On the Stability of Black Strings/Branes

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Abstract

Some issues on the stability of black string or brane solutions are summarized briefly. The stability of dS/AdS-Schwarzchild black strings has been investigated. Interestingly, the AdS-Schwarzchild black strings turn out to be stable as the horizon size becomes larger than the AdS scale. It is also shown that BTZ black strings in four dimensions are stable regardless of the horizon size. Such stable feature seems to be common for several known black strings in dimensions lower than five. Some implications of our results on the role of non-uniformity in stable black string configurations are also discussed.

I. INTRODUCTION

Black holes are solutions of Einstein gravity, which possess usually a compact null hypersurface with topology of sphere. Black strings or branes are a sort of higher dimensional generalizations of black holes. The simplest black strings or branes are the product of Schwarzchild black holes and an infinite line or a \( p \)-dimensional plane. The extra dimensions can also be compactified. Black string/brane solutions arise naturally in string theory as well. One of the most important reasons that one considers black hole solutions seriously, inspite of their odd causal structure, is that they are stable. The stability of black strings/branes was studied by Gregory and Laflamme [1]. They showed that black strings/branes are generically unstable to linearized perturbations with a large wavelength along the string. For example, the four-dimensional Schwarzchild black hole cross a circle of length \( L \) becomes unstable under linearized perturbations as \( L \) becomes larger than the order of the Schwarzchild radius \( r_h \).

One naive explanation for this instability is the entropy comparison with that of a black hole with the same mass. Namely, a black hole configuration is entropically preferable as the size of the black string \( L \) increases. However, the idea of assigning entropy to black holes

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or strings is based on the quantum behavior of such black objects which is not necessarily related to the classical stability behavior of the black string. Moreover, this argument of entropy comparison is global in nature. Recently, Gubser and Mitra have refined this global thermodynamic analysis by considering the local thermodynamic properties of the black object [2]. They conjectured that a black string/brane with noncompact translational symmetry is classically stable if, and only if, it is locally thermodynamically stable. A slightly modified version of this conjecture applicable even for compact cases is recently stated by Hubeny and Rangamani [3]. The proof of the Gubser-Mitra (GM) conjecture is sketched by Reall [4], and a more complete illustration in the simplest case of Schwarzschild black strings/branes is given by Gregory and Ross [5]. The GM conjecture practically provides a very powerful and easy way to check whether given black string/brane solutions are stable or not, compared to the usually complicated numerical analysis of classical linearized perturbations.

The black string instability found at the linear level was believed to indicate that the full nonlinear evolution of the instability would result in the fragmentation of the black string. In order to be consistent with the classical theorem of no bifurcation of the event horizon, presumably quantum effects will play some important role when the shrinkage of the string horizon reached to the stage of large enough curvature. This widely accepted idea of black string fragmentations has been used in many discussions in the literature, involving black string/brane configurations. Recently, however, Horowitz and Maeda have shown that black strings do not break up within a finite affine time in their full classical nonlinear evolution [6]. Thus, unstable black strings are likely to evolve into black string-like configurations which are probably non-uniform and stable even under linearized perturbations [7]. In addition, the recent numerical study about approximate non-uniform black string solutions by Gubser [8] indicates that such transition is not continuous, but the first order in thermodynamic nature.

In order to understand the black string instability better, it will be of interest to see whether or not stable black string/brane solutions exist, and, if it does so, to understand what makes them stable. The only known black string/brane solutions even when \( L \geq r_h \) are the extremal black \( p \)-branes carrying certain charges [9]. In this talk, I present some black string/brane solutions that reveal stable behavior under linearized perturbations. One type of them is AdS-Schwarzschild black strings in anti-de Sitter space with/without a uniform tension brane [10]. These black strings/branes become stable as the horizon radius becomes larger than the order of the AdS radius of the background geometry perpendicular to black strings/branes. The other type is black string/brane solutions in the spacetime dimensions lower than five [11]. In particular, the BTZ black strings in four dimensions turn out to be stable always, regardless of the transverse horizon size. Some physical implications of our results and some open issues are also discussed.

II. STABLE BLACK STRINGS IN ANTI-DE SITTER SPACE

In this section, I briefly summarise the results in Ref. [10] and give some further results obtained. In the five-dimensional Einstein gravity with negative cosmological constant \( \Lambda_5 = -6/l_5^2 \), one can have black string solutions whose metrics are given by

\[
ds^2 = H^{-2}(z)(\gamma_{\mu\nu}dx^\mu dx^\nu + dz^2),
\]  

(1)
\[ = H^{-2}(z) \left[ - f(r) dt^2 + \frac{1}{f(r)} dr^2 + r^2 d\Omega_3^2 + dz^2 \right], \]  

(2)

where the warping factors are

\[ H(z) = \begin{cases} 
\frac{l_4}{l_5} \sinh z/l_4, & \Lambda_4 > 0 \\
\frac{z}{l_5}, & M_4(\Lambda_4 = 0) \\
\frac{l_4}{l_5} \sin z/l_4, & \Lambda_4 < 0,
\end{cases} \]  

(3)

and \( f(r) = 1 - r_0/r - \Lambda_4 r^2/3 \). Here the four-dimensional cosmological constant \( \Lambda_4 = \pm 3/l_4^2 \) is arbitrary. When \( r_0 = 0 \), these metrics actually describe the same five-dimensional pure anti-de Sitter spacetime (AdS\(_5\)), and simply correspond to different ways of slicing it, e.g., de Sitter, flat, and anti-de Sitter slicings. If a 3-brane with uniform tension is introduced at \( z = 0 \), the resulting geometries are still described by the metrics above with replacing \( z \to |z| + c \) and now \( \Lambda_4 \) is determined by the location and the tension of the 3-brane accordingly. The stability for the flat case (i.e., \( \Lambda_4 = 0 \)) has been already studied in Refs. [2,3].

In order to check the linearized stability, let us consider small metric perturbations, \( g_{MN}(x) \to g_{MN}(x) + h_{MN}(x, z) \), about these black string background spacetimes, and see whether or not there exists any mode which is regular spatially, but grows exponentially in time. As in the usual Kaluza-Klein reduction, the massive scalar \((h_{zz})\) and vector \((h_{\mu z})\) fluctuations can always be set to be zero by using the five-dimensional diffeomorphism symmetry, leaving only four-dimensional massive spin-2 fluctuations. Moreover, the scalar and vector components of the five-dimensional linearized perturbation equations reduce to the four-dimensional transverse traceless gauge conditions for the massive spin-2 fluctuations \( h_{\mu \nu} \) [4,9]. By putting \( h_{\mu \nu}(x, z) = H^{3/2}(z)\xi(z)h_{\mu \nu}(x) \), the linearized equations turn out to be

\[ \Delta_I h_{\mu \nu}(x) \equiv \Box h_{\mu \nu}(x) + 2R_{\rho \sigma} h^{\rho \sigma}(x) = m^2 h_{\mu \nu}(x), \]  

(4)

\[ \nabla^\mu h_{\mu \nu} = 0, \quad h = \gamma^{\mu \nu} h_{\mu \nu} = 0, \]  

(5)

\[ [- \partial_z^2 + V(z)]\xi(z) = m^2 \xi(z), \quad V(z) = -\frac{3}{2} \frac{H''}{H} + \frac{15}{4} \left( \frac{H'}{H} \right)^2. \]  

(6)

Notice first that, even if the black strings we are considering are not uniform due to the warping factors along the string, the perturbation equations above become separable as in the case of translationally invariant black strings. The only difference is the spectrum of the Kaluza-Klein (KK) mass with the non-vanishing effective potential \( V(z) \) in Eq. (6). When \( m^2 = 0 \), as is pointed out in Ref. 4, Eqs. (4) and (5) are exactly same as those for perturbations about four-dimensional black hole spacetimes, which are known to be stable. Since adding a mass term usually increases stability, it had been believed for some time that black strings are stable. However, it turns out that the extra degrees of freedom coming from the massiveness could give unstable solutions [1,4].

The strongest instability is expected for the s-wave fluctuations. General, spherically symmetric perturbations can be written in canonical form as [15,1]

\[ h_{\mu \nu}(x) = e^{\Omega t} \begin{pmatrix} H_{tt}(r) & H_{tr}(r) & 0 & 0 \\
H_{tr}(r) & H_{rr}(r) & 0 & 0 \\
0 & 0 & K(r) & 0 \\
0 & 0 & 0 & K(r) \sin^2 \theta \end{pmatrix}, \]  

(7)
FIG. 1. The left figure is for the AdS case with \( r_0 = 1, 2 \) and 4. The right figure is for the dS case with \( r_0 = 1, 2, 3.5, \) and 3.8. The Nariai solution corresponds to \( r_0 \approx 3.85 \). The fixed AdS and dS radius is \( l_4 = 10 \). The straight vertical lines denote the lowest KK masses, 0.4 for AdS and 0.15 for dS.

with \( \Omega > 0 \). From the coupled equations in Eqs. (4) and (5), we can eliminate all but one variable, say \( H_{tr} \), obtaining a second order ordinary differential equation in the following form:

\[
A(r; r_0, \Lambda_4, \Omega^2, m^2) H''_{tr} + BH'_{tr} + CH_{tr} = 0.
\]  \tag{8}

With suitable boundary conditions (see the details in Ref. [10]), one can solve this equation numerically, and unstable solutions for the massive spin-2 fluctuations in Eq. (4) are shown in Fig. 1 for given dS/AdS-Schwarzschild black hole background spacetimes.

As shown in Ref. [10] explicitly, the most important observation is that adding a negative cosmological constant \( \Lambda_4 \) has a stabilization effect. As the horizon size increases, the instability becomes weak as in the case of Schwarzschild black holes [12]. For the AdS case, however, the threshold mass \( m_* \equiv m(r_0, \Lambda_4, \Omega = 0) \) vanishes even at a finite \( r_0 \), not as \( r_0 \to \infty \). The numerical search shows that this termination of unstable solutions occurs approximately at \( r_0 \approx 0.77 l_4 \) (i.e., \( r_+ \approx 0.58 l_4 \)). In addition to it, the KK mass spectrum determined by Eq. (6) has a finite mass gap as indicated by the vertical line in Fig. 1. Therefore, it turns out that AdS4-Schwarzschild black strings in AdS5 space become stable when the horizon size is \( r_+ > r_+^{ct} \approx 0.20 l_4 \). The presence of a 3-brane as in the brane world model [16] simply increases this critical value. For the dS case, on the other hand, the threshold masses remain larger than the lowest KK mass for all horizon radii bounded by the cosmological horizon. In addition, since the KK mass spectrum is continuous, all dS4-Schwarzschild black strings in AdS5 space are unstable.

The local thermodynamic stability of a segment of AdS/dS black string will be determined by the sign of the heat capacity given by \( dM/dT \sim -2\pi r_+^2 (1 - \Lambda_4 r_+^2)/(1 + \Lambda_4 r_+^2) \). For the AdS case, one can easily see that the heat capacity becomes positive for \( r_+ > l_4/\sqrt{3} \). Thus, we expect AdS black strings become stable classically when \( r_+ > l_4/\sqrt{3} \approx 0.58 l_4 \) according to the GM conjecture. The slight difference in the critical values is expected because of the non-uniformity of the AdS black string due to the warping factor [10]. On the other hand, dS black strings are expected to be unstable classically since the heat capacity is always negative for all \( r_+ < l_4/\sqrt{3} \) within the cosmological horizon.

III. LOWER DIMENSIONAL BLACK STRINGS/BRANES

It was argued in Ref. [17] that the instability of BTZ black strings in the four-dimensional brane worlds sets in when the transverse size of the black string reaches the AdS scale. Here we show that this naive expectation, which is based on the entropy comparison with a localized black hole around a 2-brane with same mass, is not true. Surprisingly, they turn out to be stable always. The metric of rotating BTZ black strings in four dimensions can be written by

\[ ds^2 = H^{-2}(z) \left[ -f dt^2 + \frac{dr^2}{f} + r^2 (d\varphi - \frac{J}{2r^2} dt)^2 + dz^2 \right], \]

where \( f(r) = -M + r^2/l_3^2 + J^2/4r^2 \) and \( H(z) = l_3/l_4 \sin z/l_3 \). The first evidence for the stable behavior can easily be obtained by applying the GM conjecture. The local thermodynamic stability of a segment of this black string is governed by that of the three-dimensional BTZ black hole. The heat capacities are given by

\[ C_J \equiv T(\partial S/\partial T)_J = 4\pi r_+ (r_+^2 - r_-^2)/(r_+^2 + 3r_-^2), \]
\[ C_M \equiv T(\partial S/\partial T)_M = 4\pi r_+ (r_+^2 - r_-^2)/(3r_+^2 + r_-^2), \]

both of which are always positive since the inner horizon radius is \( r_- \leq r_+ \). If a thermal system has rotations, the thermodynamic stability requires \((\partial \Omega_H/\partial J)_T \geq 0\) additionally since adding angular momentum is expected to increase the angular velocity of the system in average. For the rotating BTZ black hole, it turns out that

\[ \left( \frac{\partial \Omega_H}{\partial J} \right)_T = \frac{\Omega_H}{J} \frac{r_+^2 - r_-^2}{r_+^2 + 3r_-^2} \]

which is also positive always. According to the GM conjecture, therefore, we expect that the rotating BTZ black string in four dimensions is stable under linearized perturbations, regardless of the horizon size.

Now for the linearized perturbation analysis, the equations can be separated as before, and the most general form of the three-dimensional metric fluctuations is given by [11]

\[ h_{\mu\nu}(x) = e^{\Omega + i\varphi} \begin{pmatrix} H_{tt}(r) & H_{tr}(r) & i\hbar_{t\varphi}(r) \\ H_{tr}(r) & H_{rr}(r) & 0 \\ i\hbar_{t\varphi}(r) & 0 & r^2 K(r) \end{pmatrix}. \]

The decoupled equation is in the same form as Eq. (8), but it becomes much complicated and a complex equation due to the rotation parameter \( J \neq 0 \). For static BTZ black holes (i.e., \( J = 0 \)), however, one can set \( H_{t\varphi} = 0 \) by using the gauge freedom and easily solve the equation numerically. We could not find any unstable mode for various values of \( M \) and \( l_3 \). That is, massive spin-2 fluctuations do not seem to possess any instability for the three-dimensional BTZ black hole background. Therefore, we expect that the static BTZ black strings in four-dimensions are always stable under linearized perturbations. This result agrees well with the local thermodynamic stability through the GM conjecture.
IV. DISCUSSION

We have shown briefly that the AdS-Schwarzschild black strings in five dimensions become stable under linearized perturbations as the horizon size parameter (i.e., $r_+$) becomes larger than the order of the AdS$_4$ radius $l_4$. The higher dimensional extension of this result is straightforward. The AdS black strings in $(4+n)$-dimensions can be obtained simply by replacing $f(r) \rightarrow 1 - (r_0/r)^n + r^2/l_{3+n}^2$ and $d\Omega_2^2 \rightarrow d\Omega_{1+n}^2$ in Eq. (2). One can easily see that the sign of the heat capacity for such systems also becomes positive as the horizon size increases. For black $p$-branes (i.e., $dz^2 \rightarrow \delta_{ij}dz^iz^j$ for $i, j = 1, \cdots, p$), the basic feature of the stability is also same.

It should be pointed out that the essential reason for having stable AdS black strings is that AdS$_4$-Schwarzschild black hole backgrounds do not allow unstable massive spin-2 fluctuations as the horizon size increases. Furthermore, a sort of effective compactification due to the warping geometry along the string increases its stability in addition. Namely, the warping factor for the AdS case rises up as one approaches to the AdS$_5$ horizon whereas those for flat and dS cases go down to zero. Consequently, the potential in Eq. (6) for the AdS case becomes box-like with a finite mass gap, giving a confining effect for waves along the string. Notice that the proper lengths of black strings in the extra direction are infinite for all three cases. It is interesting to see that the scale of this effective compactification of AdS black strings is governed by the AdS$_4$ scale $l_4$, instead of $l_5$, because the AdS$_5$ scale does not enter in Eq. (b) at all. However, note that, when the instability sets in, the minimum horizon size of the AdS black string in proper length is indeed of order the AdS$_5$ scale, i.e., $r_+^{\text{pl}} = r_+^{\text{cr}}H_{\text{min}}^{-1} = r_+^{\text{cr}}l_5/l_4 \simeq 0.20l_5$, which is consistent with the instability mechanism argued in Ref. [13].

As mentioned above, it is suggested by Horowitz and Maeda [6] that, even in the case that a black string is unstable under linearized perturbations, its full non-linear evolution presumably ends up with a sort of non-uniform black string configuration, instead of fragmentations of the string horizon. This final state must be stationary and stable. Then it will be interesting to see what makes this configuration stable even under linearized perturbations. In other words, what is the role of the “non-uniformity”? The perturbation equations will in general consist of two parts; that is, fluctuations for black holes on “transverse” slicings of the black string and that along the string. Of course, they become separable perhaps only for special cases such as uniform black strings and those ones with overal warping factors. Now our study above seems to indicate that the stability of spin-2 fields with “bigger” degrees of freedom on black hole backgrounds of slicings is crucial generically for the stability of a non-uniform black string. However, even if such fields have “instability”, some effective compactification due to the curvature or periodicity along the black string could also make the black string stable.

For the BTZ black strings in four dimensions, we have shown that they are stable independent of the transverse horizon size since the massive spin-2 fluctuation equation itself does not possess unstable modes at all. It seems that this stable behavior for black strings/branes in dimensions lower than five is common. The thermodynamic analysis for several such solutions [18–20] shows that all of them are locally thermodynamically stable [11]. Consequently, the GM conjecture implies that they are classically stable under linearized perturbations. This special property might be related to the fact that gravitational waves in lower dimen-
sions than four do not have enough degrees of freedom for propagations. However, we point out that the massive spin-2 fluctuations in lower dimensions must have the same number of degrees of freedom as that of the four-dimensional massless gravitational waves. Further study is required concerning this issue.

Since the discovery of the Gregory-Laflamme instability in 1993, it is somewhat surprising that no specific work has been done for the stability of Kerr black strings. This is probably because the wide belief was that such rotating black string is also unstable. However, a naive application of the GM conjecture shows that this may not be the case. For instance, the heat capacity of the Kerr black hole is negative, but becomes positive as the angular momentum parameter increases towards the extremal value. Thus, the Kerr black string with large angular momentum parameter might be stable under linearized perturbations possibly due to the “centrifugal” effect of rotations in the background geometry. On the other hand, however, another thermodynamic quantity (e.g., \((\partial \Omega_H / \partial J)_T\) as in Eq. (12)) becomes negative exactly for the values of angular momentum where the heat capacity is positive. The direct analysis of linearized fluctuations in Eqs. (4)-(6) with \(H(z) = 1\) is not available at the present since the equations are not easily decoupled in this Kerr background geometry. We expect that a partial application of the four-dimensional Newman-Penrose formalism to this five-dimensional system may decouple the set of equations, resulting in a sort of “Teukolsky equations” [21]. The detailed analysis of this decoupled equation will reveal all non-trivial behaviors for the stability of Kerr black strings in the parameter space of mass and angular momentum. Consequently, it will also hint at how to extend Reall’s proof of the GM conjecture to a system with rotation.

Finally, it is also very interesting to extend the proof in Ref. [3] to the cases of Schwarzschild or dS-Schwarzschild black strings in AdS5. It is because such spacetimes have naked singularities at the AdS5 horizon outside the event horizon.

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