Curvature of (Special) Almost Hermitian Manifolds

Francisco Martín Cabrera and Andrew Swann

Abstract. We study the curvature of almost Hermitian manifolds and their special analogues via intrinsic torsion and representation theory. By deriving different formulæ for the skew-symmetric part of the $\ast$-Ricci curvature, we find that some of these contributions are dependent on the approach used, and for the almost Hermitian case we obtain tables that differ from those of Falcitelli, Farinola & Salamon. We show how the exterior algebra may used to explain some of these variations.

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1 Introduction

In [17], Tricerri and Vanhecke gave a complete decomposition of the Riemannian curvature tensor $R$ of an almost Hermitian manifold $(M, I, \langle \cdot, \cdot \rangle)$ into irreducible $U(n)$-components. These divide naturally into two groups, one forming the space $\mathcal{K} = \mathcal{K}(u(n))$ of algebraic curvature tensors for a Kähler manifold, and the other, $\mathcal{K}^\perp$, being its orthogonal complement.

In [6], Falcitelli et al. showed that the components of $R$ in $\mathcal{K}^\perp$ are linearly determined by the covariant derivative $\nabla \xi$, where $\nabla$ is the Levi-Civita connection and $\xi$ is the intrinsic torsion of the $U(n)$-structure on $M$. Gray and Hervella [10] showed that in general dimensions $\xi$ may be split into four components $\xi_1, \ldots, \xi_4$ under the action of $U(n)$. By using the minimal $U(n)$-connection $\tilde{\nabla} = \nabla + \xi$ of $M$, Falcitelli et al. display some tables which show whether or not the tensors $\tilde{\nabla} \xi_i$ and $\xi_i \odot \xi_j$ contribute to the components of $R$ in $\mathcal{K}^\perp$. This provides a unified approach to many of the curvature results obtained by Gray [8].

The present paper is motivated by the interest in extending the above results to special almost Hermitian manifolds. These are defined as almost Hermitian manifolds $(M, I, \langle \cdot, \cdot \rangle)$ equipped with a complex volume form $\Psi = \psi_+ + i \psi_-$. Equivalently they are manifolds with structure group $SU(n)$.
A detailed study of the intrinsic torsion \( \eta + \xi \) of such manifolds was made in \cite{14}, extending results of Chiossi and Salamon \cite{4}. Here \( \xi \) is the intrinsic \( U(n) \)-torsion, as above, and \( \eta \) is essentially a one-form. There is much current interest in \( SU(n) \)-structures, partly as generalisations of Calabi-Yau manifolds \cite{7,1} and partly because of the rôle played by torsion connections with holonomy \( SU(n) \) in string theory \cite{15,11}.

For \( SU(n) \) structures, the algebraic curvature tensors lie in \( K(\text{su}(n)) \) and are automatically Ricci-flat. Therefore, one may compute the Ricci curvature \( \text{Ric} \), and indeed the \( \ast \)-Ricci curvature \( \text{Ric}^\ast \), in terms of the intrinsic \( SU(n) \)-torsion \( \eta + \xi \). This enables us to find information about those \( SU(n) \)-components of the Riemannian curvature \( R \) which are determined by the tensors \( \text{Ric} \) and \( \text{Ric}^\ast \). Some of these components are contained in \( K^\perp \) and others are contained in \( K \). This will allow us, on the one hand, to get more concrete information about some components of \( R \) contained in \( K^\perp \) and, on the other hand, to enlarge the tables of Falcitelli et al. with columns related with some components contained in \( K \).

In working out these contributions, we arrived at various alternative formulae for certain curvature components purely in terms of the intrinsic \( U(n) \)-torsion \( \xi \). This leads to some entries in the tables that are different from those obtained by Falcitelli et al. To try to account for this, we consider the identity \( d^2 = 0 \) in the exterior algebra. Applying this to the Kähler 2-form \( \omega \) and considering a particular component indeed leads to a non-trivial relation between the tensors contributing to the curvature. One may view the relation \( d^2 \omega = 0 \) as one way of taking account of some of the information that the Levi-Civita connection \( \nabla = \tilde{\nabla} - \xi \) is torsion-free.

The paper is organised as follows. In \( \S 2 \) we present some preliminary material: definitions, results, notation, etc. Then in \( \S 3 \) we derive some formulæ relating curvature and intrinsic torsion. As an immediate application, we give an alternative proof of Gray’s result \cite{9} that any nearly Kähler manifold of dimension six which is not Kähler is an Einstein manifold. We then proceed to computing the contributions of different components of the intrinsic torsion and its covariant derivative to the Ricci, \( \ast \)-Ricci and Riemannian curvatures. Because of the representation theory, this behaves differently in dimensions 4 and 6 than in higher dimensions: in dimension 6, \( \xi \) splits into more \( SU(3) \)-components; in dimension 4, the space of curvature tensors is decomposed more finely under the action of \( SU(2) \). This motivates us to display results and tables in two separate sections: \( \S 4 \) for high dimensions, \( 2n \geq 8 \), and \( \S 5 \) for dimensions six and four. Finally, in \( \S 6 \) we discuss identities derived from the exterior algebra.

We remark that in this paper we will often use decompositions of tensor products without providing explicit details, since such information can be readily obtained via available computer programs.
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2 Preliminaries

An almost Hermitian manifold is a $2n$-dimensional manifold $M$, $n > 0$, with a $U(n)$-structure. This means that $M$ is equipped with a Riemannian metric $\langle \cdot, \cdot \rangle$ and an orthogonal almost complex structure $I$. Each fibre $T_m M$ of the tangent bundle can be considered as a complex vector space by defining $ix = Ix$. We will write $T_m M_C$ when we are regarding $T_m M$ as such a space.

We define a Hermitian scalar product $\langle \cdot, \cdot \rangle_C = \langle \cdot, \cdot \rangle + i \omega(\cdot, \cdot)$, where $\omega$ is the Kähler form given by $\omega(x, y) = \langle x, Iy \rangle$. The real tangent bundle $TM$ is identified with the cotangent bundle $T^*M$ by the map $x \mapsto \langle \cdot, x \rangle = x$. Analogously, the conjugate complex vector space $T_m M_C$ is identified with the dual complex space $T^*_m M_C$ by the map $x \mapsto \langle \cdot, x \rangle_C = x_C$. It follows immediately that $x_C = x + iIx$.

If we consider the spaces $\Lambda^p T^*_m M_C$ of skew-symmetric complex forms, one can check that $x_C \wedge y_C = (x + iIx) \wedge (y + iIy)$. There are natural extensions of the scalar products $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_C$ to $\Lambda^p T^*_m M$ and $\Lambda^p T^*_m M_C$, defined respectively by

$$\langle a, b \rangle = \frac{1}{p!} \sum_{i_1, \ldots, i_p=1}^{2n} a(e_{i_1}, \ldots, e_{i_p})b(e_{i_1}, \ldots, e_{i_p}),$$

$$\langle a_C, b_C \rangle_C = \frac{1}{p!} \sum_{i_1, \ldots, i_p=1}^{n} a_C(u_{i_1}, \ldots, u_{i_p})b_C(u_{i_1}, \ldots, u_{i_p}),$$

where $e_1, \ldots, e_{2n}$ is an orthonormal basis for real vectors and $u_1, \ldots, u_n$ is a unitary basis for complex vectors.

The following conventions will be used in this paper. If $b$ is a $(0, s)$-tensor, we write

$$I_{(i)}b(X_1, \ldots, X_s) = -b(X_1, \ldots, IX_i),$$

$$Ib(X_1, \ldots, X_s) = (-1)^s b(IX_1, \ldots, IX_s).$$

In [17], Tricerri and Vanhecke gave a complete decomposition of the Riemannian curvature tensor $R$ of an almost Hermitian manifold $(M, I, \langle \cdot, \cdot \rangle)$ into irreducible $U(n)$-components. As was indicated above, some of these components, constituting a $U(n)$-space denoted by $\mathcal{K} = \mathcal{K}(u(n))$, are the
only components which can occur when \( M \) is a Kähler manifold. In this text we will follow the notation used in [6] for such components. Likewise, we will adopt the formalism used in [16] and [6] for irreducible \( U(n) \)-modules. Thus, for \( n \geq 2 \),

\[
\mathcal{K} = \mathcal{K}_3 + \mathcal{K}_1 + \mathcal{K}_2,
\]

where \( \mathcal{K}_3 \cong [\sigma_0^{2,2}] \), \( \mathcal{K}_1 \cong \mathbb{R} \), \( \mathcal{K}_2 \cong [\lambda_0^{1,1}] \) and + denotes direct sum. We recall that \( \lambda_0^{p,q} \) is a complex irreducible \( U(n) \)-module coming from the \((p,q)\)-part of the complex exterior algebra and its corresponding dominant weight in standard coordinates is given by \((1, \ldots, 1, 0, \ldots, 0, -1, \ldots, -1)\), where 1 and \(-1\) are repeated \( p \) and \( q \) times respectively.

By analogy with the exterior algebra, there are also irreducible \( U(n) \)-modules \( \sigma_0^{p,q} \) with dominant weights \((p,0,\ldots,0,-q)\) coming from the symmetric algebra. The notation \([V]\) means the real vector space underlying a complex vector space \( V \) and \([W]\) denotes a real vector space which admits \( W \) as its complexification.

Moreover, let \( \text{Ric} \) and \( \text{Ric}^* \) respectively be the Ricci and \(*\)-Ricci curvatures which are defined by

\[
\text{Ric}(X,Y) = \langle R_{X,e_i} Y, e_i \rangle, \quad \text{Ric}^*(X,Y) = \langle R_{X,e_i} IY, Ie_i \rangle,
\]

where \( R_{X,Y} = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y] \) and the summation convention is used.

The components of the curvature \( R \) in \( \mathcal{K}_1 \) and \( \mathcal{K}_2 \) are determined by the trace and the trace-free components of \( \text{Ric}_H + 3 \text{Ric}^*_H \) respectively (see [17]), where \( b_H \) indicates the Hermitian part of a bilinear form \( b \), i.e., the part satisfying \( b_H(IX, IY) = b_H(X, Y) \). Note that \( \text{Ric}^*_H \) coincides with the symmetric part of \( \text{Ric}^* \).

The remaining components of \( R \), not included in \( \mathcal{K} \), are contained in a \( U(n) \)-space denoted by \( \mathcal{K}^\perp \). For \( n \geq 4 \), one has [6]

\[
\mathcal{K}^\perp = \mathcal{K}_{-1} + \mathcal{K}_{-2} + \mathcal{C}_4 + \mathcal{C}_5 + \mathcal{C}_6 + \mathcal{C}_7 + \mathcal{C}_8,
\]

where \( \mathcal{K}_{-1} \cong \mathbb{R} \), \( \mathcal{K}_{-2} \cong [\lambda_0^{1,1}] \), \( \mathcal{C}_4 \cong [\lambda_0^{2,2}] \), \( \mathcal{C}_5 \cong [U] \), \( \mathcal{C}_6 \cong [\lambda^{2,0}] \), \( \mathcal{C}_7 \cong [V] \), and \( \mathcal{C}_8 \cong [\sigma^{2,0}] \). The irreducible \( U(n) \)-modules \( U \) and \( V \) have dominant weights \((2,2,0,\ldots,0)\) and \((2,1,0,\ldots,0,-1)\) respectively. For \( n = 3 \), the decomposition of \( \mathcal{K}^\perp \) is formed by the same summands but omitting \( \mathcal{C}_4 \). Finally, when \( n = 2 \) we have to omit \( \mathcal{K}_{-2}, \mathcal{C}_4 \) and \( \mathcal{C}_7 \).

We are dealing with \( G \)-structures where \( G \) is a subgroup of the linear group \( GL(m, \mathbb{R}) \). If \( M \) possesses a \( G \)-structure, then there always exists a \( G \)-connection defined on \( M \). Moreover, if \((M^m, \langle \cdot, \cdot \rangle)\) is an orientable \( m \)-dimensional Riemannian manifold and \( G \) a closed and connected subgroup of \( SO(m) \), then there exists a unique metric \( G \)-connection \( \nabla \) such that \( \xi_x = \nabla \mathbf{x} - \nabla_x \mathbf{x} \) takes its values in \( \mathfrak{g}^\perp \), where \( \mathfrak{g}^\perp \) denotes the orthogonal complement in \( \mathfrak{so}(m) \) of the Lie algebra \( \mathfrak{g} \) of \( G \) and \( \nabla \) is the Levi-Civita connection [16], [5].
The tensor $\xi$ is the intrinsic torsion of the $G$-structure and $\tilde{\nabla}$ is called the minimal $G$-connection.

For $U(n)$-structures, the minimal $U(n)$-connection is given by $\tilde{\nabla} = \nabla + \xi$, with

$$ (2.1) \quad \xi_X Y = -\frac{1}{2} I (\nabla_X I) Y, $$

see [6]. Since $U(n)$ stabilises the Kähler form $\omega$, it follows that $\tilde{\nabla} \omega = 0$. Moreover, the equation $\xi_X (I Y) + I (\xi_X Y) = 0$ implies $\nabla \omega = -\xi \omega \in T^* M \otimes u(n)^\perp$. Thus, one can identify the $U(n)$-components of $\xi$ with the $U(n)$-components of $\nabla \omega$:

(i) if $n = 1$, $\xi \in T^* M \otimes u(1)^\perp = \{0\}$;

(ii) if $n = 2$, $\xi \in T^* M \otimes u(2)^\perp = \mathcal{W}_2 + \mathcal{W}_4$;

(iii) if $n \geq 3$, $\xi \in T^* M \otimes u(n)^\perp = \mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3 + \mathcal{W}_4$.

Here the summands $\mathcal{W}_i$ are the irreducible $U(n)$-modules given by Gray and Hervella in [10], so $\mathcal{W}_1 \cong [\lambda^3,0]$, $\mathcal{W}_2 \cong [A]$, $\mathcal{W}_3 \cong [\lambda^2,1]$ and $\mathcal{W}_4 \cong [\lambda,0]$, where $A \subset \lambda^1 \otimes \lambda^2$ is the irreducible $U(n)$-module with dominant weight $(2,1,0,\ldots,0)$. In the following, $\xi_i$ will denote the component in $\mathcal{W}_i$ of the torsion tensor $\xi$.

In [6], Falcitelli et al. proved that the components of $R$ in $\mathcal{K}^\perp$ are linearly determined by the covariant derivative $\nabla \xi$ with respect to the Levi-Civita connection $\nabla$. To prove this result, they consider the space $\mathcal{R} = \mathcal{K} + \mathcal{K}^\perp$ of curvature tensors, we recall that $\mathcal{R}$ is the kernel of the mapping $\pi = (\pi_2 \circ \pi_1) |_{\mathcal{R}} : \mathcal{R} \rightarrow \mathcal{K}^\perp$ can be expressed as the restriction to $\mathcal{R}$ of the composition map $\pi_2 \circ \pi_1$, where $\pi_1 : \Lambda^2 T^*_m M \otimes \Lambda^2 T^*_m M \rightarrow \Lambda^2 T^*_m M \otimes u(n)^\perp$ is the orthogonal projection and $\pi_2 : \Lambda^2 T^*_m M \otimes u(n)^\perp \rightarrow \mathcal{K}^\perp$ is a certain $U(n)$-equivariant homomorphism.

Since we have the identity [6]

$$ \pi_1(R)(X,Y,Z,W) = \langle (\nabla_X I \xi)_YIZ,W \rangle - \langle (\nabla_Y I \xi)_XIZ,W \rangle $$

$$ = \langle (\nabla_X \xi)_YZ,W \rangle - \langle (\nabla_Y \xi)_XZ,W \rangle + 2\langle \xi_XYZ,W \rangle - 2\langle \xi_YZ,X,W \rangle $$

with the third and fourth summands in $\Lambda^2 T^*_m M \otimes u(n)$, and $\pi_2$ is $U(n)$-equivariant, it follows that the components of $\pi^\perp(R)$ in $\mathcal{K}^\perp$ are linear functions of the components of $\nabla \xi$. Now, taking the $U(n)$-connection $\tilde{\nabla} = \nabla + \xi$ into account, one obtains

$$ (2.2) \quad \pi_1(R)(X,Y,Z,W) = \langle (\tilde{\nabla}_X \xi)_YZ,W \rangle - \langle (\tilde{\nabla}_Y \xi)_XZ,W \rangle + \langle \xi_{XY}Z,W \rangle - \langle \xi_{YZ}X,W \rangle. $$
From this equation and considering the image \( \pi_2 \circ \pi_1(R) \), Falcitelli et al. give some tables which show whether or not the tensors \( \nabla \xi_i \) and \( \xi_i \otimes \xi_j \) contribute to the components of \( R \) in \( \mathcal{K}^\perp \).

Here we also consider manifolds equipped with an \( SU(n) \)-structure. Such manifolds are called \textit{special almost Hermitian} manifolds. They are almost Hermitian \( (M, I, \langle \cdot, \cdot \rangle) \) equipped with a complex volume form \( \Psi = \psi_+ + i\psi_- \) such that \( \langle \Psi, \Psi \rangle_C = 1 \). Note that \( I_{(i)}\psi_+ = \psi_- \). See [14] for details and more exhaustive information, or [3] [13] [12].

For a special almost Hermitian \( 2n \)-manifold \( M \), we have the intrinsic torsion \( \eta + \xi \in T^*M \otimes \mathbb{R}\omega + T^*M \otimes \mathfrak{u}(n)^\perp = T^*M \otimes \mathfrak{su}(n)^\perp \) and the minimal \( SU(n) \)-connection \( \nabla = \nabla + \eta + \xi \). Since \( \nabla \) is metric and \( \eta \in T^*M \otimes \mathbb{R}\omega \), we have \( \langle Y, \eta X Z \rangle = \tilde{\eta}(X)\omega(Y, Z) \), where \( \tilde{\eta} \) is a one-form. Hence

\[
\eta X Y = \tilde{\eta}(X)IY.
\]

In [14] it is shown that the one-form \( \tilde{\eta} \) is given by

\[
-I\tilde{\eta} = \frac{1}{2n-2} * (s^d\psi_+ \wedge \psi_+ + s^d\psi_- \wedge \psi_-) - \frac{1}{2n} Id^*\omega,
\]

where * is the Hodge star operator and \( d^* \) the coderivative. This formula simplifies for \( n \geq 3 \) since then \( *d\psi_+ \wedge \psi_+ = *d\psi_- \wedge \psi_- \), and one sees that \( nI\tilde{\eta} = -\frac{1}{2} Id^*\omega \) is essentially the coefficient of \( \Psi \) in the \((n, 1)\)-part of \( d\Psi \). The other part of the intrinsic torsion \( \xi \in T^*M \otimes \mathfrak{u}(n)^\perp \) is still given by equation (2.1).

The tensors \( \omega, \psi_+ \) and \( \psi_- \) are stabilised by the \( SU(n) \)-action, and \( \nabla \omega = 0 \), \( \nabla \psi_+ = 0 \) and \( \nabla \psi_- = 0 \). Moreover, one can check \( \eta \omega = 0 \) and obtain \( \nabla \omega = -\xi \omega \in T^*M \otimes \mathfrak{u}(n)^\perp \). In general, the above mentioned \( U(n) \)-spaces \( \mathcal{W}_i \) are also irreducible as \( SU(n) \)-spaces. The only exceptions are \( \mathcal{W}_1 \) and \( \mathcal{W}_2 \) when \( n = 3 \). In fact, for that case, we have the following decompositions into irreducible \( SU(3) \)-components,

\[
\mathcal{W}_i = \mathcal{W}^+_i + \mathcal{W}^-_i, \quad i = 1, 2,
\]

where the space \( \mathcal{W}^+_i \) (\( \mathcal{W}^-_i \)) consists of those tensors \( a \in \mathcal{W}_i \subseteq T^*M \otimes \Lambda^2 T^*M \) such that the bilinear form \( r(a) \), defined by \( 2r(a)(x, y) = \langle x \wedge \psi_+, y \wedge a \rangle \), is symmetric (skew-symmetric), see [14] [4]. The components of the tensor \( \xi \) in \( \mathcal{W}^+_i \) and \( \mathcal{W}^-_i \), \( i = 1, 2 \), will be denoted by \( \xi^+_i \) and \( \xi^-_i \) respectively. Writing \( \eta \in \mathcal{W}_5 \cong T^*M \), the intrinsic \( SU(n) \)-torsion \( \xi + \eta \) is contained in \( (T^*M \otimes \mathfrak{u}(n)^\perp) + \mathcal{W}_5 \). The space \( \mathcal{W}_5 \) is always \( SU(n) \)-irreducible.

From the equations \( \nabla \psi_+ = 0 \) and \( \nabla \psi_- = 0 \), we have \( \nabla \psi_+ = -\xi \psi_+ - \eta \psi_+ \) and \( \nabla \psi_- = -\xi \psi_- - \eta \psi_- \). Moreover, for \( n \geq 2 \), in [14] it is shown that

\[
(2.3) \quad \xi X \psi_+, \xi X \psi_- \in [\Lambda^{n-2, 0}] \wedge \omega, \quad \eta X \psi_+ = n \tilde{\eta}(X)\psi_-, \quad \eta X \psi_- = -n \tilde{\eta}(X)\psi_+.
\]

When considering curvature, note that the module \( \mathcal{C}_3 = \mathcal{K}(\mathfrak{su}(n)) \) in \( \mathcal{K} \) consists of the algebraic curvature tensors for a metric with holonomy algebra \( \mathfrak{su}(n) \).
3 Some curvature formulæ

For special almost Hermitian \(2n\)-manifolds, results and tables given in [6] are still valid with respect to the tensors \(\tilde{\nabla}\xi_i\) and \(\xi_i \otimes \xi_j\). Here \(\tilde{\nabla} = \nabla - \eta\) is the minimal \(U(n)\)-connection, with \(\nabla\) denoting the minimal \(SU(n)\)-connection.

For \(SU(n)\)-structures, the additional information coming from \(\eta\) will allow us to compute the components of \(R\) in \(K_1\) and \(K_2\) in terms of the intrinsic torsion \(\eta + \xi\). To achieve this, we compute the difference between the Ricci and the \(\ast\)-Ricci curvatures. In the first instance we only need the almost Hermitian structure.

Lemma 3.1. Let \(M\) be an almost Hermitian \(2n\)-manifold, \(n \geq 2\), with minimal \(U(n)\)-connection \(\tilde{\nabla} = \nabla + \xi\), then

\[
\text{Ric}^\ast(X, Y) - \text{Ric}(X, Y) = 2\langle (\nabla_{e_i} I\xi) X Y, e_i \rangle - 2\langle (\nabla_{X} I\xi) e_i Y, e_i \rangle,
\]

\[
= 2\langle (\nabla_{e_i} \xi) X Y, e_i \rangle - 2\langle (\nabla_{X} \xi) e_i Y, e_i \rangle
\]

\[
+ 2\langle \xi_{e_i, X} Y, e_i \rangle - 2\langle \xi_{X, e_i} Y, e_i \rangle.
\]

Proof. It is straightforward to check

\[(3.1) \quad \text{Ric}^\ast(X, Y) - \text{Ric}(X, Y) = -(R_{X, e_i} \omega)(IY, e_i).
\]

However the so-called Ricci formula [2, p. 26] implies

\[(3.2) \quad -(R_{X, e_i} \omega)(IY, e_i) = \bar{a}(\nabla^2 \omega)_{X, e_i}(IY, e_i),
\]

where \(\bar{a}: T^* M \otimes T^* M \otimes \Lambda^2 T^* M \to \Lambda^2 T^* M \otimes \Lambda^2 T^* M\) is the skewing mapping.

The required identities follow from equations (3.1) and (3.2), taking into account \(\tilde{\nabla} \omega = 0\).

The components of \(R\) in \(K_{-1}\) and \(K_{-2}\) are determined by the trace and the trace-free parts of \(\text{Ric}^\ast_H - \text{Ric}_H\). Similarly, the \(C_0\)-component of \(R\) is determined by the skew-symmetric (or anti-Hermitian) part \(\text{Ric}_{AH}^\ast\) of \(\text{Ric}^\ast\). Moreover, the anti-Hermitian part \(\text{Ric}_{AH}\) of the Ricci curvature, which satisfies \(\text{Ric}_{AH}(IX, IY) = -\text{Ric}_{AH}(X, Y)\), determines the component of \(R\) in \(C_8\). These assertions motivate the expressions contained in the next lemma.

Lemma 3.2. Let \(M\) be an almost Hermitian \(2n\)-manifold, \(n \geq 2\), with minimal \(U(n)\)-connection \(\tilde{\nabla} = \nabla + \xi\), then

\[
(\text{Ric}_{AH}^\ast - \text{Ric}_H)(X, Y) = \langle (\nabla_{e_i} \xi) X Y, e_i \rangle - \langle (\nabla_{X} \xi) e_i Y, e_i \rangle
\]

\[
+ \langle (\nabla_{e_i} \xi) IX Y, e_i \rangle - \langle (\nabla_{XY} \xi) e_i IY, e_i \rangle
\]

\[
+ \langle \xi_{e_i, X} Y, e_i \rangle - \langle \xi_{X, e_i} Y, e_i \rangle
\]

\[
+ \langle \xi_{e_i, IX} Y, e_i \rangle - \langle \xi_{IX, e_i} Y, e_i \rangle.
\]
with complex volume form $\Psi = \cdots$. We now give a first result that uses the complex volume form $\Psi$.

Proof. This follows directly from Lemma\ref{lem:b} together with $\langle \xi_{\psi_i}X, e_i \rangle = \langle \xi_{\psi_i}Y, e_i \rangle$.

Up to this point, we have not said anything special about $SU(n)$-structures. We now give a first result that uses the complex volume form $\Psi$.

Lemma 3.3. Let $M$ be a special almost Hermitian $2n$-manifold, $n \geq 2$, with complex volume form $\Psi = \psi_+ + i\psi_-$ and minimal $SU(n)$-connection $\nabla = \nabla + \eta + \xi = \tilde{\nabla} + \eta$, then

$$
(3.6) \quad \text{Ric}^*(X, Y) = -n d\tilde{\eta}(X, Y) - \langle \xi_{\psi_i}X, \xi_{\psi_i}Y, e_i \rangle,
$$

$$
(3.7) \quad \text{Ric}(X, Y) = -n d\tilde{\eta}(X, Y) - \langle \xi_{\psi_i}X, \xi_{\psi_i}Y, e_i \rangle - 2\langle (\nabla_{\psi_i}X)Y, e_i \rangle
+ 2\langle (\tilde{\nabla}_{\psi_i}X)Y, e_i \rangle - 2\langle \xi_{\psi_i}X, \tilde{\nabla}_{\psi_i}Y, e_i \rangle + 2\langle \xi_{\psi_i}X, \tilde{\nabla}_{\psi_i}Y, e_i \rangle.
$$

Proof. Start by noting that $\langle R_{X, Y}\psi_+, \psi_- \rangle = -2^{n-2}\langle R_{X, Y}\psi_+, \psi_- \rangle$. Now, by the first Bianchi identity, we have

$$
(3.8) \quad \langle R_{X, Y}\psi_+, \psi_- \rangle = -2^{n-1}\text{Ric}^*(X, Y).
$$

On the other hand, using the Ricci formula $-R_{X, Y}\psi_+ = \tilde{\alpha}(\nabla^2\psi_+)(X, Y)$ and taking $\nabla = \nabla + \eta + \xi$ into account, we obtain

$$
-R_{X, Y}\psi_+ = n d\tilde{\eta}(X, Y)\psi_+ + n \tilde{\eta}(X)(\xi_{\psi_-}) - n \tilde{\eta}(Y)(\xi_{\psi_+})
+ Y_\perp(\nabla_X(\xi_{\psi_+})) - X_\perp(\nabla_Y(\xi_{\psi_+})).
$$

Using the inclusions of (29), we have $\langle \xi_{\psi_+}, \psi_- \rangle = 0$, $\langle \xi_{\psi_-}, \psi_- \rangle = 0$ and $\langle Y_\perp(\nabla_X(\xi_{\psi_+})), \psi_- \rangle = -\langle \xi_{\psi_+}, \xi_{\psi_+}, \psi_- \rangle$. This gives the following identity

$$
(3.9) \quad \langle R_{X, Y}\psi_+, \psi_- \rangle = -n 2^{n-1}d\tilde{\eta}(X, Y) - 2^{n-1}\langle \xi_{\psi_i}X, \xi_{\psi_i}Y, e_i \rangle.
$$
Using equations (3.8), (3.9) and Lemma 3.1 we obtain the required identities for \( \text{Ric}^* \) and \( \text{Ric} \).

The following theorem is an immediate consequence of the above Lemma.

**Theorem 3.4.** Let \( M \) be a special almost Hermitian \( 2n \)-manifold, \( n \geq 2 \), that is Kähler. Then \( \text{Ric}^* = \text{Ric} \) and

(i) if \( d\hat{\eta} = \lambda \omega \), for some \( \lambda \in \mathbb{R} \setminus \{0\} \), then the manifold is Einstein, or

(ii) if the one-form \( \hat{\eta} \) is closed, then the manifold is Ricci flat.

In [9], Gray proved that any nearly Kähler (type \( \mathcal{W}_1 \)) connected six-manifold which is not Kähler is Einstein. Here we give an alternative proof.

**Theorem 3.5 (Gray [9]).** Let \( M \) be a special almost Hermitian connected six-manifold of type \( \mathcal{W}_1^+ + \mathcal{W}_1^- + \mathcal{W}_5 \) which is not of type \( \mathcal{W}_5 \). Then \( M \) is an Einstein manifold such that \( \text{Ric} = 5 \text{Ric}^* = 5\alpha \langle \cdot, \cdot \rangle \), where \( \alpha = (w_1^+)^2 + (w_1^-)^2 \) with \( \nabla \omega = w_1^+ \psi_+ + w_1^- \psi_- \).

**Proof.** We already know that \( \alpha = (w_1^+)^2 + (w_1^-)^2 \) is a positive constant and the one-form \( \hat{\eta} \) is closed (see [14, Theorem 3.7]). On the other hand, since \( \nabla \omega = -\xi \omega \) and \( \nabla \omega = w_1^+ \psi_+ + w_1^- \psi_- \), we have

\[
2\langle Y, \xi_X Z \rangle = w_1^- \psi_+(X, Y, Z) - w_1^+ \psi_-(X, Y, Z).
\]

Therefore, using

\[
\langle X_\cdot \psi_+, Y_\cdot \psi_+ \rangle = \langle X_\cdot \psi_-, Y_\cdot \psi_- \rangle = 2\langle X, Y \rangle, \\
\langle X_\cdot \psi_+, Y_\cdot \psi_- \rangle = -2\omega(X, Y),
\]

we get

\[
(3.10) \quad \langle \xi_X e_i, \xi_Y e_i \rangle = \langle e_j, \xi_X e_i \rangle \langle e_j, \xi_Y e_i \rangle = \alpha \langle X, Y \rangle.
\]

Moreover, since \( \xi \in \mathcal{W}_1^+ + \mathcal{W}_1^- \) and \( \tilde{\nabla} \) is a \( U(3) \)-connection, the \( (0, 3) \)-tensors \( \langle \cdot, \xi \cdot \rangle \) and \( \langle \cdot, (\tilde{\nabla}_X \xi) \cdot \rangle \) are skew symmetric [10]. Thus, from (3.7), we get

\[
\text{Ric}(X, Y) = 5\langle \xi_X e_i, \xi_Y e_i \rangle = 5\alpha \langle X, Y \rangle.
\]

We recall that \( \langle Y, \xi_{1X} IZ \rangle = -\langle Y, \xi_X Z \rangle \), for \( \xi \in \mathcal{W}_1 \), and note that the contractions \( \langle (\tilde{\nabla}_X \xi)_e, Y, e_i \rangle \) and \( \langle (\tilde{\nabla}_e \xi)_e, Y, e_i \rangle \) both vanish. In fact, the last term is a skew-symmetric two-form and the remaining summands in the expression for \( \text{Ric} \) are symmetric. \( \square \)
Remark 3.6. Theorem 3.5 can be extended to connected almost Hermitian six-manifolds which are nearly Kähler and but not Kähler. In fact, one can define a complex volume form on an open neighbourhood \( U \) of a point where \( \nabla \omega \neq 0 \) by using the \((3,0)\)-component of this tensor. Then, \( U \) is a special almost Hermitian six-manifold of type \( \mathcal{W}_1^+ + \mathcal{W}_1^- + \mathcal{W}_5 \). Therefore, \( \text{Ric} = 5 \text{Ric}^* = 5\alpha \langle \cdot, \cdot \rangle \) on \( U \). Since the manifold is connected, it follows \( \text{Ric} = 5\alpha \langle \cdot, \cdot \rangle \) everywhere.

The expressions (3.6) and (3.7) for \( \text{Ric}^* \) and \( \text{Ric} \) allow us to compute \( 3 \text{Ric}^*_H + \text{Ric}_H \) and study the contributions of the intrinsic torsion of the \( SU(n) \)-structure to the components of \( R \) in \( K_1 \) and \( K_2 \).

Lemma 3.7. Let \( M \) be a special almost Hermitian \( 2n \)-manifold, \( n \geq 2 \), with minimal \( SU(n) \)-connection \( \nabla = \nabla + \eta + \xi = \tilde{\nabla} + \eta \), then

\[
(3.11) \quad (3 \text{Ric}^*_H + \text{Ric}_H)(X,Y) = -2n d\eta(X, IY) + 2n d\eta(IX, Y) - \langle (\tilde{\nabla}_{e_i} \xi)_X Y, e_i \rangle + \langle (\tilde{\nabla}_X \xi)_{e_i} Y, e_i \rangle + \langle (\tilde{\nabla}_{IX} \xi) e_i Y, e_i \rangle + \langle (\tilde{\nabla}_{IX} \xi) e_i Y, e_i \rangle - \langle \xi_{e_i X e_i Y, e_i} \rangle - 2\langle \xi_{e_i X e_i Y, e_i} \rangle - 2\langle \xi_{e_i X e_i Y, e_i} \rangle.
\]

(3.12)

Proof. We have

\[
-2 \text{Ric}^*(X, IY) - 2 \text{Ric}^*(IX, Y) = \langle R_{e_i e_i X, e_i IY} - \langle R_{e_i e_i X, e_i IY} \rangle - \langle R_{e_i e_i X, e_i IY} \rangle
\]

\[
= 4\langle (\tilde{\nabla}_{e_i} \xi)_{e_i} X, Y \rangle - 4\langle (\tilde{\nabla}_{e_i} \xi)_{e_i} X, Y \rangle - 4\langle (\tilde{\nabla}_{e_i} \xi)_{e_i} X, Y \rangle + 4\langle (\tilde{\nabla}_{e_i} \xi)_{e_i} X, Y \rangle
\]

from which the Lemma follows.

4 High dimensions

In this section, we consider special almost Hermitian manifolds of dimension higher than or equal to eight. For such manifolds, the decomposition into
SU(n)-irreducible modules of the space of curvature tensors $\mathcal{R}$ is the same as that coming from the action of $U(n)$. Thus,

$$\mathcal{R} = \mathcal{K} + \mathcal{K}^\perp = C_3 + C_1 + C_2 + C_{-1} + C_{-2} + C_4 + C_5 + C_6 + C_7 + C_8,$$

where all $K_i$ and $C_j$ are also SU(n)-irreducible spaces. Our aim here is to see whether or not different components of the intrinsic torsion of the SU(n)-structure contribute to the components of the curvature.

We start by studying such contributions to the SU(n)-components of the Ricci and $*$-Ricci curvatures. For $n \geq 3$, the spaces $\text{Ric}$ and $\text{Ric}^*$ of such tensors admit the following decompositions into SU(n)-irreducible modules

$$\text{Ric} = \mathbb{R}\langle \cdot, \cdot \rangle + [\lambda_0^{1,1}] + [\sigma^{2,0}], \quad \text{Ric}^* = \mathbb{R}\langle \cdot, \cdot \rangle + [\lambda_0^{1,1}] + [\lambda^{2,0}].$$

Taking into account the symmetry properties and types of the Gray-Hervella’s components $\xi_i$ of $\xi$, we obtain the following result.

**Theorem 4.1.** Let $M$ be a special almost Hermitian $2n$-manifold, $2n \geq 8$, with minimal SU(n)-connection $\nabla = \nabla + \eta + \xi = \tilde{\nabla} + \eta$. The tensors $d\eta$, $\nabla\xi$ and $\xi_i \circ \xi_j$ contribute to the components of the $*$-Ricci curvature $\text{Ric}^*$ via equation (3.10) and to the Ricci curvature $\text{Ric}$ via equation (3.11) if and only if there is a tick in the corresponding place in Table 4.1. \hfill $\blacksquare$

Using in addition $\langle \xi_{i}X_{e_{i}}, Y, e_{i} \rangle = -\langle \xi_{i}X_{e_{i}}, \xi_{i}, Y \rangle$ we get part (i) of the following theorem. Part (ii) is proved in [6].

**Theorem 4.2.** Let $M$ be a special almost Hermitian $2n$-manifold, $2n \geq 8$, with minimal SU(n)-connection $\nabla = \nabla + \eta + \xi = \tilde{\nabla} + \eta$, then

(i) Using equations (3.8), (3.13), (3.15) and (3.17), each of the tensors $\nabla\xi_i$, $\eta\xi_i$ and $\xi_i \circ \xi_j$ contributes to the components of $R$ in $K_1$, $K_2$, $K_{-1}$, $K_{-2}$, $C_6$ and $C_8$ if and only if there is a tick in the corresponding place in Table 4.2. \hfill $\blacksquare$

(ii) Taking the image $\pi_2 \circ \pi_1(R)$ into account, where $\pi_1(R)$ is given by equation (2.2), each of the tensors $\nabla\xi_i$, $\eta\xi_i$ and $\xi_i \circ \xi_j$ contributes to the components of $R$ in $C_4$, $C_5$ and $C_7$ if and only if there is a tick in the corresponding place in Table 4.2. \hfill $\blacksquare$

For part (i), we wish to emphasise that the columns for $K_{-1}$, $K_{-2}$, $C_6$ and $C_8$ are obtained by a different method to that in [6] and that for $C_7$ this even leads to a different result. In particular, we claim that the tensors $\nabla\xi_3$ and $\xi_3 \circ \xi_4$ do not contribute to the $C_6$-component of $R$, but that $\nabla\xi_1$ and $\eta\xi_1$ do. Thus the contributions of the different tensors to the distinct components of $R$ depend on the choice of the current expression that we use; different expressions may lead to different behaviour in the contributions. For the
\( \hat{d} \eta \)
\( \nabla \xi_1, \eta \xi_1 \)
\( \nabla \xi_2, \eta \xi_2 \)
\( \nabla \xi_3, \eta \xi_3 \)
\( \nabla \xi_4, \eta \xi_4 \)
\( \xi_1 \otimes \xi_1 \)
\( \xi_2 \otimes \xi_2 \)
\( \xi_3 \otimes \xi_3 \)
\( \xi_4 \otimes \xi_4 \)
\( \xi_1 \odot \xi_2 \)
\( \xi_1 \odot \xi_3 \)
\( \xi_1 \odot \xi_4 \)
\( \xi_2 \odot \xi_3 \)
\( \xi_2 \odot \xi_4 \)
\( \xi_3 \odot \xi_4 \)

Table 4.1: Ricci curvatures, \( 2n \geq 8 \)

\( C_6 \)-component of \( R \), we get a third formula from equation (3.12), which we also list in Table 4.2. A partial explanation for these different results will be given in §6. Note that the entries for \( C_6 \) in Table 4.2 only involve the intrinsic \( U(n) \)-torsion. The \([\lambda^{2.0}]\)-column of Table 4.1 provides yet another description of the \( C_6 \)-component using the \( SU(n) \)-structure.

5 Low dimensions

In this section we consider in turn special almost Hermitian manifolds of dimension six and four.

5.1 Six dimensions

The decomposition of the space of curvature tensors \( \mathcal{R} \) into irreducible \( SU(3) \)-modules has the same subspaces as that for \( U(3) \). Thus,

\[ \mathcal{R} = \mathcal{K} + \mathcal{K}^\perp = C_3 + C_1 + C_2 + C_{-1} + C_{-2} + C_5 + C_6 + C_7 + C_8, \]

with \( \mathcal{K}_i \) and \( C_j \) all \( SU(3) \)-irreducible. As we noted above, the summand \( C_4 \) is absent in this dimension. On the other hand, the \( U(3) \)-intrinsic torsion
splits under SU(3) as \( \xi = \xi_1^+ + \xi_1^- + \xi_2^+ + \xi_2^- + \xi_3 + \xi_4 \), where \( \xi_i = \xi_i^+ + \xi_i^- \), \( i = 1, 2 \). This was briefly described in §2 and more detailed information is contained in [4] and [14].

The next result concerns the contributions of the components of \( \xi \) to the components of the Ricci and the \( \ast \)-Ricci curvatures and then to the curvature components complementary to \( C_3 \).

**Theorem 5.1.** Let \( M \) be a special almost Hermitian 6-manifold with \( SU(3) \)-connection \( \tilde{\nabla} = \nabla + \eta + \xi = \tilde{\nabla} + \eta \). The tensors \( d\tilde{\eta}, \nabla\xi, \eta\xi \) and \( \xi \odot \vartheta \), for \( \xi, \vartheta = \xi_1^+, \xi_1^-, \xi_2^+, \xi_2^-, \xi_3, \xi_4 \) contribute the components of \( \text{Ric}^* \) and \( \text{Ric} \) if and only if there is a tick in the corresponding place in Table 5.1.

The corresponding contributions to the curvature components \( K_1, K_2, K_1^-, K_2^-, C_6 \) and \( C_8 \), via equations (3.3), (3.4), (3.5) and (3.11), and to the components \( C_5 \) and \( C_7 \) via \( \pi_2 \circ \pi_1(R) \) [2] are given in Table 5.2.

**5.2 Four dimensions**

The \( U(2) \)-decomposition of the space of curvature tensors \( \mathcal{R} \) is given by

\[
\mathcal{R} = \mathcal{K} + \mathcal{K}^\perp = C_3 + K_1 + K_2 + K_{1}^- + K_{2}^- + C_5 + C_6 + C_8.
\]
When we consider the $SU(2)$ action, only the modules $C_3$, $K_1$, $K_2$ and $K_{-1}$ remain irreducible. To describe the decompositions of $C_5$ and $C_6$ into $SU(2)$-irreducible modules, we will make use of tensors defined by

$$\chi(a, b) = 6 a \odot b - a \wedge b,$$

for all $a, b \in \Lambda^2 T^*M$, where $\odot$ denotes the symmetric product given by $2 a \odot b = a \otimes b + b \otimes a$. The relevant decompositions are now given by

(i) $C_5 = C_5^{++} + C_5^{+-} + C_5^{-+} + C_5^{--}$, where $C_5^{++} = \mathbb{R} \chi(\psi_+, \psi_+)$, $C_5^{--} = \mathbb{R} \chi(\psi_-, \psi_-)$ and $C_5^{+-} = \mathbb{R} \chi(\psi_+, \psi_-)$,

(ii) $C_6 = C_6^+ + C_6^-$, where $C_6^+ = \mathbb{R} \chi(\psi_+, \omega)$ and $C_6^- = \mathbb{R} \chi(\psi_-, \omega)$.

For the intrinsic torsion, the $U(2)$-decomposition of $\xi$ is given by

$$\xi = \xi_2 + \xi_4 \in \mathcal{W} = \mathcal{W}_2 + \mathcal{W}_4.$$

Under $SU(2)$, we have $\mathcal{W}_2 \cong \mathcal{W}_4 \cong T^*M$, which we will see gives rise to different choices of decompositions of $\xi$.
For an SU(2)-structure, we have $\nabla \omega \in \mathcal{W} = T^*M \otimes \psi_+ + T^*M \otimes \psi_-$. Consequently, $\nabla \omega = \xi_+ \otimes \psi_+ + \xi_- \otimes \psi_-$, where $\xi_+$ and $\xi_-$ are one-forms. Moreover,

$$2\langle Y, \xi XZ \rangle = -\xi_+(X)\psi_-(Y,Z) + \xi_-(X)\psi_+(Y,Z),$$

so $\xi = \xi_+ + \xi_-$, where

$$2\langle Y, (\xi_+)XZ \rangle = -\xi_+(X)\psi_-(Y,Z), \quad 2\langle Y, (\xi_-)XZ \rangle = \xi_-(X)\psi_+(Y,Z).$$

The two decompositions of $\xi$ are related as follows:

(i) $\xi \in \mathcal{W}_2$ if and only if $\xi_+ = I\xi_-.$

(ii) $\xi \in \mathcal{W}_4$ if and only if $\xi_+ = -I\xi_-.$

The following theorem gives information about the contributions of the components of the intrinsic torsion to the tensors $Ric^*$ and Ric. We first note that in dimension four, $Ric^*$ decomposes under SU(2) as

$$Ric^* = \mathbb{R}\langle \cdot, \cdot \rangle + [\lambda_0^{1,1}] + \mathbb{R}\psi_+ + \mathbb{R}\psi_-.$$
Theorem 5.2. Let $M$ be a special almost Hermitian $4$-manifold with minimal $\SU(2)$-connection $\nabla = \nabla + \eta + \xi = \tilde{\nabla} + \eta$. The curvature contributions corresponding to Theorems 4.1 and 4.2 via the decompositions $\xi = \xi_2 + \xi_4$ and $\eta = \eta_+ + \eta_-$ are given in Tables 5.3 and 5.4.

Proof. The absence of $\mathcal{K}_{-2}$ in the decomposition of $\mathcal{R}$ comes from the fact that

$$(5.1) \quad (\text{Ric}_H^* - \text{Ric}_H)(X,Y) = \beta \langle X,Y \rangle,$$

where $\beta = \langle \langle \tilde{\nabla}_e \xi \rangle_{e_j} e_j, e_i \rangle + \langle \xi_{e_i e_j} e_j, e_i \rangle$. Therefore, by (3.4), we have

$$(5.2) \quad (3 \text{Ric}_H^* + \text{Ric}_H)(X,Y) = -\beta \langle X,Y \rangle - 4d\tilde{\eta}(X, IY) + 4d\tilde{\eta}(IX, Y) - 2\langle \xi_X e_i, \xi_{IY} I e_i \rangle - 2\langle \xi_Y e_i, \xi_{IX} I e_i \rangle.$$

Using equations (5.1) and (5.2) the tables follow.

Remark 5.3. Let us list some direct consequences of results and tables presented here and in §4:

(a) if $\xi \in \mathcal{W}_3$, then the components of $R$ in $\mathcal{K}_{-1}$, $\mathcal{C}_5$ and $\mathcal{C}_6$ vanish;

(b) if $\xi \in \mathcal{W}_3 + \mathcal{W}_4$ and $d\tilde{\eta}$ is Hermitian, then the components of $R$ in $\mathcal{C}_5$ and $\mathcal{C}_6$ vanish;

(c) if $\xi \in \mathcal{W}_1 + \mathcal{W}_2$ and $d\tilde{\eta}$ is Hermitian, then the component of $R$ in $\mathcal{C}_6$ vanishes; and

Table 5.3: Ricci curvatures, $2n = 4$
(d) if \( n = 2 \) and \( d\hat{\eta} \) is Hermitian, then the component of \( R \) in \( C_6 \) vanishes.

There are more consequences of this sort, but they have been already pointed out in [6].

Remark 5.4. For special almost Hermitian 2-manifolds, we have the following identity, deduced in [14],

\[
K(\psi_+, \psi_-) = d\hat{\eta}(\psi_+, \psi_-) = d\eta_+(\psi_+) + d\eta_-(\psi_-) - \eta_+^2 - \eta_-^2,
\]

where \( K \) denotes the sectional curvature and \( \hat{\eta} = \eta_+ \psi_- - \eta_- \psi_+ \).

### 6 Identities from the exterior algebra

As remarked in [14] one may see different contributions to the module \( C_6 \cong [\lambda^{2,0}] \) by using different computations of the curvature. This is because of non-trivial identities relating the components of \( \tilde{\nabla}\xi_i \) and \( \xi_j \odot \xi_k \). Such an identity for the \([\lambda^{2,0}]\)-components may be obtained by comparing equations (3.4) and (3.12). However, we claim that this information may also be obtained from the exterior algebra of a \( U(n) \)-manifold.

Consider the Kähler two-form \( \omega \). Being a differential form it satisfies \( d^2 \omega = 0 \). However, since the Levi-Civita connection \( \nabla \) is torsion-free, we may compute \( d^2 \omega \) using \( \nabla \). Writing \( \nabla = \tilde{\nabla} - \xi \) and using \( \tilde{\nabla}\omega = 0 \), we have first that

\[
\frac{1}{2}d\omega(Y, Z, W) = \langle \xi_Y Z, IW \rangle + \langle \xi_W Y, IZ \rangle + \langle \xi_Z W, IY \rangle.
\]
Now $d^2 \omega = a(\nabla d\omega) - a(\xi d\omega)$, where $a: T^*M \otimes \Lambda^3 T^*M \to \Lambda^4 T^*M$ is the alternation map. One computes that these two terms are the expressions obtained respectively by summing $\varepsilon (\langle \nabla X \xi Y, IZ \rangle + \varepsilon(x_\xi Y Z, IZ)$ over all permutations of $(X, Y, Z, W)$, where $\varepsilon$ is the sign of the permutation.

We have that

$$\Lambda^4 T^*M = [\lambda^{4,0}] + [\lambda^{3,1}] + [\lambda^{2,0}] \omega + [\lambda^{0,2}] \lambda + \mathbb{R} \omega^2,$$

so in order to compute the $[\lambda^{2,0}]$-component of $d^2 \omega$ we contract with $\omega$ on the first two arguments and then take the projection to $[\lambda^{2,0}]$, which is the $(-1)$-eigenspace of $I$ acting on 2-forms. Using the symmetries of the components of $\xi$ one obtains that the $[\lambda^{2,0}]$-component of $d^2 \omega$ is

$$0 = 3 \langle \langle \nabla_{e_i} \xi_1 \rangle_{e_i} X, Y \rangle - \langle \langle \nabla_{e_i} \xi_3 \rangle_{e_i} X, Y \rangle + (n - 2) \langle \langle \nabla_{e_i} \xi_4 \rangle_{e_i} X, Y \rangle + \langle \xi_3 e_i, \xi_1 e_i, Y \rangle - \langle \xi_3 e_i, \xi_2 e_i, X \rangle - \frac{n - 5}{n - 1} \langle \xi_1 e_i, e_i, X, Y \rangle - \frac{n - 2}{n - 1} \langle \xi_2 e_i, e_i, X, Y \rangle + \langle \xi_3 e_i, e_i, X, Y \rangle.$$

We conclude that in general dimensions there is a non-trivial linear relation between the $[\lambda^{2,0}]$-components of $\nabla \xi_1, \nabla \xi_3, \nabla \xi_4, \xi_1 \otimes \xi_3, \xi_1 \otimes \xi_4, \xi_2 \otimes \xi_3, \xi_2 \otimes \xi_4$ and $\xi_3 \otimes \xi_4$. By ‘non-trivial’ we mean that no coefficient is zero, so this relation may be used to write any of the terms as a linear combination of the others. Interestingly, this relation does not involve $\xi_1 \otimes \xi_4$, when $2n = 10$.

This is sufficient to explain the difference between the ticks in the $C_6$ column in [6] and those we obtained from equation [3.12]. An extra coincidence in the coefficients explains the differences between our results from [3.1] and [3.12].

One may try to apply the above approach to the other modules that $\Lambda^4 T^*M$ has in common with the space of curvature tensors, namely $[\lambda^{2,2}], [\lambda^{0,1}] \lambda$ and $\mathbb{R} \omega^2$. However, this is not so rewarding because of the higher multiplicities that these modules have in the relevant decompositions. Indeed, $C_6 \cong [\lambda^{2,0}]$ is distinguished by occurring only with multiplicity one or zero in the modules for $\nabla \xi_i$ and $\xi_i \otimes \xi_j$.

In [3], it is pointed out that if $\xi \in \mathcal{V}_4$, then the components of $R$ in $C_4, C_5, C_6$ and $C_7$ vanish. Let us indicate how equation (6.1) gives an alternative proof of this result, for $n > 2$. In fact, using Tables [1.2] and [5.2], the vanishing of the components in $C_4, C_5$ and $C_7$ is immediate. On the other hand, equations [3.4] and [3.1] give the vanishing of the component in $C_6$.

Finally, a comparison of Tables [1.1] and [1.2] reveals another relation on special almost Hermitian manifolds: the $[\lambda^{2,0}]$-part of $d\eta$ carries all the information from the corresponding components of $\nabla \xi_i$ modulo the $[\lambda^{2,0}]$-parts of $\xi_1 \otimes \xi_3, \xi_1 \otimes \xi_4, \xi_2 \otimes \xi_3$ and $\xi_2 \otimes \xi_4$. This relation is obtainable by considering the $(n,2)$-part of the equation $d^2 \Psi = 0$, where $\Psi$ is the complex volume, cf. [14].
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Martín Cabrera: *Department of Fundamental Mathematics, University of La Laguna, 38200 La Laguna, Tenerife, Spain*. E-mail: fmartin@ull.es

Swann: *Department of Mathematics and Computer Science, University of Southern Denmark, Campusvej 55, DK-5230 Odense M, Denmark*. E-mail: swann@imada.sdu.dk