THE J-INARIANT AND TITS ALGEBRAS FOR GROUPS OF INNER TYPE $E_6$

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ABSTRACT. A connection between the indices of the Tits algebras of a split linear algebraic group $G$ and the degree one parameters of its motivic $J$-invariant was introduced by Quéguiner-Mathieu, Semenov and Zainoulline through use of the second Chern class map in the Riemann-Roch theorem without denominators. In this paper we extend their result to higher Chern class maps and provide applications to groups of inner type $E_6$.

INTRODUCTION

For a reductive group $G$, invariants known as the Tits algebras were introduced by J. Tits in [20] and have proven to be an invaluable tool for the computation of the $K$-theory of twisted flag varieties by Panin [13] and for the index reduction formulas by Merkurjev, Panin and Wadsworth [12]. Furthermore, the Tits algebras are essentially the only cohomological invariant of degree 2 (cf. [11, Ex. 31.21], and have applications to both the classification of linear algebraic groups and the study of the associated homogeneous varieties.

The $J$-invariant, as defined by Petrov, Semenov and Zainoulline in [15], is an invariant of $G$ which describes the motivic behaviour of the variety of Borel subgroups of $G$. For a prime $p$, the $J$-invariant of $G$ modulo $p$ is given by a set of integers $J_p(G) = (j_1, \ldots, j_r)$. We consider also a (possibly empty) subset $J_p^{(1)}(G) = (j_1, \ldots, j_s), s \leq r$, consisting of the parameters of the $J$-invariant of degree 1.

Motivated by the work [4], Quéguiner-Mathieu, Semenov and Zainoulline discovered a connection between these degree 1 parameters and the indices of the Tits algebras of $G$. This connection is developed in [16], through use of the second Chern class map in the Riemann-Roch theorem without denominators. The goal of this paper is to extend their result through use of higher Chern class maps (cf. Theorem [4]). We then apply this result to a group $G$ of inner type $E_6$. We provide an explicit connection between the values of $J_p^{(1)}(G)$ and the index of the Tits algebra of $G$ (see Proposition [4]). This result is used by Garibaldi, Petrov and Semenov to give finer information, summarized in their Table 10A, on the $J$-invariant for groups of type $E_6$ [5].

This paper is organized as follows. In the first section, we review the notion of characteristic classes and introduce the topological and $\gamma$-filtrations on the Grothendieck group $K_0$ of a smooth projective variety. In Section 2, we look more
closely at the \( \gamma \)-filtration of the variety \( X = G/B \) of Borel subgroups, and give a simplified definition through use of the Steinberg basis. We then consider a twisted form \( \xi X \) of \( X \) by means of a \( G \)-torsor \( \xi \in Z^1(k, G) \) and the \( \gamma \)-filtration of \( K_0(\xi X) \).

In Section 3, we recall the definition of the Tits map and its relation to our primary objects of concern, the common index \( i_c \) and the \( J \)-invariant of \( \xi \). Section 4 provides the main result of the paper, a relationship between the common index and the possible values of the indices of the \( J \)-invariant of degree 1. In the final section, we give an application of this result to a group of inner type \( E_6 \).

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1. **Gamma filtration on \( K_0 \) and the Chern class map**

In the present section we recall several useful properties of the topological filtration and the \( \gamma \)-filtration on Grothendieck’s \( K_0 \) of a smooth projective variety. The reader is advised to consult [13], [3] and [4, ch.15] for more details.

Let \( X \) be a smooth projective variety over a field \( k \). Consider the topological filtration on \( K_0(X) \) (cf. [4, Ex. 15.1.5]) given by

\[
\tau^i K_0(X) = \langle [O_V] \mid \text{codim } V \geq i \rangle,
\]

where \( O_V \) is the structure sheaf of a closed subvariety \( V \) in \( X \). We denote by \( \tau^{i+i+1} K_0(X), i \geq 0 \) the degree \( i \) component \( \tau^i K_0(X)/\tau^{i+1} K_0(X) \) of the corresponding graded ring. There is a surjection

\[
p : CH^i(X) \rightarrow \tau^{i+1} K_0(X), \quad V \mapsto [O_V],
\]

from the Chow group of codimension \( i \) cycles.

For a vector bundle \( E \) on \( X \), the total Chern class \( c(E) = 1 + c_1(E)t + c_2(E)t^2 + \ldots \) is an element of \( CH(X)[t] \), and by the Whitney sum formula, it defines a group homomorphism

\[
c : K_0(X) \rightarrow CH(X)
\]

For \( \alpha \in K_0(X) \), \( c_i(\alpha) \) is the component of \( c(\alpha) \) in \( CH^i(X) \), giving a group homomorphism

\[
c_i : \tau^i K_0(X) \rightarrow CH^i(X)
\]

defined by taking the \( i \)-th Chern class.

**1.1. Lemma.** For a smooth projective variety \( X \) over a field \( k \), \( c_i(\tau^j K_0(X)) = 0 \) for all \( 0 < i < j \), and the induced homomorphism

\[
c^+ : \tau^{i+i+1} K_0(X) \rightarrow CH^i(X)
\]

is an isomorphism over the coefficient ring \( \mathbb{Z}[1/(i+1)] \) for all \( i > 0 \).
Proof. From [4] Ex 15.3.6 it can be seen that \( c_i(\tau^{i+1} K_0(X)) = 0 \) for all \( i \in \mathbb{Z}_{>0} \),
and by the definiton of the topological filtration, \( \tau^j K_0(X) \subseteq \tau^i K_0(X) \) for all \( j > i \).
Thus, \( c_i(\tau^j K_0(X)) = 0 \) for all \( 0 < i < j \).

In [4] Ex. 15.3.6 it is shown that the composite \( c_i^* \circ p \) is simply multiplication
by \((-1)^{i-1}(i-1)!\). This result implies that \( c_i^* \) is an isomorphism for \( i \leq 2 \) and,
moreover, the map \( c_i^* \) is an isomorphism over the coefficient ring \( \mathbb{Z}[\frac{1}{(1-1)!}] \).

For any \( x \in K_0(X) \), let \( \gamma(x) = 1 + \gamma_1(x)t + \gamma_2(x)t^2 + \ldots \) denote the total characteristic class of \( x \) with values in \( K_0 \) as defined in [4] Ex. 3.2.7(b)]. We follow
the convention \( \gamma_1([L]) = 1 - [L^\vee] \) for any line bundle \( L \) over \( X \), where \( L^\vee \) denotes
the dual of \( L \).

1.2. Example. Using the Whitney sum formula we obtain the following results for \( i = 1, 2 \) by computing the total Chern classes,
\[
c(\gamma([L])) = c(1 - [L^\vee]) = \frac{1}{1-c_1(L)t} = 1 + c_1(L)t + c_1(L)^2 t^2 + \ldots
\]
This gives \( c_1(\gamma([L])) = c_1(L) \). Similarly,
\[
c(\gamma([L_1])\gamma([L_2])) = c((1 - [L_1^\vee])(1 - [L_2^\vee])) = \frac{c(1) \cdot c([L_1] \otimes [L_2])}{c([L_1]) \cdot c([L_2])}
\]
\[
= (1 - (c_1(L_1) + c_1(L_2))t)(1 + c_1(L_1)t + c_1(L_1)^2 t^2 + \ldots)(1 + c_1(L_2)t + c_1(L_2)^2 t^2 + \ldots)
\]
Hence \( c_2(\gamma([L_1])\gamma([L_2])) = -c_1(L_1)c_1(L_2) \).

By the definition of these characteristic classes (cf. [4] Ex 15.3.6]), we have in general
\[
(1) \quad c_i(\gamma([L_1]) \ldots \gamma([L_i])) = (-1)^{i-1}(i-1)! \cdot c_1(L_1) \cdot \ldots \cdot c_i(L_i).
\]

The Grothendieck \( \gamma \)-filtration on \( K_0(X) \) is defined by
\[
\gamma^i K_0(X) = \langle \gamma_{i_1}(x_1) \cdot \ldots \cdot \gamma_{i_m}(x_m) \mid i_1 + \ldots + i_m \geq i, \ x_i \in K_0(X) \rangle,
\]
(cf. [4] Ex.15.3.6], [3] Ch.3 and 5). Let \( \gamma^{i+1} K_0(X) \) denote the degree \( i \) component
\( \gamma^i K_0(X)/\gamma^{i+1} K_0(X) \) of the corresponding graded ring.

It is known that \( \gamma^i K_0(X) \) is contained in \( \tau^i K_0(X) \) for every \( i \geq 0 \), and they coincide for \( i \leq 2 \) (cf. [3] Prop.2.14]). Therefore, by Lemma [1.1] the Chern class map \( c_i \) restricted to \( \gamma^i K_0(X) \) vanishes on \( \gamma^{i+1} K_0(X) \), and induces a map
\[
c_i : \gamma^{i+1} K_0(X) \to CH^i(X).
\]

1.3. Example. For \( i = 1 \) we have \( \gamma^{1/2} K_0(X) = \tau^{1/2} K_0(X) \) and \( c_1 \circ p = id_{CH^1(X)} \), giving an isomorphism
\[
c_1 : \gamma^{1/2} K_0(X) \to CH^1(X).
\]

In the previous example, we saw that the map \( c_1 \) sends \( \gamma_1([L]) \) to \( c_1(L) \). For \( i = 2 \)
we again have \( \gamma^2 K_0(X) = \tau^2 K_0(X) \), but this time \( \gamma^3 K_0(X) \) does not necessarily
coincide with \( \tau^3 K_0(X) \). We may form an exact sequence,
\[
0 \to \tau^3 K_0(X)/\gamma^3 K_0(X) \to \gamma^{2/3} K_0(X) \to \tau^{2/3} K_0(X) \to 0.
\]
Replacing $\tau^{2/3}K_0(X)$ with $\text{CH}^2(X)$, the map $c_2 : \gamma^{2/3}K_0(X) \to \text{CH}^2(X)$ is surjective. In addition, $\ker(c_2) \cong \tau^3K_0(X)/\gamma^3K_0(X)$. Note that for all $i \geq 0$, (cf. [9 Prop. 2.14]) $\tau^iK_0(X) \otimes \mathbb{Q} \cong \gamma^iK_0(X) \otimes \mathbb{Q}$, thus $\ker(c_2)$ is torsion.

1.4. **Proposition.** The Chern class map $c_i^* : \gamma^{i+1}K_0(X) \to \text{CH}^i(X)$ is surjective over the coefficient ring $\mathbb{Z}_{(1/1)}$.

**Proof.** By Lemma 1.3, the map $c_i^* : \tau^{i+1}(X) \to \text{CH}^i(X)$ is surjective over the coefficient ring $\mathbb{Z}_{(1/1)}$. Since $\gamma^{i}K_0(X) \subseteq \tau^{i}K_0(X)$ for all $i$, we have an obvious map $\gamma^{i+1}K_0(X) \to \tau^{i+1}K_0(X)$ defined by sending $x + \gamma^{i+1}K_0(X) \mapsto x + \tau^{i+1}K_0(X)$. By definition, $c_i^*$ is the composition of these two maps.

Thus $\text{im}(c_i^*) \subseteq \text{im}(c_i^*)$, and so it remains to show that $\text{im}(c_i^*) \subseteq \text{im}(c_i^*)$ over $\mathbb{Z}_{(1/1)}$, for all $i$.

Consider an arbitrary element $x \in K_0(X)$. By the splitting principle we may assume that $x = L_1 \oplus \cdots \oplus L_n$ where $L_1, \ldots, L_n$ are line bundles over $X$. By the properties of characteristic classes, $\gamma_i(L_1 \oplus \cdots \oplus L_n) = 0$ for all $i > n$ and $\gamma_i(L_1 \oplus \cdots \oplus L_n) = s_i(\gamma_1(L_1), \ldots, \gamma_1(L_n))$ for all $0 < i \leq n$, where $s_i(\gamma_1(L_1), \ldots, \gamma_1(L_n))$ is the $i$-th elementary symmetric polynomial in variables $\gamma_1(L_1), \ldots, \gamma_1(L_n)$. Thus, taking the total Chern class, we have by (1) and Lemma 1.3

$c(\gamma(L_1 \oplus \cdots \oplus L_n)) = c(s_i(\gamma_1(L_1), \ldots, \gamma_1(L_n)))$

$$= \prod_{1 \leq j_1 < \cdots < j_i \leq n} (1 + (-1)^{i-1}(i-1)!c_1(L_{j_1}) \cdots c_1(L_{j_i})) t^i + \ldots$$

$$= 1 + (-1)^{i-1}(i-1)! s_i(c_1(L_1), \ldots, c_1(L_n)) t^i + \ldots$$

Thus, $c_i(\gamma_i(L_1 \oplus \cdots \oplus L_n)) = (-1)^{i-1}(i-1)! s_i(c_1(L_1), \ldots, c_1(L_n))$. In general,

$$c_i(\gamma_i(x)) = (-1)^{i-1}(i-1)! c_i(x) \text{ for all } x \in K_0(X).$$

□

2. Gamma filtration on twisted flag varieties

In the present section we discuss the $\gamma$-filtration on the variety of Borel subgroups. A reader is encouraged to consult [16] and [18] for further details.

Let $G$ be a split simple linear algebraic group of rank $n$ over a field $k$. We fix a split maximal torus $T$ and a Borel subgroup $B \supseteq T$ of $G$. Let $W$ denote the Weyl group of $G$ with respect to $T$. From now on, we let $X = G/B$ denote the variety of Borel subgroups of $G$. 
Let \( \{g_w\}_{w \in W} \) be the Steinberg basis of \( K_0(X) \) (cf. [19]). For each \( w \in W \), \( g_w \) is the class of a line bundle over \( X \), and together they form a \( \mathbb{Z} \)-basis of \( K_0(X) \). Combining this with the definition of the \( \gamma \)-filtration gives the following result.

2.1. Lemma. The \( i \)-th term of the \( \gamma \)-filtration on \( X = G/B \) is generated by products

\[
\gamma^{i/i+1} K_0(X) = \{ \gamma_1(g_{w_1}) \cdots \gamma_1(g_{w_i}) \mid w_1, \ldots, w_i \in W \}.
\]

Consider the twisted form \( \xi X \) of \( X \) by means of a \( G \)-torsor \( \xi \in Z^1(k, G) \). In general the group \( K_0(\xi X) \) is not generated by classes of line bundles. Hence, in the definition of \( \gamma^i K_0(\xi X) \), some higher characteristic classes \( \gamma_j(\cdot) \) may appear for \( j > 1 \).

For every field extension \( k'/k \) and variety \( Y \) over \( k \) we have a restriction map

\[
\text{res}_{k'/k} : K_0(Y) \to K_0(Y \times_k k').
\]

For every extension \( k'/k \), the map \( \text{res}_{k'/k} : K_0(X) \to K_0(X \times_k k') \) is an isomorphism. In particular, for a splitting field \( l \) of \( \xi \), \( K_0(\xi X \times_k k_l) \cong K_0(X \times_k k_l) \cong K_0(X) \), so we may consider the restriction map

\[
\text{res}_{l/k} : K_0(\xi X) \to K_0(X).
\]

The main result of [13] says that the image of \( \text{res}_{l/k} \) coincides with the sublattice

\[
\langle i_w, \xi g_w \rangle_{w \in W},
\]

where \( i_w, \xi \geq 1 \) are indices of the respective Tits algebras, which will be introduced in the next section.

Note that characteristic classes commute with restrictions, that is,

\[
\gamma_j \circ \text{res}_{k'/k} = \text{res}_{k'/k} \circ \gamma_j \quad \text{and} \quad c_j \circ \text{res}_{k'/k} = \text{res}_{k'/k} \circ c_j.
\]

We will use the following commutative diagram

\[
\begin{array}{ccc}
\gamma^{i/i+1} K_0(\xi X) & \xrightarrow{\text{res}} & \gamma^{i/i+1} K_0(X) \\
\downarrow \gamma^i \circ & & \downarrow \gamma^i \circ \\
\text{CH}^i(\xi X) & \xrightarrow{\text{res}_{\text{CH}}} & \text{CH}^i(X).
\end{array}
\]

2.2. Proposition. Consider the composite

\[
\phi_i : \gamma^{i/i+1} K_0(\xi X) \xrightarrow{\text{res}} \gamma^{i/i+1} K_0(X) \xrightarrow{\gamma^i \circ} \text{CH}^i(X).
\]

The image of \( \phi_i \) is generated by \( i_w, \xi c_1(g_w) \) for all \( w \in W \). In general, the image of \( \pi_i \) is generated by the elements

\[
(i - 1)! \left( \begin{array}{c} i \end{array} \right) \cdots \left( \begin{array}{c} i \end{array} \right) c_1(g_{w_1})^{i_1} \cdots c_1(g_{w_m})^{i_m}
\]

where \( i_1 + \cdots + i_m = i \) for all \( w_1, \ldots, w_m \in W \).
Proof. By the definitions of the restriction map and the \( \gamma \)-filtration on \( K_0(X) \), we can see that the image of \( \text{res}_\gamma(i) \) is generated by products

\[
\text{res}_\gamma(\gamma^{i+1} K_0(X)) = \{ \gamma_{i_1}(i_{w_1} g_{w_1}) \cdots \gamma_{i_m}(i_{w_m} g_{w_m}) | i_1 + \cdots + i_m = i \},
\]

where \( w_1, \ldots, w_m \in W \).

Since the \( g_w \)'s are line bundles, we have

\[
(3) \quad \gamma(i_w g_w) = \gamma(g_w)^i_w = (1 + \gamma_1(g_w)t)^i_w = \sum_{k=1}^{i_w} \binom{i_w}{k} \gamma_1(g_w)^k t^k
\]

Consider an element in \( \text{im}(\text{res}_\gamma(i)) \) of the form \( x = \gamma_{i_1}(i_{w_1} g_{w_1}) \cdots \gamma_{i_m}(i_{w_m} g_{w_m}) \) such that \( i_1 + \cdots + i_m = i \) for some \( w_1, \ldots, w_m \in W \). By (3),

\[
x = \gamma_{i_1}(i_{w_1} g_{w_1}) \cdots \gamma_{i_m}(i_{w_m} g_{w_m}) = \left( \frac{i_{w_1}}{i_1} \right) \cdots \left( \frac{i_{w_m}}{i_m} \right) \gamma_1(g_{w_1})^{i_1} \cdots \gamma_1(g_{w_m})^{i_m}.
\]

Taking the total Chern class and using Lemma [1.1] gives

\[
c(x) = c\left( \left( \frac{i_{w_1}}{i_1} \right) \cdots \left( \frac{i_{w_m}}{i_m} \right) \gamma_1(g_{w_1})^{i_1} \cdots \gamma_1(g_{w_m})^{i_m} \right)
\]

\[
= c(\gamma_1(g_{w_1})^{i_1} \cdots \gamma_1(g_{w_m})^{i_m}) \left( \frac{i_{w_1}}{i_1} \right) \cdots \left( \frac{i_{w_m}}{i_m} \right)
\]

\[
= (1 + (-1)^{i_1-1}(i-1)!c_1(\gamma_1(g_{w_1})^{i_1} \cdots \gamma_1(g_{w_m})^{i_m})t^{i_1} \cdots t^{i_m}),
\]

and so by [2],

\[
c_i(x) = (-1)^{i_1-1}(i-1)! \left( \frac{i_{w_1}}{i_1} \right) \cdots \left( \frac{i_{w_m}}{i_m} \right) c_1(\gamma_1(g_{w_1})^{i_1} \cdots \gamma_1(g_{w_m})^{i_m}).
\]

\[
\square
\]

3. Tits algebras and the \( J \)-invariant

Recall that we have defined \( G \) to be a split linear algebraic group of rank \( n \) over a field \( k \). Also, we have fixed a split maximal torus \( T \subset G \) and a Borel subgroup \( B \supset T \). Let \( T^* \) be the character group of \( T \), \( \{ \alpha_1, \ldots, \alpha_n \} \) a set of simple roots with respect to \( B \) and \( \{ \omega_1, \ldots, \omega_n \} \) the respective set of fundamental weights, so that \( \alpha_i(\omega_j) = \delta_{ij} \). We have \( \Lambda_r \subset T^* \subset \Lambda \), where \( \Lambda_r \) is the root lattice and \( \Lambda \) is the weight lattice. Consider the simply connected cover \( \tilde{G} \) of \( G \) with corresponding Borel subgroup \( \tilde{B} \) and maximal split torus \( \tilde{T} \). Given any \( \lambda \in \Lambda = \text{Hom}(T, \mathbb{G}_m) \), we can lift \( \lambda : T \to \mathbb{G}_m \) uniquely to \( \lambda : \tilde{B} \to \mathbb{G}_m \). Letting

\[
\tilde{G} \times \tilde{B} V_1 = \tilde{G} \times \mathbb{A}^1/(g, v) \sim (g \cdot b, \lambda(b)^{-1} \cdot v),
\]

the projection map \( \tilde{G} \times \tilde{B} V_1 \to \tilde{G}/\tilde{B} \) defines a line bundle \( L(\lambda) \) over \( \tilde{G}/\tilde{B} = G/B \), the variety of Borel subgroups of \( G \) (cf. [2, §1.5]).

For a fixed \( \xi \in Z^1(k, G) \), we can associate to each weight \( \lambda \) a central simple \( k \)-algebra \( A_{\xi, \lambda} \), called a Tits algebra of \( G \) (cf. [19]). We define the Tits map

\[
\beta_\xi : \Lambda/\Lambda_r \to Br(k)
\]
by sending $\bar{\lambda} \mapsto [A_{\xi, \bar{\lambda}}]$, its class in the Brauer group. This map is a group homomorphism for a fixed $\xi$, with $\bar{\lambda}_1 + \bar{\lambda}_2 \mapsto [A_{\xi, \bar{\lambda}_1}] \otimes [A_{\xi, \bar{\lambda}_2}]$.

Consider again the variety $X = G/B$. The degree 1 characteristic map in the simply connected case

$$c^{(1)}_*: \Lambda \to CH^1(X)$$

defines an isomorphism such that the cycles $h_i = c_1(L(\omega_i))$, $i = 1, \ldots, n$ form a $\mathbb{Z}$-basis of $CH^1(X)$. The degree 1 characteristic map is the restriction of this isomorphism to the character group $T^*$

$$c^{(1)}: T^* \to CH^1(X),$$

mapping $\lambda = \sum a_i \omega_i \mapsto c_1(L(\lambda)) = \sum_i a_i h_i$. In general, this defines the characteristic map

$$c: S^*(T^*) \to CH(X).$$

We denote by $\pi : CH^*(X) \to CH^*(G)$ the pull-back induced by the natural projection $G \to X$. By [7, Section 4, Rem. 2], $\pi$ is surjective and its kernel is given by the ideal $I \subset CH^*(X)$ generated by the non-constant elements in the image of the characteristic map. In particular, we have $I^{(1)} = \text{im}(c^{(1)})$, and

$$CH^1(G) \simeq CH^1(X)/(\text{im}(c^{(1)})) \simeq \Lambda/T^*.$$

Given a prime $p$, set $Ch(X) = CH(X) \otimes \mathbb{F}_p$. Taking $\mathbb{F}_p$-coefficients, we have

$$Ch^1(G) \simeq Ch^1(X)/(\text{im}(c^{(1)})) \simeq \Lambda/T^* \otimes \mathbb{F}_p.$$

It is known (cf. [8]) that $Ch(X)/I$ is isomorphic (as an $\mathbb{F}_p$-algebra, as well as a Hopf algebra) to a truncated polynomial ring of the form

$$Ch(X)/I \cong \mathbb{F}_p[x_1, \ldots, x_r]/(x_1^{k_1}, \ldots, x_r^{k_r})$$

for some integers $r$ and $k_i \geq 0$ for $i = 1, \ldots, r$, which are dependent on the group $G$. For each $i$, we let $d_i$ be the degree of the generator $x_i$. The number of generators of degree 1 is given by the dimension over $\mathbb{F}_p$ of the vector space $\Lambda/T^* \otimes \mathbb{F}_p$.

Let $s = \text{dim}_{\mathbb{F}_p}(Ch^1(G))$. Then, since $\omega_1, \ldots, \omega_n$ generate $\Lambda$ we may choose a minimal set $\{i_1, \ldots, i_s\} \subset \{1, \ldots, n\}$ such that the classes of $\omega_{i_1}, \ldots, \omega_{i_s}$ generate $\Lambda/T^* \otimes \mathbb{F}_p$. Then, $h_{i_l} = c_1(L(\omega_{i_l}))$, $l = 1, \ldots, s$ generate $Ch^1(X)$ and so we may take $x_l = \pi(h_{i_l})$, $l = 1, \ldots, s$ to be the generators of $Ch^1(G)$.

In fact, this definition of the generators $x_1, \ldots, x_s$ can be simplified using properties of the Steinberg basis. For $i = 1, \ldots, n$, $g_i := L(\omega_i) - L(\alpha_i)$. But, $\Lambda_r \subset T^*$ implies that $c_1(L(\alpha_i)) \in \text{im}(c^{(1)})$ and hence $c_1(L(\alpha_i)) \in ker(\pi)$ for all $i = 1, \ldots, n$. So, for each $l = 1, \ldots, s$, we have

$$\pi(h_{i_l}) = \pi(c_1(L(\omega_{i_l}))) = \pi(c_1(g_{i_l})) + \pi(c_1(L(\alpha_{i_l}))) = \pi(c_1(g_{i_l})).$$

Thus, we may take the generators of $Ch^1(G)$ to be $x_l = \pi(c_1(g_{i_l}))$ for $l = 1, \ldots, s$.

Let $H$ be the subgroup in $Br(k)$ generated by the classes of the Tits algebras $A_{\omega_{i_l}} = \beta_{\xi}(\bar{\omega}_{i_l})$ for $l = 1, \ldots, s$. Since $\Lambda/\Lambda_r$ is a finite abelian group, $H$ is a finite
abelian group as well. We define the common index $i_c$ of $\xi$ modulo $p$ by

$$i_c := \gcd\{\text{ind}(A_{\omega_{i_1}} \otimes \cdots \otimes A_{\omega_{i_s}}) \mid \text{at least one } a_i \text{ is coprime to } p\}.$$ 

### 3.1. Example

Let $G$ be a group of inner type $E_6$. In this case, $H$ is a cyclic group generated by a Tits algebra with index $3d$ for $d = 0, \ldots, 3$ [20 6.4.1]. Therefore, by the definition of the common index, we have $i_c = 3d$ as well.

We impose a well-ordering on the monomials $x_1^{m_1} \cdots x_r^{m_r}$ known as the DegLex order [19]. For ease of notation, we denote the monomial $x_1^{m_1} \cdots x_r^{m_r}$ by $x^M$, where $M$ is the $r$-tuple of integers $(m_1, \ldots, m_r)$, and set $|M| = \sum_{i=1}^r d_i m_i$. Given two $r$-tuples $M = (m_1, \ldots, m_r)$ and $N = (n_1, \ldots, n_r)$, we say that $x^M \leq x^N$ (or equivalently $M \leq N$) if either $|M| < |N|$, or $|M| = |N|$ and $m_i \leq n_i$ for the greatest $i$ such that $m_i \neq n_i$.

Consider the restriction map $\text{res}_{CH}: \text{CH}^*(\xi X) \rightarrow \text{CH}^*(X)$. Let $I_\xi$ denote the ideal generated by the non-constant elements in the image of $\text{res}_{CH}$. In [10 Thm. 6.4(1)] it is proven that $I_\xi \supseteq I$ and that there always exists a $\xi$ over some field extension of $k$ such that $I_\xi = I$. Such a $\xi$ is called a “generic” torsor.

This inclusion induces surjections $\text{CH}(X)/I \twoheadrightarrow \text{CH}(X)/I_\xi$ and $\text{Ch}(X)/I \twoheadrightarrow \text{Ch}(X)/I_\xi$. For each $1 \leq i \leq r$, we define $j_i$ to be the smallest integer such that $\xi_i$ contains an element $a$ of the form

$$a = x_i^{p j_i} + \sum_{x^M < x_i^{p j_i}} c_M x^M, \quad c_M \in \mathbb{F}_p.$$ 

While $r, d_i$ and $k_i$ for $i = 1, \ldots, r$ depend only on the group $G$, the values $j_1, \ldots, j_r$ depend also on the choice of $\xi$. Thus given the DegLex ordering defined above, we have a well-defined $r$-tuple $J_p,\xi(G) = (j_1, \ldots, j_r)$, called the $J$-invariant of $G$. We note that $(0, \ldots, 0) \leq (j_1, \ldots, j_r) \leq (k_1, \ldots, k_r)$ for any choice of $\xi$.

Let $J_p^{(1)} = \{j_i \mid d_i = 1\}$ be the subtuple of $J_p,\xi(G)$ consisting of only degree 1 parameters. We say that $J_p^{(1)} > m$ if for every index $j_i$ such that $k_i > m$ we have $j_i > m$.

### 4. The main result

For a fixed prime $p$, we have defined a minimal subset $\{\omega_1, \ldots, \omega_s\} \subset \Lambda$ such that the elements $x_i = \pi(c_1(g_{i_1}))$, $l = 1, \ldots, s$ generate $\text{Ch}^1(G)$. Let $I \subset \text{CH}(X)$ be the ideal generated by the non-constant elements in the image of the characteristic map $\epsilon : S^*(T^*) \rightarrow \text{CH}(X)$ and $I_\xi \subset \text{CH}(X)$ be the ideal generated by the non-constant elements in the image of the restriction map $\text{res}_{CH} : \text{CH}(\xi X) \rightarrow \text{CH}(X)$. For any integer $m$, we let $I^{(m)} \subset \text{CH}^m(X)$ and $I^{(m)}_\xi \subset \text{CH}^m(X)$ denote the homogeneous parts of these ideals of degree $m$.

### 4.1. Theorem

If $v_p(i_c) > 0$, then $I^{(1)}_\xi = I^{(1)}$. If $v_p(i_c) > 1$, then $I^{(m)}_\xi = I^{(m)}$ for $m = 2, \ldots, p$. 
Proof. Since we know already that \( I \subseteq I_\zeta \), it suffices to prove that \( I_\zeta^{(m)} \subseteq I^{(m)} \) for all \( m = 1, \ldots, p \) under the relevant hypothesis on \( \iota_\zeta \). By Proposition 1.4 and the commutative diagram in Section 2, we have that for any \( i \geq 0 \),

\[
\im(res_{CH}^{(i)}) = c_\gamma^i(\im(res_{\gamma}^{(i)})
\]

over the coefficient ring \( \mathbb{Z}[(\frac{1}{1-p})] \).

We begin first with the case \( m = 1 \). By the definition of \( I_\zeta \), we have \( I_\zeta^{(1)} = \im(res_{CH}^{(1)}) \). To show that \( I_\zeta^{(1)} \subseteq I^{(1)} \), we must prove that if \( v_p(i_\zeta) > 0 \), then for any \( w \in W \), the element \( i_w c_1(g_w) \) belongs (after tensoring with \( \mathbb{F}_p \)) to \( I^{(1)} = \im(c^{(1)}) \). Recall that \( g_w = L(\rho_w) \), and that we may write \( \rho_w = \sum_{i=1}^n a_i \omega_i \). Taking the total Chern class, we have

\[
c(g_w) = c(L(\omega_1)^{\bar{a}_1} \oplus \cdots \oplus L(\omega_n)^{\bar{a}_n}) = 1 + \left( \sum_{i=1}^n a_i c_1(L(\omega_i)) \right) t + \ldots,
\]

and hence,

\[
c_1(g_w) = \sum_{i=1}^n a_i c_1(L(\omega_i)) = \sum_{i=1}^n a_i (c_1(g_i) + c_1(L(\omega_i))) = \sum_{i=1}^n a_i c_1(g_i) \mod \im(c^{(1)}),
\]

by the results of Section 3. If all \( a_i \in \mathbb{Z} \) are divisible by \( p \), we are done. So we assume at least one \( a_i \) is coprime to \( p \).

Applying the Tits map \( \beta_\zeta \) to the class of \( \rho_w \), we get

\[
\beta_\zeta(\bar{\rho}_w) = \beta_\zeta(\sum_{i=1}^n a_i \bar{\omega}_i) = \bigotimes_{i=1}^n \beta_\zeta(\bar{\omega}_i)^{\otimes a_i} = \bigotimes_{i=1}^n [A_\xi \omega_i]^{\otimes a_i}
\]

By the assumptions that at least one of the \( a_i \) is coprime to \( p \) and that \( v_p(i_\zeta) > 0 \), we have \( \beta_\zeta(\bar{\rho}_w) \in H \setminus \{1\} \). Thus, \( p|\iota_\zeta = \im(\beta_\zeta(\bar{\rho}_w)) \), and so \( i_w c_1(g_w) = 0 \) in \( CH^1(X) \).

For the case \( m > 1 \) we work under the hypothesis that \( v_p(i_\zeta) > 1 \), and proceed by induction. We assume that the result \( I_\zeta^{(m')} \subseteq I^{(m')} \) holds for all \( m' < m \). It can be seen that

\[
I_\zeta^{(m)} = \left( \bigoplus_{j=1}^{m-1} CH^{m-j}(X) \cdot \im(res_{CH}^{(j)}) \right) \oplus \im(res_{CH}^{(m)}).
\]

By the inductive hypothesis, \( \im(res_{CH}^{(j)}) \subseteq I^{(j)} \subseteq I^{(j)} \) for \( 1 \leq j \leq m - 1 \), which implies that \( CH^{m-j}(X) \cdot \im(res_{CH}^{(j)}) \subseteq I^{(m)} \) for \( 1 \leq j \leq m - 1 \). It remains to show that \( \im(res_{CH}^{(m)}) \subseteq I^{(m)} \).

By Proposition 2.2 we know that \( \im(res_{CH}^{(m)}) = c_m(\im(res_{\gamma}^{(m)})) \), and is generated by elements of the form

\[
a = (m-1)!\left(\sum_{i=1}^{i_k} \cdot \sum_{i_k} \right) c_1(g_{w_1})^{i_1} \cdots c_1(g_{w_k})^{i_k},
\]

where \( i_1 + \cdots + i_k = m \), and \( w_1, \ldots, w_k \in W \).
If \( i_l < m \) for all \( l = 1, \ldots, k \), then \((i_{m_l})c_1(g_m)^{i_l} \in I^{(i)}\) by the inductive hypothesis. Therefore, \( a \in I^{(i_1)} \cdots I^{(i_k)} \subseteq I^{(m)} \). If, on the other hand, \( a \) is of the form \( a = (i_m)c_1(g_m)^m \), then we apply the previous argument. Namely, we have

\[
\rho_w = \sum_{i=1}^{n} a_i \omega_i,
\]

which implies

\[
(c_1(g_w))^m = \left( \sum_{i=1}^{n} a_i c_1(\omega_i) \right)^m = \left( \sum_{i=1}^{n} a_i (c_1(g_i) + c_1(\omega_i)) \right)^m.
\]

Again, \( c_1(\omega_i) \in \text{im}(c^{(1)}) \) implies \( c_1(\omega_i) \in I^{(1)} \) for all \( i = 1, \ldots, n \), and so all terms in the above expansion that are divisible by some \( c_1(\omega_i) \) are contained in \( I^{(m)} \). As in the previous case, we may write

\[
(c_1(g_w))^m = \left( \sum_{l=1}^{s} a_i c_1(g_i) \right)^m \mod I^{(m)}.
\]

If \( a_i \) is divisible by \( p \) for all \( l = 1, \ldots, s \), then we are done, so we assume that at least one \( a_i \) is coprime to \( p \). As before, this ensures that \( i_c \parallel \text{ind} (\beta_{\xi}(\bar{\rho}_w)) \), and so \( v_p(i_w) > v_p(i_c) \). It is clear that for any \( b \in \mathbb{Z}_{>0} \) if \( v_p(b) > 1 \) then \( p \mid \left( \binom{b}{l} \right) \) for all \( 1 \leq l \leq p \). Thus, under the hypothesis that \( v_p(i_c) > 1 \), we have \( p \mid \left( \binom{m}{s} \right) \), and so \( (i_m)(c_1(g_w))^m = 0 \) in \( CH^m(X) \).

With this we obtain the following result, which can be seen as a generalization of Theorem 3.8 in [10].

**4.2. Corollary.** Let \( p \) be a prime number. If \( v_p(i_c) > 0 \), then \( J_p^{(1)} > 0 \). If \( v_p(i_c) > 1 \), then \( J_p^{(1)} > 1 \).

**Proof.** Consider the diagram

\[
\begin{array}{ccc}
Ch(\xi, X) & \xrightarrow{\text{res}_{CH}} & Ch(X) \\
\downarrow{\pi} & & \downarrow{\pi} \\
Ch(G) & & Ch(G)
\end{array}
\]

We begin first with the hypothesis that \( v_p(i_c) > 0 \).

Let \( R_\xi = \text{im}(\pi \circ \text{res}_{CH}) \). Then \( \pi(a) \in R_\xi^{(1)} \) implies that \( a \in \text{im}(\text{res}_{CH}^{(1)}) = I^{(1)} \) by Theorem 4.1. Thus, \( \pi(a) = 0 \in CH^1(X)/I \) and so \( R_\xi^{(1)} = \{0\} \). Let \( x_1, \ldots, x_s \) be generators of degree 1 in \( CH^1(G) \). By the definition of the \( J \)-invariant, \( j_1 \) is the smallest non-negative integer \( m \) such that \( x_1^m \in R_\xi \). Since \( x_1 \) is non-trivial, we must have \( x_1^p = x_1 \notin R_\xi^{(1)} \) by the above argument, and so \( j_1 > 0 \).

The same argument applies for the remaining generators. Let \( 1 < i \leq s \), then if \( x^M < x_i, x^M = x_j \) for some \( j < i \). Since \( x_i + a_{i-1}x_{i-1} + \cdots + a_1x_1 \) is non-trivial for any \( a_1, \ldots, a_{i-1} \in \mathbb{F}_p \), it cannot belong to \( R_\xi^{(1)} \). Therefore \( j_i > 0 \), and so \( J_p^{(1)} > 0 \).

Under the hypothesis that \( v_p(i_c) > 1 \), suppose again that \( x_1, \ldots, x_s \) are a minimal set of generators of degree 1 in \( CH(G) \). We have the inclusion \( \text{im}(\text{res}_{CH}^{(p)}) \subset I^{(p)} \).
and by Theorem 4.1, $I_\chi^{(p)} = I^{(p)}$. Again, $R_\chi^{(p)} = \im(\pi \circ \res_{\CH}) = \{0\}$. To show that $J_p^{(1)} > 1$, we begin with the generator $x_1$. If $k_1 \leq 1$ we are done, so suppose $k_1 > 1$. Then, $x_1^{k_1} = x_1^2 \in CH^{p}(G)$ is non-trivial, and so $x_1^p \notin R_\chi^{(p)}$ implies $j_1 > 1$. Again, we extend the argument for the remaining generators. Suppose that $k_i > 1$ for some $1 < i \leq s$. Then, the element

$$\pi(a) = x_i^p + \sum_{(x^M < x_i^2) \cap (|M| = p)} a_M x^M$$

is non-trivial for any $a_M \in \mathbb{F}_p$ and hence $\pi(a) \notin R_\chi^{(p)}$, and so $j_i > 1$. Thus $J_p^{(1)} > 1$. \hfill \square

5. Applications

We now apply the results of the previous section to some $E_6$ varieties. For this, we will require the following result concerning the possible values of the $J$-invariant.

5.1. Lemma. Let $G$ be a semisimple algebraic group of inner type over $k$, $p$ a prime integer and $J_p(G) = (j_1, \ldots, j_r)$. Assume $d_i = 1$ for some $i = 1, \ldots, r$. Then $j_i \leq \max v_p(\ind A)$, where $A$ runs through all Tits algebras of $G$. Conversely, if there exists a Tits algebra $A$ of $G$ with $v_p(\ind A) > 0$, then $j_i > 0$ for at least one $i$ having $d_i = 1$.

Proof. For the first statement, we note that for any $w \in W$, $c_1(\mathcal{L}(\rho_w))^{p^b} \in I_\chi$, where $b = v_p(i_{\rho_w})$ [16, Lemma 1.12]. Since $\beta(\omega_{s_i}) = \beta(\omega_i)$ for $l = 1, \ldots, s$, this implies $i(\rho_{s_i}) = i(\omega_i)$. Letting $b_l = v_p(i_{\omega_i})$, we then have $c_1(g_{s_i})^{p^{b_l}} \in I_\chi$, and hence $j_l \leq b_l \leq \max v_p(\ind A)$ for all Tits algebras $A$ of $G$. For proof of the second statement, see [14, Prop. 4.2]. \hfill \square

5.2. Example. Let $G$ be a simple group of type $B$, $C$, $E_6$, or $E_7$. Then, $H$ is a cyclic group of order $p = 2$ or $3$, and $J_p^{(1)}$ consists of a single integer $j_1$ [1, Chap. 6]. As a consequence of Corollary 4.2 and Lemma 5.1, $j_1 = 0$ if and only if all of the Tits algebras of $G$ are split.

Let $G$ be a group of inner type $E_6$. As in the above example, $H$ is generated by a single Tits algebra $A$ of index $3^d$ for some $d = 0, \ldots, 3$. Consider the $J$-invariant of $G$ modulo $p = 3$. We note that $CH(G)$ has precisely two generators $x_1$ and $x_2$, with $d_1 = 1$ and $d_2 = 4$, where the 3-power relations are defined by $k_1 = 2$ and $k_2 = 1$ [8, Table II]. Thus

$$J_3(G) = (j_1, j_2),$$

where $j_1 = 0, 1, 2$, $j_2 = 0, 1$. These values are independent of the characteristic of the base field.

5.3. Proposition. Let $G$ be a group of inner type $E_6$ and let $A$ be its Tits algebra. Then, $\ind A = 1$ if and only if $j_1 = 0$. As well, $\ind A = 3$ if and only if $j_1 = 1$.
Proof. Since $A$ is a generator of the group $H$, the first case is a direct consequence of the above example. For the second case, suppose first that $j_1 = 1$. By Corollary 4.2 this implies that $\text{ind } A = 1$ or 3. However, by the first case, $j_1 \neq 0$ implies $\text{ind } A \neq 1$ and so $\text{ind } A = 3$. Conversely, suppose $\text{ind } A = 3$. Then $\nu_3(\text{ind } B) \leq 1$ for all $[B] \in H$. By Lemma 5.1, $j_1 \leq \max(\nu_3(\text{ind } B)) = 1$. Again by the first case, $\text{ind } A \neq 1$ implies $j_1 \neq 0$ and so $j_1 = 1$. \qed

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