BREDON COHOMOLOGY OF FINITE DIMENSIONAL $C_p$-SPACES

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Abstract. For finite dimensional free $C_p$-spaces, the calculation of the Bredon cohomology ring as an algebra over the cohomology of $S^0$ is used to prove the non-existence of certain $C_p$-maps. These are related to Borsuk-Ulam type theorems, and equivariant maps related to the topological Tverberg conjecture. For finite dimensional $C_p$-spaces which are formed out of representations, it is proved that the cohomology is a free module over the cohomology of a point. All the calculations are done for the cohomology with constant coefficients $\mathbb{Z}/p$.

1. Introduction

Bredon cohomology is usually hard to compute even for spheres. For the cyclic group $C_p$ of prime order $p$, computations of $RO(G)$-graded cohomology $H^\ast_G(S^0)$ by Stong and Lewis [16] allow us to compute the cohomology of $C_p$-spaces. In this paper, we prove some structural results about the cohomology of $C_p$-spaces with $\mathbb{Z}/p$ coefficients. In many ways, $\mathbb{Z}/p$ is the analogue of $\mathbb{Z}/p$-coefficients in ordinary cohomology in the non-equivariant case.

For coefficients in a constant Mackey functor, the integer graded cohomology is the cohomology of the orbit space. If $X$ is a $C_p$-space with free action, we prove that the $RO(G)$-graded cohomology is determined from the cohomology of the orbit space. More precisely,

$$H^\ast_{C_p}(X; \mathbb{Z}/p) \cong H^\ast(X/C_p) \otimes_{\hat{C}_p} \mathbb{Z}/p[u^\pm]$$

where $\hat{C}_p$ is set of characters of $C_p$.

We also compute the module structure of $H^\ast_{C_p}(X; \mathbb{Z}/p)$ over the cohomology of a point. The module structure allows us to rule out certain equivariant maps between free $C_p$-spaces. More precisely, we deduce Borsuk-Ulam type theorems [14]: If $V$ and $V'$ are two fixed point free $C_p$-representations, there are no $C_p$-maps $S(V) \to S(V')$ if $\dim(V) > \dim(V')$. The module structure calculations also allow us to deduce the topological Tverberg conjecture in the prime case, first proved by Bárány, Shlosman, and Szücs [1].

If $X$ is a finite dimensional free $C_p$-space, we use the module structure to obtain a numerical bound $n(X)$ such that for every $i > n(X)$ the elements of $H^i_{C_p}(S^0)$ of degree $i$ operate trivially on $H^\ast_{C_p}(X_+)$. This number is related to the Fadell-Husseini index [5].

We also prove a freeness result for certain $C_p$-spaces. This kind of theorem writes the cohomology of $X$ as a free module over the cohomology of a point. In ordinary cohomology, with $\mathbb{Z}$-coefficients this is true if $X$ has even dimensional cells, and with $\mathbb{Z}/p$ coefficients this is true for any $X$. For the group $C_p$, Lewis [16] proved a freeness result for certain even dimensional complexes. The complexes he used involved attaching cells homeomorphic to $D(V)$, a disk in a representation $V$, which are attached via maps from $S(V)$, the boundary of $D(V)$. They are even dimensional if $\dim(V)$ is even. Lewis also assumes a condition on the fixed points to deduce that the attaching maps induce the 0 map on cohomology. For this theorem, only the additive structure of the $RO(C_p)$-graded cohomology of $S^0$ is used.

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Ferland and Lewis \[6\] use the ring structure of the \(RO(C_p)\)-graded cohomology of \(S^0\) to remove the condition on the fixed points. They show that for any \(C_p\)-cell complex formed out of representations with even cells, the cohomology is a free module over the cohomology of \(S^0\). In this case, the attaching maps do not induce the 0 map on cohomology but it is proved that the cohomology of the cofibre is still free. This crucially uses the ring structure of \(H^*_{C_p}(S^0)\).

Kronholm \[15\] has proved that any \(C_2\)-space obtained inductively by attaching representation cells in increasing dimension (called \(Rep(C_2)\)-complexes), the cohomology is a free module over the cohomology of a point when the coefficients are \(\mathbb{Z}/2\). This also involves a careful analysis using the ring structure of \(H^*_{C_2}(S^0; \mathbb{Z}/2)\). In this paper, we make the analogous analysis for \(C_p\) where \(p > 2\), and prove that for a \(Rep(C_p)\)-complex, the cohomology is free as a module over the cohomology of a point when the coefficients are \(\mathbb{Z}/p\).

1.1. Organization. In section 2, we recall some definitions and results from \(RO(G)\)-graded Bredon cohomology theory. We use these methods to write down the cohomology representation complexes.

2. Preliminaries

We recall certain basic ideas and techniques in Bredon cohomology. Along the way we fix the notations used throughout the paper. The details and proofs of the stated facts may be found in \[19\]. The notation \(G\) will be used for the cyclic group \(C_p\) of prime order \(p\), though most of the facts in this section also holds for a general finite group.

For a unitary representation \(V\) of the group \(G\), let \(D(V)\) and \(S(V)\) denote the unit disc and unit representation sphere in \(V\), with the action induced from that of \(V\). A \(G\)-CW complex is a \(G\)-space \(X\) with a filtration \(\{X^{(n)}\}_{n \geq 0}\), where \(X^{(0)}\) is a disjoint union of \(G\)-orbits, and \(X^{(n)}\) is obtained from \(X^{(n-1)}\) by attaching cells of the form \(G \times_H D^n\) along maps \(G \times_H S^{n-1} \to X^{(n-1)}\) where \(H \leq G\) and the action of \(G\) on \(D^n\) and \(S^{n-1}\) are trivial. The space \(X^{(n)}\) is defined as the \(n\)-th-skeleton of \(X\). The attaching map \(G \times_H S^{n-1} \to X^{(n-1)}\) is equivalent to the map \(S^{n-1} \to (X^{(n-1)})^H = (X^H)^{(n-1)}\). One may deduce that the category of \(G\)-CW complexes is equivalent to the functor category from \(O_G\) to the category of CW-complexes, where \(O_G\) is the orbit category of \(G\) with objects are the finite \(G\)-sets and morphisms are the \(G\)-equivariant maps between \(G\)-sets.

A coefficient system for the group \(G\) is a contravariant functor from \(O_G\) to the category of Abelian groups. Since this diagram category from \(O_G\) to the Abelian group category is an additive category, therefore, we can talk about chains of coefficient systems. In particular, for a \(G\)-space we define the Bredon chain, \(C_*(X, \mathbb{Z})\), given by assignment \(G/H \to C_*(X^H, \mathbb{Z})\).

**Definition 2.1.** Let \(\underline{M}\) be a coefficient system for a group \(G\) and \(X\) be a \(G\)-space. Define the \(n\)-th-Bredon cochains of \(X\) with coefficients in \(\underline{M}\) as \(C^n_\underline{M}(X; \underline{M}) = \text{Hom}_{O_G}(C_n(X, \mathbb{Z}); \underline{M})\). The cohomology of this complex is defined as \(\mathbb{Z}\)-graded Bredon cohomology of \(X\) with coefficients in \(\underline{M}\) and denoted by \(H^n_{\underline{M}}(X, \underline{M})\).

Equivariant homotopy and cohomology theories are more naturally graded on \(RO(G)\), the Grothendieck group of finite real orthogonal representations of \(G\). To obtain this kind of theory one needs more structure on the coefficient systems. These are called Mackey functors.
Definition 2.2. A Mackey functor consists of a pair $M = (M, M^*)$ of functors from the category of finite $G$-sets to $\mathbb{A}b$, with $M$ covariant and $M^*$ contravariant. On every object $S$, $M^*$ and $M$ have the same value which we denote by $M(S)$, and $M$ carries disjoint unions to direct sums. The functors are required to satisfy that for every pullback diagram of finite $G$-sets as below

\[
\begin{array}{ccc}
P & \xrightarrow{\delta} & X \\
\gamma \downarrow & & \alpha \\
Y & \xrightarrow{\beta} & Z,
\end{array}
\]

one has $M^*(\alpha) \circ M(\beta) = M(\delta) \circ M^*(\gamma)$.

Mackey functors are naturally contravariant functors from the Burnside category $Burn_G$ of $G$ to Abelian groups. The objects of $Burn_G$ are finite $G$-sets and the morphisms are formed by group completing the monoid of correspondences. The representable functor associated to the $G$-set $G/G$ is called the Burnside ring Mackey functor $\underline{A}$. For a finite $G$-set $S$, $\underline{A}_S$ is the representable functor associated to $S$.

Example 2.3. For an Abelian group $C$, an immediate example for a Mackey functor is the constant Mackey functor $C$ defined by the assignment $C(S) = \text{Map}^G(S, C)$, the set of $G$-maps from the $G$-orbit $S$ to $C$ with trivial $G$-action.

Equivariant cohomology theories are represented by $G$-spectra. The naive $G$-spectra are those in which only the desuspension with respect to trivial $G$-spheres are allowed. Usually, what we mean by $G$-spectra are those in which desuspension with respect to all representation spheres are allowed. In the viewpoint of [17], naive $G$-spectra are indexed over a trivial $G$-universe and $G$-spectra are indexed over a complete $G$-universe. As we are allowed to take desuspension with respect to representation-spheres, the associated cohomology theories become $RO(G)$-graded.

We’ll consider the orthogonal $G$-spectra with positive complete model structure to model the equivariant stable homotopy theory, which can be read off from [11] Appendix A, B. In particular, we use $(Sp^G, \wedge, S^0)$ for the symmetric model category of orthogonal $G$-spectra. We denote the homotopy class of maps by $[\cdot, \cdot]^G$ and the equivariant function spectrum $F(\cdot, \cdot)$, a right adjoint to the smash product $\wedge$ in $Sp^G$.

Every $G$-set $S$ gives a suspension spectrum $\Sigma_S^G S$ in the category of $G$-spectra. It turns out that the category with finite $G$-sets as objects and homotopy classes of spectrum maps as morphisms is naturally isomorphic to the Burnside category. Thus, the homotopy groups of $G$-spectra are naturally Mackey functors. For an equivariant orthogonal spectrum $X$, we use $\pi_G(X)$ for its $RO(G)$-graded homotopy groups. In particular, for $\alpha = V-W \in RO(G)$,

\[\pi_\alpha(X) = [S^V, S^W \wedge X]^G.\]

In non-equivariant homotopy theory, for each Abelian group $A$, there is a construction of the Eilenberg-MacLane spectra $HA$ satisfying

\[\pi_n(HA) = \begin{cases} A, & \text{if } n = 0 \\ 0, & \text{otherwise}. \end{cases}\]

In the category of orthogonal spectra, we can also have the construction of Eilenberg-MacLane spectra for each Mackey functor.

Proposition 2.4. Let $M$ be a $G$-Mackey functor. Then there exist an equivariant Eilenberg-MacLane spectrum $H_M$, unique up to homotopy in $Sp^G$.

Proof. See [9] Theorem 5.3].
Therefore, equivariant Eilenberg MacLane spectra must arise from Mackey functors. This is a Theorem of Lewis, May, and McClure which we refer from Chapter XIII of [19]. Therefore, we may argue that the integer-graded cohomology associated to coefficient systems extends to $RO(G)$-graded cohomology theories if and only if the coefficient system has an underlying Mackey functor structure.

**Definition 2.5.** A $RO(G)$-graded cohomology theory consists of functors $E^\alpha$ for $\alpha \in RO(G)$, from reduced equivariant CW complexes to Abelian groups which satisfy the usual axioms - homotopy invariance, excision, long exact sequence and the wedge axiom.

It is interesting to note that the suspension isomorphism for $RO(G)$-graded cohomology theories takes the form $E^\alpha(X) \cong E^{\alpha+V}(S^V \wedge X)$ for every based $G$-space $X$ and representation $V$.

We recall that there are change of groups functors on equivariant spectra. The restriction functor from $G$-spectra to $H$-spectra has a left adjoint given by smashing with $G/H$. This also induces an isomorphism for cohomology with Mackey functor coefficients

$$\tilde{H}^\alpha_G(G/H_+ \wedge X; M) \cong \tilde{H}^\alpha_H(X; \text{res}_H(M))$$

The $RO(G)$-graded theories may also be assumed to be Mackey functor-valued as in the definition below.

**Definition 2.6.** Let $X$ be a pointed $G$-space, $M$ be any Mackey functor, $\alpha \in RO(G)$. Then the Mackey functor valued cohomology $\tilde{H}^\alpha_G(X; M)$ is defined as

$$\tilde{H}^\alpha_G(X; M)(G/K) = \tilde{H}^\alpha_G(G/K_+ \wedge X; M).$$

The restriction and transfer maps are induced by the appropriate maps of $G$-spectra.

The Mackey functor valued cohomology is always the reduced version, so we may only evaluate it on a pointed $G$-space. This means that we have $G$-invariant basepoint. On the other hand, the Bredon cohomology groups are defined for unpointed $G$-spaces also, so when we write the reduced version we put \(\sim\) as a superscript. This explains our notation in the rest of the document where we use the notation $\tilde{H}^\bullet_G(X_+)$. The next natural question is: *What is the structure on a Mackey functor which induces a ring structure on the cohomology of spaces?* There is a box product $\boxtimes$ on the category of Mackey functors. For two Mackey functors $M, N$, this is obtained by taking the left Kan extension along

$$\text{Burn}_G \times \text{Burn}_G \rightarrow \text{Ab}$$

The right arrow in the top row is given by $(S, T) \mapsto M(S) \times N(T)$. The left vertical arrow is given by $(S, T) \mapsto S \sqcup T$. The Mackey functors inducing ring valued cohomology theories are monoids under the box product $\boxtimes$. The Burnside ring Mackey functor $\mathbb{A}$, and the constant Mackey functors $\mathbb{Z}$, $\mathbb{Z}/p$ are monoids under $\boxtimes$. Therefore $\tilde{H}^\bullet_G(S^0; M)$ has a graded ring structure for $M = \mathbb{A}$, $\mathbb{Z}$ or $\mathbb{Z}/p$.

### 3. Cohomology of free $C_p$-spaces

In this section, we compute the Bredon cohomology of free $C_p$-spaces. There are two ingredients in this, first the ring structure on the cohomology and second the module structure over the cohomology of a point. The former is computed in Section 3.11 and the latter in Section 3.12. Along the way we also describe the method to compute $\tilde{H}_G^\bullet(S^0)$ using the Tate square, to recalculate the computations by Stong and Lewis [16].
3.1. The cohomology ring structure. Let $X$ be a free $G$-space. We prove that the cohomology ring of $X$ is obtained in a neat way from the ordinary cohomology of $X/G$. For this purpose, we set up a spectral sequence to compute the Bredon cohomology of a free $C_p$-space $X$ arising from a CW structure on $X$, which we then proceed to compute completely. This is the homotopy fixed point spectral sequence in the case $X = EC_\infty$ in [12, Proposition 2.8].

We start with a $G$-CW structure on $X$ as $X = \bigcup_s X^{(s)}$. Since $X$ is free the cells are of the type $G/e \times D^k$. We also note that $X/G$ has a CW complex structure with associated filtration $X^{(s)}/G$. Thus, we have

$$X^{(s)}/X^{(s-1)} \cong \bigvee_{e \in I(s)} G_+ \wedge S^e$$

For $\alpha \in RO(G)$, define

$$E_1^{s,t}(\alpha) = \pi^{G}_{t-s}(F(X^{(s)}/X^{(s-1)}, S^{-\alpha} \wedge H\mathbb{Z}/p)).$$

We make the following identifications

$$\begin{align*}
\pi_{t-s}^G(F(X^{(s)}/X^{(s-1)}, S^{-\alpha} \wedge H\mathbb{Z}/p)) &\cong [S^{t-s}, F(\bigvee_{e \in I(s)} G_+ \wedge S^e, S^{-\alpha} \wedge H\mathbb{Z}/p)]^G \\
&\cong \bigoplus_{e \in I(s)} [S^{t-s} \wedge G_+ \wedge S^e, S^{-\alpha} \wedge H\mathbb{Z}/p]^G \\
&\cong \bigoplus_{e \in I(s)} [S^t, S^{-\dim(\alpha)} \wedge H\mathbb{Z}/p] \\
&\cong \bigoplus_{e \in I(s)} \pi_t(S^{-\dim(\alpha)} \wedge H\mathbb{Z}/p) \\
&\cong C^*(X/G; \pi_t(S^{-\dim(\alpha)} \wedge H\mathbb{Z}/p)).
\end{align*}$$

In the case $p$ odd, all the representation spheres are orientable so that the action of $G$ on $\pi_t(S^{-\dim(\alpha)} \wedge H\mathbb{Z}/p)$ is trivial. In the case $p = 2$, also the action of $G$ on $\pi_t(S^{-\dim(\alpha)} \wedge H\mathbb{Z}/2) \cong \mathbb{Z}/2$ is trivial. It follows that the boundary $d_1$ matches with the cellular coboundary. Therefore, we have proved the following result.

**Proposition 3.2.** There is a spectral sequence

$$E_2^{s,t}(\alpha) = H^t(X/G; \pi_0(S^{-\dim(\alpha)} \wedge H\mathbb{Z}/p)) \Rightarrow \pi^{G}_{t-s}(F(X^{(*)}, H\mathbb{Z}/p))$$

with boundary $d_r : E_r^{s,t}(\alpha) \rightarrow E_r^{s+r,t-r+1}(\alpha)$. The spectral sequences assemble together for various $\alpha$ into a multiplicative $RO(C_p)$-graded spectral sequence

$$E_2^{s,\alpha} = H^s(X/G; \pi_0(S^{-\dim(\alpha)} \wedge H\mathbb{Z}/p)) \Rightarrow \pi^{G}_{-s}(F(X^{(*)}, H\mathbb{Z}/p))$$

where $s \in \mathbb{Z}$ and $\alpha \in RO(G)$.

The multiplicative structure of the spectral sequence follows verbatim from [11, Theorem 6.1] applied to $X$ instead of $EG$. We readily deduce that for every $\alpha$, the spectral sequence $E_2^{s,t}(\alpha)$ is concentrated at $t = -\dim(\alpha)$. Therefore, the spectral sequences degenerate at the second page. Also, the multiplicative structure on the spectral sequence in [3,2] implies that everything is a product of the form $H^s(X/G) \otimes \pi_0(S^\alpha \wedge H\mathbb{Z}/p)$. Now $\pi_0(S^\alpha \wedge H\mathbb{Z}/p)$ fits together as a ring $\bigotimes_{\xi \in G} \mathbb{Z}[u_\xi^\pm]$ with $|u_\xi| = 2 - \xi$. Thus we obtain
Proposition 3.3. For a free $C_p$-space $X$,
\[
\tilde{H}^*_G(X;\mathbb{Z}/p) \cong H^*(X/G;\mathbb{Z}/p) \otimes \bigotimes_{\xi \in \mathbb{G}_1(1)} \mathbb{Z}/p[u_\xi^\pm].
\]

3.4. Cohomology of $S^0$. In this section we compute the cohomology of $S^0$ with $\mathbb{Z}/p$ coefficients. This was first computed by Stong and Lewis [10]. Here, we use the Tate cohomology in $\tilde{H}^*_G(S^0;\mathbb{Z}/p)$.

Definition 3.6. (1) For an actual representation $V$ with $V^G = 0$, let $a_V \in \pi^G_v(S^0)$ be the maps $S^0 \to S^V$ which embeds $S^0$ to $S^V$ to 0 and $\infty$ in $S^V$. We will also use $a_V$ for the dual of its Hurewicz image in $\tilde{H}^V_G(S^0;\mathbb{Z}/p)$.

(2) For an actual orientable representation $W$ of dimension $n$, let $u_W$ be a generator of $\tilde{H}^n_G(S^W;\mathbb{Z}/p)(G/G)$ which restricts to the choice of orientation in $H^*_G(S^W;\mathbb{Z}/p)(G/e) \cong H_n(S^0;\mathbb{Z}/p)$.

In homotopy grading, $u_W \in \mathbb{Z}_{n-W}(H\mathbb{Z}/p)(G/G)$. We also denote its dual by $u_W$ in $\tilde{H}^{n-W}_G(S^0;\mathbb{Z}/p)$.

The cases $p$ odd and $p = 2$ are slightly different. We start with the $p$ odd case. Note that in this case there are $\frac{p-1}{2}$ real irreducible $C_p$-representations which may be listed as $\{\xi^i | 1 \leq i \leq \frac{p-1}{2}\}$ with $\xi$ the usual multiplication by the $p$th roots of 1. In the calculation of $\tilde{H}^*_G(S^0;\mathbb{Z}/p) \cong \pi_* \tilde{H}\mathbb{Z}/p$, one notes that it suffices to restrict $\star$ to the values $m + n\xi$ with $m, n \in \mathbb{Z}$. For, one notes [12] that $S^{\xi - \xi^i} \wedge H\mathbb{Z}/p \cong H\mathbb{Z}/p$, so that there are invertible classes in degrees $\xi^i - \xi^j$ which makes

\[
\tilde{H}^*_G(S^0;\mathbb{Z}/p) = \tilde{H}^*_G(S^0;\mathbb{Z}/p).
\]

From now onwards $\star$ is restricted to degrees $m + n\xi$ for $m, n \in \mathbb{Z}$, and $\star$ is restricted to integer gradings. Now Definition 3.6 gives two classes $u_\xi \in \tilde{H}^{\xi-2}_G(S^0;\mathbb{Z}/p)$ and $a_\xi \in \tilde{H}^{\xi}_G(S^0;\mathbb{Z}/p)$.

From Proposition 3.5, $\tilde{H}^*_G(EG;\mathbb{Z}/p) \cong H^*(BG;\mathbb{Z}/p) \otimes \mathbb{Z}/p[u_\xi^\pm]$. We now check that under $q$ the class $u_\xi$ maps to $u_\xi$. This is true for any free $G$-space whose cohomology is described in Proposition 3.3. From the construction of the spectral sequence in Proposition 3.2 for $X = G/e$ we have $H^*_G(G/e_{+};\mathbb{Z}/p) \cong \mathbb{Z}/p[u_\xi^\pm]$. Note that the map

\[
H^{\xi-2}_G(S^0;\mathbb{Z}/p) \cong \mathbb{Z}/p[u_\xi^\pm] \to H^{\xi-2}_G(G/e_{+};\mathbb{Z}/p) \cong A_G/e \otimes \mathbb{Z}/p
\]

induced by the projection $G/e_{+} \to S^0$ is an isomorphism at $G/G$. Therefore, the class $u_\xi \in \tilde{H}^{\xi-2}_G(S^0;\mathbb{Z}/p)$ maps to $u_\xi \in \tilde{H}^{\xi-2}_G(X_{+};\mathbb{Z}/p)$ for any $G$-space $X$.

We next calculate the image of $a_\xi \in \tilde{H}^{\xi}_G(S^0;\mathbb{Z}/p)$ in $\tilde{H}^*_G(EG_{+};\mathbb{Z}/p)$, that is the image under $q$ in the diagram 3.5. We have

\[
\tilde{H}^*_G(EG_{+};\mathbb{Z}/p) \cong \mathbb{Z}/p[x, y, u_\xi^\pm]/(x^2) \text{ where } |x| = 1, |y| = 2, |u_\xi| = \xi - 2.
\]
Observe, $\tilde{H}_G^\xi(EC_{p+}:\mathbb{Z}/p) \cong \mathbb{Z}/p\{u_\xi y\}$. Also observe that the map

$$\tilde{H}_G^\xi(EC_{p+}:\mathbb{Z}/p) \to \tilde{H}_G^\xi(S(2\xi)_+:\mathbb{Z}/p)$$

is an isomorphism (for $H \leq C_p$, the pair $(EC_{p+}^H, S(2\xi)_+^H)$ is 3-connected so, $\tilde{H}_G^\xi(EC_{p+}, S(2\xi)_+:\mathbb{Z}/p) = 0$). The cofibre sequence

$$S(2\xi)_+ \to S^0 \to S^{2\xi}$$

induces the exact sequence

$$\tilde{H}_G^{-\xi}(S^0:\mathbb{Z}/p) \to \tilde{H}_G^\xi(S^0:\mathbb{Z}/p) \to \tilde{H}_G^\xi(S(2\xi)_+:\mathbb{Z}/p) \to \tilde{H}_G^{1-\xi}(S^0:\mathbb{Z}/p)$$

One readily computes using the cofibre sequence

$$S(\xi)_+ \to S^0 \to S^\xi$$

that $\tilde{H}_G^{-\xi}(S^0:\mathbb{Z}/p) = 0$ and $\tilde{H}_G^{1-\xi}(S^0:\mathbb{Z}/p) = 0$. Therefore, we have

$$\tilde{H}_G^\xi(S^0:\mathbb{Z}/p) \cong \tilde{H}_G^\xi(S(2\xi)_+:\mathbb{Z}/p).$$

Thus, we may fix the generator $y$ such that $a_\xi \mapsto u_\xi y$. Note that as $\beta(x) = y$, this also fixes the generator $x$ (here $\beta$ refers to the Bockstein in ordinary cohomology). Therefore, we may write

$$\pi \star F(EC_{p+}, H\mathbb{Z}/p) \cong \mathbb{Z}/p[x, a_\xi, u_\xi^\pm]/(x^2).$$

Now we work around the Tate square (3.3). From [10, Proposition 1.1], it follows that the left vertical arrow is a $G$-equivalence. The maps $S^0 \to EG$ induce a localization with respect to $a_\xi$, as $EG \cong \lim_n S^n\xi$. This implies

$$\pi \star \tilde{EG} \land F(EC_{p+}, H\mathbb{Z}/p) \cong \mathbb{Z}/p[x, a_\xi^\pm, u_\xi^\pm]/(x^2).$$

Now the map

$$\mathbb{Z}/p[x, a_\xi^\pm, u_\xi^\pm]/(x^2) \to \mathbb{Z}/p[x, a_\xi^\pm, u_\xi^\pm]/(x^2)$$

is injective, so,

$$\pi \star EG_{p+} \land F(EC_{p+}, H\mathbb{Z}/p) \cong \bigoplus \mathbb{Z}/p[\Sigma^{-1}x^\epsilon u_j^\xi/a_\xi^\xi] \text{ for } \epsilon \in \{0, 1\}, j \in \mathbb{Z}, k \geq 1.$$ 

It follows that

$$\pi \star EG_{p+} \land H\mathbb{Z}/p \cong \bigoplus \mathbb{Z}/p[\Sigma^{-1}x^\epsilon u_j^\xi/a_\xi^\xi] \text{ for } \epsilon \in \{0, 1\}, j \in \mathbb{Z}, k \geq 1.$$ 

We know that $\pi \star \tilde{EG} \land H\mathbb{Z}/p$ is $a_\xi$-periodic, so it suffices to compute $\pi_n \tilde{EG} \land H\mathbb{Z}/p$ for $n \in \mathbb{Z}$ and then

$$\pi \star \tilde{EG} \land H\mathbb{Z}/p \cong \pi_n(\tilde{EG} \land H\mathbb{Z}/p)[a_\xi^\pm].$$

Now,

$$\pi_0(EG_{p+} \land H\mathbb{Z}/p) \to \pi_0(H\mathbb{Z}/p)$$

is the transfer map of $\mathbb{Z}/p$ and hence 0. Therefore, we get

$$\pi_1 \tilde{EG} \land H\mathbb{Z}/p \cong \mathbb{Z}/p[1] \oplus \mathbb{Z}/p[\Sigma^{-1}x^\epsilon u_j^\xi/a_\xi^\xi] \text{ for } \epsilon \in \{0, 1\}, k \geq 1.$$ 

We define the class $\kappa_\xi$ as $u_\xi x$. Therefore,

$$\pi \star \tilde{EG} \land H\mathbb{Z}/p \cong \mathbb{Z}/p[a_\xi^\pm, u_\xi, \kappa_\xi]/(\kappa_\xi^2).$$
We now compute the kernel of the boundary $\pi\hat{\star}E\hat{G} \cap H\mathbb{Z}/p \rightarrow \pi\hat{\star}^{-1}E\hat{G} \cap H\mathbb{Z}/p$ to be
$$\mathbb{Z}/p[a, u, \kappa]/(\kappa^2)$$
and the cokernel to be
$$\mathbb{Z}/p\{\Sigma^{-1} x^ju^{-j}a^{-k}\}$$ for $e \in \{0, 1\}, j, k > 0$.

Therefore, we obtain

**Proposition 3.8.** For $p$ odd,
$$\pi\hat{\star}(H\mathbb{Z}/p) \cong \mathbb{Z}/p[a, u, \kappa]/(\kappa^2) \oplus \mathbb{Z}/p\{\Sigma^{-1} \frac{1}{u^ja^k}\} \oplus \mathbb{Z}/p\{\Sigma^{-1} \frac{\kappa}{u^ja^k}\}$$ for $j, k > 0$.

This method was carried out in [23, Proposition 6.3] for $H\mathbb{Z}$. It is instructive to compare the results. We call the part $\mathbb{Z}/p\{\Sigma^{-1} \frac{1}{u^ja^k}\}$ as **top cone** and the divisible part $\mathbb{Z}/p\{\Sigma^{-1} \frac{\kappa}{u^ja^k}\}$ as **bottom cone**.

We deduce the $p = 2$ calculation in an analogous manner. There is only one non-trivial irreducible $C_2$-representation namely the sign representation $\sigma$. From Definition 3.6 we have two classes $u_{+}\in \hat{H}_{+}^{G}(S^0; \mathbb{Z}/2)$ and $a_{+}\in \hat{H}_{+}^{G}(S^0; \mathbb{Z}/2)$. Consider the Tate square (3.9)

(3.9) \[ \begin{array}{c}
EG_{+} \cap H\mathbb{Z}/2 \\
\cong \mathbb{Z}/2[a_{+}, u_{+}] \\
\end{array} \]

As in the case of $p$ odd, we have isomorphisms
$$\hat{H}_{+}^{G}(EG_{+}\cap H\mathbb{Z}/2) \cong \hat{H}_{+}^{G}(S^{2}\sigma_{+}; H\mathbb{Z}/2)$$

and
$$\hat{H}_{+}^{G}(S^{0}; H\mathbb{Z}/2) \cong \hat{H}_{+}^{G}(S^{2}\sigma_{+}; H\mathbb{Z}/2).$$

This allows us to write using Proposition 3.3
$$\pi\hat{\star}F(EG_{+}\cap H\mathbb{Z}/2) \cong \mathbb{Z}/2[a_{+}, u_{+}]$$
as before. This yields
$$\pi\hat{\star}E\hat{G} \cap F(EG_{+}\cap H\mathbb{Z}/2) \cong \mathbb{Z}/2[a_{+}, u_{+}]$$
and the map
$$\pi\hat{\star}F(EG_{+}\cap H\mathbb{Z}/2) \rightarrow \pi\hat{\star}E\hat{G} \cap F(EG_{+}\cap H\mathbb{Z}/2)$$
is injective. Therefore,
$$\pi\hat{\star}EG_{+}\cap H\mathbb{Z}/2 \cong \pi\hat{\star}EG_{+}\cap F(EG_{+}\cap H\mathbb{Z}/2) \cong \mathbb{Z}/2(\Sigma^{-1} \frac{1}{a_{+}^{j}})$$ for $j \in \mathbb{Z}, k \geq 1$.

Thus, in integer grading we obtain
$$\pi_{+}EG_{+}\cap H\mathbb{Z}/2 \cong \mathbb{Z}/2(\Sigma^{-1} \frac{1}{a_{+}^{j}})$$ for $j \geq 1$.

This implies
$$\pi_{+}E\hat{G} \cap H\mathbb{Z}/2 \cong \mathbb{Z}/2(1) \oplus \mathbb{Z}/2(\Sigma^{-1} \frac{u_{+}^{k}}{a_{+}}).$$
Therefore, the $a_\sigma$-periodicity of $\pi^* \tilde{E}G \wedge H\mathbb{Z}/2$ yields

$$\pi^* \tilde{E}G \wedge H\mathbb{Z}/2 \cong \mathbb{Z}/2[a_\sigma^+, u_\sigma].$$

Therefore, the kernel of the boundary

$$\pi^* \tilde{E}G \wedge H\mathbb{Z}/2 \to \pi^* E_+ \wedge H\mathbb{Z}/2$$

is $\mathbb{Z}/2[a_\sigma, u_\sigma]$, and the cokernel is

$$\oplus \mathbb{Z}/2\{\Sigma^{-1}u_\sigma^{-j}a_\sigma^{-k}\} \text{ for } j, k > 0.$$

Therefore,

**Proposition 3.10.**

$$\pi^*(H\mathbb{Z}/2) \cong \mathbb{Z}/2[a_\sigma, u_\sigma] \oplus \mathbb{Z}/2\{\Sigma^{-1}u_\sigma^{-j}a_\sigma^{-k}\} \text{ for } j, k > 0.$$

This method has been carried out for $H\mathbb{Z}$ in [23, Proposition 6.5] and [8].

**Proposition 3.11.** The action of the Böckstein $\beta$ on $\tilde{H}_G^\star(S^0)$ is given by the formula

$$\beta(\kappa_\xi) = a_\xi, \quad \beta(u_\xi) = 0, \quad \beta(a_\sigma) = a_\sigma, \quad \beta(a_\xi) = 0, \quad \beta(u_\sigma) = 0.$$

**Proof.** We know that the Böckstein homomorphism is a derivation. Therefore, we have

$$\beta(\kappa_\xi) = \beta(u_\xi x) = \beta(u_\xi)x + u_\xi \beta(x).$$

So in order to prove the first equality, it is enough to check that $\beta(u_\xi) = 0$. Note that for the cofibre sequence

$$H\mathbb{Z}/p \to H\mathbb{Z}/p^2 \xrightarrow{\pi} H\mathbb{Z}/p$$

the Böckstein homomorphism fits into the cohomology long exact sequence

$$\cdots \tilde{H}_G^{\xi-2}(S^0; \mathbb{Z}/p^2) \xrightarrow{\pi} \tilde{H}_G^{\xi-2}(S^0; \mathbb{Z}/p) \xrightarrow{\beta} \tilde{H}_G^{\xi-1}(S^0; \mathbb{Z}/p) \cdots$$

Since, the class $u_\xi \in \tilde{H}_G^\star(S^0; \mathbb{Z}/p^2)$ can be represented by the map

$$S^2 \to S^\xi \wedge H\mathbb{Z}/p^2$$

and also the class $u_\xi \in \tilde{H}_G^\star(S^0; \mathbb{Z}/p)$ represented analogously by

$$S^2 \to S^\xi \wedge H\mathbb{Z}/p$$

and they fit together into the following diagram

$$\begin{array}{ccc}
S^2 & \xrightarrow{u_\xi} & S^\xi \wedge H\mathbb{Z}/p^2 \\
\downarrow u_\xi & & \downarrow 1 \wedge \pi \\
S^\xi \wedge H\mathbb{Z}/p & & \\
\end{array}$$

Hence, we have $\pi_*(u_\xi) = u_\xi$. This yields the Böckstein sends the class $u_\xi$ to zero. Therefore, we obtain $\beta(\kappa_\xi) = a_\xi$. This completes the proof for $p$ odd.

In the case $p = 2$, we again use the fact that $\beta$ is a derivation and $\beta^2 = 0$, so that it suffices to verify that $\beta(u_\sigma) = a_\sigma$. We directly compute a cell structure on $S^n$ as $S^n \cup C_2/e \times D^1$ with the boundary map generated by identity which may be identified as $C_2/e \times S^0 \to pt \times S^0$. Therefore, the reduced Bredon homology with constant coefficients $\mathbb{Z}/k$ is computed by the two term cell complex

$$\mathbb{Z}/k \xrightarrow{2} \mathbb{Z}/k.$$
as the differential is generated by the transfer map which is multiplication by 2. Directly comparing the answers for \(k = 2\) and \(k = 4\), we see that the map \(H_1(S^\sigma;\mathbb{Z}/4) \to H_1(S^0;\mathbb{Z}/2)\) is 0. Hence it follows that \(u_\sigma\) does not lie in the image of \(H_\sigma^{-1}(S^0;\mathbb{Z}/4) \to H_\sigma^{-1}(S^0;\mathbb{Z}/2)\), and thus the only possibility is \(\beta(u_\sigma) = a_\sigma\).

\[\square\]

3.12. Module structure of \(\hat{H}^*_G(X_+;\mathbb{Z}/p)\). In this section we compute the \(\hat{H}^*_G(S^0;\mathbb{Z}/p)\)-module structure of \(\hat{H}^*_G(X_+;\mathbb{Z}/p)\). Note that this is determined by the action of the elements \(u_\sigma, a_\sigma\) for \(p = 2\), and \(u_\xi, \kappa_\xi\) and \(a_\xi\) for \(p\) odd. We write \(q : X_+ \to S^0\) for the map which quotients out \(X\) to a single point. Then, the elements in \(\hat{H}^*_G(S^0;\mathbb{Z}/p)\) act via multiplication by their images under \(q^*\) which is a ring map. Therefore it suffices to compute the images of the generators under \(q^*\). We have already done this for \(X = E\mathcal{C}_p\).

Hence, we readily obtain

**Proposition 3.13.** 1) Let \(p\) be an odd prime. Then the action of \(a_\xi, u_\xi \in \hat{H}^*_G(S^0;\mathbb{Z}/p)\) on \(\hat{H}^*_G(\mathcal{E}G_+;\mathbb{Z}/p) \cong \mathbb{Z}/p[x, a_\xi, u_\xi^\pm]/(x^2)\) is given by multiplication by the corresponding elements. The action of \(\kappa_\xi\) equals multiplication by \(xu_\xi\).

2) Let \(p = 2\). Then the action of \(a_\sigma, u_\sigma \in \hat{H}^*_G(S^0;\mathbb{Z}/2)\) on \(\hat{H}^*_G(\mathcal{E}G_+;\mathbb{Z}/2) \cong \mathbb{Z}/2[a_\sigma, u_\sigma^\pm]\) is given by multiplication by the corresponding elements.

Next we consider a general free \(G\)-space \(X\), which is non-equivariantly connected. We know that if \(X\) had some point with a non-trivial stabilizer, then \(S^0\) becomes an equivariant retract of \(X_+\), so the map \(q^*\) identifies \(\hat{H}^*_G(X_+;\mathbb{Z}/p)\) as a subring which is included in \(\hat{H}^*_G(S^0;\mathbb{Z}/p)\). So we restrict our attention to the free case, where \(X \to X/G\) is a covering space with \(G\) acting on \(X\) by Deck transformations. This induces a homomorphism \(\tau : \pi_1(X/G) \to G \cong \mathbb{Z}/p\), which is well-defined up to a unit in \(\mathbb{Z}/p\). This gives an element of \(\hat{H}^1(X/G;\mathbb{Z}/p)\) which we also denote by \(\tau\). We now have the following Proposition regarding the action of \(\hat{H}^*_G(S^0;\mathbb{Z}/p)\) on \(\hat{H}^*_G(X_+;\mathbb{Z}/p)\).

**Proposition 3.14.** 1) Let \(p\) be an odd prime. Then the action of \(u_\xi \in \hat{H}^*_G(S^0;\mathbb{Z}/p)\) on \(\hat{H}^*_G(X_+;\mathbb{Z}/p) \cong H^*(X/G;\mathbb{Z}/p) \otimes \mathbb{Z}/p[u_\xi^\pm]\) is given by multiplication by the corresponding element. The action of \(\kappa_\xi\) equals multiplication by \(\tau u_\xi\). The action of \(a_\xi\) is given by multiplication by \(\beta(\tau) u_\xi\).

2) Let \(p = 2\). Then the action of \(u_\sigma \in \hat{H}^*_G(S^0;\mathbb{Z}/2)\) on \(\hat{H}^*_G(X_+;\mathbb{Z}/2) \cong H^*(X/G;\mathbb{Z}/2) \otimes \mathbb{Z}/2[u_\sigma^\pm]\) is given by multiplication by the corresponding element. The action of \(a_\sigma\) is given by multiplication by \(\tau a_\sigma\).

**Proof.** We already know the action of \(u_\xi\) (and \(u_\sigma\) in the case \(p = 2\)) from the calculation in Proposition 3.3. For proving the rest, we determine the images of \(\kappa_\xi\) and \(a_\xi\). If \(p > 2\), the action of \(a_\xi\) follows from the action of \(\kappa_\xi\) using the Bockstein as in Proposition 3.11. Now, consider the projection map \(q : X_+ \to S^0\) and its homotopy cofibre \(C(q)\). As \(X\) is (non-equivariantly) connected, it readily follows that \(C(q)\) is non-equivariantly simply connected.

For 1), we claim that the induced map

\[q^* : \hat{H}^*_{G^{-1}}(S^0;\mathbb{Z}/p) \to \hat{H}^*_{\xi^{-1}}(X_+;\mathbb{Z}/p)\]

is injective. To prove the claim, it is enough to show that the group

\[[C(q), S^\xi^{-1} \wedge H\mathbb{Z}/p]^G = 0.\]
Note that this group fits into the long exact sequence
(3.15)
\[ \cdots \to [C(q), \Sigma^{-1} H\mathbb{Z}/p]^G \to [C(q), S^{i-1} \wedge H\mathbb{Z}/p]^G \to [C(q), S^i \wedge H\mathbb{Z}/p]^G \to \cdots \]
\[ \mathring{H}^{-1}(C(q)/G; \mathbb{Z}/p) \]
associated to the cofibre sequence
\[ S(\xi)_+ \to S^0 \to S^\xi \]
Since, the group in the left is zero, therefore, we only remain to prove the group \([C(q), S(\xi)_+ \wedge H\mathbb{Z}/p]^G\) vanishes. For this consider the cofibre sequence
\[ C_p/e_+ \to C_p/e_+ \to S(\xi)_+ \]
It gives the long exact sequence
\[ \cdots \to [C(q), C_p/e_+ \wedge H\mathbb{Z}/p]^G \to [C(q), S(\xi)_+ \wedge H\mathbb{Z}/p]^G \to [C(q), \Sigma C_p/e_+ \wedge H\mathbb{Z}/p]^G \to \cdots \]
\[ \mathring{H}^0(C(q); \mathbb{Z}/p) \to \mathring{H}^1(C(q); \mathbb{Z}/p) \]
Since, \(C(q)\) is simply connected, therefore, \([C(q), S(\xi)_+ \wedge H\mathbb{Z}/p]^G = 0\). Hence, using (3.15), we obtain \([C(q), S^i \wedge H\mathbb{Z}/p]^G = 0\). Thus, we establish the claim. It follows that \(q^*(\kappa_\xi) = \phi u_\xi\) for some \(\phi \in \mathring{H}^1(X/G) = Hom(\pi_1(X/G), \mathbb{Z}/p)\) (using Proposition 3.3) which is non-zero.

Now the above is true for any free \(G\)-space. Observe that the 1-skeleton of \(X\) is a union of copies of \(S^1\) on which \(G\) acts by multiplication by \(p^{th}\) roots of 1, which is equivariantly homeomorphic to \(S(\xi)\). We, therefore, must have that \(\phi\) pulls back non-trivially to the orbit space \(S(\xi)/G\). This is also homeomorphic to \(S^1\), with
\[ \mathring{H}^1(S(\xi)/G; \mathbb{Z}/p) \cong Hom(\pi_1(S(\xi)/G), \mathbb{Z}/p). \]
Hence, the pullback of \(\phi\) must send \(1 \in \pi_1(S(\xi)/G) \cong \mathbb{Z}\) to a unit in \(\mathbb{Z}/p\), and thus the kernel is the image of \(\pi_1(S(\xi))\). As this is true for every map from \(S(\xi)\) to \(X\), the kernel of \(\phi\) must be equal to the image of \(\pi_1(X)\). Therefore, \(\phi\) equals \(\tau\) up to a unit of \(\mathbb{Z}/p\). We may fix the choice of unit in the definition of \(\tau\) so that \(\phi = \tau\).

For 2), consider the cofibre sequence \(C_2/e_+ \to S^0 \to S^\sigma\) and its associated long exact sequence
\[ \cdots \to [C(q), H\mathbb{Z}/2]^G \to [C(q), S^\sigma \wedge H\mathbb{Z}/2]^G \to [C(q), \Sigma C_2/e_+ \wedge H\mathbb{Z}/2]^G \to \cdots \]
\[ \mathring{H}^0(C(q)/G, \mathbb{Z}/2) \to \mathring{H}^1(C(q), \mathbb{Z}/2) \]
Since, \(C(p)\) is simply connected, therefore, the group \([C(q), S^\sigma \wedge H\mathbb{Z}/2]^G\) is trivial. Thus, the map
\[ \mathring{H}^*_G(S^0, \mathbb{Z}/2) \to \mathring{H}^*_G(X^+; \mathbb{Z}/2) \]
is a monomorphism. Therefore, \(q^*(a_\sigma) = \phi u_\sigma\) for some \(\phi \in \mathring{H}^1(X/G; \mathbb{Z}/2)\). Now we may apply a similar argument as above to deduce the result. \(\square\)
4. Applications to finite dimensional $C_p$-spaces

We now restrict our attention to finite dimensional $C_p$-spaces, and focus on two kinds of examples – the free case, and the case where the space is formed by attaching cells along representation spheres. In the former case, we obtain an invariant used to prove theorems about non-existence of equivariant maps. In the latter case, we obtain a freeness theorem that the $RO(G)$-graded cohomology is a free module over the cohomology of point.

4.1. Cohomology of spheres. Let $V$ be a fixed point free $C_p$-representation, that is, $V^G = 0$. One has the cofibre sequence

$$S(V)_+ \to S^0 \to S^V,$$

where the map $S^0 \to S^V$ induces the stable map $a_V$. Therefore, we have the associated long exact sequence

$$(4.2) \quad \cdots \to H^*_G S(V)_+; \mathbb{Z}/p \to H^*_G S^0; \mathbb{Z}/p \to H^*_G S^0; \mathbb{Z}/p \to H^*_G S(V)_+; \mathbb{Z}/p \to \cdots$$

This allows us to compute $H^*_G(S(V)_+; \mathbb{Z}/p)$ from the formula for the ring structure on $H^*_G(S^0; \mathbb{Z}/p)$ in Propositions 3.8 and 3.10. We have the following useful Lemma.

**Lemma 4.3.** Let $V$ be a fixed point free representation of $C_p$. Then,

$$H^*_G(S(V)_+; \mathbb{Z}/p) = 0.$$

**Proof.** Using the exact sequence (4.2) at $\alpha = V$ and $V' = \frac{\dim(V)}{2} \xi$, we have the diagram

$$\begin{array}{ccc}
H^0_G(S^0; \mathbb{Z}/p) & \xrightarrow{a_V} & H^V_G(S^0; \mathbb{Z}/p) \\
\downarrow \cong & & \downarrow \cong \\
H^0_G(S^0; \mathbb{Z}/p) & \xrightarrow{a_{V'}} & H^V_G(S^0; \mathbb{Z}/p)
\end{array}$$

The map $H^V_G(S^0; \mathbb{Z}/p) \to H^V_{G'}(S^0; \mathbb{Z}/p)$ is induced by the space level map $S^{V'} \to S^V$ obtained by smashing together maps of the form $S^\xi \to S^{\xi'}$ given by $z \mapsto z^r$. This is an isomorphism from [Proposition 3.8, Lemma 1] which stated as for $r \not\equiv p$, $S^\xi \wedge HZ/p \cong S^{\xi'} \wedge HZ/p$.

So, it is enough to know that the map multiplication by $a_{V'}$ is an isomorphism, which is multiplication by $a_{\xi}^{(p-1)/2^{d+1}}$ and thus, follows from Proposition 3.8. For $p = 2$, $V = \dim(V)\sigma$, and we directly have $a_V = a_{\sigma}^{\dim(V)}$, so that the result is true analogously.

**Corollary 4.4.** Let $V$ and $V'$ be two fixed point free representations of $G$. Then there does not exist $G$-maps from $S(V) \to S(V')$ if $\dim(V) > \dim(V')$.

**Proof.** We note that $H^V_{G'}(S_0; \mathbb{Z}/p) = 0$, from the calculations of Propositions 3.8 and 3.10 and the identification in 3.4. If there was a $G$-map $S(V) \to S(V')$, then we obtain an induced map

$${H^\bullet}_{G}(S(V)_+; \mathbb{Z}/p) \to H^\bullet_{G}(S(V)_+; \mathbb{Z}/p)$$

of $H^\bullet_{G}(S^0; \mathbb{Z}/p)$-modules which sends 1 to 1. In degree $V'$, the left hand side is 0, and the right hand side is generated by $1 \cdot a_{V'}$. This is a contradiction.
We may also use these techniques to deduce a proof of the “topological Tverberg conjecture” in the prime case. This states that for integers \( n \geq 2, d \geq 1 \) and \( N = (d + 1)(n - 1) \), and for any continuous map \( f : \Delta^N \to \mathbb{R}^d \), there exists \( n \)-pairwise disjoint faces \( \sigma_1, \ldots, \sigma_n \) of the simplex \( \Delta^N \) such that \( f(\sigma_1) \cap \cdots \cap f(\sigma_n) \neq \emptyset \). This conjecture was first posed by Bárány, Shlosman, and Szücs \cite{1}, who proved if \( n \) is prime. Later, Özaydin \cite{20}, Sarkaria \cite{21}, and Volvikov \cite{22} using different techniques extended this result to \( n \) a power of some prime. In \cite{7}, Frick describes a counterexample when \( n \) is not a prime power and \( d \geq 3n + 1 \).

We have in earlier work observed \cite{2} that the methods in the prime power case do not help in proving weaker versions (that is, with increased values of \( N \)) if \( n \) is not a prime power.

A map \( f : \Delta^{(d+1)(n-1)} \to \mathbb{R}^d \) violating the topological Tverberg conjecture gives a \( \Sigma_n \)-equivariant map from the \( ((d + 1)(n - 1) + 1)\)-fold join \( \{1, 2, \ldots, n\}^{((d + 1)(n - 1) + 1)} \) to the representation sphere \( S(W^{\oplus d}) \) \cite{20}, where \( W \) is the standard representation of the symmetric group \( \Sigma_n \), of dimension \( (n - 1) \). For \( n = p \), restricting to the cyclic subgroup \( C_p \), we get a \( C_p \)-equivariant map from \( \{1, \ldots, p\}^{((d + 1)(n - 1) + 1)} \) to \( S(\bar{\rho}^{\oplus (d + 1)}) \), where \( \bar{\rho} \) is the reduced regular representation.

Observe that the inclusion \( EC_p^* \{((d+1)(p-1)) \subseteq EC_p^* \mathbb{Z}/p \) induces an isomorphism in \( \bar{H}_C(p, \mathbb{Z}/p) \) for \( * \leq (d + 1)(p - 1) - 1 \) and is injective for \( * = (d + 1)(p - 1) \). Since this is an inclusion of free \( C_p \)-spaces the result also holds for \( \bar{H}_C(p, \mathbb{Z}/p) \) where \( \alpha \in RO(C_p) \) with \( |\alpha| \leq (d + 1)(p - 1) \). In particular, observe for \( 1 \in \bar{H}_C^0(\mathbb{Z}/p) \), Proposition \ref{prop} yields

\[
a_{(d+1)(p-1)} \xi, 1 = u_{\xi}^{(p-1)(d+1)} y_{(p-1)(d+1)} 1 \neq 0,
\]

thus,

\[
a_{(d+1)(p-1)} \xi, 1 \neq 0 \in \bar{H}_C^{(d+1)(p-1)}(EC_p^*{((d+1)(p-1))) + \mathbb{Z}/p}.
\]

Using these formulas, the following theorem provides a key step towards the topological Tverberg conjecture in the prime case.

**Theorem 4.6.** There does not exist any \( C_p \)-map \( EC_p^*((p-1)(d+1)) \to S(\bar{\rho}^{\oplus (d+1)}) \).

**Proof.** Suppose on contrary there is \( C_p \)-map \( f : EC_p^*((p-1)(d+1)) \to S(\bar{\rho}^{\oplus (d+1)}) \). Then it induces a \( \bar{H}_C^*(S^0) \)-module map

\[
f^* : \bar{H}_C^*(S(\bar{\rho}^{\oplus (d+1)})) + \mathbb{Z}/p \to \bar{H}_C^*(EC_p^*((p-1)(d+1)) + \mathbb{Z}/p)
\]

So, using module structure, we have

\[
f^*(a_{(p-1)(d+1)} \xi, 1) = a_{(d+1)(p-1)} \xi f^*(1).
\]

Lemma \ref{lem} implies that \( a_{(p-1)(d+1)} \xi, 1 = 0 \). Therefore, this contradicts \ref{lem}. \qed

Finally, note that \( EC_p^*((p-1)(d+1)) \) is a free \( C_p \)-space of dimension \( (p - 1)(d + 1) \) and \( \{1, \ldots, p\}^{(p-1)(d+1)+1} \) has connectivity \( (p-1)(d+1) \). Therefore, by \( C_p \)-equivariant obstruction theory we have a \( C_p \)-map from \( EC_p^*((p-1)(d+1)) \to \{1, \ldots, p\}^{(p-1)(d+1)+1} \). Thus a \( C_p \)-map from \( \{1, \ldots, p\}^{(p-1)(d+1)+1} \to S(\bar{\rho}^{\oplus (d+1)}) \) induces a \( C_p \)-map \( EC_p^*((p-1)(d+1)) \to S(\bar{\rho}^{\oplus (d+1)}) \), contradicting Theorem 4.6. This contradiction implies the topological Tverberg conjecture in the odd prime case. In fact, an analogous argument may be written also in the case \( p = 2 \), but then the calculation is equivalent to Corollary 4.4

### 4.7. Invariants of finite free \( C_p \)-spaces.

Let \( X \) be a finite dimensional free \( C_p \)-space. Proposition \ref{prop} implies that \( \bar{H}_C^*(X_+; \mathbb{Z}/p) \) = 0 for \( |\alpha| \) sufficiently large. Since the classes \( \kappa_\xi \)
and $a_\xi$ raise the total degree (for an element of $RO(G)$ we refer to the total degree as the dimension) of $\tilde{H}_G^\bullet(X_+;\mathbb{Z}/p)$ by one or two respectively; so, there is minimum degree $n(X)$ of $\kappa_\xi a_\xi^2$, $\epsilon \in \{0, 1\}$ which acts trivially. That is,

$$n(X) = \begin{cases} 2j + \epsilon | \kappa_\xi a_\xi^2 \text{ acts trivially on } \tilde{H}_G^\bullet(X_+;\mathbb{Z}/p)), & \text{for } p \text{ odd} \\ \{j | a_\xi^2 \text{ acts trivially on } \tilde{H}_G^\bullet(X_+;\mathbb{Z}/2)), & \text{for } p = 2. \end{cases}$$

The number $n(X)$ behaves like an index of a $G$-space in the sense that if there is a $G$-map $X \rightarrow Y$, $n(Y) \geq n(X)$. For otherwise, the map $\tilde{H}_G^\bullet(Y_+;\mathbb{Z}/p) \rightarrow \tilde{H}_G^\bullet(X_+;\mathbb{Z}/p)$ would not be a map of $\tilde{H}_G^\bullet(S^0;\mathbb{Z}/p)$-modules.

We now relate $n(X)$ to the Fadell-Husenni index (13) for a $G$-space $X$. For a $G$-space $X$ and ring $R$, the Fadell-Husenni index of $X$ is defined to be the kernel ideal of the map $p : EG \times_G X \rightarrow BG$ in cohomology induced by the $G$-equivariant projection $X \rightarrow pt$:

$$Index_G(X; R) = Ker(p^* : H^*(BG; R) \rightarrow H^*(EG \times_G X; R)).$$

If $G = C_p$ and $R = \mathbb{Z}/p$, note that the cohomology of $BG$ is free of rank 1 in each degree. We write $i(X)$ to be the first integer $i$ where the degree $i$ part of $Index_G(X; R)$ is non-trivial.

**Theorem 4.8.** $n(X) = i(X)$.

**Proof.** For a free, finite dimensional $G$-space $X$ we have up to $G$-homotopy a unique map $\phi : X \rightarrow EG$. Also the Borel construction $X \times_G EG \simeq X/G$. Now consider the commutative triangle

$$\begin{array}{ccc}
\tilde{H}_G^\bullet(EG_+, \mathbb{Z}/p) & \xrightarrow{\phi^*} & \tilde{H}_G^\bullet(X_+;\mathbb{Z}/p) \\
\downarrow & & \downarrow \\
\tilde{H}_G^\bullet(S^0;\mathbb{Z}/p) & \xrightarrow{} & \tilde{H}_G^\bullet(X_+;\mathbb{Z}/p) \\
\end{array}$$

Since, both the space $X$ and $EG$ are free $G$-spaces, therefore, using Proposition 3.3 the horizontal map turns out to

$$p^* \otimes Id : H^*(BG) \otimes \bigotimes_{\xi \in G \setminus \{1\}} \mathbb{Z}/p[u_\xi^2] \rightarrow H^*(X/G) \otimes \bigotimes_{\xi \in G \setminus \{1\}} \mathbb{Z}/p[u_\xi^2].$$

Therefore, the result follows immediately from the Propostion 3.13 and the fact that the elements $u_\xi$ have total degree 0. $\square$

**4.9. Freeness theorem.** We use the calculations from Section 3 to prove a freeness result for a $G$-cell complex formed out of representations. We define such a complex as a $Rep(G)$-complex as below.

**Definition 4.10.** A $Rep(G)$-cell complex $X$ is a $G$-space with a filtration $\{X_n\}_{n \geq 0}$ of subspaces such that

a) $X^{(0)}$ is a finite union of disjoint copies of $G/G$.

b) For each $n$, $X^{(n+1)}$ is built up form $X^{(n)}$ by attaching cells of the form $D(V)$ along the boundary $S(V)$ with $\dim(V) = n + 1$.

c) $X = \cup_{n \geq 0}X^{(n)}$ has the colimit topology.

We will prove below that the cohomology of a $Rep(G)$-complex is a free module over $\tilde{H}_G^\bullet(S^0;\mathbb{Z}/p)$. This generalizes the analogous result for $p = 2$ in [15]. However, it needs to be clear that all finite $G$-spaces are not representable as $Rep(G)$-complexes where cells
are attached sequentially in increasing dimension. Note that for \( G \)-spheres one may have equivariant maps which increase the total dimension. For example one has the map
\[
a_\xi : S^V \to S^V \oplus \xi
\]
which is non-trivial in cohomology if \( V \) is fixed point free (Proposition 3.6). In fact the mapping cone of \( a_\xi : S^0 \to S^\xi \) is \( \Sigma S(\xi)_+ \), which is easily verified to have non-free cohomology. In fact \( \Sigma 1 \in \tilde{H}^1_G(\Sigma S(\xi)_+; \mathbb{Z}/p) \) is not divisible by \( a_\xi \) or \( u_\xi \) but satisfies \( a_\xi \cdot \Sigma 1 = 0 \).

One easily verifies analogously that the cohomologies \( \tilde{H}^\bullet_G(S(k\xi)_+; \mathbb{Z}/p) \) are all non-free, and thus the spheres inside representations are not \( Rep(G) \)-complexes as defined above.

There are plenty of examples of \( Rep(G) \)-complexes. The one point compactification \( S^V \) written as union of \( D(V) \) identifying \( S(V) \) to a point is one. Observe from [3, Section 8.1] that \( \mathbb{C}P(V) \), \( \mathbb{G}_F(V) \) are also \( Rep(G) \)-complexes for a unitary \( G \)-representation \( V \).

We begin the proof with a localization theorem, which identifies the cohomology of finite \( G \)-spaces after we invert the element \( p \). This is the case where \( X \) involves attaching a \( \nu \)-cell
\[
a_\xi \colon \tilde{H}^\bullet_G(\nu) \to \tilde{H}^\bullet_G(\nu) \oplus \tilde{H}^\bullet_G(S^0).
\]
Proof. Consider the inclusion \( X^{C_p} \to X \), which induces the map
\[
a_\xi^{-1} \tilde{H}^\bullet_G(X^+_{+}) \to a_\xi^{-1} \tilde{H}^\bullet_G(X^{C_p}_+),
\]
and that \( a_\xi^{-1} \tilde{H}^\bullet_G((\nu C_p)_+) \) is a cohomology theory. They agree on \( G \)-orbits, so the first isomorphism follows. The second isomorphism is clear as \( X^G \) is a trivial \( G \)-space. \( \Box \)

We now prove that the cohomology of a finite-dimensional \( Rep(G) \)-cell complex \( X \) is a free \( \tilde{H}^\bullet_G(S^0; \mathbb{Z}/p) \)-module. We do this in a sequence of steps. Our approach is to prove this by induction starting from the base case, where \( X^{(0)}_+ \) is a wedge of \( S^0 \). In the induction step, \( X \) involves attaching a \( \nu \)-cell \( D(\nu) \) to a \( Rep(G) \)-complex \( Y \) such that
\[
\tilde{H}^\bullet_G(Y_+) \cong \oplus_{i=1}^k \tilde{H}^\bullet_G(S^{\omega_i}) \cong \oplus_{i=1}^k \tilde{H}^\bullet_G(\omega_i(S^0)), \]
with \( \dim(\omega_i) < \dim(\nu) \). In view of the equivalence \( \mathbb{H}Z/p \wedge S^t \cong HZ/p \wedge S^t \) for \( p \nmid i, j \), we may assume that all the \( \omega_i \) and \( \nu \) only contain the trivial representations, and the representation \( \xi \). We use the long exact sequence
\[
\cdots \tilde{H}^\bullet_G(S^t) \to \tilde{H}^\bullet_G(X_+) \to \tilde{H}^\bullet_G(Y_+) \overset{d}{\to} \tilde{H}^\bullet_G(\nu(S^t)) \cdots
\]
(4.12)

We write \( \omega_i \) also for the element \( 1 \in \tilde{H}^\bullet_G(S^{\omega_i}) \) in \( \tilde{H}^\bullet_G(Y_+) \). The cohomology of \( X \) is determined by \( d(\omega_i) \in \tilde{H}^{\omega_i+1}(\nu(S^t)) \cong \tilde{H}^{\omega_i+1}(\nu(S^t)) \).

We assume that \( \dim(\omega_i) \geq \dim(\omega_j) \) if \( i \leq j \). The first case we consider is when \( \dim(\omega_1) = \dim(\nu) - 1 \). This is the case where \( d\omega_1 \) may hit the top cone of \( \tilde{H}^\bullet_G(S^t) \). In this case we will prove the Lemma below.

**Lemma 4.13.** Assume that we have a long exact sequence of \( \tilde{H}^\bullet_G(S^0) \)-modules as in (4.12).
Also assume that \( \dim(\omega_1) = \dim(\nu) - 1 \) and that \( |\omega_1^{C_p}| < |\nu^{C_p}| \). After a change of basis of
either one of the following holds 
1) $d(\omega_1) = 0$.
2) $\omega_1 = \nu - 1$, $d(\omega_1) = \nu$ and $d(\omega_1) = 0$ for all $i \geq 2$.

Proof. In order to prove the Lemma, we assume $d(\omega_1) \neq 0$ and then prove the conclusion in 2). If $\dim(\omega_1) = \dim(\nu) - 1$, we must have $\omega_1 + 1 - \nu$ is of the form $k(\xi - 2)$ for $k > 0$, so that $d(\omega_1) = u_\xi^k \nu$ up to units.

After rearranging $\omega_i$ if necessary we assume that $\omega_1$ is the one with the least value of $k$ among $\{\omega_i \mid \dim(\omega_i) = \dim(\nu) - 1, |\omega_i| < |\nu|, d(\omega_i) \neq 0\}$. Since $\omega_1$ has the least value of $k$ we have $d(u_\xi^t \omega_1) = d(\omega_1)$ for some $t$, as $\omega_i$ varies over the set above. We now change $\omega_i$ to $\omega_i - u_\xi^t \omega_1$ to assume $d(\omega_1) = 0$.

For the other $\omega_j$ we have $d(\omega_j) = 0$ or

$$d(\omega_j) = \Sigma^{-1} \frac{1}{a_\xi} \nu = d(\Sigma^{-1} \frac{1}{a_\xi} \nu + k \omega_1)$$

or,

$$d(\omega_j) = \Sigma^{-1} \frac{\kappa \xi}{a_\xi} \nu = d(\Sigma^{-1} \frac{\kappa \xi}{a_\xi} \nu + k \omega_1)$$

In each case we may add a multiple of $\omega_1$ to $\omega_j$ to ensure $d(\omega_j) = 0$. Therefore, $\omega_1$ is the only class with a non-trivial differential. We now are reduced to the simpler long exact sequence

$$(4.14) \quad \cdots \tilde{H}_G^\bullet(S^\nu) \xrightarrow{\xi} \tilde{H}_G^\bullet(X_+) \xrightarrow{i^\ast} \tilde{H}_G^\bullet(S^\nu) \xrightarrow{d} \tilde{H}_G^{\nu + 1}(S^\nu) \cdots$$

putting $\omega = \omega_1$ and suppressing the other $\omega_i$ with trivial differential. We consider (4.14) for $\bullet = \nu$, assume that the conclusion of the Proposition does not hold, and show that the boundary map $d : \tilde{H}_G^{\nu - 1}(S^\nu) \rightarrow \tilde{H}_G^\nu(S^\nu)$ is trivial. The group $\tilde{H}_G^{\nu - 1}(S^\nu)$ is generated by $\Sigma^{-1} \frac{\kappa \xi}{a_\xi} \nu$ for some $t$. If $d$ is non-trivial on this class, we must have

$$d(\Sigma^{-1} \frac{\kappa \xi}{a_\xi} \nu) = \nu$$

up to an unit in $\mathbb{Z}/p$. Note that this class is again divisible by $a_\xi$, so $d(\Sigma^{-1} \frac{\kappa \xi}{a_\xi} \nu) \in \mathbb{H}_G^{\nu - \xi}(S^\nu)$, which is zero. Now the module structure implies

$$0 = a_\xi d(\Sigma^{-1} \frac{\kappa \xi}{a_\xi} \nu) = d(\Sigma^{-1} \frac{\kappa \xi}{a_\xi} \nu) = \nu$$

which gives a contradiction. Hence, $q^*(\nu) \neq 0$. For degree reasons, also $q^*(\nu) \neq 0$.

Therefore, the class $q^*(\nu)$ would survive to the $a_\xi$-localization and have $u_\xi$ torsion, which cannot happen in the cohomology of a finite space by Lemma 4.11.

Now we prove the crucial step in the proof for a cohomological two cell complex. That is, we have the cofibre sequence

$$\Sigma^{-\nu}HZ\mathbb{Z}/p \rightarrow F(X_+, HZ\mathbb{Z}/p) \rightarrow \Sigma^{-\omega}HZ\mathbb{Z}/p$$

with $\dim(\omega) < \dim(\nu)$. This gives the long exact sequence

$$(4.15) \quad \cdots \tilde{H}_G^\bullet(S^\nu) \rightarrow \tilde{H}_G^\bullet(X_+) \rightarrow \tilde{H}_G^\bullet(S^\nu) \xrightarrow{d} \tilde{H}_G^{\bullet + 1}(S^\nu) \cdots$$

As before, we call the generators of $\tilde{H}_G^\bullet(S^\nu)$ and $\tilde{H}_G^\bullet(S^\nu)$ $\omega$ and $\nu$ respectively. Since $d$ is an $\tilde{H}_G^\bullet(S^0)$-module map, the computation is determined by $d(\omega) \in \tilde{H}_G^{\omega + 1}(S^\nu)$. 
Lemma 4.16. Let $X$ be a cohomological 2-cell complex as above. Then $\tilde{H}^\star_G(X_+) \equiv \text{free } \tilde{H}^\star_G(S^0)$-module. In particular, one of the following must holds

1. $\tilde{H}^\star_G(X_+) = 0$.
2. $\tilde{H}^\star_G(X_+) \equiv \Sigma^\infty \tilde{H}^\star_G(S^0) \oplus \Sigma^\nu \tilde{H}^\star_G(S^0)$.
3. $\tilde{H}^\star_G(X_+) \equiv \text{free with two generators one with the same dimension as } \omega \text{ and the other with the same dimension as } \nu$.

Proof. If $d(\omega) = 0$, the conclusion 2) holds. If $d(\omega) = \nu$, then 1) holds. This is the case if $\omega$ and $\nu$ satisfy the hypothesis of Lemma 4.13. We have to prove the conclusion 3) in other cases where the boundary map hits classes in the bottom cone of $\nu$. Proposition 3.8 implies that the boundary should be either

$$d(\omega) = \Sigma^{-1}\frac{1}{u_j^k}\nu$$

or

$$d(\omega) = \Sigma^{-1}\frac{k\xi}{u_j^k+1}\nu$$

Since, the Böckstein of the class $\omega$ is zero, therefore, the Böckstein of right hand side of the boundary must be zero. This forces that boundary should be one of the following form

1. $d(\omega) = \Sigma^{-1}\frac{1}{u_j^k}\nu$ for $j, k \geq 1$.
2. $d(\omega) = \Sigma^{-1}\frac{k\xi}{u_j^k+1}\nu$ for $j \geq 1$.

First, we consider the case 1). We must have the relation

$$\omega = \nu + j(2 - \xi) - k\xi.$$ 

The long exact sequence (4.15) for $\star = \alpha := \omega + j(\xi - 2)$ gives the isomorphism

$$\tilde{H}^\star_G(X_+) \xrightarrow{i^*} \tilde{H}^{\omega+j(\xi-2)}_G(S^\omega) \cong \mathbb{Z}/p\{u_j^k\omega\}.$$ 

So, there exists a class $a \in \tilde{H}^\star_G(X_+)$ such that $i^*(a) = u_j^k\omega$. Also, for $\star = \beta = \nu + j(2 - \xi) = \omega + k\xi$, the same exact sequence (4.15) gives

$$\tilde{H}^{\omega+1+k\xi}_G(S^\omega) \xrightarrow{d} \tilde{H}^{\nu+j(2-\xi)}_G(S^\nu) \xrightarrow{q^*} \tilde{H}^\star_G(X_+) \xrightarrow{i^*} \tilde{H}^{\omega+k\xi}_G(S^\omega) \xrightarrow{d} \tilde{H}^{\nu+1+j(2-\xi)}_G(S^\nu)$$

$$\mathbb{Z}/p\{a^{k-1}\kappa\xi\omega\} \xrightarrow{d} \mathbb{Z}/p\{\Sigma^{-1}\frac{k\xi}{u_j^k+1}\nu\} \xrightarrow{q^*} 0$$

Since $d(\omega) = \Sigma^{-1}\frac{1}{u_j^k}\nu$, the boundary map $d : \tilde{H}_G^{\omega+1+k\xi}(S^\omega) \to \tilde{H}_G^{\nu+j(2-\xi)}(S^\nu)$ is an isomorphism. This implies a class $b \in \tilde{H}_G(X_+)$ such that $i^*(b) = a^{k}\omega$. Observe that $\dim(a) = \dim(\omega)$ and $\dim(b) = \dim(\nu)$.

For each $i$, the class $u_i^k b$ is non zero, and hence, it'll survive $a_i$-localization. So, by Lemma 4.11 the classes $u_i^k b$ are non trivial for each $i$. From the exact sequence 4.15 at $\star = \nu$ we get the following extension

$$0 \to \mathbb{Z}/p\{\nu\} \xrightarrow{q^*} \tilde{H}^\star_G(X_+) \xrightarrow{i^*} \mathbb{Z}/p\{u_j^k u_i^k\omega\} \to 0$$

Observe that both $a_i^k a$ and $u_i^k b$ map to the same element, so that up to units,

$$q^*(\nu) = a_i^k a - u_i^k b.$$
Therefore, the classes \( a^k \xi \) and \( u \xi^j \) generate \( \tilde{H}^\star_G(X_+) \). The map \( q^\star \) is an \( \tilde{H}^\star_G(S^0) \)-module homomorphism, so that \( a \) and \( b \) generate the image of the top cone of \( \nu \) in \( X \). For the bottom cone of \( \nu \) note that (using (4.17))

\[
q^\star(\Sigma^{-1}\frac{\kappa\xi}{a^k u \xi}) = \Sigma^{-1}\frac{\kappa\xi}{a^k u \xi} \cdot a
\]

and

\[
q^\star(\Sigma^{-1}\frac{\kappa\xi}{a^k u \xi}) = -\Sigma^{-1}\frac{\kappa\xi}{a^k u \xi} \cdot b,
\]

so that these are also generated by the classes \( a^k \xi \) and \( u \xi^j \). In particular, we can construct a \( \tilde{H}^\star_G(S^0) \)-module map \( f \) from a free \( \tilde{H}^\star_G(S^0) \)-module with two generators \( \eta \) and \( \xi \) in degrees \( V + k \xi \) and \( V + j(2 - \xi) \) respectively to \( \tilde{H}^V_G(X_+) \) with \( f(\eta) = a \) and \( f(\xi) = b \). This \( f \) is surjective by the above, and by comparing degree-wise ranks we see that this is an isomorphism. Therefore, we get the conclusion 3).

Now, we consider the case \( ii) \) \( d(\omega) = \Sigma^{-1}\frac{\kappa\xi}{u \xi^j} \cdot \omega \). It gives the following relation

\[
\omega = (j + 1)(2 - \xi) - 1 + \nu.
\]

For \( \star = \alpha = \omega + j(2 - \xi) = \nu + 1 - \xi \), the long exact sequence (4.15) gives the following exact sequence

\[
0 \rightarrow \tilde{H}^\alpha_G(X_+) \rightarrow \tilde{H}^{\omega + j(2 - \xi)}(S^\omega) \rightarrow \tilde{H}^{\nu + 2 - \xi}(S^\nu) \rightarrow \ldots
\]

Since \( \kappa^2 = 0 \),

\[
d(\Sigma^{-1}\frac{\kappa\xi}{u \xi}) = 0.
\]

Therefore,

\[
d : \tilde{H}^{\omega + j(2 - \xi)}(S^\omega) \rightarrow \tilde{H}^{\nu + 2 - \xi}(S^\nu)
\]

is the zero map. Hence, we obtain the equivalence

\[
\tilde{H}^\alpha_G(X_+) \overset{i^\star}{\rightarrow} \tilde{H}^{j(2 - \xi)}(S^0)
\]

Thus, there exists a class \( c \) such that \( i^\star(c) = u \xi^j \omega \).

Next, consider \( \beta = \nu + j(2 - \xi) = \omega + \xi - 1 \), then (4.15) at \( \star = \beta \) yields

\[
\ldots \tilde{H}^{\omega + \xi - 2}(S^\omega) \rightarrow \tilde{H}^{\omega + j(2 - \xi)}(S^\nu) \rightarrow \tilde{H}^{\beta}(X_+) \rightarrow \tilde{H}^{\omega + \xi - 1}(S^\omega) \rightarrow 0
\]

Since, our \( d \) satisfies \( d(\omega) = \Sigma^{-1}\frac{\kappa\xi}{u \xi^j} \cdot \omega \). Therefore, the boundary map \( d : \tilde{H}^{\omega + \xi - 2}(S^\omega) \rightarrow \tilde{H}^{\nu + j(2 - \xi)}(S^\nu) \) is an isomorphism. So, there exists a class \( e \in \tilde{H}^{\beta}_G(X_+) \) such that \( i^\star(e) = \kappa\xi \omega \). Observe that \( \dim(e) = \dim(\omega) \) and \( \dim(e) = \dim(\nu) \). Since for each \( i \), the class \( a^k \xi e \) is non zero, therefore, it’ll survive \( a \xi \)-localization. So, by Lemma 4.11 the class \( u \xi^j e \) is non trivial for all \( i \).

Now, the exact sequence (4.15) at \( \star = \nu \) gives the short exact sequence

\[
0 \rightarrow \mathbb{Z}/p\{\nu\} \rightarrow \tilde{H}^\nu_G(X_+) \rightarrow \mathbb{Z}/p\{u \xi^j \kappa\xi \omega\} \rightarrow 0.
\]
So, the exactness yields
\begin{equation}
q^*(\nu) = u_\xi^j e - \kappa_\xi c.
\end{equation}
As in case i), using \((4.18)\) it is readily follows that the image of \(q^*\) is generated by these two elements, so that the map from the free module in two generators mapping onto \(c\) and \(e\) is surjective. By comparing ranks degree-wise, we obtain the conclusion 3).

Finally we use the 2-cell case to prove freeness for any finite \(\text{Rep}(G)\)-complex.

**Theorem 4.19.** Suppose \(X\) is a connected, locally finite, finite dimensional \(\text{Rep}(G)\)-cell complex, then \(\hat{H}_G^\star(X_+; \mathbb{Z}/p)\) is an free \(\hat{H}_G^\star(S^0; \mathbb{Z}/p)\)-module.

**Proof.** We use induction on the cellular filtration \(\{X_n\}_{n \geq 0}\) of \(X\). Since \(X\) is locally finite, it suffices to prove the case of a single cell attachment : \(X\) is obtained from \(Y\) by attaching a single cell \(D(\nu)\) for some representation \(\nu\). This yields the cofibre sequence

\[ Y \to X \to S^\nu. \]

For each \(\alpha \in RO(G)\), corresponding to the cofibre sequence above, there is a long exact sequence \((4.12)\)

\[ \cdots \to \hat{H}_G^\alpha(S^\nu) \to \hat{H}_G^\alpha(X_+) \to \hat{H}_G^\alpha(Y_+) \to \hat{H}_G^{\alpha+1}(S^\nu) \to \cdots \]

The boundary map \(d\) is a \(\hat{H}_G^\star(S^0; \mathbb{Z}/p)\)-module map. We assume that \(\hat{H}_G^\alpha(Y_+; \mathbb{Z}/p)\) is a free module over \(\hat{H}_G^\star(S^0; \mathbb{Z}/p)\) with the generators \(\omega_1, \cdots, \omega_s\). If \(d(\omega_i)\) lies in the top cone of \(\nu\) for any \(i\) we may apply Lemma 4.13 to deduce that the cohomology of \(X\) is generated by \(s - 1\) generators.

If all the \(\omega_i\) such that \(d(\omega_i) \neq 0\) map to the bottom cone of \(\nu\), we may rearrange the \(\omega_i\) such that \(\omega_1\) maps to the element with highest power of \(u_\xi\) in the denominator. Using a change of basis given by \(\{\omega_1, \omega_i - u_\xi^t \omega_1 : i \geq 2\}\) to the free module \(\hat{H}_G^\alpha(Y_+; \mathbb{Z}/p)\) we can assume \(d(\omega_i) = 0\) for all \(i \geq 2\). In the new basis possibly

\[ d(\omega_i) = u^{-1} \frac{1}{a_\xi^t u_\xi^t} \nu, \]  

which satisfies \(d(a_\xi^t \omega_i) = 0\) for some \(t'' < t\). This is possible only if \(u = 0\). Therefore, we are in the situation of Lemma 4.16. Hence, we obtain that \(\hat{H}_G^\star(X_+)\) is a free module with generators either \(a, b, \omega_3, \cdots, \omega_s\) or \(c, e, \omega_2, \cdots, \omega_s\). This completes the induction argument.

By the assumption we know \(X\) is a finite \(\text{Rep}(C_p)\)-cell complex. Hence, the conclusion follows for \(X\). \(\square\)

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