Hydrodynamic reductions and solutions of a universal hierarchy

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Abstract

The diagonal hydrodynamic reductions of a hierarchy of integrable
hydrodynamic chains are explicitly characterized. Their compatibility
with previously introduced reductions of differential type is analyzed
and their associated class of hodograph solutions is discussed.

Key words: Hydrodynamic Systems, Differential Reductions, Hodo-
graph Solutions.

MSC: 35L40,58B20.
1 Introduction

In a series of papers [1]-[4] we have considered an infinite hierarchy of integrable systems [5] which admits many interesting (1 + 1)-dimensional reductions like the Burgers, KdV and NLS hierarchies as well as different classes of energy-dependent hierarchies [6]-[9] including the Camassa-Holm model also [10]. It motivated the term universal hierarchy we proposed in [1].

This universal hierarchy can be defined in terms of a generating function \(G = G(\lambda, \mathbf{x})\) depending on a spectral parameter \(\lambda\) and an infinite set of variables \(\mathbf{x} := (\ldots, x_{-1}, x_0, x_1, \ldots)\), which admits expansions

\[
G = 1 + \frac{g_1(\mathbf{x})}{\lambda} + \frac{g_2(\mathbf{x})}{\lambda^2} + \cdots, \quad \lambda \to \infty, \quad (1)
\]

\[
G = b_0(\mathbf{x}) + b_1(\mathbf{x})\lambda + b_2(\mathbf{x})\lambda^2 + \cdots, \quad \lambda \to 0. \quad (2)
\]

The hierarchy is provided by the system of flows

\[
\partial_n G = \langle A_n, G \rangle, \quad n \in \mathbb{Z}, \quad (3)
\]

where

\[
\langle U, V \rangle := U(\partial_x V) - (\partial_x U)V,
\]

and

\[
A_n := \lambda^n + g_1(\mathbf{x})\lambda^{n-1} + \cdots + g_{n-1}(\mathbf{x})\lambda + g_n(\mathbf{x}), \quad n \geq 0 \quad (4)
\]

\[
A_{-n} := \frac{b_0(\mathbf{x})}{\lambda^n} + \frac{b_1(\mathbf{x})}{\lambda^{n-1}} + \cdots + \frac{b_{n-1}(\mathbf{x})}{\lambda}, \quad n > 0. \quad (5)
\]

In terms of the coefficients \(\{g_n(\mathbf{x})\}_{n \geq 1} \cup \{b_n(\mathbf{x})\}_{n \geq 0}\) the system (3) becomes a hierarchy of hydrodynamic chains.

An alternative and useful formulation of (3) is obtained by introducing the generating function

\[
H := \frac{1}{G}. \quad (6)
\]

Thus Eq.(3) is equivalent to

\[
\partial_n H = \partial_x \left( A_n H \right), \quad n \in \mathbb{Z}, \quad (7)
\]
which means, in particular, that the coefficients \( \{h_n(\mathbf{x})\}_{n \geq 1} \) supply an infinite set of conservation laws for (3).

The hierarchy (3) forms a compatible system. Indeed, as a consequence of (3) one derives the consistency conditions [1]-[4]

\[
\partial_n A_m - \partial_m A_n = \langle A_n, A_m \rangle, \quad n, m \in \mathbb{Z},
\]

(8)

It is important to notice that (8) implies that the pencil of differential forms

\[
\omega(\lambda) := \sum_{n \in \mathbb{Z}} (HA_n) \, dx_n,
\]

(9)

is closed with respect to the variables \( \mathbf{x} \) since it satisfies

\[
\partial_m (HA_n) - \partial_n (HA_m) = \partial_m H A_n - \partial_n H A_m + H(\partial_m A_n - \partial_n A_m)
\]

\[
= \partial_x (HA_m) A_n - \partial_x (HA_n) A_m + H \langle A_n,A_m \rangle = 0.
\]

The potential function \( Q = Q(\lambda, \mathbf{x}) \) corresponding to \( \omega \)

\[
dQ = \omega,
\]

(10)

leads to another useful formulation of our hierarchy. Indeed, according to (9)

\[
\partial_n Q = A_n \partial_x Q, \quad n \in \mathbb{Z},
\]

(11)

Moreover, (11) is completely determined by \( Q \) as

\[
G = \frac{1}{\partial_x Q}.
\]

(12)

In Section 2 of this paper the formulation (11) is applied to prove that (3) includes multidimensional models such as

\[
\begin{align*}
\frac{u_t}{u_x} & = \frac{u_x u_{yz} - u_{yx} u_z}{} , \\
\frac{u_{yy}}{u_{xx}} & = \frac{u_y u_{xz} - u_{xy} u_z}{} , \\
\frac{u_{zx}}{u_{xx}} & = \frac{u_x u_{yy} - u_{xy} u_y}{} , \\
\frac{u_t}{u_x} & = \left( \frac{u_y}{u_x} \right)_x , \\
\frac{u_t}{u_x} & = \left( \frac{u_y}{u_x} \right)_x .
\end{align*}
\]

(13-17)
Section 3 deals with the main aim of the present paper: to characterize the hydrodynamic reductions of \((3)\). These reductions are given by the solutions of \((3)\) of the form \(G = G(\lambda, R)\), where \(R = (R^1, \ldots, R^N)\) denotes a finite set of functions (Riemann invariants) satisfying a system of hydrodynamic equations of diagonal form

\[
\partial_n R_i = \Lambda_n^i(R) \partial_x R_i, \quad n \in \mathbb{Z}.
\]  

This type of reductions appeared in the context of the dispersionless KP hierarchy \([11]-[14]\) and has been also used in \([15]\) to characterize the integrability of \((2 + 1)\)-dimensional quasilinear systems. In the present paper we study these reductions for the whole set of flows of the hierarchy \((3)\) and we obtain their explicit form. Our results are summarized in Theorem 1. It should be noticed that a similar analysis for the first member \((t_1\text{-flow})\) of \((3)\) has been recently performed in \([16]\). The compatibility between hydrodynamic and differential reductions considered in part 2 of Section 3. The paper finishes with a discussion of the solutions supplied by the generalized hodograph method \([17]\).

The following notation conventions are henceforth used. Firstly, \(G_\infty\) and \(G_0\) stand for the expansions \([11],[2]\) of \(G\) as \(\lambda \to \infty\) and \(\lambda \to 0\), respectively. Furthermore, let \(V\) be the space of formal Laurent series

\[
V = \sum_{n=-\infty}^{\infty} a_n \lambda^n.
\]

We will denote by \(V_{r,s} \quad (r \leq s)\) the subspaces of elements

\[
V = \sum_{n=r}^{s} a_n \lambda^n.
\]

and by \(P_{r,s} : V \mapsto V_{r,s}\) the corresponding projectors. Given \(V \in V\) we will also denote

\[
(V)_{r,s} := P_{r,s}(V).
\]

In particular, notice that we can write

\[
A_n = \left(\lambda^n G_\infty\right)_{0, \pm \infty}, \quad A_{-n} = \left(\lambda^{-n} G_0\right)_{-\infty, -1}, \quad n \geq 1.
\]
2 Integrable models arising in the hierarchy

2.1 Multidimensional models

If we use the potential function $Q(\lambda, x)$ for the differential form

$$d Q = \sum_{n \in \mathbb{Z}} (H A_n) \, dx_n, \quad \text{(19)}$$

then from (11) one readily deduces that $Q$ admits expansions of the form

$$Q = \sum_{n \geq 0} \lambda^n x_n + \frac{q_1(x)}{\lambda} + \frac{q_2(x)}{\lambda^2} + \cdots, \quad \lambda \to \infty, \quad \text{(20)}$$

$$Q = \sum_{n \geq 1} \frac{x_n}{\lambda^n} + p_0(x) + p_1(x) \lambda + p_2(x) \lambda^2 + \cdots, \quad \lambda \to 0. \quad \text{(21)}$$

By substituting (20) into (19) and by identifying coefficients of equal powers of $\lambda$ one obtains formulas for the differentials of the functions $q_n(x)$ and $p_n(x)$ in terms of the coefficients of the expansions (1) and (2). For example, the simplest ones are

$$d q_1 = b_0 \, dx_0 + \sum_{n \geq 1} (b_n \, dx_{-n-1} - g_n \, dx_{n-1}), \quad \text{(22)}$$

$$d p_0 = \frac{1}{b_0} \left( dx_0 + \sum_{n \geq 1} (g_n \, dx_n - b_n \, dx_{-n}) \right). \quad \text{(23)}$$

They imply

$$d q_1 = \frac{1}{\partial x p_0} \left( dx_{-1} - \sum_{n \neq -1} \partial_{n+1} p_0 \, dx_n \right) \quad \text{(24)}$$

$$d p_0 = \frac{1}{\partial_{-1} q_1} \left( dx_0 - \sum_{n \neq 0} \partial_{n-1} q_1 \, dx_n \right). \quad \text{(25)}$$

Permutability of crossing derivatives of $q_1$ and $p_0$ in these identities lead at once to multidimensional nonlinear equations for the functions $g_n$ and $b_n$. For example, starting from

$$\partial_m \partial_{-1} q_1 = \partial_{-1} \partial_m q_1, \quad m \neq -1,$$
and using (24), the following nonlinear equation results
\[ \partial_n \partial_0 p_0 = \partial_x p_0 (\partial_{-1} \partial_{n+1} p_0) - (\partial_{-1} \partial_x p_0) D_{n+1} p_0, \quad n \neq -1. \]  
(26)

In the same way, from the crossing relation
\[ \partial_m \partial_n q_1 = \partial_n \partial_m q_1, \quad m, n \neq -1, \]
and (24) we get the following nonlinear equation
\[ \partial_m \left( \frac{\partial_{n+1} p_0}{\partial_x p_0} \right) = \partial_n \left( \frac{\partial_{m+1} p_0}{\partial_x p_0} \right), \quad m, n \neq -1. \]  
(27)

The same type of equations can be derived for \( q_1 \). The different choices available for \( n, m \) in the equations (26) and (27) give rise to the models (13)-(17).

### 2.2 2-dimensional integrable models

In [1]-[3] we developed a theory of differential reductions of our hierarchy based on imposing differential constraints on \( G \approx G_\infty \) of the form
\[ \left( F(\lambda, G, G_x, G_{xx}, \ldots) \right)_{-\infty, -1} = 0, \quad x := x_0. \]  
(28)

In particular the following three classes of reductions associated to arbitrary polynomials \( a = a(\lambda) \) in \( \lambda \) were characterized:

**Zero-order reductions**

\[ a(\lambda) G = U(\lambda, x), \quad U := \left( a(\lambda) G \right)_{0, +\infty}, \]  
(29)

**First-order reductions**

\[ G_x + a(\lambda) = U(\lambda, x) G, \quad U := \left( \frac{a}{G} \right)_{0, +\infty}, \]  
(30)
Second-order reductions

\[ \frac{1}{2} GG_{xx} - \frac{1}{4} G_x^2 + a(\lambda) = U(\lambda, x) G^2, \quad U := \left( \frac{a}{G^2} \right)_{0, +\infty}. \tag{31} \]

The first-order reduction for a linear function \( a(\lambda) \) determines the Burgers hierarchy. On the other hand, under the differential constraints \((31)\) the hierarchy \((18)\) describes the KdV hierarchy and its generalizations associated to energy-dependent Schrödinger spectral problems. In particular, the linear and quadratic choices for \( a(\lambda) \) lead to the KdV (Korteweg-deVries) and NLS (Nonlinear-Schrödinger) hierarchies, respectively. Indeed, if we define the functions \( \psi(\lambda, x) \) by

\[ \psi(\lambda, x) := \exp(D^{-1}_x \phi), \quad \phi := -\frac{1}{2} \frac{H_x}{H} \pm \sqrt{a(\lambda)} H, \tag{32} \]

then from \((31)\) it is straightforward to deduce that

\[ \partial_n \psi = -\frac{1}{2} (\partial_n \log H) \psi \pm \sqrt{a} A_n H \psi \]
\[ = A_n (-\frac{1}{2} D_x \log H \pm \sqrt{a} H) \psi - \frac{1}{2} A_n, \]
\[ = A_n \psi_x - \frac{1}{2} A_{n,x} \psi, \]
\[ \psi_{xx} = (\phi_x + \phi^2) \psi = (\{D_x, H\} + a H^2) \psi = U \psi. \]

In other words, the functions \( \psi \) are wave functions for the integrable hierarchies associated to energy-dependent Schrödinger problems. The evolution law of the potential function \( U \) under the flows \((18)\) can be determined from the equation

\[ \partial_n U = -\frac{1}{2} A_{n,xxx} + 2 U A_{n,x} + U_x A_n, \tag{33} \]

which arises as an straightforward consequence of \((18)\) and \((31)\).

Additional reduced hierarchies including nonlinear integrable models such as the Camassa-Holm equation can be also deduced (see \([4]\)).
3 Hydrodynamic reductions and solutions

3.1 Hydrodynamic reductions

Let us consider now the hydrodynamic reductions of (3). We look for classes of solutions

$$G = G(\lambda, \mathbf{R}),$$  \hspace{1cm} (34)

of (3) where $\mathbf{R} = (R^1, \ldots, R^N)$ satisfies an infinite system of hydrodynamic equations of diagonal form (18). Our aim is to characterize both the form of $G = G(\lambda, \mathbf{R})$ and the characteristic speeds $\Lambda^i_n$ defining the system (18). By substituting (35) into (3) then by using (18) the identification of coefficients of the derivatives $\partial_x R^i, \ i = 1, \ldots, N$ implies

$$(D_i G)\Lambda^i_n = A_n(D_i G) - (D_i A_n)G, \hspace{1cm} 1 \leq i \leq N, \ n \in \mathbb{Z} \hspace{1cm} (35)$$

where

$$D_i := \frac{\partial}{\partial R^i}.$$  

In addition to these equations we impose the requirement of the commutativity of the flows (18), which is equivalent to the following restrictions on the characteristic speeds

$$\frac{D_j \Lambda^i_n}{\Lambda^j_n - \Lambda^i_n} = \frac{D_j \Lambda^i_m}{\Lambda^j_m - \Lambda^i_m}, \hspace{1cm} i \neq j, \ m \neq n. \hspace{1cm} (36)$$

We start our analysis by considering the positive flows $n \geq 1$. From (35) we get the following system for $G \approx G_\infty$

$$(D_i G)\Lambda^i_n = (\lambda^n G)_{-\infty,0}(D_i G) - (D_i(\lambda^n G)_{-\infty,0})G, \hspace{1cm} n \geq 1. \hspace{1cm} (37)$$

By substituting in these equations the expansion $G \approx G_\infty$ and identifying the coefficients in $\frac{1}{\lambda}$ and $\frac{1}{\lambda^2}$ we get

$$D_1 g_{n+1} = \Lambda^1_n D_1 g_1, \hspace{1cm} (38)$$

$$D_1 g_{n+2} = \Lambda^1_n D_1 g_2 + g_{n+1}(D_1 g_1) - (D_1 g_{n+1})g_1. \hspace{1cm} (39)$$

As an immediate consequence it follows

$$\Lambda^i_{n+1} = g_{n+1} + \Lambda^i_n(\Lambda^i_1 - g_1), \hspace{1cm} (40)$$
which implies
\[ \Lambda_n^i = A_n \left( \lambda = \Lambda_1^i - g_1 \right), \quad n \geq 1. \quad (41) \]

We now look for a system characterizing \( \Lambda_1^i \) and \( g_1 \). To this end we differentiate (38) and find
\[ D_j D_i g_{n+1} = D_j D_i g_1 \Lambda_n^i + D_i g_1 D_j \Lambda_n^i \]
\[ = D_i D_j g_1 \Lambda_j^i + D_j g_1 D_i \Lambda_n^i, \]
so that
\[ D_{ij} g_1 = \frac{D_j \Lambda_n^i}{\Lambda_n^i - \Lambda_n^j} D_i g_1 + \frac{D_i \Lambda_n^j}{\Lambda_n^i - \Lambda_n^j} D_j g_1, \]
and from (36) we may write
\[ D_{ij} g_1 = \frac{D_j \Lambda_n^i}{\Lambda_n^j - \Lambda_n^j} D_i g_1 + \frac{D_i \Lambda_n^j}{\Lambda_n^j - \Lambda_n^j} D_j g_1, \quad i \neq j. \quad (42) \]

On the other hand according to (40)
\[ \Lambda_2^i = g_2 + \Lambda_1^i (\Lambda_1^i - g_1), \]
so that from (36) and with the help of (38) we find
\[ \frac{D_j \Lambda_1^i}{\Lambda_1^i - \Lambda_1^j} = \frac{D_j \Lambda_2^i}{\Lambda_2^j - \Lambda_2^j} = \frac{D_j \Lambda_1^i (2 \Lambda_1^i - g_1) + (\Lambda_1^j - \Lambda_2^j) D_j g_1}{(\Lambda_1^j - \Lambda_1^j)(\Lambda_1^j + \Lambda_1^i - g_1)}, \quad i \neq j, \]
which reduces to
\[ D_j \Lambda_1^i = D_j g_1, \quad i \neq j. \]

In this way
\[ \Lambda_1^i = g_1 + f^{i}(R^{i}), \quad (43) \]
where the functions \( f^{i} \) are arbitrary. By using this result in (42) it follows
\[ D_{ij} g_1 = 0, \quad i \neq j, \]
so that
\[ g_1 = \sum_{k=1}^{N} h^{k}(R^{k}), \quad (44) \]
where the functions \( h^{k} \) are arbitrary.
By using these results we may determine $G(\lambda, \mathbf{R})$ since from the equation (35) with $n = 1$

$$D_i \ln G = \frac{D_i A_i}{A_1} = \frac{\dot{h}_i(R^i)}{\lambda - f^i(R^i)},$$

where $\dot{h}_i := D_i h_i$. Thus we get

$$G(\lambda, \mathbf{R}) = \exp \left( \sum_{i=1}^{N} \int_{R^i} \frac{\dot{h}_i(R^i)}{\lambda - f_i(R^i)} d R^i \right),$$

where the indefinite integrations are determined up to a function of $\lambda$ decaying at $\lambda \to \infty$. The expression (45) coincides with the generating function of the conservation laws densities found in [16].

Let us now determine the characteristic speeds $\Lambda_i$ for the negative flows $n \leq -1$. The equations (35) imply the following system for $G \approx G_0$

$$(D_i G) \Lambda_{i-n} = (\lambda^{-n} G)_{0,\infty}(D_i G) - (D_i (\lambda^{-n} G)_{0,\infty}) G, \quad n \geq 1. \quad (46)$$

Then by inserting the expansion $G \approx G_0$ of (2) and identifying the coefficients in $\lambda^0$ and $\lambda$ we get

$$D_i b_n = (b_n + \Lambda_{i-n}) D_i \ln b_0, \quad (47)$$
$$b_0 D_i b_{n+1} = (b_n + \Lambda_{i-n}) D_i b_1 + b_{n+1}(D_i b_0) - (D_i b_n) b_1. \quad (48)$$

It implies the following recurrence relation for the characteristic speeds

$$\Lambda_{i-n} = \frac{\Lambda_i}{b_0}(b_n + \Lambda_{i-n}), \quad (49)$$

which implies

$$\Lambda_{i-n} = A_{-n}(\lambda = \frac{b_0}{\Lambda_{i-1}}), \quad n \geq 1. \quad (50)$$

The only unknown now is $\Lambda_{i-1}$ since according to (45)

$$b_0 = \exp \left( - \sum_{i=1}^{N} \int_{R^i} \frac{\dot{h}_i(R^i)}{f_i(R^i)} d R^i \right).$$

To find $\Lambda_{i-1}$ we use (49) for $n = 1$

$$\Lambda_{i-2} = \frac{\Lambda_{i-1}}{b_0}(b_1 + \Lambda_{i-1}),$$
and the commutativity condition for the \( n = -1 \) and \( n = -2 \) flows

\[
\frac{D_j \Lambda_{i-1}^i}{\Lambda_{i-1}^i - \Lambda_{i-1}^j} = \frac{D_j \Lambda_{i-2}^i}{\Lambda_{i-2}^j - \Lambda_{i-2}^i}.
\]

Thus by eliminating \( \lambda_{i-2}^j \) one finds at once

\[
D_j \ln \Lambda_{i-1}^i = D_j \ln b_0, \quad i \neq j.
\]

Therefore,

\[
\Lambda_{i-1}^i = b_0 g^i(R^i),
\]

with \( g^i \) being arbitrary functions.

At this point we have determined all the unknowns of our problem. However, in our calculation we only used a subset of the equations required, so we must prove that our solution satisfies the full system of equations (35)-(36).

Let us begin with (35) for \( n \geq 1 \)

\[
D_i \ln G = \frac{D_i A_n}{A_n - A_n^i}, \quad n \geq 1,
\]

which according to (41), (43)-(45) reads

\[
D_i A_n(\lambda) = \frac{\dot{h}^i}{\lambda - f^i} \left( A_n(\lambda) - A_n(\lambda = f^i) \right).
\]

To prove this identity we express \( A_n(\lambda) \) as

\[
A_n(\lambda) = (\lambda^n G)_{0,\infty} = \frac{1}{2\pi i} \int_{\gamma_{\infty}} \frac{\tilde{\lambda}^n G(\tilde{\lambda})}{\tilde{\lambda} - \lambda} d\tilde{\lambda} = \frac{1}{2\pi i} \int_{\gamma_{\infty}} \frac{\tilde{\lambda}^n G(\tilde{\lambda})}{\lambda - \tilde{\lambda}} d\tilde{\lambda}
\]

\[
= \frac{1}{2\pi i} \int_{\gamma_{\infty}} \frac{\tilde{\lambda}^n}{\lambda - \tilde{\lambda}} \exp \left( \sum_{i=1}^{N} \int_{\gamma_{\infty}} \frac{\dot{h}^i}{\lambda - f^i} dR^i \right) d\tilde{\lambda},
\]

where \( \gamma_{\infty} \) is a positively oriented closed loop around \( \infty \) in the complex plane of \( \tilde{\lambda} \) (\( \lambda \) and \( f^i \) are assumed to lie inside the loop). Now by differentiating this expression with respect to \( R^i \) one finds

\[
D_i A_n(\lambda) = \frac{1}{2\pi i} \int_{\gamma_{\infty}} \frac{\dot{h}^i \tilde{\lambda}^n G(\tilde{\lambda})}{(\lambda - \tilde{\lambda})(\tilde{\lambda} - f^i)} d\tilde{\lambda} = \frac{1}{2\pi i} \int_{\gamma_{\infty}} \frac{\dot{h}^i A_n(\tilde{\lambda})}{(\lambda - \tilde{\lambda})(\tilde{\lambda} - f^i)} d\tilde{\lambda}
\]

\[
= \frac{\dot{h}^i}{\lambda - f^i} \left( A_n(\lambda) - A_n(\lambda = f^i) \right) ,
\]
which proves (52). This identity leads also at once to (53) since it implies
\[
\frac{D_jA_n(\lambda = f^i)}{A_n(\lambda = f^j) - A_n(\lambda = f^i)} = \frac{\dot{h}^j}{f^j - f^i}, \quad i \neq j,
\]
which means that
\[
\frac{D_jA^i_n}{\Lambda^i_n - \Lambda^i_{-n}} = \frac{D_jA^i_1}{\Lambda^i_1 - \Lambda^i_{-1}}, \quad n > 1.
\]
Let us consider now (35) for \(n \leq -1\)
\[
D_i \ln G = \frac{D_iA_{-n}}{A_{-n} - A^i_{-n}}, \quad n \geq 1.
\]
From (45), (50) and (51) it takes the form
\[
D_iA_{-n}(\lambda) = \frac{\dot{h}^i}{\lambda - f^i} \left( A_{-n}(\lambda) - A_{-n}(\lambda = \frac{1}{g}) \right). \tag{53}
\]
In order to proof (53) we express \(A_{-n}(\lambda)\) in the form
\[
A_{-n}(\lambda) = (\lambda^{-n}G)_{-\infty, -1} = \frac{1}{2\pi i} \int_{\gamma_0} \frac{(\lambda^{-n}G(\bar{\lambda}))_{-\infty, -1}}{\bar{\lambda} - \lambda} d\bar{\lambda} = \frac{1}{2\pi i} \int_{\gamma_0} \frac{\lambda^{-n}G(\bar{\lambda})}{\bar{\lambda} - \lambda} d\bar{\lambda}
\]
\[
= \frac{1}{2\pi i} \int_{\gamma_0} \frac{\lambda^{-n}}{\lambda - \lambda} \exp \left( \sum_{i=1}^{N} \int_{R^i} \frac{\dot{h}^i}{\lambda - f^i} dR^i \right) d\bar{\lambda},
\]
where \(\gamma_0\) is a closed small loop with negative orientation around \(\bar{\lambda} = 0\) (\(\lambda\) and \(f^i\) are assumed to lie outside the loop). By differentiating with respect to \(R^i\) it yields
\[
D_iA_{-n}(\lambda) = \frac{1}{2\pi i} \int_{\gamma_0} \frac{\dot{h}^i \lambda^{-n}G(\bar{\lambda})}{(\lambda - \lambda)(\bar{\lambda} - f^i)} d\bar{\lambda}
\]
\[
= \frac{1}{2\pi i} \int_{\gamma_0} \frac{\dot{h}^i A_{-n}(\bar{\lambda})}{(\lambda - \lambda)(\bar{\lambda} - f^i)} d\bar{\lambda} = \frac{\dot{h}^i}{\lambda - f^i} \left( A_{-n}(\lambda) - A_{-n}(\lambda = f^i) \right),
\]
Hence (53) holds if we set
\[
g^i(R^i) = \frac{1}{f^i(R^i)}.
\]
The remaining commutativity conditions follow at once. Therefore we may summarize our analysis in the next theorem.

**Theorem 1.** The hydrodynamic reductions of the hierarchy are determined by

\[
G(\lambda, R) = \exp \left( \sum_{i=1}^{N} \int_{R_i}^{R_i'} \frac{h^i(R_i)}{\lambda - f^i(R_i')} \, dR_i' \right), \tag{54}
\]

\[
\partial_n R_i = \Lambda^n_i (R) \partial_x R_i, \quad \Lambda^n_i (R) := A_n(\lambda = f^i(R_i)), \tag{55}
\]

where the functions \(h^i\) and \(f^i\) are arbitrary.

Since the systems are invariant under local transformations of the form \(R_i \rightarrow \tilde{R}_i(R_i)\), without loss of generality we will henceforth set

\[
h^i(R_i) = R_i, \tag{56}
\]

so that the form of the reduced generating function is

\[
G(\lambda, R) = \exp \left( \sum_{i=1}^{N} \int_{R_i}^{R_i} \frac{dR_i}{\lambda - f^i(R_i')} \right), \tag{57}
\]

In this way some of the simplest hydrodynamic reductions are given by

\[
f^i(R_i) = c_i, \quad G = \exp \left( \sum_{i=1}^{N} \frac{R_i}{\lambda - c_i} \right), \tag{58}
\]

\[
f^i(R_i) = -\frac{R_i}{\epsilon_i}, \quad G = \prod_{i=1}^{N} \left( \frac{\epsilon_i \lambda + R_i}{\epsilon_i \lambda + \lambda_i} \right)^{\epsilon_i}. \tag{59}
\]

We also notice that the characteristic velocities of the hydrodynamic systems can be written in terms of the Schur polynomials

\[
\exp(\sum_{n \geq 1} k^nx_n) = \sum_{n \geq 0} k^n S_n(x_1, \ldots, x_n),
\]
as

\[ \Lambda^i_n(R) = \sum_{j=0}^{n} S_j(I_1, \ldots, I_j) \left( f^i(R^i) \right)^{n-j}, \quad n \geq 0, \]

\[ \Lambda_{-n}^i(R) = \exp(-I_0) \sum_{j=0}^{n-1} S_j(I_{-1}, \ldots, I_{-j}) \left( f^i(R^i) \right)^{n-j}, \quad n > 0, \]

where

\[ I_n := \text{sgn}(n) \sum_{i=1}^{N} \int R^i (f^i(R^i))^{n-1} dR^i. \]

### 3.2 Compatibility with differential reductions

A natural question is to find the hydrodynamic reductions compatible with the differential reductions (30) and (31). Let us prove the following result:

**Theorem 2.** The only hydrodynamic reductions \((57)\) compatible with either first or second order differential constraints are characterized by

\[ f^i(R^i) = -R^i + c_i, \quad i = 1, \ldots, N, \]

which correspond to generating functions of the form

\[ G(\lambda, R) = \alpha(\lambda) \prod_{i=1}^{N} (\lambda + R^i - c_i). \]

**Proof.** If we substitute \((57)\) into the differential constraint \((31)\) for second-order differential reductions we get

\[ \frac{a(\lambda)}{G^2} = U - \frac{1}{2} \sum_i \left( f^i \left( \frac{\partial_x R^i}{(\lambda - f^i)^2} + \frac{\partial_{xx} R^i}{\lambda - f^i} \right) - \frac{1}{4} \left( \sum_i \frac{\partial_x R^i}{\lambda - f^i} \right)^2. \right. \]

This means that \(G\) has a simple zero at each \(\lambda = f^i(R^i)\) so that from \((57)\) we have

\[ \exp \left( \int_{R^i}^{R_i} \frac{dR^i}{\lambda - f^i(R^i)} \right) = (\lambda - f^i(R^i))H(\lambda, R^i), \]

where \(H\) is different from zero at \(\lambda = f^i(R^i)\). If we now differentiate with respect to \(R^i\) we deduce that

\[ \frac{1}{\lambda - f^i} = -\frac{f^i}{\lambda - f^i} + O(1), \quad \lambda \to \infty. \]
Therefore the statement of the theorem for second-order differential constraints follows. The corresponding proof for first-order constraints is similar.

3.3 Hodograph solutions

The general solution of the infinite system (55) is provided by the implicit generalized hodograph formula [17]

\[
x + \sum_{n \in \mathbb{Z} - \{0\}} \Lambda_i^n(R) x_n = \Gamma^i(R), \quad i = 1, \ldots, N,
\]

where the functions \( \Gamma^i \) are the general solution of the linear system

\[
\frac{D_j \Gamma^i}{\Gamma_j - \Gamma^i} = \frac{1}{f_j - f_i}, \quad i \neq j.
\]

By introducing the potential function \( \Phi(R) \)

\[
\Gamma^i = D_i \Phi, \quad i = 1, \ldots, N,
\]

the system (61) reduces to the Laplace type form

\[
(f^i - f^j) D_i D_j \Phi = D_i \Phi - D_j \Phi, \quad i \neq j,
\]

the general solution of which depends on \( N \) arbitrary functions of one variable.

In particular it is immediate to deduce [16] that the generating function (57) provides a one-parameter family of solutions of (62). Thus we can produce important solutions of (61) from linear superpositions of \( G(\lambda, R) \).

For example

\[
\Lambda_n^i = D_i \Phi, \quad \Phi := \frac{1}{2\pi i} \int_{\gamma_\infty} \lambda^n G(\lambda, R) \, d\lambda, \quad n \geq 0,
\]

\[
\Lambda_{-n}^i = D_i \Phi, \quad \Phi := \frac{1}{2\pi i} \int_{\gamma_0} \lambda^{-n} G(\lambda, R) \, d\lambda, \quad n > 0.
\]

Furthermore, in several important cases the general solution of (62) can be written in terms of \( G(\lambda, R) \).

Examples
If we set
\[ f^i(R^i) = \frac{R^i}{n}, \quad n = 1, 2, \ldots, \]
then the generating function (non normalized as \( \lambda \to \infty \)) is
\[ G(\lambda, R) = \prod_{i=1}^{N} \left( \lambda - \frac{R^i}{n} \right)^{-n}. \]

Hence, by taking for each \( i = 1, \ldots, N \) a closed loop \( \gamma_i \) in the complex \( \lambda \)-plane with positive orientation around \( \lambda_i = \frac{R^i}{n} \), the general solution of (62) can be expressed as
\[ \Phi := \frac{1}{2\pi i} \sum_{i=1}^{N} \int_{\gamma_i} \phi_i(\lambda) G(\lambda, R) \, d\lambda, \]
where the functions \( \phi_i(\lambda) \) are arbitrary. For example, if \( n = 1 \) it takes the form
\[ \Phi = \sum_{i=1}^{N} \phi_i(R^i) \prod_{k \neq i} \frac{1}{R^i - R^k}. \]

Similarly we may deal with the case
\[ f^i(R^i) = -\frac{R^i}{n}, \quad n = 1, 2, \ldots, \]
which leads to
\[ G(\lambda, R) = \prod_{i=1}^{N} \left( \lambda + \frac{R^i}{n} \right)^{n}. \]

Now to generate the general solution of (62) we take for each \( i = 1, \ldots, N \) a path \( \gamma_i(R^i) \) in the complex \( \lambda \)-plane ending at \( \lambda_i = -R^i/n \) so that we can write
\[ \Phi := \frac{1}{2\pi i} \sum_{i=1}^{N} \int_{\gamma_i(R^i)} \phi_i(\lambda) G(\lambda, R) \, d\lambda, \]
where the functions \( \phi_i(\lambda) \) are arbitrary.

Let us consider in detail the case
\[ f^i(R^i) = -R^i, \quad i = 1, 2. \]
One finds
\[
\Phi = \left(\theta_1(-R^1) - \theta_2(-R^2)\right)(R^1 - R^2) + 2 \int_{-R^1}^{R^1} \theta_1(\lambda) \, d\lambda + 2 \int_{-R^2}^{R^2} \theta_2(\lambda) \, d\lambda,
\]
with \(\theta_i(\lambda)\) being arbitrary functions. A normalized generating function is given by
\[
G(\lambda, R) = \frac{(\lambda + R^1)(\lambda + R^2)}{(\lambda + \lambda_0)^2}, \quad \lambda_0 \neq 0.
\] (63)

We may characterize the general solution of the hydrodynamic flows (55) corresponding to \(n = -1\) and \(n = -2\) by means of the hodograph formula
\[
x + A_{-1}(\lambda = -R^i) y + A_{-2}(\lambda = -R^i) z = D_i \Phi, \quad i = 1, 2
\] (64)
where \(y := x_{-1}, \ z := x_{-2}\). From (63) one calculates
\[
b_0 = c^2 R^1 R^2, \quad b_1 = c^2 (R^1 + R^2) - 2c^3 R^1 R^2, \quad c := \frac{1}{\lambda_0},
\]
and gets that the system (64) reads
\[
(x - c^2 z) - c^2 (y - 2cz) R^2 = \dot{\theta}_1(-R^1)(R^2 - R^1) - \theta_1(-R^1) - \theta_2(-R^2),
\]
\[
(x - c^2 z) - c^2 (y - 2cz) R^1 = \dot{\theta}_2(-R^2)(R^1 - R^2) - \theta_1(-R^1) - \theta_2(-R^2).
\] (65)

In particular, it implies
\[
R = R(x - c^2 z, y - 2cz),
\]
so that \(R\) is constant on the straight lines
\[
\bar{x} := (x, y, z) = (x_0, y_0, 0) + (c^2 + 2c, 1)s.
\]

We notice that starting from these solutions we may generate solutions
\[
u(x, y, z) := \int_{y}^{y_0} b_0(R) \, d\lambda + \int_{z}^{z_0} b_1(R) \, d\lambda,
\]
of the nonlinear equation
\[
u_{yy} = u_x u_{xx} - u_y u_z.
\] 17
Acknowledgements

The authors thank Professor Eugeni Ferapontov for useful discussions. A.B. Shabat was supported by a grant of the Ministerio de Cultura y Deporte of Spain. L. Martinez Alonso was supported by the DGCYT project BFM2002-01607.

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