KÜNNETH THEOREMS FOR VIETORIS–RIPS HOMOLOGY

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Abstract. We prove a Küneth theorem for the Vietoris–Rips homology and cohomology of a semi-uniform space. We then interpret this result for graphs, where we show that the Küneth theorem holds for graphs with respect to the strong graph product. We finish by computing the Vietoris–Rips cohomology of the torus endowed with different semi-uniform structures.

1. Introduction

The Vietoris–Rips complex was first defined by Vietoris in 1927 as a way to obtain homology groups from metric spaces [13], and, somewhat later, it began to be used in the study of hyperbolic groups [8]. With the rise of topological data analysis in the last fifteen years, the Vietoris–Rips homology has become a computational, as well as theoretical, tool, and, indeed, it has become the standard invariant used in the homological analysis of data, in addition to its natural importance in the homological analysis of networks and graphs. Despite this surge of popularity, however, relatively little is known about the properties of Vietoris–Rips homology, and, until recently, even many basic results on the Vietoris–Rips homology had not been established. In a previous article [11], the first author introduced a construction of the Vietoris–Rips homology for semi-uniform spaces and proved a variant of the Eilenberg-Steenrod axioms adapted to this context.

Studying the Vietoris–Rips homology and cohomology from the point of view of semi-uniform spaces has many advantages over working directly with
clique complexes. The first is that setting up the Vietoris–Rips homology in this way enables one to use variations of classical topological arguments in settings that are very far from topological. Indeed, the results in this paper provide one example of this procedure. This, in turn, leads to another major advantage of working in the category of semi-uniform spaces. As the first author showed in [11], the Vietoris–Rips homology and cohomology are functors which act directly on semi-uniform spaces and uniformly continuous maps between them. Since there exist non-trivial uniformly continuous maps between many different kinds of spaces of interest, in particular, between topological spaces, metric spaces with a preferred scale, and graphs, this functoriality enables the construction of homomorphisms which can be used to compare the resulting Vietoris–Rips homology and cohomology groups across different classes of spaces. This is much more difficult to do if one simply studies the Vietoris–Rips homology in each setting independently of the others. Finally, using the constructions of semi-uniform spaces from Čech closure spaces given in [11], we are able to see that the Vietoris cohomology of Dowker [4], the metric cohomology of Hausmann [9], the Vietoris–Rips cohomology on metric spaces with a preferred scale introduced in [11], and the cohomology of the flag complex of a graph are all instances of the same construction on different semi-uniform spaces.

In this article, we continue the development of the algebraic topology of semi-uniform spaces started in [11], first giving an alternate definition of the Vietoris–Rips complex of a relation using simplicial sets, and then studying the Vietoris–Rips homology and cohomology of products of semi-uniform spaces with the goal of establishing Künneth theorems in this context. For the Vietoris–Rips homology as defined here, we will see that, while the Künneth theorem holds for Vietoris–Rips cohomology on arbitrary semi-uniform spaces, it is not true in general for Vietoris–Rips homology. Nonetheless, we are able to show that it does hold for semi-uniform spaces induced by graphs, which, in turn, implies a Künneth theorem for the classical Vietoris–Rips homology of graphs, which is the case of most interest to applications. Note that, while it is well-known that the Künneth theorem is false for the Vietoris–Rips homology using Cartesian products of graphs, by translating the problem into the setting of semi-uniform spaces, we see that that one should use the strong graph product instead. For a homology theory on graphs which satisfies the Künneth theorem with respect to the Cartesian graph product, see [7].

A Künneth theorem for the classical Vietoris–Rips homology on metric spaces with respect to the maximum metric on the product may also be deduced by applying the Künneth theorem for simplicial complexes to the isomorphism in Proposition 10.2 in [1]. However, with the exception of the cases treated in [11], it remains unclear for which cases the the classical and the semi-uniform Vietoris–Rips homology theories coincide. There
has also been some recent work on Künneth theorems in persistent homology [2,6], in which an expression for the persistent homology of a product is obtained, given a filtered complex constructed from a category whose homology has a Künneth formula. The main contribution of this article is that, by constructing the Vietoris–Rips homology and cohomology in the more general context of semi-uniform spaces, we are able to treat Künneth theorems for the Vietoris–Rips cohomology of graphs, metric spaces, and even topological spaces as particular instances of the same theorem. Combined with the results of [11], this allows for the computation of the Vietoris–Rips cohomology of products of spaces which are semi-uniformly homotopy equivalent to spaces whose Vietoris–Rips cohomologies are known (such as such as \((S^1 \times S^1, d_{\text{max}})\), studied in [1]), something which is unattainable using current techniques.

2. Semi-uniform spaces and the Vietoris–Rips complex

In this section, we recall the definition of semi-uniform spaces, which will be our main object of study. We begin with a few preliminary definitions.

2.1. Semi-uniform spaces.

**Definition 2.1.** Let \( U \subset X \times X \). We define

\[ U^{-1} := \{(y, x) \mid (x, y) \in U\}. \]

**Definition 2.2.** Let \( X \) be a set, and let \( \mathcal{F} \) be a non-empty collection of subsets of \( X \) with \( \emptyset \notin \mathcal{F} \). We say that \( \mathcal{F} \) is a filter iff

1. \( U \in \mathcal{F} \) and \( U \subset V \implies V \in \mathcal{F} \), and
2. \( U, V \in \mathcal{F} \implies U \cap V \in \mathcal{F} \).

**Remark 2.3.** The condition that \( \mathcal{F} \) does not contain the empty set is occasionally additional in the literature, and such filters are sometimes called proper filters. Since we will only be dealing with such filters, we see no need to make such a distinction here. Also, note that the requirement that \( \emptyset \notin \mathcal{F} \) combined with Condition (2) of Definition 2.2 above implies that the intersection of any finite collection of sets in \( \mathcal{F} \) is nonempty.

**Definition 2.4.** Let \( X \) be a set. We say that a filter \( \mathcal{U} \) on the product \( X \times X \) is a semi-uniform structure on \( X \) iff

1. each element of \( \mathcal{U} \) contains the diagonal, i.e., \( \Delta \subset U \) for all \( U \in \mathcal{U} \), and
2. if \( U \in \mathcal{U} \), then \( U^{-1} \) contains an element of \( \mathcal{U} \).

The pair \((X, \mathcal{U})\), consisting of a set \( X \) and a semi-uniform structure \( \mathcal{U} \) on \( X \), is called a semi-uniform space.
Remark 2.5. Note that, since $\mathcal{U}$ is a filter, Condition (2) in Definition 2.4 is equivalent to the condition that $U^{-1} \in \mathcal{U}$.

We give several important examples of semi-uniform spaces.

Example 2.6. Let $G = (V, E)$ be an undirected graph. We denote by $\sigma(G) := (V_G, E_G)$ the semi-uniform space where $V_G = V$ and $E_G = [E \cup \Delta_{V_G}]$, i.e. $E_G$ is the filter of subsets of $V_G \times V_G$ generated by the set $E \cup \Delta_{V_G} \subseteq V_G \times V_G$, where $\Delta_{V_G}$ is the diagonal in $V_G \times V_G$.

Since $G$ is undirected in the above definition, $E = E^{-1}$, it follows from [3, Theorem 23.A.4], that $\sigma(G)$ is a semi-uniform space.

Example 2.7. Let $(X, d)$ be a metric space. For every $q > 0$, we define

$$U_q := \{ (x, y) \subset X \times X \mid d(x, y) < q \},$$
$$U_{\leq q} := \{ (x, y) \subset X \times X \mid d(x, y) \leq q \}.$$

Now fix an $r > 0$, and define $\mathcal{U}_r$ to be the semi-uniform structure generated by the sets $U_{r+\varepsilon}$ for all $\varepsilon > 0$, and define $\mathcal{U}_{\leq r}$ to be the semi-uniform structure generated by the single relation $U_{\leq r}$. These are semi-uniform structures by [3, Theorem 23.A.4].

Example 2.8 (see also [11]). Let $X$ be a set, and let $\mathcal{P}(X)$ denote the power set of $X$. We define a Čech closure operator $c: \mathcal{P}(X) \to \mathcal{P}(C)$ to be a function such that

1. $c(\emptyset) = \emptyset$,  
2. $A \subset c(A)$ for all $A \subset X$,  
3. $c(A \cup B) = c(A) \cup c(B)$ for all $A, B \subset X$.

The pair $(X, c)$ will be called a Čech closure space. We define the interior of a set $A \subset X$ to be

$$i(A) = X - c(X - A).$$

A set $U \subset X$ is said to be a neighborhood of $A \subset X$ iff $A \subset i(U)$, and a collection $\mathcal{U} = \{U_\alpha \mid \alpha \in I\}$ is said to be an interior cover of $X$ iff $X = \bigcup_\alpha i(U_\alpha)$.

For every interior cover $\mathcal{U}$ of a Čech closure space $(X, c)$, we define two relations $V_{\mathcal{U}}$ and $R_{\mathcal{U}}$ by

$$V_{\mathcal{U}} := \{ (x, y) \mid \exists U \in \mathcal{U} : x, y \in U \},$$
$$R_{\mathcal{U}} := \{ (x, y) \mid \exists U \in \mathcal{U} : (x \in i(U) \text{ and } y \in U) \text{ or } (y \in i(U) \text{ and } x \in U) \}.$$

Let $\mathcal{I}_c$ denote the collection of interior covers of $(X, c)$, and let $\mathcal{V}_c$ and $\mathcal{R}_c$ be the filters on $X \times X$ generated by the collections $\{V_{\mathcal{U}}\}_{\mathcal{U} \in \mathcal{I}_c}$ and $\{R_{\mathcal{U}}\}_{\mathcal{U} \in \mathcal{I}_c}$, respectively. As shown in [11], $\mathcal{V}_c$ and $\mathcal{R}_c$ are semi-uniform structures on the set $X$. 

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In particular, this construction produces a semi-uniform space for each topological space \((X, c_\tau)\), where \(c_\tau\) is a closure operator with \(c_\tau^2 = c_\tau\). Note that in this case \(\mathcal{V}_{c_\tau} = \mathcal{R}_{c_\tau}\), since both are generated by open covers.

The above examples show how semi-uniform spaces are a simultaneous generalization of graphs, metric spaces with a preferred scale, and closure spaces. We refer to [3,11] for further details on semi-uniform spaces.

2.2. The Vietoris–Rips homology and cohomology of a semi-uniform space. Let \((X,U)\) be a semi-uniform space and \(U \in \mathcal{U}\). We define a simplicial set \(X^U\) by setting \(X^U_0 := \{x \in X\} = X\), and, for \(n \in \mathbb{N}\), we define

\[
X^U_n = \{ (x_0, \ldots, x_n) \mid (x_i, x_j) \in U \ \forall i < j \},
\]

with functions

\[
\partial_i : X^U_n \to X^U_{n-1}, \quad \text{where } \partial_i(x_0, x_1, \ldots, x_n) = (x_0, x_1, \ldots, \hat{x}_i, \ldots, x_n),
\]

\[
s_i : X^U_n \to X^U_{n+1}, \quad \text{where } s_i(x_0, x_1, \ldots, x_n) = (x_0, x_1, \ldots, x_i, x_i, \ldots, x_n).
\]

We emphasize that the \(x_i\) need not be distinct points of \(X\), and, in particular, since the diagonal \(\Delta\) is a member of every \(U \in \mathcal{U}\), the \(X^U_n\) will contain elements with \(x_i = x_j, i \neq j\).

**Definition 2.9.** We call the simplicial set \(X^U\) the Vietoris–Rips complex of the pair \((X,U)\).

For each \(n \in \mathbb{Z}\), let \(C_n(X^U)\) be the graded free abelian group generated by the elements of the sets \(X^U_n\), and let \(C_\ast(X^U) = \bigoplus_{n \in \mathbb{Z}} C_n(X^U)\). Define a differential \(d_n : C_n(X^U) \to C_{n-1}(X^U)\) by

\[
d_n = \sum_{i=0}^{n} (-1)^i \partial_i
\]

With these definitions, \((C_\ast(X^U), d)\) is now a chain complex, and we denote its homology by \(H_\ast(X^U)\).

Let \(\mathcal{G}\) be an \(R\)-module for a commutative ring \(R\). Viewing \(\mathcal{G}\) as the trivial chain complex concentrated in degree 0, we have that \((C_\ast(X^U) \otimes_{\mathbb{Z}} \mathcal{G}, d \otimes 1)\) is a chain complex over \(R\), and we denote its homology, the homology of \(X^U\) with coefficients in \(\mathcal{G}\), by \(H_\ast(X^U; \mathcal{G})\). Similarly, a \(q\)-dimensional cochain \(f \in C^q(X^U; \mathcal{G})\) is defined as a homomorphism \(f : C_q(X^U) \otimes_{\mathbb{Z}} R \to \mathcal{G}\) and the coboundary is given by

\[
(\delta f)(\sigma) = \sum_{i=0}^{q+1} (-1)^i f(\partial_i \sigma)
\]
for each \((q+1)\)-simplex \(\sigma\) of \(X^U\). This leads to the cohomology groups \(H^*(X^U; \mathcal{G})\).

We now consider maps between the simplicial sets corresponding to different elements of a semi-uniform structure \(\mathcal{U}\).

**Proposition 2.10.** For \(U, V \in \mathcal{U}\), \(V \subset U\), \(X^V\) is a sub-simplicial set of \(X^U\). Denote the inclusion map by \(\phi_{VU}: X^V \hookrightarrow X^U\). For \(W \subset V \subset U\), we have \(\phi_{VU} \circ \phi_{WV} = \phi_{WU}\).

**Proof.** Since \(V \subset U\), if \(\sigma = (x_0, \ldots, x_n) \in X^V_n\), then, by definition, \((x_i, x_j) \in V\) for all \(i < j\). Therefore, \((x_i, x_j) \in U\) for all \(i < j\), and \(\sigma \in X^U_n\). Define \(\phi_{VU}(\sigma) = \sigma\). Since this is both a simplicial map and an inclusion, the final statement follows, and the proof is complete. \(\square\)

We define a partial order \(\leq\) on the semi-uniform structure \(\mathcal{U}\) by writing \(U \leq V\) iff \(V \subset U\). Furthermore, since \(\mathcal{U}\) is a filter, for any \(U, V \in \mathcal{U}\), \(W = U \cap V \in \mathcal{U}\), and therefore \(\mathcal{U}\) with this partial order is a directed set. The induced maps \(\phi_{UV*}: H_* (X^V; \mathcal{G}) \to H_* (X^U; \mathcal{G})\) and \(\phi^*_{UV}: H^*(X^U; \mathcal{G}) \to H^*(X^V; \mathcal{G})\) make \((H_*(X^U; \mathcal{G}), \phi_{UV*}, U)\) and \((H^*(X^U; \mathcal{G}), \phi^*_{UV}, U)\) into inverse and direct systems of abelian groups, respectively. We finally define the Vietoris–Rips homology and cohomology of a semi-uniform space \((X, \mathcal{U})\) to be

\[
H^{VR}_*(X, \mathcal{U}; \mathcal{G}) = \lim_{\leftarrow} H_*(X^U; \mathcal{G}), \quad H^{V}_*(X, \mathcal{U}; \mathcal{G}) = \lim_{\rightarrow} H^*(X^U; \mathcal{G}).
\]

We will typically suppress the semi-uniform structure \(\mathcal{U}\) when it is unambiguous.

**Remark 2.11.** Note that, since the ordered and unordered simplicial chain complexes are chain equivalent (by [12, Theorem 4.3.8]), and since degenerate simplices in the simplicial set do not contribute to the resulting homology and cohomology (by [10, VIII. 6]), the Vietoris–Rips homology and cohomology defined here for a semi-uniform space \((X, \mathcal{U})\) are isomorphic to the those defined in [11].

**Example 2.12.** Let \(G = (V, E)\) be a graph. The Vietoris–Rips homology and cohomology of \(\sigma(G) = (V, \mathcal{E}_G)\) defined in Example 2.6 are the simplicial homology and cohomology, respectively, of the clique complex of the graph, by Theorem 5.2 below.

**Example 2.13.** Let \((X, d)\) be a metric space, and let \((X, \mathcal{U}_r)\) be the semi-uniform spaces constructed in Example 2.7. Since the sets \(U_\varepsilon, \varepsilon > 0\) are cofinal in \(\mathcal{U}_0\), it follows that the Vietoris–Rips cohomology of \((X, \mathcal{U}_0)\) is isomorphic to the metric cohomology studied by Hausmann in [9].

**Example 2.14.** Let \((X, c_\tau)\) be a topological closure space (where \(c^2_\tau = c_\tau\)), and let \((X, \mathcal{V}_{c_\tau})\) be the corresponding semi-uniform space from...
Example 2.8. By Theorems 2 and 2a in Dowker [4], the Vietoris–Rips homology and cohomology of \((X, \mathcal{V}_c)\) are isomorphic to the Čech homology and cohomology of the topological space \((X, \tau)\), where the topology \(\tau\) is generated by \(c_\tau\), i.e. a set \(U \subset X\) is open in \((X, \tau)\) iff \(c_\tau^2(X - U) = c_\tau(X - U)\).

3. Products

In this section, we recall the definitions of products for semi-uniform spaces and simplicial sets, and we then prove a theorem relating the products of Vietoris–Rips complexes which will be the basis for the Künneth Theorems to follow.

3.1. Products of semi-uniform spaces.

**Definition 3.1.** Let \(\{(X_a, \mathcal{U}_a)\}_{a \in A}\) be a family of semi-uniform spaces indexed by \(A\). Let \(X := \Pi_{a \in A} X_a\) be the Cartesian product of the sets, and denote by \(\mathcal{U}\) the filter generated by the subsets of \(X \times X\) of the form

\[
\{ (x, y) \mid (x, y) \in X \times X, a \in F \implies (\pi_ax, \pi_ay) \in U_a \},
\]

where \(F \subset A\) is some finite subset of \(A\), \(U_a \in \mathcal{U}_a\) for each \(a \in F\), and \(\pi_a : X \to X_a\) is the projection to the \(a\)-th coordinate.

**Proposition 3.2.** \((X, \mathcal{U})\) defined as above is a semi-uniform space.

**Proof.** First, let \((x, x) \in \Delta_X \times X\). Since the \(\Delta_{X_a} \subset U_a\) for every \(U_a \in \mathcal{U}_a\) and every \(a \in A\), it follows that \((x, x)\) is in every element of \(\mathcal{U}\). Since \((x, x) \in \Delta_X \times X\) was arbitrary, \(\Delta_X \times X\) is contained in every element of \(\mathcal{U}\).

Suppose now that \(U \in \mathcal{U}\). Then \(U\) contains a set \(V\) of the form (1) above. Since, for any \(U_a \in \mathcal{U}_a\) we have that \(U_a^{-1} \in \mathcal{U}_a\), we see from \(V_a^{-1} \subset U_a^{-1}\) that \(V^{-1}\) is of the form (1) as well. However, \(V^{-1} \subset U^{-1}\), so \(U^{-1} \in \mathcal{U}\), and the proof is complete. \(\square\)

**Definition 3.3.** We call \((X, \mathcal{U})\) the semi-uniform product of \(\{(X_a, \mathcal{U}_a)\}_{a \in A}\).

We will sometimes denote \((X, \mathcal{U})\) as \((\Pi_{a \in A} X_a, \Pi_{a \in A} \mathcal{U}_a)\).

3.2. Products of simplicial sets.

**Definition 3.4.** Let \(\Sigma\) and \(\Sigma'\) be simplicial sets. Then the product \(\Sigma \times \Sigma'\) is given by

\[(\Sigma \times \Sigma')_n := \{ (\sigma, \sigma') \mid \sigma \in \Sigma_n, \sigma' \in \Sigma'_n \} \].

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For any simplicial sets $\Sigma$ and $\Sigma'$, the Eilenberg-Zilber theorem [5] gives a quasi-isomorphism between the chain complexes $C_*(\Sigma \times \Sigma')$ and $C_*(\Sigma) \otimes C_*(\Sigma')$. In order to establish the Künneth theorems for Vietoris–Rips homology, we must further establish a relationship between the Vietoris–Rips complexes of products of semi-uniform spaces on the one hand, and the products of Vietoris–Rips complexes of semi-uniform spaces on the other. This follows easily from the respective definitions, and is accomplished in the following theorem. First, however, we make the following remark.

**Remark 3.5.** Let $(X, U)$ and $(Y, V)$ be semi-uniform spaces and let $U \in U$ and $V \in V$. Observe that the set $U \times V \subset (X \times X) \times (Y \times Y)$ can be seen as a subset of $(X \times Y) \times (X \times Y)$ via the isomorphism

$$\psi: (X \times Y) \times (X \times Y) \to (X \times X) \times (Y \times Y),$$

$$((x_1, y_1), (x_2, y_2)) \mapsto ((x_1, x_2), (y_1, y_2))$$

Note that the expression $(X \times Y)^{U \times V}$ is an abuse of notation, and is, more precisely, $(X \times Y)^{\psi^{-1}(U \times V)}$. Since $U \times V \cong \psi^{-1}(U \times V)$, however, we will use the first notation in place of the second throughout.

**Theorem 3.6.** Let $(X, U)$ and $(Y, V)$ be semi-uniform spaces. Then, for any $U \in U$ and $V \in V$, we have

$$(X \times Y)^{U \times V} \cong X^{U} \times Y^{V}.$$  

**Proof.** Let $\phi: (X \times Y)^{U \times V}_k \to (X^{U} \times Y^{V})_k$ be the map given by

$$\phi((x_0, y_0), \ldots, (x_k, y_k)) := ((x_0, \ldots, x_k), (y_0, \ldots, y_k)).$$

By definition, $((x_0, y_0), \ldots, (x_k, y_k)) \in (X \times Y)^{U \times V}$ iff, for all $i, j \in \{1, \ldots, k\}$, one of the following holds:

1. $\psi((x_i, y_j), (x_j, y_j)) \in U \times V$,
2. $x_i = x_j$ and $(y_i, y_j) \in V$,
3. $(x_i, x_j) \in U$ and $y_i = y_j$.

This, in turn, is true iff $(x_0, \ldots, x_k) \in X^{U}$ and $(y_0, \ldots, y_k) \in Y^{V}$. Therefore, $\phi$ gives an isomorphism between $(X \times Y)^{U \times V}$ and $X^{U} \times Y^{V}$, and the proof is complete. □

4. The Künneth theorems

The Künneth theorems for the Vietoris–Rips homology now follow from the above product relations, the Künneth theorems for simplicial sets, and the properties of exact sequences in direct and inverse limits. We begin with the results for Vietoris–Rips cohomology, where we have a Künneth formula in general.

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In order to establish this, we will first require the following lemma. We begin by recalling the following definition.

**Definition 4.1.** We say that a graded module \( \{C_q\} \) is said to be of **finite type** if it is finitely generated for every \( q \).

**Remark 4.2.** We will sometimes write the torsion product \( \text{Tor}_1(A, B) \) as \( A \ast B \).

**Lemma 4.3.** Let \((X, U)\) and \((Y, V)\) be semi-uniform spaces, \( U \in U \), and \( V \in V \). Let \( G \) and \( G' \) be modules over a principal ideal domain \( R \) such that \( G \ast G' = 0 \). If \( H_\ast(X^U; R) \) and \( H_\ast(Y^V; R) \) are of finite type or if \( H_\ast(Y^V; R) \) is of finite type and \( G' \) is finitely generated, then for any \( q \in \mathbb{Z} \) there is a natural short-exact sequence

\[
0 \to \bigoplus_{i+j=q} H^{i}_V (X^U; G) \otimes H^j_V (Y^V; G') \to H^q_V ((X \times Y)^{U \times V}; G \otimes G') \\
\to \bigoplus_{i+j=q+1} \text{Tor}_1 (H^i_V (X^U; G), H^j_V (Y^V; G')) \to 0.
\]

Furthermore, this sequence splits, but not canonically.

**Proof.** Theorem 3.6 gives \( C_\ast (X \times Y, U \times V) \cong C_\ast (X^U \times Y^U) \), and therefore \( C^\ast (X \times Y, U \times V) \cong C^\ast (X^U \times Y^U) \), and from the Eilenberg–Zilber theorem, we have \( C^\ast (X^U \times Y^V) \cong C^\ast (X^U) \otimes C^\ast (Y^V) \), where \( \cong \) indicates cochain-homotopy equivalence. The result now follows from the Künneth theorem for cochain complexes from [12, Theorem 5.5.11]. □

We now give the Künneth theorem for Vietoris–Rips cohomology.

**Theorem 4.4.** Let \((X, U)\) and \((Y, V)\) be semi-uniform spaces, and let \( G \) and \( G' \) be modules over a principal ideal domain \( R \) such that \( G \ast G' = 0 \). Suppose that \( U \) and \( V \) are generated by collections \( \{U_\alpha\}_{\alpha \in A} \) and \( \{V_\beta\}_{\beta \in B} \) such that \( H_\ast(X^{U_\alpha}; R) \) and \( H_\ast(Y^{V_\beta}; R) \) are of finite type for all \( \alpha \in A \) and \( \beta \in B \), or such that \( H_\ast(Y^{V_\beta}; R) \) is of finite type for \( \beta \in B \) and \( G' \) is finitely generated. Then for any \( q \in \mathbb{Z} \) there is a natural short-exact sequence

\[
0 \to \bigoplus_{i+j=q} H^i_V (X; G) \otimes H^j_V (Y; G') \to H^q_V (X \times Y; G \otimes G') \\
\to \bigoplus_{i+j=q+1} \text{Tor}_1 (H^i_V (X; G), H^j_V (Y; G')) \to 0.
\]

Furthermore, this sequence splits, but not canonically.

**Proof.** We first note that the family of sets in \( U \times V \) of the form \( U \times V \) are cofinal in \( U \times V \). The theorem now follows from Proposition 4.3, the fact
that Tor₁ and ⊗ commute with direct limits, and that the direct limit is an exact functor. □

For homology, the Künneth theorem does not hold in general, due to the failure of exactness of the inverse limits of exact sequences. Nonetheless, in some special cases, the exact sequences related to elements \( U \times V \in \mathcal{U} \times \mathcal{V} \) prove to be useful. We give two such situations below, beginning, as above with the following lemma.

**Lemma 4.5.** Let \((X, \mathcal{U})\) and \((Y, \mathcal{V})\) be semi-uniform spaces, and let \( \mathcal{G} \) and \( \mathcal{G}' \) be modules or a principal ideal domain such that \( \mathcal{G} \ast \mathcal{G}' = 0 \). Then for any \( U \in \mathcal{U} \), \( V \in \mathcal{V} \), and \( q \in \mathbb{Z} \), there is a natural short-exact sequence

\[
0 \to \bigoplus_{i+j=q} H_{i}^{VR}(X^U; \mathcal{G}) \otimes H_{j}^{VR}(Y^V; \mathcal{G}') \to H_{q}^{VR}((X \times Y)^{U \times V}; \mathcal{G} \otimes \mathcal{G}')
\]

\[
\to \bigoplus_{i+j=q-1} \text{Tor}_1(H_{i}^{VR}(X^U; \mathcal{G}), H_{j}^{VR}(Y^V; \mathcal{G}')) \to 0.
\]

Furthermore, this sequence splits, but not canonically.

**Proof.** Theorem 3.6 gives \( C_\ast(X \times Y, U \times V) \cong C_\ast(X^U \times Y^V) \), and from the Eilenberg-Zilber theorem \[5\], we have \( C_\ast(X^U \times Y^V) \cong C_\ast(X^U) \otimes C_\ast(Y^V) \), where \( \cong \) indicates chain-homotopy equivalence. The result now follows from the Künneth theorem for chain complexes ([12, Theorem 5.3.4]). □

**Theorem 4.6.** Let \((X, \mathcal{U})\) and \((Y, \mathcal{V})\) be semi-uniform spaces, and let \( \mathcal{G} \) and \( \mathcal{G}' \) be modules over a principal ideal domain such that \( \mathcal{G} \ast \mathcal{G}' = 0 \). Suppose \( U^\ast \) and \( V^\ast \) are maximal in \( \mathcal{U} \) and \( \mathcal{V} \), respectively, ordered by inclusion. Then \( U^\ast \times V^\ast \) is maximal in \( \mathcal{U} \times \mathcal{V} \), and for all \( q \in \mathbb{Z} \), there exists a natural short-exact sequence

\[
0 \to \bigoplus_{i+j=q} H_{i}^{VR}(X^U; \mathcal{G}) \otimes H_{j}^{VR}(Y^V; \mathcal{G}') \to H_{q}^{VR}(X \times Y; \mathcal{G} \otimes \mathcal{G}')
\]

\[
\to \bigoplus_{i+j=q-1} \text{Tor}_1(H_{i}^{VR}(X^U; \mathcal{G}), H_{j}^{VR}(Y^V; \mathcal{G}')) \to 0.
\]

Furthermore, this sequence splits, but not canonically.

**Proof.** We first show that \( U^\ast \times V^\ast \) is maximal in \( \mathcal{U} \times \mathcal{V} \). Suppose that there exists a \( W \in \mathcal{U} \times \mathcal{V} \) with \( W \subseteq U^\ast \times V^\ast \). Then there exist \( U \in \mathcal{U} \) and \( V \in \mathcal{V} \) such that \( U \times V \subseteq W \subseteq U^\ast \times V^\ast \), a contradiction. Therefore \( U^\ast \times V^\ast \) is maximal in \( \mathcal{U} \times \mathcal{V} \). For any semi-uniform space \((X, \mathcal{U})\), we have, by definition, \( H_{\ast}^{VR}(X, \mathcal{U}; \mathcal{G}) = \lim_{\leftarrow} H_{\ast}^{VR}(X^U; \mathcal{G}) \). If \( U^\ast \) is maximal in \( \mathcal{U} \),
then $H_\ast(X^U; \mathcal{G})$ is cofinal in the inverse system $\{H_\ast(X^U; \mathcal{G}), \phi_{UV_\ast}, \mathcal{U}\}$, from which it follows that

$$H_\ast^{VR}(X, \mathcal{U}; \mathcal{G}) = \lim_{\leftarrow} H_\ast^{VR}(X^U; \mathcal{G}) = H_\ast(X^U; \mathcal{G})$$

as desired. Putting these together, the result now follows from Theorem 4.5.

Although the short exact sequence does not hold in general for Vietoris–Rips homology, if the torsion term vanishes on a cofinal subset of the bases for the product semi-uniform structure, we may still conclude that there is an isomorphism of the respective homology groups, as we see from the following.

**Theorem 4.7.** Let $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ be semi-uniform spaces, and let $\mathcal{G}$ and $\mathcal{G}'$ be modules over a principal ideal domain such that $\mathcal{G} \ast \mathcal{G}' = 0$. Suppose that $\mathcal{F}' \subset \mathcal{U} \times \mathcal{V}$ is a cofinal collection of sets in of the form $U \times V$, $U \in \mathcal{U}$, $V \in \mathcal{V}$ such that, for any $U \times V \in \mathcal{F}$,

$$\bigoplus_{i+j=q-1} \text{Tor}_1(H_i^{VR}(X; \mathcal{G}), H_j^{VR}(Y; \mathcal{G}')) = 0.$$

Then

$$\bigoplus_{i+j=q} H_i^{VR}(X; \mathcal{G}) \otimes H_j^{VR}(Y; \mathcal{G}') \cong H_q^{VR}(X \times Y; \mathcal{G} \otimes \mathcal{G}') \in \mathcal{V}.$$

5. Küneth theorems for Vietoris–Rips homology on graphs

In this section, we apply the general Küneth formulae on semi-uniform spaces from Section 4 to prove the Küneth Theorem for the Vietoris–Rips homology on graphs. We begin with the following construction. All graphs are undirected.

**Definition 5.1.** Let $G = (V, E)$ and $G' = (V', E')$ be (undirected) graphs. Then the strong graph product $G \boxtimes G' = (V \boxtimes V', E \boxtimes E')$ is defined by

$$(v, v') \in V \boxtimes V' \text{ if and only if } v \in V, \ v' \in V'$$

and

$$((v_0, v'_0), (v_1, v'_1)) \in E \boxtimes E'$$

if and only if one of the following holds:

1. $(v_0, v_1) \in E$ and $(v'_0, v'_1) \in E'$,
2. $v_0 = v_1$ and $(v'_0, v'_1) \in E'$,
3. $(v_0, v_1) \in E$ and $v'_0 = v'_1$. 

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THEOREM 5.2. Let $H_*^{VR}(G; A)$ denote the Vietoris–Rips homology of the undirected graph $G = (V, E)$ with coefficients in an abelian group $A$. Then $H_*^{VR}(G; A) \cong H_*^{VR}((G; \sigma); A)$, where $\sigma(G) = (V, \mathcal{E}_G)$ is the semi-uniform space defined in Example 2.6.

PROOF. By construction, $\mathcal{E}_G$ has a maximal element $E$ consisting of the edges of the graph $G$ and the diagonal $\Delta_V$, from which it follows that $H_*^{VR}((G; \sigma); A) \cong H_*^{VR}(V^E_G; A)$. However, $H_*^{VR}(V^E_G; A)$ is exactly the homology of the simplicial set generated by the clique complex $\Sigma_G$ of $G$. Therefore,

$$H_*^{VR}(G; A) = H_*(\Sigma_G; A) \cong H_*^{VR}(V^E_G; A) \cong H_*^{VR}((G; \sigma); A),$$

and the proof is complete. \(\square\)

In order to prove the Künneth Theorem for undirected graphs, we will need the following

PROPOSITION 5.3. Let $G = (V, E)$ and $G' = (V', E')$ be graphs. Then

$$(V \times V', \mathcal{E}_{G \boxtimes G'}) = \sigma(G \boxtimes G') = \sigma(G) \times \sigma(G') = (V, \mathcal{E}_G) \times (V', \mathcal{E}_{G'}),$$

where the product on the right is the product of semi-uniform spaces.

PROOF. We must show that $\mathcal{E}_G \times \mathcal{E}_{G'} = \mathcal{E}_{G \boxtimes G'}$. First, let $U \in \mathcal{E}_G \times \mathcal{E}_{G'}$. Then, by construction, there exist sets $e \in \mathcal{E}_G$ and $e' \in \mathcal{E}_{G'}$ such that $e \times e' \subset U$. By definition of $\mathcal{E}_G$ and $\mathcal{E}_{G'}$, however, $E \subset e$ and $E' \subset e'$, so $E \times E' \subset e \times e' \subset U$. Since $E \times E' \in \mathcal{E}_{G \boxtimes G'}$ and $\mathcal{E}_{G \boxtimes G'}$ is a filter, therefore $U \in \mathcal{E}_{G \boxtimes G'}$, and we see that $\mathcal{E}_G \times \mathcal{E}_{G'} \subset \mathcal{E}_{G \boxtimes G'}$.

Now suppose $U \in \mathcal{E}_{G \boxtimes G'}$. Then $E \times E' \subset U$, and therefore $U \in \mathcal{E}_G \times \mathcal{E}_{G'}$ as well, by definition of the product semi-uniform structure. Therefore $\mathcal{E}_{G \boxtimes G'} \subset \mathcal{E}_G \times \mathcal{E}_{G'}$. It follows that $\mathcal{E}_G \times \mathcal{E}_{G'} = \mathcal{E}_{G \boxtimes G'}$, and therefore $\sigma(G \boxtimes G') = \sigma(G) \times \sigma(G')$. \(\square\)

THEOREM 5.4. Let $G = (V, E)$ and $G' = (V', E')$ be graphs, and let $\mathcal{M}$ and $\mathcal{M}'$ be modules over a principal ideal domain such that $\mathcal{M} \ast \mathcal{M}' = 0$. For every $q \in \mathbb{Z}$, there exist natural short exact sequences

$$0 \to \bigoplus_{i+j=q} H^VR_i(G; \mathcal{M}) \otimes H^VR_j(G'; \mathcal{M}') \to H^VR_q(G \boxtimes G'; \mathcal{M} \otimes \mathcal{M}') \to \bigoplus_{i+j=q+1} \text{Tor}_1(H^VR_i(G; \mathcal{M}), H^VR_j(G; \mathcal{M}')) \to 0$$

PROOF. The theorem follows immediately from Proposition 5.3, Theorem 4.6, and the definition of the classical Vietoris–Rips complex for graphs. \(\square\)
6. Applications to metric spaces

As an illustration of the above results, we examine the Vietoris–Rips homology and cohomology of the torus with different semi-uniform structures. We first recall the definition of the classical Vietoris–Rips complex on a metric space and the corresponding relations which can be used to generate semi-uniform structures.

**Definition 6.1.** Suppose that \((X, d)\) is a metric space and \(r > 0\) is a real number. The Vietoris–Rips complex \(VR_<(X; r)\) is the simplicial complex with vertex set \(X\), where a finite subset \(σ \subseteq X\) is a simplex if only if the diam(σ) < \(r\).

**Remark 6.2.** For a given \(r > 0\), note that \(VR_<(X; r)\) is the geometric realization of the simplicial set \(X^{U_r}\), and \(VR_≤(X; r)\) is the geometric realization of the simplicial set \(X^{U≤_r}\). They therefore have the same homology and cohomology groups.

We now recall following theorem from [1].

**Theorem 6.3 [1, Theorem 7.4].** Denote the circle with unit circumference by \(S^1\), and consider \(S^1\) as a metric space with the geodesic distance. For \(0 < r < \frac{1}{2}\), suppose that \(\frac{1}{2l+1} < r < \frac{l+1}{2l+3}\) for some \(l \in \{0, 1, \ldots\}\). Then we have a homotopy equivalence

\[
VR_<(S^1; r) \simeq S^{2l+1}.
\]

The following computation now follows from Theorems 6.3 and 4.4.

**Proposition 6.4.** Let \(0 < r, r' \leq \frac{1}{2}\) with \(\frac{1}{2l+1} < r < \frac{l+1}{2l+3}\) and \(\frac{1}{2l'+1} < r' < \frac{l'+1}{2l'+3}\) for some \(l, l' \in \{0, 1, 2, \ldots\}\). Let \(T^2_{r,r'}\) denote the product semi-uniform space \(T^2_{r,r} := (S^1, U_r) \times (S^1, U_{r'})\).

If \(l \neq l'\), then

\[
H^q_{VR}(T^2_{r,r'}) = \begin{cases} 
\mathbb{Z} & q = 0, 2l + 1, 2l' + 1, \text{ or } 2(l + l' + 1) \\
\{0\} & \text{otherwise.}
\end{cases}
\]

If \(l = l'\), then

\[
H^q_{VR}(T^2_{r,r}) = \begin{cases} 
\mathbb{Z} & q = 0 \text{ or } 2(2l + 1) \\
\mathbb{Z} \times \mathbb{Z} & q = 2l + 1 \\
\{0\} & \text{otherwise.}
\end{cases}
\]
By Remark 6.2 and Theorem 6.3 we have that obtain the exact sequence

\[ 0 \to \bigoplus_{i+j=q} H^i_V(S^1, U_r) \otimes H^j_V(S^1, U_{r'}) \to H^q_V(T^2_{r, r'}) \]

\[ \to \bigoplus_{i+j=q-1} \text{Tor}_1(H^i_V(S^1, U_r), H^j_V(S^1, U_{r'})) \to 0. \]

By Remark 6.2 and Theorem 6.3 we have that

\[ H^i_V(X^{U_{r+\varepsilon}}) \cong H^i(\operatorname{VR}_<(X; r + \varepsilon)) \cong H_i(S^{2l+1}) \]

for all \( \varepsilon > 0 \) sufficiently small. Since the sets \( U_{r+\varepsilon} \) are cofinal in \( U_r \), we obtain the exact sequence

\[ 0 \to \bigoplus_{i+j=q} H_i(S^{2l+1}) \otimes H_j(S^{2l'+1}) \to H^q_V(T^2_{r, r'}) \]

\[ \to \bigoplus_{i+j=q-1} \text{Tor}_1(H_i(S^{2l+1}), H_j(S^{2l'+1})) \to 0. \]

Since the torsion term in the above exact sequence is trivial, the result follows. \( \square \)

**Remark 6.5.** Note that a similar result is true for the Vietoris–Rips homology of the torus by Theorem 4.7.

Now, suppose that \( (X, d_X) \) and \( (Y, d_Y) \) metric spaces and \( (X \times Y, d) \) with \( d((x_1, y_1), (x_2, y_2)) = \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\} \). Then we have

\[ U_q^{X \times Y} = \{ (z, w) \in (X \times Y) \times (X \times Y) \mid d(z, w) < q \} \]

\[ = \{ ((x_1, y_1), (x_2, y_2)) \mid d((x_1, y_1), (x_2, y_2)) < q \} \]

\[ = \{ ((x_1, y_1), (x_2, y_2)) \mid \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\} < q \} \]

\[ = \{ ((x_1, y_1), (x_2, y_2)) \mid d_X(x_1, x_2) < r \text{ and } d_Y(y_1, y_2) < q \} \cong U_q^X \times U_q^Y \]

Note, too, that the \( U_{r+\varepsilon}^{X \times Y} \) are cofinal in \( U_r^{X \times Y} \). The following corollary follows immediately from the above comments, Theorem 4.4, and Proposition 6.4.

**Corollary 6.6.** Let \( (X, d_X) \) and \( (Y, d_Y) \) be metric spaces, and let \( d \) be the maximum metric on \( X \times Y \). Let \( \mathcal{G} \) and \( \mathcal{G}' \) be modules of a principal ideal domain such that \( \mathcal{G} \ast \mathcal{G}' = 0 \). Then we have the short exact sequence

\[ 0 \to \bigoplus_{i+j=q} H^i_V(X, U_r; \mathcal{G}) \otimes H^j_V(Y, U_{r'}; \mathcal{G}') \to H^q_V(X \times Y, U_{r}; \mathcal{G} \otimes \mathcal{G}') \]

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\[
\rightarrow \bigoplus_{i+j=q-1} \text{Tor}_1\left( H^i_{VR}(X, \mathcal{U}_r; G), H^j_{VR}(Y, \mathcal{U}_r; G') \right) \rightarrow 0.
\]

Furthermore, this sequence splits, but not canonically.

Applying this to the case of the torus, we recover the following result, which also follows from [1, Proposition 10.2].

**Corollary 6.7.** Let \( T^2 = S^1 \times S^1 \) with the maximum metric. Let \( 0 < r \leq \frac{1}{2} \) with \( \frac{1}{2l+1} < r < \frac{l+1}{2l+3} \) for some \( l = 0, 1, \ldots \). Then

\[
H_q^{VR}(T^2, \mathcal{U}_r) \cong \bigoplus_{i+j=q} H^i_{VR}(S^1, \mathcal{U}_r) \otimes H^j_{VR}(S^1, \mathcal{U}_r)
\]

\[
\cong \begin{cases} 
\mathbb{Z} & \text{if } q \in \{0, 2(2l + 1)\} \\
\mathbb{Z} \times \mathbb{Z} & \text{if } q = 2l + 1 \\
0 & \text{otherwise.}
\end{cases}
\]

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