Quenching Time for Some Semilinear Equations with A Potential

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Abstract.

This paper concerns the study of a semilinear parabolic equation subject to Neumann boundary conditions, with a potential and positive initial datum. Under some assumptions, we show that the solution of the above problem quenches in a finite time and estimate its quenching time. Finally, we give some numerical results to illustrate our analysis.

Key-words: Quenching, semilinear parabolic equation, numerical quenching time.

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1-Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial \Omega$. Consider the following initial-boundary value problem

\begin{align}
&u_t = \Delta u - c(x, t) u^{-p(x)} \text{ in } \Omega \times (0, T), \\
&\frac{\partial u}{\partial v} = 0 \text{ on } \partial \Omega \times (0, T), \\
&u(x, 0) = u_0(x) > 0 \text{ in } \bar{\Omega},
\end{align}

where $\Delta$ is the Laplacian, $v$ the exterior normal unit vector on $\partial \Omega$. We suppose that the initial datum

$u_0 \in C^2(\bar{\Omega})$ and $u_0(x) > 0$ in $\bar{\Omega}$.

Here the potential $c(x, t)$ is a nonnegative locally Hölder continuous function defined for $x \in \bar{\Omega}$ and $t \geq 0$. The exponent $p \in C^0(\bar{\Omega})$, $0 < p_0 = \inf_{x \in \bar{\Omega}} p(x) < \sup_{x \in \bar{\Omega}} p(x) = p_+$. Here $(0, T)$ is the maximal time interval of existence of the solution $u$, and by a solution, we mean the following.

**Definition 1.1.** A solution of (1)-(3) is a function $u(x, t)$ continuous in $\bar{\Omega} \times [0, T)$, $u(x, t) > 0$ in $\bar{\Omega} \times [0, T)$, and twice continuously differentiable in $x$ and once in $t$ in $\Omega \times (0, T)$.

The time $T$ may be finite or infinite. When $T$ is infinite, then we say that the solution $u$ exists globally.

When $T$ is finite, then the solution $u$ develops a quenching in a finite time, namely $\lim_{t \to T^-} u_{\min}(t) = 0$, where $u_{\min}(t) = \min_{x \in \bar{\Omega}} u(x, t)$. In this last case, we say that the solution $u$ quenches in a finite time and the time $T$ is called the quenching time of the solution $u$.

Since the pioneering work of Kawarada in 1975 (see, [25]), the study of the phenomenon of quenching for semilinear heat equations has attracted a considerable attention (see, for example [3]-[4], [6]-[8], [11], [14], [24], [26], [28]-[30], [36] and the references cited therein). More precisely, in [7] Boni has studied the problem (1)-(3) for the phenomenon of blow-up. He has given some sufficient conditions under which solutions to such equation tend to zero or blow up in a finite time. In the same way, some authors have proved the existence and uniqueness of solution (see, [16], [27]). In [8], Boni and Kouakou have treated a similar problem with variable exponent. They have estimated the quenching time and studied its continuity as a function of the initial datum $u_0$. The originality of this work is that it is the first attempt of studying the phenomenon of quenching with variable exponent and a potential depending both on space and time. Using standard methods, the local in time existence and uniqueness of solutions can be easily proved (see, [7], [16]).

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Our aim in this paper consists in showing that, under some hypotheses, the solution of (1)-(3) quenches in a finite time. If we set \( g(x, u) = c(x, t)u^{-p(x)} \), then we observe that the function \( g \) is continuous in both variables and locally Lipschitz in the second one. Let us notice that, because the initial datum of the problem considered is sufficiently regular, the solution of this problem exists and is regular. In addition, we note that the regularity of solution is as important as the regularity of the initial data, and the maximum principle holds (see, [16], [27], [33]). This paper is structured as follows. In the following section, we show that under some assumptions, the solution \( u \) of (1)-(3) quenches in a finite time and estimate its quenching time and finally, in the last section, we give some numerical results to illustrate our analysis.

2- Quenching time

In this section, using an idea of Friedman and McLeod in [17], we may prove the following result on the quenching of the solution \( u \) of (1)-(3).

**Theorem 2.1.** Suppose that there exists a constant \( A \in (0, 1) \) such that the initial datum at (3) satisfies

\[
\Delta u_0(x) - c(x, t)(u_0(x))^{-p(x)} \leq -Ac(x, t)(u_0(x))^{-p_0} \quad \text{in } \Omega.
\]

Then, the solution \( u \) of (1)-(3) quenches in a finite time \( T \) which obeys the following estimate

\[
T \leq \frac{(u_{0\min})^{p_0+1}}{AM(p_0+1)},
\]

where \( M \) is some positive constant.

**Proof.** We know that \((0, T)\) is the maximal time interval of existence of the solution \( u \). Therefore, to prove our theorem, we have to show that \( T \) is finite and satisfies the above inequality. For this fact, we introduce \( J(x, t) \) a function defined as follows

\[
J(x, t) = u_t(x, t) + Ac(x, t)(u(x, t))^{-p_0} \quad \text{in } \Omega \times [0, T).
\]

The derivative of \( J \) in \( t \) yields \( J_t = u_{tt} - p_0Ac(x, t)u^{-p_0-1}u_t \) and by a simple calculation we obtain

\[
J_t - \Delta J = (u_t - \Delta u)_t - Ap_0c(x, t)u^{-p_0-1}u_t - Ac(x, t)\Delta u^{-p_0} \quad \text{in } \Omega \times (0, T).
\]

It is not hard to see that \( \Delta u^{-p_0} = p_0(p_0+1)u^{-p_0-2} |\nabla u|^2 - p_0u^{-p_0-1}\Delta u \) in \( \Omega \times (0, T) \), which implies that \( \Delta u^{-p_0} \geq -p_0u^{-p_0} - p_0u^{-p_0-1}\Delta u \) in \( \Omega \times (0, T) \), which implies that

\[
J_t - \Delta J \leq (u_t - \Delta u)_t - Ap_0c(x, t)u^{-p_0-1}(u_t - \Delta u) \quad \text{in } \Omega \times (0, T).
\]

Use (1) and (6) to obtain

\[
J_t - \Delta J \leq c(x, t)p(x)u^{-p(x)-1}u_t + Ap_0(c(x, t))^2u^{-p_0-1-p(x)} \quad \text{in } \Omega \times (0, T).
\]

Due to the fact that \( p_0 \leq p(x) \) in \( \Omega \), we discover that

\[
J_t - \Delta J \leq c(x, t)p(x)u^{-p(x)-1}(u_t + Ac(x, t)u^{-p_0}) \quad \text{in } \Omega \times (0, T).
\]

Making use of the expression of \( J \), we derive the following inequality

\[
J_t - \Delta J \leq c(x, t)p(x)u^{-p(x)-1}J \quad \text{in } \Omega \times (0, T).
\]

The boundary condition (2) allows us to write

\[
\frac{\partial J}{\partial v} = (\frac{\partial u}{\partial v})_t - Ap_0c(x, t)u^{-p_0-1}\frac{\partial u}{\partial v} = 0 \quad \text{on } \partial \Omega \times (0, T).
\]

According to (4), we have

\[
J(x, 0) = \Delta u_0(x) - c(x, t)(u_0(x))^{-p(x)} + Ac(x, t)(u_0(x))^{-p_0} \leq 0 \quad \text{in } \Omega.
\]

One concludes by the maximum principle that \( J(x, t) \leq 0 \) in \( \Omega \times (0, T) \), that is

\[
u_t(x, t) + Ac(x, t)(u(x, t))^{-p_0} \leq 0 \quad \text{in } \Omega \times (0, T).
\]

By the definition of \( c(x, t) \), we have \( c(x, t) \leq M \) where \( M \) is some positive constant.

Thus, the estimate (7) may be rewritten as follows

\[
u^{p_0}du \leq -AMdt \quad \text{in } \Omega \times (0, T).
\]

Integrate the above inequality over \((0, T)\) to obtain

\[
T \leq \frac{(u(x, 0))^{p_0+1}}{AM(p_0+1)} \quad \text{for } x \in \Omega.
\]
We deduce that
\[ T \leq \frac{(u_{0\min})^{p_0+1}}{AM(p_0 + 1)}. \]
We observe that the quantity on the right-hand side of the above inequality is finite. Consequently, \( u \) quenches at the time \( T \) and the proof is finished. \( \square \)

3-Numerical results

To compute the numerical results, we need to consider the radial symmetric solution of the following initial-boundary value problem

\[ u_t = \Delta u - c(x,t)u^{-p(x)} \text{ in } B \times (0,T), \]
\[ \frac{\partial u}{\partial \nu} = 0 \text{ on } S \times (0,T), \]
\[ u(x,0) = u_0(x) \text{ in } B, \]

where \( c(x,t) = C(||x||,t), p(x) = \psi(||x||), u_0(x) = \varphi(||x||), B = \{x \in \mathbb{R}^N; ||x|| < 1\}, S = \{x \in \mathbb{R}^N; ||x|| = 1\}. \)

Another form of the above problem is

\[ u_t = u_{rr} + \frac{N-1}{r}u_r - C(r,t)u^{-\psi(r)}, \quad r \in (0,1), \quad t \in (0,T) \]
\[ u_r(0, t) = 0, \quad u_r(1, t) = 0, \quad t \in (0,T), \]
\[ u(r, 0) = \phi(r), \quad r \in [0,1], \]

where, we take \( C(r,t) = \frac{r+1}{r+1}, \psi(r) = 1 + \frac{r}{r+1} \text{ with } \varepsilon \in [0,1] \) and \( \phi(r) = 4 + 3 \cos(\pi r). \) In order to compute the numerical solution, we need to construct an adaptive scheme. For this fact, define the grid \( x_i = ih, \quad 0 \leq i \leq l, \)

where \( l \) is a positive integer and \( h = 1/1. \) Approximate the solution \( u \) of (9)-(11) by the solution

\[ u_h^{(n)} = \left( u_0^{(n)}, ..., u_l^{(n)} \right)^T \]

of the following explicit scheme

\[ \frac{U_0^{(n+1)} - U_0^{(n)}}{\Delta t_n} = N \left\{ \frac{2U_1^{(n)} - 2U_0^{(n)}}{h^2} - \psi_0 \right\}, \]
\[ \frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = \frac{U_{i+1}^{(n)} - 2U_i^{(n)} + U_{i-1}^{(n)}}{h^2} + \frac{(N-1)U_i^{(n)} - U_{i-1}^{(n)} - U_{i+1}^{(n)}}{2h} - C_i^{(n)} \left( U_i^{(n)} \right)^{-\psi_i}, 1 \leq i \leq l - 1, \]
\[ \frac{U_l^{(n+1)} - U_l^{(n)}}{\Delta t_n} = \frac{U_{l-1}^{(n)} - U_l^{(n)}}{h^2} + \frac{(N-1)U_l^{(n)} - U_{l-1}^{(n)}}{2h} - C_l^{(n)} \left( U_l^{(n)} \right)^{-\psi_l}, \]
\[ U_i^{(0)} = \varphi_i, \quad 0 \leq i \leq l, \]

Where \( C_i^{(n)} = \frac{ih+1}{t_n+1}, \psi_i = 1 + \frac{\varepsilon h}{ih+1} \) and \( \varphi_i = 4 + 3 \cos(\pi ih). \) For the time step we take

\[ \Delta t_n = \min \left\{ \frac{(1 - h^2)h^2}{2N}, h^2 \left( U_{h_{min}}^{(n)} \right)^{p_0+1} \right\} \]

with \( U_{h_{min}}^{(n)} = \min_{0 \leq i \leq l} U_i^{(n)} \).
This condition permits the discrete solution to reproduce the properties of the continuous one when the time \( t \) approaches the quenching time \( T \) and ensures the positivity of the discrete solution. An important fact concerning the phenomenon of quenching is that, if the solution \( u \) quenches at the time \( T \), then, when the time \( t \) approaches the quenching time \( T \), the solution \( u \) decreases to zero rapidly. We also approximate the solution \( u \) of (9)-(11) by the solution \( U_h^{(n)} \) of the implicit scheme below

\[
\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = \frac{2U_i^{(n+1)} - 2U_i^{(n)}}{h^2} - C_0^{(n)} \left( U_0^{(n)} \right)^{-\psi_0-1} U_0^{(n+1)},
\]

\[
\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = \frac{U_{i+1}^{(n+1)} - 2U_i^{(n+1)} + U_{i-1}^{(n+1)}}{h^2} + \frac{(N-1)U_{i+1}^{(n+1)} - U_{i-1}^{(n+1)}}{2h} - C_i^{(n)} \left( U_i^{(n)} \right)^{-\psi_i-1} U_i^{(n+1)},
\]

\[1 \leq i \leq l - 1,\]

\[
\frac{U_l^{(n+1)} - U_l^{(n)}}{\Delta t_n} = \frac{U_{l+1}^{(n+1)} - U_l^{(n)}}{h^2} + \frac{(N-1)U_{l+1}^{(n+1)} - U_l^{(n)}}{2h} - C_l^{(n)} \left( U_l^{(n)} \right)^{-\psi_l-1} U_l^{(n+1)},
\]

\[U_i^{(0)} = \varphi_i, \quad 0 \leq i \leq l.\]

As in the case of the explicit scheme, here again, we transform our scheme to an adaptive one by choosing

\[\Delta t_n = h^2 \left( U_h^{(n)} \right)^{p+1}.\]

Let us again remark that for the above implicit scheme, the existence and positivity of the discrete solution is also guaranteed using standard methods (see for instance [6]). It is not hard to see that

\[u_{rr}(0, t) = \lim_{r \to 0} \frac{u_r(r, t)}{r}.\]

On the other hand, according to (10), we have

\[\frac{u_r(r, t)}{r} = 0.\]

Hence, if \( r = 0 \) and \( r = 1 \), we obtain

\[u_t(0, t) = Nu_{rr}(0, t) - C(0, t)u_\psi(0, t), \quad t \in (0, T),\]

\[u_t(1, t) = Nu_{rr}(1, t) - C(1, t)u_\psi(1, t), \quad t \in (0, T).\]

These observations have been taken into account in the construction of our schemes when \( i = 0 \) and \( i = l \).

We need the following definition.

**Definition 3.1.** We say that the discrete solution \( U_h^{(n)} \) of the explicit scheme or implicit scheme quenches in a finite time if

\[
\lim_{n \to \infty} U_h^{(n)} = 0
\]

and the series \( \sum_{n=0}^{\infty} \Delta t_n \) converges. The quantity \( \sum_{n=0}^{\infty} \Delta t_n \) is called the numerical quenching time of the discrete solution \( U_h^{(n)} \).

In the following tables, in rows, we present the numerical quenching times, the numbers of iterations, the CPU times and the orders of the approximations corresponding to meshes of 16, 32, 64, 128, 256. We take for the numerical quenching time \( T_n = \sum_{j=0}^{n-1} \Delta t_j \) which is computed at the first time when

\[\Delta t_n = |T_n+1 - T_n| \leq 10^{-16}.\]

The order(s) of the method is computed from

\[s = \frac{\log\left( \frac{T_{4h} - T_{2h}}{T_{2h} - T_h} \right)}{\log(2)}.\]
Numerical experiments for $\psi_i=1 + \frac{\varepsilon_i h}{i h+1}$, $N = 2$

First case: $\varepsilon = 0$

Table 1: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

| I  | $t_n$        | n    | CPU time | s |
|----|-------------|------|----------|---|
| 16 | 2,094707    | 3975 | -        | - |
| 32 | 2,188877    | 16064| 1        | - |
| 64 | 2,239645    | 64921| 5        | 0.89 |
| 128| 2,266009    | 252408| 51      | 0.94 |
| 256| 2,279445    | 992631| 3180   | 0.97 |

Table 2: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the first implicit Euler method

| I  | $t_n$        | n    | CPU time | s |
|----|-------------|------|----------|---|
| 16 | 2,094457    | 3974 | 1        | - |
| 32 | 2,188816    | 16064| 2        | - |
| 64 | 2,239630    | 63920| 16       | 0.89 |
| 128| 2,266006    | 252408| 242     | 0.94 |
| 256| 2,279444    | 992630| 7620   | 0.98 |

Second case: $\varepsilon = 1/10$

Table 3: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

| I  | $t_n$        | n    | CPU time | s |
|----|-------------|------|----------|---|
| 16 | 2,135560    | 3973 | -        | - |
| 32 | 2,23026     | 16080| 1        | - |
| 64 | 2,285571    | 64050| 7        | 0.89 |
| 128| 2,312858    | 253140| 62      | 0.94 |
| 256| 2,326764    | 996283| 3012   | 0.97 |

Table 4: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the first implicit Euler method

| I  | $t_n$        | n    | CPU time | s |
|----|-------------|------|----------|---|
| 16 | 2,135296    | 3973 | -        | - |
| 32 | 2,232961    | 16079| 2        | - |
| 64 | 2,285554    | 64050| 17       | 0.89 |
| 128| 2,312854    | 253140| 275     | 0.94 |
| 256| 2,326763    | 996283| 7740   | 0.97 |

Third case: $\varepsilon = 1/1000$

Table 5: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

| I  | $t_n$        | n    | CPU time | s |
|----|-------------|------|----------|---|
| 16 | 2,095107    | 3975 | -        | - |
| 32 | 2,189310    | 16064| 2        | - |
| 64 | 2,240095    | 63922| 7        | 0.89 |
| 128| 2,266468    | 252414| 53      | 0.94 |
| 256| 2,279908    | 992663| 4560   | 1.08 |
Table 6: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the first implicit Euler method

| I  | \(t_n\)  | \(n\) | CPU time | s  |
|----|----------|-------|----------|----|
| 16 | 2,094857 | 3974  | -        | -  |
| 32 | 2,189248 | 16064 | 2        | -  |
| 64 | 2,240080 | 63921 | 22       | 0.89 |
| 128| 2,266451 | 252414| 256      | 0.94 |
| 256| 2,279908 | 992663| 2743     | 0.97 |

Remark 3.1. If we consider the problem (9)-(11) in the case where the potential \(C(r, t) = \frac{r+1}{t+1}\), the exponent of the nonlinear source \(\Psi(r) = 1 + \frac{er}{1+r}\) with \(e = 0\), and the initial datum \(\phi(r) = 4 + 3 \cos(\pi r)\), we see that the numerical quenching time of the discrete solution for the explicit scheme or the implicit scheme is slightly equal to that in which the exponent of the nonlinear source increases slightly, that is when \(e\) is a small positive real (see, Tables 1-6 for an illustration). This result confirms the theory established in the previous section.

In what follows, we give some plots to illustrate our analysis. In Figures 1 and 2, we can appreciate that the discrete solution quenches in a finite time. We also remark that the representation of the discrete solution when \(e=0\) is practically the same that the one when \(e=1/10\).

![Figure 1: Evolution of the explicit discrete solution: \(e = 0\)](image1)

![Figure 2: Evolution of the implicit discrete solution: \(e = 1/10\)](image2)

References

L. M. Abia, J. C. López-Marcos and J. Martínez, On the blow-up time convergence of semidiscretizations of reaction-diffusion equations, *Applied Numerical Mathematics*, 26 (1998), 399-414.

A. Acker and B. Kawohl, Remarks on quenching, *Nonlinear Analysis, Theory, Methods and Applications*, 13 (1989), 53-61.

A. Acker and W. Walter, The quenching problem for nonlinear parabolic differential equations, *Proceedings of the Fourth Conference, University Dundee, 1976*, Lecture Notes in Mathematics, Springer-Verlag, 564 (1976), 1-12.

C. Bandle and C. M. Brauner, Singular perturbation method in a parabolic problem with free boundary, *BAIL IV* (Novosibirsk 1986), Boole Press Conferences Series, Boole Dún Laoghaire, 8 (1986), 7-14.

P. Baras and L. Cohen, Complete blow-up after \(T_{\text{max}}\) for the solution of a semilinear heat equation, *Journal of Functional Analysis*, 71 (1987), 142-174.

T. K. Boni, Extinction for discretizations of some semilinear parabolic equations, *Comptes Rendus de l'Académie des Sciences de Paris, Série I, Mathématiques*, 333 (2001), 795-800.

T. K. Boni, On blow-up and asymptotic behavior of solutions to a nonlinear parabolic equation of second order, *Commentationes Mathematicae Universitatis Comenianae*, 40 (1999), 457-475.

T. K. Boni and R. K. Kouakou, Continuity of the quenching time in a semilinear parabolic equation with variable exponent, *Siandial Mathematical Seminar*, 14 (2011), pp.5-20.

C. Cortazar, M. del Pino and M. Elgueta, On the blow-up set for \(u = \Delta u^+ + u^+\), \(m > 1\), *Indiana University Mathematics Journal*, 47(2) (1998), 541-561.
C. Cortazar, M. del Pino and M. Elgueta, Uniqueness and stability of regional blow-up in a porous-medium equation, *Annale de l'Institut Henry Poincaré, Analyse Non Linéaire*, 19 (2002), 927-960.

K. Deng and H. A. Levine, On the blow-up of $u_t$ at quenching, *Proceedings of the American Mathematical Society*, 106 (1989), 1049-1056.

K. Deng and M. Xu, Quenching for a nonlinear diffusion equation with singular boundary condition, *Zeitschrift für Angewandte Mathematik und Physik*, 50 (1999), 574-584.

C. Fermanian Kammerer, F. Merle and H. Zaag, Stability of the blow-up profile of nonlinear heat equations from the dynamical system point of view, *Mathematische Annalen*, 317 (2000), 195-237.

M. Fila, B. Kawohl and H. A. Levine, Quenching for quasilinear equations, *Communications in Partial Differential Equations*, 17 (1992), 593-614.

M. Fila and H. A. Levine, Quenching on the boundary, *Nonlinear Analysis, Theory, Methods and Applications*, 21 (1993), 795-802.

A. Friedman, Partial differential equations of parabolic type, *Prentice-Hall, Englewood Cliffs, NJ*, 1964.

A. Friedman and B. McLeod, Blow-up of positive solutions of nonlinear heat equations, *Indiana University Mathematics Journal*, 34 (1985), 425-477.

V. A. Galaktionov, Boundary value problems for the nonlinear parabolic equation $u_t = \Delta u^{\alpha+1} + u^\beta$, *Differential Equation*, 17 (1981), 551-555.

V. A. Galaktionov, S. P. K. Mikhailov and A. A. Samarskii, Unbounded solutions of the Cauchy problem for the parabolic equation $u_t = (u^\alpha u) + u^\beta$, *Soviet Physics-Doklady*, 25 (1980), 458-459.

V. A. Galaktionov and J. L. Vazquez, Continuation of blow-up solutions of nonlinear heat equations in several space dimensions, *Communications on Pure and Applied Mathematics*, 50 (1997), 1-67.

V. A. Galaktionov and J. L. Vazquez, The problem of blow-up in nonlinear parabolic equation, *current developments in PDE (Temuco,1999)*, *Discrete and continuous Dynamical Systems*, 8 (2002), 399-433.

P. Groisman and J. D. Rossi, Dependence of the blow-up time with respect to parameters and numerical approximations for a parabolic problem, *Asymptotic Analysis*, 37 (2004), 79-91.

P. Groisman, J. D. Rossi and H. Zaag, On the dependence of the blow-up time with respect to the initial data in a semilinear parabolic problem, *Communications in Partial Differential Equations*, 28 (2003), 737-744.

J. Guo, On a quenching problem with Robin boundary condition, *Nonlinear Analysis, Theory, Methods and Applications*, 17 (1991), 803-809.

H. Kawarada, On solutions of initial-boundary problem for $u_t = u_{xx} + 1/(1 + u)$, *Publications of the Research Institute for Mathematical Sciences*, 10 (1975), 729-736.

C. M. Kirk and C. A. Roberts, A review of quenching results in the context of nonlinear volterra equations, *Dynamics of Continuous, Discrete and Impulsive Systems, Serie A Mathematical Analysis*, 10 (2003), 343-356.

A. Ladyzenskaya, V. A Solonnikov and N. N. Ural'ceva, Linear and quasilinear equations parabolic type, *Trans Math Monographs*, 23 AMS, Providence, RI, (1967).

H. A. Levine, The phenomenon of quenching: a survey, *Trends in the Theory and Practice of Nonlinear Analysis, North-Holland, Amsterdam*, 110 (1985), 275-286.

H. A. Levine, The quenching of solutions of linear parabolic and hyperbolic equations with nonlinear boundary conditions, *SIAM Journal of Mathematical Analysis*, 14 (1983), 1139-1152.

H. A. Levine, Quenching, nonquenching and beyond quenching for solution of some parabolic equations, *Annali di Matematica Pura ed Applicata*, 155 (1989), 243-260.

F. Merle, Solution of a nonlinear heat equation with arbitrarily given blow-up points, *Communication on Pure and Applied Mathematics*, 45 (1992), 293-300.

T. Nakagawa, Blowing up on the finite difference solution to $u_t = u_{xx} + u^2$, *Applied Mathematics and Optimization*, 2 (1976), 337-350.

C. V. Pao, Nonlinear parabolic and elliptic equations, *New York: Plenum Press*, (1992).

D. Phillips, Existence of solutions of quenching problems, *Applicable Analysis*, 24 (1987), 253-264.

M. H. Protter and H. F. Weinberger, Maximum principles in differential equations, *Prentice Hall, Inc., Englewood Cliffs, NJ*, (1967).

P. Quittner, Continuity of the blow-up time and a priori bounds for solutions in superlinear parabolic problems, *Houston Journal of Mathematics*, 29 (2003) 757-799 (electronic).

Q. Shang and A. Q. M. Khaliq, A compound adaptive approach to degenerate nonlinear quenching problems, *Numerical Methods in Partial Differential Equations*, 15 (1999), 29-47.

W. Walter, Differential-und Integral-Ungleichungen, und ihre Anwendung bei Abschätzungs-und Eindeutigkeit-problemen, (German) *Springer, Berlin*, 2 (1964).