NO ARBITRAGE AND CLOSURE RESULTS
FOR TRADING CONES WITH TRANSACTION COSTS

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Abstract. In this paper, we consider trading with proportional transaction costs as in Schachermayer’s paper of 2004. We give a necessary and sufficient condition for $\mathcal{A}$, the cone of claims attainable from zero endowment, to be closed. Then we show how to define a revised set of trading prices in such a way that firstly, the corresponding cone of claims attainable for zero endowment, $\tilde{\mathcal{A}}$, does obey the Fundamental Theorem of Asset Pricing and secondly, if $\tilde{\mathcal{A}}$ is arbitrage-free then it is the closure of $\mathcal{A}$. We then conclude by showing how to represent claims.

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1. Introduction, notation and main results

1.1. Introduction. Recollect the Fundamental Theorem of Asset Pricing in finite discrete time (see, for example, Schachermayer [10]): the fact that $\mathcal{A}$, the set of claims attainable for 0 endowment, is arbitrage-free implies and is implied by the existence of an Equivalent Martingale Measure; in addition, $\mathcal{A}$ is closed if it is arbitrage-free.

In [11], Schachermayer showed that the Fundamental Theorem of Asset Pricing fails in the context of trading with spreads/transaction costs, by giving an example of an $\mathcal{A}$ which is arbitrage-free, but whose closure does contain an arbitrage (see also Kabanov, Rasonyi and Stricker [7] and [8]). Consequently it is of interest to investigate further when the cone $\mathcal{A}$ is closed, and in cases when it is not, to find descriptions of its closure.

Schachermayer then established (Theorem 1.7 of [11]) the equivalence of two criteria associated with the no-arbitrage condition for the general set-up for trading with spreads/transaction costs: that robust no-arbitrage implies and is implied by the existence of a strictly consistent price process. Here, robust no-arbitrage means loosely that even with smaller bid-ask spreads there is no arbitrage, whilst a strictly consistent price process is one taking values in the relative interior of the set of consistent prices. In Theorem 2.1 of [11] he showed that the robust no-arbitrage condition implies the closure (in $\mathcal{L}^0$) of the set of attainable claims.

In this paper we shall first give, in Theorem 1.1 a simple necessary and sufficient condition for the set of attainable claims to be closed. We go on to show, in Theorem 1.2 how to amend the bid-ask spreads so that the new cone of attainable claims does satisfy the original Fundamental Theorem (i.e. is either arbitrage-free and closed or admits an arbitrage). Moreover, we show that in the arbitrage-free case the new cone is simply the closure of the original cone of attainable claims. Finally, in section 4

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we consider representation of attainable claims and characterize claims attainable for a given initial endowment.

1.2. Notation and main results. We are equipped with a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t : t = 0, 1, \ldots, T), \mathbb{P})\). We denote the set of non-negative, real-valued \(\mathcal{F}_t\)-measurable random variables by \(m_{\mathcal{F}_t}^+\) and the bounded non-negative, real-valued \(\mathcal{F}_t\)-measurable random variables by \(b_{\mathcal{F}_t}^+\). We denote the set of \(\mathbb{R}^d\)-valued \(\mathcal{F}_t\)-measurable random variables by \(L_{\mathcal{F}_t}^0\) and the non-negative \(\mathbb{R}^d\)-valued \(\mathcal{F}_t\)-measurable random variables by \(L_{\mathcal{F}_t}^{0+}\). More generally, we denote the set of \(\mathcal{F}_t\)-measurable random variables taking values in the (suitably measurable) random set \(S\) by \(L_{\mathcal{F}_t}(S; \mathcal{F}_t)\).

We recall the setup from Schachermayer’s paper [11] for trading with \(d\) assets. A \(d \times d\) matrix, \(\Pi\) is said to be a bid-ask matrix if

- \(\Pi_{ij} > 0\) for all \(i, j\);
- \(\Pi_{ii} = 1\);
- \(\Pi_{ij} \Pi_{jk} \geq \Pi_{ik}\).

We interpret \(\Pi_{ij}\) as the number of units of asset \(i\) required to purchase one unit of asset \(j\).

An adapted \(\mathbb{R}^{d \times d}\) process \((\pi_t : t = 0, 1, \ldots, T)\) with each \(\pi_t\) being a bid-ask matrix is known as a bid-ask process and gives the time \(t\) price for one unit of each asset in terms of each other asset. We assume that we are given a fixed bid-ask process, \(\pi\).

Next we define, for a fixed bid-ask matrix, \(\Pi\), the solvency cone, \(K(\Pi)\), as the convex cone in \(\mathbb{R}^d\) spanned by the canonical basis vectors of \(\mathbb{R}^d\), \((e_i)_{1 \leq i \leq d}\), together with the vectors \(\Pi_{ij} e_i - e_j\). The solvency cone thus consists of all those holdings which can be traded to a non-negative holding at the prices specified by \(\Pi\).

The cone of portfolios available at price zero under the bid-ask matrix \(\Pi\) is \(-K(\Pi)\).

The time \(t\) trading cone consists of all those portfolios (including those attainable by the “burning” of assets) which are available at time \(t\) from zero endowment. A moment’s thought will show that the set of trades which will be available at time \(t\) is the convex cone \(L_{\mathcal{F}_t}^0(-K(\pi_t); \mathcal{F}_t) \overset{df}{=} -K_t\).

The fundamental object of study is the cone of claims attainable from zero endowment, which will be denoted by \(\mathcal{A}\), and is defined to be

\[-K_0 + \ldots + (-K_T).\]

We also consider

\[\mathcal{C}_t \overset{df}{=} \{ X \in L_{\mathcal{F}_t}^0 : cX \in \mathcal{A} \text{ for all } c \in b_{\mathcal{F}_t}^+ \}.\]

We say a few words on the interpretation of \(\mathcal{C}_t\) versus \(-\mathcal{K}_t\). It is clear that \(-\mathcal{K}_t \subseteq \mathcal{C}_t \subseteq \mathcal{A}\), thus we have the equality

\[\mathcal{A} = \mathcal{C}_0 + \ldots + \mathcal{C}_T.\]

We can think of \(\mathcal{C}_t\) as consisting of those trades which are available on terms that are known at time \(t\) but which may require trading at later times to be realised.

Although each \(-\mathcal{K}_t\) is closed in \(L_{\mathcal{F}_t}^0\), this is not enough to ensure that \(\mathcal{A}\) is closed in \(L_{\mathcal{F}_t}^0\). In contrast we find the following necessary and sufficient condition for the closure of \(\mathcal{A}\):

\textbf{Theorem 1.1.} \(\mathcal{A}\) is closed in \(L_{\mathcal{F}_t}^0\) if and only if each \(\mathcal{C}_t\) is closed.
Let $\bar{A}$ denote the closure of $A$ in $\mathcal{L}_T^0$. Unlike in a classical market, $A$ can be arbitrage-free, that is to say
\[ A \cap \mathcal{L}_T^{0,+} = \{0\}, \]
yet not closed. It is then natural to ask for a description of the closure, $\bar{A}$.

**Theorem 1.2.** There is an adjusted bid-ask process $\bar{\pi}$ (see Definition 3.6) such that the associated cone of claims $\bar{A}$ satisfies $A \subseteq \bar{A} \subseteq \bar{A}$. Moreover, either $\bar{A}$ contains an arbitrage or it is arbitrage-free and closed. In the former case, $\bar{A}$ also contains an arbitrage, while in the latter case $\bar{A} = \bar{A}$.

2. Results on the closedness of $A$

As we have remarked already, $A$ can be arbitrage-free but not closed. Recall that Schachermayer gives a sufficient condition for the closedness of $A$ in terms of robust arbitrage.

Schachermayer defines the bid-ask spreads as the (random) intervals $[\pi_{t,j}^{i}, \pi_{t,j}^{i+}]$, for $i, j \in \{1, \ldots, d\}$ and $t = 0, \ldots, T$, and defines robust no-arbitrage as follows:

- the bid-ask process $\pi$ satisfies robust no-arbitrage if there is a bid-ask process $\bar{\pi}$ with smaller bid-ask spreads than $\pi$ (i.e. one whose bid-ask spreads almost surely fall in the relative interiors, in $\mathbb{R}$, of the bid-ask spreads for $\pi$) whose cone of admissible claims is arbitrage-free.

**Theorem 2.1** of Schachermayer [11] then states that robust no-arbitrage implies that the cone $A$ is closed — as the remark after the proof states, the proof relies only on the collection of null strategies (see Definition 2.5) being a closed vector space. However it is easy to find an example where $A$ is closed and arbitrage-free but robust no-arbitrage fails.

Consider the following example.

**Example 2.1.** Suppose that $T = 1$, $d = 2$, $\pi_{0}^{1,2} = 1$, $\pi_{0}^{2,1} = 2$ whilst $\pi_{1}^{1,2} = 1$ for each pair $i, j$. Take $\Omega = \mathbb{N}$, $\mathcal{F}_0$ trivial and $\mathcal{F}_1 = 2^n$ with $\mathbb{P}$ given by $\mathbb{P}(n) = 2^{-n}$.

It is immediately clear that robust no-arbitrage cannot hold, since any bid-ask process $\bar{\pi}$ with smaller bid-ask spreads than $\pi$ must have $\bar{\pi}_{0}^{1,2} = (\frac{1}{2}, 1)$ and $\bar{\pi}_{1}^{2,1} = 1$. There is then an arbitrage in the corresponding cone $\bar{A}$ since $e_2 - \bar{\pi}_{0}^{1,2} e_1 + e_1 - \bar{\pi}_{1}^{2,1} e_2$ must be a positive multiple of $e_1$.

**Remark 2.2.** With the setup of Example 2.1 it is clear from the bid-ask prices that
\[ -K_0 = \{(x, y) : x + y \leq 0 \text{ and } x + 2y \leq 0\} \]
and
\[ -K_1 = \{(X, Y) \in \mathcal{L}_T^0 : X + Y \leq 0 \mathbb{P} \text{ a.s.}\} \]
and so (since $-K_0 \subset -K_1$ and $A = -K_0 + -K_1$)
\[ A = \{(X, Y) \in \mathcal{L}_T^0 : X + Y \leq 0 \mathbb{P} \text{ a.s.}\}. \]

We can then see that $C_0 = \{(x, y) : x + y \leq 0\}$, while $C_1 = A = \{(X, Y) \in \mathcal{L}_T^0 : X + Y \leq 0 \mathbb{P} \text{ a.s.}\}$.

It is tempting to speculate that if $A$ is not closed, then $\bar{A}$ contains an arbitrage. The following example (compare with example 1.3 in Grigoriev [4]) shows that this is false.
Example 2.3. Suppose that $T = 1, d = 2, \pi_{1}^{1,2} = 1, \pi_{1}^{2,1} = 2$ whilst $\pi_{0}^{i,j} = 1$ for each pair $i, j$. Take $\Omega = \mathbb{N}$, $\mathcal{F}_{0}$ trivial and $\mathcal{F}_{1} = 2^{\mathbb{N}}$ with $\mathbb{P}$ given by $\mathbb{P}(n) = 2^{-n}$.

Then we have

$$A = \{(X, Y) \in \mathcal{L}_{1}^{0}: X + Y \leq 0 \, \mathbb{P} \mbox{ a.s.}\},$$

whereas

$$A = \{(X, Y) \in \mathcal{L}_{1}^{0}: X + Y \leq 0 \, \mathbb{P} \mbox{ a.s. and } 2X + Y \mbox{ is a.s. bounded above}\}.$$

Lemma 2.4. For each $t$, $C_{t}$ is a convex cone in $\mathcal{L}_{t}^{0}$ and

$$A = C_{0} + \ldots + C_{T}.$$

Proof. Convexity for $C_{t}$ is inherited from $A$ as is stability under multiplication by positive scalars. The decomposition result follows from the fact that

$$-K_{t} \subseteq C_{t}$$

and the fact that $C_{t} \subseteq A$. \hfill \Box

Definition 2.5. For any decomposition of $A$ as a sum of convex cones:

$$A = M_{0} + \ldots + M_{T},$$

we call elements of $M_{0} \times \ldots \times M_{T}$ which almost surely sum to 0, null-strategies (with respect to the decomposition $M_{0} + \ldots + M_{T}$) and denote the set of them by $\mathcal{N}(M_{0} \times \ldots \times M_{T})$. For convenience we denote $(-K_{0}) \times \ldots \times (-K_{T})$ by $\mathbb{K}$ and $C_{0} \times \ldots \times C_{T}$ by $\mathbb{C}$.

In what follows we shall often use the lemma below (Lemma 2 in Kabanov et al [8]):

Lemma 2.6. Suppose that

$$A = M_{0} + \ldots + M_{T}$$

is a decomposition of $A$ into convex cones with $M_{t} \subseteq \mathcal{L}_{t}^{0}$ and $bF_{t}^{\pm} M_{t} \subseteq M_{t}$ for each $t$; then $A$ is closed if $\mathcal{N}(M_{0} \times \ldots \times M_{T})$ is a vector space and each $M_{t}$ is closed.

Lemma 2.7. Suppose that $A = M_{0} + \ldots + M_{T}$, where for each $t$, $M_{t} \subseteq \mathcal{L}_{t}^{0}$ and $bF_{t}^{\pm} M_{t} \subseteq M_{t}$, then

$$M_{t} \subseteq C_{t}.$$

Moreover, for each $0 \leq t \leq T$,

$$A_{t}(C) \overset{\text{def}}{=} C_{0} + \ldots + C_{t} = A \cap \mathcal{L}_{t}^{0}.$$

Proof. The inclusion $M_{t} \subseteq C_{t}$ follows immediately from the fact that $M_{t} \subseteq A$; the stability under multiplication by $bF_{t}^{\pm}$; and the definition of $C_{t}$.

To prove the equality (2.1), suppose $X \in A \cap \mathcal{L}_{t}^{0}$. Let

$$X = \xi_{0} + \ldots + \xi_{T},$$

be a decomposition of $X$ with $\xi \in \mathbb{C}$. It follows from the fact that $X \in \mathcal{L}_{t}^{0}$ and $\xi_{s} \in \mathcal{L}_{t}^{0}$ for each $s < t$ that

$$Y = \xi_{t} + \ldots + \xi_{T} \in \mathcal{L}_{t}^{0}.$$

Therefore, it is sufficient to show that

$$(C_{t} + \ldots + C_{T}) \cap \mathcal{L}_{t}^{0} \subseteq C_{t}.$$

Now take $Y \in (C_{t} + \ldots + C_{T}) \cap \mathcal{L}_{t}^{0}$ and $c \in bF_{t}^{\pm}$: clearly $cY \in A \cap \mathcal{L}_{t}^{0}$ and hence, by definition, $Y \in C_{t}$. \hfill \Box
We may now give the **Proof of Theorem 1.1**

First assume that $\mathcal{A}$ is closed and $(X_n)_{n \geq 1}$ is a sequence in $\mathcal{C}_t$ converging in $L^0$ to $X$. It follows immediately from the assumption that $c X_n \xrightarrow{L^0} c X \in \mathcal{A}$ for all $c \in b \mathcal{F}_t^+$, hence $X \in \mathcal{C}_t$.

For the reverse implication we shall show that $\mathcal{N}(\mathcal{C})$ is a vector space and the result will then follow from Lemma 2.6.

Now suppose $(\xi_0, \ldots, \xi_T) \in \mathcal{N}(\mathcal{C})$ and $c \in b \mathcal{F}_t^+$ with almost sure upper bound $B$; then, defining $\zeta_s = B \xi_s$ for $s \neq t$ and $\zeta_t = (B - c) \xi_t$, it is clear (from the definition of $\mathcal{C}_s$) that $(\zeta_0, \ldots, \zeta_T) \in \mathcal{C}$, with

$$\sum_0^T \zeta_s = -c \xi_t.$$

It follows that $-c \xi_t \in \mathcal{A}$, $\forall c \in b \mathcal{F}_t^+$ and so $-\xi_t \in \mathcal{C}_t$ for each $t$ so that $\mathcal{N}(\mathcal{C})$ is a vector space as required. $\square$

**Remark 2.8.** In the proof above we used the fundamental property of null strategies: if $(\xi_t)_{0 \leq s \leq T}$ is a null strategy then $-\xi_t \in \mathcal{C}_t$. A null strategy allows one to eliminate friction in any of its component trades. In what follows we shall generalize this idea to more general sequences of strategies.

### 3. A revised fundamental theorem of asset pricing

We return to Example 2.3:

**Example 3.1.** Recall that $T = 1, d = 2, \pi_{1,2} = 1, \pi_{2,1} = 2$ whilst $\pi_{ij} = 1$ for each pair $i, j$; $\Omega = \mathbb{N}$, $\mathcal{F}_0$ is trivial and $\mathcal{F}_1 = 2^\Omega$ with $\mathbb{P}$ given by $\mathbb{P}(n) = 2^{-n}$.

We leave it as an exercise for the reader to show, as claimed above, that $\bar{\mathcal{A}} = \{(X, Y) \in L_1^0 : X + Y \leq 0 \ \mathbb{P} \ a.s.\}$ and hence corresponds to an adjusted bid-ask process, which is identically equal to 1. To do so, one may consider the null strategy $\xi$ given by $\xi_0 = e_1 - e_2$ and $\xi_1 = e_2 - e_1$.

In this section we shall show that $\bar{\mathcal{A}}$, if arbitrage-free, can always be represented by some adjusted bid-ask process. However, the next example, which is a minor adaptation of one of the key examples in Schachermayer [11], shows that it is necessary to consider more than just null strategies when seeking the appropriate adjusted prices.

**Definition 3.2.** We define $\mathcal{C}_t(\bar{\mathcal{A}})$ by analogy with $\mathcal{C}_t(\mathcal{A})$:

$$\mathcal{C}_t(\bar{\mathcal{A}}) \overset{\text{def}}{=} \{ X \in L_1^0 : c X \in \bar{\mathcal{A}} \text{ for all } c \in b \mathcal{F}_t^+ \}.$$  

**Example 3.3.** Suppose that $T = 1, d = 4, \Omega = \mathbb{N}$, $\mathcal{F}_0$ is trivial and $\mathcal{F}_1 = 2^\Omega$. The bid-ask process at time 0 satisfies $\pi_{0,1} = 1, \pi_{0,3} = 3$ whilst $\pi_{ij} = 1$ for each other pair
$i, j$ with $i \neq j$. At time 1, we have $\pi_{1,4}^1 = \omega = \frac{1}{\pi_{1,1}^4}$ and $\pi_{2,3}^2 = \omega = \frac{1}{\pi_{2,1}^3}$, whilst $\pi_{4,3}^4 = 1$ and $\pi_{3,4}^3 = 3$. All other entries are defined implicitly by the criterion

$$\pi_{1}^{i,j} = \min_{i=0,\ldots,n=j} \pi_{1}^{0,i_{1}} \cdots \pi_{1}^{n-1,n_{1}}.$$  

We shall show that $e_4 - e_3, e_2 - e_1, e_1 - e_2 \in C_1(\bar{A})$ even though there is no null strategy, $\xi$, with $\xi_0 = e_1 - e_2$ or with $\xi_0 = e_2 - e_1$ or with $\xi_0 = e_3 - e_4$.

First, define a sequence of strategies $\xi^N$ as follows: $\xi_0^N = N(e_4 - e_2)$ and

$$\xi_1^N = \frac{N(\omega e_4 - \omega e_1)}{\omega} - \frac{1(N_{\geq \omega})}{(N_{\geq \omega})}(e_3 - e_4) + N(e_2 - \frac{1}{\omega} e_3),$$

which means that $\xi^N_1 = N(e_4 - e_1) + 1(N_{\geq \omega})(e_4 - e_3)$.

Notice that $\sum_{t=0}^1 \xi^N_t = 1(N_{\geq \omega})(e_4 - e_3) \xrightarrow{\text{a.s.}} e_4 - e_3$ as $N \to \infty$, so we conclude that $e_4 - e_3 \in C_0(\bar{A})$. However, $e_3 - e_4 \in \mathcal{K}_1$ and so $((e_4 - e_3), (e_3 - e_4))$ is null for $\mathcal{C}(\bar{A})$ and hence $e_4 - e_3 \in C_1(\bar{A})$.

Now, given an element $X$ of $b\mathcal{F}_1^+$ with a.s. bound $B$, consider the strategy $((N + B)(e_1 - e_2) + (e_3 - e_4), (N + (B - X)(e_1 - e_2) + 1(N+(B-X)_{\geq \omega})(e_4 - e_3))$, which sums to $X(e_1 - e_2) - 1(N+(B-X)_{< \omega})(e_4 - e_3) \xrightarrow{\text{a.s.}} X(e_1 - e_2)$ as $N \to \infty$. This shows that $e_1 - e_2 \in C_1(\bar{A})$ and so is also in $C_0(\bar{A})$.

Lastly, consider the strategy

$$(N(e_2 - e_1) + (e_3 - e_4), (N + X))(e_4 - e_3) \xrightarrow{\text{a.s.}} X(e_2 - e_1)$$

as $N \to \infty$. This shows that $e_2 - e_1 \in C_1(\bar{A})$ and so is also in $C_0(\bar{A})$.

It follows that $\bar{A}$ corresponds to the adjusted bid-ask process $\bar{\pi}$ given, for $t = 0$, by: $\bar{\pi}_{0}^{1,2} = \bar{\pi}_{0}^{2,1} = \bar{\pi}_{0}^{3,4} = \bar{\pi}_{0}^{4,3} = 1$, $\bar{\pi}_{0}^{i,j} = \bar{\pi}_{0}^{j,i} = 3$ for $i \in \{1, 2\}$ and $j \in \{3, 4\}$; and for $t = 1$ by: $\bar{\pi}_{1}^{1,4} = \omega = \frac{1}{\pi_{1,1}^4} = \bar{\pi}_{1}^{2,3} = \frac{1}{\pi_{1,1}^2}$, whilst $\bar{\pi}_{1}^{4,3} = \bar{\pi}_{1}^{3,4} = \bar{\pi}_{1}^{1,2} = \bar{\pi}_{1}^{2,1} = 1$.

To see this, notice that the inclusion $\mathcal{A} \subset \bar{A}$ is obvious, while $\bar{A}$ is closed (by robust no-arbitrage) and the inclusion $\mathcal{A} \subset \bar{A}$ follows from the arguments above.

In order to prove our new version of the Fundamental Theorem we first define the adjusted bid-ask process, $\bar{\pi}$. This process will either be equal to the original bid-ask process or frictionless ($\omega$ by $\omega$ and for a given pair $(i, j)$).

**Definition 3.4.** Given a bid-ask process $\pi$, we define for each $(i, j, t)$,

$$z_{t}^{i,j} \overset{\text{def}}{=} e_j - \pi_{t}^{i,j} e_i$$

and

$$R_{t}^{i,j} \overset{\text{def}}{=} \{ B \in \mathcal{F}_t : -z_{t}^{i,j} 1_B \in \mathcal{A} \}.$$  

**Lemma 3.5.** If $B \in \mathcal{F}_t$ then

$$-z_{t}^{i,j} 1_B \in \mathcal{A} \iff -z_{t}^{i,j} 1_B \in \mathcal{C}_t(\bar{A}).$$

**Proof.** Clearly the RHS implies the LHS a fortiori.

To prove the reverse implication, first note that, by definition of $-\mathcal{K}_t$,

$$kz_{t}^{i,j} \in -\mathcal{K}_t$$

for any $k \in m\mathcal{F}_t^+$,

which in turn implies that

$$kz_{t}^{i,j} \in \mathcal{C}_t$$

for any $k \in m\mathcal{F}_t^+$. 

since \(-\mathcal{K}_t \subset \mathcal{C}_t\). Now suppose that \(c \in b\mathcal{F}_t^+\) with bound \(M\), and set
\[
Z \overset{\text{def}}{=} c(-z_t^{i,j}1_B) = M(-z_t^{i,j}1_B) + (M - c)z_t^{i,j}1_B.
\]
The first term on the right hand side of (3.3) is in \(\tilde{\mathcal{A}}\) since \(M\) is a positive constant, \(-z_t^{i,j}1_B\) is in \(\tilde{\mathcal{A}}\) by assumption and \(\tilde{\mathcal{A}}\) is a cone. The second term is in \(\tilde{\mathcal{A}}\) by (3.2) and, since \(\tilde{\mathcal{A}}\) is a convex cone, \(Z \in \tilde{\mathcal{A}}\). The result follows. \(\square\)

Now observe that the collection \(R_t^{i,j}\) is closed under countable unions. To see this, observe first that, since \(\tilde{\mathcal{A}}\) is a closed cone, \(R_t^{i,j}\) is closed under countable, disjoint, unions. Now notice that, from Lemma 3.5, if \(B \in R_t^{i,j}\) and \(D \in \mathcal{F}_t\), then \(B \cap D \in R_t^{i,j}\). It follows that if \((B_n)_{n \geq 1}\) is a sequence in \(R_t^{i,j}\), then \(B_n \cap \left(\bigcup_{k=1}^{n-1} B_k\right) \in R_t^{i,j}\). We then deduce, by the usual exhaustion argument, that there exists a \(\mathbb{P}\text{-a.s.}\) maximum, which we denote by \(B_t^{i,j}\); that is to say that
\[
B \in R_t^{i,j} \text{ and } B_t^{i,j} \subseteq B \implies \mathbb{P}(B \setminus B_t^{i,j}) = 0.
\]

**Definition 3.6.** We define the adjusted bid-ask process \(\tilde{\pi}\) as follows:

for each pair \(i \neq j\) and for each \(t\), \(\tilde{\pi}_t^{j,i} \overset{\text{def}}{=} 1_{\pi_t^{i,j}1_B} + 1_{(B_t^{i,j})^c} \cdot \tilde{z}_t^{j,i}
\]

**Remark 3.7.** \(\tilde{\pi}\) need not satisfy the condition:
\[
\tilde{\pi}_t^{ik} \leq \tilde{\pi}_t^{ij} \tilde{\pi}_t^{jk},
\]
but we may still define the corresponding trading cone and apply Lemma 2.6.

We denote the corresponding trading cones and cone of attainable claims by \((-\tilde{\mathcal{K}}_t)_{0 \leq t \leq T}\) and \(\tilde{\mathcal{A}}\) respectively. Throughout the rest of the paper we denote \(e_j - \tilde{\pi}_t^{i,j} e_i\) by \(\tilde{z}_t^{i,j}\).

We now give the

**Proof of Theorem 1.2**

We first show that \(\mathcal{A} \subseteq \tilde{\mathcal{A}} \subseteq \tilde{\mathcal{A}}\), and then show that \(\tilde{\mathcal{A}}\) is closed if it is arbitrage-free.

**Proof that \(\mathcal{A} \subseteq \tilde{\mathcal{A}}\):**

Since \(\pi_t^{ij} \pi_t^{ji} \geq 1\), it follows from the definition that \(\tilde{\pi}_t \leq \pi_t\) for each \(t\) and so
\[-\mathcal{K}_t \subseteq -\tilde{\mathcal{K}}_t,
\]
and hence
\[
\mathcal{A} \subseteq \tilde{\mathcal{A}}.
\]

**Proof that \(\tilde{\mathcal{A}} \subseteq \tilde{\mathcal{A}}\):**

we show this by demonstrating that
\[-\mathcal{K}_t \subseteq \tilde{\mathcal{A}}.
\]
for each \(0 \leq t \leq T\).

This, in turn, is achieved by showing that
\[
(3.4) \quad d \tilde{z}_t^{j,i} \in \tilde{\mathcal{A}}, \text{ for all } d \in m\mathcal{F}_t^+.
\]

From the definition of the adjusted bid-ask process, we obtain:
\[
\tilde{z}_t^{j,i} = -\tilde{\pi}_t^{j,i} z_t^{i,j} 1_{B_t^{i,j}} + z_t^{j,i} 1_{(B_t^{i,j})^c}.
\]
Observe that $-z_{i,j}^t 1_{B_i^t,j} \in C_t(\tilde{A})$ by definition of the set $B_i^t,j$ and (3.3), so

$$-d\tilde{z}_{i,j}^t 1_{B_i^t,j} \in C_t(\tilde{A}) \subset \tilde{A},$$

and

$$d z_{i,j}^t 1_{(B_i^t,j)^c} \in -\mathcal{K}_t \subset \tilde{A}$$

by definition of $-\mathcal{K}_t$, so that $d \tilde{z}_{i,j}^t \in \tilde{A}$ as required.

**Proof that ($\tilde{A}$ is closed if $\tilde{A}$ is arbitrage-free):**

We prove this by showing that the nullspace $\tilde{N} \overset{\text{def}}{=} N \left( (-\mathcal{K}_0) \times \ldots \times (-\mathcal{K}_T) \right)$ is a vector space and then appealing to Lemma 2.6.

Let $\xi \in \tilde{N}$. Then, defining $C_t(\tilde{A})$ analogously to $C_t(A)$, for each $t$ we have, by Remark 2.8, $-\xi_t \in C_t(\tilde{A})$, because $\xi$ is null for $\tilde{A}$.

Now, since $\xi_t \in -\mathcal{K}_t$ we may write it as

$$\xi_t = \sum_{i,j} \alpha_t^{i,j} z_{i,j}^t - \sum_k \beta_t^k e_k,$$

for suitable $\alpha_t^{i,j}$ and $\beta_t^k$ in $h\mathcal{F}^+$. Moreover, $-\xi_t \in C_t(\tilde{A})$ and since $\sum_{i,j} \alpha_t^{i,j} z_{i,j}^t \in \tilde{A}$ we conclude that $\sum_k \beta_t^k e_k \in \tilde{A}$. Now, since, by assumption, $\tilde{A}$ is arbitrage-free, we conclude that $\sum_k \beta_t^k e_k = 0$ a.s., so

$$\xi_t = \sum_{i,j} \alpha_t^{i,j} z_{i,j}^t,$$

and consequently $-\sum_{i,j} \alpha_t^{i,j} z_{i,j}^t \in C_t(\tilde{A})$. Since $C_t(\tilde{A})$ is a convex cone and $\alpha_t^{i,j} z_{i,j}^t \in -\mathcal{K}_t \subset C_t(\tilde{A})$ for each $(i,j)$, we may deduce that, for each pair $(i,j)$:

$$-\alpha_t^{i,j} z_{i,j}^t \in C_t(\tilde{A}).$$

Now, multiplying by the positive, bounded and $\mathcal{F}_t$-measurable r.v. $\frac{1}{\alpha_t^{i,j}} 1_{\{\alpha_t^{i,j} \neq 0\}}$ we see that

$$-z_{i,j}^t 1_{\{\alpha_t^{i,j} > 0\} \cap (B_i^t,j)^c} = -\tilde{z}_{i,j}^t 1_{\{\alpha_t^{i,j} > 0\} \cap (B_i^t,j)^c} \in \tilde{A} \subset \tilde{A}.$$

Then, by definition of the set $B_i^t,j$, for each $n$ the subset $D_i^t,j(n) \overset{\text{def}}{=} \{\alpha_t^{i,j} > \frac{1}{n}\} \cap (B_i^t,j)^c \subset B_i^t,j$. Now, by taking the union over $n$, we see that

$$D_i^t,j \overset{\text{def}}{=} \{\alpha_t^{i,j} > 0\} \cap (B_i^t,j)^c = \bigcup_n D_i^t,j(n) \subset B_i^t,j,$$

and we obtain therefore that

$$\tilde{\pi}_t^{i,j} = \pi_t^{i,j} = \frac{1}{\tilde{\pi}_t^{i,j}}$$

on the subset $D_i^t,j$. We deduce that

$$-\tilde{z}_t^{i,j} 1_{D_i^t,j} = -z_t^{i,j} 1_{D_i^t,j} = \tilde{z}_t^{i,j} z_t^{i,j} 1_{D_i^t,j} \in -\mathcal{K}_t,$$

and

$$-\tilde{z}_t^{i,j} 1_{\{\alpha_t^{i,j} > 0\} \cap (B_i^t,j)^c} = \tilde{z}_t^{i,j} z_t^{i,j} 1_{\{\alpha_t^{i,j} > 0\} \cap (B_i^t,j)^c} \in -\mathcal{K}_t \subset -\mathcal{K}_t.$$
4. Decompositions of \( \mathcal{A} \), representation and dual cones

4.1. Decompositions of \( \mathcal{A} \) and consistent price processes. We have given a necessary and sufficient condition for \( \mathcal{A} \) to be closed in terms of the \( C_t(\mathcal{A}) \) and we have shown how to amend the bid-ask prices so that the new cone attainable with zero endowment is \( \mathcal{A} \) (if \( \mathcal{A} \) is arbitrage-free). It is natural to ask whether the resulting trading cones \( (-\tilde{K}_t)_{0 \leq t \leq T} \) coincide with the \( C_t(\tilde{\mathcal{A}}) \)'s. The following example shows that this is far from the case.

**Example 4.1.** Suppose that \( T = 1 \), \( d = 4 \), \( \Omega = \{1, 2\} \), \( \mathcal{F}_0 \) is trivial and \( \mathcal{F}_1 = 2^\Omega \). The bid-ask process at time 0 satisfies \( \pi_0^{4,3} = \pi_0^{4,2} = 1 \) whilst, for all other pairs \( i \neq j \), \( \pi_0^{ij} = 4 \); the bid-ask process at time \( t = 1 \) satisfies \( \pi_1^{2,1}(1) = 4/3 = 2 - \pi_1^{3,1}(1) = 2 - \pi_1^{1,2}(2) = \pi_1^{3,1}(2) \) whilst, for all other pairs \( i \neq j \), \( \pi_1^{ij} = 4 \). By considering the strategy \( \xi \) given by \( \xi_0 = \frac{1}{2}(e_3 + e_2) - e_4 \) and \( \xi_1 = e_1 - \frac{1}{2}(e_3 + e_2) \), we see that \( e_1 - e_4 \in A \) and hence is in \( C_0 \). Now \( \Omega \) is finite so \( \mathcal{A} \) is closed and it is now easy to check that \( \bar{\pi} = \pi \), yet \( e_1 - e_4 \notin -\tilde{K}_0 \) and so \( -\tilde{K}_0 \neq C_0 \).

In the rest of this section we shall show that nevertheless, the \( C_t \)'s and their ‘duals’ behave like the original trading cones.

Whereas each trading cone, being generated by a finite set of random vectors, can clearly be identified as \( \mathcal{L}^0(S; \mathcal{F}_t) \) for a suitable random cone \( S \), the same is not evidently true of the \( C_t \)'s. Thus, we first need some abstract results relating to cones of random variables.

**Remark 4.2.** We denote by \( \mathcal{D} \), the collection of all closed subsets of \( \mathbb{R}^d \). The standard Borel structure on \( \mathcal{D} \), known as the Effros-Borel structure, and denoted \( \mathcal{B}(\mathcal{D}) \), is as follows: for any set \( B \) in \( \mathbb{R}^d \) define \( \mathcal{D}(B) \) by

\[
\mathcal{D}(B) = \{ C \in \mathcal{D} : C \cap B \neq \emptyset \},
\]

then \( \mathcal{B}(\mathcal{D}) = \sigma(\pi) \), where

\[
\pi = \{ \mathcal{D}(B) : B \text{ open in } \mathbb{R}^d \}.
\]

**Definition 4.3.** Let us consider a map \( \Lambda : \Omega \rightarrow \mathcal{D} \). We say that \( \Lambda \) is Effros-Borel measurable if for all open sets \( U \subset \mathbb{R}^d \), we have \( \{ \omega : \Lambda(\omega) \cap U \neq \emptyset \} \in \mathcal{F} \). We denote by \( \Upsilon \), the set of all Effros-Borel measurable maps. We also refer to any \( \Lambda \in \Upsilon \) as a random closed set.

**Lemma 4.4.** For any \( X \in \mathcal{L}^0(\mathbb{R}^d; \mathcal{F}) \) and \( \Lambda \in \Upsilon \),

\[
(X \in \Lambda) \overset{\text{def}}{=} \{ \omega : X(\omega) \in \Lambda(\omega) \} \in \mathcal{F}.
\]

**Proof.** First, by the fundamental measurability theorem of Himmelberg [5], there is a sequence of \( \mathbb{R}^d \)-valued random variables \( (X_n)_{n \geq 1} \) such that a.s

\[
\Lambda(\omega) = \{ X_n(\omega) : n \geq 1 \}.
\]

Then, the set \( \{ \omega : X(\omega) \in \Lambda(\omega) \} = \bigcap \bigcup \{ \omega : |X_i(\omega) - X(\omega)| < \frac{1}{n} \} \in \mathcal{F} \). \( \square \)

**Remark 4.5.** In what follows we call a map \( D \in \Upsilon \) with values in the set of closed convex cones in \( \mathbb{R}^d \) a random closed cone.

**Theorem 4.6.** Abstract closed convex cones theorem. Let \( C \) be a closed convex cone in \( \mathcal{L}^0(\mathbb{R}^d; \mathcal{F}) \), then

\[
C \text{ is stable under multiplication by (scalar) elements of } b\mathcal{F}^+.
\]
iff there is a map $\Lambda \in \Upsilon$ such that

\begin{equation}
\mathcal{C} = \mathcal{L}^0(\Lambda; \mathcal{F}).
\end{equation}

In this case, the map $\Lambda$ is a random closed cone.

Proof. The implication $(4.3) \Rightarrow (4.2)$ is obvious.

To prove the direct implication: we consider the family:

$\Upsilon_C = \{\Gamma \in \Upsilon : \mathcal{L}^0(\Gamma; \mathcal{F}) \subset \mathcal{C}\}$.

From Valadier [13] and [14], there is an essential supremum $\Lambda \in \Upsilon$ of this family $\Upsilon_C$, i.e.:

1. for all $\Gamma \in \Upsilon_C$, we have $\Gamma \subset \Lambda$ a.s.;
2. if $\Sigma \in \Upsilon$ is such that for all $\Gamma \in \Upsilon_C$, we have $\Gamma \subset \Sigma$ a.s, then $\Lambda \subset \Sigma$ a.s.

Moreover there is a countable subfamily $(\Gamma_n)_{n \geq 1} \subset \Upsilon_C$ such that $\Lambda = \bigcup_{n \geq 1} \Gamma_n$ a.s.

We want to prove that $\mathcal{C} = \mathcal{L}^0(\Lambda; \mathcal{F})$. To do this, first we remark that $\mathcal{C}(\Lambda) = \bigcup_{n \geq 1} \mathcal{C}(\Gamma_n)$. Then $\mathcal{L}^0(\Lambda; \mathcal{F}) \subset \mathcal{C}$ and so $\Lambda \in \Upsilon_C$. Now let $\xi \in \mathcal{C}$ and define the map $\Gamma(\omega) = \Lambda(\omega) \cup \{\xi(\omega)\}$. For $X \in \Gamma$ a.s and $B = \{\xi = X\}$ we have $X1_B \in \Lambda$ and then $X1_B \in \mathcal{C}$ and $X1_B = \xi1_B \in \mathcal{C}$. So $X \in \mathcal{C}$. We deduce that $\mathcal{L}^0(\Gamma; \mathcal{F}) \subset \mathcal{C}$ and then $\Gamma \in \Upsilon_C$. By the essential supremum property of $\Lambda$, we have $\Gamma \subset \Lambda$ and then $\xi \in \Lambda$ a.s.

Now suppose that $(4.3)$ is satisfied and consider the sequence $(X_n)_{n \geq 1}$ that generates $\Lambda$. For any $\alpha \in \mathbb{R}^n$, define

$$Y_{n,\alpha} = \sum_{i=1}^{n} \alpha_i X_i.$$ 

Notice that, denoting the non-negative rationals by $\mathbb{Q}_+$, the collection

$$S \overset{\text{def}}{=} \{Y_{n,\alpha} : \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Q}_+^n\}$$

is countable.

Define the map $\tilde{\Lambda}$ by:

$$\tilde{\Lambda}(\omega) = \overline{\{Y(\omega) : Y \in S\}}^{\mathbb{R}^d}.$$ 

From the convex cone property of $\mathcal{C}$, we have each $Y \in \mathcal{C}$ and then, from $(4.3)$, $\mathbb{P}(Y \in \Lambda) = 1$. We deduce that $\tilde{\Lambda} \subset \Lambda$ a.s and then (since $X_n \in S$ for each $n$) that $\Lambda = \tilde{\Lambda}$ a.s.

Definition 4.7. Given a closed convex cone $\mathcal{C}$ in $\mathcal{L}^1_0$ satisfying $(4.2)$ (with respect to the $\sigma$-algebra $\mathcal{F}$) we denote the corresponding random convex cone in $(4.3)$ by $\Lambda(\mathcal{C}; \mathcal{F})$.

Corollary 4.8. Suppose that $0 \leq p \leq \infty$ and let $\mathcal{C}$ be a convex cone in $\mathcal{L}^p(\mathbb{R}^d; \mathcal{F})$ with $\mathcal{C}$ closed in $\mathcal{L}^p(\mathbb{R}^d; \mathcal{F})$ if $0 \leq p < \infty$, and with $\mathcal{C}$ $\sigma(\mathcal{L}^\infty(\mathbb{P}), \mathcal{L}^1(\mathbb{P}))$-closed if $p = \infty$. Then, $\mathcal{C}$ is stable under multiplication by (scalar) elements of $b\mathcal{F}^+$ iff there exists a random closed cone $D$ such that

$$\mathcal{C} = \mathcal{L}^p(D; \mathcal{F}).$$

Proof. First suppose that $0 \leq p < \infty$ and consider $\overline{\mathcal{C}}^0 \overset{\text{def}}{=} \overline{\mathcal{C}^0}$, the closure of $\mathcal{C}$ in $\mathcal{L}^0$. It is clear that $\overline{\mathcal{C}}^0$ inherits stability under multiplication by $b\mathcal{F}^+$ from $\mathcal{C}$ so, by Theorem 4.6

$$\overline{\mathcal{C}}^0 = \mathcal{L}^0(D; \mathcal{F}),$$

where $D = \Lambda(\overline{\mathcal{C}}^0; \mathcal{F})$. It suffices then to prove that $\mathcal{C} = \overline{\mathcal{C}}^0 \cap \mathcal{L}^p$. The inclusion $\mathcal{C} \subset \overline{\mathcal{C}}^0 \cap \mathcal{L}^p$ is obvious. Now let $X \in \overline{\mathcal{C}}^0 \cap \mathcal{L}^p$, so there exists a sequence $Y^n \in \mathcal{C}$
which converges a.s to \( X \). Take a sequence \( (\phi_m)_{m \geq 1} \) of continuous functions on \( \mathbb{R} \) with compact support such that \( \phi_m \) tends pointwise to 1 as \( m \to \infty \), then, by the Bounded Convergence Theorem, \( Y^n_m \overset{\text{d}}{=} Y^n m(|Y^n|) \in C \) converges to \( Y_m \overset{\text{d}}{=} X \phi_m(|X|) \) in \( \mathcal{L}^p \). So \( Y_m \in C \) and, by letting \( m \uparrow \infty \), we obtain the result that \( X \in C \).

In the case where \( p = \infty \), given \( X \in C' \cap \mathcal{L}^\infty \) again take a sequence \( (Y^n) \) in \( C \) such that \( Y^n \overset{\text{a.s.}}{\to} X \). Then, for any \( f \in \mathcal{L}^1(\mathbb{R}^d; \mathcal{F}) \) and any \( m \), we have that \( f.Y^n \phi_m(|Y^n|) \overset{\text{a.s.}}{\to} f.X\phi_m(|X|) \), and then \( f.Y^n \phi_m(|Y^n|) \overset{\text{L}^\infty}{\to} f.X\phi_m(|X|) \) by the Dominated Convergence Theorem. We conclude that \( X\phi_m(|X|) \in C \) and hence, again letting \( m \uparrow \infty \), we obtain the inclusion \( C_0 \cap \mathcal{L}^\infty \subset C \), since \( C \) is closed in \( \sigma(\mathcal{L}^\infty, \mathcal{L}^1) \) and hence in \( \mathcal{L}^\infty \).

**Lemma 4.9.** Let \( C \) be a closed convex cone in \( \mathcal{L}^0(\mathbb{R}^d; \mathcal{F}) \), stable under multiplication by (scalar) elements of \( b\mathcal{F}^+ \), let \( 1 \leq p < \infty \), and \( \Lambda = \Lambda(\mathcal{C}; \mathcal{F}) \) be as defined before, then defining

\[
C^p = C \cap \mathcal{L}^p,
\]

the polar of \( C^p \) is given by

\[
(C^p)^* = \mathcal{L}^q(\Lambda^*; \mathcal{F}),
\]

where \( q \) is the conjugate of \( p \) and \( \Lambda^* \) is the polar of \( \Lambda \) in \( \mathbb{R}^d \).

**Proof.** This parallels the second half of the proof of Theorem 4.6 \( \Box \)

**Definition 4.10.** An adapted sequence of random closed cones in \( \mathbb{R}^d \), \( (M_t)_{t=0,...,T} \), is called a trading decomposition of \( \mathcal{A} \) if

\[
\mathcal{A} = \mathcal{L}^0(M_0; \mathcal{F}_0) + \ldots + \mathcal{L}^0(M_T; \mathcal{F}_T).
\]

For such a decomposition, set \( \mathcal{M}_t = \mathcal{L}^0(M_t; \mathcal{F}_t) \) and, recalling that \( \mathcal{M} \) denotes \( \mathcal{M}_0 \times \ldots \times \mathcal{M}_T \), set

\[
\mathcal{A}_t(\mathcal{M}) \overset{\text{def}}{=} \mathcal{M}_0 + \ldots + \mathcal{M}_t.
\]

For any trading decomposition \( (M_t)_{t=0,...,T} \), we define a consistent price process (with respect to \( (M_t)_{t=0,...,T} \)) to be a martingale, \( Z \), with \( Z_t \) taking values in \( M_t^* \setminus \{0\} \) for each \( t \). Thus, a consistent price process is nothing but a martingale selection of the set-valued process \( (M_t^* \setminus \{0\}) \).

Let \( \phi : \Omega \to (0,1] \) be an \( \mathcal{F}_T \)-measurable positive random variable. We denote by \( \mathcal{L}^1_\phi \) the Lebesgue space associated to the norm defined by

\[
||f||_{\mathcal{L}^1_\phi} \overset{\text{def}}{=} \mathbb{E}\{|f|\mathbb{R}^d\}.
\]

Its dual, denoted by \( \mathcal{L}^\infty_\psi \), with \( \psi = \frac{1}{\phi} \), is associated with the norm

\[
||f||_{\mathcal{L}^\infty_\psi} = \text{ess sup}\{\psi |f|\mathbb{R}^d\}.
\]

**Theorem 4.11.** \( \hat{\mathcal{A}} \), the closure of \( \mathcal{A} \) in \( \mathcal{L}^0 \), is arbitrage-free iff there is a consistent (for some and then for any trading decomposition \( (M_t)_{t=0,...,T} \) of \( \mathcal{A} \)) price process \( Z \), and in this case, for every strictly positive \( \mathcal{F}_T \)-measurable \( \phi : \Omega \to (0,1] \) we may find a consistent price process \( Z \) such that \( |Z_T| \leq \phi \) for some positive constant \( c \). In particular, taking \( \phi = 1 \), we can find a bounded consistent price process iff \( \hat{\mathcal{A}} \) is closed.
We conclude that

\[ ((1) \begin{proof} \text{Proof. This follows very closely the proof of Theorem 1.7 (assuming Theorem 2.1) of Schachermayer [11], ignoring references to ‘robust’ and ‘strict’. A sketch proof is as follows: under the assumption that } \mathcal{A} \text{ is arbitrage-free, an exhaustion argument (see [15]), establishes the existence of a strictly positive element, } Z, \text{ of the polar to } \hat{A} \cap \mathcal{L}_0^1, \text{ whilst Lemma 4.9 and the fact that } \mathcal{M}_t \subset \mathcal{A} \text{ establishes that } Z_t \overset{\text{def}}{=} \mathbb{E}[Z|\mathcal{F}_t] \in \Lambda^*(\mathcal{M}_t; \mathcal{F}_t). \text{ Conversely, given a consistent } Z, \text{ we define a frictionless bid-ask process } \hat{\pi} \text{ by} \]

\[ \hat{\pi}_{ij}^t = \frac{Z_i^j}{Z_t^i}. \]

\text{Taking } Z^1 \text{ as numéraire and observing that } Q \text{ given by } \frac{dQ}{d\mathbb{P}} \text{ is an EMM for the corresponding discounted asset prices, we see, by applying the fundamental theorem for frictionless trading, that } \hat{A} \text{ is closed and arbitrage-free. Now it is clear, since } Z \text{ is a consistent price process, that } \mathcal{M}_t \subset -\hat{K}_t = \{X \in \mathcal{L}_t^0 : Z_tX \leq 0 \text{ a.s.}\} \text{ and hence it follows that } \hat{A} \text{ is arbitrage-free.} \]

\[ \square \]

\text{Similar results were proved in Stricker [12], Jouini and Kallal [9], Schachermayer [11] and Grigoriev [4].}

\text{We denote } \mathcal{A} \cap \mathcal{L}_0^1 \text{ by } \mathcal{A}^\phi \text{ and by } \mathcal{A}_t^\psi \text{ its polar cone. We denote the consistent price processes with } Z_T \in \mathcal{A}_t^\psi \text{ by } \mathcal{A}_t^\psi, \text{ and the sets } \{X : X = Z_t \text{ for some } Z \in \mathcal{A}_t^\psi\} \text{ and } \{X : X = Z_t \text{ for some } Z \in \mathcal{A}_t^\psi\} \text{ by } \mathcal{A}_t^\psi \text{ and } \mathcal{A}_t^\psi \text{ respectively.}

\textbf{Remark 4.12.} Notice that if } \mathcal{A}_t^\psi \text{ is non-empty, then, identifying martingales with their terminal values, } \mathcal{A}_t^\psi \text{ is the closure in } L_\infty^\psi \text{ of } \mathcal{A}_t^\psi. \text{ This is a standard argument, following from the fact that if } X \in \mathcal{A}_t^\psi \text{ and } Y \in \mathcal{A}_t^\psi, \text{ then } X + \epsilon Y \in \mathcal{A}_t^\psi \text{ for every } \epsilon > 0. \text{ It also follows that } \mathcal{A}_t^\psi \text{ is the closure in } L_\infty^\psi \text{ of } \mathcal{A}_t^\psi.

\textbf{Remark 4.13.} Note that in Theorem 4.11 we do not need to assume that } \mathcal{A} \text{ is decomposed as a sum of } -\hat{K}_t \text{'s, but merely that it admits a trading decomposition.}

\textbf{Lemma 4.14.} Let } X \in \mathcal{L}_0^1. \text{ Then the following assertions are equivalent.}

\begin{enumerate}
    \item[(1)] \( X \in \mathcal{C}_t^\phi \overset{\text{def}}{=} \mathcal{C}_t \cap \mathcal{L}_0^1. \)
    \item[(2)] \( X \in \mathcal{L}_0^\psi(\mathcal{F}_t) \text{ and } Z_tX \leq 0 \text{ a.s. for all } Z \in \mathcal{A}_t^\psi. \)
    \item[(3)] \( \mathbb{E}[(W.X)|\mathcal{F}_t] \leq 0 \text{ for all } W \in L_\infty^\psi^+ \text{ such that } \mathbb{E}[W|\mathcal{F}_t] \in \mathcal{A}_t^\psi. \)
\end{enumerate}

\textbf{Proof.} \((1) \Rightarrow (2)\)

\text{Clearly, if } X \in \mathcal{C}_t^\phi, \text{ then } X \in \mathcal{L}_0^\psi(\mathcal{F}_t). \text{ Now, for } Z \in \mathcal{A}_t^\psi \text{ and } f \in b\mathcal{F}_t^+ \text{ we have:} \]

\[ \mathbb{E}f(Z_tX) = \mathbb{E}Z_t.(fX) = \mathbb{E}Z_T.(fX) \leq 0, \]

\text{since } Z_T \in \mathcal{A}_t^\psi \text{ and } fX \in \mathcal{A}^\phi. \text{ Since } f \text{ is arbitrary it follows that } Z_tX \leq 0 \text{ a.s.}

\((2) \Rightarrow (1)\)

\text{Now let } f \in b\mathcal{F}_t^+ \text{ and } X \text{ satisfy (2). We need only prove that } fX \in \mathcal{A}.

\text{Let } Z \in \mathcal{A}_t^\psi, \text{ then}

\[ \mathbb{E}Z_T.(fX) = \mathbb{E}Z_t.(fX) = \mathbb{E}f(Z_tX) \leq 0. \]

\text{Therefore, given } Z \in \mathcal{A}_t^\psi, \text{ by taking a sequence } (Z_n)_{n \geq 1} \text{ in } \mathcal{A}_t^\psi \text{ converging in } L_\infty \text{ to } Z \text{ we conclude that } \mathbb{E}Z_T.(fX) \leq 0 \text{ and hence } fX \in \mathcal{A}^\phi \subset \mathcal{A}. \]

\((2) \Rightarrow (3)\)
We remark that for $X$ satisfying (2) we have, for every $W \in \mathcal{L}_\psi^{\infty,+}$ such that $E[W|\mathcal{F}_t] \in \mathcal{A}^{\ast,\psi}_t$ and $f \in b\mathcal{F}_t^+$,

$$E(f \, (W.X)) = E(f \, E(W|\mathcal{F}_t).X) \leq 0.$$ 

Since $f$ is an arbitrary element of $b\mathcal{F}_t^+$,

$$E[(W.X)|\mathcal{F}_t] \leq 0.$$ 

$((3) \Rightarrow (2))$

Take an $X$ satisfying (3). We prove first that $X \in \mathcal{L}^1_\phi(\mathcal{F}_t)$.

From (3) we deduce that for every $W \in \mathcal{L}_\psi^{\infty,+}$ we have $E[(W - E(W|\mathcal{F}_t)).X] = 0$ since

$$E[(W - E(W|\mathcal{F}_t))|\mathcal{F}_t] = 0 \in \mathcal{A}^{\ast,\psi}_t.$$ 

Consequently for every $W \in \mathcal{L}_\psi^{\infty,+}$ we get

$$EW.(X - E(X|\mathcal{F}_t)) = E(W - E(W|\mathcal{F}_t)).X = 0.$$ 

Since $W$ is an arbitrary element of $\mathcal{L}_\psi^{\infty,+}$ we may deduce that $X = E(X|\mathcal{F}_t)$. Let $Z_t \in \mathcal{A}^{\ast,\psi}_t$, then

$$Z_t.X = E(Z_t.X|\mathcal{F}_t) \leq 0.$$ 

$\square$

4.2. Representation. The following is an easy modification of Theorem 4.1 of Schachermayer \[11\] and Theorem 4.2 of Delbaen, Kabanov and Valkeila \[3\]:

**Theorem 4.15.** Suppose that $\theta \in \mathcal{L}^0_T$ and $\mathcal{A}$ is closed and arbitrage-free. The following are equivalent:

(i) There is a self-financing process $\eta$ such that

$$\theta \leq \eta_T,$$

i.e. $\theta \in \mathcal{A}$.

(ii) For every consistent pricing process $Z$ such that the negative part $(\theta.Z_T)_-$ of the random variable $\theta.Z_T$ is integrable, we have

$$E[\theta. Z_T] \leq 0.$$ 

**Proof.** The proof is a much simplified version of the proof of Theorem 4.1 of Schachermayer \[11\]. We give a sketch of the proof.

$(i) \Rightarrow (ii)$

It is easy to check that Remark 2.4 of Schachermayer \[11\] remains valid if we replace the assumption there that $\pi$ satisfies the robust no-arbitrage assumption by the assumption that $\mathcal{A}$ is closed and arbitrage-free, or indeed, merely the assumption that there is a consistent price process. With this change, we have the forward implication.

$(ii) \Rightarrow (i)$

Fix $\theta$ and suppose that (i) does not hold. Now choose a $\phi$ such that $\theta \in \mathcal{L}^1_\phi$. Note that $\mathcal{A}^\phi$ is a closed, convex cone in $\mathcal{L}^1_\phi$. Since $\theta \notin \mathcal{A}^\phi$, there exists a separating continuous linear functional $Z \in \mathcal{L}_\psi^{\infty}$ such that $Z|_{\mathcal{A}^\phi} \leq 0$ and $<Z, \theta> = E[Z.\theta] > 0$. It follows from the first of these properties that $Z_t = E[Z|\mathcal{F}_t]$ is a consistent price process, and then the second shows that (ii) fails. $\square$

We may now consider representation of elements of $\mathcal{A}$:
Theorem 4.16. Suppose $\theta \in A^\phi$ and $\eta$ is an adapted $\mathbb{R}^d$-valued process in $L_\phi^1$ with $\eta_T = \theta$, and define $\xi = (\xi_0, ..., \xi_T)$ by $\xi_t = \eta_t - \eta_{t-1}$ with $\eta_{-1} \equiv 0$. Then $\xi \in \prod_{t=0}^T C_t^\phi$ if and only if for all $Z \in A_\psi^0$, the process $M^Z$ defined by $M^Z_t = \eta_t.Z_t$, is a supermartingale and $M^Z_T \geq \eta.T$.

Proof. Let $\xi \in \prod_{t=0}^T C_t^\phi$ and $Z \in A_\psi^0$. Then
$$E(M^Z_{t+1} | \mathcal{F}_t) = E(\eta_t.Z_{t+1} | \mathcal{F}_t) = \eta_t.Z_t = M^Z_t + \xi_t.Z_t \leq M^Z_t,$$
since $\xi_t \in C_t^\phi$ and $Z \in A_\psi^0$. Moreover we have
$$M^Z_T = \eta_{T-1}.Z_T = -\xi_T.Z_T + \eta.T \geq \eta.T,$$
by the same argument. Conversely, we prove that for every $t$, $\xi_t \in C_t^\phi$: by Lemma 4.14 we need to prove that $Z_t.\xi_t \leq 0$ a.s for every $Z \in A_\psi^0$ which is the case since, for $t \leq T-1$,
$$\xi_t.Z_t = E(M^Z_{t+1} | \mathcal{F}_t) - M^Z_t \leq 0,$$
and for $t = T$ we have
$$\xi_T.Z_T = \eta.T - M^Z_T \leq 0.$$
\hfill $\Box$

Problem 4.17. We would like to show that
$$A^\phi = C_0^\phi + \ldots + C_T^\phi,$$
or just that
$$A^\phi = \bigoplus_{t=0}^T C_t^\phi.$$
but a proof of either statement eludes us.

We conjecture that (4.4) is true.

Remark 4.18. We can consider $\eta$’s only defined for $t \leq T - 1$ in the theorem above to obtain the following:

Corollary 4.19. Suppose that $\eta$ is adapted to $(\mathcal{F}_t : 0 \leq t \leq T - 1)$. Then $\xi \in \prod_{t=0}^{T-1} C_t^\phi$ if and only if the process $M^Z$ is a supermartingale for all $Z \in D^{0,\psi}$. We may close $\eta$ on the right by $\theta$ if and only if $M^Z_T \geq \eta.Z$ for all $Z \in D^{0,\psi}$.

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