Global well-posedness and scattering for the radial, defocusing, cubic wave equation with almost sharp initial data

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ABSTRACT
In this article, we prove that the cubic wave equation is globally well-posed and scattering for radial initial data lying in a weighted, slightly supercritical space. This space of functions is slightly smaller than the general critical space.

1. Introduction
The defocusing, cubic nonlinear wave equation
\[ \partial_t^2 u - \Delta u = -u^3, \quad u(0, \cdot) = u_0, \quad u_t(0, \cdot) = u_1, \] (1.1)
is invariant under the scaling
\[ u(t, x) \mapsto \lambda u(\lambda t, \lambda x). \] (1.2)
That is, for any \( \lambda > 0 \), (1.2) solves (1.1) with initial data \((\lambda u_0(\lambda x), \lambda^2 u_1(\lambda x))\). This scaling preserves the \( H^{1/2} \times \dot{H}^{-1/2} \) norm of the initial data:
\[ ||\lambda u_0(\lambda x)||_{\dot{H}^{1/2}(\mathbb{R}^3)} = ||u_0||_{\dot{H}^{1/2}(\mathbb{R}^3)}, \quad \text{and} \quad ||\lambda^2 u_1(\lambda x)||_{\dot{H}^{-1/2}(\mathbb{R}^3)} = ||u_1||_{\dot{H}^{-1/2}(\mathbb{R}^3)}. \] (1.3)
Because of this fact, \( H^{1/2} \times \dot{H}^{-1/2} \) is called the critical Sobolev space for (1.1). A complete dichotomy has been proved for the local theory of (1.1). On the negative side, [1] proved that (1.1) is locally ill-posed for \( u_0 \in \dot{H}^s, \ u_1 \in \dot{H}^{s-1} \) for any \( s < \frac{1}{2} \). The proof exploited the scaling in (1.2) in order to prove the existence of a solution that was not continuous in time at \( t = 0 \). On the positive side, [2] proved that (1.1) is locally well-posed for initial data lying in the Sobolev space \( H^{1/2} (\mathbb{R}^3) \times \dot{H}^{-1/2} (\mathbb{R}^3) \). Moreover, for data lying in \( H^s \times \dot{H}^{s-1} \), \( \frac{1}{2} < s < \frac{3}{2} \), [2] proved that (1.1) is locally well-posed, with the time of existence depending only on the size of the initial data.

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Remark: In this article, well-posedness refers to the standard definition that a unique solution that is continuous in time exists, and that the solution depends continuously on the initial data. A natural question in light of the result of [2] is when can local well-posedness results be extended to global well-posedness. It is a well-known fact from ordinary differential equations theory that this does not always hold. For example, consider the equation

\[ u_{tt} = u^3. \] (1.4)

The solution \( u(t) = \sqrt{2}t^{-1} \) to (1.4) certainly blows up in finite time. The article [3] proved that solutions to

\[ u_{tt} - \Delta u = |u|^{p-1}u, \] (1.5)

for \( 1 < p < 5, d = 3 \), also can exhibit such ordinary differential equations type growth. On the other hand, for the ordinary differential equation

\[ u_{tt} = -u^3, \] (1.6)

\( u(t) \) is concave down in time when \( u \) is positive, and is concave up in time when \( u \) is negative, so the solution to (1.6) should not blow up, but rather be global. It is for this reason, as well as due to the fact that

\[ \Delta u - u^3 = 0, \] (1.7)

has no nonzero solutions, that it is conjectured that

**Conjecture 1.1.** (1.1) is globally well-posed and scattering in \( \dot{H}^{1/2}(\mathbb{R}^3) \times \dot{H}^{-1/2}(\mathbb{R}^3) \).

Scattering also has the usual definition.

**Definition 1.1** (Scattering). A solution to (1.1) is said to scatter forward in time if the solution exists for all \( t > 0 \) and there exists \( (u_0^+, u_1^+) \in \dot{H}^{1/2}(\mathbb{R}^3) \times \dot{H}^{-1/2}(\mathbb{R}^3) \) such that

\[
\lim_{t \to -\infty} ||u(t) - S(t)(u_0^+, u_1^+)||_{\dot{H}^{1/2}(\mathbb{R}^3)} = 0,
\]

\[
\lim_{t \to -\infty} ||\partial_t u(t) - \partial_t S(t)(u_0^+, u_1^+)||_{\dot{H}^{-1/2}(\mathbb{R}^3)} = 0,
\] (1.8)

where \( S(t) \) is the solution operator to the linear wave equation \( u_{tt} - \Delta u = 0 \). Similarly, \( u \) is said to scatter backward in time if there exists \( (u_0^-, u_1^-) \in \dot{H}^{1/2} \times \dot{H}^{-1/2} \) such that (1.8) holds for \( t \to -\infty \).

The counterpart to Conjecture 1.1 holds for the quintic problem.

**Theorem 1.1.** The defocusing, quintic wave equation problem

\[ \partial_t^2 u - \Delta u = -u^5, u(0, \cdot) = u_0, u_t(0, \cdot) = u_1, \] (1.9)

is globally well-posed and scatters both forward and backward in time for any \( (u_0, u_1) \in \dot{H}^1 \times L^2 \).

**Proof.** See [4].

**Remark:** In this case (1.9) is invariant under the scaling \( \lambda^{1/2}u(\lambda t, \lambda x) \), which preserves the \( \dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \).
Equation (1.9) is called an energy-critical problem because a solution to (1.9) conserves the energy
\[ E(u(t)) = \frac{1}{2} \int (u_t(x) + 1) (u_t(x)) dx + \frac{1}{6} \int (u(x))^6 dx = E(u(0)) \] (1.10)
for the entire time of its existence. This quantity is unchanged by the scaling \( \lambda^{1/2} u(\lambda t, \lambda x) \), controls the \( \dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \), and by the Sobolev embedding theorem
\[ E(u(0)) \leq \|u_0\|_{\dot{H}^1(\mathbb{R}^3)}^2 + \|u_0\|_{\dot{H}^1(\mathbb{R}^3)}^6 + \|u_1\|_{L^2(\mathbb{R}^3)}^2. \] (1.11)

However, for the cubic initial value problem there does not exist a conserved quantity which controls the \( \dot{H}^{1/2} \times \dot{H}^{-1/2} \) norm of \((u(t),u_t(t))\). In fact, for radial data, the existence of such a quantity would prove Conjecture 1.1.

**Theorem 1.2.** Suppose \( u_0 \in \dot{H}^{1/2}(\mathbb{R}^3), u_1 \in \dot{H}^{-1/2}(\mathbb{R}^3) \) are radial functions, and \( u \) solves (1.1) on a maximal interval \( 0 \in I \subset \mathbb{R} \) with
\[ \sup_{t \in I} \|u(t)\|_{\dot{H}^{1/2}(\mathbb{R}^3)} + ||u_1(t)||_{\dot{H}^{-1/2}(\mathbb{R}^3)} < \infty. \] (1.12)

Then \( I = \mathbb{R} \) and the solution \( u \) scatters both forward and backward in time.

**Proof.** See [5]. \( \square \)

In this article, we prove that (1.1) is globally well-posed and scattering for \( u_0, u_1 \) contained in a subset of \( \dot{H}^{1/2}(\mathbb{R}^3) \times \dot{H}^{-1/2}(\mathbb{R}^3) \).

**Theorem 1.3.** Suppose there exists a positive constant \( \epsilon > 0 \) such that
\[ \|u_0\|_{\dot{H}^{1/2+\epsilon}(\mathbb{R}^3)} + \|x\|^{2\epsilon} u_0\|_{\dot{H}^{1/2+\epsilon}(\mathbb{R}^3)} \leq A < \infty, \] (1.13)
and
\[ \|u_1\|_{\dot{H}^{-1/2+\epsilon}(\mathbb{R}^3)} + \|x\|^{2\epsilon} u_1\|_{\dot{H}^{-1/2+\epsilon}(\mathbb{R}^3)} \leq A < \infty. \] (1.14)

Then (1.1) has a global solution and there exists some \( C(A, \epsilon) < \infty \) such that
\[ \int_{\mathbb{R}} (u(t,x))^4 dxdt \leq C(A, \epsilon), \] (1.15)
which proves that \( u \) scatters both forward and backward in time.

**Remark:** Theorem 1.3 with \( \epsilon = 0 \) would imply Conjecture 1.1.

The proof of Theorem 1.3 is based on two previous results.

**Theorem 1.4** (Global well-posedness). For any \( \epsilon > 0 \), if \( u_0 \) and \( u_1 \) are radial functions and \( u_0 \in \dot{H}^{1/2}(\mathbb{R}^3) \cap \dot{H}^{1/2+\epsilon}(\mathbb{R}^3), u_1 \in \dot{H}^{-1/2}(\mathbb{R}^3) \cap \dot{H}^{-1/2+\epsilon}(\mathbb{R}^3) \), then (1.1) is globally well-posed.

**Proof.** See [6]. \( \square \)

**Remark:** The initial data in Theorem 1.3 satisfies \( u_0 \in \dot{H}^{1/2+\epsilon}(\mathbb{R}^3) \cap \dot{H}^{1/2-\epsilon}(\mathbb{R}^3) \) and \( u_1 \in \dot{H}^{-1/2+\epsilon}(\mathbb{R}^3) \cap \dot{H}^{-1/2-\epsilon}(\mathbb{R}^3) \), and thus (1.1) has a global solution under such initial data.
The proof of Theorem 1.4 used the I-method, which is an improvement over the Fourier truncation method. For example, using the I-method, [7] improved the results of [8] for the nonlinear Schrödinger equation. On the wave equation side, [9] extended the results of [10] for the cubic wave equation to $s > \frac{13}{18}$ and to $s > \frac{7}{10}$ if $u$ has radial symmetry. Perhaps more importantly, [11] proved a well-posedness result which was technically unattainable via the Fourier truncation method. See [6] for a more detailed discussion of the history of the I-method.

The second result utilized the conformal transformation and energy methods.

**Theorem 1.5.** Assume that $A$, $\epsilon$ are positive constants. Let $(u_0, u_1) \in \dot{H}^1 \times L^2$ be radial initial data so that

$$\int |\nabla u_0(x)|^2 (1 + |x|)^{1+2\epsilon} dx + \int |u_1(x)|^2 (1 + |x|)^{1+2\epsilon} dx \leq A^2,$$  

(1.16)

then the solution to (1.1) scatters in both time directions with

$$||u||_{L_{t,x}^4(\mathbb{R} \times \mathbb{R}^3)} \leq C(A, \epsilon) < \infty.$$  

(1.17)

**Proof.** See [12].

The proof in [12] follows four steps.

1. Since $u_0 \in \dot{H}^1$ and $u_1 \in L^2$, then by the conservation of energy (1.1) has a global solution under the initial data in Theorem 1.5.

2. Define the conformal transformation of $u$,

$$v(\tau, y) = \frac{\sinh|y|}{|y|} e^\tau u \left( e^\tau \frac{\sinh|y|}{|y|} \cdot y, t_0 + e^\tau \cosh|y| \right), \quad (\tau, y) \in \mathbb{R} \times \mathbb{R}^3,$$  

(1.18)

$t_0 < 0$ is a fixed constant. Then [12] shows that $v(0, y)$ has finite energy.

3. Prove that if $v$ solves the conformal wave equation,

$$||v||_{L_{t,x}^4(\mathbb{R} \times \mathbb{R}^3)} \approx E(v(0)).$$  

(1.19)

4. Show that (1.19) implies that $||u||_{L_{t,x}^4(\mathbb{R} \times \mathbb{R}^3)} < \infty$, which proves scattering.

The conformal energy is invariant under the scaling (1.2). Because the conformal energy is conserved, to prove Theorem 1.3 it suffices to prove that “most” (in a sense that will be more fully defined later) of a solution $u$ to (1.1) with initial data lying in (1.13) and (1.14) has uniformly bounded conformal energy. This implies scattering.

By Theorem 1.4, step one also holds for the initial data in Theorem 1.3. In place of step two, we will prove that $v(0, y) \in \dot{H}^{-1/2+\epsilon}(\mathbb{R}^3)$ and $v(0, y) \in \dot{H}^{-1/2+\epsilon}(\mathbb{R}^3)$. Next, as in [6], we will utilize the I-method to prove that $||v||_{L_{t,x}^4(\mathbb{R} \times \mathbb{R}^3)} < \infty$. Step four may be copied directly from [12], proving Theorem 1.3.

2. Linear estimates and harmonic analysis

In this section we will collect several estimates concerning the linear wave equation and harmonic analysis. These estimates will be utilized throughout the article. None of the results in this section are new, in fact all are very well-known.
**Definition 2.1** (Littlewood-Paley partition of unity). Suppose \( \psi \in C_0^\infty(\mathbb{R}^3) \) is a radial, decreasing function supported on \( |x| \leq 2, \psi = 1 \) on \( |x| \leq 1 \). Then for any \( j \in \mathbb{Z} \) we define the Littlewood-Paley projection

\[
(P_j f)(x) = \mathcal{F}^{-1} \left( \left( \psi \left( \frac{2^j \xi}{2} \right) - \psi \left( \frac{2^{j+1} \xi}{2} \right) \right) \hat{f}(\xi) \right)(x),
\]

where \( \mathcal{F} \) is the Fourier transform

\[
\mathcal{F}f(\xi) = (2\pi)^{3/2} \int e^{-i \xi \cdot x} f(x) dx,
\]

and

\[
\mathcal{F}^{-1} \left( \hat{f}(\xi) \right)(x) = (2\pi)^{-3/2} \int e^{i \xi \cdot x} \hat{f}(\xi) d\xi.
\]

Also define the operators

\[
(P_{\leq j} f)(x) = \sum_{l \leq j} P_l f,
\]

and \( P_{> j} = 1 - P_{\leq j} \).

**Remark:** Since \( \psi \) is a \( C_0^\infty(\mathbb{R}^3) \) function, \( P_j f \) is the convolution of \( f \) with a Schwartz function \( \mathcal{F}^{-1} P_j(x) \) that satisfies

\[
\mathcal{F}^{-1} P_j(x) = \hat{\psi}_j(x) \approx i2^j (1 + 2^l |x|)^{-l}, \text{ for any } l \in \mathbb{Z}.
\]

By direct computation this gives the Sobolev embedding theorem

\[
||P_j f||_{L^q(\mathbb{R}^3)} \lesssim 2^{j(l-q/2)} ||P_j f||_{L^p(\mathbb{R}^3)},
\]

when \( q \geq p \). There is also the radial Sobolev embedding theorem

\[
|||x| (P_j f)||_{L^\infty} \lesssim 2^{j/2} ||P_j f||_{L^2}.
\]

Next recall the energy estimate

**Theorem 2.1** (Energy estimate). If \( u \) solves \( u_{tt} - \Delta u = 0 \) on an interval \( I \), with \( t_0 \in I \), then

\[
||\nabla u(t)||_{L^2(I \times \mathbb{R}^3)} + ||u_t(t)||_{L^2(I \times \mathbb{R}^3)} = ||\nabla u(t_0)||_{L^2(\mathbb{R}^3)} + ||u_t(t_0)||_{L^2(\mathbb{R}^3)}.
\]

Also recall the Strichartz estimates of [13].

**Theorem 2.2** (Strichartz estimate). If \( u \) solves \( u_{tt} - \Delta u = 0 \) on an interval \( I \), with \( t_0 \in I \), then

\[
||u(t)||_{L^4(I \times \mathbb{R}^3)} \lesssim ||u(t_0)||_{H^{1/2}(\mathbb{R}^3)} + ||u_t(t_0)||_{H^{-1/2}(\mathbb{R}^3)}.
\]

Also recall the endpoint Strichartz estimate of [14] for the radial wave equation.
**Theorem 2.3** (Endpoint Strichartz estimate). If $u$ solves $u_{tt} - \Delta u = 0$ on an interval $I$, with $t_0 \in I$, then
\[
\|u(t)\|_{L_t^p L_x^q(I \times \mathbb{R}^3)} \lesssim \|u(t_0)\|_{H^1(I)} + \|u_t(t_0)\|_{L^2(I)}.
\] (2.10)

**Remark:** Then by interpolation, for any $0 < \sigma < 1$, if $u$ is a radial solution to $u_{tt} - \Delta u = 0$, $0 < \sigma < 1$,
\[
\frac{1}{p} + \frac{3}{q} = \frac{3}{2} - \sigma,
\] (2.11)

then
\[
\|u\|_{L_t^p L_x^q(I \times \mathbb{R}^3)} \lesssim \|u(t_0)\|_{H^{\sigma}(I)} + \|u_t(t_0)\|_{H^{\sigma-1}(I)}.
\] (2.12)

**Remark:** This fact is also true when $u$ is nonradial. See [15].

**Theorem 2.4** (Local energy estimate). Let $B_R = \{ x : |x| \leq R \}$ and let $A_R = B_{2R} \setminus B_R = \{ x : R \leq |x| \leq 2R \}$. Also let $B_1 = B_2$ and $A_1 = A_2$. Then define the norms
\[
\|f\|_{L_t^p L_x^q(I \times \mathbb{R}^3)} = \sup_{l \in \mathbb{Z}} 2^{-l/2} \|f\|_{L_{t,x}^p(I \times B_l)}.
\] (2.13)

Also let
\[
\|g\|_{L_t^p L_x^q(I \times \mathbb{R}^3)} = \sum_{l \in \mathbb{Z}} 2^{l/2} \|g\|_{L_{t,x}^p(I \times A_l)}.
\] (2.14)

Also, for any $j \in \mathbb{Z}$ let
\[
\|f\|_{L_t^p L_x^q(I \times \mathbb{R}^3)} = \sup_{l \in \mathbb{Z}, l \geq -j} 2^{-l/2} \|f\|_{L_{t,x}^p(I \times B_l)}.
\] (2.15)

Then if $u$ is a radial solution to $u_{tt} - \Delta u = 0$ on an interval $I$, with $t_0 \in I$,
\[
\|\nabla u\|_{L_t^p L_x^q(I \times \mathbb{R}^3)} + \|u_t\|_{L_t^p L_x^q(I \times \mathbb{R}^3)} \lesssim \|\nabla u(t_0)\|_{L^2(I)} + \|u_t(t_0)\|_{L^2(I)}.
\] (2.16)

**Proof.** The proof follows by the sharp Huygens principle. Without loss of generality suppose $t_0 = 0$, $u(0) = f$, and $u_t(0) = g$. By time reversal symmetry we can assume $t \geq 0$. If $t \leq R$, then by the energy estimate (Theorem 2.1),
\[
\|\nabla u(t)\|_{L_t^p L_x^q([-R,R] \times \mathbb{R}^3)} + \|u_t(t)\|_{L_t^p L_x^q([-R,R] \times \mathbb{R}^3)} \lesssim R^{1/2} \|f\|_{H^{1/2}(I)} + R^{1/2} \|g\|_{L^2(I)}.
\] (2.17)

Next, since $u$ is radial, if $r \leq R$ and $t > R$,
\[
ru(t, r) = \frac{1}{2} (r + t)f(r + t) - \frac{1}{2} (t-r)f(t-r) + \frac{1}{2} \int_{t-r}^{t+r} sg(s) ds.
\] (2.18)
Now then,
\[
\partial_t(\rho u(t, r)) = \frac{1}{2} f(r + t) + \frac{1}{2} (r + t) f'(r + t) + \frac{1}{2} f(t - r) + \frac{1}{2} (t - r) f'(t - r)
+ \frac{1}{2} (t + r) g(t + r) + \frac{1}{2} (t - r) g(t - r),
\]
\[
\partial_t(\rho u(t, r)) = \frac{1}{2} f(r + t) + \frac{1}{2} (r + t) f'(r + t) - \frac{1}{2} f(t - r) - \frac{1}{2} (t - r) f'(t - r)
+ \frac{1}{2} (t + r) g(t + r) - \frac{1}{2} (t - r) g(t - r).
\]

(2.19)

Now by Fubini’s theorem, Hölder’s inequality, and Hardy’s inequality,
\[
\int_0^\infty \int_0^R (r + t)^2 \, dr \, dt + \int_0^\infty \int_0^R f(t - r)^2 \, dr \, dt
\leq R \int_0^\infty f(s)^2 \, ds \leq R \left\| \frac{1}{|x|} \right\|_{L^2_x(V)}^2 \leq R \left\| V \right\|_{L^2_x(V)}^2,
\]
while
\[
\int_0^\infty \int_0^R (r + t)^2 \, dr \, dt + \int_0^\infty \int_0^R f'(t - r)^2 \, dr \, dt
\leq R \int_0^\infty f'(s)^2 \, ds \leq R \left\| V \right\|_{L^2_x(V)}^2.
\]
and
\[
\int_0^\infty \int_0^R (t + r)^2 \, dr \, dt + \int_0^\infty \int_0^R (t - r)^2 \, dr \, dt
\leq R \left\| g \right\|_{L^2_x(V)}^2.
\]

(2.22)

Since \( \rho u_t = \partial_t(\rho u) \) and \( \rho u_r = \partial_r(\rho u) - u \), to complete the proof of Theorem 2.4 it only remains to show that
\[
\int_0^\infty \int_0^R u(t, r)^2 \, dr \, dt
\leq R \left\| V \right\|_{L^2_x(V)}^2 + R \left\| g \right\|_{L^2_x(V)}^2.
\]

(2.23)

However this follows directly from the endpoint Strichartz estimate in Theorem 2.3 and Hölder’s inequality.

\( \square \)

**Theorem 2.5.** If \( u \) is a radial solution to
\[
u_t - \Delta u = F_1 + F_2 + F_3 + F_4, \quad u(0) = u_0, \quad u_t(0) = u_1,
\]
then for any \( j \)
\[
\left\| P_j u \right\|_{L^2_x(L^\infty_t(J \times \mathbb{R}^3))} + 2^{j/2} \left\| P_j u \right\|_{L^1_x(L^\infty_t(J \times \mathbb{R}^3))} + 2^j \left\| P_j u \right\|_{L^2_x(L^\infty_t(J \times \mathbb{R}^3))} + \left\| P_j u \right\|_{L^2_x(L^2_t(J \times \mathbb{R}^3))}
+ \left\| P_j u \right\|_{L^2_x(L^2_t(J \times \mathbb{R}^3))} \leq 2^j \left\| P_j u_0 \right\|_{L^2(\mathbb{R}^3)} + \left\| P_j u_1 \right\|_{L^2(\mathbb{R}^3)} + \left\| P_j F_1 \right\|_{L^2_x(L^2_t(J \times \mathbb{R}^3))}
+ \left\| P_j F_2 \right\|_{L^2_x(L^2_t(J \times \mathbb{R}^3))} + \left\| P_j F_3 \right\|_{L^2_x(L^2_t(J \times \mathbb{R}^3))} + 2^j \left\| P_j F_4 \right\|_{L^2_x(L^2_t(J \times \mathbb{R}^3))}.
\]

(2.25)
To simplify notation let
\[
S(j, u) = ||P_j u||_{L^2_{x, t}(\mathbb{R}^3)} + 2^{j/2} ||P_j u||_{L^2_{x, t}(\mathbb{R}^3)} + 2^j ||P_j u||_{L^\infty_{x, t}(\mathbb{R}^3)} + ||P_j u||_{L^2_{x, t}(\mathbb{R}^3)}.
\] (2.26)

**Proof.** Let \( \tilde{P}_j = P_{j-1} + P_j + P_{j+1} \), so that, by definition 2.1, \( \tilde{P}_j P_j = P_j \). By Duhamel’s principle
\[
u(t) = S(t)(u_0, u_1) + \int_0^t S(t-\tau)(0, F_1 + F_2 + F_3 + F_4)\,d\tau
\] (2.27)
solves (2.24), and
\[
P_j u(t) = S(t)(P_j u_0, P_j u_1) + \int_0^t S(t-\tau)(0, P_j F_1 + P_j F_2 + P_j F_3 + P_j F_4)\,d\tau
\] (2.28)

By Theorems 2.1–2.3,
\[
||P_j S(t)(u_0, u_1)||_{L^2_{x, t}(\mathbb{R}^3)} + 2^{j/2} ||P_j S(t)(u_0, u_1)||_{L^2_{x, t}(\mathbb{R}^3)} + 2^j ||P_j S(t)(u_0, u_1)||_{L^\infty_{x, t}(\mathbb{R}^3)} + ||P_j S(t)(u_0, u_1)||_{L^2_{x, t}(\mathbb{R}^3)} \leq ||\nabla P_j u_0||_{L^2(R^3)} + ||P_j u_1||_{L^2(R^3)} \leq 2^j ||P_j u_0||_{L^2(R^3)} + ||P_j u_1||_{L^2(R^3)}.
\] (2.29)

Meanwhile, by Theorem 2.4,
\[
||\nabla P_j S(t)(u_0, u_1)||_{L^\infty_{x, t}(\mathbb{R}^3)} \leq 2^j ||P_j u_0||_{L^2(R^3)} + ||P_j u_1||_{L^2(R^3)}.
\] (2.30)

To simplify notation let \( v_j = P_j S(t)(u_0, u_1) \). Now, by (2.5), if \( R \geq 2^{-j}, |x| \leq R \), by Taylor’s formula and \( \int \overline{\psi}_j(x)\,dx = 0 \),
\[
P_j v_j(t, x) = 2^{3j} \int \overline{\psi}_j(2^j (x-y))\,v_j(t, y)\,dy = 2^{3j} \int \overline{\psi}_j(2^j (x-y))\,[v_j(t, y) - v_j(t, x)]\,dy
\]
\[
= 2^{3j} \int_0^1 \overline{\psi}_j(2^j (x-y))\,(y-x)\cdot \nabla v_j(t, x + \tau(y-x))\,dy\,d\tau,
\] (2.31)
and so by (2.5), if \( R = 2^{-j+m}, m \geq 0 \),
\[
R^{-1/2} ||\tilde{P}_j P_j u||_{L^2_{x, t}(\mathbb{R}^3)} \leq R^{-1/2} \sum_{l \geq 0} 2^{-10(l+m)} 2^{-j} R^{-1/2} ||\nabla v_j||_{L^2_{x, t}(\mathbb{R}^3)}
\]
\[
\leq ||P_j u_0||_{L^2(R^3)} + 2^{-j} ||P_j u_1||_{L^2(R^3)}.
\] (2.32)

Thus,
\[
S(j, S(t)(u_0, u_1)) \leq 2^j ||P_j u_0||_{L^2(R^3)} + ||P_j u_1||_{L^2(R^3)}.
\] (2.33)

Next, by the principle of superposition combined with (2.33), if
\[
u^1(t) = \int_0^t S(t-\tau)(0, F_1)\,d\tau,
\] (2.34)
then
\[
S(j, u^1(t)) \leq ||P_j F_1||_{L^1_{x, t}(\mathbb{R}^3)}.
\] (2.35)

Next, by the Christ-Kiselev lemma, duality, Theorem 2.2, and the sine addition formulas (see [15]), if
\[ u^2(t) = \int_0^t S(t-\tau)(0, P_j F_2) d\tau, \quad S(j, u^2(t)) \leq 2^{j/2} ||P_j F_2||_{L^2_{k,j}(\mathbb{R}^3)} \] (2.36)

Next, let
\[ u^3(t) = \int_0^t S(t-\tau)(0, P_j F_3) d\tau. \] (2.37)

First suppose that \( F_3 \) is supported on \( r \leq R \). By the fundamental solution of the wave equation,
\[ r u^3(t, r) = \int_0^t \int_{r-\tau-r}^{r+(t-\tau)} s F_3(s, \tau) ds d\tau + \int_{\sup(0, t-r)}^{\sup(0, t-r)} \int_{r-\tau-r}^{(t-\tau)+r} s F_3(s, \tau) ds d\tau. \] (2.38)

By the support properties of \( F_3 \), the integrals
\[ \int_{r-\tau-r}^{r+(t-\tau)} s F_3(s, \tau) ds, \quad \text{and} \quad \int_{(t-\tau)-r}^{(t-\tau)+r} s F_3(s, \tau) ds, \] (2.39)
are only nonzero if \( |(t-\tau)-r| \leq R \). Thus, if for some \( k \in \mathbb{Z}, kR \leq (t-r) < (k+1)R \), (2.39) is zero unless \( (k-1)R \leq \tau \leq (k+1)R \). Thus, the supports of
\[ \int_{\inf((k+1)R, t)}^{\sup((k+1)R, t)} S(t-\tau)(0, F_3) d\tau \] (2.40)
are finitely intersecting. Now, by Hölder’s inequality in time,
\[ ||F_3||_{L^1_{t} L^2_{x}([kR, (k+1)R] \times \mathbb{R}^3)} \leq R^{1/2} ||F_3||_{L^2_{k,x}([kR, (k+1)R] \times \mathbb{R}^3)}, \] (2.41)
so by Theorems 2.1–2.4,
\[ ||u^3(t)||_{L^p_{t} L^q_{x}(\mathbb{R}^3)} + ||\nabla u^3(t)||_{L^p_{t} L^q_{x}(\mathbb{R}^3)} + ||\partial_t u^3(t)||_{L^p_{t} L^q_{x}(\mathbb{R}^3)} + ||\nabla u^3(t)||_{L^p_{t} L^q_{x}(\mathbb{R}^3)} + ||\partial_t u^3(t)||_{L^p_{t} L^q_{x}(\mathbb{R}^3)} \leq R^{1/2} ||F_3||_{L^2_{k,x}(\mathbb{R}^3)}. \] (2.42)

If \( F_3 \) is compactly supported then \( P_j F_3 \) need not be, but making a calculation similar to (2.30)–(2.33) proves that
\[ S(j, u^3(t)) \leq ||P_j F_3||_{L^2_{k,j}(\mathbb{R}^3)}. \] (2.43)

Finally let
\[ u^4(t) = \int_0^t S(t-\tau)(0, F_4) d\tau. \] (2.44)

Since \( P_j = \tilde{P}_j P_j \),
\[ P_j u^4(t) = P_j \int_0^t S(t-\tau)(0, \tilde{P}_j F_4) d\tau. \] (2.45)

Now, by the Sobolev embedding theorem,
\[ ||\tilde{P}_j F_4||_{L^2_{k,j}(\mathbb{R}^3)} \leq 2^{3j/2} ||\tilde{P}_j F_4||_{L^2_{k,j}(\mathbb{R}^3)}, \] (2.46)
so in particular,
Also, by the radial Sobolev embedding theorem,
\[ \|x|\tilde{P}_j F_4 \|_{L^2_j (J \times \mathbb{R}^3)} \leq 2^{j/2} \|\tilde{P}_j F_4 \|_{L^2_j (J \times \mathbb{R}^3)}. \]
(2.48)

Therefore,
\[ \sum_{l > j} 2^{j/2} \|\tilde{P}_j F_4 \|_{L^2_j (J \times A_l)} \leq 2^{j/2} \sum_{l > j} 2^{-j/2} \|\tilde{P}_j F_4 \|_{L^2_j (J \times \mathbb{R}^3)} \leq 2 \|\tilde{P}_j F_4 \|_{L^2_j (J \times \mathbb{R}^3)}. \]
(2.49)

Thus, by (2.43),
\[ S(j, u^j(t)) \leq 2 \|\tilde{P}_j F_4 \|_{L^2_j (J \times \mathbb{R}^3)} = 2 \|\tilde{P}_j F_4 \|_{L^2_j (J \times \mathbb{R}^3)}. \]
(2.50)

This completes the proof of the theorem.

\[ \square \]

**Theorem 2.6.** If \( u \) is a radial solution to the wave equation
\[ u_{tt} - \Delta u = F_1 + F_2 + F_3, \quad u(t_0) = u_0, \quad u_t(t_0) = u_1, \]
on the interval \( J \) with \( t_0 \in I \), and \( 0 < \sigma < 1 \) satisfies

\[ \frac{1}{p} = 1 - \frac{\sigma}{2}, \quad \frac{1}{q} = \frac{1 - \sigma}{2}, \]
(2.52)

\[ \|\nabla u\|_{L^p_{j=0} (J \times \mathbb{R}^3)} + \|u_t\|_{L^p_{j=0} (J \times \mathbb{R}^3)} \leq \|u_0\|_{H^1(\mathbb{R}^3)} + \|u_1\|_{L^2(\mathbb{R}^3)} \]
\[ + \|F_1\|_{L^p_{j=0} (J \times \mathbb{R}^3)} + \|\nabla |F_2|\|_{L^p_{j=0} (J \times \mathbb{R}^3)} + \|F_3\|_{L^p_{j=0} (J \times \mathbb{R}^3)}. \]
(2.53)

**Proof.** By the endpoint Strichartz estimate (Theorem 2.3) and the local energy-estimate (Theorem 2.4),
\[ \|\nabla S(t)(u_0, u_1)\|_{L^p_{j=0} (J \times \mathbb{R}^3)} + \|\partial_t S(t)(u_0, u_1)\|_{L^p_{j=0} (J \times \mathbb{R}^3)} \]
\[ + \|S(t)(u_0, u_1)\|_{L^p_{j=0} (J \times \mathbb{R}^3)} \leq \|u_0\|_{H^1(\mathbb{R}^3)} + \|u_1\|_{L^2(\mathbb{R}^3)}. \]
(2.54)

Also by the principle of superposition, if \( u^1(t) = \int_0^t S(t-\tau)F_1(\tau)d\tau, \)
\[ \|\nabla u^1\|_{L^p t=0 (J \times \mathbb{R}^3)} + \|\partial_t u^1\|_{L^p t=0 (J \times \mathbb{R}^3)} \leq \|F_1\|_{L^p_{j=0} (J \times \mathbb{R}^3)}. \]
(2.55)

Next, by the Christ-Kiselev lemma (again see [15]), when \( \sigma < 1 \), if \( u^2(t) = \int_0^t S(t-\tau)F_2(\tau)d\tau, \)
\[ \|u^2\|_{L^p t=0 (J \times \mathbb{R}^3)} + \|\partial_t u^2\|_{L^p t=0 (J \times \mathbb{R}^3)} + \|u^2\|_{L^p_{j=0} (J \times \mathbb{R}^3)} \leq \|\nabla |F_2|\|_{L^p_{j=0} (J \times \mathbb{R}^3)}. \]
(2.56)

Finally, as in (2.40), if \( F_3 \) is supported on \( |x| \leq R \), then the supports of
\[ \int_{[kR, (k+1)R]^R} S(t-\tau)(0, F_3)(\tau)d\tau, \]
(2.57)
are finitely intersecting, so since by Hölder’s inequality in time,
\[ \|F_3\|_{L^p_{j=0} ([kR, (k+1)R] \times \mathbb{R}^3)} \leq R^{1/2} \|F_3\|_{L^p_{j=0} ([kR, (k+1)R] \times \mathbb{R}^3)}, \]
(2.58)
which proves that if \( u^3(t) = \int_0^t S(t-\tau)(0, F_3)d\tau, \)
\[ ||u^2||_{L^2_{t,s}(\mathbb{R}^3)} + ||\partial_t u^3||_{L^2_{t,s}(\mathbb{R}^3)} + ||u^3||_{L^2\infty_{t,s}(\mathbb{R}^3)} \leq ||F_3||_{L^2_{t,s}(\mathbb{R}^3)}. \] (2.59)

### 3. The conformal symmetry for the wave equation

In this section we outline the proof of the main theorem, Theorem 1.3. The proof of the theorem depends on several propositions, which will then be proved in subsequent sections. First, observe that by Theorem 1.4, if \( u_0 \in H^{1/2+\epsilon}(\mathbb{R}^3) \cap H^{1/2}(\mathbb{R}^3) \) and \( u_1 \in H^{-1/2+\epsilon}(\mathbb{R}^3) \cap \dot{H}^{-1/2}(\mathbb{R}^3) \) are radial functions, then (1.1) has a global solution. We will show in Lemma 4.1 that this is true.

Now let \( v = Tu \), where \( T \) is the conformal transformation

\[
(Tu)(y,\tau) = \frac{\sinh|y|}{|y|} e^{\tau} u \left( \tilde{T}(y,\tau) \right) = \frac{\sinh|y|}{|y|} e^{\tau} u \left( e^{-\tau} \sinh|y|, y, t_0 + e^{\tau} \cosh(|y|) \right), (y,\tau) \in \mathbb{R}^3 \times \mathbb{R},
\] (3.1)

where \( t_0 < 0 \) will be defined later. Since \( u \) is radially symmetric, taking

\[
v(s,\tau) = \frac{\sinh s}{s} e^{\tau} u(e^{-\tau} \sinh s, t_0 + e^{\tau} \cosh s).
\] (3.2)

Then by direct computation, taking \( w(r,t) = ru(r,t) \),

\[
v_{\tau\tau} - \Delta v = v_{\tau\tau} - vs - \frac{2}{s} vs = \frac{1}{s} \left[(sv)_{\tau\tau} - (sv)_{ss}\right] = -\left(\frac{s}{\sinh s}\right)^2 v^3 = -\phi(s)v^3.
\] (3.3)

**Remark:** See section five of [12] for more details concerning the calculation.

We prove

\[
\int_0^\infty \int_{\mathbb{R}} \left( v(s,\tau) \right)^4 \phi(s) s^2 ds d\tau < \infty
\] (3.4)

in four steps.

**Proposition 3.1** (Initial data). There exists some constant \( C_0(A,\epsilon) < \infty \) such that

\[
||v(s,0)||_{H^{1/2+\epsilon}(\mathbb{R}^3)} + ||\partial_t v(s,0)||_{\dot{H}^{-1/2+\epsilon}(\mathbb{R}^3)} \leq A,\epsilon 1.
\] (3.5)

Now let \( N = 2^k_0 \) and define the I-operator

\[
I = P_{\leq k_0} + \sum_{j>0} 2^{-j(1+\epsilon)} P_{j+k_0}.
\] (3.6)

Then

\[
\frac{1}{2} ||\nabla Iv||_{L^2(\mathbb{R}^3)}^2 + \frac{1}{2} ||Iv||_{L^2(\mathbb{R}^3)}^2 \leq A,\epsilon N^{1-2\epsilon}.
\] (3.7)

Also, by the Sobolev embedding theorem and Hölder’s inequality,

\[
\frac{1}{4} \int \phi(s) |v(s,0)|^4 s^2 ds \leq A,\epsilon N^{1-2\epsilon}.
\] (3.8)
Therefore, there exists some $C_0(A, \epsilon)$ such that
\[ E(Iv(0)) \leq C_0(A, \epsilon)N^{1-2\epsilon}. \] (3.9)

**Proposition 3.2** (Long time Strichartz estimate). Suppose $\mathcal{J}$ is an interval on which
\[ \int_{\mathcal{J}} \int J_0 \phi(x)(Iv(x,t))^4 dx dt \leq C_1 N^{1-2\epsilon}, \] (3.10)
and $\sup_{t \in \mathcal{J}} E(Iv(t)) \leq 2C_0(A, \epsilon)N^{1-2\epsilon}$. Then there exists $k_0(C_1, A, \epsilon)$ sufficiently large such that,
\[ \sup_{j \geq k_0-7} 2^{j(e-1/2)} S(j, \nu) \leq C_0(A, \epsilon). \] (3.11)

**Remark:** It is important to observe that the implicit constant in (3.11) crucially does not depend on $C_1$.

**Proposition 3.3** (Almost conservation of energy). Suppose $\mathcal{J}$ is an interval on which
\[ \int_{\mathcal{J}} \int J_0 \phi(s) \left( \frac{\cosh s}{\sinh s} \right) (Iv(s, \tau))^4 s^2 ds d\tau \leq C_1 N^{1-2\epsilon}, \] (3.12)
and $\sup_{t \in \mathcal{J}} E(Iv(t)) \leq 2C_0(A, \epsilon)N^{1-2\epsilon}$, and $E(Iv(0)) \leq C_0(A, \epsilon)N^{1-2\epsilon}$. Then for $k_0(C_1, A, \epsilon)$ sufficiently large,
\[ \sup_{t \in \mathcal{J}} E(Iv(t)) \leq \frac{3}{2} C_0(A, \epsilon)N^{1-2\epsilon}. \] (3.13)

**Proposition 3.4** (Almost Morawetz estimate). If $\mathcal{J}$ is an interval on which $E(Iv(\tau)) \leq 2C_0(A, \epsilon)N^{1-2\epsilon}$, and
\[ \int_{\mathcal{J}} \int J_0 \phi(s) \left( \frac{\cosh s}{\sinh s} \right) (Iv(s, \tau))^4 s^2 ds d\tau, \] (3.14)
then for $k_0(C_1, A, \epsilon)$ sufficiently large,
\[ \int_{\mathcal{J}} \int J_0 \phi(s) \left( \frac{\cosh s}{\sinh s} \right) (Iv(s, \tau))^4 s^2 ds d\tau \leq C_0(A, \epsilon)N^{1-2\epsilon}. \] (3.15)

Armed with Propositions 3.1–3.4,
\[ \int_{\mathbb{R}} \int_{0}^{\infty} \phi(s) v(s, \tau)^4 s^2 ds d\tau < \infty \] (3.16)
may be proved by a bootstrap argument. Choose some $C_1(A, \epsilon) \gg C_0(A, \epsilon)$ and define the set
\[ J_0 = \left\{ T \in \mathbb{R}: \int_{-T}^{T} \int_{0}^{\infty} \phi(s) \left( \frac{\cosh s}{\sinh s} \right) (Iv(s, \tau))^4 s^2 ds d\tau \leq C_1 N^{1-2\epsilon}, \text{and} \ sup_{t \in [-T,T]} E(Iv(t)) \leq 2N^{1-2\epsilon} \right\}. \] (3.17)
By the dominated convergence theorem, \( J_0 \) is a closed set. Also, clearly \( 0 \in J_0 \).

Therefore, it suffices to prove that \( J_0 \) is open, which would then imply that \( J_0 = \mathbb{R} \).

Now by Proposition 3.3,

\[
E(Iv(\tau)) \leq \frac{3}{2} C_0(A, \epsilon) N^{1-2\epsilon}
\]

for all \( \tau \in J_0 \). Also, Proposition 3.4 implies

\[
\int _J \phi(s) \left( \frac{\cosh s}{\sinh s} \right) (Iv(s, \tau))^4 s^2 dsd\tau \leq C_0(A, \epsilon) N^{1-2\epsilon}.
\]

Therefore, by local well-posedness (Lemma 3.5) there exists some open interval \( J_1 \) that contains \( J_0 \), such that

\[
\int _J \int _0^{\infty} \phi(s) \left( \frac{\cosh s}{\sinh s} \right) (Iv(s, \tau))^4 s^2 dsd\tau \leq C_1 N^{1-2\epsilon},
\]

and

\[
\sup _{t \in J_1} E(Iv(t)) \leq 2C_0(A, \epsilon) N^{1-2\epsilon}.
\]

Therefore \( J_0 \) is both open and closed in \( \mathbb{R} \), and since \( J_0 \) is non-empty, \( J_0 = \mathbb{R} \), and

\[
\int _\mathbb{R} \int _0^{\infty} \phi(s) \left( \frac{\cosh s}{\sinh s} \right) (Iv(s, \tau))^4 s^2 dsd\tau \leq C_0(A, \epsilon) N^{1-2\epsilon}.
\]

Meanwhile, by Proposition 3.2, (3.22), and the fact that \( \frac{\sinh s}{\sinh s} \geq 0 \),

\[
\sum _{j \geq k_0} ||P_j v||_{L_t^1 (\mathbb{R} \times \mathbb{R}^3)} \lesssim C_0(A, \epsilon),
\]

so

\[
\int _\mathbb{R} \int _0^{\infty} \phi(s) \left( (1-I)v(s, \tau) \right)^4 s^2 dsd\tau \leq C_0(A, \epsilon)^4.
\]

Therefore, we have proved

\[
\int _\mathbb{R} \int _\mathbb{R}^3 \phi(s) v(s, \tau)^4 s^2 dsd\tau \lesssim_{A, \epsilon} 1.
\]

Now we can follow the argument in section 6.3 of [12] to complete the proof of Theorem 1.3. Let \( \chi(x) \) be a smooth function,

\[
\chi(x) = \begin{cases} 
1 & \text{if } |x| \geq 1 \\
0 & \text{if } |x| \leq \frac{1}{2}
\end{cases}
\]

Fixing \( \delta > 0 \) to be a small, fixed constant, \( R(A, \epsilon) \) may be chosen to be sufficiently large so that

\[
||\chi \left( \frac{x}{R(A, \epsilon)} \right) u_0||_{\dot{H}^{1/2}(\mathbb{R}^3)} + ||\chi \left( \frac{x}{R(A, \epsilon)} \right) u_1||_{\dot{H}^{-1/2}(\mathbb{R}^3)} \leq \delta.
\]
Then by small data arguments,
\[ w_{tt} - \Delta w = -w^3, \quad w(0, x) = \chi u_0, \quad w_t(0, x) = \chi u_1, \tag{3.28} \]
has a solution
\[ ||w||_{L^4_t(L^4)} \leq \delta. \tag{3.29} \]
Also, by finite propagation speed this implies \( u = w \) for \( |x| \geq R + |t| \), and thus
\[ \int_0^\infty \int_{|x| \geq R+t} u(x, t)^4 \, dx \, dt \leq \delta. \tag{3.30} \]
Choose \( t_0 < 0 \) in (3.1) to satisfy \( t_0^2 > R^2 + 1 \). Then
\[ \{ (t, r) : t \geq 0, 0 \leq r \leq R + t \} \subset \Omega = \{ (x, t) : |x|^2 < (t-t_0)^2 - 1, t > t_0 \} = \tilde{T}(\{ (y, \tau) : \tau > 0 \}). \tag{3.31} \]
Now by the change of variables formula,
\[ dx \, dt = 4\pi r^2 \, dr \, dt = e^{4\tau} \left( \frac{\sinh |y|}{|y|} \right)^2 \, dy \, d\tau, \tag{3.32} \]
so
\[ \int_{\Omega} \int u(x, t)^4 \, dx \, dt = \int_0^\infty \int_0^\infty e^{4\tau} \left( \sinh s \right)^2 \left( u(e^\tau \sinh s, t_0 + e^\tau \cosh s) \right)^4 s^2 \, ds \, d\tau \\
= \int_0^\infty \int_0^\infty \left( \frac{s}{\sinh s} \right)^2 \left( e^\tau \sinh s \right)^4 u(e^\tau \sinh s, t_0 + e^\tau \cosh s) \, s^2 \, ds \, d\tau \\
= \int_0^\infty \int_0^\infty \phi(s) \nu(s, \tau) \, s^2 \, ds \, d\tau \approx \mu, \tag{3.33} \]
This finally proves
\[ \int_0^\infty \int_{R^3} u(x, t)^4 \, dx \, dt < \infty. \tag{3.34} \]

**Theorem 1.3** then follows by time reversal symmetry. \( \square \)

It only remains to pay off the debt incurred in Propositions 3.1–3.4 as well as a local well-posedness result. The proof of local well-posedness is relatively straightforward. Assume throughout that the \( N \) in (3.6) is large.

**Lemma 3.5** (First local result). Suppose \( E(I\nu(0)) \leq N^{1-2\epsilon} \). Then the conformal wave equation
\[ \nu_{tt} - \Delta \nu = -\phi(x) \nu^3, \tag{3.35} \]
is locally well-posed on some interval.

**Proof.** By Strichartz estimates, Hölder’s inequality, and Bernstein’s inequality, since \( P_{>k_0}|\nabla|^{1/2} I \sim P_{>k_0} N^{1/2-\epsilon}|\nabla|^{\epsilon} \),
\[ |\nabla|^{1/2}Iv|_{L^2_x(J \times \mathbb{R}^3)} + |Iv|_{L^2_t L^\infty_x(J \times \mathbb{R}^3)} + |Iv|_{L^\infty_t L^2_x(J \times \mathbb{R}^3)} \]
\[ \leq |\nabla Iv(0)|_{L^2(\mathbb{R}^3)} + |Iv(0)|_{L^2(\mathbb{R}^3)} + |J|^{1/2}|\nabla^2 Iv|_{L^2_t L^\infty_x(J \times \mathbb{R}^3)} |Iv|_{L^2_t L^2_x(J \times \mathbb{R}^3)} \]
\[ + N^{1/2-\epsilon}|\nabla^2 [((1-I)v)_{L^2_t L^\infty_x(J \times \mathbb{R}^3)}] \|
\]
\[ \leq N^{1/2-\epsilon} + |J|^{1/2}|Iv|_{L^2_t L^2_x(J \times \mathbb{R}^3)} |Iv|_{L^\infty_t L^2_x(J \times \mathbb{R}^3)} + N^{-1}|\nabla|^{1/2}Iv|_{L^2_t L^2_x(J \times \mathbb{R}^3)} \|.
\]  \( (3.36) \)

Taking \(|J| \leq \frac{1}{N^2}\) proves
\[ |\nabla|^{1/2}Iv|_{L^2_x(J \times \mathbb{R}^3)} + |Iv|_{L^2_t L^\infty_x(J \times \mathbb{R}^3)} + |Iv|_{L^\infty_t L^2_x(J \times \mathbb{R}^3)} \leq N^{1/2-\epsilon}. \]  \( (3.37) \)

This gives local well-posedness.

\[ \square \]

**Lemma 3.6** (Second local result). Suppose that \( \phi(x) = \left( \frac{|x|}{\sinh|x|} \right)^2 \), and \( v \) solves the equation
\[ \nu_t - \Delta v = -\phi(x) v^3, E(Iv(0)) \leq N^{1-2\epsilon}, \]  \( (3.38) \)
on the interval \( J_k \) and
\[ \int_J \int \phi(x)(Iv(x,t))^4 \, dx \, dt \leq \delta^4, \]  \( (3.39) \)
for some small \( \delta > 0 \). (\( \delta \) may be independent of \( N \)). Then
\[ |Iv|_{L^\infty_t L^2_x(J \times \mathbb{R}^3)} + \sup_j S(j, Iv) \leq N^{1/2-\epsilon}. \]  \( (3.40) \)

**Proof.** Suppose \( J_k = [a_k, b_k] \). Following an argument similar to Lemma 3.5,
\[ |Iv|_{L^2_t L^\infty_x(J \times \mathbb{R}^3)} + |\nabla|^{1/2}Iv|_{L^2_x(J \times \mathbb{R}^3)} \leq |\nabla Iv(a_k)|_{L^2(\mathbb{R}^3)} + |Iv(a_k)|_{L^2(\mathbb{R}^3)} \]
\[ + \|Iv\|_{L^2_t L^\infty_x(J \times \mathbb{R}^3)} \|\phi(x)^{1/2} Iv\|_{L^2_x(J \times \mathbb{R}^3)} \]
\[ + N^{1/2-\epsilon} \|\nabla^2 [((1-I)v)_{L^2_t L^\infty_x(J \times \mathbb{R}^3)}] \|
\]
\[ \leq N^{1/2-\epsilon} + \delta^2 |Iv|_{L^2_t L^\infty_x(J \times \mathbb{R}^3)} + N^{-1} |\nabla|^{1/2}Iv|_{L^2_t L^2_x(J \times \mathbb{R}^3)} \|^3.
\]  \( (3.41) \)

The theorem then follows by the contraction mapping principle and Theorem 2.5.  \( \square \)

**Corollary 3.7.** Suppose \( J \) is an interval on which
\[ \int \int \phi(x)(Iv(x,t))^4 \, dx \, dt \leq CN^{2(1-\epsilon)}, \]  \( (3.42) \)
and \( E(Iv)(t) \leq CN^{2(1-\epsilon)} \) for all \( t \in J \). Then
\[ |Iv|_{L^\infty_t L^2_x(J \times \mathbb{R}^3)} + \sup_j S(j, Iv) \leq C^{1/2}N^{1-2\epsilon}. \]  \( (3.43) \)

**Proof.** Partition \( J \) into \( \approx N^{1-2\epsilon} \) subintervals \( J_k \) such that
\[ \int_J \int \phi(x)(Iv(x,t))^4 \, dx \, dt \leq \delta^4. \]  \( (3.44) \)
Then apply the previous lemma.  \( \square \)
4. Initial data

Now it only remains to prove Propositions 3.1–3.4. In this section, we prove Proposition 3.1. In order to utilize (3.1), it is first necessary to prove that the solution to (1.1) is global, which follows directly from proving that the initial data $u_0$ and $u_1$ lie in the set of data prescribed in [6]. Throughout this section, $f \leq g$ denotes $f \leq C(A, \epsilon)g$.

**Lemma 4.1.** Suppose that for some $\epsilon > 0$, $u_0$, $u_1$ are radial functions with $u_0 \in \dot{H}^{1+\epsilon}(\mathbb{R}^3)$ and $u_1 \in \dot{H}^{\epsilon-1}(\mathbb{R}^3)$. Also suppose that $|x|^{2\epsilon}u_0 \in \dot{H}^{1+\epsilon}(\mathbb{R}^3)$ and $|x|^{2\epsilon}u_1 \in \dot{H}^{\epsilon-1}(\mathbb{R}^3)$. Then this implies

$$ u_0 \in \dot{H}^{1/2+\epsilon}(\mathbb{R}^3) \cap \dot{H}^{1/2-\epsilon}(\mathbb{R}^3), \text{ and } u_1 \in \dot{H}^{-1/2+\epsilon}(\mathbb{R}^3) \cap \dot{H}^{-1/2-\epsilon}(\mathbb{R}^3).$$

**Proof.** For any $0 \leq s < 1$,

$$g \in \dot{H}^s(\mathbb{R}^3) \Rightarrow |x|^{-2\epsilon}g \in \dot{H}^{s-2\epsilon}(\mathbb{R}^3).$$

When $s = 0$, (4.2) follows from Hardy’s inequality. When $0 < s < 1$ we adapt the proof of Hardy’s inequality (see for example [16]). Let $\psi(x) \in C_0^\infty(\mathbb{R}^3)$, $\psi(x) = 1$ for $|x| \leq 1$, and $\psi(x) = 0$ for $|x| > 2$. Then let

$$\chi_j(x) = \psi(2^{-j-1}x) - \psi(2^{-j}x),$$

and make the partition of unity, when $x \neq 0$,

$$1 = \sum_j \chi_j(x) = \psi(2^{k_0}x) + \sum_{j \geq k_0} \chi_j(x).$$

$k_0$ could be any integer. Combining the Littlewood-Paley partition of unity with the spatial partition of unity,

$$|x|^{-2\epsilon}g = \sum_m \sum_l |x|^{-2\epsilon}\chi_l(x)(P_mg).$$

By Bernstein’s inequality,

$$|||x|^{-2\epsilon}\chi_l(x)(P_mg)||_{L^2(\mathbb{R}^3)} \leq 2^{-2\epsilon} \sum_{m \geq j} 2^{-ms}||P_mg||_{H^s},$$

so by Young’s inequality,

$$\sum_j 2^{2j(s-2\epsilon)}||P_j\left((1-\psi(2^jx))(P_{\leq j}f)\right)||_{L^2}^2 \leq \sum_j 2^{2j(s-2\epsilon)}\left(\sum_{l \geq j} \sum_{m \geq j} 2^{-2\epsilon} 2^{-ms}||P_mg||_{H^s}\right)^2 \leq \sum_m ||P_mg||_{H^s}^2 \leq ||g||_{H^s}^2.$$ 

Meanwhile, by Hölder’s inequality, the Sobolev embedding theorem, and Bernstein’s inequality,

$$||P_j\left(|x|^{-2\epsilon}\chi_l(x)(P_{\geq j}g)\right)||_{L^2(\mathbb{R}^3)} \leq 2^{\epsilon|||x|^{-2\epsilon}\chi_l(x)(P_{\geq j}g)||_{L^{2\epsilon}(\mathbb{R}^3)}} \leq 2^{\epsilon 2^{(1-2\epsilon)}\sum_{m \geq j} 2^{-ms}||P_mg||_{H^s}}.$$ 

so again by Young’s inequality,
\[
\sum_j 2^{j(s-2\epsilon)} |P_j\left(\psi(2^j x)(P \lesssim f)\right)||_{L^2}^2 \\
\leq \sum_j 2^{j(s-2\epsilon)} \left(\sum_{l \geq -j} \sum_{m \leq j} 2^{2-2m} 2^{l(1-2\epsilon)} ||P_m g||_{H^l}\right)^2 \leq ||g||_{H^s}^2. \tag{4.9}
\]

Also, by Bernstein’s inequality,
\[
||P_j\left(|x|^{-2\epsilon} \chi_l(x)(P \lesssim g)\right)||_{L^2(R^3)} \leq 2^{-j} ||\nabla \left(|x|^{-2\epsilon} \chi_l(x)(P \lesssim g)\right)||_{L^2(R^3)} \\
\leq 2^{-j} ||\nabla \left(|x|^{-2\epsilon} \chi_l(x)\right)||_{L^\infty} ||P \lesssim g||_{L^2} + 2^{-j} |||x|^{-2\epsilon} \chi_l(x)||_{L^\infty} ||\nabla P \lesssim g||_{L^2}. \tag{4.10}
\]

Again by Young’s inequality this implies
\[
\sum_j 2^{j(s-2\epsilon)} |P_j\left((1-\psi(2^j x))(P \lesssim f)\right)||_{L^2}^2 \\
\leq \sum_j 2^{j(s-2\epsilon)} \left(\sum_{l \geq -j} \sum_{m \leq j} 2^{-j} 2^{-2l} 2^{m(1-s)} ||P_m g||_{H^l}\right)^2 \leq ||g||_{H^s}^2. \tag{4.11}
\]

Finally, by Hölder’s inequality plus the Sobolev embedding theorem,
\[
|||x|^{-2\epsilon} \chi_l(x)P \lesssim g||_{L^2(R^3)} \leq ||P \lesssim g||_{L^2} 2^{l(1-2\epsilon)} \sum_{m \leq j} 2^{m(1-s)} ||P_m g||_{H^l}, \tag{4.12}
\]

and by Young’s inequality,
\[
\sum_j 2^{j(s-2\epsilon)} |P_j\left(\psi(2^j x)(P \lesssim f)\right)||_{L^2}^2 \\
\leq \sum_j 2^{j(s-2\epsilon)} \left(\sum_{l \geq -j} \sum_{m \leq j} 2^{m(1-s)} 2^{l(1-2\epsilon)} ||P_m g||_{H^l}\right)^2 \leq ||g||_{H^s}^2. \tag{4.13}
\]

This proves (4.2), which directly implies \(u_0 \in H^{s-\epsilon} \). By duality this also implies \(u_1 \in H^{-1/2+\epsilon}(R^3) \). Indeed, suppose \(g \in H^{-1/2+\epsilon}(R^3) \). Then
\[
\int u_1(x)g(x)dx = \int \left(|x|^{2\epsilon} u_1(x)\right)\left(|x|^{-2\epsilon} g(x)\right)dx \\
\leq |||x|^{2\epsilon} u_1||_{H^{-1/2+\epsilon}(R^3)} |||x|^{-2\epsilon} g||_{H^{1/2+\epsilon}(R^3)} \tag{4.14}
\]

This proves the lemma. \(\square\)

Thus, by [6] we know that the wave equation (1.1) with initial data prescribed in (1.13) and (1.14) has a global solution. Now let \(\nu\) be the conformal transformation of \(u\),
\[
\nu(s, \tau) = e^{\tau} \sinh s u(e^{\tau} \sinh s, t_0 + e^\tau \cosh s). \tag{4.15}
\]

**Proposition 4.2** (Initial data). \(\nu(s, 0) \in H^{1+\epsilon}(R^3) \) and \(\partial_\nu \nu(s, 0) \in H^{-1-\epsilon}(R^3) \).

**Proof.** Choose \(s_0\) so that \(\cosh s_0 + t_0 = 0\). Make a linear-nonlinear decomposition of \(u\),
\[
u(t, x) = S(t)(u_0, u_1) + (u(t, x) - S(t)(u_0, u_1)) = u_l(t, x) + u_nl(t, x), \tag{4.16}
\]
and let
\[ v_1(s, \tau) = \frac{e^s \sinh s}{s} u_i(e^s \sinh s, e^s \cosh s + t_0), \tag{4.17} \]
and
\[ v_2(s, \tau) = \frac{e^s \sinh s}{s} u_0(e^s \sinh s, e^s \cosh s + t_0). \tag{4.18} \]

Recalling the free evolution of the radial wave equation in \( \mathbb{R}^3 \), since \( t_0 + \cosh s_0 = 0 \) and \( \sinh s_0 - (t_0 + \cosh s_0) = -e^{-s_0} - t_0 > -1 - t_0 \), if \( s \geq s_0 \),
\[ s v_1(s, \tau) = \frac{1}{2} \left[ u_0(t_0 + e^s)^2(s_0) + u_0(0) - e^{s_0}(0 - e^{s_0}) \right] + \frac{1}{2} \int_{-t_0-e^{s_0}}^{t_0+e^{s_0}} u_1(r) \, dr. \tag{4.19} \]

We will first show that
\[
\left( 1 - \psi \left( \frac{s}{s_0} \right) \right) \frac{1}{s} (\partial_s + \partial_t)(s v_1)(s, \tau) \big|_{\tau=0}
= \left( 1 - \psi \left( \frac{s}{s_0} \right) \right) \frac{1}{s} \left[ u_0'(t_0 + e^s)(t_0 + e^s) e^s + u_0(t_0 + e^s)e^s + e^s u_1(t_0 + e^s)(t_0 + e^s) \right]
\] \tag{4.20}

lies in \( \dot{H}^{-1/2+\varepsilon}(\mathbb{R}^3) \). By duality it suffices to estimate
\[
\int_0^\infty g(s) \left( 1 - \psi \left( \frac{s}{s_0} \right) \right) \left[ (\partial_s + \partial_t)(s v_1)(s, \tau) \big|_{\tau=0} \right] \, ds,
\tag{4.21}
\]
for some radial \( g \in \dot{H}^{1/2-\varepsilon}(\mathbb{R}^3) \). Now if \( g \in \dot{H}^\sigma(\mathbb{R}^3) \) is a radial function, \(-1 \leq \sigma \leq 1\), then by direct calculation,
\[
|x| g(|x|) = C \int_0^\infty \sin (|x| r) g(r) r \, dr,
\tag{4.22}
\]
which can be extended to an odd function lying in \( \dot{H}^\sigma(\mathbb{R}) \).

Next we will make use of a technical lemma.

**Lemma 4.3.** Suppose \( \chi(x) = \chi(|x|) \) is a radial function, \( \chi \in C_0^\infty(\mathbb{R}^3) \), and \( \chi(x) \) supported on \( \frac{1}{2} \leq |x| \leq 2 \). Then for any \(-1 < \sigma < 1\),
\[
||| \chi(2^{-k} x) f |||_{\dot{H}^\sigma(\mathbb{R}^3)} \lesssim 2^{-2k \varepsilon} ||| x |^{2\varepsilon} f |||_{L^2(\mathbb{R}^3)}.
\tag{4.23}
\]

**Proof.** First take \( 0 \leq \sigma \leq 1 \). By the product rule in [17] and the Sobolev embedding theorem, if \( \frac{1}{p} = \frac{1}{2} - \frac{\sigma}{3} \),
\[
||| \nabla^\sigma \chi(2^{-k} x) f |||_{L^2(\mathbb{R}^3)} \lesssim ||| \nabla^\sigma (|x|^{2\varepsilon} f) |||_{L^2(\mathbb{R}^3)} ||| \chi(2^{-k} x)|x|^{-2\varepsilon}|||_{L^\infty(\mathbb{R}^3)}^\sigma
+||| x |^{2\varepsilon} f |||_{L^p(\mathbb{R}^3)} ||| \nabla (\chi(2^{-k} x)|x|^{-2\varepsilon}) |||_{L^1(\mathbb{R}^3)} \lesssim 2^{-2k \varepsilon} ||| x |^{2\varepsilon} f |||_{\dot{H}^\sigma(\mathbb{R}^3)}.
\tag{4.24}
\]

The case when \(-1 \leq \sigma < 0\) follows by duality. \( \square \)
Now make the decomposition

$$1 - \psi\left(\frac{s}{s_0}\right) = \sum_{k \geq s_0} \chi_k(x), \quad (4.25)$$

where each $\chi_k(x), k \geq s_0$ is supported on $k \leq |x| \leq k + 2$.

Now for $-\frac{1}{2} \leq \sigma \leq \frac{1}{2}$ we have the scaling identity,

$$||u(\lambda x)||_{\dot{H}^\sigma(R)} = \lambda^{\sigma-1/2} ||u||_{\dot{H}^\sigma(R)}. \quad (4.26)$$

Also by the product rule and Hardy’s inequality,

$$||x|^{2\sigma} u_0(x)||_{\dot{H}^{-1/2+\varepsilon}(R)} \leq |||x|^{2\sigma} u_0||_{\dot{H}^{-1/2+\varepsilon}(R)} + |||x|^{2\sigma-1} u_0||_{\dot{H}^{-1/2+\varepsilon}(R)}. \quad (4.27)$$

Therefore, by Lemma 4.3, (4.22), and (4.26),

$$||u_k(s) u'_0(t_0 + e^s)(t_0 + e^s) e^s||_{\dot{H}^{-1/2+\varepsilon}(R)} + ||u_k(s) u_0(t_0 + e^s) e^s||_{\dot{H}^{-1/2+\varepsilon}(R)}$$

$$\lesssim e^{-c(k)} \left( |||x|^{2\sigma} u_0||_{\dot{H}^{1/2+\varepsilon}(R)} + |||x|^{2\sigma} u_1||_{\dot{H}^{-1/2+\varepsilon}(R)} \right). \quad (4.28)$$

Therefore, by (4.22),

$$\int_0^\infty \psi\left(\frac{s}{s_0}\right) g(s) \left[ (\partial_\tau + \partial_s) v_1(s, \tau) \right]_{\tau=0} s ds$$

$$\leq \sum_{k \geq s_0} 2^{-c(k)} ||g||_{\dot{H}^{1/2+\varepsilon}(R)} \left( |||x|^{2\sigma} u_0||_{\dot{H}^{1/2+\varepsilon}(R)} + |||x|^{2\sigma} u_1||_{\dot{H}^{-1/2+\varepsilon}(R)} \right)$$

$$\leq ||g||_{\dot{H}^{1/2+\varepsilon}(R)} \left( |||x|^{2\sigma} u_0||_{\dot{H}^{1/2+\varepsilon}(R)} + |||x|^{2\sigma} u_1||_{\dot{H}^{-1/2+\varepsilon}(R)} \right). \quad (4.29)$$

Next,

$$(\partial_\tau - X_0 (s v_1(s, \tau))) = u'_0(-t_0+e^{-s})(-t_0+e^{-s})e^{-s} + u_0(-t_0+e^{-s})e^{-s} + u_1(-t_0+e^{-s})(-t_0+e^{-s})e^{-s}.$$ \quad (4.30)

This time, for any $k \geq s_0$, by (4.26),

$$||u_k(s) u'_0(-t_0+e^{-s})(-t_0+e^{-s}) e^{-s}||_{\dot{H}^{-1/2}(R)} + ||u_k(s) u_0(-t_0+e^{-s}) e^{-s}||_{\dot{H}^{-1/2}(R)}$$

$$\lesssim e^{-c(k)} \left( ||u_0||_{\dot{H}^{-1/2+\varepsilon}(R)} + ||u_1||_{\dot{H}^{-1/2+\varepsilon}(R)} \right). \quad (4.31)$$

Finally, by the radial Sobolev embedding theorem,

$$||x| u_1||_{L^\infty_t L^1_x (R, |x| > |x| + R)} \leq ||u_1||_{L^\infty_t L^{1/2+\varepsilon}_x (R, |x|)} + ||u_1||_{L^\infty_t L^{1/2-\varepsilon}_x (R, |x|)}$$

$$\lesssim ||u(0)||_{\dot{H}^{1/2+\varepsilon}(R)} + ||u(0)||_{\dot{H}^{1/2-\varepsilon}(R)} + ||u_1(0)||_{\dot{H}^{-1/2+\varepsilon}(R)} + ||u_1(0)||_{\dot{H}^{-1/2-\varepsilon}(R)} \lesssim 1. \quad (4.32)$$

Therefore, by (4.17),

$$|v_1(s, 0)| \leq \frac{1}{s^2}, \quad (4.33)$$

and
\[ \int_{s_0}^{\infty} |v_1(s, 0)|^\frac{3}{s^2} ds < \infty. \] (4.34)

Therefore, we have proved
\[
||\partial_t \left( \left( 1 - \psi \left( \frac{s}{s_0} \right) \right) v_1 \right) (s, \tau) ||_{\dot{H}^{-1/2+\epsilon}(\mathbb{R}^3)} + ||\partial_t \left( \left( 1 - \psi \left( \frac{s}{s_0} \right) \right) v_1 \right) (s, \tau) ||_{\dot{H}^{-1/2+\epsilon}(\mathbb{R}^3)} \\
\leq ||u_0||_{\dot{H}^{-1/2+\epsilon}(\mathbb{R}^3)} + ||u_1||_{\dot{H}^{-1/2+\epsilon}(\mathbb{R}^3)} + |||x|^2 u_0||_{\dot{H}^{-1/2+\epsilon}(\mathbb{R}^3)} + |||x|^2 u_1||_{\dot{H}^{-1/2+\epsilon}(\mathbb{R}^3)}. \] (4.35)

Next let us consider \((1 - \psi(\frac{s}{s_0}))v_2(s, \tau)\). Now for \(s \geq s_0, t_0 + \cosh s > -1-t_0 > 0\), and \(\sinh s(1 - t_0 + \cosh s) > -t_0 - 1 > R\). By (3.29), small data arguments, finite propagation speed, and the radial Sobolev embedding theorem,
\[
|||x||_{L^\infty_{t}^\infty(\mathbb{R}^3)} \leq ||u_0||_{\dot{H}^{-1/2+\epsilon}} + ||u_1||_{\dot{H}^{-1/2+\epsilon}} + ||u_1||_{\dot{H}^{-1/2+\epsilon}}. \] (4.36)

Therefore,
\[
|||x||_{1/2+\epsilon}^2 u^3 ||_{L^1_t L^2_x(\mathbb{R}^3)} \leq \int_{\mathbb{R}} \left( \int_{R + |t|} s^{1-\epsilon} s^2 \frac{ds}{s^6} \right)^{1/2} dt \leq \int_{\mathbb{R}} \left( \frac{1}{R}\right)^{1+\epsilon/2} dt < \infty. \] (4.37)

By direct computation
\[
(\partial_t + \partial_x) \int_{t_0+e^{-t}+t}^{t_0+e^{-t}+t} u^3(r) r dr |_{t=0} = (t_0 + e^{-t} - t) u^3(t_0 + e^{-t} - t) e^t. \] (4.38)

By the Sobolev embedding theorem and duality, \(L^1_t(\mathbb{R}) \subset \dot{H}^{-1/2+\epsilon}(\mathbb{R})\), so
\[
\int_0^\infty \left( \int_{s_0}^{\infty} |(t_0 + e^{-t} - t) u^3(t_0 + e^{-t} - t) e^t|^{1-\epsilon} ds \right) dt < \infty, \] (4.39)

which implies that \((\partial_t + \partial_x) v_2 |_{t=0} \in \dot{H}^{-1/2+\epsilon}(\mathbb{R}^3)\). Also,
\[
(\partial_t - \partial_x) \int_{t_0+e^{-t}+t}^{t_0+e^{-t}+t} u^3(r) r dr |_{t=0} = (-t_0 - e^{-t} - t) u^3(-t_0 - e^{-t} + t) e^{-t}, \] (4.40)

and
\[
\int_0^\infty \left( \int_{s_0}^{\infty} |(-t_0 - e^{-t} - t) u^3(-t_0 - e^{-t} + t) e^{-t}|^{1-\epsilon} ds \right) dt < \infty. \] (4.41)

Then again by the finite propagation speed, \(|v_2(s, 0)| \leq \frac{1}{s}\) when \(s \geq s_0\), which proves \(\frac{1}{s} v_2(s, 0) \leq \frac{1}{s^2}\), and therefore
\[
\left\| \left( 1 - \psi \left( \frac{s}{s_0} \right) \right) v(s, \tau) \right\|_{\dot{H}^{1/2+\epsilon}(\mathbb{R}^3)} + \left\| \partial_t \left( 1 - \psi \left( \frac{s}{s_0} \right) \right) v(s, \tau) \right\|_{\dot{H}^{1/2+\epsilon}(\mathbb{R}^3)} \leq 1. \] (4.42)

Now we turn to \(\psi(\frac{s}{s_0}) v(s, \tau) |_{t=0}\). Making a different linear-nonlinear decomposition of \(u\),
\begin{align*}
u(t, x) &= S(t-1-t_0)(u(-t_0-1, x), u_t(-t_0-1, x)) \\
&+ \left[ u(t, x) - S(t-1-t_0)(u(-t_0-1, x), u_t(-t_0-1, x)) \right] = u(t, x) + u_h(t, x). \quad (4.43) 
\end{align*}

Again by the fundamental solution to the linear wave equation for radial data, if \( v_1 \) is given by (4.17) under the new \( u(t, x) \),
\begin{align*}
sv_1(s, \tau) &= \frac{1}{2} u(-1-t_0, e^{\tau+t} - 1)(e^{\tau+t} - 1) + \frac{1}{2} u(-1-t_0, 1-e^{\tau+t})(1-e^{\tau+t}) \\
&+ \frac{1}{2} \int_{1-e^{\tau-t}}^{e^{\tau+t}} u_t(-1-t_0, r)rdr. \quad (4.44) 
\end{align*}

By the global well-posedness result of [6],
\begin{equation}
\| u(-1 + t_0, x) \|_{H^{1/2+\epsilon}} + \| u_t(-1 + t_0, x) \|_{H^{-1/2+\epsilon}} \lesssim 1. \quad (4.45)
\end{equation}

Therefore, by direct computation,
\begin{align*}
(\partial_t + \partial_\tau)(sv_1(s, \tau))|_{\tau=0} &= \frac{1}{2} u'(-1-t_0, e^{\tau} - 1)(e^{\tau} - 1)e^\tau \\
&+ \frac{1}{2} u(-1-t_0, e^{\tau} - 1)e^\tau + \frac{1}{2} u_t(-1-t_0, e^{\tau} - 1)(e^{\tau} - 1)e^\tau, \quad (4.46)
\end{align*}

and
\begin{align*}
(\partial_t - \partial_\tau)(sv_1(s, \tau))|_{\tau=0} &= -\frac{1}{2} u'(-1-t_0, 1-e^{-\tau})(1-e^{-\tau})e^{-\tau} \\
&- \frac{1}{2} u(-1-t_0, 1-e^{-\tau})e^{-\tau} + \frac{1}{2} u_t(-1-t_0, 1-e^{-\tau})(1-e^{-\tau})e^{-\tau}. \quad (4.47)
\end{align*}

Since \( \psi(s) - (s - 1) e^{s} \) and \( \psi(s) - (s - 1) e^{-s} \) and all their derivatives are uniformly bounded above and below, by (4.45) and Hardy’s inequality,
\begin{equation}
\left\| \frac{1}{s} (\partial_t + \partial_\tau)(sv_1(s, \tau)) \right\|_{H^{-1/2}(\mathbb{R}^3)} + \left\| \frac{1}{s} (\partial_t - \partial_\tau)(sv_1(s, \tau)) \right\|_{H^{-1/2}(\mathbb{R}^3)} \lesssim 1. \quad (4.48)
\end{equation}

Also,
\begin{align*}
\frac{1}{s} v_1(s, \tau) \bigg|_{\tau=0} &= \frac{1}{2s^2} u(-1-t_0, e^{\tau} - 1)(e^{\tau} - 1) \\
&+ \frac{1}{2s^2} u(-1-t_0, 1-e^{-\tau})(1-e^{-\tau}) + \frac{1}{2s^2} \int_{1-e^{-\tau}}^{e^{-\tau}} u_t(-1-t_0, r)rdr. \quad (4.49)
\end{align*}

By Hardy’s inequality,
\begin{align*}
\left\| \frac{1}{2s} \psi \left( \frac{s}{s_0} \right) u(-1-t_0, e^{\tau} - 1) \left( \frac{e^{\tau} - 1}{s} \right) \right\|_{H^{-1/2+\epsilon}(\mathbb{R}^3)} + \left\| \frac{1}{2s} \left( 1 - \psi \left( \frac{s}{s_0} \right) \right) u(-1-t_0, 1-e^{-\tau}) \left( \frac{1-e^{-\tau}}{s} \right) \right\|_{H^{-1/2+\epsilon}(\mathbb{R}^3)} \lesssim 1. \quad (4.50)
\end{align*}

Finally, for any \( 0 < \theta < 1 \) we have a uniform bound
\begin{equation}
\left\| u_t(-1-t_0, \theta(e^{\tau} - 1) + (1-\theta)(1-e^{-\tau})) \frac{\theta(e^{\tau} - 1) + (1-\theta)(1-e^{-\tau})}{s} \right\|_{H^{-1/2+\epsilon}(\mathbb{R}^3)} \lesssim 1, \quad (4.51)
\end{equation}

and since \( \frac{\sinh s}{s} \psi(s) \) and all its derivatives are uniformly bounded, making a change of variables,
and by the radial Sobolev embedding theorem,

\[
\|P_j u\|_{L^\infty_t([t_0,0] \times A_R)} \leq \inf \left( R^{-1+\epsilon} 2^{3j} \|u\|_{L^2_t H^1([t_0,0] \times \mathbb{R}^3)}, R^{-1+2\epsilon} 2^k \|u\|_{L^\infty_t H^{1/2}([t_0,0] \times \mathbb{R}^3)} \right).
\]

Therefore,

\[
\left\| |x|^{1/2-\epsilon} u^3 \right\|_{L^1_t L^2_x([t_0,0] \times \{x:|x| \leq 2e^{t_0}+|t_0|\})} \leq 1.
\]

Now for any \( t_0 \leq t \leq e^{t_0} \),

\[
(\partial_t + \partial_s) \int_{t_0+e^{t_0}+(t-t_0+1)}^{t_0+e^{t_0}+t} ru_3^3(t,r) dr |_{\tau=0} = -e^t u_3^3(t_0+e^t+(-1+t_0+t), t)(t_0+e^t+(-1+t_0+t)).
\]

Then by (4.57),

\[
\int_{t_0}^{e^{t_0}} \left( \int_0^{2e^{t_0}} |e^t u_3^3(t_0+e^t+(-1+t_0+t), t)(t_0+e^t+(-1+t_0+t))|^{1-\epsilon} ds \right) dt \leq 1.
\]

Also,

\[
(\partial_t - \partial_s) \int_{t_0+e^{t_0}+(t_0-1-t)}^{t_0+e^{t_0}+(t_0-1-t)} ru_3^3(t,r) dr |_{\tau=0} = -e^{-t} u_3^3(-t_0+e^{-t}+(t_0-1-t), t)(-t_0+e^{-t}+(t_0-1-t)),
\]

so again by (4.57),
Then 5. Multi-linear estimates

Recall the spatial partition of unity in (4.3) and (4.4). By Hölder's inequality and the Sobolev embedding theorem,

\[
\int_{t_0}^{e^0} \left( \int_0^{2\delta_0} \left| e^{-s} u^3 \left( -t_0 + e^{-s} + (t_0 - 1 - t), t \right) \left( -t_0 + e^{-s} + (t_0 - 1 - t) \right) \right|^{\frac{1}{1-\epsilon}} \, ds \right)^{1-\epsilon} \, dt \leq 1. \tag{4.61}
\]

Finally, as in (4.51)–(4.53), since for any 0 < \theta < 1,

\[
\int_{t_0}^{e^0} \left( \int_0^{2\delta_0} \left| e^{e^i} \left( \theta(t_0 + e^i + (1 + t_0 + t)) + (1-\theta)(-t_0 + e^{-s} + (t_0 - 1 - t)), t \right) \times (\theta(t_0 + e^i + (1 + t_0 + t)) + (1-\theta)(-t_0 + e^{-s} + (t_0 - 1 - t)))^{\frac{1}{1-\epsilon}} \, dt \right) \leq 1,
\]

for any 0 < \theta < 1, we have proved

\[
\left\| \partial_t \left( \frac{s}{s_0} \psi \right) v_2(s, \tau) \right\|_{H^{-1/2-\epsilon}(\mathbb{R}^3)} + \left\| \partial_t \left( \frac{s}{s_0} \psi \right) v_2(s, \tau) \right\|_{H^{-1/2-\epsilon}(\mathbb{R}^3)} \leq 1. \tag{4.63}
\]

This finally proves Proposition 4.2.

\[\square\]

5. Multi-linear estimates

The proofs of Propositions 3.2–3.4 will utilize several multi-linear estimates, which will be proved in this section. All implicit constants in this section may depend on the \(\epsilon > 0\) in Theorem 1.3. To simplify notation let \(s = \frac{1}{2} + \epsilon\). \(I\) is the operator defined in (3.6).

**Theorem 5.1.** Suppose \(|M(x)| \leq e^{-c|x|}\) for some \(c > 0\), and \(f, g, h\) are radial functions. Then

\[
\|I(M(x)f)g\|_{L^1_t(J \times \mathbb{R}^3)} + \|M(x)f(g)\|_{L^1_t(J \times \mathbb{R}^3)} \leq 2k_0/2 \|f\|_{L^1_{k_0}(J \times \mathbb{R}^3)} \|g\|_{L^1_{k_0}(J \times \mathbb{R}^3)} \|h\|_{L^1_{k_0}(J \times \mathbb{R}^3)}. \tag{5.1}
\]

Also,

\[
\|I(M(x)f)g\|_{L^1_{k_0}(J \times \{x:|x| > 2^{k_0-1}\})} + \|M(x)f(g)\|_{L^1_{k_0}(J \times \{x:|x| > 2^{k_0-1}\})} \leq |k_0| \|f\|_{L^1_{k_0}(J \times \mathbb{R}^3)} \|g\|_{L^1_{k_0}(J \times \mathbb{R}^3)}. \tag{5.2}
\]

**Proof.** Recall the spatial partition of unity in (4.3) and (4.4). By Hölder’s inequality and the Sobolev embedding theorem,

\[
\|\psi(2^{k_0}x)M(x)f\|_{L^1_{k_0}(J \times \mathbb{R}^3)} \leq \|f\|_{L^2_{k_0}(J \times \mathbb{R}^3)} \|g\|_{L^2_{k_0}(J \times \mathbb{R}^3)} \|P_{\leq k_0} h\|_{L^1_{k_0}(J \times \mathbb{R}^3)} \leq 2k_0/2 \|f\|_{L^1_{k_0}(J \times \mathbb{R}^3)} \|g\|_{L^1_{k_0}(J \times \mathbb{R}^3)} \|h\|_{L^1_{k_0}(J \times \mathbb{R}^3)}. \tag{5.3}
\]

Also, by the radial Sobolev embedding theorem,

\[
\sup_{k \geq -k_0} \|\psi(x)M(x)f\|_{L^1_{k_0}(J \times \mathbb{R}^3)} \leq \sup_{k \geq -k_0} \|f\|_{L^2_{k_0}(J \times A_1)} \|g\|_{L^2_{k_0}(J \times A_1)} \|P_{\leq k_0} h\|_{L^1_{k_0}(J \times A_1)} \leq 2k_0/2 \|f\|_{L^1_{k_0}(J \times \mathbb{R}^3)} \|g\|_{L^1_{k_0}(J \times \mathbb{R}^3)} \|h\|_{L^1_{k_0}(J \times \mathbb{R}^3)}. \tag{5.4}
\]
Therefore, since $|M(x)| \leq e^{-c|x|}$,
\[
\sum_{j \geq -k_0} \|\xi_j(x)M(x)\hat{f}(P_{\leq k_0}h)\|_{L^1_t(L^2_x)} 
\leq |k_0| 2^{5/2} \|f\|_{L^2_{t_0}L^2_x(J \times \mathbb{R}^d)} \|g\|_{L^2_{t_0}L^2_x(J \times \mathbb{R}^d)} \|h\|_{L^\infty_{t_0}L^2_x(J \times \mathbb{R}^d)}. \tag{5.5}
\]

$\||I(M(x)\hat{f})(P_{\leq k_0}h)\|_{L^1_{t_j}}$ may be estimated with a similar argument combined with the fact that the Littlewood-Paley kernels are rapidly decreasing. Indeed, by (2.5), if $I(x)$ is the kernel of $I$, then for any $M > 0$,
\[
I(x) \leq \frac{1}{(2^{k_0} |x|)^M}. \tag{5.6}
\]

So if $j, l \geq -k_0$, then by the radial Sobolev embedding theorem,
\[
\||\xi_j(x)I(\xi_l(x)M(x)\hat{f})(P_{\leq k_0}h)\|_{L^1_{t_j}(L^2_x)} 
\leq 2^{-10j-l} \||\xi_j(x)M(x)\hat{f}\|_{L^1_{t_j}(L^2_x)} \||\xi_l(x)(P_{\leq k_0}h)\|_{L^\infty_{t_j}L^2_x(J \times \mathbb{R}^d)} \tag{5.7}
\]

Therefore, by (5.3), (5.5), (5.7), and Young’s inequality, the proof of (5.1) is complete.

The proof of (5.2) is similar. Because $|M(x)| \leq e^{-c|x|}$,
\[
\|((1-\psi(2^{k_0}x)) \frac{1}{|x|} M(x)\hat{f})\|_{L^1_{t_x}(L^2_x)} \leq \sum_{j \geq -k_0} 2^{-j} \||\xi_j(x)M(x)\hat{f}\|_{L^1_{t_x}(L^2_x)} \tag{5.8}
\]

Also, since $\frac{1}{|x|} \leq 2^{k_0}$ when $|x| \geq 2^{-k_0}$,
\[
\|((1-\psi(2^{k_0}x)) \frac{1}{|x|} I(\psi(2^{k_0}x)M(x)\hat{f}))\|_{L^1_{t_x}(L^2_x)} \approx 2^{k_0} \|f\|_{L^2_{t_0}L^2_x(J \times B_{1-k_0})} \|g\|_{L^2_{t_0}L^2_x(J \times B_{1-k_0})}. \tag{5.9}
\]

Also, by (5.6), (5.8), and Young’s inequality,
\[
\sum_{j, l \geq -k_0} \|\xi_j(x) \frac{1}{|x|} I(\xi_l(x)M(x)\hat{f})\|_{L^1_{t_x}(L^2_x)} \leq \sum_{j, l \geq -k_0} 2^{-10j-l} 2^{-l} \||\xi_j(x)M(x)\hat{f}\|_{L^1_{t_x}(L^2_x)} \tag{5.10}
\]

This concludes the proof of (5.2). 
\[\square\]

**Theorem 5.2.** Suppose that $M(x)$ is some function satisfying $|M(x)| \leq e^{-c|x|}$ for some $c > 0$. Then
\[
\|I(M(x)\hat{f}h)\|_{L^1_{t_x}(L^2_x)} \leq |k_0| \|g\|_{L^2_{t_0}L^2_x(J \times \mathbb{R}^d)} \|f\|_{L^2_{t_0}L^2_x(J \times \mathbb{R}^d)} \tag{5.11}
\]

\[
\times \left(2^{-k_0/2} \|h\|_{L^\infty_{t_0}L^2_x(J \times \mathbb{R}^d)} + \|x\|^{1/2} \|h\|_{L^\infty_{t_0}L^2_x(J \times \mathbb{R}^d)} \right).
\]

**Proof.** Again by (4.4),

$$I(M(x)gh)f = \psi(2^{k_0}x)I(\psi(2^{k_0}x)M(x)gh)f + \sum_{l \geq -k_0} \chi_l(x)I(\psi(2^{k_0}x)M(x)gh)f$$

$$+ \sum_{j \geq -k_0} \psi(2^{k_0}x)I(\chi_j(x)M(x)gh)f + \sum_{j,l \geq -k_0} \chi_l(x)I(\chi_j(x)M(x)gh)f.$$  \hspace{0.5cm} (5.12)

By Hölder’s inequality and the support properties of \(\psi(Nx)\),

$$||\psi(Nx)I(\psi(2^{k_0}x)M(x)gh)f||_{l^1_{x}(J \times R^3)} \leq ||f||_{l^2_{x}(J \times A_{-k_0})} ||g||_{l^2_{x}(J \times R^3)} ||h||_{l^\infty_{x}(J \times R^3)}$$

$$\leq 2^{-k_0/2} ||f||_{l^\infty_{x}(J \times R^3)} ||g||_{l^2_{x}(J \times R^3)} ||h||_{l^\infty_{x}(J \times R^3)}.$$  \hspace{0.5cm} (5.13)

Also,

$$\sum_{j \geq -k_0} ||\psi(2^{k_0}x)I(\chi_j(x)M(x)gh)f||_{l^1_{x}(J \times R^3)}$$

$$\leq \sum_{j \geq -k_0} ||f||_{l^2_{x}(J \times B_{-k_0})} ||g||_{l^2_{x}(J \times R^3)} ||\chi_j(x)h||_{l^\infty_{x}(J \times R^3)}$$

$$\leq 2^{-k_0/2} ||f||_{l^\infty_{x}(J \times R^3)} ||g||_{l^2_{x}(J \times R^3)} \sum_{j \geq -k_0} ||\chi_j(x)h||_{l^\infty_{x}(J \times R^3)}$$

$$\leq ||f||_{l^\infty_{x}(J \times R^3)} ||g||_{l^2_{x}(J \times R^3)} ||x|^{1/2}h||_{l^\infty_{x}(J \times R^3)}.$$  \hspace{0.5cm} (5.14)

Next, by (5.6),

$$\sum_{l \geq -k_0} ||\chi_l(x)I(\psi(2^{k_0}x)M(x)gh)f||_{l^1_{x}(J \times R^3)}$$

$$\leq \sum_{l \geq -k_0} 2^{-10(l+k_0)} ||f||_{l^2_{x}(J \times A_{l})} ||g||_{l^2_{x}(J \times R^3)} ||h||_{l^\infty_{x}(J \times R^3)}$$

$$\leq \sum_{l \geq -k_0} 2^{-10(l+k_0)} 2^{l/2} ||f||_{l^\infty_{x}(J \times R^3)} ||g||_{l^2_{x}(J \times R^3)} ||h||_{l^\infty_{x}(J \times R^3)}$$

$$\leq 2^{-k_0/2} ||f||_{l^\infty_{x}(J \times R^3)} ||g||_{l^2_{x}(J \times R^3)} ||h||_{l^\infty_{x}(J \times R^3)}.$$  \hspace{0.5cm} (5.15)

Finally by (5.6) and \(|M(x)| \leq e^{-c|x|}\),

$$\sum_{j,l \geq -k_0} ||\chi_l(x)I(\chi_j(x)M(x)gh)f||_{l^1_{x}(J \times R^3)}$$

$$\leq \sum_{j,l > 0} 2^{-10(j-l)} ||f||_{l^2_{x}(J \times A_{l})} ||g||_{l^2_{x}(J \times R^3)} ||\chi_j(x)M(x)h||_{l^\infty_{x}(J \times R^3)}$$

$$\leq \sum_{j,l > 0} 2^{-10(j-l)} 2^{l/2-j/2} e^{-2l} ||f||_{l^\infty_{x}(J \times R^3)} ||g||_{l^2_{x}(J \times R^3)} ||x|^{1/2}h||_{l^\infty_{x}(J \times R^3)}$$

$$\leq |k_0||f||_{l^\infty_{x}(J \times R^3)} ||g||_{l^2_{x}(J \times R^3)} ||x|^{1/2}h||_{l^\infty_{x}(J \times R^3)}.$$  \hspace{0.5cm} (5.16)

This proves the theorem. \(\square\)
**Theorem 5.3.** If $M(x)$ is a function satisfying $|M(x)| \leq e^{-c|x|}$, then
\[
\|I(M(x)(P_{\geq k_0-\gamma}f)^3)\|_{L^2_{1/2}(\mathbb{R}^1)} \lesssim 2^{k_0(1-s-\epsilon)} \left( \sup_{j > k_0 - 7} 2^{-j(1-s)} S(j,f) \right)^3 \tag{5.17}
\]
\[
+ 2^{k_0(1-s-\epsilon)} \left( \sup_{j > k_0 - 7} 2^{-j(1-s)} S(j,f) \right)^2 \|f\|_{L^2_{1/2}H'(\mathbb{R}^1)}.
\]

**Proof.** By the Sobolev embedding theorem,
\[
\|P_{\leq k_0} \left( M(x)(P_{\geq k_0-\gamma}f)^3 \right)\|_{L^2_{1/2}(\mathbb{R}^1)} \lesssim 2^{3k_0/2} \|M(x)(P_{\geq k_0-\gamma}f)^3\|_{L^2_{1,1}(\mathbb{R}^1)} \tag{5.18}
\]
\[
\lesssim 2^{3k_0/2} \sum_{k_0 - 7 \leq j_1 \leq j_2 \leq j_3} \|M(x)(P_{j_1}f)(P_{j_2}f)(P_{j_3}f)\|_{L^2_{1,1}(\mathbb{R}^1)} \tag{5.19}
\]
Now, by the radial Sobolev embedding theorem $|||x|^{-1}f|||_{L^\infty} \leq ||f||_{H'}$ and Bernstein’s inequality, for any $l \in \mathbb{Z},$
\[
|||P_{j_2}f|||_{L^2_{2,1}(\mathbb{R}^1)} \lesssim |||P_{j_2}f|||_{L^2_{2,1}(\mathbb{R}^1)} \lesssim 2^k |||P_{j_2}f|||_{L^2_{2,1}(\mathbb{R}^1)} |||P_{j_2}f|||_{L^2_{2,1}(\mathbb{R}^1)} \tag{5.20}
\]
Now by Hölder’s inequality, for any $l \in \mathbb{Z},$
\[
2^{-l/2}|||f|||_{L^2_{2,1}(\mathbb{R}^1)} \leq 2^l |||f|||_{L^2_{2,1}(\mathbb{R}^1)}, \tag{5.21}
\]
so in particular, when $l \leq -j,$
\[
2^{-l/2}|||f|||_{L^2_{2,1}(\mathbb{R}^1)} \lesssim 2^{-j} S(j,f). \tag{5.22}
\]
Combining this fact with (5.20) and the fact that $|M(x)| \lesssim e^{-cR/2}$ for $\frac{R}{2} \leq |x| \leq R,$
\[
(5.19) \lesssim 2^{3k_0/2} \sum_{k_0 - 7 \leq j_1 \leq j_2 \leq j_3} \|P_{j_1}f\|_{L^2_{1,1}H'(\mathbb{R}^1)} \|P_{j_2}f\|_{L^2_{1,1}(\mathbb{R}^1)} \|P_{j_3}f\|_{L^2_{1,1}(\mathbb{R}^1)} \tag{5.23}
\]
\[
\lesssim \left( \sup_{j > k_0 - 7} 2^{-j(1-s)} S(j,f) \right)^2 \|f\|_{L^2_{1,1}H'(\mathbb{R}^1)} \cdot 2^{3k_0/2} \sum_{k_0 - 7 \leq j_1 \leq j_2 \leq j_3} 2^{-j_2} 2^{-j_3} \tag{5.23}
\]
\[
\lesssim 2^{k_0(1-s-\epsilon)} \left( \sup_{j > k_0 - 7} 2^{-j(1-s)} S(j,f) \right)^2 \|f\|_{L^2_{1/2}H'(\mathbb{R}^1)} \tag{5.23}
\]
When estimating
\[
\sum_{j > k_0} 2^{k_0(1-s-\epsilon)} 2^{-j(1-s)} |||P_{j}(M(x)(P_{\geq k_0-\gamma}f)^3)\|_{L^2_{1/2}(\mathbb{R}^1)} \tag{5.24}
\]
the terms in which $j_2 \leq j$ and the terms in which $j_2 \geq j$ will be analyzed in two different manners, where once again $j_1 \leq j_2 \leq j_3$. Again by the Sobolev embedding theorem, (5.20), (5.22), and $|M(x)| \lesssim e^{-c|x|}$,
\[
\sum_{k_0 < j \leq j_0} \sum_{k_0 - 7 < j \leq j_0} 2^{k_0(1-s)} 2^{-j(1-s)} \|P_j (M(x) (P_j f)(P_j f)(P_j f))\|_{L^2_t L^2_x (\mathbb{R}^3)} \\
\leq \sum_{k_0 < j \leq j_0} \sum_{k_0 - 7 < j \leq j_0} 2^{k_0(1-s)} 2^{(j(1/2+s))} \|P_j f\|_{L^\infty_t H^s (\mathbb{R}^3)} \|P_j f\|_{L^\infty_t L^2_x (\mathbb{R}^3)} \|P_j f\|_{L^\infty_t L^2_x (\mathbb{R}^3)} \\
\leq 2^{k_0(1-s-c)} \left( \sup_{j > k_0 - 7} 2^{-j(1-s)} S(j, f) \right)^2 \|f\|_{L^\infty_t H^4 (\mathbb{R}^3)}. 
\]

(5.25)

Meanwhile,
\[
\sum_{k_0 - 7 \leq j \leq k_0} \sum_{k_0 - 7 \leq j \leq j_0} 2^{k_0(1-s)} 2^{-j(1-s)} \|P_j f\|_{L^2_t L^2_x (\mathbb{R}^3)} \|P_j f\|_{L^2_t L^2_x (\mathbb{R}^3)} \|P_j f\|_{L^2_t L^2_x (\mathbb{R}^3)} \\
\leq \left( \sup_{j \geq k_0 - 7} 2^{j} \|P_j f\|_{L^2_t L^2_x (\mathbb{R}^3)} \right)^2 \\
\times \sum_{k_0 - 7 \leq j \leq j_0} \sum_{k_0 - 7 \leq j \leq j_0} 2^{-j(1-s)} 2^{-2j} 2^{j(1-s)} \left( \sup_{j > k_0 - 7} 2^{-j(1-s)} \|P_j f\|_{L^\infty_t L^2_x (\mathbb{R}^3)} \right) \\
\leq 2^{k_0(1-s-2e)} \left( \sup_{j > k_0 - 7} 2^{-j(1-s)} S(j, f) \right)^3.
\]

This proves the theorem. \[\square\]

**Proposition 5.4** (Bilinear estimate). Suppose \(|M(x)| \leq e^{-c|x|}.\) Then
\[
\|M(x)(P_j f)g\|_{L^2_t L^2_x (\mathbb{R}^3)} \leq |k_0| 2^{-j} S(j, f) \left( N^{-1/2} \|g\|_{L^\infty_t L^2_x (\mathbb{R}^3)} + \|g\|_{L^\infty_t H^4 (\mathbb{R}^3)} \right).
\]

(5.27)

**Proof.** First, by Hölder’s inequality,
\[
\|(P_j f)g\|_{L^2_t L^2_x (\mathbb{R}^3)} \leq 2^{-j} \|P_j f\|_{L^2_t L^2_x (\mathbb{R}^3)} \|g\|_{L^\infty_t L^1_x (\mathbb{R}^3)} \leq 2^{-j} S(j, f) \|g\|_{L^\infty_t H^1 (\mathbb{R}^3)}. 
\]

(5.28)

Next, for any \(-j \leq l \leq 1-k_0,\)
\[
\|(P_j f)g\|_{L^2_t L^2_x (\mathbb{R}^3)} \leq 2^{l/2} \|P_j f\|_{L^2_t L^2_x (\mathbb{R}^3)} \|g\|_{L^\infty_t L^2_x (\mathbb{R}^3)} \leq 2^{-j} 2^{l/2} S(j, f) \|g\|_{L^\infty_t L^2_x (\mathbb{R}^3)}. 
\]

(5.29)

Summing up,
\[
\sum_{-j \leq l \leq 1-k_0} 2^{-j} 2^{l/2} S(j, f) \|g\|_{L^\infty_t L^2_x (\mathbb{R}^3)} \leq 2^{-k_0/2} 2^{-j} S(j, f) \|g\|_{L^\infty_t L^2_x (\mathbb{R}^3)}. 
\]

(5.30)

Now by the radial Sobolev embedding theorem,
\[
\sum_{-k_0 \leq l \leq 0} \|g\|_{L^2_t L^2_x (\mathbb{R}^3)} \leq |k_0| \sup_{-k_0 \leq j \leq 0} \left( 2^{-j/2} \|f\|_{L^2_t L^2_x (\mathbb{R}^3)} \right) \left( 2^{j/2} \|g\|_{L^\infty_t L^2_x (\mathbb{R}^3)} \right) \\
\leq |k_0| \|f\|_{L^\infty_t L^2_x (\mathbb{R}^3)} \|g\|_{L^\infty_t H^4 (\mathbb{R}^3)}. 
\]

(5.31)

Finally, by the radial Sobolev embedding combined with the fact that \(|M(x)| \leq e^{-c|x|},\) for some constant \(c > 0,\)
\[
\sum_{j \geq 0} ||M(x)g||_{L^2_{t,x}(I \times A_j)} \leq \sum_{j \geq 0} e^{-c2^j} \left( \sup_j 2^{-j/2} ||f||_{L^2_{t,x}(I \times A_j)} \right) \left( 2^{j/2} ||g||_{L^\infty_{t,x}(I \times A_j)} \right)
\]

(5.32)

This proves the proposition.

We conclude this section with a tri-linear estimate close to the origin.

**Theorem 5.5.** If \( |M(x)| \leq e^{-c|x|} \) for some \( c > 0 \) and \( \psi(2^k x) \) is as in (4.4),

\[
||I(M(x)fg)\psi(2^k x)\frac{1}{|x|^{3/4}}||_{L^{4/3}_{t,x}(I \times \mathbb{R}^d)} \leq 2^{-k_0^2} \left( \sup_j 2^{-j(1-s)} S(j,f) \right) \frac{1}{|x|^{3/4}} \left( \sum_{j > k_0} \right) \left( \sum_{l \geq -k_0} \left| P_{\leq k_0} (M(x)) \psi(2^k x) \frac{1}{|x|^{3/4}} \right|_{L^{4/3}_{t,x}(I \times \mathbb{R}^d)} \right)
\]

(5.33)

**Proof.** Combining the Littlewood-Paley decomposition, (3.6), and (4.4),

\[
||I(M(x)fg)\psi(2^k x)\frac{1}{|x|^{3/4}}||_{L^{4/3}_{t,x}(I \times \mathbb{R}^d)} \leq \left| P_{\leq k_0} (M(x))fg \right| \frac{1}{|x|^{3/4}} \psi(2^k x) \frac{1}{|x|^{3/4}} \left| P_{\geq k_0} (M(x)) \psi(2^k x) \frac{1}{|x|^{3/4}} \right|_{L^{4/3}_{t,x}(I \times \mathbb{R}^d)}
\]

(5.34)

\[
\leq \left| P_{\leq k_0} (M(x)) \psi(2^k x) \frac{1}{|x|^{3/4}} \right|_{L^{4/3}_{t,x}(I \times \mathbb{R}^d)} \frac{1}{|x|^{3/4}} \psi(2^k x) \frac{1}{|x|^{3/4}} \left| P_{\geq k_0} (M(x)) \psi(2^k x) \frac{1}{|x|^{3/4}} \right|_{L^{4/3}_{t,x}(I \times \mathbb{R}^d)}
\]

(5.35)

\[
\leq \sum_{j_2 \leq k_0} \left| P_{\leq k_0} (M(x)) \psi(2^k x) \left( P_{j_2} f \right) \right| \frac{1}{|x|^{3/4}} \psi(2^k x) \frac{1}{|x|^{3/4}} \left| P_{\geq k_0} (M(x)) \psi(2^k x) \frac{1}{|x|^{3/4}} \right|_{L^{4/3}_{t,x}(I \times \mathbb{R}^d)}
\]

(5.36)

\[
+ \sum_{j_2 > k_0} \left| P_{\leq k_0} (M(x)) \psi(2^k x) \left( P_{j_2} f \right) \right| \frac{1}{|x|^{3/4}} \psi(2^k x) \frac{1}{|x|^{3/4}} \left| P_{\geq k_0} (M(x)) \psi(2^k x) \frac{1}{|x|^{3/4}} \right|_{L^{4/3}_{t,x}(I \times \mathbb{R}^d)}
\]

(5.37)
Therefore, (5.33) is bounded by the right hand side of (5.33).

Next, by Hölder’s inequality,

\[
\|P_{\leq k_0} \left( M(x) \psi \left( 2^{k_0} x \right) \right) \|_{L^{4/3}_x (\mathbb{R}^3)} \lesssim 2^{-3k_0/4} \|P_{\leq k_0} \|_{L^1_x (\mathbb{R}^3)} \|g\|_{L^4_x (\mathbb{R}^3)} \|h\|_{L^6_x (\mathbb{R}^3)} (5.40)
\]

so then

\[
\sum_{j_2 \geq k_0} (5.40) \lesssim 2^{-k_0} \left( \sup_{j_2} 2^{j_2(s-1)} S(j_2,f) \right) \|g^2\|_{L^4_{k_0} L^1_{j_0} (\mathbb{R})} \|h\|_{L^6_{j_0} (\mathbb{R})}. (5.41)
\]

Next, by Hölder’s inequality and the Sobolev embedding theorem,

\[
\|P_{\leq k_0} \left( M(x) \psi \left( 2^{k_0} x \right) \right) \|_{L^{4/3}_x (\mathbb{R}^3)} \lesssim 2^{3k_0/4} \|P_{j_2} f\|_{L^2_x (\mathbb{R}^3)} \|g\|_{L^4_x (\mathbb{R}^3)} \|h\|_{L^6_x (\mathbb{R}^3)} (5.42)
\]

so

\[
\sum_{j_2 > k_0} (5.42) \lesssim 2^{-k_0} \left( \sup_{j_2 > k_0} 2^{j_2(s-1)} S(j_2,f) \right) \|g^2\|_{L^4_{k_0} L^1_{j_0} (\mathbb{R})} \|h\|_{L^6_{j_0} (\mathbb{R})}. (5.43)
\]

Therefore, (5.36) is bounded by the right hand side of (5.33).

Next, by Hölder’s inequality and (4.4),

\[
\|P_{\leq k_0} \left( Z_i(x) M(x) \psi \left( 2^{k_0} x \right) \right) \|_{L^{4/3}_x (\mathbb{R}^3)} \lesssim 2^{-10(j+k_0)} 2^{-3k_0/4} \|P_{j_2} f\|_{L^2_x (\mathbb{R}^3)} \|g\|_{L^4_x (\mathbb{R}^3)} \|h\|_{L^6_x (\mathbb{R}^3)} (5.44)
\]

so
\[
\sum_{l \geq -k_0} \sum_{j_2 \leq k_0} (5.44) \leq 2^{-k_0s} \left( \sup_{j_2} 2^{j_2(s-1)} S(j_2, f) \right) \|g^2\|^{1/2}_{L^0_k L^1_x(J \times R^d)} \|h\|_{L^\infty_k(J \times R^d)}. \quad (5.45)
\]

Also, by Hölder’s inequality, the Sobolev embedding theorem, and (4.4),
\[
\|P_{\leq k_0} (x^l M(x) (P_{j_2} f) g h) \|_{L^0_k L^1_x(J \times R^d)} \leq \frac{1}{|x|^3/4} \psi(2^k x) \left( \sup_{j_2} 2^{j_2(s-1)} S(j_2, f) \right) \|g^2\|^{1/2}_{L^0_k L^1_x(J \times R^d)} \|h\|_{L^\infty_k(J \times R^d)} \quad (5.46)
\]

so
\[
\sum_{l \geq -k_0} \sum_{j_2 > k_0} (5.46) \leq 2^{-k_0s} \left( \sup_{j_2} 2^{j_2(s-1)} S(j_2, f) \right) \|g^2\|^{1/2}_{L^0_k L^1_x(J \times R^d)} \|h\|_{L^\infty_k(J \times R^d)}. \quad (5.47)
\]

Thus (5.37) is bounded by the right hand side of (5.33).

Next, by Hölder’s inequality and the Sobolev embedding theorem,
\[
2^{k_0(1-s)} 2^{-j(1-s)} \|P_{j} (M(x) \psi(2^k x) (P_{j_2} f) g h) \|_{L^0_k L^1_x(J \times R^d)} \leq 2^{k_0(3-\delta)} 2^{-j(1-s)} \|P_{j_2} f\|^{3/4}_{L^0_k L^4_x(J \times R^d)} \|P_{j_2} f\|^{1/4}_{L^0_k L^4_x(J \times R^d)} \|g\|^{1/4}_{L^0_k L^4_x(J \times R^d)} \|h\|_{L^\infty_k(J \times R^d)} \quad (5.48)
\]

and therefore,
\[
\sum_{j > k_0} \sum_{j_2 \leq j} (5.48) \leq 2^{-k_0s} \left( \sup_{j_2} 2^{j_2(s-1)} S(j_2, f) \right) \|g^2\|^{1/2}_{L^0_k L^1_x(J \times R^d)} \|h\|_{L^\infty_k(J \times R^d)}. \quad (5.49)
\]

Also by the Sobolev embedding theorem and Hölder’s inequality, when \( j_2 \geq j \geq k_0, \)
\[
2^{k_0(1-s)} 2^{-j(1-s)} \|P_{j} (M(x) \psi(2^k x) (P_{j_2} f) g h) \|_{L^0_k L^1_x(J \times R^d)} \leq 2^{k_0(\delta-\delta)} 2^{-j(1-s)} \|P_{j_2} f\|^{3/6}_{L^0_k L^4_x(J \times R^d)} \|g\|^{1/6}_{L^0_k L^4_x(J \times R^d)} \|h\|_{L^\infty_k(J \times R^d)} \quad (5.50)
\]

so then
\[
\sum_{j > k_0} \sum_{j_2 > j} (5.50) \leq 2^{-k_0s} \left( \sup_{j_2} 2^{-j_2(s-1)} S(j_2, f) \right) \|g^2\|^{1/2}_{L^0_k L^1_x(J \times R^d)} \|h\|_{L^\infty_k(J \times R^d)}. \quad (5.51)
\]

Therefore, (5.38) is bounded by the right hand side of (5.33).
Finally, by Hölder’s inequality and (4.4),

\[
2^{k_0(1-s)}2^{-j(1-s)}\|P_j(M(x)\xi_l(x)(P_{ji}f)gh)\|_{L_t^4_x(l, R^3)}\|L_t^{3/2}(J \times R^3)} \leq 2^{-10(k_0+1)2^{k_0(\frac{2}{3}-s)}2^{-j(1-s)}}\|P_{j}f\|_{L_t^{3/2}(J \times R^3)}\|P_{j}f\|_{L_t^{1/2}(J \times R^3)}\|g\|_{L_t^1_x(J \times R^3)}\|h\|_{L_t^\infty_x(J \times R^3)}
\]

\[
\leq 2^{-10(k_0+1)2^{3l/8}2^{k_0(\frac{2}{3}-s)}2^{-j(1-s)}2^{-j/4}}\left(\sup_{j_2} 2^{-j(1-s)}S(j_2, f)\right)\|g\|_{L_t^{1/2}}\|L_t^{\infty}(J \times R^3)}\|h\|_{L_t^{\infty}(J \times R^3)}\]  

(5.52)

and therefore,

\[
\sum_{j > k_0} \sum_{j_2 \leq j} \sum_{l \geq -k_0} (5.52) \leq 2^{-k_0s} \left(\sup_{j_2} 2^{-j(1-s)}S(j_2, f)\right)\|g\|_{L_t^{1/2}}\|L_t^{\infty}(J \times R^3)}\|h\|_{L_t^{\infty}(J \times R^3)}\].

(5.53)

Also by the Sobolev embedding theorem, Hölder’s inequality, \(j_2 \geq j \geq k_0\), and (4.4),

\[
2^{k_0(1-s)}2^{-j(1-s)}\|P_j(M(x)\xi_l(x)(P_{ji}f)gh)\|_{L_t^4_x(l, R^3)}\|L_t^{3/2}(J \times R^3)} \leq 2^{-10(k_0+1)2^{k_0(\frac{2}{3}-s)}2^{-j(1-s)}}2^{5j/6}\|P_{j}f\|_{L_t^{3/2}(J \times R^3)}\|g\|_{L_t^1_x(J \times R^3)}\|h\|_{L_t^\infty_x(J \times R^3)}
\]

\[
\leq 2^{-10(k_0+1)2^{3l/4}2^{k_0(\frac{2}{3}-s)}2^{-j(1-s)}}2^{5j/6}\|P_{j}f\|_{L_t^{3/2}(J \times R^3)}\|g\|_{L_t^{1/2}}\|L_t^{\infty}(J \times R^3)}\|h\|_{L_t^{\infty}(J \times R^3)}\]

\[
\leq 2^{-10(k_0+1)2^{3l/4}2^{k_0(\frac{2}{3}-s)}2^{-j(1-s)}}2^{-j/4}\left(\sup_{j_2} 2^{j(1-s)}S(j_2, f)\right)\|g\|_{L_t^{1/2}}\|L_t^{\infty}(J \times R^3)}\|h\|_{L_t^{\infty}(J \times R^3)}\]

(5.54)

so then

\[
\sum_{j > k_0} \sum_{j_2 \leq j} \sum_{l \geq -k_0} (5.54) \leq 2^{-k_0s} \left(\sup_{j_2} 2^{j(1-s)}S(j_2, f)\right)\|g\|_{L_t^{1/2}}\|L_t^{\infty}(J \times R^3)}\|h\|_{L_t^{\infty}(J \times R^3)}\].

(5.55)

Therefore, (5.39) is bounded by the right hand side of (5.33). This completes the proof of the theorem.

\[\square\]

6. Long time Strichartz estimates

In this section, we prove Proposition 3.2. Recalling the proposition,

**Proposition 6.1** (Long time Strichartz estimates). Suppose \(N = 2^{k_0}, s = \frac{1}{2} + \epsilon\), \(J\) is an interval on which

\[
\int_J \int \phi(x)(Iv(x, t))^4 dx \leq C_1 N^{2(1-s)},
\]

(6.1)

and \(E(Iv)(t) \leq 2C_0(A, \epsilon)N^{2(1-s)}\) for all \(t \in J\). Then for \(k_0(C_0, A, \epsilon)\) sufficiently large,

\[
\sup_{j \geq k_0} 2^{-j(1-s)}S(j, v) \leq C_0(A, \epsilon).
\]

(6.2)

**Proof.** To prove this we use an induction on frequency estimate combined with a bootstrap argument. Let \(J \subset J\) be an interval such that
\[ \|P_{> k_0} v\|_{L^4_x(L^2_t(x \times \mathbb{R}^3))} \leq N^{-\epsilon/2}. \] (6.3)

By the dominated convergence theorem combined with Corollary 3.7, such an interval exists. We then make a bootstrap argument and show that this implies
\[ \|P_{> k_0} v\|_{L^4_x(L^2_t(x \times \mathbb{R}^3))} \leq N^{-\epsilon}, \] (6.4)
which then by the dominated convergence theorem shows that \( I \subset J \) can be extended to all of \( J \).

By Theorem 2.5,
\[
S(j, v) \leq 2^j \|P_j v_0\|_{L^2_t(L^2_x)} + \|P_j v_1\|_{L^2_t(L^2_x)} + \|\phi(x)(P_{> j-7} v)(P_{\leq k_0} v)(P_{\leq j} v)\|_{L^2_t(L^2_x)}(x \times \mathbb{R}^3)
+ 2 \|\phi(x)(P_{> j-10} v)(P_{\leq k_0} v)(P_{> j} v)\|_{L^2_t(L^2_x)} + 2^{j/2} \|\phi(x)(P_{> j-7} v)(P_{> k_0} v)^2\|_{L^4_t(L^2_x)(x \times \mathbb{R}^3)}
+ \|P_j(\phi(x)(P_{\leq j} v)^3)\|_{L^1_t(L^2_x)(x \times \mathbb{R}^3)}.
\]

Since \( E(Iv(t)) \leq 2C_0(A, \epsilon)N^{2(1-s)} \) for all \( t \in I \),
\[ 2^j \|P_j v_0\|_{L^2_t(L^2_x)} + \|P_j v_1\|_{L^2_t(L^2_x)} \leq C_0(A, \epsilon)(2^{(1-s)} + N^{1-s}). \] (6.6)

Next, by Proposition 5.4, the Sobolev embedding theorem, and the fact that \( \phi(x)^{1/2} = \left(\frac{|x|}{\sinh|x|}\right) \),
\[
\|\phi(x)^{1/2}(P_{> j-7} v)(P_{\leq k_0} v)(P_{\leq j} v)\|_{L^2_t(L^2_x)} \leq \sum_{l > j-7} \cdot 2^{-l} S(l, v)(P_{\leq k_0} v)\|_{L^\infty H^1_t(\mathbb{R}^3)}
\leq |k_0|2^{k_0(1-s)2^{-j}\beta} \left( \sup_{l > j-7} 2^{-l(1-s)} S(l, v) \right) \cdot C_0(A, \epsilon)
= \ln(N)N^{1-s}2^{-j}\beta \left( \sup_{l > j-7} 2^{-l(1-s)} S(l, v) \right) \cdot C_0(A, \epsilon). \] (6.7)

Next, by Bernstein’s inequality, (6.6), (6.7), and \( \phi \in L^\infty \),
\[
2^j \|\phi(x)(P_{> j-7} v)(P_{\leq k_0} v)(P_{> j} v)\|_{L^2_t(L^2_x)} \leq 2^j \|\phi(x)^{1/2}(P_{> j} v)(P_{\leq k_0} v)\|_{L^2_t(L^2_x)} \|P_{> j} v\|_{L^\infty L^2_t(L^2_x)}
\leq 2^{(1-s)} \ln(N)N^{1-s} \left( \sup_{l > j-7} 2^{-l(1-s)} S(l, v) \right) \cdot C_0(A, \epsilon) \|P_{> j} v\|_{L^\infty L^2_t(L^2_x)}
\leq 2^{(1-s)} \ln(N)N^{1-s} \left( \sup_{l > j-7} 2^{-l(1-s)} S(l, v) \right) \cdot C_0(A, \epsilon)^2(2^{-j}N^{1-s} + 2^{-j\beta}). \] (6.8)

Next, by the Sobolev embedding theorem, the radial Sobolev embedding theorem, the fact that \( \phi(x)^{1/2} = \left(\frac{|x|}{\sinh|x|}\right) \), and bounds on the energy \( E(Iv(t)) \),
\[
2^{-j/2} \|P_{\leq j} v\|_{L^\infty_x(L^2_t(B_{2j}))} + \sum_{l > j} 2^{j/2} \|\phi(x)^{1/2}(P_{\leq j} v)\|_{L^\infty_t(L^2_x)(A_{2j})}
\leq (|j| + 1) \|P_{\leq j} v\|_{L^\infty H^1_t(x \times \mathbb{R}^3)} \leq (|j| + 1)C_0(A, \epsilon)(N^{1-s} + 2^{(1-s)}),
\] (6.9)
so by Hölder’s inequality and (6.7),
\[ \left\lVert \phi(x)(P_{j-\gamma}v)(P_{\leq k_0}v)(P_{\leq j}v) \right\rVert_{L^2_x(L^4_t)} \lesssim (6.9) \cdot \left\lVert \phi(x)^{1/2}(P_{j-\gamma}v)(P_{\leq k_0}v) \right\rVert_{L^2_x(L^4_t)} \lesssim \left( \sup_{t > j-\gamma} 2^{-t(1-s)} S(l, v) \right) \cdot C_0(A, \epsilon)^2 2^{(j-1-s)} \ln(N)(|l| + 1) N^{1-s} \left( 2^{-j} N^{1-s} + 2^{-j} \right). \] 

(6.10)

Next, by the bootstrap assumption \[ \left\lVert P_{k_0}v \right\rVert_{L^4_t(L^4_x)} \leq N^{-\epsilon/2}, \]
\[ 2^{i/2} \left\lVert \phi(x)(P_{j-\gamma}v)(P_{k_0}v)^2 \right\rVert_{L^4_t(L^4_x)} \leq 2^{i/2} \left\lVert P_{j-\gamma}v \right\rVert_{L^4_t(L^4_x)} \left\lVert P_{k_0}v \right\rVert_{L^4_t(L^4_x)} \leq 2^{(j-1-s)} N^{-\epsilon} \left( \sup_{t > j-\gamma} 2^{-t(1-s)} S(l, v) \right). \]

(6.11)

Finally, decompose
\[ \phi(x) = \left( \left( P_{j-\gamma} \phi(x) \right)^{1/2} (P_{j-\gamma} \phi(x)) \right) + \left( \left( P_{\leq j-\gamma} \phi(x)^{1/2} \right) \left( P_{\leq j-\gamma} \phi(x)^{1/2} \right) \right) + 2 \left( \left( P_{j-\gamma} \phi(x)^{1/2} \right) \phi(x)^{1/2} \right) - 2 \left( \left( P_{j-\gamma} \phi(x)^{1/2} \right) \phi(x)^{1/2} \right) \right). \]

(6.12)

Because \[ \left( \frac{|x|}{\sinh|x|} \right) \text{ and all its derivatives are smooth and rapidly decreasing, for any } x, k, j > 0, \]
\[ |P_{j-\gamma} \phi(x)^{1/2}| \lesssim k 2^{-j}(1 + |x|)^{-k}, \]
so by the radial Sobolev embedding theorem and a crude summation of local bounds (in particular Corollary 3.7),
\[ \left\lVert \left( \left( P_{j-\gamma} \phi(x)^{1/2} \right)^2 \right) \phi(x)^{1/2} \right\rVert_{L^4_t(L^4_x)} \lesssim 2^{-2i/2} \left\lVert P_{\leq j-\gamma} v \right\rVert_{L^4_t(L^4_x)} \left\lVert \phi(x)^{1/2} \right\rVert_{L^4_t(L^4_x)} \left\lVert P_{\leq j-\gamma} v \right\rVert_{L^4_t(L^4_x)} \lesssim 2^{-2i/2} \left( 1 + \frac{2^{(j-1-s)}}{N^{1-s}} \right)^{3} \left( 1 + \frac{2^{(j-1-s)}}{N^{1-s}} \right)^{3} \right. \]

(6.14)

Next, using (6.1), Corollary 3.7, and splitting \[ P_{\leq j-\gamma} v = Iv - P_{j-\gamma} Iv + P_{\leq j-\gamma} (1-I)v, \]
\[ \left| \left\lVert P_{j-\gamma} \phi(x)^{1/2} \right\rVert_{L^4_t(L^4_x)} \right| \leq \left\lVert \phi(x)^{1/2} \right\rVert_{L^4_t(L^4_x)} \left\lVert P_{\leq j-\gamma} v \right\rVert_{L^4_t(L^4_x)} \left\lVert P_{\leq j-\gamma} \phi(x)^{1/2} \right\rVert_{L^4_t(L^4_x)}, \]

(6.15)

and by Bernstein’s inequality,
\[ \left| \left\lVert P_{j-\gamma} \phi(x)^{1/2} \right\rVert_{L^4_t(L^4_x)} \right| \lesssim 2^{-2} \left\lVert Iv \right\rVert_{L^4_t(L^4_x)} \left\lVert P_{\leq j-\gamma} v \right\rVert_{L^4_t(L^4_x)} \left\lVert P_{\leq j-\gamma} Iv \right\rVert_{L^4_t(L^4_x)}, \]

(6.16)
Also, by definition of the \( I \)-operator and the bootstrap assumption,
\[
\| \phi(x)^{1/2} \left( P_{\geq j-\gamma} \phi(x)^{1/2} \right) (P_{\leq j-\gamma}(1-I)v)^3 \|_{L^4_{x}(\mathbb{R}^4)} \\
\leq \| P_{\geq j-\gamma} \phi(x)^{1/2} \|_{L^2_{x}(\mathbb{R}^4)} \| P_{\leq j-\gamma}(1-I)v \|_{L^4_{x}(\mathbb{R}^4)} \| P_{\geq k_0} v \|_{L^4_{x}(\mathbb{R}^4)}^2 \\
\leq 2^{-j} C_1(A, \epsilon) C_0(A, \epsilon) \left( 1 + \frac{2j^{(1-s)}}{N^{1-s}} \right) N^{2(1-s)} N^{-\epsilon}.
\]

Since
\[
P_j \left( (P_{\leq j-\gamma}v)^3 \right) (P_{\leq j-\gamma} \phi(x)^{1/2})^2 = 0,
\]
then by (6.5), (6.6), (6.8), (6.10), (6.11), (6.14), (6.15),
\[
2^{-j(1-s)} S(j, v) \leq C_0(A, \epsilon) N^{1-s} + N^{-\epsilon} \left( \sup_{l \geq j-7} 2^{-l(1-s)} S(l, v) \right) \\
+ \ln(N) (|j| + 1) C_0(A, \epsilon)^2 \left( 2^{-j} N^{2(1-s)} + 2^{-j} N^{1-s} \right) \left( \sup_{l \geq j-7} 2^{-l(1-s)} S(l, v) \right) \\
+ 2^{-2j} 2^{-j(1-s)} C_1^2 C_0(A, \epsilon)^3 N^{3(1-s)} \left( 1 + \frac{2j^{(1-s)}}{N^{1-s}} \right) \\
+ 2^{-j} 2^{-j(1-s)} C_0(A, \epsilon) N^{3(1-s)} \left( 1 + \frac{2j^{(1-s)}}{N^{1-s}} \right).
\]

Choosing \( N(C_1, A, \epsilon) \) sufficiently large, then for \( j \) such that \( 2^j \geq N^{1-s} \),
\[
\ln(N) (|j| + 1) C_0(A, \epsilon)^2 \left( 2^{-j} N^{2(1-s)} + 2^{-j} N^{1-s} \right) + N^{-\epsilon} \leq N^{-\epsilon/10},
\]
and
\[
2^{-2j} C_1^2 C_0(A, \epsilon)^3 N^{5(1-s)} \left( 1 + \frac{2j^{(1-s)}}{N^{1-s}} \right)^3 + 2^{-j} C_1^2 C_0(A, \epsilon) N^{5(1-s)} \left( 1 + \frac{2j^{(1-s)}}{N^{1-s}} \right) \leq 1,
\]
so
\[
\left( \sup_{l \geq j} 2^{-l(1-s)} S(l, v) \right) \leq C_0(A, \epsilon) \left( 1 + 2^{-j(1-s)} N^{1-s} \right) + N^{-\epsilon/10} \left( \sup_{l \geq j-7} 2^{-l(1-s)} S(l, v) \right).
\]

Then by induction on \( j \), starting with \( j \geq k_0(1-\epsilon) \), and thus \( 2^j \geq N^{1-s} \), and Corollary \( 3.7 \), there exists some \( c > 0 \) such that
\[
\left( \sup_{l \geq k_0(1-\epsilon)} 2^{-l(1-s)} S(l, v) \right) \leq C_0(A, \epsilon) + N^{-c \ln(N)} C_0(A, \epsilon) C_1 N^{2(1-s)},
\]
which proves the theorem for \( J \subset J \). But then this implies (6.4) holds for \( J \subset J \), and thus by local well-posedness (Lemma 3.5) there exists a larger open interval \( J \subset I_1 \subset J \) for which (6.3) holds on then closure of \( I_1 \). By the usual bootstrap arguments, this proves the theorem for all of \( J \).
Corollary 6.2. For \( k_0(C_1, A, \epsilon) \) sufficiently large, \( N = 2k_0 \),
\[
\| IP_{\geq k_0-7}v \|_{L^3_{x,t}(\mathcal{J} \times \mathbb{R}^3)} + \| \nabla IP_{\geq k_0-7}v \|_{L^2_{x,t}(\mathcal{J} \times \mathbb{R}^3)} + \| IP_{\geq k_0-7}v_t \|_{L^2_{x,t}(\mathcal{J} \times \mathbb{R}^3)} \leq C_0(A, \epsilon)N^{1-s}.
\] (6.24)

Proof. Let \((p, q)\) be an \( \dot{H}^{1+\epsilon} \)-admissible pair
\[
\frac{1}{p} = \frac{1}{4} + \frac{\epsilon}{2}, \quad \frac{1}{q} = \frac{1}{4} - \frac{\epsilon}{2}.
\] (6.25)

By interpolation and Proposition 6.1,
\[
\sup_{j \geq k_0-14} \| P_j v \|_{L^3_{x,t}(\mathcal{J} \times \mathbb{R}^3)} \leq C_0(A, \epsilon).
\] (6.26)

Then let
\[
\frac{1}{p} = \frac{1}{p} + \frac{1}{2}, \quad \frac{1}{q} = \frac{1}{q} + \frac{1}{2}.
\] (6.27)

Combining Proposition 6.1 with (6.26),
\[
\| (P_{> k_0-14}v)^3 \|_{L^3_{x,t}(\mathcal{J} \times \mathbb{R}^3)} \leq \sum_{k_0-14 \leq j_1 \leq j_2 \leq j_3} \| P_{j_1} v \|_{L^3_{x,t}(\mathcal{J} \times \mathbb{R}^3)} \| P_{j_2} v \|_{L^3_{x,t}(\mathcal{J} \times \mathbb{R}^3)} \| P_{j_3} v \|_{L^3_{x,t}(\mathcal{J} \times \mathbb{R}^3)}
\leq \sum_{k_0-14 \leq j_1 \leq j_2 \leq j_3} 2^{-j_2} 2^{-j_3} \leq N^{-2\epsilon}.
\] (6.28)

Meanwhile, by (6.10) and the conclusion of Proposition 6.1,
\[
\sum_{j > k_0-7} \| \phi(x)(P_{> j-7}v)(P_{\leq k_0}v)^2 \|_{L^3_{x,t}(\mathcal{J} \times \mathbb{R}^3)} \leq C_0(A, \epsilon)^3 \sum_{j > k_0-7} 2^{-2j} N^{1-s} \leq C_0(A, \epsilon)N^{1-s}.
\] (6.29)

Finally by (6.14)–(6.17),
\[
\| IP_{> k_0-7}(\phi(x)(P_{\leq k_0-14}v)^3) \|_{L^3_{x,t}(\mathcal{J} \times \mathbb{R}^3)} \leq \sum_{j > k_0-7} \| \tilde{P}_j(\phi(x)(P_{\leq k_0-14}v)^3) \|_{L^3_{x,t}(\mathcal{J} \times \mathbb{R}^3)} \leq C_0(A, \epsilon)N^{1-s}.
\] (6.30)

Therefore, since
\[
P_{> k_0-7}I \sim N^{1-s}\|
\nabla \|^{s-1},
\] (6.31)
by theorem 2.6, (6.28)–(6.30),
\[
\| \nabla IP_{> k_0-7}v \|_{L^2_{x,t}(\mathcal{J} \times \mathbb{R}^3)} + \| IP_{> k_0-7}v_t \|_{L^2_{x,t}(\mathcal{J} \times \mathbb{R}^3)} + \| IP_{> k_0-7}v_t \|_{L^2_{x,t}(\mathcal{J} \times \mathbb{R}^3)} \leq C_0(A, \epsilon)N^{1-s}.
\] (6.32)

This proves the theorem. \(\square\)

7. Change of energy

Next recall Proposition 3.3.
Theorem 7.1. Suppose $\mathcal{J}$ is an interval on which
\[
\int_{\mathcal{J}} \left( \frac{\cosh|x|}{\sinh|x|} \right) \phi(x)(Iv(x,t))^4 dx dt \leq C_1 N^{1-2\epsilon} = C_1 N^{2(1-s)},
\] (7.1)
and $\sup_{t \in \mathcal{J}} E(Iv(t)) \leq 2C_0(A, \epsilon)N^{2(1-s)}$. Then for $k_0(C_1, A, \epsilon)$ sufficiently large,
\[
\sup_{t \in \mathcal{J}} E(Iv(t)) \leq \frac{3}{2} C_0(A, \epsilon)N^{2(1-s)}.
\] (7.2)

Proof. By Proposition 6.1 and Corollary 6.2, we can choose $k_0(C_1, A, \epsilon)$ sufficiently large so that
\[
\sup_{j \geq k_0-7} 2^{-j(1-s)} S(j, v) \leq C_0(A, \epsilon),
\] (7.3)
and
\[
\left\| \nabla IP \right\|_{L^2_{t,x}}(\mathcal{J} \times \mathbb{R}^3) + \left\| IP \right\|_{L^2_{t,x}}(\mathcal{J} \times \mathbb{R}^3) + \left\| IP \right\|_{L^\infty_{t,x}}(\mathcal{J} \times \mathbb{R}^3) \leq C_0(A, \epsilon)N^{1-s}.
\] (7.4)
The change of the modified energy
\[
E(Iv(t)) = \frac{1}{2} \int |\nabla Iv(x,t)|^2 dx + \frac{1}{2} \int (Iv_t(x,t))^2 dx + \frac{1}{4} \int \phi(x)(Iv(x,t))^4 dx,
\] (7.5)
is given by
\[
\frac{d}{dt} E(Iv(t)) = \int (Iv_t(x,t)) \cdot \left[ \phi(x)(Iv(x,t))^3 - I(Iv(x,t))v(x,t)^3 \right] dx.
\] (7.6)

Remark: When $I = 1$ then the energy is clearly conserved.
First observe that (7.1), (7.3), and the fact that $\frac{1}{|x|} \leq \frac{\cosh|x|}{\sinh|x|}$ imply
\[
\left\| \phi(x)^{1/2}(P_{\geq k_0} v)^2 \right\|_{L^2_{t,x}}(\mathcal{J} \times \mathbb{R}^3) \leq \left\| \phi(x)^{1/4} (Iv) \right\|_{L^4_{t,x}}(\mathcal{J} \times \mathbb{R}^3) + N^{1/2} \left\| P_{\geq k_0} Iv \right\|_{L^2_{t,x}}(\mathcal{J} \times \mathbb{R}^3)
\] (7.7)
\[
\leq C_1^{1/2} N^{1-s} + C_0(A, \epsilon)N^{1/2-2\epsilon},
\]
and
\[
\left\| (P_{> k_0} v)^2 \right\|_{L^2_{t,x}}(\mathcal{J} \times \mathbb{R}^3) + \left\| (P_{> k_0} Iv)^2 \right\|_{L^2_{t,x}}(\mathcal{J} \times \mathbb{R}^3) \leq C_0(A, \epsilon)N^{1/2-2\epsilon}.
\] (7.8)
Therefore, by Theorem 5.1, Proposition 6.1, the bootstrap Assumption (7.1), and the decay of $\phi(x)^{1/2}$,
\[
\int_{\mathcal{J}} \phi(x)(P_{> k_0-7} Iv)(IP_{\leq k_0} v)^2 dx dt \leq \ln(N) N^{1/2} \left\| \phi(x)^{1/2}(Iv)^2 \right\|_{L^2_{t,x}}(\mathcal{J} \times \mathbb{R}^3) \left\| P_{> k_0-7} Iv \right\|_{L^2_{t,x}}(\mathcal{J} \times \mathbb{R}^3) \left\| P_{\leq k_0-7} Iv \right\|_{L^\infty_{t,x}}(\mathcal{J} \times \mathbb{R}^3)
\] (7.9)
\[
\leq \ln(N) N^{1/2-s} C_0(A, \epsilon)^2 C_1^{1/2} N^{2(1-s)}.
\]
Also, by Theorem 5.1, Proposition 6.1, and (7.1),

\[
\int_I \int_I (\phi(x)(P_{>\gamma})^2)(IP_{\leq k_0} v_t) \, dxdt \\
\leq \ln(N)N^{1/2}||P_{\leq k_0} v_t||_{L_{t,x}^2(J \times R^3)} ||\phi(x)^{1/2}v^2||_{L_{t,x}^2(J \times R^3)} ||P_{\geq k_0 - \gamma} v||_{L_{t,x}^2(J \times R^3)}
\]

\[
\leq \ln(N)N^{1/2}||Iv||_{L_{t,x}^2(J \times R^3)} \left(C_1^{1/2}N^{1-s} + C_0(A, \epsilon)N^{1/2-2s} \right) \sum_{j>k_0 - \gamma} 2^{-j} S(j, v)
\]

\[
\leq \ln(N)N^{1/2-j} C_0(A, \epsilon)^2 N^{1-s} \left(C_1^{1/2}N^{1-s} + C_0(A, \epsilon)N^{1/2-2s} \right).
\]

Also, by (5.2), Proposition 6.1, Corollary 6.2, (7.1), the radial Sobolev embedding theorem, and Bernstein’s inequality

\[
\int_I \int_{|x| \geq \frac{t}{\sqrt{N}}} \phi(x)(P_{>\gamma})^2(Iv)^2(IP_{>k_0} v_t) \, dxdt \\
\leq \ln(N)||\phi(x)^{1/2}(Iv)^2||_{L_{t,x}^2(J \times R^3)} ||P_{>k_0} v_t||_{L_{t,x}^2(J \times R^3)} ||x||_{L_t^\infty} \, dxdt
\]

\[
\leq \ln(N)C_0(A, \epsilon)C_1^{1/2}N^{2(1-s)} \sum_{j>k_0 - \gamma} 2^{-j/2} ||Iv||_{L_t^\infty H^s(J \times R^3)}
\]

\[
\leq \ln(N)C_0(A, \epsilon)^2 C_1^{1/2}N^{2(1-s)}N^{1-s},
\]

while

\[
\int_I \int_{|x| \leq \frac{t}{\sqrt{N}}} \phi(x)(P_{>\gamma})^2(Iv)^2(IP_{>k_0} v_t) \, dxdt \\
\leq ||Iv||_{L_{t,x}^2(J \times R^3)} ||P_{>\gamma} v_t||_{L_{t,x}^2(J \times R^3)} ||IP_{>k_0} v_t||_{L_{t,x}^2(J \times R^3)}
\]

\[
\leq C_0(A, \epsilon)^2 N^{2(1-s)} N^{1/2} \left||Iv\right||_{L_{t,x}^2(J \times R^3)} \\
\leq C_0(A, \epsilon)^2 C_1^{1/2}N^{2-2s}N^{1-s}.
\]

Meanwhile, by the Sobolev embedding theorem and the bounds on the energy,

\[
||P_{\leq k_0 - \gamma} v||_{L_{t,x}^\infty H^1(J \times R^3)} + N^{-1/2}||P_{\leq k_0 - \gamma} v||_{L_{t,x}^\infty(J \times R^3)} \approx \sup_{t \in J} E(Iv(t))^{1/2} \leq C_0(A, \epsilon)N^{1-s},
\]

so by (5.11), Proposition 6.1, Corollary 6.2, and Proposition 5.4,

\[
\int_I \int_I (\phi(x)v(x, t) (P_{>\gamma} v)(P_{\leq k_0} v)) (x, t)(IP_{>k_0} v_t) (x, t) \, dxdt \\
\leq \ln(N)C_0(A, \epsilon)N^{1-s} ||P_{\geq k_0} v_t||_{L_{t,x}^2(J \times R^3)} ||\phi(x)^{1/2}v^2||_{L_{t,x}^2(J \times R^3)}
\]

\[
\leq \ln(N)N^{2(1-s)} C_0(A, \epsilon)^2 \left( \sum_{j>k_0 - \gamma} 2^{-j} S(j, v) \right) ||P_{\leq k_0} v||_{L_{t,x}^\infty H^1} + N^{-1/2} ||P_{\leq k_0} v||_{L_{t,x}^\infty} + ||P_{>k_0 - \gamma} v||_{L_{t,x}^2(J \times R^3)}
\]

\[
\leq \ln(N)C_0(A, \epsilon)^4 N^{2(1-s)} (\ln(N)N^{1-2s} + N^{-2\epsilon}).
\]
Also, by Proposition 6.1 and Theorem 5.3,
\[
\|I\left(\phi(x)(P_{>k_0-\gamma}v)^3\right)(P_{\leq k_0}v_t)\|_{L^2_t(L^2_x(J\times R^3))} \\
\leq \|I\left(\phi(x)(P_{>k_0-\gamma}v)^3\right)\|_{L^2_t(L^2_x(J\times R^3))} \|v_t\|_{L^\infty_tL^2_x(J\times R^3)} \\
\leq C_0(A,\epsilon)^3N^{1-s-\epsilon}\left(\sup_{t\in J}E(Iv(t))\right)^{1/2} \leq C_0(A,\epsilon)^4N^{2(1-s)}N^{-\epsilon}.
\] (7.15)

Now then,
\[
\phi(x)(P_{\leq k_0-\gamma}v(x,t))^3 - I\left(\phi(x)(P_{\leq k_0-\gamma}v(x,t))^3\right) = (1-I)\left(\phi(x)(P_{\leq k_0-\gamma}v(x,t))^3\right).
\] (7.16)

Recall from (6.14)–(6.17) that
\[
\left\|Iv_t\right\|_{L^\infty_tL^2_x(J\times R^3)} \leq C_0(A,\epsilon)N^{1-s},
\] (7.18)

Therefore we have proved that for \(k_0(C_1, A, \epsilon)\) sufficiently large, \(N = 2^{k_0}\),
\[
\int_{J}\left|\frac{d}{dt}E(Iv(t))\right|dt \leq N^{2(1-s)}N^{-\epsilon/2}C_1^2C_0(A,\epsilon)^4 \ll N^{2(1-s)}.
\] (7.19)

This finally proves
\[
\int_{J}\left|\frac{d}{dt}E(Iv(t))\right|dt < N^{2(1-s)},
\] (7.20)

and therefore for all \(t \in J\),
\[
|E(Iv(t)) - E(Iv(0))| \ll N^{2(1-s)},
\] (7.21)

and therefore since \(E(Iv(0)) \leq C_0(A,\epsilon)N^{2(1-s)}\), \(\sup_{t \in J}E(Iv(t)) \leq \frac{3}{2}C_0(A,\epsilon)N^{2(1-s)}\).

\[\blacksquare\]

8. Morawetz estimates

Finally we prove Proposition 3.4.

**Proposition 8.1** (Morawetz estimates). Suppose \(v\) solves the conformal wave equation on \(J\) with \(\sup_{t \in J}E(Iv(t)) \leq 2C_0(A,\epsilon)N^{2(1-s)}\) and
\[
\int_{J}\phi(x)\left(\frac{\cosh|\gamma|}{\sinh|\gamma|}\right)|Iv(x,t)|^4dxdt \leq C_1N^{2(1-s)}.
\] (8.1)

Then for \(k_0(C_1, A, \epsilon)\) sufficiently large, \(N = 2^{k_0}\), if \(\phi(x) = \left(\frac{|\gamma|}{\sinh|\gamma|}\right)^2\),
\[
\int_{J}\phi(x)\left(\frac{\cosh|\gamma|}{\sinh|\gamma|}\right)|Iv(x,t)|^4dxdt \leq C_0(A,\epsilon)N^{2(1-s)}.
\] (8.2)
\textbf{Proof.} Let }a(x) = |x|\text{ and let }

\[ M(t) = \int_{\mathcal{F}} \left( \nabla \mathcal{V}(x, t) \cdot \nabla a(x) + \frac{1}{2} \Delta a(x) \mathcal{V}(x, t) \right) dx. \tag{8.3} \]

By Hardy’s inequality,

\[ \sup_{t \in \mathcal{F}} |M(t)| \leq ||\nabla \mathcal{V}||_{L^2_t(\mathcal{F} \times \mathbb{R}^n)} ||\mathcal{V}||_{L^2_t(\mathcal{F} \times \mathbb{R}^n)} \leq C_0(\mathcal{A}, \varepsilon) N^{2(1-\varepsilon)}. \tag{8.4} \]

Then following the computations of [12],

\[ \int_{\mathcal{F}} \phi(x) \left( \frac{\cosh|x|}{\sinh|x|} \right) (\mathcal{V}(x, t))^4 dx dt \leq \left| \int_{\mathcal{F}} \frac{d}{dt} M(t) dt \right| + \left| \int_{\mathcal{F}} \mathcal{E}(t) dt \right|. \tag{8.5} \]

with error terms are given by

\[ \mathcal{E}(t) = \int (I(\phi(x) \mathcal{V}^3(x, t)) - \phi(x)(\mathcal{V})^3(x, t)) \left( \nabla \mathcal{V}(x, t) \cdot \nabla a(x) + \frac{1}{2} \Delta a(x) \mathcal{V}(x, t) \right) dx, \tag{8.6} \]

By the fundamental theorem of calculus,

\[ \left| \int_{\mathcal{F}} \frac{d}{dt} M(t) dt \right| \leq \sup_{t \in \mathcal{F}} |M(t)| \leq C_0(\mathcal{A}, \varepsilon) N^{2(1-\varepsilon)}. \tag{8.7} \]

Therefore it remains to estimate

\[ \left| \int_{\mathcal{F}} \mathcal{E}(t) dt \right|. \tag{8.8} \]

Split the error into two terms,

\[ \mathcal{E}_1(t) = \int (I(\phi(x) \mathcal{V}^3(x, t)) - \phi(x)(\mathcal{V})^3(x, t)) (\nabla \mathcal{V}(x, t) \cdot \nabla a(x)) dx, \]

\[ \mathcal{E}_2(t) = \int (I(\phi(x) \mathcal{V}^3(x, t)) - \phi(x)(\mathcal{V})^3(x, t)) \frac{1}{2} \Delta a(x) \mathcal{V}(x, t) dx. \tag{8.9} \]

By direct calculation, \( \nabla a(x) = \frac{x}{|x|^2} \), and also by definition of the energy and Corollary 6.2, \( \nabla \mathcal{V} \) and \( \mathcal{V} \) have the same estimates, and thus the terms in \( \mathcal{E}_1(t) \) may be estimated in a manner which is exactly analogous to the corresponding terms in the previous section.

Indeed, as in (7.9),

\[ \int_{\mathcal{F}} \phi(x) (P > \kappa - \gamma \mathcal{V})(\mathcal{V})^2 (\nabla a(x) \cdot \nabla IP \leq \kappa \mathcal{V}) dx dt \]

\[ \leq \ln(N) N^{1/2} ||\phi(x)^{1/2} (\mathcal{V})^2||_{L^\infty_t L^2_x(\mathcal{F} \times \mathbb{R}^n)} ||P > \kappa - \gamma \mathcal{V}||_{L^\infty_t L^2_x(\mathcal{F} \times \mathbb{R}^n)} \]

\[ \leq \ln(N) N^{1/2-s} C_0(\mathcal{A}, \varepsilon)^2 C_1^{1/2} N^{2(1-\varepsilon)}. \tag{8.10} \]
Next, as in (7.10),
\[
\int J \int (\phi(x)(P > k_0 - \gamma)v) (\nabla a(x) \cdot \nabla IP_{\leq k_0} v) dx dt
\]
\[
\leq \ln(N)N^{1/2} ||\nabla P_{\leq k_0} iv||_{L^\infty_t L^2_x(\mathcal{J} \times \mathbb{R}^3)} ||\phi(x)_{1/2}v^2||_{L^\infty_{t_0} L^2_{x_0}(\mathcal{J} \times \mathbb{R}^3)} ||P_{\geq k_0 - \gamma} v||_{L^\infty_{t_0} L^2_{x_0}(\mathcal{J} \times \mathbb{R}^3)} (8.11)
\]
\[
\leq \ln(N)N^{1/2} \left( C_1^{1/2} N^{1-s} + C_0(A, \epsilon) N^{1/2 - 2\epsilon} \right) ||\nabla iv||_{L^\infty_t L^2_x(\mathcal{J} \times \mathbb{R}^3)} \sum_{j > k_0 - \gamma 2^{-j/\delta}} S(j, \nu)
\]
\[
\leq \ln(N)N^{1/2-s} C_0(A, \epsilon)^2 N^{1-s} \left( C_1^{1/2} N^{1-s} + C_0(A, \epsilon) N^{1/2 - 2\epsilon} \right).
\]

As in (7.12),
\[
\int J \int |x| \leq \frac{k_0}{2} \phi(x)(P > k_0 - \gamma) iv (iv)^2 (\nabla a(x) \cdot \nabla IP > k_0 v) dx dt
\]
\[
\leq ||iv||_{L^\infty_t L^2_x (\mathcal{J} \times B_{k_0}/2)} ||P > k_0 - \gamma iv||_{L^2_t L^\infty_x (\mathcal{J} \times \mathbb{R}^3)} ||\nabla IP > k_0 v||_{L^\infty_t L^2_x(\mathcal{J} \times \mathbb{R}^3)} (8.12)
\]
\[
\leq C_0(A, \epsilon)^2 N^{2(1-s)} N^{-1/2} ||(iv)^2||_{L^\infty_t L^2_x (\mathcal{J} \times \mathbb{R}^3)}
\]
\[
\leq C_0(A, \epsilon)^2 C_1^{1/2} N^{2-s} N^{1-s}.
\]

Also, as in (7.11),
\[
\int J \int |x| \geq \frac{k_0}{2} \phi(x)(P > k_0 - \gamma) iv (iv)^2 (\nabla a(x) \cdot \nabla IP > k_0 v) dx dt
\]
\[
\leq \ln(N)||\phi(x)_{1/2}(iv)^2||_{L^\infty_{t_0} L^2_{x_0}(\mathcal{J} \times \mathbb{R}^3)} ||\nabla IP > k_0 v||_{L^\infty_{t_0} L^2_{x_0}(\mathcal{J} \times \mathbb{R}^3)} ||x||_{L^\infty_{t_0} L^2_{x_0}(\mathcal{J} \times \mathbb{R}^3)} (8.13)
\]
\[
\leq \ln(N)C_0(A, \epsilon) C_1^{1/2} N^{2(1-s)} \sum_{j > k_0 - \gamma 2^{-j/2}} ||P_j iv||_{L^\infty_t H^1(\mathcal{J} \times \mathbb{R}^3)}
\]
\[
\leq \ln(N)C_0(A, \epsilon)^2 C_1^{1/2} N^{2(1-s)} N^{1-2\epsilon}.
\]

As in (7.15),
\[
\int J \int (I(\phi(x)(P > k_0 - \gamma)v)^3) (\nabla a(x) \cdot \nabla IP_{\geq k_0} v) dx dt
\]
\[
\leq ||I(\phi(x)(P > k_0 - \gamma)v)^3||_{L^1_t L^2_x (\mathcal{J} \times \mathbb{R}^3)} ||\nabla iv||_{L^\infty_t L^2_x(\mathcal{J} \times \mathbb{R}^3)} (8.14)
\]
\[
\leq C_0(A, \epsilon)^3 N^{1-s-\epsilon} \left( \sup_{t \in \mathcal{J}} E(iv(t)) \right)^{1/2} \leq C_0(A, \epsilon)^4 N^{2(1-s)} N^{-\epsilon}.
\]

Next, following (7.14),
\[
\int J \int I(\phi(x)v(x, t)(P > k_0 - \gamma) (P_{\leq k_0} v)(x, t) (\nabla IP > k_0 iv)(x, t) dx dt
\]
\[
\leq \ln(N)C_0(A, \epsilon) N^{1-s} ||\nabla IP_{\geq k_0} v||_{L^\infty_{t_0} L^2_{x_0}(\mathcal{J} \times \mathbb{R}^3)} ||\phi(x)_{1/2}(P > k_0 - \gamma)v||_{L^\infty_{t_0} L^2_{x_0}(\mathcal{J} \times \mathbb{R}^3)}
\]
\[
\leq \ln(N)N^{2(1-s)} C_0(A, \epsilon)^2 \left( \ln(N)N^{1-s} ||P > k_0 - \gamma v||_{L^\infty_{t_0} L^2_{x_0}(\mathcal{J} \times \mathbb{R}^3)} + ||P > k_0 - \gamma v||_{L^2_{t_0} L^2_{x_0}(\mathcal{J} \times \mathbb{R}^3)} \right)
\]
\[
\leq \ln(N)C_0(A, \epsilon)^4 N^{2(1-s)} (\ln(N)N^{1-2s} + N^{-2\epsilon}).
\]
Finally, by (7.17),
\[
\int_J \int (I-1) \left( \phi(x)(P_{k_0-\gamma}^v)^3 \right) \left( \nabla a(x) \cdot \nabla I\nu(x,t) \right) \, dx \, dt \\
\leq \left\| (I-1) \left( \phi(x)(P_{k_0-\gamma}^v)^3 \right) \right\|_{L^2(J \times \mathbb{R})} \left\| \nabla I\nu \right\|_{L^2(J \times \mathbb{R})} \leq C_1^2 C_0^4 (A, \epsilon) (N^{4-6s} + N^{3-4s}).
\] (8.16)

Therefore, we have proved that
\[
\int_J |E_1(t)| \, dt \leq N^{2(1-s)} N^{-\epsilon/2} C_1^2 C_0^4 (A, \epsilon).
\] (8.17)

Also by (7.17), \(\Delta a(x) \leq \frac{1}{|x|}\), and Hardy’s inequality,
\[
\int_J \int (I-1) \left( \phi(x)(P_{k_0-\gamma}^v)^3 \right) \left[ \frac{1}{2} \Delta a(x) I\nu(x,t) \right] \, dx \, dt \\
\leq \left\| (I-1) \left( \phi(x)(P_{k_0-\gamma}^v)^3 \right) \right\|_{L^2(J \times \mathbb{R})} \left\| \frac{1}{|x|} I\nu \right\|_{L^2(J \times \mathbb{R})} \\
\leq C_1^2 C_0^4 (A, \epsilon) (N^{4-6s} + N^{3-4s}).
\] (8.18)

Therefore, it only remains to estimate
\[
\int_J \int \phi(x) I\nu^2 (P_{k_0-\gamma}^v) \frac{1}{|x|} \, (I\nu) \, dx \, dt,
\] (8.19)

and
\[
\int_J \int I \left( \phi(x) v^2 (P_{k_0-\gamma}^v) \right) \frac{1}{|x|} \, (I\nu) \, dx \, dt.
\] (8.20)

First, by (5.2), Proposition 6.1, (7.7), (7.8), and the radial Sobolev embedding theorem,
\[
\int_J \int |x| > \frac{1}{N} \left| I \left( \phi(x)(P_{k_0-\gamma}^v)^2 \right) \frac{1}{|x|} \, (I\nu) \, dx \, dt \\
\leq N^{1/2} \ln(N) \left| P_{k_0-\gamma}^v \right|_{L^{\infty}(J \times \mathbb{R})} \left| \phi(x)^{1/2} \nu \right|_{L^2(J \times \mathbb{R})} \left| |x|^{1/2} I\nu \right|_{L^2(J \times \mathbb{R})} \leq N^{3/2-2s} \ln(N) C_0 (A, \epsilon)^2 C_1^{1/2} N^{1-s} + C_0 (A, \epsilon)^2 N^{1/2-2s}.
\] (8.21)

Next, following (7.15), by Hardy’s inequality,
\[
\int_J \int |x| \leq \frac{1}{N} \left| I \left( \phi(x)(P_{k_0-\gamma}^v)^3 \right) \frac{1}{|x|} \, (I\nu) \, dx \, dt \\
\leq \left| I \left( \phi(x)(P_{k_0-\gamma}^v)^3 \right) \right|_{L^2(J \times \mathbb{R})} \left| \frac{1}{|x|} I\nu \right|_{L^2(J \times \mathbb{R})} \leq C_0 (A, \epsilon)^4 N^{-\epsilon} N^{2(1-s)}.
\] (8.22)

Also, by theorem 5.5, (7.7), (7.8), the fact that \(\Delta a(x) \leq \frac{1}{|x|}\), the support of \(\psi(2^k_0 x)\), and Proposition 6.1,
\[
\int_J \int I \left( \phi(x)(P_{k_0-\gamma}^v)(P_{k_0-\gamma}^v)^v \right) \frac{1}{|x|} \psi(2^k_0 x) \, (I\nu) \, dx \, dt \\
\leq \left| I \left( \phi(x)(P_{k_0-\gamma}^v)(P_{k_0-\gamma}^v)^v \right) \psi(2^k_0 x) \frac{1}{|x|^{3/4}} \right|_{L^2(J \times \mathbb{R})} \left| \phi(x)^{1/4} \right|_{L^2(J \times \mathbb{R})} \left| I\nu \right|_{L^2(J \times \mathbb{R})} \leq C_0 (A, \epsilon)^4 N^{2(1-s)}.
\] (8.23)
\[ \leq N^{-\varepsilon} C_0(A, \varepsilon) C_1^{1/4} \mathbb{N}^{\frac{2\varepsilon}{1 + 2\varepsilon}} \| v \|^2 \| P_{\leq k_0-7} v \|_{L^8_t L^2_x(\mathcal{J} \times \mathbb{R}^3)} \leq C_0(A, \varepsilon)^2 C_1^{1/4} \mathbb{N}^{\frac{2\varepsilon}{1 + 2\varepsilon}} N^{1-\varepsilon} \left( C_1^{1/4} \mathbb{N}^{\frac{2\varepsilon}{1 + 2\varepsilon}} + C_0(A, \varepsilon) N^{1-4-\varepsilon} \right). \]  

Finally, by \( \Delta a(x) \leq \frac{1}{|x|}; |x| \leq \left( \frac{\cosh |x|}{\sinh |x|} \right) \), Hardy’s inequality and Proposition 6.1,

\[
\int \int \phi(x) (iv(x,t))^2 (IP_{> k_0-7} v)(x,t) (\Delta a(x) iv(x,t)) dx dt \leq \left\| \frac{\phi(x)^{1/4}}{|x|^{1/4}} iv \right\|_{L^4_t L^2_x(\mathcal{J} \times \mathbb{R}^3)} \left\| \frac{1}{|x|^{1/2}} (IP_{\geq k_0-7} v) \right\|_{L^2_t L^\infty_x(\mathcal{J} \times \mathbb{R}^3)} \left\| IP_{\geq k_0-7} v \right\|_{L^2_t L^\infty_x(\mathcal{J} \times \mathbb{R}^3)} \leq N^{-\varepsilon/2} N^{\frac{\varepsilon}{2}} \left\| \frac{\phi(x)^{1/4}}{|x|^{1/4}} iv \right\|_{L^4_t L^2_x(\mathcal{J} \times \mathbb{R}^3)} \leq C_3^{1/4} N^{-\varepsilon} N^{2(1-\varepsilon)}. \tag{8.25} \]

Therefore, by (8.21)–(8.25),

\[
\int_{\mathcal{J}} |\mathcal{E}_2(t)| dt \ll N^{2(1-\varepsilon)}, \tag{8.26} \]

which proves the proposition.

\[ \square \]

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