Composite BPS configurations of p-branes in 10 and 11 dimensions

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Abstract

We give an overview of various composite BPS configurations of string theory and M-theory p-branes represented as classical supergravity solutions. Type II string backgrounds can be obtained by S- and T- dualities from the NS-NS configurations corresponding to exact conformal sigma models. The single-center solutions can be also generated from the Schwarzschild solution by applying a sequence of boosts, ‘smearings’ in some number of dimensions, dualities and taking the extremal limit. The basic ‘marginal’ backgrounds representing threshold BPS bound states of branes are parametrised by a number of independent harmonic functions, one for each brane. ‘Non-marginal’ BPS configurations in $D = 10$ can be constructed from the marginal ones using U-duality and thus are parametrised, in addition to harmonic functions, by a finite number of $O(d,d)$ and $SL(2,R)$ ‘angles’. Some of them can be viewed as dimensional reductions of coordinate-transformed (boosted or rotated) marginal configurations of M-branes. We present a new more general class of configurations in which some of the branes or their intersection spaces are localised on other branes. In particular, we find the supergravity background describing the type II BPS configuration of a 3-brane, RR 5-brane and NS-NS 5-brane, and related ‘localised’ 2-5-5 $D = 11$ solution. We also consider the classical action for a 3-brane probe moving in such type IIB backgrounds and determine the structure of the corresponding moduli space metrics.

February 1997

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1. Marginal and non-marginal BPS configurations of branes: an overview

1.1. Introduction

Viewing the $D = 10$ and $D = 11$ supergravities as the low-energy limits of the superstring theories and M-theory, it is important to have a better understanding of the structure of the space of their classical BPS solutions. Being supersymmetric, such solutions are expected to encode useful information about the corresponding states of the full quantum theory.

The existence of classical solutions describing BPS configurations of branes indicates a possibility of existence of the corresponding quantum bound states. The structure of actions of classical $p$-brane probes propagating in supergravity backgrounds produced by configurations of other branes gives (at least partial) information about related quantum theories. T- and S- duality connections between configurations of branes in $D = 10$ imply certain relations between their counterparts in $D = 11$ so that their study may help to identify hidden symmetries of (quantum) M-theory which are not explicit in the $D = 11$ supergravity action. Finally, intersections of branes wrapped over internal spaces represent lower-dimensional black holes and thus guide the studies of the black-hole properties by suggesting which configurations of ‘microscopic’ branes admitting quantum-mechanical description should be considered.

The stationary supergravity solutions representing composite BPS configurations of branes can be classified into ‘marginal’ (or ‘threshold’) and ‘non-marginal’ ones. The marginal backgrounds are the basic ones while the $D = 10$ non-marginal configurations fall into families of descendants of a ‘core’ marginal solution to which they are related by U-duality.\footnote{In what follows U-duality \cite{1} will mean a superposition of T- (i.e. $O(d, d)$) and S- (i.e. $SL(2, R)$) duality transformations. T-duality will be assumed to act in all possible isometric directions, including time. We shall discuss composite configurations of branes in $D = 10$ and 11; for reviews of $p$-brane solutions in various dimensions see \cite{2}.}

The marginal solutions are parametrised by a number $N$ of independent harmonic functions $H_i(x)$ which is equal to the number of branes in the configuration (counting also a possible wave along null direction).\footnote{The simplest example of a composite BPS solution parametrised by \textit{two} independent harmonic functions is the superposition of the fundamental string \cite{3} and a plane wave \cite{4}, representing, in the 1-center case, a BPS string state with a momentum flow along the string \cite{5,6}. Another example is a superposition of a fundamental string and a solitonic 5-brane $1||5$ \cite{7,8}. The existence of such NS-NS sector solutions parametrised by several harmonic functions follows directly from the conformal invariance condition on the string sigma-model (‘chiral null model’ \cite{9}). The $1||5$ configuration served as a starting point for the construction of various intersecting \cite{9} brane configurations in $D = 10, 11$ which are parametrised by several harmonic functions according to the ‘harmonic function rule’ \cite{10}.}
branes when all internal dimensions of the branes are isometries (so that the configuration can be viewed as an ‘anisotropic’ brane [11]) the functions $H_i$ satisfy the free flat-space Laplace equation with respect to the common transverse space coordinates. In more general cases of ‘localised’ intersections of branes discussed below some of $H_i$ may depend on internal coordinates of some branes and satisfy curved-space Laplace equations. When all of the harmonic functions have singularities at the same center, the mass of the marginal configuration is proportional to the sum of the ‘charges’, $M = Q_1 + \ldots + Q_N$.

Marginal configurations with the same number of (families of parallel) branes $N$ (that means, typically, with the same amount $1/2^N$ of unbroken supersymmetry) belong to one universality class being related by simple discrete T- and S- duality transformations combined with an operation of ‘smearing’ (or ‘delocalisation’, or forming an infinite periodic array) in some number of transverse dimensions. Starting with a marginal configuration, one can also apply a sequence of T- and S- duality transformations with arbitrary continuous (at the classical level) parameters. The result is a non-marginal configuration of branes which is parametrised by $N$ harmonic functions of its ‘parent’ marginal solution and a finite number of U-duality parameters (angles and boosts of $O(d,d)$ duality [12] and entries of the $SL(2,R)$ matrix of S-duality [13]). Since T- and S-dualities preserve supersymmetry, these non-marginal solutions have the same amount of unbroken supersymmetry as their ‘parent’. From the lower-dimensional point of view, they correspond to U-dual versions of black holes obtained by wrapping all isometric internal coordinates of a composite $p$-brane configuration over a torus. In the case of single-center harmonic functions the non-marginal solution will represent a configuration with $\tilde{N} > N$ charges and its mass will be typically of the form $M = \sum \sqrt{Q_1^2 + \ldots + Q_N^2}$, indicating a non-vanishing ‘binding’ energy.

The marginal configurations in $D = 10$ have direct counterparts in $D = 11$. One way to construct non-marginal configurations of branes in $D = 11$ is to lift up the $D = 10$ non-marginal solutions found by U-duality from the marginal type II theory ones. Some of them turn out to be just rotations and finite boosts of marginal M-brane intersections [14], but there are also other non-marginal $D = 11$ solutions (e.g., the 2 + 5 combination [15]). The general rule of constructing non-marginal $D = 11$ configurations from marginal ones (i.e. a counterpart of U-duality in $D = 10$ which applies directly in $D = 11$ (i.e. is a certain transformation of the $D = 11$ metric and 3-index tensor) remains to be explicitly formulated. While the S-duality acts in $D = 11$ simply as a coordinate transformation ‘mixing’ the directions of dimensional reduction and T-duality from IIA to IIB theory [16,17], the lift to $D = 11$ of the action of $O(d,d)$ duality on the space of $D = 10$ solutions was not yet described in general. This may give an important hint about new symmetries of M-theory.

In Section 1.2 we shall discuss the construction of various composite BPS configurations of type II and M-theory $p$-branes represented as classical supergravity solutions.
Some type IIB solutions (in particular, a non-marginal combination of a fundamental string and an instanton) will be described in Section 1.3. In Section 2 we shall consider in more detail the non-marginal configurations of branes in $D = 10$ and $D = 11$, and present some new examples of such solutions.

Section 3 will be devoted to a more general class of ‘localised’ configurations which have less isometries than ‘smeared’ intersections of branes found previously in [8]. We shall show how they can be constructed by applying dualities to the ‘fundamental string plus 5-brane’ type NS-NS backgrounds corresponding to exact conformal ‘chiral null models’. In particular, we shall find a supergravity background representing a type IIB intersection of a 3-brane, RR 5-brane and NS-NS 5-brane (the existence of such BPS configuration was pointed out in [19]). We shall also discuss some related solutions in $D = 10$ and $D = 11$, e.g., a localised configuration of a 2-brane and two 5-branes in $D = 11$. In Section 4 we shall study the classical actions for $p$-brane probes moving in such composite BPS backgrounds.

1.2. Construction of solutions and examples

The BPS configurations with single-center harmonic functions have ‘non-extremal’ generalisations (corresponding, upon compactification of isometric internal dimensions, to U-duality families of non-extremal black holes). It is remarkable that to construct such solutions there is no need to solve the classical equations explicitly [20]: all one needs to know is (i) the vacuum Schwarzschild solution, and (ii) T- and S-duality transformation rules of type II supergravity fields [16]. The one-center BPS solutions can then be obtained by taking the extremal limit. Indeed, starting with the neutral black string (i.e. ‘Schwarzschild $\times R_y$’), boosting it along isometric $y$-direction and applying T-duality one finds the non-extremal version of the fundamental string background [21]. S-duality relates it to the RR string of IIB theory. Adding extra isometries (i.e. smearing in transverse directions) and applying T-duality leads to all other $p > 1$ RR $p$-branes [23]. Acting by S-duality on the RR 5-brane one finds the non-extremal version of NS-NS 5-brane of type IIB theory, which, in turn, is related by T-duality in longitudinal direction to the (identical) 5-brane background of type IIA theory. More general U-duality transformations lead to non-extremal versions of non-marginal BPS configurations with 1/2 of supersymmetry.

To find composite brane solutions with two charges which become marginal 1/4 supersymmetric configurations in the extremal limit one may start with a black fundamental string ($ds_{10}^2 = H^{-1}(r)[-f(r)dt^2 + dy^2] + f^{-1}(r)dr^2 + r^2d\Omega_7^2$, $f = 1 - \mu/r^6$, etc.) and

\[\text{Infinite boost combined with sending the mass of the Schwarzschild solution } \mu \text{ to zero gives a plane wave background [22] which is T-dual to the extremal fundamental string.}\]
apply a boost. In the limit of the infinite boost and $\mu \to 0$ this gives a superposition of a fundamental string with a wave or $1 \uparrow$.

Similar more general extremal solution is parametrised by two independent harmonic functions $H_1$ and $H_w = 1 + K$

$$ds^2_{10} = H_1^{-1}(x)[-dt^2 + dz^2 + K(x)(dt - dz)^2] + dx_m dx_m,$$

(1.1)

$$e^{2\phi} = H_1^{-1}, \quad dB = dH_1^{-1} \wedge dt \wedge dz .$$

The existence of this solution follows from conformal invariance of the corresponding ‘chiral null model’ [4]. This configuration serves as a starting point for the construction of other marginal and non-marginal $1/4$ supersymmetric compositions of branes. The $SL(2,R)$ duality converts NS-NS objects into the RR ones and T-duality relates all of the RR $p$-branes. For example, we get the following U-duality sequence of solutions (adding extra transverse isometries by ‘smearing’ the string solution): $1_{NS} \uparrow \rightarrow 1_R \uparrow \rightarrow 5_R \uparrow \rightarrow 5_{NS} \uparrow \rightarrow 5_{NS} \parallel 1_{NS} \rightarrow 5_R \parallel 1_R \rightarrow 3 \perp 3 \rightarrow 4 + 0$, etc. Another sequence is $1_{NS} \uparrow \rightarrow 1_R \uparrow \rightarrow 1 \parallel 0$, etc.

In particular, the $1/4$ supersymmetric NS-NS background corresponding to $5_{NS} \parallel 1_{NS}$ can be obtained from the above one $1 \uparrow$ (1.1) just by using the standard T- and S-duality transformation rules. This solution describing a fundamental string smeared over

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4 We shall use the following notation for the bound states of branes. $p \parallel p'$ will denote a marginal composition of a $p$-brane and $p'$-brane in which their internal dimensions are parallel (in the single-center case one brane is on top of the other; in the case of multi-center harmonic functions this is a collection of parallel branes localised at different points). $p \perp p'$ will denote a marginal configuration representing orthogonal intersection of two branes (they may share some number $q(< p, p')$ of spatial dimensions and can be separated in transverse dimensions in the case of multi-center choice of the two independent harmonic functions). For most of the solutions discussed in Sections 1 and 2 the configuration of several branes will be ‘delocalised’ in all internal dimensions. $p \uparrow$ will stand for a $1/4$ supersymmetric bound state of a $p$-brane (with $p > 1$) and a plane wave, i.e. a configuration of a $p$-brane with a momentum flow along one of its longitudinal directions (it can be obtained, e.g., as a limit of a non-extremal $p$-brane infinitely boosted in the longitudinal direction). $p + p'$ will denote a non-marginal configuration representing a bound state of two branes which cannot be separated in the transverse dimensions. This is an interpolation between a $p$-brane and a $p'$-brane configurations and is parametrised by a single harmonic function specifying the common center(s) of the two branes forming the bound state. $(...)_n$ will indicate that the configuration is smeared over $n$ transverse directions, i.e. has $n$ extra isometries. All $D = 10$ metrics below are the string-frame metrics.
a solitonic 5-brane $5_{NS} || 1_{NS}$ was originally found directly, by using conformal sigma model considerations (which also imply its exactness to all orders in $\alpha'$) \[8\]

\[
ds_{10}^2 = H_5(x)[H_1^{-1}(x)H_5^{-1}(x)(-dt^2 + dz^2) + H_5^{-1}(x)dy_n dy_n + dx_m dx_m] ,
\]

\[1.2\]

\[
e^{2\phi} = H_5H_1^{-1} , \quad dB = dH_1^{-1} \wedge dt \wedge dz + *dH_5 ,
\]

where $(z, y_n)$ are the internal coordinates of the 5-brane $(n, m = 1, 2, 3, 4)$.

To construct marginal configurations with more than two charges one is to start again with non-extremal solution, add extra isometries, apply boost and T-duality, and take the extremal limit. In this way one finds the explicit form of the 3-charge configurations like $5_{NS} || 1_{NS}$, $3_{NS} \perp 3_{NS}$, etc. This procedure (i.e. U-duality) automatically determines the rules of intersections of branes which are consistent with the marginal BPS property. It also implies the ‘harmonic function rule’ dictating the dependence of background fields on harmonic functions. Since the U-duality transformations preserve supersymmetry, all $p$-branes dual to the fundamental string have $1/2$ of supersymmetry, all BPS combinations of branes dual to $1_{NS} \perp 1_{NS}$ have $1/4$ of supersymmetry and configurations dual to $1_{NS} || 5_{NS}$ have $1/8$ of supersymmetry. One is also guaranteed to have the same amount of supersymmetry for the $D = 11$ solutions obtained by ‘lifting up’ the type IIA backgrounds.

An alternative approach to determining the rules of constructing marginal configurations of $p$-branes (i.e. the intersection rule and the harmonic function rule) which applies to all possible $p$-brane choices in $D = 10$ and in $D = 11$ is based on consideration of an action of a $p$-brane probe moving in the supergravity background produced by another $p'$-brane and imposing the condition of the vanishing of a force on a static probe (marginal BPS state condition). This determines the relative orientation of the $p$-brane probe with respect to the $p'$-brane source and thus the intersection rule \[24\]. Thus the knowledge of single-brane solutions and the basic terms in the actions for their collective coordinates which follow from the supergravity actions makes possible to construct composite configurations of branes.\[3\] The conclusion is that in $D = 11$ the following intersections represent marginal BPS configurations (as originally suggested in \[3,27\]): $2_{NS} \perp 2(0)$, $5_{NS} \perp 5(3)$, $2_{NS} \perp 5(1)$ (figure in brackets is the number of common spatial directions of the two orthogonally intersecting branes). In $D = 10$ one finds that the following configurations are possible: (i) NS-NS intersections: $1_{NS} || 5_{NS}$, $5_{NS} \perp 5(3)$; (ii) RR intersections: $p_{NS} \perp q(n)$, $n = \frac{1}{2}(p + q) - 2$, i.e. $n = 0$: $4_{NS} || 0, 3_{NS} \perp 2_{NS}$; $n = 1$: $1_{NS} || 5_{NS} \perp 2, 3_{NS} \perp 3_{NS}$; $n = 2$: $6_{NS} || 5_{NS} \perp 4_{NS}$, etc.; (iii) ‘mixed’ intersections: for any RR $p$-brane $p_{NS}$ the following intersections are possible: $1_{NS} || p_{NS}(0)$ and $5_{NS} \perp p_{NS}(n)$, $n = p - 1$. Examples are $1_{NS} \perp 1_{NS}$ (which is T-dual to $(1 + 0)$ or $2 \uparrow$) and $5_{NS} \perp 1_{NS}$, $5_{NS} \perp 2$, $5_{NS} \perp 5_{NS}$.

\[5\] The intersection rules can be found also directly from the basic field equations \[25,26\].
The non-marginal solutions are obtained by applying more general T- and S-duality transformations. They depend on extra U-duality parameters and thus ‘interpolate’ between marginal configurations with the same amount of supersymmetry (and the same number of isometries). For example, applying $SL(2, R)$ transformation to the fundamental string $1_{NS}$ one finds the string-string bound state $1_{NS} + 1_{R}$. U-duality then relates this solution to other 1/2 supersymmetric non-marginal bound states; for example, $1_{NS} + 1_{R} \rightarrow 1_{NS} + 3 \rightarrow 1_{R} + 3 \rightarrow 0 + 2$, or $1_{NS} + 1_{R} \rightarrow 1_{NS} + 5_{R} \rightarrow 1_{R} + 5_{NS} \rightarrow 0 + 5$. The 2 + 0 and 5 + 0 non-marginal bound states can be obtained also as dimensional reductions of the $D = 11$ 2-brane and 5-brane finitely boosted in transverse 11-th dimension, 2 $\leftrightarrow$ and 5 $\leftrightarrow$ [14]. An alternative way to construct these configurations is to apply $O(d, d)$ duality transformations to single $p$-branes with a number of transverse isometries. For example, starting with a 0-brane ‘smeared’ over a line, finitely boosting it along the isometric direction and performing T-duality one finds again the $1_{NS} + 1_{R}$ solution [14]. Starting with a RR $p$-brane smeared over one transverse dimension and applying T-duality in the direction rotated by an angle in the plane formed by $y$ and an internal $p$-brane coordinate one finds the bound state of a RR $(p - 1)$-brane and a RR $(p + 1)$-brane, for example, $(1_{R})_1 \rightarrow 0 + 2$ or $(2)_1 \rightarrow 1_{R} + 3$ [28]. More general $O(d, d)$ duality transformations with several parameters lead to more complicated 1/2 supersymmetric configurations; for example, starting with 2-brane with extra 2 or 4 transverse isometries one finds $(2)_2 \rightarrow 4 + 2 + 0$ [28] or $(2)_3 \rightarrow 6 + 4 + 2 + 0$ [10].

To find non-marginal configurations with 1/4 of supersymmetry (depending on two harmonic functions and a number of parameters) one may start with a marginal solution $1 \uparrow$ (or $1 + 0$, or $2 \perp 2$, etc.) and apply U-duality with arbitrary ‘angles’ and ‘boosts’. This leads, in particular, to the explicit form of the supergravity background representing the $(4 + 2 + 0)\parallel 0$ configuration (see Section 2). Similarly, one can construct families of non-marginal configurations with 1/8 of supersymmetry (three harmonic functions) by starting with $N > 2$ marginal bound states like $2 \perp 2 \perp 2$, etc. They will include as special cases the non-marginal configurations depending on $N' < N$ different harmonic functions but the same number of U-duality parameters.

Lifting the marginal type IIA solutions to $D = 11$ one finds the marginal composite configurations of M-branes: $2, 5, 2 \perp 2, 2 \perp 5, 2 \perp 2 \perp 2$, etc. [14, 29, 30]. The existence of configurations with longitudinal momentum waves like $2 \uparrow, 5 \uparrow, 2 \perp 5 \uparrow, 5 \perp 5 \uparrow$ [10, 29, 30] can be viewed as a consequence of the existence of the corresponding $D = 10$ solutions in the NS-NS sector, $1 \uparrow, 5 \uparrow, 1 \parallel 5 \uparrow$ which are described by conformal chiral null models

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6 This symbolic notation indicates the nonvanishing charges present in the non-marginal bound state and also branes which are special cases of this more general configuration corresponding to limiting values of the angular parameters (this notation ignores the fact that there are actually several 2-brane charges corresponding to different orthogonal planes).
These solutions (and their generalisations to the case when the wave harmonic function depends on the null coordinate \( u = z - t \)) can be also constructed directly in \( D = 11 \) following the approach of [31].

Reducing the \( D = 11 \) solutions to \( D = 10 \) along different directions leads to several \( D = 10 \) marginal backgrounds which thus have a common origin in \( D = 11 \) (for various examples and a classification of such solutions see [10,29,18,32,33]). In particular, \( 5 \uparrow \) and \( 4 + 0 \) in \( D = 10 \) are the reductions of \( 5 \uparrow \) in \( D = 11 \). More complicated example is \( 2 \perp 5 \) which has the following counterparts in \( D = 10 \): (i) \( 2 \perp 4 \) which is T-dual to \( 1_\text{R} \parallel 5_\text{R} \) or \( 3 \perp 3 \); (ii) \( 1 \perp 4 \) which is T-dual to \( 5_\text{R} \uparrow \); (iii) \( 1 \parallel 5 \) which is T-dual to \( 5_\text{NS} \uparrow \) and \( 1_\text{NS} \parallel 5_\text{NS} \). These T-dual configurations are related by simple S-duality; this is consistent with the \( D = 11 \) interpretation of S-duality of IIB theory as a coordinate transformation that interchanges the directions of dimensional reduction and T-duality between type IIA and type IIB theories [16,13].

Thus part of U-duality relating different configurations of type IIA branes in \( D = 10 \) becomes simply a coordinate transformation in \( D = 11 \). Dimensionally reduced to \( D = 10 \), the \( D = 11 \) solutions with the same \( N \) become connected by U-duality. However, there is no known analogue of U-duality which would relate them directly in \( D = 11 \).

Similar conclusions follow from consideration of non-marginal backgrounds. Some of \( D = 10 \) non-marginal solutions can be viewed as reductions of coordinate-transformed (rotated or boosted) marginal configurations in \( D = 11 \) [14] (for example, as already mentioned above, \( 2 + 0 \) and \( 5 + 0 \) are reductions of finitely boosted M-branes). The coordinate transformations applied to marginal configurations in \( D = 11 \) do not lead, however, to the full U-duality families of non-marginal bound states in \( D = 10 \). In addition to coordinate transformations of the marginal configurations there are also other non-trivial non-marginal \( D = 11 \) bound states (like \( 2 + 5 \) [17] and its generalisations [34]) which are lifts of other ‘parts’ of U-duality families in \( D = 10 \). We shall discuss examples of new \( D = 11 \) solutions of that kind in Section 2 below.

1.3. Some type IIB solutions

As follows from the above discussion, all 1/2 supersymmetric BPS configurations of \( p \geq 0 \) branes in \( D = 10 \) can be constructed by starting with a 0-brane background, smearing it in some number of dimensions and applying U-duality. At the same time, the basic object of lowest space-time dimensionality is the type IIB instanton [35]

\[
ds_{10}^2 = H_{-1}^{1/2}(x) dx_\mu dx_\mu , \quad e^\phi = H_{-1} , \quad C = (H_{-1})^{-1} - 1 \quad (1.3)
\]

Here \( \mu = 0, 1, ..., 9 \) and \( H_{-1} \) is the harmonic function \((= 1 + Q/x^8 \) for a single instanton\). We consider the type IIB theory with euclidean time \( x_0 = it \) so that \( C = iC \), where \( C \) is the RR scalar. Taking a distribution of instantons along \( x_0 \), so that \( x_0 \) becomes an
isometry, $ds^2_{10} = H_0^{1/2}(dx^2_0 + dx_m dx_m)$, $H_0 = 1 + Q/x^7$, and applying T-duality in this direction one finds the 0-brane solution [23] of type IIA theory. The backgrounds for all other RR $p$-branes can be constructed in a similar way by spreading instantons over a $(p + 1)$-dimensional space-time and performing T-duality transformations. The NS-NS branes can then be obtained by S-duality (for example, the type IIA 5-brane is T-S-T dual of the instanton solution smeared over a 6-plane).

Starting with a stationary type IIA $p$-brane configuration and including the T-duality along the isometric time direction into the set of possible $O(d,d)$ transformations, one is able to construct various ‘instanton-type’ solutions of type IIB theory. Some of such backgrounds will have the (euclidean) time direction being orthogonal to the other RR directions one finds the 0-brane solution [23] of type IIA theory. The backgrounds for all other $O$-planes will have the metric [37]

$$ds^2 = (H_1 H_3)^{1/2} (H_3^{-1} dx_k dx_k + dx_m dx_m), \quad iC = H_0^{-1} - 1, \quad e^\phi = H_0.$$

A non-trivial example of a marginal euclidean type IIB configuration parametrised by two harmonic functions is a bound state of a (smeared) instanton and a 3-brane $3\parallel(-1)$ which has the metric [37]

$$ds^2_{10} = (H_{-1} H_3)^{1/2} (H_3^{-1} dx_k dx_k + dx_m dx_m),$$

where $k = 1, ..., 4, m = 5, ..., 10$, and $H_{-1}$ and $H_3$ are the harmonic functions depending only on $x_m$. This solution is T-dual to $4\parallel0$ or $5R\parallel1R$.

Simplest non-marginal type IIB configurations can be found by applying $O(d,d)$ duality to a 0-brane with one extra transverse isometry. Starting with $0 \mapsto$, i.e. the 0-brane finitely boosted in the isometric direction to the velocity $v = \cos \theta$

$$ds^2_{10A} = \tilde{H}^{1/2} (\ - \tilde{H}^{-1} d\tilde{t}^2 + d\tilde{y}^2 + dx_m dx_m),$$

$$dA = d\tilde{H}^{-1} \wedge d\tilde{t}, \quad e^{2\phi} = \tilde{H}^{3/2}, \quad \tilde{H} - 1 = \sin^2 \theta (H - 1), \quad H = H_0,$n

$$\tilde{t} \equiv (\sin \theta)^{-1} (t - \cos \theta y), \quad \tilde{y} \equiv (\sin \theta)^{-1} (y - \cos \theta t),$$

and performing T-duality in $y$ gives [14] the string-string solution $1_{NS} + 1_R$ of [13]

$$ds^2_{10B} = \tilde{H}^{1/2} [H^{-1} (-dt^2 + dy^2) + dx_m dx_m], \quad e^{2\phi} = H^{-1} \tilde{H}^2,$$

$$C = \sin \theta \cos \theta (H - 1)\tilde{H}^{-1}, \quad dB + i dC_2 = (\cos \theta + i \sin \theta) dH^{-1} \wedge dt \wedge dy.$$
T-duality in $t$ direction produces the same background with $\cos \theta \rightarrow \frac{1}{\cos \theta}$, $\sin \theta \rightarrow \frac{\sin \theta}{\cos \theta}$ (note that the boosted configuration (1.6) is ‘symmetric’ in $t$ and $y$), i.e. (1.7) and

$$C = \frac{\sin \theta}{\cos^2 \theta} (H - 1) \tilde{H}^{-1}, \quad dB + i dC_2 = \frac{1 + i \sin \theta}{\cos \theta} dH^{-1} \wedge dt \wedge dy. \quad (1.9)$$

If we set $H - 1 = \cos^2 \theta (H_{-1} - 1)$, $\tilde{H} - 1 = \frac{\sin^2 \theta}{\cos^2 \theta} (H - 1) = \sin^2 \theta (H_{-1} - 1)$ then this non-marginal type IIB configuration can be interpreted as a bound state of a fundamental string and an instanton, $1_{NS} + (-1)$. Indeed, in the zero-boost limit ($\theta = \frac{\pi}{2}$, $H = 1$, $\tilde{H} = H_{-1}$) this becomes the instanton (1.3) (smeread over a 2-plane) while in the infinite boost limit ($\theta = 0$, $H = H_{-1}$, $\tilde{H} = 1$) we get the fundamental string background. Since both $1_{NS} + 1_R$ and $1_{NS} + (-1)$ are dual to $0 \mapsto \infty$ they are related by $O(2,2)$ duality.

Treating $\theta$ as a complex parameter one finds the background which formally interpolates between all three limiting cases: $1_{NS}$, $1_R$ and $-1$. To clarify why such interpolation is possible, let us start with unboosted 0-brane with an extra isometry $y$ and rotated time direction $x_0 = it$. T-duality along $x_0$ gives the instanton, while T-duality along $y$ gives the RR string (continued to euclidean time). T-duality along a rotated direction $y' = \cos \psi y + \sin \psi x_0$ then produces the non-marginal $1_R + (-1)$ background. It has the same structure as (1.7), (1.8) but now with $t \rightarrow -ix_0$, $\tilde{H} = H_0$, $H - 1 = \cos^2 \psi(H_0 - 1)$ and $\cos \psi = \frac{1}{\sin \theta}$, $\sin \psi = i \frac{\cos \theta}{\sin \theta}$ ($\psi$ plays the role of an imaginary boost parameter). The configuration $1_{NS} + (-1)$ may be of interest in connection with instanton matrix model discussed in [38].

2. Non-marginal BPS configurations in $D = 10$ and in $D = 11$

To find non-marginal solutions in $D = 10$ one may apply $O(d,d)$ and $SL(2,R)$ dualities to the marginal configurations. The basic transformations of $O(d,d)$ duality are ‘boost + T-duality’ and ‘rotation + T-duality’. As discussed above, boosting a 0-brane in an isometric direction and applying T-duality one finds $1_{NS} + 1_R$ type IIB configuration. Starting with the RR string $1_R$ along $y_1$ with an extra transverse isometry $y_2$ and applying T-duality along rotated direction in the $(y_1, y_2)$ plane one obtains the non-marginal $2 + 0$ type IIA configuration.

Below we shall consider more complicated examples of such non-marginal solutions which ‘interpolate’ between their marginal limiting cases and study their lifts to $D = 11$. The idea is to find $D = 11$ solutions which are parametrised by a number of harmonic

---

7 Though the solution under discussion is a non-marginal one, it is interesting to note that the metric (1.7) has the same structure as that of a would-be marginal superposition of an instanton and a fundamental string constructed according to the harmonic function rule with independent functions $\tilde{H}$ and $H$. 

9
functions \( H_i \) and parameters \( \varphi_i \) such that the variation of the ‘angles’ connects various marginal \( D = 11 \) solutions with the same number of independent harmonic functions and thus the same amount of supersymmetry. One way to construct such solutions is to start with any of limiting marginal \( D = 11 \) backgrounds, reduce it down in isometric directions to some lower \( D \), apply U-duality and then lift the resulting configuration back to \( D = 11 \). Studying such general families of solutions may help to learn how T-duality acts directly in \( D = 11 \): having a generalisation of a simple marginal solution ‘dressed’ by U-duality parameters, one may be able to extract the transformation rule of the \( D = 11 \) metric and the \( C_3 \) field that generates it. It is not clear a priori whether this procedure leads to new solutions, or, as it happens in \( D = 10 \), they are related to the basic marginal ones by a symmetry.

Let us start with simplest examples. The fundamental string \( 1_{NS} \) along \( y_1 \) in a space with one extra isometry \( y_2 \) is transformed by T-duality along a rotated direction \((y_2' = \cos \varphi \, y_1 + \sin \varphi \, y_2)\) into \( 1_{NS} \) finitely boosted in that angled direction (so that it has \( H \to \tilde{H} = 1 + \sin^2 \varphi(H - 1) \) and \( t \) and \( y_2' \) ‘mixed up’ with velocity \( \cos \varphi \)). Lift to \( D = 11 \) gives a 2-brane finitely boosted in an angled direction with the second internal direction being \( y_{11} \)

\[
ds_{11}^2 = \tilde{H}^{1/3}[\tilde{H}^{-1}(-dt^2 + dy_1^2 + dy_{11}^2) + d\tilde{y}_2^2 + dx_md_m] ,
\]

\[
\tilde{t} = (\sin \varphi)^{-1}(t - \cos \varphi \, y_2') , \quad \tilde{y}_2 = (\sin \varphi)^{-1}(y_2' - \cos \varphi \, t) .
\]

Similarly, applying T-duality at angle to \( 1_R \) one finds \( 2 + 0 \) configuration which is lifted to a 2-brane boosted along the orthogonal \( y_{11} \) direction

\[
ds_{11}^2 = \tilde{H}^{1/3}[\tilde{H}^{-1}(-dt^2 + dy_1^2 + dy_2^2) + d\tilde{y}_{11}^2 + dx_md_m] ,
\]

\[
\tilde{t} = (\sin \varphi)^{-1}(t - \cos \varphi \, y_{11}) , \quad \tilde{y}_{11} = (\sin \varphi)^{-1}(y_{11} - \cos \varphi \, t) .
\]

These two backgrounds correspond to the type IIB solutions related by discrete \((\theta = \frac{\pi}{2})\) \( SL(2,R) \) rotation, which in \( D = 11 \) thus corresponds to interchanging the direction of dimensional reduction \( y_{11} \) with the T-duality direction \( y_2' \). More generally, if we start with the solution \( 1_{NS} + 1_R \) (parametrised by the \( SL(2,R) \) angle \( \theta \)) and apply T-duality at an angle \( \varphi \) we get, after lifting the background to \( D = 11 \), the 2-brane solution where \( y_{11} \)

---

\(^8\) The \( D = 10 \) type II supergravity actions are known to be invariant under (or related by) T-duality transformations along abelian isometries. Though T-duality may look accidental at the level of the supergravity action, it has a microscopic explanation based on 2d duality on the string world sheet and the fact that \( D = 10 \) supergravity actions are the low-energy effective actions of string theories. When this symmetry is lifted up to the level of \( D = 11 \) there is no known general microscopic explanation for it.
and \( y'_1 \) are rotated by \( \theta \). The resulting boosted and rotated 2-brane solution is parametrised by the harmonic function and the two angles \((\theta, \varphi)\).

It may seem that there is a correspondence between rotation in type IIB theory and boost in \( D = 11 \) theory: a rotation and T-duality applied to \( 1_R \) leads to the same type IIA configuration \( 2 + 0 \) as a finite boost and dimensional reduction applied to the \( D = 11 \) 2-brane. This relation is not, however, universal: for example, T-duality applied to rotated 3-brane gives the \( 4 + 2 \) solution \([28]\) which is the reduction of static \( 5 + 2 \) configuration \([13]\) discussed below. At the same time, the reduction of finitely boosted M5-brane \( 5 \to \to \) gives the \( 5 + 0 \) non-marginal configuration in \( D = 10 \) \([14]\) which is not T-dual to a rotated type IIB solution. Another example is \( 6 + 4 \) configuration which is T-dual to rotated \( 5_R \) and is a dimensional reduction of \( 7_{KK} + 5 \), i.e. of a \( 1/2 \) supersymmetric non-marginal configuration which is an interpolation between the \( D = 11 \) ‘Kaluza-Klein 7-brane’ (or a KK monopole \([39]\) ) and a 5-brane.

Instead of boosting M-branes one can also rotate them. The reduction along the rotated direction also produces non-marginal configurations in \( D = 10 \) \([14,40]\). For example, a plane wave along generic cycle of 2-torus in \( D = 11 \) leads to finitely boosted 0-brane \( 0 \to \to \) in \( D = 10 \) and 2-brane reduced along rotated direction becomes \( 2 + 1 \) bound state in \( D = 10 \). The T-duality relations in \( D = 10 \) are \( 2 + 1 \to 1_{NS} + 1_R \to 0 \to \to \). In general, finitely boosting a RR \( p \)-brane \( p_R \) smeared in one transverse direction and applying T-duality along this direction leads to \( (p + 1)_R + 1_{NS} \) configuration, i.e. a non-marginal bound state of a RR \( (p + 1) \)-brane and a fundamental string \([14,40]\).

This illustrates how some of U-duality parameters in \( D = 10 \) are simply the parameters of coordinate transformations (boosts and rotations) in \( D = 11 \): in general, a coordinate transformation of a \( D = 11 \) solution and its dimensional reduction leads to the same type IIA configuration as certain coordinate transformation and T-duality applied to a type IIB solution.

Turning to more complicated examples, let us consider the non-marginal solution \( 5 + 2 \) \([13]\) depending on harmonic function and one extra angle \( \theta \) (in the single-center case it is parametrised by two charges). It ‘interpolates’ between the basic marginal \( D = 11 \) solutions – the 2-brane \( (\theta = 0) \) and the 5-brane \( (\theta = \frac{\pi}{2}) \). The corresponding metric is

\[
 ds^2_{11} = H^{1/3} \tilde{H}^{1/3} \left[ H^{-1}(-dt^2 + dy_1^2 + dy_2^2) + \tilde{H}^{-1}(dy_3^2 + dy_4^2 + dy_5^2) + dx_m dx_m \right], \quad (2.3)
\]

where \( \tilde{H} = 1 + \sin^2 \theta (H - 1) \).

The (static) marginal configurations with two harmonic functions are \( 2 \perp 2, 2 \perp 5 \) and \( 5 \perp 5 \). To find how to embed \( 2 \perp 2 \) and \( 2 \perp 5 \) into a single family of \( D = 11 \) solutions let us note that the reduction of \( 2 \perp 5 \) to \( D = 10 \) is \( 2 \perp 4 \) which is T-dual to \( 1 \perp 3 \) or \( 2 \perp 2 \). Let us start with \( 1 \perp 3 \) with one extra isometry \( (y_5) \) and do T-duality along the rotated direction in the \( (y_4, y_5) \) plane, where \( y_4 \) is one of the 3-brane’s directions. If the rotation angle is
zero, i.e. the T-duality is along \( y_5 \), we get 2\( \perp \)4 which lifts up to 2\( \perp \)5. If the angle is \( \frac{\pi}{2} \) we get (2\( \perp \)2)\( \perp \) which lifts up to 2\( \perp \)2. The resulting \( D = 11 \) background which interpolates between 2\( \perp \)2 and 2\( \perp \)5 thus has the following metric

\[
\begin{align*}
\begin{aligned}
ds_{11}^2 &= \tilde{H}_3^{1/3} H_1^{1/3} \left[ -H_1^{-1} H_3^{-1} dt^2 + H_1^{-1} dy_1^2 + H_3^{-1} (dy_2^2 + dy_3^2) \right]\\+
&\quad + \tilde{H}_3^{-1} H_3^{-1} dy_5^2 + \tilde{H}_3^{-1} (dy_4^2 + dy_{11}^2) + dx_m dx_m \end{aligned}
\end{align*}
\]

where \( \tilde{H}_3 = 1 + \sin^2 \varphi (H_3 - 1) \) (in the single-center case \( H_i = 1 + Q_i/x^2 \)). This solution can be generalised to include two more angles that will ‘connect’ 2\( \perp \)2 to 2\( \perp \)5. As a result, one finds a family of 1/4 supersymmetric non-marginal \( D = 11 \) backgrounds which is parametrised by two independent harmonic functions and three angles, and which contains the marginal configurations 2 \( \perp \) 5, 2 \( \perp \) 2 and 2 \( \perp \) 5 as special cases (equivalent (2 \( \perp \) 5) \( \perp \) solution appeared in [34]). Similar construction can be carried out by starting with 5 \( \perp \) 5 \( \perp \) configuration and finding a non-marginal family of solutions that will include 2 \( \perp \) 2, 2 \( \perp \) 2 \( \perp \) and 5 \( \perp \) 5 \( \perp \) as special limiting cases. Some examples of composite non-marginal \( D = 11 \) configurations with 1/4 and 1/8 of supersymmetry (\( (2 + 5) \perp (2 + 5) \), (2 + 5) \( \perp \) (2 + 5), etc.) were constructed in [34].

To find a non-marginal 1/4 supersymmetric \( D = 11 \) background which will include 2\( \perp \)2 and 5 \( \uparrow \) as limiting cases let us start with the \( D = 10 \) type IIA 2\( \perp \)2 solution with \((y_1, y_2)\) and \((z_1, z_2)\) as the internal spaces of the two 2-branes and apply T-duality twice in the angled \((\varphi, \psi)\) directions in the planes \((y_1, z_1)\) and \((y_2, z_2)\). Lifting the resulting \( D = 10 \) configuration (which can be denoted symbolically as 4 \( \perp \) 2 \( \perp \) 0 or as \((4 + 2 \perp 0) \perp 0\)) to \( D = 11 \) gives the following 1/4 supersymmetric background parametrised by two harmonic functions and two rotation angles

\[
\begin{align*}
ds_{11}^2 &= H_\varphi^{1/3} H_\psi^{1/3} \left[ -H_1^{-1} H_2^{-1} dt^2 + H_\varphi^{-1} (dy_1^2 + dz_1^2) + H_\psi^{-1} (dy_2^2 + dz_2^2) \right] \quad (2.5)
\end{align*}
\]

9 The special case of this construction when the harmonic function of one of the two 2-branes was set equal to one was first considered in [28]. Another obvious generalisation is to apply T-duality at angles to 2 \( \perp \) 2 \( \perp \) configuration. In the case when only one of the three harmonic functions is non-trivial (i.e. when one 2-brane is ‘smeared’ over 4 orthogonal directions) one finds the 1/2 supersymmetric non-marginal bound state ‘6 + 4 + 2 + 0’ considered from D-brane point of view in [41]-[42]. Related non-marginal \( D = 11 \) configurations were discussed in [43].

10 Such 1/4 supersymmetric non-marginal bound state was considered in D-brane description in [44] where the existence of the corresponding supergravity solution was also conjectured. It was noted there that taking a 4-brane with a self-dual \( F_{mn} \) background (i.e. with only one independent parameter) and adding a 0-brane one finds a trivial static potential, implying that there should exist a 1/4 supersymmetric BPS bound state of this generalised ‘4-brane’ with an extra 0-brane.
\[ + H_1H_2H_\varphi^{-1}H_\psi^{-1}(dy_{11} - Adt)^2 + dx_m dx_m \],

\[ H_\varphi \equiv 1 + (H_1 - 1)\cos^2 \varphi + (H_2 - 1)\sin^2 \varphi , \quad H_\psi \equiv 1 + (H_1 - 1)\cos^2 \psi + (H_2 - 1)\sin^2 \psi , \]

\[ A = H_1^{-1}\sin \varphi \sin \psi + H_2^{-1}\cos \varphi \cos \psi , \]

\[ C_3 = (H_1 - H_2)[\sin \varphi \cos \varphi H_\varphi^{-1}dy_1 \wedge dz_1 - \sin \psi \cos \psi H_\psi^{-1}dy_2 \wedge dz_2] \wedge dy_{11} + \ldots \]

In the single-center case \( H_i = 1 + Q_i/x^3 \), \( i = 1, 2, \varphi, \psi \), and \( Q_\varphi = Q_1\cos^2 \varphi + Q_2\sin^2 \varphi \), \( Q_\psi = Q_1\cos^2 \psi + Q_2\sin^2 \psi \). We have written down explicitly only those terms in \( C_3 \) which vanish for \( \varphi = \psi = 0 \). Other terms (which are found using the T-duality transformation rules in the RR sector [16]) include the \( 2\perp 2 \)-type structures \((H_1^{-1}dy_1 \wedge dz_1 \wedge dt + H_2^{-1}dy_2 \wedge dz_2 \wedge dt)\) as well as a ‘magnetic’ 5-brane type term.

Some special cases of the solution (2.5) are: \( \varphi, \psi = 0 : 5 \uparrow \); \( \varphi = 0, \psi = \pi/2 \) : \( (2\perp 2)_1 \); \( H_2 = 1, \psi = 0 : 2 + 5 \); \( H_2 = 1, \psi = \pi/2 \) : \( (2 \mapsto 3)_3 \). Other special cases are found by using the symmetries between \( Q_1, Q_2, \varphi, \psi \). For example, the metric of \((2 \mapsto 3)_3\) (a 2-brane smeared in 3 isometric directions and finitely boosted along of them) is indeed given by (2.3) with \( \psi = \pi/2 \), \( H_\psi = 1 \), \( H_2 = 1 \)

\[ ds_{11}^2 = H_\varphi^{1/3}[-H_1^{-1}dt^2 + H_\varphi^{-1}(dy_1^2 + dz_1^2)] + dy_2^2 + dz_2^2 \quad (2.6) \]

\[ + H_1H_\varphi^{-1}(dy_{11} - H_1^{-1}\sin \varphi dt)^2 + dx_m dx_m \].

We thus get an interpolation between the \( 1/2 \) supersymmetric \( 2 + 5 \) configuration and the transversely boosted 2-brane in space with three extra isometries \( (2 \mapsto 3)_3 \), and the \( 1/4 \) supersymmetric infinitely boosted 5-brane \( 5 \uparrow \) and \( 2\perp 2 \) in the space with one extra isometry. The supersymmetry is increased only when one of the two independent harmonic functions is set equal to 1. The appearance of 5-brane is not surprising as \( 2\perp 2 \) is T-dual (with T-duality applied twice along one of two 2-branes) to \( 4 + 0 \) which can be lifted up to \( 5 \uparrow \) in \( D = 11 \).

As already mentioned above, it would be important to describe the rules of constructing similar complicated non-marginal solutions directly in terms of \( D = 11 \) theory. In view of the relation via dimensional reduction, the set of non-marginal configurations in \( D = 11 \) is in one-to-one correspondence with the set of non-marginal configurations in \( D = 10 \). However, in contrast to the \( D = 10 \) case, the action of T-duality on the \( D = 11 \) set remains to be understood.
3. More general ‘localised’ configurations of branes in $D = 10$ and $D = 11$

3.1. String localised on 5-brane and related solutions

It was noted in the previous sections that essentially all marginal (and thus also non-marginal) composite BPS configurations of branes in $D = 10$ can be constructed by applying S- and T- dualities to the basic NS-NS backgrounds $1 \uparrow$, or $1 \parallel 5$, or $1 \parallel 5 \uparrow$ which correspond to exact conformal sigma-models. These backgrounds are parametrised by several harmonic functions satisfying the flat-space Laplace equations. As we shall discuss below, starting with more general fundamental string type NS-NS backgrounds parametrised by functions satisfying curved-space Laplace equations one is able to construct more general composite $p$-brane configurations by applying S-duality and T-duality in isometric directions. While the intersecting brane solutions in [10] and the previous sections were isometric in all internal directions of the branes (i.e. the position of the intersection was ‘smeared’ over the branes) and thus can be interpreted also as single anisotropic $p$-branes [11] these more general backgrounds correspond to ‘localised’ intersections.

As was shown in [4], the following string sigma-model $(u, v = z \mp t)$

$$L = H_1^{-1}(X)\partial u \bar{\partial} v + L_\perp(X) - \frac{1}{2} \alpha' \sqrt{g_2} R_2 \ln H_1(X) ,$$

(3.1)

is conformal to all orders in $\alpha'$ provided the transverse theory defined by $L_\perp = (G_{ij} + B_{ij})\partial X^i \bar{\partial} X^j + \alpha' \sqrt{g_2} R_2 \phi_\perp(X)$ is conformal and $H_1$ satisfies the marginality condition, or the generalised Laplace equation, $\nabla^i (e^{-2\phi_\perp} \partial_i) H_1 + ... = 0$. Dots stand for higher-order terms which are absent when the transverse theory is $(4,4)$ supersymmetric as will always be the case in the examples discussed below. The ‘chiral null model’ (3.1) admits a generalisation where one includes also the terms like $K(X)\partial u \bar{\partial} u$ and $A_i(X) \partial u \bar{\partial} X^i$.

The standard fundamental string solution [3] corresponds to the trivial choice $L_\perp = \partial X^i \bar{\partial} X^i$ so that $H_1(X)$ is a flat-space harmonic function. Examples of more complicated solitonic 5-brane-type [44] choices of exact (super)conformal $L_\perp$ where considered in [7,8]. The resulting solitonic backgrounds may be interpreted as a string lying on a 5-brane superposed with a Kaluza-Klein monopole [4], a string lying on a solitonic 5-brane [8], or a string lying on a superposition of two solitonic 5-branes [8]. In the simplest case when the 8-dimensional transverse space part of (3.1) represents a 5-brane wrapped over a flat 4-torus one finds the following background $(X = (y_n, x_m), \ n, m = 1, ..., 4, \text{cf.}(1.2))$ [8]

$$ds_{10}^2 = H_1^{-1}(x, y)(-dt^2 + dz^2) + dy_n dy_n + H_5(x) dx_m dx_m ,$$

(3.2)

$$dB = dH_1^{-1} \wedge dt \wedge dz + *dH_5 , \ e^{2\phi} = H_1^{-1} H_5 ,$$

14
where \((z, y_n)\) are the internal dimensions of the 5-brane, \(H_5(x)\) is the harmonic function \((\partial^m \partial_m H_5 \equiv \partial_x^2 H_5 = 0)\) defining the position of 5-brane(s) and the string function \(H_1(x, y)\) satisfies

\[
[\partial_x^2 + H_5(x)\partial_y^2]H_1(x, y) = 0 .
\] (3.3)

The same equation should be satisfied by the function \(K(x, y)\) of the longitudinal wave. An obvious special solution is found by taking \(H_1\) as a product (or a sum) of the two special harmonic functions, \(H_1(x, y) = H_1(x)H_1'(y)\). Such factorised solution does not match, however, onto a (sum of) delta-function string source(s) \(-\mu \delta^{(4)}(x)\delta^{(4)}(y)\) which should be present in the r.h.s. of (3.3) for a localised string solution. Though it is not important for what follows, it is natural to assume that the solution \(H_1(x, y)\) of (3.3) should be chosen in such a way that in the limit \(H_5 \to 1\) it becomes a free fundamental string one with the harmonic function having isolated singularities in the 8-dimensional transverse space. Unfortunately, it turns out that such a solution does not have a simple expression in terms of elementary functions even for the one-center choice of the 5-brane function, \(H_5 = 1 + Q/x^2\).

Ref. [8] concentrated on the special solution for which \(H_1\) does not depend on the 5-brane coordinates \(y_n\) transverse to the string. The main reason was that such a background directly corresponds to an extremal black hole in \(D = 5\) upon dimensional reduction. In this special case \((H_1(x, y) = H_1(x), \partial_x^2 H_1 = 0)\) the string is smeared over the 5-brane so that the background has 5 spatial isometries and is parametrised by the flat-space harmonic functions \(H_5(x)\) and \(H_1(x)\). More general solutions representing a string localised on the 5-brane, i.e. with \(H_1\) having non-trivial dependence on \(y_n\) were recently discussed in [45,46]. Similar ‘localised’ generalisations exist for the conformal models in [7] which, in the ‘smeared’ case, describe extremal black holes in \(D = 4\) which have regular horizons.

The localised solutions have the same amount of supersymmetry and the same BPS marginality property as the ‘smeared’ ones. These properties are universally determined by the special holonomy of the generalised connection of the corresponding chiral null model (3.1) (implying also its \((4, 4)\) supersymmetry in the case of type II superstring theory). They are the consequences of the special choice of the transverse theory (and of the chiral null structure of (3.1)) and do not depend on the form of \(H_1(x, y)\). However, in contrast to the delocalised solution where the two harmonic functions \(H_1(x)\) and \(H_5(x)\) appear on an equal footing and specify the position(s) of the string(s) and 5-brane(s) (which, in general, are independent and arbitrary), the roles of \(H_5(x)\) and \(H_1(x, y)\) are obviously asymmetric in the localised case (3.3). To distinguish the localised intersections from the smeared ones we shall put ‘hats’ on the symbols \(\parallel, \perp\), i.e. the configuration described by (3.2) will be denoted as \(1_{NS} \parallel 5_{NS}\).

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11 I am grateful to J. Maldacena for pointing out an error in this equation in the original version of [8] and useful discussions of related localised p-brane solutions.
Applying S-duality to $1_{NS} \parallel 5_{NS}$ as a type IIB solution we find the $1_{R} \parallel 5_{R}$ configuration describing a RR string localised on RR 5-brane (which generalises the ‘smeared’ configuration used in [17]). T-duality along the isometric $z$-direction gives the $4 \parallel 0$ configuration with the position of the 0-brane (determined by $H_{1}(x, y)$) being localised on the 4-brane. If we assume that $H_{1}(x, y)$ does not depend on one of the four 5-brane coordinates, i.e. that the string is smeared over, e.g., $y_{1}$, then applying T-duality along $y_{1}$ we find the $2 \perp 4$ type IIA configuration in which the position of the intersection string is localised on the 4-brane $(z, y_{2}, y_{3}, y_{4})$ but not on the 2-brane $(z, y_{1})$. This asymmetry becomes even more apparent if we delocalise the solution also in $y_{2}$ and again apply T-duality along the resulting isometric direction. Using T-duality transformation rules [16] one finds the following $3 \perp 3$ configuration (which generalises the ‘smeared’ $3 \perp 3$ solution [8])

$$ds^{2}_{10} = (H_{3}H_{3}')^{1/2} \left[ (H_{3}H_{3}')^{-1}(-dt^{2} + dz^{2}) + H_{3}^{-1}(dy_{1}^{2} + dy_{2}^{2}) + H_{3}^{-1}(dy_{3}^{2} + dy_{4}^{2}) + dx_{m}dx_{m} \right],$$

$$dC_{4} = dt \wedge dz(dH_{3}^{-1} \wedge dy_{1} \wedge dy_{2} + H_{3}'^{-1} \wedge dy_{3} \wedge dy_{4})$$

$$+ * d_{x}H_{3} \wedge dy_{1} \wedge dy_{2} + * dH_{3}' \wedge dy_{3} \wedge dy_{4} + H_{3}' \wedge dy_{3} \wedge dx_{1} \wedge dx_{2} + dx_{3} \wedge dx_{4},$$

$$H_{3} = H_{1}(x, y_{3}, y_{4}), \quad H_{3}' = H_{5}(x), \quad e^{2\phi} = 1,$$

where $* d_{y} \equiv dy_{4} \partial y_{3} - dy_{3} \partial y_{4}$. It seems unlikely that there exists a generalisation of this solution in which the two 3-brane functions $H_{3}$ and $H_{3}'$ appear symmetrically (i.e. $H_{3}'$ depends on $y_{1}, y_{2}$) and which still has a BPS property. T-duality along $z$-direction then would give the localised $2 \perp 2$ solution smeared only in one transverse direction[12].

More localised intersections can be constructed by first smearing in some of the transverse directions and then using T-duality. For example, applying T-duality to the above $3 \perp 3$ configuration along $x_{1}$ one finds $4 \perp 4$ solution where the intersection 2-brane is localised only on one of the two 4-branes. Another example is obtained by starting with $1 \parallel 5$ smeared in one transverse direction, i.e. having isometry in $x_{4}$ coordinate in (3.2). T-duality along $x_{4}$ then converts the solitonic 5-brane part of (3.2) into the ‘Kaluza-Klein 5-brane’ (or KK monopole) part, which is a purely gravitational (Euclidean Taub-NUT) background

$$ds^{2}_{10} = H_{1}^{-1}(-dt^{2} + dz^{2}) + dy_{n}dy_{n} + H_{5}^{-1}(dx + B_{1}dx_{m})^{2} + H_{5}dx_{m}dx_{m},$$

\[12\] Such solution would not have an obvious analogue in the NS-NS sector. The structure of the metric of the intersecting $p$-brane solutions seems to be rather rigidly fixed by the BPS condition [24]. Also, eq. (3.3) does not seem to admit a non-trivial generalisation to a system of two equations for $H_{1}$ and $H_{5}$, both depending on $y_{n}$ (related observations were made in [48]).
\[ dB = dH_1^{-1} \wedge dt \wedge dz \] \[ e^{2\phi} = H_1^{-1} \] \[ dB = *dH_5 \] \[ H_1 = H_1(\vec{x}, y) \] \[ H_5 = H_5(\vec{x}) \]

We have set \( x_n = (x_m, x_4) \) and used \( x \) to denote the coordinate dual to \( x_4 \). Starting instead with \( 5R\parallel_1 \parallel_1 \) solution and applying T-duality along \( x_4 \) and \( y_4 \) we find the \( 5R\parallel_1 \parallel_3 \) (and, by S-duality, \( 5NS\parallel_1 \parallel_3 \)) configuration, i.e. the intersection of a 5-brane (which is smeared over \( x \)) with a 3-brane over a 2-space which is localised on the 5-brane. More general backgrounds including these as special cases will be considered below.

Lifting the \( 1\parallel_1 \parallel_5 \) solution (3.2) to \( D = 11 \) by adding the isometric direction \( y_{11} \) one finds the M-brane intersection \( 2\parallel_1 \parallel_5 \) where the intersection string is localised on the 5-brane \((z, y_1, y_2, y_3, y_4)\) but not on the 2-brane \((z, y_{11})\).

\[ ds_{11}^2 = H_2^{1/3} H_5^{2/3} [H_2^{-1} H_5^{-1} (-dt^2 + dz^2) + H_2^{-1} dy_{11}^2 + H_5^{-1} dy_n dy_n + dx_m dx_m] \] \[ dC_3 = (dH_2^{-1} \wedge dt \wedge dz + *dH_5) \wedge dy_{11} \] \[ H_2 = H_1(x, y) \] \[ H_5 = H_5(x) \]

Similarly, lifting the configuration \( 4\parallel_1 \parallel_4 \) (T-dual to \( 3\parallel_1 \parallel_3 \) as mentioned above) to \( D = 11 \) one finds the \( 5\parallel_1 \parallel_2 \) solution, again with asymmetric localisation of the intersection 3-brane on only one of the two 5-branes. Starting with \( 3\parallel_1 \parallel_3 \) smeared in \( y_3 \) and applying T-duality along this coordinate one finds the \( 2\parallel_1 \parallel_4 \) solution with the intersection string localised on the 2-brane \((H_4 = H_3(x, y_4), H_2 = H_2(x))\). Lifting this solution to \( D = 11 \) gives the \( 5\parallel_1 \parallel_2 \) solution similar to (3.6) where \( H_2 = H_2(x) \), \( H_5 = H_4(x, y_4) \), i.e. the intersection is localised on the 2-brane instead of the 5-brane. These 1/4 supersymmetric localised intersecting M-brane solutions were independently found by J. Gauntlett.

Localised BPS configurations of branes parametrised by three harmonic functions can be constructed in an analogous way by starting with a generalisation of (3.2) with an additional function \( K = H'(x, y) - 1 \) satisfying (3.3) and representing the longitudinal momentum wave along the string localised on the 5-brane, \( 5\parallel_1 \parallel_1 \). One can then construct various intersecting configurations by relaxing localisation in some of the internal 5-brane coordinates and/or smearing in some of the transverse dimensions and applying S- and T-duality. Lifting the resulting backgrounds to \( D = 11 \) leads to 1/8 supersymmetric intersections of M-branes with (varied amount of) localisation, e.g., \( 2\parallel_1 \parallel_5 \parallel_5 \), etc.

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13 One may consider a periodic array of 5-branes in \( x \)-direction, all intersected by a 3-brane having \( x \) as the dimension transverse to the 5-brane. Similar configurations (but with localisation in \( x \)) were discussed in [19].

14 This localised solution may be of interest in the context of discussions [40] of 3-branes as intersections of two M5-branes.

15 Let us note also that applying T-duality to \( 3\parallel_1 \parallel_3 \) along the intersection string direction gives \( 2\parallel_1 \parallel_2 \) solution with one transverse isometry, which is lifted to a \( 2\parallel_1 \parallel_2 \) solution in \( D = 11 \) with two transverse isometries. There does not seem to exist a similar localised \( 2\parallel_1 \parallel_2 \) solution with no transverse isometries.
3.2. String localised on intersection of two 5-branes and related solutions

One straightforward generalisation of the above discussion is obtained by replacing the product of the 4-torus and curved ‘5-brane’ 4-space which was used as an 8-dimensional transverse conformal theory in (3.1),(3.2) by the direct product of the two 5-brane theories \[ [50] \] (which is obviously conformally invariant being described by the sum of two independent conformal sigma-model actions). The resulting conformal background is \[ [8] \]

\[ ds_{10}^2 = H_1^{-1}(x,y)(-dt^2 + dz^2) + H_5'(y)dy_n dy_n + H_5(x)dx_m dx_m , \]  

\[ dB = dH_1^{-1} \wedge dt \wedge dz + *dH_5 + *dH_5' , \quad e^{2\phi} = H_1^{-1} H_5 H_5' , \]

where the harmonic functions \( H_5(x) \) and \( H_5'(y) \) \((\partial_x^2 H_5 = 0, \partial_y^2 H_5' = 0)\) define the positions of the two 5-branes \((z, y_n)\) and \((z, x_m)\) and the string function \( H_1(x, y) \) satisfies

\[ [H_5'(y)\partial_x^2 + H_5(x)\partial_y^2]H_1(x, y) = 0 \].

In the special case of \( H_1 = 1 \) the background (3.7) represents the configuration of two 5-branes which share the string direction and are ‘localised’ with respect to each other. We shall denote this configuration as \( 5_{NS} \cap 5_{NS} \) following \[ [18] \] where this ‘overlapping 5-brane’ interpretation of the solution of \[ [50] \] was suggested and it was lifted to \( 5 \cap 5 \) in \( D = 11 \). Another special case with \( H_1 = H_1(x)H_1'(y) \) corresponds to the ‘dyonic string’ generalisation of the solution of \[ [50] \] which was found in \[ [52] \].

We shall use the notation \( 5 \cap 5 \| 1 \) for the general NS-NS solution (3.7),(3.8). Starting with this configuration and applying S-duality and T-duality along \( z \) we get the related solutions \( 5_R \cap 5_R \| 1_R \) and \( 4 \cap 4 \| 0 \). Lifting \( 5 \cap 5 \| 1 \) to \( D = 11 \) we find the \( 5 \cap 5 \| 2 \) M-brane configuration which generalises the \( 5 \| 2 \) solution (3.6) to the presence of an additional 5-brane (and generalises \( 5 \cap 5 \) of \[ [18] \) to the presence of the 2-brane)

\[ ds_{11}^2 = H_2^{1/3} H_5^{2/3} H_5'^{2/3} [ (H_2 H_5 H_5')^{-1} (-dt^2 + dz^2) + H_2^{-1} dy_1^2 + H_5^{-1} dy_n dy_n + H_5'^{-1} dx_m dx_m ] , \]

\[ dC_3 = (dH_2^{-1} \wedge dt \wedge dz + *dH_5 + *dH_5') \wedge dy_1 , \]

\[ H_2 = H_1(x, y) , \quad H_5 = H_5(x) , \quad H_5' = H_5'(y) . \]

Other related \( D = 10 \) solutions can be constructed by relaxing localisation in some of the \((x_n, y_m)\) coordinates and applying T-duality along these isometric directions. For example,

\[ \text{\textsuperscript{16}} \text{ It is possible also to consider more general non-direct-product 8-dimensional conformal models based on hyper-Kähler metrics (and their generalisations including antisymmetric tensor background). This leads to more general intersecting brane solutions constructed in [51].} \]
let us split the coordinates in the 3+1 way, \( x_m = (x_i, x_4) \), \( y_m = (y_i, y_4) \), \( i = 1, 2, 3 \), and assume that the functions \( H_1, H_5, H'_5 \) do not depend on \( x_4, y_4 \). Starting with \( 5_R \cap 5_R \parallel 1_R \) type IIB solution delocalised in \((x_4, y_4)\)

\[
ds_{10}^2 = (H_1 H_5 H'_5)^{1/2}[(H_1 H_5 H'_5)^{-1}(-dt^2 + dz^2) + H_5^{-1}(dy_4^2 + d\bar{y}^2) + H'_5^{-1}(d\bar{x}_4^2 + d\bar{x}^2)] ,
\]

\[
e^{2\phi} = H_1 (H_5 H'_5)^{-1} , \quad dC_2 = dH_1^{-1} \wedge dt \wedge dz + *dH_5 \wedge dx_4 + *dH'_5 \wedge dy_4 , \quad (3.10)
\]

\[
H_1 = H_1(\vec{x}, \vec{y}) , \quad H_5 = H_5(\vec{x}) , \quad H'_5 = H'_5(\vec{y}) ,
\]

and applying T-duality along \( x_4 \) and \( y_4 \) one finds the \( 5_R \cap 5_R \parallel 3 \) configuration with the metric which has the expected ‘harmonic function rule’ form

\[
ds_{10}^2 = (H_3 H_5 H'_5)^{1/2}[(H_3 H_5 H'_5)^{-1}(-dt^2 + dz^2) + (H_3 H_5 H'_5)^{-1}dx^2 + (H_3 H_5 H'_5)^{-1}dy_4^2 + H'_5^{-1}d\bar{y}^2 + H'_5^{-1}d\bar{x}^2] , \quad (3.11)
\]

\[
H_3 = H_1(\vec{x}, \vec{y}) , \quad H_5 = H_5(\vec{x}) , \quad H'_5 = H'_5(\vec{y}) .
\]

Here \( x, y \) are dual to \( x_4, y_4 \) and the 3-brane coordinates are \((z, x, y)\). Each of the two 5-branes \((z, x, y)\) and \((z, y, x)\) (which share one string direction) intersects with the 3-brane over a 2-space (note that \( x, y \) effectively interchanged places compared to \( x_4, y_4 \)). The 3-brane is localised on each of the 5-branes only in 2 out of 3 coordinates. This generalises the \( 5_R \parallel 3 \) solution mentioned above.

S-duality then leads to the solution \( 5_{NS} \cap 5_{NS} \parallel 3 \) where the RR 5-branes are replaced by the NS-NS ones. It is possible also to construct a solution representing the S-‘self-dual’ configuration \( 5_{NS} \cap 5_R \parallel 3 \) with the two different types of 5-branes, which intersect each other and the 3-brane over a 2-space (the existence of such BPS configuration was pointed out in [19]). The corresponding background can be constructed by applying U-duality to \( 5_{NS} \cap 5_{NS} \parallel 1_{NS} \) delocalised in \( x_4 \) and \( y_4 \). Indeed, T-duality along \( x_4 \) transforms first \( 5_{NS} \) \((z, \vec{y}, y_4)\) into KK 5-brane \( 5_{KK} \) (described by euclidean Taub-NUT metric, cf.(3.3)). S-duality then converts \( 5_{KK} \cap 5_{NS} \parallel 1_{NS} \) into \( 5_{KK} \cap 5_{R} \parallel 1_R \) (being a purely gravitational background, \( 5_{KK} \) is invariant under S-duality). Applying T-duality twice along \( x_4 \) and \( y_4 \) leads to \( 5_{NS} \cap 5_{R} \parallel 3 \) solution.

Alternatively, one may start with its \( D = 11 \) counterpart \( 5 \cap 5 \parallel 2 \) (3.3) ‘smeared’ in \( x_4 \) and \( y_4 \) directions so that it has \( dC_3 = (dH_1^{-1} \wedge dt \wedge dz + *dH_5 \wedge dx_4 + *dH'_5 \wedge dy_4) \wedge dy_{11} \). Since \( x_4 \) is now an isometry, one may reduce this solution down to \( D = 10 \) along \( x_4 \) obtaining the \( 4 \cap 5 \parallel 2 \) type IIA solution. Applying T-duality along \( y_4 \) then leads to the \( 5_{NS} \cap 5_{R} \parallel 3 \) type IIB background

\[
ds_{10}^2 = (H_3 H'_5)^{1/2}H_5[(H_3 H_5 H'_5)^{-1}(-dt^2 + dz^2 + dy_4^2) + H_3^{-1}dx^2 + H_5^{-1}d\bar{y}^2 + H'_5^{-1}d\bar{x}^2] , \quad (3.12)
\]
\[ e^{2 \phi} = H_5 H_5'^{-1}, \quad dB = *dH_5 \wedge dx, \quad dC_2 = *dH_5' \wedge dx, \]
\[ dC_4 = dH_3^{-1} \wedge dt \wedge dz \wedge dx \wedge dy + H_5 *d_y H_3 \wedge dx_1 \wedge dx_2 \wedge dx_3 + H_5' *d_x H_3 \wedge dy_1 \wedge dy_2 \wedge dy_3, \]
\[ H_3 = H_1(\vec{x}, \vec{y}), \quad H_5 = H_5(\vec{x}), \quad H_5' = H_5'(\vec{y}). \]

Here \( y \) denotes the coordinate dual to \( y_4, x \equiv y_{11} \) and \(*d_x = \frac{1}{2} \epsilon_{ijk} dx_j \wedge dx_k \partial_{x_i}, *d_y = \frac{1}{2} \epsilon_{ijk} dy_j \wedge dy_k \partial_{y_i}\). The coordinates of the branes are 3: \((z, y, x)\), \(5_{NS}: (z, y, y_i)\), \(5_R: (z, y, x_i)\). S-duality maps this background into itself with \( H_5 \leftrightarrow H_5', \vec{x} \leftrightarrow \vec{y} \). The 3-brane is localised only relative to the \((x_i, y_i)\) 3-spaces of the two 5-branes, and all branes are delocalised in the common transverse direction \( x \equiv y_{11} \). It is not clear if there exists a similar static solution describing the branes localised in \( x \), i.e. the configuration considered in [19].

For comparison, let us note that there exists another marginal \(5_{NS} \perp 5_R \perp 3\) type IIB configuration which is covariant under S-duality. This is the delocalised intersection where the 5-branes intersect over a 4-brane and each intersects the 3-brane over a 2-space. This configuration is T-dual (along direction parallel to \(5_{NS}\)) to \(5_{NS} \perp 4 \perp 4\) which is a dimensional reduction of the configuration \(5_5 \perp 5_5 \perp 3\) of three orthogonal 5-branes in \(D = 11\) [110].

The metric of \(5_{NS} \perp 5_R \perp 3\) is (cf. (3.12))
\[ ds^2_{10} = (H_3 H_5'_{-1} (H_3 H_5 H_3')^{-1} (-dt^2 + dz^2) + (H_5 H_5')^{-1} (dy_1^2 + dy_2^2 + dy_3^2) \]
\[ + (H_3 H_5)^{-1} dy_4^2 + (H_3 H_5')^{-1} dy_5^2 + dx_i dx_i \]  
(3.13)

where \(H_5, H_5', H_3\) depend only on \(x_i\).

4. Actions for brane probes in backgrounds of composite \(p\)-brane configurations

An advantage of knowing explicitly the supergravity backgrounds representing composite configurations of different type of branes is that one can easily determine the structure of classical actions of \(p\)-brane probes moving in closed string backgrounds produced by the corresponding systems of brane sources. In the case of supersymmetric D-brane configurations this determines the form of the second-derivative terms in the action which

\text{17} On possibility (suggested by the expressions for the moduli metrics in [19]) is that a solution localised in \(x\) will have \(H_5, (H_5')\) replaced by the sum of harmonic function in \(x\) and a harmonic function in \(x_i (y_i)\), i.e. \(H_5 = q|x - x_0| + \frac{q}{|x - x_0|}\). Such function is obviously a solution of 5-brane conformal invariance condition in (3.7) \((\partial^n \partial_m H_5 = 0, with x = x_4)\) but it is not clear that such an ansatz is fully consistent as one is no longer able to apply T-duality in \(x\)-direction to relate various configurations as was done above.
appear as 1-loop corrections in the open string theory description \cite{53,54,55}. This classical approach is particularly useful in the case when (some of) the sources are the NS-NS branes for which there is no simple analogue of a perturbative D-brane description.

For example, let us consider a fundamental string probe moving in the $5_{NS} \cap 5_{NS} \parallel 1$ background \cite{B7}. Orienting the probe along $(t, z)$ and choosing the static gauge ($X^a = \sigma^a$) one finds that the string action

$$I_1 = T_1 \int d^2 \sigma [\sqrt{- \det (G_{MN}(X) \partial_a X^M \partial_b X^N)} + \frac{1}{2} B_{MN}(X) \epsilon^{ab} \partial_a X^M \partial_b X^N]$$

takes the following form

$$I_1 = T_1 \int d^2 \sigma [(- \det [H^{-1}(x, y) \eta_{ab} + H_5^i(y)] \partial_a y_n \partial_b y_n$$

$$+ H_5(x) \partial_a x_m \partial_b x_m)]^{1/2} - H^{-1}(x, y)]$$

$$= \frac{1}{2} T_1 \int d^2 \sigma [H_5^i(y) \partial_a y_n \partial_a y_n + H_5(x) \partial_a x_n \partial_a x_n + ...] . \quad (4.1)$$

The vanishing of the static potential indicates that this is a BPS configuration. The dependence on $H_1(x, y)$ cancels out in the second-derivative approximation. This is related to the absence of velocity-squared corrections to the force (i.e. to the flatness of the moduli space) in the system of parallel strings. In general, the moduli space metric in (4.1) is the same as the 8-dimensional hyper-Kähler metric in (3.7) (in the single-center case $H_5 = 1 + Q / x^2$, $H_5^i = 1 + Q' / y^2$). The same result is found by considering the S-dual situation – a RR string probe moving in $5_R \cap 5_R \parallel 1_R$ background.

Let us now consider the bosonic terms in the action for a 3-brane probe moving in a type IIB supergravity background (see, e.g., \cite{56})

$$I_3 = T_3 \int d^4 \sigma [e^{-\phi} \sqrt{- \det (\hat{G} + \hat{F}) + \frac{1}{4!} \epsilon^{abcd} \hat{C}_{abcd} + \frac{1}{2} \hat{F}^{*ab} \hat{C}_{ab} + \frac{1}{4} C \hat{F}^{*ab} \hat{F}_{ab}]} , \quad (4.2)$$

where $\hat{F}_{ab} = F_{ab} + \hat{B}_{ab}$, $F = dA$, $F^{*ab} = \frac{1}{2} \epsilon^{abcd} F_{cd}$, $\hat{G}_{ab} = G_{MN} \partial_a X^M \partial_b X^N$, etc., and $a = 0, 1, 2, 3$.

It is important to note that the gauge field $A_a$ will not, in general, decouple from the background and should be taken into consideration in discussing the low-energy (moduli-space) approximation.

\textsuperscript{18} To make this action manifestly covariant under $SL(2, Z)$ duality \cite{57,58} one should add a $B \wedge C_2$ term so that $\hat{F}^{*ab} \hat{C}_{ab}$ becomes $F^{*ab} \hat{C}_{ab}$. Equivalently, this corresponds to choosing $\hat{C}_4$ as $\hat{C}_{abcd} = \hat{C}'_{abcd} - 6 \hat{B}_{[ab} \hat{C}_{cd]}$ where $C'_4$ is invariant under $SL(2, Z)$. This subtlety will not be important in what follows as $B \wedge C_2$ will vanish for the backgrounds we shall consider.
For example, if the background is produced by an NS-NS 5-brane smeared over one transverse \((x_4)\) direction, the action for a 3-brane probe positioned parallel to \((z, x, y)\) directions is (here \(y_n = (y_i, y_4 \equiv y)\), \(x_i = (x_i, x_4 \equiv x)\))

\[
I_3 = T_3 \int d^4 \sigma \sqrt{\text{det} \left[ \delta^\alpha_\beta + \kappa^{ac}(\partial_\alpha y_i \partial_\beta y_i + H_5 \partial_c x_i \partial_b x_i + F_{cb}) \right]} , \quad (4.3)
\]

\[
\kappa_{ac} = \text{diag}(-1, 1, 1, H_5) , \quad F = F + B_4(\bar{x}) dx_i \wedge dx , \quad dB = *dH_5 , \quad H_5 = H_5(\bar{x}) .
\]

Expanding in powers of derivatives we get

\[
I_3 = T_3 \int d^4 \sigma \left[ 1 + \frac{1}{2} \kappa^{ac}(\partial_\alpha y_i \partial_\beta y_i + H_5 \partial_c x_i \partial_b x_i) + \frac{1}{2} \kappa^{ac} \kappa^{bd} F_{ab} F_{cd} + \ldots \right] . \quad (4.4)
\]

Let us split the world-volume indices as \(a = (k, 3)\), \(k = 0, 1, 2\), take \(A_a = (A_k, A_3 \equiv \theta)\) and assume that all the fields do not depend on \(\sigma_3\) (equal to \(x\) in the static gauge). Then

\[
\kappa^{ac} \kappa^{bd} F_{ab} F_{cd} = F_{kl}^2 + 2 H_5^{-1}(\partial_k \theta + B_i(\bar{x}) \partial_k x_i)^2 \quad \text{so that}
\]

\[
I_3 = \frac{1}{2} T_3 \int d^4 \sigma \left( 2 + \partial_k y_i \partial_k y_i + H_5(\bar{x}) \partial_k x_i \partial_k x_i + H_5^{-1}(\bar{x}) \left[ \partial_k \theta + B_i(\bar{x}) \partial_k x_i \right]^2 \right) + \ldots , \quad (4.5)
\]

where dots stand also for the decoupled \(F_{kl}\)-terms. Keeping the component \(A_3 = \theta\) of the world-volume gauge field is important as it does not decouple from the background. This leads to the following moduli space metric (for future comparison, we include also the contribution of the decoupled component \(A_2 \equiv \theta'\))

\[
ds^2 = dy_i^2 + d\theta'^2 + H_5(\bar{x}) dx_i dx_i + H_5^{-1}(\bar{x}) \left[ d\theta + B_i(\bar{x}) dx_i \right]^2 , \quad (4.6)
\]

\[
dB = *dH_5 , \quad H_5 = 1 + \sum_s \frac{Q_s}{|\bar{x}_s - \bar{x}_0|} .
\]

Its curved part is the same hyper-Kähler metric as in the Kaluza-Klein 5-brane background (cf.\((3.3)\)) which is related to the solitonic 5-brane by T-duality along \(x = x_4\). Here the role of the coordinate dual to \(x_4\) is played by the component \(A_3 = \theta\) of the gauge field (\(\theta'\) corresponds to the dual of \(y_4\)). This should not be surprising as T-duality applied to the whole system including the probe should transform the corresponding gauge field component into a D-brane collective coordinate. Related discussion appeared in \([19]\) where the 5-brane (or a collection of parallel 5-branes) was assumed to be localised in \(x = x_4\), while here we are considering a ‘smeared’ case.

The same action is found if the \(5_{\text{NS}}\) background is replaced by the \(5_R\) one. This follows from the \(SL(2, R)\) covariance of the 3-brane action in a type IIB supergravity background \([57,58]\). Here the role of an ‘extra’ coordinate is played by the magnetic counterpart of the ‘electric’ gauge field variable \(\theta\). Indeed, in the RR 5-brane background \(B = 0\) and so \(F = F\) but instead there is the \(F \wedge \hat{C}_2\) coupling with \(C_2 = B \wedge dx\). Adding this term to
the $H_5(\vec{x})\kappa^{ac}\kappa^{bd}F_{ab}F_{cd}$ one coming from expansion of the Born-Infeld action (which has an extra factor of $H_5$ compared to (4.5)) and introducing $\epsilon_{kl\sigma}\partial_\sigma\tilde{\theta}$ as the dual ‘monopole’ part of $F_{kl}$ (or performing $d = 3$ duality, see below) one finishes, after decoupling of other gauge field components, with an equivalent action, where $\tilde{\theta}$ is playing the role of $\theta$, so that the moduli space metric contains the term $H_5^{-1}(d\tilde{\theta} + B_i dx_i)^2$.

The cases of more complicated configurations like $5_{NS}\cap 5_{NS}$ or $5_{NS}\cap 5_R\perp 3$ are treated in a similar way. For example, putting a 3-brane probe in the $5_{NS}\cap 5_{NS}\perp 3$ background leads to the following action (the 3-brane probe is oriented along $z, x, y$, cf. (3.11))

$$I_3 = T_3 \int d^4 \sigma \left[ H_3^{-1} \left( \det[\delta_a^a + \kappa^{ac}(H_3H'_3\partial_a y_i\partial_c y_i)
+ H_3H_3\partial_a x_i\partial_c x_i + H_3^{1/2}F_{ac}] \right)^{1/2} - H_3^{-1} \right],$$

(4.7)

where $H_3 = H_1(\vec{x}, \vec{y})$, $H_5 = H_5(\vec{x})$, $H'_5 = H'_5(\vec{y})$,

$$\kappa_{ab} = \text{diag}(-1, 1, H_5, H'_5), \quad F = F + B \wedge dx + B' \wedge dy,$$

and the $-H_3^{-1}$ term in the potential came from the $C_4$-background produced by the 3-brane source. As in (1.1) the dependence on the 3-brane source function $H_3(\vec{x}, \vec{y})$ disappears in the moduli space approximation. Introducing $A_2 = \theta'$ and $A_3 = \theta$ and decoupling the remaining components of the gauge field one finds the following moduli space metric

$$ds^2 = H'_5(\vec{y})dy_i dy_i + H'^{-1}_5(\vec{y})[d\theta' + B'_i(\vec{y})dy_i]^2$$

(4.8)

$$+ H_5(\vec{x})dx_i dx_i + H^{-1}_5(\vec{x})[d\theta + B_i(\vec{x})dx_i]^2,$$

which is the direct product of the two 4-d Euclidean Taub-NUT metrics in (1.6). This 8-dimensional hyper-Kähler metric is related to the transverse metric of the $5_{NS}\cap 5_{NS}$ background by T-duality in $x_4$ and $y_4$ (cf. (3.11)).

Finally, let us consider the most interesting case of the ‘mixed’ $5_{NS}\cap 5_R\perp 3$ background (3.12) which is ‘self-dual’ under the $SL(2, Z)$. The corresponding 3-brane action is (cf. (4.7))

$$I_3 = T_3 \int d^4 \sigma \left[ H_3^{-1} \left( \det[\delta_a^a + \kappa^{ac}(H_3H'_3\partial_b y_i\partial_c y_i + H_3H_3\partial_b x_i\partial_c x_i)
+ (H_3H'_3)^{1/2}F_{cb}] \right)^{1/2} - H_3^{-1} + \frac{1}{2}F^{*ab}\hat{C}_{ab} + ... \right],$$

(4.9)

where $H_3 = H_1(\vec{x}, \vec{y})$, $(H_5)_{NS} = H_5(\vec{x})$, $(H_5)_R = H'_5(\vec{y})$,

$$\kappa_{ab} = \text{diag}(-1, 1, H_5H'_5), \quad B = B_i dx_i \wedge dx, \quad C_2 = dB'_i dy_i \wedge dx,$$
and $\partial_i B_j = \frac{1}{2} \epsilon_{ijk} \partial_k H_5$, \ $\partial_i B'_j = \frac{1}{2} \epsilon_{ijk} \partial_k H'_5$. Dots in (4.10) stand for higher-order terms coming from the $C_4$ background in (3.12). The leading terms in low-energy expansion are

$$I_3 = \frac{1}{2} T_3 \int d^4 \sigma \left[ k^{ab} (H'_5 \partial_a y_i \partial_b y_i + H_5 \partial_a x_i \partial_b x_i) \right]$$

$$+ \frac{1}{2} H_5 k^{ac} k^{bd} F_{ab} F_{cd} + F^{*ab} \dot{C}_{ab} + \ldots \right] .$$

Assuming as above that the fields depend only on the first three world-volume coordinates $\sigma_k \ (k = 0, 1, 2)$ and setting $A_3 \equiv \theta$ we find

$$I_3 = \frac{1}{2} T_3 \int d^4 \sigma \left[ H'_5 (\vec{y}) \partial_k y_i \partial_k y_i + H_5 (\vec{x}) \partial_k x_i \partial_k x_i \right]$$

$$+ \frac{1}{2} H_5 (\vec{\bar{y}}) F_{kl} F_{kl} + H'_5^{-1} (\vec{y}) [\partial_k \theta + B_i (\vec{x}) \partial_k x_i]^2 + \epsilon_{kls} F_{kl} B'_i (\vec{y}) \partial_s y_i + \ldots \right] .$$

Compared to the previous $5_{NS} \cap 5_{NS}$ example, here the $5_{NS}$ monopole potential $B$ couples to $F_{ab}$ electrically while the $5_{R}$ one $B'$ – magnetically. To decouple the $F_{kl}$ components of the field strength we introduce the ‘magnetic’ variable $\dot{A}_3 \equiv \dot{\theta}$ as a Lagrange multiplier by adding the term $\epsilon_{kls} F_{kl} \partial_s \dot{\theta}$ which imposes the $dF = 0$ condition (this is equivalent to performing the $d = 3$ duality transformation $A_k \rightarrow \dot{A}_3$). Integrating out (or redefining) $F_{kl}$ we finish with (note that $\epsilon_{kls} \epsilon^{klr} = -\delta^r_s$, $\eta_{kl} = diag(-1, 1, 1)$)

$$I_3 = \frac{1}{2} T_3 \int d^4 \sigma \left[ H'_5 (\vec{y}) \partial_k y_i \partial_k y_i + H_5 (\vec{x}) \partial_k x_i \partial_k x_i \right]$$

$$+ H'_5^{-1} (\vec{y}) [\partial_k \theta + B_i (\vec{x}) \partial_k x_i]^2 + H_5^{-1} (\vec{x}) [\partial_k \dot{\theta} + B'_i (\vec{y}) \partial_k y_i]^2 + \ldots \right] .$$

As expected, the action is manifestly covariant under the S-duality transformation, i.e. under $\vec{x} \leftrightarrow \vec{y}$, $H_5 \leftrightarrow H'_5 \ (B_i \leftrightarrow B'_i)$, combined with world-volume duality $\theta \leftrightarrow \dot{\theta}$. This invariance (extended to the full quantum level) was related in [19] to a mirror symmetry of $N = 4$ supersymmetric $d = 3$ gauge theories.

The moduli space metric corresponding to the $5_{NS} \cap 5_{R}$ configuration as measured by a classical 3-brane probe is thus

$$ds^2 = H'_5 (\vec{y}) dy_i dy_i + H_5^{-1} (\vec{x}) [d \theta + B'_i (\vec{y}) dy_i]^2$$

$$+ H_5 (\vec{x}) dx_i dx_i + H'_5 (\vec{y}) [d \theta + B_i (\vec{x}) dx_i]^2 .$$

In contrast to the metric (4.8) which appeared in the $5_{NS} \cap 5_{NS}$ or $5_{R} \cap 5_{R}$ cases, this is a non-trivial $D = 8$ hyper-Kähler metric [59] which does not factorise into a direct product of independent $D = 4$ Euclidean Taub-NUT metrics.
One can repeat the above discussion for a $D = 11$ 2-brane probe moving in the $5 \cap 5 \hat{2}$ background (3.3). The resulting action for a 2-brane parallel to $(z, y_{11})$ is

$$I_2 = T_2 \int d^3 \sigma \left[ (- \det \hat{G})^{1/2} + \frac{1}{6} \epsilon^{abc} \hat{C}_{abc} \right]$$

(4.14)

$$= T_2 \int d^3 \sigma \left( H_2^{-1} \sqrt{\det[\delta_a^b + \kappa^{ac}(H_2 H'_5 \partial_c y_n \partial_b y_n + H_2 H_5 \partial_c x_m \partial_b x_m)]} \right.$$

$$- H_2^{-1} + \frac{1}{2} \epsilon^{kl} [B_{mn}(x) \partial_k x^m \partial_l x^n + B'_{mn}(y) \partial_k y^m \partial_l y^n] \right),$$

$$\kappa_{ac} = \text{diag}(-1, 1, H_5 H'_5), \quad dB = *dH_5, \quad dB' = *dH'_5,$$

where $a, b = 0, 1, 2, \quad k, l = 0, 1$. Assuming that the fields do not depend on $\sigma_2 = y_{11}$ and expanding in powers of derivatives we finish with an action which is a direct sum of the two 5-brane (super)conformal 2d models

$$I_2 = \frac{1}{2} T_2 \int d^3 \sigma \left[ H'_5(x) \partial_c y_n \partial_b y_n + \epsilon^{kl} B'_{mn}(y) \partial_k y^m \partial_l y^n \right.$$  

$$\left. + H_5(x) \partial_c x_m \partial_b x_m + \epsilon^{kl} B_{mn}(x) \partial_k x^m \partial_l x^n + \ldots \right].$$

This is the expected result as the $D = 11$ $5 \cap 5$ configuration reduces to the two 5-brane NS-NS background (cf. (3.7)).

5. Acknowledgements

I am grateful to M. Cvetić, J. Gauntlet, A. Hanany, I. Klebanov, J. Maldacena and P. Townsend for useful and stimulating discussions. This work was presented at the Tel Aviv Workshop on Duality, January 9-12, 1997, and I would like to thank the organisers – C. Sonnenschein and S. Yankielowicz for their kind hospitality. I acknowledge the support of PPARC and the European Commission TMR programme ERBFMRX-CT96-0045.
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