Ultralight Axion Dark Matter and Its Impact on Dark Halo Structure in N-body Simulations

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Abstract

Ultralight axion is a dark matter candidate with mass $O(10^{-22})$ eV and de Broglie wavelength of order kiloparsec. Such an axion, also called fuzzy dark matter (FDM), thermalizes via gravitational force and forms a Bose–Einstein condensate. Recent studies suggested that the quantum pressure from FDM can significantly affect structure formation in small scales, thus alleviating the so-called “small-scale crisis.” In this paper, we develop a new technique to discretize the quantum pressure and illustrate the interactions among FDM particles in an $N$-body simulation that accurately simulates the formation of the dark matter halo and its inner structure in the region outside the softening length. In a self-gravitationally bound virialized halo, we find a constant density solitonic core, which is consistent with theoretical prediction. The existence of the solitonic core reveals the nonlinear effect of quantum pressure and impacts structure formation in the FDM model.

Key words: astroparticle physics – dark matter – Galaxy: halo – methods: numerical

1. Introduction

There have been many compelling pieces of evidence for the existence of cold dark matter (CDM), which successfully explains the rotation curves of spiral galaxies, the cosmic microwave background power spectrum, the Bullet Cluster, and the large-scale structure formation of the universe. The best-motivated model for CDM is weakly interacting massive particles (WIMPs), whose mass range is from the sub-GeV to 100 TeV owing to the relic density requirement. Unfortunately, up to now, no compelling evidence of WIMPs has been found by different types of searches, such as collider searches (Aaboud et al. 2016; CMS 2016), underground detections (Akerib et al. 2016; Tan et al. 2016), and astronomical observations (Ackermann et al. 2015; Aartsen et al. 2016). The null signals reported in all these experiments have shrunk the parameter space of many WIMP models to some finely tuned regions. Also, the next generation of CDM searches are shifting their focus to different mass regions.

The CDM model, through detailed $N$-body simulations, although successfully explaining the observations on large scales, fails to account for the observations on relatively smaller scales; this is known as the “small-scale crisis,” which includes (i) the missing satellites problem (Klypin et al. 1999; Moore et al. 1999), (ii) the cusp–core problem (de Blok 2010), and (iii) the too-big-to-fail problem (Boylan-Kolchin et al. 2011). In order to alleviate these problems, two types of mechanisms are introduced. Some authors believe that the problems can be alleviated by considering baryonic feedback carefully (Schaller et al. 2015a, 2015b). Others believe that a new mechanism for velocity boost is needed for the DM momentum exchange beyond the collisionless picture of the CDM. Examples include strongly self-interacting DM (Tulin et al. 2013) and fuzzy dark matter (FDM; Hu et al. 2000). Both of them share the feature of smoothing out the cuspy matter distribution.

The ultralight axion (ULA) or the so-called FDM is then a good candidate for CDM (Turner 1983; Press et al. 1990; Sin 1994; Goodman 2000; Peebles 2000; Matos et al. 2002; Guzmán & Ureña-López 2003; Amendola & Barbieri 2006; Calabrese & Spergel 2016; Kim & Marsh 2016). It not only retains the success of CDM in dealing with large-scale issues, but also provides a possible solution to the small-scale crisis. The FDM is a scalar boson with an extremely light mass, $\gtrsim 10^{-22}$ eV, which is required by a recent observation on the reionization history of the universe (Bozek et al. 2015). A recent discussion about the constraints on the FDM mass from the CMB can be found in Hložek et al. (2017). With a common velocity on the order of $100 \text{ km s}^{-1}$, the de Broglie wavelength of the FDM is very long, $\sim O(\text{kpc})$ (Sin 1994; Hu et al. 2000; Ferrer & Grifols 2004; Boehmer & Harko 2007; Mielke & Perez 2009; Sikivie & Yang 2009; Lee & Lim 2010; Chavanis 2011; Chavanis & Delfini 2011; Dev et al. 2017; Marsh 2016). Some authors suggested that the FDM is in Bose–Einstein condensate (BEC; Schive et al. 2014a). However, this is still an open question, because the BEC transition temperature is higher than the cosmological temperature. Hence, in this paper, we will not join such a debate but instead focus on numerically solving the structure formation of the FDM model efficiently. The FDM is nonthermally produced and thus still in the nonrelativistic regime and behaves like CDM. However, due to the quantum nature of the FDM on small scales—about tens or hundreds of kiloparsecs—the FDM is remarkably distinguishable from normal CDM. One can find these differences in the matter power spectrum (Veltmaat & Niemeyer 2016), halo mass function (HMF; Du et al. 2016), and halo structure (Schwabe et al. 2016). The most striking feature of the FDM is that the halo has a solitonic core of size $\sim O(\text{kpc})$ resulting from the quantum pressure of the FDM particles, which can be larger than their self-gravity. Hence, the quantum pressure plays an essential role in solving the small-scale crisis. Moreover, if the halo’s small-scale structure can be measured more accurately in the future, the FDM particle mass can be constrained. The latest reviews are given in Marsh (2016) and Hui et al. (2017).

In this work, we propose a new scheme—the effective particle–particle (PP) interaction—for simulating the FDM model, by
which one can compute the quantum effect of the FDM in an N-body simulation with high resolution. In a self-gravitationally bound virialized halo, we find a constant density core—the solitonic core—of size of around 1 kpc, but with a lower density than that in the conventional CDM model. The result shows a nontrivial quantum pressure effect on the structure formation. We present the effects of linear and nonlinear power-spectrum growth, especially on the nonlinear effect in high-density regions. For small-scale structures, particularly for scales smaller (larger) than one de Broglie wavelength, the quantum pressure is attractive (repulsive). This leads to nontrivial structure formation in high-density regions. However, in early age, low overdensity regions, the linear effect is mainly from the repulsive quantum pressure, which suppresses the matter power spectrum at scales smaller than the Jeans scale.

We have developed a simulation scheme independently and completely different from several previous studies, which were based on direct cosmological simulations with the Schrödinger–Poisson equations (Schive et al. 2014a), smoothed particle hydrodynamic (SPH) scheme simulation technique (Mocz & Succi 2015), and particle–mesh (PM) scheme simulation technique (Veltmaat & Niemeyer 2016). We cross-check our results with previous simulations and find them consistent with each other. Nevertheless, our implementation of the quantum effect with a simple PP method helps us to explain some of the previously unclear behavior and understand self-consistently how quantum pressure affects structure formation.

The main advantage of our scheme is its high running efficiency. From all of the tests that we have performed, the CPU time required for the FDM simulations is only two to three times longer than the corresponding CDM simulations, which is expected for our method described here. Thus, we can perform cosmological simulations on a larger scale than previous literature with the same amount of computational resources (Schive et al. 2014a; Mocz & Succi 2015; Veltmaat & Niemeyer 2016).

This paper is arranged as follows. In Section 2, we introduce the theoretical approach of the effective PP interaction. In Section 3, we describe the simulation setup and discuss the results. Finally, we summarize our outcomes in Section 4.

2. Methodology

2.1. Schrödinger–Poisson Equations

The nature of the FDM can be well-described by the Schrödinger–Poisson equations,

$$i\hbar \frac{d\Psi}{dt} = -\frac{\hbar^2}{2m_\Lambda} \nabla^2 \Psi + m_\Lambda V \Psi$$

and

$$\nabla^2 V = 4\pi Gm_\Lambda |\Psi|^2.$$ (2)

Here, $\hbar$, $m_\Lambda$, and $V$ are the Planck constant, particle mass, and the gravitational potential acting on a particle, respectively. The wave function $\Psi$ can be written as

$$\Psi = \frac{\rho}{\sqrt{m_\Lambda}} \exp \left( \frac{iS}{\hbar} \right)$$ (3)

in terms of the number density $\frac{\rho}{m_\Lambda}$, while we can define the gradient of $S$ to be the DM momentum,

$$\nabla S = m_\Lambda \mathbf{v}.$$ (4)

After solving the Schrödinger–Poisson equations, from the real and imaginary parts of the solution, one can obtain the continuity equation,

$$\frac{d\rho}{dt} + \nabla \cdot (\rho \mathbf{v}) = 0,$$ (5)

and the momentum-conservation equation,

$$\frac{dv}{dt} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla (Q + V),$$ (6)

where we have defined the quantum pressure as

$$Q = -\frac{\hbar^2}{2m_\Lambda} \nabla^2 \sqrt{\rho}.$$ (7)

Equations (5) and (6) are known as the Madelung equations (Spiegel 1980; Uhlemann et al. 2014; Marsh 2015). One can see that such a pressure is only related to the mass density $\rho$ and can be treated as a new force on the particles in addition to gravity. Later, we shall focus on this pressure term and discuss how to obtain the acceleration information by using Hamiltonian field theory. To discuss the effect of quantum pressure, we can start with the Hamiltonian without the gravity term,

$$H = \int \frac{\hbar^2}{2m_\Lambda} |\nabla \Psi|^2 d^3x = \frac{\hbar^2}{2} \int |\mathbf{v}|^2 d^3x + \int \frac{\hbar^2}{2m_\Lambda} (\nabla \sqrt{\rho})^2 d^3x.$$ (8)

We can write the kinetic energy term in discretized form with the particle index $j$,

$$T = \int \frac{\hbar^2}{2} |\mathbf{v}|^2 d^3x = \sum_j \frac{1}{2} m_j \left( \frac{dq_j}{dt} \right)^2,$$ (9)

where $q_j$ is the coordinate of the $j$th particle, and the effective potential energy is from the quantum pressure,

$$K_\rho = \int \frac{\hbar^2}{2m_\Lambda} (\nabla \sqrt{\rho})^2 d^3x.$$ (10)

Note that we have not discretized $K_\rho$ here but delay it to the next subsection because it will require some effort to do so. Based on $T$ and $K_\rho$, the Lagrangian of the system without gravity is

$$L = T - K_\rho = \sum_j \frac{1}{2} m_j \left( \frac{dq_j}{dt} \right)^2 - \int \frac{\hbar^2}{2m_\Lambda} (\nabla \sqrt{\rho})^2 d^3x,$$ (11)

and the Euler–Lagrangian equation becomes

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0 \implies m_j \ddot{q}_j = -\frac{\partial K_\rho}{\partial q_j}.$$ (12)

One can see that the $\rho$ in $K_\rho$ is a continuous function, which cannot be used in the PP method. Therefore, the major task is to further discretize the continuous function $\partial K_\rho/\partial q_j$, which we shall describe in more detail in the next subsection. We clarify
here that the indices $i$ and $j$ here represent the simulation particles rather than single “ultralight axion” particles.

2.2. Particle–Particle Implementation of Quantum Pressure

For a PP interaction system, the number density for each individual particle is a delta function. Intuitively, the mass density $\rho$ can be discretized as

$$\rho(r) = \sum_i m_i \delta(r - r_i),$$

(13)

where the summation over the index $i$ means adding up all the particles. Numerically, the treatment of a delta function is a difficult computational problem because of the sampling coverage issue. However, conventionally, one can approximate a delta function by a narrow Gaussian kernel function, as long as the width is small enough. The reasons for and advantages to using the Gaussian smoothing kernel are listed as follows. (i) It naturally keeps the kernel smooth, differentiable, and spherically symmetric. (ii) The PP interaction can naturally avoid the singularity at zero-density positions. Because of the finite grid size, such a singularity can numerically make the results unphysical. Specifically, we write down the delta function as

$$\delta(r - r_i) = \frac{2\sqrt{2}}{\lambda^3 \pi^{3/2}} \exp(-\frac{2|r - r_i|^2}{\lambda^2}),$$

(14)

with a narrow width $\lambda$. Note that the value of $\lambda$ is not arbitrary and should be the same as the de Broglie wavelength because one FDM particle has to be found with a high probability within a Gaussian wave packet, motivated by the wave function Equation (3). In our work, the probability of finding an FDM in one wavelength is set to 95%. Taking the FDM mass around $O(10^{-31})$ GeV as an example, its wavelength is of order kiloparsec. Inserting Equation (14) back to the $\mathbf{(\nabla \sqrt{\rho(r)})}^2$ term in Equation (10), the expression can be expanded by using the kernel function,

$$\mathbf{(\nabla \sqrt{\rho(r)})}^2 \approx \frac{4}{4\rho(r)} \left[ \sum_i m_i \left( \nabla \delta(r - r_i) \right)^2 \right],$$

$$= \frac{4}{4\rho(r)} \left[ \sum_i m_i \delta(r - r_i)(-\frac{4}{\lambda^2})(r - r_i) \right]^2,$$

$$= \frac{4}{\lambda^4 \rho(r)} \left[ \sum_i m_i \delta(r - r_i)(r - r_i) \right]^2.$$  

(15)

In the simulation, FDM particles such as axions will be grouped into a big mass clump in space which can be treated as an imaginary particle point (neglecting the size of the clump on the cosmological scale), and the mass density, Equation (13), becomes

$$\rho(r) = \sum_j \sum_i m_i \delta(r - r_j),$$

(16)

with the index $j$ for each imaginary particle clump. Mathematically, one can think of the mass density as being expanded around $r_j$ to include all FDM particles, $r \to r - r_j$ and $r_i \to r_i - r_j$. Given such a consideration, the summation of individual FDM particles is effectively the same as summing over all of the imaginary particle points, and Equation (15) can be further polished to be

$$\mathbf{(\nabla \sqrt{\rho(r)})}^2 \approx \frac{4}{\lambda^4} \left[ \sum_j m_j \delta(r - r_j)(r - r_j) \right]^2,$$

$$\times \left[ \sum_j m_j \delta(r - r_j) \right]^{-1}.$$  

(17)

It is worth mentioning that the form in Equation (17) is identical to that in Equation (15) but their meanings should not be confused. Hence, we leave the different indices here. At this stage, we have successfully converted the Gaussian wave packet into an imaginary particle-smoothing kernel. To completely discretize $\partial K_i/\partial q_i$, we still need to integrate Equation (17) over all space. Due to the nature of the delta function, we just need to focus on the volume surrounding the imaginary particle points. Therefore, the integration together with kernel approach gives

$$\int (\mathbf{(\nabla \sqrt{\rho(r)})}^2) dV \approx \int \frac{4dV}{\lambda^4} \left[ \sum_j m_j \delta(r - r_j)(r - r_j) \right]^2,$$

$$\times \left[ \sum_j m_j \delta(r - r_j) \right]^{-1},$$

(18)

$$\approx 4\lambda^4 \sum_j m_j \delta(r - r_j)(r - r_j)^2 \Delta V_j B_j,$$

(19)

$$\approx 4\lambda^4 \sum_j m_j \frac{\Delta V_j B_j}{\lambda^3 \pi^{3/2}} \exp\left[-\frac{(r - r_j)^2}{\lambda^2}\right](r - r_j)^2,$$  

(20)

where the parameters $\Delta V_j$ and $B_j$ are the effective volume and correction factor of the $j$th simulation particle, which will be described in more detail below.

We propose a correction factor $B_j$ for the $j$th simulation particle in order to numerically take care of the different integration results between the delta function and Gaussian kernel. In other words, when we treat a delta function as a Gaussian kernel with width equal to one matter wavelength, it does not behave like a delta function in the region where the distance between two kernel centers is less than one wavelength. In such a short range, the overlap between two Gaussian tails can also contribute significantly, especially when performing integration with high particle density. For more detailed explanations and the fitting formula for $B_j$, Equation (30), see Appendix A.

Theoretically, the effective volume $\Delta V_j$ for each simulation particle $j$ is of the order of $\lambda^3 \pi^{3/2}$, resulting from a Gaussian kernel integral. However, the exact value of $\Delta V_j$ can differ from system to system because of the complexity of the inner kernel region. Hence, we treat it as a phenomenological free parameter (a constant for simplicity), but we adjust its value to match the result inside the solitonic core obtained by other approaches that have better resolution in the region less than one wavelength, such as in Schive et al. (2014a). Finally,
Equation (10) can be simply rearranged as
\[
\sum_j \frac{\partial K_p}{\partial y_j} = \frac{4\hbar^2}{m_j^3 \lambda^4} \sum_j m_j \Delta V_j B_j \exp \left[ -\frac{2|\mathbf{r} - \mathbf{r}_j|^2}{\lambda^2} \right] 
\times (1 - \frac{2|\mathbf{r} - \mathbf{r}_j|^2}{\lambda^2})(\mathbf{r} - \mathbf{r}_j),
\]
and the equation of motion, Equation (12), becomes
\[
\sum_j m_j \ddot{\mathbf{r}}_j = -\frac{4\hbar^2}{m_j^3 \lambda^4} \sum_j m_j \Delta V_j B_j \exp \left[ -\frac{2|\mathbf{r} - \mathbf{r}_j|^2}{\lambda^2} \right] 
\times (1 - \frac{2|\mathbf{r} - \mathbf{r}_j|^2}{\lambda^2})(\mathbf{r} - \mathbf{r}_j).
\]
Substituting \( q \) with \( r \), the additional acceleration from the quantum pressure used in the simulation can be written as
\[
\dddot{r} = \frac{4\hbar^2}{M_0 m^2 \lambda^4} \sum_j B_j \exp \left[ -\frac{2|\mathbf{r} - \mathbf{r}_j|^2}{\lambda^2} \right] 
\times (1 - \frac{2|\mathbf{r} - \mathbf{r}_j|^2}{\lambda^2})(\mathbf{r}_j - \mathbf{r}).
\]
Here, \( M \) is the mass of the simulation particle and \( M_0 \) is a normalization factor accounting for the size of \( \Delta V_j \), which we choose to be \( 10^6 M_0 \). Interestingly, if we put any “one” test particle around some quantum pressure sources, the additional energy injected by the quantum pressure term in to the system is zero. Namely, the total work, the integration of Equation (23) from \( r = 0 \) to \( r = \infty \), vanishes.

To illustrate the effect of quantum pressure, let us consider a two-particle system \( \langle B_j = 1 \rangle \) separated by a distance of order \( O(\text{kpc}) \). The acceleration caused by quantum pressure will be \( O\left( \frac{\hbar^2}{m^2 \lambda^4} \right) \sim O(10^{-10} \text{ m s}^{-2}) \). In the left panel of Figure 1, we demonstrate the effect of quantum pressure in the plane of \( (r, \dot{r}) \). The acceleration from quantum pressure, gravity, and their sum are shown by the black dashed line, blue line, and red line, respectively. Clearly, the quantum pressure can be attractive (positive sign in our definition) \footnote{Here, the terminology “attractive” and “repulsive” are just simple phrases to describe the quantum pressure being positive or negative, respectively.} if the distance between two particles is less than \( \lambda / \sqrt{2} \). However, it becomes repulsive (negative sign in our definition) if the distance is greater than \( \lambda / \sqrt{2} \). To understand this, we refer back to the quantum pressure definition in Equation (7). The pressure term \( Q \) is proportional to the second derivative of the mass density, namely, the curvature of the density, which can have negative, positive, or zero values, physically corresponding to attractive, repulsive, and zero forces.

In the right panel of Figure 1, we show the position \( x(t) \) of one of the particles in this two-particle system as a function of time by solving Equation (23). Here, the origin is located at the center of mass such that \( r(t) = 2x(t) \). The red solid, black dashed, green solid, and blue dashed–dotted lines represent the cases with initial positions \( x(0) \) at \( 0.3 \lambda, \lambda / \sqrt{8}, 0.4 \lambda, \) and \( 2 \lambda \), respectively. Also, the position of the other particle is drawn in the corresponding lighter colors for reference. Interestingly, when the distance between the two particles is smaller than \( \lambda / \sqrt{2} \), the attractive force will bound them. Note that this also demonstrates the phenomenon of Bose–Einstein condensation. One should bear in mind that for fermionic particles, such short-range attractive forces would not exist because of Pauli’s Exclusion Principle. However, the repulsive force will push these two particles away for \( \frac{\lambda}{\sqrt{2}} < r(0) < \sqrt{2} \lambda \). At \( r(0) = \lambda / \sqrt{2} \), the node represents zero interaction. For the \( x(0) \gg \sqrt{2} \lambda \) case, the two particles barely feel the force from each other and so the they will continue with their initial velocities, though we set these to be zero.
This reveals the quantum pressure to be a short-range interaction, shown in Equation (23), with an exponentially decaying term.

3. Numerical Result

Gadget2 (Springel 2005) is a TreePM hybrid N-body code. However, in order to describe continuous quantum pressure, the interaction term has to be discretized as we have discussed in the previous section. Therefore, we modified Gadget2 to compute the contribution from quantum pressure. In this section, we first describe the details of what we used in our simulation and then we present our simulation results.

3.1. The Simulation Setup

The PM method is mostly useful for cosmological simulations with periodic boundary conditions. As we have seen from the previous section, the quantum pressure behaves like a short-range interaction, we therefore keep the original PM code, which takes care of the long-range force calculation. However, we have to use the tree method to take care of the short-range force calculation, and so the part of the Tree force calculation in Gadget2 has to be modified to include the quantum pressure term. Finally, we do not need to set a softening for the quantum pressure acceleration since it is finite in the \((r_f - r) \approx 0\) region.

To study the differences between the CDM and FDM, we set up a self-collapsing system. A cubic box with a side length of 400 kpc is generated with a total of \(10^8\) simulation particles homogeneously distributed inside the box with the vacuum boundary condition. The box can be understood as an overdensity region that will collapse under its self-gravity. Each simulation particle is of \(10^8\) solar masses. All particles start from rest, and the system collapses due to self-gravity, forming a stable self-gravitationally bound virialized halo at the center. It is sufficient to stop the simulation around 10 Gyr. The final stable virialized halo in the FDM model depends sensitively on the initial slight perturbation given to the system.

Two kinds of Gadget2 simulations are performed: one with the modified code for the FDM quantum pressure effect and the other one with the original Gadget2 code. In both simulations, we use the same initial conditions, and the gravitational softening length is chosen to be 0.89 kpc. The ULA mass and wavelength are fixed to be \(2.5 \times 10^{-22}\) eV and 1.4 kpc, respectively. We further run several simulations with different softening lengths in order to make sure that the final density and velocity distributions of the halos have converged.

3.2. Simulation Results

In Figure 2, we present the particle distributions of our simulations for a self-gravitationally collapsing system. The three panels on the upper (lower) row are for the CDM (FDM) scenario. The panels from left to right show two-dimensional slices at the evolution times of 0, 2, and 3 Gyr. Clearly, the distribution of FDM is more smooth and spread out than that of the CDM case, especially at later times. This is exactly the novel feature of quantum pressure.

To further illustrate the effect of quantum pressure on the halo density profile, we plot the mass enclosed in an 8 kpc region from the center in the left panel of Figure 3, which shows the core mass evolution along the time direction in units of gigayears for the CDM (blue line) and FDM (red solid line) cases. For reference, we also plot two curves in pink (thin dashed) for the \(B_j = 1\) case.\(^8\) Both CDM and FDM systems will be in a state of equilibrium after 3–4 Gyr when the mass within the 8 kpc region no longer changes significantly. Note that the FDM halo possesses slightly less (about 80% of the) mass inside 8 kpc than the CDM one, due to quantum pressure. The FDM halo is also slowly losing mass inside 8 kpc even after 4 Gyr while the CDM is already stable. However, both FDM halo and CDM halo are fully virialized after 4 Gyr, and their overall density profiles are stable.

In the right panel of Figure 3, we show the rotational curves for the FDM (for reference, the red solid line is for the FDM including the correction \(B_j\) and the pink thin dashed lines is for the FDM without correction) and CDM (blue line) halo after 10 Gyr of evolution. We take this halo as an example because it is fully virialized. We can see that inside 3 kpc, the circular velocity for the FDM halo is always smaller than that of the CDM halo. This implies that the FDM halo has a lower density in the inner core than the CDM halo. Note that the green shaded region represents the softening length 0.89 kpc, and we should treat it as our numerical simulation resolution boundary, inside which we should not trust the results.

We plot two density profiles at 3 and 10 Gyr separately in Figure 4. In the left panel, we show the density profiles for the FDM (red crosses) and CDM (blue circles) halos at 3 Gyr. Again, we plot the two solutions for the FDM without the correction factor (pink thin solid lines) as reference. The green shaded region represents the region with radius smaller than the softening length 0.89 kpc. The halo has still not reached equilibrium at 3 Gyr time, and there is no significant difference between the CDM and FDM. We also plot the Einasto profile fitting in the final virialized state with the blue dashed line for comparison. At the region around 100 kpc, some particles are still bouncing out so that the tail is not fitted well to the Einasto profile. In the right panel, we show the density profiles at 10 Gyr. The fitting function of the solitonic core is shown by the red dashed line, the formula for which was given in Schive et al. (2014b),

\[
\rho_\text{s}(r) \approx \rho_0 [1 + 0.091 \left(\frac{r}{r_c}\right)^2]^{-8},
\]

with an additional fitting parameter \(\rho_0\), and

\[
\rho_0 \approx 3.1 \times 10^6 \left(\frac{2.5 \times 10^{-22} \text{eV}}{m_\chi}\right) \left(\frac{r}{r_c}\right)^4 M_\odot \text{ kpc}^{-3}.
\]

We set \(\rho_0 = 5000\) and \(r_c = 3\) kpc in our plot. Beyond the solitonic core, the density profiles of the FDM halos are essentially the same as those of the CDM halos and the Einasto profile.

We confirm the existence of a solitonic core slowly emerging from the FDM simulation, with a size of 3 kpc. Inside the solitonic core, our result agrees with the previous result obtained in Schive et al. (2014b), which was based on a grid-based numerical solution of the Schrödinger–Poisson equations. Outside the solitonic core, our result agrees very

\(^8\) Because of the algorithm of the tree method used in the message passing interface (MPI), the boundary of the simulation grid can be slightly different. Without implementing the correction factor \(B_j\), such small differences from Gaussian tails can accumulate and develop into two different but stable results by using several different node assignments.
well with the Einasto profile, which is also a well-known property of the FDM halo. In a previous study (Veltmaat & Niemeyer 2016), a boosted power on small scales was also reported for their small-box comoving coordinate simulation. Comparing to the CDM simulation, Veltmaat & Niemeyer (2016) found at most 10% more power in the FDM simulation. We can now easily understand such a result by considering the quantum pressure on small scales.

We also compare the velocity dispersion profiles of the FDM (red lines) and CDM (blue line) halos after 3 Gyr (left panel) and after 10 Gyr (right panel) of evolution in Figure 5. At 3 Gyr (left panel), we do not see significant differences between the FDM and CDM halos, except in the small central region less than the softening length. However, the FDM velocity dispersion is smaller than that of the CDM halo between 3 kpc and 20 kpc, at both 3 Gyr and 10 Gyr, which can be understood as being due to the lower mass enclosed within 3 kpc in the FDM halos, as discussed in Figure 3.

4. Summary and Outlook

In summary, we proposed using a Gaussian kernel function to discretize the quantum pressure term to simulate FDM in the PP method for N-body simulations. We note that the quantum pressure does not provide additional energy to the system, but it will certainly change the halo’s inner structure. In order to understand the quantum pressure effect, we study a two-body system and find that the force between the two particles is always attractive if the distance between them is less than $\lambda/\sqrt{2}$, but it will turn repulsive if the distance between them is larger than $\lambda/\sqrt{2}$. On small scales, the quantum pressure contribution can be even larger than that of gravity.

With our discretized quantum pressure approach to the PP method, we have constructed two N-body simulations, one for the CDM and the other one for the FDM, in a collapsing system with identical initial conditions. We found that the FDM halo center can be clearly distinguished from that of the CDM, and we also confirmed that a solitonic core forms at the FDM halo center, where the mass density is very flat, similar to isothermal or Burkert profiles.

We have also compared the FDM halo evolution with the CDM case based on the mass enclosed within 8 kpc from the center. We confirmed that the FDM halo will reach equilibrium slightly later than the CDM halo while its galaxy formation history is consistent with current data (Hui et al. 2017). As implied by the rotation curves, the FDM halo evolves into an inner solitonic core with either higher or lower density than the CDM if they started with identical initial conditions; similar results were be found by Veltmaat & Niemeyer (2016).

The solitonic core we found in this work may not be able to quantitatively solve the cusp–core problem, but it suggests that
it may provide a solution if a full-scale cosmological simulation is performed. One may worry that the nonlinear effect from the attractive quantum pressure at the region less than one wavelength scale can bring higher mass density back to the center. This can be understood as the linear power-spectrum growth from the FDM model suppressing any power smaller than the FDM Jeans scale, 

\[ \frac{2(\pi G \rho)^{1/4} m_\lambda^{1/2}}{\rho^{1/2}}. \]

However, we started our simulations for the CDM and FDM with identical initial conditions, and the quantum pressure of the FDM suppresses the density in scales larger than one wavelength albeit it boosts the density up in scales smaller than one wavelength. These two effects—the suppression of matter power spectrum from redshift \( z = 100 \) to \( z = 16 \) on small scales, but a boost of matter power spectrum at around \( z = 2 \) on small scales—were also found in Veltmaat & Niemeyer (2016).

We do not consider the cosmological simulation in this work but would like to return to this in the near future.

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Appendix A

Correction of High Number-density Environment

In the low-density limit, it is reasonable to assume that particles are dominantly under a two-body force without a contribution from massive overlap terms. However, in a high number-density environment, our two-body interaction picture is no longer sufficient because of massive overlap contributions. In this section, we discuss our strategy to correct this issue.

From Equations (18) and (19), if we take \( B_j = 1 \), we can see that the approximation

\[ \sum_j m_j \delta(r - r_j)(r - r_j)^2 \]

will not hold when we replace the delta function with the Gaussian kernel. Clearly, the approximation only works when the two-body interaction is dominant. In reality, a high number-density system, e.g., in the center of a halo, has to be considered as a more complicated system.

Let us start with a simple setup. Assuming \( N \) particles occupy a volume \( V \), the average number density is \( N/V \) and the average distance \( D \) can be defined as \( (N/V)^{1/3} \). In the limit of the average distance being much larger than the matter wavelength \( D \gg \lambda \), the probability for any particle pair to appear within one wavelength is given by

\[ \mathcal{P}_0 \propto (\lambda/D)^3. \]

Extending the system to three particles, the probability of two particles being within one wavelength from one another will be proportional to \( \mathcal{P}_0^2 \). Similarly, the probability will be proportional to \( \mathcal{P}_0^{N-1} \) for an \( N \)-particle system. Thus, we can conclude that the sum of all pair-interactions can be a very good approximation (leading order) for an \( N \)-body system as long as \( D \gg \lambda \).

On the other hand, if we consider the Gaussian kernel \( \mathcal{G}(r - r_i) \) in the limit of \( D \ll \lambda \), it is no longer accurate to
ignore the overlap between the Gaussian tails. Such next-to-leading order contributions can be very significant if we add up all the tail interactions of the $N$-body system. Hence, we propose a numerical correction factor $B_j$ to account for such additional interactions. To estimate the value of $B_j$, whose index $j$ denotes the label of the tree node, we have to unfold the tree method back to Equation (16). Therefore, we can first define $B_j$ (for each $j$) to be the net effect of the Gaussian tails of all individual simulation particles $i$,

$$
\left[ \sum_i m_i \tilde{g}(r-r_i) (r-r_i) \right] \left[ \sum_i m_i \tilde{g}(r-r_i) \right]^{-1} = B_j \sum_i m_i \tilde{g}(r-r_i) (r-r_i)^2.
$$

We can rewrite the left-hand side of Equation (28) to be

$$
\sum_{j=1}^{N} m_j^3 \tilde{g}^2(r-r_i)(r-r_i)^2 + 2 \sum_{j<i,j}\cdot m_j m_i \tilde{g}(r-r_j) \tilde{g}(r-r_i)(r-r_i) \cdot (r-r_j).
$$

We can see that the major difference between the left-hand side (LHS) and right-hand side (RHS) of Equation (28) mainly comes from the overlap terms, which can be amplified for $N$ particles. Thus, we can guess that the factor $B_j$ is proportional to $N \sim (D/\lambda)^3$. However, finding the exact form for $B_j$ is not so trivial. Here, we perform a Monte Carlo simulation in the range of $D/\lambda$ from zero to infinity to compute exactly the value of the LHS and RHS of Equation (28), and then a proper fitting function for $B_j$ can be obtained.

In the following, we list the steps of such a Monte Carlo simulation.

1. Generate a large number of particles randomly distributed in a cubic box. The length of the box side is much larger than the matter wavelength $\lambda$.
2. Calculate the values of the LHS and RHS of Equation (28) with $B_j = 1$ at the center of the box.
3. Compare the two values computed in step 2 and calculate the value of $B_j$ with respect to $D/\lambda$.
4. Repeat the previous steps with different numbers of particles, different box sizes, and different $\lambda$’s to ensure the convergence of the results.

We present our simulation results in Figure 6 and fit the data with a formula,

$$
B_j = (D/\lambda)^3/(10 + (D/\lambda)^3).
$$

The error bars of each data point originate from the standard deviation of different scans. As expected, $B_j$ approaches the value of about 1 at the region of very large $D/\lambda$, but is almost linearly increasing in the log–log scale in the region $D < 2\lambda$. The fitting formula (30) provides the correction of the quantum pressure in the densest region in the simulation, for example, at the center of halos.

### A.1. Numerical Implementation of Quantum Pressure and Its Corrections in the Tree Algorithm

Gadget2 (Springel 2005) is a TreePM hybrid $N$-body code and the correction factor $B_j$ has to be implemented in the
Tree data structure. In the following, we list the basic procedures of the new implementation.

1. We build a tree structure to save the information of the particle distribution.
2. For each simulation particle, we calculate the gravitational force from the tree nodes.
3. The quantum pressure from the same tree nodes is also computed in the same time and same numerical loop.
4. The local particle number-density information can be taken from the tree so that the correction factors $B_i$ can be calculated accordingly.
5. To avoid breaking Newton’s third law, we also take the particle number density for the father node of the particle into account. Such an action properly keeps the density around the particle and the tree node smooth without sharp artificial cuts, which could break Newton’s third law.

In principle, the computational time for an FDM simulation is about twice as much as that for a classical gravity only CDM simulation. In fact, among our tests, the FDM simulation costs two to three times as much as the CDM one. When particles are close to each other, the quantum pressure can be larger than gravity to maintain the accuracy of acceleration, velocity, and position calculation. It turns out that all time steps are adaptively shorter than the one without quantum pressure.

Appendix B
Softening Length and Kernel Size

The gravitational softening length (option SofteningHalo in Gadget2) is a critical numerical parameter for N-body simulations because it can prevent singularity from being approached when the distance between two particles becomes very small. In addition, some softening lengths within a certain volume can help maintain the correct gravitational force. Ideally, if one could use an infinite number of particles to represent the fluid, then the softening length would be zero. However, in our FDM simulations, the softening length can neither be zero nor a completely artificial parameter because the density distribution is defined by the Gaussian kernel. Unlike CDM, the FDM softening length is strongly related to the matter wavelength. In this appendix, we estimate the softening length as a function of the wavelength $\lambda$. 

Figure 5. Left (right) panel: the velocity dispersion profiles at 3(10) Gyr. The two solutions for the FDM without the correction factor $B_i$ (pink thin solid lines) are shown for reference. The green shaded region represents the radius of the halo smaller than the softening length 0.89 kpc. Before final virialization, there are no significant differences between the FDM and CDM halos. The slight differences between the CDM and FDM halos around 10 kpc are still clear.

Figure 6. Simulation results are shown by the blue error bars. The red solid line is the power law with power index $= 3$. The green solid line is the fitting formula we give in Equation (30). The error bars in the x-axis are given by the bin size. The error bars in the y-axis are given by the standard deviation of $B_i$ in the bins.
For a distribution of mass with Gaussian density distribution, the gravitational acceleration acting on a test particle is

\[ \ddot{r} = \frac{GM(<r)}{r^3}, \]

where \( M(<r) \) is the mass enclosed in the radius \( r \) from the center of the Gaussian kernel. The mass enclosed in the radius \( r \) can be re-parameterized as

\[ \frac{M(<r)}{M(r = \infty)} = \int_0^r \exp\left(-\frac{r^2}{2\lambda^2}\right)4\pi r^2 dr \]

\[ = \frac{1}{\sqrt{2\pi}} \int_0^\infty \exp\left(-\frac{x^2}{2}\right)4\pi x^2 dx \]

\[ = 1.13 \exp\left(-\frac{r^2}{2\lambda^2}\right) \frac{\sqrt{2}}{\lambda}. \]

In Figure 7, we present various gravitational accelerations (blue dashed lines) by taking into account different softening lengths; also see Equation (4) in Springel (2005).

We also show the analytical curve (red solid line) that is computed by assuming the Gaussian kernel density distribution described in Equation (32). Clearly, the softening length \( \text{SOFT} = 0.89 \) kpc gives the best fit to the analytical curve, yet \( \text{SOFT} = 0.8 \) kpc or \( \text{SOFT} = 1 \) kpc differs by less than 50\%. Therefore, we use the best-fit value of the softening length, 0.89 kpc, in the simulation. The wavelength \( \lambda = 1.4 \) kpc has been chosen in this work; nevertheless, \( \lambda \) and the softening length \( \text{SOFT} = 0.89 \) kpc are correlated in this context.

Appendix C
Verification of Particle-Particle Simulation

In this section, our PP method is verified with some simple one-dimensional simulations by comparing with other methods such as the PM method and the approach introduced in Veltmaat & Niemeyer (2016). We first perform the simulations based on the PP and PM methods. The same initial conditions are applied: two particles located 10 distance units away from each other and the density distribution smoothed by Gaussian kernels. For the sake of simplicity, we turn off the gravitational potential so that the acceleration of the particles is only caused by the Quantum Pressure (QP).

In the PP simulation, we adopt the acceleration of the particle described by Equation (23) and then evolve the particles using the iteration relations

\[ x_{t+\delta t} = x_t + v_t \delta t + \frac{1}{2} a_t (\delta t)^2, \quad v_{t+\delta t} = v_t + \frac{1}{2} a_t \delta t, \]

where \( x_t, v_t, \) and \( a_t \) are the position, velocity, and acceleration of the particle at time \( t \), respectively. The size of the time step is given by an adjustable value of \( \delta t \) in order to achieve adequate accuracy.

In the PM simulation, the acceleration \( a_t \) is \( -\nabla Q \) based on the original definition. The acceleration \( -\nabla Q \) will be computed for each bin. To handle the \( \nabla^2 \sqrt{\rho} \) in the QP given by Equation (7), the three-point function is engaged to numerically calculate second order derivative,

\[ f''(x) = \lim_{h \to 0} \frac{f(x + h) - 2f(x) + f(x - h)}{h^2}, \]

where \( h \) is the bin size. However, we adopt the two-point function

\[ f''(x) = \lim_{h \to 0} \frac{f(x + h) - f(x - h)}{2h} \]

to calculate \( -\nabla Q \). We have carefully checked the convergence of the simulations with different values of \( \delta t \) and \( h \). With the values \( \delta t = 0.01 \) units of time and \( h = 0.05 \) units of distance, the result is stable for illustration purposes.

The comparison of the PP and PM results for the correlation between the acceleration and position of a particle is demonstrated in Figure 8. The thick red solid line presents...
Figure 9. Comparison of the acceleration from our method with that from the method in Veltmaat & Niemeyer (2016) and using the kernel in Veltmaat & Niemeyer (2016). The red solid line is the result using our PP method, the black dashed line is the result using the method in Veltmaat & Niemeyer (2016) with a Gaussian kernel, and the blue dashed–dotted line is the result using the method in Veltmaat & Niemeyer (2016) with a spline kernel (the kernel used in Veltmaat & Niemeyer 2016).

The PM result, and the thick cyan solid line is for the PP result. One can decompose the PM simulation result into the lowest frequency mode (N = 1), high frequency mode (N = 2), and noise, corresponding to the dark green dotted–dashed line, purple dashed line, and light green dashed line, respectively. From Appendix B of Hui et al. (2017), we recognize that the lowest frequency mode is actually related to the lowest eigenstate of the Schrödinger–Poisson equation (ground state N = 1), and the contribution of the lowest eigenstate of Schrödinger–Poisson equation is larger than that of other eigenstates in the density of the halo. Importantly, we found that the PP simulation result is dominated by the lowest frequency mode, which can be seen from the small difference between the thick blue solid line and dark green dashed line.

There is no doubt that the lowest frequency mode is more significant than the higher frequency modes because the higher frequency modes are nearly averaged out in each interval. This fact implies that our PP method for FDM simulation is only able to capture the ground-state component; nevertheless, it is the most dominant component. Hence, such a PP method for FDM simulation is a good approximation and helps to reduce the simulation cost and complexity. There could be a very small error near the boundary of the softening length due to the omitted higher frequency modes when using our PP method. However, it is just a tiny effect on the large-scale structure in a cosmological simulation because there is much weaker QP due to the larger simulation particle mass.

Next, we compare our PP method with the method developed in Veltmaat & Niemeyer (2016). Again, we place two particles symmetrically and give them a tiny initial velocity to make them move toward each other. We record the acceleration of one of the particles in the process as shown in Figure 9. In the simulation, we use the leapfrog algorithm to impose the acceleration on each particle and follow their movement.

The red solid line is the acceleration versus position curve recorded with the PP method. The black dashed and blue dashed–dotted lines are recorded with the method developed in Veltmaat & Niemeyer (2016) but using the Gaussian and spline kernels, respectively. For convenience, we called the kernel used in Veltmaat & Niemeyer (2016) the spline kernel. Our PP method gives a similar acceleration versus position curve to that in Veltmaat & Niemeyer (2016), despite the different choices of kernels. We also found that the Gaussian kernel is more suitable for QP calculation since the spline kernel is not smooth after performing the first-order derivative.

We conclude that our PP method is consistent with a previous PM method (Veltmaat & Niemeyer 2016) with respect to QP calculation. Moreover, our PP method captures the lowest frequency mode of the QP, the resolution and accuracy of which are enough for a cosmological simulation. Further improvement of our method will lie in considering additional factors such as an anisotropic kernel and higher frequency modes. We will study these possible improvements in the future.

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