Differential technique for the covariant orbital angular momentum operators

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February 23, 2018

Abstract:

The orbital angular momentum operator expansion turns to be a powerful tool to construct the fully covariant partial wave amplitudes of hadron decay reactions and hadron photo- and electroproduction processes.

In this paper we consider a useful development of the orbital angular momentum operator expansion method. We present the differential technique allowing the direct calculation of convolutions of two orbital angular momentum operators with an arbitrary number of open Lorentz indices. This differential technique greatly simplifies calculations when the reaction subject to the partial wave analysis involves high spin particles in the initial and/or final states. We also present a useful generalization of the orbital angular momentum operators.
1 Introduction

Study of the strong interaction at low and intermediate energies provides us the detailed information on the spectrum, properties and structure of strongly interacting particles - hadrons. This knowledge is indispensable for improving our understanding of QCD in the nonperturbative regime. It also brings crucial tests both for the QCD-inspired phenomenological strong interaction models and for the lattice QCD calculations.

The resent discoveries in the heavy quark sector \[1, 2, 3, 4\] brought evidences for the existence of the tetraquark mesons and pentaquark baryons (see e.g. \[5\] for a recent review and complete set of references). This definitely forces us to go beyond the naive quark model that describes mesons and baryons as bound states of quark-antiquark and three quarks respectively. In the light quark meson sector the situation is yet not so clear, however a number of collaborations have reported the observation of the so-called exotic mesons with quantum numbers forbidden for the \(q\bar{q}\)-system. For the light quark baryon sector it worths mentioning the long standing problem \[6\] of structure of \(\Lambda(1405)\) \(J^P = \frac{1}{2}^-\) resonance which is considered to be a candidate for the \(K\bar{N}\) bound state \[7, 8\]. Another example is the controversial \(uudd\bar{s}\ \theta^+(1540)\) state \[9\] widely discussed in the 2000s (see \[10\] for a discussion) and its non-strange partners (see e.g. \[11\] and references therein).

A usual pattern for the exotic states decays are the channels involving three or more particles in the final state. This also turns to be the case for the radial excitations of the non-exotic states. Signals from such states can be extracted from experimental data by means of a rather involved partial wave analysis. This analysis must correctly account for all correlations in the multidimensional phase space between amplitudes corresponding to all possible decay chains.

In a number of analyzes (see e.g. \[4\]) the corresponding partial wave amplitudes are constructed by means of the two step procedure. Firstly, one calculates amplitudes for transition between all possible two particle channels. The angular dependence of such amplitudes is described by the well known non-relativistic spherical functions. At the second step, the complete amplitude for a given decay chain is constructed as a product of the corresponding two-particle amplitudes, which are rotated and boosted into a particular reference frame. Then, the amplitudes for all possible decay chains, in turn, are rotated and boosted to one selected reference frame to get the correct interference pattern. However, this approach is plagued with ambiguities in the boosting procedure. It seems that a more straightforward way is to use the covariant approach from the very beginning. By construction, the covariant approach is independent of the reference frame and can be directly applied to any reaction with multiparticle final states.

The development of the covariant approach for the partial wave analysis of the experimental data has a long history \[12, 13, 14, 15, 16\]. The systematic method, which allow to construct partial wave amplitudes for arbitrary value of resonance spin and arbitrary spins of decay particles, was suggested in \[17\] and saw further development in \[18, 19\]. It is based on the spin-orbital classification of partial wave amplitudes. One of the main components of this approach is the construction of the orbital-angular-momentum-operators.
The OAM-operators are the Lorentz tensors with rank which corresponds to the orbital angular momentum \( n \) of the resonating two particle system. These operators can be constructed with the help of recurrent relations expressing higher order OAM-operators through the operators of lower ranks.

Let us consider a resonance decay into an intermediate state and a spectator state of particular orbital angular momentum. The corresponding partial wave amplitudes are constructed as convolutions of several OAM-operators. These convolutions may have open Lorentz indices, which are further convoluted with the polarization vectors of the initial and final state particles. Calculations of OAM-operators convolutions with open Lorentz indices, in many cases, represent a rather complicated mathematical task.

In this paper we present the differential technique allowing a direct calculation of convolutions of two OAM-operators with an arbitrary number of open Lorentz indices. We consider this procedure as a significant step in the development of the covariant approach for the partial wave analysis of the reactions with multiparticle final states. It also will help to develop the fully electromagnetically gauge invariant description of hadron photo- and electroproduction reactions.

2 Properties of orbital-angular-momentum-operators

\( X_{\mu_1...\mu_n}^{(n)} \)

In this Section we review the basic properties of the OAM-operators \( X_{\mu_1...\mu_n}^{(n)} \). These operators occur in the description of the decay of a composite particle with integer spin \( n \) and momentum \( P = k_1 + k_2 \) \((P^2 = s)\) into two spinless particles with momenta \( k_1 \) and \( k_2 \). They also serve as building blocks for the description of more involved cases.

In order to ensure that the operators \( X_{\mu_1...\mu_n}^{(n)} \) correspond to the appropriate spin-\( n \) irreducible representations of the Lorentz group these operators should satisfy the following list of properties [17]:

- Symmetry with respect to permutation of indices:
  \[ X_{\mu_1...\mu_i...\mu_j...\mu_n}^{(n)} = X_{\mu_1...\mu_j...\mu_i...\mu_n}^{(n)}; \] (1)

- Orthogonality to the total momentum \( P \):
  \[ P^{\mu_i} X_{\mu_1...\mu_i...\mu_n}^{(n)} = 0; \] (2)

- The tracelessness property over any pair of indices:
  \[ g^{\mu_i\mu_j} X_{\mu_1...\mu_i...\mu_j...\mu_n}^{(n)} = 0. \] (3)

In order to satisfy the orthogonality condition (2) the OAM-operators are constructed from the relative momentum \( k_{\mu}^{\perp} \) and the orthogonal metric tensor \( g_{\mu\nu}^{\perp} \):

\[
k_{\mu}^{\perp} = g_{\mu\nu}^{\perp} \frac{1}{2} (k_1 - k_2)_{\nu}; \quad g_{\mu\nu}^{\perp} = g_{\mu\nu} - \frac{P_{\mu} P_{\nu}}{s}.
\] (4)
Note that the trace of the orthogonal metric tensor $g_{\mu \nu}^{\perp}$ defined in (3) is

$$g_{\mu}^{\perp \mu} \equiv 3.$$ 

The operator for $n = 0$ is a scalar and the $n = 1$ operator is just $k_{\mu}^{\perp}$:

$$X^{(0)}(k_{\perp}) = 1; \quad X^{(1)}(k_{\perp}) = k_{\mu}^{\perp}. \quad (5)$$

The operators $X_{\mu_{1}...\mu_{n}}^{(n)}(k_{\perp})$ for $n > 1$ can be constructed from the recurrence relation

$$X_{\mu_{1}...\mu_{n}}^{(n)}(k_{\perp}) = k_{\perp}^{\alpha} Z_{\mu_{1}...\mu_{n},\alpha}^{(n-1)}, \quad \text{where}$$

$$Z_{\mu_{1}...\mu_{n},\alpha}^{(n-1)} = \frac{2n - 1}{n^2} \sum_{i=1}^{n} g_{\mu_{i}^\perp \alpha} X_{\mu_{1}...\mu_{i-1}\mu_{i+1}...\mu_{n}}^{(n-1)}(k_{\perp})$$

$$- \frac{2}{n^2} \sum_{i,j=1 \atop i < j}^{n} g_{\mu_{i} \mu_{j}}^{\perp} X_{\mu_{1}...\mu_{i-1}\mu_{i+1}...\mu_{n}}^{(n-1)}(k_{\perp})$$

$$- \frac{2}{n^2} \sum_{i,j=1 \atop i < j}^{n} g_{\mu_{i} \mu_{j}}^{\perp} X_{\mu_{1}...\mu_{i-1}\mu_{i+1}...\mu_{n}}^{(n-1)}(k_{\perp}). \quad (6)$$

It is straightforward to check that $X_{\mu_{1}...\mu_{n}}^{(n)}(k_{\perp})$ defined from (3) satisfies the properties listed in Eqs. (1), (2), (3).

The following convolution identity is valid

$$k_{\perp}^{\mu_{n}} X_{\mu_{1}...\mu_{n}}^{(n)}(k_{\perp}) = |k_{\perp}|^{2} X_{\mu_{1}...\mu_{n-1}}^{(n-1)}(k_{\perp}), \quad |k_{\perp}| \equiv \sqrt{k_{\perp}^{2}}. \quad (7)$$

By iterating (3) one can work out the explicit expression for the operator $X_{\mu_{1}...\mu_{n}}^{(n)}(k_{\perp})$:

$$X_{\mu_{1}...\mu_{n}}^{(n)}(k_{\perp})$$

$$= \alpha(n) \left[ k_{\mu_{1}}^{\perp} \ldots k_{\mu_{n}}^{\perp} - \frac{|k_{\perp}|^{2}}{2n - 1} \left( \sum_{i,j=1 \atop i < j}^{n} g_{\mu_{i} \mu_{j}}^{\perp} k_{\mu_{1}}^{\perp} \ldots k_{\mu_{i}}^{\perp} \wedge \ldots k_{\mu_{j}}^{\perp} \wedge \ldots k_{\mu_{n}}^{\perp} \right) - \frac{|k_{\perp}|^{4}}{(2n - 1)(2n - 3)} \right.$$  

$$\times \left( \sum_{i,j=1 \atop k < l}^{n} \sum_{k,l=1 \atop k < l, k \neq i,j}^{n} g_{\mu_{i} \mu_{j} \mu_{k} \mu_{l}}^{\perp} k_{\mu_{1}}^{\perp} \ldots k_{\mu_{i}}^{\perp} \wedge \ldots k_{\mu_{j}}^{\perp} \wedge \ldots k_{\mu_{k}}^{\perp} \wedge \ldots k_{\mu_{l}}^{\perp} \right) + \ldots \right], \quad (8)$$

where throughout this paper $k_{\mu_{i}}^{\perp}$ denotes that the $i$-th entry $k_{\mu_{i}}^{\perp}$ is omitted in the $k_{\mu_{1}}^{\perp} \ldots k_{\mu_{n}}^{\perp}$ range:

$$k_{\mu_{1}}^{\perp} \ldots k_{\mu_{i}}^{\perp} \ldots k_{\mu_{n}}^{\perp} \equiv k_{\mu_{1}}^{\perp} \ldots k_{\mu_{i-1}}^{\perp} k_{\mu_{i+1}}^{\perp} \ldots k_{\mu_{n}}^{\perp};$$

and $\alpha(n)$ stands for the normalization factor.

Employing the convolution identity (7) together with the recurrence relation (3) one can work out the normalization of the OAM-operators:

$$X_{\mu_{1}...\mu_{n}}^{(n)}(k_{\perp}) X_{\mu_{1}...\mu_{n}}^{(n)}(k_{\perp}) = \alpha(n) |k_{\perp}|^{2n}; \quad \alpha(n) = \frac{(2n - 1)!!}{n!}. \quad (9)$$
One can check that the following generalization of the convolution relation (10) is valid for \( n \geq l \):

\[
X_{\mu_1 \ldots \mu_n}^{(n)}(k_\perp) X_{\mu_1 \ldots \mu_l}^{(l)}(k_\perp) = \alpha(l)|k_\perp|^2 X_{\mu_1 \ldots \mu_{n-l}}^{(n-l)}(k_\perp). \tag{10}
\]

Note that for \( l = 1 \) one recovers the convolution relation (9) while \( l = n \) leads to the normalization condition (8).

The convolution of two spin-\( n \) OAM-operators with the momenta \( k \) and \( q \) is given by

\[
X_{\mu_1 \ldots \mu_n}^{(n)}(k_\perp) X_{\mu_1 \ldots \mu_n}^{(n)}(q_\perp) = \alpha(n)|q_\perp|^n |k_\perp|^n P_n(z); \tag{11}
\]

where \( z = \frac{(q_\perp, k_\perp)}{|q_\perp||k_\perp|} \), and \( P_n(z) \) are the Legendre polynomials.

### 3 Differential technique for OAM-operators

In this section we adopt the covariant version of C. Zemach’s O(3) differential technique [12] for the case of OAM-operators \( X_{\mu_1 \ldots \mu_n}^{(n)}(k_\perp) \). We consider the derivative operation \( D_{\mu}^{(n+1)}(k_\perp) \):

\[
D_{\mu}^{(n+1)}(k_\perp) = \frac{1}{n+1} \left( (2n+1)\frac{k_\mu^\perp}{|k_\perp|^2} - \frac{k_\mu^\perp}{|k_\perp|} \frac{\partial}{\partial |k_\perp|} - \frac{\partial}{\partial k_\perp^\mu} \right). \tag{12}
\]

We would like to show that

\[
D_{\mu}^{(n+1)}(k_\perp) \{ |k_\perp|^2 X_{\mu_1 \ldots \mu_n}^{(n)}(k_\perp) \} = X_{\mu_1 \ldots \mu_n}^{(n+1)}(k_\perp). \tag{13}
\]

This derivative operation can be seen as the “inversion” of the convolution formula (10). It allows the iterative construction of the spin-\((n+1)\) OAM-operator from the spin-\(n\) OAM-operator.

For our proof we employ the expression (8) for OAM-operators obtained by iterating the recurrence expression (9) for \( X_{\mu_1 \ldots \mu_n}^{(n)} \). This expansion is a polynomial in \(|k_\perp|^2 \). For given \( n \) the highest power is \(|k_\perp|^n \) for \( n \)-even and \(|k_\perp|^{n-1} \) for \( n \)-odd. Let us consider the action of the derivative operator (12) on the OAM-operator

\[
D_{\mu}^{(n+1)}(k_\perp)|k_\perp|^2 X_{\mu_1 \ldots \mu_n}^{(n)}(k_\perp). \tag{14}
\]

For example, consider \(|k_\perp|^0\) and \(|k_\perp|^2\) terms in (8):

\[
D_{\mu}^{(n+1)}(k_\perp)|k_\perp|^2 \frac{\alpha(n) k_\mu^\perp \ldots k_{\mu_n}^\perp}{n+1} = \frac{1}{n+1} (2n+1) \frac{\alpha(n) k_\mu^\perp \ldots k_{\mu_n}^\perp}{n+1} - \frac{|k_\perp|^2}{n+1} \alpha(n+1) \sum_{i=1}^n g_{\mu_i}^{\perp} k_{\mu_1}^\perp \ldots k_{\mu_{i-1}}^\perp k_{\mu_{i+1}}^\perp \ldots k_{\mu_n}^\perp
\]

\[
= \alpha(n+1) k_\mu^\perp \ldots k_{\mu_n}^\perp - \frac{|k_\perp|^2}{2n+1} \alpha(n+1) \sum_{i=1}^n g_{\mu_i}^{\perp} k_{\mu_1}^\perp \ldots k_{\mu_{i-1}}^\perp k_{\mu_{i+1}}^\perp \ldots k_{\mu_n}^\perp. \tag{15}
\]

\[
|k_\perp|^0 \text{ term for } X_{\mu_1 \ldots \mu_n}^{(n+1)}(k_\perp), \quad \text{part of } |k_\perp|^2 \text{ term for } X_{\mu_1 \ldots \mu_n}^{(n+1)}(k_\perp).
\]
for the calculation of convolutions of OAM-operators with several open indices. This proof of eq. (17) is extremely convenient for the calculation of convolutions of OAM-operators with several open indices. This technique turns to be fully equivalent to the covariant differential technique for the so-called contracted projectors suggested by M. Scadron in Ref. [13] (see also Chapter I of Ref. [14] and Appendix A of Ref. [20] for a short review of the contracted projectors method). In fact,

\[ X^{(n)}_{\alpha_1...\alpha_n}(k_\perp)X^{(n)}_{\alpha_1...\alpha_n}(k_\perp) = \mathcal{P}^n(q, p, P), \tag{17} \]

where \( \mathcal{P}^n(q, p, P) \) stands for the contracted projectors (the numerator of the bosonic spin sum contracted with the initial and final relative momenta):

\[ \mathcal{P}^n(q, p, P) \equiv \alpha(n)^2 q_{\mu_1} \cdots q_{\mu_n} O^{\nu_1...\nu_n}_{\mu_1...\mu_n} k^{\nu_1} \cdots k^{\nu_n}, \tag{18} \]

\[ -D^{(n+1)}(k_\perp) \frac{|k_\perp|^4}{2n-1} \alpha(n) \left( \sum_{i,j=1 \atop i < j}^n g_{\mu_1,i_1}\mu_1 k_{\mu_1}^1 \cdots k_{\mu_j}^1 \cdots k_{\mu_n}^1 \right) \]

\[ = -\alpha(n) \frac{1}{n+1} \frac{2(n+1) + 1 - 4}{2n-1} |k_\perp|^2 \left( \sum_{i,j=1 \atop i < j}^n g_{\mu_1,i_1}\mu_1 k_{\mu_1}^1 \cdots k_{\mu_j}^1 \cdots k_{\mu_n}^1 \right) \]

\[ + \alpha(n) \frac{1}{n+1} \frac{1}{2n-1} |k_\perp|^4 \left( \sum_{i,j=1 \atop i < j}^n g_{\mu_1,i_1}\mu_1 k_{\mu_1}^1 \cdots k_{\mu_j}^1 \cdots k_{\mu_n}^1 \right) \]

\[ = -\alpha(n+1) \frac{|k_\perp|^2}{2n+1} \left( \sum_{i,j=1 \atop i < j}^n g_{\mu_1,i_1}\mu_1 k_{\mu_1}^1 \cdots k_{\mu_j}^1 \cdots k_{\mu_n}^1 \right) \]

\[ + \alpha(n+1) \frac{|k_\perp|^4}{(2n+1)(2n-1)} \left( \sum_{i,j=1 \atop i < j}^n \sum_{k=1 \atop k \neq i,j}^n g_{\mu_1,i_1}\mu_1 g_{\mu_1,j_1}\mu_1 k_{\mu_1}^1 \cdots k_{\mu_j}^1 \cdots k_{\mu_n}^1 \right) ; \tag{16} \]

and analogously for all higher order terms.

Therefore, we conclude that the result of action of the operator (12) on \(|k_\perp|^2 \times \{ O(|k_\perp|^m) \} \) term in \( X^{(n)}_{\mu_1...\mu_n}(k_\perp) \) (2 \( \leq m < n \), \( m \) - even) contributes into the expansion (11) of \( X^{(n+1)}_{\mu_1...\mu_n}(k_\perp) \) at orders \( O(|k_\perp|^m) \) and \( O(|k_\perp|^{m+1}) \) with the proper coefficients\footnote{The highest order \( O(|k_\perp|^n) \) term for even \( n \) is somewhat special. In this case there is a contribution only into \( O(|k_\perp|^n) \) term coming from it.}. By combining the two contributions at each order in \(|k_\perp| \) we recover the complete result for \( X^{(n+1)}_{\mu_1...\mu_n}(k_\perp) \) coinciding with that from the recurrence relation (8). This finalizes the proof of eq. (13).
where $O_{\mu_1...\mu_n}^{\nu_1...\nu_n}$ stands for the bosonic projection operator\footnote{For the explicit form and properties of the bosonic projection operator $O_{\mu_1...\mu_n}^{\nu_1...\nu_n}$ see e.g. Sec.3.2 of Ref.\cite{19}.} that projects an arbitrary rank-$n$ Lorentz tensor into a tensor, which satisfies the conditions (1)-(3).

For example, let us consider the convolution of OAM-operators with one open index. It is straightforward to check that

$$X_{\mu_{\alpha_1}...\alpha_n}(q_{\perp})X_{\alpha_1...\alpha_n}(k_{\perp}) = D_{\mu}^{(n+1)}(q_{\perp}) \left[ \alpha(n) |q_{\perp}|^{n+2} |k_{\perp}|^{n} P_n(z) \right]$$

$$= \frac{\alpha(n)}{(n+1)} |q_{\perp}|^{n+2} |k_{\perp}|^{n} \sum_{N=0}^{1} C_{\mu}^{(N;1,0)}(n; q_{\perp}, k_{\perp}) P_{n}^{(N)}(z),$$

where by $P_{n}^{(N)}(z)$ we denote the $N$-th derivative of the $n$-th Legendre polynomial and

$$C_{\mu}^{(0;1,0)}(n; q_{\perp}, k_{\perp}) = (n+1) \frac{q_{\mu \perp}}{|q_{\perp}|^2}; \quad C_{\mu}^{(1;1,0)}(n; q_{\perp}, k_{\perp}) = -\frac{dz}{dq_{\mu \perp}}.$$

Note that the tensor structures (20) possess the simple convolution properties

$$q_{\mu}^{\mu} C_{\mu}^{(0;1,0)}(n; q_{\perp}, k_{\perp}) = (n+1); \quad q_{\perp}^{\mu} C_{\mu}^{(1;1,0)}(n; q_{\perp}, k_{\perp}) = 0.$$

This ensures the validity of the identity

$$q_{\mu}^{\mu} X_{\mu_{\alpha_1}...\alpha_n}(q_{\perp})X_{\alpha_1...\alpha_n}(k_{\perp}) = |q_{\perp}|^{2} X_{\alpha_1...\alpha_n}(q_{\perp})X_{\alpha_1...\alpha_n}(k_{\perp}),$$

that is the consequence of (7).

Employing the explicit expression (B.8) for $\frac{dz}{dq_{\mu \perp}}$ together with the well known recurrence relation for the derivative of the Legendre polynomial

$$P'_{n+1}(z) = zP'_n(z) + (n + 1)P_n(z)$$

we check that we indeed recover the eq. (B.1) of Appendix B of Ref. \cite{18}:

$$D_{\mu}^{(n+1)}(q_{\perp}) \left[ \alpha(n) |q_{\perp}|^{n+2} |k_{\perp}|^{n} P_n(z) \right]$$

$$= \frac{\alpha(n)}{(n+1)} |q_{\perp}|^{n+1} |k_{\perp}|^{n} \left\{ - \frac{k_{\perp}^{\mu \perp}}{|k_{\perp}|} P'_n(z) + \frac{q_{\mu \perp}}{|q_{\perp}|} P'_{n+1}(z) \right\}.$$

The application of this technique for the more involved cases is presented in the Appendix \cite{14}.\footnote{For the explicit form and properties of the bosonic projection operator $O_{\mu_1...\mu_n}^{\nu_1...\nu_n}$ see e.g. Sec.3.2 of Ref.\cite{19}.}
4 Some generalization of OAM-operators

Formally the OAM-operators $X_{\mu_1...\mu_n}^{(n)}(k_\perp)$ are the most general spin-$n$ operators satisfying the list of requirements (1)–(3) constructed from a sole vector $k$. This can be most clearly seen from the formula employing the bosonic projection operator:

$$k_{\mu_1} \ldots k_{\mu_n} O_{\mu_1 \ldots \mu_n}^{\mu_1 \ldots \mu_n} = \frac{1}{\alpha(n)} X_{\mu_1 \ldots \mu_n}^{(n)}(k_\perp).$$  \hspace{1cm} (24)

A natural generalization, which was already considered within the non-covariant formalism of C. Zemach [12], are the operators constructed out of two, three and more independent vectors. For example, let us consider the spin-$n$ operators $X_{\mu_1 \ldots \mu_n}^{(n)}(k_\perp, q_\perp)$ constructed out $n - l$ entries of the four-vector $k$ and $l$ entries of the four-vector $q$ with $0 \leq l \leq n$:

$$\prod_{n-l \text{ entries }} k_{\mu_1} \ldots k_{\mu_{n-l}} q_{\mu_{n-l+1}} \ldots q_{\mu_n} O_{\mu_1 \ldots \mu_n}^{\mu_1 \ldots \mu_n} \equiv \frac{1}{\alpha(n)} X_{\mu_1 \ldots \mu_n}^{(n,l)}(k_\perp, q_\perp).$$  \hspace{1cm} (25)

One can check that

$$X_{\mu_1 \ldots \mu_n}^{(n,l)}(k_\perp, q_\perp) = \left[ \prod_{i=0}^{l-1} \frac{1}{n-i} \left( q_\perp^\mu \frac{d}{dk_\perp^\mu} \right) \right] X_{\mu_1 \ldots \mu_n}^{(n)}(k_\perp).$$  \hspace{1cm} (26)

Also it is straightforward to verify that for $l \leq n$

$$X_{\mu_1 \ldots \mu_n}^{(n)}(k_\perp) X_{\mu_1 \ldots \mu_n}^{(n,l)}(k_\perp, q_\perp) = \alpha(n)|k_\perp|^{2n-l} q_\perp^l P_l(z).$$  \hspace{1cm} (27)

Obviously for $l = n$ we get

$$X_{\mu_1 \ldots \mu_n}^{(n,l=n)}(k_\perp, q_\perp) = X_{\mu_1 \ldots \mu_n}^{(n)}(q_\perp)$$

and we recover (17) from (27).

One also can work out the following equation

$$(2n + 1) z X_{\mu_1 \ldots \mu_n}^{(n)}(k_\perp) = n \frac{|k_\perp|}{|q_\perp|} X_{\mu_1 \ldots \mu_n}^{(n,1)}(k_\perp, q_\perp) + (n + 1) \frac{q_\perp^\mu}{|k_\perp||q_\perp|} X_{\mu_1 \ldots \mu_n}^{(n+1)}(k_\perp).$$  \hspace{1cm} (28)

Let us sketch the proof of eq. (29). To get the first relation for the coefficients of the two tensors in the r.h.s. of eq. (29) one has to set $q_\perp = p_\perp$. To get the second relation for the coefficients one has to contract (29) with $X_{\mu_1 \ldots \mu_n}^{(n)}(k_\perp)$ and employ (1) together with (27) for $l = 1$ and (17).

Now, by contracting (29) with $X_{\mu_1 \ldots \mu_n}^{(n)}(k_\perp)$, one can see explicitly the property of the OAM-operators related to the well-known recurrence relation for the Legendre polynomials:

$$(2n + 1) z X_{\mu_1 \ldots \mu_n}^{(n)}(k_\perp) X_{\mu_1 \ldots \mu_n}^{(n)}(q_\perp) = n \frac{|k_\perp|}{|q_\perp|} \frac{X_{\mu_1 \ldots \mu_n}^{(n,1)}(k_\perp, q_\perp)}{\alpha(n)|k_\perp| P_{n-1}(z)} X_{\mu_1 \ldots \mu_n}^{(n)}(q_\perp) + (n + 1) \frac{q_\perp^\mu}{|k_\perp||q_\perp|} \frac{X_{\mu_1 \ldots \mu_n}^{(n+1)}(k_\perp)}{\alpha(n)|k_\perp|^{n+1}|q_\perp| P_{n+1}(z)}.$$

$$+(n + 1) \frac{1}{|k_\perp||q_\perp|} \frac{X_{\mu_1 \ldots \mu_n}^{(n+1)}(k_\perp) q_\perp^\mu}{\alpha(n+1) |k_\perp|^{n+1}|q_\perp| P_{n+1}(z)} X_{\mu_1 \ldots \mu_n}^{(n)}(q_\perp).$$  \hspace{1cm} (30)
Obviously this is nothing but the familiar relation for the Legendre polynomials

\[(2n + 1)zP_n(z) = nP_n(z) + (n + 1)P_{n+1}(z)\]  \hspace{1cm} (31)

known as Bonnet’s recursion formula. Note that in the last term in (31) we used the identity

\[O_{\nu_1\ldots\nu_n}^{\mu_1\ldots\mu_n} q_\mu X^{(n)}_{\mu_1\ldots\mu_n} (q_\perp) = \frac{n + 1}{2n + 1} X^{(n+1)}_{\nu_1\ldots\nu_n} (q_\perp),\]  \hspace{1cm} (32)

that can be easily established from eq. (3).

The generalized OAM-operators introduced in this Section can be useful for the development of covariant and fully electromagnetically gauge invariant description of hadron photo- and electroproduction reactions.

5 Conclusions

Convolutions of OAM-operators with several open Lorentz indices occur in the description of photo- and electroproduction of mesons off nucleons within the covariant approach for construction of partial wave amplitudes. The differential technique developed in the present paper considerably simplifies the calculation of such OAM-operator convolutions. These findings will greatly help the development of the fully electromagnetically gauge invariant description of hadron photo- and electroproduction reactions within the covariant OAM-operator expansion approach.

Acknowledgements

K.S. is grateful to V. Vereshagin for very instructive discussions on the method of contracted projectors. The paper was supported by grant RSF 16-12-10267.

A Useful convolutions of orbital-angular-momentum-operators

In this Appendix we present some useful convolutions of OAM-operators with several open indices occurring in the calculation of baryon electroproduction amplitudes with the use of the set of effective vertices worked out in [21]. This can be easily done with the help of the derivative operation (13).
A.1 Case of 3 open indices

To derive the explicit expression for the convolution of OAM-operators with 3 open indices we apply the $D_\nu^{(n+1)}(k_\perp)$ operator to the expression for the convolution of OAM-operators with 2 open indices given by eq. (B.3) of Ref. [18]:

$$|k_\perp|^2 X_{\mu_1\mu_2\alpha_1...\alpha_n}^{(n+2)}(q_\perp) X_{\alpha_1...\alpha_n}^{(n)}(k_\perp).$$

We get

$$D_\nu^{(n+1)}(k_\perp) \left[ |k_\perp|^2 \sum_{N=0}^{3} C^{(N;\mu_1\nu)}_{\mu_1\mu_2\nu}(n; q_\perp, k_\perp) P_n^{(N)}(z) \right]$$

where

$$C^{(0;2,1)}_{\mu_1\mu_2\nu}(n; q_\perp, k_\perp) = (n+1)^2 \left( \frac{q_{\mu_1}^\perp q_{\mu_2}^\perp}{|q_\perp|^4} - \frac{g_{\mu_1\mu_2}^\perp}{|q_\perp|^2} \right) \frac{k_\perp^{\perp}}{|k_\perp|^2};$$

$$C^{(1;2,1)}_{\mu_1\mu_2\nu}(n; q_\perp, k_\perp) = -d^3z \frac{dk_\perp^{\perp}}{dq_{\mu_1}^\perp dq_{\mu_2}^\perp dk_{\nu}^{\perp}} \left( n+1 \right)^2 \frac{q_{\mu_1}^\perp}{|q_\perp|^2|k_\perp|^2};$$

$$C^{(2;2,1)}_{\mu_1\mu_2\nu}(n; q_\perp, k_\perp) = -d^3z \frac{dz}{dq_{\mu_1}^\perp dq_{\mu_2}^\perp dk_{\nu}^{\perp}} \left( n+1 \right)^2 \frac{q_{\mu_1}^\perp}{|q_\perp|^2|k_\perp|^2};$$

$$C^{(3;2,1)}_{\mu_1\mu_2\nu}(n; q_\perp, k_\perp) = -d^3z \frac{dz}{dq_{\mu_1}^\perp dq_{\mu_2}^\perp dk_{\nu}^{\perp}} \left( n+1 \right)^2 \frac{q_{\mu_1}^\perp}{|q_\perp|^2|k_\perp|^2};$$

Note that all the tensor structures $C^{(N;\mu_1\nu)}_{\mu_1\mu_2\nu}(n; q_\perp, k_\perp)$ are symmetric under the permutation $\mu_1 \leftrightarrow \mu_2$, and possess the simple convolution properties:

$$k_\perp^\nu C^{(N;2,1)}_{\mu_1\mu_2\nu}(n; q_\perp, k_\perp) = (n+1) C^{(N;2,0)}_{\mu_1\mu_2}(n; q_\perp, k_\perp) \quad \text{for} \quad N = 0, 1, 2;$$

$$k_\perp^\nu C^{(3;2,1)}_{\mu_1\mu_2\nu}(n; q_\perp, k_\perp) = 0;$$

$$q_\perp^{\mu_2} C^{(N;2,1)}_{\mu_1\mu_2\nu}(n; q_\perp, k_\perp) = (n+2) C^{(N;1,1)}_{\mu_1\nu}(n; q_\perp, k_\perp) \quad \text{for} \quad N = 0, 1, 2;$$

$$q_\perp^{\mu_2} C^{(3;2,1)}_{\mu_1\mu_2\nu}(n; q_\perp, k_\perp) = 0. \quad (A3)$$
A.2 Case of 4 open indices

Finally, for the previously poorly known convolution with 4 open indices

\[ X_{\mu_1 \nu_1}^{(n+2)}(q_\perp) X_{\mu_2 \nu_2}^{(n+2)}(k_\perp) \]

we can write the following formula

\[ X_{\mu_1 \nu_1}^{(n+2)}(q_\perp) X_{\mu_2 \nu_2}^{(n+2)}(k_\perp) = D_{\nu_2}^{(n+2)}(k_\perp) \left[ |k_\perp|^2 X_{\mu_1 \nu_2}^{(n+1)}(q_\perp) X_{\mu_2 \nu_1}^{(n+1)}(k_\perp) \right] \]  

Expressed by eq. [X]

Performing explicitly the derivative operation we get

\[ X_{\mu_1 \nu_1}^{(n+2)}(q_\perp) X_{\mu_2 \nu_2}^{(n+2)}(k_\perp) = \frac{\alpha(n)}{(n+2)^2(n+1)^2} |q_\perp|^4 |k_\perp|^4 + \sum_{N=0}^{4} C_{\mu_1 \nu_1 \nu_2}^{(N; 2, 2)} (n; q_\perp, k_\perp) P_n^{(N)}(z), \]  

where

\[ C_{\mu_1 \nu_1 \nu_2}^{(0; 2, 2)} (n; q_\perp, k_\perp) = (n+1)^2 \left( (n+3) \frac{q_\perp^4}{|q_\perp|^4} - \frac{g_{\mu_1 \mu_2}^4}{|q_\perp|^2 k_\perp^2} \right) \]  

\[ C_{\mu_1 \nu_1 \nu_2}^{(1; 2, 2)} (n; q_\perp, k_\perp) = \frac{d^4 z}{dq_\perp^4 dq_\perp^2 dk_\perp^4 dk_\perp^2} \]  

\[ - (n+1) \left( \frac{k_\perp^4}{|k_\perp|^2} \frac{d^2 z}{dq_\perp^2 dk_\perp^2} \right) + \{ \nu_1 \leftrightarrow \nu_2 \} - (n+1) \left( \frac{q_\perp^4}{|q_\perp|^2} \frac{d^2 z}{dq_\perp^2 dk_\perp^2} + \{ \mu_1 \leftrightarrow \mu_2 \} \right) \]  

\[ + (n+1) \left( (n+3) \frac{k_\perp^4 k_\perp^4}{|k_\perp|^4} - \frac{g_{\nu_1 \nu_2}^4}{|k_\perp|^2} \right) \frac{d^2 z}{dq_\perp^2 dk_\perp^2} + (n+1) \left( (n+3) \frac{q_\perp^4}{|q_\perp|^4} - \frac{g_{\mu_1 \mu_2}^4}{|q_\perp|^2} \right) \frac{d^2 z}{dk_\perp^4 dk_\perp^2} \]  

\[ + (n+1)^2 \left( \frac{q_\perp^4}{|q_\perp|^2} \frac{d^2 z}{dk_\perp^2 dk_\perp^2} + \{ \nu_1 \leftrightarrow \nu_2 \} \right) + (n+1)^2 \left( \frac{q_\perp^4}{|q_\perp|^2} \frac{d^2 z}{dk_\perp^2 dk_\perp^2} + \{ \nu_1 \leftrightarrow \nu_2 \} \right) \]  

\[ - (n+1)^2 \left( \frac{d z}{dq_\perp^4} \left( (n+3) \frac{k_\perp^4 k_\perp^4}{|k_\perp|^4} - \frac{g_{\mu_1 \nu_2}^4}{|q_\perp|^2} \right) + \{ \mu_1 \leftrightarrow \mu_2 \} \right) \]  

\[ - (n+1)^2 \left( \frac{d z}{dk_\perp^4} \left( (n+3) \frac{q_\perp^4 k_\perp^4}{|q_\perp|^4} - \frac{k_\perp^4 g_{\mu_1 \mu_2}^4}{|q_\perp|^2} \right) + \{ \nu_1 \leftrightarrow \nu_2 \} \right); \]
\[ C_{\mu_1, \mu_2, \nu_1, \nu_2}^{(2; 2, 2)}(n; q, k) \]

\[ = \left( \frac{d^3 z}{dq^m_1 dq^m_2 dk^{\nu_1}_1} \frac{dz}{dk^{\nu_2}_1} + \{\nu_1 \leftrightarrow \nu_2\} \right) + \left( \frac{d^3 z}{dq^m_1 dk^{\nu_1}_1 dk^{\nu_2}_1 dq^m_2} + \{\mu_1 \leftrightarrow \mu_2\} \right) \]

\[ + \frac{d^2 z}{dq^m_1 dq^m_2 dk^{\nu_1}_1 dk^{\nu_2}_1} + \frac{d^2 z}{dq^m_1 dk^{\nu_1}_1 dk^{\nu_2}_1 dq^m_2} + \frac{d^2 z}{dq^m_1 dk^{\nu_1}_1 dq^m_2 dk^{\nu_2}_1} \]

\[-(n + 1) \left( \frac{d^2 z}{dq^m_1 dq^m_2} \left( \frac{k^{\nu_1}_1}{|k_1|^2} \frac{dz}{dk^{\nu_1}_1} + \frac{k^{\nu_2}_1}{|k_1|^2} \frac{dz}{dk^{\nu_2}_1} \right) + \{\mu \leftrightarrow \nu; q \leftrightarrow k_1\} \right) \]

\[-(n + 1) \left( \frac{d^2 z}{dq^m_1 dk^{\nu_1}_1 dq^m_2} \left( \frac{k^{\nu_1}_1}{|k_1|^2} \frac{dz}{dk^{\nu_1}_1} + \frac{q^{\nu_2}_1}{|k_1|^2} \frac{dz}{q^{\nu_2}_1} \right) + \{\mu_1 \leftrightarrow \mu_2\} \right) \]

\[-(n + 1) \left( \frac{d^2 z}{dq^m_1 dq^m_2 dk^{\nu_1}_1} \left( \frac{k^{\nu_2}_1}{|k_1|^2} \frac{dz}{dk^{\nu_1}_1} + \frac{q^{\nu_1}_1}{|k_1|^2} \frac{dz}{q^{\nu_1}_1} \right) + \{\nu_1 \leftrightarrow \nu_2\} \right) \]

\[+(n + 1)^2 \left( \frac{dz}{dq^m_1 dq^m_2} \left( \frac{q^{\nu_1}_1 k^{\nu_2}_1}{|q_1|^2 |k_1|^2} + \frac{q^{\nu_2}_1 k^{\nu_1}_1}{|q_1|^2 |k_1|^2} \right) + \frac{dz}{dq^m_1 dk^{\nu_1}_1 dq^m_2} \right) \]

\[+ \frac{dz}{dq^m_1 dq^m_2} (n + 1) \left( n + 3 \right) \frac{k^{\nu_1}_1 k^{\nu_2}_1}{|k_1|^4} - \frac{g^{\nu_1}_1 g^{\nu_2}_1}{|q_1|^4} - \frac{g^{\nu_1}_1 g^{\nu_2}_1}{|q_1|^2} \left( n + 1 \right) \left( n + 3 \right) \frac{q^{\nu_1}_1 q^{\nu_2}_1}{|q_1|^4} \]

\[ C_{\mu_1, \mu_2, \nu_1, \nu_2}^{(3; 2, 2)}(n; q, k) = \frac{d^2 z}{dq^m_1 dq^m_2 dk^{\nu_1}_1} \frac{dz}{dk^{\nu_2}_1} + \frac{d^2 z}{dk^{\nu_1}_1 dk^{\nu_2}_1 dq^m_2 dq^m_1} \]

\[+ \left( \frac{d^2 z}{dq^m_1 dk^{\nu_1}_1 dq^m_2} \frac{dz}{dk^{\nu_2}_1} + 3 \text{ permutations} \right) \]

\[-(n + 1) \left( \frac{dz}{dq^m_1 dq^m_2} \left( \frac{dz}{dq^m_1} + \frac{dz}{dq^m_2} + \frac{dz}{q^{\nu_1}_1} \right) \right) \]

\[-(n + 1) \left( \frac{dz}{dq^m_1 dq^m_2} \left( \frac{dz}{dq^m_1} + \frac{dz}{q^{\nu_1}_1} \right) \right) \]

\[ C_{\mu_1, \mu_2, \nu_1, \nu_2}^{(4; 2, 2)}(n; q, k) = \frac{dz}{dq^m_1 dq^m_2} \frac{dz}{dk^{\nu_1}_1} \frac{dz}{dk^{\nu_2}_1} \frac{dz}{q^{\nu_1}_1} \frac{dz}{q^{\nu_2}_1} \]

Note that the coefficients \( C^{(N; 2, 2)} \) are symmetric under \( \mu_1 \leftrightarrow \mu_2, \nu_1 \leftrightarrow \nu_2 \) and \( q \leftrightarrow k_1 \) permutations.
These coefficient also possess the nice convolution properties. Namely,

\[ k_1^{\nu_1} C^{(N;2,2)}_{\mu_1 \mu_2 \nu_1 \nu_2}(n; q_\perp, k_\perp) = (n + 2)C^{(N;2,1)}_{\mu_1 \mu_2}(n; q_\perp, k_\perp); \quad N = 0, 1, 2, 3; \]

\[ k_1^{\nu_1} C^{(4;2,2)}_{\mu_1 \mu_2 \nu_1 \nu_2}(n; q_\perp, k_\perp) = 0; \]

\[ k_1^{\nu_1} k_2^{\nu_2} C^{(N;2,2)}_{\mu_1 \mu_2 \nu_1 \nu_2}(n; q_\perp, k_\perp) = (n + 2)(n + 1)C^{(N;0,0)}_{\mu_1}(n; q_\perp, k_\perp); \quad N = 0, 1; \]

\[ k_1^{\nu_1} k_1^{\mu_1} q_1^{\nu_2} C^{(0;2,2)}_{\mu_1 \mu_2 \nu_1 \nu_2}(n; q_\perp, k_\perp) = (n + 2)^2(n + 1)^2. \]  

(A7)

## B Miscellaneous

In this Appendix we summarize the explicit expressions for the set of tensor structures occurring in the tensor coefficients \( C^{(N;i,j)}_{\mu_1 \ldots \mu_i \nu_1 \ldots \nu_j} \) within the convolutions of angular momentum operators summarized in Appendix A.

\[ \frac{dz}{dq_\perp^\mu} = \frac{k_\perp^\mu}{|q_\perp||k_\perp|} - z \frac{q_\perp^\mu}{|q_\perp|^2}. \]  

(B8)

Note that

\[ \frac{dz}{dq_\perp^\mu} q_\perp^\mu = 0. \]

\[ \frac{d^2z}{dq_\perp^{\mu_1}dq_\perp^{\mu_2}} = 3z q_\perp^{\mu_1} q_\perp^{\mu_2} - \frac{q_\perp^{\mu_1} k_\perp^{\mu_2} + q_\perp^{\mu_2} k_\perp^{\mu_1}}{|q_\perp|^4|k_\perp|^2} - z \frac{g_{\mu_1 \mu_2}}{|q_\perp|^2}. \]  

(B9)

The tensor is symmetric under \( \mu_1 \leftrightarrow \mu_2 \) permutation and satisfies

\[ \frac{d^2z}{dq_\perp^{\mu_1}dq_\perp^{\mu_2}} q_\perp^{\mu_2} = - \frac{dz}{dq_\perp^{\mu_1}}. \]

\[ \frac{d^2z}{dq_\perp^{\mu_1}dk_\perp^{\nu_1}} = z \frac{q_\perp^{\mu_1} k_\perp^{\nu_1}}{|q_\perp|^2|k_\perp|^2} - \frac{q_\perp^{\mu_1} q_\perp^{\nu_1}}{|q_\perp|^3|k_\perp|} - \frac{k_\perp^{\mu_1} k_\perp^{\nu_1}}{|q_\perp||k_\perp|^3} + g_{\mu_1 \nu_1}. \]  

(B10)

This tensor is obviously symmetric under \( q_\perp \leftrightarrow k_\perp \) interchange and satisfies

\[ \frac{d^2z}{dq_\perp^{\mu_1}dk_\perp^{\nu_1}} k_\perp^{\nu_1} = 0; \quad \frac{d^2z}{dq_\perp^{\mu_1}dk_\perp^{\nu_1}} q_\perp^{\mu_1} = 0; \]

\[ \frac{d^3z}{dq_\perp^{\mu_1}dq_\perp^{\mu_2}dk_\perp^{\nu}} = 3z q_\perp^{\mu_1} q_\perp^{\mu_2} q_\perp^{\nu} - 3z q_\perp^{\mu_1} q_\perp^{\mu_2} k_\perp^{\nu} + \frac{q_\perp^{\mu_1} q_\perp^{\nu} k_\perp^{\mu_2}}{|q_\perp|^3|k_\perp|^2} + \frac{q_\perp^{\mu_2} k_\perp^{\mu_1} k_\perp^{\nu}}{|q_\perp|^3|k_\perp|^3} \]

\[ - g_{\mu_1 \mu_2} q_\perp^{\nu} + g_{\mu_2 \nu} q_\perp^{\mu_1} q_\perp^{\nu} - g_{\mu_1 \mu_2} q_\perp^{\nu} \frac{q_\perp^{\mu_1}}{|q_\perp|^3|k_\perp|} + z \frac{g_{\mu_1 \mu_2} k_\perp^{\nu}}{|q_\perp|^2|k_\perp|^2}. \]  

(B11)

(B12)
The tensor is symmetric under $\mu_1 \leftrightarrow \mu_2$ permutation and satisfies

$$ \frac{d^2 z}{dq_1^{\mu_1} dq_2^{\mu_2} dk_1^\nu k_2^\nu} = 0; \quad (B13) $$

$$ \frac{d^3 z}{dq_1^{\mu_1} dq_2^{\mu_2} dk_1^\nu dk_2^\nu q_1^\mu} = - \frac{d^2 z}{dq_1^{\mu_1} dk_1^\nu}. \quad (B14) $$

Finally,

$$ \frac{d^4 z}{dq_1^{\mu_1} dq_2^{\mu_2} dk_1^\nu dk_2^\nu dk_3^\nu dk_4^\nu} = 9 g_{\mu_1 \nu_1} g_{\mu_2 \nu_2} k_{\nu_1}^\nu k_{\nu_2}^\nu \left[ q_1^{\nu_1} k_1^\nu \right]^4 - 4 \left( k_{\mu_1} g_{\mu_2 \nu_1} k_{\nu_2}^\nu + k_{\mu_2} g_{\mu_1 \nu_2} k_{\nu_1}^\nu + k_{\mu_1} k_{\mu_2} + k_{\nu_1} k_{\nu_2} \right) \left[ q_1^{\nu_1} k_1^\nu \right]^3 \left[ q_1^{\nu_2} k_1^\nu \right]^3 $$

$$ - 3 \left( g_{\mu_1 \nu_1} g_{\mu_2 \nu_2} + g_{\mu_1 \nu_2} g_{\mu_2 \nu_1} \right) \left[ q_1^{\nu_1} k_1^\nu \right]^3 \left[ q_1^{\nu_2} k_1^\nu \right]^3 + \frac{g_{\mu_1 \nu_1} g_{\mu_2 \nu_2} g_{\mu_1 \mu_2}}{q_1^{\nu_1} k_1^\nu \left[ q_1^{\nu_2} k_1^\nu \right]^3} $$

$$ + \frac{g_{\mu_1 \nu_1} k_{\mu_1} g_{\mu_2 \nu_2} k_{\mu_2} k_{\nu_2}}{q_1^{\nu_1} k_1^\nu \left[ q_1^{\nu_2} k_1^\nu \right]^3} $$

$$ + \frac{g_{\mu_1 \nu_1} k_{\mu_1} k_{\nu_1} g_{\mu_2 \nu_2}}{q_1^{\nu_1} k_1^\nu \left[ q_1^{\nu_2} k_1^\nu \right]^3} $$

Finally,

$$ \frac{d^4 z}{dq_1^{\mu_1} dq_2^{\mu_2} dk_1^\nu dk_2^\nu dk_3^\nu} = - \frac{d^3 z}{dq_1^{\mu_1} dk_1^\nu}. \quad (B16) $$

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