do not contribute to the entropy because the action of an abelian flat connection is zero. The non-compact $SL(2,\mathbb{R})_L \times SL(2,\mathbb{R})_R$ gauge group factor comes from the fact that three dimensional gravity can be written in terms of Chern-Simons variables with the gauge group being determined by the isometries of $AdS_3$. Supersymmetry acts on the right, that is, on the $SL(2,\mathbb{R})_R \times SU(2)_R$ factor, with $SU(2)_R$ the R-symmetry. Furthermore, one has an $SU(2)_L$ factor, which arises from gauging the isometries of a local $S^3$ in the full geometry.

The Chern-Simons action contains the following terms

$$S = -\frac{i\tilde{k}_L}{4\pi}I[\tilde{A}_L] + \frac{i\tilde{k}_R}{4\pi}I[\tilde{A}_R] - \frac{ik_R}{4\pi}I[A_R] + \frac{ik_L}{4\pi}I[A_L],$$ (3.4)
where $\tilde{A}_{L,R}$ are respectively the $SL(2,\mathbb{R})_{L,R}$ connections and $A_{L,R}$ are the $SU(2)_{L,R}$ connections. We weight the path integral with $\exp S$. Due to supersymmetry the Chern-Simons levels $\tilde{k}_R$ and $k_R$ are equal. Nevertheless, the levels $\tilde{k}_L$ and $k_L$ remain independent. We have denoted the Chern-Simons action by $I[A]$, which we define as

$$I[A] = \int_M \text{Tr} \left( A \wedge dA + \frac{2}{3} A^3 \right),$$

with the trace in the fundamental representation.

To compute the Chern-Simons action on the solid torus we follow [24, 4]. A flat connection $A_f$ is always pure gauge and as such we can write it as

$$A_f = -dg g^{-1}, g \in G,$$

(3.6)

where $G = SL(2), SU(2)$ is the gauge group. As explained in [24] the gauge transformation $g$ can be brought to the "normal" form

$$g = f(x_c, r)e^{-\frac{i}{2} \beta \sigma^3 x_{nc}}.$$  

(3.7)

The coordinates $(x_c, r)$ with $x_c \in [0, 2\pi]$ and $r \in [0, 1]$, parametrize the disk $D$, while $x_{nc}$ parametrizes the circle $S^1$. The function $f(x_c, r) \in G$ maps points on the disk to elements of the gauge group. One has the condition that $f(x_c, r = 1) = e^{-\frac{i}{2} \alpha \sigma^3 x_{c}}$ at the boundary of the disk, and moreover it is constant at the origin, so that the gauge field is well defined at that point. The constant $\beta$ fixes the holonomy along the non-contractible cycle. With this parametrization the holonomies become diagonal in $G$, up to conjugation.

The Chern-Simons action of the flat connection can be computed using the Stoke’s theorem [24]. This gives

$$I[A_f] = \frac{2\pi^2}{4\pi^2} \alpha \beta.$$  

(3.8)

Furthermore, in the path integral with Chern-Simons action we need to introduce the boundary action [25, 26, 4]

$$S_{\text{bnd}} = \frac{1}{4\pi} \int_{\partial M} \text{Tr} A_1 A_2,$$

(3.9)

where $A_1$ is the component of $A$ along the cycle $C_1$ in the boundary torus and similarly for $A_2$. This boundary action ensures that the variational problem is well posed. These boundary values are determined as follows. In the black hole problem, the leading contribution to the entropy comes from the geometry $M_{1,0}$. In this case, the cycle $C_1$ bounds the $AdS_2$ disk and it is parametrized by the euclidean time, whereas, the M-theory circle corresponds to the cycle $C_2$, and it is parametrized by the coordinate $y$. The boundary conditions are such that the component along $C_2$ is fixed, while the component along $C_1$, that is, $A_1$ is allowed to fluctuate. In [4], this choice has been shown to be consistent with the $AdS_2$ microcanonical boundary conditions of the quantum entropy formalism [5].
For the manifold $M_{(c,d)}$, $\alpha$ and $\beta$, which parametrize respectively the holonomies along the contractible and non-contractible cycles, are determined as in [4], that is
\[
2\pi i \frac{\sigma^3}{2} \alpha \equiv \oint_{C_c} A_f = c \oint_{C_1} A_f + d \oint_{C_2} A_f, \tag{3.10}
\]
and
\[
2\pi i \frac{\sigma^3}{2} \beta \equiv \oint_{C_{nc}} A_f = a \oint_{C_1} A_f + b \oint_{C_2} A_f, \tag{3.11}
\]
computed at $\partial M = T^2$. Define
\[
\oint_{C_1} A_f = 2\pi i \gamma \sigma \frac{\sigma}{2}, \quad \oint_{C_2} A_f = 2\pi i \delta \sigma \frac{\sigma}{2}, \tag{3.12}
\]
then from equations (3.10) and (3.11) we obtain
\[
\alpha = c\gamma + d\delta, \quad \beta = a\gamma + b\delta. \tag{3.13}
\]
The Chern-Simons action of the flat connection together with the boundary term (3.9) is
\[
S + S_{\text{bnd}} = \frac{i\pi}{2} k\alpha\beta - \frac{i\pi}{2} k\gamma\delta, \tag{3.14}
\]
where we have reintroduced the Chern-Simons level $k$.

### 3.1 Flat connections from M2 branes

In [18], it was proposed that the quantum entropy path integral of M-theory on $AdS_2 \times S^1 \times S^2 \times M_6$, with $M_6$ a Calabi-Yau manifold, receives the contribution of a finite number of off-shell backgrounds. The contribution of these saddles to the path integral can be computed using localization, and it turns out, that they can be identified with the polar Bessel functions of the Rademacher expansion. It is argued that after turning on singular fluxes on the Calabi-Yau, the full back-reacted geometry is a solution of five dimensional supergravity with renormalized $c_2$ coefficient, which parameterizes the mixed gauge-gravitational Chern-Simons terms in five dimensions. When the fluxes are absent, $c_2$ is the second Chern-class (tangent bundle) of the Calabi-Yau. The presence of fluxes can be interpreted equivalently in terms of M2 and anti-M2 branes wrapping holomorphic cycles in the Calabi-Yau. The renormalization is such that the effective Chern-Simons levels are
\[
\tilde{k}_R = \frac{p_3}{6} + \hat{c}_2 \cdot \frac{p}{12}, \quad \tilde{k}_L = \frac{p_3}{6} + \hat{c}_2 \cdot \frac{p}{6}, \tag{3.15}
\]
and similarly for $k_R$, which is equal to $\tilde{k}_R$ by supersymmetry. The parameter $\hat{c}_2$ denotes the effective renormalized value of $c_2$, which in terms of the fluxes $f_a$ and $\bar{f}_a$ is given by
\[
\hat{c}_{2a} = c_{2a} - 12(f_a + \bar{f}_a), \quad f_a, \bar{f}_a \in \mathbb{Z}^+. \tag{3.16}
\]
where the subscript $a$ parameterizes a basis of two cycles. In the M2 brane picture, $f_a$ and $\overline{f}_a$ are the number of M2 and anti-M2 branes respectively. The range of $f_a, \overline{f}_a$ is not arbitrary. The renormalization leads to a correction of the physical size of the geometry which puts a bound on $f_a, \overline{f}_a$. In [18], this bound was shown to be the same as the one imposed by the stringy exclusion principle.

In addition, such fluxes induce a large gauge transformation on the $U(1)$ gauge fields of five dimensional supergravity, as

$$A^a \rightarrow A^a - 2(f^a - \overline{f}^a)dx_c,$$

(3.17)

where we defined $f_a = D_{ab}f^b$ with $D_{ab} = D_{abc}p^c$, and $D_{abc}$ is the Calabi-Yau intersection matrix; $p^a$ are the magnetic fluxes on the sphere and map to the configuration of M5 branes wrapping a four cycle on the Calabi-Yau [27]. Note that the gauge transformation is singular at the origin since it is proportional to $dx_c$, which is the disk angle, and $A^a$ vanishes there. Physically this singularity is expected because there are M2 branes sitting at the origin, with $f^a - \overline{f}^a$ the total charge.

For the $\mathcal{N} = 4$ theory, which is our primary interest, we have

$$\Delta f^1 = \Delta f^1 = -\frac{p^1}{P^2}(f_1 - \overline{f}_1), \quad \Delta f^a = 0, \ a \neq 1$$

(3.18)

where we have defined $\Delta f^a \equiv (f^a - \overline{f}^a)$; we also have $D_{abc} = D_{iab} = D_{aib} = D_{abi} = C_{ab}$. We can show that the $U(1)$ gauge field $A^1$ corresponds to a $U(1)$ truncation of the $SU(2)_L$ gauge field after dimensional reduction on the sphere [4]. The precise map is $A_L = i\sigma^3A^1/2$ [4, 22]. Following [4], we compute the $SU(2)_L$ Chern-Simons contribution. In this work we consider $p^1 = 1$ for simplicity. It would be important to generalize these results for arbitrary $p^1$, though we believe the results will not suffer significant changes. The boundary conditions for $A_L$ can be determined from the attractor equations. We have

$$\oint_{C_2} A_L = 2\pi i \frac{Q.P}{P^2},$$

(3.19)

where $Q.P = -q_1p^1 + q_0p^a$ and $P^2 = C_{ab}p^ap^b$ with $a = 2 \ldots n_v$, and $n_v$ the number of vectors. We have denoted $i \equiv i\sigma^3$. A key aspect of our construction when compared with the Kloosterman sum computation of [4], is that the Wilson line along the contractible cycle receives the contribution of the M2 branes that are sitting at the origin. Essentially, the gauge transformation (3.18) leads to the Wilson line

$$\pi i \alpha = \oint_{C_c} A_L = \pi i \left(2n + \frac{\nu}{k_L}\right), \quad \nu = \epsilon k_L + f_1 - \overline{f}_1,$$

(3.20)

with $\epsilon = \pm 1$, $k_L = \frac{P^2}{2}$ and $n \in \mathbb{Z}$. We have introduced $\epsilon$ such that in the absence of M2 branes the holonomy is $-1$; this equals the holonomy of the $SU(2)_R$ connection [22], which
is necessary to ensure that the geometry corresponds to the R sector of the dual CFT. Given \( \alpha = c \gamma + d \delta \) and \( \beta = a \gamma + b \delta \) (3.13) we determine

\[
\gamma = \frac{1}{c} \left( 2n + \frac{\nu}{k_L} \right) - \frac{d Q.P}{c k_L},
\]

and hence

\[
\beta = \frac{a}{c} \left( 2n + \frac{\nu}{k_L} \right) - \frac{ad Q.P}{c k_L} + b Q.P - \frac{b}{k_L},
\]

with \( ad - bc = 1 \). Therefore the total Chern-Simons action plus boundary terms (3.14) is

\[
I_{CS+Bad} = \pi i \frac{a}{2k_L c} \left( \nu + 2nk_L \right)^2 - \pi i \frac{Q.P(\nu + 2k_L n) + \frac{d}{2k_L c}(Q.P)^2 + 2\pi i\mathbb{Z}}{c k_L}.
\]

(3.23)

It is easy to see that the exponential of \( I_{CS+Bad} \) is invariant under \( n \to n + c \mathbb{Z} \) and so we have to truncate the sum of \( n \) to lie in \( \mathbb{Z}/c \mathbb{Z} \). Similarly we can show (see appendix A) that the exponential is invariant under \( Q.P \to Q.P + 2k_L \mathbb{Z} \), and so we can write \( Q.P = \mu \) with \( \mu \in \mathbb{Z}/2k_L \mathbb{Z} \).

Now we consider the gravitational \( SL(2)_{L,R} \) and \( SU(2)_R \) Chern-Simons terms. On the supersymmetric side the contributions coming from the \( SL(2, \mathbb{R})_R \) and \( SU(2)_R \) terms cancel each other as pointed out in [4]. The holonomy of \( A_R \) is such that the orbifold preserves the localization supercharge. Nevertheless, the gravitational \( SL(2, \mathbb{R})_L \) contribution is non-trivial and equals (equation 4.46 in [4])

\[
\exp \left( -\frac{\pi i}{2c} \frac{k_L a}{\bar{k}_L} + \frac{\pi i}{2c} \frac{d}{k_L} - \Delta \right),
\]

(3.24)

where \( R \) is the asymptotic value of the radius of the M-theory circle and \( \bar{k}_L \) is the renormalized \( SL(2, \mathbb{R})_L \) Chern-Simons level (3.15). The radius \( R \) can be computed for \( \Delta f = 0 \), which gives

\[
R^2 = \frac{\Delta}{k_L k_L},
\]

(3.25)

after solving the attractor equations, or equivalently, from extremizing the quantum entropy function. Here \( \Delta = Q^2 P^2 - (Q.P)^2 \) is the quartic invariant charge combination. Hence, (3.24) becomes

\[
\exp \left( -\frac{\pi i}{2c} \frac{k_L a}{\bar{k}_L} + \frac{\pi i}{2c} \frac{d}{k_L} - \Delta \right).
\]

(3.26)

The Rademacher expansion predicts, nonetheless, the phase

\[
-\frac{\pi i}{2c} \left( \bar{k}_L - 2(\Delta f)^2 \right) + \frac{\pi i}{2c} \frac{\Delta d}{k_L}.
\]

with the combination \( \bar{k}_L - 2(\Delta f)^2 \) being the polarity. We have defined \( (\Delta f)^2 = D^{ab} \Delta f_a \Delta f_b \).

In the five dimensional theory [18], the term \( (\Delta f)^2 \) in the polarity arises indirectly from a delta
function contribution of the $U(1)$ gauge field strength, as explained in [18]. To be more precise, from the localization computation of [18] we find the entropy function

$$-2\pi \hat{q}_0 \frac{\Delta}{R} + \frac{\pi}{2} R\Delta f = \frac{\pi}{4} RD_{ab}(\phi^a + q^a)q^b(\phi^b + q^b) - 2\pi i \frac{\phi^a}{q^0} \Delta f_a - 2\pi i q_a \Delta f^a,$$

(3.27)

with $\hat{q}_0 = q_0 - D_{ab} q^a q^b/2 = -\Delta/4k_L$ and $R = 2/\phi^0$. The terms proportional to $\Delta f_a$ arise from the delta function induced by the large gauge transformation. When $\Delta f_a = 0$, we can identify, at the on-shell level, the term $-2\pi \frac{\Delta}{R} + \frac{\pi}{2} R\Delta f$ with the Chern-Simons action of the flat connection on $M(1,0)$. Integrating out $\phi^a$, the gaussian induces a correction $(\Delta f)^2$ to $\Delta f$, and one effectively obtains the entropy

$$-2\pi \hat{q}_0 \frac{\Delta}{R} + \frac{\pi}{2} R(\Delta f - 2(\Delta f)^2).$$

(3.28)

Since in the effective three dimensional Chern-Simons theory one assumes that both the $SL(2, \mathbb{R})_L$ and $SU(2)_L$ factors are decoupled, the effective Chern-Simons level for the $SL(2, \mathbb{R})_L$ factor should be in fact $\Delta f - 2(\Delta f)^2$. On the other hand, from the attractor equations we have now $R^2 = \Delta/k_L(\Delta f - 2(\Delta f)^2)$. Proceeding as before, now we find the phase

$$\exp \left( -\pi i \frac{\Delta f}{2} \frac{\Delta f_a}{c} + \pi i \frac{\Delta f}{2k_L c} \right).$$

(3.29)

We can identify $\Delta f - 2(\Delta f)^2$ with the polarity $\Delta f_a - m > 0$ in the Rademacher expansion (2.10). That is, in terms of the fluxes that polarity has the form

$$\frac{\Delta f}{4} = \left( \frac{P^2}{2} - (f_1 - \bar{f}_1) \right)^2 - \bar{f}_1 + n_p$$

$$= \frac{\nu^2}{4k_L} - m$$

(3.30)

with $\nu = f_L - f_L - (f_1 - \bar{f}_1)$ and $m = \bar{f}_1 - n_p$, and $n_p = 0,1$ for the $T^4$, K3 CHL orbifold compactifications respectively [18].

Assembling the different pieces, the phase (3.29) then gives the term that multiplies $M^{-1}(\gamma)$ in (2.12). The charge combination $\Delta = Q^2 P^2 - (Q,P)^2$ can always be written in form $4nk_L - \mu^2$ with $Q,P = \mu \mod(2k_L)$ and $n \in \mathbb{Z}$. So we identify $\Delta/4k_L$ with $n - \Delta f$ in (2.12). Similarly we have $\Delta f_a = \nu^2/4k_L$ in the same expression. Furthermore, integration over the $\phi^a$ gives rise to a term proportional to $1/\sqrt{\det D_{ab}}$, which for the $N = 4$ compactifications is proportional to $1/\sqrt{k_L}$, after setting $p^1 = 1$. The exponential of the $SU(2)_L$ Chern-Simons contribution (3.23), together with the factor $1/\sqrt{k_L}$, reproduces the analytic formula for the matrix $M^{-1}(\gamma)^{-1}$, except for a dependence on $c$ in the normalization factor, which we fix in the next section.

3.2 Measure dependence on $|Z_\nu|$

The measure of the finite dimensional integral that one obtains using localization can be fixed by one-loop computation in Chern-Simons theory. The original computation [22] focused on
the $AdS_2 \times S^1$ geometry but we can easily generalize it for the $AdS_2 \times S^1/Z_c$ orbifolds. We also point the reader to the discussion in section 5.1 of [18].

The result for the partition function in the unorbifolded theory is the integral

$$
\int dR \prod_{a=1}^{b+1} d\phi^a \frac{1}{R} \exp \left[ -2\pi \hat{g}_0 \frac{R}{\hat{k}_L} + \frac{\pi}{2} R\hat{R}_L - \frac{\pi}{4} RD_{ab}\phi^a\phi^b \right].
$$

(3.31)

We are neglecting a factor dependent on the physical size of the geometry, which does not play any role for what we want to say. The measure $1/R$ follows from a one-loop computation in Chern-Simons theory, which gives

$$
Z_{\text{CS}}^{\text{1-loop}} \propto \frac{R^{-b/2-1}}{\sqrt{\prod k_i}},
$$

(3.32)

where $k_i$ runs through the $SL(2,\mathbb{R})_L \times SU(2)_L \times U(1)^b$ Chern-Simons levels. The one-loop contribution (3.32) comes entirely from the zero modes of the gauge fields whose measure in the path integral is determined using an ultra-locality argument. Essentially, one imposes an ultra-local measure of the form

$$
\int D[A] \exp [k \int \text{Tr} A \wedge *A] = 1,
$$

(3.33)

for the non-abelian gauge fields, and similarly for the $U(1)$ gauge fields. This normalization defines the measure $D[A]$. In Chern-Simons theory we have to pick a metric to define this measure, and this is the reason why the integral over the zero modes gives factors of $R$ in (3.32).

We can repeat the same logic but now for the orbifold $AdS_2 \times S^1/Z_c$. This will give a dependence of $D[A]$ on the order of the orbifold $|Z_c| = c$. We have to remark, nevertheless, that for the $U(1)$ gauge fields the dependence of the measure $D[A]$ on $c$ is ambiguous. The reason is that from the Chern-Simons point of view, the $U(1)$ gauge fields are free fields and so we could have absorbed a $1/c$ dependence of $\int \text{Tr} A \wedge *A$ in the normalization (3.33), in a rescaling of the gauge fields. In contrast, the non-abelian gauge fields are interacting fields and a rescaling leads effectively to a change in the interaction term. Therefore, we understand that for the $U(1)$ gauge fields, we have the choice to exclude from $D[A]$ the $c$ dependence that one obtains from the ultra-locality argument. We will see this leads to the desired result. Nevertheless, it would be important to check this more explicitly.

In the unorbifolded theory, the normalization (3.33) gives a dependence of $(kR)^{1/2}$ in the measure for each non-abelian gauge group factor, while in the orbifold case we have $(kR/c)^{1/2}$, where the $1/c$ factor is the result of the quotient. For the remaining $U(1)$ factors we obtain a $(kR)^{1/2}$ dependence with no $c$ factor, as argued. Repeating the one-loop computation in the orbifold geometry, which is a volume over the zero modes modes, we find

$$
Z_{\text{CS}}^{\text{1-loop}}|_{M(c,d)} \propto c \frac{R^{-b/2-1}}{\sqrt{\prod k_i}},
$$

(3.34)
where the $c$ factor comes from the $SL(2,\mathbb{R})_L \times SU(2)_L$ gauge fields. On the other hand, the entropy function for the orbifold geometry is

$$-2\pi \frac{\hat{q}_0}{cR} + \frac{\pi R}{2c} \hat{k}_L - \frac{\pi R}{4c} D_{ab} \phi^a \phi^b,$$

(3.35)

where the factor of $1/c$ is due to the $\mathbb{Z}_c$ quotient. For purpose of computing the localization measure we have set $\Delta f = 0$. Comparing the Chern-Simons computation with the one-loop correction that we obtain from extremizing the entropy function, we find that the measure in (3.31) acquires an additional factor of $c^{-b/2}$.

Nonetheless, this computation only takes into account the fluctuations around a particular flat connection in the geometry $M_{(c,d)}$. The path integral contains for fixed $c$ a sum over geometries $M_{(c,d)}$, and holonomies $\text{Hol}(A_L)$, which are parameterized respectively by the integers $d$ and $n$ valued in $\mathbb{Z}/c\mathbb{Z}$ (3.23). From the Chern-Simons point of view, the sum over geometries and holonomies is characterized by the sum over the Wilson lines $\oint_{C_1} \tilde{A}_L$ and $\oint_{C_1} A_L$ respectively, which are allowed to fluctuate. Since these holonomies are $\mathbb{Z}_c$ valued, each sum over $\oint_{C_1} \tilde{A}_L$ and $\oint_{C_1} A_L$ must be accompanied by a factor of $1/c$, which ensures that the volume of the gauge group is correctly factored out. This gives an additional $1/c^2$ factor.

The finite dimensional integral then has the form

$$\frac{1}{c^{b/2+2}} \int dR \prod_{a=1}^{b+1} d\phi^a \frac{1}{R} \exp \left[ -2\pi \frac{\hat{q}_0}{cR} + \frac{\pi R}{2c} \hat{k}_L - \frac{\pi R}{4c} D_{ab} \phi^a \phi^b \right].$$

(3.36)

Performing the various gaussian integrals, we obtain the Bessel answer

$$\frac{1}{c^{\sqrt{\mathcal{K}_L}} \sqrt{\det(D_{ab})}} \int \frac{dR}{R^{3/2+b/2}} \exp \left[ -2\pi \frac{\hat{q}_0}{cR} + \frac{\pi R}{2c} \hat{k}_L \right].$$

(3.37)

The term $\sqrt{\det(D_{ab})}$ is proportional to $\sqrt{\mathcal{K}_L}$. The factor $1/\sqrt{\mathcal{K}_L}$ joins the $SU(2)_L$ Kloosterman sum (3.23) to give the normalization found in the the matrix $M^{-1}(\gamma)$ (2.13). The remaining $1/c$ factor can be identified with the one that multiplies the Kloosterman sums in the Rademacher formula (2.10). We therefore obtain an exact matching with the microscopic formula. By construction, we now have guaranteed that for large $c$ this measure ensures that the sum over the $M_{(c,d)}$ geometries is not divergent.

Given this result we can also try to reproduce the measure that one obtains for modular forms. In this case the Kloosterman sums reduce to the classical case (1.1). From the bulk point of view we also do not expect $SU(2)_L$ Chern-Simons terms. Following the same reasoning, we find a single factor of $1/c$ from the zero mode volume, since now we only have a sum over the $\oint_{C_1} \tilde{A}_L$ holonomies parameterized by $d \in \mathbb{Z}/c\mathbb{Z}$. Again, this result agrees with the Rademacher prediction.

4. Discussion and Conclusion

In this work, we have considered the contribution of $AdS_2 \times S^1 \times S^2/\mathbb{Z}_c$ orbifolds to the quantum entropy path integral following the proposal [18]. We have provided a generalization of the
non-perturbative corrections studied in [4] for one-quarter BPS black holes in four dimensional $\mathcal{N} = 4$ string theory. To this end the main results are:

- **Generalized Kloosterman sums**: we have derived generalized Kloosterman sums from the gravity point of view. A key aspect of this construction is the contribution of non-trivial flat connections to the Chern-Simons action, which arise after considering M2 and $\mathbb{M}_2$-branes wrapping cycles on the Calabi-Yau [18]. The result of the bulk computation is in perfect agreement with an analytic formula for the generalized Kloosterman sums of Jacobi forms of arbitrary index.

- **Integers from quantum gravity**: we derive the exact dependence of the localization measure on the order of the orbifold $|Z_c| = c$. Together with the Kloosterman sums and the Bessel functions that we obtain by supersymmetric localization, we can show that the $AdS_2$ path integral reproduces the Rademacher expansion at all orders in the charges.

Our results constitute a very important piece of evidence in favor of the proposal put forward in [18]. It is important to stress the key aspect of that proposal which is at the heart of the construction presented in this work: the fact that we can associate to each polar Bessel function a different saddle geometry. Then, the inclusion of the orbifold geometries becomes straightforward from the path integral point of view and the Chern-Simons computation follows as originally shown in [4].

As a byproduct of our results it would be important to understand if the Kloosterman sums obey special arithmetic properties that can explain the black hole degeneracy of more general charge configurations. From the microscopic side we already have a good understanding of the degeneracy of non-primitive dyons for both $\mathcal{N} = 8$ and $\mathcal{N} = 4$ compactifications [28, 29, 30]. On the macroscopic side, there is partial understanding [31, 32] in terms of $AdS_2 \times S^2$ orbifolds in the quantum entropy. However, this holds only at the on-shell level, and so it would be important to extend such results to the quantum level, as we did in this work. If such arithmetic properties exist, one may be able to solve a puzzle related to U-duality invariance of the $\mathcal{N} = 8$ answer raised in the beginning of this work.

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**A. An elementary derivation of the multiplier matrix**

In this section we use a trick by Zagier and Skoruppa [33] to derive an analytic formula for the multiplier matrix.
We start with the definition of the theta functions

\[ \theta_{m,\rho}(\tau, z) = \sum_{l=\rho \mod(2m)} q^{l^2/4m} y^l \]  
\[ = \sum_{n \in \mathbb{Z}} q^{(\rho+2mn)^2/4m} y^{(\rho+2mn)}, \]

with \( \rho \) a representative of the equivalence class \( \mathbb{Z}/2m\mathbb{Z} \). These are modular functions with weight \( 1/2 \) and level \( m \), that is, under modular transformations, one has

\[ \theta_{m,\rho}(a\tau + b, c\tau + d, z) = \left(c\tau + d\right)^{1/2} e^{2\pi im \frac{cs^2}{c^2 \tau + d^2}} \sum_{\sigma \mod 2m} K_{\rho\sigma}(\gamma) \theta_{m,\sigma}(\tau, z) \]  
\[ \theta_{m,\rho}(\tau, z + \lambda \tau + \mu) = e^{-2\pi i m (\lambda^2 \tau + 2\lambda z)} \theta_{m,\rho}(\tau, z), \ (\lambda, \mu) \in \mathbb{Z}^2, \]

with \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \). For convenience, define

\[ (\theta_{m,\rho}|\gamma)(\tau, z) \equiv \left(c\tau + d\right)^{-1/2} e^{-2\pi i m \frac{cs^2}{c^2 \tau + d^2}} \theta_{m,\rho}(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}), \]

which by (A.3), is equivalent to

\[ (\theta_{m,\rho}|\gamma)(\tau, z) = \sum_{\sigma \mod 2m} K_{\rho\sigma}(\gamma) \theta_{m,\sigma}(\tau, z). \]  

By definition, we have the equality

\[ (\theta_{m,\rho}|\gamma)(\tau, z) = \sum_{s \in \mathbb{Z}} y^s \int_0^1 e^{-2\pi i x s} (\theta_{m,\rho}|\gamma)(\tau, x) dx. \]  

Introducing the expression (A.5) in this integral and using the Fourier expansion of the theta function, we obtain

\[ (\theta_{m,\rho}|\gamma)(\tau, z) = \sum_{s \in \mathbb{Z}} y^s (c\tau + d)^{-1/2} \int_0^1 e^{-2\pi i x s} e^{-2\pi i m \frac{sx^2}{c^2 \tau + d^2}} \theta_{m,\rho}(\frac{a\tau + b}{c\tau + d}, \frac{x}{c\tau + d}) dx \]
\[ = \sum_{s \in \mathbb{Z}} y^s (c\tau + d)^{-1/2} \sum_{r=\rho \mod(2m)} \int_0^1 \exp \left[ 2\pi i \left( -xs - m \frac{cx^2}{c\tau + d} + \gamma(\tau) \frac{r^2}{4m} + \frac{x}{c\tau + d} \right) \right] dx, \]  

with \( \gamma(\tau) = a\tau + b/c\tau + d \). Using \( \frac{a\tau + b}{c\tau + d} = \frac{a}{c} - \frac{1}{c(c\tau + d)} \) we write

\[ -xs - m \frac{cx^2}{c\tau + d} + \gamma(\tau) \frac{r^2}{4m} + \frac{x}{c\tau + d} = \frac{a}{c} \frac{r^2}{4m} - \frac{sr}{2mc} + \frac{d^2}{4m} + \frac{s^2}{4m} + \frac{mc}{c\tau + d} \left( x - \frac{r}{2mc} + \frac{s}{2mc} \right)^2, \]
and hence equation (A.8) becomes

\[
(\theta_{m,\rho}\mid \gamma)(\tau, z) = \sum_{s \in \mathbb{Z}} \sum_{r = \rho \mod(2m)} q^{s^2/4m} g^r(c\tau + d)^{-1/2} e^{2\pi i \left( \frac{\tau^2}{4m} + \frac{sr}{2mc} + \frac{d}{c} \frac{s^2}{4m} \right)} \int_{0}^{1} e^{-2\pi i \frac{mc}{c\tau + d} \left( x - \frac{r}{2mc} + \frac{x^2 + d}{2mc} \right)^2} \, dx
\]

\[
= \sum_{\sigma \mod 2m} \sum_{n \in \mathbb{Z}} \sum_{r = \rho \mod(2m)} q^{(\sigma+2mn)^2/4m} g^{(\sigma+2mn)} e^{2\pi i \left( \frac{\tau^2}{4m} + \frac{sr}{2mc} + \frac{d}{c} \frac{(\sigma+2mn)^2}{4m} \right)}
\]

\[
\times (c\tau + d)^{-1/2} \int_{0}^{1} e^{-2\pi i \frac{mc}{c\tau + d} \left( x - \frac{r}{2mc} + (\sigma+2mn)\frac{x^2 + d}{2mc} \right)^2} \, dx. \tag{A.9}
\]

Using the decomposition \( r = \rho + 2m\alpha + 2ml \) with \( 0 \leq \alpha \leq c - 1 \) and \( l \in \mathbb{Z} \), we obtain

\[
\sum_{r = \rho \mod(2m)} e^{2\pi i \left( \frac{\rho^2}{4m} - \frac{s\rho}{2mc} + \frac{d}{c} \frac{s^2}{4m} \right)} \int_{0}^{1} e^{-2\pi i \frac{mc}{c\tau + d} \left( x - \frac{r}{2mc} - n\frac{d}{c} + (\sigma+2mn)\frac{x^2 + d}{2mc} \right)^2} \, dx
\]

\[
= (c\tau + d)^{1/2} \frac{1}{(2mci)^{1/2}} \sum_{\alpha = 0}^{c-1} e^{2\pi i \left( \frac{\rho^2}{4m} - \frac{s\rho}{2mc} + \frac{d}{c} \frac{s^2}{4m} \right)}, \tag{A.10}
\]

with \( s = \sigma + 2mn \), and we have used the fact that the sum over \( r \) splits into a sum, first, over \( \alpha \) and then over \( l \), with the later being absorbed in a redefinition of \( x \), extending the integral to the real line. On the other hand, we can show that the sum

\[
\sum_{\alpha = 0}^{c-1} \exp \left[ 2\pi i \left( \frac{\rho^2}{4m} - \frac{s\rho}{2mc} + \frac{d}{c} \frac{s^2}{4m} \right) \right],
\]

with \( s = \sigma + 2mn \), is invariant under either \( \rho \to \rho + 2ml_1 \) or \( s \to s + 2ml_2 \), with \( l_1, l_2 \in \mathbb{Z} \). To see this, note that a shift of \( \rho \) by \( 2ml \) in (A.11) is equivalent to a shift of \( \alpha \) by \( l \). Since we are summing over the equivalence class \( \alpha \in \mathbb{Z}/c\mathbb{Z} \), this shift is innocuous in the sum over \( \alpha \). To show invariance under \( s \to s + 2ml \), the first step is to rewrite the exponential in (A.11) as

\[
2\pi i \left( \frac{\rho^2}{4m} - \frac{s\rho}{2mc} + \frac{d}{c} \frac{s^2}{4m} \right) =
\]

\[
2\pi i \left( \frac{\rho^2}{4m} - \frac{s\rho}{2mc} + \frac{d}{c} \frac{s^2}{4m} \right) + \frac{b\sigma(\rho + 2m\alpha)}{2m} - \frac{bd}{4m} \sigma^2. \tag{A.12}
\]

The shift of \( s \) by \( 2ml \) can be absorbed in \( \alpha \), shifting by \( -dl \). Since the sum over \( \alpha \) is independent of these shifts, as explained previously, we find that the sum (A.11) only depends on the equivalence class of \( s \), that is, on \( \sigma \).

Introducing the result (A.10) back in (A.9) we obtain finally

\[
(\theta_{m,\rho}\mid \gamma)(\tau, z) = \sum_{\sigma \mod 2m} \sum_{n \in \mathbb{Z}} \sum_{r = \rho \mod(2m)} q^{(\sigma+2mn)^2/4m} g^{(\sigma+2mn)} \frac{1}{(2mci)^{1/2}} \sum_{\alpha = 0}^{c-1} e^{2\pi i \left( \frac{\rho^2}{4m} - \frac{s\rho}{2mc} + \frac{d}{c} \frac{s^2}{4m} \right)} \theta_{m,\sigma}(\tau, z), \tag{A.13}
\]
and hence by (A.6) we conclude
\[ K_{\rho,\sigma}(\gamma) = \frac{1}{(2mc)^{1/2}} \sum_{\alpha=0}^{c-1} e^{2\pi i \left( \frac{a (\rho + 2m\alpha)^2}{4m} - \frac{\sigma (\rho + 2m\alpha)}{2m} + \frac{d^2}{4m} \right)}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}). \quad (A.14) \]

### A.1 Some properties of $K(\gamma)_{\rho\sigma}$

In the following, we show explicitly that the Kloosterman formula (A.14) obeys the group property
\[ \sum_{\lambda=1}^{2m} K_{\rho,\lambda}(\gamma) K_{\lambda,\sigma}(\gamma') = K_{\rho,\sigma}(\gamma\gamma'), \quad \gamma, \gamma' \in SL(2, \mathbb{Z}), \quad (A.15) \]
as expected from the definition (A.6).

To do that, we need a few other properties of the formula (A.14). From the previous analysis of (A.11), we can conclude
\[ K_{\rho,\sigma}(\gamma) = K_{\rho+2ml,\sigma}(\gamma), \quad K_{\rho,\sigma}(\gamma) = K_{\rho,\sigma+2ml}(\gamma), \quad l \in \mathbb{Z}. \quad (A.16) \]
That is, the Kloosterman sums $K_{\rho,\sigma}$ (A.14) only depend on the equivalence class of $\rho \in \mathbb{Z}/2m\mathbb{Z}$ and $\sigma \in \mathbb{Z}/2m\mathbb{Z}$.

Given (A.16), we can rewrite
\[ \sum_{\lambda=1}^{2m} K_{\rho,\lambda}(\gamma) K_{\lambda,\sigma}(\gamma') = \frac{1}{cc'} \sum_{\lambda=1}^{2mcc'} K_{\rho,\lambda}(\gamma) K_{\lambda,\sigma}(\gamma'), \quad (A.17) \]
with
\[ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \gamma' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in SL(2, \mathbb{Z}). \]

Then we have
\[ \sum_{\lambda=1}^{2m} K_{\rho,\lambda}(\gamma) K_{\lambda,\sigma}(\gamma') = \]
\[ = \frac{1}{cc'} \frac{1}{2mi\sqrt{cc'}} \sum_{\alpha=0}^{c-1} \sum_{\beta=0}^{c'-1} \exp \left[ \pi i \frac{a \alpha}{2mc} (\rho + 2m\alpha)^2 + 2\pi i m \frac{a'}{c} \beta^2 - 2\pi i \frac{\sigma \beta}{c'} + \frac{\pi i \frac{d'}{cc'} \lambda^2}{2m c'^2} \right] \]
\[ \times \sum_{\lambda=1}^{2mcc'} \exp \left[ \pi i \frac{\frac{d''}{cc'} \lambda^2}{2mcc'} + \pi i \lambda \left( 2\frac{a'}{c'} \beta - \frac{1}{mc} (\rho + 2m\alpha) - \frac{\sigma}{mc'} \right) \right], \quad (A.18) \]
with
\[ \gamma\gamma' = \begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix} \in SL(2, \mathbb{Z}). \]

In the sum (A.18) we can use Gauss’s reciprocity formula
\[ \sum_{\lambda \mod(n)} \exp \left[ \pi i \frac{m^2 \lambda^2}{n} + 2\pi i \psi \lambda \right] = \sqrt{\frac{in}{m}} \sum_{\lambda \mod(m)} \exp \left[ -\pi i \frac{n}{m} (\lambda + \psi)^2 \right], \quad (A.19) \]
with \( n, m \in \mathbb{Z} \), \( nm \in 2 \mathbb{Z} \) and \( n\psi \in \mathbb{Z} \). Note that the sum over \( \lambda \) in (A.18) only depends on \( \lambda \mod (2mc') \) and hence, using the reciprocity formula we can write

\[
\sum_{\lambda=1}^{2mcc'} \exp \left[ \pi i \frac{e''}{2mcc'} \lambda^2 + \pi i \lambda \left( 2 \frac{a'}{c'} \beta - \frac{1}{mc} (\rho + 2ma) - \frac{\sigma}{mc'} \right) \right] = \\
\sqrt{2imcc'} \sum_{\lambda=0}^{c''-1} \exp \left[ -2\pi i \frac{mcc'}{c''} \left( \lambda + \frac{a'}{c'} \beta - \frac{1}{2mc} (\rho + 2ma) - \frac{\sigma}{2mc'} \right)^2 \right].
\]

Thus we find

\[
\sum_{\lambda=1}^{2m} K_{\rho,\lambda}(\gamma) K_{\lambda,\sigma}(\gamma') = \\
= \frac{1}{cc'} \frac{1}{\sqrt{2mc''}} \sum_{\alpha=0}^{c-1} \sum_{\beta=0}^{c'-1} \exp \left[ \pi i \frac{a}{2mc} (\rho + 2ma)^2 + 2\pi i \frac{a'}{c'} \beta^2 - 2\pi i \sigma \beta \right] + \frac{2\pi im}{2mc} \sigma^2 \\
\times \sum_{\lambda=0}^{c''-1} \exp \left[ -2\pi i \frac{mcc'}{c''} \left( \lambda + \frac{a'}{c'} \beta - \frac{1}{2mc} (\rho + 2ma) - \frac{\sigma}{2mc'} \right)^2 \right]. \tag{A.20}
\]

Summing over \( \lambda \) is the same as summing over equivalence classes of \( \alpha \) since \( \lambda - \alpha/c = (c\lambda - \alpha)/c \). Therefore, we can absorb the sum over \( \lambda \) in \( \alpha \) to obtain

\[
\sum_{\lambda=1}^{2m} K_{\rho,\lambda}(\gamma) K_{\lambda,\sigma}(\gamma') = \\
= \frac{1}{cc'} \frac{1}{\sqrt{2mc''}} \sum_{\alpha=0}^{c-1} \sum_{\beta=0}^{c'-1} \exp \left[ \pi i \frac{a}{2mc} (\rho + 2ma)^2 + 2\pi i \frac{a'}{c'} \beta^2 - 2\pi i \sigma \beta \right] + \frac{2\pi im}{2mc} \sigma^2 \\
\times \exp \left[ -2\pi i \frac{mcc'}{c''} \left( \frac{a'}{c'} \beta - \frac{1}{2mc} (\rho + 2ma) - \frac{\sigma}{2mc'} \right)^2 \right].
\]

Similarly, since \( \beta a'/c' - \sigma/2mc' = (2ma' \beta - \sigma)/2mc' \), the sum over \( \beta \) can be traded by a sum over equivalence classes of \( \sigma \). Moreover, under \( \sigma \to \sigma + 2ma' \beta \) we have

\[
2\pi i \frac{a'}{c'} \beta^2 - 2\pi i \frac{\sigma \beta}{c'} + \frac{2\pi im}{2mc} \sigma^2 \to \frac{2\pi i}{2mc} \sigma^2 + 2\pi i \mathbb{Z}. \tag{A.21}
\]

Since a shift of \( \sigma \) by \( 2ml \) with \( l \in \mathbb{Z} \) is innocuous in \( K_{\lambda,\sigma}(\gamma) \), the sum over \( \beta \) gives an exact \( c' \)
factor. Therefore we get

\[ \sum_{\lambda=1}^{2m} K_{\rho,\lambda}(\gamma)K_{\lambda,\sigma}(\gamma') = \]

\[ = \frac{1}{c} \frac{1}{\sqrt{2mc''}} \sum_{\alpha=0}^{c''-1} \exp \left[ \frac{\pi i}{2mc} \left( \frac{a}{2m} \rho + 2m\alpha \right)^2 + \frac{\pi i}{2mc} \omega' \right] \]

\[ \times \exp \left[ -2\pi i \frac{mc}{c''} \left( \frac{1}{2mc} \left( \rho + 2m\alpha \right) + \frac{\sigma}{2mc} \right)^2 \right] \]

\[ = \frac{1}{c} \frac{1}{\sqrt{2mc''}} \sum_{\alpha=0}^{c''-1} \exp \left[ 2\pi i \left( \frac{a'' (\rho + 2m\alpha)^2}{c''} - \frac{\sigma (\rho + 2m\alpha)}{2mc''} + \frac{d'' \sigma^2}{c'' 4m} \right) \right]. \]

(A.22)

We now use the property that the sum over \( \alpha \) in (A.22) depends only on the equivalence class \( \alpha \mod(c'') \). This gives a factor of \( c \). We obtain finally

\[ \sum_{\lambda=1}^{2m} K_{\rho,\lambda}(\gamma)K_{\lambda,\sigma}(\gamma') = \]

\[ = \frac{1}{\sqrt{2mc''}} \sum_{\alpha=0}^{c''-1} \exp \left[ 2\pi i \left( \frac{a'' (\rho + 2m\alpha)^2}{c''} - \frac{\sigma (\rho + 2m\alpha)}{2mc''} + \frac{d'' \sigma^2}{c'' 4m} \right) \right] = K_{\rho\sigma}(\gamma\gamma'), \]

(A.23)

as we wanted to show.

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