The noncommutative QED threshold energy versus the optimum collision energy

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Abstract

Möller Scattering and Bhabha Scattering on noncommutative space-time is restudied. It is shown that the noncommutative correction of scattering cross sections is not monotonous enhancement with the total energy of colliding electrons, there is an optimum collision energy to get the greatest noncommutative correction. Most surprisingly, there is a linear relation between the noncommutative QED threshold energy and the optimum collision energy.

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1 Introduction

The interest in formulating field theories on noncommutative spaces is relatively old[1]. It has been revived recently due to developments connected to string theories in which the noncommutativity of space-time is an important characteristic of D-Brane dynamics at low energy limit[2, 3, 4]. Much attention on noncommutative field theory has been attracted[5, 6, 7, 8, 9, 10, 11, 12, 13, 14]. It is shown that field theories on noncommutative space-time are well defined quantum theories[3]. From the point of view of string theory, we need urgently to set up one theory that is in conformity with existing elementary particle theory and can be examined within the attainable range of energy scale in experiment at present or in near future. It was generally thought that prediction of string theory and noncommutative effect can only be examined at Plank energy scale or at Great Unification energy scale. However, Witten et al. propose recently that the effect of the string maybe appear at the energy scale of TeV, the threshold value of commutativity of space, as D-brane dynamic main nature in low energy

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limit, might as well be thought that its energy scale is also at scale of TeV, or not far from that at least\cite{15}. In other hand, the energy scale of TeV is just typical running energy scale of collide machine of next generation\cite{16}. However, till now only a little attention has been paid on whether noncommutative effects can be examined\cite{17, 18} in experiments. In this letter, we restudy the Möller scattering and Bhabha scattering in noncommutative quantum electrodynamics (NCQED) to establish the relation between NCQED threshold energy and the most optimum collision energy while noncommutative correction of scattering cross section arrives at the maximum and to discuss the method to determine the NCQED threshold energy. Both Möller and Bhabha scattering are pure lepton process. Their Standard Model result is in accord with QED, so we only need discuss these process in QED even if at the energy scale of TeV.

The noncommutative space can be realized by the coordinate operators satisfying

\[
[X_\mu, X_\nu] = i\theta_{\mu\nu} = \frac{1}{\Lambda_{NC}^2} c_{\mu\nu}
\]

where \(\theta_{\mu\nu}\) are the noncommutative parameters, which is real, anti-symmetric and commutes with space-time coordinate \(X_\mu\); \(c_{\mu\nu}\) are dimensionless parameters, and \(\Lambda_{NC}\) denotes noncommutative threshold energy. A noncommutative version of an ordinary field theory can be obtained by replacing all ordinary products with Moyal \(\star\) products defined by

\[
(f \star g)(x) = \exp \left( \frac{1}{2} i\theta_{\mu\nu} \partial_x^\mu \partial_y^\nu \right) f(x)g(y) \big|_{y=x}
\]

we can get the NCQED Lagrangian by using above recipe as

\[
\mathcal{L} = \frac{1}{2} i(\bar{\psi} \gamma^\mu D_\mu \psi - (D_\mu \bar{\psi}) \star \gamma^\mu \psi) - m\bar{\psi} \star \psi - \frac{1}{4} F_{\mu\nu} \star F^{\mu\nu}
\]

where \(D_\mu \psi = \partial_\mu \psi - ieA_\mu \star \psi\), \((D_\mu \bar{\psi}) = \partial_\mu \bar{\psi} + ie\bar{\psi} \star A_\mu\), \(F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ie(A_\mu \star A_\nu - A_\nu \star A_\mu)\). This Lagrangian can be used to obtain the Feynman rules for perturbative calculations. The propagators for the free fermions and gauge fields of NCQED are the same as that in the case of ordinary QED. But for interaction terms, every interaction vertex has a phase factor correction depending on in and out 4-momentum. Here we give out the interaction vertices of NCQED as that in Fig.1, where \(p_1 \theta p_2\) is the abbreviation for \(p_1^\mu \theta_{\mu\nu} p_2^\nu\).

\[2\] NCQED correction of cross section for Möller Scattering

The Feynman graphs of Möller scattering considered the contribution of the lowest rank tree graphs is shown as in Fig.2, which consists of u-channel and t-channel. According to the Feynman rules of NCQED, we can get the scattering
\[ = 2g \sin(p_1 \theta / 2) \left[ (p_1 - p_2) g_{\mu \nu} + (p_2 - p_3) g_{ij} + (p_3 - p_1) g_{\mu \rho} \right] \]

\[ = 4i g^2 \left[ (g^{\mu \nu} g_{\rho \sigma} - g^{\mu \rho} g_{\nu \sigma} - g^{\mu \sigma} g_{\nu \rho}) \sin(p_1 \theta p_2 / 2) \sin(p_2 \theta p_3 / 2) \right. \]

\[ + (g^{\mu \rho} g_{\nu \sigma} - g^{\mu \sigma} g_{\nu \rho}) \sin(p_3 \theta p_1 / 2) \sin(p_2 \theta p_3 / 2) \]

\[ + (g^{\mu \sigma} g_{\nu \rho} - g^{\mu \rho} g_{\nu \sigma}) \sin(p_1 \theta p_4 / 2) \sin(p_2 \theta p_3 / 2) \]

Figure 1: Feynman rules for interaction vertices of NCQED

Figure 2: Feynman graph of Möller scattering
amplitudes:

\[ M_u = -\bar{u}r^s(k_2)(-ig\gamma^\mu \exp(ip_1^\mu k_2^\nu/2)u^{s_1}(p_1)) - ig\mu\nu \]
\[ \cdot \bar{u}r^s(k_1)(-ig\gamma^\nu \exp(ip_2^\nu k_1^\mu/2)u^{s_2}(p_2)), \quad (4) \]

\[ M_t = \bar{u}r^s(k_1)(-ig\gamma^\mu \exp(ip_1^\mu k_1^\nu/2)u^{s_1}(p_1)) - ig\mu\nu \]
\[ \cdot \bar{u}r^s(k_2)(-ig\gamma^\nu \exp(ip_2^\nu k_2^\mu/2)u^{s_2}(p_2)). \quad (5) \]

By the similar method used in the ordinary QED, we can obtain the scattering section on noncommutative space at ultra-relativistic limit as

\[ \frac{d\sigma_{NC}}{d\Omega} = \frac{2\alpha^2}{s} \left[ \frac{8}{\sin^2 \theta} - \frac{6}{\sin^2 \theta} + \frac{1}{2} + \frac{2\cos(\phi_u - \phi_t)}{\sin^2 \theta} \right] \quad (6) \]

with Mandelstam variables

\[ \phi_u = \frac{1}{2}(p_1^\mu k_1^\nu + p_2^\mu k_2^\nu), \quad \phi_t = \frac{1}{2}(p_1^\mu k_1^\nu + p_2^\mu k_2^\nu), \]
\[ s = (p_1 + p_2)^2 = (k_1 + k_2)^2. \]

Let

\[ p_1^\mu = \frac{\sqrt{s}}{2}(1, -1, 0, 0) \]
\[ p_2^\mu = \frac{\sqrt{s}}{2}(1, 1, 0, 0). \]
\[ k_1^\mu = \frac{\sqrt{s}}{2}(1, -\cos \theta, -\sin \theta \cos \phi, -\sin \theta \sin \phi) \]
\[ k_2^\mu = \frac{\sqrt{s}}{2}(1, \cos \theta, \sin \theta \cos \phi, \sin \theta \sin \phi) \]
\[ \phi_M \equiv \phi_u - \phi_t = \frac{s}{2\Lambda_{NC}^2}(c_{12} \cos \phi - c_{31} \sin \phi) \sin \theta. \quad (7) \]

Comparing with the differential scattering cross section for commutative case

\[ \frac{d\sigma_C}{d\Omega} = \frac{2\alpha^2}{s} \left[ \frac{8}{\sin^2 \theta} - \frac{6}{\sin^2 \theta} + \frac{1}{2} + \frac{2\cos(\phi_u - \phi_t)}{\sin^2 \theta} \right], \quad (8) \]

we get the correction for the differential scattering cross section as

\[ \Delta \left( \frac{d\sigma}{d\Omega} \right) = \frac{2\alpha^2}{s} \left[ \frac{2}{\sin^2 \theta} (1 - \cos \phi_M) \right] \quad (9) \]

It is shown that \( \Delta \left( \frac{d\sigma}{d\Omega} \right) \) is not a monotonous function of \( s \), but appears some extremum when \( \phi_M \) is not very small. The maximum value of \( \Delta \left( \frac{d\sigma}{d\Omega} \right) \) appears when \( \sqrt{s} \) satisfy following relations

\[ \cos(x) + x \sin(x) - 1 = 0, \quad \text{and} \quad \cos(x) < 0 \quad (10) \]

where \( x = \left( \frac{\sqrt{s}}{\Lambda_{NC}} \right)^2 \frac{(c_{12} \cos \phi - c_{31} \sin \phi) \sin \theta}{2} \). The solutions of eqs(10) is \( x = 2.33, \ 9.21, \ 15.58, \ 21.9, \ldots \).
It means that the collision energy $\sqrt{s}$ corresponding to the maximum $\Delta \left( \frac{d\sigma}{d\Omega} \right)$ is direct proportion to the noncommutative threshold energy $\Lambda_{NC}$. The minimum of the ratio is 2.16 when we take $c_{12} = 1$, $c_{31} = 0$ and $\phi = 0$, $\theta = \frac{\pi}{2}$.

Moreover, one can expect the maximum correction of the total scattering cross section $\Delta \sigma$ will appear and the optimum colliding energy will also be direct proportion to the noncommutative threshold energy $\Lambda_{NC}$.

The correction curves for total scattering cross section $\Delta \sigma$ vs colliding energy $\sqrt{s}$ as shown in Fig.3 in which we take $\Lambda_{NC} = 500$ GeV, $c_{12} = 1$, $c_{31} = 0$ and an angular cut of $|\cos \theta| \leq |z| = 0.9(0.7, 0.5)$.

From Fig.3, we really find that $\Delta \sigma$ does not always increase with $\sqrt{s}$, but exists as a kurtosis distribution. Under the condition of $\Lambda_{NC} = 500$ GeV, a maximum appears when the energy of collision particles $\sqrt{s_o} = 1288.8$ GeV. This implies that there is an optimum collision energy to observe the noncommutative effect in Möller scattering, if the NCQED threshold energy is fixed. However, we do not know the exact value of the NCQED threshold energy. We have to determine the NCQED threshold energy $\Lambda_{NC}$ at the first. To do this, it is useful to establish the relation between the NCQED threshold energy and optimum collision energy $\sqrt{s_o}$, which is shown as in Fig.4. After curve fitting, we get

$$\sqrt{s_o} = 0.000001 + 2.5776\Lambda_{NC}. \quad (11)$$

This means that the optimum collision energy is about two and half times of the NCQED threshold energy. If we determine the optimum collision energy $\sqrt{s_o}$ for noncommutative effect in the Möller scattering in next generation colliding experiment by increasing gradually the energy of collision particles, we can
Figure 4: The optimum collision energy $\sqrt{s}$ vs the NCQED threshold energy $\Lambda_{NC}$ for Möller scattering
determine the NCQED threshold energy $\Lambda_{NC}$ by relation eq.(11).

3 NCQED correction of cross section for Bhabha Scattering

In order to know if the linear relation between the NCQED threshold energy and the optimum collision energy is universal, we restudy Bhabha scattering. Its Feynman graphs with the lowest rank tree graphs is shown as in Fig.5, which consists of t-channel and s-channel. The scattering amplitudes of two graphs are respectively

$$M_t = -ig^2 \exp(i\phi_t) \frac{(\bar{u}^{s_2}(k_1)\gamma^\mu u^{\tau_1}(p_1))\bar{v}^{\tau_2}(p_2)\gamma_\mu v^{r_2}(k_2)}{(p_1 - k_1)^2}$$  \hspace{1cm} (12)

$$M_s = ig^2 \exp(i\phi_s) \frac{(\bar{v}^{\tau_1}(p_2)\gamma^\mu u^{\tau_1}(p_1))\bar{u}^{s_2}(k_1)\gamma_\mu v^{r_2}(k_2)}{(p_1 + p_2)^2}$$  \hspace{1cm} (13)

where

$$\phi_t = \frac{1}{2}(p_1^\mu \theta_\mu \nu k_1^\nu - p_2^\mu \theta_\mu \nu k_2^\nu), \ \ \phi_s = \frac{1}{2}(p_2^\mu \theta_\mu \nu p_1^\nu - k_2^\mu \theta_\mu \nu k_1^\nu).$$  \hspace{1cm} (14)
The differential scattering cross section for Bhabha scattering on noncommutative space at ultra-relativistic limit is

\[ \frac{d\sigma_{NC}}{d\Omega} = \frac{\alpha^2}{4s} \left(3 + \cos^2 \theta + 2 \frac{(1 + \cos \theta)(4 - \sin^2 \theta \cos \phi_b)}{(1 - \cos \theta)^2} \right) \]  \( (15) \)

with \( \phi_b \equiv \phi_s - \phi_t \).

To take

\[
\begin{align*}
p_1^\mu &= \sqrt{s}(1, -1, 0, 0), \\
k_1^\mu &= \frac{\sqrt{s}}{2\Lambda_{NC}^2}(1, -\cos \theta, -\sin \theta \cos \phi, -\sin \theta \sin \phi), \\
p_2^\mu &= \frac{\sqrt{s}}{2\Lambda_{NC}^2}(1, 1, 0, 0), \\
k_2^\mu &= \frac{\sqrt{s}}{2\Lambda_{NC}^2}(1, \cos \theta, \sin \theta \cos \phi, \sin \theta \sin \phi),
\end{align*}
\]

then

\[ \phi_b = -\frac{s}{2\Lambda_{NC}^2} [c_{01}(1 - \cos \theta) - c_{02} \sin \theta \cos \phi - c_{03} \sin \theta \sin \phi]. \]  \( (16) \)

Comparing with the differential scattering cross section of Bhabha scattering in ordinary QED case

\[ \frac{d\sigma_C}{d\Omega} = \frac{\alpha^2}{4s} \left[3 + \cos^2 \theta + 2 \frac{(1 + \cos \theta)(4 - \sin^2 \theta)}{(1 - \cos \theta)^2} \right]. \]  \( (17) \)

we obtain the correction of differential scattering cross section for noncommutative effect as

\[ \Delta \left( \frac{d\sigma}{d\Omega} \right)_B = \frac{\alpha^2}{4s} \left[2 (1 + \cos \theta)^2 \frac{(1 - \cos \phi_b)}{(1 - \cos \theta)(1 - \cos \phi_b)} \right]. \]  \( (18) \)
Figure 6: The relation between $\Delta\sigma$ and $\sqrt{s}$ for Bhabha scattering with $c_{01} = 1$, $c_{02} = c_{03} = 0$.

It is noted that the noncommutative effect increase the cross section in the Bhabha scattering in opposition to the case in the Möller scattering. The maximum value of $\Delta\left(\frac{d\sigma}{d\Omega}\right)_B$ will also appear when $\sqrt{s}$ satisfy following relations

$$\cos(x) + x \sin(x) - 1 = 0, \quad \text{and} \quad \cos(x) < 0 \quad (19)$$

where $x = \left(\frac{\sqrt{s}}{\Lambda_{NC}}\right)^2 \frac{c_{01}(1-\cos\theta) - c_{02}\sin\theta \cos\phi - c_{03}\sin\theta \sin\phi}{2}$. Because the eq.(19) is the same as eq.(10) besides the difference of the expression of $x$, the linear relation between the NCQED threshold energy and the optimum collision energy will still exist.

The correction curves for total scattering cross section $\Delta\sigma$ vs colliding energy $\sqrt{s}$ as shown in Fig.6 in which we take $\Lambda_{NC} = 500$ GeV, $c_{01} = 1$, $c_{02} = c_{03} = 0$ and an angular cut of $|\cos\theta| \leq |z| = 0.9(0.7, 0.5)$.

From Fig.6, we also find that $\Delta\sigma$ does not always increase with $\sqrt{s}$, but exist as a kurtosis distribution. Under the condition of $\Lambda_{NC} = 500$ GeV, a maximum appears when the energy of collision particles $\sqrt{s_o} = 1276.5$ GeV. It show that there also is an optimum collision energy to observe the noncommutative effect in Bhabha scattering. The relation between the NCQED threshold energy and optimum collision energy is shown as in Fig.7.

After curve fitting, we obtain

$$\sqrt{s_o} = -0.0035 + 2.5529\Lambda_{NC} \quad (20)$$

This means that there really is a linear relation between the optimum collision energy and the NCQED threshold energy $\Lambda_{NC}$. Moreover, its slope is almost the
same as that in Möller scattering. If we determine the optimum collision energy for noncommutative effect in the Bhabha or Möller scattering in next generation colliding experiment, we can determine the NCQED threshold energy $\Lambda_{NC}$ by relations eqs.(11,20). In other hand, we shall choose the optimum collision energy as the running energy of the next generation collider to explore the noncommutative effect.

4 Summary and discussions

Firstly, we find that it is not the easier to explore noncommutative effect, the higher collision energy in both Möller and Bhabha scattering, but there is an optimum collision energy to get the greatest noncommutative correction for total scattering cross section. Secondly, we find the linear relations eqs.(11,20) between the optimum collision energy and the NCQED threshold energy $\Lambda_{NC}$, the ratio is about 2.5 for both Möller and Bhabha scattering. If the optimum collision energy is determined by increasing gradually the energy of collision particles from Hundreds GeV to several TeV in the next generation collider, then the NCQED threshold energy can be determined from the above linear relations. The ordinary QED scattering cross section is inverse proportion to square of the collision energy $s$, the NCQED correction of the cross section is direct proportion to $[1 - \cos(\Delta \phi_{NC})]$ where $\Delta \phi_{NC}$ is the noncommutative phase factor correction which is direct proportion to $s$, so we argue that the
linear relation between $\sqrt{s_0}$ and $\Lambda_{NC}$ will be satisfied by most QED scattering process. It will be the most efficiency to explore the noncommutative effects by choosing the optimum collision energy $\sqrt{s_0}$ as the operating energy of the colliding experiment.

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References

[1] H.S. Snyder, Phys. Rev. D 71(1947)38.
[2] A. Connes, M.R. Douglas and A. Schwarz, JHEP 02(1998) 003.
[3] M.R. Douglas and C. Hull, JHEP 02(1998) 008.
[4] N. Seiberg and E. Witten, JHEP 09(1999) 032.
[5] L. Dolan, C.R. Nappi, hep-th/0302122.
[6] M.M. Sheikh-Jabbari, JHEP 06(1999) 015.
[7] C.P. Martin, D. Sanchez-Ruiz, Phys. Rev. Lett. 83(1999) 476.
[8] N. Ishibashi, S. Iso, H. Kawai and Y. Kitazawa, Nucl. Phys. B573(2000) 573.
[9] I.Ya. Aref’eva, D.M. Belov, A.S. Koshelev, Phys. Lett. B476(2000) 431.
[10] F.J. Petriello, Nucl. Phys. B601(2001)169.
[11] S.M. Carroll, J.A. Harvey et al., Phys. Rev. Lett. 87 (2001) 141601.
[12] N. Seiberg, L. Susskind and Nicolaos, JHEP 06(2000)004.
[13] M.M. Sheikh-Jabbari, Phys. Rev. Lett. 84(2000) 5265.
[14] Ihab. F. Riad and M.M. Sheikh-Jabbari, JHEP08(2000) 045.
[15] E. Witten, Nucl. Phys. B471(1996)135; P. Horava and E. Witten, Nucl. Phys. B460(1996) 506.
[16] V.I. Telnov, Nucl. Instr. Methods A294(1990) 72.
[17] Prakash Mathews, Phys. Rev. D63(2001) 075007.
[18] J. Hewett, F.J. Petriello and T.G. Rizzo, Phys. Rev. D64(2001) 075012; Phys. Rev. D66(2002) 036001.