SUFFICIENT CONDITIONS AND RADIUS PROBLEMS FOR A STARLIKE CLASS INVOLVING A DIFFERENTIAL INEQUALITY

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Abstract. Let $A_n$ be the class of analytic functions $f(z)$ of the form $f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k$, $n \in \mathbb{N}$ and let

$$\Omega_n := \left\{ f \in A_n : |zf'(z) - f(z)| < \frac{1}{2}, z \in \mathbb{D} \right\}.$$ 

We make use of differential subordination technique to obtain sufficient conditions for the class $\Omega_n$, and then employ these conditions to construct functions which involve double integrals and members of $\Omega_n$. We also consider a subclass $\hat{\Omega}_n \subset \Omega_n$ and obtain subordination results for members of $\hat{\Omega}_n$ besides a necessary and sufficient condition. Writing $\Omega_1 = \Omega$, we obtain inclusion properties of $\Omega$ with respect to functions defined on certain parabolic regions and as a consequence, establish a relation connecting the parabolic starlike class $S_p$ and the uniformly starlike $UST$. Various radius problems for the class $\Omega$ are considered and the sharpness of the radii estimates is obtained analytically besides graphical illustrations.

1. Introduction

Let $C$ be the set of complex numbers and let $H = H(\mathbb{D})$ be the totality of functions $f(z)$ that are analytic in the open unit disc $\mathbb{D} := \{ z \in \mathbb{C} : |z| < 1 \}$. For $a \in \mathbb{C}$ and $n \in \mathbb{N} := \{1, 2, 3, \ldots \}$, we define the function classes $H_n(a)$ and $A_n$ as follows:

$$H_n(a) := \left\{ f \in H : f(z) = a + \sum_{k=n}^{\infty} a_k z^k, a_k \in \mathbb{C} \right\}$$

and

$$A_n := \left\{ f \in H : f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k, a_k \in \mathbb{C} \right\}.$$

In particular, we write $A := A_1$. For $0 \leq \alpha < 1$, let $S^*(\alpha)$ and $C(\alpha)$ be the subclasses of $A$ which consist of functions that are, respectively, starlike and convex of order $\alpha$. Analytically,

$$S^*(\alpha) := \left\{ f : \text{Re}\left(\frac{zf''(z)}{f'(z)}\right) > \alpha \right\} \text{ and } C(\alpha) := \left\{ f : \text{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \right\}.$$

Further, $S^* := S^*(0)$ and $C := C(0)$ are the well-known classes of starlike and convex functions in $\mathbb{D}$. We note that $f \in C$ if and only if $zf' \in S^*$ and $C \subseteq S^* \subseteq S$, where $S$ is the collection of all functions $f \in A$ that are univalent in $\mathbb{D}$. For further details related to these classes, we refer to the monograph of Duren [5]. In our discussion, we make use of the following concepts from the literature.

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Definition 1.1 (Subordination [10]). Let \( f, g \in \mathcal{H} \). We say that \( f \) is subordinate to \( g \), written as \( f \prec g \), if there exists a function \( w \), analytic in \( D \) with \( w(0) = 0 \) and \(|w(z)| < 1\), such that \( f(z) = g(w(z)) \).

Moreover, if the function \( g(z) \) is univalent in \( D \), then \( f \prec g \) if and only if \( f(0) = g(0) \) and \( f(D) \subset g(D) \).

Definition 1.2 (Hadamard product). Let \( f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in \mathcal{A} \) and \( g(z) = z + \sum_{k=2}^{\infty} b_k z^k \in \mathcal{A} \), then the Hadamard product (convolution) of \( f \) and \( g \) is denoted by \( f \ast g \) and is defined as the analytic function
\[
h(z) = (f \ast g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k, \quad z \in D.
\]

Under the operation of Hadamard product, the function \( \ell(z) = z/(1-z) = z + \sum_{k=2}^{\infty} z^k \), which maps \( D \) onto the right-half plane \( \text{Re}(w) > -1/2 \), plays the role of identity element. That is, for any function \( f \in \mathcal{A} \),
\[
(f \ast \ell)(z) = f(z) = (\ell \ast f)(z).
\]

Definition 1.3 (Subordinating Factor Sequence [19]). A sequence \( \{s_k\}_{k=1}^{\infty} \) of complex numbers is said to be a subordinating factor sequence if for every convex function \( \nu(z) = z + \sum_{k=2}^{\infty} c_k z^k \), \( z \in D \) we have the subordination given by
\[
\sum_{k=1}^{\infty} s_k c_k z^k \prec \nu(z), \quad (c_1 = 1).
\]

Let \( \psi : \mathbb{C}^2 \times D \to \mathbb{C} \) be a complex function, and let \( h : D \to \mathbb{C} \) be univalent. If \( p \in \mathcal{H} \) satisfies the first-order differential subordination
\[
\psi(p, zp' ; z) \prec h(z), \quad z \in D,
\]
then \( p \) is called a solution of the differential subordination. If \( q \) is univalent and \( p \prec q \) for all \( p \) satisfying (1.1), then \( q(z) \) is said to be a dominant of (1.1). A dominant \( \hat{q} \) that satisfies \( \hat{q} \prec q \) for all dominants \( q \) of (1.1) is called the best dominant of (1.1). The best dominant is unique up to the rotations of \( D \). For further details related to results on differential subordinations, we refer to the monograph of Miller and Mocanu [14] (see also [3]).

Lemma 1.1 ([14, Theorem 3.1b, p. 71]). Let \( h(z) \) be a convex function in \( D \) with \( h(0) = a \), \( \gamma \neq 0 \) and \( \text{Re}\gamma \geq 0 \). If \( p(z) \in \mathcal{H}_n(a) \) and
\[
p(z) + \gamma^{-1} zp'(z) \prec h(z),
\]
then
\[
p(z) \prec q(z) \prec h(z),
\]
where
\[
q(z) = \frac{\gamma}{n z^{n-1}} \int_0^z h(\xi)\xi^{n-1} d\xi.
\]

The function \( q(z) \) is convex and is the best dominant.
Given $n \in \mathbb{N}$, we define a new function class $\Omega_n$ as follows

$$\Omega_n := \left\{ f \in A_n : |zf'(z) - f(z)| < \frac{1}{2}, \ z \in D \right\}.$$ 

For $n = 1$, the class $\Omega_1 := \Omega$ was recently introduced and studied by Peng and Zhong [16]. The authors in [16] have shown that $\Omega$ is a subset of $S^*$, and hence established that the members of $\Omega$ are univalent in $D$. Besides discussing several geometric properties of the members of $\Omega$, they [16] proved that the radius of convexity for $\Omega$ is $\frac{1}{2}$, and that $\Omega$ is closed under the Hadamard product, i.e., if $f_1, f_2 \in \Omega$, then $f_1 * f_2 \in \Omega$. The closedness under Hadamard product makes the class $\Omega$ more important, as this is not true, in general, for several subclasses of $S$ (e.g., $S^*$). Recently, Obradović and Peng [15] considered the class $\Omega$ and gave two sufficient conditions for functions $f \in A$ to belong to the class $\Omega$.

In this paper, we consider the class $\Omega_n$, which is in some sense, a natural generalization of $\Omega$. The paper is organized as follows: In Section 2, we use differential subordination to obtain sufficient conditions for the functions $f \in A_n$ to be in the class $\Omega_n$. Moreover, we use these results to construct functions of the form

$$f(z) = \int_{0}^{1} J(s, t, z) ds dt,$$

and obtain conditions on the kernel function $J$ so that $f \in \Omega_n$. In Section 3, we consider a subclass $\Omega_n \subset \Omega_n$, for which the sufficient conditions obtained in Section 2 become necessary also, and prove a subordination result for the elements of $\Omega_n$. In Section 4 and Section 5 we restrict ourselves to the class $\Omega$ (i.e., we fix $n = 1$). In Section 4 inclusion relations between $\Omega$, the parabolic starlike class $S_p$ and the uniformly starlike class $UST$ are studied, and as a consequence a remarkable result connecting $S_p$ and $UST$ is derived. In Section 5 several newly constructed starlike classes are introduced and the corresponding radius problems for the class $\Omega$ are settled. Also, sharpness of the radii estimates is illustrated graphically. Interesting problems for future work are proposed in Section 6.

2. SUFFICIENT CONDITIONS FOR THE CLASS $\Omega_n$

In this section, we consider some conditions on the functions $f \in A_n$, so that they belong to the class $\Omega_n$.

**Theorem 2.1.** Let $n \in \mathbb{N}$ and $\gamma \geq 1$. If $f \in A_n$ satisfies

$$\left| zf''(z) + (\gamma - 1) \left( f'(z) - \frac{f(z)}{z} \right) \right| < \frac{n + \gamma}{2},$$

then $f \in \Omega_n$. The result is sharp for the function

$$\hat{f}_{n, \mu}(z) = z + \frac{\mu}{2n} z^{n+1}, \ |\mu| = 1.$$ 

**Proof.** We rewrite the inequality (2.1) in terms of subordination as

$$zf''(z) + (\gamma - 1) \left( f'(z) - \frac{f(z)}{z} \right) < \frac{n + \gamma}{2} z.$$ 

(2.2)

Setting

$$p(z) = f'(z) - \frac{f(z)}{z} = \sum_{k=n}^{\infty} (ka_{k+1})z^k \in H_n(0),$$

and obtain conditions on the kernel function $J$ so that $f \in \Omega_n$. In Section 3, we consider a subclass $\Omega_n \subset \Omega_n$, for which the sufficient conditions obtained in Section 2 become necessary also, and prove a subordination result for the elements of $\Omega_n$. In Section 4 and Section 5 we restrict ourselves to the class $\Omega$ (i.e., we fix $n = 1$). In Section 4 inclusion relations between $\Omega$, the parabolic starlike class $S_p$ and the uniformly starlike class $UST$ are studied, and as a consequence a remarkable result connecting $S_p$ and $UST$ is derived. In Section 5 several newly constructed starlike classes are introduced and the corresponding radius problems for the class $\Omega$ are settled. Also, sharpness of the radii estimates is illustrated graphically. Interesting problems for future work are proposed in Section 6.
the subordination (2.2) takes the form
\[ \gamma p(z) + zp'(z) < \frac{n + \gamma}{2} z := h(z). \]

It can be easily seen that \( h(z) \) is convex in \( \mathbb{D} \) and \( h(0) = 0 = p(0) \). Hence it follows from Lemma 1.1 that
\[ p(z) < \frac{1}{nz^n} \int_0^z \left( \frac{n + \gamma}{2} \xi \frac{\xi^{-1}}{2} \right) d\xi = \frac{1}{2} z. \]

This further implies that
\[ \left| f'(z) - f(z) \right| < \frac{1}{2}. \]

(2.3)

Now making use of (2.3), and the fact that \( f(0) = 0 \), it follows that
\[ |zf'(z) - f(z)| = |z| \left| f'(z) - f(z) \right| < \frac{1}{2}. \]

This proves that \( f \in \Omega_n \). For the function \( \hat{f}_{n,\mu}(z) = z + \frac{\mu}{2n} z^{n+1} \) with \( |\mu| = 1 \), we have
\[ \left| z \hat{f}_{n,\mu}'(z) + (\gamma - 1) \left( \hat{f}_{n,\mu}'(z) - \frac{\hat{f}_{n,\mu}(z)}{z} \right) \right| = \left| \frac{n + \gamma}{2} \mu z^n \right| < \frac{n + \gamma}{2}. \]

That is, \( \hat{f}_{n,\mu}(z) \) satisfies the condition of Theorem 2.1 and hence belongs to \( \Omega_n \). Indeed, for \( z \in \mathbb{D} \), we have
\[ \left| z \hat{f}_{n,\mu}'(z) - \hat{f}_{n,\mu}(z) \right| = \left| \left( z + \mu \frac{n + 1}{2n} z^{n+1} \right) - \left( z + \mu \frac{1}{2n} z^{n+1} \right) \right| = \left| \frac{1}{2} z^{n+1} \right| < \frac{1}{2}. \]

Letting \( \gamma = 1 \) in Theorem 2.1 yields the following result.

**Corollary 2.1.** If \( f \in A_n \) satisfies
\[ |zf^n(z)| < \frac{n + 1}{2}, \]
then \( f \in \Omega_n \). The result is sharp.

Fixing \( n = 1 \), and then taking \( \gamma = 1 \) and \( \gamma = 2 \) in Theorem 2.1 respectively, we obtain the following sufficient conditions proved by Obradović and Peng [15].

**Corollary 2.2** ([15] Theorem 2]). If \( f \in A \) satisfies \(|zf^n(z)| < 1\), then \( f \in \Omega \). The number 1 is best possible.

**Corollary 2.3** ([15] Theorem 3]). Let \( f \in A \). If
\[ \left| z^2 f''(z) + zf'(z) - f(z) \right| < \frac{3}{2}, \]
then \( f \in \Omega \). The number \( \frac{3}{2} \) is best possible.

**Note.** For brevity, we fix \( \hat{f}_{n,\mu}(z) = z + \frac{\mu}{2n} z^{n+1} (|\mu| = 1) \) and write \( \hat{f}_1(z) = \hat{f}_{1,1}(z) = z + \frac{1}{2} z^2 \).

**Theorem 2.2.** Let \( \gamma \geq 1, n \in \mathbb{N} \), and let
\[ f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \in A_n, \quad z \in \mathbb{D}. \]
If
\[ \sum_{k=n+1}^{\infty} (k-1)(k+\gamma-1)|a_k| \leq \frac{n+\gamma}{2}, \tag{2.4} \]
then \( f \in \Omega_n \). Equality holds for the function \( \tilde{f}_{n,\mu}(z) \).

**Proof.** Suppose that (2.4) holds, then for \( z \in \mathbb{D} \), we have
\[
\left| z f''(z) + (\gamma - 1) \left( \frac{f'}{f} - \frac{f(z)}{z} \right) \right| = \left| \sum_{k=n+1}^{\infty} k(k-1)a_k z^{k-1} + (\gamma - 1) \left( \sum_{k=n+1}^{\infty} (k-1)a_k z^{k-1} \right) \right|
= \left| \sum_{k=n+1}^{\infty} (k-1)[k+\gamma-1]a_k z^{k-1} \right|
< \sum_{k=n+1}^{\infty} (k-1)[k+\gamma-1]|a_k|
\leq \frac{n+\gamma}{2}.
\]
Now, \( f(z) \) is in the class \( \Omega_n \) follows from Theorem 2.1.

It is clear that for \( \tilde{f}_{n,\mu}(z) \), we have \( a_{n+1} = \frac{\mu}{2n} \) and \( a_k = 0 \) for all \( k \geq n+2 \), so that
\[
\sum_{k=n+1}^{\infty} (k-1)[k+\gamma-1]|a_k| = (n+1-1)[n+1+\gamma-1] \frac{1}{2n} = \frac{n+\gamma}{2}.
\]

For \( \gamma = 1 \), Theorem 2.2 gives...
Corollary 2.4. If \( f(z) = z + \sum_{k=n+1}^\infty a_k z^k \in A_n \) satisfies 
\[
\sum_{k=n+1}^\infty k(k-1)|a_k| \leq \frac{n+1}{2},
\]
then \( f \in \Omega_n \). The result is sharp.

If we fix \( n = 1 \), and then take \( \gamma = 1 \) and \( \gamma = 2 \) in Theorem 2.2 respectively, we obtain the following sufficient conditions for the function class \( \Omega \).

**Corollary 2.5.** Let \( f(z) = z + \sum_{k=2}^\infty a_k z^k \in A \). If 
\[
\sum_{k=2}^\infty k(k-1)|a_k| \leq 1,
\]
then \( f \in \Omega \) and the result is sharp.

**Corollary 2.6.** Let \( f(z) = z + \sum_{k=2}^\infty a_k z^k \in A \) satisfies 
\[
\sum_{k=2}^\infty (k^2-1)|a_k| \leq 3/2.
\]
Then \( f \in \Omega \) and the result is sharp.

**Remark 1.** In both the results, Corollary 2.5 and Corollary 2.6, the equality holds for the function \( \hat{f}_{1,\mu}(z) = z + \frac{\mu}{2}z^2 \) with \( |\mu| = 1 \).

**Example 2.1.** Consider the function 
\[
\vartheta_n(z) = z - \frac{3}{20(n+1)}z^{n+1} + \frac{6}{25(n+2)}z^{n+2} + \frac{1}{10(n+3)}z^{n+3}.
\]
We have 
\[
\sum_{k=n+1}^\infty k(k-1)|a_k| = n(n+1)\left[ -\frac{3}{20(n+1)} + (n+1)(n+2)\left\{ \frac{6}{25(n+2)} \right\} 
+ (n+2)(n+3)\left\{ \frac{1}{10(n+3)} \right\} \right]
= \frac{3n}{20} + \frac{6(n+1)}{25} + \frac{(n+2)}{10}
= \left( \frac{n+1}{2} \right) \left( \frac{3}{10} + \frac{12}{25} + \frac{1}{5} \right)
+ \left( \frac{1}{10} - \frac{3}{20} \right)
= \left( \frac{n+1}{2} \right) \left( \frac{49}{50} \right) - \frac{1}{20}
< \frac{n+1}{2}.
\]
That is \( \vartheta_n(z) \) satisfies the conditions of Corollary 2.4 and hence \( \vartheta_n \in \Omega_n \). Indeed, for \( z \in \mathbb{D} \), we have 
\[
|z\vartheta_n'(z) - \vartheta_n(z)| = \left| -\frac{3n}{20(n+1)}z^{n+1} + \frac{6(n+1)}{25(n+2)}z^{n+2} + \frac{(n+2)}{10(n+3)}z^{n+3} \right|
< \frac{3n}{20(n+1)} + \frac{6(n+1)}{25(n+2)} + \frac{(n+2)}{10(n+3)}
\leq \frac{3}{20} + \frac{6}{25} + \frac{1}{10} = \frac{49}{100} < \frac{1}{2}.
\]
We now use Theorem 2.1 to construct functions involving double integrals that are members of the function class $\Omega_n$.

**Theorem 2.3.** Let $\gamma \geq 1$, $n \in \mathbb{N}$, and let $\mathcal{J}(z)$ be analytic in $\mathbb{D}$ such that

$$|\mathcal{J}(z)| \leq \frac{n + \gamma}{2}. \tag{2.5}$$

Then the function

$$f(z) = z + z^{n+1} \int_0^1 \int_0^1 \mathcal{J}(stz)s^{n-1}t^{n+\gamma-1} dsdt \tag{2.6}$$

belongs to the class $\Omega_n$. Moreover, if equality holds in (2.5), then the function in (2.6) becomes $\hat{f}_{n,\mu} \in \Omega_n$.

**Proof.** Let us consider the function $f \in A_n$ satisfying the second-order differential equation

$$zf''(z) + (\gamma - 1) \left( f' - \frac{f(z)}{z} \right) = z^n \mathcal{J}(z). \tag{2.7}$$

From (2.5) and (2.7), we have

$$\left| zf''(z) + (\gamma - 1) \left( f' - \frac{f(z)}{z} \right) \right| < \frac{n + \gamma}{2}.$$

Therefore, in view of Theorem 2.1, it follows that the solution of the differential equation (2.7) must lie in the function class $\Omega_n$. We show that the solution of (2.7) is the function given in (2.6). Writing

$$q(z) = f'(z) - \frac{f(z)}{z},$$

the equation (2.7) reduces to the form

$$z^{1-\gamma} (z^\gamma q(z))' = z^n \mathcal{J}(z).$$

This, on solving, gives

$$q(z) = z^n \int_0^1 \mathcal{J}(tz)t^{n+\gamma-1} dt,$$

or, equivalently

$$f'(z) - \frac{f(z)}{z} = z^n \int_0^1 \mathcal{J}(tz)t^{n+\gamma-1} dt. \tag{2.8}$$

Now the differential equation (2.8) can be written as

$$z \left( \frac{f(z)}{z} - 1 \right)' = z^n \int_0^1 \mathcal{J}(tz)t^{n+\gamma-1} dt,$$

whose solution is

$$f(z) = z + z^{n+1} \int_0^1 \int_0^1 \mathcal{J}(stz)s^{n-1}t^{n+\gamma-1} dsdt.$$

If equality holds in (2.5), then we have

$$\mathcal{J}(z) = \mu \frac{n + \gamma}{2} \text{ with } |\mu| = 1.$$

Substituting this in (2.6), we obtain the function $\hat{f}_{n,\mu}(z)$. \qed
**Corollary 2.7.** Let $J \in \mathcal{H}$ such that $|J(z)| \leq 1$. Then the function

$$f(z) = z + z^2 \int_0^1 J(stz)tdsdt$$

belongs to the class $\Omega$.

We conclude this section by showing that for each $n \in \mathbb{N}$ and for each $\mu \in \mathbb{C}$ with $|\mu| = 1$, the function $\hat{f}_{n,\mu}(z)$ is an extreme point of $\Omega_n$. We prove it by showing that $\hat{f}_{n,\mu}(z)$ satisfies the condition established by Peng and Zhong [16, Theorem 3.14]. Observe that

$$\hat{f}_{n,\mu}(z) = z + \frac{\mu}{2n}z^{n+1} = z + \frac{1}{2}z \int_0^\pi \phi(\xi),$$

where $\phi(\xi) = \mu \xi^{n-1}$ satisfies

$$\int_0^{2\pi} \log \left[1 - |\phi(e^{i\theta})|\right] d\theta = -\infty.$$
where
\[ \tau = \frac{n}{2n + 1}. \]

The number \( \tau \) is the best possible estimate.

Proof. Since \( f \in \hat{\Omega}_n \), we have
\[ \sum_{k=n+1}^{\infty} k(k-1)|a_k| \leq \frac{n + 1}{2}. \]  
(3.5)

Also, from the given representations of \( f(z) \) and \( \nu(z) \), and the definition of Hadamard product, we have
\[ \tau(f * \nu)(z) = \tau \left( z + \sum_{k=n+1}^{\infty} a_k c_k z^k \right) = \sum_{k=1}^{\infty} s_k c_k z^k, \]
where \( c_1 = 1 \) and
\[ s_k = \begin{cases} 
\tau & \text{for } k = 1 \\
0 & \text{for } 2 \leq k \leq n \\
\tau a_k & \text{for } k \geq n + 1.
\end{cases} \]

Clearly, the subordination (3.3) will hold if we prove that \( \{s_k\}_{k=1}^{\infty} \) is a subordinating factor sequence. In view of Lemma 3.1, it is sufficient to prove that
\[ \text{Re} \left( 1 + 2 \sum_{k=1}^{\infty} s_k z^k \right) > 0. \]

Now using (3.5) and the fact that the sequence \( \{k(k-1)\}_{k=n+1}^{\infty} \) is an increasing sequence, we have for \( |z| = r < 1 \),
\[ \text{Re} \left( 1 + 2 \sum_{k=1}^{\infty} s_k z^k \right) = \text{Re} \left( 1 + 2\tau z + 2 \sum_{k=n+1}^{\infty} \tau a_k z^k \right) \geq 1 - 2\tau |z| - 2\tau \sum_{k=n+1}^{\infty} |a_k||z|^k \]
\[ = 1 - 2\tau r - 2\tau \sum_{k=n+1}^{\infty} |a_k| r^k \]
\[ \geq 1 - 2\tau r - 2\tau \frac{1}{n(n+1)} \sum_{k=n+1}^{\infty} k(k-1)|a_k| r \]
\[ \geq 1 - 2\tau r - 2\tau \frac{1}{n(n+1)} \frac{n + 1}{2} r \]
\[ = 1 - r \left( 2\tau + \frac{\tau}{n} \right) \]
\[ = 1 - r > 0. \]

Thus (3.3) holds true for every function \( f \in \hat{\Omega} \). If we choose the function \( \nu(z) \) as the convex function \( \ell(z) = z/(1 - z) \) the inequality (3.4) follows. A simple observation shows that the function
\[ \hat{f}_{n-1}(z) = z - \frac{1}{2n} z^{n+1} \in \Omega_n \]
which guarantees that the number \( \tau \) cannot be replaced by any larger one. \( \square \)
The following subordination result for $\Omega$ is an immediate consequence of Theorem 3.1.

**Corollary 3.1.** Let $f \in \hat{\Omega}$. Then for every convex function $\nu(z)$ in $\mathbb{D}$, we have

$$
\frac{1}{3} (f * \nu)(z) \prec \nu(z)
$$

and

$$
\text{Re}(f(z)) > \frac{3}{2}.
$$

(3.6)

The sharpness of the estimate $\frac{1}{3}$ is guaranteed by the function $\hat{f}_{1,-1}(z) = z - z^2/2 \in \Omega$ (see Figure 2).

**Remark 2.** The validity of the inequality (3.6) for functions in $\Omega$ is evident from the growth theorem [16, Theorem 3.1].

![Figure 2](image)

**Figure 2.** $C_1$ is the Boundary curve of $\hat{f}_{1,-1}(\mathbb{D})$ and $C_2$ is that of $\frac{1}{3} \hat{f}_{1,-1}(\mathbb{D})$, where $\hat{f}_{1,-1}(z) = z - z^2/2$.

4. **Inclusion Properties of $\Omega$**

We start this section by stating a necessary condition for a function $f \in \mathcal{A}$ to be in the function class $\Omega$.

**Lemma 4.1** ([16, Corollary 3.12]). If $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ is in $\Omega$, then

$$
|a_k| \leq \frac{1}{2(k - 1)}, \quad k \geq 2.
$$

(4.1)
As mentioned earlier, \( \Omega \subset S^* \) \cite[Theorem 3.1]{16}, the fact that the Koebe function \( z/(1 - z)^2 = z + \sum_{k=2}^{\infty} k z^k \in S^* \) does not satisfy the necessary condition (4.1) implies that this containment is proper. In this section, we discuss some inclusion type relations existing between \( \Omega, S_p \) and \( \text{UST} \).

**Definition 4.1** (Parabolic Starlike Functions \((S_p)\) \cite{17}). A function \( f \in A \) is said to be in the class \( S_p \subset S^* \) if and only if

\[
\Re \left( \frac{zf'(z)}{f(z)} \right) > \left| \frac{z}{f(z)} - 1 \right|, \quad z \in \mathbb{D}.
\]

These functions were introduced by Ronning \cite{17} and later studied and generalized by many authors (see \cite{2, 8, 9}). Geometrically, \( f \in S_p \) if and only if all the values taken by the expression \( zf'(z)/f(z) \) lie in the parabolic region

\[
R_p := \{ w = u + iv \in \mathbb{C} : v^2 < 2u - 1 \}.
\]

**Lemma 4.2** (\cite[Theorem 3]{17}). The function \( f_k(z) = z + a_k z^k \) is in \( S_p \) if and only if

\[
|a_k| \leq \frac{1}{(2k - 1)}, \quad k \geq 2.
\]

**Lemma 4.3** (\cite[Corollary 2.4]{9}). Let \( f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in A \). If

\[
\sum_{k=2}^{\infty} (2k - 1)|a_k| \leq 1,
\]

then \( f \in S_p \).

**Lemma 4.4.** The function \( f_k(z) = z + a_k z^k \) is in \( \Omega \) if and only if

\[
|a_k| \leq \frac{1}{2(k - 1)}, \quad k \geq 2. \quad (4.2)
\]

**Proof.** The necessary part easily follows from Lemma 4.1. We now suppose (4.2) holds, then

\[
|zf_k'(z) - f_k(z)| = |(k - 1)a_k z^k| < (k - 1)|a_k| \leq \frac{1}{2}.
\]

Hence, \( f_k \in \Omega \).

\( \square \)

If we consider the function \( \hat{f}_{1, \mu}(z) = z + \frac{\mu}{2} z^2 \in \Omega \), then it easily follows from Lemma 4.2 that \( \Omega \not\subset S_p \). For the other direction, we give the following result.

**Theorem 4.1.** If \( f_k(z) = z + a_k z^k \) belongs to \( S_p \), then \( f_k \in \Omega \) for every \( k \geq 2 \).

**Proof.** As \( f_k(z) = z + a_k z^k \in S_p \), Lemma 4.2 gives that

\[
|a_k| \leq \frac{1}{(2k - 1)}, \quad k \geq 2.
\]

Since,

\[
\frac{1}{(2k - 1)} < \frac{1}{2(k - 1)} \quad \text{for all } k \geq 2,
\]

the desired result follows from Lemma 4.4.

\( \square \)

The following result holds for the functions in \( \hat{\Omega} \subset \Omega \) with certain restrictions.
Theorem 4.2. Let \( f(z) = z + \sum_{k=3}^{\infty} a_k z^k \in \hat{\Omega} \) (i.e., \( a_2 = 0 \)). Then \( f \in S_p \).

Proof. \( f \in \hat{\Omega} \) gives

\[
\sum_{k=2}^{\infty} k(k-1)|a_k| \leq 1. \tag{4.3}
\]

Making use of (4.3), we have

\[
\sum_{k=2}^{\infty} (2k-1)|a_k| = \sum_{k=3}^{\infty} (2k-1)|a_k| \leq \sum_{k=3}^{\infty} k(k-1)|a_k| = \sum_{k=2}^{\infty} k(k-1)|a_k| \leq 1.
\]

Therefore, it follows from Lemma 4.3 that \( f \in S_p \). \( \square \)

Definition 4.2 (Uniformly Starlike Functions (UST) [7]). A function \( f \in A \) is said to be in the class \( UST \subset S^* \) if and only if

\[
\text{Re} \left( \frac{(z - \xi)f'(z)}{f(z) - f(\xi)} \right) \geq 0
\]

for every pair \( (\xi, z) \in \mathbb{D} \times \mathbb{D} \).

This class of functions was introduced by Goodman [7]. These functions have the property that for every circular arc \( \gamma \) contained in \( \mathbb{D} \), with center \( \zeta \) also in \( \mathbb{D} \), the arc \( f(\gamma) \) is starlike with respect to \( f(\zeta) \).

Lemma 4.5 ([13, Theorem 5]). If

\[
|a_k| \leq \sqrt{\frac{k+1}{2k^3}}, \quad k \geq 2,
\]

then the function \( f_k(z) = z + a_k z^k \) is in \( UST \).

Using Lemma 4.5, we prove the following theorem.

Theorem 4.3. Let \( f_k(z) = z + a_k z^k \) be in \( \Omega \). Then, for all \( k \geq 3 \), \( f(z) \) is in \( UST \).

Proof. Given \( f_k(z) = z + a_k z^k \in \Omega \), we have from Lemma 4.4 that

\[
|a_k| \leq \frac{1}{2(k-1)}, \quad k \geq 2.
\]

Since,

\[
\frac{1}{2(k-1)} \leq \sqrt{\frac{k+1}{2k^3}} \text{ for all } k \geq 3,
\]

it follows from Lemma 4.5 that \( f_k(z) = z + a_k z^k \) is in \( UST \) for all \( k \geq 3 \). \( \square \)

We note that \( S_p \not\subset UST \) and \( UST \not\subset S_p \) (cf. [1, page 21]). In view of Theorem 4.1 and Theorem 4.3, we remark the following important result which is not available in the literature. This result gives a kind of inclusion relation between \( S_p \) and \( UST \).

Remark 3. If \( f_k(z) = z + a_k z^k \) is in \( S_p \), then \( f_k \in UST \) for all \( k \geq 3 \).
5. Radii Problems for $\Omega$

By a radius problem, we mean the following: For two families $\mathcal{F}_1, \mathcal{F}_2$ in $A$, we say that the number $\rho (0 < \rho \leq 1)$ is the $\mathcal{F}_1$-radius for $\mathcal{F}_2$, if $\rho$ is the largest number such that for every $r$ satisfying $0 < r \leq \rho$ we have

$$\frac{1}{r}f(rz) \in \mathcal{F}_1$$

for all $f \in \mathcal{F}_2$.

We note that it has been proved [16, Theorem 3.4] that the $C$-radius for $\Omega$ is $\frac{1}{2}$. In this section, we will prove some more radii results for the class $\Omega$. Before proceeding, we list out some lemmas that are useful for our discussion.

Lemma 5.1 ([16, Theorem 3.1]). If $f \in \Omega$, then

$$|z| - \frac{1}{2}|z|^2 \leq |f(z)| \leq |z| + \frac{1}{2}|z|^2,$$

(5.1)

and

$$1 - |z| \leq |f'(z)| \leq 1 + |z|.$$

(5.2)

Further, for each $0 \neq z \in \mathbb{D}$, equality occurs in both the estimates if and only if

$$f(z) = \hat{f}_{1,\mu}(z) = z + \frac{\mu}{2}z^2$$

with $|\mu| = 1$.

(5.3)

Lemma 5.2. Let $f \in \Omega$. Then for $|z| = r < 1$, we have the sharp estimate

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \frac{r}{2 - r}.$$

Proof. Since $f \in \Omega$, we have

$$|zf'(z) - f(z)| < \frac{1}{2}.$$ 

This can be equivalently written in the equation form as

$$zf'(z) - f(z) = \frac{1}{2}z^2 \phi(z),$$

where $\phi(z) \in \mathcal{H}$ and $|\phi(z)| \leq 1$. This further implies that

$$|zf'(z) - f(z)| \leq \frac{1}{2}|z|^2.$$ 

(5.4)

Inequality (5.4) along with (5.1) yields

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| = \frac{1}{|f(z)|} |zf'(z) - f(z)| \leq \frac{\frac{1}{2}|z|^2}{|z| - \frac{1}{2}|z|^2} = \frac{r}{2 - r}.$$

The sharpness of the estimate follows from Lemma 5.1.

$S^*(\alpha)$-radius of $\Omega$.

Theorem 5.1. The $S^*(\alpha)$-radius for the class $\Omega$ is $r(\alpha) = \frac{2(1-\alpha)}{2-\alpha}$, where $0 \leq \alpha < 1$.

Proof. It is easy to see that

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1 - \alpha \implies \text{Re} \frac{zf'(z)}{f(z)} > \alpha.$$
If \( f \in \Omega \) and \( |z| = r \), then we conclude from Lemma 5.2 that
\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1 - \alpha \quad \text{if} \quad \frac{r}{2 - r} \leq 1 - \alpha.
\]
The later inequality holds true if \( r \leq 2(1 - \alpha)/(2 - \alpha) = r(\alpha) \). For sharpness, we consider the function \( \hat{f}_1(z) = z + z^2/2 \in \Omega \). At the point \( z_0 = -2(1 - \alpha)/(2 - \alpha) \) lying on the circle \( |z| = r(\alpha) \), we have
\[
\frac{z_0\hat{f}_1'(z_0)}{\hat{f}_1(z_0)} = \frac{1 + z_0}{1 + z_0/2} = \alpha.
\]
This proves that the result is sharp. \( \square \)

**Remark 4.** The \( S^* \left( \frac{1}{2} \right) \)-radius for the class \( \Omega \) is \( \frac{2}{3} \).

**\( S_p \)-radius of \( \Omega \).**

**Theorem 5.2.** The \( S_p \)-radius for the class \( \Omega \) is \( \frac{2}{3} \).

![Figure 3](https://example.com/figure3.png)

**Figure 3.** Sharpness of \( S_p \)-radius: \( \hat{F}_1(z) = z\hat{f}_1(z)/\hat{f}_1(z) \) with \( \hat{f}_1(z) = z + z^2/2 \).

**Proof.** Let \( f \in \Omega \). Then by a well known result [1] p. 21 we get that
\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| < \frac{1}{2} \implies f \in S_p.
\]
In view of Lemma 5.2, this inequality is true if \( r/(2-r) < 1/2 \), that is if \( r < 2/3 \). Further, we note that \( f \in S_p \) implies (see [17])

\[
\text{Re} \left( \frac{zf'(z)}{f(z)} \right) > \frac{1}{2}.
\]

We now show that there exists at least one function \( f \in \Omega \) for which \( \text{Re} (zf'(z)/f(z)) = 1/2 \) at some point on the circle \( |z| = 2/3 \). This can be easily shown if we take \( f(z) = \hat{f}_1(z) = z + z^2/2 \in \Omega \) and \( z = -2/3 \). Hence the result is sharp (see Figure 3).

**Remark 5.** For \( \Omega \), we have \( S^* \left( \frac{1}{2} \right) - \text{radius} = S_p - \text{radius} = \frac{2}{3} \).

In 1992, Ma and Minda [11] introduced a general method of constructing function classes \( S^*(\varphi) \subset S^* \) as

\[
S^*(\varphi) := \left\{ f \in A : \frac{zf'(z)}{f(z)} < \varphi(z) \right\},
\]

where the function \( \varphi : D \to \mathbb{C} \) satisfies (i) \( \varphi(z) \) is univalent with positive real part, (ii) \( \varphi(z) \) maps \( D \) onto a region that is starlike with respect to \( \varphi(0) = 1 \), (iii) \( \varphi(D) \) is symmetric about the real axis and (iv) \( \varphi'(0) > 0 \). In the recent past, using this approach a number of starlike classes have been introduced and studied along with their geometric properties. Here, we first mention a few of them and then solve the corresponding radius problem for the class \( \Omega \).

**S\(^*\)e-radius of \( \Omega \).** In 2015, the class \( S^*_e = S^*(e^z) \) was introduced by Mendiratta et al. [12]. Thus, \( S^*_e \) is the collection of all functions \( f \in A \) satisfying \( zf'(z)/f(z) < e^z \). Equivalently, the function \( f \in S^*_e \) if and only if \( zf'(z)/f(z) \) lies in the region

\[
R_e := \{ w \in \mathbb{C} : |\log w| < 1 \}.
\]

Further, if \( f \in S^*_e \), then

\[
\frac{1}{e} < \text{Re} \left( \frac{zf'(z)}{f(z)} \right) < e.
\]

**Lemma 5.3 ( [12, Lemma 2.2]).** For \( 1/e < a < e \), let \( r_a \) be given by

\[
r_a = \begin{cases} 
  a - \frac{1}{e}, & \frac{1}{e} < a \leq \frac{1}{2}(e + \frac{1}{e}) \\
  e - a, & \frac{1}{2}(e + \frac{1}{e}) \leq a < e.
\end{cases}
\]

Then \( \{ w \in \mathbb{C} : |w - a| < r_a \} \subset R_e \).

**Theorem 5.3.** The \( S^*_e \)-radius for the class \( \Omega \) is \( r_e = \frac{2(1-e^{-1})}{2-e^{-1}} \approx 0.774600032643 \).

**Proof.** Let \( f \in \Omega \). Then, for \( |z| = r < 1 \), Lemma 5.2 gives

\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \frac{r}{2-r},
\]

which is a disk centered at \( (1,0) \) and radius \( r/(1-r) \). It follows from Lemma 5.3 that this disk will be contained in \( R_e \) if and only if

\[
\frac{r}{2-r} < 1 - \frac{1}{e}.
\]
This is true if and only if
\[ r < \frac{2 \left( \frac{1 - \frac{1}{r}}{r} \right)}{2 - \frac{1}{e}} = r_e. \]

If we consider the function \( \hat{f}_1(z) = z + z^2/2 \in \Omega \), then it is easy to see that at the point \( z = -r_e \), we have
\[ \frac{z \hat{f}_1'(z)}{\hat{f}_1(z)} = \frac{1}{e}. \]

This proves that the result is sharp (see Figure 4).

**Figure 4.** Sharpness of \( S_e^* \)-radius: \( \hat{F}_1(z) = z \hat{f}_1'(z)/\hat{f}_1(z), \hat{f}_1(z) = z + z^2/2. \)

**S\(_e^*\)-radius of \( \Omega \).** In 2016, Sharma et al. \[18\] introduced and discussed the class \( S_e^* \subset S^* \) defined as

\[
S_e^* := \left\{ f \in A : \frac{zf'(z)}{f(z)} < 1 + \frac{4}{3} z + \frac{2}{3} z^2 \right\}.
\]

A function \( f \in A \) is in the class \( S_e^* \) if and only if \( zf'(z)/f(z) \) lies in the region bounded by the cardioid \( (9u^2 - 18u + 9v^2 + 5)^2 - 16(9u^2 - 6u + 9v^2 + 1) = 0 \). We let

\[
R_C := \left\{ u + iv : \left(9u^2 - 18u + 9v^2 + 5\right)^2 - 16\left(9u^2 - 6u + 9v^2 + 1\right) < 0 \right\}.
\]

**Lemma 5.4 (\[18\] Lemma 2.5).** For \( 1/3 < a < 3 \), let \( r_a \) be given by

\[
r_a = \begin{cases} 
\frac{3a-1}{3}, & \frac{1}{3} < a \leq \frac{5}{3} \\
3 - a, & \frac{5}{3} \leq a < 3.
\end{cases}
\]

Then \( \{ w \in \mathbb{C} : |w - a| < r_a \} \subset R_C \).
Theorem 5.4. The $S^*_C$-radius for the class $\Omega$ is $\frac{4}{5}$.

Proof. In view of Lemma 5.2 and Lemma 5.4, it follows that for any $f \in \Omega$, the disk (5.5) will lie inside the region $R_C$ if and only if
\[ \frac{r}{2 - r} < \frac{2}{3} \iff r < \frac{4}{5}. \]

Again, from Lemma 5.4, it can be easily seen that the largest disk with center at $(1,0)$ and lying completely inside $R_C$ is $\{ w : |w - 1| < \frac{2}{3} \}$.

Clearly the left diametric end point of this disk is $\frac{1}{3}$. The sharpness of our result will follow if we can find at least one function $f \in \Omega$ and a point on the circle $|z| = \frac{4}{5}$, say $z_0$, such that the value of $zf'(z)/f(z)$ at $z_0$ is $\frac{1}{3}$. We see that one such function in $\Omega$ is $\hat{f}_1(z) = z + \frac{z^2}{2}$, and the corresponding point on the circle $|z| = \frac{4}{5}$ is $z_0 = -\frac{4}{5}$ (see Figure 5). $\square$

\[ \gamma_C : \left(9u^2 - 18u + 9v^2 + 5\right)^2 - 16 \left(9u^2 - 6u + 9v^2 + 1\right) = 0. \]

\[ \gamma_0 : |w - 1| = \frac{2}{3}. \]

\[ D_C : \hat{F}_1 \left( |z| < \frac{4}{5} \right), \text{ where } \hat{F}_1(z) = \frac{zf_1(z)}{f(z)} \]

with $f_1(z) = z + \frac{z^2}{2}$.

$A = 1/3$.

Figure 5. Sharpness of $S^*_C$-radius.

$S^*_S$-radius of $\Omega$. In 2019, using the same Ma-Minda’s construction [11], Cho et al. [4] introduced another important class of starlike functions $S^*_S$ defined as

\[ S^*_S := \left\{ f \in A : \frac{zf'(z)}{f(z)} < 1 + \sin z \right\}. \]

Let us set $R_S := \varphi_S(\mathbb{D})$, where $\varphi_S(z) = 1 + \sin z$.

Lemma 5.5 ( [4, Lemma 3.3]). For $1 - \sin 1 \leq a \leq 1 + \sin 1$, let $r_a$ be given by

\[ r_a = \sin 1 - |a - 1|. \]

Then $\{ w \in \mathbb{C} : |w - a| < r_a \} \subset R_S$.

Using Lemma 5.5, the following theorem can be proved.
Theorem 5.5. The $S^*$-radius for the class $\Omega$ is $r_S = \frac{2\sin 1}{1+\sin 1} \approx 0.91391174962$ (see Figure 6).

$\gamma_S$: Boundary curve of $\varphi_s(\mathbb{D})$, where $\varphi_s(z) = 1 + \sin z$.

$D_S : \hat{F}_1(|z| < r_S), \hat{F}_1(z) = \frac{\hat{f}_1(z)}{f_1(z)}$ with $\hat{f}_1(z) = z + \frac{z^2}{2}$.

$B = 1 - \sin 1$.

Figure 6. Sharpness of $S^*$-radius.

$S^*$-radius of $\Omega$. Lately, Goel and Kumar [6] introduced the starlike class $S^*_SG = S^*(\varphi)$, where $\varphi(z) = 2/(1 + e^{-z})$ is the modified sigmoid function. The following radius result can be easily verified by applying [6, Lemma 2.2].

Theorem 5.6. The $S^*_SG$-radius for the class $\Omega$ is $r_{SG} = \frac{e^{-1}}{e} \approx 0.632120558828577$.

6. Concluding Remarks and Some Open Problems

Since the members of $\Omega$ are starlike, it easily follows from the Alexander’s theorem that the members of the class

$$\Upsilon := \left\{ f \in A : |z^2 f''(z)| < \frac{1}{2} \right\}.$$

are convex.

The following result of Kanas and Wisniowska [8, Corollary 3.2] follows immediately.

Corollary 6.1. If $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in A$ satisfies

$$\sum_{k=2}^{\infty} k(k-1)|a_k| \leq \frac{1}{2},$$

(6.1)

then $f(z)$ is convex.

Remark 6. If $\hat{\Upsilon}$ denotes all functions $f \in A$ satisfying (6.1), then we have the inclusion $\hat{\Upsilon} \subset C$. Also, from the definition (3.2), it follows that $\hat{\Upsilon} \subset \Omega \subset \Omega$. 
The fact that $\Omega$ is closed with respect to Hadamard product [16, Theorem 3.7], which is true for the convex class $\mathcal{C}$ also [5, Theorem 8.6, p. 247], gives the intuition of some interesting relationship between $\Omega$ and $\mathcal{C}$. Also, we have been successful in finding a class $\hat{\mathcal{T}}$ such that $\hat{\mathcal{T}} \subset \mathcal{C}$ and $\hat{\mathcal{T}} \subset \Omega$ (see, Remark [4]), which further strengthens the above intuitional claim of some kind of relationship between the two. We conjecture the following result regarding the Hadamard product between the members of $\Omega$.

**Conjecture.** Let $f_1, f_2 \in \Omega$. Then $f_1 \ast f_2 \in \mathcal{C}$.

Using the sufficiency condition (6.1), the truthfulness of the above result can be easily verified for the function $\hat{\tilde{f}}_{1,\mu}(z) = z + \frac{\mu}{2}z^2 \in \Omega$ with $|\mu| = 1$, which serves as the extremal for many problems in $\Omega$ [16] (see also [15]). Indeed, for $\hat{\tilde{g}}_{\mu}(z) = \hat{\tilde{f}}_{1,\mu}(z) \ast \hat{\tilde{f}}_{1,\mu}(z) = z + \frac{\mu^2}{4}z^2$,

we have

$$\sum_{k=2}^{\infty} k(k-1)|a_k| = 2\left(\frac{1}{4}\right) = \frac{1}{2}.$$

Hence $\hat{\tilde{g}}_{\mu}(z) \in \mathcal{C}$.

**Strongly Starlike Functions.** A function $f \in \mathcal{A}$ is said to be strongly starlike of order $\beta$ ($0 < \beta \leq 1$) if and only if

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\beta \pi}{2}, \quad z \in \mathbb{D}.$$

We usually denote this class of functions by $SS^*(\beta)$. Observe that $SS^*(1) = S^*$, and for $0 < \beta < 1$, the class $SS^*(\beta)$ consists only of bounded starlike functions, and hence in this case the inclusion $SS^*(\beta) \subset S^*$ is proper.

**Problem.** To find the $SS^*(\beta)$-radius for the class $\Omega$.

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