Getting inflationary models using the deformation method

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Abstract
In this paper we show that the dynamics associated with slow-roll models of inflation can be investigated through a method called the deformation procedure. Using the latter, we explicitly derive an expression linking two slow-roll inflationary potentials, applying the resulting framework to show how to construct an eternal inflation from chaotic inflation, or even, a natural inflation from hilltop inflation, easily.

Keywords: inflaton, slow-roll, eternal inflation, chaotic inflation, natural inflation, hilltop inflation

1. Introduction
The Standard Cosmological Model successfully accounts for some of the main observed properties of our Universe today but, unfortunately, it still leaves many unsolved puzzles to be dealt with [1, 2]. In this sense, the most promising attempt to solve these problems is presented in the context of the theory of inflation, where a scalar field, called inflaton, is responsible for driving the evolution of the very early Universe into a very short period of accelerated expansion [3–5]. In particular, the choice of the potential associated with the inflaton field is of great interest since it may provide the correct dynamics to ensure the success of the model, bringing the period of rapid expansion to an end (grateful exit) at the right moment and with the right conditions so that a Universe consistent with the one we observe today can be generated [6–9]. Based on that, the search for models with suitable potentials that allow us to fit the parameter models with the observational bounds is needed.

Here we consider an alternative approach to investigate models of inflation using a method of deformation for the potentials associated with the inflaton field. This method was studied in many other contexts [13–17] and may offer an easier and more fruitful way to study the dynamics of new models of inflation. As an initial application, we will consider slow-roll inflation models (for a review see [10–12] and the references therein), where the slow-roll parameters are given in terms of the potential that drives the inflation. This class of models is suitable for us since it is well known in the literature and, in the slow-roll limit, presents first-order differential equations, making the application straightforward. Note that this does not in any sense restrict the power of the deformation method because it can be equally well applied to frameworks where higher-order differential equations are found. In fact, a higher-order differential equation can be reduced to a first-order differential equation, implying that at linear order any solution found in one case is equal to the solution in the other case [18].

Furthermore, the deformation procedure can be applied to more complicated models such as multi-field theories, non-canonical dynamics, f(R) theories, or even other cosmological phases. Thus, the application we intend to perform here could open interesting branches of research. One should notice that the primordial perturbations are very sensitive to the choice of the potential and, in this limit, the deformation procedure is not applied (see [19, 20] for a discussion about the non-Gaussianities, and primordial perturbations at linear order with multi-fields).

The present work is organized as follows. In section 2 we quickly review the method of deformation, presenting some of the main aspects associated with it. The line element describing the flat Friedman-Robertson-Walker Universe is found in section 3, along with a review of the slow-roll model of inflation. Our application is performed in section 4, where we apply the deformation procedure to show how a new solution can be obtained from a given inflation solution. In
section 5 we dedicate a few examples to illustrate the new framework from the previous Section. Finally, the conclusion is presented in section 6.

Throughout this paper we use units in which \( c = 4\pi G = H_0 = 1 \), where \( c \) is the speed of light, \( G \) is Newton’s gravitational constant, \( H \) is the Hubble parameter and the subscript ‘0’ refers to the initial time.

2. Deformation procedure in field theory

Let us now briefly review the main aspects of the deformation method before applying it to the slow-roll models of inflation we aim to consider.

As a result of advances in research in high energy physics, a vast class of models described by a scalar field has been proposed. However, the difficulty lies in the fact that many of these models do not provide an analytical description of the studied system, which hinders the complete understanding of it. It is thus necessary to find a method that can generate potentials with analytical solutions and that are of physical interest. A rather effective method is the so-called deformed procedure proposed by Bazeia et al (original paper) and consists in generating new solutions from a given potential, describing a known model, with the aid of a proper deformation function. As a consequence, the new model obtained through the deformation method provides new features, which can be studied analytically without having to use computational methods or numerical analysis, offering the advantage of a simpler framework to work with. The relationship between the potential of the original model \( V(x) \), and the potential of the obtained model, \( V(\phi) \), is given by

\[
V(\phi) = \frac{V(x \rightarrow f(\phi))}{f_\phi^2(\phi)},
\]

where \( f(\phi) \) is the deformation function and \( f_\phi = \partial_\phi f(\phi) \). Thereby, if \( \chi(x) \) is a static solution of the starting model, then we get

\[
\phi(x) = f^{-1}(\chi(x)),
\]

with \( \phi(x) \) being the solution of the new deformed model. We can see that for the case of topological solutions a deformed defect, \( \phi(x) \), connects the corresponding minima of the field \( \chi(x) \) of the original model, given by \( \partial_i = f^{-1}(v_i) \), \( i = 1, 2, 3, \ldots, n \).

3. Cosmological background

We shall consider an action describing the dynamics of a scalar field, \( \chi(x) \), minimally coupled to the Einstein–Hilbert action accounting for gravity, i.e

\[
S = \int d^4x \sqrt{-g} \left( -\frac{1}{4} R + \mathcal{L}(\chi, X) \right),
\]

where \( R \) is the scalar curvature, \( \mathcal{L}(\chi, X) \) the Lagrangian of the model, \( X = \chi_{,\mu} \chi^{,\mu}/2 \) its kinematic’s term and comma stands for partial derivatives. In the following it will be assumed that \( \chi \) plays the role of the inflaton field.

Now let us consider the flat Friedmann–Robertson–Walker Universe described by the line element

\[
d\tau^2 = dt^2 - a^2(t) \left( dx^2 + dy^2 + dz^2 \right),
\]

where \( t \) is the physical time, \( x, y \) and \( z \) are comoving spatial coordinates and \( a(t) \) is the scale factor.

We assume that the energy-momentum tensor for the inflaton field can be written as

\[
T_{\mu \nu} = (\rho + p) u^\mu u^\nu - pg^\mu \nu, \tag{5}
\]

which describes a perfect fluid and where

\[
\begin{align*}
    u_\mu &= \frac{\chi_{,\mu}}{\sqrt{2\chi}}, \\
    \rho &= 2X\mathcal{L}_\chi - \mathcal{L}, \\
    p &= \mathcal{L}(X, \chi).
\end{align*}
\]

In equation (5), \( u^\mu \) is the four-velocity field describing the motion of the fluid (for timelike \( \chi_{,\mu} \)), while \( \rho \) and \( p \) are its proper energy density and pressure, respectively.

By considering the line element above, the action in equation (3) provides the following equations of motion

\[
H^2 = \frac{2}{3} \rho, \tag{7}
\]

and

\[
\dot{a}/a = -\frac{1}{3}(\rho + 3p), \tag{8}
\]

where \( H = \dot{a}/a \) and the dot stands for derivative with respect to the physical time \( t \).

If we consider the standard dynamics, described by Lagrangian density

\[
\mathcal{L} = \frac{1}{2} X_{,\mu} X^{,\mu} - V(\chi), \tag{9}
\]

the continuity equation can be written as

\[
\dot{\chi} + 3H\chi + V_\chi = 0, \tag{10}
\]

the index \( _\chi \) stands for derivative with respect to the field \( \chi \). The pressure and energy density are given by

\[
\rho = \frac{1}{2} \chi^2 + V, \quad p = \frac{1}{2} \chi^2 V. \tag{11}
\]

Thus, we can rewrite equations (7) and (8) as

\[
H^2 = \frac{1}{3} \chi^2 + \frac{2}{3} V, \tag{12}
\]

and

\[
\dot{H} = -\dot{\chi}^2. \tag{13}
\]

In the slow-roll approximation, that is, when \( V \gg \dot{\chi} \), models of inflation which describe an early Universe with an energy density dominated by the potential term, \( V(\chi) \), are of particular interest. In fact, in this approximation, the inflaton field does not vary too rapidly and we can neglect the kinetic
term in equation (12) to get the first-order equations

\[ H^2 \approx \frac{2}{3} V(\chi), \tag{14} \]

\[ 3H\dot{\chi} + V_{,\chi} = 0. \tag{15} \]

These equations show that the choice of the potential allows us to apply limits to the inflationary parameters. The number \( N \) of e-folds, written as \( N = \ln \left( \frac{a_{\text{end}}}{a} \right) \), where \( a_{\text{end}} \) is the scalar factor in the end of inflation, can be obtained through the expression

\[ a = a_0 \exp \left( \int_0^\nu \left( \frac{2}{3} V(\chi) \right)^{1/2} \text{d}t \right), \tag{16} \]

or equivalently \( N = \int_0^{\text{end}} H \text{d}t \).

In addition, in the slow-roll approximation, the slow-roll parameters can be found to be \([21, 22]\)

\[ \epsilon = \frac{1}{4} \left( \frac{V_{,\chi}}{V} \right)^2, \quad \eta = \frac{1}{2} \frac{V_{,\chi \chi}}{V}. \tag{17} \]

In order to establish the flatness condition, the expressions above have to obey the constraints \( |\epsilon| < 1 \) and \( |\eta| < 1 \). Therefore, we can see that all parameters are sensitive to the choice of the potential.

The deformation procedure provides a new class of analytical potentials, which could be an easier way to study potential-dependent parameters. In the next section we will see how this procedure can be applied to the slow-roll inflation scenario.

4. Deforming slow-roll inflationary models

Initially we consider that the inflaton has its dynamics described by the Lagrangian density

\[ \mathcal{L} = \frac{1}{2} \chi_{,\mu} \chi^{\mu} - V(\chi), \tag{18} \]

where \( V(\chi) \) presents the potential field. The continuity equation in this dynamics takes the form

\[ \rho_{,\chi} + 3H\dot{\chi} = 0. \tag{19} \]

Since we know that

\[ H^2 = \frac{2}{3} \rho \tag{20} \]

and squared, (19) we obtain the useful relation

\[ 6\rho\dot{\chi}^2 = \rho_{,\chi}^2. \tag{21} \]

Now we consider another dynamics for the inflaton evolution described by the Lagrangian density

\[ \mathcal{L} = \frac{1}{2} \phi_{,\mu} \phi^{\mu} - \tilde{V} (\phi). \tag{22} \]

Similarly to the previous model results

\[ 6\rho\dot{\phi}^2 = \rho_{,\phi}^2. \tag{23} \]

The key point of this description is to redefine the dynamics field via the relation

\[ \chi = f (\phi), \tag{24} \]

where \( f (\phi) \) is a so called deformation function. As a direct consequence of this definition, we can write

\[ \dot{\phi} = \frac{\dot{\chi}}{f_{,\phi}}, \tag{25} \]

in which \( f_{,\phi} = df/d\phi \). By using (21) and (23) we come to

\[ \frac{\ddot{\rho}_{,\phi}}{\rho} = \frac{1}{f_{,\phi}^2} \left( \frac{\rho_{,\phi}^2}{\rho} \right)_{\chi = f(\phi)}. \tag{26} \]

This presents a generic correspondence between two energy densities describing two dynamics scalar fields. This is a natural way to deform the energy density for two fluids in the same background provided that these fluids have a conserved energy-momentum tensor, or better, provided that they satisfy a continuity equation.

The slow-roll condition applied to the equation of motion of the scalar field allows us to rewrite it as

\[ 3H\dot{\chi} = -V_{,\chi}, \tag{27} \]

now

\[ 6\dot{\chi}^2 = V_{,\chi}^2, \tag{28} \]

as well as

\[ 6\dot{\phi}^2 = \tilde{V}_{,\phi}^2, \tag{29} \]

and we have

\[ \frac{\tilde{V}_{,\phi}^2}{\tilde{V}} = \frac{1}{f_{,\phi}^2} \left( \frac{V_{,\phi}^2}{V} \right)_{\chi = f(\phi)}. \tag{30} \]

This presents a generic correspondence between two potentials describing the inflaton dynamics, under slow-roll approximation.

The solutions in the new model are obtained with \( \phi = f^{-1}(\chi) \), this is the inverse deformation function calculated with the solutions of the original model. An important implication of this framework is based under the possibility of obtaining an analytical description for new inflation solutions and allows us to analyze the parameters for these solutions, provided we know the parameters for the original inflationary model, minimizing or even annulling the numerical techniques, making the search for more complicated vacuum configurations more accessible.
The limit in that slow-roll condition ceases to be valid \( \chi^2/2 = V(\chi) \) depends on the potential chosen and we have

\[
\int_{\chi_{\text{ini}}}^{\chi_{\text{end}}} \frac{d\chi}{\sqrt{2V}} = \frac{1}{2} (\chi_{\text{end}} - \chi_{\text{ini}})^2.
\]

which leads to \( t_{\text{end}} - t_{\text{ini}} \neq \tilde{t}_{\text{end}} - \tilde{t}_{\text{ini}} \), since that (30) is a valid relation between the potential of the original model and the potential of the deformed model. In this sense, the deformation procedure does not show the correlation between two slow-roll sectors, but between two potentials that have a slow-roll regime by construction, which can be seen by analyzing the deformation of the slow-roll parameters.

5. Applications

To illustrate this framework initially, we deal with a model based in the chaotic inflation [23–25]. In this model the dynamics field is driven by the quadratic potential \( V(\chi) = V_0\chi^2 \). The deformation procedure can lead us directly to an eternal inflation model [26–28], described by potential \( \tilde{V}(\phi) = \tilde{V}_0\phi^p \), where we choose \( p > 2 \), assuming the deformation function

\[
f(\phi) = \chi = \frac{4\sqrt{V_0} \phi^{p/2} \tan \left(\frac{\phi}{\sqrt{V_0}}\right)}{(p-4)r^2 \arctanh(\cos((\phi)))},
\]

applied to potential chaotic inflation and using (30). We can observe the behavior of both models in figure 1.

Once the potentials are known, we can obtain the slow-roll parameters. To the original potential we have \( \epsilon = \eta = \chi^{-2} \). It applying the deformation procedure, these parameters are obtained in the other frame and now we write

\[
\tilde{\epsilon} = \frac{p^2}{4\phi^2}, \quad \tilde{\eta} = \frac{p(p-1)}{2\phi^2}
\]

The end of inflation occurs now with an additional choice of parameter \( p \) for the deformed model, while only conditions for fields are necessary for the original model.

The e-fold number can be estimated in this framework as

\[
N = \frac{1}{2} (\chi_{\text{end}} - \chi_{\text{ini}}^2)
\]

and to the deformed model

\[
\tilde{N} = \frac{1}{p} (\phi_{\text{end}}^2 - \phi_{\text{ini}}^2).
\]

We note that the case \( p = 4 \) must be analyzed separately. To this choice, we write \( \tilde{V}(\phi) = 2\phi^4 \) and following the previous results, the deformation function is

\[
f(\phi) = \chi = \frac{\sqrt{V_0} \ln \phi}{\lambda^{2/p}}
\]

Now the slow-roll parameters are

\[
\tilde{\epsilon} = \frac{4}{\phi^2}, \quad \tilde{\eta} = \frac{6}{\phi^2}
\]

To this case, the e-fold number is given by the same \( \tilde{N} \) for \( p = 4 \).

We now consider as the initial model the hilltop inflation [29–31], being

\[
V(\chi) = \left(V_0 - \frac{2}{p} \chi^{p/2}\right)^2
\]

This potential can be deformed directly in the natural inflation potential [32–34], given by \( \tilde{V}(\phi) = \tilde{V}_0 \cos^2(\phi) \), when we choose the following deformation function

\[
f(\phi) = \chi = \left(\frac{4\sqrt{V_0} r^2}{(p-4)r^2 \arctanh(\cos((\phi)))}\right)^{2/(p-4)}
\]

being the integration constant such that \( f(\phi = \pi/(2r)) = 0 \).

Here, the slow-roll parameters for the original potential are

\[
\epsilon = \frac{1}{4} \frac{\chi^{p-2}}{(p-4)^2 \left(\frac{V_0}{V} - \frac{V_0}{\chi}\right)^2},
\]

\[
\eta = \frac{p-1}{2p} \chi^{p-2} - \frac{1}{4} \frac{V_0}{V} (p-2)(\chi^{p-4}/2)^2
\]

In the other way, for the deformation potential we have

\[
\tilde{\epsilon} = r^2 \tan^2(\phi), \quad \tilde{\eta} = r^2 \left(\tan^2(\phi) - 1\right).
\]
See figure 2. The e-fold number can be computed as
\[
N = \left[ \chi^{2} \left( \frac{1}{p} + 4 \frac{V_{0}}{\lambda} \frac{p}{p - 4} \right) \right]_{\phi_{\text{end}}}^{-\phi_{\text{ini}}}
\] (43)
and for the deformed model we have
\[
\tilde{N} = \frac{1}{r^{2}} \ln \left( \frac{\sin(\phi_{\text{ini}})}{\sin(\phi_{\text{end}})} \right)
\] (44)
For \( p = 4 \) we come to
\[
f(\phi) = \chi = \left( \frac{\sin(\phi)}{1 + \cos(\phi)} \right)^{2/\sqrt{V_{0} r^{4}}}
\] (45)
and the integration constant is such that \( f(\phi = \pi/(2r)) = 1 \). The slow-roll parameters \( \epsilon \) and \( \eta \) are the same as above, with \( p = 4 \) in their respective expressions.

The e-fold number is now
\[
N = \left[ \chi^{2} \left( \frac{1}{4} - 2 \frac{V_{0}}{\lambda} \ln \chi \right) \right]_{\phi_{\text{end}}}^{-\phi_{\text{ini}}}
\] (46)

In both cases, we can see that the parameters in the slow-roll inflation regime of the new model are constructed, through the deformation procedure, considering known results of the original model. We take special deformation functions only to illustrate the procedure, exploiting well-established results in the literature. However, if we take the choice of arbitrary deformation functions, we can generate new potentials, expanding the analytical range of solutions in the slow-roll inflation regime. Once again we call attention to the valid limits of this application and we reiterate that it is not valid to after-primordial perturbation generated during inflation, since the choice of potential is very sensitive to the inflaton, clearly presented in the plots above—see [19, 20].

6. Summary

In this paper we have shown how a method called the deformation procedure can be applied to slow-roll models of inflation. This class of models easily favors the application of the method since it presents first-order differential equations. This does not impose a constraint on the procedure because we can reduce the order of higher order differential equations to first order differential equations. Thus, the method of deformation can be used to further investigate more general models of inflation. This is the focus of our present investigations, which will be presented in future papers.

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