Spectral triples and differential calculi related to the Kronecker foliation

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Abstract

Following ideas of Connes and Moscovici, we describe two spectral triples related to the Kronecker foliation, whose generalized Dirac operators are related to first and second order signature operators. We also consider the corresponding differential calculi \( \Omega_D \), which are drastically different in the two cases. As a side-remark, we give a description of a known calculus on the two-dimensional noncommutative torus in terms of generators and relations.

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1 Introduction

In Connes’ approach to noncommutative differential geometry, the notion of a spectral triple plays an essential role, see [1]. It encodes the differential and Riemannian structure of the noncommutative space as well as its dimension. From the physical point of view, spectral triples have been used to construct unified field theoretical models, in particular the standard model (see [1], [2]), and also models including gravitation ([3], [4], [5]). From the mathematical point of view, only a few types of noncommutative spaces have been used in these examples: commutative algebras of smooth functions on a manifold ([6]), finite dimensional algebras (for a classification of spectral triples in this case see [7] and [8]) and products of both. In [8] it was shown that it is not straightforward to define spectral triples related to covariant differential calculi on quantum groups. Explicit examples of spectral triples have also been described for the irrational rotation algebra and higher dimensional noncommutative tori ([9], [10]). For these examples, the data of the triple were chosen according to physical needs or taking advantage of special structures available in the underlying algebra. An important part of the information needed for physical purposes is the explicit form of the differential calculus of a spectral triple. Such calculi have been analyzed in the above-mentioned cases ([11], [12], [13], [14]). In [12] it has been shown...
that the extra structure of a finitely generated projective module allows to introduce
the graded algebra of differential-form-valued endomorphisms which gives a natural
mathematical language to build unified field theoretical models in the spirit of the
Mainz-Marseille approach \[13\]. In \[14\] and \[15\], the notion of spectral triple itself
has been modified and enriched using ideas from supersymmetric quantum theory.
One arrives at noncommutative structures generalizing classical geometrical struc-
tures (Riemannian, symplectic, Hermitian, Kähler, ... structures). Physical hopes
are mainly directed to superconformal field theories (with noncommutative target
spaces).

Recently, see \[16\], Connes and Moscovici have described a method which makes
it possible to construct spectral triples in a systematic way for crossed product al-
gebras related to foliations. Let \((M, F)\) be a regular foliation of a smooth manifold
\(M\) with Euclidean structures on both the corresponding distribution and the nor-
mal bundle. There is an associated spectral triple for the crossed product algebra
\(C^\infty(M) \rtimes \Gamma\), where \(\Gamma\) is a group of diffeomorphisms preserving these structures.
The corresponding Dirac operator is a hypoelliptic operator which is closely related
to the signature operator of the foliated manifold. This signature operator is a
modification of the standard signature operator in differential geometry, see \[17\].

In this paper, we construct explicitly two spectral triples related to the Kro-
necker foliation. We choose as diffeomorphism group the group \(\mathbb{R}\) which defines the
foliation by its action on \(\mathbb{T}^2\) and obviously preserves natural translation invariant
Euclidean structures. Thus we arrive at the algebra \(C^\infty(\mathbb{T}^2) \rtimes \mathbb{R}\), whose \(C^*\) version
is known to be Morita equivalent to the irrational rotation algebra (noncommu-
tative torus), see \[18\], \[19\]. The Dirac operator of the first spectral triple (which
has dimension 2) is closely related to the ordinary signature operator on \(\mathbb{T}^2\). For
the construction of the second triple (of dimension three) we follow the strategy
proposed in \[16\]. The corresponding signature operators and henceforth also the
Dirac operators can be diagonalized explicitly in both cases. Then we pass to the
differential calculi associated to the spectral triples constructed before. It turns
out that for the triple related to the first order signature operator the differential
calculus can be completely determined. Restricted to \(C^\infty(\mathbb{T}^2)\) it projects down to
the de Rham calculus on \(\mathbb{T}^2\). The analysis of the differential calculus for the second
triple turns out to be much more involved. We show that for the restriction of this
triple to the subalgebra \(C^\infty(\mathbb{T}^2)\) (i.e. choosing the trivial diffeomorphism group)
the corresponding one forms give just the universal calculus on \(C^\infty(\mathbb{T}^2)\).

In the appendix we have added the explicit description of the differential calculus
for the spectral triple related to the irrational rotation algebra, see \[1, 9\], which has
properties similar to the calculus associated to the linear signature operator.

2 The spectral triple related to a foliation

For the convenience of the reader, we recall here the definition of a spectral triple
and the differential calculus related to such a triple \([10, 11]\):

**Definition 1** A spectral triple \((A, \mathcal{H}, D)\) consists of a \(*\)-algebra \(A\), a Hilbert space
\(\mathcal{H}\) and an unbounded operator \(D\) on \(\mathcal{H}\), such that

(i) \(A\) acts by a \(*\)-representation \(\pi\) in the algebra \(B(\mathcal{H})\) of bounded operators on
\(\mathcal{H}\),

(ii) the commutators \([D, \pi(a)], a \in A\), are bounded and

\[2\]
(iii) the operator $D$ has discrete spectrum with finite multiplicity.

A spectral triple is said to have dimension $n$, if the eigenvalues (with multiplicity) $\mu_k$ of $|D|$ fulfill $\lim_{k \to \infty} \frac{\mu_k}{B^k} = C \neq 0$.

We will have no need to refer to gradings or real structures usually included in the definition of a spectral triple, and also not to more general notions of dimension.

The representation $\pi$ of $A$ in $B(\mathcal{H})$ can be extended to a representation $\pi^* : \Omega(A) \to B(\mathcal{H})$ of the universal differential calculus $\Omega(A)$ by

$$\pi^*(\sum_k a_k^0 da_1^k \cdots da_n^k) = \sum_k \pi(a_k^0)[D, \pi(a_k^1) \cdots [D, \pi(a_k^n)]].$$

If $J_0 := \oplus_n \ker \pi^n$, then $J := J_0 + dJ_0$ is a differential ideal, and one arrives at the differential calculus $\Omega_D(A)$,

$$\Omega^n_D(A) := \Omega^n(A)/J.$$

Note that, if $\pi$ is faithful, there are isomorphisms

$$\Omega^1_D(A) \simeq \pi^1(\Omega^1(A)) \quad (2.1)$$

and

$$\Omega^2_D(A) \simeq \pi^2(\Omega^2(A))/\pi^2(dJ_0^1). \quad (2.2)$$

Now we review shortly the procedure given in [14], which relates a spectral triple to a regular foliation of a smooth manifold. Let $M$ be a compact manifold with a foliation given by an integrable distribution $V \subset TM$. The normal bundle of the foliation is $N := TM/V$, with canonical projection $\rho : TM \to N$. Assume further that both $V$ and $N$ are equipped with Euclidean fibre metrics and with an orientation (i.e. there are distinguished nowhere vanishing sections $\omega_V$, $\omega_N$ of the exterior bundles $\Lambda^v V$, $\Lambda^n N$ ($v = \dim V$, $n = \dim N$)). Furthermore, $\omega_V$ and $\omega_N$ also define a nonvanishing section of $\Lambda^v V^* \otimes \Lambda^n N^* \simeq \Lambda^{v+n} T^*M$, i.e. a volume form on $M$. The bundle of interest for us is

$$E = \bigwedge V_c^* \otimes \bigwedge N_c^*.$$

Obviously, the metrics on $V$ and $N$ give rise to Hermitian metrics on $\bigwedge V_c^*$ and $\bigwedge N_c^*$ and thus also on $E$. The orientations $\omega_V$ and $\omega_N$ can be mapped by means of the metrics to sections $\gamma_V$ of $\bigwedge V_c^*$ and $\gamma_N$ of $\bigwedge N_c^*$, which can be used, together with the metrics, to define an analogue of the Hodge star on the exterior bundles $\bigwedge V_c^*$ and $\bigwedge N_c^*$. We choose a variant of the $*$-operation such that $^{*2}_{V_c} = 1$ and $^{*2}_{N_c} = 1$, i.e. $^*_{V_c}$ and $^*_{N_c}$ can be considered as $\mathbb{Z}_2$-grading operators (cf. [14]).

Thus, the space of sections of $E$ has a natural inner product, and we denote by $\mathcal{H} = L^2(M, E)$ the Hilbert space of square integrable sections of this bundle. From now on, we always consider complexified vector bundles, but omit the subscript $C$.

In order to construct a generalized Dirac operator, a longitudinal differential $d_L$ and a transversal differential operator $d_H$ have to be defined. The differential $d_L$ is defined canonically by means of the Bott connection ([2]) given as the partial covariant derivative $\nabla : \Gamma(V) \times \Gamma(N) \to \Gamma(N)$ defined by

$$\nabla_X Y = \rho \left( [X, \tilde{Y}] \right),$$

for $X \in \Gamma(V), Y \in \Gamma(N)$ and $\tilde{Y} \in \Gamma(TM)$ such that $\rho(\tilde{Y}) = Y$. By a standard procedure (using the Leibniz rule and duality) $\nabla$ is extended to a differential $d_L$:
\[ \Gamma(E) \rightarrow \Gamma(E) \text{ defined by linear mappings } \Gamma(\Lambda^k V^* \otimes \Lambda^l N^*) \rightarrow \Gamma(\Lambda^{k+1} V^* \otimes \Lambda^{l} N^*), \]

\[ d_L \alpha(X_0, \ldots, X_k) = \sum_{i=0, \ldots, k} (-1)^i \nabla_{X_i} (\alpha(X_0, \ldots, \hat{X}_i, \ldots, X_k)) \]

\[ + \sum_{i<j} (-1)^{i+j} \alpha([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k), \]

\[ X_i \in \Gamma(V). \] Since the Bott connection is flat, we have \( d_L^2 = 0. \)

In order to define a transversal differential operator one has to choose a sub-
\[ H \subset TM \text{ complementary to } V. \] This defines a bundle isomorphism \( j_H : \Lambda V^* \otimes \Lambda N^* \rightarrow \Lambda T^* M \) in the following way: Let us denote by \( \text{pr}_V \) and \( \text{pr}_H \) the projections corresponding to the decomposition \( TM^* = V^* \oplus H^* \), by \( \rho_H : H \rightarrow N \) the restriction of \( \rho \) to \( H \) and by \( \rho_H^* \) its transposed map. Then \( j_H \) is defined as the following composition:

\[ \Lambda V^* \otimes \Lambda N^* \xrightarrow{\text{id} \otimes \rho_H^*} \Lambda V^* \otimes \Lambda H^* \xrightarrow{\text{pr}_V \otimes \text{pr}_H^*} \Lambda T^* M \otimes \Lambda T^* M \xrightarrow{\Sigma} T^* M, \]

where \( \otimes \rightarrow \wedge \) denotes the replacement of the tensor product by the wedge product. Now, the transversal operator \( d_H \) is obtained from the exterior differential \( d \) by transporting with \( j_H \) and projecting to a certain homogeneous component: \( \Lambda V^* \otimes \Lambda N^* \) has an obvious bigrading, and denoting by \( \pi(r,s) \) the projector to the homogeneous component of bidegree \( (r, s) \), one defines

\[ d_H \alpha = \pi^{(r,s+1)}(j_H^{-1} \circ d \circ j_H(\alpha)) \]

for \( \alpha \in \Gamma(\Lambda V^* \otimes \Lambda N^*). \) The operator \( d_H \) is a graded derivation of the \( \mathbb{Z}_2 \)-graded algebra \( \Gamma(\Lambda V^* \otimes \Lambda N^*). \)

In a foliation chart, \( d_L \) and \( d_H \) look as follows. Let \( (x^i, y^k), i = 1, \ldots, v, k = 1, \ldots, n \) be local coordinates of \( M \) such that \( x^i \) are coordinates on the leaf (foliation chart). The corresponding coordinate vector fields \( \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^k} \right) \) form a local frame of \( TM \) and \( \left( \frac{\partial}{\partial y^k} \right) \) a frame of \( V. \) The corresponding dual frame of \( T^* M \) consists of the differentials \( (dx^i, dy^k). \) We define \( \theta^i \in \Gamma(V^*) \) by \( \theta^i(\frac{\partial}{\partial x^j}) = \delta^i_j \) \( (i, j = 1, \ldots, v). \) It is immediate from the definition of \( N \) that the elements \( n_k := \frac{\partial}{\partial y^k} + V \) \( (k = 1, \ldots, n) \) form a local frame of \( N. \) The elements of the corresponding dual frame of \( N^* \) are denoted by \( n^k. \) Finally, we choose a local frame \( h_k \) of the transversal space \( H. \) This frame is fixed by assuming \( \rho_H(h_k) = n_k. \) This leads to

\[ h_k = h_k^i \frac{\partial}{\partial x^i} + \frac{\partial}{\partial y^k}, \]

with coefficient functions \( h_k^i \) characterizing \( H. \) Then, the elements \( \theta^{i_1} \wedge \cdots \wedge \theta^{i_r} \otimes n^{j_1} \wedge \cdots \wedge n^{j_s} \) form a local frame of \( E, \) and one can show that \( d_L \) and \( d_H \) are given by the following local formulas:

\[ d_L(\alpha_{i_1 \ldots i_r, j_1 \ldots j_s} \theta^{i_1} \wedge \cdots \wedge \theta^{i_r} \otimes n^{j_1} \wedge \cdots \wedge n^{j_s}) = \frac{\partial \alpha_{i_1 \ldots i_r, j_1 \ldots j_s}}{\partial x^i} \theta^{i_2} \wedge \cdots \wedge \theta^{i_r} \otimes n^{j_1} \wedge \cdots \wedge n^{j_s} + \]

\[ \alpha_{i_1 \ldots i_r, j_1 \ldots j_s} \sum_{i=1}^{r} \frac{\partial h_{k}^{i}}{\partial x^i} \theta^{i_1} \wedge \cdots \wedge \theta^{i_i} \otimes n^{j_1} \wedge \cdots \wedge n^{j_s}, \]

\[ (-1)^r \left( \frac{\partial}{\partial y^k} + h_k^i \frac{\partial}{\partial x^i} \right) \alpha_{i_1 \ldots i_r, j_1 \ldots j_s} \theta^{i_1} \wedge \cdots \wedge \theta^{i_r} \otimes n^{j_1} \wedge \cdots \wedge n^{j_s} + \]

\[ \alpha_{i_1 \ldots i_r, j_1 \ldots j_s} \sum_{i=1}^{r} \frac{\partial h_{k}^{i}}{\partial x^i} \theta^{i_1} \wedge \cdots \wedge \theta^{i_i} \otimes n^{j_1} \wedge \cdots \wedge n^{j_s}, \] (2.3)
(where $\theta^t$ at position $t$ replaces $\theta^i$). The longitudinal differential $d_L$ acts as a differential in leaf direction, whereas $d_H$ is a sum of a principal part, which differentiates in transversal direction, and a zero order part. As examples, let us give formulae for $d_H$ acting on functions, $(1,0)$-, $(0,1)$- and $(1,1)$-forms:

\[
\begin{align*}
    d_H f &= \ h_k(f)n^k, \\
    d_H(\alpha_i\theta^i) &= -h_k(\alpha_i)\theta^i \otimes n^k - \alpha_i \frac{\partial h_k}{\partial x^j} \theta^j \otimes n^k, \\
    d_H(\alpha_k n^k) &= \ h_l(\alpha_k) n^l \wedge n^k, \\
    d_H(\alpha_{ik} \theta^i \wedge n^k) &= -h_l(\alpha_{ik}) \theta^i \otimes n^l \wedge n^k - \alpha_{ik} \frac{\partial h_l}{\partial x^j} \theta^j \otimes n^l \wedge n^k,
\end{align*}
\]

$(d_H(n^k) = 0)$. For the adjoint operators $d^*_L$ and $d^*_H$ (in $\mathcal{H}$) it is difficult to write down explicit formulae. One can show

\[
\begin{align*}
d^*_L \alpha &= - *_V d_L *_V + \text{term of order zero},
\end{align*}
\]

where $*_V$ is the (partial) Hodge operator related to the Euclidean metric and the orientation of $V$. Since $d^*_H$ lowers the $N^*$-degree one has for $\alpha \in \Gamma(\wedge^r V^* \otimes \wedge^0 N^*)$

\[
d^*_H \alpha = 0.
\]

Explicit formulae for $d^*_H$ become rather complicated as, e.g., the case of $(0,1)$-forms shows:

\[
d^*_H(\alpha_i \theta^i) = -g^k_l(h_k(\alpha_i) - \alpha_m \Gamma_{N^m}^l + \alpha_l \left( \frac{\partial h_k}{\partial x^i} + \frac{1}{2} g^i_{ij} h_k(g_{V^i j}) \right)),
\]

(2.4)

where $g^k_l = g_N(n^k, n^l)$, $g_{V^i j} = g_V(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$, $g^i_{ij} = g_V(\theta^i, \theta^j)$ are the local components of the fibre metrics (and their duals), and $\Gamma_{N^m}^l$ are the Christoffel symbols corresponding to $g_{N^k l}$.

In [16], using $d_H$ and $d_L$, two differential operators were introduced by

\[
\begin{align*}
    Q_L &= d_L d^*_L - d^*_L d_L \\
    Q_H &= d_H + d^*_H 
\end{align*}
\]

and the mixed signature operator $Q$ for $M$, acting on a form with $N$-degree $\partial_N$, was defined by

\[
Q = Q_L (-1)^{\partial_N} + Q_H.
\]

(2.5)

As noted in [16], $Q$ is selfadjoint. Finally, a generalized Dirac operator $D$ is defined as the unique selfadjoint operator such that

\[
D|D| = Q. \tag{2.6}
\]

If zero is not an element of the spectrum of $Q$, it is given as

\[
D = Q|Q|^{-1/2} = Q(Q^2)^{-1/4}, \tag{2.7}
\]

as shows a straightforward argument using the spectral decomposition of $Q$.

One motivation for choosing a second order longitudinal part is the following: the index of the signature operator should not depend on the choice of the transversal subbundle $H$. Usually, the index of a pseudodifferential operator only depends on its principal symbol. However, as follows from the local formulae [2,3] and [2,4],
its principal part explicitly depends on $H$, the dependence being in the coefficients of the partial derivatives with respect to leaf coordinates. It turns out that one can get rid of this dependence by introducing a modified notion of pseudodifferential operators ($\psi DO'$) which assigns a degree 2 to transversal coordinates and a degree 1 to longitudinal ones. To have a contribution also from $Q_L$, one has to pass to a second order operator. In [16] a homotopy argument was given to show that this does not affect the longitudinal signature class.

Let $\Gamma$ be any group of diffeomorphisms of $M$ which preserves the distribution $V$ and the Euclidean metrics on both $V$ and $N$. Then $\psi \in \Gamma$ acts via the pull back as unitary operator $U^*_\psi$ on $\mathcal{H}$, whereas functions from $C_\infty(M)$ act there as multiplication operators. The crossed product algebra $\mathcal{A} := C_\infty(M) \rtimes \Gamma$ can be defined as the $*$-subalgebra of $B(\mathcal{H})$ generated by these two types of operators. Due to $U^*_\psi f = (f \circ \psi) U^*_\psi$ every element of $\mathcal{A}$ is a finite sum of elements $f U^*_\psi$. Then we have, see [16]:

**Theorem 1** $(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple of dimension $v + 2n$.

Our main aim is to describe explicitly this spectral triple for the Kronecker foliation of the two-torus.

## 3 Spectral triples for the Kronecker foliation

### 3.1 The crossed product algebra for the Kronecker foliation

Let us start with some conventions and notations. We consider the two-torus as the quotient $\mathbb{T}^2 = \mathbb{R}^2/2\pi \mathbb{Z}^2$. Thus, we have natural local coordinates $0 < \vartheta_1, \vartheta_2 < 2\pi$. Consider the $\mathbb{R}$-manifold $(\mathbb{T}^2, \mathbb{R}, \psi)$, with group action

$$\psi : \mathbb{T}^2 \times \mathbb{R} \to \mathbb{T}^2,$$

given by

$$\psi((\vartheta_1, \vartheta_2), t) = (\vartheta_1 + at, \vartheta_2 + bt),$$

with $a, b \in \mathbb{R}$ such that $a > 0$, $a^2 + b^2 = 1$ and $\theta = \frac{b}{a}$ being irrational. The foliation of $\mathbb{T}^2$ by the orbits of $\psi$ is called the Kronecker foliation. It is well known, see [21], that each leaf of this foliation is diffeomorphic to $\mathbb{R}$ and lies dense in $\mathbb{T}^2$.

The coordinate transformation

$$x = a \vartheta_1 + b \vartheta_2$$
$$y = b \vartheta_1 - a \vartheta_2$$

is orthogonal and leads to coordinates $(x, y)$ of a foliation chart. In these coordinates, $\mathbb{R}$ acts as follows

$$\psi((x, y), t) = (x + t, y).$$

To be more precise, this is the lifted action of $\mathbb{R}$ on $\mathbb{T}^2$, applied to global coordinates $(x, y)$ obtained from global coordinates $(\vartheta_1, \vartheta_2)$ by the orthogonal transformation.

It is well known, see [23], that associated to the action of a locally compact group $K$ on a manifold $M$ there is a transformation groupoid $G$. For the Kronecker foliation, we have $G = \mathbb{T}^2 \times \mathbb{R}$ with range and source maps $r$ and $s$ given by

$$r(p, t) = \psi(p, t)$$
$$s(p, t) = p,$$
$p \in \mathbb{T}^2$, the space of units being $\mathbb{T}^2$. The associated crossed product algebra

$$\mathcal{O} = \mathcal{O}(\mathbb{T}^2) \rtimes \mathbb{R}$$

is the $*$-algebra generated by the unitary operators $U_1, U_2$ and $V_t$ acting in the Hilbert space $L^2(\mathbb{T}^2)$ given by

$$(U_1 \xi)(\vartheta_1, \vartheta_2) = e^{i\vartheta_1} \cdot \xi(\vartheta_1, \vartheta_2)$$
$$(U_2 \xi)(\vartheta_1, \vartheta_2) = e^{i\vartheta_2} \cdot \xi(\vartheta_1, \vartheta_2)$$
$$(V_t \xi)(\vartheta_1, \vartheta_2) = \xi(\vartheta_1 + at, \vartheta_2 + bt), \quad (3.1)$$

$\forall \xi \in L^2(\mathbb{T}^2)$. Let $e_{kl} = e^{i(k\vartheta_1 + l\vartheta_2)}$ ($k, l \in \mathbb{Z}$) be the basis of trigonometric polynomials of $L^2(\mathbb{T}^2)$. Obviously from (3.1) it follows that

$$U_1 e_{kl} = e_{k+1,l}$$
$$U_2 e_{kl} = e_{k,l+1}$$
$$V_t e_{kl} = e^{i(ak+bl)t} e_{kl}, \quad (3.2)$$

It is now immediate to show

**Proposition 1** The unitary operators $U_1, U_2, V_t$ satisfy

$$U_1 U_2 = U_2 U_1, \quad (3.3)$$
$$V_t U_1 = e^{i\omega t} U_1 V_t, \quad (3.4)$$
$$V_t U_2 = e^{ibt} U_2 V_t, \quad (3.5)$$
$$V_t V_s = V_{t+s}, \quad t,s \in \mathbb{R}. \quad (3.6)$$

**Remark 1** For rational $\frac{a}{b} = \frac{m}{n}$, $m, n$ relative prime, there is an additional relation

$$V_{2\pi \sqrt{m^2+n^2}} = V_0 = 1;$$

$2\pi \sqrt{m^2+n^2}$ is the smallest value of $t$ such that $V_t = V_0 = 1$ and any other such $t$ is an integer multiple of it.

**Proposition 2** The $*$-algebra $\mathcal{O}(\mathbb{T}^2) \rtimes \mathbb{R}$ is isomorphic to $\mathbb{C}(u_1, u_2, v_t)/J$, where $\mathbb{C}(u_1, u_2, v_t)$ is the free associative unital $*$-algebra generated by $u_1, u_2$ and $v_t$, $t \in \mathbb{R}$, and $J$ is the $*$-ideal generated by (3.3)–(3.6) and unitarity conditions for the generators.

**Proof:** By universality of $\mathbb{C}(u_1, u_2, v_t)/J$, there exists a homomorphism $\pi$ of this algebra onto $\mathcal{O}(\mathbb{T}^2) \rtimes \mathbb{R}$ sending the corresponding generators into each other. Now, by (3.3)–(3.6), a general element $a$ of $\mathbb{C}(u_1, u_2, v_t)/J$ is a linear combination of the monomials $v_t u_1^k u_2^l$, for some $t_j \in \mathbb{R}$ (with $j \neq j' \Rightarrow t_j \neq t_j'$), $k, l \in \mathbb{Z}$, $a = \sum a_{jkl} v_t u_1^k u_2^l$ (finite sum). We first show that all $V_t$ are independent. $V_t = \sum b_j V_{t_j}$ is equivalent to $1 = \sum b_j e^{i(ax(t_j-t))}$, $\forall k, l$. Since $\frac{a}{b}$ is irrational, $\{ak+bl | k, l \in \mathbb{Z}\}$ is dense in $\mathbb{R}$, and one can conclude $1 = \sum b_j e^{i(kx(t_j-t))}$, $\forall x \in \mathbb{R}$. But 1 and $e^{ix(t_j-t)}$ are orthogonal as almost-periodic functions, see [23], and therefore linearly independent. Then also $V_t U_1^k U_2^l$ and $V_{t'} U_1^k U_2^l$ ($t \neq t'$) are linearly independent. Since $V_t U_1^k U_2^l$ and $V_{t'} U_1^k U_2^l$ ($k, l \neq (k', l')$) shift a different number of steps in the above basis $e_{kl}$ they are obviously independent. In other words, the monomials $V_t U_1^k U_2^l$ constitute
a basis in \( \mathcal{O}(\mathbb{T}^2) \times \mathbb{R} \). If \( \pi(a) = \sum a_{jkl}V_jU_1^kU_2^l = 0 \) we have \( a_{jkl} = 0 \) and \( \pi \) is a bijection. □

In analogy with the definition given before Theorem 1, putting \( M = \mathbb{T}^2 \) and \( \Gamma = \mathbb{R} \), we define the crossed product

\[
\mathcal{A} := C^\infty(\mathbb{T}^2) \rtimes \mathbb{R}
\]

as a \( * \)-subalgebra of \( B(L^2(\mathbb{T}^2)) \).

**Remark 2** We can introduce a set of seminorms on \( \mathcal{O} = \mathcal{O}(\mathbb{T}^2) \rtimes \mathbb{R} \) as follows.

Let \( \mathcal{O}(\mathbb{T}^2) \subset \mathcal{O} \) be the \( * \)-subalgebra generated by \( U_1 \) and \( U_2 \). We define a family \( (p_n)_{n \in \mathbb{N}} \) of seminorms on this subalgebra by

\[
p_n(\sum_{jk} a_{jk}U_1^jU_2^k) = \sup_{j,k \in \mathbb{Z}} (1 + |j| + |k|)^n|a_{jk}|.
\]

It is well-known ([24, 22.19.2.4.]) that the completion of \( \mathcal{O}(\mathbb{T}^2) \) in the corresponding Fréchet topology is \( C^\infty(\mathbb{T}^2) \). Now we define seminorms on \( \mathcal{O} \) by

\[
p_n^\times(\sum_{jkl} a_{jkl}V_jU_1^kU_2^l) = p_n(\sum_{jkl} a_{jkl}U_1^kU_2^l).
\]

Then it is easy to show that \( \mathcal{A} \) is the completion \( \mathcal{O} \) with respect to the Fréchet topology defined by the family of seminorms \( p_n^\times \). To this end, one first notes that every element of \( \mathcal{A} \) is a finite sum of products \( fV_t \), \( f \in C^\infty(\mathbb{T}^2) \). By the above, \( f \) is a limit of elements \( f_k \) of \( \mathcal{O}(\mathbb{T}^2) \) with respect to \( p_n \), and by the definition of \( p_n^\times \) it is obvious that \( fV_t \) is the limit of \( f_kV_t \).

Let us now describe the Hilbert space of the spectral triple of Theorem 1 for the Kronecker foliation.

Here, both \( V \) and \( N \) are one-dimensional, with local frames consisting each of one vector \( \frac{\partial}{\partial y} \) and \( \bar{\nu} = \frac{\partial}{\partial y} + V \) respectively. Let \( \tau \) and \( \nu \) denote the corresponding elements of the dual frames. Then \( E = \bigwedge V^* \hat{\otimes} \bigwedge N^* \) consists of four one-dimensional subspaces of elements of degrees \( (0,0), (1,0), (0,1), (1,1) \), with local frames \( 1, \tau, \nu, \tau \otimes \nu \), respectively. The natural choice of translation invariant (under the natural action of \( \mathbb{R}^2 \) on \( \mathbb{T}^2 \)) Euclidean fibre metrics makes these frame elements mutually orthogonal unit vectors in \( L^2(\mathbb{T}^2, E) \). We may identify

\[
L^2(\mathbb{T}^2, E) = L^2(T^2) \oplus L^2(T^2) \oplus L^2(T^2) \oplus L^2(T^2),
\]

with \( e_{kl}1 \rightarrow (e_{kl},0,0,0), \ldots, e_{kl}\tau \otimes \nu \rightarrow (0,0,0,e_{kl}) \).

**Remark 3** Since the generators act, according to ([7.4]), componentwise in \( L^2(\mathbb{T}^2, E) \) the crossed product algebra of Theorem 1 coincides with ([3.7]).

We choose the transversal subspace \( H \) in the simplest way, i.e. we put \( h^1_2 = 0 \), i.e. \( H \) is generated by the coordinate vector field \( \frac{\partial}{\partial y} \). Then the general formulae of the foregoing section lead (with some easy computations for the adjoints) to the
following expressions:

|      | $f$   | $f_\tau$ | $f_\nu$ | $f_\tau \otimes \nu$ |
|------|-------|----------|---------|----------------------|
| $d_L$ | $\frac{\partial f}{\partial x}$ | 0        | $\frac{\partial f}{\partial x} \otimes \nu$ | 0                     |
| $d_H$ | $\frac{\partial f}{\partial y}$ | $-\frac{\partial f}{\partial y} \otimes \nu$ | 0       | 0                    |
| $d_L^*$ | 0      | $-\frac{\partial f}{\partial x}$ | 0       | $-\frac{\partial f}{\partial x} \nu$ |
| $d_H^*$ | 0      | 0        | $-\frac{\partial f}{\partial y}$ | $\frac{\partial f}{\partial y} \tau$ |

with $f \in C^\infty(\mathbb{T}^2)$. To prove, e.g.,

$$d_H^*(f_\tau \otimes \nu) = \frac{\partial f}{\partial y} \tau,$$

we denote by $(\cdot|\cdot)$ the scalar product in $L^2(\mathbb{T}^2, E)$ and observe that

$$(g_\tau|d_H^*(f_\tau \otimes \nu)) = (d_H(g_\tau)|f_\tau \otimes \nu) = (-\frac{\partial g}{\partial y} \otimes \nu|f_\tau \otimes \nu)$$

$$= -\int \frac{\partial g(x,y)}{\partial y} f(x,y) dxdy = \int g(x,y) \frac{\partial f(x,y)}{\partial y} dxdy$$

$$= (g_\tau|\frac{\partial f}{\partial y} \tau).$$

Note that all the above operators can also be written as matrix differential operators.

### 3.2 The first order signature operator as Dirac operator

We will first show that $(C^\infty(\mathbb{T}^2) \times \mathbb{R}, L^2(\mathbb{T}^2, E), \tilde{Q})$, with $\tilde{Q}$ being the linear signature operator

$$\tilde{Q} = d_L + d_L^* + d_H + d_H^*,$$

is a spectral triple of dimension 2. Using the foliation chart $(x,y)$ and the local frame $\{1, \tau, \nu, \tau \otimes \nu\}$, this operator can be written as

$$\tilde{Q} = \begin{pmatrix}
0 & -\frac{\partial}{\partial x} & -\frac{\partial}{\partial y} & 0 \\
\frac{\partial}{\partial x} & 0 & 0 & \frac{\partial}{\partial y} \\
\frac{\partial}{\partial y} & 0 & 0 & -\frac{\partial}{\partial x} \\
0 & -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0
\end{pmatrix}. $$

The eigenvalue problem for $\tilde{Q}$ is most easily solved by considering its square

$$\tilde{Q}^2 = \begin{pmatrix}
-\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} & 0 & 0 & 0 \\
0 & -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} & 0 & 0 \\
0 & 0 & -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} & 0 \\
0 & 0 & 0 & -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}
\end{pmatrix}. $$
In coordinates \((\vartheta_1, \vartheta_2)\) one is immediately led to the eigenvalue equations
\[
\left(-\left(\frac{a}{\partial \vartheta_1} + b \frac{\partial}{\partial \vartheta_2}\right)^2 - \left(\frac{b}{\partial \vartheta_1} - \frac{a}{\partial \vartheta_2}\right)^2\right) f_j = \lambda^2 f_j,
\]
for the four components of an eigenvector \(f = f_1 + f_2 \tau + f_3 \nu + f_4 \tau \otimes \nu\) of \(\tilde{Q}^2\). It is now straightforward to see that
\[
e^1_{kl} = \begin{pmatrix} e_{kl} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad e^2_{kl} = \begin{pmatrix} 0 \\ e_{kl} \\ 0 \\ 0 \end{pmatrix}, \quad e^3_{kl} = \begin{pmatrix} 0 \\ 0 \\ e_{kl} \\ 0 \end{pmatrix}, \quad e^4_{kl} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ e_{kl} \end{pmatrix}
\]
are eigenvectors of \(\tilde{Q}^2\) to the eigenvalue
\[
\lambda^2_{kl} = (ak + bl)^2 + (al - bk)^2.
\]
The operator \(\tilde{Q}\) itself acts in this basis of \(L^2(\mathbb{T}^2, E)\) as follows:
\[
\begin{align*}
\tilde{Q}(e^1_{kl}) &= (ak + bl)e^2_{kl} + (al - bk)e^3_{kl} \\
\tilde{Q}(e^2_{kl}) &= (-ak - bl)e^1_{kl} + (-al + bk)e^4_{kl} \\
\tilde{Q}(e^3_{kl}) &= (-al + bk)e^1_{kl} + (ak + bl)e^4_{kl} \\
\tilde{Q}(e^4_{kl}) &= (al - bk)e^2_{kl} - (ak + bl)e^3_{kl}.
\end{align*}
\]
Thus, we have
\[
\lambda^\pm_{kl} = \pm \sqrt{(ak + bl)^2 + (al - bk)^2} \quad (3.8)
\]
and
\[
\begin{align*}
e^{+1}_{kl} &= \frac{bk - al}{\lambda^+_{kl}} e^1_{kl} + \frac{ak + bl}{\lambda^+_{kl}} e^4_{kl}, \\
e^{+2}_{kl} &= \frac{-ak + bl}{\lambda^+_{kl}} e^1_{kl} + \frac{-al + bk}{\lambda^+_{kl}} e^4_{kl}, \\
e^{-1}_{kl} &= \frac{ak + bl}{\lambda^-_{kl}} e^2_{kl} + \frac{-al - bk}{\lambda^-_{kl}} e^3_{kl}, \\
e^{-2}_{kl} &= \frac{-al - bk}{\lambda^-_{kl}} e^2_{kl} + \frac{ak + bl}{\lambda^-_{kl}} e^3_{kl} + e^4_{kl},
\end{align*}
\]
form a complete set of eigenvectors of \(\tilde{Q}\).

**Proposition 3** \((C^\infty(\mathbb{T}^2) \times \mathbb{R}, L^2(\mathbb{T}^2, E), \tilde{Q})\) is a spectral triple of dimension 2.

**Proof:** The eigenvalues \((3.8)\) of \(\tilde{Q}\) have finite multiplicity, tend to infinity for \(k, l \to \infty\) and have no finite accumulation point. Thus, the resolvent of \(\tilde{Q}\) is compact. Since
\[
[\tilde{Q}, V_i] = 0, \quad (3.9)
\]
boundedness of the commutators of \(\tilde{Q}\) with elements of the algebra follows from the fact that
\[
[\tilde{Q}, fV_i] = f[\tilde{Q}, V_i] + [\tilde{Q}, f]V_i = \begin{pmatrix}
0 & -\frac{\partial f}{\partial x} & -\frac{\partial f}{\partial y} & 0 \\
\frac{\partial f}{\partial x} & 0 & 0 & \frac{\partial f}{\partial y} \\
\frac{\partial f}{\partial y} & 0 & 0 & -\frac{\partial f}{\partial x} \\
0 & -\frac{\partial f}{\partial y} & \frac{\partial f}{\partial x} & 0
\end{pmatrix} V_i,
\]
is a bounded matrix multiplication operator in $L^2(\mathbb{T}^2, E)$. In order to see that the $n$-th eigenvalue of $|\tilde{Q}|$ is of order $\sqrt{n}$ notice first that the eigenvalues of $|\tilde{Q}|$ are $\lambda^2_{kl}$, with multiplicity $4 \times$ (number of $(k, l) \in \mathbb{Z}^2$ leading to the same $\lambda^2_{kl}$). The number of eigenvalues with absolute value less than some $R > 0$ is then $4 \times$ (number of integer lattice points inside a circle of radius $R$), i.e. equal to $4 \times (\text{the area } \pi R^2)$ up to lower order terms in $R$. (Recall that $(x, y) \mapsto (\vartheta_1, \vartheta_2)$ is orthogonal.) This proves the claim. 

Note that the commutators of $\tilde{Q}$ with the generators $U_1$ and $U_2$ are explicitly given by

$$
\begin{align*}
[\tilde{Q}, U_1] e^1_{kl} &= ae^2_{k+1,l} - be^3_{k+1,l}, \\
[\tilde{Q}, U_1] e^2_{kl} &= -ae_{k+1,l} + be^4_{k+1,l}, \\
[\tilde{Q}, U_1] e^3_{kl} &= be_{k+1,l} + ae^4_{k+1,l}, \\
[\tilde{Q}, U_1] e^4_{kl} &= -be_{k+1,l} - ae^3_{k+1,l}
\end{align*}
$$

and

$$
\begin{align*}
[\tilde{Q}, U_2] e^1_{kl} &= be^2_{k,l+1} + ae^3_{k,l+1}, \\
[\tilde{Q}, U_2] e^2_{kl} &= -be^1_{k,l+1} - ae^4_{k,l+1}, \\
[\tilde{Q}, U_2] e^3_{kl} &= -ae^1_{k,l+1} + be^4_{k,l+1}, \\
[\tilde{Q}, U_2] e^4_{kl} &= ae^2_{k,l+1} - be^3_{k,l+1}.
\end{align*}
$$

In order to describe the differential algebra $\Omega_{\tilde{Q}}(\mathcal{O}(\mathbb{T}^2) \times \mathbb{R})$, we denote, as in formulae (2.3) and (2.2), by $\pi^1$ and $\pi^2$ the extensions of $\pi$ to universal one and two forms. Since $\pi$ is faithful by Proposition 2, $\Omega_{\tilde{Q}}^1(\mathcal{O}(\mathbb{T}^2) \times \mathbb{R})$ is isomorphic to $\pi^1(\Omega^1(\mathcal{O}(\mathbb{T}^2) \times \mathbb{R}))$, with $du_j \mapsto [\tilde{Q}, U_j]$, $dv_t \mapsto [\tilde{Q}, V_t]$, and $\Omega_{\tilde{Q}}^2(\mathcal{O}(\mathbb{T}^2) \times \mathbb{R}) = \Omega^2(\mathcal{O}(\mathbb{T}^2) \times \mathbb{R})/(\text{ker } \pi^2 + d(\text{ker } \pi^1)) \simeq \pi^2(\Omega^2(\mathcal{O}(\mathbb{T}^2) \times \mathbb{R})) / \pi^2(d(\text{ker } \pi^1))$.

Let us first note that, under the identification $L^2(\mathbb{T}^2, E) \simeq \mathbb{C}^4 \otimes L^2(\mathbb{T}^2)$ given by

$$
e (0) \otimes e_{kl}, e^2_{kl} \mapsto (0) \otimes e_{kl}, e^3_{kl} \mapsto (0) \otimes e_{kl}, e^4_{kl} \mapsto (0) \otimes e_{kl},
$$

$U_1, U_2, V_t$ and the above commutators can be written as follows:

$$
U_1 = 1 \otimes s_1, \; U_2 = 1 \otimes s_2, \; V_t = 1 \otimes v_{abt}, \quad (3.11)
$$

where $s_1 e_{kl} = e_{k+1,l}$, $s_2 e_{kl} = e_{k,l+1}$, $v_{abt} e_{kl} = e^{(ak+bt)} e_{kl}$, and

$$
[\tilde{Q}, U_1] = \begin{pmatrix}
0 & a & -b & 0 \\
-a & 0 & 0 & b \\
b & 0 & 0 & a \\
0 & -b & -a & 0
\end{pmatrix} \otimes s_1, \quad [\tilde{Q}, U_2] = \begin{pmatrix}
0 & b & a & 0 \\
-b & 0 & 0 & -a \\
-a & 0 & 0 & b \\
0 & a & -b & 0
\end{pmatrix} \otimes s_2. \quad (3.12)
$$

Using this representation, together with $[s_1, s_2] = 0$, $s_1 v_{abt} = e^{iat} v_{abt} s_1$, $s_2 v_{abt} = e^{ibt} v_{abt} s_2$, it is easy to show
Lemma 1

\[ U_j[\tilde{Q}, U_k] = [\tilde{Q}, U_k]U_j, \quad \forall j, k \in \{1, 2\}, \quad (3.13) \]

\[ V_t[\tilde{Q}, U_1] = e^{iat}[\tilde{Q}, U_1]V_t, \quad V_t[\tilde{Q}, U_2] = e^{ibt}[\tilde{Q}, U_2]V_t, \quad (3.14) \]

\[ [\tilde{Q}, U_1][\tilde{Q}, U_2] = -[\tilde{Q}, U_2][\tilde{Q}, U_1]. \quad (3.15) \]

Explicitly, we have

\[
[\tilde{Q}, U_1][\tilde{Q}, U_2] = \begin{pmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix} \otimes s_1 s_2 . \quad (3.16)
\]

Proposition 4

(i) \( \Omega^1_{\tilde{Q}}(\mathcal{O}(\mathbb{T}^2) \rtimes \mathbb{R}) \) is a free left (and right) \( \mathcal{O}(\mathbb{T}^2) \rtimes \mathbb{R} \)-module with basis \( \{du_1, du_2\} \). Its bimodule structure is determined by

\[ u_j du_k = du_k u_j, \quad \forall j, k \in \{1, 2\}, \quad (3.17) \]

\[ v_i du_1 = e^{iat} du_1 v_i, \quad v_i du_2 = e^{ibt} du_2 v_i. \quad (3.18) \]

Moreover,

\[ du_1 = 0. \quad (3.19) \]

(ii) \( \Omega^2_{\tilde{Q}}(\mathcal{O}(\mathbb{T}^2) \rtimes \mathbb{R}) \) is a free left (and right) \( \mathcal{O}(\mathbb{T}^2) \rtimes \mathbb{R} \)-module with basis \( \{du_1 du_2\} \), with

\[ du_1 du_2 = -du_2 du_1. \quad (3.20) \]

(iii) \( \Omega^k_{\tilde{Q}}(\mathcal{O}(\mathbb{T}^2) \rtimes \mathbb{R}) = 0 \) for \( k \geq 3 \).

Proof: Since the representation \( \pi \) of \( \mathcal{O}(\mathbb{T}^2) \rtimes \mathbb{R} \) in \( L^2(\mathbb{T}^2, E) \) is faithful, the equations (3.17), (3.18) and (3.20) follow from (3.13)–(3.15), whereas (3.19) comes from (3.4). Now it is sufficient to show that every element of \( \pi^1(\Omega^1(\mathcal{O}(\mathbb{T}^2) \rtimes \mathbb{R})) \) is of the form \( a_1[\tilde{Q}, U_1] + a_2[\tilde{Q}, U_2], \quad a_1, a_2 \in \pi(\Omega^1(\mathcal{O}(\mathbb{T}^2) \rtimes \mathbb{R})) \) and that from \( a_1[\tilde{Q}, U_1] + a_2[\tilde{Q}, U_2] = 0 \) follows \( a_1 = a_2 = 0 \). The first claim is immediate from the fact that \( \pi(\mathcal{O}(\mathbb{T}^2) \rtimes \mathbb{R}) \) is generated by \( U_1, U_1^*, U_2, U_2^*, V_t \) and from (3.9), (3.13)–(3.15). (Note that commutators \( [\tilde{Q}, U_j] \) can be reduced to \( [\tilde{Q}, U_j] \) using the Leibniz rule: From 0 = \( [\tilde{Q}, U_j] [\tilde{Q}, U_j] = U_j^*[\tilde{Q}, U_j] + [\tilde{Q}, U_j^*] U_j \) follows \( [\tilde{Q}, U_j^*] = -U_j^*[\tilde{Q}, U_j] U_j^* \). On the other hand, multiplying \( [\tilde{Q}, U_j] U_j = U_j[\tilde{Q}, U_j] \) from both sides with \( U_j^* \) gives \( U_j^*[\tilde{Q}, U_j] = [\tilde{Q}, U_j] U_j^* \).) It remains to show linear independence. Assume

\[
\sum_{ij} \left( a_{ij} V_{t_{ij}} U_1^* U_2^* [\tilde{Q}, U_1] + b_{ij} V_{t_{ij}} U_1^* U_2^* [\tilde{Q}, U_2] \right) = 0 ,
\]

\[ a_{ij}, b_{ij} \in \mathbb{C}, \quad \text{finite summation over } i, j \in \mathbb{Z}. \]

Acting with this expression on the basis vector \( e_{kl}^1 \) gives

\[ a_{ij} e^{it_{ij}((k+i+1)a+(l+j)b)} (ae_{k+i+1,l+j}) - be_{k+i+1,l+j} \]

\[ = b_{i+1,j} e^{-it_{ij}((k+i+1)a+(l+j)b)} (be_{k+i+1,l+j}) + ae_{k+i+1,l+j}, \]

(now for fixed \( i, j \), which means

\[ a_{ij} e^{it_{ij}((k+i+1)a+(l+j)b)} - bb_{i+1,j} e^{-it_{ij}((k+i+1)a+(l+j)b)} = 0 \]

\[ ba_{ij} e^{it_{ij}((k+i+1)a+(l+j)b)} + ab_{i+1,j} e^{-it_{ij}((k+i+1)a+(l+j)b)} = 0 . \]
Since
\[
\det \begin{pmatrix}
  a e^{i t_j ((k+i+1)a+(l+j)b)} & -b e^{i t_{j-1}((k+i+1)a+(l+j)b)} \\
  b e^{i t_j ((k+i+1)a+(l+j)b)} & a e^{i t_{j-1}((k+i+1)a+(l+j)b)}
\end{pmatrix} = (a^2 + b^2) e^{i (t_j + t_{j-1})((k+i+1)a+(l+j)b)} = e^{i (t_j + t_{j-1})((k+i+1)a+(l+j)b)} \neq 0,
\]
this system has only the trivial solution \(a_{ij} = b_{i+1,j-1} = 0\). Thus (i) is proven.

To prove (ii) note that differentiating (3.17) for \(j = k\) leads immediately to \(du_1 du_1 = du_1 du_2 = 0\). Moreover, (3.17) yields \(du_1 du_2 = -du_2 du_1\), so that \(\Omega^2(\mathcal{Q}(T^2) \times \mathbb{R})\) is generated by \(du_1 du_2\). It remains to show that it is freely generated. To this end we have to determine \(\ker \pi^1\). From (3.13) and (3.14) it follows that \(\ker \pi^1\) contains the bimodule generated by the elements \(u_j du_k - du_k u_j\), \(v_t du_1 - e^{i t} du_1 v_t\), \(v_t du_2 - e^{i t} du_2 v_t\). On the other hand, this bimodule also contains \(\ker \pi^1\): Let
\[
\alpha = \sum a^r_{t,j,k,l,m,n,q} v_t u^k_j u^l_2 du_r v_m u^n_1 u^q_2 \in \ker \pi^1.
\]
Using the commutation rules (3.13) and (3.14) and the basis property of the \([\mathcal{Q}, u_i]\) already proved in (i), one concludes from \(\pi^1(\alpha) = 0\) that
\[
\sum a^r_{t,j,k,l,m,n,q} v_t u^k_j + \sum u_1 u^l_2 e^{i((k+1)a+lb)t} = 0, \quad r = 1, 2.
\]
Now, making use of the basis property of the monomials \(v_t u^k_j u^l_2\) (Proposition 3), one has
\[
\sum_{t_j + t_m = T_k + n = K, l + q = L} a^r_{j,k,l,m,n,q} e^{i((k+1)a+lb)t} = 0, \quad r = 1, 2, \tag{3.21}
\]
for every fixed \(T, K, L\). Now fix (for \(r = 1\)) \(k_0, l_0, t_{j_0}\) and write equation (3.21) as
\[
a^1_{t_0,k_0,l_0,T-t_{j_0},K-k_0,L-l_0} = \sum_{t_j \neq t_{j_0}, k \neq k_0, l \neq l_0} a^1_{t,j,k,l,T-t_j,K-k,L-L} e^{i((k+1)a+lb)(T-t_j)}.
\]
Inserting this into \(\alpha\) one obtains
\[
\alpha = \sum a^1_{t,j,k,l,T-t_j,K-k,L-L} v_t u^k_1 u^l_2 du_1 v_{T-t_j} u^k_1 u^l_2 - e^{i((k+1)a+lb)(T-t_{j_0})} \sum_{(t,j,k,l) \neq (t_0,k_0,l_0)} a^1_{t,j,k,l,T-t_j,K-k,L-L} v_t u^k_1 u^l_2 du_1 v_{T-t_j} u^k_1 u^l_2 - e^{i((k+1)a+lb)(T-t_{j_0})} \cdot e^{i((k+1)a+lb)(T-t_j)} + \text{(a similar term for } r = 2).\]
Now, fix \(t_j > t_{j_0}, k > k_0, l > l_0\), and reduce the power of \(u_2\) in front of \(du_1\) in the first term by subtracting and adding \(du_1 u_2\), thus producing a term in the bimodule (with \(u_2 du_1 - du_1 u_2\) in the middle) and a new term with a new \(c\)-factor of the old kind. Iterating this procedure removes all superfluous powers of \(u_2\). One can do the same for \(u_1\) and \(v_t\) and finally ends up with an expression which turns out to be zero (up to a lot of terms all lying in the bimodule). We leave the detailed computation to the reader. Thus we have shown that \(\ker \pi^1\) is also contained in the bimodule generated by the elements (3.17) and (3.18).
Therefore, a general element of \( \ker \pi^1 \) is of the form

\[
j = \sum a_k \alpha_k b_k
\]

with \( a_k, b_k \in \mathcal{O}(\mathbb{T}^2) \times \mathbb{R} \), \( \alpha_k \) one of the above generating elements of \( \ker \pi^1 \). Then

\[
\pi^2(dj) = \sum \pi(a_k) \pi^2(d\alpha_k) \pi(b_k),
\]

because the other terms appearing according to the Leibniz rule contain a factor \( \pi^1(\alpha_k) = 0 \). We have to determine \( \pi^2 \circ d(u_j du_k - du_k u_j) = \pi^2(du_j du_k + du_k du_j) = [\hat{Q}, U_j][\hat{Q}, U_k] + [\hat{Q}, U_k][\hat{Q}, U_j] \). A trivial calculation using \((3.12)\) and \((3.11)\) shows that \([\hat{Q}, U_j][\hat{Q}, U_j] = -U_j^2\), whereas \([\hat{Q}, U_1][\hat{Q}, U_2] + [\hat{Q}, U_2][\hat{Q}, U_1] = 0\) by \((3.15)\). It follows that

\[
\pi^2(d \ker \pi^1) = \pi(\mathcal{O}(\mathbb{T}^2) \times \mathbb{R}),
\]

and it remains to show that from \( \pi(a)[\hat{Q}, U_1][\hat{Q}, U_2] \in \pi(\mathcal{O}(\mathbb{T}^2) \times \mathbb{R}) \) follows \( a = 0 \). Indeed, this follows from the fact that algebra elements have the diagonal form \((3.11)\) whereas \([\hat{Q}, U_1][\hat{Q}, U_2]\) is antidiagonal \((3.16)\), which is preserved under multiplication with a diagonal element \( \pi(a) \).

(iii) is a trivial consequence of the fact that in a form of degree \( \geq 3 \) at least two differentials of the same generator \( u_j \) will meet to produce 0. \( \square \)

**Remark 4** One can define a first order differential calculus for the algebra \( \mathcal{A} = C^\infty(\mathbb{T}^2, \mathbb{R}) \) in the following way: Let \( \Omega^1_Q(\mathcal{A}) \) be the free left \( \mathcal{A} \)-module with basis \( \{du_1, du_2\} \). Equipped with the product of the topologies defined by the sequence of seminorms \( p^n \), \( \Omega^1_Q(\mathcal{A}) \) is a free left topological \( \mathcal{A} \)-module. One turns \( \Omega^1_Q(\mathcal{A}) \) into a right \( \mathcal{A} \)-module by defining \( a_j du_j u_k := a_j u_k du_j \) for \( j, k \in 1, 2 \), \( a_1 du_1 v_t := e^{iat} a_1 v_t du_1 \) and \( a_2 du_2 v_t := e^{iat} a_2 v_t du_2 \) \( (a_j \in \mathcal{A}) \), and extending this by continuity. This gives \( \Omega^1_Q(\mathcal{A}) \) the structure of a topological bimodule containing \( \Omega^1_Q(\mathcal{O}) \) as a dense submodule. It is also not difficult to see that the differential can be extended to a continuous map \( d : \mathcal{A} \rightarrow \Omega^1_Q(\mathcal{A}) \). Analogously, one can define a topological \( \mathcal{A} \)-bimodule \( \Omega^2_Q(\mathcal{A}) \) such that the natural mappings \( \Omega^1_Q(\mathcal{A}) \times \Omega^1_Q(\mathcal{A}) \rightarrow \Omega^2_Q(\mathcal{A}) \) and \( d : \Omega^1_Q \rightarrow \Omega^2_Q(\mathcal{A}) \) are continuous. We conjecture that the differential calculus \( \Omega^2_Q(\mathcal{A}) \) so constructed coincides with the calculus (to be denoted by the same symbol) resulting from the spectral triple \( (\mathcal{A}, L^2(\mathbb{T}^2, E), \hat{Q}) \).

### 3.3 The mixed signature operator

Let us now consider the mixed signature operator \( Q \) given by formula \((2.5)\). In matrix representation, we have

\[
Q = \begin{pmatrix}
\frac{\partial^2}{\partial x^2} & 0 & \frac{\partial}{\partial y} & 0 \\
0 & -\frac{\partial^2}{\partial x^2} & 0 & -\frac{\partial}{\partial y} \\
-\frac{\partial}{\partial y} & 0 & -\frac{\partial^2}{\partial x^2} & 0 \\
0 & \frac{\partial}{\partial y} & 0 & \frac{\partial^2}{\partial x^2}
\end{pmatrix}.
\]

In order to diagonalize this operator, we have to solve the eigenvalue problem

\[
Q \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = \lambda \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix}
\]

(3.22)
with \( f_i \in L^2(T^2, E) \). \( Q \) is already block-diagonal and acts in the same way in the space of \((0,0)\)- and \((0,1)\)-forms and in the space of \((1,1)\)- and \((1,0)\)-forms. It suffices to diagonalize one block. Defining
\[
g = f_1 + f_3 \\
h = f_1 - f_3
\]
one arrives at
\[
\frac{\partial^2 h}{\partial x^2} + \frac{\partial h}{\partial y} = \lambda g \\
\frac{\partial^2 g}{\partial x^2} - \frac{\partial g}{\partial y} = \lambda h.
\]
Returning to the original coordinates \((\vartheta_1, \vartheta_2)\), the foregoing equations read
\[
a^2 \frac{\partial^2 h}{\partial \vartheta_1^2} + 2ab \frac{\partial^2 h}{\partial \vartheta_1 \partial \vartheta_2} + b^2 \frac{\partial^2 h}{\partial \vartheta_2^2} - b \frac{\partial h}{\partial \vartheta_1} + a \frac{\partial h}{\partial \vartheta_2} = \lambda g, \\
a^2 \frac{\partial^2 g}{\partial \vartheta_1^2} + 2ab \frac{\partial^2 g}{\partial \vartheta_1 \partial \vartheta_2} + b^2 \frac{\partial^2 g}{\partial \vartheta_2^2} + b \frac{\partial g}{\partial \vartheta_1} - a \frac{\partial g}{\partial \vartheta_2} = \lambda h.
\]
The ansatz
\[
g = \sum_{k,l \in \mathbb{Z}} \eta_{kl} e^{i(k\vartheta_1 + l\vartheta_2)} \\
h = \sum_{k,l \in \mathbb{Z}} \chi_{kl} e^{i(k\vartheta_1 + l\vartheta_2)}
\]
leads to
\[
\left((a^2k^2 + 2abkl + b^2l^2)^2 + (bk - al)^2\right) \chi_{kl} = \lambda^2 \chi_{kl},
\]
which gives the eigenvalues
\[
\lambda_{kl} = \pm \sqrt{(ak + bl)^2 + (bk - al)^2}.
\]
One easily concludes that eigenvectors to the eigenvalues \(\lambda_{kl} \pm\) are of the form
\[
h_{kl} = e_{kl}, \quad g_{kl} = \gamma_{kl} \pm e_{kl}
\]
with
\[
\gamma_{kl} = \frac{-(ak + bl)^2 + i(al - bk)}{\lambda_{kl}}.
\]
The eigenvectors of the original problem (3.22) are
\[
f_{1kl} = \frac{1}{2} (g_{kl} + h_{kl}) = \frac{1}{2} (1 + \gamma_{kl} \pm) e_{kl} \\
f_{3kl} = \frac{1}{2} (g_{kl} - h_{kl}) = \frac{1}{2} (\gamma_{kl} \pm - 1) e_{kl},
\]
or, written as elements of \(L^2(T^2, E)\),
\[
e_{kl}^{(1)} = \frac{1}{2} e_{kl} \left((\gamma_{kl} \pm + 1)1 + (\gamma_{kl} \pm - 1)\nu\right).
\]
If we assume that the metrics are chosen so that the frame elements $1, \tau, \nu, \tau \otimes \nu$ are orthonormal, these vectors are already orthonormal (note that $|\gamma_{kl\pm}| = 1$.) The same argument yields another set
\[ e_{kl\pm}^{(2)} = \frac{1}{2} e_{kl} ((\gamma_{kl\pm} + 1)\tau \otimes \nu + (\gamma_{kl\pm} - 1)\tau) \]
of eigenvectors to the same eigenvalues $\lambda_{kl\pm}$. Note that the eigenvalue 0 appears only for $k = l = 0$. In that case, equations (3.23) and (3.24) decouple, and we get four independent eigenvectors $1, \tau, \nu, \tau \otimes \nu$. In order to see that these vectors together with the $e_{kl\pm}^{(1)}$ form an orthonormal basis of $L^2(T^2, E)$, it is sufficient to see that all the vectors $e_{kl}1, e_{kl}\tau, e_{kl}\nu, e_{kl}\tau \otimes \nu$ are linear combinations of the foregoing vectors. This follows from the fact that the matrix
\[
\begin{pmatrix}
\gamma_{kl+} + 1 & \gamma_{kl-} - 1 \\
\gamma_{kl-} + 1 & \gamma_{kl+} - 1
\end{pmatrix}
\]
is always invertible (its determinant being $-4\gamma_{kl+}$).

Thus we have found the spectral decomposition of the selfadjoint operator $Q$. Its unboundedness is reflected in the unboundedness of the $\lambda_{kl\pm}$. It is now easy to write down also the spectral decomposition of the corresponding Dirac operator $D$: Applying (2.7) for nonzero eigenvalues gives
\[De_{kl\pm}^{(1,2)} = \pm \sqrt{\lambda_{kl}} e_{kl\pm}^{(1,2)},\]

where $\lambda_{kl}$ is the positive root $\lambda_{kl+}$. Putting $e_{00+}^{(1)} = 1, e_{00-}^{(1)} = \nu, e_{00+}^{(2)} = \tau \otimes \nu$ and $e_{00-}^{(2)} = \tau$, the formula defines $D$ also on the kernel of $Q$ (cf. 2.6), and gives the spectral decomposition of $D$.

The unitary operators $U_1, U_2$ and $V_t$ act by (3.2) on the basis vectors $e_{kl\pm}^{(1,2)}$ as follows
\[
\begin{align*}
U_1 e_{kl\pm}^{(1,2)} &= \frac{1}{2} \left\{ \left( 1 + \frac{\gamma_{kl\pm}}{\gamma_{k+l+1}} \right) e_{k+l,1+}^{(1,2)} + \left( 1 + \frac{\gamma_{kl\pm}}{\gamma_{k+l-1}} \right) e_{k+l,1-}^{(1,2)} \right\}, \\
U_2 e_{kl\pm}^{(1,2)} &= \frac{1}{2} \left\{ \left( 1 + \frac{\gamma_{kl\pm}}{\gamma_{k+l+1}} \right) e_{k+l,1+}^{(1,2)} + \left( 1 + \frac{\gamma_{kl\pm}}{\gamma_{k+l-1}} \right) e_{k+l,1-}^{(1,2)} \right\}, \\
V_t e_{kl\pm}^{(1,2)} &= e^{i(ka+lb)t} e_{kl\pm}^{(1,2)}.
\end{align*}
\]

Defining
\[ \eta_{kl\pm}^{(1,2)} := \frac{1}{2} \left( e_{kl+}^{(1,2)} \pm e_{kl-}^{(1,2)} \right), \]
one finds
\[
\begin{align*}
U_1 \eta_{kl+}^{(1,2)} &= \eta_{k+l,1+}, \\
U_1 \eta_{kl-}^{(1,2)} &= \frac{\gamma_{kl}}{\gamma_{k+l}} \eta_{k+l,1-}, \\
U_2 \eta_{kl+}^{(1,2)} &= \eta_{k+l,1+}, \\
U_2 \eta_{kl-}^{(1,2)} &= \frac{\gamma_{kl}}{\gamma_{k+l}} \eta_{k+l,1-}, \\
V_t \eta_{kl+}^{(1,2)} &= e^{i(ka+lb)t} \eta_{kl+}^{(1,2)}, \\
D \eta_{kl\pm}^{(1,2)} &= \sqrt{\lambda_{kl}} \eta_{kl\pm}^{(1,2)}.
\end{align*}
\]

From Theorem [1] or by direct computation using (3.25)–(3.30) one gets
**Proposition 5** \( (C^\infty(\mathbb{T}^2) \times \mathbb{R}, L^2(\mathbb{T}^2, E), D) \) is a spectral triple of dimension 3.

Next, one would like to describe the differential calculus \( \Omega_D \) related to this spectral triple. Unfortunately, we have no definite result about \( \Omega_D \). We will however show that the first order calculus for the restriction of the spectral triple to the subalgebra \( C^\infty(\mathbb{T}^2) \) is the universal one, supporting our conjecture that also the first order calculus for the full triple is universal, up to some relations involving \( V_t \).

To begin with, we have

**Lemma 2** Let \( p, q, r, s \in \mathbb{Z} \). Then we have

\[
U_1^* U_2^p [D, U_1^q U_2^q] \eta_{kl\pm}^{(1,2)} = \frac{\sqrt{\lambda_{k+s,l+q} \gamma_{k+s,l+q}} - \sqrt{\lambda_{kl} \gamma_{kl}} \eta_{kl\pm}^{(1,2)}}{\gamma_{k+r+s,l+p+q}} \eta_{k+r+s,l+p+q\mp}.
\]

Moreover,

\[
[D, V_t] = 0.
\]

**Proof:** By direct computation using (3.25)–(3.30). \( \square \)

From Theorem 1 we know that the particular choice \( \Gamma = 1 \) gives rise to a spectral triple over \( C^\infty(\mathbb{T}^2) \). Let us now first investigate the corresponding differential calculus \( \Omega_D(\mathcal{O}(\mathbb{T}^2)) \). By faithfulness of the representation, we can again identify \( \Omega_D(\mathcal{O}(\mathbb{T}^2)) \) with a subspace of \( B(L^2(\mathcal{H})) \). We have

**Proposition 6** The first order differential calculus \( \Omega_D(\mathcal{O}(\mathbb{T}^2)) \) is freely generated by the elements \([D, U_1^s U_2^q]\) \((s, q \in \mathbb{Z})\).

**Proof:** We show that no nontrivial relations between \( U_1, U_2 \) and \( D \) can exist. Let us first consider relations between \( D \) and \( U_1 \) only. From Lemma 2 it follows for \( p = q = 0 \) that

\[
U_1^* [D, U_1^s] \eta_{kl\pm}^{(1,2)} = \frac{\sqrt{\lambda_{k+s,l} \gamma_{k+s,k} - \sqrt{\lambda_{kl} \gamma_{kl}}} \eta_{kl\pm}^{(1,2)}}{\gamma_{k+r+s,k+1}} \eta_{k+r+s,k+1\mp}.
\]

Using the Leibniz rule and the fact that different overall powers of \( U_1 \) are independent we find that nontrivial relations would be of the form

\[
\sum_{m=0}^{s-1} a_m U_1^m [D, U_1^{s-m}] = 0,
\]

for \( s \in \mathbb{N} \). Applying (3.32) to \( \eta_{n0\pm}^{(1,2)} \) \((n = k, \ldots, k + s - 1)\) we get the following system of equations

\[
\sum_{j=0}^{s-1} a_j \left( \sqrt{\lambda_{k+j+1,0} \gamma_{k+j+1,0}} - \sqrt{\lambda_{k,0} \gamma_{k,0}} \right) = 0
\]

\[
\vdots
\]

\[
\sum_{j=0}^{s-1} a_j \left( \sqrt{\lambda_{k+s-1,0} \gamma_{k+s-1,0}} - \sqrt{\lambda_{k+s-1,0} \gamma_{k+s-1,0}} \right) = 0.
\]

For the discussion of this system of equations it is useful to define a function \( h \) on \( \mathbb{Z} \) putting

\[
h(i) = \sqrt{\lambda_{0 \gamma_{0}} - \sqrt{\lambda_{1 \gamma_{-1}}}}.
\]
Lemma 3 We have
\[
\begin{vmatrix}
h(i_0) & \cdots & h(i_0 + k) \\
\vdots & \ddots & \vdots \\
h(i_k) & \cdots & h(i_k + k)
\end{vmatrix} \neq 0,
\]
for all \(k \in \mathbb{N}\) and \(i_0, \ldots, i_k \in \mathbb{Z}\).

Proof: See appendix A. \qed

Thus, there are no relations between \(U_1\) and \(D\) besides the ones coming from the Leibniz rule. In the general case we are looking for \(a_{mn} \in \mathbb{C}\) such that
\[
\sum_{m=0}^{s-1} \sum_{n=0}^{q-1} a_{mn} U_1^m U_2^n \left[ D, U_1^{s-m} U_2^{q-n} \right] = 0.
\]

Again, we are led to the consideration of a homogeneous linear system of equations for the \(a_{mn}\). The corresponding matrix of coefficients is an \((sq) \times (sq))\)-matrix with general matrix element
\[
C_{k,(m,n)} = \left( \sqrt{\lambda_{k+s-m,q-n}\gamma_{k+s-m,q-n} - \sqrt{\lambda_{k0}\gamma_{k0}}} \right),
\]
\((k = 1, \ldots, sq).\) In analogy to the case discussed above we have

Lemma 4 Let \(s, q \in \mathbb{N}\) be fixed. Then we have
\[
\det \left( C_{k,(m,n)} \right) \neq 0.
\]

Proof: The proof is a straightforward generalization of the proof of Lemma 3 to the case of functions, defined on \(\mathbb{Z}^2\), see [26]. \qed

The proof of the proposition follows now immediately from the fact that between the elements \([D, U_1^s U_2^q]\) there are no relations besides the ones coming from the Leibniz rule. \qed

We were not able to derive more relations of the type (3.31) between commutators of \(D\) with some generator and other generators (up to such relations resulting from applying \([D, \cdot]\) to (3.3)–(3.6) and the unitarity condition, using the derivation property). This seems to be due to the fact that \(\lambda_{kl}\) and \(\gamma_{kl}\) contain second and fourth powers of \(k\) and \(l\) under the square root. Therefore we

Conjecture 1 The bimodule \(\Omega^1_D(C^\infty(\mathbb{T}^2) \times \mathbb{R})\) is generated by \(du_1\) and \(du_2\) and is described by two relations,
\[
v_t du_1 = e^{iat} du_1 v_t, \quad v_t du_2 = e^{ibt} du_2 v_t.
\]

It seems that these difficulties in the end come from the quadratic part in the signature operator \(Q\).

Let us note that we could choose another diffeomorphism group, restricting the action of \(\mathbb{R}\) to the subgroup \(\mathbb{Z}\). Then, the generators \(V_t\) (or \(v_t\)) would be reduced to one generator \(V_1 = V(v_t = v)\), and all the above formulae remain, replacing always \(V_t\) (or \(v_t\)) by some power of \(V\) (or \(v\)). However, we would not get rid of the difficulties related to the differential calculus.
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A Proof of Lemma 3

The proof of this lemma rests on the following characterization of functions \( f \) defined by determinants of Hankelian type, see [23], such that

\[
\begin{vmatrix}
  f(i_0) & \cdots & f(i_0 + k) \\
  \vdots & \ddots & \vdots \\
  f(i_k) & \cdots & f(i_k + k)
\end{vmatrix} = 0 ,
\]

\( \forall k \in \mathbb{N} \) and \( i_0, \ldots, i_k \in \mathbb{Z} \). We have

**Theorem 2** A function \( f \) defined on \( \mathbb{Z} \) fulfils (A.1) if and only if it is of one of the following two types

\[
f_1(i) = \beta^i \sum_{j=0}^{k-1} \alpha_j i^j
\]

\[
f_2(i) = \sum_{j=1}^{k} \alpha_j \beta_j^i ,
\]

with \( \alpha, \beta \) and \( \beta_j \in \mathbb{C} \).

**Proof:** Let us first show by induction that \( f_1 \) and \( f_2 \) fulfil (A.1). For \( f_1(i) = \beta^i \sum_{j=0}^{k-1} \alpha_j i^j \) and \( k = 1 \) we have

\[
\begin{vmatrix}
  \alpha_0 \beta^0 & \alpha_0 \beta^{i+1} \\
  \alpha_0 \beta^i & \alpha_0 \beta^{i+1}
\end{vmatrix} = 0 ,
\]

\( \forall i, j \in \mathbb{Z} \). Let, now (A.1) be valid for \( k = n \). Then we have for \( k = n + 1 \)

\[
\begin{vmatrix}
  \beta^{i_0} \sum_{j=0}^{n} \alpha_j i_j^0 & \beta^{i_0+1} \sum_{j=0}^{n} \alpha_j (i_0 + 1)^j & \cdots & \beta^{i_0+n+1} \sum_{j=0}^{n} \alpha_j (i_0 + n + 1)^j \\
  \beta^{i_0+1} \sum_{j=0}^{n} \alpha_j i_j^1 & \beta^{i_0+1} \sum_{j=0}^{n} \alpha_j (i_1 + 1)^j & \cdots & \beta^{i_0+1+n+1} \sum_{j=0}^{n} \alpha_j (i_1 + n + 1)^j \\
  \vdots & \vdots & \ddots & \vdots \\
  \beta^{i_0+n+1} \sum_{j=0}^{n} \alpha_j i_j^{n+1} & \beta^{i_0+n+1} \sum_{j=0}^{n} \alpha_j (i_{n+1} + 1)^j & \cdots & \beta^{i_0+n+1+n+1} \sum_{j=0}^{n} \alpha_j (i_{n+1} + n + 1)^j \\
  \sum_{j=0}^{n} \alpha_j i_j^0 & \sum_{j=0}^{n} \alpha_j (i_0 + 1)^j & \cdots & \sum_{j=0}^{n} \alpha_j (i_0 + n + 1)^j \\
  \sum_{j=0}^{n} \alpha_j i_j^1 & \sum_{j=0}^{n} \alpha_j (i_1 + 1)^j & \cdots & \sum_{j=0}^{n} \alpha_j (i_1 + n + 1)^j \\
  \vdots & \vdots & \ddots & \vdots \\
  \sum_{j=0}^{n} \alpha_j i_j^{n+1} & \sum_{j=0}^{n} \alpha_j (i_{n+1} + 1)^j & \cdots & \sum_{j=0}^{n} \alpha_j (i_{n+1} + n + 1)^j
\end{vmatrix} = 0.
\]
But
\[
\begin{align*}
&\sum_{j=0}^{n} \alpha_j i_0^j \quad \sum_{j=0}^{n} \alpha_j (i_0 + 1)^j \quad \cdots \quad \sum_{j=0}^{n} \alpha_j (i_0 + n + 1)^j \\
&\sum_{j=0}^{n} \alpha_j i_1^j \quad \sum_{j=0}^{n} \alpha_j (i_1 + 1)^j \quad \cdots \quad \sum_{j=0}^{n} \alpha_j (i_1 + n + 1)^j \\
&\vdots \quad \vdots \quad \cdots \end{align*}
\]
\begin{align*}
&\sum_{j=0}^{n} \alpha_j i_{n+1}^j \quad \sum_{j=0}^{n} \alpha_j (i_{n+1} + 1)^j \quad \cdots \quad \sum_{j=0}^{n} \alpha_j (i_{n+1} + n + 1)^j \\
&\sum_{j=0}^{n} \alpha_j i_0^j \quad \sum_{j=1}^{n} \alpha_j \left[(i_0 + 1)^j - i_0^j\right] \quad \cdots \quad \sum_{j=1}^{n} \alpha_j \left[(i_0 + n + 1)^j - (i_0 + n)^j\right] \\
&\sum_{j=0}^{n} \alpha_j i_1^j \quad \sum_{j=1}^{n} \alpha_j \left[(i_1 + 1)^j - i_1^j\right] \quad \cdots \quad \sum_{j=1}^{n} \alpha_j \left[(i_1 + n + 1)^j - (i_1 + n)^j\right] \\
&\vdots \quad \vdots \quad \cdots \\
&\sum_{j=0}^{n} \alpha_j i_{n+1}^j \quad \sum_{j=1}^{n} \alpha_j \left[(i_{n+1} + 1)^j - i_{n+1}^j\right] \quad \cdots \quad \sum_{j=1}^{n} \alpha_j \left[(i_{n+1} + n + 1)^j - (i_{n+1} + n)^j\right]
\end{align*}
\begin{align*}
&\sum_{j=0}^{n} \alpha_j i_0^j \quad \sum_{j=1}^{n} \alpha_j \sum_{l=0}^{j-1} \left(\binom{j}{i} i_0^l\right) \quad \cdots \quad \sum_{j=1}^{n} \alpha_j \sum_{l=0}^{j-1} \left(\binom{j}{i} (i_0 + n)^l\right) \\
&\sum_{j=0}^{n} \alpha_j i_1^j \quad \sum_{j=1}^{n} \alpha_j \sum_{l=0}^{j-1} \left(\binom{j}{i} i_1^l\right) \quad \cdots \quad \sum_{j=1}^{n} \alpha_j \sum_{l=0}^{j-1} \left(\binom{j}{i} (i_1 + n)^l\right) \\
&\vdots \quad \vdots \quad \cdots \\
&\sum_{j=0}^{n} \alpha_j i_{n+1}^j \quad \sum_{j=1}^{n} \alpha_j \sum_{l=0}^{j-1} \left(\binom{j}{i} i_{n+1}^l\right) \quad \cdots \quad \sum_{j=1}^{n} \alpha_j \sum_{l=0}^{j-1} \left(\binom{j}{i} (i_{n+1} + n)^l\right)
\end{align*}
\begin{align*}
&\sum_{j=0}^{n} \alpha_j i_0^j \quad \sum_{j=1}^{n} \alpha_j \sum_{l=0}^{j-1} \left(\binom{j}{i} i_0^l\right) \quad \cdots \quad \sum_{j=1}^{n} \alpha_j \sum_{l=0}^{j-1} \left(\binom{j}{i} (i_0 + n)^l\right) \\
&\sum_{j=0}^{n} \alpha_j i_1^j \quad \sum_{j=1}^{n} \alpha_j \sum_{l=0}^{j-1} \left(\binom{j}{i} i_1^l\right) \quad \cdots \quad \sum_{j=1}^{n} \alpha_j \sum_{l=0}^{j-1} \left(\binom{j}{i} (i_1 + n)^l\right) \\
&\vdots \quad \vdots \quad \cdots \\
&\sum_{j=0}^{n} \alpha_j i_{n+1}^j \quad \sum_{j=1}^{n} \alpha_j \sum_{l=0}^{j-1} \left(\binom{j}{i} i_{n+1}^l\right) \quad \cdots \quad \sum_{j=1}^{n} \alpha_j \sum_{l=0}^{j-1} \left(\binom{j}{i} (i_{n+1} + n)^l\right)
\end{align*}
\begin{align*}
&(-1)^{n+1} \sum_{j=0}^{n} \alpha_j i_{n+1}^j
\end{align*}
\begin{align*}
&\sum_{j=1}^{k} \alpha_j \beta_j^i \quad \sum_{j=1}^{k} \alpha_j \beta_j^{i+1} \\
&\sum_{j=1}^{k} \alpha_j \beta_1^i \quad \sum_{j=1}^{k} \alpha_j \beta_1^{i+1} = 0,
\end{align*}

by assumption. Analogously, for \( f_2(i) = \sum_{j=1}^{k} \alpha_j \beta_j^i \), we have for \( k = 1 \)

\[
\begin{vmatrix}
\alpha_1 \beta_1^i & \alpha_1 \beta_1^{i+1} \\
\alpha_1 \beta_1^i & \alpha_1 \beta_1^{i+1} \\
\end{vmatrix} = 0,
\]
\( \forall i, j \in \mathbb{Z} \). Let us now assume the validity of (A.1) for \( k = n \). Then we have for \( k = n + 1 \)

\[
\begin{vmatrix}
\sum_{j=1}^{n+1} \alpha_j \beta_j^{i_0} & \sum_{j=1}^{n+1} \alpha_j \beta_j^{i_0+1} & \cdots & \sum_{j=1}^{n+1} \alpha_j \beta_j^{i_0+n+1} \\
\sum_{j=1}^{n+1} \alpha_j \beta_j^{i_1} & \sum_{j=1}^{n+1} \alpha_j \beta_j^{i_1+1} & \cdots & \sum_{j=1}^{n+1} \alpha_j \beta_j^{i_1+n+1} \\
\vdots & \vdots & \ddots & \vdots \\
\sum_{j=1}^{n+1} \alpha_j \beta_j^{i_{n+1}} & \sum_{j=1}^{n+1} \alpha_j \beta_j^{i_{n+1}+1} & \cdots & \sum_{j=1}^{n+1} \alpha_j \beta_j^{i_{n+1}+n+1}
\end{vmatrix}
\]

\[
= \begin{vmatrix}
\sum_{j=1}^{n+1} \alpha_j \beta_j^{i_0} + n (\beta_j - \beta_{n+1}) & \sum_{j=1}^{n+1} \alpha_j \beta_j^{i_0+n} (\beta_j - \beta_{n+1}) \\
\sum_{j=1}^{n+1} \alpha_j \beta_j^{i_1} (\beta_j - \beta_{n+1}) & \sum_{j=1}^{n+1} \alpha_j \beta_j^{i_1+n} (\beta_j - \beta_{n+1}) \\
\vdots & \vdots \\
\sum_{j=1}^{n+1} \alpha_j \beta_j^{i_{n+1}} (\beta_j - \beta_{n+1}) & \sum_{j=1}^{n+1} \alpha_j \beta_j^{i_{n+1}+n} (\beta_j - \beta_{n+1})
\end{vmatrix}
\]

\[
= + \cdots + \begin{vmatrix}
\sum_{j=1}^{n+1} \alpha_j \beta_j^{i_1} (\beta_j - \beta_{n+1}) & \sum_{j=1}^{n+1} \alpha_j \beta_j^{i_1+n} (\beta_j - \beta_{n+1}) \\
\vdots & \vdots \\
\sum_{j=1}^{n+1} \alpha_j \beta_j^{i_{n+1}} (\beta_j - \beta_{n+1}) & \sum_{j=1}^{n+1} \alpha_j \beta_j^{i_{n+1}+n} (\beta_j - \beta_{n+1})
\end{vmatrix}
\]

\[
= 0,
\]

by assumption.

Let us now assume that a function \( f \) defined on \( \mathbb{Z} \) fulfils (A.1) for some \( k \in \mathbb{N} \). We choose \( i_1 = i_0 + 1, \ldots, i_k = i_0 + k \) and let \( f(i_0), \ldots, f(i_0 + 2k - 1) \) denote the corresponding values of \( f \). Then \( f(i_0 + 2k) \) has to fulfil

\[
\begin{vmatrix}
\sum_{j=1}^{n+1} \alpha_j \beta_j^{i_0} & \sum_{j=1}^{n+1} \alpha_j \beta_j^{i_0+1} & \cdots & \sum_{j=1}^{n+1} \alpha_j \beta_j^{i_0+n+1} \\
\sum_{j=1}^{n+1} \alpha_j \beta_j^{i_1} & \sum_{j=1}^{n+1} \alpha_j \beta_j^{i_1+1} & \cdots & \sum_{j=1}^{n+1} \alpha_j \beta_j^{i_1+n+1} \\
\vdots & \vdots & \ddots & \vdots \\
\sum_{j=1}^{n+1} \alpha_j \beta_j^{i_{n+1}} & \sum_{j=1}^{n+1} \alpha_j \beta_j^{i_{n+1}+1} & \cdots & \sum_{j=1}^{n+1} \alpha_j \beta_j^{i_{n+1}+n+1}
\end{vmatrix}
\]

provided that

\[
\begin{vmatrix}
\sum_{j=1}^{n+1} \alpha_j \beta_j^{i_0} (\beta_j - \beta_{n+1}) & \sum_{j=1}^{n+1} \alpha_j \beta_j^{i_0+n} (\beta_j - \beta_{n+1}) \\
\sum_{j=1}^{n+1} \alpha_j \beta_j^{i_1} (\beta_j - \beta_{n+1}) & \sum_{j=1}^{n+1} \alpha_j \beta_j^{i_1+n} (\beta_j - \beta_{n+1}) \\
\vdots & \vdots \\
\sum_{j=1}^{n+1} \alpha_j \beta_j^{i_{n+1}} (\beta_j - \beta_{n+1}) & \sum_{j=1}^{n+1} \alpha_j \beta_j^{i_{n+1}+n} (\beta_j - \beta_{n+1})
\end{vmatrix}
\]

\(= 0\),

(We may assume without loss of generality that (A.4) holds. In [26] it is shown that in the other case one is led to the case \( k - 1 \).) Proceeding further we find that the
2k values \(f(i_0), \ldots, f(i_0 + 2k - 1)\) determine \(f\) completely. Now we show that this function is either of type (A.3) or (A.2).

Let first the constants \(f(i_0), \ldots, f(i_0+2k-1)\) be such that the following condition holds

\[
\sum_{j=0}^{l+1} \beta^{l+1-j} f(i+j) (-1)^j \binom{l+1}{j} = 0 \quad ,
\]

(A.5)

for some \(\beta \in \mathbb{C}, l \in \{0, \ldots, k-1\}\) and all \(i = i_0, \ldots, i_0 + 2k - l - 1\). We show that the corresponding function on \(\mathbb{Z}\) is of the form (A.2). Suppose that \(\beta \in \mathbb{C}\) is a solution of (A.3). Then we find constants \(\alpha_i\) as follows. Defining \(g(i) := \frac{f(i)}{i!} (\beta \neq 0)\), we can always find \(\alpha_i\) \((i = 0, \ldots, l)\) as solutions of the following linear system of equations

\[
\begin{align*}
g(i_0) &= \alpha_0 + \alpha_1 i_0 + \cdots + \alpha_l i_0^l \\
g(i_0 + 1) &= \alpha_0 + \alpha_1 (i_0 + 1) + \cdots + \alpha_l (i_0 + 1)^l \\
& \quad \vdots \\
g(i_0 + l) &= \alpha_0 + \alpha_1 (i_0 + l) + \cdots + \alpha_l (i_0 + l)^l
\end{align*}
\]

by

\[
\alpha_i = \frac{1}{\Delta} \begin{vmatrix}
1 & i_0 & \cdots & i_0^{l-1} \\
1 & i_0 + 1 & \cdots & (i_0 + 1)^{l-1} \\
\vdots & \vdots & \ddots & \vdots \\
1 & i_0 + l & \cdots & (i_0 + l)^{l-1}
\end{vmatrix} g(i_0) \begin{vmatrix}
\cdots & \cdots & \cdots \\
\cdots & (i_0 + 1)^l & \cdots \\
\cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots
\end{vmatrix} = (-1)^{l(l+1)/2} \prod_{j=1}^{l} j! \neq 0 .
\]

Now that we have chosen the constants \(\beta\) and \(\alpha_0, \ldots, \alpha_l\) such that

\[f(i_0 + j) = f_1(i_0 + j)\]

is fulfilled, for all \(j = 0, \ldots, l\), it remains to be shown that we also have

\[f(i_0 + l + 1) = f_1(i_0 + l + 1) .\]

But

\[
\sum_{j=0}^{r} (-1)^j \binom{r}{j} j^s = 0 \quad ,
\]

\(\forall s = 0, \ldots, r - 1\) (which follows from evaluating the \(s\)-th derivative of \(f(x) = (x - 1)^r = \sum_{j=0}^{r} \binom{r}{j} (-1)^j x^{r-j}\) at \(x = 1\)). Now we find

\[
\begin{align*}
\sum_{j=0}^{l+1} \beta^{l+1-j} (-1)^j \binom{l+1}{j} f_1(i+j) &= \sum_{j=0}^{l+1} \beta^{l+1-j} \beta^{i+j} (-1)^j \binom{l+1}{j} \sum_{m=0}^{l} \alpha_m (i+j)^m \\
&= \beta^{l+i+1} \sum_{j=0}^{l+1} \sum_{m=0}^{l} \sum_{n=0}^{m} \alpha_m (-1)^j \binom{l+1}{j} \binom{m}{n} i^n j^{m-n} \\
&= \beta^{l+i+1} \sum_{m=0}^{l} \alpha_m \sum_{n=0}^{m} \binom{m}{n} i^n \sum_{j=0}^{l+1} (-1)^j \binom{l+1}{j} j^{m-n} = 0 .
\end{align*}
\]
Therefore, we have
\[ \sum_{j=0}^{l+1} (-1)^j \beta^{l+1-j} \binom{l+1}{j} (f(i_0 + j) - f_1(i_0 + j)) = (-1)^{l+1} (f(i_0 + l + 1) - f_1(i_0 + l + 1)) = 0 \, , \]
i.e.
\[ f(i_0 + l + 1) = f_1(i_0 + l + 1) \, . \]
Let us now consider the general case (A.3). Suppose, that \( f(i_0), \ldots, f(i_0 + 2k - 1) \) are chosen such that (A.3) does not hold. Then we have to solve the following system of algebraic equations (where we have chosen \( i_0 = 0 \))
\[
\begin{align*}
  f(0) &= C_1 + \cdots + C_k \\
  f(1) &= C_1 \beta_1 + \cdots + C_k \beta_k \\
  \vdots \\
  f(2k - 1) &= C_1 \beta_1^{2k-1} + \cdots + C_k \beta_k^{2k-1} 
\end{align*}
\]
which can always be done using Gröbner basis techniques, see [26]. \( \square \)

**Remark 5** If the parameters \( f(i_0), \ldots, f(i_0 + 2k - 1) \) satisfy
\[ f(i_0 + l) = \frac{f(i_0 + 1)^l}{f(i_0)^{l-1}} \, , \]
\( \forall l = 0, \ldots, 2k - 1 \), then one easily checks that
\[ \beta = \frac{f(i_0 + 1)}{f(i_0)} \]
fulfills (A.3) and the constants \( \alpha_i \) are given by
\[ \alpha_0 = \frac{f(i_0)^{i_0+1}}{f(i_0 + 1)^{i_0}} \, , \quad \alpha_1 = \cdots = \alpha_{k-1} = 0 \, . \]
The proof of Lemma 3 follows now immediately from the observation that the function
\[ h(i) = \sqrt{\lambda_{i_0} \gamma_{i_0}} - \sqrt{\lambda_{i-1} \gamma_{i-1}} \, , \]
is obviously not of the form (A.2) or (A.3). \( \square \)

**B** The differential algebra for the irrational rotation algebra

Let us first recall, see [18, 1], that the algebra of the noncommutative torus is generated by two unitaries \( u, v \) subject to the relation
\[ uv = e^{-2\pi i \theta} vu \, . \]
The algebra can be considered on the purely *-algebraic level (Laurent polynomials in \( u, v \)) where a general element is a finite linear combination of ordered polynomials \( u^k v^l \), \( k, l \in \mathbb{Z} \), on the level of smooth functions, where the general element is a series

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\[ \sum a_{kl} u^k v^l \] with coefficients \( a_{kl} \) subject to the condition that \((|k|^n + |l|^n)|a_{kl}|\) are bounded for all \( n > 0 \). Finally, there is also the \( C^* \)-version, defined e.g. by using irreducible representations for performing a norm closure of the polynomial algebra.

It is well-known that both the polynomial and the \( C^* \)-algebra can be interpreted as convolution algebras of the reduced holonomy groupoid of the Kronecker foliation, with \( \theta \) being the angle defining the direction of the leaves. We denote the \( C^* \)-algebra by \( A_\theta \), the smooth algebra by \( A_\theta \) and the polynomial algebra by \( \mathcal{O}_\theta \). There exists a tracial state \( \tau \) on \( A_\theta \), given by

\[ \tau(\sum a_{kl} u^k v^l) = a_{00}, \]

and there are two canonical derivations \( \delta_1 \) and \( \delta_2 \) on \( A_\theta \) defined by

\[ \delta_1(u^k v^l) = 2\pi iku^k v^l, \quad \delta_2(u^k v^l) = 2\pi ilu^k v^l. \]

With these data, the well-known spectral triple is defined as follows: First, the tracial state \( \tau \) is used to define the GNS Hilbert space \( \mathcal{H}_\tau \). Secondly, the derivations \( \delta_1 \) and \( \delta_2 \) give rise to unbounded operators on \( \mathcal{H}_\tau \), whose domain of definition is the image of \( A_\theta \) in \( \mathcal{H}_\tau \) (under the GNS procedure). The same is true for \( \partial := \frac{1}{2\pi i}(\delta_1 - i\delta_2) \). Now take \( \mathcal{H} := \mathcal{H}_\tau \oplus \mathcal{H}_\tau \) and \( D := \begin{pmatrix} 0 & \partial \\ \partial^* & 0 \end{pmatrix} \) as Hilbert space and generalized Dirac operator of a spectral triple over \( A_\theta \). The dimension of this spectral triple is known to be two. The corresponding differential calculus \( \Omega_D \) was described by Connes in terms of elements of \( \mathcal{H} \). We have the following description of \( \Omega_D(O_\theta) \) in terms of relations between the generators of the algebra and their differentials:

**Proposition 7** (i) \( \Omega_D^1(O_\theta) \) is a free left (or right) \( O_\theta \)-module with basis \( \{du, dv\} \). The bimodule structure of \( \Omega_D^1(O_\theta) \) is given by

\begin{align*}
udu &= duu, \quad u^* du = du^* u, \quad u^* du^* = du^* u^*, \quad (B.1) \\
vdu &= dvv, \quad v^* dv = dv^* v, \quad v^* dv^* = dv^* v^*, \quad (B.2) \\
vdu &= e^{2\pi i\theta} du, \quad udv = e^{-2\pi i\theta} du, \quad (B.3) \\
vdu^* &= e^{-2\pi i\theta} du^* v, \quad u^* dv = e^{2\pi i\theta} du^* v^*, \quad (B.4) \\
v^* du &= e^{-2\pi i\theta} dv^* u, \quad udv^* = e^{2\pi i\theta} dv^* u. \quad (B.5)
\end{align*}

(ii) \( \Omega_D^2(A_\theta) \) is a free left (or right) \( A_\theta \)-module with basis \( \{dvd\} \). The relation

\[ dudv = -e^{2\pi i\theta} dvdu \quad (B.6) \]

is fulfilled.

(iii) \( \Omega_D^k(A_\theta) = 0 \) for \( k \geq 3 \).

**Proof:** (i) \( \tau \) is a faithful state, thus the GNS representation \( \pi \) is faithful. Consequently, \( \Omega_D^1(O_\theta) \cong \pi(\Omega^1(O_\theta)) \), where the isomorphism sends differentials to commutators with \( D \). To verify the relations \([B.1]-[B.3]\) it is therefore sufficient to consider the images of these expressions under \( \pi \). If we denote by \( g \) the element corresponding to \( a \in O_\theta \) in \( \mathcal{H}_\tau \), it is immediately verified that the \( e_{kl} := u^k v^l \) form an orthonormal basis in \( \mathcal{H}_\tau \). From this basis we obtain in an obvious way an orthonormal basis \( \{e_{kl}^+, e_{kl}^-\} \) of \( \mathcal{H}_\tau \oplus \mathcal{H}_\tau \) (to be precise, \( e_{kl}^+ = (e_{kl}, 0) \), \( e_{kl}^- = (0, e_{kl}) \)). In this basis, \( U := \pi \oplus \pi(u) \), \( V := \pi \oplus \pi(v) \), \( D \) act as follows:

\[ U(e_{kl}^+) = e_{k+1,l}^+, \quad V(e_{kl}^-) = e_{2\pi i\theta} e_{k,l+1}^+ \quad (B.7) \]
\[ D(e_{kl}^\pm) = \sqrt{2\pi}(\pm ik + l)e_{kl}^\pm. \] (B.8)

Now, it is straightforward to verify the relations \([B.1]-[B.5]\) (with \(U, V, [D, \cdot]\) instead of \(u, v, d\)). From these and the Leibniz rule (also taking into account unitarity of the generators \(u, v\)) it is obvious that \([D, U]\) and \([D, V]\) generate \(\pi(\Omega^1(\mathcal{O}_\theta))\) as a left (or right) \(\mathcal{O}_\theta\)-module. To prove that it is a freely generated left module, assume \(P[D, U] + Q[D, V] = 0\) with \(P, Q \in \pi \oplus \pi(\mathcal{O}_\theta)\). It follows from \([B.7]\) and \([B.8]\) that

\[
[D, U](e_{kl}^\pm) = \pm i\sqrt{2\pi}e_{k+1,l}^\pm \quad (B.9)
\]

\[
[D, V](e_{kl}^\pm) = e^{2\pi i k \theta} \sqrt{2\pi} e_{k,l+1}^\pm. \quad (B.10)
\]

Therefore, terms in \(P[D, U] + Q[D, V]\) can only compensate if they contain the same overall number of \(U\) and of \(V\). This means that it is sufficient to consider terms of the form \(\alpha = pU^nV^{m+1}[D, U] + qU^nV^m[D, V], p, q \in \mathbb{C}\). Acting on \(e_{kl}^+\), we obtain

\[
\alpha e_{kl}^+ = \sqrt{2\pi}(e^{2\pi i (k+1)(m+1)\theta} + e^{2\pi i km\theta})e_{k+n+1,l+m+1}^- = 0,
\]

which is equivalent to

\[
p e^{2\pi i (k+m+1)} + q = 0.
\]

Since this should be true for all \(k\), it follows that \(p = q = 0\).

(ii) Differentiating the relations \([B.1]\) and \([B.2]\) gives immediately \(dudu = dxdv = du^*du = dv^*dv = 0\). Analogously, \([B.3]\) leads to \([B.6]\). Thus, we know already that \(dudv\) generates the two forms as a left (or right) \(A_\theta\)-module. It remains to show that it is freely generated.

Let us recall that

\[ \Omega^2_D(\mathcal{O}_\theta) \cong \pi(\Omega^2(\mathcal{O}_\theta))/\pi(\mathcal{O}_\theta), \]

where \(J^1 = \ker \pi \cap \Omega^1(\mathcal{O}_\theta)\). Thus, any relation true in \(\pi(\Omega^2(\mathcal{O}_\theta))\) is also true in \(\pi(\Omega^2_D(\mathcal{O}_\theta))\). Now, a similar argument as in the proof of Proposition \[3\] shows that (i) implies that \(J^1\) coincides with the \(\mathcal{O}_\theta\)-submodule generated the elements corresponding to the relations \([B.1]-[B.5]\). It follows that \(dJ^1\) is a finite sum of elements of the form \(a db c\) with \(a, c \in \mathcal{O}_\theta\) and \(b\) one of the elements \([B.1]-[B.5]\). Now, one shows by a direct computation that \(\pi(db) \in \pi(\mathcal{O}_\theta)\) if \(b\) is one of the elements \([B.1]-[B.2]\) whereas \(\pi(db) = 0\) if \(b\) is one of the remaining elements. It follows that \(\pi(dJ^1) = \pi(\mathcal{O}_\theta)\). It remains to show that from \(\pi(d)[D, U][D, V] \in \pi(\mathcal{O}_\theta)\) it follows that \(d = 0\). From the above formulae it is now immediate that any element of the algebra acts on the \(e_{kl}^\pm\) in a way not depending on + or −, \(\pi(a)e_{kl}^\pm = \sum \lambda_{ij}e_{ij}^\pm\), \(\lambda_{ij}\) independent on + or −. On the other hand,

\[
[D, U][D, V]e_{kl}^\pm = \pm 2\pi i e^{2\pi i \theta} e_{k+1,l+1}^\pm,
\]

from which (ii) follows immediately. \(\square\)

Remark 6 As in Remark \[3\], we can construct a topological version \(\Omega_D(A_\theta)\) of this calculus (using seminorms \(q_n(\sum a_{kl}u^kv^l) = \sup_{k,l}(1 + |k|^n + |l|^n)|a_{kl}|\)). A comparison with the results of \([7]\) shows that this gives indeed the calculus \(\Omega_D(A_\theta)\) of the spectral triple \((A_\theta, \mathcal{H}, D)\).

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