STRUCTURE OF THE FUNDAMENTAL GROUPS OF ORBITS OF
SMOOTH FUNCTIONS ON SURFACES

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Abstract. Let $M$ be a smooth compact connected surface, $P$ be either the real line
$\mathbb{R}$ or the circle $S^1$ and $f : M \to P$ be a Morse map. Denote by $S(f)$ and $O(f)$ the
respective stabilizer and orbit of $f$ with respect to the right action of the group
$\mathcal{D}(M)$ of diffeomorphisms of $M$. In a series of papers the author described homotopy
types of $S(f)$ and computed higher homotopy groups of $O(f)$. The present paper
describes the structure of the remained fundamental group $\pi_1 O(f)$ for the case when
$M$ is orientable and differs from 2-sphere and 2-torus.

The result holds as well for a larger class of smooth maps $f : M \to P$ having the
following property: the germ of $f$ at each of its critical points is smoothly equivalent
to a homogeneous polynomial $\mathbb{R}^2 \to \mathbb{R}$ without multiple factors.

1. Introduction

Let $M$ be a smooth compact connected surface and $P$ be either the real line $\mathbb{R}$
or the circle $S^1$. For each closed subset $X \subset M$ let $\mathcal{D}(M,X)$ be the group of $C^\infty$-
diffeomorphisms fixed on $X$ and

$$S(f, X) = \{ h \in \mathcal{D}(M, X) \mid f \circ h = f \}, \quad O(f, X) = \{ f \circ h \mid h \in \mathcal{D}(M, X) \}$$

be respectively the stabilizer and the orbit of $f \in C^\infty(M, P)$ under the standard right
action of $\mathcal{D}(M, X)$ on $C^\infty(M, P)$.

We will endow $\mathcal{D}(M, X)$ and $C^\infty(M, P)$ with $C^\infty$ Whitney topologies. These topologies
induce certain topologies on the spaces $S(f, X)$ and $O(f, X)$. Denote by $\mathcal{D}_{id}(M, X)$ and
$S_{id}(f, X)$ the identity path components of $\mathcal{D}(M, X)$ and $S(f, X)$ respectively and by
$O_f(f)$ the path component of $O(f)$ containing $f$. If $X = \emptyset$, we will omit $X$ from
notation, e.g. we write $\mathcal{D}(M)$ instead of $\mathcal{D}(M, \emptyset)$, and so on.

In [7] [8] [10] [11] [12] for a large class of smooth maps $f : M \to P$ and certain “$f$-
adopted submanifolds” $X \subset M$ the author described the homotopy types of $S(f, X)$,
computed the higher homotopy groups of $O(f, X)$, and obtained certain information
about $\pi_1 O(f, X)$, see Theorem [11.4] below.

The main result of this paper, Theorem [1.10] gives a complete description of the
structure of $\pi_1 O(f, X)$ for the case when $M$ is orientable and differs from 2-sphere and
2-torus. It expresses $\pi_1 O(f, X)$ in terms of special wreath product with $\mathbb{Z}$ over some
finite cyclic groups.
1.1. Preliminaries. Let $C^\infty_\partial(M,P)$ be the subset of $C^\infty(M,P)$ consisting of maps $f$ satisfying the following axiom:

**Axiom (B).** The map $f : M \to P$ takes constant values on connected components of $\partial M$, and the set $\Sigma_f$ of critical points of $f$ is contained in the interior $\text{Int} M$.

Denote by $\text{Morse}(M,P) \subset C^\infty_\partial(M,P)$ the subset consisting of Morse maps, i.e. maps having only non-degenerate critical points. It is well known that $\text{Morse}(M,P)$ is open and everywhere dense in $C^\infty_\partial(M,P)$.

Let also $\mathcal{F}(M,P)$ be the subset of $C^\infty_\partial(M,P)$ consisting of maps $f$ satisfying the following additional axiom:

**Axiom (L).** For each critical point $z$ of $f$ the germ of $f$ at $z$ is smoothly equivalent to some homogeneous polynomial $f_z : \mathbb{R}^2 \to \mathbb{R}$ without multiple factors. By Morse lemma a non-degenerate critical point of a map $f : M \to P$ is smoothly equivalent to a homogeneous polynomial $\pm x^2 \pm y^2$ which, evidently, has no multiple factors, and so satisfies (L). This implies that

\[
\text{Morse}(M,P) \subset \mathcal{F}(M,P).
\]

Notice that every critical point satisfying Axiom (L) is isolated. Moreover such a point $z$ is non-degenerate if and only if the corresponding homogeneous polynomial $f_z$ has degree $\geq 3$, see e.g. \[8\] §7.

**Definition 1.2.** \[12\]. Let $f \in \mathcal{F}(M,P)$, $X \subset M$ be a compact submanifold, not necessarily connected, and whose connected components may have distinct dimensions. Let also $X^i$, $i = 0, 1, 2$, be the union of connected components of $X$ of dimension $i$. Then $X$ will be called $f$-**adopted** if the following conditions hold true:

(a) $X \cap \Sigma_f \subset X^0 \cup \text{Int} X^2$;

(b) $f$ takes constant value at each connected component of $X^1 \cup \partial X^2$.

The following lemma gives examples of $f$-adopted submanifolds. We left it to the reader.

**Lemma 1.3.** Let $X,Y \subset M$ be two submanifolds.

1. If $X$ is $f$-adopted, then so is every connected component of $X$.
2. If $X$ and $Y$ are $f$-adopted and disjoint, then $X \cup Y$ is $f$-adopted as well.
3. Suppose every connected component of $X$ has dimension 2 Then $X$ is $f$-adopted if and only if the restriction $f|_X$ satisfies axioms (B) and (L).
4. Let $a,b \in P$ be two distinct regular values of $f \in \mathcal{F}(M,P)$, and $[a,b] \subset P$ the closed segment between them. Then $X = f^{-1}[a,b]$ and any family of connected components of $X$ is $f$-adopted. \(\square\)

Let $f \in \mathcal{F}(M,P)$ and $X \subset M$ be an $f$-adopted submanifold. Denote

\[
S'(f) = S(f) \cap \mathcal{D}_{id}(M), \quad S'(f,X) = S(f) \cap \mathcal{D}_{id}(M,X).
\]  

(1.1)

In a sequel all the homotopy groups of $\mathcal{O}(f,X)$ will have $f$ as a base point, and so the notation $\pi_k \mathcal{O}(f,X)$ will always mean $\pi_k(\mathcal{O}(f,X),f)$. Notice that the latter group is also isomorphic with $\pi_k(\mathcal{O}(f,X),f)$. The following theorem summarizes the results concerning the homotopy types of $S_{id}(f,X)$ and $\mathcal{O}_f(f,X)$.
Theorem 1.4. \([7, 10, 11, 12]\). Let \(f \in \mathcal{F}(M, P)\) and \(X \subset M\) be an \(f\)-adopted submanifold. Then the following statements hold true.

1) \(\mathcal{O}_f(f, X) = \mathcal{O}_f(f, X \cup \partial M)\).

2) The map \(p : D(M, X) \rightarrow \mathcal{O}(f, X)\) defined by \(p(h) = f \circ h\) is a Serre fibration.

3) Suppose that either \(f\) has at least one critical point being not a non-degenerate local extreme or \(M\) is non-orientable. Then \(\mathcal{S}_\text{id}(f)\) is contractible, \(\pi_n \mathcal{O}(f) = \pi_n M\) for \(n \geq 3\), \(\pi_2 \mathcal{O}_f(f) = 0\), and we also have the following short exact sequence:

\[
1 \rightarrow \pi_1 \mathcal{D}_\text{id}(M) \xrightarrow{p} \pi_1 \mathcal{O}(f) \xrightarrow{\partial} \pi_0 \mathcal{S}'(f) \rightarrow 1. \tag{1.2}
\]

If \(M\) is orientable and distinct from \(S^2\) and \(T^2\) then \(\pi_1 \mathcal{O}(f)\) is solvable.

4) Suppose that the Euler characteristic \(\chi(M)\) is less than the number of points in \(X\). This holds for instance when either \(\dim X > 0\) or \(\chi(M) < 0\). Then both \(\mathcal{D}_\text{id}(M, X)\) and \(\mathcal{S}_\text{id}(f, X)\) are contractible, \(\pi_i \mathcal{O}(f, X) = 0\) for \(i \geq 2\), and we have an isomorphism:

\[
\pi_1 \mathcal{O}(f, X) \cong \pi_0 \mathcal{S}'(f, X).
\]

Moreover, there exist finitely many mutually disjoint \(f\)-adopted subsurfaces \(B_1, \ldots, B_n \subset M \setminus (X^1 \cup X^2)\) each diffeomorphic either to a 2-disk \(D^2\), or a cylinder \(S^1 \times I\), or a Möbius band \(Mo\), and such that if we denote

\[
f_i := f|_{B_i} : B_i \rightarrow P, \quad \hat{B}_i := (B_i \cap X^0) \cup \partial B_i
\]

for \(i = 1, \ldots, n\), then the following isomorphisms hold:

\[
\pi_1 \mathcal{O}_f(f, X) \cong \pi_0 \mathcal{S}'(f, X) \cong \times_{i=1}^{n} \pi_0 \mathcal{S}'(f_i, \hat{B}_i) \cong \times_{i=1}^{n} \pi_0 \mathcal{O}_f(f_i, \hat{B}_i).
\]

5) Let \(U\) be any open neighbourhood of \(X^1 \cup X^2\). Then there exists an \(f\)-adopted submanifold \(N \subset M\) such that every connected component of \(N\) has dimension 2,

\[
X^0 \cap N = \emptyset, \quad X^1 \cup X^2 \subset \text{Int} N \subset N \subset U
\]

and the inclusion \(\mathcal{S}'(f, X^0 \cup N) \subset \mathcal{S}'(f, X)\) is a homotopy equivalence.

Proof. Statements 1) and 5) are proved in \([12\) Corollaries 2.1 \& 6.2\] respectively. Statement 2) is a general result initially established in the paper by F. Sergeraert \([15\] for smooth functions of finite codimension on arbitrary closed manifolds. In particular, all singularities satisfying Axiom (L) have finite codimensions, \([11\) Lemma 12\]. This covers the case \(X = \emptyset\). The proof for \(X = \Sigma_f\) was given in \([7\) §11\], and for arbitrary \(f\)-adopted submanifold \(X\) in \([12\) Theorem 5.1\].

Statement 3) is proved in \([7\) Theorems 1.3, 1.5\] for Morse maps, and extended to the class \(\mathcal{F}(M, P)\) in \([11\). Solvability result is obtained in \([13\].

Statement 4) was initially established in \([10\) Theorem 1.7\] for \(X = \emptyset\), and extended to the general case in \([12\) Theorem 2.4\]. \(\square\)

1.5. Wreath products \(A \wr Z_m^m\) and \(A \wr Z_m\). Let \(A\) be any group and \(Z_m\), \(m \geq 1\), be a finite cyclic group of order \(m\). Denote by \(A^{Z_m}\) the set of all maps \(Z_m \rightarrow A\) (being not necessarily homomorphisms). Then \(Z_m\) naturally acts on \(A^{Z_m}\) from the right and therefore one can define the corresponding semidirect product \(A^{Z_m} \rtimes Z_m\) which is denoted by

\[
A \wr Z_m.
\]
and called \textit{wreath product} of $A$ and $Z_m$.

More generally, notice that the group $Z$ acts on $Z_m$ by the rule
\[ z \ast k = z + k \mod m \]
for $z \in Z$ and $k \in Z_m$. This action induces a right action of $Z$ on $A^{Z_m}$ and therefore one can define the corresponding semidirect product $A^{Z_m} \rtimes Z$ which will be denoted by
\[ A \rtimes_{Z_m} Z \]
and called the \textit{wreath product} of $A$ and $Z$ over $Z_m$.

Thus $A \rtimes_{Z_m} Z$ is the set $A^{Z_m} \times Z$ with the multiplication defined as follows. Let $(\alpha, a), (\beta, b) \in A^{Z_m} \times Z$. Define $\gamma : Z_m \to A$ by the formula $\gamma(i) = \alpha(i + b) \cdot \beta(i)$, $i \in Z_m$, where $\cdot$ is the multiplication in $A$. Then, by definition, the product $(\alpha, a)$ and $(\beta, b)$ in $A \rtimes_{Z_m} Z$ is
\[ (\alpha, a)(\beta, b) := (\gamma, a + b) \]
If $\epsilon : Z_m \to A$ is the constant map into the unit of $A$, then $(\epsilon, 0)$ is the unit of $A \rtimes_{Z_m} Z$.

Evidently, for $m = 1$ the group $A \rtimes_{Z_m} Z$ is isomorphic with the direct product $A \times Z$.

Also if $A = \{1\}$, then $A \rtimes_{Z_m} Z \cong Z$ for all $m \geq 1$. Finally notice that there is a natural epimorphism
\[ q : A \rtimes_{Z_m} Z \to A \rtimes_{Z_m} Z, \quad q(\alpha, n) = (\alpha, n \mod m). \]

1.6. \textbf{Action of $S(f)$ on the Kronrod-Reeb graph.} For $f \in \mathcal{F}(M, P)$ denote by $\Gamma(f)$ the Kronrod-Reeb graph of $f$, i.e. the factor-space of $M$ obtained by shrinking every connected component of every level-set $f^{-1}(c)$ of $f$ to a point. This graph is very useful for understanding the structure of $f$, see e.g. [4, 11, 3, 16].

Notice that there is a natural action of $S(f)$ on $\Gamma(f)$ defined as follows. Let $h \in S(f)$, so $f \circ h = f$. Then $h(f^{-1}(c)) = f^{-1}(c)$ for all $c \in P$. In particular, $h$ interchanges connected components of $f^{-1}(c)$ being points of $\Gamma(f)$, and therefore it yields a certain homeomorphism $\lambda(h)$ of $\Gamma(f)$, such that the correspondence $h \mapsto \lambda(h)$ is a homomorphism $\lambda : S(f) \to \text{Aut}(\Gamma(f))$ into the group of all automorphisms of $\Gamma(f)$. Let
\[ G(f) := \lambda(S'(f)) \]
be the group of automorphisms of $\Gamma(f)$ induced by isotopic to the identity diffeomorphisms of $M$ preserving $f$.

\textbf{Definition 1.7.} \[13]. \textit{Let $\mathcal{R}$ be the minimal class of all finite groups satisfying the following conditions:}

\begin{enumerate}
  \item the unit group $\{1\}$ belongs to $\mathcal{R}$;
  \item if $A, B \in \mathcal{R}$ then $A \times B \in \mathcal{R}$;
  \item if $A \in \mathcal{R}$ and $m \geq 1$, then $A \rtimes Z_m \in \mathcal{R}$.
\end{enumerate}
Theorem 1.8. Let $M$ be a compact orientable surface distinct from $S^2$ and $T^2$. Then the class $R$ coincides with each of the following classes of groups:

$$\{ G(f) \mid f \in \text{Morse}(M, P) \}, \quad \{ G(f) \mid f \in \mathcal{F}(M, P) \}.$$ 

In other words, for each $f \in \mathcal{F}(M, P)$ the group $G(f)$ can be obtained from the unit group $\{1\}$ by finitely many operations of direct products and wreath products from the top with certain finite cyclic groups. Conversely, for any group $G \in R$ one can find $f \in \mathcal{F}(M, P)$ which can be assumed even Morse, such that $G \cong G(f)$.

Since $G(f)$ is a finite group, $\lambda$ reduces an epimorphism $\lambda : \pi_0 S'(f) \to G(f)$. Also notice that we have a surjective boundary homomorphism $\partial_1 : \pi_1 O(f) \to \pi_0 S'(f)$, see Eq. (1.2). Therefore

$$G(f) = \lambda \circ \partial_1(\pi_1 O(f))$$

is a factor group of $\pi_1 O(f)$. Thus Theorem 1.8 says that the factor group $G(f)$ of $\pi_1 O(f)$ can be described in terms of wreath products $A \wr \mathbb{Z}_m \mathbb{Z}$ being factor groups of $A \wr \mathbb{Z}$.

Our main result, Theorem 1.10 below, shows that $\pi_1 O(f)$ itself can be described in terms of wreath products $A \wr \mathbb{Z}$.

Definition 1.9. Let $\mathcal{P}$ be the minimal class of groups satisfying the following conditions:

1. the unit group $\{1\}$ belongs to $\mathcal{P}$;
2. if $A, B \in \mathcal{P}$, then $A \times B \in \mathcal{P}$;
3. if $A \in \mathcal{P}$ and $m \geq 1$, then $A \wr \mathbb{Z} \in \mathcal{P}$.

Theorem 1.10. Let $M$ be a connected compact orientable surface distinct from 2-sphere and 2-torus. Then the class $\mathcal{P}$ coincides with each of the following two classes of fundamental groups:

$$\{ \pi_1 O(f) \mid f \in \text{Morse}(M, P) \}, \quad \{ \pi_1 O(f) \mid f \in \mathcal{F}(M, P) \}.$$ 

It means that for each $f \in \mathcal{F}(M, P)$ the group $\pi_1 O(f)$ can be obtained from $\{1\}$ by finitely many operations of direct products and wreath products from the top with $\mathbb{Z}$ over certain finite cyclic groups. Conversely, for any group $G \in \mathcal{P}$ one can find $f \in \mathcal{F}(M, P)$, which can be assumed even Morse, such that $G \cong \pi_1 O(f)$.

The proof will be given in §4.

Remark 1.11. If $f$ is generic, that is every critical level set of $f$ contains exactly one critical point, then by [7], $G(f) = \{1\}$ and $\pi_1 O(f) \cong \mathbb{Z}^k$ for some $k \geq 0$. In particular, $\pi_1 O(f) \in \mathcal{P}$.

Remark 1.12. It is proved in [13] that each group $A \in \mathcal{R}$ is solvable. By similar arguments the same statement can be established for the class $\mathcal{P}$. Therefore Theorem 1.10 gives another proof of solvability result in 3) of Theorem 1.4. We leave the details for the reader.

Remark 1.13. Theorem 1.10 holds for certain classes of smooth functions on $T^2$, see [14].
2. Critical level sets

Let \( f \in \mathcal{F}(M, P) \), \( c \in P \) and \( K \) be a connected component of the level set \( f^{-1}(c) \). We will call \( K \) critical if it contains critical points of \( f \), and regular otherwise.

Assume that \( K \) is critical. Then due to Axiom (L) \( K \) has a structure of a 1-dimensional CW-complex whose 0-cells are critical points of \( f \) belonging to \( K \).

Choose \( \varepsilon > 0 \) and let \( N \) be a connected component of \( f^{-1}[c - \varepsilon, c + \varepsilon] \) containing \( K \). We will call \( N \) an \( f \)-regular neighbourhood of \( K \) if the following two conditions hold, see Figure 2.1:

- \( N \cap \partial M = \emptyset \),
- \( N \cap \Sigma_f = K \cap \Sigma_f \).

\[
\begin{align*}
N & \cap \partial M = \emptyset, \\
N & \cap \Sigma_f = K \cap \Sigma_f.
\end{align*}
\]

**Figure 2.1.** Critical component of level-set of \( f \)

Let also \( \mathcal{S}_{\text{inv}}(f, K) = \{ h \in \mathcal{S}(f) \mid h(K) = K \} \) be the subgroup of \( \mathcal{S}(f) \) consisting of diffeomorphism leaving \( K \) invariant. We will now define two equivalence relations \( \sim_K \) and \( \sim_{\partial N} \) on \( \mathcal{S}_{\text{inv}}(f, K) \).

Given a homeomorphism \( h : M \rightarrow M \) and a connected orientable submanifold \( X \subset M \) we will say that \( X \) is positively invariant for \( h \) whenever the following conditions hold true:

- \( h(X) = X \), and
- if \( X \) is not a point, then the restriction map \( h|_X : X \rightarrow X \) is a preserving orientation homeomorphism.

Let \( h \in \mathcal{S}_{\text{inv}}(f, K) \). Since \( h \) preserves the set of critical points of \( f \), it follows that the restriction \( h|_K : K \rightarrow K \) is a cellular homeomorphism of \( K \). We will say that \( h \) is \( K \)-trivial, if each cell of \( K \) is positively invariant for \( h \). Denote by \( \mathcal{T}(f, K) \) the subgroup of \( \mathcal{S}_{\text{inv}}(f, K) \) consisting of \( K \)-trivial diffeomorphisms. Evidently, \( \mathcal{T}(f, K) \) is normal in \( \mathcal{S}_{\text{inv}}(f, K) \). Given \( g \in \mathcal{S}_{\text{inv}}(f, K) \) we will write \( g \sim_K h \) whenever \( g^{-1} \circ h \in \mathcal{T}(f, K) \).

Furthermore, let \( N \) be an \( f \)-regular neighbourhood of \( K \). Then \( h(N) = N \) and so \( h \) yields a certain permutation of connected components of \( \partial N \). Say that \( h \) is \( \partial N \)-trivial, if each connected component of \( \partial N \) if positively invariant for \( h \). Denote by \( \mathcal{T}(f, \partial N) \) the normal subgroup of \( \mathcal{S}_{\text{inv}}(f, K) \) consisting of \( \partial N \)-trivial diffeomorphisms. Again for \( g \in \mathcal{S}_{\text{inv}}(f, K) \) we write \( g \sim_{\partial N} h \) whenever \( g^{-1} \circ h \in \mathcal{T}(f, \partial N) \).

**Lemma 2.1.** see [7, Theorems 6.2, 7.1]. Let \( g, h \in \mathcal{S}_{\text{inv}}(f, K) \) and \( N \) be an \( f \)-regular neighbourhood of \( K \). Then the following statements hold.
(1) Suppose there exists at least one edge $\delta$ being positively invariant for $g^{-1} \circ h$. Then all cells of $K$ are also positively invariant for $g^{-1} \circ h$, and in particular, $g \sim_{\delta} h$.

(2) Let $W$ be an open neighbourhood of $N$. If $g \sim_{\delta} h$, then there exists an isotopy of $g$ in $S_{\text{inv}}(f, K)$ supported in $W$ to some $g'$ such that $g' = h$ on $N$.

In particular, every $K$-trivial diffeomorphism is isotopic in $S_{\text{inv}}(f, K)$ via an isotopy supported in $W$ to a diffeomorphism fixed on $N$.

(3) $T(f, K) \subset T(f, \partial N)$, and so the relation $g \sim_{\delta} h$ always implies $g \sim_{\partial N} h$.

(4) Suppose that either

- $g$ and $h$ are isotopic as diffeomorphisms of $M$ or
- $N$ can be embedded into $\mathbb{R}^2$.

Then $T(f, K) = T(f, \partial N)$ and so the relations $g \sim_{\delta} h$ and $g \sim_{\partial N} h$ are equivalent.

Proof. Statement (1) is a consequence of [7, Claim 7.1.1], (2) follows from [7, Theorem 6.2 & Lemma 6.4] for Morse maps and from [11, Theorem 5] for all $f \in F(M, P)$, (3) follows from (2), and (4) from [7, Theorem 7.1].

Now let $B$ be a connected component of $\partial N$, and $\hat{B}$ be a regular component of some level-set of $f$ such that $B$ and $\hat{B}$ bound a cylinder $C$ containing no critical points of $f$, see Figure 2.2. Denote also $\hat{N} = N \cup C$.

![Figure 2.2. Extended neighbourhood of K](image)

**Lemma 2.2.** see [7, Lemma 4.14]. Suppose $K$ is not a local extreme of $f$. If $g, h \in S(f, \hat{B})$ are such that

- $g = h$ on some neighbourhood of $\hat{N}$ and
- $g$ and $h$ are isotopic in $S(f)$,

then they are isotopic in $S(f)$ relatively to some neighbourhood of $\hat{N}$.

Proof. It suffices to prove this lemma for the case when $h = \text{id}_M$, and so we are in the situation when $g$ belongs to $S_{\text{id}}(f)$ and is also fixed on $N$.

Suppose $M$ is orientable. In this case one can define a smooth flow $F : M \times \mathbb{R} \to M$ such that $g \in S_{\text{id}}(f)$ if and only if there exists a $C^\infty$-function $\alpha_g : M \to \mathbb{R}$ satisfying the identity: $g(x) = F(x, \alpha_g(x))$ for all $x \in M$, see [11, Theorem 3]. Moreover, this function is unique on any connected $f$-adopted subsurface containing at least one critical point being not a non-degenerate local extreme of $f$. By assumption $\hat{N}$ contains such points and $g$ is fixed on $\hat{N}$. Hence $\alpha_g = 0$ on $\hat{N}$. Then the isotopy between $g$ and $\text{id}_M$ in $S_{\text{id}}(f)$ can be given by the formula $g_t(x) = F(x, t\alpha_g(x))$, $t \in [0, 1]$, see [7, Lemma 4.14].
If $M$ is non-orientable, the proof follows by the arguments similar to the proof of [11, Theorem 3] for non-orientable case.

Lemma 2.3. Suppose $K$ is not a local extreme of $f$. Then there exists an epimorphism $\eta : S(f, \widehat{B}) \to \mathbb{Z}$ having the following properties.

1. $T(f, K) = \eta^{-1}(m\mathbb{Z})$ for some $m \geq 1$. In particular, $S(f, \widehat{B})/T(f, K) \cong \mathbb{Z}_m$.

2. Let $W$ be any open neighbourhood of $\widehat{N}$ and $g, h \in S(f, \widehat{B})$. Then $\eta(g) = \eta(h)$ if and only if there exists an isotopy of $g$ in $S(f, \widehat{B})$ supported in $W$ to some $g'$ such that $g' = h$ on $\widehat{N}$.

Proof. Let $V$ be the connected component of $\widehat{N} \setminus K$ containing $\widehat{B}$ and $Q = \overline{\mathbb{V}} \setminus \Sigma_f$. It is easy to see that $Q$ is diffeomorphic to $S^1 \times [0, 1] \setminus F$, where $F$ is a finite subset of $S^1 \times 1$, see Figures 2.2 and 2.3. Let $p : \tilde{Q} \to Q$ be the universal covering map for $Q$. Evidently, $\tilde{Q}$ is diffeomorphic with $\mathbb{R} \times [0, 1] \setminus \mathbb{Z} \times 1$, see Figure 2.3. Let $b > 0$ be the number of points in $F$. It also equals to the number of connected components of $S^1 \times 1 \setminus F$. Denote $J_i = (i, i + 1) \times 1$, $i \in \mathbb{Z}$.

Then

$$p(J_i) = p(J_{i+b}), \; i \in \mathbb{Z}. \tag{2.1}$$

Now let $h \in S(f, \widehat{B})$. Since $h$ is fixed on $\widehat{B}$, it follows that $h|_Q$ lifts to a unique diffeomorphism $\tilde{h}$ of $\tilde{Q}$ fixed on $\mathbb{R}^1 \times 0$. Then $\tilde{h}$ “shifts” open intervals $\{J_i\}_{i \in \mathbb{Z}}$ preserving their linear order. In other words, there exists a unique $k \in \mathbb{Z}$ such that $\tilde{h}(J_i) = J_{i+k}$ for all $i \in \mathbb{Z}$. Define a map $\eta' : S(f, \widehat{B}) \to \mathbb{Z}$ by

$$\eta'(h) = k.$$

It is easy to check that $\eta'$ is in fact a homomorphism.

It follows from Eq. (2.1) that $h \in T(f, K)$ if and only if $\eta'(h)$ is divided by $b$. In other words

$$T(f, K) = (\eta')^{-1}(b\mathbb{Z}). \tag{2.2}$$

Let us show that $\eta'$ is a non-trivial homomorphism. Recall that there exists a Dehn twist $\tau \in S(f, \widehat{B})$ supported in $\mathcal{C}$, see [7, § 6]. Then it is easy to see that $\eta'(<\tau) = b$ or $-b$. In particular, $\tau \in T(f, K)$.

Hence the image of $\eta'$ is also a non-zero subgroup of $\mathbb{Z}$, so $\eta'(S(f, \widehat{B})) = n\mathbb{Z}$ for some $n \geq 1$. In particular, due to Eq. (2.2) $n$ must divide $b$. Therefore the map $\eta : S(f, \widehat{B}) \to \mathbb{Z}$ defined by

$$\eta(h) = \eta'(h)/n$$
is an epimorphism.

Property (1) for \( \eta \) now follows from Eq. (2.2) with \( m = b/n \). It remains to check (2).

Let \( g, h \in S(f, \hat{B}) \) and \( \hat{g}, \hat{h} : \hat{Q} \to \hat{Q} \) be unique liftings of \( g|_Q \) and \( h|_Q \) respectively fixed on \( \mathbb{R}^1 \times 0 \).

Suppose there exists an isotopy \( \{g_t\}_{t \in [0, 1]} \) in \( S(f, \hat{B}) \) such that \( g_0 = g \) and \( g_1 = h \) on \( \hat{N} \). We claim that \( \eta(g) = \eta(h) \).

Indeed, let \( \hat{g}_t \) be the lifting of \( g_t|_Q \) fixed on \( \mathbb{R}^1 \times 0 \). Then \( \{\hat{g}_t\}_{t \in [0, 1]} \) is an isotopy between \( \hat{g} = \hat{g}_0 \) and \( \hat{h} = \hat{g}_1 \). Hence all \( \hat{g}_t \) shift boundary components \( \{(i, i + 1) \times 1\}_{i \in \mathbb{Z}} \) of \( \hat{Q} \) in the same way, and so

\[
\eta(g) = \eta(g_0) = \eta(g_1) = \eta(h).
\]

Conversely, suppose \( \eta(g) = \eta(h) \). Then \( g \) and \( h \) interchange edges of \( K \) in the same way, and so by (1) of Lemma 2.1 \( g \sim h \). Moreover, by (2) of that lemma \( g \) is isotopic to a diffeomorphism \( g' \) such that \( g' \sim h \) on \( N \). Hence \( g' \circ h^{-1}|_\hat{N} \) is supported in a cylinder \( C \), see Figures 2.2 and 2.3, and so it is isotopic relatively to \( \partial C \) to some degree \( a \) of the Dehn twist \( \tau \) mentioned above. Therefore \( \eta(g' \circ h^{-1}) = \eta(\tau^a) = ak/n \). However

\[
\eta(g' \circ h^{-1}) - \eta(h) = \eta(g) - \eta(h) = 0,
\]

whence \( a = 0 \). This means that \( g' \circ h^{-1}|_C \) is isotopic to \( \tau^0|_C = \mathrm{id}_C \) relatively to \( \partial C \). Hence by [7, Lemma 4.12(3)] that isotopy can be made \( f \)-preserving. Thus \( g' \) (and therefore \( g \)) is isotopic in \( S(f, \hat{B}) \) to some \( g'' \) such that \( g'' \sim h \) on \( \hat{N} \). Lemma is completed. \( \square \)

3. Functions on 2-disks and cylinders

In this section we assume that \( M \) is either a 2-disk or a cylinder, \( f \in \mathcal{F}(M, P) \), and \( \hat{B} \) is a connected component of \( \partial M \). Our aim is to establish the following key result which will be proved in §3.7.

**Proposition 3.1.** The group \( \pi_0 S'(f, \partial M) \) belongs to class \( \mathcal{P} \).

For the proof we need some preliminary statements.

**Lemma 3.2.**

1. \( S'(f, \hat{B}) = S(f, \hat{B}) \)
2. \( \pi_0 S'(f, \hat{B}) = \pi_0 S'(f, \partial M) \).

**Proof.** (1) Recall that by definition \( S'(f, \hat{B}) := S(f, \hat{B}) \cap \mathcal{D}_{id}(M, \hat{B}) \). Therefore we should only prove that each \( h \in S(f, \hat{B}) \) is isotopic to \( \mathrm{id}_M \) relatively to \( \partial M \). By 5) of Theorem 1.4 \( h \) is isotopic in \( S(f, \hat{B}) \) to a diffeomorphism \( h' \) fixed on some neighbourhood of \( \hat{B} \). Since \( M \) is either a 2-disk or a cylinder, it follows from [17, 2] that then \( h' \) is isotopic to \( \mathrm{id}_M \) relatively to \( \hat{B} \).

(2) If \( M = D^2 \), then \( \hat{B} = \partial M \) and the statement is trivial. Suppose \( M = S^1 \times I \). Then by 1) and 4) of Theorem 1.4 we have the following isomorphisms:

\[
\pi_0 S'(f, \hat{B}) \cong \pi_1 O(f, \hat{B}) \cong \pi_1 O(f, \partial M) \cong \pi_0 S'(f, \partial M).
\]

Lemma is completed. \( \square \)
Thus due to (2) for the proof of Proposition 3.1 it suffices to show that \( \pi_0 \mathcal{S}(f, \hat{B}) \in \mathcal{P} \). Of course, this replacement is non-trivial only for \( M = S^1 \times I \).

Let \( Z \) be the union of all critical components of all level sets of \( f \), \( U \) be a connected component of \( M \setminus Z \) containing \( \hat{B} \), and \( K \) be that unique critical component from \( Z \) which intersects \( \overline{U} \). Roughly speaking, \( K \) is the “closest” to \( \hat{B} \) critical component of some level set of \( f \).

Let also \( N \) be an \( f \)-regular neighbourhood of \( K \) that does not contain \( \hat{B} \) and

\[
\hat{N} = N \cup U, \quad C = U \setminus N.
\]

Then we are in the notations and under assumptions of Lemma 2.3 for a special case when \( \hat{B} \) is a boundary component of \( \partial M \).

By (1) or Lemma 3.2 each \( h \in \mathcal{S}(f, \hat{B}) \) is isotopic to \( \text{id}_M \), whence by (4) of Lemma 2.1 we get that \( \mathcal{T}(f, K) = \mathcal{T}(f, \partial N) \). Moreover, by Lemma 2.2 there exists an epimorphism

\[
\eta: \mathcal{S}(f, \hat{B}) \longrightarrow \mathbb{Z}
\]

satisfying \( \mathcal{T}(f, K) = \eta^{-1}(m\mathbb{Z}) \) for some \( m \geq 1 \), and so

\[
\mathcal{S}(f, \hat{B})/\mathcal{T}(f, K) = \mathcal{S}(f, \hat{B})/\mathcal{T}(f, \partial N) \cong \mathbb{Z}_m.
\]

**Lemma 3.3.** Let \( g \in \mathcal{S}(f, \hat{B}) \) be such that \( g(Y) \cap Y = \emptyset \) for some connected component \( Y \) of \( M \setminus \hat{N} \). If \( \eta(g) = 1 \), then \( g^i(Y) \cap Y = \emptyset \) for \( i = 1, \ldots, m - 1 \), and \( g^m(Y) = Y \).

**Proof.** Notice that under assumption of lemma \( g \not\in \mathcal{T}(f, \partial N) \), whence \( m > 1 \). Moreover, \( \eta(g^m) = m \in m\mathbb{Z} \), and so \( g^m \in \mathcal{T}(f, \partial N) \). Therefore \( g^m(Y) = Y \).

It remains to consider the case \( i \in \{1, \ldots, m - 1\} \). Let \( \tilde{M} \) be a closed surface obtained by gluing every connected component of \( \partial M \) with a 2-disk. Since \( M \) is either a 2-disk or a cylinder, we obtain that \( \tilde{M} \) is a 2-sphere. Then \( \tilde{M} \setminus K \) if a union of open 2-disks, and so we have a cellular subdivision of \( \tilde{M} \) by 0- and 1-cells of \( K \) and 2-cells being connected components of \( \tilde{M} \setminus K \).

Let \( h \in \mathcal{S}(f, \tilde{N}) \). Since \( h \) leaves invariant boundary components of \( \partial M \), it extends to a certain homeomorphism \( \tilde{h} \) of all of \( \tilde{M} \) which preserves orientation of \( \tilde{M} \), and therefore is also homotopic to \( \text{id}_{\tilde{M}} \). Then by [10] Proposition 5.4] either

(a) all cells are positively invariant for \( \tilde{h} \), or

(b) the number of positively invariant cells of \( \tilde{h} \) is equal to the Euler characteristic of \( \tilde{M} \), i.e. to 2.

In particular this holds for \( h = g^i \), \( i = 1, \ldots, m - 1 \).

Now let \( \delta_0, \delta_1 \) be positively invariant cells of \( \widehat{g} \). Then they are also positively invariant for \( \widehat{g}^i \), \( i = 1, \ldots, m - 1 \). Notice also that these cells do not intersect \( Y \), since \( g(Y) \cap Y = \emptyset \).

Therefore if we assume that \( g^i(Y) = Y \) for some \( i = 1, \ldots, m - 1 \), then \( \widehat{g}^i \) would have at least 3 positively invariant cells of \( \tilde{M} \), and by (a) all other cells of \( \tilde{M} \) must also be \( \widehat{g}^i \)-invariant. But this would mean that \( g^i \in \mathcal{T}(f, \partial N) \) which is possible only if \( i \) is a multiple of \( m \). We get a contradiction with the assumption \( i \in \{1, \ldots, m - 1\} \). Hence \( g^i(Y) \cap Y = \emptyset \) for \( 1 \leq i \leq m - 1 \). \( \square \)
Fix any \( g \in \mathcal{S}(f, \hat{B}) \) with \( \eta(g) = 1 \) and let
\[
C = X_0, X_1, \ldots, X_a
\]
be all \( g \)-invariant connected components of \( M \setminus \overline{N} \),
\[
X = X_1 \cup \cdots \cup X_a
\]
be the union of all these components except for \( C \), and
\[
S_i = X_i \cap \partial N, \quad S = X \cap \partial N;
\]
see Figure 3.1. By (1) of Lemma 2.3 these notation does not depend on a particular choice of such \( g \).

**Lemma 3.4.** There exists \( g \in \mathcal{S}(f, \hat{B}) \) fixed near \( X \) and satisfying \( \eta(g) = 1 \).

**Proof.** Let \( h \in \mathcal{S}(f, \hat{B}) \) be any element with \( \eta(h) = 1 \). Then \( h \) leaves invariant every connected component of \( S \), and preserves their orientation. Therefore \( h \) is isotopic in \( \mathcal{S}(f, \hat{B}) \) to a diffeomorphism \( h' \) fixed on some neighbourhood of \( S \). Now change \( h' \) on \( X \) by the identity and denote the obtained diffeomorphism by \( g \). Then \( g \in \mathcal{S}(f, \hat{B}) \) and \( \eta(g) = 1 \). \( \square \)

Let \( g \in \mathcal{S}(f, \hat{B}) \) be such that \( \eta(g) = 1 \). It follows from Lemma 3.3 that connected components of \( M \setminus \overline{N} \) that are not \( g \)-invariant can be enumerated as follows:
\[
\begin{align*}
Y_{0,1} & \quad Y_{0,2} & \cdots & \quad Y_{0,b} \\
Y_{1,1} & \quad Y_{1,2} & \cdots & \quad Y_{1,b} \\
& \cdots & \cdots & \cdots \\
Y_{m-1,1} & \quad Y_{m-1,2} & \cdots & \quad Y_{m-1,b}
\end{align*}
\]
so that
\[
g(Y_{j,q}) = g(Y_{j+1 \mod m, q})
\]
for all \( j, q \). In other words, \( g \) cyclically shifts down the rows of Eq. (3.2), see Figure 3.1.

Denote
\[
Y_j = Y_{j,1} \cup Y_{j,2} \cup \cdots \cup Y_{j,b}, \quad Y = \bigcup_{j=0}^{m-1} Y_j,
\]
\[
T_{j,q} = \partial Y_{j,q} \cap N, \quad T_j = \partial Y_j \cap N.
\]
Then
\[ g^j(Y_0) = Y_j, \quad Y_j \cap Y_{j'} = \emptyset \]
for \( j \neq j' = 0, \ldots, m - 1 \). Consider also the restrictions
\[ f_X = f|_X : X \to P, \quad f_{Y_j} = f|_{Y_j} : Y_j \to P. \]

**Lemma 3.5.** In the notation above there exists an isomorphism
\[ \psi : \pi_0\mathcal{S}(f, \hat{B}) \to \pi_0\mathcal{S}(f_X, S) \times \left( \pi_0\mathcal{S}(f_{Y_0}, T_0) \right. \left. \oplus \mathbb{Z} \right). \tag{3.3} \]

For \( m = 1 \), \( \psi \) reduces to an isomorphism
\[ \psi : \pi_0\mathcal{S}(f, \hat{B}) \to \pi_0\mathcal{S}(f_X, S) \times \mathbb{Z}. \]

**Proof.** Choose \( g \in \mathcal{S}(f, \hat{B}) \) fixed near \( X \) and satisfying \( \eta(g) = 1 \), see Lemma 3.4.

Let \( \gamma \in \pi_0\mathcal{S}(f, \hat{B}) \). By (1) of Lemma 2.3 we can take a representative \( h \in \gamma \) such that \( g^{-\eta(h)} \circ h \) is fixed on some neighbourhood of \( \hat{N} \). As \( g \) is fixed near \( X \), we have that
\[ g^{-\eta(h)} \circ h(X) = h(X) = X, \quad g^{-\eta(h)} \circ h(Y_j) = Y_j, \]
for all \( j \), whence
\[ h|_X \in \mathcal{S}(f_X, S), \quad g^{j - \eta(h)} \circ h \circ g^j|_{Y_0} \in \mathcal{S}(f_{Y_0}, T_0) \]
for \( j = 1, \ldots, m \). Therefore we obtain a function
\[ \sigma_h : \mathbb{Z}_m \to \pi_0\mathcal{S}(f_{Y_0}, T_0), \quad \sigma(j) = \left[ g^{j - \eta(h)} \circ h \circ g^j|_{Y_0} \right] \]
for \( j = 0, \ldots, m - 1 \).

Consider the following element belonging to \( \pi_0\mathcal{S}(f_X, S) \times \left( \pi_0\mathcal{S}(f_{Y_0}, T_0) \oplus \mathbb{Z} \right) \):
\[ \psi(\gamma) = \left( [h|_X], \sigma_h, \eta(h) \right). \]

We claim that the correspondence \( \gamma \mapsto \psi(\gamma) \) is the desired isomorphism (3.3).

**Step 1.** First we show that \( \psi(\gamma) \) does not depend on a particular choice of a representative \( h \in \gamma \) such that \( g^{-\eta(h)} \circ h \) is fixed on some neighbourhood of \( \hat{N} \).

Indeed, let \( h' \in \gamma \) be another element such that \( g^{-\eta(h')} \circ h' \) is fixed near \( \hat{N} \). Then \( h' = h = g^{\eta(h)} \) near \( \hat{N} \) and \( h' \) is isotopic to \( h \) in \( \mathcal{S}_{\text{id}}(f, \hat{B}) \).

In particular, it follows from (1) of Lemma 2.3 that \( \eta(h) = \eta(h') \).

Moreover, by Lemma 2.2 \( h \) and \( h' \) are isotopic in \( \mathcal{S}_{\text{id}}(f, \hat{B}) \) relatively some neighbourhood of \( \hat{N} \). This implies that \( h|_X \) is isotopic to \( h'|_X \) relatively some neighbourhood of \( S \), and for each \( j = 0, \ldots, m - 1 \) the restriction \( g^{j - \eta(h)} \circ h \circ g^j|_{Y_0} \) is isotopic to \( g^{j - \eta(h')} \circ h' \circ g^j|_{Y_0} \) relatively some neighbourhood of \( T_0 \). In other words,
\[ [h|_X] = [h'|_X] \in \pi_0\mathcal{S}'(f_X, S), \]
\[ [g^{j - \eta(h)} \circ h \circ g^j|_{Y_0}] = [g^{j - \eta(h')} \circ h' \circ g^j|_{Y_0}] \in \pi_0\mathcal{S}(f_{Y_0}, T_0), \]
for \( j = 0, \ldots, m - 1 \). Hence \( \psi(\gamma) \) does not depend on a particular choice of such \( h \).

**Step 2.** \( \psi \) is a homomorphism. Let \( h_0, h_1 \in \mathcal{S}(f, \hat{B}) \). We have to show that
\[ \psi([h_0 \circ h_1]) = \psi([h_0]) \cdot \psi([h_1]). \]
Put \( k_i = \eta(h_i), \ i = 0, 1 \). Since \( \eta \) is a homomorphism, \( \eta(h_0 \circ h_1) = k_0 + k_1 \).

By Step 1 we can assume that \( g^{-k_0} \circ h_i \) is fixed on \( \hat{N} \), \( i = 0, 1 \). Define the following four functions

\[
\sigma_0, \sigma_1, \sigma, \hat{\sigma} : \mathbb{Z}_m \rightarrow \pi_0 \mathcal{S}(f_{Y_0}, T_0)
\]

by

\[
\sigma_0(j) = [g^{-j-k_0} \circ h_i \circ g^j|_{Y_0}], \quad \sigma_1(j) = [g^{-j-k_1} \circ h_i \circ g^j|_{Y_0}],
\]

\[
\sigma(j) = [g^{-j-k-0-k_1} \circ h_0 \circ h_1 \circ g^j|_{Y_0}], \quad \hat{\sigma}(j) = \sigma_0(j + k_1) \circ \sigma_1(j)
\]

for \( j = 0, \ldots, m - 1 \). Then

\[
\psi([h_i]) = ([h_i|_X], \sigma_i, k_i), \quad i = 0, 1,
\]

\[
\psi([h_0 \circ h_1]) = ([h_0 \circ h_1|_X], \sigma, k_0 + k_1),
\]

and by the definition of multiplication

\[
\psi([h_0]) \circ \psi([h_1]) = ([h_0|_X], \sigma_0, k_0) ([h_1|_X], \sigma_1, k_1)
\]

\[
= ([h_0|_X] \circ [h_1|_X], \hat{\sigma}, k_0 + k_1) = ([h_0 \circ h_1|_X], \sigma, k_0 + k_1).
\]

It remains to show that \( \hat{\sigma} = \sigma \). Let \( j = 0, \ldots, m - 1 \), then

\[
\sigma(j) = [g^{-j-k_0-k_1} \circ h_0 \circ h_1 \circ g^j|_{Y_0}]
\]

\[
= [g^{-j-k_0-k_1} \circ h_0 \circ g^{j+k_1}|_{Y_0}] \circ [g^{-j-k_1} \circ h_1 \circ g^j|_{Y_0}]
\]

\[
= \sigma_0(j + k_1) \circ \sigma_1(j) = \hat{\sigma}(j).
\]

Thus \( \psi \) is a homomorphism.

**Step 3.** \( \psi \) is a monomorphism. Let \( h \in \mathcal{S}(f, \hat{B}) \) be such that \( g^{-\eta(h)} \circ h \) is fixed near \( \hat{N} \), and suppose that \( [h] \in \ker(\psi) \). This means that

\[
[h|_X] = [\text{id}_X] \in \pi_0 \mathcal{S}(f_X, S),
\]

\[
[g^{-j} \circ h \circ g^j|_{Y_0}] = [\text{id}_{Y_0}] \in \pi_0 \mathcal{S}(f_{Y_0}, T_0),
\]

\[
\eta(h) = 0,
\]

for \( j = 0, \ldots, m - 1 \). In other words, \( h|_X \) is isotopic in \( \mathcal{S}_{id}(f_X, S) \) to \( \text{id}_X \), and \( h|_{Y_j} \) is isotopic in \( \mathcal{S}_{id}(f_{Y_j}, T_j) \) to \( \text{id}_{Y_j} \). These isotopies give an isotopy between \( h \) and \( \text{id}_M \) in \( \mathcal{S}(f, \hat{B}) \). Hence \( [h] = [\text{id}_M] \in \mathcal{S}(f, \hat{B}) \), and so \( \ker(\psi) \) is trivial.

**Step 4.** \( \psi \) is surjective. Let \( \hat{h} \in \mathcal{S}(f_X, S), \sigma : \mathbb{Z}_m \rightarrow \pi_0 \mathcal{S}(f_{Y_0}, T_0), \) and \( k \in \mathbb{Z} \). We have to find \( h \in \mathcal{S}(f, \hat{B}) \) with \( \psi([h]) = ([\hat{h}], \sigma, k) \). For each \( j \in \mathbb{Z}_m \) choose \( h_j \in \mathcal{S}(f_{Y_0}, T_0) \) such that \( \sigma(j) = [h_j] \). Due to 5) of Theorem 4.4 we can assume that \( \hat{h} \) is fixed near \( S \) and each \( h_j \) is fixed near \( T_0 \). Define \( h \) by the formula:

\[
h(x) = \begin{cases} 
g^k(x), & x \in N, 
g^k \circ \hat{h}(x), & x \in X, 
g^{j+k} \circ h_j \circ g^{-j}(x), & x \in Y_j, \ j = 0, \ldots, m - 1. \end{cases}
\]
Then it is easy to check that \( h \in S(f, \hat{B}) \) and \( \psi([h]) = ([\hat{h}], \sigma, k) \). Lemma 3.5 is completed.

**Lemma 3.6.** 1) Let \( f \in \mathcal{F}(S^1 \times I, P) \) be a map without critical points. Then 
\[
\pi_0 S'(f, S^1 \times 0) = \pi_0 S'(f, \partial(S^1 \times I)) = 0.
\]

2) Let \( f \in \mathcal{F}(D^2, P) \) be a map having exactly one critical point, which therefore must be a local extreme.

(a) If \( z \) is a non-degenerate local extreme of \( f \), then \( \pi_0 S'(f, \partial D^2) = 0 \).

(b) Suppose \( z \) is a degenerate local extreme of \( f \). Then \( \pi_0 S'(f, \partial D^2) \cong \mathbb{Z} \).

**Proof.** These statements are contained in the previous papers by the author, though they were not explicitly formulated. In fact, statement 1) follows from [7, Lemma 4.12(2,3)], statement 2(a) from [5, Eq (25)] or from results of [9, 6], and statement 2(b) from results of [8]. We leave the details to the reader.

### 3.7. Proof of Proposition 3.1

Due to (2) of Lemma 3.2 it suffices to prove that \( \pi_0 S'(f, \hat{B}) \in \mathcal{P} \).

If \( K \) is either empty or consists of a unique point, then by Lemma 3.6 \( \pi_0 S'(f, \hat{B}) \) is either trivial or isomorphic with \( \mathbb{Z} \). Therefore it belongs to the class \( \mathcal{P} \).

Suppose now that \( K \) consists of more than one point, and let \( n \) be the total number of critical points of \( f \) in all of \( M \). We will use induction on \( n \).

If \( n = 0 \), then we are in the case 1) of Lemma 3.6 which is already considered. Suppose Proposition 3.1 is proved for all \( n < k \) for some \( k \geq 1 \). Let us establish it for \( n = k \).

Preserving notation of Lemma 3.5 let \( K \) be the “closest” to \( \hat{B} \) critical component of some level set of \( f \), see beginning of §3. Since \( X \) is a disjoint union of surfaces \( X_i, i = 1, \ldots, a \), as well as \( Y_0 \) is a disjoint union of \( Y_{0,q}, q = 1, \ldots, b \), it follows that

\[
\pi_0 S(f_X, S) \cong \prod_{i=1}^{a} \pi_0 S(f_{X_i}, S_i), \quad \pi_0 S'(f_{Y_0}, S_0) \cong \prod_{q=1}^{b} \pi_0 S(f_{Y_{0,q}}, T_{0,q}),
\]

whence from Lemma 3.5 we get an isomorphism

\[
\pi_0 S(f, \hat{N}) \cong \left(\prod_{i=1}^{a} \pi_0 S(f_{X_i}, S_i)\right) \times \left(\prod_{q=1}^{b} \pi_0 S(f_{Y_{0,q}}, T_{0,q})\right) \cong \mathbb{Z}.
\]

As each pair \((M, \hat{B}), (X_i, S_i)\), and \((Y_{j,q}, T_{j,q})\) is diffeomorphic either with \((D^2, \partial D^2)\) or with \((S^1 \times I, S^1 \times 0)\), it follows from (1) of Lemma 3.2 that

\[
S'(f, \hat{N}) = S(f, \hat{N}), \quad S'(f_{X_i}, S_i) = S(f_{X_i}, S_i), \quad S'(f_{Y_{0,q}}, T_{0,q}) = S(f_{Y_{0,q}}, T_{0,q}),
\]

so we also have an isomorphism

\[
\pi_0 S'(f, \hat{N}) \cong \left(\prod_{i=1}^{a} \pi_0 S'(f_{X_i}, S_i)\right) \times \left(\prod_{q=1}^{b} \pi_0 S'(f_{Y_{0,q}}, T_{0,q})\right) \cong \mathbb{Z}.
\]

Notice that each of the restrictions \( f|_{X_i} \) and \( f|_{Y_{0,q}} \) has less critical points than \( n \), whence by inductive assumption \( \pi_0 S'(f_{X_i}, S_i) \) and \( \pi_0 S'(f_{Y_{0,q}}, T_{0,q}) \) belong to the class \( \mathcal{P} \). Hence \( \pi_0 S'(f, \hat{N}) \in \mathcal{P} \) as well. Proposition 3.1 is completed.
4. Proof of Theorem 1.10

Let $M$ be a compact orientable surface distinct from $S^2$ and $T^2$. Then $D_{	ext{id}}(M, \partial M)$ is contractible, and by 1) and 4) of Theorem 1.4 for each $f \in \mathcal{F}(M, P)$ we have the following isomorphisms

$$\pi_1 \mathcal{O}(f) \cong \pi_1 \mathcal{O}(f, \partial M) \cong \pi_0 \mathcal{S}^1(f, \partial M).$$

Therefore it suffices to prove that the class $\mathcal{P}$ coincides with each of the following classes of groups:

$$\{ \pi_0 \mathcal{S}^1(f, \partial M) \mid f \in \text{Morse}(M, P) \}, \quad \{ \pi_0 \mathcal{S}^1(f, \partial M) \mid f \in \mathcal{F}(M, P) \}.$$

**Lemma 4.1.** For each $f \in \mathcal{F}(M, P)$ the group $\pi_0 \mathcal{S}^1(f, \partial M)$ belongs to $\mathcal{P}$.

**Proof.** By 4) of Theorem 1.4 there exist finitely many disjoint subsurfaces $X_1, \ldots, X_n \subset M$ each $X_i$ is diffeomorphic either with $D^2$ or with $S^1 \times I$, and such that

$$\pi_0 \mathcal{S}^1(f, \partial M) \cong \bigotimes_{i=1}^n \pi_0 \mathcal{S}^1(f_{X_i}, \partial X_i).$$

But by Proposition 3.1 $\pi_0 \mathcal{S}^1(f_{X_i}, \partial X_i) \in \mathcal{P}$ for all $i$, whence $\pi_0 \mathcal{S}^1(f, \partial M) \in \mathcal{P}$ as well. \qed

For the converse statement we make a remark concerning the structure of groups from $\mathcal{P}$. By definition a group $G$ belongs to the class $\mathcal{P}$ if and only if it can be obtained from the unit group $\{1\}$ by finitely many operations of direct product $\times$ and wreath product $\wr \mathbb{Z}$ from the top with $\mathbb{Z}$. We will call such a presentation of $G$ a $\mathcal{P}$-presentation.

A priori a $\mathcal{P}$-presentation of $G$ is not unique, e.g. $\mathbb{Z} \cong \mathbb{Z} \wr \mathbb{Z} \cong \mathbb{Z} \wr \mathbb{Z}$. Given a $\mathcal{P}$-presentation $\xi_G$ of $G$ denote by $\mu(\xi_G)$ the total number of signs $\times$ and $\wr \mathbb{Z}$ for some $m \geq 1$, used in $\xi_G$. For example, a group $G = \mathbb{Z}^2 \times (\mathbb{Z} \wr \mathbb{Z})$ has a $\mathcal{P}$-presentation

$$\xi_G : G \cong (\mathbb{Z} \wr \mathbb{Z}) \times (1 \wr \mathbb{Z}) \times ((1 \wr \mathbb{Z}) \wr \mathbb{Z}).$$

with $\mu(\xi_G) = 6$.

**Lemma 4.2.** For each $G \in \mathcal{P}$ then there exists an $f \in \text{Morse}(M, P)$ such that

$$\pi_0 \mathcal{S}^1(f, \partial M) \cong G.$$

**Proof.** Case $M = D^2$ or $S^1 \times I$. If $G = \{1\}$ is a unit group, we take $f$ to be a Morse map from 1) or 2a) of Lemma 3.6 according to $M$. Then $\pi_0 \mathcal{S}^1(f, \partial M) \cong G = \{1\}$.

Suppose that we proved our lemma for all groups $A \in \mathcal{P}$ having a $\mathcal{P}$-presentation $\xi_A$ with $\mu(\xi_A) < n$ and let $G \in \mathcal{P}$ be a group having a $\mathcal{P}$-presentation $\xi_G$ with $\mu(\xi_G) = n$. It follows from the definition of class $\mathcal{P}$ that then either

(i) there exist $A, B \in \mathcal{P}$ and $m \geq 2$, such that $G \cong A \times (B \wr \mathbb{Z})$, where $A$ and $B$ have $\mathcal{P}$-presentations $\xi_A$ and $\xi_B$ with $\mu(\xi_A), \mu(\xi_B) < \mu(\xi_G)$, or

(ii) there exist $A \in \mathcal{P}$ such that $G \cong A \times \mathbb{Z}$, where $A$ has a $\mathcal{P}$-presentation $\xi_A$ with $\mu(\xi_A) < \mu(\xi_G)$. 

First assume that $M = D^2$.

(i) Suppose $G \cong A \times (B \wr \mathbb{Z})$. Define a Morse function $\varphi : M \to P$, as it is shown in Figure 4.1(a) for $m = 3$. So $\varphi$ has one local minimum $x$ and $m$ local maximums $y_0, \ldots, y_{m-1}$ satisfying $\varphi(y_0) = \cdots = \varphi(y_{m-1})$ and there exists a diffeomorphism $g \in S(f, \partial M)$ that cyclically interchange these points, i.e. $g(y_j) = y_{j+1 \mod m}$. Let $X$ be a $\varphi$-regular disk neighbourhood of $x$, $Y_0$ be a $\varphi$-regular disk neighbourhood of $y_0$, and $Y_j = g^j(y_j)$, $j = 1, \ldots, m - 1$. As $\mu(\xi_A), \mu(\xi_B) < \mu(\xi_G)$, we have by induction that there exist $\alpha \in \text{Morse}(X, P)$ and $\beta \in \text{Morse}(Y_0, P)$ such that

$$A \cong \pi_0 S'(\alpha, \partial X), \quad B \cong \pi_0 S'(\beta, \partial Y_0).$$

Not loosing generality, one can assume that $\alpha = \varphi$ in a neighbourhood of $\partial X$ and $\beta = \varphi$ in a neighbourhood of $\partial Y$. Replace $\varphi$ with $\alpha$ on $X$, with $\beta_j = \beta \circ g^{-j}$ on $Y_j$, $j = 0, \ldots, m - 1$, and denote the obtained new map by $f$, see Figure 4.1(b). Then $f \in \text{Morse}(M, P)$ and it follows from Proposition 3.1 that

$$\pi_0 S'(f, \partial M) \cong A \times (B \wr \mathbb{Z}) \cong G.$$

(ii) Suppose now $G \cong A \times \mathbb{Z}$. Define a Morse function $\varphi : M \to P$ having two local maximums $x$ and $y$ such that $\varphi(x) \neq \varphi(y)$, see Figure 4.2(a). Let $Y$ be a $\varphi$-regular disk neighbourhood of $y$ such that $\varphi(x) \notin \varphi(Y)$. Since $\mu(\xi_A) < \mu(\xi_G)$, it follows by induction that there exist $\beta \in \text{Morse}(Y, P)$ such that $A \cong \pi_0 S'(f, \partial Y)$ and $\alpha = \varphi$ near $\partial Y$. Now

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure4_1}
\caption{$M = D^2$. Case (i)}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure4_2}
\caption{$M = D^2$. Case (ii)}
\end{figure}
replace \( \varphi \) with \( \beta \) on \( Y \) and denote the obtained map by \( f \). Then \( f \in \text{Morse}(M, P) \) and it follows from Proposition 3.1 and Lemma 3.6 2(a) that
\[
\pi_0 S'(f, \partial M) \cong A \times \mathbb{Z} \cong G.
\]

For \( M = S^1 \times I \), the proof of the cases (i) and (ii) is similar to the case of \( D^2 \), and is illustrated in Figure 4.3. We leave the details for the reader.

\[\text{Case (i)}\]

\[\text{Case (ii)}\]

**Figure 4.3.** \( M = S^1 \times I \)

Now let \( M \) be an arbitrary compact orientable surface distinct from \( S^2, T^2, D^2 \), and \( S^1 \times I \). Choose a Morse function \( \varphi : M \to P \) such that

- all critical points of \( \varphi \) of index 1 belong to the same critical level-set of \( \varphi \);
- the values of \( \varphi \) at distinct boundary components and distinct local extremes of \( \varphi \) are distinct,

see Figure 4.4. Fix some local extreme \( y \) of \( \varphi \) and let \( Y \) be a \( \varphi \)-regular disk neighbourhood

**Figure 4.4.** General case
Let $G \in \mathcal{P}$. Since the theorem is already proved for a disk $D^2 \simeq Y$, there exists $\beta \in \mathcal{F}(Y, P)$ with $\pi_0 S'(\beta, \partial Y) \cong G$ and $\beta = \varphi$ in some neighbourhood of $\partial Y$. Replace $\varphi$ with $\beta$ on $Y$ and denote the obtained map by $f$. Then $f \in \text{Morse}(M, P)$ and it follows from Proposition 3.1 that $\pi_0 S'(\varphi, \partial M) \cong \pi_0 S'(f, \partial Y) \cong G$. Lemma 4.2 and Theorem 1.10 completed.

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