Bell’s Theorem: A New Derivation That Preserves Heisenberg and Locality

Michael Clover
Science Applications International Corporation
San Diego, CA

(Dated: February 9, 2020)

By implicitly assuming that all measurements occur simultaneously, Bell’s Theorem only applied to local theories that violated Heisenberg’s Uncertainty Principle. By explicitly introducing time into our derivation of Bell’s theorem, an extra term related to the time-ordering of actual measurements is found to augment (i.e. weaken) the upper bound of the inequality. Since the same locality assumptions hold for this rederivation as for the original, we conclude that only classical measurement-order independent local hidden variable theories are constrained by Bell’s inequality; time dependent, non-classical local theories (i.e. theories respecting Heisenberg’s Uncertainty Principle) can satisfy this new bound while exceeding Bell’s limit. Unconditional nonlocality is only expected to occur with Bell parameters between $2\sqrt{2}$ and $4$. This weakening of Bell’s inequality is seen for the quantum Bell operator (squared) as an extra term involving the commutators of local measurement operators. We note that a factorizable second-quantized wavefunction can reproduce experimental measurements; because such wavefunctions allow local de Broglie-Bohm hidden variable modelling, we have another indication that violation of Bell’s inequality does not require an acceptance of non-locality.

PACS numbers: 03.65.-w, 03.65.Ud

INTRODUCTION

John Bell, in his book, “Speakable and Unspeakable in Quantum Mechanics” showed that all prior attempts to “prove” the completeness of QM made invalid assumptions. He also showed that the interpretation offered by de Broglie and later expanded upon by Bohm evaded all those “proofs”, but when applied to two particles involved a troubling feature: nonlocality or “action at a distance.” With the realization that the de Broglie-Bohm (dBB) interpretation resulted in nonlocal forces between particles, Bell developed a mathematical relationship that is taken to show that nonlocality is an essential aspect of reality, or at least an essential part of hidden variable theories. In the last 35-40 years, experiments (e.g. ) have universally discovered that his bound is violated.

We show that Bell’s analysis implicitly assumed a compatibility or simultaneity of measurements in his analysis. This goes beyond assuming the simultaneous existence of hidden variables to assuming their simultaneous measureability, and this means his derivation denies Heisenberg’s Uncertainty Principle (HUP). Our derivation removes that assumption by explicitly accounting for the fact that each measurement has to occur at a separate time and ends with a weaker inequality that is not violated by Quantum Mechanics, experiment, nor local non-classical hidden variable theories.

Bell’s Derivation of his Theorem

Bell’s derivation of the Bell/CHSH inequality starts by calculating the difference of two (theoretical) averages or correlation coefficients,

$$\langle AB \rangle - \langle AB' \rangle = \int d\lambda \rho(\lambda)A(a, \lambda)B(b, \lambda) - \int d\lambda \rho(\lambda)A(a, \lambda)B(b', \lambda) , $$

where $A, B, B' = \pm 1$ are the results of measurements and depend on the orientation of various filters ($a, b$ or $b'$) and may also depend on hidden variables, $\lambda$, which are assumed to have some distribution, $\rho(\lambda)$. Experimental averages have a similar form, for example, $\langle AB \rangle \equiv N^{-1} \sum \lambda A_i(a)B_i(b)$, and provide a better notation if we wish to apply this theorem to, say, the Copenhagen Interpretation of quantum mechanics, where $\rho(\lambda) = \delta(\lambda - \Psi)$, i.e. where the measurements only depend on the wavefunction.

The derivation begins by pulling the integral out of the difference,

$$\langle AB \rangle - \langle AB' \rangle = \langle AB - AB' \rangle ,$$

followed by further mathematical manipulations, finally leading to

$$|\langle AB \rangle - \langle AB' \rangle| + |\langle A'B' \rangle + \langle A'B \rangle| \leq 2 ,$$

the famous Bell inequality.

The act of undistributing the integral at equation requires us to assume that $B(b, \lambda)$ measurement result is
known at the same (timeless) moment that \( B(b', \lambda) \) measurement result is known, the theoretical equivalent of assuming that Heisenberg’s Uncertainty Principle doesn’t apply to these hidden variables or wavefunctions (i.e. that \( \sigma_x \) and \( \sigma_y \) can both be known simultaneously). This undistribution has also been termed “counterfactual” because experimentally, Bob cannot make both measurements at the same time – simultaneous magnetic fields at zero and ninety degrees compose a single field at 45 degrees, etc.

**BELL’S THEOREM FORCED TO CORRESPOND WITH REALITY**

We re-derive Bell’s theorem consistent with the Uncertainty Principle by introducing a measure of time into the derivation, imagining that the four experiments \((\langle AB \rangle, \langle AB' \rangle, \langle A'B \rangle, \langle A'B' \rangle)\) are measured sequentially at times \( t_1, t_2, t_3 \) and \( t_4 \), respectively. We will make the same assumption of locality as Bell, that Alice’s result, \( A \), is only a function of her setting, \( a \), and is independent of Bob’s setting, \( b \) (i.e. \( A = A(a, \lambda) \neq A(a, b, \lambda) \)) and vice versa. We differ from Bell in that we write \( A(a(t), \lambda(t)) \) in general. The detector settings, \( a(t) \), \( b(t) \) will each be constant for the periods of time corresponding to the different correlation measurements, so that we will write \( a_i^2 \) during the second time interval, and \( b_i \) during the third time interval, etc.

For typographical convenience, we will assume that any theoretical averages that would have been written as \( \int \, d\lambda \rho(\lambda) A(a, \lambda) B(b, \lambda) \) can be converted into a normalized sum over (an arbitrarily large number of) events, \( N^{-1} \sum_{i=1}^{N} a_i \lambda_i A_i B(b_i, \lambda_i), \) by ensuring that \( \lambda_i \) occurs with a frequency proportional to \( \rho(\lambda) \).

Beginning with this kind of data model, we can write,

\[
\langle AB \rangle_1 - \langle AB' \rangle_2 = N^{-1} \sum_{i=1}^{N} A(a_i, \lambda_i) B(b_i, \lambda_i) \\
- N^{-1} \sum_{i=2}^{N} A(a_i, \lambda_i) B(b_i, \lambda_i) ,
\]

\[
\equiv N^{-1} \sum_{i=1}^{N} A_1 B_1 - N^{-1} \sum_{i=2}^{N} A_2 B_2' ,
\]

where we have introduced the further abbreviation \( A(a_i', \lambda_i') \equiv A_i' \), etc. We now assume a reordering of the elements within each ensemble so that the hidden variables exactly correspond to each other (or at least arbitrarily closely), allowing us to associate a particular element from one ensemble with a particular element from another. Completing Bell’s first step (absorbing factors of \( N^{-1} \) into the summation sign),

\[
\langle AB \rangle_1 - \langle AB' \rangle_2 = \sum_{i=1}^{N} A_1 B_1 - A_2 B_2' = \langle AB - AB' \rangle ,
\]

allows us to add and subtract terms. Thus,

\[
\langle AB - AB' \rangle = \sum [A_1 B_1 \pm A_1 A_2' B_3' - A_2 B_2' + A_2 B_2' A_4' B_3 + A_2 B_2' A_4' B_4] ,
\]

factoring terms,

\[
\langle AB - AB' \rangle = \sum A_1 B_1 [1 \pm A_2 B_3' - B_2' B_4] + \sum A_2 B_2' A_4' B_4 ,
\]

Bell’s terms (but with subscripts)

\[
\sum [A_1 A_2' B_3 - A_2 A_2' B_4] .
\]

Adding and subtracting another term and refactoring,

\[
\langle AB \rangle_1 - \langle AB' \rangle_2 = Bell’s \ terms \\
\pm \sum A_1 A_2' [B_1 B_3' - B_2' B_4] + \sum [A_1 A_2' B_3 - A_2 A_2' B_4] ,
\]

so that by taking the absolute value of the left and right hand sides (\(|A| \leq 1, |B| \leq 1 \), we have

\[
|\langle AB \rangle_1 - \langle AB' \rangle_2| \leq 1 \pm \langle A'B \rangle_4 + 1 \pm \langle A'B' \rangle_3
\]

\[
+ \sum |[B_1 B_3' - B_2' B_4]|
\]

\[
+ \sum |[A_1 A_2' - A_2 A_2']| .
\]

If individual terms in either of the last two sums are negative, we can replace any \(|x-y|\) with \((x-y)+2(y-x)\), collecting the latter (a minority of terms) into \([...]\). In that case, our inequality becomes

\[
|\langle AB \rangle_1 - \langle AB' \rangle_2| \leq 1 \pm \langle A'B \rangle_4 + 1 \pm \langle A'B' \rangle_3
\]

\[
+ \sum |[B_1 B_3' - B_2' B_4]| + [...]
\]

\[
+ \sum |[A_1 A_2' - A_2 A_2']| + [...] ,
\]

where the last term is identically zero since it is the difference of a correlation function, \( \{A(a, t)A(a', t+\Delta t)\} \), with itself. Skipping some algebra that Bell skipped, and introducing a factor, \( f \geq 1 \), to account for the \([...]\) terms, we get:

\[
|\langle AB \rangle_1 - \langle AB' \rangle_2| + |\langle A'B \rangle_3 + \langle A'B \rangle_4| \\
\leq 2 + f \sum |[B_1 B_3' - B_2' B_4] ,
\]

where the last term is related to whether \( b \) is measured earlier or later than \( b' \) in the constructed correlation coefficients. If the order of measurements is irrelevant, we recover Bell’s limit of 2.
The notion of “earlier” and “later” would not make much sense if it only referred to elements from two independent ensembles – Bertie’s answer is presumably independent of Bernie’s answer, and neither has anything to do with Barney’s or Bob’s. However, because we reordered the elements of the ensembles, it is the case that Bernie1 is an identical clone with Bernie2, Bernie3 and Bernie4 – the same hidden variables, the same names – the same everything except wall-clock time and detector setting. The new term in equation 4 can therefore be interpreted as cumulating the difference between making the b independent measurement and the b measurement on the same element of an ensemble at different times.

Our earlier reordering of elements in the ensembles required us to assume that λ1 = “matched” λ2; our derivation now presents an alternative. If the hidden variables are time independent, then λ3 will also be numerically identical to λ1 and λ1 = λ2 and it would seem reasonable that both correlation functions would be numerically identical and that Bell’s inequality would be satisfied. If the hidden variables are dynamic, and if we take seriously the clone-equivalence of Bernie’s evolution from the intersection, we would expect to find that Bell’s inequality would be satisfied. If the hidden variables are dynamic, and if we take seriously the clone-equivalence of Bernie and Bernie’s evolution from λ1 after the b measurement was made (and similarly for λ4’s evolution from λ2).

It is not enough that the hidden variables be time dependent; their dynamical behavior must be such that measuring the b′ measurement makes whatever value λ1 = λ2 had after the b measurement was made (and similarly for λ4’s evolution from λ2).

If we make the identification that the horizontally polarized state [H] corresponds to |1⟩, with σx = +1, and the vertically polarized state [V] corresponds to |0⟩, with σz = −1, then the operator σz corresponds to σy, assuming A and B measure mixtures of σx and σz.

We can now ask what the matrix element evaluates to for unentangled and entangled photons. If Ou and Mandel’s experiment [5] were performed without the beam splitter, or Wehls et al.‘s experiment [6] sampled paired photons from anywhere on the two cones except where they intersected, we would expect to find

\[ \langle H_1 V_2 | \hat{S}_{Bell}^2 | H_1 V_2 \rangle = 4, \]

since \( \langle \sigma_z | \sigma_y \rangle = 0 \). This is consistent with observation [7].

If we take the entangled singlet wavefunction, \( |\psi_e\rangle = \frac{1}{\sqrt{2}}(H_1 V_2 - V_1 H_2) \) then our result is

\[ \langle \psi_e | \hat{S}_{Bell}^2 | \psi_e \rangle = 8, \]

due to non-zero cross-terms (e.g. \( H \sigma_y V = i \)). This is also consistent with observation [5, 6].

Let us now consider the case of Ou and Mandel, [5] who used a second quantized (QED) wavefunction to describe their data (where \( |\psi_p\rangle \equiv |\lambda_{photon}^{d_{detector polarization}}\rangle \)),

\[ |\psi\rangle = (T_x T_y)^{1/2}|1_{x1}, 1_{y2}\rangle + (R_x R_y)^{1/2}|1_{y1}, 1_{x2}\rangle - i(R_y T_x)^{1/2}|1_{x1}, 1_{y2}\rangle + i(R_x T_y)^{1/2}|1_{x2}, 1_{y2}\rangle , \]

and creation-annihilation operators to build a joint intensity operator. Then, even though the wavefunction can be rewritten in a factorized manner,

\[ |\psi\rangle \equiv (\sqrt{T_x}|1_{x1}\rangle + i\sqrt{R_x}|1_{x2}\rangle) (\sqrt{T_y}|1_{y2}\rangle - i\sqrt{R_y}|1_{y1}\rangle) = |\psi_e\rangle |\psi_e\rangle . \]  (6)

The coincidence operator \( E_1^{(+)} E_2^{(-)} E_1^{(+)} E_2^{(+)} \), where

\[ E_1^{(+)} = \cos \theta_1 \sqrt{T_x} a_x^\dagger - i \sin \theta_1 \sqrt{R_x} a_y^\dagger , \]
\[ E_2^{(+)} = i \cos \theta_2 \sqrt{R_y} a_x^\dagger + \sin \theta_2 \sqrt{T_y} a_y^\dagger . \]
will ensure that both $|\langle \hat{S} \rangle | \sim 2\sqrt{2}$ and $\langle \hat{S}^2 \rangle \sim 8$. (The components of the wavefunction that would send two particles to Alice or Bob do not contribute to coincidences.) Since this wavefunction is of a product form, a de Broglie-Bohm \cite{11} model for each photon would result in local forces and trajectories.

**CONCLUSIONS**

The implicit assumption of simultaneous measurability or temporal order-independence of measurements at different orientations means that Bell’s claim of a locality bound is also a time independence or Heisenberg irrelevance constraint. Classical local hidden variable theories are precluded by experiment, but non-classical (i.e. non-commutative), dynamic local hidden variable theories are not precluded by experiment, but to Bell’s original limit, but to the weaker Cirel’son \cite{10} limit of $2\sqrt{2}$; nonlocality presumably only shows up with the violation of that weaker limit. The additional terms of equations \textit{4} or \textit{5} only contribute if non-classical effects occur locally at both locations; they do not require a distant particle to instantaneously affect a nearby particle’s behavior in any way.

To the extent that the Copenhagen interpretation ($\lambda_i \equiv \psi(t_i)$ in the derivations) and the de Broglie-Bohm interpretation ($\lambda_i = \psi(t_i)$+hidden variables) both give the same answers in all experimental situations, our result shows that neither interpretation needs to be non-local in order to explain the data.

For a single particle, the de Broglie-Bohm interpretation is a non-classical, dynamic hidden variable theory. It shows how variables evolve in a manner that makes it clear why they can’t be measured simultaneously or give the same result if measured in a different order. If applied to a factorizable wavefunction of multiple particles, it is also a local HVT.

For the case of entangled particles that violate Bell’s inequality, the entanglement is a result of the “post-selection” of only coincidences between Alice and Bob. Ou and Mandel’s wavefunction will generate local dBB trajectories for each particle, showing that dBB is non-problematic even for multiple “entangled” particles.

Similar arguments should apply to experiments like that of Weihs, \textit{et al.} \cite{3}, where the entanglement is “pre-selected” before being fed into the optical fibers (throwing away the $\sim 99\%$ of the photons in the non-intersecting parts of the down-conversion cones).

Using a “post-selection” wavefunction, as is usual in first quantization analyses, dBB will generate non-local forces on each trajectory, but each trajectory will contribute to a coincidence event; using a factorized “pre-selection” wavefunction appropriate to second quantization will generate local forces along each trajectory, but only some trajectories will satisfy the coincidence conditions. A model may be nonlocal, but it doesn’t have to be nonlocal to explain the data. Computational efficiency hardly seems like an adequate reason to give up locality.

If Bell’s inequality depends on $\sim HUP$ and $LOC$, then violation of his inequality requires us to accept $HUP$ or $\sim LOC$. Nonlocality is not a choice, however, since our new derivation, which assumes $HUP$ and $LOC$, generates a weaker inequality that is respected by all experiments.

* Electronic address: michael.r.clover@saic.com

[1] J. Bell, in Speakable and Unspeakable in Quantum Mechanics (Cambridge University Press, 1971), pp. 29–39.
[2] D. Bohm, Phys. Rev. 85, 166 (1952).
[3] Z. Y. Ou and L. Mandel, Phys. Rev. Lett. 61, 50 (1988).
[4] A. Aspect, J. Dalibard, and G. Roger, Phys. Rev. Lett. 49, 1804 (1982).
[5] G. Weihs, T. Jennewein, C. Simon, H. Weinfurter, and A. Zeilinger, Phys. Rev. Lett. 81, 5039 (1998).
[6] A. Rizzi, The meaning of bell’s theorem (2003), quant-ph/0310098v1.
[7] W. D. Baer, A. Mann, and M. Revzen, Found. Phys. 29, 67 (1999).
[8] L. J. Landau, Phys. Lett. A 120, 54 (1987).
[9] M. Clover, Quantum mechanics and reality are really local (2003), quant-ph/0312198v1.
[10] B. S. Cirel’son, Lett. Math. Phys. 4, 93 (1980).
[11] P. R. Holland, The Quantum Theory of Motion, An Account of the de Broglie-Bohm Causal Interpretation of Quantum Mechanics (Cambridge University Press, 1993).
[12] Modern experiments make individual measurements in a random order, but the essential point is that they are not at the identical time; there is no loss of generality in the temporal binning used here.
[13] Instead of regarding this as two perfectly matched (“cloned”) ensembles, one could instead imagine this as a single ensemble with the same particle undergoing two consecutive measurements, which has it’s own experimental implications.
[14] Different temporal assignments would allow us to derive an alternative equation with, for example, $\sum \langle A_1 A_1' - A_2 A_2' \rangle$ on the right hand side of equation \textit{4}.
[15] Everything in this derivation is also consistent with a single particle encountering a follow-on detector in a tandem experimental configuration.
[16] If Bob and Alice measure angles with the same convention, Bob must set his to $-22.0^\circ, -67.5^\circ$ if Alice uses $0^\circ, 45^\circ$.
[17] G. Weihs, priv. comm., June 2004