TIME EVOLUTION OF VORTEX RINGS WITH LARGE RADIUS AND VERY CONCENTRATED VORTICITY

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ABSTRACT. We study the time evolution of an incompressible fluid with axial symmetry without swirl when the vorticity is sharply concentrated on $N$ annuli of radii $\approx r_0$ and thickness $\varepsilon$. We prove that when $r_0 = |\log \varepsilon|^{\alpha}$, $\alpha > 2$, the vorticity field of the fluid converges as $\varepsilon \to 0$ to the point vortex model, at least for a small but positive time. This result generalizes a previous paper that assumed a power law for the relation between $r_0$ and $\varepsilon$.

1. Introduction

In the present paper we study the motion of an incompressible inviscid fluid with an axial symmetry without swirl (for the exact definition see later on) when the initial vorticity is very concentrated on $N$ annuli of radii $\approx r_0$ and thickness $\varepsilon$. We prove the relation of this motion with the so-called point vortex system in the plane when $r_0 \to \infty$ as $\varepsilon \to 0$.

The motion of an incompressible inviscid fluid is governed by the Euler equations, that for a fluid of unitary density in three dimensions read:

\begin{align}
\left( \partial_t + (u \cdot \nabla) \right) \omega &= (\omega \cdot \nabla) u, \quad (1.1) \\
\nabla \cdot u &= 0 \quad \text{(continuity equation)}, \quad (1.2) \\
u(x, 0) &= u_0(x) \quad \text{(velocity field)}, \quad \text{and boundary conditions}. \quad \text{From now on we suppose that the velocity vanishes as } |x| \to \infty. \quad \text{This assumption allows to reconstruct the velocity from the vorticity:}
\end{align}

\begin{equation}
u(x, t) = -\frac{1}{4\pi} \int \frac{x - y}{|x - y|^3} \wedge \omega(y, t) \, dy. \quad (1.3)
\end{equation}

We use now cylindrical coordinates $(z, r, \theta)$ and suppose that the initial velocity field has the form (axial symmetry without swirl):

\begin{equation}
u(x, t) = (u_z, u_r, u_\theta) = (u_z(z, r, t), u_r(z, r, t), 0). \quad (1.4)
\end{equation}

The time evolution conserves this symmetry. In this case the vorticity is

\begin{equation}\omega = \nabla \wedge u = (0, 0, \omega_\theta) = (0, 0, \partial_z u_r - \partial_r u_z), \quad (1.5)
\end{equation}

and the Euler equations become

\begin{align}
\left( \partial_t + (u_z \partial_z + u_r \partial_r) \right) \omega_\theta - \frac{u_r \omega_\theta}{r} &= 0, \quad (1.6) \\
\partial_z (r u_z) + \partial_r (r u_r) &= 0. \quad (1.7)
\end{align}

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From now on we denote \( \omega_0 \) by \( \omega \). Finally, by (1.3),

\[
\begin{align*}
\omega'(z, r, t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dz' \int_{0}^{\infty} dr' \\
\int_{0}^{\pi} d\theta \frac{\omega(z', r', t)[r \cos \theta - r']}{[(z - z')^2 + (r - r')^2 + 2rr'(1 - \cos \theta)]^{3/2}},
\end{align*}
\]

(1.8)

\[
\begin{align*}
u_r(z, r, t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dz' \int_{0}^{\infty} dr' \\
\int_{0}^{\pi} d\theta \frac{\omega(z', r', t)[z - z'] \cos \theta}{[(z - z')^2 + (r - r')^2 + 2rr'(1 - \cos \theta)]^{3/2}},
\end{align*}
\]

(1.9)

Hence, the axially symmetric solutions to the Euler equations are given by the solutions to eqs. (1.5) - (1.9). Eq. (1.6) means that the quantity \( \omega/r \) remains constant along the flow generated by the velocity field, i.e.

\[
\frac{\omega(z(t), r(t), t)}{r(t)} = \frac{\omega(z(0), r(0), t)}{r(0)}, \tag{1.10}
\]

where \( (z(t), r(t)) \) solve

\[
\dot{z}(t) = u_z(z(t), r(t), t), \quad \dot{r}(t) = u_r(z(t), r(t), t). \tag{1.11}
\]

It is possible to introduce an equivalent weak formulation of (1.6) that allows to consider non-smooth initial data; by a formal integration by parts we obtain indeed

\[
\frac{d}{dt} \omega_i(f) = \omega_t [u_z \partial_z f + u_r \partial_r f + \partial_t f], \tag{1.12}
\]

where \( f = f(z, r, t) \) is a bounded smooth test function and

\[
\omega_i(f) := \int_{-\infty}^{\infty} dz \int_{0}^{\infty} dr \omega(z, r, t) f(z, r, t). \tag{1.13}
\]

It is known that a global (in time) existence and uniqueness of a weak solution to the associate Cauchy problem holds when the initial vorticity is a bounded function with compact support contained in the open half-plane \( \Pi := \{(z, r) : r > 0\} \), see for instance [27] pag.91 or the Appendix of [7]. In particular, it can be shown that the support of the vorticity remains in the open half-plane \( \Pi \) at any time. A point in the half-plane \( \Pi \) denotes a circumference in the whole space. The special class of axisymmetric without swirl solutions are called sometimes smoke rings, because there exist particular solutions whose shape remains constant in time (the so-called steady vortex ring) and translate in the \( z \)-direction with constant speed (see for instance [9]). The existence and the properties of these solutions is an old question. For a rigorous proof by means of variational methods see [1][10]. For references on axially symmetric solution without swirl see also the review paper [29].

Denote \( x = (x_1, x_2) := (z, r - r_0) \). We assume that initially the vorticity is concentrated in \( N \) blobs of the form

\[
\omega_{\varepsilon}(x, 0) = \sum_{i=1}^{N} \omega_{i, \varepsilon}(x, 0) \tag{1.14}
\]

where \( \omega_{i, \varepsilon}(x, 0) \) are functions with a definite sign such that, denoting by \( \Sigma(\xi|\rho) \) the open disk of center \( \xi \) and radius \( \rho \) in \( \mathbb{R}^2 \),

\[
\Lambda_{i, \varepsilon} := \supp \omega_{i, \varepsilon} \subset \Sigma(z_i|\varepsilon); \quad \Sigma(z_i|\varepsilon) \cap \Sigma(z_j|\varepsilon) = \emptyset \quad \forall i \neq j \tag{1.15}
\]
being $\varepsilon > 0$ a small parameter and $z_1, \ldots, z_N$, points contained in a bounded region of $\mathbb{R}^2$ such that

$$\min_{i \neq j} |z_i - z_j| > \rho_m$$

for a positive constant $\rho_m$ independent of $\varepsilon$. Moreover we assume that, for any $i = 1, \ldots, N$,

$$\int dx \omega_{i,\varepsilon}(x,0) := a_i \in \mathbb{R}$$

independent of $\varepsilon$ and

$$|\omega_{i,\varepsilon}(x,0)| \leq M \varepsilon^{-\gamma}, \quad M > 0, \quad \gamma > 0.$$  \hfill (1.17)

We will discuss if, in some cases and for small $\varepsilon$, the time evolution of these states has the same form. In Theorem 2.1 we will prove that, as the assumption $r_0(\varepsilon) = |\log \varepsilon|^{\alpha}$ $(\alpha > 2)$ is fulfilled, the evolved state $\omega_{\varepsilon}(x,t)$ can be written as

$$\omega_{\varepsilon}(x,t) = \sum_{i=1}^{N} \omega_{i,\varepsilon}(x,t),$$

where $\omega_{i,\varepsilon}(x,t)$ are functions with definite sign such that

$$\Lambda_{i,\varepsilon}(t) := \text{supp} \omega_{i,\varepsilon}(\cdot,t) \subset \Sigma(z_i(t)|r_i(\varepsilon)), \quad \forall i \neq j,$$

with

$$\Sigma(z_i(t)|r_i(\varepsilon)) \cap \Sigma(z_j(t)|r_i(\varepsilon)) = \emptyset \quad \forall i \neq j,$$  \hfill (1.20)

being $r_i(\varepsilon)$ a positive function, vanishing for $\varepsilon \to 0$, and $z_i(t) \in \mathbb{R}^2$ solution to the point-vortex model, that is the dynamical system defined by the following differential equations:

$$\dot{z}_i(t) = \sum_{j=1}^{N} a_j K(z_i(t) - z_j(t)), \quad z_i(0) = z_i$$  \hfill (1.21)

for $i = 1, \ldots, N$, and

$$K(x) = \frac{-1}{2\pi} \nabla^\perp \log |x|, \quad \nabla^\perp = (\partial_2, -\partial_1),$$  \hfill (1.22)

where $-1/2\pi \log |x|$ is the fundamental solution of the Laplace operator in $\mathbb{R}^2$. When all the $a_i$ have the same sign there is a global solution, otherwise there is a finite time at which a collapse (that is two $z_i$ arriving at the same point) or a $z_i$ going to infinity can occur. However (see for instance [27]) these events are exceptional.

Few words on this dynamical system: it has been introduced by Helmholtz as particular solution of the Euler equations [12] and investigated by many authors [14,15,28]. It has been used to investigate the time evolution of irregular initial data and it produces an approximation method (called vortex method) in which $N \to \infty$ and $a_i \to 0$ (for more information see for instance the textbook [27] or [17] and references in [8]).

Even if solutions of (1.21) cannot be a solution of the Euler equations, they can be an average of different solutions that in $\mathbb{R}^3$ are clusters of straight lines of vorticity, as it is discussed in [13,21,25,26,30]. In [22] it is shown that the same happens when the straight lines are changed into large enough annuli, with radius of the order $r_0(\varepsilon) = \varepsilon^{-\alpha}$, for any $\alpha > 0$ and for any finite time, and recently for long times in [7].
In the present paper we assume a weaker dependence of \( r_0 \) on \( \varepsilon \) and we will show that the relation with the point vortex model remains valid at least for a finite but positive time. We consider radii of the order \( |\log \varepsilon|^{\alpha} \), \( \alpha > 2 \) (this lower bound on \( \alpha \) appears for a technical reason which occurs in eq. (3.41)), hence the rings are less distant from the axis, where the effects of curvature become stronger. For laws \( r_0 = f(\varepsilon) > \varepsilon^{-\alpha} \) the convergence to the point-vortex model happens faster, while for the previous logarithmic law the convergence is slower. This imposes a more careful strategy to prove the convergence, since we need an iterative method as the one used in [5] in a different context, which produces a result only for bounded times. This reflects the difficulty of the convergence when the distance from the axis approaches the scale \( |\log \varepsilon|^{\alpha} \), \( 0 \leq \alpha \leq 1 \), for which only for \( \alpha = 0 \) (for one vortex alone [2] and \( N \) vortices [5]) and for \( \alpha = 1 \) for one vortex alone [24] some results are available. For \( \alpha = 0 \) the dynamics of the vortices converges to simple translations parallel to the symmetry axis with constant speed. For \( \alpha = 1 \) we can conjecture that the convergence of the dynamics is not to the point-vortex model, but to the dynamical system defined by

\[
\dot{z}_i(t) = \sum_{j=1 \atop j \neq i}^N a_j K(z_i(t) - z_j(t)) + e_1 a_i, \quad z_i(0) = z_i, \quad e_1 = (1, 0).
\] (1.23)

This is proved rigorously for one vortex alone in [24], but for \( N \) vortices it is an open problem and the correspondence with (1.23) is established only at a heuristic level.

We also mention that in literature it is discussed how the point-vortex model behaves under a viscosity perturbation [3, 7, 11, 16, 19, 20, 23, 31], but this topic is out of the scope of the present analysis.

2. Main result

A warning on the notation. Hereafter in the paper we denote by \( C \) a generic positive constant (eventually changing from line to line) which is independent of the parameter \( \varepsilon \) and the time \( t \).

We define a suitable scaling of variables, in order to get the convergence to the point-vortex dynamics, in such a way that the rings increase their radius while their support becomes smaller. When the radius increases, the interaction of the \( N \) vortices with the axis becomes negligible in the limit \( \varepsilon \to 0 \), which permits to obtain a convergence to the point-vortex dynamics. Denoting by \((r, z, \theta)\) the cylindrical coordinates in \( \mathbb{R}^3 \), we recall the previously mentioned coordinates

\[
x = (x_1, x_2) := (z, r - r_0)
\] (2.1)

and consider an initial vorticity as specified in (1.14)-(1.17), whose evolution at time \( t \) can be expressed as in (1.18). Knowing the velocity field \( u(\cdot, t) := u(t) \), we can define the trajectory of a fluid element starting at \( x \) as the solution of the integral equation

\[
\phi_t(x) = x + \int_0^t u_s(\phi_s(x))ds.
\]

Since the quantity \( \omega/r \) remains constant along the flow generated by the velocity field, we have

\[
\omega_{t,\varepsilon}(x, t) := \frac{r_0 + x_2}{r_0 + (\phi_{-t}(x))_2} \omega_{t,\varepsilon}(\phi_{-t}(x), 0).
\] (2.2)
Moreover $\omega_{i,\varepsilon}(x, t)$ preserves the initial sign and the total mass $a_i$, as immediately follows from the definitions (see also Lemma 3.3).

Furthermore, for each index $i$, we can decompose the velocity field $u$ as follows:

$$u(x, t) = u^i(x, t) + F^i_t(x, t),$$

where

$$u^i(x, t) = \int dy \ G(x, y) \ \omega_{i,\varepsilon}(y, t)$$

is the velocity field generated by the vortex $\omega_{i,\varepsilon}$, and

$$F^i_t(x, t) = \sum_{j \neq i} \int dy \ G(x, y) \ \omega_{j,\varepsilon}(y, t) \quad (2.3)$$

is the one generated by the remaining $N - 1$ vortices; here $G(x, y)$ denotes the integral kernel appearing in (1.8)-(1.9) in the new coordinates (2.1).

Let us call \( \{z_i(t)\}_{i=1, \ldots, N} \) the solution to point vortex dynamics (1.21) with intensities $a_i$ and initial data $z_i$ (with $z_i \neq z_j$ for $i \neq j$). As already discussed, such dynamics is well defined globally in time apart from a zero measure set of initial data. Even in this last case in which a collapse can occur, since our results hold for times smaller than a positive constant, we can consider such constant (let’s call it $T^*$) much smaller than the first collapse time. With this viewpoint we define, for a constant $\bar{R} > 0$,

$$T_\omega := \sup\{t > 0 : \text{supp} \omega_{i,\varepsilon}(s) \subseteq \Sigma(z_i(s)|\bar{R}) \ \forall i = 1, \ldots, N, \ \forall s \in [0, t]\},$$

(2.4)

and

$$\bar{T} = \min\{T_\omega, T^*\},$$

(2.5)

and

$$R_m := \min_{i \neq j} \inf_{t \in [0, \bar{T}]} |z_i(t) - z_j(t)| > 0.$$ \quad (2.6)

We ask

$$\bar{R} < R_m / 4 \quad (2.7)$$

(it will be used in the sequel). Observe that such requirement is non-empty: for $\bar{R} = \varepsilon$ it results $\bar{T} = 0$ (by the initial data (1.14)-(1.17)), and (2.7) is obviously fulfilled (for small $\varepsilon$). Considering then a small positive $\bar{R}$ (but independent of $\varepsilon$), we obtain consequently a small $\bar{T}$, and for a continuity argument (2.7) can still be satisfied. In the next Theorem we state a better result, the size of the support of $\omega_i$ at time $t$ (for short times) is a quantity which vanishes for $\varepsilon \to 0$. The result is the following.

**Theorem 2.1.** Consider initial vorticity as in (1.14)-(1.17) and $r_0 = |\log \varepsilon|^\alpha$, for any $\alpha > 2$. Then there is a $T > 0$ such that

$$\text{supp} \omega_{i,\varepsilon}(s) \subseteq \Sigma(z_i(s)|C_T|\log \varepsilon|^{-k}) \ \forall i = 1, \ldots, N, \ \forall s \in [0, T]$$

where $C_T$ is a positive constant, $k = (\alpha - 2)/2$, $\varepsilon \in (0, \varepsilon_0)$, with $\varepsilon_0 < 1$ solution to

$$C_T|\log \varepsilon|^{-k} = R_m / 4.$$ 

We remark that with the previous definition of $\varepsilon_0$ it results $C_T|\log \varepsilon|^{-k} < R_m / 4$. 


3. Proof of Theorem 2.1

We give the general strategy of the proof, which is rather technical and composed of many auxiliary Lemmas and Propositions. We study the motion of a tagged vortex (with index \(i\)) under the influence of the remaining \(N-1\) vortices. The field generated by the remaining \(N-1\) vortices has the features of a given external bounded field, since in the time interval \([0, \bar{T}]\) the minimum distance between any two distinct vortices remains greater than a positive constant. We make use then of a fundamental estimate on the growth in time of the moment of inertia of the tagged vortex, and we estimate the vorticity mass far from the center of vorticity, showing that it is negligible when \(\varepsilon\) is small, by means of an iterative method. Putting together these (and other technical) results we achieve the proof. Some of these tools are similar to those of previous papers [4–7,13,22], and we write them again for completeness.

We discuss then the preliminary results we need, starting with the estimate of the convolution kernel \(G\) (in the new coordinates (2.1)), showing that, under suitable assumptions, this is near to \(K\) (of the planar case).

Making use of (1.8)-(1.9), written with respect to the new coordinate system (2.1), we get

\[
 u(x, t) = \int_{\mathbb{R}^2} G(x, y) \omega(y, t) dy
\]

where the convolution kernel \(G(x, y)\) is defined by:

\[
 G_1(x, y) = \frac{1}{2\pi} \int_{0}^{\pi} d\theta \frac{(r_0 + y_2)(r_0 + x_2) - (r_0 + x_2) \cos \theta}{\{|x - y|^2 + 2(r_0 + x_2)(r_0 + y_2)(1 - \cos \theta)|^{3/2}}
\]

\[
 G_2(x, y) = \frac{1}{2\pi} \int_{0}^{\pi} d\theta \frac{(r_0 + y_2)(x_1 - y_1) \cos \theta}{\{|x - y|^2 + 2(r_0 + x_2)(r_0 + y_2)(1 - \cos \theta)|^{3/2}}
\]

We now want to give an estimate for this convolution kernel, in particular we want to show that, for small enough \(\varepsilon\), the vector field \(u\) is near to the vector field \(\tilde{u}\) corresponding to the planar case, namely

\[
 \tilde{u}(x, t) = \int_{\mathbb{R}^2} K(x - y) \omega(y, t) dy
\]

We need the following lemma, whose proof is contained in [7] and reported here for completeness.

**Lemma 3.1.** Let us define, for \(a > 0\):

\[
 I_1(a) := \int_{0}^{\pi} d\theta \frac{\cos \theta}{|a^2 + 2(1 - \cos \theta)|^{3/2}},
\]

\[
 I_2(a) := \int_{0}^{\pi} d\theta \frac{1 - \cos \theta}{|a^2 + 2(1 - \cos \theta)|^{3/2}}.
\]

Denoting by \(\chi_{(0,1)}(\cdot)\) the characteristic function of the interval \((0, 1)\), the following equalities hold:

\[
 I_1(a) = a^{-2} + R_1(a), \quad I_2(a) = -\frac{1}{2} \log a \cdot \chi_{(0,1)}(a) + R_2(a),
\]

where \(a \cdot R_1(a)\) is bounded and \(|R_2(a)| \leq C \min(1, \frac{1}{a})\).
and hence it can be bounded by

\[ I = \int_0^\pi \frac{2|\sin(\theta/2)|^2 \cos(\theta/2)}{a^2 + 4|\sin(\theta/2)|^2} \frac{1}{3/2} + \int_0^\pi \frac{2|\sin(\theta/2)|^2(1 - \cos(\theta/2))}{a^2 + 4|\sin(\theta/2)|^2} \frac{1}{3/2}. \tag{3.4} \]

By the substitution \( z = 2\sin(\theta/2) \), for the first integral in the right hand side of (3.4) we have

\[ \int_0^2 \frac{dz}{z^2} \frac{z^2}{2[a^2 + z^2]^{3/2}} = \frac{1}{2} \left[ \log(\sqrt{a^2 + z^2} + z) - \frac{z}{\sqrt{a^2 + z^2}} \right]_{z=0}^{z=2} = -(a^2 + 4)^{-1/2} + \frac{1}{2} \log(2 + \sqrt{a^2 + 4}) - \frac{1}{2} \log a. \]

We deduce that for \( a \to 0 \) this quantity is equal to \(-\frac{1}{2} \log a\) plus a bounded rest, while for \( a \to \infty \) it behaves like \( a^{-1} \).

For the second integral in (3.4), first of all we have

\[ \int_0^\pi \frac{d\theta}{a^2 + 4|\sin(\theta/2)|^2} \left[ \frac{1}{3/2} - \frac{1 - \cos(\theta/2)}{\sin(\theta/2)} \right] \leq \frac{1}{4} \int_0^\pi d\theta \frac{1 - \cos(\theta/2)}{\sin(\theta/2)} \]

which is a bounded integral; on the other hand

\[ \int_0^\pi d\theta \frac{2|\sin(\theta/2)|^2(1 - \cos(\theta/2))}{a^2 + 4|\sin(\theta/2)|^2} \frac{1}{3/2} \leq \frac{2}{a^3} \int_0^\pi d\theta \sin^2(\theta/2)(1 - \cos(\theta/2)) \leq C a^{-3}. \]

These estimates for the two integrals in (3.4) show that the equality for \( I_2 \) in (3.3) holds with \( R_2(a) \) bounded by a constant for small \( a \), and by \( C a^{-1} \) for large \( a \).

We evaluate now \( I_1 \), first when \( a < 1 \), by decomposing the integral as

\[ I_1 = \int_0^\pi \frac{\cos(\theta/2)}{a^2 + 2|\sin(\theta/2)|^2} \frac{1}{3/2} + \int_0^\pi \frac{\cos \theta - \cos(\theta/2)}{a^2 + 2|\sin(\theta/2)|^2} \frac{1}{3/2}. \tag{3.5} \]

The first integral in the right hand side of (3.5) can be computed as before with the substitution \( z = 2\sin(\theta/2) \):

\[ \int_0^2 \frac{dz}{z^2} \frac{1}{a^2 + z^2} = \left[ \frac{z}{a^2 \sqrt{a^2 + z^2}} \right]_{z=0}^{z=2} = \frac{2}{a^2 \sqrt{a^2 + 4}} \]

which, for \( a \to 0 \), is equal to \( a^{-2} \) plus a bounded rest. The second integral in (3.5) can be bounded by noticing that

\[ 0 \leq \cos(\theta/2) - \cos \theta \leq 1 - \cos \theta \quad \text{for } 0 \leq \theta \leq \pi \]

and hence it can be bounded by \( I_2 \).

We analyse now the case \( a \geq 1 \), observing that

\[ |I_1(a)| \leq a^{-3} \int_0^\pi d\theta |\cos \theta| = \frac{2}{a^3} \]

and so \( |R_1(a)| \leq 2a^{-3} + a^{-2} \). In both cases, \( a < 1 \) and \( a \geq 1 \), we have that \( a \cdot R_1(a) \) is bounded, as it goes to zero like \( a \log a \) when \( a \to 0 \), and behaves like \( a^{-1} \) when \( a \to \infty \). \qed

**Proposition 3.2.** Consider \( x, y \) such that:

\[ |x_2| \leq \frac{r_0}{2} \quad |y_2| \leq \frac{r_0}{2} \]
and let \( r_0 = |\log \varepsilon|^\alpha \). Then, for \( \varepsilon \) small enough:

\[
|G(x, y) - K(x - y)| \leq \frac{C}{|\log \varepsilon|^\alpha} \left( 1 + \log |\log \varepsilon| + |\log |x - y|| \cdot \chi_{(0,1)}(|x - y|) \right).
\]

(3.6)

**Proof.** Define \( a := |x - y| (r_0 + x_2)^{-1/2}(r_0 + y_2)^{-1/2} \).

\[
2\pi G_1(x, y) = \int_0^{2\pi} d\theta \frac{(r_0 + y_2)(y_2 - x_2 \cos \theta + r_0(1 - \cos \theta))}{(r_0 + x_2)^{3/2}(r_0 + y_2)^{3/2} \{a^2 + 2(1 - \cos \theta)\}^{3/2}}
\]

\[
= y_2 \cdot (I_1(a) + I_2(a)) - x_2 \cdot I_1(a) + r_0 \cdot I_2(a)
\]

(3.7)

where \( \frac{y_2 - x_2}{r_0 + x_2} \) is the first component of \( 2\pi K(x - y) \), so we subtract this quantity and estimate \( |G_1 - K_1| \).

Let us put \( A = \sqrt{\frac{r_0 + y_2}{r_0 + x_2}} \), hence we have

\[
|A - 1| = \left| \frac{A^2 - 1}{A + 1} \right| = \left( \frac{|y_2 - x_2|}{r_0 + x_2} \right) (1 + A)^{-1} \leq \frac{2|x - y|}{r_0}
\]

where in the last inequality we have used the assumption \( x_2 \geq -\frac{r_0}{2} \). Furthermore

\[
\sqrt{\frac{r_0 + y_2}{(r_0 + x_2)^3}} \leq \frac{1}{(r_0 + x_2)^{1/2}(r_0 + y_2)^{1/2}} \cdot \frac{r_0 + y_2}{r_0 + x_2} \leq \frac{1}{(r_0 + x_2)^{1/2}(r_0 + y_2)^{1/2}} \cdot \frac{r_0 + x_2}{r_0 + x_2} \leq \frac{2}{r_0} (1 + a).
\]

Collecting these estimates into \( |D_1| \), and calling \( D := G - K \), we get:

\[
2\pi |D_1(x, y)| \leq \frac{2|x - y| |y_2 - x_2|}{r_0 |x - y|^2} + \frac{a \cdot R_1(a)}{r_0 + x_2} + \frac{2}{r_0} (1 + a) \cdot I_2(a)
\]

\[
\leq \frac{2}{r_0} + \frac{2C}{r_0} + \frac{2}{r_0} \left( C - \frac{1}{2} \log a \cdot \chi_{(0,1)}(a) \right).
\]

For \( a \in (0, 1) \)

\[
0 \leq - \log a = \log(r_0 + x_2)^{1/2} + \log(r_0 + y_2)^{1/2} - \log |x - y|
\]

\[
\leq \log r_0 + \log(3/2) + |\log |x - y|| \cdot \chi_{(0,1)}(|x - y|)
\]

where, by assumption, \( \frac{1}{2}r_0 \leq r_0 + x_2 \leq \frac{3}{2}r_0 \). Since \( r_0 = |\log \varepsilon|^\alpha \) we obtain

\[
2\pi |D_1(x, y)| \leq \frac{C}{|\log \varepsilon|^\alpha} \left[ 1 + \log |\log \varepsilon| + |\log |x - y|| \cdot \chi_{(0,1)}(|x - y|) \right].
\]
We analyse now the second component:
\[
2\pi G_2(x, y) = \int_0^\pi \frac{(r_0 + y_2)(x_1 - y_1) \cos \theta}{(r_0 + x_2)^{3/2} (r_0 + y_2)^{3/2} \{a^2 + 2(1 - \cos \theta)\}^{3/2}} dx_1 - y_1 \\
= \frac{x_1 - y_1}{(r_0 + x_2)^{3/2} (r_0 + y_2)^{1/2}} I_1(a) \\
= \int \frac{r_0 + y_2}{r_0 + x_2} \cdot \frac{x_1 - y_1}{|x - y|^2} + \frac{x_1 - y_1}{(r_0 + x_2)^{3/2} (r_0 + y_2)^{1/2}} R_1(a).
\]

We proceed as before,
\[
2\pi |D_2(x, y)| \leq \frac{2|x - y| |x_1 - y_1|}{r_0 |x - y|^2} + \frac{a \cdot R_1(a)}{r_0 + x_2} \leq \frac{2}{r_0^2} (1 + C).
\]

Since \( |D_1 - D_2| \leq |D_1| + |D_2| \), we get
\[
|D(x, y)| \leq \frac{C}{|\log \varepsilon|^\alpha} (1 + \log |\log \varepsilon| + |\log |x - y|| \cdot \chi_{(0,1)}(|x - y|))
\]
and the Proposition is thus proved. \(\square\)

The next lemma states the conservation of the \(L^1\) norm of \(\omega\) and a bound on its \(L^\infty\) norm.

**Lemma 3.3.** Let \(\omega_{i, \varepsilon}\) as in (1.14) - (1.18). Then
\[
\int_{\mathbb{R}^2} dx \omega_{i, \varepsilon}(x, t) = \int_{\mathbb{R}^2} dx \omega_{i, \varepsilon}(x, 0) = a_i \tag{3.8}
\]
and for each time \(\omega_{i, \varepsilon}(x, t)\) has the same sign of \(\omega_{i, \varepsilon}(x, 0)\). Moreover for \(t \leq \bar{T}\) and for small enough \(\varepsilon\),
\[
|\omega_{i, \varepsilon}(x, t)| \leq 3M \varepsilon^{-\gamma}. \tag{3.9}
\]

**Proof.** Equation (3.8) is a direct consequence of the conservation of \(\omega/r\), after integrating in \(\mathbb{R}^3\) adopting cylindrical coordinates. The conservation of sign is evident by the definition of \(\omega_{i, \varepsilon}(x, t)\). To obtain (3.9) we observe that, if \(x \in \Lambda_{i, \varepsilon}(0) := \text{supp} \omega_{i, \varepsilon}(0)\), then \(|x - z_i| \leq \varepsilon\) and
\[
|r_0 + x_2| = || \log |\log \varepsilon|^{\alpha} + x_2 | \geq | \log |\log \varepsilon|^{\alpha} - |z_i| - \varepsilon \geq \frac{1}{2} |\log |\log \varepsilon|^{\alpha}.
\]
Since \(\phi_t(x) \in \Lambda_{i, \varepsilon}(t)\), it results \(|\phi_t(x) - z_i(t)| \leq \bar{R}\) and then
\[
|r_0 + \phi_t^i(x)| \leq |\log |\log \varepsilon|^{\alpha} + |z_i(t)| + \bar{R}.
\]
Moreover
\[
|z_i(t)| \leq |z_i| + \frac{1}{2\pi} \int_0^t ds \sum_{j \neq i} \frac{|a_j|}{|z_i(s) - z_j(s)|} \leq |z_i| + \frac{t}{2\pi R_m} \sum_{j \neq i} |a_j|,
\]
therefore, for \(t \leq \bar{T}\) and for \(\varepsilon\) small enough,
\[
|r_0 + \phi_t^i(x)| \leq \frac{3}{2} |\log |\log \varepsilon|^{\alpha}.
\]
The bound (3.9) follows from the equality
\[
\omega_{i, \varepsilon}(\phi_t(x), t) = \frac{r_0 + \phi_t^i(x)}{r_0 + x_2} \omega_{i, \varepsilon}(x, 0).
\]

\(\square\)
We are now able to prove that the difference \( u - \tilde{u} \) is small.

**Proposition 3.4.** Let \( \omega_\varepsilon \) as in (1.14)-(1.18), and let \( t \leq \bar{T}, \ x \in \Lambda_\varepsilon(t) \). Then, if \( \varepsilon \) is small enough,

\[
|u(x,t) - \tilde{u}(x,t)| \leq \frac{C}{|\log \varepsilon|^{\alpha-1}}.
\]

(3.10)

**Proof.** Note that, if \( x \in \Lambda_{i,\varepsilon}(t) \) (for any \( i \)), then \( |x_2| \leq r_0/2 \), as seen in the proof of Lemma 3.3. We can then apply Proposition 3.2 to bound \( |u(x,t) - \tilde{u}(x,t)| \), and the worst term to treat is

\[
\int_{|x-y|<1} dy \left| \log |x-y| \right| |\omega_\varepsilon(y,t)| \leq C |\log \varepsilon|^{\alpha-1}.
\]

(3.11)

We use a classical trick, that is a rearrangement: we bound this integral with the one obtained by concentrating as much as possible the vorticity around the singularity of \( \log |x-y| \), namely \( y = x \) (and by the characteristic function we integrate only in the domain \( |x-y| < 1 \)). Let us proceed with a fixed \( i \) in the summation in (3.11). Since the integral of \( \omega_{i,\varepsilon} \) is constant in time, and its \( L^\infty \) norm is less or equal to \( 3M\varepsilon^{-\gamma} \), we get the rearrangement replacing \( \omega_{i,\varepsilon} \) with the function equal to the constant \( 3M\varepsilon^{-\gamma} \) in the disk of centre \( x \) and radius \( r \), and equal to zero outside this disk. The radius \( r \) is chosen such that the total mass of vorticity is \( |a_i| \), so

\[
\int_{|x-y|<1} dy \left| \log |x-y| \right| \sum_{i=1}^N |\omega_{i,\varepsilon}(y,t)| \leq C + C |\log \varepsilon|.
\]

which inserted in (3.6) gives (3.10). □

We give here a useful split of the field \( F_{\varepsilon} \).

**Lemma 3.5.** Recalling the definition of \( F_{\varepsilon} \) given in (2.3), we can write, for \( t \leq \bar{T} \)

\[
F_{\varepsilon} = F_{\varepsilon,1} + F_{\varepsilon,2}
\]

where \( F_{\varepsilon,1} \) is Lipschitz and bounded uniformly in \( \varepsilon \), and \( F_{\varepsilon,2} \) is small, i.e.

\[
\|F_{\varepsilon,2}\|_{L^\infty} \leq \frac{C}{|\log \varepsilon|^{\alpha-1}}.
\]
Proof. Let us define
\[ F_{i,1}^i(x,t) = \sum_{j \neq i} \int dy K(x - y) \omega_{j,\varepsilon}(y,t), \]
\[ F_{i,2}^i(x,t) = \sum_{j \neq i} \int dy |G(x,y) - K(x - y)| \omega_{j,\varepsilon}(y,t). \]

It results that \( F_{i,1}^i \) is Lipschitz and bounded uniformly in \( \varepsilon \) by the properties of \( K \) outside the disk \( \Sigma(0 \frac{R_m}{2}) \) (it is Lipschitz and bounded); in fact, if \( x \in \Lambda_{i,\varepsilon}(t) \) and \( y \in \Lambda_{j,\varepsilon}(t) \), for \( i \neq j \), we have
\[ |x - y| \geq |z_i(t) - z_j(t)| - |x - z_i(t)| - |y - z_j(t)| \geq R_m - 2\bar{R} \geq \frac{R_m}{2} \]
for \( 4\bar{R} < R_m \). The smallness of \( F_{i,2}^i \) is achieved by Proposition 3.4.

We define now the center of vorticity and the moment of inertia,
\[ B_{i}^i(t) := a_i^{-1} \int_{\mathbb{R}^2} dx x \omega_{i,\varepsilon}(x,t) \]
\[ I_{i}^i(t) := \int_{\mathbb{R}^2} dx |x - B_{i}^i(t)|^2 |\omega_{i,\varepsilon}(x,t)| \]
whose properties will be exploited in the following.

In the next lemmas we omit for simplicity the index \( i \) from the notation and we assume, without lost of generality, \( a_i = 1 \). This is equivalent to consider a “reduced system” with only one vortex moving in an external field acting on it, which has the properties stated in Lemma 3.6. In this case it is easily verified that the following equation holds:
\[ \frac{d}{dt} \omega_t(f) = \omega_t \left[ ((u + F_{i}) \cdot \nabla) f + \partial_t f \right]. \] (3.13)

The results we will prove hold obviously for each \( i \).

Lemma 3.6. For \( t \leq \bar{T} \) and for small enough \( \varepsilon \),
\[ I_{\varepsilon}(t) \leq \frac{C}{\log \varepsilon|^{2(\alpha - 1)}}. \] (3.14)

Proof. We estimate the derivative of \( I_{\varepsilon}(t) \), using (3.13):
\[ \frac{d}{dt} I_{\varepsilon}(t) = \int dx \omega_{\varepsilon}(x,t) \left[ (u + F_{\varepsilon}) \cdot (x - B_{\varepsilon}(t)) - \dot{B}_{\varepsilon}(t) \cdot (x - B_{\varepsilon}(t)) \right]. \]

Moreover
\[ \frac{d}{dt} B_{\varepsilon}(t) = \int dx \omega_{\varepsilon}(x,t) (u(x,t) + F_{\varepsilon}(x,t)), \]
then
\[ \frac{d}{dt} I_{\varepsilon}(t) = 2 \int dx \omega_{\varepsilon}(x,t) \left[ u(x,t) - \int dy \omega_{\varepsilon}(y,t) u(y,t) \right] \cdot (x - B_{\varepsilon}(t)) + 2 \int dx \omega_{\varepsilon}(x,t) \left[ F_{\varepsilon}(x,t) - \int dy \omega_{\varepsilon}(y,t) F_{\varepsilon}(y,t) \right] \cdot (x - B_{\varepsilon}(t)). \]

Consider first the term containing \( F_{\varepsilon} \): we note that, by the definition of \( B_{\varepsilon}(t) \),
\[ \int dx \omega_{\varepsilon}(x,t)(x - B_{\varepsilon}(t)) \cdot \int dy \omega_{\varepsilon}(y,t) F_{\varepsilon}(y,t) = 0 \]
\[ \int dx \omega_\varepsilon(x, t)(x - B_\varepsilon(t)) \cdot F_{\varepsilon,1}(B_\varepsilon(t), t) = 0. \]

We then obtain:

\[
2 \left| \int dx \omega_\varepsilon(x, t) \left[ F_{\varepsilon}(x, t) - \int dy \omega_\varepsilon(y, t) F_{\varepsilon}(y, t) \right] \cdot (x - B_\varepsilon(t)) \right|
\]

\[
= 2 \left| \int dx \omega_\varepsilon(x, t) \left[ F_{\varepsilon,1}(x, t) - F_{\varepsilon,1}(B_\varepsilon(t), t) \right] \cdot (x - B_\varepsilon(t)) \right|
\]

\[
+ 2 \left| \int dx \omega_\varepsilon(x, t) F_{\varepsilon,2}(x, t) \cdot (x - B_\varepsilon(t)) \right|
\]

\[
\leq 2 \int dx \omega_\varepsilon(x, t) |x - B_\varepsilon(t)|^2 + \frac{C}{|\log \varepsilon|^{\alpha - 1}} \int dx |x - B_\varepsilon(t)| \omega_\varepsilon(x, t)
\]

\[
\leq 2L I_\varepsilon(t) + \frac{C}{|\log \varepsilon|^{\alpha - 1}} |I_\varepsilon(t)|^{1/2}
\]

where, in the last line, we used Cauchy-Schwarz inequality, and \( L \) is the Lipschitz constant of \( F_{\varepsilon,1} \).

For the term containing \( u \), we have analogously:

\[
\int dx \omega_\varepsilon(x, t)(x - B_\varepsilon(t)) \cdot \int dy \omega_\varepsilon(y, t) u(y, t) = 0.
\]

Moreover, by the antisymmetry of \( K \),

\[
\int dx \omega_\varepsilon(x, t) \tilde{u}(x, t) = \int dx \int dy \omega_\varepsilon(x, t) \omega_\varepsilon(y, t) K(x - y) = 0 \tag{3.15}
\]

and recalling that, by definition, \((x - y) \cdot K(x - y) = 0\), we get

\[
\int dx \omega_\varepsilon(x, t) x \cdot \tilde{u}(x, t) = \int dx \int dy \omega_\varepsilon(x, t) \omega_\varepsilon(y, t) x \cdot K(x - y)
\]

\[
= \int dx \int dy \omega_\varepsilon(x, t) \omega_\varepsilon(y, t) y \cdot K(x - y)
\]

hence this integral is zero as well, by the antisymmetry of \( K \). Using Proposition \( \ref{prop:antisymmetry} \) we get then:

\[
2 \left| \int dx \omega_\varepsilon(x, t) \left[ u(x, t) - \int dy \omega_\varepsilon(y, t) u(y, t) \right] \cdot (x - B_\varepsilon(t)) \right|
\]

\[
\leq 2 \int dx \omega_\varepsilon(x, t) |u(x, t) - \tilde{u}(x, t)| |x - B_\varepsilon(t)|
\]

\[
\leq \frac{C}{|\log \varepsilon|^{\alpha - 1}} \int dx \omega_\varepsilon(x, t) |x - B_\varepsilon(t)| \leq \frac{C}{|\log \varepsilon|^{\alpha - 1}} |I_\varepsilon(t)|^{1/2}
\]

where Cauchy-Schwarz inequality has been used again in the last line.

Hence we have

\[
|I_\varepsilon(t)| \leq 2L I_\varepsilon(t) + \frac{C}{|\log \varepsilon|^{\alpha - 1}} |I_\varepsilon(t)|^{1/2}.
\]

Defining \( M_\varepsilon(t) := |I_\varepsilon(t)|^{1/2} \), by Gronwall’s inequality and using the fact, by the initial data, \( I_\varepsilon(0) \leq 4\varepsilon^2 \), we get

\[
M_\varepsilon(t) \leq \left( 2 \varepsilon + \frac{C}{2L|\log \varepsilon|^{\alpha - 1}} \right) e^{Lt} \leq \frac{C}{|\log \varepsilon|^{\alpha - 1}} e^{Lt}. \tag{3.16}
\]
From the previous bound we finally obtain, recalling that \( t \leq \bar{T} \),
\[
I_\varepsilon(t) \leq \frac{C}{|\log \varepsilon|^{\alpha - 1}}.
\]

**Lemma 3.7.** Let us put
\[
R_t := \max\{|x - B_\varepsilon(t)| : x \in \Lambda_\varepsilon(t)\}
\]
and choose \( x_0 \in \Lambda_\varepsilon(0) \) such that, at time \( t \leq \bar{T} \),
\[
\frac{3}{4}R_t \leq |\phi_\varepsilon(x_0) - B_\varepsilon(t)| \leq R_t.
\]

Then at this time \( t \) the following inequality holds:
\[
\frac{d}{dt}|\phi_\varepsilon(x_0) - B_\varepsilon(t)| \leq 2LR_t + \frac{CL_\varepsilon(t)}{R_t^2} + \sqrt{\frac{3M\varepsilon^{-\gamma}m_\varepsilon(R_t/2,t)}{\pi}} + \frac{C}{|\log \varepsilon|^{\alpha - 1}}
\]
where the function \( m_\varepsilon \) is defined by:
\[
m_\varepsilon(R,t) := \int_{|y - B_\varepsilon(t)| > R} dy \omega_\varepsilon(y,t) \quad \text{for } R \in (0, +\infty).
\]

**Proof.** Let us put \( x = \phi_\varepsilon(x_0) \). We have:
\[
\frac{d}{dt}|\phi_\varepsilon(x_0) - B_\varepsilon(t)| = |u(x,t) + F_\varepsilon(x,t) - \dot{B}_\varepsilon(t)| \frac{x - B_\varepsilon(t)}{|x - B_\varepsilon(t)|}
\]
\[
= \left[ \int dy (F_\varepsilon(x,t) - F_\varepsilon(y,t)) \omega_\varepsilon(y,t) \right] \frac{x - B_\varepsilon(t)}{|x - B_\varepsilon(t)|}
\]
\[
+ \left[ \int dy (u(x,t) - u(y,t)) \omega_\varepsilon(y,t) \right] \frac{x - B_\varepsilon(t)}{|x - B_\varepsilon(t)|}.
\]
The term involving \( F_\varepsilon \) is easily bounded using Proposition 3.5 since \( F_{\varepsilon,1} \) is Lipschitz and \( F_{\varepsilon,2} \) is small:
\[
\int dy |F(x,t) - F(y,t)| \omega_\varepsilon(y,t) \leq L \int dy |x - y| \omega_\varepsilon(y,t) + \frac{C}{|\log \varepsilon|^{\alpha - 1}}
\]
\[
\leq 2LR_t + \frac{C}{|\log \varepsilon|^{\alpha - 1}}.
\]

For the second integral containing the difference of the velocity field, we split it into three terms, recalling that
\[
\int dy \bar{u}(y) \omega_\varepsilon(y) = 0:
\]
\[
|u(x,t) - \bar{u}(x,t)| \leq \frac{C}{|\log \varepsilon|^{\alpha - 1}},
\]
\[
\left| \int dy (u(y,t) - \bar{u}(y,t)) \omega_\varepsilon(y,t) \right| \leq \frac{C}{|\log \varepsilon|^{\alpha - 1}}.
\]
The third (non trivial) term is
\[
\bar{u}(x,t) \cdot \frac{x - B_\varepsilon(t)}{|x - B_\varepsilon(t)|} = \frac{x - B_\varepsilon(t)}{|x - B_\varepsilon(t)|} \cdot \int dy K(x - y)\omega_\varepsilon(y,t).
\]
The integration domain can be decomposed into two regions: \( A_1 := \Sigma(B_\varepsilon(t)|R_t/2) \) and \( A_2 := \mathbb{R}^2 \setminus A_1 \). We call \( H_1 \) and \( H_2 \) the resultant integrals. We follow for \( H_1 \) the proof of [4 Lemma 2.5]; recalling (1.22) and the notation \( x^\perp = (x_2, -x_1) \) for
$x = (x_1, x_2)$, after introducing the new variables $x' = x - B_\varepsilon(t)$, $y' = y - B_\varepsilon(t)$, and using that $x' \cdot (x' - y')^\perp = -x' \cdot y^\perp$, we get,

$$H_1 = \frac{1}{2\pi} \int_{|y'| \leq R_t/2} dy' \frac{x' \cdot y^\perp}{|x'| |x' - y'|^2} \omega_\varepsilon(y' + B_\varepsilon(t)).$$

(3.24)

By definition of center of vorticity (3.12), $\int dy' \frac{x^\perp}{|x'|} \omega_\varepsilon(y' + B_\varepsilon(t)) = 0$, so that

$$H_1 = H_1' - H_1''.$$

(3.25)

where

$$H_1' = \frac{1}{2\pi} \int_{|y'| \leq R_t/2} dy' \frac{x' \cdot y^\perp}{|x'|} \frac{y' \cdot (2x' - y')}{|x' - y'|^2} \omega_\varepsilon(y' + B_\varepsilon(t)),$$

$$H_1'' = \frac{1}{2\pi} \int_{|y'| > R_t/2} dy' \frac{x' \cdot y^\perp}{|x'|} \omega_\varepsilon(y' + B_\varepsilon(t)).$$

From (3.18) we have $|x'| \geq 3R_t/4$, and hence $|y'| \leq R_t/2$ implies $|x' - y'| \geq R_t/4$ and $|2x' - y'| \leq |x' - y'| + |x'| \leq |x' - y'| + R_t \leq 5|x' - y'|$, so that

$$|H_1'| \leq \frac{C}{R_t} \int_{|y'| \leq R_t/2} dy' \frac{|y'|}{|x'|} \omega_\varepsilon(y' + B_\varepsilon(t)) \leq \frac{C I_\varepsilon(t)}{R_t^3}.$$

To bound $H_1''$ we note that, in view of (3.17), the integration is restricted to $|y'| \leq R_t$, so that (using the lower bound in (3.18))

$$|H_1''| \leq \frac{C}{R_t} \int_{|y'| > R_t/2} dy' \omega_\varepsilon(y' + B_\varepsilon(t)) \leq \frac{C I_\varepsilon(t)}{R_t^3},$$

where in the last inequality we used the Chebyshev’s inequality. By (3.25) and the previous estimates we conclude that

$$|H_1| \leq \frac{C I_\varepsilon(t)}{R_t^3}.$$

(3.26)

We analyse now $H_2$. We first note that

$$|H_2| \leq \frac{1}{2\pi} \int_{|y - B_\varepsilon(t)| > R_t/2} dy \frac{1}{|x - y|} \omega_\varepsilon(y, t)$$

so we can bound $H_2$ using again rearrangement, as in the proof of Proposition 3.4, we bound the integral taking a vorticity concentrated, as much as possible, around the singularity of $1/|x - y|$. Therefore, the rearrangement is achieved defining $\omega_\varepsilon$ equal to $3 M \varepsilon^{-\gamma}$ for $|x - y| < r$, and equal to 0 for $|x - y| \geq r$, where $r$ is chosen such that $3 M \varepsilon^{-\gamma} \cdot \pi r^2 = m_\varepsilon(R_t/2, t)$ (which is the “total mass” of $\omega_\varepsilon$ in the integration domain $A_2$). Then

$$|H_2| \leq \frac{3 M \varepsilon^{-\gamma}}{2\pi} \int_{|z| < r} \frac{dz}{z} = 3 M \varepsilon^{-\gamma} \int_0^r \frac{\rho}{\rho} d\rho = 3 M \varepsilon^{-\gamma} r = \sqrt{\frac{3 M m_\varepsilon(R_t/2)}{\pi \varepsilon^\gamma}}.$$

Collecting this last estimate, (3.21), (3.22), (3.23), and (3.26), we obtain (3.19).

We investigate now the behavior near to 0 of the function $m_\varepsilon(\cdot, t)$ introduced in (3.20).
Lemma 3.8. There exists $T > 0$ such that, for each $\ell > 0$,
\[
\lim_{\varepsilon \to 0} \varepsilon^{-\ell} m_\varepsilon \left( \frac{1}{\log |r|}, t \right) = 0 \tag{3.27}
\]
for any $t \in [0, T]$ and $k = (\alpha - 2)/2$.

Proof. Given $R \geq 2h > 0$, let $W_{R,h}(x)$, $x \in \mathbb{R}^2$, be a non-negative smooth function, depending only on $|x|$, such that
\[
W_{R,h}(x) = \begin{cases} 
1 & \text{if } |x| \leq R, \\
0 & \text{if } |x| \geq R + h,
\end{cases} \tag{3.28}
\]
and, for some $C > 0$,
\[
|\nabla W_{R,h}(x)| < \frac{C}{h}, \tag{3.29}
\]
\[
|\nabla W_{R,h}(x) - \nabla W_{R,h}(x')| < \frac{C}{h^2} |x - x'|. \tag{3.30}
\]

We define the quantity
\[
\mu_t(R, h) = \int dx \left[ 1 - W_{R,h}(x - B_\varepsilon(t)) \right] \omega_\varepsilon(x, t), \tag{3.31}
\]
which is a mollified version of $m_\varepsilon$, satisfying
\[
\mu_t(R, h) \leq m_\varepsilon(R, t) \leq \mu_t(R - h, h). \tag{3.32}
\]
In particular, it is enough to prove the claim with $\mu_t$ instead of $m_\varepsilon$.

The convenience is that the function $\mu_t$ is differentiable (with respect to $t$); therefore we compute its derivative:
\[
\frac{d}{dt} \mu_t(R, h) = - \int dx \nabla W_{R,h}(x - B_\varepsilon(t)) \cdot \left[ u(x, t) + F_\varepsilon(x, t) - \dot{B}_\varepsilon(t) \right] \omega_\varepsilon(x, t)
\]
\[
= - H_3 - H_4 - H_5
\]
with
\[
H_3 = \int dx \nabla W_{R,h}(x - B_\varepsilon(t)) \cdot \tilde{u}(x, t) \omega_\varepsilon(x, t)
\]
\[
H_4 = \int dx \nabla W_{R,h}(x - B_\varepsilon(t)) \cdot \left[ F_{\varepsilon,1}(x, t) - \int dy F_{\varepsilon,1}(y, t) \omega_\varepsilon(y, t) \right] \omega_\varepsilon(x, t)
\]
\[
H_5 = \int dx \nabla W_{R,h}(x - B_\varepsilon(t)) \cdot \left[ u(x, t) - \tilde{u}(x, t) - \int dy [u(y, t) - \tilde{u}(y, t)] \omega_\varepsilon(y, t) \right] \omega_\varepsilon(x, t)
\]
\[
+ \int dx \nabla W_{R,h}(x - B_\varepsilon(t)) \cdot \left[ F_{\varepsilon,2}(x, t) - \int dy F_{\varepsilon,2}(y, t) \omega_\varepsilon(y, t) \right] \omega_\varepsilon(x, t)
\]
since $\dot{B}_\varepsilon(t) = \int dy \omega_\varepsilon(y, t) [F_\varepsilon(y, t) + u(y, t) - \tilde{u}(y, t)]$. We immediately observe that, thanks to Proposition 3.4 to Lemma 3.5 and to the fact that $\nabla W_{R,h}(z)$ is zero if $|z| \leq R$,
\[
|H_5| \leq \frac{C}{h} \cdot \frac{C}{|\log |r||^{\alpha - 1}} \cdot m_\varepsilon(R, t). \tag{3.33}
\]
Following the proof of [5, Proposition 3.4] we find (we postpone the estimates of \(|H_3|\) and \(|H_4|\) in the Appendix):

\[
|H_3| \leq \frac{C}{R^3} I_e(t) m_\epsilon(R, t) ;
\]

\[
|H_4| \leq C \left(1 + \frac{2R}{h}\right) m_\epsilon(R, t) + \frac{C}{R^2} I_e(t) m_\epsilon(R, t) .
\]

Recalling (3.14), from estimates (3.33), (3.34), and (3.35), we have:

\[
\frac{d}{dt} \mu_t(R, h) \leq A_\epsilon(R, h) m_\epsilon(R, t)
\]

for any \(t \leq \bar{T}\), where

\[
A_\epsilon(R, h) = C \left( \frac{1}{R^3 |\log \epsilon|^{2(\alpha - 1)}} + \frac{1}{R^2 h |\log \epsilon|^{2(\alpha - 1)}} + \frac{1}{h |\log \epsilon|^{\alpha - 1}} + \frac{2R}{h} + 1 \right).
\]

Therefore, by (3.32) and (3.36),

\[
\mu_t(R, h) \leq \mu_0(R, h) + A_\epsilon(R, h) \int_0^t ds \mu_s(R - h, h)
\]

for any \(t \leq \bar{T}\) and \(\epsilon\) sufficiently small. We iterate the last inequality \(n = |\log \epsilon|\) times (where \([a]\) denotes the integer part of the positive number \(a\)), from

\[
R_0 = \frac{1}{|\log \epsilon|^k} \quad \text{to} \quad R_n = \frac{1}{2|\log \epsilon|^k},
\]

where \(R_n = R_0 - nh\), and consequently

\[
h = \frac{1}{2n |\log \epsilon|^k}.
\]

In this range for \(R\) the quantity \(A_\epsilon(R, h)\) is bounded by \(C|\log \epsilon|\), in fact

\[
\frac{2R}{h} \leq C \frac{|\log \epsilon|^{k+1}}{|\log \epsilon|^k},
\]

\[
\frac{1}{h |\log \epsilon|^{\alpha - 1}} \leq C \frac{|\log \epsilon|^{k+1}}{|\log \epsilon|^k} \leq C |\log \epsilon|
\]

since \(k = (\alpha - 2)/2 < \alpha - 1\),

\[
\frac{1}{R^2 h |\log \epsilon|^{2(\alpha - 1)}} \leq C \frac{|\log \epsilon|^{3k+1}}{|\log \epsilon|^{2(\alpha - 1)}} \leq C |\log \epsilon|
\]

since \(3k < 2(\alpha - 1)\),

\[
\frac{1}{R h^3 |\log \epsilon|^{2(\alpha - 1)}} \leq C \frac{|\log \epsilon|^{4k+3}}{|\log \epsilon|^{2(\alpha - 1)}} \leq C |\log \epsilon|
\]

since \(4k + 2 = 2(\alpha - 1)\). Note that in this point we need to choose \(k = (\alpha - 2)/2\) and \(\alpha > 2\).

Then, for any \(t \in [0, \bar{T}]\), it results \(A_\epsilon(R, h) \leq C|\log \epsilon|\) and

\[
\mu_t(R_0 - h, h) \leq \mu_0(R_0 - h, h) + \sum_{j=1}^{n-1} \mu_0(R_j, h) \frac{(C|\log \epsilon|)^j}{j!} + \frac{(C|\log \epsilon|)^n}{(n-1)!} \int_0^t ds \left( t - s \right)^{n-1} \mu_s(R_n, h).
\]
Since \( \Lambda_{\varepsilon}(0) \subseteq \Sigma(\varepsilon) \), we can determine \( \varepsilon \) so small such that \( \mu_{0}(R_{j}, h) = 0 \) for any \( j = 0, \ldots, n \), so that, for any \( t \in [0, \widetilde{T}] \),
\[
\mu_{t}(R_{0} - h, h) \leq \frac{(C|\log \varepsilon|)^{n}}{(n-1)!} \int_{0}^{t} \frac{ds}{(t-s)^{n-1}} \mu_{s}(R_{n}, h) \leq \frac{(C|\log \varepsilon|)^{n}}{n!}, \tag{3.42}
\]
where the obvious estimate \( \mu_{s}(R_{n}, h) \leq 1 \) has been used in the last inequality. In conclusion, using also (3.32),
\[
m_{\varepsilon}(R_{0}, t) \leq \mu_{t}(R_{0} - h, h) \leq (Ct)^{|\log \varepsilon|} \quad \forall t \in [0, \widetilde{T}],
\]
which implies (3.27) for \( t \leq T \) and \( T \) suitably small. \( \square \)

We are now ready to prove Theorem 2.1

**Proof of Theorem 2.1** With the previous results we can prove now that, for all \( t \in [0, T] \),
\[
\Lambda_{\varepsilon}(t) \subseteq \Sigma \left( B_{\varepsilon}(t) | C| \log \varepsilon |^{-k} \right) . \tag{3.43}
\]
Recalling the definition of \( R_{t} \) given in (3.17), if at time \( t \in [0, T] \) we have, for a certain \( x_{0} \in \Lambda_{\varepsilon}(0) \),
\[
\frac{3}{4} R_{t} \leq |\phi_{t}(x_{0}) - B_{\varepsilon}(t)| \leq R_{t} \tag{3.44}
\]
then the time derivative of \( |\phi_{t}(x_{0}) - B_{\varepsilon}(t)| \) is bounded by (3.19), that is (considering also (3.13))
\[
\frac{d}{dt} |\phi_{t}(x_{0}) - B_{\varepsilon}(t)| \leq 2 L R_{t} + \frac{C}{R_{t}^{2}|\log \varepsilon|^{2(\alpha-1)}} + C \sqrt{\varepsilon^{-\gamma} m_{\varepsilon}(R_{t}/2, t)} + \frac{C}{|\log \varepsilon|^{\alpha-1}}\tag{3.45}
\]
for each index \( i \) of the \( N \) vortices, omitted to simplify the notation.

Let \( t_{0} \) be the first time at which \( R_{0} = |\log \varepsilon|^{-k} \) and \( R_{t} = |\log \varepsilon|^{-k} \) for \( t \geq t_{0} \); of course if such \( t_{0} \) does not exist (3.43) is already achieved, as well as if there are time intervals after \( t_{0} \) for which \( R_{0} \leq |\log \varepsilon|^{-k} \). (3.43) is achieved in these time intervals.

In the worst case in which a fluid particle fulfills (3.44) in the whole time interval \([t_{0}, T]\), then we can bound the right hand side of (3.45) in the following way,
\[
\frac{d}{dt} |\phi_{t}(x_{0}) - B_{\varepsilon}(t)| \leq C|\phi_{t}(x_{0}) - B_{\varepsilon}(t)|, \tag{3.46}
\]

since the other terms are negligible with respect to the first one, by Lemma 3.8 (which holds for \( R_{t} \geq |\log \varepsilon|^{-k} \) with \( \ell > \gamma \) and by the following
\[
\frac{C}{R_{t}^{2}|\log \varepsilon|^{2(\alpha-1)}} \leq C R_{t} \quad \iff \quad R_{t} \geq \frac{C}{|\log \varepsilon|^{\alpha-1}};\tag{3.47}
\]
which hold true since \( R_{t} \geq |\log \varepsilon|^{-k} \) and \( k = (\alpha - 2)/2 \). From (3.46) we immediately obtain
\[
|\phi_{t}(x_{0}) - B_{\varepsilon}(t)| \leq |\phi_{t}(x_{0}) - B_{\varepsilon}(t_{0})| e^{C(t-t_{0})} \leq |\log \varepsilon|^{-k} e^{C(t-t_{0})}\tag{3.48}
\]
hence \( |\phi_{t}(x_{0}) - B_{\varepsilon}(t)| \leq C|\log \varepsilon|^{-k} \) for any fluid particle satisfying (3.44) \( \forall t \in [t_{0}, T] \). We can now deduce the same bound for any fluid particle in \( \Lambda_{\varepsilon}(t) \), that is
\[
\Lambda_{\varepsilon}(t) \subseteq \Sigma(B_{\varepsilon}(t)|C| \log \varepsilon |^{-k}), \tag{3.49}
\]
or equivalently $R_t \leq C|\log \varepsilon|^{-k}$. Suppose in fact that for a fluid particle, which does not satisfy $3.44$ for $t = t_0$, the quantity $|\phi_t(x_0) - B_\varepsilon(t)|$ reaches at a certain $t^* > t_0$ the value $3R_\varepsilon/4$; at this time $t^*$ the quantity $|\phi_t(x_0) - B_\varepsilon(t^*)|$ is clearly bounded by the analogous quantity for a fluid particle which satisfies $3.44$ for any $t \in [t_0, T]$. If the first fluid particle for $t \geq t^*$ enters the region $3.44$, then $|\phi_t(x_0) - B_\varepsilon(t)|$ has a time derivative which is bounded by $3.44$, hence its successive growth is controlled by $C|\log \varepsilon|^{-k}$.

If there is not a single fluid particle which satisfies $3.44$ for any $t \in [t_0, T]$, the same argument can be applied in subintervals $[t_0, t_1], [t_1, t_2], \ldots, [t_n, T]$, in any of which certainly there is a fluid particle for which $3.44$ holds. Therefore for any fluid particle in $\Lambda_\varepsilon(t)$ we have $|\phi_t(x_0) - B_\varepsilon(t)| \leq C|\log \varepsilon|^{-k}$, that is $R_t \leq C|\log \varepsilon|^{-k}$. Thus $3.43$ is proved.

It remains to prove that

$$|B^i_\varepsilon(t) - z_i(t)| \leq \frac{C}{|\log \varepsilon|^k} \quad \text{for each } i, \quad (3.49)$$

since with this bound, together with $3.43$, Theorem 2.1 follows. In this last point we reintroduce the index $i$. We have:

$$\dot{B}^i_\varepsilon(t) - \dot{z}_i(t) = a_i^{-1} \int dx \left( u^i(x, t) + F^i_\varepsilon(x, t) \right) \omega_{i, \varepsilon}(x, t)$$

$$- \sum_{j \neq i} a_j K(z_i(t) - z_j(t)).$$

We add and subtract appropriate terms, then by the the splitting of $F^i_\varepsilon$ given in Lemma 3.5 we get:

$$\dot{B}^i_\varepsilon(t) - \dot{z}_i(t) = a_i^{-1} \int dx u^i(x, t) \omega_{i, \varepsilon}(x, t) + a_i^{-1} \int dx F^i_{\varepsilon, 2}(x, t) \omega_{i, \varepsilon}(x, t)$$

$$+ a_i^{-1} \int dx \left[ F^i_{\varepsilon, 1}(x, t) - F^i_{\varepsilon, 1}(B^i_\varepsilon(t), t) \right] \omega_{i, \varepsilon}(x, t)$$

$$+ \sum_{j \neq i} \int dy \left[ K(B^i_\varepsilon(t) - y) - K(B^i_\varepsilon(t) - B^j_\varepsilon(t)) \right] \omega_{j, \varepsilon}(y, t)$$

$$+ \sum_{j \neq i} a_j \left[ K(B^i_\varepsilon(t) - B^j_\varepsilon(t)) - K(B^i_\varepsilon(t) - z_j(t)) \right]$$

$$+ \sum_{j \neq i} a_j \left[ K(B^i_\varepsilon(t) - z_j(t)) - K(z_i(t) - z_j(t)) \right].$$

The first term on the right hand side is controlled by adding and subtracting $\bar{u}^i(x, t)$ inside the integral, using Proposition 3.4 and (3.15). For the other terms we use Lemma 3.3 and the Lipschitz property of $K$ outside the disk $\Sigma(0, R_{\min}/2)$ (we call $L_1$ the Lipschitz constant of $K$ in this region) obtaining

$$|\dot{B}^i_\varepsilon(t) - \dot{z}_i(t)| \leq |a_i|^{-1} \frac{C}{|\log \varepsilon|^{a-1}} + |a_i|^{-1} L_1 F^i_\varepsilon(t)^{1/2} |a_i|^{1/2}$$

$$+ \sum_{j \neq i} L_1 F^j_\varepsilon(t)^{1/2} |a_j|^{1/2} + \sum_{j \neq i} |a_j| L_1 |B^j_\varepsilon(t) - z_j(t)|$$

$$+ \sum_{j \neq i} |a_j| L_1 |B^j_\varepsilon(t) - z_i(t)|.$$
Defining $\Delta(t) \equiv \max_{i=1,\ldots,N} |B_i(t) - z_i(t)|$, then

$$\Delta(t) \leq \max_{i=1,\ldots,N} |B_i(t) - \tilde{z}_i(t)|$$

$$\leq \frac{C}{|\log \varepsilon|^{n-1}} + C \sum_{j=1}^{N} \sqrt{I_j^2(t)} + 2L_1 \sum_{j=1}^{N} |a_j| \Delta(t).$$

By definition of $F_{t,1}$ it can be immediately seen that $L \geq L_1 \sum_j |a_j|$. By integration of the previous inequality we obtain:

$$\Delta(t) \leq \Delta(0)e^{2Lt} + C \int_0^t \sum_{j=1}^{N} \sqrt{I_j^2(s)}e^{L(t-s)} \, ds + \frac{C}{|\log \varepsilon|^{n-1}} \cdot (e^{2Lt} - 1).$$

Using the bound (3.16) we get, for $t \leq T$,

$$\Delta(t) \leq \frac{C}{|\log \varepsilon|^{n-1}},$$

so (3.49) is achieved. This concludes the proof of Theorem 2.1. □

**Appendix**

In this appendix we derive estimates (3.34) and (3.35).

$$H_3 = \int dx \nabla W_{R,h}(x - B_x(t)) \cdot \int dy K(x - y) \omega_x(x, t) \omega_x(x, t)$$

$$= \frac{1}{2} \int dx \int dy \left[ \nabla W_{R,h}(x - B_x(t)) - \nabla W_{R,h}(y - B_x(t)) \right]$$

$$\cdot K(x - y) \omega_x(x, t) \omega_x(y, t)$$

$$H_4 = \int dx \nabla W_{R,h}(x - B_x(t)) \cdot \int dy \left[ F_{\varepsilon,1}(x, t) - F_{\varepsilon,1}(y, t) \right] \omega_x(x, t) \omega_x(x, t),$$

where the second expression of $H_3$ is due to the antisymmetry of $K$.

Concerning $H_3$, we introduce the new variables $x' = x - B_x(t)$, $y' = y - B_x(t)$, define $\tilde{\omega}_x(z, t) := \omega_x(z + B_x(t), t)$, and let

$$f(x', y') = \frac{1}{2} \tilde{\omega}_x(x', t) \tilde{\omega}_x(y', t) \left[ \nabla W_{R,h}(x') - \nabla W_{R,h}(y') \right] \cdot K(x' - y'),$$

so that $H_3 = \int dx' \int dy' f(x', y')$. We observe that $f(x', y')$ is a symmetric function of $x'$ and $y'$ and that, by (3.28), a necessary condition to be different from zero is if either $|x'| \geq R$ or $|y'| \geq R$. Therefore,

$$H_3 = \left[ \int_{|x'| > R} dx' \int dy' + \int_{|y'| > R} dy' \int_{|x'| > h} dx' \int_{|y'| > R} dy' \right] f(x', y')$$

$$= 2 \int_{|x'| > R} dx' \int_{|y'| > R} dy' f(x', y') - \int_{|x'| > h} dx' \int_{|y'| > R} dy' f(x', y')$$

$$= H_3' + H_3'' + H_3'''$$.
with
\[
H'_3 = 2 \int_{|x'| > R} \, dx' \int_{|y'| \leq R-h} \, dy' \, f(x', y') ,
\]
\[
H''_3 = 2 \int_{|x'| > R} \, dx' \int_{|y'| > R-h} \, dy' \, f(x', y') ,
\]
\[
H'''_3 = - \int_{|x'| > R} \, dx' \int_{|y'| > R} \, dy' \, f(x', y') .
\]
By the assumptions on \( W_{R,h} \), we have \( \nabla W_{R,h}(z) = \eta_h(|z|)z/|z| \) with \( \eta_h(|z|) = 0 \) for \( |z| \leq R \). In particular, \( \nabla W_{R,h}(y') = 0 \) for \( |y'| \leq R - h \), hence
\[
H'_3 = \int_{|x'| > R} \, dx' \, \bar{\omega}_z(x', t) \eta_h(|x'|) \frac{x'}{|x'|} \cdot \int_{|y'| \leq R-h} \, dy' \, K(x' - y') \bar{\omega}_z(y', t) .
\]
In view of (3.29), \( |\eta_h(|z|)| \leq C/h \), so that
\[
|H'_3| \leq \frac{C}{h} m_+(R, t) \sup_{|x'| > R} |A_3(x')| ,
\]
with
\[
A_3(x') = \frac{x'}{|x'|} \cdot \int_{|y'| \leq R-h} \, dy' \, K(x' - y') \bar{\omega}_z(y', t) .
\]
Now, recalling (1.22) and using that \( x' \cdot (x' - y')^\perp = -x' \cdot y'^\perp \), we get,
\[
A_3(x') = \frac{1}{2\pi} \int_{|y'| \leq R-h} \, dy' \, \frac{x' \cdot y'^\perp}{|x'|^2 |x' - y'|^2} \bar{\omega}_z(y', t) .
\]
By (3.12), \( \int dy' y'^\perp \bar{\omega}_z(y', t) = 0 \), so that
\[
A_3(x') = A'_3(x') - A''_3(x') ,
\]
where
\[
A'_3(x') = \frac{1}{2\pi} \int_{|y'| \leq R-h} \, dy' \, \frac{x' \cdot y'^\perp}{|x'|^2 |x' - y'|^2} \bar{\omega}_z(y', t) ,
\]
\[
A''_3(x') = \frac{1}{2\pi} \int_{|y'| > R-h} \, dy' \, \frac{x' \cdot y'^\perp}{|x'|^2} \bar{\omega}_z(y', t) .
\]
We notice that if \( |x'| > R \) then \( |y'| \leq R - h \) implies \( |x' - y'| \geq h \) and \( |2x' - y'| \leq |x' - y'| + |x'| \). Therefore, for any \( |x'| > R \geq 2h \),
\[
|A'_3(x')| \leq \frac{1}{2\pi} \left[ \frac{1}{|x'|^2} \cdot \frac{1}{|x'|^2} \right] \int_{|y'| \leq R-h} \, dy' \, |y'|^2 \bar{\omega}_z(y', t) \\
\leq \frac{I_z(t)}{2\pi} \left[ \frac{1}{2h^2} \cdot \frac{1}{R^2} \right] \leq \frac{3I_z(t)}{4\pi Rh^2} .
\]
To bound \( A''_3(x') \), by Chebyshev’s inequality, for any \( |x'| > R \geq 2h \) we have,
\[
|A''_3(x')| \leq \frac{1}{2\pi} \int_{|y'| > R-h} \, dy' \, |y'| \bar{\omega}_z(y', t) \leq \frac{I_z(t)}{2\pi R^2 (R-h)} \leq \frac{I_z(t)}{2\pi Rh^2} .
\]
From Eqs. (3.31) and (3.33), the previous estimates, and \( R \geq 2h \), we conclude that
\[
|H'_3| \leq \frac{5C I_z(t)}{4\pi Rh^3} m_+(R, t) .
\]
Now, by \((3.30)\) and then applying the Chebyshev’s inequality and again \(R \geq 2h\),
\[
|H''_n| + |H''_n'|
\leq \frac{C}{\pi h^2} \int_{|x'| \geq R} dx' \int_{|y'| \geq R-h} dy' \tilde{\omega}_\epsilon(y', t) \tilde{\omega}_\epsilon(x', t)
= \frac{C}{\pi h^2} m_\epsilon(R, t) \int_{|y'| \geq R-h} dy' \tilde{\omega}_\epsilon(y', t) \leq \frac{4C I_\epsilon(t)}{\pi R^2 h^2} m_\epsilon(R, t).
\]
In conclusion, recalling \(R \geq 2h\),
\[
|H_3| \leq \frac{13C I_\epsilon(t)}{4\pi Rh^3} m_\epsilon(R, t).
\]
(3.55)

Concerning \(H_4\), we observe that by \((3.28)\) the integrand is different from zero only if \(R \leq |x - B_\epsilon(t)| \leq R + h\). Therefore, by Lemma 3.5 and \((3.29)\) we have, using again the variables \(x' = x - B_\epsilon(t), y' = y - B_\epsilon(t)\),
\[
|H_4| \leq \frac{C}{h} \int_{|x'| \geq R} dx' \tilde{\omega}_\epsilon(x', t) \int_{|y'| \geq R} dy' \tilde{\omega}_\epsilon(y', t)
+ \frac{C}{h} \int_{R \leq |x'| \leq R + h} dx' \tilde{\omega}_\epsilon(x', t) \int_{|y'| \leq R} dy' |x' - y'| \tilde{\omega}_\epsilon(y', t).
\]
Since \(|x' - y'| \leq 2R + h\) in the domain of integration of the last integral and using the Chebyshev’s inequality in the first one we get,
\[
|H_4| \leq \frac{C I_\epsilon(t)}{R^2 h} m_\epsilon(R, t) + C \left(1 + \frac{2R}{h}\right) m_\epsilon(R, t).
\]
(3.56)

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