COLLAPSED 5-MANIFOLDS WITH PINCHED POSITIVE SECTIONAL CURVATURE

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\textbf{Abstract.} Let $M$ be a closed 5-manifold of pinched curvature $0 < \delta \leq \sec M \leq 1$. We prove that $M$ is homeomorphic to a spherical space form if $M$ satisfies one of the following conditions: (i) $\delta = 1/4$ and the fundamental group is a non-cyclic group of order $\geq C$, a constant. (ii) The center of the fundamental group has index $\geq w(\delta)$, a constant depending on $\delta$. (iii) The ratio of the volume and the maximal injectivity radius is $< \epsilon(\delta)$. (iv) The volume is less than $\epsilon(\delta)$ and the fundamental group $\pi_1(M)$ has a center of index at least $w$, a universal constant, and $\pi_1(M)$ is either isomorphic to a spherical 5-space group or has an odd order.

0. Introduction

The sphere theorem asserts that if a manifold $M$ admits a quarter pinched metric, $\frac{1}{4} < \sec M \leq 1$, then its universal covering space is homeomorphic to a sphere. A natural problem is to determine whether $M$ is homeomorphic to a spherical space form, $S^n/\Gamma, \Gamma \subset O(n+1)$? We will call $\Gamma$ a spherical $n$-space group. This problem is (wildly) open in all odd-dimension $n \geq 5$ (cf. [Ha], [GKR], [IHR]). Clearly, a positive answer implies that the fundamental group $\pi_1(M)$ is isomorphic to a spherical space group and the $\pi_1(M)$-action on the universal covering is conjugate to a linear action. A subtlety is that neither of these holds without a positive curvature condition: in every odd dimension $n \geq 5$ ([Ha]), (0.1) There are infinitely many non-spherical space groups acting freely on an $n$-sphere. (0.2) There are infinitely many distinct free actions on an $n$-sphere by a spherical space group which do not conjugate to any linear action. Obviously, the quotient manifolds in (0.1) and (0.2) are not homeomorphic to any spherical space form.

In this paper, as a first step we investigate the case of dimension 5. We give positive answers (Theorems A-D) for a $\delta$-pinched 5-manifold $M$ whose fundamental group is not small (equivalently, whose volume is small). In particular, we rule out (0.1) and (0.2) in our circumstances via studying certain symmetry structure on $M$ discovered by Cheeger-Fukaya-Gromov ([CFG], [Ro1]).

We now begin to state the main results in this paper.
Theorem A.

Let $M$ be a closed 5-manifold with $\frac{1}{4} < \sec M \leq 1$. If the fundamental group $\pi_1(M)$ is a non-cyclic group of order $\geq C$ (a constant), then $M$ is homeomorphic to a spherical space form.

For any $\delta$-pinching, we can generalize Theorem A (whose assumption is slightly stronger than the case of $\delta = 1/4$ in Theorem B).

Theorem B.

Let $M$ be a closed 5-manifold with $0 < \delta \leq \sec M \leq 1$. If the center of $\pi_1(M)$ has index $\geq w(\delta)$ (a constant depending on $\delta$), then $M$ is homeomorphic to a spherical space form.

Note that $M$ satisfies that $\text{diam}(M) \leq \pi/\sqrt{\delta}$ (Bonnet theorem) and $\text{vol}(M) \leq \text{vol}(S^5_\delta)/w(\delta) \ll 1$ (volume comparison), where $S^5_\delta$ denotes the sphere of constant curvature $\delta$. According to [CFG] (cf. [Ro1]), $M$ admits local compatible isometric $T^k$-actions with $k \geq 1$ of some nearby positively curved metric (details will be given shortly). In our circumstance, we show that $k \geq 2$, and thus the universal covering of $M$ is diffeomorphic to a sphere ([Ro2]). The main work in the proofs of Theorems A and B is to show, using the local symmetry structure, that $\pi_1(M)$ is isomorphic to a spherical space group and the $\pi_1(M)$-action is conjugate to a linear one (see Theorem E, compare to (0.1) and (0.2)).

Observe that any $\delta$-pinched 5-manifold satisfies that $\text{diam}(M) \leq \pi/\sqrt{\delta}$ (Bonnet theorem) and $\text{vol}(M) \leq \text{vol}(S^5_\delta)/w(\delta) \ll 1$ (volume comparison), where $S^5_\delta$ denotes the sphere of constant curvature $\delta$. According to [CFG] (cf. [Ro1]), $M$ admits local compatible isometric $T^k$-actions with $k \geq 1$ of some nearby positively curved metric (details will be given shortly). In our circumstance, we show that $k \geq 2$, and thus the universal covering of $M$ is diffeomorphic to a sphere ([Ro2]). The main work in the proofs of Theorems A and B is to show, using the local symmetry structure, that $\pi_1(M)$ is isomorphic to a spherical space group and the $\pi_1(M)$-action is conjugate to a linear one (see Theorem E, compare to (0.1) and (0.2)).

Theorem C.

For $0 < \delta \leq 1$, there exists a small number, $\epsilon(\delta) > 0$, such that if a closed 5-manifold $M$ satisfies

$$0 < \delta \leq \sec M \leq 1, \quad \frac{\text{vol}(M)}{\max \text{injrad}(M,x)} < \epsilon(\delta),$$

then $M$ is homeomorphic to a spherical space form.

Note that Theorems B and C do not hold if one replaces the condition “$\delta > 0$” with “$\delta \geq 0$” (see Example 2.7). We intend to discuss the classification with “$\delta \geq 0$” elsewhere.

Consider a collapsed $\delta$-pinched 5-manifold $M$ close in the Gromov-Hausdorff distance to a metric space $X$ of dimension 4 (equivalently, $k = 1$). A new trouble is to determine the topology of the universal covering space, or equivalently the topology of $X$. This looks quite difficult; it seems to require a classification of positively curved 4-manifolds in the case when $X$ is smooth (and thus the fundamental group of $M$ is cyclic). The following result says that $M$ is homeomorphic to a spherical space form when $\pi_1(M)$ is certain non-cyclic group.

Theorem D.

For $0 < \delta \leq 1$, there exists $\epsilon(\delta) > 0$ such that if a closed 5-manifold $M$ satisfies

$$0 < \delta \leq \sec 1, \quad \text{vol}(M) < \epsilon(\delta),$$

then $M$ is homeomorphic to a spherical space form.
then $M$ is homeomorphic to a spherical space form, provided $\pi_1(M)$ has a center of index at least $w > 0$, a constant (independent of $\delta$), and $\pi_1(M)$ is a spherical 5-space group or $|\pi_1(M)|$ is odd.

We mention that there are infinitely many spherical 5-space forms satisfying the conditions of Theorems A-D (see Example 2.6).

A natural question is when $M$ in Theorems A-D is diffeomorphic to a spherical 5-space form? We mention the following: a spherical 5-space form $S^5/\Gamma$ admits exactly one or two different smooth structures depending on $|\Gamma|$ odd or even ([KS], [Wa]). Moreover, both smooth structures may allow a non-negatively curved metric (e.g., there are exactly four smooth manifolds homotopy equivalent to $\mathbb{R}P^5$, all of them admit metrics of non-negative sectional curvature ([GZ]), and two of them are not homeomorphic to each other).

A question of Yau ([Yau]) is if in a given homotopy type contains at most finitely many diffeomorphism types that can support a metric of positive sectional curvature. A positive answer is known only in dimensions 2 and 3 ([Ha]). When restricting to the class of pinched metrics, positive answers are known in even dimensions and the class of manifolds (odd-dimensions) with finite second homotopy groups ([FR1], [PT]).

In view of the above, Theorem B has the following corollary.

**Corollary 0.3.**

Let $M$ be a close $\delta$-pinched 5-manifold. Then the homotopy type of $M$ contains at most $c(\delta)$ many diffeomorphism types that support a $\delta$-pinched metric, provided $\pi_1(M)$ has a center with index $\geq w(\delta)$.

As mentioned earlier, our approach to Theorems A-D is based on the fibration theorem of Cheeger-Fukaya-Gromov on collapsed manifolds with bounded sectional curvature and diameter ([CFG], [CG1,2]). In our circumstances, the fibration theorem asserts that there is a constant $v(n, \delta) > 0$ such that if a $\delta$-pinched $n$-manifold $M$ has a volume less than $v(n, \delta)$, then $M$ admits a pure $F$-structure all whose orbits are of positive dimensions. By the Ricci flows technique, one can show that there is invariant metric which is at least $\delta/2$-pinched ([Ro1]).

In the case of a finite fundamental group, the notion of a pure $F$-structure is equivalent to that of a $\pi_1$-invariant torus $T^k$-action on a manifold $M$, which is defined by an effective $T^k$-action on the universal covering space $\tilde{M}$ of $M$ such that it extends to a $T^k \rtimes \rho_1(M)$-action, where $\rho : \pi_1(M) \to \text{Aut}(T^k)$ is a homomorphism from the fundamental group to the automorphism group of $T^k$. Clearly, the $T^k$-action on $\tilde{M}$ is the lifting of a $T^k$-action on $M$ if and only if $\rho$ is trivial or equivalently, the $T^k$-action and the $\pi_1(M)$-action commute. Hence, the notion of a $\pi_1$-invariant torus action generalizes that of a global torus action and the $T^k$-orbit structure on $\tilde{M}$ projects onto $M$ so that each orbit is a flat submanifold.

Consider $M$ as in Theorems A-D. In view of the above, we may assume that $M$ admits a $\pi_1$-invariant isometric $T^k$-action ($k \geq 1$).

**Theorem E.**

Let $M$ be a closed 5-manifold of positive sectional curvature. If $M$ admits a $\pi_1$-invariant isometric $T^k$-action with $k > 1$, then $M$ is homeomorphic to a spherical space form.
Theorem E is known in the following special cases: $M$ (itself) admits an isometric $T^3$-action ([GS]) or $M$ is simply connected ([Ro2]).

We show that under the assumptions of Theorems B and C, $k > 1$ and thus Theorem E implies Theorems B and C. In the case $k = 1$, the $T^1$-action on the universal covering $\tilde{M}$ of $M$ is free if $\tilde{M}$ is a sphere and if $\pi_1(M)$ is not cyclic. We then complete the proof of Theorem A by proving the following topological result: Let a finite group $\Gamma$ act freely on $S^5$. If $S^5$ admits a free $\Gamma$-invariant $T^1$-action such that the induce $\Gamma$-action on $S^5/T^1$ is pseudo-free, then $S^5/\Gamma$ is homeomorphic to a spherical space form (see Proposition 1.4).

In view of the above, Theorem D follows from Theorem F.

Let $M$ be a closed 5-manifold of positive sectional curvature which admits a $\pi_1$-invariant fixed point free isometric $T^1$-action. Then the universal covering $\tilde{M}$ is diffeomorphic to $S^5$, provided $\pi_1(M)$ has a center of index $\geq w$, and $\pi_1(M)$ is a spherical 5-space group or $|\pi_1(M)|$ is odd.

It is worth to point it out that every spherical 5-space form admits a $\pi_1$-invariant isometric $T^3$-action and a free isometric $T^1$-action and a $\pi_1(M)$-invariant isometric $T^2$-action ([Wo], p.225).

We would like to put Theorems E and F in a little perspective. In the study of positive sectional curvature, due to the obvious ambiguity the class of positively curved manifolds with (large) symmetry has frequently been a focus of the investigations. According to K. Grove, this also serves as a strategy of searching for new examples and obstructions. There has been significant progress in the last decade on classification of simply connected manifolds with large symmetry rank (the rank of the isometry group), cf. [GS], [FR1,2], [HK], [Ro1-3], [Wi1,2]. However, not much is known for non-simply connected manifolds with large symmetry rank. Theorems E and F may be treated as an attempt in this direction.

We now give an outline of the proof of Theorems E and F.

By the compact transformation group theory ([Bre]), the topology of a $T^k$-space $M$ is closely related to that of the singular set (the union of all non-principal orbits) and the orbit space $M/T^k$. In the presence of an invariant metric of positive sectional curvature, the singular set and the orbit space are very restricted, and this is the ultimate reason for a possible classification. In our proofs, we will thoroughly investigate the structure of the singularity and the orbit space.

The proof of Theorem E divides into two situations: $k = 3$ (Theorem 3.1) and $k = 2$, and the main work is in the case of $k = 2$. When $k = 2$, we first prove Theorem E at the level of fundamental groups (Theorem 4.1). Then we divide the proof into two cases: the $\pi_1$-invariant $T^2$-action is pseudo-free (section 5) and not pseudo-free (section 6). In the former case, we study the $T^2$-action on the universal covering space, $\tilde{M} \simeq S^5$, which has a singular set, $\mathcal{S}$, consisting of three isolated circle orbits. It suffices to show that the $(\pi_1(M), T^2)$-bundle, $\tilde{M} - \mathcal{S} \to (\tilde{M} - \mathcal{S})/T^2$, is conjugate to a standard linear model from spherical space forms. This can be done following [FR2] if one can assume that the orbit space, $\tilde{M}^* = \tilde{M}/T^2$, is a homeomorphic sphere. The problem is that we only know that $X$ is a homotopy 3-sphere. We overcome this difficulty by combining the above with tools from the $s$-cobordism theory in dimension 5. In the non-pseudo-free case, the singular set has dimension 3, and by analyzing the singular structure, we are able to view $M$
as a gluing of standard pieces.

In the proof of Theorem F, by studying the induced $\pi_1(M)$-action on $\tilde{M}^* = \tilde{M}/T^1$ (which is not trivial because $\pi_1(M)$ is not cyclic), we bound from above the Euler characteristic of $M^*$ via the technique of $\eta$-extent (Proposition 9.5, cf. [Gr], [Ya]). With this constraint, we show that the condition on the fundamental groups implies that $\tilde{M}$ is a homology sphere, the $T^1$-action is free and the $\pi_1(M)$-action on $\tilde{M}^*$ is pseudo-free (note that the standard free linear $T_1$-action on $S^5$ is preserved by any spherical 5-space group, [Wo]). By employing results in [HL] and [Wi1,2] on pseudo-free actions by finite groups on a homeomorphic complex projective plane, we show that the $\pi_1(M)$-action is homeomorphically conjugate to a linear action.

Remark 0.5.

By [AM], the $1_4$-pinching in Theorem A may be replaced by a slightly weaker pinching constant $1_4 - \epsilon$.

Remark 0.6.

Theorem F actually holds with a weak restrictions on $\pi_1(M)$ (see Section 3).

The rest of the paper is organized as follows:

Part I. Proofs of Theorems A-D by assuming Theorems E and F.

§1. Proof of Theorem A by assuming Theorem E.

§2. Proof of Theorems B and C by assuming Theorem E.

§3. Proof of Theorem D by assuming Theorems E and F.

Part II. Proofs of Theorems E and F

§4. Preparations

§5. Proof of Theorem E for $k = 3$.

§6. Proof of Theorem E at the level of fundamental groups.

§7. Proof of Theorem E for pseudofree $T^2$-actions.

§8. Completion of the proof of Theorem E.

§9. Proof of Theorem F.

Appendix: The $s$-cobordism theory.

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Part I. Proof of Theorems A-D by assuming Theorems E and F

1. Proof of Theorem A by Assuming Theorem E

Let $M$ be as in Theorem A. Then the universal covering space $\tilde{M}$ is homeomorphic to $S^5$ (the sphere theorem). Because the volume of $M$ is small, $\tilde{M}$ admits a $\pi_1$-invariant isometric $T^k$-action (Theorems 1.1 and 1.2). We will use this structure to show that $M$ is homeomorphic to a spherical space form (compare to (0.1) and (0.2)). By Theorem E, we may assume that $k = 1$. Because $\pi_1(M)$ is non-cyclic, we show that the isometric $T^1$-action must be free and commute with the $\pi_1(M)$-action such that the induced $\pi_1(M)$-action on $\tilde{M}^*$ is pseudo-free (Lemma 1.10).
These properties are all that is required to prove, based on a result (Theorem 1.5, c.f. [HL], [Wil1,2]), that $\pi_1(M)$ is isomorphic to a spherical 5-space group (Lemma 1.7) and the $\pi_1(M)$-action is conjugate to a linear action (Proposition 1.4).

**a. $\pi_1$-invariant isometric $T^k$-actions on collapsed manifolds.**

According to [CFG], if a closed $n$-manifold $M$ with bounded curvature and diameter has a small volume, then $M$ admits a pure nilpotent Killing structure whose orbits are infra-nilmanifolds. When the fundamental group of $M$ is finite, an orbit is the actually a flat manifold, and the nilpotent Killing structure is equivalent to a $\pi_1$-invariant almost isometric $T^k$-action ([Ro1]). The $\pi_1$-invariance implies that the $T^k$-orbits on the universal covering descend to $M$, also denoted by $T^k(x), x \in M$. A $T^k$-orbit on $M$ is called regular, if it has a tubular neighborhood in which $T^k$-orbits form a fiber bundle. Let $S$ denote the set of all non-regular orbits. Then $M - S$ is an open dense subset. For a small number $\eta > 0$, let $U_{-\eta}(S) = \{ x \in M : d(x,S) > \eta \}$.

**Theorem 1.1 ([CFG]).**

Given $n, d > 0$, there exist constants, $\epsilon(n,d), c(n) > 0$, such that if a closed $n$-manifold $M$ of finite fundamental group satisfying

$$|\text{sec}_M| \leq 1, \quad \text{diam}(M) \leq d, \quad \text{vol}(M) < \epsilon \leq \epsilon(n,d),$$

then $M$ admits a $\pi_1$-invariant $T^k$-action satisfying

1. (1.1.1) Every $T^k$-orbit has a positive dimension and diameter $< \epsilon$.
2. (1.1.2) Any orbit in $U_{-\eta}(S)$ has a second fundamental form, $|II| \leq c(n)\eta^{-1}$.
3. (1.1.3) There is a $T^k$-invariant metric of sectional curvature bounded by one which is $\epsilon$-close to the original metric in $C^1$-norm.

Note that one can always assume a small constant $\eta = \eta(n) > 0$ such that $U_{-\eta} \neq \emptyset$ (if $U_{-\eta} = \emptyset$, then $\text{diam}(M) < \eta$ and thus $M$ is almost flat ([Gr1]). Then $S = \emptyset$ and therefore $U_{-\eta} = M$, a contradiction). Using the Ricci flows technique ([Ha]), one can obtain a nearby metric as follows:

**Theorem 1.2 ([Ro1]).**

Let $(M,g)$ satisfy the conditions of Theorem 1.1. For $\epsilon > 0$, there is a $T^k$-invariant metric $g_\epsilon$ such that

$$|g - g_\epsilon|_{C^1} < \epsilon, \quad \min\{|\text{sec}_g\} - \epsilon \leq \text{sec}_{g_\epsilon} \leq \max\{|\text{sec}_g\} + \epsilon.$$

**b. Spherical 5-space forms.**

Before presenting proofs of Theorems A-D, it may help to review some basic facts about a spherical 5-space form.

Given a spherical 5-space form, $S^5(1)/\Gamma, \Gamma \subset O(6)$, $\Gamma$ is either cyclic or is generated by two elements,

$$A = \begin{pmatrix} R(1/m) & 0 & 0 \\ 0 & R(r/m) & 0 \\ 0 & 0 & R(r^2/m) \end{pmatrix}, \quad B = \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ R(3\ell/n) & 0 & 0 \end{pmatrix},$$

where
where $R(\theta)$ denotes the standard $2 \times 2$ rotation matrix with rotation angle $2\pi\theta$, $I$ is the $2 \times 2$ identity matrix and $r, n \in \mathbb{Z}$ satisfy $n \equiv 0 \mod 9$, $(n(r-1), m) = 1$ and $r^2 + r + 1 \equiv 0 \mod m \ ([Wo])$. Clearly, $A^m = B^n = 1$ and $BAB^{-1} = A^r$. Note that $\Gamma$ may have linear actions $S^5(1)$ which are pairwise non-conjugate.

Let’s now construct on $S^5(1)$ a $\Gamma$-invariant isometric $T^3$-, free $T^1$- and pseudo-free $T^2$-action. First, the standard $T^3$-action on $S^5(1)$ is clearly $\Gamma$-invariant and the free diagonal circle subgroup $T^1$-action commutes with the $\Gamma$-action. Define a pseudo-free $T^2$-action on $S^5(1) = \{(z_1, z_2, z_3) \in \mathbb{C}^3, |z_1|^2 + |z_2|^2 + |z_3|^2 = 1\}$:

$$(e^{i\theta}, e^{i\phi})(z_1, z_2, z_3) = (e^{i(\theta+\phi)}z_1, e^{i(\theta-2\phi)}z_2, e^{i(-2\theta+\phi)}z_3)$$

with a principal isotropy group $\mathbb{Z}_3$ generated by $(e^{\frac{2\pi i}{3}}, e^{\frac{4\pi i}{3}})$, where $T^2 = T^1_\theta \times T^1_\phi$.

The holonomy representation, $\rho : \Gamma \to \text{Aut}(T^2)$ is defined by

$$\rho(A) = \text{id}_{T^2}, \quad \rho(B) = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix},$$

where the matrix is with respect to the standard basis $\theta, \phi$.

We conclude this section with following results (which will be referred several times through the rest of the paper). Let $M$ be a closed 5-manifold of positive sectional curvature.

(1.3.1) If $M$ admits an isometric $T^3$-action, then $M$ is diffeomorphic to a lens space ([GS]).

(1.3.2) If $M$ is simply connected and if $M$ admits an isometric $T^2$-action, then $M$ is diffeomorphic to $S^5$ ([Ro3]).

c. A criterion of a pseudo-free linear action on 5-spheres.

From now on, we will use $S^5$ to denote a homeomorphic 5-sphere and $S^5(1)$ a sphere of constant curvature one. Consider a finite group $\Gamma$ acting freely on $S^5$. As mentioned in (0.1) and (0.2), $\Gamma$ may not be isomorphic to any spherical 5-space group, nor, even assuming $\Gamma$ isomorphic to a spherical 5-space group, the $\Gamma$-action on $S^5$ may not conjugate to any linear action. Hence, additional conditions are required for a $\Gamma$-action to conjugate to a linear action.

We will give a criterion, Proposition 1.4, and use it to prove Theorem A. Note that this criterion will be also used in the proof of Theorem F.

Spherical space forms have been completely classified, see [Wo]. From [Wo], p225, we observe that if a finite group $\Gamma \subset SO(6)$ acts freely on $S^5(1)$ by isometries, then $\Gamma$ commutes with a standard free linear $T^1$-action on $S^5(1)$. If, in addition, $\Gamma$ is not cyclic, then the induced $\Gamma$-action on $S^5/T^1$ is pseudo-free i.e., any non-trivial element has only isolated fixed points.

The above properties are sufficient for a free $\Gamma$-action on $S^5$ to conjugate to a linear action.

**Proposition 1.4.**

Let a finite group $\Gamma$ act freely on $S^5$. If $\Gamma$ commutes with a free $T^1$-action on $S^5$ such that the induced $\Gamma$-action on $S^5/T^1$ is pseudo-free, then the $\Gamma$-action is homeomorphically conjugate to a linear action.
A $G$-action is called locally linear, if each singular point has an invariant neighborhood which is equivariantly homeomorphic to a neighborhood of 0 in a real representation space. In particular, smooth actions are locally linear.

The following result (cf. [HL], [Wi1-2]) plays a crucial role in the proof of Proposition 1.4.

**Theorem 1.5.**

Any pseudo-free locally linear action by a finite group on a 4-manifold homeomorphic to $\mathbb{C}P^2$ is topologically conjugate to the linear action of a subgroup of $PSU(3)$ on $\mathbb{C}P^2$.

Note that $PSU(3) = SU(3)/Z_3$, where $Z_3$ is the center of $SU(3)$. It is perhaps useful to recall some details on what finite subgroups of $PSU(3)$ can act linearly and pseudo-freely on $\mathbb{C}P^2$. By [Wi2] (also [HL]), such a group is either cyclic $\mathbb{Z}_n = \langle x \rangle$, or noncyclic with a presentation

\[(1.6) \quad \{x, y : yxy^{-1} = x^r, x^n = y^3 = 1, \text{ where } r^2 + r + 1 = 0 \mod (n)\}\]

The linear action of the group on $\mathbb{C}P^2$ is given by

\[x[z_0, z_1, z_2] = [\omega z_0, \omega^{-r} z_1, z_2]; \quad y[z_0, z_1, z_2] = [z_1, z_2, z_0]\]

where $\omega = e^{\frac{2\pi i}{n}}$ is the $n$-th root of the unit.

Observe that the group in (1.6) is $\mathbb{Z}_3 \oplus \mathbb{Z}_3 = \langle x, y \rangle$ if $n = 3$, and $n$ can not be an integral multiple of 9. Therefore, (1.6) can never be a 5-dimensional spherical space form group if $n > 1$ (cf. [Wo] page 225). However, Petrie [Pe] constructed a free action of (1.6) on $S^5$ if $n = 7$ and $r = 2$.

**Lemma 1.7.**

Under the assumptions of Proposition 1.4, let $\Gamma_0 \subset \Gamma$ denote the subgroup which acts trivially on $S^5/T^1$. Then

(1.7.1) $\Gamma_0 \subseteq C(\Gamma)$, the center of $\Gamma$.

(1.7.2) $\Gamma$ is isomorphic to a subgroup of $SU(3)$.

**Proof.** (1.7.1) Because $\Gamma_0$ acts trivially on $S^5/T^1$, $\Gamma_0 \subset T^1$, and thus $\Gamma_0 \subseteq C(\Gamma)$. Let $\Gamma_0 \cong \mathbb{Z}_\ell$.

(1.7.2) By Theorem 1.5 and the above discussion, $\Gamma^* = \Gamma/\Gamma_0$ is either cyclic, or a noncyclic group as in (1.6). The desired result follows in the former case, because $\Gamma$ must be abelian and thus must be cyclic because $\Gamma$ acts freely on $S^5$.

Let $\Gamma^*$ be a group in (1.6). Then $\Gamma^*$ has a trivial center. Moreover, $n$ must be coprime to 3 because otherwise $\Gamma^*$ contains $\mathbb{Z}_3 \oplus \mathbb{Z}_3$ as a subgroup. By analyzing the central extension $1 \rightarrow \mathbb{Z}_\ell \rightarrow \Gamma \rightarrow \Gamma^* \rightarrow 1$ restricted on $\mathbb{Z}_3 \oplus \mathbb{Z}_3$ we conclude that $\Gamma$ also contains a subgroup isomorphic to $\mathbb{Z}_3 \oplus \mathbb{Z}_3$. This is absurd since $\Gamma$ acts freely on $S^5$. Similarly, the central extension $\Gamma$, when restricted on $\langle x \rangle$ and $\langle y \rangle$, gives rise to cyclic groups of order $\ell n$ and $3\ell$ respectively. Hence $\Gamma$ contains $\mathbb{Z}_{\ell n}$ as a cyclic subgroup of index 3, and every Sylow group in $\Gamma$ is cyclic. By the Burnside Theorem (cf. [Wo] Theorem 5.4.1, p.163), $\Gamma$ is generated by two elements $A$ and $B$ with relations

\[A^k = B^s = 1, \quad BAB^{-1} = A^r\]
where \(|\Gamma| = ks\), \(((r - 1)s, k) = 1\) and \(r^s \equiv 1(\text{mod } k)\). Since \(\Gamma\) contains an index 3 normal subgroup, we conclude that \(s\) is divisible by 3, and \(\{A, B^3\}\) generates the normal cyclic subgroup, \(r^3 \equiv 1(\text{mod } k)\). Note that the center of \(\langle A, B \rangle\) is generated by \(B^3\) which has order \(s/3\). Hence \(s = 3\ell\), and \(k = n\). Therefore, \(\Gamma\) may be realized as the subgroup of \(SU(3)\) generated by the matrices

\[
\begin{pmatrix}
R(1/n) & 0 & 0 \\
0 & R(r/n) & 0 \\
0 & 0 & R(r^2/n)
\end{pmatrix},
\begin{pmatrix}
0 & I & 0 \\
0 & 0 & I \\
R(1/\ell) & 0 & 0
\end{pmatrix},
\]

where \(R(\theta)\) denote the standard \(2 \times 2\) rotation matrix with rotation angle \(2\pi\theta\), and \(I\) the \(2 \times 2\) identity matrix. The desired result follows. \(\square\)

**Proof of Proposition 1.4.**

By Freedman [Fr], \(S^5/T^1\) is homeomorphic to \(\mathbb{C}P^2\). Consider the induced \(\Gamma\)-action on \(S^5/T^1\). Let \(\Gamma_0\) and \(\Gamma_0^*\) be defined in the proof of Lemma 1.7. Then \(\Gamma_0 \cong \mathbb{Z}_\ell\) is a subgroup of \(T^1\). By Theorem 1.5, the \(\Gamma^*\)-action is conjugate to a linear \(\Gamma^*\)-action on \(\mathbb{C}P^2\) (and thus \(\Gamma^*\) is identified with a subgroup of \(PSU(3)\), denoted by \(\Gamma^\ell\)) by an equivariant homeomorphism \(f : (S^5/T^1, \Gamma^*) \to (\mathbb{C}P^2, \Gamma^\ell)\).

By Lemma 1.7, \(\Gamma \cong \Gamma^\ell \subset SU(3)\), which acts linearly on \(S^5(1)\) that is the lifting of the \(\Gamma^\ell\)-action on \(\mathbb{C}P^2\) (but we should note that the \(\Gamma^\ell\)-action may not be free a priorily). For the sake of convenience, let us identify \(\Gamma^\ell\) with \(\Gamma\). It remains to prove that, the free \(\Gamma\)-action on \(S^5\) is conjugate to the linear \(\Gamma^\ell\)-action on \(S^5(1)\).

Consider the \((\Gamma, T^1)-\) (resp. \((\Gamma^\ell, T^1)\)) principal bundle \(S^5 \to S^5/T^1\).

By Theorem 4.5 (see Section 4) it suffices to prove that the induced principal \(T^1\)-bundle

\[(1.8)\quad T^1 \to E\Gamma \times_\Gamma S^5 \to E\Gamma \times_\Gamma \mathbb{C}P^2\]

is equivalent to the corresponding principal \(T^1\)-bundle of \((\Gamma^\ell, T^1)\)-bundle on \(S^5(1)\). Clearly, \(\pi_1(E\Gamma \times_\Gamma \mathbb{C}P^2) = \Gamma\), and \(\pi_2(E\Gamma \times_\Gamma \mathbb{C}P^2) \cong \mathbb{Z}\). Therefore,

\[H^2(E\Gamma \times_\Gamma \mathbb{C}P^2; \mathbb{Z}) \cong \mathbb{Z} \oplus H_1(\Gamma)\]

where the free part may be regarded as \(\text{Hom}(H_2(E\Gamma \times_\Gamma \mathbb{C}P^2); \mathbb{Z})\) by the universal coefficients theorem.

Let \(e_\Gamma\) denote the Euler class of the principal \(T^1\)-bundle in (1.8). By the homotopy exact sequence one sees that \(e_\Gamma\) is a primitive element of \(H^2(E\Gamma \times_\Gamma \mathbb{C}P^2; \mathbb{Z})\), i.e. modulo the torsion group \(H_1(\Gamma)\) it generates the group. Moreover, with the notions in the proof of Lemma 1.7, \(H_1(\Gamma) \cong \mathbb{Z}_{3\ell}\).

Let \(\mathbb{Z}_{3\ell} := \langle B \rangle \subset \Gamma\), which acts on \(\mathbb{C}P^2\) with three isolated fixed points. Let \([p] \in \mathbb{C}P^2\) be such a fixed point. Consider the orbit \(\Gamma [p] \subset \mathbb{C}P^2\). The restriction of the fiber bundle (1.8) on \(E\Gamma \times_\Gamma \Gamma [p] = E\mathbb{Z}_{3\ell} \times_{\mathbb{Z}_{3\ell}} [p] = B\mathbb{Z}_{3\ell}\) is equivalent to the principal bundle

\[(1.9)\quad T^1 \to E\mathbb{Z}_{3\ell} \times_{\mathbb{Z}_{3\ell}} T^1 \to E\mathbb{Z}_{3\ell} \times_{\mathbb{Z}_{3\ell}} [p] = B\mathbb{Z}_{3\ell}\]

whose total space is homotopy equivalent to \(T^1/\mathbb{Z}_{3\ell} = T^1\). Therefore, the Euler class \(e_\Gamma\), restricts to the Euler class of (1.8), which is clearly a generator of \(9\).
c. Proof of Theorem A by assuming Theorem E.

We need the following lemma to apply the criterion in Proposition 1.4.

**Lemma 1.10.**

Let $M$ be a closed 5-manifold of positive curvature with universal covering space $\tilde{M} \approx S^5$. Assume that $M$ admits a $\pi_1$-invariant isometric $T^1$-action. If $\pi_1(M)$ is not cyclic, then

(1.10.1) the $T^1$-action on $\tilde{M}$ is free;
(1.10.2) the $T^1$-action and the $\pi_1(M)$-action on $\tilde{M}$ commute.
(1.10.3) the induced $\pi_1(M)$-action on $\tilde{M}^*$ is pseudo-free.

**Proof.** If (1.10.1) does not hold, there is an isotropy group $\mathbb{Z}_p \subset T^1$ whose fixed point set in $\tilde{M}$ is either a circle or a homotopy 3-sphere (see Theorem 4.3 in Section 4). Clearly, $\pi_1(M)$ preserves the fixed point set $\tilde{M}^{Z_p}$. This trivially implies that $\pi_1(M)$ is cyclic if $\dim(M^{Z_p}) = 1$, a contradiction. If $\dim(M^{Z_p}) = 3$ we get a totally geodesic 3-manifold $\tilde{M}^{Z_p}/\pi_1(M)$ in $M$. By Theorem 4.3 and [Ha], we see that $\pi_1(M)$ is a spherical 3-space group which acts freely on $S^5$. We conclude once again that $\pi_1(M)$ is cyclic, a contradiction.

If (1.10.2) is false, then there is an element $\gamma \in \pi_1(M)$ such that the holonomy image $\rho(\gamma) = -1 \in \text{Aut}(T^1) = \mathbb{Z}_2$. In other words, for any $t \in T^1$ and $x \in \tilde{M}$ it holds that $t\gamma x = \gamma t^{-1}x$. Clearly the induced action of $\gamma$ on $\tilde{M}^* = \mathbb{C}P^2$ has at least a fixed point for a pure topological reason. Thus, $\gamma$ preserves a $T^1$-orbit, saying $T^1 \cdot x$. Choose $t \in T^1$ so that $t^2x = \gamma x$. Note that $\gamma tx = t^{-1}\gamma x = tx$. This implies that $\gamma$ fixes the point $tx \in \tilde{M}$, a contradiction to the fact that $\pi_1(M)$ acts freely on $\tilde{M}$.

If (1.10.3) is false, then there is an element $\gamma \in \pi_1(M)$ whose action on $\tilde{M}^* = \mathbb{C}P^2$ has a 2-dimensional fixed point component $F_0$. Observe that, for any $x \in \tilde{M}$ with image $x^* \in F_0$, there exists an element $t_x^* \in T^1$ (depending smoothly on
\(x^* \in F_0\) so that \(t_{x^*} \gamma(x) = x\). Clearly \(t_{x^*} \in T^1\) has order \(|\gamma|\) since \(\Gamma\) acts freely on the orbit \(T^1 \cdot x\). Thus \(t_{x^*}\) is constant for all \(x^* \in F_0\) since \(F_0\) is connected. This implies that \(t_{x^*} \gamma\) contains the preimage of \(F_0\) in its fixed point set, which has to be homeomorphic to \(S^3\) by the Smith theory and Hamilton's work once again. Therefore, \(F_0\) must be a 2-sphere. The centralizer \(C_{\langle \gamma \rangle}(\pi_1(M))\) of the subgroup \(\langle \gamma \rangle \subset \pi_1(M)\) acts freely on fixed point component \((\approx S^3)\) of \(t_{x^*} \gamma\). For the same reasoning above we know that \(C_{\langle \gamma \rangle}(\pi_1(M))\) is cyclic.

On the other hand, for any \(\eta \in \pi_1(M)\), by Frankel's theorem we then have that \(\eta(F_0) \cap F_0 \neq \emptyset\), i.e. there is \(x^* \in F_0\) such that \(\gamma(\eta(x^*)) = \eta(x^*)\). This implies that \(\eta^{-1} \gamma \eta(x^*) = x^*\) and hence \(\eta^{-1} \gamma \eta\) and \(\gamma\) both preserve the circle orbit over \(x^*\). Thus \(\eta^{-1} \gamma \eta\) and \(\gamma\) generate a cyclic subgroup of \(\pi_1(M)\). In particular, \(\eta^{-1} \gamma \eta\) is in the centralizer \(C_{\langle \gamma \rangle}(\pi_1(M))\). Since \(C_{\langle \gamma \rangle}(\pi_1(M))\) is cyclic, clearly \(\eta^{-1} \gamma \eta\) (resp. \(\gamma\)) generates the unique cyclic subgroup of order \(|\gamma|\) in \(C_{\langle \gamma \rangle}(\pi_1(M))\). Therefore, \(\eta^{-1} \gamma \eta \in \langle \gamma \rangle\). This proves that \(\langle \gamma \rangle\) itself is a normal subgroup in \(\pi_1(M)\), and hence \(\pi_1(M)\) acts freely on the \(S^3\) as above, which implies that \(\pi_1(M)\) is cyclic. A contradiction. \(\square\)

**Proof of Theorem A by assuming Theorem E.**

Consider \(M\) as in Theorem A, whose Riemannian universal covering space \(\tilde{M}\) is homeomorphic to \(S^5\) (the sphere theorem). Let \(S^5_\delta\) denote a sphere of constant curvature \(\delta\). By the volume comparison,

\[
\text{vol}(M) = \frac{\text{vol}(\tilde{M})}{|\pi_1(M)|} \leq \frac{\text{vol}(S^5_{1/4})}{C} < \epsilon
\]

is small, and thus without loss of generality we may assume \(\tilde{M}\) admits a \(\pi_1(M)\)-invariant isometric \(T^k\)-action (Theorems 1.1 and 1.2). By Theorem E, we may further assume that \(k = 1\). By Lemma 1.10, we can apply Proposition 1.4 to conclude the desired result. \(\square\)

2. PROOF OF THEOREMS B AND C BY ASSUMING THEOREM E

d. Proof of Theorem B by assuming Theorem E.

Consider \(M\) in Theorem B. As in the proof of Theorem A, we may assume that the volume of \(M\) is small and thus \(\tilde{M}\) admits a \(\pi_1(M)\)-invariant isometric \(T^k\)-action. We will prove that \(k > 1\) and then apply Theorem E.

Consider a \(\pi_1\)-invariant \(T^1\)-action on \(\tilde{M}\). The kernel of the holonomy representation, \(\rho : \pi_1(M) \to \text{Aut}(T^1) \cong \mathbb{Z}_2\), is a normal subgroup of index at most two, and thus the \(T^1\)-action on \(\tilde{M}\) descends to a \(T^1\)-action on \(\tilde{M}/ \ker(\rho)\), which is either \(M\) or a double covering of \(M\). In particular, a \(T^1\)-orbit on \(M\) has a tube on which there is induced \(T^1\)-action. We will call isotropy groups of the local \(T^1\)-action isotropy groups of the \(\pi_1\)-invariant \(T^1\)-action.

Let \(\text{Met}\) denote the collection of isometric classes of compact metric spaces. Equipped with the Gromov-Hausdorff distance \(d_{GH}\), \(\text{Met}\) becomes a complete metric space. The Gromov compactness ([Gr]) asserts that any sequence of closed \(n\)-manifolds with Ricci curvature uniformly bounded from below and diameter uniformly bounded above contains a convergent subsequence with respect to \(d_{GH}\) (note that the limit may not a be a Riemannian manifold).
Lemma 2.1.

Assume a sequence of closed $n$-manifolds $M_i \xrightarrow{dGH} X$ such that $|\text{sec}_{M_i}| \leq 1$, where $X$ is a compact metric space. If $\dim(X) = (n - 1)$, then there is a uniform upper bound on the order of isotropy groups of the $\pi_1$-invariant $T^1$-action on $M_i$.

Proof. We argue by contradiction, assuming that $x_i \in M_i$ such that the isotropy group $T^1_{x_i} \cong \mathbb{Z}_{h_i}$ with $h_i \to \infty$ (see (1.1.1)). Passing to a subsequence if necessary, we may assume that $x_i \to x \in X$. Note that an open neighborhood of $x$ is homeomorphic to a cone over the limit of $S^1_{x_i}/T^1_{x_i}$, where $S^1_{x_i}$ is the unit sphere in the normal space to $T^1(x_i)$, and $T^1_{x_i}$ acts on $S^1_{x_i}$ via the isotropy representation. Because $h_i \to \infty$, the limit of $S^1_{x_i}/T^1_{x_i}$ has dimension $\leq n - 3$, and thus the cone has dimension $\leq n - 2$, a contradiction to $\dim(X) = n - 1$. □

Consider an exceptional $T^1$-orbit $T^1(x)$ in $M$, with isotropy group $\mathbb{Z}_{h_i}$. Then there is a lower bound on $h$ that is related to the fundamental group. Let $\gamma$ denote the homotopy class of $T^1(x)$ with order $r$, and let $\sigma$ be the homotopy class of a principal $T^1$-orbit with order $s$. By analyzing the covering map from a component in $\tilde{M}$ of the preimage of a tube of $T^1(x)$ in $M$, one sees that $h \geq r/s$.

In the proof of Theorem B, we will use the following result on fundamental groups of positively curved manifolds ([Ro3]). A cyclic subgroup of $\pi_1(M)$ is called maximal, if it is not properly contained in any cyclic subgroup of $\pi_1(M)$.

Theorem 2.2.

Let $M$ be a closed $n$-manifold of positive sectional curvature. If $M$ admits a $\pi_1$-invariant isometric $T^k$-action, then any maximal normal cyclic subgroup of $\pi_1(M)$ has index $\leq w(n)$.

Proof of Theorem B by assuming Theorem E.

By the volume comparison, $\text{vol}(M) = \text{vol}(\tilde{M})/|\pi_1(M)| \leq \text{vol}(S^3_\delta)/w(\delta)$. We may assume that $w(\delta)$ is large so that $\text{vol}(M) < \epsilon$. By Theorems 1.1 and 1.2, without loss of generality we may assume that $M$ admits a $\pi_1$-invariant isometric $T^k$-action. By Theorem E, it suffices to show that $k > 1$.

We argue by contradiction: assuming a sequence, $M_i$, satisfying the above with $w_i(\delta) \to \infty$, and $k = 1$. By the Gromov’s compactness, we may assume that $M_i \xrightarrow{dGH} X$. Because $k = 1$, it follows that $\dim(X) = 4$, and thus any isotropy group of the $\pi_1$-invariant $T^1$-action on $M_i$ has order $\leq c$ (Lemma 2.1).

To get a contradiction, we will find an isotropy group of order $> c$. Take any maximal normal cyclic subgroup $H_i = \langle \gamma_i \rangle \subset \pi_1(M_i)$. By Theorem 2.2, we obtain

$$w_i \leq [\pi_1(M_i) : \text{cent}(\pi_1(M_i))]$$

$$\leq [\pi_1(M_i) : H_i \cap \text{cent}(\pi_1(M_i))]$$

$$= [\pi_1(M_i) : H_i] \cdot [H_i : H_i \cap \text{cent}(\pi_1(M_i))]$$

$$\leq w(5) \cdot [H_i : H_i \cap \text{cent}(\pi_1(M_i))]$$

(note that the above implies that $H_i$ is not trivial). Clearly, for $i$ large we may assume that $[H_i : H_i \cap \text{cent}(\pi_1(M_i))] > c$. Let $\sigma_i$ denote the homotopy class of a principal $T^1$-orbit on $M_i$. Then $\sigma_i$ is in the center of $\pi_1(M_i)$. Assume that $\gamma_i$ preserves some $T^1$-orbit $T^1(\tilde{x})$ (see Theorem 4.7 in Section 4). Because $\sigma$ preserves
all $T^1$-orbits, $\gamma_i$ and $\sigma_i$ generate a normal cyclic subgroup, and the maximality of $H_i$ implies that $\sigma_i \in H_i$. Note that $\gamma_i$ is a multiple of the homotopy class of the projection of $T^1(\tilde{x})$ in $M$. Then the isotropy group of $\pi(\tilde{T}^1(\tilde{x})$ has order at least $[H_i : \langle \sigma_i \rangle] \geq [H_i : H_i \cap \text{cent}(\pi_1(M_i))] > c$, a contradiction. □

e. Proof of Theorem C by assuming Theorem E.

Similar to the proof of Theorem B, Theorem C is a consequence of the following proposition and Theorem E.

**Proposition 2.3.**

Let $M$ be a closed $n$-manifold of finite fundamental group satisfying

$$|\text{sec}_M| \leq 1, \quad \text{diam} \ (M) \leq d, \quad \frac{\text{vol} \ (M)}{\max \{\text{injrad}(M, z)\}} < \epsilon_1(n, d).$$

Then $M$ admits a $\pi_1$-invariant isometric $T^k$-action with $k > 1$.

For a motivation of Proposition 2.3, consider the metric product of a unit sphere and a flat $\epsilon$-torus, $M_\epsilon = S^n \times \epsilon^2 T^k$. Then $\frac{\text{vol} \ (M_\epsilon)}{\max \{\text{injrad}(M_\epsilon, z)\}} \to 0$ (resp. is proportional to $\text{vol} \ (S^n)$) if $k > 1$ (resp. $k = 1$), and the $\pi_1$-invariant structure in Theorem 1.1 is the multiplication on the $T^k$-factor.

**Proof of Proposition 2.3.**

We may assume that $\epsilon_1(n, d)$ is small so that $\text{vol} \ (M) < \epsilon(n, d)$, and thus $M$ admits a $\pi_1$-invariant almost isometric $T^k$-action (Theorem 1.1). Without the loss of generality, we may assume that the metric is $T^k$-invariant (Theorem 1.2).

We argue by contradiction; assuming a sequence, $M_i$, as in the above such that $\text{vol}(M_i)/\max\{\text{injrad}(M_i, z)\} \to 0$ and $k = 1$. Without loss of generality, we may assume that $M_i \xrightarrow{\partial M_i} X$. Let $x_i \in M_i$ such that $\text{injrad}(M_i, x_i) = \max\{\text{injrad} \ (M_i, z)\}$. We claim that there is a constant, $\eta > 0$, such that, for all $i$, $T^1(x_i)$ is contained in the $\eta$-tube $U_i$ of some $T^1$-orbit, $T^1(y_i)$. Assuming the claim (whose proof is given at the end), we will bound $\frac{\text{vol}(U_i)}{\max\{\text{injrad} \ (M_i, x_i)\}}$ from below by a positive constant (depending on $\eta$), a contradiction.

Because $T^1$ acts isometrically on $U_i$,

$$\text{vol}(U_i) = \text{length}(T^1(y_i)) \cdot \text{area}(D_i^\perp), \quad (2.3.1)$$

where $D_i^\perp$ denotes a normal slice of $U_i$. We shall bound $\text{area}(D_i^\perp)$ from below, and bound length($T^1(y_i)$) in terms of length($T^1(x_i)$).

Let $T^1_{y_i}$ be the isotropy group of $T^1(y_i)$; and let $p_i : \tilde{U}_i \to U_i$ denote the Riemannian $|T^1_{y_i}|$-covering map, where $\tilde{U}_i = T^1 \times D_i^\perp$, $U_i = T^1 \times T^1_{y_i} D_i^\perp$ (the Slice lemma) and the lifting $T^1$-action acts on $\tilde{U}_i$ by the rotation of the $T^1$-factor. By the Gray-O'Neill Riemannian submersion formula, the sectional curvature on $\tilde{U}_i/T^1$ is upper bounded by a constant $c_1(n)$ (cf. (1.1.2)). By the volume comparison, we conclude that $\text{vol}(D_i^\perp) = \text{area}(\tilde{U}_i/T^1) \geq \text{vol}(B_\eta)$, where $B_\eta$ is a $\eta$-ball in the $(n - 1)$-space form of constant curvature $c_1(n)$. From (2.3.1), we get

$$\text{vol}(U_i) \geq \text{length}(T^1(y_i)) \cdot \text{vol}(B_\eta). \quad (2.3.2)$$
By (1.1.2) we may assume that the second fundamental group of all $T^1$-orbits on $\tilde{U}_i$ are uniformly bounded by a constant $c(n)\eta^{-1}$. Then we may assume a constant $c(n,r)$ such that $\text{length}(T^1(\tilde{x}_i)) \leq c(n,\eta) \cdot \text{length}(T^1(\tilde{y}_i))$, where $p_i(\tilde{x}_i) = x_i$ and $p_i(\tilde{y}_i) = y_i$. Then

$$\text{(2.3.3)} \quad \text{length}(T^1(x_i)) \leq c(n,\eta) \cdot |T^1_{y_i}| \cdot \text{length}(T^1(y_i)).$$

Recall that the $T^1$-orbit at any point represents all the collapsed directions of the metric (cf. [CFG]). In particular, we may assume that

$$\text{(2.3.4)} \quad \text{injrad}(M_i,x_i) \leq \frac{1}{2} \text{length}(T^1(x_i)).$$

Using (2.3.2)-(2.3.4), we derive

$$\text{(2.3.5)} \quad \frac{\text{vol}(M_i)}{\text{injrad}(M_i,x_i)} \geq \frac{\text{vol}(U_i)}{\frac{1}{2}\text{length}(T^1(x_i))} \geq \frac{\text{length}(T^1(y_i)) \cdot \text{vol}(B_\eta)}{\frac{1}{2}\text{length}(T^1(x_i))} \geq \frac{2 \cdot \text{ vol}(B_\eta)}{c(n,\eta) \cdot |T^1_{y_i}|}.$$

By Lemma 2.1, we may assume that $|T^1_x| \leq h(n,d)$ and thus see a contradiction in (2.3.5) because the left hand side converges to zero.

We now verify the claim. Recall that $M_i/T^1$ is homeomorphic and $\epsilon_i$-isometric $X$, $\epsilon_i \to 0$, and the projection of the singular set on $M_i$ into $X$ converging to the singular set of $X$ (with respect to the Hausdorff distance). On the other hand, $X$ is the metric quotient, $X = Y/O(n)$, where $Y$ is a Riemannian manifold on which $O(n)$ acts isometrically (cf. [CFG]). We can now pick up $\eta$ from the stratification structure on $(Y,O(n)$ i.e. there are $O(n)$-invariant subsets,

$$Y = S_0 \supset S_1 \supset \cdots \supset S_r, \quad \tilde{S}_i = \bigcup_{j \geq i} S_j$$

such that each component of $S_i$ has a unique isotropy group, and $S_r$ is a closed totally geodesic submanifold, where $\tilde{A}$ denote the closure of a subset $A$. We can choose a sequence of numbers, $1 > \eta_r >> \eta_{r-1} >> \cdots >> \eta_1$ such that each $x \in S_i = \bigcup_{j \geq i} T_{\eta_j}(S_j)$ satisfies that $O(n)(x)$ has a $\eta_i$-tube. We then choose $\eta = \eta_1/2$. \(\Box\)

**Remark 2.4.**

Observe that the above proof goes through if one replaces the assumption, "$\frac{\text{vol}(M)}{\max \text{injrad}(M,z)} < \epsilon_1(n,d)$" by "$\frac{\text{vol}(M)}{\text{injrad}(M)} < \epsilon_1(n,d)$". However, the later is not a valid assumption since every spherical 5-space form satisfies $\frac{\text{vol}(S^5/T)}{\text{injrad}(S^5/T)} \geq \frac{\text{vol}(S^5)}{\pi}$, which include the case ‘$k = 1$’ (see the proof of Theorem C). On the other hand, given $\epsilon > 0$, there are infinitely many spherical 5-spaces satisfying $\frac{\text{vol}(S^5/T)}{\max \text{injrad}(S^5/T,z)} < \epsilon$ (see Example 2.6).

The following may be viewed as a converse to Proposition 2.3.
Lemma 2.5.

Let \( M_i \xrightarrow{d_{GH}} X \) such that \(|\sec_{M_i}| \leq 1\) and \(\text{diam}(M_i) \leq d\). If \(\dim(X) \leq n-2\), then
\[
\frac{\text{vol}(M_i)}{\text{max}\{\text{injrad}(M_i,z)\}} \rightarrow 0.
\]

Proof. We argue by contradiction; without loss of generality we may assume that
\[
\frac{\text{vol}(M_i)}{\text{max}\{\text{injrad}(M_i,z)\}} \geq c > 0 \text{ for all } i.
\]

By the Cheeger’s lemma ([CE]), the ratio,
\[
c \leq \frac{\text{vol}(M_i)}{\text{injrad}(M)} \cdot \frac{\text{injrad}(M_i)}{\text{max}\{\text{injrad}(M_i,z)\}} \leq \frac{\text{vol}(M_i)}{\text{max}\{\text{injrad}(M_i,z)\}} \leq c(n,d),
\]
and thus
\[
1 \leq \frac{\text{max}\{\text{injrad}(M_i,z)\}}{\text{injrad}(M_i)} \leq \frac{c(n,d)}{c}.
\]

Let \( S_i \) denote the singular set of the \(\pi_1(M_i)\)-invariant isometric \(T^k\)-action on \(M_i\), and let \( U_i \) denote the \(\epsilon\)-tube of \(S_i\). Then the orbit projection, \( p_i : M_i - U_i \rightarrow M_i^* - U_i^* \) (\(x^* = p_i(x)\)) is a Riemannian submersion with fiber a flat manifold \(F_i\). We may choose \(\epsilon\) small so that
\[
c \geq \frac{\text{vol}(M_i - U_i)}{\text{max}\{\text{injrad}(M_i,z)\}} = \frac{\int_{M_i^* - U_i^*} \text{vol}(p_i^{-1}(x^*)) d\text{vol}^*}{\text{max}\{\text{injrad}(M_i,z)\}}.
\]

Because \(\text{diam}(p_i^{-1}(x^*)) \rightarrow 0\) uniformly as \(i \rightarrow \infty\), again by the Cheeger’s lemma we may assume that
\[
\text{vol}(p_i^{-1}(x^*)) \leq \text{injrad}(p_i^{-1}(x^*)) \epsilon_i,
\]
where \(\epsilon_i \rightarrow 0\) as \(i \rightarrow \infty\).

Because the fiber \(p_i^{-1}(x^*)\) points all collapsed directions ([CFG]), we may assume that \(\text{injrad}(M_i,x) \simeq \text{injrad}(p_i^{-1}(x^*))\). Then
\[
c \geq \frac{\text{vol}(M_i - U_i)}{\text{max}\{\text{injrad}(M_i,z)\}} \leq \frac{\int_{M_i^* - U_i^*} \text{2injrad}(M_i,z)}{\text{max}\{\text{injrad}(M_i,z)\}} \epsilon_i d\text{vol}^* \leq \frac{2c(n,d)}{c} \text{vol}(M_i^* - U_i^*) \epsilon_i \rightarrow 0,
\]
a contradiction. \(\square\)

We now apply Lemma 2.5 to construct spherical 5-space forms satisfying Theorems B and C.

Example 2.6.

We first construct lens spaces, \(S^5/\mathbb{Z}_p\). Consider a semi-free linear \(T^2\)-action on \(S^5\). Let \(T^1_i \subset T^2\) such that \(\text{diam}(T^2/T^1_i) \leq i^{-1}\) and \(T^1_i\) has no fixed point. We then choose a large prime \(p_i\) so that \(\mathbb{Z}_{p_i}(\subset T^1_i)\) acts freely on \(S^5\) and \(\text{length}(T^1_i/\mathbb{Z}_{p_i}) \leq i^{-1}\) (because the \(T^1_i\)-action has only finitely many isotropy groups). Clearly, the Gromov-Hausdorff distance, \(d_{GH}(S^5/\mathbb{Z}_{p_i}, S^5/T^2) \rightarrow 0\). By Lemma 2.5, \(S^5/\mathbb{Z}_{p_i}\) satisfies the conditions of Theorem C.

We now construct non-lens spaces. Let \(A,B\) be the generators of a spherical group as in Subsection b. Clearly, the center of \(\Gamma\) is generated by \(\gamma^3_2\) and \([\Gamma : \langle \gamma^3_2 \rangle] \geq m\). Then \(S^5/\Gamma\) satisfies conditions of Theorems B and C when \(m\) large.
Example 2.7.

We will construct examples showing that Theorems B and C are false if relaxing “$\delta > 0$” to “$\delta \geq 0$”.

According to [Wo], there is a sequence of non-cyclic spherical 3-space groups, $\Gamma_i$, such that $S^3/\Gamma_i$ converges to a closed interval $I$ and $[\Gamma_i : \text{cent}(\Gamma_i)] \to \infty$. Then $M_i = S^2/\Gamma_i \times S^2$ converges to $I \times S^2$ with $0 \leq \sec_M \leq 1$. However, $\Gamma_i$ cannot act freely on $S^5$ (any finite group acting freely both on $S^3$ and $S^5$ must be cyclic, cf. [Bro]). Note that by Lemma 2.5, $\frac{\text{vol}(M_i)}{\max\{\text{injrad}(M_i,z)\}} \to 0$.

3. Proof of Theorem D by Assuming Theorems E and F

We first state a generalization of Theorem F. We call an abelian subgroup $c_s$ of a finite group $\Gamma$ a semi-center, if its centralizer has index at most two in $\Gamma$ and if $|c_s|$ is ‘maximal’ among all such abelian subgroups. Obviously, a semi-center contains the center, and coincides with the center when $|\Gamma|$ is odd. However, a semi-center may not be unique when $|\Gamma|$ is even.

Theorem F'.

Let $M$ be a closed 5-manifold of positive sectional curvature which admits a $\pi_1$-invariant fixed point free isometric $T^1$-action. Then the universal covering $\tilde{M}$ is diffeomorphic to $S^5$, provided $\pi_1(M)$ satisfies the following conditions:

(F1) $\pi_1(M)$ has a semi-center of index at least $w > 0$, a universal constant.

(F2) $\pi_1(M)$ does not contain any index $\leq 2$ subgroup isomorphic to a spherical 3-space group.

We claim that Theorem F' implies Theorem F. Let $\pi_1(M)$ satisfies the assumptions of Theorem F. Then $\pi_1(M)$ is not cyclic. We shall check that $\pi_1(M)$ satisfies (F1) and (F2). If $\pi_1(M)$ is isomorphic to a spherical 5-space group, then $\pi_1(M)$ contains an index 3 normal cyclic subgroup and any semi-center of $\pi_1(M)$ coincides with its center (see p.225, [Wo]). Note that $\pi_1(M)$ contains no spherical 3-space group of index at most 2; otherwise, the spherical 3-space group is cyclic because it also acts freely $S^5$ ([Bro]), and thus $\pi_1(M)$ has a cyclic subgroup of index 2, a contradiction. If $\pi_1(M)$ has an odd order, then any semi-center has to coincide with its center, and $\pi_1(M)$ has to be isomorphic to a spherical 3-space group which is cyclic because its order is odd, a contradiction.

Proof of Theorem D by assuming Theorems E and F'.

Let $S^5_\delta$ denote a sphere of constant curvature $\delta$. By the volume comparison, we may assume

$$\text{vol}(M) = \frac{\text{vol}(\tilde{M})}{|\pi_1(M)|} \leq \frac{\text{vol}(S^5_\delta)}{w(\delta)} < \epsilon,$$

and thus we may assume, without loss of generality, that $\tilde{M}$ admits a $\pi_1(M)$-invariant isometric $T^k$-action (Theorems 1.1 and 1.2). If $k > 1$, then by Theorem E conclude the desired result. If $k = 1$, then by Theorem F' we see that the Riemannian universal covering of $M$ is diffeomorphic to a sphere, and thus by Proposition 1.4 and Lemma 1.10 we conclude the desired result. □
Part II. Proofs of Theorems E and F

4. Preparations

In this section, we supply materials that will be used in the proofs of Theorems E and F in the rest of this paper.

a. Fixed point set of abelian groups.

Consider a compact Lie group $G$ acting isometrically on a closed manifold $M$. Let $M^G$ denote the set of $G$-fixed points. Then each component of $M^G$ is a closed totally geodesic submanifold. If $G = T^k$, then $F$ has even codimension. For a generic compact Lie group, the topology of $M$ may not be well related to the topology of $M^G$. However, the opposite situation occurs when $G$ is abelian (cf. [Bre], p.163).

**Theorem 4.1.**

Let $M$ be a compact $\mathbb{Z}_h$-space. If $h = p$ is a prime, then

$$\chi(M) = \chi(M^{\mathbb{Z}_p}) \mod p.$$ 

If $\mathbb{Z}_p$ ($p$ is a prime) acts trivially on the homology group $H_*(M; \mathbb{Z})$, then

$$\chi(M) = \chi(M^{\mathbb{Z}_p}).$$

The last assertion of the above lemma is from [Bre] III exercise 13 (p.169).

**Theorem 4.2.**

Let a compact abelian Lie group $G$ act effectively on a closed manifold $M$, and let $N$ denote an invariant subset. Then

$$\text{rank}(H^*(M^G, N^G; \ell)) \leq \text{rank}(H^*(M, N; \ell)),$$

where $G = T^k$ and $\ell = \mathbb{Q}$ or $G = \mathbb{Z}_p^k$ and $\ell = \mathbb{Z}_p$.

A consequence of Theorem 4.2 is

**Theorem 4.3 (Smith).**

Let a torus $T^k$ act effectively on a closed manifold $M$. If $M$ is a rational homology sphere, then $N^{T^k}$ is a rational homology sphere.

The $T^k$-action on a sphere without fixed points is well understood.

**Theorem 4.4 ([Bre], P. 164).**

Let $M$ be a $n$-dimensional homology sphere and admit a $T^k$ action with no fixed point. If $H \subset T^k$ is a subtorus of dimension $k - 1$, let $r(H)$ denote that integer, for which $M^H$ is a homology $r(H)$-sphere. Then with $H$ ranging over all subtori of dimension $k - 1$ and $r(H) \geq 0$, we have

$$n + 1 = \sum_H (r(H) + 1).$$
b. A generalized Lashof-May-Segal theorem.

Let $G$ denote a compact Lie group ($G$ can be finite). For two $G$-spaces, $Y$ and $Z$, a map, $f : Y \to Z$, is called an $G$-map if $f(g \cdot y) = g \cdot f(y)$ for all $y \in Y$ and $g \in G$.

A principal $(G,T^k)$-bundle is a principal $T^k$-bundle, $T^k \to E \xrightarrow{p} B$, such that $E$ and $Y$ are $G$-spaces, $p$ is a $G$-map and the $G$-action on $E$ preserves the structural group of the bundle. Note that the $G$-action and $T^k$-action may not commute. Two principal $(G,T^k)$-bundles are called equivalent, if there is a $G$-equivariant bundle equivalent map.

Let $B((G,T^k))(B)$ be the set of equivalence classes of principal $(G,T^k)$-bundles over $B$. When $G = \{1\}$, we will skip $G$ from the notation.

Let $EG$ be the infinite join of $G$, a contractible free $G$-CW complex (cf. [Hu]). Put $B_G = EG \times_G B$ and $E_G = EG \times_G E$. There is a natural transformation,

$$\Phi : B((G,T^k))(B) \to B((T^k))(B_G)$$

by sending a principal $(G,T^k)$-bundle $p : E \to B$ to the principal $T^k$-bundle $p_G : E_G \to B_G$. The following theorem is a special case of [FR2] Theorem 3.3 which generalizes the Lashof-May-Segal theorem.

**Theorem 4.5.**

Let $B,G$ be as in the above. Then $\Phi : B((G,T^k))(B) \to B((T^k))(B_G)$ is a bijection.

Theorem 4.5 can be used in the following situation (see Section 7): Let $M$ be a closed manifold of finite fundamental group which admits a pseudo-free $T^k$-action. Let $\bar{M}_0 = \bar{M} - \mathcal{S}$, where $\mathcal{S}$ is the union of singular orbits. Then $\bar{M}_0 \to \bar{M}_0^*$ is a $(\pi_1(M),T^k)$-bundle.

c. Positive curvature and isometric torus actions.

In the rest of this section, we will consider an isometric $T^k$-action on a closed manifold $M$ of positive sectional curvature. As seen in Theorems 4.1–4.4, the topology of $M$ is closely related to the singular structure and the orbit space of the $T^k$-action. In the presence of a positive curvature, the singular structure and the orbit space is very restricted.

A basic constraint on the singular structure is given by following Berger’s vanishing theorem ([Ro1], also [GS], [Su]).

**Theorem 4.6.**

Let a torus $T^k$ act isometrically on a closed manifold $M$ of positive sectional curvature. Then there is a $T^k$-orbit which is a circle. Moreover, the fixed point set is not empty when $\dim(M)$ is even.

Theorem 4.6 implies, via the isotropy representation at a circle orbit, that large $k$ yields closed totally geodesic submanifolds of small codimension.

In the study of the fundamental group of a positively curved manifold on which $T^k$ acts isometrically, the following result is a useful tool.
Theorem 4.7 ([Ro4]).

Let $M$ be a closed manifold of positive sectional curvature on which $T^k$ acts isometrically, and let $\phi$ be an isometry on $M$ which commutes with the $T^k$-action. Then $\phi$ preserves some $T^k$-orbit which is a circle.

We now illustrate a situation where Theorem 4.7 may be applied. Let $T^k$ act isometrically on a closed manifold $M$ of finite fundamental group, and let $\pi: \tilde{M} \to M$ denote the Riemannian universal covering. Let $p: \tilde{M} \to \tilde{M}^*$ be the orbit projection, where $\tilde{T}^k$ denotes the covering torus of $T^k$ acting on $\tilde{M}$. For any $\gamma \neq 1 \in \pi_1(M)$, because the $\gamma$-action commutes with the $\tilde{T}^k$-action, $\gamma$ induces an isometry of $\tilde{M}^*$, denoted by $\gamma^*$. Obviously, (4.8.1) $\gamma^*$ is trivial if and only if $\gamma \in H$, the subgroup generated by loops in a principal $T^k$-orbit.

(4.8.2) $\gamma$ preserves an orbit, say $\tilde{T}^k(\tilde{x})$, if and only $\gamma^*$ fixes $\tilde{x} = p(\tilde{x})$.

(4.8.3) If $k = 1$ and $\gamma$ preserves $\tilde{T}^1(\tilde{x})$, then $T^1(\pi(\tilde{x}))$ is an exceptional orbit, whose isotropy group contains a subgroup $\mathbb{Z}_h$ with $h$ the order of $\gamma$.

Corollary 4.9.

Let $M$ be a closed manifold of positive sectional curvature. If $M$ admits an isometric $T^k$-action, then the subgroup generated by loops in a principal $T^k$-orbit is cyclic, say $\langle \alpha \rangle$. Moreover, for any $\gamma \in \pi_1(M)$, $\alpha$ and $\gamma$ generate a cyclic subgroup.

Proof. Let $H$ denote the subgroup generated by loops in a principal $T^k$-orbit. Let $\tilde{T}^k$ denote the lifting $T^k$ that acts on the Riemannian universal covering space $\tilde{M}$. Then $H$ preserves all principal $\tilde{T}^k$-orbits and preserves all $\tilde{T}^k$-orbit. Because there is a circle $\tilde{T}^k$-orbit, $H = \langle \alpha \rangle$ is cyclic. Because any $\gamma \in \pi_1(M)$ preserves some circle orbits, $\alpha$ and $\gamma$ generate a cyclic group. \[\square\]

The following connectedness theorem of Wilking provides a useful tool to contract information on homotopy groups from the existence of a closed totally geodesic submanifold of small codimension (cf. see [FMR]).

A map from $N$ to $M$ is called $(i+1)$-connected, if it induces an isomorphism up to the $i$-th homotopy groups and a surjective homomorphism on the $(i+1)$-th homotopy groups.

Theorem 4.10 ([Wi]).

Let $M$ be a closed $n$-manifold of positive sectional curvature, and let $N$ be a closed totally geodesic submanifold of dimension $m$. If there is a Lie group $G$ that acts isometrically on $M$ and fixes $N$ pointwisely, then the inclusion map is $(2m - n + 1 + C(G))$-connected, where $C(G)$ is the dimension of a principal orbit of $G$.

5. Proof of Theorem E for $k = 3$

Consider the case $k = 3$ in Theorem E. By (1.3.1), we may assume that the holonomy representation $\rho: \pi_1(M) \to \text{Aut}(T^3) = \text{GL}(\mathbb{Z}, 3)$ is not trivial and $\tilde{M}/\ker(\rho)$ is diffeomorphic to a lens space (in particular, $\ker(\rho)$ is cyclic). The goal of this section is to prove
Theorem 5.1.

Let $M$ be a closed 5-manifold of positive sectional curvature. If $M$ admits a $\pi_1$-invariant isometric $T^3$-action, then $M$ is homeomorphic to a spherical space form.

By Proposition 1.4, the following two lemmas imply Theorem 5.1.

Lemma 5.2.

Let $M$ be a closed 5-manifold of positive sectional curvature. Suppose that $M$ admits a $\pi_1$-invariant isometric $T^3$-action. Then $T^3$ has a circle subgroup $T^1$ which acts freely on $\tilde{M}$ and which commutes with the $\pi_1(M)$-action.

Proof. From the above, $\tilde{M} \approx S^5$ and $\text{ker}(\rho)$ is cyclic. We claim that $[\pi_1(M) : \text{ker}(\rho)] = 3$. It is easy to see that $\tilde{M}/T^3$ is homeomorphic to a simplex $\Delta^2$ as stratified set such that the three vertices are the projection of isolated three circle orbits in $\tilde{M}$ and the three edges are the projection of three components of $T^2$-orbits (cf. [FR3]). If $\gamma \in \pi_1(M)$ acts trivially on the orbit space $\Delta^2$, the holonomy $\rho(\gamma) \in \text{Aut}(T^3)$ preserves all isotropy groups of the $T^3$-action in $\tilde{M}$. In particular, $\rho(\gamma)$ preserves the three circle isotropy groups, $H_1 \cap H_2, H_1 \cap H_3, H_2 \cap H_3$, where $H_1, H_2, H_3$ are 2-dimensional isotropy groups. By the argument in the proof of (1.9.2) we know that $\rho(\gamma)$ is the identity on all $H_1 \cap H_2, H_1 \cap H_3, H_2 \cap H_3$. Therefore, $\rho(\gamma) = I \in \text{GL}(Z, 3)$. On the other hand, an element $\gamma \in \text{ker}(\rho)$ acts on $\Delta^2$ preserving all vertices and edges, and thus the identity in $\Delta^2$. This clearly implies that $\pi_1(M)/\text{ker}(\rho) \cong Z_3$ acts effectively on $\Delta^2$ by rotating the three vertices.

Since $\rho(\pi_1(M)) \subset \text{GL}(Z, 3)$ is non-trivial, by the above we know that $\rho(\pi_1(M)) \cong Z_3$. Let $A$ denote a generator of $\rho(\pi_1(M))$. It is easy to see that $A$ must have 1 as an eigenvalue. Let $v$ be an eigenvector of eigenvalue 1, and let $T^s = \exp_{\epsilon} \dot{v}$ be the closed subgroup generated by the vector $v$. Then $T^s$ acts on $\tilde{M}$ commuting with the $\pi_1(M)$-action.

We claim that $T^s$ acts freely on $\tilde{M}$ and thus $s = 1$. If the claim is false, there is an isotropy group $H \neq 1 \subset T^s$ with fixed point set $\tilde{M}^H \neq \emptyset$. Note that $\tilde{M}^H$ is either a circle or a totally geodesic three sphere (Theorems 4.2 and 4.10). In either cases, $\tilde{M}^H$ is invariant by the $T^3$-action and thus it contains one or two circle orbits of the $T^3$-action. By the commutativity, $\pi_1(M)$ also preserves $\tilde{M}^H$ and so $\pi_1(M)$ preserves one or two vertices of $\Delta^2$. Therefore, $\pi_1(M)$ has at least a fixed point among the three vertices of $\Delta^2$. A contradiction, since we have just shown $\pi_1(M)$ rotates the three vertices. \qed

Lemma 5.3.

Let $T^1$ be as in Lemma 5.2. If $\rho : \pi_1(M) \to \text{GL}(Z, 3)$ is non-trivial, then the induced $\pi_1(M)$-action on $\tilde{M}^* \simeq \tilde{M}/T^1$ is pseudo-free.

Note that Lemma 1.10 implies Lemma 5.3 if $\pi_1(M)$ is not cyclic.

Proof of Lemma 5.3.

Because $\tilde{M}$ is diffeomorphic to $S^5$ on which $T^1$ acts freely, $\tilde{M}^* \simeq CP^2$ (cf. [Fr]). We argue by contradiction, assuming a $\gamma \in \pi_1(M)$ which has a 2-dimensional fixed point set in $\tilde{M}^*$. Let $p : \tilde{M} \to \tilde{M}/T^3 \simeq \Delta^2$ be the orbit projection. By the proof of Lemma 5.2 we know that $\pi_1(M)/\text{ker}(\rho) \cong Z_3$. 

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If $\rho(\gamma)$ is non-trivial, as we have seen in the proof of Lemma 5.2, $\gamma$ rotates the vertices (and also edges) of $\Delta^2$ and hence it has only an isolated fixed point in the interior of $\Delta^2$. The preimage of this fixed point in $\tilde{M}/T^1$ is a 2-torus which must contain the fixed point set of the $\gamma$-action on $CP^2$. On the other hand, the fixed point set of the $\gamma$-action on $\tilde{M}^* = CP^2$, which is a totally geodesic 2-dimensional submanifold (a sphere or $RP^2$). A contradiction.

If $\rho(\gamma)$ is trivial, $\gamma$ acts trivially on $\Delta^2$ by the proof of Lemma 5.2. For a 2-dimensional connected component $F_0$ in the fixed point set of $\gamma$-action in $\pi_1(M)$, there is an element $t_0 \in T^1$ so that the fixed point set of $t_0 \gamma \in T^3 \ltimes \pi_1(M)$ contains a 3-dimensional totally geodesic submanifold $F \subset \tilde{M}$ (cf. the proof of (1.10.3)). Since $\rho(\gamma) = I$, $F$ is $T^3$-invariant with principal isotropy group a circle subgroup $C \subset T^3$. Since $\ker(\rho) \subset \pi_1(M)$ is normal cyclic, $\langle \gamma \rangle$ is also a normal subgroup. Therefore $\rho(\gamma)(C) = C$ and by the proof of Lemma 5.2, $C = T^1$. A contradiction, since $T^1$ acts freely on $\tilde{M}$. □

6. Proof of Theorem E at The Level of Fundamental Groups

In this section, we will prove Theorem E at the level of fundamental groups. The main result in this section is the following:

**Theorem 6.1.**

Let $M$ be a closed 5-manifold of positive sectional curvature. If $M$ admits a $\pi_1$-invariant isometric $T^2$-action, then the fundamental group of $M$ is isomorphic to that of a spherical 5-space form.

By Theorem 5.1, it suffices to prove Theorem 6.1 for $k = 2$.

**Lemma 6.2.**

A finite non-cyclic group $\Gamma$ is isomorphic to the fundamental group of a spherical 5-space form, if $\Gamma$ satisfies the following conditions:

(6.2.1) Every subgroup of order $3p$ is cyclic, for any prime $p$.

(6.2.2) $\Gamma$ has a normal cyclic subgroup of index 3.

**Proof.** By [Wo] Theorem 5.3.2 (in page 161), (6.2.1) and (6.2.2) implies that every Sylow subgroup of $\Gamma$ is cyclic. By [Wo] Theorem 5.4.1 (in page 163), $\Gamma$ is generated by two elements $A$ and $B$ with relations

$$A^m = B^n = 1, \quad BAB^{-1} = A^r$$

where $((r - 1)n, m) = 1$ and $r^n \equiv 1(\text{mod } m)$. The order $|\Gamma| = mn$. By (6.2.2), $n \equiv 0(\text{mod } 3)$, and $\{A, B^3\}$ generates a normal cyclic subgroup of index 3. Thus $r^3 \equiv 1(\text{mod } m)$. By [Wo] page 225, it only remains to prove that $n \equiv 0(\text{mod } 9)$. Suppose not, $(\frac{m}{9}, 3) = 1$. For a prime factor $p|m$, $A^\frac{m}{p}$ is of order $p$. By (6.2.1) the subgroup generated by $A^\frac{m}{p}$ and $B^\frac{m}{p}$ is cyclic of order $3p$. On the other hand, this group is a normal cyclic extension over a cyclic group satisfying the relation

$$B^\frac{m}{p} A^\frac{m}{p} B^{-\frac{m}{p}} = A^\frac{m}{p} r^\frac{m}{p}$$
This implies that

\[(6.2.3) \quad r^2 - 1 \equiv 0 \pmod{p}\]

Recall that \((r - 1, p) = 1\) and \(r^3 \equiv 1 \pmod{p}\). By considering \(\frac{r^2}{3} \pmod{3}\) and (6.2.3) we get that \(r^2 - 1 \equiv 0 \pmod{p}\) and hence \(r \equiv -1 \pmod{p}\). A contradiction, since \(r^2 + r + 1 \equiv 0 \pmod{p}\). \(\square\)

In the proof of Theorem 6.1, we will establish (6.2.1) and (6.2.2).

Recall that a \(T^k\)-action \((k > 1)\) is pseudo-free, if all singular orbits are isolated.

**Lemma 6.3.**

Let \(M\) be a closed 5-manifold of positive sectional curvature with a \(\pi_1\)-invariant isometric \(T^2\)-action. If the \(T^2\)-action on \(\tilde{M}\) is not pseudo-free, then \(\pi_1(M)\) is cyclic.

**Proof.** By (1.3.2), \(\tilde{M}\) is diffeomorphic to \(S^5\). Consider all circle subgroups of \(T^2\) with nonempty fixed point sets: By Theorem 4.4, there are three possibilities, accordingly we divide the proof into three cases:

Case (1). The fixed point set \(\tilde{M}T^2 \neq \emptyset\);

Because \(\pi_1(M)\) preserves the fixed point set, it suffices to show that \(\tilde{M}T^2\) is a circle. By the Smith theory (cf. Theorems 4.3) \(\tilde{M}T^2\) is a homology sphere. Since \(T^2\) acts effectively on the normal space of \(\tilde{M}T^2\), \(\tilde{M}T^2\) is a circle.

Case (2). There are two distinct circle subgroups: \(T_1, T_2\) such that \(\dim(\tilde{M}T_1) = 3\) and \(\tilde{M}T_2\) is a circle;

For any \(\gamma \in \pi_1(M)\), observe that \(F(\rho(\gamma)(T_1), \tilde{M}) = \gamma(\tilde{M}T_1), \rho(\gamma)(T_1) = T_1\) where \(\rho\) is the holonomy representation (otherwise, there two distinct circle subgroups with fixed point sets of dimension 3). Consequently, \(\rho(\gamma)(T_2) = T_2\) and thus \(\pi_1(M)\) preserves \(\tilde{M}T_2\). Therefore, \(\pi_1(M)\) is cyclic.

Case (3). There are three distinct circle subgroups with fixed points set of dimension 1, but there are non-trivial finite isotropy groups.

Let \(Z_p \subset T^2\) (\(p\) is a prime) be a finite isotropy group. Note that \(\dim(\tilde{M}Z_p) = 3\). If \(\rho(\gamma)(Z_p) = Z_p\) for all \(\gamma \in \pi_1(M)\), then \(\pi_1(M)\) preserves \(\tilde{M}Z_p\). Because \(\tilde{M}Z_p\) contains exactly two of the three circle orbits in Case (3), \(\pi_1(M)\) must preserve the unique circle orbit outside \(\tilde{M}Z_p\) and thus \(\pi_1(M)\) is cyclic.

If there is \(\gamma \in \pi_1(M)\) such that \(\rho(\gamma)(Z_p) \neq Z_p\), then \(\gamma(\tilde{M}Z_p) = \tilde{M}^{\rho(\gamma)(Z_p)}\), and \(F_0 = \tilde{M}Z_p \cap \tilde{M}^{\rho(\gamma)(Z_p)}\) is a circle which is the fixed point set of \(Z_p^2 \subset T^2\) (Theorem 4.3). Because \(\rho(\gamma)(Z_p^2) = Z_p^2\) for all \(\gamma \in \pi_1(M), \pi_1(M)\) preserves \(F_0\), and thus \(\pi_1(M)\) is cyclic. \(\square\)

**Lemma 6.4.**

Let \(M\) be a closed 5-manifold of positive sectional curvature. Suppose that \(M\) admits a \(\pi_1\)-invariant isometric \(T^2\)-action with an empty fixed point set. If the \(T^2\)-action on \(\tilde{M}\) is pseudo-free, then either \(\pi_1(M)\) is cyclic or satisfies (6.2.1) and (6.2.2).

Combining Lemmas 6.2-6.4, we obtain the case of Theorem 6.1 for \(k = 2\), and together with Theorem 5.1 we obtain Theorem 6.1.
For a pseudofree $T^2$-action on $\tilde{M} \approx S^5$, by Theorem 4.4 again there are exactly three isolated circle orbits. Moreover, $\tilde{M}^*$ is a topological manifold and thus a homotopy 3-sphere because $\tilde{M}^*$ is simply connected. Because $\gamma \in \pi_1(M)$ maps a singular orbit to a singular orbit, we may view $\pi_1(M)$ acting on three points (singular orbits) in $\tilde{M}^*$ by permutations. This defines a homomorphism, $\phi : \pi_1(M) \rightarrow S_3$, the permutation group of three letters. The kernel of $\phi$ is a normal subgroup acting trivially on the three points (or equivalently, which preserving every circle orbits). Thus $\ker(\phi)$ is cyclic.

In the proof of Lemma 6.4 we need:

**Lemma 6.5.**

Let $M$ be as in Lemma 6.4. Then $\phi$ is trivial if and only if the holonomy representation $\rho : \pi_1(M) \rightarrow \text{Aut}(T^2)$ is trivial.

**Proof.** Let $H_i, i = 1, 2, 3$, denote the three isotropy groups of the isolated singular orbits of the pseudofree $T^2$-action. Note that, for any $\gamma \in \pi_1(M)$, and $x \in \tilde{M}$ with isotropy group $I_x$, the isotropy group of $\gamma(x)$, $I_{\gamma(x)} = \rho(\gamma)(I_x)$. Therefore, if $\rho$ is trivial, then $\pi_1(M)$ preserves the isotropy groups, and so preserves every singular orbits, i.e., $\phi$ is trivial.

Conversely, if $\phi$ is trivial, $\pi_1(M)$ preserves the three singular orbits. In particular, $\pi_1(M)$ is cyclic. Therefore, $\rho(\gamma)(H_i) = H_i$ for any $\gamma \in \pi_1(M)$. It is easy to see that $H_i$, $i = 1, 2, 3$, generate $T^2$. Therefore, in the Lie algebra of $T^2$, $\mathbb{R}^2$, the automorphism $\rho(\gamma) \in \text{GL}(\mathbb{Z}, 2)$ has three different eigenvectors whose eigenvalues are 1 or $-1$. This implies that $\rho(\gamma) = I$ or $-I$, where $I$ is the unit matrix. If $\rho(\gamma) = -I$, i.e, for any $x \in \tilde{M}$, $\gamma^{-1}x = t\gamma x$. For a point $x$ in a singular orbit, it holds $\gamma x = t^2 x$ for some $t \in T^2$. Hence $\gamma tx = t^{-1} \gamma x = tx$. This implies that $tx$ is a fixed point of $\gamma$, a contradiction. \hfill $\square$

Our proof of Lemma 6.4 involves a homotopy invariant, ‘the first $k$-invariant’. This invariant can be used to distinguish two connected spaces whose first and second homotopy groups are the same. Let’s now briefly recall its definition ([Wh]).

Let $X$ be a connected space, and $K(\pi_i(X), \ell)$ denote the Eilenberg-Maclane space. Corresponding to each map, $k_1 : K(\pi_1(X), 1) \rightarrow K(\pi_2(X), 3)$, there is a unique fibration,

$$K(\pi_2(X), 2) \longrightarrow E_{k_1} \xrightarrow{f} K(\pi_1(X), 1) \xrightarrow{k_1} K(\pi_2(X), 3)$$

with fiber $K(\pi_2(X), 2)$. The total space $E_{k_1}$ is unique and the classifying map $f$ has a lifting, $\tilde{f} : X \rightarrow E_{k_1}$,

$$E_{k_1} \longrightarrow K(\pi_1(X), 1) \xrightarrow{f} X$$

so that $\tilde{f}_* : \pi_i(X) \rightarrow \pi_i(E_{k_1})$ is an isomorphism for $i = 1, 2$. The corresponding cohomology class $k_1 \in H^3(K(\pi_1(X), 1); \pi_2(X))$ is called the first $k$-invariant of $X$. Clearly, the first $k$-invariant is a homotopy invariant.
Let $L_\ell = S^3/\mathbb{Z}_\ell$ denote a lens space. It is well-known that the punctured lens space has non-trivial first $k$-invariant, i.e., for $p \in L_\ell$, $k_1(L_\ell - \{p\}) \neq 0$ (cf. [EM]).

**Proof of Lemma 6.4.**

Let $\phi : \pi_1(M) \to S_3$ be the homomorphism as above. We first claim that:

(6.4.1) the image of $\phi$ must be either $\{1\}$ or $\mathbb{Z}_3$.

Obviously, if $\text{Im}(\phi) \cong \{1\}$, then $\pi_1(M)$ fixes all isolated circle orbits, thus $\pi_1(M)$ acts freely on every circle orbit, and so $\pi_1(M)$ is cyclic. (6.2.1) clearly follows by (6.4.1).

To prove (6.4.1) it suffices to rule out the case $\text{Im}(\phi) \cong \mathbb{Z}_2$, since the case $\text{Im}(\phi) = S_3$ may be ruled out by taking the pre-image $\phi^{-1}(\mathbb{Z}_2)$ of a order 2 subgroup $\mathbb{Z}_2 \subset S_3$ instead of $\pi_1(M)$.

Assuming an element $\gamma \in \pi_1(M)$ so that $\phi(\gamma)$ is of order 2. By definition $\gamma$ preserves a unique singular circle orbit, with isotropy group $H \cong S^1$, and permutes the rest two singular circle orbits. Since $\tilde{M}^*$ is a homotopy 3-sphere, the induced action of $\gamma$ on $\tilde{M}^*$ has at least a fixed point which is not the isolated singular points (the singular orbits). This implies that $\gamma$ preserves a principal orbit $T^2 \cdot x$ and acts freely on. By Lemma 6.5 $\rho(\gamma)$ is also nonzero, of order 2. It is easy to show that the $\gamma$-action and the transitive $T^2$-action on $T^2 \cdot x$ do not commute. Therefore, the free $\gamma$-action on $T^2 \cdot x$ has a quotient space the Klein bottle. This implies that, up to conjugation, the action of $\gamma$ on the orbit is given by the composition of the multiplication $\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \in T^2$ with the rotation $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \rho(\gamma)$ on $T^2$. Thus, for $t_0 = \begin{bmatrix} \alpha_{1}^{-1} \\ \alpha_{2}^{-1} \end{bmatrix} \in T^2$, the fixed point set of $t_0 \gamma$ on $T^2 \cdot x$ contains two disjoint circles, $S^1 \times \{\pm 1\}$. Therefore, the fixed point set $F$ of $t_0 \gamma$ on $\tilde{M}$ has dimension 3, a homology 3-sphere, which intersects with every principal orbit either empty, or two disjoint circles.

Observe that $F$ projects to the fixed point set $\gamma$ on $\tilde{M}^*$. Therefore, $\gamma$ acts on $\tilde{M}^*$ with fixed point set a 2-dimensional homology sphere. If $F$ does not intersept with the circle orbit with isotropy group $H$ (preserved by $\gamma$), the fixed point set of $\gamma$ on $\tilde{M}^*$ is not connected, a contradiction. Since $H$ is preserved by $\rho(\gamma)$, i.e. $\rho(\gamma)(H) = H$, $H$ is either $S^1 \times \{1\}$, or $\{1\} \times S^1$ in the standard coordinate for $T^2$. Now the intersection of $F$ with the singular circle orbit consists of either two points, or the whole singular orbit, depending on whether the reduced automorphism $\rho(\gamma) \in \text{Aut}(T^2/H)$ is trivial or not. In the former case, $H$ acts $F$ semifreely, with two isolated fixed points. A contradiction, since $F$ is a homotopy $3$-sphere by Theorem 4.10, because otherwise $H$ acts on two points punctured 3-sphere freely, absurd by Euler characteristic reasoning. For the latter case, the quotient $F/H$ is a 2-disk, with an action of $\{\pm 1\} \subset 1 \times S^1$ freely in the interior of the disk. A contradiction again by the Brower fixed point theorem.

It remains to prove (6.2.2). If $\pi_1(M)$ contains a non-cyclic subgroup $G \subset \pi_1(M)$ of order $3q$ where $q$ is a prime, by (6.2.1) $\phi(G) \cong \mathbb{Z}_3$ and thus $G$ contains a subgroup $\mathbb{Z}_3$ acting pseudo-freely on $\tilde{M}^*$ (denoted by $\Sigma$), a homotopy 3-sphere. Let $\tilde{M}_0$ denote the complement of the three isolated circle orbits on $\tilde{M}$. Consider the $T^2$-bundle,

$$T^2 \to \tilde{M}_0/\mathbb{Z}_3 \to \tilde{M}_0^*/\mathbb{Z}_3,$$

and the classifying map $f : \tilde{M}_0^*/\mathbb{Z}_3 \to B(T^2 \times \mathbb{Z}_3)$ of the principal $T^2 \times \mathbb{Z}_3$ bundle.
Theorem 7.1.

Let the assumptions be as in Theorem E. If \( k = 2 \) and the \( T^2 \)-action is pseudo-free, then \( M \) is homeomorphic to a spherical space form.

Before starting a proof, it may be helpful to look at a linear model in Theorem 7.1. Consider a linear pseudo-free \( T^2 \)-action as in \( b \) of Section 1, and a commuting linear \( \mathbb{Z}_{k\ell} \)-action on \( S^5(1) \). Then \( \mathbb{Z}_{k\ell} \) also acts on the orbit space \( S^5(1)/T^2 = S^3 \). Let \( \mathbb{Z}_k \) denote the principal isotropy group of the reduced linear action on \( S^3 \). We will call the \( \mathbb{Z}_k \)-action along the \( T^2 \)-orbits. The reduced \( \mathbb{Z}_\ell = \mathbb{Z}_{k\ell}/\mathbb{Z}_k \) action on \( S^3 \) has a fixed point set \( S^1 \), which spans a plane of \( \mathbb{R}^4 \). The condition of the linear \( \mathbb{Z}_k \) action (defined by multiplying \( (e^\frac{2\pi}{k}i, e^\frac{2\pi}{\ell}i, e^\frac{2\pi}{\ell}i) \)) along the \( T^2 \)-orbits can be written as \( a + b + c = 0 \pmod{2k} \). Note that in the above model, the action by \( A \) is always along the \( T^2 \)-orbits.

In the following context we will continue to use \( \Gamma_\ell \) (resp. \( T^2_\ell \)) to mean a linear \( \Gamma \)-action (\( T^2 \)-action) on \( S^5(1) \) so that it extends to a linear action of \( T^2_\ell \times \Gamma \).

Consider a pseudofree linear \( T^2 \)-action on \( S^5 \) defined in the above. Let \( S^5_0 \) denote the complement of small open tubes (\( \simeq D^4 \times T^1 \)) around the three isolated circle orbits on \( S^5_0 \). Let \( M \) be as in Theorem 7.1. By Theorem 4.4 the pseudofree \( T^2 \)-action on \( \tilde{M} \approx S^5 \) has exactly three isolated circle orbits. Let \( \tilde{M}_0 \) denote the complement of small open tubes of the three isolated circle orbits. By Theorem 6.1 \( \Gamma = \pi_1(M) \) is a spherical 5-space group. Our main effort is to show that \( M_0 = \tilde{M}/\Gamma \) is homeomorphic to \( S^5_0/\Gamma_0 \). By [SW] the gluing of a handle \( D^4 \times T^1 \) is unique up to homeomorphism, and therefore \( M \) is homeomorphic to \( S^5/\Gamma_0 \).

Let \( \rho : \Gamma \to \text{Aut}(T^2) \) denote the holonomy representation of the \( \pi_1 \)-invariant action. By Lemma 6.6 we know that \( \ker(\rho) \) is cyclic, and the image \( \rho(\Gamma) \) is either trivial or isomorphic to \( \mathbb{Z}_3 \). Let \( \mathbb{Z}_k \) denote the principal isotropy group of the reduced \( \Gamma \)-action on \( \tilde{M}^* \). By definition one sees that \( \mathbb{Z}_k \) acts on \( \tilde{M} \) through the \( T^2 \)-orbits.
Now let us consider the principal $T^2$-bundle $T^2 \to \tilde{M}_0 \to \tilde{M}_0^*$. Assuming the Poincare conjecture, $\tilde{M}_0^* = S^3_0$ is the complement of $S^3$ by removing three small 3-disks. Note that $\Gamma$ acts on this principal bundle, and moreover, the sub-action of $\ker(\rho)$ commutes with the $T^2$-action. The following lemma is immediate:

**Lemma 7.2.**

The principal $T^2$-bundle is unique up to weak equivalence. Therefore, every pseudofree $T^2$-action on $S^5$ is conjugate to a linear $T^2$-action.

**Proof.** Note that the bundle is uniquely determined by its Euler class, an element in $H^2(\tilde{M}_0^*; \mathbb{Z}^2) \cong \text{Hom}(\mathbb{Z}^2; \mathbb{Z}^2)$. Considering the Euler class as a $2 \times 2$ matrix (given by its classifying map to $BT^2$), its determinant is $\pm 1$ since the total space $\tilde{M}_0$ is 2-connected by the transversality theorem. Therefore, up to the left action by $GL(2, \mathbb{Z})$, i.e. up to an automorphism of $T^2$, the bundle is unique. The desired result follows. $\Box$

By Lemma 7.2 the sub-action by $\mathbb{Z}_k$ on $\tilde{M} \approx S^5$ is conjugately linear. Therefore $\tilde{M}/\mathbb{Z}_k$ is diffeomorphic to a lens space $S^5/\mathbb{Z}_k$. Of course one should note that there are possibly many different ways to embed $\mathbb{Z}_k$ in $T^2$ which acts freely on $S^5$, and consequently the lens space may not be unique.

Let us consider the reduced principal $T^2$-bundle $T^2/\mathbb{Z}_k \to \tilde{M}_0/\mathbb{Z}_k \to \tilde{M}_0^*$, regarded as a $\Gamma/\mathbb{Z}_k$-equivariant bundle.

**Lemma 7.3.**

If the $\Gamma$-action and $T^2$-action on $\tilde{M}$ commute, then the above $\Gamma$-equivariant principal $T^2$-bundle is $\Gamma/\mathbb{Z}_k$-equivariantly equivalent to a linear $T^2$-bundle $T^2/\mathbb{Z}_k \to S^5/\mathbb{Z}_k \to S^3$.

**Proof.** Note that $\Gamma$ is cyclic. Let us write $\Gamma = \mathbb{Z}_{k\ell}$ where $\mathbb{Z}_k$ is as above. The effective action on $\mathbb{Z}_{\ell}$ on $\Sigma \approx S^3$ has fixed point e.g., the three isolated marked points representing the three singular orbits. By a deep theorem of [BLP] this action of $\mathbb{Z}_{\ell}$ on $\Sigma$ is conjugate to a linear action on $S^3$. Therefore, by Theorem 4.5 it suffices to prove that the associated principal $T^2$-bundle

$$T^2/\mathbb{Z}_k \to E\mathbb{Z}_{\ell} \times_{\mathbb{Z}_{\ell}} \tilde{M}_0/\mathbb{Z}_k \to E\mathbb{Z}_{\ell} \times_{\mathbb{Z}_{\ell}} \tilde{M}_0^*$$

is unique up to weak equivalence. We need only to show its Euler class $e(\gamma) \in H^2(E\mathbb{Z}_{\ell} \times_{\mathbb{Z}_{\ell}} \Sigma_0; \mathbb{Z}^2)$ can be realized by the Euler class of a linear $\Gamma_0$-equivariant principal $T^2$-bundle over $S^3_0$ with total space $S^5_0/\mathbb{Z}_k$, where $\mathbb{Z}_k$ acts linearly on $S^5$ along the $T^2$-orbits. By some standard calculation we get that $H^2(E\mathbb{Z}_{\ell} \times_{\mathbb{Z}_{\ell}} \tilde{M}_0^*; \mathbb{Z}^2) \cong \text{Hom}(\mathbb{Z}^2; \mathbb{Z}^2)$. By restricting the bundle to $E\mathbb{Z}_{\ell} \times_{\mathbb{Z}_{\ell}} [p]$ where $[p] \in \tilde{M}_0^*$ is a fixed point of the $\mathbb{Z}_{\ell}$-action, one sees that $e(\gamma)$ restricting to $H^2(B\mathbb{Z}_{\ell}; \mathbb{Z}^2) \cong \mathbb{Z}^2_{\ell}$ is an element of order $\ell$.

By comparing with the linear model discussed at the beginning of this section, it is straightforward to check that every pair $(a, b) \in \mathbb{Z}^2_{\ell}$ generating an order $\ell$ element can be realized as the torsion component of the Euler class of a linear $\mathbb{Z}_{\ell}$-equivariant principal $T^2$-bundle on $S^3_0$ with total space $\tilde{M}_0/\mathbb{Z}_{k\ell}$. The torsion free part of $e(\gamma)$ is uniquely determined by its lifting to $H^2(E\mathbb{Z}_{\ell} \times \tilde{M}_0^*; \mathbb{Z}^2) \cong \text{Hom}(\mathbb{Z}^2, \mathbb{Z}^2)$, which is the Euler class of the forgetful principal $T^2$-bundle on $\tilde{M}_0^*$, regarded as a non-equivariant bundle. By Lemma 7.2 this Euler class is uniquely determined by the
total space $\tilde{M}_0/\mathbb{Z}_k$, or equivalently, by the conjugacy class of the embedding of $\mathbb{Z}_k$ in $T^2$. This proves the desired result. □

Next let us consider the case where the holonomy $\rho : \Gamma \to \text{Aut}(T^2)$ is non-trivial.

**Lemma 7.4.**

Let $M$ be as in Theorem 7.1. If the holonomy $\rho : \Gamma \to \text{Aut}(T^2)$ is non-trivial, then the $\Gamma$-equivariant principal $T^2$-bundle $\tilde{M}_0/\mathbb{Z}_k \to M_0^*$ is $\Gamma/\mathbb{Z}_k$-equivariantly equivalent to a linear $T^2$-bundle $\tilde{T^2}/\mathbb{Z}_k \to S^3_0/\mathbb{Z}_k \to S^3_0$.

We first need some preparation. By Lemma 6.5 the image $\rho(\Gamma) \cong \mathbb{Z}_3 \subset \text{GL}(2, \mathbb{Z}) = \text{Aut}(T^2)$. Recall that $\text{SL}(2, \mathbb{Z}) \cong \mathbb{Z}_4 \ast_{\mathbb{Z}_2} \mathbb{Z}_6$ has a subgroup of order 3, unique up to conjugation, which is generated by

$$\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}.$$ 

By Lemma 6.5 there exists an element $\gamma \in \Gamma$ such that $\phi(\gamma)$ has order 3.

Let $\mathbb{Z}_k \subset \Gamma$ denote the principal $\gamma \in \Gamma$ of the induced action on $\Sigma$. The quotient group $\Gamma/\mathbb{Z}_k$ acts effectively on $\Sigma \cong S^3$, and which is not free, unless $\Gamma/\mathbb{Z}_k \cong \mathbb{Z}_3$. By Theorem 6.1 it is easy to see that $\Gamma/\mathbb{Z}_k \cong \mathbb{Z}_3$ only if $\Gamma$ is cyclic. Therefore, by [BLP] once again the reduced action on $\Gamma/\mathbb{Z}_k$ on $\Sigma \cong S^3$ is conjugate to a linear action on $S^3$, unless $\Gamma/\mathbb{Z}_k = \mathbb{Z}_3$. In the latter case (where $\Gamma/\mathbb{Z}_k = \mathbb{Z}_3$) we may replace the ”conjugation” by ”s-cobordism” (cf. Appendix), and everything goes through. For the sake of simplicity we now assume that $\Gamma/\mathbb{Z}_k$ acts linearly on $\Sigma \cong S^3$. Therefore, $\Gamma/\mathbb{Z}_k$ is a subgroup of $\text{SO}(4)$.

**Sublemma 7.5.**

$\Gamma/\mathbb{Z}_k$ is cyclic.

**Proof.** We need only to consider the case where $\Gamma$ is not cyclic. By Theorem 6.1 $\Gamma = \{A, B : A^n = B^n = 1, BAB^{-1} = A^r\}$ is a spherical 5-space group where $n \equiv 0(\text{mod } 9)$, $(n(r-1), m) = 1$, and $r^3 \equiv 1(\text{mod } m)$. Observe that $m$ must be odd. Hence, any possible 2-subgroup of $\Gamma$ is cyclic, saying $\Gamma_2$, such that $\Gamma = \Gamma_2 \times \Gamma/\Gamma_2$, where $\Gamma/\Gamma_2$ is again a spherical 5-space group of odd order. Since every odd order subgroup of $\text{SO}(4)$ is also a subgroup of $\text{Spin}(4) = S^3 \times S^3$, by the classification of finite subgroups of $S^3$ one concludes that a odd order subgroup of $S^3 \times S^3$ is abelian of rank at most 2. Applying this we know that $\Gamma/\mathbb{Z}_k$ is abelian. By the presentation of $\Gamma$ above this implies that $\Gamma/\mathbb{Z}_k$ is cyclic. □

Let us write $\Gamma/\mathbb{Z}_k = \mathbb{Z}_{3^t}$.

**Sublemma 7.6.**

If $k \neq 1$, then $\Gamma$ is not cyclic and has a normal cyclic subgroup whose quotient group is cyclic, i.e., $1 \to \mathbb{Z}_m \to \Gamma \to \mathbb{Z}_n \to 1$.

**Proof.** Recall that $\mathbb{Z}_k$-acts in $\tilde{M}$ as a sub-action of the pseudo-free $T^2$-action. Since the $T^2$-action is $\Gamma$-invariant, for any $\alpha \in \Gamma$, it holds that $\alpha \beta \alpha^{-1} = \rho(\alpha)(\beta)$, where $\beta \in \mathbb{Z}_k \subset T^2$ (Notice $\mathbb{Z}_k$ is invariant by $\rho(\alpha)$). If $\Gamma$ is abelian, then $\rho(\alpha)(\beta) = \beta$ even if $\rho(\alpha) = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \in \text{Aut}(T^2)$. A contradiction, since 1 is not an eigenvalue of the matrix. It is clear that $\Gamma$ satisfies the desired property. □
Proof of Lemma 7.4.

By the discussion at the beginning of this section, there is a linear (and free) \( \Gamma \)-action on \( S^3(1) \) so that the matrix \( A \) has order \( k \) which acts along the \( T^2 \)-orbits (with the pseudofree \( T^2 \)-action as in the beginning), and \( B \) has order \( 3\ell \), where \( \ell \) is divisible by 3. It is easy to see that the linear \( T \)-morphisms. Therefore, the forgetful homomorphism \( \text{Hom}(\tilde{M}_0^*;\mathbb{Z}_\rho) \rightarrow \text{GL}(\mathbb{Z},2) \), the latter is characterized by its Euler class, an element in the local cohomology group \( H^2(\mathbb{Z}_\rho ; \tilde{M}_0^*;\mathbb{Z}_\rho^2) \). To prove lemma 7.4 we only need to verify that the Euler class of the affine bundle is uniquely determined by Euler class of the principal \( T^2 \)-bundle \( \tilde{M}_0/\mathbb{Z}_k \rightarrow \Sigma_0 \), that implies \( \tilde{M}_0/\mathbb{Z}_k \rightarrow M_0^* \) is \( \Gamma \)-equivariantly equivalent to the data in the linear model case.

By the short exact sequence \( 1 \rightarrow \mathbb{Z}_\rho \rightarrow \mathbb{Z}[\mathbb{Z}_3] \rightarrow \mathbb{Z} \rightarrow 1 \) we can calculate the local cohomology group

\[
1 \rightarrow H^2(E\mathbb{Z}_\rho;\tilde{M}_0^*;\mathbb{Z}_\rho^2) \rightarrow H^2(E\mathbb{Z}_\rho;\tilde{M}_0^*;\mathbb{Z}) \rightarrow H^2(E\mathbb{Z}_\rho;\tilde{M}_0^*;\mathbb{Z})
\]

where the middle space \( E\mathbb{Z}_\rho;\tilde{M}_0^* \) is the three fold covering of \( E\mathbb{Z}_\rho;\tilde{M}_0^* \). In the above exact sequence, the middle term is isomorphic to \( \mathbb{Z}_3^2 \oplus \mathbb{Z}_\rho \), and the last term is isomorphic to \( \mathbb{Z}_\rho \). By the universal coefficients theorem it is readily to see that the torsion part of the middle term goes injectively into the last term. Therefore, the local cohomology group \( H^2(E\mathbb{Z}_\rho;\tilde{M}_0^*;\mathbb{Z}_\rho^2) \) is torsion free of rank 2. Hence, by the universal coefficients theorem again, \( H^2(E\mathbb{Z}_\rho;\tilde{M}_0^*;\mathbb{Z}_\rho^2) \) is given by \( \text{Hom}(H_2(\tilde{M}_0^*;\mathbb{Z}_\rho^2),\mathbb{Z}) \cong \text{Hom}(\mathbb{Z}_3^2,\mathbb{Z}_\rho^2) \), where \( \text{Hom} \) denotes the \( \rho \)-invariant homomorphisms. Therefore, the forgetful homomorphism \( \text{Hom}(\mathbb{Z}_3^2,\mathbb{Z}_\rho^2) \rightarrow \text{Hom}(\mathbb{Z}_3^2,\mathbb{Z}_\rho^2) \) sends the Euler class of the \( \mathbb{Z}_3^\rho \)-equivariant \( T^2 \)-bundle on \( \tilde{M}_0^* \) to the Euler class of the principal \( T^2 \)-bundle (forgetting the \( \mathbb{Z}_3^\rho \)-action), which is clearly injective. The desired result follows. \( \square \)

Remark 7.7.

By the proof of Lemma 7.4 and the comments after Lemma 7.2, we know that the \( \Gamma \)-equivariant principal \( T^2 \)-bundle in Lemma 7.4 is uniquely determined by the total space \( \tilde{M}_0/\mathbb{Z}_k \) and so by the lens space \( \tilde{M}/\mathbb{Z}_k \), which depends only on the embedding of \( \mathbb{Z}_k \) in \( T^2 \).

Proof of Theorem 7.1.

Let \( M \) be as in Theorem 7.1. By Theorem 6.1 \( \pi_1(M) = \Gamma \) is a spherical 5-space group. By Lemmas 7.3 and 7.4 we know that \( \tilde{M}_0/\Gamma := M_0 \) is diffeomorphic to \( S^5/\Gamma \). Since \( M \) is obtained by gluing three handles \( S^1 \times D^4 \) along the boundary components \( S^1 \times S^3 \). Because every self diffeomorphism of \( S^1 \times S^3 \) extends to a self homeomorphism of \( S^1 \times D^4 \) (cf. [SW]), the homeomorphism type does not depend on the gluing. Therefore, \( M \) is homeomorphic to \( S^5(1)/\Gamma \). The desired result follows. \( \square \)
8. Completion of The proof of Theorem E

After the works in Sections 6-7, we are ready to finish the remaining case in the proof of Theorem E, i.e., the case that a $\pi_1$-invariant $T^2$ action on $M$ is not pseudofree.

Proof of Theorem E.

First, by Theorems 7.1 we only need to consider a non-pseudo-free $T^2$-action, i.e. the $T^2$-action has a non-empty fixed point, a 3-dimensional stratum with circle isotropy group, or a nontrivial finite isotropy group but without fixed point. In all cases, $\pi_1(M) := \Gamma$ is cyclic (Lemmas 6.3 and 6.4). Recall that $\tilde{M} = S^5$ with an action of $T^2 \ltimes \rho \Gamma$.

Case 1. The $T^2$-action has a non-empty fixed point set.

Note that the fixed point set must be a circle (Theorem 4.2). By local isotropy representation of $T^2$ at the fixed point set, there are two circle isotropy groups with three dimensional fixed point sets, two totally geodesic $S^3$. Observe that the $T^2$-action on $M$ is free outside the union of the two 3-dimensional strata, and the quotient space $\tilde{M}^*$ is homeomorphic to the 3-ball $D^3$, whose boundary $S^2 = D^2_+ \cup D^2_-$, where $D^2_\pm$ is the image of the two 3-dimensional strata and $D^2_+ \cap D^2_- \{0\}$ is the image of the fixed point set of $T^2$. We claim that the $T^2$-action and $\Gamma$-action commute (equivalently, $\rho$ is trivial). By now it is easy to see that $M$ is diffeomorphic to a lens space (cf. [GS]).

Identify $\tilde{M}^*$ with $D^3$. Observe that $\Gamma$ acts isometrically on $\tilde{M}^*$ and preserves the boundary $\partial(\tilde{M}^*) = \partial D^3$. If the commutativity fails, $\Gamma$ acts non-trivially on $\tilde{M}^*$. By the well-known Brouwer fixed point theorem, $\Gamma$ has at least a fixed point in the interior of $D^3$, which represents a principal orbit, say $T^2 \cdot x$. As in the proof of Lemma 6.4, $\Gamma$ acts on $T^2 \cdot x$ with quotient a Klein bottle, since $\rho(\Gamma)$ is not trivial. Therefore, the same argument in the proof of Lemma 6.4 implies an element $t_0 \in T^2$ so that $t_0 \gamma$ has a 3-dimensional fixed point set $F$ in $\tilde{M}$. By the Frankel’s theorem, $F$ intersects with the two 3-dimensional strata, of circle isotropy groups. Clearly, $F$ projects to the fixed point set of $\Gamma$ in $D^3$. Recall that $F$ intersects with a principal orbit in two circles. This together shows that the fixed point set of $\Gamma$ in $D^3$ is a 2-dimensional, and so a disk, with non-empty intersections with both $D^2_+$ and $D^2_-$. Therefore, the fixed point set $(D^3)^\Gamma$ contains at least a point of $D^2_+ \cap D^2_\gamma \{0\}$, the fixed point set of the $T^2$-action. For any such a point $[x]$, its preimage $x \in \tilde{M}$ satisfies $\gamma x = x$. A contradiction, since $\Gamma$ acts freely on $\tilde{M}$.

Case 2. The $T^2$-action has no fixed point, but it has a 3-dimensional stratum with circle isotropy group.

Let $T^1 \subset T^2$ denote the unique circle isotropy group with 3-dimensional fixed point set. Since $\Gamma$ preserves the strata, $\Gamma$ preserves the isotropy group $T^1$, that is, for any $g \in \Gamma$, $\rho(g)(T^1) = T^1$. Therefore, $T^1 \times \Gamma$ acts on $\tilde{M}$. We claim that the $T^1$-action and $\Gamma$-action commute. Then $M$ admits a $T^1$-action with three dimensional fixed point set. By [GS] again we know that $M$ is diffeomorphic to a lens space.

It is clear that the $T^1$-action on $\tilde{M}$ is semi-free with fixed point set a totally geodesic $S^3$, and the orbit space $\tilde{M}^*$ is homeomorphic to $D^4$. Note that $\Gamma$ acts freely on $\partial(\tilde{M}^*) = S^3$. Therefore $\Gamma$ acts on $D^4$ with a unique fixed point in the interior, saying $0 \in D^4$, which is a principal orbit for the $T^1$-action on $\tilde{M}$. If the commutativity fails, then there is a generator $\gamma \in \Gamma$ such that $\rho(\gamma) \in \text{Aut}(T^1)$ is
given by the inverse automorphism. Let $T^1 \cdot x_0$ denote the principal $T^1$-orbit over $0 \in D^4$. Since $\gamma(x_0) \neq x_0$, let $t \in T^1$ satisfy the equation $t^2 x_0 = \gamma x_0$. Then $x = tx_0$ satisfies the equation $\gamma x = \gamma t x_0 = \rho(\gamma)(t)\gamma x_0 = t^{-1} \gamma x_0 = x$. A contradiction, since $\gamma$ acts freely on $T^1 \cdot x_0$.

Case 3. The $T^2$-action has only isolated singular orbits but has finite order isotropy groups.

Assume that $\mathbb{Z}_p \subset T^2$ is an isotropy group of order $p$, whose fixed point set $\tilde{M}^{\mathbb{Z}_p}$ is a totally geodesic 3-sphere. Note that $\Gamma$ preserves $\mathbb{Z}_p$ and it acts freely on $\tilde{M}^{\mathbb{Z}_p}$. By Theorem 4.4 the $T^2$-action has three isolated singular orbits, and the orbit space $\tilde{M}^*$ is a homotopy 3-sphere. We first claim that the $\Gamma$-action and the $T^2$-action commute and thus $T^2$ acts on $M$. Consequently, $\Gamma$ is cyclic. We argue by contradiction. Suppose not, there is a nontrivial $\Gamma$-action on the orbit space $M^*$ acting transitively on the set of three singular orbits (by Lemma 6.5). Therefore, there is an element $\gamma \in \Gamma$ which moves the three points transitively. Note that the image of $\tilde{M}^{\mathbb{Z}_p}/T^2$ is an interval, $[0,1]$, connecting two singular orbits. Let $\rho : \Gamma \to \text{Aut}(T^2)$ denote the holonomy. By Lemma 6.5 $\rho(\gamma)$ is non-trivial. Note that $\mathbb{Z}_p$, $\rho(\gamma)(\mathbb{Z}_p)$, $\rho(\gamma^2)(\mathbb{Z}_p)$ are all isotropy groups of the $T^2$-action on $\tilde{M}$ such that they have pairwisely different fixed point set. However, since any two of $\mathbb{Z}_p$, $\rho(\gamma)(\mathbb{Z}_p)$, $\rho(\gamma^2)(\mathbb{Z}_p)$ generate the same subgroup isomorphic to $\mathbb{Z}_p \oplus \mathbb{Z}_p$, of rank 2 in $T^2$, so the fixed point set of $\mathbb{Z}_p \oplus \mathbb{Z}_p$ is the union of three isolated singular circle orbits in $\tilde{M} \approx S^5$. A contradiction, by Theorem 4.3.

Consider the $T^2$-action on $M$. For a finite isotropy group $\mathbb{Z}_p \subset T^2$, the 3-dimensional fixed point component $F \subset \tilde{M}^{\mathbb{Z}_p}$ is a lens space since it is invariant by the isometric $T^2$-action (by [GS]). By Theorem 4.10 the fundamental group of $F$ is isomorphic to $\Gamma$. Since the orbit space $M/T^2$ is again a homotopy 3-sphere with exactly three isolated singular circle orbits, $M/T^2 - F/T^2$ homotopically retracts to the unique singular orbit (a point in the orbit space) outside $F/T^2$. Thus, $M - F$ is homotopy equivalent to the circle orbit outside $F$. This implies that $M = D(\nu) \cup \partial S^1 \times D^4$, where $D(\nu)$ is the normal disk bundle of $F$ in $M$, and $S^1 \times D^4$ is a disk tubular neighborhood of the circle orbit. By the Gysin exact sequence for the unit circle bundle $S(\nu)$ over the lens space $F = S^3/\Gamma$ we conclude that the Euler class $e(\nu) \in H^2(F) \cong \Gamma$ is a generator. Thus, the bundle $D(\nu)$ is uniquely up to conjugation. Using the same fact about the gluing along $S^1 \times S^3$ as in the proof of Theorem 7.1 we conclude that $M$ is diffeomorphic to a lens space, the desired result follows. □

9. Proof of Theorem F

As we noticed in Section 3, Theorem F' implies Theorem F. The goal of this section is to prove Theorem F'. Recall that the algebraic conditions on the fundamental group $\pi_1(M)$ in Theorem F' are essentially as follows:

(9.1.1) Any index $\leq 2$ normal subgroup $\Gamma \lhd \pi_1(M)$ has a center $C(\Gamma)$ of index at least $w$.

(9.1.2) Any index $\leq 2$ normal subgroup $\Gamma \lhd \pi_1(M)$ is not a spherical 3-space group.

In fact (9.1.2) may be replaced by $\Gamma$ is neither cyclic, nor generalized quaternionic group, and binary dihedral group.

Lemma 9.2.
Let $M$ be a closed 5-manifold of positive sectional curvature which admits a $\pi_1$-invariant fixed point free isometric $T^1$-action. If $\pi_1(M)$ satisfies (9.1.1) and (9.1.2), then

(9.2.1) Every non-principal $T^1$-orbit is isolated, and $\tilde{M}^*$ is a simply connected orbifold with only isolated singularities.

(9.2.2) $H_2(\tilde{M};\mathbb{Z})$ is torsion free and has rank equal to $b_2(\tilde{M}^*) - 1$.

**Proof.** (9.2.1) If there is a nontrivial subgroup $\mathbb{Z}_p \subset T^1$ with a fixed point component $\tilde{M}^p$ of dimension 3, note that the fixed point set $\tilde{M}^p$ is invariant by the free $\pi_1(M)$-action. Thus $\pi_1(M)$ is isomorphic to the fundamental group of a 3-dimensional spherical space form, by Theorem 4.10. A contradiction to the algebraic condition (9.1.2).

(9.2.2) First, $H_2(\tilde{M};\mathbb{Z}) \cong \pi_2(\tilde{M})$ (by the Hurewicz theorem). Let $\tilde{M}_0$ denote the union of all principal $T^1$-orbits. By (9.2.1), $\pi_i(\tilde{M}) \cong \pi_i(\tilde{M}_0)$, $i = 1, 2$. Because $\tilde{M}_0$ is obtained by removing some isolated circle orbits, by the transversality $\tilde{M}_0$ is simply connected and thus $\tilde{M}_0^* = \tilde{M}_0/T^2$ is simply connected. Since every singularity in $\tilde{M}^*$ is a conical point whose neighborhood in $\tilde{M}^*$ is a cone over a lens space $S^3/\mathbb{Z}_p$, it is easy to see that $H_2(\tilde{M}^*;\mathbb{Z}) = H_2(\tilde{M}_0^*;\mathbb{Z}) \cong \pi_2(\tilde{M}_0^*)$ (the last isomorphism is from Hurewicz theorem). From the homotopy exact sequence of the fibration,

$$1 \rightarrow \pi_2(\tilde{M}_0) \rightarrow \pi_2(\tilde{M}_0^*) \rightarrow \mathbb{Z} \rightarrow 1,$$

it suffices to show that $H_2(\tilde{M}_0^*;\mathbb{Z})$ is torsion free. This is true because $H_2(\tilde{M}_0^*;\mathbb{Z}) \cong H^2(\tilde{M}_0^*,\partial\tilde{M}_0^*;\mathbb{Z})$ (the Lefschetz duality) whose torsion is isomorphic to the torsion of $H_1(\tilde{M}_0^*,\partial\tilde{M}_0^*;\mathbb{Z}) = 0$ (the universal coefficient theorem). \qed

b. **Estimate the Euler characteristic of $\tilde{M}^*$.**

To determine the topology of $\tilde{M}$, we will first estimate the Euler characteristic of $\tilde{M}^*$ using the constraint on the fundamental groups.

**Proposition 9.3.**

Let the assumptions and notations be as in Lemma 9.2. Then $\chi(\tilde{M}^*) = 2 + b_2(\tilde{M}^*) \leq 5$ or equivalently, $b_2(\tilde{M}^*) = b_2(\tilde{M}) + 1 \leq 3$.

Let $p : \tilde{M} \rightarrow \tilde{M}^*$ denote the orbit projection. Then $\pi_1(M)$ acts on $\tilde{M}^*$ by isometries. For $\gamma \in \pi_1(M)$, we will use $\gamma^*$ to denote the isometry on $\tilde{M}^*$ induced by $\gamma$ (cf. §2). By Theorem 4.7, $(\tilde{M}^*)^{\gamma^*} \neq \emptyset$, if $\gamma$ commutes with the $T^1$-action. To estimate the characteristic of $\tilde{M}^*$, we will first estimate the number of isolated $\gamma^*$-fixed points (see Theorem 4.1) via the technique of $q$-extent estimate ([GM], [Ya]).

The $q$-extent $xt_q(X)$, $q \geq 2$, of a compact metric space $(X,d)$ is, by definition, given by the following formula:

$$xt_q(X) = \left(\frac{q}{2}\right)^{-1}\max\left\{\sum_{1 \leq i < j \leq q} d(x_i, x_j) : \{x_i\}_{i=1}^q \subset X\right\}$$

Given a positive integer $n$ and integers $k, l \in \mathbb{Z}$ coprime to $n$, let $L(n; k, l)$ be the 3-dimensional lens space, the quotient space of a free isometric $\mathbb{Z}_n$-action on $S^3$.   

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defined by
\[ \psi_{k,l} : \mathbb{Z}_n \times S^3 \to S^3; \quad g(z_1, z_2) = (\omega^k z_1, \omega^l z_2) \]
with \( g \in \mathbb{Z}_n \) a generator, \( \omega = e^{i \frac{2\pi}{n}} \) and \( (z_1, z_2) \in S^3 \subset \mathbb{C}^2 \).

Note that \( L(n; k, l) \) and \( L(n; -k, l) \) (resp. \( L(n; l, k) \)) are isometric (cf. [Ya] p.536). Obviously \( L(n; -k, l) \) and \( L(n; n - k, l) \) are isometric. Therefore, up to isometry we may always assume \( k, l \in (0, n/2) \) without loss of generality. The proof of Lemma 7.3 in [Ya] works identically for \( L(n; k, l) \) with \( 0 < k, l < n/2 \) to prove

**Lemma 9.4 ([Ya]).**

Let \( L(n; k, l) \) be a 3-dimensional lens space of constant sectional curvature one. Then

\[
xt_q(L(n; k, l)) \leq \arccos \left\{ \cos(\alpha_q) \cos \frac{\pi n}{q} - \frac{1}{2} \left\{ (\cos \frac{\pi n}{q} - \cos \frac{\pi}{n})^2 + \sin^2(\alpha_q)(n^2 \sin \frac{\pi}{n} - \sin \frac{\pi n}{q})^2 \right\}^{1/2} \right\}
\]

where \( \alpha_q = \pi / (2(2 - [(q + 1)/2]^{-1})) \), where \( \lfloor x \rfloor \) means the integer part of \( x \).

**Corollary 9.5.**

Let \( L(n; k, l) \) be a 3-dimensional lens space of constant sectional curvature one. If \( n \geq 61 \), then \( xt_5(L(n; k, l)) < \pi/3 \).

**Corollary 9.6.**

If the exponent of \( \gamma^* \) is at least 61, then \( (\tilde{M}^*)^{\gamma^*} \) contains at most five isolated fixed points.

**Proof.** We argue by contradiction, assuming \( x_1^*, \ldots, x_6^* \) are six isolated \( \gamma^* \)-fixed points. Let \( X = \tilde{M}^*/\langle \gamma^* \rangle \). Connecting each pair of points by a minimal geodesic in \( X \), we obtain a configuration consisting of twenty geodesic triangles. Because \( X \) has positive curvature in the comparison sense ([Pe]), the sum of the interior angles of each triangle is \( > \pi \) and thus the sum of total angles of the twenty triangles, \( \sum \theta_i > 20\pi \). We then estimate the sum of the total angles in the following way, first estimate from above of the ten angles around each \( x_i^* \) and then sum up over the six points. We claim that the sum of angles at \( x_i^* \) is bounded above by

\[
10 \cdot xt_5(x_i^*) \leq 10 \frac{\pi}{3} \text{ and thus } \sum \theta_i \leq 6(10 \cdot \frac{\pi}{3}) = 20\pi, \text{ a contradiction.}
\]

Let \( \tilde{x}_i \in \tilde{M} \) such that \( p(\tilde{x}_i) = x_i^* \), and let \( \tilde{t} \in T^1 \) such that \( \tilde{t} \cdot \gamma \) fixes \( T^1(\tilde{x}_i) \). Let \( S^3_\tilde{x}_i \) denote the unit 3-sphere in the normal space of \( T^1(\tilde{x}_i) \). If the isotropy group at \( \tilde{x}_i \) is trivial, then the space of directions at \( x_i^* \) in \( X \) is isometric \( S^3_\tilde{x}_i / \langle \tilde{t} \cdot \gamma \rangle \) which is a lens space with a fundamental group of order \( |\gamma^*| \). By Corollary 9.4, we conclude that the sum of the ten angles is bounded above by

\[
\left( \begin{array}{c} 5 \\ 2 \end{array} \right) xt_5(L) = 10 \cdot \frac{\pi}{3}.
\]

If the isotropy group at \( \tilde{x}_i \) is not trivial, then the above estimate still holds because the 5-extent only gets smaller when passing to the quotient of \( S^3_\tilde{x}_i \) by the isotropy group. □
Lemma 9.7.
Let $\tilde{X}$ be a simply connected topological space of dimension 4 with a finite group $G$-action. Assume that the total Betti number of $\tilde{X}$ is bounded above by a constant $N$. Then $G$ has a normal subgroup of order at least $|G|/c(N)$ which acts trivially on the homology $H^*(\tilde{X})$, where $c(N)$ is a function depending only on $N$.

Proof. Let $\rho : G \to \text{Aut}(H^*(\tilde{X}))$ denote the homomorphism induced by the $G$-action on $H^*(\tilde{X})$. Because the torsion subgroup of $\text{Aut}(H^*(\tilde{X}))$ is a subgroup of $\text{GL}(\mathbb{Z}, N)$, which is bounded above by a constant depending only on $N$ (cf. [Th]), say $c(N)$, the conclusion follows. \qed

Proof of Proposition 9.3.
Because $\tilde{M}^*$ is a simply connected orbifold with isolated singularities (Lemma 9.2), we can apply the Poincaré duality on homology groups with rational coefficients to conclude that $\chi(\tilde{M}^*) = 2 + b_2(\tilde{M}^*)$. Let $\Gamma_0$ be the principal isotropy group of $\pi_1(M)$ on $\tilde{M}^*$. By Theorem 4.9 $\Gamma_0$ is cyclic and belongs to the center of a certain index at most two normal subgroup (the subgroup of orientation preserving isometries in $\pi_1(M)$). By the assumption (9.1.1), the index $|\pi_1(M) : \Gamma_0| \geq w$.

On the other hand, by the Betti number bound in [Gr] and Lemma 9.7 we may assume that the normal subgroup $G \subset \pi_1(M)/\Gamma_0$ has order $\geq w/k_0$, and thus acts trivially on $H^*(\tilde{M}^*; \mathbb{Z})$.

For any $\beta^* \in G$, by Theorem 4.8 the fixed point set $(\tilde{M}^*)^{\beta^*} \neq \emptyset$ and by Theorem 4.10 we may assume that $(\tilde{M}^*)^{\beta^*}$ is a finite set (otherwise $\tilde{M}$ contains a 3-dimensional totally geodesic submanifold, and so Theorem 4.10 implies that it is a homotopy sphere). If $\beta^*$ is of prime order and $|\beta^*| \geq 61$, by Theorem 4.1

$$\chi(\tilde{M}^*) = \chi(F(\beta^*, \tilde{M}^*)).$$

Therefore, by Corollary 9.6 we conclude $\chi(\tilde{M}^*) \leq 5$.

Now we assume that $|G|$ has all prime factors $\leq 60$. Since there are at most 17 primes less than 60, there is a prime $p \leq 60$ so that $G$ has a $p$-Sylow subgroup $G_p$ of order $\geq \frac{w}{17k_0}$. Let $\beta^*_0 \in G_p$ be an element of order $p$ generating a normal subgroup of $G_p$ (by finite group theory [Se], $G_p$ is nilpotent), we may assume that $\beta^*_0$ has only isolated fixed points for the same reasoning as above, say $p_1, \ldots, p_n$. By Theorem 4.1, $n = \chi((\tilde{M}^*)^{\beta^*_0}) = \chi(\tilde{M}^*)$. Observe that $n \leq b = b(4)$. Now $G_p$ acts on the fixed point set $(\tilde{M}^*)^{\beta^*_0}$, thus we get a homomorphism $h : G_p \to S_n$, where $S_n$ is the permutation group of $n$-words. Therefore, the kernel of $h$ has order at least $\frac{w}{17k_0n} \geq \frac{w}{17k_0b}$. By a well-known result of Gromov $\pi_1(M)$ may be generated by a bounded number of generators, so is $G_p$, generated by $c$ elements. Hence ker$(h)$ contains an element, say $\beta^*_1$, of order at least 61, if $\frac{w}{17k_0b}$ is sufficiently large. Since $\beta^*_1$ fixes the set $\{p_1, \ldots, p_n\}$ pointwisely, by Corollary 9.6 $n \leq 5$. Therefore, $\chi(\tilde{M}^*) = n \leq 5$. \qed

c. The completion of the proof of Theorem F’.

Lemma 9.8.
Let $M$ be a closed 5-manifold of positive sectional curvature which admits a $\pi_1$-invariant isometric $T^1$-action. If $\pi_1(M)$ satisfies (9.1.1), then the $T^1$-action on $\tilde{M}$ is free.
Proof. We argue by contradiction, assuming the $T^1$-action is not free. Then there is at least a finite isotropy group $\mathbb{Z}_p \subset T^1$. By Theorem 4.10 we may assume that $\dim(F(\mathbb{Z}_p, \tilde{M})) = 1$, since otherwise, $\tilde{M}$ contains a totally geodesic 3-manifold. Then $\tilde{M}^{\mathbb{Z}_p}$ consists of at most two components (circles), if $b_2(\tilde{M}) \leq 1$, or three components if $b_2(\tilde{M}) \leq 2$ (Theorem 4.2).

Let $H$ denote the subgroup of $\pi_1(M)$ preserving all components of $\tilde{M}^{\mathbb{Z}_p}$. Then $H$ is a cyclic normal subgroup such that the quotient $\pi_1(M)/H$ acting effectively on the set of exceptional orbits. If $b_2(\tilde{M}) \leq 1$, by counting the number of components $\pi_1(M)/H$ has order at most 2. A contradiction to (9.1.1).

If $b_2(\tilde{M}) = 2$, and $\tilde{M}^{\mathbb{Z}_p}$ has exactly three components, it is easy to see that $H$ acts on $\tilde{M}^*$ has at most five isolated fixed points, three of them are the exceptional $T^1$-orbits with isotropy group $\mathbb{Z}_p$. Because $H$ is normal in $\pi_1(M)$, thus $\pi_1(M)$ acts on the fixed point set and it sends an exceptional orbit to an exceptional orbit. Therefore, $\pi_1(M)$ acts on the union of the rest at most two isolated $T^1$-orbits fixed by $H$. This implies once again that $\pi_1(M)$ has an index $\leq 2$ cyclic normal subgroup. The desired result follows. □

Lemma 9.9.

Let $M$ be as in Lemma 9.8. Then the induced $\pi_1(M)$-action on $\tilde{M}^*$ is pseudo-free.

Proof. By Lemma 9.8, $\tilde{M}^*$ is a closed 4-manifold. If there is an element $\gamma \in \pi_1(M)$ with a fixed point component $F \subset \tilde{M}^*$ of dimension 2, there exists an element $t \in T^1$ so that $t\gamma$ has a 3-dimensional fixed point set in $\tilde{M}$. By Theorem 4.10 this implies that $\tilde{M} \approx S^5$. By Lemma 1.10, we conclude that $\pi_1(M)$ acts on $\tilde{M}^*$ pseudofreely, a contradiction. □

Lemma 9.10.

Let $M$ be a closed 5-manifold of positive sectional curvature which admits a $\pi_1$-invariant fixed point free isometric $T^1$-action. If $\pi_1(M)$ satisfies (9.1.1) and (9.1.2), then $\tilde{M}$ is a homotopy sphere.

Proof. Because the $T^1$-action on $\tilde{M}$ is free (Lemma 9.7), $\tilde{M}^*$ is a closed simply connected smooth 4-manifold of positive sectional curvature. Because $b_2(\tilde{M}^*) = b_2(\tilde{M}) + 1 \geq 1$, by Proposition 9.3, $3 \leq 2 + b_2(\tilde{M}^*) = \chi(\tilde{M}^*) \leq 5$. If $\chi(\tilde{M}^*) = 3$, then $b_2(\tilde{M}) = b_2(\tilde{M}^*) - 1 = 0$. This together with (9.2.2) implies that $\tilde{M} \approx S^5$.

It suffices to rule out the cases $\chi(\tilde{M}^*) = 4$ and $\chi(\tilde{M}^*) = 5$.

Case (a) If $\chi(\tilde{M}^*) = 4$;

Note that $\tilde{M}^*$ is homeomorphic to $S^2 \times S^2$, $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ or $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$, up to a possible orientation reversing ([Fr]). By the classification of simply connected 5-manifolds (cf. [Ba]) $\tilde{M} = S^2 \times S^3$ or $S^2 \tilde{S}S^3$. Because $\pi_1(M)$ preserves the Euler class of the principal circle bundle $T^1 \to \tilde{M} \to \tilde{M}^*$, the natural homomorphism given by the $\pi_1(M)$-action on the second homology group, $\alpha : \pi_1(M) \to \text{Aut}(H^2(\tilde{M}^*; \mathbb{Z}); I)$, where $\text{Aut}(H^2(\tilde{M}^*; \mathbb{Z}); I)$ is the automorphism group preserving the intersection form.

Let $\Gamma_0$ (resp. $\Gamma$) be the principal isotropy group of $\pi_1(M)$-action on $\tilde{M}^*$ (resp. orientation preserving subgroup of $\pi_1(M)$). Recall that $\Gamma \subset \pi_1(M)$ is a normal
subgroup of index at most 2, and \( \Gamma_0 \) is in the center of \( \Gamma \) which generates a cyclic subgroup with any element of \( \Gamma \) (cf. 4.10) Let \( G = \Gamma / \Gamma_0 \). The homomorphism \( \alpha \) reduces to a homomorphism \( \bar{\alpha} : G \to \text{Aut}(H^2(\hat{M}^*; \mathbb{Z}); I) \). It is easy to see that the image of \( \bar{\alpha} \) has order at most 2 (cf. [Mc2]).

Subcase (a1) If \( \text{ker}(\bar{\alpha}) \) has odd order;

Consider the action of \( \text{ker}(\bar{\alpha}) \) on \( \hat{M}^* \). Observe that the action is pseudo-free, and every isotropy group is cyclic. By [Mc2] Lemma 7.5 and the first paragraph in the proof of Lemma 4.7 [Mc2] (which identically extends to our case) it follows that, \( \text{ker}(\alpha^*) \) is cyclic of odd order. If \( \text{Im}(\alpha^*) \) is not trivial and \( \hat{M}^* \approx S^2 \times S^2 \), by [Bre] VII Corollary 7.5 there is an involution on \( \hat{M}^* \) with a 2-dimensional fixed point set. A contradiction by Lemma 9.8. If \( \hat{M}^* \approx \mathbb{C}P^2 \# \mathbb{C}P^2 \) or \( \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \), in the former case every self homeomorphism of \( \hat{M}^* \) is orientation preserving, and for the latter \( \text{Im}(\alpha^*) = 0 \). Therefore, in either cases, \( \pi_1(M) \) has an image in \( \text{Aut}(H^2(\hat{M}^*; \mathbb{Z}); I) \) of order at most 2, and by Corollary 4.10 it contains a normal cyclic subgroup of index at most 2, a contradiction to (9.1.1).

Subcase (a2) If \( \text{ker}(\alpha^*) \) has even order;

By [Bre] VII Lemma 7.4, any involution in \( \text{ker}(\alpha^*) \) has a 2-dimensional fixed point set on \( \mathbb{C}P^2 \# \mathbb{C}P^2 \) or \( \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \). Therefore, by Lemma 9.9 we may assume that \( \hat{M}^* = S^2 \times S^2 \). By [Mc2] Theorem 3.3, \( \text{ker}(\alpha^*) \) is a polyhedral. Therefore \( \text{ker}(\alpha^*) \) is either cyclic, dihedral, or is a non-abelian group of order of order 12 (Tetrahedral group and two others) or \( \text{ker}(\alpha^*) \) is Octahedral group (of order 24), or Icosahedral group (of order 60). Thus, \( \text{ker}(\alpha^*) \) is cyclic or dihedral if the order \( |G| > 120 \). We may require our constant \( w \) in Theorem E is larger than 120. By [Mc2] Theorems 3.9 and 4.10 \( G \) is listed as follows:

(9.10.1) a cyclic, or dihedral group;
(9.10.2) \( Q_{2^k m} \times \mathbb{Z}_n \); where \( Q_{2^k m} \) is the generalized quaternionic group, \( m, n \) are coprime odd integers;
(9.10.3) \( D_{2^k m} \times \mathbb{Z}_n \), where \( D_{2^k m} = \mathbb{Z}_m \rtimes \mathbb{Z}_{2^k} \) and \( \mathbb{Z}_{2^k} \) acts on \( \mathbb{Z}_m \) by inverse automorphism, \( m, n \) are coprime odd integers, and \( k \geq 2 \);
(9.10.4) A non-splitting extension of a dihedral group by \( \mathbb{Z}_2 \).

Since \( \Gamma \) is a center extension of a cyclic group by \( G \), by Lemma 9.11 below the group \( \Gamma \) satisfies \( 2p \)-condition, i.e., for any prime \( p \), a subgroup of order \( 2p \) is cyclic (cf. [Mi]). By group extension theory, it is not hard to verify, the dihedral groups in (9.10.1), (9.10.2) and (9.10.4) must be reduced from a quaternionic subgroup of \( \Gamma \), and the group \( D_{2^k m} \) in (9.10.3) is reduced from \( D_{2^{k'} m} \), where \( k' \geq 2 \). Moreover, for \( G \) in (9.10.1), (9.10.2) and (9.10.3), \( \Gamma \) is isomorphic to a 3-dimensional spherical space form group (compare [Mi] Theorem 2), and so \( \pi_1(M) \) contains an index at most 2 subgroup isomorphic to a 3-dimensional spherical space form group, a contradiction by the assumption. If \( G \) is a group in (9.10.4), and \( [\pi_1(M) : \Gamma] = 2 \), there is a maximal cyclic subgroup \( \langle \gamma \rangle \) of \( \pi_1(M) \) of index 8. Therefore, \( \pi_1(M) \) acts on the 4 fixed points of \( \gamma \) on \( S^2 \times S^2 \) without any even order isotropy group by the maximality of \( \langle \gamma \rangle \). A contradiction, since the permutation group \( S_4 \) does not contain cyclic subgroup of order 8. This shows that \( \Gamma = \pi_1(M) \), and it contains a quaternionic subgroup of index 2. A contradiction again to the assumption.

Case (b) If \( \chi(\hat{M}^*) = 5 \);

Since the Euler characteristic \( \chi(\hat{M}^*) \) is odd, by [Bre] VII Corollary 7.6 any involution on \( X \) has a 2-dimensional fixed point set. Therefore, by Lemma 9.9 we may assume that \( \pi_1(M)/\Gamma_0 \) has odd order. In particular, \( \pi_1(M) = \Gamma \), i.e.
any element of $\pi_1(M)$ preserves the orientation of $\tilde{M}^*$. For the homomorphism $\alpha^* : G \to \text{Aut}(H_2(\tilde{M}^*; \mathbb{Z}))$, by [Mc1] the kernel $G_0 = \ker(\alpha)$ is an abelian subgroup of $T^2$ with non-empty fixed point set. Therefore, $G_0$ must be cyclic by combining Lemma 9.9, otherwise, there exists a cyclic subgroup of $G_0$ with a 2-dimensional fixed point set. Therefore $\pi_1(M)/G_0$ is isomorphic to an odd order subgroup of the automorphism group $\text{Aut}(H_2(\tilde{M}^*; \mathbb{Z})) = \text{GL}(\mathbb{Z}, 3)$. By [Th] the finite subgroup of $\text{GL}(\mathbb{Z}, 3)$, up to possible 2-torsion, is a subgroup of $\text{GL}(\mathbb{Z}_2, 3)$ which has order $(2^3 - 1)(2^3 - 2)(2^3 - 4)$, which is coprime to 5. Note that $\pi_1(M)$ preserves the fixed point set $(\tilde{M}^*)^{G_0}$, which consists of 5 isolated points. Therefore $\pi_1(M)/G_0$ is an odd order subgroup of $S_5$ and $\pi_1(M)/G_0 \neq \mathbb{Z}_5$. This implies that $\pi_1(M)$ fixes at least 2 points among $(\tilde{M}^*)^{G_0}$ and so $\pi_1(M)$ is cyclic. A contradiction.

Lemma 9.11.

Let $S^2 \times S^3 \to S^2 \times S^2$ be a $G$-equivariant principal $T^1$-bundle. If $G$ acts freely on $S^2 \times S^3$, pseudofreely on $S^2 \times S^2$. Then there is no order 2 element of $G$ acting non-trivially on $S^2 \times S^2$ inducing the identity on homology groups.

Proof. If not, for an order 2 element $\beta$, its fixed point set $F$ in $S^2 \times S^2$ consists of 4 points (by Theorem 4.1). For such a fixed point $[x] \in S^2 \times S^2$ with $x \in S^2 \times S^3$, there is an element $t \in T^1$ so that $\beta x = tx$. The freeness of $T^1$-action and $\beta$-action implies that the order of $t$ is the same as $\beta$, of order 2. Since $T^1$ has only one element of order 2, say $t_0$, this proves that $\beta t_0^{-1}$ has fixed point set the union of 4 circles. A contradiction by Theorem 4.2.

Appendix: s-cobordism theory

In the proof of Theorem E for pseudofree $T^2$-action (Section 7) we assumed the Poincaré conjecture holds. The goal of this appendix is to give an account how this assumption can be removed by using the s-cobordism theory.

We call two closed $n$-manifolds $M_1$ and $M_2$ are $s$-cobordant if there is an $(n+1)$-manifold $W$ with boundary $M_1 \sqcup M_2$ so that the inclusions $i_1 : M_1 \to W$ and $i_2 : M_2 \to W$ are both simple homotopy equivalences (cf. [Ke]). The manifold $W$ is called an $s$-cobordism. The deep s-cobordism theorem (originally due to Smale for simply connected manifold, and extended to non-simply connected case by Barden-Mazur-Stallings, cf. [Ke] for a detailed account) asserts that an $s$-cobordism is diffeomorphic (or homeomorphic if manifolds are TOP.) to the product $M_1 \times [0, 1]$, provided $n \geq 5$. Hence, $M_1$ and $M_2$ are diffeomorphic (resp. homeomorphic). The dimension assumption is crucial. In fact, counterexamples exist for $n = 3, 4$, by Cappell-Shaneson, and Donaldson’s theory. An $s$-cobordism theorem for simply connected 4-manifold (i.e. $n = 3$) would imply the 3-dimensional Poincaré conjecture.

It is a well-known fact that every simply connected 3-manifold is $s$-cobordant (by terminology should be called $h$-cobordant) to $S^3$. The key issue of our solution in the proof Theorem E without assuming the Poincaré conjecture is to replace the homeomorphism (equivalently diffeomorphism) by $s$-cobordism. If we could
obtain an $s$-cobordism between our 5-manifold $M$ with a spherical space form, the $s$-cobordism theorem applies to imply

our desired result. Note that the dimension shifting is important and this can be obtained by using the additional $\pi_1$-invariant $T^2$-action on the manifold.

**Definition A.1.**

Let $M_1$ (resp. $M_2$) be a closed 5-manifold with a smooth action by $T^2 \times G$ where the $T^2$-action is pseudofree and the $G$-action is free. We call that $W$ is an $s$-cobordism between $(M_1, T^2 \times G)$ and $(M_2, T^2 \times G)$ if

(A.1.1) $W$ is an $s$-cobordism between $M_1$ and $M_2$;

(A.1.2) There is an action by $T^2 \times G$ on $W$ whose restriction on $M_1$ (resp. $M_2$) gives $(M_1, T^2 \times G)$ (resp. $(M_1, T^2 \times G)$), and the $T^2$-action is pseudofree and the $G$-action is free.

By the $s$-cobordism theory (cf. [Ke]), $W$ is homeomorphic to the product $M_i \times [0, 1]$ when $\dim(M_i) \geq 5$. However, the induced $T^2$-action on $M_i \times [0, 1]$ may not preserve the slice $M_i \times \{t\}$ ($i = 1, 2$, $t \in (0, 1)$, so that one may not conclude that $(M_1, T^2)$ is conjugate to $(M_2, T^2)$.

As we have seen before, if $T^2$ acts pseudofree on $S^5$, the orbit space $S^5/T^2$ is a manifold homotopy equivalent to $S^3$. Hence, if $W$ is an $s$-cobordism between two pseudofree $T^2$-actions on $S^5$, obviously, the orbit space $W/T^2$ gives an $s$-cobordism between the two orbit spaces (two homotopy 3-spheres) of the actions on $S^5$. We will show below the converse also holds.

**Lemma A.2.**

A pseudo-free $T^2$-action on $S^5$ is $s$-cobordant to a pseudo-free linear $T^2$-action.

**Proof.** As in Section 7 we make use the convention that $T^2_0$ indicates a linear $T^2$-action on $S^5$. Let $V$ be an $s$-cobordism of the homotopy 3-sphere $S^5/T^2 := \Sigma$ and $S^5/T^2_0 = S^3$. By Theorem 4.4 $T^2$ (resp. $T^2_0$) has three isolated circle orbits, denoted by $T^1_i \times D^4$ (resp. $T^1_i \times D^4$) respectively. Let $p_i$ (resp. $q_i$) denote the projections of these circle orbits in $S^5/T^2$ (resp. $S^5/T^2_0$), $i = 1, 2, 3$, respectively. Connecting $p_i$ and $q_i$ by a simple disjoint paths in $V$, $i = 1, 2, 3$, let $V_0$ denote the complement of the three simple paths. Observe that $V_0$ is homotopy equivalent to $S^3_0 = S^3 - q_1 \cup q_2 \cup q_3$ (resp. $\tilde{M}_0^* = \tilde{M}_0^* - p_1 \cup p_2 \cup p_3$). It is easy to see that $H^2(S^3_0; \mathbb{Z}^2) \cong \text{Hom}(\mathbb{Z}^2, \mathbb{Z}^2)$. Consider the Euler classes of principal $T^2$-bundles on $\Sigma_0$ and $S^3_0$ as $2 \times 2$ matrices in the above group. Hence, the determinants of the matrices are $\pm 1$ because the 2-connectedness of the complement of the singular orbits. Therefore, modifying by an automorphism of $T^2$ if necessary (in other words, up to weak equivalence), we may assume that the two Euler classes are homotopic, regarded as maps to the classifying space $BT^2$. The homotopy gives a map $h : V_0 \to BT^2$ whose restrictions on $S^3_0$ and $\tilde{M}_0^*$ correspond to the classifying maps of the principal $T^2$-bundles. Let $W_0$ denote the total space of the $T^2$-bundle over $V_0$, which is a 6-manifold with boundary.

Finally, we can attach equivariantly three copies of $T^1 \times D^4 \times [0, 1]$ to $W_0$ along the boundaries, so that the three copies of $T^1 \times D^4 \times \{0\}$ fill the three singular orbits in $S^5$. This gives a $T^2$-manifold $W$ with boundary $S^5 \sqcup S^5$, which yields the desired $s$-cobordism between $(S^5, T^2)$ and $(S^5, T^2_0)$. $\square$
In Lemmas 7.3 and 7.4 the induced Γ-action on the homotopy 3-sphere Σ (the orbit space) may not be trivial. We need to find an equivariant s-cobordism between (Σ, Γ) with a standard linear action on (S³, Γ). Recently, a deep result (cf. [BLP]) on 3-orbifold implies that every finite group acting non-freely on S³ is conjugate to a linear action. Let Γ be a finite group. For a smooth or locally linear nonfree Γ-action on a homotopy 3-sphere (possibly reducible) Σ, the result of [BLP] implies that (Σ, Γ) is Γ-equivariantly diffeomorphic to (S³, Γ) # (|Γ| · M⁰, Γ), where Γ acts linearly on S³, and acts freely on |Γ| · M⁰ and M⁰ is a homotopy 3-sphere (we thank Porti for pointing out this fact to us). Therefore, it is easy to get a 4-dimensional s-cobordism (V, Γ) between (Σ, Γ) and (S³, Γ). Indeed, we may take (V, Γ) = (S³ × [0, 1], Γ) # (|Γ| · B₀, Γ), where B₀ is a contractible 4-manifold with boundary M⁰, and # is the boundary connected sum along the boundary piece S³ × {1}. An exceptional case is Γ = Z₃ which acts freely on the homotopy 3-sphere Σ. In this case the main result of [BLP] does not apply. However, it is easy to see that Σ/Γ is simple homotopy equivalent to the lens space S³/Z₃ (unique), because the Whitehead torsion Wh(Z₃) = 0 (cf. [Coh]). Hence, there is an s-cobordism between S³/Z₃ and Σ/Z₃, which gives exactly a Z₃-equivariant s-cobordism between (S³, Z₃) and (Σ, Z₃).

Lemma A.3.

The Γ-equivariant principal T²-bundle over M⁰ over Σ in Lemmas 7.3 and 7.4 is Γ-equivariantly s-cobordant to a linear principal T²-bundle over S³. Hence, M⁰ is diffeomorphic to the linear model S³/Γ₀.

Proof. Because the proofs of Lemmas 7.3 and 7.4 involve only homotopy theory, using the Γ-equivariant s-cobordism V defined above, there is a well-defined Γ-equivariant principal T²-bundle over V₀, where V₀ is obtained by removing certain Γ-equivariant three simple paths, e.g., in the case Γ acts non-freely on Σ, just take the product (p₁ ∪ p₂ ∪ p₃) × [0, 1] ⊂ V; in the case Γ acts freely on Σ, take the preimage of any simple path joining the singular point of S³/Z₃ and Σ/Z₃. The total space of the principal T²-bundle over V₀ gives a Γ-equivariant s-cobordism between (M₀, Γ) and (S³, Γ₀), which implies the diffeomorphism between S³/Γ₀ and M₀. □

References

[AM] U. Abresch; W. T. Meyer, Pinching below 1/4, injectivity radius estimate, and sphere theorems, J. Diff. Geom 40 (1994), 643-691.
[Ba] D. Barden, Simply connected five manifolds, Ann. of Math. 82 (1965), 365-385.
[Bre] G. Bredon, Introduction to compact transformation groups, Academic Press 48 (1972).
[Bro] K. S. Brown, Cohomology of groups, Springer-Verlag New York (1982).
[BLP] M. Boileau; B. Leeb; J. Porti, Geometrization of 3-dimensional orbifolds, Annals of Math. 162 (2005), 195-290.
[CFG] J. Cheeger; K. Fukaya; M. Gromov, Nilpotent structures and invariant metrics on collapsed manifolds, J. A.M.S 5 (1992), 327-372.
[Coh] M. Cohen, A course in simple-homotopy theory. Graduate Texts in Mathematics, 10, Springer-Verlag, New York-Berlin, (1973).
S. Eilenberg; S. MacLane, *On the groups $H(\pi, n)$, III*, Ann. of Math. **60** (1954), 513-557.

F. Fang; S. Mendonca; X. Rong, *A connectedness principle in the geometry of positive curvature*, Comm. Anal. Geom. **13 No.2**. (2005), 479-501.

F. Fang; X. Rong, *Positive curvature, volume and second Betti number*, Geom. Funct. Anal. **9** (1999), 641-674.

F. Fang; X. Rong, *The second twisted Betti numbers and the convergence of collapsing Riemannian manifolds*, Invent. Math. **150** (2002), 61-109.

F. Fang; X. Rong, *Homeomorphic classification of positively curved manifolds with almost maximal symmetry rank*, Math. Ann. **332** (2005), 81-101.

T. Frankel, *Manifolds of positive curvature*, Pacific J. Math. **11** (1961), 165-174.

M. Freedman, *Topology of Four Manifolds*, J. of Diff. Geom. **28** (1982), 357-453.

M. Gromov, *Structures metriques pour les varietes riemanniennes. Edited by J. Lafontaine and P. Pansu*, Textes Mathématiques, 1 CEDIC, Paris (1981).

K. Grove; H. Karcher; E. Ruh, *Group actions and curvature*, Invent. Math. **23** (1974), 31-48.

K. Grove, S. Markvosen, *New extremal problems for the Riemannian recognition program via Alexandrov geometry*, J. Amer. Math. Soc. **8** (1995), 1-28.

K. Grove, C. Searle, *Positively curved manifolds with maximal symmetry-rank*, J. Pure Appl. Alg. **91** (1994), 137-142.

K. Grove, W. Ziller, *Curvature and symmetry of Milnor spheres*, Ann. of Math. **152** (2000), 331-367.

K. Grove; A. Petrunin; W. Tuschmann, *Nonnegative pinching, moduli spaces and bundles with infinitely many souls*, Preprint (2005).

V. Kapovitch; A. Petrunin; W. Tuschmann, *Nonnegative pinching, moduli spaces and bundles with infinitely many souls*, Preprint (2005).

M. Kervaire, *Le théorème de Barden-Mazur-Stallings*, Comment. Math. Helv. **40** (1965), 31-42.

W. Klingenberg; T. Sakai, *Remarks on the injectivity radius estimate for almost $1/4$-pinched manifolds*, Lecture Notes in Math. Springer **1201** (1986), 156-164.

M. Mccoey, M. Symmetry groups of four manifolds, Topology **41** (2002), 835-851.

M. Mccoey, M., *Groups which acts pseudo-freely on $S^2 \times S^2$*, preprint arXiv:math.GT/9906159 (1999).

J. Milnor, *Groups which acts on $S^n$ without fixed points*, Amer. J. Math. **79** (1957), 623-630.

T. Petrie, *Free metacyclic group actions on spheres*, Ann. of Math. **94** (1971), 108-124.

A. Petrunin; W. Tuschmann, *Diffeomorphism finiteness, positive pinching and second homotopy*, Geom. Funct. Anal. **9** (1999), 736-774.

X. Rong, *On the fundamental group of manifolds of positive sectional curvature*, Ann. of Math **143** (1996), 397-411.

X. Rong, *Positively curved manifolds with almost maximal symmetry rank*, Geom. Dedicata **95** (2002), 157-182.

X. Rong, *On fundamental groups of positively curved manifolds with local torus actions*, Asian J. Math., **9** (2005), 545-559.

J.P. Serre, *Linear representations of finite groups*, GTM **42**, Springer-Verlag, New York-Heidelberg (1977).

R. Stong; Z. Wang, *Self-homeomorphisms of 4-manifolds with fundamental group $\mathbb{Z}$*, Topology. Appl. **106** (2000), 49-56.

W. Thurston, *Three-dimensional geometry and topology, Edited by Silvio Levy*, Princeton Mathematical Series, 35. Princeton University Press, Princeton, NJ **1** (1997).
[Wi1] Wilczynski, D, Periodic maps on simply connected four manifolds, Topology 30 (1991), 55-65.

[Wi2] Wilczynski, D, Group actions on complex projective planes, Trans. Amer. Math. Soc. 303 (1987), 707-731.

[Wi] B. Wilking, Torus actions on manifolds of positive sectional curvature, Acta Math (2003), 259-297.

[Wo] J. A. Wolf, The spaces of constant curvature, McGraw-Hill series in higher mathematics (1976).

[Ya] D. Yang, On the topology of non-negatively curved simply connected 4-manifolds with discrete symmetry, Duke Math. J. (1994), 531-545.

[Yau] S.-T. Yau, Open problems in differential geometry, Proc. Sympo. in pure Math. 54 (1993), 1-28.

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