RESTRICTED SIMPLE LIE (SUPER)ALGEBRAS IN CHARACTERISTIC 3

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ABSTRACT. We give explicit formulas proving restrictedness of the following Lie (super)algebras: known exceptional simple vectorial Lie (super)algebras in characteristic 3, deformed Lie (super)algebras with indecomposable Cartan matrix, and (under certain conditions) their simple subquotients over an algebraically closed field of characteristic 3, as well as one type of the deformed divergence-free Lie superalgebra with any number of indeterminates in any characteristic.

1.1. Introduction. Recall that a Lie algebra $g$ over a field $K$ of characteristic $p > 0$ is called restricted or with a $p$-structure $x \mapsto x^p$, if for any $x \in g$, we have $(\text{ad}_x)^p = \text{ad}_{x^p}$ for some $x^p \in g$. A module $M$ over a restricted Lie algebra $g$, and representation $\rho$ defining $M$ are called restricted if $\rho(x^p) = \rho(x)^p$. A Lie superalgebra $g = g_0 \oplus g_1$ is restricted if $g_0$ is restricted and $g$ is a restricted $g_0$-module. Thanks to squaring, i.e., the map $x \mapsto x^2(= \frac{1}{2}[x,x]$ if $p \neq 2$) for any $x \in g_1$, any restricted Lie superalgebra has a $2p$-structure, i.e., a map $x \mapsto x^{2p}$ for any $x \in g_1$.

In his Appendix to [LL], P. Deligne advised us to investigate first of all the restricted Lie (super)algebras and their restricted modules as related to geometry and hence of interest. This note is an addendum to [BLLS], in which several general statements on restrictedness valid for any $p > 0$ are formulated, to [BGL2], where the Cartan matrices and Chevalley generators for modular Lie superalgebras are defined, and to [BGL1, GL3, BGLLS, BGLLS1] describing Lie (super)algebras considered here. The main result of [BLLS] deals with $p = 2$; here we give examples for $p \neq 2$, mainly for $p = 3$. The ground field is algebraically closed.

Classification [BW] is implicit: to explicitly define $p|2p$-structure on a simple Lie superalgebra $g$ it suffices to give expressions of $w^p$ (resp. $w^{2p}$) for all even (resp. odd) elements of any basis of $g$. We give, at last, the explicit answer in case $\text{svect}_{(1+\bar{u})}(m; 1|2s)$, see (4); the deforms of series $\mathfrak{h}$ will be considered elsewhere.

No classification of simple Lie superalgebras is yet available for any $p > 0$, or of simple Lie algebras for $p = 3$ and 2, except for Lie (super)algebras with indecomposable Cartan matrix, and their simple subquotients, see [BGL2], whose $p|2p$-structure, if exists, is given explicitly. These Lie (super)algebras are “symmetric”, i.e., have a symmetric root system. For the classification of true deformations, i.e., results of deformations which are neither trivial nor semitrivial, of symmetric Lie (super)algebras whose restrictedness we establish here, see [BLW, BGL3].

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We also consider vectorial Lie (super)algebras. Following Bourbaki we use Gothic font for Lie (super)algebras; \( 1 := (1, \ldots, 1) \) is the shearing vector with the smallest heights of divided powers. Proofs of lemmas, Fact \((3)\), formulas \((1)\) and \((4)\) are obtained with the help of the SuperLie code, see [Gr].

1.2. Deforms of Lie (super)algebras with indecomposable Cartan matrix. In Lemmas \(1.2.2\) and \(1.2.3\) the cocycles \( c_k \) and the elements of the Chevalley basis \( x_i \) (resp. \( y_i \)) corresponding to the positive (resp. negative) roots are given for the Cartan matrix given in [BGL3], let the \( h_j := \{x_j, y_j\} \) be the elements of the maximal torus.

1.2.1. Deforms of \( \mathfrak{o}(5) \) for \( p = 3 \). Recall that the contact bracket of two divided powers \( f, g \in \mathcal{O}(p, q; t; \mathbb{N}) \) is defined to be
\[
\{f, g\}_{k.b.} = \triangle f \cdot \partial_t g - \partial_t f \cdot \triangle g + \partial_p f \cdot \partial_q g - \partial_q f \cdot \partial_p g \text{ with } \triangle f = 2f - p\partial_p f - q\partial_q f.
\]

A basis of \( \mathcal{L}(\varepsilon, 0, 0) \) is expressed in terms of generating functions of \( \mathfrak{t}(3; 1) \) and root vectors of \( \mathfrak{o}(5) \) as follows, see [BLW] Prop. 3.2

| deg | the element's weight \( \sim \) its generating function (=Chevalley basis vector) |
|-----|------------------------------------------------------------------------------|
| -2  | \( E_{-2\alpha-\beta} = [E_{-\alpha}, E_{-\alpha-\beta}] \sim 1 (= y_1) \);          |
| -1  | \( E_{-\alpha} \sim p(= y_2) \); \( E_{-\alpha-\beta} = [E_{-\beta}, E_{-\alpha}] \sim q(= y_3) \); |
| 0   | \( H_\alpha \sim -\varepsilon t + pq(= h_2) \); \( H_\beta \sim -pq(= h_1) \); \( E_\beta \sim p^2(= y_1) \); \( E_{-\beta} \sim -q^2(= x_1) \); |
| 1   | \( E_\alpha \sim -(1 + \varepsilon) pq^2 + \varepsilon q t(= x_2) \); \( E_{\alpha+\beta} = [E_\beta, E_\alpha] \sim (1 + \varepsilon) p^2 q + \varepsilon p t(= x_3) \); |
| 2   | \( E_{2\alpha+\beta} = [E_\alpha, E_{\alpha+\beta}] \sim \varepsilon(1 + \varepsilon) p^2 q^2 + \varepsilon^2 t^2(= x_4) \). |

Nonzero values of the deformed bracket with parameters \( \delta \) and \( \rho \) are as follows:
\[
\begin{align*}
[E_{-2\alpha-\beta}, E_\beta] &= \delta E_\alpha, & [E_{-2\alpha-\beta}, E_\alpha] &= \delta E_{-\beta}, & [E_{-\alpha}, E_\beta] &= -\delta \varepsilon E_{2\alpha+\beta}, \\
[E_{-2\alpha-\beta}, E_{-\alpha-\beta}] &= \rho E_\beta, & [E_{-\alpha-\beta}, E_\beta] &= -\rho E_{2\alpha+\beta}, & [E_{-2\alpha-\beta}, E_{-\beta}] &= -\rho E_{\alpha+\beta}.
\end{align*}
\]

As proved in [Kos], in the family \( \mathcal{L}(\varepsilon, \delta, \rho) \), only \( \mathcal{L}(2, 0, 2) \) and Brown algebras \( \mathfrak{br}(2; \varepsilon) := \mathcal{L}(\varepsilon, 0, 0) \) for \( \varepsilon \neq 0 \) represent classes of non-isomorphic Lie algebras up to isomorphisms \( \mathfrak{br}(2; \varepsilon) \sim \mathfrak{br}(2; \varepsilon') \) if and only if \( \varepsilon \varepsilon' = 1 \) for \( \varepsilon \neq \varepsilon' \); observe that \( \mathfrak{br}(2; -1) \simeq \mathfrak{o}(5) \simeq \mathfrak{sp}(4) \).

1.2.1a. Lemma. The 3-structure on \( \mathcal{L}(\varepsilon, \delta, \rho) \) is given by the formulas
\[
\begin{align*}
h_1^{[3]} &= h_1, & h_2^{[3]} &= \varepsilon^2 h_2, & y_2^{[3]} &= \delta(1 + 2\varepsilon^2)h_1 + \frac{\delta}{\varepsilon}(1 + 2\varepsilon^2)h_2, & y_3^{[3]} &= \frac{\varepsilon}{\delta}h_2, \\
y_4^{[3]} &= \varepsilon \delta \rho(2 + \varepsilon^2) y_1, & y_1^{[3]} &= x_1^{[3]} = x_2^{[3]} = x_3^{[3]} = x_4^{[3]} = 0.
\end{align*}
\]

In [Kos], Rudakov’s claim “\( \mathcal{L}(\varepsilon, \delta, \rho) \) is restricted” is cited but the explicit formulas \((1)\) were never published, as far as we know.

1.2.2. Lemma. Let \( \mathfrak{g}_{c_k} \) be the deform with even parameter \( \lambda \) corresponding to the cocycle \( c_k \) of \( \mathfrak{g} = \mathfrak{br}(3) \) or \( \mathfrak{brj}(2; 3) \). The Lie (super)algebras \( \mathfrak{g}_{c_k} \), and those symmetric to them \((x \leftrightarrow y)\), are restricted. For any \( k \), the \( p^2 \)p-maps vanish on all weight vectors, except the following ones: \( h_i^{[3]} = h_i \) for all \( i \) and \( x_3^{[3]} = -\lambda h_3 \) for \( \mathfrak{br}(3)_{c=3} \), and also \( x_5^{[3]} = \lambda(h_2 + h_3) \) for \( \mathfrak{br}(3)_{c=5} \), and also \( x_5^{[3]} = -\lambda(h_1 + h_2 + h_3) \) for \( \mathfrak{br}(3)_{c=9} \); and \( x_1^{[3]} = \lambda(h_1 - h_2) \) for \( \mathfrak{br}(3)_{c=18} \). Besides, for \( \mathfrak{brj}(2; 3)_{c=12} \), we have \( x_6^{[3]} = 2\lambda h_1 \) and for \( \mathfrak{brj}(2; 3)_{c=6} \), we have \( x_3^{[3]} = \lambda(h_1 - h_2) \).

1.2.3. Lemma. Let \( \mathfrak{g}_{c_k} \) be the deform with odd parameter \( \tau \) corresponding to the cocycle \( c_k \) of \( \mathfrak{g} = \mathfrak{g}(1, 6) \) or \( \mathfrak{g}(4, 3) \) or \( \mathfrak{g} = \mathfrak{g}(2, 3) \). The Lie superalgebras \( \mathfrak{g}_{c_k} \), corresponding
to the cocycles $c_k$, and those symmetric to them $(x \leftrightarrow y)$, are restricted. For any $k$, the $p|2p$-maps vanish on all weight vectors, except the following ones: $h_i^{[3]} = h_i$ for all $i$ for $g = \mathfrak{g}(1,6)$ and $g(4,3)$, and also $d^{[3]} = d$ for $g = \mathfrak{g}(2,3)$ modulo the central element $c = h_1 - h_2$.

1.3. Fact. Let $g_0$ be a restricted Lie (super)algebra and $g_{-1}$ an irreducible restricted $g_0$-module that generates the Lie superalgebra $g_0 = \oplus_{-d < i < 0} g_i$. Let vectorial Lie (super)algebra $g$ (sdim; $N$), where sdim is the superdimension of $g_-$, be the prolong, i.e., the result of generalized Cartan prolongation, see [Shch], of the pair $(g_{-1}, g_0)$. It is easy to see that the Lie superalgebra $g$ (sdim; $N$) is not restricted if $N \neq 1\over 2$, see [BLLS]; the proof of this statement for Lie algebras was first published in [KFD] Th.2.

Fact. If $\mathbb{Z}$-graded vectorial Lie (super)algebra $g := \mathfrak{g}(\text{sdim}; 1) —$ the generalized Cartan prolong of its non-positive components, see [Shch], $—$ is restricted, and the $i$-th derived (super)algebra $g^{(i)}$ of $g$ contains a maximal torus of $g$, then $g^{(i)}$ is restricted

1.3.1. New examples. The left column in (3) shows where the simple Lie (super)algebras in the right column are described for any $N$; for them eqs. (2) hold:

| $[BL]$, $[BGL]$, $[BGLS]$ | $\mathfrak{so}^{(1)}(10; 1)$, $\mathfrak{so}^{(1)}(7; 1)$, $\mathfrak{so}^{(1)}(7; 1)$, $\mathfrak{m}(6; 1)$, $\mathfrak{s}(6; 1)$, $\mathfrak{c}^{(1)}(3; 1)$, $\mathfrak{f}(3; 1)$ |
| $\mathfrak{B}(1; 1|7)$, $\mathfrak{M}(3; 1|3)$, $\mathfrak{B}(3; 1|5)$, $\mathfrak{N}(4; 1|5)$, $\mathfrak{B}(1; 3|4)$, $\mathfrak{B}(1; 1|3)$ |

1.3.2. New formulas. The deformations of simple derived superalgebras of Lie superalgebras of the form $\mathfrak{g}(\text{sdim}; N|2s)$ are not classified, we consider only one example. Let $\mathfrak{svect}(m; N|2s)$ be the Lie superalgebra preserving the volume element vol in even indeterminates $u_1, \ldots, u_m$ and odd ones $u_{m+1}, \ldots, u_{m+2s}$, let $\bar{u} = u_1(\partial_{u_1} - 1) \cdots u_m(\partial_{u_m} - 1) u_{m+1} \cdots u_{m+2s}$ and $\mathfrak{svect}(1+\bar{u})(m; N|2s)$ be the deformation preserving $(1 + \bar{u})$ vol. For $\mathfrak{svect}(1+\bar{u})(m; 1|2s)$ and $\mathfrak{svect}(1+\bar{u})(m; 1|2s)$ in characteristic $p > 0$, we have (only distinctions with (2) are given):

$$((1 - \bar{u})\partial_t)[p] = -(\partial_t^{p-1} \bar{u})\partial_t,$$

where $\partial_t$ is even.

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