ON CERTAIN PROBABILISTIC PROPERTIES OF POLYNOMIALS OVER THE RING OF $p$-ADIC INTEGERS

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ABSTRACT. In this article, we study several probabilistic properties of polynomials defined over the ring of $p$-adic integers under the Haar measure. First, we calculate the probability that a monic polynomial is separable, generalizing a result of Polak. Second, we introduce the notion of two polynomials being strongly coprime and calculate the probability of two monic polynomials being strongly coprime. Finally, we explain how our method can be used to extrapolate other probabilistic properties of polynomials over the ring of $p$-adic integers from polynomials defined over the integers modulo powers of $p$.

1. Introduction.

Let $K$ be a field. We say that a polynomial $f \in K[x]$ is separable if its roots in an algebraic closure of $K$ are distinct. This is equivalent to $(f, f') = K[x]$, where $f'$ denotes the formal derivative of $f$. More abstractly, this is also equivalent to $A = K[x]/f$ being a separable $K$-algebra (meaning that $A$ is projective as an $A \otimes_K A$-module). For example, $x(x + 1)$ is separable, but $(x + 1)^2$ is not. We may generalize this definition to commutative rings. Let $R$ be a commutative ring. We say that a polynomial $f \in R[x]$ is separable if $A = R[x]/f$ is a separable $R$-algebra (meaning that $A$ is projective as an $A \otimes_R A$-module). When $f$ is monic, this turns out to be equivalent to $(f, f') = R[x]$ as in the case of polynomials defined over a field (we refer the readers to [1, 5, 11] for details). If $f, g \in R[x]$, we say that $f$ and $g$ are relatively prime or coprime if there is no monic polynomial of positive degree in $R[x]$ that divides both $f$ and $g$.

The motivation for this article comes from two articles, namely [11], where the proportion of separable monic polynomials in $(\mathbb{Z}/p^k\mathbb{Z})[x]$ is derived from a previous result of Carlitz [3] on separable polynomials in $(\mathbb{Z}/p\mathbb{Z})[x]$, and [7], where certain formulae for the probability that two monic polynomials in $(\mathbb{Z}/p^k\mathbb{Z})[x]$ are relatively prime are found. The results we are interested in are as follows.

Theorem 1 (Polak, [11]). Let $p$ be a prime number and $k \geq 1$ an integer. The proportion of monic polynomials of degree $d \geq 2$ that are separable in $(\mathbb{Z}/p^k\mathbb{Z})[x]$ is $1 - p^{-1}$.

Theorem 2 (Hagedorn and Hatley, [7]). Let $p$ be an odd prime number and $m, k \geq 1$ integers. The probability that two randomly chosen monic polynomials of degrees $m$ and $2$ in $(\mathbb{Z}/p^k\mathbb{Z})[x]$ are relatively prime is given by

$$P_{\mathbb{Z}/p^k\mathbb{Z}}(m, 2) = 1 - \frac{f_k(p)}{p^{3k}},$$

where $f_k(x) \in \frac{1}{2}\mathbb{Z}[x]$ is an explicit monic polynomial of degree $2k$. 

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Results similar to Theorem 2 have been found for multiple polynomials over a finite field in [2, 4]. Other similar problems on polynomials over finite fields can be found in [8] and [6], where multivariate polynomials and common divisors of multiple polynomials are studied, respectively.

Let \( \mathbb{Z}_p = \lim_{\leftarrow k} \mathbb{Z}/p^k \mathbb{Z} \) denote the ring of \( p \)-adic integers. We caution readers that some authors write \( \mathbb{Z}_p \) for \( \mathbb{Z}/p \mathbb{Z} \). In the present article, \( \mathbb{Z}_p \) is very different from \( \mathbb{Z}/p \mathbb{Z} \). It is the ring of sequences \((a_k)_{k \geq 1}\), where \( a_k \in \mathbb{Z}/p^k \mathbb{Z} \) is such that the image of \( a_{k+1} \) under the natural projection map \( \mathbb{Z}/p^{k+1} \mathbb{Z} \to \mathbb{Z}/p^k \mathbb{Z} \) equals \( a_k \) for all \( k \geq 1 \). There is a natural bijection between \( \mathbb{Z}_p \) and the set of formal sums

\[
\left\{ \sum_{i \geq 0} c_ip^i : c_i \in \{0, 1, \ldots, p-1\} \right\}
\]

In particular, there is a natural projection map \( \mathbb{Z}_p \to \mathbb{Z}/p^k \mathbb{Z} \) for all \( k \geq 1 \). We shall see in the main part of the article that the inverse limit definition of \( \mathbb{Z}_p \) is fundamental for our theoretical proofs, whereas representing elements of \( \mathbb{Z}_p \) by formal sums is more convenient for dealing with explicit examples.

We observe that Polak’s result, Theorem 1, does not depend on \( k \), and Hagedorn and Hatley’s formula in Theorem 2 converges to 1 as \( k \to \infty \). It therefore seems natural to speculate that the probability that a monic polynomial over \( \mathbb{Z}_p \) is separable should be \( 1 - p^{-1} \), whereas the probability that two monic polynomials of degree \( m \) and \( 2 \) are relatively prime over \( \mathbb{Z}_p \) should be 1.

In order to make sense of probabilities on polynomials over \( \mathbb{Z}_p \), in this article we consider the Haar measure on \( \mathbb{Z}_p \), which can be extended to the set of polynomials over \( \mathbb{Z}_p \) in a natural way. This is akin to [12], where the distribution of splitting types and Galois groups of polynomials over extensions of \( \mathbb{Z}_p \) are studied. Our first result is the following generalization of Theorem 1.

**Theorem 3.** Let \( p \) be a prime number and \( d \geq 2 \) an integer. With respect to the Haar measure, the probability that a degree \( d \) monic polynomial over \( \mathbb{Z}_p \) is separable is given by \( 1 - p^{-1} \).

This is Theorem 9 below. After our manuscript was first submitted to the journal, we learned that our result recovers the first part of [12, Theorem 1.1] when \( K_p = \mathbb{Q}_p \). In this article, we give two proofs of this theorem. The first one is based on discriminants, which were also used by Weiss in [12], and builds on results in [11] for \( \mathbb{Z}/p^k \mathbb{Z} \). The second proof is based on a lifting lemma for monic polynomials that we prove in Section 2. More specifically, we show that if \( f, g \in \mathbb{Z}_p[x] \) are monic, then the ideal generated by \( f \) and \( g \) equals \( \mathbb{Z}_p[x] \) if and only if their images under the canonical projection modulo \( p \) generate \( (\mathbb{Z}/p \mathbb{Z})[x] \). We make use of the fact that a monic polynomial \( f \) defined over a commutative ring \( R \) is separable if and only if \( (f, f') = R[x] \), where \( f' \) denotes the formal derivative of \( f \) (see [10, §1.4]). Therefore, our lemma allows us to translate the separability of a monic polynomial in \( \mathbb{Z}_p[x] \) to the separability of its image in \( (\mathbb{Z}/p \mathbb{Z})[x] \). We can then apply Carlitz’s result on separable polynomials over \( \mathbb{Z}/p \mathbb{Z} \) to obtain a new proof of Theorem 3 without using discriminants. The second method we present here is very different from the work of Weiss. It would be interesting to see whether our lifting technique can be generalized to give an alternative proof of Weiss’s result in the full generality. It would also be interesting to investigate whether our method can be used to recover other results in [12].
Our lifting lemma leads us to define the following new notion on polynomials. We say that two polynomials \( f \) and \( g \) defined over a commutative ring \( R \) are \textit{strongly coprime} if they generate \( R[x] \). This is a stronger condition than being relatively prime (having no common nonunit factor). Our lifting lemma allows us to prove the following theorem (which is Theorem 11 below).

**Theorem 4.** Let \( p \) be a prime number and \( d, e \geq 1 \) integers. With respect to the Haar measure, the probability that two polynomials \( f, g \in \mathbb{Z}_p[x] \) of degree \( d \) and \( e \), respectively, are strongly coprime is given by \( 1 - p^{-1} \).

Finally, we show in Theorem 13 that our method can be used to extrapolate the formulae of Theorem 2 to calculate the probability of monic polynomials over \( \mathbb{Z}_p[x] \) to be relatively prime.

**Theorem 5.** Let \( p \) be an odd prime number and \( m \geq 1 \) an integer. With respect to the Haar measure, the probability that two randomly chosen monic polynomials of degrees \( m \) and \( 2 \) in \( \mathbb{Z}_p[x] \) are relatively prime is equal to 1.

The methods used in our proofs are inspired by Hensel’s lemma, which says that the roots of a polynomial \( f \in \mathbb{Z}_p[x] \) can be found by lifting the roots of \( \tilde{f} \in (\mathbb{Z}/p\mathbb{Z})[x] \) recursively, where \( \tilde{f} \) is the polynomial obtained from \( f \) by projecting its coefficients from \( \mathbb{Z}_p \) to \( \mathbb{Z}/p\mathbb{Z} \). To quote Neal Koblitz in [9],

Hensel’s lemma is often called the \( p \)-adic Newton’s lemma because the approximation technique used to prove it is essentially the same as Newton’s method for finding a real root of a polynomial equation with real coefficients. . . . In one respect the \( p \)-adic Newton’s method is much better than Newton’s method in the real case. In the \( p \)-adic case, it’s guaranteed to converge to a root of a polynomial. In the real case, Newton’s method usually converges, but not always. . . . Such perverse silliness is impossible in \( \mathbb{Q}_p \).

2. Lifting of Polynomials from \( \mathbb{Z}/p^k\mathbb{Z} \) to \( \mathbb{Z}_p \).

We begin with the following lemma on Euclidean division for polynomials over a commutative ring (rather than a field).

**Lemma 6** (Division algorithm). Let \( R \) be a commutative ring and \( f, g \in R[x] \) such that the leading coefficient of \( g \) is a unit of \( R \). Then there exist \( q, r \in R[x] \) such that

\[
f = qg + r,
\]

where either \( r = 0 \) or \( \deg(r) < \deg(g) \).

**Proof.** Since the lemma is a simple generalization of the usual division algorithm for polynomials defined over a field, we only give a sketch of a proof here.

If \( \deg(f) < \deg(g) \), then we may simply take \( q = 0 \) and \( r = f \). So, we may assume that \( \deg(f) \geq \deg(g) \). Let \( i = \deg(f) - \deg(g) \geq 0 \) and write \( a \) and \( u \) for the leading coefficients of \( f \) and \( g \), respectively. Since \( u \) is a unit of \( R \), there exists \( b \in R \) such that \( a = ub \). Then the leading terms of both \( f \) and \( bX^i g \) are \( aX^{\deg(f)} \). In particular,

\[
\deg(f - bX^i g) < \deg(f).
\]
If the left-hand side is smaller than \( \deg(g) \), then we may take \( q = bX^i \) and \( r = f - bX^i g \) and we are done. Otherwise, we may repeat this procedure with \( f \) replaced by \( f - bX^i g \), which will produce another polynomial whose degree is strictly smaller than that of \( f - bX^i g \). We may keep on subtracting appropriate multiples of \( g \) and eventually this will produce a polynomial of degree strictly less than \( \deg(g) \) or the zero polynomial, as required. \( \square \)

If \( f \in \mathbb{Z}_p[x] \), we write \( \tilde{f}_k \) for its natural image in \( (\mathbb{Z}/p^k\mathbb{Z})[x] \) under the reduction map induced by \( \mathbb{Z}_p \rightarrow \mathbb{Z}/p^k\mathbb{Z} \). Note that if \( f \) is monic, then so is \( \tilde{f}_k \). Furthermore, \( \deg(f) = \deg(\tilde{f}_k) \).

**Theorem 7.** Let \( f, g \in \mathbb{Z}_p[x] \) be two monic polynomials of degree at least 1. Then the \( \mathbb{Z}_p[x] \)-ideal generated by \( f \) and \( g \) equals \( \mathbb{Z}_p[x] \) if and only if the \( (\mathbb{Z}/p\mathbb{Z})[x] \)-ideal generated by \( \tilde{f}_1 \) and \( \tilde{g}_1 \) equals \( (\mathbb{Z}/p\mathbb{Z})[x] \).

**Proof.** It suffices to show the following equivalence:

\[
\exists \alpha, \beta \in \mathbb{Z}_p[x], \alpha f + \beta g = 1 \iff \exists \alpha_0, \beta_0 \in (\mathbb{Z}/p\mathbb{Z})[x], \alpha_0 \tilde{f}_1 + \beta_0 \tilde{g}_1 = 1.
\]

(\( \Rightarrow \)) Given \( f, g, \alpha, \beta \in \mathbb{Z}_p[x] \) such that \( \alpha f + \beta g = 1 \), one can take \( \alpha_0 \) and \( \beta_0 \) to be \( \tilde{\alpha}_1 \) and \( \tilde{\beta}_1 \), respectively. Then we have trivially \( \alpha_0 \tilde{f}_1 + \beta_0 \tilde{g}_1 = 1 \).

(\( \Leftarrow \)) Given \( f, g \in \mathbb{Z}_p[x] \) and \( \alpha_0, \beta_0 \in (\mathbb{Z}/p\mathbb{Z})[x] \) such that

\[
\alpha_0 \tilde{f}_1 + \beta_0 \tilde{g}_1 = 1,
\]

we will construct a sequence \((r_i, s_i)_{i \in \mathbb{Z}_{>0}}\), where \( r_i, s_i \in (\mathbb{Z}/p^2\mathbb{Z})[x] \) such that \( \deg(r_i) < \deg(g) \), \( \deg(s_i) < \deg(f) \), and

\[
r_i \tilde{f}_{2^i} + s_i \tilde{g}_{2^i} = 1.
\]

We will construct this sequence by induction.

**Base case:** Since \( g \) is monic, so is \( \tilde{g}_1 \) and the division algorithm of Lemma 6 yields \( q_0, r_0 \in (\mathbb{Z}/p\mathbb{Z})[x] \) such that \( \deg(r_0) < \deg(g) \) and

\[
\alpha_0 = q_0 \tilde{g}_1 + r_0.
\]

We may rewrite (2) as

\[
r_0 \tilde{f}_1 + s_0 \tilde{g}_1 = 1,
\]

where \( s_0 = q_0 \tilde{f}_1 + \beta_0 \). Note that

\[
\deg(s_0) + \deg(g) = \deg(s_0) + \deg(\tilde{g}_1)
= \deg(s_0 \tilde{g}_1)
= \deg(r_0 \tilde{f}_1)
= \deg(r_0) + \deg(f) < \deg(g) + \deg(f).
\]

Hence, we deduce that \( \deg(s_0) < \deg(f) \) as required.

**Inductive hypothesis:** We assume that there exist \( r_i, s_i \in (\mathbb{Z}/p^{2^i}\mathbb{Z})[x] \) such that \( \deg(r_i) < \deg(g) \), \( \deg(s_i) < \deg(f) \), and

\[
r_i \tilde{f}_{2^i} + s_i \tilde{g}_{2^i} = 1.
\]
In particular, $Q$ on projecting to $\mathbb{Z}$ one can verify that, in $\mathbb{Z}$, this concludes the proof of (\Leftrightarrow). This proof gives a constructive algorithm for a linear combination in $\mathbb{Z}/p^{s+1}\mathbb{Z}$, respectively. This gives

$$\alpha_{i+1} \tilde{f}_{2^{i+1}} + \beta_{i+1} \tilde{g}_{2^{i+1}} = 1$$

inside $(\mathbb{Z}/p^{2^{i+1}}\mathbb{Z})[x]$. By the same argument as in the base case, we may replace $\alpha_{i+1}$ and $\beta_{i+1}$ by polynomials $r_{i+1}$ and $s_{i+1}$ such that $\deg(r_{i+1}) < \deg(g)$ and $\deg(s_{i+1}) < \deg(f)$ using the division algorithm of Lemma 6, as required.

Since $\mathbb{Z}_p$ is compact and $\deg(r_i)$ and $\deg(s_i)$ are of bounded degrees, $(r_i, s_i)$ admits a subsequence that converges to a pair of polynomials $(r_\infty, s_\infty)$ in $\mathbb{Z}_p[x]$ satisfying

$$r_\infty f + s_\infty g = 1.$$ 

This concludes the proof of (\Leftrightarrow).

**Remark 1.** Note that the same proof would go through if we replace $\mathbb{Z}/p\mathbb{Z}$ by $\mathbb{Z}/p^k\mathbb{Z}$ for an arbitrary integer $k \geq 1$. This proof gives a constructive algorithm for a linear combination in $\mathbb{Z}_p[x]$.

We give an explicit example that illustrates the inductive step of the algorithm.

**Example 1.** Take $p = 5$ and consider elements of $\mathbb{Z}_5$ as formal sums

$$c_0 + c_1 \cdot 5 + c_2 \cdot 5^2 + \cdots, \quad c_i \in \{0, 1, 2, 3, 4\}$$

as in (1). We consider the following monic polynomials

$$f := x^2 + (3 + 4 \cdot 5 + 2 \cdot 5^2 + \cdots)x + (2 + 3 \cdot 5 + 4 \cdot 5^2 + \cdots),$$

$$g := x + (4 + 2 \cdot 5 + 4 \cdot 5^2 + \cdots).$$

One can verify that, in $(\mathbb{Z}/5\mathbb{Z})[x]$,

$$f_1 + (4x + 1)g_1 = 1.$$ 

Taking the lifts $\tilde{r}_0$ and $\tilde{s}_0$ in $\mathbb{Z}_5[x]$ to be 1 and $4x + 1$ respectively, we have

$$\tilde{r}_0 f + \tilde{s}_0 g = (5 + 5^2 + \cdots)x + (1 + 5 + 4 \cdot 5^2 + \cdots)$$

$$= 1 + 5 \cdot (x^2 + (1 + 5 + \cdots)x + (1 + 4 \cdot 5 + \cdots)).$$

In particular, $Q_0$ is given by $x^2 + (1 + 5 + \cdots)x + (1 + 4 \cdot 5 + \cdots)$.

Multiplying $\tilde{r}_0$ and $\tilde{s}_0$ by $1 - 5 \cdot Q_0$ yields

$$(1 - 5 \cdot Q_0)\tilde{r}_0 = (4 \cdot 5 + \cdots)x^2 + (4 \cdot 5 + \cdots)x + (1 + 4 \cdot 5 + \cdots),$$

$$(1 - 5 \cdot Q_0)\tilde{s}_0 = (5 + \cdots)x^3 + (\cdots)x^2 + (4 + \cdots)x + (1 + 4 \cdot 5 + \cdots).$$

On projecting to $(\mathbb{Z}/5^2\mathbb{Z})[x]$, we obtain

$$\alpha_1 = (4 \cdot 5)x^2 + (4 \cdot 5)x + (1 + 4 \cdot 5), \quad \beta_1 = 5x^3 + 4x + (1 + 4 \cdot 5).$$
The division algorithm of Lemma 6 (with $R = \mathbb{Z}/5^2\mathbb{Z}$) gives
\[ \alpha_1 = ((4 \cdot 5)x + 3 \cdot 5) \tilde{g}_2 + 2 \cdot 5 + 1. \]

Therefore, we obtain
\[ r_1 = 2 \cdot 5 + 1, \quad s_1 = ((4 \cdot 5)x + 3 \cdot 5) \tilde{f}_2 + \beta_1 = (2 \cdot 5 + 4)x + 1. \]

### 3. Probability of separable polynomials over $\mathbb{Z}_p$.

We recall that if $R$ is a commutative ring, a polynomial $f \in R[x]$ is said to be **separable** if $R[x]/f$ is a separable $R$-algebra. When $f$ is monic, this is equivalent to
\[ (f, f') = R[x], \]
where $f'$ denotes the formal derivative of $f$ (see [10, §1.4]). Let us write $S^d_d R$ for the set of separable monic polynomials of degree $d$ in $R[x]$. Let $p$ be a fixed prime number. In [11], Polak showed that the proportion of separable degree $d$ monic polynomials in $(\mathbb{Z}/p^k \mathbb{Z})[x]$ is given by $1 - p^{-1}$ for all integers $k \geq 1$ and $d \geq 2$. In particular, we have
\[ \# S^d_d (\mathbb{Z}/p^k \mathbb{Z}) = p^d (1 - p^{-1}). \]

The goal of this section is to generalize Polak’s result to polynomials defined over the ring of $p$-adic integers $\mathbb{Z}_p$.

Let $P_d(\mathbb{Z}_p)$ denote the set of monic polynomials of degree $d$ defined over $\mathbb{Z}_p$. In particular, if $f = X^d + a_{d-1}X^{d-1} + \cdots + a_0 \in P_d(\mathbb{Z}_p)$, we may identify $f$ with the $d$-tuple $(a_0, \ldots, a_{d-1}) \in \mathbb{Z}_p^d$. We equip $P_d(\mathbb{Z}_p) = \mathbb{Z}_p^d$ with the product measure, denoted by $\mu^d_{\text{Haar}}$, coming from the unique Haar measure on $\mathbb{Z}_p$. When $d = 1$, we omit $d$ from the notation and simply write $\mu_{\text{Haar}}$.

**Lemma 8.** Let $f \in (\mathbb{Z}/p^k \mathbb{Z})[x]$ be a monic polynomial of degree $d$. Write $[f] \subset P_d(\mathbb{Z}_p)$ for the preimage of $f$ under the natural projection $\mathbb{Z}_p[x] \to (\mathbb{Z}/p^k \mathbb{Z})[x]$. Then
\[ \mu^d_{\text{Haar}} ([f]) = \frac{1}{p^{kd}}. \]

**Proof.** By definition, $\mu_{\text{Haar}}(a + p^k \mathbb{Z}_p) = \frac{1}{p^k}$ for all $a \in \mathbb{Z}_p$. Hence, the preimage of any element in $\mathbb{Z}/p^k \mathbb{Z}$ in $\mathbb{Z}_p$ has measure $\frac{1}{p^k}$. Hence the lemma follows by considering the preimage of each coefficient of $f$. \( \square \)

**Theorem 9.** Let $p$ be a prime number and $d \geq 2$ an integer. Then
\[ \mu^d_{\text{Haar}} (S^d_d (\mathbb{Z}_p)) = 1 - p^{-1}, \]
where $S^d_d (\mathbb{Z}_p)$ denotes the set of separable monic polynomials of degree $d$ in $\mathbb{Z}_p[x]$.

We give two proofs. The first one is an easy generalization of the proof given in [11] using discriminants, whereas the second one is based on Theorem 7.

**Proof 1.** Let us first recall from [11, Proposition 2.1] the following result.

**Proposition 10.** Let $R$ be a commutative ring with no nontrivial idempotents and let $f \in R[x]$ be a monic polynomial. Then $f$ is separable if and only if the discriminant $\text{disc}(f) \in R$ is a unit.
For a polynomial of a fixed degree, \( \text{disc}(f) \) can be realized as a polynomial in the coefficients of \( f \), which is independent of the ring \( R \). In particular, given any \( f \in \mathbb{Z}_p[x] \), we have
\[
\text{disc}(f) \equiv \text{disc}(\tilde{f}_1) \pmod{p}.
\]
In particular, \( f \) is separable in \( \mathbb{Z}_p[x] \) if and only if \( \tilde{f}_1 \) is separable in \( (\mathbb{Z}/p\mathbb{Z})[x] \). Therefore,
\[
S^d_{\mathbb{Z}_p} = \bigsqcup_{f \in S^d_{\mathbb{Z}/p\mathbb{Z}}} [f],
\]
where \([f]\) denotes the preimage of \( f \) in \( \mathbb{Z}_p[x] \) under the natural projection and \( \bigsqcup \) denotes the disjoint union of sets. By Lemma 8 and (4), we deduce that
\[
\mu^d_{\text{Haar}}(S^d_{\mathbb{Z}_p}) = \mu^d_{\text{Haar}}(\bigsqcup_{f \in S^d_{\mathbb{Z}/p\mathbb{Z}}} [f]) = \frac{\#S^d_{\mathbb{Z}/p\mathbb{Z}}}{p^d} = \frac{p^d(1 - p^{-1})}{p^d} = 1 - p^{-1},
\]
as required. \( \square \)

Proof 2. Let \( f \in \mathbb{Z}_p[x] \). Recall that its natural image in \( (\mathbb{Z}/p\mathbb{Z})[x] \) is denoted by \( \tilde{f}_1 \). Theorem 7 tells us that
\[
(f, f') = \mathbb{Z}_p[x] \iff (\tilde{f}_1, \tilde{f}'_1) = (\mathbb{Z}/p\mathbb{Z})[x].
\]
Thus, by the criterion of separability given in (3), \( f \) is separable in \( \mathbb{Z}_p[x] \) if and only if \( \tilde{f}_1 \) is separable in \( (\mathbb{Z}/p\mathbb{Z})[x] \). As in Proof 1, we may now deduce Theorem 9 from the fact that \( \#S^d_{\mathbb{Z}/p\mathbb{Z}} = p(1 - p^{-1}) \) as given by (4), which is a special case of the main result of [11] (see Theorem 1). \( \square \)

4. Probability of strongly coprime polynomials over \( \mathbb{Z}_p \).

If \( K \) is a field, then a polynomial \( f \in K[x] \) is separable if and only if
\[
(f, f') = K[x].
\]
There is a similar notion for relatively prime polynomials. That is, if \( f, g \in K[x] \), then \( f \) and \( g \) are relatively prime if and only if
\[
(f, g) = K[x].
\]

However, if we replace \( K \) by a commutative ring \( R \), then it is possible for two relatively prime polynomials \( f, g \in R[x] \) to be such that \( f \) and \( g \) generate a proper ideal of \( R[x] \). We define the following new notion that allows us to extrapolate information on relatively prime polynomials in \( (\mathbb{Z}/p\mathbb{Z})[x] \) to polynomials in \( \mathbb{Z}_p[x] \).
Definition. Let $R$ be a commutative ring. We say that two polynomials $f, g \in R[x]$ are strongly coprime if $(f, g) = R[x]$.

Theorem 11. Let $p$ be a prime number and $d, e \geq 1$ integers. With respect to the Haar measures on $P_d(Z_p)$ and $P_e(Z_p)$, the probability that two random polynomials in $P_d(Z_p)$ and $P_e(Z_p)$ are strongly coprime is given by $1 - p^{-1}$.

Proof. Given a ring $R$, let us write $\mathcal{R}^{d,e}_{R}$ for the set of pairs of relatively prime polynomials $(f, g)$, where $f, g \in R[x]$ such that $\deg(f) = d$ and $\deg(g) = e$.

Recall from Theorem 7 that $(f, g) = Z_p[x] \iff (\tilde{f}_1, \tilde{g}_1) = (Z/pZ)[x]$. In other words, $f$ and $g$ are strongly coprime if and only if their projections are. Hence, $\mathcal{R}^{d,e}_{Z_p} = \bigcup_{(f,g) \in \mathcal{R}^{d,e}_{Z/pZ}} [f] \times [g]$.

Let $\mu_{\text{Haar}}^{d,e} := \mu_{\text{Haar}}^{d} \times \mu_{\text{Haar}}^{e}$ be the product measure on $P_d(Z_p) \times P_e(Z_p)$. Then, Lemma 8 tells us that

$$\mu_{\text{Haar}}^{d,e}(\mathcal{R}^{d,e}_{Z_p}) = \sum_{(f,g) \in \mathcal{R}^{d,e}_{Z/pZ}} \mu_{\text{Haar}}^{d,e}([f] \times [g]) = \frac{\# \mathcal{R}^{d,e}_{Z/pZ}}{p^{d+e}}.$$

It was proved in [2] that $\# \mathcal{R}^{d,e}_{Z/pZ} = p^{d+e}(1 - p^{-1})$, hence the result follows. \hfill \Box

Note that we may deduce a similar statement for polynomials defined over $Z/p^kZ$ for any integers $k \geq 1$.

Corollary 12. Let $p$ be a prime number and $d, e, k \geq 1$ integers. The probability that two monic polynomials of degree $d$ and $e$ in $Z/p^kZ[x]$ are strongly coprime is given by $1 - p^{-1}$.

Proof. As explained in Remark 1, we have the equivalence $(f, g) = Z_p[x] \iff (f_1, g_1) = (Z/p^kZ)[x] \iff (\tilde{f}_1, \tilde{g}_1) = (Z/pZ)[x]$. Hence, the calculation in the proof of Theorem 11 goes through if we replace $Z_p$ by $Z/p^kZ$. \hfill \Box

5. Probability of relatively prime polynomials over $Z_p$.

For a polynomial over $Z_p$, being square-free is weaker than being separable. In fact, as remarked in [12], the set of square-free polynomials over $Z_p$ has full Haar measure, whereas that of separable polynomials has measure $1 - p^{-1}$ by Theorem 9. Similarly, two polynomials over $Z_p$ being relatively prime polynomials (i.e., having no common nonunit factor) is weaker than being strongly coprime. In this section we illustrate how a similar calculation to what we made in the previous sections
allows us to extrapolate Hagedorn and Hatley’s formulae in Theorem 2 to a result on relatively prime polynomials over $Z_p$.

**Theorem 13.** Let $p$ be an odd prime number and $m \geq 1$ an integer. With respect to the Haar measure, the probability that two randomly chosen monic polynomials in $P_m(Z_p)$ and $P_2(Z_p)$ are relatively prime is 1.

**Proof.** Let $f, g \in Z_p[x]$. Observe that if $\tilde{f}_k$ and $\tilde{g}_k$ are relatively prime as polynomials in $(Z/p^kZ)[x]$, then $f$ and $g$ are also relatively prime since any common nonunit factors between them would give common nonunit factors of $\tilde{f}_k$ and $\tilde{g}_k$.

For $R = Z_p$ or $Z/p^kZ$, let us write $R_{mZ}$ for the set of relatively prime polynomials $(f, g)$ with $\deg(f) = m$ and $\deg(g) = 2$. Recall that, by our notation, the measure on $R_{mZ}$ will be $\mu_{Haar}^{m,2}$. Then our observation above tells us that

$$R_{mZ_p} \supset \bigsqcup_{(f, g) \in R^m_{Z/p^kZ}} [f] \times [g].$$

Therefore, on applying Lemma 8 and Theorem 2, we obtain the following inequality:

$$\mu_{Haar}^{m,2} \left( R_{mZ_p} \right) \geq \mu_{Haar}^{m,2} \left( \bigsqcup_{(f, g) \in R^m_{Z/p^kZ}} [f] \times [g] \right)$$

$$= \frac{\# R^m_{Z/p^kZ}}{p^{m(2+3k)}}$$

$$= 1 - \frac{f_k(p)}{p^{3k}}.$$

From [7, p. 224] we have that the polynomial $f_k$ is of degree $2k$ and its coefficients have absolute value at most 2. Therefore,

$$1 - \frac{f_k(p)}{p^{3k}} \to 1, \quad \text{as } k \to \infty.$$

But $\mu_{Haar}^{m,2} \left( R_{mZ_p} \right)$ is at most 1 by definition. This forces

$$\mu_{Haar}^{m,2} \left( R_{mZ_p} \right) = 1,$$

as required. \qed

To conclude, we would like to outline a number of future problems that may be studied using our method.

(i) We may calculate the probability of two polynomials over $Z_p$ of any degrees to be relatively prime by estimating the number of polynomials over $Z/p^kZ$ of the same degrees that are relatively prime.

(ii) On generalizing techniques of [7], we may generalize results of [2, 4, 6] to multiple, potentially multivariate, polynomials over $Z/p^kZ$ (instead of fields). Once this is achieved, we may then apply our method to study multiple, potentially multivariate, polynomials over $Z_p$.

(iii) We may, as in [12], consider the ring of integers of a finite extension of $Q_p$ and generalize our results on coprime and strongly coprime polynomials.
(iv) We may give a new proof of Weiss’s formula on the distribution of splitting types of irreducible polynomials defined over the ring of integers of a finite extension of \( \mathbb{Q}_p \).

(v) It would also be interesting to investigate whether we may refine the different estimates on the distribution of Galois groups of irreducible polynomials in [12, Theorems 1.2–1.6] using our lifting technique.

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