NONLINEAR WAVES IN THERMOELASTIC DIELECTRICS

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ABSTRACT. This paper is addressed to the analysis of wave propagation in electroelastic materials. First the balance equations are reviewed and the entropy inequality is established. Next the constitutive equations are considered for a deformable and heat-conducting dielectric. To allow for discontinuity wave propagation, an appropriate objective rate equation of the heat flux is considered. The thermodynamic consistency of the whole set of constitutive equations is established. Next the nonlinear evolution equations so determined are tested in relation to wave propagation properties. Waves are investigated in the form of weak discontinuities and the whole system of equations for the jumps is obtained. As a particular simple case the propagation into an unperturbed region is examined. Both the classical electromagnetic waves and the thermal waves are found to occur. In both cases the mechanical term is found to be induced by the electrical or the thermal wave discontinuity.

1. Introduction. The modelling of electromechanical interactions within nonlinear continuum physics is a subject of interest in many respects. As with continuum mechanics, the Eulerian and Lagrangian descriptions of continuum physics differ for the appropriate functions representing forces and stresses, in addition to the difference about the use of the current configuration or the reference configuration. The two descriptions of the electromagnetic fields provide the magnetic field \(\mathbf{H}\), the magnetic induction \(\mathbf{B}\), the electric field \(\mathbf{E}\) and the electric displacement \(\mathbf{D}\) in the Lagrangian description while the corresponding description of the polarization (density) \(\mathbf{P}\) and the magnetization \(\mathbf{M}\) are not uniquely defined. This is so because the balance equations of electromagnetism in integral form involve \(\mathbf{H}\) and \(\mathbf{E}\) through line integrals and \(\mathbf{B}\) and \(\mathbf{D}\) through surface integrals.

The electromechanical interactions are described by the balance equations. Yet the literature shows differences among various representations of force, couple, and power in an electromagnetic continuum [14, 6, 1, 5]. In addition, contributions to the electromechanical interactions arise from the selection of the independent variables in the constitutive equations. In an electroelastic material we might think that \(\mathbf{E}\) is the natural variable of electric character. However, by the principle of objectivity the effect of \(\mathbf{E}\) on any scalar function should be the same as that supplied by the vector \(\mathbf{Q}\mathbf{E}\), \(\mathbf{Q}\) being any time-dependent rotation tensor. A natural hint is that we assume the dependence on \(\mathbf{E}\) via the invariant scalar \(\mathbf{E} \cdot \mathbf{E}\). Yet we might observe that there are also invariant vectors namely \(\mathbf{F}^T \mathbf{E}\), \(\mathbf{F}^{-1} \mathbf{E}\), and \(\mathbf{JF}^T \mathbf{E}\), \(\mathbf{JF}^{-1} \mathbf{E}\), where \(\mathbf{F}\) is the deformation gradient, the superposed \(T\) means transpose, and \(\mathbf{J} = \det \mathbf{F}\).

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Accordingly the use of any of these vectors as the appropriate variable of electric character seems to be equally justified. To fix ideas, in this paper we use $\mathbf{F}^T \mathbf{E}$ as the variable of electrical character. By means of different arguments about objectivity, the same choice is made in [8] and [5], where $\mathbf{F}^T \mathbf{E}$ is termed Lagrangian field.

This paper is addressed to the analysis of wave propagation in thermo-electroelastic materials. We first review the balance equations and hence establish the entropy inequality. Next we look at the constitutive equations. In addition to accounting for the deformation and the electric field, the body is supposed to be a heat-conducting material. Accordingly, the wave propagation is allowed and the thermodynamic consistency holds provided an appropriate modelling is considered for the heat flux vector. It is well known that Fourier’s law is inadequate to allow wave propagation and that various constitutive equations for heat conduction have been established that are compatible with wave propagation (see e.g. [13, 2, 11]). Consistent with the invariance of $\mathbf{F}^T \mathbf{w}$, for any vector $\mathbf{w}$, here the constitutive equation for the heat flux $\mathbf{q}$ is considered by letting $\mathbf{F}^T \mathbf{q}$ be the pertinent variable accounting for heat conduction. As a consequence it follows that $\mathbf{q}$ satisfies a rate equation involving an objective time derivative.

The thermodynamic consistency of the whole set of constitutive equations is established. Next the nonlinear evolution equations are tested in relation to wave propagation properties. Waves are investigated in the form of weak discontinuities and the whole system of equations for the jumps is obtained. As a particular simple case the propagation into an unperturbed region is examined.

**Notation.** We consider a body occupying the time dependent region $\mathcal{R}_t \subset \mathbb{E}^3$. The motion is described by means of the function $\chi(\mathbf{X}, t)$ providing the position vector $\mathbf{x} \in \mathcal{R}_t$ in terms of the position vector $\mathbf{X}$, in a reference configuration $\mathcal{R}$, and the time $t$, so that $\mathcal{R}_t = \chi(\mathcal{R}, t)$. The deformation is described by means of the deformation gradient $\mathbf{F} = \partial \mathbf{x} / \partial \mathbf{X}$, $\mathbf{F}_{ik} = \partial \chi_{ik} / \partial \chi_i$. The symbol $\nabla$ denotes the gradient in the current configuration $\Omega$, $\varepsilon$ the internal energy density (per unit mass), $\mathbf{T}$ the Cauchy stress, $\mathbf{L}$ the velocity gradient, $\mathbf{q}$ the heat flux vector, $r$ the (external) heat supply, $\rho$ the mass density. Owing to the polar character of polarizable media, the stress $\mathbf{T}$ need not be symmetric.

2. **Second law inequality.** Since the electric field $\mathbf{E}$ and the polarization $\mathbf{P}$ are dependent on the frame of reference, for definiteness we let $\mathbf{E}$ and $\mathbf{P}$ be the fields at the local frame of reference at rest with the material.

Relative to general models of electromagnetic solids we assume the magnetization and the electric current are zero. Moreover, no magnetic field is applied to the body. The balance equations of linear momentum, angular momentum, and energy can be written in the form\(^1\) (see, e.g., [14, 12])

$$\rho \dot{\mathbf{v}} = \nabla \cdot \mathbf{T} + \rho \mathbf{b} + q \mathbf{E} + \rho \mathbf{f}_p,$$

$$\text{skw} \mathbf{T} = \text{skw}(\mathbf{P} \otimes \mathbf{E}),$$

$$\rho \dot{\mathbf{e}} = \rho \mathbf{\pi} \cdot \mathbf{E} + \mathbf{T} \cdot \mathbf{L} - \nabla \cdot \mathbf{q} + \rho r,$$

where $\mathbf{\pi} = \mathbf{P} / \rho$ is the polarization per unit mass, $\mathbf{b}$ is the mechanical body force density (per unit mass), and $\mathbf{f}_p$ is the body force density on the electric dipoles.

\(^1\)In components, $(\nabla \cdot \mathbf{T})_i = \partial_{x_j} T_{ij}$. Comparisons with approaches where $(\nabla \cdot \mathbf{T})_i = \partial_{x_j} T_{ji}$ are possible by considering the transpose.
Particle-like arguments about the action of the field on single dipoles justify the assumption \( \rho f_r = (P \cdot \nabla)E + v \times (P \cdot \nabla)B + \rho \pi \times B \), \( B \) being possibly induced by \( E \) via Ampère’s law. For definiteness and simplicity here we let
\[
\rho f_r = (P \cdot \nabla)E,
\]
this being the dominant term for weak magnetic fields \( B \). The entropy per unit mass \( \eta \) is assumed to satisfy the inequality
\[
\rho \dot{\eta} + \nabla \cdot (q/\theta) - \rho r/\theta + \nabla \cdot k \geq 0,
\]
where \( k \) is the extra-entropy flux. Substitution of \( \nabla \cdot q - \rho r \) from the energy equation (2) gives
\[
\rho \theta \dot{\eta} - \rho \dot{\varepsilon} + \rho \pi \cdot E + T \cdot L - \frac{1}{\theta} q \cdot \nabla \theta + \theta \nabla \cdot k \geq 0.
\]
Let
\[
\phi = \varepsilon - \theta \eta - E \cdot \pi.
\]
Upon some rearrangements we can write the inequality in the form
\[
- \rho (\dot{\phi} + \eta \dot{\theta}) - P \cdot \dot{E} + T \cdot L - \frac{1}{\theta} q \cdot \nabla \theta + \theta \nabla \cdot k \geq 0. \tag{4}
\]
We state the second law of thermodynamics by saying that inequality (4) has to hold for all possible processes compatible with the balance equations.

In [3] and [5] the total stress tensor is considered as the sum of the elastic stress tensor, here \( T \), and the Maxwell stress tensor,
\[
\tau_m = E \otimes D - \frac{1}{2} \epsilon_0 (E \cdot E) I.
\]
The stress \( \tau_m \) is used systematically to express the boundary conditions and the use is allowed by the identity
\[
(P \cdot \nabla)E = \nabla \cdot \tau_m. \tag{5}
\]
The proof of (5) is based on the assumptions
\[
\nabla \times E = 0, \quad \nabla \cdot D = 0,
\]
the vanishing of \( \nabla \times E \) being a consequence of the assumed time independence of the pertinent fields. For, since \( \nabla \cdot D = 0 \) and \( D = \epsilon_0 E + P \) then
\[
\nabla \cdot (E \otimes D) = E \nabla \cdot D + (D \cdot \nabla)E = \epsilon_0 (E \cdot \nabla)E + (P \cdot \nabla)E.
\]
Since \( \nabla \times E = 0 \) then
\[
(E \cdot \nabla)E = \frac{1}{2} \nabla (E \cdot E)
\]
and hence (5) follows. This in turn shows that the use of \( \tau_m \) is less general than the use of (3) as the body force. In addition, as remarked also in [5], there are different definitions of the Maxwell stress. That is why we prefer to employ the body force \( (P \cdot \nabla)E \) rather than the stress \( \tau_m \).
3. **Constitutive assumptions and objectivity.** To describe the evolution of a thermoelastic dielectric we let the response functions of the material be determined by the deformation gradient $F$, the temperature $\theta$, and the electric field $E$. The dependence on the gradient $\nabla E$ is allowed so that nonlocal properties can be described. Moreover the dependence on the time derivative $\dot{E}$ is allowed so that appropriate evolution features can be modelled. Heat conduction is also allowed and this is thought to be modelled via the dependence both on $\nabla \theta$ and on the heat flux $q$. Hence we might say that the constitutive equations have the form
\[
\phi = \tilde{\phi}(F, \theta, \nabla \theta, \nabla E, \dot{E}, q)
\]
and the like for the other constitutive functions. Yet the constitutive functions are required to satisfy the principle of objectivity.

Let $F, F^*$ be two frames of reference. The vector positions $x$ and $x^*$ of a point, relative to $F$ and $F^*$, are related by ([16], §17)
\[
x^* = c(t) + Q(t)x,
\]
where $c(t)$ is an arbitrary vector-valued function while $Q(t)$ is a proper orthogonal tensor function, $\det Q = 1$. We denote by
\[
\Omega = QQ^T, \quad \Omega \in \text{Skw},
\]
the spin tensor associated with $Q$. A tensor function $A(x, t), x \in \Omega$, is objective if, under the change of frame $F \to F^*$, transforms as
\[
A^* = QAQ^T.
\]
The material time derivative of an objective tensor is not objective in that
\[
\dot{A}^* \neq Q\dot{A}Q^T.
\]
The analogue holds for an objective vector. Instead, a tensor function $A(x, t), x \in \Omega$, is invariant if, under the change of frame $F \to F^*$, satisfies the invariance property
\[
A^* = A.
\]
The deformation gradient $F$ is an objective vector (see [7]),
\[
F^* = QF.
\]
Hence the Cauchy-Green tensor
\[
C = F^T F
\]
is invariant in that
\[
C^* = (F^*)^T F^* = F^T Q^T QF = F^T F = C.
\]
Likewise, for any objective vector $u$ the vector $F^T u$ is invariant,
\[
(F^*)^T u^* = F^T Q^T Qu = F^T u.
\]
Hence, letting $E$ and $q$ be objective we can say that the vectors
\[
\Xi = F^T E, \quad Q = F^T q
\]
are invariant; incidentally, $\Xi$ is just the field $E_1$ of [4]. This in turn implies that
\[
\dot{C}, \quad \Xi, \quad Q
\]
are invariant.
Let $f$ be any invariant scalar or vector. The gradient $\nabla f$, in the current configuration, satisfies the relation

$$(\nabla f)^* = \partial_{x^*} f = Q \partial_x f = Q \nabla f$$

and hence $\nabla f$ is objective (vector). As a consequence $F^T \nabla f$ is invariant,

$$(F^T \nabla f)^* = (F^*)^T \nabla^* f = F^T Q^T Q \nabla f = F^T \nabla f.$$ 

Indeed,

$$F^T \nabla = \nabla_R$$

is the (invariant) gradient in the reference configuration.

Based on these properties, next we investigate the model of nonlinear deformable dielectrics by letting

$$\Gamma = (C, \theta, \Xi, \nabla_R \theta, \nabla_R \Xi, \dot{\Xi}, Q)$$

be the set of independent variables.

It is worth remarking that objectivity and invariance need not single out a unique set of independent variables. First we observe that $J F^T u$ is invariant too. Yet $J F^T u$ cannot be viewed as the gradient in the reference configuration. A further invariant field is $J F^{-1} u$ in that

$$(J F^{-1} u)^* = J^*(F^*)^{-1} u^* = J F^{-1} Q^{-1} Qu = J F^{-1} u.$$ 

Indeed, $J F^{-1} u$ is the material vector provided by Nanson’s formula [9, 10]. Moreover, $J F^{-1} P$ is the vector variable used in [15] to describe elastic dielectrics. The appropriate choice of physical variables is related to the constitutive properties under consideration.

The modelling of heat conduction hinges on the by-now standard view that Fourier’s law is not compatible with wave propagation and that, instead, equations of the Maxwell-Cattaneo type should be considered to obtain models allowing for finite wave speed. In this regard we observe that, since $\dot{F} = LF$,

$$\dot{Q} = \dot{F}^T q + F^T \dot{q} = F^T (\dot{q} + D q - W q),$$

where $D = \text{sym} L$, $W = \text{skw} L$. Also, since $\dot{j} = J \nabla \cdot v$,

$$(J F^T q)^* = J F^T (\dot{q} + D q - W q + (\nabla \cdot v) q)$$

and, since $\dot{F}^{-T} = -F^{-1} \dot{F} F^{-1}$,

$$(J F^{-1} q)^* = J F^{-1} (\dot{q} - D q - W q + (\nabla \cdot v) q).$$

Accordingly, the three derivatives are, respectively, $F^T$, $J F^T$, and $J F^{-1}$ times appropriate objective derivatives. Indeed,

$$\dot{q} + D q - W q \quad \text{and} \quad \dot{q} - D q - W q + (\nabla \cdot v) q$$

are the Cotter-Rivlin derivative and the Truesdell derivative [11] of $q$. 
4. Thermodynamic restrictions. We assume $\phi, \eta, T, \dot{\mathbf{Q}},$ and $k$ are functions of $\Gamma$. Indeed, we suppose $\phi$ is differentiable while the remaining functions are continuous. Evaluation of $\phi$ and substitution in (4) gives
\[ -\rho(\partial_{\phi} + \eta)\dot{\theta} - \rho \partial C \phi \cdot \dot{C} - \rho \partial \mathbf{a} \phi \cdot \dot{\mathbf{a}} - \rho \partial_{T} \phi \cdot \dot{\mathbf{a}} + \rho \partial_{\mathbf{Q}} \phi \cdot \dot{\mathbf{Q}} = -\rho \partial_{\mathbf{a}} \phi \cdot \dot{\mathbf{a}} - \rho \partial_{\mathbf{Q}} \phi \cdot \dot{\mathbf{Q}} - \mathbf{P} \cdot \dot{\mathbf{E}} + T \cdot \mathbf{L} - \frac{1}{\theta} \mathbf{q} \cdot \nabla \theta + \theta \nabla \cdot \mathbf{k} \geq 0. \] (6)
The arbitrariness and the linearity of $\dot{\mathbf{a}}, \dot{\mathbf{Q}},$ and $\dot{\theta}$ imply
\[ \partial_{\mathbf{a}} \phi = 0, \quad \partial_{T} \phi = 0, \quad \eta = -\partial_{\phi}. \] (7)
Hence we have
\[ -\rho \partial C \phi \cdot \dot{C} - \rho \partial \mathbf{a} \phi \cdot \dot{\mathbf{a}} - \rho \partial_{\mathbf{Q}} \phi \cdot \dot{\mathbf{Q}} - \mathbf{P} \cdot \dot{\mathbf{E}} + T \cdot \mathbf{L} - \frac{1}{\theta} \mathbf{q} \cdot \nabla \theta + \theta \nabla \cdot \mathbf{k} \geq 0. \] (8)
Some identities allow a better understanding of the structure of (8). Since $\dot{\mathbf{F}} = \mathbf{L} \mathbf{F}$ then
\[ \partial C \phi \cdot \dot{C} = \partial C \phi \cdot \dot{\mathbf{F}} + (\dot{\mathbf{F}}^T \mathbf{F} + \dot{\mathbf{F}}^T \dot{\mathbf{F}}) = 2(\mathbf{F} \partial C \phi \mathbf{F}^T) \cdot \mathbf{D}. \]
Moreover
\[ \partial \mathbf{a} \phi \cdot \dot{\mathbf{a}} = (\mathbf{F} \partial \mathbf{a} \phi) \cdot \dot{\mathbf{E}} + (\mathbf{E} \otimes \partial \mathbf{a} \phi) \cdot \dot{\mathbf{F}}, \quad (\mathbf{E} \otimes \partial \mathbf{a} \phi) \cdot \dot{\mathbf{F}} = [\mathbf{E} \otimes (\mathbf{F} \partial \mathbf{a} \phi)] \cdot \mathbf{L}. \]
Further,
\[ J \nabla \cdot \mathbf{k} = \nabla_{\mathbf{r}} \cdot \mathbf{k}, \quad \mathbf{k} = : J^{-1} \mathbf{F} \mathbf{F} \cdot \mathbf{D}. \]
As a consequence, upon multiplication by $J/\theta$, inequality (8) becomes
\[ \frac{1}{\theta} [J \mathbf{T} - 2 \rho \mathbf{G} \partial_{C} \phi \mathbf{F}^T - \rho \mathbf{E} \otimes (\mathbf{F} \partial \mathbf{a} \phi)] \cdot \mathbf{D} + [J \mathbf{T} - \rho \mathbf{E} \otimes (\mathbf{F} \partial \mathbf{a} \phi)] \cdot \mathbf{W} \]
\[ = -\frac{1}{\theta} (J \mathbf{P} + \rho \mathbf{F} \partial \mathbf{a} \phi) \cdot \dot{\mathbf{E}} - \frac{\rho_{B}}{\theta} \partial_{T} \mathbf{a} \phi \cdot \nabla_{\mathbf{r}} \Xi - \frac{1}{\theta} \mathbf{q} \cdot \nabla \theta + \theta \nabla \cdot \mathbf{k} \geq 0. \] (9)
Now,
\[ \nabla_{\mathbf{r}} \Xi = \nabla_{\mathbf{r}} \dot{\mathbf{a}} \]
and
\[ -\frac{\rho_{B}}{\theta} \partial_{T} \mathbf{a} \phi \cdot \nabla_{\mathbf{r}} \Xi = -\nabla_{\mathbf{r}} \cdot \frac{\rho_{B}}{\theta} \partial_{T} \mathbf{a} \phi \cdot \dot{\mathbf{a}} = \dot{\mathbf{a}} \cdot \frac{\rho_{B}}{\theta} \partial_{T} \mathbf{a} \phi \cdot \dot{\mathbf{a}} \]
where
\[ \dot{\mathbf{a}} = \nabla_{\mathbf{r}} \cdot \frac{\rho_{B}}{\theta} \partial_{T} \mathbf{a} \phi. \]
Moreover,
\[ \dot{\mathbf{a}} \cdot (\dot{\mathbf{F}}^T \mathbf{E} + \dot{\mathbf{F}}^T \dot{\mathbf{E}}) = (\mathbf{E} \otimes \mathbf{F} \dot{\mathbf{a}}) \cdot \mathbf{L} + (\mathbf{F} \dot{\mathbf{a}}) \cdot \dot{\mathbf{E}}. \]
Hence we can write inequality (9) in the form
\[ \frac{1}{\theta} [J \mathbf{T} - 2 \rho \mathbf{G} \partial_{C} \phi \mathbf{F}^T - \rho \mathbf{E} \otimes (\mathbf{F} \partial \mathbf{a} \phi) + \mathbf{E} \otimes \mathbf{F} \partial \mathbf{a} \phi] \cdot \mathbf{D} \]
\[ + \frac{1}{\theta} [J \mathbf{T} - \rho \mathbf{E} \otimes (\mathbf{F} \partial \mathbf{a} \phi) + \mathbf{E} \otimes (\mathbf{F} \partial \mathbf{a} \phi)] \cdot \mathbf{W} - \frac{1}{\theta} \mathbf{q} \cdot \nabla \theta \]
\[ -\frac{1}{\theta} (J \mathbf{P} + \rho \mathbf{F} \partial \mathbf{a} \phi - \rho \mathbf{E} \otimes \mathbf{F} \partial \mathbf{a} \phi) \cdot \dot{\mathbf{E}} - \frac{\rho_{B}}{\theta} \partial_{T} \mathbf{a} \phi \cdot \dot{\mathbf{Q}} + \nabla_{\mathbf{r}} \cdot (\mathbf{k} - \frac{\rho_{B}}{\theta} \partial_{T} \mathbf{a} \phi \cdot \dot{\mathbf{a}}) \geq 0. \]
This inequality suggests that we let
\[ \mathbf{k} = \frac{\rho_{B}}{\theta} \partial_{T} \mathbf{a} \phi \cdot \dot{\mathbf{a}}. \] (10)
The arbitrariness and the linearity of $\mathbf{D}, \mathbf{W}, \dot{\mathbf{E}}$ imply that
\[ \mathbf{T} = 2 \rho \mathbf{F} \partial_{C} \phi \mathbf{F}^T + \rho \mathbf{E} \otimes (\mathbf{F} \partial \mathbf{a} \phi) - \theta \mathbf{E} \otimes (J^{-1} \mathbf{F} \partial \mathbf{a} \phi), \] (11)
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\[
P = -\rho F \partial_\xi \phi + \theta J^{-1} F \xi.
\]  (12)

Upon multiplying by \(\theta/J\) the remaining inequality we find

\[
-\frac{1}{\theta} \mathbf{q} \cdot \nabla \theta - \rho \partial_\phi \mathbf{Q} \cdot \mathbf{Q} \geq 0.
\]  (13)

Equations (7), (10)-(13) are sufficient for the validity of the second law inequality.

Further conclusions follow if the function \(\dot{\mathbf{Q}}\) is given an explicit form. For definiteness we let

\[
\dot{\mathbf{Q}} = -\frac{1}{\tau} [\mathbf{Q} + \kappa \nabla \mu \theta],
\]  (14)

\(\tau\) and \(\kappa\) being possibly dependent on \(J, \theta\). We can then solve equation (14) on the interval \([t_0, t]\) and find that

\[
\mathbf{Q}(t) = \mathbf{Q}(t_0) \exp\left(\int_{t_0}^{t} \frac{\kappa}{\tau} (s) \exp(\int_{s}^{t} \tau^{-1}(y) dy) \nabla \mu \theta(s) ds\right).
\]  (15)

We now examine the consistency between equation (14) and inequality (13). Upon substitution we have

\[
-\frac{1}{\theta} \mathbf{q} \cdot \nabla \theta + \frac{\rho}{\tau} \partial_\phi \mathbf{Q} \cdot (\mathbf{Q} + \kappa \nabla \mu \theta) \geq 0.
\]  (16)

Since

\[
\mathbf{q} \cdot \nabla \theta = (\mathbf{F}^{-T} \mathbf{Q}) \cdot (\mathbf{F}^{-T} \nabla \mu \theta) = (\mathbf{F}^{-1} \mathbf{F}^{-T} \mathbf{Q}) \cdot \nabla \mu \theta = (\mathbf{C}^{-1} \mathbf{Q}) \cdot \nabla \mu \theta,
\]

we can write inequality (16) in the form

\[
(-\frac{1}{\theta} \mathbf{C}^{-1} \mathbf{Q} + \frac{\kappa \rho}{\tau} \partial_\phi \mathbf{Q}) \cdot \nabla \mu \theta + \frac{\rho}{\tau} \partial_\phi \mathbf{Q} \cdot \mathbf{Q} \geq 0.
\]

The arbitrariness of \(\nabla \mu \theta\) implies that the inequality holds if and only if

\[
-\frac{1}{\theta} \mathbf{C}^{-1} \mathbf{Q} + \frac{\kappa \rho}{\tau} \partial_\phi \mathbf{Q} = 0,
\]  (17)

\[
\frac{\rho}{\tau} \partial_\phi \mathbf{Q} \cdot \mathbf{Q} \geq 0.
\]  (18)

A direct integration of (17) gives

\[
\phi = \frac{\tau}{2 \kappa \rho \theta} \mathbf{Q} \cdot \mathbf{C}^{-1} \mathbf{Q} + \hat{\phi}(\mathbf{C}, \theta, \Xi, \nabla \mu \Xi).
\]  (19)

Inequality (18) then becomes

\[
\frac{1}{\kappa \theta} \mathbf{Q} \cdot \mathbf{C}^{-1} \mathbf{Q} \geq 0.
\]  (20)

The positive definiteness of \(\mathbf{C}^{-1}\) implies that inequality (20) holds if and only if \(\kappa > 0\). Accordingly, as with the classical Fourier’s law, the second law provides the positive valuedness of the heat conductivity.

It is worth pointing out that the rate equation (14) is invariant under a change of reference configuration. Let \(\mathcal{R}_1\) and \(\mathcal{R}_2\) be two reference configurations and let \(\mathbf{F}_0\) be the (constant) deformation gradient associated with the deformation \(\mathcal{R}_1 \rightarrow \mathcal{R}_2\). Let \(\mathbf{F}_1\) and \(\mathbf{F}_2\) be the deformation gradients relative to \(\mathcal{R}_1\) and \(\mathcal{R}_2\). Hence

\[
\mathbf{F}_1(t) = \mathbf{F}_2(t) \mathbf{F}_0,
\]
at any \(X \in \mathcal{R}_1\). Now, if \(F\) is the deformation gradient then
\[
\nabla_\theta \psi = F^T \nabla \theta.
\]
Relative to \(\mathcal{R}_1\), eq. (14) becomes
\[
(F^T_1 q)' = - \frac{1}{\tau} [F^T_1 q + \kappa F^T_1 \nabla \theta].
\]
Replacing \(F_1 = F_2 F_0\) we obtain
\[
F^T_0 (F^T_2 q)' = - \frac{1}{\tau} F^T_0 [F^T_2 q + \kappa F^T_2 \nabla \theta],
\]
whence
\[
(F^T_2 q)' = - \frac{1}{\tau} [F^T_2 q + \kappa F^T_2 \nabla \theta].
\]
As a consequence
\[
(F^T_1 q)' = - \frac{1}{\tau} [F^T_1 q + \kappa F^T_1 \nabla \theta] \iff (F^T_2 q)' = - \frac{1}{\tau} [F^T_2 q + \kappa F^T_2 \nabla \theta]
\]
for any pair of deformation gradients connected by \(F_2 = F_1 F_0\). The same invariance property follows from the solution (15).

5. **Evolution equations.** The equations governing the evolution of the body are now established by means of the balance equations and the constitutive properties. To get a more tractable model we neglect the dependence on \(\nabla_\theta \Xi\) and let \(q = 0\). We also neglect the body force \(\rho b\) and the heat supply \(\rho r\).

The equation of motion (1) reduces to
\[
\rho \dot{v} = \nabla \cdot T + (P \cdot \nabla) E. \tag{21}
\]
By (11)-(12) we can write the Cauchy stress \(T\) in the form
\[
T = \mathcal{T} - E \otimes P, \quad \mathcal{T} := 2 \rho \dot{F} \partial_C \phi F^T.
\]
Since
\[
\nabla \cdot (E \otimes P) = E \nabla \cdot P + (P \cdot \nabla) E
\]
and
\[
0 = q = \nabla \cdot (\varepsilon_0 E + P)
\]
than equation (21) becomes
\[
\rho \dot{v} = \nabla \cdot \mathcal{T} + \varepsilon_0 E \nabla \cdot E. \tag{22}
\]
By definition, the energy density \(\varepsilon\) is given by
\[
\varepsilon = \phi + \eta \theta + E \cdot \pi.
\]
Hence
\[
\dot{\varepsilon} = (\partial_\theta \phi + \eta) \dot{\theta} + \theta \dot{\eta} + \dot{E} \cdot \pi + E \cdot \dot{\pi} + \partial_J f \dot{J} Q \cdot C^{-1} Q + \partial_C \phi \dot{C} + \partial_{\mathcal{E}} \phi \cdot \dot{\mathcal{E}}.
\]
Since
\[
\partial_{\mathcal{E}} \phi \cdot \dot{\mathcal{E}} = (E \otimes \partial_{\mathcal{E}} \phi) \cdot \dot{F} + (F \partial_{\mathcal{E}} \phi) \cdot \dot{E}
\]
and, by (7) and (12),
\[
\partial_\theta \phi + \eta = 0, \quad F \partial_{\mathcal{E}} \phi + E = 0,
\]
we find
\[
\dot{\varepsilon} = \dot{\theta} \dot{\eta} + E \cdot \dot{\pi} + \partial_J f \dot{J} Q \cdot C^{-1} Q + \partial_C \phi \cdot C + (E \otimes \partial_{\mathcal{E}} \phi) \cdot \dot{F}.
\]
The balance of energy (2) then becomes
\[ \rho \theta \dot{\eta} + \partial f \oint J \cdot \nabla (C^{-1} Q + \partial C \phi \cdot \dot{C} + (E \otimes \partial \phi) \cdot \dot{F}) \cdot (T \cdot L - \nabla \cdot q). \] (23)

Moreover
\[ T \cdot L = T \cdot L - (E \otimes P) \cdot L = (TF^{-T} - E \otimes F^{-1} P) \cdot \dot{F}. \]
and
\[ \partial C \phi \cdot \dot{C} + (E \otimes \partial \phi) \cdot \dot{F} = (2F \partial C \phi + E \otimes \partial \phi) \cdot \dot{F}. \]
In view of (11)-(12) it follows that
\[ \partial C \phi \cdot \dot{C} + (E \otimes \partial \phi) \cdot \dot{F} = \dot{C} \cdot \dot{F}. \]
and hence (23) reduces to
\[ \rho \theta \dot{\eta} = -\nabla \cdot q. \] (24)

To evaluate \( \dot{\eta} \) it is convenient to consider \( q \) as a variable, rather than \( Q \). For definiteness, based on (19) we assume \( \phi \) takes the form
\[ \phi = f(\theta)Q \cdot C^{-1} Q + \Phi(\theta) + \dot{\phi}(\mathbf{C}, \mathbf{\Xi}). \]
Hence observe
\[ Q \cdot C^{-1} Q = q^2. \]
As a consequence the entropy per unit mass \( \eta \) can be written
\[ \eta = -f'\theta q^2 - \Phi'\theta \]
whence
\[ \dot{\eta} = -[f''\theta q^2 + \Phi''\theta] \dot{\theta} - 2f'\theta q \cdot \dot{q}. \]
At \( q = 0 \), it is natural to assume that \( \dot{\eta} \) is positive if \( \dot{\theta} \) is positive or rather \( \eta \) is an increasing function of \( \theta \). This is the case for an ideal gas ([18], §9.2). Accordingly we let
\[ \Phi'' < 0. \]
The value \( \dot{q} \) is determined by the rate equation (14). Since
\[ \dot{Q} = F^T q + F^T \dot{q} \]
then equation (14) becomes
\[ F^T q + F^T \dot{q} = -\frac{1}{\tau} (F^T q + \kappa F^T v) \cdot \nabla \theta \]
and then
\[ \dot{q} = -L^T q - \frac{1}{\tau} q - \frac{\kappa}{\tau} q \cdot \nabla \theta. \] (25)
As a consequence,
\[ q \cdot \dot{q} = -q \cdot Dq - \frac{1}{\tau} q^2 - \frac{\kappa}{\tau} q \cdot \nabla \theta. \]
Substitution in (24) results in
\[ -\rho \theta \{(f''q^2 + \Phi''\theta) \dot{\theta} - 2f'q \cdot Dq + \frac{1}{\tau} q^2 + \frac{\kappa}{\tau} q \cdot \nabla \theta \} + \nabla \cdot q = 0. \] (26)
The mass density \( \rho \) is subject to the continuity equation
\[ \dot{\rho} + \rho \nabla \cdot v = 0. \] (27)
The occurrence of \( \nabla E \) in the equation of motion (22) requires that we consider also Maxwell’s equations. Since we are dealing with non-magnetic materials then we let \( B = \mu_0 H \). In addition we let the electric current be zero. Since
\[ |E| = (F^{-T} \Xi \cdot F^{-T} \Xi)^{1/2} = (\Xi \cdot C^{-1} \Xi)^{1/2}, \]
Hence we can write Maxwell’s equations in the form
\[ \nabla \times \mathbf{E} = \rho, \quad \nabla \cdot \mathbf{H} = 0, \]
where \( \rho \) and \( \mathbf{H} \) are continuous across \( \sigma \). Hence we can write Maxwell’s equations in the form
\[ \mathbf{D} = \epsilon(\rho, |\mathbf{E}|)\mathbf{E}, \quad \epsilon = \epsilon_0 + \frac{\rho h'}{|\mathbf{E}|}. \]
As a consequence
\[ \mathbf{D} = \epsilon(\rho, |\mathbf{E}|)\mathbf{E}, \quad \epsilon = \epsilon_0 + \frac{\rho h'}{|\mathbf{E}|}. \]

Finally, in suffix notation, the equation of motion (22) becomes
\[ \rho \dot{\mathbf{v}}_i = \partial C_{\mathbf{K}} T_{ij} \partial x_j \mathbf{C}_{\mathbf{K}} + \partial \rho T_{ij} \partial x_j \mathbf{q} + \partial \mathbf{H}_j \partial x_j \mathbf{K}_{\mathbf{K}} + \epsilon_0 \mathbf{E}_i \partial x_k \mathbf{E}_k. \]

Equations (25)-(30) constitute the system of governing equations in the unknown functions \( \rho, \mathbf{v}, \mathbf{q}, \mathbf{H}, \mathbf{E} \). Some properties of the system (25)-(30) are now investigated in connection with weak discontinuity waves.

6. Weak discontinuity waves. A propagating surface \( \sigma(t) \) is a weak discontinuity wave if the fields \( \rho, \mathbf{v}, \mathbf{F}, \mathbf{q}, \mathbf{H}, \mathbf{E} \), with \( \mathbf{v} = \dot{\mathbf{X}}, \mathbf{F} = \nabla \dot{\mathbf{X}} \), have the following properties.

1) \( \rho, \mathbf{v}, \mathbf{F}, \mathbf{q}, \mathbf{H}, \mathbf{E} \) are continuous functions of \( \mathbf{x} \) and \( t \) jointly for all \( \mathbf{x} \in \mathbb{R}_t, t \in \mathbb{R} \);
2) \( \rho, \nabla \rho, \mathbf{v}, \mathbf{F}, \nabla \mathbf{F}, \mathbf{q}, \partial \mathbf{H}, \nabla \mathbf{H}, \mathbf{E}, \nabla \mathbf{E} \) and all higher order derivatives suffer, at most, jump discontinuities across \( \sigma \) but are continuous in \( \mathbf{x} \) and \( t \) jointly everywhere else.

As an immediate consequence it follows that
\[ \mathbf{T} = 2 \rho \mathbf{F} \partial C_{\dot{\phi}} (\mathbf{C}, \Xi, \mathbf{Q}) \mathbf{F}^T \]
is continuous across \( \sigma \).

Let \( \Omega_+ \) and \( \Omega_- \) be the subregions of \( \mathbb{R}_t \) ahead of and behind \( \sigma \), \( \mathbb{R}_t = \Omega_- \cup \sigma \cup \Omega_+ \). Also let \( \mathbf{n} \) be the unit normal to \( \sigma \) pointing into \( \Omega_+ \). Let \( [g] \) denote the discontinuity of \( g \) across \( \sigma \). By definition,
\[ [g] = g_- - g_+, \]
where \( g_- \) and \( g_+ \) are the limit values of \( g(\mathbf{x}, t) \) as \( \mathbf{x} \) approaches the pertinent point on \( \sigma(t) \) while remaining within \( \Omega_+ \) and \( \Omega_- \), respectively.

At a weak discontinuity the jumps associated with the system (25)-(30) satisfy
\[ [\dot{\mathbf{q}}] = -[\mathbf{L}] [\mathbf{q}] - \frac{\epsilon}{\tau} [\nabla \mathbf{E}], \tag{31} \]
\[ [\mathbf{q} + 2 \rho (f' q^2 + \mathbf{F} \cdot \mathbf{n}) \hat{\mathbf{r}} + 2 \rho \mathbf{F} \cdot [\mathbf{D}] \mathbf{q} + \frac{2 \rho \mathbf{F} \cdot f' \mathbf{q}}{\tau} \mathbf{q} \cdot [\nabla \mathbf{E}] + [\nabla \cdot \mathbf{q}] = 0, \tag{32} \]
\[ \epsilon [\mathbf{L} \cdot \mathbf{E}] + [\mathbf{E}] = 0, \quad [\mathbf{L} \cdot \mathbf{E}] = 0, \tag{33} \]
\[ [\mathbf{E} \cdot \mathbf{H}] = -\mu_0 (\dot{\mathbf{H}}) - [\mathbf{E} \cdot \mathbf{H}] + [\mathbf{E} \cdot \mathbf{H}] = 0, \tag{34} \]
\[ [\mathbf{L} \cdot \mathbf{H}] = \epsilon (\dot{\mathbf{E}}) = -[\mathbf{L} \cdot \mathbf{E}] + [\mathbf{E} \cdot \mathbf{E}] = 0, \tag{35} \]
\[ \rho [\dot{\mathbf{v}}_i] = \partial C_{\mathbf{K}} T_{ij} [\partial x_j \mathbf{C}_{\mathbf{K}}] + \partial \rho T_{ij} [\partial x_j \mathbf{q}] + \partial \mathbf{H} T_{ij} [\partial x_j \mathbf{K} + E_i [\nabla \cdot \mathbf{E}], \tag{37} \]
The jumps are subject to the geometrical and kinematical conditions of compatibility ([17], chapter C). Denote by
\[
\partial_n g = \mathbf{n} \cdot \nabla g
\]
the normal derivative of a field \(g\). By the geometrical conditions of compatibility if the vector field \(\mathbf{w}\) is continuous across \(\sigma\) then
\[
[\nabla \cdot \mathbf{w}] = \mathbf{n} \cdot [\partial_n \mathbf{w}], \quad [\nabla \times \mathbf{w}] = \mathbf{n} \times [\partial_n \mathbf{w}],
\]
while
\[
[\partial_{\mathbf{x}} w_i] = [\partial_n w_i] n_j.
\]
By the kinematical conditions it follows that \(u_n\) being the speed of displacement of \(\mathbf{n}\). Since \(\dot{g} = \partial_t g + \mathbf{v} \cdot \nabla g\) and \([\mathbf{v}] = 0\) then
\[
\dot{g} = -U[\partial_n g], \quad U = u_n - \mathbf{v} \cdot \mathbf{n},
\]
\(U\) being referred to as the local speed of propagation of \(\sigma\).

We now investigate the system (31)-(37) in the unknowns \([\partial_n \theta]\), \([\partial_n \rho]\), \([\partial_n \mathbf{v}]\), \([\partial_n \mathbf{q}]\), \([\partial_n \mathbf{E}]\), \([\partial_n \mathbf{H}]\). Look first at (34)-(36). By (34) it follows
\[
\epsilon \mathbf{n} \cdot [\partial_n \mathbf{E}] + \partial_\rho \epsilon \mathbf{E} \cdot \mathbf{n} [\partial_n \rho] + \partial_\mathbf{E} \epsilon \mathbf{E} \cdot \mathbf{n} [\partial_n \mathbf{E}] / |\mathbf{E}| = 0, \quad \mathbf{n} \cdot [\partial_n \mathbf{H}] = 0.
\]
Since
\[
[\nabla \cdot \mathbf{v}] = \mathbf{n} \cdot [\partial_n \mathbf{v}], \quad [\mathbf{L}] = [\partial_n \mathbf{v}] \otimes \mathbf{n}
\]
then by applying the geometrical and kinematical conditions of compatibility to eqs (35) and (36) we obtain
\[
\mathbf{n} \times [\partial_n \mathbf{E}] = \mu_0 (U [\partial_n \mathbf{H}] + \mathbf{n} \cdot \mathbf{H} [\partial_n \mathbf{v}] - \mathbf{H} \cdot [\partial_n \mathbf{v}] - [\partial_n \mathbf{v}] [\partial_n \mathbf{E}]),
\]
\[
\mathbf{n} \times [\partial_n \mathbf{H}] = -\epsilon (U [\partial_n \mathbf{E}] + \mathbf{n} \cdot \mathbf{E} [\partial_n \mathbf{v}] - \mathbf{E} \cdot [\partial_n \mathbf{v}] - [\partial_n \mathbf{v}] [\partial_n \mathbf{H}] - [\partial_n \mathbf{v}] [\partial_n \mathbf{H}]).
\]
Evaluation of the curl of both equations and some rearrangements result in
\[
(1 - \mu_0 \epsilon_0 \epsilon_r U^2) [\partial_n \mathbf{E}] = \mu_0 \epsilon_0 \epsilon_r U (\mathbf{n} \cdot \mathbf{E} [\partial_n \mathbf{v}] - \mathbf{n} \cdot [\partial_n \mathbf{v}] \mathbf{E})
\]
\[
-\mu_0 (\mathbf{n} \times [\partial_n \mathbf{v}] \mathbf{n} \cdot \mathbf{H} - \mathbf{n} \cdot [\partial_n \mathbf{v}] \mathbf{n} \times \mathbf{H}),
\]
\[
(1 - \mu_0 \epsilon_0 \epsilon_r U^2) [\partial_n \mathbf{H}] = \mu_0 \epsilon_0 \epsilon_r U (\mathbf{n} \cdot [\partial_n \mathbf{v}] \mathbf{H} - \mathbf{n} \cdot \mathbf{H} [\partial_n \mathbf{v}] - \mathbf{n} \cdot [\partial_n \mathbf{v}] \mathbf{H}),
\]
\[
+\epsilon_0 \epsilon_r (\mathbf{n} \cdot \mathbf{E} \mathbf{n} \times [\partial_n \mathbf{v}] - \mathbf{n} \times \mathbf{E} \mathbf{n} \cdot [\partial_n \mathbf{v}]).
\]
The discontinuities \([\partial_n \mathbf{E}]\) and \([\partial_n \mathbf{H}]\) are much affected by the mechanical discontinuity \([\partial_n \mathbf{v}]\) and the state ahead of the wave, via \(\mathbf{E}\) and \(\mathbf{H}\). Yet, \(U^2 \neq 1 / \mu_0 \epsilon_0 \epsilon_r\) then \(\mathbf{n} \cdot [\partial_n \mathbf{E}] = 0, \mathbf{n} \cdot [\partial_n \mathbf{H}] = 0\).

We now apply the compatibility relations (38), (39), (41) to obtain the system for the jumps of the normal derivatives. By (31) and (32) we obtain
\[
U [\partial_n \mathbf{q}] = \mathbf{n} \cdot \mathbf{q} [\partial_n \mathbf{v}] + \frac{\kappa}{\tau} [\partial_n \theta]
\]
and
\[
\rho \theta U (f'' q^2 + \Phi'') [\partial_n \theta] + 2 \rho \theta f' q \cdot \mathbf{n} \cdot [\partial_n \mathbf{v}] + \frac{2 \rho \theta f' \kappa}{\tau} \mathbf{q} \cdot \mathbf{n} [\partial_n \theta] + \mathbf{n} \cdot [\partial_n \mathbf{q}] = 0.
\]
Moreover, by (33) we find
\[
U [\partial_n \rho] = \rho \mathbf{n} \cdot [\partial_n \mathbf{v}] .
\]
Equation (37) can be written
\[-\rho U [\partial_n v_i] = \partial_{\varepsilon_{ij}} T_{ij} n_j [\partial_n C_{iK}] + \partial_\theta T_{ij} n_j [\partial_n \theta_i] + \partial_{\varepsilon_{ij}} T_{ij} n_j E_h [\partial_n F_{iK}] + \partial_{\varepsilon_{ij}} T_{ij} n_j E_h [\partial_n E_h] + e_0 E_r n_h [\partial_n E_h], \] (46)

We now determine \([\partial_n v_i], [\partial_n F_{iK}], \text{ and } [\partial_n C_{iK}]\) in terms of \([\partial_n^2 u_i]\). Since
\[F_{iK} = \delta_{ij} + \partial_{x_{x_i}} u_h = \delta_{ij} + \partial_{x_{x_i}} u_h F_{jK}\]
and
\[\partial_n F_{iK} = [\partial_n \partial_{x_{x_i}} u_h] F_{jK} + \partial_{x_{x_i}} u_h [\partial_n F_{jK}]\]
then
\[(\delta_{ij} - \partial_{x_{x_i}} u_h)[\partial_n F_{jK}] = F_{jK} [\partial_n \partial_{x_{x_i}} u_h] = F_{jK} n_i [\partial_n^2 u_h].\]
Letting
\[A_{hj} = \delta_{jh} - \partial_{x_{x_i}} u_h\]
we obtain
\[\partial_n F_{iK} = A_{hj}^{-1} F_{jK} n_p [\partial_n^2 u_h].\]

As a consequence,
\[\partial_n C_{iK} = [\partial_n F_{iH}] F_{iK} + F_{iH} [\partial_n F_{iK}]\]
can be expressed in terms of \([\partial_n^2 u_i]\). Finally,
\[\partial_n v_i = [\partial_n (\partial_{x_{x_i}} u + v \cdot \nabla u_i)] = [\partial_n \partial_{x_{x_i}} u_i] + [\partial_n v] \cdot \nabla u_i + v \cdot \nabla [\partial_n u_i]\]
\[= (\partial_{x_{x_i}} u_i + v \cdot \nabla u_i) [\partial_n^2 u_i] + [\partial_n v_i] \partial_{x_{x_i}} u_i\]
whence
\[\partial_n v_i = - U A_{hj}^{-1} [\partial_n^2 u_i].\]

If the region ahead is undeformed then
\[F_{iK} = \delta_{ij}, \partial_{x_{x_i}} u_h = 0, [\partial_n F_{iK}] = n_K [\partial_n^2 u_h], [\partial_n C_{iK}] = n_H [\partial_n^2 u_K] + n_K [\partial_n^2 u_H].\]

A simple scheme arises if we restrict attention to wave propagation in an unperterbed region where \(E = 0, H = 0, q = 0, \text{ and } F = 1, \nabla u = 0.\) In this case eq. (40) reduces to
\[n \cdot [\partial_n E] = 0, \quad n \cdot [\partial_n H] = 0,\]
whence \([\partial_n E] \text{ and } [\partial_n H] \text{ are orthogonal to } n.\) Moreover eqs. (42)-(43) simplify to
\[1 - \mu_0 \varepsilon_\rho \varepsilon_r U^2 [\partial_n E] = 0, \quad (47)\]
\[1 - \mu_0 \varepsilon_\rho \varepsilon_r U^2 [\partial_n H] = 0. \quad (48)\]

Equations (44)-(45) take the simple form
\[U [\partial_n q] = \frac{\kappa}{r} n [\partial_n \theta], \quad \rho \theta U \Phi'' [\partial_n \theta] + n \cdot [\partial_n q] = 0,\]
whence
\[(\rho \theta |\Phi''| U^2 - \frac{\kappa}{r}) [\partial_n \theta] = 0. \quad (49)\]

Finally, eq. (46) reduces to
\[\rho U^2 [\partial_n^2 u_i] = 2 \sigma C_{iK} T_{ij} n_j n_l [\partial_n^2 u_K] + \partial_\theta T_{ij} n_j [\partial_n \theta] + \partial_{\varepsilon_{ij}} T_{ij} n_j [\partial_n E_K]. \quad (50)\]

If \(T_{ij}\) is nonlinear in \(\mathbf{E}\) then \(\partial_{\varepsilon_{ij}} T_{ij} = 0\) as \(E = 0.\)

By (47), as well as by (48), it follows that an electromagnetic wave occurs, \([\partial_n E] \neq 0\), only if
\[U^2 = \frac{1}{\mu_0 \varepsilon_\rho \varepsilon_r}. \quad (51)\]
which is the classical result for electromagnetic waves in undeformable media. In this case, by (49) it follows \( [\partial_n \theta] = 0 \) and the wave can be viewed as homothermal. The jump \( [\partial^2_n u] \) instead is nonzero if \( \partial_{zK} T_{ij} \neq 0 \). Indeed, it follows from (50) that
\[
(\rho U^2 \delta_{iK} - 2\partial_{C:HK} T_{ij} n_j n_H) [\partial^2_n u_K] = \partial_{zK} T_{ij} n_j [\partial_n E_K],
\]
which shows how \( [\partial^2_n u_K] \) is produced by \( [\partial_n E_K] \).

Discontinuities of different character occur if \( U \) is not given by (51). By (49) it follows that \( [\partial_n \theta] \) is nonzero only if
\[
U^2 = \frac{k}{\rho \theta T |\Phi|^2}.
\]
If \( U \) is given by (49) then \( [\partial_n E] = 0 \) and (50) supplies
\[
(\rho U^2 \delta_{iK} - 2\partial_{C:HK} T_{ij} n_j n_H) [\partial^2_n u_K] = \partial_{\theta} T_{ij} n_j [\partial_n \theta].
\]

Owing to the dependence of \( T \) on the temperature, the thermal discontinuity \( [\partial_n \theta] \) induces the mechanical discontinuity \( [\partial^2_n u_K] \).

To sum up, if the state ahead of the discontinuity surface is unperturbed then two types of waves are allowed: electromagnetic waves, with \( U \) given by (51), and thermal waves, with \( U \) given by (52). Equation (53) shows how a mechanical jump \( [\partial^2_n u] \) is produced by a thermal jump \( [\partial \theta] \).

In [4] homothermal plane waves are investigated in finitely deformed electroelastic solids, with no magnetic effects. A comparison is then in order between eq. (46) and the propagation condition (4.18) of [4]. For simplicity we let \( F = 1 \) and \( \partial_{\theta} T_{ij} = 0 \). To make the comparison significant we now let \( E \neq 0 \). Also we let \( U^2 \neq 1/\mu_0 c_0 \epsilon_r \); by (42) it follows that \( n \cdot [\partial_n E] = 0 \). Equation (46) becomes
\[
\rho U^2 [\partial^2_n u_i] - 2\partial_{C:HK} T_{ij} n_j n_H [\partial^2_n u_K] = \partial_{zK} T_{ij} n_j [\partial^2_n u_K] = \partial_{zK} T_{ij} n_j F_{hK} [\partial_n E_k].
\]
By (42) we can express \( [\partial_n E] \) in terms of \( [\partial^2_n u] \) to obtain the propagation condition for the jump \( [\partial^2_n u] \). The more involved form of the present propagation condition, relative to (4.12) of [4] is motivated by the different models adopted. In [4],
\[
\nabla \times E = 0, \quad \dot{T} = A F + \Gamma \dot{D},
\]
so that the variation of \( T \) is linear with respect to the variations \( \dot{F} \) of \( F \) and \( \dot{D} \) of \( D \). The linearity then much simplifies the propagation condition.

7. Conclusions. Nonlinear dielectrics are modelled as thermo-electroelastic materials. The natural dependence of constitutive properties on the electric field \( E \) is questioned in relation with the objectivity principle. The invariance of \( F^T E \), as well as \( J F^T E, F^{-1} E, JF^{-1} E \), suggests that we let the dependence on \( E \) hold via \( F^T E \). The corresponding thermodynamic restrictions are then derived and in particular it follows that
\[
\text{skw} T = \text{skw} (P \otimes E),
\]
as is required by the balance of angular momentum. This would not be the case with the dependence on \( E \) rather than on \( F^T E \).

If the material is assumed to conduct in the Maxwell-Cattaneo sense then wave propagation becomes a possibility; of course, its constitutive properties are still required to be consistent with thermodynamics and the principle of objectivity. Here this is accomplished by letting \( F^T q \) be the appropriate variable for the heat flux and the constitutive equation is assumed in the rate-equation form. The whole set of thermodynamic restrictions is then derived.
The nonlinear model so established is investigated in connection with the propagation of weak discontinuity waves. In this regard some new compatibility conditions are derived that are associated with the simultaneous occurrence of material and spatial derivatives such as $\dot{v}, F$ and $L, \nabla \theta$. After deriving the whole system of equation in the unknown discontinuities, the particular case of propagation in an unperturbed region is examined. It follows that two types of waves are allowed: electromagnetic waves and thermal waves with local speed of propagation given by (51) and (52). In both cases the mechanical jump $[\partial^2_n u]$ is nonzero and can be viewed as induced by $[\partial_n E]$ or $[\partial_n \theta]$.

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