On the $f$-biharmonic Maps and Submanifolds

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Abstract. In this paper, we prove that every $f$-biharmonic map from a complete Riemannian manifold into a Riemannian manifold with non-positive sectional curvature, satisfying some condition, is $f$-harmonic. Also we present some properties for the $f$-biharmonicity of submanifolds of $S^n$, and we give the classification of $f$-biharmonic curves in 3-dimensional sphere.

1. Introduction

Let $f : (M^m, g) \to (0, +\infty)$ be a smooth function. An $f$-harmonic map is a map $\varphi : (M^m, g) \to (N^n, h)$ between two Riemannian manifolds that is a critical point of the $f$-energy

$$E_f(\varphi; D) = \frac{1}{2} \int_D f|d\varphi|^2_v g,$$

for any compact domain $D$, where $v_g$ is the volume element (see [2],[5],[10]).

The $f$-tension field of $\varphi$ is given by

$$\tau_f(\varphi) = \text{trace}_g \nabla f d\varphi = f \tau(\varphi) + d\varphi(\text{grad}^{M} f).$$

The $f$-tension field of $\varphi$ vanishes ($\tau_f(\varphi) = 0$) means that $\varphi$ is a harmonic map.

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For any compact domain $D \subseteq M$, the $f$-bienergy is defined by

\[
E_{2,f}(\varphi; D) = \frac{1}{2} \int_{D} |\tau_{f}(\varphi)|^{2} v_{g},
\]

\(\varphi\) is called $f$-biharmonic if $\varphi$ is a critical point of the $f$-bienergy functional for any compact domain $D$. The Euler-Lagrange equation associated to the $f$-bienergy functional is given by

\[
\tau_{2,f}(\varphi) \equiv -f \text{trace}_{g} R^{N}(\tau_{f}(\varphi), d\varphi) d\varphi - \text{trace}_{g} \left( \nabla^{g} f(\nabla^{g} \tau_{f}(\varphi)) - f \nabla^{g} M \tau_{f}(\varphi) \right) = 0,
\]

for an orthonormal frame \(\{e_{1}, ..., e_{m}\}\), we have

\[
\text{trace}_{g} R^{N}(\tau_{f}(\varphi), d\varphi) d\varphi = \sum_{i=1}^{m} R^{N}(\tau_{f}(\varphi), d\varphi(e_{i})) d\varphi(e_{i}),
\]

\[
\text{trace}_{g} \left( \nabla^{g} f(\nabla^{g} \tau_{f}(\varphi)) - f \nabla^{g} M \tau_{f}(\varphi) \right) = \sum_{i=1}^{m} \left\{ \nabla^{g}_{e_{i}} f(\nabla^{g}_{e_{i}} \tau_{f}(\varphi)) - f \nabla^{g}_{M e_{i}} \tau_{f}(\varphi) \right\}.
\]

\(\tau_{2,f}(\varphi)\) is called the $f$-bitension field of $\varphi$. The solutions of the equation (1.2) are the biharmonic maps ([5],[10]).

The $f$-harmonic and $f$-biharmonic concept is a natural generalization of harmonic maps (Eells and Sampson [7]), $p$-harmonic and exponentially harmonic maps (Eells [6]) and biharmonic maps (Jiang [8]). In mathematical physics, $f$-harmonic maps, relate to the equations of the motion of a continuous system of spins ([4]) and the gradient Ricci-soliton structure ([11]).

In this paper, we prove that every $f$-biharmonic map from a complete Riemannian manifold into a Riemannian manifold with non-positive sectional curvature, satisfies some condition, is $f$-harmonic (Theorem 2.3). Also we present some properties for the $f$-biharmonicity of submanifolds of $\mathbb{S}^{n}$ (Theorem 3.1), and we give the classification of $f$-biharmonic curves in 3-dimensional sphere (Theorem 4.1).

2. Some Results on $f$-harmonic and $f$-biharmonic Maps

Let $\varphi : (M^{m}, g) \to (N^{n}, h)$ be a smooth immersed map between two Riemannian manifolds. Then $\varphi^{-1}(TN)$ is decomposed into direct sum $T(M) \oplus N(M)$ of tangent fiber $T(M)$ and normal fiber $N(M)$. If $f \in C^{\infty}(M)$ be a smooth positive function, then

\[
(\nabla^{g}_{X} f d\varphi)(Y) = \nabla^{g}_{X} (f d\varphi(Y)) - f d\varphi(\nabla^{g}_{X} Y) = B_{f}(X, Y)
\]

for all vector fields $X, Y \in \Gamma(TM)$. For any orthonormal frame $\{e_{1}, ..., e_{m}\}$ on $M$, we have

\[
\tau_{f}(\varphi) = \sum_{i=1}^{m} B_{f}(e_{i}, e_{i}) \in N(M).
\]
Proposition 2.1. For any vector fields \( \xi \in N(M) \) and \( X, Y \in \Gamma(TM) \), we have

\[
(2.3) \quad h(B_f(X, Y), \xi) = fh(A_\xi(X), d\varphi(Y))
\]

where \( A_\xi \) denote the shape operator with respect to \( \xi \) defined by:

\[
(2.4) \quad \nabla^N_X \xi = -A_\xi(X) + \nabla^\perp_X \xi.
\]

Proof. Since \( \xi \in N(M) \) and \( d\varphi(Y) \in T(M) \), then we have:

\[
h(\xi, fd\varphi(Y)) = 0.
\]

Whence

\[
0 = \nabla_X h(\xi, fd\varphi(Y)) = h(\nabla^N_X \xi, fd\varphi(Y)) + h(\xi, \nabla^\perp_X fd\varphi(Y)) = h(-A_\xi(X) + \nabla^\perp_X \xi, fd\varphi(Y)) + h(\xi, B_f(X, Y)) = -h(A_\xi(X), fd\varphi(Y)) + h(\xi, B_f(X, Y))
\]

Proposition 2.2. If \( \nabla^N_X \tau_f(\varphi) = 0 \) for any vector field \( X \in \Gamma(TM) \), then \( \tau_f(\varphi) = 0 \).

Proof. From the formula (2.4) with \( \xi = \tau_f(\varphi) \), we have \( A_{\tau_f(\varphi)}(X) = 0 \).

By the Proposition 2.1, we obtain:

\[
h(B_f(X, Y), \tau_f(\varphi)) = fh(A_{\tau_f(\varphi)}(X), d\varphi(Y)) = 0,
\]

and

\[
|\tau_f(\varphi)|^2 = \sum_{i=1}^{m} h(B_f(e_i, e_i), \tau_f(\varphi)) = 0,
\]

where \( X, Y \in \Gamma(TM) \) and \( \{e_1, \ldots, e_m\} \) is an orthonormal frame on \( M \).

Theorem 2.3. Let \((M^m, g)\) be a complete Riemannian manifold with infinite volume and \((N^n, h)\) be a Riemannian manifold with non-positive sectional curvature. Let \( f \) be a smooth positive function on \( M \). If \( \varphi : (M^m, g) \to (N^n, h) \) is an \( f \)-biharmonic map with finite \( f \)-bienergy, satisfying:

\[
(2.5) \quad \text{trace}_g \nabla^\varphi f \nabla^\varphi \varphi - f \text{trace}_g \nabla^\varphi \nabla^\varphi \leq 0
\]

then \( \varphi \) is \( f \)-harmonic.
The inequality (2.5) means that for any vector field \( X \in \Gamma(\varphi^{-1}(TN)) \) and any orthonormal frame \((e_i)_i\) on \(M\), we have:

\[
\sum_{i=1}^{m} h(\nabla_{e_i} f \nabla_{e_i} X, X) - fh(\nabla_{e_i} \nabla_{e_i} X, X) = \sum_{i=1}^{m} h(e_i(f) \nabla_{e_i} X, X) \leq 0.
\]

**Proof.** Assume that \( \varphi : (M^m, g) \to (N^n, h) \) is \( f\)-biharmonic. Fix a point \( x \in M \) and let \( \{e_1, ..., e_m\} \) be an orthonormal frame with respect to \( g \) on \( M \), such that \( \nabla_{e_i} e_j = 0 \), at \( x \) for all \( i, j = 1, ..., m \).

From formula (1.2), we obtain:

\[
-f \sum_{i=1}^{m} R^N(\tau_f(\varphi), d\varphi(e_i))d\varphi(e_i) - \sum_{i=1}^{m} \nabla_{e_i} f \nabla_{e_i} \tau_f(\varphi) = 0,
\]

then, we have

\[
h(\sum_{i=1}^{m} R^N(\tau_f(\varphi), d\varphi(e_i))d\varphi(e_i), \tau_f(\varphi), \tau_f(\varphi)) = 0,
\]

and

\[
-h(\sum_{i=1}^{m} \nabla_{e_i} \nabla_{e_i} \tau_f(\varphi), \tau_f(\varphi)) = -h(\sum_{i=1}^{m} \nabla_{e_i} \nabla_{e_i} \tau_f(\varphi), \tau_f(\varphi)) + h(\sum_{i=1}^{m} R^N(\tau_f(\varphi), d\varphi(e_i))d\varphi(e_i), \tau_f(\varphi))
\]

\[
+ h(\sum_{i=1}^{m} \nabla_{e_i} f \nabla_{e_i} \tau_f(\varphi), \tau_f(\varphi)).
\]

(2.6)

Since, \( \text{trace}_g h(\nabla^g f \nabla^g \tau_f(\varphi), \tau_f(\varphi)) - \text{trace}_g h(\nabla^g \nabla^g \tau_f(\varphi), \tau_f(\varphi)) \leq 0 \), and the sectional curvature of \( N \) is non-positive, from (2.6) we deduce

\[
- h(\sum_{i=1}^{m} \nabla_{e_i} \nabla_{e_i} \tau_f(\varphi), \tau_f(\varphi)) \leq 0.
\]

(2.7)

Let \( \rho \) be a smooth function with compact support on \( M \), by (2.7) we have

\[
- h(\sum_{i=1}^{m} \nabla_{e_i} \nabla_{e_i} \tau_f(\varphi), \rho^2 \tau_f(\varphi)) \leq 0,
\]
is equivalent to

\[ -(\sum_{i=1}^{m} e_i \left( h(\nabla_{e_i}^2 \tau_f(\varphi), \rho^2 \tau_f(\varphi)) \right) ) + \sum_{i=1}^{m} h(\nabla_{e_i}^2 \tau_f(\varphi), \nabla_{e_i}^2 \rho^2 \tau_f(\varphi)) \leq 0. \]

by the Stoke’s theorem, we have

\[ \int_M \sum_{i=1}^{m} e_i \left( h(\nabla_{e_i}^2 \tau_f(\varphi), \rho^2 \tau_f(\varphi)) \right) v_g = 0. \]

From formulae (2.8) and (2.9), we obtain

\[ \int_M \rho^2 |\nabla_{e_i}^2 \tau_f(\varphi)|^2 v_g + \int_M \sum_{i=1}^{m} 2 \rho e_i(\rho) h(\nabla_{e_i}^2 \tau_f(\varphi), \tau_f(\varphi)) v_g \leq 0. \]

By the Young’s inequality we have

\[ -2h(\rho \nabla_{e_i}^2 \tau_f(\varphi), e_i(\rho) \tau_f(\varphi)) \leq \epsilon \rho^2 |\nabla_{e_i}^2 \tau_f(\varphi)|^2 + \frac{1}{\epsilon} e_i(\rho)^2 |\tau_f(\varphi)|^2. \]

From (2.10) and (2.11) we deduce the inequality

\[ \int_M \sum_{i=1}^{m} \rho^2 |\nabla_{e_i}^2 \tau_f(\varphi)|^2 v_g \leq \epsilon \int_M \sum_{i=1}^{m} \rho^2 |\nabla_{e_i}^2 \tau_f(\varphi)|^2 v_g + \frac{1}{\epsilon} \int_M \sum_{i=1}^{m} e_i(\rho)^2 |\tau_f(\varphi)|^2 v_g. \]

Let \( \epsilon = \frac{1}{2} \), by (2.12) we have

\[ \frac{1}{2} \int_M \sum_{i=1}^{m} \rho^2 |\nabla_{e_i}^2 \tau_f(\varphi)|^2 v_g \leq 2 \int_M \sum_{i=1}^{m} e_i(\rho)^2 |\tau_f(\varphi)|^2 v_g. \]

consider the smooth function \( \rho = \rho_R \), such that, \( \rho \leq 1 \) on \( M \), \( \rho = 1 \) on the ball \( B(x, R) \), \( \rho = 0 \) on \( M \setminus B(x, 2R) \) and \( |\text{grad}^M \rho| \leq \frac{2}{R} \). Then

\[ \frac{1}{2} \int_M \rho^2 \sum_{i=1}^{m} |\nabla_{e_i}^2 \tau_f(\varphi)|^2 v_g \leq \frac{8}{R^2} \int_M |\tau_f(\varphi)|^2 v_g. \]

Since \( E_{2, f}(\varphi) = \frac{1}{2} \int_M |\tau_f(\varphi)|^2 v_g < \infty \), when \( R \to \infty \), we obtain

\[ \frac{1}{2} \int_M \sum_{i=1}^{m} |\nabla_{e_i}^2 \tau_f(\varphi)|^2 v_g = 0, \]

therefore

\[ \nabla_{e_i}^2 \tau_f(\varphi) = 0, \]
for all $i = 1, \ldots, m$, using the Proposition 2.2, we deduce $\tau_f(\varphi) = 0$. □

3. $f$-biharmonic Submanifolds in Spheres

Let $M$ be a submanifold of $\mathbb{S}^n$ of dimension $m$, $i : M \rightarrow \mathbb{S}^n$ be the canonical inclusion and $f \in C^\infty(M)$ be a smooth positive function. We denote by $B$ the second fundamental form of the submanifold $M$, by $A$ the shape operator, by $H$ the mean curvature vector field of $M$.

**Theorem 3.1.** The map $i$ is $f$-biharmonic if and only if

$$
(m - 1)f \text{grad}^M f + 3mfA_H(\text{grad}^M f) + \frac{1}{2} \text{grad}^M(\|\text{grad}^M f\|^2)
$$

$$
- \frac{m^2}{2} f^2 \text{grad}^M(\|H\|^2) + 2mf^2 \sum_{i=1}^m A(\nabla_{e_i}^\mathbb{S}^n H)(e_i) + f \text{Ricci}^M(\text{grad}^M f)
$$

$$
+ f \text{grad}^M(\Delta^M f) + f \sum_{i=1}^m A_B(e_i, \text{grad}^M f)(e_i) = 0,
$$

where $\{e_1, \ldots, e_m\}$ be an orthonormal frame.

To prove the Theorem 3.1, we need the following lemma:

**Lemma 3.2.** Let $\Delta^\perp$ the Laplacian in the normal bundle of $M$, then

$$
\text{trace} \left( \nabla^\mathbb{S}^n \right)^2 H = -\frac{m}{2} \text{grad}^M(\|H\|^2) + 2m \sum_{i=1}^m A(\nabla_{e_i}^\mathbb{S}^n H)(e_i)
$$

$$
+ m \sum_{i=1}^m B(e_i, A_H(e_i)) + \Delta^\perp H.
$$

**Proof.** Let $\{e_1, \ldots, e_m\}$ be an orthonormal frame such that $\nabla^M e_j = 0$ at $x \in M$ for
all $i, j = 1, \ldots, m$. Then calculating at $x$

$$
\sum_{i=1}^{m} \nabla_{e_i}^{S^n} \nabla_{e_i}^{S^n} H = \sum_{i=1}^{m} \nabla_{e_i}^{S^n} \left(A_H(e_i) + \left(\nabla_{e_i}^{S^n} H \right)^\perp\right)
= \sum_{i=1}^{m} \nabla_{e_i}^{M} A_H(e_i) + \sum_{i=1}^{m} B(e_i, A_H(e_i)) \\
+ \sum_{i=1}^{m} A\left(\nabla_{e_i}^{S^n} H \right)^\perp (e_i) + \sum_{i=1}^{m} \left(\nabla_{e_i}^{S^n} \left(\nabla_{e_i}^{S^n} H \right)^\perp\right)^\perp.
$$

(3.1)

Since $\langle A_H(X), Y \rangle = -\langle B(X, Y), H \rangle$ for all $X, Y \in \Gamma(TM)$, we get

$$
\sum_{i=1}^{m} \nabla_{e_i}^{M} A_H(e_i) = \sum_{i,j=1}^{m} \langle \nabla_{e_i}^{M} A_H(e_i), e_j \rangle e_j
= \sum_{i,j=1}^{m} e_i(\langle A_H(e_i), e_j \rangle) e_j
= -\sum_{i,j=1}^{m} e_i(\langle B(e_i, e_j), H \rangle) e_j
= -\sum_{i,j=1}^{m} e_i(\langle \nabla_{e_i}^{S^n} e_i, H \rangle) e_j.
$$

Since $\nabla_{X}^{S^n} \nabla_{Y}^{S^n} = R^{S^n}(X, Y)Z + \nabla_{Y}^{S^n} \nabla_{X}^{S^n} + \nabla_{[X, Y]}^{S^n} Z$ for all $X, Y, Z \in \Gamma(TS^n)$,

$$
\sum_{i=1}^{m} \nabla_{e_i}^{M} A_H(e_i) = -\sum_{i,j=1}^{m} \langle \nabla_{e_i}^{S^n} \nabla_{e_j}^{S^n} e_i, H \rangle e_j - \sum_{i,j=1}^{m} \langle \nabla_{e_j}^{S^n} e_i, \nabla_{e_i}^{S^n} H \rangle e_j
= -\sum_{i,j=1}^{m} \langle R_{e_i}^{S^n}(e_i, e_j) e_i, H \rangle e_j - \sum_{i,j=1}^{m} \langle \nabla_{e_j}^{S^n} \nabla_{e_i}^{S^n} e_i, H \rangle e_j
- \sum_{i,j=1}^{m} \langle B(e_i, e_j), \left(\nabla_{e_i}^{S^n} H \right)^\perp \rangle e_j.
$$
Since $R^{S^n}(X,Y)Z = < Y, Z > X - < X, Z > Y$ for all $X,Y,Z \in \Gamma(TS^n)$,
\[
\sum_{i=1}^{m} \nabla^M_{e_i^j} A_H(e_i) = - \sum_{i,j=1}^{m} e_j(\langle \nabla^{S^n}_{e_i} e_i, H \rangle) e_j + \sum_{i,j=1}^{m} \langle \nabla^{S^n}_{e_i} e_i, \nabla^{S^n}_{e_j} H \rangle e_j
\]
\[
+ \sum_{i,j=1}^{m} A(\nabla^{S^n}_{e_i} H)^\perp(e_i) e_j
\]
\[
= -m \sum_{j=1}^{m} e_j(\langle H, H \rangle) e_j + m \sum_{j=1}^{m} \langle H, \nabla^{S^n}_{e_j} H \rangle e_j
\]
\[
+ \sum_{i=1}^{m} A(\nabla^{S^n}_{e_i} H)^\perp(e_i)
\]
\[
(3.2)
\]
By (3.1) and (3.2) the Lemma 3.2. follows.  

\textit{Proof of Theorem 3.1.} The $f$-tension field of $i$ is given by
\[
\tau_f(i) = f \tau(i) + dH(\nabla^M f)
\]
\[
= m f H + \nabla^M f.
\]
Let \{e_1, ..., e_m\} be an orthonormal frame such that $\nabla^M_{e_i} e_j = 0$ at $x \in M$ for all $i, j = 1, ..., m$. Then calculating at $x$
\[
\sum_{i=1}^{m} R^{S^n}(\tau_f(i), dH(e_i))dH(e_i) = m f \sum_{i=1}^{m} R^{S^n}(H, e_i)e_i
\]
\[
+ \sum_{i=1}^{m} R^{S^n}(\nabla^M f, e_i)e_i.
\]
Since $\langle H, e_i \rangle = 0$ and $R^{S^n}(X,Y)Z = < Y, Z > X - < X, Z > Y$, we get
\[
(3.3)
\sum_{i=1}^{m} R^{S^n}(\tau_f(i), dH(e_i))dH(e_i) = m^2 f H + (m - 1) \nabla^M f.
\]
We compute
\[
(3.4)
\sum_{i=1}^{m} \nabla^i e_i f \nabla^i e_i \tau_f(i) = \nabla^i \nabla^M f \tau_f(i) + f \sum_{i=1}^{m} \nabla^i e_i \nabla^i e_i \tau_f(i).\]
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The first term on the left-hand side of (3.4) is

\[
\nabla_{\text{grad}^M f}^1 f \tau_f (i) = m \nabla_{\text{grad}^M f}^1 f H + \nabla_{\text{grad}^M f}^1 f \text{grad}^M f \\
= m |\text{grad}^M f|^2 H + m f \nabla_{\text{grad}^M f}^M f H + \nabla_{\text{grad}^M f}^M f \text{grad}^M f \\
= m |\text{grad}^M f|^2 H + m f A_H(\text{grad}^M f) + m f (\nabla_{\text{grad}^M f}^M f H) \perp \\
+ \nabla_{\text{grad}^M f}^M f \text{grad}^M f + B(\text{grad}^M f, \text{grad}^M f) \\
= m |\text{grad}^M f|^2 H + m f A_H(\text{grad}^M f) + m f (\nabla_{\text{grad}^M f}^M f H) \perp \\
+ \frac{1}{2} \text{grad}^M (|\text{grad}^M f|^2) + B(\text{grad}^M f, \text{grad}^M f).
\]

(3.5)

The second term on the left-hand side of (3.4) is

\[
f \sum_{i=1}^{m} \nabla_{e_i}^1 \nabla_{e_i}^1 f \tau_f (i) = m f \sum_{i=1}^{m} \nabla_{e_i}^1 (e_i(f) H + f \nabla_{e_i}^1 H) \\
+ f \sum_{i=1}^{m} \nabla_{e_i}^1 (\nabla_{e_i}^M f + B(e_i, \text{grad}^M f)) \\
= m f (\Delta^M f) H + 2 m f \nabla_{\text{grad}^M f}^M f H \\
+ m f^2 \sum_{i=1}^{m} \nabla_{e_i}^1 \nabla_{e_i}^1 H + f \sum_{i=1}^{m} \nabla_{e_i}^M \nabla_{e_i}^1 \text{grad}^M f \\
+ f \sum_{i=1}^{m} B(e_i, \nabla_{e_i}^M f) + \sum_{i=1}^{m} A_B(e_i, \text{grad}^M f) (e_i) \\
+ f \sum_{i=1}^{m} (\nabla_{e_i}^M B(e_i, \text{grad}^M f)) \perp.
\]

(3.6)

By the Lemma 3.2 we have

\[
\sum_{i=1}^{m} \nabla_{e_i}^1 \nabla_{e_i}^1 H = \frac{m}{2} \text{grad}^M (|H|^2) + 2 \sum_{i=1}^{m} A_B(\nabla_{e_i}^M H) \perp (e_i) \\
+ \sum_{i=1}^{m} B(e_i, A_H(e_i)) + \Delta^2 H.
\]

(3.7)

From the equation

\[
\text{trace}(\nabla^M)^2 \text{grad}^M f = \text{Ricci}^M(\text{grad}^M f) + \text{grad}^M (\Delta^M f),
\]

and the formulae (3.3), (3.4), (3.5), (3.6) and (3.7), the Theorem 3.1 follows. \( \square \)
Example 3.3. We consider 

\[ M = S^m \left( \frac{1}{\sqrt{2}} \right) \times \left\{ \frac{1}{\sqrt{2}} \right\} = \left\{ \left( x_1, \ldots, x_{m+1}, \frac{1}{\sqrt{2}} \right) \in \mathbb{R}^{m+2} \mid \sum_{i=1}^{m+1} x_i^2 = \frac{1}{2} \right\}, \]

a parallel hypersphere of \( S^{m+1} \). The second fundamental form of \( M \) is

\[ B(X, Y) = \nabla d_i(X, Y) = \left\langle X, Y \right\rangle_H, \]

where \( H = -\eta \) and \( \eta = (x_1, \ldots, x_{m+1}, -\frac{1}{\sqrt{2}}) \) ([1], [9]).

From direct calculations we have

\[ A_H X = -X, \left( \nabla^m_X H \right)^\perp = 0 \text{ for all } X \in \Gamma(TM) \text{ and } \Delta H = 0. \]

Let \( f \in C^\infty(M) \) be a smooth positive function and \( \{e_1, \ldots, e_m\} \) be an orthonormal frame, then

\[ \sum_{i=1}^{m} A_{B(e_i, \nabla^M f)}(e_i) = -\nabla^M f, \quad \sum_{i=1}^{m} B(e_i, A_H(e_i)) = -m H \]

\[ \sum_{i=1}^{m} (\nabla^{m-1}_e B(e_i, \nabla^M f))^\perp = (\Delta^M f) H. \]

According to Theorem 3.1, the inclusion map \( i \) is \( f \)-biharmonic if and only if

\[
\begin{cases}
-2(m+1) f \nabla^M f + \frac{1}{2} \nabla^M (| \nabla^M f |^2) \\
+ f \text{ Ricci}^M(\nabla^M f) + f \nabla^M (\Delta^M f) = 0, \\
(m+1) | \nabla^M f |^2 + (m+2) f (\Delta^M f) = 0.
\end{cases}
\]

4. \( f \)-biharmonic Curves in \( S^3 \)

**Theorem 4.1.** Let \( \gamma : I \subset \mathbb{R} \rightarrow S^3 \) be a differentiable curve parametrized by arc length, and let \( f : I \rightarrow (0, \infty) \) be a smooth function. Then, the curve \( \gamma \) is \( f \)-biharmonic if and only if

\[
\begin{cases}
f f''' + f' f'' - 4 k^2 f f' - 3 k k' f^2 = 0, \\
3 k f f''' + 4 k' f f'' + 2 k (f')^2 + k'' f^2 - k^2 f^2 - k \tau f^2 + k f^2 = 0, \\
4 k \tau f' + 2 k' \tau f + k \tau' f = 0,
\end{cases}
\]

where \( k \) is the geodesic curvature and \( \tau \) is the geodesic torsion of \( \gamma \).
Proof. Let \( \{T, N, B\} \) be an orthonormal frame field tangent to \( S^3 \) along \( \gamma \), where \( T = d\gamma(d/dt) \) is the unit vector field tangent to \( \gamma \), \( N \) is the unit normal vector field in the direction of \( \nabla_{S^3} T \) and \( B \) is chosen so that \( \{T, N, B\} \) is a positive oriented basis. Then we have the following Frenet equations

\[
\begin{cases}
\nabla_{S^3} T = k N, \\
\nabla_{S^3} N = -k T + \tau B, \\
\nabla_{S^3} B = -\tau N.
\end{cases}
\] (4.1)

The tension field of the curve \( \gamma \) is given by

\[
\tau(\gamma) = \nabla^\gamma_{\frac{d}{dt}} d\gamma(d/dt) = \nabla_{S^3} T.
\] (4.2)

By (4.1) and (4.2), the \( f \)-tension field of the curve \( \gamma \) is given by

\[
\tau_f(\gamma) = f' T + k f N.
\] (4.3)

The curve \( \gamma \) is \( f \)-biharmonic if and only if

\[
f R^S(\tau_f(\gamma), d\gamma(d/dt)) + \nabla^\gamma_{\frac{d}{dt}} f \nabla^\gamma_{\frac{d}{dt}} \tau_f(\gamma) = 0.
\] (4.4)

By (4.3), the first term on the left-hand side of (4.4) is

\[
f R^S(\tau_f(\gamma), d\gamma(d/dt)) = k f^2 N,
\] (4.5)

here \( R^S(X,Y)Z = < Y, Z > X - < X, Z > Y \) for all \( X, Y, Z \in \Gamma(TS^3) \).

By (4.1) and (4.3), the second term on the left-hand side of (4.4) is

\[
\nabla^\gamma_{\frac{d}{dt}} f \nabla^\gamma_{\frac{d}{dt}} \tau_f(\gamma) = (f f'' + f' f'' - 4k^2 f f' - 3kk' f^2) T
\]
\[
+ (3k f f'' + 4k' f f' + 2k (f')^2
\]
\[
+ k'' f^2 - k^3 f^2 - k \tau^2 f^2) N
\]
\[
+ (4k \tau f f' + 2k' \tau f^2 + k \tau' f^2) B.
\] (4.6)

The Theorem 4.1 follows from (4.5) and (4.6). \( \square \)
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