MAPS WITH DIMENSIONALLY RESTRICTED FIBERS

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Abstract. We prove that if \( f : X \rightarrow Y \) is a closed surjective map between metric spaces such that every fiber \( f^{-1}(y) \) belongs to a class of space \( S \), then there exists an \( F_{\sigma} \)-set \( A \subset X \) such that \( A \in S \) and \( \dim f^{-1}(y) \setminus A = 0 \) for all \( y \in Y \). Here, \( S \) can be one of the following classes: (i) \( \{ M : e - \dim M \leq K \} \) for some CW-complex \( K \); (ii) \( C \)-spaces; (iii) weakly infinite-dimensional spaces. We also establish that if \( S = \{ M : \dim M \leq n \} \), then \( \dim f \triangle g \leq 0 \) for almost all \( g \in C(X, I^{n+1}) \).

1. Introduction

All spaces in the paper are assumed to be paracompact and all maps continuous. By \( C(X, M) \) we denote all maps from \( X \) into \( M \). Unless stated otherwise, all function spaces are endowed with the source limitation topology provided \( M \) is a metric space.

The paper is inspired by the results of Pasynkov [11], Torunczyk [15], Sternfeld [14] and Levin [8]. Pasynkov announced in [11] and proved in [12] that if \( f : X \rightarrow Y \) is a surjective map with \( \dim f \leq n \), where \( X \) and \( Y \) are finite-dimensional metric compacta, then \( \dim f \triangle g \leq 0 \) for almost all maps \( g \in C(X, I^n) \) (see [10] for a non-compact version of this result). Torunczyk [15] established (in a more general setting) that if \( f, X \) and \( Y \) are as in Pasynkov’s theorem, then for each \( 0 \leq k \leq n - 1 \) there exists a \( \sigma \)-compact subset \( A_k \subset X \) such that \( \dim A_k \leq k \) and \( \dim f|\langle X \setminus A_k \rangle \leq n - k - 1 \).

Next results in this direction were established by Sternfeld and Levin. Sternfeld [14] proved that if in the cited above results \( Y \) is not-necessarily finite-dimensional, then \( \dim f \triangle g \leq 1 \) for almost all \( g \in C(X, I^n) \) and there exists a \( \sigma \)-compact subset \( A \subset X \) such that \( \dim A \leq n - 1 \) and \( \dim f|\langle X \setminus A \rangle \leq 1 \). Levin [8] improved Sternfeld’s results by showing that \( \dim f \triangle g \leq 0 \) for almost all \( g \in C(X, I^{n+1}) \), and has shown that

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this is equivalent to the existence of an $n$-dimensional $\sigma$-compact subset $A \subset X$ with $\dim f| (X \setminus A) \leq 0$.

The above results of Pasynkov and Torunczyk were generalized in [17] for closed maps between metric space $X$ and $Y$ with $Y$ being a $C$-space (recall that each finite-dimensional paracompact is a $C$-space [6]). But the question whether the results of Pasynkov and Torunczyk remain valid without the finite-dimensionality assumption on $Y$ is still open.

In this paper we provide non-compact analogues of Levin’s results for closed maps between metric spaces.

We say that a topological property of metrizable spaces is an $S$-property if the following conditions are satisfied: (i) $S$ is hereditary with respect to closed subsets; (ii) if $X$ is metrizable and $\{H_i\}_{i=1}^{\infty}$ is a sequence of closed $S$-subsets of $X$, then $\bigcup_{i=1}^{\infty} H_i \in S$; (iii) a metrizable space $X \in S$ provided there exists a closed surjective map $f: X \to Y$ such that $Y$ is a 0-dimensional metrizable space and $f^{-1}(y) \in S$ for all $y \in Y$; (iv) any discrete union of $S$-spaces is an $S$-space.

Any map whose fibers have a given $S$-property is called an $S$-map.

Here are some examples of $S$-properties (we identify $S$ with the class of spaces having the property $S$):

- $S = \{X : \dim X \leq n\}$ for some $n \geq 0$;
- $S = \{X : \dim_G X \leq n\}$, where $G$ is an Abelian group and $\dim_G$ is the cohomological dimension;
- more generally, $S = \{X : e - \dim X \leq K\}$, where $K$ is a CW-complex and $e - \dim$ is the extension dimension, see [4], [5];
- $S = \{X : X$ is weakly infinite-dimensional$\}$;
- $S = \{X : X$ is a $C$-space$\}$.

To show that the property $e - \dim \leq K$ satisfies condition (iii), we apply [3, Corollary 2.5]. For weakly infinite-dimensional spaces and $C$-spaces this follows from [7].

**Theorem 1.1.** Let $f: X \to Y$ be a closed surjective $S$-map with $X$ and $Y$ being metrizable spaces. Then there exists an $F_{\sigma}$-subset $A \subset X$ such that $A \in S$ and $\dim f^{-1}(y) \setminus A = 0$ for all $y \in Y$. Moreover, if $f$ is a perfect map, the conclusion remains true provided $S$ is a property satisfying conditions (i) – (iii).

Theorem 1.1 was established by Levin [9, Theorem 1.2] in the case $X$ and $Y$ are metric compacta and $S$ is the property $e - \dim \leq K$ for a given CW-complex $K$. Levin’s proof of this theorem remains valid for arbitrary $S$-property, but it doesn’t work for non-compact spaces.

We say that a map $f: X \to Y$ has a countable functional weight (notation $W(f) \leq \aleph_0$), see [10]) if there exists a map $g: X \to \prod^{\aleph_0}$
such that \( f \triangle g : X \to Y \times \mathbb{I}^{\aleph_0} \) is an embedding. For example [12, Proposition 9.1], \( W(f) \leq \aleph_0 \) for any closed map \( f : X \to Y \) such that \( X \) is a metrizable space and every fiber \( f^{-1}(y) \), \( y \in Y \), is separable.

**Theorem 1.2.** Let \( X \) and \( Y \) be paracompact spaces and \( f : X \to Y \) a closed surjective map with \( \dim f \leq n \) and \( W(f) \leq \aleph_0 \). Then \( C(X, \mathbb{I}^{n+1}) \) equipped with the uniform convergence topology contains a dense subset of maps \( g \) such that \( \dim f \triangle g \leq 0 \).

It was mentioned above that this corollary was established by Levin [8, Theorem 1.6] for metric compacta \( X \) and \( Y \). Levin’s arguments don’t work for non-compact spaces. We are using the Pasynkov’s technique from [10] to reduced the proof of Theorem 1.2 to the case of \( X \) and \( Y \) being metric compacta.

Our last results concern the function spaces \( C(X, \mathbb{I}^n) \) and \( C(X, \mathbb{I}^{\aleph_0}) \) equipped with the source limitation topology. Recall that this topology on \( C(X, M) \) with \( M \) being a metrizable space can be described as follows: the neighborhood base at a given map \( h \in C(X, M) \) consists of the sets \( B_\rho(h, \epsilon) = \{ g \in C(X, M) : \rho(g, h) < \epsilon \} \), where \( \rho \) is a fixed compatible metric on \( M \) and \( \epsilon : X \to (0,1] \) runs over continuous positive functions on \( X \). The symbol \( \rho(h(x), g(x)) < \epsilon(x) \) for all \( x \in X \). It is well known that for paracompact spaces \( X \) this topology doesn’t depend on the metric \( \rho \) and it has the Baire property provided \( M \) is completely metrizable.

**Theorem 1.3.** Let \( f : X \to Y \) be a perfect surjection between paracompact spaces and \( W(f) \leq \aleph_0 \).

(i) The maps \( g \in C(X, \mathbb{I}^{\aleph_0}) \) such that \( f \triangle g \) embeds \( X \) into \( Y \times \mathbb{I}^{\aleph_0} \) form a dense \( G_\delta \)-set in \( C(X, \mathbb{I}^{\aleph_0}) \) with respect to the source limitation topology;

(ii) If there exists a map \( g \in C(X, \mathbb{I}^n) \) with \( \dim f \triangle g \leq 0 \), then all maps having this property form a dense \( G_\delta \)-set in \( C(X, \mathbb{I}^n) \) with respect to the source limitation topology.

**Corollary 1.4.** Let \( f : X \to Y \) be a perfect surjection with \( \dim f \leq n \) and \( W(f) \leq \aleph_0 \), where \( X \) and \( Y \) are paracompact spaces. Then all maps \( g \in C(X, \mathbb{I}^{n+1}) \) with \( \dim f \triangle g \leq 0 \) form a dense \( G_\delta \)-set in \( C(X, \mathbb{I}^{n+1}) \) with respect to the source limitation topology.

Corollary 1.4 follows directly from Theorem 1.2 and Theorem 1.3(ii). Corollary 1.5 below follows from Corollary 1.4 and [2, Corollary 1.1], see Section 3.

**Corollary 1.5.** Let \( X, Y \) be paracompact spaces and \( f : X \to Y \) a perfect surjection with \( \dim f \leq n \) and \( W(f) \leq \aleph_0 \). Then for every metrizable ANR-space \( M \) the maps \( g \in C(X, \mathbb{I}^{n+1} \times M) \) such
that \( \dim g(f^{-1}(y)) \leq n + 1 \) for all \( y \in Y \) form a dense \( G_\delta \)-set \( E \) in \( C(X, \mathbb{I}^{n+1} \times M) \) with respect to the source limitation topology.

Finally, let us formulate the following question concerning property S (an affirmative answer of this question yields that (strong) countable-dimensionality is an S-property):

**Question 1.6.** Suppose \( f : X \to Y \) is a perfect surjection between metrizable spaces such that \( \dim Y = 0 \) and each fiber \( f^{-1}(y) \), \( y \in Y \), is (strongly) countable-dimensional. Is it true that \( X \) is (strongly) countable-dimensional?

2. S-properties and maps into finite-dimensional cubes

This section contains the proofs of Theorem 1.1 and Theorem 1.2.

**Proof of Theorem 1.1.** We follow the proof of [18, Proposition 4.1]. Let us show first that the proof is reduced to the case \( f \) is a perfect map. Indeed, according to Vainstein’s lemma, the boundary \( \text{Fr} f^{-1}(y) \) of every fiber \( f^{-1}(y) \) is compact. Defining \( F(y) \) to be \( \text{Fr} f^{-1}(y) \) if \( \text{Fr} f^{-1}(y) \neq \emptyset \), and an arbitrary point from \( f^{-1}(y) \) otherwise, we obtain a set \( X_0 = \bigcup \{ F(y) : y \in Y \} \) such that \( X_0 \subseteq X \) is closed and the restriction \( f|X_0 \) is a perfect map. Moreover, each \( f^{-1}(y) \setminus X_0 \) is open in \( X \) and has the property S (as an \( F_\sigma \)-subset of the S-space \( f^{-1}(y) \)). Hence, \( X \setminus X_0 \) being the union of the discrete family \( \{ f^{-1}(y) \setminus X_0 : y \in Y \} \) of S-set is an S-set. At the same time \( X \setminus X_0 \) is open in \( X \). Consequently, \( X \setminus X_0 \) is the union of countably many closed sets \( X_i \subseteq X \), \( i = 1, 2, \ldots \) Obviously, each \( X_i \), \( i \geq 1 \), also has the property S. Therefore, it suffices to prove Theorem 1.1 for the S-map \( f|X_0 : X_0 \to Y \).

So, we may suppose that \( f \) is perfect. According to [10], there exists a map \( g : X \to \mathbb{I}^{\omega_0} \) such that \( g \) embeds every fiber \( f^{-1}(y) \), \( y \in Y \). Let \( g = \Delta_{i=1}^\infty g_i \) and \( h_i = f\Delta g_i : X \to Y \times I_i \), \( i \geq 1 \). Moreover, we choose countably many closed intervals \( I_i \) such that every open subset of \( I \) contains some \( I_i \). By [17, Lemma 4.1], for every \( j \) there exists a 0-dimensional \( F_\sigma \)-set \( C_j \subseteq Y \times I_j \) such that \( C_j \cap (\{y\} \times I_j) \neq \emptyset \) for every \( y \in Y \). Now, consider the sets \( A_{ij} = h_i^{-1}(C_j) \) for all \( i, j \geq 1 \) and let \( A \) be their union. Since \( f \) is an S-map, so is the map \( h_i \) for any \( i \). Hence, \( A_{ij} \) has property S for all \( i, j \). This implies that \( A \) has also the same property.

It remains to show that \( \dim f^{-1}(y) \setminus A \leq 0 \) for every \( y \in Y \). Let \( \dim f^{-1}(y_0) \setminus A > 0 \) for some \( y_0 \). Since \( g|f^{-1}(y_0) \) is an embedding, there exists an integer \( i \) such that \( \dim g_i(f^{-1}(y_0) \setminus A) > 0 \). Then \( g_i(f^{-1}(y_0) \setminus A) \) has a nonempty interior in \( I \). So, \( g_i(f^{-1}(y_0) \setminus A) \) contains some \( I_j \). Choose \( t_0 \in I_j \) with \( c_0 = (y_0, t_0) \in C_j \). Then there exists \( x_0 \in \)}
onto small modification of \[10, \text{Theorem 8.1}\]. For any map \(f \times \) is easily seen that the formula

\[
d\text{ence between } C \text{ between compact spaces with } n > \]

We fix a map \(f \) maps \(g \) such that \(f = \pi_n \circ g \), where \(\pi_n : Y \times \mathbb{I}^{n+1} \to Y \) is the projection onto \(Y \). We also consider the other projection \(\varpi_n : Y \times \mathbb{I}^{n+1} \to \mathbb{I}^{n+1} \). It is easily seen that the formula \(g \to \varpi_n \circ g \) provides one-to-one correspondence between \(C(X, Y \times \mathbb{I}^{n+1}, f)\) and \(C(X, \mathbb{I}^{n+1})\). So, we may assume that \(C(X, Y \times \mathbb{I}^{n+1}, f)\) is a metric space isometric with \(C(X, \mathbb{I}^{n+1})\), where \(C(X, \mathbb{I}^{n+1})\) is equipped with the supremum metric.

**Proposition 2.1.** Let \(f : X \to Y\) be an \(n\)-dimensional surjective map between compact spaces with \(n > 0\) and \(\lambda : X \to Z\) a map into a metric compactum \(Z\). Then the maps \(g \in C(X, Y \times \mathbb{I}^{n+1}, f)\) satisfying the condition below form a dense subset of \(C(X, Y \times \mathbb{I}^{n+1}, f)\): there exists a compact space \(H\) and maps \(\varphi : X \to H\), \(h : H \to Y \times \mathbb{I}^{n+1}\) and \(\mu : H \to Z\) such that \(\lambda = \mu \circ \varphi\), \(g = h \circ \varphi\), \(W(h) \leq \aleph_0\) and \(\dim h = 0\).

**Proof.** We fix a map \(g_0 \in C(X, Y \times \mathbb{I}^{n+1}, f)\) and \(\epsilon > 0\). Let \(g_1 = \varpi_n \circ g_0\). Then \(\lambda \Delta g_1 \in C(X, Z \times \mathbb{I}^{n+1})\). Consider also the constant maps \(f' : Z \times \mathbb{I}^{n+1} \to Pt\) and \(\eta : Y \to Pt\), where \(Pt\) is the one-point space. So, we have \(\eta \circ f = f' \circ (\lambda \Delta g_1)\). According to Pasynkov’s factorization theorem [13, Theorem 13], there exist metrizable compacta \(K, T\) and maps \(f^* : K \to T\), \(\xi_1 : X \to K\), \(\xi_2 : K \to Z \times \mathbb{I}^{n+1}\) and \(\eta^* : Y \to T\) such that:

- \(\eta^* \circ f = f^* \circ \xi_1\);
- \(\xi_2 \circ \xi_1 = \lambda \Delta g_1\);
- \(\dim f^* \leq \dim f \leq n\).

If \(p : Z \times \mathbb{I}^{n+1} \to Z\) and \(q : Z \times \mathbb{I}^{n+1} \to \mathbb{I}^{n+1}\) denote the corresponding projections, we have

\[
p \circ \xi_2 \circ \xi_1 = \lambda \text{ and } q \circ \xi_2 \circ \xi_1 = g_1.
\]

Since \(\dim f^* \leq n\), by Levin’s result [8, Theorem 1.6], there exists a map \(\phi : K \to \mathbb{I}^{n+1}\) such that \(\phi\) is \(\epsilon\)-close to \(q \circ \xi_2\) and \(\dim f^* \Delta \phi \leq 0\). Then the map \(\phi \circ \xi_1\) is \(\epsilon\)-close to \(g_1\), so \(g = f \Delta (\phi \circ \xi_1)\) is \(\epsilon\)-close to \(g_0\).

Denote \(\varphi = f \Delta \xi_1\), \(H = \varphi(X)\) and \(h = (id_Y \times \phi)|H\). If \(\varpi_H : H \to K\) is the restriction of the projection \(Y \times K\) to \(K\) on \(H\), we have

\[
\lambda = p \circ \xi_2 \circ \xi_1 = p \circ \xi_2 \circ \varpi_H \circ \varphi, \text{ so } \lambda = \mu \circ \varphi, \text{ where } \mu = p \circ \xi_2 \circ \varpi_H.
\]

Moreover, \(g = f \Delta (\phi \circ \xi_1) = (id_Y \times \phi) \circ (f \Delta \xi_1) = h \circ \varphi\). Since \(K\) is a metrizable compactum, \(W(\phi) \leq \aleph_0\). Hence, \(W(h) \leq \aleph_0\).
To show that \( \dim h \leq 0 \), it suffices to prove that \( \dim h \leq \dim f^* \Delta \phi \).

To this end, we show that any fiber \( h^{-1}((y,v)) \), where \( (y,v) \in Y \times \Pi^{n+1} \), is homeomorphic to a subset of the fiber \( (f^* \Delta \phi)^{-1}(\eta^*(y,v)) \). Indeed, let \( \pi_Y \) be the restriction of the projection \( Y \times K \to Y \) on the set \( H \). Since \( \eta^* \circ f = f^* \circ \xi_1 \), \( H \) is a subset of the pullback of \( Y \) and \( K \) with respect to the maps \( \eta^* \) and \( f^* \). Therefore, \( \omega_H \) embeds every fiber \( \pi_Y^{-1}(y) \) into \( (f^*)^{-1}(y) \), \( y \in Y \). Let \( a_i = (y_i,k_i) \in H \subset Y \times K \), \( i = 1,2 \), such that \( h(a_1) = h(a_2) \). Then \( (y_1,\phi(k_1)) = (y_2,\phi(k_2)) \), so \( y_1 = y_2 = y \) and \( \phi(k_1) = \phi(k_2) = v \). This implies \( \omega_H(a_i) = k_i \in (f^*)^{-1}(\eta^*(\pi_Y(a_i))) = (f^*)^{-1}(\eta^*(y)), i = 1,2 \). Hence, \( \omega_H \) embeds the fiber \( h^{-1}((y,v)) \) into the fiber \( (f^* \Delta \phi)^{-1}(\eta^*(y,v)) \). Consequently, \( \dim h \leq \dim f^* \Delta \phi = 0 \). \( \square \)

We can prove now Theorem 1.2. It suffices to show every map from \( C(X,Y \times \Pi^{n+1},f) \) can be approximated by maps \( g \in C(X,Y \times \Pi^{n+1},f) \) with \( \dim g \leq 0 \). We fix \( g_0 \in C(X,Y \times \Pi^{n+1},f) \) and \( \epsilon > 0 \). Since \( W(f) \leq \aleph_0 \), there exists a map \( \lambda: X \to \Pi^{\aleph_0} \) such that \( f \Delta \lambda \) is an embedding. Let \( \beta f: \beta X \to \beta Y \) be the Čech-Stone extension of the map \( f \). Then \( \dim \beta f \leq n \), see [13, Theorem 15]. Consider also the maps \( \beta \lambda: \beta X \to \Pi^{\aleph_0} \) and \( \bar{g}_0 = \beta f \Delta g_1 \), where \( g_1 = \omega_n \circ g_0 \). According to Proposition 2.1, there exists a map \( \bar{g} \in C(\beta X, \beta Y \times \Pi^{n+1}, \beta f) \) which is \( \epsilon \)-close to \( \bar{g}_0 \) and satisfies the following conditions: there exists a compact space \( H \) and maps \( \varphi: \beta X \to H \), \( h: H \to \beta Y \times \Pi^{n+1} \) and \( \mu: H \to \Pi^{\aleph_0} \) such that \( \beta \lambda = \mu \circ \varphi, \bar{g} = h \circ \varphi, W(h) \leq \aleph_0 \) and \( \dim h = 0 \). We have the following equalities

\[
\beta f \Delta \beta \lambda = (\pi_n \circ \bar{g}) \Delta (\mu \circ \varphi) = (\pi_n \circ h \circ \varphi) \Delta (\mu \circ \varphi) = ((\pi_n \circ h) \Delta \mu) \circ \varphi,
\]

where \( \pi_n \) denotes the projection \( \beta Y \times \Pi^{n+1} \to \beta Y \). This implies that \( \varphi \) embeds \( X \) into \( H \) because \( f \Delta \lambda \) embeds \( X \) into \( Y \times \Pi^{\aleph_0} \). Let \( g \) be the restriction of \( \bar{g} \) over \( X \). Identifying \( X \) with \( \varphi(X) \), we obtain that \( h \) is an extension of \( g \). Hence, \( \dim g \leq \dim h = 0 \). Observe also that \( g \) is \( \epsilon \)-close to \( g_0 \), which completes the proof. \( \square \)

3. Proof of Theorem 1.3 and Corollary 1.5

Proof of Theorem 1.3(ii). We first prove condition (ii). Since \( W(f) \leq \aleph_0 \), there exists a map \( \lambda: X \to \Pi^{\aleph_0} \) such that \( f \Delta \lambda \) embeds \( X \) into \( Y \times \Pi^{\aleph_0} \). Choose a sequence \( \{ \gamma_k \}_{k \geq 1} \) of open covers of \( \Pi^{\aleph_0} \) with \( \text{mesh}(\gamma_k) \leq 1/k \), and let \( \omega_k = \lambda^{-1}(\gamma_k) \) for all \( k \). We denote by \( C(\omega_k,0)(X,\Pi^n,f) \) the set of all maps \( g \in C(X,\Pi^n) \) with the following property: every \( z \in (f \Delta g)(X) \) has a neighborhood \( V_z \) in \( Y \times \Pi^n \) such that \( (f \Delta g)^{-1}(V_z) \) can be represented as the union of a disjoint open in \( X \) family refining the cover \( \omega_k \). According to [17, Lemma 2.5], each of the sets
$C_{(\omega_k,0)}(X,\mathbb{I}^n, f), k \geq 1,$ is open in $C(X, \mathbb{I}^n)$ with respect to the source limitation topology. It follows from the definition of the covers $\omega_k$ that $\bigcap_{k \geq 1} C_{(\omega_k,0)}(X, \mathbb{I}^n, f)$ consists of maps $g$ with $\dim f \Delta g \leq 0$. Since $C(X, \mathbb{I}^n)$ with the source limitation topology has the Baire property, it remains to show that any $C_{(\omega_k,0)}(X, \mathbb{I}^n, f)$ is dense in $C(X, \mathbb{I}^n)$.

To this end, we fix a cover $\omega_m$, a map $g_0 \in C_{(\omega_m,0)}(X, \mathbb{I}^n, f)$ and a function $\epsilon: X \to (0, 1]$. We are going to find $h \in C_{(\omega_m,0)}(X, \mathbb{I}^n, f)$ such that $\rho(g_0(x), h(x)) < \epsilon(x)$ for all $x \in X$, where $\rho$ is the Euclidean metric on $\mathbb{I}^n$. Then, by [1, Lemma 8.1], there exists an open cover $U$ of $X$ satisfying the following condition: if $\alpha: X \to K$ is a $U$-map into a paracompact space $K$ (i.e., $\alpha^{-1}(\omega)$ refines $U$ for some open cover $\omega$ of $K$), then there exists a map $q: G \to \mathbb{I}^n$, where $G$ is an open neighborhood of $\alpha(X)$ in $K$, such that $g_0$ and $q \circ \alpha$ are $\epsilon/2$-close with respect to the metric $\rho$. Let $U_1$ be an open cover of $X$ refining both $U$ and $\omega_m$ such that $\inf \{\epsilon(x) : x \in U\} > 0$ for all $U \in U_1$.

Since $\dim f \Delta g \leq 0$ for some $g \in C(X, \mathbb{I}^n)$, according to [1, Theorem 6] there exists an open cover $V$ of $Y$ such that for any $V$-map $\beta: Y \to L$ into a simplicial complex $L$ we can find a $U_1$-map $\alpha: X \to K$ into a simplicial complex $K$ and a perfect $PL$-map $p: K \to L$ with $\beta \circ f = p \circ \alpha$ and $\dim p \leq n$. We can assume that $V$ is locally finite. Take $L$ to be the nerve of the cover $V$ and $\beta: Y \to L$ the corresponding natural map. Then there exist a simplicial complex $K$ and maps $p$ and $\alpha$ satisfying the above conditions. Hence, the following diagram is commutative.

$$
\begin{array}{ccc}
X & \xrightarrow{\alpha} & K \\
\downarrow f & & \downarrow p \\
Y & \xrightarrow{\beta} & L
\end{array}
$$

Since $K$ is paracompact, the choice of the cover $U$ guarantees the existence of a map $\varphi: G \to \mathbb{I}^n$, where $G \subset K$ is an open neighborhood of $\overline{\alpha(X)}$, such that $g_0$ and $h_0 = \varphi \circ \alpha$ are $\epsilon/2$-close with respect to $\rho$. Replacing the triangulation of $K$ by a suitable subdivision, we may additionally assume that no simplex of $K$ meets both $\alpha(X)$ and $K \setminus G$. So, the union $N$ of all simplexes $\sigma \in K$ with $\sigma \cap \overline{\alpha(X)} \neq \emptyset$ is a subcomplex of $K$ and $N \subset G$. Moreover, since $N$ is closed in $K$, $p_N = p|N: N \to L$ is a perfect map. Therefore, we have the following commutative diagram:
Using that $\alpha$ is a $U_1$-map and $\inf \{ \epsilon(x) : x \in U \} > 0$ for all $U \in \mathcal{U}_1$, we can construct a continuous function $\epsilon_1 : N \to (0,1]$ and an open cover $\gamma$ of $N$ such that $\epsilon_1 \circ \alpha \leq \epsilon$ and $\alpha^{-1}(\gamma)$ refines $\mathcal{U}_1$. Since $\dim p_N \leq \dim p \leq n$ and $L$, being a simplicial complex, is a $C$-space, we can apply [17, Theorem 2.2] to find a map $\varphi : p \to \mathbb{P}$. Then $h = \varphi \circ \alpha$. Hence, $h_0$ is $\epsilon/2$-close to $g_0$. Therefore, $h$ is $\epsilon$-close.

It remains to show that $h \in C(\omega_0, \mathbb{P}, f)$. To this end, fix a point $z = (f(x), h(x)) \in (f(h))_0(X) \subset Y \times \mathbb{P}$ and let $y = f(x)$. Then $w = (p_N \triangle \varphi_1)(\alpha(x)) = (\beta(y), h(x))$. Since $\varphi_1 \in C(\gamma, \mathbb{P}, p_N)$, there exists a neighborhood $V_w$ of $w$ in $L \times \mathbb{P}$ such that $W = (p_N \triangle \varphi_1)^{-1}(V_w)$ is a union of a disjoint open family in $N$ refining $\gamma$. We can assume that $V_w = V_{\beta(y)} \times V_{h(x)}$, where $V_{\beta(y)}$ and $V_{h(x)}$ are neighborhoods of $\beta(y)$ and $h(x)$ in $Y$ and $\mathbb{P}$, respectively. Consequently, $(f \triangle h)^{-1}(\Gamma) = \alpha^{-1}(W)$, where $\Gamma = \beta^{-1}(V_{\beta(y)}) \times V_{h(x)}$. Finally, observe that $\alpha^{-1}(W)$ is a disjoint union of an open in $X$ family refining $\omega_m$. Therefore, $h \in C(\omega_0, \mathbb{P}, f)$.}

Proof of Theorem 1.3(i). Let $\lambda$ and $\omega_k$ be as in the proof of Theorem 1.3(i). Denote by $C_{\omega_k}(X, \mathbb{P}, f)$ the set of all $g \in C(X, \mathbb{P})$ such that $f \triangle g$ is an $\omega_k$-map. It can be shown that every $C_{\omega_k}(X, \mathbb{P}, f)$ is open in $C(X, \mathbb{P})$ with the source limitation topology (see [16, Proposition 3.1]). Moreover, $\bigcap_{k \geq 1} C_{\omega_k}(X, \mathbb{P}, f)$ consists of maps $g$ with $f \triangle g$ embedding $X$ into $Y \times \mathbb{P}$. So, we need to show that each $C_{\omega_k}(X, \mathbb{P}, f)$ is dense in $C(X, \mathbb{P})$ equipped with the source limitation topology.

To prove this fact we follow the notations and the arguments from the proof of Theorem 1.3(ii) (that $C(\omega_0, \mathbb{P}, f)$ are dense in $C(X, \mathbb{P})$) by considering $\mathbb{P}$ instead of $\mathbb{P}$). We fix a cover $\omega_m$, a map $g_0 \in C(X, \mathbb{P})$ and a function $\epsilon \in C(X, (0,1])$. Since $W(f) \leq \mathbb{P}$, we can apply Theorem 6 from [11] to find an open cover $\mathcal{U}$ of $Y$ such that for any $V$-map $\beta : Y \to L$ into a simplicial complex $L$ there exists a $U_1$-map $\alpha : X \to K$ into a simplicial complex $K$ and a perfect PL-map $p : K \to L$ with $\beta \circ f = p \circ \alpha$. Proceeding as before, we find a map $h = \varphi_1 \circ \alpha$ which is $\epsilon$-close to $g_0$, where $\varphi_1 \in C(\gamma, \mathbb{P}, p_N)$. It is easily seen that
\( \varphi_1 \in C_\gamma(N, \mathbb{P}^0, p_N) \) implies \( h \in C_{\omega_m}(X, \mathbb{P}^0, f) \). So, \( C_{\omega_m}(X, \mathbb{P}^0, f) \) is dense in \( C(X, \mathbb{P}^0) \).

**Proof of Corollary 1.5.** It follows from [2, Proposition 2.1] that the set \( E \) is \( G_\delta \) in \( C(X, \mathbb{P}^{n+1} \times M) \). So, we need to show it is dense in \( C(X, \mathbb{P}^{n+1} \times M) \). To this end, we fix \( g^0 = (g_1^0, g_2^0) \in C(X, \mathbb{P}^{n+1} \times M) \) with \( g_1^0 \in C(X, \mathbb{P}^{n+1}) \) and \( g_2^0 \in C(X, M) \). Since, by Corollary 1.4, the set

\[ G_1 = \{ g_1 \in C(X, \mathbb{P}^{n+1}) : \dim f \triangle g_1 \leq 0 \} \]

is dense in \( C(X, \mathbb{P}^{n+1}) \), we may approximate \( g_1^0 \) by a map \( h_1 \in G_1 \). Then, according to [2, Corollary 1.1], the maps \( g_2 \in C(X, M) \) with \( \dim g_2((f \triangle h_1)^{-1}(z)) = 0 \) for all \( z \in Y \times \mathbb{P}^{n+1} \) form a dense subset \( G_2 \) of \( C(X, M) \). So, we can approximate \( g_2^0 \) by a map \( h_2 \in G_2 \). Let us show that the map \( h = (h_1, h_2) \in C(X, \mathbb{P}^{n+1}) \times M \) belongs to \( E \). We define the map \( \pi_h : (f \triangle h_1)(X) \to (f \triangle h_1)(X), \pi_h(f(x), h_2(x)) = (f(x), h_1(x)) \), \( x \in X \). Because \( f \) is perfect, so is \( \pi_h \). Moreover, \( (\pi_h)^{-1}(f(x), h_1(x)) = h_2(f^{-1}(f(x)) \cap h_1^{-1}(h_1(x))) \), \( x \in X \). So, every fiber of \( \pi_h \) is 0-dimensional. We also observe that \( \pi_h(h(f^{-1}(y))) = (f \triangle h_1)(f^{-1}(y)) \) and the restriction \( \pi_h|h(f^{-1}(y)) \) is a perfect surjection between the compact spaces \( h(f^{-1}(y)) \) and \( (f \triangle h_1)(f^{-1}(y)) \) for any \( y \in Y \). Since \( (f \triangle h_1)(f^{-1}(y)) \subset \{ y \} \times \mathbb{P}^{n+1} \), \( \dim(f \triangle h_1)(f^{-1}(y)) \leq n + 1 \), \( y \in Y \). Consequently, applying the Hurewicz’s dimension-lowering theorem [6] for the map \( \pi_h|h(f^{-1}(y)) \), we have \( \dim h(f^{-1}(y)) \leq n + 1 \). Therefore, \( h \in E \), which completes the proof. \( \square \)

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