A VIEW ON MULTIPLE RECURRENCE

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Dedicated to Jacob (Jaap) Korevaar
on the occasion of his approaching 100th birthday

Abstract. In this note we present a proof of multiple recurrence for ergodic systems (and thereby of Szemerédi’s theorem) being a mixture of three known proofs. It is based on a conditional version of the Jacobs-de Leeuw-Glicksberg decomposition and properties of the Gowers-Host-Kra uniformity seminorms.

1. Introduction

The celebrated Szemerédi theorem [32] from 1975 asserts that every subset of the natural numbers with positive upper density contains arithmetic progressions of arbitrary length. This means that as soon as a set does not vanish asymptotically it has certain structure inside. In 1977 Furstenberg [17] presented his groundbreaking ergodic theoretic proof of Szemerédi’s theorem based on multiple recurrence which has had enormous impact. In particular, various related results in additive number theory were proven using ergodic theoretic methods such as the Furstenberg-Katznelson multidimensional Szemerédi theorem [19], the Bergelson-Leibman polynomial Szemerédi theorem [1], the Green-Tao theorem on the existence of arithmetic progressions in the primes [23, 24], the Tao-Ziegler polynomial Green-Tao theorem [35, 36, 38] and the multidimensional Green-Tao theorem [37, 5, 11] (where [5] does not use ergodic theory). For some related works see also [3, 14, 2, 16, 39, 15, 13, 31, 4].

In this note we present a proof of multiple recurrence for ergodic systems (and hence of Szemerédi’s theorem) being a mixture of three known proofs, namely Tao’s modification [34] of the classical Furstenberg-Katznelson proof [19], the original Furstenberg proof [17] and (the beginning of) the proof of multiple convergence by Host and Kra [26]. For a finitary quantitative proof of Szemerédi’s theorem with similar philosophy see Tao [33].

Like the original Furstenberg proof [17] (and in contrast to the classical Furstenberg-Katznelson proof), our proof does not use transfinite induction and deals for multiple recurrence of order $k$ with a tower of $k - 1$ factors, namely Furstenberg’s distal factors of order less than or equal to $k - 1$. We give an alternative proof of Furstenberg’s result [17] that the distal factor of order $k - 1$ is characteristic for $k$-term multiple ergodic averages by showing that this factor is an extension

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of the Host-Kra-Ziegler factor of order $k - 1$ (via working with the corresponding subspaces of functions). Thereby we do not need any information on the Host-Kra-Ziegler factors other than the description of the corresponding subspaces via the Gowers-Host-Kra seminorms, not even the fact that they are factors. Note that the original proof of Furstenberg of the characteristic property of the distal factors of finite order is based on a description of conditional eigenfunctions of fibered product systems (see [17, Theorem 7.1]) and analysis of diagonal measures, with a subsequent simplification by Frantzikinakis [12, Theorem 5.2], still based on [17, Theorem 7.1], using the van der Corput trick.

Two decompositions will play a central role, namely a conditional version of the Jacobs-de Leeuw-Glicksberg decomposition and the Host-Kra decomposition. The first one, decomposing the $L^2$-space into conditionally almost periodic and conditionally weakly mixing functions, is a stronger version due to Tao [34, Chapter 2] of the dichotomy between almost periodic and weakly mixing extensions due to Furstenberg, Katznelson [19]. The Host-Kra decomposition decomposes, for every $k$, the $L^2$-space into a part where the Gowers-Host-Kra seminorm of order $k + 1$ vanishes and the orthogonal complement known as the Host-Kra-Ziegler factor of order $k$. Then properties of the Gowers-Host-Kra seminorms and the conditional Jacobs-de Leeuw-Glicksberg decomposition quickly imply that the distal factors of finite order are extensions of the Host-Kra-Ziegler factors of the corresponding order, see Proposition 7.1 below. (Host, Kra [20, Lemma 6.2], [27, Lemma 18.2] and Ziegler [40, Thm. 6.1] showed a stronger property of the Host-Kra-Ziegler factors, namely that they are compact extensions of each other. For an alternative proof using the conditional Jacobs-de Leeuw-Glicksberg decomposition see Zorin-Kranich [42, Lemma 10.1].)

The rest of the argument is standard. By definition, the distal factors of finite order are maximal compact extensions of each other. By induction, starting with the fixed (one-point) factor and using the fact that compact extensions preserve multiple recurrence (Proposition 7.4 below), one has multiple recurrence for the distal factors of all orders. Since the distal factor of order $k - 1$ is an extension of the Host-Kra-Ziegler factor of order $k - 1$, it is characteristic for $k$-term multiple ergodic averages by the generalized von Neumann theorem (Proposition 6.3 and Corollary 6.5 below), completing the proof.

As indicated above, the structure of the proof is essentially implicit in Host, Kra [20], Ziegler [40] as well as in Furstenberg [17] combined with Furstenberg, Katznelson, Ornstein [20], cf. Tao [33, Remark 4.7]. We present here a view based on the conditional Jacobs-de Leeuw-Glicksberg decomposition and properties of the Gowers-Host-Kra seminorms.

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1 in Furstenberg’s terminology, generalized eigenfunctions
2 Host and Kra showed that the Host-Kra-Ziegler factors are isometric extensions (as defined by Furstenberg [17]) of each other. See, e.g., Glasner [22, Theorem 9.14] for the characterization of isometric extensions as those being generated by conditional eigenfunctions implying almost periodicity of such extensions.
2. Preliminaries

Throughout this article we call a triple \((X, \mu, \varphi)\) a measure-preserving system if \((X, \mu)\) is a probability space and \(\varphi\) is a measurable invertible transformation on \(X\) with measurable inverse which is \(\mu\)-preserving, i.e., \(\mu(\varphi^{-1}A) = \mu(A)\) holds for every measurable set \(A \subset X\). For a measure-preserving system \((X, \mu, \varphi)\) we call the linear invertible isometry \(T : L^1(X, \mu) \rightarrow L^1(X, \mu)\) defined by \(Tf := f \circ \varphi\) the Koopman operator of the system. The Koopman operator acts as an invertible isometry on \(L^p(X, \mu)\) for every \(p \in [1, \infty]\) and is unitary on \(L^2(X, \mu)\). Since we mostly work with the Koopman operator \(T\) instead of the transformation \(\varphi\), we often write \((X, \mu, T)\) instead of \((X, \mu, \varphi)\).

A measure-preserving system \((X, \mu, \varphi)\) is called ergodic if it has no non-trivial invariant sets, i.e., if every measurable set satisfying \(\varphi^{-1}A \subset A\) up to a null set (which is equivalent to \(\varphi^{-1}A = A\) and hence, by the invertibility of \(\varphi\), to \(A = \varphi(A)\) up to a null set) has measure zero or one. For the corresponding Koopman operator \(T\) on every \(L^p(X, \mu)\) this is equivalent to \(\text{Fix}(T) = \mathbb{C}1\) (i.e., \(T\) has only constant invariant functions).

We now define multiple recurrence and multiple convergence. Thereby, \(f > 0\) means \(f \geq 0\) and \(f \neq 0\) (as element of \(L^\infty(X, \mu)\)).

Definition 2.1. Let \((X, \mu, T)\) be a measure-preserving system and \(k \in \mathbb{N}\). We say that \((X, \mu, T)\) satisfies

- **multiple recurrence of order** \(k\) (shortly \(MR_k\)) if for every \(0 < f \in L^\infty(X, \mu)\)

  \[
  \liminf_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \int_X f \cdot T^n f \cdot T^{2n} f \cdots T^{kn} f \, d\mu > 0.
  \]

- **multiple convergence of order** \(k\) if for every \(f_1, \ldots, f_k \in L^\infty(X, \mu)\) the limit of multiple ergodic averages

  \[
  \frac{1}{N} \sum_{n=1}^{N} T^n f_1 \cdot T^{2n} f_2 \cdots T^{kn} f_k
  \]

exists in \(L^2(X, \mu)\).

We say that \((X, \mu, T)\) satisfies multiple recurrence (shortly MR) if it satisfies \(MR_k\) for all \(k \in \mathbb{N}\) and define analogously multiple convergence.

Note that, given multiple convergence of order \(k\), the property \(MR_k\) means that every function \(f \in L^\infty(X, \mu)\) with \(f > 0\) correlates with the limit of the averages \(\frac{1}{N} \sum_{n=1}^{N} T^n f \cdots T^{kn} f\). Moreover, for functions of the form \(f = 1_A\) for some \(A \subset X\) with \(\mu(A) > 0\) property 4 implies for the underlying transformation \(\varphi\) that \(\mu(A \cap \varphi^{-n}A \cap \ldots \cap \varphi^{-kn}A) > 0\) holds for many \(n \in \mathbb{N}\) explaining the name “multiple recurrence”\(^3\).

\(^3\)In this case \(\varphi^{-1}\) is clearly \(\mu\)-preserving as well. The invertibility assumption is technical and will assure that the conditional expectation commutes with the transformation. Since in our context one can work with invertible systems without loss of generality, we assume all systems to be invertible.
We now state Szemerédi’s theorem which inspired the study of multiple recurrence and multiple convergence.

**Theorem 2.2** (Szemerédi [32]). Let $C \subset \mathbb{N}$ have positive upper density, i.e., satisfy $d(C) := \limsup_{N \to \infty} \frac{|C \cap \{1, \ldots, N\}|}{N} > 0$. Then $C$ contains arbitrarily long arithmetic progressions, i.e., for every $k \in \mathbb{N}$ there exist $a, n \in \mathbb{N}$ with $a, a+n, \ldots, a+kn \in C$.

Furstenberg [17] divided his ergodic theoretic proof of Szemerédi’s theorem into two parts. The difficult part was to show multiple recurrence for ergodic systems.

**Theorem 2.3** (Furstenberg, multiple recurrence for ergodic systems). Every ergodic measure-preserving system satisfies MR.

The easy part was to build the following bridge between number theory and ergodic theory, see, e.g., Furstenberg [18, p. 77], Kra [29, Section 2.2] or [9, Section 20.1] for details.

**Proposition 2.4** (Furstenberg, correspondence principle). Multiple recurrence for ergodic systems implies the assertion of Szemerédi’s theorem. More precisely, for every $C \subset \mathbb{N}$ there exists an ergodic measure-preserving system $(X, \mu, \phi)$ and a measurable set $A \subset X$ with $\mu(A) \geq d(C)$ such that, for every $k \in \mathbb{N}$, the set $C$ contains an arithmetic progression of length $k+1$ if and only if there exists $n \in \mathbb{N}$ such that the set $A \cap \phi^{-n}A \cap \ldots \cap \phi^{-kn}A$ has positive measure.

In particular, it suffices to prove that for every $f \in L^{\infty}(X, \mu)$ with $f > 0$ there exists $n \in \mathbb{N}$ such that $\int_X f \cdot T^n f \cdot T^{2n} f \cdot \cdots \cdot T^{kn} f \, d\mu > 0$ which is a formally weaker property but turns out to be equivalent to MR$k$. For a discussion and a deduction of multiple recurrence back from Szemerédi’s theorem see, e.g., [9, Section 20.2].

Multiple convergence remained open until it was proved by Host, Kra [26] in 2005, with a subsequent alternative proof by Ziegler [40].

**Theorem 2.5** (Multiple convergence, Host, Kra [26], Ziegler [40]). Every measure-preserving system satisfies multiple convergence.

We now collect some definitions.

We do not need to assume systems to be separable (i.e., the underlying $\sigma$-algebra to be countably generated up to null sets). So we work with the following abstract notion of a factor. A measure-preserving system $(Y, \nu, S)$ is called a (Markov) factor of $(X, \mu, T)$ if there exists a Markov homomorphism $J : L^1(Y, \nu) \to L^1(X, \mu)$ which

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4For our purposes ergodic systems suffice by Proposition 2.3. Note that multiple recurrence holds for general (also not necessarily invertible) measure-preserving systems, see, e.g., Einsiedler, Ward [7] pp. 177–178.

5Furstenberg [17] showed a stronger property than MR where the lower Cesáro limit $\operatorname{liminf}_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \{1, \ldots, N\}$ is replaced by the lower Banach limit $\operatorname{liminf}_{N \to \infty} \frac{1}{N-M+1} \sum_{n=M}^{N} \{1, \ldots, N\}$.

6By the same argument one can easily replace the upper density $d(C)$ by the upper Banach density $d^*(C) := \limsup_{N \to \infty} \frac{|C \cap \{M+1, \ldots, N\}|}{N-M}$ leading to the corresponding stronger version of Szemerédi’s theorem proved by Furstenberg [17].

7An operator $J : L^1(Y, \nu) \to L^1(X, \mu)$ is called Markov if it is positive (i.e., $f \geq 0$ implies $Jf \geq 0$) and satisfies $J1 = 1$ as well as $Jf = \int_Y Jf \, dv$ holds for every
respects the Koopman operators, i.e., satisfies $J_S = TJ$. In this case $(X, \mu, T)$ is called a *(Markov)* extension of $(Y, \nu, S)$ and $J$ is called the *(Markov)* factor map.

**Remark 2.6.** (a) For a (Markov) factor $(Y, \nu, S)$ of a measure-preserving system $(X, \mu, T)$ with factor map $J : L^1(Y, \nu) \to L^1(X, \nu)$ the set $JL^\infty(Y, \nu)$ is a conjugation invariant, $T$- and $T^{-1}$-invariant subalgebra of $L^\infty(X, \mu)$ containing $1$, and there is a (up to isomorphism one-to-one) correspondence between (Markov) factors of $(X, \mu, T)$ and conjugation invariant, $T$- and $T^{-1}$-invariant subalgebras of $L^\infty(X, \mu)$ containing $1$. Every such subalgebra $\mathcal{A}$ of functions further corresponds to a $\varphi$- and $\varphi^{-1}$-invariant sub-$\sigma$-algebra of the original $\sigma$-algebra on $X$ for the underlying transformation $\varphi$, namely the smallest $\varphi$- and $\varphi^{-1}$-invariant sub-$\sigma$-algebra such that all functions from $\mathcal{A}$ are measurable with respect to it. We will work with the above characterization of factors via subalgebras of functions.

(b) For separable systems every (Markov) factor map is induced by a *point factor map*, i.e., $J$ is of the form $Jf = f \circ \pi$ for a morphism $\pi : X \to Y$, i.e., a measure-preserving map such that $\pi \circ \varphi = \psi \circ \pi$ holds $\mu$-a.e. for the underlying transformations $\varphi$ and $\psi$ on $X$ and $Y$, respectively. For details see, e.g., [3 Chapter 7]. The readers who prefer point factors to Markov factors can assume without loss of generality that the system is separable by passing to the $\varphi$- and $\varphi^{-1}$-invariant sub-$\sigma$-algebra generated by the function $f$ for multiple recurrence and by the functions $f_1, \ldots, f_k$ for multiple convergence of order $k$, respectively.

In particular, for a measure-preserving system $(X, \mu, T)$ with factor $(Y, \nu, S)$ the corresponding factor map $J : L^1(Y, \mu) \to L^1(X, \nu)$ acts as a contraction w.r.t. the $L^p$-norm for every $p \in [1, \infty]$. The adjoint operator $J'$ extends to a Markov operator

$$E_Y := J' : L^1(X, \mu) \to L^1(Y, \nu)$$

called the *conditional expectation operator*. It also acts as a contraction w.r.t. the $L^p$-norm for every $p \in [1, \infty]$. Recall that the conditional expectation operator satisfies

$$\int_Y E_Y f \, d\nu = \int_X f \, d\mu, \quad E_Y (Jg \cdot f) = gE_Y f \quad \forall f \in L^1(X, \mu) \forall g \in L^\infty(Y, \nu).$$

Since we assume factors to be invertible, one also has

$$E_Y T = SE_Y.$$

The operator

$$P_Y := JE_Y : L^2(X, \mu) \to L^2(X, \mu)$$

is the orthogonal projection onto the subspace $JL^2(Y, \nu)$. We call $P_Y$ the *projection onto the factor Y*. It extends to a Markov operator acting as a contraction w.r.t. every $L^p$-norm, $p \in [1, \infty]$. Another important property of $P_Y$ is

$$P_Y f > 0 \quad \text{for every } f > 0.$$

(Indeed, the Markov property of $J$ and $E_Y$ implies $P_Y f \geq 0$ and $\int_X P_Y f \, d\mu = \int_X f \, d\mu > 0$ ensuring $P_Y f \neq 0$.)

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$f \in L^1(Y, \nu)$. A Markov operator is called *Markov homomorphism* (or *Markov embedding*) if it preserves the absolute value or, equivalently, is multiplicative on $L^\infty(Y, \nu)$, see, e.g., [3] Theorem 7.29.
We say that a subspace $A$ of $L^2(X, \mu)$ is \textit{induced by a factor} $(Y, \nu, S)$ of $(X, \mu, T)$ if $A = JL^2(Y, \nu)$ for the corresponding Markov embedding $J$ or, equivalently, $A = L^2(X, \Sigma', \mu)$ for the $\varphi$-invariant sub-$\sigma$-algebra $\Sigma'$ corresponding to $(Y, \nu, S)$ for the underlying transformation $\varphi$ on $X$. By Remark 2.6(a), this is the case if and only if $A \cap L^\infty(X, \mu)$ is a conjugation invariant, $T$- and $T^{-1}$-invariant subalgebra of $L^\infty(X, \mu)$ containing $1$.

3. Single recurrence and von Neumann’s decomposition

We recall how the mean ergodic theorem and the property MR$_1$ rely on the von Neumann decomposition

$$H = \text{Fix}T \oplus \text{rg}(I - T)$$

for contractions $T$ on a Hilbert space $H$, see, e.g., [9, Theorem 8.6]. Let $(X, \mu, T)$ be a measure-preserving system and $f \in L^2(X, \mu) =: H$. By the telescopic sum argument the second part of (6) does not contribute to the limit of the ergodic averages $\frac{1}{N} \sum_{n=1}^{N} T^n f$ and $T$ acts as the identity operator on the first part of (6). Thus if $f > 0$ then

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \int_X f \cdot T^n f \, d\mu = \left( \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} T^n f, f \right) = \|P_{\text{Fix}T}f\|^2_{L^2(X, \mu)} > 0.$$ 

The last inequality follows from the measure preserving property of $T$ (which implies $\int_X P_{\text{Fix}T}f \, d\mu = \int_X f \, d\mu \neq 0$) and reflects the fact that Fix$T$ is induced by a factor, namely the fixed factor.

In other words, the second part in the von Neumann decomposition (6) does not contribute to the limit of the case $L^2(X) = \text{Fix}T$ where the assertion is trivial due to the very structured behavior of the orbits (being one point).

4. Double recurrence and the classical Jacobs-de Leeuw-Glicksberg decomposition

To illustrate the argument for higher $k$ we now briefly discuss the case $k = 2$ which is analogous to the case $k = 1$. For a detailed exposition see, e.g., [5, Section 8.3].

\textbf{Definition 4.1.} Let $T$ be a contraction on a Hilbert space $H$. We call $f \in H$ \textit{weakly mixing} if $\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} |(T^n f, f)| = 0$.

\textbf{Remark 4.2.} By the Koopman-von Neumann lemma, see, e.g., [5, Lemma 9.16], for a contraction $T$ on a Hilbert space $H$, $f \in H$ is weakly mixing if and only if there exists a set $J \subset \mathbb{N}$ with density 1 (i.e., satisfying $d(J) := \lim_{N \to \infty} \frac{|\{1, \ldots, N\} \cap J|}{N} = 1$) such that $\lim_{n \to \infty, n \in J} (T^n f, f) = 0$.

Double convergence and double recurrence rely on the following decomposition replacing the von Neumann decomposition, where $\mathbb{T}$ denotes the unit circle.

\footnote{For $k = 1$ one does not need $f$ to be bounded.}
Theorem 4.3 (Jacobs-de Leeuw-Glicksberg decomposition). Let $T \in L(H)$ be a contraction on a Hilbert space $H$. Then one has the orthogonal decomposition

\begin{equation}
H = \overline{\text{lin}} \{ f \in H : Tf = \lambda f \text{ for some } \lambda \in \mathbb{T} \} \oplus \{ f \in H : f \text{ is weakly mixing} \}.
\end{equation}

Let now $(X, \mu, T)$ be a measure-preserving system and $f, g \in L^\infty(X, \mu)$. One shows via the so-called van der Corput trick that the double averages

\[
\frac{1}{N} \sum_{n=1}^{N} T^n f \cdot T^n g
\]

converge to zero whenever $f$ or $g$ is weakly mixing. Since the first part of the decomposition (7) is induced by a factor, namely the Kronecker factor, we have that $P_{kr} f, P_{kr} g \in L^\infty(X, \mu)$ for the corresponding projection $P_{kr}$. Thus double convergence reduces to showing it for eigenfunctions for which it is immediate. For double recurrence let $g := f > 0$. Then $P_{kr} f > 0$ by (5) and we can assume by (7) and the above observation that $(X, \mu, T)$ coincides with its Kronecker factor.

To summarize, the second part in the decomposition (7) does not contribute to the limit and, using the fact that the first part is induced by a factor, we can assume that this factor coincides with the original system. Then one uses the very structured behavior of the orbits.

Next we consider two generalizations of (7), the conditional Jacobs-de Leeuw-Glicksberg decomposition and the Host-Kra decomposition, see (8) and (19) below.

5. **Conditional Jacobs-de Leeuw-Glicksberg decomposition**

We now present the abstract conditional setting introduced by Tao [34, Chapter 2]. To simplify the notation we will shorten $L^p(X, \mu)$ to $L^p(X)$ for a probability space $(X, \mu)$. Moreover, by writing $cf$ for $c \in L^\infty(Y)$ and $f \in L^1(X)$ we identify $c$ with its image under $J : L^1(Y) \to L^1(X)$.

**Definition 5.1.** Let $(X, \mu, T)$ be a measure-preserving system with factor $(Y, \nu, S)$. For $f, g \in L^2(X)$ we call the function

\[
(f, g)_{L^2(X|Y)} := E_Y (f \cdot g) \in L^1(Y)
\]

the **conditional scalar product** of $f$ and $g$ w.r.t. $Y$ and the function

\[
\|f\|_{L^2(X|Y)} := ((f, f)_{L^2(X|Y)})^{\frac{1}{2}} = (E_Y (|f|^2))^{\frac{1}{2}} \in L^2(Y)
\]

the **conditional norm** of $f$ w.r.t. $Y$. Moreover, the **conditional $L^2$-space** is defined by

\[
L^2(X|Y) := \{ f \in L^2(X) : \|f\|_{L^2(X|Y)} \in L^\infty(Y) \}.
\]

A function $f \in L^2(X|Y)$ is called **conditionally almost periodic** if for every $\varepsilon > 0$ there exists a finitely generated module zonotope $Z$ (i.e., a set of the form

\[
\{ c_1 f_1 + \ldots + c_m f_m : \|c_1\|_{L^\infty(Y)}, \ldots, \|c_m\|_{L^\infty(Y)} \leq 1 \}
\]
A function $f \in L^2(X)$ is called \textit{conditionally weakly mixing}\footnote{We follow here Zorin-Kranich \cite{Zorin-Kranich:conditional_almost_periodic} Section 8. Tao \cite{Tao:conditional_mixing} Section 2.14.1 calls a function $f \in L^2(X|Y)$ conditionally weakly mixing if $\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \|T^n f\|_{L^2(X|Y)}^2 = 0$. It is easy to see that these two definitions are equivalent on $L^2(X|Y)$.} if

$$
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \|T^n f\|_{L^2(X|Y)}^2 = 0.
$$

We denote the set of all such functions by $W(X|Y)$.

Remark 5.2. (a) It follows directly from the definition that

$$
\|f\|_{L^2,\infty(X|Y)} := \|f\|_{L^2(X|Y)}\|f\|_{L^\infty(X|Y)}
$$

holds for every $f \in L^\infty(X)$ and $g \in L^{2,\infty}(X|Y)$.

(b) The space $L^2(X|Y)$ is clearly an $L^\infty(Y)$-module (i.e., a linear subspace closed under multiplication by $L^\infty(Y)$-functions) with

$$
L^\infty(X) \subset L^2(X|Y) \subset L^2(X).
$$

Moreover, $L^2(X|Y)$ is $T$-invariant by \footnote{See Zorin-Kranich \cite{Zorin-Kranich:conditional_almost_periodic} Lemma 8.1] based on Tao \cite{Tao:conditional_mixing} Exercises 2.14.1, 2.14.2.} with $T$ acting as a contraction w.r.t. the norm $\| \cdot \|_{L^2,\infty(X|Y)}$.

(c) It is easy to see that $A(X|Y)$ is a $T$- and $T^{-1}$-invariant sub-$L^\infty(Y)$-module. Moreover, it is closed in $L^2(X|Y)$ w.r.t. the norm $\| \cdot \|_{L^2,\infty(X|Y)}$.

(d) It is easy but somewhat longer to show that $W(X|Y)$ is a closed linear subspace of $L^2(X)$.

(e) Analogously to Remark 4.2, $f \in L^2(X|Y)$ is conditionally weakly mixing if and only if there exists a set $J \subset \mathbb{N}$ with density 1 such that

$$
\lim_{n \to \infty, n \in J} \|T^n f\|_{L^2(X|Y)} = 0.
$$

(f) For the trivial one-point factor $(Y, \nu, S)$ the conditional definitions coincide with the unconditional ones. In particular, in this case $f \in W(X|Y)$ if and only if $f$ is weakly mixing.

Remark 5.3 (Connection to conditional almost periodicity in measure). A function $f \in L^2(X|Y)$ is called \textit{conditionally almost periodic in measure} if for every $\varepsilon > 0$ there exists a measurable set $M \subset Y$ with $\nu(M) > 1 - \varepsilon$ such that $1_M f \in A(X|Y)$, see Tao \cite{Tao:conditional_mixing} Def. 2.13.7, cf. Furstenberg, Katznelson \cite{Furstenberg-Katznelson} Section 2 and Furstenberg \cite{Furstenberg} Section 6.3]. Denote the set of all such functions by $AM(X|Y)$. This set satisfies

$$
AM(X|Y) = \overline{A(X|Y)} \cap L^2(X|Y),
$$

for some $m \in \mathbb{N}$, $f_1, \ldots, f_m \in L^2(X|Y))$ such that for every $n \in \mathbb{Z}$ there exists $g \in \mathbb{Z}$ with

$$
\| T^n f - g \|_{L^2(X|Y)} < \varepsilon.
$$

We denote the space of all conditionally almost periodic functions by $A(X|Y)$. We denote the space of all conditionally almost periodic functions by $A(X|Y)$. We denote the space of all conditionally almost periodic functions by $A(X|Y)$.
where the closure is taken in $L^2(X)$, see, e.g., Zorin-Kranich \cite{Zorin-Kranich}, cf. Furstenberg, Katznelson \cite{Furstenberg, Katznelson} Thm. 2.1 and Furstenberg \cite{Furstenberg} Thm. 6.13. For the reader’s convenience we present here the short argument. The inclusion \( \subset \) follows directly from the definition of conditional almost periodicity. For the converse inclusion let \( f \in L^2(X|Y) \) such that \( f = \lim_{n \to \infty} f_n \) in \( L^2(X) \) for some sequence \( (f_n) \subset A(X|Y) \).

By (3) we have
\[
\int_Y E_Y(|f_n - f|^2) \, d\nu = \|f_n - f\|_{L^2(X)}^2 \to 0 \quad \text{as} \quad n \to \infty.
\]

Hence we can assume without loss of generality (passing to a subsequence if necessary) that \( E_Y(|f_n - f|^2) \to 0 \) as \( n \to \infty \) holds \( \nu \)-a.e. By Egorov’s theorem there exists a measurable set \( M \subset Y \) with \( \nu(M) > 1 - \varepsilon \) such that \( \|1_M E_Y(|f_n - f|^2)\|_{L^\infty(Y)} \to 0 \), or, equivalently by (3),
\[
\|1_M f_n - 1_M f\|_{L^2(X|Y)} \to 0 \quad \text{as} \quad n \to \infty.
\]

Since for every \( n \in \mathbb{N} \) the function \( 1_M f_n \) is conditionally almost periodic (by using the same module zonotopes as for \( f_n \)), so is \( 1_M f \) by Remark 5.2(c).

The following conditional version of Theorem 4.3 is due to Tao \cite[Section 2.14.2]{Tao}, cf. Furstenberg, Katznelson \cite[Prop. 2.2]{Furstenberg, Katznelson} and Furstenberg, Katznelson, Ornstein \cite[Thm. 10.1]{Furstenberg, Katznelson, Ornstein}. Here and later we mean the closure in \( L^2(X) \) if not specified otherwise.

**Theorem 5.4** (Conditional Jacobs-de Leeuw-Glicksberg decomposition). Let \((X, \mu, T)\) be a measure-preserving system with factor \((Y, \nu, S)\). Then one has the orthogonal decomposition
\[
L^2(X) = A(X|Y) \oplus W(X|Y).
\]

**Remark 5.5.** Zorin-Kranich \cite{Zorin-Kranich} showed that for regular systems with ergodic factor
\[
A(X|Y) = E(X|Y)
\]
holds for the set \( E(X|Y) \) of conditional eigenfunctions, see Jamneshan \cite[Theorem 4.1]{Jamneshan} for the general case in the context of arbitrary group actions. Here \( f \in L^2(X|Y) \) is called a conditional eigenfunction if its orbit is contained in a \( T \)-invariant finitely generated sub-\( L^\infty(Y) \)-module (i.e., a set of the form
\[
\{c_1 f_1 + \ldots + c_m f_m : c_1, \ldots, c_m \in L^\infty(Y)\}
\]
for some \( m \in \mathbb{N} \), \( f_1, \ldots, f_m \in L^2(X|Y) \). Thus we also have the orthogonal decomposition
\[
L^2(X) = E(X|Y) \oplus W(X|Y),
\]
cf. Zimmer \cite[Cor. 7.10]{Zimmer}. For two different recent abstract approaches to \cite{Jamneshan} and \cite{Edeko, Haase, Kreidler}.

The following lemma is crucial, see Tao \cite[Exercise 2.13.6]{Tao} combined with Remark 5.3. We assume ergodicity of the factor in order to simplify the argument and follow the proof by Zorin-Kranich \cite{Zorin-Kranich}.

\[\text{\textsuperscript{11}}\text{This definition due to Furstenberg, Weiss \cite{Furstenberg, Weiss}, see also Zimmer \cite[Section 7]{Zimmer}, is equivalent to the one of generalized eigenfunctions introduced by Furstenberg \cite{Furstenberg} for ergodic factors \((Y, \nu, S)\).}\]
Lemma 5.6. Let \((X, \mu, T)\) be a measure-preserving system with an ergodic factor \((Y, \nu, S)\). Then \(A(X|Y)\) is induced by a factor.

Proof. We proceed in three steps.

Step 1. We first show that for every \(f \in A(X|Y)\) one can take bounded generators of the module zopotopes in the definition of conditional almost periodicity. Let \(\varepsilon > 0\) and let \(f_1, \ldots, f_m \in L^2(X|Y)\) satisfy
\[
\text{orb}(f) := \{T^n f : n \in \mathbb{Z}\} \subset U_\varepsilon(Z(f_1, \ldots, f_m)),
\]
where we denote by \(Z(f_1, \ldots, f_m)\) the module zotope generated by \(f_1, \ldots, f_m\) and by \(U_\varepsilon(Z)\) the \(\varepsilon\)-neighborhood of a set \(Z \subset L^2(X|Y)\) with respect to the norm \(\| \cdot \|_{L^2,\infty(X|Y)}\).

Fix \(j \in \{1, \ldots, m\}\) and consider for every \(k \in \mathbb{N}\) the function \(f_{j,k} := f_j 1_{\{|f_j| \leq k\}}\) which is bounded by \(k\). Since the functions \(f_{j,k}\) approximate \(f_j\) in \(L^2(X)\), \(E_Y(|f_j - f_{j,k}|^2)\) converge to zero in \(L^1(Y)\) as \(k \to \infty\). By Egorov’s theorem (passing first to a subsequence if necessary), we see that there exists \(k \in \mathbb{N}\) and a set \(A_j \subset Y\) with \(\nu(A_j) > 1 - \frac{\varepsilon}{2m}\) such that \(E_Y(|f_j - f_{j,k}|^2) < \frac{\varepsilon}{m}\) on \(A_j\). Write now \(f_j = g_j + b_j + r_j\) with \(g_j := f_{j,k}\) being bounded by \(k\), \(b_j := (f_j - f_{j,k})1_{Y \setminus A_j}\) and the rest \(r_j := (f_j - f_{j,k})1_{A_j}\). Since \(\|r_j\|_{L^2,\infty(X|Y)} \leq \frac{\varepsilon}{m}\) by the definition of \(k\), we have
\[
\text{orb}(f) \subset U_{2\varepsilon}(Z(g_1, \ldots, g_m, b_1, \ldots, b_m)).
\]

Denote by \(\varphi\) and \(\psi\) the underlying transformations on \(X\) and \(Y\), respectively. Consider the “good” set \(B := \cap_{j=1}^m A_j\) satisfying \(\nu(B) > 1 - \varepsilon\). By (11) and since the functions \(b_j\) vanish on \(B\), for every \(l \in \mathbb{N}_0\) there exist \(s_l \in Z(g_1, \ldots, g_m)\) (which is then bounded by \(M := km\)) such that
\[
\|1_B T^l f - s_l\|_{L^2,\infty(X|Y)} < 2\varepsilon
\]
which we can rewrite as
\[
\|1_{\psi^l B} f - T^{-l} s_l\|_{L^2,\infty(X|Y)} < 2\varepsilon.
\]

Consider now the function
\[
\tilde{f} = \sum_{l=0}^\infty 1_{\psi^l B} T^{-l} s_l 1_{Y \setminus (\cup_{j=0}^{l-1} \psi^j(B))}.
\]

Clearly, \(\tilde{f}\) is bounded by \(M\). Moreover, since \(\cup_{l=0}^\infty \psi^l(B) = Y\) up to a null set by the ergodicity of \((Y, \nu, S)\), (12) implies
\[
\|f - \tilde{f}\|_{L^2,\infty(X|Y)} \leq 2\varepsilon.
\]
In particular, \(\tilde{f}\) satisfies by (11)
\[
\text{orb}(\tilde{f}) \subset U_{4\varepsilon}(Z(g_1, \ldots, g_m, b_1, \ldots, b_m)).
\]

We now truncate the new generators as follows. Let \(N := \frac{M}{\varepsilon} \sum_{j=1}^m \|b_j\|_{L^2,\infty(X|Y)}\) and set \(D := \cap_{j=1}^m \{|b_j| \leq N\} \subset X\). By \(1_{X \setminus D} \leq \sum_{j=1}^m \frac{|b_j|}{N}\) we have
\[
\|1_{X \setminus D}\|_{L^2,\infty(X|Y)} \leq \frac{1}{N} \sum_{j=1}^m \|b_j\|_{L^2,\infty(X|Y)} = \frac{\varepsilon}{M}.\]
By \((14)\) for every \(j \in \{1, \ldots, m\}\) and \(n \in \mathbb{Z}\) there exist \(c_{j,n}, d_{j,n} \in L^\infty(Y)\) bounded by 1 such that

\[
\left\| T^n \hat{f} - \sum_{j=1}^{m} c_{j,n} g_j - \sum_{j=1}^{m} d_{j,n} b_j \right\|_{L^2,\infty(X|Y)} < 4\varepsilon.
\]

It follows, by adding and subtracting \(1_D T^n \hat{f}\) and using \((15)\) and Remark \((5,2)\) a, that

\[
\left\| T^n \hat{f} - \sum_{j=1}^{m} c_{j,n} g_j - \sum_{j=1}^{m} d_{j,n} b_j \right\|_{L^2,\infty(X|Y)} \leq \left\| T^n \hat{f} \right\|_{L^\infty(X)} \left\| 1_{X \setminus D} \right\|_{L^2,\infty(X|Y)} + \left\| T^n \hat{f} \right\|_{L^\infty(X)} \left\| 1_{X \setminus D} \right\|_{L^2,\infty(X|Y)} < 4\varepsilon + \frac{M \varepsilon}{M} = 5\varepsilon.
\]

This implies by \((15)\)

\[
\text{orb}(f) \subset U_{5\varepsilon}(Z(g_1 1_D, \ldots, g_m 1_D, b_1 1_D, \ldots, b_m 1_D)),
\]

with every generator of this zonotope being bounded by \(\max\{k, N\}\). This completes the proof of Step 1.

**Step 2.** We now prove that \(A(X|Y) \cap L^\infty(X)\) is dense in \(A(X|Y)\). To do this it is enough to show that for every \(f \in A(X|Y)\) the function \(f^+ := \max\{0, f\}\) belongs to \(A(X|Y)\) which by shifting, multiplying with \(-1\) and applying the argument again implies conditional almost periodicity of the function \(f 1_{\{|f| \leq N\}}\) for every \(N \in \mathbb{N}\). Let \(f \in A(X|Y)\) and \(\varepsilon > 0\). By Step 1 we can assume that the orbit of \(f\) belongs to an \(\varepsilon\)-neighborhood \(\text{w.r.t.} \| \cdot \|_{L^2,\infty(X|Y)}\) of a module zonotope with bounded generators \(f_1, \ldots, f_m\). Using \(\| \cdot \|_{L^\infty(X|Y)} \leq \| \cdot \|_{L^\infty(X)}\), by changing to a \(2\varepsilon\)-neighborhood we can assume that each \(f_j\) is a finite linear combination of characteristic functions. Moreover, by enlarging \(m\) we can further assume that each \(f_j\) is a positive multiple of a characteristic function with supports being disjoint (up to null sets). After these modifications, by

\[
\left\| T^n f^+ - \sum_{j=1}^{m} c_j^+ f_j \right\|_{L^2,\infty(X|Y)} \leq \left\| T^n f - \sum_{j=1}^{m} c_j f_j \right\|_{L^2,\infty(X|Y)}
\]

we see that the orbit of \(f^+\) belongs to the same neighborhood of the same module zonotope as the orbit of \(f\) finishing the argument.

**Step 3.** It is clear that the set \(A(X|Y) \cap L^\infty(X)\) is conjugation invariant, contains \(1\) and is invariant under both \(T\) and \(T^{-1}\). Moreover, by Step 2 it is dense in \(A(X|Y)\). It thus remains to show that it is a subalgebra of \(L^\infty(X)\). Let \(f, g \in A(X|Y) \cap L^\infty(X)\) and assume without loss of generality that \(\|g\|_\infty \leq 1\). Let further \(\varepsilon > 0\) and take \(f_1, \ldots, f_m \in L^2(X|Y)\) with \(\text{orb}(f) \in U_\varepsilon(Z(f_1, \ldots, f_m))\). By Step 1 we can assume that \(f_1, \ldots, f_m \in L^\infty(X)\) and denote \(M := \max\{\|f_1\|_\infty, \ldots, \|f_m\|_\infty\}\). Let further \(g_1, \ldots, g_l \in L^2(X|Y)\) satisfy \(\text{orb}(g) \in U_{\frac{\varepsilon}{M}}(Z(g_1, \ldots, g_l))\). Then for
every \( n \in \mathbb{Z} \) and appropriate \( a_n \in \mathbb{Z}(f_1, \ldots, f_m) \) and \( b_n \in \mathbb{Z}(g_1, \ldots, g_l) \) we have by the triangle inequality and Remark 5.2(a)

\[
\|T^n(fg) - a_nb_n\|_{L^{2,\infty}(X|Y)} \leq \|T^n f - a_n\|_{L^{2,\infty}(X|Y)} \|g\|_{L^{\infty}(X)} + \|a_n\|_{L^{\infty}(X)} \|T^n g - b_n\|_{L^{2,\infty}(X|Y)} < \frac{\varepsilon}{2} + M \frac{\varepsilon}{2mM} = \varepsilon.
\]

Since each \( a_nb_n \) belongs to the module zonotope generated by \( f_jg_k, j \in \{1, \ldots, m\}, k \in \{1, \ldots, l\} \), this shows \( fg \in A(X|Y) \cap L^{\infty}(X) \). The proof is complete. \( \square \)

**Remark 5.7.** The factor inducing \( A(X|Y) \) is the maximal compact extension of \( Y \) in \( X \) in the sense of Definition 7.3 below. Moreover, by (9) it coincides with the maximal isometric extension of \( Y \) in \( X \) introduced by Furstenberg, Weiss [21] for ergodic systems.

As a corollary we obtain the following property of \( W(X|Y) \).

**Lemma 5.8.** Let \((X, \mu, T)\) be a measure-preserving system with factor \((Y, \nu, S)\). Then \( W(X|Y) \cap L^{\infty}(X) \) is dense in \( W(X|Y) \) w.r.t. the \( L^2 \)-norm.

**Proof.** Let \( f \in W(X|Y) \) and take a sequence \((f_n) \subset L^{\infty}(X)\) converging to \( f \) in \( L^2(X) \). Consider the sequence \((P_{W(X|Y)}f_n)\) where \( P_{W(X|Y)} \) denotes the orthogonal projection onto \( W(X|Y) \). This sequence clearly belongs to \( W(X|Y) \) and converges to \( f \) by the Pythagoras theorem. Moreover, for every \( n \in \mathbb{N} \) we have

\[ P_{W(X|Y)}f_n = f_n - P_{A(X|Y)}f_n \in L^{\infty}(X) \]

by (8) and Lemma 5.6 (Recall that the projection onto a factor is contractive w.r.t. the \( L^{\infty} \)-norm.) \( \square \)

We now introduce the distal factors of finite order defined by Furstenberg [17] using conditional eigenfunctions instead of conditionally almost periodic functions (cf. Remark 5.2).

**Definition 5.9.** Let \((X, \mu, T)\) be an ergodic measure-preserving system. We construct the sequence of factors inductively as follows. Start with \( D_0 := \text{Fix} T = \mathbb{C}1 \) and the corresponding fixed (one-point) factor \( D_0 \) and for every \( k \in \mathbb{N} \) denote by \( D_k \) the factor inducing \( D_k := A(X|D_{k-1}) \) (see Lemma 5.6). We call \( D_k \) the distal factor of order \( k \) of \((X, \mu, T)\).

Furstenberg [17] showed that for regular, ergodic systems for each \( k \in \mathbb{N} \) the factor \( D_k \) (defined via conditional eigenfunctions) is characteristic for \( k \)-term multiple ergodic averages [2] in the sense that both convergence and the limit of the averages remain unchanged if we replace every function by its projection onto this factor. See also Frantzikinakis [12] Theorem 5.2 who deduced it from Furstenberg [17] Theorem 7.1] using the van der Corput trick. We will give an alternative proof of this fact here, see Propositions 6.3 and 7.1 below.

6. Gowers-Host-Kra seminorms

The uniformity seminorms were introduced by Gowers [25] in his proof of Szemerédi’s theorem via higher order Fourier analysis for rotations on cyclic groups \( \mathbb{Z}_N \).
and were extended by Host and Kra \cite{Host_Kra_2005} to arbitrary ergodic measure-preserving systems in their proof of multiple convergence.

**Definition 6.1** (Gowers-Host-Kra (uniformity) seminorms). Let \((X, \mu, T)\) be an ergodic measure-preserving system and \(f \in L^\infty(X, \mu)\). The Gowers-Host-Kra (or uniformity) seminorms are defined inductively by

\[
\|f\|_{U_1} := \left| \int_X f \, d\mu \right|,
\]

\[
\|f\|_{U_{l+1}}^{2l+1} := \limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \|T^n f \cdot \mathcal{T}^l f\|_{U_1}^2, \quad l \in \mathbb{N}.
\]

**Remark 6.2** (Second uniformity seminorm). For an ergodic measure-preserving system \((X, \mu, T)\) the second uniformity seminorm satisfies

\[
\|f\|_{U_2}^4 = \limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} |\langle T^n f, f \rangle|^2.
\]

In particular, \(f \in L^\infty(X)\) is weakly mixing if and only if \(\|f\|_{U_2} = 0\).

It is easy to see that the Gowers-Host-Kra seminorms are increasing and satisfy

\[
\|f\|_{U_k} \leq \|f\|_{L^\infty(X)} \quad \forall k \in \mathbb{N}.
\]

We need the following stronger property of these seminorms, see, e.g., \cite[Equation (12)]{Gowers_2001}.

**Lemma 6.3.** Let \((X, \mu, T)\) be an ergodic measure-preserving system and \(k \in \mathbb{N}\). Then for \(p_k := \frac{2^k}{k+1}\) one has \(\|f\|_{U_k} \leq \|f\|_{L^{p_k}(X)}\) for every \(f \in L^\infty(X)\).

Let \((X, \mu, T)\) be an ergodic measure-preserving system. Consider for every \(k \in \mathbb{N}_0\) the orthogonal decomposition

\[
L^2(X) = Z_k \oplus \{ f \in L^\infty(X) : \|f\|_{U_{k+1}} = 0 \}
\]

known as the Host-Kra decomposition of order \(k\), where \(Z_k\) is at first defined as the orthogonal complement of the second part. In particular, we have \(Z_0 = C1 = \text{Fix}T\) by ergodicity and the decomposition \(\text{(19)}\) for \(k = 0\) coincides with the von Neumann decomposition \(\text{(6)}\). Moreover, for \(k = 1\) \(\text{(19)}\) coincides with the Jacobs-de Leeuw-Glicksberg decomposition \(\text{(7)}\) by Remark 6.2 and Lemma 5.8 applied to the one-point factor.

The following property shows the relevance of the uniformity seminorms for multiple convergence and recurrence. It is an easy consequence of the definition of the Gowers-Host-Kra seminorms and the van der Corput inequality, see, e.g., \cite[Section 14.1]{Host_Kra_2005}.

**Proposition 6.4** (Generalized von Neumann theorem). Let \((X, \mu, T)\) be an ergodic measure-preserving system, \(k \in \mathbb{N}\) and \(f_1, \ldots, f_k \in L^\infty(X)\). Then

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} T^n f_1 \cdot T^{2n} f_2 \cdot \cdots \cdot T^{kn} f_k \leq \min_{j=1;\ldots,k} \|f_j\|_{U_k}.
\]

\(\text{Remark 12}\) Initially Host, Kra \cite{Host_Kra_2005} defined the seminorms using cube measure spaces. The two definitions are easily shown to be equivalent, see, e.g., Kra \cite[Lemma 7.4]{Kra_2000} or \cite[Section 14.2]{Gowers_2001}.
Corollary 6.5 (The subspace $Z_{k-1}$ is characteristic for $k$-term multiple averages).
Let $(X, \mu, T)$ be an ergodic measure-preserving system, $k \in \mathbb{N}$ and $f_1, \ldots, f_k \in L^\infty(X)$. Then
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^N T^n f_1 \cdot T^{2n} f_2 \cdots T^{kn} f_k = 0 \quad \text{in } L^2(X)
\]
holds whenever $f_j \perp Z_{k-1}$ for some $j \in \{1, \ldots, k\}$.

Proof. Assume without loss of generality that $\|f_1\|_\infty \leq 1, \ldots, \|f_k\|_\infty \leq 1$ and let $f_j \perp Z_{k-1}$ for some $j \in \{1, \ldots, k\}$. By the Host-Kra decomposition (19) there exists a sequence $(g_m) \subset L^\infty(X)$ satisfying $\|g_m\|_{U^k} = 0$ for all $m \in \mathbb{N}$ with $\lim_{m \to \infty} \|f_j - g_m\|_2 = 0$. Thus for every $m \in \mathbb{N}$, by decomposing $f_j = (f_j - g_m) + g_m$, we have by (20) and the triangle inequality
\[
\limsup_{N \to \infty} \left\| \frac{1}{N} \sum_{n=1}^N T^n f_1 \cdot T^{2n} f_2 \cdots T^{kn} f_k \right\|_2 \leq \|f_j - g_m\|_{L^2(X)}.
\]
Letting $m \to \infty$ proves the assertion. \hfill \Box

The following property of the subspaces $Z_k$ can be shown via constructing the factors using cube measure spaces, see Host, Kra [26], [27, Chapter 9] and, e.g., [8, Section 14.4]. We will not need this fact and include it here for completeness.

Proposition 6.6. For every $k \in \mathbb{N}_0$ the subspace $Z_k$ is induced by a factor $Z_k$, called the Host-Kra-Ziegler factor of order $k$.

In particular, $Z_0$ is the one-point factor and $Z_1$ is the Kronecker factor. The Host-Kra-Ziegler factors were introduced by Host and Kra [26] and subsequently independently by Ziegler [40] (see Leibman [30] for the equality of the two constructions) in their proofs for multiple convergence. We refer to Host, Kra [26, 27] for a detailed analysis of these factors and the deep structure theorem which states that for ergodic regular systems each $Z_k$ is an inverse limit of nilsystems of step $k$.

Again, we will not need anything from this theory here.

Remark 6.7 (Multiple recurrence for weakly mixing systems). For weakly mixing systems\footnote{A measure-preserving system $(X, \mu, \varphi)$ is called weakly mixing if $\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^N |\mu(A \cap \varphi^{-n}(B)) - \mu(A)\mu(B)| = 0$ for every measurable $A, B \subset X$. This is equivalent to the orthogonal decomposition $L^2(X) = C1 \oplus \{\text{weakly mixing functions}\}$.} one easily shows using Remark 6.2 and induction that all Gowers-Host-Kra seminorms are equal to the first seminorm (see, e.g., Kra [29, Section 7.3]). Thus in this case $Z_k = C1$ holds for every $k \in \mathbb{N}$ and both multiple convergence and multiple recurrence follow from Corollary 6.5 with
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^N \int_X f_0 \cdot T^n f_1 \cdot T^{2n} f_2 \cdots T^{kn} f_k \, d\mu = \prod_{j=0}^{k-1} \int_X f_j \, d\mu
\]
recovering a result of Furstenberg [17, Equation (1)].
7. Host-Kra-Ziegler factors versus distal factors

We now connect the Host-Kra-Ziegler factors to distal factors of the corresponding order using the conditional Jacobs-de Leeuw-Glicksberg decomposition (Theorem 5.4) and Lemma 6.3.

**Proposition 7.1** (Distal factors are extensions of Host-Kra-Ziegler factors). Let \((X, \mu, T)\) be an ergodic measure-preserving system. Then \(Z_k \subset D_k\) for every \(k \in \mathbb{N}\).

**Proof.** Let \(k \in \mathbb{N}_0\). By the orthogonal decomposition

\[
L^2(X) = D_{k+1} \oplus W(X|D_k)
\]

from the conditional Jacobs-de Leeuw-Glicksberg Theorem 5.4 and the Host-Kra decomposition (19) for \(k + 1\) we need to show

\[
W(X|D_k) \subset \{ f \in L^\infty(X) : \| f \|_{U^{k+2}} = 0 \}.
\]

By Lemma 5.8 it suffices to show

\[
\| f \|_{U^{k+2}} = 0 \quad \text{for every} \quad f \in W(X|D_k) \cap L^\infty(X).
\]

We show (22) for every \(k \in \mathbb{N}_0\) by induction on \(k\). For \(k = 0\) let \(f \in W(X|D_0) \cap L^\infty(X)\), where \(D_0\) is the one-point factor. Then \(f\) is weakly mixing by Remark 5.2(f) and hence \(\| f \|_{U^2} = 0\) by Remark 6.2. Assume now that \(k \in \mathbb{N}\) and that (22) holds for \(k - 1\). Let \(f \in W(X|D_k)\) be bounded and assume without loss of generality \(\| f \|_\infty \leq 1\). Take \(n \in \mathbb{N}\). By the triangle inequality and the decomposition (21) for \(k - 1\)

\[
\| T^n f \cdot \tilde{f} \|_{U^{k+1}} \leq \| P_{D_k}(T^n f \cdot \tilde{f}) \|_{U^{k+1}} + \| P_{W(X|D_{k-1})}(T^n f \cdot \tilde{f}) \|_{U^{k+1}}
\]

for the orthogonal projections \(P_{D_k}\) and \(P_{W(X|D_{k-1})}\) onto the factor \(D_k\) and the subspace \(W(X|D_{k-1})\), respectively. (Recall that the first projection and hence also the second, complementary, projection maps bounded functions to bounded functions.) The last summand in (23) equals zero by the induction hypothesis. Moreover, recall that \(P_{D_k} = J E_{D_k}\) for the corresponding Markov factor map \(J\) and that both \(J\) and \(E_{D_k}\) act as contractions w.r.t. the \(L^p\)-norm for every \(p \in [1, \infty]\).

So we have by (23), Lemma 6.3 and \(\| f \|_\infty \leq 1\) denoting \(p_{k+1} := \frac{2}{k+2}\)

\[
\| T^n f \cdot \tilde{f} \|_{U^{k+1}} \leq \| P_{D_k}(T^n f \cdot \tilde{f}) \|_{U^{k+1}} \leq \| P_{D_k}(T^n f \cdot \tilde{f}) \|_{L^p(X)}
\]

\[
\leq \| E_{D_k}(T^n f \cdot \tilde{f}) \|_{L^1(Y)} \leq \| E_{D_k}(T^n f \cdot \tilde{f}) \|_{L^1(Y)}^{1/p_{k+1}}.
\]

Since \(\| T^n f \cdot \tilde{f} \|_{U^{k+1}} \leq 1\) by \(\| f \|_\infty \leq 1\) and (18), this implies

\[
\| T^n f \cdot \tilde{f} \|_{U^{k+1}} \leq \| P_{D_k}(T^n f \cdot \tilde{f}) \|_{U^{k+1}} \leq \| E_{D_k}(T^n f \cdot \tilde{f}) \|_{L^1(Y)}
\]

and, by \(f \in W(X|D_k)\),

\[
\| f \|_{U^{k+2}} = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \| T^n f \cdot \tilde{f} \|_{U^{k+1}}^2 = 0
\]

follows. This proves (22) and completes the proof. \(\square\)
Remark 7.2. As mentioned in the introduction, as a corollary of Proposition 6.4 and Proposition 7.1 we obtain an alternative proof of the fact proved by Furstenberg [17], see also Frantzikinakis [12, Theorem 5.2] based on Furstenberg [17, Theorem 7.1], that the distal factor $D_{k-1}$ of order $k-1$ is characteristic for $k$-term multiple recurrence.

We now define compact extensions, cf. Furstenberg, Katznelson [19, Def. 3.1] and Furstenberg, Katznelson, Ornstein [20, Section 9].

**Definition 7.3.** A measure-preserving system $(X, \mu, T)$ is called a compact extension of its factor $(Y, \nu, S)$ if $A(X|Y) = L^2(X)$. In particular, for an ergodic measure-preserving system $(X, \mu, T)$ and every $k \in \mathbb{N}_0$ the distal factor $D_{k+1}$ is by definition a compact extension of $D_k$, namely the maximal compact extension of $D_k$ in $(X, \mu, T)$.

The last ingredient of the proof of Theorem 2.5 is the following property of compact extensions, see Tao [34, Theorem 2.13.11] combined with Remark 5.3, cf. Furstenberg, Katznelson [19, Prop. 3.5], Furstenberg, Katznelson, Ornstein [20, Thm. 9.1] and Einsiedler, Ward [7, Section 7.9].

**Proposition 7.4 (Compact extensions preserve MR).** Let $(X, \mu, T)$ be a compact extension of $(Y, \nu, S)$. If $(Y, \nu, S)$ satisfies MR then so does $(X, \mu, T)$. 

Proof of multiple recurrence for ergodic systems (Theorem 2.5). Let $(X, \mu, T)$ be an ergodic measure-preserving system and $k \in \mathbb{N}$. Denoting by $P_{D_{k-1}}$ the orthogonal projection onto $D_{k-1}$, for $f \in L^\infty(X)$ with $f > 0$ the function $P_{D_{k-1}}f$ is also bounded and satisfies $P_{D_{k-1}}f > 0$ by Lemma 5.6 and (5). Therefore Proposition 7.1 and Corollary 6.5 imply

$$\lim_{N \to \infty} \left\| \frac{1}{N} \sum_{n=1}^{N} T^n f \cdots T^{kn} f - \frac{1}{N} \sum_{n=1}^{N} T^n P_{D_{k-1}} f \cdots T^{kn} P_{D_{k-1}} f \right\|_{L^2(X)} = 0$$

and we can assume without loss of generality that $L^2(X) = D_{k-1}$. So we have the chain of factors

$$X = D_{k-1} \to D_{k-2} \to \ldots \to D_1 \to D_0 = \{\cdot\}$$

where $\{\cdot\}$ denotes the one-point factor, and every factor in this chain is a compact extension of the next one by definition. Finally, the one-point factor clearly satisfies MR. Thus every $D_j$, and hence also $X = D_{k-1}$, satisfies MR by Proposition 7.4. The proof is complete. □

**References**

[1] V. Bergelson, A. Leibman, *Polynomial extensions of van der Waerden’s and Szemerédi’s theorems*, J. Amer. Math. Soc. 9 (1996), 725–753.

[2] V. Bergelson, A. Leibman, E. Lesigne, *Intersective polynomials and the polynomial Szemerédi theorem*, Adv. Math. 219 (2008), 369–388.
[3] V. Bergelson, R. McCutcheon, *An ergodic IP polynomial Szemerédi theorem*, Mem. Am. Math. Soc. 146 (695) (2000), viii+106.

[4] V. Bergelson, J. Moreira, F. K. Richter, *Single and multiple recurrence along non-polynomial sequences*, Adv. Math. 368 (2020), 107146, 69 pp.

[5] B. Cook, Á. Magyar, T. Titichetrakun, *A multidimensional Szemerédi theorem in the primes via combinatorics*, Ann. Comb. 22 (2018), 711–768.

[6] N. Edelen, M. Haase, H. Kreidler, *A decomposition theorem for unitary group representations on Kaplansky-Hilbert modules and the Furstenberg-Zimmer structure theorem*, preprint, 2021, available at https://arxiv.org/abs/2104.04865.

[7] M. Einsiedler, T. Ward, *Ergodic Theory with a View Toward Number Theory*. Graduate Texts in Mathematics, 259. Springer-Verlag London, Ltd., London, 2011.

[8] T. Eisner, B. Farkas, *Operator Theoretic Aspects of Ergodic Theory*. Graduate Texts in Mathematics, vol. 272, Springer, Cham, 2015.

[9] T. Eisner, B. Farkas, M. Haase, R. Nagel, *Ergodic Theorems*. Book manuscript, submitted.

[10] T. Eisner, T. Tao, *Large values of the Gowers-Host-Kra seminorms*, J. Anal. Math. 117 (2012), 133–186.

[11] J. Fox, Y. Zhao, *A short proof of the multidimensional Szemerédi theorem in the primes*, Amer. J. Math. 137 (2015), 1139–1145.

[12] N. Frantzikinakis, *The structure of strongly stationary systems*, J. Anal. Math. 93 (2004), 359–388.

[13] N. Frantzikinakis, *A multidimensional Szemerédi theorem for Hardy sequences of different growth*, Trans. Am. Math. Soc. 367 (8) (2015), 5653–5692.

[14] N. Frantzikinakis, B. Host, B. Kra, *Multiple recurrence and convergence for sequences related to the prime numbers*, J. Reine Angew. Math. 611 (2007), 131–144.

[15] N. Frantzikinakis, B. Host, B. Kra, *The polynomial multidimensional Szemerédi theorem along shifted primes*, Israel J. Math. 194 (2013), 331–348.

[16] N. Frantzikinakis, M. Wierdl, *A Hardy field extension of Szemerédi’s theorem*, Adv. Math. 222 (2009), 1–43.

[17] H. Furstenberg, *Ergodic behavior of diagonal measures and a theorem of Szemerédi on arithmetic progressions*, J. Analyse Math. 31 (1977), 204–256.

[18] H. Furstenberg, *Recurrence in Ergodic Theory and Combinatorial Number Theory*. M. B. Porter Lectures. Princeton University Press, Princeton, N.J., 1981.

[19] H. Furstenberg, Y. Katznelson, *An ergodic Szemerédi theorem for commuting transformations*, J. Analyse Math. 34 (1978), 275–291 (1979).

[20] H. Furstenberg, Y. Katznelson, D. Ornstein, *The ergodic theoretical proof of Szemerédi’s theorem*, Bull. Amer. Math. Soc. (N.S.) 7 (1982), 527–552.

[21] H. Furstenberg, B. Weiss, *A mean ergodic theorem for \(\frac{1}{N}\sum_{n=1}^{N} f(T^n x)g(T^{2n} x)\)*, Convergence in ergodic theory and probability (Columbus, OH, 1993), 193–227, Ohio State Univ. Math. Res. Inst. Publ., 5, de Gruyter, Berlin, 1996.

[22] E. Glasner, *Ergodic Theory via Joinings*. Mathematical Surveys and Monographs, 101. American Mathematical Society, Providence, RI, 2003.

[23] B. Green, T. Tao, *The primes contain arbitrarily long arithmetic progressions*. Ann. of Math. (2) 167 (2008), 481–547.

[24] B. Green, T. Tao, *Linear equations in primes*, Ann. of Math. (2) 171 (2010), 1753–1850.

[25] W. T. Gowers, *A new proof of Szemerédi’s theorem*, Geom. Funct. Anal. 11 (2001), 465–588.

[26] B. Host, B. Kra, *Nonconventional ergodic averages and nilmanifolds*, Ann. of Math. (2) 161 (2005), 397–488.

[27] B. Host, B. Kra, *Nilpotent Structures in Ergodic Theory*. Mathematical Surveys and Monographs, 236. American Mathematical Society, Providence, RI, 2018.

[28] A. Jamneshan, *An uncountable Furstenberg-Zimmer structure theory*, preprint, 2021, available at https://arxiv.org/abs/2103.17167.

[29] B. Kra, *Ergodic methods in additive combinatorics*, Additive combinatorics, 103–143, CRM Proc. Lecture Notes, 43, Amer. Math. Soc., Providence, RI, 2007.

[30] A. Leibman, *Host-Kra and Ziegler factors and convergence of multiple averages*, Handbook of Dynamical Systems, vol. 1B, B. Hasselblatt and A. Katok, eds., Elsevier (2005), 841–853.

[31] J. Moreira, F. K. Richter, *A spectral refinement of the Bergelson-Host-Kra decomposition and new multiple ergodic theorems*, Ergodic Theory Dynam. Systems 39 (2019), 1042–1070.
[32] E. Szemerédi, *On sets of integers containing no k elements in arithmetic progression*. Acta Arith. 27 (1975), 199–245.

[33] T. Tao, *A quantitative ergodic theory proof of Szemerédi’s theorem*, Electron. J. Combin. 13 (2006), no. 1, Research Paper 99, 49 pp.

[34] T. Tao, Poincaré’s legacies, pages from year two of a mathematical blog. Part II. American Mathematical Society, Providence, RI, 2009.

[35] T. Tao, T. Ziegler, *The primes contain arbitrarily long polynomial progressions*, Acta Math. 201 (2008), 213–305.

[36] T. Tao, T. Ziegler, *Erratum to “The primes contain arbitrarily long polynomial progressions”*, Acta Math. 210 (2013), 403–404.

[37] T. Tao, T. Ziegler, *A multi-dimensional Szemerédi theorem for the primes via a correspondence principle*, Israel J. Math. 207 (2015), 203–228.

[38] T. Tao, T. Ziegler, *Polynomial patterns in the primes*, Forum Math. Pi 6 (2018), e1, 60 pp.

[39] T. D. Wooley, T. Ziegler, *Multiple recurrence and convergence along the primes*, Amer. J. Math. 134 (2012), 1705–1732.

[40] T. Ziegler, *Universal characteristic factors and Furstenberg averages*, J. Amer. Math. Soc. 20 (2007), 53–97.

[41] P. Zorin-Kranich, *Compact extensions are isometric*, unpublished note, 2011. Available at https://www.math.uni-bonn.de/~pzorin/.

[42] P. Zorin-Kranich, *Ergodic Theory*, lecture notes, 2015/16. Available at https://www.math.uni-bonn.de/~pzorin/.

[43] R. J. Zimmer, *Ergodic actions with generalized discrete spectrum*, Illinois J. Math. 20 (1976), 555–588.

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