Electromagnetic wave refraction at an interface of a double wire medium

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Abstract

Plane-wave reflection and refraction at an interface with a double wire medium is considered. The problem of additional boundary conditions (ABC) in application to wire media is discussed and an ABC-free approach, known in the solid state physics, is used. Expressions for the fields and Poynting vectors of the refracted waves are derived. Directions and values of the power density flow of the refracted waves are found and the conservation of the power flow through the interface is checked. The difference between the results, given by the conventional model of wire media and the model, properly taking into account spatial dispersion, is discussed.

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I. INTRODUCTION

Wire medium (WM) is an artificial medium formed by a lattice of ideally conducting thin wires. Recently, we have observed a growing interest to such artificial media of possible new physical phenomena and potential applications. This medium at low frequencies is usually described as a uniaxial crystal, whose permittivity tensor components are expressed by the plasma model. It has been shown [1], that if the wavevector in a WM has a nonzero component along the wires, the plasma model should be corrected introducing spatial dispersion (SD). Similarly, spatial dispersion is inherent to the double WM (DWM) formed by two mutually orthogonal lattices of thin ideally conducting straight wires, see Fig. 1.

For consideration of waves in double wire media let us take a case where the wires are perpendicular to each other in the \( y \) and \( z \) directions. The waves in unbounded space filled with a DWM medium were studied in [2] numerically, in [3] using a semi-analytical approximation of the local field, and in [4] both numerically and using the effective medium (EM) approach. In the last paper a very good agreement between the results given by the EM and full-wave theories for all types of waves in DWM (if the wires are thin) has been demonstrated.

In this paper we consider the plane-wave reflection and refraction at an interface of DWM using the effective medium approach. We assume that the two orthogonal wire arrays are identical, the period of the lattice is equal to \( L \) in the \( x, y, \) and \( z \) directions, and the radius of the wires is equal to \( r_0 \). In this case the wire lattice is square in the plane of the wires, i.e., the \((yz)\) plane. We assume also that the interface of DWM lies in the \((xy)\) plane and the incident wave vector lies in the \((yz)\) plane (see Fig. 2).

In the next sections we will give basic expressions, obtained in the framework of the EM approach, and formulate the wave refraction problem, demonstrating necessity of additional boundary conditions (ABC’s) both for interface problems with single and double wire media. Then we will discuss some approaches, applying in solid state physics in order to overcome the ABC problem and look what may be useful for us in application to WM. Using the ABC-free approach we will find the reflection coefficient, amplitudes and Poynting vectors for refracted waves. We will
compare directions of the group velocity and the energy density flow found from the expression for the Poynting vector containing additional terms inherent for media with spatial dispersion. Conservation of the normal component of the power flow vector at passing through the interface is checked.

II. FIELD EQUATIONS AND EIGENWAVES IN DOUBLE UNBOUNDED WIRE MEDIUM

Assuming space-time dependence of fields as $e^{i(\omega t - k_y y - k_z z)}$, there are non-zero wave vector components parallel to wires. Anisotropy appears in this electromagnetic crystal with square lattice and DWM behaves as a biaxial crystal with the relative permittivity dyadic

$$\bar{\epsilon} = \epsilon_x u_x u_x + \epsilon_y u_y u_y + \epsilon_z u_z u_z,$$

with

$$\epsilon_x = \epsilon_h, \quad \epsilon_y = \epsilon_h \left( 1 - \frac{k_y^2}{k^2 - k_y^2} \right), \quad \epsilon_z = \epsilon_h \left( 1 - \frac{k_z^2}{k^2 - k_z^2} \right),$$

where $k = \frac{\omega}{c} \sqrt{\epsilon_h}$, $c$ is the speed of light, and $\epsilon_h$ is the relative permittivity of the host medium. Note, that model 1 works both for real and imaginary $k_y$ (for propagating and evanescent waves, respectively), see 2.

In the wire medium the Maxwell equations

$$\nabla \times \mathbf{E} = -j \omega \mu_0 \mathbf{H},$$

$$\nabla \times \mathbf{H} = j \omega \varepsilon_0 \bar{\epsilon} \cdot \mathbf{E}$$

split into two separate subsystems describing wave propagation of fields with TE and TM polarizations:

$$-j(k_y u_y + k_z u_z) \times (E_x u_x + E_y u_y + E_z u_z) = -j \omega \mu_0 (H_x u_x + H_y u_y + H_z u_z)$$

$$-j(k_y u_y + k_z u_z) \times (H_x u_x + H_y u_y + H_z u_z) = j \omega \varepsilon_0 (\epsilon_x E_x u_x + \epsilon_y E_y u_y + \epsilon_z E_z u_z).$$
For ordinary (TE) waves this leads to the wave equation

\[ k^2 - k_y^2 - k_z^2 E_x = 0. \] (7)

The same equations hold for \( H_y \) and \( H_z \). There are no effects due to wires, and ordinary waves propagate as in any isotropic dielectric medium.

Whereas for extraordinary (TM) waves the wires affect the propagation, and we obtain the wave equation

\[ k^2 \epsilon_y - k_x^2 k_y^2 \epsilon_y \epsilon_z H_x = 0. \] (8)

The same equation can be written for \( E_y \) and \( E_z \).

In order to solve the wave reflection problem we need to evaluate the eigenwaves which are outgoing from the interface of DWM. It means that we have to find \( k_z \) under fixed \( k \) and \( k_y \). Let \( k_y = k \sin \theta \), where \( \theta \) is the incidence angle.

It follows from (8) that the dispersion equation has the form

\[ T(k_z, \omega) = k_z^2 - \left[ k^2 \epsilon_y(k_y) - k^2 \epsilon_y(k_y) \right] = 0, \] (9)

and its solution is:

\[ k_{z, \pm}^2 = \frac{2k^4 - 2k^2k_p^2 - 3k^2k_y^2 + 2k_p^2k_y^2 + k_y^4 \pm k_y \sqrt{(k_y^2 - k^2)((2k_p^2 + k_y^2)^2 - k^2(4k_p^2 + k_y^2))}}{2(k^2 - k_p^2)}. \] (10)

Two waves propagating or attenuating in both directions follow from the effective medium theory (10). The conventional isotropic plasma model leads to only two waves for a certain direction, namely, \( k_z = \pm \sqrt{k_0^2 \epsilon - k_y^2}, \epsilon = \epsilon_h(1 - k_p^2/k^2) \), where \( k_0 = \omega/c \).

For the following consideration of the refraction problem we will need to know properties of waves propagating in the \( z \) direction. Here we briefly revise these properties (see our previous paper [4] for more details). The real and imaginary parts of \( k_z \) versus the normalized frequency \( k/k_p \) are presented in Fig. 3 for \( \theta = \pi/4 \). Let us assume that \( \epsilon_h = 1 \). In the simple case of the conventional model (dotted curve) \( k_z \) is imaginary if \( k < K_2 = k_p / \cos \theta \), and it is real if \( k > K_2 \). The solution of the conventional model is completely wrong between \( K_1 \) and \( K_2 \) because it gives an imaginary value of \( k_z \) instead of a real one which is obtained from the correct model. The correct, more complicated solutions, follow from Eq. (10). Analyzing Eq. (10), one can see that there exist three frequency regions, corresponding to different kinds of solution.

The first one is the low frequency band \( k < K_1 \), where

\[ K_1 = k_p \frac{\sqrt{2}}{\sin \theta} \sqrt{\frac{1 - \cos \theta}{\cos \theta}} \] (11)

is the stop band edge: Beyond this wave number the waves are propagating. There the propagation constant \( k_z \) is complex despite the fact that we have assumed the medium to be lossless (see Fig. 3). Actually, there are two complex conjugate solutions for each Re(\( k_z \)) > 0.
FIG. 3: Real and imaginary parts of $k_z$, calculated using the electrodynamical model (solid curves) and the EM theory (dashed curves). The dotted curve shows $k_z$ given by the conventional plasma model.

The second frequency area is $K_1 < k < K_2$. At point $K_2$ one of the solutions is zero and within the range $K_1 < k < K_2$ we have a forward wave and a backward wave with respect to the interface. It means that one of the waves has a positive projection of the wave vector on the interface inner normal, while the other wave has a negative projection.

Finally, for $k > K_2$ both of the waves are propagating forward waves. Electrodynamical calculations [4] (using the three dimensional Green’s function) confirm the results of the effective medium theory with a high accuracy in a wide spectral range including the regions of evanescent and propagating waves (see solid and dashed curves in Fig. 3). Note that point $K_2$ corresponds to the edge of the passband in the framework of the old model. Thus, the model taking into account spatial dispersion leads to a considerably more complicated structure of eigenwaves than the conventional model of an isotropic plasma, and it is in very good agreement with the results of the full-wave analysis.

III. WAVE REFLECTION FROM A WIRE MEDIUM INTERFACE AND THE PROBLEM OF ADDITIONAL BOUNDARY CONDITIONS

As it was shown above, there exist two extraordinary waves with the wave vector and the electric field in the $(yz)$ plane. Assuming the $y$-component of the electric field of the incident wave to be equal to unity, and applying the continuity conditions of the tangential field components results in the reflection problem formulated as follows:

\[ 1 + R_E = E_+ + E_- \]
\[ (1 - R_E)/Z_0 = E_+/Z_+ + E_-/Z_- \]

where $R_E$ is the unknown reflection coefficient for electric field, $E_+$, $E_-$ are the unknown amplitudes of refracted waves in the wire medium, $Z_0$ is the wave impedance (TM) in free space and
$Z_\pm$ are the wave impedances of the refracted waves. Thus the problem becomes similar to one appearing in crystallooptics, where excitons arise and spatial dispersion cannot be neglected \[7\]. The main difficulty here is the necessity to invoke additional boundary conditions (ABC) in order to match solutions at the interface of media. It was pointed out first by S. Pekar \[8\] (1956), that the well-known Maxwell’s boundary conditions (12) are not sufficient to connect the amplitudes of the incident and transmitted waves in adjoining media, if more than one independent wave can propagate in any medium.

Probably, the first ABC were proposed by S. Pekar \[8\], and his ABC stay among the more often used in the theory of media with SD. The simplest phenomenological Pekar’s ABC are the following:

\[
\begin{align*}
    P &= 0, \\
    \frac{\partial P}{\partial z} &= 0 \\
\end{align*}
\]

(13)
at the boundary, where $P$ is the polarization vector. Let us consider if we can apply to the WM the first condition of (13) in the region of evanescent waves. For the $y$-component of the polarization vector one can write

\[
P_y = \chi_{y1}E_1 + \chi_{y2}E_2 = 0,
\]

(14)
where the susceptibilities $\chi_{y1}$ and $\chi_{y2}$ relate to different waves, but $\chi_{y1} = \chi_{y2}$. Hence we come to $E_1 + E_2 = 0$, which means that the reflection coefficient for the electric field $R_E = -1$. In other words, a semi-infinite WM behaves as a perfect metal plane. However, we know that it is not so even if only evanescent waves are excited in the WM. The reflected wave changes its phase due to penetration into WM, and this phase is not equal to $\pi$. The same result is obtained considering condition $P_z = 0$. The second Pekar’s condition (13) (for derivatives) leads to a similar result. Thus, the conventional Pekar’s ABC are useless for solving our problem.

After pioneering Pekar’s publication different ABC were proposed for problems of crystallooptics, as well as semiconductor and plasma electrodynamics. All of these works relate to specific media and take into account the properties of a sub-surface layer at the media interface (see \[7, 9\] and the bibliography presented there). Besides, phenomenological assumptions and experimental data are used in these theories. Since WM strongly differs from a solid-state crystal, only general approaches to ABC which do not concerned with particular media, may be interesting for us.

Many authors (see, for example, \[10, 11\]) derived ABC from a given model of medium with an explicit specification of its surface. After simplifications these models give nothing more than Pekar’s ABC. Among different theories of SD media so-called “ABC-Free” theories attract our attention. One of the ways for ABC derivation is a concept of ”exciton dead layer”, proposed by Hopfield and Thomas. Actually, there is a layer in which exciton wave function has evanescent form. Kikuo Cho \[12\] has pointed out that there is a certain limit of transition layer thickness, below which the ABC theory is unnecessary. However, Cho theory is not yet free from some parameters determined by specific surfaces. B. Chen and D.F. Nelson declare that their work \[13\] solves the macroscopic ABC problem completely. Despite that the authors of \[13\] used a complicated quantum mechanical model, their derivation found that the fully macroscopic solution
is equivalent to the use of Pekar’s ABC $P(z = 0) = 0$. Thus, such an approach also is not suitable for us. A. Vinogradov and co-authors, considering in [14] effects of SD in composite metamaterials, have obtained ABCs using an assumption of existence of additional waves in free space which are the same as in the metamaterial but have evanescent nature. This assumption leads to additional equations at the interface.

Another way to solve the problem was proposed recently by K. Henneberger [15]. It is based on the assumption of an abrupt transition from medium to vacuum. It is assumed that the incident wave excites a source $s(z, \omega)$, located within a sub-surface layer $0 < z < 2a$, and its thickness is assumed to be negligible. This approach is appropriate for our problem of wire media interface. Indeed, it is known for problems of diffraction by single semi-infinite wire grids that the induced currents deviate from the regular amplitudes far from the interface only in a very narrow region whose width is of the order of the grating period [16]. This conclusion holds for grids of wires both parallel and perpendicular to the edge. For this reason we assume that for the wire medium interface the transition layer has the thickness of only a few periods of the lattice. This thickness is much smaller than the wavelength and negligible as compared with the length of the wires (wires are infinite in our model).

Applying this approach to our interface problem of free space and the wire medium, the wave equation for $H_x$ in unbounded medium (Eq. (8) written for $H_x$) should be replaced by an inhomogeneous equation

$$\frac{\partial^2 H_x}{\partial z^2} + \left[k_0^2 \varepsilon_y(k_y) - k_y^2 \varepsilon_y(k_y) \varepsilon_z(k_z)\right] H_x = s(z, \omega), \quad (15)$$

where $H_x$ is the refracted field. It means that any propagating wave has to be created by a source. The proper source of the penetrating wave in the wire medium is the incident wave and the polarization induced by it in the medium. Such an externally controlled source can be identified with some polarization additionally induced to the one already described by $\bar{\varepsilon}$. Therefore it is located only on the surface and in the transition region, where the induced polarization deviates from that in the bulk medium. After Fourier transform of Eq. (15) one obtains

$$H_x(z, \omega) = \int_{-\infty}^{\infty} \frac{dq}{2\pi} \frac{s(q, \omega)e^{iqz}}{T(q, \omega)}, \quad (16)$$

where $s(q, \omega)$ is the Fourier transform of $s(z, \omega)$ and $T(q, \omega)$ is determined by Eq. (11). Assuming an abrupt transition from the medium to vacuum, we can present the source as a delta function $s(z, \omega) = s_0(\omega)\delta(z)$, then $s(q, \omega) = s_0(\omega)$. If $T(q, \omega)$ is an analytical function, the integration in Eq. (16) can be performed using the residue method. Residues can be found by presenting $1/T$ in the form

$$\frac{1}{T(k_z, \omega)} = \frac{1}{k_z^2 - k_0^2 \varepsilon_y(k_y) - k_y^2 \varepsilon_y(k_y) \varepsilon_z(k_z)} = \frac{\beta_+}{k_2^2 - k_{z+}^2} + \frac{\beta_-}{k_2^2 - k_{z-}^2}, \quad (17)$$

where the coefficients are

$$\beta_+ = \frac{k_2^2 - k_{z+}^2 - k_{z+}^2}{k_2^2 - k_{z+}^2}$$

$$\beta_- = \frac{k_2^2 - k_{z-}^2 - k_{z-}^2}{k_2^2 - k_{z-}^2} \quad (18)$$
\[
\beta_- = -\frac{k^2 - k^2_y - k^2_z}{k^2_z - k^2_{z+}}. \tag{19}
\]

The residues (the relative amplitudes of the transmitted field components) read

\[
R_+ = \frac{\beta_+}{2k_{z+}} = \frac{k^2 - k^2_y - k^2_{z+}}{2(k^2_z - k^2_{z+})k_{z+}}, \tag{20}
\]

\[
R_- = \frac{\beta_-}{2k_{z-}} = -\frac{k^2 - k^2_y - k^2_{z-}}{2(k^2_z - k^2_{z+})k_{z-}}. \tag{21}
\]

Finally, the field component \(H_x\) in the wire medium is

\[
H_x = s_0 \left[R_+ e^{-jk_{z+}z} + R_- e^{-jk_{z-}z}\right] = s_0 \left[\frac{\beta_+}{2k_{z+}} e^{-jk_{z+}z} + \frac{\beta_-}{2k_{z-}} e^{-jk_{z-}z}\right]. \tag{22}
\]

The expression for \(E_y\) and \(E_z\) can be obtained from the Maxwell equations as

\[
k_z H_x = -k_0 \varepsilon_y E_y, \quad -k_y H_x = -k_0 \varepsilon_z E_z, \tag{23}\]

which gives us the electric field components in the wire medium

\[
E_y = -\frac{s_0}{k_0 \varepsilon_y} \left[\beta_+ e^{-jk_{z+}z} + \beta_- e^{-jk_{z-}z}\right] \tag{24}
\]

\[
E_z = \frac{k_y s_0}{\omega \varepsilon_o} \left[\frac{\beta_+}{2k_{z+} \varepsilon_{z+}} e^{-jk_{z+}z} + \frac{\beta_-}{2k_{z-} \varepsilon_{z-}} e^{-jk_{z-}z}\right]. \tag{25}
\]

Now we have expressions for all field components induced in the wire medium for the TM polarization.

In free space there exist incident and reflected TM waves. Magnetic field components are

\[
H_y^i = H_0 e^{-jk_y y} e^{-j\beta_0 z}, \quad H_y^r = H_r e^{-jk_y y} e^{j\beta_0 z} = R_H H_0 e^{-jk_y y} e^{j\beta_0 z}, \tag{26}
\]

where \(\beta_0 = \sqrt{k_0^2 - k_y^2}\) and the electric field components are

\[
E_y^i = -\frac{\beta_0}{k_0} H_0 e^{-jk_y y} e^{-j\beta_0 z}, \quad E_y^r = R_H \frac{\beta_0}{k_0} H_0 e^{-jk_y y} e^{j\beta_0 z} \tag{27}
\]

\[
E_z^i = \frac{k_y}{k_0} H_0 e^{-jk_y y} e^{-j\beta_0 z}, \quad E_z^r = R_H \frac{k_y}{k_0} H_0 e^{-jk_y y} e^{j\beta_0 z}. \tag{28}
\]

At the interface \(z = 0\) the continuity of the tangential field components leads to relations

\[
H_0 + R_H H_0 = \frac{s_0}{2} \left[\frac{\beta_+}{k_{z+}} + \frac{\beta_-}{k_{z-}}\right], \tag{29}
\]

\[
-\frac{\beta_0}{k_0} H_0 + \frac{\beta_0}{k_0} R_H H_0 = -\frac{s_0}{2k_0 \varepsilon_y} [\beta_+ + \beta_-]. \tag{30}
\]
from which the reflection coefficient for magnetic field is obtained:

\[ R_H = \left( \frac{\beta_+ + \beta_-}{k_{z+} + k_{z-}} \right) - \frac{1}{\varepsilon_y} \left( \frac{\beta_+ + \beta_-}{\beta_0 + \beta_-} \right). \]  \hspace{1cm} (31)

Finally, the explicit expression for the coefficient \( s_0 \) (the transmission source) is obtained:

\[ s_0 = \left( 1 + \frac{R_H}{2} \right) H_0 = \frac{4H_0}{\left( \frac{\beta_+ + \beta_-}{k_{z+} + k_{z-}} \right) + \frac{1}{\varepsilon_y} \left( \frac{\beta_+ + \beta_-}{\beta_0 + \beta_-} \right)}. \]  \hspace{1cm} (32)

In the region of complex waves \( k/k_p < K_1 \) we have to chose the branches of the square roots for \( k_{z\pm} \) having positive imaginary parts. In the region \( K_1 < k/k_p < K_2 \) it is necessary to take for the backward wave ("-" wave) the root branch with the opposite sign. In the above derivation we evaluated the reflection coefficient for the magnetic field. The reflection coefficient for the electric field is \( R_E = -R_H \cos \theta \).

A. Reflection from a single wire medium interface

The plane wave reflection coefficient from an interface of a single wire medium where wires are along the \( z \) axis is easily obtained as a special case of the previously considered double wire medium reflection problem. In a single wire medium the permittivity components are \( \varepsilon_x = \varepsilon_y = \varepsilon_h \) and \( \varepsilon_z = \varepsilon_h \left( 1 - \frac{k_z^2}{k^2} \right) \). Evaluating the dispersion equation we have as solutions \( k_{z+} = k \), which is the propagation factor for the TEM mode and \( k_{z-} = \sqrt{k^2 - k_y^2 - k_p^2} \) for the TM mode. Thus, in the single wire medium the two extraordinary eigenwaves are TEM and TM polarized.

We can use exactly the same expressions for the reflection coefficient as in the case of the double wire medium simply substituting \( k_{z+} = k \) and \( k_{z-} = \sqrt{k^2 - k_y^2 - k_p^2} \). This leads to expressions for the coefficients

\[ \beta_+ = \frac{k_p^2}{k_y^2 + k_p^2} \]  \hspace{1cm} (33)

\[ \beta_- = \frac{k_y^2}{k_y^2 + k_p^2} \]  \hspace{1cm} (34)

and residues

\[ R_+ = \frac{k_p^2}{2k(k_y^2 + k_p^2)} \]  \hspace{1cm} (35)

\[ R_- = \frac{k_y^2}{2\sqrt{k^2 - k_y^2 - k_p^2(k_y^2 + k_p^2)}}. \]  \hspace{1cm} (36)

IV. DISCUSSION

Fig. 4 shows the phase of the reflected wave \( \phi = \arctan \{ \text{Im}(R_H)/\text{Re}(R_H) \} \) versus the incidence angle \( \theta \), calculated for different \( k/k_p \), corresponding to the region of complex waves (see Fig. 3).
Here $\text{Re}(R_H)$ and $\text{Im}(R_H)$ are the real and imaginary parts of the reflection coefficient $R_H$. The calculations have been performed using the model (2), taking into account spatial dispersion, and the conventional one,

$$
\epsilon_y = \epsilon_z = \epsilon_h \left( 1 - \frac{k_p^2}{k^2} \right).
$$

(37)

As it is expected, both models give the modulus of the reflection coefficient equal to unity due to the absence of losses and propagating waves in wire medium. It is remarkable, that the spatial dispersion results in a weak dependence of the phase of the reflected wave on the incidence angle. The parameters of the wire medium are taken as above in the eigenvalues calculations.

The real and imaginary parts of the reflection coefficient in a wide spectral range cover areas with complex waves, FW and BW waves and FW waves only, are presented in Fig. 5. The incidence angle is taken to be $\pi/4$. For understanding of these characteristics it is useful to compare them with the eigenvalue dispersion (Fig. 3). Distinctive points $K_1$, where propagating waves appear in the framework of new model, and $K_2$, where both of the waves become forward ones. Results, given by the old model, are also shown here. The most important feature is that the area of propagating waves shifts by $\Delta k/k_p = K_2 - K_1$ in comparison with that given by the old model.

![Normalized phase of the reflected wave \( \phi/\pi \) versus the incidence angle \( \theta/\pi \), calculated at different \( k/k_p \): curves 1 correspond to \( k/k_p=0.3 \) and curves 2 correspond to \( k/k_p=0.7 \). Solid and dashed curves show the reflection phase given by the new and conventional models, respectively.](image)

**FIG. 4**

V. GROUP VELOCITY, POYNTING VECTOR AND REFRACTED WAVES IN DWM

In this section we will discuss the group velocity and Poynting vectors of waves excited in a double wire medium and check the power conservation at the interface. It is well known that the
FIG. 5: Real and imaginary parts of the reflection coefficient versus the normalized wavevector \( k/k_p \). The solid line shows \( \text{Re}(R_E) \), the dashed line shows \( \text{Im}(R_E) \) (the new model), the dash-dot and dot lines correspond to \( \text{Re}(R_E) \) and \( \text{Im}(R_E) \), respectively, obtained in the framework of the old model.

The group velocity is defined as

\[
v_{gr} = \nabla_{\kappa} \omega.
\] (38)

The Poynting vector that determines the energy density flow for media with spatial dispersion has the form [17]:

\[
S = \frac{1}{2} \text{Re}\{E \times H^*\} - \frac{\omega}{4} \frac{\partial \epsilon_{ik}}{\partial k} E_i^* E_k,
\] (39)

where the permittivity dyadic components are expressed by formulae [2] for DWM. Their partial derivatives read

\[
\frac{\partial \epsilon_x}{\partial k_x} = 0, \quad \frac{\partial \epsilon_y}{\partial k_y} = -\frac{2k_p^2k_y}{(k^2 - k_y^2)^2}, \quad \frac{\partial \epsilon_z}{\partial k_z} = -\frac{2k_p^2k_z}{(k^2 - k_z^2)^2}.
\] (40)

Let us derive expressions for the Poynting vector in free space and in the wire medium. In free space the field expressions are

\[
H_1 = H_0[e^{-j\beta_0z} + R_H e^{j\beta_0z}]e^{-jk_y y}u_x
\] (41)

\[
E_1 = \beta_0 H_0 \left[-e^{-j\beta_0z} + R_H e^{j\beta_0z}\right]e^{-jk_y y}u_y + \frac{k_y H_0}{k_0} \left[e^{-j\beta_0z} + R_H e^{j\beta_0z}\right]e^{-jk_y y}u_z.
\] (42)

The fields of the waves, marked by + and −, in the wire medium are

\[
H_{2\pm} = \frac{s_0}{2} \frac{\beta_\pm}{k_{z\pm}} e^{-jk_{z\pm}z} e^{-jk_y y} u_x,
\] (43)

\[
E_{2\pm} = -\frac{s_0 \eta}{2k_0 \epsilon_y} \beta_\pm e^{-jk_{z\pm}z} e^{-jk_y y} u_y + \frac{k_y s_0 \eta}{2k_0} \frac{\beta_\pm}{k_{z\pm} \epsilon_{z\pm}} e^{-jk_{z\pm}z} e^{-jk_y y} u_z.
\] (44)
Now we can write the Poynting vector in free space and in the wire medium (at the interface):

\[
S_1(0) = \frac{1}{2} \text{Re}\{E_1 \times H_1^*\} = \frac{|H_0|^2 \eta}{2k_0} \left[ \beta_0 (1 - |R_H|^2) u_z + k_y (1 + 2|R_H| \cos \phi + |R_H|^2) u_y \right]
\]

(45)

with the notation \(R_H = |R_H| e^{j\phi}\). In the wire medium, the cross product term is

\[
S_0^0(0) = \frac{1}{2} \text{Re}\{E_2 \times H_2^*\} = \frac{|s_0|^2 \eta}{2k_0} \frac{1}{4} \text{Re}\left\{ \frac{1}{\epsilon_y} \frac{\beta_0 \beta_0^* - \beta_0^* \beta_0}{k_{z \pm}^2} u_z + k_y \frac{\beta_0 \beta_0^*}{k_{z \pm}^2 k_{z \pm}^* \epsilon_{z \pm}} u_y \right\}.
\]

(46)

and the spatial dispersive term is

\[
S_2^d(0) = \frac{|s_0|^2 \eta}{2k_0} \frac{1}{4} \left[ \frac{k_p^2 k_y^2 k_{z \pm}^2 \beta_0^*}{(k^2 - k_{z \pm}^2)^2 k_{z \pm}^* \epsilon_{z \pm} \epsilon_{z \pm}^*} u_z + \frac{k_p^2 k_y^2}{(k^2 - k_{z \pm}^2)^2 \epsilon_y^2} \beta_0^* \beta_0^* u_y \right].
\]

(47)

The total Poynting vector in wire medium is

\[
S_{2\pm}(0) = S_{2\pm}^0(0) + S_{2\pm}^d(0).
\]

(48)

As an example, we consider excitation of modes, located above the plasma frequency \[2\]. For the taken parameters of DWM, the wire radius \(r_0 = 0.01\) cm, and the period \(L = 1\) cm, the plasma wavenumber is \(k_p \approx 1.38\) cm\(^{-1}\). Dispersion diagram in form of isofrequencies and directions of the Poynting vector (the energy velocity) for these modes with respect to the normal to the interface (\(z\)-axis) are shown in Fig. 6. The calculations were performed at \(k/k_p = 1.35\). The value of the tangential component \(k_y\) is determined by the incidence angle \(\theta\), namely, \(k_y = k \sin \theta\). The direction of the group velocity, found by numerical differentiation of the dispersion characteristics, exactly coincides with the direction of \(S_{2\pm}\), calculated using formulae (46)–(48). Disregarding the term that takes into account spatial dispersion leads to a strongly incorrect result, illustrated by dashed curves.

\[\text{FIG. 6: 1. Isofrequencies, dotted curves. 2. Solid curves show the angle } \psi \text{ between the energy velocity and the } z \text{ axis versus } k_y. 3. Dashed curves show the angle between } S_{2\pm}^0(0) \text{ and the } z \text{ axis versus } k_y.\]

Next, let us consider the power conservation at the interface of free space and the wire medium. It is important to check this properly because in the wire medium region there exist two waves, and the wire medium is spatially dispersive. In the frequency range considered here the parameter
values are assumed to be real. The normal components of the Poynting vector on both sides of
the interface are (using the continuity condition of the tangential field components)

\[ S_{z1}(0) = \frac{|s_o|^2}{8\omega\epsilon_o\epsilon_y} \left( \frac{\beta_+}{k_{z+}} + \frac{\beta_-}{k_{z-}} \right), \] (49)

and

\[ S_{z2}(0) = \frac{|s_o|^2}{8\omega\epsilon_o} \left[ \frac{1}{\epsilon_y} \left( \frac{\beta_+^2}{k_{z+}} + \frac{\beta_-^2}{k_{z-}} \right) + \frac{k_p^2k_y^2\beta_+^2}{(k^2 - k_{z+}^2)^2\epsilon_{z+}^2k_{z+}} + \frac{k_p^2k_y^2\beta_-^2}{(k^2 - k_{z-}^2)^2\epsilon_{z-}^2k_{z-}} \right]. \] (50)

Subtracting these power density components leads to expression

\[ S_{z1}(0) - S_{z2}(0) = \frac{|s_o|^2}{8\omega\epsilon_o} \left( \frac{1}{k_{z+}} + \frac{1}{k_{z-}} \right) \left[ \frac{\beta_+\beta_-}{\epsilon_y} - \frac{k_p^2k_y^2}{(k^2 - k_{z-}^2)^2\epsilon_{z-}^2k_{z-}} \right]. \] (51)

Using the expressions for \( \beta_{\pm}, k_{z\pm}^2 \) and \( \epsilon_y \), we find that the term inside the square brackets vanishes.
The normal component of the Poynting vector is continuous across the interface, which means that
the power conservation law is satisfied.

The normal to the interface components of the Poynting vectors of the refracted waves as well
as the reflection coefficient (which is purely real beyond \( K_1 \)) are shown in Fig. 7. The values of
\( S_{z+} \) and \( S_{z-} \) are normalized to the power density of the incident plane wave \( |S_i| = \frac{1}{2}\eta H_0^2 \cos \theta \).
These results agree with those shown in Fig. 3, namely, \( S_{z-} \) becomes zero at point \( k/k_p = K_2 \). The point \( K_2 \) is a transition point of the “ – ” wave because it is a backward wave with respect
to the interface if \( k/k_p < K_2 \), and a forward wave if \( k/k_p > K_2 \).

![FIG. 7: Reflection coefficient \( R_H \) (dotted curve), \( S_{z+} \) (solid curve) and \( S_{z-} \) (dashed curve) versus the normalized wave vector.](image)

Angular dependence of the normal components of the Poynting vector is presented in Fig. 8. Both refracted modes are TM modes in this case. At low frequencies (\( k/k_p < 1 \)) a plane wave
cannot excite propagating TM modes at any incidence angle \( \theta \), so we present here the results for
\( k/k_p = 1.5 \). There exists a critical angle \( \theta_c \approx 0.3\pi \), such that \( S_{z1} = S_{z2} = 0 \) if \( \theta > \theta_c \). As in the
previous case, \( R_H = +1 \) if \( \theta = \theta_c \), and \( R_H \rightarrow 1 \) if \( \theta \rightarrow \pi/2 \), so DWM is an electric wall for grazing incidence.
A. Poynting vector in single wire media

The Poynting vector in a single wire medium can be simply deduced from the expressions for the double wire media. In this case the partial derivatives of the permittivity components read

$$\frac{\partial \varepsilon_x}{\partial k_x} = 0, \quad \frac{\partial \varepsilon_y}{\partial k_y} = 0, \quad \frac{\partial \varepsilon_z}{\partial k_z} = -\frac{2k_p^2 k_z}{(k^2 - k_z^2)^2}. \quad (52)$$

The additional term arises now only in the z component of the Poynting vector, and it is equal to

$$S_{2\pm}^d = |s_o|^2 \eta \frac{k_p^2}{2k_o} u_z \left[ k_y + k_z \left( \frac{1}{\varepsilon_y} + \frac{k_p^2 k_y^2}{(k^2 - k_z^2)^2} \right) u_z \right]. \quad (53)$$

Thus, the total Poynting vector reads for real parameter values

$$S_{2\pm} = \frac{|s_o|^2 \eta k_p^2}{2k_o} \beta_{\pm} \frac{k_z}{\varepsilon_{\pm}} + \frac{k_z^2}{\varepsilon_{\pm}} \left[ k_y u_y + k_x \left( \frac{1}{\varepsilon_y} + \frac{k_p^2 k_y^2}{(k^2 - k_z^2)^2} \right) u_z \right]. \quad (54)$$

This gives us after substituting $k_{z\pm}$ and $\beta_{\pm}$

$$S_{2+} = \frac{|s_o|^2 \eta k_p^2}{2k_o} \frac{k k_z^2}{4(k_y^2 + k_p^2)} u_z \quad (55)$$

and

$$S_{2-} = \frac{|s_o|^2 \eta k_y^2 + k_p^2}{2k_o} \frac{k_p^2}{4(k^2 - k_z^2 - k_p^2)} u_z. \quad (56)$$

It is known that for each eigenwave the group velocity in single wire media is parallel to the phase velocity. From the above expressions one can see that the Poynting vector is also in the same direction as the group velocity (and the phase velocity). The continuity condition of the power flow across the interface (continuity of the normal component of the Poynting vector) can be easily checked similarly as for double wire media.

Fig. 9 illustrates the angular dependence of the reflection coefficient $R_H$ and the normal to the interface component of the Poynting vector $S_z$ for the TEM mode ($S_{TEM}$). Parameters of the WM are taken the same as in the case of DWM. The amplitude of the magnetic field of the incident
FIG. 9: The real and imaginary parts of $R_H$ (dashed and dotted curves, respectively), $S_{TEM}$ (solid curves) versus the incidence angle, calculated at $k/k_p = 0.5$.

FIG. 10: The real and imaginary parts of $R_H$ (dashed and dotted curves, respectively), $S_{TEM}$ and $S_{zTM}$ (solid curves) versus the incidence angle, calculated at $k/k_p = 1.5$.

wave is $H_0 = 1$, and the calculated Poynting vector is normalized to the incident wave power flow density. Wavenumber $k/k_p$ corresponds to the frequency below the plasma resonance and the TM mode cannot be excited at any $\theta$. For grazing incidence $R_H \to -1$. It is obvious that in this case the surface must behave as a wall, because no traveling waves in the medium can be excited. Actually, it behaves as a magnetic wall, because in this geometry the wires are orthogonal to the interface, and the normal component of the electric field is zero.

More complicated is the case when the wavenumber is taken beyond the plasma resonance, i.e. $k/k_p = 1.5$, see Fig. 10. However, it follows from Eq. (2), that for oblique wave incidence, the “effective” plasma wavenumber becomes $k_p' = k_p/\cos \theta$ because $k_y = k \sin \theta$. For grazing incidence $R_H \to -1$, similarly to the previous case.

For small incidence angles both TEM and TM modes are excited. There exists an incidence angle $\theta'$, which is the angle of total reflection. It takes place when $k = k_p'$, so $\cos \theta' = 2/3$. Neither TEM nor TM modes are excited, and $R_H \to 1$, which corresponds to the case of an electric wall. For larger $\theta$ the TM mode disappears.
VI. CONCLUSIONS

Considering the problem of a plane wave refraction at the interface of a double wire medium, exhibiting strong spatial dispersion, we have shown that Pekar’s additional boundary conditions are not applicable for its solution. We have analyzed the literature discussion on the ABC problem and have come to the point of view, that Hennenberger’s approach \[15\] for SD media can be applied for some kinds of metamaterials in the microwave range including single and double wire media.

Despite of criticism of the used ABC-free method, see \[18, 19\] and also replies \[20\], we suppose that application of this approach has allowed us to overcome the problem of additional waves appearing in media with spatial dispersion and to obtain reasonable results for reflection coefficient and for the power densities and group velocities of the refracted waves. Fulfilment of the conservation of the power, passing through the interface, can be considered as an evidence of the correctness of this approach.

Possible applications of DWM include antenna structures, low-frequency filters, frequency selective radomes, double negative metamaterials.

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