Patterns with prescribed numbers of critical points on topological tori

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ABSTRACT
We study the existence of critical points of stable stationary solutions to reaction–diffusion problems on topological tori. Stable nonconstant stationary solutions are often called patterns. We construct topological tori and patterns with prescribed numbers of critical points whose locations are explicit.

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1. Introduction
Let $M$ be a topological torus equipped with a Riemannian metric $g$. For $u = u(x, t)$ on $M$, we consider the reaction–diffusion problem:

$$\partial_t u = \Delta_g u + f(u) \quad \text{in } M \times (0, \infty),$$

where $f \in C^1(\mathbb{R})$ is a function of $u$ and $\Delta_g$ denotes the Laplace–Beltrami operator on $M$,

$$\Delta_g u = \text{div}(\nabla_g u) = \sum_{i=1}^2 \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i} \left( \sqrt{|g|} (\nabla_g u)^i \right).$$

A stationary solution $U$ of (1) is said to be stable in the sense of Lyapunov if for each $\epsilon > 0$, there exists a $\delta > 0$ such that, for every initial data $u_0$ with $\|u_0 - U\|_\infty < \delta$ we have $\|u(\cdot, t) - U\|_\infty < \epsilon$ for every $t > 0$. Throughout, we will refer to stable nonconstant stationary solutions as patterns.

The existence and nonexistence of patterns on surfaces of revolution and more general compact $d$-dimensional Riemannian manifolds have been studied in [1–7]. Among these,
in [3, Theorem 2] Jimbo introduced manifolds and nonlinearities $f$ having complex patterns whose construction is analogous to that in [8], and in [1] Bandle et al. constructed a class of surfaces of revolution with non-empty boundary and nonlinear terms $f$ having patterns with the Neumann boundary condition by solving some ordinary differential equations with the aid of an idea introduced by Yanagida in [9] to construct the nonlinear terms $f$.

In this paper, we study the stability of patterns of (1) where $M$ is a small perturbation $T^2_\epsilon$ of the standard tori $T^2$. Our purpose is to find complex patterns of (1) on topological tori with exact numbers of critical points.

In our previous work [10], we constructed topological tori $M$ together with patterns on $M$ having at least $4n$ critical points for sufficiently large $n$. In the beginning, we slightly perturb the surfaces of revolution $D$ in [1] in such a way that each centre curve of each new surface $M_\kappa$ is just a circular arc with sufficiently small curvature $\kappa$, where $M_0 = D$. The perturbed patterns correspond to the reaction–diffusion problem with the Neumann boundary condition

$$\begin{aligned}
\partial_t u &= \Delta_g u + f(u) \quad \text{in } M_\kappa \times (0, \infty), \\
\frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial M_\kappa \times (0, \infty),
\end{aligned} \tag{3}$$

where $\nu$ denotes the outward unit normal vector to the boundary $\partial M_\kappa$. With the aid of the implicit function theorem, the patterns of (3) exist on $M_\kappa$ with no critical point in the interior of $M_\kappa$. Furthermore, we attach a sufficiently large even number $2n$ of copies of $M_\kappa$ together with the perturbed pattern on $M_\kappa$ to each other in such a way that the centre curve of the new closed surface $M$ is just a whole circle with curvature $\kappa$. As a consequence, the stationary solution of (1) on $M$ is constructed as a symmetric function on $M$. The stability of the constructed solution on $M$ having at least $4n$ critical points follows from the symmetry coming from the construction. We can only show that there exist at least two critical points on each boundary $\partial M_\kappa$ in $M$ and hence the constructed pattern has at least $4n$ critical points.

The objective of this paper is to show that patterns of (1) exist on topological tori with an exact number $4n$ of critical points by introducing another simple way to construct topological tori together with its patterns. In [10] we start with a small piece of topological tori $M_\kappa$ and we then arrive at the whole tori $M$ by attaching a sufficiently large even number of copies of $M_\kappa$. On the other hand, in this paper we directly perturb the standard tori $T^2$ to obtain new topological tori $T^2_\epsilon$ with a small parameter $\epsilon$, where $T^2_0 = T^2$. Topologically, $T^2_\epsilon$ is the same as $M$ constructed in [10], but geometrically, they are different from each other. The reason why we are able to give an exact number of critical points on $T^2_\epsilon$ is simply because we deal with the explicit perturbation $T^2_\epsilon$ of the standard torus $T^2$ in this paper.

We start with considering the upper half of a standard torus $T^2$. By some adjustment, using [1, Theorem 4.1, p. 41] yields patterns of problem (3) where $M_\kappa$ is replaced by the upper half of $T^2$. Subsequently, we prove that patterns of (1) exist on $T^2$. Next, we slightly perturb $T^2$ by simply changing the radius of the tube from a constant into a periodic function to obtain new topological tori $T^2_\epsilon$ with a small parameter $\epsilon$. Then the patterns of (1) on $T^2$ together with the implicit function theorem yield patterns of (1) on $T^2_\epsilon$. Moreover,
the stability of each pattern enables us to examine the critical points of the pattern in $T^2_e$. We summarize our result in the following theorem.

**Theorem 1.1:** There exist a nonlinearity $f$ and a number $N \in \mathbb{N}$ such that, for each $n \geq N$, a perturbation $M$ of a standard torus $T^2$ together with a pattern $U$ of (1) is constructed in such a way that $U$ has exactly $4n$ critical points.

We organize the paper as follows. In Section 2, we introduce patterns on standard tori with the aid of a result in [1]. In Section 3, we start with the construction of perturbed tori and give a proof of Theorem 1.1. We also mention the exact locations of the critical points of the patterns.

**2. Standard tori**

Let $T^2$ be a standard torus properly embedded in $\mathbb{R}^3$ and parameterized by

$$
\begin{align*}
x_1 &= (R + r \cos \varphi) \cos \theta, \\
x_2 &= (R + r \cos \varphi) \sin \theta, \\
x_3 &= r \sin \varphi,
\end{align*}
$$

where $R$, $r$ are constants and $R > r > 0$. Set $x^1 = \varphi, x^2 = \theta$. Then $T^2$ is a 2-dimensional Riemannian manifold with metric $ds^2$ and area element $d\sigma$ given by

$$
\begin{align*}
ds^2 &= \sum_{i,j=1}^2 g_{ij} \, dx^i \, dx^j = r^2 \, d\varphi^2 + (R + r \cos \varphi)^2 \, d\theta^2, \\
d\sigma &= \sqrt{|g|} \, d\varphi \, d\theta = r(R + r \cos \varphi) \, d\varphi \, d\theta.
\end{align*}
$$

The Riemannian gradient $\nabla_g u$ of $u$ with respect to $g$ on $T^2$ is given by

$$
\nabla_g u = \left( \begin{array}{c} 
\frac{1}{r^2} \frac{\partial \varphi u}{\partial \varphi} \\
\frac{1}{(R + r \cos \varphi)^2} \frac{\partial \theta u}{\partial \theta}
\end{array} \right),
$$

and the Laplace–Beltrami operator $\Delta_g$ on $T^2$ is expressed as

$$
\Delta_g u = \frac{1}{r^2} u_{\varphi \varphi} + \frac{1}{(R + r \cos \varphi)^2} u_{\theta \theta} - \frac{\sin \varphi}{r(R + r \cos \varphi)} u_{\varphi}.
$$

Geometrically, $T^2$ is a surface of revolution obtained by revolving a circle with radius $r$ with centre $(R, 0, 0)$ in $x_1x_3$ plane about the $x_3$-axis.

For a surface of revolution $D$, in [1] Bandle et al. studied the existence and nonexistence of patterns of the reaction–diffusion problem on $D$ with the Neumann boundary condition

$$
\begin{align*}
\frac{\partial t u}{\partial t} &= \Delta_g u + f(u) \quad \text{in } D \times (0, \infty), \\
\frac{\partial u}{\partial v} &= 0 \quad \text{on } \partial D \times (0, \infty),
\end{align*}
$$

where $v$ denotes the outward unit normal vector to $\partial D$. 

Consider the eigenvalue problem linearized at a stationary solution $U$ of (1) (or (6)):
\[
\begin{cases}
\Delta_g \phi + f'(U)\phi = -\lambda \phi & \text{in } M \text{ (or } D), \\
\frac{\partial \phi}{\partial \nu} = 0 & \text{on } \partial D \text{ for problem (6)}.
\end{cases}
\tag{7}
\]

The Rayleigh quotient of problem (7) can be taken on in terms of the principal eigenvalue
\[
\lambda_1 = \inf_{\phi \neq 0} \frac{\int_M (|\nabla g\phi|^2 - f'(U)\phi^2) \, d\sigma}{\int_D \phi^2 \, d\sigma}. \tag{8}
\]

If $\phi_1$ is the normalized eigenfunction that corresponds to the principal eigenvalue $\lambda_1$, then it satisfies
\[
\begin{cases}
\Delta_g \phi_1 + f'(U)\phi_1 = -\lambda_1 \phi_1 & \text{in } M \text{ (or } D), \\
\frac{\partial \phi_1}{\partial \nu} = 0 & \text{on } \partial D \text{ for problem (6)}, \\
\|\phi_1\|_{L^2(M) \text{ (or } L^2(D))} = 1, \quad \phi_1 > 0 & \text{in } M \text{ (or } D), \\
\lambda_1 = \int_M (|\nabla g\phi_1|^2 - f'(U)\phi_1^2) \, d\sigma.
\end{cases}
\tag{9}
\]

Note that the normalized eigenfunction that corresponds to the principal eigenvalue $\lambda_1$ is uniquely determined. It is well known that the sign of the principal eigenvalue determines the stability of a stationary solution, with the following stability criterion (see [11, Theorem 5.1.1, p. 98, and Theorem 5.1.3, p. 102] for instance):

- $U$ is stable, if $\lambda_1 > 0$,
- $U$ is unstable, if $\lambda_1 < 0$,
- The stability of $U$ is undetermined, if $\lambda_1 = 0$.

In particular, we suppose that $D$ is parameterized by
\[
\begin{cases}
x_1 = \psi(\rho) \cos \theta, \\
x_2 = \psi(\rho) \sin \theta, \\
x_3 = \chi(\rho),
\end{cases}
\tag{10}
\]

where $L > 0$ is a constant and $\psi, \chi \in C^3([0, L])$ satisfy that
\[
\psi > 0 \text{ and } (\psi')^2 + (\chi')^2 = 1 \text{ on } [0, L].
\]

Recall a theorem of [1] which states

**Theorem 2.1:** ([1, Theorem 4.1, p. 41]) Suppose that for some $\rho_0 \in (0, L)$
\[
\left( \frac{\psi'}{\psi'} \right)' > 0 \quad \text{at } \rho = \rho_0. \tag{11}
\]

Then, there exists $f \in C^1(\mathbb{R})$ such that problem (6) admits a pattern $Z = Z(\rho)$, where the principal eigenvalue $\lambda_1$ of the eigenvalue problem linearized at $Z$ is positive.
The pattern $Z = Z(\rho)$ of (6) on $D$ is a positive function of one variable $\rho$ (see [1, (4.8),(4.9), p.43 and Proposition 3.2., p. 39]). The nonlinear term $f = f(Z)$ changes its sign and is defined by [1, (4.10), p. 43] in such a way that for any $\rho \in (0, L)$

$$f[Z(\rho)] = -\frac{(\psi Z')'}{\psi}(\rho). \quad (12)$$

Let us set

$$\rho = r\varphi, \; L = r\pi, \; \psi(\rho) = R + r \cos(r^{-1}\rho) \text{ and } \chi(\rho) = r \sin(r^{-1}\rho).$$

Then $D$ corresponds to $T^+$, the upper half of the standard torus $T^2$ (see Figure 1). If $\varphi \in (0, \pi)$ is sufficiently close to $\pi$, then it follows that

$$\left(\frac{\psi'}{\psi}\right)' = -\frac{r + R \cos \varphi}{r(R + r \cos \varphi)^2} > 0,$$

which guarantees (11). Hence, by Theorem 2.1, a pattern $Z(\rho) = U^*(\varphi)$ of (6) exists on $T^+$ for $f$ given by (12). Moreover, we have from [1, proof of Lemma 4.4, pp. 43–44] that

(a) $U^*(\varphi)$ is positive in $(0, \pi]$;
(b) $U^*(\varphi)$ is strictly increasing in $(0, \pi)$;
(c) $U^*(\varphi)$ has no critical points in the interior of $T^+$.

We add one remark in order to apply Theorem 2.1 ([1, Theorem 4.1, p. 41]) to our surface of revolution $T^+$. Bandle et al. assume another condition [1, (2.13), p. 37] which is not
satisfied for our $T^+$, but the following condition holds true for $T^+$:
\[
\frac{\partial}{\partial v} = \frac{\partial}{\partial \rho} \quad \text{on} \{\varphi = \pi\} \quad \text{and} \quad \frac{\partial}{\partial v} = -\frac{\partial}{\partial \rho} \quad \text{on} \{\varphi = 0\},
\]
as they mention it in [1, just after (2.13), p. 37] and use it in [1, line 14, p. 44]. Thus, we are able to apply Theorem 2.1 to our $T^+$.

Consider the eigenvalue problem (7) linearized at the stationary solution $U^*$ of (6) for $D = T^+$, and let $\phi_1^*$ be the normalized eigenfunction that corresponds to the principal eigenvalue $\lambda_1$. Since $U^*$ is a function of one variable $\varphi$ and the normalized eigenfunction is uniquely determined, we see that $\phi_1^*$ is also a function of one variable $\varphi$.

### 2.1. Patterns on standard tori

Let $T^-$ be the lower half of $T^2$. Then
\[
\partial T^+ = \partial T^- = T^2 \cap \{x_3 = 0\} \quad \text{and} \quad T^2 = T^+ \cup \partial T^+ \cup T^-.
\]
Moreover, $\partial T^+$ consists of the two horizontal circles $C_{\max}, C_{\min}$ whose radii are the maximum $R + r$ and the minimum $R - r$, respectively, and hence
\[
T^2 \cap \{x_3 = 0\} = C_{\max} \cup C_{\min}.
\]

**Theorem 2.2:** For the standard torus $M = T^2$, there exists a nonlinearity $f$ together with a pattern $U$ of (1) with $\lambda_1 > 0$ such that the set of critical points of $U$ equals $C_{\max} \cup C_{\min}$.

**Proof:** Since $U^*$ satisfies the Neumann boundary condition on $\partial T^+$, we can define a stationary solution $U = U(\varphi)$ of (1) for $M = T^2$ by
\[
U(\varphi) = \begin{cases} U^*(\varphi) & \text{if } \varphi \in [0, \pi], \\ U^*(2\pi - \varphi) & \text{if } \varphi \in (\pi, 2\pi). \end{cases}
\]
Observe that
\[
\begin{align*}
(\text{i}) & \quad U \text{ is symmetric with respect to each component of } T^2 \cap \{x_3 = 0\}; \\
(\text{ii}) & \quad U(\varphi) \text{ is positive in } (0, 2\pi); \\
(\text{iii}) & \quad U(\varphi) \text{ is strictly increasing in } (0, \pi) \text{ and strictly decreasing in } (\pi, 2\pi); \\
(\text{iv}) & \quad \text{The set of critical points of } U \text{ equals } T^2 \cap \{x_3 = 0\}, \text{ and } U \text{ achieves its positive maximum on } C_{\min} \text{ and zero minimum on } C_{\max}.
\end{align*}
\]
Thus, it suffices to prove

\[\blacksquare\]

**Lemma 2.3:** The principal eigenvalue $\lambda_1$ of the eigenvalue problem linearized at $U$ is positive and hence $U$ is stable.

**Proof:** Consider the eigenvalue problem (7) linearized at the stationary solution $U$ of (1) for $M = T^2$. We may define the normalized eigenfunction $\phi_1 = \phi_1(\varphi)$ that corresponds to
the same principal eigenvalue $\lambda_1 (> 0)$ as for $T^+$, by

$$\phi_1(\varphi) = \begin{cases} 
\frac{1}{\sqrt{2}} \phi_1^*(\varphi) & \text{if } \varphi \in [0, \pi], \\
\frac{1}{\sqrt{2}} \phi_1^*(2\pi - \varphi) & \text{if } \varphi \in (\pi, 2\pi).
\end{cases} \quad (14)$$

By the uniqueness of the normalized eigenfunction that corresponds to the principal eigenvalue, this $\phi_1$ is exactly the eigenfunction we want and hence the principal eigenvalue $\lambda_1$ is positive.

3. Standard tori with perturbation

In this section we will slightly perturb $T^2$ and analyse its stability along with the existence of critical points. Let $T^2_\epsilon$ denote a perturbation of $T^2$ where $T^2_0 = T^2$. To be precise, $T^2_\epsilon$ is parameterized by

$$\begin{align*}
x_1 &= (R + r_\epsilon(\theta) \cos \varphi) \cos \theta, \\
x_2 &= (R + r_\epsilon(\theta) \cos \varphi) \sin \theta, \\
x_3 &= r_\epsilon(\theta) \sin \varphi,
\end{align*} \quad ((\varphi, \theta) \in I := S^1 \times S^1), \quad (15)$$

where $n \in \mathbb{N}$, $r_\epsilon(\theta) = r + \epsilon \sin(n\theta)$, and the constants $R, r, \epsilon$ satisfy $R > r + |\epsilon| = \max_{\theta \in S^1} r_\epsilon(\theta)$ (see Figure 2).

Set $x^1 = \varphi, x^2 = \theta$. Then the corresponding Riemannian metric $d s^2_\epsilon$ and area element $d \sigma^\epsilon$ for $T^2_\epsilon$ are given by

$$\begin{align*}
ds^2_\epsilon &= \sum_{i,j=1}^2 g^\epsilon_{ij} \, dx^i \, dx^j = r_\epsilon^2(\theta) \, d\varphi^2 + \left[ (R + r_\epsilon(\theta) \cos \varphi)^2 + (r_\epsilon'(\theta))^2 \right] \, d\theta^2, \\
\, d\sigma^\epsilon &= \sqrt{|g^\epsilon|} \, d\varphi \, d\theta = r_\epsilon(\theta) \sqrt{(R + r_\epsilon(\theta) \cos \varphi)^2 + (r_\epsilon'(\theta))^2} \, d\varphi \, d\theta.
\end{align*}$$

Figure 2. $T^2_\epsilon$ with $R = 5$, $\epsilon = 0.2$, and $n = 15.$
The Riemannian gradient $\nabla_{g^\epsilon} u$ of $u$ with respect to $g^\epsilon$ on $T^2_\epsilon$ is given by

$$\nabla_{g^\epsilon} u = \begin{pmatrix} \frac{1}{r^2(\theta)} \partial_\varphi u \\ \frac{1}{(R + r_\epsilon(\theta) \cos \varphi)^2 + (r'_\epsilon(\theta))^2} \partial_\theta u \end{pmatrix},$$

(16)

and the Laplace–Beltrami operator $\Delta_{g^\epsilon}$ on $T^2_\epsilon$ is expressed as

$$\Delta_{g^\epsilon} u = \frac{1}{r^2(\theta)} u_{\varphi\varphi} + \frac{1}{\Phi(\varphi, \theta)} u_{\theta\theta} + \frac{\Phi(\varphi, \theta)}{r^2(\theta)} u_{\varphi} + \frac{r'_\epsilon(\theta) \Phi - r_\epsilon(\theta) \Phi_\varphi}{r_\epsilon(\theta) \Phi^3} u_\theta,$$

(17)

where we set $\Phi = \Phi(\varphi, \theta) = \sqrt{(R + r_\epsilon(\theta) \cos \varphi)^2 + (r'_\epsilon(\theta))^2}$.

### 3.1. Existence of patterns

The arguments in this section follow those used in [10, Section 4.1]. We only show the main points of arguments in each proof. For further detail see [10, Section 4.1].

**Theorem 3.1:** Let $U$ be the pattern of (1) for $M = T^2$ given by Theorem 2.2. There exists $\epsilon_0 > 0$ such that for each $|\epsilon| \in (0, \epsilon_0)$, a pattern $U^\epsilon$ of (1) for $M = T^2_\epsilon$ exists.

**Lemma 3.2:** Let $0 < \alpha < 1$. There exists $\epsilon_1 > 0$ such that for each $|\epsilon| \in (0, \epsilon_1)$, a stationary solution $U^\epsilon$ of (1) for $M = T^2_\epsilon$ and $\delta_1(\epsilon) > 0$ with $\lim_{\epsilon \to 0} \delta_1(\epsilon) = 0$ exist and satisfy

$$\|U^\epsilon - U\|_{C^{2,\alpha}(I)} < \delta_1(\epsilon),$$

if $|\epsilon| \in (0, \epsilon_1)$,

where $I = S^1 \times S^1$ is given in (15).

**Proof:** Set

$$X = \left(-\frac{1}{2}(R - r), \frac{1}{2}(R - r)\right) \subset \mathbb{R} \text{ and } Y = C^{2,\alpha}(I).$$

Let $F$ be a mapping from $X \times Y$ to $C^{\alpha}(I)$ defined by

$$F(\epsilon, v) = \Delta_{g^\epsilon}(U + v) + f(U + v) \text{ for } (\epsilon, v) \in X \times Y.$$

With the aid of the implicit function theorem, we solve $F = 0$ near the point $(0, 0)$. We notice that $F(0, 0) = 0$ and $F$ is of class $C^1$. The partial Fréchet derivative of the mapping $F(\epsilon, v)$ with respect to $v$ at $(0, 0)$ is expressed as

$$\frac{\partial F}{\partial v}(0, 0)q = \Delta_{g^\epsilon} q + f'(U) q \text{ for } q \in Y.$$

Let us show that $\frac{\partial F}{\partial v}(0, 0)$ is invertible. For each $h \in C^{\alpha}(I)$, we consider the following problem for $q$:

$$\Delta_{g^\epsilon} q + f'(U) q = h \text{ in } T^2.$$

(18)

Since $\lambda_1 > 0$, the standard theory of elliptic partial differential equations of second order provides us a unique solution $q \in Y$ of (18) and a constant $C > 0$ independent of $q$ and $h$.
satisfying

\[
\left\| \frac{\partial F}{\partial v}(0,0) \right\|_{C^{2,\alpha}(I)}^{-1} h = \| q \|_{C^{2,\alpha}(I)} \leq C \| h \|_{C^{\alpha}(I)}
\]

for each \( h \in C^\alpha(I) \). Thus, by the implicit function theorem (see [12, Theorem 15.1, p.148] or [13, Theorem 2.7.2, p.34]), there exist \( \epsilon_1 \in (0, \frac{1}{2}(R-r)) \) with \( \mathcal{N} = (-\epsilon_1, \epsilon_1) \) and a unique \( C^1 \) mapping \( v : \mathcal{N} \to Y \) such that \( v(0) = 0 \) and for every \( \epsilon \in \mathcal{N} \)

\[
F(\epsilon, v(\epsilon)) = \Delta g^\epsilon(U^\epsilon) + f(U^\epsilon) = 0,
\]

where we set \( U^\epsilon = U + v(\epsilon) \). This yields the conclusion. ■

Now, we are in position to prove the stability of the stationary solution \( U^\epsilon \) of (1) for \( M = T^2_{\epsilon} \). Let \( \lambda_1^\epsilon \) be the principal eigenvalue with the normalized eigenfunction \( \phi_1^\epsilon \in H^1(I) \) of (7) linearized at \( U^\epsilon \) for \( M = T^2_{\epsilon} \). Then we have

\[
\begin{cases}
\Delta g^\epsilon \phi_1^\epsilon + f'(U^\epsilon)\phi_1^\epsilon = -\lambda_1^\epsilon \phi_1^\epsilon \text{ in } T^2_{\epsilon}, \\
\|\phi_1^\epsilon\|_{L^2(T^2_{\epsilon})} = 1, \quad \phi_1^\epsilon > 0 \text{ in } T^2_{\epsilon},
\end{cases}
\]

where

\[
\lambda_1^\epsilon = \inf_{\phi \neq 0, \phi \in H^1(T^2_{\epsilon})} \frac{\int_{T^2_{\epsilon}} \left( |\nabla g^\epsilon \phi|^2 - f'(U^\epsilon)\phi^2 \right) \, d\sigma^\epsilon}{\int_{T^2_{\epsilon}} \phi^2 \, d\sigma^\epsilon} = \frac{\int_{T^2_{\epsilon}} \left[ |\nabla g^\epsilon \phi_1^\epsilon|^2 - f'(U^\epsilon)(\phi_1^\epsilon)^2 \right] \, d\sigma^\epsilon}{\int_{T^2_{\epsilon}} \phi_1^\epsilon^2 \, d\sigma^\epsilon}.
\]

Lemma 3.3: For every \( |\epsilon| \in (0, \epsilon_1) \), there exists \( \delta_2(\epsilon) > 0 \) with \( \lim_{\epsilon \to 0} \delta_2(\epsilon) = 0 \) such that

(i) \( \| f'(U) - f'(U^\epsilon) \|_{\infty} \leq \delta_2(\epsilon) \),
(ii) \( (1 - \delta_2(\epsilon)) \, d\sigma \leq \, d\sigma^\epsilon \leq (1 + \delta_2(\epsilon)) \, d\sigma \),
(iii) \( (1 - \delta_2(\epsilon)) |\nabla g \phi_1^\epsilon|^2 \leq |\nabla g \phi_1^\epsilon|^2 \leq (1 + \delta_2(\epsilon)) |\nabla g \phi_1^\epsilon|^2 \).

Proof: Assertion (i) comes from Lemma 3.2 and the continuity of \( f' \) for some \( \delta_2(\epsilon) > 0 \) with \( \lim_{\epsilon \to 0} \delta_2(\epsilon) = 0 \). Next, represent \( \sqrt{|g^\epsilon|} \) in assertion (ii) and \( |\nabla g \phi_1^\epsilon|^2 \) in assertion (iii) by using the Taylor expansion with respect to \( \epsilon \) at \( \epsilon = 0 \). By the continuity of \( \sqrt{|g^\epsilon|} \) and \( |\nabla g \phi_1^\epsilon|^2 \), we can generate a chain of inequalities on each expansion and choose \( \delta_2(\epsilon) \) smaller to obtain both assertion (ii) and assertion (iii) ■

Lemma 3.4: There exists a constant \( C^* > 0 \) such that if \( |\epsilon| < \epsilon_1 \) then

\[
\int_{T^2_{\epsilon}} |\nabla g \phi_1^\epsilon|^2 \, d\sigma^\epsilon \leq C^*.
\]
**Proof**: From (20), we have that

\[
\int_{T_2^\epsilon} \left| \nabla g^\epsilon \phi_1^\epsilon \right|^2 \, d\sigma^\epsilon = \lambda_1^\epsilon + \int_{T_2^\epsilon} f'(U^\epsilon)(\phi_1^\epsilon)^2 \, d\sigma^\epsilon. \tag{22}
\]

First, we need to prove that \(\lambda_1^\epsilon\) is bounded from above. By (21), every \(\phi \in H^1(I)\) satisfies

\[
\lambda_1^\epsilon \leq \frac{\int_{T_2^\epsilon} \left( |\nabla g^\epsilon \phi|^2 - f'(U^\epsilon) \phi^2 \right) \, d\sigma^\epsilon}{\int_{T_2^\epsilon} \phi^2 \, d\sigma^\epsilon}.
\]

Choose \(\phi \equiv 1\) and use assertion (i) to obtain

\[
\lambda_1^\epsilon \leq \delta_2(\epsilon) + \max_{T_2} |f'(U)|.
\]

Since \(||\phi_1^\epsilon||_{L^2(T_2^\epsilon)} = 1\), assertion (i) gives

\[
\int_{T_2^\epsilon} f'(U^\epsilon)(\phi_1^\epsilon)^2 \, d\sigma^\epsilon = \int_{T_2^\epsilon} (f'(U^\epsilon) - f'(U))(\phi_1^\epsilon)^2 \, d\sigma^\epsilon + \int_{T_2^\epsilon} f'(U)(\phi_1^\epsilon)^2 \, d\sigma^\epsilon \leq \delta_2(\epsilon) + \max_{T_2} |f'(U)|.
\]

Then, (22) yields the conclusion. ■

**Lemma 3.5**: \(\lambda_1^\epsilon \to \lambda_1\) as \(\epsilon \to 0\).

**Proof**: This follows directly from the same argument as in the proof of [10, Lemma 4.5, pp. 11–12]. From (8), we have

\[
\lambda_1 \leq \frac{\int_{T_2} \left( |\nabla g \phi_1|^2 - f'(U) \phi_1^2 \right) \, d\sigma}{\int_{T_2} \phi_1^2 \, d\sigma}.
\]

With the aid of Lemmas 3.3 and 3.4, we infer that there exists \(\delta_3(\epsilon) > 0\) with \(\lim_{\epsilon \to 0} \delta_3(\epsilon) = 0\) satisfying

\[
\lambda_1 \leq \lambda_1^\epsilon + \delta_3(\epsilon). \tag{23}
\]

By proceeding similarly, from (21), we have

\[
\lambda_1^\epsilon \leq \frac{\int_{T_2^\epsilon} \left( |\nabla g^\epsilon \phi_1|^2 - f'(U^\epsilon) \phi_1^2 \right) \, d\sigma^\epsilon}{\int_{T_2^\epsilon} \phi_1^2 \, d\sigma^\epsilon},
\]

and hence we infer that there exists \(\delta_4(\epsilon) > 0\) with \(\lim_{\epsilon \to 0} \delta_4(\epsilon) = 0\) satisfying

\[
\lambda_1^\epsilon \leq \lambda_1 + \delta_4(\epsilon). \tag{24}
\]

Combining (23) and (24) yields the conclusion. ■

**Proof**: (Proof of Theorem 3.1) From Lemma 3.2, we conclude that the stationary solution \(U^\epsilon\) exists in the neighbourhood of \(\epsilon = 0\). Then, by Lemma 3.5, the principal eigenvalue \(\lambda_1^\epsilon\) is positive for sufficiently small \(|\epsilon|\). This completes the proof. ■
3.2. Proof of Theorem 1.1

First of all, we mention the symmetry of \( U^\epsilon \) coming from the fact that \( U^\epsilon \) is uniquely determined in a neighbourhood of \( U \) by the implicit function theorem. To be precise, the implicit function theorem together with the symmetry of both \( U \) and \( T_2^\epsilon \) gives us the symmetry of \( U^\epsilon \) with respect to \( T_2^\epsilon \cap H \) for the following \( n + 1 \) planes \( H \):

\[
H = \{ x_3 = 0 \}, \{ -x_1 \sin \theta_k + x_2 \cos \theta_k = 0 \} \text{ with } k = 0, 1, \ldots, n - 1,
\]

(25)

where \( \theta_k = \frac{2k+1}{2n} \pi \). Thus we have in particular that if \( |\epsilon| < \epsilon_0 \) then

\[
\frac{\partial U^\epsilon}{\partial \varphi} = 0 \text{ for every } (\varphi, \theta) \in [0, \pi] \times S^1,
\]

(26)

\[
\frac{\partial U^\epsilon}{\partial \theta} = 0 \text{ for every } (\varphi, \theta) \in S^1 \times \{ \theta_k | k = 0, 1, \ldots, 2n - 1 \}.
\]

(27)

These imply that if \( |\epsilon| < \epsilon_0 \), then \( U^\epsilon \) has at least \( 4n \) critical points in \( T_2^\epsilon \) corresponding to the points in the following finite set \( C \):

\[
C = \{ 0, \pi \} \times \{ \theta_k | k = 0, 1, \ldots, 2n - 1 \}.
\]

(28)

In view of (5), we recall that \( U^0 = U = U(\varphi) \) satisfies

\[
\frac{1}{r^2} U_{\varphi\varphi} - \frac{\sin \varphi}{r(R + r \cos \varphi)} U_\varphi + f(U) = 0 \text{ in } T^2.
\]

(29)

Hence we have from (26)

\[
\frac{1}{r^2} U_{\varphi\varphi} + f(U) = 0 \text{ at } \varphi = 0, \pi.
\]

Therefore, since \( U \) is nonconstant and achieves its maximum at \( \varphi = \pi \) and its minimum at \( \varphi = 0 \), from the unique solvability of the Cauchy problem for the ordinary differential equation (29) we must have that

\[
f(U(0)) < 0 < f(U(\pi)) \text{ and hence } U_{\varphi\varphi}(0) > 0 > U_{\varphi\varphi}(\pi).
\]

(30)

Since the set of critical points of \( U \) equals \( T^2 \cap \{ x_3 = 0 \} \), it follows from Lemma 3.2, (26) and (30) that there exists \( \tau_1 \in (0, \epsilon_0) \) satisfying that if \( |\epsilon| < \tau_1 \), the set of critical points of \( U^\epsilon \) is contained in \( T_2^\epsilon \cap \{ x_3 = 0 \} \). Thus, in order to determine all the critical points of \( U^\epsilon \), we need to examine whether the derivative of \( U^\epsilon \) with respect to \( \theta \) vanishes or not for \( \varphi = 0, \pi \).
Observe that as $\epsilon \to 0$,
\[
U^\epsilon = U + v(\epsilon) = U + \epsilon \frac{\partial U}{\partial \epsilon} \bigg|_{\epsilon=0} + o(\epsilon). \tag{31}
\]

Set $V = \frac{\partial U^\epsilon}{\partial \epsilon} \bigg|_{\epsilon=0}$. By differentiating (31) with respect to $\theta$ twice, we see that as $\epsilon \to 0$
\[
\frac{\partial U^\epsilon}{\partial \theta} = \epsilon V_\theta + o(\epsilon) \quad \text{and} \quad \frac{\partial^2 U^\epsilon}{\partial \theta^2} = \epsilon V_{\theta\theta} + o(\epsilon). \tag{32}
\]

We will examine $V$ and evaluate $V_\theta$, $V_{\theta\theta}$ for $\varphi = 0, \pi$. Recall that
\[
F(\epsilon, v(\epsilon)) = \Delta g U^\epsilon + f(U^\epsilon) = 0 \text{ for every } (\varphi, \theta) \in I = S^1 \times S^1. \tag{33}
\]

Differentiating (33) with respect to $\epsilon$ yields
\[
0 = \frac{\partial}{\partial \epsilon} (F(\epsilon, v(\epsilon))) = \frac{1}{\epsilon^2} U_{\varphi\varphi}^\epsilon + \frac{\partial}{\partial \epsilon} \left( \frac{1}{\epsilon^2} \right) U_{\varphi}^\epsilon + \frac{1}{\Phi^2} U_{\theta\theta}^\epsilon + \frac{\partial}{\partial \epsilon} \left( \frac{1}{\Phi^2} \right) U_{\theta}^\epsilon
\[
+ \frac{\Phi_{\varphi}}{\epsilon^2 \Phi} U_{\varphi}^\epsilon + \frac{\partial}{\partial \epsilon} \left( \frac{\Phi_{\varphi}}{\epsilon^2 \Phi} \right) U_{\varphi}^\epsilon
\[
+ \frac{r_\epsilon' - r_\epsilon \Phi_{\theta}}{r_\epsilon \Phi_3} U_{\theta}^\epsilon + \frac{\partial}{\partial \epsilon} \left( \frac{r_\epsilon' - r_\epsilon \Phi_{\theta}}{r_\epsilon \Phi_3} \right) U_{\theta}^\epsilon + f'(U^\epsilon) U_{\epsilon}^\epsilon.
\]

Since $U^0$ depends only on $\varphi$, by setting $\epsilon = 0$, we have
\[
\Delta_g V + f'(U) V = \frac{2}{r} \sin(n\theta) \left[ -f(U) + \frac{R \sin \varphi}{2r(R + r \cos \varphi)^2} U_{\varphi} \right]. \tag{34}
\]

Observe that the right-hand side of (34) is infinitely differentiable in $\theta$ and all the coefficients of the left-hand side of (34) are independent of $\theta$. Then, by the standard regularity theory for elliptic partial differential equations (see [14]), we may differentiate (34) with respect to $\theta$ twice to obtain
\[
\Delta_g \left( \frac{V_{\theta\theta}}{n^2} + V \right) + f'(U) \left( \frac{V_{\theta\theta}}{n^2} + V \right) = 0. \tag{35}
\]

Then the function $\frac{V_{\theta\theta}}{n^2} + V$ might be an eigenfunction of problem (7) linearized at $U$ for $M = T^2$ which corresponds to eigenvalue $0$. Since the principal eigenvalue $\lambda_1$ is positive, $\frac{V_{\theta\theta}}{n^2} + V$ must vanish identically. Thus, it is easy to express $V$ as
\[
V(\varphi, \theta) = C_1(\varphi) \cos(n\theta) + C_2(\varphi) \sin(n\theta) \text{ for every } (\varphi, \theta) \in I \tag{36}
\]
for some functions $C_1(\varphi), C_2(\varphi)$ of class $C^2$. It remains to examine $C_1(\varphi)$ and $C_2(\varphi)$.

Substituting (36) into (34) yields the following two ordinary differential equations:
\[
C_1''(\varphi) - \frac{r \sin \varphi}{R + r \cos \varphi} C_1'(\varphi) - B(\varphi) C_1(\varphi) = 0, \tag{37}
\]
\[ C_2'(\varphi) - \frac{r \sin \varphi}{R + r \cos \varphi} C_2'(\varphi) - B(\varphi) C_2(\varphi) = A(\varphi), \tag{38} \]

where we set

\[ A(\varphi) = 2r \left[ -f(U) + \frac{R \sin \varphi}{2r(R + r \cos \varphi)^2} U \right] \quad \text{and} \quad B(\varphi) = r^2 \left[ \frac{n^2}{(R + r \cos \varphi)^2} - f'(U) \right]. \]

We choose the number \( N \in \mathbb{N} \) in Theorem 1.1 as

\[ N^2 > \max_{\varphi \in S^1} |f'(U)|(R + r)^2. \tag{39} \]

Let us assume that \( n \geq N \) from now on. Then \( B(\varphi) \) is positive everywhere. This fact together with (37) yields that \( C_1(\varphi) \equiv 0 \), since the maximum principle implies that \( C_1 \) achieves neither its positive maximum nor its negative minimum. Therefore, under (39) we have from (36)

\[ V(\varphi, \theta) = C_2(\varphi) \sin(n\theta) \text{ for every } (\varphi, \theta) \in I. \tag{40} \]

From (26), we have that \( U^e|_{\varphi=0,\pi} = \epsilon V|_{\varphi=0,\pi} + o(\epsilon) = 0 \). Hence,

\[ 0 = V(\varphi, \theta) = C_2'(\varphi) \sin(n\theta) \text{ for every } (\varphi, \theta) \in \{0, \pi\} \times S^1. \]

Then we obtain

\[ C_2'(0) = C_2'(\pi) = 0. \tag{41} \]

Let us first show that \( C_2(0) \neq 0 \). Multiplying (38) by \( R + r \cos \varphi \) yields that

\[ \left[ (R + r \cos \varphi)C_2'(\varphi) \right]' = (R + r \cos \varphi) \left[ B(\varphi) C_2(\varphi) + A(\varphi) \right]. \tag{42} \]

Then, with the aid of (41), by integrating (42) in \( \varphi \) from 0 to \( \pi \), we have

\[ \int_0^{\pi} (R + r \cos \varphi) \left[ B(\varphi) C_2(\varphi) + A(\varphi) \right] \, d\varphi = 0. \tag{43} \]

Multiplying (29) by \( r^2(R + r \cos \varphi) \) yields that

\[ \left[ (R + r \cos \varphi)U'(\varphi) \right]' = -r^2(R + r \cos \varphi)f(U). \tag{44} \]

Since \( U'(0) = U'(\pi) = 0 \), by integrating (44) in \( \varphi \) from 0 to \( \pi \), we have

\[ \int_0^{\pi} (R + r \cos \varphi)f(U) \, d\varphi = 0. \tag{45} \]

Then (45) yields that

\[ \int_0^{\pi} (R + r \cos \varphi)A(\varphi) \, d\varphi = \int_0^{\pi} \frac{R \sin \varphi}{R + r \cos \varphi} U' \, d\varphi > 0, \]

where we used that \( U' > 0 \) in \((0, \pi)\), and hence by (43) we conclude that

\[ \int_0^{\pi} (R + r \cos \varphi)B(\varphi) C_2(\varphi) \, d\varphi < 0. \tag{46} \]
Suppose that \( C_2(0) = 0 \). Since \( C''_2(0) = 0 \), substituting \( \varphi = 0 \) into (38) yields that
\[
C''_2(0) = A(0) = -2rf(U(0)) > 0,
\]
where we used (30). Therefore there exists \( \delta \in (0, \pi) \) satisfying
\[
C_2(\varphi) > 0 \text{ and } C'_2(\varphi) > 0 \text{ for every } \varphi \in (0, \delta).
\]
On the other hand, it follows from (46) and the positivity of \( B(\varphi) \) that \( C_2(\varphi) \) must be negative at some point in \( (0, \pi) \). Thus, we may find \( \delta_* \in [\delta, \pi) \) such that
\[
C'_2(\delta_*) = 0 \text{ and } C_2(\varphi) > 0 \text{ for every } \varphi \in (0, \delta_*). \tag{47}
\]
Then, replacing \( \pi \) by \( \delta_* \) and using the same arguments as in getting (43) and (45) yield that
\[
\int_0^{\delta_*} (R + r \cos \varphi) [B(\varphi)C_2(\varphi) + A(\varphi)] \, d\varphi = \int_0^{\delta_*} (R + r \cos \varphi)f(U) \, d\varphi = 0.
\]
Also, by combining these and using that \( U' > 0 \) in \( (0, \pi) \), we arrive at
\[
\int_0^{\delta_*} (R + r \cos \varphi)B(\varphi)C_2(\varphi) \, d\varphi < 0,
\]
which contradicts (47) and the positivity of \( B(\varphi) \). Thus, this concludes that \( C_2(0) \neq 0 \).
Since from (41) \( C'_2(\pi) = C'_2(2\pi) = 0 \), by employing the same argument on the interval \( (\pi, 2\pi) \) as on \( (0, \pi) \), we may also have that \( C_2(\pi) \neq 0 \). Hence it follows from (40) that for \( (\varphi, \theta) \in \{0, \pi\} \times [0, 2\pi) \)
\[
V_\theta \neq 0 \text{ if } \theta \notin \{\theta_k \mid k = 0, 1, \ldots, 2n - 1\} \text{ and } V_{\theta\theta} \neq 0 \text{ if } \theta \in \{\theta_k \mid k = 0, 1, \ldots, 2n - 1\},
\]
where \( \theta_k = \frac{2k+1}{2n} \pi \) are given in (25). Therefore, by combining this with (27) and (32), we find \( \tau_2 \in (0, \tau_1) \) such that if \( |\epsilon| < \tau_2 \), the set of critical points of \( U^\epsilon \) correspond to \( \mathcal{C} \) given by (28), which consists of exactly \( 4n \) points in \( T^2_\epsilon \). This completes the proof of Theorem 1.1.

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