Some new results in extremal graph theory

Vladimir Nikiforov *
Department of Mathematical Sciences, University of Memphis, Memphis, TN 38152
e-mail: vnikifrv@memphis.edu

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Abstract

In recent years several classical results in extremal graph theory have been improved in a uniform way and their proofs have been simplified and streamlined. These results include a new Erdős-Stone-Bollobás theorem, several stability theorems, several saturation results and bounds for the number of graphs with large forbidden subgraphs.

Another recent trend is the expansion of spectral extremal graph theory, in which extremal properties of graphs are studied by means of eigenvalues of various matrices. One particular achievement in this area is the casting of the central results above in spectral terms, often with additional enhancement. In addition, new, specific spectral results were found that have no conventional analogs.

All of the above material is scattered throughout various journals, and since it may be of some interest, the purpose of this survey is to present the best of these results in a uniform, structured setting, together with some discussions of the underpinning ideas.

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1 Introduction

The purpose of this survey is to give a systematic account of two recent lines of research in extremal graph theory. The first one, developed in [14], [15], [16], [63, 68], improves a number of classical results grouped around the theorem of Turán. The main progress is along the following three guidelines: replacing fixed parameters by variable ones; giving explicit conditions for the validity of the statements; developing and using tools of general scope. Among the results obtained are a new Erdős-Stone-Bollobás theorem (see Section 2.2), several stability theorems (see Section 2.4), several saturation results, and bounds for the number of graphs without given large subgraphs.

The second line of research, developed in [13], [69, 82], can be called spectral extremal graph theory, where connections are sought between graph properties and the eigenvalues of certain matrices associated with graphs. As a result of this research, much of classical extremal graph theory has been translated into spectral statements, and this translation has also brought enhancement. Among the results obtained are spectral forms of the Turán theorem and the Erdős-Stone-Bollobás theorem, several stability theorems, along with new bounds for the Zarankiewicz problem (What is the maximum number of edges in a graph with no $K_{s,t}$?).

In the course of this work a few tools were developed, which help to cast systematically some classical results and their proofs into spectral form. The use of this machinery is best exhibited in [66], where we gave a new stability theorem and also its spectral analog - Theorems 2.19 and 3.10 below. As an illustration, in Section 5 we outline the proofs of these two results.

We believe that ultimately the spectral approach to extremal graph theory will turn out to be more fruitful than the conventional one, albeit it is also more difficult, and is still underdeveloped. Indeed, most statements in conventional terms can be cast and proved in spectral terms, but in addition to that, there are a lot of specific spectral results (say, Theorem 3.22) with no conceivable conventional setting.

The rest of the survey is organized as follows. To keep the beginning straightforward, the bulk of the necessary notation and the basic facts have been shifted to Section 6 although some definitions are given also where appropriate. Section 2 covers the conventional, nonspectral problems, while Section 3 presents the spectral results. In Section 4 we have collected some basic and more widely applicable statements, which we have found useful on more than one occasion. Finally Section 5 presents some proof techniques for illustration, and in fact these are the only proofs in this survey.

2 New results on classical extremal graph problems

In extremal graph theory one investigates how graph properties depend on the value of various graph parameters. In a sense almost all of graph theory deals with extremal problems, but there is a bundle of results grouped around Turán’s theorem [89], that undoubtedly constitutes the core of extremal graph theory. To state this celebrated theorem, which has stimulated researchers for more than six decades, recall that for $n \geq r \geq 2$, the Turán graph $T_r(n)$ is the complete $r$-partite graph of order $n$ whose class sizes differ by at most one. We let $t_r(n) = e(T_r(n))$.

**Theorem 2.1** If $G$ is a graph of order $n$, with no complete subgraph of order $r + 1$, then $e(G) \leq t_r(n)$ with equality holding only when $G = T_r(n)$.
Here is a more popular, but slightly weaker version, which we shall call the concise Turán theorem:

If \( G \) is a graph of order \( n \), with \( e(G) > (1 - 1/r) n^2 / 2 \), then \( G \) contains a complete subgraph of order \( r + 1 \).

No doubt, Turán’s theorem is a nice combinatorial statement and it is not too difficult to prove as well. However, its external simplicity is incomparable with its real importance, since this theorem is a cornerstone on which rest much more general statements about graphs. Thus, in this survey, we shall meet the Turán graph \( T_r(n) \) and the numbers \( t_r(n) \) on numerous occasions.

2.1 The extremal problems that are studied

Among the many questions motivated by Turán’s theorem, the ones that we will discuss in Section 2 fall into the following three broad classes:

1. Which subgraphs are present in a graph \( G \) of order \( n \) whenever \( e(G) > t_r(n) \) and \( n \) is sufficiently large?

As we shall see, here the range of \( e(G) - t_r(n) \) determines different problems: when \( e(G) - t_r(n) = o(n) \) we have saturation problems, and when \( e(G) - t_r(n) = o(n^2) \), we have Erdős-Stone type problems.

Other questions that we will be interested in give rise to the so called stability problems, concerning near-maximal graphs without forbidden subgraphs.

2. Suppose that \( H_n \) is a graph which is present in any graph \( G \) of order \( n \) whenever \( e(G) > t_r(n) \), but \( H_n \) is not a subgraph of the Turán graph \( T_r(n) \). We can ask the following questions:

- What can be the structure of an \( H_n \)-free graph \( G \) of order \( n \) if \( e(G) > t_r(n) - f(n) \), where \( f(n) \geq 0 \) and \( f(n) = o(n^2) \)?

- What can be the structure of an \( H_n \)-free graph \( G \) of order \( n \), with minimum degree \( \delta(G) > (1 - c) \delta(t_r(n)) \) for some sufficiently small \( c > 0 \)?

Obviously these two general questions have lots of variations, many of which are intensively studied due to their applicability in other extremal problems.

Finally, recall that a long series of results deals with the number of graphs having some monotone or hereditary properties. Here we will discuss a similar and natural question which, however, goes beyond this paradigm:

3. Let \( \{H_n\} \) be a sequence of graphs with \( v(H_n) = o(\log n) \). How many \( H_n \)-free graphs of order \( n \) are there?

2.2 Erdős-Stone type problems

We write \( K_r(s_1, \ldots, s_r) \) for the complete \( r \)-partite graph with class sizes \( s_1, \ldots, s_r \), and set for short

\[ K_r(p) = K_r(p, \ldots, p) \quad \text{and} \quad K_r(p; q) = K_r(p, \ldots, p, q). \]

Let us recall the fundamental theorem of Erdős and Stone [42].

Theorem 2.2 For all \( c > 0 \) and natural \( r, p \), there is an integer \( n_0(p, r, c) \) such that if \( G \) is a graph of order \( n > n_0(p, r, c) \) and \( e(G) \geq (1 - 1/r + c) n^2 / 2 \), then \( G \) contains a \( K_{r+1}(p) \).
Noting that \( t_r(n) \approx (1 - 1/r)n^2/2 \), we see the close relation of Theorem 2.2 to Turán’s theorem. In fact, Theorem 2.2 answers a fairly general question: what is the maximum number of edges \( e(n, H) \) in a graph of order \( n \) that does not contain a fixed \((r + 1)\)-chromatic subgraph \( H \)? Theorem 2.2 immediately implies that \( e(n, H) \leq (1 - 1/r + o(1))n^2/2 \). On the other hand, \( T_r(n) \) contains no \((r + 1)\)-chromatic subgraphs, and so, \( e(n, H) = (1 - 1/r + o(1))n^2/2 \).

Write \( g(n, r, c) \) for the maximal \( p \) such that every graph \( G \) of order \( n \) with \( e(G) \geq (1 - 1/r + c)n^2/2 \) contains a \( K_{r+1}(p) \). For almost 30 years the order of magnitude of \( g(n, r, c) \) remained unknown; it was established first by Bollobás and Erdős in [8], as given below. This simplest quantitative form of the Erdős-Stone theorem we call the Erdős-Stone-Bollobás theorem.

**Theorem 2.3** There are constants \( c_1, c_2 > 0 \) such that
\[
c_1 \log n \leq g(n, r, c) \leq c_2 \log n.
\]

Subsequently the function \( g(n, r, c) \) was determined with great precision in [9], [21], [10], [52], to name a few milestones. However, since Szemerédi’s Regularity Lemma is a standard tool in this research, the results are confined to fixed \( c \), and \( n \) extremely large.

To overcome these restrictions, in [63], we proposed a different approach, based on the expectation that the presence of many copies of a given subgraph \( H \) must imply the existence of large blow-ups of \( H \). As a by-product, this approach gave results in other directions as well, which otherwise do not seem too close to the Erdős-Stone theory; two such topics are outlined in 2.2.2 and 2.2.3.

### 2.2.1 Refining the Erdős-Stone-Bollobás theorem

The general idea above is substantiated for cliques in the following two theorems, given in [63].

**Theorem 2.4** Let \( r \geq 2 \), let \( c \) and \( n \) be such that
\[
0 < c < 1/r! \quad \text{and} \quad n \geq \exp(c^{-r}),
\]
and let \( G \) be a graph of order \( n \). If \( k_r(G) > cn^r \), then \( G \) contains a \( K_r(s; t) \) with \( s = \lfloor c^r \log n \rfloor \) and \( t > n^{1-c^{-r}} \).

In a nutshell, Theorem 2.4 says that if a graph contains many \( r \)-cliques, then it has large complete \( r \)-partite subgraphs. Hence, to obtain Theorem 2.3 all we need to prove is that the hypothesis of the Erdős-Stone-Bollobás theorem implies the existence of sufficiently many \( r \)-cliques. This implication is fairly standard, and so we obtain the following explicit version of the Erdős-Stone-Bollobás theorem.

**Theorem 2.5** Let \( r \geq 2 \), let \( c \) and \( n \) be such that
\[
0 < c < 1 \quad \text{and} \quad n \geq \exp\left(\left(\frac{r}{c}\right)^{r+1}\right),
\]
and let \( G \) be a graph of order \( n \). If \( e(G) \geq (1 - 1/r + c)n^2 \), then \( G \) contains a \( K_r(s; t) \) with
\[
s = \left\lfloor \left(\frac{c}{r^r}\right)^{r+1} \log n \right\rfloor \quad \text{and} \quad t > n^{1-(c/r^r)^r}.
\]
In the two theorems above, we would like to emphasize the three principles outlined in the introduction: first, the fundamental parameter \( c \) may depend on \( n \), e.g., letting \( c = \frac{1}{\log \log n} \), the conclusion is meaningful for sufficiently large \( n \); note that this fact can be verified precisely because the conditions for validity are stated explicitly. Also, the proof of these theorems relies on more basic statements of wider applicability - Lemma 4.1 and Lemma 4.2.

Another observation about this setup is the peculiarity of the graphs \( K_r(s; t) \) in the conclusions of the above theorems: if the statement holds for some \( c \), then it holds also for all positive \( c' < c \) as long as \( n \) is large enough. That is to say, when \( n \) increases, in addition to the graphs \( K_r(s; t) \) guaranteed by the theorems, we can find other, larger and more lopsided graphs \( K_r(s'; t') \) with \( s' < s \) and \( t' > t \). This same observation can be made on numerous other occasions below, and usually we shall omit it to avoid repetition.

Let us note that Theorem 2.3 implies also the following assertion, which strengthens the observation of Erdős and Simonovits [43]:

**Theorem 2.6** Let \( r \geq 3 \) and let \( F_1, F_2, \ldots \) be \((r + 1)\)-chromatic graphs satisfying \( v(F_n) = o(\log n) \). Then

\[
\max \{ e(G) : G \in \mathcal{G}(n) \text{ and } F_n \not\subseteq G \} = \frac{r-1}{2r} n^2 + o(n^2).
\]

Thus Theorem 2.6 solves asymptotically the Turán problem for families of forbidden subgraphs whose order grows not too fast with \( n \). Moreover, the condition \( v(F_n) = o(\log n) \) can be sharpened further using the bounds given by Ishigami in [52].

### 2.2.2 Graphs with many copies of a given subgraph

In this subsection we shall apply the basic idea above to arbitrary subgraphs of graphs, including induced ones.

Let us first define a blow-up of a graph \( H \): given a graph \( H \) of order \( r \) and positive integers \( k_1, \ldots, k_r \), we write \( H(k_1, \ldots, k_r) \) for the graph obtained by replacing each vertex \( u \in V(H) \) with a set \( V_u \) of size \( k_u \) and each edge \( uv \in E(H) \) with a complete bipartite graph with vertex classes \( V_u \) and \( V_v \).

We are interested in the following generalization of Theorem 2.4: Suppose that a graph \( G \) of order \( n \) contains \( cn^r \) copies of a given subgraph \( H \) on \( r \) vertices. How large a “blow-up” of \( H \) must \( G \) contain?

The following theorem from [64] is an analog of Theorem 2.4 for arbitrary subgraphs.

**Theorem 2.7** Let \( r \geq 2 \), let \( c \) and \( n \) be such that

\[
0 < c < \frac{1}{r!} \quad \text{and} \quad n \geq \exp \left( e^{2} \right),
\]

and let \( H \) be a graph of order \( r \). If \( G \in \mathcal{G}(n) \) and \( G \) contains more than \( cn^r \) copies of \( H \), then \( G \) contains an \( H(s, \ldots, s, t) \) with \( s = \left\lfloor c^{r} \log n \right\rfloor \) and \( t > n^{1-c^{-}} \).

A similar theorem is conceivable for induced subgraphs, but note the obvious bump: the complete graph \( K_n \) has \( \Theta(n^2) \) edges, i.e. \( K_2 \)'s, but contains no induced 4-cycle, i.e. \( K_2(2) \). To come up with a meaningful statement, we need the following more flexible version of a blow-up:
We say that a graph $F$ is of type $H(k_1, \ldots, k_r)$, if $F$ is obtained from $H(k_1, \ldots, k_r)$ by adding some (possibly zero) edges within the sets $V_u, u \in V(H)$.

This definition in hand, we can state the induced graph version of Theorem 2.7 also from [64].

**Theorem 2.8** Let $r \geq 2$, let $c$ and $n$ be such that

$$0 < c < 1/r! \quad \text{and} \quad n \geq \exp \left(\frac{c^2}{r}\right),$$

and let $H$ be a graph of order $r$. If $G \in \mathcal{G}(n)$ and $G$ contains more than $cn^r$ induced copies of $H$, then $G$ contains an induced subgraph of type $H(s, \ldots, s, t)$, where $s = \left\lfloor \frac{c^2 \log n}{r}\right\rfloor$ and $t > n^{1-c^{-1}}$.

For constant $c$, the above theorems give the correct order of magnitude of the subgraphs of type $H(s, \ldots, s, t)$, namely, $\log n$ for $s$ and $n^{1-o(c)}$ for $t$. When $c$ depends on $n$, the best bounds on $s$ and $t$ are apparently unknown.

### 2.2.3 Complete $r$-partite subgraphs of dense $r$-graphs

In this subsection, graph stands for $r$-uniform hypergraph for some fixed $r \geq 3$. We use again $K_r(s_1, \ldots, s_r)$ to denote the complete $r$-partite $r$-graph with class sizes $s_1, \ldots, s_r$.

In the spirit of the previous topics, it is natural to ask: Suppose that a graph $G$ of order $n$ contains $cn^r$ edges. How large a subgraph $K_r(s)$ must $G$ contain? As shown by Erdős and Stone [92] and Erdős [32], $s \geq a \left(\log n\right)^{1/(r-1)}$ for some $a = a(c) > 0$, independent of $n$.

In [65], this fundamental result was extended in three directions: $c$ may depend on $n$, the complete $r$-partite subgraph may have vertex classes of variable size, and the graph $G$ is taken to be an $r$-partite $r$-graph with equal classes. The last setup is obviously more general than just taking $r$-graphs.

The following three theorems are given in [65].

**Theorem 2.9** Let $r \geq 3$, let $c$ and $n$ be such that

$$0 < c \leq r^{-3} \quad \text{and} \quad n \geq \exp \left(\frac{1}{c^r-1}\right),$$

and let the positive integers $s_1, \ldots, s_{r-1}$ satisfy $s_1s_2\cdots s_{r-1} \leq c^{-1} \log n$. Then every graph with $n$ vertices and at least $cn^r/r!$ edges contains a $K_r(s_1, \ldots, s_{r-1}, t)$ with $t > n^{1-c^{-2}}$.

Instead of this theorem it is easier and more effective to prove a more general one for $r$-partite $r$-graphs.

**Theorem 2.10** Let $r \geq 3$, let $c$ and $n$ be such that

$$0 < c \leq r^{-3} \quad \text{and} \quad n \geq \exp \left(\frac{1}{c^r-1}\right),$$

and let the positive integers $s_1, \ldots, s_{r-1}$ satisfy $s_1s_2\cdots s_{r-1} \leq c^{-1} \log n$. Let $U_1, \ldots, U_r$ be sets of size $n$ and $E \subseteq U_1 \times \cdots \times U_r$ satisfy $|E| \geq cn^r$. Then there exist $V_1 \subseteq U_1, \ldots, V_r \subseteq U_r$ satisfying $V_1 \times \cdots \times V_r \subseteq E$ and

$$|V_1| = s_1, \ldots, |V_{r-1}| = s_{r-1}, \quad |V_r| > n^{1-c^{-2}}.$$
In turn, Theorem 2.10 is deduced from a counting result about \( r \)-partite \( r \)-graphs, which generalizes the double counting argument of Kövari, Sós and Turán for bipartite graphs \[57\].

**Theorem 2.11** Let \( r \geq 2 \) and let \( c \) and \( n \) be such that

\[
2^r \exp \left( -\frac{1}{r} (\log n)^{1/r} \right) \leq c \leq 1.
\]

Let \( G \) be an \( r \)-partite \( r \)-graph with parts \( U_1, \ldots, U_r \) of size \( n \), and with edge set \( E \subseteq U_1 \times \cdots \times U_r \) satisfying \( |E| \geq cn^r \). If the positive integers \( s_1, s_2, \ldots, s_r \) satisfy \( s_1 s_2 \cdots s_r \leq \log n \), then \( G \) contains at least

\[
\left( \frac{c}{2^r} \right)^{rs_1 \cdots s_r} \binom{n}{s_1} \cdots \binom{n}{s_r}.
\]

complete \( r \)-partite subgraphs with precisely \( s_i \) vertices in \( U_i \) for every \( i = 1, \ldots, r \).

Following Erdős \[32\] and taking a random \( r \)-graph \( G \) of order \( n \) and density \( 1 - \varepsilon \), a straightforward calculation shows that with probability tending to 1, \( G \) does not contain a \( K_r(\varepsilon, \ldots, \varepsilon) \) for \( s > A (\log n)^{1/(r-1)} \), where \( A = A(\varepsilon) \) is independent of \( n \). That is to say, Theorems 2.9 and 2.10 are essentially tight.

### 2.3 Saturation problems

Saturation problems concern the type of subgraphs one necessarily finds in graphs of order \( n \), with \( t_r(n) + o(n^2) \) edges. Among all possible saturation problems we will consider only the most important case: which subgraphs necessarily occur in graphs of order \( n \) and size \( t_r(n) + 1 \)? Turán’s theorem says that such graphs contain a \( K_{r+1} \), but one notes that they contain much larger supergraphs of \( K_{r+1} \).

Our first theorem completes an unfinished investigation started by Erdős in 1963, in \[31\]. We also present several results related to joints - a class of important subgraphs, whose study was also initiated by Erdős.

#### 2.3.1 Unavoidable subgraphs of graphs in \( G(n, t_r(n) + 1) \)

Let \( s_1 \geq 2 \), and write \( K^+_r(s_1, s_2, \ldots, s_r) \) for the graph obtained from \( K_r(s_1, s_2, \ldots, s_r) \) by adding an edge to the first part. For short, we also set

\[
K^+_r(p) = K^+_r(p, \ldots, p) \quad \text{and} \quad K^+_r(p; q) = K^+_r(p, \ldots, p, q).
\]

In \[31\] Erdős gave the following result:

**Theorem 2.12** For every \( \varepsilon > 0 \), there exist \( c = c(\varepsilon) > 0 \) and \( n_0(\varepsilon) \) such that if \( G \) is a graph of order \( n > n_0(\varepsilon) \) and \( e(G) > \lceil n^2/4 \rceil \), then \( G \) contains a

\[
K^+_2(\lfloor c \log n \rfloor, \lfloor n^{1-\varepsilon} \rfloor).
\]

For some time there was no generalization of this result for \( K^+_r(s; t) \) until Erdős and Simonovits \[41\] came up with a similar assertion valid for all \( r \geq 2 \).

**Theorem 2.13** Let \( r \geq 2 \), \( q \geq 1 \), and let \( n \) be sufficiently large. If \( G \) is a graph of order \( n \) with \( t_r(n) + 1 \) edges, then \( G \) contains a \( K^+_r(q) \).
In a sense Theorem 2.13 is best possible as any graph \( H \) that necessarily occurs in all sufficiently large graphs \( G \in G(n, t_r(n) + 1) \) can be imbedded in \( K^+_r(q) \) for \( q \) sufficiently large. To see this, just add an edge to the Turán graph \( T_r(n) \) and note that all \( (r + 1) \)-partite subgraphs of this graph are edge-critical with respect to the chromatic number. However, Theorem 2.12 suggests that stronger statements are possible, and indeed, in [67], we extended both Theorems 2.12 and 2.13 to the following one.

**Theorem 2.14** Let \( r \geq 2 \), let \( c \) and \( n \) be such that
\[
0 < c \leq r^{-\frac{r+7}{r+1}} \quad \text{and} \quad n \geq e^{2/c},
\]
and let \( G \) be a graph of order \( n \). If \( e(G) > t_r(n) \), then \( G \) contains a
\[
K^+_r \left( \left[ c \log n \right]; \left\lceil n^{1-\sqrt{c}} \right\rceil \right).
\]

As usual, in Theorem 2.14 \( c \) may depend on \( n \) within the given confine. Note also that if the conclusion holds for some \( c \), it holds also for positive \( c' < c \), provided \( n \) is sufficiently large. This implies Erdős’s Theorem 2.12.

### 2.3.2 Joints and books

Erdős [35] proved that if \( r \geq 2 \) and \( n > n_0(r) \), every graph \( G = G(n, t_r(n) + 1) \) has an edge that is contained in at least \( n^{r-1}/(10r)^{6r} \) cliques of order \( (r + 1) \). This fundamental fact seems so important, that in [14] we found it necessary to give the following definition:

An \( r \)-**joint** of size \( t \) is a collection of \( t \) distinct \( r \)-cliques sharing an edge.

Note that two \( r \)-cliques of an \( r \)-joint may share up to \( r - 1 \) vertices and that for \( r > 3 \) there may be many nonisomorphic \( r \)-joints of the same size. We shall write \( js_r(G) \) for the maximum size of an \( r \)-joint in a graph \( G \); in particular, if \( 2 \leq r \leq n \) and \( r \) divides \( n \), then \( js_r(T_r(n)) = \left( \frac{n}{r} \right)^{r-2} \).

In this notation, the above result of Erdős reads: if \( r \geq 2 \), \( n > n_0(r) \), and \( G \in G(n, t_r(n) + 1) \), then
\[
js_{r+1}(G) \geq \frac{n^{r-1}}{(10r)^{6r}}.
\]  

(1)

In fact, the study of \( js_3(G) \), also known as the booksize of \( G \), was initiated by Erdős even earlier, in [30], and was subsequently generalized in [34] and [35]; it seems that he foresaw the importance of joints when he restated his general result in 1995, in [36]. A quintessential result concerning joints is the “triangle removal lemma” of Ruzsa and Szemerédi [87], which can be stated as a lower bound on the booksize \( js_3(G) \) when \( G \) is a graph of a particular kind.

In fact joints help to obtain several of the results mentioned in this survey, e.g., the general stability Theorem 2.19 and its spectral version, Theorem 3.10. Later, we shall give also spectral conditions for the existence of large joints, in Theorem 3.8.

In [14], Bollobás and the author enhanced the bound of Erdős (1) to the following explicit one.

**Theorem 2.15** Let \( r \geq 2 \), \( n > r^8 \), and let \( G \) be a graph of order \( n \). If \( e(G) \geq t_r(n) \), then
\[
js_{r+1}(G) > \frac{n^{r-1}}{r^{r+5}}
\]

unless \( G = T_r(n) \).
In \cite{16}, an analogous theorem is given in the case when $G$ has many $r$-cliques, rather than edges. More precisely, letting $k_r(G)$ stand for the number of $r$-cliques of a graph $G$, we have

**Theorem 2.16** Let $r \geq 2$, $n > r^8$, and let $G$ be a graph of order $n$. If $k_s(G) \geq k_s(T_r(n))$ for some $s$, $(2 \leq s \leq r)$, then

$$ js_{r+1}(G) \geq \frac{n^{r-1}}{r^{2r+12}} $$

unless $G = T_r(n)$.

Note that Theorems \ref{thm:2.15} and \ref{thm:2.16} cannot be improved too much, as shown by the graph $G$ obtained by adding an edge to $T_r(n)$: we have $k_s(G) \geq k_s(T_r(n))$ but $js_{r+1}(G) \leq \lceil n/r \rceil^{r-1}$. However, the best bound in Theorem \ref{thm:2.15} is known only for 3-joints. Usually a 3-joint of size $t$ is called a book of size $t$. Edwards \cite{28}, and independently Khadziivanov and Nikiforov \cite{56} proved the following theorem.

**Theorem 2.17** If $G$ is a graph of order $n$ with $e(G) > \lfloor n^2/4 \rfloor$, then it contains a book of size greater than $n/6$.

This theorem is best possible in view of the following graph. Let $n = 6k$. Partition $[n]$ into 6 sets $A_{11}, A_{12}, A_{13}, A_{21}, A_{22}, A_{23}$ with $|A_{11}| = |A_{12}| = |A_{13}| = k - 1$ and $|A_{21}| = |A_{22}| = |A_{23}| = k + 1$. For $1 \leq j < k \leq 3$ join every vertex of $A_{ij}$ to every vertex of $A_{ik}$ and for $j = 1, 2, 3$ join every vertex of $A_{1j}$ to every vertex of $A_{2j}$. The resulting graph has size $> \lfloor n^2/4 \rfloor + 1$ and its booksize is $k + 1 = n/6 + 1$.

A more recent presentation of these results can be found in \cite{12}.

### 2.4 Stability problems

This subsection has three parts. First we sharpen the classical stability theorem of Erdős \cite{33,34} and Simonovits \cite{88}, which gives information about the structure of graphs without fixed forbidden subgraphs and whose size is close to the maximum possible. Second, we give several specific stability theorems for specific forbidden subgraphs, where stronger conclusions are possible. Lacking a better term, we call such cases **strong stability**.

Finally, we discuss the structure of $K_r$-free graphs of large minimum degree. This is a rich area with many results and a long history. It is not customary to consider it in the context of stability problems, but we believe this is the general category where this area belongs, since most of its statements can be phrased so that large minimum degree of a $K_r$-free graph implies a certain structure.

#### 2.4.1 A general stability theorem

Let $F$ be a fixed $(r + 1)$-partite graph $F$ and $G$ be a graph of order $n$. The theorem of Erdős and Stone implies that if $\varepsilon > 0$ and $e(G) > (1 - 1/r + \varepsilon) n^2/2$, then $G$ contains $F$, when $n$ is sufficiently large. On the other hand, $T_r(n)$ is $r$-partite and therefore does not contain $F$, although

$$ e(T_r(n)) = t_r(n) \approx (1 - 1/r) n^2/2. $$

Erdős and Simonovits \cite{33,44,88} noticed that if a graph $G$ of order $n$ contains no copy of $F$ and has close to $(1 - 1/r) n^2/2$ edges, then $G$ is similar to $T_r(n)$.
Theorem 2.18 Let \( r \geq 2 \) and let \( F \) be a fixed \((r + 1)\)-partite graph. For every \( \delta > 0 \), there is an \( \varepsilon > 0 \) such that if \( G \) is a graph of order \( n \) with \( e(G) > (1 - 1/r - \varepsilon)n^2/2 \), then either \( G \) contains \( F \) or \( G \) differs from \( T_r(n) \) in fewer than \( \delta n^2 \) edges.

A closer inspection of this statement reveals that \( \varepsilon \) depends both on \( \delta \) and on \( F \). To investigate this dependence, we simplify the picture by assuming that \( F \) is a complete \((r + 1)\)-graph. Moreover, radically departing from the setup of fixed \( F \), we assume that \( F = K_{r+1}\left(\lfloor c \log n \rfloor; \lceil n^{1-\sqrt{\varepsilon}} \rceil \right) \) for some \( c > 0 \). Note that for a given \( n \) the single real parameter \( c \) characterizes \( F \) completely. It turns out with this selection of \( F \) we still can get an enhancement of Theorem 2.18 as proved in [60].

Theorem 2.19 Let \( r \geq 2 \), let \( c, \varepsilon \) and \( n \) be such that

\[
0 < c < r^{-3(r+14)(r+1)}, \quad 0 < \varepsilon < r^{-24}, \quad n > e^{1/c},
\]

and let \( G \) be a graph of order \( n \). If \( e(G) > (1 - 1/r - \varepsilon)n^2/2 \), then one of the following statements holds:

(a) \( G \) contains a \( K_{r+1}\left(\lfloor c \log n \rfloor; \lceil n^{1-\sqrt{\varepsilon}} \rceil \right) \);

(b) \( G \) differs from \( T_r(n) \) in fewer than \( (e^{1/3} + c^{1/(3r+3)})n^2 \) edges.

Note that, as usual, \( c \) may depend on \( n \). A natural question is how tight Theorem 2.19 is.

The complete answer seems difficult since two parameters, \( \varepsilon \) and \( c \), are involved. First, the factor \( (e^{1/3} + c^{1/(3r+3)}) \) in condition (b) is far from the best one, but is simple. However for fixed \( c \) condition (a) is best possible up to a constant factor. Indeed, let \( \alpha > 0 \) be sufficiently small. A randomly chosen graph of order \( n \) with \((1 - \alpha)n^2/2\) edges contains no \( K_2(\lfloor c' \log n \rfloor; \lfloor c' \log n \rfloor) \) and differs from \( T_r(n) \) in more that \( c''n^2 \) edges for some positive \( c' \) and \( c'' \), independent of \( n \).

2.4.2 Strong stability

For certain forbidden graphs condition (ii) of Theorem 2.19 can be strengthened. Such particular stability theorems can be of interest in applications, e.g., Ramsey problems. We start with a theorem in [84], which gives a particular stability condition for \( K_{r+1} \)-free graphs.

Theorem 2.20 Let \( r \geq 2 \) and \( 0 < \varepsilon \leq 2^{-10r-6} \), and let \( G \) be a \( K_{r+1} \)-free graph of order \( n \). If \( e(G) > (1 - 1/r - \varepsilon)n^2/2 \), then \( G \) contains an induced \( r \)-partite graph \( H \) of order at least \((1 - 2\sqrt{\varepsilon})n\) and with minimum degree \( \delta(H) \geq (1 - 1/r - 4\sqrt{\varepsilon})n\).

Note that the stability condition in this theorem is stronger than condition (b) of Theorem 2.19. Indeed, the classes of \( H \) are almost equal, it is almost complete, and contains almost all vertices of \( G \). This type of conclusion is the purpose of the three theorems below. In the first two of them the premise “\( K_{r+1} \)-free” will be further weakened; but Theorem 2.20 is still of interest, because it is proved for all conceivable \( n \).

The following two theorems have been proved in [67] and [14].

Theorem 2.21 Let \( r \geq 2 \), let \( c, \varepsilon \) and \( n \) be such that

\[
0 < c < r^{-(r+7)(r+1)/2}, \quad 0 < \varepsilon < r^{-8}/8, \quad n > e^{2/c},
\]
and let $G$ be a graph of order $n$. If $e(G) > (1 - 1/r - \varepsilon)n^2/2$, then one of the following statements holds:

(a) $G$ contains a $K_r^+$ \( [\lceil c \log n \rceil ; \lceil n^{1-2\sqrt{2\varepsilon}} \rceil] \);

(b) $G$ contains an induced $r$-partite subgraph $H$ of order at least \((1 - \sqrt{2\varepsilon})n\), with minimum degree

$$\delta(H) > \left(1 - 1/r - 2\sqrt{2\varepsilon}\right)n.$$

**Theorem 2.22** Let $r \geq 2$, let $c$ and $n$ be such that

$$r \geq 2, \quad 0 < \varepsilon < r^{-8}/32, \quad n > r^8,$$

and let $G$ be a graph of order $n$. If $e(G) > (1 - 1/r - \varepsilon)n^2/2$, then one of the following statements holds:

(a) $j_{s_{r+1}}(G) > (1 - 1/r^3)n^{r-1}/r^{r+5}$;

(b) $G$ contains an induced $r$-partite subgraph $H$ of order at least \((1 - 4\sqrt{\varepsilon})n\), with minimum degree

$$\delta(H) > \left(1 - 1/r - 6\sqrt{\varepsilon}\right)n.$$ 

As one can expect, the analogous statement for books is quite close to the best possible \cite{12}.

**Theorem 2.23** Let $0 < \varepsilon < 10^{-5}$ and let $G$ be a graph of order $n$. If $e(G) > (1/4 - \varepsilon)n^2$, then either $G$ contains a book of size at least \((1/6 - 2\sqrt{\varepsilon})n\) or $G$ contains an induced bipartite graph $H$ of order at least \((1 - \sqrt{\varepsilon})n\) and with minimal degree $\delta(H) \geq (1/2 - 4\sqrt{\varepsilon})n$.

### 2.4.3 $K_r$-free graphs with large minimum degree

A famous theorem of Andrásfai, Erdős and Sós \cite{1} shows that if $r \geq 2$ and $G$ is a $K_{r+1}$-free graph of order $n$ and with minimum degree satisfying

$$\delta(G) > \left(1 - \frac{3}{3r - 1}\right)n,$$

then $G$ is $r$-partite. They also gave an example showing that equality in (2) is not sufficient to get the same conclusion.

In particular, for $r = 2$ this statement says that every triangle-free graph of order $n$ with minimum degree $\delta(G) > 2n/5$ is bipartite. On the other hand, Hajnal \cite{11} constructed a triangle-free graph of order $n$ with arbitrary large chromatic number and with minimum degree $\delta(G) > (1/3 - \varepsilon)n$. In view of Hajnal’s example, Erdős and Simonovits \cite{11} conjectured that all $K_3$-free graphs of order $n$ with $\delta(G) > n/3$ are 3-chromatic. However, this conjecture was disproved by Häggkvist \cite{40}, who described for every $k \geq 1$ a 10k-regular, 4-chromatic, triangle-free graph of order $29k$. The example of Häggkvist is based on the Mycielski graph $M_3$, also known as the Grötzsch graph, which is a 4-chromatic triangle-free graph of order 11. To construct $M_3$, let $v_1, \ldots, v_5$ be the vertices of a 5-cycle and choose 6 other vertices $u_1, \ldots, u_6$. Join $u_i$ to the neighbors of $v_i$ for all $i = 1, \ldots, 5$, and finally join $u_6$ to $u_1, \ldots, u_5$.

Other graphs that are crucial in these questions are the triangle-free, 3-chromatic Andrásfai graphs $A_1, A_2, \ldots,$ first described in \cite{3}: set $A_1 = K_2$ and for every $i \geq 2$ let $A_i$ be the complement of the $(i - 1)$-th power of the cycle $C_{3i-1}$.
To state the next structural theorems we need the following definition: a graph $G$ is said to be homomorphic to a graph $H$, if there exists a map $f : V(G) \rightarrow V(H)$ such that $uv \in E(G)$ implies that $f(u)f(v) \in E(H)$.

In [53], Jin generalized the case $r = 2$ of the theorem of Andrásfai, Erdős and Sós and a result of Hägkvist from [49] in the following theorem.

**Theorem 2.24** Let $1 \leq k \leq 9$, and let $G$ be a triangle-free graph of order $n$. If

$$\delta(G) > \frac{k+1}{3k+2} n,$$

then $G$ is homomorphic to $A_k$.

Note that this result is tight: taking the graph $A_{k+1}$, and blowing it up by a factor $t$, we obtain a triangle-free graph $G$ of order $n = (3k+2)t$ vertices, with $\delta(G) = (k+1)n/(3k+2)$, which is not homomorphic to $A_k$.

Note also that all graphs satisfying the premises of Theorem 2.24 are 3-chromatic. Addressing this last issue, Jin [54], and Chen, Jin and Koh [22] gave a finer characterization of all $K_3$-free graphs with $\delta > n/3$.

**Theorem 2.25** Let $G$ be a triangle-free graph of order $n$, with $\delta(G) > n/3$. If $\chi(G) \geq 4$, then $M_3 \subset G$. If $\chi(G) = 3$ and

$$\delta(G) > \frac{k+1}{3k+2} n,$$

then $G$ is homomorphic to $A_k$.

Finally, Brandt and Thomassé [20] gave the following ultimate result.

**Theorem 2.26** Let $G$ be a triangle-free graph of order $n$. If $\delta(G) > n/3$, then $\chi(G) \leq 4$.

It is natural to ask the same questions for $K_r$-free graphs with large minimum degree. Contrary to expectation, the answers are by far easier. First, the graphs of Andrásfai, Hajnal and Hägkvist are easily generalized by joining them with appropriately chosen complete $(r-3)$-partite graphs.

In particular, for every $\varepsilon$ there exists a $K_{r+1}$-free graph of order $n$ with

$$\delta(G) > \left(1 - \frac{2}{2r-1} - \varepsilon\right) n$$

and arbitrary large chromatic number, provided $n$ is sufficiently large.

Hence, the main question is: how large $\chi(G)$ can be when $G$ is a $K_{r+1}$-free graph of order $n$ with $\delta(G) > (1 - 2/(2r-1))n$. The answer is:

**Theorem 2.27** Let $r \geq 2$ and let $G$ be a $K_{r+1}$-free graph of order $n$. If

$$\delta(G) > \left(1 - \frac{2}{2r-1}\right) n,$$

then $\chi(G) \leq r + 2$. 

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This theorem leaves only two cases to investigate, viz., $\chi(G) = r + 1$ and $\chi(G) = r + 2$. As one can expect, when $\delta(G)$ is sufficiently large, we have $\chi(G) = r + 1$. The precise statement extends Theorem 2.24 as follows.

**Theorem 2.28** Let $r \geq 2$, $1 \leq k \leq 9$, and let $G$ be a $K_{r+1}$-free graph of order $n$. If

$$\delta(G) > \left(1 - \frac{2k - 1}{(2k - 1)r - k + 1}\right)n$$

then $G$ is homomorphic to $A_k + K_{r-2}$.

As a corollary, under the premises of Theorem 2.28, we find that $\chi(G) \leq r + 1$. Also Theorem 2.28 is best possible in the following sense: for every $k$ and $n$, there exists an $(r + 1)$-chromatic $K_{r+1}$-free $G$ of order $n$ with

$$\delta(G) \geq \left(1 - \frac{2k - 1}{(2k - 1)r - k + 1}\right)n - 1$$

that is not homomorphic to $A_k + K_{r-2}$.

Using the example of Häggkvist, we construct for every $n$ an $(r + 2)$-chromatic, $K_{r+1}$-free graph $G$ with

$$\delta(G) \geq \left(1 - \frac{19}{19r - 9}\right)n - 1,$$

which shows that the conclusion of Theorem 2.28 does not necessarily hold for $k \geq 10$.

To give some further structural information, we extend Theorem 2.26 as follows.

**Theorem 2.29** Let $r \geq 2$ and $G$ be a $K_{r+1}$-free graph of order $n$ with

$$\delta(G) > \left(1 - \frac{2}{2r - 1}\right)n.$$ 

If $\chi(G) \geq r + 2$, then $M_3 + K_{r-2} \subset G$. If $\chi(G) \leq r + 1$ and

$$\delta(G) > \left(1 - \frac{2k - 1}{(2k - 1)r - k + 1}\right)n$$

then $G$ is homomorphic to $A_k + K_{r-2}$.

This result is best possible in view of the examples described prior to Theorem 2.29.

We deduce the proofs of Theorems 2.27, 2.28, and 2.29 by induction on $r$ from Theorems 2.26, 2.24, and 2.25 respectively. The induction step, carried out uniformly in all the three proofs, is based on the crucial Lemma 4.5. This lemma can be applied immediately to extend other results about triangle-free graphs.

The new results in this subsection, together with Lemma 4.5, have been published in [68]. Since the first version of that paper was made public, the author learned that similar research has been undertaken independently by W. Goddard and J. Lyle [48].
2.5 The number of graphs with large forbidden subgraphs

An intriguing question is how many graphs with given properties there are. For certain natural properties such as “$G$ is $K_r$-free” or “$G$ has no induced graph isomorphic to $H$” satisfactory answers have been obtained. Thus, given a graph $H$, write $P_n(H)$ for the set of all labelled graphs of order $n$ not containing $H$. A classical result of Erdős, Kleitman and Rothschild states that

$$
\log_2 |P_n(K_{r+1})| = (1 - 1/r + o(1)) \left( \frac{n}{2} \right).
$$

(3)

Ten years later, Erdős, Frankl and Rödl showed that the conclusion in (3) remains valid if $K_{r+1}$ is replaced by an arbitrary fixed $(r+1)$-chromatic graph $H$.

In fact, as shown in [15], the conclusion in (3) also remains valid if $K_{r+1}$ is replaced by a sequence of forbidden graphs whose order grows with $n$. Until recently such results seemed to be out of reach; however, the framework laid out in [63] and [64] has opened new possibilities.

Here is the theorem that directly generalizes the Erdős-Frankl-Rödl result.

**Theorem 2.30** Given $r \geq 2$ and $0 < \varepsilon \leq 1/2$, there exists $\delta = \delta(\varepsilon) > 0$ such that for $n$ sufficiently large,

$$
(1 - 1/r) \left( \frac{n}{2} \right) \leq \log_2 |P_n(K_{r+1})| \left( \left\lfloor \delta \log n \right\rfloor ; \left\lceil n^{1 - \sqrt{\varepsilon}} \right\rceil \right) \leq \left( 1 - 1/r + \varepsilon \right) \left( \frac{n}{2} \right).
$$

(4)

Note that the real contribution of Theorem 2.30 is the upper bound in (4) since the lower bound follows by counting the labelled spanning subgraphs of the Turán graph $T_r(n)$. Let us mention that the proof of Theorem 2.30 does not use Szemerédi’s Regularity Lemma, which is a standard tool for such questions.

Similar statements can be proved for forbidden induced subgraphs, where the role of the chromatic number is played by the coloring number $\chi_c$ of a graph property, introduced first in [17], and defined below.

Let $0 \leq s \leq r$ be integers and let $\mathcal{H}(r,s)$ be the class of graphs whose vertex sets can be partitioned into $s$ cliques and $r - s$ independent sets. Given a graph property $\mathcal{P}$, the coloring number $\chi_c(\mathcal{P})$ is defined as

$$
\chi_c(H) = \max \{ r : \mathcal{H}(r,s) \subset \mathcal{P} \text{ for some } s \in [r] \}.
$$

Also, given a graph $H$, let us write $P_n^*(H)$ for the set of graphs of order $n$ not containing $H$ as an induced subgraph. Alexeev [2] and, independently Bollobás and Thomason [17], [18] proved that the exact analog of the result of Erdős, Frankl and Rödl holds:

If $H$ is a fixed graph and $r = \chi_c(P_n^*(H))$, then

$$
\log_2 |P_n^*(H)| = (1 - 1/r + o(1)) \left( \frac{n}{2} \right).
$$

(5)

This result also can be extended by replacing $H$ with a sequence of forbidden graphs whose order grows with $n$. To this end, recall the definition of a graph of type $H(k_1, \ldots, k_h)$, where $H$ is a fixed labelled graph of order $h$ and $k_1, \ldots, k_h$ are positive integers (Subsection 2.2.2). Informally, a graph of type $H(k_1, \ldots, k_h)$ is obtained by first “blowing-up” $H$ to $H(k_1, \ldots, k_h)$.
and then adding (possibly zero) edges to the vertex classes of the “blow-up” but keeping intact
the edges across vertex classes.

Now, given a labelled graph $H$ and positive integers $p$ and $q$, let $\mathcal{P}_n(H; p, q)$ be the set of
labelled graphs of order $n$ that contain no induced subgraph of type $H (p, \ldots, p, q)$.

Here is the result for forbidden induced subgraphs, also from [15].

**Theorem 2.31** Let $H$ be a labelled graph and let $r = \chi_c(\mathcal{P}_n^*(H))$. For every $\varepsilon > 0$, there is
$\delta = \delta(\varepsilon) > 0$ such that for $n$ sufficiently large

$$
(1 - 1/r) \left(\frac{n}{2}\right) \leq \log_2 \left| \mathcal{P}_n(H; [\delta \log n], \left\lfloor n^{1-\sqrt{r}} \right\rfloor) \right| \leq (1 - 1/r + \varepsilon) \left(\frac{n}{2}\right).
$$

In a sense Theorems 2.30 and 2.31 are almost best possible, in view of the following simple
observation, which can be proved by considering the random graph $G_{n,p}$ with $p \rightarrow 1$.

Given $r \geq 2$ and $\varepsilon > 0$, there exists $C > 0$ such that the number $S_n$ of labelled graphs which
do not contain $K_2([C \log n], [C \log n])$ satisfies $S_n \geq (1 - \varepsilon) 2^{(\frac{n}{2})}$.

## 3 Spectral extremal graph theory

Generally speaking, spectral graph theory investigates graphs using the spectra of various ma-
trices associated with graphs, such as the adjacency matrix. For an introduction to this topic
we refer the reader to [23].

Given a graph $G$ with vertex set $\{v_1, \ldots, v_n\}$, the adjacency matrix of $G$ is a matrix $A = [a_{ij}]$
of size $n$ given by

$$
a_{ij} = \begin{cases} 
1, & \text{if } (v_i, v_j) \in E(G); \\
0, & \text{otherwise}.
\end{cases}
$$

Note that $A$ is symmetric and nonnegative, and much is known about the spectra of such matrices. For instance, the eigenvalues of $A$ are real numbers, which we shall denote by $
\mu_1(G), \ldots, \mu_n(G)$, indexed in non-increasing order. The value $\mu(G) = \mu_1(G)$ is called the
spectral radius of $G$ and has maximum absolute value among all eigenvalues.

Another matrix that we shall use is the Laplacian matrix $L$, defined as $D(G) - A(G)$, where $D(G)$ is the diagonal matrix of the row-sums of $A$, i.e., the degrees of $G$. The eigenvalues of the
Laplacian are denoted by $\lambda_1(G), \ldots, \lambda_n(G)$, indexed in non-decreasing order.

A third matrix associated with graphs is the $Q$-matrix or the “signless Laplacian”, defined as $Q = D + A$. The eigenvalues of $Q$ are denoted by $q_1(G), \ldots, q_n(G)$, indexed in non-increasing order. The $Q$-matrix has received a lot of attention in recent years, see, e.g., [24], [25] and [26].
The Laplacian and the $Q$-matrix are positive semi-definite matrices, and $\lambda_1(G) = 0$.

### 3.1 The spectral problems that are studied

How large can be the spectral radius $\mu(G)$ when $G$ is a $K_r$-free graph of order $n$? Such questions
come easily to the mind when one studies extremal graph problems. In fact, with any extremal
problem of the type “What is the maximum number of edges in a graph $G$ of order $n$ with property $\mathcal{P}$?”
goes a spectral analog: “What is the maximum spectral radius of a graph $G$ of
order $n$ with property $P$? This is not merely a superficial analogy since if we have a solution of the spectral problem, then by the fundamental inequality

$$
\mu(G) \geq 2e(G)/n,
$$

we immediately obtain an upper bound on $e(G)$ as well. The use of this implication is illustrated on several occasions below; in particular, for the Zarankiewicz problem we obtain the sharpest bounds on $e(G)$ known so far.

On the other hand, inequality (7) suggests a way to conjecture spectral results by taking known nonspectral extremal statements that involve the average degree of a graph and replacing the average degree by $\mu(G)$. More often than not, the resulting statement is correct and even stronger, but of course it needs its own proof. To create suitable proof tools we painstakingly built several technical but rather flexible statements such as Theorem 4.8 and Lemma 4.6.

This smooth machinery is sufficient to prove spectral analogs of most of the extremal problems discussed in Section 2 and of several others as well. Among these results are: various forms of Turán’s theorem, the Erdős-Stone-Bollobás theorem, conditions for large joints and for odd cycles; a general stability theorem and several strong stability theorems, an asymptotic solution of the general extremal problem for nonbipartite forbidden subgraphs, the Zarankiewicz problem, sufficient conditions for paths and cycles, sufficient conditions for Hamilton paths and cycles.

Despite these successful translations, more can be expected from spectral extremal graph theory, which seems inherently richer than the conventional one. Indeed, we give also a fair number of spectral results that have no conventional analog, for example, results involving the smallest eigenvalue of the adjacency matrix or the spectral radius of the Laplacian matrix.

### 3.2 Spectral forms of the Turán theorem

In 1986, Wilf [90] showed that if $G$ is a graph of order $n$ with clique number $\omega(G) = \omega$, then

$$
\mu(G) \leq (1 - 1/\omega)n.
$$

Note first that in view of the inequality $\mu(G) \geq 2e(G)/n$, (8) implies the concise Turán theorem:

$$
e(G) \leq (1 - 1/\omega)n^2/2.
$$

However, inequality (8) opens many other new possibilities. Indeed, if we combine (8) with other lower bounds on $\mu(G)$, e.g., with

$$
\mu^2(G) \geq \frac{1}{n} \sum_{u \in V(G)} d^2(u),
$$

we obtain other forms of (9). An infinite class of similar lower bounds is given in [70].

Below we sharpen inequality (8) in two ways.

**A concise spectral Turán theorem**

In 1970 Nosal [85] showed that every triangle-free graph $G$ satisfies $\mu(G) \leq \sqrt{e(G)}$. This result was extended in [69] and [75] in the following theorem, conjectured by Edwards and Elphick in [29]:

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Theorem 3.1 If $G$ is a graph of order $n$ and $\omega(G) = \omega$, then

$$\mu^2(G) \leq 2 \left(1 - 1/\omega\right) e(G).$$  \hfill (10)

If $G$ has no isolated vertices, then equality is possible if and only if one of the following conditions holds:

(a) $\omega = 2$ and $G$ is a complete bipartite graph;
(b) $\omega \geq 3$ and $G$ is a complete regular $\omega$-partite graph.

In view of (9), we see that

$$\mu^2(G) \leq 2 \left(1 - 1/\omega\right) m \leq 2 \left(1 - 1/\omega\right) (1 - 1/\omega) n^2/2 = ((1 - 1/\omega) n)^2,$$

and so (10) implies (8).

As shown in [69], inequality (10) follows from the celebrated result of Motzkin and Straus [61]:

Let $G$ be a graph of order $n$ with cliques number $\omega(G) = \omega$. If $(x_1, \ldots, x_n)$ is a vector with nonnegative entries, then

$$\sum_{uv \in E(G)} x_u x_v \leq \frac{\omega - 1}{2\omega} \left( \sum_{u \in V(G)} x_u \right)^2.$$  \hfill (11)

On the other hand, this result follows in turn from the concise Turán theorem, as shown in [71]. The implications

(10) $\implies$ (9) $\implies$ MS $\implies$ (10)

justify regarding inequality (10) as a flexible spectral form of the concise Turán theorem.

Next we extend Theorem 3.1 in a somewhat unexpected direction. Recall that, a $k$-walk in a graph $G$ is a sequence of vertices $v_1, \ldots, v_k$ of $G$ such that $v_i$ is adjacent to $v_{i+1}$ for $i = 1, \ldots, k-1$; write $w_k(G)$ for the number of $k$-walks in $G$. Observing that $2e(G) = w_2(G)$, we see that the following theorem, given in [70], extends inequality (10).

Theorem 3.2 If $r \geq 1$ and $G$ is a graph with clique number $\omega(G) = \omega$, then

$$\mu^r(G) \leq (1 - 1/\omega) w_r(G).$$  \hfill (12)

Suppose that $G$ has no isolated vertices and equality holds for some $r \geq 1$.

(i) If $r = 1$, then $G$ is a regular complete $\omega$-partite graph.
(ii) If $r \geq 2$ and $\omega > 2$, then $G$ is a regular complete $\omega$-partite graph.
(iii) If $r \geq 2$ and $\omega = 2$, then $G$ is a complete bipartite graph, and if $r$ is odd, then $G$ is regular.

It is somewhat surprising that for $r \geq 2$ the number of vertices of $G$ is not relevant in this theorem.
A precise spectral Turán theorem
Since Wilf’s inequality (8) becomes equality only when \( \omega \) divides \( n \), one can expect that some fine tuning is still possible. Indeed, in [72] we sharpened inequality (8), bringing it the closest possible to the Turán theorem.

**Theorem 3.3** If \( G \) is a graph of order \( n \) with no complete subgraph of order \( r + 1 \), then \( \mu(G) \leq \mu(T_r(n)) \). Equality holds if and only if \( G = T_r(n) \).

Here is an equivalent, shorter form of this statement: If \( G \in \mathcal{G}(n) \) and \( \omega(G) = \omega \), then \( \mu(G) < \mu(T_\omega(n)) \) unless \( G = T_\omega(n) \).

Note also that \( \mu(T_2(n)) = \sqrt{\lfloor n^2/4 \rfloor} \); for \( \omega \geq 3 \) there is also a closed expression for \( \mu(T_\omega(n)) \), but it is somewhat cumbersome.

**Spectral radius and independence number**
One wonders if there is a theorem about the independence number \( \alpha(G) \), similar to the Turán theorem. One obvious answer is obtained by restating the concise Turán theorem in complementary terms

\[
2e(G) \geq n^2/\alpha(G) - n,
\]

which immediately implies that \( \mu(G) \geq n/\alpha(G) - 1 \) as well. Note that here the spectral statement follows from the conventional one. However, a proof by induction on \( \alpha \) gives the following sharper result.

**Theorem 3.4** If \( G \in \mathcal{G}(n) \) and \( \alpha(G) = \alpha \), then \( \mu(G) \geq \lfloor n/\alpha \rfloor - 1 \).

In a different direction, for connected graphs and some special values of \( \alpha \), more specific results have been proved in [91].

Also, by the well-known inequality \( q_1(G) \geq 2\mu_1(G) \), Theorem 3.4 proves Conjecture 27 from [50].

### 3.3 A spectral Erdős-Stone-Bollobás theorem
Having seen various spectral forms of the Turán theorem, one can expect that many other results that surround it can be cast in spectral form as well; and this is indeed the case. The following theorem, given in [75], is the spectral analog of the Erdős-Stone-Bollobás theorem, more precisely, of Theorem 2.5.

**Theorem 3.5** Let \( r \geq 3 \), let \( c \) and \( n \) be such that

\[
0 < c < 1/r!, \quad n \geq \exp((r^r/c)^r),
\]

and let \( G \) be a graph of order \( n \). If

\[
\mu(G) \geq (1 - 1/(r - 1) + c)n,
\]

then \( G \) contains a \( K_r(s;t) \) with \( s \geq \lfloor (c/r^r)^r \log n \rfloor \) and \( t > n^{1-c^{r-1}} \).
Let us emphasize that the functionality of Theorem 2.5 is entirely preserved: in particular, $c$ may depend on $n$, e.g., letting $c = 1/\log \log n$, the conclusion is meaningful for sufficiently large $n$.

Since $\mu(G) \geq 2e(G)/v(G)$, Theorem 3.5 in fact, implies Theorem 2.5. Other lower bounds on $\mu(G)$, such as those given in [70], imply other new versions of this theorem.

Suppose that $c$ is a sufficiently small constant. Choosing randomly a graph $G$ of order $n$ with $\left\lfloor (1 - 1/(r - 1) + 2c) n^2/2 \right\rfloor$ edges, we have $\mu(G) \geq (1 - 1/(r - 1) + c)n$, but $G$ contains no $K_2([C\log n],[C\log n])$ for some $C > 0$, independent of $n$. Hence, for constant $c$, Theorem 3.5 is best possible up to a constant factor.

We close this topic with a consequence of Theorem 3.5, given in [78], that solves asymptotically the following general extremal problem: Given a family $\mathcal{F}$ of nonbipartite forbidden subgraphs, what is the maximum spectral radius of a graph of order $n$ containing no member of $\mathcal{F}$.

**Theorem 3.6** Let $r \geq 3$ and let $F_1, F_2, \ldots$ be $r$-partite graphs satisfying $v(F_n) = o(\log n)$. Then

$$\max \{\mu(G) : G \in \mathcal{G}(n) \text{ and } F_n \not\subseteq G\} = \left(1 - \frac{1}{r - 1}\right)n + o(n).$$

It is likely that in the setup of Theorem 3.6 the condition $v(F_n) = o(\log n)$ can be sharpened.

### 3.4 Saturation problems

The precise spectral Turán theorem implies that if $G$ is a graph of order $n$ with $\mu(G) > \mu(T_r(n))$, then $G$ contains a $K_{r+1}$. Since this setting is analogous to the case when $e(G) > t_r(n)$, one would expect much larger supergraphs of $K_{r+1}$. In fact, as we shall see, all results from Subsection 2.3 have their spectral analogs. In addition, it is not difficult to show that if $G$ is a graph of order $n$, then the inequality $e(G) > e(T_r(n))$ implies the inequality $\mu(G) > \mu(T_r(n))$. Therefore, the spectral theorems below imply the corresponding nonspectral extremal results, albeit with somewhat narrower ranges of the parameters.

We start with the spectral analog of Theorem 2.14 given in [76].

**Theorem 3.7** Let $r \geq 2$, let $c$ and $n$ be such that

$$0 < c \leq r^{-(2r+9)(r+1)} \quad n \geq \exp(2/c),$$

and let $G$ be a graph of order $n$. If $\mu(G) > \mu(T_r(n))$, then $G$ contains a $K_r^+\left(\lceil c\log n \rceil, \left\lceil n^{1-\sqrt{\varepsilon}} \right\rceil\right)$.

Theorem 3.7 is essentially best possible since for every $\varepsilon > 0$, choosing randomly a graph $G$ of order $n$ with $e(G) = \left\lfloor (1 - \varepsilon)n^2/2 \right\rfloor$, we see that $\mu(G) > (1 - \varepsilon)n$, but $G$ contains no $K_2([C\log n])$ for some $C > 0$, independent of $n$.

The theorem corresponding to Theorem 2.15 is given in [76]. We state it here in a somewhat refined form.

**Theorem 3.8** Let $r \geq 2$, $n > r^{15}$, and let $G$ be a graph of order $n$. If $\mu(G) \geq \mu(T_r(n))$, then $j s_{r+1}(G) > n^{r-1}/r^{2r+4}$ unless $G = T_r(n)$.
Theorem 3.8 and its stability complement Theorem 3.12 are crucial in the proof of several other spectral extremal results.

It is easy to see the Turán graph $T_2(n)$ contains no odd cycles and that $\mu(T_2(n)) = \sqrt{\lfloor n^2/4 \rfloor}$. Hence the following theorem gives a sharp spectral condition for the existence of odd cycles.

**Theorem 3.9** Let $G$ be a graph of sufficiently large order $n$. If $\mu(G) > \sqrt{\lfloor n^2/4 \rfloor}$, then $G$ contains a cycle of length $t$ for every $t \leq n/320$.

This theorem, given in [79], is motivated by the following result of Bollobás ([6], p. 150): if $G$ is a graph of order $n$ with $e(G) > \lfloor n^2/4 \rfloor$, then $G$ contains a cycle of length $t$ for every $t = 3, \ldots, \lceil n/2 \rceil$.

### 3.5 Stability problems

We shall show that most stability results from Subsection 2.4 have their spectral analogs. However, we could not find spectral analogs of the stability problems that involve minimum degree (Subsection 2.4.3).

We first state a general spectral stability result, and then two stronger versions for specific graphs. We give here only Theorems 3.11 and 3.12 since they are important for various applications, but our machinery helped to deduce many others of somewhat lesser importance, and they can be found in [76] and [79].

The following analog of Theorem 2.19 was given in [76].

**Theorem 3.10** Let $r \geq 2$, let $c$, $\varepsilon$ and $n$ be such that

$$0 < c < r^{-8(r+21)(r+1)} \quad 0 < \varepsilon < 2^{-36}r^{-24}, \quad n > \exp(1/c),$$

and let $G$ be a graph of order $n$. If $\mu(G) \geq (1 - 1/r - \varepsilon)n$, then one of the following statements holds:

(a) $G$ contains a $K_{r+1} \left(\lceil c \log \rho \rceil; \lfloor n^{1-\sqrt{\varepsilon}} \rfloor \right)$;

(b) $G$ differs from $T_r(n)$ in fewer than $(\varepsilon^{1/4} + c^{1/(8r+8)})n^2$ edges.

The proofs of Theorem 2.19 and 3.10 given in [76] illustrate the isomorphism between the sets of tools developed for the spectral and nonspectral problems. The texts of the two proofs are almost identical, while the differences come from the use of different tools. We refer the reader to Section 5 for more details.

The next two theorems are crucial for several applications. The first one, proved in [13], is a spectral equivalent of Theorem 2.20.

**Theorem 3.11** Let $r \geq 2$ and $0 \leq \varepsilon \leq 2^{-10r^{-6}}$, and let $G$ be a $K_{r+1}$-free graph of order $n$. If

$$\mu(G) \geq (1 - 1/r - \varepsilon)n, \quad (15)$$

then $G$ contains an induced $r$-partite graph $H$ of order at least $(1 - 3\alpha^{1/3})n$ and minimum degree

$$\delta(H) > \left(1 - 1/r - 6\varepsilon^{1/3}\right)n.$$

Finally, we have a spectral stability theorem for large joints, proved in [76].
Theorem 3.12 Let \( r \geq 2 \), let \( \varepsilon \) and \( n \) be such that
\[
0 < \varepsilon < 2^{-10}r^{-6}, \quad n \geq r^{20},
\]
and let \( G \) be a graph of order \( n \). If \( \mu(G) > (1 - 1/r - \varepsilon) n \), then \( G \) satisfies one of the conditions:
(a) \( j_{s+1}^r(G) > n^{r-1}/r^{2r+5} \);
(b) \( G \) contains an induced \( r \)-partite subgraph \( H \) of order at least \((1 - 4\varepsilon^{1/3}) n \) with minimum degree \( \delta(H) > (1 - 1/r - 7\varepsilon^{1/3}) n \).

3.6 The Zarankiewicz problem

What is the maximum spectral radius of a graph of order \( n \) with no \( K_{s,t} \)? This is a spectral version of the famous Zarankiewicz problem: what is the maximum number of edges in a graph of order \( n \) with no \( K_{s,t} \)? Except for few cases, no complete solution to either of these problems is known. For instance, Babai and Guiduli [5] have shown that
\[
\mu \leq \left(\frac{s}{t} - 1\right) + o(1) n^{1-1/t}.
\]
Using a different method, in [82] we improved this result as follows:

Theorem 3.13 Let \( s \geq t \geq 2 \), and let \( G \) be a \( K_{s,t} \)-free graph of order \( n \).
(i) If \( t = 2 \), then
\[
\mu(G) \leq 1/2 + \sqrt{(s - 1)(n - 1) + 1}/4.
\]
(ii) If \( t \geq 3 \), then
\[
\mu(G) \leq (s - t + 1)^{1/t} n^{1-1/t} + (t - 1) n^{1-2/t} + t - 2.
\]
On the other hand, in view of the inequality \( 2e(G) \leq \mu(G) n \), we see that if \( G \) is a \( K_{s,t} \)-free graph of order \( n \), then
\[
e(G) \leq \frac{1}{2} (s - t + 1)^{1/t} n^{2-1/t} + \frac{1}{2} (t - 1) n^{2-2/t} + \frac{1}{2} (t - 2) n.
\]
This is a slight improvement of a result of Füredi [16] and this seems the best known bound on \( e(G) \) so far.

For some values of \( s \) and \( t \) the bounds given by (16) and (17) are tight as we now demonstrate.

The case \( t = 2 \)

For \( s = t = 2 \) inequality (16) shows that every \( K_{2,2} \)-free graph \( G \) of order \( n \) satisfies
\[
\mu(G) \leq 1/2 + \sqrt{n - 3}/4.
\]
This bound is tight because equality holds for the friendship graph (a collection of triangles sharing a single common vertex).

Also, Erdős-Rényi [10] showed that if \( q \) is a prime power, the polarity graph \( ER_q \) is a \( K_{2,2} \)-free graph of order \( n = q^2 + q + 1 \) and \( q(q+1)^2/2 \) edges. Thus, its spectral radius \( \mu(ER_q) \) satisfies
\[
\mu(ER_q) \geq \frac{q^3 + 2q^2 + q}{q^2 + q + 1} > q + 1 - \frac{1}{q} = 1/2 + \sqrt{n - 3/4} - \frac{1}{\sqrt{n - 1}}.
\]
which is also close to the upper bound.

For \( s > 2 \), equality in (16) is attained when \( G \) is a strongly regular graph in which every two vertices have exactly \( s - 1 \) common neighbors. There are examples of strongly regular graphs of this type; here is a small selection from Gordon Royle’s webpage:

| \( s \) | \( n \) | \( \mu(G) \) |
|---|---|---|
| 3  | 45  | 12  |
| 4  | 96  | 20  |
| 5  | 175 | 30  |
| 6  | 36  | 15  |

We are not aware whether there are infinitely many strongly regular graphs in which every two vertices have the same number of common neighbors. However, Füredi [47] has shown that for any \( n \) there exists a \( K_{s,t} \)-free graph \( G_n \) of order \( n \) such that

\[
e(G_n) \geq \frac{1}{2} n^{(s-1)/t} + O(n^{1/t})
\]

and so,

\[
\mu(G_n) \geq \sqrt{sn} + O(n^{1/3})
\]

thus (16) is tight up to low order terms.

**The case** \( s = t = 3 \)

The bound (17) implies that if \( G \) is a \( K_{3,3} \)-free graph of order \( n \), then

\[
\mu(G) \leq n^{2/3} + 2n^{1/3} + 1.
\]

On the other hand, a construction due to Alon, Rönyai and Szabó [4] implies that for all \( n = q^3 - q^2 \), where \( q \) is a prime power, there exists a \( K_{3,3} \)-free graph \( G_n \) of order \( n \) with

\[
\mu(G_n) \geq n^{2/3} + \frac{2}{3} n^{1/3} + C
\]

for some constant \( C > 0 \). Thus, the bound (17) is asymptotically tight for \( s = t = 3 \). The same conclusion can be obtained from Brown’s construction of \( K_{3,3} \)-free graphs [19].

**The general case**

As proved in [4], there exists \( c > 0 \) such that for all \( t \geq 2 \) and \( s \geq (t - 1)! + 1 \), there is a \( K_{s,t} \)-free graph \( G_n \) of order \( n \) with

\[
e(G_n) \geq \frac{1}{2} n^{2-1/t} + O(n^{2-1/t-c}).
\]

Hence, for such \( s \) and \( t \) we have

\[
\mu(G) \geq n^{1-1/t} + O(n^{1-1/t-c});
\]

thus, the bound (17) and also the bound of Babai and Guiduli give the correct order of the main term.
3.7 Paths and cycles

We give now some results about the maximum spectral radius of graphs of order \( n \) without paths or cycles of specified length. Writing \( C_k \) and \( P_k \) for the cycle and path of order \( k \), let us define the functions

\[
\begin{align*}
  f_l(n) &= \max \{ \mu(G) : G \in \mathcal{G}(n) \text{ and } C_l \not\subseteq G \}; \\
  g_l(n) &= \max \{ \mu(G) : G \in \mathcal{G}(n) \text{ and } C_l \not\subseteq G, \text{ and } C_{l+1} \not\subseteq G \}; \\
  h_l(n) &= \max \{ \mu(G) : G \in \mathcal{G}(n) \text{ and } P_l \not\subseteq G \}.
\end{align*}
\]

For these functions we shall show below some exact expressions or at least good asymptotics. It should be noted that except for \( f_1(n) \) when \( l \) is odd, these questions are quite different from their nonspectral analogs.

The lower bounds on \( f_{2l}(n), g_l(n) \) and \( h_l(n) \) are given by two families of graphs, which for sufficiently large \( n \) give the exact values of \( h_l(n) \), and perhaps also of \( f_{2l}(n) \) and \( g_l(n) \).

Suppose that \( 1 \leq k < n \).

(1) Let \( S_{n,k} \) be the graph obtained by joining every vertex of a complete graph of order \( k \) to every vertex of an independent set of order \( n - k \), that is, \( S_{n,k} = K_k \cup \overline{K_{n-k}} \).

(2) Let \( S_{n,k}^+ \) be the graph obtained by adding one edge within the independent set of \( S_{n,k} \).

Note that \( P_{l+1} \not\subseteq S_{n,k} \) and \( C_l \not\subseteq S_{n,k} \) for \( l \geq 2k + 1 \). Likewise, \( P_{l+1} \not\subseteq S_{n,k} \) and \( C_l \not\subseteq S_{n,k} \) for \( l \geq 2k + 2 \).

Therefore,

\[
\begin{align*}
  h_{2k}(n) &\geq \mu(S_{n,k}) = (k - 1) / 2 + \sqrt{kn - (3k^2 + 2k - 1) / 4}, \\
  h_{2k+1}(n) &\geq \mu(S_{n,k}^+) = (k - 1) / 2 + \sqrt{kn - (3k^2 + 2k - 1) / 4} + 1/n + O\left(n^{-3/2}\right), \\
  g_{2k}(n) &\geq \mu(S_{n,k}) = (k - 1) / 2 + \sqrt{kn - (3k^2 + 2k - 1) / 4}, \\
  g_{2k+1}(n) &\geq \mu(S_{n,k}^+) = (k - 1) / 2 + \sqrt{kn - (3k^2 + 2k - 1) / 4} + 1/n + O\left(n^{-3/2}\right), \\
  f_{2k+2}(n) &\geq \mu(S_{n,k}^+) = (k - 1) / 2 + \sqrt{kn - (3k^2 + 2k - 1) / 4} + 1/n + O\left(n^{-3/2}\right).
\end{align*}
\]

Below we shall give also rather close upper bounds for these functions.

Forbidden odd cycle

In view of Theorem 3.9 we find that if \( l \) is odd and \( n > 320l \), then

\[
f_l(n) = \sqrt{n^2 / 4}.
\]

The smallest ratio \( n/l \) for which this equation is still valid is not known.

Clearly, for odd \( l \) we have \( f_l(n) \sim n/2 \), which is in sharp contrast to the value of \( f_l(n) \) for even \( l \).

Forbidden cycle \( C_4 \)

The value of \( f_4(n) \) is essentially determined in [72]:

24
Let \( G \) be a graph of order \( n \) with \( \mu(G) = \mu \). If \( C_4 \not\subseteq G \), then
\[
\mu^2 - \mu \leq n - 1. \tag{19}
\]
Equality holds if and only if every two vertices of \( G \) have exactly one common neighbor, i.e., when \( G \) is the friendship graph.

An easy calculation implies that
\[
q_4(n) = 1/2 + \sqrt{n - 3/4 + O(1/n)},
\]
where for odd \( n \) the \( O(1/n) \) term is zero. Finding the precise value of \( q_4(n) \) for even \( n \) is an open problem.

Here is a considerably more involved bound on the spectral radius of a \( C_4 \)-free graph of given size, given in \cite{77}.

**Theorem 3.14** Let \( m \geq 9 \) and \( G \) be a graph with \( m \) edges. If \( \mu(G) > \sqrt{m} \), then \( G \) has a 4-cycle.

This theorem is tight, for all stars are \( C_4 \)-free graphs with \( \mu(G) = \sqrt{m} \). Also, let \( S_{n,1} \) be the star of order \( n \) with an edge within its independent set. The graph \( S_{n,1} \) is \( C_4 \)-free and has \( n \) edges, but \( \mu(G) > \sqrt{n} \) for \( 4 \leq n \leq 8 \), while \( \mu(S_{9,1}) = 3 \).

**Forbidden cycle \( C_{2k} \)**

The inequality \cite{19} can be generalized for arbitrary even cycles as follows: if \( C_{2k+2} \not\subseteq G \), then
\[
\mu^2 - (k - 1) \mu \leq k (n - 1).
\]

In fact, a slightly stronger assertion was proved in \cite{81}.

**Theorem 3.15** Let \( k \geq 1 \) and \( G \) be a graph of order \( n \). If
\[
\mu(G) > k/2 + \sqrt{kn + (k^2 - 4k)/4},
\]
then \( C_{2l+2} \subseteq G \) for every \( l = 1, \ldots, k \).

In view of the graph \( S_{n,k}^+ \), Theorem \cite{3.15} implies that
\[
(k - 1)/2 + \sqrt{kn + o(n)} \leq p_{2k+2}(n) \leq k/2 + \sqrt{kn + o(n)}. \tag{20}
\]
The exact value of \( p_{2k+2}(n) \) is not known for \( k \geq 2 \), and finding this value seems a challenge. Nevertheless, the precision of \cite{20} is somewhat surprising, given that the asymptotics of the maximum number of edges in \( C_{2k+2} \)-free graphs of order \( n \) is not known for \( k \geq 2 \).
Forbidden pair of cycles \(\{C_{2k}, C_{2k+1}\}\)

Let us consider now the function \(g_l(n)\). To begin with, Favaron, Mahéo, and Saclé [45] showed that if a graph \(G\) of order \(n\) contains neither \(C_3\) nor \(C_4\), then \(\mu(G) \leq \sqrt{n-1}\). Since the star of order \(n\) has no cycles and its spectral radius is \(\sqrt{n-1}\), we see that

\[
g_3(n) = \sqrt{n-1}.
\]

We do not know the exact value of \(g_l(n)\) for \(l > 3\), but we have the following theorem from [81].

**Theorem 3.16** Let \(k \geq 1\) and \(G\) be a graph of order \(n\). If

\[
\mu(G) > \left(\frac{k-1}{2} + \frac{k\sqrt{n} + (k+1)^2}{4}\right),
\]

then \(C_{2k+1} \subset G\) or \(C_{2k+2} \subset G\).

Theorem 3.16 together with the graphs \(S_{n,k}\) and \(S_{n,k}^+\), gives

\[
\frac{k-1}{2} + \frac{k\sqrt{n} + o(n)}{4} \leq g_{2k}(n) \leq \frac{k}{2} + \frac{k\sqrt{n}}{4} + o(n),
\]

\[
g_{2k}(n) = \frac{k-1}{2} + \frac{k\sqrt{n}}{4} + o\left(n^{-1/2}\right).
\]

**Forbidden path \(P_k\)**

The function \(h_k(n)\) is completely known for large \(n\). As proved in [81]:

**Theorem 3.17** Let \(k \geq 1\), \(n \geq 2^{4k}\) and let \(G\) be a graph of order \(n\).

(i) If \(\mu(G) \geq \mu(S_{n,k})\), then \(G\) contains a \(P_{2k+2}\) unless \(G = S_{n,k}\).

(ii) If \(\mu(G) \geq \mu(S_{n,k}^+)\), then \(G\) contains a \(P_{2k+3}\) unless \(G = S_{n,k}^+\).

Theorem 3.17 together with the graphs \(S_{n,k}\) and \(S_{n,k}^+\), implies that for every \(k \geq 1\) and \(n \geq 2^{4k}\), we have

\[
h_{2k}(n) = \mu(S_{n,k}) = \frac{k-1}{2} + \frac{k\sqrt{n} - (3k^2 + 2k - 1)}{4},
\]

\[
h_{2k+1}(n) = \mu(S_{n,k}^+) = \frac{k-1}{2} + \frac{k\sqrt{n} - (3k^2 + 2k - 1)}{4} + \frac{1}{n} + O\left(n^{-3/2}\right).
\]

### 3.8 Hamilton paths and cycles

In [86], Ore found the following sufficient condition for the existence of Hamilton paths and cycles.

**Theorem 3.18** Let \(G\) be a graph of order \(n\). If

\[
e(G) \geq \binom{n-1}{2}
\]

then \(G\) contains a Hamiltonian path unless \(G = K_{n-1} + K_1\). If the inequality (21) is strict, then \(G\) contains a Hamiltonian cycle unless \(G = K_{n-1} + e\).
In the line above and further, $K_{n−1} + e$ denotes the complete graph $K_{n−1}$ with a pendent edge.

Recently, Fiedler and Nikiforov [44] deduced a spectral version of Ore’s result.

**Theorem 3.19** Let $G$ be a graph of order $n$. If

$$\mu(G) \geq n - 2,$$  \hspace{1cm} (22)

then $G$ contains a Hamiltonian path unless $G = K_{n−1} + K_1$. If the inequality (22) is strict, then $G$ contains a Hamiltonian cycle unless $G = K_{n−1} + e$.

A subtler spectral condition for Hamiltonicity was obtained using the spectral radius of the complement of a graph.

**Theorem 3.20** Let $G$ be a graph of order $n$ and $\mu(\overline{G})$ be the spectral radius of its complement. If

$$\mu(\overline{G}) \leq \sqrt{n - 1},$$

then $G$ contains a Hamiltonian path unless $G = K_{n−1} + K_1$. If

$$\mu(\overline{G}) \leq \sqrt{n - 2},$$

then $G$ contains a Hamiltonian cycle unless $G = K_{n−1} + e$.

Zhou [92], adopting the same technique, proved a similar result for the signless Laplacian, which has been subsequently refined in [83] to the following one:

**Theorem 3.21** Let $G$ be a graph of order $n$ and $q(\overline{G})$ be the spectral radius of the $Q$-matrix of its complement.

(i) If

$$q(\overline{G}) \leq n,$$  \hspace{1cm} (23)

then $G$ contains a Hamiltonian path unless $G$ is the union of two disjoint complete graphs or $n$ is even and $G = K_{n/2−1,n/2+1}$.

(ii) If

$$q(\overline{G}) \leq n - 1,$$  \hspace{1cm} (24)

then $G$ contains a Hamiltonian cycle unless $G$ is a union of two complete graphs with a single common vertex or $n$ is odd and $G = K_{[n/2],[n/2]}$.

Note that if the inequality in (23) or (24) is strict, then the corresponding conclusion holds with no exception. Also, as it turns out, Theorem 3.21 considerably strengthens the classical degree conditions for Hamiltonicity by Ore [86].
3.9 Clique number and eigenvalues

If a triangle-free graph is sufficiently dense, then it contains large independent sets and the modulus of its smallest eigenvalue cannot be very small. A more general statement of such type has been proved in [11] for graphs of bounded clique number. Somewhat later, the following explicit dependence was found in [70].

**Theorem 3.22** If \( G \in \mathcal{G}(n, m) \) and \( \omega(G) = \omega \), then

\[
\mu_n(G) < -\frac{2}{\omega} \left( \frac{2m}{n^2} \right)^{\omega} n. \tag{25}
\]

Inequality (25) captures pretty well the situation in dense graphs, that is, if \( G \) is a dense graph with \( \mu_n(G) = O(n^{1-c}) \) for some \( c \in (0, 1/2) \), then \( G \) contains cliques of order \( \Omega(\log n) \).

Moreover, as shown in [70], inequality (25) is tight up to a constant factor for several classes of sparse graphs, but complete investigation of this issue seems difficult.

In [75], inequality (25) was used to derive a lower bound on \( \alpha(G) \), thus giving other cases of tightness.

**Theorem 3.23** Let \( G \in \mathcal{G}(n, m) \), \( d = 2m/n \), and \( \tau = |\mu_n(G)| \). If \( d \geq 2 \), then

\[
\alpha(G) > \left( \frac{n}{d + 1} - 1 \right) \left( \log \frac{d + 1}{\tau} - \log \log (d + 1) \right).
\]

Inequality (25) is concise, but it is difficult to use because the right-hand side is exponential in \( \omega(G) \). The following two somewhat simpler bounds, given in [74], stem from Turán’s theorem and some inequalities that will be given in the next subsection.

**Theorem 3.24** Let \( G \in \mathcal{G}(n, m) \), \( d = 2m/n \), and \( \omega(G) = \omega \). Then

\[
\omega \geq 1 + \frac{dn}{(n - d)(d - \mu_n(G))}.
\]

Equality holds if and only if \( G \) is a complete regular \( \omega \)-partite graph.

Similar inequalities [72] exist also for the Laplacian eigenvalues.

**Theorem 3.25** Let \( G = \mathcal{G}(n, m) \), \( d = 2m/n \) and \( \omega(G) = \omega \). Then

\[
\omega \geq 1 + \frac{dn}{\lambda_n(G)(n - d)},
\]

with equality holding if and only if \( G \) is a regular complete \( \omega \)-partite graph.

Also,

\[
\alpha(G) \geq 1 + \frac{(n - 1 - d)n}{(n - \lambda_2(G))(1 + d)},
\]

with equality holding if and only if \( G \) is the union of \( \alpha(G) \) disjoint cliques of equal order.

Note that both bounds in the last theorem imply the concise Turán theorem.
3.10 Number of cliques and eigenvalues

It turns out that the numbers of various cliques of a graph are closely related to its most important eigenvalues. Bollobás and Nikiforov \[13\] proved the following chain of inequalities, which were useful on several occasions.

**Theorem 3.26** Let $G$ be a graph with $\omega (G) = \omega \geq 2$ and $\mu (G) = \mu$. For every $r = 2, \ldots, \omega$,

$$\mu^{r+1} \leq (r+1) k_{r+1} (G) + \sum_{s=2}^{r} (s-1) k_{s} (G) \mu^{r+1-s}.$$  

Observe that, with $r = \omega - 1$, Theorem 3.26 gives the following inequality from \[69\]; it has been applied to obtain a two line proof of the spectral precise Turán theorem in \[72\].

**Theorem 3.27** If $G$ is a graph with $\omega (G) = \omega \geq 2$ and $\mu (G) = \mu$, then

$$\mu^{\omega} \leq k_{2} (G) \mu^{\omega-2} + 2k_{3} (G) \mu^{\omega-3} + \cdots + (\omega - 1) k_{\omega} (G).$$

Another important consequence of Theorem 3.26, also in \[13\], gives a lower bound on the number of cliques of any order as stated below.

**Theorem 3.28** If $r \geq 2$ and $G \in G (n)$, then

$$k_{r+1} (G) \geq \left( \frac{\mu (G)}{n} - 1 + \frac{1}{r} \right) \frac{r (r-1)}{r+1} \left( \frac{n}{r} \right)^{r+1}.$$  

The remaining two theorems of this subsection are given in \[74\] and have multiple uses. The first one relates the numbers of triangles, edges and vertices of a graph with the smallest eigenvalue of its adjacency matrix.

**Theorem 3.29** If $G \in G (n, m)$, then

$$\mu_{n} (G) \leq \frac{3n^{3}k_{3} (G) - 4m^{3}}{nm (n^{2} - 2m)}.$$  

with equality if and only if $G$ is a regular complete multipartite graph.

Inequality (26) should be regarded as a multifaceted relation that can be used for different purposes. By way of illustration, let us restate it as a lower bound on $k_{3} (G)$, getting

$$k_{3} (G) \geq \frac{\mu_{n} (G) (nm (n^{2} - 2m) + 4m^{3}}{3n^{3}},$$  

with equality holding for regular complete multipartite graphs. However, for all dense quasi-random graphs we have $\mu_{n} (G) = o (n)$ and $3k_{3} (G) = 4 (1 + o (1)) m^{3}/n^{2}$. This implies that

$$\frac{4m^{3}}{3n^{3}} \leq k_{3} (G) \geq o (1) mn + \frac{4m^{3}}{3n^{3}},$$

and we reach the somewhat paradoxical conclusion that inequality (27) is tight up to low order additive terms for almost all graphs, since almost all graphs are dense and quasi-random.

Statements similar to Theorem 3.29 have been obtained in \[74\] for the largest Laplacian eigenvalue $\lambda_{n} (G)$ as well.
Theorem 3.30 If $G \in \mathcal{G}(n, m)$, then
\[
6nk_3(G) \geq (n + \lambda_n(G)) \sum_{u \in V(G)} d^2(u) - 2nm\lambda_n(G)
\]
with equality if and only if $G$ is a complete multipartite graph, and
\[
\lambda_n(G) \geq \frac{2m^2 - 3nk_3(G)}{m(n^2 - 2m)}n,
\]
with equality if and only if $G$ is a regular complete multipartite graph.

3.11 Chromatic number

Let $G$ be a graph of order $n$. One of the best known results in spectral graph theory is the inequality of A.J. Hoffman [51]
\[
\chi(G) \geq 1 + \frac{\mu_1(G)}{-\mu_n(G)},
\] (28)

However, it seems that there is a lot more to find in this area. Indeed, in [73] we proved the following alternative bound.

Theorem 3.31 For every graphs of order $n$,
\[
\chi(G) \geq 1 + \frac{\mu_1(G)}{\lambda_n(G) - \mu_1(G)}.
\] (29)

Equality holds if and only if every two color classes of $G$ induce a regular bipartite graph of degree $|\mu_n(G)|$.

When $G$ is obtained from $K_n$ by deleting an edge, inequality (29) gives $\chi(G) = n - 1$, while (28) gives only $\chi(G) \geq n/2 + 2$. By contrast, for a sufficiently large wheel $W_{1,n}$, i.e., a vertex joined to all vertices of a cycle of length $n$, (29) gives $\chi \geq 2$, while (28) gives $\chi \geq 3$.

However, such comparisons are not too informative since, in [73], both (29) and (28) have been deduced from the same matrix theorem.

4 Some useful tools

In this section we present some results that we have found useful on multiple occasions. The selection and the arrangement of these results does not follow any particular pattern.

We start with an inequality stated by Moon and Moser in [60]; it seems that Khadžiivanov and Nikiforov [55] were the first to publish its complete proof, see also [58], Problem 11.8. The inequality has been used in many questions, say in the proof of Theorem 2.5.

Lemma 4.1 Let $1 \leq s < t < n$, and let $G$ be a graph of order $n$, with $k_t(G) > 0$. Then
\[
\frac{(t + 1)k_{t+1}(G)}{tk_t(G)} - \frac{n}{t} \geq \frac{(s + 1)k_{s+1}(G)}{sk_s(G)} - \frac{n}{s},
\] (30)
The following two simple lemmas were used to obtain a number of results in Section 2. The first one was proved in [63], and the second one in [67].

**Lemma 4.2** Let \( r \geq 2 \), let \( c, n, m, s \) be such that 
\[
0 < c \leq 1/2, \quad n \geq \exp\left(c^{-r}\right), \quad s = \lfloor c^r \log n \rfloor \leq (c/2) m + 1,
\]
and let \( G \) be a bipartite graph with parts \( A \) and \( B \) of size \( m \) and \( n \). If \( e(G) \geq cmn \), then \( G \) contains a \( K_2(s, t) \) with parts \( S \subset A \) and \( T \subset B \) such that \( |S| = s \) and \( |T| = t > n^{1-c^{r-1}} \).

**Lemma 4.3** Let \( \alpha, c, n, m \) be such that 
\[
0 < \alpha \leq 1, \quad 1 \leq c \log n \leq \alpha m/2 + 1,
\]
and let \( G \) be a bipartite graph with parts \( A \) and \( B \) of size \( m \) and \( n \). If \( e(G) \geq \alpha mn \), then \( G \) contains a \( K_2(s, t) \) with parts \( S \subset A \) and \( T \subset B \) such that \( |S| = \lfloor c \log n \rfloor \) and \( |T| = t > n^{1-c \log \alpha/2} \).

The following lemma, given in [78], strengthens a classical condition for the existence of paths given by Erdős and Gallai [39]. It has been used to obtain results about forbidden cycles and elsewhere.

**Lemma 4.4** Suppose that \( k \geq 1 \) and let the vertices of a graph \( G \) be partitioned into two sets \( U \) and \( W \).

(i) If 
\[
2e(U) + e(U,W) > (2k - 2) |U| + k |W|,
\]
then there exists a path of order \( 2k \) or \( 2k + 1 \) with both ends in \( U \).

(ii) If 
\[
2e(U) + e(U,W) > (2k - 1) |U| + k |W|,
\]
then there exists a path of order \( 2k + 1 \) with both ends in \( U \).

The following lemma from [68] was used to prove Theorems 2.27, 2.28 and 2.29 but may be used to carry over other stability results from triangle-free graphs to \( K_r \)-free graphs for \( r > 3 \).

**Lemma 4.5** Let \( r \geq 3 \) and let \( G \) be a maximal \( K_{r+1} \)-free graph of order \( n \). If 
\[
\delta(G) > \left(1 - \frac{2}{2r - 1}\right) n,
\]
then \( G \) has a vertex \( u \) such that the vertices not joined to \( u \) are independent.

The following lemma, given in [79], bounds the minimal entry of eigenvectors to the spectral radius of the adjacency matrix. This can be useful in various situations, e.g., in conjunction with Lemma 4.7 from [81] and Theorem 4.8 it can be used to prove upper bounds on \( \mu(G) \) by induction. Both lemmas have been used to prove several results in Section 3.
Lemma 4.6 Let $G$ be a graph of order $n$ with minimum degree $\delta(G) = \delta$ and $\mu(G) = \mu$. If $(x_1, \ldots, x_n)$ is a unit eigenvector to $\mu$, then
\[
\min \{x_1, \ldots, x_n\} \leq \sqrt{\frac{\delta}{\mu^2 + \delta(n - \delta^2)}}.
\]

Lemma 4.7 Let $G$ be a graph of order $n$ and let $(x_1, \ldots, x_n)$ be a unit eigenvector to $\mu(G)$. If $u$ is a vertex satisfying $x_u = \min \{x_1, \ldots, x_n\}$, then
\[
\mu(G - u) \geq \mu(G) \frac{1 - 2x_u^2}{1 - x_u^2}.
\]

The theorem below, given in [79], has been used to prove the spectral analog of several nonspectral results.

Theorem 4.8 Let $\alpha, \beta, \gamma, K$ and $n$ be such that
\[
0 < 4\alpha \leq 1, \quad 0 < 2\beta \leq 1, \quad 1/2 - \alpha/4 \leq \gamma < 1, \quad K \geq 0, \quad n \geq (42K + 4)/\alpha^2\beta,
\]
and let $G$ be a graph of order $n$. If
\[
\mu(G) > \gamma n - K/n \quad \text{and} \quad \delta(G) \leq (\gamma - \alpha) n,
\]
then there exists an induced subgraph $H \subset G$ with $|H| \geq (1 - \beta) n$, satisfying one of the following conditions:
(a) $\mu(H) > \gamma (1 + \beta\alpha/2) |H|$;
(b) $\mu(H) > \gamma |H|$ and $\delta(H) > (\gamma - \alpha) |H|$.

The abundance of parameters in Theorem 4.8 may obstruct its understanding. In summary, the theorem can be applied when one has to prove that if $\mu(G)$ is sufficiently large then $G$ contains some subgraph $F$. If $\delta(G)$ is not large enough, by tossing away not too many low degree vertices, one gets a graph $H$ in which either both $\mu(H)$ and $\delta(H)$ are large enough or $\mu(H)$ is considerably above the expected average. Most likely, either of these properties will help to find a copy of $F$ in $H$. The many parameters ensure greater flexibility.

In [11], using interlacing, Bollobás and Nikiforov gave the following inequality, which has been used to prove several results involving the minimum eigenvalue of the adjacency matrix, e.g., Theorem 3.22.

Theorem 4.9 If $G \in \mathcal{G}(n, m)$, then for every partition $V(G) = V_1 \cup V_2$,
\[
\mu_n(G) \leq \frac{2e(V_1)}{|V_1|} + \frac{2e(V_2)}{|V_2|} - \frac{2m}{n}.
\]

Note that this inequality is analogous to the well-known inequality for the Laplacian (see Mohar, [59]):
\[
\lambda_n(G) \geq \frac{e(V_1, V_2)}{|V_1||V_2|} n,
\]
and in fact for regular graphs both inequalities are identical.
5 Illustration proofs

The purpose of this section is to illustrate the use of the tools developed for translating non-spectral into spectral results. To this end we shall sketch the proofs of Theorems 2.19 and 3.10.

The structure of both proofs is identical. In both proofs we shall use Theorem 2.4 from Section 2.2. The main difference comes from the fact that in the proof of Theorem 2.19 we use Theorem 2.22 while in the proof of Theorems 3.10 we use the analogous spectral result Theorem 3.12.

Proof of Theorem 2.19

Let \( G \) be a graph of order \( n \) with \( e(G) > (1 - 1/r - \varepsilon) n^2/2 \). Define the procedure \( P \) as follows:

While \( js_{r+1}(G) > n^{r-1/r^{r+6}} \) do

Select an edge contained in \( \lceil n^{r-1/r^{r+6}} \rceil \) cliques of order \( r + 1 \) and remove it from \( G \).

Set for short \( \theta = c^{1/(r+1)} r^{r+6} \) and assume first that \( P \) removes at least \( \lceil \theta n^2 \rceil \) edges before stopping. Then

\[
kr+1(G) \geq \theta n^{r-1/r^{r+6}} = c^{1/(r+1)} n^{r+1},
\]

and Theorem 2.4 implies that

\[
K_{r+1} \left( \lceil c \ln n \rceil, \ldots, \lceil c \ln n \rceil, \left\lfloor n^{1-\sqrt{2}} \right\rfloor \right) \subset G.
\]

Thus, in this case condition (a) holds, completing the proof.

Assume therefore that \( P \) removes fewer than \( \lceil \theta n^2 \rceil \) edges before stopping. Writing \( G' \) for the resulting graph, we see that

\[
e(G') > e(G) - \theta n^2 > (1 - 1/r - \varepsilon - \theta) n^2/2
\]

and \( js_{r+1}(G') < n^{r-1/r^{r+6}} \). Here we want to apply Theorem 2.22 and so we check for its prerequisites. First, from \( \log n \geq 1/c \geq r^{3(r+14)/(r+1)} \) we easily get \( n > r^8 \). Also,

\[
\varepsilon + \theta < r^{-8/8}.
\]

Now, Theorem 2.22 implies that \( G' \) contains an induced \( r \)-partite subgraph \( G_0 \) satisfying

\[
|G_0| \geq \left( 1 - \sqrt{2(\varepsilon + \theta)} \right) n \text{ and } \delta(G_0) > \left( 1 - 1/r - 2\sqrt{2(\varepsilon + \theta)} \right) n.
\]

By routine calculations we find that \( G \) differs from \( T_r(n) \) in fewer than

\[
\left( \theta + (2r^2 - r) \sqrt{2(\varepsilon + \theta)} \right) n^2 < \left( \varepsilon^{1/3} + c^{1/(3r+3)} \right) n^2
\]

edges, and condition (b) follows, completing the proof of Theorem 2.19. \( \square \)

Proof of Theorem 3.10

Let \( G \) be a graph of order \( n \) with \( \mu(G) > (1 - 1/r - \varepsilon) n \). Define the procedure \( P \) as follows:

While \( js_{r+1}(G) > n^{r-1/r^{2r+5}} \) do

Select an edge contained in \( \lceil n^{r-1/r^{2r+5}} \rceil \) cliques of order \( r + 1 \) and remove it from \( G \).
Set for short \( \theta = c^{1/(r+1)}r^{2r+5} \) and assume first that \( \mathcal{P} \) removes at least \( \lceil \theta n^2 \rceil \) edges before stopping. Then

\[
k_{r+1}(G) \geq \theta n^{r-1}/r^{2r+5} = c^{1/(r+1)}n^{r+1},
\]
and Theorem 2.4 implies that

\[
K_{r+1}\left(\lceil c \ln n \rceil, \ldots, \lceil c \ln n \rceil, \lceil n^{1-\sqrt{2}} \rceil\right) \subsetneq G.
\]

Thus, in this case condition (a) holds, completing the proof.

Assume now that \( \mathcal{P} \) removes fewer than \( \lceil \theta n^2 \rceil \) edges before stopping. Write \( G' \) for the resulting graph; we obviously have \( j_{s_{r+1}}(G') \leq n^{r-1}/r^{2r+5} \). Letting \( \mu(X) \) be the largest eigenvalue of a Hermitian matrix \( X \), recall Weyl’s inequality

\[
\mu(B) \geq \mu(A) - \mu(A-B),
\]
holding for any Hermitian matrices \( A \) and \( B \). Also, recall that \( \mu(H) \leq \sqrt{2e(H)} \) for any graph \( H \). Applying these results to the graphs \( G \) and \( G' \), we find that

\[
\mu(G') \geq \mu(G) - \sqrt{2\theta} n \geq \left(1 - 1/r - \varepsilon - \sqrt{2\theta}\right)n.
\]

Here we want to apply Theorem 3.12 and so we check for its prerequisites. First, from \( \log n \geq 1/c \geq r^8(r+21)(r+1) \) we easily get \( n > r^{20} \). Also,

\[
\varepsilon + \sqrt{2\theta} < 2^{-10}r^{-6}.
\]

Now, Theorem 3.12 implies that \( G' \) contains an induced \( r \)-partite subgraph \( G_0 \), satisfying

\[
|G_0| \geq \left(1 - 4\left(\varepsilon + \sqrt{2\theta}\right)^{1/3}\right)n \quad \text{and} \quad \delta(G_0) > \left(1 - 1/r - 7\left(\varepsilon + \sqrt{2\theta}\right)^{1/3}\right)n.
\]

By routine calculations we find that \( G \) differs from \( T_r(n) \) in fewer than

\[
\left(\theta + (7r^2 - 3r)\left(\varepsilon + \sqrt{2\theta}\right)^{1/3}\right)n^2 < \left(\varepsilon^{1/4} + c^{1/(8r+8)}\right)n^2
\]
edges, and condition (b) follows, completing the proof of Theorem 3.10.

\[\square\]

6 Notation and basic facts

Throughout the survey our notation generally follows [7]. Given a graph \( G \), we write:

- \( V(G) \) for the vertex set of \( G \);
- \( E(G) \) for the edge set of \( G \) and \( e(G) \) for \( |E(G)| \);
- \( \alpha(G) \) for the independence number of \( G \) (see below);
- \( \delta(G) \) and \( \Delta(G) \) for the minimum and maximum degrees of \( G \);
- \( \omega(G) \) for the clique number of \( G \) (see below);
- \( k_s(G) \) for the number of \( s \)-cliques of \( G \) (see below);
- \( G - u \) for the graph obtained by removing the vertex \( u \in V(G) \);
- \( \Gamma(u) \) for the set of neighbors of a vertex \( u \), and \( d(u) \) for \( |\Gamma(u)| \);
- \( e(X) \) for the number of edges induced by a set \( X \subset V(G) \);
- $e(X,Y)$ for the number of edges joining vertices in $X$ to vertices in $Y$, where $X$ and $Y$ are disjoint subsets of $V(G)$;

We write $\mathcal{G}(n)$ for the set of graphs of order $n$ and $\mathcal{G}(n,m)$ for the set of graphs of order $n$ and size $m$.

Also, $[n]$ stands for the set $\{1,2,\ldots,n\}$.

**Mini glossary**

- **clique** - a subgraph that is complete. An $s$-clique has $s$ vertices; $k_s(G)$ stands for the number of $s$-cliques of $G$;
- **clique number** - the size of the largest clique of $G$, denoted by $\omega(G)$;
- **chromatic number** - the minimum number of independent sets that partition $V(G)$, denoted by $\chi(G)$;
- **disjoint union** of two graphs $G$ and $H$ is the union of two vertex disjoint copies of $G$ and $H$. The disjoint union of $G$ and $H$ is denoted by $G + H$;
- **independent set** - a set of vertices of $G$ that induces no edges;
- **independence number** - the size of the largest independent set of $G$, denoted by $\alpha(G)$;
- **join** of two vertex disjoint graphs $G$ and $H$ is the union of $G$ and $H$ together with all edges between $G$ and $H$. The join of $G$ and $H$ is denoted by $G \vee H$;
- **joint** - a set of cliques of the same order sharing an edge. An $r$-joint of size $t$ consists of $t$ cliques of order $r$;
- **book of size** $t$ - a 3-joint of size $t$, that is to say, a collection of $t$ triangles sharing an edge;
- **homomorphic graph** - a graph $G$ is said to be homomorphic to a graph $H$, if there exists a map $f : V(G) \to V(H)$ such that $uv \in E(G)$ implies $f(u)f(v) \in E(H)$;
- **graph property** - a family of graphs closed under isomorphisms;
- **hereditary property** - graph property closed under taking induced subgraphs;
- **monotone property** - graph property closed under taking subgraphs;
- **$H$-free graph** - a graph that has no subgraph isomorphic to $H$;
- **friendship graph** - a collection of triangles sharing a single common vertex;
- **$k$-th power of a cycle** $C_n$ - a graph with vertices $\{1,2,\ldots,n\}$, and $(i,j)$ is an edge if $i - j = \pm 1, \pm 2, \ldots, \pm k \mod n$;
- $K_r$ and $\overline{K}_r$ - the complete and the edgeless graph of order $r$;
- $K_r(s_1, s_2, \ldots, s_r)$ - the complete $r$-partite graph with class sizes $s_1, s_2, \ldots, s_r$. We set for short $K_r(p) = K_r(p,\ldots,p)$ and $K_r(p;q) = K_r(p,\ldots,p,q)$;

- **$r$-uniform hypergraph** - a hypergraph whose edges are subsets of $r$ vertices;
- **Turán graph** $T_r(n)$ - given $n \geq r \geq 2$, this is the complete $r$-partite graph whose class sizes differ by at most one. We let $t_r(n) = e(T_r(n))$. If $t$ is the remainder of $n$ mod $r$, then

$$t_r(n) = \frac{r-1}{2r} \left( n^2 - t^2 \right) + \binom{t}{2},$$

which in turn implies that

$$\frac{r-1}{2r}n^2 - \frac{r}{8} \leq t_r(n) \leq \frac{r-1}{2r}n^2;$$

- **Turán problem** - given a family of graphs $F$, find the maximum number of edges in a graph of order $n$, having no subgraph belonging to $F$;
**quasi-random graph** - informally, an almost regular graph, in which the second largest in modulus eigenvalue is much smaller than the spectral radius;

**spectral radius of a graph** - in general, the spectral radius of a matrix is the largest modulus of its eigenvalues. For a graph, this is usually the spectral radius of its adjacency matrix, which is an eigenvalue itself;

**Laplacian matrix** - the matrix $L = D - A$, where $A$ is the adjacency matrix and $D$ is the diagonal matrix of the row-sums of $A$, that is the degrees of $G$;

**Q-matrix,** also known as **signless Laplacian** - the matrix $Q = D + A$;

**Szemerédi’s Regularity Lemma** - an important result of analytical graph theory, which states that every graph can be approximated by graphs of bounded order. For background on this lemma we refer the reader to [7], Section IV.5;

**Zarankiewicz problem** - a class of problems aiming to determine the maximum number of edges in a graph with no $K_{s,t}$. There are several variations, most of which are only partially solved. See [7] for details.

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