A construction of slice knots
via annulus twists

Tetsuya Abe (RIMS)
joint work with
Motoo Tange (Tsukuba University)

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The definitions of slice knots and ribbon knots

\[ S^3 : \text{3-sphere} \]
\[ B^4 : \text{4-ball s.t } \partial B^4 = S^3 \]

**A knot \( K \) in \( S^3 \) is slice**

if it bounds a properly embedded disk in \( B^4 \).

**A knot \( K \) in \( S^3 \) is ribbon**

if it bounds an immersed disk in \( S^3 \) with only ribbon singularities.
The slice-ribbon conjecture

**Fact.** Any ribbon knot is a slice knot.

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**The slice-ribbon conjecture [Fox]**

Any slice knot is a ribbon knot.

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**Partial results.**

2-bridge knots (Lisca, 2007)

Some Pretzel knots (Greene-Jabuka, 2011)

Some Montesinious knots (Lecuona, 2012)

Roughly speaking, they proved that some knots are **not slice** using Gauge theory.

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We prove some slice knots are ribbon.
Typical examples

- $12_{a990}$ is slice (A. and Tange)
- Omae's knot

Fact.

- Omae’s knot is slice (A. and Tange)
- $12_{a990}$ is slice (Herald-Kirk-Livingston)
Typical examples

- Omae’s knot is slice (A. and Tange)
- $12_{a990}$ is slice (Herald-Kirk-Livingston)

Fact.
Today’s contents

1. Construction of slice knots via $t_n$-moves

We explain a construction of slice knots via $t_n$-moves. In fact, this construction gives ribbon knots. This tells us

• why $12_{a990}$ is slice, and

• why $12_{a990}$ is ribbon.

2. Construction of slice knots via annulus twists

We explain a construction of slice knots via annulus twists. In fact, this construction gives ribbon knots in our case. This tells us

• why Omae’s knot is slice, and

• why Omae’s knot is ribbon.
1. Construction of slice knots via $t_n$-moves

**A characterization of a slice knot.**
A knot $K$ is slice if and only if $K \#^3 R$ is ribbon, where $R$ is a ribbon knot.

This is essentially due to Fox. The slice-ribbon conjecture implies that we can take $R$ to be the trivial knot.

$T_{p,q}$: a torus knot of type $(p,q)$

**Example (Herald-Kirk-Livingston).**
They proved that $12_{a990} \# T_{2,3} \# T_{2,-3}$ is ribbon.

Therefore $12_{a990}$ is a slice knot.
A construction of slice knots

A $t_n$-move is a tangle replacement as follows.

Lemma.

If we obtain the 3-comp. unlink from a knot $K$ by applying a $t_{2n+1}$- and $t_{-(2n+1)}$-move, then $K \# T_{2,2n+1} \# T_{2,-(2n+1)}$ is a ribbon knot.

In particular, $K$ is a slice knot.

This is essentially due to Herald-Kirk-Livingston.
The idea of Proof

$\text{add } T(2, 2n+1)$

$t_{2n+1} - \text{move}$

$2n+1 \text{ half twists}$

$\text{add a band}$
Theorem 1[A.-Tange].

If we obtain the 3-comp. unlink from a knot $K$ by applying a $t_{2n+1}$- and $t_{-(2n+1)}$-move, then $K$ is ribbon.

**Proof.** By the assumption, there exist two trivial tangles s.t. if we apply a $t_{2n+1}$-move and a $t_{-(2n+1)}$-move, then we obtain the 3-comp. unlink. We can assume that two trivial tangles are connected as follows.
Now we add two bands along dashed arcs as follows.

\[
\begin{array}{c}
\includegraphics[width=0.5\textwidth]{chart.png}
\end{array}
\]

**Claim.**
The resulting link is the link obtained from \( K \) by applying a \( t_{2n+1} \)-move and a \( t_{-(2n+1)} \)-move.

By this claim and the assumption, the resulting link is the 3-comp. unlink. This means that \( K \) is a ribbon knot.
The reason why $12_{a990}$ is ribbon

If we apply a $t_3$-move and a $t_{-3}$-move, then $12_{a990}$ is unlinked as follows.

Therefore $12_{a990}$ is a slice knot. Furthermore, $12_{a990}$ is a ribbon knot. By the proof of the theorem, we obtain the following.
2. Construction of slice knots via annulus twists

We want slice knots which are not obviously ribbon.

We will show that $K_n$ is such a knot.

(i.e. a slice knot which is not obviously ribbon)
Some properties of the knots $K_n$

- $8_{20}$ is a ribbon knot.
- $\mathcal{M}(8_{20},0) \approx \mathcal{M}(K_n,0)$ (due to Osolinach)

The two properties suggest that $K_n$ is slice.

**Remark**

$$X(8_{20},0) \approx X(K_n,0)$$ (due to [AJOT])
Lemma

Let $K$ and $K'$ be knots s.t. $M(K, 0) \approx M(K', 0)$.

If $K$ is ribbon, $K'$ bounds a properly emb. disk in $W$ s.t. $\partial W \approx S^3$ and $W$ is homotopic to $B^4$.

The proof of this lemma is similar to that of $\Delta(K) \doteq 1 \iff K$ is topologically slice.

This lemma is essentially due to Akbulut and explained in [AJOT].

Abe, Jong, Omae and Takeuchi, Annulus twist and diffeomorphic 4-manifolds, preprint (2012).

If $W \approx B^4$, then $K'$ is slice.
A handle diagram of $W_n$

Since $8_{20}$ is ribbon and $M(8_{20},0) \approx M(K_n,0)$, by the lemma, $K_n$ bounds a properly embedded disk in a 4-manifold $W_n$ s.t. $\partial W_n \approx S^3$ and $W_n$ is homotopic to $B^4$.

Lemma[A.-Tange]. $W_n$ has the following handle diagram.
Theorem 2 [A.-Tange].

\[ W_n \approx B^4. \] In particular, \( K_n \) is a slice knot.

**Sketch of a proof.** By handle calculus, we prove

\[ W_n \approx W_{n-1} \approx W_{n-2} \approx \cdots \approx W_0 \approx B^4. \]

Note that handle moves include **adding canceling pairs of 2-3 handles** as in

S. Akubulut, Cappell-Shaneson homotopy spheres are standard, Ann. of Math (2010)

**Remark**

We can also prove \( W_n \approx W_{n-1} \) using a log transformation.
Let $\text{HD} = h^0 \cup h^1_1 \cup \ldots \cup h^1_n \cup h^2_1 \cup \ldots \cup h^2_n$ be a handle diagram which represents $B^4$.

If there exists a handle move from HD to the empty handle diagram $\emptyset$ without adding canceling pairs of 2-3 handles, then the boundary of cocore disk of $h^2_i$ ($i = 1, 2, \ldots, n$) is ribbon.
A technical lemma

In the proof of Theorem 2, we used handle moves from this handle diagram of \( W_n \) to the empty handle diagram \( \emptyset \) which include adding canceling pairs of 2-3 handles.

**Lemma[A.-Tange]**

We can obtain handle moves from this handle diagram into \( \emptyset \) without adding canceling pairs of 2-3 handles.
$K_n$ is a ribbon knot.

**Theorem 3 [A.-Tange].**

$K_n$ is a ribbon knot.