Estimates of Green Function for some perturbations of fractional Laplacian

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Abstract

Suppose that $Y(t)$ is a $d$-dimensional Lévy symmetric process for which its Lévy measure differs from the Lévy measure of the isotropic $\alpha$-stable process ($0 < \alpha < 2$) by a finite signed measure. For a bounded Lipschitz set $D$ we compare the Green functions of the process $Y$ and its stable counterpart. We prove a few comparability results either one sided or two sided. Assuming an additional condition about the difference of the densities of the Lévy measures, namely that it is of order of $|x|^{-d+\varepsilon}$ as $|x| \to 0$, where $\varepsilon > 0$, we prove that the Green functions are comparable, provided $D$ is connected.

These results apply for example to $\alpha$-stable relativistic process. This process was studied in [R, CS3], where the bounds for its Green functions were proved for $d > \alpha$ and smooth sets. In the paper we also considered one dimensional case for $\alpha \geq 1$ and proved that the Green functions for an open and bounded interval are comparable.

1 Introduction

The purpose of the paper is to study estimates of the Green functions of bounded open sets of a symmetric Lévy process $Y_t$, which lives on $\mathbb{R}^d$. We assume that its Lévy measure is close in some sense, which we specify later, to the Lévy measure of the isotropic $\alpha$-stable process. From the point of view of infinitesimal generators, the generator of the semigroup corresponding to $Y_t$ can be considered as a perturbation of the fractional Laplacian by a bounded linear operator. The potential theory of the stable process was extensively investigated in the recent years (see [Bo1], [BB], [CS1], [K2]) and there are several results providing the estimates of the Green functions of $C^{1,1}$ bounded sets (see [K1] and [CS2]) or even bounded Lipschitz sets ([J], [Bo2]). We intend to make a comparison of the Green function of the process $Y_t$ and its stable counterpart. One of the first results in this direction...
was contained in $[R]$, where so called relativistic $\alpha$-stable process was considered. This is a process which characteristic function is of the form
\[
E^0 e^{iz \cdot Y_t} = e^{-t \left( |z|^2 + m^2/\alpha \right)^{\alpha/2} - m}, \quad z \in \mathbb{R}^d,
\]
where $0 < \alpha < 2$ and $m > 0$ is a parameter. Observe that for $m = 0$ it reduces to the isotropic $\alpha$-stable process. The main result of $[R]$ says that the Green function of $C^{1,1}$ bounded set was comparable to the Green function of the isotropic $\alpha$-stable process if $d > \alpha$. Later on that result was derived by a different method in $[CS3]$. In the present paper we develop methods from $[R]$ to derive several extensions of the results proved therein. The main results are contained in the following two theorems.

**Theorem 1.1.** Let $D \subset \mathbb{R}^d$ be a Lipschitz connected and bounded open set. Suppose that $Y_t$ is a symmetric purely jump Lévy process in $\mathbb{R}^d$ with $d \geq 1$ and $\nu^Y(x)$ is the density of its Lévy measure. By $\tilde{\nu}(x)$ we denote the density of the Lévy measure of the isotropic stable process and by $\tilde{G}_D$ its Green function of $D$. Assume that $\sigma(x) = \tilde{\nu}(x) - \nu^Y(x) \geq 0$, $x \in \mathbb{R}^d$, and $\sigma(x) \leq c|x|^{\alpha - d}$ for $|x| \leq 1$, where $c, \varrho > 0$. Then there exists a constant $C = C(d, \alpha, D, \varrho, c)$, such that
\[
C^{-1} \tilde{G}_D(x, y) \leq G_D^Y(x, y) \leq C \tilde{G}_D(x, y),
\]
for all $x, y \in D$.

In the next theorem we remove the assumption about positivity of the function $\sigma$ at the cost of some mild assumption about the behaviour of the density of the Lévy measure.

**Theorem 1.2.** With the same notation as in the previous theorem assume that there are positive constants $c$ and $\varrho$ such that $|\sigma(x)| \leq c|x|^{\alpha - d + \varrho}$ for $|x| \leq 1$, and $\nu^Y(x)$ is bounded on $B^c(0, 1)$. Then there is a constant $C = C(d, \alpha, D, \varrho, \sigma)$ such that for any $x, y \in D$,
\[
C^{-1} \tilde{G}_D(x, y) \leq G_D^Y(x, y) \leq C \tilde{G}_D(x, y).
\]

Observe that in the first theorem the assumption about the positivity of $\sigma$ enables us not to assume anything about the behaviour of $\nu^Y(x)$ away from the origin except it has to be dominated by $\tilde{\nu}$. For example $\nu^Y(x)$ can vanish outside some neighborhood of the origin. Of course that assumptions are readily checked for the relativistic process (see $[R]$ for the description of the Lévy measure), so the theorem extends to Lipschitz bounded domains the main result of $[R]$ (see also $[CS3]$). In addition, note that it covers the one-dimensional case for $\alpha \geq 1$, which was not treated in the neither papers cited above. Actually both papers assumed $d \geq 2$ but the proofs remain valid for $d > \alpha$. To our best knowledge the one dimensional result is a new one which fills the gap in the potential theory of the relativistic process.

The methods we apply are elementary and are based on the fact that for any two pure jump processes such that the difference of their Lévy measures is a positive and finite measure one can represent one of the processes as a sum of the other and an independent compound
Poisson process. A different approach in taken in [CS3], where the problem in $C^{1,1}$ case was tackled by so called drift transform technique. After obtaining the main results of the present paper the authors found on the website of Panki Kim a paper of Kim and Lee [KL] with similar results as ours but even for more general sets (so called $\kappa$-fat sets). The method they use is essentially designed in [CS3], so our methods and results can be viewed as an alternative approach to the problem of comparing the Green functions. Moreover our method can handle the situation when a Lévy measure vanishes outside some neighborhood of the origin which seems not be an option in the other method used in [CS3] or [KL].

The paper is organized in the following way. In Section 2 we set up the notation and provide necessary definitions and basic facts needed in the sequel. At first we do not assume that $Y_t$ is compared with the stable process but we sometimes work in slightly more general setup. Namely some of the results are formulated in such a way that $Y_t$ is compared with another Lévy process $X_t$ under the appropriate assumptions about their Lévy measures. In Section 3 we prove the main estimates along with some other related results. To prove Theorem 1.2 we first prove the estimates for sets of small diameter and then use it to prove Boundary Harnack Principle (BHP) for the process $Y_t$ in the case when its Lévy measure dominates the Lévy measure of the isotropic $\alpha$-stable process.

## 2 Preliminaries

In $\mathbb{R}^d$, $d \geq 1$, we consider a symmetric Lévy processes $X_t$ such that its characteristic triplet is equal to $(0, \nu, 0)$, where $\nu$ is its (nonzero) Lévy measure. That is its characteristic function is given by

$$E^0 e^{iz \cdot X_t} = e^{-t \int_{\mathbb{R}^d} (1 - \cos(z \cdot w)) \nu(dw)}, \quad z \in \mathbb{R}^d.$$ 

If the measure $\nu$ is absolutely continuous with respect to the Lebesgue measure then by $\nu(x)$ we denote its density. By $p(t, x, y)$ we denote the transition densities of $X_t$, which are assumed to be bounded and defined for every $x, y \in \mathbb{R}^d$. The potential kernel for $X_t$ is given by

$$U(x, y) = U(x - y) = \int_0^\infty p(t, x - y) dt.$$ 

We use the notation $C = C(\alpha, \beta, \gamma, \ldots)$ to denote that the constant $C$ depends on $\alpha, \beta, \gamma, \ldots$. Usually values of constants may change from line to line, but they are always strictly positive and finite. Sometimes we skip in notation that constants depend on usual quantities (e.g. $d, \alpha$). Next, we give some definitions. We use $f \approx g$ on $D$ to denote that the functions $f$ and $g$ are comparable, that is there exists a constant $C$ such that

$$C^{-1} f(x) \leq g(x) \leq C f(x), \quad x \in D.$$ 

Let $D \subset \mathbb{R}^d$ be an open set. By $\tau_D$ we denote the first exit time from $D$ that is

$$\tau_D = \inf\{t > 0 : X_t \notin D\}.$$ 

Next, we investigate boundness of the first moment of $\tau_D$. 


Lemma 2.1. For any bounded open set $D$ there exists a constant $C = C(D)$ such that

$$\sup_{x \in \mathbb{R}^d} E^x \tau_D \leq C.$$ 

Proof. The proof of this lemma follows by the same arguments as in the classical case for the Brownian motion (see [CZ]). The argument therein requires the existence of $t_0 > 0$ such that $\sup_{x \in \mathbb{R}^d} P^x(X_{t_0} \in D) < 1$. However, repeating the steps from Lemma 48.3 in [S], one can obtain that

$$\sup_{x \in \mathbb{R}^d} P^x(X_t \in D) = O(t^{-1/2}), \quad t \to \infty.$$ 

In order to study the killed process on exiting of $D$ we construct its transition densities by the classical formula

$$p_D(t, x, y) = p(t, x, y) - r_D(t, x, y),$$

where

$$r_D(t, x, y) = E^x[t \geq \tau_D; p(t - \tau_D, X_{\tau_D}, y)].$$

The arguments used for Brownian motion (see eg. [CZ]) will prevail in our case and one can easily show that $p_D(t, x, y)$, $t \geq 0$, satisfy the Chapman-Kolmogorov equation (semigroup property). Moreover the transition density $p_D(t, x, y)$ is a symmetric function $(x, y)$ a.s. Assuming some other mild conditions on the transition densities of the (free) process one can actually show that $p_D(t, x, y)$ can be chosen as continuous functions of $(x, y)$. Next, we define the Green function of the set $D$,

$$G_D(x, y) = \int_0^\infty p_D(t, x, y) dt.$$ 

Let us see that the integral is well defined, because

$$\int_D G_D(x, y) dy = \int_D \int_0^\infty p_D(t, x, y) dt dy = \int_0^\infty P^x(\tau_D > t) dt = E^x \tau_D < \infty.$$ 

Hence for every $x \in \mathbb{R}^d$ the Green function $G_D(x, y)$ is well defined (y) a.s. Again under the assumptions which make $p_D(t, x, y)$, $t > 0$, continuous functions in arguments $x, y$ one can show that the Green function is a continuous (in extended sense) function on $D \times D$.

It is well known that if the Lévy measure is absolutely continuous with respect to the Lebesgue measure then the distribution of $X_{\tau_D}$ restricted to $D^c$ is absolutely continuous as well (see Ikeda Watanabe) and the density is given by so called Ikeda-Watanabe formula:

$$P_D(x, z) = \int_D G_D(x, y) \nu(y - z) dy, \quad (x, z) \in D \times D^c$$

We call $P_D(x, z)$ the Poisson kernel. Under some other mild conditions $X_{\tau_D}$ has zero probability of belonging to the boundary of $D$ so in this case the Poisson kernel fully describe the distribution of the exiting point.
We say that measurable function $u$ is harmonic with respect to $X_t$ in an open set $D$ if for every bounded open set $U$ satisfying $U \subset D$,

$$u(x) = E^x u(X_{\tau_U}), \ x \in U.$$  

Whereas if

$$u(x) = E^x u(X_{\tau_D}), \ x \in D,$$

then we say that $u$ is regular harmonic with respect to $X_t$ in an open set $D$.

The following lemma is a simple consequence of Lemma 2.1 and boundness of $p(t,x)$.

**Lemma 2.2.** For any $x \in D$ and $t \geq 1$ we have

$$p_D(t, x, y) \leq C(X) \frac{E^x \tau_D E^y \tau_D}{t^2} \ (y) \ a.s. \ .$$

**Proof.** Observe that for $s \geq 0$,

$$\sup_{x, y \in D} p_D(s + 1/2, x, y) \leq \sup_{x, y \in \mathbb{R}^d} p(s + 1/2, x - y) = \sup_{x \in \mathbb{R}^d} p(1/2, \cdot) * p(s, x) \leq \sup_{x \in \mathbb{R}^d} p(1/2, x) = C_1.$$

Hence, by the Chapman-Kolmogorov equation we obtain for $t \geq 1$ and $(y)$ a.s.

$$p_D(t, x, y) = \int_D p_D(t/2, x, z)p_D(t/2, z, y)dz \leq C_1 P^x(\tau_D > t/2).$$

Applying again the Chapman-Kolmogorov equation together with the above inequality we get

$$p_D(t, x, y) \leq C_1 P^x(\tau_D > t/4) \int_D p_D(t/2, z, y)dz = C_1 P^x(\tau_D > t/4) P^y(\tilde{\tau}_D > t/2),$$

where $\tilde{\tau}_D = \inf\{t > 0 : -X_t \in D\}$. But the process $X_t$ is symmetric, so $\{X_t\} \overset{d}{=} \{-X_t\}$. Hence

$$P^y(\tilde{\tau}_D > t/2) = P^y(\tau_D > t/2).$$

Therefore, we have

$$p_D(t, x, y) \leq C_1 P^x(\tau_D > t/4) P^y(\tau_D > t/2).$$

The application of Chebyshev’s inequality completes the proof. \hfill \Box

**Remark 2.3.** If $X_t$ is isotropic stable process then by similar arguments we have for $t > 0$ and $x, y \in D$,

$$p_D(t, x, y) \leq C(\alpha, d) \frac{E^x \tau_D E^y \tau_D}{t^{2+d/\alpha}}.$$  

In one of our general results (Theorem 3.1) we require the following property which exhibits a relation between moments of the exiting times and the Green function.
Suppose that there is a constant $c = c(D)$ such that

$$E^x \tau_D E^y \tau_D \leq c G_D(x, y), \quad x, y \in D.$$ 

At the first glance the above condition looks a bit restrictive but actually it holds in the stable case ([K2], [CS1], [B]) and usually it is derived as a consequence of the intrinsic ultracontractivity of the killed process. In the recent paper of the first author (see [G]) the intrinsic ultracontractivity is studied under much broader assumptions. For example the above property holds if $p_D(t, \cdot, \cdot)$ is continuous in $x, y$ and the Lebesgue measure is absolute continuous with respect to the Lévy measure.

From now we consider two symmetric Lévy processes $Y_t$ and $X_t$ such that a signed measure $\sigma = \nu^Y - \nu^X$ is finite, where $\nu^Y, \nu^X$ are Lévy measures of $Y_t$ and $X_t$ respectively. We use that notational convention throughout the whole paper, e.g. we denote the transition density of $X_t$ by $p^X(t, x)$ and the transition density of $Y_t$ by $p^Y(t, x)$. Later on we specify one of the processes, say $X_t$, to be the isotropic stable process. The aim of this paper is to provide some comparisons between the two processes in various aspects of which the relationship of the Green functions is our main target. Some of the results are general but our typical situation is a comparison between the isotropic stable process and another process with the Lévy measures sufficiently close to each other.

With the assumption that $\sigma = \nu^X - \nu^Y$ is finite we can write the following formula comparing infinitesimal generators on $L^1(\mathbb{R}^d)$ of these processes

$$\mathcal{A}^Y = \mathcal{A}^X - P,$$

where $P \varphi(x) = \sigma \ast \varphi(x) - \sigma(\mathbb{R}^d)\varphi(x)$.

The fact that $P$ is a bounded operator implies that the domains of these generators coincide.

As mentioned above, very often the process $X_t$ is taken to be the isotropic $\alpha$-stable process, $0 < \alpha < 2$. To emphasize its role we denote it by $\tilde{X}_t$. That process has the following characteristic function:

$$E^{0} e^{itz} \tilde{X}_t = e^{-t|z|^\alpha}, \quad z \in \mathbb{R}^d.$$

From now on, we will use the tilde sign to denote functions, measures and etc. corresponding to $\tilde{X}_t$. For example its Lévy measure is given by the formula

$$\tilde{\nu}(B) = \int_B \mathcal{A}(-\alpha, d)|x|^{-d-\alpha} dx,$$

where $\mathcal{A}(\rho, d) = \frac{\Gamma((d-\rho)/2)}{\pi^{d/2} |\Gamma(\rho/2)| \rho^{d/2}}$. The potential kernel which is well defined for $\alpha < d$ is given by

$$\tilde{U}(x) = \mathcal{A}(\alpha, d)|x|^{\alpha-d}, \quad x \in \mathbb{R}^d.$$

The next two lemmas provide basic tools for examining the relationship between the Green functions. In the first we compare the moments of exiting times only under the assumption that $\sigma = \nu^X - \nu^Y$ is a finite signed measure, while in the second we require that
σ is nonnegative. This assumption provides us with a nice inequality involving the transition densities. However the both lemmas already appeared in [R] under some additional assumptions, we deliver the proofs for the reader convenience.

**Lemma 2.4.** Let $D$ be a bounded open set and $σ = ν^X − ν^Y$ be finite. Then we have on $D$,

$$E^xτ^X_D ≈ E^xτ^Y_D.$$ 

**Proof.** Suppose that the Jordan decomposition of $σ = σ_+−σ_−$. Let $V_t$ be a compound Poisson process independent of $X_t$ with the Lévy measure $σ_+$ and $V_t'$ be a compound Poisson process independent of $Y_t$ with the Lévy measure $σ_-$. We put $Z_t = X_t + V_t$, then of course we have \{Z_t\} = \{Y_t + V_t\}. Hence, it’s enough to show that $E^xτ^Z_D ≈ E^xτ^X_D$.

Let us define a stopping time $T$ by $T = \inf\{t > 0 : V_t ≠ 0\}$. The processes $X_t$ and $V_t$ are mutually independent, therefore $X_t$ and $T$ are independent as well. Besides, $Z_t = X_t$ for $0 ≤ t < T$. We set $m = σ_-(\mathbb{R}^d)$.

First, we claim that $E^x(τ^X_D) ≤ 2E^x(τ^X_D − t)$ for $t$ large enough. Indeed, by the Markov Property and Lemma 2.1 we have

$$E^xτ^X_D = E^x(τ^X_D ∧ t) + E^x(τ^X_D ∧ t; τ^X_D − t) = E^x(τ^X_D ∧ t) + E^x(τ^X_D > t; E^{X_t}τ^X_D) \leq E^x(τ^X_D ∧ t) + C E^xτ^X_D t,$$

which proves our claim for $t ≥ 2C$.

Because $τ^Z_D ∧ T = τ^X_D ∧ T$, so by independence $T$ and $X_t$ we get

$$E^xτ^Z_D ≥ E^x(τ^Z_D ∧ T) = E^x(τ^X_D ∧ T) = \int_0^∞ E^x(τ^X_D ∧ t) me^{−mt}dt \geq \int_{2C}^∞ E^x(τ^X_D ∧ t) me^{−mt}dt ≥ \frac{1}{2}e^{−2Cm}E^xτ^X_D.$$

Now, we prove the upper bound

$$E^xτ^Z_D = E^x(τ^Z_D ∧ T) + E^x(τ^Z_D > T; τ^Z_D − T) \leq E^xτ^X_D + E^x(τ^Z_D > T; E^{Z^T}_τ^Z_D) \leq E^xτ^X_D + C P^x(τ^Z_D > T),$$

but

$$P^x(τ^Z_D > T) ≤ P^x(τ^X_D ≥ T) = m \int_0^∞ P^x(τ^X_D ≥ t)e^{−mt}dt ≤ mE^xτ^X_D,$$

which ends the proof. □

**Lemma 2.5.** Suppose that $σ = ν^X − ν^Y$ is a nonnegative finite measure and $D$ is an open set. Then for any $x ∈ D$ and $t > 0$,

$$p^Y_D(t, x, ·) ≤ e^{mt}p^X_D(t, x, ·) \ a.s. .$$

If, in addition, we assume that $p^Y(t, ·)$ and $p^X(t, ·)$ are continuous then we have for $x, y ∈ D$,

$$r^Y_D(t, x, y) ≤ e^{2mt}r^X_D(t, x, y).$$
Proof. We put \( m = \sigma(\mathbb{R}^d) < \infty \), and define a compound Poisson process \( V_t \) with the Lévy measure \( \sigma \) independent of \( Y_t \). A random variable

\[
T = \inf\{t \geq 0 : V_t \neq 0\}
\]

has the exponential distribution with intensity \( m \). Then \( Y_t \) and \( T \) are independent and for \( 0 \leq t < T \) we have \( X_t = Y_t \).

Let \( A \) be a Borel subset of \( D \). Since \( Y_t = X_t \), for \( t < T \) we infer that \( \tau_D^Y \cap \{ T > t \} = \tau_D^X \cap \{ T > t \} \). By independence of \( Y_t \) and \( T \)

\[
P^x(t < \tau_D^Y; Y_t \in A)P^x(T > t) = P^x(t < \tau_D^X; X_t \in A; T > t)
\]

so we obtain that (y) a.s.,

\[
p_D^Y(t, x, y)P^x(T > t) \leq p_D^X(t, x, y).
\]

But \( T \) has the exponential distribution with intensity \( m \), that is \( P^x(T > t) = e^{-mt} \).

The second inequality is proved analogously, using the first with \( D = \mathbb{R}^d \) in the intermediate step. Moreover the continuity of \( p^Y(t, \cdot) \) and \( p^X(t, \cdot) \) is required to justify the last step:

\[
r_D^Y(t, x, y)e^{-mt} = E^x[t \geq \tau_D^Y; p^Y(t - \tau_D^Y, Y_{\tau_D^Y}, y)]P^x(T > t)
= E^x[\tau_D^Y \leq t < T; p^Y(t - \tau_D^Y, Y_{\tau_D^Y}, y)]
= E^x[\tau_D^X \leq t < T; p^X(t - \tau_D^X, X_{\tau_D^X}, y)]
\leq e^{mt}E^x[\tau_D^X \leq t; p^X(t - \tau_D^X, X_{\tau_D^X}, y)]
= e^{mt}r_D^X(t, x, y).
\]

The next lemma is a sort of a comparison between transition densities in that sense that a ”nice” behaviour of them for one process implies that the transition densities of the second are uniformly bounded away from zero. The ”nice” behaviour for example is present if the first process is the isotropic stable process. We use that result in the sequel to assure that the transition densities of the killed process are continuous and to assure the property A. We define an exponent of a signed finite measure \( \sigma \) by

\[
\exp\{\sigma\}(A) = e^{-\sigma(\mathbb{R}^d)}\sum_{n=0}^{\infty} \frac{\sigma^n(A)}{n!}, \quad \text{where} \ A \subset \mathbb{R}^d \ \text{is a Borel set.}
\]
Lemma 2.6. Suppose that $\nu^X$ and $\nu^Y$ are absolutely continuous and $\sigma(x) = \nu^X(x) - \nu^Y(x)$ is an integrable function such that $|p^X(t, \cdot) \ast \sigma(x)| + |\sigma(x)| \leq c_1$ for $|x| \geq \delta$ and $t \leq 1$. If $p^X(t, x) \leq c_2 e^{-\zeta}$ for $t \leq 1$, where $\zeta > 0$, and $p^X(t, x) \leq c_3(\delta)$ for $|x| \geq \delta$, then there is a constant $C$ such that

$$p^Y(t, x) \leq C, \quad |x| \geq (|\zeta| \lor 1) \delta \text{ and } t > 0.$$  

Proof. Suppose that $\int_{\mathbb{R}^d} |\sigma(x)| dx = M < \infty$. We put $\int_{\mathbb{R}^d} \sigma(x) dx = m$. We can write

$$p^Y(t, x) = p^X(t, \cdot) \ast \exp\{-t\sigma\} = p^X(t, x) e^{tm} + \sum_{n=1}^{\infty} \frac{(-t)^n p^X(t, \cdot) \ast \sigma^n(x)}{n!} e^{tm}.$$  

Observe that $|p^X(t, \cdot) \ast \sigma^n(x)| \leq \sup_{y \in \mathbb{R}^d} p^X(t, y) M^n \leq c_2 M^n e^n$, so for $t \leq 1$ we have

$$\left| \sum_{n \geq \zeta} \frac{(-t)^n p^X(t, \cdot) \ast \sigma^n(x)}{n!} e^{tm} \right| \leq C \sum_{n \geq \zeta} \frac{t^{n-\zeta} M^n}{n!} = C e^{M} < \infty.  \tag{2}$$  

Now, we show that if $|p^X(t, \cdot) \ast \sigma(x)| + |\sigma(x)| \leq c(1)$ for $|x| \geq \delta$ and $t \leq 1$ then

$$|p^X(t, \cdot) \ast \sigma^n(x)| \leq c(n), \quad |x| \geq n\delta. \tag{3}$$  

We assume (3) for $n$ and we prove it for $n + 1$. Observe that

$$|p^X(t, \cdot) \ast \sigma^{n+1}(x)| \leq \int_{B^c(x, n\delta)} |p^X(t, \cdot) \ast \sigma^n(x - y)||\sigma(y)| dy + \int_{B(x, n\delta)} |p^X(t, \cdot) \ast \sigma^n(x - y)||\sigma(y)| dy \leq c(n) M + c_1 M^n,$$

because if $y \in B(x, n\delta)$ then $|y| \geq |x| - |x - y| \geq \delta$. Combining (2) and (3) and using that $p^X(t, x) \leq c(\delta)$ for $|x| \geq \delta$ we end the proof for $t \leq 1$.

Next, for $t > 1$ we have

$$\sup_{x \in \mathbb{R}^d} p^Y(t, x) = \sup_{x \in \mathbb{R}^d} p^Y(1, \cdot) \ast p^Y(t - 1, x) \leq \sup_{x \in \mathbb{R}^d} p^Y(1, x) = C,$$

which proves the conclusion for $t > 1$.  

The following lemma is an attempt to find a condition under which the potential kernel of a process is comparable at the vicinity of the origin with the stable potential kernel. It will play an important role in proving the upper bound for the Green function $G_D^Y$ by its stable counterpart (see Theorem 3.22).

Lemma 2.7. Let $d > \alpha$. Let $-\sigma = \nu^Y - \tilde{\nu}$ be a nonnegative finite measure such that $\tilde{U} \ast (-\sigma)(x) \leq C \tilde{U}(x)$ for $|x| \leq 1$ then for some constant $C > 1$,

$$C^{-1} \tilde{U}(x) \leq U^Y(x) \leq C \tilde{U}(x), \quad |x| \leq 1.$$  

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Proof. Suppose that $-\sigma = \nu^Y - \tilde{\nu} \geq 0$. Let $-\sigma(\mathbb{R}^d) = m > 0$. We can write

$$p^Y(t, x) = \tilde{p}(t, \cdot) \ast \exp\{-t\sigma\} = \tilde{p}(t, x) e^{-tm} + \sum_{n=1}^{\infty} \frac{t^n \tilde{p}(t, \cdot) \ast (-\sigma)^n(x)}{n!} e^{-tm}. $$

Observe that $\tilde{p}(t, \cdot) \ast (-\sigma)^n(x) \leq \sup_{y \in \mathbb{R}^d} \tilde{p}(t, y) m^n = C \frac{m^n}{\omega^{d/\alpha}}$ so for $n > d/\alpha - 1$ we have

$$\int_0^\infty \frac{t^n \tilde{p}(t, \cdot) \ast (-\sigma)^n(x)}{n!} e^{-tm} dt \leq C \int_0^\infty \frac{t^{n-d/\alpha} m^n}{n!} e^{-tm} dt \leq C \frac{\Gamma(n+1-d/\alpha)}{n!} m^{d/\alpha+1} \leq C \frac{m^{d/\alpha+1}}{n^{d/\alpha}}.$$

This implies that

$$\int_0^\infty \sum_{n>d/\alpha-1} \frac{t^n \tilde{p}(t, \cdot) \ast (-\sigma)^n(x)}{n!} e^{-tm} \leq C \sum_{n>d/\alpha-1} \frac{m^{d/\alpha+1}}{n^{d/\alpha}} = c(\alpha, m, d) < \infty. \quad (4)$$

Next estimating $t^n e^{-tm} \leq C(n, m) < \infty$ we have

$$\int_0^\infty \frac{t^n \tilde{p}(t, \cdot) \ast (-\sigma)^n(x)}{n!} e^{-tm} dt \leq C(n, m) \tilde{U} \ast (-\sigma)^n(x).$$

Let $\tilde{U}(x) = \frac{\omega}{|x|^{d-\alpha}}$. If we assume that $\tilde{U} \ast (-\sigma)(x) \leq C\tilde{U}(x)$ for $|x| \leq 1$ then we claim that

$$\tilde{U} \ast (-\sigma)^n(x) \leq C(n)\tilde{U}(x), \quad |x| \leq 1. \quad (5)$$

We check this for $n = 2$ since the general case will follow by induction.

$$\tilde{U} \ast \sigma^2(x) = \int_{B(x, 1)} \tilde{U} \ast (-\sigma)(x - y) (-\sigma)(dy) + \int_{B^c(x, 1)} \tilde{U}(x - y) \sigma^2(dy) \leq C \int_{B(x, 1)} \tilde{U}(x - y) (-\sigma)(dy) + \omega \cdot m^2 \leq C^2 \tilde{U}(x) + \omega \cdot m^2 \leq C(2) \tilde{U}(x),$$

because $\lim_{|x| \to 0} \tilde{U}(x) = \infty$. By (4) and (5) we conclude that $U^Y(x) \leq C\tilde{U}(x), \quad |x| \leq 1$.

Getting the reverse inequality is almost immediate since $\tilde{p}(t, x) \leq e^{t\omega} p^Y(t, x)$ (Lemma 2.5 with the fact that $\tilde{p}(t, \cdot)$ and $p^Y(t, \cdot)$ are continuous). The following estimate is well known:

$$\tilde{p}(t, x) \leq C(d, \alpha) \left( t^{-d/\alpha} \wedge \frac{t}{|x|^{d+\alpha}} \right). \quad (6)$$

Hence for $|x| \leq 1$,

$$\tilde{U}(x) \leq C \int_0^1 \tilde{p}(t, x) dt,$$
for some constant $C = C(d, \alpha)$. Therefore

$$
\tilde{U}(x) \leq \int_0^1 \tilde{p}(t, x) dt \leq e^m \int_0^1 p^Y(t, x) dt \leq e^m U^Y(x),
$$

for $|x| \leq 1$. \hfill \square

**Remark 2.8.** If $-\sigma(x)$ is a nonnegative density of a finite measure and

$$
-\sigma(x) \leq C|x|^{-d+\varrho}, \quad |x| \leq 1,
$$

where $\varrho > 0$ then the condition $\tilde{U} \ast (-\sigma)(x) \leq C\tilde{U}(x)$ for $|x| \leq 1$ is satisfied.

The last lemma in this section is intended to treat the one-dimensional recurrent case while comparing two processes of which one is a stable one. This case is different from the transient one and requires somewhat different arguments.

**Lemma 2.9.** Let $d = 1$, $\alpha \geq 1$ and $0 < t_0 \leq 1$. Suppose that $\sigma = \tilde{\nu} - \nu^Y$ is a finite measure. Then there exists a constant $C = C(m, M)$ such that

$$
\int_0^{t_0} \left| \tilde{p}(t, x) - e^{-2mt} p^Y(t, x) \right| dt \leq C t_0^{2-1/\alpha},
$$

where $m = \sigma(\mathbb{R})$ and $M = |\sigma|_1(\mathbb{R})$.

**Proof.** Let $\sigma(\mathbb{R}) = m$ and $|\sigma|_1(\mathbb{R}) = M > 0$. We can write

$$
p^Y(t, x) = \tilde{p}(t, \cdot) \exp\{-t\sigma\} = \tilde{p}(t, x) e^{tm} + \sum_{n=1}^{\infty} \frac{(-t)^n \tilde{p}(t, \cdot) \ast \sigma^n(x)}{n!} e^{tm}.
$$

Next $|\tilde{p}(t, \cdot) \ast \sigma^n(x)| \leq \sup_{y \in \mathbb{R}} \tilde{p}(t, y) M^n = C M^n_{1/\alpha}$. Using this estimate we obtain

$$
|\tilde{p}(t, x) - e^{-2mt} p^Y(t, x)| = \left| \tilde{p}(t, x)(1 - e^{-mt}) - \sum_{n=1}^{\infty} \frac{(-t)^n \tilde{p}(t, \cdot) \ast \sigma^n(x)}{n!} e^{-mt} \right|
$$

$$
\leq \tilde{p}(t, x)(1 - e^{-mt}) + C \frac{M^n}{t^{1/\alpha}} \sum_{n=1}^{\infty} \frac{(tM)^n}{n!} e^{-mt}
$$

From the above it easily follows that there is a constant $C = C(m, M)$ such that

$$
|\tilde{p}(t, x) - e^{-2mt} p^Y(t, x)| \leq C t^{1-1/\alpha}, \quad t \leq 1.
$$

Now the conclusion follows by integration. \hfill \square
3 Comparability of the Green functions

In this section we prove our main results. We start with a general one-sided estimate of Green functions.

**Theorem 3.1.** Let \( D \) be a bounded open set and a finite measure \( \sigma = \nu - \nu \) be nonnegative. Suppose that for one of the processes \( X_t \) or \( Y_t \) its Green function satisfies the property A. Then there exists a constant \( C = C(\sigma, D, \alpha, d) \) such that for \( x \in D \),

\[
G_Y^D(x, y) \leq CG_X^D(x, y) \quad (y) \text{ a.s.}
\]

**Proof.** Denote \( \sigma(\mathbb{R}^d) = m \). From Lemmas 2.2 and 2.5 we get (y) almost surely

\[
G_Y^D(x, y) = \int_0^{t_0} p_Y^D(t, x, y)dt + \int_{t_0}^{\infty} p_Y^D(t, x, y)dt
\]

\[
\leq e^{mt_0} \int_0^{t_0} p_X^D(t, x, y)dt + C_1 \int_{t_0}^{\infty} t^{-2}E^x_{\tau_D^Y}E^y_{\tau_D^Y} dt,
\]

for \( t_0 \geq 1 \). Hence

\[
G_Y^D(x, y) \leq cG_X^D(x, y) + \frac{C_1}{t_0}E^x_{\tau_D^Y}E^y_{\tau_D^Y}.
\]

If \( Y_t \) satisfies

\[
E^x_{\tau_D^Y}E^y_{\tau_D^Y} \leq C_2 G_D^Y(x, y),
\]

then for \( t_0 = \max\{1, 2C_1C_2\} \) we get

\[
G_Y^D(x, y) \leq 2cG_D^Y(x, y).
\]

Now, suppose that (7) holds for \( X_t \). Then by Lemma 2.4 we have

\[
G_Y^D(x, y) \leq cG_D^X(x, y) + C_3 E^x_{\tau_D^Y}E^y_{\tau_D^Y} \leq CG_X^D(x, y),
\]

which ends the proof. \( \square \)

Kulczycki in \([K2]\) showed that for the isotropic \( \alpha \)-stable process the property A is satisfied for any bounded open set \( D \), so we obtain the following.

**Corollary 3.2.** Let \( D \) be a bounded open set. If \( \sigma = \tilde{\nu} - \nu \) is a nonnegative and finite measure then there is a constant \( C \) such that

\[
G_Y^D(x, y) \leq CG_Y^D(x, y).
\]

If \( \nu - \tilde{\nu} \) is a nonnegative and finite measure then

\[
G_Y^D(x, y) \leq CG_Y^D(x, y).
\]
Suppose that \( p^X_D(t,x,\cdot) \) and \( p^Y_D(t,\cdot,x) \) are continuous for any \( x \in D \). If the Lebesgue measure is absolutely continuous with respect to the Lévy measure of \( X_t \), then the following theorem is true for any bounded open set \( D \). Whereas if there exists a radius \( r > 0 \) such that density \( \nu^X_{ac} \) of the absolute continuous part of the Lévy measure satisfies
\[
\inf_{x \in B(0,r)} \nu^X_{ac}(x) > 0,
\]
then the following theorem holds for any bounded and connected Lipschitz domain \( D \) (see [G]).

**Theorem 3.3.** For every \( t > 0 \) there is a constant \( c = c(t,D,\alpha) \) such that
\[
cE^x\tau^X_D E^y\tau^X_D \leq p^X_D(t,x,y), \quad x, y \in D.
\]

If we integrate the above inequality with respect to \( dt \) we get the property A for \( X_t \)
\[
C E^x\tau^X_D E^y\tau^X_D \leq G^X_D(x,y).
\]

Therefore from Theorem 3.1 we infer that

**Corollary 3.4.** Let \( p^X_D(t,\cdot,\cdot) \) be continuous for every \( t > 0 \), and let a finite measure \( \sigma = \nu^X - \nu^Y \) be nonnegative. Suppose that the Lebesgue measure is absolutely continuous with respect to \( \nu^X \). Then for any bounded open set \( D \) there exists a constant \( C = C(\sigma,D,\alpha,d) \) such that for \( x \in D \),
\[
G^Y_D(x,y) \leq CG^X_D(x,y), \quad (y) \text{ a.s.}.
\]

Our next goal is to reverse the above estimate. We are not able to do it under the above assumptions but this will be done under some additional assumptions through several steps. In the first one we take advantage of the following lemma which can be proved similarly as Lemma 7 in [R].

**Lemma 3.5.** Let \( \sigma = \nu^X - \nu^Y \) be a nonnegative finite measure. Suppose that \( G^X_D(x,\cdot) \) and \( G^Y_D(x,\cdot) \) are continuous then
\[
G^X_D(x,y) \leq G^Y_D(x,y) + E^x[\tau^X_D > T; G^X_D(X_T,y)],
\]
where \( T \) is defined by (I).

This lemma can be rewritten in the way which is more useful for further analysis.

**Corollary 3.6.** Suppose that \( \sigma = \nu^X - \nu^Y \) is a nonnegative finite measure, \( G^X_D(x,\cdot) \) and \( G^Y_D(x,\cdot) \) are continuous. Then
\[
G^X_D(x,y) \leq G^Y_D(x,y) + \int_D \int_{D-w} G^Y(x,w)G^X(w+z,y)\sigma(dz)dw.
\]
Proof. See the proof of Lemma 9 in [R].

From now on we assume that \( X_t = \tilde{X}_t \) and that the measure \( \sigma = \tilde{\nu} - \nu^Y \) is finite and absolutely continuous. We will use the following notational convention: in the case when a measure \( \mu \) is absolutely continuous we denote its density by \( \mu(x) \). That is \( \sigma(x) \) is the density of \( \tilde{\nu} - \nu^Y \) Moreover we assume a particular behavior of \( \sigma(x) \) near 0, that is we suppose there exist \( q > 0 \) and \( C \) such that

\[
|\sigma(x)| \leq C|x|^{q-d}, \quad |x| \leq 1.
\]

In addition we assume that \( \sigma(x) \) is bounded on \( B^c(0,1) \), which obviously is equivalent to boundness of \( \nu^Y(x) \) on \( B^c(0,1) \).

For example the above conditions are satisfied by the Lévy measure of the relativistic process (see [R]) and the Lévy measure of the \( \alpha \)-stable process truncated to \( B(0,1) \) \( (\nu^Y(x) = 1_{B(0,1)}(x)\overline{v}(x)) \).

With these assumptions we have that the characteristic function of \( Y_t \) is integrable, so \( p^Y(t,\cdot) \) is bounded and continuous. Moreover, by \([\Box]\) we get that for any \( \delta > 0 \),

\[
\tilde{p}(t,x) \leq C(\delta), \quad |x| \geq \delta.
\]

Therefore from Lemma 2.6 we obtain that the transition density of \( Y_t \) also satisfies

\[
p^Y(t,x) \leq C(\delta) \quad |x| \geq \delta.
\]

This property enables us to prove, similarly as for the Brownian motion in [CZ], that \( p^D_D(t,x,\cdot) \) and \( p^Y(t,\cdot,y) \) are continuous, moreover \( G^Y_D(x,\cdot) \) and \( G^Y_D(\cdot,y) \) are continuous, too. Hence under the present assumptions, in all claims of the results proved so far, we have that the estimates hold for every \( y \) not for almost all.

Furthermore, we have that there exists a radius \( r \) and a constant \( c \) such that \( \tilde{\nu}(x) \leq c\nu^Y(x) \) on \( B(0,r) \). So, \( \inf_{x \in B(0,r)} \nu^Y(x) > 0 \). Therefore from Theorem 3.3 we have that for any bounded and connected Lipschitz domain the process \( Y_t \) satisfies property A. That is we have the following corollary.

**Corollary 3.7.** Let \( \sigma(x) = \tilde{\nu}(x) - \nu^Y(x) \) be an integrable function satisfying \([\Box]\). Moreover let \( \sigma \) be bounded on \( B^c(0,1) \). Then the property A holds for \( Y_t \) and any bounded connected Lipschitz domain. Whereas if we assume that \( \nu^Y \geq \tilde{\nu} \) then the property A holds for \( Y_t \) and any bounded open set.

Let \( D \subset \mathbb{R}^d \) be a bounded Lipschitz domain with Lipschitz character \( (r_0,\lambda) \) (see [J], [Bo1] for the definitions). We need to introduce some additional notation related to \( D \). We assume that \( D \) is a nonempty, open and bounded set. We put \( r_0 = \frac{\dim(D)}{4\sqrt{1+\lambda^2}} \). The set \( \{ x \in D : \delta_D(x) \geq r_0/2 \} \) is nonempty. We choose one of its elements and denote by \( x_0 = x_0(D) \). Besides we fix a point \( x_1 \) such that \( |x_0 - x_1| = r_0/4 \). For any \( x,y \in D \) let \( r = r(x,y) = \delta_D(x) \lor \delta_D(y) \lor |x-y| \). If \( r \leq r_0/32 \) we put \( A_{x,y} \) as an element of the following set

\[
B(x,y) = \{ A \in D : B(A,\kappa r) \subset D \cap B(x,3r) \cap B(y,3r) \},
\]

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and if \( r > r_0/32 \) we set \( A_{x,y} = x_1 \).

For Lipschitz domains Jakubowski [J] proved the following theorem about estimates of the Green function for the isotropic \( \alpha \)-stable process in the case \( d \geq 2 \). If \( d = 1 \), then analogous theorem is true as well for \( \alpha < 1 \) (see e.g. [ByB]).

**Theorem 3.8.** Let \( D \) be a bounded Lipschitz domain and \( d > \alpha \). There is a constant \( C_1 = C_1(d, \lambda, r_0, \text{diam}(D), \alpha) \) such that for every \( x, y \in D \) we have

\[
C_1^{-1} \frac{\tilde{\phi}_D(x) \tilde{\phi}_D(y)}{\tilde{\phi}_D^2(A_{x,y})} |x - y|^\alpha \leq \tilde{G}_D(x,y) \leq C_1 \frac{\tilde{\phi}_D(x) \tilde{\phi}_D(y)}{\tilde{\phi}_D^2(A_{x,y})} |x - y|^\alpha
\]

where \( \tilde{\phi}_D(x) = \tilde{G}_D(x, x_0) \land \mathcal{A}(d, \alpha) r_0^{\alpha - d} \).

From the scaling property of the Green function for the isotropic \( \alpha \)-stable process we have the following remark.

**Remark 3.9.** The constant \( C_1 \) depends on \( r_0 \) and \( \text{diam}(D) \) only by their ratio \( r_0 \).

Now, we recall estimates for the Green function of the isotropic \( \alpha \)-stable process if \( 1 = d \leq \alpha \). Their proof can be found e.g. in [ByB].

**Theorem 3.10.** Let \( d = 1 \) and \( D \) be an open interval. Then we have on \( D \times D \),

\[
\tilde{G}_D(x, y) \approx \begin{cases} 
\log \left( \frac{\delta_D(x) \delta_D(y)}{|x - y|} \right)^{1/2} + 1, & \alpha = 1, \\
\min \left\{ \frac{\delta_D(x) \delta_D(y)}{|x - y|} \right\}^{(\alpha - 1)/2}, & 1 < \alpha.
\end{cases}
\]

The consequence of Lemma 13 and 15 from [J] is the following lemma.

**Lemma 3.11.** There are constants \( \gamma = \gamma(d, \lambda, \alpha) < \alpha < d \) and \( C = C(d, \lambda, \alpha, r_0) \) such that for every \( x, y, z, w \in D \) we have

\[
\frac{\tilde{\phi}_D^2(A_{x,y})}{\tilde{\phi}_D(A_{x,w}) \tilde{\phi}_D(A_{z,y})} \leq C \max \left\{ 1, \frac{|x - y|^{\gamma}}{|x - w|^{\gamma}}, \frac{|x - y|^2}{|x - w|^2}, \frac{|x - y|}{|x - w|}, \frac{|x - y|}{|z - y|} \right\}.
\]

**Proof.** First, we assume that \( |x - y| \leq |x - w| \). Then it can be proved using similar methods as in Lemma 13 of [J] that

\[
\tilde{\phi}_D(A_{x,y}) \leq C(d, \lambda, \alpha, r_0) \tilde{\phi}_D(A_{x,w}). \tag{9}
\]

Now, let \( |x - w| \leq |x - y| \). Then from the proof of Lemma 15 in [J] we infer that

\[
\tilde{\phi}_D(A_{x,y}) \leq C(d, \lambda, \alpha, r_0) \frac{|x - y|^{\gamma}}{|x - w|^{\gamma}} \tilde{\phi}_D(A_{x,w}), \tag{10}
\]

for some \( 0 < \gamma < \alpha \). Combining (9) and (10) ends the proof. \( \square \)
Lemma 3.12. Let \( x \neq y \in D, -d < q \) and \( 0 < a, b \). Then there exists a constant \( C = C(d, a, b, q) \) such that

\[
\int_D \int_D |y-z|^{a-d}|z-w|^{q}|w-x|^{b-d}dwdz \leq C \begin{cases} |x-y|^{a+q+b}, & a + q + b < 0, \\
1 + \log \left( \frac{\text{diam}(D)}{|x-y|} \right), & a + q + b = 0, \\
(q \text{diam}(D))^a \left( 1 + \log \left( \frac{\text{diam}(D)}{|x-y|} \right) \right), & a = b = -q, \\
(q \text{diam}(D))^{a+q+b}, & \text{otherwise.}
\end{cases}
\]

Proof. By changing variables: \( u = \frac{z-y}{|x-y|} \) and \( v = \frac{w-x}{|x-y|} \) we get

\[
\int_D \int_D |y-z|^{a-d}|z-w|^{q}|w-x|^{b-d}dwdz = |x-y|^{a+q} \int_{D-y} |u|^{a-d} |v|^{b-d} |u-v-q|^{q} du dv,
\]

where \( q = \frac{x-y}{|x-y|} \).

For \( q + a < 0 \) we have

\[
\int_{\mathbb{R}^d} |u|^{a-d} |u - v - q|^{q} du = C_{d,a,q} |v + q|^{a+q},
\]

and for \( q + a + b < 0 \),

\[
\int_{\mathbb{R}^d} |v|^{b-d} |v + q|^{a+q} dv = C_{d,a,b,q},
\]

which proves the first case. When \( q + a + b = 0 \), then we have

\[
\int_{D-y} |v|^{b-d} |v + q|^{a+q} dv \leq \int_{B(0,2)} |v|^{b-d} |v + q|^{a+q} dv + 2^{-q-a} \int_{B(0, \text{diam}(D)/|x-y|) \setminus B(0,2)} |v|^{-d} dv \\
= C(d, a, b, q) + C(d, a, q) \left( \log \left( \frac{\text{diam}(D)}{|x-y|} \right) - \log(2) \right) \vee 0 \\
\leq C(d, a, b, q) \left\{ 1 + \log \left( \frac{\text{diam}(D)}{|x-y|} \right) \right\}.
\]

If \( 0 < q + a + b < b \) then

\[
\int_{D-y} |v|^{b-d} |v + q|^{a+q} dv \leq \int_{B(0,2)} |v|^{b-d} |v + q|^{a+q} dv + 2^{-q-a} \int_{B(0, \text{diam}(D)/|x-y|) \setminus B(0,2)} |v|^{q+a+b-d} dv \\
\leq C(d, a, b, q) \left\{ 1 + \left( \frac{\text{diam}(D)}{|x-y|} \right)^{q+a+b} \right\}.
\]

The remaining cases can be proved in the same way.
Lemma 3.13. Let $d > \alpha$. Suppose that there is a positive $\varrho$ and $c_1 = c_1(\text{diam}(D))$ such that $|\sigma(x)| \leq c_1|x|^{d-}\varrho$ for $|x| \leq \text{diam}(D)$. Then there exists a constant $C = C(d, \lambda, r_0, \alpha, \varrho)$ such that for all $x, y \in D$,

$$\int_D \int_D \tilde{G}_D(y, z)|\sigma(z - w)|\tilde{G}_D(w, x)dwz \leq c_1C(\text{diam}(D))^{\zeta_1}|x - y|^{\zeta_2}\tilde{G}_D(x, y),$$

for some $\zeta_1 \geq 0$ and $\zeta_2 > 0$.

Proof. From Theorem 3.8 and Lemma 13 in [6] we obtain

$$\frac{\tilde{G}_D(x, w)\tilde{G}_D(z, y)}{\tilde{G}_D(x, y)} \approx \left( \frac{|x - y|}{|x - w||y - z|} \right)^{d-\alpha} \frac{\tilde{G}_D(w)\tilde{G}_D(z)\tilde{G}_D^2(A_{x,y})}{\tilde{G}_D^2(A_{x,w})\tilde{G}_D^2(A_{z,y})} \approx \left( \frac{|x - y|}{|x - w||y - z|} \right)^{d-\alpha} \frac{\tilde{G}_D^2(A_{x,y})}{\tilde{G}_D(A_{x,w})\tilde{G}_D(A_{z,y})}.$$ 

Because $|\sigma(x)| \leq c_1|x|^{d-}\varrho$ for $|x| \leq \text{diam}(D)$ we get $|\sigma(w - z)| \leq c_1|w - z|^{d-}\varrho$ on $D \times D$. So, from Lemma 3.11 it’s enough to prove that for some $\zeta_1 \geq 0$ and $\zeta_2 > 0$,

$$|x - y|^{d-\alpha + \rho_1 + \rho_2} \int_D \int_D |x - w|^{\alpha - \rho_1 - \rho_2}|w - z|^{d-}\varrho|z - y|^{\alpha - \rho_2 - d}dwz \leq C(\text{diam}(D))^{\zeta_1}|x - y|^{\zeta_2},$$

for some $C = C(d, \rho_1, \rho_2, \varrho)$, where $\rho_1, \rho_2 \in \{0, \gamma\}$. Recall that $\gamma < \alpha$, hence the above inequality is a consequence of Lemma 3.12.

By inspecting the estimates from Theorem 3.10 one can check that the following remark is true.

Remark 3.14. In the case $d = 1 < \alpha$ the above lemma does not hold. This is a reason why the proof below of Theorem 1.1 in the one-dimensional case for $\alpha \geq 1$ needs to employ some other arguments then in the general case.

### 3.1 Proof of Theorem 1.1

Throughout this subsection we assume that $\sigma = \tilde{\nu} - \nu^Y$ is a finite nonnegative absolutely continuous measure and its density satisfies

$$\sigma(x) \leq C|x|^{\varrho-d}, \quad |x| \leq 1,$$

for some positive $\varrho$. Then there is also a constant $c = c(C, d, \alpha, \text{diam}(D))$ such that $\sigma(x) \leq c|x|^{\varrho-d}$ for $|x| \leq \text{diam}(D)$. Let $D$ be a bounded connected Lipschitz domain. Then the property A holds for $Y_t$ by Theorem 3.3.

The corollaries 3.2 and 3.6 allow us to write the following inequality

$$C_1^{-1}G^Y_D(x, y) \leq \tilde{G}_D(x, y) \leq G^Y_D(x, y) + C_1\tilde{R}_D(x, y), \quad (11)$$

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where $\tilde{R}_D(x, y) = \int_D \int_D \tilde{G}_D(x, w)\sigma(w - z)\tilde{G}_D(z, y)dwdz$.

From Theorems 3.8 and 3.10 we obtain that for $|x - y| \geq \theta > 0$

$$\tilde{G}_D(x, y) \leq C(\theta)E^x\tilde{\tau}_D E^y\tilde{\tau}_D.$$ 

Hence, by the property A and Lemma 2.4 we get

$$\tilde{G}_D(x, y) \leq C(\theta)G^Y_D(x, y), \quad |x - y| \geq \theta > 0$$

What remains it is to show that $\tilde{R}_D(x, y) \leq \frac{1}{2}\tilde{G}_D(x, y)$ if $|x - y|$ is small enough. But for $d > \alpha$ this is a consequence of Lemma 3.13. This completes the proof for $d > \alpha$.

Now, we deal with the case $1 = d \leq \alpha$. We need to show that $\tilde{G}_D(x, y) \leq CG^Y_D(x, y)$ if $|x - y|$ is small enough. Recall that in this case $D$ is a bounded open interval.

**Lemma 3.15.** Let $d = 1$. Then there is a constant $C = C(\alpha, D, m)$ such that for any $x, y \in D$,

$$\tilde{R}_D(x, y) \leq C(\delta_D(x)\delta_D(y))^{\alpha/2} \frac{1}{|x - y|^{1 - \rho}}.$$ 

**Proof.** From Theorem 3.10 it is easy to see that

$$\tilde{G}_D(x, y) \leq C(\delta_D(x)\delta_D(y))^{\alpha/2} \frac{1}{|x - y|}.$$ 

Hence, for $\rho < 1$ we can prove in the same way as in Lemma 8 in [R] that

$$\int_D \tilde{G}_D(x, w)\frac{dw}{|w - y|^{1 - \rho}} \leq C(\delta_D(x))^{\alpha/2} \frac{1}{|x - y|^{1 - \rho}}.$$ 

(13)

From the above

$$\int_D \tilde{G}_D(x, w)\sigma(z - w)dw \leq C \int_D \tilde{G}_D(x, w)\frac{dw}{|w - z|^{1 - \rho}} \leq C \frac{\delta_D(x)^{\alpha/2}}{|x - z|^{1 - \rho}}.$$ 

If $\rho \geq 1$ then $\sigma$ is bounded and one knows that $E^x\tilde{\tau}_D \approx (\delta_D(x))^{\alpha/2}$, so

$$\int_D \tilde{G}_D(x, w)\sigma(z - w)dw \leq CE^x\tilde{\tau}_D \leq c\delta_D(x)^{\alpha/2}.$$ 

Now, we use symmetry of the Green function and the inequality (13) again to get

$$\tilde{R}_D(x, y) \leq C(\delta_D(x)\delta_D(y))^{\alpha/2} \frac{1}{|x - y|^{1 - \rho}}.$$ 

Finally, we are able to prove the lower bound of the Green function for $1 = d \leq \alpha$. 

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Proposition 3.16. Let $D$ be a bounded and open interval. Let $\alpha \geq 1$. Then there exists a constant $C = C(m, d, \alpha, D)$ such that for any $x, y \in D$, 

$$\tilde{G}_D(x, y) \leq CG^{Y}_D(x, y).$$

Proof. Note that we only need to consider the case $|x - y| \leq \theta$ for some sufficiently small $\theta > 0$. First, we assume that $\delta_D(x)\delta_D(y) < |x - y|^2$. By Theorem 3.10 this implies that 

$$\frac{(\delta_D(x)\delta_D(y))^{\alpha/2}}{|x - y|} \leq CG_D(x, y).$$

Then apply Lemma 3.15 to obtain 

$$\tilde{R}_D(x, y) \leq C|x - y|^\alpha \tilde{G}_D(x, y),$$

for some constant $C$. So from (11) it follows that 

$$\tilde{G}_D(x, y) \leq G^{Y}_D(x, y) + \tilde{C}|x - y|^{\alpha \vee 1} \tilde{G}_D(x, y). \quad (14)$$

By the estimates of $\tilde{p}_D(t, x, y)$ (Remark 2.3) we have 

$$\int_{t_0}^\infty \tilde{p}_D(t, x, y)dt \leq Ct_0^{-1/\alpha} (\delta_D(x)\delta_D(y))^{\alpha/2}. \quad (15)$$

Next, from Lemma 2.5 for $X = \tilde{X}$ we have 

$$\tilde{p}_D(t, x, y) \leq p^{Y}_D(t, x, y) + \tilde{p}(t, x, y) - e^{-2mt} p^{Y}(t, x, y), \quad (16)$$

so integrating over $[0, t_0]$, where $t_0 = (\delta_D(x)\delta_D(y))^{\alpha/6} \leq 1$, using Lemma 2.9, and combining with (15) we obtain 

$$\tilde{G}_D(x, y) = \int_{0}^{t_0} \tilde{p}_D(t, x, y)dt + \int_{t_0}^{\infty} \tilde{p}_D(t, x, y)dt$$

$$\leq G^{Y}_D(x, y) + \int_{0}^{t_0} (\tilde{p}(t, x, y) - e^{-2mt} p^{Y}(t, x, y))dt + Ct_0^{-1/\alpha} (\delta_D(x)\delta_D(y))^{\alpha/2}$$

$$\leq G^{Y}_D(x, y) + \tilde{C}t_0^{-1/\alpha} + Ct_0^{-1/\alpha} (\delta_D(x)\delta_D(y))^{\alpha/2}$$

$$= G^{Y}_D(x, y) + C(\delta_D(x)\delta_D(y))^{\alpha/2}. \quad (17)$$

Now assume that $|x - y|^2 \leq \delta_D(x)\delta_D(y)$ and take into account that in this case $\tilde{G}_D(x, y) \geq C(\delta_D(x)\delta_D(y))^{(\alpha-1)/2}$, so we can rewrite (17) as 

$$\tilde{G}_D(x, y) \leq G^{Y}_D(x, y) + c(\delta_D(x)\delta_D(y))^{\rho} \tilde{G}_D(x, y). \quad (18)$$

where $\rho = \frac{2-\alpha}{6} > 0$. Observe that (18) in the case $|x - y|^2 \leq \delta_D(x)\delta_D(y) \leq \theta$, and (14) in the case $\delta_D(x)\delta_D(y) \leq |x - y|^2 \leq \theta$ for $\theta$ sufficiently small provide the conclusion. From the
remaining cases $\delta_D(x)\delta_D(y) \geq \theta$ or $|x - y|^2 \geq \theta$ only the first needs to be considered and can be handled in a very simple way. Indeed, in this situation
\[
(\delta_D(x)\delta_D(y))^{\frac{2\nu - 1}{\alpha}} \leq (\delta_D(x)\delta_D(y))^{\frac{\nu - \frac{\alpha + 1}{\alpha}}{\theta}} \leq C\theta^{\frac{\alpha - 1}{\alpha}} G^Y_D(x, y),
\]
where the last step follows from the fact that $Y_t$ has the property A and Lemma 2.4. Hence the conclusion holds by (17). This completes the proof.

\[3.2 \text{ Case } \nu^Y \geq \tilde{\nu}\]

Throughout this subsection we assume that $\nu^Y \geq \tilde{\nu}$ and in addition let $D$ be a bounded Lipschitz domain. Note that in this case by the result of Sztonyk [Sz] the process $Y$ does not hit the boundary on exiting $D$, so if $u$ is regular harmonic on $D$ with respect to the process $Y$ then
\[
u^Y(x) = E^x u(Y_{\tau_D}) = \int_{D^c} u(z)P^Y_D(x, z)dz, \quad x \in D.
\]

The aim of this section is to prove that the Green functions are comparable, first for $D$ with small diameter and then for arbitrary bounded Lipschitz domains. The result for $D$ of small diameter allows us to prove a version of the Boundary Harnack Principle under the following assumptions:

**G1** $\nu^Y(x) \geq \tilde{\nu}(x)$ for $x \in \mathbb{R}^d \setminus \{0\}$,

**G2** for some $R > 0$ there are constants $c_1(R)$ and $\gamma$ such that
\[
|\sigma(x)| = |\tilde{\nu}(x) - \nu^Y(x)| \leq c_1|x|^{\alpha - d} \text{ for } |x| \leq R,
\]

**G3** there is a constant $c_2 = c_2(R)$ such that
\[
\nu^Y(x) \leq c_2\nu^Y(y) \quad \text{for any } x, y \in \mathbb{R}^d \text{ such that } |x - y| \leq R/2 \text{ and } |x|, |y| \geq R/2.
\]

Then after establishing BHP we show that we can remove the assumption about the diameter of the set $D$.

We start with the iteration of the inequality from Corollary (3.6) to obtain for $G^Y_D(x, \cdot)$ continuous,
\[
G^Y_D(x, y) \leq \tilde{G}_D(x, y) + \sum_{k=1}^n [\langle H^k_D \rangle^k G^Y_D(\cdot, y)][x] + [\langle H^k_D \rangle_1 G^Y_D(\cdot, y)][x],
\]
where $H^k_D : L^1(D) \to L^1(D)$ is given by
\[
[H^k_D f(\cdot)](x) = \int_D \int_D \tilde{G}_D(x, w)|\sigma(w - z)|f(z)dwdz.
\]

We now prove comparability of Green functions for sets of small diameter. Note that the constant $C$ in the conclusion of the following Proposition depends on $D$ through $r_0$ and $\lambda$. This feature is crucial for our future applications.
Proposition 3.17. Let \( d > \alpha \). Let \( D \) be a Lipschitz domain and \( G_D^Y(x, \cdot) \) be continuous and \( v^Y \) satisfies \( G1 \) and \( G2 \), then there exist constants \( R_0 = R_0(d, \alpha, \lambda, r_0, \sigma) \leq R \) and \( C = C(R_0) \) which has the following property: if \( \text{diam}(D) \leq R_0 \) then
\[
C^{-1}\tilde{G}_D(x, y) \leq G_D^Y(x, y) \leq C\tilde{G}_D(x, y), \quad x, y \in D.
\]

Proof. If \( \text{diam}(D) \leq R \) by Lemma 3.13 we get that
\[
[H_D^p\tilde{G}_D(\cdot, y)](x) \leq C_1 \text{diam}(D)^\theta \tilde{G}_D(x, y),
\]
for some constant \( C_1 = C_1(d, \alpha, \lambda, r_0, \sigma) \) and \( \theta > 0 \). Iterating the above inequality we obtain that \([H_D^p]^k\tilde{G}_D(\cdot, y)](x)\) is bounded by
\[
(C_1 \text{diam}(D)^\theta)^k \tilde{G}_D(x, y).
\]

Setting
\[
R_0 = \frac{1}{2} C_1^{-1/\theta} \wedge R
\]
we obtain for \( \text{diam}(D) \leq R_0 \) that
\[
[H_D^p\tilde{G}_D(\cdot, y)](x) \leq \theta \tilde{G}_D(x, y), \quad \text{(21)}
\]
for some \( \theta \leq 1/2 \).

Next, we show that for any \( x \neq y \in D \)
\[
\lim_{n \to \infty} [(H_D^p)^nG_D^Y(\cdot, y)](x) = 0.
\]

Indeed, let us observe that for a positive \( f \in L^1(D) \) we have from (21) that
\[
[(H_D^p)^2f](x) = \int_D \int_D \int_D \int_D \tilde{G}_D(x, u)\sigma(u-v)|\tilde{G}_D(v, w)|\sigma(w-z)f(z)dzdwdu
\]
\[
= \int_D \int_D [(H_D^p)^2\tilde{G}_D(\cdot, w)](x)\sigma(w-z)f(z)dzdw
\]
\[
\leq \theta \int_D \int_D \tilde{G}_D(x, w)\sigma(w-z)f(z)dzdw
\]
\[
= \theta[H_D^p]^2f(x).
\]

Iterating we obtain \([H_D^p]^{n+1}G_D^Y(\cdot, y)](x) \leq \theta^n[(H_D^p)^nG_D^Y(\cdot, y)](x)\). So it is enough to prove that \([H_D^p]^{n+1}G_D^Y(\cdot, y)](x)\) is finite. But from Lemma 2.4 we obtain that there is a constant \( C \) such that \( G_D^Y(x, y) \leq C\tilde{U}(x - y) \). Hence by Lemma 3.12 we get
\[
[(H_D^p)^nG_D^Y(\cdot, y)](x) \leq C \int_D \tilde{U}(x - w)\sigma(w-z)\tilde{U}(z-y)dwdz < \infty.
\]

Finally, we infer from (20) that if \( \text{diam}(D) \leq R_0 \) then
\[
G_D^Y(x, y) \leq \frac{\theta}{1-\theta} \tilde{G}_D(x, y),
\]
which together with Corollary 3.2 ends the proof. \( \square \)
Remark 3.18. The constant $C(R_0)$ in the above theorem converges to 1 if $\text{diam}(D)$ converges to 0.

The next result shows that the Poisson kernels for $D$ are comparable under the assumptions of the preceding result. This in consequence provides necessary tools to establish BHP, which is employed to show comparability of Green functions for sets of arbitrary finite diameter.

**Proposition 3.19.** Let $d > \alpha$ and $D$ be a bounded Lipschitz domain. Assume that $\nu^Y$ satisfies assumptions $G1$ and $G2$ and is bounded on $B^c(0, R)$. There exist constants $R_0 = R_0(d, \alpha, \lambda, r_0, \sigma) \leq R/2$ and $C = C(R_0)$ which satisfy for $D$ such that $\text{diam}(D) \leq R_0$

$$C^{-1}P_D(x, z) \leq P_D^Y(x, z) \leq CP_D(x, z),$$

for any $x \in D$ and $z \in D^c : \delta_D(z) \leq R_0$. Moreover, if we suppose that $\nu^Y$ satisfies assumption $G3$ with $R = 2R_0$, then there exists a constant $C(R_0)$ such that

$$C^{-1}\nu^Y(z - x)E^{x\tau_D} \leq P_D^Y(x, z) \leq C\nu^Y(z - x)E^{x\tau_D},$$

for $x \in D$ and $z \in D^c : \delta_D(z) > R_0$.

**Proof.** By Proposition 3.17 there are constants $\overline{R}_0 \leq R/2$ and $C_1(\overline{R}_0)$ such that

$$C_1^{-1}G_D(x, y) \leq G_D^Y(x, y) \leq C_1G_D(x, y),$$

for $D$ with $\text{diam}(D) \leq \overline{R}_0$. Next, from Theorem 1 in [W] we have the following formula

$$P_D^Y(x, z) = \int_D \nu^Y(z - y)G_D^Y(x, y)dy.$$

But $|\sigma(w)| \leq c_1|w|^{-d+\varrho} = c_1\varphi(-\alpha, d)^{-1}\overline{\nu}(w)|w|^\varrho\alpha$. So for $z \in D^c : \delta_D(z) \leq R_0$ we have

$$|\sigma(z - y)| \leq c_1\varphi(-\alpha, d)^{-1}(2R_0)^{\varrho+\alpha}\overline{\nu}(z - y).$$

Hence, we put $R_0 = \overline{R}_0 \wedge 1/2\left(\frac{\varphi(-\alpha, d)}{2c_1}\right)^{1/(\alpha+\varrho)}$ and then

$$|\sigma(z - y)| \leq \frac{1}{2}\overline{\nu}(x).$$

By the above inequality we obtain

$$P_D^Y(x, z) \leq C_1 \int_D \nu^Y(z - y)\overline{G}_D(x, y)dy
\leq C_1 \left(\int_D \overline{\nu}(z - y)\overline{G}_D(x, y)dy + \int_D \sigma(z - y)\overline{G}_D(x, y)dy\right)
\leq C_1\overline{P}_D(x, y) + C_1 \int_D |\sigma(z - y)|\overline{G}_D(x, y)dy
\leq \frac{3}{2}C_1\overline{P}_D(x, y),$$

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and

\[ P_D^Y(x, z) \geq C_1^{-1} \int_D \nu^Y(z - y) \tilde{G}_D(x, y) dy \]

\[ \geq C_1^{-1} \tilde{P}_D(x, y) - C_1^{-1} \int_D |\sigma(z - y)| \tilde{G}_D(x, y) dy \]

\[ \geq \frac{C_1^{-1}}{2} \tilde{P}_D(x, y), \]

which ends the proof of the first claim of the theorem.

Now, suppose that there is a constant \( c = c(R_0) \) such that \( \nu^Y(x) \leq c \nu^Y(y) \) for all \( |x|, |y| \geq R_0 \) such that \( |x - y| \leq R_0 \). Assume that \( z \in \overline{D} : \delta_D(z) > R_0 \). For \( x, y \in D \) we have

\[ |x - z| \geq \delta_D(z) \geq R_0 \] and of course \( |x - y| \leq \text{diam}(D) \leq R_0 \).

Hence, we get

\[ P_D^Y(x, z) \leq c C_1 \nu^Y(x - z) \int_D \tilde{G}_D(x, y) dy \]

\[ = c C_1 \nu^Y(x - z) E^{x \tilde{\tau}_D}. \]

Similarly the lower bound is

\[ P_D^Y(x, z) \geq (c C_1)^{-1} \nu^Y(x - z) E^{x \tilde{\tau}_D}. \]

\[ \square \]

**Theorem 3.20.** *(Boundary Harnack Principle-BHP)* Let \( d > \alpha \) and \( D \) be a bounded Lipschitz domain. Suppose that \( \nu^Y \) satisfies \( \text{G1-G3} \). Let \( Z \in \partial D \). Then there exists a constant \( \rho_0 = \rho_0(D) \) such that for any \( \rho \in (0, \rho_0] \) and two functions \( u \) and \( v \) which are nonnegative in \( \mathbb{R}^d \) and positive, regular harmonic in \( D \cap B(Z, \rho) \). If \( u \) and \( v \) vanish on \( D^c \cap B(Z, \rho) \), then for \( x, y \in D \cap B(Z, \rho) \)

\[ \frac{u(x)}{v(x)} \leq C \frac{u(y)}{v(y)}, \]

for some constant \( C = C(D, \alpha, \sigma) \) and \( \beta(d, \lambda) \in (0, 1) \).

**Proof.** There is a constant \( R_1 = R_1(d, \lambda) \geq 1 \) (see e.g. [Bo1]) such that for all \( Z \in \partial D \) and \( r \in (0, r_0) \), there exists a Lipschitz domain \( \Omega(r) \) with the Lipschitz constant \( \lambda R_1 \) and the localization radius \( \text{diam}(D) r_0 / R_1 \), having the property

\[ D \cap B(Z, r/R_1) \subset \Omega(r) \subset D \cap B(Z, r). \]

The proof consists of showing that there are constants \( C = C(D, \alpha, \sigma) \) and \( \rho_0 \) such that for \( \rho < \rho_0 \) and \( z \in \Omega(\rho)^c \cap B^c(Z, \rho/R_1) \),

\[ P^Y_{\Omega(\rho)}(x, z) \leq C \frac{E^{x \tilde{\tau}_{\Omega(\rho)}}}{E^{y \tilde{\tau}_{\Omega(\rho)}}} P^Y_{\Omega(\rho)}(y, z), \quad (22) \]
where \(x, y \in D \cap B(Z, \rho/(R_12))\). It is worth mentioning that the constant \(C\) is universal for all sets \(\Omega(\rho), \rho \leq \rho_0\). This would give the conclusion with \(\beta = 1/(2R_1)\) since by (19) we have

\[
    u(x) = E^x u(Y_{\Omega(\rho)}) = \int_{\Omega(\rho)^c} u(z) P^y_{\Omega(\rho)}(x, z) dz
    = \int_{\Omega(\rho)^c \setminus B(Z, \rho/R_1)} u(z) P^y_{\Omega(\rho)}(x, z) dz
    \leq C \frac{E^x \tau_{\Omega(\rho)}}{E^y \tau_{\Omega(\rho)}} \int_{\Omega(\rho)^c \setminus B(Z, \rho/R_1)} u(z) P^y_{\Omega(\rho)}(y, z) dz
    = C \frac{E^x \tau_{\Omega(\rho)}}{E^y \tau_{\Omega(\rho)}} u(y),
\]

which would imply

\[
    \frac{u(x) v(y)}{u(y) v(x)} \leq C \frac{E^x \tau_{\Omega(\rho)}}{E^y \tau_{\Omega(\rho)}} C \frac{E^y \tau_{\Omega(\rho)}}{E^x \tau_{\Omega(\rho)}} = C^2.
\]

Now we prove (22). From Proposition 3.19 we obtain that there exist some constant \(C\) and \(\rho_0 < r_0(D)\) and \(C_1 = C_1(\rho_0)\) such that for any \(\rho \leq \rho_0\)

\[
    C_1^{-1} \tilde{P}_{\Omega(\rho)}(x, z) \leq P^y_{\Omega(\rho)}(x, z) \leq C_1 \tilde{P}_{\Omega(\rho)}(x, z),
\]

if \(\delta_{\Omega(\rho)}(z) \leq \rho_0\). Note that \(C_1\) is universal for all \(\Omega(\rho)\).

By Theorem 2 in [J] we have that there is some \(C_2 = C_2(\alpha, d, \lambda, r_0)\) such that for any \(x, y \in D\) and \(z \in \overline{D}\)

\[
    \tilde{P}_{\Omega(\rho)}(x, z) \leq C_2 \frac{E^x \tau_{\Omega(\rho)}}{E^y \tau_{\Omega(\rho)}} \frac{\tilde{\phi}_{\Omega(\rho)}(A_{y, z'})}{\tilde{\phi}_{\Omega(\rho)}(A_{x, z'})} |y - z|^{d-\alpha} \tilde{P}_{\Omega(\rho)}(y, z),
\]

where \(z' \in \{A \in D : B(A, \kappa\delta_{\Omega(\rho)}(z)) \subset D \cap B(S, \delta_{\Omega(\rho)}(z))\}\) if \(\delta_{\Omega(\rho)}(z) \leq r_0/32\) and \(z' = x_1\) if \(\delta_{\Omega(\rho)}(z) > r_0/32\) for \(S\) such that \(|z - S| = \delta_{\Omega(\rho)}(z)\). If \(x, y \in D \cap B(Z, \rho/(R_12))\) and \(z \in \Omega(\rho)^c \cap B^c(Z, \rho/R_1)\) then

\[
    \frac{|y - z|}{|x - z|} \leq \frac{|x - z| + |x - y|}{|x - z|} \leq (1 + \frac{\rho/R_1}{\rho/(2R_1)}) = 3.
\]

Now, suppose that \(\delta_{\Omega(\rho)}(z) \leq \rho/32\) then we obtain

\[
    |x - z'| \geq |x - z| - |z - z'| \geq |x - z| - |z - S| - |z' - S| \geq \frac{\rho}{2} - 2\delta_{\Omega(\rho)}(z) \geq \frac{7}{16}\rho > \frac{r_0}{32},
\]

while if \(\delta_{\Omega(\rho)}(z) > \rho/32\) then \(z' = x_1\), so \(\delta_{\Omega(\rho)}(z') \geq r_0/4\). Therefore \(A_{x, z'} = x_1 = A_{y, z'}\) and of course \(\frac{\tilde{\phi}_{\Omega(\rho)}(A_{y, z'})}{\tilde{\phi}_{\Omega(\rho)}(A_{x, z'})} = 1\). Hence for \(x, y \in D \cap B(Z, \rho/(R_12))\) and \(z \in \Omega(\rho)^c \cap B^c(Z, \rho/R_1)\)
such that $\delta_{\Omega(\rho)}(z) \leq \rho_0$ we get

$$P^Y_{\Omega(\rho)}(x, z) \leq C_1^2 C_2 3^{d-\alpha} \frac{E^x \tilde{\tau}_{\Omega(\rho)}}{E^y \tilde{\tau}_{\Omega(\rho)}} P^Y_{\Omega(\rho)}(y, z).$$

Next, observe that $G_1 - G_3$ imply that for $r \leq R$ there is a constant $c = c(r)$ such that $\nu^Y(x) \leq c \nu^Y(y)$ for all $x$ and $y$ such that $|x - y| \leq r$ and $|x|, |y| \geq r$. Hence for $\delta_{\Omega(\rho)}(z) \geq \rho_0$ we have

$$P^Y_{\Omega(\rho)}(x, z) \leq C_3(\rho_0) \nu^Y(z-x) E^x \tilde{\tau}_{\Omega(\rho)} \leq C_3(\rho_0) c(\rho_0) \nu^Y(z-y) E^x \tilde{\tau}_{\Omega(\rho)} \leq c C_3^2 \frac{E^x \tilde{\tau}_{\Omega(\rho)}}{E^y \tilde{\tau}_{\Omega(\rho)}} P^Y_{\Omega(\rho)}(y, z).$$

This completes the proof of (22) and hence the theorem. \(\square\)

For regular harmonic functions, which vanish on $D^c$ we infer the following remark.

**Remark 3.21.** Suppose $\nu^Y$ satisfies $G_1, G_2$ and is bounded on $B^c(0, R)$. Let $Z \in \partial D$. Then there exists a constant $\rho_0 = \rho_0(D)$ such that for any $\rho \in (0, \rho_0]$ and two functions $u$ and $v$ which are nonnegative in $\mathbb{R}^d$ and positive, regular harmonic in $D \cap B(Z, \rho)$. If $u$ and $v$ vanish on $D^c$, then for $x, y \in D \cap B(Z, \rho)$

$$\frac{u(x)}{v(x)} \leq C \frac{u(y)}{v(y)},$$

for some constant $C = C(D, \alpha, \sigma)$ and $\beta(d, \lambda) \in (0, 1)$.

**Theorem 3.22.** Let $d > \alpha$ and $D$ be a bounded Lipschitz domain. Assume that $\nu^Y$ satisfies assumptions $G_1, G_2$ and is bounded on $B^c(0, R)$. Then for $x, y \in D$ we have

$$C^{-1} \tilde{G}_D(x, y) \leq G^Y_D(x, y) \leq C \tilde{G}_D(x, y),$$

for some constant $C = C(d, \lambda, r_0, \sigma)$.

**Proof.** Observe that for $|x - y| \leq N(\delta_D(x) \wedge \delta_D(y))$,

$$G^Y_D(x, y) \geq G^Y_{B(x, \delta_D(x) \wedge \delta_D(y) \wedge R_0)}(x, y) \geq C \tilde{G}_{B(x, \delta_D(x) \wedge \delta_D(y) \wedge R_0)}(x, y),$$

where $R_0$ is such that $G^Y_{B(0, R_0)}(x, y) \approx \tilde{G}_{B(0, R_0)}(x, y)$ (such $R_0$ exists from Proposition 3.17). Next, it is easy to see from Theorem 3.4 in [K1] that

$$c(N)|x - y|^{\alpha - d} \leq \tilde{G}_{B(x, \delta_D(x) \wedge \delta_D(y) \wedge R_0)}(x, y) \leq C G^Y_D(x, y). \quad (23)$$

From Lemma 2.7 we have

$$G^Y_D(x, y) \leq U^Y(x - y) \leq C \tilde{U}(x - y) = C|x - y|^{\alpha - d}. \quad (24)$$
We define similarly as in Theorem 3.8 the truncated Green function for $Y_t$ by

$$\phi^Y_D(x) = G^Y_D(x_1, y) \wedge A(d, \alpha)r_0^{d+\alpha}. $$

Using Remark 3.21 we can repeat the arguments from Lemma 17 in [1] to show that

$$\phi^Y_D(x) \approx E^x\tau^Y_D. $$

Next, by Lemma 2.3 we get

$$E^x\tau^Y_D \approx E^x\tau^Y_D. $$

Therefore

$$\phi^Y_D(x) \approx \tilde{\phi}_D(x). \tag{25}$$

By the above and [24] we infer that there is a constant $r$ such that $\phi^Y_D(x) = G^Y_D(x, x_0)$ for $x \in D \cap B^c(x_0, r)$. Hence by Harnack’s inequality for $\alpha$-stable harmonic functions we obtain

$$G^Y_D(x_1, x_0) = \phi^Y_D(x) \approx \tilde{\phi}_D(x) \leq C(N) \tilde{\phi}_D(y) \approx \phi^Y_D(y) = G^Y_D(y, x_0). \tag{26}$$

Using BHP for $Y_t$ (Remark 3.21), and taking into account (23), (24) and (26) we can prove a version of Theorem 3.8 with $G^Y_D$ instead of $\tilde{G}_D$ (see the proof of Theorem 1 in [1]), that is

$$C_1^{-1}\frac{\phi^Y_D(x_1)\phi^Y_D(y)}{(\tilde{\phi}_D(A_{x,y}))^2}|x-y|^{\alpha-d} \leq G^Y_D(x, y) \leq C_1\frac{\phi^Y_D(x_1)\phi^Y_D(y)}{(\tilde{\phi}_D(A_{x,y}))^2}|x-y|^{\alpha-d}. \tag{27}$$

Applying (25) and then comparing the above estimate with the bound from Theorem 3.8 we get the conclusion. \hfill \square

### 3.3 Proof of Theorem 1.2

Let $d > \alpha$ and $D$ be a connected Lipschitz domain. Suppose that $|\sigma(x)| \leq c_3|x|^{-d+\varrho}$ for $|x| \leq 1$, where $\varrho > 0$ and $\nu^Y(x)$ is bounded on $B^c(0,1)$. Then the property A holds for $Y_t$ from Corollary 3.7.

Let $\{Z_t\}$ be a Lévy process with the Lévy measure, which density is equal to $\nu(x) \vee \tilde{\nu}(x)$. Then of course the process $Z_t$ and the set $D$ satisfies the assumptions of Theorem 3.22. So, we obtain that there is a constant $C_1$ such that

$$C_1^{-1}\tilde{G}_D(x, y) \leq G^Z(x, y) \leq C_1\tilde{G}_D(x, y). \tag{28}$$

Therefore we have that

$$C_2^{-1}\tilde{\phi}_D(x)\tilde{\phi}_D(y)\frac{1}{(\tilde{\phi}_D(A_{x,y}))^2}|x-y|^{\alpha-d} \leq G^Z_D(x, y) \leq C_2\tilde{\phi}_D(x)\tilde{\phi}_D(y)\frac{1}{(\tilde{\phi}_D(A_{x,y}))^2}|x-y|^{\alpha-d}. \tag{29}$$

Moreover, the property A holds for $Y_t$, that is

$$CE^x\tau^Y_{t_D}E^y\tau^Y_{t_D} \leq G^Y_D(x, y). \tag{30}$$
Having (28) and (29) hold, we can repeat the proof of Theorem 1.1 for \( d > \alpha \). Hence there exists a constant \( C_3 \) which satisfies
\[
C_3^{-1}G^Y(x, y) \leq G^Z(x, y) \leq C_3G^Y(x, y).
\] (30)
Combining (27) and (30) give us
\[
C^{-1}\tilde{G}_D(x, y) \leq G^Y_D(x, y) \leq C\tilde{G}_D(x, y),
\]
which completes the proof.

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