PRIMARY DECOMPOSITION WITH DIFFERENTIAL OPERATORS

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Abstract. We introduce differential primary decompositions for ideals in a commutative ring. Ideal membership is characterized by differential conditions. The minimal number of conditions needed is the arithmetic multiplicity. Minimal differential primary decompositions are unique up to change of bases. Our results generalize the construction of Noetherian operators for primary ideals in the analytic theory of Ehrenpreis-Palamodov, and they offer a concise method for representing affine schemes. The case of modules is also addressed. We implemented an algorithm in Macaulay2 that computes the minimal decomposition for an ideal in a polynomial ring.

1. Introduction

Macaulay’s theory of inverse systems [13] employs differential operators to characterize membership in an ideal that is primary to the maximal ideal in a power series ring or to the maximal irrelevant ideal in a polynomial ring. The number of operators needed is the multiplicity of the ideal. Building on [4], this description was extended to primary ideals in our previous papers [7,8]. The present article develops a minimal such representation for arbitrary ideals in a commutative \( \mathbb{k} \)-algebra that is essentially of finite type over a perfect field \( \mathbb{k} \). We introduce differential primary decompositions. These are differential conditions that characterize ideal membership.

Example 1.1. We describe an ideal \( I \) in \( R = \mathbb{Q}[x,y,z] \). A polynomial \( f \) lies in \( I \) if and only if

(a) both \( f \) and \( x \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} \) vanish on the \( x \)-axis,
(b) both \( f \) and \( y \frac{\partial f}{\partial z} + \frac{\partial f}{\partial x} \) vanish on the \( y \)-axis,
(c) both \( f \) and \( z \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \) vanish on the \( z \)-axis, and
(d) \( \frac{\partial^3 f}{\partial x \partial y \partial z} + \frac{\partial^3 f}{\partial y^2 \partial z} + \frac{\partial^3 f}{\partial y^2 \partial z} \) vanishes at the origin \((0,0,0)\).

More familiar formats would be a list of ideal generators or a minimal primary decomposition:

\[
I = \langle xy^2z, z^2y^3z, 2xyz - xz^2 + y^3z, 2xyz - x^3z, 2xyz - y^2z + x^2 \rangle
\]

We notice four associated primes: \( p_1 = \langle y, z \rangle \), \( p_2 = \langle x, z \rangle \), \( p_3 = \langle x, y \rangle \), and \( p_4 = \langle x, y, z \rangle \). In (a)-(d) we characterized membership in \( I \) by a set of linear differential operators for each prime:

\( \mathfrak{A}_1 = \{ 1, x \partial_y + \partial_z \} \), \( \mathfrak{A}_2 = \{ 1, y \partial_z + \partial_x \} \), \( \mathfrak{A}_3 = \{ 1, z \partial_x + \partial_y \} \), \( \mathfrak{A}_4 = \{ \partial_x \partial_y \partial_z + \partial_x^2 \partial_y + \partial_y^2 \partial_x + \partial_z^2 \partial_x \} \).

The primary ideal \( \langle y^2, z^2, y - xz \rangle \) is a famous example due to Palamodov [22]. He showed that membership in this ideal cannot be described by differential operators with constant coefficients.

We now discuss the issues that are addressed in this paper. Consider an ideal \( I \) in an essentially of finite type \( \mathbb{k} \)-algebra \( R \). Suppose that its set of associated primes is \( \text{Ass}(R/I) = \{ p_1, \ldots, p_k \} \subset \text{Spec}(R) \). We wish to characterize ideal membership in \( I \) by means of differential operators. The natural place to look for these is the ring \( \text{Diff}_{\mathbb{k}}(R, R) \) of \( \mathbb{k} \)-linear differential operators on \( R \).

As in Example 1.1, we hope to find finite subsets \( \mathfrak{A}_1, \ldots, \mathfrak{A}_k \subset \text{Diff}_{\mathbb{k}}(R, R) \) such that

\[
I = \{ f \in R \mid \delta(f) \in p_i \text{ for all } \delta \in \mathfrak{A}_i \text{ and } i = 1, 2, \ldots, k \}.
\]
Such differential primary decompositions exist for ideals \( I \) in a polynomial ring \( R = \mathbb{K}[x_1, \ldots, x_n] \). An instance with \( n = 3 \), \( k = 4 \), \( |\mathfrak{A}_1| = |\mathfrak{A}_2| = |\mathfrak{A}_3| = 2 \) and \( |\mathfrak{A}_4| = 1 \) was shown in Example 1.1. This existence result is the Fundamental Principle of Palamodov-Ehrenpreis, which is important in analysis [3, 10, 17, 22]. The elements \( \delta \in \mathfrak{A}_i \) are known as Noetherian operators. They are the key to solving linear partial differential equations with constant coefficients. The computation of Noetherian operators was addressed in [4–8, 21] and continues to be of interest for both PDE and computer algebra. For recent work along these lines see [1, 15]. If each \( p_i \) is a rational maximal ideal then we are dealing with inverse systems [13] and identifying the \( \mathfrak{A}_i \) is standard textbook material [20, Theorem 3.27]. An algorithm for the case when \( R \) is a polynomial ring and \( I \) is primary was given recently in [7], where \( \text{Diff}_{R/\mathbb{K}}(R,R) \) is the Weyl algebra \( \mathbb{K}(x_1, \ldots, x_n, \partial_{x_1}, \ldots, \partial_{x_n}) \), and punctual Hilbert schemes play a major role.

In this article, we present answers to the following two questions:

(a) What is the analog of the representation (1) in \( \mathbb{K} \)-algebras other than polynomial rings?
(b) If a representation (1) exists for an ideal \( I \subset R \), what is the minimal size of the sets \( \mathfrak{A}_i \)?

To motivate Problem (a), we note that (1) fails for \( \mathbb{K} \)-algebras \( R \) that are not regular. The noncommutative ring \( \text{Diff}_{R/\mathbb{K}}(R,R) \) can be very complicated. It is usually not Noetherian. A classical example is the cubic cone that was studied by Bernstein, Gel’fand and Gel’fand in [2]:

\[
R = \mathbb{C}[x,y,z]/(x^3 + y^3 + z^3).
\]

If \( I \) is a power of the maximal ideal \( \langle x, y, z \rangle \) in \( R \) then a representation (1) does not exist. This follows from [8, Example 5.2]. The issue is that there are too few differential operators on \( R \).

Problem (b) is motivated by a foundational question in computational algebraic geometry: how to measure the complexity of a subscheme in affine space or projective space? Our answer is drawn from [23]. For any \( p_i \in \text{Ass}(R/I) \) let \( \text{mult}_I(p_i) \) denote the length of the largest ideal of finite length in \( R_{p_i}/IR_{p_i} \). We define the arithmetic multiplicity of the ideal \( I \) to be the sum

\[
\text{amult}(I) = \text{mult}_I(p_1) + \text{mult}_I(p_2) + \cdots + \text{mult}_I(p_k).
\]

If \( I \) is a monomial ideal in \( R = \mathbb{K}[x_1, \ldots, x_n] \) then \( \text{amult}(I) \) is the number of standard pairs \( (\delta, p_i) \), by [23, Lemma 3.3]. In such a pair, \( p_i \) is a monomial prime and we can identify \( \delta \) with a differential operator \( \prod_{x_j \in p_i} \partial_{x_j}^{x_j} \). The set \( \mathfrak{A}_i \) of such operators describes the contribution of the coordinate subspace \( V(p_i) \) to the scheme \( V(I) \), and this yields the minimal representation (1).

Example 1.2. Let \( R = \mathbb{Q}[x, y, z] \) and \( I = \langle x^2 y, x^2 z, y^2 x, xyz^2 \rangle \) as in [23, eqn (1.5)]. This ideal has \( k = 4 \) associated primes, namely \( p_1 = \langle x \rangle \), \( p_2 = \langle y, z \rangle \), \( p_3 = \langle x, y \rangle \) and \( p_4 = \langle x, y, z \rangle \). We see in [23, eqn (3.3)] that membership in \( I \) is characterized by amult(\( I \)) = 5 Noetherian operators:

\[
\mathfrak{A}_1 = \{1\}, \mathfrak{A}_2 = \{1\}, \mathfrak{A}_3 = \{\partial_z\} \text{ and } \mathfrak{A}_4 = \\{\partial_x \partial_y, \partial_x \partial_y \partial_z\}.
\]

Our contribution is a general theory that resolves both problems (a) and (b). We propose a variant of (1) where \( \mathfrak{A}_i \) consists of operators in \( \text{Diff}_{R/\mathbb{K}}(R,R/p_i) \). Such differential primary decompositions always exist, they satisfy \( |\mathfrak{A}_i| \geq \text{mult}_I(p_i) \), and equality is attained for all \( i \).

The presentation is organized as follows. In Section 2 we fix the set-up and we review basics on differential operators in commutative algebra. In Section 3 we introduce differential primary decompositions. Our main result on their existence and minimality appears in Theorem 3.6. Section 4 generalizes this result from ideals to modules. We then specialize to formally smooth \( \mathbb{K} \)-algebras \( R \), where differential operators in \( \text{Diff}_{R/\mathbb{K}}(R,R) \) suffice and (1) is valid as stated.

In Section 5 we turn to the case of polynomial rings, which is most relevant for applications. Theorem 5.3 extends the results on primary ideals in [7]. We show how to compute the pairs \( (p_i, \mathfrak{A}_i) \) for any ideal \( I \) in \( R = \mathbb{K}[x_1, \ldots, x_n] \) with char(\( \mathbb{K} \)) = 0. We discuss our Macaulay2 implementation, we present non-trivial examples, and we reflect on applications to linear PDE as in [3, 10, 17, 22]. Further practical tools for solving such linear PDE can be found in [1, 15].
2. Differential Operators in Commutative Algebra

Throughout this paper we assume the setup below. In this section, we fix notation and we recall some foundational results to be used (for further details, the reader is referred to [14, §16]).

Setup 2.1. Let \( \mathbb{k} \) be a field and let \( R \) be a \( \mathbb{k} \)-algebra essentially of finite type over \( \mathbb{k} \). Let \( A \) be a \( \mathbb{k} \)-subalgebra of \( R \) such that \( R \) is essentially of finite type over \( A \). This means that \( R \) is the localization of a finitely generated \( A \)-algebra. We are mostly interested in the case \( A = \mathbb{k} \).

For any prime \( p \in \text{Spec}(R) \), we denote by \( k(p) := R_p/R_p = \text{Quot}(R/p) \).

For two \( R \)-modules \( M \) and \( N \), we regard \( \text{Hom}_A(M, N) \) as an \( (R \otimes_A R) \)-module, by setting
\[
((r \otimes_A s)\delta)(w) = r\delta(sw) \quad \text{for all } \delta \in \text{Hom}_A(M, N), \ w \in M, \ r, s \in R.
\]
We use the bracket notation \([\delta, r](w) = \delta(rw) - r\delta(w)\) for \( \delta \in \text{Hom}_A(M, N), r \in R \) and \( w \in M \).

Unless specified otherwise, whenever we consider an \( (R \otimes_A R) \)-module as an \( R \)-module, we do so by letting \( R \) act via the left factor of \( R \otimes_A R \). We now introduce our most relevant module.

Definition 2.2. Let \( M, N \) be \( R \)-modules. The \( m \)-th order \( A \)-linear differential operators, denoted \( \text{Diff}^m_{R/A}(M, N) \subseteq \text{Hom}_A(M, N) \), form an \( (R \otimes_A R) \)-module that is defined inductively by
1. \( \text{Diff}^0_{R/A}(M, N) := \text{Hom}_R(M, N) \).
2. \( \text{Diff}^m_{R/A}(M, N) := \left\{ \delta \in \text{Hom}_A(M, N) \mid [\delta, r] \in \text{Diff}^{m-1}_{R/A}(M, N) \text{ for all } r \in R \right\} \).

The set of all \( A \)-linear differential operators from \( M \) to \( N \) is the \( (R \otimes_A R) \)-module
\[
\text{Diff}_{R/A}(M, N) := \bigcup_{m=0}^{\infty} \text{Diff}^m_{R/A}(M, N).
\]

Subsets \( \mathcal{E} \subseteq \text{Diff}_{R/A}(M, N) \) are viewed as differential equations. Their solution spaces are
\[
\text{Sol}(\mathcal{E}) := \left\{ w \in M \mid \delta(w) = 0 \text{ for all } \delta \in \mathcal{E} \right\} = \bigcap_{\delta \in \mathcal{E}} \ker(\delta).
\]

Example 2.3. If \( R = \mathbb{k}[x_1, \ldots, x_n] \) is a polynomial ring over a field \( \mathbb{k} \) of characteristic zero, then \( \text{Diff}_{R/\mathbb{k}}(R, R) \) is the Weyl algebra \( D_n = R(\partial x_1, \ldots, \partial x_n) \). For a derivation see e.g. [7, Lemma 1].

To describe differential operators, one uses the module of principal parts. Consider the multiplication map \( \mu : R \otimes_A R \rightarrow R, \ r \otimes_A s \mapsto rs \). The kernel of this map is the ideal \( \Delta_{R/A} \subset R \otimes_A R \).

Definition 2.4. Let \( M \) be an \( R \)-module. The module of \( m \)-th principal parts is defined as
\[
P^m_{R/A}(M) := \frac{R \otimes_A M}{\Delta_{m+1}^{R/A} (R \otimes_A M)}.
\]
This is a module over \( R \otimes_A R \) and thus over \( R \). For simplicity of notation, set \( P^m_{R/A} := P^m_{R/A}(R) \).

For any \( R \)-module \( M \), we consider the universal map \( d^m : M \rightarrow P^m_{R/A}(M), w \mapsto 1 \otimes_A w \).

The following result is a fundamental characterization of the modules of differential operators.

Proposition 2.5 ([14, Proposition 16.8.4], [16, Theorem 2.2.6]). Let \( M \) and \( N \) be \( R \)-modules and let \( m \geq 0 \). Then, the following map is an isomorphism of \( R \)-modules:
\[
(d^m)^* : \text{Hom}_R\left( P^m_{R/A}(M), N \right) \xrightarrow{\cong} \text{Diff}^m_{R/A}(M, N), \quad \varphi \mapsto \varphi \circ d^m.
\]

The next lemma records basic facts about the localization of differential operators.

Lemma 2.6. Let \( M, N \) be \( R \)-modules with \( M \) finitely generated. Given \( W \subset R \) multiplicatively closed, we set \( S = W^{-1}R, M' = W^{-1}M \) and \( N' = W^{-1}N \). Then the following statements hold:
Remark 2.7. Given \( m \), this is well-defined by the induction hypothesis and the fact that \( \delta, r \), the vertical map on the right is injective. Hence \( \text{Ker}(\delta) = \text{Ker}(\delta') \cap M \), and the claim follows. □

For the sake of completeness, we describe the localization of differential operators explicitly.

Remark 2.7. Given \( \delta \in \text{Diff}^m_{R/A}(M, N) \), we extend it to an element \( \delta' \in \text{Diff}^{m+1}_{R/A}(M', N') \).

Proof. (i) See, e.g., [8, Lemma 2.7].

(ii) The proof follows verbatim that of [8, Proposition 3.6 (ii)]. Let \( \delta \in \mathfrak{A} \) and consider its extension \( \delta' \in \mathfrak{A}' \). We have the following commutative diagram of \( A \)-linear maps:

\[
\begin{array}{ccc}
M & \xrightarrow{\delta} & N \\
\downarrow & & \downarrow \\
M' & \xrightarrow{\delta'} & N'
\end{array}
\]

The vertical map on the right is injective. Hence \( \text{Ker}(\delta) = \text{Ker}(\delta') \cap M \), and the claim follows. □

Here is another basic remark regarding the solutions to differential operators of order \( m \).

Remark 2.8. Let \( M \) be an \( R \)-module, \( p \in \text{Spec}(R) \) a prime ideal and \( \delta \in \text{Diff}^m_{R/A}(M, R/p) \). Then, by induction on \( m \), it follows that \( \text{Sol}(\delta) \supseteq p^{m+1}M \). If \( m = 0 \), then \( \delta \in \text{Hom}_R(M, N) \) and the result is clear. If \( m > 0 \), for \( r \in p \) and \( \beta \in p^mM \), then we obtain the identity

\[
\delta(r\beta) = r\delta(\beta) + [\delta, r](\beta) = 0.
\]

The latter equation follows from \([\delta, r] \in \text{Diff}^{m-1}_{R/A}(M, R/p)\) and the induction hypothesis.

The following lemma will be used for a process of lifting differential operators.

Lemma 2.9 ([8, Proposition 3.12]). Suppose that \( R \) is formally smooth over \( A \). Let \( p \in \text{Spec}(R) \) be a prime ideal and \( F \) a free \( R \)-module of finite rank. Then, the canonical map \( \text{Diff}^m_{R/A}(F, R) \to \text{Diff}^m_{R/A}(F, R/p) \) is surjective for all \( m \geq 0 \).

For ease of notation, we fix the following piece of data for the rest of this section.

Notation 2.10. Assume Setup 2.1 with \( A = \mathbb{K} \). Let \( p \in \text{Spec}(R) \) with residue field \( \mathbb{F} = k(p) \).

Let \( \mathcal{M} \) be the kernel of the multiplication map \( \mathbb{F} \otimes_k R_p \to \mathbb{F} \). Then \( \mathcal{M} = \Delta_{R/k}(\mathbb{F} \otimes_k R) \) is the extension of \( \Delta_{R/k} \subset R \otimes_k R \) into \( \mathbb{F} \otimes_k R \). The ideal \( \mathcal{M} \) is maximal with \( (\mathbb{F} \otimes_k R_p) / \mathcal{M} \cong \mathbb{F} \).

The following lemma will be used in the proof of Theorem 3.6.

Lemma 2.11 ([8, Lemma 3.8, Lemma 3.14]). Let \( M \) be a finitely generated \( R \)-module.

(i) There is an isomorphism of \( (R_p \otimes_k R_p) \)-modules

\[
\text{Diff}^m_{R_p/k}(M_p, \mathbb{F}) \cong \text{Hom}_F \left( \frac{\mathbb{F} \otimes_k M_p}{\mathcal{M}^{m+1}(\mathbb{F} \otimes_k M_p)}, \mathbb{F} \right).
\]
Lemma 2.6. The setup below is set throughout.

Definition 3.2. For all \( p \in \text{Ass}(R/I) \), \( p \) is a \( R \)-primary ideal.

The next proposition contains a technical result of fundamental importance for our approach.

Proposition 2.13 ([4, Proposition 4.1], [8, Proposition 3.9]). If \( R \) is a separable algebraic extension, then we have an isomorphism of local rings

\[
\varepsilon_{p,m} : R_p/p^mR_p \congto (R \otimes_K R_p)/\mathcal{M}^m
\]

for all \( m \geq 1 \).

3. Arithmetic multiplicity and Noetherian operators

In this section we introduce a notion of primary decomposition that is based on differential operators. We use the notation and results in Section 2. The setup below is set throughout.

Setup 3.1. Let \( K \) be a field and \( R \) be a \( K \)-algebra essentially of finite type over \( K \).

Our definition rests on localizing along associated prime ideals \( p_i \). This is an essential feature.

Definition 3.2. Let \( I \subseteq R \) be an ideal with \( \text{Ass}(R/I) = \{ p_1, \ldots, p_k \} \subseteq \text{Spec}(R) \). A differential primary decomposition of \( I \) is a list of pairs \( (p_1, A_1), \ldots, (p_k, A_k) \), where \( A_i \subseteq \text{Diff}_{R/K}(R, R/p_i) \) is a finite set of differential operators, such that the following equation holds for each \( p \in \text{Ass}(R/I) \):

\[
I_p = \bigcap_{1 \leq j \leq k} \{ f \in R_p \mid \delta^j(f) = 0 \text{ for all } \delta \in A_i \}.
\]

Here \( \delta^j \in \text{Diff}_{R/K}(R_p, R_p/p_iR_p) \) denotes the image of an operator \( \delta \in A_i \) under Lemma 2.6 (i).

The next lemma records fundamental properties of differential primary decompositions.

Lemma 3.3. For a differential primary decomposition as in Definition 3.2, the following holds:

(i) The ideal \( I \) satisfies (1), i.e. \( I = \{ f \in R \mid \delta(f) = 0 \text{ for all } \delta \in A_i \text{ and } 1 \leq i \leq k \} \).

(ii) For each index \( i \in \{ 1, \ldots, k \} \), the set \( A_i \) of differential operators is non-empty.

Proof. Fix a minimal primary decomposition \( I = Q_1 \cap \cdots \cap Q_k \) where \( Q_i \) is a \( p_i \)-primary ideal.

(i) By applying Lemma 2.6 (ii) to the localization at \( p \in \text{Ass}(R/I) \), we get

\[
\bigcap_{i : p_i \subseteq p} Q_i = \bigcap_{i : p_i \subseteq p} \{ f \in R \mid \delta(f) = 0 \text{ for all } \delta \in A_i \}.
\]

The equality on the left holds because \( Q_i \) is a \( p_i \)-primary ideal (see, e.g., [19, Theorem 4.1]). By intersecting the equation above over all \( p \in \text{Ass}(R/I) \), we obtain the desired identity (1).

(ii) Suppose that \( A_i = \emptyset \) for some \( i \in \{ 1, \ldots, k \} \), and set \( q = p_i \). By intersecting (6) for all \( p \in \text{Ass}(R/I) \) with \( p \subseteq q \), we obtain

\[
\bigcap_{i : p_i \subseteq q} Q_i = \bigcap_{i : p_i \subseteq q} \{ f \in R \mid \delta(f) = 0 \text{ for all } \delta \in A_j \}.
\]
Here each $p_j$ is strictly contained in $q$. Since $\mathfrak{A}_i = \emptyset$, by definition, we conclude $\bigcap_{j : p_j \subseteq q} Q_j = IR_q \cap R$. This contradicts the hypothesis $q \in \text{Ass}(R/I)$, and so the result follows. \hfill $\Box$

**Remark 3.4.** Differential primary decompositions always exist. Let $I = Q_1 \cap \cdots \cap Q_k$ where each $Q_i$ is a $p_i$-primary ideal. By [8, Theorem A], there is a finite set $\mathfrak{B}_i \subset \text{Diff}_{R_p / k}(R_{p_i}, k(p_i))$ of Noetherian operators such that $Q_i R_{p_i} = \{ f \in R_{p_i} \mid \delta(f) = 0 \text{ for all } \delta \in \mathfrak{B}_i \}$. By lifting $\mathfrak{B}_i$ to a set of operators $\mathfrak{B}_i \subset \text{Diff}_{R/k}(R, R/p_i)$, it follows that $(p_1, \mathfrak{B}_1), \ldots, (p_k, \mathfrak{B}_k)$ is a differential primary decomposition of $I$. Hence the requirement in Definition 3.2 can always be achieved.

What we are looking for here is more ambitious. We wish to represent any ideal $I$ by a minimal differential primary decomposition. This should reflect the intrinsic complexity of the affine scheme defined by $I$. This brings us back to the notion from [23] we saw in the Introduction.

**Definition 3.5.** For an ideal $I \subset R$, its arithmetic multiplicity is the positive integer

$$\text{mult}(I) := \sum_{p \in \text{Ass}(R/I)} \text{length}_{R_p}(H^0(R_p/IR_p)) = \sum_{p \in \text{Ass}(R/I)} \text{length}_{R_p}( (IR_p : R_p (pR_p) \infty ) / IR_p).$$

In our Introduction and in [23], the length inside the sum was denoted $\text{mult}_I(p)$ and called the multiplicity of $I$ along $p$. It is the length of the largest ideal of finite length in the ring $R_p/IR_p$.

The next theorem is our main result in this section. An ideal $I$ always has a differential primary decomposition whose total number of operators is equal to the arithmetic multiplicity. Moreover, $\text{mult}(I)$ is a lower bound on the size of any differential primary decomposition.

**Theorem 3.6.** Assume Setup 3.1 with $k$ perfect. Fix an ideal $I \subset R$ with $\text{Ass}(R/I) = \{p_1, \ldots, p_k\} \subset \text{Spec}(R)$. The size of a differential primary decomposition is at least $\text{mult}(I)$, and this upper bound is tight. More precisely:

(i) $I$ has a differential primary decomposition $(p_1, \mathfrak{A}_1), \ldots, (p_k, \mathfrak{A}_k)$ such that $|\mathfrak{A}_i| = \text{mult}_I(p_i)$.

(ii) If $(p_1, \mathfrak{A}_1), \ldots, (p_k, \mathfrak{A}_k)$ is a differential primary decomposition for $I$, then $|\mathfrak{A}_i| \geq \text{mult}_I(p_i)$.

The proof of Theorem 3.6 appears further below. We start with a proposition that transfers the approximation technique used in [7,8] to ideals that are not necessarily primary.

**Proposition 3.7.** Fix an ideal $I \subset R$ and an associated prime $p \in \text{Ass}(R/I)$. Let $J = (I :_R p^\infty)$ and let $\mathbb{F} = k(p)$ be the residue field of $p$. Assume that $k \hookrightarrow \mathbb{F}$ is a separable algebraic extension. Then the following statements hold:

(i) There exists a positive integer $m_0$ such that, for all $m \geq m_0$, we have the isomorphism

$$J_p / I_p \cong J_p + p^m R_p / I_p + p^m R_p.$$

(ii) Using the canonical map $\gamma_p : R_p \rightarrow \mathbb{F} \otimes_k R_p$, we define the ideals

$$\mathfrak{a}_m = \gamma_p(I_p) + \mathcal{M}^m \quad \text{and} \quad \mathfrak{b}_m = \gamma_p(J_p) + \mathcal{M}^m.$$

Then, if we choose $m \geq m_0$ as in part (i), we obtain the following isomorphism

$$J_p / I_p \cong \mathfrak{b}_m / \mathfrak{a}_m.$$

(iii) Let $\mathcal{E}_m$ be the $(R_p \otimes_k R_p)$-submodule of $\text{Diff}_{R_p/k}^{m-1}(R_p, \mathbb{F})$ that is determined by the $\mathcal{M}$-primary ideal $\mathfrak{a}_m$ as in Lemma 2.11 (iii). Then the localized ideal $I_p$ is recovered as follows:

$$I_p = \bigcap_{m=1}^{\infty} \text{Sol}(\mathcal{E}_m).$$
Proof. (i) We have a canonical surjection \( J_p \twoheadrightarrow J_p \cap (I_p + p^m R_p) \). From this we obtain the isomorphism

\[
\frac{J_p}{J_p \cap (I_p + p^m R_p)} \cong \frac{J_p + p^m R_p}{I_p + p^m R_p}.
\]

We must show that \( J_p \cap (I_p + p^m R_p) = I_p \) for \( m \gg 0 \). The left hand side contains the right hand side for any \( m \geq 0 \). On the other hand, since \( J_p \supset I_p \), we have \( J_p \cap (I_p + p^m R_p) = I_p + J_p \cap p^m R_p \). By localizing we get \( J_p = (I_p :_{R_p} (p R_p)^\infty) \), and this implies \( J_p \cap p^m R_p \subset I_p \) for \( m \gg 0 \).

(ii) From Proposition 2.13 we have the isomorphisms

\[
R_p/(I_p + p^m R_p) \xrightarrow{\cong} (F \otimes_k R_p)/a_m \quad \text{and} \quad R_p/(J_p + p^m R_p) \xrightarrow{\cong} (F \otimes_k R_p)/b_m.
\]

Hence the result is obtained by combining these isomorphisms with that in part (i).

(iii) Lemma 2.11 (iv) and Proposition 2.13 give the equality \( \bigcap_{m=1}^{\infty} \text{Sol}(E_m) = \bigcap_{m=1}^{\infty} (I_p + p^m R_p) \).

Finally, by Krull’s Intersection Theorem [19, Theorem 8.10], the right hand side equals \( I_p \).

□

Proof of Theorem 3.6. We write \( I = Q_1 \cap \cdots \cap Q_k \), where \( Q_i \) is a \( p_i \)-primary ideal, and \( p_i \neq p_j \) for \( i \neq j \). We also assume that the \( k \) indices are ordered such that \( p_j \subset p_i \) implies \( j < i \).

(i) We proceed by induction. Fix \( i \in \{1, \ldots, k\} \) and assume the following hypotheses to hold:

(a) there exist \( A_1, \ldots, A_{i-1} \) with \( A_j \subset \text{Diff}_{R/(R/p_j)}(R, R/p_j) \) and \( |A_j| = \text{mult}_I(p_j) \) for \( 1 \leq j \leq i-1 \);

(b) for all \( 1 \leq j \leq i-1 \), the following identity holds:

\[
\bigcap_{1 \leq \ell \leq j} Q_{\ell} R_{p_j} = \bigcap_{1 \leq \ell \leq j} \{ f \in R_{p_j} \mid E_{p_j}(\delta)(f) = 0 \text{ for all } \delta \in A_\ell \}.
\]

These hold vacuously for the base case \( i = 1 \). To simplify notation, we set \( p = p_i \) and \( F = k(p) \).

Our aim is to find \( A = A_i \subset \text{Diff}_{R/(R/p)}(R, R/p) \) such that \( |A| = \text{mult}_I(p) \) and \( (7) \) holds with \( j = i \). For each \( \xi \in \text{Diff}_{R/(R/p)}(R, F) \), Lemma 2.6 yields \( \delta \in \text{Diff}_{R/(R/p)}(R, R/p) \) and \( r \in R \setminus p \) such that \( \xi = \frac{\omega_{\delta} r}{\delta} \). So, we localize at \( p \) and we consider operators in \( \text{Diff}_{R/(R/p)}(R, F) \). These can be lifted. Lemma 2.12 gives a field extension \( k \subset L \subset R_p \) such that \( L \hookrightarrow F \) is separable and algebraic. By [8, Lemma 2.7 (ii)], we have \( \text{Diff}_{R/(R/p)}(R, F) \subset \text{Diff}_{R/(R/p)}(R, F) \). We now set \( k = L \) and this makes Proposition 3.7 applicable. Setting \( J = (I :_{R} p^\infty) \), we have the primary decompositions

\[
I_p = \bigcap_{1 \leq \ell \leq i} Q_{\ell} R_p \quad \text{and} \quad J_p = \bigcap_{1 \leq \ell \leq i-1} Q_{\ell} R_p,
\]

where \( J_p = R_p \) if \( p \) is a minimal prime of \( I \). We now divide the proof into three shorter steps.

**Step 1.** Let \( a_m = \gamma_p(I_p) + \mathcal{M}^m \) and \( b_m = \gamma_p(J_p) + \mathcal{M}^m \) as in Proposition 3.7 (ii). Following Lemma 2.11 (iii), let \( E_m \) be the \((R_p \otimes_{k} R_p)\)-submodule determined by the inclusion

\[
E_m \cong \text{Hom}_{R_p} \left( \frac{F \otimes_{k} R_p}{a_m}, F \right) \hookrightarrow \text{Hom}_{R_p} \left( \frac{F \otimes_{k} R_p}{\mathcal{M}^m}, F \right) \cong \text{Diff}_{R/(R/p)}(R, F).
\]

Since \( I_p = \bigcap_{m=1}^{\infty} \text{Sol}(E_m) \), by Proposition 3.7 (iii), we restrict ourselves to studying the \( E_m \)'s.

**Step 2.** The idea is to “delete or not take into account” the differential conditions for describing \( J_p \) that are available by induction. Let \( m \geq 0 \). We have the short exact sequence

\[
0 \to b_m a_m \to F \otimes_k R_p a_m \to F \otimes_k R_p b_m \to 0.
\]
By dualizing with the functor $\text{Hom}_F(-, F)$, we obtain the short exact sequence
\begin{equation}
0 \to \text{Hom}_F\left(\frac{\mathcal{F} \otimes_K R_p}{b_m}, F\right) \to \mathcal{E}_m \to \text{Hom}_F\left(\frac{b_m}{a_m}, F\right) \to 0.
\end{equation}

Notice that $\text{Hom}_F\left(\frac{\mathcal{F} \otimes_K R_p}{b_m}, F\right)$ is isomorphic to an $(R_p \otimes_K R_p)$-submodule $\mathcal{H}_m \subset \text{Diff}_{R_p/K}(R_p, F)$ such that $\text{Sol}(\mathcal{H}_m) = J_p + p^m R_p$. Again, this follows from Lemma 2.11 and Proposition 2.13.

By (8), we can find an $F$-basis $B_m = \{\omega_1^{(m)}, \ldots, \omega_m^{(m)}, \xi_1^{(m)}, \ldots, \xi_{s_m}^{(m)}\}$ of the $F$-vector space $\mathcal{E}_m$ such that $\{w_1^{(m)}, \ldots, \omega_m^{(m)}\}$ is an $F$-basis of the subspace $\mathcal{H}_m \subset \mathcal{E}_m$, and $s_m = \text{dim}_F(b_m/a_m)$.

**Step 3.** Although the problem now seems to be of an infinite nature, it is not: we can bound the order $m$. By Proposition 3.7 (i),(ii), there exists $m_0 \geq 1$ such that $J_p/I_p \cong b_m/a_m$ is an isomorphism for $m \geq m_0$. Hence, for $m \geq m_0$, we obtain the following commutative diagram:

$$
\begin{array}{c}
0 & \longrightarrow & \mathcal{H}_m & \longrightarrow & \mathcal{E}_m & \longrightarrow & \text{Hom}_F\left(\frac{b_m}{a_m}, F\right) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{H}_{m+1} & \longrightarrow & \mathcal{E}_{m+1} & \longrightarrow & \text{Hom}_F\left(\frac{b_{m+1}}{a_{m+1}}, F\right) & \longrightarrow & 0
\end{array}
$$

The rows are exact, the two left vertical maps are inclusions, and the right one is an isomorphism. We choose the $F$-bases $B_m$ and $B_{m+1}$ above in such a way that $\{\xi_1^{(m)}, \ldots, \xi_{s_m}^{(m)}\} = \{\xi_1^{(m+1)}, \ldots, \xi_{s_{m+1}}^{(m+1)}\}$. This set stabilizes and we denote it by $\{\xi_1, \ldots, \xi_s\} \subset \mathcal{E}_{m_0}$. Its cardinality is $s = \text{dim}_F(b_{m_0}/a_{m_0}) = \text{length}_{R_p}(J_p/I_p) = \text{mult}_I(p)$.

Using Krull’s Intersection Theorem and the induction hypothesis (b), we find
\begin{equation}
\bigcap_{m=1}^{\infty} \text{Sol}(\mathcal{H}_m) = \bigcap_{m=1}^{\infty} (J_p + p^m R_p) = J_p = \bigcap_{1 \leq \ell \leq s-1, \rho \leq p} \left\{ f \in R_p \mid \mathcal{L}_p(\delta)(f) = 0 \text{ for all } \delta \in \mathcal{A}_\ell \right\}.
\end{equation}

Setting $\mathcal{A}' = \{\xi_1, \ldots, \xi_s\}$ and using the way the $F$-bases $B_m$ were chosen, we conclude
\begin{equation}
\begin{aligned}
I_p & = \bigcap_{m=m_0}^{\infty} \text{Sol}(\mathcal{E}_m) = \bigcap_{m=m_0}^{\infty} \text{Sol}(B_m) = \left( \bigcap_{m=m_0}^{\infty} \text{Sol}(\mathcal{H}_m) \right) \cap \text{Sol}(\mathcal{A}') \\
& = \bigcap_{1 \leq \ell \leq s-1, \rho \leq p} \left\{ f \in R_p \mid \mathcal{L}_p(\delta)(f) = 0 \text{ for all } \delta \in \mathcal{A}_\ell \right\} \cap \text{Sol}(\mathcal{A}')
\end{aligned}
\end{equation}

So, we obtained the identity that proves part (i).

(ii) Consider any differential primary decomposition $(p_1, \mathcal{A}_1), \ldots, (p_k, \mathcal{A}_k)$ for the given ideal $I$. Fix $1 \leq i \leq k$, and set $p = p_i$, $\mathbb{F} = k(p)$, $\mathcal{A} = \mathcal{A}_i$, and $J = (I :_{R_p} p^\infty)$. By assumption, we have
\begin{equation}
\begin{aligned}
I_p & = \bigcap_{1 \leq \ell \leq s-1, \rho \leq p} \left\{ f \in R_p \mid \mathcal{L}_p(\delta)(f) = 0 \text{ for all } \delta \in \mathcal{A}_\ell \right\}.
\end{aligned}
\end{equation}

The map $\iota : J_p \hookrightarrow R_p$ induces a canonical map $\tau : \text{Diff}_{R_p/K}(R_p, \mathbb{F}) \to \text{Diff}_{R_p/K}(J_p, \mathbb{F})$, $\delta \mapsto \delta \circ \iota$. Since $I_p$ is a $p$-primary submodule of $J_p$, we can assume $\mathcal{A} \subset \text{Diff}_{R_p/K}(R_p, F)$ and $p^{m+1}J_p \subseteq I_p$.

Let $\tilde{\mathcal{A}} \subset \text{Diff}_{R_p/K}(J_p, F)$ be the image of $\{\mathcal{L}_p(\delta) \mid \delta \in \mathcal{A}\}$ under $\tau$. It follows from (11) that
\begin{equation}
I_p = \left\{ f \in J_p \mid \tilde{\delta}(f) = 0 \text{ for all } \tilde{\delta} \in \tilde{\mathcal{A}} \right\}.
\end{equation}

Applying $- \otimes_{R_p} J_p$ to $R_p/p^{m+1}R_p \to (\mathbb{F} \otimes_K R_p)/M^{m+1}$, we obtain the map
\begin{equation}
J_p/p^{m+1}J_p \cong R_p/p^{m+1}R_p \otimes_{R_p} J_p \to \frac{\mathbb{F} \otimes_K R_p}{M^{m+1}} \otimes_{R_p} J_p \cong \frac{\mathbb{F} \otimes_K J_p}{M^{m+1} (\mathbb{F} \otimes_K J_p)} =: \mathcal{Q}.
\end{equation}
The module Q appears in Lemma 2.11 (i). Let \( \mathcal{G} \subset \text{Hom}_F(Q,F) \) be the \((R_p \otimes_K R_p)\)-module generated by \( \mathfrak{A} \) in \( \text{Hom}_F(Q,F) \). Note that \( \mathcal{G} \) is a finitely generated module over the Artinian local ring \((F \otimes_K R_p)/M^{m+1}\). Let \( V \subset F \otimes_K J_p \) such that \begin{equation}
abla M^{m+1}(F \otimes_K J_p) \cong \{ w \in Q \mid \eta(w) = 0 \text{ for all } \eta \in \mathcal{G} \}.\end{equation}
Dualizing the inclusion \( \mathcal{G} \subset \text{Hom}_F(Q,F) \), we get the short exact sequence \begin{equation}0 \rightarrow Z \rightarrow Q \rightarrow \text{Hom}_F(\mathcal{G},F) \rightarrow 0,\end{equation}\( Z = \{ w \in Q \mid \eta(w) = 0 \text{ for all } \eta \in \mathcal{G} \} \). We get the isomorphism \( \text{Hom}_F(\mathcal{G},F) \cong \frac{F \otimes_K J_p}{\mathcal{V}} \) by combining (14) and (15). By (12) and (13), we get an inclusion \( J_p/I_p \hookrightarrow \frac{\mathcal{V}}{\mathcal{V} \otimes_K J_p} \). This implies \[ |\mathfrak{A}| = |\tilde{\mathfrak{A}}| \geq \dim_F(\text{Hom}_F(\mathcal{G},F)) = \dim_F\left(\frac{F \otimes_K J_p}{\mathcal{V}}\right) \geq \text{length}_{R_p}(J_p/I_p) = \text{mult}_I(p).\] This is the desired inequality, which completes the proof of Theorem 3.6.

\textbf{Remark 3.8.} After substituting \( K \) by an intermediate field \( K \subset \mathbb{L} \subset R_p \) such that \( \mathbb{L} \hookrightarrow F \) is separable and algebraic, the minimal differential primary decomposition arose from a compatible basis for the \( F \)-vector spaces \( \mathcal{H}_m \subset \mathcal{E}_m \) in (8). Every such basis gives a set of operators for \( \mathfrak{A}_i \). In that sense, the minimal differential primary decomposition is unique up to choices of bases.

\section{From Ideals to Modules}

We now turn to differential primary decompositions for modules. The methods to be used for modules are the same as for ideals. We continue using the setup and notation in Section 3. We fix a finitely generated \( R \)-module \( M \). In applications, this is typically a free module \( M = R^p \).

\textbf{Definition 4.1.} Let \( U \subset M \) be an \( R \)-submodule with \( \text{Ass}(M/U) = \{ p_1, \ldots, p_k \} \subset \text{Spec}(R) \). A differential primary decomposition of \( U \) is a list of pairs \( (p_1, \mathfrak{A}_1), \ldots, (p_k, \mathfrak{A}_k) \) such that \( \mathfrak{A}_i \subset \text{Diff}_{R/p}(M, R/p_i) \) is a finite set of differential operators, and for each \( p \in \text{Ass}(M/U) \) we have
\[ U_p = \bigcap_{1 \leq i \leq k, p_i \subset p} \{ w \in M_p \mid \delta'(w) = 0 \text{ for all } \delta \in \mathfrak{A}_i \}.\] Here \( \delta' \in \text{Diff}_{R/p}(M_p, R_p/p, R_p) \) denotes the image of an operator \( \delta \in \mathfrak{A}_i \) under Lemma 2.6 (i).

We begin with the analogous to Lemma 3.3, Definition 3.5 and Proposition 3.7.

\textbf{Lemma 4.2.} For a differential primary decomposition as in Definition 4.1, the following holds:
(i) The submodule is recovered as \( U = \{ w \in M \mid \delta(w) = 0 \text{ for all } \delta \in \mathfrak{A}_i \text{ and } 1 \leq i \leq k \}. \)
(ii) For each index \( i \in \{ 1, \ldots, k \} \), the set \( \mathfrak{A}_i \) of differential operators is non-empty.

\textbf{Proof.} Essentially verbatim to the proof of Lemma 3.3. \( \square \)

\textbf{Definition 4.3.} For a submodule \( U \subset M \), its arithmetic multiplicity is the positive integer
\[ \text{amult}(U) := \sum_{p \in \text{Ass}(M/U)} \text{length}_{R_p}(\mathcal{H}_p(U/M/U)) = \sum_{p \in \text{Ass}(R/I)} \text{length}_{R_p}\left(\frac{(U_p:M_p (pR_p)^{\infty})}{U_p}\right).\]

\textbf{Example 4.4.} Primary decompositions for submodules have been studied in computer algebra (cf. [18]), but explicit examples are rare, even over a polynomial ring. Working with their differential operators is unfamiliar, and the development of numerical algorithms is highly desirable.

Let us consider \( R = \mathbb{Q}[x,y,z] \), \( M = R^2 \) and \( U = \text{image}_R\left[\begin{array}{ccc} x^2 & x & x^2 \\ y^2 & y & y^2 \\ z & z & z^2 \end{array}\right] = U_1 \cap U_2 \cap U_3 \), where
\[ U_1 = \text{image}_R\left[\begin{array}{ccc} 0 & x \\ 1 & 0 \end{array}\right], \quad U_2 = \text{image}_R\left[\begin{array}{ccc} x & y^2 \\ z & xz - y^2 \end{array}\right], \quad U_3 = \text{image}_R\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & y^2 & 0 \\ 0 & 0 & z^2 \end{array}\right]. \]
The modules $U_i$ are primary with associated primes $p_i = \langle x \rangle$, $p_2 = \langle xz - y^2 \rangle$, $p_3 = \langle y, z \rangle$. Each prime has multiplicity one in $U$, so $\text{ann}U = 3$. A minimal differential primary decomposition is given by $U = \{u \in R^2 \mid \delta_i(u) \in p_i \text{ for } i = 1, 2, 3\}$ where $\delta_1 = (1,0)$, $\delta_2 = (z, -x)$, $\delta_3 = (0, \partial_z)$.

We verified this example with the homological methods described in [11, Section 1].

The next proposition is our approximation result for the case of modules.

**Proposition 4.5.** Fix an $R$-submodule $U \subset M$ and an associated prime $p \in \text{Ass}(M/U)$. Let $V = (U :_M p^\infty)$ and $F = k(p)$ be the residue field of $p$. Assume that $\mathbb{K} \hookrightarrow F$ is a separable algebraic extension. Then the following statements hold:

(i) There exists a positive integer $m_0$ such that, for all $m \geq m_0$, we have the isomorphism

$$\frac{V_p}{U_p} \cong \frac{V_p + p^m M_p}{U_p + p^m M_p}.$$

(ii) Let $\gamma_{p,M} : M_p \to F \otimes_{\mathbb{K}} M_p$ be the map induced by $\gamma_p$ in Proposition 3.7 (ii), and define the $(R_p \otimes_{\mathbb{K}} R_p)$-modules

$$a_m = \gamma_{p,M}(U_p) + M^m(\mathbb{F} \otimes_{\mathbb{K}} M_p) \quad \text{and} \quad b_m = \gamma_{p,M}(V_p) + M^m(\mathbb{F} \otimes_{\mathbb{K}} M_p).$$

Then, if we choose $m \geq m_0$ as in part (i), we obtain the following isomorphism

$$\frac{V_p}{U_p} \cong \frac{b_m}{a_m}.$$

(iii) Let $E_m$ be the $(R_p \otimes_{\mathbb{K}} R_p)$-submodule of $\text{Diff}^{m-1}_{R_p/k}(M_p, F)$ determined by the $\mathcal{M}$-primary submodule $a_m$ as in Lemma 2.11 (iii). The localized module $U_p$ is recovered as follows:

$$U_p = \bigcap_{m=1}^{\infty} \text{Sol}(E_m).$$

**Proof.** Part (i) is obtained identically to Proposition 3.7 (i), and similarly for part (ii). For part (ii) we use Proposition 2.13. Taking the tensor product $- \otimes_{R_p} M_p$, we obtain the isomorphism

$$\frac{M_p}{p^m M_p} \cong \frac{R_p}{p^m R_p} \otimes_{R_p} M_p \cong \frac{F \otimes_{\mathbb{K}} R_p}{\mathcal{M}^m} \otimes_{R_p} M_p \cong \frac{F \otimes_{\mathbb{K}} M_p}{\mathcal{M}^m(\mathbb{F} \otimes_{\mathbb{K}} M_p)}$$

for all $m \geq 1$.

From this we get the isomorphisms:

$$M_p/(U_p + p^m M_p) \cong (\mathbb{F} \otimes_{\mathbb{K}} M_p)/a_m \quad \text{and} \quad M_p/(V_p + p^m M_p) \cong (\mathbb{F} \otimes_{\mathbb{K}} M_p)/b_m.$$ 

So, the proof is analogous to that of Proposition 3.7 (ii). \qed

The next theorem is an extension of our main result (Theorem 3.6) to the case of modules.

**Theorem 4.6.** Assume Setup 3.1 with $\mathbb{K}$ perfect. For any submodule $U \subset M$ with $\text{Ass}(M/U) = \{p_1, \ldots, p_k\} \subset \text{Spec}(R)$, we have:

(i) $U$ has a differential primary decomposition $(p_1, \mathfrak{A}_1), \ldots, (p_k, \mathfrak{A}_k)$ such that

$$|\mathfrak{A}_i| = \text{length}_{R_{p_i}}(H^0_{p_i}(M_{p_i}/U_{p_i})).$$

(ii) If $(p_1, \mathfrak{A}_1), \ldots, (p_k, \mathfrak{A}_k)$ is any differential primary decomposition for $U$, then

$$|\mathfrak{A}_i| \geq \text{length}_{R_{p_i}}(H^0_{p_i}(M_{p_i}/U_{p_i})).$$

Thus, the size of a differential primary decomposition is at least $\text{ann}U$. 
Proof. Fix a primary decomposition $U = N_1 \cap \cdots \cap N_k$ where $N_i \subset M$ is a $p_i$-primary submodule of $M$. Without any loss of generality, we order the primary submodules $N_1, \ldots, N_k$ in such a way that $p_j \subsetneq p_i$ implies $j < i$. For the proof of (i) we proceed as in Theorem 3.6 (i). Namely, we use induction on $i = 1, 2, \ldots, k$ to derive the representation by differential operators:

$$\bigcap_{1 \leq i \leq j, p_j \subsetneq p_i} (N_i)_{p_j} = \bigcap_{1 \leq i \leq j} \{ w \in M_{p_i} \mid \mathcal{L}_{p_i}(\delta)(w) = 0 \text{ for all } \delta \in \mathfrak{A}_i \}.$$

Here $\mathfrak{A}_i \subset \text{Diff}_{R/K}(M, R/p_i)$ is carefully constructed to satisfy $|\mathfrak{A}_i| = \text{length}_R (H^0_{p_i} (M_{p_i}/U_{p_i}))$.

The main change is that we now use Proposition 4.5 instead of Proposition 3.7.

The proof of the lower bound in part (ii) mirrors that of Theorem 3.6 (ii). \qed

In Theorem 3.6 and Theorem 4.6 we used differential operators that take values in $R/p_i$ rather than in $R$. This was necessary for the existence of a differential primary decomposition. Indeed, the representation (1) is generally not available for differential operators that map into $R/p_i$.

One way to remedy this is to restrict the class of rings $R$. To this end, we now assume that the $K$-algebra $R$ is formally smooth over the ground field $K$, and we work in a free module $M = R^p$. For the definition and basic properties of formally smooth algebras we refer to [24, Tag 00TH].

We are interested in a notion of differential primary decomposition that utilizes operators in $\text{Diff}_{R/K}(R, R)$ and $\text{Diff}_{R/K}(M, R)$ respectively. Consider an ideal $I \subset R$ or a submodule $U \subset M$. The representation in Definition 3.2 is called a strong differential primary decomposition of $I$ if we can choose $\mathfrak{A}_i \subset \text{Diff}_{R/K}(R, R)$ for $i = 1, \ldots, k$. The representation in Definition 4.1 is called a strong differential primary decomposition of $M$ if $\mathfrak{A}_i \subset \text{Diff}_{R/K}(M, R)$ for $i = 1, \ldots, k$.

The following important result follows as a corollary from our previous developments.

**Corollary 4.7.** Assume Setup 3.1 with $K$ perfect. Let $R$ be formally smooth over $K$ and $M = R^p$ a free module of finite rank. For any submodule $U \subset M$ with $\text{Ass}(M/U) = \{p_1, \ldots, p_k\} \subset \text{Spec}(R)$, we have:

(i) $U$ has a strong differential primary decomposition $(p_1, \mathfrak{A}_1), \ldots, (p_k, \mathfrak{A}_k)$ such that

$$|\mathfrak{A}_i| = \text{length}_R (H^0_{p_i} (M_{p_i}/U_{p_i})).$$

(ii) If $(p_1, \mathfrak{A}_1), \ldots, (p_k, \mathfrak{A}_k)$ is a strong differential primary decomposition for $U$, then

$$|\mathfrak{A}_i| \geq \text{length}_R (H^0_{p_i} (M_{p_i}/U_{p_i})).$$

Thus, the size of a strong differential primary decomposition is at least $\text{mult}(U)$.

**Proof.** This follows directly from Lemma 2.9 and Theorem 4.6. \qed

The next example shows that the hypothesis of $K$ being perfect cannot be avoided. The minimality result in Theorem 3.6 (i) and Theorem 4.6 (i) may fail without that assumption.

**Example 4.8.** Fix a prime $p \in \mathbb{N}$ and the rational function field $K = \mathbb{F}_p(t)$ over $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. Consider the maximal ideal $p = (x^p - t)$ in the polynomial ring $R = K[x]$. It is well known that $\text{Diff}_{R/K}(R, R) = \bigoplus_{m=0}^{\infty} RD^m_x$ where $D^m_x = \partial^m_x/m!$ is the differential operator determined by

$$D^m_x(x^\beta) = \binom{\beta}{m} x^{\beta-m} \quad \text{for all } \beta \geq 0.$$

The multiplicity of $p^2$ is $\text{mult}_{p^2}(p) = 2$. However, the minimal number of Noetherian operators required to describe the $p$-primary ideal $p^2$ is equal to $p + 1$. An explicit minimal description is

$$p^2 = \{ f \in R \mid D^m_{x,f}(f) \in p \text{ for all } 0 \leq m \leq p \}.$$

The right hand side is a $p$-primary ideal by [8, Proposition 3.5]. It must equal $p^2$ since the only $p$-primary ideals in $R$ are powers of $p$, and $D^m_{p^2}(p) \subset p$ for $m \leq p - 1$. See also [8, Example 5.3].
We close with noting that a differential primary decomposition in a polynomial ring can be pushed forward to a quotient algebra. The result below is proved by the methods in Section 2.

**Proposition 4.9.** Let $S = \mathbb{K}[x_1, \ldots, x_n]$ and $R = S/\mathfrak{R}$ for an ideal $\mathfrak{R} \subset S$. Given an ideal $J \subset S$ with $J \supseteq \mathfrak{R}$, it represents an ideal $I = J/\mathfrak{R}$ in $R$.

(i) We have a canonical inclusion $\operatorname{Diff}_{R/\mathfrak{R}}^m(\mathbb{R}, \mathbb{R}/I) \hookrightarrow \operatorname{Diff}_{S/\mathfrak{R}}^m(S, S/J)$ for all $m \geq 0$.

(ii) If $\delta \in \operatorname{Diff}_{S/\mathfrak{R}}^m(S, S/J)$ and $\operatorname{Ker}(\delta) \supseteq \mathfrak{R}$, then $\delta$ is in the image of the inclusion in part (i).

The promised pushforward works as follows. This is specially important from a practical point of view, since Algorithm 5.4 below is restricted to polynomial rings $S$ with char($\mathbb{K}$) = 0.

**Corollary 4.10.** Let $(p_1, \mathfrak{A}_1), \ldots, (p_k, \mathfrak{A}_k)$ be a differential primary decomposition for $J$ as in Proposition 4.9. Then $\mathfrak{A}_i \subset \operatorname{Diff}_{S/\mathfrak{R}}^m(S, S/p_i)$ can be identified with a set of differential operators $\overline{\mathfrak{A}}_i \subset \operatorname{Diff}_{R/\mathfrak{R}}^m(R, R/p_i)$, and $(\overline{p}_i, \overline{\mathfrak{A}}_i), \ldots, (\overline{p}_k, \overline{\mathfrak{A}}_k)$ is a differential primary decomposition for $I$.

**Proof.** Pick $m$ such that $\mathfrak{A}_i \subset \operatorname{Diff}_{S/\mathfrak{R}}^m(S, S/p_i)$. We have $p_i \supseteq \mathfrak{R}$ and $\operatorname{Ker}(\delta_i) \supseteq \mathfrak{R}$ for all $\delta_i \in \mathfrak{A}_i$.

By applying Proposition 4.9 with $p_i$ and $\overline{p_i}$, we can take $\overline{\mathfrak{A}}_i$ to be the preimage of $\mathfrak{A}_i$ under the canonical inclusion $\operatorname{Diff}_{R/\mathfrak{R}}^m(R, R/p_i) \hookrightarrow \operatorname{Diff}_{S/\mathfrak{R}}^m(S, S/p_i)$. This implies the assertion. $\square$

We conclude with an illustration for the non-smooth ring (2) discussed in the Introduction.

**Example 4.11.** Let $S = \mathbb{Q}[x, y, z]$, $R = S/(x^3 + y^3 + z^3)$, $I = (y^2, y^3 + z^2, x^2 y) \subset R$ and $J = (x^2 z, y^3 + z^3, x^2 y, x^3 + y^3 + z^3) \subset S$. Minimal primary decompositions for the two ideals are

\begin{equation}
J = Q_1 \cap Q_2 \cap Q_3 \quad \text{and} \quad I = \overline{Q_1} \cap \overline{Q_2} \cap \overline{Q_3},
\end{equation}

where $Q_1 = \langle y + z, x^2 \rangle$, $Q_2 = \langle y^2 - y z + z^2, x^2 \rangle$ and $Q_3 = \langle y + z, z^2, x^2 z, x^3 \rangle$. Their radicals are $p_1 = \langle y + z, x \rangle$, $p_2 = \langle x, y^2 - y z + z^2 \rangle$ and $p_3 = \langle x, y, z \rangle$, with $\operatorname{mult}(p_1) = 2$, $\operatorname{mult}(p_2) = 2$ and $\operatorname{mult}(p_3) = 1$. Note that $Q_2$ and $p_2$ would break into two components over $C$, the setting in (2), but we here use $\mathbb{Q}$. A minimal strong differential primary decomposition for $J$ in $S$ equals

\begin{equation}
(p_1, \{1, \partial_x\}), \quad (p_2, \{1, \partial_x\}) \quad \text{and} \quad (p_3, \{\partial_x^2\}).
\end{equation}

By Corollary 4.10, we can interpret (17) as a differential primary decomposition for $I$. However, there is no strong differential primary decomposition for the ideal $I$ in the non-regular ring $\mathbb{R}$. To be precise, using the method in [8, Example 5.2], it can be shown that the contribution of the $\overline{p}_3$-primary ideal $\overline{Q_3}$ cannot be described by using differential operators in $\operatorname{Diff}_{R/\mathfrak{R}}^m(R, R)$.

## 5. POLYNOMIAL RINGS

In this section, we fix a field $\mathbb{K}$ of characteristic zero and $R = \mathbb{K}[x_1, \ldots, x_n]$. Here, Corollary 4.7 holds. Differential operators live in the Weyl algebra $\operatorname{Diff}_{R/\mathfrak{R}}^m(R, R) = \mathbb{K}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \rangle$. For a free module $M = R^p$ we have $\operatorname{Diff}_{R/\mathfrak{R}}^m(M, R) = \operatorname{Diff}_{R/\mathfrak{R}}^m(R, R)^p = \mathbb{K}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \rangle^p$. In words, a differential operator on $M = R^p$ is a $p$-tuple of elements in the Weyl algebra. In what follows we focus on explicit descriptions and computations. We will employ a framework similar to that in [7]. One goal is to present an algorithm which we implemented in Macaulay2 [12].

**Remark 5.1.** We begin with a convenient formula for the multiplicity of an associated prime:

$$\operatorname{mult}(P) = \deg(\operatorname{saturate}(I, P)/I)/\deg(P)$$

This means that the arithmetic multiplicity $\operatorname{amult}(I)$ can be computed in Macaulay2 as follows:

```macaulay2
sum apply(ass(I), P -> degree(saturate(I,P)/I)/degree(P))
```

As an illustration, we list the four associated primes in Example 1.2 with their multiplicities:

```
R = QQ[x,y,z]; I = ideal(x^2*y, x^2*z, x*y^2, x*y*z^2);
apply(ass(I), P -> {P,degree(saturate(I,P)/I)/degree(P)})
```
The output shows that the primes \( p_1, p_2, p_3, p_4 \) have multiplicities 1, 1, 1, 2 in this affine scheme.

We now turn to differential primary decompositions. Let \( S = \{x_{i_1}, \ldots, x_{i_t}\} \) be a subset of the variables in \( R = \mathbb{K}[x_1, \ldots, x_n] \). Consider the polynomial subring \( \mathbb{K}[S] := \mathbb{K}[x_{i_1}, \ldots, x_{i_t}] \subseteq R \) and the field of rational functions \( \mathbb{K}(S) := \mathbb{K}(x_{i_1}, \ldots, x_{i_t}) \). As in [7], we work with the relative Weyl algebra, which is defined as the ring of \( \mathbb{K}[S] \)-linear differential operators on \( R \):

\[
D_n(S) := \text{Diff}_{\mathbb{K}[S]}(R, R) = R\langle \partial_{x_{i}} \mid x_{i} \not\in S \rangle \subseteq R\langle \partial_{x_1}, \ldots, \partial_{x_n} \rangle.
\]

Every differential operator \( \delta \in D_n(S) \) is a unique \( \mathbb{K} \)-linear combination of standard monomials \( x^\alpha \partial^\beta = x_{i_1}^{\alpha_1} \cdots x_{i_t}^{\alpha_t} \left( \prod_{i \in S} \partial_{x_i}^{\alpha_i} \right), \) where \( \alpha_i \in \mathbb{N}, \beta_i \in \mathbb{N} \). Differential operators \( \delta \in D_n(S) \) act on polynomials \( f \in R \) in the familiar way, which is given by \( x_i(f) = x_i \cdot f \) and \( \partial_{x_i}(f) = \partial f / \partial x_i \).

Let \( p \in \text{Spec}(R) \) be a prime ideal of dimension \( \dim(R/p) = d \). We say that the set of variables \( S \) is a basis modulo \( p \) if \( |S| = d \) and \( \mathbb{K}[S] \cap p = \{0\} \). This specifies the bases of the algebraic matroid of the prime \( p \). In the notation of [20, Example 13.2], this is the algebraic matroid associated with the generators \( E = \{\tau_1, \ldots, \tau_n\} \) of the field extension \( K = k(p) \) over \( F = k \).

We propose the following more refined notion of strong differential primary decomposition. This following definition for polynomial rings differs from Definitions 3.2 and 4.1 in that every component is now a triplet, with the new entry being a choice of subset of \( \{x_1, \ldots, x_n\} \).

**Definition 5.2.** Let \( I \) be an ideal in the polynomial ring \( R = \mathbb{K}[x_1, \ldots, x_n] \), with associated primes \( \text{Ass}(R/I) = \{p_1, \ldots, p_k\} \). A differential primary decomposition of \( I \) is a list of triplets

\[
(p_i, S_i, \mathfrak{A}_i), \quad (p_2, S_2, \mathfrak{A}_2), \quad \ldots, \quad (p_k, S_k, \mathfrak{A}_k),
\]

where \( S_i \) is a basis modulo \( p_i \) and \( \mathfrak{A}_i \) is a finite set of differential operators in \( D_n(S_i) \) such that

\[
I_p \cap R = \bigcap_{i \in \mathfrak{A}_i, \mathfrak{A}_i \neq p} \{ f \in R \mid \delta(f) \in p_i \text{ for all } \delta \in \mathfrak{A}_i \} \quad \text{for each } p \in \text{Ass}(R/I).
\]

This condition implies (1), so we get the desired test of membership in \( I \) by differential operators.

What follows is our main result on differential primary decompositions over polynomial rings.

**Theorem 5.3.** Fix a polynomial ideal \( I \subset R \) with \( \text{Ass}(R/I) = \{p_1, \ldots, p_k\} \subset \text{Spec}(R) \). The size of a differential primary decomposition satisfies \( k \geq \text{amult}(I) \), and this is tight. More precisely,

(i) \( I \) has a differential primary decomposition \( \{(p_i, S_j, \mathfrak{A}_j)\}_{i=1, \ldots, k} \) such that \( |\mathfrak{A}_j| = \text{mult}_j(p_i) \).

(ii) If \( \{(p_i, S_j, \mathfrak{A}_j)\}_{i=1, \ldots, k} \) is any differential primary decomposition for \( I \), then \( |\mathfrak{A}_j| \geq \text{mult}_j(p_i) \).

**Proof.** As before, we fix a primary decomposition \( I = Q_1 \cap \cdots \cap Q_k \) where \( \text{rad}(Q_i) = p_i \) for \( i = 1, \ldots, k \). We assume that \( p_j \subset p_i \) implies \( j < i \). Let \( S_i \subset \{x_1, \ldots, x_n\} \) be a basis modulo \( p_i \).

(i) We proceed by induction. By Lemma 2.9, we may consider differential operators in \( \text{Diff}_{\mathbb{K}[S]}(R, R/p_i) \). Fix \( i \in \{1, \ldots, k\} \) and assume that the following induction hypotheses hold:

(a) There exist \( \mathfrak{A}_{i-1}, \mathfrak{A}_{i-1} \) with \( \mathfrak{A}_j \subset D_n(S_j) \) and \( |\mathfrak{A}_j| = \text{mult}_j(p_i) \) for all \( 1 \leq j \leq i-1 \).

(b) The identity (7) holds for \( 1 \leq j \leq i-1 \), with \( \mathfrak{A}_i \) identified with its image in \( \text{Diff}_{\mathbb{K}[S]}(R, R/p_i) \).

Set \( p = p_i, \mathfrak{F} = k(p) \) and \( S = S_i \). We invoke the same steps as in the proof of Theorem 3.6 (i) and arrive at the conclusions of (10). We now work over the field \( \mathbb{K}(S) \) instead of \( \mathbb{K} \), and we construct a subset \( \mathfrak{B} = \{\xi_1, \ldots, \xi_\ell\} \subset \mathbb{K}(S) \) with \( s = \text{mult}(p) \) such that

\[
I_p = \bigcap_{1 \leq \ell \leq \text{amult}(I)} \{ f \in p \mid \Sigma_{\ell}(\delta(f)) = 0 \text{ for all } \delta \in \mathfrak{B} \} \bigcap \text{Sol}(\mathfrak{B}).
\]

Since \( S \) is a basis modulo \( p \), we have \( R/p \otimes_{\mathbb{K}[S]} \mathbb{K}(S) = \mathfrak{F} \), and we obtain canonical isomorphisms

\[
\text{Diff}_{\mathbb{K}[S]}(R/p, \mathfrak{F}) \cong \text{Diff}_{\mathbb{K}[S]}(R, R/p, \mathfrak{F}) \cong \mathbb{K}(S) \otimes_{\mathbb{K}[S]} \text{Diff}_{\mathbb{K}[S]}(R, R/p).
\]
See [8, Lemma 2.7 (iii)] for the isomorphism on the left. For each $1 \leq h \leq s$, we now write $\xi_h$ as 
\[ \omega_{\langle p \rangle} \]
where $\delta_h \in \text{Diff}_R[k][S](R, R/p)$ and $r_h \in k[S]\{0\}$. After lifting $\{\delta_1, \ldots, \delta_s\}$ into $A_1 \subset D_n(S_1)$, we obtain the desired decomposition for $I_p$.

(ii) Suppose that $(p_1, S_1, A_1), \ldots, (p_k, S_k, A_k)$ is a differential primary decomposition for $I$. Fix $p = p_i$ with $1 \leq i \leq k$, and set $\mathcal{I} = \mathbb{K}[S] \{0\}$, $S = S_i$, $W = k[S] \{0\}$, $S = W^{-1}R$, $\mathcal{I} = I_p \cap R$ and $J = (\mathcal{I} : R p_i^{\infty})$. From Definition 5.2 and by analogy with (12), we obtain
\[ \mathcal{I} = \{ f \in J \mid \tilde{\delta}(f) = 0 \text{ for all } \tilde{\delta} \in \tilde{\mathbb{A}} \}. \]

Here $\tilde{\mathbb{A}}$ denotes the image of $\mathbb{A}$ under the canonical map
\[ D_n(S) = \text{Diff}_{R/k}[S](R, R) \rightarrow \text{Diff}_{R/k}[S](J, R/p), \quad \delta \mapsto \pi \circ \delta \circ \iota \]
determined by $\iota : J \hookrightarrow R$ and $\pi : R \rightarrow R/p$. Since the operators in $\mathbb{A} \subset D_n(S)$ are $k[S]$-linear,
\[ \mathcal{I}S = \{ f \in JS \mid \mathcal{L}_W(\tilde{\delta})(f) = 0 \text{ for all } \tilde{\delta} \in \tilde{\mathbb{A}} \}. \]

Using the isomorphism in (18), we can proceed as in Theorem 3.6 (ii) to infer $|\mathbb{A}| \geq \text{mult}(I)$.

Finally, we present our algorithm for computing a minimal differential primary decomposition. The algorithm is correct because it realizes the steps in the proof of Theorem 5.3 (i). We use the representation in [7, Theorem 2.1] and the method for Noetherian operators in [7, Algorithm 8.1].

**Algorithm 5.4** (Differential primary decomposition for an ideal in a polynomial ring).

**Input:** An ideal $I$ in $R = \mathbb{K}[x_1, \ldots, x_n]$, where $\text{char}(\mathbb{K}) = 0$.

**Output:** A differential primary decomposition for $I$ of minimal size $\text{amult}(I)$.

1. Compute the set of associated prime ideals, $\text{Ass}(R/I) = \{ p_1, \ldots, p_k \}$.
2. For $i$ from 1 to $k$ do:
   1. Compute a basis $S_i$ modulo $p_i$, and let $\mathcal{I}_i = k(p_i)$ be the residue field of $p_i$.
   2. Compute the ideal $\mathcal{I} = I_p \cap R$ — this is the intersection of all primary components of $I$ whose radical is contained in $p_i$.
   3. Compute the ideal $J = \mathcal{I} : R p_i^{\infty}$ — this is the intersection of all primary components of $I$ whose radical is strictly contained in $p_i$.
   4. Find $m > 0$ giving the isomorphism in Proposition 3.7 (i): $J/\mathcal{I} \cong (J + p_i^m)/(I + p_i^m)$.
   5. By using [7, Theorem 2.1, Algorithm 8.1], compute the $\mathbb{F}_r$-vector subspaces $\mathcal{E}$ and $\mathcal{H}$ of the Weyl-Noether module $\mathcal{F}_i \otimes_R D_n(S_i) \cong \mathcal{F}_i \langle \partial_x \mid x_j \notin S_i \rangle$. These are $(R \otimes_k k[S_i])$-modules that correspond to the $p_i$-primary ideals $I + p_i^m$ and $J + p_i^m$ respectively.
   6. Compute an $\mathbb{F}_r$-vector subspace $\mathcal{G} \subset \mathcal{E}$ complementary to $\mathcal{H}$ in $\mathcal{E}$, i.e., we have the direct sum $\mathcal{E} = \mathcal{H} \oplus \mathcal{G}$. Compute an $\mathbb{F}_r$-basis $\mathcal{F}_i$ of $\mathcal{G}$.
   7. Lift the basis $\mathcal{F}_i$ to a subset $A_i \subset D_n(S_i)$ in the corresponding relative Weyl algebra.
3. Output the triples $(p_1, S_1, A_1), \ldots, (p_k, S_k, A_k)$.

We implemented this in Macaulay2 [12]. Our code is made available at mathrepo.mis.mpg.de. This augments the package for primary ideals that is described in [5], and which rests on [6, 7].

The two commands in our implementation are called `solvePDE` and `getPDE`. The command `solvePDE` takes as its input an ideal $I$ in a polynomial ring and it creates a list of pairs $(p_i, A_i)$ such that (1) holds and $|A_i| = \text{mult}_{p_i}(I)$ for all $i$. The command `getPDE` reverses that process. It starts from a list of Noetherian operators and computes ideal generators. That reverse process does not check whether the given differential operators satisfy the conditions stipulated in [7,
Example 5.5. We run our two Macaulay2 commands on the ideal given in Example 1.2:

\[
\text{load "noetherianOperatorsCode.m2"}\\
\text{R = QQ\{x,y,z\}; I = ideal}(x^2y,x^2z,x^2y^2,x^2y^2z^2)\\
\text{solvePDE(I)}\\
\text{getPDE(oo)}
\]

The output of the command \text{solvePDE(I)} is the list of four pairs \((p_i,A_i)\) that realizes (1):

\[
\{(\text{ideal } x,\{1\}), (\text{ideal } y,z,\{1\}), (\text{ideal } x,y,\{dx\}), (\text{ideal } x,y,z,\{dx*dy,dx*dy*dz\})\}
\]

Our choice of the name \text{solvePDE} is a reference to the dual interpretation of the ideal \(I\), namely as a system of linear partial differential equations with constant coefficients. The Noetherian operators in \(A_i\) can be interpreted as polynomials in \(2n\) variables, called \textit{Noetherian multipliers}, as in [7, eqn (20)]. With this reinterpretation, our theory describes a minimal integral representation for all solutions to the given PDE. The command \text{solvePDE} computes all solutions to the PDE in the sense of the Ehrenpreis-Palamodov Fundamental Principle [7, Theorem 3.3]. We illustrate this for the binomial ideal in [9, Example 5.1], which served in a statistics application.

Example 5.6. Consider the following system of linear PDE for an unknown function \(f : R^4 \rightarrow R\):

\[
(19) \quad \frac{\partial^5 f}{\partial x_1^3 \partial x_3^2} = \frac{\partial^5 f}{\partial x_2^3 \partial x_3}, \quad \frac{\partial^5 f}{\partial x_1^2 \partial x_3^3} = \frac{\partial^5 f}{\partial x_2^2 \partial x_3}, \quad \frac{\partial^5 f}{\partial x_1 \partial x_2 \partial x_3^3} = \frac{\partial^5 f}{\partial x_1 \partial x_2}.
\]

We wish to describe all sufficiently differentiable functions \(f\) that satisfy these four PDE. The system (19) corresponds to an ideal \(I\) in \(R = \text{Q}[x_1,x_2,x_3,x_4]\). We enter this into Macaulay2:

\[
\text{R = QQ\{x1,x2,x3,x4\};}\\
\text{I = ideal}( x1^3*x3^2-x2^5, x2^2*x4^3-x3^5, x1^5*x4^2-x2^7, x1^2*x4^5-x3^7 );
\]

This has four associated primes. The one minimal prime is the toric ideal \(p_1 = \langle x_1 x_2 x_3 x_4 \rangle^\infty\).

The three embedded primes are \(p_2 = \langle x_1, x_2, x_3 \rangle\), \(p_3 = \langle x_2, x_3, x_4 \rangle\), \(p_4 = \langle x_1, x_2, x_3, x_4 \rangle\). Using \text{primaryDecomposition(I)}, we obtain a primary decomposition, where the primary components have multiplicities 1, 67, 60, 916. The methods in [5–7] would compute 1044 Noetherian operators to describe \(I\). However, Remark 5.1 reveals that \(\text{amult}(I) = 1 + 18 + 18 + 170\) suffice. Our command \text{solvePDE(I)} computes a minimal list of 170 Noetherian operators in under ten minutes. For instance, the last of the 18 Noetherian operators \(\delta\) displayed for the prime \(p_2\) is

\[
x4^5*dx1*dx2*dx3^8 + 1120*x4^2*dx1*dx2^3*dx3^3 + 6720*x1^3*dx2*dx3
\]

This translates into the following integral representation for certain special solutions to (19):

\[
f(x_1,x_2,x_3,x_4) = \int (x_1 x_2 x_3^8 t^5 + 1120 x_1 x_2^2 x_3^3 t^2 + 6720 x_1^3 x_2 x_3) d\mu(t).
\]

Here the notation is as in [7, Theorem 3.3]. Our Macaulay2 output furnishes 170 such formulas. The (differential) primary decomposition gives insight into connectivity of random walks in [9].

The command \text{solvePDE} can be used to compute solutions for arbitrary homogeneous linear PDE with constant coefficients. We believe that our results offer a recipe for putting the Ehrenpreis-Palamodov theory from [3,10,17,22] into real-world practise. One crucial ingredient for this endeavor will be the development of numerical methods. The advantage of numerical algorithms over symbolic ones was highlighted by Chen et al. [6], and we strongly agree with their assessment. A natural next step is the development of an efficient numerical method whose input is a list of polynomials and whose output is a minimal differential primary decomposition. Likewise, it would be desirable to develop a numerical algorithm for primary fusion, whose input consists of two ideals \(I\) and \(J\) and whose output is the intersection \(I \cap J\). Here, each of the three ideals is encoded by a minimal differential primary decomposition (1). Primary fusion...
will describe the scheme-theoretic union of two affine schemes in terms of differential operators. Further developments in the context of applications were obtained in the subsequent work [1].

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