MAPS ON POSETS, AND BLOCKERS

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ABSTRACT. An order-theoretic generalization of Seymour relations describing the connection between the set-theoretic blocker, deletion, and contraction maps on clutters, is presented.

1. Introduction

The aim of this note is to present an order-theoretic generalization of Seymour relations [13] which describe the set-theoretic blocker, deletion, and contraction maps on clutters, see (1.2) below. Those relations are a powerful tool of discrete mathematics, see, e.g., [5, 6].

A set $H$ is called a blocking set (cover, system of representatives, transversal) for a nonempty family $G = \{G_1, \ldots, G_m\}$ of nonempty subsets of a finite set if it holds $|H \cap G_k| > 0$, for each $k \in \{1, \ldots, m\}$. The family of all inclusion-minimal blocking sets for $G$ is called the blocker of $G$, see, e.g., [8, Chapter 8]. We denote the blocker of $G$ by $B(G)$.

A family of subsets of a finite ground set $S$ is called a clutter or a Sperner family if no set from that family contains another. The empty clutter $\emptyset$ containing no subsets of $S$, and the clutter $\{\hat{0}\}$ whose unique set is the empty subset $\hat{0}$ of $S$, are called the trivial clutters on $S$. The set-theoretic blocker map assigns to a nontrivial clutter its blocker, and this map alternates the trivial clutters: $B(\emptyset) := \{\hat{0}\}$ and $B(\{\hat{0}\}) := \emptyset$, see, e.g., [4].

Let $X \subseteq S$. The set-theoretic deletion ($\setminus X$), and contraction ($/X$) maps on clutters are defined in the following way: if $G$ is a nontrivial clutter on $S$ then the deletion $G \setminus X$ is the family $\{G \in G : |G \cap X| = 0\}$, and the contraction $G/X$ is the family of all inclusion-minimal sets from the family $\{G - X : G \in G\}$. One often says that the clutters $G \setminus X$ and $G/X$ are those on the ground set $S - X$. The trivial clutters do not change under the deletion and contraction maps: $\emptyset \setminus X = \emptyset/X := \emptyset$ and $\{\hat{0}\} \setminus X = \{\hat{0}\}/X := \{\hat{0}\}$.

Let $G$ be a clutter on the ground set $S$. We have

$$B(B(G)) = G,$$

(1.1)

see [7, 9]; given a subset $X \subseteq S$, it holds (see [13]):

$$B(G \setminus X) = B(G/X) \quad \text{and} \quad B(G)/X = B(G \setminus X).$$

(1.2)
2. A GENERALIZATION OF RELATIONS

We refer the reader to [14, Chapter 3] for information and terminology in the theory of posets. See, e.g., [1, Chapter IV] on the Galois correspondence and (co)closure operators.

**Theorem 2.1.** Let $L$ be a finite poset. Let $\delta : L \to L$ be an order-preserving map, and let $\gamma : L \to L$ be an order-preserving map such that
\begin{equation}
\gamma(x) \geq x , \tag{2.1}
\end{equation}
for all $x \in L$. Let $\beta : L \to L$ be an order-reversing map such that
\begin{equation}
\beta(\beta(x)) \geq x , \tag{2.2}
\end{equation}
for all $x \in L$. Either of the relations (for all $x \in L$):
\begin{equation}
\beta(\delta(\beta(x))) \geq \gamma(x) , \tag{2.3}
\end{equation}
\begin{equation}
\beta(\gamma(\beta(x))) \geq \delta(x) \tag{2.4}
\end{equation}
implies
\begin{equation}
\delta(\beta(z)) \leq \beta(\gamma(z)) \leq \beta(z) \leq \gamma(\beta(z)) \leq \beta(\delta(z)) , \tag{2.5}
\end{equation}
for any $z \in L$. Moreover, if $\beta(\beta(x)) = x$, for all $x \in L$, then either of the equalities $\beta(\delta(\beta(x))) = \gamma(x)$ and $\beta(\gamma(\beta(x))) = \delta(x)$, for all $x \in L$, implies
\begin{equation}
\delta(\beta(z)) = \beta(\gamma(z)) \leq \beta(z) \leq \gamma(\beta(z)) = \beta(\delta(z)) , \tag{2.6}
\end{equation}
for any $z \in L$.

**Proof.** Relation (2.1) implies
\begin{equation}
\beta(\gamma(z)) \leq \beta(z)
\end{equation}
because the map $\beta$ is order-reversing; moreover, we have
\begin{equation}
\beta(z) \leq \gamma(\beta(z)) .
\end{equation}

We now prove implication (2.3) $\implies$ (2.5).

On the one hand, with respect to (2.2), we have $\delta(\beta(z)) \leq \beta(\delta(\beta(z)))$. On the other hand, since $\beta$ is order-reversing, relation (2.3) implies $\beta(\beta(\delta(\beta(z)))) \leq \beta(\gamma(z))$. We obtain
\begin{equation}
\delta(\beta(z)) \leq \beta(\gamma(z)) . \tag{2.7}
\end{equation}

Further, on the one hand, relation (2.3) implies $\gamma(\beta(z)) \leq \beta(\delta(\beta(z)))$. On the other hand, since $\beta(\beta(z)) \geq z$ by (2.2), and $\delta$ is order-preserving, and $\beta$ is order-reversing, we obtain $\beta(\delta(\beta(z))) \leq \beta(\delta(z))$. We conclude that
\begin{equation}
\gamma(\beta(z)) \leq \beta(\delta(z)) , \tag{2.8}
\end{equation}
and we are done.

We now prove implication (2.4) $\implies$ (2.5).
On the one hand, with respect to (2.4), we have \( \delta(\beta(z)) \leq \beta(\gamma(\beta(\beta(z)))) \).

On the other hand, since \( \beta \) is order-reversing, and \( \gamma \) is order-preserving, relation (2.2) implies \( \beta(\gamma(\beta(\beta(z)))) \leq \beta(\gamma(z)) \). We obtain (2.7).

Further, on the one hand, relation (2.2) implies \( \gamma(\beta(z)) \leq \beta(\gamma(\beta(\beta(z)))) \).

On the other hand, since \( \beta \) is order-reversing, relation (2.4) implies \( \beta(\gamma(\beta(\beta(z)))) \leq \beta(\delta(z)) \). We come to (2.8), and we are done.

The proof of relation (2.6) is now straightforward, with respect to the argument above. \( \square \)

Note that since the map \( \beta \) in Theorem 2.1 is order-reversing, and (2.2) holds, it is a consequence of [1, Proposition 4.36(iii)] that we have

\[
\beta(\beta(\beta(x))) = \beta(x),
\]

for any \( x \in L \).

To illustrate Theorem 2.1, we give a comment to (1.2). Let \( P \) be a finite bounded poset of cardinality greater than one, whose least element is denoted \( \hat{0}_P \). We denote by \( \mathcal{I}(A) \) and \( \mathcal{F}(A) \) the order ideal and filter of \( P \) generated by an antichain \( A \subseteq P \), respectively. The atoms of \( P \) are the elements covering \( \hat{0}_P \); we denote the set of all atoms of \( P \) by \( P_a \).

The antichains in \( P \) compose a distributive lattice, denoted \( \mathcal{A}(P) \). In the present note, the antichains are ordered in the following way: if \( A' \), \( A'' \) \( \in \mathcal{A}(P) \) then we set

\[
A' \leq A'' \text{ iff } \mathcal{F}(A') \subseteq \mathcal{F}(A'') .
\]

We call the least element \( \hat{0}_{\mathcal{A}(P)} \) and greatest element \( \hat{1}_{\mathcal{A}(P)} \) of \( \mathcal{A}(P) \) the trivial antichains in \( P \) because, in the context of the present note, those antichains are counterparts of the trivial clutters. Here \( 0_{\mathcal{A}(P)} \) is the empty antichain in \( P \), and \( \hat{1}_{\mathcal{A}(P)} \) is the one-element antichain \{\( \hat{0}_P \)\}.

- If \( \{a\} \) is a nontrivial one-element antichain in \( P \) then the order-theoretic blocker \( b(a) \) of \( \{a\} \) in \( P \) is the antichain

\[
b(a) := \mathcal{I}(a) \cap P^a .
\]

- If \( A \) is a nontrivial antichain in \( P \) then the order-theoretic blocker \( b(A) \) of \( A \) in \( P \) is the following meet in \( \mathcal{A}(P) \):

\[
b(A) := \bigwedge_{a \in A} b(a) .
\]  

(2.10)

- The order-theoretic blockers of the trivial antichains in \( P \) are:

\[
b(\hat{0}_{\mathcal{A}(P)}) := \hat{1}_{\mathcal{A}(P)} , \quad b(\hat{1}_{\mathcal{A}(P)}) := 0_{\mathcal{A}(P)} .
\]

See [2] [3] [10] [11] [12] on blockers in posets.

The map \( b : \mathcal{A}(P) \to \mathcal{A}(P) \) is called the order-theoretic blocker map on \( \mathcal{A}(P) \). That map is order-reversing, with the property \( b(b(A)) \geq A \), for all \( A \in \mathcal{A}(P) \). Equality (2.9) implies

\[
b(b(b(A))) = b(A) ,
\]
Proof. There is nothing to prove if $A$ is trivial.

Let $\{a\}$ be a nontrivial one-element antichain in $P$.

1. Suppose that $|b(a) \cap X| = 0$. In this case we have

$$b(b(a) \setminus X) = b(b(a)) = \{a\}/X$$

and

$$b(b(a)/X) = b(b(a')) \geq \{a\} = \{a\}/X.$$
2. Suppose that $|b(a') \cap X| > 0$ and $b(a') \not\subseteq X$. In this case we have 
$$b(b(a')\setminus X) = b(b(a') - X) = \{a'\}/X$$
and
$$b(b(a')/X) = b(\hat{1}_{\mathfrak{A}(P)}) = \hat{0}_{\mathfrak{A}(P)} = \{a'\}\setminus X.$$ 
3. If $b(a') \subseteq X$ then we have 
$$b(b(a')\setminus X) = b(b(a') - X) = b(\hat{0}_{\mathfrak{A}(P)}) = \hat{1}_{\mathfrak{A}(P)} = \{a'\}/X$$
and
$$b(b(a')/X) = b(\hat{1}_{\mathfrak{A}(P)}) = \hat{0}_{\mathfrak{A}(P)} = \{a'\}\setminus X.$$ 

Now, let $A$ be an arbitrary nontrivial antichain in $P$. On the one hand, we by definition (2.10) have
$$b(b(A)\setminus X) = b\left(\bigwedge_{a \in A} b(a)\right) \setminus X$$
and
$$b(b(A)/X) = b\left(\bigwedge_{a \in A} b(a)\right) / X$$
in $\mathfrak{A}(P)$. On the other hand, for any element $a' \in A$, we have
$$b\left(\bigwedge_{a \in A} b(a)\right) \setminus X \geq b(b(a')\setminus X) \geq \{a'\}/X$$
and
$$b\left(\bigwedge_{a \in A} b(a)\right) / X \geq b(b(a')/X) \geq \{a'\}\setminus X$$
in $\mathfrak{A}(P)$. Definitions (2.11) now imply relations (2.12) and (2.13). \qed

With the help of relation (2.5) and Lemma 2.2 we come to the following conclusion:

**Corollary 2.3** ([11], Theorem 2.6). **For any antichain $A$ in $P$, the relation**
$$b(A)\setminus X \leq b(A/X) \leq b(A) \leq b(A)/X \leq b(A)\setminus X$$
**holds in $\mathfrak{A}(P)$.**

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