CERTAIN UNITARY LANDLINGS-VOGAN PARAMETERS FOR SPECIAL ORTHOGONAL GROUPS I

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Abstract. A Langlands parameter, in the Langlands dual group, can be decomposed into a product of a tempered parameter and a positive quasi-character. Fixing a tempered parameter, Arthur conjectured that positive quasi-characters corresponding to certain weighted Dynkin diagrams for the centralizer of the tempered parameter will yield unitary representations. In this paper, we treat one basic case in Arthur’s conjecture for the special orthogonal groups. We establish the unitarity for a class of Langlands-Vogan parameters in Arthur’s packet.

Introduction

Let $G$ be a semisimple Lie group. Let $\Pi_u(G)$ be the unitary dual of $G$. Let $\Pi(G)$ be the admissible dual of $G$. Fix a minimal parabolic subgroup $P_m$. Let $P$ be a parabolic subgroup containing $P_m$. Let $MAN$ be the Langlands decomposition of $P$. Langlands showed that $\Pi(G)$ is in one-to-one correspondence with a triple $(P, \sigma, v)$, where $\sigma$ is the (infinitesimal) equivalence class of an irreducible tempered representation of $M$ and $v$ is a complex-valued linear functional on $a$ such that its real part $\Re(v)$ is in the open positive Weyl chamber of $a^*$. The representation corresponding to $(P, \sigma, v)$ is $J(P, \sigma, v)$, the Langlands quotient. By a theorem of Harish-Chandra, $\Pi_u(G)$ can be regarded as a subset of $\Pi(G)$. We shall say that a Langlands parameter $(P, \sigma, v)$ is unitary or unitarizable if $J(P, \sigma, v)$ is unitarizable.

Irreducible tempered representations are all unitarizable. Therefore their equivalence classes, the tempered dual, can be regarded as a subset of $\Pi_u(G)$. The classification of the tempered dual is completed by Knapp and Zuckerman (KZ). Fix a tempered parameter $\sigma$, the problem of determining the unitary dual $\Pi_u(G)$ can be approached by determining those $v$ such that $J(P, \sigma, v)$ is unitarizable. This set of $v$ is necessarily bounded. Arthur conjectured that if $v$ corresponds to a certain type of weighted Dynkin diagrams, $J(P, \sigma, v)$ will be unitary. In this paper, we shall treat one basic case in Arthur’s conjecture for special orthogonal groups. We shall also give some indication how other cases can be studied.

Before we state our main results. Let me introduce some notations. Let $G = SO(p, q)$. Suppose that $0 < p \leq q$. Fix a maximal compact subgroup $K$. Then $K \cong SO(p)SO(q)$. $K$ is disconnected. Let $(\xi) \in \Pi_u(SO(p))$ and $(\eta) \in \Pi_u(SO(q))$ as in $2.2$. If $(\xi) \otimes (\eta)$ extends to a representation of $K$, then we obtain two such extensions $(\xi, \eta, \pm) \in \Pi_u(K)$. We use the convention set in [KV] to mark these two different extensions. If $(\xi) \otimes (\eta)$ does not extend to a representation of $K$, then there is a representation $([\xi], |\eta|, +) \in \Pi_u(K)$ such that $([\xi], |\eta|, +)|_{SO(p)SO(q)}$ contains a copy of $(\xi) \otimes (\eta)$. The unitary dual of $SO(p)SO(q)$ can be parametrized by $(\xi, \eta, \pm)$ and $([\xi], |\eta|, +)$. If $p = 0$, we will write $(\eta)$ as $(0, \eta, +)$. See $2.2$ for details.

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1This research is partially supported by the NSF grant DMS 0700809.
2keywords: Unitary representations, invariant tensor product, leading exponents, tempered representations, orthogonal groups, Langlands Classification, degenerate principal series, minimal K-type, Arthur’s Packet, Vogan subquotient, Howe’s correspondence, dual reductive pair, unitary dual, admissible dual, Harish-Chandra Module
Given a real vector \( \mu = (\mu_1, \mu_2, \ldots, \mu_n) \), let \( \Omega \) be the rearrangement of \( \mu \) in descending order
\[
\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n.
\]
Let \( |\mu| = (|\mu_1|, |\mu_2|, \ldots, |\mu_n|) \). For any constant \( C \), let \( c \) be the constant vector \( (c, c, \ldots, c) \) of a proper dimension. All general linear groups in this paper are real general linear groups, denoted by \( GL(n) \).

**Theorem 0.1.** Let \( \sigma \) be an irreducible tempered representation of \( SO(p-d,q-d) \) such that one of its minimal \( S(O(p-d)O(q-d)) \)-types is \( (\xi, \eta, +) \) (Vogan). Then the Langlands-Vogan subquotient of \( \text{Ind}^{SO(p,q)}_{SO(p-d,q-d)GL(1)^dN} \sigma \otimes \prod_{i=1}^{d} |\det|^{\nu_i} \) is unitary where

1. \( \nu \) is the semisimple element in \( so(2d, \mathbb{C}) \) corresponding to \( \text{diag}(\frac{1}{2}, -\frac{1}{2}) \) under a Lie algebra homomorphism \( sl(2, \mathbb{C}) \to so(2d, \mathbb{C}) \) when \( p+q \) is even;
2. \( \nu \) is the semisimple element in \( sp(2d, \mathbb{C}) \) corresponding to \( \text{diag}(\frac{1}{2}, -\frac{1}{2}) \) under a Lie algebra homomorphism \( sl(2, \mathbb{C}) \to sp(2d, \mathbb{C}) \) when \( p+q \) is odd.

Here \( \nu \) is parametrized by \( \mathbb{R}^d \) and \( |\det| \) is simply the absolute value of \( g \in GL(1) \).

If \( d \neq 0 \), the Langlands-Vogan subquotient is defined to be the irreducible subquotient of
\[
\text{Ind}^{SO(p,q)}_{SO(p-d,q-d)GL(1)^dN} \sigma \otimes \prod_{i=1}^{d} |\det|^{\nu_i}
\]
containing a minimal \( K \)-type \( (|\xi| \oplus 0, |\eta| \oplus 0, +) \). If \( (p, q) \neq (d, d) \), it is equivalent to the Langlands quotient of
\[
\text{Ind}^{SO(p,q)}_{SO(p-d,q-d)GL(1)^dN} \sigma \otimes \prod_{i=1}^{d} |\det|^{\Omega_i}
\]
When \( \sigma \) is trivial, we obtain unipotent representations. In this case, our result overlaps with Baru (for \( p = q, q - 1 \)) and Quan.

This paper is organized as follows. In Section 1, we define the invariant tensor product of two representations weakly as distributions (Definitions \[1.1\] \[1.9\]). We prove that invariant tensor product may inherit Lie group actions (Lemma \[1.11\]), Lie algebra actions (Lemma \[1.16\]) and Hermitian forms (Prop. \[1.20\]). In Section 2, we review the basic theory of induced representations, Langlands classification, Vogan’s subquotients and growth of the matrix coefficients. In Section 3, we review the known results on the degenerate principal series \( I_n(s) \) and its unitarizable subquotients, due to Johnson (\[HO\]), later reworked by Sahi (\[Sahi2\]), Zhang (\[ZH\]) and Lee (\[LL\]). We give some additional analysis on the small constituents \( E_m(n) \) with \( 0 \leq m \leq \lfloor \frac{n}{2} \rfloor \).

In Section 4, we define the quantum induction \( Q(2m)(\pi) \) on the Harish-Chandra module level to be \( V(E_m(p+q+d)) \otimes SO(p,q) V(\pi) \). Here \( V(\cdot) \) stands for the Harish-Chandra module and \( \pi \in \Pi(SO(p,q)) \) satisfies a certain growth condition. When \( \pi \) is unitary, \( Q(2m)(\pi) \) inherits a Harish-Chandra module structure of \( SO(q+d, p+d) \) from \( E_m(p+q+d) \). Suppose that \( p+q \leq 2m+1 \leq p+q+d \). The main task in Section 4 is to show that the canonical invariant Hermitian form for \( Q(2m)(\pi) \) is positive definite under a growth condition on \( \pi \) (Theorem \[4.2\]).

In Section 5, we show that \( Q(2m)(\pi) \) is a subrepresentation of \( \text{Ind}^{SO(q+d,p+d)}_{SO(q,p)GL(1)^dN} \pi \otimes |\det|^{m - \frac{p+q}{2}} \) if we regard \( \pi \) as a representation of \( SO(q,p) \) (Theorem \[5.7\]).

In Section 6, we relate \( Q(2m) \) to the composition of Howe’s correspondence (\[Ho89\]). Some caution is taken to handle the difference between \( SO(p,q) \) and \( O(p,q) \). Suppose that \( p+q \leq 2m+1 \leq p+q+d \).
We apply Kudla’s preservation principle and results of Moeglin and Adam-Barbasch to show the nonvanishing of 
\( Q(2m) \pi \) (\([\text{Ku}] \ [\text{MO}] \ [\text{AB}] \ [\text{Henon}]\)). See Theorem 6.5. We then apply Howe’s theory (\([\text{Ho89}] \) and results of \([\text{MO}] \ [\text{AB}] \ [\text{Paul}]\)) to determine a minimal \( K \)-type of \( Q(2m) \pi \). Our assumption that \( \pi \) has a \( K \)-type of the form \( (\xi, \eta, +) \) is essential to guarantee that the minimal \( K \)-type is a \( K \)-type of minimal degree. This allows us to determine the Langlands-Vogan parameter of \( Q(2m) \pi \). See Theorem 6.6.

Finally, in Section 7, we review some basic facts about Arthur’s packets and show that quantum induction can be applied inductively to obtain unitary Langlands-Vogan parameters. Perhaps, the following theorem is worth mentioning.

**Theorem 0.2.** Suppose that \( p + q \leq 2m + 1 \leq p + q + d \) and \( p \leq q \). Let \( \pi \) be an irreducible unitary representation of \( SO(q, p) \) such that its every \( K \) finite matrix coefficient \( f(g) \) satisfies the condition that

\[
|f(k_1 \exp H(g) k_2)| \leq C_f \exp \left(m + 2 - p - q - \epsilon, m + 3 - p - q, \ldots, m + 1 - q\right) |H(g)|
\]

for some \( \epsilon > 0 \). Here \( k_1, k_2 \in S(O(q)O(p)) \) and \( H(g) \in a_m \cong \mathbb{R}^p \). Then there exists a unitaizable subrepresentation of \( \text{Ind}_{SO(q+p)}^{SO(q+d,p+d)}(\pi \otimes |\det|^{m - p - q - \frac{d}{2}}) \).

Here \( p \leq q \) is not essential. See Theorem A in Section 7 for more detail.

0.1. **Notations.** Write \( \mu \prec \lambda \) if for every \( k \in [1, n] \)

\[
\sum_{i=1}^{k} \mu_i < \sum_{i=1}^{k} \lambda_i.
\]

Write \( \mu \preceq \lambda \) if for every \( k \in [1, n] \)

\[
\sum_{i=1}^{k} \mu_i \leq \sum_{i=1}^{k} \lambda_i.
\]

Let \( \mathbb{Z} \) be the set of integers. Let \( \mathbb{Z} + \frac{1}{2} \) be the set of half integers.

All topological groups and topological vector spaces are assumed to be Hausdorff. Let \( \mathcal{X} \) be a topological vector space (TVS). Let \( \mathcal{L}(\mathcal{X}) \) be the space of continuous linear operators on \( \mathcal{X} \). A linear representation \( \pi \) of \( G \) on \( \mathcal{X} \), is said to be continuous if the group action \( G \times \mathcal{X} \to \mathcal{X} \) is continuous. The TVS \( \mathcal{X} \) here needs not to be complete.

If \( \mathcal{X} \) is a Hilbert space, then \((\pi, \mathcal{X})\) is called a Hilbert representation. All Hilbert spaces in this paper are assumed to be separable. All Hilbert representations are assumed to be continuous.

For any complex vector space \( V \), let \( V^c \) be the vector space \( V \), regarded as a real vector space, equipped with the conjugated complex multiplication,

\[
c.u \in V^c = \overline{u} \in V \quad (c \in \mathbb{C}, u \in V^c).
\]

Then \( V^c \) is a complex linear vector space. \( V^c = V \) as real vector spaces. But \( V^c \) and \( V \) have different complex structures.

A Maximal compact subgroup of a semisimple Lie group may be universally denoted by \( K \). The nilradical of a parabolic subgroup of a semisimple group will be universally denoted by \( N \). \( C \) will be used as a universal constant. Identity operators or matrices will be denoted by \( I \) or \( I_n \) where \( n \).
is the dimension.

All the important conventions will be highlighted by boldfaced letters. All inner products or Hermitian forms will be denoted by \((,\)\). The definition of these forms will be clear within the context.

Let \(G\) be a unimodular Lie group. Let \(X\) be a principal \(G\)-bundle with a \(G\)-invariant measure. Let \(H\) be a unitary representation of \(G\). Then \(L^2(X \times_G H, X/G)\) is a Hilbert space. The Hilbert inner product is given by

\[
(\phi, \psi) = \int_{[x] \in X/G} (\phi(x), \psi(x))d[x].
\]

Whenever we have an integral of this form, the integrand does not depend on the choices of \(x\) in \([x]\).

1. Invariant Distributions and Invariant Tensor Product

1.1. Invariant Distributions and Averaging Operators. Let \(G\) be a locally compact group. Let \((\pi, X)\) be a linear representation of \(G\). Let \(u \in X\). \(u\) is called a \(G\)-invariant vector if \(\pi(g)u = u\) for any \(g \in G\). We denote the space of invariants of \(X\) by \(X^G\).

Let \(\pi\) be a continuous representation of \(G\) on a locally convex TVS \(X\). Let \(X^*\) be the dual space of \(X\) equipped with the weak-* topology. The continuous representation \((\pi, X)\) induces a continuous representation \((\pi^*, X^*)\) as follows. For any \(\delta \in X^*, v \in X, g \in G\), define

\[
\pi^*(g)(\delta)(v) = \delta(\pi(g)^{-1}v).
\]

This is the dual representation. Obviously, \(\pi^*(g)(\delta) \in X^*\) and \(G \times X^* \rightarrow X^*\) is continuous.

Given \(\delta \in X^*\) and \(v \in X\), the matrix coefficient

\[
g \rightarrow \delta(\pi(g)v)
\]

is a continuous function on \(G\). Denote it by \(\pi_{v,\delta}(g)\).

**Definition 1.1.** Let \(G\) be a locally compact unimodular group with finite center. Suppose that there is a subspace \(Y\) of \(X^*\), and a subspace \(X_0\) of \(X\) such that \(\pi_{v,\delta}(g) \in L^1(G)\) for any \(\delta \in Y\) and any \(v \in X_0\). Then we define a map

\[
A_{G,Y}: X_0 \rightarrow \text{Hom}(Y, \mathbb{C})
\]

by \(\forall u \in X_0,\)

\[
A_{G,Y}(u)(\delta) = \int_G \delta(\pi(g)u)dg, \quad (\delta \in Y).
\]

We call \(A_{G,Y}\) the averaging operator with respect to \(Y\).

**Lemma 1.2.** If \(Y\) is \(G\)-stable, then the image of the averaging operator

\[
A_{G,Y}(X_0) \subseteq \text{Hom}(Y, \mathbb{C})^G.
\]

Proof: For any \(u \in X_0, \delta \in Y, g \in G\), we have

\[
A_{G,Y}(u)(\pi^*(g)\delta) = \int_{h \in G} \pi^*(g)\delta(\pi(h)u)dh = \int_{h \in G} \delta(\pi(g^{-1})\pi(h)u)dh = \int_{h \in G} \delta(\pi(h)u)dh = A_{G,Y}(u)(\delta).
\]
We shall remark that whether \( \mathcal{Y} \) is \( G \)-stable or not is not essential. Consider

\[ ^G\mathcal{Y} = \text{span}\{\pi^*(g)(\delta) \mid g \in G, \delta \in \mathcal{Y}\}. \]

Then \( \mathcal{A}_{G, \mathcal{Y}}(\chi_0) \) is well-defined and

\[ \mathcal{A}_{G, \mathcal{Y}}(\chi_0) \subseteq \text{Hom}(^G\mathcal{Y}, \mathbb{C})^G. \]

1.2. \textbf{\( L^1 \)-mutually Dominated Subspaces}. Naturally, \( \mathcal{A}_{G, \mathcal{Y}}(\chi_0) \) shall be interpreted as a linear subspace of linear functions on \( \mathcal{Y} \). The averaging operator will then depend on the choices of the subspace \( \mathcal{Y} \) in \( \mathcal{X}^* \). This is inconvenient in applications. There is an interpretation of \( \mathcal{A}_G \) that is somewhat independent of the choices of \( \mathcal{Y} \) as I shall explain.

Notice that

\[ \mathcal{A}_{G, \mathcal{Y}}(\chi_0) \cong \chi_0 / \ker(\mathcal{A}_{G, \mathcal{Y}}). \]

If for two different choices \( \mathcal{Y}_1 \) and \( \mathcal{Y}_2 \), we can show that

\[ \ker(\mathcal{A}_{G, \mathcal{Y}_1}) = \ker(\mathcal{A}_{G, \mathcal{Y}_2}), \]

then the image \( \mathcal{A}_{G, \mathcal{Y}_1}(\chi_0) \) can be identified with \( \mathcal{A}_{G, \mathcal{Y}_2}(\chi_0) \).

\textbf{Definition 1.3.} Fix \( \chi_0 \subseteq \mathcal{X} \). Let \( \mathcal{Y}_1, \mathcal{Y}_2 \) be two subspaces of \( \mathcal{X}^* \). We say that \( \mathcal{Y}_1 \) is \( L^1 \)-dominated by \( \mathcal{Y}_2 \) (with respect to \( \chi_0 \)), if for every \( u \in \chi_0 \) and \( \delta \in \mathcal{Y}_1 \), there is a sequence \( \{f_n\} \subset \mathcal{Y}_2 \) such that

\[ \pi_{u, f_n}(g) \to \pi_{u, \delta}(g) \quad (\forall g \in G) \]

and \( \{\pi_{u, f_n}(g)\} \) are uniformly dominated by an \( L^1 \)-function on \( G \). If \( \mathcal{Y}_2 \) is also \( L^1 \)-dominated by \( \mathcal{Y}_1 \), we say that \( \mathcal{Y}_1 \) and \( \mathcal{Y}_2 \) are mutually \( L^1 \)-dominated subspaces of \( \mathcal{X}^* \) with respect to \( \chi_0 \).

\textbf{Remark 1.4.} If \( \mathcal{Y}_1 \) are the (finite) linear span of a set of vectors, it suffices that the condition in Definition 1.3 holds for \( \delta \) in the spanning set. Same is true for a spanning set of vectors in \( \chi_0 \).

\textbf{Theorem 1.5.} Suppose that \( \mathcal{Y}_1 \) and \( \mathcal{Y}_2 \) are mutually \( L^1 \)-dominated subspaces of \( \mathcal{X}^* \) with respect to \( \chi_0 \). Then

\[ \ker(\mathcal{A}_{G, \mathcal{Y}_1}) = \ker(\mathcal{A}_{G, \mathcal{Y}_2}), \]

and \( \mathcal{A}_{G, \mathcal{Y}_1}(\chi_0) \) can be identified with each other cannonically.

\textbf{Proof:} Suppose \( u \in \ker(\mathcal{A}_{G, \mathcal{Y}_2}) \subset \chi_0 \). Then for any \( f \in \mathcal{Y}_2 \), we have

\[ \mathcal{A}_{G, \mathcal{Y}}(u)(f) = \int_G f(\pi(g)u)dg = \int \pi_{u, f}(g)dg = 0. \]

Let \( \delta \) be arbitrary in \( \mathcal{Y}_1 \). Since \( \mathcal{Y}_1 \) is \( L^1 \)-dominated by \( \mathcal{Y}_2 \), there is a sequence \( \{f_n\} \subset \mathcal{Y}_2 \) such that \( \pi_{u, f_n}(g) \to \pi_{u, \delta}(g) \) and \( \pi_{u, f_n}(g) \) are uniformly dominated by an \( L^1 \)-function on \( G \). Since

\[ \int_G \pi_{u, f_n}(g)dg = 0, \] by the dominated convergence theorem, \( \int_G \pi_{u, \delta}(g)dg = 0 \). Hence \( u \in \ker(\mathcal{A}_{G, \mathcal{Y}_1}) \).

We have shown that

\[ \ker(\mathcal{A}_{G, \mathcal{Y}_2}) \subseteq \ker(\mathcal{A}_{G, \mathcal{Y}_1}). \]

The converse can be proved in the same way. Our assertions follow. \( \square \)
1.3. Averaging Operator $\mathcal{A}_G$ for Hermitian Representations.

**Definition 1.6.** Let $(\pi, \mathcal{X})$ be a continuous representation of $G$ on a TVS $\mathcal{X}$. If $\mathcal{X}$ is equipped with a nondegenerate Hermitian form, we say that $(\pi, \mathcal{X})$ is a Hermitian representation of $G$. Let $V$ be a subspace of $\mathcal{X}$ such that for any $v_1, v_2 \in V$,

$$\pi_{v_1,v_2} : g \in G \to (\pi(g)v_1,v_2)$$

is in $L^1(G)$. Define

$$\mathcal{A}_G : V \to \text{Hom}(V^c, \mathbb{C})$$

by

$$\mathcal{A}_G(v_1)(v_2) = \int_G (\pi(g)v_1,v_2)dg \quad (v_1 \in V, v_2 \in V^c).$$

Let $(\pi, \mathcal{H})$ be a (continuous) Hilbert representation of $G$. Let $(\pi^*, \mathcal{H}^*)$ be the contragredient Hilbert representation. Let $\mathcal{H}^c$ be $\mathcal{H}$, equipped with the conjugate complex structure

$$c.u \in \mathcal{H}^c = \overline{u} \in \mathcal{H} \quad (c \in \mathbb{C}).$$

Clearly, $\pi(g)$ acts on $\mathcal{H}^c$ preserving the complex structure. So it defines a Hilbert representation, denoted by $(\pi^c, \mathcal{H}^c)$. If $\pi$ is unitary, we have

$$(\pi^c, \mathcal{H}^c) \cong (\pi^*, \mathcal{H}^*),$$

essentially by the Riesz representation theorem.

**Remark 1.7.** Suppose that there is a subspace $V \subseteq \mathcal{H}$ such that for any $v_1, v_2 \in V$,

$$\pi_{v_1,v_2} : g \in G \to (\pi(g)v_1,v_2)$$

is in $L^1(G)$. Then $\mathcal{A}_G : V \to \text{Hom}(V^c, \mathbb{C})$ is well-defined. On the other hand, by Riesz representation theorem, the inner product on $\mathcal{H}$ induces a topological embedding $V^c \hookrightarrow \mathcal{H}^*$. Thus $\mathcal{A}_G(V)$ is well-defined. Clearly, $\mathcal{A}_G(V) = \mathcal{A}_{G,V^c}(V)$. Notice here that the group $G$ acts on $V^c$ via $\pi^c$. Unless stated otherwise, $V^c$ will be equipped with the action of $\pi^c(G)$.

If $V$ is $G$-invariant, then $\mathcal{A}_G(V) = \mathcal{A}_{G,V^c}(V)$ will be a subspace of $\text{Hom}(V^c, \mathbb{C})$. Generally speaking $V^c$ will not be $\pi^c(G)$-invariant unless $(\pi|_V, V)$ is unitary. So the image $\mathcal{A}_G(V)$ may not be in the $G$-invariant subspace of $\text{Hom}(V^c, \mathbb{C})$.

**Example 1.8.** Let $G$ be a compact group equipped with the probability measure. Let $(\pi, \mathcal{H})$ be a unitary representation of $G$. Then $\mathcal{A}_G$ is defined for the whole space $\mathcal{H}$. The space

$$\mathcal{A}_G(\mathcal{H}) = \text{Hom}(\mathcal{H}^c, \mathbb{C})^G.$$

The right hand side can be identified with $\mathcal{H}^G$. The operator $\mathcal{A}_G$ is the projection operator onto the trivial-isotypic subspace. Consequently, for $G$ compact, the averaging integral $\mathcal{A}_G(v)$ can simply be defined as $\int_G \pi(g)v dg$. This can be regarded as the strong form of the averaging operator.

1.4. Invariant Tensor Products.

**Definition 1.9.** Let $G$ be a unimodular group. Let $(\pi, \mathcal{X})$ and $(\tau, \mathcal{Z})$ be two continuous representations of $G$. Suppose that there are subspaces $\mathcal{X}_0 \subseteq \mathcal{X}, \mathcal{Z}_0 \subseteq \mathcal{Z}$ and subspaces $\mathcal{Y} \subseteq \mathcal{X}^*, \mathcal{W} \subseteq \mathcal{Z}^*$ such that the matrix coefficients

$$\pi_{\mathcal{X}_0,\mathcal{Y}}(g)\tau_{\mathcal{Z}_0,\mathcal{W}}(g) \subseteq L^1(G).$$

We define

$$\mathcal{X}_0 \otimes_{G,\mathcal{Y} \otimes \mathcal{W}} \mathcal{Z}_0 = \mathcal{A}_{G,\mathcal{Y} \otimes \mathcal{W}}(\mathcal{X}_0 \otimes \mathcal{Z}_0) \subseteq \text{Hom}(\mathcal{Y} \otimes \mathcal{W}, \mathbb{C}).$$

We call $\mathcal{X}_0 \otimes_{G,\mathcal{Y} \otimes \mathcal{W}} \mathcal{Z}_0$ the invariant tensor product of $\mathcal{X}_0$ and $\mathcal{Z}_0$ with respect to $\mathcal{Y} \otimes \mathcal{W}$.

Similarly, we can define $\mathcal{Y} \otimes_{G,\mathcal{X}_0 \otimes \mathcal{Z}_0} \mathcal{W}$ since $\mathcal{X}_0 \subseteq \mathcal{X} \subseteq (\mathcal{X}^*)^*$ and $\mathcal{Z}_0 \subseteq \mathcal{Z} \subseteq (\mathcal{Z}^*)^*$. 


Lemma 1.10. For any $x_i \in \mathcal{X}_0, z_i \in Z_0, y_j \in Y, w_j \in W$,

$$\sum_{i=1}^{t} x_i \otimes_{G,Y \otimes W} z_i (\sum_{j=1}^{s} y_j \otimes w_j) = (\sum_{j=1}^{s} y_j \otimes_{G,\mathcal{X}_0 \otimes Z_0} w_j)(\sum_{i=1}^{t} x_i \otimes z_i).$$

This induces a nondegenerate bilinear form

$$(\mathcal{X}_0 \otimes_{G,Y \otimes W} \mathcal{Z}_0, Y \otimes_{G,\mathcal{X}_0 \otimes Z_0} W) \rightarrow \mathbb{C},$$

namely

$$\sum_{i=1}^{t} x_i \otimes_{G,Y \otimes W} z_i (\sum_{j=1}^{s} y_j \otimes w_j) = (\sum_{j=1}^{s} x_j \otimes_{G,Y \otimes W} z_i)(\sum_{i=1}^{t} y_j \otimes w_j).$$

Proof: The first equation follows directly from definition:

$$\sum_{i=1}^{t} x_i \otimes_{G,Y \otimes W} z_i (\sum_{j=1}^{s} y_j \otimes w_j) = \int_{G} \sum_{j=1}^{s} y_j(\pi(g)x_i)w_j(\tau(g)z_i)dg.$$ 

Notice that $(\sum_{i=1}^{t} x_i \otimes_{G,Y \otimes W} z_i)(\sum_{j=1}^{s} y_j \otimes w_j) = 0$ for all $\{x_i\} \subseteq \mathcal{X}_0, \{z_i\} \subseteq Z_0$ if and only if $\mathcal{A}_{G,\mathcal{X}_0 \otimes Z_0}(\sum_{j=1}^{s} y_j \otimes w_j) = 0$. So one obtains a form

$$\sum_{i=1}^{t} x_i \otimes_{G,Y \otimes W} z_i (\sum_{j=1}^{s} y_j \otimes w_j) = (\sum_{j=1}^{s} x_j \otimes_{G,Y \otimes W} z_i)(\sum_{i=1}^{t} y_j \otimes w_j).$$

It is easy to see that this form is nondegenerate. \(\square\).

Definition 1.11. If $(\pi, \mathcal{X})$ and $(\tau, \mathcal{Z})$ are Hermitian representations and $\mathcal{X}_0 \subseteq \mathcal{X}, \mathcal{Z}_0 \subseteq \mathcal{Z}$ such that

$$(\pi(g)x, y)(\tau(g)z, w) \in L^1(G) \quad (x, y \in \mathcal{X}_0, z, w \in \mathcal{Z}_0),$$

we define the canonical

$$\mathcal{X}_0 \otimes_G \mathcal{Z}_0 = \mathcal{A}_{G}(\mathcal{X}_0 \otimes \mathcal{Z}_0).$$

Explicitly,

$$(x \otimes_G z)(y \otimes w) = \int_{G} (\pi(g)x, y)(\tau(g)z, w)dg, \quad (x, y \in \mathcal{X}_0; y, w \in \mathcal{Z}_0).$$

Whenever we use the notation $\mathcal{X}_0 \otimes_G \mathcal{Z}_0$, we assume that $G$ is unimodular and $\mathcal{X}_0 \otimes_G \mathcal{Z}_0$ is well-defined.

In the case that $(\pi, \mathcal{X})$ and $(\tau, \mathcal{Z})$ are Hilbert representations, then $\mathcal{X}_0^c \hookrightarrow \mathcal{X}^*$ and $\mathcal{Z}_0^c \hookrightarrow \mathcal{Z}^*$. We have

$$\mathcal{X}_0 \otimes_G \mathcal{Z}_0 = \mathcal{X}_0 \otimes_{G,\mathcal{X}_0 \otimes \mathcal{Z}_0} \mathcal{Z}_0.$$

1.5. A Strong form of ITP—the Geometric Realization.

Theorem 1.12. Let $G$ be a unimodular Lie group and $X$ be a smooth manifold. Let $G$ act on $X$ smoothly and freely from the right. Suppose that $X$ has a nontrivial $G$-invariant measure $dx$ given by a smooth nowhere vanishing volume form. Denote the action of $G$ on $L^2(X, dx)$ from the right by $R$. Then $L^2(X)$ is a unitary representation of $G$. Let $C_c^\infty(X)$ be the space of smooth function on $X$ with compact support. Let $(\pi, \mathcal{H})$ be a unitary representation of $G$. Then there is an injection $\mathcal{I}$
from \( C_c^\infty(X) \otimes_G H \) to \( C_c^\infty(X \times_G H) \) (the smooth and compactly supported sections of the Hilbert vector bundle \( X \times_G H \to X/G \)) defined by

\[
I(f \otimes u)(x) = \int_G f(xg) \pi(g) u dg, \quad (f \in C_c^\infty(X), \, u \in H).
\]

In addition, for any dense subspace \( V \) of \( H \), \( I(C_c^\infty(X) \otimes_G V) \) is dense in \( L^2(X \times_G H) \).

**Remark 1.13.** Regard \( X \) as a principal \( G \)-fibration. This theorem says that \( A_G \) can be interpreted as integration along the fiber in a proper sense. This is a strong form of invariant tensor product.

Proof: Let \( f_1, f_2 \in C_c^\infty(X) \) and \( u_1, u_2 \in H \). Then \((R(g)f_1, f_2)_{L^2(X)} \) must also have compact support on \( G \). We have

\[
(f_1 \otimes_G u_1)(f_2 \otimes u_2) = \int_G \int_X f_1(xg) f_2(x) dx |(\pi(g)u_1, u_2)dg|
\]

\[
= \int_G \int_{X \times X} f_1(xg) f_2(x) (\pi(g)u_1, u_2)dx dg
\]

\[
= \int_G \int_{X \times X} f_1(xg) f_2(x) (\pi(g)u_1, u_2)dx dg
\]

\[
= \int_{X/G} \int_{G \times G} f_1(xg) f_2(x) (\pi(g)u_1, u_2) dx dg [x]
\]

\[
= \int_{X/G} \int_{G \times G} f_1(xg) f_2(x) (\pi(g)u_1, u_2) dx dg [x]
\]

\[
= \int_{X/G} (\int_G f_1(xg) \pi(g)u_1 dg_1, \int_G f_2(x) \pi(h)u_2 dh) [x]
\]

(2)

Here the measure on \( X/G \) is normalized so that \( dx = d|x|dg \). Since the second line converges absolutely, by Fubini’s Theorem, we can interchange integrals. Observe that the function

\[
I(f_1 \otimes u_1) : X \ni x \to \int_G f_1(xg) \pi(g)u_1 dg_1
\]

is well-defined, compactly supported and satisfies that for every \( g_2 \in G \),

\[
I(f_1 \otimes u_1)(xg_2) = \pi(g_2^{-1})I(f_1 \otimes u_1)(x) \in H.
\]

So \( I(f_1 \otimes u_1) \) is a smooth section of the Hilbert vector bundle

\[
X \times_G H \to X/G
\]

with compact support. We have

\[
(f_1 \otimes_G u_1)(f_2 \otimes u_2) = (I(f_1 \otimes u_1), I(f_2 \otimes u_2))_{L^2(X \times_G H)}.
\]

Let \( V \) be a dense subspace of \( H \). We would like to show that

\[
\text{span}\{I(f \otimes u) \mid f \in C_c^\infty(X), \, u \in V\}
\]

is dense in \( L^2(X \times_G H) \). Suppose otherwise. Then there exists a \( \Psi \in L^2(X \times_G H) \) such that for any \( f \in C_c^\infty(X), \, u \in V \), we have \( (I(f \otimes u), \Psi) = 0 \). Notice that

\[
(I(f \otimes u), \Psi) = \int_{X/G} \int_G f(xg) \pi(g) u dg, \Psi(x) dx = \int_X f(x)(u, \Psi(x)) dx = 0.
\]
Hence for any fixed $u \in V$, $(u, \Psi(x)) = 0$ almost everywhere. Let $v \in \mathcal{H}$ be an arbitrary vector in $\mathcal{H}$. Since $\mathcal{H}$ is separable and $V$ is dense in $\mathcal{H}$, there exists a sequence $\{u_i\}_{1}^{\infty} \subset V$ such that $u_i \rightarrow v$. Now for each $u_i$, $(u_i, \Psi(x)) = 0$ almost everywhere. Since the set $\{u_i\}$ is countable, $\{u_i\} \perp \Psi(x)$ almost everywhere. Hence $(v, \Psi(x)) = 0$ almost everywhere. Choose an orthonormal basis $\{v_j\}_{j=1}^{\infty}$ for $\mathcal{H}$. For each $v_j$, $(v_j, \Psi(x)) = 0$ almost everywhere. Hence $(\mathcal{H}, \Psi(x)) = 0$ almost everywhere. It follows that $\Psi(x) = 0$ almost everywhere. Therefore, the set $\{\mathcal{I}(f \otimes u) \mid f \in C^{\infty}_{c}(X)\}$ spans a dense subspace of $L^2(X \times_G \mathcal{H})$.

If $\sum_{i=1}^{t} f_i \otimes_G u_i = 0$, then $(\sum_{i=1}^{t} \mathcal{I}(f_i \otimes u_i), \mathcal{I}(f \otimes u)) = 0$ for every $f \in C^{\infty}_{c}(X)$ and $u \in \mathcal{H}$. So $\sum_{i=1}^{t} \mathcal{I}(f_i \otimes u_i) = 0$. Conversely, if $\sum_{i=1}^{t} \mathcal{I}(f_i \otimes u_i) = 0$, $\sum_{i=1}^{t} f_i \otimes_G u_i = 0$. We can identify $C^{\infty}_{c}(X) \otimes_G \mathcal{H}$ with its image in $L^2(X \times_G \mathcal{H}, dx)$. □

1.6. Representations obtained from invariant tensor product.

**Lemma 1.14.** Let $G_1$ be a unimodular group. Let $(\sigma_{1,2}, \mathcal{H}_{1,2})$ be a Hilbert representation of $G_1 \times G_2$ and $(\sigma_1, \mathcal{H}_1)$ a Hilbert representation of $G_1$. Let $V_{1,2}$ be a $G_2$-stable subspace of $\mathcal{H}_{1,2}$ and $V_1$ be a subspace of $\mathcal{H}_1$. Suppose that $V^{\ast}_{1,2}$ as a subspace of $\mathcal{H}^{\ast}_{1,2}$ is a $\sigma^{\ast}_{1,2}(G_2)$-invariant subspace. Suppose that $V_{1,2} \otimes_{G_1} V_1$ is well-defined. Then $V_{1,2} \otimes_{G_1} V_1$ inherits a $G_2$-action from $V_{1,2}$, namely

$$g : v_{1,2} \otimes_{G_1} v_1 \rightarrow \sigma_{1,2}(g)v_{1,2} \otimes_{G_1} v_1, \quad (v_{1,2} \in V_{1,2}, v_1 \in V_1).$$

Hence $V_{1,2} \otimes_{G_1} V_1$ is a linear representation of $G_2$.

Proof: Consider $\mathcal{A}_{G_1} : V_{1,2} \otimes V_1 \rightarrow \text{Hom}(V^{\ast}_{1,2} \otimes V^{\ast}_1, \mathbb{C})$. It suffice to show that the subspace $\ker(\mathcal{A}_{G_1})$ is $G_2$-invariant. Suppose that $\sum_{j=1}^{t} v^{1}_{1,2} \otimes v^{1}_1 \in \ker(\mathcal{A}_{G_1})$. Then for any $u_{1,2} \in V^{\ast}_{1,2}, u_1 \in V^{\ast}_1$, we have

$$\int_{G_1} \sum_{j=1}^{t} (\sigma_{1,2}(g_1)v^{1,2}_{1,2}, u_{1,2})(\sigma_1(g_1)v^{1}_1, u_1)dg_1 = 0.$$

Since $V^{\ast}_{1,2}$ is a $\sigma^{\ast}_{1,2}(G_2)$-invariant subspace, for any $g_2 \in G_2$ and $u_{1,2} \in V^{\ast}_{1,2}, \sigma^{\ast}_{1,2}(g_2^{-1})u_{1,2} \in V^{\ast}_{1,2} \subset \mathcal{H}^{\ast}_{1,2}$. We have

$$\int_{G_1} \sum_{j=1}^{t} (\sigma_{1,2}(g_1)\sigma_{1,2}(g_2)v^{1,2}_{1,2}, u_{1,2})(\sigma_1(g_1)v^{1}_1, u_1)dg_1 = \int_{G_1} \sum_{j=1}^{t} (\sigma_{1,2}(g_1)v^{1,2}_{1,2} \sigma^{\ast}_{1,2}(g_2^{-1})u_{1,2})(\sigma_1(g_1)v^{1}_1, u_1)dg_1 = 0.$$

Hence $\sum_{j=1}^{t} \sigma_{1,2}(g_2)v^{1,2}_{1,2} \otimes v^{1}_1 \in \ker(\mathcal{A}_{G_1})$. □

**Remark 1.15.**

1. If $\sigma_{1,2}|_{G_2}$ is unitary and $V_{1,2}$ is $G_2$-invariant, then $V^{\ast}_{1,2}$ will automatically be $\sigma^{\ast}_{1,2}(G_2)$-invariant. $G_2$-action on $V_{1,2} \otimes_{G_1} V_1$ is compatible with the action of $G_2$ on $\text{Hom}(V^{\ast}_{1,2} \otimes V^{\ast}_1, \mathbb{C})$.

2. For $G_2$ a semisimple Lie group, let $K_2$ be a maximal compact subgroup of $G_2$. The same statement holds for $(g_2, K_2)$-module $V_{1,2}$.

Similarly, we have the following two lemma.

**Lemma 1.16.** Let $G_2$ be a semisimple Lie group. Let $K_2$ be a maximal compact subgroup of $G_2$. Let $(\sigma_{1,2}, X_{1,2})$ be a continuous representation of $G_1 \times G_2$ and $(\sigma_1, X_1)$ be a continuous representation of $G_1$. Let $V_{1,2}$ be a $(g_2, K_2)$-module in $X_{1,2}$ and $V_1$ a subspace of $X_1$. Let $U_{1,2}$ be a $(g_2, K_2)$-module in $X^{\ast}_{1,2}$ and $U_1$ a subspace of $X^{\ast}_1$. Then $V_{1,2} \otimes_{G_1} U_{1,2} \otimes U_1 \otimes U_1$ inherits a $(g_2, K_2)$-module structure from $V_{1,2}$. 

\[ \]
Lemma 1.17. Let $G_2$ be a semisimple Lie group. Let $K_2$ be a maximal compact subgroup of $G_2$. Let $(\sigma_1, X_{1,2})$ be a continuous representation of $G_1 \times G_2$ equipped with a $(g_2, K_2)$ invariant Hermitian form and $(\sigma_1, X_1)$ be a Hermitian representation of $G_1$. Let $V_{1,2}$ be a $(g_2, K_2)$-module in $X_{1,2}$. Let $V_1$ be a subspace of $X_1$. Then $V_{1,2} \otimes G_1 V_1$ inherits a $(g_2, K_2)$-module structure from $V_{1,2}$.

Example 1.18 (Injectivity). Let

$$V_1 \subseteq V_2 \subseteq \ldots \subseteq V_n \subseteq X$$

be a sequence of subspaces of a Hilbert representation $(\pi, X)$ of $G$. Let $U$ be a subspace of the Hilbert representation $(\sigma, Y)$. Suppose that $V_n \otimes_G U$ is well-defined. Then we have

$$V_1 \otimes_{G, V_n^c \otimes U^c} U \subseteq V_2 \otimes_{G, V_n^c \otimes U^c} U \subseteq \ldots \subseteq V_n \otimes G U.$$  

1.7. Invariant Hermitian Forms for ITP. Suppose that the representations $(\sigma_1, X_{1,2})$ and $(\sigma_1, X_1)$ are equipped with $G_1$-invariant Hermitian forms $(\cdot, \cdot)$. Notice that

$$\sum_{j=1}^s (v^1_{1,2} \otimes G_1 v^1_1)(\sum_{i=1}^t u^1_{1,2} \otimes u^1_i) = \sum_{i,j} \int_{G_1} (\sigma_1, (g_1)v^1_{1,2}, u^1_1) (\sigma_1, (g_1)u^1_i, u^1_i) dg_1$$

$$= \sum_{i,j} \int_{G_1} (v^1_{1,2}, (g_1^{-1})u^1_1) (v^1_1, (g_1^{-1})u^1_i) dg_1$$

$$= \sum_{i,j} \int_{G_1} (v^1_{1,2}, (g_1)u^1_1) (v^1_1, (g_1)u^1_i) dg_1$$

$$= \sum_{i,j} \int_{G_1} (\sigma_1, (g_1)u^1_1, v^1_{1,2}, (\sigma_1, (g_1)u^1_i, v^1_i)) dg_1$$

$$= \sum_{i=1}^t u^1_{1,2} \otimes G_1 v^1_1 (\sum_{j=1}^s u^1_{1,2} \otimes v^1_j).$$

Here $v^1_{1,2} \in V_{1,2}$, $v^1_1 \in V_1$, $u^1_{1,2} \in V^c_{1,2}$ and $u^1_i \in V^c_i$. By Lemma 1.13, $(\sum_{j=1}^s v^1_{1,2} \otimes G_1 v^1_1)(\sum_{i=1}^t u^1_{1,2} \otimes u^1_i)$ yields a pairing between $\sum_{j=1}^s v^1_{1,2} \otimes G_1 v^1_1$ and $\sum_{i=1}^t u^1_{1,2} \otimes G_1 u^1_i$. By our computation, this pairing produces a Hermitian form on $V_{1,2} \otimes G_1 V_1$. We give the following

Definition 1.19. Suppose that the representations $(\sigma_1, X_{1,2})$ and $(\sigma_1, X_1)$ are equipped with $G_1$-invariant Hermitian forms $(\cdot, \cdot)$. Define a Hermitian form on $V_{1,2} \otimes G_1 V_1$ as follows

$$(v_{1,2} \otimes G_1 v_1, u_{1,2} \otimes G_1 u_1) = (v_{1,2} \otimes G_1 v)(u_{1,2} \otimes u_1) = \int_{G_1} (\sigma_1, (g_1)v_{1,2}, u_1) (\sigma_1, (g_1)u_1, u_1) dg_1.$$

Proposition 1.20. Let $G_1, G_2$ be two semisimple Lie groups. Let $K_i$ be a maximal compact subgroup of $G_i$ respectively. Let $(\sigma_1, X_{1,2})$ be a continuous representation of $G_1 \times G_2$ equipped with a $G_1 \times G_2$-invariant Hermitian form $(\cdot, \cdot)$ and $(\pi_1, X_1)$ be a continuous representation of $G_1$ equipped with a $G_1$-invariant Hermitian form $(\cdot, \cdot)$. Let $V_{1,2}$ be a $(g_2, K_2)$-module in $X_{1,2}$. Let $V_1$ be a subspace of $X_1$. Then the Hermitian form on $V_{1,2} \otimes G_1 V_1$ is non-degenerate, invariant under the action of both $K_2$ and $g_2$. 

Proof: Let $u_{1,2}, v_{1,2} \in V_{1,2}$ and $u_1, v_1 \in V_1$. Let $k_2 \in K_2$. Then

$$
(\sigma_{1,2}(k_2)u_{1,2} \otimes_{G_1} u_1, \sigma_{1,2}(k_2)v_{1,2} \otimes_{G_1} v_1)
= \int_{G_1} (\sigma_{1,2}(g_1 k_2)u_{1,2}, \sigma_{1,2}(k_2)v_{1,2})(\sigma_1(g_1)u_1, v_1)
= \int_{G_1} (\sigma_{1,2}(g_1)u_{1,2}, v_{1,2})(\sigma_1(g_1)u_1, v_1)
= (u_{1,2} \otimes_{G_1} u_1, v_{1,2} \otimes_{G_1} v_1)
$$

(4)

The invariance of $g_2$-action is similar. □.

Corollary 1.21. Under the hypothesis of Prop. 1.20, the averaging operator

$$
\mathcal{A}_{G_1} : V_{1,2} \otimes V_1 \to \text{Hom}(V_{1,2}^c \otimes V_1^c, \mathbb{C})
$$

is a $(g_2, K_2)$-module homomorphism.

2. Representation Theory of $SO(p,q)$

From now on all representations are assumed to be continuous. Let $G$ be a semisimple Lie group. Fix a maximal compact subgroup $K$. A Hilbert representation $(\pi, \mathcal{H})$ of $G$ is said to be admissible if it restricts to a unitary representation of $K$ and the isotypic subspaces $\mathcal{H}_\sigma$ where $\sigma \in \hat{K}$ are finite dimensional. Let $\mathcal{H}_K$ be the $K$-finite vectors of $\mathcal{H}$. Then $\mathcal{H}_K$ is a $(g, K)$-module with certain compatibility conditions. We call $\mathcal{H}_K$ the Harish-Chandra module of $\mathcal{H}$. We may also call a representation admissible, if (1) the $K$-finite vectors are dense in the representation, (2) the $K$-finite vectors yield a $(g, K)$-module and (3) each $K$-type has finite multiplicity. In all cases, $V_K$ will be used to denote the $K$-finite vectors of a representation $V$ of $G$. Two admissible representations are said to be infinitesimally equivalent if their underlying Harish-Chandra modules are equivalent. In this paper, we will mainly be interested in infinitesimal equivalences.

Let $U(g)$ be the universal enveloping algebra. Let $Z$ be the center of $U(g)$. Let $I : Z \to \mathbb{C}$ be a character of $Z$. We say that a $(g, K)$-module $V$ has infinitesimal character $I$ if $Z$ acts on $V$ by the character $I$. We use the Harish-Chandra isomorphism to identify the spectrum of $Z$ with the complex dual of $\mathfrak{h}$, a maximal Cartan subalgebra of $g_C$. This identification is unique up to the action of the Weyl group $W(g_C, \mathfrak{h})$. In other words, $I$ and $wI$ are the same infinitesimal character if $w \in W(g_C, \mathfrak{h})$. If $(\pi, V)$ is an admissible representation of $G$ and $V_K$ has infinitesimal character $\lambda$, then we say that $\pi$ has infinitesimal character $\lambda$.

Let $G = SO_n(p,q)$ be the identity component of $SO(p,q)$. Suppose that $pq \neq 0$. So $G$ is noncompact. We often assume $q \geq p$ so the noncompact component of the KAK-decomposition is in $\mathbb{R}^q$. This is not necessary, but convenient. If $q < p$, then the noncompact component of the KAK decomposition is in $\mathbb{R}^p$.

Let $\chi$ be the unique unitary character of $SO(p,q)$ that maps the nonidentity component to $-1$. We extend $\chi$ to a character of products of real general linear groups and $SO(p,q)$ by defining

$$
\chi|_{GL} = \frac{\det}{|\det|}
$$

Define $\chi$ on $SO(k,0)$ to be the trivial character. Take $\chi$ to be the universal character of $SO$, $GL$ and their products.
2.1. **Representations of $SO(p,q)$**. Let $(\pi, V)$ be an irreducible unitary representation of $G$. Let $h$ be an element in the nonidentity component of $SO(p, q)$ such that $h^2 = I$. Then $SO(p, q)$ can be identified with a semidirect product of $\{I, h\}$ and $G$. Define an unitary representation $(\pi^h, V)$ as

$$\pi^h(g) = \pi(h^{-1}gh) \quad (g \in G).$$

Then $\pi^h \cong \pi$ or $\pi^h \not\cong \pi$.

2.1.1. **The case $\pi^h \cong \pi$**. If $\pi^h \cong \pi$, then there is an intertwining operator $i : V \to V$ such that

$$\pi(g)i = i\pi^h(g) = i\pi(h^{-1}gh), \quad (g \in G).$$

It follows that

$$\pi(g)i = i\pi(h^{-1}gh)i = i\pi(h^{-2}gh^2) = i\pi(g).$$

Since $\pi$ is irreducible, by Schur’s Lemma, $ii = \lambda \in \mathbb{C} - \{0\}$. Normalize $i$ so that $i^2 = I$. Define a representation $(\overline{\pi}, V)$ such that

$$\overline{\pi}(h) = i, \quad \overline{\pi}(g) = \pi(g) (\forall g \in G).$$

It is easy to check that $\overline{\pi}$ is an unitary irreducible representation of $SO(p, q)$. $\overline{\pi}$ has the following properties:

1. $\overline{\pi}|_G \cong \pi$;
2. $\overline{\pi} \otimes \chi \not\cong \overline{\pi}$;
3. $\overline{\pi}$ and $\overline{\pi} \otimes \chi$ come from two different choices of normalization.

2.1.2. **The case $\pi^h \not\cong \pi$**. If $\pi^h \not\cong \pi$, then we can define an unitary representation $(\overline{\pi}, V \oplus V)$ as follows:

$$\overline{\pi}(h)(v_1, v_2) = (v_2, v_1), \quad \overline{\pi}(g)(v_1, v_2) = (\pi(g)v_1, \pi(h^{-1}gh)v_2) \quad (v_1, v_2 \in V, g \in G).$$

Then

$$\overline{\pi}(h^{-1}\overline{\pi}(g)\overline{\pi}(h)(v_1, v_2) = \overline{\pi}(h)\overline{\pi}(g)(v_1, v_2) = \overline{\pi}(h)(\pi(g)v_1, \pi(h^{-1}gh)v_2) = (\pi(h^{-1}gh)v_1, \pi(g)v_2);$$

$$\overline{\pi}(h^{-1}gh)(v_1, v_2) = (\pi(h^{-1}gh)v_1, \pi(h^{-1}h^{-1}gh)v_2) = (\pi(h^{-1}gh)v_1, \pi(g)v_2).$$

Hence $\overline{\pi}(h^{-1})\overline{\pi}(g)\overline{\pi}(h) = \overline{\pi}(h^{-1}gh)$. Obviously, $\overline{\pi}(h)^2 = I$. So $\overline{\pi}$ is an unitary representation of $SO(p, q)$. It is irreducible by Mackey analysis. $\overline{\pi}$ has the following properties:

1. $\overline{\pi}|_G = \pi \oplus \pi^h$ and $\pi \not\cong \pi^h$;
2. $\overline{\pi} \otimes \chi \cong \overline{\pi}$.

Similar statements holds for irreducible unitary representations of the compact group $O(m)$ and the general linear group $GL(n)$. Unless otherwise specified, if $\pi$ is an irreducible unitary representation of $G$, $\overline{\pi}$ will be used to denote the one, or one of the two representations defined in this section.

2.2. **Representations of $SO(p)$ and $O(p)$**. Given a compact Lie group $K$, let $\hat{K}$ be the the equivalence classes of irreducible unitary representations of $K$. When $K$ is connected, $\hat{K}$ is parametrized by integral dominant weights upon a choice of a positive root system. If $\xi$ is the highest weight of $\pi$, we write $(\pi_\xi, V_\xi)$ or sometimes just $(\xi)$ for $\pi$.

Let $K = SO(p) \times SO(q)$. The irreducible unitary representations of $K$ are products of irreducible unitary representations of $SO(p)$ and $SO(q)$.
2.2.1. Odd Orthogonal Groups. For $p$ odd, $\widehat{SO}(p)$ is parameterized by an integral vector $\xi$ of dimension $\frac{p}{2}$: 
\[ \xi_1 \geq \xi_2 \geq \ldots \geq \xi_{\frac{p}{2}-1} \geq \xi_{\frac{p}{2}} \geq 0. \]
Denote this set of integral vectors by $\Pi_p(\mathbb{Z})^+$. In this case, $-I_p \in O(p) - SO(p)$ and $O(p) \cong SO(p) \times \{ \pm I_p \}$. The unitary dual $\widehat{O}(p)$ can be parametrized by $(\xi, \pm)$ with $\xi \in \Pi_p(\mathbb{Z})^+$. Here $\pi_{\xi, \pm}|_{SO(p)}$ is $\pi_{\xi}$ and $\pi_{\xi, \pm}(-I_p)$ is $\pm$ identity.

When $p = 1$, there is only a trivial representation of $SO(1)$. We denote it by $(0)$. There are two irreducible unitary representations of $O(1)$. We denote them by $(0, +)$ and $(0, -)$.

2.2.2. Even Orthogonal Group. For $p$ even, $\widehat{SO}(p)$ is parametrized by an integral vector $\xi$ of dimension $\frac{p}{2}$:
\[ \xi_1 \geq \xi_2 \geq \ldots \geq \xi_{\frac{p}{2}-1} \geq |\xi_{\frac{p}{2}}|. \]
Denote this set of integral vectors by $\Pi_p(\mathbb{Z})^+$. Fix an element $h \in O(p) - SO(p)$ such that $h^2 = I_p$.

1. If $\xi_{\frac{p}{2}} \neq 0$, then $\pi_h^\pm \cong \pi_{-\xi}$ with $\xi^- = (\xi_1, \xi_2, \ldots, \xi_{\frac{p}{2}-1}, -\xi_{\frac{p}{2}})$. We obtain an irreducible unitary representation of $O(p)$ such that $\pi|_{SO(p)} = \pi \oplus \pi^h$. Following $[KV]$, we parametrize such $\pi$ by $(|\xi|, +)$.

2. If $\xi_{\frac{p}{2}} = 0$, then $\pi_h^\pm \cong \pi_{\xi}$. So $\pi$ extends to two representations of $O(p)$, differing by det. we shall follow the convention set in (6.10 $[KV]$) and parametrize these two representations by $(\xi, +)$ and $(\xi, -)$.

The distinction between $(\xi, +)$ and $(\xi, -)$ will become crucial when we apply Howe’s theory of dual pair correspondence.

2.2.3. Representations of $S(O(p)O(q))$. Let $S(O(p)O(q)) = \{(k_1, k_2) \in O(p)O(q) \mid \det k_1 \det k_2 = 1\}$. $S(O(p)O(q))$ is a maximal compact subgroup of $SO(p, q)$. The unitary dual of $S(O(p)O(q))$ can be parametrized by $(\xi, \eta, \pm)$ if $(\xi) \otimes (\eta)$ extends to a representation of $O(p)O(q)$; by $(|\xi|, |\eta|, +)$ if $(\xi) \otimes (\eta)$ does not extend to a representation of $O(p)O(q)$.

2.3. Basic Structure Theory of $G$. Suppose now that $p \leq q$. Fix a maximal compact subgroup $K = SO(p)SO(q)$ in $G$. Fix an Iwasawa decomposition $K A_m N_m$. Let $\Delta(g, a_m)$ be the restricted roots and $\Delta^+(g, a_m)$ be the positive restricted roots defined by $N_m$. Let $\{e_1, e_2, \ldots, e_p\}$ be the standard basis for $a_m^*$ such that
\[ \Delta(g, a_m) = \{ \pm e_i \pm e_j \mid i > j, i, j \in [1, p]\} \quad (p = q); \]
\[ \Delta(g, a_m) = \{ \pm e_i \pm e_j \mid i > j, i, j \in [1, p]\} \cup \{ \pm e_i \mid i \in [1, p]\} \\ \cup \{ e_i \mid i \in (q-p) } \quad (q > p). \]
Here $q - p$ denotes the multiplicities. Let $\rho(p, q)$ be the half sum of positive restricted roots. Then
\[ \rho(p, q) = \frac{q + p}{2} - 1, \frac{q + p}{2}, \ldots, \frac{q - p}{2}. \]

The basis $\{e_i\}_{i=1}^p$ defines a coordinate system for $a_m$. By this coordinate system, we identify $a_m$ with $\mathbb{R}^p$. Equip $a_m$ and $a_m^*$ with the standard inner product of $\mathbb{R}^p$. The open Weyl chamber is
\[ a_m^+ = \{ H_1 > H_2 > \ldots > H_p > 0 \mid H \in \mathbb{R}^p\} \quad (q > p); \]
\[ a_m^- = \{ H_1 > H_2 > \ldots > |H_p| \mid H \in \mathbb{R}^p\} \quad (q = p). \]
For $q > p$, the Weyl group $W(g, a_m)$ is generated by permutations and sign changes. For $q = p$, the Weyl group $W(g, a_m)$ is generated by permutations and even number of sign changes. Let $\overline{a_m^+} = \exp a_m^+$. 

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Consider now the Cartan decomposition $KA_mK$. Since $W(g,a_m) = N_K(a_m)/Z_K(a_m)$, $G$ can be decomposed as $K\overline{A}_mK$. Here $N_K(a_m)$ is the normalizer of $a_m$ in $K$ and $Z_K(a_m)$ is the centralizer of $a_m$ in $K$. For any $g \in G$, write $g = k_1 \exp H^+(g)k_2$ with $H^+ \in \overline{A}_m^+$. $H^+(g)$ is uniquely determined by $g$.

Let $M_m = Z_K(a_m)$. Then $M_m$ can be identified with

$$\{(\epsilon_1, \epsilon_2, \ldots, \epsilon_p, h) \mid \epsilon_i = \pm 1, h \in \text{SO}(q-p), \prod_{i=1}^{p} \epsilon_i = 1\}.$$

Let $P_m = M_mA_mN_m$. $P_m$ is a minimal parabolic subgroup.

2.4. Induced Representations. Let $P \supseteq P_m$ be a parabolic subgroup of $G$. Let $P = LN$ be the Levi decomposition. Then $L$ has the following form

$$\{\chi(g_0) \prod_{i=1}^{l} \det h_i = 1 \mid (h_1, h_2, \ldots, h_l, g_0) \in GL(r_1)GL(r_2) \ldots GL(r_l)SO(p - \sum_{i=1}^{l} r_i, q - \sum_{i=1}^{l} r_i)\},$$

where $r_i \geq 1(i \in [1, l]), p \geq \sum_{i=1}^{l} r_i$. Let $P = MAN$ be the Langlands decomposition. Let $SL^\pm(r_i) = \{g \mid g = \pm 1 \mid g \in GL(r_i)\}$. Then $A \cong (\mathbb{R}^+)^l$ and

$$M = \{\chi(g_0) \prod_{i=1}^{l} \det h_i = 1 \mid (h_1, h_2, \ldots, h_l, g_0) \in \prod_{i=1}^{l} SL^\pm(r_i) \times SO(p - \sum_{i=1}^{l} r_i, q - \sum_{i=1}^{l} r_i)\}.$$

The connected component of $M$, $M_0 \cong SL(r_1)SL(r_2) \ldots SL(r_l)SO_0(p - \sum_{i=1}^{l} r_i, q - \sum_{i=1}^{l} r_i)$.

For the special orthogonal group $SO(p, q)$, the Levi factor of a parabolic subgroup must be of the form

$$GL(r_1)GL(r_2) \ldots GL(r_l)SO(p - \sum_{i=1}^{l} r_i, q - \sum_{i=1}^{l} r_i);$$

and $M \cong \prod_{i=1}^{l} SL^\pm(r_i) \times SO(p - \sum_{i=1}^{l} r_i, q - \sum_{i=1}^{l} r_i)$.

Let $\Delta^+(g, a)$ be the positive roots from $N_m$. Let $\rho$, or sometimes $\rho_P$ be the half sum of the positive roots (with multiplicities) in $\Delta^+(g, a)$. Notice that $\Delta(g, a)$ in general is not a root system. Nevertheless, $\Delta^+(g, a)$ still defines an open positive Weyl chamber in $a^*$, namely

$$[a^*]^+ = \{v \in a^* \mid (v, \alpha) > 0, \forall \alpha \in \Delta^+(g, a)\}.$$

Let $(\sigma, \mathcal{H}_\sigma)$ be an admissible Hilbert representation of $M$ with respect to $M_K = M \cap K$. Let $v \in (a^*)_C$. Let $\text{Ind}_P^G \sigma \otimes \mathbb{C}_v$ be the normalized induced representation. The Hilbert space is given by

$$\{f : G \to \mathcal{H}_\sigma \mid f(gman) = \exp -(v + \rho)(\log a)\sigma(m)^{-1} f(g), f|_K \in L^2(K, \mathcal{H}_\sigma)\}.$$

If $(\sigma, V_\sigma)$ is an admissible representation of $M$, we can take continuous functions

$$\{f : G \to V_\sigma \text{ continuous} \mid f(gman) = \exp -(v + \rho)(\log a)\sigma(m)^{-1} f(g)\}.$$

Again, we obtain an admissible representation of $G$. Whether taking $L^2$-sections or continuous sections, the underlying Harish-Chandra module is the same, namely, $K$-finite vectors in $\text{Ind}_M^G [V_\sigma]_{M \cap K}$. So the induced representations are infinitesimally equivalent.

Parabolically induced representations have the following nice properties:
as a direct summand of $J$ poses into a direct sum of irreducibles. Every irreducible $H$ arish-Chandra module can be constructed certainty intertwining operator. In this case, $J$ Matrix Coefficients.

2.6.

Langlands quotient without assuming that $\Re v = 0$ and $\sigma$ is unitary, the $\Ind_P^G \sigma \otimes \mathbb{C}_v$ is unitary. This type of induction is called unitary parabolic induction.

(2) If $\sigma$ has infinitesimal character $\lambda$, then $\Ind_P^G \sigma \otimes \mathbb{C}_v$ has infinitesimal character $\lambda \oplus v$.

(3) If $\sigma|_{M \cap K}$ has a composition series of finite length, then $(\Ind_P^G \sigma \otimes \mathbb{C}_v)_K$ will also have a composition series of finite length.

(4) $\Ind_P^G \sigma \otimes \mathbb{C}_v|_{K} = \Ind_{M \cap K}^K (V_\sigma)|_{M \cap K}$. The action of $K$ is independent of $v$.

(5) Every irreducible admissible representation occurs infinitesimally as a subrepresentation of a principal series representation, namely those induced representation with $P = P_m$ and $\sigma$ irreducible.

2.5. Langlands Classification. We shall now review some basic facts about Langland classification. Langlands classification reduces the classification of the infinitesimal equivalence classes of irreducible admissible representations to the classification of irreducible tempered representations (LA [KN]). Then Knapp and Zuckerman complete the classification of irreducible tempered representations, which are necessarily unitarizable. To a large extent, Langlands classification is based on the asymptotes of matrix coefficients of irreducible admissible representations.

Theorem 2.1 (Langlands Classification). Fix a minimal parabolic subgroup $P_m$. Then the infinitesimal equivalence classes of irreducible admissible representations of $G$ is in one-to-one correspondence to the following triples

(1) $P = MAN$, a parabolic subgroup containing $P_m$;

(2) the equivalence class of $\sigma$, where $\sigma$ is an irreducible tempered representation of $M$;

(3) $v \in (\mathfrak{a}^*)_C$ such that $\Re (v) \in [\mathfrak{a}^*]^+$.

The irreducible admissible representation $J(P, \sigma, v)$ is given by the unique irreducible admissible quotient of $\Ind_{MAN}^G \sigma \otimes \mathbb{C}_v$.

In particular, Langlands classification says that there is a unique irreducible quotient for $\Ind_{MAN}^G \sigma \otimes \mathbb{C}_v$. So $J(P, \sigma, v)$ is often called the Langlands quotient. Notice that if $\Re (v) \notin [\mathfrak{a}^*]^+$ or $\sigma$ not tempered, there could be many irreducible quotients. At the end of this section, we will “isolate” some Langlands quotient without assuming that $\Re (v)$ is in the open Weyl chamber, using Vogan’s lowest $K$-types.

Remark 2.2. There is a revised Langlands classification based on cuspidal parabolic subgroups. I shall only state some facts that will be used later. Let $P = MAN$ be a cuspidal parabolic subgroup containing $P_m$. Let $\sigma$ be a discrete series representation of $M$. Suppose that $\Re (v)$ is in the closed positive Weyl chamber. Then there is a Langlands quotient $J(P, \sigma, v)$, defined to be the image of certain intertwining operator. In this case, $J(P, \sigma, v)$ may not be irreducible. Nevertheless, it decomposes into a direct sum of irreducibles. Every irreducible Harish-Chandra module can be constructed as a direct summand of $J(P, \sigma, v)$.

2.6. Matrix Coefficients. Now we shall list a few important properties, that more or less characterize Langlands quotients. We will start with irreducible tempered representations. These representations are all unitarizable. Let $\Xi_\lambda(g)$ be Harish-Chandra’s spherical function. We write $\Xi(g)$ for $\Xi_0(g)$. Notice that $\Xi_\lambda(g) = \Xi_{w \lambda}(g)$ for any $w \in W(g, a_m)$.

Theorem 2.3. Let $\pi$ be an irreducible admissible representation of $G$. The following are equivalent.

(1) $(\pi, \mathcal{H})$ is an irreducible tempered representation.

(2) All $K$-finite matrix coefficients of $\pi$ are bounded by a multiple of $\Xi(g)$.

(3) All $K$-finite matrix coefficients of $\pi$ are in $L^{2+}(G)$ for every $\epsilon > 0$.

(4) $(\pi, \mathcal{H})$ is infinitesimally equivalent to a subrepresentation of $\Ind_P^G \sigma \otimes \mathbb{C}_v$ with $\sigma$ a discrete series and $\Re (v) = 0$. 

(5) All leading exponent $v$ of $\pi$ satisfies $\Re(v + \rho(p,q))(\mathfrak{a}_m^+ - \mathfrak{a}_m^-) \leq 0$.

If $(\pi, \mathcal{H})$ is assumed to be unitary, then the conditions above are equivalent to the condition that $(\pi, \mathcal{H})$ is weakly contained in $L^2(G)$ (Wallach). Now we make some remarks about Langlands' quotient.

Remark 2.4. (1) The $K$-finite matrix coefficients of $J(P,\sigma,v)$ are bounded by multiples of $\Xi_{\Re(v) \geq 0}(g)$. Here we embed $\mathfrak{a}^* \to \mathfrak{a}_m^*$. 

(2) The $K$-finite matrix coefficients of $\text{Ind}_{P}^{G} \sigma \otimes \mathbb{C}_v$ are bounded by multiples of $\Xi_{\Re(v) \geq 0}(g)$ (see Prop 7.14 [KN]).

(3) If $\Re(v) \in -[\mathfrak{a}^*]^+$, then there is a unique irreducible admissible subrepresentation in $\text{Ind}_{P}^{G} \sigma \otimes \mathbb{C}_v$. This representation is the dual representation of $J(P,\sigma,-v)$.

Langlands' classification can be established by studying the leading exponents of the asymptotic expansion of matrix coefficients. We quote some result from [KN].

Theorem 2.5. Let $\pi$ be an irreducible admissible representation of $G$. Let $v - \rho(p,q)$ be a leading exponent of $\pi$. Let $\mathfrak{t}_m$ be a maximal Cartan subalgebra of $\mathfrak{o}(q-p)$. Then $v$ is related to the infinitesimal character $I(\pi) \in \langle \mathfrak{a}_m \oplus \mathfrak{t}_m \rangle_{\mathbb{C}}$ by the following equation

$$v = wI(\pi)|_{\mathfrak{a}_m}$$

where $w$ is an element in the Weyl group of the complex Lie algebra. If there is a $\lambda$ such that each leading exponent $v - \rho(p,q)$ satisfies the condition that

$$|\rho(p,q) - v| \geq -\lambda$$

then every $K$-finite matrix coefficient $f(k_1 \exp H^+ k_2)$ is bounded by a multiple of

$$(1 + (H^+, H^+))^Q \exp(\lambda(|H^+|))$$

for some $Q > 0$.

2.7. Vogan’s Subquotient. In practice, Langlands classification tells little about the algebraic structure of the Harish-Chandra module. It is not easy to apply Langlands classification, for example, to determine the composition factors of an admissible quasisimple representation. In [Vogan79], Vogan took a more algebraic approach and studied the $K$-types of an irreducible admissible representation. Vogan proved that in each irreducible admissible representation each minimal $K$-type appears with multiplicity one.

Definition 2.6 (Vogan). Let $\pi$ be an admissible representation of $G$. Fix a maximal compact subgroup $K$. Let $V(\pi)$ be the Harish-Chandra module of $\pi$. Suppose that $\mu$ is a lowest $K$-type with multiplicity one in $V(\pi)$. Let $V_0(\pi,\mu)$ be the $(\mathfrak{g},K)$-module generated by $\mu$ in $V(\pi)$. Let $V_1(\pi,\mu)$ be the maximal $(\mathfrak{g},K)$-submodule of $V_0(\pi,\mu)$ not containing $\mu$. We call $V_0(\pi,\mu)/V_1(\pi,\mu)$ a Vogan subquotient of $V(\pi)$. If Vogan subquotient is unique in $V(\pi)$, we say that $V_0(\pi,\mu)/V_1(\pi,\mu)$ is the Vogan subquotient of $V(\pi)$.

We shall make several remarks here.

(1) First, by definition, a Vogan subquotient is an irreducible $(\mathfrak{g},K)$-module.

(2) The same definition can be generalized to allow lowest $K$-types with multiplicities. Then $V_0(\pi,\mu)/V_1(\pi,\mu)$ may not be irreducible.

(3) The lowest $K$-type may not be unique, even for irreducible Harish-Chandra modules.

Theorem 2.7 ([Vogan79]). Let $P$ be a cuspidal parabolic subgroup of $G$ containing the minimal parabolic subgroup $P_0$. Let $\sigma$ be a discrete series representation of $G$. Suppose that $\Re(v)$ is in the closed positive Weyl chamber. Then $J(P,\sigma,v)$ is equivalent to the direct sum of Vogan subquotients of $\text{Ind}_{P}^{G} \sigma \otimes \mathbb{C}_v$. 
Definition 2.8. Let $G$ be a real reductive group. Given a cuspidal parabolic subgroup $P$, a tempered representation $\sigma$, any $v \in \mathfrak{a}_c^\circ$, if $\text{Ind}^G_P \sigma \otimes \mathbb{C}_v$ has a unique Vogan subquotient $\pi$, we call $(P, \sigma, v)$ the Langlands-Vogan parameter of $\pi$. Here $v$ is not unique.

3. Degenerate Principal Series $I_n(s)$

Let $s \in \mathbb{R}$. Consider $SO_0(n,n)$. Let $I_n(s) = \text{Ind}^{SO_0(n,n)}_{GL(n)_{2n}} \det |^s$. Notice that
\[
[\text{Ind}^{SO_0(n,n)}_{GL(n)_{2n}} \det |^s \otimes \mathbb{C}_\pm]_{SO_0(p,q)} \cong \text{Ind}^{SO_0(n,n)}_{GL(n)_{2n}} \det |^s,
\]
Here $\mathbb{C}_+$ is the trivial representation of $SL^+(n)$ and $\mathbb{C}_-$ is the sign character of $SL^+(n)$. In addition
\[
\text{Ind}^{SO_0(n,n)}_{GL(n)_{2n}} \det |^s \otimes \mathbb{C}_- \cong \chi \otimes [\text{Ind}^{SO_0(n,n)}_{GL(n)_{2n}} \det |^s \otimes \mathbb{C}_+].
\]

Essentially, for Siegel parabolic subgroups, there is only one degenerate principal series of $SO(n,n)$ under study. If one considers the group $Spin_0(n,n)$, there is another degenerate principal series. The composition factors, $K$-types and unitarity of the composition factors are determined by K. Johnson ([JO]) and reworked by several authors in a more general context ([Sahi2], [Zhang], [LL]). In this section, our quotients and subrepresentations shall be interpreted in the category of Harish-Chandra modules.

3.1. Reducibility, $K$-types, composition series and Unitarizability. Recall that the infinitesimal character of $I_n(s)$,
\[
\mathcal{I}(I_n(s)) = \left( \frac{n-1}{2} + s, \frac{n-3}{2} + s, \ldots, \frac{n-3}{2} + s, \frac{n-1}{2} + s \right).
\]
Here the infinitesimal character is unique only up to the action of Weyl group. Due to the non-degenerate pairing between $I_n(s)$ and $I_n(-s)$, we have $I_n(s) \cong I_n(-s)^\ast$. There is an intertwining operator between $I_n(s)$ and $I_n(-s)$. The composition factors of $I_n(s)$ and $I_n(-s)$ are the same. As shown in [JO], $I_n(s)$ is irreducible if $s \notin \mathbb{Z} + \frac{n-1}{2}$.

Theorem 3.1 (Johnson, see also [Sahi2], [Zhang], [LL]). Suppose that $s \in \frac{n-1}{2} + \mathbb{Z}$.

1. The $K$-types of $I_n(s)$, independent of $s$, are multiplicity free and are parameterized by \{$(\lambda, \lambda) \mid \lambda \in \Pi_n(\mathbb{Z})^\ast$\}. More precisely, $K$-types in $I_n(s)$ are of the following pairs of integral highest weights
\[
(\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{n_0}) \geq 0, (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{n_0}) \geq 0, \quad (K = SO(2n_0 + 1)SO(2n_0 + 1));
\]
\[
(\lambda_1 \geq \lambda_2 \geq \ldots \geq |\lambda_{n_0}|), (\lambda_1 \geq \lambda_2 \geq \ldots \geq |\lambda_{n_0}|), \quad (G = SO(2n_0)SO(2n_0)).
\]
This is essentially the Peter-Weyl Theorem.

2. For $n$ odd, $I_n(s)$ has at most $\frac{n-1}{2} + 1$ composition factors, namely, $V_i(s)(i \in [0, \frac{n-1}{2}])$. If $i \geq \frac{n-1}{2} - |s|$, the composition factor $V_i(s)$ exists. It has the following $K$-types
\[
\{ (\lambda, \lambda) \mid \lambda_i \geq |s| - \frac{n-1}{2} + i \geq \lambda_{i+1} \}.
\]
If $i \neq \frac{n-1}{2}$, $V_i(s)$ is called a small constituent. $V_{n-1}(s)$ is called a large constituent.

3. For $n$ even, $I_n(s)$ has at most $\frac{n}{2} + 2$ composition factors, namely,
\[
V_0(s), v_1(s), \ldots, v_{\frac{n}{2}}(s), v_{\frac{n}{2}+1}(s), v_{\frac{n}{2}}(s)^\pm.
\]
(a) Let $i \in [0, \frac{n}{2} - 1]$. If $i \geq \frac{n-1}{2} - |s|$, the composition factor $V_i(s)$ exists. It has the following $K$-types
\[
\{ (\lambda, \lambda) \mid \lambda_i \geq |s| - \frac{n-1}{2} + i \geq \lambda_{i+1} \}.
\]
These composition factors are called small constituents.
(b) The composition factors $V_{\varphi}(s)^\pm$ always exist. They are called large constituents. They have the following $K$-types:
\[
\{ (\lambda, \lambda) \mid \pm \lambda \varphi \geq |s| + \frac{1}{2} \}.
\]

(4) $V_1(s)$ is unitary only in the following two situations,
(a) If $\frac{n-1}{2} - |s| \in [0, [\frac{n-1}{2}]]$, $V_{\varphi_1}(-|s|)$ is always unitary.
(b) If $n$ is even, $V_{\varphi}(s)^\pm$ is always unitary for any $s \in \frac{n-1}{2} + \mathbb{Z}$, i.e., $s$ a half integer.

Theorem 3.2. Let $V_1(s)$ be given by Corollary 3.3.

If $n$ is even, $V_{\varphi}(s)^\pm$ is always unitary for any $s \in \frac{n-1}{2} + \mathbb{Z}$, i.e., $s$ a half integer.

The composition factors $V_{\varphi}(s)^\pm$ yield an irreducible unitary representation of $SO(n,n)$.

(5) If $m = \frac{n-1}{2} - s$ and $|\frac{n-1}{2}| \geq m \geq 0$, $V_m(s)$ occurs as the unique quotient of $I_n(s)$ and as the unique subrepresentation of $I_n(-s)$.

3.2. Langlands-Vogan Parameter and Growth of $V_m(\frac{n-1}{2} - m)$. Now suppose $s = \frac{n-1}{2} - m$ and $m \in [0, [\frac{n-1}{2}]]$. Clearly, Vogan’s lowest $K$-type of $V_1(s)(i \geq m)$ is $(\lambda, \lambda)$ with
\[
\lambda = (\lambda_1 = \ldots = \lambda_i = -m + i) \geq \lambda_{i+2} = \ldots = \lambda_\varphi(= 0)).
\]

In particular, when $i = m$, the lowest $K$-type is trivial. Hence the small constituent $V_m(\frac{n-1}{2} - m)$ is spherical. We obtain

**Theorem 3.2.** Suppose $s = \frac{n-1}{2} - m$ and $m \in [0, [\frac{n-1}{2}]]$. Then $V_m = V_m(\frac{n-1}{2} - m)$ has the following properties:

1. $V_m$ is spherical. The $K$-types of $V_m$ are of the form $(\lambda, \lambda)$ with
\[
\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m = 0) = \lambda_{m+2} = \ldots = \lambda_\varphi(= 0)).
\]

In particular, if $m = 0$, we obtain the “smallest” small constituent, the trivial representation.  

2. The infinitesimal character $\mathcal{I}(V_m) = \eta(n - m - 1, m)$ where
\[
\eta(n - m - 1, m) = (n - m - 1, n - m - 2, \ldots, m, m - 1, m - 1, \ldots, 1, 1, 0).
\]

3. $V_m$ is the Langlands quotient of $\text{Ind}_G K \text{triv} \otimes \mathbb{C}_{\eta(n - m - 1, m)}$.

4. $V_m$ is the Vogan subquotient of $\text{Ind}_{GL(n)_{nN}} \mathbb{C}_{\eta(n - m - 1, m)} \otimes||\det||^{\frac{m-1}{2}}$.

5. The $K$-finite matrix coefficients of $V_m$ and of $I_n(\frac{n-1}{2} - m)$ are all bounded by a multiple of $\Xi_{\eta(n - m - 1, m)}(g)$, and $\Xi_{\eta(n - m - 1, m)}(\exp H^+)$ is bounded by
\[
(1 + (H^+, H^+))^{2\frac{n-2m}{2}} \exp(-m, -m, \ldots, -m, -m + 1, -m + 1, -m + 2, -m + 2, \ldots, -1, -1, 0, 0)(H^+)
\]
for some positive $Q$.

**Corollary 3.3.** Let $u, v \in I_n(\frac{n-1}{2} - m)(m \in [0, [\frac{n-1}{2}]]$ such that $u|_K$ and $v|_K$ are continuous. Then the matrix coefficient $(I_n(\frac{n-1}{2} - m)(g)u, v)$ is bounded by
\[
\sup(|u(k)|, k \in K) \sup(|v(k)|, k \in K) \Xi_{\eta(n - m - 1, m)}(g).
\]

Proof: Since $\eta(n - m - 1, m)$ is real, $\Xi_{\eta(n - m - 1, m)}(g)$ is a positive function. On the compact picture, the function $|u(k)|$ and $|v(k)|$ are bounded by multiples of the spherical vector–the constant function $1_K$. Since $\frac{n-1}{2} - m$ is real,
\[
|I_n(\frac{n-1}{2} - m)(g)u, v)| \leq \sup(|u(k)|, k \in K) \sup(|v(k)|, k \in K)|I_n(\frac{n-1}{2} - m)(g)1_K, 1_K = \sup(|u(k)|, k \in K) \sup(|v(k)|, k \in K) \Xi_{\eta(n - m - 1, m)}(g).
\]
3.3. **Howe’s Duality Correspondence and Small constituent** $V_m(\frac{n-1}{2} - m)$. The representation $I_n(s)$ of $SO_0(n, n)$ can be extended to a representation of $SO(n, n)$ in two ways. By abusing notation, we keep $I_n(s)$ to denote the representation $\text{Ind}_{GL(n,N)}^{SO(n,n)} \det |^s$. All $V_i(s)$ now inherits a $SO(n, n)$ action from $I_n(s)$, except that $V_\pm(\frac{s}{2}) \oplus V_\mp(\frac{s}{2})$ constitute an irreducible representation of $SO(n, n)$ when $n$ is even. So from now on, the small constituent $V_m(\frac{n-1}{2} - m)$ will carry a $SO(n, n)$ action.

In fact, $V_m(\frac{n-1}{2} - m)$ can be made into a representation of $O(n, n)$ as follows. We define an action of $O(n) \times O(n)$ on each $SO(n) \times SO(n)$-type $(\xi, \eta)$ of $(\xi, \eta)$ via $(\xi, +) \otimes (\xi, +)$. Then $V_m(\frac{n-1}{2} - m)$ becomes a representation of $O(n, n)$. Again, by abusing notation, use $V_m(\frac{n-1}{2} - m)$ to denote this representation of $O(n, n)$.

Howe’s duality correspondence is a one-to-one correspondence between a subset of the admissible dual of $O(p, q)$ and a subset of the admissible dual of $Sp_{2k}(\mathbb{R})$ ([H1089]). Let $(O(p, q), Sp_{2k}(\mathbb{R}))$ be a dual reductive pair in $Sp_{2k}(\mathbb{R})$. Let $\tilde{Sp}_{2k}(\mathbb{R})$ be the unique double covering of $Sp_{2k}(\mathbb{R})$. We say that $\pi_1$ and $\pi_2$ be irreducible admissible representations of $O(p, q)$ and $\tilde{Sp}_{2k}(\mathbb{R})$. The Harish-Chandra module is $P_{\pi_2}(\mathbb{R})$ is in the stable range. By a theorem of Lee ([LL]), the small constituent $V_m(\frac{n-1}{2} - m)$ corresponds to a one dimensional character under Howe’s duality correspondence. By the theory of stable range of local theta correspondence $\theta(2m; n, n)(\text{triv}) \cong \omega(n, n; 2m) \otimes Sp_{2m}(\mathbb{R}) \text{triv}$ ([L189]). Notice here we must have $m \leq \frac{n-1}{2}$. So the dual pair $(O(n, n), Sp_{2m}(\mathbb{R}))$ is in the stable range.

**Theorem 3.4** ([LL], [L189]). Suppose that the integer $0 \leq m \leq \frac{n-1}{2}$. The representation $\theta(2m; n, n)(\text{triv})$ is infinitesimally equivalent to $V_m(\frac{n-1}{2} - m)$. In addition, 

$$V_m(\frac{n-1}{2} - m) \cong \omega(n, n; 2m) \otimes Sp_{2m}(\mathbb{R}) \text{triv}.$$ 

Notice that when $p + q$ is even, $\theta(p, q; 2k)$ is a correspondence between representations of the linear group $O(p, q)$ and representations of the linear group $Sp_{2k}(\mathbb{R})$.

3.4. **Matrix Coefficients of the Oscillator Representation and $\Xi_{n(m-n-1, m)}(g)$**. Consider the oscillator representation $(\omega, L^2(\mathbb{R}^N))$ of $Sp_{2N}(\mathbb{R})$ as in [He10], [Heq]. Choose the standard maximal compact subgroup $K$. Then $K$ is the double covering of $U(N)$. There is a canonical unitary character $\sqrt{\det}$ on $K$. The Harish-Chandra module is $P(x) \exp -\frac{1}{2} ||x||^2$ with $P(x)$ the polynomial algebra on $x \in \mathbb{R}^N$. Let $P_k(x)$ be homogeneous polynomials of degree $k$. Let $P_{\leq k}(x)$ be the polynomials of degree less or equal to $k$. Then $P_k(x) \exp -\frac{1}{2} ||x||^2$ is not a $K$-invariant subspace, except when $k = 0$. But $P_{\leq k}(x) \exp -\frac{1}{2} ||x||^2$ is a $K$-invariant subspace. $\omega$ is a lowest weight module with the lowest weight vector $\exp -\frac{1}{2} ||x||^2$. The group $K$ acts on $\exp -\frac{1}{2} ||x||^2$ by scalar $\sqrt{\det}$. We sometimes call $\exp -\frac{1}{2} ||x||^2$ an almost spherical vector.
Lemma 3.5. Write \( g = k_1 \exp H(g) k_2 \) with \( k_1, k_2 \in K \) and \( H = (H_1, H_2, \ldots, H_N) \in \mathbb{R}^N \). Then

\[
F_N(g) = (\omega(k_1 \exp H k_2) \exp -\frac{1}{2}\|x\|^2, \exp -\frac{1}{2}\|x\|^2) = C_N(\sqrt{\det(k_1 k_2)}) \prod_{i=1}^{N} (\exp H_i + \exp -H_i)^{-\frac{1}{2}},
\]

where \( C_N \) is a positive constant depending only on \( N \). Obviously, if \( \sqrt{\det(k_1 k_2)} = 1 \), then \( F_N(g) > 0 \).

Now let \( N = 2nm \). Embed the dual pair \((O(n, n), Sp_{2m}(\mathbb{R}))\) as in [Heq]. Essentially, this amounts to a compactible \( KAK \) decomposition for the dual pair in \( Sp_{2N}(\mathbb{R}) \). In this particular situation, we actually have \((O(n, n), Sp_{2m}(\mathbb{R}))\) acting on \( L^2(\mathbb{R}^{2n} \otimes \mathbb{R}^m) \). Let \( k \in U(m) \). Then \( k \) acts on \( \exp -\frac{1}{2}\|x\|^2 \) by \( \sqrt{\det(k)}^n \sqrt{\det(k)}^m = 1 \). Obviously the compact group \( O(n)O(n) \) acts on \( \exp -\frac{1}{2}\|x\|^2 \) trivially. Hence we obtain

\[
F_N|_{O(n,n)Sp_{2m}(\mathbb{R})} > 0.
\]

Let 1 be a unit vector in the trivial representation of \( Sp_{2m}(\mathbb{R}) \). By Howe’s theory ([Ho89]), Theorem 3.4 and [Heq], the invariant tensor \( \exp -\frac{1}{2}\|x\|^2 \otimes Sp_{2m}(\mathbb{R}) \) 1 is a spherical vector of \( V_m(\frac{n}{2} - m) \). Hence we obtain

Theorem 3.6. Let \((O(n, n), Sp_{2m}(\mathbb{R}))\) be a dual pair in \( Sp_{4nm}(\mathbb{R}) \) as in [Heq]. Then the matrix coefficient

\[
(\omega(g_1, g_2) \exp -\frac{1}{2}\|x\|^2, \exp -\frac{1}{2}\|x\|^2) > 0 \quad ((g_1, g_2) \in (O(n, n), Sp_{2m}(\mathbb{R})))
\]

and

\[
\Xi_{\eta(n-m-1,m)}(g_1) = C \int_{\mathbb{R}^{2m}} (\omega(g_1, g_2) \exp -\frac{1}{2}\|x\|^2, \exp -\frac{1}{2}\|x\|^2) dg_2.
\]

I shall add one well-known lemma concerning the matrix coefficients of the oscillator representation.

Lemma 3.7. Let \( \omega, L^2(\mathbb{R}^N) \) be the oscillator representation of \( \tilde{Sp}_{2N}(\mathbb{R}) \). Let \( \phi, \psi \in \mathcal{P}(x) \exp -\frac{1}{2}\|x\|^2 \) be a \( K \)-finite vector. Then there exists a positive constant \( K_{\phi, \psi} \) such that

\[
|\omega(k_1 \exp H k_2)\phi, \psi) | \leq K_{\phi, \psi}|F_N(k_1 \exp H k_2)| = K_{\phi, \psi}C_N \prod_{i=1}^{N} (|H_i^{-1}| + |H_i|)^{-\frac{1}{2}}.
\]

Proof: Suppose that all \( K \)-translations of \( \phi \) and \( \psi \) are uniformly bounded by \( C(1+\|x\|^2)^M \exp -\frac{1}{2}\|x\|^2 \). This is possible because \( K \)-translations of \( \phi \) and \( \psi \) are in a compact subset in a finite dimension subspace of \( \mathcal{P}(x) \exp -\frac{1}{2}\|x\|^2 \). Clearly \( C(1+\|x\|^2)^M \exp -\frac{1}{2}\|x\|^2 \) is bounded by \( C_1 \exp -\frac{1}{2}\|x\|^2 \). An easy computation shows that

\[
|\omega(k_1 \exp H k_2)\phi, \psi) | \leq C_1^2 (\omega(\exp H) \exp -\frac{1}{4}\|x\|^2, \exp -\frac{1}{4}\|x\|^2) = K_{\phi, \psi}|F_N(k_1 \exp H k_2)|.
\]

\square

4. Quantum Induction and Positivity of the Invariant Hermitian Form

Let \( \pi \) be an admissible representation of a semisimple Lie group \( H \). Fix a maximal compact subgroup \( K_H \) of \( H \). **We use** \( V(\pi) \) **to denote the Harish-Chandra module of** \( \pi \). In most part of this paper, the maximal compact subgroup is specified explicitly or implicitly and \( V(\pi) \) is well-defined.

Let \( n = p + q + d \). Equip the vector space \( \mathbb{R}^{n,n} = \mathbb{R}^{2n} \) with the standard quadratic form of signature \((n, n)\) and the standard basis \( \{e_i, f_j \mid i \in [1, n], j \in [1, n]\} \). Decompose \( \mathbb{R}^{n,n} \) canonically.
as $\mathbb{R}^{p,q} \oplus \mathbb{R}^{q+d,p+d}$. This induces a canonical embedding of $SO(p, q)SO(q+d,p+d)$ into $SO(n, n)$. The maximal compact subgroups are chosen to be the standard ones.

Let $0 \leq m \leq \frac{n}{2}$. Denote the unitary representation $\theta(2m; n, n)(\text{triv})$ by $\mathcal{E}_m(n)$. Then $\mathcal{E}_m(n)$ is infinitesimally equivalent to $V_m(n-m)$. We also use $\mathcal{E}_m(n)$ to denote the group action and Lie algebra action. Let $(\pi, \mathcal{H}_\pi)$ be an irreducible unitary representation of $G = SO(p, q)$. Suppose that $V(\mathcal{E}_m(n)) \otimes_G V(\pi)$ is well-defined and nonzero. The nonvanishing of $V(\mathcal{E}_m(n)) \otimes_G V(\pi)$ will be established in Theorem 6.5.

From Lemma 1.17, we see that $V(\mathcal{E}_m(n)) \otimes_G V(\pi)$ is a $(g, K)$-module for $SO(q+d, p+d)$. Recall from Prop. 1.20 that $V(\mathcal{E}_m(n)) \otimes_G V(\pi)$ is equipped with a canonical $(\sigma(q+d, p+d), O(q+d)O(p+d))$ invariant Hermitian form

$$\left( \phi_1 \otimes_G u_1, \phi_2 \otimes_G u_2 \right) = \int_G \left( \mathcal{E}_m(n)(g) \phi_1, \phi_2 \right)(\pi(g) u_1, u_2) dg.$$

Fix a nonzero vector $u \in V(\pi)$. In this section, we shall show that this Hermitian form on $V(\mathcal{E}_m(n)) \otimes_G u$ is positive definite. Therefore, the $(\sigma(q+d, p+d), SO(q+d)SO(p+d))$-module $V(\mathcal{E}_m(n)) \otimes_G u$ has an invariant pre-Hilbert structure.

We call the process: $\mathcal{Q}(2m) : V(\pi) \to V(\mathcal{E}_m(n)) \otimes_G V(\pi)$ quantum induction. $\mathcal{Q}(2m)$ is sometimes written more completely as $\mathcal{Q}(p, q; 2m; q+d, p+d)$ ([Quan]). Quantum induction can be defined for all classical groups ([Quan]).

4.1. A Positivity Theorem. Let $V$ be a vector space over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ equipped with a nondegenerate sesquilinear form. Let $V = V_1 \oplus V_2$ and $V_1 \perp V_2$. Let $G(V)$ be the isometry group of $V$. We say a unitary representation of a unimodular group $G$ is almost square integrable if the matrix coefficients with respect to a dense subspace are in $L^{2+\epsilon}(G)$ for any $\epsilon > 0$.

Theorem 4.1. Let $G = G(V_1 \oplus V_2)$ and $G_1 = G(V_1)$. Let $(\sigma, \mathcal{H}_\sigma)$ be an almost square integrable representation of $G$ and $(\pi, \mathcal{H}_\pi)$ be an unitary representation of $G_1$. Here $\sigma$ is often not irreducible. Let $K$ be a maximal compact subgroup of $G$. Let $\Xi$ be Harish-Chandra’s basic spherical function for $G$. Let $\psi \in \mathcal{H}_\pi$ such that

$$\int_{G_1} |\Xi(g_1)(\pi(g_1) \psi, \psi)| dg_1 < \infty.$$

Then the invariant tensor product $(\mathcal{H}_\sigma)_K \otimes_{G_1} \mathbb{C}\psi$ is well-defined and its canonical invariant Hermitian form is positive definite.

Proof: Fix any $\phi \in (\mathcal{H}_\sigma)_K$. Since $\sigma$ is almost square integrable, by a Theorem of Cowling-Haagerup-Howe ([CHH]), for any $\phi_1 \in (\mathcal{H}_\sigma)_K$, there is a constant $C$ such that $|\langle \sigma(g) \phi, \phi_1 \rangle| \leq C \Xi(g)$. Since $\int_{G_1} |\Xi(g_1)(\pi(g_1) \psi, \psi)| dg_1 < \infty$, $(\sigma(g_1) \phi, \phi_1)(\pi(g_1) \psi, \psi) \in L^1(G_1)$. So $(\mathcal{H}_\sigma)_K \otimes_{G_1} \mathbb{C}\psi$ is well-defined. Now it suffices to show that $\langle \phi \otimes_G \psi, \phi \otimes_G \psi \rangle \geq 0$.

Since $\phi$ is $K$-finite, let $S \subset \hat{K}$ be the finite set of $K$-types occurring in the $K$-module generated by $\phi$. Let $C_c^\infty(G)(S)$ be the subspace of $C_c^\infty(G)$ such that the $K$-action from the left decomposes into the $K$-types in $S$. By our assumption, $(\sigma(g) \phi, \phi)$ is almost square integrable. So $(\sigma, \mathcal{H}_\sigma)$ is weakly contained in $L^2(G)$ ([CHH]). We can construct a sequence of $K$-finite functions $\phi_i$ in $C_c^\infty(G)(S)$ such that $(\sigma(g) \phi, \phi)$ can be approximated by $(L(g)\phi_i, \phi_i)$ uniformly on compacta. In particular, taking $g$ be the identity, we obtain

$$\|\phi_i\|_{L^2} \to \|\phi\|_{\mathcal{H}_\sigma}.$$
Hence $\|\phi_i\|_{L^2}$ is bounded uniformly for all $i$. Now by a Theorem of Cowling-Haagerup-Howe (CHH),

$$\|(L(g)\phi_i, \phi_i)\| \leq C \Xi(g) \quad (\forall g \in G),$$

where $C$ only depends on the norm $\|\phi_i\|$ and the $K$-types in $S$. Thus $C$ can be chosen uniformly for all $\phi_i$. Observe that

$$\int_{G_1} (L(g_1)\phi_i, \phi_i)(\pi(g_1)\psi, \psi)dg_1$$

$$= \int_{G_1} \int_G \phi_i(g_1^{-1}g)\overline{\phi_i(g)}dg_1\overline{(\pi(g_1)\psi, \psi)}dg_1$$

[absolutely integrable, since $\phi_i$ is compactly supported]

$$= \int_{G_1} \int_{G_1 \setminus G} \int_{G_1} \phi_i(g_1^{-1}h_1g)\overline{\phi_i(h_1g)}h_1d[G_1g](\pi(g_1)\psi, \psi)dg_1$$

$$= \int_{G_1 \setminus G} \int_{G_1 \times G_1} \phi_i(g_1g)\phi_i(h_1g)h_1d[G_1g](\pi(g_1)\psi, \psi)dg_1$$

$$= \int_{G_1 \setminus G} (\int_{G_1} \phi_i(g_1)\pi(g_1^{-1})\psi dg_1)(\int_{G_1} \phi_i(h_1g)\pi(h_1^{-1})\psi dh_1)d[G_1g]$$

$$\geq 0$$

Now $\{(L(g_1)\phi_i, \phi_i)(\pi(g_1)\psi, \psi)\}$ are uniformly bounded by an integrable function $|C \Xi(g_1)(\pi(g_1)\psi, \psi)|$. By the Dominated Convergence Theorem,

$$\int (\sigma(g_1)\phi, \phi)(\pi(g_1)\psi, \psi)dg_1 = \int \lim_{i \to \infty} (L(g_1)\phi_i, \phi_i)(\pi(g_1)\psi, \psi)dg_1$$

(7)

$$= \lim_{i \to \infty} \int_{G_1} (L(g_1)\phi_i, \phi_i)(\pi(g_1)\psi, \psi)dg_1 \geq 0.$$

Hence $(\phi \otimes_{G_1} \psi, \phi \otimes_{G_1} \psi) \geq 0$. □

4.2. **Positivity of the Hermitian Form.** Now we shall prove that for each $u \in V(\pi)$, the canonical Hermitian form on $V(\mathcal{E}_m(n)) \otimes_G u$ is positive definite. We shall apply Theorem 4.1.1

By Theorem 4.2, the matrix coefficients of $V(\mathcal{E}_m(n))$ are bounded by a multiple of

$$(1 + (H^+(g), H^+(g)))^Q \exp(-m, -m, \ldots, -m, -m + 1, -m + 1, -m + 2, -m + 2 \ldots, -1, -1, 0, 0)(H^+(g))$$

Since $n - 2m \geq 1$, their restrictions onto $SO(k, 2m + 2 - k)$ are almost square integrable. Hence $\mathcal{E}_m(n)|_{SO(k, 2m + 2 - k)}$ is almost square integrable. We can now apply Theorem 4.1 with $G = SO(k, 2m + 2 - k)$ and $G_1 = SO(p, q)$. The necessary and sufficient condition for the existence of a $G_1 \subseteq G$ is that $p + q \leq 2m + 2$. 
**Theorem 4.2.** Suppose that $p + q \leq 2m + 2 \leq n + 1$ and $p \leq q$. Let $\pi$ be an irreducible unitary representation of $SO(p, q)$ such that its every $K$ finite matrix coefficient $f(g)$ satisfies the condition that

$$|f(g)| \leq C_j \exp(m + 2 - p - q - \epsilon, m + 3 - p - q, \ldots, m + 1 - q)(|H^+(g)|).$$

for some $\epsilon > 0$. Then for any $u \in V(\pi)$, the invariant tensor product $V(\mathcal{E}_m(n)) \otimes_{SO(p, q)} u$ is well-defined and the canonical Hermitian form on $V(\mathcal{E}_m(n)) \otimes_{SO(p, q)} u$ is positive definite.

Proof: Suppose that $f(g) = (\pi(g)u, u)$ satisfies the condition that

$$|f(g)| \leq C \exp(m + 2 - p - q - \epsilon, m + 3 - p - q, m + 4 - p - q, \ldots, m + 1 - q)(|H^+(g)|).$$

Recall that $\Xi(x)$ for the group $SO(p, 2m + 2 - p)$ is bounded by

$$C_1(1 + (H^+(x), H^+(x)))^Q \exp(-m, -m + 1, -m + 2, \ldots, -m + p - 1)(H^+(x)).$$

Then $\Xi|_{SO(p, q)}(g)|f(g)|$ is bounded by

$$C_1C(1 + (H^+(x), H^+(x)))^Q \exp(2 - p - q - \epsilon, 4 - p - q, 6 - p - q, \ldots, 0)(|H^+(g)|),$$

which is clearly in $L^1(SO(p, q))$. So $\Xi|_{SO(p, q)}(g)|f(g)|$ is in $L^1(SO(p, q))$. We have seen that $\mathcal{E}_m(n)|_{SO(p, 2m + 2 - p)}$ is almost square integrable. By Theorem 4.1, the invariant tensor product $V(\mathcal{E}_m(n)) \otimes_{SO(p, q)} u$ is well-defined. In addition, for any $\phi \in V(\mathcal{E}_m(n))$, we have

$$(\phi \otimes_{SO(p, q)} u, \phi \otimes_{SO(p, q)} u) \geq 0.$$ 

Hence the invariant Hermitian form on $V(\mathcal{E}_m(n)) \otimes_{SO(p, q)} u$ is positive definite. \qed

**4.3. Independence of the Hilbert Structure.** Let $s = \frac{n - 1}{2} - m \geq 0$. Recall that $V_m(-s)$ is a subrepresentation of $I_n(-s)$ and it inherits a pre-Hilbert space structure and module structure from $I_n(-s)$. By Lemma 1.14 and Remark 1.15, $V_m(-s) \otimes_{SO(p, q)} V(\pi)$ inherits a $(\phi(q + d, p + d), SO(q + d)O(p + d))$-module structure from $V_m(-s)$. We have seen that on the Harish-Chandra module level $\mathcal{E}_m(n)$ can be identified with $V_m(-s)$. But their inner products are different. So given a Hilbert representation $(\pi, H)$ of $G$, can we identify $V(\mathcal{E}_m(n)) \otimes_{SO(p, q)} V(\pi)$ with $V_m(-s) \otimes_{SO(p, q)} \otimes_{SO(p, q)} V(\pi)$ on the Harish-Chandra module level?

Let $j : V(\mathcal{E}_m(n)) \rightarrow V_m(-s)$ be the identification of the Harish-Chandra module. Notice that both inner products on $V(\mathcal{E}_m(n))$ and $V_m(-s)$ are $K$-invariant. Let $(\cdot, \cdot)_1$ be the inner product on $V(\mathcal{E}_m(n))$ and $(\cdot, \cdot)_2$ be the inner product on $V_m(-s)$. We can construct a map $A : V(\mathcal{E}_m(n)) \rightarrow V_m(-s)$ such that for any $\phi, \psi \in V(\mathcal{E}_m(n))$

$$(\phi, \psi)_1 = (j(\phi), A(\psi))_2.$$ 

Since both inner products are nondegenerate, $A$ must be a bijection on each $K$-type. So $A$ is one-to-one and onto. We have

$$(\mathcal{E}_m(n)(g)\phi, \psi)_1 = (I_n(-s)(g)j(\phi), A(\psi))_2.$$

In particular $\sum_i (\phi_i \otimes_{SO(p, q)} v_i \in V(\mathcal{E}_m(n)) \otimes_{SO(p, q)} V(\pi)$ vanishes if and only if

$$\int_{SO(p, q)} ((\mathcal{E}_m(n)(g)\phi_i, \psi)_1(\pi_n(g)v_i, u)dg = 0 \quad (\forall \psi \in V(\mathcal{E}_m(n)), u \in V(\pi)), $$
Lemma 5.1. \(X\) acts on \(\mathbb{R}\{\text{ Fix the standard basis } G\}\)

Theorem 4.3. We see that the kernels of the two averaging operators are the same. We obtain.

\[\int_{SO(p,q)} ((I_n(-s)(g)j(\phi_i), A(\psi))_2(\pi_1(g)v_1, u)d\psi = 0 \quad (\forall \psi \in V(E_m(n)), u \in V(\pi)).\]

We see that the kernels of the two averaging operators are the same. We obtain.

**Theorem 4.3.** Let \(n = p + q + d\) and \(2m + 1 \leq n\). Let \((\pi, H)\) be an irreducible admissible Hilbert representation of \(G = SO(p,q)\). As Harish-Chandra modules of \(SO(q+d,p+d)\), we have \(V(E_m(n)) \otimes_G V(\pi) \cong V_m(m - \frac{n-1}{2}) \otimes_G V(\pi)\). Suppose that there is an invariant Hermitian form attached to the smooth vectors \(H^\infty\). Let \(hV(\pi)\) be the Harish-Chandra module equipped with this Hermitian structure. Then as Harish-Chandra modules of \(SO(q+d,p+d)\),

\[V(E_m(n)) \otimes_G V(\pi) \cong V(E_m(n)) \otimes_G hV(\pi)\]

I shall remark that this theorem holds for any semisimple group \(G\) and any irreducible unitary Harish-Chandra modules \(V(\pi)\) of a bigger semisimple group \(H \supseteq G\).

5. **Quantum Induction: Subrepresentation Theorem**

Let \(n = p + q + d\), \(0 \leq m \leq \frac{n-1}{2}\) and \(s = \frac{n-1}{2} - m\). Let \(\eta = \eta(n - m - 1,m)\). In this section, we shall show that \(Q(2m)(V(\pi))\) is a subrepresentation of the induced module \(Ind_{SO(q,p)GL(d)N}^{SO(q+d,p+d)} hV(\pi) \otimes |\det|^{-\frac{n-1}{2} + m}\) after we identify \(SO(q,p)\) with \(SO(p,q)\). Hence \(Q(2m)(V(\pi))\) is admissible and quasisimple.

Let us consider the \(SO(p,q)SO(q+d,p+d)\) action on \(I_n(s)\). \(I_n(s)\) is a quasisimple admissible representation of \(SO(n,n)\). The \(K\)-finite subspace \(V(I_n(s))\) is a Harish-Chandra module. Vectors in \(I_n(s)\) can be regarded as functions on \(X \cong SO(n,n)/P_0\), one branch of the maximal isotropic Grassmanian of \(\mathbb{R}^{n,n}\). Here \(P_0\) is the Siegel parabolic subgroup. The symmetric subgroup \(SO(p,q)SO(q+d,p+d)\) acts on \(X\) with a unique open dense orbit \(X_0\). The action on this open dense orbit can be described as follows.

Given two \(n\)-dimensional Euclidean spaces \(\mathbb{R}^n\) equipped with the standard inner products, let \(\mathbb{R}^{n,n} = \mathbb{R}^n \oplus \mathbb{R}^n\) be a 2\(n\)-dimensional real vector space equipped with the symmetric form

\[(x_1, y_1), (x_2, y_2)) = (x_1, x_2) - (y_1, y_2), \quad (x_1, x_2, y_1, y_2) \in \mathbb{R}^n.\]

Fix the standard basis \(\{e_1, e_2, \ldots, e_n, f_1, f_2, \ldots, f_n\}\). Let \(SO(n,n)\) be the special linear group preserving the symmetric form (\(\cdot, \cdot\)). We decompose \(\mathbb{R}^{n,n} = \mathbb{R}^{p,q} \oplus \mathbb{R}^{q+d,p+d}\) where \(\mathbb{R}^{p,q} = \text{span}(e_1, e_2, \ldots, e_p, f_1, f_2, \ldots, f_q)\), \(\mathbb{R}^{q+d,p+d} = \text{span}(e_{p+1}, e_{p+2}, \ldots, e_n, f_{q+1}, f_{q+2}, \ldots, f_n)\).

We obtain a diagonal embedding from \(SO(p,q)SO(q+d,p+d)\) into \(SO(n,n)\). Let

\[W_1 = \text{span}\{e_1 + f_{q+1}, e_2 + f_{q+2}, \ldots, e_p + f_{q+p}; f_1 + e_{p+1}, f_2 + e_{p+2}, \ldots, f_q + e_{p+q}\}\]
\[W_0 = \text{span}\{e_{p+q+1} + f_{p+q+1}, \ldots, e_{p+q+d} + f_{p+q+d}\}.\]

**Lemma 5.1.**

1. Let \(W = W_1 \oplus W_0\). Then \(W\) is a maximal isotropic subspace of \(\mathbb{R}^{n,n}\).
2. Fix \(W\) as the base point in \(X\). Let \(LN\) be the parabolic subgroup of \(SO(q+d,p+d)\) that preserves \(W_0\). Then \(L \cong SO(q,p)GL(d)\) and \(X_0 \cong SO(q+d,p+d)/GL(d)N\).
3. \(W\) induces a negative isometry between \(\mathbb{R}^{p,q}\) and \(\mathbb{R}^{q,p} \subseteq \mathbb{R}^{q+d,p+d}\).
4. \(W\) induces an identification of \(SO(p,q)\) with \(SO(q,p)\), denote this by \(g_1 \mapsto \hat{g}_1\).
5. Then \((SO(p,q)SO(q+d,p+d))w = \{(g_1, \hat{g}) \mid g_1 \in SO(p,q)\}GL(d)N\).
From now on, \( SO(p, q) \) will be identified with \( SO(q, p) \) via \( g \to \hat{g} \). Let \( \det \) be the character of \( P_n \) obtained by taking the absolute value of the determinant character on the Levi factor. The restriction of \( \det \) onto \( (SO(p, q)SO(q+d, p+d))_W \) is simply the absolute value of the determinant character of the \( GL(d) \) factor.

Let \( C_c^\infty(X_0, s) \) be smooth sections in \( I_n(s) \) with compact support in \( X_0 \). Notice that both \( C_c^\infty(X_0, s) \) and \( V(I_n(s)) \) are dense subspaces of the Hilbert representation \( I_n(s) \).

5.1. **Realization of** \( C_c^\infty(X_0, s) \otimes_{SO(p, q), C_c^\infty(X_0, -s)} V(\pi^*) \) \( V(\pi) \). The main result of this subsection is Theorem 5.2. This is the vector bundle version of Theorem 1.12.

Recall that there is a \( SO(n, n) \)-invariant complex linear pairing between \( I_n(s) \) and \( I_n(-s) \):

\[
(\phi, \psi) = \int_{S(O(n)O(n))/O(n)} \phi(k)\psi(k)d[k], \quad (\phi \in I_n(s), \psi \in I_n(-s)).
\]

This pairing induces a pairing between \( C_c^\infty(X_0, s) \) and \( C_c^\infty(X_0, -s) \). \( C_c^\infty(X_0, s) \), regarded as a smooth representation of \( SO(p, q)SO(q+d, p+d) \), consists of smooth sections of the homogeneous line bundle:

\[
SO(q+d, p+d) \times_{GL(d)N} \det^{-\frac{d+p}{2}} \to X_0,
\]

with compact support.

Let \( g_1 \in SO(p, q) \) and \( x \in X_0 \). By Lemma 5.1, \( x \) can be written as \( g_2W \) with \( g_2 \in SO(q+d, p+d) \). Then

\[
g_1g_2W = g_2g_1W = g_2g_1^{-1}\hat{g}_1^{-1}W = g_2\hat{g}_1^{-1}W.
\]

Hence left \( g_1 \) action on \( X_0 \cong SO(q+d, p+d)/GL(d)N \) is simply the right \( \hat{g}_1^{-1} \) action. In particular

\[
(I_n(s)g_1)f(g_2) = f(\hat{g}_1g_2) = f(g_2\hat{g}_1) = (R(\hat{g}_1)f)(g_2).
\]

Now equip \( X_0 \) with a left \( S(O(q+d)O(p+d)) \) and right \( SO(q, p) \) invariant measure. From the homogeneous line bundle structure \( [19] \), we see that the action of \( S(O(q+d)O(p+d)) \times SO(q, p) \) on \( X_0 \) is transitive. The stabilizer \( T \) is isomorphic to \( O(d)\Delta(S(O(q)O(p))) \). Here

\[
\Delta(S(O(q)O(p))) = \{(k, k) \mid k \in S(O(q)O(p))\} \subseteq S(O(q+d)O(p+d)) \times SO(q, p).
\]

Applying coordinate transformation to Eq. (8), we have

\[
(\phi, \psi) = \int_{S(O(q+d)O(p+d)) \times SO(q, p)/T} \phi(g)\psi(g)\Delta([g])d[g], \quad (\phi \in I_n(s), \psi \in I_n(-s)).
\]

Here \( \Delta([g]) \) is the Jacobian of the coordinate transformation from

\[
X_0 \ni [k] \to [g] \in S(O(q+d)O(p+d)) \times SO(q, p)/T.
\]

Notice that \( \phi(g)\psi(g) \) only depends on \( [g] \). Since this form is invariant under the action of \( S(O(q+d)O(p+d)) \times SO(q, p) \), \( \Delta([g]) \) must be a constant. We normalize the measure on \( X_0 \) so that \( \Delta([g]) = 1 \).

We identify \( SO(p, q) \) with \( SO(q, p) \subseteq SO(q+d, p+d) \) by \( g \to \hat{g} \). Let \( (\pi, H_\pi) \) be an irreducible unitary representation of \( SO(p, q) \). By Lemma 1.10 for any \( u \in V(\pi), v \in V(\pi^*) \), \( \phi \in C_c^\infty(X_0, s) \)
and \( \psi \in C^\infty_c(X_0, -s) \), we have
\[
(\phi \otimes \text{SO}(p,q), C_\pi(X_0, -s) \otimes V(\pi^*) \ u, \psi \otimes \text{SO}(p,q), C_\pi(X_0, s) \otimes V(\pi) \ v)
= \int_{\text{SO}(p,q)} (I_n(s)(g)\phi, \psi)(\pi(g)u, v)dg
= \int_{\text{SO}(p,q)} \int_{S(O(q+d)O(p+d)) \times \text{SO}(q,p)} \phi(xg)\psi(x)dg\int_{\text{SO}(q,p)} \phi(xh)\psi(xh)dh\int_{\text{SO}(q,p)} \phi(xh')\psi(xh(\pi^{-1}(h)^{-1}h')u, v)dh'dh\int_{\text{SO}(q,p)} \psi(xh)\pi^*(h)vdh'dx.
\]
(10)
It is easy to check that all these integrals are well-defined and converge absolutely. \textbf{Now put}
\[
\mathcal{I}(\phi \otimes u)(x) = \int_{\text{SO}(q,p)} \phi(xh)\pi(h)udh \quad (x \in \text{SO}(q + d, p + d)).
\]
The map \( \mathcal{I} \) is essentially integration on the right \( \text{SO}(q, p) \) fiber for
\[
\text{SO}(q, p) \to \text{SO}(q + d, p + d)/\text{GL}(d)N \to \text{SO}(q + d, p + d)/\text{SO}(q, p)\text{GL}(d)N.
\]
From the structure of the bundle [1], \( \mathcal{I}(\phi \otimes u)(x) \) can be identified with a smooth section of
\[
\text{SO}(q + d, p + d) \times \text{SO}(q,p)\text{GL}(d)N \mathcal{H}_\pi \otimes |\det|^{-s} \to \text{SO}(q + d, p + d)/\text{SO}(q, p)\text{GL}(d)N.
\]
Similarly, \( \mathcal{I}(\psi \otimes v)(x) \) can be identified with a smooth section of
\[
\text{SO}(q + d, p + d) \times \text{SO}(q,p)\text{GL}(d)N \mathcal{H}_\pi \otimes |\det|^{-s} \to \text{SO}(q + d, p + d)/\text{SO}(q, p)\text{GL}(d)N.
\]
In addition, the complex linear pairing \( (\phi \otimes \text{SO}(p,q), C_\pi(X_0, -s) \otimes V(\pi^*) \ u, \psi \otimes \text{SO}(p,q), C_\pi(X_0, s) \otimes V(\pi) \ v) \) is exactly
\[
(\mathcal{I}(\phi \otimes u), \mathcal{I}(\psi \otimes v))|_{S(O(q+d)O(p+d))/O(d)S(O(q)O(p))}.
\]
Notice here that \( \text{SO}(q + d, p + d)/\text{SO}(q, p)\text{GL}(d)N \cong S(O(q + d)O(p + d))/O(d)S(O(q)O(p)). \)
Recall that for \( P = \text{SO}(q, p)\text{GL}(d)N \subseteq \text{SO}(q + d, p + d), \) the half sum of positive roots in \( \Delta(\phi + d, p + d), \) a is exactly \( \frac{d(p+q+d-1)}{2} = \frac{d(n-1)}{2}. \) This corresponds to the character \( |\det|^{\frac{d(n-1)}{2}}. \)
Hence, we see that \( \mathcal{I}(C^\infty_c(X_0, s) \otimes V(\pi)) \) can be identified with certain smooth sections of
\[
\text{Ind}_{\text{SO}(q,p)\text{GL}(d)N}^{\text{SO}(q+d,p+d)} \mathcal{H}_\pi \otimes |\det|^s.
\]
**Theorem 5.2.** \textbf{Regard} \( \pi \) \textbf{as also as a representation of} \( \text{SO}(q, p) \) \textbf{by identifying} \( \text{SO}(q, p) \) \textbf{with} \( \text{SO}(q, p) \) \textbf{as in Lemma 5.1.} \textbf{For any} \( u \in V(\pi), \ v \in V(\pi^*), \ \phi \in C^\infty_c(X_0, s) \) \textbf{and} \( \psi \in C^\infty_c(X_0, -s) \), \textbf{we have}
\[
(\phi \otimes \text{SO}(p,q), C_\pi(X_0, -s) \otimes V(\pi^*) \ u, \psi \otimes \text{SO}(p,q), C_\pi(X_0, s) \otimes V(\pi) \ v) = (\mathcal{I}(\phi \otimes u), \mathcal{I}(\psi \otimes v))
\]
where the right hand side is the complex linear pairing between \( \text{Ind}_{\text{SO}(q,p)\text{GL}(d)N}^{\text{SO}(q+d,p+d)} \pi \otimes |\det|^s \) \textit{and} \( \text{Ind}_{\text{SO}(q,p)\text{GL}(d)N}^{\text{SO}(q+d,p+d)} \pi^* \otimes |\det|^{-s}. \) \textbf{In addition,} \( \mathcal{I} \) \textbf{induces an} \( \text{SO}(q + d, p + d) \)-\textbf{invariant isomorphism from}
\[
C^\infty_c(X_0, s) \otimes \text{SO}(p,q), C^\infty_c(X_0, -s) \otimes V(\pi^*) V(\pi).
\]
onto a dense subspace of smooth vectors in $\text{Ind}_{\text{SO}(q,p)\text{GL}(d),\pi}^{\text{SO}(q+d,p+d)} \otimes |\det|^s$.

Proof:
(1) The first statement is clear from the argument preceding the theorem.
(2) Next, we show that $\{I(\phi \otimes u) | \phi \in C_c^\infty(X_0, s), u \in V(\pi)\}$ is dense in $\text{Ind}_{\text{SO}(q,p)\text{GL}(d),\pi}^{\text{SO}(q+d,p+d)} \otimes |\det|^s$. Recall that the vector bundle $D$ is a $GL(d)_N$-principal bundle. Choose a local trivialization on an open $SO(q,p)$-invariant $\Omega \subset X_0$ for the vector bundle $D$. This amounts to choosing a local trivialization for the bundle

$$SO(q + d, p + d) \times_{SO(q,p)GL(d),N} \text{triv} \otimes |\det|^s \rightarrow SO(q + d, p + d)/SO(q,p)GL(d),N.$$ 

Then we have $C_c^\infty(\Omega, s) \cong C_c^\infty(\Omega, C)$. By Theorem 1.12, $I(C_c^\infty(\Omega, C) \otimes V(\pi))$ is dense in $L^2(\Omega \times_{SO(q,p)} \mathcal{H}_\pi)$. By partition of unity, we obtain the desired result.

(3) Finally, we want to show that $I$ induces an isomorphism from $C_c^\infty(X_0, s) \otimes SO(p,q),C_c^\infty(X_0,-s) \otimes V(\pi^*)$ onto its $\text{SO}(q+p,d)$-isotypic subspace, which will be finite dimensional. Let $C_c^\infty(X_0, s)_\sigma$ be the $\sigma$-isotypic subspace. Then

$$P_\sigma I(C_c^\infty(X_0, s) \otimes V(\pi)) = I(C_c^\infty(X_0, s)_\sigma \otimes V(\pi)).$$

By Theorem 5.2, the left hand side must be the full finite dimensional $P_\sigma(\text{Ind}_{\text{SO}(q,p)\text{GL}(d),\pi}^{\text{SO}(q+d,p+d)}) \otimes |\det|^s$. Otherwise, $I(C_c^\infty(X_0, s) \otimes V(\pi)))$ will not be dense in $\text{Ind}_{\text{SO}(q,p)\text{GL}(d),\pi}^{\text{SO}(q+d,p+d)} \otimes |\det|^s$. Hence

$$I(C_c^\infty(X_0, s)_\sigma \otimes V(\pi)) = P_\sigma(\text{Ind}_{\text{SO}(q,p)\text{GL}(d),\pi}^{\text{SO}(q+d,p+d)} \otimes |\det|^s).$$

Let $D(X_0, s)$ be the $S(O(q + d)O(p + d))-\text{finite subspace of} C_c^\infty(X_0, s)$. We have

Corollary 5.3. $I$ induces an equivalence of Harish-Chandra modules of $SO(q + d, p + d)$:

$$D(X_0, s) \otimes_{SO(q,p),C_c^\infty(X_0,-s)\otimes V(\pi^*)} V(\pi) \rightarrow V(\text{Ind}_{\text{SO}(q,p)\text{GL}(d),\pi}^{\text{SO}(q+d,p+d)} \otimes |\det|^s).$$

5.2. Two Invariant Tensor Products.

Theorem 5.4. Let $s = \frac{m-1}{2} - m$, $n = p + q + d$ and $\eta = \eta(n - m - 1, m)$. Let $\pi$ be an irreducible unitary representation of $SO(p,q)$ such that every leading exponent $\mu$ satisfies

$$\eta + (\mu, 0_{q+d}) - (d + 1, d + 2, \ldots, d + p, 0_{q+d}) < 0.$$ 

Then $D(X_0, -s) \otimes V(\pi^*)$ and $V(I_n(s))$ are two mutually $L^1$-dominated subspaces of $I_n(-s) \otimes \mathcal{H}_\pi^*$, with respect to $V(I_n(s)) \otimes V(\pi)$. We have
(1) as \( \langle \mathfrak{o}(q + d, p + d), S(O(q + d)O(p + d)) \rangle \)-modules,
\[
V(I_n(s)) \otimes_{SO(p,q)} V(\pi) \cong V(I_n(s)) \otimes_{SO(p,q), D(X_0, -s) \otimes V(\pi^*)} V(\pi).
\]
(2) as \( SO(q + d, p + d) \) representations,
\[
D(X_0, -s) \otimes_{SO(p,q), C_c^\infty(X_0, s) \otimes V(\pi)} V(\pi)^* \cong D(X_0, -s) \otimes_{SO(p,q), V(I_n(s)) \otimes V(\pi)} V(\pi)^*.
\]
(3) There is a nondegenerate \( \langle \mathfrak{o}(q + d, p + d), S(O(q + d)O(p + d)) \rangle \)-invariant complex linear pairing \( (\cdot, \cdot) \):
\[
(V(I_n(s)) \otimes_{SO(p,q)} V(\pi), D(X_0, -s) \otimes_{SO(p,q), C_c^\infty(X_0, s) \otimes V(\pi)} V(\pi^*)) \rightarrow \mathbb{C}.
\]

Proof: Fix \( K = S(O(n)O(n)) \). For each \( \phi \in D(X_0, -s) \), \( \phi|_K \) is continuous and thus bounded on \( K \). Let \( \psi \in V(I_n(s))^c \subseteq I_n(-s) \) such that \( \psi|_K \) approaches \( \phi|_K \) uniformly. This is always possible by essentially Stone-Weierstrass Theorem. Then for any \( \phi_0 \in V(I_n(s)) \) or \( \phi_0 \in D(X_0, s) \)
\[
(I_n(s)(g)\phi_0, \psi_i) \rightarrow (I_n(s)(g)\phi_0, \phi)
\]
and by Cor. 5.3 all \( \|I_n(s)(g)\phi_0, \psi_i\| \) are uniformly bounded by
\[
\|\phi_0|_K\| \sup_i \|\psi_i|_K\| \sup \Xi_{\eta}(g).
\]
By our assumption on \( \mu \) and Theorem 2.5 \( \Xi_{\eta}(g)|_{SO(p,q)}(\pi(g)u, v) \) is in \( L^1(SO(p,q)) \). By Definition 1.3 and Remark 1.4 \( D(X_0, -s) \otimes V(\pi^*) \) is \( L^1 \)-dominated by \( V(I_n(s))^c \otimes V(\pi)^c \) with respect to both \( D(X_0, s) \otimes V(\pi) \) and \( V(I_n(s)) \otimes V(\pi) \). Similarly, \( V(I_n(s))^c \otimes V(\pi)^c \) is also \( L^1 \)-dominated by \( D(X_0, -s) \otimes V(\pi^*) \).

By Theorem 1.5 we have
\[
V(I_n(s)) \otimes_{SO(p,q)} V(\pi) \cong V(I_n(s)) \otimes_{SO(p,q), D(X_0, -s) \otimes V(\pi^*)} V(\pi).
\]

By Lemma 1.4 and Lemma 1.10 both spaces inherits the same \( \langle \mathfrak{o}(q + d, p + d), S(O(q + d)O(p + d)) \rangle \)-module structure from \( V(I_n(s)) \). (1) follows. (2) follows similarly.

Now consider \( D(X_0, -s) \) as a dense subspace of the Hilbert representation \( I_n(-s) \), which is the dual of the Hilbert representation \( I_n(s) \). Let \( (\cdot, \cdot) \) be the pairing. By Lemma 1.10 we obtain a nondegenerate pairing
\[
(V(I_n(s)) \otimes_{SO(p,q), D(X_0, -s) \otimes V(\pi^*)} V(\pi), D(X_0, -s) \otimes_{SO(p,q), V(I_n(s)) \otimes V(\pi)} V(\pi^*)) \rightarrow \mathbb{C}.
\]
This pairing is \( \langle \mathfrak{o}(q + d, p + d), S(O(q + d)O(p + d)) \rangle \)-invariant, due to the structure of \( I_n(s) \) and \( I_n(-s) \). Combining with (1) and (2), we obtain a nondegenerate invariant pairing between \( V(I_n(s)) \otimes_{SO(p,q)} V(\pi) \) and \( D(X_0, -s) \otimes_{SO(p,q), C_c^\infty(X_0, s) \otimes V(\pi)} V(\pi^*) \). \( \square \)

Obviously, the same statements in Theorem 5.4 hold for \(-s\).

5.3. Subrepresentation Theorem. By Theorem 5.4 (3) and Corollary 5.3 we obtain a map
\[
\mathcal{I}_1 : V(I_n(s)) \otimes_{SO(p,q)} V(\pi) \rightarrow \text{Hom}(V(\text{Ind}_{SO(q+p,d)}^{SO(q+d,p+d)} GL(d)N^N \pi^* \otimes |\det|^{-s}), \mathbb{C})
\]
that preserves the \( \langle \mathfrak{o}(q + d, p + d), S(O(q + d)O(p + d)) \rangle \)-actions. Since all vectors in \( V(I_n(s)) \) are \( S(O(q + d)O(p + d)) \)-finite, all vectors in \( V(I_n(s)) \otimes_{SO(p,q)} V(\pi) \) are \( S(O(q + d)O(p + d)) \) finite. We obtain an \( \langle \mathfrak{o}(q + d, p + d), S(O(q + d)O(p + d)) \rangle \) isomorphism
\[
\mathcal{I}_1 : V(I_n(s)) \otimes_{SO(p,q)} V(\pi) \rightarrow \text{Hom}(V(\text{Ind}_{SO(q+p,d)}^{SO(q+d,p+d)} GL(d)N^N \pi^* \otimes |\det|^{-s}), \mathbb{C})_{SO(q+d)SO(p+d)} \cong V(\text{Ind}_{SO(q+p,d)}^{SO(q+d,p+d)} GL(d)N^N \pi \otimes |\det|^s).
\]
Notice that the pairing between \( V(I_n(s)) \otimes_{SO(p,q)} V(\pi) \) and \( V(Ind_{SO(q+pGL(d))N}^{SO(q+d,p+d)} \pi \otimes |\det|^s) \) is nondegenerate and \( S(O(q+d)O(p+d)) \)-invariant. Each \( S(O(q+d)O(p+d)) \) representation \( \sigma \) has only finite multiplicity in \( V(Ind_{SO(q+pGL(d))N}^{SO(q+d,p+d)} \pi \otimes |\det|^s) \). So the multiplicity of \( \sigma^* \) in \( V(I_n(s)) \otimes_{SO(p,q)} V(\pi) \) must be the same. We have shown that \( Z_1 \) is onto. We obtain

**Theorem 5.5.** Let \( s = \frac{n-1}{2} - m, n = p+q+d-1 \) and \( \eta = \eta(n-m-1,m) \). Let \( \pi \) be an irreducible unitary representation of \( SO(p,q) \) such that all leading exponent \( \mu \) satisfies

\[
\eta + (\mu,0_{q+d}) - (d+1,d+2,\ldots,d+p,0_{q+d}) < 0.
\]

Then \( V(I_n(s)) \otimes_{SO(p,q)} V(\pi) \cong V(Ind_{SO(q+pGL(d))N}^{SO(q+d,p+d)} \pi \otimes |\det|^s) \).

**Corollary 5.6.** Under the same hypothesis as Theorem 5.5, \( V_m(-s) \otimes_{SO(p,q)} V(\pi) \) can be identified as a subrepresentation of \( Ind_{SO(q+pGL(d))N}^{SO(q+d,p+d)} \pi \otimes |\det|^s \).

Proof: Recall from Theorem 3.4 that \( V_m(-s) \) is a submodule of \( V(I_n(-s)) \). Hence

\[
V_m(-s) \otimes_{SO(p,q),V(I_n(-s))} V(\pi) \rightarrow V(I_n(-s)) \otimes_{SO(p,q)} V(\pi).
\]

By Theorem 5.5 \( V_m(-s) \otimes_{SO(p,q),V(I_n(-s))} V(\pi) \) can be identified with a Harish-Chandra submodule of \( V(Ind_{SO(q+pGL(d))N}^{SO(q+d,p+d)} \pi \otimes |\det|^s) \).

Let \( W(-s) \) be the subspace of \( V(I_n(-s)) \) consisting of the \( K \)-types outside of \( V_m(-s) \). Notice that \( W(-s) \) is not a Harish-Chandra module of \( SO(n,n) \). Nevertheless, matrix coefficients \( (I_n(-s)(g)V_m(-s),W(-s)) \) is zero. In addition, \( V(I_n(-s)) = V_m(-s) \oplus W(-s) \). Hence

\[
V_m(-s) \otimes_{SO(p,q),V(I_n(-s))} V(\pi) \cong V_m(-s) \otimes_{SO(p,q),V(I_n(-s))} V(\pi) \cong V_m(-s) \otimes_{SO(p,q)} V(\pi).
\]

So \( V_m(-s) \otimes_{SO(p,q)} V(\pi) \) can be identified as a subrepresentation of \( Ind_{SO(q+pGL(d))N}^{SO(q+d,p+d)} \pi \otimes |\det|^s \).

By Theorem 4.3 we have

**Theorem 5.7.** Under the same hypothesis as Theorem 5.5, \( Q(2m)(V(\pi)) = V(\mathcal{E}_m(n)) \otimes_{SO(p,q)} V(\pi) \) can be identified as a subrepresentation of \( Ind_{SO(q+pGL(d))N}^{SO(q+d,p+d)} \pi \otimes |\det|^s \).

### 6. Howe’s Correspondence: Nonvanishing Theorem and Minimal \( K \)-types

In this section, we shall relate \( Q(2m) \) to the composition \( \theta(2m; q+d, p+d) \theta(q, p; 2m) \). The nonvanishing of \( Q(2m)(V(\pi)) \) then follows from Kudla’s preservation principle (Ku). We will then apply Howe’s technique to compute minimal \( K \)-types in \( Q(2m)(V(\pi)) \) ([Adams AH MO Paul]). In the cases when a Vogan’s minimal \( K \)-type ([Vogan79]) is of minimal degree in the sense of Howe ([Ho89]), we obtain the Langlands-Vogan parameter for \( Q(2m)(V(\pi)) \). This technique does not allow us to treat all \( Q(2m)(V(\pi)) \) because minimal \( K \)-types in the sense of Vogan may not be of minimal degree.

Suppose that \( 2m + 1 \leq n \). In this section, \( \mathcal{E}_m(n) \) will be a unitary representation of \( O(n,n) \), namely \( \theta(2m; n,n)(\text{triv}) \).

### 6.1. Associativity of Invariant Tensor Products

Let us recall the following theorem.

**Theorem 6.1 ([He00]).** Let \( (G_1, G_2) \) be a dual reductive pair in \( Sp \). Let \( \omega \) be the oscillator representation of \( Sp \). Let \( (G_1, G_2) \) be the induced covering from the metaplectic covering. Let \( \pi \) be an irreducible unitary representation of \( \hat{G}_1 \). Then \( V(\omega) \otimes_{\hat{G}_1} V(\pi) \), whenever defined, is equivalent to \( V(\theta(\pi^*)) \).
In the case of \((O(p, q), S_{2m}(\mathbb{R}))\), \(\theta\) can always be regarded as a correspondence between representations of \(O(p, q)\) and \(\widetilde{S}_{2m}(\mathbb{R})\). This will be our viewpoint from now on.

**Theorem 6.2.** Let \(n = p + q + d + 2m \leq n + 1\). Let \(\pi\) be an irreducible unitary representation of \(O(p, q)\). Suppose that \(V(\mathcal{E}_m(n)) \otimes_{O(p, q)} V(\pi)\) is well-defined. Then \(V(\omega(p + d + 2m)) \otimes_{\widetilde{S}_p(\mathbb{R})} (V(\omega(p, q; 2m)) \otimes_{O(p, q)} V(\pi))\) is well-defined and as Harish-Chandra modules of \(O(p + d + 2m)\).

\[ V(\mathcal{E}_m(n)) \otimes_{O(p, q)} V(\pi) \cong V(\omega(p + d + 2m)) \otimes_{\widetilde{S}_p(\mathbb{R})} (V(\omega(p, q; 2m)) \otimes_{O(p, q)} V(\pi)). \]

Recall from Theorem 3.6 that
\[ V(\mathcal{E}_m(n)) \cong V(\omega(n, n; 2m)) \otimes_{S_{2m}(\mathbb{R})} \text{triv} \cong [V(\omega(p + d + 2m)) \otimes V(\omega(p, q; 2m))] \otimes_{\widetilde{S}_p(\mathbb{R})} \text{triv}. \]

So when we restrict our representation onto \(O(p, q)O(p + d + 2m)\), we have
\[ V(\mathcal{E}_m(n))|_{O(p, q)O(p + d + 2m)} \cong V(\omega(p, q; 2m)) \otimes_{\widetilde{S}_p(\mathbb{R})} V(\omega(p + d + 2m)). \]

Then it suffices to prove the associativity of invariant tensor products
\[ V(\omega(p + d + 2m)) \otimes_{\widetilde{S}_p(\mathbb{R})} [V(\omega(p, q; 2m)) \otimes_{O(p, q)} V(\pi)] \]
\[ \cong [V(\omega(p + d + 2m)) \otimes_{\widetilde{S}_p(\mathbb{R})} V(\omega(p, q; 2m))] \otimes_{O(p, q)} V(\pi). \]

Theorem 3.6 and Lemma 3.7 allow us to apply Fubini’s theorem. Then we can interchange two integrals defined by the averaging operator \(A_{\widetilde{S}_p(\mathbb{R})}\) and \(A_{O(p, q)}\).

**Proof:** Suppose that \(V(\mathcal{E}_m(n)) \otimes_{O(p, q)} V(\pi)\) is well-defined.

1. Let \(\exp -\frac{1}{2}||x||^2\) and \(\exp -\frac{1}{2}||y||^2\) be the lowest weight vectors in the Schrödinger model of \(\omega(p, q; 2m)\) and \(\omega(q + d + 2m)\). Then \(\exp -\frac{1}{2}||x||^2 \otimes_{\widetilde{S}_p(\mathbb{R})} \exp -\frac{1}{2}||y||^2\) yields a nonzero spherical vector in \(\mathcal{E}_m(n)\). We obtain, \(\forall \ g_1 \in O(p, q), \ g_2 \in O(q + d + 2m)\),

\[ \Xi_{\omega}(g_1, g_2) = C \int_{h \in S_{2m}(\mathbb{R})} (\omega(p, q; 2m)(g_1, h) \exp -\frac{1}{2}||x||^2, \exp -\frac{1}{2}||x||^2) \]
\[ (\omega(q + d + 2m)(g_2, h) \exp -\frac{1}{2}||y||^2, \exp -\frac{1}{2}||y||^2) dh. \]

By Theorem 3.3, the integrand is always positive. Since the vector \(\exp -\frac{1}{2}||x||^2 \otimes_{\widetilde{S}_p(\mathbb{R})} \exp -\frac{1}{2}||y||^2 \otimes_{O(p, q)} V\) is well-defined for all \(v \in V(\pi)\), we have \(\Xi_{\omega}(g_1)((\pi(g_1)u, v) \in L^1(\Omega(p, q))\).

By Fubini’s theorem,
\[ (\omega(p, q; 2m)(g_1, h) \exp -\frac{1}{2}||x||^2, \exp -\frac{1}{2}||x||^2)(\omega(q + d + 2m)(h) \exp -\frac{1}{2}||y||^2, \exp -\frac{1}{2}||y||^2)((\pi(g_1)u, v) \]

is integrable on \((g_1, h) \in O(p, q) \times \widetilde{S}_{2m}(\mathbb{R})\).

2. In particular, for almost all \(h \in \widetilde{S}_{2m}(\mathbb{R})\), we have
\[ |(\omega(p, q; 2m)(g_1, h) \exp -\frac{1}{2}||x||^2, \exp -\frac{1}{2}||x||^2)(\pi(g_1)u, v) | \in L^1(\Omega(p, q)). \]

From Lemma 3.3, we have the exact form of the function \(|(\omega(p, q; 2m)(g_1, h) \exp -\frac{1}{2}||x||^2, \exp -\frac{1}{2}||x||^2)| : \]
\[ \prod_{i=1}^{m} \prod_{j=1}^{m} (a_i^2 + b_j^{-2} + b_j^{-2})^{\frac{1}{2}d} \prod_{j=1}^{m} (b_j + b_j^{-1})^{-\frac{d+2}{2}}, \]

where \(g_1 = k_1 a_k k_2\) and \(h = u_1 b u_2\) are the KAK decompositions for \(O(p, q)\) and \(\widetilde{S}_{2m}(\mathbb{R})\). See [He2] for more details. The integrability of \((14)\) for one \(h \in \widetilde{S}_{2m}(\mathbb{R})\) implies the integrability...
of (14) for the identity element $e$. This is essentially due to the fact that for fixed $b$,
\[
\frac{1}{a_i^2 + a_i^{-2} + b_j^2 + b_j^{-2}} \geq \frac{2}{(b_j^2 + b_j^{-2})(a_i^2 + a_i^{-2} + 2)}.
\]
Since
\[
\frac{1}{a_i^2 + a_i^{-2} + b_j^2 + b_j^{-2}} \leq \frac{1}{(a_i + a_i^{-1})(a_i + a_i^{-1})},
\]
the integrability of (14) at $e$ then implies the integrability of (14) for every $h \in \widetilde{Sp}_{2m}(\mathbb{R})$. Now by Lemma 6.3, for any $\phi_1, \phi_2 \in \mathcal{P}(x)$ and $\psi_1, \psi_2 \in \mathcal{P}(y)$, we have for every $h \in \widetilde{Sp}_{2m}(\mathbb{R})$,
\[
|\omega(p, q; 2m)(g_1, h)(\phi_1(x) \exp -\frac{1}{2}\|x\|^2), \phi_2 \exp -\frac{1}{2}\|x\|^2)|(|\phi(g_1)u, v)| \in L^1(O(p, q)).
\]
It follows that $V(\omega(p, q; 2m)) \otimes_{O(p, q)} V(\pi)$ is well-defined. By Theorem 6.1
\[
V(\omega(p, q; 2m)) \otimes_{O(p, q)} V(\pi) = V(\theta(p, q; 2m)(\pi^*)).
\]
(3) Now $V(\omega(p, q; 2m)) \otimes_{O(p, q)} V(\pi)$ is a Harish-Chandra module for a continuous representation $\theta(p, q; 2m)(\pi^*)$. Since $\omega(p, q; 2m)$ and $\pi$ are both unitary, by Prop. 1.20 the canonical Hermitian form is $(sp_{2m}(\mathbb{R}), U(m))$-invariant. By Lemma 6.7 and Part (1) of the proof, we have
\[
(\omega(p, q; 2m)(g_1, h)(\phi_1(x) \exp -\frac{1}{2}\|x\|^2), \phi_2 \exp -\frac{1}{2}\|x\|^2)
\]
(\omega(q + d, p + d, 2m)(h)(\psi_1(y) \exp -\frac{1}{2}\|y\|^2), \psi_2(y) \exp -\frac{1}{2}\|y\|^2)|(|\psi(g_1)u, v)|
\]
is integrable on $O(p, q) \times \widetilde{Sp}_{2m}(\mathbb{R})$. So $V(\omega(q + d, p + d; 2m)) \otimes_{\widetilde{Sp}_{2m}(\mathbb{R})} [V(\omega(p, q; 2m)) \otimes_{O(p, q)} V(\pi)]$ is well-defined.

By Fubini’s theorem, we have
\[
V(E_m(n)) \otimes_{O(p, q)} V(\pi) \cong V(\omega(q + d, p + d; 2m)) \otimes_{\widetilde{Sp}_{2m}(\mathbb{R})} (V(\omega(p, q; 2m)) \otimes_{O(p, q)} V(\pi)).
\]
In addition, both spaces inherit the same $(o(q + d, p + d), O(q + d)O(p + d))$-module structure from $V(\omega(q + d, p + d; 2m))$. Our theorem follows immediately.

\[
\square
\]

6.2. Nonvanishing of Quantum Induction. Identify $O(p, q)$ with $O(q, p)$ as in Lemma 5.1. This amounts to essentially rearranging the coordinates. A representation $\pi$ of $SO(p, q)$ will also be regarded as a representation of $SO(q, p)$. We would like to relate $Q(2m)(V(\pi))$ to $\theta(2m; q + d, p + d)\theta(q, p; 2m)(\text{Ind}_{SO(q, p)}^{\widetilde{Sp}_{2m}(\mathbb{R})})$.

Lemma 6.3. As unitary representations of $(O(p, q), \widetilde{Sp}_{2m}(\mathbb{R}))$, $\omega(p, q; 2m) \cong \omega(q, p; 2m)^* \cong \omega(q, p; 2m)^c$. In addition, $\theta(p, q; 2m)(\pi^c) \cong [\theta(q, p; 2m)(\pi)]^c$.

Proof: Let $\omega_m$ be the oscillator representation of $\widetilde{Sp}_{2m}(\mathbb{R})$. Then $\omega(p, q; 2m)$ can be modelled by $[\otimes^p \omega_m] \otimes [\otimes^q \omega_m]$. So $\omega(p, q; 2m)^c$ can be modelled by $[\otimes^p \omega_m^c] \otimes [\otimes^q \omega_m^c]$. By reordering of coordinates, we have $\omega(p, q; 2m)^c \cong \omega(q, p; 2m)$.

If $\pi \otimes \theta(p, q; 2m)(\pi)$ appears as a quotient of $\omega(p, q; 2m)$, then $\pi^c \otimes [\theta(p, q; 2m)(\pi)]^c$ appears as a quotient of $\omega(p, q; 2m)^c \cong \omega(q, p; 2m)$. So $\theta(p, q; 2m)(\pi^c) \cong [\theta(q, p; 2m)(\pi)]^c$. Equivalently,
\[
\theta(p, q; 2m)(\pi^c) \cong [\theta(q, p; 2m)(\pi)]^c.
\]

\[
\square
\]
Theorem 6.4. If \(2m + 1 \geq p + q\), then either \(\theta(2m; q + d, p + d)(\theta(q, p; 2m)(\pi)) \neq 0\) or \(\theta(2m; q + d, p + d)(\theta(q, p; 2m)(\pi \otimes \text{det})) \neq 0\).

Proof: By a theorem of Moeglin ([MO]) and a theorem of Adams-Barbasch ([AB]), we have either \(\theta(q, p; 2m)(\pi) \neq 0\) or \(\theta(q, p; 2m)(\pi \otimes \text{det}) \neq 0\). See also [Henon]. By Kudla’s preservation principle ([KV]), \(\theta(2m; q + d, p + d)(\theta(q, p; 2m)(\pi)) \neq 0\) if \(\theta(q, p; 2m)(\pi) \neq 0\). Our theorem then follows. □

Now suppose that \(\pi\) is an irreducible unitary representation of \(O(p, q)\), then \(\pi^* \cong \pi\). By a Theorem of Przebinda [P2], \(\theta(\pi)\) will have an invariant Hermitian structure, i.e., \([\theta(\cdot)(\pi)]^* \cong [\theta(\cdot)(\pi)]^\circ\). We obtain
\begin{align}
V(\mathcal{E}_m(n)) \otimes_{O(p-q)} V(\pi) \\
\cong V(\omega(q + d, p + d; 2m)) \otimes_{\mathbb{S}P_{2m}(\mathbb{R})} (V(\omega(p, q; 2m)) \otimes_{O(p,q)} V(\pi)) \\
\cong V(\omega(q + d, p + d; 2m)) \otimes_{\mathbb{S}P_{2m}(\mathbb{R})} V(\theta(q, p; 2m)(\pi^*)) \\
\cong V(\theta(2m; q + d, p + d)(\theta(q, p; 2m)(\pi^*))) \\
\cong V(\theta(2m; q + d, p + d)(\theta(q, p; 2m)(\pi))) \\
\end{align}
\[(16)\]

Theorem 6.5. Suppose that \(2m + 1 \geq p + q\). Let \(\pi\) be an irreducible unitary representation of \(SO(p, q)\). Then \(Q(2m)(V(\pi)) \neq 0\). As Harish-Chandra modules of \(SO(q + d, p + d), Q(2m)(V(\pi))\) is equivalent to \(\theta(2m; q + d, p + d)(\theta(q, p; 2m)(\pi))\) if \(\pi\) is unique, is equivalent to
\[\theta(2m; q + d, p + d)(\theta(q, p; 2m)(\pi)) \oplus \theta(2m; q + d, p + d)(\theta(q, p; 2m)(\pi \otimes \text{det}))\]
if \(\pi\) is not unique.

Proof: First of all, we have
\[(17)\]
\[V(\mathcal{E}_m(n)) \otimes_{SO(p-q)} V(\pi) \cong V(\mathcal{E}_m(n)) \otimes_{O(p,q)} (V(\text{Ind}_{SO(p,q)}^{O(p,q)} \pi)).\]

If \(\text{Ind}_{SO(p,q)}^{O(p,q)} \pi\) is irreducible, then \(\pi \cong \text{Ind}_{SO(p,q)}^{O(p,q)} \pi\) and \(\text{Ind}_{SO(p,q)}^{O(p,q)} \pi \otimes \text{det} \cong \text{Ind}_{SO(p,q)}^{O(p,q)} \pi\). By the previous theorem and Eq. (16)
\[Q(2m)(V(\pi)) \cong \theta(2m; q + d, p + d)(\theta(q, p; 2m)(\pi)) \neq 0.\]

If \(\text{Ind}_{SO(p,q)}^{O(p,q)} \pi\) is not irreducible, then it must contain two subrepresentations \(\pi\) and \(\pi \otimes \text{det}\), where \(\pi\) is a unitary representation of \((O(p,q), \text{det})\). By the previous theorem,
\[Q(2m)(V(\pi)) \cong \theta(2m; q + d, p + d)(\theta(q, p; 2m)(\pi)) \oplus \theta(2m; q + d, p + d)(\theta(q, p; 2m)(\pi \otimes \text{det})) \neq 0.\]

6.3. Langlands-Vogan Parameter of Quantum Induced Module. Howe’s correspondence uniquely determines a one to one correspondence between the \(K\)-types of the lowest degrees in \(\pi\) and \(\theta(\pi)\) ([Ho89]). We denote this \(K\)-types correspondence by \(\theta_0\). For any \(K\)-type \(\tau\), let \(d(\tau, \omega)\) be the lowest degree of \(\tau\)-isotypic subspace in the Fock-Segal-Bargman model of \(\omega\). \(\theta_0\) is often called the correspondence of joint harmonics. \(\theta_0\) is known explicitly ([KV] [MO] [Adams]). In the instances that a minimal \(K\)-type in the sense of Vogan is a \(K\)-type of lowest degree, Howe’s correspondence can be determined in terms of the Langlands-Vogan parameter. This has been done mostly in the equal rank case ([MO] [AB] [Paul]). With the appearance of \((\xi, -)\) for \(O(p)\) or \(O(q)\), a minimal \(K\)-type may not be a \(K\)-type of the lowest degree and Howe’s correspondence is difficult to compute explicitly.

Two necessary conditions for defining nonvanishing quantum induction in Theorems 5.7 and 6.3 are \(p + q \leq 2m + 1\) and \(s = \frac{p+q+d-1}{2} - m \geq 0\). We suppose that \(p + q \leq 2m + 1 \leq p + q + d\).
Suppose that \(((ξ_1, +), (ξ_2, +))\) is a minimal $O(p)O(q)$ type of an irreducible constituent $\pi$ of $\text{Ind}^{O(p)O(q)}_{SO(p,q)} \tau$. The minimal degree of $((ξ_2, +), (ξ_1, +))$ in $ω(q, p; 2m)$ is the sum of all entries of $ξ_1$ and $ξ_2$ [MO [AB [Adams]]. The key observation is that $d(((ξ_2, +), (ξ_1, +)), ω(q, p; 2m))$ is independent of $m$ as long as $p + q ≤ 2m + 1$. By the results of Moeglin, Adam-Barbasch and Paul, for the equal size cases,

$$d(((ξ_2, +), (ξ_1, +)), ω(q, p; 2m)) = \min\{d(τ, ω(q, p; 2m)) \mid τ ⊆ V(π), τ ∈ O(q)O(p)\},$$

i.e., the minimal $K$-type $(ξ_2, +) ⊗ (ξ_1, +)$ is a $K$-type of minimal degree. It follows that for any $2m ≥ p + q - 1$, we have

$$d(((ξ_2, +), (ξ_1, +)), ω(q, p; 2m)) = \min\{d(τ, ω(q, p; 2m)) \mid τ ⊆ V(π), τ ∈ O(q)O(p)\}. $$

Now $θ(q, p; 2m)(π)$ is an irreducible representation of $SO_{2m}(R)$. By Howe’s theory ([Ho89]), the corresponding $K$-type

$$θ_0(q, p; 2m)((ξ_2, +) ⊗ (ξ_1, +)) = (ξ_2, 0, -ξ_1) + \frac{q-p}{2}. $$

This $K$-type in $θ(q, p; 2m)(π)$ has a similar property:

$$d(((ξ_2, 0, -ξ_1) + \frac{q-p}{2}), ω(q, p; 2m)) = \min\{d(τ', ω(q, p; 2m)) \mid τ' ⊆ V(θ(q, p; 2m)(π))\}. $$

Observe that $d(τ', ω(q, p; 2m)) = d(τ', ω(q+d, p+d; 2m))$. So the statement above holds if we replace $ω(q, p; 2m)$ by $ω(q + d, p + d; 2m)$. It follows that the $K$-type $(ξ_2, 0, -ξ_1) + \frac{q-p}{2}$ of $θ(q, p; 2m)(π)$ is also of the minimal degree in $ω(2m; q + d, p + d)$. Hence

$$θ_0(2m; q + d, p + d)((ξ_2, 0, -ξ_1) + \frac{q-p}{2}) ⊆ V(θ(2m; q + d, p + d)θ(q, p; 2m)(π)). $$

The left hand side is equal to $((ξ_2 ⊕ 0, +), (ξ_1 ⊕ 0, +)).$ Hence the $K$-type $((ξ_2 ⊕ 0, +), (ξ_1 ⊕ 0, +))$ occurs in $θ(2m; q + d, p + d)θ(q, p; 2m)(π)$. Notice that $((ξ_2 ⊕ 0), (ξ_1 ⊕ 0), +)$ is the minimal $K$-type of $Ind^{SO(q+d,p+d)}_{SO(q,p)GL(d,N)}π ⊗ |det|^s$ (Ch 8. [GW]).

**Theorem 6.6.** Suppose that $p + q ≤ 2m + 1 ≤ p + q + d$. Suppose that $(ξ_1, ξ_2, +)$ is a minimal $S(O(p)O(q))$ type of $π$. Then $Q_{2m}(V(π))$ contains the irreducible Vogan subquotient of $Ind^{SO(q+d,p+d)}_{SO(q,p)GL(d,N)}π ⊗ |det|^s$. Here $s = \frac{d}{2} - m = \frac{p+q+d-2m-1}{2}.$

7. Unitary Langlands-Vogan Parameters

The main purpose of this section is to determine the unitarity of certain Langlands-Vogan parameter. By a theorem of Harish-Chandra, an admissible representation $π$ is unitarizable if $V(π)$ has a $(g, K)$-invariant pre-Hilbert structure. By Prop. 1.20, $Q(2m)(V(π))$ has an invariant Hermitian form. Theorem 4.2 allows us to determine when this Hermitian form will be positive definite. We have the following

**Theorem 7.1** (Theorem A). Suppose that $p + q ≤ 2m + 1 ≤ p + q + d$ and $p ≤ q$. Let $π$ be an irreducible unitary representation of $SO(p, q)$ such that its every $K$ finite matrix coefficient $f(g)$ satisfies the condition that

$$|f(g)| ≤ C_f \exp(m + 2 - p - q - \epsilon, m + 3 - p - q, ..., m + 1 - q)(|H^+(g)|).$$

for some $\epsilon > 0$. Then

1. $Q(p, q; 2m; q + d, p + d)(V(π))$ is well-defined;
(2) \( \mathcal{Q}(p, q; 2m; q + d, p + d)(V(\pi)) \) is nonvanishing;
(3) \( \mathcal{Q}(p, q; 2m; q + d, p + d)(V(\pi)) \) is a subrepresentation of \( \text{Ind}_{SO(q,p)GL(d)N}^{SO(q+d,p+d)} \pi \otimes |\det|^{-\frac{m-p+q-d-1}{2}} \) upon identifying \( SO(p, q) \) with \( SO(q, p) \);
(4) \( \mathcal{Q}(p, q; 2m; q + d, p + d)(V(\pi)) \) is unitarizable.

Proof: (1) follows from Theorem 4.2, (2) follows from Theorem 5.5, (3) follows from Cor 5.6. We only need to prove (4). Notice that

\[
V(\xi_m(n)) \otimes_{SO(p,q)} V(\pi) = V(\xi_m(n)) \otimes_{O(p,q)} V(\text{Ind}_{SO(p,q)}^{O(p,q)} \pi).
\]

is either irreducible or decompose into two irreducible admissible representations of \( O(q + d, p + d) \). Each irreducible one will be of the form \( \text{Ind}_{\xi_m(n)}^{SO(p,q)} u \) for some \( u \in V(\text{Ind}_{SO(p,q)}^{O(p,q)} \pi) \). Applying Theorem 4.2 to \( O(p, q) \), the invariant Hermitian form on \( V(\xi_m(n)) \otimes_{O(p,q)} u \) is positive definite. By Harish-Chandra’s theorem, \( \mathcal{Q}(p, q; 2m; q + d, p + d)(V(\pi)) \) is unitarizable. □

**Theorem 7.2** (Theorem B). Suppose that \( p + q \leq 2m + 1 \leq p + q + d \) and \( p + q \leq q \). Let \( \pi \) be an irreducible unitary representation of \( SO(q, p) \) such that its every \( K \) finite matrix coefficient \( f(g) \) satisfies the condition that

\[
|f(g)| \leq C_\eta \exp\left(m + 2 - p - q - \epsilon, m + 3 - p - q, \ldots, m + 1 - q\right)(|H^+(g)|).
\]

for some \( \epsilon > 0 \). Suppose that the minimal \( S(O(q)O(p)) \)-types of \( \pi \) contain a \( (\xi, \eta, +) \). Then the Vogan subquotient of

\[
\text{Ind}_{SO(q,p)GL(1)^N}^{SO(q+d,p+d)} \pi \otimes |\det|^{-\frac{m-p+q-d-1}{2}} \] \( \otimes |\det|^{-\frac{m-1-q}{2}} \ldots |\det|^{-\frac{m-2+q}{2}} \)

is unitary.

Proof: Since

\[
m - \frac{p + q + d - 1}{2} - \rho(SL_d(\mathbb{R})) = (m - \frac{p + q}{2} - d + 1, m - \frac{p + q}{2} - d + 2, \ldots, m - \frac{p + q}{2}),
\]

the Vogan subquotient of \( \text{Ind}_{SO(q,p)GL(1)^N}^{SO(q+d,p+d)} \pi \otimes |\det|^{-\frac{2m-p+q-d-1}{2}} \) is exactly the Vogan subquotient of

\[
\text{Ind}_{SO(q,p)GL(1)^N}^{SO(q+d,p+d)} \pi \otimes |\det|^{-\frac{m-p+q-d-1}{2}} \ldots |\det|^{-\frac{m-2+q}{2}} \]

Our theorem follows from Theorem 6.6 and Theorem [A] (4). □

7.1. Arthur’s Packet. Let \( G \) be an inner form of an algebraic reductive group. Let \( ^L G \) be the Langlands dual group. Let \( W_\mathbb{R} = \mathbb{Z}_2 \times \mathbb{C}^\mathbb{R} \) be the Weil group. A Langlands parameter \( \phi : W_\mathbb{R} \to ^L G \) can be decomposed into a product of a compact part \( \phi_0 \) and a noncompact part \( \phi_+ \). \( \phi_0 \) and \( \phi_+ \) commute with each other. Arthur’s parameter is a map

\[
\psi : W_\mathbb{R} \times SL(2, \mathbb{C}) \to ^L G,
\]

such that \( \psi|_{W_\mathbb{R}} \) is a tempered parameter. Since \( \psi(W_\mathbb{R}) \) and \( \psi(SL(2, \mathbb{C})) \) commute, \( \psi(SL(2, \mathbb{C})) \) must be in the centralizer of \( \psi(W_\mathbb{R}) \), in fact, the identity component of the centralizer of \( \psi(W_\mathbb{R}) \). Let \( C(\psi(W_\mathbb{R}))_0 \) be the identity component of the centralizer. \( C(\psi(W_\mathbb{R}))_0 \) is a reductive group. Then Arthur defines a Langlands parameter

\[
\phi_\psi : \phi_\psi(w) = \psi(w, H_{|w|}), (w \in W_\mathbb{R})
\]

where \( H_{|w|} = \text{diag}(|w|^\frac{1}{2}, |w|^{-\frac{1}{2}}) \in SL(2, \mathbb{C}) \). Arthur conjectured that the representations in \( \phi_\psi \) are unitary ([AP83]).
7.2. Type D: \( p+q \) even. In the case \( G = SO(p, q) \) with \( p+q \) even, \( L^G_0 = SO(p+q, \mathbb{C}) \). We consider only those \( \psi \) with \( \psi(w) \) a tempered representation of \( SO(p-k, q-k) \) and \( \psi(SL(2, \mathbb{C}))_0 \subseteq SO(2k, \mathbb{C}) \). We recall the following facts ([CM]).

(1) Group homomorphisms from \( SL(2, \mathbb{C}) \) to \( SO(2k, \mathbb{C}) \) are in one-to-one correspondence to Lie algebra homomorphisms from the standard triple \( \{ H, X, Y \} \) to the Lie algebra \( \mathfrak{so}(2n, \mathbb{C}) \).

(2) The \( SO(2n, \mathbb{C}) \)-conjugacy classes of Lie algebra homomorphisms from the standard triple \( \{ H, X, Y \} \) to the Lie algebra \( \mathfrak{so}(2n, \mathbb{C}) \) are in one-to-one correspondence with nilpotent adjoint orbits of \( SO(2n, \mathbb{C}) \), namely the \( SO(2n, \mathbb{C}) \) conjugacy class of \( X \).

(3) Nilpotent adjoint orbits of \( O(2k, \mathbb{C}) \) are in one-to-one correspondence with orthogonal Young diagrams of size \( 2k \). A Young diagram is called orthogonal if each even part occurs with even multiplicity.

Denote the nilpotent adjoint orbit of \( O(2k, \mathbb{C}) \) corresponding to \( \mathcal{O}_D \) by \( \mathcal{O}_D \). We say \( \mathcal{O}_D \) is very even if only even parts occur in \( \mathcal{O}_D \). Almost all nilpotent adjoint orbits \( \mathcal{O}_D \) of \( O(2k, \mathbb{C}) \) are nilpotent adjoint orbits of \( SO(2k, \mathbb{C}) \), except for very even \( \mathcal{O}_D \). For very even \( \mathcal{O}_D \), \( \mathcal{O}_D \) splits into two nilpotent orbits of \( SO(2k, \mathbb{C}) \). We denote them by \( \mathcal{O}_{D, \pm} \).

Let \( \mathcal{O}_D \) be an orthogonal Young diagram, given by the partition of \( 2k \):

\[
\sum_{i=1}^{r} d_i = 2k, \quad (d_1 \leq d_2 \leq d_3 \ldots \leq d_r).
\]

In order to state Arthur’s conjecture, we must know \( s_D \), the semisimple element in \( SO(2k, \mathbb{C}) \) corresponding to \( \text{diag}(\frac{1}{2}, -\frac{1}{2}) \) in \( \mathfrak{sl}(2, \mathbb{C}) \). Denote the conjugacy class of \( s_D \) by \( v_D \). \( v_D \) shall be identified with an element in \( \mathbb{R}^k \) up to the action of the Weyl group of \( SO(2k, \mathbb{C}) \). \( v_D \) can be constructed as a direct sum in the following way (Pg. 78-79 [CM]).

(1) If \( d_i = d_{i+1} = d \), define \( v_{(d,d)} = (d_1-\frac{1}{2}, d_2-\frac{3}{2}, \ldots, 1-d) \). We construct \( v_{(d,d)} \) for all pairs. Since even \( d \) appears with even multiplicities, we will only be left with odd \( d \)'s. There are even number of them.

(2) If \( (d_i, d_j) \) is an odd pair, define \( v_{(d_i,d_j)} = (d_1-\frac{1}{2}, 1, \ldots, 1, 0, -1, \ldots, \frac{d_i-1}{2}) \). Due to the Weyl group action, \( v_{(d_i,d_j)} \) is equivalent to \( (d_1-\frac{1}{2}, \frac{d_i-1}{2}, -1, \ldots, 1, 0, 1, 2, \ldots, \frac{d_i-1}{2}) \).

(3) If \( \mathcal{O}_D \) is not very even, there must be at least two odd \( d_i \)'s. Then \( 0 \) appears in \( v_D \). Due to the Weyl group action, we can use \( |v_D| \) to represent the semisimple element \( s_D \). If \( \mathcal{O}_D \) is very even, we can apply the Weyl group action to make the entries of \( v_D \) all positive, except possibly one entry. Then there are two nonequivalent \( v_{D, \pm} \); \( v_{D, +} \) with even number of negative numbers and \( v_{D, -} \) with odd number of negative numbers. Obviously, \( \mathcal{O}_{D, \pm} \) are related to each other by the group action \( O(2k, \mathbb{C})/SO(2k, \mathbb{C}) \).

Now we can restate Theorem B for \( p + q \) even.

**Corollary 7.3.** Suppose that \( p+q \leq 2m+1 \leq p+q+d, p \leq q \) and \( p+q \) even. Let \( \pi \) be an irreducible unitary representation of \( SO(p, q) \) such that its every \( K \) finite matrix coefficient \( f(g) \) satisfies the condition

\[
|f(g)| \leq C_f \exp(m + 2 - p - q - \epsilon, m + 3 - p - q, \ldots, m + 1 - q)(|H^+(g)|).
\]

for some \( \epsilon > 0 \). Suppose that the minimal \( S(O(p)O(q)) \)-types contain a \( (\xi, \eta, +) \). Then Vogan subquotient of \( \text{Ind}_{SO(p,q)GL(1)^d}^{SO(p+d,q+d)} \pi \otimes (\text{triv} \otimes \mathbb{C}^{r(2m+1-p-q, p+q+2d-2m-1)}) \) is unitary.
Remark 7.4. Now let $2m + 1 - p - q = s$ and $p + q + 2d - 2m - 1 = t$. Then $s, t$ are odd. The growth condition becomes

$$|f(g)| \leq C_f \exp(-\frac{p+q}{2} + \frac{s+3}{2} - \frac{p+q}{2} + \frac{s+5}{2} - \frac{q+p}{2} + \frac{s+1}{2})((H^+(g))).$$

The assertion is that the Vogan subquotient of $\text{Ind}_{SO(p+q-d)N}^{SO(p+d,q+d)}(\text{triv} \otimes C_{v_{t,\iota}})$ is unitary.

Theorem 7.5 (Theorem C). Let $p+q$ be even. Let $D$ be an orthogonal Young diagram of size $2k$. Let $\sigma$ be an irreducible tempered representation of $SO(p-k, q-k)$ with a minimal $SO(0)SO(q-k)$-type $(\xi, \eta, +)$. Then the Langlands-Vogan parameter $(SO(p-k, q-k)GL(1)^{k}N, \sigma \otimes \text{triv}, v_D)$ is unitary, with a minimal $K$-type $(\xi \oplus 0, \eta \oplus 0, +)$.

Proof: Clearly, a minimal $K$-type of $\text{Ind}_{SO(p-k,q-k)GL(1)^{K}N}^{SO(p,q)GL(1)^{K}N} \sigma \otimes (\text{triv} \otimes C_{v_D})$ is $(\xi \oplus 0, \eta \oplus 0, +)$. Let $D$ be defined by the partition

$$d_1 \leq d_2 \leq d_3 \ldots \leq d_r \quad (\sum_{i=1}^{r} d_i = 2k).$$

We apply induction on $r$. If $r = 0$, we have the tempered representation $\sigma$ which is unitarizable.

If there is a pair $d_i = d_{i+1} = d$, let $D_0$ be the Young diagram obtained by deleting $d_i, d_{i+1}$. Then by induction hypothesis, the Langlands-Vogan parameter $(SO(p-k, q-k)GL(1)^{K}N, \sigma \otimes \text{triv}, v_D)$ is unitary. Denote this representation by $\pi_0$. Then $\text{Ind}_{SO(p-d,q-d)GL(d)N}^{SO(p,q)GL(d)N} \sigma_0 \otimes \text{triv}$ is unitary. It is easy to see that the Langlands-Vogan parameter $(SO(p-k, q-k)GL(1)^{K}N, \sigma \otimes \text{triv}, v_D)$ is a subrepresentation of $\text{Ind}_{SO(p-d,q-d)GL(d)N}^{SO(p,q)GL(d)N} \sigma_0 \otimes \text{triv}$, hence is unitary.

If there are no repeats in $d_i$, then $d_1, d_2, \ldots, d_r$ must be all odd and $r$ must be even. If $r = 2$, set $d_0 = 0$. Let $D_0$ be the Young diagram obtained from $D$ by deleting $d_r$ and $d_{r-1}$. By induction hypothesis, the Langlands-Vogan parameter $(SO(p-k, q-k)GL(1)^{K}N, \sigma \otimes \text{triv}, v_D)$ is unitary. Denote this unitary representation of $SO(p-d_r, q-d_{r-1})$ by $\pi_0$. Then by Remark 7.4, $K$-finite matrix coefficients of $\pi_0$ are bounded by multiples of $\varepsilon_{v_D \oplus 0_{p-k}}(g)$. Notice that

$$\text{(18)}$$

$$\langle v_{D_0} \oplus 0_{p-k}, -\rho(p - \frac{d_r + d_{r-1}}{2}) - q - \frac{d_r + d_{r-1}}{2} \rangle$$

$$\leq \frac{d_r - 2}{2} \left( \frac{p+q-d_r-2r+1}{2} - 1, \frac{p+q-d_r-2r+1}{2} - 2, \ldots, \frac{q-p}{2} \right)$$

$$\leq \left( \frac{p+q-d_r-2r+1}{2} + \frac{d_{r-1} + 3}{2} - \frac{p+q-d_r-2r+1}{2} + \frac{d_{r-1} + 3}{2} - 1, \ldots, \frac{q-p}{2} + \frac{d_{r-1} + 1}{2} \right)$$

since $\frac{d_r - 2}{2} + 1 \leq \frac{d_r - 2 + 3}{2} - \epsilon$. Also notice that the minimal $K$-type of $\pi_0$ is of the form $(\xi \oplus 0, \eta \oplus 0, +)$. By Cor. 7.3 and Remark 7.4, the Vogan subquotient of

$$\text{Ind}_{SO(p-d_r, q-d_{r-1})GL(1)^{K}N}^{SO(p-k, q-k)GL(1)^{K}N} \sigma_0 \otimes (\text{triv} \otimes C_{v_{t,\iota}})$$

is unitary with a minimal $K$-type $(\xi \oplus 0, \eta \oplus 0, +)$. By double induction formula, the Langlands-Vogan parameter $(SO(p-k, q-k)GL(1)^{K}N, \sigma \otimes \text{triv}, v_D)$ is unitary. \(\Box\)
Let $D$ be a symplectic Young diagram of size $2k$. It can be described as a partition of $2k$:

$$\sum_{i=1}^{r} d_i = 2k, \quad (d_1 \leq d_2 \ldots d_{r-1} \leq d_r).$$

Let $s_D$ be the semisimple element corresponding to diag($\frac{1}{2}, -\frac{1}{2}$). Use $v_D \in \mathbb{R}^k$ to parametrize the conjugacy class of $s_D$. Then $v_D$ will be unique up to sign changes and permutations. $v_D$ can be constructed as a direct sum in the following way (Pg 78 [CM]).

1. If $d_i = d_{i+1} = d$, then $v_{(d,d)} = (\frac{d-1}{2}, \frac{d-3}{2}, \ldots, -\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1-d}{2})$. We do this for every pair of integers. Then we are only left with even $d_i$'s.
2. If $d_i$ is even, let $v_{(d_i)} = (\frac{d_i-1}{2}, \frac{d_i-3}{2}, \ldots, \frac{1}{2})$. We do this for every even $d_i$ that is not in an identical pair.

Now we can restate Theorem B for $p+q$ odd.

**Corollary 7.6.** Suppose that $p+q \leq 2m+1 \leq p+q+d$, $p \leq q$ and $p+q$ odd. Let $\pi$ be an irreducible unitary representation of $SO(p,q)$ such that its every $K$ finite coefficient $f(g)$ satisfies the condition that

$$|f(g)| \leq C_f \exp(\sqrt{m+2-p-q-\epsilon, m+3-p-q\ldots m+1-q}(|H^+(g)|)).$$

for some $\epsilon > 0$. Suppose that the minimal $SO(p)O(q)$-types of $\pi$ contains a $(\xi, \eta, +)$. Then the Vogan subquotient of $\text{Ind}^{SO(p,d+q)}_{SO(p,q)O(1)^d} \pi \otimes (\text{triv} \otimes \mathbb{C}^p_{v(2m+1-p-q)} \oplus C_{v(p+q+2d-2m-1)})$ is unitary.

**Remark 7.7.** Now let $2m+1-p-q = s$ and $p+q+2d-2m-1 = t$. Then $s$, $t$ are even. The growth condition becomes

$$|f(g)| \leq C_f \exp(-\frac{p+q}{2} + \frac{s+3}{2} - \epsilon, -\frac{p+q}{2} + \frac{s+5}{2}, \ldots -\frac{q+p}{2} + \frac{s+1}{2})(|H^+(g)|).$$

The assertion is that the Vogan subquotient of $\text{Ind}^{SO(p+d,q)}_{SO(p,q)GL(1)^d} \pi \otimes (\text{triv} \otimes C_{v(s)} \oplus C_{v(t)})$ is unitary.

**Theorem 7.8** (Theorem D). Let $p+q$ be odd. Let $D$ be a symplectic Young diagram of size $2k$. Let $\sigma$ be an irreducible tempered representation of $SO(p-k,q-k)$ such that $\sigma$ has a minimal $K$-type of the form $(\xi, \eta, +)$. Then the Langlands-Vogan parameter $(SO(p-k,q-k)GL(1)^kN, \sigma \otimes \text{triv}, v_D)$ is unitary, with a minimal $K$-type $(\xi \oplus 0, \eta \oplus 0, +)$.

Proof: Clearly, a minimal $K$-type of $(SO(p-k,q-k)GL(1)^kN, \sigma \otimes \text{triv}, v_D)$ is of the form $(\xi \oplus 0, \eta \oplus 0, +)$. Let $D$ be defined by the partition

$$d_1 \leq d_2 \leq d_3 \ldots \leq d_r \quad (\sum_{i=1}^{r} d_i = 2k).$$

We apply induction on $r$. If $r = 0$, we have the tempered representation $\sigma$ which is unitarizable.
If there is a pair $d_i = d_i+1 = d$, let $\mathbf{D}_0$ be the Young diagram obtained by deleting $d_i, d_{i+1}$. Then by induction hypothesis, the Langlands-Vogan parameter $(SO(p-k,q-k)GL(k-d)N, \sigma \otimes \text{triv}, v_{\mathbf{D}_0})$ is unitary. Denote this representation by $\pi_0$. Then $\text{Ind}^{SO(p,q)}_{SO(p-d,q-d)GL(d)N} \pi_0 \otimes \text{triv}$ is unitary. It is easy to see that the Langlands-Vogan parameter $(SO(p-k,q-k)GL(1)^k N, \sigma \otimes \text{triv}, v_{\mathbf{D}})$ is a subrepresentation of $\text{Ind}^{SO(p,q)}_{SO(p-d,q-d)GL(d)N} \pi_0 \otimes \text{triv}$, hence is unitary.

If there are no repeats in $d_i$, then $d_1, d_2, \ldots, d_r$ must be all even. Now pair all the $d_i$’s in descending order, namely, $(d_r, d_{r-1}), (d_{r-2}, d_{r-3})$ and so on. Add $d_0 = 0$ if necessary. Let $\mathbf{D}_0$ be the Young diagram obtained from $\mathbf{D}$ by deleting $d_r$ and $d_{r-1}$. By induction hypothesis, the Langlands-Vogan parameter $(SO(p-k,q-k)GL(1)^k N, \sigma \otimes \text{triv}, v_{\mathbf{D}_0})$ is unitary. Denote this unitary representation of $SO(p-d, q-d), q - d_r + d_{r-1}$ by $\pi_0$. Then by Remark 2.4, $K$-finite matrix coefficients of $\pi_0$ are bounded by multiples of $\Xi_{d_0} \equiv a_{p-k}(g)$. Notice that

\begin{equation}
(v_{\mathbf{D}_0} \oplus 0_{p-k}) - \rho(SO(p - \frac{d_r + d_{r-1}}{2}, q - \frac{d_r + d_{r-1}}{2})) \\
\geq \frac{d_r - 2}{2} - \frac{p + q - d_r - d_{r-1}}{2} - 1, \frac{p + q - d_r - d_{r+1}}{2} - 2, \ldots, q - p)
\end{equation}

since $\frac{d_r - 2}{2} + 1 \leq \frac{d_{r-1} - 3}{2} - \epsilon$. Also notice that the minimal $K$-type of $\pi_0$ is of the form $(\xi \oplus 0, \eta \oplus 0, +)$.

By Cor. 7.6 and Remark 7.7, the Vogan subquotient of

\begin{equation}
\text{Ind}^{SO(p,q)}_{SO(p-d,q-d)GL(1)^k N} \pi_0 \otimes (\text{triv} \otimes \Xi_{d_0}^{-1} \oplus v_{\mathbf{D}})
\end{equation}

is unitary. By double induction formula, the Langlands-Vogan parameter $(SO(p-k,q-k)GL(1)^k N, \sigma \otimes \text{triv}, v_{\mathbf{D}})$ is unitary. □

\textbf{References}

[Adams] J. Adams, “The Theta Correspondence over R,” Harmonic analysis, Group representations, Automorphic forms and invariant theory Ed. J.-S. Li, E.-C. Tan, N. Wallach, C.-B. Zhu, World Scientific (2007)

[AB] J. Adams, D. Barbasch, “Generic Representations of the Metaplectic Group” Compositio Math. (vol 113), 1998 (23-66).

[ABV] J. Adams, D. Barbasch, D. Vogan The Langlands Classification and Irreducible Characters for Real Reductive Groups, Progress in Mathematics, Birkhäuser, 1992.

[Ar83] J. Arthur, “On some Problems suggested by the trace formula ”, Lecture Notes in Mathematics, Vol 1041, Springer, 1984, (1-50).

[Baru] D. Barbasch, “The Unitary Spherical Spectrum for split classical groups,” J. Inst. Math. Jussieu. (V. 9 No. 2), 2010 (265-356).

[CM] D. Collingwood, M. McGovern, Nilpotent orbits on Semisimple Lie algebras, Van Nostrand Reinhold (1994).

[CHH] M. Cowling, U. Haagerup, R. Howe, “Almost $L^2$ matrix coefficients” J. reine angew. Math. (v 387), 1988, (97-110).

[GW] R. Goodman, N. Wallach, Representations and Invariants of the Classical Groups, 1998, Cambridge University Press.

[HHi] H. Huang and H. He “Symmetric Group Action on Isotropic Grassmanian” preprint (2009).

[He06] Hongyu He, “Theta Correspondence I-Semistable Range: Construction and Irreducibility”, Communications in Contemporary Mathematics (Vol 2), 2000, (255-283).

[Heon] Hongyu He, “Nonvanishing of Certain Sesquilinear Form in Theta Correspondence,” AMS Journal of Representation Theory.

[Heu] Hongyu He, “Unitary Representations and Theta Correspondence for Classical Groups of Type I”, Journal of Functional Analysis, 2003.

[Heq] Hongyu He, “Composition of Theta Correspondences ”, Adv. in Math. 190 (2005), 225-263.
[Quan] Hongyu He, “Unipotent Representations and Quantum Induction” preprint (2002).
[Quan1] Hongyu He, “Quantum Induction for Classical Groups of Type I,” in process.
[Ho89] R. Howe, “Transcending Classical Invariant Theory,” Journal of American Mathematical Society (v2), 1989 (535-552).
[JO] K. Johnson, “Degenerate Principal Series and Compact Groups,” Math. Ann., 287 (1990), 703-718.
[KZ] A. W. Knapp, G. Zuckerman “Classification of irreducible tempered representations of semisimple groups,” Ann. of Math. 116 No. 2, 1982, (389-455).
[KV] M. Kashiwara, M. Vergne, “On the Segal-Shale-Weil Representations and Harmonic Polynomials”, Invent. Math. (44), 1978, (1-47).
[KN] A. Knapp, Representation Theory on Semisimple Groups: An Overview Based on Examples Princeton University Press, 1986.
[Ku] S. Kudla, “On the local theta-correspondence,” Invent. Math. 83 (1986), no. 2, 229–255.
[LA] R. Langlands, “On the Classification of Irreducible Representations of Real Algebraic Groups,” Representation Theory and Harmonic Analysis on Semisimple Lie Groups, Math. Surveys Monographs. 31, Amer. Math. Soc., Providence, RI 1989, (101-170).
[LI89] J-S. Li, “Singular Unitary Representation of Classical Groups,” Inventiones Mathematicae (V. 97), 1989 (237-255).
[LL] H. Y. Loke, “Howe Quotients of Unitary Characters and Unitary Lowest Weight Modules,” Representation Theory, 10 (2006) 21-47.
[MO] C. Mœglin, “Howe Correspondence for Dual Reductive Pairs: some calculations in the Archimedean case”, Journal of Functional Analysis, (v. 85), 1989, (1-85).
[Paul] A. Paul On the Howe correspondence for symplectic-orthogonal dual pairs. J. Funct. Anal. 228 (2005), no. 2, 270-310.
[Pr] T. Przebinda “On Howe’s Duality Theorem,” Journal of Functional Analysis (v 81), 1988, (160-183).
[Sahi2] S. Sahi, “Jordan algebras and degenerate principal series,” J. Reine Angew. Math. 462 (1995), 1–18.
[Seph] B. Speh, “The Unitary Dual of GL(3, R) and GL(4, R),” Math. Ann. 258 (1981), 113-133.
[Vogan] D. Vogan, “The Unitary Dual of GL(n) over an Archimedean field” Invent. Math 83 (1986), 449-505.
[Vogan79] David Vogan, “The Algebraic Structure of the Representation of semisimple Lie groups” Annals of Mathematics 109, 1979, (1-60).
[Wallach] N. Wallach Real Reductive Groups II Academic Press (1992)
[Zhang] G. Zhang, “Jordan Algebras and Generalized Principal Series Representations,” Math. Ann. 302 (1995) 773-786.
[ZH] Chen-Bo Zhu, Huang-Jing Song, “On Certain Small Representations of Indefinite Orthogonal Groups”, Representation Theory (Vol. 1), (1997), 190-206.

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