EIGENFUNCTION EXPANSIONS FOR THE SCHRÖDINGER EQUATION WITH AN INVERSE-SQUARE POTENTIAL

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Dedicated to Professor I.V. Tyutin on the occasion of his 75th birthday

Abstract. We consider the one-dimensional Schrödinger equation $-f'' + \frac{\kappa^2 - 1/4}{r^2}f = Ef$ on the positive half-axis with the potential $q_\kappa(r) = (\kappa^2 - 1/4)r^{-2}$. For each complex number $\vartheta$, we construct a solution $u_{\kappa, \vartheta}(E)$ of this equation that is analytic in $\kappa$ in a complex neighborhood of the interval $(-1, 1)$ and, in particular, at the “singular” point $\kappa = 0$. For $-1 < \kappa < 1$ and real $\vartheta$, the solutions $u_{\kappa, \vartheta}(E)$ determine a unitary eigenfunction expansion operator $U_{\kappa, \vartheta} : L_2(0, \infty) \to L_2(\mathbb{R}, \mathcal{V}_{\kappa, \vartheta})$, where $\mathcal{V}_{\kappa, \vartheta}$ is a positive measure on $\mathbb{R}$. We show that every self-adjoint realization of the formal differential expression $-\partial_r^2 + q_\kappa(r)$ for the Hamiltonian is diagonalized by the operator $U_{\kappa, \vartheta}$ for some $\vartheta \in \mathbb{R}$. Using suitable singular Titchmarsh-Weyl $m$-functions, we explicitly find the measures $\mathcal{V}_{\kappa, \vartheta}$ and prove their continuity in $\kappa$ and $\vartheta$.

1. Introduction

This paper is devoted to eigenfunction expansions connected with the one-dimensional Schrödinger equation

$$-\partial_r^2 f(r) + \frac{\kappa^2 - 1/4}{r^2}f(r) = Ef(r), \quad r > 0,$$

where $\kappa$ and $E$ are real parameters. It is easy to see that the function $f(r) = r^{1/2} J_\kappa(E^{1/2}r)$, where $J_\kappa$ is the Bessel function of the first kind of order $\kappa$, is a solution of (1) for every $E > 0$ and $\kappa \in \mathbb{R}$ (this follows immediately from the fact that $J_\kappa$ satisfies the Bessel equation). These solutions can be used to expand square-integrable functions on the positive half-axis $\mathbb{R}^+ = (0, \infty)$. More precisely, given $\kappa > -1$ and a square-integrable complex function $\psi$ on $\mathbb{R}^+$ that vanishes for large $r$, we can define the function $\hat{\psi}$ on $\mathbb{R}^+$ by setting

$$\hat{\psi}(E) = \frac{1}{\sqrt{2}} \int_0^\infty \sqrt{r} J_\kappa(\sqrt{Er}) \psi(r) \, dr, \quad E > 0.$$

The map $\psi \to \hat{\psi}$ up to a change of variables then coincides with the well-known Hankel transformation $\mathbb{T}$ and induces a uniquely determined unitary operator in $L_2(\mathbb{R}^+)$. Since the development of a general theory of singular Sturm-Liouville problems by Weyl [2], this transformation has been used by many authors to illustrate various approaches to eigenfunction expansions for this kind of problem [3, 4, 9].

If $\kappa \geq 1$, then transformation $\mathbb{T}$ is the unique eigenfunction expansion associated with $\mathbb{T}$ up to normalization of eigenfunctions. On the other hand, for $|\kappa| < 1$,
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a one-parametric family of different expansions can be constructed using solutions of (1) (see Chap. 4 in [4]). The reason for this ambiguity is that the formal differential expression for the Hamiltonian

\[ -\frac{\partial^2}{\partial r^2} + \frac{\kappa^2 - 1/4}{r^2} \]

does not uniquely determine the quantum-mechanical problem for \(|\kappa| < 1\) and admits various self-adjoint realizations in \(L_2(\mathbb{R}_+)\) that yield different eigenfunction expansions. In [9], all self-adjoint realizations of (3) were characterized using suitable asymptotic boundary conditions and the corresponding eigenfunction expansions were explicitly found.

In both [4] and [9], the cases \(0 < |\kappa| < 1\) and \(\kappa = 0\) were treated separately and eigenfunction expansions for \(\kappa = 0\) could not be obtained from those for \(0 < |\kappa| < 1\) by taking the limit \(\kappa \to 0\). This situation is not quite satisfactory from the physical standpoint. In particular, self-adjoint operators associated with (3) can be used to construct self-adjoint realizations of Aharonov-Bohm Hamiltonian [10], in which case zero and nonzero \(\kappa\) correspond to integer and noninteger values of the dimensionless magnetic flux through the solenoid. Hence, the existence of a well-defined limit \(\kappa \to 0\) is necessary to ensure the continuous transition between integer and noninteger values of the flux in the Aharonov-Bohm model. Here, we propose a parametrization of self-adjoint realizations of (3) and corresponding eigenfunction expansions that is continuous in \(\kappa\) on the interval \((-1, 1)\) (and, in particular, at \(\kappa = 0\)).

We now formulate our main results. Let \(\lambda\) denote the Lebesgue measure on \(\mathbb{R}\) and \(C_0^\infty(\mathbb{R}_+)\) be the space of all smooth functions on \(\mathbb{R}_+\) with compact support. Given a \(\lambda\)-a.e. defined function \(f\) on \(\mathbb{R}_+\), we let \([f]\) denote the equivalence class of \(f\) with respect to the Lebesgue measure on \(\mathbb{R}_+\) (i.e., the restriction of the measure \(\lambda\) to \(\mathbb{R}_+\)). For every \(\kappa \in \mathbb{R}\), differential expression (3) naturally determines the operator \(\hat{h}_\kappa\) in \(L_2(\mathbb{R}_+)\) whose domain \(D_{\hat{h}_\kappa}\) consists of all elements \([f]\) with \(f \in C_0^\infty(\mathbb{R}_+)\):

\[ D_{\hat{h}_\kappa} = \{ [f] : f \in C_0^\infty(\mathbb{R}_+) \} , \]

\[ \hat{h}_\kappa [f] = [-f'' + q_\kappa f] , \quad f \in C_0^\infty(\mathbb{R}_+) . \]

Here, \(q_\kappa\) denotes the potential term in (3).

\[ q_\kappa(r) = \frac{\kappa^2 - 1/4}{r^2} , \quad r \in \mathbb{R}_+ . \]

The operator \(\hat{h}_\kappa\) is obviously symmetric and hence closable. The closure of \(\hat{h}_\kappa\) is denoted by \(\tilde{h}_\kappa\),

\[ \tilde{h}_\kappa = \overline{\hat{h}_\kappa} . \]

The self-adjoint extensions of \(\hat{h}_\kappa\) (or, equivalently, of \(\tilde{h}_\kappa\)) can be naturally interpreted as self-adjoint realizations of formal expression (3) (cf. Remark 5 below).

For any \(z, \kappa \in \mathbb{C}\), we define the function \(u^\kappa(z)\) on \(\mathbb{R}_+\) by the relation:

\[ u^\kappa(z) = r^{1/2+\kappa} X_\kappa(r^2 z) , \quad r \in \mathbb{R}_+ , \]

1Throughout the paper, a.e. means either “almost every” or “almost everywhere.”
2For brevity, we let \(u^\kappa(z|r)\) denote the value of the function \(u^\kappa(z)\) at a point \(r\): \(u^\kappa(z|r) = (u^\kappa(z))(r)\).
where the entire function $\mathcal{X}_\kappa$ is given by

$$
\mathcal{X}_\kappa(\zeta) = \frac{1}{2\kappa} \sum_{n=0}^{\infty} \frac{(-1)^n\zeta^n}{\Gamma(\kappa + n + 1)n!2^n}, \quad \zeta \in \mathbb{C}.
$$

The function $\mathcal{X}_\kappa$ is closely related to Bessel functions: for $z \neq 0$, we have

$$
\mathcal{X}_\kappa(\zeta) = \zeta^{-\kappa/2} J_\kappa(\zeta^{1/2}).
$$

Because $J_\kappa$ satisfies the Bessel equation, it follows that

$$
-\partial_r^2 u^{\pm\kappa}(z|r) + q_\kappa(r)u^{\pm\kappa}(z|r) = zu^{\pm\kappa}(z|r), \quad r \in \mathbb{R}_+,
$$

for every $\kappa \in \mathbb{C}$ and $z \neq 0$. By continuity, this also holds for $z = 0$. In particular, $u^{\pm\kappa}(E)$ are solutions of spectral problem (11) for every $\kappa, E \in \mathbb{R}$.

Given a positive Borel measure $\sigma$ on $\mathbb{R}$ and a $\sigma$-measurable complex function $g$, we let $T^\sigma_g$ denote the operator of multiplication by $g$ in $L_2(\mathbb{R}, \sigma)$.

It is well known (see, e.g., [5, 6, 9]) that the operator $h_\kappa$ is self-adjoint and can be diagonalized by Hankel transformation (2) for $\kappa \geq 1$. In terms of functions $u^\kappa(z)$, this result can be formulated as follows.

**Theorem 1.** Let $\kappa > -1$ and the measure $\mathcal{V}_\kappa$ on $\mathbb{R}$ be defined by (11). Then there is a unique unitary operator $U_\kappa: L_2(\mathbb{R}_+) \rightarrow L_2(\mathbb{R}, \mathcal{V}_\kappa)$ such that

$$
(U_\kappa \psi)(E) = \int_0^\infty u^\kappa(E|r)\psi(r) dr, \quad \psi \in L_2^0(\mathbb{R}_+),
$$

for $\mathcal{V}_\kappa$-a.e. $E$. The operator $U_\kappa^{-1}T^\kappa_\psi U_\kappa$, where $\iota$ is the identity function on $\mathbb{R}$ (i.e., $\iota(E) = E$ for all $E \in \mathbb{R}$), is a self-adjoint extension of $h_\kappa$ that coincides with $h_\kappa$ for $\kappa \geq 1$.

By (7) and (9), we have $u^\kappa(E|r) = E^{-\kappa/2}r^{1/2} J_\kappa(E^{1/2}r)$, $r \in \mathbb{R}_+$, for every $E > 0$. The operator $U_\kappa$ hence coincides with transformation (2) up to normalization of eigenfunctions. We note that $h_\kappa = h_{|\kappa|}$ for all $\kappa \in \mathbb{R}$ and $h_\kappa$ is therefore diagonalized by $U_{|\kappa|}$ for all real $\kappa$ such that $|\kappa| \geq 1$. If $0 \leq \kappa < 1$, then $U_\kappa^{-1}T^\kappa_\psi U_\kappa$ is the Friedrichs extension of $h_\kappa$ (see [11]).

We now turn to parametrizing all self-adjoint extensions of $h_\kappa$ in the case $-1 < \kappa < 1$. Let

$$
\mathcal{O} = \{ \kappa \in \mathbb{C} : \kappa \neq \pm 1, \pm 2, \ldots \}.
$$

For $\kappa \in \mathcal{O}$ and $\vartheta, z \in \mathbb{C}$, we define the function $u^\kappa_\vartheta(z)$ on $\mathbb{R}_+$ by setting

$$
u^\kappa_\vartheta(z) = \frac{u^\kappa(z) \sin(\vartheta + \vartheta_\kappa) - u^{-\kappa}(z) \sin(\vartheta - \vartheta_\kappa)}{\sin \pi \kappa}, \quad \kappa \in \mathcal{O} \setminus \{0\},
$$

Here and hereafter, we assume that the function $q_\kappa$ on $\mathbb{R}_+$ is defined by [5] for all $\kappa \in \mathbb{C}$.

More precisely, $T^\sigma_g$ is the operator in $L_2(\mathbb{R}, \sigma)$ whose graph consists of all pairs $(\varphi_1, \varphi_2)$ such that $\varphi_1, \varphi_2 \in L_2(\mathbb{R}, \sigma)$ and $\varphi_2(E) = g(E)\varphi_1(E)$ for $\sigma$-a.e. $E$.

We recall that a Borel measure $\sigma$ on $\mathbb{R}$ is called a Radon measure on $\mathbb{R}$ if $\sigma(K) < \infty$ for every compact set $K \subset \mathbb{R}$. 
The equality
\[ \pi Y \quad \text{and} \quad \delta (18) \]
and
\[ \Gamma \quad \text{and} \quad \gamma (13) \]
the entire function \( Y \) and 
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the Bessel equation.

\[ (14) \]
\[ \vartheta \]

\[ (15) \]
\[ C \]
\[ \{ \}

\[ (17) \]
\[ d\bar{V}_{\kappa,\vartheta}(E) = \frac{\Gamma(E) \sin^2 \pi \kappa}{2 E^{-\kappa} \sin^2(\vartheta + \vartheta_\kappa) - 2 \cos \pi \kappa \sin(\vartheta + \vartheta_\kappa) \sin(\vartheta - \vartheta_\kappa) + E^\kappa \sin^2(\vartheta - \vartheta_\kappa)} dE \]
and \( \delta_{E_{\kappa,\vartheta}} \) is the Dirac measure at the point

\[ (18) \]
\[ E_{\kappa,\vartheta} = - \left( \frac{\sin(\vartheta + \vartheta_\kappa)}{\sin(\vartheta - \vartheta_\kappa)} \right)^{1/\kappa}. \]

To compute the limit of \( u_0^\kappa(z|r) \) as \( \kappa \to 0 \), we must apply L'Hôpital's rule and use the equality 
\[ \Gamma'(1 + n)/\Gamma(1 + n) = c_n - \gamma \quad \text{(see formula (9) in Sec. 1.7.1 in [12])}. \]

Alternatively, we can express \( u_0^\kappa(z|r) \) in terms of the Bessel functions \( J_0 \) and \( Y_0 \) by means of the equality 
\[ \kappa \gamma(\zeta) = 2 \left( \gamma + \log \frac{2}{\zeta} \right) J_0(\zeta) - 2 Y(\zeta^2) \quad \text{(see formula (33) in Sec. 7.2.4 in [12]) and use the Bessel equation.} \]
For $\kappa = 0$, the measure $\mathcal{V}_{\kappa, \vartheta}$ is defined by taking the limit $\kappa \to 0$ in formulas (16)–(18). This yields

$$\mathcal{V}_{0, \vartheta} = \begin{cases} \tilde{\mathcal{V}}_{0, \vartheta}, & \vartheta \in \pi \mathbb{Z}, \\ \frac{\pi |E_{0, \vartheta}|}{2 \sin^2 \vartheta} \delta_{E_{0, \vartheta}} + \tilde{\mathcal{V}}_{0, \vartheta}, & \vartheta \notin \pi \mathbb{Z}, \end{cases}$$

where

$$E_{0, \vartheta} = -e^{\pi \cot \vartheta}$$

and the positive Radon measure $\tilde{\mathcal{V}}_{0, \vartheta}$ on $\mathbb{R}$ is given by

$$d\tilde{\mathcal{V}}_{0, \vartheta}(E) = \frac{1}{2} \left( \Theta(E) \right) \frac{\Theta(E)}{(\cos \vartheta - \pi^{-1} \log E \sin \vartheta)^2 + \sin^2 \vartheta} dE.$$

The next theorem describes self-adjoint extensions of $h_\kappa$ for $-1 < \kappa < 1$ in terms of their eigenfunction expansions.

**Theorem 3.** Let $-1 < \kappa < 1$. For every $\vartheta \in \mathbb{R}$, there is a unique unitary operator $U_{\kappa, \vartheta} : L_2(\mathbb{R}^+) \to L_2(\mathbb{R}, \mathcal{V}_{\kappa, \vartheta})$ such that

$$(U_{\kappa, \vartheta} \psi)(E) = \int_0^\infty u^\kappa_{\vartheta}(E|r) \psi(r) \, dr, \quad \psi \in L_2^c(\mathbb{R}^+),$$

for $\mathcal{V}_{\kappa, \vartheta}$-a.e. $E$. The operator

$$h_{\kappa, \vartheta} = U_{\kappa, \vartheta}^{-1} T_i^{\mathcal{V}_{\kappa, \vartheta}} U_{\kappa, \vartheta},$$

where $i$ is the identity function on $\mathbb{R}$, is a self-adjoint extension of $h_\kappa$. Conversely, every self-adjoint extension of $h_\kappa$ is equal to $h_{\kappa, \vartheta}$ for some $\vartheta \in \mathbb{R}$. Given $\vartheta, \vartheta' \in \mathbb{R}$, we have $h_{\kappa, \vartheta} = h_{\kappa, \vartheta'}$ if and only if $\vartheta - \vartheta' \in \pi \mathbb{Z}$.

For $\vartheta = \vartheta_0$, we have $\mathcal{V}_{\kappa, \vartheta_0} = \mathcal{V}_\kappa$ and $u^\kappa(z) = u^\kappa_0(z)$ for all $z \in \mathbb{C}$, and the operator $U_{\kappa, \vartheta}$ therefore coincides with the Hankel transformation $U_\kappa$.

The expansions described by Theorem 3 have the advantage that neither the eigenfunctions $u^\kappa_{\vartheta}(E)$ nor the spectral measures $\mathcal{V}_{\kappa, \vartheta}$ have any discontinuities at $\kappa = 0$. This follows from Lemma 2 and the next theorem.

**Theorem 4.** Let $\varphi$ be a continuous function or a bounded Borel function on $\mathbb{R}$ with compact support. Then $(\kappa, \vartheta) \to \int \varphi(E) d\mathcal{V}_{\kappa, \vartheta}(E)$ is respectively a continuous function or a Borel function on $(-1, 1) \times \mathbb{R}$ that is bounded on $[-\alpha, \alpha] \times \mathbb{R}$ for every $0 \leq \alpha < 1$.

Our main results are Theorems 3 and 4. We also give a new proof of Theorem 1 based on locally defined singular $m$-functions (see below).

To prove Theorems 1 and 3 we use a recently developed variant of the Titchmarsh-Weyl-Kodaira theory [6, 8]. In those papers, a generalization of the notion of the Titchmarsh-Weyl $m$-function was proposed that is applicable not only to problems with a regular endpoint but also to a broad class of Schrödinger operators with two singular endpoints. Using such singular $m$-functions leads to a notable simplification in the treatment of eigenfunction expansions in comparison with the general theory [13, 5] based on matrix-valued measures (but we note that the results

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8In this paper, the term “spectral measure” always refers to a certain positive measure on $\mathbb{R}$ whose precise definition is given in Proposition 14. This usage differs from that adopted in [10], where this term was applied to projection-valued measures in a Hilbert space.
in [6] [8] for eigenfunction expansions can be easily derived from Kodaira’s general approach [13]; see Remark [16] below).

The paper is organized as follows. In Sec. 2, we give the general theory concerning self-adjoint extensions of one-dimensional Schrödinger operators and their eigenfunction expansions. The main statement in that section, Proposition 13, is similar to Theorem 3.4 in [8], but unlike the latter gives a local version of the formula for the spectral measures. This allows using different $m$-functions for different regions of the spectral parameter. In Sec. 3 we give a proof of Theorem 1 illustrating this local approach to finding spectral measures and establish Theorem 5. Section 4 is devoted to the proof of Theorem 3.

2. ONE-DIMENSIONAL SCHRODINGER OPERATORS

In this section, we recall basic facts [5] [14] [15] concerning self-adjoint extensions of one-dimensional Schrödinger operators and briefly describe the approach to eigenfunction expansions developed in [6] [8]. A distinctive feature of the subsequent exposition is that it uses the notion of a boundary space (see Definition 6 below) that can be viewed as a formalization of the concept of a self-adjoint boundary condition. Using boundary spaces allows treating the limit point and limit circle cases on equal footing whenever possible, which makes the presentation of results clearer.

Let $-\infty \leq a < b \leq \infty$, $\lambda_{a,b}$ be the restriction to $(a,b)$ of the Lebesgue measure $\lambda$ on $\mathbb{R}$, and $D$ be the space of all complex continuously differentiable functions on $(a,b)$ whose derivative is absolutely continuous on $(a,b)$ (i.e., absolutely continuous on every segment $[c,d]$ with $a < c \leq d < b$). Let $q$ be a complex locally integrable function on $(a,b)$. Given $z \in \mathbb{C}$, we let $l_{q,z}$ denote the linear operator from $D$ to the space of complex $\lambda_{a,b}$-equivalence classes such that

$$
(l_{q,z}f)(r) = -f''(r) + q(r)f(r) - zf(r)
$$

for $\lambda$-a.e. $r \in (a,b)$ and set

$$
l_q = l_{q,0}.
$$

For every $f \in D$ and $z \in \mathbb{C}$, we have $l_{q,z}f = l_qf - z[f]$, where $[f] = [f]_{\lambda_{a,b}}$ is the $\lambda_{a,b}$-equivalence class of $f$. For every $c \in (a,b)$ and all complex numbers $z$, $\zeta_1$, and $\zeta_2$, there is a unique solution $f$ of the equation $l_{q,z}f = 0$ such that $f(c) = \zeta_1$ and $f'(c) = \zeta_2$. This implies that solutions of $l_{q,z}f = 0$ constitute a two-dimensional subspace of $D$. The Wronskian $W_r(f,g)$ at a point $r \in (a,b)$ of any functions $f,g \in D$ is defined by the relation

$$
W_r(f,g) = f(r)g'(r) - f'(r)g(r).
$$

Clearly, $r \rightarrow W_r(f,g)$ is an absolutely continuous function on $(a,b)$. If $f$ and $g$ are such that $r \rightarrow W_r(f,g)$ is a constant function on $(a,b)$ (in particular, this is the case when $f$ and $g$ are solutions of $l_{q,z}f = l_{q,z}g = 0$ for some $z \in \mathbb{C}$), then its value is denoted by $W(f,g)$. It follows immediately from (23) that the identities

$$
W_r(f_1, f_2)W_r(f_3, f_4) + W_r(f_1, f_3)W_r(f_4, f_2) + W_r(f_2, f_3)W_r(f_1, f_4) = 0,
$$

$$
W_r(f_1, f_2, f_3, f_4) = f_1(r)f_2(r)W_r(f_3, f_4) + W_r(f_1, f_3)f_2(r)f_4(r) + W_r(f_1, f_4)f_2(r)f_3(r)
$$

hold for any $f_1, f_2, f_3, f_4 \in D$ and $r \in (a,b)$.

In the rest of this section, we assume that $q$ is real. Let

$$
D_q = \{f \in D : f \text{ and } l_qf \text{ are both square-integrable on } (a,b)\}.
A \( \lambda_{a,b} \)-measurable complex function \( f \) is said to be left or right square-integrable on \( (a, b) \) if respectively \( \int_{a}^{c} |f(r)|^2 \, dr < \infty \) or \( \int_{c}^{b} |f(r)|^2 \, dx < \infty \) for any \( c \in (a, b) \). The subspace of \( D \) consisting of left or right square-integrable on \( (a, b) \) functions \( f \) such that \( l_q f \) is also respectively left or right square-integrable on \( (a, b) \) is denoted by \( D^q_L \) or \( D^q_R \). We obviously have \( D_q = D^q_L \cap D^q_R \). It follows from (22) by integrating by parts that

\[
\int_{c}^{d} ((l_q z f)(r)g(r) - f(r)(l_q z g)(r)) \, dr = W_q(f, g) - W_q(f, g)
\]

for every \( f, g \in D, z \in \mathbb{C} \), and \( c, d \in (a, b) \). This implies the existence of limits \( W_a(f, g) = \lim_{d \to a} W_q(f, g) \) and \( W_b(f, g) = \lim_{c \to b} W_q(f, g) \) respectively for every \( f, g \in D_q \) and \( f, g \in D_q^r \). Moreover, it follows that

\[
\langle l_q f, [g] \rangle - \langle [f], l_q g \rangle = W_b(\bar{f}, g) - W_a(\bar{f}, g)
\]

for any \( f, g \in D_q \), where \( \langle \cdot, \cdot \rangle \) is the scalar product in \( L_2(a, b) \).

For any linear subspace \( Z \) of \( D_q \), let \( L_q(Z) \) be the linear operator in \( L_2(a, b) \) defined by the relations

\[
D_{L_q(Z)} = \{ [f] : f \in Z \},
L_q(Z)[f] = l_q f, \quad f \in Z.
\]

We define the minimal operator \( L_q \) by setting

\[
L_q = L_q(D_q^0)
\]

where

\[
D_q^0 = \{ f \in D_q : W_q(f, g) = W_b(f, g) = 0 \text{ for any } g \in D_q \}.
\]

By (23), the operator \( L_q(Z) \) is symmetric if and only if \( W_a(\bar{f}, g) = W_b(\bar{f}, g) \) for any \( f, g \in Z \). In particular, \( L_q \) is a symmetric operator. Moreover, \( L_q \) is closed and densely defined, and its adjoint \( L_q^* \) is given by

\[
L_q^* = L_q(D_q)
\]

(see Lemma 9.4 in [14]). If \( T \) is a symmetric extension of \( L_q \), then \( L_q^* \) is an extension of \( T^* \) and hence of \( T \). In view of (30), this implies that \( T \) is of the form \( L_q(Z) \) for some subspace \( Z \) of \( D_q \).

**Remark 5.** Self-adjoint operators of the form \( L_q(Z) \) can be naturally viewed as self-adjoint realizations of the differential expression \(-d^2/dr^2 + q\). If \( L_q(Z) \) is self-adjoint, then equality (30) and the closedness of \( L_q \) imply that \( L_q(Z) \) is an extension of \( L_q \) because \( L_q(D_q^0) \) is an extension of \( L_q(Z) \). Therefore, the self-adjoint realizations of the expression \(-d^2/dr^2 + q\) are precisely the self-adjoint extensions of the minimal operator \( L_q \).

**Definition 6.** We say that a linear subspace \( X \) of \( D_q^0 \) is a left boundary space if

1. if \( W_a(\bar{f}, g) = 0 \) for any \( f, g \in X \) and
2. if \( g \in X \) whenever \( g \in D_q^0 \) satisfies the equality \( W_a(\bar{f}, g) = 0 \) for all \( f \in X \).

Replacing \( D_q^0 \) with \( D_q^r \) and \( a \) with \( b \), we obtain the definition of a right boundary space.
Definition 7. If \( W_a(f, g) = 0 \) for any \( f, g \in D_q^l \), then \( q \) is said to be in the limit point case (l.p.c.) at \( a \). Otherwise \( q \) is said to be in the limit circle case (l.c.c.) at \( a \). Similarly, \( q \) is said to be in the l.p.c. at \( b \) if \( W_b(f, g) = 0 \) for any \( f, g \in D_q^r \) and to be in the l.c.c. at \( b \) otherwise.

Clearly, \( q \) is in the l.p.c. at \( a \) or \( b \) if and only if \( D_q^l \) or \( D_q^r \) is the respective unique left or right boundary space. Given \( f \in D_q^l \) or \( f \in D_q^r \), we set
\[
D_{q,f}^l = \{ g \in D_q^l : W_a(\bar{f}, g) = 0 \}, \quad D_{q,f}^r = \{ g \in D_q^r : W_b(\bar{f}, g) = 0 \}.
\]

For every \( E \in \mathbb{R} \), we let \( S_{q,E}^l \) and \( S_{q,E}^r \) denote the respective sets of all nontrivial real elements \( f \) of \( D_q^l \) and \( D_q^r \) such that \( l_{q,E} f = 0 \).

The next proposition reformulates well-known results concerning self-adjoint extensions of \( L_q \) (see, e.g., Sec. 9.2 in [13]) in the language of boundary spaces.

Proposition 8. Let \( q \) be a real locally integrable function on \((a, b)\). Then the following statements hold:

1. Let \( X \) and \( Y \) respectively be left and right boundary spaces. Then the operator \( L_q(X \cap Y) \) is a self-adjoint extension of \( L_q \).
2. Let \( L_q(X \cap Y) = L_q(\bar{X} \cap \bar{Y}) \) for some left boundary spaces \( X \) and \( \bar{X} \) and right boundary spaces \( Y \) and \( \bar{Y} \). Then we have \( X = \bar{X} \) and \( Y = \bar{Y} \).
3. Let \( E \in \mathbb{R} \) and \( f \in S_{q,E}^l \) or \( f \in S_{q,E}^r \). Then \( D_{q,f}^l \) or \( D_{q,f}^r \) is respectively a left or right boundary space.
4. Let \( z \in \mathbb{C} \). Then \( q \) is in l.c.c. at \( a \) or at \( b \) if and only if every \( f \in D \) such that \( l_{q,z} f = 0 \) is respectively left or right square-integrable on \((a, b)\).
5. If \( q \) is in l.p.c. either at \( a \) or at \( b \), then every self-adjoint extension of \( L_q \) is equal to \( L_q(X \cap Y) \) for some left boundary space \( X \) and right boundary space \( Y \).
6. Let \( q \) be in l.c.c. at \( a \) or \( b \) \( E \in \mathbb{R} \). Then every left or right boundary space is respectively equal to \( D_{q,f}^l \) or \( D_{q,f}^r \) for some \( f \in S_{q,E}^l \) or \( f \in S_{q,E}^r \).

The operators of the form \( L_q(X \cap Y) \), where \( X \) and \( Y \) are left and right boundary spaces, are called self-adjoint extensions of \( L_q \) with separated boundary conditions.

Remark 9. As mentioned above, boundary spaces can be thought of as self-adjoint boundary conditions. In this sense, the domain of \( L_q(X \cap Y) \) consists of (the \( \lambda_{a,b} \)-equivalence classes of) all elements of \( D_q \) satisfying the self-adjoint boundary conditions \( X \) and \( Y \) on the respective left and right.

Remark 10. Let \( f \) and \( g \) be linear independent solutions of \( l_{q,z} f = l_{q,z} g = 0 \), where \( \text{Im} \ z \neq 0 \). Suppose \( f \) satisfies a self-adjoint boundary condition at \( a \) (i.e., belongs to some left boundary space). Let \( A \) denote the set of all \( \zeta \in \mathbb{C} \) such that \( q + \zeta f \) belongs to some right boundary space. Then \( A \) is either a one-point set or a circle depending on whether \( q \) is in the l.p.c. or l.c.c. at \( b \). Moreover, \( A \) is the limit of the circles \( A_\alpha \) obtained by replacing \( b \) with a regular endpoint \( c \in (a, b) \) in the definition of \( A \). Such a limit procedure was originally used by Weyl [2] to distinguish between the l.p.c. and l.c.c.

If \( q \) is in the l.p.c. at both \( a \) and \( b \), then statement 1 in Proposition 8 implies that the operator \( L_q(D_q) \) is self-adjoint. In view of (30), it follows that \( L_q \) is self-adjoint.

For every \( f \in D_q^l \), we set
\[
L_q^l = L_q(D_q^l \cap D_q^r).
\]
Lemma 11. Let $E \in \mathbb{R}$ and $q$ be in the l.c.c. at $a$ and in the l.p.c. at $b$. Then the self-adjoint extensions of $L_q^f$, where $f \in S_{q,E}^1$, are precisely the operators $L_q^g$, where $g \in S_{q,E}^1$. For $f, g \in S_{q,E}^1$, the equality $L_q^f = L_q^g$ holds if and only if $g = cf$ for some real $c \neq 0$.

Proof. The first statement follows immediately from statements 1, 3, 5, and 6 in Proposition 8. Let $f, g \in S_{q,E}^1$. If $g = cf$, then we have

$$D_{q,f}^l = D_{q,g}^l$$

by (31) and therefore $L_q^f = L_q^g$. Conversely, if $L_q^f = L_q^g$, then statements 2 and 3 in Proposition 8 imply equality (33). Because $f \in D_{q,f}^l$ by (31), we conclude that $f \in D_{q,g}^l$ and hence $W_a(f,g) = 0$. Because $l_{q,z}f = l_{q,z}g = 0$, it follows that $W(g,f) = 0$, whence $g = cf$. \hfill \Box

We now consider the eigenfunction expansions associated with $L_q$.

Let $O \subset \mathbb{C}$ be an open set. We say that a map $u : O \to \mathcal{D}$ is a $q$-solution in $O$ if $l_{q,z}u(z) = 0$ for every $z \in O$. A $q$-solution $u$ in $O$ is said to be analytic if the functions $z \rightarrow u(z|r)$ and $z \rightarrow \partial_z u(z|r)$ are analytic in $O$ for any $r \in (a,b)$. A $q$-solution $u$ in $O$ is said to be nonvanishing if $u(z) \neq 0$ for every $z \in O$ and is said to be real if $u(E)$ is real for every $E \in O \cap \mathbb{R}$.

Definition 12. A triple $(q,Y,u)$ is called an expansion triple if $q$ is a real locally integrable function on $(a,b)$, $Y$ is a right boundary space, and $u$ is a real nonvanishing analytic $q$-solution in $\mathbb{C}$ satisfying the following conditions:

1. $u(z) \in D_{q}^l$ for all $z \in \mathbb{C}$ and
2. there exists $E \in \mathbb{R}$ such that $W_a(u(E),u(z)) = 0$ for all $z \in \mathbb{C}$.

Lemma 13. Let $t = (q,Y,u)$ be an expansion triple. Then there is a unique left boundary space $X^t$ such that $u(z) \in X^t$ for all $z \in \mathbb{C}$. For every $E \in \mathbb{R}$, we have $X^t = D_{q,u(E)}^l$.

Proof. Let $E \in \mathbb{R}$ and $X$ be a left boundary space containing $u(E)$. By (31) and condition (1) in Definition 6 we have $X \subset D_{q,u(E)}^l$. On the other hand, if $g \in D_{q,u(E)}^l$, then we have $W_a(f,g) = 0$ for every $f \in X$ because $D_{q,u(E)}^l$ is a left boundary space by statement 3 in Proposition 8. In view of condition (2) in Definition 6 we conclude that $g \in X$ and hence $X = D_{q,u(E)}^l$. This implies that $X^t$ (if it exists) is unique and equal to $D_{q,u(E)}^l$ for all $E \in \mathbb{R}$. By (31) and Definition 12 there exists $E \in \mathbb{R}$ such that $u(z) \in D_{q,u(E)}^l$ for all $z \in \mathbb{C}$. This proves the existence of $X^t$. \hfill \Box

Let $t = (q,Y,u)$ be an expansion triple, $\tilde{u}$ be a real analytic $q$-solution in $\mathbb{C}$ such that $W(u(z),\tilde{u}(z)) \neq 0$ for every $z \in \mathbb{C}$, and $v$ be a nonvanishing analytic $q$-solution in $\mathbb{C}_+$ such that $v(z) \in Y$ for all $z \in \mathbb{C}_+$ (such $\tilde{u}$ and $v$ always exist; see Lemma 2.4 in 8 and Lemma 9.8 in 14). Then $W(v(z),u(z)) \neq 0$ for every $z \in \mathbb{C}_+$ because we would otherwise have $u(z) \in X^t \cap Y$ and hence the self-adjoint operator $L_q(X^t \cap Y)$ would have an eigenvalue in $\mathbb{C}_+$. We define the analytic function $M_{g}^t$ in $\mathbb{C}_+$ by the relation

$$M_{g}^t(z) = \frac{1}{\pi} \frac{W(v(z),\tilde{u}(z))}{W(v(z),u(z))W(u(z),\tilde{u}(z))}$$

As usual, $\mathbb{C}_+$ denotes the open upper half-plane of the complex plane: $\mathbb{C}_+ = \{ z \in \mathbb{C} : \text{Im } z > 0 \}$. 
(this definition is obviously independent of the choice of $v$). Following [8], we call such functions singular Titchmarsh-Weyl $m$-functions. Below, we see that it is sometimes useful to consider $q$-solutions $\tilde{u}$ that are defined on an open set $O \subset \mathbb{C}$ other than the entire complex plane. In that case, we assume that $\mathcal{M}_0^q$ is defined on $O \cap \mathbb{C}_+$. Let $L_2^q(a, b)$ denote the subspace of $L_2(a, b)$ consisting of all its elements vanishing $\lambda$-a.e. outside some compact subset of $(a, b)$. The next proposition gives a way of constructing eigenfunction expansions for self-adjoint extensions of $L_q$ with separated boundary conditions.

**Proposition 14.** Let $t = (q, Y, u)$ be an expansion triple. Then the following statements hold:

1. There exists a unique positive Radon measure $\sigma$ on $\mathbb{R}$ (called the spectral measure for $t$) such that
   \[
   \int \varphi(E) \text{Im} \mathcal{M}_0^q(E + i\eta) \, dE \to \int \varphi(E) \, d\sigma(E) \quad (\eta \downarrow 0)
   \]
   for every continuous function $\varphi$ on $\mathbb{R}$ with compact support and every real analytic $q$-solution $\tilde{u}$ in $\mathbb{C}$ such that $W(u(z), \tilde{u}(z)) \neq 0$ for every $z \in \mathbb{C}$.

2. Let $\sigma$ be the spectral measure for $t$. There is a unique unitary operator $U : L_2(a, b) \to L_2(\mathbb{R}, \sigma)$ (called the spectral transformation for $t$) such that
   \[
   (U\psi)(E) = \int_a^b u(E|r)\psi(r) \, dr, \quad \psi \in L_2^q(a, b),
   \]
   for $\sigma$-a.e. $E$.

3. Let $\sigma$ and $U$ be the spectral measure and transformation for $t$, and let the left boundary space $X^\downarrow$ be as in Lemma [13]. Then we have
   \[
   L_q(X^\downarrow \cap Y) = U^{-1}T^\downarrow U,
   \]
   where $\iota$ is the identity function on $\mathbb{R}$.

4. Let $\sigma$ be the spectral measure for $t$, $O \subset \mathbb{C}$ be an open set, and $\tilde{u}$ be a real analytic $q$-solution in $O$ such that $W(u(z), \tilde{u}(z)) \neq 0$ for every $z \in O$. Then we have
   \[
   \int_{\text{supp} \varphi} \varphi(E) \text{Im} \mathcal{M}_0^q(E + i\eta) \, dE \to \int_{O \cap \mathbb{R}} \varphi(E) \, d\sigma(E) \quad (\eta \downarrow 0)
   \]
   for every continuous function $\varphi$ on $O \cap \mathbb{R}$ with compact support (supp $\varphi$ denotes the support of $\varphi$).

**Proof.** Statements 1–3 are a straightforward reformulation of the corresponding results in [8] in the language of boundary spaces. Let $O$ and $\tilde{u}$ satisfy the conditions in statement 4 and $\theta$ be a real analytic $q$-solution in $\mathbb{C}$ such that $W(u(z), \theta(z)) \neq 0$ for every $z \in \mathbb{C}$. Substituting $f_1 = u(z)$, $f_2 = v(z)$, $f_3 = \tilde{u}(z)$, and $f_4 = \theta(z)$ in (24) and dividing the result by $\pi W(u(z), \theta(z))W(u(z), \tilde{u}(z))$ yields
\[
\mathcal{M}_0^q(z) = \mathcal{M}_0^q(z) + \frac{1}{\pi W(u(z), \theta(z))W(u(z), \tilde{u}(z))} \frac{W(u(z), \theta(z))}{W(u(z), \tilde{u}(z))}
\]
for any $z \in O \cap \mathbb{C}_+$. Statement 4 now follows from statement 1 because the last term in the right-hand side is analytic in $O$ and real on $O \cap \mathbb{R}$. \qed
Corollary 15. Let $\sigma$ and $U$ be the spectral measure and transformation for an expansion triple $t = (q, Y, u)$. Then we have

$$
(U^{-1}\varphi)(r) = \int u(E|r)\varphi(E) \, d\sigma(E), \quad \varphi \in L_2^\circ(\mathbb{R}, \sigma),
$$

for $\lambda$-a.e. $r \in (a, b)$. If $\sigma\{E\} \neq 0$ for some $E \in \mathbb{R}$, then $[u(E)]$ is an eigenfunction of $L_q(X^j \cap Y)$. If $\sigma\{E\} = 0$ we have $U^{-1}[\chi(E)] = \sigma\{E\}[u(E)]$, where $\chi(E)$ is the characteristic function of the one-point set $\{E\}$.

Proof. Given $\varphi \in L_2^\circ(\mathbb{R}, \sigma)$ and $r \in (a, b)$, let $\tilde{\varphi}(r)$ denote the right-hand side of (35). By statement 2 in Proposition 14, we have

$$
\langle \psi, U^{-1}\varphi \rangle = \langle U\psi, \varphi \rangle = \int d\sigma(E)\varphi(E) \int_a^b \overline{\psi(r)}u(E|r) \, dr = \int_a^b \overline{\psi(r)}\tilde{\varphi}(r) \, dr
$$

for any $\psi \in L_2^\circ(a, b)$, whence (35) follows. In particular, we have $U^{-1}[\chi(E)] = \sigma\{E\}[u(E)]$, where $\chi(E)$ is the characteristic function of the one-point set $\{E\}$. By statement 3 in Proposition 14 this implies that $[u(E)]$ is an eigenfunction of $L_q(X^j \cap Y)$ if $\sigma\{E\} \neq 0$. \□

Remark 16. While the above proof of Proposition 14 refers to [8], this result can also be easily derived using Kodaira’s general approach [13] based on matrix-valued measures. Indeed, if we set $s_1(z) = \tilde{u}(z)/W(u(z), \tilde{u}(z))$ and $s_2(z) = u(z)$ for $z \in \mathbb{C}$, then the only non-real entry $M_{22}(z)$ of the characteristic matrix $M$ defined by formula (1.13) in [13] is equal to $\pi M_2^\circ(\bar{z})$ and statements 1–3 in Proposition 14 hence essentially coincide with Theorem 1.3 in [13] in this case. The simple direct proof given in [8] employs a single $m$-function and does not involve matrix-valued measures. It essentially relies on the technique developed in [6], where potentials in the l.p.c. at both endpoints were considered (a treatment in the same spirit for the l.c.c. at one of the endpoints can be found in [16]). A similar approach to finding spectral measures was also proposed in [9] in the context of the Schrödinger equation with the inverse-square potential.

If $q$ is locally square-integrable on $(a, b)$, then formulas (28) and (29) imply that the space $C_0^\infty(a, b)$, of smooth functions on $(a, b)$ with compact support is contained in $D_q^0$ and $L_q$ is an extension of $L_q(C_0^\infty(a, b))$. The proof of the next lemma is given in Appendix B.

Lemma 17. Let $q$ be a real locally square-integrable function on $(a, b)$. Then $L_q$ is the closure of $L_q(C_0^\infty(a, b))$.

3. Eigenfunction expansions for inverse-square potential

We now assume that $a = 0$ and $b = \infty$ and apply the above general theory to the potential $q_\kappa$ given by (34). It follows immediately from (4) and (27) that $h_\kappa = L_{q_\kappa}(C_0^\infty(\mathbb{R}^+))$. In view of (6) and Lemma 17 this implies that

$$
h_\kappa = L_{q_\kappa}, \quad \kappa \in \mathbb{R}.
$$

The equation $l_{q_\kappa} f = 0$ has linearly independent solutions $r^{1/2 \pm \kappa}$ for $\kappa \neq 0$ and $r^{1/2}$ for $\kappa = 0$. We conclude that by statement 4 in Proposition 8

1. $q_\kappa$ is in the l.p.c. at both 0 and $\infty$ for real $\kappa$ such that $|\kappa| \geq 1$
2. $q_\kappa$ is in the l.p.c. at $\infty$ and in the l.c.c. at 0 for $-1 < \kappa < 1$. 


Hence, the operator \( h_\kappa \) is self-adjoint for \(|\kappa| \geq 1\) and has multiple self-adjoint extensions for \(-1 < \kappa < 1\).

For any \( \kappa \in \mathbb{C} \), let the map \( u^{\kappa} : \mathbb{C} \to \mathcal{D} \) be defined by (7). By (10), we have
\[
l_{q_\kappa, z} u^{\pm \kappa}(z) = 0, \quad z, \kappa \in \mathbb{C}.
\]

In what follows, we systematically use notation (15) for the complex plane with a cut along a ray. We let \( \log \) denote the branch of the logarithm in \( \mathbb{C}_{3\pi/2} \) satisfying the condition \( \log 1 = 0 \) and set \( z^\rho = e^{\rho \log z} \) for all \( z \in \mathbb{C}_{3\pi/2} \) and \( \rho \in \mathbb{C} \).

For any \( \kappa \in \mathbb{C} \), we define the map \( v^{\kappa} : \mathbb{C}_{3\pi/2} \to \mathcal{D} \) by the relation
\[
v^{\kappa}(z|r) = \frac{i\pi}{2} e^{i\pi \kappa/2 r^{1/2}} H^{(1)}_\kappa(r^{3/2}), \quad r \in \mathbb{R}_+, \ z \in \mathbb{C}_{3\pi/2},
\]
where \( H^{(1)}_\kappa \) is the first Hankel function of order \( \kappa \). Because \( H^{(1)}_\kappa \) is a solution of the Bessel equation, we have
\[
l_{q_\kappa, z} v^{\kappa}(z) = 0
\]
for every \( z \in \mathbb{C}_{3\pi/2} \) and \( \kappa \in \mathbb{C} \). It follows from the relation \( H^{(1)}_{-\kappa} = e^{i\pi \kappa} H^{(1)}_\kappa \) (formula (9) in Sec. 7.2.1 in [12]) that
\[
v^{-\kappa}(z) = v^{\kappa}(z), \quad \kappa \in \mathbb{C}, \ z \in \mathbb{C}_{3\pi/2}.
\]

The well-known asymptotic form of \( H^{(1)}_\kappa(\zeta) \) for \( \zeta \to \infty \) (see formula (1) in Sec. 7.13.1 in [12]) implies that
\[
v^{\kappa}(z|r) \sim 2^{-1} \sqrt{\pi(i + 1)z^{-1/4} e^{i\pi z^{1/2}}}, \quad r \to \infty,
\]
for every \( \kappa \in \mathbb{C} \) and \( z \in \mathbb{C}_{3\pi/2} \) and hence \( v^{\kappa}(z) \) is right square-integrable for all \( \kappa \in \mathbb{C} \) and \( z \in \mathbb{C}_+ \). Using the expression for the Wronskian of Bessel functions (formula (29) in Sec. 7.11 in [12]),
\[
W_z(J_\kappa, H^{(1)}_\kappa) = \frac{2i}{\pi z},
\]
and taking (10) into account, from (7), (9), and (38), we derive that
\[
W(u^{\kappa}(z), u^{\kappa}(z)) = z^{-\kappa/2} e^{i\pi \kappa/2}, \quad W(v^{\kappa}(z), u^{-\kappa}(z)) = z^{\kappa/2} e^{-i\pi \kappa/2}
\]
for any \( \kappa \in \mathbb{C} \) and \( z \in \mathbb{C}_{3\pi/2} \).

**Lemma 18.** Let \( \kappa > -1 \). Then \( u^{\kappa}(z) \) is a nontrivial element of \( \mathcal{D}^{\kappa}_{q_\kappa} \) for every \( z \in \mathbb{C} \), and we have \( W_0(u^{\kappa}(z), u^{\kappa}(z')) = 0 \) for all \( z, z' \in \mathbb{C} \).

**Proof.** Because \( \kappa > -1 \), it follows from (7) that \( u^{\kappa}(z) \) is left square-integrable for all \( z \in \mathbb{C} \). In view of (37), this implies that \( u^{\kappa}(z) \in \mathcal{D}^{\kappa}_{q_\kappa} \) for all \( z \in \mathbb{C} \). By (7), \( u^{\kappa}(z) \) is nontrivial for \( z \neq 0 \) because otherwise \( \lambda^{\kappa}_r \) would be identically zero. Because \( u^{\kappa}(0)r = 2^{-\kappa r^{1/2} + \pi \kappa}/\Gamma(\kappa + 1) \) by (7) and (8), we conclude that \( u^{\kappa}(0) \) is nontrivial for \( \kappa > -1 \). By (7) and (25), we have
\[
W_r(u^{\kappa}(z), u^{\kappa}(z')) = 2r^{2+2\kappa}(z' \lambda^{\kappa}_r(r^{2} z) \lambda^{\kappa}_r(r^{2} z) - z \lambda^{\kappa}_r(r^{2} z) \lambda^{\kappa}_r(r^{2} z'))
\]
and hence \( W_0(u^{\kappa}(z), u^{\kappa}(z')) = 0 \) for all \( z, z' \in \mathbb{C} \).

By (37), \( u^{\kappa} \) is a real analytic \( q_\kappa \)-solution in \( \mathbb{C} \) for every \( \kappa \in \mathbb{R} \). Because \( q_\kappa \) is in the l.p.c. at \( \infty \), \( \mathcal{D}^{\kappa}_{q_\kappa} \) is a right boundary space for all \( \kappa \in \mathbb{R} \). Definition [12] and Lemma [13] therefore imply that \( t_\kappa = (q_\kappa, \mathcal{D}^{\kappa}_{q_\kappa}, u^{\kappa}) \) is an expansion triple for every \( \kappa > -1 \). Let \( \sigma_\kappa \) denote the spectral measure for \( t_\kappa \).
Lemma 19. Let \( \kappa > -1 \). Then \( \sigma_\kappa = \nu_\kappa \), where \( \nu_\kappa \) is the measure on \( \mathbb{R} \) defined by (11).

Proof. By (38), (39), and (41), \( v^\kappa \) is a nonvanishing analytic \( q_\kappa \)-solution in \( \mathbb{C}_{3\pi/2} \) such that \( v^\kappa(z) \in D^\kappa_q \) for every \( z \in \mathbb{C}_\kappa \). Let \( \tilde{u}_1 \) be the restriction \( v^\kappa|_O \) of \( v^\kappa \) to the domain \( O = \{ z \in \mathbb{C} : \text{Re} z < 0 \} \). In view of (38), we have \( \tilde{u}_1(E) = r^{1/2}K_\kappa(r\sqrt{E}) \) for \( E < 0 \), where \( K_\kappa \) is the modified Bessel function of the second kind of order \( \kappa \) (formula (15) in Sec. 7.2.2 in [12]). Hence, \( \tilde{u}_1(E) \) is real for \( E < 0 \). By (42), we have \( W(\tilde{u}_1(z), v^\kappa(z)) \neq 0 \) for all \( z \in O \). Substituting \( t = t_\kappa, v = v^\kappa|_C \), and \( \tilde{u} = \tilde{u}_1 \) in (34) yields \( M_{\tilde{u}_1}(z) = 0 \) for all \( z \in O \cap C_+ \). By statement 4 in Proposition 14, we conclude that \( \sigma_\kappa \) vanishes and hence coincides with \( \nu_\kappa \) on \( (-\infty, 0) \). Let the map \( \tilde{u}_2 : C_\kappa \to D \) be given by \( \tilde{u}_2(z|r) = r^{1/2}Y_\kappa(rz^{1/2}) \), where \( Y_\kappa \) is the Bessel function of the second kind of order \( \kappa \). We have \( l_{q_\kappa, z}\tilde{u}_2(z) = 0 \) for any \( z \in C_\kappa \) because \( Y_\kappa \) satisfies the Bessel equation, and \( \tilde{u}_2 \) is therefore an analytic \( q_\kappa \)-solution in \( C_\kappa \). Because \( Y_\kappa \) is real for positive real arguments, \( \tilde{u}_2(E) \) is real for \( E > 0 \). Because \( H^{(1)}_{\kappa} = J_{\kappa} + i\nu_\kappa \), if follows from (12) that

\[
W_z(J_\kappa, \nu_\kappa) = W_z(H^{(1)}_{\kappa}, \nu_\kappa) = -iW_z(J_\kappa, H^{(1)}_{\kappa}) = \frac{2}{\pi z}.
\]

By (7), (9), and (38), we obtain \( W(u^\kappa(z), \tilde{u}_2(z)) = 2z^{-\kappa/2/\pi} \neq 0 \) for \( z \in C_\kappa \) and \( W(v^\kappa(z), \tilde{u}_2(z)) = i\pi \kappa z^{\kappa/2} \) for \( z \in C_+ \). In view of (43), substituting \( t = t_\kappa, v = v^\kappa|_C \), and \( \tilde{u} = \tilde{u}_2 \) in (34) yields

\[
M_{\tilde{u}_2}(z) = \frac{i\pi \kappa z^{\kappa/2}}{2}, \quad z \in C_+.
\]

Statement 4 in Proposition 14 therefore ensures that \( \sigma_\kappa \) coincides with \( \nu_\kappa \) on \( (0, \infty) \). It remains to note that \( \sigma_\kappa(\{0\}) = 0 \) because otherwise \( u^\kappa(0) \) would be square-integrable by Corollary 15.

Proof of Theorem 1. It follows from statement 2 in Proposition 14 and Lemma 19 that the operator \( U_\kappa \) exists and is equal to the spectral transformation for \( t_\kappa \). Statement 1 in Proposition 8, statement 3 in Proposition 14, formula (36), and Lemma 19 therefore imply that \( U_{\kappa}^{-1}T^{V_\kappa}U_\kappa \) is a self-adjoint extension of \( h_\kappa \). For \( \kappa \geq 1, h_\kappa \) is self-adjoint and hence coincides with its self-adjoint extension \( U_{\kappa}^{-1}T^{V_\kappa}U_\kappa \).

Remark 20. As mentioned in Sec. 4, the operator \( U_\kappa \) essentially coincides with the Hankel transformation. In [6, 8], where this transformation was treated similarly, the second solution \( \hat{u} \) used to calculate the spectral measure was required to be globally defined. This required distinguishing between integer and noninteger values of \( \kappa \). Using a locally defined \( \hat{u} \) in the proof of Lemma 19 allows treating all values of \( \kappa \) uniformly.

Given \( \kappa \in \mathcal{O} \) and \( \vartheta \in \mathbb{C} \), let the map \( u^\kappa_0 : \mathbb{C} \to D \) be defined by (12) and (13). Because (10) is satisfied for \( u^\kappa_0(z) \) in place of \( u^\pm\kappa(z) \) (see Sec. 1), we have

\[
l_{q_\kappa, z}u^\kappa_0(z) = 0, \quad \kappa \in \mathcal{O}, \quad \vartheta, z \in \mathbb{C}.
\]

By (12) and (13), we have

\[
W(u^\kappa(z), u^\kappa_0(z)) = \frac{z^{-\kappa/2}e^{i\pi \kappa/2}}{\sin \pi \kappa} (\sin(\vartheta + \vartheta_\kappa) - e^{-i\pi \kappa}z^\kappa \sin(\vartheta - \vartheta_\kappa))
\]
for all $\kappa \in \mathcal{O} \setminus \{0\}$, $\vartheta \in \mathbb{C}$, and $z \in \mathbb{C}_{3\pi/2}$. By Lemma \ref{lemma:analytic} \( \kappa \to W(v^\kappa(z), u^\kappa_\vartheta(z)) \) is an analytic function in $\mathcal{O}$ for fixed $\vartheta$ and $z$ and we can therefore find $W(v^0(z), u^0_\vartheta(z))$ by taking the limit $\kappa \to 0$ in (45). As a result, we obtain

1. \[ W(v^0(z), u^0_\vartheta(z)) = \cos \vartheta + (i - \pi^{-1} \log z) \sin \vartheta, \quad \vartheta \in \mathbb{C}, \ z \in \mathbb{C}_{3\pi/2}. \]

For every $\kappa \in \mathcal{O}$ and $z \in \mathbb{C}$, we set

2. \[ w^\kappa(z) = u^\kappa_{\vartheta+\vartheta,\kappa}(z), \]

where $\vartheta, \kappa$ is given by (14). It follows from (12) and (13) that

3. \[ w^\kappa(z) = \frac{u^\kappa(z) \cos \pi \kappa - u^{-\kappa}(z)}{\sin \pi \kappa}, \quad \kappa \in \mathcal{O} \setminus \{0\}, \]

4. \[ w^0(z|r) = \frac{2}{\pi} \left[ \left( \log \frac{r}{2} + \gamma \right) w_0^0(z|r) - \sqrt{r} Y(r^2 z) \right], \quad r \in \mathbb{R}_+, \]

for every $z \in \mathbb{C}$ and

5. \[ u^\kappa_{\vartheta,\kappa}(z) = u^\kappa(z) \cos(\vartheta - \vartheta, \kappa) + w^\kappa(z) \sin(\vartheta - \vartheta, \kappa) \]

for all $\kappa \in \mathcal{O}$ and $\vartheta, z \in \mathbb{C}$. By (14) and (17), we have

6. \[ l_{\kappa, z} w^\kappa(z) = 0, \quad \kappa \in \mathcal{O}, \ z \in \mathbb{C}. \]

**Lemma 21.** Let $-1 < \kappa < 1$. Then $u^\kappa(z), w^\kappa(z) \in \mathcal{D}^I_{\kappa, z}$ for every $z \in \mathbb{C}$, and

7. \[ W_0(u^\kappa(z), w^\kappa(z')) = 0, \quad W_0(u^\kappa(z), w^\kappa(z')) = \frac{2}{\pi} \]

for every $z, z' \in \mathbb{C}$.

**Proof.** Because $q_k$ is in the l.c.c. at 0 for $-1 < \kappa < 1$, statement 4 in Proposition \ref{proposition} and equalities (57) and (51) imply that $u^\kappa(z), w^\kappa(z) \in \mathcal{D}^I_{q_k}$ for every $z \in \mathbb{C}$. Given $z \in \mathbb{C}$ and $-1 < \kappa < 1$, we define a smooth function $a^\kappa_{\vartheta}(z)$ by setting $a^\kappa_{\vartheta}(r) = X_{\kappa}(r^2 z)$, where $X_{\kappa}$ is given by (68). For $r \in \mathbb{R}_+$, we have $u^\kappa(z|r) = r^{1/2 \pi \kappa} a^\kappa_{\vartheta}(r)$. In view of (25), it follows that

8. \[ W_r(u^\kappa(z), u^{-\kappa}(z')) = r W_r(a^\kappa_{\vartheta}, a^{-\kappa}_{\vartheta}) - 2 \kappa a^\kappa_{\vartheta}(r) a^{-\kappa}_{\vartheta}(r) \]

for every $r \in \mathbb{R}_+$ and $z, z' \in \mathbb{C}$. Because $a^\kappa_{\vartheta}(0) = 2^{-\kappa}/\Gamma(1 + \kappa)$ for any $z \in \mathbb{C}$, we obtain $W_0(u^\kappa(z), u^{-\kappa}(z')) = -2 \sin \pi \kappa(z)/\pi$. The statement of the lemma for $0 < |z| < 1$ now follows from (18) and Lemma 18. Given $z \in \mathbb{C}$, we define the smooth function $b_z$ on $\mathbb{R}$ by setting

9. \[ b_z(r) = (\gamma - \log 2) X_0(r^2 z) - Y(r^2 z). \]

By (49), we have

10. \[ \pi w^0(z|r)/2 = r^{1/2} \log r a^0_\vartheta(r) + r^{1/2} b_z(r) \]

for every $r \in \mathbb{R}_+$. In view of (25), it follows that

11. \[ \frac{\pi^2}{2} W_r(w^0(z), w^0(z')) = r W_r(a^0_\vartheta, b_z) + r \log r W_r(a^0_\vartheta, a^0_\vartheta) + a^0_\vartheta(r) a^0_\vartheta(r), \]

12. \[ \frac{\pi^2}{4} W_r(w^0(z), w^0(z')) = r \log^2 r W_r(a^0_\vartheta, a^0_\vartheta) + r \log r (W_r(a^0_\vartheta, b_z) + W_r(b_z, a^0_\vartheta)) + r W_r(b_z, b_z) + b_z(r) a^0_\vartheta(r) - a^0_\vartheta(r) b_z(r) \]

for every $r \in \mathbb{R}_+$ and $z, z' \in \mathbb{C}$. Because $a^0_\vartheta(0) = 1$ and $b_z(0) = \gamma - \log 2$ for any $z \in \mathbb{C}$, these equalities imply the required statement for $\kappa = 0$. \qed
In view of (37) and (51), Lemma 21 implies that
\[ W(u^\kappa(z), w^\kappa(z)) = \frac{2}{\pi} \]
for every \( z \in \mathbb{C} \) and \(-1 < \kappa < 1 \) (and hence for all \( z \in \mathbb{C} \) and \( \kappa \in \mathcal{O} \)), and it follows from (50) that
\[ W(u^\kappa_\vartheta(z), u^\kappa_{\vartheta-\pi/2}(z)) = -\frac{2}{\pi}, \quad \vartheta, z \in \mathbb{C}, \ \kappa \in \mathcal{O}. \]

Let \(-1 < \kappa < 1 \) and \( \vartheta \in \mathbb{R} \). By (44) and (53), \( u^\kappa_\vartheta \) is a real nonvanishing analytic \( q_\kappa \)-solution in \( \mathbb{C} \). In view of (51), Lemma 18 and Lemma 21, we have \( u^\kappa_\vartheta(z) \in \mathcal{D}_{q^\kappa}^\kappa \) for all \( z \in \mathbb{C} \) and \( W_0(u^\kappa_\vartheta(z), u^\kappa_\vartheta(z')) = 0 \) for all \( z, z' \in \mathbb{C} \). Because \( \mathcal{D}_{q^\kappa}^\kappa \) is a right boundary space, it follows from Definition 12 that \( t_{\kappa,\vartheta} = (q_\kappa, \mathcal{D}_{q^\kappa}^\kappa, u^\kappa_\vartheta) \) is an expansion triple. Let \( \sigma_{\kappa,\vartheta} \) denote the spectral measure for \( t_{\kappa,\vartheta} \).

**Lemma 22.** Let \(-1 < \kappa < 1 \) and \( \vartheta \in \mathbb{R} \). Then \( \sigma_{\kappa,\vartheta} = \mathcal{V}_{\kappa,\vartheta} \), where \( \mathcal{V}_{\kappa,\vartheta} \) is the measure on \( \mathbb{R} \) defined by formulas (16) - (21).

**Proof.** By (49), (50), and (41), \( v^\kappa \) is a nonvanishing analytic \( q_\kappa \)-solution in \( \mathbb{C}_{3\pi/2}^\kappa \) such that \( v^\kappa(z) \in \mathcal{D}_{q^\kappa}^\kappa \) for every \( z \in \mathbb{C}_+ \). Let the meromorphic function \( \mathcal{M}_{\kappa,\vartheta} \) in \( \mathbb{C}_{3\pi/2}^\kappa \) be defined by the relation
\[ \mathcal{M}_{\kappa,\vartheta}(z) = -\frac{1}{2} \frac{W(v^\kappa(z), u^\kappa_{\vartheta-\pi/2}(z))}{W(v^\kappa(z), u^\kappa_\vartheta(z))}, \quad z \in \mathbb{C}_{3\pi/2}^\kappa. \]

Substituting \( t = t_{\kappa,\vartheta} \) and \( v = v^\kappa|_{\mathbb{C}_+} \) in (54) and taking (53) into account, we conclude that \( \mathcal{M}_{\kappa,\vartheta} \) coincides on \( \mathbb{C}_+ \) with the singular Titchmarsh-Weyl \( m \)-function \( \mathcal{M}_{\kappa,\vartheta}^{t,\kappa} \) for \( \bar{u} = u^\kappa_{\vartheta-\pi/2} \). By statement 1 in Proposition 14, we have
\[ \int \varphi(E) \, d\sigma_{\kappa,\vartheta}(E) = \lim_{\eta \to 0} \int \varphi(E) \, \text{Im} \, \mathcal{M}_{\kappa,\vartheta}(E+i\eta) \, dE \]
for any continuous function \( \varphi \) on \( \mathbb{R} \) with compact support. We note that \( \sigma_{\kappa,\vartheta}(\{0\}) = 0 \) because otherwise \( u^\kappa_\vartheta(0) \) would be square-integrable by Corollary 15. It therefore suffices to show that \( \sigma_{\kappa,\vartheta} \) and \( \mathcal{V}_{\kappa,\vartheta} \) coincide on the intervals \((-\infty, 0)\) and \((0, \infty)\). This can be easily done using representation (55) for \( \sigma_{\kappa,\vartheta} \). Because the explicit expressions for \( \sigma_{\kappa,\vartheta} \) differ for \( 0 < |\kappa| < 1 \) and \( \kappa = 0 \), we consider these cases separately.

1. The case \( 0 < |\kappa| < 1 \): In view of (49) and (54), we have
\[ \mathcal{M}_{\kappa,\vartheta}(z) = \frac{1}{2} \frac{\cos(\vartheta + \vartheta_\kappa) - e^{-i\pi \kappa z^\kappa} \cos(\vartheta - \vartheta_\kappa)}{2 \sin(\vartheta + \vartheta_\kappa) - e^{-i\pi \kappa z^\kappa} \sin(\vartheta - \vartheta_\kappa)}. \]

It is easy to see that \( \mathcal{M}_{\kappa,\vartheta} \) has no singularities on \((0, \infty)\) and
\[ \text{Im} \, \mathcal{M}_{\kappa,\vartheta}(E) = \frac{\sin^2 \pi \kappa}{2 E^{-\kappa} \sin^2 \vartheta_+ - 2 \cos \pi \kappa \sin \vartheta_+ \sin \vartheta_- + E^\kappa \sin^2 \vartheta_-}, \quad E > 0, \]
where \( \vartheta_\pm = \vartheta \pm \vartheta_\kappa \). By (16), (17), and (55), we conclude that \( \sigma_{\kappa,\vartheta} \) coincides with \( \mathcal{V}_{\kappa,\vartheta} \) on \((0, \infty)\). For \( \vartheta \in [-|\vartheta_\kappa|, |\vartheta_\kappa|] + \pi \mathbb{Z} \), \( \mathcal{M}_{\kappa,\vartheta} \) is real on \((-\infty, 0)\) and has no singularities on this set. Formula (55) therefore implies that \( \sigma_{\kappa,\vartheta} \) is zero on \((-\infty, 0)\) for such \( \vartheta \). If \( \vartheta \in ([|\vartheta_\kappa|, |\vartheta_\kappa|] + \pi \mathbb{Z}) \), then \( \mathcal{M}_{\kappa,\vartheta} \) has a simple pole at the point \( E_{\kappa,\vartheta} \) given by (18) and, hence, is representable in the form
\[ \mathcal{M}_{\kappa,\vartheta}(z) = g(z) + \frac{A}{E_{\kappa,\vartheta} - z}. \]
where \( g \) is a function analytic in \( \mathbb{C}_{3\pi/2} \) and real on \((-\infty, 0)\) and

\[
A = \lim_{z \to E_{\kappa, \vartheta}} (E_{\kappa, \vartheta} - z) \mathcal{M}_{\kappa, \vartheta}(z) = \frac{\sin \pi\kappa |E_{\kappa, \vartheta}|}{2\sin(\vartheta + \pi\kappa) \sin(\vartheta - \pi\kappa)}.
\]

It therefore follows from (55) that \( \sigma_{\kappa, \vartheta} \) is equal to \( \pi A \delta_{E_{\kappa, \vartheta}} \) on \((-\infty, 0)\). Hence, \( \sigma_{\kappa, \vartheta} \) coincides with \( \nu_{\kappa, \vartheta} \) on \((-\infty, 0)\) for all \( \vartheta \).

2. The case \( \kappa = 0 \): In view of (46) and (54), we have

\[
\mathcal{M}_{0, \vartheta}(z) = \frac{1}{2} \left( i - \pi^{-1} \log z \right) \cos \vartheta - \sin \vartheta,
\]

It is easy to see that \( \mathcal{M}_{0, \vartheta} \) has no singularities on \((0, \infty)\) and

\[
\text{Im} \mathcal{M}_{0, \vartheta}(E) = \frac{1}{2} \frac{1}{(\cos \vartheta - \log E \sin \vartheta/\pi)^2 + \sin^2 \vartheta}, \quad E > 0.
\]

By (12), (21), and (49), we conclude that \( \sigma_{0, \vartheta} \) coincides with \( \nu_{0, \vartheta} \) on \((0, \infty)\). For \( \vartheta \in \pi \mathbb{Z} \), \( \mathcal{M}_{0, \vartheta} \) is real on \((-\infty, 0)\) and has no singularities on this set. Formula (55) therefore implies that \( \sigma_{0, \vartheta} \) is zero on \((-\infty, 0)\) for such \( \vartheta \). If \( \vartheta \notin \pi \mathbb{Z} \), then \( \mathcal{M}_{0, \vartheta} \) has a simple pole at the point \( E_{0, \vartheta} \) given by (21) and is hence representable in the form

\[
\mathcal{M}_{0, \vartheta}(z) = g(z) + \frac{A}{E_{0, \vartheta} - z},
\]

where \( g \) is a function analytic in \( \mathbb{C}_{3\pi/2} \) and real on \((-\infty, 0)\) and

\[
A = \lim_{z \to E_{0, \vartheta}} (E_{0, \vartheta} - z) \mathcal{M}_{0, \vartheta}(z) = \frac{\pi |E_{0, \vartheta}|}{2\sin^2 \vartheta}.
\]

It therefore follows from (55) that \( \sigma_{0, \vartheta} \) is equal to \( \pi A \delta_{E_{0, \vartheta}} \) on \((-\infty, 0)\). Therefore, \( \sigma_{0, \vartheta} \) coincides with \( \nu_{0, \vartheta} \) on \((-\infty, 0)\) for all \( \vartheta \).

**Proof of Theorem 3.** It follows from statement 2 in Proposition 14 and Lemma 22 that the operator \( U_{\kappa, \vartheta} \) exists and is equal to the spectral transformation for \( t_{\kappa, \vartheta} \). Statement 3 in Proposition 14 and Lemma 22 therefore imply that \( h_{\kappa, \vartheta} \) is equal to \( L_q(X^{r_{\kappa, \vartheta}} \cap \mathcal{D}_q^\dagger) \). By statement 1 in Proposition 8 and formula (56), we conclude that \( h_{\kappa, \vartheta} \) is a self-adjoint extension of \( h_{\kappa} \). In view of Lemma 13 and (52), we have

\[
h_{\kappa, \vartheta} = L_{q_n}^{u_{\vartheta}(0)}.
\]

By (51) and (52), every real \( f \in \mathcal{D} \) satisfying \( l_q f = 0 \) is proportional to \( u_{\vartheta}(0) \) for some \( \vartheta \in \mathbb{R} \). By Lemma 11 and formulas (50) and (57), it follows that every self-adjoint extension of \( h_{\kappa} \) is equal to \( h_{\kappa, \vartheta} \) for some \( \vartheta \in \mathbb{R} \). Let \( \vartheta, \vartheta' \in \mathbb{R} \). By Lemma 11 and (57), we have \( h_{\kappa, \vartheta} = h_{\kappa, \vartheta'} \) if and only if \( u_{\vartheta}(0) = c u_{\vartheta'}(0) \) for some real \( c \neq 0 \). In view of (51) and (52), the last condition holds if and only if \( \vartheta - \vartheta' \in \pi \mathbb{Z} \).

**Remark 23.** We note that the function \( q_{\kappa} \) given by (5) is real not only for real \( \kappa \) but also for imaginary \( \kappa \). A complete description of eigenfunction expansions in this case can be found in [9]. It is easy to see that \( t_{\kappa, \vartheta} = (q_{\kappa}, \mathcal{D}_q^{r_{\kappa}}, u_{\vartheta}) \) remains an expansion triple for imaginary \( \kappa \) and the spectral measure for \( t_{\kappa, \vartheta} \) can again be calculated using formulas (55) and (56). An analogue of Theorem 3 for imaginary \( \kappa \) can thus be obtained.
4. Continuity of spectral expansions

In this section, we prove Theorem 4.1

Let the continuous function \( \Phi \) on \((-1, 1) \times \mathbb{R}_+\) be defined by setting

\[
(58) \quad \Phi(\kappa, E) = -\frac{\log E}{\pi \sin(\pi \kappa)} \sin \left( \frac{i \kappa}{2} \log E \right),
\]

where the entire function sinc is defined by the equality

\[
sinc \zeta = \begin{cases} 
\zeta^{-1} \sin \zeta, & \zeta \in \mathbb{C} \setminus \{0\}, \\
1, & \zeta = 0.
\end{cases}
\]

It follows that

\[
(59) \quad \Phi(\kappa, E) = \begin{cases} 
-\log E/\pi, & \kappa = 0, \\
((\sin \pi \kappa)^{-1})(E^{-\kappa/2} - E^{\kappa/2}), & 0 < |\kappa| < 1.
\end{cases}
\]

For every \( \vartheta \in \mathbb{R} \) and \(-1 < \kappa < 1\), we define the function \( t_{\kappa, \vartheta} \) on \( \mathbb{R}_+ \) by the formula

\[
(60) \quad t_{\kappa, \vartheta}(E) = 2 + \Phi(\kappa, E)^2(1 - \cos 2\vartheta \cos \pi \kappa) + \Phi(\kappa, E)(E^{-\kappa/2} + E^{\kappa/2}) \sin 2\vartheta.
\]

It follows from (17), (21), and (59) by a straightforward calculation that

\[
(61) \quad d\tilde{\nu}_{\kappa, \vartheta}(E) = t_{\kappa, \vartheta}(E)^{-1} \Theta(E) dE
\]

for all \( \vartheta \in \mathbb{R} \) and \(-1 < \kappa < 1\). By the Cauchy–Bunyakovsky inequality, we have

\[
| -c \cos 2\vartheta + d \sin 2\vartheta | \leq \sqrt{c^2 + d^2}
\]

for any \( c, d \in \mathbb{R} \). Applying this bound to \( c = \Phi(\kappa, E)^2 \cos \pi \kappa \) and

\[
d = \Phi(\kappa, E)(E^{-\kappa/2} + E^{\kappa/2}) = \Phi(\kappa, E)\sqrt{\Phi(\kappa, E)^2 \sin^2 \pi \kappa + 4},
\]

from (60), we deduce that \( t_{\kappa, \vartheta}(E) \geq f(\Phi(\kappa, E)^2) \), where \( f(y) = 2 + y - \sqrt{y^2 + 4y} \), \( y \geq 0 \). Because

\[
f(y) = \frac{4}{2 + y + \sqrt{y^2 + 4y}} \geq \frac{2}{2 + y}, \quad y \geq 0,
\]

we conclude that \( t_{\kappa, \vartheta}(E)^{-1} \leq 1 + \Phi(\kappa, E)^2/2 \) for all \( E > 0 \), \(-1 < \kappa < 1\), and \( \vartheta \in \mathbb{R} \).

By (55), the function \( \kappa \to \Phi(\kappa, E)^2 \) is even and increases on \([0, 1)\) for every \( E > 0 \). Let \( 0 < \alpha < 1 \). In view of (59), it follows that

\[
(62) \quad t_{\kappa, \vartheta}(E)^{-1} \leq 1 + \frac{1}{2} \Phi(\alpha, E)^2 \leq \frac{1}{2 \sin^2 \pi \alpha} (E^\alpha + E^{-\alpha})
\]

for all \( E > 0 \), \( \vartheta \in \mathbb{R} \), and \(-\alpha \leq \kappa \leq \alpha \). Let \( \varphi \) be a bounded Borel function on \( \mathbb{R} \) with compact support and \( B = (-1, 1) \times \mathbb{R} \). Because the function \((\kappa, \vartheta) \to t_{\kappa, \vartheta}(E)^{-1} \varphi(E)\) is continuous on \( B \) for every \( E > 0 \), relations (61) and (62) and the dominated convergence theorem imply that \((\kappa, \vartheta) \to \int \varphi(E) d\tilde{\nu}_{\kappa, \vartheta}(E)\) is a continuous function on \( B \) that is bounded on \([-\alpha, \alpha] \times \mathbb{R} \) for every \( 0 \leq \alpha < 1 \). Let \( B' = \{(\kappa, \vartheta) \in B : \vartheta \in (|\vartheta_{\kappa}|, \pi - |\vartheta_{\kappa}|) + \pi \mathbb{Z}\} \).

It follows from (16) and (19) that

\[
\int \varphi(E) d\tilde{\nu}_{\kappa, \vartheta}(E) = \int \varphi(E) d\tilde{\nu}_{\kappa, \vartheta}(E) + b_{\varphi}(\kappa, \vartheta),
\]

where \( b_{\varphi}(\kappa, \vartheta) \) is a bounded Borel function on \( \mathbb{R} \).
where the function $b_{\varphi}$ on $B$ is defined by the relation
\begin{equation}
(63) \quad b_{\varphi}(\kappa, \vartheta) = \begin{cases} \Phi(\kappa, |E_{\kappa, \vartheta}|)\varphi(E_{\kappa, \vartheta}), & (\kappa, \vartheta) \in B', \\ 0, & (\kappa, \vartheta) \in B \setminus B', \end{cases}
\end{equation}
and the continuous function $\Phi$ on $(-1, 1) \times \mathbb{R}_+$ is given by
\[
\Phi(\kappa, E) = \frac{1}{2}E\pi^2 \text{sinc}(\pi\kappa) \left( \Phi(\kappa, E) + \frac{1}{\cos^2 \vartheta_\kappa} \right).
\]
For every $(\kappa, \vartheta) \in B'$, we have $|\cot \vartheta \tan \vartheta_\kappa| < 1$, and it follows from (18) and (20) that
\[
E_{\kappa, \vartheta} = -\exp \left[ \pi \cot \vartheta \frac{\text{sinc}(\vartheta_\kappa)}{2 \cos \vartheta_\kappa} (\cot \vartheta \tan \vartheta_\kappa) \right],
\]
where $g$ is a continuous function on $(-1, 1)$ such that $g(y) = y^{-1} \log(1+y)(1-y)^{-1}$ for $y \neq 0$ and $g(0) = 2$. Hence, $(\kappa, \vartheta) \rightarrow E_{\kappa, \vartheta}$ is a continuous function on $B'$, and $b_{\varphi}$ is therefore a Borel function on $B$. Estimating $\Phi(\kappa, E)^2$ as above, we obtain
\begin{equation}
(64) \quad \Phi(\kappa, E) \leq \frac{\pi^2 E}{2 \sin^2 \pi\alpha} (E^\alpha + E^{-\alpha}), \quad (\kappa, E) \in [-\alpha, \alpha] \times \mathbb{R}_+,
\end{equation}
for every $0 < \alpha < 1$. In view of (63), this implies that $b_{\varphi}$ is bounded on $[-\alpha, \alpha] \times \mathbb{R}$ for every $0 \leq \alpha < 1$. To complete the proof, it remains to show that $b_{\varphi}$ is continuous on $B$ if $\varphi$ is continuous. Let $-1 < \kappa < 1$. It follows from (18) and (20) that $|E_{\kappa, \vartheta}|$ strictly decreases from $\infty$ to $0$ as $\vartheta$ varies from $|\vartheta_\kappa|$ to $\pi - |\vartheta_\kappa|$. Hence, for every $E > 0$, there is a unique $\tau_E(\kappa) \in (|\vartheta_\kappa|, \pi - |\vartheta_\kappa|)$ such that $|E_{\kappa, \tau_E(\kappa)}| = E$. The continuity of $E_{\kappa, \vartheta}$ in $(\kappa, \vartheta)$ implies that $\tau_E$ is a continuous function on $(-1, 1)$ for every $E > 0$. Let $\beta > 0$ be such that $\varphi(E) = 0$ for every $E \leq -\beta$. Given $0 < \alpha < 1$ and $0 < \delta$, we define the open subset $B_{\alpha, \delta}$ of $B$ by setting
\[
B_{\alpha, \delta} = \{ (\kappa, \vartheta) \in (-\alpha, \alpha) \times \mathbb{R} : \vartheta \in (\tau_\delta(\kappa) - \pi, \tau_\beta(\kappa)) + \pi\mathbb{Z} \}.
\]
If $\delta < 1$, then it follows from (63) and (64) that
\[
|b_{\varphi}(\kappa, \vartheta)| \leq \frac{\pi^2 \delta^{1-\alpha}}{2 \sin^2 \pi\alpha \sup_{E \in \mathbb{R}} |\varphi(E)|}, \quad (\kappa, \vartheta) \in B_{\alpha, \delta}.
\]
Given $(\kappa, \vartheta) \in B \setminus B'$ and $\varepsilon > 0$, we pick an arbitrary $\alpha \in (|\kappa|, 1)$ and choose $\delta > 0$ so small that the right-hand side of the last inequality is less than $\varepsilon$. Then $B_{\alpha, \delta}$ is a neighborhood of $(\kappa, \vartheta)$ where the absolute value of $b_{\varphi}$ is less than $\varepsilon$. This proves that $b_{\varphi}$ is continuous at every point of $B \setminus B'$. Because $b_{\varphi}$ is obviously continuous on $B'$, the theorem is proved.

**Appendix A. Proof of Lemma 2**

Let $\text{Log}$ be the branch of the logarithm in $\mathbb{C}_\pi$ satisfying $\text{Log} 1 = 0$ and $p$ be the analytic function in $\mathbb{C} \times \mathbb{C}_\pi$ defined by the relation $p(\kappa, r) = e^{\kappa \text{Log} r}$ (hence $p(\kappa, r) = r^\kappa$ for $r \in \mathbb{R}_+$). Let $G$ be the analytic function in $\mathbb{C} \times \mathbb{C} \times \mathbb{C}_\pi$ such that
\[G(\kappa, z, r) = p(1/2 + \kappa, r) \chi_{\kappa}(r^2 z) \quad \text{for all} \quad \kappa, z \in \mathbb{C} \quad \text{and} \quad r \in \mathbb{C}_\pi.\]
We then have $G(\kappa, z, r) = u^\kappa(z|r)$ for all $\kappa, z \in \mathbb{C}$ and $r \in \mathbb{C}_\pi$. We define the function $F$ on $\Theta \times \mathbb{C} \times \mathbb{C} \times \mathbb{C}_\pi$ by setting
\[
F(\kappa, \vartheta, z, r) = \frac{G(\kappa, z, r) \sin(\vartheta + \vartheta_\kappa) - G(-\kappa, z, r) \sin(\vartheta - \vartheta_\kappa)}{\sin \pi \kappa}, \quad \kappa \in \Theta \setminus \{0\},
\]
\[
F(0, \vartheta, z, r) = G(0, z, r) \cos \vartheta + \frac{2}{\pi} \left[ \left( \log \frac{r}{2} + \gamma \right) G(0, z, r) - p(1/2, r) \mathcal{V}(z r^2) \right] \sin \vartheta
\]
\[
\int_{\Theta} F(\kappa, \vartheta, z, r) d\kappa = 2 \pi \left( \log \frac{r}{2} + \gamma \right) G(0, z, r) - \int_{\Theta} p(1/2, r) \mathcal{V}(z r^2) d\kappa,
\]
for every \( z, \vartheta \in \mathbb{C} \) and \( r \in \mathbb{C}_\pi \). It follows immediately from (12), (13), and the definition of \( F \) that \( F(\kappa, \vartheta, z, r) = w_0^2(z|r) \) for every \( \vartheta, z \in \mathbb{C}, \kappa \in \mathcal{O}, \) and \( r \in \mathbb{R}_+ \). The function \( (\vartheta, z, r) \rightarrow F(\kappa, \vartheta, z, r) \) is obviously analytic in \( \mathbb{C} \times \mathbb{C} \times \mathbb{C}_\pi \) for every fixed \( \kappa \in \mathcal{O} \). The function \( \kappa \rightarrow F(\kappa, \vartheta, z, r) \) is analytic in \( \mathcal{O} \setminus \{0\} \) and continuous at \( \kappa = 0 \) (this is ensured by the same calculation as used to find the limit in (13)) and is therefore analytic in \( \mathcal{O} \) for every fixed \( \vartheta, z \in \mathbb{C} \) and \( r \in \mathbb{C}_\pi \). Hence, \( F \) is analytic in \( \mathcal{O} \times \mathbb{C} \times \mathbb{C} \times \mathbb{C}_\pi \) by the Hartogs theorem.

**Appendix B. Proof of Lemma 17**

Let \( T = L_q(C^\infty_0(a, b)) \). Because \( L_q \) is closed, it suffices to show that \( T^* = L^*_q \). For this, we only need to prove that \( D_{T^*} \subset D_{L^*_q} \) because \( L_q \) is an extension of \( T \) and \( T^* \) is hence an extension of \( L^*_q \). By (30), \( D_{L^*_q} \) consists of all elements \([g]\) with \( g \in D_q \). Therefore, for every \( \phi \in D_{T^*} \), we must find \( g \in D_q \) such that \( \phi = [g] \). Let \( \psi = T^*\phi \). We then have

\[
\langle T[f], \phi \rangle = \langle [f], \psi \rangle, \quad f \in C^\infty_0(a, b).
\]

Because \( (T[f])(r) = -f''(r) + q(r)f(r) \) for \( \lambda \)-a.e. \( r \in (a, b) \), we obtain

\[
- \int_a^b f''(r)\phi(r) \, dr = \int_a^b f(r)(\psi(r) - q(r)\phi(r)) \, dr, \quad f \in C^\infty_0(a, b).
\]

Because both \( q \) and \( \phi \) are locally square-integrable on \( (a, b) \), the function \( q\phi \) is locally integrable on \( (a, b) \). We choose \( c \in (a, b) \) and define \( h \in D \) by setting

\[
h(r) = \int_c^r \rho \int_c^\rho (\psi(t) - q(t)\phi(t)) \, dt.
\]

We obviously have \( h''(r) = \psi(r) - q(r)\phi(r) \) for \( \lambda \)-a.e. \( r \in (a, b) \). Integrating by parts, we obtain

\[
\int_a^b f(r)(\psi(r) - q(r)\phi(r)) \, dr = \int_a^b f''(r)h(r) \, dr
\]

and therefore

\[
\int_a^b f''(r)(\phi(r) + h(r)) \, dr = 0, \quad f \in C^\infty_0(a, b).
\]

This means that the second derivative of \( \phi + h \) in the sense of generalized functions is equal to zero. Hence, there are \( A, B \in \mathbb{C} \) such that \( \phi(r) + h(r) = Ar + B \) for \( \lambda \)-a.e. \( r \in (a, b) \). Let \( g \in D \) be defined by the relation \( g(r) = Ar + B - h(r), r \in (a, b) \). Then we obviously have \([g] = \phi \). Because \( g''(r) = -\psi(r) + q(r)\phi(r) \), we have \( l_qg)(r) = \psi(r) - q(r)\phi(r) + q(r)g(r) = \psi(r) \) for \( \lambda \)-a.e. \( r \in (a, b) \) and therefore \( l_qg = \psi \). This implies that both \( g \) and \( l_qg \) are square-integrable and hence \( g \in D_q \).

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