On the Exceptional Gauged WZW Theories

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Abstract

We consider two different versions of gauged WZW theories with the exceptional groups and gauged with any of theirs null subgroups. By constructing suitable automorphism, we establish the equivalence of these two theories. On the other hand our automorphism, relates the two dual irreducible Riemannian globally symmetric spaces with different characters based on the corresponding exceptional Lie groups.

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In the last few years, duality transformation as an abelian or non-abelian symmetry of conformal field theories has been studied extensively [1, 2, 3]. In WZW models, the duality transformations are given by the automorphisms of the group $G$ of the models. In particular, it has been shown that the target space of the gauged WZW model with the group $G$ and vector gauged abelian subgroup is dual to the corresponding target space of the axial gauged WZW model gauged by the same abelian subgroup. Moreover the duality transformation is implemented by an automorphism of the gauged subgroup [4, 5, 6, 7]. When the gauged subgroup is not semisimple, and in particular when it is null, the target space of vector gauged WZW model reduces to a space with less dimensions [8, 9, 10, 11]. In other words, it was shown that in the usual vector gauging of classical Lie group $G$ WZW model, by its maximal null subgroup $H$, the effective action of the gauged model reduces to that of a Toda theory which the number of Toda fields is less than $\text{Dim}G-\text{Dim}H$.

An explanation for the reduction of the degrees of freedom of the target space of the vector gauged model is related to the obvious dimensional reduction of the corresponding chiral gauged model when the left and right gauge actions are independent and in two different subgroups isomorphic to vector null gauged subgroup.

In ref. [12], it was proved that the vector and chiral gauged theories with any classical Lie groups and any null gauged subgroup are equivalent to each other. The equivalence map was constructed for every classical Lie groups and it was shown that these maps were in fact involutive automorphisms of corresponding Lie algebras. These involutive automorphisms could be used for real form construction of simple Lie algebras and also gave us the pairs of dual Riemannian globally symmetric spaces [13].

In this paper, we consider the vector and chiral gauged WZW models which based on the exceptional groups and gauged with any of theirs null subgroups and prove that these models are equivalent to each other. We will find that the equivalence map which relates the two models, is the involutive automorphism of the algebra and in the context of Riemannian geometry relates the two dual irreducible Riemannian globally symmetric spaces based on the exceptional Lie group. In particular, for the exceptional group $G_2$, we give the calculation in detail, and in the case of other exceptional groups, because of complexity, we present the final results. The work done in this paper completes the equivalence of vector and chiral gauged WZW models with any Lie groups.

$G_2$ Gauged WZW Models

Let’s recall the structure of gauged WZW models based on the exceptional Lie group $G_2$. The vector gauged action is given by [14, 13]

$$S_V(g, A, \bar{A}) = S(g) + \frac{k}{2\pi} \int d^2 z \text{Tr}(-\bar{A}g^{-1}g + A\bar{A}g^{-1} - A\bar{A} + g\bar{A}g^{-1}A),$$

$$S(g) = \frac{k}{4\pi} \int d^2 z \text{Tr}(g^{-1}\bar{\partial}g^{-1}\bar{\partial}g) - \frac{k}{12\pi} \int \text{Tr}(g^{-1}dg)^3,$$

which is invariant under the gauge transformations

$$g \rightarrow h^{-1}gh, A \rightarrow h^{-1}(A - \partial)h, \bar{A} \rightarrow h^{-1}(\bar{A} - \bar{\partial})h,$$

(2)
where \( h = h(z, \bar{z}) \) is a group element in subgroup \( H \) of \( G_2 \) and \( A \) and \( \bar{A} \) take their values in the algebra \( \mathcal{L}(H) \) of subgroup \( H \). On the other hand, the chiral WZW action \[ S_C(g, A, \bar{A}) = S(g) + \frac{k}{2\pi} \int d^2z \text{Tr}(-\bar{A}g^{-1}\partial g + A\partial gg^{-1} + g\bar{A}g^{-1}A), \] \( (3) \) is invariant under the following transformations

\[ g \rightarrow h^{-1}g\bar{h}, \quad A \rightarrow h^{-1}(A - \partial)h, \quad \bar{A} \rightarrow \bar{h}^{-1}(\bar{A} - \bar{\partial})\bar{h}, \] \( (4) \)

where \( \bar{h} = h(\bar{z}) \) belongs to the subgroup \( H_1 \), and \( \bar{h} \) belongs to another subgroup \( H_2 \) of \( G_2 \). \( A \) takes its value in \( \mathcal{L}(H_1) \) and \( \bar{A} \) in \( \mathcal{L}(H_2) \).

Now, we impose following transformations on the \( g \) field and the gauge fields of the chiral theory

\[ g' = g\theta, \quad \bar{A}' = \theta^{-1}\bar{A}\theta, \quad A' = A, \] \( (5) \)

and demand that the \( g' \) and \( \bar{A}' \) become the corresponding fields in the vector theory.

After straightforward but lengthy calculations, one can find that

\[ \theta = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \] \( (6) \)

It is obvious that \( \theta^2 = 1 \). For proving the equivalence, it is also necessary that the involutive automorphism \( \theta \) belongs to the Lie group \( G_2 \). According to a well known theorem \[ [13] \], an involutive automorphism of a compact semisimple Lie algebra \( A \) belongs to the adjoint group of \( A \) (inner automorphism) if and only if the rank of the set of fixed points of the automorphism is equal to the rank of the Lie algebra \( A \). Let us take \( E \) to be in the Lie algebra \( g_2 \), according to

\[ E = 2\sqrt{6}(e_1E_1 + \tilde{e}_1E_{-1} + e_3E_3 + \tilde{e}_3E_{-3} + e_5E_5 + \tilde{e}_5E_{-5} + e_6E_6 + \tilde{e}_6E_{-6}) + 2\sqrt{2}(e_2E_2 + \tilde{e}_2E_{-2} + e_4E_4 + \tilde{e}_4E_{-4}) + 4\sqrt{3}h_1H_1 + 4h_2H_2, \] \( (7) \)

where \( E \)'s, \( H_1 \) and \( H_2 \) are given in the Appendix. Then, the invariant set of involutive automorphism \( \theta \) is \( f = \frac{E + \theta E\theta}{2} \), which with the redefinition of variables

\[ \epsilon_1 = \frac{e_1 + \tilde{e}_1}{2}, \quad \epsilon_2 = \frac{e_2 - \tilde{e}_2}{2}, \quad \epsilon_3 = \frac{e_3 + \tilde{e}_3}{2}, \quad \epsilon_4 = \frac{e_4 + \tilde{e}_4}{2}, \quad \epsilon_5 = \frac{e_5 - \tilde{e}_5}{2}, \quad \epsilon_6 = \frac{e_6 + \tilde{e}_6}{2}, \] \( (8) \)
can be written as

\[
f = \begin{pmatrix}
0 & \epsilon_1 & \epsilon_4 & 0 & -\epsilon_5 & \sqrt{2}\epsilon_3 & \epsilon_2 \\
\epsilon_1 & 0 & 0 & \epsilon_4 & -\epsilon_6 & \sqrt{2}\epsilon_5 & -\epsilon_3 \\
\epsilon_4 & 0 & 0 & \epsilon_1 & \epsilon_3 & -\sqrt{2}\epsilon_6 & \epsilon_6 \\
0 & \epsilon_4 & \epsilon_1 & 0 & -\epsilon_2 & -\sqrt{2}\epsilon_3 & \epsilon_5 \\
\epsilon_5 & -\epsilon_6 & \epsilon_3 & \epsilon_2 & 0 & \sqrt{2}\epsilon_1 & 0 \\
\sqrt{2}\epsilon_3 & -\sqrt{2}\epsilon_5 & \sqrt{2}\epsilon_5 & -\sqrt{2}\epsilon_3 & \sqrt{2}\epsilon_1 & 0 & \sqrt{2}\epsilon_1 \\
-\epsilon_2 & -\epsilon_3 & \epsilon_6 & -\epsilon_5 & 0 & \sqrt{2}\epsilon_1 & 0 \\
\end{pmatrix}.
\]

(9)

It can easily be shown that \( f \) produces an algebra and its rank is two. More precisely, the algebra \( f \) is isomorphic to \( \text{su}(2) \otimes \text{su}(2) \). Hence, \( \theta \) belongs to \( G_2 \) Lie group and establishes the equivalence of vector and chiral theories with \( G_2 \) group. It is also interesting to note the relation of our constructed automorphism \( \theta \) and the dual Riemannian globally symmetric spaces based on the \( G_2 \) group manifold. The anti-invariant set of involutive automorphism \( \theta \) is \( p = \frac{E - \theta E\theta}{2} \) and is an eight-parametric space. The form of \( p \) is given by

\[
p = \begin{pmatrix}
\rho_1 & \eta_1 & \eta_4 & 0 & \eta_5 & \sqrt{2}\eta_3 & \eta_2 \\
-\eta_1 & -\rho_2 & 0 & \eta_4 & -\eta_6 & -\sqrt{2}\eta_5 & -\eta_3 \\
-\eta_4 & 0 & \rho_2 & \eta_1 & -\eta_3 & -\sqrt{2}\eta_5 & -\eta_6 \\
0 & -\eta_4 & -\eta_1 & -\rho_1 & \eta_2 & \sqrt{2}\eta_3 & \eta_5 \\
\eta_5 & \eta_6 & \eta_3 & \eta_2 & \rho_1 + \rho_2 & \sqrt{2}\eta_1 & 0 \\
-\sqrt{2}\eta_3 & -\sqrt{2}\eta_5 & -\sqrt{2}\eta_5 & -\sqrt{2}\eta_3 & -\sqrt{2}\eta_1 & 0 & \sqrt{2}\eta_1 \\
\eta_2 & \eta_3 & \eta_6 & \eta_5 & 0 & -\sqrt{2}\eta_1 & -\rho_1 - \rho_2 \\
\end{pmatrix},
\]

(10)

where

\[
\begin{align*}
\rho_1 &= 2(h_1 + h_2), \quad \rho_2 = 2(h_1 - h_2), \\
\eta_1 &= \frac{\epsilon_1 - \bar{\epsilon}_1}{2}, \quad \eta_2 = \frac{\epsilon_2 + \bar{\epsilon}_2}{2}, \quad \eta_3 = \frac{\epsilon_3 - \bar{\epsilon}_3}{2}, \quad \eta_4 = \frac{\epsilon_4 - \bar{\epsilon}_4}{2}, \quad \eta_5 = \frac{-\epsilon_5 - \bar{\epsilon}_5}{2}, \quad \eta_6 = \frac{\epsilon_6 - \bar{\epsilon}_6}{2}.
\end{align*}
\]

(11)

Moreover, we have \([p, p] \subset f\) and \([f, p] \subset p\). These properties of \( \theta \), in the context of dual Riemannian symmetric spaces is enough for construction of the unique pair of dual spaces. In fact, the \( \theta \)-invariant algebra \( f \) is the maximal compact subalgebra of \( g_2^* \) and gives rise to the compact-noncompact dual Riemannian symmetric pairs \( \frac{G_2}{F_2} \) and \( \frac{G_2^*}{F_2^*} \).

**Other Exceptional Gauged WZW Models**

In this section, we give the involutive automorphisms \( \theta \) which according to equations (5), convert the \( g \) and \( \bar{A} \) fields of the vector gauged \( F_4, E_{6,7,8} \) WZW models to the corresponding chiral models,

\[
\begin{align*}
\theta_{F_4} &= \sum_{i=1}^{26} (\epsilon_{i,i+26} - \epsilon_{i+26,i}), \\
\theta_{E_6} &= \sum_{i=1}^{39} (\epsilon_{i,79-i} - \epsilon_{i+39,40-i}),
\end{align*}
\]

(12)
\[ \theta_{E_7} = \sum_{i=1}^{67} e_{i,68-i} - \sum_{i=1}^{66} e_{i+67,134-i}, \]
\[ \theta_{E_8} = \sum_{i=1}^{124} (e_{i,i+124} - e_{i+124,i}), \quad (12) \]

where \( e_{ij} \) is the matrix with one at the \((i,j)\) entry and zero elsewhere. It can easily be seen that the above automorphisms satisfy \( \theta^2 = 1 \). Moreover, the set of fixed points of the above involutive automorphisms is,

\( f_{F_4} = sp(3) \otimes su(2), f_{E_6} = su(6) \otimes su(2), f_{E_7} = su(8), f_{E_8} = so(16) \) which shows that \( \theta_{F_4}, \theta_{E_6}, \theta_{E_7}, \theta_{E_8} \) belongs to the \( F_4, E_6, E_7, E_8 \) Lie group respectively.

**Concluding Remarks**

The vector gauged WZW theory with any exceptional group \( G \) gauged with any of its null subgroups is equivalent to the chiral gauged WZW theory. The equivalence map is the inner automorphism of algebra \( \mathcal{L}(G) \) which gives the irreducible Riemannian globally symmetric spaces. For example, in the case of \( G_2 \) group, the equivalence map gives the irreducible Riemannian globally symmetric spaces \( \frac{G_2}{SU(2) \otimes SU(2)} \) and \( \frac{G_2}{SU(2) \otimes SU(2)} \). Previously, the equivalence of the vector and chiral gauged WZW theories with any classical Lie groups were established. In all the classical series, the corresponding map is an involutive automorphism of the Lie algebra [12].

**Appendix**

We take the generators of \( g_2 \) Lie algebra as in [17],

\[ E_1 = \frac{1}{2\sqrt{6}}(e_{12} + e_{34}) + \frac{1}{2\sqrt{3}}(e_{56} + e_{67}), \]
\[ E_2 = \frac{1}{2\sqrt{2}}(e_{17} + e_{54}), \]
\[ E_3 = \frac{1}{2\sqrt{3}}(e_{16} - e_{64}) + \frac{1}{2\sqrt{6}}(e_{53} - e_{27}), \]
\[ E_4 = \frac{1}{2\sqrt{2}}(e_{13} + e_{24}), \]
\[ E_5 = -\frac{1}{2\sqrt{6}}(e_{15} + e_{74}) + \frac{1}{2\sqrt{3}}(e_{26} + e_{63}), \]
\[ E_6 = \frac{1}{2\sqrt{2}}(e_{73} - e_{25}), \quad (13) \]

and \( E_{-i} = \tilde{E}_i \). The generators of Cartan subalgebra are given by

\[ H_1 = \frac{1}{4\sqrt{3}}(e_{11} - e_{22} + e_{33} - e_{44}) + \frac{1}{2\sqrt{3}}(e_{55} - e_{77}), \]
\[ H_2 = \frac{1}{4}(e_{11} + e_{22} - e_{33} - e_{44}). \quad (14) \]

We note the null property of generators as follows,

\[ E_1^3 = E_2^2 = E_3^2 = E_4^2 = E_5^3 = E_6^2 = Tr(E_1^2) = Tr(E_5^2) = 0. \quad (15) \]
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