Almost scalar-flat Kähler metrics on affine algebraic manifolds

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Abstract

In this paper, we study the existence of a complete Kähler metric whose scalar curvature is flat away from some ample divisor and arbitrarily small near it on certain affine algebraic manifold. Such a metric is obtained by gluing the solution of the degenerate complex Monge-Ampère equation and the complete Kähler metric whose scalar curvature decays at infinity.

1 Introduction

A fundamental problem in Kähler geometry is the existence of constant scalar curvature Kähler metrics on complex manifolds. If a complex manifold is noncompact, there are many positive result in this problem. In 1979, Calabi [6] showed the existence of the complete Ricci-flat Kähler metric on the total space of the canonical line bundle over the Fano manifold with a Kähler Einstein metric. In addition, there exist following generalizations of Calabi’s result [6]. In 1990, Bando-Kobayashi [4] showed the existence of the complete Ricci-flat Kähler metric on the complement of the Kähler Einstein Fano hypersurface. In 1991, Tian-Yau [17] showed the existence of the complete Ricci-flat Kähler metric on the complement of the Calabi-Yau hypersurface. On the other hand, as a scalar curvature version of Calabi’s result [6], in 2002, Hwang-Singer [10] showed the existence of the complete scalar-flat Kähler metric on the total space of some line bundle over the compact complex manifold with a constant nonnegative scalar curvature Kähler metric. However, the similar generalization of Hwang-Singer [10] which is like [4] or [17] is unknown. In this paper, toward the scalar curvature version of results [4] and [17], we study the existence of a complete Kähler metric on some affine algebraic manifold whose scalar curvature is almost flat.

Let \((X, L_X)\) be a polarized manifold of dimension \(n\), i.e., \(X\) is an \(n\)-dimensional compact complex manifold with an ample line bundle \(L_X\). Assume that there exists a smooth hypersurface \(D \subset X\) with

\[ D \in |L_X|. \]
Set an ample line bundle $L_D := L_X|_D$ over $D$. From the ampleness of $L_X$, there exists a Hermitian metric $h_X$ on $L_X$ which defines a Kähler metric $\theta_X$ on $X$. Assume that the restriction of $h_X$ to $L_D$ defines also a Kähler metric $\theta_D$ on $D$. Let $\hat{S}_D$ be an average of scalar curvature $S(\theta_D)$ of $\theta_D$;

$$
\hat{S}_D := \frac{\int_D S(\theta_D)\theta_D^{n-1}}{\int_D \theta_D^{n-1}} = \frac{(n-1)c_1(K_D^{-1}) \cup c_1(L_D)^{n-2}}{c_1(L_D)^{n-1}},
$$

where $K_D^{-1}$ is the anti-canonical line bundle over $D$. Note that $\hat{S}_D$ is a topological invariant. In this paper, we assume that $\hat{S}_D > 0$.

Then, following [4], we can define the complete Kähler metric on $X \setminus D$ whose scalar curvature decays near $D$ by

$$
\omega_0 := \frac{n(n-1)}{\hat{S}_D} \sqrt{-1} \partial \bar{\partial} \exp \left( \frac{\hat{S}_D}{n(n-1)} t \right).
$$

Take positive integers $l > n$ and $m$ such that the line bundle $K_X^{-l} \otimes L_X^m$ is very ample. Let $F \in |K_X^{-l} \otimes L_X^m|$ be a smooth hypersurface defined by a holomorphic section $\sigma_F \in H^0(X, K_X^{-l} \otimes L_X^m)$ such that the divisor $D + F$ is simple normal crossing. For a defining section $\sigma_D \in H^0(X, L_X)$ of $D$, set

$$
\xi := \sigma_F \otimes \sigma_D^{-m}
$$

From Yau [18, Theorem 7], we know that the following degenerate complex Monge-Ampère equation is solvable:

$$(\theta_X + \sqrt{-1} \partial \bar{\partial} \varphi)^n = \xi^{-1/l} \wedge \bar{\xi}^{-1/l}.$$ 

Moreover, from a priori estimate due to Kołodziej [11], we know that the solution $\varphi$ above is bounded on $X$. In this paper, we will give explicit estimates of higher order derivatives of $\varphi$. Namely, we show that

**Theorem 1.1.** Take local holomorphic coordinates $(z_i^j)_{i=1}^n = (z^1, z^2, \ldots, z^{n-2}, w_F, w_D)$ such that $\{w_F = 0\} = F$ and $\{w_D = 0\} = D$. Then, there exists a constant $a(n)$ depending only on the dimension $n$ such that

$$
\left| \frac{\partial^2}{\partial z^i \partial \bar{z}^j} \varphi \right| = O\left( |w_D|^{-2m/l} |w_F|^{-2/l} a(n) \right),
$$

$$
\left| \frac{\partial^4}{\partial w_F^i \partial w_F^j} \varphi \right| = O\left( |w_D|^{-2a(n)m/l} |w_F|^{-2-2a(n)/l} \right),
$$

$$
\left| \frac{\partial^4}{\partial w_D^i \partial w_D^j} \varphi \right| = O\left( |w_D|^{-2-2a(n)m/l} |w_F|^{-2a(n)/l} \right),
$$

as $|w_F|, |w_D| \to 0$, for any $1 \leq i, j \leq n-2$ and multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $0 \leq \sum_i \alpha_i \leq 2$. 

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By applying Theorem 1.1, we have the main result in this paper:

**Theorem 1.2.** Assume that there exist positive integers \( l > n \) and \( m \) such that
\[
\frac{a(n)m}{2l} < \frac{\hat{S}_D}{n(n-1)}
\]
and the line bundle \( K^{-l}_X \otimes L^m_X \) is very ample. Take a smooth hypersurface \( F \in |K^{-l}_X \otimes L^m_X| \) such that \( D + F \) is simple normal crossing. Then, for any relatively compact domain \( Y \subset X \setminus (D \cup F) \), there exists a complete Kähler metric \( \omega \) on \( X \setminus D \) whose the scalar curvature \( S(\omega) = 0 \) on \( Y \) and is arbitrarily small on the complement of \( Y \).

From [1, Theorem 1.5], we know that if there exists a complete Kähler metric which defines the asymptotically conicalness on \( X \setminus D \) with a sufficiently small scalar curvature, \( X \setminus D \) admits a complete scalar flat Kähler metric. In fact, Theorem 1.2 gives a Kähler metric whose scalar curvature is under control, but unfortunately, the Kähler metric in Theorem 1.2 is not of asymptotically conical geometry near the intersection of \( D \) and \( F \).

The organization of this paper is following. In Section 2, we constructs Kähler potentials whose scalar curvatures are under control. In addition, we recall the gluing technique of plurisubharmonic functions. In Section 3, we prove Theorem 3. To show this, we study the dependency of higher derivatives of the solution of the degenerate complex Monge-Ampère equation on the maximal ratio of eigenvalues. In Section 4, we prove Theorem 1.2.

**Acknowledgment.** The author would like to thank Professor Ryoichi Kobayashi who first brought the problem in this paper to his attention for many helpful comments. In particular, the author learned the idea of applying the complex Monge-Ampère equation to scalar curvatures from him.

2 Plurisubharmonic functions with small scalar curvature

To prove Theorem 1.2 we prepare Kähler potentials, i.e., strictly plurisubharmonic functions, whose scalar curvatures are under control.

2.1 Kähler potential near \( D \) and scalar curvature

In this subsection, we consider the Kähler potential near \( D \) and study the scalar curvature of it. Recall that \((X, L_X)\) is a polarized manifold of dimension \( n \) and \( D \) is a smooth hypersurface in \( X \) with
\[
D \in |L_X|.
\]
Set an ample line bundle \( L_D := L_X|_D \) over \( D \) and take a Hermitian metric \( h_X \) on \( L_X \) which defines a Kähler metric \( \theta_X \) on \( X \). Assume that the restriction \( h_D \) of \( h_X \) to \( D \) defines
also a Kähler metric $\theta_D$ on $D$. Let $\hat{S}_D$ be an average of scalar curvature $S(\theta_D)$ of $\theta_D$;

$$\hat{S}_D := \frac{\int_D S(\theta_D)\theta_D^{n-1}}{\int_D \theta_D^{n-1}} = \frac{(n-1)c_1(K_D^{-1}) \cup c_1(L_D)^{n-2}}{c_1(L_D)^{n-1}},$$

where $K_D^{-1}$ is the anti-canonical line bundle over $D$ and assume that

$$\hat{S}_D > 0.$$

Set $t := \log ||\sigma_D||^{-2}$. Note that $\theta_X = \sqrt{-1} \partial \bar{\partial} t = \sqrt{-1} \partial \bar{\partial} \log ||\sigma_D||^{-2}$ on $X \setminus D$. Set

$$\Theta(t) = \frac{n(n-1)}{\hat{S}_D} \exp \left( \frac{\hat{S}_D}{n(n-1)} t \right).$$

Following [4], we can define a complete Kähler metric by

$$\omega_0 := \sqrt{-1} \partial \bar{\partial} \Theta(t) = \frac{n(n-1)}{\hat{S}_D} \sqrt{-1} \partial \bar{\partial} \exp \left( \frac{\hat{S}_D}{n(n-1)} t \right)$$

on $X \setminus D$. Let us start computing the scalar curvature of $\omega_0$.

**Lemma 2.1.** The Ricci form of $\omega_0$ is given by

$$\text{Ric}(\omega_0) = \text{Ric}(\theta_X) - \frac{\hat{S}_D}{n-1} \theta_X - \sqrt{-1} \partial \bar{\partial} \log \left( 1 + \frac{\hat{S}_D}{n(n-1)} ||\partial t||^2_{\theta_X} \right).$$

**Proof.** To show this lemma, it is enough to see that the volume form of $\omega_0$, From the definition of $\omega_0$, we have

$$\omega_0 = \exp \left( \frac{\hat{S}_D}{n(n-1)} t \right) \left( \theta_X + \frac{\hat{S}_D}{n(n-1)} \sqrt{-1} \partial t \wedge \bar{\partial} t \right).$$

So, the following identity

$$\sqrt{-1} \partial t \wedge \bar{\partial} t \wedge \theta_X^{n-1} = \frac{1}{n} ||\partial t||^2_{\theta_X} \theta_X^n$$

implies that the volume form of $\omega_0$ is given by

$$\omega_0^n = \exp \left( \frac{\hat{S}_D}{n-1} t \right) \left( 1 + \frac{\hat{S}_D}{n(n-1)} ||\partial t||^2_{\theta_X} \right) \theta_X^n.$$

Recall that the Ricci form is given by $\text{Ric}(\omega_0) = -\sqrt{-1} \partial \bar{\partial} \log \omega_0^n$. Thus, the lemma follows.

Then, immediately we have
Lemma 2.2. The scalar curvature $S(\omega_0)$ can be written as
\[ S(\omega_0) = O\left(\|\sigma_D\|^2 S_D/n(n-1)\right) \]
as $\sigma_D \to 0$.

Remark 2.3. Moreover, from \cite{1} Theorem 1.1, if $\theta_D$ is cscK, we have the following strong result:
\[ S(\omega_0) = O\left(\|\sigma_D\|^{2+2 S_D/n(n-1)}\right) \]
as $\sigma_D \to 0$.

2.2 Kähler potential with a small scalar curvature near $F$

In this subsection, we consider constructing a Kähler metric on $X$ whose scalar curvature is small near the smooth hypersurface $F$. For some Hermitian metric on $K_X^{-l} \otimes L_X^m$, set $b := \log \|\sigma_F\|^{-2}$. Since the holomorphic line bundle $K_X^{-l} \otimes L_X^m$ is very ample, we may assume that $\sqrt{-1} \partial \bar{\partial} b$ is a smooth Kähler metric on $X$ by abuse of notation. For parameters $v > 0$ and $\beta \in \mathbb{Z}_{>0}$, define a function
\[ G_v^\beta(b) := \int_{b_0}^b \left(\frac{1}{e^{-y} + v}\right)^{1/\beta} dy \]
for some fixed $b_0 \in \mathbb{R}$. Note that $G_v^\beta(b)$ is defined smoothly outside $F$ since $\lim_{b \to \infty} G_v^\beta(b) = +\infty$ for any fixed $v > 0$.

Lemma 2.4. For $\beta \geq 1$, the Kähler metric $\gamma_v^\beta := \sqrt{-1} \partial \bar{\partial} G_v^\beta(\beta b)$ is defined as a Kähler metric on $X$ by abuse of notation.

Proof. In fact,
\[ \sqrt{-1} \partial \bar{\partial} G_v^\beta(\beta b) = \beta \sqrt{-1} \partial \left[ \left(\frac{1}{e^{-\beta b} + v}\right)^{1/\beta} \bar{\partial} b \right] \]
\[ = \left(\frac{1}{e^{-\beta b} + v}\right)^{1/\beta} \left(\beta \sqrt{-1} \partial \bar{\partial} b + \frac{e^{-\beta b}}{e^{-\beta b} + v} \sqrt{-1} \partial b \wedge \bar{\partial} b \right). \]
Note that the last term
\[ \frac{e^{-\beta b}}{e^{-\beta b} + v} \sqrt{-1} \partial b \wedge \bar{\partial} b \]
is defined smoothly on $X$. Since $\sqrt{-1} \partial \bar{\partial} b$ is defined on $X$, we have finished proving.

Next, the scalar curvature of $\gamma_v^\beta$ is given by

Lemma 2.5. For $\beta \geq 3$, we obtain
\[ S(\gamma_v^\beta) = S(\sqrt{-1} \partial \bar{\partial} G_v^\beta(\beta b)) = O((\|\sigma_F\|^{2\beta} + v)^{1/\beta}) \]
as $\|\sigma_F\| \to 0$. 

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Proof. The lemma follows from the similar way in the computation of the scalar curvature of $\omega_0$. In fact, since
\[
((\sqrt{-1}\partial\bar{\partial}G_{\nu}^{\beta}(\beta b))^{n} = \beta^{n} \left( \frac{1}{e^{-\beta b} + v} \right)^{n/\beta} \left( 1 + \frac{\beta e^{-\beta b}}{e^{-\beta b} + v} ||\partial b||^2 \right) ) (\sqrt{-1}\partial\bar{\partial}b)^n,
\]
we have
\[
\text{Ric}(\sqrt{-1}\partial\bar{\partial}G_{\nu}^{\beta}(\beta b)) = \text{Ric}(\sqrt{-1}\partial\bar{\partial}b) - \sqrt{-1}\partial\bar{\partial} \log \left( 1 + \frac{n\beta e^{-\beta b}}{e^{-\beta b} + v} ||\partial b||^2 \right)
\]
\[
+ \frac{n}{\beta} \left( \frac{1}{e^{-\beta b} + v} \sqrt{-1}\partial\bar{\partial}e^{-\beta b} + \frac{\beta}{(e^{-\beta b} + v)^2} \sqrt{-1}\partial e^{-\beta b} \wedge \bar{\partial}e^{-\beta b} \right).
\]
Note that second and last terms above are zero on $F$. Thus, when we consider the scalar curvature $S(\gamma^{\beta}_0)$, it is enough to see the Kähler metric $1/(e^{-\beta b} + v)^{1/\beta} \sqrt{-1}\partial\bar{\partial}b$ and the Ricci curvature $\text{Ric}(\sqrt{-1}\partial\bar{\partial}b)$. Therefore the desired result is obtained. \qed

\section{2.3 Ricci-flat Kähler metric away from $D \cup F$}

In this subsection, we study the (incomplete) Ricci-flat Kähler metric away from the support of the divisor $D + F$. Recall that the setting in Theorem \[\text{1.2}\]. Let $l > n$ and $m$ be positive integers such that there exists a holomorphic section $\sigma_F \in H^0(K_X^{-l} \otimes L_X^m)$ which defines a smooth hypersurface $F \subset X$, i.e., $(\sigma_F)_0 = F$. It follows from the hypothesis of the average value $\hat{S}_D$ of the scalar curvature that divisors $D$ and $F$ intersect to each other. Set
\[
\xi := \sigma_F \otimes \sigma_D^{-m}.
\]
Note that $\xi$ is a meromorphic section of $K_X^{-l}$. Then, define the singular and degenerate volume form $V$ by
\[
V := \xi^{-1/l} \wedge \xi^{-1/l}
\]
From the construction above, $V$ has a finite volume on $X$ and its curvature form, i.e., Ricci form, is zero on the complement of $D \cup F$. For the Kähler metric $\theta_X$ on $X$, write
\[
V = f \theta^n_X
\]
for some non-negative function $f$ on $X$ with the normalized condition
\[
\int_X V = \int_X f \theta^n_X = \int_X \theta^n_X.
\]
We know that $f$ is smooth away from $D \cup F$. From Yau \[\text{[18, Theorem 7]}\], recall the solvability of the meromorphic complex Monge-Ampère equation:

\textbf{Theorem 2.6.} Let $L_1$ and $L_2$ be holomorphic line bundles over a compact Kähler manifold $(X, \theta_X)$. Let $s_1$ and $s_2$ be nonzero holomorphic sections of $L_1$ and $L_2$, respectively. Let $F$ be a smooth function on $X$ such that $\int_X |s_1|^{2k_1}|s_2|^{2k_2} \exp(F)\theta_X = \text{Vol}(X)$, where $k_1 \geq 0$ and $k_2 \geq 0$. Suppose that $\int_X |s_2|^{2nk_2} < \infty$ for $n = \dim X$.

Then, we can solve the following equation
\[
(\theta_X + \sqrt{-1}\partial\bar{\partial}\varphi)^n = |s_1|^{2k_1}|s_2|^{2k_2} \exp(F)\theta^n_X
\]
so that $\varphi$ is smooth outside divisors of $s_1$ and $s_2$ with $\sup_X \varphi < +\infty$. \[\text{\hfill 6}\]
Then, we can solve the following complex Monge-Ampère equation
\[(\theta_X + \sqrt{-1} \partial \bar{\partial} \varphi)^n = f \theta_X^n.\]
with \(\varphi \in C^\infty(X \setminus D \cup F)\). Thus, we obtain a Ricc-flat Kähler metric \(\theta_X + \sqrt{-1} \partial \bar{\partial} \varphi\) on the complement of \(D \cup F\). For this solution \(\varphi\), we obtain the following a priori estimate due to Kolodziej\[11\] (see also \[9\]):

**Theorem 2.7.** If \(f\) is in \(L^p(\theta_X^n)\) for some \(p > 1\), we have
\[\text{Osc}_X \varphi \leq C\]
for some \(C > 0\) depending only on \(\theta_X\) and \(||f||_{L^p}\).

### 2.4 Gluing plurisubharmonic functions

In this subsection, following [7, Chapter I], we consider gluing Kähler potentials, i.e., plurisubharmonic functions, obtained in previous subsections. Let \(\rho \in C^\infty(\mathbb{R}, \mathbb{R})\) be a nonnegative function with support in \([-1, 1]\) such that \(\int_{\mathbb{R}} \rho(h)dh = 1\) and \(\int_{\mathbb{R}} h\rho(h)dh = 0\).

**Lemma 2.8.** For arbitrary \(\eta = (\eta_1, ..., \eta_p) \in (0, +\infty)^p\), the function
\[M_\eta(t_1, ..., t_p) = \int_{\mathbb{R}^p} \max\{t_1 + h_1, ..., t_p + h_p\} \prod_{1 \leq j \leq p} \eta_j^{-1} \rho(h_j/\eta_j)dh_1...dh_p\]
possesses the following properties:

a) \(M_\eta(t_1, ..., t_p)\) is non decreasing in all variables, smooth and convex on \(\mathbb{R}^p\);

b) \(\max\{t_1, ..., t_p\} \leq M_\eta(t_1, ..., t_p) \leq \max\{t_1 + \eta_1, ..., t_p + \eta_p\}\);

c) \(M_\eta(t_1, ..., t_p) = M_\eta(u_1, ..., u_p)\) if \(t_j + \eta_j \leq \max_{k \neq j}\{t_k - \eta_k\}\);

d) \(M_\eta(t_1 + a, ..., t_p + a) = M_\eta(t_1, ..., t_p) + a\);

e) if \(u_1, ..., u_p\) are plurisubharmonic and satisfy \(H(u_j)(\xi) \geq \gamma_z(\xi)\) where \(z \mapsto \gamma_z\) is a continuous hermitian form on \(TM\), then \(u = M_\eta(u_1, ..., u_p)\) is a plurisubharmonic and satisfies \(Hu_z(\xi) \geq \gamma_z(\xi)\).

**Remark 2.9.** Lemma 2.8 is a key in the proof of Richberg theorem (see [7]). In our case, we have already prepared three plurisubharmonic functions and must compute the Ricci form of the glued Kähler metric later. Therefore, we need the explicit formula of the glued function.

In addition, we obtain easily

**Lemma 2.10.** There exists a constant \(C > 0\) such that
\[\left| \frac{\partial^{|\alpha|} M_\eta}{\partial \theta^\alpha}(t) \right| \leq C \min\{\eta_j|\alpha_j \neq 0\} \prod_{\alpha_i \neq 0} \eta_i^{-\alpha_i}\]
for any multi index \(\alpha = (\alpha_i)_i\) with \(1 \leq |\alpha| \leq 4\).
Recall that the Kähler potential of $\omega_0$ is given by

$$\Theta(t) = \frac{n(n-1)}{S_D} \exp \left( \frac{S_D}{n(n-1)} t \right).$$

For $\delta \in (0, 1)$, set $\tilde{G}_v^\beta(b) := G_v^\beta(\beta b) + \delta \Theta(t)$. We need

**Lemma 2.11.** For the complete Kähler metric $\sqrt{-1\partial\overline{\partial}(\tilde{G}_v^\beta(b))}$ on $X \setminus D$, we have

$$S(\sqrt{-1\partial\overline{\partial}(\tilde{G}_v^\beta(b))) = \begin{cases} O(||\sigma_D||^{2\beta_d/\beta(n-1)}) & \text{near } D, \\ O(||\sigma_F||^{2\beta + v}) & \text{near } F. \end{cases} \quad (2.1)$$

**Proof.** First, we see the behavior of the scalar curvature near $D$. Since

$$||\sigma_D||^{2+2\beta_d/(n-1)} \left( \sqrt{-1\partial\overline{\partial}(\tilde{G}_v^\beta(b))} \right)^n$$

is a smooth volume form on $X$, the Ricci form of $\sqrt{-1\partial\overline{\partial}(\tilde{G}_v^\beta(b))}$ given by

$$\text{Ric}(\sqrt{-1\partial\overline{\partial}(\tilde{G}_v^\beta(b))) = - \left( \frac{\hat{S}}{n-1} + 1 \right) \theta_X - \sqrt{-1\partial\overline{\partial} \log ||\sigma_D||^{2+2\beta_d/(n-1)} \left( \sqrt{-1\partial\overline{\partial}(\tilde{G}_v^\beta(b))} \right)^n}$$

is defined smoothly on $X$. Recall that

$$\sqrt{-1\partial\overline{\partial}(\tilde{G}_v^\beta(b))) = \delta \omega_0 + \gamma_v^\beta.$$ 

Thus, we have the desired result near $D$. Similarly, the volume form

$$\left( ||\sigma_F||^{2\beta + v} \right)^{n/\beta} \left( \sqrt{-1\partial\overline{\partial}(\tilde{G}_v^\beta(b))} \right)^n$$

is smooth near $F \setminus (D \cap F)$. Then, the following identity

$$\text{Ric}(\sqrt{-1\partial\overline{\partial}(\tilde{G}_v^\beta(b))) = \frac{n}{\beta} \left( \frac{1}{e^{-\beta b} + v} \sqrt{-1\partial\overline{\partial} e^{-\beta b} + \frac{\beta}{(e^{-\beta b} + v)^2} \sqrt{-1\partial\overline{\partial} e^{-\beta b} \wedge \overline{\partial} e^{-\beta b}} \right)$$

$$- \sqrt{-1\partial\overline{\partial} \log||(\sigma_F||^{2\beta + v})^{n/\beta} \left( \sqrt{-1\partial\overline{\partial}(\tilde{G}_v^\beta(b))} \right)^n}$$

implies the desire result near $F$. \hfill $\square$

From Lemma 2.8, we immediately have

**Proposition 2.12.** For parameters $c, v, \eta$ and $\delta > 0$, a function defined by

$$M_{c,v,\eta} := M_\eta \left( \Theta(t), \tilde{G}_v^\beta(b), t + \varphi + c \right)$$

is a strictly plurisubharmonic function on $X \setminus (D \cup F)$.

**Remark 2.13.** From [11], the solution $\varphi$ is bounded on $X$. Thus, by taking $c > 0$ sufficiently large, $\varphi$ does not affect the value of $M_{c,v,\eta}$. 

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By taking sufficiently large $c > 0$, we have

$$M_{c,v,\eta} = \begin{cases} 
\Theta(t) & \text{near } D \text{ and away from } F, \\
\tilde{G}_v^\beta(b) & \text{near } F \text{ and away from } D, \\
t + \varphi + c & \text{away from } F \text{ and } D.
\end{cases} \tag{2.2}$$

Set

$$\omega_{c,v,\eta} := \sqrt{-1} \partial \bar{\partial} M \left( \Theta(t), \tilde{G}_v^\beta(b), t + \varphi + c \right).$$

The reason that we consider the second Kähler potential contains the term $\delta \Theta(t)$ is that we want to make $\omega_{c,v,\eta}$ complete on $X \setminus D$. The function $M_{c,v,\eta}$ is defined on $X \setminus (D \cup F)$. But, $\omega_{c,v,\eta}$ is defined on $X \setminus D$ because the Kähler metric $\sqrt{-1} \partial \bar{\partial}b$ defined on $X$ by abuse of notation. From (2.2), we know that the scalar curvature of $\omega_{c,v,\eta}$ is small on three regions above (in particular, away from $D$ and $F$, $S(\omega_{c,v,\eta}) = 0$ since $t + \varphi + c$ is a Kähler potential whose Ricci form is flat).

The explicit formula of $\omega_{c,v,\eta}$ is written as

$$\omega_{c,v,\eta} = \frac{\partial M_{c,v,\eta}}{\partial t_1} \omega_0 + \frac{\partial M_{c,v,\eta}}{\partial t_2} (\gamma_v^\beta + \delta \omega_0) + \frac{\partial M_{c,v,\eta}}{\partial t_3} \sqrt{-1} \partial \bar{\partial}(t + \varphi)
+ \left[ \frac{\partial \Theta(t)}{\partial t} \frac{\partial \tilde{G}_v^\beta(b)}{\partial (t + \varphi)} \right] \left[ \frac{\partial^2 M_{c,v,\eta}}{\partial t_i \partial t_j} \right] \left[ \frac{\partial \Theta(t)}{\partial t} \frac{\partial \tilde{G}_v^\beta(b)}{\partial (t + \varphi)} \right]^t.$$

Thus, when we compute the scalar curvature of $\omega_{c,v,\eta}$, higher order derivatives of $\varphi$ arise in Ricci tensors of $\omega_{c,v,\eta}$. So, we must study the behavior of higher order derivatives of $\varphi$ near $D \cup F$.

## 3 Proof of Theorem 1.1

To study the behavior of higher order derivatives of $\varphi$, the elliptic operator defined by the Kähler metric $\theta_X + \sqrt{-1} \partial \bar{\partial} \varphi$ plays an important role. More precisely, the ellipticity of such operator is crucial. From [14] (see also [8], [9]), the $C^2$-estimate is given by

**Theorem 3.1.** Let $dV$ be a smooth volume form. Assume that $\varphi \in \text{PSH}(X, \theta_X)$ satisfies

$$(\theta_X + \sqrt{-1} \partial \bar{\partial} \varphi)^n = e^{\psi_+ - \psi_-} dV$$

with $\int_X \varphi \theta_X^n = 0$. Assume that we are given $C > 0$ and $p > 1$ such that

(i) $\sqrt{-1} \partial \bar{\partial} \psi_+ \geq -C \theta_X$ and $\sup_X \psi_+ \leq C$.

(ii) $\sqrt{-1} \partial \bar{\partial} \psi_- \geq -C \theta_X$ and $\| e^{-\psi_-} \|_{L^p} \leq C$.

Then there exists $A > 0$ depending only on $\theta_X$, $p$ and $C$ such that

$$0 \leq \theta_X + \sqrt{-1} \partial \bar{\partial} \varphi \leq A e^{-\psi_-} \theta_X.$$
In our case, we obtain that

$$0 \leq \theta_X + \sqrt{-1} \partial \bar{\partial} \varphi \leq A||\sigma_D||^{2m/l}||\sigma_F||^{-2/l} \theta_X.$$ 

Since

$$(\theta_X + \sqrt{-1} \partial \bar{\partial} \varphi)^n = \xi^{-1/l} \wedge \xi^{-1/l}$$

and $\xi = \sigma_F \otimes \sigma_D^{-m}$, we can estimate the eigenvalues of $\theta_X + \sqrt{-1} \partial \bar{\partial} \varphi$. Namely, the maximal eigenvalue $\Lambda$ and the minimal eigenvalue $\lambda$ of the Kähler metric $\theta_X + \sqrt{-1} \partial \bar{\partial} \varphi$ are estimated by

$$\Lambda = O(||\sigma_F||^{-2/l}), \quad \lambda^{-1} = O(||\sigma_D||^{-2m/l}).$$

To consider third and forth order derivatives, we recall the $C^{2, \epsilon}$-estimate of $\varphi$.

### 3.1 Preliminaries

This subsection follows from [9, Chapter 14]. Let $\mathcal{H}$ denote the set of all $n \times n$ Hermitian matrices and set

$$\mathcal{H}_+ := \{A \in \mathcal{H}|A > 0\}.$$ 

In addition, for $0 < \lambda < \Lambda < \infty$, let $S(\lambda, \Lambda)$ be the subset of $\mathcal{H}_+$ whose eigenvalues lie in the interval $[\lambda, \Lambda]$. First, recall the following result from linear algebra (see [13], [9]):

**Lemma 3.2.** One can find unit vectors $\zeta_1, \ldots, \zeta_N \in \mathbb{C}^n$ and $0 < \lambda_s < \Lambda_s < \infty$, depending only on $n, \lambda$ and $\Lambda$, such that every $A \in S(\lambda, \Lambda)$ can be written as

$$A = \sum_{k=1}^{N} \beta_k \zeta_k \otimes \overline{\zeta}_k,$$

where $\beta_k \in [\lambda_s, \Lambda_s]$. The vectors $\zeta_1, \ldots, \zeta_N \in \mathbb{C}^n$ can be chosen so that they contain a given orthonormal basis of $\mathbb{C}^n$.

**Proof.** The space $\mathcal{H}$ is of real dimension $n^2$. Every $A \in \mathcal{H}$ can be written as

$$A = \sum_{k=1}^{n} \lambda_k w_k \otimes \overline{w}_k,$$

where $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ are the eigenvalues and $w_1, \ldots, w_n \in \mathbb{C}^n$ are the corresponding unit eigenvectors. It follows that there exist unit vectors $\zeta_1, \ldots, \zeta_n \in \mathbb{C}^n$ such that the matrices $\zeta_k \otimes \overline{\zeta}_k$ span $\mathcal{H}$ over $\mathbb{R}$. For such sets of vectors we consider the sets of matrices

$$U(\zeta_1, \ldots, \zeta_n) := \left\{ \sum_{k} \beta_k \zeta_k \otimes \overline{\zeta}_k | 0 < \beta_k < 2\Lambda \right\}.$$

They forms an open covering of $S(\lambda/2, \Lambda)$, a compact subset of $\mathcal{H}$. Choosing a finite subcovering we get unit vectors $\zeta_1, \ldots, \zeta_N$ such that

$$S(\lambda/2, \Lambda) \subset \left\{ \sum_{k=1}^{N} \beta_k \zeta_k \otimes \overline{\zeta}_k | 0 \leq \beta_k < 2\Lambda \right\}.$$
For $A \in S(\lambda, \Lambda)$, we have

$$A - \frac{\lambda}{2N} \sum_{k=1}^{N} \beta_k \zeta_k \otimes \zeta_k \in S(\lambda, \Lambda)$$

and the lemma follows. Observe that we can take arbitrary $\lambda_* < \lambda/N$ and $\Lambda_* > \Lambda$.  

**Remark 3.3.** It follows from the form of the covering $U(\zeta_1, \ldots, \zeta_{n^2})$ that the number $N$ in the previous proof is depending only on the dimension $n$.

### 3.2 Refinement of $C^{2,\epsilon}$-estimate

Take local holomorphic coordinates $(z^i)_{i=1}^n = (z^1, z^2, \ldots, z^{n-2}, w_F, w_D)$ such that $\{w_F = 0\} = F$ and $\{w_D = 0\} = D$. We consider the complex Monge-Ampère equation

$$\det(u_{i\overline{j}}) = f$$

on an open subset $\Omega \subset \mathbb{C}^n \setminus (D \cup F)$. It follows from our construction that we may assume that the function $f$ is a form of

$$f = |w_F|^{-2/l} |w_D|^{2m/l}.$$

Fix an unit vector $\zeta \in \mathbb{C}^n$. Differentiating the following equation:

$$\log \det(u_{i\overline{j}}) = \log f,$$

we have

$$u^i_{\overline{\overline{j}}} u_{\zeta, i} u_{\zeta, \overline{j}} = (\log f)_{\zeta, \zeta} + u^i_{\overline{\overline{j}}} u^{k\overline{l}} u_{\zeta, i} u_{\zeta, k} u_{\zeta, l} \geq (\log f)_{\zeta, \zeta} = 0.$$

Here we use the standard Einstein convention and the notation $(u^i_{\overline{\overline{j}}}) = ((u_{i\overline{j}})^t)^{-1}$. Set

$$a^i_{\overline{j}} = fu^i_{\overline{\overline{j}}}.$$

Then, for any $i$, we have

$$(a^i_{\overline{j}})_{\zeta} = f u^i_{\overline{j}} u_{\zeta, i} u_{\zeta, \overline{j}} = f u^{k\overline{l}} u_{\zeta, k} u_{\zeta, l} u_{\zeta, i} u_{\zeta, \overline{j}} \geq 0.$$

Thus, we obtain

$$(a^i_{\overline{j}} u_{\zeta, i})_{\zeta} = (a^i_{\overline{j}})_{\zeta} u_{\zeta, i} + a^i_{\overline{j}} u_{\zeta, i} \geq f (\log f)_{\zeta, \zeta} = 0.$$

Note that $u_{\zeta, i} u_{\zeta, i}$ is a subsolution of the equation $Lv = 0$, where $Lv := \sum_{i,j} (a^i_{\overline{j}} v_i v_j)_{\overline{j}}$. The assumption of $u$ and the later lemma ensure that the operator $L$ is uniformly elliptic (in the real sense). Therefore, from [13, Theorem 8.18], the weak Harnack inequality yields

$$r^{-2n} \int_{B_r} \left( \sup_{B_{4r}} u_{\zeta, \overline{\zeta}} - u_{\zeta, \overline{\zeta}} \right) \leq C_H (\sup_{B_r} u_{\zeta, \overline{\zeta}} - \sup_{B_r} u_{\zeta, \overline{\zeta}}),$$

where $B_{4r} := B(z_0, 4r) \subset \Omega$ with $d(z_0, \partial \Omega) > 4r$.  

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Remark 3.4. From the proof of [13, Theorem 8.18], we know that the optimal Harnack constant $C_H$ is estimated by

$$C_H = C_n \sqrt{\Lambda / \lambda},$$

where $C_n$ is depending only on $n$.

But in our case, we will only consider the behavior of $\varphi$ in the neighborhood of $D \cup F$ and the $C^2$-estimate of $\varphi$ implies that

$$u_x^+ = O(||\sigma_F||^{-2/l})$$
$$u_x^- = O(||\sigma_D||^{-2m/l})$$

as $||\sigma_F|| \to 0$ and $||\sigma_D|| \to 0$. So, we have

Lemma 3.5. In our case, the constant $C_H$ in Harnack inequality is estimated by

$$C_H = O(\Lambda / \lambda)$$

Set $U := (u_{ij})$. For $x, y \in B_{4r}$, we obtain

$$a^{ij}(y)u_{ij}(x) = f(y) u^{ij}(y) u_{ij}(x) = f(y) \text{tr}(U(y)^{-1}U(x)).$$

In particular, $a^{ij}(y)u_{ij}(y) = nf(y)$. Since $\det(f(y)^{1/n}U(y)^{-1}) = 1$, we have

$$a^{ij}(y)u_{ij}(x) = f(y)^{1-1/n} \text{tr}(f(y)^{1/n}U(y)^{-1}U(x))$$
$$\geq nf(y)^{1-1/n} \det(U(x))^{1/n}$$
$$= nf(y)^{1-1/n} f(x)^{1/n}.$$

Here, we use the following lemma (see [9]):

Lemma 3.6. For any $A \in \mathcal{H}_+$, we have

$$(\det A)^{1/n} = \frac{1}{n} \inf \{\text{tr}(AB) | B \in \mathcal{H}_+, \det B = 1\}.$$

Therefore, for any $x, y \in B_{4r}$ and $\epsilon \in (0, 1)$, we have

$$a^{ij}(y)(u_{ij}(y) - u_{ij}(x)) \leq nf(y) - nf(y)^{1-1/n} f(x)^{1/n}$$
$$= nf(y)^{1-1/n} (f(y)^{1/n} - f(x)^{1/n})$$
$$\leq C(\epsilon)_4 |x - y|^\epsilon,$$

where

$$C(\epsilon)_4 := n \sup_{\Omega} (f^{1-1/n}) \text{Hö}l_\epsilon^{\Omega}(f^{1/n})$$

and $\text{Hö}l_\epsilon$ denotes a $\epsilon$-Hölder constant.

Remark 3.7. In [9], they used the Lipschitz constant of $f$. But in our case, it is enough to use the Hölder constant of $f$ for sufficiently small $\epsilon$. 

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Set $\lambda, \Lambda > 0$ so that the eigenvalues of $(a^{ij}(\cdot))(y)$ lie in the interval $[\lambda, \Lambda]$. We can find unit vectors $\zeta_1, \ldots, \zeta_N \in \mathbb{C}^n$ such that for any $x, y \in \Omega$,

$$a^{ij}(y)(u_{ij}(y) - u_{ij}(x)) = \sum_{k=1}^{N} \beta_k(y)(u_{\zeta_k}(y) - u_{\zeta_k}(x)),$$

where $\beta_k(y) \in [\lambda_*, \Lambda_*]$ and $\lambda_*, \Lambda_* > 0$.

Thus, we have

$$\sum_{k=1}^{N} \beta_k(y)(u_{\zeta_k}(y) - u_{\zeta_k}(x)) \leq C(\epsilon)|x - y|^\epsilon.$$

Set

$$M_{k,r} := \sup_{B_r} u_{\zeta_k}, m_{k,r} := \inf_{B_r} u_{\zeta_k},$$

and

$$\eta(r) := \sum_{k=1}^{N} (M_{k,r} - m_{k,r}).$$

To establish the Hölder condition

$$\eta(r) \leq C r^{\tilde{\epsilon}}$$

for some $0 < \tilde{\epsilon} < 1$, we need the following lemma from [13]:

**Lemma 3.8.** Let $\eta$ and $\sigma$ be non-decreasing functions defined on the interval $(0, R_0]$ such that there exist $\tau, \alpha \in (0, 1)$ satisfying

$$\eta(\tau r) \leq \alpha \eta(r) + \sigma(r)$$

for all $r \in (0, R_0]$. Then, for any $\mu \in (0, 1)$, we have

$$\eta(R) \leq \frac{1}{\alpha} \left( \frac{R}{R_0} \right)^{(1-\mu)(\log \alpha/\log \tau)} + \frac{\sigma(R_0^{1-\mu})}{1 - \alpha}.$$

So, it suffices to show that

$$\eta(r) \leq \delta \eta(4r) + C r^\epsilon, 0 < r < r_0,$$

where $\delta, \epsilon \in (0, 1)$ and $r_0 > 0$.

For fixed $k$, Harnack inequality implies that

$$r^{-2n} \int_{B_r} \sum_{l \neq k} (M_{l,4r} - u_{\zeta_l,\zeta_l}) = \sum_{l \neq k} r^{-2n} \int_{B_r} (M_{l,4r} - u_{\zeta_l,\zeta_l}) \leq \sum_{l \neq k} C_H(M_{l,4r} - M_{l,r}) \leq \sum_{l \neq k} C_H(\eta(4r) - \eta(r)) = (N - 1)C_H(\eta(4r) - \eta(r)).$$

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For \( x \in B_{4r} \) and \( y \in B_r \), we have

\[
\beta_k(y)(u_{\xi_k \zeta_k}(y) - u_{\xi_k \zeta_k}(x)) \leq C(\epsilon)A|x - y|^r + \sum_{l \neq k} \beta_l(y)(u_{\xi_l \zeta_l}(x) - u_{\xi_l \zeta_l}(y)) \\
\leq 5C(\epsilon)A^r + \Lambda^* \sum_{l \neq k} (M_{l,Ar} - u_{\xi_l \zeta_l}(y)).
\]

Thus, for all \( y \in B_r \), we have

\[
u_{\xi_k \zeta_k}(y) - m_{k,Ar} \leq \frac{1}{\lambda^*} \left( 5C(\epsilon)A^r + \Lambda^* \sum_{l \neq k} (M_{l,Ar} - u_{\xi_l \zeta_l}(y)) \right).
\]

Therefore,

\[
r^{-2n} \int_{B_r} (u_{\xi_k \zeta_k}(y) - m_{k,Ar}) \leq r^{-2n} \int_{B_r} \frac{1}{\lambda^*} \left( 5C(\epsilon)A^r + \Lambda^* \sum_{l \neq k} (M_{l,Ar} - u_{\xi_l \zeta_l}(y)) \right) \\
\leq \frac{5C(\epsilon)A^r}{\lambda^*} + \frac{\Lambda^*}{\lambda^*} r^{-2n} \int_{B_r} \sum_{l \neq k} (M_{l,Ar} - u_{\xi_l \zeta_l}) \leq \frac{5C(\epsilon)A^r}{\lambda^*} + \frac{\Lambda^*}{\lambda^*} (N-1)C_H(\eta(4r) - \eta(r)).
\]

Using Harnack inequality again, we have

\[
M_{k,Ar} - m_{k,Ar} = r^{-2n} \int_{B_r} (\sup_{B_{4r}} u_{\xi_k \zeta_k} - u_{\xi_k \zeta_k}(y) - m_{k,Ar}) \\
\leq C_H(M_{k,Ar} - M_k) + \frac{5C(\epsilon)A^r}{\lambda^*} + \frac{\Lambda^*}{\lambda^*} (N-1)C_H(\eta(4r) - \eta(r)) \\
\leq \left( C_H + \frac{\Lambda^*}{\lambda^*} (N-1)C_H \right) \eta(4r) - \left( C_H + \frac{\Lambda^*}{\lambda^*} (N-1)C_H \right) \eta(r) + \frac{5C(\epsilon)A^r}{\lambda^*}.
\]

Summing over \( k \), we have

\[
\eta(4r) \leq N \left( C_H + \frac{\Lambda^*}{\lambda^*} (N-1)C_H \right) \eta(4r) - N \left( C_H + \frac{\Lambda^*}{\lambda^*} (N-1)C_H \right) \eta(r) + N \frac{5C(\epsilon)A^r}{\lambda^*}.
\]

Thus, we obtain

\[
\eta(r) \leq \frac{N \left( C_H + \frac{\Lambda^*}{\lambda^*} (N-1)C_H \right) - 1}{N \left( C_H + \frac{\Lambda^*}{\lambda^*} (N-1)C_H \right)} \eta(4r) + \frac{5C(\epsilon)A^r}{\lambda^*} r^{\mu}.
\]

Since we can take arbitrary \( \lambda^* N < \lambda \) and \( \Lambda^* > \Lambda \), we may assume that \( \lambda^* N = \lambda \) and \( \Lambda^* = \Lambda \). From the interior Hölder estimate for solutions of Poisson’s equation [13, Theorem 4.6], by taking a suitable and small \( \epsilon > 0 \) and \( \mu \in (0, 1) \) which is close to 0, we have
Lemma 3.9. By taking $\epsilon \leq \min \{2/l, 4m/nl\}$, there exists $0 < \tilde{\epsilon} < \epsilon$ with

$$||u||_{C^{2,\tilde{\epsilon}}} = O \left( \left( \frac{\Lambda}{\lambda} \right) C_H \right).$$

Thus, applying Lemma 3.5, we have

Proposition 3.10.

$$||\varphi||_{C^{2,\tilde{\epsilon}}} = O \left( \left( ||\sigma_D||^{-2m/l} ||\sigma_F||^{-2/l} \right)^2 \right).$$

3.3 Third and forth order estimates

This subsection also follows from [9, Chapter 14]. To consider higher order estimates, we recall Schauder estimate. The complex Monge-Ampère operator

$$F(D^2u) = \det(u_{i,j})$$

is elliptic if the $2n \times 2n$ real symmetric matrix $A := (\partial F/\partial u_{p,q})$ is positive (we denote here by $u_{p,q}$ the element of the real Hessian $D^2u$). The matrix $A$ is determined by

$$\frac{d}{dt} F(D^2u + tB)|_{t=0} = \text{tr}(A'B).$$

From [5] (see also [9, Exercise 14.8]), we have

Lemma 3.11. One has

$$\lambda_{\min}(\partial F/\partial u_{p,q}) = \frac{\det(u_{i,j})}{4\lambda_{\max}(u_{i,j})}, \lambda_{\max}(\partial F/\partial u_{p,q}) = \frac{\det(u_{i,j})}{4\lambda_{\min}(u_{i,j})},$$

where $\lambda_{\min}(\partial F/\partial u_{p,q})$ and $\lambda_{\max}(\partial F/\partial u_{p,q})$ denote minimal and maximal eigenvalue of the matrix $(\partial F/\partial u_{p,q})_{p,q}$ respectively.

Then, we can estimate the ellipticity in the real sense. Consider applying the standard elliptic theory to the equation

$$F(D^2u) = f.$$

For a fixed unit vector $\zeta$ and small $h > 0$, we consider

$$u_h(x) := \frac{u(x + h\zeta) - u(x)}{h}$$

and

$$a^p_q(x) := \int_0^1 \frac{\partial F}{\partial u_{p,q}} (tD^2u(x + h\zeta) + (1 - t)D^2u(x)) dt.$$

Thus, we have

$$a^p_q(x)u_h = \frac{1}{h} \int_0^1 \frac{d}{dt} F(tD^2u(x + h\zeta) + (1 - t)D^2u(x)) dt = f^h(x).$$

From the definition of $a^p_q$, we obtain

$$||a^p_q||_{C^{0,\tilde{\epsilon}}} \leq C||u||_{C^{2,\tilde{\epsilon}}}^{n-1} = O((\Lambda/\lambda)^{2(n-1)})$$

for sufficiently small $h > 0$. Schauder estimate implies
Proposition 3.12. There exists $C_S > 0$ such that
\[
||u^h||_{C^2,\bar{z}} \leq C_S(||f^h||_{C^0,\bar{z}} + ||u^h||_{C^0})
\]
for any $h > 0$.

Therefore, we can obtain the estimate of derivatives of the solution $\varphi$ in the desired direction by taking a suitable vector $\zeta$ and tending $h \to 0$. Seeing the proof of [13, Lemma 6.1 and Theorem 6.2] (see also Appendix of this paper), we have
\[
C_S = O(\lambda^{-2-\tilde{\epsilon}}(\Lambda/\lambda)^{2(n-1)(2+\tilde{\epsilon})}).
\]

As $h \to 0$, we have following third order estimates of $\varphi$:

Proposition 3.13. For any multi-index $\alpha = (\alpha_1, ..., \alpha_n)$ satisfying $\sum_i \alpha_i = 2$, we have
\[
\left| \frac{\partial}{\partial z^i} \partial^\alpha \varphi \right| = O\left(C_S|w_D|^{-4m/l}|w_F|^{-4/l}\right),
\]
\[
\left| \frac{\partial}{\partial w_F} \partial^\alpha \varphi \right| = O\left(C_S|w_D|^{-4m/l}|w_F|^{-1-4/l}\right),
\]
\[
\left| \frac{\partial}{\partial w_D} \partial^\alpha \varphi \right| = O\left(C_S|w_D|^{-1-4m/l}|w_F|^{-4/l}\right),
\]
as $|w_D|, |w_F| \to 0$.

Let $\dot{a}^{p,q}_h$ be a differential of $a^{p,q}_h$ in some direction. From the definition of $a^{p,q}_h$, we know that
\[
||\dot{a}^{p,q}_h||_{C^0,\bar{z}} \leq C||\dot{u}||_{C^2,\bar{z}}||u||_{C^{2,\bar{z}}}^{-2}
\]
Thus, differentiating the equation $a^{p,q}_h(x)u^h_{p,q}(x) = f^h(x)$, Schauder estimate implies again the following inequality:
\[
||\dot{u}^h||_{C^2,\bar{z}} \leq C_S(||\dot{f}^h - \dot{a}^{p,q}_h u^h_{p,q}||_{C^0,\bar{z}} + ||\dot{u}^h||_{C^0})
\]
Thus, we have finished proving Theorem 1.1 by taking a suitable vector $\zeta$ and tending $h \to 0$. \hfill \Box

Remark 3.14. Seeing the proof above, we can find that
\[
a = a(n) = O(n^2).
\]

4 Proof of Theorem 1.2

In this section, we prove Theorem 1.2. To compute the scalar curvature of the Kähler metric, we have to consider the inverse matrix. Namely, we need
Lemma 4.1. For the matrix
\[ T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \]
if \( A \) and \( S := D - CA^{-1}B \) are invertible, \( T \) is invertible and its inverse matrix can be written as
\[ T^{-1} = \begin{bmatrix} A^{-1} + A^{-1}BS^{-1}CA^{-1} & -A^{-1}BS^{-1} \\ -S^{-1}CA^{-1} & S^{-1} \end{bmatrix}. \]

Proof. Directly, we have
\[
\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} I & A^{-1}B \\ O & I \end{bmatrix}^{-1} \begin{bmatrix} A & O \\ O & S \end{bmatrix}^{-1} \begin{bmatrix} I & O \\ CA^{-1} & I \end{bmatrix}^{-1} = \begin{bmatrix} I & -A^{-1}B \\ O & I \end{bmatrix} \begin{bmatrix} A^{-1} & O \\ O & S^{-1} \end{bmatrix} \begin{bmatrix} I & O \\ -CA^{-1} & I \end{bmatrix} = \begin{bmatrix} A^{-1} + A^{-1}BS^{-1}CA^{-1} & -A^{-1}BS^{-1} \\ -S^{-1}CA^{-1} & S^{-1} \end{bmatrix}.
\]

\( \square \)

Since we assume that the divisor \( D + F \) is simple normal crossing, we can choose block matrices in suitable directions in local holomorphic coordinates defining hypersurfaces \( D \) and \( F \).

To prove Theorem 1.2 we consider the case that the parameter \( \eta = (\eta_1, \eta_2, \eta_3) \) depends on \( c > 0 \). More precisely, set \( \eta_i := a_i c \) for \( i = 1, 2 \) for \( a_i \in (0, 1) \) and let \( \eta_3 \) is a fixed positive real number. We use many parameters, i.e., \( c, v, \beta, \delta, \eta, a_i \). In these parameters, when we want to make the scalar curvature \( S(\omega_{c,v,\eta}) \) small, we consider that \( c \to \infty \) and \( v \to 0 \). On the other hand, other parameters \( \beta, \delta, \eta, a_i \) will not tend to \( \infty, 0 \) or 1. Settings of these bounded parameters will be given later.

Proof of Theorem 1.2. Take a relatively compact domain \( Y \subset \subset X \setminus (D \cup F) \). Since \( G_v^\beta(\beta b) \to \beta e^{\beta b} \) as \( v \to 0 \), we can find a sufficiently large number \( c_0 = c_0(Y) > 0 \) so that
\[ Y \subset \subset \left\{ t + \varphi + c_0 > \max\{ \Theta(t), \tilde{G}_v^\beta(b) \} \right\} \subset \subset X \setminus (D \cup F) \]
for any \( v > 0 \). For simplicity, we write \( \varphi + c_0 \) by the same symbol \( \varphi \).

Directly, we have
\[
\omega_{c,v,\eta} = \sqrt{-1} g_{\nu \eta} \frac{dz^i}{\partial \theta_1} \wedge \frac{dz^j}{\partial \theta_2} = \frac{\partial M_{c,v,\eta}}{\partial t_1} \omega_0 + \frac{\partial M_{c,v,\eta}}{\partial t_2} (\gamma_v^\beta + \delta \omega_0) + \frac{\partial M_{c,v,\eta}}{\partial t_3} \sqrt{-1} \partial \overline{\partial} (t + \varphi)
+ \left[ \partial \Theta(t) \quad \partial \tilde{G}_v^\beta(b) \quad \partial (t + \varphi) \right] \left[ \frac{\partial^2 M_{c,v,\eta}}{\partial t_3} \partial \Theta(t) \quad \overline{\partial} \tilde{G}_v^\beta(b) \quad \overline{\partial} (t + \varphi) \right]^t.
\]

It follows from the convexity of \( M \) that the last term is semi-positive.

When we compute the scalar curvature of \( \omega_{c,v,\eta} \), the difficulty comes from terms \( \partial \Theta(t) \wedge \overline{\partial} \Theta(t) \) and \( \partial \tilde{G}_v^\beta(b) \wedge \overline{\partial} \tilde{G}_v^\beta(b) \). For these terms, since functions \( t \) and \( b \) are defined by
Hermitian norms of holomorphic sections, it is enough to focus on derivatives in normal directions of smooth hypersurfaces $D$ and $F$ by taking suitable local trivializations of line bundles $L_X$ and $K_X^{-1} \otimes L_X^{m}$. The reason that scalar curvatures of two Kähler metrics $\omega_0, \gamma_v^\beta$ are under control is that Ricci curvatures are bounded and Kähler metrics grow asymptotically near hypersurfaces. Thus, it is enough that we focus on derivatives of $\varphi$ and $M_\eta$ arising in Ricci tensors. Higher order derivatives of $\varphi$ are estimated in the previous section (Theorem 2.11). In addition, the definition of a parameter $\eta$ and Lemma 2.10 imply that higher order derivatives in first or second variable of $M_\eta$ are estimated by some negative power of $c > 0$.

**Claim 1.** On the region defined by

$$ (t + \varphi + c) + \eta_3 < \max\{\Theta(t) - \eta_1, \tilde{G}_v^\beta(b) - \eta_2\}, $$

$$ |\Theta(t) - \tilde{G}_v^\beta(b)| < \eta_1 + \eta_2, $$

we can make the scalar curvature $S(\omega_{c,v})$ small arbitrarily as $c \to \infty$.

**Proof.** On this region, we can write as

$$ \omega_{c,v,\eta} = \frac{\partial M_{c,v,\eta}}{\partial t_1} \omega_0 + \frac{\partial M_{c,v,\eta}}{\partial t_2} (\gamma_v + \delta \omega_0) + \left[ \partial \Theta(t) \quad \partial \tilde{G}_v^\beta(b) \right] \left[ \frac{\partial^2 M_{c,v,\eta}}{\partial t_1^j} \right] \left[ \partial \Theta(t) \quad \partial \tilde{G}_v^\beta(b) \right]^t. $$

**Lemma 4.2.** Take a point $p \in D \cap F$ and local holomorphic coordinates $(z^1, ..., z^{n-2}, w_F, w_D)$ centered at $p$ satisfying $D = \{w_D = 0\}$ and $F = \{w_F = 0\}$. By taking suitable local trivializations of $L_X$ and $K_X^{-1} \otimes L_X^{m}$, we may assume that if $(z^1, ..., z^{n-2}, w_F, w_D) = (0, ..., 0, w_F, w_D)$, we have

$$ \partial \Theta(t) \wedge \bar{\partial} \Theta(t) = O(|w_F|^2 |w_D|^{-4S_D/(n-1)})dw_F \wedge dw_F + O(|w_F||w_D|^{-1-4S_D/(n-1)})dw_D \wedge dw_F^c + dw_F \wedge dw_D, $$

$$ \partial G_v^\beta(b) \wedge \bar{\partial} G_v^\beta(b) = O(|w_F|^{2\beta} v^{-2/\beta}|w_F|^{-2})dw_F \wedge dw_F + O(|w_F|^{-1}|w_D|(|w_F|^{2\beta} v^{-2/\beta}|w_F|^{-2})dw_D \wedge dw_F + dw_F \wedge dw_D + dw_D. $$

From the definition of the region we considering, we obtain

$$ \omega_{c,v,\eta} = \begin{pmatrix}
g_1,\bar{1} & \cdots & g_1,\bar{n-2} & g_1,\bar{n-1} & g_1,\bar{n} \\
\vdots & \ddots & \vdots & \vdots & \vdots \\
g_{n-2,\bar{1}} & \cdots & g_{n-2,\bar{n-2}} & g_{n-2,\bar{n-1}} & g_{n-2,\bar{n}} \\
g_{n-1,\bar{1}} & \cdots & g_{n-1,\bar{n-2}} & g_{n-1,\bar{n-1}} & g_{n-1,\bar{n}} \\
g_n,\bar{1} & \cdots & g_n,\bar{n-2} & |w_F|^{-1} |w_D| \left(|w_F|^{2\beta} v^{-2/\beta}|w_F|^{-2} \right) & |w_D|^{-1} |w_D| \left(|w_F|^{2\beta} v^{-2/\beta}|w_F|^{-2} \right) \\
\end{pmatrix} $$

as $w_D, w_F \to 0$. 

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In particular, coefficients $g_{i,j}$ for $1 \leq i, j \leq n-2$ come from Kähler metrics $\omega_0$ and $\gamma_\beta^\alpha$. Thus,

$$
\begin{bmatrix}
  g_{1,1} & \cdots & g_{1,n-2} \\
  \vdots & \ddots & \vdots \\
  g_{n-2,1} & \cdots & g_{n-2,n-2}
\end{bmatrix} = O(|w_D|^{-2\delta_D/n(n-1)} + (|w_F|^2 + v)^{-1/\beta}).
$$

For other blocks, we similarly have

$$
\begin{bmatrix}
  g_{1,n-1} & g_{1,n} \\
  \vdots & \vdots \\
  g_{n-2,n-1} & g_{n-2,n}
\end{bmatrix} = O(|w_D|^{-2\delta_D/n(n-1)} + (|w_F|^2 + v)^{-1/\beta}).
$$

From Lemma 1.1 we have

$$
g^{i,j} =
\begin{bmatrix}
  g^{1,1} & \cdots & g^{1,n-2} & g^{1,n-1} & g^{1,n} \\
  \vdots & \ddots & \vdots & \vdots & \vdots \\
  g^{n-1,1} & \cdots & g^{n-1,n-2} & c(|w_F|^{2\beta} + v)^{2/\beta}|w_F|^2 & c|w_D|^{3+4\delta_D/n(n-1)}|w_F| \\
  g^{n,n-1} & \cdots & g^{n,n-2} & c|w_D|^{3+4\delta_D/n(n-1)}|w_F| & c|w_D|^{2+4\delta_D/n(n-1)}
\end{bmatrix}
$$

as $w_D, w_F \to 0$. Since metric tensors $g^{i,j}$ with $i, j \neq n-1, n$ come from Kähler metrics $\omega_0$ and $\gamma_\beta^\alpha$ whose scalar curvature have been already known. Thus, it is enough to study the case that $i = n-1, n$ and $j = n-1, n$. Recall that Ricci tensors are defined by $R_{i,j} := g^{p,q}\partial p\partial q / \partial z^i / \partial z^j + g^{k,r}g^{p,j}(\partial g_{k,l} / \partial z^i)(\partial g_{l,j} / \partial z^j)$. So, the Ricci form $\text{Ric}(\omega_{c,v,\eta})$ is written as

$$
\begin{bmatrix}
  R_{1,1} & \cdots & R_{1,n-2} & R_{1,n-1} & R_{1,n} \\
  \vdots & \ddots & \vdots & \vdots & \vdots \\
  R_{n-2,1} & \cdots & R_{n-2,n-2} & R_{n-2,n-1} & R_{n-2,n} \\
  R_{n-1,1} & \cdots & R_{n-1,n-2} & c^{-3}(|w_F|^{2\beta} + v)^{-2/\beta}|w_F|^{-2} & c^{-3}|w_F|^{-1}|w_D|(|w_F|^{2\beta} + v)^{-2/\beta} \\
  R_{n,1} & \cdots & R_{n,n-2} & c^{-3}|w_F|^{-1}|w_D|(|w_F|^{2\beta} + v)^{-2/\beta} & c^{-3}|w_D|^{-2+4\delta_D/n(n-1)}
\end{bmatrix}
$$

as $w_D, w_F \to 0$ and other Ricci tensors $R_{i,j}$ for $1 \leq i \leq n-2$ are under control.

By taking a trace, we obtain the following:

$$S(\omega_{c,v,\eta}) = O(c^{-2}).$$

\[\square\]

**Claim 2.** Consider the region defined by

$$\hat{C}_\omega^\beta(b) + \eta_2 < \max\{\Theta(t) - \eta_1, (t + \varphi + c) - \eta_3\},$$

$$|\Theta(t) - (t + \varphi + c)| < \eta_1 + \eta_3.$$

By choosing parameters $\eta, \delta$ so that

$$(1 - \delta)c + \delta\eta_1 - \eta_2 = (1 - \delta + \delta a_1 - a_2)c = 0$$

for any $c > 0$ and taking sufficiently large $\beta$, we can make the scalar curvature $S(\omega_{c,v})$ small arbitrarily as $c \to \infty$. 
On this region, since
\[ M_{c,v,\eta} = M(\Theta(t), t + \varphi + c) \]
from Lemma 2.8 we have
\[
\omega_{c,v,\eta} = \frac{\partial M_{c,v,\eta}}{\partial t_1} \omega_0 + \frac{\partial M_{c,v,\eta}}{\partial t_3} \sqrt{-1} \partial \bar{\partial}(t + \varphi) + \left[ \partial \Theta(t) \partial(t + \varphi) \right] \left[ \frac{\partial^2 M_{c,v,\eta}}{\partial t_i \partial t_j} \right] \left[ \overline{\Theta(t)} \bar{\partial}(t + \varphi) \right]^t.
\]

From the hypothesis of this claim, we have
\[
G^3(\beta b) < (t + \varphi + c) + \eta_3 - \delta \Theta(t) - \eta_2 < (1 - \delta)(t + \varphi + c) + \delta(\eta_1 + \eta_3) + \eta_3 - \eta_2 = (1 - \delta)(t + \varphi) + (1 + \delta)\eta_3.
\]

By taking small \( v > 0 \) and suitable \( b_0 \) in the definition of the function \( G^3_v(\beta b) \), we may assume that
\[
\beta b < G^3_v(\beta b).
\]

From [11] again, recall that \( \varphi \) is bounded on \( X \). So, on this region, we have the following inequality:
\[
||\sigma_F||^{-2\beta/(1-\delta)} < C||\sigma_D||^{-2}
\]
for some constant \( C > 0 \) depending only on the \( C^0\)-norm of \( \varphi \). By taking a sufficiently large number \( \beta \) which depends on \( \delta, m, l \) and \( a = a(n) \) in Theorem [11], we may assume that
\[
||\sigma_F||^{-2 - 2n/l} < C||\sigma_D||^{-2a_m/l}.
\]

Thus, on this region, the growth of derivatives of \( \varphi \) can be controlled by the Kähler metric \( \omega_0 \). Take a point in \( D \setminus (D \cap F) \) and local holomorphic coordinates \( (z^1, \ldots, z^{n-1}, w_D) \) satisfying \( D = \{ w_D = 0 \} \). Then, we have
\[
\left| \frac{\partial^2}{\partial z_i \partial \overline{z_j}} \partial_a \varphi \right| = O\left( |w_D|^{-2a_m/l} \right)
\]
if \( 1 \leq i, j \leq n - 1 \) and
\[
\left| \frac{\partial^2}{\partial w_D \partial \overline{w_D}} \partial_a \varphi \right| = O\left( |w_D|^{-2-2a_m/l} \right).
\]

Similarly, we have

**Lemma 4.3.** By taking a suitable local trivialization of \( L_X \), we may assume that if \( (z^1, \ldots, z^{n-1}, w_D) = (0, \ldots, 0, w_D) \), we have
\[
\partial \Theta(t) \wedge \overline{\Theta(t)} = O( |w_D|^{-2-4S_D/n(n-1)} ) dw_D \wedge \overline{dw_D},
\]

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Recall the hypothesis
\[ \frac{am}{2l} < \frac{\tilde{S}_D}{n(n-1)}. \]
In addition, Theorem 1.1 and Lemma 4.1 show that higher order derivatives including \( \partial^4 \varphi/\partial w^2 \partial \pi^2 \) are controlled by taking a trace with respect to \( \omega_{c,v,\eta} \). Therefore, we can ignore derivatives of \( \varphi \) arising in Ricci tensors. The computation of the scalar curvature is given by the similar way in the previous claim. \( \square \)

Claim 3. Consider the region defined by
\[
\Theta(t) + \eta_1 < \max \{ \tilde{G}_v^\beta(b) - \eta_2, (t + \varphi + c) - \eta_3 \},
\]
\[
|\tilde{G}_v^\beta(b) - (t + \varphi + c)| < \eta_2 + \eta_3.
\]
By choosing sufficiently small number \( v > 0 \) so that
\[
(||\sigma_F||^2 + v)^{2/3} < ||\sigma_F||^{4a/l}
\]
holds on this region, we can make the scalar curvature \( S(\omega_{c,v}) \) small arbitrarily as \( c \to \infty \).

The reason that we can find a sufficiently small number \( v > 0 \) satisfying the statement in this claim is that \( \min \{ ||\sigma_D|| \} \) on this region increasing as \( v \to 0 \) and \( 4a/l < 4 \). We need the following

Lemma 4.4. By taking suitable local trivialization of \( K_X^{-1} \otimes L_X^n \), we may assume that if \((z^1, ..., z^{n-2}, w_F, z_n) = (0, ..., 0, w_F, 0)\), we have
\[
\partial G_v^\beta(\beta b) \land \overline{\partial} G_v^\beta(\beta b) = O((||w_F||^2 + v)^{-2/3}||w_F||^{-2})dw_F \land d\overline{w_F}.
\]
Thus, we can prove this claim by the same way in previous sections. \( \square \)

The remained case is the region defined by
\[
|\Theta(t) - \tilde{G}_v^\beta(b)| < \eta_1 + \eta_2,
\]
\[
|\tilde{G}_v^\beta(b) - (t + \varphi + c)| < \eta_2 + \eta_3,
\]
\[
|\Theta(t) - (t + \varphi + c)| < \eta_1 + \eta_3.
\]
On this region, we can apply the same way in previous sections. Thus, we have finished proving Theorem 1.2. \( \square \)

5 Appendix

In this appendix, we recall and refine the Schauder estimate for the 2-nd order elliptic linear operator defined by the Kähler metric \( \theta_X + \sqrt{-1} \partial \bar{\partial} \varphi \). Let \( \Omega \subset \mathbb{R}^n \) be an open subset.

Following [13] p.61, we define for \( u \in C^k, C^{k,\alpha} \) the following quantities
\[
[u]_{k,0;\Omega}^* = [u]_{k,;\Omega}^* = \sup_{x \in \Omega, |\beta| = k} d_x^k |D^\beta u(x)|;
\]
\[ |u|_{k,0,\Omega} = |u|_{k,\Omega}^* = \sum_{j=0}^{k} |u|_{j,\Omega}^*; \]
\[
[u]_{k,\alpha,\Omega}^* = \sup_{x,y \in \Omega, |y|=k} d^k_{x,y} \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^\alpha};
\]
\[
|u|^*_{k,\alpha,\Omega} = |u|^*_{k,\Omega} + |u|^*_{k,\alpha,\Omega};
\]
\[
|f|^*_{l_0,\alpha,\Omega} = \sup_{x \in \Omega} d^k_x |f(x)| + \sup_{x,y \in \Omega} d^k_{x,y} \frac{|f(x) - f(y)|}{|x-y|^\alpha},
\]
where \(d_x := \text{dist}(x, \partial \Omega)\) and \(d_{x,y} := \min\{d_x, d_y\} \).

In this appendix, we consider the following equation
\[
Lu = a^{ij}(x)D_{ij}u = f(x), \quad a^{ij} = a^{ji}
\]
where \(a^{ij}\) and \(f\) defined in \(\Omega\). First, we consider the equation with constant coefficient by following [13, Lemma 6.1].

**Lemma 5.1.** In the equation
\[
L_0u = A^{ij}D_{ij}u = f(x), \quad A^{ij} = A^{ji}
\]
where \([A^{ij}]\) is a constant matrix such that there are positive constants \(\lambda, \Lambda\) such that
\[
\lambda|\xi|^2 \leq A^{ij}\xi_i\xi_j \leq \Lambda|\xi|^2
\]
for any \(\xi \in \mathbb{R}^n\).

Let \(u \in C^2(\Omega), f \in C^\alpha(\Omega)\) satisfies \(L_0u = f(x)\) in \(\Omega\). Then,
\[
|u|^*_{l_0,\alpha,\Omega} \leq C(\Lambda/\lambda)^{2+\alpha}(|u|_{0,\Omega} + |f|^*_{l_0,\alpha,\Omega})
\]
where \(C = C(n)\).

**Proof.** Let \(P\) be a constant matrix which defines a nonsingular linear transformation \(y = Px\) from \(\mathbb{R}^n\) to \(\mathbb{R}^n\). Set \(\tilde{u}(y) := u(P^ty) = u(x)\) for \(y \in \tilde{\Omega} := \tilde{P}(\Omega)\). Under this transformation, one verify easily that
\[
A^{ij}D_{ij}^\alpha u(x) = \tilde{A}^{ij}D_{ij}^\alpha \tilde{u}(y),
\]
where \(\tilde{A} = P^tAP\). For a suitable orthogonal matrix \(P\), \(\tilde{A}\) is a diagonal matrix whose diagonal elements are the eigenvalues \(\lambda_i\) of \(A\). For the diagonal matrix \(D := [\lambda_i\delta_{ij}],\) set \(Q := PD\). Then, the transformation defined by \(y = Qx\) takes \(L_0u = f(x)\) into the Poisson equation \(\Delta \tilde{u}(y) = \tilde{f}(y)\). Since \(P\) is a orthogonal matrix, we have
\[
\Lambda^{-1/2}|x| \leq |y| = |Qx| \leq \lambda^{-1/2}|x|.
\]
So, we have
\[
c(k,n)^{-1}(\Lambda/\lambda)^{-(k+\alpha)/2}|v|^*_{k,\alpha,\Omega} \leq \tilde{|v}|^*_{k,\alpha,\tilde{\Omega}} \leq c(k,n)(\Lambda/\lambda)^{(k+\alpha)/2}|v|^*_{k,\alpha,\Omega}
\]
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and 
\[ c(k, n)^{-1} \Lambda^{-\alpha/2} \lambda^{(k+\alpha)/2} |v|^{(k)}_{0, \alpha; \Omega} \leq |\tilde{v}|^{(k)}_{0, \alpha; \tilde{\Omega}} \leq c(k, n) \lambda^{-\alpha/2} \Lambda^{(k+\alpha)/2} |v|^{(k)}_{0, \alpha; \Omega}. \]

In this paper, we may assume that \( \lambda < 1 \) and \( \Lambda > 1 \). Applying \([13, \text{Theorem 4.8}]\) to variables \( y \in \tilde{\Omega} \), we have
\[
|v^*_{2, \alpha; \Omega}| \leq c(n)(\Lambda/\lambda)^{(2+\alpha)/2} |\tilde{v}|^*_{2, \alpha; \tilde{\Omega}} \\
\leq c(n)(\Lambda/\lambda)^{(2+\alpha)/2} (|\tilde{u}|_{0, \tilde{\Omega}} + |\tilde{f}|^{(2)}_{0, \alpha; \tilde{\Omega}}) \\
\leq c(n)^2 (\Lambda/\lambda)^{2+\alpha} (|u|_{0; \Omega} + |f|^{(2)}_{0, \alpha; \Omega}).
\]

Next, we recall that the basic interior Schauder estimate \([13, \text{Theorem 6.2}]\).

**Theorem 5.2.** Let \( u \in C^{2, \alpha}(\Omega) \) be a bounded solution in an open subset \( \Omega \subset \mathbb{R}^n \) of the equation
\[ Lu = a^{ij}(x) D_{ij} u = f \]
where \( f \in C^{\alpha}(\Omega) \) and there are positive constants \( \lambda, \Lambda \) such that the coefficients satisfy
\[ a^{ij}(x) \xi_i \xi_j \geq \lambda |\xi|^2 \]
for any \( x \in \Omega, \xi \in \mathbb{R}^n \) and
\[ |a^{ij}|^{(0)}_{0, \alpha; \Omega} \leq \Lambda. \]
Then
\[ |u^*_{2, \alpha; \Omega}| \leq C(\Lambda/\lambda)^{2+\alpha} (|u|_{0; \Omega} + |f|^{(2)}_{0, \alpha; \Omega}). \]

**Proof.** Consider new coefficients defined by
\[ \tilde{a}^{ij} := \frac{1}{\Lambda} a^{ij}. \]
It follows that
\[ \tilde{a}^{ij}(x) \xi_i \xi_j \geq (\Lambda/\lambda)^{-1} |\xi|^2 \]
for any \( x \in \Omega, \xi \in \mathbb{R}^n \) and
\[ |\tilde{a}^{ij}|^{(0)}_{0, \alpha; \Omega} \leq 1. \]
Then, the proof of \([13, \text{Theorem 6.2}]\) gives the desired result.

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