On necessary and sufficient conditions for strong hyperbolicity in systems with constraints

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Abstract
In this work, we study constant-coefficient first order systems of partial differential equations and give necessary and sufficient conditions for those systems to have a well-posed Cauchy problem. In many physical applications, due to the presence of constraints, the number of equations in the PDE system is larger than the number of unknowns, thus the standard Kreiss conditions can not be directly applied to check whether the system admits a well-posed initial value formulation. In this work, we find necessary and sufficient conditions such that there exists a reduced set of equations, of the same dimensionality as the set of unknowns, which satisfy Kreiss conditions and so are well defined and properly behaved evolution equations. We do that by studying the systems using the Kronecker decomposition of matrix pencils and, once the conditions are met, finding specific families of reductions which render the system strongly hyperbolic. We show the power of the theory in some examples: Klein Gordon, the ADM, and the BSSN equations by writing them as first order systems, and studying their Kronecker decomposition and general reductions.

Keywords: strong hyperbolicity, evolution equations, ADM and BSSN equations, Kronecker decomposition, constraint equations, hyperbolic reductions, partial differential equations

1. Introduction

In [1] Geroch introduces a general setting for dealing with first order systems of partial differential equations. The novelty of his approach was that by keeping the description covariant, that is without choosing an evolution time nor a time-space splitting, several features of the
underlying structure of these systems became apparent: first, there is a notion of constraint equations which is well defined and does not depend on the introduction of any preferred hyper-surface, and second, there is, in general, a non-natural notion of an ‘evolution system’. Constraints are certain linear combinations of the equations in the system that satisfy some property, while evolution equations are other linear combinations which we shall call reductions; when these reductions give rise to a well-posed set of evolution equations we call them hyperbolizers. Well-posedness, the assertion that solutions depend continuously on their initial data, is a necessary condition on any physical theory to have predictability. Well-posedness, in particular, becomes crucial when trying to find numerical solutions, see for instance [2]. In this work it will be necessary to enlarge the class of allowed reductions, they would not just be multiplicative linear combinations, but we shall also allow pseudo-differential ones (keeping their degree to zero). It is in this extended class that we can find necessary and sufficient conditions for the existence of hyperbolizers. This extension arises naturally, and an extensive literature about the theory of pseudo-differential operators can be consulted [3–17], etc.

The problems in the cases with no constraints present (a well defined statement), and where the system is consistent, that is, the number of equations coincides with the number of unknowns, have been fully understood in the celebrated Kreiss Matrix theorem [18–20]. This theorem does so by stating several equivalent conditions for the system to admit a hyperbolizer, which in the general cases is some pseudo-differential operator. Once one of these hyperbolizers is found, it is used for the construction of energy estimates, which in turn are used for establishing well-posedness ([4, 21, 22]). Recently in [23] a new necessary condition was found for this particular class of systems, namely systems without constraints. It involves the use of the singular value decomposition (SVD) of the principal symbol of the system. The strength of this new condition manifests itself in the fact that, contrary to the others in Kreiss theorem, it can be applied to generic first order systems, that is, not necessarily square ones. If that condition were to fail for a first order set of equations, then there would be no reduction which would make it strongly hyperbolic. Thus, a powerful tool has been developed to easily rule out theories which fail it. In this work, we refine the condition for it to also becomes a sufficient condition.

The theory we shall develop can be quite general, but in this article, we shall restrict to linear, constant-coefficient systems. This restriction would allow us to make simpler assertions and, correspondingly, simpler proofs. The general theory will be spelled out in a more technical paper. Nevertheless, most of the material here introduced applies to generic first order quasi-linear systems.

Our approach consists of: first, choosing a hyper-surface, at each point of it, the covector normal to it allows us to transform the principal symbol into a matrix pencil, second applying the Kronecker decomposition to it, thus obtaining the intrinsic structure of the differential equations and finally building a specific reduction of the system. The Kronecker decomposition allows us to recognize in its blocks the evolution of the physically relevant fields and constraint parts of the differential equations. The blocks related to constraint propagation admit many different reductions, in particular, it is possible to build reductions with arbitrary finite constraint propagation speeds. A similar technique for the case in which the space-time is two-dimensional was used by Motloch et al [25].
In section 2 we introduce Geroch’s formalism to fix notation. We then define, in this covariant setting strong hyperbolicity, and so introduce the hyperbolizers. The definitions we introduce are such that these reductions, in more than $1+1$ dimensions, can be pseudo-differential, namely, they can depend not only on the sections of the bundle but also in the co-tangent bundle of the base manifold. For generic quasi-linear systems well-posedness, as we understand it today, needs as a sufficient condition smoothness with respect to this co-tangent bundle. Since in this work the theory is restricted to linear constant-coefficient systems, only an algebraic condition suffices, that is, no smoothness condition is needed. We finally state the main theorem of the theory. We do it in steps, first, we state a theorem asserting the equivalence of our new conditions to those of the Kreiss matrix theorem. A new feature of this new condition is that we only need to look at certain matrix pencils in a neighborhood of their generalized eigenvalues. With this tool at hand, we can easily state our main theorem for generic systems.

In section 3 we build the necessary ingredients for proving our main theorem. Essentially we look for a Kronecker decomposition of the principal symbol at certain points of the characteristic surfaces and show that the hyperbolicity condition limits the possible Kronecker blocks to only two types. Of those two blocks, one is of Jordan type. Under the condition on the angles stated in the main theorem, these Jordan blocks can only be diagonal. Once this is established a hyperbolizer can be easily constructed. We still need to prove the uniformity of our construction. For that we use the same condition, which is a uniformity condition on the angles that two kernel subspaces form between each other, to infer the uniformity needed for strong hyperbolicity. In practical applications, for a given system, there is a simple algorithm to compute such angles, so this condition is really helpful in understanding possible new theories. The general theory provided by the Kronecker decomposition allows more general types of constraints than those appearing in Geroch’s formalism. They essentially reflect the existence of constraints that contain higher-order derivatives of the fields. They appear as higher-dimensional blocks in that decomposition. Nevertheless, we have found that all of them can be readily taken care of with an appropriate reduction. Unfortunately, we do not have any physically relevant example of these types of constraints. Nature seems to prefer the lowest order ones.

Finally in section 4 we introduce some examples, the Klein Gordon, the ADM, and the BSSN equations, which illustrate the power of the theory. We finish the work with several appendices where, besides proving the new Kreiss condition, we have included preliminary material and notation.

2. The setting and the main theorem

We consider constant-coefficient first order systems of the form

$$\gamma^{\alpha \beta}_{\nu} \nabla_a \phi^{\alpha} = 0$$  \hspace{1cm} (1)

over a real manifold $M$, with $x^\alpha$ a point of $M$ and $\dim M = n + 1$. We follow the notation of [1, 23]. Here the fields $\phi^{\gamma}$ are the unknown fields and $\gamma^{\alpha \beta}_{\nu}$ is a given constant\footnote{With respect to a global flat connection.} tensor field that depends on the particular physical theory under study. They are sections on a bundle with a vector fiber which we shall denote by $\mathcal{F}_E$. Lower letters $a, b, c$ represent space-time indices, Greek indices $\alpha, \beta, \gamma$ represent field indices $|\alpha| := \dim (\alpha) = u$, and capital letters $A, B, \ldots$ represent multi-tensorial indices on the fiber space of equations $|A| := \dim (A) = e$. We shall
denote its vector space by $\Psi_L$ and consider systems that have at least the same number of equations than fields, so $e \geq u$.

We are only interested in strongly hyperbolic systems. Since they are stable under the addition of lower-order terms, in the analysis covariant derivatives can be exchanged for partial derivatives. For the same reason, we set any lower-order term to zero.

In our description we shall introduce a particular local covector field, $n_a$. It is convenient to adapt a coordinate system to it in such a way that $n_a = \nabla_a t$, where $t$ is called the time coordinate and it is a function such that its level surfaces define a local foliation of $M$ by hyper-surfaces $\Sigma_t$. Then the set of coordinates $x^a = (t, x^1, \ldots, x^n)$ define Gaussian normal coordinates adapted to this foliation. Consider the vector $t^a = (1, 0, \ldots, 0)$ such that $t^a n_a = 1$ and $t^a \partial_a = \partial_t$ and the projector $m^a_b; = \delta^a_b - t^a n_b$ (where $\delta^a_b$ is the identity map) such that $m^a_b t^b = 0$ and $m^a_b n_a = 0$. Then equation (1) could be written as

$$N^A_{\alpha} n_a \partial_t \phi^\alpha + i N^A_{\alpha} k_a \phi^\alpha = 0,$$

Notice that the term $m^a_b \partial_b$ has no temporal partial derivatives.

Since we are considering constant-coefficient problems, we can Fourier transform in spatial coordinates and reduce the system to the following equivalent system

$$N^A_{\alpha} n_a \partial_t \hat{\phi}^\alpha + i N^A_{\alpha} k_a \hat{\phi}^\alpha = 0,$$

where $k_a t^a = 0, k_a m^a_b = k_b$ and with initial data over the hyper-surface $t = 0$

$$\left. \phi^\alpha \right|_{t=0} = \hat{\phi}^\alpha_{\text{initial}} e^{ik_a x^a}.$$

Since the frequency $k_a$ in the initial data is fixed but arbitrary, we look for solutions of equation (2) for all $k_a$ not proportional to $n_a$.

In general, the equation system (2) has more equations than fields, in particular, there are $c$ linear combinations of equations without time derivatives. They are called differential constraints, for a formal and geometrical definition see [1]. We are going to restrict consideration to those systems where the number of equations satisfies $e = u + c$ where $c$ is the number of constraints. In Geroch’s terminology, they are called complete.

These constraints restrict the available initial data and for consistency, it must be shown that if initially satisfied they remain so along with the evolution. We shall not deal with this problem in this work, assuming this is so, since it involves integrability conditions which depend on lower-order terms.

While in Geroch’s formalism constraint equations are singled out, evolution equations are not. They are not unique and further structure must be introduced to single out a particular set of them. Given a particular set of evolution equations, linear combinations of constraints can be added to generate another equivalent system. They are not naturally unique. To single out a particular set we introduce a new tensor field $h^\alpha_A$ that reduces the system to a set of purely evolution equations,

$$h^\alpha_A N^A_{\alpha} n_a \partial_t \hat{\phi}^\alpha + ih^\alpha_A N^A_{\alpha} k_a \hat{\phi}^\alpha = 0.$$

This set has $u$ independent equations, as many as there are fields. We shall refer to $h^\alpha_A$ as a reduction. In general, it will depend on the wave number vector $k_a$.

We shall call system (1) strongly hyperbolic if there is a reduction such that system (3) is so, using the usual definition, namely definition 2 below.
We first need to introduce another definition, assuming that $h^\alpha_A \gamma^\nu_\omega \alpha \omega$ is invertible (a necessary condition for hyperbolicity), we define

$$A^{\alpha \gamma}\kappa_A := \left( h^\alpha_A \gamma^\nu_\omega \alpha \omega \right)^{-1} h^\alpha_A \gamma^\nu_\omega \alpha \omega k_A. \quad (4)$$

In addition, in the following definitions when we say ‘for all $k_A$’ we mean ‘all $k_A$ not proportional to $n_A$ and $|k| = 1$, with $|\cdot|$ some positive definite norm’.

**Definition 1.** System (3) is called hyperbolic if for all $k_A$, $A^{\alpha \gamma}\kappa_A$ has only real eigenvalues.

This definition means that all propagation velocities are real, so no exponential growth with frequency can be expected, although a polynomial growth is possible. This is not by itself sufficient for stability and well-posedness but it is certainly necessary. Following the Kreiss’s Matrix theorem [18, 20] we now state several necessary and sufficient conditions for strong hyperbolicity of evolution equations:

**Definition 2.** We call system (3) strongly hyperbolic if any of the following four equivalent conditions hold:

(a) System (3) is hyperbolic and $A^{\alpha \gamma}\kappa_A$ is uniformly diagonalizable: that is, for all $k_A$ there exist $S_\alpha^\gamma(k)$, and $C > 0$ such that $A^{\alpha \gamma}\kappa_A = S_\alpha^\gamma(k) \Lambda^\gamma_\eta(k) (S^{-1})^\gamma_\eta(k)$ with $\Lambda^\gamma_\eta(k)$ being diagonal and $|S(k)| |S^{-1}(k)| \leq C$.

(b) For all $k_A$ and $x \in \mathbb{C}$ with $\text{Im}(s) > 0$, there exists a constant $C > 0$ such that,

$$\left| A^{\alpha \gamma}\kappa_A - s^\alpha_\gamma \right|^{-1} \leq \frac{C}{\text{Im}(s)}. \quad (5)$$

(c) For all $k_A$, there exists a positive definite Hermitian form $H(k)_{\alpha \beta}$ and a constant $C > 0$ such that,

1. $H(k)_{\alpha \eta}A^{\alpha \gamma}\kappa_A$ is a Hermitian form, i.e. $H(k)_{\alpha \eta}A^{\alpha \gamma}\kappa_A = H(k)_{\beta \gamma}A^{\alpha \gamma}\kappa_A$.
2. $\frac{1}{2}H^0_{\beta \gamma} \geq H(k)_{\beta \gamma} \geq CH^0_{\beta \gamma} > 0$ for all $k_A$, where $H^0_{\beta \gamma}$ is a positive definite Hermitian form that does not depend on $k_A$.

(d) For all $k_A$ and $t \geq 0$, there exists $C > 0$ such that $\left| e^{i\lambda t} A^{\alpha \gamma}\kappa_A \right| \leq C$.

For real equation systems, Hermiticity has to be understood by symmetry in the corresponding indices.

The question is then: **under which circumstances do there exist reductions which make the system (3) strongly hyperbolic?** Clearly the conditions for the existence of such reductions, $h^\alpha_A$, which we shall call from now on hyperbolizers, depends only on the properties of the principal symbol, in particular on the behavior of $\Omega^{\omega
u}_\gamma(k)$ along the set of planes $S_\alpha^\gamma = \{(\lambda)s := -\lambda n_A + k_A\}$, for all $k_A$ not proportional to $n_A$ with $|k| = 1$ and $\lambda \in \mathbb{C}$. More specifically, we shall concentrate on neighborhoods of real lines ($\lambda \in \mathbb{R}$) of these planes. We suppress the complex symbol $\mathbb{C}$ from the notation, $S_\alpha^\gamma$, to refer to the set of these lines. The condition that $k_A$ is not proportional to $n_A$ implies that those planes and lines do not cross the origin for any $\lambda$. Each complex plane depends on some $k_A$ but we shall call them generically $l_\alpha(k)$ in order not to obfuscate the notation.

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3 For non-constant-coefficients systems that growth can even become exponential, see for instance [20].

4 The $(s^0_\gamma)$ tensor $H_{\gamma \eta}$ is a Hermitian form if $H_{\gamma \eta} = \overline{H_{\eta \gamma}}$. It is positive definite if $\overline{\lambda}^\gamma H_{\gamma \eta} \lambda^\eta > 0$ for all $\lambda^\gamma \in T_x M$ and for all $x \in M$. 

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Notice that if we propose a plane wave solution, \( \hat{\phi}^\alpha = \delta \phi^\alpha e^{i(-\lambda a) x^a} \) for (3), we arrive at an equation for the right kernel of the principal symbol,

\[
h_A^{ab} \hat{\phi}^{\alpha b}(\lambda) \delta \phi^\alpha = \left[ \lambda \left( -h_A^{ab} \hat{\phi}^{\alpha b} n_b \right) + \left( h_A^{ab} \hat{\phi}^{\alpha b} k_b \right) \right] \delta \phi^\alpha = 0,
\]

(6)

where \( l(\lambda)_b = -\lambda n_a + k_a \in S^C_{\alpha a} \). Here the unknowns are \( \delta \phi^\alpha \) and \( \lambda \). The completeness of these plane wave solutions is key to understanding well-posedness. Thus, we shall next study the kernels of the principal symbol \( \hat{\rho}^{\alpha b} \eta^b \) along \( S^C_{\alpha a} \). We shall call the subspace of vectors \( \delta \phi^\eta \) such that \( (\hat{\rho}^{\alpha b} \eta^b) \delta \phi^\eta = 0 \) the right kernel and the subspace of covectors \( \chi_b \) such that \( \chi_b (\hat{\rho}^{\alpha b} \eta^b) = 0 \) the left kernel of equations (5).

In analogy to our definition 1, we define hyperbolicity for the whole system equation (1). It will become clear that, in this more general case, this condition is also necessary for the hyperbolicity of any reduced system (3).

**Definition 3.** System (1) is called hyperbolic if there exists a covector field \( n_a \) such that

(a) \( \hat{\rho}^{\alpha b} n_b \) has only trivial right kernel.

(b) For each plane \( l(\lambda)_a \in S^C_{\alpha a} \), if \( \hat{\rho}^{\alpha b} l(\lambda)_b \) has a non-trivial right kernel, then \( \lambda \in \mathbb{R} \).

Condition 1, is necessary in order for the existence of \( h_A^{ab} \) such that \( h_A^{ab} \hat{\rho}^{\alpha b} n_b \) is invertible. It also implies that the dimension of the left kernel of \( \hat{\rho}^{\alpha b} n_b \) is \( c = e - u \).

Condition 2 is obviously necessary as otherwise, the exponent in the plane wave solution would be real and for some values of \( k_a \) would imply an unbounded growth.

As shown in [23] the system is hyperbolic if and only if for any fixed positive definite pair of Hermitian forms \( G_{AB} \) and \( G^{\gamma \alpha} \) in \( \Psi_L \) and \( \Phi_R \) respectively, the following polynomial equation in \( \lambda \) and \( \bar{\lambda} \)

\[
p(l(\lambda)_a) := \det \left( G^{\gamma \alpha} \hat{\rho}^{\gamma \beta} l(\lambda)_b G_{AB} \hat{\rho}^{\alpha b} l(\lambda)_b \right) = 0
\]

(7)

has only real roots. This result does not depend on the particular pair \( G_{AB} \) and \( G^{\gamma \alpha} \) of metrics used, nevertheless we shall need, to define uniformity, to choose some, constant, but otherwise arbitrary pair of these metrics.

In addition, for a given direction \( n_a \) we let the generalized eigenvalues be the elements \( \lambda_i = \lambda_i (k) \) of the set of roots of equation (7) i.e. \( p(l(\lambda)_a) = 0 \); and characteristic covectors the corresponding set of covectors \( l(\lambda)_b \in S^C_{\alpha a} \). So hyperbolicity means that equation (6) has only real generalized eigenvalues. They are the physical characteristics of the system. That is, along them the physical degrees of freedom propagate. Notice furthermore that the matrix \( A^{\alpha b} k_a \) inherits the generalized eigenvalues of the principal symbol. This matrix will, in general, have further eigenvalues, as we shall show in the following sections, they would depend on the particular reduction employed.

Our main theorem establishes which conditions on the principal symbol are necessary and sufficient for the system (1) to be strongly hyperbolic. To formulate it, we first need to introduce further notation related to the angles between vector subspaces. Introductions to these topics are given, for instance, in [26, 27], however, we give here a brief definition of these angles.

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5 Right kernel will be vectors that contract with down indices on the operator, and left kernel will be covectors that contract to up indices.
Let $V$ a vectorial space with a positive definite metric $\langle \cdot , \cdot \rangle$ and with the standard definition of the angle between two vectors $u, w \in V$ given by $\angle (u, w) = \arccos \left( \frac{\langle u, w \rangle}{\|u\| \|w\|} \right)$, with $\|u\| = \sqrt{\langle u, u \rangle}$. Let now $U, W$ two subspaces spaces of $V$, such that $p := \dim (U) \leq \dim (W)$, then the definition of the $p$ angles (or canonical angles) between $U$ and $W$ is given recursively: the first one is

$$\theta_1 = \min \{ \angle (u, w) : u \in U, w \in W, \|u\| = \|w\| = 1 \},$$

where $u_1 \in U$ and $w_1 \in W$ are some vectors realizing the minimization. The general expression for the angles is then given by

$$\theta_i = \min \{ \angle (u, w) : u \in U, w \in W, \|u\| = \|w\| = 1, \langle u, u_1 \rangle = \ldots = \langle u, u_{i-1} \rangle = 0, \langle w, w_1 \rangle = \ldots = \langle w, w_{i-1} \rangle = 0 \},$$

where again $u_i \in U$ and $w_i \in W$ realize the minimization’s. We notice that the index $i$ runs from 1 to $\dim (U)$, since the process finishes after $p$ steps. For details see [26], where an alternative and explicit form for calculating these angles is given. This alternative form will be used later (see equation (8)).

Continuing with the relevant definitions, we call $\tau_i (k)$ with $i \in F(k) = \{ 1, \ldots, w(k) \}$ the different eigenvalues of $\Lambda_{\gamma}^{\alpha} k_{\alpha}$, and $\Phi_{R}^{\tau_i (k)}$ the right vector eigen-subspace (see equation (6)) and the left covector eigen-subspace of $\Lambda_{\gamma}^{\alpha} k_{\alpha}$ respectively. Finally we call $\Upsilon_{L}^{\tau_i (k)}$ the subspace obtained by raising the index to the covectors of $\left( \Upsilon_{L}^{\tau_i (k)} \right)'$ with $G^\gamma$. Now, since $\Upsilon_{L}^{\tau_i (k)}$ and $\Phi_{R}^{\tau_i (k)}$ are subspaces of $\Phi_{R}$ and we have the positive definite metric $G^\gamma$, it is possible to define geometric angles between these subspaces that measure how close they are to each other. The number of angles is equal to the smallest dimension of the subspaces, since here $r_{\tau_i (k)} := \dim \Upsilon_{L}^{\tau_i (k)} = \dim \Phi_{R}^{\tau_i (k)}$, there are $r_{\tau_i (k)}$ angles.

With all the background given we are now in the position to give another equivalent condition to the Kreiss Matrix theorem, which is expressed in terms of the angles between the subspaces. The proof of this result is given in appendix A, and we just quote it here. A result by Strang [28] is used in the proof. This condition will allow us to prove our main theorem 2.

**Theorem 1.** System (3) is strongly hyperbolic if and only if it is hyperbolic with respect to $n_k$ and, for all $i \in F(k)$, and all $k_{\alpha}$ non proportional to some $n_k$ with $|k| = 1$, there is a constant maximum angle $\tilde{\theta} < \frac{\pi}{2}$ such that all angles between $\Upsilon_{L}^{\tau_i (k)}$ and $\Phi_{R}^{\tau_i (k)}$ are less than or equal to it.

For each $\tau_i (k)$, the cosines of these angles turn out to be the $r_{\tau_i (k)}$ singular values of the square matrix

$$\left( T_{\gamma}^{\tau_i (k)} \right)^{i} j = v^{\alpha} G_{\gamma \alpha} \delta \phi_{j}^{\gamma},$$

\[\text{(8)}\]

\[\text{6The } \tau_i (k) \text{ include the generalized eigenvalues } \lambda_i (k) \text{ and the other eigenvalues obtained after the reduction takes place. Since all quantities depend on } k_{\alpha} \text{ we explicitly put that dependence.}\]

\[\text{7From now on, we denote the dual space associated to some space with the } \prime \text{ symbol.}\]
where \( \{ \psi^{\alpha} \in \mathcal{Y}_L^{(k)} \} \) and \( \{ \delta \phi_j \} \in \Phi_R^{(k)} \), with \( i, j \in F(k) \), are orthonormal bases of the corresponding subspaces. Thus, these angles can be easily computed in examples.

We turn now to the main result of this work, obtaining necessary and sufficient conditions for the existence of a well-posed reduction. To that end, we shall use our previous theorem as a guide. We shall consider the right and left kernel of the complete (previous to any reduction) principal symbol, project them, and obtain a condition between the angles of these subspaces.

We fix an \( n_a \) for which the system is hyperbolic and consider the line \( l(\lambda) \subset S_{n_a} \). As before, the number and the geometric multiplicity (i.e. the dimension of the right kernel) of the generalized eigenvalues depends on \( k_a \). For each \( k_a \) we shall call \( \lambda_i (k) \) with \( i \in D(k):= \{ 1, 2, \ldots, q(k) \} \) the different generalized eigenvalues and \( r_{\lambda_i (k)} \), the geometric multiplicity of the corresponding \( \lambda_i (k) \). Notice that in the quasi-linear case \( \lambda_i (k), q(k) \) and \( r_{\lambda_i (k)} \) could depend on the space-time points \( x \) and the fields \( \phi^\alpha(\tau) \), but since we are considering the constant-coefficients case this dependence does not appear.

At each of the generalized eigenvalues, \( \lambda_i (k) \), we have a left and right kernel of the principal part. We shall call then \( \Psi_L^{(\lambda_i (k))} \) and \( \Phi_R^{(\lambda_i (k))} \). They have dimensions \( r_{\lambda_i (k)} + e - u \) and \( r_{\lambda_i (k)} \), respectively. Using \( G^{\alpha\nu} \), we can map \( \Psi_L^{(\lambda_i (k))} \) into \( \Phi_R^{(k)} \) (the dual space of \( \Phi_R \)) and call it \( \Phi_L^{(\lambda_i (k))} \). It is not possible to know, in a generic way, its dimension. We only know that the left kernel of \( G^{\alpha\nu} \) is \( e - u \), thus we can bound the dimension as \( r_{\lambda_i (k)} \leq \dim (\Phi_L^{(\lambda_i (k))})' \leq e - u + r_{\lambda_i (k)} \).

These concepts allow us to state the result of [23] in the simple equivalent form

\[
\left( \Phi_L^{(\lambda_i (k))} \right)' \mid_{\Phi_R^{(\lambda_i (k))}} = \left( \Phi_L^{(\lambda_i (k))} \right)'.
\]

Consider now the subspace obtained by raising the index of the elements of \( \left( \Phi_L^{(\lambda_i (k))} \right)' \) with \( G^{\nu\nu} \), and calling that subspace \( \Phi_L^{(\lambda_i (k))} \) and \( \Phi_R^{(\lambda_i (k))} \), and so we can define the angles between \( \Phi_L^{(\lambda_i (k))} \) and \( \Phi_R^{(\lambda_i (k))} \). For each \( \lambda_i (k) \), \( k_a \) we shall call \( \theta^{(\lambda_i (k))} \) with \( j \in I_{\lambda_i (k)} := \{ 1, \ldots, r_{\lambda_i (k)} \} \) these angles, they are geometric quantities and the answer to our problem is given in terms of them. We are now in position to formulate our theorem:

**Theorem 2.** The constant-coefficient system (1) is strongly hyperbolic (admits at least one hyperbolizer) if and only if it is hyperbolic with respect to some direction \( n_a \) and, for all \( i \in D(k) \) and all \( k_a \) non proportional to \( n_a \), with \( |k| = 1 \), there is a constant maximum angle \( \vartheta < \frac{\pi}{2} \) between \( \Phi_L^{(\lambda_i (k))} \) and \( \Phi_R^{(\lambda_i (k))} \).

It is possible to show that the necessary condition (9) implies that for each \( k_a \) there exists a metric such that all the angles between \( \Phi_L^{(\lambda_i (k))} \) and \( \Phi_R^{(\lambda_i (k))} \) vanish. But this is not sufficient for the theorem, since we need a global metric (independent on \( k \)) such that a lower-order bound (the existence of \( \vartheta \)) exists. If we write it in terms of the cosine of the angles, we obtain:

\[
\min_{i \in D(k) \cup I_{\lambda_i (k)}, |I_{\lambda_i (k)}| = 1} \cos \theta^{(\lambda_i (k))}_j \geq \cos \vartheta > 0
\]

for all \( k_a \) non proportional to \( n_a \).

The angles are computed in the same way as before, using orthonormal bases of the spaces \( \Phi_L^{(\lambda_i (k))} \) and \( \Phi_R^{(\lambda_i (k))} \) and building a new matrix \( (T^{\nu\gamma})^j_i \) resulting from contracting these bases with the metric. Notice that this matrix will in general be rectangular, since any basis of \( \Phi_L^{(\lambda_i (k))} \) has \( r_{\lambda_i (k)} \) or more vectors.

Assuming that the conditions of theorem 2 hold, we shall show how to build reductions \( h^\alpha_A \) in such a way that the degeneracy of the generalized eigenvalues does not change, and the new
eigenvalues introduced by $h^\alpha_A$ can be chosen to be simple and different to the ones of the whole system. These reductions will comply with the hypothesis of theorem 1 and so conclude the proof.

3. Theorem proof

The proof of the theorem splits into five subsections. First, we introduce, in section 3.1, the Kronecker decomposition of pencils, which will be applied to the principal symbol and later used to build certain reductions. Next, in section 3.2, after introducing notation for the bases of the corresponding subspaces of left and right kernels of the principal symbol, we prove lemma 1 which connects the hypothesis in our theorem with the necessary condition in [23]. In section 3.3, we prove lemma 2 which gives a set of equivalent necessary and sufficient conditions for the existence of reductions $h^\alpha_A$ which gives diagonalizable $A^\alpha_k$, and explicitly display all possible reductions. In section 3.4 we complete the proof by showing, in lemma 3, that it is possible to find reductions such that the extra eigenvalues of $A^\alpha_k$, the propagation velocities of the constraints, are simple and different to the physical propagation velocities. Finally in section 3.5, we further restrict the reductions, $h^\alpha_A$, so that the angles between $\Phi_{\lambda}^{\lambda (k)}$, $\Phi_R^{\lambda (k)}$, and $\Psi_L^{\lambda (k)}$, $\Phi_R^{\lambda (k)}$ are equal. Applying theorem 1 to the reduced system we conclude the proof.

3.1. Kronecker decomposition of pencils

Consider now the principal symbol matrix pencil equation

$$\Omega = \lambda (\Omega + \Omega_k)$$

for fixed $n_a$ and $k_a$ in the line $l(\lambda) \in S_{n_a}$. The intrinsic structure of this matrix pencil, at each one of these points, determines the strong hyperbolicity of the system. This structure will become apparent via the Kronecker decomposition [29, 30]. It consists of a change of basis for the field and equation spaces, which depend on $k_a$, $n_a$, but are independent of the parameter $\lambda$.

The new basis transforms the symbol (11) into simple blocks. In this form, the study of right and left kernel becomes easy.

Since the hyperbolicity condition restricts $\Omega$ to have only trivial right kernel, the allowed blocks of the Kronecker decomposition of (11) simplifies into just two types of possible blocks, namely:

(a) $J_m(\lambda_i)$-Jordan blocks:

$$J_m(\lambda_i) = \begin{pmatrix} \lambda - \lambda_i & 1 & 0 & 0 \\ 0 & \lambda - \lambda_i & \ldots & 0 \\ 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \lambda - \lambda_i \end{pmatrix} \in \mathbb{C}^{m \times m}$$

with $\lambda_i$ the generalized eigenvalues introduced in section 2;

---

8 Actually, since the symbol is linear in $k_a$ the basis can only depend on the direction of $k_a$ and not on its magnitude, that is the reason why we only consider lines where $k_a$ is of unit length for some arbitrary metric.
The $L^T_0$ cases are vanishing rows.

As an example

$$N^{\lambda_1}_{\eta_1}(\lambda) = \begin{pmatrix} J_1 (\lambda_1) & 0 & 0 & 0 \\ 0 & J_3 (\lambda_2) & 0 & 0 \\ 0 & 0 & L_2 & 0 \\ 0 & 0 & 0 & L_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathbb{C}^{11 \times 7}. \quad (12)$$

In general the Kronecker decomposition includes other blocks (see appendix B), which do not appear here.

The Kronecker structure of a particular symbol is unique, however, in general, there will exist different bases that lead to it. In appendix C we shall show how to find the different blocks of $N^{\lambda_1}_{\eta_1}(\lambda)$. It is important to notice that most physical systems have only $L^T_1$-blocks and $L^T_0$-rows. That is the case for instance for Maxwell electrodynamics, non-linear electrodynamics, force-free electrodynamics, ideal, charged and conformal fluids, etc. Indeed, all constraints in Geroch’s sense [1] are related just to $L^T_0$-rows and $L^T_1$-blocks.

### 3.2. Necessary condition for strong hyperbolicity

Let $l(\lambda) = -\lambda n_a + k_a \in S_{n_a}$, the Kronecker decomposition asserts that the subspace $\Psi_L$ of left kernel of the principal symbol $N^{Bb}_{\eta}(\lambda)$ is expanded by a set of $e - u$ unique vectors $\chi^s_\lambda$ where $s \in C_\chi = \{1, \ldots, e - u\}$ for any $\lambda$, and it increases when $\lambda = \lambda_i(k)$. We shall choose a set of arbitrary $r_\lambda(k)$ new covectors $(\upsilon^l_\lambda)_A$ with $l \in I_\lambda(k)$ to complete a basis for $\Psi_L^{\lambda_i}$. We shall refer to $(\upsilon^l_\lambda)_A$ as the generalized eigencovectors. Thus, for each $\lambda = \lambda_i(k)$,

$$\Psi^{\lambda_i}_L = \text{span} \{ (\chi^s_\lambda)_A, (\upsilon^l_\lambda)_A \} \quad \text{with } s \in C_\chi \text{ and } l \in I_\lambda(k) \quad (13)$$

where $(\chi^s_\lambda)_A \equiv \chi^s_\lambda(\lambda_i(k))$.

On the other hand, $N^{Bb}_{\eta}(\lambda)$ has non-trivial right kernel when $\lambda = \lambda_i(k)$, this subspace $\Phi^{\lambda_i}_R$ is expanded by

$$\Phi^{\lambda_i}_R = \text{span} \{ (\delta \phi^j_\lambda)_A^s \} \quad \text{with } j \in I_\lambda(k). \quad (14)$$

Notice that $\dim \Psi^{\lambda_i}_L = e - u + r_\lambda(k)$ and $\dim \Phi^{\lambda_i}_R = r_\lambda(k)$.

As in [23] we shall now look at the singular value decomposition (SVD) of the principal symbol at $\lambda = \lambda_i$. For this we introduce two scalar products, the positive definite Hermitian
forms $G_{AB}$ and $G_{\alpha \gamma}$ on each of the spaces. From the discussion above there are $r_{\lambda, (k)}$ vanishing singular values

$$\sigma_{u+1-j} [\gamma^{\lambda \alpha} l(\lambda_s (k))_u] = 0,$$

with $j \in I_{\lambda, (k)}$.

Consider now the extended two-parameter line $I_{\lambda, (\theta)} = -\varepsilon e^{i\theta} n_u + l(\lambda)_u$ with $\varepsilon$ real, $\theta \in [0,2\pi]$, $0 < |\varepsilon| < 1$ and with $l(\lambda)_u \in S_{n_u}$. Then, as it is shown in [31, 32], the perturbed singular values (singular values of the perturbed principal symbol) are expanded in the following way:

$$\sigma_{u+1-j} [\gamma^{\lambda \alpha} l_{\lambda, (\theta)}(\lambda_s (k))_u] = (\rho_{\lambda, (k)})_j \varepsilon + O(\varepsilon^2), \quad (15)$$

where $(\rho_{\lambda, (k)})_j$ are the singular values of the matrix

$$(R^{\lambda (k)})^l_m = \left( \begin{array}{c} \tilde{e}_l^{(k)} \cr \tilde{e}_l^{(k)} \cr \end{array} \right)_A \gamma^{\lambda \alpha} n_a^{(k)} \left( \begin{array}{c} \tilde{e}_m^{(k)} \cr \tilde{e}_m^{(k)} \cr \end{array} \right)^\alpha$$

with $I = (l,s)$, and where $\{(\tilde{e}_l^{(k)})_A, (\tilde{e}_m^{(k)})_A \}$ is an orthonormalized basis with respect to the metric $G^{\lambda \alpha}$, of $\Psi^{\lambda (k)}_L$, while $\{(\tilde{e}_m^{(k)})_A \}$ is an orthonormalized basis with respect to the metric $G_{\alpha \gamma}$, of $\Phi^{\lambda (k)}_L$.

In [23] was shown that a necessary condition for the existence of a reduction is that the singular values of $\gamma^{\lambda \alpha} l_{\lambda, (\theta)}(\lambda_s (k))_u$ are either of order $O(\varepsilon^1)$ or $O(\varepsilon^0)$ which is equivalent to the condition that none of the singular values $(\rho_{\lambda, (k)})_j$ of $R^{\lambda (k)}$ vanish. In the following lemma we shall prove that condition (10) in theorem 2 implies that $(\rho_{\lambda, (k)})_j > 0$ and so that the necessary condition in [23] holds.

**Lemma 1.** For all $i \in D_{\lambda, (k)}$, $j \in I_{\lambda, (k)}$, with $k_a$ not proportional to $n_u$ and $|k| = 1$, if equation (10) holds, then $(\rho_{\lambda, (k)})_j > 0$.

**Proof.** Recalling that $\Phi^{\lambda (k)}_L$ is the map of $\Psi^{\lambda (k)}_L$ into $\Phi_R$ by $\gamma^{\lambda \alpha} n_a G^{\alpha \gamma}$, a particular set that spans this subspace is

$$\Phi^{\lambda (k)}_L = \text{span}\left\{ (\tilde{e}_l^{(k)})^\gamma, (\tilde{e}_m^{(k)})^\gamma \right\} \quad \text{with} \ s \in C_{\lambda} \ \text{and} \ l \in I_{\lambda, (k)}. \quad (16)$$

Notice that there might be linearly dependent vectors among the $(\tilde{e}_l^{(k)})^\gamma$ and $(\tilde{e}_m^{(k)})^\gamma$. However, consider first the case that they are linearly independent. To calculate the angles $\theta^{(k)}_{\lambda}$ between $\Phi^{\lambda (k)}_L$ and $\Phi^{\lambda (k)}_R$ we need to use orthonormalized bases on these subspaces. Calling $Q'$, with $J = (s,l)$ and $I = (m,n)$, the square matrix that connects the bases $\{(\tilde{e}_l^{(k)})^\gamma, (\tilde{e}_m^{(k)})^\gamma\}$ with a new orthonormalized basis $\{(\tilde{e}_l^{(k)})^\gamma, (\tilde{e}_m^{(k)})^\gamma\}$, in the metric $G_{\alpha \gamma}$, of $\Phi^{\lambda (k)}_L$,

$$\left( \begin{array}{c} (\tilde{e}_l^{(k)})^\gamma \\
(\tilde{e}_m^{(k)})^\gamma \
\end{array} \right) = Q' \left( \begin{array}{c} (\tilde{e}_l^{(k)})^\gamma \\
(\tilde{e}_m^{(k)})^\gamma \
\end{array} \right).$$

$^9$ Recall that the singular values are ordered in such a way that $\sigma_1 [\gamma^{\lambda \alpha} n_a (\lambda_s (k))_u] \geq \sigma_2 [\gamma^{\lambda \alpha} n_a (\lambda_s (k))_u] \geq \ldots \geq \sigma_n [\gamma^{\lambda \alpha} n_a (\lambda_s (k))_u].$
The cosines of the angles $\theta_{k,j}^{L(k)}$ between $\Phi_{L}^{\lambda(k)}$ and $\Phi_{R}^{\lambda(k)}$ are the singular values of the matrix

$$
(T^{\lambda(k)})_{ij} = \left( \begin{array}{c}
\tilde{\chi}_{j,A}^{k} \\
\tilde{\psi}_{j,A}
\end{array} \right) G_{\gamma l} \left( \delta \phi_{m}^{\lambda} \right) \eta = Q'_{ij} \left( R^{\lambda(k)} \right)_{ij}
$$

(17)

Since the singular values of $T^{\lambda(k)}$ do not vanish by hypothesis (equation (10)), then $R^{\lambda(k)}$ has only trivial right kernel and its singular values can not vanish. In case the of $\left( \tilde{\chi}_{j,A}^{k} \right)$ and $\left( \tilde{\psi}_{j,A} \right)$ being linear dependent, some of them shall be removed until obtaining a basis for defining $\left( T^{\lambda(k)} \right)_{ij}$. Next, there exists a rectangular matrix $Q'_{ij}$ such that equation (17) holds and the conclusion is the same. Thus, we conclude the proof of the lemma.

3.3. Building reductions

In this subsection, we prove a lemma which gives a set of equivalent conditions and shows how to build, using the Kronecker decomposition of the principal part, the general reduction $h_{j}^{\lambda}$ giving a diagonalizable reduced matrix. It is important to notice that if any $J_{n}$ Jordan block with $m \geq 2$ appears in the Kronecker decomposition, then the system is intrinsically weakly hyperbolic. However, if these blocks do not appear, this condition is not sufficient for strong hyperbolicity since two problems can be present. The first one is: a reduction can introduce a $J_{m}$ Jordan block with $m \geq 2$, from a $L^{T}$ block. This $L^{T}$ block will be associated with the constraints propagation, reducing the system to a weakly hyperbolic one. This would give an ill-posed subsidiary system for the constraint propagation. The second one is: a reduction for which $A_{\eta}^{\alpha} k_{\eta}$ is diagonalizable, but not uniformly diagonalizable, then the systems will also be ill-posed. To solve these problems we shall use, in the next subsections, the results of lemma 3 and the lower bound condition equation (10).

**Lemma 2.** Let system (1) be hyperbolic for $n_{\eta}$, then the following conditions are equivalent:

for each line $l(\lambda) = -\lambda n_{\eta} + k_{\eta}$ in $S_{n_{\eta}}$

(a) There exists a reduction $h_{\lambda}$, homogeneous of degree 0 in $k_{\eta}$, such that $A_{\eta}^{\alpha} k_{\eta}$ is diagonalizable.

(b) The Kronecker decomposition of the principal symbol pencil (11) has all its Jordan blocks of dimension 1.

(c) The singular value decomposition of the principal symbol pencil $\mathcal{R}_{\alpha}^{\alpha_{\lambda}}(\alpha, \beta)$ along any extended line $l(\alpha, \beta)$ has only singular values of orders $O\left( |\epsilon|^{0} \right)$ and $O\left( |\epsilon|^{i} \right)$ i.e. $\rho_{\alpha}^{\beta}(\text{l}) > 0$ for all $\lambda \in D_{\alpha}$, and $j \in L_{\alpha}(\text{k})$.

**Proof.** The Kronecker decomposition of the principal symbol (see [29, 30]) is:

$$
\mathcal{R}_{\alpha}^{\alpha_{\lambda}}(\alpha, \beta) = \lambda \left( -\mathcal{R}_{\alpha}^{\alpha_{\lambda}} n_{\beta} \right) + \left( \mathcal{R}_{\alpha}^{\alpha_{\lambda}} k_{\beta} \right),
$$

$$
= Y^{A_{\beta}}(x, \phi, n, k) K_{\alpha}^{B_{\lambda}}(\alpha) W^{\alpha_{\lambda}}(x, \phi, n, k),
$$

(18)

where $l(\lambda) = -\lambda n_{\beta} + k_{\beta}$ in $S_{n_{\beta}}$, $Y, W$ are invertible operators and

$$
K_{\alpha}^{B_{\lambda}}(\lambda) := \lambda I_{\alpha}^{B_{\lambda}} + M_{\alpha}^{B_{\lambda}},
$$

is the Kronecker matrix. It is a block matrix with $J_{m}(\lambda)$-Jordan blocks, $L_{\alpha}^{T}$ blocks and trivial $L_{\alpha}^{T}$-rows. The operators $I$ and $M$ are unique but in general they could change for different values.
of \((n,k)\)\(^{10}\) although that does not seem to occur for the standard physical examples. However, in general \(Y\) and \(W\) could be chosen in different ways even at the same point. Notice that,

\[-\mathcal{GR}^{\alpha \beta}_{\eta} n_{\beta} = Y^{A}_{B} f^{B}_{\alpha} W^{\alpha}_{\eta} \quad \text{and} \quad \mathcal{GR}^{\alpha \beta}_{\eta} k_{\beta} = Y^{A}_{B} M^{B}_{\alpha} W^{\alpha}_{\eta},\]

(b) \Rightarrow (a) We propose the following ansatz for a reduction:

\[h^{\mu}_{C} = s^{\alpha \beta} W^{\alpha}_{\alpha} H_{\alpha C} (Y^{-1})_{D}^{C},\]

with \(S\) being any invertible bilinear form and \(H\) another one that depends on the explicit form of \(K\), and which will be given explicitly later on. With this ansatz the reduced system simplifies to:

\[h^{\mu}_{C} \mathcal{GR}^{\alpha \beta}_{\eta}(\lambda)_{\beta} = \lambda \left(s^{\alpha \beta} W^{\alpha}_{\alpha} H_{\alpha C} \gamma_{\eta} (Y^{-1})_{D}^{C} + s^{\alpha \beta} W^{\alpha}_{\alpha} H_{\alpha C} M^{C}_{\alpha} W^{\alpha}_{\eta}\right).\]

Thus, assuming for a moment that:

(a) \(H_{\alpha C} F^{\alpha}_{\eta}\) is a positive definite Hermitian form and
(b) \(H_{\alpha C} M^{C}_{\alpha}\) is a Hermitian form,

we conclude that \(W^{\alpha}_{\alpha} H_{\alpha C} F^{\alpha}_{\eta}\) and \(W^{\alpha}_{\alpha} H_{\alpha C} M^{C}_{\alpha} W^{\alpha}_{\eta}\) are Hermitian forms, the first being positive definite. From this and recalling the definition of \(A^{\alpha \beta}_{\eta} k_{\alpha}\) in equation (4), we have

\[A^{\alpha \beta}_{\eta} k_{\alpha} = -\left((SW^{\alpha}_{\eta} H_{\alpha C} F^{\alpha}_{\eta})^{-1}\right)^{\nu}_{\tau} \left(s^{\alpha \beta} W^{\alpha}_{\alpha} H_{\alpha C} \gamma_{\eta} (Y^{-1})_{D}^{C} + s^{\alpha \beta} W^{\alpha}_{\alpha} H_{\alpha C} M^{C}_{\alpha} W^{\alpha}_{\eta}\right),\]

(20)

where \(\left((SW^{\alpha}_{\eta} H_{\alpha C} F^{\alpha}_{\eta})^{-1}\right)^{\nu}_{\tau} = \left(s^{\alpha \beta} W^{\alpha}_{\alpha} H_{\alpha C} \gamma_{\eta} (Y^{-1})_{D}^{C} + s^{\alpha \beta} W^{\alpha}_{\alpha} H_{\alpha C} M^{C}_{\alpha} W^{\alpha}_{\eta}\right)^{-1}\nu}_{\tau}. \]

Thus, \(A^{\alpha \beta}_{\eta} k_{\alpha}\) is Hermitizable (or symmetrizable), and therefore diagonalizable. Furthermore it has only real eigenvalues. Notice that \(S\) introduces more degrees of freedom in \(h_{\mu C}\) that can be chosen arbitrarily as long as \(S\) is invertible. But they turn out not to be relevant, since they do not appear in \(A^{\alpha \beta}_{\eta} k_{\alpha}\). Thus, if we find \(H\) satisfying (a) and (b) the implication (b) \Rightarrow (a) will be proven. Using the hypothesis (b) in the theorem we shall conclude (a) by building \(H_{\alpha C}\). We shall propose a specific\(^{11}\) \(H\) for each block of \(K^{A}_{\eta}(\lambda)\). Consider first the Jordan blocks. The \(J_{n_{a}}(\lambda_{a})\)-Jordan blocks have kernel when \(\lambda\) is equal to the generalized eigenvalues \(\lambda_{a}\), then hyperbolicity with respect to \(n_{a}\) implies that these eigenvalues are real. In addition, condition (b) implies these blocks are one-dimensional, therefore in \(K^{A}_{\eta}(\lambda)\) there appear \(m \times m\) identity blocks multiplied by \((\lambda - \lambda_{a})\) that we call \(I_{m_{a}}(\lambda - \lambda_{a})\). For these blocks we choose \(H_{I_{m_{a}}}\) to be any positive definite Hermitian form of size \(m \times m\). We now turn to the generic \((I_{m_{i}}^{s})^{j}\) blocks with \(i = 1, \ldots, m + 1\) and \(j = 1, \ldots, m\). We propose a particular \((H_{I_{m_{i}}}^{s})^{j}\) with \(s = 1, \ldots, m\) of the following form:

\[
\begin{pmatrix}
  (H_{I_{m_{i}}}^{s})^{j}_{ii} & (I_{m_{j}}^{T})^{j}_{i}
\end{pmatrix}
= \begin{pmatrix}
  a_{1} & a_{2} & \ldots & a_{m} & a_{m+1} \\
  a_{2} & \ldots & a_{m} & a_{m+1} & a_{m+2} \\
  \vdots & \ddots & \ddots & \ddots & \ddots \\
  a_{m} & a_{m+1} & a_{m+2} & \ldots & a_{2m}
\end{pmatrix}
\begin{pmatrix}
  \lambda & 0 & 0 & 0 \\
  1 & \lambda & 0 & 0 \\
  0 & 1 & \ldots & 0 \\
  0 & 0 & \ldots & \lambda \\
  0 & 0 & 0 & 1
\end{pmatrix}
\]

(21)

\(^{10}\)(\(x, \phi, n, k\)) in the quasilinear case.

\(^{11}\)The most general form of \(H\), with a not necessarily diagonalizable reduced principal symbol, is presented in the ADM example.
with all components real. Notice that $L^T_m$ can be split into $L^T_m = (\lambda L^T_m + M^T_{lm})$ with

$$I^T_m = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad M^T_{lm} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

then

$$\left( H^T_{lm} \right)^j_i \left( L^T_m \right)^j_i (\lambda) = \begin{pmatrix} a_1 & a_2 & \ldots & a_m & a_{m+1} \\ a_2 & \ldots & a_m & a_{m+1} & a_{m+2} \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ a_{m-1} & a_{m-2} & \ldots & a_m & a_{m+1} \\ a_m & a_{m+1} & \ldots & a_{m+2} & a_{2m} \end{pmatrix} \begin{pmatrix} \lambda & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix},$$

$$= \lambda \left( g_m \right)_{\delta \alpha} + \left( l_m \right)_{\delta \alpha}$$

with

$$\left( g_m \right)_{\delta \alpha} = \begin{pmatrix} a_1 & a_2 & \ldots & a_m \\ a_2 & \ldots & a_m & a_{m+1} \\ \vdots & \ddots & \ddots & \ddots \\ a_{m-1} & a_{m-2} & \ldots & a_m \\ a_m & a_{m+1} & \ldots & a_{2m-1} \end{pmatrix} \quad \text{and} \quad \left( l_m \right)_{\delta \alpha} = \begin{pmatrix} a_2 & \ldots & a_m & a_{m+1} \\ \ldots & a_m & a_{m+1} & a_{m+2} \\ a_m & a_{m+1} & a_{m+2} & \ldots \\ \ldots & a_{2m-1} & a_{2m} \end{pmatrix}.$$  \hspace{1cm} (22)

Notice the cascade form of $H^T_{lm}$ and that both, $g_m$, and $l_m$ are symmetric. It is not so difficult to see that this is the most general form of $H^T_{lm}$ fulfilling that condition. To satisfy conditions (a) and (b) we only need to show that $g_m$ can be made to be positive definite by choosing appropriately the coefficients $a_i$ with $i = 1, \ldots, 2m - 1$. We do this by induction in $m$. When $m = 1$ the positivity condition is just $a_1 > 0$. Assuming the inductive hypothesis: $g_m$ is positive definite (as in equation (22)), we enlarge the Hermitian form to $g_{m+1}$ by adding a new column and a new row

$$\left( g_{m+1} \right)_{\delta \alpha} = \begin{pmatrix} a_1 & a_2 & \ldots & a_m & a_{m+1} \\ a_2 & \ldots & a_m & a_{m+1} & a_{m+2} \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ a_{m-1} & a_{m-2} & \ldots & a_m & a_{m+1} \\ a_m & a_{m+1} & \ldots & a_{2m-1} & a_{2m} \end{pmatrix}.$$  \hspace{1cm} (22)

Thus there appear just two new coefficients $a_{2m}$ and $a_{2m+1}$ (they are the only new coefficients in $g_{m+1}$ that are not in $g_m$). We need to show that there exists a possible choice of these coefficients such that $g_{m+1}$ is positive definite. Since $g_m$ as in equation (22) is positive definite, by Sylvester’s criterion, we only need to show that $\det (g_{m+1}) > 0$. Expanding the determinant along the last column, the condition becomes:

$$\det (g_{m+1}) = a_{2m+1} \det (g_m) + f (a_1, \ldots, a_{2m}) > 0$$
for some function \( f \) that does not depend on \( a_{2m+1} \). Thus, choosing any \( a_{2m} \), \( f(a_1, \ldots, a_{2m}) \) becomes known, and since \( \det (g_m) > 0 \) we just need to take \( a_{2m+1} \) so that,

\[
a_{2m+1} > \frac{f(a_1, \ldots, a_{2m})}{\det (g_m)},
\]

to obtain a positive definite \( g_{m+1} \). Finally, for the vanishing rows of \( K^A_{\eta} (\lambda) \), the \( L^T_m \) rows, we choose for \( H \) arbitrary columns. They do not seem to play any role. The resulting structure for \( H \) becomes as shown in the following example:

\[
(H)_{\lambda\lambda} K^A_{\eta} (\lambda) = \begin{pmatrix}
H_{tailx} & 0 & 0 & 0 & A_1 & B_1
0 & H_{tailx} & 0 & 0 & A_2 & B_2
0 & 0 & H_{T^2} & 0 & A_3 & B_3
0 & 0 & 0 & H_{T^1} & A_4 & B_4
\end{pmatrix}
\]

\[
\times \begin{pmatrix}
\text{Id}_{m_3} (\lambda - \lambda_1) & 0 & 0 & 0
0 & \text{Id}_{m_4} (\lambda - \lambda_2) & 0 & 0
0 & 0 & H_{T^2} & 0
0 & 0 & 0 & H_{T^1}
0 & 0 & 0 & 0
0 & 0 & 0 & 0
\end{pmatrix},
\]

\[
= \begin{pmatrix}
H_{tailx} & 0 & 0 & 0 & A_1 & B_1
0 & H_{tailx} & 0 & 0 & A_2 & B_2
0 & 0 & H_{T^2} & 0 & A_3 & B_3
0 & 0 & 0 & H_{T^1} & A_4 & B_4
\end{pmatrix},
\]

\[
\times \begin{pmatrix}
\text{Id}_{m_3} (\lambda - \lambda_1) & 0 & 0 & 0
0 & \text{Id}_{m_4} (\lambda - \lambda_2) & 0 & 0
0 & 0 & H_{T^2} & 0
0 & 0 & 0 & H_{T^1}
0 & 0 & 0 & 0
0 & 0 & 0 & 0
\end{pmatrix},
\]

\[
= \begin{pmatrix}
H_{tailx} & 0 & 0 & 0 & A_1 & B_1
0 & H_{tailx} & 0 & 0 & A_2 & B_2
0 & 0 & H_{T^2} & 0 & A_3 & B_3
0 & 0 & 0 & H_{T^1} & A_4 & B_4
\end{pmatrix},
\]

\[
(23)
\]

with \( H_{T^1}, H_{T^2} \) as in equation (21). The \( H_{tailx}, H_{tailx} \) blocks are arbitrary positive definite Hermitian forms, and \( A_i, B_i \) for \( i = 1, 2, 3, 4 \) are arbitrary blocks. Notice that this example satisfies conditions (a) and (b). As we mentioned before, in the Geroch formalism, which incorporates most physical examples, there are no \( L^T_m \) blocks with \( m \geq 2 \). Only \( L^T_1 \) and \( L^T_0 \)-rows appear. These kind of blocks allow the introduction of other reductions linking different \( L^T_1 \) blocks. This particular case will be discussed in appendix D.

(b) \( \leftrightarrow \) (c) Since the order of the perturbed singular values are invariant under change of bases and choice of scalar products [23], we shall make some fixed, arbitrary, choice for them to make the calculation concrete. However, the result will be general. We choose bases for which \( \mathcal{N}^b (\lambda) = K^B_{\eta} (\lambda) \), and consider, in that bases, the positive definite Hermitian forms \( G_{AB} = (1, \ldots, 1) \) and \( G_{\alpha\gamma} = (1, \ldots, 1) \) to define the adjoint operator. With this choice, the computation of the singular values decouples into blocks. Thus we only need to check the form of the perturbed singular values for each of the \( J_l (\lambda) \) and \( L^T_m \) blocks. The \( l \)-dimensional Jordan block has perturbed singular values of order \( O (\varepsilon^l) \) (see [23]) therefore the perturbed singular values are order \( O (\varepsilon^0) \) if and only if the Jordan blocks are one-dimensional. On the other hand, the \( L^T_m \)-blocks have only trivial right kernel for any \( \lambda \), therefore their perturbed singular values are order \( O (\varepsilon^0) \) (non vanishing). Indeed, using the determinant relation in the SVD, we get:

\[
\det \left( \left( (L^T_m)^* \right)_i \left( (L^T_m) \right)_j \right) = \sigma_1^2 \left[ L^T_m \right] \ldots \sigma_m^2 \left[ L^T_m \right],
\]

\[
= \left( |\lambda|^2 \right)^{m-1} + \left( |\lambda|^2 \right)^{m-2} + \ldots + |\lambda|^2 + 1 > 0,
\]
where the $\sigma_j [L^T_{m}]$ are the singular values of $L^T_m$. Thus they never vanish, and when perturbed there is always a neighborhood in which they remain positive. Thus they are order $O(\varepsilon^0)$. As an example

$$\det\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \end{pmatrix} = (|\lambda|^2)^2 + (|\lambda|^2) + 1.$$  

(a) $\Rightarrow$ (c) This implication was established in work [23]. This completes the proof of the lemma.

### 3.4. Choosing extra eigenvalues

In the proof of the lemma above we constructed families of reductions that make the system Hermitizable. It means, reductions $h^\rho_C$ such that

$$A^\rho_k = \left( (h^\rho_C \Omega_{\gamma}^C n_a)^{-1} \right) \Delta_B^\rho \Omega^{\delta^B a} k_b$$

is diagonalizable with only real eigenvalues for all $k_a$ not proportional to $n_a$.

Notice that if $h^\rho_C$ is one of these reductions, then so is $\tilde{h}^\gamma_B = \left( (h^\rho_C \Omega_{\gamma}^C n_a)^{-1} \right) \Delta_B^\rho$, since it gives the same matrix (24), (here $\delta^B a$ is the identity matrix). When $\tilde{h}^\gamma_B$ is written in term of the Kronecker decomposition (18) it assumes a simpler form,

$$\tilde{h}^\gamma_B = -W^{-1} \left( (H_{\delta C} \Omega_{\gamma}^C)^{-1} \right) \Delta^\rho B C.$$  

This $\tilde{h}^\gamma_B$ does not depend on the $S$ matrix, showing that these degrees of freedom do not play any role in the reductions. We shall use it in what follows. Recall that when a reduction is applied to the system, the kernel of the resulting operator will increase, as there will be $m$ more elements from each $L^T_m$ block. We shall denote the values of $\lambda$ for which the kernels appear as \{ $\pi_i (k)$ \} with $i \in E(k) := \{ 1, 2, \ldots, u - \sum_{j \in D(k)} r_j (k) \}$. Notice also that since the $L^T$ blocks can change from point to point, (in $k_a$), the new eigenvalues can also change, nevertheless the diagonalizability of $A$ implies that at all points the number of the generalized eigenvalues plus these new elements equals the dimension of the field space.

**Lemma 3.** Assume condition (c) of lemma 2 holds, then there exists $\tilde{h}^\gamma_B$ as equation (25) such that:

All the $\pi_i (k)$ are different among each other and also different from the generalized eigenvalues $\lambda_i (k)$, with $i \in D(k)$, and have algebraic multiplicity equal to 1.

**Proof.** Consider the form of $\tilde{h}^\gamma_B$ in equation (25). The corresponding reduction block of the $L^T_m$ block is given by

$$\tilde{H}_{T,m}^\gamma = \begin{pmatrix} 1 & 0 & \ldots & 0 & \tilde{a}_{m+1} \\
0 & 1 & \ldots & 0 & \tilde{a}_{m+2} \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & 1 & \tilde{a}_{2m} \end{pmatrix}.$$
with
\[
\begin{pmatrix}
\tilde{a}_{m+1} \\
\tilde{a}_{m+2} \\
\vdots \\
\tilde{a}_{2m}
\end{pmatrix} = g_m^{-1}
\begin{pmatrix}
a_{m+1} \\
a_{m+2} \\
\vdots \\
a_{2m}
\end{pmatrix},
\]
where \(g_m\) and \(a_i\) where defined in equation (22). Among all possible reductions we shall now look for very special ones, namely those for which all eigenvalues of the block are different. To find them we just need to give values for some of the generic coefficients \(a_i\). Notice that if we find a reduction for which the block has different eigenvalues, then the block will be diagonalizable, and so there will be coefficients \(a_i\) satisfying the positivity condition required.

But for the rest of the construction we shall not need to find them. Indeed, if \(\tilde{h}_c^{\gamma}(\lambda)\) has an \(L^*_\gamma\) block in its Kronecker decomposition, then, \(\det(\tilde{h}_c^{\gamma}(\lambda))\) is proportional to
\[
\det(\tilde{H}_{L^*_\gamma}^{T}) = \det\left(
\begin{pmatrix}
1 & 0 & \ldots & 0 \\
0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 1
\end{pmatrix}
\begin{pmatrix}
\tilde{a}_{m+1} \\
\tilde{a}_{m+2} \\
\vdots \\
\tilde{a}_{2m}
\end{pmatrix}
\begin{pmatrix}
\lambda & 0 & 0 & 0 \\
1 & \lambda & 0 & 0 \\
0 & 1 & \ldots & 0 \\
0 & 0 & \ldots & \lambda
\end{pmatrix}
\right),
\]
\[
= \det\left(
\begin{pmatrix}
\lambda & 0 & 0 & \tilde{a}_{m+1} \\
1 & \ldots & 0 & \tilde{a}_{m+2} \\
0 & \ldots & \lambda & \ldots \\
0 & 0 & 1 & \lambda + \tilde{a}_{2m}
\end{pmatrix}
\right),
\]
\[
= \lambda^m + \tilde{a}_{2m}\lambda^{m-1} - \tilde{a}_{2m-1}\lambda^{m-2} + \cdots - \tilde{a}_{m+1}\lambda^2 + \tilde{a}_{m+2}\lambda - \tilde{a}_{m+1}.
\]

Given any set of \(m\) different real numbers it is easy to choose the coefficients \(\tilde{a}_i\) so that the polynomial has them as roots. This will fix the desired reduction. \(\Box\)

### 3.5. Uniform lower bound and strong hyperbolicity

Finally, with the help of the particular reductions that we have constructed (equation (25)), we shall use theorem 1 to conclude the proof. Notice that the reductions \(\tilde{h}_c^{\gamma}\) depend on the bases \(W\) and \(Y\). These bases are not unique, and we shall use this freedom to choose an appropriate reduction \(\tilde{h}_c^{\gamma}\).

The reduced system is
\[
\tilde{h}_c^{\gamma}(\lambda) = -\lambda \delta^{\gamma} + A^{\gamma}k_a = -\lambda \delta^{\gamma} - (W^{-1})^{\gamma} \left(H_{IC}f_C^{\gamma}(\gamma)^{-1}\right)^{\gamma} H_{IC}M_\xi W^{\alpha}, \tag{26}
\]
where we still have freedom in choosing \(W\) for our advantage.

\(W\) has kernel when \(\lambda = \pi_j(k)\) and \(\lambda = \lambda_i(k)\). In order to apply theorem 1, we need to calculate the cosines of the angles \(\cos \theta^{\pi_j(k)}\), \(\cos \theta^{\lambda_i(k)}\) between its kernels, \(\gamma^{\pi_j(k)}\), \(\gamma^{\lambda_i(k)}\), \(\Phi^{\pi_j(k)}\), and \(\Phi^{\lambda_i(k)}\) respectively and show that they are uniformly bounded from below.

Since each \(\pi_j(k)\) is a simple eigenvalue of \(A^{\gamma}k_a\), using the implicit function theorem it is possible to show that \(\cos \theta^{\pi_j(k)}\) is continuous in \(k_a\) and since \(k_a\) belongs to a compact set, then \(\cos \theta^{\pi_j(k)}\) reaches its lower bound on that set. But since they are simple, their cosines can not vanish for any \(j\), (when perturbed, the corresponding vanishing singular values must be
of order 1) therefore the lower bound has to be positive. Notice that for this conclusion we do not need any information about the $W$ transformation.

To finish the proof we only need to calculate $\cos \theta_j^{\lambda(k)}$ for each $\lambda_j(k)$. Notice that if any of the $\lambda_i$ are simple, then we can use the above argument, so the interesting case is when we have non-trivial blocks. Given any one of them, since the $\pi_j \neq \lambda_j$, the right kernel subspace $\Phi_{R}^{(k)}$ is invariant under the application of the reduction of the corresponding Jordan-block. But notice that the left kernel of $(26)$, $\Psi_{L}^{(k)}$, depends on $W$. We now need to accommodate $W$ so that the angles we are looking for coincide with the angles of the unreduced system. For that we look now for the left kernel of the whole system, $\Psi_{L}^{\lambda}$. This kernel has an invariant subspace whose dimension is independent of $\lambda$, we call it $\Delta(\lambda)$. This subspace is uniquely defined in the Kronecker decomposition of the principal symbol, and it is the span of a set of particular vectors which are linear combinations with coefficients which are power laws in $\lambda$, they are introduced in appendix C. The kernel increases its dimension by $r_i$ for each specific $\lambda_i$. The $\Delta(\lambda)$ subspace has the important property that when projected with $\Omega^{Ah}_{nh}G^{(\gamma)}$ is orthogonal to the right kernel $\Psi_{R}^{\lambda(k)}$ (see appendix E). So we now project into $\Psi_{R}$ the whole kernel with $\Omega^{Ah}_{nh}G^{(\gamma)}$ and call the resulting subspace $\Psi_{L}^{\lambda}$. The projected image of $\Delta(\lambda)$ will be called $\Delta_{\Omega}(\lambda)$. Using the metric $G_{\gamma \gamma}$ we write $\Phi_{L}^{\lambda}$ as the direct sum of $\Delta_{\Omega}(\lambda)$ and its perpendicular inside it, $(\Phi_{L}^{\lambda})^\perp$. Since $\Delta_{\Omega}(\lambda)$ is perpendicular to $\Phi_{R}^{(k)}$ and to $(\Phi_{L}^{\lambda})^\perp$, the angles between $(\Phi_{L}^{\lambda})^\perp$ and $\Phi_{R}^{\lambda}$ are the same as the angles between $\Phi_{L}^{\lambda}$ and $\Phi_{R}^{\lambda}$, which are the ones that appear in our theorem’s hypothesis, and so their cosines are bounded away from zero. We want to find now a $W$ such that $\Psi_{L}^{\lambda}$ coincides with $(\Phi_{L}^{\lambda})^\perp$, and so do their respective angles. To do that we choose a set of $r_i$ linearly independent vectors $\{v_i^A\}$ in $\Psi_{L}^{\lambda}$ such that $\text{span}\{v_i^A \Omega^{Ah}_{nh}G^{(\gamma)}\} = (\Phi_{L}^{\lambda})^\perp$. Taking now the set of canonical $e - u$ vectors in $\Delta(\lambda)$, $\{\chi_A^\lambda\}$ we obtain a basis for $\Psi_{L}^{\lambda}$, $\{(\chi_A^\lambda, v_i^f)\}$ (this basis defines $W$, see appendix C). Indeed assume that the above vectors are not linearly independent, that is, we can write a vector in $\Delta(\lambda)$ as a linear combination of the other vectors, $\chi_A = a v_i^A$. Contracting with $\Omega^{Ah}_{nh}G^{(\gamma)}$ we get, $\chi_A \Omega^{Ah}_{nh}G^{(\gamma)} = a v_i^A \Omega^{Ah}_{nh}G^{(\gamma)}$. Now, the LHS is an element of $\Delta_{\Omega}(\lambda)$ while the RHS is an element of $(\Phi_{L}^{\lambda})^\perp$. Since these spaces are perpendicular to each other we conclude that both must vanish. But since by assumption the $v_i^A \Omega^{Ah}_{nh}G^{(\gamma)}$ are linearly independent we conclude that the $a_i$ must vanish, and so reach a contradiction. Using this basis, it is straightforward to see that: the resulting $W$ has the property that $(\Phi_{L}^{\lambda})^\perp$ is the left kernel of the reduced system, thus $(\Phi_{L}^{\lambda})^\perp$ coincides with $\Psi_{L}^{\lambda}$. Therefore the angles between $\Psi_{L}^{\lambda}$ and $\Phi_{R}^{\lambda}$, and $(\Phi_{L}^{\lambda})^\perp$ and $\Phi_{R}^{\lambda}$ are the same. This ends the proof of the main theorem.

3.5.1. Practical applications of the proof. We conclude the subsection showing how to apply this construction in practical examples. Given a principal symbol we perform its Kronecker decomposition. This provides us with a basis

$$\{\chi_A^\lambda(\lambda), (v_i^f)^A\}$$  (27)

where $\chi_A^\lambda(\lambda) := (\theta_{m,-1})_A - \lambda(\theta_{m,-2})_A + \lambda^2(\theta_{m,-3})_A - \cdots - \lambda^m(\theta_{m,0})_A$ are a set of vectors which are in the left kernel for all $\lambda$, while the rest of the basis vectors are only kernels for particular values of $\lambda$. The subspace spanned by $\{\chi_A^\lambda(\lambda)\}$ is what we called $\Delta(\lambda)$ above. For each $\lambda_i$ the span of the whole set of these vectors $\{\chi_A^\lambda(\lambda_i), (v_i^f)^A\}$ is the left kernel $\Psi_{L}^{\lambda_i}$. But this basis is not unique, we shall change the elements $\{(v_i^f)^A\}$ to find simple reductions $h^\nu_{A'}$. From equation (25) we see that once we fix the $\{(v_i^f)^A\}$ the reduction is also fixed, for $r$ also
depends only on these vectors. As mention above for each $i \in I_{\lambda_{(k)}}$ we now choose $r_{\lambda_{(k)}}$ new vectors $\{ (\tilde{v}^A_{(k)}_{\lambda_i}) \}$ so that when they are projected into $\Phi_R$, they span the subspace $(\Phi^\perp_{\lambda_{(k)}})$. 

4. Examples

4.1. Klein Gordon

In this subsection, we study the Klein Gordon equation in the Minkowski four-dimensional space-time. We show that the Kronecker decomposition of the principal symbol is $1 \times J_1(1)$, $1 \times J_1(-1)$, $3 \times L_1^T$ and $3 \times L_0^T$, with generalized eigenvalues $\pm 1$. As it is shown in [1], this system is symmetric hyperbolic, hence it is strongly hyperbolic. We shall show the possible reductions of the systems.

The Klein Gordon equation is

$$g^{ab} \nabla_b \nabla_a \phi = 0,$$

with the Minkowski metric $g^{ab}$. This equation can be written in first order form introducing new variables,

$$\phi_a := \nabla_a \phi.$$  \hspace{1cm} (28)

We obtain eleven equations for five variables ($\phi, \phi_a$). They are

$$\nabla_b \phi_a - \phi_b = 0,$$

$$\nabla^b \phi_a = 0,$$

$$\nabla_a [\phi_b] = 0.$$

Taking the Fourier transform of them we obtain the principal symbol,

$$\left( \begin{array}{cc} \delta & 0 \\ 0 & g_{ab} \\ 0 & \delta \end{array} \right) \lambda d \left( \begin{array}{c} \delta \phi \\ \delta \phi_a \\ \delta \phi_{(k)} \end{array} \right).$$

Choosing a time-like covector $n_a$ and lines $l_a(\lambda) = -\lambda n_a + k_a \in S^C_m$, we obtain the matrix pencil form of the principal part,

$$-\lambda \left( \begin{array}{ccc} \delta & 0 & 0 \\ 0 & g_{ab} & 0 \\ 0 & \delta \end{array} \right) n_d + \left( \begin{array}{ccc} \delta & 0 & 0 \\ 0 & g_{ab} & 0 \\ 0 & \delta \end{array} \right) k_d.$$

Where we are considering $n \cdot n = \epsilon_{ab} n^a n_b = -1, k \cdot k = 1$, and $n \cdot k = 0$. Following the appendix B, we calculate the left kernel of this pencil to show explicitly its Kronecker decomposition. It is spanned by the covectors $\{ (\tilde{\theta})_{0A}, (\tilde{\theta})_{1A}, -\lambda \theta_{0A} + \theta_{1A}, -\lambda (\tilde{\theta})_{0A} + (\tilde{\theta})_{1A} \}$ with $i = 1, 2$, which span the subspace $\Delta(\lambda)$, and the eigenvectors $\{ \upsilon_{1A}, \upsilon_{2A} \}$ associated to the generalized eigenvalues $\lambda = \pm 1$. The Kronecker left basis is then

$$\left( \begin{array}{c} \tilde{\theta} \\ \tilde{\theta} \end{array} \right)_{0A} = (0 \ 0 \ \epsilon_{k_h b_d} n_d n_b k_h) \rightarrow 1,$$

$$\left( \begin{array}{c} \tilde{\theta} \\ \tilde{\theta} \end{array} \right)_{0A} = (\epsilon_{k_h b_d} n_d n_b k_h \ 0 \ 0) \rightarrow 2,$$
\[ \theta_{0A} = \left( \varepsilon^{kla}l_{1d}l_{2n_{k_{1}}} 0 0 \right) \rightarrow 1, \]
\[ \theta_{1A} = \left( \varepsilon^{kla}l_{1d}l_{2n_{k_{1}}} 0 0 \right) \rightarrow 1, \]
\[ \left( \tilde{\theta}_{0} \right)_{0A} = \left( 0 0 \varepsilon^{kla}(l_{1})_{n_{k_{1}}} \right) \rightarrow 2, \]
\[ \left( \tilde{\theta}_{1} \right)_{1A} = \left( 0 0 \varepsilon^{kla}(l_{1})_{n_{k_{1}}} \right) \rightarrow 2, \]
\[ \nu_{iA} = \left( 0 - \frac{1}{2} \lambda_{i} n^{b}k_{c} \right) \rightarrow 2, \]

where \( l_{i,k} = l_{i,n} = 0 \) and \( l_{i,j} = \delta_{ij} \) with \( i,j = 1,2 \). With this set we build the matrices \( Y_{B}^{A}(n,k) \), \( K_{\alpha}^{B}(\lambda) \) and \( W_{\eta}^{\alpha}(n,k) \) as in equation (18).

\[ Y_{B}^{A}(n,k) = \left( \begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & n_{a} & -k_{a} & -l_{1a} & l_{2a} & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-n_{[b]c_{1}} & n_{[b]c_{1}} & -k_{[b]d_{1}} & -n_{[b]d_{2}} & k_{[b]d_{2}} & 0 & 0 & 0 & 0 & l_{[b]d_{2}} & 0 \\
\end{array} \right), \]

\[ K_{\alpha}^{B}(\lambda) = \left( \begin{array}{cccccc}
\lambda - 1 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda + 1 & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{array} \right), \quad W_{\eta}^{\alpha}(n,k) = \left( \begin{array}{cccccc}
0 & \frac{1}{2}(n^{i} + k^{i}) \\
0 & \frac{1}{2}(-n^{i} + k^{i}) \\
0 & -l_{1} \\
0 & l_{2} \\
0 & 0 \\
-1 & 0 \\
\end{array} \right). \]

They realize the Kronecker decomposition of the principal symbol.

Following equation (25) the reductions are:

\[ \tilde{h} = W^{-1} \left( \begin{array}{cccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & e_{1} & d_{1} & f_{1} \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & e_{2} & d_{2} & f_{2} \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & e_{3} & d_{3} & f_{3} \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & e_{4} & d_{4} & f_{4} \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & e_{5} & d_{5} & f_{5} \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & e_{6} & d_{6} & f_{6} \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & e_{7} & d_{7} & f_{7} \\
\end{array} \right) Y^{-1}, \]

where the coefficients in \( \tilde{h}_{\gamma} \) are arbitrary complex function of \( n \) and \( k \), with the exception of \( a_{1}, b_{1} \) and \( c_{3} \) which are real.

The pseudo-differential evolution equations (principal part) are:

\[ \left( \frac{\partial_{\lambda} \tilde{\phi}}{\partial_{\phi} \tilde{\phi}} \right) = \left( \begin{array}{cccc}
\varepsilon^{kla}n^{b}R^{e} & \varepsilon^{m} & k^{i}n^{a} & \varepsilon^{m} & \varepsilon^{m} & n_{i} & n_{j} & -a_{2i} & -a_{2i} & -a_{2i} \end{array} \right) \left( \begin{array}{c}
\phi \\
\phi \\
\phi \\
\phi \\
\phi \\
\phi \\
\phi \\
\phi \\
\phi \\
\phi \\
\end{array} \right), \]

with \( a_{2} = a_{2} + ia_{2} \), \( S_{\eta}^{\alpha} = -a_{2i} \left( l_{1}d_{1} - l_{2}d_{2} \right) \left( l_{1}e_{1} - l_{2}e_{2} \right) \left( l_{1}f_{1} - l_{2}f_{2} \right) \), and \( R^{e} \) is any complex vector.
It is instructive to look now at the possible differential reductions. In Cartesian adapted coordinates the Klein Gordon system becomes,
\[ \partial_t \phi = \phi_0, \]
\[ \partial_t \phi_0 = -\partial^i \phi_i, \]
\[ \partial_t \phi_i = \partial_i \phi_0, \]
\[ C_i := \partial_t \phi_i - \phi_i = 0, \]
\[ C_{ij} := \partial_t \phi_{ij} = 0, \]
where the last two equations are clearly the constraints.

The most general differential hyperbolization of this system is obtained by setting \( S^\alpha_\beta = 0 \) and \( e_3 = L' k_i \). The equations for the principal part become,
\[ \partial_t \phi = L'^i \partial_i \phi - R_k \epsilon^{ijk} \partial_j \phi_j = L'C_i - R_k \epsilon^{ijk} C_{ij}, \]
\[ \partial_t \phi_0 = -\partial^i \phi_i, \]
\[ \partial_t \phi_i = \partial_i \phi_0 + R_k \epsilon^{ijk} \partial_j \phi_j = \partial_i \phi_0 + R_k \epsilon^{ijk} C_j + i a_{32} \epsilon^{ij} C_{ij}. \]
This expression clearly shows that the freedom in the choice of reductions is the addition of arbitrary linear combination of constraints to some of the equations.

4.2. Einstein equations

In the following two sections we study in a pseudo-differential form the linearized, densitized ADM and a version of the BSSN equations of general relativity. Next, we present their Kronecker decompositions and show hyperbolizers for them. We do not explain the details of how to arrive at the pseudo-differential equations, the complete theory can be found in the work of Nagy, Reula, and Ortiz [5]. We show that the principal symbol of these systems have the following Kronecker structure: they both share a diagonal part of \( 2 \times J_1(0), 2 \times J_1(1), 2 \times J_1(-1), 1 \times J_1(\sqrt{b}) \) and \( 1 \times J_1(-\sqrt{b}) \). This \( 8 \times 8 \) block corresponds to the physical propagation. Also, the ADM equations have \( 4 \times L_1^1 \) blocks and the BSSN equations have \( 7 \times L_1^1 \) and \( 6 \times L_0^0 \) blocks. Since the systems do not have any \( J_m \) Jordan block with \( m \geq 2 \), lemma 2 implies that there exist diagonalizable reductions of the systems. We will present some of them, check the uniformity condition and so conclude that they are proper hyperbolizations.

In addition, we show that in the non-densitized BSSN theory (\( b = 0 \)), the Kronecker decomposition has a two-dimensional Jordan block, and therefore that there exist no hyperbolizers for it. This is because of the \( 1 \times J_1(\sqrt{b}) \) and \( 1 \times J_1(-\sqrt{b}) \) blocks collapse to a \( 1 \times J_2(0) \). The same calculation can be made in the non-densitized ADM equations, with similar conclusions.

4.2.1. ADM. The \( 3 + 1 \) principal part of the ADM equations, including the constraints equations, for the linearized fields \( (h_{ij}, k_{ij}) \) on a flat background are:
\[ \frac{1}{N} \partial_t h_{ij} = -2 k_{ij}, \]
\[ \frac{1}{N} \partial_t k_{ij} = \frac{1}{2} \epsilon^{ij} \left( -\partial_k \partial_i h_{kj} - (1 + b) \partial_i \partial_j h_{kl} + 2 \partial_k \partial_i h_{jkl} \right), \]
\[ 0 = (-\Delta h + \partial^i \partial_i h_{ij}), \]
\[ 0 = \partial^i k_i - \partial_k k. \]
Here $e_0$ is the hypersurface-induced flat background metric and $\partial_t$ its connection. The lapse is densitized as $N = h^2 Q(x)$, where $b$ is the constant densitization parameter and $Q(x)$ is an arbitrary function that do not depend on the unknown fields. In addition, the shift vector and some lower-order terms are taken to be zero. More details can be found in [5].

Taking a Fourier transform $(h_{ij}, k_{ij}) \rightarrow (\hat{h}_{ij}, \hat{k}_{ij}) e^{i(-\lambda r + k_i x^i)}$ and defining $\hat{l}_{ij} = i|k_i| \hat{h}_{ij}$ with $|k|_e = \sqrt{\epsilon^j k_i k_j}$ we obtain the following principal symbol:

$$\mathcal{N}_{ij}^{ab}(\lambda)_{\alpha} \delta \phi^a = (\lambda E^A + B^A_{\alpha}) \begin{pmatrix} \hat{l}_{sr} \\ \hat{k}_{sr} \end{pmatrix},$$

$$= \begin{pmatrix} 2 \left( \delta^r_{i} \delta^r_{j} + (1 + b) k_r k_r e^{\epsilon r} - 2 k_0 k_i k_j \right) & \lambda \delta^r_{i} \delta^r_{j} \\ -\alpha q^r_{i} & 0 \end{pmatrix} \begin{pmatrix} \hat{l}_{sr} \\ \hat{k}_{sr} \end{pmatrix}$$

for the variables $\hat{l}_{sr}, \hat{k}_{sr}$.

Notice that we are considering the lines $-\lambda n_a + k_i \in S_{n_a}$ with $n_a = \nabla \epsilon t$ and $t$ the time coordinate. Also $\lambda := \frac{1}{N \epsilon t}$ and $q^r := e^{\epsilon r} - k^r k^r$ is the projector orthogonal to $k^r$. We denote $k_i$ to be the covector such that $|k|_e = 1$. For details about the pseudo-differential structure see [5] (a different notation is used here; with respect to that article, $\omega_i \rightarrow k_i$ and $\alpha = 1$). The characteristic speeds are solutions to (7), which becomes,

$$\sqrt{\det \left( G^{sr} \left( \lambda E^A + B^A_{\alpha} \right) G_{AB} \left( \lambda E^B_{\rho} + B^B_{\rho} \right) \right)} = |\lambda|^2 \left| (\lambda^2 - 1) \right|^2 |(b - \lambda^2)| \sqrt{p_1(\lambda, \lambda)} = 0.$$

Here $G^{sr}$ and $G_{AB}$ are any positive definite Hermitian forms, and $p_1(\lambda, \lambda)$ a positive polynomial in $\lambda, \bar{\lambda}$. The bar means conjugation. Therefore the mode propagation velocities, or equivalently the generalized eigenvalues of the complete system are: $\lambda = 0$, $\lambda = \pm 1$ and $\lambda = \pm \sqrt{b}$.

The left kernel of $M_{ij}^A$ are vectors to the form $\delta X_A = (\alpha^i, \gamma^i, \rho, \tau^i)$. It is useful to define the vectors $x_\pm$ with $x_+ x_- = \delta_\pm$ (the Kronecker delta) and $\hat{k} x_\pm = 0$. A convenient basis for the generalized left kernel is:

For the generalized eigenvalue $\lambda = 0$, we have two eigenvectors

$$\upsilon_{1,\pm A} = \left( 2 k^l x_\pm^l, 0, 0, -4 x_\pm^l \right).$$

For $\lambda = \pm 1$, we have four eigenvectors

$$\upsilon_{2,\pm A} = \left( \mp \frac{1}{2} \left( x_+ x_+^l - x_- x_-^l \right), \left( x_+ x_+^l + x_- x_-^l \right), 0, 0 \right),$$

$$\upsilon_{3,\pm A} = \left( \mp \frac{1}{2} x_+^{l'} x_-^{l'}, x_+^{l'} x_-^{l'}, 0, 0 \right).$$

For $\lambda = \pm \sqrt{b}$, we have two eigenvectors

$$\upsilon_{4,\pm A} = \left( \mp \frac{1}{2} \sqrt{b} k^l \bar{k}^{l'}, \bar{k}^l k^{l'}, \frac{1}{2}(b + 1), 0 \right).$$

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The left kernel associated to the $4 \times L_1^T$ blocks are of order $1$ in $\lambda$, they are given by

\[
(\theta)_1A - \lambda(\theta)_0A = \left( \frac{1}{2} q^{ij}, 0, \frac{1}{2} \lambda, \tilde{k}^i \right),
\]

\[
(\theta)_2A - \lambda(\theta)_0A = \left( 0, \frac{1}{2} q^{ij}, \frac{1}{4}, \frac{1}{2} \lambda \tilde{k}^i \right),
\]

\[
(\theta)_3A - \lambda(\theta)_0A = \left( 0, 2\tilde{k}^i x^j, 0, -2\lambda x^j \right).
\]

Therefore the Kronecker decomposition has the $5$ generalized eigenvalues with their corresponding eigenvectors that defines an $8 \times 8$ diagonal block and $4 \times L_1^T$ blocks:

\[
K = \begin{pmatrix}
\lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda - 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda - 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda + 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda + 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda - \sqrt{b} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda + \sqrt{b}
\end{pmatrix}.
\]

The most general reduction diagonal in time is:

\[
\left( (H_{3C} F_{NC})^{-1} \right)^{\rho5} H_{3C} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p_1 & 0 & r_1 & 0 & s_1 & 0 & w_1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & p_2 & 0 & r_2 & 0 & s_2 & 0 & w_2 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & p_3 & 0 & r_3 & 0 & s_3 & 0 & w_3 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & p_4 & 0 & r_4 & 0 & s_4 & 0 & w_4 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & p_5 & 0 & r_5 & 0 & s_5 & 0 & w_5 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & p_6 & 0 & r_6 & 0 & s_6 & 0 & w_6 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & p_7 & 0 & r_7 & 0 & s_7 & 0 & w_7 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & p_8 & 0 & r_8 & 0 & s_8 & 0 & w_8 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & a_1 & c_1 & c_2 & c_3 & b_1 & a_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_2 & 0 & f_1 & a_3 & m_1 & b_3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_4 & 0 & f_2 & l_1 & a_4 & b_3
\end{pmatrix}.
\]

This takes the form of the reductions considered in appendix D when setting $p_i = r_i = s_i = w_i = 0$ and

\[
A = \begin{pmatrix}
a_1 & c_1 & c_2 & c_3 \\
b_1 & a_2 & g_1 & g_2 \\
b_2 & f_1 & a_3 & m_1 \\
b_3 & f_2 & l_1 & a_4
\end{pmatrix}.
\]
is a diagonalizable matrix with real and simple eigenvalues different from the generalized ones, namely \( \{0, -1, 1, \sqrt{B}, -\sqrt{B}\} \). The coefficients \( a_i, b_i, c_i, f_i, m_i, l_i, p_i, r_i, s_i, w_i \) parametrize the remaining degrees of freedom of the most general reductions 
\[
\tilde{h}_B = (W^{-1})_\rho^\gamma \left( (H_C t_\gamma^C)^{-1} \right)^{\alpha i} H_C (W^{-1})^C_B,
\]
where \( W^{-1} \) and \( W \) are building using the basis above. Then,

\[
\tilde{h}_B = \begin{pmatrix} \delta'_\rho \delta'_i & 0 & A_{3sr} & A_{4sr} \\ 0 & \delta'_\rho \delta'_i & B_{3sr} & B_{4sr} \end{pmatrix},
\]

\[
A_{3sr} = -\frac{1}{2} p_1 \tilde{k}_s(x_{r+}) \frac{1}{2} p_2 \tilde{k}_s(x_{r-}) + \left( \frac{1}{4} p_4 - \frac{1}{4} p_6 \right) x_{s+}(x_{r-}) + \left( \frac{1}{4} p_3 - \frac{1}{4} p_5 - \frac{1}{2} a_1 \right) x_{s+} x_{r+}
\]

\[
= \left( \frac{1}{4} p_3 - \frac{1}{4} p_5 + \frac{1}{2} a_1 \right) x_{s+} x_{r-} + \frac{1}{4} p_7 - \frac{1}{2} p_8 \right) \frac{1}{\sqrt{b}} \tilde{k}_r \tilde{k}_s,
\]

\[
B_{3sr} = +\frac{1}{4} (2b + 2 - p_7 - p_8) \tilde{k}_r \tilde{k}_s - \frac{1}{8} (p_3 + p_5 - 2 + 4 b_1) x_{s+} x_{r+}
\]

\[
+ \frac{1}{8} (p_3 + p_5 - 2 - 4 b_1) x_{s-} x_{r-} - \frac{1}{8} (p_4 + p_6) x_{s+}(x_{r-}) - \frac{1}{2} b_2 \tilde{k}_s(x_{r-}) - \frac{1}{2} b_3 \tilde{k}_s(x_{r-}),
\]

\[
A_{4sr} = +\frac{1}{2} \sqrt{b} \tilde{k}_r \left( \frac{1}{2} (r_7 - r_8) \tilde{k}' + 2 (s_8 - 2 s_7) x'_+ + 2 (w_8 - w_7) x'_- \right)
+ \frac{1}{2} \tilde{k}_s(x_{r+}) \left( -\frac{1}{2} r_1 \tilde{k}' + (-4 + 2 s_1) x'_+ + 2 w_1 x'_- \right)
+ \frac{1}{2} \tilde{k}_s(x_{r-}) \left( -\frac{1}{2} r_2 \tilde{k}' + 2 s_2 x'_+ + (-4 + 2 w_2) x'_- \right)
+ \frac{1}{2} x_{s+}(x_{r+}) \left( \left( -\frac{1}{2} r_3 + \frac{1}{2} r_5 + (2 - c_1) \right) \tilde{k}' + (2 s_5 - 2 s_3 + 4 c_2) x'_+ 
+ (2 w_5 - 2 w_3 + 4 c_3) x'_- \right) - \frac{1}{2} x_{s-}(x_{r-}) \left( \left( -\frac{1}{2} r_5 + \frac{1}{2} r_7 - (2 - c_1) \right) \tilde{k}' + (2 s_5 - 2 s_3 + 4 c_2) x'_+ 
+ (2 w_5 - 2 w_3 + 4 c_3) x'_- \right)
+ \frac{1}{2} \tilde{k}_s(x_{r-}) \left( \frac{1}{2} (r_4 - r_6) \tilde{k}' + 2 (s_6 - s_8) x'_+ + 2 (w_6 - w_8) x'_- \right),
\]

\[
B_{4sr} = +\frac{1}{2} \sqrt{b} \tilde{k}_r \left( -\frac{1}{2} (r_7 + r_8) \tilde{k}' + 2 (s_7 + s_8) x'_+ + 2 (w_7 + w_8) x'_- \right)
+ \frac{1}{2} \tilde{k}_s(x_{r+}) \left( -\frac{1}{2} (4 a_2 + r_3 + r_5) \tilde{k}' + 2 (s_3 + s_5 + 4 g_1) x'_+ + 2 (w_3 + w_5 + 4 g_2) x'_- \right)
+ \frac{1}{2} \tilde{k}_s(x_{r-}) \left( \frac{1}{2} (4 a_2 - (r_3 + r_5)) \tilde{k}' + 2 (s_3 + s_5 - 4 g_1) x'_+ + 2 (w_3 + w_5 - 4 g_2) x'_- \right),
\]

\[
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\]

\[
F Abalos and O Reula
\]
4.2.2. BSSN.

The 3 + 1 linearized, principal part of the BSSN equations, for the fields \((h_{ij}, k_{ij}, f_i)\) with \(i, j = 1, 2, 3\) are:

\[
\frac{1}{N} \partial_t h_{ij} = -2 k_{ij}, \\
\frac{1}{N} \partial_t k_{ij} = -\frac{1}{2} \epsilon^{ijl} (\partial_l \partial_t h_{ij} + b \partial_l \partial_t h_{ij}) + \partial_l f_j, \\
\frac{1}{N} \partial_t f_i = -2 \epsilon^{i lj} \partial_l k_{ij} + \epsilon^{ijl} \partial_l k_{ij}, \\
0 = -\epsilon^{ijl} \Delta h_{ij} + \partial^j \partial_i h_{ji}, \\
0 = \partial^i k_{ij} - \epsilon^{ijl} \partial_l k_{ij}, \\
0 = \partial_i f_j - \partial_j \partial^i h_{ij} + \frac{1}{2} \epsilon^{ijl} \partial_l \partial_i h_{ij},
\]

where we are using the notation of the ADM case. Therefore the principal symbol is:

\[
\begin{align*}
+ \frac{1}{4} x_{+(x_{-r})} \left( -\frac{1}{2} r_{ij} \tilde{k}^i + 2 s_i x_i^+ + 2 u_i x_i^- \right) + \frac{1}{4} x_{+(x_{-r})} \left( -\frac{1}{2} r_{ij} \tilde{k}^i + 2 s_i x_i^+ + 2 u_i x_i^- \right) \\
+ \tilde{k}_{i(x_{-r})} \left( -\frac{1}{2} f_{ij} \tilde{k}^i + 2 a_{ij} x_i^+ + 2 m_{ij} x_i^- \right) + \tilde{k}_{i(x_{-r})} \left( -\frac{1}{2} f_{ij} \tilde{k}^i + 2 l_{ij} x_i^+ + 2 a_{ij} x_i^- \right).
\end{align*}
\]

This expression includes all possible reductions, the pseudo-differential, and the differential ones (if they exist). Also, we obtain a set of uniform hyperbolizations, making zero all coefficients exception for \(a_{ij}\), and choosing them real, different among them and different to the generalized eigenvalues. To show this, we consider these particular reductions, obtain the reduced system, and check the conditions between the canonical angles in theorem 1. Indeed, using equation (A.6) in appendix A, we conclude that equation (10) holds. The eigen-projectors are:

\[
p^{\lambda=0}(\hat{k}) = \begin{bmatrix}
\tilde{k}_{i(x_{+r})} & \tilde{k}_{i(x_{-r})} \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
2\tilde{k}^i x_i^+ & 0 \\
2\tilde{k}^i x_i^- & 0
\end{bmatrix} = \begin{bmatrix}
2\tilde{k}_{i(x_{+r})} \tilde{k}^i x_i^+ + 2\tilde{k}_{i(x_{-r})} \tilde{k}^i x_i^- & 0 \\
0 & 0
\end{bmatrix},
\]

\[
|p^{\lambda=0}| = \frac{1}{\cos \theta_2} = 1,
\]

\[
p^{\lambda=\pm 1}(\hat{k}) = \begin{bmatrix}
\frac{1}{4} \left( x_{+(x_{+r})} - x_{+(x_{-r})} \right) \left( x_i^+ x_i^- - x_i^- x_i^+ \right) \\
\frac{1}{8} \left( x_{+(x_{+r})} - x_{+(x_{-r})} \right) \left( x_i^+ x_i^- - x_i^- x_i^+ \right)
\end{bmatrix} = \begin{bmatrix}
\frac{1}{2} \left( x_{+(x_{+r})} - x_{+(x_{-r})} \right) \left( x_i^+ x_i^- - x_i^- x_i^+ \right) \\
\frac{1}{4} \left( x_{+(x_{+r})} - x_{+(x_{-r})} \right) \left( x_i^+ x_i^- - x_i^- x_i^+ \right)
\end{bmatrix},
\]

\[
|p^{\lambda=\pm 1}| = \frac{1}{\cos \theta_2} = \sqrt{\frac{1}{8} \frac{34}{32} + \frac{25}{32} \approx 1.2289},
\]

this proves that the cosines of the principal angles are uniformly lower bounded.
for the variables $\hat{l}_{sr}, \hat{k}_{sr}, \hat{f}_{sr}$. For the details about the pseudo-differential structure see [5]. Analogously to the ADM case,

$$\sqrt{\det \left( G^{\alpha \beta} \left( \lambda E^\alpha_\rho + B^\alpha_\rho \right) G_{AB} \left( \left( \lambda E^B_\rho + B^B_\rho \right) \right) \right)} = |\lambda|^2 |(\lambda^2 - 1)^2 |(b - \lambda^2)| \sqrt{p_2(\lambda, \bar{\lambda})} = 0.$$

Again, $G^{\alpha \beta}$ and $G_{AB}$ are positive definite Hermitian forms, and $p_2(\lambda, \bar{\lambda})$ is a positive polynomial in $\lambda, \bar{\lambda}$. Thus, the mode’s propagation-velocities, or equivalently the generalized eigenvalues are the same to the ADM case $\lambda = 0, \lambda = \pm 1$ and $\lambda = \pm \sqrt{b}$.

We present the Kronecker structure of this system, but we study two cases separately, the densitized $b \neq 0$ and non-densitized $b = 0$ lapse.

**4.3. Case $b \neq 0$**

The left kernel of $M^A_\alpha$ are covectors of the form $\delta X_\delta = (\alpha^i, \gamma^{ij}, \eta^i \epsilon^{ij}, \rho, \tau^j)$ with the index $i,j = 1,2,3$. We use the same definition as before for $x_\pm$ with $x_+ x_- = \delta_\pm$ and $\hat{k}_+ x_- = 0$. In addition we define the projector $q^j := \epsilon^{ij} - \hat{k}^j$.

One basis of the generalized left kernel is:

For the generalized eigenvalue $\lambda = 0$ we have two eigenvectors

$$v_{1, \pm} = \left(0, 0, x_\pm^2, 0, 0, -2x_\pm^1 \right).$$

For the eigenvalues $\lambda = \pm 1$ we have four eigenvectors

$$v_{2, \pm} = \left( \pm \frac{1}{2} (x_+^1 x_+^1 - x_-^1 x_-^1), \left( x_+^1 x_+^1 - x_-^1 x_-^1 \right), 0, 0, 0, 0 \right),$$

$$v_{3, \pm} = \left( \pm \frac{1}{2} x_+^1 x_-^1, x_+^1 x_-^1, 0, 0, 0, 0 \right).$$

For $\lambda = \pm \sqrt{b}$ we have two eigenvectors

$$v_{4, \pm} = \left( \pm \frac{1}{2} \sqrt{bk^i \tilde{k}^j} \tilde{k}^i \tilde{k}^j 0 \tilde{k}^i \tilde{k}^j \frac{1}{2} (b + 1) 0 \right).$$

A basis of the left kernel, for all $\lambda$, is composed by a set of vectors, of order 1 in $\lambda$ and
another set of order 0 in $\lambda$. The first ones define $7 \times L_1^T$ blocks, they are:

$$
(\theta_1)_{1A} - \lambda(\theta_0)_{1A} = (q^{ij} 0 0 0 \lambda 2\tilde{k}^i),
$$

$$
(\theta_1)_{2A} - \lambda(\theta_0)_{2A} = \begin{pmatrix} 0 & q^{ij} & 0 & 0 & \frac{1}{2} \lambda \tilde{k}^i \\ 0 & 0 & 0 & 0 & 0 \\ \end{pmatrix},
$$

$$
(\theta_1)_{3A} - \lambda(\theta_0)_{3A} = \begin{pmatrix} -\epsilon^{ij} & 0 & -2\omega^i & 0 & 0 \\ -\omega^i & 0 & 0 & 0 & 0 \\ \end{pmatrix},
$$

$$
(\theta_1)_{4\pm A} - \lambda(\theta_0)_{4\pm A} = \begin{pmatrix} \tilde{k}^i x^j_{\pm} & 0 & -x^i_{\pm} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \end{pmatrix},
$$

$$
(\theta_1)_{5\pm A} - \lambda(\theta_0)_{5\pm A} = \begin{pmatrix} 0 & -\tilde{k}^i x^j_{\pm} & 0 & -\frac{1}{2} \tilde{k}^i x^j_{\pm} & 0 & \lambda x^j_{\pm} \\ 0 & 0 & 0 & 0 & 0 \\ \end{pmatrix}.
$$

The remaining vectors define $6 \times L_0^T$ blocks, they are:

$$
(\theta_0)_{1A} = \begin{pmatrix} 0 & 0 & 0 & x^i_{+} x^j_{+} - x^i_{-} x^j_{-} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \end{pmatrix},
$$

$$
(\theta_0)_{2A} = \begin{pmatrix} 0 & 0 & 0 & x^i_{+} x^j_{+} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \end{pmatrix},
$$

$$
(\theta_0)_{3A} = \begin{pmatrix} 0 & 0 & 0 & x^i_{+} x^j_{-} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \end{pmatrix},
$$

$$
(\theta_0)_{4\pm A} = \begin{pmatrix} 0 & 0 & 0 & \tilde{k}^i x^j_{\pm} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \end{pmatrix},
$$

$$
(\theta_0)_{5A} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \end{pmatrix}.
$$

Therefore the Kronecker decomposition is $2 \times J_1(0), 2 \times J_1(1), 2 \times J_1(-1), 1 \times J_1(\sqrt{b}), 1 \times J_1(-\sqrt{b}), 7 \times L_1^T$ and $6 \times L_0^T$ (six null rows).

Studying the explicit form of the reductions that appear in Nagy et al work [5], we notice that in these reductions, the case $b = 1$ leads to a weakly hyperbolic system for $c \neq 2$. The problem is the particular reduction that they are considering. Indeed, as can be easily checked, for the left-kernel basis presented before, the Kronecker decomposition does not change when $b = 1$. This case has nothing special from the Kronecker structure perspective. Therefore, there are other reductions that lead to strongly hyperbolic systems, for example, the one obtained by choosing $c = 2$.

### 4.4. Non-densitized case $b = 0$

In this case, the generalized eigenvalues $\pm \sqrt{b}$ go to 0, and the corresponding eigenspace degenerate, since $\upsilon_{6A}$ collapses to $\upsilon_{7A}$ and no other eigenvector appears. We will show that the Kronecker decomposition of $M$ has a Jordan block of order 2 associated to this degeneration. We follow theorem 7 in [23] and show that the operator $L$ has a non-trivial right kernel.

To construct $L$ we calculate a basis of the right kernel of $M$ for $\lambda = 0$, it is:

$$
\chi_{q}^\alpha = \begin{pmatrix} 2\tilde{k}_{(s)} x_{+} & 2\tilde{k}_{(s)} x_{-} & \tilde{k}_{(s)} \tilde{k}_{(s)} & 0 \\ 0 & 0 & 0 & 0 \\ x_{+} & 0 & x_{-} & 0 \\ \end{pmatrix},
$$

each column is an vector of the basis, indices by $q = 1, 2, 3$. A basis of the left kernel is:
Each row is a vector of the basis, indices by \( u = 1, \ldots, 16 \). Then \( L_q^u \) is:

\[
L_q^u = \Xi_A \lambda^{ij} x_q^u = \left( \begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array} \right),
\]

which has non-trivial right kernel, therefore the Kronecker decomposition has at least an \( l \)-Jordan block with \( l \geq 2 \). In this particular case, using the dimension of the left kernel for the other generalized eigenvalues, it is possible to conclude that a J_2(0) Jordan block appears. So this system is intrinsically ill-posed, i.e. there is not a strongly hyperbolic reduction as follows from lemma 2.

5. Discussion

In this work, we have found a necessary and sufficient condition that a first order system has to satisfy in order to have a reduction which is a hyperbolization (see theorem 2). In the case
of a constant-coefficient system, this hyperbolization implies that the reduced system has a well-posed initial value formulation. That is, given the values of the unknowns in an appropriate hyper-surface (the initial data) a unique solution to the reduced system exists and it depends continuously on that data. Contrary to the classical treatment, the reduced system is not a partial differential system, but in general, it is a pseudo-differential system. Nevertheless, the usual theory applies in the sense that energy norms can be constructed and the corresponding estimates obtained. It is important to realize that once initial data is given the solution of the reduced system is unique, so if the complete system has a solution for that data, it must be that one. This, in general, does not need to be the case, even in the event the data chosen satisfies initially the whole system. There remains, in this more general setting, to show constraint propagation is consistent. But this problem involves looking at lower-order terms of the system, the integrability conditions, which we are avoiding here and which in general are difficult to deal with in general. At the principal symbol level, two approaches that can be taken regarding this problem: one is to define constraint quantities, that is, linear combinations of the fields which vanish when the constraints are satisfied and check that they also satisfy hyperbolic equations provided some consistency conditions are satisfied. This has been done partially in [33] for the case of algebraic reductions. The present case can be dealt with the same machinery we have used here and will be developed in a future paper. Another approach is to extend the system, adding new variables, so that it no longer has constraints. This scheme has many advantages, in particular for obtaining systems with better numerical behavior [34–38]. The present machinery allows telling when a given system would admit a hyperbolic extension, in general pseudo-differential. Again, these results will be presented elsewhere.

As mentioned, the resulting set of evolution equations that a hyperbolization selects is, in general, a pseudo-differential system. Thus, it is not clear whether this system has a causal propagation, that is whether there exists a maximal propagation speed. Clearly, the eigenvalues are all finite, but that does not necessarily mean that a solution for a compactly supported initial data would remain so. In fact if the reduced system is not analytic in $k_\alpha$, then the solution cannot have compact support. Indeed, assume data of compact support, $\phi_0$, then its Fourier transform, $\hat{\phi}_0$ is analytic. Writing the system as

$$\frac{d\hat{\phi}_\alpha}{dt} = iA^\alpha_{\beta}(k)\hat{\phi}_\beta,$$

the solution would be,

$$\hat{\phi}_\alpha(k, t) = e^{iA(k)t}\hat{\phi}_0(k),$$

but if $A^\alpha_{\beta}(k)$ is not analytic, neither would be the solution for any finite $t$. Thus, for non-analytic reductions, the solution cannot have compact support at any time before or after the initial slice. We believe that causality would follow for analytic reductions, a possible way to see this is using the ideas in [39]. In any case, it is then important to develop a theory spelling out necessary and sufficient conditions equivalent to the existence of analytic reductions. This would not only be important for causality, but also for extending the results of this work to variable coefficients or quasi-linear system. In that case, with the present technology, smoothness of the principal symbol is needed to obtain the energy estimates used for showing existence for such systems.

We now turn to variable coefficient systems, an intermediate step to treat general quasi-linear systems. Given any point in the cotangent-bundle ($p, k$) we can perform the Kronecker decomposition and find suitable hyperbolic reductions. But in general, the reduction would also depend on the point and the bundle. Thus, when going back to space-time we would
end up with an operator equation. It is not clear in what sense that equation is related to the original partial differential equation. Further ideas are needed to establish some equivalence of operators at the pseudo-differential level, and perhaps an intermediate operator which can relate our reduction to the original system.

In the case of variable coefficient or quasi-linear systems we can still consider reductions which do not depend on $k$, $\tilde{\gamma} \tilde{\nu}^k$, namely a differential reduction. These $\tilde{\gamma} \tilde{\nu}^k$ selects a set of evolution equations as:

$$\tilde{\gamma} \tilde{\nu}^k \nabla_x \phi^a(x, \phi) = \tilde{\gamma} \tilde{\nu}^k F^k(x, \phi).$$

(29)

Assume our system satisfies all the uniformity conditions for all points and furthermore that we can find a differential reduction $\tilde{\gamma} \tilde{\nu}^k (x, \phi_0)$ among all possible, with $\phi_0^a$ a background solution, then the system (29) is uniformly diagonalizable for all points $(p, k_a)$. Furthermore, if the symmetrizer, $\tilde{\gamma} \tilde{\nu}^k H_{\nu c} F^c, W_{\gamma}^\gamma$ is smooth in their variables $(p, \phi^a, k_a)$, then system (29) is strongly hyperbolic, see [21]. In addition if $\tilde{\gamma} \tilde{\nu}^k (p, \phi_0)$ is a differential reduction and $\tilde{\gamma} \tilde{\nu}^k H_{\nu c} F^c, W_{\gamma}^\gamma$ is independent of $k_a$ and smooth in their variables, the systems is symmetric hyperbolic.

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Appendix A. Proof of theorem 1

Consider the resolvent Kreiss condition of the matrix theorem equation (5). We shall use it in the following equivalent form:

$$\frac{1}{C} \varepsilon \leq \min_{j \in \{1, \ldots, a\}} [\sigma^j \left[A^a_{a'} k_a - \lambda_R \delta^a_{a'} - i \varepsilon \delta^a_{a'} \right]]$$

(A.1)

where $s = \lambda_R + i \varepsilon$ and we have used that for any invertible $B \in \mathbb{C}^{n \times n}$ matrix, $|B^{-1}| = \frac{1}{\min_{j \in \{1, \ldots, a\}} |\sigma_j[B]|}$, with $\sigma_j[B]$ the singular values of $B$.

We are going to prove that equation (A.1) holds for all $\lambda_R \in \mathbb{R}$ and all $\varepsilon > 0$ if and only if all the eigenvalues $\tau_i(k)$ of $A^a_{a'} k_a$ are real, and for all $i \in F_{\nu}(k) = \{1, \ldots, w_{\nu}(k)\}$ and all $k_a$ non proportional to $n_a$, with $|k| = 1$, the angles $\theta_l^{\gamma(k)}$ between $T_L^{\gamma(k)}$ and $\Phi_R^{\gamma(k)}$ satisfy the lower bound condition

$$\cos \theta_l^{\gamma(k)} \geq \cos \vartheta > 0.$$  

(2.2)

(\Leftarrow) Consider the right-hand side of equation (A.1), with $\lambda_R = \tau_i(k)$ and $0 \leq \varepsilon \ll 1$. In that case, as it was explained in [23], the vanishing (at $\varepsilon = 0$) singular values have the following $\varepsilon$ dependence:

$$\sigma_{a - r_i(k) + l} = \varepsilon \sigma_l[T^{\gamma(k)}] + O(\varepsilon^2)$$

(\Rightarrow) Consider the right-hand side of equation (A.1), with $\lambda_R = \tau_i(k)$ and $0 \leq \varepsilon \ll 1$. In that case, as it was explained in [23], the vanishing (at $\varepsilon = 0$) singular values have the following $\varepsilon$ dependence:

$$\sigma_{a - r_i(k) + l} = \varepsilon \cos \theta_l^{\gamma(k)} + O(\varepsilon^2),$$

(1.3)

with $l \in I_{\tau_i(k)} := \{1, \ldots, r_i(k)\}$ and $(T^{\gamma(k)})^l_j$ as given in equation (8). That is, $\sigma_l[T^{\gamma(k)}] = \cos \theta_l^{\gamma(k)}$, where $\theta_l^{\gamma(k)}$ are the angles between the subspaces $Y_L^{\gamma(k)}$ and $\Phi_R^{\gamma(k)}$. Since by the
hypothesis above these cosines are bounded by below (equation (A.2)), for all \( \tau_i, l \) and \( k_a \), this means that the singular values of \( A^a_{\gamma} k_a - \tau_i \delta^a_{\gamma} - i\varepsilon \delta^a_{\gamma} \) are all of order \( O(\varepsilon^0) \) or \( O(\varepsilon^1) \). So by theorem 3.3 in [23] the matrix \( A^a_{\gamma} k_a \) is diagonalizable for all \( k_a \) not proportional to \( n_a \) and \( |k| = 1 \). Thus equation (A.1) holds with \( \frac{1}{\varepsilon} = \cos \theta \) and \( 0 \leq \varepsilon < 1 \). We now extend the proof for all \( \varepsilon \). To do that, we shall take the limit when \( \varepsilon \) goes to zero in equation (5), in two different ways, resulting in an upper bound of the eigen-projectors of \( A^a_{\gamma} k_a \) (as Strang showed [28]) that then we shall use to conclude the first part of the theorem. We first take,\n\n\[
\lim_{\varepsilon \to 0} \left| (A^a_{\gamma} k_a - \tau_i (k) \delta^a_{\gamma} - i\varepsilon \delta^a_{\gamma})^{-1} \right| \varepsilon = \lim_{\varepsilon \to 0} \min_{l \in \mathbb{Z}} \sigma_l \left[ A^a_{\gamma} k_a - \tau_i (k) \delta^a_{\gamma} - i\varepsilon \delta^a_{\gamma} \right], \tag{A.4}
\]
\[
\lim_{\varepsilon \to 0} \left| (A^a_{\gamma} k_a - \tau_i (k) \delta^a_{\gamma} - i\varepsilon \delta^a_{\gamma})^{-1} \right| \varepsilon = \lim_{\varepsilon \to 0} \min_{l \in \mathbb{Z}} \sigma_l \left[ A^a_{\gamma} k_a - \tau_i (k) \delta^a_{\gamma} - i\varepsilon \delta^a_{\gamma} \right], \tag{A.5}
\]
On the other hand, since we know that \( A^a_{\gamma} k_a \) is diagonalizable, it can be written in terms of their eigen-projectors \( (P^i (k))^\gamma \)
\[
A^a_{\gamma} k_a = \sum_{i \in F(k)} \tau_j (k) (P^i (k))^\gamma, \nonumber
\]
where \( (P^i (k))^\gamma (P^j (k))^\eta = \sum_{j \in F(k)} \delta^j (P^j (k))^\alpha \) and \( \sum_{j \in F(k)} (P^j (k))^\alpha = \delta^\gamma \). Since
\[
\lim_{\varepsilon \to 0} \left| (A^a_{\gamma} k_a - \tau_i (k) \delta^a_{\gamma} - i\varepsilon \delta^a_{\gamma})^{-1} \right| = \lim_{\varepsilon \to 0} \left| \sum_{j \in F(k)} \frac{\tau_j (k) - \tau_i (k) - i\varepsilon}{(P^j (k))^\gamma} \right| = \left| (P^i (k))^\gamma \right|,
\]
we conclude
\[
\left| (P^i (k))^\gamma \right| = \frac{1}{\min_{l \in \mathbb{Z}} \sigma_l \theta^a_{l(k)}}, \tag{A.6}
\]
and so, by equation (A.2),
\[
\left| (P^i (k))^\gamma \right| \leq \frac{1}{\cos \theta} = C.
\]
Thus, for any \( \varepsilon > 0 \)
\[
\left| (A^a_{\gamma} k_a - \lambda R \delta^a_{\gamma} - i\varepsilon \delta^a_{\gamma})^{-1} \right| \varepsilon = \left| \sum_{j \in F(k)} \frac{\varepsilon}{\tau_j (k) - \lambda R - i\varepsilon} (P^j (k))^\gamma \right|,
\]
\[
\leq \sum_{j \in F(k)} \left| \frac{\varepsilon}{\tau_j (k) - \lambda R - i\varepsilon} \right| (P^j (k))^\gamma,
\]
\[
\leq \sum_{j \in F(k)} (P^j (k))^\gamma,
\]
\[
\leq uC.
\]
where in third line we have used that $\left|\sum_{i=1}^{n} \frac{1}{\sqrt{|r_i|}}\right| \leq 1$.

$(\Rightarrow)$ If Kreiss resolvent matrix condition holds we know by definition 2 that $A^{\alpha \beta}k$ is diagonalizable with real eigenvalues for all $k$. Since now equation (A.4) is bounded above by $C$, we conclude, using equation (A.5), that

$$\min_{l \in F_{(k)}} \cos \theta l_{(i)}^{\alpha \beta} \leq C$$

for all $i \in F_{(k)}$ and all $k$. This concludes the proof of theorem 1.

**Appendix B. General Kronecker decomposition**

For presentation in what follows we shall define:

$$E^A_\eta \eta := (N^A \eta \eta n), B^A_\alpha \eta := (N^A \eta \eta k)$$

The full Kronecker decomposition of any pair of matrices $(E^A_\eta, B^A_\alpha)$ in the pencil form $\lambda E^A_\eta + B^A_\alpha$ is given by

$$\lambda E^A_\eta + B^A_\alpha = Y^A_B K^A_\alpha (\lambda) W^\eta_\alpha,$$

where $Y^A_B$ and $W^\eta_\alpha$ are invertible independent of $\lambda$, and $K^A_\alpha (\lambda)$ is a block matrix.

As we mention in section 3.1, when $E^A_\eta$ has only trivial right kernel, the blocks of $K^A_\alpha (\lambda)$ are: $J_m (\lambda)$-Jordan Blocks, $L^T_m$-blocks and $L^L_0$-vanished rows. In the general case, when $E^A_\eta$ has non-trivial right kernel, new blocks appear:

The zero blocks

$$O_{m \times l} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \ldots & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathbb{C}^{m \times l},$$

the $L_m$ blocks

$$L_m = \begin{pmatrix} \lambda & 1 & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 & 0 \\ 0 & 0 & \ldots & \ldots & 0 \\ 0 & 0 & 0 & \lambda & 1 \end{pmatrix} \in \mathbb{C}^{m \times m+1},$$

and the $N_m$ blocks

$$N_m = \begin{pmatrix} 1 & \lambda & 0 & 0 & 0 \\ 0 & 1 & \lambda & 0 & 0 \\ 0 & 0 & \ldots & \ldots & 0 \\ 0 & 0 & 0 & 1 & \lambda \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in \mathbb{C}^{m \times m}.$$
Notice that the right kernel of $E^A_\eta$ is given by the non-trivial kernel of the zero blocks,
\[
\begin{pmatrix}
0 \\
0 \\
\vdots \\
1
\end{pmatrix} \in \mathbb{C}^{m+1 \times 1}
\text{ for } L_m \text{ and }
\begin{pmatrix}
1 \\
0 \\
\vdots \\
0
\end{pmatrix} \in \mathbb{C}^{m \times 1}
\text{ for } N_m.
\]

**Appendix C. Kronecker decomposition of hyperbolic systems**

In this appendix, we shall show how to construct the basis in which the system reduces to Kronecker blocks.

Consider $l(\lambda)_b \in S^C_{n_a}$ for some $n_a$. We shall use the left kernel as the principal ingredient for building the basis. It is given by equation,
\[
X_A \eta \eta l(\lambda)_b = X_A \left[ \lambda \left( -\eta l(\lambda)_b \right) + (\eta l(\lambda)_b) \right] = 0.
\]

Since we are dealing with hyperbolic systems for which we already have shown that the Jordan blocks are one-dimensional we shall restrict consideration to $-\eta l(\lambda)_b$ of maximal range and $\eta l(\lambda)_b$ with only one-dimensional Jordan blocks.

As appendix B, in what follows we shall call these matrices, $E^A_\eta$:
\[
E^A_\eta := \left( -\eta l(\lambda)_b \right), \quad B^A_\alpha := \left( \eta l(\lambda)_b \right).
\]

First consider the case $e = u$, where $e = \dim(A)$ and $u = \dim(\alpha)$ for some $x, \phi, n, k$, then $\eta l(\lambda)_b$ has a trivial Jordan decomposition, therefore
\[
\eta l(\lambda)_b = \sum_{i=1}^u \eta l(\lambda)_b (\lambda - \lambda_i),
\]
\[
= \eta l(\lambda)_b (\lambda - \lambda_1) + \eta l(\lambda)_b (\lambda - \lambda_2) + \cdots + \eta l(\lambda)_b (\lambda - \lambda_u),
\]
\[
= \left( \begin{array}{cccc}
\lambda - \lambda_1 & 0 & 0 & 0 \\
0 & \lambda - \lambda_2 & 0 & 0 \\
0 & 0 & \ldots & 0 \\
0 & 0 & 0 & \lambda - \lambda_u
\end{array} \right)
\left( \begin{array}{c}
\eta l(\lambda)_b (\lambda - \lambda_1) \\
\eta l(\lambda)_b (\lambda - \lambda_2) \\
\vdots \\
\eta l(\lambda)_b (\lambda - \lambda_u)
\end{array} \right).
\]

Here $\{ \eta l(\lambda)_b \}$ is a basis of left eigenvectors associated to the eigenvalues $\lambda_i$ (or the left kernel of $\eta l(\lambda)_b$) i.e.
\[
u l(\lambda)_b (\lambda)_b = 0,
\]
and $\{ \eta l(\lambda)_b \}$ is the co-basis such that
\[
u l(\lambda)_b \eta l(\lambda)_b = \delta_i^j.
\]

In equation (C.2) the vectors $\{ \eta l(\lambda)_b \}$ are in column form and the co-vectors $\nu l(\lambda)_b$ in row form. Notice that
\[
\sum_{i=1}^u \eta l(\lambda)_b (\lambda - \lambda_i) = \left( -\eta l(\lambda)_b \right),
\]
\[
\sum_{i=1}^u \nu l(\lambda)_b (\lambda - \lambda_i) = \left( \eta l(\lambda)_b \right).
\]
and it is possible to find the eigenvalues $\lambda_i$, solving the polynomial equation $\det(\mathcal{M}_n^{\Delta}(\lambda)) = 0$ for $\lambda$.

In the case in which $e > u$ for some $(x, \phi, n, k)$, the decomposition has, besides the Jordan blocks, additional blocks usually denoted as $L^m_n$-blocks. The maximal range condition on $\mathcal{M}_n^{\Delta}$ prevents other Kronecker blocks from appearing.

As before we use the left kernel to compute the decomposition. For arbitrary $\lambda$, and fixed $\kappa$, there is a left kernel subspace which is of fixed dimension $(e - u)$. This subspace, which we call $\Delta(\lambda)$, depends in a polynomial way with respect to $\lambda$ in fact it is generated by a set of linearly independent vectors $\{\chi^i_\lambda(\lambda)\}$ with $i = 1, \ldots, e - u$ such that $\chi^i_\lambda(\lambda) \mathcal{M}_n^{\Delta}(\lambda) = 0 \forall \lambda \in \mathbb{C}$.

The coefficients of the $\chi^i_\lambda(\lambda)$ are polynomial in $\lambda$. Among all the possible bases of $\Delta(\lambda)$ we take the non-zero $\chi^1_\lambda(\lambda)$ such that it has the least degree, that we call $m_1$, in $\lambda$. It is

$$\chi^1_\lambda(\lambda) = ((\theta_{m_1})_A - \lambda(\theta_{m_1 - 1})_A + \lambda^2(\theta_{m_1 - 2})_A - \cdots - \lambda^{m_1}\theta_{m_1})_A,$$

(C.4)

with $(\theta_i)\lambda$ independent of $\lambda$. We continue by choosing $\chi^2_\lambda(\lambda)$ independent of $\chi^1_\lambda(\lambda)$ and of degree too, such that $m_1 \leq m_2$.

We perform this process until there are no linearly-independent vectors in $\Delta(\lambda)$. Thus we finish with a set of scalars that are called minimal indices for the rows of $\mathcal{M}_n^{\Delta}(\lambda)$, which are

$$m_1 \leq m_2 \leq \cdots \leq m_{e - u}.$$

They define the $L^m_n$ blocks of $\mathcal{M}_n^{\Delta}(\lambda)$. In particular for each $\chi^i_\lambda$ as in equation (C.4), from equation (C.1), we have the following:

$$(\theta_{m_1})_A B^1_\alpha = 0, \quad (\theta_{m_1 - 1})_A B^2_\alpha = (\theta_{m_1})_A E^2_\eta, \cdots$$

$$\quad (\theta_0)_A B^m_\alpha = (\theta_1)_A E^m_\eta, \quad (\theta_0)_A E^m_\eta = 0. \quad \text{(C.5)}$$

The example in equation (12) has $m_1 = 0$, $m_2 = 0$ because the two null rows and $m_3 = 1$, $m_4 = 2$ since it has an $L^1_n$ and $L^2_n$ blocks.

In addition, when $\lambda = \lambda_i$ the left kernel increases with a set of new independent covectors called $\{v_i\}$. These covectors are only defined up to members of $\{\chi^i_\lambda(\lambda)\}$.

Collecting all the $\{v_i\}$ associated to the different generalized eigenvalues, and the covectors $\{\chi^i_\lambda(\lambda)\}$, it is possible to compute the Kronecker decomposition as in the following example:

$$\mathcal{M}_n^{\Delta}(\lambda) = (v^{1A} v^{2A} v^{3A} \theta^{3A} \theta^{2A} \theta^{1A} \theta^{0A} \tilde{\theta}^{0A} \tilde{\theta}^{1A} \tilde{\theta}^{2A})$$

$$\times \begin{pmatrix}
\lambda - \lambda_1 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda - \lambda_2 & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda - \lambda_3 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda & 0 & 0 \\
0 & 0 & 0 & 1 & \lambda & 0 \\
0 & 0 & 0 & 0 & 1 & \lambda \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
v_1 E^C_\alpha \\
v_2 E^C_\alpha \\
v_3 E^C_\alpha \\
\theta_1 E^C_\alpha \\
\theta_2 E^C_\alpha \\
\theta_3 E^C_\alpha \\
\theta_4 E^C_\alpha \\
\end{pmatrix}$$

12 We recall that the $\lambda_i$ can be calculated by solving equation (7).
\[ \begin{align*}
&= \sum_{i=1}^{3} v^{iA} (v_{iA} E^{A}) (\lambda - \lambda_i) + \lambda \sum_{i=1}^{m_2} \theta^{iA} (\theta_i C E^{C}) + \sum_{i=0}^{m_3} \theta^{iA} (\theta_{i+1} C E^{C}), \\
&\quad + \theta^{iA} \left( \theta_i C E^{C} \right) + \theta^{iA} \left( \theta_{i+1} C E^{C} \right)
\end{align*} \]

Here \( m_1 = 0, m_2 = 0, m_3 = 3 \) and \( v_{iA} \) are the left eigenvectors associated to the generalized eigenvalues \( \lambda_i \), \( \theta_{0A} \) define the \( L_{m_1}^{T} \), \( \theta_{0A} \) the \( L_{m_2}^{T} \) and \( \theta_{iA} \) with \( i = 0, \ldots, 3 \) define the \( L_{m_3}^{T} \) block. Here the \( \lambda_i \) can be degenerate, i.e. they can take the same values.

The vectors with the raised indices define a co-basis,

\[ v^{iA} v_{jA} = \delta_{ij} \quad \text{with } i, j = 1, 2, 3, \]

\[ \theta^{iA} \theta_{jA} = \delta_{ij} \quad \text{with } i, j = 1, 2, 3, \]

\[ \bar{\theta}^{iA} \bar{\theta}_{jA} = 1, \]

\[ \bar{\theta}^{iA} \bar{\theta}_{0A} = 1, \]

and any other contractions vanish.

### Appendix D. Reductions in \( L_{T}^{T} \) case

Consider a set of \( L_{T}^{T} \)-blocks in the Kronecker decomposition of a principal symbol, we can reduce all of them together and find more general reductions with diagonalizable reduced principal symbols, for instance, given the following structure,

\[
\begin{pmatrix}
L_{1}^{T} & 0 & 0 \\
0 & L_{1}^{T} & 0 \\
0 & 0 & \ldots \\
0 & 0 & 0 & L_{1}^{T}
\end{pmatrix}
= \lambda \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
= \lambda \left( \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} + \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \right).
\]

For these cases, more general \( H \) are possible, they are given by:

\[
\begin{pmatrix}
b_1 & 0 & \tilde{b}_2 & 0 & \ldots & 0 & b_{s} & 0 \\
b_2 & 0 & \tilde{b}_3 & 0 & \ldots & 0 & b_{2s-1} & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
b_{s} & 0 & b_{2s-1} & 0 & \ldots & 0 & b_{3n+1} & 0
\end{pmatrix}
+ \begin{pmatrix}
0 & c_1 & 0 & \tilde{c}_2 & 0 & \ldots & 0 & \tilde{c}_s \\
0 & c_2 & 0 & c_{s+1} & 0 & \ldots & 0 & \tilde{c}_{2s-1} \\
0 & \ldots & 0 & \ldots & 0 & \ldots & 0 & \ldots \\
0 & c_{s} & 0 & c_{2s-1} & 0 & \ldots & 0 & c_{3n+1}
\end{pmatrix}
\]

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Indeed,
\[
\left( H_{LT}^T \right)_{\delta A} \begin{pmatrix}
L_T^T & 0 & 0 & 0 \\
0 & L_T^T & 0 & 0 \\
0 & 0 & \ldots & 0 \\
0 & 0 & 0 & L_T^T
\end{pmatrix} = \lambda(g_1)_{\delta A} + (g_2)_{\delta A},
\]
with
\[
(g_1)_{\delta A} = \begin{pmatrix}
b_1 & \bar{b}_2 & \ldots & \bar{b}_s \\
b_2 & b_{r+1} & \ldots & \bar{b}_{2r-1} \\
\vdots & \vdots & \ddots & \vdots \\
b_r & b_{2s-1} & \ldots & b_{\frac{m+1}{2}n}
\end{pmatrix},
(g_2)_{\delta A} = \begin{pmatrix}
c_1 & \bar{c}_2 & \ldots & \bar{c}_s \\
c_2 & c_{r+1} & \ldots & \bar{c}_{2r-1} \\
\vdots & \vdots & \ddots & \vdots \\
c_r & c_{2s-1} & \ldots & c_{\frac{m+1}{2}n}
\end{pmatrix}.
\]

Since both are Hermitian we only need to assert positivity of \((g_1)_{\delta A}\) to conclude the reduction gives rise to a diagonalizable reduced system. This can be done by choosing appropriately the \(b\) coefficients.

**Appendix E. Orthogonality of kernel subspaces**

Here we show that the subspaces \(\Phi^\lambda_{\eta^k}\) and \(\Delta^\lambda_{\eta^k}\) are orthogonal among each other.

As before we use the notation, \(\Omega^\lambda_{\eta^k} n(\lambda)\) = \((\lambda L^A_{\eta^k} + B^A_{\eta^k})\) with \(l(\lambda)\) \(\in S_{\eta^k}\). We are going to show that
\[
\chi^\lambda_{\eta^k}(\lambda) = 0.
\]

As it was explained in appendix C, for each \(L^A_{\eta^k}\) block of the principal symbol \(\Omega^\lambda_{\eta^k} n(\lambda)\), there are \(m+1\) covectors \(\{\theta_i\}\) such that
\[
\chi^\lambda_{\eta^k}(\lambda) = \left( (\theta_m)_{\lambda} - \lambda(\theta_{m-1})_{\lambda} + \lambda^2(\theta_{m-2})_{\lambda} - \cdots - \lambda^m(\theta_0)_{\lambda} \right)
\]
is in the left kernel for all \(\lambda \in \mathbb{R}\) and the covectors \(\{\theta_i\}\) fulfill the equation (C.5). This set of covectors expanded the subspace \(\Delta^\lambda(\lambda)\). On the other hand when \(\lambda = \lambda_{\eta^k}(k)\) the right kernel elements satisfy:
\[
(\lambda_{\eta^k}(k)L^A_{\eta^k} + B^A_{\eta^k}) \left( \delta \phi_{\lambda} \right)_{\eta^k} = 0.
\]

In order to show equation (E.1), it is enough to prove that \((\theta_{\eta^k})_{\lambda} E^A_{\eta^k} \left( \delta \phi_{\lambda} \right)_{\eta^k} = 0\) for any \(s\) in any \(L^T\)-block. Using the equations (C.5) and (E.3) several times in sequence,
\[
(\theta_{\eta^k})_{\lambda} E^A_{\eta^k} \left( \delta \phi_{\lambda} \right)_{\eta^k} = (\theta_{\eta^k})_{\lambda} E^A_{\eta^k} \left( \delta \phi_{\lambda} \right)_{\eta^k},
= (-1)^s \lambda_{\eta^k}(k) (\theta_{\eta^k})_{\lambda} E^A_{\eta^k} \left( \delta \phi_{\lambda} \right)_{\eta^k},
= \cdots
= 0,
\]
we prove the assertion.
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