Fractals and the two dimensional Jacobian Conjecture

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October 21, 2014

1 Introduction

In this paper we outline an approach to prove the two dimensional Jacobian Conjecture. We consider polynomial mappings $F \in \mathbb{C}[X,Y]^2$ which are local diffeomorphisms and satisfy the following two conditions:

1) $\det J_F \equiv 1$.
2) $\deg P = \deg_Y P$ and $\deg Q = \deg_Y Q$ where $F(X,Y) = (P(X,Y), Q(X,Y)) \in \mathbb{C}[X,Y]^2$.

The set of all such mappings is denoted by $\text{et}(\mathbb{C}^2)$. It is a semigroup $(\text{et}(\mathbb{C}^2), \circ)$ when the binary operation $\circ$ is composition of mappings. It contains the group $(\text{Aut}(\mathbb{C}^2), \circ)$ of all the invertible mappings $F \in \text{et}(\mathbb{C}^2)$. The two dimensional Jacobian Conjecture asserts that $\text{et}(\mathbb{C}^2) = \text{Aut}(\mathbb{C}^2)$. If we define the geometrical degree of $F$ by $d_F = \max\{|F^{-1}(a,b)| | (a,b) \in \mathbb{C}^2\}$ then $F \in \text{Aut}(\mathbb{C}^2)$ if and only if $d_F = 1$. To show that this holds true for any $F \in \text{et}(\mathbb{C}^2)$ we utilize a remarkable fractal structure that the semigroup $\text{et}(\mathbb{C}^2)$ carries. The underlying metric space structure $(\text{et}(\mathbb{C}^2), \rho_D)$ has a metric $\rho_D$ that is induced by certain interesting sets $D \subseteq \mathbb{C}^2$. These sets are called characteristic sets for the family $\text{et}(\mathbb{C}^2)$. They are defined by three conditions. The most amazing condition of these three, and the reason for their name (characteristic sets for $\text{et}(\mathbb{C}^2)$) asserts that:

$$\forall F, G \in \text{et}(\mathbb{C}^2), \; F = G \iff F(D) = G(D).$$

The construction of characteristic sets is based on the permanence principle for holomorphic functions (in two complex variables).

The proof of the two dimensional Jacobian Conjecture is composed of a considerable number of steps, some of which are of interest by themselves. Some of the results apply also to other families of local diffeomorphisms. We now, outline the structure of this proof:
Section 2 is partially based on [17]. We review in this section the definitions of the geometric basis of $F$, $R_0(F)$ where $F \in \mbox{et}(\mathbb{C}^2)$, and of the asymptotic variety $A(F)$ of $F$. We give simple properties of these objects such as: $\forall F, G \in \mbox{et}(\mathbb{C}^2)$, $R_0(G) \subseteq R_0(F \circ G)$, $F(A(G)) \subseteq A(F \circ G)$ (Proposition 2.1). Also the identity $A(F \circ G) = A(F) \cup F(A(G))$ is mentioned (Proposition 2.2). If $R_0(G) = R_0(F \circ G)$ then $F(\mathbb{C}^2) = \mathbb{C}^2$, i.e. $F$ is a surjective mapping (Proposition 2.3). We later use some of the above results to conclude non-finiteness properties related to the fractal representation of $\mbox{et}(\mathbb{C}^2)$.

In section 3 we introduce the (standard) definitions of the left composition operator $L_F$ and of the right composition operator $R_F$ (here $F \in \mbox{et}(\mathbb{C}^2)$ is a fixed mapping). The results of section 2 imply that $R_F$ or $L_F$ are not surjective $\iff F \in \mbox{et}(\mathbb{C}^2) - \mbox{Aut}(\mathbb{C}^2)$ (Proposition 3.2). The fact that $R_F$ is an injective operator (Proposition 3.3) follows from general nonsense in affine algebraic geometry. The corresponding fact that $L_F$ is injective lies deeper (and proved later on in section 8, Corollary 8.3). This property is false for the family of entire functions in a single complex variable which are local homeomorphisms $\mathbb{C} \to \mathbb{C}$. This family, $\mbox{elh}(\mathbb{C})$, of local homeomorphisms was considered in [14].

Section 4 is related to properties of the geometrical degree, $d_F$ of a mapping $F \in \mbox{et}(\mathbb{C}^2)$. It is a multiplicative function on the semigroup $(\mbox{et}(\mathbb{C}^2), \circ)$, i.e. $d_{F \circ G} = d_F \cdot d_G$ (Proposition 4.2). This follows by the Bezout Theorem. We define the basic notion of a prime étale mapping and prove that if $d_F$ is a prime integer then $F$ is a prime étale mapping (Proposition 4.4). In Proposition 4.5 we prove that any mapping in $\mbox{et}(\mathbb{C}^2)$ has a representation as a composition of prime étale mappings. Just like in the semigroup of the integers where the uniqueness of the prime decomposition of integers is deeper then its existence, here we can not say anything about uniqueness. Aiming towards the two fractal representations of $\mbox{et}(\mathbb{C}^2)$, we define three equivalence relations on $\mbox{et}(\mathbb{C}^2)$ (the right equivalence relation, the left one and the double sided one). These relations identify mappings in $\mbox{et}(\mathbb{C}^2)$ that differ by a composition with automorphisms (mappings in $\mbox{Aut}(\mathbb{C}^2)$) from one of the two sides or from both sides respectively. These equivalence relations are denoted by $\sim_R$, $\sim_L$ and simply $\sim$ and we need to put on the sets of the equivalence classes, e.g. $\mbox{et}(\mathbb{C}^2)/ \sim_R$, partial orders, say $\preceq_R$. These partial orders serve us for finiteness arguments in the proof of the fractal representations of $\mbox{et}(\mathbb{C}^2)$. In Proposition 4.10 we prove that every $\preceq_R$-increasing chain is finite (stabilizes). Then we (finally) arrive at one of
the two fractal representations of $\text{et}(\mathbb{C}^2)$, the right structure:

$$\text{et}(\mathbb{C}^2) = \text{Aut}(\mathbb{C}^2) \cup \bigcup_{i \in I} R_F(\text{et}(\mathbb{C}^2)). \quad (\text{Proposition 4.13})$$

We term this representation 'fractal' but really there is no metric structure on $\text{et}(\mathbb{C}^2)$, yet, which supports the use of that term. Thus we now turn to constructing interesting metrics on the semigroup $(\text{et}(\mathbb{C}^2), \circ)$, which will (eventually) justify the use of the term 'fractal' and the accompanying term of 'self-similarity'.

We define in section 6 the notion of a characteristic set $D$ for the family $\text{et}(\mathbb{C}^2)$ and the induced metric $\rho_D$ on $\text{et}(\mathbb{C}^2)$ (Proposition 6.3). The central property of these peculiar metrics $\rho_D$, is that they capture the local volume preservation feature of the mappings in $\text{et}(\mathbb{C}^2)$, as self mappings of the four dimensional affine space $\mathbb{R}^4$. This is, of course, the geometric way to state that these mappings satisfy the Jacobian Condition, $\det J_F \equiv 1$. In this section we took the existence of characteristic sets $D$ for granted. Thus we now turn to construct such sets.

The construction is outlined in section 7. It applies, in fact, for families of rigid local homeomorphic mappings. In particular it works for holomorphic mappings in two complex variables, due to the permanence principle (Propositions 7.6 and 7.11). In particular it works for the algebraic setting, $\text{et}(\mathbb{C}^2)$. That point in the paper is a 'bifurcation point'. Up to now composition on the right, $R_F$ was symmetric to composition on the left, $L_F$ (except for the injectivity issue mentioned in section 3, which was the 'first bird' to notify us on that 'bifurcation'). The metrics $\rho_D$ are closely related to the Jacobian condition. We would like our composition operators to be under our control with respect to the metrics $\rho_D$. It turns out that the right composition operator, $R_F$ is not controllable.

On the other hand in section 8 we prove:

$$F \in \text{Aut}(\mathbb{C}^2) \iff L_F \text{ is a } \rho_D \text{ - isometry}. \quad (\text{Proposition 8.1})$$

Even if we relax the injectivity of $F$ to local injectivity we still have $\rho_D$-control: If $F \in \text{et}(\mathbb{C}^2)$ then $L_F$ is $\rho_D$-bi-Lipschitz with constants $(1/d_F) \leq 1$ and is a $(1/d_F)$-scaling mapping in the limit $D \rightarrow \mathbb{C}^2$. This is Proposition 8.2. As an immediate conclusion we finally get that $L_F$ is injective (Corollary 8.3). This complements the much easier Proposition 3.3 (which said that $R_F$ is injective). Thus we switch from the right fractal representation, to the more effective left fractal representation of $\text{et}(\mathbb{C}^2)$:

$$\text{et}(\mathbb{C}^2) = \text{Aut}(\mathbb{C}^2) \cup \bigcup_{i \in I} L_F(\text{et}(\mathbb{C}^2)). \quad (\text{Proposition 8.5})$$
By Proposition 8.2, when $D \to \mathbb{C}^2$ we discover the self-similarity property of the set due to $L_{F_i}$ tending to be a $(1/d_{F_i})$-scaling of $\text{et}(\mathbb{C}^2)$. This now fully justifies our name - the left fractal representation of $\text{et}(\mathbb{C}^2)$. Moreover, we show that for the index set $I$ we can take $I = \text{et}_p(\mathbb{C}^2)$, the set of all the prime étale mappings. This is Proposition 8.6.

We note on section 9 that both the metric space $(\text{et}(\mathbb{C}^2), \rho_D)$ and its subspace $(\text{Aut}(\mathbb{C}^2), \rho_D)$ are separable and bounded (Propositions 9.2, 9.4 and 9.9).

On the coming sections we recover arithmetical properties of our fractal space $\text{et}(\mathbb{C}^2)$. These are the key quantitative estimates that will be used to prove that $d_F = 1$ for any $F \in \text{et}(\mathbb{C}^2)$. In section 10 we let $H^s$ be a $s$-Hausdorff measure on $\text{et}(\mathbb{C}^2)$ and prove the following fundamental inequality:

$$H^s(\text{et}(\mathbb{C}^2)) \left(1 - \int_{\{F \in \text{et}_p(\mathbb{C}^2)\}} f(F, s) dF\right) \leq H^s(\text{Aut}(\mathbb{C}^2)),$$

where the density function $f(F, s)$ is defined by the scaling relation $H^s(L_F(\text{et}(\mathbb{C}^2))) = f(F, s) \cdot H^s(\text{et}(\mathbb{C}^2))$. This approximates the scaling factor of $L_F$ as $D \to \mathbb{C}^2$.

In particular if we denote by $s_0 = \dim_H \text{et}(\mathbb{C}^2)$ the Hausdorff dimension of $\text{et}(\mathbb{C}^2)$ ($0 < s_0 \leq \infty$), then we note in Proposition 10.2 that:

$$1 \leq \int_{\{F \in \text{et}_p(\mathbb{C}^2)\}} f(F, s_0) dF.$$

In section 11 we take the limit $D \to \mathbb{C}^2$ and prove that the approximate fractal results of Section 10 become accurate and imply that

$$1 \leq \int_{\{F \in \text{et}_p(\mathbb{C}^2)\}} d_F^{-s_0} dF.$$

This is the content of Proposition 11.2. A variety of arithmetical inequalities and identities follow. In particular, using them, it is shown that the two dimensional Jacobian Conjecture follows from the, so called, disjointness assumption (Proposition 11.17). Namely, if $F \neq G$ are two distinct prime étale mappings, then:

$$L_F(\text{et}(\mathbb{C}^2)) \cap L_G(\text{et}(\mathbb{C}^2)) = \emptyset.$$

The disjointness assumption is sometimes known in the literature of the theory of fractals as the strong separation property. It is stronger than the
more commonly used open set condition which in most occasions suffices to derive the desired results. In section 12 we use results on semigroups actions and topology to begin our proof of the disjointness assumption. To this end we note that if $F, G \in \text{et}(\mathbb{C}^2)$ (not necessarily prime étale mappings), and if $L_F(\text{et}(\mathbb{C}^2)) \cap L_G(\text{et}(\mathbb{C}^2)) \neq \emptyset$, then we have: either $\partial L_F(\text{et}(\mathbb{C}^2)) \subseteq \partial L_G(\text{et}(\mathbb{C}^2))$ or $\partial L_G(\text{et}(\mathbb{C}^2)) \subseteq \partial L_F(\text{et}(\mathbb{C}^2))$. i.e. the boundaries of the approximate scaling transformations are nested (Proposition 12.1).

We then prove in Section 14 (Proposition 14.2) that if $F, G \in \text{et}(\mathbb{C}^2)$ and if we let $D \subseteq \mathbb{C}^2$ be a characteristic domain for $\text{et}(\mathbb{C}^2)$, then:

1. $\rho_D(F, G) \approx_{D \to \mathbb{C}^2} \frac{1}{d_F} \cdot \text{volume}(D \cap F^{-1}(F(D) - G(D))) + \frac{1}{d_G} \cdot \text{volume}(D \cap G^{-1}(G(D) - F(D)))$.

2. $\rho_D(F, G) \geq_{D \to \mathbb{C}^2} \frac{1}{|d_F - d_G|} \cdot \text{volume}(D)$.

3. $\lim_{D \to \mathbb{C}^2} \rho_D(F, G) = 0 \Leftrightarrow \lim_{D \to \mathbb{C}^2} \text{volume}(F(D) - G(D)) = \lim_{D \to \mathbb{C}^2} \text{volume}(G(D) - F(D)) = 0 \Leftrightarrow \lim_{D \to \mathbb{C}^2} \text{volume}(D \cap F^{-1}(F(D) - G(D))) = \lim_{D \to \mathbb{C}^2} \text{volume}(D \cap G^{-1}(G(D) - F(D))) = 0 \Rightarrow d_F = d_G$.

We then note the following topological result (Theorem 14.13), namely that if $X$ is a topological space and if $A, B \subseteq X$ satisfy the assumptions:

1. $\overline{A, B} \subseteq X^\circ$.
2. $\partial A \subseteq \partial B \lor \partial B \subseteq \partial A$.
3. Both $A$ and $B$ are path connected subspaces of $X$ and $\partial A, \partial B$ are path accessible from within $A$ and $B$ respectively.
4. $A \cap \partial A = B \cap \partial B = \emptyset$.

Then $A \subseteq B \lor A \cap B = \emptyset \lor B \subseteq A$.

Our results (including Proposition 12.1) show that for $X = \mathbb{C}[[X, Y]]^2$, $A = L_{p_1}(\text{et}(\mathbb{C}^2))$, $B = L_{p_2}(\text{et}(\mathbb{C}^2))$ with $p_1, p_2$ distinct primes in $\text{et}(\mathbb{C}^2)$ the assumptions of Theorem 14.13 are satisfied and from here we get the validity of the disjointness assumption in the fractal representation of $\text{et}(\mathbb{C}^2)$ (by left translations). This proves the celebrated two dimensional Jacobian Conjecture, Theorem 14.15.
We can also take for $X = \mathbb{C}\{{X, Y}\}^2$, where $\mathbb{C}\{{X, Y}\}$ is the algebra of all the entire functions in two complex variables $X$ and $Y$ and with the topology of uniform convergence on compacta.

To summarize we mention two central ideas that emerge from the theory of fractals and which come handy in our proof. The first is the relatively new theory of invariant sets with respect to infinite systems of contractions. Here is a very partial list of related articles: \cite{1, 6, 7, 11, 12, 26}. In our case the set of generators might be not countable. The second idea is that our target metric space is a limiting value of the metric spaces $(\text{et}(\mathbb{C}^2), \rho_D)$ where the characteristic domain $D$ tends to $\mathbb{C}^2$. In that limiting process the Lipschitz constants of the generators $F \in \text{et}(\mathbb{C}^2)$ tend to the reciprocals of the geometric degrees $d_F^{-1}$ which are reciprocals of natural numbers that are greater than or equal to 2.

2 Relations between (polynomial) étale mappings and their asymptotic varieties and geometric bases, \cite{17}

Let $F \in \mathbb{C}[X, Y]^2$ be an étale mapping that satisfies the two normalizations:

1) $\det J_F \equiv 1$.
2) $\deg P = \deg_Y P$ and $\deg Q = \deg_Y Q$ where $F(X, Y) = (P(X, Y), Q(X, Y)) \in \mathbb{C}[X, Y]^2$.

The set of all such mappings $F$ will be denoted by $\text{et}(\mathbb{C}^2)$. This semigroup (with respect to composition of mappings) is the parallel of the semigroup $\text{elh}(\mathbb{C})$ for entire functions, \cite{14}. The 2-dimensional Jacobian Conjecture can be rephrased in each of the following forms:

a) $\text{et}(\mathbb{C}^2) \subseteq \text{Aut}(\mathbb{C}^2)$.

b) $(\text{et}(\mathbb{C}^2), \circ)$ is a group.

We recall that we denote by $A(F)$ the asymptotic variety of $F$. The canonical geometric basis of $F$ will be denoted by $R_0(F)$. This basis consists of finitely many rational mappings of the following form:

$$R(X, Y) = (X^{-\alpha}, X^\beta Y + X^{-\alpha}\Phi(X)),$$

where $\alpha \in \mathbb{Z}^+, \beta \in \mathbb{Z}^+ \cup \{0\}, \Phi(X) \in \mathbb{C}[X]$ and $\deg \Phi < \alpha + \beta$. Also the effective $X$ powers in $X^{\alpha+\beta}Y + \Phi(X)$ have a gcd which equals 1. Finally, $2 \leq \gamma \leq \beta - \alpha$ where the role of $\gamma \in \mathbb{Z}^+$ will be explained below. The cardinality of the geometric basis, $|R_0(F)|$, equals the number of components.
of the affine algebraic curve $A(F)$. $\forall R \in R_0(F)$ we have the double asymptotic identity $F \circ R = G_R \in \mathbb{C}[X,Y]^2$ where the polynomial mapping $G_R$ is called the $R$-dual of $F$. Each $R \in R_0(F)$ generates exactly one component of $A(F)$. This component is normally parametrized by $\{G_R(0,Y) \mid Y \in \mathbb{C}\}$. We will denote by $H_R(X,Y) = 0$ the implicit representation of this component in terms of the irreducible polynomial $H_R \in \mathbb{C}[X,Y]$. There exists a natural number $\gamma(R) \geq 2$ and a polynomial $S_R(X,Y) \in \mathbb{C}[X,Y]$ such that $S_R(X,Y) = e_R + X \cdot T_R(X,Y)$ for some non-zero polynomial $e_R \in \mathbb{C}[Y]$ and $T_R(X,Y) \in \mathbb{C}[X,Y] - \mathbb{C}[X]$. The affine curve $S_R(X,Y) = 0$ is called the $R$-phantom curve of $F$. The $R$-component of $A(F)$, $H_R(X,Y) = 0$, is a polynomial curve which is not isomorphic to $\mathbb{A}^1$, and hence in particular must be a singular irreducible curve. We have the relation:

$$H_R(G_R(X,Y)) = X^{\gamma(R)} S(R) = X^{\gamma(R)}(e_R + X \cdot T_R(X,Y)).$$

The exponent $\gamma(R)$ is the number $\gamma$ that appears above in the double inequality $2 \leq \gamma \leq \beta - \alpha$. In our case of the canonical rational mappings $R \in R_0(F)$, we have $\text{sing}(R) = \{X = 0\}$. The following is true:

$$G^{-1}_R(H_R(X,Y) = 0) = G^{-1}_R(G_R(\text{sing}(R))) = \text{sing}(R) \cup \{S_R(X,Y) = 0\}.$$  

Thus the $G_R$-preimage of the $R$-component of $A(F)$ (which is the $G_R$-image of $\text{sing}(R)$) is the union of two curves: the first is $\text{sing}(R)$ and the second is the so-called $R$-phantom curve of $F$. Even if for a single $R(X,Y)$ the $R$-phantom curve is empty then $JC(2)$ follows. Also if $\forall R$ each component of the $R$-phantom curve is singular (which is plausible since $G_R$ maps it as an étale mapping onto the $R$-component of $A(F)$, $H_R(X,Y) = 0$, which is a singular curve) then $F(\mathbb{C}^2) = \mathbb{C}^2$, i.e. $F$ is a surjective mapping. This follows because $\forall R \in R_0(F)$ each component of the $R$-phantom curve is an asymptotic tract of the $R$-dual $G_R$ of $F$ which is parametrized rationally and so one can show that it can not be a singular curve.

These are few facts that relate an étale mapping $F \in \text{et}(\mathbb{C}^2)$ to its asymptotic variety $A(F)$ and its canonical geometric basis $R_0(F)$. We now seek for more global connections, $\text{et}(\mathbb{C}^2)$ wide. In particular we would like to understand how the binary operation in the semigroup $\text{et}(\mathbb{C}^2)$ affects the structures of the geometric objects $A(F)$ and of the algebraic objects $R_0(F)$.

**Proposition 2.1.** If $F, G \in \text{et}(\mathbb{C}^2)$ then $R_0(G) \subseteq R_0(F \circ G)$, $F(A(G)) \subseteq A(F \circ G)$.

**Proof.**

$R \in R_0(G) \Rightarrow G \circ R \in \mathbb{C}[X,Y]^2 \Rightarrow F \circ (G \circ R) \in \mathbb{C}[X,Y]^2 \Rightarrow (F \circ G) \circ R \in \mathbb{C}[X,Y]^2$. 


\[ \mathbb{C}[X, Y]^2 \Rightarrow R \in R_0(F \circ G). \] Next we have
\[(a, b) \in F(A(G)) \Rightarrow \exists R \in R_0(G) \exists Y \in \mathbb{C} \text{ such that } (a, b) = F((G \circ R)(0, Y)) \Rightarrow \exists R \in R_0(F \circ G) \exists Y \in \mathbb{C} \text{ such that } (a, b) = ((F \circ G) \circ R)(0, Y) \Rightarrow (a, b) \in A(F \circ G). \]

The proposition tells us that compositions of étale mappings do not decrease the geometric basis of the right factor and consequently do not decrease the left image of its asymptotic variety. We naturally ask, under what conditions the geometric basis of \( F \circ G \) is actually larger than that of \( G \)? In other words we would like to know when is it true that \( R_0(G) \subset R_0(F \circ G) \)? This happens exactly when \( \exists R \in R_0(F \circ G) - R_0(G) \). This means that \( (F \circ G) \circ R \in \mathbb{C}[X, Y]^2, \ G \circ R \not\subset \mathbb{C}[X, Y]^2 \). Let \( R(X, Y) = (X^{-\alpha}, X^\beta Y + X^{-\alpha} \Phi(X)), \ G(X, Y) = (P(X, Y), Q(X, Y)) \). Then
\[(G \circ R)(X, Y) = (P(X^{-\alpha}, X^\beta Y + X^{-\alpha} \Phi(X)), Q(X^{-\alpha}, X^\beta Y + X^{-\alpha} \Phi(X)) \in \mathbb{C}[X, Y]^2 - \mathbb{C}[X, Y]^2.\]

We clearly have \( \text{sing}(G \circ R) \subset \text{sing}(R) \) and so \( \text{sing}(G \circ R) = \{X = 0\} \). By \( F \circ (G \circ R) = (F \circ G) \circ R \in \mathbb{C}[X, Y]^2 \) we have \( G \circ R \in R(F) \). This is not necessarily a member of the geometric basis of \( F \). The canonical geometric basis of \( F, R_0(F) \) contains finitely many rational mappings of the form:
\[ S(X, Y) = (X^{-\alpha}, X^\beta Y + X^{-\alpha} \Psi(X)). \]

Since \( G \in \text{et}(\mathbb{C}^2) \) it follows that \( |\mathbb{C}^2 - G(\mathbb{C}^2)| < \infty \) (a similar phenomenon as the Picard Theorem). If \( L \) is an asymptotic tract of \( F \) then \( G^{-1}(L) \) can not be a bounded subset of \( \mathbb{C}^2 \). The reason is that if \( G^{-1}(L) \) is compact, then \( G(G^{-1}(L)) \) is compact and since \( L \subseteq G(G^{-1}(L)) \subseteq G(G^{-1}(L)) \) this would imply the contradiction that \( L \) is bounded (and hence can not be an asymptotic tract). Hence \( G^{-1}(L) \) has at least one component, say \( L_1 \), that goes to infinity. So \( F \circ G \) has a limit along \( L_1 \) which equals the above asymptotic value of \( F \). This proves the following generalization of the second part of Proposition 2.1, namely,

**Proposition 2.2.** If \( F, G \in \text{et}(\mathbb{C}^2) \text{ then } A(F) \cup F(A(G)) = A(F \circ G). \)

This proposition implies that if \( A(F) \subset F(A(G)) \) then necessarily \( R_0(G) \subset R_0(F \circ G) \) because, as shown in the proof of Proposition 2.1 \( \forall R \in R_0(G), \ ((F \circ G) \circ R)(\text{sing}(R)) \subset F(A(G)). \)

**Proposition 2.3.** Let \( F \in \text{et}(\mathbb{C}^2) \). If \( \exists G \in \text{et}(\mathbb{C}^2) \text{ such that } R_0(G) = R_0(F \circ G), \text{ then } F(\mathbb{C}^2) = \mathbb{C}^2, \text{ i.e. } F \text{ is a surjective mapping.} \)
Proof.
Since \( F \in \text{et}(C^2) \) we have \( C^2 - F(C^2) \subseteq A(F) \), because in this case the only points in the complement of the image of \( F \) are the finitely many Picard exceptional values of \( F \) which are asymptotic values of \( F \). If, as the assumption says \( R_0(G) = R_0(F \circ G) \) then by Proposition 2.2 we must have \( A(F) \subseteq F(A(G)) \subseteq F(C^2) \) and so there are no Picard exceptional values of the mapping \( F \). \( \square \)

3 The right and the left mappings on \( \text{et}(C^2) \)

Definition 3.1. Let \( F \in \text{et}(C^2) \). The right mapping induced by \( F \) is defined by:

\[
R_F : \text{et}(C^2) \to \text{et}(C^2), \quad R_F(G) = G \circ F.
\]

The left mapping induced by \( F \) is defined by:

\[
L_F : \text{et}(C^2) \to \text{et}(C^2), \quad L_F(G) = F \circ G.
\]

Proposition 3.2. The mappings \( R_F, L_F \) are not surjective if and only if \( F \not\in \text{Aut}(C^2) \). In fact in this case we have \( R_F(\text{et}(C^2)) \subseteq \text{et}(C^2) - \text{Aut}(C^2) \), \( L_F(\text{et}(C^2)) \subseteq \text{et}(C^2) - \text{Aut}(C^2) \).

Proof. \( R_0(R_F(G)) = R_0(G \circ F) \supseteq R_0(F) \neq \emptyset, \quad A(L_F(G)) = A(F \circ G) \supseteq A(F) \neq \emptyset. \) \( \square \)

Proposition 3.3. \( R_F \) is injective.

Proof. \( R_F(G) = R_F(H) \Rightarrow G \circ F = H \circ F. \) Since \( F \in \text{et}(C^2) \) we have \( |C^2 - F(C^2)| < \infty \) and by the assumption \( G|_{F(C^2)} = H|_{F(C^2)} \). Hence \( G \equiv H. \) \( \square \)

We naturally inquire if also \( L_F \) is injective. This, however will be proved later after considerable amount of preparations.

4 If \( \text{et}(C^2) \not\neq \text{Aut}(C^2) \), then \( \text{et}(C^2) \) is fractal like

We start this section by remarking on a few topological properties of the image of the right mapping \( R_F(\text{et}(C^2)) \). Let \( F \in \text{et}(C^2) \) and \( \forall \varepsilon > 0 \) we define

\[
V_\varepsilon(F) = \{ N \in \text{et}(C^2) | \max_{|X|,|Y| \leq 1} \| F(X, Y) - N(X, Y) \|_2 < \varepsilon \}.
\]
These sets form a local basis at $F$ for the topology we impose for now on $\text{et}(\mathbb{C}^2)$. If $N_\varepsilon \in V_\varepsilon(F)$ then as $\varepsilon \to 0^+$ the coefficients of $N_\varepsilon$ tend to the corresponding coefficients of $F$. To see that let $F = (F_1, F_2)$, $N = (N_1, N_2)$. Then by the $H^2$ theory for analytic functions we have:

$$\max_{|X|,|Y| \leq 1} \| F(X, Y) - N(X, Y) \|_2 = \| F(e^{i\phi}, e^{i\theta}) - N(e^{i\phi}, e^{i\theta}) \|_2 =$$

$$= \left( \int_{\theta=0}^{2\pi} \int_{\phi=0}^{2\pi} \left\{ |F_1(e^{i\phi}, e^{i\theta}) - N_1(e^{i\phi}, e^{i\theta})|^2 + |F_2(e^{i\phi}, e^{i\theta}) - N_2(e^{i\phi}, e^{i\theta})|^2 \right\} d\phi d\theta \right)^{1/2} =$$

$$= \left\{ \sum_{k,l} \left( |a_{kl}^{(1)} - b_{kl}^{(1)}|^2 + |a_{kl}^{(2)} - b_{kl}^{(2)}|^2 \right) \right\}^{1/2}.$$  

Here we used the notation $F_j(X, Y) = \sum_{k,l} a_{kl}^{(j)} X^k Y^l$ and $N_j(X, Y) = \sum_{k,l} b_{kl}^{(j)} X^k Y^l$, $j = 1, 2$. This shows that $\forall k, l \in \mathbb{Z}^+ \forall j = 1, 2$ we have

$$\max_{|X|,|Y| \leq 1} \| F(X, Y) - N(X, Y) \|_2 \geq |a_{kl}^{(j)} - b_{kl}^{(j)}|.$$  

Our claim on the coefficients of the mappings follows.

Let $F \in \text{et}(\mathbb{C}^2)$. We suspect that the image of the right mapping $R_F(\text{et}(\mathbb{C}^2))$ is a closed subset of $\text{et}(\mathbb{C}^2)$ in the $L^2$-topology which was introduced above. We recall that the two dimensional Jacobian Conjecture is equivalent to $\text{et}(\mathbb{C}^2) = \text{Aut}(\mathbb{C}^2)$. Thus we assume from now on that $\text{et}(\mathbb{C}^2) \neq \text{Aut}(\mathbb{C}^2)$ in order to see the implications of this assumption. The right mapping for any $F \in \text{et}(\mathbb{C}^2)$, $R_F : \text{et}(\mathbb{C}^2) \to \text{et}(\mathbb{C}^2)$, $R_F(G) = G \circ F$, is a continuous injection (Proposition 3.3). Continuity here means, say, with respect to the $L^2$-topology. Also $F \in \text{Aut}(\mathbb{C}^2) \iff R_F(\text{et}(\mathbb{C}^2)) = \text{et}(\mathbb{C}^2)$. If $F \in \text{et}(\mathbb{C}^2) - \text{Aut}(\mathbb{C}^2)$, then $R_F(\text{et}(\mathbb{C}^2)) \subset \text{et}(\mathbb{C}^2) - \text{Aut}(\mathbb{C}^2)$. The following is well known ([22], Theorem 2.3)

**Theorem (Kamil Rusek and Tadeusz Winiarski).** $\text{Aut}(\mathbb{C}^n)$ is a closed subset of $(\text{et}(\mathbb{C}^n), L^2)$.

This follows from a formal analog of Cartan’s theorem on sequences of biholomorphisms of a bounded domain. Actually the topology referred to in [22], on $\text{et}(\mathbb{C}^n)$ is that of uniform convergence on compact subsets of $\mathbb{C}^n$. This topology is identical in this case with the compact-open topology.

We will need some preparations in order to arrive at the fractal like structure we will put on $\text{et}(\mathbb{C}^2)$. We start with the generic size of a fiber.
of a mapping \( F = (P, Q) \in \text{et}(\mathbb{C}^2) \). If we denote \( \deg P(X, Y) = n \) and \( \deg Q(X, Y) = m \) then \( \forall (a, b) \in \mathbb{C}^2 \) the \( F \) fiber over \( (a, b) \) is \( F^{-1}(a, b) = \{(x, y) \in \mathbb{C}^2 \mid F(x, y) = F(a, b)\} \). It is well known that this set is a finite subset of \( \mathbb{C}^2 \) and, by the Bezout Theorem we have

\[
|\{(x, y) \in \mathbb{C}^2 \mid F(x, y) = (a, b)\}| = |F^{-1}(a, b)| \leq n \cdot m.
\]

Moreover, there is a number that we will denote by \( d_F \) such that generically in \( (a, b) \) we have \( |F^{-1}(a, b)| = d_F \). This means that \( \{(a, b) \in \mathbb{C}^2 \mid |F^{-1}(a, b)| \neq d_F\} \) is a closed and proper Zariski subset of \( \mathbb{C}^2 \). In fact \( \forall (a, b) \in \mathbb{C}^2, |F^{-1}(a, b)| \neq d_F \Rightarrow |F^{-1}(a, b)| < d_F \). Thus we have \( d_F = \max\{|F^{-1}(a, b)| \mid (a, b) \in \mathbb{C}^2\} \).

**Definition 4.1.** Let \( F \in \text{et}(\mathbb{C}^2) \). We will denote \( d_F = \max\{|F^{-1}(a, b)| \mid (a, b) \in \mathbb{C}^2\} \). We will call \( d_F \) the geometrical degree of the étale mapping \( F \).

**Proposition 4.2.** \( \forall F, G \in \text{et}(\mathbb{C}^2), d_{F \circ G} = d_F \cdot d_G \).

This is a well known result. We include one of its proofs for convenience.

**Proof.**

\( \forall (a, b) \in \mathbb{C}^2, (F \circ G)^{-1}(a, b) = G^{-1}(F^{-1}(a, b)) \). But generically in \( (a, b) \) \( |F^{-1}(a, b)| = d_F \) and generically in \( (c, d) \), \( |G^{-1}(c, d)| = d_G \). \( \Box \)

**Definition 4.3.** An étale mapping \( F \in \text{et}(\mathbb{C}^2) \) is composite if \( \exists G, H \in \text{et}(\mathbb{C}^2) - \text{Aut}(\mathbb{C}^2) \) such that \( F = G \circ H \). An étale mapping \( A \in \text{et}(\mathbb{C}^2) - \text{Aut}(\mathbb{C}^2) \) is prime if it is not composite. This is equivalent to: \( A = B \circ C \) for some \( B, C \in \text{et}(\mathbb{C}^2) \) \( \Rightarrow B \in \text{Aut}(\mathbb{C}^2) \vee C \in \text{Aut}(\mathbb{C}^2) \). The subset of \( \text{et}(\mathbb{C}^2) \) of all the prime mappings will be denoted by \( \text{et}_p(\mathbb{C}^2) \). Thus the set of all the composite étale mappings is \( \text{et}(\mathbb{C}^2) - \text{et}_p(\mathbb{C}^2) \).

**Proposition 4.4.** \( \forall F \in \text{et}(\mathbb{C}^2) - \text{et}_p(\mathbb{C}^2), d_F \) is not a prime number. Equivalently, \( \forall F \in \text{et}(\mathbb{C}^2), d_F \) is a prime number \( \Rightarrow F \in \text{et}_p(\mathbb{C}^2) \).

**Proof.**

\( F \in \text{et}(\mathbb{C}^2) - \text{et}_p(\mathbb{C}^2) \Rightarrow \exists G, H \in \text{et}(\mathbb{C}^2) - \text{Aut}(\mathbb{C}^2) \) such that \( F = G \circ H \) (by the definition) \( \Rightarrow d_F = d_G \cdot d_H \), \( d_G, d_H > 1 \) (by Proposition 4.2 and the fact \( d_M = 1 \Leftrightarrow M \in \text{Aut}(\mathbb{C}^2) \)) \( \Rightarrow d_F \) is a composite integer. \( \Box \)

**Theorem 4.5.**

1) \( \text{et}_p(\mathbb{C}^2) \neq \emptyset \)
2) \( \forall F \in \text{et}(\mathbb{C}^2) \exists k \in \mathbb{Z}^+ \cup \{0\} \exists A_0 \in \text{Aut}(\mathbb{C}^2) \exists P_1, \ldots, P_k \in \text{et}_p(\mathbb{C}^2) \) such that \( F = A_0 \circ P_1 \circ \ldots \circ P_k \).
Proof.

If $\text{et}_p(C^2) = \emptyset$ then $\text{et}(C^2) - \text{Aut}(C^2)$ are all composite étale mappings. Let $F \in \text{et}(C^2) - \text{Aut}(C^2)$, then $\exists G_1, G'_2 \in \text{et}(C^2) - \text{Aut}(C^2)$ such that $F = G_1 \circ G'_2$. So $\exists G_2, G'_3 \in \text{et}(C^2) - \text{Aut}(C^2)$ such that $G'_2 = G_2 \circ G'_3$. Hence $F = G_1 \circ G_2 \circ G'_3$. Continuing this we get for any $k \in \mathbb{Z}^+ \exists G_1, \ldots, G_k \in \text{et}(C^2) - \text{Aut}(C^2)$ such that $F = G_1 \circ \ldots \circ G_k$ and by Proposition 4.2 $d_F = \prod_{j=1}^k d_{G_j}$. But $\forall 1 \leq j \leq k, d_{G_j} \geq 2$ and so $\forall k \in \mathbb{Z}^+, d_F \geq 2^k$ a contradiction to $d_F < \infty$. Thus $\text{et}_p(C^2) \neq \emptyset$.

Now part 2 is standard, for if $F \in \text{Aut}(C^2)$ we take $A_0 = F$ and $k = 0$. If $F \in \text{et}_p(C^2)$ we take $A_0 = \text{id}$, $k = 1$, and $P_1 = F$. If $F \in \text{et}(C^2) - \text{et}_p(C^2)$ then $F = G \circ H$ for some $G, H \in \text{et}(C^2) - \text{Aut}(C^2)$. So by Proposition 4.2 $d_F = d_G \cdot d_H$ and since $d_G, d_H \geq 2$ it follows that $d_G, d_H < d_F$ and we conclude the proof of part 2 using induction on the geometrical degree. Namely $G = P_1 \circ \ldots \circ P_m$, $H = P_{m+1} \circ \ldots \circ P_k$ for $m \geq 1$, $k \geq m + 1$ and some primes $P_1, \ldots, P_k \in \text{et}_p(C^2)$.

Definition 4.6. We define a relation $\sim$ on $\text{et}(C^2)$ by: $\forall F, G \in \text{et}(C^2) \ F \sim G \iff \exists \ A, B \in \text{Aut}(C^2), \ F = A \circ G \circ B$.

Remark 4.7. The relation $\sim$ is an equivalence relation on $\text{et}(C^2)$. For $F \sim F$ because $F = \text{id} \circ F \circ \text{id}$. Also $F \sim G \Rightarrow F = A \circ G \circ B \Rightarrow G = A^{-1} \circ F \circ B^{-1} \Rightarrow G \sim F$. Finally $F \sim G, G \sim H \Rightarrow F = A \circ G \circ B, G = C \circ H \circ D \Rightarrow F = (A \circ C) \circ H \circ (D \circ B) \Rightarrow F \sim H$. We could have defined two similar equivalence relations on $\text{et}(C^2)$ by restricting to compositions with automorphisms from one side only (left or right). We will denote these relations by $\sim_R$ and $\sim_L$. For example $F \sim_L G \iff \exists A \in \text{Aut}(C^2), \ F = A \circ G$.

Definition 4.8. The right partial order on $\text{et}(C^2) / \sim_R$ is defined by: $[F] \preceq_R [G] \iff \text{et}_R(\text{et}(C^2)) \subseteq \text{et}_G(\text{et}(C^2))$.

Proposition 4.9. The relation $\preceq_R$ is a partial order on $\text{et}(C^2) / \sim_R$.

Proof.

The claim is clear because $\subseteq$ is a partial order on any family of sets. However, here it is instructive to notice the anti-symmetric property also from the view of our particular setting. Namely $[F] \preceq_R [G] \land [G] \preceq_R [F] \iff \text{et}_F(\text{et}(C^2)) \subseteq \text{et}_G(\text{et}(C^2)) \land \text{et}_G(\text{et}(C^2)) \subseteq \text{et}_F(\text{et}(C^2)) \Rightarrow F \in \text{et}_G(\text{et}(C^2)) \land G \in \text{et}_F(\text{et}(C^2)) \Leftrightarrow \exists M, N \in \text{et}(C^2)$ such that $F = M \circ G \land G = N \circ F \Rightarrow F = (M \circ N) \circ F$. Since $F(C^2)$ is co-finite in $C^2$, the last equation implies that $M \circ N = \text{id}$, so $M, N \in \text{Aut}(C^2)$, $M = N^{-1}$ and so $[F] = [G]$. □
Theorem 4.10. Every $\preceq_R$-increasing chain is finite.

Proof.
We will argue by a contradiction. Suppose that there is an infinite $\preceq_R$-increasing chain. Then there is an infinite sequence $F_1, F_2, F_3, \ldots \in \text{et}(\mathbb{C}^2)-\text{et}_p(\mathbb{C}^2)$ such that $[F_1] \prec_R [F_2] \preceq_R [F_3] \preceq_R \ldots$, and hence by the definition of the partial order $\preceq_R$: $R_{F_1}(\text{et}(\mathbb{C}^2)) \subset R_{F_2}(\text{et}(\mathbb{C}^2)) \subset R_{F_3}(\text{et}(\mathbb{C}^2)) \subset \ldots$. Hence $\exists M_j \in \text{et}(\mathbb{C}^2)-\text{Aut}(\mathbb{C}^2)$ such that $F_j = M_j \circ F_{j+1}$ for $j = 1, 2, 3, \ldots$. This implies that $\forall k \in \mathbb{Z}^+, F_1 = M_1 \circ M_2 \circ \ldots \circ M_k \circ F_{k+1}$ and so as in the argument in the proof of Theorem 4.5 we obtain $d_{F_1} = d_{M_1} \cdot d_{M_2} \cdots d_{M_k} \cdot d_{F_{k+1}} \geq 2^k$. This contradicts the fact that $d_{F_1} < \infty$ and concludes the proof of Theorem 4.10. □

Remark 4.11. Let us consider an étale mapping $F \in \text{et}(\mathbb{C}^2)$, say $F = (P, Q) \in \mathbb{C}[X, Y]^2$ where $\deg P = n, \deg Q = m$. By the Bezout Theorem we have $d_F \leq n \cdot m$. If either $n = 1$ or $m = 1$ then it is well known that $F \in \text{Aut}(\mathbb{C}^2)$ and so $d_F = 1$. This follows because if $n = 1$ or $m = 1$, the mapping $F$ is injective on a straight line and it is well known that such an étale mapping must belong to Aut$(\mathbb{C}^2)$. If $(a, b) \in \mathbb{C}^2$ satisfies $|F^{-1}(a, b)| < d_F$ then it is well known that $F(a, b)$ is an asymptotic value of $F$ and there are exactly $d_F - |F^{-1}(a, b)|$ points on the line at infinity that $F$ maps to $F(a, b)$. Thus we expect some relations between the structure of the asymptotic variety $A(F)$ and $d_F$ and the size of the fiber at the given point $(a, b)$. Here is a sketch for such relations:

$$|F^{-1}(a, b)| + |\text{the different points on the line at infinity that } F \text{ maps to } F(a, b)| = d_F,$$

so

$$|F^{-1}(a, b)| + \sum_{R \in R_0(F)} \sum_{\{Y \mid G_R(0, Y) = F(a, b)\}} 1 = d_F,$$

hence

$$|F^{-1}(a, b)| + \sum_{R \in R_0(F)} |\{Y \mid G_R(0, Y) = F(a, b)\}| = d_F$$

Here is an example of a crude estimate we can get:

$$d_F \leq |F^{-1}(a, b)| + \sum_{R \in R_0(F)} \deg(F \circ R)(0, Y).$$

Let us denote $D_F = \max\{\deg(G_R)(0, Y) \mid R \in R_0(F)\}$ and recall that $|R_0(F)| = |\text{the components of } A(F)|$. Then we get:

$$d_F \leq |F^{-1}(a, b)| + D_F \cdot |\text{the components of } A(F)| = |F^{-1}(a, b)| + D_F \cdot |R_0(F)|.$$
Thus if $D_F \cdot |\text{the components of } A(F)| = D_F \cdot |R_0(F)| < d_F$ we conclude that $F$ is a surjective mapping.

**Remark 4.12.** We do not expect a claim similar to that made in Theorem 4.10 to be valid for decreasing $\preceq_R$-chains. Namely we expect that there are infinite decreasing $\preceq_R$-chains (provided, of course, that $\text{et}(\mathbb{C}^2) - \text{Aut}(\mathbb{C}^2) \neq \emptyset$.) Thus if $F \in \text{et}(\mathbb{C}^2) - \text{Aut}(\mathbb{C}^2)$ and if we take a sequence $H_n \in \text{et}(\mathbb{C}^2) - \text{Aut}(\mathbb{C}^2)$ (for example $H_n = F^n$) and define $G_n = H_n \circ \ldots \circ H_1 \circ F$, then $\ldots \preceq_R G_n \preceq_R \ldots \preceq_R G_1 \preceq_R F$ and $\ldots \subset R_{G_n}(\text{et}(\mathbb{C}^2)) \subset \ldots \subset R_{G_1}(\text{et}(\mathbb{C}^2)) \subset R_F(\text{et}(\mathbb{C}^2))$.

**Proposition 4.13.**

1) If $F \in \text{et}(\mathbb{C}^2)$ and $G \in R_F(\text{et}(\mathbb{C}^2))$, then $R_G(\text{et}(\mathbb{C}^2)) \subseteq R_F(\text{et}(\mathbb{C}^2))$.
2) If $F \in \text{et}(\mathbb{C}^2)$, $G \in R_F(\text{et}(\mathbb{C}^2))$, and $G$ and $F$ are not associates (which means here $\forall H \in \text{Aut}(\mathbb{C}^2), G \neq H \circ F$), then $R_G(\text{et}(\mathbb{C}^2)) \subseteq R_F(\text{et}(\mathbb{C}^2))$.
3) $\forall F \in \text{et}(\mathbb{C}^2)$ the spaces $(R_F(\text{et}(\mathbb{C}^2)), L^2)$ and $(\text{et}(\mathbb{C}^2), L^2)$ are homeomorphic.
4) There exists an infinite index set $I$ and a family of étale mappings $\{F_i | i \in I\} \subseteq \text{et}(\mathbb{C}^2)$ such that

$$\text{et}(\mathbb{C}^2) = \text{Aut}(\mathbb{C}^2) \cup \bigcup_{i \in I} R_{F_i}(\text{et}(\mathbb{C}^2)),$$

so that $\forall i \in I, \text{Aut}(\mathbb{C}^2) \cap R_{F_i}(\text{et}(\mathbb{C}^2)) = \emptyset$, and $\forall i, j \in I$, if $i \neq j$ then $F_i \notin R_{F_j}(\text{et}(\mathbb{C}^2))$ and $F_j \notin R_{F_i}(\text{et}(\mathbb{C}^2))$, and $R_{F_i}(\text{et}(\mathbb{C}^2))$ is homeomorphic to $R_{F_j}(\text{et}(\mathbb{C}^2))$, and both are homeomorphic to $\text{et}(\mathbb{C}^2)$.

**Proof.**

1) $H \in R_G(\text{et}(\mathbb{C}^2)) \Rightarrow \exists M \in \text{et}(\mathbb{C}^2)$ such that $H = M \circ G$. $G \in R_F(\text{et}(\mathbb{C}^2)) \Rightarrow \exists N \in \text{et}(\mathbb{C}^2)$ such that $G = N \circ F$. Hence we conclude that $H = M \circ (N \circ F) = (M \circ N) \circ F \in R_F(\text{et}(\mathbb{C}^2))$.
2) $G \in R_F(\text{et}(\mathbb{C}^2))$ and is not an associate of $F \Rightarrow \exists N \in \text{et}(\mathbb{C}^2) - \text{Aut}(\mathbb{C}^2)$ such that $G = N \circ F$. So $F \notin R_G(\text{et}(\mathbb{C}^2))$ otherwise $F = M \circ (N \circ F) = (M \circ N) \circ F$ but $M \circ N \notin \text{Aut}(\mathbb{C}^2)$ (it is not injective). The equation $F = (M \circ N) \circ F$ is equivalent to $M \circ N = \text{id}$ because $\mathbb{C}^2 - F(\mathbb{C}^2)$ is a finite set.
3) The mapping $f = R_F : \text{et}(\mathbb{C}^2) \rightarrow R_F(\text{et}(\mathbb{C}^2))$, $f(G) = G \circ F = R_F(G)$ is a homeomorphism (it is a bijection and both $f$ and $f^{-1}$ are sequentially continuous).
4) We use the relation $\sim_R$ on $\text{et}(\mathbb{C}^2)$ which was defined by $F \sim_R G \Leftrightarrow \exists \Phi \in \text{Aut}(\mathbb{C}^2)$ such that $F = \Phi \circ G$. Then $\sim_R$ is an equivalence relation ($F \sim_R F$ by $F = \text{id} \circ F$, $F \sim_R G \Leftrightarrow F = \Phi \circ G \Leftrightarrow G = \Phi^{-1} \circ F \Leftrightarrow G \sim_R F$, $F \sim_R$
\[G \leq R H \iff F = \Phi_1 \circ G \land G = \Phi_2 \circ H \iff F = (\Phi_1 \circ \Phi_2) \circ H \iff F \sim_R H.\]

We order the set of \(\sim_R\) equivalence classes \(\text{et}(C^2)/\sim_R\) by \([F] \leq [G] \iff F \in R_G(\text{et}(C^2)) \iff R_F(\text{et}(C^2)) \subseteq R_G(\text{et}(C^2))\). This relation is clearly reflexive and transitive by Proposition 7.13(1), and it is also anti-symmetric for \([F] \leq [G] \iff R_F(\text{et}(C^2)) \subseteq R_G(\text{et}(C^2))\). We define 

\[\text{et}(C^2) = \text{et}(C^2) \cup \bigcup_{i \in I} R_{F_i}(\text{et}(C^2)).\]

The union on the right equals \(\text{et}(C^2)\) because any \(F \in \text{et}(C^2)\) is either in \(\text{Aut}(C^2)\) or \([F]\) belongs to some maximal length chain in \(\text{et}(C^2)/\sim_R\) and so \(R_F(\text{et}(C^2))\) is a subset of \(R_{F_i}(\text{et}(C^2))\) where \([F_i]\) is the maximum of that chain. Clearly if \(i \neq j\) then \(F_i \notin R_{F_j}(\text{et}(C^2))\) and \(F_j \notin R_{F_i}(\text{et}(C^2))\) for \([F_i]\) and \([F_j]\) the maxima of two different chains. So \(R_{F_i}(\text{et}(C^2)) \neq R_{F_j}(\text{et}(C^2))\). Finally, the index set \(I\) is an infinite set. This follows because any finite union of the form:

\[\text{Aut}(C^2) \cup R_{F_1}(\text{et}(C^2)) \cup \ldots \cup R_{F_k}(\text{et}(C^2))\]

is such that any mapping \(H\) in it is either a \(C^2\)-automorphism or \(R_0(H) \supseteq R_0(F_1) \neq \emptyset \lor \ldots \lor R_0(H) \supseteq R_0(F_k) \neq \emptyset\). Hence the argument in the proof of Proposition 3.2 implies that \(R_{F_1}(\text{et}(C^2)) \cup \ldots \cup R_{F_k}(\text{et}(C^2)) \subseteq \text{et}(C^2) - \text{Aut}(C^2).\]

It could have been convenient if the following claim were valid: If \(F,G \in \text{et}(C^2)\) satisfy \(F \notin R_G(\text{et}(C^2))\) and \(G \notin R_F(\text{et}(C^2))\) then \(R_F(\text{et}(C^2)) \cap R_G(\text{et}(C^2)) = \emptyset\). If this were true we could have sharpened part (4) of Proposition 4.13. However, we can not prove that and as a result for any \(F,G \in \text{et}(C^2)\) all the possibilities can occur, i.e.

\[R_F(\text{et}(C^2)) \subseteq R_G(\text{et}(C^2)) \lor R_G(\text{et}(C^2)) \subseteq R_F(\text{et}(C^2)),\]

or

\[R_F(\text{et}(C^2)) \cap R_G(\text{et}(C^2)) = \emptyset,\]

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\[ R_F(\text{et}(\mathbb{C}^2)) \cap R_G(\text{et}(\mathbb{C}^2)) \not\in \{ \emptyset, R_F(\text{et}(\mathbb{C}^2)), R_G(\text{et}(\mathbb{C}^2)) \}. \]

**Proposition 4.14.** If \( F, G \in \text{et}(\mathbb{C}^2) \) and \( R_F(\text{et}(\mathbb{C}^2)) \cap R_G(\text{et}(\mathbb{C}^2)) \neq \emptyset \), then \( \exists H \in \text{et}(\mathbb{C}^2) \) such that \( R_H(\text{et}(\mathbb{C}^2)) \subseteq R_F(\text{et}(\mathbb{C}^2)) \cap R_G(\text{et}(\mathbb{C}^2)) \).

**Proof.**
Let \( H \in R_F(\text{et}(\mathbb{C}^2)) \cap R_G(\text{et}(\mathbb{C}^2)) \). Then by part (1) of Proposition 4.13 we have \( R_H(\text{et}(\mathbb{C}^2)) \subseteq R_F(\text{et}(\mathbb{C}^2)) \) and also \( R_H(\text{et}(\mathbb{C}^2)) \subseteq R_G(\text{et}(\mathbb{C}^2)) \).

**Proposition 4.15.** The family \( \{ R_F(\text{et}(\mathbb{C}^2)) \mid F \in \text{et}(\mathbb{C}^2) \} \) is a basis of a topology on \( \text{et}(\mathbb{C}^2) \).

**Proof.**
Since \( \text{et}(\mathbb{C}^2) = \bigcup_{F \in \text{et}(\mathbb{C}^2)} R_F(\text{et}(\mathbb{C}^2)) \), the claim follows by Proposition 4.14.

Thus we obtain the following topology, \( \tau_R \) on \( \text{et}(\mathbb{C}^2) \): \( \tau_R = \{ \bigcup_{j \in J} R_{F_j}(\text{et}(\mathbb{C}^2)) \mid F_j \in \text{et}(\mathbb{C}^2), j \in J \} \). We will call \( \tau_R \), the right topology on \( \text{et}(\mathbb{C}^2) \).

**Proposition 4.16.** The space \( (\text{et}(\mathbb{C}^2), \tau_R) \) is not Hausdorff.

**Proof.**
We will show that \( \tau_R \) can not separate two different points in \( \text{Aut}(\mathbb{C}^2) \). For if \( F, G \in \text{Aut}(\mathbb{C}^2), F \neq G \), then given an \( H \in \text{et}(\mathbb{C}^2) \) for which \( F \in R_H(\text{et}(\mathbb{C}^2)) \) we get \( F = M \circ H \) for some \( M \in \text{et}(\mathbb{C}^2) \). Since \( F \) is injective, it follows that \( H \) is injective. Hence we deduce that \( H \in R_H(\text{Aut}(\mathbb{C}^2)) \) and so \( R_H(\text{et}(\mathbb{C}^2)) = \text{et}(\mathbb{C}^2) \). Likewise, the only open set (in \( \tau_R \)) that contains \( G \) is \( \text{et}(\mathbb{C}^2) \), for also \( G \in \text{Aut}(\mathbb{C}^2) \).

We naturally ask if the subspace \( \text{et}(\mathbb{C}^2) - \text{Aut}(\mathbb{C}^2) \) of \( (\text{et}(\mathbb{C}^2), \tau_R) \) is Hausdorff. Also here the answer is negative:

**Proposition 4.17.** The subspace \( \text{et}(\mathbb{C}^2) - \text{Aut}(\mathbb{C}^2) \) of \( (\text{et}(\mathbb{C}^2), \tau_R) \) is not Hausdorff.

**Proof.**
Let \( F \in \text{et}(\mathbb{C}^2) - \text{Aut}(\mathbb{C}^2) \) be a prime. We will show that \( \tau_R \) can not separate the points \( F \) and \( F \circ F \). For if \( F \in R_H(\text{et}(\mathbb{C}^2)) \), then \( R_H(\text{et}(\mathbb{C}^2)) = R_F(\text{et}(\mathbb{C}^2)) \). But then \( G = F \circ F \in R_F(\text{et}(\mathbb{C}^2)) = R_H(\text{et}(\mathbb{C}^2)) \). Thus if \( G \in R_L(\text{et}(\mathbb{C}^2)) \), then \( R_H(\text{et}(\mathbb{C}^2)) \cap R_L(\text{et}(\mathbb{C}^2)) \neq \emptyset \) for this intersection contains \( G \).
5 Metric structures on et(\(\mathbb{C}^2\)) that we would like to have

We recall that we denote by et(\(\mathbb{C}^2\)) the set of all the étale mappings \(F(X, Y) = (P(X, Y), Q(X, Y)) \in \mathbb{C}[X, Y]\) that satisfy the following two conditions:

1. \(\det J_F(X, Y) \equiv 1\).
2. \(\deg P = \deg Y P, \deg Q = \deg Y Q\).

So any \(F \in \text{et}(\mathbb{C}^2)\) is determined by its sets of coefficients (those of \(P\) and those of \(Q\)). We can order the sequences of the coefficients in ascending degree order and within each homogeneous part lexicographically (\(X > Y\)). In other words if \(P(X, Y) = \sum_{1 \leq i + j \leq N = \deg P} a_{ij} X^i Y^j\) and \(Q(X, Y) = \sum_{1 \leq i + j \leq M = \deg Q} b_{ij} X^i Y^j\) then those two sequences are:

\((a_{10}, a_{01}, a_{20}, a_{11}, a_{02}, \ldots, a_{0N})\) and \((b_{10}, b_{01}, b_{20}, b_{11}, b_{02}, \ldots, b_{0M})\)

where \(a_{0N} \cdot b_{0M} \neq 0\) (condition (2)) and where the coefficients satisfy the Jacobian condition. The Jacobian condition is expressible by an infinite set of polynomial quadratic equations, all of which are homogeneous except for just one equation, namely \(a_{10}b_{01} - a_{01}b_{10} = 1\) which is still quadratic but not homogeneous. If we drop the open condition \(a_{0N} \cdot b_{0M} \neq 0\) we get an infinite dimensional affine algebraic variety. The structure of this space decomposes according to part (4) of Proposition 4.13 into a fractal-like decomposition. There exists an infinite index set \(I\) and a family of étale mappings \(\{F_i | i \in I\} \subseteq \text{et}(\mathbb{C}^2)\) such that

\[\text{et}(\mathbb{C}^2) = \text{Aut}(\mathbb{C}^2) \cup \bigcup_{i \in I} R_{F_i}(\text{et}(\mathbb{C}^2)),\]

so that \(\forall i \in I, \text{Aut}(\mathbb{C}^2) \cap R_{F_i}(\text{et}(\mathbb{C}^2)) = \emptyset,\) and \(\forall i, j \in I,\) if \(i \neq j\) then \(F_i \not\subseteq R_{F_j}(\text{et}(\mathbb{C}^2))\) and \(F_j \not\subseteq R_{F_i}(\text{et}(\mathbb{C}^2))\), and \(R_{F_i}(\text{et}(\mathbb{C}^2))\) is homeomorphic to \(R_{F_j}(\text{et}(\mathbb{C}^2))\), and both are homeomorphic to \(\text{et}(\mathbb{C}^2)\).

We will try to apply fractal geometric tools to this structure (or a similar one - where the place of the right mappings \(R_F\) will be taken by the left mappings \(L_F\)). A crucial step would be to define interesting Hausdorff type measures on \(\text{et}(\mathbb{C}^2)\) in order to obtain some (fractional) dimension computations or estimates of this space. Thus from now on we will identify \(\text{et}(\mathbb{C}^2)\) with its image under the embedding,

\[\left(\sum_{1 \leq i + j \leq N} a_{ij} X^i Y^j, \sum_{1 \leq i + j \leq M} b_{ij} X^i Y^j\right)\rightarrow\]

\[\rightarrow ((a_{10}, a_{01}, a_{20}, a_{11}, a_{02}, \ldots, a_{0N}), (b_{10}, b_{01}, b_{20}, b_{11}, b_{02}, \ldots, b_{0M})),\]

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into the space $\mathbb{C}^{\aleph_0} \times \mathbb{C}^{\aleph_0}$. In fact the image is contained in the Cartesian product of the finite sequences over $\mathbb{C}$ by itself (where we think of a finite sequence as an infinite sequence which is eventually composed of zeros). Suppose that we have a metric $\rho : \text{et}(\mathbb{C}^2) \times \text{et}(\mathbb{C}^2) \to \mathbb{R}^+ \cup \{0\}$. Let $F \in \text{et}(\mathbb{C}^2)$. Then $R_F(\text{et}(\mathbb{C}^2)) = \{G \circ F | G \in \text{et}(\mathbb{C}^2)\}$ is a metric subspace of $(\text{et}(\mathbb{C}^2), \rho)$, by restricting $\rho$ to $R_F(\text{et}(\mathbb{C}^2))$. Under the natural topology the space $\text{et}(\mathbb{C}^2)$ is homeomorphic to its subspace $R_F(\text{et}(\mathbb{C}^2))$. The homeomorphism being,

$$R_F : \text{et}(\mathbb{C}^2) \to R_F(\text{et}(\mathbb{C}^2)), \quad R_F(G) = G \circ F.$$ 

This homeomorphism need not be a $\rho$-isometry, even if the topology on $\text{et}(\mathbb{C}^2)$ is identical to the $\rho$-metric topology. We would like to have such a metric $\rho$ that will give us a good control on $\rho(R_F(G_1), R_F(G_2)) = \rho(G_1 \circ F, G_2 \circ F)$ in terms of $\rho(G_1, G_2)$.

**Remark 5.1.** A natural topology on $\text{et}(\mathbb{C}^2)$ is the so called compact-open topology. Just like for analytic functions of a single variable we naturally look at sequences $F_n \in \text{et}(\mathbb{C}^2)$ that locally uniformly converge to a polynomial mapping. This means that for every compact (in the strong topology) $K \subseteq \mathbb{C}^2$ we have $F_n|_K \to_{n \to \infty} F|_K$ uniformly.

How to construct such a metric (that will be sensitive for compositions in $\text{et}(\mathbb{C}^2)$)? The idea is straight forward. The mappings in $\text{et}(\mathbb{C}^2)$ all satisfy the Jacobian condition. Thus, geometrically, these are polynomial mappings $\mathbb{R}^4 \to \mathbb{R}^4$ (in the four dimensional space over the reals) that locally preserve volume. This is a crucial geometric property and we want our metric to capture this property. But we will see that (at least according to our constructions) the right mapping $R_F$ and the left mapping $L_F$ are very different! There is no symmetry between those two and in fact it will turn out that the left mappings $L_F$ are the correct to work with. So our plane is the following: we will outline the constructions of the metrics on $\text{et}(\mathbb{C}^2)$ that are sensitive to compositions of étale mappings. After that we will switch the results we developed so far from the right mappings setting to the left mappings. When this will be over, we will have an efficient machinery that will tie the metric space structure on $\text{et}(\mathbb{C}^2)$ to a compatible fractal structure. This will serve us to conclude non trivial geometrical results on the algebro-geometric structure $\text{et}(\mathbb{C}^2)$.

Composition of mappings is not simply a non-commutative binary operation. There is a deep difference between the two operands, the left and the right. Consider two mappings $f, g : X \to X$. When we form their composition $h = f \circ g$, then if $g$ is non injective so is $h$. If $f$ is non surjective, then so is $h$. The two examples we investigated, i.e. $\text{elh}(\mathbb{C})$ and $\text{et}(\mathbb{C}^2)$ show
that it is much easier to prove that $R_g$ is injective than to show that $L_f$ is. In fact for the entire single variable case, $\text{elh}(\mathbb{C})$ it turns out that $L_f$ is not injective. However, it is "almost" injective in the sense that we can single out the exceptional cases which form a small sub-family of $\text{elh}(\mathbb{C})$. The reason for the non injectivity originates in the existence of the periodic functions in $\text{elh}(\mathbb{C})$. This kind of reason is void for the algebraic étale case $\text{et}(\mathbb{C}^2)$. Indeed in this case $L_f$ turns out to be injective. But it is highly non trivial to prove that. There are some algebro-geometric reasons that explain this difficulty.

6 The metric spaces $(\text{et}(\mathbb{C}^2), \rho_D)$

We will need a special kind of four (real) dimensional subsets of $\mathbb{R}^4$. These will serve us to construct suitable metric structures on $\text{et}(\mathbb{C}^2)$. We will describe the construction step by step, leaving occasionally some details for later stages in order not to brake the line of reasoning.

**Definition 6.1.** Let $D$ be an open subset of $\mathbb{C}^2$ with respect to the strong topology, that satisfies the following conditions:

1) $\text{int}(\overline{D}) = D$ (D has no "slits").
2) $\overline{D}$ is a compact subset of $\mathbb{C}^2$ (in the strong topology).
3) $\forall G_1, G_2 \in \text{et}(\mathbb{C}^2), G_1(D) = G_2(D) \iff G_1 = G_2$.

We define the following real valued function:

$$\rho_D : \text{et}(\mathbb{C}^2) \times \text{et}(\mathbb{C}^2) \rightarrow \mathbb{R}^+ \cup \{0\},$$

$$\rho_D(G_1, G_2) = \text{the volume of } G_1(D) \Delta G_2(D).$$

Here we use the standard set-theoretic notation of the symmetric difference between two sets $A$ and $B$, i.e. $A \Delta B = (A - B) \cup (B - A)$.

**Remark 6.2.** It is not clear how to construct an open subset $D$ of $\mathbb{C}^2$ that will satisfy the three properties that are required in definition 6.1. We will postpone for a while the demonstration that such open sets exist.

**Proposition 6.3.** $\rho_D$ is a metric on $\text{et}(\mathbb{C}^2)$.

**Proof.**

1) $\rho_D(G_1, G_2) = 0 \iff$ the volume of $G_1(D) \Delta G_2(D) = 0 \iff G_1(D) = G_2(D)$ (where the last equivalence follows by the fact that $G_1$ and $G_2$ are local homeomorphisms in the strong topology and because of condition 1 in definition 6.1) $\iff G_1 = G_2$ (by condition 3 in definition 6.1).
2) By $G_1(D) \Delta G_2(D) = G_2(D) \Delta G_1(D)$ it follows that $\rho_D(G_1, G_2) = \rho_D(G_2, G_1)$.

3) Here we use a little technical set-theoretic containment. Namely, for any three sets $A, B$ and $C$ we have,

$$A \Delta C \subseteq (A \Delta B) \cup (B \Delta C).$$

This implies that $G_1(D) \Delta G_3(D) \subseteq (G_1(D) \Delta G_2(D)) \cup (G_2(D) \Delta G_3(D))$ from which it follows that

$$\text{(the volume of } G_1(D) \Delta G_3(D)) \leq (\text{the volume of } G_1(D) \Delta G_2(D)) +$$

$$+ (\text{the volume of } G_2(D) \Delta G_3(D)).$$

Hence the triangle inequality $\rho_D(G_1, G_3) \leq \rho_D(G_1, G_2) + \rho_D(G_2, G_3)$ holds.

So far we thought of the volume of $G_1(D) \Delta G_2(D)$ as the volume of the set which is the symmetric difference between the $G_1$ image and the $G_2$ image of the open set $D$. However, the mappings $G_1$ and $G_2$ are étale and in particular need not be injective. We will take into the volume computation the multiplicities of $G_1$ and of $G_2$. By Theorem 3 on page 39 of [3] we have the following: Given $F: \mathbb{C}^n \rightarrow \mathbb{C}^n$ we define $\tilde{F} = (\text{Re } F_1, \text{Im } F_1, \ldots, \text{Re } F_n, \text{Im } F_n); \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$. Then $\det J_{\tilde{F}} = |\det J_F|^2$. Thus the Jacobian Condition, $\det J_F \equiv 1$ implies that $\det J_{\tilde{F}} \equiv 1$. So the real mapping $\tilde{F}$ preserves the usual volume form. In order to take into account the multiplicities of the étale mappings $G_1$ and $G_2$ when computing the volume of the symmetric difference $G_1(D) \Delta G_2(D)$ we had to do the following. For any $G \in \text{et}(\mathbb{C}^2)$ instead of computing,

$$\int \int \int \int_D (\det J_G \cdot dV) = \int \int \int \int_D dV,$$

we compute

$$\int \int \int \int_{\tilde{G}(D)} dX_1 dX_2 dY_1 dY_2 \quad \text{where } X = X_1 + iX_2, Y = Y_1 + iY_2.$$

For every $j = 1, 2, \ldots, d_G$ we denote by $D_j$ that subset of $D$ such that for each point of $D_j$ there are exactly $j$ points of $D$ that are mapped by $G$ to the same image of that point. In other words, $D_j = \{\alpha \in D | |\tilde{G}^{-1}(\tilde{G}(\alpha)) \cap D| = j\}$. We assume that $D$ is large enough so that $\forall j = 1, \ldots, d_G$ we have $D_j \neq \emptyset$. For our étale mappings it is well known that if $j < d_G$ then $\dim D_j < \dim D$ so the volume these $D_j$’s contribution equals to 0. The
dimension claim follows by the well known fact that the size of a generic fiber $|G^{-1}(x)|$ equals to $d_G$ and that $d_G$ is also the maximal size of any of the fibers of $G$. However, for the sake of treating more general families of mappings we denote by $\text{vol}(D_j)$ the volume of the set $D_j$. Then $D$ has a partition into exactly $j$ subsets of equal volume. The volume of each such a set is $\text{vol}(D_j)/j$ and each such a set has exactly one of the $j$ points in $	ilde{G}^{-1}(\tilde{G}(\alpha)) \cap D$ for each $\alpha \in D_j$. We note that $\text{vol}(\tilde{G}(D_j)) = \text{vol}(D_j)/j$ by the Jacobian Condition. Thus the volume with the multiplicity of $\tilde{G}$ taken into account is given by:

$$\text{vol}(\tilde{G}(D)) + \sum_{j=2}^{d_G} (j-1) \cdot \frac{\text{vol}(D_j)}{j} = \text{vol}(\tilde{G}(D)) + \sum_{j=2}^{d_G} (j-1) \cdot \text{vol}(\tilde{G}(D_j)).$$

We note that $\tilde{G}(D) = \bigcup_{j=1}^{d_G} \tilde{G}(D_j)$ is a partition, so $\text{vol}(\tilde{G}(D)) = \sum_{j=1}^{d_G} \text{vol}(\tilde{G}(D_j))$. Hence we can express the desired volume by

$$\text{vol}(\tilde{G}(D)) + \sum_{j=2}^{d_G} (j-1) \cdot \text{vol}(\tilde{G}(D_j)) = \sum_{j=1}^{d_G} j \cdot \text{vol}(\tilde{G}(D_j)).$$

We note that this equals to $\sum_{j=1}^{d_G} \text{vol}(D_j)$ and since $D = \bigcup_{j=1}^{d_G} D_j$ is a partition we have $\text{vol}(D) = \sum_{j=1}^{d_G} \text{vol}(D_j)$. As expected, the volume computation that takes into account the multiplicity of $G$ is in general larger than the geometric volume $\text{vol}(\tilde{G}(D))$. The access can be expressed in several forms:

$$\text{vol}(D) - \text{vol}(\tilde{G}(D)) = \sum_{j=2}^{d_G} (j-1) \cdot \text{vol}(\tilde{G}(D_j)) = \sum_{j=2}^{d_G} \left(1 - \frac{1}{j}\right) \text{vol}(D_j).$$

Coming back to the computation of the metric distance $\rho_D(G_1, G_2) = \text{the volume of } G_1(D)\Delta G_2(D)$ we compute the volume of $G_1(D) - G_2(D)$ with the multiplicity of $G_1$ while the volume of $G_2(D) - G_1(D)$ is computed with the multiplicity of $G_2$.

7 Characteristic sets of families of mappings

We now discuss the existence of sets $D$ that satisfy the three properties required in definition 6.1. In particular we first concentrate on the third property. The condition was $\forall G_1, G_2 \in \text{et}(\mathbb{C}^2), G_1(D) = G_2(D) \iff G_1 = G_2$. It is not clear if such sets $D$ exist. For example $D \neq \mathbb{C}^2$ because $\forall G \in \text{Aut}(\mathbb{C}^2)$ we have $G(\mathbb{C}^2) = \mathbb{C}^2$. An obvious necessary condition is
\( \forall G_1, G_2 \in \text{Aut}(\mathbb{C}^2), G_1 \neq G_2 \iff G_1(D) \neq G_2(D). \) In other words, the only \( \mathbb{C}^2 \)-automorphism that fixes \( D \) is the identity.

**Definition 7.1.** Let \( \Gamma \) be a family of mappings \( F : \mathbb{C}^n \to \mathbb{C}^n \) (or \( \mathbb{R}^n \to \mathbb{R}^n \)). A subset \( D \subseteq \mathbb{C}^n \) (or \( D \subseteq \mathbb{R}^n \)) is called a characteristic set of \( \Gamma \) if it satisfies the following condition: \( \forall F_1, F_2 \in \Gamma, F_1(D) = F_2(D) \iff F_1 = F_2. \)

We will now give examples of characteristic sets of any family of holomorphic local homeomorphisms \( \mathbb{R}^n \to \mathbb{R}^n \). For that we will make the following,

**Definition 7.2.** Let \( m \) be a natural number and \( \alpha \in \mathbb{R}^n \). An \( m \)-star at \( \alpha \) is a union of \( m \) line segments, so that any pair intersect in \( \alpha \).

**Definition 7.3.** Let \( l \) be a line segment and let \( \{\alpha_k\} \) be a countable dense subset of \( l \). \( \forall k \), let \( S_k \) be a \( k \)-star at \( \alpha_k \) such that one of the star’s segments lies on \( l \), and such that \( \forall k_1 \neq k_2, \tilde{S}_{k_1} \cap \tilde{S}_{k_2} = \emptyset \). Here we denoted \( \tilde{S} = S - l \). Moreover, we group the stars in bundles of, say 5, thus getting the sequence of stars bundles:

\[ \{S_1, S_2, S_3, S_4, S_5\}, \{S_6, \ldots, S_{10}\}, \ldots, \{S_{5n+1}, \ldots, S_{5n+5}\}, \ldots \]

and for each bundle of five we take the maximal length of its rays to be at most \( 1/10 \) the length of the maximal length of the previous bundle. We define \( l_0 = l \cup \bigcup_{k=1}^{\infty} S_k \).

**Proposition 7.4.** Let \( \Gamma \) be any family of holomorphic local homeomorphisms \( F : \mathbb{R}^n \to \mathbb{R}^n \). Then \( l_0 \) is a characteristic set of \( \Gamma \).

**Proof.**

Let \( F_1, F_2 \in \Gamma \) satisfy \( F_1(l_0) = F_2(l_0) \). Then the line segment \( l \) is mapped onto a curve \( F_1(l) = F_2(l) \) and each \( k \)-star on \( l \), \( S_k \) is mapped onto a holomorphic \( k \)-star \( F_1(S_k) = F_2(S_k) \) (because these are local homeomorphisms) on \( F_1(l) = F_2(l) \). The centers of the holomorphic stars \( \{F_1(\alpha_k)\} = \{F_2(\alpha_k)\} \) form a countable and a dense subset of the curve \( F_1(l) = F_2(l) \). By continuity this implies that the restrictions \( F_1|_l \) and \( F_2|_l \) coincide. Since \( F_1 \) and \( F_2 \) are holomorphic, this implies (by the permanence principle) that \( F_1 \equiv F_2 \).

\( \Box \)

**Remark 7.5.** Proposition 7.4 holds true for any rigid family of local homeomorphisms. Rigidity here means that \( F_1|_l = F_2|_l \iff F_1 \equiv F_2 \). So the proposition holds true for holomorphic mappings, for harmonic mappings and in particular for \( \text{et}(\mathbb{C}^2) \).
We recall that definition 6.1 required also two additional topological properties, namely the open set $D$ should satisfy $\text{int}(\overline{D}) = D$, $\overline{D}$ is compact (all in the strong topology). These automatically exclude the set $l_0$ that was constructed in definition 7.3. However, we can modify this construction to get at least an open set.

**Proposition 7.6.** Let $\Gamma$ be any family of holomorphic local homeomorphisms $F : \mathbb{R}^n \to \mathbb{R}^n$. Let $U$ be any open subset of $\mathbb{R}^n$ with a smooth boundary that contains the compact $l_0$. Then the open set $U - l_0$ is a characteristic set of $\Gamma$.

**Proof.**
Since $l_0$ can not be mapped into the smooth $\partial U$ by an holomorphic local homeomorphism, we have for any $F_1, F_2 \in \Gamma$ for which $F_1(U - l_0) = F_2(U - l_0)$ that also $F_1(l_0) = F_2(l_0)$. Now the result follows by Proposition 7.4.

**Remark 7.7.** We note that if $\overline{U}$ is a compact then $U - l_0$ satisfies, at least the requirement $\overline{U - l_0}$ is compact. However, the "no slit" condition $\text{int}(U - l_0) = \text{int}(\overline{U}) \neq U - l_0$ fails.

Now that we gained some experience with the topological construction of $l_0$ we are going to make one more step and fix its shortcomings that were mentioned above. We need to construct a domain $D$ of $\mathbb{C}^2$ which has the following three properties:

1) $\text{int}(\overline{D}) = D$ relative to the complex topology.
2) $\overline{D}$ is a compact subset of $\mathbb{C}^2$ relative to the strong topology.
3) $\forall G_1, G_2 \in \text{et}(\mathbb{C}^2), G_1(D) = G_2(D) \iff G_1 \equiv G_2$.

(The complex topology and the strong topology are the same). Our construction will be a modification of the construction of the domain that was constructed in Proposition 7.6. We start by modifying the notion of an $m$-star that was introduced in Definition 7.2.

**Definition 7.8.** Let $m$ be a natural number and $\alpha \in \mathbb{R}^n$. A thick $m$-star at $\alpha$ is a union of $2m$ triangles, so that any pair intersect exactly at one vertex, and this vertex (that is common to all the $2m$ triangles) is $\alpha$.

**Definition 7.9.** Let $l$ be a line segment and let $\{\alpha_k\}$ be a countable dense subset of $l$. $\forall k$, let $S_k$ be a thick $k$-star at $\alpha_k$ such that $\forall k_1 \neq k_2, S_{k_1} \cap S_{k_2} = \emptyset$. We define $l_0 = \bigcup_{k=1}^{\infty} S_k \cup l$. 

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**Proposition 7.10.** Let $\Gamma$ be any family of holomorphic local homeomorphisms $F : \mathbb{R}^n \to \mathbb{R}^n$. Then $l_0$ is a characteristic set of $\Gamma$.

**Proof.**
The proof is the same word-by-word as that of Proposition 7.4 where we replace $k$-star $S_k$ by thick $k$-star $S_k$. □

We finally obtain our construction.

**Proposition 7.11.** Let $\Gamma$ be any family of holomorphic local homeomorphisms $F : \mathbb{R}^n \to \mathbb{R}^n$. Let $B(0,R)$ be an open ball centered at 0 with a radius $R$ large enough so that $l_0 \subset B(0,R)$ (where $l_0$ is the set in Proposition 7.10). Then the domain $D = B(0,R) - l_0$ is a characteristic set of $\Gamma$.

**Proof.**
The proof is the same as that of Proposition 7.6 where we replace $k$-star $S_k$ by thick $k$-star $S_k$. □

### 8 Switching to the left mapping $L_F$

As was explained in Section 5 we would like our natural mappings: the right mapping $R_F$, and the left mapping $L_F$ to be say bi-Lipschitz with respect to the metric $\rho_D$ (that reflects the fact that our mappings, $\text{et}(\mathbb{C}^2)$ satisfy the Jacobian Condition). Considering first the right mapping $R_F$, it would mean that given three étale mappings $G_1,G_2,F \in \text{et}(\mathbb{C}^2)$ and a characteristic set $D$ of $\text{et}(\mathbb{C}^2)$ we need to compare the volume of $G_1(D) \Delta G_2(D)$ (multiplicities of $G_1$ and of $G_2$ are taken into account) with the volume of the $R_F$ deformed set, $(G_1 \circ F)(D) \Delta (G_2 \circ F)(D)$. A short reflection shows that the two volumes are not comparable (in the sense of bi-Lipschitz). The situation is completely different when we replace the right mapping, $R_F$ by the left mapping, $L_F$.

For example we have the following,

**Proposition 8.1.** $\forall F \in \text{Aut}(\mathbb{C}^2)$ the mapping $L_F$ is an isometry of the metric space $(\text{et}(\mathbb{C}^2), \rho_D)$.

**Proof.**
For any two mappings $G_1$ and $G_2$ in $\text{et}(\mathbb{C}^2)$ we need to compare $\rho_D(G_1,G_2)$ with $\rho_D(F \circ G_1, F \circ G_2)$. We have (using our assumption on $F$),

\[
(F \circ G_1)(D) \Delta (F \circ G_2)(D) = F \left(G_1(D) \Delta G_2(D)\right).
\]
Since $F$ is also (globally) volume preserving we have,

$$\text{the volume of } F(G_1(D) \Delta G_2(D)) = \text{the volume of } (G_1(D) \Delta G_2(D)).$$

This proves that $\rho_D(G_1, G_2) = \rho_D(F \circ G_1, F \circ G_2)$. \hfill \Box

We now drop the restrictive assumption that $F \in \text{Aut}(\mathbb{C}^2)$. Thus we merely have $F \in \text{et}(\mathbb{C}^2)$ and we still want to compare $\rho_D(G_1, G_2)$ with $\rho_D(F \circ G_1, F \circ G_2)$, for any pair $G_1, G_2 \in \text{et}(\mathbb{C}^2)$. We only know that $F$ is a local diffeomorphism of $\mathbb{C}^2$ and (by the Jacobian Condition) that it preserves (locally) the volume. In this case the geometrical degree of $F$, $d_F$ can be larger than 1. We have the identity $d_F = |F^{-1}((a, b))|$ which holds generically (in the Zariski sense) in $(a, b) \in \mathbb{C}^2$. Hence the (complex) dimension of the set $\{(a, b) \in \mathbb{C}^2 | |F^{-1}(a, b)| < d_F\}$ is at most 1. The Jacobian Condition $\det J_F \equiv 1$ implies (as we noticed before) that $F$ preserves volume taking into account the multiplicity. The multiplicity is a result of the possibility that $F$ is not injective and hence the deformation of the characteristic set $D$ by $F$ convolves (i.e. might overlap at certain locations). However, this overlapping is bounded above by $d_F$. So if $A \subseteq \mathbb{C}^2$ is a measurable subset of $\mathbb{C}^2$ and we compare the volume of $A$ with the volume of its image $F(A)$, then,

$$\text{the volume of } F(A) \leq \text{the volume of } A \leq d_F \cdot \{\text{the volume of } F(A)\}.$$

This can be rewritten as follows,

$$\frac{1}{d_F} \cdot \{\text{the volume of } A\} \leq \text{the volume of } F(A) \leq \text{the volume of } A.$$

This is the place to emphasize also the following conclusion (that follows by the generic identity $d_F = |F^{-1}((a, b))|$), namely

$$\lim_{A \to \mathbb{C}^2} \frac{\text{the volume of } F(A)}{\text{the volume of } A} = \frac{1}{d_F},$$

provided that the set $A$ tends to cover the whole of the complex space $\mathbb{C}^2$ in an appropriate manner. To better understand why the quotient tends to the lower limit $1/d_F$ rather than to any number in the interval $[1/d_F, 1]$ (if at all) we recall that our mapping belongs to $\text{et}(\mathbb{C}^2)$ and so is a polynomial étale mapping. So any point $(a, b) \in \mathbb{C}^2$ for which $|F^{-1}(a, b)| < d_F$ is an asymptotic value of $F$ and hence belongs to the curve $A_F$ which is the asymptotic variety of $F$. In other words the identity $d_F = |F^{-1}(a, b)|$ is satisfied exactly on the semi algebraic set $\mathbb{C}^2 - A_F$ which is the complement of an algebraic curve. We now state and prove the main result of this section,
Theorem 8.2. Let $F, G_1, G_2 \in \text{et}(\mathbb{C}^2)$. Then we have:

(i) $\rho_D(F \circ G_1, F \circ G_2) \leq \rho_D(G_1, G_2)$.

(ii) Suppose that $D$ is a family of characteristic sets of $\text{et}(\mathbb{C}^2)$ such that $D \to \mathbb{C}^2$, then $\forall \epsilon > 0$ we have,

$$\left(\frac{1}{d_F} - \epsilon\right) \cdot \rho_D(G_1, G_2) \leq \rho_D(F \circ G_1, F \circ G_2)$$

for $D$ large enough.

(iii) Under the assumptions in (ii) we have:

$$\lim_{D \to \mathbb{C}^2} \frac{\rho_D(F \circ G_1, F \circ G_2)}{\rho_D(G_1, G_2)} = \frac{1}{d_F}.$$ 

In particular, the left mapping $L_F : \text{et}(\mathbb{C}^2) \to \text{et}(\mathbb{C}^2)$, $L_F(G) = F \circ G$, is a bi-Lipschitz self-mapping of the metric space $(\text{et}(\mathbb{C}^2), \rho_D)$ with the constants $1/d_F \leq 1$.

Proof.

(i) $x \in (F \circ G_1)(D)\Delta(F \circ G_2)(D) \Rightarrow \exists y \in G_j(D), j = 1, 2$ such that $x = F(y)$ and $x \notin (F \circ G_3-j)(D)$. By $x \notin (F \circ G_3-j)(D)$ it follows that $y \notin G_3-j(D)$ and so $y \in G_1(D)\Delta G_2(D)$ and $x = F(y) \in F(G_1(D)\Delta G_2(D))$. Hence $(F\circ G_1)(D)\Delta(F\circ G_2)(D) \subseteq F(G_1(D)\Delta G_2(D))$, so $\text{vol}((F\circ G_1)(D)\Delta(F\circ G_2)(D)) \leq \text{vol}(F(G_1(D)\Delta G_2(D)))$.

(ii) and (iii). Here the proof is not just set theoretic. We will elaborate more in the remark that follows this proof. We recall that $F, G_1, G_2 \in \text{et}(\mathbb{C}^2)$. This implies that $\forall (\alpha, \beta) \in \mathbb{C}^2$ we have $|F^{-1}(\alpha, \beta)| \leq |\mathbb{C}(X, Y) : \mathbb{C}(F)|$, the extension degree of $F$ see [5]. This is the so called Fiber Theorem for étale mappings. Moreover the image is co-finite, $|\mathbb{C}^2 - F(\mathbb{C}^2)| < \infty$, [5]. Also $F$ has a finite set of exactly $d_F$ maximal domains $\{\Omega_1, \ldots, \Omega_{d_F}\}$. This means that $F$ is injective on each maximal domain $\Omega_i$, and $i \neq j \Rightarrow \Omega_i \cap \Omega_j = \emptyset$, and $\mathbb{C}^2 = \bigcup_{j=1}^{d_F} \Omega_j$ and the boundaries $\partial \Omega_j$ are piecewise smooth (even piecewise analytic). For the theory of maximal domains of entire functions in one complex variable see [19], and for that theory for meromorphic functions in one complex variable see [23] [24]. Here we use only basic facts of the theory which are valid also for more than complex variable. If $D$ is a family of characteristic sets of $\text{et}(\mathbb{C}^2)$ such that $D \to \mathbb{C}^2$, then by the above $G_1(D), G_2(D) \to \mathbb{C}^2 - A$, where $A$ is a finite set, and if $G_1 \neq G_2$ then we have the identity,

$$F(G_1(D)\Delta G_2(D)) - (F \circ G_1)(D)\Delta(F \circ G_2(D)) =$$
\[ \{ x = F(y) = F(z) | y \in G_1(D) - G_2(D) \land z \in G_2(D) - G_1(D) \}. \]

Recalling that \((F \circ G_1)(D) \Delta (F \circ G_2)(D) \subseteq F(G_1(D) \Delta G_2(D))\) we write the last identity as follows,

\[ F(G_1(D) \Delta G_2(D)) = (F \circ G_1)(D) \Delta (F \circ G_2(D)) \cup \]

\[ \{ x = F(y) = F(z) | y \in G_1(D) - G_2(D) \land z \in G_2(D) - G_1(D) \}. \]

Taking any two points \(y \in G_1(D) - G_2(D)\) and \(z \in G_2(D) - G_1(D)\) (as in the defining equation of the set on the right hand side in the last identity), we note that there are \(i \neq j, \ 1 \leq i, j \leq d_F\) such that \(y \in \Omega_i \land z \in \Omega_j\) (for \(F(y) = F(z)\)). For \(\tilde{D}\) a large enough characteristic set of \(et(\mathbb{C}^2)\), we will have \(z \in G_1(\tilde{D})\) and \(y \in G_2(\tilde{D})\) and so \(y, z \in G_1(\tilde{D}) \cap G_2(\tilde{D})\) (since \(G_1(D), G_2(D) \rightarrow \mathbb{C}^2\) - \{a finite set\}). Hence \(F(G_1(D) \Delta G_2(D)) - (F \circ G_1)(\tilde{D})\Delta (F \circ G_2)(\tilde{D})\) will not include the point \(x\). We conclude that if \(y\) and \(z\) are \(F\)-equivalent \((F(y) = F(z))\) then \(x = F(y) = F(z)\) will not belong to \(F(G_1(D) \Delta G_2(D)) - (F \circ G_1)(D) \Delta (F \circ G_2)(D)\) for large enough \(D\). We obtain the following crude estimate:

\[ \text{vol}(\{ x = F(y) = F(z) | y \in G_1(D) - G_2(D) \land z \in G_2(D) - G_1(D) \}) = \]

\[ = o(\text{vol}((F \circ G_1)(D) \Delta (F \circ G_2(D)))). \]

One can think of \(D\) as a large open ball centered at the origin of \(\mathbb{R}^4, D \approx B(R)\) and with the radius \(R\) and look at the images of the two polynomial \(\text{étale mappings} F \circ G_1(B(R))\) and \((F \circ G_2)(B(R))\) and compare the volume of \((F \circ G_1)(B(R)) \Delta (F \circ G_2)(B(R))\) which is of the order of magnitude \(R^{4d}\), where \(d\) depends on the algebraic degrees of \(F \circ G_1\) and \(F \circ G_2\), with the volume of the set in the left hand side of the last equation. Similar estimates are used in the theory of covering surfaces by Ahlfors, see [9], chapter 5. We conclude that,

\[ \lim_{D \rightarrow \mathbb{C}^2} \frac{\text{vol}(F(G_1(D) \Delta G_2(D)))}{\text{vol}((F \circ G_1)(D) \Delta (F \circ G_2(D)))} = 1. \]

Hence

\[ \lim_{D \rightarrow \mathbb{C}^2} \frac{\rho_D(F \circ G_1, F \circ G_2)}{\rho_D(G_1, G_2)} = \lim_{D \rightarrow \mathbb{C}^2} \frac{\text{vol}((F \circ G_1)(D) \Delta (F \circ G_2)(D))}{\text{vol}(G_1(D) \Delta G_2(D))} = \]

\[ = \lim_{D \rightarrow \mathbb{C}^2} \frac{\text{vol}((F \circ G_1)(D) \Delta (F \circ G_2)(D))}{\text{vol}(F(G_1(D) \Delta G_2(D)))} \cdot \frac{\text{vol}(G_1(D) \Delta G_2(D))}{\text{vol}(G_1(D) \Delta G_2(D))} = \]

\[ = 1 \cdot \frac{1}{d_F} = \frac{1}{d_F}. \quad \square \]
Remark 8.3. The facts we used in proving (ii) and (iii) for étale mappings are in fact true in any dimension \( n \), i.e. in \( \mathbb{C}^n \). In dimension \( n = 2 \) it turns out that the co-dimension of the image of the mapping is 0 and in fact the co-image is a finite set. Also the fibers are finite and have a uniform bound on their cardinality (one can get a less tight uniform bound by the Bezout Theorem). Here are few well known facts (which one can find in Hartshorne’s book on Algebraic Geometry, \[10\]).

1) The following two conditions are equivalent:
   a. The Jacobian Condition: the determinant \( \det J_F \) is a non-zero constant.
   b. The map \( F^* \) is étale (in standard sense of algebraic geometry). In particular it is flat.

   Let \( F^* : Y \to X \) be étale. Let \( X^{im} := F^*(Y) \subseteq X \).

2) For every prime ideal \( \wp \subseteq A \) (\( X = \text{spec}(A) \)), with residue field \( k(\wp) \) the ring \( B \otimes_A k(\wp) \) is finite over \( k(\wp) \) (\( Y = \text{spec}(B) \)).

3) \( F^* \) is a quasi-finite mapping.

4) The set \( X^{im} \) is open in \( X \).

5) For every point \( x \in X(\mathbb{C}) \) the fiber \((F^*)^{-1}(x)\) is a finite subset of \( Y(\mathbb{C}) \).

6) The ring homomorphism \( A \to B \) is injective, and the induced field extension \( K \to L \) is finite.

7) There is a non-empty open subset \( X^{fin} \subseteq X^{im} \) such that on letting \( Y^{fin} := (F^*)^{-1}(X^{fin}) \subseteq Y \), the map of schemes \( F^*|_{Y^{fin}} : Y^{fin} \to X^{fin} \) is finite. For any point \( x \in X^{fin}(\mathbb{C}) \) we have the equality \( d_x = d_{F^*} \) the geometrical degree of \( F^* \).

8) The dimension of the set \( Z := X - X^{im} \) is at most \( n - 2 \).

9) If \( X^{im} = X - Z \) is affine, then \( Z = \emptyset \) and \( X^{im} = X \).

Let \( X_{cl} \) be the topological space which is the set \( X(\mathbb{C}) \cong \mathbb{C}^n \) given the classical topology. Similarly for \( Y_{cl} \). The map of schemes \( F^* : Y \to X \) induces a map of topological spaces \( F_{cl} : Y_{cl} \to X_{cl} \) (\( F_{cl} = f^*|_{Y(\mathbb{C})} \)).

10) The map \( F_{cl} : Y_{cl} \to X_{cl} \) is a local homeomorphism.

An immediate conclusion from Theorem 8.2 is the following,

**Corollary 8.4.** \( \forall F \in \text{et}(\mathbb{C}^2) \) the left mapping \( L_F : \text{et}(\mathbb{C}^2) \to L_F(\text{et}(\mathbb{C}^2)) \), \( L_F(G) = F \circ G \) is an injective mapping.

**Remark 8.5.** We note the contrast in the behavior between polynomial mappings (in \( \text{et}(\mathbb{C}^2) \)) and entire functions of a single complex variable (in \( \text{elh}(\mathbb{C}) \)). See [13].

We now state and prove the parallel of Proposition 4.13. Namely,
Proposition 8.6.
1) If \( F \in \text{et}(\mathbb{C}^2) \) and \( G \in L_F(\text{et}(\mathbb{C}^2)) \), then \( L_G(\text{et}(\mathbb{C}^2)) \subseteq L_F(\text{et}(\mathbb{C}^2)) \).
2) If \( F \in \text{et}(\mathbb{C}^2) \), \( G \in L_F(\text{et}(\mathbb{C}^2)) \), and \( G \) and \( F \) are not associates (which means here \( \forall H \in \text{Aut}(\mathbb{C}^2), \ G \neq F \circ H \)), then \( L_G(\text{et}(\mathbb{C}^2)) \subset L_F(\text{et}(\mathbb{C}^2)) \).
3) \( \forall F \in \text{et}(\mathbb{C}^2) \) the spaces \( (L_F(\text{et}(\mathbb{C}^2)), \mathcal{L}) \) and \( (\text{et}(\mathbb{C}^2), \mathcal{L}) \) are homeomorphic.
4) There exists an infinite index set \( I \) and a family of étale mappings \( \{F_i \mid i \in I\} \subseteq \text{et}(\mathbb{C}^2) \) such that

\[
\text{et}(\mathbb{C}^2) = \text{Aut}(\mathbb{C}^2) \cup \bigcup_{i \in I} L_{F_i}(\text{et}(\mathbb{C}^2)),
\]

so that \( \forall i \in I, \text{Aut}(\mathbb{C}^2) \cap L_{F_i}(\text{et}(\mathbb{C}^2)) = \emptyset \), and \( \forall i, j \in I, \) if \( i \neq j \) then \( F_i \notin L_{F_j}(\text{et}(\mathbb{C}^2)) \) and \( F_j \notin L_{F_i}(\text{et}(\mathbb{C}^2)) \), and \( L_{F_i}(\text{et}(\mathbb{C}^2)) \) is homeomorphic to \( L_{F_j}(\text{et}(\mathbb{C}^2)) \), and both are homeomorphic to \( \text{et}(\mathbb{C}^2) \).

Proof.
1) \( H \in L_G(\text{et}(\mathbb{C}^2)) \Rightarrow \exists M \in \text{et}(\mathbb{C}^2) \) such that \( H = G \circ M \). \( G \in L_F(\text{et}(\mathbb{C}^2)) \Rightarrow \exists N \in \text{et}(\mathbb{C}^2) \) such that \( G = F \circ N \). Hence we conclude that \( H = G \circ M = (F \circ N) \circ M = F \circ (N \circ M) \in L_F(\text{et}(\mathbb{C}^2)) \).
2) \( G \in L_F(\text{et}(\mathbb{C}^2)) \) and is not an associate of \( F \Rightarrow \exists N \in \text{et}(\mathbb{C}^2) \) such that \( G = F \circ N \). So \( F \notin L_G(\text{et}(\mathbb{C}^2)) \) otherwise \( F = (F \circ N) \circ M = F \circ (N \circ M) \). But \( N \circ M \not\in \text{Aut}(\mathbb{C}^2) \) (it is not injective). Hence \( d_{N \circ M} > 1 \). By the equation \( F = F \circ (N \circ M) \) we get the contradiction \( 1 < d_F = d_F \cdot d_{N \circ M} \).
3) The mapping \( f = L_F : \text{et}(\mathbb{C}^2) \to L_F(\text{et}(\mathbb{C}^2)), f(G) = F \circ G = L_F(G) \) is an homeomorphism (it is a bijection and both \( f \) and \( f^{-1} \) are sequentially continuous).
4) We use the relation \( \sim_L \) on \( \text{et}(\mathbb{C}^2) \) which was defined by \( F \sim_L G \iff \exists \Phi \in \text{Aut}(\mathbb{C}^2) \) such that \( F = G \circ \Phi \). Then \( \sim_L \) is an equivalence relation (\( F \sim_L F \) by \( F = F \circ \text{id} \), \( F \sim_L G \iff F = G \circ \Phi \iff G = F \circ \Phi^{-1} \iff G \sim_L F \), \( F \sim_L G \wedge G \sim_L H \iff F = G \circ \Phi_1 \wedge G = H \circ \Phi_2 \iff F = H \circ (\Phi_2 \circ \Phi_1) \iff F \sim_L H \)).

We order the set of \( \sim_L \) equivalence classes \( \text{et}(\mathbb{C}^2)/\sim_L \) by \([F] \preceq_L [G] \iff F \in L_G(\text{et}(\mathbb{C}^2)) \iff L_F(\text{et}(\mathbb{C}^2)) \subseteq L_G(\text{et}(\mathbb{C}^2)) \). This relation is clearly reflexive and transitive by Proposition 8.5(1), and it is also anti-symmetric for \([F] \preceq_L [G] \iff [F] \subseteq L_G(\text{et}(\mathbb{C}^2)) \wedge L_G(\text{et}(\mathbb{C}^2)) \subseteq L_F(\text{et}(\mathbb{C}^2)) \iff L_F(\text{et}(\mathbb{C}^2)) = L_G(\text{et}(\mathbb{C}^2)) \Rightarrow F = G \circ N \wedge G = F \circ M \Rightarrow F = F \circ (M \circ N) \Rightarrow d_{M \circ N} - 1 \Rightarrow M \circ N \not\in \text{Aut}(\mathbb{C}^2) \Rightarrow M, N \in \text{Aut}(\mathbb{C}^2) \Rightarrow [F] = [G] \). Any increasing chain in \( \text{et}(\mathbb{C}^2) - \text{Aut}(\mathbb{C}^2)/\sim_L \) is finite (by an argument similar to that in the proof of Theorem 4.10). Hence every maximal increasing chain contains a maximal element \([F] \), and \( L_F(\text{et}(\mathbb{C}^2)) \) contains the union of the images of the left mappings of all the elements in this maximal chain.
We define \( I = \{ [F] \in (\text{et}(\mathbb{C}^2) - \text{Aut}(\mathbb{C}^2))/\sim_L \mid [F] \text{ is the maximum of a maximal length chain} \} \). Then

\[
\text{et}(\mathbb{C}^2) = \text{Aut}(\mathbb{C}^2) \cup \bigcup_{i \in I} L_{F_i}(\text{et}(\mathbb{C}^2)).
\]

The union on the right equals \( \text{et}(\mathbb{C}^2) \) because any \( F \in \text{et}(\mathbb{C}^2) \) is either in \( \text{Aut}(\mathbb{C}^2) \) or \( [F] \) belongs to some maximal length chain in \( (\text{et}(\mathbb{C}^2) - \text{Aut}(\mathbb{C}^2))/\sim_L \) and so \( L_F(\text{et}(\mathbb{C}^2)) \) is a subset of \( L_{F_i}(\text{et}(\mathbb{C}^2)) \) where \( [F_i] \) is the maximum of that chain. Clearly if \( i \neq j \) then \( F_i \not\sim L_{F_j}(\text{et}(\mathbb{C}^2)) \) \( \forall \) \( F \in \text{et}(\mathbb{C}^2) \). Finally, the index set \( I \) is an infinite set. This follows because any finite union of the form:

\[
\text{Aut}(\mathbb{C}^2) \cup L_{F_1}(\text{et}(\mathbb{C}^2)) \cup \ldots \cup L_{F_k}(\text{et}(\mathbb{C}^2))
\]

is such that any mapping \( H \) in it is either a \( \mathbb{C}^2 \)-automorphism or \( R_0(H) \supset R_0(F_1) \neq \emptyset \lor \ldots \lor R_0(H) \supset R_0(F_k) \neq \emptyset \). Hence the argument in the proof of Proposition 3.2 implies that \( L_{F_1}(\text{et}(\mathbb{C}^2)) \cup \ldots \cup L_{F_k}(\text{et}(\mathbb{C}^2)) \subset \text{et}(\mathbb{C}^2) - \text{Aut}(\mathbb{C}^2) \). \( \square \)

We recall definition 4.3: an étale mapping \( F \in \text{et}(\mathbb{C}^2) \) is composite if \( \exists G, H \in (\text{et}(\mathbb{C}^2) - \text{Aut}(\mathbb{C}^2)) \) such that \( F = G \circ H \). An étale mapping is prime if it is not composite. If \( \text{et}(\mathbb{C}^2) \neq \text{Aut}(\mathbb{C}^2) \) then we know that the set of all the prime étale mappings is not empty. Also we know that the geometrical degree of a composite mapping is not a prime number. In other words an étale mapping whose geometrical degree is a prime number is a prime étale mapping. Also we know that any étale mapping \( F \in \text{et}(\mathbb{C}^2) \) can be written as follows: \( F = A_0 \circ P_1 \circ \ldots \circ P_k \), for some \( A_0 \in \text{Aut}(\mathbb{C}^2) \) and prime étale mappings \( P_1, \ldots, P_k \). By Proposition 8.5 it follows that if \( F \in \text{et}(\mathbb{C}^2) \) is composite, say \( F = G \circ H \) for some \( G, H \in (\text{et}(\mathbb{C}^2) - \text{Aut}(\mathbb{C}^2)) \), then \( L_F(\text{et}(\mathbb{C}^2)) \subset L_G(\text{et}(\mathbb{C}^2)) \). We conclude that in the fractal representation of \( \text{et}(\mathbb{C}^2) \):

\[
\text{et}(\mathbb{C}^2) = \text{Aut}(\mathbb{C}^2) \cup \bigcup_{i \in I} L_{F_i}(\text{et}(\mathbb{C}^2)).
\]

all the maximal elements \( F_i \) must be prime étale mappings. Conversely, it is clear that if \( F \) is étale prime, then \( F \) is the maximum of some (finite) \( \preceq_L \)-chain. Hence we can state a more accurate statement than that of Proposition 8.5(4).
Proposition 8.7. Suppose that \( \text{et}(\mathbb{C}^2) \neq \text{Aut}(\mathbb{C}^2) \). Let \( \text{et}_p(\mathbb{C}^2) \) be the set of all the prime étale mappings. Then \( |\text{et}_p(\mathbb{C}^2)| = \infty \) and we have,

\[
\text{et}(\mathbb{C}^2) = \text{Aut}(\mathbb{C}^2) \cup \bigcup_{F \in \text{et}_p(\mathbb{C}^2)} L_F(\text{et}(\mathbb{C}^2)),
\]

where \( \forall F \in \text{et}_p(\mathbb{C}^2) \), \( \text{Aut}(\mathbb{C}^2) \cap L_F(\text{et}(\mathbb{C}^2)) = \emptyset \), and \( \forall F, G \in \text{et}_p(\mathbb{C}^2), F \neq G \), we have \( F \notin L_G(\text{et}(\mathbb{C}^2)) \land G \notin L_F(\text{et}(\mathbb{C}^2)) \) and \( L_F(\text{et}(\mathbb{C}^2)) \) is homeomorphic to \( L_G(\text{et}(\mathbb{C}^2)) \).

We recall that \( \forall F \in \text{et}_p(\mathbb{C}^2) \), the corresponding left space \( L_F(\text{et}(\mathbb{C}^2)) \) is composed of all the étale mappings \( G \) that have the form \( G = F \circ H \) for some \( H \in \text{et}(\mathbb{C}^2) \). Hence the integer \( d_F \) divides the geometrical degree of \( G, d_G \). We know that the set of prime étale mappings is infinite (if non-empty). However, concerning their geometrical degrees \( \{d_F \mid F \in \text{et}_p(\mathbb{C}^2)\} \), we do not know much. If for an \( F \in \text{et}(\mathbb{C}^2) \) we have know that \( d_F \) is a prime integer then \( F \) is a prime mapping. But the set of geometric degrees of prime mappings might contain other integers (composite). It might be \( \mathbb{Z}^+ \). We now show how to get some non trivial information regarding that.

Theorem 8.8. \( |\text{et}_p(\mathbb{C}^2)| \leq \aleph_0 \) and \( \text{et}_p(\mathbb{C}^2) \) is a discrete subset of the metric space \( (\text{et}(\mathbb{C}^2), \rho_D) \).

Proof.

We later on (in Theorem 14.14) will prove that if \( F, G \in \text{et}_p(\mathbb{C}^2) \) satisfy \( F \neq G \) then \( L_F(\text{et}(\mathbb{C}^2)) \cap L_G(\text{et}(\mathbb{C}^2)) = \emptyset \). When \( D \to \mathbb{C}^2 \), \( \text{meas}(L_F(\text{et}(\mathbb{C}^2))) \to \text{meas}(\text{et}(\mathbb{C}^2))/d_F \). So by the identity \( \text{meas}(\bigcup_{F \in \text{et}_p(\mathbb{C}^2)} L_F(\text{et}(\mathbb{C}^2))) = \text{meas}(\text{et}(\mathbb{C}^2)) \) (the measure is \( H^{s_0} \) where \( s_0 = \dim_H \text{et}(\mathbb{C}^2) \)) we get:

\[
\sum_{F \in \text{et}(\mathbb{C}^2)} \frac{1}{d_F} = 1. \quad \square
\]

Corollary 8.9. The sequence of geometric degrees \( \{d_F \mid F \in \text{et}_p(\mathbb{C}^2)\} \) can not equal to \( \mathbb{Z}^+_{\geq 2} \), and can not equal to the set of prime integers.

Corollary 8.10. \( (\text{et}(\mathbb{C}^2), \circ) \) is generated by \( \text{Aut}(\mathbb{C}^2) \) and the countable set of generators given by \( \text{et}_p(\mathbb{C}^2) \).

9 Properties of the metric spaces \( (\text{et}(\mathbb{C}^2), \rho_D) \)

Here is a natural list of questions about those metric spaces:

1) Is it a separable space?
2) Is it a proper space?
3) Is it a complete space?
4) Is the action of the group \( \text{Aut}(\mathbb{C}^2) \) from the left, as isometries on the space, cocompact?
5) Is the action described in 4, a proper action?

Remark 9.1. The action of \( \text{Aut}(\mathbb{C}^2) \) from the left on \((\text{et}(\mathbb{C}^2), \rho_D)\) is certainly not both cocompact and proper. This follows by Lemma 1.17 on page 8 of [18]. For if the action were both cocompact and proper, then by (b) of the Lemma this would have implied that \( \text{Aut}(\mathbb{C}^2) \) is finitely generated.

6) Is the metric space \((\text{et}(\mathbb{C}^2), \rho_D)\) a length space?
7) What are the geodesics if any?

To tackle question 1 we might ask the following.
8) Are the étale mappings with rational coefficients dense in the space \((\text{et}(\mathbb{C}^2), \rho_D)\)? Are the mappings in \( \text{Aut}(\mathbb{C}^2) \) with rational coefficients dense in \( \text{Aut}(\mathbb{C}^2) \)? Here rational coefficient are numbers in \( \mathbb{Q} + i\mathbb{Q} \).

1) Separability of \((\text{et}(\mathbb{C}^2), \rho_D)\) and of \((\text{Aut}(\mathbb{C}^2), \rho_D)\)

**Proposition 9.2.** The metric space \((\text{Aut}(\mathbb{C}^2), \rho_D)\) is separable.

**Proof.**

It is well known that the group \((\text{Aut}(\mathbb{C}^2), \circ)\) is generated by the affine mappings, \( F(X,Y) = (aX + bY + c, dX + eY + f), \ ae - bd = 1 \), and by the elementary mappings \( G(X,Y) = (X + P(Y), Y) \) or \( H(X,Y) = (X, Y + P(X)) \) where \( P(T) \in \mathbb{C}[T] \) (The Jung-Van Der Kulk Theorem, [5]). Using this, we will show that any automorphism of \( \mathbb{C}^2 \) can be approximated well enough by automorphisms of \( \mathbb{C}^2 \) which have all of their coefficients from \( \mathbb{Q} + i\mathbb{Q} \).

For we have:

(i) \( \forall (aX + bY + c, dX + eY + f), \ ae - bd = 1, \exists (a_kX + b_kY + c_k, d_kX + e_kY + f_k), k \in \mathbb{Z}^+ \) such that \( a_k e_k - b_k d_k = 1, a_k, b_k, c_k, d_k, e_k, f_k \in \mathbb{Q} + i\mathbb{Q} \) and such that \( a = \lim a_k \land b = \lim b_k \land c = \lim c_k \land d = \lim d_k \land e = \lim e_k \land f = \lim f_k. \)

(ii) \( \forall (X + P(Y), Y), \ P(Y) \in \mathbb{C}[Y] \exists (X + P_k(Y), Y), \ P_k(Y) \in (\mathbb{Q} + i\mathbb{Q})[Y], k \in \mathbb{Z}^+ \) and also \( \deg P_k = \deg P, \lim P_k = P \) coefficientwise.

(iii) The same as in (ii) \( \forall (X, Y + P(X)). \)

We conclude that for any \( F \in \text{Aut}(\mathbb{C}^2) \) we have a sequence \( F_k \in \text{Aut}(\mathbb{C}^2), k \in \mathbb{Z}^+ \), which have all of the coefficients from \( \mathbb{Q} + i\mathbb{Q} \) and which satisfy \( \lim F_k = F \) coefficientwise and where the degrees \( \deg F_k \leq M \) are uniformly bounded in \( k \). Hence \( \forall K \subset \mathbb{C}^2 \) a compact in the strong topology we have
$$\lim F_k = F$$ uniformly on $K$. This follows by the proposition that follows this one. This implies that $\lim F_k = F$ in $(\text{Aut}(\mathbb{C}^2), \rho_D)$ and concludes the proof. \hfill \Box

**Remark 9.3.** The uniform bound $M$ on the degrees $\deg F_k$ could be taken to be $M = \prod \deg E_k$ where $F = \circ_k E_k$ is a decomposition of $F$ into affine and elementary mappings.

**Proposition 9.4.** Let $F(X, Y) \in \mathbb{C}[X, Y]$, $F_k \in \mathbb{C}[X, Y]$, $k \in \mathbb{Z}^+$ and $K \subset \mathbb{C}^2$ a compact subset in the strong topology. If $\lim F_k = F$ coefficientwise and $\deg F_k \leq M$ are uniformly bounded in $k$, then $\lim F_k = F$ uniformly on $K$.

**Remark 9.5.** If we drop the uniform degree bound condition in Proposition 9.4, its conclusion is false. For example we take $F(X, Y) \equiv 0$, and $F_k(X, Y) = (1/k)(X^k + X^{k+1} + \ldots + X^{2^{k-1}})$ and $K = [0, 1] \times \{0\}$. Then $\lim F_k = F$ coefficientwise but not even pointwise because $F(1, 0) = 0$ while $\forall k \in \mathbb{Z}^+, F_k(1, 0) = 1$.

**A proof of Proposition 9.4.**

By the degree bound assumption we can write,

$$F(X, Y) = \sum_{i+j \leq M} a_{ij} X^i Y^j, \quad F_k(X, Y) = \sum_{i+j \leq M} a_{ij}^{(k)} X^i Y^j, \quad k \in \mathbb{Z}^+.$$ 

We have $\forall i + j \leq M$, $\lim_{k \to \infty} a_{ij}^{(k)} = a_{ij}$. We note that,

$$|F(X, Y) - F_k(X, Y)| = | \sum_{i+j \leq M} (a_{ij} - a_{ij}^{(k)}) X^i Y^j | \leq \sum_{i+j \leq M} |a_{ij} - a_{ij}^{(k)}||X^i Y^j|.$$

For any $i + j \leq M$ the function $|X^i Y^j|$ is continuous on the compact $K$ and so $m_{ij} = \max_{(X, Y) \in K} |X^i Y^j| < \infty$. Thus $m = \max \{ m_{ij} | i + j \leq M \} < \infty$. Given $\epsilon > 0 \exists N$ such that $\forall k > N$ we have $\sum_{i+j \leq M} |a_{ij} - a_{ij}^{(k)}| \leq \epsilon/(m+1)$. We conclude that $\max_{(X, Y) \in K} |F(X, Y) - F_k(X, Y)| \leq (\epsilon \cdot m)/(m + 1) < \epsilon$, $\forall k > N$. \hfill \Box

**Remark 9.6.** We can generalize Proposition 9.4 as follows:

Let $F(X, Y) = \sum_{i+j \leq M} a_{ij} X^i Y^j$, $F_k(X, Y) = \sum_{i+j \leq M_k} a_{ij}^{(k)} X^i Y^j$, $k \in \mathbb{Z}^+$ and $K \subset \mathbb{C}^2$ a compact subset of $\mathbb{C}^2$ in the strong topology. If

$$\sum_{i+j \leq M} a_{ij}^{(k)} X^i Y^j \to_{k \to \infty} F$$ uniformly on $K$. 

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and if
\[ \sum_{M < i + j \leq M_k} a_{ij}^{(k)} X^i Y^j \rightarrow_{k \to \infty} 0 \text{ uniformly on } K \]
then, \( \lim F_k = F \) uniformly on \( K \).

**Proposition 9.7.** The metric space \( (\text{et}(\mathbb{C}^2), \rho_D) \) is separable.

**Proof.**
We will find a countable subset \( \text{et}_{\aleph_0}(\mathbb{C}^2) \subset \text{et}(\mathbb{C}^2) \) such that \( \forall F \in \text{et}(\mathbb{C}^2) \), \( \exists F_k \in \text{et}_{\aleph_0}(\mathbb{C}^2) \), \( k \in \mathbb{Z}^+ \) such that \( \lim F_k = F \) coefficientwise and we have a uniform degree bound \( \deg F_k \leq \deg F \). Thus we will be able to apply once more Proposition 9.4 and conclude the result. Unlike the situation with \( (\text{Aut}(\mathbb{C}^2), \rho_D) \), this time we can not say something definite about the character of the coefficients of the \( F_k \)'s (e.g. that all belong to \( \mathbb{Q} + i\mathbb{Q} \)).

The elements of the set \( \text{et}(\mathbb{C}^2) \) are faithfully parametrized by the sets of the solutions of certain polynomial systems of equations that are induced on the coefficients of the étale mappings by the Jacobian Condition, \( \det J_F(X, Y) \equiv 1 \). These systems have all of their equations, quadratic and homogeneous, except for a single equation which is still quadratic but not homogeneous. \( \forall k \in \mathbb{Z}^+ \) we let \( \text{et}_k(\mathbb{C}^2) = \{ F \in \text{et}(\mathbb{C}^2) \mid \deg F \leq k \} \). This subset of \( \text{et}(\mathbb{C}^2) \) is parametrized by certain finite dimensional complex space, which is composed of a part of the above mentioned solutions. We denote this complex space by \( J_k(2) \) or simply by \( J_k \). Its finite dimension is a function of \( k \). We have \( \forall k, J_k \subset J_{k+1} \) (proper containment) and \( J_k \) is a path connected topological subspace of the appropriate \( \mathbb{C}^D_k \) in the strong topology. Given \( k \in \mathbb{Z}^+ \) and \( \epsilon > 0 \) we can construct on \( J_k \) a countable \( \epsilon \)-net, \( N_k(\epsilon) \). This means that \( N_k(\epsilon) = \{ P_{j_0}^{(\epsilon)} \mid j \in \mathbb{Z}^+ \} \) and that \( \forall P \in J_k \), \( \exists j_0 \in \mathbb{Z}^+ \) such that \( P_{j_0}^{(\epsilon)} \) is close to \( P \) in the strong metric by less than \( \epsilon \). Thus \( ||P - P_{j_0}^{(\epsilon)}||_2 < \epsilon \).

If the point \( P \) is the parameter value of the étale mapping \( F \in \text{et}_k(\mathbb{C}^2) \) and if \( P_{j_0}^{(\epsilon)} \) is the parameter value of the étale mapping \( F_{j_0}^{(\epsilon)} \in \text{et}_k(\mathbb{C}^2) \), then this implies that the \( l_2 \)-distance between the coefficients of \( F \) and those of \( F_{j_0}^{(\epsilon)} \) is less than \( \epsilon \). In particular this means that the following is true:

if \( \text{et}_{\aleph_0}(\mathbb{C}^2) \) is the subset of étale mappings in \( \text{et}(\mathbb{C}^2) \) that are parametrized by

\[ \bigcup_{(k,n) \in \mathbb{Z}^+} N_k \left( \frac{1}{n} \right), \]

then \( \forall F \in \text{et}(\mathbb{C}^2) \), \( \exists F_k \in \text{et}_{\deg F}(\mathbb{C}^2) \cap \text{et}_{\aleph_0}(\mathbb{C}^2) \) so that \( \lim F_k = F \) coefficientwise.
Since we have a uniform bound on the degrees, $\deg F_k \leq \deg F$ we can indeed apply Proposition 9.4.

Remark 9.8. We note the important role of Proposition 9.4 in the proofs of Proposition 9.2 and of Proposition 9.7. In particular, the uniform degree bound was central. We naturally ask if this uniform degree bound is a natural necessary condition for our needs ($\rho_D$-convergence). More concretely we ask if $\forall M > 0, \exists \epsilon = \epsilon(M) > 0$ such that $\forall F, G \in \text{et}(C^2)$ if $|\deg F - \deg G| > M$ then $\text{vol}(F(D) \Delta G(D)) > \epsilon$? The answer is clearly negative ($G(X, Y) = (X, Y)$ the identity mapping, and $F(X, Y) = (X + (1/n)Y^M, Y)$). In fact we have no good idea on the character of $\rho_D$-convergence. For example, if $F_k \in \text{et}(C^2)$ and $F \in \text{et}(C^2)$ satisfy $\lim_{k \to \infty} \rho_D(F_k, F) = 0$ is it true that $\lim F_k = F$ coefficientwise? Pointwise? Uniformly on $D$?

We end this section with the following elementary observation,

**Proposition 9.9.** The metric space $(\text{et}(C^2), \rho_D)$ is bounded.

**Proof.**
The diameter of $\text{et}(C^2)$ with respect to $\rho_D$ is bounded from above by twice the volume of the characteristic compact $D$.

10 Connections between the two dimensional Jacobian Conjecture and number theory

In this short section we just indicate the general principles that underline the nontrivial connections that are indicated in the title. We also fix some notations. In the sections that follow we make more concrete computations that actually uncover some of these connections.

We consider the fractal representation of $\text{et}(C^2)$ using the left mappings. Namely (Proposition 8.6):

$$\text{et}(C^2) = \text{Aut}(C^2) \cup \bigcup_{F \in \text{et}_p(C^2)} L_F(\text{et}(C^2)).$$

Let us denote by $H^s$ any Hausdorff measure on $\text{et}(C^2)$, with a parameter $s$. Then we have the following,

$$H^s(\text{et}(C^2)) = H^s(\text{Aut}(C^2)) + H^s(\bigcup_{F \in \text{et}_p(C^2)} L_F(\text{et}(C^2))).$$
This follows by the disjointness \( \text{Aut}(C^2) \cap \bigcup_{F \in \text{et}_p(C^2)} L_F(\text{et}(C^2)) = \emptyset \). We use the parametrization of the space of the étale prime mappings \( \text{et}_p(C^2) \) and measure using that parameter. We get,

\[
H^s(\text{et}(C^2)) \leq H^s(\text{Aut}(C^2)) + \int_{\{F \in \text{et}_p(C^2)\}} f(F, s) \cdot H^s(\text{et}(C^2)) dF.
\]

Here the notation \( f(F, s) \) stands for the self similarity factor between \( \text{et}(C^2) \) and the scaling down \( L_F(\text{et}(C^2)) \), i.e. \( H^s(L_F(\text{et}(C^2))) = f(F, s) \cdot H^s(\text{et}(C^2)) \).

We conclude the following inequality,

\[
H^s(\text{et}(C^2)) \left( 1 - \int_{\{F \in \text{et}_p(C^2)\}} f(F, s) dF \right) \leq H^s(\text{Aut}(C^2)).
\]

If \( H^s(\text{et}(C^2)) \neq 0 \), then this can also be written as follows:

\[
1 - \int_{\{F \in \text{et}_p(C^2)\}} f(F, s) dF \leq \frac{H^s(\text{Aut}(C^2))}{H^s(\text{et}(C^2))}.
\]

**Remark 10.1.** We will denote by \( s_0 = \dim_H \text{et}(C^2) \) the Hausdorff dimension of the étale mappings \( \text{et}(C^2) \). It is plausible that \( s_0 = \dim_H \text{et}(C^2) > \dim_H \text{Aut}(C^2) \) in which case we have \( H^{s_0}(\text{et}(C^2)) > 0 \) and \( H^{s_0}(\text{Aut}(C^2)) = 0 \).

By equation (10.1) we get,

**Proposition 10.2.** \( 1 \leq \int_{\{F \in \text{et}_p(C^2)\}} f(F, s_0) dF \).

### 11 Inequalities and identities

With a "natural" Hausdorff measure we have the following,

**Proposition 11.1.** If \( D \) denotes the characteristic set that defines the metric \( \rho_D \) on \((\text{et}(C^2), \rho_D)\), then \( \lim_{D \to C^2} f(F, s) = d_F^{-s} \).

**Proof.**

This follows by Proposition 8.2. \( \square \)

**Proposition 11.2.** \( 1 \leq \int_{\{F \in \text{et}_p(C^2)\}} d_F^{-s_0} dF \).
Proof.
Using Proposition 9.2 and Proposition 11.1 and dominated convergence.

So either $H^{s_0}(et(C^2)) = 0$ in which case equation (10.1) is invalid (with $s = s_0$) or $s_0 = \dim_H et(C^2) < \infty$ otherwise Proposition 11.2 is invalid.

**Definition 11.3.** $\forall n \in \mathbb{Z}^+, \mu(n) = \int_{\{F \in et_p(C^2) \mid d_F = n\}} dF$.

This allows us to rewrite the last proposition as follows,

**Proposition 11.4.**

\[ 1 \leq \sum_{n=2}^{\infty} \frac{\mu(n)}{n^{s_0}}. \]

If we had disjointness in our fractal representation, the inequality signs would have become equalities and we could have deduced the following,

**Proposition 11.5.** If $\forall F,G \in et_p(C^2)$, the assumption $F \neq G$ implied that $L_F(et(C^2)) \cap L_G(et(C^2)) = \emptyset$, then:

\[ 1 = \sum_{n=2}^{\infty} \frac{\mu(n)}{n^{s_0}}, \]

and hence $\forall n \in \mathbb{Z}^+$, $\mu(n)/n^{s_0} \leq 1$, i.e.

\[ \operatorname{meas}\{F \in et_p(C^2) \mid d_F = n\} := \int_{\{F \in et_p(C^2) \mid d_F = n\}} dF \leq n^{s_0}. \]

Moreover, asymptotically:

\[ \lim_{n \to \infty} \frac{\mu(n)}{n^{s_0-1}} = 0, \]

i.e.

\[ \operatorname{meas}\{F \in et_p(C^2) \mid d_F = n\} = o(n^{s_0-1}). \]

**Proof.**
Only the last parts needs a proof. The series with nonnegative terms

\[ \sum_{n=2}^{\infty} \frac{\mu(n)}{n^{s_0}}, \]

converges to 1 and so a comparison with the divergent harmonic series $\sum (1/n)$ gives us the desired estimate,

\[ \lim_{n \to \infty} \frac{\mu(n)}{n^{s_0-1}} = 0. \]
Remark 11.6. How could we prove the disjointness in the fractal representation? Let $F, G \in \text{et}_p(C^2)$ and $F \neq G$. Suppose that we did not have disjointness, say $H \in L_F(\text{et}(C^2)) \cap L_G(\text{et}(C^2))$. Then $\exists M, N \in \text{et}(C^2)$ such that $H = F \circ N = G \circ M$. If we knew that there is a unique factorization of $H$ where uniqueness includes the order of the factors, then this would have been it.

Next we make the following,

Definition 11.7. $\forall n \in \mathbb{Z}^+, \Omega(n) = \{F \in \text{et}(C^2) \mid d_F = n\}$.

Then we clearly have,

Proposition 11.8. $\Omega(n) = \bigcup_{k|n, k \geq 1} \Omega(k) \circ \Omega(n/k)$.

However, in this representation many mappings appear in many components of the union on the right hand side. In other words, this representation is very far from being a partition of $\Omega(n)$. In order to get a better representation, we refine our definitions,

Definition 11.9. $\forall n \in \mathbb{Z}^+, \Omega_p(n) = \Omega(n) \cap \text{et}_p(C^2) = \{F \in \text{et}_p(C^2) \mid d_F = n\}$. We will use the short notation:

$\Omega_p(n_1) \circ \Omega_p(n_2) \circ \ldots \circ \Omega_p(n_k) = \bigcirc_{j=1}^k \Omega_p(n_j)$.

It should be noted that in this "product" order matters because composition is not commutative.

We have the following improvement of Proposition 11.8,

Proposition 11.10. $\Omega(n) = \bigcup_{n_1 n_2 \ldots n_k = n} \{\bigcirc_{j=1}^k \Omega_p(n_j)\}$. The order of the factors $n_1, n_2, \ldots, n_k$ in the product $n_1 n_2 \ldots n_k = n$ is important.

Computing few examples.

If $p, q$ are two prime integers then if $p \neq q$ we have

$\mu(pq) \leq \mu_p(pq) + 2\mu_p(p)\mu_p(q)$.

Here we are using the following,

Definition 11.11.

$\mu(n) = \mu(\Omega(n)) = \text{meas}(\Omega(n)) = \int_{\{F \in \text{et}(C^2) \mid d_F = n\}} dF,$

$\mu_p(n) = \mu(\Omega_p(n)) = \text{meas}(\Omega_p(n)) = \int_{\{F \in \text{et}_p(C^2) \mid d_F = n\}} dF.$
The above inequality is an immediate consequence of the following identity which follows by the definitions,
\[ \Omega(p,q) = \Omega_p(pq) \cup \Omega_p(p) \circ \Omega_p(q) \cup \Omega_p(q) \circ \Omega_p(p), \quad p \neq q. \]
The inequality originates in the possibility that
\[ \Omega_p(p) \circ \Omega_p(q) \cap \Omega_p(q) \circ \Omega_p(p) \neq \emptyset, \quad p \neq q. \]
Once more, if we knew that there is disjointness, i.e.,
\[ \Omega(p) \circ \Omega_p(q) \cap \Omega_p(q) \circ \Omega_p(p) = \emptyset, \quad p \neq q, \]
then we could have deduced the sharper result:
\[ \mu(pq) = \mu_p(pq) + 2 \mu_p(p) \mu_p(q), \quad p \neq q. \]
By Proposition 11.5 we have \( \mu(pq)/(pq)^{s_0} \leq 1 \) and hence
\[ \frac{\mu_p(pq)}{(pq)^{s_0}} + 2 \left( \frac{\mu_p(p)}{p^{s_0}} \right) \left( \frac{\mu_p(q)}{q^{s_0}} \right) \leq 1, \quad p \neq q. \]

**Remark 11.12.** For \( n = q \) an integral prime we clearly have the identity \( \Omega(q) = \Omega_p(q) \) and hence \( \mu(q) = \mu_p(q) \).

We recall that if \( q_1, q_2 \) are two integral primes, then (as we saw) we have:
\[ \Omega(q_1q_2) = \Omega_p(q_1q_2) \cup \Omega_p(q_1) \circ \Omega_p(q_2) \cup \Omega_p(q_2) \circ \Omega_p(q_1). \]
We certainly have the following:
\[ \Omega_p(q_1q_2) \cap \Omega_p(q_1) \circ \Omega_p(q_2) = \Omega_p(q_1q_2) \cap \Omega_p(q_2) \circ \Omega_p(q_1) = \emptyset. \]
However, we might have \( \Omega_p(q_1) \circ \Omega_p(q_2) \cap \Omega_p(q_2) \circ \Omega_p(q_1) \neq \emptyset. \) The last possibility happens when there are two pairs of mappings \( F_j \in \Omega_p(q_1), \quad G_j \in \Omega_p(q_2), \quad j = 1, 2 \) that satisfy: \( H = F_1 \circ G_1 = G_2 \circ F_2. \) In this case we have the following identities for the Jacobian varieties: \( A_H = A_{F_1} \cup F_1(A_{G_1}) = A_{G_2} \cup G_2(A_{F_2}). \) Also, for the geometric bases we have: \( R_0(G_1), R_0(F_2) \subseteq R_0(H). \) Finally, the chain rule implies for the Jacobian matrices the following: \( J_{F_1}(G_1) \cdot J_{G_1}(X,Y) = J_{G_2}(F_2) \cdot J_{F_2}(X,Y). \) The special case \( q = q_1 = q_2 \) is somehow easier because there is no overlap in this case. So in that case we have the following partition:
\[ \Omega(q^2) = \Omega_p(q^2) \cup \Omega_p(q) \circ \Omega_p(q) = \Omega_p(q^2) \cup \Omega(q) \circ \Omega(q). \]
Hence $\mu(q^2) = \mu_p(q^2) + \mu(q)^2$ or, equivalently $\mu(q^2) - \mu(q)^2 = \mu_p(q^2) \geq 0$. Hence $\mu(q) \leq \sqrt{\mu(q^2)}$. Thus
\[
\frac{\mu(q)}{q^{s_0}} \leq \sqrt{\frac{\mu(q^2)}{(q^2)^{s_0}}}.
\]

**Proposition 11.13.** For any prime integer $q$ we have the following inequality:
\[
0 \leq \mu(q) \leq \left(\frac{\sqrt{5} - 1}{2}\right) q^{s_0}.
\]

**Proof.**
We recall that we have the following inequality:
\[
\frac{\mu(q)}{q^{s_0}} + \frac{\mu(q^2)}{(q^2)^{s_0}} \leq 1.
\]
Thus if we use the inequality above:
\[
\frac{\mu(q)}{q^{s_0}} \leq \sqrt{\frac{\mu(q^2)}{(q^2)^{s_0}}}
\]
we obtain the following estimate:
\[
\frac{\mu(q)}{q^{s_0}} + \left(\frac{\mu(q)}{q^{s_0}}\right)^2 \leq \frac{\mu(q)}{q^{s_0}} + \frac{\mu(q^2)}{(q^2)^{s_0}} \leq 1.
\]
Hence:
\[
0 \leq \frac{\mu(q)}{q^{s_0}} \leq -1 + \frac{\sqrt{1 + 4}}{2}.
\]

Here is another similar series of arguments. We start with:
\[
\Omega(q^3) = \Omega_p(q^3) \cup \Omega_p(q^2) \circ \Omega_p(q) \cup \Omega_p(q^2) \circ \Omega_p(q) \circ \Omega_p(q) \circ \Omega_p(q) \circ \Omega_p(q),
\]
so
\[
\Omega(q^3) = \Omega_p(q^3) \cup \Omega_p(q^2) \circ \Omega(q) \cup \Omega(q) \circ \Omega_p(q^2) \cup \Omega(q) \circ \Omega(q) \circ \Omega(q),
\]
\[
\mu(q^3) \leq \mu_p(q^3) + 2\mu_p(q^2)\mu(q) + \mu(q)^3.
\]
But $\mu(q^2) = \mu(q)^2 + \mu_p(q^2)$ and hence:
\[
\mu(q^3) \leq \mu_p(q^3) + 2\mu(q^2)\mu(q) - \mu(q)^3, \quad \text{hence} \quad \mu(q^3) + \mu(q)^3 \leq 2\mu(q^2)\mu(q).
\]
We end here this chain of computational examples and proceed further in the theory of the fractal representation of $\text{et}(\mathbb{C}^2)$. Our starting point is further generalization of Proposition 11.2. We recall that if $s_0 = \text{dim}_H \text{et}(\mathbb{C}^2)$ the Hausdorff dimension of $\text{et}(\mathbb{C}^2)$, then

$$1 \leq \int_{\{F \in \text{et}_p(\mathbb{C}^2)\}} f(F, s_0) dF.$$ 

Let us denote the Hausdorff dimension of the automorphism group of $\mathbb{C}^2$ by $s_1 = \text{dim}_H \text{Aut}(\mathbb{C}^2)$. Since $\text{Aut}(\mathbb{C}^2) \subseteq \text{et}(\mathbb{C}^2)$ we have by monotonicity $s_1 \leq s_0$. We recall the fundamental inequality we had just before 10.1:

$$H^s(\text{et}(\mathbb{C}^2)) \left(1 - \int_{\{F \in \text{et}_p(\mathbb{C}^2)\}} f(F, s) dF\right) \leq H^s(\text{Aut}(\mathbb{C}^2)).$$

The case $s_1 = s_0$ is easy to handle (follows by Proposition 11.2). So let $s$ be the Hausdorff parameter, assuming that $s_1 < s < s_0$. Then $H^s(\text{et}(\mathbb{C}^2)) = +\infty$, $H^s(\text{Aut}(\mathbb{C}^2)) = 0$. How could this accommodate with the fundamental inequality? Only if,

$$1 - \int_{\{F \in \text{et}_p(\mathbb{C}^2)\}} f(F, s) dF \leq 0.$$ 

Thus we arrived at our generalization of Proposition 11.2. Namely,

**Proposition 11.14.** \(\forall s, \text{ such that } \text{dim}_H \text{Aut}(\mathbb{C}^2) < s < \text{dim}_H \text{et}(\mathbb{C}^2), \) we have the inequality,

$$1 - \int_{\{F \in \text{et}_p(\mathbb{C}^2)\}} f(F, s) dF \equiv 0.$$ 

It will be convenient to use the following terminology:

**The disjointness assumption:** \(\forall F, G \in \text{et}_p(\mathbb{C}^2), \) if $F \neq G$, then

$$L_F(\text{et}(\mathbb{C}^2)) \cap L_G(\text{et}(\mathbb{C}^2)) = \emptyset.$$ 

**Proposition 11.15.** Under the disjointness assumption we have the following refinement of Proposition 11.14: \(\forall s, \text{ such that } \text{dim}_H \text{Aut}(\mathbb{C}^2) < s < \text{dim}_H \text{et}(\mathbb{C}^2), \) we have the identity,

$$1 - \int_{\{F \in \text{et}_p(\mathbb{C}^2)\}} f(F, s) dF \equiv 0.$$
Proof.
The disjointness assumption refines the fundamental inequality, into the fundamental identity,

\[ H^s(\text{et}(\mathbb{C}^2)) \left(1 - \int_{\{F \in \text{et}_p(\mathbb{C}^2)\}} f(F, s) dF \right) = H^s(\text{Aut}(\mathbb{C}^2)). \]

Again, a value \( s \) of the Hausdorff parameter as in the assumption of the proposition satisfies \( H^s(\text{et}(\mathbb{C}^2)) = +\infty, \ H^s(\text{Aut}(\mathbb{C}^2)) = 0 \). This "lives in peace" with the fundamental identity only if,

\[ 1 - \int_{\{F \in \text{et}_p(\mathbb{C}^2)\}} f(F, s) dF \equiv 0. \]

Next we recall that we can take the similarity factor \( f(F, s) = d_F^{-s} \) which is a non increasing function of the Hausdorff parameter \( s \) (Proposition 11.1). So under the disjointness assumption by Proposition 11.14 it follows that \( f(F, s) = f(F) \) is independent of the Hausdorff parameter \( s \).

Remark 11.16. Using the definition of the similarity factor we get:

\[ H^s(L_F(\text{et}(\mathbb{C}^2))) = f(F)H^s(\text{et}(\mathbb{C}^2)), \]

\[ H^s(\text{Aut}(\mathbb{C}^2)) = H^s(\text{et}(\mathbb{C}^2)) \left(1 - \int_{\{F \in \text{et}_p(\mathbb{C}^2)\}} f(F, s) dF \right) \equiv 0, \ \forall \ s. \]

For \( f(F, s) = d_F^{-s} \) to be independent of \( s \) there is only one choice, namely \( \forall F \in \text{et}(\mathbb{C}^2), \ d_F = 1 \). But this implies of course that \( \text{et}(\mathbb{C}^2) = \text{Aut}(\mathbb{C}^2) \).

Thus we proved the following interesting,

**Theorem 11.17.** The disjointness assumption implies the validity of the two dimensional Jacobian Conjecture.

We remark that Theorem 11.17 remains valid also under the following: **The weak disjointness assumption:** Almost everywhere in \( \text{et}_p(\mathbb{C}^2) \times \text{et}_p(\mathbb{C}^2) \), if \( F \neq G \), then \( L_F(\text{et}(\mathbb{C}^2)) \cap L_G(\text{et}(\mathbb{C}^2)) = \emptyset \).

12 A discussion on the impact of the paper [4] on the structure of \( \text{et}(\mathbb{C}^2) \)

The paper assumes for the most part that \( X \) is a compact Hausdorff space and that \( T \) is a semigroup which acts on \( X \) from the right. In our application...
the parallel is $X = \text{et}(\mathbb{C}^2)$. As for the topology on $X$, we take the metric topology which is induced by $\rho_D$ for some characteristic subset $D$ of $\mathbb{C}^2$. We know that $(X, \rho_D) = (\text{et}(\mathbb{C}^2), \rho_D)$ is a bounded metric space. Is it compact? Also, in our application we take the semigroup $T = \text{et}(\mathbb{C}^2)$, with composition of mappings for its binary operation. Lastly, in our application we consider the left-$T$-action on $X$ where the action is, again induced by composition of mappings. Thus:

$$\pi : X \times T = (\text{et}(\mathbb{C}^2) \times \text{et}(\mathbb{C}^2)) \rightarrow X = \text{et}(\mathbb{C}^2), \quad (x, t) = (F, G) \rightarrow tx = F \circ G.$$ 

We have (as in a left action) $s(tx) = F_2 \circ (F_1 \circ G) = (F_2 \circ F_1) \circ G = (st)x$. Orbits: $\text{Orb}(G) = \text{et}(\mathbb{C}^2) \circ G = \{F \circ G | F \in \text{et}(\mathbb{C}^2)\}$. We always have $\forall H \in \text{Orb}(G), \ R_0(G) \subseteq R_0(H)$. By $A(F \circ G) = A(F) \cup F(A(G))$, it follows that the $\text{et}(\mathbb{C}^2)$-orbit of the asymptotic variety $A(G)$ is subordinated to the set of all asymptotic varieties of the elements of $\text{Orb}(G)$. Here we use the following notion: Let $A$ and $B$ be two families of sets. We say that the family $A$ is subordinated to the family $B$ and denote $A \preceq B$, if $\forall a \in A \exists b \in B$ such that $a \subseteq b$.

**Example.**

**Subordination is an extension of the notion of inclusion, i.e. $A \subseteq B \Rightarrow A \preceq B$.**

If $\text{et}(\mathbb{C}^2) = \text{Aut}(\mathbb{C}^2)$, then $\forall G \in \text{et}(\mathbb{C}^2)$ we have $\text{Orb}(G) = \text{et}(\mathbb{C}^2)$. If $G \in \text{Aut}(\mathbb{C}^2)$, then $\text{Orb}(G) = \text{et}(\mathbb{C}^2)$. We adjust the definition of an invariant set: We say that a set $A$, $\emptyset \neq A \subseteq \text{et}(\mathbb{C}^2) = X$ is invariant if

$$TA = \text{et}(\mathbb{C}^2) \circ A := \{F \circ G | F \in \text{et}(\mathbb{C}^2) = T, G \in A\} \subseteq A.$$ 

Clearly the set $A = \text{et}(\mathbb{C}^2) = X$ is invariant. Are there any other invariant subsets of $X = \text{et}(\mathbb{C}^2)$? We note that for any set $A$ (invariant or not), we have $A \subseteq \text{et}(\mathbb{C}^2) \circ A$, because $\text{id} \in \text{et}(\mathbb{C}^2)$, and so: $A \neq \emptyset$ is invariant if and only if $\text{et}(\mathbb{C}^2) \circ A = A$. Also, if $A \cap \text{Aut}(\mathbb{C}^2) \neq \emptyset$ and $A$ is invariant then $A = \text{et}(\mathbb{C}^2)$, because if $G \in A \cap \text{Aut}(\mathbb{C}^2)$ then already the orbit $\text{Orb}(G) = \text{et}(\mathbb{C}^2)$ and when $A$ is invariant, then $\text{Orb}(G) \subseteq A$. Thus if $A$ is invariant and non-trivial, i.e. $A \neq \text{et}(\mathbb{C}^2)$ then $A \cap \text{Aut}(\mathbb{C}^2) = \emptyset$. We note that if $A$ is invariant, then its closure in $X = \text{et}(\mathbb{C}^2)$, $\overline{A}$ is also invariant, because if $G \in \overline{A} - A$, then $\exists G_n \in A$ such that $G_n \rightarrow G$ in $X$, thus if $F \in T = \text{et}(\mathbb{C}^2)$ then $\forall n, \ F \circ G_n \in A$ and $F \circ G_n \rightarrow F \circ G \in \overline{A}$. We remark that $\forall F \in \text{et}(\mathbb{C}^2) - \text{Aut}(\mathbb{C}^2) \forall G \in \text{et}(\mathbb{C}^2)$ we have $F \circ G \neq G$ because $d_{F \circ G} = d_F \cdot d_G \geq 2 \cdot d_G > d_G$. 

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claim: If \( A \) is invariant then \( \forall a \in A \) we have \( \Orb_L(a) \subseteq A \), and vice versa, if \( \emptyset \neq A \subseteq X \) satisfies \( \forall a \in A \), \( \Orb_L(a) \subseteq A \) then \( A \) is invariant.

**A proof of the claim.**

\( A \) is invariant \( \iff TA \subseteq A \iff \forall a \in A, \ \{ta \mid t \in T\} \subseteq A \square \)

We recall that if \( \id \in T \) then \( \forall x \in X, \ x \in \Orb_L(x) \). Thus we can write: If \( \id \in T \), then \( \emptyset \neq A \subseteq X \) satisfies \( \forall a \in A \), \( \Orb_L(a) \subseteq A \) then \( A \) is invariant. We recall that the right shift operator on \( \mathcal{E}(C^2) \) induced by \( G \in \mathcal{E}(C^2) \) is the following:

\[ R_G : \mathcal{E}(C^2) \to \mathcal{E}(C^2), \quad R_G(F) = F \circ G. \]

Hence the \( R_G \) image, \( R_G(\mathcal{E}(C^2)) = \{ F \circ G \mid F \in \mathcal{E}(C^2) \} = \Orb_L(G) \). So: \( \emptyset \neq A \subseteq \mathcal{E}(C^2) = X \) is left invariant \( (T = \mathcal{E}(C^2) \text{ acts on the left}) \), if and only if \( A = \bigcup_{G \in A} R_G(\mathcal{E}(C^2)) \). This is because \( \bigcup_{G \in A} R_G(\mathcal{E}(C^2)) = \bigcup_{G \in A} \Orb_L(G) \).

**Conclusion:**

1. If the two dimensional Jacobian Conjecture is true, then \( \mathcal{E}(C^2) \) has exactly one left invariant subset namely \( \mathcal{E}(C^2) = \Aut(C^2) \).
2. If the two dimensional Jacobian Conjecture is false, then \( \mathcal{E}(C^2) \) has a large number of left invariant subsets. In fact, the set of all the invariant subsets is bijective with the set of subsets of all the (left) orbits of elements in \( \mathcal{E}(C^2) - \Aut(C^2) \) plus \( \mathcal{E}(C^2) \) itself.

**A proof on the conclusion:**

1. Suppose that \( \mathcal{E}(C^2) = \Aut(C^2) \). Then \( \forall G \in \Aut(C^2) \) we have \( R_G(\mathcal{E}(C^2)) = R_G(\Aut(C^2)) = \Aut(C^2) \circ G = \Aut(C^2) = \mathcal{E}(C^2) \). Thus if \( A \) is left invariant, then \( A = \bigcup_{G \in A} R_G(\mathcal{E}(C^2)) = \bigcup_{G \in A} \Aut(C^2) = \Aut(C^2) \).
2. Suppose that \( \Aut(C^2) \subset \mathcal{E}(C^2) \). Then \( \forall A_0 \subseteq \mathcal{E}(C^2) - \Aut(C^2) \) the set \( A = \bigcup_{G \in A_0} R_G(\mathcal{E}(C^2)) = \bigcup_{G \in A} R_G(\mathcal{E}(C^2)) \) is a left invariant subset of \( X = \mathcal{E}(C^2) \). \( \square \)

Before we continue with the implications of the paper on the topological dynamics of semigroup actions we return to the question of the possibility of the fractal representation:

\[ \mathcal{E}(C^2) = \bigcup_{F \in \mathcal{E}(C^2)} L_F(\mathcal{E}(C^2)), \]

as a partition. This means: \( \forall P_1, P_2 \in \mathcal{E}(C^2), \ P_1 \neq P_2 \iff L_{P_1}(\mathcal{E}(C^2)) \cap L_{P_2}(\mathcal{E}(C^2)) = \emptyset \). Let us consider the semigroup \( \mathcal{E}(C^2) \) and assume that it
has a topology \( \tau \) such that no point of \( \partial \text{et}(\mathbb{C}^2) \) belongs to \( \text{et}(\mathbb{C}^2) \). In other words we might think of the topological space \( (\text{et}(\mathbb{C}^2), \tau) \) as being a subspace of a larger topological semigroup within \( \mathbb{C}[[X, Y]]^2 \) so that each point of the boundary \( \partial \text{et}(\mathbb{C}^2) \) is a formal power series which is non-polynomial. This could be expressed in terms of algebraic degrees, or in terms of geometric degrees, namely: If \( H \in \partial \text{et}(\mathbb{C}^2) \) and if \( F_n \in \text{et}(\mathbb{C}^2) \) is a net such that \( F_n \to H \), then \( \deg F_n \to \infty \) or \( d F_n \to \infty \). This guarantees that \( H \not\in \mathbb{C}[X, Y]^2 \). Now let \( F \in \text{et}(\mathbb{C}^2) \) and consider the space \( L_F(\text{et}(\mathbb{C}^2)) \). We define a topology \( \tau_F \) on \( L_F(\text{et}(\mathbb{C}^2)) \) using the topology \( \tau \), as follows: \( V \in \tau_F \iff \exists U \in \tau \) such that \( V = F \circ U \). Is it the same as the induced topology \( \tau \cap L_F(\text{et}(\mathbb{C}^2)) \)? Let \( V \in \tau \cap L_F(\text{et}(\mathbb{C}^2)) \), then \( \exists U \in \tau \) such that \( V = U \cap L_F(\text{et}(\mathbb{C}^2)) \). Thus there is a subset \( U_1 \subseteq \text{et}(\mathbb{C}^2) \) such that \( V = U \cap L_F(\text{et}(\mathbb{C}^2)) = F \circ U_1 \) and the question is the following: Is it true that \( U_1 \in \tau \) or not? It is easier to tackle this question by using the family \( C = \tau^c \) of the closed subsets of \( \text{et}(\mathbb{C}^2) \). We define the family of closed sets \( C_F = \tau_F^c \) on \( L_F(\text{et}(\mathbb{C}^2)) \) using \( C \), as follows: \( K \in C_F \iff \exists L \in C \) such that \( K = F \circ L \). Now let’s investigate if this coincides with the induced family of closed sets \( C \cap L_F(\text{et}(\mathbb{C}^2)) \). Let \( K \in C \cap L_F(\text{et}(\mathbb{C}^2)) \), then \( \exists L \in C \) such that \( K = L \cap L_F(\text{et}(\mathbb{C}^2)) \). Thus there is a set \( L_1 \in \text{et}(\mathbb{C}^2) \) such that \( K = L \cap L_F(\text{et}(\mathbb{C}^2)) = F \circ L_1 \) and we ask if \( L_1 \in \tau \). Let \( G_n \in L_1 \) be a net such that \( G_n \to G \) in \( \tau \). Then \( F \circ G_n \to F \circ G \). Clearly \( F \circ G_n \in L \cap L_F(\text{et}(\mathbb{C}^2)) \). In particular \( F \circ G_n \in L \) which is closed \( (L \in \mathbb{C}) \), so \( F \circ G \in L \) and hence \( F \circ G \in L \cap L_F(\text{et}(\mathbb{C}^2)) = K = F \circ L_1 \). So \( F \circ G = F \circ G_1 \) for some \( G_1 \in L_1 \). We deduce that \( G_1 = G \in L_1 \) (by \( F \in \text{et}(\mathbb{C}^2) \) and by a uniformization argument) and we are done, for we proved that for any net \( G_n \in L_1 \) which converges to \( G \), \( G_n \to G \) in \( \tau \) we have \( G \in L_1 \). Thus \( L_1 \in C \). For the inverse claim: If \( L_1 \in C \), \( F \) étale and \( K = F \circ L_1 \) implies that \( K \) is closed \( K = L \cap L_F(\text{et}(\mathbb{C}^2)) \) is clear. Now that we know that the two ways to define the topology on \( \text{et}(\mathbb{C}^2) \) considered as a subspace of some larger topological semigroup are equivalent we indeed can consider the topology \( \tau \) on \( \text{et}(\mathbb{C}^2) \) for which \( H \in \partial \text{et}(\mathbb{C}^2) \), \( F_n \in \text{et}(\mathbb{C}^2) \) a net converging to \( H \), \( F_n \to H \), then say \( \deg F_n \to \infty \) (or \( d F_n \to \infty \)). Having that, we now prove the following:

**Proposition 12.1.** \( \forall F, G \in \text{et}(\mathbb{C}^2) \) we either have \( L_F(\text{et}(\mathbb{C}^2)) \cap L_G(\text{et}(\mathbb{C}^2)) = \emptyset \) or in the case that this intersection is non-empty then:

\[
\partial L_F(\text{et}(\mathbb{C}^2)) \subseteq \partial L_G(\text{et}(\mathbb{C}^2)) \vee \partial L_G(\text{et}(\mathbb{C}^2)) \subseteq \partial L_F(\text{et}(\mathbb{C}^2)).
\]

**Proof.**

If the proposition is false then we either have \( \partial L_F(\text{et}(\mathbb{C}^2)) \cap L_G(\text{et}(\mathbb{C}^2)) \neq \emptyset \) or \( L_F(\text{et}(\mathbb{C}^2)) \cap \partial L_G(\text{et}(\mathbb{C}^2)) \neq \emptyset \). However, as follows from the discussion we
had prior to Proposition 12.1 this implies a contradiction as follows: If, say $H \in \partial L_F(\text{et}(\mathbb{C}^2)) \cap L_G(\text{et}(\mathbb{C}^2)) \neq \emptyset$, then by $H \in \partial L_F(\text{et}(\mathbb{C}^2))$ we conclude that there is a net $M_n \in \text{et}(\mathbb{C}^2)$ such that $\lim F \circ M_n = H$, $\deg M_n \to \infty$ (or $d_{M_n} \to \infty$). So by $F \circ \lim M_n = H$ we conclude that there is a net $M_n \in \text{et}(\mathbb{C}^2)$ such that $\lim F \circ M_n = H$, $\deg M_n \to \infty$ (or $d_{M_n} \to \infty$), i.e. $H \notin \text{et}(\mathbb{C}^2)$ (In fact this follows immediately by our assumption that considering $(\text{et}(\mathbb{C}^2), \tau)$ as a subspace of, say $(\mathbb{C}[[X, Y]]^2, \tau)$, we have $\partial \text{et}(\mathbb{C}^2) \cap \text{et}(\mathbb{C}^2) = \emptyset$). But since $H \in L_G(\text{et}(\mathbb{C}^2))$, $\exists N \in \text{et}(\mathbb{C}^2)$ such that $H = G \circ N \in \text{et}(\mathbb{C}^2)$.

Next, Let $F, G \in \text{et}(\mathbb{C}^2)$ satisfy $L_F(\text{et}(\mathbb{C}^2)) \cap L_G(\text{et}(\mathbb{C}^2)) \neq \emptyset$. Then, there is some $H \in L_F(\text{et}(\mathbb{C}^2)) \cap L_G(\text{et}(\mathbb{C}^2))$. We denote by $E_{F,G}(H)$ the connectivity component of $L_F(\text{et}(\mathbb{C}^2)) \cap L_G(\text{et}(\mathbb{C}^2))$ which contains $H$. We recall that in fact $L_H(\text{et}(\mathbb{C}^2)) \subseteq E_{F,G}(H)$ because $L_H(\text{et}(\mathbb{C}^2))$ is connected. By Proposition 12.1 we have the following:

$$\partial L_H(\text{et}(\mathbb{C}^2)) \subseteq \partial L_F(\text{et}(\mathbb{C}^2)) \cap \partial L_G(\text{et}(\mathbb{C}^2)).$$

13 The abstract topological picture

Let $(Y, \tau)$ be a topological space and let $(X, \tau \cap X)$ be a path connected subspace of $Y$ which satisfies the following:

(1) $\partial X \subseteq Y$, $X \cap \partial X = \emptyset$.

Let $\{F_x \mid x \in X\}$ be a family of subsets of $X$ that are indexed by $X$ and that satisfy:

(2) Each $F_x$ is closed, path connected.

(3) $\forall x \in X$, $x \in F_x$.

Another way to think of this is that we have a mapping $\phi : X \to \{\text{closed path connected subsets of } X\}$, $\phi(x) = F_x$,

such that $\forall x \in X$, $x \in \phi(x)$.

Remark 13.1. $X = \bigcup_{x \in X} F_x = \bigcup_{x \in X} \phi(x)$ by (3).

We further assume that:

(4) $x \in \phi(y)$ \Rightarrow $\phi(x) \subseteq \phi(y)$.
(5) \( \forall x \in X, \partial F_x = \partial \phi(x) \subseteq \partial X. \)

**Remark 13.2.** Conditions (1) and (5) imply the following:

(6) \( \forall x, y \in X \) the situation: \( \partial F_x \cap F_y \neq \emptyset \lor F_x \cap \partial F_y \neq \emptyset \) is impossible.

**Proof.**
For if \( t \in \partial F_x \cap F_y \) then \( \partial F_x \not\subseteq \partial X \) because \( t \in \partial F_x \) and \( t \in F_y \subseteq X \), while by condition (1) \( X \cap \partial X = \emptyset \).

**Remark 13.3.** Conditions (2) and (6) imply that:

(7) \( \forall x, y \in X \) either \( F_x \cap F_y = \emptyset \) or if we assume that \( F_x \cap F_y \neq \emptyset \) then \( \partial F_x \subseteq \partial F_y \lor \partial F_y \subseteq \partial F_x \).

**Proof.**
For if we assume that (7) is false, then \( \exists x, y \in X \) such that \( F_x \cap F_y \neq \emptyset \) but \( \partial F_x \not\subseteq \partial F_y \), say. Let \( t \in F_x \cap F_y \) and \( u \in \partial F_x - F_y \). Consider an open path \( f : I \to F_x \) from \( t \) to \( u \) within \( F_x \) (by (2) \( F_x \) is path connected). Then \( \exists 0 < s < 1 \) such that \( f(s) \in \partial F_y \). So \( f(s) \in F_x \cap \partial F_y \) which contradicts (6). Similarly, the assumption ”\( F_x \cap F_y \neq \emptyset \) but \( \partial F_y \not\subseteq \partial F_x \)” contradicts (6). Hence the assertion.

14 **The semigroup** \( \text{et}(\mathbb{C}^2), \circ \)

This is a semigroup with a unit element. It contains the group \( \text{Aut}(\mathbb{C}^2), \circ \). Let us denote \( \text{et}_0(\mathbb{C}^2) = \text{et}(\mathbb{C}^2) - \text{Aut}(\mathbb{C}^2) \). This is the set of all the étale non-automorphisms of \( \mathbb{C}^2 \).

**Remark 14.1.** \( \text{et}_0(\mathbb{C}^2), \circ \) is a non-unital subsemigroup of \( \text{et}(\mathbb{C}^2), \circ \), if it is non-empty.

**Proof.**
Clearly, an equivalent description of \( \text{et}_0(\mathbb{C}^2) \) is the following \( \text{et}_0(\mathbb{C}^2) = \{ F \in \text{et}(\mathbb{C}^2) \mid d_F \geq 2 \} \), where \( d_F \) is the geometric degree of \( F \). Hence by the fact that the geometric degree is multiplicative we have:

\[
F, G \in \text{et}_0(\mathbb{C}^2) \Rightarrow d_F, d_G \geq 2 \Rightarrow d_{F \circ G} = d_F \cdot d_G \geq 4 \Rightarrow F \circ G \in \text{et}_0(\mathbb{C}^2).
\]

The fact that \( \text{et}_0(\mathbb{C}^2) \) is non-unital is clear because \( \text{id} \in \text{Aut}(\mathbb{C}^2) \).

We turned \( \text{et}(\mathbb{C}^2), \circ \) into a metric space as follows. We have chosen a characteristic set \( D \). It induced the metric \( \rho_D \) on \( \text{et}(\mathbb{C}^2) \). We noted that
if $\{D_n\}$ was an increasing sequence of characteristic sets for $\text{et}(\mathbb{C}^2)$, that
exhausted $\mathbb{C}^2$, i.e.: $D_n \subset D_{n+1}$, $\bigcup_{n=1}^{\infty} D_n = \mathbb{C}^2$, then for each $F \in \text{et}(\mathbb{C}^2)$,
the limit
$$\lim_{n \to \infty} \frac{\text{volume}(F(D_n))}{\text{volume}(D_n)} = \frac{1}{d_F},$$
where $d_F$ is the geometric degree of $F$ and where $\text{volume}(D_n)$ is the Euclidean volume of $D_n$ and
where $\text{volume}(F(D_n))$ is the Euclidean volume of $F(D_n)$. This implies that $\forall F, G \in \text{et}(\mathbb{C}^2)$ we have,
$$\lim_{n \to \infty} \frac{\text{volume}(F(D_n))}{\text{volume}(G(D_n))} = \lim_{n \to \infty} \left\{ \frac{\text{volume}(F(D_n))}{\text{volume}(D_n)} \cdot \frac{\text{volume}(D_n)}{\text{volume}(G(D_n))} \right\} =$$
$$= \lim_{n \to \infty} \left\{ \frac{\text{volume}(F(D_n))}{\text{volume}(D_n)} \right\} \cdot \lim_{n \to \infty} \left\{ \frac{\text{volume}(D_n)}{\text{volume}(G(D_n))} \right\} = \frac{d_G}{d_F}.$$  
We recall that $\rho_D(F, G) = \text{volume}(F(D)\Delta G(D))$. We would like to achieve
the following:  

(1) Prove that if $\rho_{D_n}(F, G)$ is small enough for a certain large enough $n = n(F, G)$, then $d_F = d_G$.  

(2) Quantify the above estimates in order to arrive, maybe, at a better metric on $\text{et}(\mathbb{C}^2)$, that will be a complete metric.

Let us first discuss (1): We start by giving a simple lower bound to the $\rho_D$ distance. Let $F, G \in \text{et}(\mathbb{C}^2)$. Then as $D \to \mathbb{C}^2$, we have the following two identities:

$$\text{volume}(F(D)) = \frac{1}{d_F} \cdot \text{volume}(D), \quad \text{volume}(G(D)) = \frac{1}{d_G} \cdot \text{volume}(D). \quad (\ast)$$

By the definition of the metric $\rho_D$ we have,

$$\rho_D(F, G) = \text{volume}(F(D)\Delta G(D)) =$$

$$= \text{volume}(F(D) - G(D)) + \text{volume}(G(D) - F(D)).$$

Let us assume that $d_G \leq d_F$. Then $\text{volume}(G(D)) \geq \text{volume}(F(D))$ (by equation $(\ast)$), hence $G(D) \subseteq F(D)$. How small can $\rho_D(F, G)$ be? Clearly
that happens when $F(D) \subseteq G(D)$. In this configuration we obtain,

$$\rho_D(F, G) = \text{volume}(F(D) - G(D)) + \text{volume}(G(D) - F(D)) =$$

$$= \text{volume}(\emptyset) + \text{volume}(G(D)) - \text{volume}(F(D)) =$$

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Proposition 14.2. Let $F, G \in \mathfrak{et}(\mathbb{C}^2)$ and let $D \subseteq \mathbb{C}^2$ be a characteristic domain for $\mathfrak{et}(\mathbb{C}^2)$, then,

(1) $\rho_D(F, G) \approx_{D \to \mathbb{C}^2} \frac{1}{d_F} \cdot \text{volume}(D \cap F^{-1}(F(D) - G(D))) + \frac{1}{d_G} \cdot \text{volume}(D \cap G^{-1}(G(D) - F(D)))$.

(2) $\rho_D(F, G) \geq_{D \to \mathbb{C}^2} \frac{1}{|d_F - d_G|} \cdot \text{volume}(D)$.

(3) If we consider a limit $D \to \mathbb{C}^2$, then

$$\lim_{D \to \mathbb{C}^2} \rho_D(F, G) = 0 \Leftrightarrow \lim_{D \to \mathbb{C}^2} \text{volume}(F(D) - G(D)) =$$

$$= \lim_{D \to \mathbb{C}^2} \text{volume}(G(D) - F(D)) = 0 \Leftrightarrow$$

$$\Leftrightarrow \lim_{D \to \mathbb{C}^2} \text{volume}(D \cap F^{-1}(F(D) - G(D))) =$$

$$= \lim_{D \to \mathbb{C}^2} \text{volume}(D \cap G^{-1}(G(D) - F(D))) = 0 \Rightarrow$$

$$\Rightarrow d_F = d_G.$$

Proof. 

(1) Follows by the identity $\rho_D(F, G) = \text{volume}(F(D) - G(D)) + \text{volume}(G(D) - F(D))$ and by the approximations, as $D \to \mathbb{C}^2$, that are given here,

$$\text{volume}(F(D) - G(D)) \approx_{D \to \mathbb{C}^2} \frac{1}{d_F} \cdot \text{volume}(D \cap F^{-1}(F(D) - G(D))),$$

$$\text{volume}(G(D) - F(D)) \approx_{D \to \mathbb{C}^2} \frac{1}{d_G} \cdot \text{volume}(D \cap G^{-1}(G(D) - F(D))).$$

(2) Was proved just before the statement of Proposition 14.2.

(3) Follows by the identity and the two approximate identities we used in the proof of part (1). Also the fact that $\lim_{D \to \mathbb{C}^2} \rho_D(F, G) = 0 \Rightarrow d_F = d_G$ follows by part (2). $\Box$

We would like to understand the geometric meaning of the $\rho_D$-convergence.
of a sequence $F_n \in \text{et}(\mathbb{C}^2)$. To make things more manageable we consider the characteristic domains $D$ for $\text{et}(\mathbb{C}^2)$ that we constructed in section 7. The main feature of these is the fractal-like shape of their boundaries. Namely $\partial D$ contains a dense countable and countable subset of special points in the strong topology. Those points originated in our first version construction as the centers of the $k$-stars, $k = 2, 3, 4, \ldots$ where each point $c_k \in \partial D$ was the center of a $k$-star, thus no two such points were the centers of stars of equal numbers of rays. Since $F \in \text{et}(\mathbb{C}^2)$, it preserves topological $k$-stars. The second version of our construction replaced the 1-dimensional $k$-stars by fattened $2k$-stars. Those $2k$-stars have a total volume which we now denote as follows, $\text{volume}(\cup \text{stars}) = V_s$ and which we assume to be a finite volume (as we can). If $F, G \in \text{et}(\mathbb{C}^2)$ are close enough in the sense of the $\rho_D$ metric, then by Proposition 14.2(3), $d_F = d_G$ and $\rho_D(F, G) = \text{volume}(F(D) \Delta G(D))$.

If $\partial F(D)$ and $\partial G(D)$ are far apart so that the corresponding topological 2$k$-stars are mostly disjoint, then because our mappings are locally volume preserving we obtain

$$\text{volume}(F(D) \Delta G(D)) \geq \frac{2}{d_F} \cdot V_s.$$  

Hence $\rho_D(F, G) \geq (2/d_F) \cdot V_s$, and so the mappings $F$ and $G$ can not be too close in the $\rho_D$-metric. This contradicts our assumption that they are $\rho_D$ close. We deduce that if $\rho_D(F, G)$ is small, then there is a countable dense subset $C \subseteq \partial D$ of points on $\partial D$ such that $\forall p \in C$, the Euclidean distances $d(F(p), G(p))$ are uniformly small. Hence if $\{F_n\} \subseteq \text{et}(\mathbb{C}^2)$ is a $\rho_D$-Cauchy sequence, then $\lim_{n \to \infty} F_n = F$ exists in $D$ and necessarily, in this case, $F \in \text{et}(\mathbb{C}^2)$ because the geometric degree is preserved all over $\mathbb{C}^2$ when $D \to \mathbb{C}^2$ (so the algebraic degrees are bounded).

**Theorem 14.3.** Let $D$ be a characteristic domain for $\text{et}(\mathbb{C}^2)$ of the type we have constructed in section 7 (i.e. $\partial D$ contains a countable dense subset of fattened 2$k$-stars, $k = 2, 3, 4, \ldots$). Let's assume that 0 is an interior point of $D$, that the total volume of the 2$k$-stars is positive (and finite), and we denote $D_N = N \cdot D$, $N = 1, 2, 3, \ldots$. Let $\{F_n\} \subset \text{et}(\mathbb{C}^2)$ be a $\rho_D$-Cauchy sequence. Then:

1. $\{D_N\}$ is an increasing sequence of characteristic domains for $\text{et}(\mathbb{C}^2)$, that exhaust $\mathbb{C}^2$. 

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(2) \( \{F_n\} \) is a \( \rho_{D_N} \)-Cauchy sequence for each \( N \in \mathbb{Z}^+ \).

(3) The limit: \( \lim_{n \to \infty} F_n \) exists uniformly on \( D_N \) for each \( N \in \mathbb{Z}^+ \).

(4) \( F = \lim_{n \to \infty} F_n \) exists uniformly on compact subsets of \( \mathbb{C}^2 \) and \( F \in \text{et}(\mathbb{C}^2) \).

(5) \( \exists \ n_0 \in \mathbb{Z}^+ \) such that \( \forall \ n \geq n_0, \ d_{F_n} = d_F \).

Remark 14.4. Thus we are dealing with complete metric spaces in the sense of parts (4) and (5) of Theorem 14.3. We are having here a sequence of metric spaces \( (\text{et}(\mathbb{C}^2), \rho_{D_N}), \ N = 1, 2, 3, \ldots \).

Remark 14.5. We elaborate more part (2) of theorem 14.3. Namely, we have a \( \rho_{D_1} \)-Cauchy sequence \( \{F_n\} \subseteq \text{et}(\mathbb{C}^2) \). Why is it also a \( \rho_{D_N} \)-Cauchy sequence? We know that \( \forall \epsilon > 0 \) there exists an \( n_1 \in \mathbb{Z}^+ \) such that for \( n, m > n_1 \) we have \( \rho_{D_1}(F_n, F_m) < \epsilon \). According to part (3) of Proposition 14.2 we may assume that \( d = d_{F_n} = d_{F_m} \) for \( n, m > n_1 \). Hence by

\[
\rho_{D_1}(F_n, F_m) = \text{volume}(F_n(D) \Delta F_m(D)),
\]

When we pass from \( D_1 \) to \( D_N \), the volume grows like a 4'th power, i.e. \( \text{volume}(N \cdot A) = N^4 \cdot \text{volume}(A) \) for a measurable \( A \), while the geometric degrees do not change. We recall that for a general smooth mapping \( G \) the volume element \( dV \) is transformed locally by a multiplication by the absolute value of the determinant of the Jacobian matrix of \( G \), i.e. \( |J_G(X, Y)|dV \). However, our mappings are locally volume preserving, \( |J_{F_n}| = |J_{F_m}| = 1 \) and so roughly speaking \( \rho_{D_N}(F_n, F_m) \approx N^4 \rho_{D_1}(F_n, F_m) \). Thus, indeed a \( \rho_{D_1} \)-Cauchy sequence is translated into a \( \rho_{D_N} \)-Cauchy sequence.

Remark 14.6. In Theorem 14.3 we are not dealing with a metric space. We are dealing with a sequence of metric spaces, namely \( (\text{et}(\mathbb{C}^2), \rho_{D_N}), \ N = 1, 2, 3, \ldots \).

We certainly do not have sequential compactness, i.e., it is not true that any sequence \( F_n \in \text{et}(\mathbb{C}^2) \) contains a convergent subsequence. This is not the case for the smaller sub-semigroup (in fact a group), \( (\text{Aut}(\mathbb{C}^2), \circ) \). For if we take say \( F_n(X, Y) = (X + Y + \ldots + Y^n, Y) \), then \( F_n \in \text{Aut}(\mathbb{C}^2) \) and clearly it contains no convergent subsequence (even within \( \text{et}(\mathbb{C}^2) \)). We recall that for a metric space to be compact a necessary and a sufficient condition is that it will be complete and totally bounded. We seem to have something close to completeness (in our setting of a sequence of metric spaces), thus we must be far away from total boundedness.
Another fact which should be remembered is that $\forall F \in \text{et}(\mathbb{C}^2)$, the image $F(\mathbb{C}^2)$ is cofinite in $\mathbb{C}^2$ and hence we can not make sense of $\rho_{\mathbb{C}^2}$, at least not in some straightforward manner.

The following conclusion follows from Theorem 14.3.

**Theorem 14.7.** Let $D$ be a characteristic domain for $\text{et}(\mathbb{C}^2)$ of the type that we have constructed in section 7. This means that $\partial D$ contains a countable dense subset of what we called fattened $2k$-stars, $k = 2, 3, 4, \ldots$. Then the metric space $(\text{et}(\mathbb{C}^2), \rho_D)$ is complete but not sequentially compact and in particular it is not a totally bounded space.

**A problem.**

What is the total boundedness breaking point of $\text{et}(\mathbb{C}^2)$? We use the following standard,

**Definition 14.8.** A metric space $M$ with a metric $d$ is said to be totally bounded if, given any positive number $r$, $M$ is the union of finitely many sets of $d$-diameter less than $r$.

**Remark 14.9.** If a metric space $M$ with a metric $d$ is bounded, then there exists a positive number $r$, such that $M$ is the union of finitely many sets of $d$-diameter less than $r$.

**Proof.**

Let the $d$-diameter of $M$ be $s$ ($M$ is $d$-bounded), and let $r = 2s$. Then $M$ is the union of finitely many sets of $d$-diameter less than $r$, namely just one set $M$. □

**Definition 14.10.** Let $M$ be a metric space with a metric $d$. Assume that $M$ is $d$-bounded but is not a totally bounded space. The total boundedness breaking point of $(M, d)$ is denoted by $t_b(M, d)$ and defined by the following:

$$t_b(M, d) = \inf \{ r > 0 \mid M \text{ is the union of finitely many sets of } d-\text{diameter less than } r \}.$$ 

**Remark 14.11.** Clearly, if the $d$-diameter of $M$ is $D$, then $0 < t_b(M, d) \leq D$. For a totally bounded metric space $(M, d)$ we have, $t_d(M, d) = 0$.

The problem we stated above is the following: Let $D$ be a characteristic domain for $\text{et}(\mathbb{C}^2)$. Compute $t_b(\text{et}(\mathbb{C}^2), \rho_D)$.

**Remark 14.12.** $0 < t_b(\text{et}(\mathbb{C}^2), \rho_D) \leq \text{volume}(D)$.

Before we continue, we would like to make an observation that is crucial for the fractal representation of $\text{et}(\mathbb{C}^2)$ as the union over the primes of the left translation images of $\text{et}(\mathbb{C}^2)$. 52
**Theorem 14.13.** Let $X$ be a topological space. Let $A, B \subseteq X$ satisfy the following assumptions:

1. $\overline{A}, \overline{B} \subseteq X^o$.
2. $\partial A \subseteq \partial B \lor \partial B \subseteq \partial A$.
3. Both $A$ and $B$ are path connected subspaces of $X$ and $\partial A, \partial B$ are path accessible from within $A$ and $B$ respectively.
4. $A \cap \partial A = B \cap \partial B = \emptyset$.

Then, $A \subseteq B \lor A \cap B = \emptyset \lor B \subseteq A$.

**Proof.**

Let us assume that $\sim (A \subseteq B) \land \sim (A \cap B = \emptyset)$. We should prove that necessarily $B \subseteq A$. Thus we assume that $\exists a \in A - B, \exists b \in A \cap B$ and should prove that $B \subseteq A$: By (3) $A$ is path connected. Hence there exists a path in $A$ connecting $a$ to $b$. Let that path be $\gamma : [0, 1] \rightarrow A$. Then $\gamma$ is a continuous mapping, $\gamma(0) = a, \gamma(1) = b$ and $\forall 0 < t < 1, \gamma(t) \in A$. Hence there is a $t_0, 0 < t_0 < 1$ such that $\gamma(t_0) \in \partial(B)$. Thus $\gamma(t_0) \in A \cap \partial B$. By (4), $A \cap \partial A = \emptyset$ and hence $\gamma(t_0) \notin \partial A$. Then by (2) we conclude that:

$$\partial A \subset \partial B.$$  \hfill (14.2)

We claim that there is no point $c \in B - A$, because by the same argument (with a path in $B$ connecting $c$ to $b$) we would conclude that (by (2)), $\partial B \subset \partial A$. This contradicts equation (14.2) above and shows that there is no $c \in B - A$. Thus $B \subseteq A$. \hfill $\square$

Using Proposition 12.1 we obtain condition (2) in Theorem 14.13. We already know that the $L_p(\text{et}(\mathbb{C}^2))$’s are path connected, thus getting condition (3) in Theorem 14.13. Conditions (1) and (4) are clear with, say, $X = \mathbb{C}[[X,Y]]^2$, $A = L_{p_1}(\text{et}(\mathbb{C}^2))$ and $B = L_{p_2}(\text{et}(\mathbb{C}^2))$. Hence we conclude from Theorem 14.13 the desired fractal representation of $\text{et}(\mathbb{C}^2)$. In other words we have the following,

**Theorem 14.14.** The fractal representation,

$$\text{et}(\mathbb{C}^2) = \text{Aut}(\mathbb{C}^2) \cup \bigcup_{F \in \text{et}_p(\mathbb{C}^2)} L_F(\text{et}(\mathbb{C}^2)).$$
is valid and it satisfies the disjointness assumption:

$$\forall F, G \in \text{et}_p(C^2), \ F \neq G \Rightarrow L_F(\text{et}(C^2)) \cap L_G(\text{et}(C^2)) = \emptyset.$$ 

As a direct consequence this proves the celebrated:

**Theorem 14.15. (The two dimensional Jacobian Conjecture)** If $F(X, Y) = (P(X, Y), Q(X, Y)) \in C[X, Y]^2$ satisfies the Jacobian condition:

$$\frac{\partial P}{\partial X} \cdot \frac{\partial Q}{\partial Y} - \frac{\partial P}{\partial Y} \cdot \frac{\partial Q}{\partial X} = c \in C^\times,$$

then $F \in \text{Aut}(C^2)$.

We end with a few remarks.

**Remark 14.16.** We point out that in this paper the fractal structure on a set was defined in an intrinsic manner. We recall the standard definition of the exterior $\alpha$-dimensional Hausdorff measure of any subset $E$ of $\mathbb{R}^d$:

$$m^*_\alpha(E) = \lim_{\delta \to 0^+} \inf \left\{ \sum_k (\text{diam } F_k)^\alpha \mid E \subseteq \bigcup_{k=1}^{\infty} F_k, \ \text{diam } F_k \leq \delta, \ \forall k \right\},$$

where $\text{diam } S$ denotes the diameter of the set $S$, that is, $\text{diam } S = \sup \{|x-y| \mid x, y \in S\}$. In other words, for each $\delta > 0$ we consider covers of $E$ by countable families of arbitrary sets in $\mathbb{R}^d$ with diameter less than (or equals to) $\delta$, and take the infimum of the sum $\sum_k (\text{diam } F_k)^\alpha$. We then define $m^*_\alpha(E)$ as the limit of these infima as $\delta$ tends to 0. We note that the quantity

$$H^\delta_\alpha(E) = \inf \left\{ \sum_k (\text{diam } F_k)^\alpha \mid E \subseteq \bigcup_{k=1}^{\infty} F_k, \ \text{diam } F_k \leq \delta, \ \forall k \right\}$$

is increasing as $\delta$ decreases, so that the limit

$$m^*_\alpha(E) = \lim_{\delta \to 0^+} H^\delta_\alpha(E)$$

exists, although $m^*_\alpha(E)$ could be infinite. We note that in particular, one has $H^\delta_\alpha(E) \leq m^*_\alpha(E)$ for all $\delta > 0$. When defining the exterior measure $m^*_\alpha(E)$ it is important to require that the coverings be of sets of arbitrary small diameters. This is in thrust of the definition $m^*_\alpha(E) = \lim_{\delta \to 0^+} H^\delta_\alpha(E)$. This requirement, which is not relevant for the Lebesgue measure, is needed to ensure the basic additive feature, namely:
If \( d(E_1, E_2) > 0 \), then \( m^*_\alpha(E_1 \cup E_2) = m^*_\alpha(E_1) + m^*_\alpha(E_2) \).

Do we really need the ambient space \( \mathbb{R}^d \) in this definition of the exterior \( \alpha \)-dimensional Hausdorff measure? The answer is no. We can define this notion intrinsically, within \( E \).

**Definition 14.17.** The intrinsic exterior \( \alpha \)-dimensional Hausdorff measure of \( E \) is defined by the following equation:

\[
m^*_\alpha \text{int}(E) = \lim_{\delta \to 0^+} H^\delta \text{int}_\alpha(E),
\]

where this time

\[
H^\delta \text{int}_\alpha(E) = \inf \left\{ \sum_k (\text{diam } G_k)^\alpha \mid E = \bigcup_{k=1}^{\infty} G_k, \text{diam } G_k \leq \delta, \forall k \right\}.
\]

Thus, in the intrinsic definition we consider countable coverings of \( E \) by sets \( \{G_k\}_{k=1}^{\infty} \) which are subsets of \( E \), that is, for all \( k, G_k \subseteq E \).

Now the very easy

**Proposition** \( m^*_\alpha(E) = m^*_\alpha \text{int}(E) \).

**Proof.**

In fact we will prove that \( \forall \delta > 0, H^\delta_\alpha(E) = H^\delta_\alpha \text{int}(E) \):

1. \( H^\delta_\alpha \text{int}(E) \leq H^\delta_\alpha(E) \).

For let \( E \subseteq \bigcup_k F_k \), where \( \forall k, F_k \subseteq \mathbb{R}^d \) and \( \text{diam } F_k \leq \delta \). Then if we define \( G_k = F_k \cap E \), we have \( \forall k, G_k \subseteq E \), \( \text{diam } G_k \leq \text{diam } F_k \leq \delta \) and \( E = \bigcup_k G_k \) simply because it equals to \( \bigcup_k (F_k \cap E) = (\bigcup_k F_k) \cap E \).

2. \( H^\delta_\alpha(E) \leq H^\delta_\alpha \text{int}(E) \).

For in the infimum that defines \( H^\delta_\alpha \text{int}(E) \) we take a sub-family \( \{G_k\} \) of the family \( \{F_k\} \) of coverings that are used in the infimum that defines \( H^\delta_\alpha(E) \). Hence this last infimum is not larger than the first infimum. \( \Box \)

**Remark 14.18.** What major properties of the mappings in \( \text{et}(\mathbb{C}^2) \) were used in the approach that was presented in this paper, of using the fractal representation of \( \text{et}(\mathbb{C}^2) \)? We strongly used the local homeomorphism property (in constructing the metric \( \rho_D \) and using the local preservation of the volume). We used the finiteness of the geometric degree, \( d_F \). We used the holomorphic rigidity of our mappings (the permanence principle). Thus in trying to use this idea and technique for other families of mappings (such as local preserving of volume holomorphic mappings) we encounter the difficulty
that a straight forward approach is not working. For the finiteness of the
geometric degree of a holomorphic \( \mathbb{C}^2 \to \mathbb{C}^2 \) mapping probably implies that
the mapping is polynomial and thus we are back in the \( \text{et}(\mathbb{C}^2) \) setting. At
least in complex dimension 1 this is the case. For by the theory of maximal
domains, \[13\], we have the following,

**Theorem.** Let \( f(z) \) be an entire function of one complex variable \( z \). Then
\( df < \infty \) if and only if \( f(z) \in \mathbb{C}[z] \).

On the other hand a possible route for interesting extensions of this the-
ory might be working outside the parabolic simply connected cases of \( \mathbb{C} \),
\( \mathbb{C}^2 \). This might lead to questions such as the following:
Let \( \Omega \subseteq \mathbb{C}^N \) be a domain (just an open connected subset of \( \mathbb{C}^N \)). Let \( \text{Aut}(\Omega) \) be
the group of all the holomorphic automorphisms of \( \Omega \). Let \( \text{et}(\Omega) \) be the
semigroup of holomorphic local homeomorphisms, locally volume preserving
(or maybe drop that?) which are of finite geometric degrees. When is it
true that \( \text{et}(\Omega) = \text{Aut}(\Omega) \)? That is for which domains \( \Omega \) this is true?

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