Do algebraic numbers follow Khinchin’s Law?*

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Abstract
The coefficients of the regular continued fraction for random numbers are distributed by the Gauss-Kuzmin distribution according to Khinchin’s law. Their geometric mean converges to Khinchin’s constant and their rational approximation speed is Khinchin’s speed. It is an open question whether these theorems also apply to algebraic numbers of degree > 2. Since they apply to almost all numbers it is, however, commonly inferred that it is most likely that non quadratic algebraic numbers also do so. We argue that this inference is not well grounded. There is strong numerical evidence that Khinchin’s speed is too fast. For Khinchin’s law and Khinchin’s constant the numerical evidence is unclear. We apply the Kullback Leibler Divergence (KLD) to show that the Gauss-Kuzmin distribution does not fit well for algebraic numbers of degree > 2. Our suggestion to truncate the Gauss-Kuzmin distribution for finite parts fits slightly better but its KLD is still much larger than the KLD of a random number. So, if it converges the convergence is non uniform and each algebraic number has its own bound. We conclude that there is no evidence to apply the theorems that hold for random numbers to algebraic numbers.

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1 Introduction

The coefficients of the regular continued fraction of a uniformly distributed random number follow Khinchin’s Law. Already Gauss had guessed this and Kuzmin (1928) as well as Khinchin (1963), p. 64, theorem 33 proved it and gave an error bound for the convergence.

It is a common folklore that it is likely that algebraic numbers of degree $> 2$ also follow Khinchin’s Law, cf., e.g., the quote from the backcover of Lang’s Lang (1995):

One general idea is that algebraic numbers will exhibit a behaviour that is the same as almost all numbers in a probabilistic sense, except under very specific structural conditions, namely quadratic numbers. Results for almost all numbers (due to Khinchin) show an interplay between calculus and number theory.

Lang & Trotter (1972) made $\chi^2$-tests and concluded that the results are as expected. Bombieri & van der Poorten (1975), p. 141 even go so far to say:

There is no reason to believe that the continued fraction expansions of non-quadratic algebraic irrationals generally do anything other than to faithfully follow Khinchin’s Law as detailed below. Indeed experiment suggests that this is even true for parts, short relative to the length of the period, of the expansions of quadratic irrationals.

In contrast, we argue that the belief that Khinchin’s Law carries over to nonquadratic algebraic irrationals is not well grounded. Section 2 provides an overview of our main reasons for this claim. The following sections substantiate the reasons in more detail. Section 3 discusses continued fractions of algebraic numbers. Section 4 measures the goodness of fit for our newly proposed truncated Gauss-Kuzmin distribution in comparison to the standard Gauss-Kuzmin distribution. Section 5 shows that each algebraic number seems to have its own special behaviour regarding Khinchin’s constant. Section 6 discusses conjectures regarding the role of Khinchin’s Law, Khinchin’s constant and Khinchin’s approximation speed for algebraic numbers. Section 7 draws the conclusion.

2 Reasons for Believing that Algebraic Numbers do not follow Khinchin’s Law

For the following reasons, we question that Khinchin’s Law carries over to algebraic numbers with degree $> 2$.

1. Parts of the period of quadratic irrationals show that the $\chi^2$ tests are not reliable.

2. The Kuzmin distribution is unbounded, while the coefficients of finite parts of the regular continued fraction of algebraic numbers are bounded due to Liouville’s
theorem. Furthermore, it is known that the Liouville bounds can be improved. Each number has its own bound and this should be reflected in the distribution or the error bound of its convergence, if it converges. It can presently not be ruled out that all coefficients of the regular continued fraction of algebraic numbers are bounded. This is an old, open question.

3. Since this is an open question, an account is needed that can also deal with the case of bounded coefficients. We suggest to truncate the Gauss-Kuzmin distribution for finite parts of nonquadratic algebraic numbers to deal with this. The results show that their KLD is much larger than the KLD of a random number.

4. Even if the distribution converges to the Gauss-Kuzmin distribution, the numerical evidence shows that the distribution is non uniform. Without clear-cut calculable error bounds this does not help much for finite parts. The error bounds for algebraic numbers, however, are different from those for the random numbers and each algebraic number has its own error bound.

5. The convergence behaviour of the geometric means of the coefficients of the regular continued fractions is very different for each algebraic number. If it converges the convergence is non uniform and each number has its own specific error bound. For a uniformly distributed random number, however, it converges quite fast to Khinchin’s constant.

6. For a typical random number \( r_n = \frac{B(n)}{B(n-1) \log(B(n-1))} > 1 \) has an infinite number of solutions (\( \frac{A(n)}{B(n)} \) are the convergents of the random number). But for algebraic numbers numerical evidence by Lang & Trotter (1972), p. 117, Table B does not support this. For the given 6 algebraic numbers of the 3rd degree there are only 8 such cases for \( n = 1000 \) to 3000 and for \( \sqrt[3]{7} \) there is not a single such case. This is worse than for the first 1000 convergents, where \( \sqrt[3]{2} \) already has 6 such cases. This is numerical evidence that the distribution does not converge to the Gauss-Kuzmin distribution within the margin of the error bounds of random numbers. If it converges, the rate of convergence is slower.

7. There is strong numerical evidence that for any algebraic number \( a \) a constant \( K(a) > 0 \) exists with \( b_n < K(a)n \) for all natural numbers \( n \) with \( b_n \) being the coefficients of the regular continued fraction of \( a \). So, the arithmetic means of the \( b_n \) of algebraic numbers are most probably bounded (like for quadratic irrationals), while for random numbers they diverge to infinity.

8. If such a constant \( K(a) > 0 \) exists, then it can be proven that Levy’s error bound for the convergence for random numbers to the Gauss-Kuzmin distribution is not valid for algebraic numbers.
3 Continued fractions of algebraic numbers

This section considers the distribution of the coefficients of regular continued fractions of algebraic numbers with degree $> 2$. Lang & Trotter (1972) aim to show that these coefficients follow a Gauss-Kuzmin distribution. The distribution is defined as follows. Consider a continued fraction expansion of a random number $x$ uniformly distributed in $(0,1)$:

$$ x = \frac{1}{k_1 + \frac{1}{k_2 + \ldots}}. $$

Then the following holds asymptotically for the distribution of the coefficients $k_n$:

$$ \lim_{n \to \infty} P(k_n = k) = - \log_2(1 - \frac{1}{(k+1)^2}). $$

The following error bound with a constant $C > 0$ was given by Lévy (1029). With

$$ x_n = \frac{1}{k_n + \frac{1}{k_{n+1} + \ldots}} $$

the probability $|P(x_n \leq s) - \log_2(1 + s)|$ converges for all $0 \leq s \leq 1$ to 0 and Levy proved the error bound

$$ |P(x_n \leq s) - \log_2(1 + s)| \leq C \cdot 0.7^n. $$

This distribution is known to be the Gauss-Kuzmin distribution. The approximation of the Gauss-Kuzmin distribution is an asymptotic result for uniformly distributed random numbers. Lang & Trotter (1972) claim that this result carries over to algebraic numbers of degree $> 2$. But Lang & Trotter (1972) does not give error bounds for the convergence. This is problematic as the Gauss-Kuzmin distribution is unbounded by definition while the coefficients of finite parts of the continued fraction expansion are bounded by Liouville’s theorem.

Liouville’s theorem says that for an algebraic number $a$ of a degree $n > 2$ there exists a positive constant $C(a)$ with

$$ |a - \frac{p}{q}| > \frac{C(a)}{q^n} $$

for all natural numbers $p$ and $q$.

Liouville’s bound is very rough. In general, each algebraic number of degree $> 2$ has its own bound. Because the Thue Siegel Roth theorem (proven by Roth (1955)) provides only an upper bound for a measure of the speed of the best rational approximation, it is an open question to what extent it can be improved for specific algebraic numbers with effectively computable bounds. Voutier (2007) investigates this for $\sqrt[3]{2}$ and mentions the following famous result by Korobov (1990)
| $\sqrt{2} - \frac{p}{q} | > \frac{1}{q^{5.5}}$

for all natural numbers $p$ and $q$ with the exception of 1 and 4 for $q$.

Hence, effectively computable bounds exist, which are much sharper than Liouville’s. These sharper bounds could be used to truncate the Gauss-Kuzmin distribution for algebraic numbers, logarithms of rational numbers and other numbers for which the method with the hypergeometric series, explained by Voutier (2007), applies. It is still an open question, whether all periods have such bounds Waldschmidt (2006), p. 437, question 2. A period is a number that can be expressed as an integral of an algebraic function over an algebraic domain. It is difficult to obtain general results because each period has its very own behaviour. It seems to be especially difficult to obtain an optimal lowest upper bound. For certain periods like $\sqrt{2}$ Voutier (2007), $\zeta(2)$ Zudilin (2014) and $\pi$ Zeilberger & Zudilin (2020) this has been dealt with many times, always slightly improving the results and in all cases research is still going on. The history for $\pi$ is especially fascinating. Mahler (1953) started with the upper bound 42 for the effective irrationality measure (in Korobov’s formula this is 2.5 as exponent of $q$) and currently Zeilberger & Zudilin (2020) hold the record with 7.103205334137. Waldschmidt (2006), p. 437 poses a further problem:

A more ambitious goal would be to ask whether real or complex periods behave, from the point of view of Diophantine approximation by algebraic numbers, like almost all real or complex numbers.

In addition to study differences between random and algebraic numbers, it is an important, though even more difficult endeavour to investigate differences between random numbers and periods. At least for certain classes of periods, it should be possible to obtain results.

Since the Gauss-Kuzmin distribution does not hold for finite parts of the continued fractions of algebraic numbers, we suggest to take this into account by a truncated version of the Gauss-Kuzmin distribution. The truncated Gauss-Kuzmin distribution for a finite part is defined by the probability function

$$P_{\text{trunc}K}(k_n = k) = \frac{P_k(k)}{1 - \log_2(\frac{1}{1 + \text{maxn}})}.$$ (1)

$P_k(k)$ denotes the probability function of the standard Gauss-Kuzmin distribution for $k \leq \text{maxn}$ and 0 for $k > \text{maxn}$, where $\text{maxn}$ is the maximum of the coefficients of the regular continued fraction of the algebraic number for that finite part.

By truncating the Gauss-Kuzmin distribution, we therefore open up the possibility that the coefficients of the regular continued fraction representation of algebraic numbers are bounded in general. For infinite unbounded continued fractions our distribution approaches the standard Gauss-Kuzmin distribution, which is thus also embedded in our account.
4 Measuring the goodness of fit of the truncated Kuzmin distribution

Lang & Trotter (1972) used $\chi^2$ goodness of fit tests to show that the coefficients of continued fractions of algebraic numbers follow a Gauss-Kuzmin distribution. They draw their conclusions from non-rejections of the $\chi^2$ tests. However, as is well known, the non-rejection of a null hypothesis is no proof for the hypothesis to be true since a type II error with unknown error probability occurs. In addition to this, the $\chi^2$ test is known to be unreliable. It is justified mainly because it is easy to use, particularly, when the access to computer power is limited.

Nevertheless the data from the table from Lang & Trotter (1972), p. 220 are compatible with our approach. The first 5 columns are as follows.

| Number | $\chi^2$ (n=1000) | P   | $\chi^2$ (n=3000) | P   |
|--------|------------------|-----|------------------|-----|
| $\sqrt{2}$ | 4.61            | 0.13| 5.59             | 0.22|
| $\sqrt{3}$ | 8.41            | 0.51| 10.33            | 0.68|
| $\sqrt{4}$ | 8.47            | 0.51| 7.71             | 0.44|
| $\sqrt{5}$ | 8.07            | 0.47| 9.48             | 0.61|
| $\sqrt{7}$ | 10.22           | 0.67| 13.32            | 0.85|

Table 1: $\chi^2$ test by Lang and Trotter

P is the approximate probability that $\chi^2$ for a random sample would not be larger. The hypothesis that random behaviour cannot be rejected is in line with our expectation. But we claim that if it converges, the convergence is non uniform. In 4 of 5 cases, P is further away from 0.5 for n=3000 than for n=1000. So it seems to be very difficult to get precise error bounds, if it converges.

As $\chi^2$ test are not very reliable, we revisit this question by using the Kullback Leibler Divergence (KLD) instead. We do not merely test for one possible distribution whether it fits the data but also compare two different distributions in how they fit the data.

In the following, log is used and 2 is the base throughout. The Kullback Leibler Divergence for the discrete distributions $P$ and $Q$ is defined by

$$KLD(P,Q) = \sum_x P(x) \log \frac{P(x)}{Q(x)}.$$  \hspace{1cm} (2)

We have calculated the KLD for 1000 coefficients for algebraic numbers and the standard Gauss-Kuzmin distribution. Table 2 shows the results for some roots of 2.

The values in Table 2 can be compared to the KLDs of the regular continued fractions of pseudo random numbers. We choose six pseudo random numbers with 1000 decimal digits and calculated the KLDs rounded to 4 digits: 0.0836, 0.0603, 0.0836, 0.0573, 0.0802, 0.0718. This, as well as our calculations of the KLDs for further algebraic numbers, indicates that the Gauss-Kuzmin distribution fits much better to the pseudo random numbers than to the algebraic numbers, which shows that the error bounds of
the algebraic numbers are worse, if it converges. Therefore, we suggest to truncate the Gauss-Kuzmin distribution as defined in equation (1) for finite parts of the regular continued fraction of each algebraic number with a different bound using the properties of the specific number. Note that this truncation of the distribution is different from the truncation of the Gauss-Kuzmin law, which is used by Hensley (1988).

Table 3 provides our calculations of the KLDs for the truncated distribution for 1000 coefficients of the numbers from Table 2.

| Algebraic number | KLD   |
|------------------|-------|
| $\sqrt{2}$      | 0.0953|
| $\sqrt{2}$      | 0.0730|
| $\sqrt{2}$      | 0.0884|
| $\sqrt{2}$      | 0.1102|
| $\sqrt{2}$      | 0.1114|
| $\sqrt{2}$      | 0.0920|

Table 3: KLD for the truncated Kuzmin distribution rounded to 4 digits

For the same 6 pseudo random numbers as before we obtain the KLDs 0.0832, 0.0592, 0.0825, 0.0563, 0.0787, 0.0669. The KLDs of the truncated Gauss-Kuzmin distribution are always lower than those of the Gauss-Kuzmin distribution. Note that even the truncated Gauss-Kuzmin distribution still has a quite large KLD compared to the random numbers. So, it seems likely to consider each number’s own distribution and error bounds, if it converges.

The KLDs of the truncated Gauss-Kuzmin distribution also indicate that the Gauss-Kuzmin distribution itself is slightly too large for coefficients of nonquadratic algebraic numbers because the KLDs of the truncated Gauss-Kuzmin distribution are always better. This can also be concluded from the fact that both, the standard and the truncated Gauss-Kuzmin distribution, are only defined for positive numbers. Thus, the probability mass of the standard Gauss-Kuzmin distribution of values greater than our truncation is redistributed over the finite interval of our truncated version according to the Gauss-Kuzmin distribution.
Furthermore, it is not even clear whether the truncated distributions converge to the Gauss-Kuzmin distribution, because it is still an open question whether the coefficients are bounded or not. Adamczewski & Bugeaud & Davison (2006) discuss this and give more sources.

For these reasons, we suggest to use the truncated Gauss-Kuzmin distribution. This might also be interesting for $\pi$, $\log(2)$ and other numbers. This is related to Lang’s open conjecture that the approximation speed $B(n)^2 \log(B(n))$ is the best possible for his classical numbers as defined in Lang (1971), p. 635. This means that if $A(n)/B(n)$ with $B(n) > 0$ is a rational sequence converging to the classical number $a$, then for every $\varepsilon > 0$ the product $|a - A(n)/B(n)| B(n)^2 (\log(B(n)))^{1+\varepsilon}$ is always bounded away from 0 Lang (1971), p. 664.

One may go even further and conjecture that this speed is already too fast and that $|a - A(n)/B(n)| B(n)^2 (\log(B(n)))$ is always bounded away from 0 for Lang’s classical numbers. For quadratic irrationals, for which $B(n)^2$ is the well-known approximation speed, and for Euler’s number Lang (1995), p. 76, theorem 5 this is proven but otherwise it is an open question.

5 Khinchin’s constant

Another indication that the speed of the rational approximation of each algebraic number of degree $> 2$ should be investigated separately is Khinchin’s constant $K_0$. Bailey&Borwein&Crandall (1997), p. 423 write:

It is remarkable that, even though a random fraction’s limiting geometric mean exists and furthermore equals the Khinchin constant with probability one, not a single explicit real number (e.g., a real number cast in terms of fundamental constants) has been demonstrated to have elements whose geometric mean equals $K_0$.

Let us abbreviate the truncated Gauss-Kuzmin distributions at $\max n$ by $GKT(\max n)$. To test the hypothesis that finite parts of the regular continued fraction of algebraic numbers of degree $> 2$ are distributed with $GKT(\max n)$, we first calculate the value $KC(\max n)$ of the limit of the geometric means of the coefficients of the regular continued fraction of random numbers distributed with $GKT(\max n)$:

$$KC(1) = 1, \quad KC(\max n) = 2^{GKT(2)} \cdots \max n^{GKT(\max n)} \text{ for } \max n \geq 2.$$

This sequence converges for $\max n \to \infty$ monotonously increasing to Khinchin’s constant $K_0$ as expected.

This theoretical result for random numbers can now be compared to numerical calculations of the geometrical mean for algebraic numbers. As Table 4 shows for our test cases, they do not behave like it is expected for the case of random numbers distributed with a truncated Gauss-Kuzmin distribution, since the geometric mean is larger than $K_0$.
| Number          | Geometric mean |
|-----------------|----------------|
| random number   | 2.685          |
| \(\sqrt{3}\)   | 2.735          |
| \(\sqrt[4]{3}\) | 2.742          |
| \(\sqrt{3}\)   | 2.671          |
| \(\sqrt[5]{3}\) | 2.696          |
| \(\sqrt[6]{3}\) | 2.711          |
| \(\sqrt[7]{3}\) | 2.692          |

Table 4: Geometric mean for the first 10000 coefficients

So, if the geometric means \(GM\) converge, then this is non uniform and each number has its own error bound. Bailey&Borwein&Crandall (1997), p. 425 conjecture the following error bound for their example:

\[
|K_0 - GM(n)| < \frac{C}{n^{0.3}}
\]

with \(C > 0\).

In the case of \(\sqrt{3}\) this leads to \(C = 5.7\) for \(n = 10000\). This seems reasonable for algebraic numbers, if their geometric means converge. Numerical evidence supports that the convergence for random numbers is much faster.

Similar results are known from other numbers, cf. Shiu (1995), p. 1315 for \(\pi\):

\[...\text{we wish to point out that even if there is a convergence, the rate has to be very slow. It is easy to see that, with } n = 10000, \text{ the change in value of any single partial quotient will have an effect on the third decimal digit for the value of } K(a,n). \]

In fact we found that \(K(\pi,10000)\) differs from \(K\) by more than \(K(\pi,100)\) does.

So the convergence behaviour to Khinchin’s constant shows that the truncated distributions do not fit better than the Gauss-Kuzmin distribution for algebraic numbers.

6 The Role of Randomness

The following theorems can be proven for random numbers by probabilistic means:

**T1:** Khinchin’s Law holds for the distribution of the coefficients of the regular continued fraction Khinchin (1963), pp. 92f.

**T2:** Khinchin’s constant is the limit of the geometric means of those coefficients Khinchin (1963), p. 93.

**T3:** Khinchin’s approximation speed \(B(n)^2 \log(B(n))\) holds Khinchin (1963), p. 69. The convergents of the random number are \(A(n)/B(n)\).
But what role do these theorems play for nonquadratic algebraic numbers? We state the following conjectures for nonquadratic algebraic numbers:

**C1:** Khinchin’s Law is near the distribution and might be valid but precise error bounds for each number are needed as the convergence is non uniform and slower than for random numbers.

**C2:** Khinchin’s constant is near the geometric means for large \( n \) and might be the limit but precise error bounds are needed as the convergence is much slower than for random numbers.

**C3:** Khinchin’s approximation speed \( B(n)^2 \log(B(n)) \) does not hold; it is an unreachable upper bound.

According **C1**, there is an upper bound for the coefficients for finite parts. And even when the Gauss-Kuzmin distribution is truncated for finite parts, it is still only near the real distribution. Using the KLDs, it might be interesting to investigate further what “near” means exactly in this context and to search for precise error bounds for each specific number.

According **C2**, it seems to be very difficult to be more precise here.

**C3** means that if \( A(n)/B(n) \) with \( B(n) > 0 \) is a rational sequence converging to the algebraic number \( a \), then \( |a - A(n)/B(n)| B(n)^2 \log(B(n)) \) always diverges to infinity. \( **C3** \) can even be generalised for periods and Lang’s classical numbers. It can be proven for quadratic irrationals. For Euler’s number, Adams (1966) proved the speed \( B(n)^2 \frac{\log(B(n))}{\log(\log(B(n)))} \), see also Lang (1995), p. 74. For nonquadratic algebraic numbers, the question is open.

It is even open, whether the coefficients are bounded at all. Our calculation seems to indicate that Khinchin’s speed is too fast for algebraic numbers, which agrees with the experimental evidence reported by Lang & Trotter (1972), p. 116:

The tables [for the first 3000 terms of the continued fractions of the cubic numbers] therefore suggest that the type may in fact not be bigger than a constant times the logarithm, and may even be of an order of magnitude smaller than the logarithm.

We suggest the new speed \( B(n)^2 \frac{\sqrt{\log(B(n))}}{\log\log(B(n))} \) and have tested it together with Khinchin’s speed for \( \sqrt{2} \) for the first 3000 convergents:

So, there is strong numerical evidence that Khinchin’s speed is too fast, while the new speed fits much better. The 27 measurements larger than 100 merely indicate that the convergents are relatively bad.

Our result basically agrees with Lang & Trotter (1972), p. 118, Table I and p. 122 Table III, where the authors also investigate other algebraic numbers and come to the conclusion that good measurements with Khinchin’s speed are very rare. They investigate \( r_n = \frac{B(n)}{B(n-1)\log(B(n-1))} \) and observe that the values for \( r_n \) basically decrease.
Table 5: Speed measurements

Another indication for C3 is the connection of the speed for algebraic numbers with the solutions of diophantine equations. Experimental evidence shows that the speed is slow, when only small solutions exist, cf. Smart (1998), p. 135:

Folklore. If a diophantine equation has only finitely many solutions then those solutions are small in ‘height’ when compared to the parameters of the equation. This folklore is, however, only widely believed because of the large amount of experimental evidence which now exists to support it.

From the strongest possible form of the ABC conjecture it follows that only the speed $B(n)^2 \exp(\sqrt[3]{\log(B(n))})$ is reachable by nonquadratic algebraic numbers of degree Frankenhuysen (1999), p. 46, formula(1.1). Yet, here all ABC equalities are considered and the prime factors contribute to the quality, too. We conjecture that no proper large solutions to resulting equations exist because each fixed algebraic numbers can only have finitely many large ABC hits. Hence, Roth’s theorem can be sharpened further. Van Frankenhuysen observes that there is numerical evidence for $O(\log(b_n)) = O(\log(n))$ in his example. This would imply that C3 is valid for this case.

The data of Lang & Trotter (1972), p. 118, Table I and p. 122 Table III indicate that for all algebraic numbers $a$ there exists a constant $K(a)$ with $b_n < K(a)n$ for all natural numbers $n$. From their data the following values for $K(a)$ can be calculated:

Table 6: K measurements until n=1000

All $K(a)$ are relatively small, which is numerical evidence for C3. It follows that the distribution cannot converge to the Kuzmin distribution within the error bounds for random numbers. This can be seen as follows. Set in Levy’s uniform error bound

$| P(x_n <= s) - \log_2(1 + s) | <= C \cdot 0.7^n$

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Table 7: K measurements from n=1000 until n=3000

| Number | K(a)  | n of maximum | r(n) |
|--------|-------|--------------|------|
| √2     | 10.694| 1191         | 5.5  |
| √3     | 3.971 | 2407         | 3.3  |
| √4     | 3.226 | 1974         | 2.7  |
| √5     | 15.807| 1196         | 13.8 |

the sequence \( s_n = \frac{1}{K(a)n} \). Then \( P(x_n <= s_n) = 0 \) and \( \log_2(1+s_n) \) is unbounded. So \( C \) does not exist and the convergence behaviour must be different from the random numbers. Furthermore, all \( K(a) \) strongly decrease. This is also evidence that C3 is valid. For random numbers, on the other hand, the inequality \( b_n > n \cdot \log(n) \) is valid for infinitely many \( n \). Thus, for random numbers the arithmetic means of the coefficients diverges to infinity (see Khinchin (1963), p.93f), while for algebraic numbers we conjecture that the arithmetic means are bounded due to the numerical evidence.

It is possible to prove C3 for numbers such as quadratic irrationals, \( e \) and other numbers with regular continued fractions of Hurwitz’ type. For numbers resulting from differential equations \( y' = \frac{P(x,y)}{Q(x,y)} \) (or \( y'' \) etc.) with rational polynomials \( P(x,y) \) and \( Q(x,y) \), C3 can be proven for many cases using the solution via power series and C fractions, which are then transformed into continued fractions with positive natural numbers \( a_n \) and \( b_n \). If all \( a_n = 1 \), C3 is valid unconditionally. Yet, this cannot always be achieved by this method. One example is the hypergeometric differential equation, where Gauss gave the continued fractions of the solutions. Especially interesting are the confluent hypergeometric functions (see Jones & Thron (1980), chapter 6.1.2, p. 205-211). In the resulting continued fractions with positive natural \( a_n \) and \( b_n \), the \( b_n \) are linear in \( n \) and so Khinchin’s speed is too fast. One good example is tanh(z) with \( a_0 = z \), \( a_n = z^2 \), \( b_n = 2n + 1 \) (formula 6.1.56 of Jones & Thron (1980)). For all natural numbers \( z = m \) this is the regular continued fraction and Khinchin’s speed is too fast.

Due to the numerical evidence and the fact that there are no known counterexamples to C3, be it as a conjecture applied to algebraic numbers, periods or Lang’s classical numbers, we conjecture it for these kind of numbers as a challenge for future research.

The conjectures C1 to C3 compare, in a way, real randomness with pseudo randomness and state that the pseudo randomness applying to specific numbers defined by equations is near real randomness but does not reach it exactly. This is known from other generators of pseudo random numbers as well.

7 Conclusion

This paper revisits the question whether certain properties of random numbers carry over to algebraic numbers. For random numbers the coefficients of the regular continued fraction asymptotically follow a Gauss-Kuzmin distribution and the convergence has clear error bounds. For algebraic numbers of degree > 2 this seems implausible because
numerical evidence shows that the convergence is non uniform, if it converges. When the coefficients of the regular continued fractions are bounded, it does not converge at all. We therefore propose a new truncated Gauss-Kuzmin distribution to model the distribution of the coefficients of finite parts of the regular continued fraction of algebraic numbers of degree $> 2$. We apply the Kullback-Leibler Divergence to show that our truncated Gauss-Kuzmin distribution gives a better fit than the standard Gauss-Kuzmin distribution. This finding is underpinned by simulation results for a variety of algebraic numbers of degree $> 2$. Yet, the KLDs are still quite large. Furthermore, the convergence behaviour of the geometric means is very different. It is not even clear, whether the means converge to Khinchin’s constant or not. Likewise, it is still an open question, whether Khinchin’s speed applies to algebraic number of degree $> 2$. There is, in fact, strong numerical evidence that Khinchin’s speed is too fast. These questions are also interesting to consider for larger classes of numbers like periods or Lang’s classical numbers. Probably each such number has its own distribution and error bound. In any case, great care is required when laws that apply to random numbers and, thus, to almost all numbers from a probabilistic point of view, are considered to carry over to specific numbers.

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