HARMONIC FUNCTIONS ON RANK ONE
ASYMPTOTICALLY HARMONIC MANIFOLDS

GERHARD KNIEPER AND NORBERT PEYERIMHOFF

Abstract. Asymptotically harmonic manifolds are simply connected complete Riemannian manifolds without conjugate points such that all horospheres have the same constant mean curvature $h$. In this article we present results for harmonic functions on rank one asymptotically harmonic manifolds $X$ with mild curvature boundedness conditions. Our main results are (a) the explicit calculation of the Radon-Nykodym derivative of the visibility measures, (b) an explicit integral representation for the solution of the Dirichlet problem at infinity in terms of these visibility measures, and (c) a result on horospherical means of bounded eigenfunctions implying that these eigenfunctions do not admit non-trivial continuous extensions to the geometric compactification $\overline{X}$.

CONTENTS

1. Introduction 1
2. Uniform divergence of geodesics 4
3. The geometric compactification 6
4. Gromov hyperbolicity 9
5. Visibility measures and their Radon-Nykodym derivative 12
6. Solution of the Dirichlet problem at infinity 18
7. Polynomial volume growth of horospheres 21
8. Horospherical means and bounded eigenfunctions 22
References 27

1. INTRODUCTION

Manifolds with asymptotically harmonic metrics were first introduced by Ledrappier ([Led, Thm. 1]) in the special case of negative curvature in connection with rigidity of measures related to the Dirichlet problem (harmonic measure) and the dynamics of the geodesic flow (Bowen-Margulis measure). One of the equivalent characterisations of asymptotically harmonic metrics there was that all horospheres have constant mean curvature $h \geq 0$. We express this geometric property in terms of Jacobi tensors (see Definition

Date: December 23, 2014.

2010 Mathematics Subject Classification. Primary 53C25, Secondary 37D20, 53C23, 53C40.

Key words and phrases. asymptotically harmonic manifolds, harmonic functions, visibility measures, Gromov hyperbolicity, Dirichlet problem at infinity, mean value property at infinity.
Let \((X, g)\) be a complete Riemannian manifold without conjugate points and let \(\pi : SX \to X\) be the canonical footpoint projection from the unit tangent bundle. For \(v \in SX\) let \(c_v : \mathbb{R} \to X\) be the unique geodesic given by \(c'_v(0) = v\). Let \(S_{v,r}\) and \(U_{v,r}\) be the orthogonal Jacobi tensors along \(c_v\), defined by \(S_{v,r}(0) = U_{v,r}(0) = \text{id}\) and \(S_{v,r}(r) = 0\) and \(U_{v,r}(-r) = 0\). Note that we have \(U_{v,r}(t) = S_{-v,r}(-t)\). The stable and unstable Jacobi tensors \(S_v\) and \(U_v\) are then defined as the Jacobi tensors along \(c_v\) with initial conditions \(S_v(0) = U_v(0) = \text{id}\) and \(S'_v(0) = \lim_{r \to \infty} S'_{v,r}(0)\) and \(U'_v(0) = \lim_{r \to \infty} U'_{v,r}(0)\). They are related by \(U_v(t) = S_{-v}(-t)\).

**Definition 1.1.** An asymptotically harmonic manifold \((X, g)\) is a complete, simply connected Riemannian manifold without conjugate points such that for all \(v \in SX\) we have \(\text{tr} U(v) = h\) for a constant \(h \geq 0\).

The manifolds considered in this article are rank one asymptotically harmonic manifolds. The notion of rank has been introduced by Ballmann, Brin and Eberlein in [BBE] for nonpositively curved manifolds as the dimension of the parallel Jacobi fields along geodesics. Since we do not assume nonpositive curvature, the notion of rank has to be understood in the following generalized sense given in [Kn2, Def. 3.1]:

**Definition 1.2.** Let \((X, g)\) be a complete simply connected Riemannian manifold without conjugate points. For \(v \in SX\) let \(D(v) = U(v) - S(v)\) and we define

\[
\text{rank}(v) = \dim(\ker D(v)) + 1
\]

and

\[
\text{rank}(X) = \min\{\text{rank}(v) \mid v \in SX\}.
\]

In [KnPe2], we proved equivalence of the following four properties for asymptotically harmonic manifolds \(X\) under the mild curvature boundedness condition

\[
(1.1) \quad \|R\| \leq R_0 \quad \text{and} \quad \|\nabla R\| \leq R'_0
\]

for some constants \(R_0, R'_0 > 0\): (a) \(X\) has rank one, (b) \(X\) has Anosov geodesic flow, (c) \(X\) is Gromov hyperbolic, and (d) \(X\) has purely exponential volume growth with growth rate \(h_{\text{vol}} = h\). These equivalences were first proved for noncompact harmonic manifolds in [Kn2] and for asymptotically harmonic manifolds admitting compact quotients in [Zi1]. Besides negatively curved symmetric spaces, Damek-Ricci spaces provide examples of rank one harmonic and therefore also asymptotically harmonic manifolds, since they all have purely exponential volume growth. As a consequence, all Damek-Ricci spaces are Gromov hyperbolic. (Note that all non-symmetric Damek-Ricci spaces admit zero-curvature.) In this article, we use the above equivalences to study harmonic functions on rank one asymptotically harmonic manifolds \((X, g)\) satisfying (1.1). Let us discuss the results of this paper in more detail.

In Sections 2 and 3, we introduce the geometric boundary \(X(\infty)\) via equivalence classes of geodesic rays and the canonical maps \(\varphi_p : S_pX \to X(\infty)\), \(\varphi_p(v) = c_v(\infty)\). These maps have natural extensions \(\tilde{\varphi}_p\) to the geometric
compactification \( \overline{X} = X \cup X(\infty) \), and we show that these extensions are homeomorphisms. The visibility measures \( \{\mu_p\} \) on \( X(\infty) \) are then defined as follows:

**Definition 1.3.** Let \( M_1(X(\infty)) \) denote the space of Borel probability measures on \( X(\infty) \). For every \( p \in X \), let \( \mu_p \in M_1(X(\infty)) \) be defined by

\[
\int_{X(\infty)} f(\xi) \, d\mu_p(\xi) = \frac{1}{\omega_n} \int_{S_pX} f(\varphi_p(v)) \, d\theta_p(v) \quad \forall f \in C(X(\infty)),
\]

where \( n = \dim(X) \) and \( \omega_n \) is the volume of the \((n-1)\)-dimensional standard unit sphere and \( d\theta_p \) is the volume element of \( S_pX \) induced by the Riemannian metric. \( \mu_p \) is called the visibility measure of \((X,g)\) at the point \( p \).

Sections 4 and 5 are concerned with the explicit calculation of the Radon-Nykodym derivative of the visibility measures. To state the result (Theorem 1.4 below), we need Busemann functions. Let \( v \in S_qX \) and \( \xi = c_v(\infty) \in X(\infty) \). Then the Busemann function (associated to \( v \in S_qX \) or to \((q,\xi) \in X \times X(\infty)\)) is defined as

\[
b_v(p) = b_{q,\xi}(p) = \lim_{t \to \infty} d(c_v(t),p) - t.
\]

**Theorem 1.4.** Let \((X,g)\) be a rank one asymptotically harmonic manifold satisfying (1.1). Let \( \{\mu_p\}_{p \in X} \) be the associated family of visibility measures. Then these measures are pairwise absolutely continuous and we have

\[
\frac{d\mu_p}{d\mu_q}(\xi) = e^{-hb_{q,\xi}(p)}.
\]

An analogous result on the Radon-Nykodym derivative for asymptotically harmonic manifolds in the case of pinched negative curvature was given in [CaSam, Prop. 6.1].

Since our rank one asymptotically harmonic manifolds \((X,g)\) are Gromov hyperbolic and have positive Cheeger constants (see [KnPe2, Prop. 5.3]), the general theory of Ancona [Anc1, Anc2] implies that the geometric boundary and the Martin boundary agree and that the Dirichlet problem at infinity can be solved. In Section 6 we give an alternative direct proof of this latter fact and give an explicit integral representation for the solution of the Dirichlet problem at infinity in terms of the visibility measures:

**Theorem 1.5.** Let \((X,g)\) be a rank one asymptotically harmonic manifold satisfying (1.1). Let \( f : X(\infty) \to \mathbb{R} \) be a continuous function. Then there exists a unique harmonic function \( H_f : X \to \mathbb{R} \) such that

\[
\lim_{x \to \xi} H_f(x) = f(\xi).
\]

Moreover, \( H_f \) has the following integral presentation:

\[
H_f(x) = \int_{X(\infty)} f(\xi) \, d\mu_x(\xi),
\]

where \( \{\mu_x\}_{x \in X} \subset M_1(X(\infty)) \) are the visibility probability measures.

A related result in the setting of harmonic manifolds can be found in Zimmer [Zi2, Thm. 1]. Moreover, the solution of the Dirichlet problem
at infinity for general nonpositively curved rank one manifolds admitting compact quotients was shown by Ballmann [Ba].

In Section 8 we consider eigenfunctions $\Delta f + \lambda f = 0$, $\lambda \in \mathbb{R} \setminus \{0\}$ on rank one asymptotically harmonic manifolds $X$ satisfying (1.1). We show that if such an eigenfunction $f \in C^\infty(X, \mathbb{C})$ has a continuous extension to the boundary $X(\infty)$, then the extension must be necessarily trivial, in contrast to Theorem 1.5 for harmonic functions. The proof is based on taking horospherical means. Since horospheres $\mathcal{H}$ are noncompact, the averages have to be taken via compact exhaustions $\{K_j\}$ with smooth boundaries $\partial K_j$. We first observe in Section 8 (see Theorem 8.1) that, for continuous functions $f : X = X \cup X(\infty) \rightarrow \mathbb{R}$ and horospheres $\mathcal{H}$ centered at $\xi \in X(\infty)$ with compact exhaustion $\{K_j\}$, we have

$$\lim_{j \to \infty} \int_{K_j} f(x) dx \over \text{vol}_{n-1}(K_j) = f(\xi).$$

The expression (1.4) is called the horospherical mean of $f$ with respect to the exhaustion $\{K_j\}$. In Section 7, we prove that all horospheres in these spaces have polynomial volume growth, which implies that they admit (compact) isoperimetric exhaustions $\{K_j\}$, that is,

$$\frac{\text{vol}_{n-2}(\partial K_j)}{\text{vol}_{n-1}(K_j)} \to 0 \text{ as } j \to \infty.$$

The main result of Section 8 is that, for all $\lambda \in \mathbb{R} \setminus \{0\}$, the horospherical means (with respect to isoperimetric exhaustions) of bounded eigenfunctions are zero.

**Theorem 1.6.** Let $(X, g)$ be a rank one asymptotically harmonic manifold of dimension $n$ satisfying (1.1) and $h > 0$ be the mean curvature of all horospheres. Let $\lambda \neq 0$ be a real number and $f \in C^\infty(X)$ be a bounded function satisfying $\Delta f + \lambda f = 0$ and $\mathcal{H} \subset X$ be a horosphere with isoperimetric exhaustion $\{K_j\}$. Then we have

$$\lim_{j \to \infty} \int_{K_j} f(x) dx \over \text{vol}_{n-1}(K_j) = 0.$$

This result leads to the following above mentioned fact, complementing Theorem 1.5.

**Theorem 1.7.** Let $(X, g)$ be a rank one asymptotically harmonic manifold satisfying (1.1). Let $\lambda \in \mathbb{R} \setminus \{0\}$ and $f \in C^\infty(X)$ be an eigenfunction $\Delta f + \lambda f = 0$. If $f$ has a continuous extension $F \in C(X)$ then we have necessarily $F|_{X(\infty)} \equiv 0$.

**Acknowledgement:** The authors would like to thank Evangelia Samiou for bringing the references [ItSa1, ItSa2] to their attention.

2. **Uniform divergence of geodesics**

In this section, we prove that for every distance $d > 0$ and any angle $\alpha > 0$ there exists a $t_0 > 0$, such that any two unit speed geodesics $c_1, c_2$ starting at the same point and differing by an angle $\geq \alpha$ will diverge uniformly in
the sense that \( d(c_1(t), c_2(t)) \geq d \) for all \( t \geq t_0 \). For the proof, we start with the following lemma.

**Lemma 2.1.** Let \((X, g)\) be a manifold without conjugate points and, for \( v \in SX\), let \( A_v \) be the orthogonal Jacobi tensor along \( c_v \) satisfying \( A_v(0) = 0 \) and \( A'_v(0) = id \). Then we have

\[
\begin{align*}
(\text{i}) & \quad A_v(t) = U_v(t) \int_0^t (U_v^*U_v)^{-1}(u)du, \\
(\text{ii}) & \quad (U'_v(0) - S'_{v,t}(0))^{-1} = \int_0^t (U_v^*U_v)^{-1}(u)du.
\end{align*}
\]

**Proof.** Since the endomorphism \( U_v(u) \) is non-singular and Lagrangian for all \( u \in \mathbb{R} \), we conclude from \cite{KnPe2} Prop. 2.1 that

\[
A_v(t) = U_v(t) \left( \int_0^t (U_v^*U_v)^{-1}(u)du \right) C_1 + C_2
\]

with suitable constant tensors \( C_1 \) and \( C_2 \). Evaluating and differentiating this identity at \( t = 0 \) yields \( C_2 = 0 \) and \( C_1 = id \), finishing the proof of (i). The statement (ii) can be found in \cite{KnPe2} Lemma 2.3.

**Proposition 2.2.** Let \((X, g)\) be a rank one asymptotically harmonic manifold satisfying (1.1). Then there exist constants \( a, \rho > 0 \) such that

\[
\|A_v(t)x\| \geq ae^{\frac{\rho}{2}t}\|x\|
\]

for all \( v \in SX \), \( x \in v^\perp \subset TX \) and \( t \geq 1 \).

**Proof.** We conclude from \cite{KnPe2} Thm. 1.3] that there exists \( \rho > 0 \) such that \( D(v) = U(v) - S(v) \geq \rho \cdot id \). Using \cite{KnPe2} Prop. 2.5 and the fact that \( S_v(t) \) is non-singular for \( t \geq 0 \), we conclude that there exists \( a_2 > 0 \) such that \( \|S_{v}^{-1}(t)y\| \geq \frac{1}{a_2}e^{\frac{\rho}{2}t}\|y\| \) for all \( y \in (\Phi^t v)^\perp \), where \( \Phi^t : SX \to SX \) denotes the geodesic flow. Using \( S_{\Phi^t v}(u)y_u = S_{w}(u + t)(S_{v}^{-1}(t)y_u) \) where \( y_u \) is the parallel translation of \( y \in (\Phi^t w)^\perp \) along \( c_v \) with \( u = -t \) and \( -v = \Phi^t w \) yields

\[
\|U_v(t)y\| = \|S_{-t}(-t)y\| = \|S_{-\Phi^t v}(t)y_t\| \geq \frac{1}{a_2}e^{\frac{\rho}{2}t}\|y\|.
\]

Lemma 2.1 yields

\[
\|A_v(t)(U'_v(0) - S'_{v,t}(0))y\| = \|U_v(t)y\| \geq \frac{1}{a_2}e^{\frac{\rho}{2}t}\|y\|,
\]

i.e.,

\[
\|A_v(t)x\| \geq \frac{1}{a_2\|U'_v(0) - S'_{v,t}(0)\|}e^{\frac{\rho}{2}t}\|x\| \geq \frac{1}{a_2(\|U'_v(0)\| + \|S'_{v,t}(0)\|)}e^{\frac{\rho}{2}t}\|x\|.
\]

The proposition follows now from \( \|U'_v(0)\| \leq \sqrt{R_0} \) and \( \|S'_{v,t}(0)\| = \|A'_v(t)A^{-1}_v(t)\| \leq \sqrt{R_0}\coth(\sqrt{R_0}) \), which can be found in \cite{KnPe2} Lem. 2.2.

Using this we derive the uniform divergence of geodesics described above.
Corollary 2.3. Let \( c_v : [0, \infty) \to X \) and \( c_w : [0, \infty) \to X \) be two geodesics with \( v, w \in S_pX \). Then

\[
d(c_v(t), c_w(t)) \geq a(t) \angle(v, w)
\]

where \( a : [0, \infty) \to [0, \infty) \) is a function (not depending on \( p \in X \)) with \( \lim_{t \to \infty} a(t) = \infty \).

Proof. Let \( c : [0, 1] \to X \) be a geodesic connecting \( c_v(t) \) with \( c_w(t) \). Then \( c \) is given by

\[
c(s) = \exp_p r(s)v(s),
\]

where \( v(s) \in S_pX \) and \( r(s) > 0 \) for all \( 0 \leq s \leq 1 \), and \( v(0) = v, v(1) = w \) and \( r(0) = r(1) = t \). Then

\[
c'(s_0) = D\exp_p(r(s_0)v(s_0))(r'(s_0)v(s_0) + r(s_0)v'(s_0))
\]

\[
= r'(s_0)c_v'(s_0)(r(s_0)) + A_v(s_0)(r(s_0))(v'(s_0)).
\]

Since \( c_v'(s_0) \perp A_v(s_0)(r(s_0))(v'(s_0)) \), we obtain

\[
\|c'(s_0)\|^2 = (r'(s_0))^2 + |A_v(s_0)(r(s_0))v'(s_0)|^2
\]

\[
\geq |A_v(s_0)(r(s_0))v'(s_0)|^2.
\]

If there exists \( s_0 \in [0, 1] \) such that \( r(s_0) \leq \frac{t}{\pi} \) then using the triangle inequality we have \( d(c_v(t), c_w(t)) \geq t \geq (t/\pi) \angle(v, w) \). If this is not the case, we obtain for all \( t > 0 \)

\[
d(c_v(t), c_w(t)) = \text{length}(c) \geq \int_0^1 |A_v(s)(r(s))v'(s)|ds
\]

\[
\geq a e^{\frac{\pi t}{2}} \int_0^1 |v'(s)|ds
\]

\[
\geq a e^{\frac{\pi t}{2}} \angle(v, w).
\]

The corollary follows now with the choice

\[
a(t) = \min \left\{ \frac{t}{\pi}, ae^{\frac{\pi t}{2}} \right\}.
\]

Remark. Note that the proof shows that the function \( a(t) \) describing the divergence of geodesics has at least linear growth.

3. The geometric compactification

Let \((X, g)\) be a rank one asymptotically harmonic manifold satisfying (1.1). The geometric boundary \( X(\infty) \) is the set of equivalence classes of asymptotic geodesic rays. Two geodesic rays \( c_1, c_2 : [0, \infty) \to X \) are called asymptotic, if there exists \( C > 0 \) with \( d(c_1(t), c_2(t)) \leq C \) for all \( t \geq 0 \). The equivalence class of a geodesic ray \( c \) is denoted by \( c(\infty) \).

Let \( p \in X \) and consider the map \( \varphi_p : S_pX \to X(\infty) \) with \( \varphi_p(v) = c_v(\infty) \). Uniform divergence of geodesics implies that \( \varphi_p \) is injective. Our next aim is to prove surjectivity of \( \varphi_p \). For this, we first prove general results which
will also be useful later on. The first result requires besides no conjugate points only a lower bound on the sectional curvature of \( X \).

**Proposition 3.1.** Let \( p, q_0, p_0 \in X \) be three different points such that \( 1 < r = d(p, p_0) = d(q_0, p_0) \) and \( d(p, q_0) < r - 1 \). Let \( \beta \) be the radial projection (from \( p_0 \)) of the geodesic connecting \( p \) and \( q_0 \) into \( S_r(p_0) \). Then there is a function \( b : \mathbb{R} \to (0, \infty) \) such that

\[
d_{S_r(p_0)}(p, q_0) \leq \text{length}(\beta) \leq \left( \max_{|s| \leq r} b(s) \right) d(p, q_0),
\]

where \( d_{S_r(p_0)} \) is the intrinsic distance of the sphere \( S_r(p_0) \).

**Proof.** Let \( \gamma : [0, d(p, q_0)] \to X \) be the geodesic connecting \( p \) and \( q_0 \). We first write \( \gamma \) and \( \beta \) in polar coordinates, i.e.,

\[
\gamma(t) = \exp_{p_0}(d(t)v(t)), \quad \beta(t) = \exp_{p_0}(rv(t))
\]

with \( d(t) = d(p_0, \gamma(t)) > 1 \) and \( v : [0, d(p, q_0)] \to S_{p_0}X \). Then we have

\[
\gamma'(t) = d'(t)c_{v(t)}(d(t)) + A_v(d(t))(v'(t))
\]
and \( \beta'(t) = A_v(r)v'(t) \). Note that the lower bound on sectional curvature yields the existence of a function \( b : \mathbb{R} \to [0, \infty) \) such that for all \( v \in SX \) and \( r \geq 1 \) we have \( \|S_{v,r}(t)\| \leq b(t) \) (see proof of Lemma 2.16 in [Kn1]). Using \( A_v(r)A_v^{-1}(x) = S_{v,x}(x-r) \) we therefore obtain

\[
\|\beta'(t)\| = \|A_v(r)v'(t)\| \leq \|A_v(r)(A_v^{-1}(d(t))v'(t))\| \leq b(d(t) - r)\sqrt{\|A_v(d(t))v'(t)\|^2 + \|d'(t)c_{v(t)}(d(t))\|^2} \\
\leq \left( \max_{|s| \leq r} b(s) \right) \|\gamma'(t)\|.
\]

The last inequality above follows from \( r - d(p, q_0) \leq d(t) = d(p_0, \gamma(t)) \leq r + d(p, q_0) \) and \( d(p, q_0) \leq r \).

For the next result, we need to introduce for every \( v \in SX \) and \( r > 0 \) the function \( b_{v,r}(p) = d(c_v(r), p) - r \) and the Busemann function \( b_v(p) = \lim_{r \to \infty} b_{v,r}(p) \). Since we also use a uniform bound on on the norm of Jacobi tensor \( S_{v,r}(t) \) for all \( t \geq 0 \) and \( r \geq 1 \) as has been derived in [KnPe2, Cor. 2.6] we need the assumption on \( X \) made at the beginning of this section.

**Corollary 3.2.** Let \( p, q \in X, r > 2d(p, q) + 1, v \in S_pX \) and \( w = -\text{grad} b_{v,r}(q) \in S_qX \). Then there exists a constant \( C = C(r) > 0 \) such that

\[
d(c_v(t), c_w(t)) \leq (1 + 2Ce^{-\frac{t^2}{2}})d(p, q) \quad \text{for all } 0 \leq t \leq r.
\]

**Proof.** Let \( p_0 = c_v(r), q_0 = c_w(d(p_0, q) - r) \in S_r(p_0) \) and \( w_0 = c'_w(d(p_0, q) - r) \). Then we have

\[
d(p, q_0) \leq d(p, q) + d(q, q_0) \leq 2d(p, q) < r - 1.
\]

Let \( \beta : [0, 1] \to S_r(p_0) \) be the intrinsic geodesic in \( S_r(p_0) \) connecting \( p \) and \( q_0 \). Let \( d_{p_0}(x) = d(p_0, x) \) and \( N(x) = -\text{grad} d_{p_0}(x) \) for \( x \neq p_0 \). Let
\[
\beta_t : [0, 1] \rightarrow S_{r-t}(p_0) \text{ defined by } \beta_t(s) = c_{N(\beta_t(s))}(t) \text{ for } t \in [0, r). \text{ Then } \\
\beta_t(s) = S_{N(\beta_t(s)),r}(t)(\beta'(s))_t, \text{ which implies, using } [KnPe2 \text{ Cor. 2.6}], \\
\|\beta'(s)\| \leq \|S_{N(\beta(s)),r}(t)\| \cdot \|\beta'(s)\| \leq a_2 e^{-\frac{2t}{r}} \|\beta'(s)\|. 
\]
Consequently,
\[
d(c_v(t), c_{w_0}(t)) = \text{length}(\beta_t) \leq a_2 e^{-\frac{2t}{r}} d_{S_\infty(p_0)}(p, q_0) \leq C e^{-\frac{2t}{r}} d(p, q_0) 
\]
with \( C = a_2 \max_{|s| \leq r} b(s) \), using Proposition 3.1. This implies
\[
d(c_v(t), c_{w}(t)) \leq d(c_v(t), c_{w_0}(t)) + d(c_{w_0}(t), c_{w}(t)) \\
\leq C e^{-\frac{2t}{r}} d(p, q_0) + d(q, q_0) \\
\leq (1 + 2C e^{-\frac{2t}{r}}) d(p, q).
\]

Now we prove surjectivity of \( \varphi_p \): Let \( c : [0, \infty) \rightarrow X \) be a geodesic ray with \( w = c'(0) \in S_q X \). Let \( v = -\grad b_w(p) \in S_p X \). Then \( c_v \) is asymptotic to \( c_w \) by Corollary 3.2 with \( r = \infty \). Therefore \( \varphi_p(v) = c(\infty) \) and \( \varphi_p \) is surjective.

We define \( \overline{X} = X \cup X(\infty) \) and introduce for every \( p \in X \) the following bijective map \( \overline{\varphi}_p : B_1(p) \rightarrow \overline{X} \), where \( B_1(p) \subset T_p X \) is the closed ball of radius 1:
\[
\overline{\varphi}_p(v) = \begin{cases} 
\varphi_p(v) & \text{if } \|v\| = 1, \\
\exp_p \left( \frac{1}{1-\|v\|} v \right) & \text{if } \|v\| < 1.
\end{cases}
\]

We define a topology on \( \overline{X} \) such that the bijective map \( \overline{\varphi}_p : B_1(p) \rightarrow \overline{X} \) is a homeomorphism. Next we show that this topology on \( \overline{X} \) does not depend on the reference point \( p \). For that we need to show that \( \overline{\varphi}_{p,q} = \overline{\varphi}_q^{-1} \circ \overline{\varphi}_p : B_1(p) \rightarrow B_1(q) \) is a homeomorphism.

For the continuity of \( \overline{\varphi}_{p,q} \) note first that
\[
\overline{\varphi}_{p,q}(v) = \begin{cases} 
-\grad b_v(q) & \text{if } \|v\| = 1, \\
\exp_q^{-1} \left( \exp_p \left( \frac{1}{1-\|v\|} v \right) \right) & \text{if } \|v\| < 1.
\end{cases}
\]
Let \( v_n \in B_1(p) \) such that \( v_n \rightarrow v \in B_1(p) \). If \( \|v\| < 1 \), the continuity of \( \overline{\varphi}_{p,q} \) at \( v \) follows from the continuity of the exponential maps. If \( \|v\| = 1 \), it suffices to consider two cases: in the first case we have \( 0 \neq \|v_n\| < 1 \) for all \( n \) and \( \|v_n\| \rightarrow 1 \), and in the second case we have \( \|v_n\| = 1 \) for all \( n \). We present the prove of the first case, the second case goes analogously: Note that we have
\[
\exp_q^{-1} \left( \exp_p \left( \frac{1}{1-\|v_n\|} v_n \right) \right) = \\
= -d(q, c_{v_n}(f(\|v_n\|))) \frac{\grad b_{v_n}}{1 + d(q, c_{v_n}(f(\|v_n\|))} f(\|v_n\|) = w_n,
\]
where \( f(x) = \frac{x}{1-x} \). We need to show that \( w_n \rightarrow -\grad b_v(q) \). Choose a convergent subsequence \( w_{n_j} \in S_q X \) with limit \( w \in S_q X \). Then there exists a constant \( a > 0 \) such that for all sufficiently large \( n \in \mathbb{N} \)
\[
d(c_{v_n}(t), c_{w_n}(t)) \leq a \quad \text{for all } 0 \leq t \leq f(\|v_n\|) = r_n.
\]
by Corollary 3.2. Note that \( r_n \to \infty \). This implies that
\[
d(c_v(t), c_w(t)) \leq 1 \quad \text{for all } t \geq 0,
\]
i.e., \( c_v \) and \( c_w \) are asymptotic geodesic rays. By Corollary 3.2, \( c_v \) and \( c_{-\grad_w(q)} \) are also asymptotic. Therefore, by the injectivity of \( \varphi_q \), we have \( w = -\grad_w(q) \). This finishes the proof that \( \bar{\varphi}_{p,q} \) is a homeomorphism.

This topology on \( \overline{X} \) was first introduced for Hadamard manifolds by Eberlein-O’Neill \([\text{EON}]\) and is called cone topology. The points in \( X(\infty) \subset \overline{X} \) are called points at infinity. Note that a sequence \( x_n \in X \) converges in the cone topology to a point at infinity if and only if for every \( p \in X \) we have \( d(x_n, p) \to \infty \) and for every \( \epsilon > 0 \) there exists \( n(\epsilon) \) such that \( \angle_p(x_n, x_m) < \epsilon \) for all \( n, m \geq n(\epsilon) \). We write “\( \angle_p(x_n, x_m) \to 0 \) as \( n, m \to \infty \)” for the latter.

4. GROMOV HYPERBOLICITY

We start this section by introducing the Gromov product.

**Proposition 4.3.** (see \([\text{CDP}]\) Chapter 1, Prop. 3.6) Let \( (X, d) \) be a \( \delta \)-hyperbolic space. Then we have for all \( x, y, z \in X \):
\[
(x|z)_x \geq \min\{ (x|z)_x, (y|z)_x \} - 8\delta.
\]

**Proposition 4.4.** (see \([\text{CDP}]\) Chapter 3, Lem. 2.7) Let \( (X, d) \) be a \( \delta \)-hyperbolic space. Then we have for all \( x, y \in X \):
\[
(x|y)_x \leq d(x, \sigma_{xy}) \leq (x|y)_x + 32\delta.
\]

Now assume that \( X \) is a rank one asymptotically harmonic manifold satisfying \([1.1]\) and, therefore, a Gromov hyperbolic space, by \([\text{KnPe2}]\) Thm. 1.5. We show now that two sequences \( \{x_n\}, \{y_n\} \subset X \) have the same limiting behavior at infinity in the cone topology if and only if
\[
\lim_{n \to \infty} (x_n|y_n)_p = \infty.
\]
We note that condition \( \lim_{n,m \to \infty} (x_n|x_m)_p = \infty \) is used for general Gromov hyperbolic space as a definition for convergence to infinity (see \([\text{BS}]\) Section 2.2).
Theorem 4.5. Let $X$ be a rank one asymptotically harmonic manifold satisfying (1.1). Let $p \in X$ and $\{x_n\}, \{y_n\}$ be two sequences in $X$. The following are equivalent.

(a) We have $d(x_n, p), d(y_n, p) \to \infty$ and $\angle_p(x_n, y_n) \to 0$ for $n \to \infty$.

(b) $(x_n|y_n)_p \to \infty$ for $n \to \infty$.

Proof. $(b) \Rightarrow (a)$: $X$ is $\delta$-hyperbolic for some $\delta \geq 0$. Let $(x_n|y_n)_p \to \infty$. We know from Proposition 4.4 that $d(p,x_n), d(p,y_n) \geq (x_n|y_n)_p$, which shows that $d(p,x_n), d(p,y_n) \to \infty$ as $n \to \infty$. It remains to show that $\angle_p(x_n, y_n) \to 0$. Let $U_{px_n}, U_{py_n}$ be $\delta$-tubes around the geodesic arcs $\sigma_{px_n}$ and $\sigma_{py_n}$. Then the geodesic $\sigma_{x_ny_n}$ must contain a point $p_1 \in U_{px_n} \cap U_{py_n}$. We conclude from Proposition 4.4 that

$$d(p_1,p) \geq d(\sigma_{x_ny_n}, p) \geq (x_n|y_n)_p.$$

Let $\gamma_1$ and $\gamma_2$ be the shortest curves connecting $p_1$ with $\sigma_{px_n}$ and $\sigma_{py_n}$ at the points $\hat{x}_n$ and $y'_n$, see Figure 1. Then $d(p_1, \hat{x}_n), d(p_1, y'_n) \leq \delta$, which implies $d(\hat{x}_n, y'_n) \leq 2\delta$ and

$$d(\hat{x}_n, p), d(y'_n, p) \geq (x_n|y_n)_p - \delta.$$

![Figure 1. Illustration of the proof of (b) ⇒ (a) in Theorem 4.5](image)

We assume, without loss of generality, that $d(\hat{x}_n, p) \geq d(y'_n, p)$. Let $\hat{y}_n \in \sigma_{py_n}$ be such that $d(p, \hat{y}_n) = d(p, \hat{x}_n)$. This implies that

$$d(\hat{x}_n, p) = d(\hat{y}_n, p) \geq (x_n|y_n)_p - \delta.$$

Since

$$d(y'_n, p) \leq d(\hat{y}_n, p) = d(\hat{x}_n, p) \leq d(y'_n, p) + d(\hat{x}_n, y'_n) \leq d(y'_n, p) + 2\delta,$$
and since $y'_n, \hat{y}_n$ lie on the same geodesic arc $\sigma_{p y_n}$, we have $d(y'_n, \hat{y}_n) \leq 2\delta$.
This implies that
\[
d(\hat{y}_n, \hat{x}_n) \leq d(y'_n, \hat{x}_n) + d(\hat{y}_n, y'_n) \leq 2\delta + 2\delta = 4\delta.
\]
Using Corollary 2.3 we conclude that
\[
v_0 \geq \text{length}(\sigma_{\hat{x}_n \hat{y}_n}) \geq a(d(\hat{x}_n, p))\angle_p(x_n, y_n).
\]
Since $d(\hat{x}_n, p) \to \infty$, we also have $a(d(\hat{x}_n, p)) \to \infty$, which implies that
\[
\angle_p(x_n, y_n) \to 0.
\]

(a) $\Rightarrow$ (b): Assume $\angle_p(x_n, y_n) \to 0$ and $d(x_n, p), d(y_n, p) \to \infty$ for $n \to \infty$. For all $R > 0$, there exists \(n_0(R) \geq 0\), such that for all $n \geq n_0(R)$:
\[
d(p, x_n), d(p, y_n) \geq R \quad \text{and} \quad d(c_{p x_n}(R), c_{p y_n}(R)) \leq 1,
\]
since $\angle_p(x_n, y_n) \to 0$ for $n \to \infty$. Note that the constant $n_0(R)$ does not depend on $p$, but only on the values $d(p, x_n), d(p, y_n)$ and $\angle_p(x_n, y_n)$, since $X$ has a uniform lower curvature bound.

We show now the following: The geodesic arc $\sigma_{x_n y_n}$ has empty intersection with the open ball $B_{R-\frac{1}{2}}(p)$ for all $n \geq n_0(R)$.

If $\sigma_{x_n y_n} \cap B_R(p) = \emptyset$, there is nothing to prove. If $\sigma_{x_n y_n} \cap B_R(p) \neq \emptyset$, there exists a first $t_0 > 0$ and a last $t_1 > 0$ such that
\[
q_1 = \sigma_{x_n y_n}(t_0), \quad q_2 = \sigma_{x_n y_n}(t_1) \in S_R(p),
\]
where $S_R(p)$ denotes the sphere of radius $R > 0$ around $p$ (see Figure 2).
Then we have
\[
d(q_1, q_2) = l(\sigma_{x_n y_n}) - d(x_n, q_1) - d(y_n, q_2).
\]

Figure 2. Illustration of the proof of (a) $\Rightarrow$ (b) in Theorem 4.5
Using (4.1), we have

\[
\begin{align*}
  l(x_n y_n) & \leq d(x_n, \sigma_{px_n}(R)) + d(\sigma_{px_n}(R), \sigma_{py_n}(R)) + d(\sigma_{py_n}(R), y_n) \\
  & \leq d(x_n, \sigma_{px_n}(R)) + d(y_n, \sigma_{py_n}(R)) + 1,
\end{align*}
\]

which implies that

\[
l(q_1, q_2) \leq (d(x_n, \sigma_{px_n}(R)) - d(x_n, q_1)) \\
  + (d(y_n, \sigma_{py_n}(R)) - d(y_n, q_2)) + 1.
\]

(4.2)

Since \(d(p, x_n) = R + d(\sigma_{px_n}(R), x_n) \leq d(q_1, x_n) + R\) (by the triangle inequality), we obtain \(d(x_n, q_1) - d(x_n, \sigma_{px_n}(R)) \geq 0\), and similarly \(d(y_n, q_2) - d(y_n, \sigma_{py_n}(R)) \geq 0\). This, together with (4.2) shows \(d(q_1, q_2) \leq 1\). But then the geodesic segment of \(x_n y_n\) between \(q_1\) and \(q_2\) cannot enter the ball \(B_{R-\frac{1}{2}}(p)\).

Therefore, we have for all \(n \geq n_0(R)\),

\[R - \frac{1}{2} \leq d(p, x_n y_n) \leq (x_n|y_n)_p + 32\delta,
\]

using Proposition 4.4. This shows that \((x_n|y_n)_p \to \infty\) as \(n \to \infty\).

\[\square\]

5. Visibility measures and their Radon-Nykodym derivative

Let \((X, g)\) be a rank one asymptotically harmonic manifold of dimension \(n\). The boundary \(X(\infty) \subset X\) is homeomorphic to the sphere \(S^{n-1}\) and equipped with the relative topology of the cone topology. Moreover, we have a family of visibility measures \(\{\mu_p \in M_1(X(\infty))\}_{p \in X}\), which were introduced in Definition 1.3. We will see that any two visibility measures \(\mu_p, \mu_q \in M_1(X(\infty))\) are absolutely continuous, by calculating their Radon-Nykodym derivative via a limiting process. Similar calculations were carried out in [CaSam] Section 6.1 for asymptotically harmonic manifolds with pinched negative curvature.

Lemma 5.1. For all \(p, q \in X\) there exists \(t(p, q) > 0\) such that for all \(t \geq t(p, q)\) and all \(v \in S_q X\) the geodesic ray \(c_v : [0, \infty) \to X\) intersects \(S_t(p)\) in a unique point \(F_t(v)\) (see Figure 3). In particular, the map \(F_t : S_q X \to S_t(p)\) is bijective.

Proof. Let \(a(t)\) be as in Corollary 2.3. Choose \(t_0\) such that for all \(t \geq t_0\) we have \(2d(p, q) \leq a(t)\). Define

\[t(p, q) = \max\{d(p, q) + 1, t_0\}.
\]

In particular, \(q\) lies in the ball of radius \(t\) around \(p\), for all \(t \geq t(p, q)\), and hence for all \(v \in S_q X\) the geodesic ray \(c_v : [0, \infty) \to X\) intersects \(S_t(p)\). Let \(t \geq t(p, q)\), and assume that \(q' = c_v(t_1)\) is an intersection point of \(c_v([0, \infty))\) and \(S_t(p)\) such that \(c_v'(t_1)\) is either pointing into \(B_t(p)\) or is tangent to \(S_t(p)\), i.e.,

\[\angle(c_v'(t_1), c_v'(t)) \geq \pi/2,
\]
where \( w \in S_p X \) is the unique vector such that \( c_w(t) = q' \). Using the triangle inequality we obtain
\[
 t - d(p, q) \leq t_1 \leq t + d(p, q)
\]
Using Corollary 2.3, we obtain for all \( s \geq 0 \)
\[
 d(c_v(t_1 - s), c_w(t - s)) \geq a(s)\pi/2.
\]
In particular for \( s = t \) this yields
\[
a(t)\pi/2 \leq d(c_v(t_1 - t), p) \leq d(c_v(t_1 - t), q) + d(q, p) \leq 2d(p, q) \leq a(t),
\]
which is a contradiction. Hence, a second intersection point between the geodesic ray \( c_v([0, \infty)) \) and \( S_t(p) \) cannot occur. \( \square \)

**Proposition 5.2.** Let \((X, g)\) be a complete, simply connected noncompact manifold without conjugate points and \( p, q \in X \). Consider the map \( F_t : S_q X \to S_t(p) \), where \( F_t(v) \) is the first intersection point of the geodesic ray \( c_v : [0, \infty) \to X \) with \( S_t(p) \). If \( q \) is contained in the ball of radius \( t \) about \( p \), this map is well defined. Then the Jacobian of \( F_t \) is given by
\[
(5.1) \quad \text{Jac} \; F_t(v) = \frac{\det A_v(d(q, F_t(v)))}{\langle N_p(F_t(v), N_q(F_t(v)) \rangle},
\]
where \( N_x(y) = (\text{grad} \; d_x)(y) \) and \( d_x(y) = d(x, y) \).

Note that (5.1) agrees with [CaSam] (6.3)]. For convenience of the readers, we provide our own proof of this formula.

**Proof.** Choose a curve \( \gamma : (-\epsilon, \epsilon) \to S_q X \) with \( \gamma(0) = v \in S_q X \). Then
\[
 F_t(\gamma(s)) = \exp_q(d(q, F_t(\gamma(s))) \cdot \gamma(s)),
\]
and, using the chain rule and the product rule,
\[
 DF_t(v)(\gamma'(0)) =
 D \exp_q(d(q, F_t(v))) \cdot v)((N_q(F_t(v)), DF_t(v)\gamma'(0))v + d(q, F_v(v)) \cdot \gamma'(0)).
\]
Moreover, \( N_0 \). This shows that and consequently \( \langle \cdot, \cdot \rangle_{(e_0)^\perp} \). In particular, we have

\[
D \exp_q(t v)(t w) = Y(t)(w) = J(t),
\]

where \( J \) is the Jacobi tensor along \( c_v \) with \( Y(0) = 0 \) and \( Y'(0) = \text{id} \), and therefore \( J \) is a Jacobi field along \( c \) satisfying \( J(0) = 0 \) and \( J'(0) = w \).

Note that \( \text{Jac} \) \( \theta \) is the Jacobi tensor along \( c \), consequently, \( (5.2) \) can be rewritten as \( (5.3) \).

\[
(5.2) \quad DF_t(v)(\gamma'(0)) = \langle N_q(F_t(v)), DF_t(v)\gamma'(0) \rangle N_q(F_t(v)) + A_v(d(q,F_t(v)))(\gamma'(0)).
\]

Consequently,

\[
(5.3) \quad L_{F_t(v)} \circ DF_t(v) = A_v(d(q,F_t(v))).
\]

To finish the proof of the above Proposition, we need the following lemma.

**Lemma 5.3.** \( \text{Jac} \, L_x = \lVert \langle N_p(x), N_q(x) \rangle \rVert \).

**Proof.** Consider

\[
N_p(x)^\perp \cap N_q(x)^\perp = \{w \in T_x \mathcal{X} \mid \langle w, N_p(x) \rangle = 0 \text{ and } \langle w, N_q(x) \rangle = 0 \}.
\]

Then \( N_p(x)^\perp \cap N_q(x)^\perp \) has co-dimension one in \( N_p(x)^\perp \) and \( L_x \) is the identity on \( N_p(x)^\perp \cap N_q(x)^\perp \). Let

\[
w_0 = N_q(x) - \langle N_q(x), N_p(x) \rangle N_p(x) \in N_p(x)^\perp.
\]

The vector \( w_0 \) is orthogonal to \( N_p(x)^\perp \cap N_q(x)^\perp \) since for all \( w \in N_p(x)^\perp \cap N_q(x)^\perp \) we have \( \langle w, N_p(x) \rangle = 0 \) and \( \langle w, N_q(x) \rangle = 0 \), and therefore

\[
\langle w, w_0 \rangle = \langle w, N_q(x) \rangle - \langle N_q(x), N_p(x) \rangle \langle w, N_p(x) \rangle = 0.
\]

Moreover, \( L_x w_0 \) is also orthogonal to \( N_p(x)^\perp \cap N_q(x)^\perp \):

\[
L_x w_0 = w_0 - \langle w_0, N_q(x) \rangle N_q(x)
\]

\[
= \langle N_p(x), N_q(x) \rangle (\langle N_p(x), N_q(x) \rangle N_q(x) - N_p(x)),
\]

and consequently \( \langle w, L_x w_0 \rangle = 0 \) for all \( w \) satisfying \( \langle w, N_p(x) \rangle = \langle w, N_q(x) \rangle = 0 \). This shows that

\[
\text{Jac} \, L_x = \frac{\lVert L_x w_0 \rVert}{\lVert w_0 \rVert}.
\]

Since

\[
\lVert L_x w_0 \rVert^2 = \langle N_p(x), N_q(x) \rangle^2 (1 - \langle N_p(x), N_q(x) \rangle^2)
\]

and

\[
\lVert w_0 \rVert^2 = 1 - \langle N_p(x), N_q(x) \rangle^2,
\]

we have

\[
\text{Jac} \, L_x = \frac{\lVert L_x w_0 \rVert}{\lVert w_0 \rVert} = \frac{\langle N_p(x), N_q(x) \rangle}{\sqrt{1 - \langle N_p(x), N_q(x) \rangle^2}}.
\]

Thus, \( \text{Jac} \, L_x = \lVert \langle N_p(x), N_q(x) \rangle \rVert \) as desired.
we obtain
\[
\text{Jac } L_x = \left( \frac{(N_p(x), N_q(x))^2(1 - (N_p(x), N_q(x))^2)}{1 - (N_p(x), N_q(x))^2} \right)^{1/2}
\]
\[= |(N_p(x), N_q(x))|,
\]
which yields the lemma. □

Finally, (5.3) implies that
\[
\text{Jac } F_t(v) = \frac{\det A_u(d(q,F_t(v)))}{\text{Jac } L_{F_t}(v)} = \frac{\det A_u(d(q,F_t(v)))}{(N_p(F_t(v)), N_q(F_t(v)))},
\]
finishing the proof of the proposition. □

**Figure 4.** Illustration of the map \( B_t : S_qX \rightarrow S_pX \)

**Corollary 5.4.** Let \((X,g)\) be a complete, simply connected noncompact manifold without conjugate points and \( p,q \in X \). Let \( B_t : S_qX \rightarrow S_pX, v \mapsto \frac{1}{t} \exp_p^{-1} \circ F_t(v) \) (see Figure 4). Then we have
\[
\text{Jac } B_t(v) = \frac{\det A_u(d(q,F_t(v)))}{\det A_u(t)} \cdot \frac{1}{(N_p(F_t(v)), N_q(F_t(v)))},
\]
where \( u = B_t(v) \).

**Proof.** Let \( u \in S_pX \). Then \( D \exp_p(tu) : u^+ \rightarrow T_{\exp_p(tu)}S_t(p) \) is given by \( D \exp_p(tu)(w) = \frac{1}{t} A_u(t)(w) \), and therefore with \( u = B_t(v) \),
\[
\text{Jac } B_t(v) = \frac{1}{\det A_u(t)} \cdot \text{Jac } F_t(v)
\]
\[= \frac{\det A_u(d(q,F_t(v)))}{\det A_u(t)} \cdot \frac{1}{(N_p(F_t(v)), N_q(F_t(v)))}.\]
□
From now on, \((X, g)\) denotes a rank one asymptotically harmonic manifold satisfying (1.1) with \(n = \dim(X)\). Let \(f \in C(X(\infty))\). We know from Lemma 5.4 that \(B_t : S_qX \rightarrow S_pX\) is a bijection, for \(t > 0\) large enough. Then we have with \(f_1 = f \circ \varphi_p\):

\[
\int_{X(\infty)} f(\xi) \, d\mu_p(\xi) = \frac{1}{\omega_n} \int_{S_pX} f_1(w) \, d\theta_p(w) = \frac{1}{\omega_n} \int_{S_qX} (f_1 \circ B_t)(v)(\text{Jac} \, B_t)(v) \, d\theta_q(v).
\]

We will show that

(i) \(\lim_{t \to \infty} B_t = (\varphi_p)^{-1} \circ \varphi_q\),

(ii) There exist constants \(t_0 > 0\) and \(C > 0\) such that

\(|\text{Jac} \, B_t(v)| \leq C\quad \forall v \in S_qX, \; t \geq t_0\).

(iii) We have, for all \(v \in S_qX\),

\[
\lim_{t \to \infty} \text{Jac} \, B_t(v) = e^{-h_b(v)}
\]

where \(b_v\) is the Busemann function introduced in (1.2). Having these facts, we conclude with Lebesgue’s dominated convergence that

\[
\int_{X(\infty)} f(\xi) \, d\mu_p(\xi) = \lim_{t \to \infty} \frac{1}{\omega_n} \int_{S_qX} (f_1 \circ B_t)(v)(\text{Jac} \, B_t)(v) \, d\theta_q(v) = \frac{1}{\omega_n} \int_{S_qX} (f \circ \varphi_q)(v)e^{-h_b(v)} \, d\theta_q(v)
\]

\[
= \int_{X(\infty)} f(\xi)e^{-h_{b_v}(\xi)} \, d\mu_q(\xi),
\]

with \(b_v, \xi = b_v\) with \(\xi = c_v(\infty)\) and \(v \in S_qX\). This proves Theorem 1.4 from the Introduction:

**Theorem 1.4.** Let \((X, g)\) be a rank one asymptotically harmonic manifold satisfying (1.1). Let \((\mu_p)_{p \in X}\) be the associated family of visibility measures. Then these measures are pairwise absolutely continuous and we have

\[
\frac{d\mu_p}{d\mu_q}(\xi) = e^{-h_{b_v}(\xi)(p)}.
\]

It remains to prove properties (i), (ii) and (iii) listed above.

**Proof of (i):** Let \(t_n \to \infty\) and \(s_n \geq 0, w_n = B_{t_n}(v) \in S_pX\) such that \(y_n = \exp_g(s_nv) = \exp_g(t_nw_n)\). We obviously have \(s_n \to \infty\) and \(y_n \to \varphi_q(v)\). Let \(w_{n_j}\) be a convergent subsequence of \(w_n = B_{t_n}(v)\) with limit \(w \in S_pX\). Then we have \(y_{n_j} \to \varphi_p(w)\) and

\[
\varphi_q(v) = \varphi_p(w).
\]

This shows that \(\lim_{n \to \infty} B_{t_n}(v) = (\varphi_p)^{-1} \circ \varphi_q(v)\). \(\square\)

For the proof of (ii), we need the following lemma:

**Lemma 5.5.** For every \(\epsilon > 0\), there exists \(t_0 > 0\) such that we have for all \(v \in S_qX\)

\[
|(N_p(F_t(v)), N_q(F_t(v))| - 1| < \epsilon\quad \forall \, t \geq t_0.
\]
Proof. This is an easy consequence of Corollary 2.3.

Proof of (ii): We start with the formula (see [Kn2, p. 676])
\[
\det A_v(t) = \frac{\det U_v(t)}{\det(U_v'(0) - S'_{v,t}(0))} = e^{ht}.
\]
Then
\[
\frac{\det A_v(d(q,F_t(v)))}{\det A_{u_t}(t)} = \frac{e^{hd(q,F_t(v))}}{\det(U_v'(0) - S'_{v,t}(0))} e^{ht},
\]
where \(u_t = B_t(v) \in S_{p}X\).

Let \(\epsilon > 0\) be chosen. Since \(\det(U_v'(0) - S'_{v,t}(0))\) converges monotonically to a universal constant \(A > 0\) (see [KnPe2, Theorem 1.3] and use the fact that \(X\) is rank one), we conclude with Dini that the convergence is uniform on compact sets. Therefore, there exists \(t_0 \geq 0\) such that
\[
A \leq \det(U_{u_t}'(0) - S'_{u_t,t}(0)) \leq A + \epsilon \quad \text{for all} \quad w \in S_{p}X \cup S_{q}X \quad \text{and} \quad t \geq t_0.
\]
Using Lemma 5.5 and increasing \(t_0 > 0\), if necessary, we can also assume that
\[
\langle N_p(F_t(v)), N_q(F_t(v)) \rangle \geq \frac{1}{2}
\]
for all \(t \geq t_0\). Since \(d(q,F_t(v)) \leq t + d(p,q)\), we conclude from Corollary 5.4 for all \(t \geq t_0\) and all \(v \in S_{q}X\),
\[
|\text{Jac} B_t(v)| \leq 2A + \epsilon e^{hd(p,q)}.
\]

Proof of (iii): This is an immediate consequence of Lemma 5.5 and the following Lemma:

**Lemma 5.6.** Using the notation above we have that
\[
\lim_{t \to \infty} \frac{\det A_v(d(q,F_t(v)))}{\det A_{u_t}(t)} = e^{-hb_v(p)}.
\]
where \(u_t = B_t(v)\).

Proof. We recall that
\[
\frac{\det A_v(d(q,F_t(v)))}{\det A_{u_t}(t)} = e^{h(d(q,F_t(v)) - t)} \frac{\det(U_v'(0) - S'_{u_t,t}(0))}{\det(U_v'(0) - S'_{v,d(q,F_t(v))}(0))}
\]
and
\[
\frac{\det(U_v'(0) - S'_{u_t,t}(0))}{\det(U_v'(0) - S'_{v,d(q,F_t(v))}(0))} \to \frac{A}{A} = 1.
\]
Now the lemma follows from
\[
\lim_{t \to \infty} d(q,F_t(v)) - t = \lim_{t \to \infty} d(q,F_t(v)) - d(p,F_t(v)) = \lim_{s \to \infty} d(q,c_v(s)) - d(p,c_v(s)) = \lim_{s \to \infty} s = d(p,c_v(s)) = -b_v(p).
\]
Remark. Theorem 1.4 has an analogue for simply connected, noncompact harmonic manifolds \((X, g)\) without the rank one condition and replacing the geometric boundary \(X(\infty)\) by the Busemann boundary (see [KnPe1, Theorem 12.6]). There, we have \(\det A_v(t) = f(t)\) for all \(v \in SX\), where \(f(t)\) is the volume density function, and \(f(t)\) is an exponential polynomial. Moreover, the uniform divergence of geodesics (Corollary 2.3) holds there without the rank one condition. These results are not known for general asymptotically harmonic manifolds.

6. Solution of the Dirichlet problem at infinity

Since rank one asymptotically harmonic manifolds \((X, g)\) satisfying (1.1) are Gromov hyperbolic with positive Cheeger constant (see [KnPe2]), general results of Ancona yield that the Martin boundary and the geometric boundary coincide ([Anc2 Théorème 6.2]) and that the Dirichlet problem at infinity has a solution ([Anc2 Théorème 6.7]). In this section we give an alternative direct proof that the Dirichlet problem at infinity has a solution for these manifolds by providing a concrete integral formula of the solution using the visibility measures. Moreover, this shows that the visibility measures coincide with the harmonic measures on \(X(\infty)\).

A crucial step for our result of this section is to show that \(\lim_{x \to \xi} \mu_x = \delta_\xi\), where \(\delta_\xi\) is the \(\delta\)-distribution at \(\xi\). This abstract condition will follow from the next proposition. To state it, we introduce for \(v_0 \in S_pX\) and \(\delta > 0\) the cone
\[
C(v_0, \delta) = \{c_v(t) \mid t \in [0, \infty], \angle(v_0, v) \leq \delta\}.
\]
Note that the set of all truncated cones \(C(v_0, \delta) \cap B_p(p)\) together with all open balls \(B_r(q)\) define a basis of the cone topology of the geometric compactification \(X\).

Proposition 6.1. Let \((X, g)\) be a rank one asymptotically harmonic manifold satisfying (1.1). Let \(p \in X\) and \(\delta > 0\). Then there exists a constant \(C_1 = C_1(\delta) > 0\) such that for all \(v \in S_pX\)
\[
b_v(q) \geq d(p, q) - C_1 \quad \text{for all } q \in X \setminus C(v, \delta).
\]

Proof. Let \(p \in X\) and \(\delta > 0\) be given. Then there exists a constant \(C_1 > 0\) such that
\[
0 \leq 2(c_v(t)|q)_p \leq C_1 \quad \forall t \geq 0 \quad \forall v \in S_pX \quad \forall q \in X \setminus C(v, \delta),
\]
where \((\cdot|\cdot)_p\) is the Gromov product introduced in Definition 1.1. If this were false, then we could find sequences \(t_n \geq 0, v_n \in S_pX\) and \(q_n \in X \setminus C(v_n, \delta)\) such that
\[
(c_v_n(t_n)|q_n)_p \to \infty.
\]
Let \(q_n = c_{w_n}(r_n)\) with \(w_n \in S_pX\) and \(r_n = d(q_n, p)\). This would mean, by Theorem 1.5, that \(d(p, q_n) \to \infty\) and \(\angle_p(v_n, w_n) \to 0\), which is a contradiction to \(q_n \in X \setminus C(v_n, \delta)\).

(6.1) means that
\[
d(p, q) - (d(c_v(t), q) - t) \leq C_1 \quad \forall t \geq 0.
\]

Taking the limit \(t \to \infty\), we obtain
\[
d(p, q) - b_v(q) = d(p, q) - \lim_{t \to \infty} (d(c_v(t), q) - t) \leq C_1.
\]
Remark. The statement of the proposition includes the fact that any horoball $\mathcal{H}$, centered at $\xi = c_v(\infty) \in X(\infty)$, ends up inside any given cone $C(v, \delta)$, when being translated to a horoball $\tilde{\mathcal{H}}$ along the stable direction (see the illustration in Figure 5). (Note that the horoballs centered at $\xi$ can be described by $\{q \in X \mid b_v(q) \leq -C\}$, and that these horoballs become smaller and shrink towards the limit point $\xi$, as $C \in \mathbb{R}$ increases to infinity.)

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure5.png}
\caption{Geometric property necessary for the solution of the Dirichlet problem at infinity}
\end{figure}

Remark. Proposition 6.1 does not hold if $(X, g)$ is the Euclidean space. In this case, every horoball is a halfspace, which lies never inside a given cone.

Now we state our main result of this section, namely, the solution of the Dirichlet problem at infinity for rank one asymptotically harmonic manifolds satisfying (1.1) via an explicit integral formula involving the visibility measures (see Theorem 1.5 from the Introduction).

**Theorem 1.5.** Let $(X, g)$ be a rank one asymptotically harmonic manifold satisfying (1.1). Let $f : X(\infty) \to \mathbb{R}$ be a continuous function. Then there exists a unique harmonic function $H_f : X \to \mathbb{R}$ such that

$$\lim_{x \to \xi} H_f(x) = f(\xi).$$

Moreover, $H_f$ has the following integral presentation:

$$H_f(x) = \int_{X(\infty)} f(\xi) d\mu_x(\xi),$$

where $\{\mu_x\}_{x \in X} \subset \mathcal{M}_1(X(\infty))$ are the visibility probability measures.
Proof. (a) We show first that \( \int_{X(\infty)} f(\xi) d\mu_\xi(\xi) \) is a harmonic function. Let \( p \in X \). Then

\[
\Delta_x \int_{X(\infty)} f(\xi) d\mu_\xi(\xi) = \Delta_x \int_{X(\infty)} f(\xi) e^{-hb_{p,\xi}(x)} d\mu_p(\xi).
\]

Let \( K \subset X \) be a compact set. Then \( x \mapsto f(\xi) e^{-hb_{p,\xi}(x)} \) is bounded for all \( x \in K \) and all \( \xi \in X(\infty) \), because of \( |b_{p,\xi}(x)| \leq d(p, x) \). Moreover \( \Delta_x f(\xi) e^{-hb_{p,\xi}(x)} = 0 \) and \( b_{p,\xi}(x) \) is smooth, because of \( \Delta_x b_{p,\xi} = h \). Therefore,

\[
\Delta_x \int_{X(\infty)} f(\xi) d\mu_\xi(\xi) = \int_{X(\infty)} f(\xi) \Delta_x e^{-hb_{p,\xi}(x)} d\mu_p(\xi) = 0.
\]

(b) Now we prove

\[
\lim_{x \to \xi_0} \int_{X(\infty)} f(\xi) d\mu_\xi(\xi) = f(\xi_0).
\]

Let \( \xi_0 = c_{v_0}(\infty) \) with \( v_0 \in S_pX \). Without loss of generality, we can assume that \( f(\xi_0) = 0 \) (by subtracting a constant if necessary). Let \( \epsilon > 0 \) be given. Then there exists \( \delta > 0 \), such that

\[
|f(c_v(\infty))| \leq \epsilon \quad \forall v \in S_pX \text{ with } \angle_p(v_0, v) \leq \delta.
\]

We split the integral representing \( H_f(x) \) in the following way:

\[
\omega_n |H_f(x)| \leq \left| \int_{S_pX \setminus \{v \mid \angle(v_0,v) \leq \delta\}} f(c_v(\infty)) e^{-hb_v(x)} d\theta_p(v) \right| + \left| \int_{\{v \mid \angle(v_0,v) \leq \delta\}} f(c_v(\infty)) e^{-hb_v(x)} d\theta_p(v) \right|.
\]

Now, using Proposition 6.1, we obtain for all \( x \in C(v_0, \delta/2) \) and \( C_1 = C_1(\delta/2) \)

\[
\omega_n |H_f(x)| \leq \|f\|_\infty \int_{S_pX \setminus \{v \mid \angle(v_0,v) \leq \delta\}} e^{-h(d(p,x)-C_1)} d\theta_p(v) + \epsilon \int_{\{v \mid \angle(v_0,v) \leq \delta\}} e^{-hb_v(x)} d\theta_p(v) \leq \|f\|_\infty \omega_n e^{hC_1} e^{-hd(p,x)} + \epsilon \int_{S_pX} e^{-hb_v(x)} d\theta_p(v) \leq \omega_n \left( \|f\|_\infty e^{hC_1} e^{-hd(p,x)} + \epsilon + \|f\|_\infty e^{hC_1} e^{-hd(p,x)} \right).
\]

Let \( x_n = c_{v_0}(r_n) \) with \( v_n \in S_pX \) and \( r_n \geq 0 \) be a sequence converging to \( \xi_0 \in X(\infty) \). Then we have \( r_n = d(p, x_n) \to \infty \) and \( \angle_p(v_0, v_n) \to 0 \). Since \( \epsilon > 0 \) was arbitrary, the above estimate shows that

\[
H_f(x) \to 0 \quad \text{for } x \to \xi_0.
\]

(c) Uniqueness of the solution follows from the maximum principle. \( \square \)
The above considerations show that rank one asymptotically harmonic manifolds $(X,g)$ with reference point $x_0 \in X$ satisfying \eqref{Eq:1.1} admit Poisson kernels of the form $P(x,\xi) = e^{-h_{x_0,x}(x)}$.

These Poisson kernels can be used to define a map $\varphi : X \ni x \rightarrow P(x,\xi)d\mu_{x_0}(\xi) \in P(X(\infty))$, where $P(X(\infty))$ is the space of all probability measures on $\partial X$ which are absolutely continuous to $\mu_{x_0}$. $P(X(\infty))$ carries a natural Riemannian metric $G$, called the Fisher-Information metric (see \cite{Fr} or \cite{ItSa1} for more details). The following was proved in \cite[Prop. 1]{ItSa1} for homogeneous Hadamard manifolds of dimension $n$:

If $(X,g)$ admits Poisson kernels of the form $P(x,\xi) = e^{-cb_{x_0,x}(x)}$ with $c > 0$, then the Poisson kernel map $\varphi : X \rightarrow P(X(\infty))$ satisfies $\varphi^*G = c^2/n g$, i.e., that $\varphi$ is a homothety. Examples of such spaces are rank one symmetric spaces of non-compact type and Damek-Ricci spaces. Conversely, the following was shown in \cite[Thm 1.3]{ItSa2}:

If $(X,g)$ is an $n$-dimensional Hadamard manifold admitting a Poisson kernel map $\varphi : X \rightarrow P(X(\infty))$, which is both a homothety with constant $c^2/n$, $c > 0$ and minimal, then $(X,g)$ is necessarily asymptotic harmonic with horospheres of mean curvature $c$. These results provide an interesting characterization of asymptotic harmonic manifolds via the Poisson kernel map.

### 7. Polynomial volume growth of horospheres

Let $(X,g)$ be a rank one asymptotically harmonic manifold satisfying \eqref{Eq:1.1}. Let $W^s(v) \subset SX$ be a strong stable manifold through $v \in SX$. Its projection $H_v = \pi W^s(v) \subset X$ is a horosphere orthogonal to $v$. Let $p = \pi(v)$. Consider a curve

$$\beta : [0,1] \rightarrow H_v$$

with length($\beta$) $\leq r$. Let $\gamma : [0,1] \rightarrow W^s(v)$ be the lift of $\beta$ in the strong stable manifold and $\beta_t = \pi \Phi^t \gamma$, where $\Phi^t$ is the geodesic flow on $SX$. We conclude from \cite[Corollary 2.6]{KnPe2} that

$$\text{length}(\beta_t) \leq a_2re^{-\frac{ct}{\rho}}$$

for all $t \geq 0$. Hence $\text{length}(\beta_t) \leq 1$ for all $t \geq t_0 := \frac{2\log(a_2r)}{\rho}$.

Since the curvature of $X$ and the second fundamental form of horospheres are bounded, the Gauss equation implies that the sectional curvatures of horospheres are bounded, as well. Therefore, by the volume comparison theorem, any ball of radius 1 in any horosphere has an intrinsic volume bounded by some constant $A > 0$:

$$\text{vol}_H(B_1(q)) \leq A \quad \forall \ H \text{ horospheres} \forall q \in H.$$ 

This implies that

$$\text{vol}_{H_v}(B_r(p)) \leq \text{vol}_{H_v}(\Phi^{-t_0}B_1(\pi \Phi^{t_0}(v))) \leq e^{ht_0} \text{vol}_{H_{\Phi^{t_0}(v)}}(B_1(\pi \Phi^{t_0}(v))) \leq Ae^{ht_0} = A' r^{\frac{2n}{\rho}},$$
with $A' = A a_2^{\frac{n}{2}}$. This proves that all horospheres have polynomial volume growth in $(X,g)$. \qed

8. HOROSPHERICAL MEANS AND BOUNDED EIGENFUNCTIONS

In this section, we are mainly concerned with horospherical means of bounded eigenfunctions on rank one asymptotically harmonic manifolds $X$ satisfying (1.1). Before we consider the special class of eigenfunctions, we first state a general result for all continuous functions on the geometric compactification $X$. The underlying space is also more general than just rank one asymptotically harmonic manifolds.

**Theorem 8.1.** Let $(X,g)$ be a complete, simply connected Riemannian manifold without conjugate points of dimension $n$. Assume that the geometric compactification $\overline{X} = X \cup X(\infty)$ carries a topology such that the maps $\tilde{\phi}_p : B_1(p) \to \overline{X}$ are homeomorphisms for all $p \in X$ (see Section 3 for details). Moreover, we assume that the following holds for every horosphere $\mathcal{H} \subset X$:

(a) We have $\text{vol}_{n-1}(\mathcal{H}) = \infty$.

(b) For every ball $B_r(p)$ of radius $r > 0$ around $p \in X$, we have $\text{vol}_{n-1}(\mathcal{H} \cap B_r(p)) < \infty$.

(c) The closure of $\mathcal{H}$ in the geometric compactification $\overline{X}$ satisfies

$$\overline{\mathcal{H}} = \mathcal{H} \cup \{\xi\},$$

where $\xi \in X(\infty)$ is the center of $\mathcal{H}$.

Then we have for every horosphere $\mathcal{H} \subset X$ centered at $\xi \in X(\infty)$, every compact exhaustion $\{K_j\}$, and every continuous function $f : \overline{X} \to \mathbb{R}$:

$$\lim_{j \to \infty} \frac{\int_{K_j} f(x)dx}{\text{vol}_{n-1}(K_j)} = f(\xi).$$

**Figure 6.** Contraction of the geodesic flow on stable horospheres

\[ \mathcal{H} = \pi W_v^s \]

\[ \mathcal{H}_{\Phi^t(v)} = \pi W_{\pi^t(v)}^s \]
Proof. Let \( \mathcal{H} \) be centered at \( \xi \in X(\infty) \) and \( p_0 \in X \). We show first indirectly that for every open neighbourhood \( U \subset X \) of \( \xi \) there exists \( R > 0 \) such that
\[
\mathcal{H} \subset B_R(p_0) \cup U.
\]
Assume that there exists \( x_n \in \mathcal{H} \) with \( x_n \notin U \) and \( d(p_0, x_n) \to \infty \). Then, after choosing a subsequence if necessary, we have \( x_n \to \xi' \in X(\infty) \) with \( \xi' \neq \xi \). But this is ruled out by (c).

Let \( \{ K_j \} \) be a compact exhaustion and \( \epsilon > 0 \) be given. Then there exists an open neighbourhood \( U \subset X \) of \( \xi \) such that
\[
|f(q) - f(\xi)| < \epsilon \quad \text{for all } q \in U.
\]
Let \( R > 0 \) such that (8.2) is satisfied. Let \( K_{j,0} = K_j \cap B_\epsilon(p_0) \) and \( K_{j,1} = K_j \setminus K_{j,0} \subset U \). Then (a) and (b) yield \( 1/\text{vol}(K_j) \int_{K_{j,0}} f \to 0 \) and \( \text{vol}(K_{j,1})/\text{vol}(K_j) \to 1 \), which imply
\[
f(\xi) - \epsilon \leq \liminf_{j \to \infty} \frac{\int_{K_j} f(x) dx}{\text{vol}_{n-1}(K_j)} \leq \limsup_{j \to \infty} \frac{\int_{K_j} f(x) dx}{\text{vol}_{n-1}(K_j)} \leq f(\xi) + \epsilon.
\]
This shows (8.1), since \( \epsilon > 0 \) was arbitrary. \( \square \)

The following proposition states that Theorem 8.1 is applicable in our setting of rank one asymptotically harmonic manifolds.

**Proposition 8.2.** Let \((X, g)\) be a rank one asymptotically harmonic manifold of dimension \( n \) satisfying (1.2). Then every horosphere \( \mathcal{H} \subset X \) satisfies properties (a), (b), and (c) in Theorem 8.1.

Before we present the proof of the proposition, we first introduce some useful notation. Let \( \mathcal{H} \subset X \) be a horosphere. Then there exists \( p_0 \in \mathcal{H} \) and \( v \in S_p_0 X \) such that \( \mathcal{H} = b_v^{-1}(0) \). Let \( \mathcal{H}_t = b_v^{-1}(t) \) and \( \eta_t : X \to X \) be the flow associated to \( \text{grad } b_v \). Then \( \mathcal{H} = \mathcal{H}_0 \), \( \eta_t : \mathcal{H}_0 \to \mathcal{H}_t \) and, for every \( A \subset \mathcal{H}_0 \) and \( A(t) = \eta_t(A) \subset \mathcal{H}_t \) we have (see [PeSa, Prop. 3.1])
\[
\text{vol}_{n-1}(A(t)) = e^{ht} \text{vol}_{n-1}(A).
\]

**Proof.** (a) Assume there is a horosphere \( \mathcal{H} \subset X \) with \( \text{vol}_{n-1}(\mathcal{H}) < \infty \). Using the above notation associated to \( \mathcal{H} \), we see that the horoball
\[
\mathcal{B} = \bigcup_{t \leq 0} \mathcal{H}_t = b_v^{-1}((-\infty, 0])
\]
must also be of finite volume, since
\[
\text{vol}_n(\mathcal{B}) = \int_{-\infty}^{0} e^{ht} dt \text{vol}_{n-1}(\mathcal{H}) = \frac{1}{h} \text{vol}_{n-1}(\mathcal{H}).
\]
But \( \mathcal{B} \) contains the balls \( B_r(\xi_v(r)) \subset X \) with arbitrarily large radii \( r > 0 \), whose volumes become arbitrarily large because of Proposition 2.2. This is a contradiction.

(b) Let \( \mathcal{H} \) be a horosphere and \( B_r(p) \subset X \) a ball. Let \( A = \mathcal{H} \cap B_r(p) \) and assume that \( \text{vol}_{n-1}(A) = \infty \). Let \( A_1 = \bigcup_{0 \leq t \leq 1} A(t) \). Then we also have \( \text{vol}_n(A_1) = \infty \), by (8.3). But \( A_1 \subset B_{r+1}(p) \), and \( B_{r+1}(p) \) has finite volume. This is, again, a contradiction.

(c) Since \( \mathcal{H} = b_v^{-1}(0) \) is closed in \( X \), we only need to show that \( \mathcal{H} \) has no other accumulation points in \( X(\infty) \) other than \( \xi \). We proceed indirectly.
Assume there exist \( x_n \in \mathcal{H} \) with \( d(p, x_n) \to \infty \) and \( \lim x_n = \xi' \in X(\infty) \) and \( \xi' \neq \xi \). Then we can find \( \delta > 0 \) such that \( \xi' = c_\nu(\infty) \) for some \( w \in S_{p_0}X \) with \( \angle(w, v) > \delta \). Using the remark after Proposition 6.1 we know that there exists \( s < 0 \) such that \( \mathcal{H}_s = \eta_s(\mathcal{H}) \subset C(v, \delta) \). Let \( x_n(s) = \eta_s(x_n) \in \mathcal{H}_s \).

Since \( d(x_n, x_n(s)) = s \), we still have \( x_n(s) \to \xi' \) and \( x_n(s) \in \mathcal{H}_s \subset C(v, \delta) \) and, therefore, \( \angle(w, v) \leq \delta \), which is a contradiction. \( \square \)

Next we prove the main result of this section for bounded eigenfunctions (see Theorem 1.6 in the Introduction). The proof is similar to the proof of Theorem 1 in [KP].

**Theorem 1.6.** Let \((X, g)\) be a rank one asymptotically harmonic manifold of dimension \( n \) satisfying (1.1) and \( h > 0 \) be the mean curvature of all horospheres. Let \( \lambda \neq 0 \) be a real number and \( f \in C^\infty(X) \) be a bounded function satisfying \( \Delta f + \lambda f = 0 \) and \( \mathcal{H} \subset X \) be a horosphere with isoperimetric exhaustion \( \{K_j\} \). Then we have

\[
\lim_{j \to \infty} \frac{\int_{K_j} f(x) dx}{\text{vol}_{n-1}(K_j)} = 0. 
\]

**Remark.** Since horospheres have polynomial volume growth, the intrinsic balls of suitably chosen increasing radii \( r_j \) satisfy

\[
\frac{\text{vol}_{n-2}(\partial B_{r_j}(r_j))}{\text{vol}_{n-1}(B_{r_j}(r_j))} \to 0.
\]

A suitable choice of sets \( K_j \) are regularized spheres, as explained in [KP, p. 665]. But there might be many more increasing sets satisfying this asymptotic isoperimetric property.

**Proof.** We give an indirect proof. Assume that (8.4) is not satisfied. Then we can assume – by replacing \( \{K_j\} \) by a subsequence, if needed – that there exists \( c \neq 0 \) such that

\[
\lim_{j \to \infty} \frac{\int_{K_j} f(x) dx}{\text{vol}_{n-1}(K_j)} = c.
\]

Let \( \eta_t : \mathcal{H}_0 \to \mathcal{H}_t \) be again the flow defined above after Proposition 8.2.

Let \( K_j(t) = \eta_t(K_j) \subset \mathcal{H}_t \). Recall that we have

\[
\text{vol}_{n-1}(K_j(t)) = e^{ht} \text{vol}_{n-1}(K_j).
\]

Since \( X \) has a lower sectional curvature bound, there exists \( C > 0 \) such that

\[
\text{vol}_{n-2}(\partial K_j(t)) \leq e^{C|t|} \text{vol}_{n-2}(\partial K_j).
\]

This implies that, on every compact interval \( I \subset [0, \infty) \), we have

\[
\left\| \frac{\text{vol}_{n-2}(\partial K_j(\cdot))}{\text{vol}_{n-1}(K_j(\cdot))} \right\|_{\infty, I} \to 0, \; \text{as} \; j \to \infty.
\]

Define

\[
g_j(t) = \frac{\int_{K_j(t)} f(x) dx}{\text{vol}_{n-1}(K_j(t))} \quad \forall \; t \in \mathbb{R}.
\]

Since \( \|g_j\|_\infty \leq \|f\|_\infty \), using diagonal arguments, we find a subsequence \( g_{j_k} \) such that \( g_{j_k}(t) \to g(t) \), for all rational \( t \in \mathbb{Q} \). Since \( |\nabla f| \) is uniformly
bounded by Yau’s gradient estimate [Yau, Theorem 3], \( f \) is uniformly continuous and therefore, the sequence \( g_{jk} \) is equicontinuous. This implies that we have \( g_{jk} \to g \) pointwise to a continuous limit and \( g(0) = c \neq 0 \).

Next we show that \( g \) satisfies

\[
(8.5) \quad g'' + hg' + \lambda g = 0,
\]

in the distributional sense. Let \( \psi \in C_0^\infty(\mathbb{R}) \) be a test function. Then we have

\[
\int_{-\infty}^{\infty} g_j(t) \psi''(t) - h\psi'(t) + \lambda \psi(t) dt = \int_{-\infty}^{\infty} \frac{\int_{K_j(t)} f(x) dx}{\text{vol}_{n-1}(K_j(t))} (\psi''(t) - h\psi'(t) + \lambda \psi(t)) dt.
\]

Let \( \tilde{f} : \mathcal{H} \times (-\infty, \infty) \to \mathbb{R} \) be defined as \( \tilde{f}(x, t) := f(\eta(x)) \). The transformation formula yields:

\[
\int_{K_j(t)} f(x) dx = \int_{K_j} f \circ \eta(x) \text{Jac} \eta(x) dx = e^{ht} \int_{K_j} \tilde{f}(x, t) dx.
\]

Therefore, we have \( g_j(t) = 1/\text{vol}_{n-1}(K_j) \int_{K_j} f(\eta x) dx \), and

\[
g_j''(t) + hg_j'(t) + \lambda g_j = \frac{1}{\text{vol}_{n-1}(K_j)} \int_{K_j} \frac{d^2}{dt^2} f(\eta x) + h \frac{d}{dt} f(\eta x) + \lambda f(\eta x) dx
\]

\[
= \frac{1}{\text{vol}_{n-1}(K_j(t))} \int_{K_j(t)} \Delta_x f(x) + \lambda f(x) - \Delta \eta_t f(x) dx
\]

\[
= -\frac{1}{\text{vol}_{n-1}(K_j(t))} \int_{K_j(t)} \Delta \eta_t f(x) dx
\]

\[
= \frac{1}{\text{vol}_{n-1}(K_j(t))} \int_{\partial K_j(t)} \langle \text{grad} \eta_t f(x), \nu_x \rangle dx,
\]

where \( \nu_x \) denotes the outward unit vector of \( \partial K_j(t) \subset \mathcal{H}_t \). Since \( \text{supp} \ \psi \subset \mathbb{R} \) is compact, we have

\[
\int_{-\infty}^{\infty} g_j(t) \psi''(t) - h\psi'(t) + \lambda \psi(t) dt = \int_{-\infty}^{\infty} (g_j''(t) + hg_j'(t) + \lambda g_j(t)) \psi(t) dt
\]

\[
= \int_{-\infty}^{\infty} \frac{1}{\text{vol}_{n-1}(K_j(t))} \int_{\partial K_j(t)} \langle \text{grad} \eta_t f(x), \nu_x \rangle dx \psi(t) dt.
\]
Taking absolute value and using, again, Yau’s gradient estimate [Yau Theorem 3], we obtain
\[
\left| \int_{-\infty}^{\infty} g_j(t)(\psi''(t) - h\psi'(t) + \lambda\psi(t))dt \right|
\leq \int_{\text{supp} \psi} \frac{\text{vol}_{n-2}(\partial K_j(t))}{\text{vol}_{n-1}(K_j(t))} \| \text{grad}_X f \|_\infty \| \psi \|_\infty \ dt \to 0,
\]
as \ j \to \infty. By Lebesgue’s dominated convergence, and since \( \| g \|_\infty, \| g_j \|_\infty \leq \| f \|_\infty \), we conclude that
\[
\int_{-\infty}^{\infty} g(t)(\psi''(t) - h\psi'(t) + \lambda\psi(t))dt = 0,
\]
i.e., the continuous function \( g \) satisfies (8.5) in the distributional sense. Therefore, \( g \) is smooth and satisfies \( g'' + hg' + \lambda g = 0 \) in the classical sense. This implies that \( g \) is of the general form
\[
(8.6) \quad g(t) = c_1 e^{-\frac{t}{2} + \sqrt{\left(\frac{h}{2}\right)^2 - \lambda}} t + c_2 e^{-\frac{t}{2} - \sqrt{\left(\frac{h}{2}\right)^2 - \lambda}} t
\]
if \( \lambda \neq (h/2)^2 \) and
\[
(8.6) \quad g(t) = c_1 e^{-\frac{t}{2}} t + c_2 e^{-\frac{t}{2}} t
\]
if \( \lambda = (h/2)^2 \). It is straightforward to check for \( \lambda \neq 0 \) that every choice of \( (c_1, c_2) \neq (0, 0) \) leads to an unbounded function \( g(t) \). But \( g \) must be bounded because of \( \| g \|_\infty \leq \| f \|_\infty \). Therefore we conclude that \( (c_1, c_2) = (0, 0) \) in contradiction to \( g(0) = c \neq 0 \), finishing the indirect proof. \( \square \)

**Examples of bounded eigenfunctions.** (a) Let \( X \) be a rank one symmetric space of non-compact type and \( M = X/\Gamma \) be a compact quotient. Then every non-constant \( \Delta_M \)-eigenfunction \( f \in C^\infty(M) \) gives rise to a bounded lift \( \tilde{f} \in C^\infty(X) \) which is also a \( \Delta_X \)-eigenfunction to the same eigenvalue. Since \( \tilde{f} \) is non-constant and \( \Gamma \)-periodic, it does not admit a continuous extension to the compactification \( \overline{X} \).

(b) Let \( X^{(p,q)} \) be a Damek-Ricci space with \( p, q \) defined as in [Rou]. Then \( X^{(p,q)} \) is an asymptotically harmonic manifold with \( h = p/2 + q \) and there exist radial eigenfunctions \( \varphi_\mu \in C^\infty(X^{(p,q)}) \) satisfying
\[
\Delta \varphi_\mu + \left( \mu^2 + \left(\frac{h}{2}\right)^2 \right) \varphi_\mu = 0 \quad \text{and} \quad \varphi_\mu(e) = 1,
\]
where \( \mu \in \mathbb{C} \) and \( e \in X^{(p,q)} \) denotes the neutral element in the Damek-Ricci space considered as a solvable group. If \( 0 < i\mu < h/2 \), we have (see [Rou p. 78])
\[
\varphi_\mu(r) \sim c(\mu) e^{(i\mu-h/2)r} \quad \text{as} \ r \to \infty
\]
with suitable constants \( c(\mu) \in \mathbb{R}\setminus\{0\} \). This means that \( \varphi_\mu \) is a bounded eigenfunction with trivial continuous extension to the compactification \( \overline{X^{(p,q)}} \).
Now we are in a position to prove our final result (see Theorem 1.7 in the Introduction) which states that the above examples are the only two possible cases with regards to continuous extensions of bounded eigenfunctions $f$: either $f$ cannot be extended to $\overline{X}$ or the extension is trivial.

**Theorem 1.7.** Let $(X,g)$ be a rank one asymptotically harmonic manifold satisfying (1.1). Let $\lambda \in \mathbb{R}\backslash\{0\}$ and $f \in C^\infty(X)$ be an eigenfunction $\Delta f + \lambda f = 0$. If $f$ has a continuous extension $F \in C(\overline{X})$ then we have necessarily $F|_{\partial X} \equiv 0$.

**Proof.** Assume that $\lambda \neq 0$ and that an eigenfunction $\Delta f + \lambda f = 0$ has a continuous extension $F$ on the compactification $\overline{X}$. Then we know from Theorem 8.1 that all horospherical means of $f$ over horospheres centered at $\xi \in X(\infty)$ agree with $F(\xi)$. On the other hand, we conclude from Theorem 1.6 that all horospherical means with isoperimetric exhaustions have to vanish. Moreover, every horosphere in $X$ has polynomial volume growth and, therefore, admits isoperimetric exhaustions. This implies that $F|_{\partial X} \equiv 0$. □

**References**

[Anc1] A. Ancona. *Positive harmonic functions and hyperbolicity*, in J. Král et al (eds.) "Potential theory—surveys and problems (Prague, 1987)", Lecture Notes in Math. 1344, 1–23, Springer-Verlag, Berlin, 1988.

[Anc2] A. Ancona. *Théorie du potentiel sur les graphes et les variétés*, in École d’été de Probabilités de Saint-Flour XVIII—1988, Lecture Notes in Math. 1427, 1–112, Springer-Verlag, Berlin, 1990.

[BBE] W. Ballmann, M. Brin, P. Eberlein. *Structure of manifolds of nonpositive curvature I.*, Ann. of Math. 122 (1) (1985), 171–203.

[Ba] W. Ballmann. *On the Dirichlet problem at infinity for manifolds of nonpositive curvature*, Forum Math. 1 (2) (1989), 201–213.

[BS] S. Buyalo, V. Schroeder. *Elements of asymptotic geometry*, European Mathematical Society (EMS), Zürich, 2007.

[CaSam] P. Castillon, A. Sambusetti. *On asymptotically harmonic manifolds of negative curvature*, Math. Z. DOI 10.1007/s00209-014-1293-7, 18 March 2014, see also arXiv:1203.2182, 12 March 2012.

[CDP] M. Coornaert, T. Delzant, A. Papadopoulos. *Géométrie et théorie des groupes*, Lecture Notes in Mathematics 1441, Springer-Verlag, Berlin, 1990.

[EON] P. Eberlein, B. O’Neill. *Visibility manifolds*, Pacific J. Math. 46 (1973), 45–109.

[Fr] Th. Friedrich. *Die Fisher-Information und symplektische Strukturen*, Math. Nachr. 153 (1991), 273–296.

[ItoSa1] M. Itoh, H. Satoh. *Information geometry of Poisson kernels on Damek-Ricci spaces*, Tokyo J. Math. 33 (2010), 129–144.

[ItoSa2] M. Itoh, H. Satoh. *Fisher information geometry, Poisson kernel and asymptotical harmonicity*, Differential Geom. Appl. 29 (2011), suppl. 1, S107–S115.

[KP] L. Karp, N. Peyerimhoff. *Horospherical means and uniform distribution of curves of constant geodesic curvature*, Math. Z. 231 (1999), 655–677.

[Kn1] G. Knieper. *Hyperbolic Dynamics and Riemannian Geometry*, in Handbook of Dynamical Systems, Vol. 1A 2002, Elsevier Science B., eds. B. Hasselblatt and A. Katok, (2002), 453–545.

[Kn2] G. Knieper. *New results on noncompact harmonic manifolds*, Comment. Math. Helv. 87 (2012), 669–703, see also arXiv:0910.3872, 20 October 2009.

[KnPe1] G. Knieper and N. Peyerimhoff. *Noncompact harmonic manifolds*, Oberwolfach Preprint OWP 2013 - 08, see also arXiv:1302.3841, 15 February 2013.

[KnPe2] G. Knieper and N. Peyerimhoff. *Geometric properties of rank one asymptotically harmonic manifolds*, to appear in J. Differential Geom., see also arXiv:1307.6029, 7 January 2014.
[Led] F. Ledrappier. *Harmonic measures and Bowen-Margulis measures*, Israel J. Math. 71 (3) (1990), 275–287.

[PeSa] N. Peyerimhoff and E. Samiou. *Integral geometric properties of non-compact harmonic spaces*, J. Geom. Anal. DOI 10.1007/s12220-013-9416-7, 12 April 2013, see also arXiv:1210.3957 15 October 2012.

[Rou] F. Rouvière. *Espaces de Damek-Ricci, géométrie et analyse*, Séminaires & Congrès 7 (2003), 45–100.

[Yau] S. T. Yau. *Harmonic functions on complete Riemannian manifolds*, Comm. Pure Appl. Math. 28 (1975), 201–228.

[Zi1] A. M. Zimmer. *Compact asymptotically harmonic manifolds*, J. Mod. Dynamics. 6 (3) (2012), 377–403, see also arXiv:1205.2271 16 October 2012.

[Zi2] A. M. Zimmer. *Boundaries of non-compact harmonic manifolds*, Geom. Dedicata 168 (2014), 339–357, see also arXiv:1208.4802 16 December 2012.

FACULTY OF MATHEMATICS, RUHR UNIVERSITY BOCHUM, 44780 BOCHUM, GERMANY
E-mail address: gerhard.knieper@rub.de

DEPARTMENT OF MATHEMATICAL SCIENCES, DURHAM UNIVERSITY, DURHAM DH1 3LE, UK
E-mail address: norbert.peyerimhoff@dur.ac.uk