Convergence of Cubic Spline Super Fractal Interpolation Functions

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Abstract

In the present work, the notion of Cubic Spline Super Fractal Interpolation Function (SFIF) is introduced to simulate an object that depicts one structure embedded into another and its approximation properties are investigated. It is shown that, for an equidistant partition points of \([x_0, x_N]\), the interpolating Cubic Spline \(g_\sigma(x) \equiv g_\sigma^{(0)}(x)\) and their derivatives \(g_\sigma^{(j)}(x)\) converge respectively to the data generating function \(y(x) \equiv y^{(0)}(x)\) and its derivatives \(y^{(j)}(x)\) at the rate of \(h^{2-j+\epsilon}(0 < \epsilon < 1)\), \(j = 0, 1, 2\), as the norm \(h\) of the partition of \([x_0, x_N]\) approaches zero. The convergence results for Cubic Spline SFIF found here show that any desired accuracy can be achieved in the approximation of a regular data generating function and its derivatives by a Cubic Spline SFIF and its corresponding derivatives.

Key Words: Fractal Interpolation Function, Spline, Super Fractals, Convergence

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1 Introduction

Barnsley [1] introduced Fractal Interpolation Function (FIF) using the theory of Iterated Function System (IFS). Later, Barnsley [2, 3, 4] introduced the class of super fractal sets constructed by using multiple IFSs to simulate such objects. Massopust [5] constructed super fractal functions and V-variable fractal functions by joining pieces of fractal functions which are attractors of finite family of IFSs.

FIF, constructed as attractor of a single Iterated Function System (IFS) by virtue of self-similarity alone, is not rich enough to describe an object found in nature or output of a certain scientific experiment. The objects of nature generally reveal one or more structures embedded in to another. Similarly, the outcomes of several scientific experiments exhibit randomness and variation at various stages. Therefore, more than one IFSs are needed to model such objects. A solution of fractal interpolation problem based on several IFS to model such objects is introduced in [6] by introducing the notion of Super Fractal Interpolation Function (SFIF). The construction of SFIF use more than one IFS wherein, at each level of iteration, an IFS is chosen from a pool of several IFS. This approach ensured desired randomness and variability needed to facilitate better geometrical modeling of objects found in nature and results of certain scientific experiments.

Spline functions, introduced by Schoenberg [7], find vast applications in areas like data fitting [8], computer aided geometric design [9, 10], numerical solutions of differential equations [11], etc. A piecewise polynomial function \( \vartheta \) on an interval \([x_0, x_N]\), which is composed of subintervals \([x_{i-1}, x_i] \), \( i = 1, 2, \ldots, N \), is called a Spline of order \( n \) if (i) \( \vartheta(x) \) is a continuous polynomial of degree atmost \( n-1 \) in each subintervals \([x_{i-1}, x_i] \), \( i = 1, 2, \ldots, N \), and (ii) the derivatives \( \vartheta^{(m)} \), \( 0 \leq m \leq n-2 \), are continuous on \([x_0, x_N]\). A Cubic Spline is a Spline of degree 3. For a data set \( \{x_i\} \) of \( n+1 \) points, a Cubic Spline is constructed with \( n \) piecewise cubic polynomials between the data points. If \( \vartheta \) represents a Cubic Spline approximating the function \( y \in C^4[x_0, x_N] \), then \( \vartheta \) is twice continuously differentiable and \( \vartheta(x_i) = y(x_i) \).
Navascues and Sebastian [12] considered a Cubic Spline FIF as a generalization of classical Spline and obtained estimates on error in approximation of the data generating function by a Cubic Spline FIF. However, their Cubic Spline FIF was constructed using a single IFS and so it is not equipped enough to simulate an object that depicts one structure embedded into another. To approximate an object by a spline-like FIF, the concept of Cubic Spline SFIF is introduced in the present work and its approximation properties are investigated. The convergence results for Cubic Spline SFIF found here show that any desired accuracy can be achieved in the approximation of a regular data generating function and its derivatives by a Cubic Spline SFIF and its corresponding derivatives.

The organization of the present chapter is as follows: In Section 2, a brief review on the construction of Super Fractal Interpolation Function for a given finite set of data is given. The notion of Cubic Spline SFIF is introduced in Section 3. It is proved in this section that, for an equidistant partition points of \([x_0, x_N]\), the interpolating Cubic Spline SFIF \(g_\sigma(x) \equiv g_\sigma^{(0)}(x)\) and their derivatives \(g_\sigma^{(j)}(x)\) converge respectively to the data generating function \(y(x) \equiv y^{(0)}(x)\) and its derivatives \(y^{(j)}(x)\) at the rate of \(h^{2-j+\epsilon}(0 < \epsilon < 1), j = 0, 1, 2,\) as the norm \(h\) of the partition of \([x_0, x_N]\) approaches zero.

2 Construction of SFIF

In this section, a brief introduction on the construction of Super Fractal Interpolation Function (SFIF) is given.

Let \(S_0 = \{(x_i, y_i) \in \mathbb{R}^2 : i = 0, 1, \ldots N\}\) be the set of given interpolation data. The contractive homeomorphisms \(L_n : I \rightarrow I_n\) for \(n = 1, \ldots, N\), are defined by

\[
L_n(x) = a_n x + b_n
\]

where, \(a_n = \frac{x_n-x_{n-1}}{x_N-x_0}\) and \(b_n = \frac{x_Nx_{n-1}-x_0x_n}{x_N-x_0}\). For \(k = 1, 2, \ldots, M, M > 1\) and \(n = 1, 2, \ldots, N\),
let the functions $G_{n,k} : I \times \mathbb{R} \to \mathbb{R}$ defined by

$$G_{n,k}(x, y) = \gamma_{n,k} y + q_{n,k}(x)$$  \hspace{1cm} (2.2)

satisfy the join-up conditions

$$G_{n,k}(x_0, y_0) = y_{n-1} \quad \text{and} \quad G_{n,k}(x_N, y_N) = y_n. \quad (2.3)$$

Here, $\gamma_{n,k}$ are free parameters chosen such that $|\gamma_{n,k}| < 1$ and $\gamma_{n,k} \neq \gamma_{n,l}$ for $k \neq l$.

The Super Iterated Function System (SIFS) that is needed to construct SFIF corresponding to the set of given interpolation data $S_0$ is defined as the pool of IFS

$$\left\{ \{\mathbb{R}^2; \omega_{n,k} : n = 1, 2, \ldots, N\}, \quad k = 1, 2, \ldots, M \right\} \quad (2.4)$$

where, the functions $\omega_{n,k} : I \times \mathbb{R} \to I \times \mathbb{R}$ are given by

$$\omega_{n,k}(x, y) = (L_n(x), G_{n,k}(x, y)) \text{ for all } (x, y) \in \mathbb{R}^2 \quad (2.5)$$

By (2.3), it is observed that $\omega_{n,k}$ are continuous functions.

To introduce a SFIF associated with SIFS (2.4), let $\{W_k : H(\mathbb{R}^2) \to H(\mathbb{R}^2), \quad k = 1, 2, \ldots, M\}$, be a collection of continuous functions defined by

$$W_k(G) = \bigcup_{n=1}^{N} \omega_{n,k}(G), \text{ where } \omega_{n,k}(G) = \{\omega_{n,k}(x, y) \text{ for all } (x, y) \in G\}. \quad (2.6)$$

Then, $\{H(\mathbb{R}^2); \quad W_1, \ldots, W_M\}$ is a hyperbolic IFS, since $h(W_k(A), W_k(B)) \leq \max_{1 \leq n \leq N} \gamma_{n,k} h(A, B)$, where $h$ is Hausdorff metric on $H(\mathbb{R}^2)$. Hence, by Banach fixed point theorem, there exists an attractor $\mathcal{A} \in H(H(\mathbb{R}^2))$. 


Let \( \Lambda \) be the code space on \( M \) natural numbers \( 1, 2, \ldots, M \). In the construction of SFIF, for a \( \sigma = \sigma_1\sigma_2\ldots \in \Lambda \), let the action of SIFS (2.4) at the iteration level \( j \) be defined by \( S_j = W_{\sigma_j}(S_{j-1}) \), where \( S_0 \) is the set of given interpolation data. For a fixed \( \sigma \in \Lambda \), define

\[
G_\sigma \equiv \lim_{k \to \infty} W_{\sigma_k} \circ \ldots \circ W_{\sigma_1}(S_0) = \lim_{k \to \infty} S_k. \tag{2.7}
\]

The following proposition is instrumental for precise definition of a SFIF:

**Proposition 2.1** Let \( G_\sigma \) be defined by (2.7). Then, \( G_\sigma \) is the attractor of SIFS (2.4) for \( \sigma = \sigma_1\sigma_2\ldots\sigma_k\ldots \in \Lambda \) and is graph of a continuous function \( g_\sigma : I \to \mathbb{R} \) such that \( g_\sigma(x_n) = y_n \) for all \( n = 0, 1, \ldots, N \).

Super Fractal Interpolation Function (SFIF) is defined using Proposition 2.1 as follows:

**Definition 2.1** The **Super Fractal Interpolation Function (SFIF)** for the given interpolation data \( \{(x_i, y_i) : i = 0, 1, \ldots, N\} \) is defined as the function \( g_\sigma \) whose graph \( G_\sigma \) is the attractor of SIFS (2.4).

### 3 Cubic Spline SFIF

The convergence of Cubic Spline SFIF is investigated here using the conditions of differentiability found in [6]. Throughout in this section, the interpolation data \( \{(x_n, y_n) : n = 0, 1, \ldots, N\} \) is assumed to be such that \( x_n - x_{n-1} = h \), for \( n = 1, 2, \ldots, N \) and \( x_0 = 0 \). Also, throughout in the sequel, the SIFS (2.4) is chosen such that \( \gamma_{n,k} = \gamma_k, n = 1, 2, \ldots, N \), for some \( \gamma_k, 0 < \gamma_k < 1, k \in \{1, 2, \ldots, M\} \).

**Definition 3.1** A SFIF \( g_\sigma \), associated with SIFS (2.4), is called **Cubic Spline SFIF** if \( q_{n,k}, \) given in (2.2), are cubic polynomials for \( n = 1, 2, \ldots, N \) and \( k = 1, 2, \ldots, M \).
It is observed that, if \( q_{n,k}(x) = q_{n,k,3}x^3 + q_{n,k,2}x^2 + q_{n,k,1}x + q_{n,k,0} \), the coefficients \( q_{n,k,i}, i = 0,1,2,3 \), depend upon \( \gamma_k \) due to (2.3), necessitating in the sequel, the use of notation \( q_n(\gamma_k, x) \) in place of \( q_{n,k}(x) \). Throughout in this section, it is assumed that, for some \( A_0 \geq 0 \), the polynomials \( q_n \) satisfy

\[
\left| q_n(\gamma_k, x) - q_n(\gamma_l, x) \right| \leq A_0, \quad (3.1)
\]

for \( n = 1,2,\ldots,N \), \( k,l = 1,2,\ldots,M \) and \( x \in [x_0,x_N] \).

Let \( \gamma_{k_0} \) be such that \( |\gamma_{k_0}| < 1 \), \( \beta_{k_0} = \max_{1 \leq l \leq M} |\gamma_l - \gamma_{k_0}| < 1 \) and \( \varsigma = \varsigma_1 \varsigma_2 \ldots \varsigma_j \ldots \in \Lambda \) be such that \( \varsigma_j = k_0 \) for all \( j \in \mathbb{N} \). Consider the family of continuous functions

\[
\mathcal{G} = \{ f : I \to \mathbb{R} \text{ such that } f \text{ is continuous, } f(x_0) = y_0 \text{ and } f(x_N) = y_N \} \quad (3.2)
\]

with metric \( d_G(f,g) = \max_{x \in I} |f(x) - g(x)| \). For Read-Bajraktarevic operator \( T : \Lambda \times \mathcal{G} \to \mathcal{G} \) defined by

\[
T(\sigma, g)(x) = \lim_{k \to \infty} \left\{ G_{i_k, \sigma_k} \left( L_{i_k}^{-1}(x), G_{i_{k-1}, \sigma_{k-1}} \left( L_{i_{k-1}}^{-1} \circ L_{i_k}^{-1}(x), G_{i_{k-2}, \sigma_{k-2}} \left( \ldots \right) \right), G_{i_1, \sigma_1} \left( L_{i_1}^{-1} \circ \ldots \circ L_{i_k}^{-1}(x), g(L_{i_1}^{-1} \circ \ldots \circ L_{i_k}^{-1}(x)) \ldots \right) \right) \right\} \quad (3.3)
\]

where, \( \Lambda \) is the code space on \( M \) natural numbers \( 1,2,\ldots,M \) and \( \mathcal{G} \) is given by (3.2), the following proposition gives a bound on \( \|T(\sigma, g)(x) - T(\varsigma, g)(x)\| \) for \( \sigma, \varsigma \in \Lambda \):

**Proposition 3.1** Let \( g \in \mathcal{G} \) and inequality (3.1) be satisfied. Then,

\[
\|T(\sigma, g) - T(\varsigma, g)\|_\infty \leq \beta_{k_0} \left\{ \|g\|_\infty + \frac{A_0}{1 - \gamma_*} + \frac{B_0}{1 - \beta_{k_0}} \right\} \quad (3.4)
\]

where \( \sigma, \varsigma \in \Lambda \), \( \max_{n=1,2,\ldots,N} |q_n(\gamma_{k_0}, x)| = B_0 \), \( \gamma_* = \max_{1 \leq l \leq M} |\gamma_l| < 1 \) and \( \beta_{k_0} = \max_{1 \leq l \leq M} |\gamma_l - \gamma_{k_0}| < 1 \).
\textbf{Proof} By the definition of \( T(\sigma, g)(x) \) (c.f. (3.3)),

\[
|T(\sigma, g)(x) - T(\varsigma, g)(x)| \\
\leq \lim_{k \to \infty} \left\{ \prod_{j=1}^{k} |\gamma_{\sigma_j} - \gamma_{k_0}| \|g(L_{i_1}^{-1} \circ \cdots \circ L_{i_k}^{-1}(x))| \\
+ \sum_{p=1}^{k} \left( \prod_{j=p+1}^{k} |\gamma_{\sigma_j}| \right) |q_p(\gamma_{\sigma_p}, L_{i_p}^{-1} \circ \cdots \circ L_{i_k}^{-1}(x)) - q_p(\gamma_{k_0}, L_{i_p}^{-1} \circ \cdots \circ L_{i_k}^{-1}(x))| \\
+ \sum_{p=1}^{k-1} \left( \prod_{j=p+1}^{k} |\gamma_{\sigma_j} - \gamma_{k_0}| \right) \|q_p(\gamma_{\sigma_p}, L_{i_p}^{-1} \circ \cdots \circ L_{i_k}^{-1}(x))\| \right\}. \tag{3.5}
\]

Since \( q_n(\gamma_{k_0}, x) \) are cubic polynomials defined on compact set \([x_0, x_N]\), there exists a \( B_0 > 0 \) such that \( \max_{x \in [x_0, x_N]} |q_n(\gamma_{k_0}, x)| = B_0 \). Therefore, by (3.1) and (3.5), it follows that

\[
|T(\sigma, g)(x) - T(\varsigma, g)(x)| \leq \lim_{k \to \infty} \left\{ \left( \prod_{j=1}^{k} |\gamma_{\sigma_j} - \gamma_{k_0}| \right) \|g\|_{\infty} \\
+ \sum_{p=1}^{k} \left( \prod_{j=p+1}^{k} |\gamma_{\sigma_j}| \right) |\gamma_{\sigma_p} - \gamma_{k_0}| A_0 + \sum_{p=1}^{k-1} \left( \prod_{j=p+1}^{k-1} |\gamma_{\sigma_j} - \gamma_{k_0}| \right) B_0 \right\}.
\]

Since \( \beta_{k_0} = \max_{1 \leq l \leq M} |\gamma_l - \gamma_{k_0}| < 1 \) and \( \gamma_* = \max_{1 \leq l \leq M} |\gamma_l| < 1 \), (3.4) follows from the above inequality.

The following proposition gives a bound on \( \|g_\sigma - g_\varsigma\|, \sigma, \varsigma \in \Lambda \):

\textbf{Proposition 3.2} Let \( g_\sigma, g_\varsigma \in \mathcal{G} \) and (3.1) be satisfied. Then,

\[
\|g_\sigma - g_\varsigma\|_{\infty} \leq \frac{\beta_{k_0}}{1 - \gamma_*} \left( \|g_\varsigma\|_{\infty} + \frac{A_0}{1 - \gamma_*} + \frac{B_0}{1 - \beta_{k_0}} \right) \tag{3.6}
\]

where \( \gamma_*, \beta_{k_0} \) and \( B_0 \) are as in Proposition 3.1.
Proof Since (3.1) is satisfied, \( g_\sigma = T(\sigma, g_\sigma) \) and \( g_\varsigma = T(\varsigma, g_\varsigma) \) for Read-Bajraktarevic operator \( T \), defined by (3.3),
\[
\|g_\sigma - g_\varsigma\|_\infty \leq \gamma_* \|g_\sigma - g_\varsigma\|_\infty + \beta_{k_0} \left( \|g_\varsigma\|_\infty + \frac{A_0}{1 - \gamma_*} + \frac{B_0}{1 - \beta_{k_0}} \right).
\]
The inequality (3.6) now follows from the above inequality.

\[\square\]

Remark 3.1 For \( \gamma_{k_0} = 0 \), \( \gamma_* = \beta_{k_0} \). Thus, inequality (3.6) for \( \gamma_{k_0} = 0 \) implies
\[
\|g_\sigma - g_\varsigma\|_\infty \leq \frac{\gamma_*}{1 - \gamma_*} \left( \|g_\varsigma\|_\infty + \frac{A_0 + B_0}{1 - \gamma_*} \right).
\]

By Hall and Meyer’s theorem [13], \( \|g_\varsigma\|_\infty \leq K_0 h^4 + J_0 \). Consequently, Proposition 3.2 for \( \gamma_{k_0} = 0 \) gives,
\[
\|g_\sigma - g_\varsigma\|_\infty \leq \frac{\gamma_*}{1 - \gamma_*} \left( K_0 h^4 + J_0 + \frac{A_0 + B_0}{1 - \gamma_*} \right). \tag{3.7}
\]

Using inequality (3.7), the order of approximation of data generating function \( y(x) \) by SFIF \( g_\sigma \) is given by the following theorem:

Theorem 3.1 Let \( y(x) \in C^4[x_0, x_N] \) be a data generating function and \( g_\sigma \in \mathcal{G} \) be a SFIF associated with SIFS (2.4) such that \( \gamma_*(h) = \max_{i=1,2,\ldots,N} |\gamma_i| \leq \frac{h^{2+s}}{|I|^{2+s}} \), for some \( s \), \( 0 < s < 1 \), where \( h = |x_i - x_{i-1}|, \ i = 1, 2, \ldots, N \) and \( |I| = |x_N - x_0| \). Then, for \( 0 < \epsilon < s \),
\[
\|y - g_\sigma\|_\infty = o(h^{2+\epsilon}). \tag{3.8}
\]

provided (3.1) holds.
Proof Since (3.1) holds, an application of inequality (3.7) gives,

\[
\|y - g_\sigma\|_\infty \leq \|y - g_\varsigma\| + \|g_\varsigma - g_\sigma\|_\infty \\
\leq K_0 h^4 + \frac{\gamma_*(h)}{1 - \gamma_*(h)} \left( K_0 h^4 + J_0 + \frac{A_0 + B_0}{1 - \gamma_*(h)} \right) \\
\leq \frac{1}{1 - \gamma_*(h)} \left( K_0 h^4 + \gamma_*(h) J_0 + \frac{\gamma_*(h) (A_0 + B_0)}{1 - \gamma_*(h)} \right). \tag{3.9}
\]

Using \(|\gamma_*(h)| \leq \frac{h^{2+s}}{2+s}\), the inequality (3.9) implies

\[
\|y - g_\sigma\|_\infty \leq \frac{|I|^{(2+s)}}{|I|^{(2+s)} - h^{(2+s)}} \left\{ K_0 h^4 + \left( \frac{J_0}{|I|^{(2+s)}} \right) \right\} + \left( \frac{(A_0 + B_0) h^{(2+s)}}{|I|^{(2+s)} - h^{(2+s)}} \right). \tag{3.10}
\]

The order of approximation error given by (3.8) follows from the above inequality.

Remark 3.2 It follows from inequality (3.10) that, in fact, \(\|y - g_\sigma\|_\infty = O(h^{2+s})\).

Remark 3.3 If \(M = 1\) in SIFS (2.4), then \(g_\sigma\) reduces to a FIF. The convergence result for a Cubic Spline FIF [12] follows as a particular case of Theorem 3.1.

The order in approximation of derivatives of data generating function by corresponding derivatives of SFIF is now investigated. It is known [6] that \(g_\sigma^{(1)}(x)\) and \(g_\sigma^{(2)}(x)\) are SFIFs associated with SIFSs \(\left\{ \{\mathbb{R}^2; \omega_{i,k,j}(x, y) = (L_i(x), G_{i,k,j}(x, y)) : i = 1, 2, \ldots, N\}; k = 1, 2, \ldots, M \right\}\) for \(j = 1, 2\) respectively. Here, the functions \(G_{i,k,1}(x, y)\) and \(G_{i,k,2}(x, y)\) are given by

\[
G_{i,k,1}(x, y) = N \gamma_{k} y + N q_i^{(1)}(\gamma_k, x) \quad \text{and} \quad G_{i,k,2}(x, y) = N^2 \gamma_{k} y + N^2 q_i^{(2)}(\gamma_k, x).
\]

Let, for some \(A_j \geq 0\), the polynomials \(q_n\) satisfy

\[
\frac{|q_n^{(j)}(\gamma_k, x) - q_n^{(j)}(\gamma_l, x)|}{|\gamma_k - \gamma_l|} \leq A_j, \quad j = 0, 1, 2, \tag{3.11}
\]

for all \(n = 1, 2, \ldots, N, k, l = 1, 2, \ldots, M\) and \(x \in [x_0, x_N]\)
For $j = 1, 2$, define the Read-Bajraktarevic operator $T_j : \Lambda \times \mathcal{G} \to \mathcal{G}$ by

$$T_j(\sigma, g)(x) = \lim_{k \to \infty} \left\{ G_{i_k, \sigma, j} \left( L_{i_k}^{-1}(x), G_{i_{k-1}, \sigma_{k-1}, j} \left( L_{i_{k-1}}^{-1} \circ L_{i_k}^{-1}(x), G_{i_{k-2}, \sigma_{k-2}, j} \left( \ldots, G_{i_1, \sigma_1, j} \left( L_{i_1}^{-1} \circ \ldots \circ L_{i_k}^{-1}(x), g(L_{i_1}^{-1} \circ \ldots \circ L_{i_k}^{-1}(x)) \right) \right) \right) \right) \right\} \quad (3.12)$$

where, $\Lambda$ is the code space on $M$ natural numbers $1, 2, \ldots, M$ and $\mathcal{G}$ is given by $(3.2)$. To find the order of approximation of derivatives of data generating function $y(x)$ by corresponding derivatives of SFIF $g_{\sigma}$, the bounds on $\|T_j(\sigma, g)(x) - T_j(\varsigma, g)(x)\|$ similar to $(3.4)$ are needed. Such a bound on $\|T_j(\sigma, g)(x) - T_j(\varsigma, g)(x)\|$ for $\sigma, \varsigma \in \Lambda$, is given by the following proposition:

**Proposition 3.3** Let $g \in \mathcal{G}$ and inequality $(3.11)$ be satisfied. Then, for $j = 1, 2$,

$$\|T_j(\sigma, g) - T_j(\varsigma, g)\| \leq N^j \beta_{k_0} \left\{ \|g\|_\infty + \frac{A_j}{1 - N^j \gamma_s} + \frac{N^j B_j}{1 - N^j \beta_{k_0}} \right\} \quad (3.13)$$

where, $\sigma, \varsigma \in \Lambda$, $\max_{x \in [x_0, x_N]} |q_n^{(j)}(\gamma_{k_0}, x)| \leq B_j$, $\gamma_s = \max_{1 \leq l \leq M} |\gamma_l| < \frac{1}{N^2}$

and $\beta_{k_0} = \max_{1 \leq l \leq M} |\gamma_l - \gamma_{k_0}| < \frac{1}{N^2}$.

**Proof** By $(3.12)$,

$$|T_j(\sigma, g)(x) - T_j(\varsigma, g)(x)|$$

$$\leq \lim_{k \to \infty} \left\{ N^{jk} \left( \prod_{n=1}^k |\gamma_{\sigma_n} - \gamma_{k_0}| \right) |g(L_{i_1}^{-1} \circ \ldots \circ L_{i_k}^{-1}(x))| \right.$$ 

$$+ \sum_{m=1}^k N^{j(k-m+1)} \left( \prod_{n=m+1}^k |\gamma_{\sigma_n}| \right) \times$$

$$\times |q_m^{(j)}(\gamma_{\sigma_m}, L_{i_m}^{-1} \circ \ldots \circ L_{i_k}^{-1}(x)) - q_m^{(j)}(\gamma_{k_0}, L_{i_m}^{-1} \circ \ldots \circ L_{i_k}^{-1}(x))| \right.$$ 

$$+ \sum_{m=1}^{k-1} N^{j(k-m+1)} \left( \prod_{n=m+1}^k |\gamma_{\sigma_n} - \gamma_{k_0}| \right) |q_m^{(j)}(\gamma_{k_0}, L_{i_m}^{-1} \circ \ldots \circ L_{i_k}^{-1}(x))| \right\}. \quad (3.14)$$
Since $q_n^{(j)}(x)$ are polynomials defined on compact set $[x_0, x_N]$, there exists a $B_j > 0$ such that $\max_{x \in [x_0, x_N]} |q_n^{(j)}(x)| \leq B_j$. Therefore, by (3.11) and (3.14), it follows that

$$|T_j^{(\sigma, g)}(x) - T_j^{(\varsigma, g)}(x)| \leq \lim_{k \to \infty} \left\{ N_j^k \left( \prod_{n=1}^{k} |\gamma_{\sigma_n} - \gamma_{\varsigma_0}| \right) \|g\|_{\infty} + \sum_{m=1}^{k-1} N_j^{j(m-k)} \left( \prod_{n=m+1}^{k} |\gamma_{\sigma_n}| \right) |\gamma_{\sigma^m} - \gamma_{\varsigma_0}| A_j \right\}.$$ 

Since $\beta_{k_0} = \max_{1 \leq l \leq M} |\gamma_{l} - \beta| < \frac{1}{N^2}$ and $\gamma_* = \max_{1 \leq l \leq M} |\gamma_{l}| < \frac{1}{N^2}$, (3.13) follows from the above inequality.

Similar to Proposition 3.2, the following proposition gives a bound on $\|g_{\sigma}^{(j)} - g_{\varsigma}^{(j)}\|$ for $\sigma, \varsigma \in \Lambda$:

**Proposition 3.4** Let $g_{\sigma}, g_{\varsigma} \in G$ and (3.11) be satisfied. Then, for $j = 1, 2$,

$$\|g_{\sigma}^{(j)} - g_{\varsigma}^{(j)}\|_{\infty} \leq \frac{N_j^{j \beta_{k_0}}}{1 - N_j \gamma_*} \left\{ \|g_{\sigma}^{(j)}\|_{\infty} + \frac{A_j}{1 - N_j \gamma_*} + \frac{N_j B_j}{1 - N_j \beta_{k_0}} \right\} \tag{3.15}$$

where, $\gamma_*, \beta_{k_0}$ and $B_j$ are as in Proposition 3.3.

**Proof** Since (3.11) is satisfied, $g_{\sigma}^{(j)} = T_j^{(\sigma, g_{\sigma}^{(j)})}$ and $g_{\varsigma}^{(j)} = T_j^{(\varsigma, g_{\varsigma}^{(j)})}$ for Read-Bajraktarevic operator $T_j$, defined by (3.12),

$$\|g_{\sigma}^{(j)} - g_{\varsigma}^{(j)}\|_{\infty} \leq N_j^{j \gamma_*} \|g_{\sigma}^{(j)} - g_{\varsigma}^{(j)}\|_{\infty} + N_j^{j \beta_{k_0}} \left\{ \|g_{\sigma}^{(j)}\|_{\infty} + \frac{A_j}{1 - N_j \gamma_*} + \frac{N_j B_j}{1 - N_j \beta_{k_0}} \right\}.$$ 

The inequality (3.15) now follows from the above inequality.
Remark 3.4 For $\gamma_{k_0} = 0$, $\gamma_\ast = \beta_{k_0}$. Thus, inequality (3.15) implies

$$\|g^{(j)}_\sigma - g^{(j)}_\varsigma\|_\infty \leq \frac{N^j \gamma_\ast}{1 - N^j \gamma_\ast} \left\{ \|g^{(j)}_\varsigma\|_\infty + \frac{A_j + N^j B_j}{1 - N^j \gamma_\ast} \right\}.$$ 

By Hall and Meyer’s theorem [13], $\|g^{(j)}_\varsigma\|_\infty \leq K_j h^4 + J_j$. Consequently, Proposition 3.4 for $\gamma_{k_0} = 0$, gives,

$$\|g^{(j)}_\sigma - g^{(j)}_\varsigma\|_\infty \leq \frac{N^j \gamma_\ast}{1 - N^j \gamma_\ast} \left\{ K_j h^4 + J_j + \frac{A_j + N^j B_j}{1 - N^j \gamma_\ast} \right\}. \quad (3.16)$$

Using inequality (3.16), the orders of approximation of derivatives of data generating function $y(x)$ by corresponding derivatives of SFIF $g_\sigma$ are given by the following theorem:

**Theorem 3.2** Let $y(x) \in C^4[x_0, x_N]$ be a data generating function and $g_\sigma \in \mathcal{G}$ be a SFIF associated with SIFS (2.4) such that $\gamma_\ast(h) = \max_{i=1,2,\ldots,N} |\gamma_i| \leq \frac{h^{2+s}}{|I|^{s+1}}$, for some $s$, $0 < s < 1$, where $h = x_i - x_{i-1}$, $i = 1, 2, \ldots, N$ and $|I| = x_N - x_0$. Then, for $j = 1, 2$ and $0 < \epsilon < s$,

$$\|y^{(j)} - g^{(j)}_\sigma\|_\infty = o(h^{2-j+\epsilon}) \quad (3.17)$$

provided (3.11) holds.

**Proof** Since (3.11) holds, an application of inequality (3.16) gives,

$$\|y^{(j)} - g^{(j)}_\sigma\|_\infty \leq \frac{1}{1 - N^j \gamma_\ast(h)} \left\{ K_j h^{4-j} + \frac{N^j \gamma_\ast(h)}{1 - N^j \gamma_\ast(h)} \left[ A_j + N^j B_j \right] \right\}.$$ 

Using inequality (3.16), the orders of approximation of derivatives of data generating function $y(x)$ by corresponding derivatives of SFIF $g_\sigma$ are given by the following theorem:

**Theorem 3.2** Let $y(x) \in C^4[x_0, x_N]$ be a data generating function and $g_\sigma \in \mathcal{G}$ be a SFIF associated with SIFS (2.4) such that $\gamma_\ast(h) = \max_{i=1,2,\ldots,N} |\gamma_i| \leq \frac{h^{2+s}}{|I|^{s+1}}$, for some $s$, $0 < s < 1$, where $h = x_i - x_{i-1}$, $i = 1, 2, \ldots, N$ and $|I| = x_N - x_0$. Then, for $j = 1, 2$ and $0 < \epsilon < s$,

$$\|y^{(j)} - g^{(j)}_\sigma\|_\infty = o(h^{2-j+\epsilon}) \quad (3.17)$$

provided (3.11) holds.
Using $|\gamma_s(h)| \leq \frac{h^{2+s}}{|h|^{2+s}}$, the inequality (3.18) implies

$$\|y^{(j)} - g^{(j)}_\sigma\|_\infty \leq |I|^{(2+s-j)} \left( K_j h^{4-j} + \frac{J_j h^{(2+s-j)}}{|I|^{(2+s-j)}} + \frac{h^{(2+s-j)}}{|I|^{(2+s-j)} - h^{(2+s-j)}} \left[ A_j + N^j B_j \right] \right).$$

(3.19)

The order of approximation error given by (3.17) follows from the above inequality.

Remark 3.5 It follows from inequality (3.19) that, in fact, $\|y^{(j)} - g^{(j)}_\sigma\|_\infty = O(h^{2+s-j})$, $j = 1, 2$.

Remark 3.6 If $M = 1$ in SIFS $\left\{ \{\mathbb{R}^2; \omega_{i,k,j}(x, y) = (L_i(x), G_{i,k,j}(x, y)) : i = 1, 2, \ldots, N \}, k = 1, 2, \ldots, M \right\}$, $j = 1, 2$, then $g^{(j)}_\sigma$, $j = 1, 2$, are FIFs. The convergence results for derivatives of a Cubic Spline FIFs [12] follow as a particular case of Theorem 3.2.

4 Conclusions

In the present work, the notion of Cubic Spline SFIF is introduced for an efficient approximation of the data generating function. The approximation properties of a Cubic Spline SFIF are investigated and it is proved that the order of approximation of the data generating function $y(x) \equiv y^{(0)}(x)$ and its derivatives $y^{(j)}(x)$ by an interpolating Cubic Spline SFIF $g_\sigma \equiv g^{(0)}_\sigma$ and its corresponding derivatives $g^{(j)}_\sigma$ respectively, is $o(h^{2-j+\epsilon})$, $0 < \epsilon < 1$, $j = 0, 1, 2$. These convergence results show that it is possible to approximate any regular data generating function by Cubic Spline SFIF with arbitrary accuracy. Our study of Cubic Spline SFIF is likely to have wide applications like pattern-forming alloy solidification in chemistry, blood vessel patterns in biology, signal processing, fragmentation of thin plates in engineering, stock markets in finance, wherein significant randomness and variability is observed in simulation of various processes.
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