Integrable initial boundary value problems

Ismagil Habibullin

Ufa Institute of Mathematics, Russian Academy of Science,
Chernyshevskii Str., 112, Ufa, 450077, Russia

Abstract

The problem of searching boundary value problems for soliton equations consistent with the integrability property is discussed. A method of describing integrals of motion for the integrable initial boundary value problems for the KP equation is suggested via Green identity.

Keywords: Korteweg-de Vries equation, Kadomtsev-Petviashvili equation, Green identity, integrals of motion

PACS number: 02.30.Ik

1 Introduction

Consider the IBV problem of the general form for the NLS equation

\[ iq_t = q_{xx} + c|q|^2q, \quad x > 0, \quad t > 0, \]  
\[ a_1q_x + a_2q|_{x=0} = f(t), \]  
\[ q|_{t=0} = q_0(x), \quad q_0(x)|_{x \to +\infty} \to 0, \]

which was studied by many authors. The usual scattering matrix \( s(\lambda, t) \) of the corresponding Dirac operator on the half-line \( x > 0 \) depends on \( t \) in a very implicit way. Namely, it satisfies the following matrix equation

\[ s_t = 2i\lambda^2[s, \sigma_3] + Z(q(0, t), q_x(0, t), \lambda)s. \]  

Equation contains unknowns \( s, q \) and \( q_x \). How to study such kind of equations? At the first glance it contains an extra unknown and it is underdetermined. But some implicit requirement should be valid: \( s(\lambda, t) \) preserves its
analytical properties on the upper and lower half planes $Im\lambda > 0$ and $Im\lambda < 0$. So really the equation is correctly defined.

Different approaches to study the equation (4) are discussed in the literature, for instance, recently were proposed the global relation method (Fokas) and the elimination by restriction method (Degasperis, Manakov, Santini). The following result allows to understand the essence of the problem (1), (2), (3).

**Theorem**, [1]. The entries $\alpha, \beta$ of the scattering matrix $s(\lambda, t)$ satisfy the following system of equations

$$
\alpha_t(k,t) = \int_{-\infty}^{\infty} \frac{dk'}{k' - k - i0} F_1(\alpha(k', t), \beta(k', t), f(t)) \\
\beta_t(k,t) = \int_{-\infty}^{\infty} \frac{dk'}{k' - k - i0} F_2(\alpha(k', t), \beta(k', t), f(t))
$$

Generally this system of equations with the variable coefficients is nonlinear ($F_1, F_2$ – are second degree polynomials), it is integrable only if $f(t) \equiv 0$. Thus, the IBV problem is equivalent to the Cauchy problem for a pseudodifferential equation with two independent variables (generally nonintegrable). Integrability is lost when $f(t) \neq 0$. Only in the homogeneous case the DMS equation is integrable (it becomes linear). The IBV problem for $f(t) \equiv 0$ is studied in details (M.Ablowitz, H.Segur 1975 when $a_1 = 0$ or $a_2 = 0$ and R.Bikbaev, V.Tarasov, I.Habibullin 1990-91 if $a_1a_2 \neq 0$). It admits soliton solutions. The asymptotics for the large values of time are obtained for an arbitrary initial value.

If $f(t)$ is not identically zero then no hope to find exact solutions to the IBV problem. One of the ways here is to introduce a small parameter $0 < \epsilon \to 0$, i.e. replace $f(t)$ by $\epsilon f(t)$ and to study the influence of the boundary by using the appropriately developed perturbation theory.

To extract integrable cases one can apply the integrability test to the system for the scattering matrix like (5)-(6). But we will testify the boundary condition using directly the Lax pair. Suppose the equation

$$
q_t = f(q, q_x, q_{xx}, \ldots)
$$

admits the Lax pair of the form

$$
\psi_x = U(q, \lambda)\psi, \quad \psi_t = V(q, q_x, \ldots \lambda)\psi
$$
Let a boundary condition of the form

\[ F(t, q, q_x, \ldots) = 0 \]  

is imposed at the point \( x = 0 \). Substitute the BC (9) into the second equation in (8): \( W([q], t, \lambda) = V(q, q_x, \ldots \lambda)|_{F(t,q,q_x,\ldots)=0} \) and find

\[ \psi_t = W([q], t, \lambda)\psi \]  

along the line \( x = 0 \)

The BC (9) is consistent with the Lax pair (8) if the linear equation (10) admits an additional discrete symmetry such that there exists a matrix valued function \( H([q], t, \lambda) \) and an involution \( h = h(\lambda) \) such that the transformation \( \psi \rightarrow \bar{\psi} = H\psi \) converts a solution \( \psi \) of the equation (10) into a solution. In terms of the potentials this requirement reads as

\[ H_t(\lambda) = W(\lambda)H(\lambda) - H(\lambda)W(h(\lambda)) \]  

**Example 1**, see [2]. Consider the Korteweg-de Vries equation \( u_t = u_{xxx} - 6uu_x \). The coefficient matrices for the Lax pair are defined as

\[
U = \begin{pmatrix}
0 & 1 \\
u - \lambda & 0
\end{pmatrix},
\]

\[
V = \begin{pmatrix}
u_x & -4\lambda - 2u \\
u_{xx} - (4\lambda + 2u)(u - \lambda) & -u_x
\end{pmatrix}.
\]

Suppose the BC imposed at \( x = 0 \) is of the form

\[ u = F_1(u_x, t), \quad u_{xx} = F_2(u_x, t). \]

To look for the discrete symmetry we must solve the equation

\[
\frac{dH}{dt} = \begin{pmatrix}
F_1 & -4\lambda - 2u \\
F_2 - (4\lambda + 2u)(u - \lambda) & -F_1
\end{pmatrix}H - H
\begin{pmatrix}
F_1 & -4h(\lambda) - 2u \\
F_2 - (4h(\lambda) + 2u)(u - h(\lambda)) & -F_1
\end{pmatrix}
\]

Here unknowns \( F_1, F_2, H = H(u, u_x, u_{xx}, \ldots), h = h(\lambda) \) are uniquely found. The answer is

\[
H = \begin{pmatrix}
2\lambda + a & 0 \\
a - \lambda + \sqrt{3a^2 - b - 3\lambda^2} & a - \lambda - \sqrt{3a^2 - b - 3\lambda^2}
\end{pmatrix},
\]
\[ h(\lambda) = \frac{-\lambda + \sqrt{3a^2 - b - 3\lambda^2}}{2}. \]

The BC is of the form
\[ u|_{x=0} = a, \quad u_{xx}|_{x=0} = b, \]
where \( a \) and \( b \) are arbitrary constants.

**Example 2, [2].** The Harry Dym equation \( u_t + u^3 u_{xxx} = 0 \) admits two kinds of BC:

i) \( u|_{x=0} = 0, \quad u_x|_{x=0} = b, \)
\[ H = \begin{pmatrix} 1 & 0 \\ e^{4\lambda bt} & 1 \end{pmatrix}, \quad h(\lambda) = \lambda; \]

ii) \( u_x|_{x=0} = au, \quad u_{xx}|_{x=0} = a^2 u/2 + b/u, \)
\[ H = \begin{pmatrix} \lambda & 0 \\ (\lambda - h(\lambda))a/2 & h(\lambda) \end{pmatrix}, \]
\[ h(\lambda) = \frac{-b - 2\lambda + \sqrt{b^2 - 4b\lambda - 12\lambda^2}}{4}; \]

where \( a, b \) are constants.

**Example 3, [3].** The discrete Heisenberg model
\[ (T_{m+1} - 1) \frac{1}{q - q_{-1,0}} = (T_n - 1) \frac{1}{q - q_{0,-1}}, \quad (12) \]
has the following Lax pair
\[ L = \begin{pmatrix} \lambda - \frac{q_{-1,0}}{q - q_{-1,0}} & -\frac{qq_{-1,0}}{q - q_{-1,0}} \\ \frac{1}{q - q_{-1,0}} & \lambda + \frac{q}{q - q_{-1,0}} \end{pmatrix}, \]
\[ A = \begin{pmatrix} \lambda - \frac{q_{0,-1}}{q - q_{0,-1}} & -\frac{qq_{0,-1}}{q - q_{0,-1}} \\ \frac{1}{q - q_{0,-1}} & \lambda + \frac{q}{q - q_{0,-1}} \end{pmatrix}. \]

In this case the discrete involution and the cutting off condition are found from the equation
\[ H(m + 1, \lambda)A(m, N, \lambda) = A(m, N, h(\lambda))H(m, \lambda). \quad (13) \]
The BC reads as
\[ q_{m,0} = \frac{cq_{m,1} + (-1)^m a}{c + (-1)^m bq_{m,1}}, \quad (14) \]
where \( a, b, c \) are arbitrary constants and \( a^2 + b^2 \neq 0 \). The matrix \( H \) takes the form
\[
H(m, \lambda) = \begin{pmatrix}
1 & (-1)^m ac(2\lambda + 1) \\
(-1)^mbc(2\lambda + 1) & 1
\end{pmatrix},
\]
and the involution is \( h(\lambda) = -\lambda - 1 \).

2 How to use the discrete symmetry in the ISM

Let us discuss how to use the discrete symmetry when constructing solutions of the corresponding IBV problems. Take the IBV problem for the KdV equation with vanishing BC (see, [4])
\[
\begin{align*}
&u_t = u_{xxx} - 6uu_x, \quad x > 0, \ t > 0, \\
&u|_{x=0} = 0, \quad u_{xx}|_{x=0} = 0, \\
&u|_{t=0} = u_0(x), \quad u_0(x)|_{x\to+\infty} \to 0.
\end{align*}
\]
In this case \( H \) is the unity matrix and \( h(\lambda) = \lambda^\frac{-1+i\sqrt{3}}{2} = \lambda\omega \). Actually, the discrete symmetry reflects only the fact that the evolution of the scattering matrix
\[
s_t = 4i\lambda^3[s, \sigma_3] + u_x(0, t)\sigma_1 s
\]
is invariant under the change \( \lambda \to \omega\lambda \). This equation is really nonlinear but the discrete symmetry allows one to linearize it. Put \( z = \lambda^3 \) and define the matrices
\[
\begin{align*}
c_+(z, t) &= (s_1(\omega\lambda, t), s_2(\lambda, t)) \\
c_-(z, t) &= \sigma_1 \bar{c}_+(\bar{z}, t)\sigma_1
\end{align*}
\]
These two matrices satisfy the Riemann problem
\[
c_+(z, t) = c_-(z, t)p(z, t),
\]
where \( p(z, t) = e^{-4i\sigma_3 t}p(z, 0)e^{4i\sigma_3 t} \). Now it is easy to see that the scattering matrix \( s(\lambda, t) \) is found from the linear equation (19). Using this fact one can prove

**Theorem.** Let the initial value satisfy the conditions
1) \( u(x, 0) = u_0(x) \) is smooth and vanishes;
2) the associated Sturm-Liouville operator has no discrete eigenvalues,
3) the scattering matrix is unbounded at \( \lambda = 0 \)
then the problem (15), (16), (17) is uniquely solvable for all \( t > 0 \). The function \( u_x(0, t) \) satisfies the following representation
\[
u_x(0, t) = \frac{1}{t} + o\left(\frac{1}{t}\right), \quad t \to \infty\]

In this case two of three functions \( u(0, t), u_x(0, t), u_{xx}(0, t) \) are zero and the third one slowly decays. It is not ever in \( L_1 \), only in \( L_2 \).

Unfortunately, there is no regular soliton-like solutions of the KdV equation with the vanishing boundary conditions. If the parameters \( a \) and \( b \) are different from zero then regular exact soliton-like solutions (as well as finite-gap solutions) exist approaching \( C = \sqrt{a^2 - b/3} \) at \( x = \infty \) and satisfying the BC at \( x = 0 \). They are described in [6]. In this case time evolution of the scattering matrix is reduced to a Riemann problem on a Riemann surface defined by the function \( h(\lambda) = \frac{-\lambda + \sqrt{3a^2 - b - 3\lambda^2}}{2} \).

3 Discrete symmetry and BC in multidimensional case

Consider the well known 2D-Toda chain
\[
u_{xt}(n) = \exp\{u(n - 1) - u(n)\} - \exp\{u(n) - u(n + 1)\}, \quad (20)
\]
with the following Lax pair
\[
\phi(n + 1) = (D_x + u_x(n)) \phi(n), \quad (21)
\]
\[
\phi_{xt}(n) = -u_x(n) \phi_t(n) - \exp\{u(n - 1) - u(n)\} \phi(n). \quad (22)
\]
Impose a cutting off constraint at \( n = 0 \)
\[
f(u(-1), u(0)) = 0. \quad (23)
\]
How to find all integrable cases only by using the equation (22)? To this end it is necessary to study the discrete symmetries of the equation (22), appearing under the BC. But now there is no \( \lambda \) and we are to find some generalization of
the involution $\lambda \to h(\lambda)$. It is evident that the Toda chain is invariant under transform $x \leftrightarrow t$, so the following pair of equation

$$\psi(n + 1) = (D_t + u_t(n)) \psi(n),$$
(24)

$$\psi_{xt}(n) = -u_t(n) \psi_x(n) - \exp\{u(n - 1) - u(n)\} \psi(n).$$
(25)
gives also a Lax pair to the Toda chain.

**Proposition**, [5]. Suppose that there exists such an operator $M = aD^2_x + bD_x + c$ that for $n = 0$ for any solution $\psi$ of the equation (25) the function $\phi = M \psi$ is a solution of (22). Then the BC (23) takes one of the forms below

1) $e^{u(-1)} = 0,$
2) $u(-1) = 0,$
3) $u(-1) = u(0),$
4) $u_x(-1) = -u_t(0)e^{-u(0)-u(-1)}.$
(26)

The corresponding operator $M$ is respectively of the form

1) $M_1 = a_0e^uD^2_x + (b_0e^u + a_0u_xe^u)D_x,$
2) $M_2 = e^uD^2_x + u_xe^uD_x,$
3) $M_3 = e^uD_x,$
4) $M_4 = e^uD^2_x + u_xe^uD_x + e^{-u}.$
(27)

where $a_0, b_0$ - arbitrary constant parameters, and $u = u(0)$. Notice that the operator $M_1$ is a linear combination of the operators $M_2$ and $M_3$.

All of the cutting off conditions above are known to be consistent with the integrability. Initiated by this example we formulate the discrete involution test for multidimensional equations. Two Lax pairs which are not connected by any conjugation transformation, should become conjugate after imposing the BC.

Apply now the test to look for boundary conditions to the KP equation

$$v_\tau + v_{xxx} - 6vv_x = 3w_y,$$
(28)

$$w_x = v_y,$$

admitting the Lax pair

$$\phi_{xx} = i\phi_y + v\phi,$$
(29)

$$\phi_\tau = -4\phi_{xxx} + 6v\phi_x + 3(v_x + iw)\phi.$$
The equation (28) is invariant under the change $y \rightarrow -y$, $w \rightarrow -w$ and by this reason the following system of equations is also a Lax pair for the KP

$$\psi_{xx} = -i\psi_y + v\psi,$$

$$\psi_{\tau} = -4\psi_{xxx} + 6v\psi_x + 3(v_x - iw)\psi. \quad (32)$$

**Proposition, [5].** Suppose that there exists a differential operator $M = aD_x^2 + bD_x + c$ such that for $y = 0$ for any solution $\psi$ of the equation (30) the function defined as $\phi = M\psi$ is a solution to (32). Then one of the following equations holds

1) $w|_{y=0} = 0$,  
2) $(v_x - iw)|_{y=0} = 0$,  
3) $(w_\tau - 2v_{xxy} + 6iv_{yyx} + 6v_xw - 6vw_x - 6iw^2 - 12cv_y)|_{y=0} = 0,$

where $c = c(x, \tau)$ is a solution of the equation $c_x = (-v_x + \frac{i}{2}w)|_{y=0}$. The corresponding operator $M$ is of the form

1) $M = 1,$  
2) $M = D_x,$  
3) $M = D_x^2 + c.$  

(33)

If one replaces $\psi \leftrightarrow \phi$ one gets one more constraint

$$4) \quad (v_x + iw)|_{y=0} = 0 \quad (35)$$

4 Integrals of motion

To look for integrals of motion we will use the Green identities which are in this case as follows

$$\frac{d}{dx}(\phi_x\psi - \phi\psi_x) = i\frac{d}{dy}(\phi\psi) \quad (36)$$

and

$$\frac{d}{d\tau}(\phi_x\psi - \phi\psi_x) = 4\frac{d}{dy}(\psi\phi_y - \psi_y\phi - \frac{i}{2}v\phi\psi + i\phi_x\psi_x). \quad (37)$$

4 Integrals of motion
Suppose that eigenfunctions $\psi$ and $\phi$ are defined as follows

$$\phi(x, y, \tau, k) = e^{-ik^2y+kx-4k^3\tau}(1 + \sum_{j=1}^{\infty} k^{-j} \phi_j), \quad (38)$$

$$\psi(x, y, \tau, k) = e^{ik^2y-kx+4k^3\tau}(1 + \sum_{j=1}^{\infty} k^{-j} \psi_j) \quad (39)$$

for $k \to \infty$, and satisfy the asymptotic requirements

$$\phi e^{ik^2y-kx+4k^3\tau}, \psi e^{-ik^2y+kx-4k^3\tau} \to 1$$

for $x \to -\infty$ and for $x \to +\infty$, respectively, then the function $F(k)$

$$F(k) = \int_{-\infty}^{\infty} (\phi_x \psi - \phi \psi_x - 2k)\, dy \quad (40)$$

is a generating function of the conserved quantities. Actually, by using the Green identities (36), (37) one gets

$$\frac{\partial}{\partial \tau} F(k) = \int_{-\infty}^{\infty} \frac{\partial}{\partial \tau} (\phi_x \psi - \phi \psi_x - 2k)\, dy = \int_{-\infty}^{\infty} \frac{d}{dy} (\phi \psi_y - \phi_y \psi + \frac{i}{2} \psi \phi - i \phi_x \psi_x)\, dy = 0.$$

Equations (29)-(32) admits one more Green identity

$$i \frac{\partial}{\partial \tau} (\phi \psi) = 4 \frac{\partial}{\partial x} (\psi \phi_y - \phi \psi_y - \frac{i}{2} \psi \phi + i \phi_x \psi_x), \quad (41)$$

which allows one to find the generating function of integrals of motion for the initial boundary value problem on the half-plane

$$F_1(k) = \int_{0}^{\infty} (\phi_x \psi - \phi \psi_x - 2k)\, dy + i \int_{-\infty}^{x} (\phi \psi - C(k))|_{y=0}\, ds, \quad (42)$$

Here the integrand in the first integral is taken at $(x, y, \tau, k)$, and in the second at $(s, 0, \tau, k)$. Really, by means of the identities (36), (37), (41) one gets

$$\frac{\partial F_1}{\partial \tau} = 0.$$ Taking the first coefficients gives

**Proposition.** The KP equation on the half-plane $y > 0$, $-\infty < x < \infty$ with any of BC (33.1), (33.2), (35) preserves the energy

$$J_2 = \int_{0}^{\infty} \int_{-\infty}^{\infty} v^2(x, y)\, dx\, dy = \text{const}$$

**References**

[1] A.Degasperis, S.V.Manakov, P.M.Santini, *Initial-Boundary Value Problems for Linear and Soliton PDEs*, Theoretical and Mathematical Physics, Volume 133, Number 2 (2002), 1475-1489, DOI: 10.1023/A:1021138525261 arXiv:nlin.SI/020530 v1.
[2] I.T.Habibullin, A.N.Vil’danov, *Boundary conditions consistent with L-A pairs*, Proceeding of the International Conference MOGRAN 2000, Ufa, Russia, 27 September-03 October, 2000

[3] I.T.Habibullin, T.G.Kazakova, J.Phys.A: Math. Gen. 34(2001) 10369

[4] I.T.Habibullin, *Initial Boundary Value Problem for the KdV Equation on a Semiaxis with Homogeneous Boundary Conditions*, Theoretical and Mathematical Physics, Volume 130, Number 1 (2001), 25-44, DOI: 10.1023/A:1013824330433

[5] E. V. Gudkova and I. T. Habibullin, *Kadomtsev-Petviashvili Equation on the Half-Plane*, Theoretical and Mathematical Physics, 2004, Volume 140, Number 2, Pages 1086-1094

[6] V. E. Adler, I. T. Habibullin and A. B. Shabat, *Boundary value problem for the KDV equation on a half-line*, Theoretical and Mathematical Physics, Volume 110, Number 1 (1997), 78-90, DOI: 10.1007/BF02630371