ON A CONJECTURE ABOUT UNIVALENT POLYNOMIALS
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Abstract. In this paper we prove a recent conjecture formulated by Dmitrishin, Smorodin
and Stokolos about that certain polynomials are univalent in the unit disk. As a conse-
quence we get an upper estimate for the Koebe radius of univalent polynomials.

Let \( \mathbb{D} = \{ z : |z| < 1 \} \). Denote by \( S \) the class of analytic and univalent in the disk \( \mathbb{D} \) functions whose Taylor’s series begin with \( f(z) = z + \cdots \). The famous Koebe Quarter Theorem says that if \( f \in S \) then \( f(\mathbb{D}) \) contains a disc of radius \( 1/4 \). It is easy to see
that if \( f \) is a quadratic polynomial from \( S \) then the Koebe radius (the radius of maximal
disk centered at the origin that covered by \( f(\mathbb{D}) \)) is equal to \( 1/2 > 1/4 \). In general, for
polynomials of degree \( n \), the Koebe radius is \( \geq (1 + c/n^2)/4 \) for some absolute constant
\( c > 0 \). This estimate can be easily derived from a classical fact proved by Pick [1]. It says
that the Koebe radius of a bounded function from the class \( S \) is greater than

\[
2M^2 - M - 2M \sqrt{M(M-1)} = \frac{1}{4} + \frac{1}{8M} + O(1/M^2)
\]

where \( M = \sup_{z \in \mathbb{D}} |f(z)| \). In view of de Branges’s theorem, we have the estimates \( |a_n| \leq n \)
which imply that for a polynomial of degree \( n \) the following inequality holds \( M \leq 1 + 2 +
\cdots + n = n(n+1)/2 \). Consequently, the Koebe radius for polynomials from \( S \) is not less
than \( (1/4)(1 + 1/(2n^2)) \).

In 2002 Dimitrov [3] posed the following problem: Find an extremal polynomial that
minimizes the Koebe radius among all polynomials of a fixed degree \( n \) from the class \( S \).

It seems that this problem is too hard to solve at this time. At first, one may ask about
a natural candidate that resolves the problem.

Such the candidate was found, but, surprisingly, it did not coincide with any of classical
Suffridge polynomials [2] for \( n > 2 \). Namely, Dmitrishin, Smorodin and Stokolos ( [4],
Conjecture 1) formulated the following

**Conjecture.** The polynomials

\[
P(z) := \csc \frac{2 \pi}{n+2} \csc \frac{\pi}{n+2} \sum_{k=1}^{n} \left( (n-k+3) \sin \frac{(k+1)\pi}{n+2} - (n-k+1) \sin \frac{(k-1)\pi}{n+2} \right) \sin \frac{k\pi}{n+2} z^k
\]

are univalent in the disk \( \mathbb{D} \), i.e. \( P_n \in S \) for all \( n \geq 2 \).

The Koebe radius for these polynomials is equal to \( (1/4) \sec^2 \frac{\pi}{n+2} \) which is not too far
from being the best possible because asymptotically it behaves like \( (1/4)(1 + c/n^2) \).

Dmitrishin, Dillies, Skrinnik, Smorodin and Stokolos ( [5], [6]) proved this conjecture
for polynomials of degree \( n \leq 6 \). Dmitrishin, Dyakonov and Stokolos [7] found the optimal

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Koebe radius for polynomials of degree \( n = 3 \). Also they demonstrated that the Suffridge polynomials cannot be extremal for all \( n > 2 \).

Our main result is the following

**Theorem.** The polynomials \( P(z) \) are univalent in \( \mathbb{D} \) for \( n \geq 10 \).

According to Theorem 1 from the paper \[6\] the polynomials can be presented in the form

\[
P(e^{it}) = \frac{1}{2} \left( \cos t - \cos \frac{2\pi}{n+2} \right) + \frac{1 - \cos \frac{2\pi}{n+2}}{(n+2)(1-\cos t)} \sin t \sin \frac{n+2}{2} t \cos \frac{n+2}{2} t e^{\frac{n+2}{2} it}.
\]

We define

\[
u(t) := Re(P(e^{it})) = \frac{1}{2} \left( \cos t - \cos \frac{2\pi}{n+2} \right) + \frac{1 - \cos \frac{2\pi}{n+2}}{(n+2)(1-\cos t)} \sin t \sin \frac{n+2}{2} t \cos \frac{n+2}{2} t,
\]

\[
u(t) := Im(P(e^{it})) = \frac{1 - \cos \frac{2\pi}{n+2}}{(n+2)(1-\cos t)} \sin t \sin \frac{n+2}{2} t \sin \frac{n+2}{2} t.
\]

Further we will assume that \( n \geq 10 \).

We are going to show that the set \( \Gamma := \{ P(e^{it}) : t \in [0, \pi] \} \) is a simple curve. To do this, we cut this curve into three curves: \( \Gamma_1 = \{ P(e^{it}) : t \in [0, \frac{2\pi}{n+2}] \} \), \( \Gamma_2 = \{ P(e^{it}) : t \in [\frac{2\pi}{n+2}, \frac{4\pi}{n+2}] \} \), \( \Gamma_3 = \{ P(e^{it}) : t \in (\frac{4\pi}{n+2}, \pi] \} \). We notice that the curves \( \Gamma_1 \) and \( \Gamma_3 \) are standard curves but without one endpoint.

To prove the main theorem we need

**Lemma 1.** The following inequalities are valid

1. \( \sin y \leq y \quad \forall y \geq 0 \),
2. \( \cos y \geq \cos x + (x-y) \sin x - \frac{(x-y)^2}{2} \cos x, \quad 0 \leq x \leq y \leq \pi \),
3. \( \sin y \geq \sin x - (x-y) \cos x - \frac{(x-y)^2}{2} \sin x + \frac{1}{6} (x-y)^3 \cos x, \quad 0 \leq x \leq y \leq \pi \),
4. \( \cos 2y \leq 1 - 2y^2 + \frac{2y^4}{3} \quad \forall y \geq 0 \),
5. \( \cos y + \cos 2y \geq -\frac{9}{8}, \quad \forall y \geq 0 \),
6. \( \sin y \leq \sin x - (x-y) \cos x, \quad 0 \leq y \leq x \leq \pi \),
7. \( \cos y \leq \cos x + (x-y) \sin x - \frac{(x-y)^2}{2} \cos x, \quad 0 \leq y \leq x \leq \pi \),
8. \( \cos y \leq \cos x - (y-x) \sin x, \quad 0 \leq x \leq y \leq \pi \),
9. \( \sin y \geq \sin x - (x-y) \cos x - \frac{(x-y)^2}{2} \sin x, \quad 0 \leq y \leq x \leq \frac{\pi}{2} \).
On univalent polynomials

(10) \[ \cos ky \geq \cos kx + k(x - y) \sin kx - k^2 \frac{(x - y)^2}{2} \cos kx - k^3 \frac{(x - y)^3}{6} \sin kx, \quad 0 \leq y \leq x \leq \frac{\pi}{2k}, \]

(11) \[ \tan \frac{\pi y}{x} \leq \frac{\pi}{x}(y - x), \quad 0 \leq \frac{x}{2} \leq y \leq x \leq \pi, \]

(12) \[ \tan \frac{\pi y}{x} \geq \frac{\pi}{x}(y - x), \quad 0 \leq x \leq y \leq \frac{3x}{2} \leq \pi, \]

(13) \[ \cot \frac{\pi y}{x} \geq \frac{x}{\pi(y - x)} - \frac{\pi(y - x)}{3x}, \quad 0 \leq \frac{x}{2} \leq y \leq x \leq \pi, \]

(14) \[ \cot y \leq \frac{1}{y - \frac{y}{3}}, \quad 0 \leq y \leq \frac{\pi}{2}, \]

(15) \[ 2y \cos y - 3 \sin y \leq 0, \quad 0 \leq y \leq \frac{\pi}{2}. \]

**Proof.** Let \( y \geq 0. \)

(1) This is a classical fact.

(2) \[ g(y) = \cos y - \cos x + (y - x) \sin x + \frac{(y - x)^2}{2} \cos x. \]

\[ g'(y) = -\sin y + \sin x + (y - x) \cos x. \]

\[ g''(y) = -\cos y + \cos x \geq 0 \Rightarrow g'(y) \geq g'(x) = 0 \Rightarrow g(y) \geq g(x) = 0. \]

(3) \[ g(y) = \sin y - \sin x - (y - x) \cos x + \frac{(y - x)^2}{2} \sin x + \frac{1}{6}(y - x)^3 \cos x. \]

\[ g'(y) = \cos y - \cos x + (y - x) \sin x + \frac{(y - x)^2}{2} \cos x \geq 0 \text{ from (2)} \Rightarrow g(y) \geq g(x) = 0. \]

(4) \[ g(y) = \cos 2y - 1 + 2y^2 - \frac{2y^4}{3}. \]

\[ g'(y) = -2(\sin 2y - 2y + \frac{4y^3}{3}) \leq 0 \text{ from (3) for } x = 0, y \to 2y, \Rightarrow g(y) \leq g(0) = 0. \]

(5) \[ \cos y + \cos 2y + \frac{3}{8} = 2 \cos y + \cos y + \frac{1}{8} = \frac{1}{8}(1 + 4 \cos x)^2 \geq 0. \]

(6) \[ g(y) = \sin y - \sin x + (x - y) \cos x. \]

\[ g'(y) = \cos y - \cos x \geq 0 \Rightarrow g(y) \leq g(x) = 0. \]

(7) \[ g(y) = \cos y - \cos x - (x - y) \sin x + \frac{(x - y)^2}{2} \cos x. \]

\[ g'(y) = -\sin y + \sin x - (x - y) \cos x \geq 0 \text{ from (6)} \Rightarrow g(y) \leq g(x) = 0. \]

(8) \[ g(y) = \cos y - \cos x + (y - x) \sin x. \]

\[ g'(y) = -\sin y + \sin y + \sin x \leq 0 \Rightarrow g(y) \leq g(x) = 0. \]

(9) \[ g(y) = \sin y - \sin x + (x - y) \cos x + \frac{(x - y)^2}{2} \sin x. \]

\[ g'(y) = \cos y - \cos x - (x - y) \sin x. \]

\[ g''(y) = -\sin y + \sin y + \sin x \geq 0 \Rightarrow g'(y) \leq g'(x) = 0 \Rightarrow g(y) \geq g(x) = 0. \]

(10) \[ g(y) = \cos kx - kx \cos kx + k^2 \frac{(x - y)^2}{2} \sin kx + k^3 \frac{(x - y)^3}{6} \sin kx. \]

\[ g'(y) = -k(\sin ky - \sin kx + k(x - y) \cos kx + \frac{k^2(x - y)^2}{2} \sin kx) \leq 0 \text{ from (9) for } y \to ky, x \to kx \Rightarrow g(y) \geq g(x) = 0. \]

(11) \[ g(y) = \tan \frac{\pi y}{x} - \frac{\pi}{x}(y - x). \]

\[ g'(y) = \frac{\pi \tan^2 x}{x} \geq 0 \Rightarrow g(y) \leq g(x) = 0. \]
(12) \[ g(y) = \tan \frac{xy}{x} - \frac{x}{x} (y - x). \]
\[ g'(y) = \frac{\pi \tan^2 \frac{xy}{x}}{2} \geq 0 \Rightarrow g(y) \geq g(x) = 0. \]

(13) \[ g(y) = \cot \frac{xy}{x - \frac{1}{x} (y - x)} + \pi (y - x) = \frac{3\pi x (x - y) \cos \frac{xy}{x} - (3 + x^2) y^2 + 2\pi x} {3\pi x (x - y) \sin \frac{xy}{x}} =: \frac{\text{num}(y)}{\text{den}(y)}. \]

The denominator is positive, so we study the sign of the numerator.
\[ \text{num}(y)' = -\frac{\pi^2 (x - y) (y - x) \cos \frac{xy}{x} + x \sin \frac{xy}{x}} {x} \leq 0 \text{ from (11)} \Rightarrow \text{num}(y) \geq \text{num}(x) = 0. \]

(14) \[ g(y) = \cot y - \frac{1}{y} - 3 \leq \frac{3x \cos 3x + x^2 \sin x}{3x \sin x} =: \frac{\text{num}(y)}{\text{den}(y)}. \]

The denominator is non-negative, then we consider the sign of the numerator.
\[ \text{num}(y)' = x(\cos x - \sin x). \] Further \((x \cos x - \sin x)' = -x \sin x \leq 0 \Rightarrow \text{num}(y) \leq \text{num}(0) = 0. \]

(15) \[ g(y) = 2y \cos y - 3 \sin y. \]
\[ g'(y) = -\cos y - 2y \sin y \leq 0 \Rightarrow g(y) \leq g(0) = 0. \]

Lemma 2. The set \( \Gamma_1 \) is a simple curve.

Proof. Let us consider
\[ \frac{u(t)}{v(t)} = \cot \frac{(2 + n)t}{2} - \frac{1}{4} (2 + n) \left( \cos \frac{2\pi}{2 + n} - \cos t \right) \csc \frac{2\pi}{2 + n} \csc \frac{(2 + n)t}{2} \tan \frac{t}{2}. \]

Its derivative is equal to
\[ \frac{1}{16} (2 + n) \csc^2 \frac{(2 + n)t}{2} \csc^2 \frac{\pi}{2 + n} \sec^2 \frac{t}{2} \left[ -3 + \cos t \left( 2 \cos \frac{2\pi}{2 + n} + \cos t \right) + \right. \]
\[ \left. + 2(2 + n) \left( \cos \frac{2\pi}{2 + n} - \cos t \right) \cot \frac{(2 + n)t}{2} \sin t - \sin^2 t \right]. \]

Further we will consider the sign of the derivative. Therefore, non-negative factors in front of the square brackets can be ignored. We will consider two cases.

Case 1. \( t \in \left[ 0, \frac{\pi}{2+n} \right]. \)

The following inequality is valid
\[ d(t) := -3 + \cos t \left( 2 \cos \frac{2\pi}{2 + n} + \cos t \right) + 2(2 + n) \left( \cos \frac{2\pi}{2 + n} - \cos t \right). \]
\[ \cdot \cot \frac{(2 + n)t}{2} \sin t - \sin^2 t \leq -2 - \frac{25t^2}{36} + 2 \cos \frac{2\pi}{2 + n} + \]
\[ + \frac{5}{144} \left( \pi + (2 + n)t \right) \left( 24 + (\pi - (2 + n)t)^2 \right) \left( -2 + t^2 + 2 \cos \frac{2\pi}{2 + n} \right). \]

To prove it we have used the estimates which follow from the inequalities (2), (3), (10) and (1) from Lemma [1]
\[ 1 - \frac{t^2}{2} \leq \cos t \leq 1, \quad \sin t \geq t - \frac{t^3}{6} \geq \frac{5t^5}{6}, \quad \sin^2 t \geq t^2 \left( 1 - \frac{t^2}{6} \right)^2 \geq \frac{25t^2}{36}, \]
\[ \cos \frac{(2 + n)t}{2} \geq \frac{1}{2} (\pi - (2 + n)t) - \frac{1}{48} (\pi - (2 + n)t)^3, \quad \sin \frac{(2 + n)t}{2} \leq \frac{(2 + n)t}{2}. \]
Let $\frac{2\pi}{n+2} = x \in (0, \frac{\pi}{6}]$. From the inequality (17) we have
\[
d(t) \leq -2 - \frac{25t^2}{36} + 2\cos x - \frac{5}{144x^3}(-2t + x)(-24\pi x^2 + \pi^3(-2t + x)^2)(-2 + t^2 + 2\cos x)
\]
where $t \in [0, \frac{\pi}{2}]$. Simple calculations show that
\[
-2 - \frac{25t^2}{36} + 2\cos x < 0, \quad -2t + x \geq 0,
\]
\[
-2 + t^2 + 2\cos x \leq -2 + \frac{x^2}{4} + 2\cos x \leq 0,
\]
\[
-24\pi x^2 + \pi^3(-2t + x)^2 = 4\pi^3 t(t - x) + \pi(\pi^2 - 24)x^2 < 0.
\]
From here we see that the expression (16) is negative. It means that the function $\frac{u(t)}{v(t)}$ is strictly decreasing on the segment $t \in [0, \frac{\pi}{n+2}]$.

**Case 2.** $t \in \left[\frac{\pi}{n+2}, \frac{2\pi}{n+2}\right)$.

In this case we use the estimates (9), (6) and (13) from Lemma 1 and the following consequence of (7):
\[
\cos t < \cos x + (x - t)\sin x,
\]
where $x = \frac{2\pi}{n+2} \in (0, \frac{\pi}{6}]$. To show that the expression (16) is negative, it is enough to verify the following inequality
\[
(18) \quad \frac{(t - x)^2}{24x^2} \left[-8\pi^2 \left(2 + (t - x)^2\right) - 3 \left(-12 + (t - x)^2\right)x^2 + \left(-8\pi^2 \left(-2 + (t - x)^2\right) + 3 \left(-4 + (t - x)^2\right)x^2\right)\cos 2x + 12(t - x)(-2\pi^2 + x^2)\sin 2x\right] =: \frac{(t - x)^2}{24x^2} h(t, x) < 0.
\]
It is easy to show that the expression $h(t, x)$ can be presented in the form $a(x)t^2 + b(x)t + c(x)$ where
\[
a(x) = -8\pi^2 - 8\pi^2 \cos 2x - 3x^2(1 - \cos 2x) \leq 0.
\]
From here we see that the branches of the parabola with respect to $t$ are going down. Let us find the sign of $b(x)$. We have
\[
b(x) = 16\pi^2 x + 6x^3 + 2x \left(8\pi^2 - 3x^2\right)\cos 2x + 12 \left(-2\pi^2 + x^2\right)\sin 2x.
\]
From the inequalities $\cos 2x \leq 1$ and (3) for $y = 2x$ it follows that
\[
b(x) \leq 8x \left(-2\pi^2 + 3x^2 + 4\pi^2 x^2 - 2x^4\right) \leq 0.
\]
Therefore, the vertex of parabola (18) lies to the left of the segment under consideration. Hence the maximum is achieved at the point $t = \frac{\pi}{2}$. Consequently,
\[
h(t, x) \leq 36x^2 - \frac{3x^4}{4} - 2\pi^2 \left(8 + x^2\right) + \frac{1}{4} \left(64\pi^2 - 8 \left(6 + \pi^2\right)x^2 + 3x^4\right)\cos 2x - 6x \left(-2\pi^2 + x^2\right)\sin 2x.
\]
Using estimates (1) for $y = 2x$ and (4) we obtain
\[
h(t, x) \leq \frac{1}{6} x^2 \left(3x^4 \left(-19 + x^2\right) + 72 \left(2 + x^2\right) - 8\pi^2 \left(9 - 11x^2 + x^4\right)\right) < 0.
\]
Hence, the expression $\frac{u(t)}{v(t)}$ is a decreasing on the segment $[0, \frac{2\pi}{n+2}]$ function. Lemma 2 is proved.

Lemma 3. The set $\Gamma_2$ is a simple curve.

Proof. To prove this lemma we will investigate the function

$$v(t) = \frac{(1 - \cos \frac{2\pi}{n+2}) \sin t \sin^2 \frac{n+2}{2} t}{(n + 2)(1 - \cos t) \left( \cos t - \cos \frac{2\pi}{n+2} \right)^2}$$

on the segment $\left[\frac{2\pi}{n+2}, \frac{4\pi}{n+2}\right]$.

Again we will consider two cases:

Case 1. $t \in \left[\frac{2\pi}{n+2}, \frac{3\pi}{n+2}\right]$.

Let us note that the function $\frac{\sin t}{1 - \cos t}$ decreases on the segment $\left[\frac{2\pi}{n+2}, \frac{4\pi}{n+2}\right]$. Let us consider

$$(19) \quad \left( \frac{\sin \frac{n+2}{2} t}{\cos t - \cos \frac{2\pi}{n+2}} \right)'_t = -\frac{1}{2} \left(2 + n\right) \left( \cos \frac{2\pi}{2 + n} - \cos t \right) \cos \frac{(2+n)t}{2} + \sin t \sin \frac{(2+n)t}{2}$$

We have to show that

$$-\frac{1}{2} \left(2 + n\right) \left( \cos \frac{2\pi}{2 + n} - \cos t \right) + \sin t \tan \frac{(2+n)t}{2} \geq 0.$$  

We will use the estimates (2), (3) and (12) for $x = \frac{2\pi}{n+2} \in (0, \frac{\pi}{6}]$. It remains to show that

$$(t, x) := \cos x - t^2 \cos x - x^2 \cos x + t(2x \cos x - 3 \sin x) + 3x \sin x > 0$$

The equation $\delta(t, x) = 0$ describes a parabola of $t$ directed downward by branches. From the inequality (15) we see that the vertex projection is $< \frac{2\pi}{n+2}$. It means that the minimum value is achieved at the point $t = \frac{3}{2}x$. We get

$$\delta(t, x) \geq \frac{1}{4} \left(12 - x^2\right) \cos x - 6x \sin x$$

The derivative of this expression with respect to $x$ is equal to

$$\frac{1}{4} \left(-8x \cos x + (-18 + x^2) \sin x\right) < 0.$$  

Consequently, the minimum is achieved at the point $x = \frac{\pi}{6}$ and hence

$$\delta(t, x) \geq \frac{1}{4} \left(12 - x^2\right) \cos x - 6x \sin x > 0, \quad x \in \left(0, \frac{\pi}{6}\right).$$

Therefore, the expression (19) is negative. Hence non-negative function $\frac{\sin \frac{n+2}{2} t}{\cos t - \cos \frac{2\pi}{n+2}}$ is a decreasing function. From here we see that $v(t)$ is a decreasing function for $t \in \left[\frac{2\pi}{n+2}, \frac{3\pi}{n+2}\right]$. 

Case 2. $t \in \left[\frac{3\pi}{n+2}, \frac{4\pi}{n+2}\right]$.  

On this segment the function \(1 - \cos t\) increases, while \(\sin^2 \frac{n+2}{2} t\) decreases. The derivative of the remaining part

\[
\left( \frac{(1 - \cos \frac{2\pi}{n+2}) \sin t}{(n+2)(\cos t - \cos \frac{2\pi}{n+2})^2} \right)' = \frac{(\cos \frac{2\pi}{n+2} - 1) (-3 + 2 \cos \frac{2\pi}{n+2} \cos t + \cos 2t)}{2(n+2)(\cos t - \cos \frac{2\pi}{n+2})^3} < 0.
\]

Hence this function also decreases. It means that the function \(v(t)\) decreases for \(t \in [\frac{3\pi}{n+2}, \frac{4\pi}{n+2}]\).

Consequently, \(v(t)\) is a decreasing function on the segment \(12 \pi n + 2, 4 \pi n + 2\). Lemma 3 is proved.

**Lemma 4.** The set \(\Gamma_3\) is a simple curve.

**Proof.** To prove this Lemma it will be sufficient to show that \(u(t)\) increases on this segment. We take the derivative:

\[
(20) \quad 2u'(t) \left( \cos \frac{2\pi}{n+2} - \cos t \right)^3 = \left( \cos \frac{2\pi}{n+2} - \cos t \right) \left( 2 \cos((2 + n) t) \cot \frac{t}{2} \right.
\]

\[
\cdot \sin^2 \frac{\pi}{n+2} + \sin t + \left( -\cos \frac{2\pi}{n+2} + \cos t - 2 \sin^2 t \right) \sin^2 \frac{\pi}{n+2} \sin((2 + n) t)
\]

\[
\left. + \frac{(-2 \cos^2 \frac{\pi}{n+2} + \cos t + \cos 2t) \sin^2 \frac{\pi}{n+2}}{(n+2) \sin^2 \frac{t}{2}} \right).
\]

**Case 1.** \(t \in (\frac{4\pi}{n+2}, \frac{\pi}{2}]\).

Using the inequalities \(\cos((2 + n) t) \geq -1, \sin((2 + n) t) \leq 1\), we see that the expression \(20\) is not less than

\[
\left( \cos \frac{2\pi}{n+2} - \cos t \right) \left( -2 \cot \frac{t}{2} \sin^2 \frac{\pi}{n+2} + \sin t \right) + \frac{(-2 \cos^2 \frac{\pi}{n+2} + \cos t + \cos 2t) \sin^2 \frac{\pi}{n+2}}{(n+2) \sin^2 \frac{t}{2}}.
\]

We use the following substitutions: \(\cot \frac{t}{2} = y, y \in [1, \cot \frac{2\pi}{n+2}]\) and \(s = \frac{\pi}{n+2}, s \in (0, \frac{\pi}{12}]\). After a simplification we see that the numerator is a polynomial of \(y\):

\[
\rho(y, s) := -\frac{s \sin^2 2s}{2\pi} + 4y \cos^4 s - \frac{1}{\pi} \left( 3sy^2(3 + \cos 2s) \sin^2 s \right) + y^3(-1 + \cos 4s) - \frac{1}{\pi} \left( sy^4(7 + 3 \cos 2s) \sin^2 s \right) + 4y^5 \sin^4 s + \frac{2}{\pi} sy^6 \sin^4 s.
\]

Certainly, positive members of the fifth and sixth degree can be ignored. Further we apply the estimates, which follow from the inequalities \(1\) and \(2\):

\[
7 + 3 \cos 2s \leq 3(3 + \cos 2s), \quad \cos s \leq 1, \quad \sin^2 s \leq s^2, \quad \cos^4 s \geq \left( 1 - \frac{s}{2} \right)^4.
\]
We note that this completes the proof in the Case 1.

These estimates imply that

\[ \rho(y,s) \geq 4y - 8sy - \frac{7s^6y^3}{4} + s^2(6y - 8y^3) + \frac{s^4}{4}(y + 29y^3) - \frac{2s^3}{\pi}(1 + \pi y + 6y^2 + 6y^4). \]

We note that \(-\frac{2s^3}{\pi} \geq -\frac{2s^3y^2}{\pi}\). Thus,

\[ (21) \quad 4\pi \rho(y,s) \geq r(y,s) := \pi(-2 + s)^4 + \pi s^2(-32 + 29y^2 - 7s^4)y^2 - (8s^3(1 + 6(y + y^2))). \]

Now, let us show \(r(y,s) > 0\). The derivative is equal to

\[ r(y,s)' = -48s^3 + 2(-32\pi s^2 + 29\pi s^4 - 7\pi s^6)y - 144s^2y^2. \]

It is easy to see that this expression is negative. So it is enough to show that the expression \(r(y,s)\) is positive for \(y = \cot 2s\). After the substitution we have

\[ (22) \quad r(y,s) \geq \pi(-2 + s)^4 - 8s^3 - 48s^2\cot 2s - (32\pi s^2 - 29\pi s^4 + 7\pi s^6)\cot^2 2s - 48s^3\cot^3 2s. \]

From (14) it follows that \(\cot 2s \leq \frac{1}{2\pi} - \frac{2\pi}{3} \leq \frac{1}{2\pi}\). Consequently,

\[ r(y,s) \geq -6 + 8\pi - 32\pi s + \frac{631\pi s^2}{12} - 8(\pi + 1)s^3 - \left(32 + \frac{1235\pi}{36}\right)s^4 + \frac{1}{9}(128 + 158\pi)s^6 - \frac{28\pi s^6}{9}. \]

This polynomial is positive for \(s \in (0, \frac{\pi}{12})\). This fact can be checked by taking derivatives up to the fifth order. Hence, the expression (21) and, therefore, (20) is positive as well and this completes the proof in the Case 1.

**Case 2.** \(t \in \left[\frac{\pi}{2}, \pi\right]\).

We use the estimates \(\cos((2 + n)t) \geq -1, \sin((2 + n)t) \leq (n + 2)(\pi - t)\) and obtain that the expression (20) is not less than

\[ (23) \quad m(t) := \left(\cos\frac{2\pi}{n + 2} - \cos t\right)\left(-2\cot\frac{t}{2}\sin^2\frac{\pi}{n + 2} + \sin t\right) + \frac{(-2\cos^2\frac{\pi}{n + 2} + \cos t + \cos 2t)}{\sin^2\frac{t}{2}}\sin^2\frac{\pi}{n + 2}(\pi - t). \]

From the estimate \(\cos t \leq 0\) and the inequality (5) it follows that

\[ m(t) \geq \cos\frac{2\pi}{n + 2} \left(-2\cot\frac{t}{2}\sin^2\frac{\pi}{n + 2} + \sin t\right) - \left(2\cos^2\frac{\pi}{n + 2} + \frac{9}{8}\right)\sin^2\frac{\pi}{n + 2}(\pi - t). \]

Consequently,

\[ m(t)\sin^2\frac{t}{2} \geq -\frac{1}{8}(\pi - t)\left(17 + 8\cos\frac{2\pi}{2 + n}\right)\sin^2\frac{\pi}{2 + n} + \frac{1}{2}\cos\frac{2\pi}{2 + n}\left(\cos\frac{2\pi}{2 + n} - \cos t\right)\sin t \geq -\frac{1}{8}(\pi - t)\left(17 + 8\cos\frac{2\pi}{2 + n}\right)\sin^2\frac{\pi}{2 + n} + \frac{1}{2}\cos^2\frac{2\pi}{2 + n}\left(\pi - t - \frac{1}{6}(\pi - t)^3\right). \]
Now we divide the last inequality by \( \pi - t \). We see that
\[
\frac{m(t) \sin^2 \frac{t}{2}}{\pi - t} \geq \sigma(q) := -\frac{9}{16} \left(1 - \cos \frac{2\pi}{2 + n}\right) + \frac{1}{2} \cos \frac{4\pi}{2 + n} - \frac{q}{24} \left(1 + \cos \frac{4\pi}{2 + n}\right),
\]
where \( q = (\pi - t)^2 \in \left(0, \frac{\pi^2}{4}\right] \). This linear function decreases with respect to \( q \), hence it attains its minimum at \( q = \frac{\pi^2}{4} \). Thus, we obtain
\[
\sigma(q) \geq \frac{1}{48} \left(p^2(48 - \pi^2) + 27p - 51\right),
\]
where \( p = \cos \frac{2\pi}{2 + n} \in \left[\sqrt{\frac{3}{2}}, 1\right] \). Simple calculations show that this expression is positive on the segment. This proves the positivity of \( m(t) \) and, therefore, the inequality \( u'(t) > 0 \).

Hence, the inequality is proved and \( u(t) \) increases on the segment \( \left[\frac{4\pi}{6 + 2}, \pi\right) \). Lemma 4 is proved.

Now we are able to prove Theorem.

Proof. We have already proved the statement for three separate segments. To finish the proof we need to show, that \( \Gamma_1, \Gamma_2 \) and \( \Gamma_3 \) do not intersect each other.

First of all, we note that on the segment \( \left[\frac{2\pi}{6 + 2}, \frac{4\pi}{6 + 2}\right] \) the following inequalities are valid
\[
\cos t \leq \cos \frac{2\pi}{n + 2} - \left(t - \frac{2\pi}{n + 2}\right) \sin \frac{2\pi}{n + 2},
\]
\[
\sin(2 + n)t \leq \left(n + 2\right) \left(t - \frac{2\pi}{n + 2}\right), \quad \cot \frac{t}{2} \leq \cot \frac{\pi}{n + 2}.
\]
The first of them follows from \( 8 \), the second follows from \( 1 \) and the third is evident.

Hence
\[
2 \left(\cos \frac{2\pi}{n + 2} - \cos t\right)^2 u(t) = \cos t - \cos \frac{2\pi}{n + 2} + \frac{2}{n + 2} \cot \frac{t}{2} \frac{n + 2}{n + 2} \sin(2 + n)t \leq 0.
\]

On the segment \( \left[\frac{4\pi}{6 + 2}, \pi\right) \) from the proof of Lemma 4 it follows that \( u(t) < u(\pi) = -\frac{1}{2(1 + \cos \frac{2\pi}{6 + 2})} < 0 \).

For \( t \in \left[0, \frac{\pi}{6 + 2}\right] \) the inequality \( u(t) > 0 \) holds, so we can exclude it from consideration, because it was shown that on the second and third segments the values of \( u(t) \) are negative.

Further, we will consider only such values of \( t \), for which \( u(t) \leq 0 \). Since \( \frac{u(t)}{v(t)} \) is a decreasing function on the first segment, we see that \( u(t) \leq 0 \) for \( t \in [t_0, \pi] \) where \( t_0 = \min\{t \in [0, \pi] : u(t) = 0\} \).

We note that
\[
(24) \quad u(t) = \frac{1}{2} \left(\cos t - \cos \frac{2\pi}{n + 2}\right) + v(t) \cot \frac{n + 2}{2} t.
\]

From here on the segment \( \left[t_0, \frac{2\pi}{n + 2}\right] \) we obtain
\[
(25) \quad \frac{1}{2} \left(\cos t - \cos \frac{2\pi}{n + 2}\right) + v(t) \cot \frac{n + 2}{2} t \leq 0 \Rightarrow v(t) \geq -\frac{\tan \frac{n + 2}{2} t}{2 \left(\cos t - \cos \frac{2\pi}{n + 2}\right)}.
\]
On the other hand, on the segment \([\frac{2\pi}{n+2}, \pi]\) the following inequality holds. 

\[ v(t) \leq v\left( \frac{2\pi}{n+2} \right) = \frac{n+2}{4} \csc \frac{2\pi}{n+2}. \]

(26)

It remains to show, that for \(t \in [t_0, \frac{2\pi}{n+2}]\) the following inequality is valid

\[ 0 < -\frac{\tan \frac{n+2}{2} t}{2 (\cos t - \cos \frac{2\pi}{n+2})} - \frac{n+2}{4} \csc \frac{2\pi}{n+2} = \frac{- (n+2) (\cos t - \cos \frac{2\pi}{n+2}) \csc \frac{2\pi}{n+2} - 2 \tan \left( \frac{n+2}{2} t \right)}{4 (\cos t - \cos \frac{2\pi}{n+2})}. \]

This inequality easily follows from the inequality

\[ - (n+2) (\cos t - \cos \frac{2\pi}{n+2}) \csc \frac{2\pi}{n+2} - 2 \tan \left( \frac{n+2}{2} t \right) > 0 \]

which is an easy consequence of the inequalities (7) and (11).

Therefore, from the inequalities (25) and (26) it follows that \(\Gamma_1\) and \(\Gamma_2\) do not intersect.

Finally, we need to show, that \(\Gamma_3\) does not intersect \(\Gamma_1\) and \(\Gamma_2\). For this purpose we will prove that \(u'(t) > 0\) on the segment \([\frac{3\pi}{n+2}, \frac{4\pi}{n+2}]\). From the equality (24) we get

\[ u'(t) = \frac{\sin t}{2 (\cos \frac{2\pi}{n+2} - \cos t)^2} - v(t) \left( 1 + \frac{n}{2} \right) \csc^2 \frac{(n+2)}{2} + v'(t) \cot \frac{(n+2)t}{2}. \]

According to the proof of Lemma 3 on this segment \(v'(t) < 0\), so that

\[ u'(t) > \frac{\sin t}{2 (\cos \frac{2\pi}{n+2} - \cos t)^2} - v(t) \left( 1 + \frac{n}{2} \right) \csc^2 \frac{(n+2)}{2} = \frac{\cot \frac{t}{2}}{2 (\cos \frac{2\pi}{n+2} - \cos t)} \geq 0. \]

(27)

From the inequality (27) and the proof of Lemma 4 it follows that \(u'(t) > 0\) on the segment \([\frac{3\pi}{n+2}, \pi]\). Now we note, that for \(t \in [\frac{4\pi}{n+2}, \pi]\) the following estimate holds:

\[ \frac{(n+2)v(t)}{1 - \cos \frac{2\pi}{n+2}} = \frac{\sin t \sin^2 \frac{n+2}{2} t}{(1 - \cos t) (\cos t - \cos \frac{2\pi}{n+2})^2} \leq \frac{\sin t}{(1 - \cos t) (\cos t - \cos \frac{2\pi}{n+2})^2} < \]

\[ \frac{\sin \frac{3\pi}{n+2}}{(1 - \cos \frac{3\pi}{n+2}) (\cos \frac{4\pi}{n+2} - \cos \frac{2\pi}{n+2})^2} = v \left( \frac{3\pi}{n+2} \right), \]

where the last estimate is true, because both of the functions \(\frac{\sin t}{1 - \cos t}\) and \(\frac{1}{(\cos t - \cos \frac{2\pi}{n+2})^2}\) decrease.

From this fact and the proof of Lemma 3 we see, that \(\Gamma_1\), \(\Gamma_2\) and \(\Gamma_3\) do not intersect each other. It means that \(\Gamma\) is a simple curve.

Let \(\overline{\Gamma} = \{ z : z \in \Gamma \}\) and let \(\Omega\) be the interior of the curve \(\Gamma \cup \overline{\Gamma}\). It is easy to see that this domain is simply connected. Using the argument principle it is easy to show that the polynomials \(P(z)\) are univalent in the disk \(\mathbb{D}\) and that they maps the disk onto \(\Omega\). This concludes the proof of Theorem.
On univalent polynomials

On the graph one can see the behavior of the function on the segment $[0, \pi)$ for $n = 12$. The curve $\Gamma_1$ is indicated by green, $\Gamma_2$ is blue and $\Gamma_3$ is red. Also we have marked the images of the points $\{\frac{\pi}{n+2}, \frac{2\pi}{n+2}, \frac{3\pi}{n+2}, \frac{4\pi}{n+2}\}$.

**Figure 1.** Case $n = 12$. The curve $\Gamma$ with the curves $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$.

**Remark.** There are no doubts that Theorem is also true for $n \leq 9$. The present scheme (with some modifications) is also working in this case and we are able to prove Theorem for $n \leq 9$. However, it would be nice to find a short (or even an elegant, as it was done in [8] for $n = 4$) proof in this case.

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