On the existence of plane curves with imposed multiple points

Joaquim Roé *
Departament d’Àlgebra i Geometria, Universitat de Barcelona,
Gran Via, 585, E-08007, Barcelona.
e-mail: jroevell@cerber.mat.ub.es

July 30, 2018

Abstract

We prove that a plane curve of degree $d$ with $r$ points of multiplicity $m$ must have

$$d \geq m (r - 1) \prod_{i=2}^{r-1} \left( 1 - \frac{i}{i^2 + r - 1} \right)$$

$$d > (\sqrt{r - 1} - \frac{\pi}{8}) m$$

1991 Mathematics Subject Classification: Primary 14C20. Secondary 14J26, 14M20, 14H20.

1 Introduction

In [11], Nagata showed a counterexample to the fourteenth problem of Hilbert; in his construction, he proved that, for $n > 3$, a plane curve going with multiplicity at least $m$ through $n^2$ points in general position must have degree strictly bigger than $nm$. Moreover, he conjectured that this result should also hold for a non-square number of points, that is, a curve with multiplicity $m$ at $r \geq 10$ points in general position must have degree strictly bigger than $\sqrt{rm}$.

This conjecture has been proved only in some particular cases. In [4], Evain proves it for $m$ small enough, concretely for $r > \left( \frac{8m - 1}{4m - 1} (m + 1) \right)^2$. In the case of irreducible reduced curves, Xu proved in [13] the inequalities $d > \sqrt{r - 1} m - \frac{1}{2 \sqrt{r - 1}}$ and $d > \sqrt{r - 1} m$. As far as we know, the best known bound for the general case is what follows from Nagata’s result, $d > \lfloor \sqrt{r} \rfloor m$, where $\lfloor \cdot \rfloor$ denotes the integral part.

*Partially supported by CIRIT 1997FI-00141, CAICYT PB95-0274 and “AGE-Algebraic Geometry in Europe” contract no. ERB940557.
In this work we prove the inequalities

\[ d \geq m(r - 1) \prod_{i=2}^{r-1} \left(1 - \frac{i}{i^2 + r - 1}\right) \]

\[ d > \left(\sqrt{r-1} - \frac{\pi}{8}\right)m \]

for all \( r \geq 10 \). This is better than the known bound for \( r \) in any interval \(((n + \frac{2}{3})^2 + 1, (n + 1)^2)\), \( n \in \mathbb{Z} \). Our approach is based on a specialization of the scheme consisting of \( r \) points in general position with multiplicity \( m \) to an appropriate cluster scheme supported at a single point.

We would like to thank the referee for his/her very helpful suggestions.

\[ \text{2 Definitions} \]

Given an algebraic variety \( Z \) over an algebraically closed field \( k \), and a closed subvariety \( Z' \) of \( Z \), we will write \( b : \text{Bl}(Z, Z') \rightarrow Z \) for the blowing-up of \( Z \) with center \( Z' \).

Let \( p_1 \in S_0 = \mathbb{P}^2, p_2 \in S_1 = \text{Bl}(S_0, \{p_1\}), \ldots, p_r \in S_{r-1} = \text{Bl}(S_{r-2}, \{p_{r-1}\}) \). The set \( \{p_1, p_2, \ldots, p_r\} \) is called a \textit{cluster} (see [4]) and the sequence \( K = (p_1, p_2, \ldots, p_r) \) is an \textit{ordered cluster}. Here we will be concerned only with ordered clusters and we will call them simply clusters. Note that some of the points of a cluster can be identified to proper points of \( \mathbb{P}^2 \), whereas others may lie infinitely near to preceding points. A \textit{system of multiplicities} for a cluster \( K = (p_1, p_2, \ldots, p_r) \) is a sequence of integers \( (m) = (m_1, m_2, \ldots, m_r) \), and a pair \((K, m)\) where \( K \) is a cluster and \((m)\) a system of multiplicities is called a \textit{weighted cluster}. We review now briefly some known results on clusters; for the proofs, refer to [2], [3], having in mind the minor change that we do not require all points in a cluster to be infinitely near to the first one.

Given a weighted cluster, we have an ideal sheaf and a zero-dimensional subscheme of \( \mathbb{P}^2 \) associated to it. Write \( S_K = \text{Bl}(S_{r-1}, \{p_r\}) \) and denote by \( \pi_K \) the composition \( S_K \rightarrow \mathbb{P}^2 \) of the blowing-ups of the points of \( K \). Let \( E_i \) be the pullback (total transform) in \( S_K \) of the exceptional divisor of blowing up \( p_i \). Then the ideal sheaf

\[ \mathcal{H}_{K,m} = (\pi_K)_* \mathcal{O}_{S_K}(-m_1E_1 - m_2E_2 - \cdots - m_mE_r) \]

defines a zero-dimensional subscheme of \( \mathbb{P}^2 \), and the stalks of \( \mathcal{H}_{K,m} \) are complete ideals in the stalks of \( \mathcal{O}_{\mathbb{P}^2} \). Conversely, if \( I \) is a coherent sheaf of ideals on \( \mathbb{P}^2 \) defining a zero-dimensional scheme whose stalks are complete ideals then there is a weighted cluster \((K, m)\) such that \( I = \mathcal{H}_{K,m} \). We will call such schemes \textit{cluster schemes}. Remark that a plane curve contains the cluster scheme defined by \((K, m)\) if and only if it goes (virtually, as in [4]) through \((K, m)\). This notion has already been considered by Greuel-Lossen-Shustin in [1] (with the name \textit{generalized singularity scheme}) and also by Harbourne in [8] (with the name \textit{generalized fat point scheme}).

Given two points \( p_i, p_j \) in a cluster \( K \) with \( j > i \), we say that \( p_j \) is \textit{proximate} to \( p_i \) if and only if \( j = i + 1 \) and \( p_j \) lies on the exceptional divisor \( E \subset S_i \) of blowing up \( p_i \), or \( j > i + 1 \) and \( p_j \) lies on the \textit{strict} transform of \( E \). The
proximity inequality at \( p_i \) is

\[
m_i \geq \sum_{p_j \text{ prox. to } p_i} m_j.
\]

A cluster satisfying the proximity inequalities at all its points is called consistent. It happens that different weighted clusters \((K_1, m^{(1)})\) and \((K_2, m^{(2)})\) define the same cluster scheme. In this case \( \mathcal{H}_{K_1, m^{(1)}} = \mathcal{H}_{K_2, m^{(2)}} \) and we will say that the two clusters are equivalent. For example, if \( p_2 \) is infinitely near \( p_1 \) then the weighted clusters

\[
K_1 = (p_1) \quad m^{(1)} = (1) \\
K_2 = (p_1, p_2) \quad m^{(2)} = (0, 1)
\]

are equivalent. However, if we ask that \( m^{(i)} > 0 \) for all \( i \) and \((K, m)\) be consistent, then the cluster scheme determines the weighted cluster, but for the ordering of points.

Given an arbitrary weighted cluster \((K, m)\) there is a procedure called unloading (see [4], [IV.II], or [1]) which gives a new system of multiplicities \((m')\) such that \((K, m')\) is consistent and equivalent to \((K, m)\). In each step of the procedure, one unloads some amount of multiplicity on a point \( p_i \) whose proximity inequality is not satisfied, from the points proximate to it. This means that there is an integer \( n > 0 \) such that, increasing the multiplicity of \( p_i \) by \( n \) and decreasing the multiplicity of every point proximate to \( p_i \) by \( n \), the resulting weighted cluster is equivalent to \((K, m)\) and satisfies the proximity inequality at \( p_i \). In other words, if \( \tilde{E}_i \subset S_K \) is the strict transform of the exceptional divisor of blowing-up \( p_i \), \( D = -m_1E_1 - m_2E_2 - \cdots - m_rE_r \) and and \( \tilde{E}_i \cdot D < 0 \) then one chooses \( n \) as the minimal integer with \( \tilde{E}_i \cdot (D - n\tilde{E}_i) \geq 0 \) and replaces \( D \) by \( D - n\tilde{E}_i \). A finite number of unloading steps lead to the desired equivalent consistent cluster.

Let \( T \) be a variety, which for the moment we will think of as a fixed base for our constructions. Let \( p : X \to T \) be a smooth morphism of relative dimension \( n \), and let \( i : Y \to X \) be a smooth embedding over \( p \).

Let us consider the diagonal morphism \( \Delta := Id_Y \times_T i \) which makes the following diagram commutative:

\[
\begin{array}{ccc}
Y & \overset{\Delta}{\longrightarrow} & Y \times_T X \\
\downarrow{Id_Y} & & \downarrow{p_X} \\
Y & \underset{p_Y}{\longrightarrow} & X \\
\end{array}
\]

The image \( \Delta(Y) \) is a closed smooth subvariety isomorphic to \( Y \). Consider the blowing-up

\[
BF(X, Y, T) := Bl(Y \times_T X, \Delta(Y)) \xrightarrow{b} Y \times_T X,
\]
and the commutative diagram

\[
\begin{array}{ccc}
BF(X,Y,T) & \xrightarrow{p_X \circ b} & X \\
\downarrow_{p_Y \circ b} & & \downarrow \\
Y & \rightarrow & T
\end{array}
\]

We call \( \pi = p_X \circ b \) and \( q = p_Y \circ b \). As \( \Delta \) is a smooth embedding over \( p \), it follows that \( q \) is smooth, of relative dimension \( n \) (see [6, 19.4]). We call \( BF(X,Y,T) \) the family of blowing up \( X \) at the points of \( Y \). We are going to see that the morphism \( BF(X,Y,T) \xrightarrow{\pi} X \), makes the fibers of \( q \) into ordinary blowings up, hence the name. Given \( y \in Y \), with \( p(y) = t \), call \( BF(X,Y,T)_y = BF(X,Y,T) \times_Y \{ y \} \) and \( X_t = X \times_T \{ t \} \). Note that \( y \in X_t \).

**Proposition 2.1.** For every point \( y \in Y \), and \( t = p(y) \in T \) consider the blowing-up \( b : Bl(X_t, \{ y \}) \rightarrow X_t \). Then there is a unique isomorphism

\[
Bl(X_t, \{ y \}) \xrightarrow{\psi} BF(X,Y,T)_y
\]

satisfying \( b = \pi|_{BF(X,Y,T)_y} \circ \psi \).

**Proof.** Follows from [10, 2.4], as \( \Delta(Y) \) is obviously a local complete intersection, flat over \( Y \). \( \square \)

### 3 Varieties of clusters

Take now \( X_{-1} = Spec \, k \), \( X_0 = \mathbb{P}^2_k \), \( p_0 : \mathbb{P}^2_k \rightarrow Spec \, k \), and define recursively \( X_i \), \( p_i \) as the blowing-up family

\[
X_i = BF(X_{i-1}, X_{i-1}, X_{i-2}) \xrightarrow{p_i} X_{i-1}.
\]

The morphisms \( p_i \) are in this case projective and smooth of relative dimension 2, so their fibers are projective smooth surfaces. We have also morphisms \( \pi_i : X_i \rightarrow X_{i-1} \) whose restrictions to the fibers of \( p_i \) are by proposition 2.1 the blowing-ups of the points of the fibers of \( p_{i-1} \). To simplify notations, let us say \( \pi_{r,i} = \pi_{i+1} \circ \pi_{i+2} \circ \cdots \circ \pi_r \), \( p_{r,i} = p_i \circ p_{i+1} \circ \cdots \circ p_r \). If there is no confusion possible on \( r \), we will also write \( p_i \) for \( p_{r,i} \), so \( p_i(x) \) is a point in \( X_{i-1} \), defined for all \( x \) in \( X_r \), \( r \geq i \). For any point \( x \in X_i \), we will call \( S_x = (X_i)_{p_i(x)} = X_i \times X_{i-1} \{ p_i(x) \} \) the surface containing \( x \). Recall that for any cluster \( K \), \( \pi_K : S_K \rightarrow \mathbb{P}^2 \) is the composition of the blowing-ups of the points in \( K \).

The following proposition makes the set of all clusters with \( r \) points into an algebraic variety.

**Proposition 3.1.** For every \( r \geq 1 \) there is a bijection

\[
X_{r-1} \xrightarrow{K} \{ \text{clusters of } r \text{ points} \}
\]

and, for every \( x \in X_{r-1} \), a unique isomorphism \( \psi_x : S_{K(x)} \rightarrow (X_r)_x \) such that \( \pi_K = \pi_{r,0}|_{(X_r)_x} \circ \psi_x \).

---

Note on notation: The commutative diagram illustrates the relationship between the blowing-up morphisms and the target spaces. The proposition 2.1 establishes the existence of a unique isomorphism \( \psi \) that satisfies the given conditions. The proposition 3.1 extends this concept to a set of clusters, providing a bijection and isomorphism that defines the structure of these clusters.
Proof. Follows from [7, 1.2], since there is an obvious bijection

\[
\{\text{ordered blowing-ups at } r \text{ points}\} \longrightarrow \{\text{ordered clusters of } r \text{ points}\}
\]

\[S_K \mapsto K\]

Notice that the ordering of points in clusters is essential in proposition 3.1. If two clusters differing only in the order of points were considered equal, as in [2], then injectivity would fail. From now on identify the set of clusters of \(r\) points to the variety \(X_{r-1}\).

For every pair of integers \(1 \leq i < j \leq r\) there is a subset of \(X_{r-1}\) containing exactly those clusters \(K = (x_1, x_2, \ldots, x_r)\) for which \(x_j\) is proximate to \(x_i\). It can be proved that these subsets are constructible subsets of \(X_{r-1}\); we will focus on some of them which are irreducible closed varieties.

Call \(F_i\) the exceptional divisor of

\[X_i \xrightarrow{b_i} X_{i-1} \times_{X_{i-2}} X_{i-1}.\]

Because of proposition 2.1 the pullback of \(F_i\) to \((X_i)_{p_i}\) is the exceptional divisor \(E_i\) of blowing up \(p_i\) in \(S_{p_i}\). It is clear that \(p_i(K)\) is proximate to \(p_{i-1}(K)\) if and only if \(p_i(K) \in F_{i-1}\). So there is a closed subvariety

\[Y_{r-1} := \bigcap_{i=2}^{r} p_i^{-1}(F_{i-1}) \subset X_{r-1}\]

containing exactly those clusters \(K\) for which \(p_{i+1}(K)\) is proximate to \(p_i(K)\) for all \(i\). It is also clear that \(p_{r-1}(Y_{r-1}) = Y_{r-2}\), if we allow \(Y_0 = \mathbb{P}^2\).

Lemma 3.2. For all \(r\), there is a closed immersion

\[BF(X_{r-1}, Y_{r-1}, X_{r-2}) \xrightarrow{i} X_r\]

such that \(Y_r\) is the image of the exceptional divisor \(F'_r\) of

\[BF(X_{r-1}, Y_{r-1}, X_{r-2}) \xrightarrow{b} X_{r-1} \times_{X_{r-2}} Y_{r-1}\]

Proof. The closed immersion \(i\) is the strict transform of the closed immersion

\[Y_{r-1} \times_{X_{r-2}} X_{r-1} \longrightarrow X_{r-1} \times_{X_{r-2}} X_{r-1}\]

(see [3, II.7.15]). By definition of the \(Y_r\) we know that \(Y_r = F_r \cap p_{r-1}^{-1}(Y_{r-1})\), and obviously \(i(F'_r) \subset F_r\), and

\[(p_r \circ i)(BF(X_{r-1}, Y_{r-1}, X_{r-2})) = Y_{r-1},\]

so \(i(F'_r) \subset Y_r\). on the other hand, if \(y_r \in Y_r\) then \(p_r(y_r) = y_{r-1} \in Y_{r-1}\), so

\[y_r \in S_{y_r} \cong Bl(S_{y_{r-1}}, \{y_{r-1}\}) \cong BF(X_{r-1}, Y_{r-1}, X_{r-2})_{y_{r-1}},\]

which implies \(y_r \in i(F'_r)\). So \(Y_r \subset i(F'_r)\), and the proof is complete. \(\square\)
Corollary 3.3. For all \( r \), \( Y_r \) together with the restricted morphism \( p_r : Y_r \to Y_{r-1} \) is a \( \mathbb{P}^1 \)-bundle, and \( Y_r \) is irreducible.

To deal with the proximity relations between points \( p_i \) and \( p_j \) where \( j > i + 1 \) we need some control on the strict transforms of the exceptional divisor of blowing up \( p_i \). In contrast to what we have seen in the case \( j = i + 1 \), there is no variety \( \tilde{F}_i \subset X_{j-1} \) whose pullback to \( (X_{j-1})_{p_{j-1}(K)} \) is the desired strict transform for all \( K \). To overcome this difficulty we restrict ourselves to clusters in \( Y_{r-1} \) and define varieties \( D_{i,j} \subset X_{j-1} \) whose pullback to \( (X_{j-1})_{p_{j-1}(K)} \) is the strict transform of the exceptional divisor of blowing up \( p_i(K) \) if \( p_{j-1}(K) \) is proximate to \( p_i(K) \) and empty in any other case. Let first

\[
D'_{i,i+1} = D_{i,i+1} = Y_i.
\]

Suppose now we have defined \( D_{i,j-1} \subset X_{j-2} \) and \( D'_{i,j-1} = D_{i,j-1} \cap Y_{j-2} \), such that the morphism \( p_{j-2}|_{D_{i,j-1}} \) is smooth of relative dimension 1 (observe that for \( D_{i,i+1} = Y_i \) this is so). As there is a closed immersion \( D_{i,j-1} \times X_{j-3} \to X_{j-2} \times X_{j-3} \) there is also a closed immersion (its strict transform)

\[
D_{i,j} = \text{BF} \left( D_{i,j-1}, D'_{i,j-1}, X_{j-3} \right) \overset{i}{\to} X_{j-1}
\]

which we take as the definition of \( D_{i,j} \). Moreover, as \( p_{j-2}|_{D_{i,j-1}} \) is smooth of relative dimension 1, \( \Delta(D_{i,j-1}) \) has codimension 1 in \( D_{i,j-1} \times X_{j-3} \) and

\[
\text{BF} \left( D_{i,j-1}, D'_{i,j-1}, X_{j-3} \right) \overset{b}{\to} D_{i,j-1} \times X_{j-3} D'_{i,j-1}
\]

is an isomorphism. We have

\[
D'_{i,j} = D_{i,j} \cap Y_{j-1} \cong \Delta(D'_{i,j-1}) \subset \text{BF} \left( D_{i,j-1}, D'_{i,j-1}, X_{j-3} \right).
\]

So \( D'_{i,j} \) is isomorphic to \( D'_{i,j-1} \), and \( p_{j-1}|_{D_{i,j}} \) is smooth of relative dimension 1.

We will call \((i,j)\)-proximity variety the subvariety \( P_{i,j} = p_j^{-1}(D'_{i,j}) \subset Y_{r-1} \).

Lemma 3.4. In a cluster \( K \in Y_{r-1} \) the points \( p_{i+1}, p_{i+2}, \ldots, p_j \) are proximate to \( p_i \) if and only if \( K \) lies in the \((i,j)\)-proximity variety. Furthermore, the proximity varieties are irreducible and there are inclusions

\[
P_{i,i+1} \supset P_{i,i+2} \supset \cdots \supset P_{i,r}.
\]

Proof. The first part will clearly be proved if we show that

\[
D_{i,j} \times X_{j-1} \{p_{j-1}\} \subset S_{p_j}
\]

is the strict transform of \( E_i \). This comes out easily by induction on \( j - i \). For \( j - i = 1 \), it is immediate by proposition 2.1. For \( j - i > 1 \), proposition 2.1 gives

\[
D_{i,j} \times X_{j-1} \{p_{j-1}\} = \text{Bl}(D_{i,j-1} \times X_{j-2} \{p_{j-2}\}, p_{j-1}),
\]

that is, the strict transform in \( S_{p_j} \) of \( D_{i,j-1} \times X_{j-2} \{p_{j-2}\} \), which by the induction hypothesis is the strict transform of \( E_i \) in \( S_{p_{j-1}} \), so we are done.

From their own definition, the \( D'_{i,j} \) are all isomorphic to \( Y_i \), which is irreducible. Induction on \( r - j \) gives the irreducibility of the \( P_{i,j} \). Indeed, if \( P_{i,j} \) is irreducible then its preimage by \( p_{r-1}|_{Y_{r-1}} \) must be irreducible also, because \( Y_{r-1} \to Y_{r-2} \) is a projective space bundle.

The inclusions between the \( P_{i,j} \) are clear, from the first part of the lemma.
Lemma 3.4 shows that there are subsets $U_{1,i}$ open and dense in $P_{1,i}$ which contain all those clusters $K$ with

- $p_j(K)$ proximate to $p_{j-1}(K)$, $j = 2, \ldots, r$
- $p_j(K)$ proximate to $p_1(K)$, $2 \leq j \leq i$

and no other proximity relations.

**Lemma 3.5.** Let $(m) = (m_1, m_2, \ldots, m_r)$ be a system of multiplicities, and call $M = \sum_{j=2}^{r} m_j$. Define $\alpha_i = \frac{i-1}{i-1}$ and $\beta_i = 1 - \frac{i-1}{(i-1)^2 + r-1}$. Suppose that for some $i \in \{2, 3, \ldots, r\}$ and $A \in \mathbb{R}$ the inequalities

$$\frac{(i-2)m_1 + M}{(i-2)\alpha_{i-1} + 1} \geq A,$$

$$m_1 \geq \alpha_{i-1} A,$$

are satisfied. Then there is a system of multiplicities $(m')$ which is equivalent to $(m)$ for all clusters in $U_{1,i}$ and satisfies

$$\frac{(i-1)m_1' + M'}{(i-1)\alpha_i + 1} \geq \beta_i A,$$  \hspace{1cm} (1)

$$m_1' \geq \alpha_i \beta_i A.$$  \hspace{1cm} (2)

**Proof.** We know that for a given cluster of $r$ points $K$ there is a system of multiplicities $(m')$, consistent and equivalent to $(m)$, which is obtained from $(m)$ by the unloading procedure. The unloading procedure depends only on the multiplicities and the proximity relations, and so it is the same for all clusters in $U_{1,i}$.

Due to the proximity relations which hold for the points of a cluster in $U_{1,i}$, when an unloading step is applied to the point $p_j$, $1 < j < r$ the only point whose multiplicity is decreased is $p_{j+1}$, so $m_1$ and $M$ remain unchanged. When an unloading step is applied to $p_1$, the points whose multiplicity is decreased are $\{p_2, p_3, \ldots, p_i\}$, so if $m_1$ is increased by $n$, $M$ is decreased by $(i-1)n$. In both cases, the quantity $(i-1)m_1 + M$ remains the same. When an unloading step is applied to $p_r$, which happens only when its multiplicity has become negative, then one replaces it by zero, so $(i-1)m_1 + M$ might increase, but does never decrease. After the complete unloading procedure we get

$$(i-1)m_1' + M' \geq (i-1)m_1 + M = (i-2)m_1 + M + m_1 \geq ((i-2)\alpha_{i-1} + 1)A + \alpha_{i-1}A = ((i-1)\alpha_i + 1)\beta_iA.$$

This proves (1). To see (2), we multiply this inequality by $\alpha_i$, so we get

$$\alpha_i ((i-1)m_1' + M') \geq ((i-1)\alpha_i + 1)\alpha_i \beta_i A.$$  \hspace{1cm} (3)

On the other hand, as $(m')$ is consistent, $(K, m)$ must satisfy all the proximity inequalities, and these imply easily

$$m_1' - \alpha_i M' \geq 0.$$  \hspace{1cm} (4)

If we add both inequalities, we obtain (2).
4  The bound

Let $F_i^{(r)}$ be the pullback of $F_i$ by $\pi_{r,i} : X_r \rightarrow X_i$. Let $[F_0]^{(r)}$ be the pullback to $X_r$ by $\pi_{r,0}$ of the class of a line in $\mathbb{P}^2$. For any cluster $K \in X_{r-1}$ and $i > 0$, the pullback to the surface $S_K$ of $F_i^{(r)}$ by the inclusion is obviously the same as the pullback $E_i$ of the class of the exceptional divisor of blowing up $p_i$ in $S_{p_i(K)}$ by $\pi_{r,i}|S_K$. Similarly, the pullback of $[F_0]^{(r)}$ to $S_K$ is the same as the pullback $[E_0]$ of the class of a line by $\pi_{r,0}|S_K$. All together, we have

$$\mathcal{O}_{X_r}(F_i^{(r)}) \otimes \mathcal{O}_{X_{r-1}} k(K) = \mathcal{O}_{X_r}(E_i)$$

for all $i$. Given an integer $d$ we define

$$\mathcal{J}_{d,m} = \mathcal{O}_{X_r}(dF_0^{(r)} - m_1F_1^{(r)} - m_2F_2^{(r)} - \cdots - m_rF_r^{(r)}).$$

Equality (3) and the projection formula show that, for every cluster $K \in X_{r-1}$,

$$\mathcal{H}_{K,m}(d) = \mathcal{H}_{K,m} \otimes \mathcal{O}_{\mathbb{P}^2}(d) = (\pi_K)_* (\mathcal{J}_{d,m} \otimes \mathcal{O}_{X_{r-1}} k(K))$$

and $H^0(\mathcal{H}_{K,m}(d)) \cong H^0(\mathcal{J}_{d,m} \otimes \mathcal{O}_{X_{r-1}} k(K))$.

In our specialization, we start from a cluster $K$ consisting of $r$ points in general position, to specialize it, step by step, to the closed subvarieties $P_{1,i}$. We obtain the following theorem:

**Theorem 4.1.** If a plane curve of degree $d$ passes with multiplicity $m$ through $r$ points in general position, then

$$d \geq m(r - 1) \prod_{i=2}^{r-1} \left(1 - \frac{i}{i^2 + r - 1}\right)$$

(4)

**Proof.** Let $\mathcal{J}$ and $\mathcal{H}$ be the sheaves defined above. We start from the system of multiplicities $(m) = (m, m, \ldots, m)$. We have to prove that for general $K \in X_{r-1}$, the inequality

$$H^0((\mathcal{J}_{d,m}) \otimes \mathcal{O}_{X_{r-1}} k(K)) \cong H^0(\mathcal{H}_{K,m}(d)) \neq 0$$

implies (2), so assume this inequality holds for general $K$. As $X_r \rightarrow X_{r-1}$ is smooth, the invertible sheaf $\mathcal{J}_{d,m}$ is flat over $X_{r-1}$, so by the semicontinuity theorem [9, III,12.8] we have

$$H^0(\mathcal{H}_{K,m}(d)) \neq 0$$

for all $K \in P_{1,i}$ and any $i$.

Now for $K \in P_{1,3}$ the system of multiplicities $(m)$ is not consistent. We can find by unloading multiplicities a consistent system $(m^{(3)})$ which is equivalent to $(m)$ for all clusters in $U_{1,3}$. Applying lemma [13] with $(m) = (m, m, \ldots, m)$, $M = (r - 1) m$ and $i = 3$, we have

$$\frac{(i-2)m_1 + M}{(i-2)\alpha_{i-1} + 1} = \frac{m + (r - 1) m}{\alpha_2 + 1} = m(r - 1)$$

8
so we can take $A = A_2 = m(r - 1)$ and the lemma gives

$$\frac{2m_1^{(3)} + M^{(3)}}{2\alpha_3 + 1} \geq m(r - 1)\beta_3$$

$$m_1^{(3)} \geq m(r - 1)\alpha_3 \beta_3.$$  

As $(m^{(3)})$ is equivalent to $(m)$ for all clusters in $U_{1,3}$, we have

$$H^0(\mathcal{H}_{K,m^{(3)}}(d)) = H^0(\mathcal{H}_{K,m}(d)) \neq 0$$

if $K \in U_{1,3}$. As $U_{1,3}$ is open and dense in $P_{1,3}$, and $P_{1,4} \subset P_{1,3}$, the semicontinuity theorem applied to the new sheaf $\mathcal{J}_{d,m^{(3)}}$ implies

$$H^0(\mathcal{H}_{K,m^{(3)}}(d)) \neq 0$$

for all $K \in P_{1,4}$. The new system of multiplicities need not be (but in fact could be) consistent for $K \in P_{1,4}$. In any case we can find a new system $(m^{(4)})$ (which could be equal to $(m^{(3)})$) to use here. We apply lemma 3.5 to the new situation, with $A_3 = m(r - 1)\beta_3$, and we obtain

$$\frac{3m_1^{(4)} + M^{(4)}}{3\alpha_4 + 1} \geq m(r - 1)\beta_3 \beta_4$$

$$m_1^{(4)} \geq m(r - 1)\alpha_4 \beta_3 \beta_4.$$  

Iterating the process we finally get a system $(m^{(r)}) = (m_1^{(r)}, m_2^{(r)}, \ldots, m_r^{(r)})$, with

$$m_1^{(r)} \geq m(r - 1)\alpha_r \prod_{i=3}^{r} \beta_i = m(r - 1)\prod_{i=2}^{r-1} \left(1 - \frac{i}{i^2 + r - 1}\right)$$

and

$$H^0(\mathcal{H}_{K,m^{(r)}}(d)) \neq 0$$

for all $K \in P_{1,r}$. It is clear that this implies $d \geq m_1^{(r)}$. \qed

The reader may notice that the proof of theorem 4.1 is valid for any divisor class on an irreducible smooth projective surface $S$, except for the last step, namely $d \geq m_1^{(r)}$, which assumes $C \subset S = \mathbb{P}^2$. The specialization of a set of multiple points to a cluster scheme containing a point of multiplicity $m' \geq m(r - 1)\prod_{i=2}^{r-1} \left(1 - \frac{i}{i^2 + r - 1}\right)$ holds thus on any such surface.

5 A calculation

The aim of this section is to compare the bound of theorem 4.1 with Nagata’s conjecture (which reads $d > m\sqrt{r}$), and with previously known results. We obtain the following:

**Proposition 5.1.** Let $n \geq 9$ be a natural number. Then

$$n \prod_{i=2}^{n} \left(1 - \frac{i}{i^2 + n}\right) > \sqrt{n} - \frac{\pi}{8}.$$
This has an immediate corollary:

**Corollary 5.2.** If a plane curve of degree \( d \) passes with multiplicity \( m \) through \( r \geq 10 \) points in general position, then

\[
d > m \left( \sqrt{r - 1} - \frac{\pi}{8} \right)
\]

**Proof of proposition 5.1.** The goal is to bound

\[
b = n \prod_{i=2}^{n} \left( 1 - \frac{i}{i^2 + n} \right) = n \prod_{i=1}^{n-1} \left( 1 - \frac{i}{i^2 + n} \right) = n \prod_{i=1}^{n-1} \frac{n + i^2 - i}{i^2 + n}
\]

below. This can be rewritten as

\[
n \left( \frac{1}{n} \right) \prod_{i=2}^{n} \left( n + i^2 - i \right) = \prod_{i=1}^{n-1} \frac{n + (i + 1)^2 - (i + 1)}{i^2 + n} = \prod_{i=1}^{n-1} \left( 1 + \frac{i}{i^2 + n} \right).
\]

We thus have

\[
b^2 = n \prod_{i=1}^{n-1} \left( 1 - \frac{i}{i^2 + n} \right) \prod_{i=1}^{n-1} \left( 1 + \frac{i}{i^2 + n} \right) = n \prod_{i=1}^{n-1} \left( 1 - \frac{i}{i^2 + n} \right)^2.
\]

Let \( 1 - \epsilon = \prod_{i=1}^{n-1} (1 - (i/(i^2 + n))^2) \) and let \( 1 + \delta = \prod_{i=1}^{n-1} (1 + (i/(i^2 + n))^2) \). Then \( \epsilon = 1 - \prod_{i=1}^{n-1} (1 - (i/(i^2 + n))^2) \) and \( \delta = -1 + \prod_{i=1}^{n-1} (1 + (i/(i^2 + n))^2) \) both involve the same terms, except that they occur with signs in \( \epsilon \), so \( 0 < \epsilon < \delta \). Thus \( 1 - \epsilon < 1 - \delta \), and so \( b^2 > n(1 - \delta) \).

We can bound \( b^2 \) (and hence \( b \)) below by bounding \( 1 + \delta \) (and hence \( \delta \)) above. But \( \log(1 + x) \leq x \) so \( \log \prod_{i=1}^{n-1} (1 + (i/(i^2 + n))^2) \leq \sum_{i=1}^{n-1} (i/(i^2 + n))^2 \).

The Fourier series for \( \sinh \sqrt{n\pi} \) on \( [-\pi, \pi] \) is

\[
\frac{2}{\pi} \left( \sinh \sqrt{\pi} \right) \sum_{i \geq 1} (-1)^i \frac{-i}{i^2 + n} \sin ix
\]

so Parseval’s identity gives

\[
\left( \frac{\pi}{2 \sinh \sqrt{\pi}} \right)^2 \frac{1}{\pi} \int_{-\pi}^{\pi} \sin^2 \sqrt{n\pi} \, dx = \sum_{i \geq 1} \left( \frac{i}{i^2 + n} \right)^2.
\]

The integral can be exactly evaluated; we get

\[
\int_{-\pi}^{\pi} \sin^2 \sqrt{n\pi} \, dx = -\pi + \frac{1}{2\sqrt{n}} \sinh 2\sqrt{n\pi}.
\]

Also,

\[
\sum_{i \geq n} \left( \frac{i}{i^2 + n} \right)^2 \geq \sum_{i \geq n} \left( \frac{i}{i^2 + 1} \right)^2 \geq \int_{n}^{\infty} \left( \frac{1}{x + 1} \right)^2 \, dx = \frac{1}{n + 1}.
\]

Thus we have

\[
\sum_{i=1}^{n-1} \left( \frac{i}{i^2 + n} \right)^2 \leq \left( \frac{\pi}{2 \sinh \sqrt{\pi}} \right)^2 \frac{1}{\pi} \left( -\pi + \frac{1}{2\sqrt{n}} \sinh 2\sqrt{n\pi} \right) - \frac{1}{n + 1} \leq \frac{\pi \sinh 2\sqrt{n\pi}}{8\sqrt{n} \sinh^2 \sqrt{n\pi}} - \frac{1}{n + 1}.
\]
Define $t = e^{\sqrt{n\pi}}$, so
\[
\frac{\sinh 2\sqrt{n\pi}}{2 \sinh^2 \sqrt{n\pi}} = \frac{t + 1/t}{t - 1/t} = \left(1 + \frac{1}{t^2}\right) \left(1 + \frac{1}{t^2} + \frac{1}{t^4} + \cdots\right) = \left(1 + \frac{1}{t^2}\right) \left(1 + \frac{1}{t^2} + \frac{1.5}{t^4} + \cdots\right) \leq \left(1 + \frac{3}{t^2}\right)
\]
because $n \geq 9$, and
\[
\sum_{i=1}^{n-1} \left(\frac{i}{t^2 + n}\right)^2 \leq \frac{\pi}{4\sqrt{n}} + \frac{1}{t^2} - \frac{1}{n + 1}.
\]
But $e^{\sqrt{n\pi}} \geq 3n$ (look at the tangent line to $e^{\sqrt{n\pi}}$ at $n = 9$), so $e^{2\sqrt{n\pi}} \geq 3n^2 \geq (n + 2)(n + 1)$ hence $1/t^2 \leq 1/((n + 2)(n + 1))$; therefore
\[
\sum_{i=1}^{n-1} \left(\frac{i}{t^2 + n}\right)^2 \leq \frac{\pi}{4\sqrt{n}} - \frac{1}{n + 2}.
\]
This means $\delta \leq -1 + e^{\pi/(4\sqrt{n}) - 1/((n + 2)}$, hence
\[
b^2 \geq n(1 - \delta) \geq n(2 - e^{\pi/(4\sqrt{n}) - 1/((n + 2)} = n \left(1 - \frac{\pi}{4\sqrt{n}} + \frac{1}{n + 2} - \frac{1}{2!} \left(\frac{\pi}{4\sqrt{n}} - \frac{1}{n + 2}\right)^2 - \cdots\right) \geq n \left(1 - \frac{\pi}{4\sqrt{n}} + \frac{1}{n + 2} - \frac{1}{2!} \left(\frac{\pi}{4\sqrt{n}}\right)^2 - \cdots\right)
\]
and by comparison with a geometric series, this last is at least as big as
\[
n - \frac{\pi\sqrt{n}}{4} + \frac{n}{n + 2} - \frac{2u^2}{1 - u},
\]
where $u = (\pi/(4\sqrt{n}))/2 \leq \pi/24$, so $1/(1 - u) \leq 1/(1 - (\pi/24)) \leq 1.2$, so $-n2u^2/(1 - u) \geq -2.4nu^2 \geq -4$; i.e., $b^2 \geq n - \pi\sqrt{n}/4 + n/(n + 2) - .4 \geq n - \pi\sqrt{n}/4 + 9/11 - .4$. Finally, $(\sqrt{n} - \pi/8)^2 = n - \pi\sqrt{n}/4 + \pi^2/64$, and $9/11 - .4 > \pi^2/64$, so $b \geq \sqrt{n} - \pi/8$, as required.

References

[1] Casas-Alvero, E., Infinitely near imposed singularities and singularities of polar curves Math. Ann. 287 (1990), 429-454.

[2] Casas-Alvero, E., Singularities of plane curves (1997), to appear.

[3] Enriques, F. - Chisini, O., Lezioni sulla teoria geometrica delle equazioni e delle funzioni algebriche, N. Zanichelli, Bologna 1915.

[4] Evain, L., Une minoration du degr des courbes planes singularites impos es Preprint ENS Lyon, 212 (1997), 1-17.
[5] Greuel, G.M. - Lossen, C. - Shustin, E., Plane curves of minimal degree with prescribed singularities (1997) to appear in Inv. Math.

[6] Grothendieck, A. - Dieudonn, J., EGA IV, 4, Elments de gomtrie algrique, Inst. Hautes Etudes Sci. Publ. Math. 32 (1967).

[7] Harbourne, B., Iterated blow-ups and moduli for rational surfaces, in: A. Holme and R. Speiser, eds., Algebraic Geometry Sundance 1986, LNM 1311, Springer 1988, pp. 101-117.

[8] Harbourne, B., Free resolutions of fat point ideals on $\mathbb{P}^2$ J. Pure App Algebra 125 (1998) 213-234.

[9] Hartshorne, B., Algebraic Geometry, GTM 52, Springer 1977.

[10] Kleiman, S.L., Multiple-point formulas I: Iteration, Acta Math. 147 (1981), 13-49.

[11] Nagata, N., On the fourteenth problem of Hilbert Amer. J. Math. 81 (1959), 766-772.

[12] Ran, Z., Curvilinear enumerative geometry, Acta Math. 155 (1985), 81-101.

[13] Xu, G., Curves in $\mathbb{P}^2$ and symplectic packings, Math. Ann. 299 (1994), 609-613.