ON THE FIXATIC NUMBER OF GRAPHS

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ABSTRACT. The fixing number of a graph $G$ is the smallest cardinality of a set of vertices $F \subseteq V(G)$ such that only the trivial automorphism of $G$ fixes every vertex in $F$. Let $\Pi = \{F_1, F_2, \ldots, F_k\}$ be an ordered $k$-partition of $V(G)$. Then $\Pi$ is called a fixatic partition if for all $i; 1 \leq i \leq k$, $F_i$ is a fixing set for $G$. The cardinality of a largest fixatic partition is called the fixatic number of $G$. In this paper, we study the fixatic numbers of graphs. Sharp bounds for the fixatic number of graphs in general and exact values with specified conditions are given. Some realizable results are also given in this paper.

1. Introduction

Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. The neighborhood of a vertex $v$ of $G$ is the set $N(v) = \{u \in V(G) : uv \in E(G)\}$. The number of vertices in $N(v)$ is the degree of $v$, denoted by $d(v)$. A vertex of degree one is called a leaf or a pendant vertex. If two distinct vertices $u$ and $v$ of $G$ have the property that $N(u) - \{v\} = N(v) - \{u\}$, then $u$ and $v$ are called twin vertices (or simply twins) in $G$. If for a vertex $u$ of $G$, there exists a vertex $v \neq u$ in $G$ such that $u, v$ are twins in $G$, then $u$ is said to be a twin in $G$. A set $T \subseteq V(G)$ is said to be a twin-set in $G$ if any two of its elements are twins.

An automorphism $\alpha$ of $G$, $\alpha : V(G) \rightarrow V(G)$, is a bijective mapping such that $\alpha(u)\alpha(v) \in E(G)$ if and only if $uv \in E(G)$. Thus, each automorphism $\alpha$ of $G$ is a permutation of the vertex set $V(G)$ which preserves adjacencies and non-adjacencies. The automorphism group of a graph $G$, denoted by $\Gamma(G)$, is the set of all automorphisms of a graph $G$. A connected graph $G$ is symmetric if $\Gamma(G) \neq \{id\}$. The stabilizer of a vertex $v \in V(G)$, denoted $\Gamma_v(G)$, is the set $\{\alpha \in \Gamma(G) : \alpha(v) = v\}$. The stabilizer of a set of vertices $F \subseteq V(G)$ is $\Gamma_F(G) = \{\alpha \in \Gamma(G) : \alpha(v) = v, \forall v \in F\}$. Note that $\Gamma_F(G) = \bigcap_{v \in F} \Gamma_v(G)$. For a vertex $v$ of a graph $G$, the orbit of $v$, denoted $\mathcal{O}(v)$, is the set of all vertices $\{u \in V(G) : \alpha(v) = u \text{ for some } \alpha \in \Gamma(G)\}$. Two vertices $u$ and $v$ are similar if they belong to the same orbit.

A vertex $v$ is fixed by a group element $g \in \Gamma(G)$ if $g \in \Gamma_v(G)$. A set of vertices $F \subseteq V(G)$ is a fixing set of $G$ if $\Gamma_F(G)$ is trivial. The fixing number of a graph $G$, denoted by $fix(G)$, is the smallest cardinality of a fixing set of $G$ [8]. The graphs

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with $\text{fix}(G) = 0$ are the rigid graphs [11], which have trivial automorphism group. Every graph $G$ has a fixing set (the set of vertices of $G$ itself). It is also clear that, any set containing all but one vertex is a fixing set. Thus, for a graph $G$ on $n$ vertices, $0 \leq \text{fix}(G) \leq n - 1$ [11].

The fixing number of a graph $G$ was defined by Erwin and Harary [8] for the first time. Boutin introduced the concept of determining set and defined it as follows: A subset $D$ of vertices in a graph $G$ is called a determining set if whenever $g, h \in \Gamma(G)$ with the property that $g(u) = h(u)$ for all $u \in D$, then $g(v) = h(v)$ for all $v \in V(G)$ [3]. The minimum cardinality of a determining set is called the determining number. In [9], it was shown that fixing set and determining set are equivalent. A considerable literature has been developed in this field (see [4, 9, 11, 15]). The concept of fixing number originates from the idea of breaking symmetries in graphs, which have applications in the problem of programming a robot to manipulate objects [13].

Given an ordered set $W = \{w_1, w_2, ..., w_k\}$ of vertices of a connected graph $G$ and a vertex $v$ of $G$, the locating code of $v$ with respect to $W$ is $(d(v, w_1), d(v, w_2), ..., d(v, w_k))$, denoted by $c_W(v)$. The set $W$ is called a locating set for $G$ if distinct vertices have distinct codes. The location number of a graph $G$, denoted by $\text{loc}(G)$, is the minimum cardinality of a locating set for $G$. This notion was introduced by Slater in [15]. Independently, Harary and Melter [12] studied this notion and used the term metric dimension rather than the location number.

A partition of vertex set of a graph is the partition of the vertices into subsets with some specific conditions. In graph theory, a number of partitions of the vertices of a graph have been introduced. In 1977, Cockayne and Hedetnieme [7] introduced the concept of a domatic partition in which each subset is a dominating set of a graph. In 2000, Chartrand, Salehi and Zhang [10] defined a resolving partition which corresponds to resolving sets of a graph. In 2002, Chartrand, Erwin, Henning, Slater and P. Zhang [5] introduced the concept of a locating-chromatic partition in which every class is a color class. In 2013, Salman, Javaid and Chaudhary [14] defined a locatic partition in which each subset is a locating set. The maximum number of classes in a locatic partition is called the locatic number of $G$ and is denoted by $\mathcal{L}(G)$.

In this paper, we define the fixatic partition and the corresponding fixatic number of any connected graph $G$ of order $n$ as follows:

Let $\Pi = \{F_1, F_2, ..., F_k\}$ be an ordered $k$-partition of $V(G)$. Then $\Pi$ is called a fixatic partition if for all $i; 1 \leq i \leq k$, $F_i$ is a fixing set for $G$. The fixatic number of $G$, denoted by $F_{xt}(G)$, is the maximum number of classes in a fixatic partition. We use $\Pi_t$ to denote the number of possible fixatic partitions having $F_{xt}(G)$ number of classes.

Unless otherwise specified, all the graphs $G$ considered in this paper are simple, connected and symmetric.
This paper is organized as follows. Section 2 provides the study of fixatic number of graphs and join graphs. Some bounds on fixatic number with some certain conditions are also given. Section 3 establishes the connections between the fixing and fixatic number of graphs in the form of realizable results.

2. Fixatic Number of Graphs

We begin this section with an example to explain the concept of this new parameter, that is, we will discuss the technique to find the fixatic partition and the corresponding fixatic number of a connected graph $G$.

**Example 2.1.** (co-rising sun graph)

![Figure 1. The graph with $fix(G) = 1$.](image)

From the graph $G$ of Figure 1, we note that, each vertex of $G$ forms a fixing set except $v_4$, so $F_{xt}(G) = 6$. Total number of fixatic partitions $\Pi$ of $V(G)$ into $F_{xt}$ classes is 6.

Every class in a fixatic partition of $V(G)$ is a fixing set (not necessarily minimum fixing set) for $G$. Let $\Pi = \{F_1, F_2, \ldots, F_k\}$ be a fixatic partition of $V(G)$ and $fix(G)$ is the fixing number of $G$, then $|F_i| \geq fix(G)$ for all $i; 1 \leq i \leq k$. Each $F_i$ in $\Pi$ will be referred to as fixatic class. Note that, if $fix(G) > \left\lfloor \frac{n}{2} \right\rfloor$ for all $i; 1 \leq i \leq k$, then a fixatic partition $\Pi$ of $V(G)$ will have only one class. However, $F_{xt}(G) = 1$, does not imply that $fix(G) \neq \left\lfloor \frac{n}{2} \right\rfloor$. For example, $F_{xt}(K_{1,3}) = 1$ though $fix(G) = 2 = \left\lfloor \frac{n}{2} \right\rfloor$.

Note that, if $F_{xt}(G) = n$, then there exists a unique fixatic partition $\Pi$ of $V(G)$ that contains $n$ disjoint classes each of cardinality one, and hence $|\Gamma_v(G)| = 1$ for all $v \in V(G)$. Again, if $|\Gamma_v(G)| = 1$ for all $v \in V(G)$, then $fix(G) = 1$, and hence $F_{xt}(G) = n$. Thus, we have the following proposition:

**Proposition 2.2.** Let $G$ be a connected graph, then $F_{xt}(G) = n$ if and only if the stabilizer of each of its vertex is trivial.

An automorphism $\beta \in \Gamma_{V(G)-\{u,v\}}(G)$ interchanges two vertices $u$ and $v$ of a graph $G$ if $\beta(u) = v$ and $\beta(v) = u$.

**Lemma 2.3.** Let $u, v \in V(G)$ and $\beta \in \Gamma(G)$ interchanges $u$ and $v$, then $F_{xt}(G) \geq 2$ if and only if there exist two fixatic classes $F_i$ and $F_j$, $i \neq j$ in any fixatic partition $\Pi$ such that $u \in F_i$ and $v \in F_j$. 
Let $\text{Proposition 2.5.}$

$\text{multinomial number } n \text{ into } k \text{ partitions of cycle } C \text{ into } k \text{ classes.}$

| $\text{suppose that, } F \text{ to two disjoint fixatic classes for otherwise if they belong to the same fixatic class say } F_k, \text{ then } F_{xt}(G) \geq 2 \text{ implies that } F_l (l \neq k) \text{ is not a fixing set for } G.$ Conversely, suppose that, $F_{xt}(G) < 2.$ Then there is a unique fixatic partition $\Pi$ that contains all the vertices of $G$, which is contradiction. \hfill \square

We can find the total number of surjections $g$ from an $n$-set to a $k$-set by using the following theorem.

$\text{Theorem 2.4.}[2] \text{ Let } S_r \text{ denotes the set of surjections from an } n \text{-set to a } k \text{-set, then } |S_r| = k! S(n,k), \text{ where } S(n,k) \text{ denotes the number of partitions of an } n \text{-set into } k \text{ classes.}$

The number of surjections from an $n$-set to a $k$-set $\{x_1, x_2, \ldots, x_k\}$, with the property that $n_1$ objects go into the first set $x_1$, $n_2$ go into $x_2$ and so on, is a multinomial number $\binom{n}{n_1, n_2, \ldots, n_k}$. We denote, the number of partitions of an $n$-set into $k$ classes with each class of cardinality $i$ by $S(n,k(i))$.

The following proposition provides the fixatic number and total number of fixatic partitions of cycle $C_n, n \geq 3$ on odd number of vertices by using the Theorem $[2,4]$

$\text{Proposition 2.5.}$

Let $C_n$ be a cycle on $n \geq 3$ odd number of vertices, then $F_{xt}(C_n) = \left\lfloor \frac{n}{2} \right\rfloor$ and $\Pi_t = \binom{n}{(\frac{n}{2}-1)\cdot(\frac{n}{3}) \cdot 2^{\frac{n-3}{(\frac{2}{2})-1)}} \cdot 2\cdot 3^{\frac{n-3}{(\frac{2}{2})-1}}.$

$\text{Proof.}$ Note that $\text{fix}(C_n) = 2$ and all pair of vertices form a fixing set. Hence, $F_{xt}(C_n) = \left\lfloor \frac{n}{2} \right\rfloor$.

Any partition of $V(G)$ of cardinality $\left\lfloor \frac{n}{2} \right\rfloor$ contains $\left\lfloor \frac{n}{2} \right\rfloor - 1$ classes of order two and 1 class of order three. The class having three vertices can be find in $\binom{n}{3}$ different possible ways and $\left\lfloor \frac{n}{2} \right\rfloor - 1$ classes of order two in $S(n-3, (\left\lfloor \frac{n}{2} \right\rfloor - 1)(2))$ different possible ways: the number of partitions of an $(n-3)$-set, say $\{v_1, \ldots, v_i, v_{i+4}, \ldots, v_n\}$ into $\left(\left\lfloor \frac{n}{2} \right\rfloor - 1\right)$ classes each of order 2.

For this, we find the number of surjections from an $(n-3)$-set, $U = \{v_1, \ldots, v_i, v_{i+4}, \ldots, v_n\}$ to a $(\left\lfloor \frac{n}{2} \right\rfloor - 1)$-set $X = \{x_1, x_2, \ldots, x_k\}$ which distributes the vertices of $U$ into the sets of $X$, in such a way that, each set $x_i, 1 \leq i \leq k$ receives exactly two vertices from $U$. Let $S_r$ denotes the set of surjections from an $(n-3)$-set $U$ to a $(\left\lfloor \frac{n}{2} \right\rfloor - 1)$-set $X$, with the property that, two vertices go into the first set $x_1$, two go into $x_2$, and so on, then $|S_r| = \binom{n}{(\frac{n}{2}-1)\cdot 2^{\frac{n-3}{(\frac{2}{2})-1}}}$.

By Theorem $[2,4]$ we have $|S_r| = S(n-3, (\left\lfloor \frac{n}{2} \right\rfloor - 1)(2)) \cdot (\left\lfloor \frac{n}{2} \right\rfloor - 1)!$ which implies that, $S(n-3, (\left\lfloor \frac{n}{2} \right\rfloor - 1)(2)) = \frac{|S_r|}{(\left\lfloor \frac{n}{2} \right\rfloor - 1)!}$. Hence, $\Pi_t = \frac{1}{(\left\lfloor \frac{n}{2} \right\rfloor - 1)!}$ $|S_r|.$\hfill \square
Sharp upper and lower bounds for fixatic number of a connected graph in terms of the fixing and location numbers, are given in the following lemma:

**Lemma 2.6.** Let \( G \) be a connected graph of order \( n \geq 2 \) and \( \mathcal{L}(G) \) be the locatic number of \( G \), then \( \mathcal{L}(G) \leq \text{fix}_t(G) \leq \left\lfloor \frac{n}{\text{fix}(G)} \right\rfloor \). Both bounds are sharp.

**Proof.** For \( \text{fix}_t(G) = k \), \( n = \sum_{i=1}^{k} |F_i| \geq k \text{fix}(G) \) which implies that \( \text{fix}_t(G) \leq \left\lfloor \frac{n}{\text{fix}(G)} \right\rfloor \).

Lower bound follows from the fact that \( \text{fix}(G) \leq \text{loc}(G) \).

For the sharpness of the upper bound, take \( G = P_{2n}, n \geq 1 \), we have a single fixatic partition \( \Pi \) having \( n \) maximum number of classes.

For the sharpness of the lower bound, take \( G = K_n, n \geq 2 \), then \( \text{fix}(G) = n - 1 = \text{loc}(G) \), and hence we have a single fixatic partition \( \Pi \) having one class that consists of all the vertices of a graph \( G \). \( \square \)

Since \( \text{fix}(G) = \text{fix}(\overline{G}) \), so by Lemma 2.6, we have the following corollary:

**Corollary 2.7.** Let \( G \) be a connected graph of order \( n \geq 2 \), then \( 2 \leq \text{fix}_t(G) + \text{fix}_t(\overline{G}) \leq 2n \). Both bounds are sharp.

A vertex \( v \) of a graph \( G \) is called saturated if it is adjacent to all other vertices of \( G \). Let \( e \) be an edge of a connected graph \( G \). If \( G - e \) is disconnected, then \( e \) is called a bridge or a cut-edge of \( G \).

Following result shows, another upper bound for the fixatic number of a connected graph \( G \) by using the definition of a saturated vertex.

**Proposition 2.8.** Let \( G \) be a connected symmetric graph of order \( n \geq 2 \), let \( n' \) be the number of its saturated vertices, then \( \text{fix}_t(G) \leq n - n' + 2 \). This bound is sharp.

**Proof.** Suppose that, \( \text{fix}_t(G) > n - n' + 2 \). From Lemma 2.6 \( n - n' + 2 < \text{fix}_t(G) \leq \left\lfloor \frac{n}{\text{fix}(G)} \right\rfloor \). Note that, \( n - n' + 2 \neq 0 \) because \( n' \leq n \). Thus, \( \text{fix}(G) < \frac{n}{n - n' + 2} \). Now, if \( G \) is connected graph of order \( n \geq 4 \) with two saturated vertices, then \( \text{fix}(G) < 1 \) in other words \( \text{fix}(G) = 0 \), which is contradiction because \( G \) is symmetric.

For the sharpness of this bound, take \( G = K_2 \). Hence, \( \text{fix}_t(G) \leq n - n' + 2 \). \( \square \)

**Proposition 2.9.** Let \( T \) be a twin set of a connected graph \( G \) with \(|T| \geq 3\), then \( \text{fix}_t(G) \leq \text{fix}_t(G - u) \), for all \( u \in T \).

**Proof.** Suppose that, \( u \in T \). Then \( G - u \) is an induced subgraph of \( G \). Since \(|T| \geq 3\), so \( \text{fix}(G - u) < \text{fix}(G) \). Since \( \text{fix}(G) < n \), so \( \frac{n}{\text{fix}(G - u)} > \frac{n}{\text{fix}(G)} \). Thus, by Lemma 2.6 \( \text{fix}_t(G) \leq \text{fix}_t(G - u) \), for all \( u \in T \). \( \square \)

Following this Proposition, we establish the following theorem:

**Theorem 2.10.** Let \( T \) be a twin set of a connected graph \( G \) with \(|T| \geq 3\), then \( \text{fix}_t(G) \leq \text{fix}_t(G - B) \) for every subset \( B \subseteq T \) with \(|B| \leq |T| - 2 \).
Lemma 2.11. Let \( G \) be a connected graph of order \( n \geq 2 \), then \( n \leq \text{fix}(G) + F_{xt}(G) \leq n + 1 \). Both bounds are sharp.

Proof. Whenever \( \text{fix}(G) = 1 \), then \( F_{xt}(G) \leq n \), and hence \( \text{fix}(G) + F_{xt}(G) \leq n + 1 \). Note that \( F_{xt}(G) = 1 \) whenever \( \left\lceil \frac{n}{2} \right\rceil < \text{fix}(G) \leq n - 1 \). Take \( \text{fix}(G) = n - 1 \), then \( n \leq \text{fix}(G) + F_{xt}(G) \). Hence, \( n \leq \text{fix}(G) + F_{xt}(G) \leq n + 1 \).

The upper bound is sharp whenever \( G = P_{2k}, k \geq 1 \) and the lower bound is sharp whenever \( G = K_n, n \geq 3 \). Hence, \( n \leq \text{fix}(G) + F_{xt}(G) \leq n + 1 \). \( \square \)

A fixing vertex is a vertex which forms a fixing set of a connected graph \( G \). Let \( G \) be a connected graph and \( y \in V(G) \) be a fixing vertex. Now, if \( G - y \) is a connected symmetric graph, then \( \text{fix}(G) = \text{fix}(G - y) \), and hence we have the following result:

Proposition 2.12. Let \( G \) be a connected graph of order \( n \geq 3 \), and \( y \in V(G) \) a fixing vertex. Let \( G - y \) be a connected symmetric graph, then \( \text{fix}(G) \geq \text{fix}(G - y) \).

This bound is sharp.

Suppose that \( G_1 = (V_1, E_1) \) and \( G_2 = (V_2, E_2) \) be two graphs with disjoint vertex sets \( V_1 \) and \( V_2 \) and disjoint edge sets \( E_1 \) and \( E_2 \). The union of \( G_1 \) and \( G_2 \) is the graph \( G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2) \). The join of \( G_1 \) and \( G_2 \) is the graph \( G_1 + G_2 \) that consists of \( G_1 \cup G_2 \) and all edges joining all vertices of \( V_1 \) with all vertices of \( V_2 \).

Theorem 2.13. For any two connected graphs \( G_1 \) and \( G_2 \), \( \text{fix}(G_1 + G_2) \geq \text{fix}(G_1) + \text{fix}(G_2) \). This bound is sharp.

Proof. Suppose that, \( \text{fix}(G_1) = n_1 \), \( \text{fix}(G_2) = n_2 \) and \( \text{fix}(G_1 + G_2) = n_3 \). Let \( F_3 = \{v_1, v_2, ..., v_{n_3}\} \) be any fixing set of \( G_1 + G_2 \) of minimum cardinality. Suppose to the contrary, that \( n_3 < n_1 + n_2 \). Without loss of generality, take \( n_3 = n_1 + n_2 - 1 \), that is, for any \( v_i \in F_3, 1 \leq i \leq n_3 \), \( F_4 = F_3 - \{v_i\} \) is a fixing set of \( G_1 + G_2 \).

If \( v_i \) belongs to any fixing set of minimum cardinality of \( G_1 \) (or of \( G_2 \)), then there exists a vertex \( v_j \in V(G_1), j \neq i \) (or \( \in V(G_2) \)) such that \( v_j \in \mathcal{O}(v_i) \), and hence \( F_4 \) is not a fixing set of \( G_1 + G_2 \).

If \( v_i \) does not belong to any fixing set of minimum cardinality of \( G_1 \) and of \( G_2 \), then \( v_i \in V(G_1 + G_2) \) is a saturated vertex. Now, if \( v_i \in V(G_1) \), then there must exists a saturated vertex \( v_j \in V(G_2) \), such that \( \Gamma_{\{v_i, v_j\}}(G_1 + G_2) \) is non trivial. Thus, \( F_4 \) is not a fixing set of \( G_1 + G_2 \), which is contradiction. Hence, \( \text{fix}(G_1 + G_2) > \text{fix}(G_1) + \text{fix}(G_2) \). Sharpness of this bound follows, when we take, \( G_1 = K_2 \) and \( G_2 = C_n, n \geq 4 \). Hence, \( \text{fix}(G_1 + G_2) \geq \text{fix}(G_1) + \text{fix}(G_2) \). \( \square \)

Now, we establish the sharp upper bound of a join graph by using Theorem 2.13.

Theorem 2.14. Let \( G_1 \) and \( G_2 \) be the connected graphs of order \( n_1, n_2 \geq 2 \), respectively and \( G = G_1 + G_2 \) be the join graph of order \( n_3 \geq 5 \), then \( F_{xt}(G) \leq \min\{F_{xt}(G_1), F_{xt}(G_2)\} \). This bound is sharp.
Lemma 3.3. For all integers \( k = F_{xt}(G) > F_{xt}(G_1) \), where \( F_{xt}(G_1) = \min \{ F_{xt}(G_1), F_{xt}(G_2) \} \). Then there will \( k \) classes consisting of fixing sets of \( G \), that is, which will fix vertices of \( G_1 \) as well as \( G_2 \). This means vertices of \( G_1 \) will be partitioned into more classes than \( F_{xt}(G) \) which will \( fix(G) \) but this is contradiction with the definition of \( F_{xt}(G) \). For the sharpness of this bound, take \( G_1 = K_2 \) and \( G_2 = P_{2n}, n \geq 2 \). Hence, \( F_{xt}(G) \leq \min \{ F_{xt}(G_1), F_{xt}(G_2) \} \).

Proposition 2.15. Let \( G_1 = K_1 \) and \( G_2 \) be any connected graph and \( G = G_1 + G_2 \) with \( |G_2| \geq 2 \), then \( F_{xt}(G) \leq F_{xt}(G_2) \). This bound is sharp.

3. Some Realizable results

In the following lemma, we show that every integer \( t \geq 2 \) is realizable as the fixing number and the fixatic number of some connected graphs:

Lemma 3.1. For any integer \( t \geq 2 \), there exists a connected graph \( G \) such that \( fix(G) = t = F_{xt}(G) \).

Proof. Consider \( G \) be a connected graph obtained by a path \( P_2 : uv \) by joining 2 paths \( P_{t-1}^1 : u_i, u_{i,2}, \ldots, u_{i,t-1}, 1 \leq i \leq 2 \) each of order \( t-1 \) with vertex \( u \) and \( t \) paths \( P_{t-1}^t : v_j, v_{j,2}, \ldots, v_{j,t-1}, 1 \leq j \leq t \) each of order \( t-1 \) with vertex \( v \) of a path \( P_2 \) respectively. Let \( A_i = \{ u_{i,1}, u_{i,2}, \ldots, u_{i,t-1} \}, 1 \leq i \leq 2 \) and \( B_j = \{ v_{j,1}, v_{j,2}, \ldots, v_{j,t-1} \}, 1 \leq j \leq t \). Note that, any minimum fixing set \( F \) of \( G \) must contains exactly one vertex from either \( A_1 \) or \( A_2 \) and one vertex from all \( B_j, 1 \leq j \leq t \), except one. This implies that \( fix(G) = t \). We also observe that, the sets \( B_{j} = \{ v_{i,j} \} \), \( 1 \leq j \leq t-1, 1 \leq i \neq j \leq t \) and \( B^* = \{ v_{i,j}, u_{i,2} \}, 1 \leq i = j \leq t-1 \) are the maximum number of fixing sets of minimum cardinality of \( G \). Hence, each fixatic partition \( \Pi \) contains \( t-1 \) classes of cardinality \( t \), and one class of cardinality \( 2t \) because \( t-2 \) vertices \( \{ u_{i,j}, i \neq j \} \) of \( P^* \) and two vertices of \( P_2 \) do not form a fixing set. It follows that \( F_{xt}(G) = t \).

Proposition 3.2. Let \( G \) be a connected graph. If \( fix(G) = F_{xt}(G) \), then no class of any fixatic partition \( \Pi \) of \( V(G) \) can be a singleton set.

Proof. Contrary suppose that, a partition \( \Pi \) contains a singleton class, \( F_j = \{ v \}, v \in V(G) \). Note that, \( v \) is a fixing vertex, and hence \( F_{xt}(G) = 1 \), which implies that \( \Gamma(G) = \{ e \} \), a contradiction.

A rooted tree is a tree with a labeled vertex called a root.

Lemma 3.3. For all integers \( t \geq 2 \), there exists a connected graph \( G \) such that \( fix(G) = t \) and \( F_{xt}(G) = t + 1 \).

Proof. Consider a rooted tree with \( v \) as its root obtained by joining \( t + 1 \) paths \( P_i^t : v_{i,1}, v_{i,2}, \ldots, v_{i,t}, 1 \leq i \leq t + 1 \) each of order \( t \) with the root. Note that, the minimum fixing set for \( G \) contains exactly one vertex from each path \( P_i^t, 1 \leq i \leq t \), which
Lemma 3.4. For any integer $t \geq 3$, there exists a connected graph $G$ such that $fix(G) + F_{xt}(G) = t$.

Proof. Consider a path $P_{t-2}: v_1, v_2, ..., v_{t-2}$. Join a set $\{v'_i, v''_i\}$, $1 \leq i \leq t-2$ of leaves with each $v_i$, $1 \leq i \leq t-2$ of the path $P_{t-2}$. Note that, the graph $G$ has $t-2$ twin sets of vertices, and hence $fix(G) = t-2$. We also observe that, each fixatic partition $\Pi$ contains two maximum number of classes, one class of order $t-2$ and the other class of order $2t-4$. Hence, $F_{xt}(G) = 2$ and the result follows. □

A vertex $v$ of degree at least three in a connected graph $G$ is called a major vertex. Two paths rooted from the same major vertex and having the same length are called the twin stems.

Lemma 3.5. For any integer $t \geq 1$, there exists a connected graph $G$ such that $F_{xt}(G) - fix(G) = t$.

Proof. Consider a rooted tree as $G$ with $v$ as its root obtained by joining $t+1$ paths $P^i_t: v_{i,1}v_{i,2}...v_{i,2t-1}$, $1 \leq i \leq t+1$ each of order $2t-1$ with the root. Since all $t+1$ paths are in fact the twin stems, $fix(G) = t$. Further $A = \{v_{i,j}\}$, $1 \leq i = j \leq t$, $B_j = \{v_{i,j}\}$, $1 \leq j \leq t$, $1 \leq i \neq j \leq t+1$, $C_k = \{v_{i,j}\}$, $1 \leq k \leq t-1$, $j = k+t$, $1 \leq i \leq t$ are the disjoint fixing sets of minimum cardinality. Each fixatic partition $\Pi$ contains $2t$ maximum number of classes in such a way that, $2t-1$ classes are of order $t$ and one class of order $2t$. Hence, $F_{xt}(G) = 2t$. □

Lemma 3.5 can also be stated as follows:

Lemma 3.6. For any integer $t \geq 1$, there exists a connected graph $G$ such that $fix(G) = t$ and $F_{xt}(G) = 2t$.

Consider a path $P_2 = uv$. For all integers $t \geq 3$, construct a connected graph $G$ as follows: If $t$ is even, then join $\frac{t+2}{2}$ leaves with $u$ and $\frac{t+2}{2}$ leaves with $v$ of the path $P_2$. If $t$ is odd, then join $\left\lfloor \frac{t}{2} \right\rfloor + 1$ leaves with $u$ and same number of leaves with vertex $v$.

From this type of construction of $G$, we have a useful result, stated as follows:

Lemma 3.7. For any integer $t \geq 3$, there exists a connected graph $G$ such that $fix(G) - F_{xt}(G) = t$.

Let $G$ be a connected graph. A matching $M$ in a graph $G$ is a set of edges such that no two edges have a common vertex. The size of matching is the number of edges in it. A vertex contained in an edge of $M$ is covered by $M$. A matching that
covers every vertex of the graph $G$ is called a perfect matching [10]. Note that, each element in a matching is an edge which is two vertices subset of the vertex set $V(G)$.

**Proposition 3.8.** Let $G$ be a connected graph of order $n \geq 4$ with $F_{xt}(G) = 2$, then each partite of every fixatic partition $\Pi$ of $V(G)$ induces an edge if and only if $G \in \{C_4, K_{2,2}, K_1 + P_3, K_4 - e\}$, where $K_4 - e$ be the graph obtained by deleting any one edge from $K_4$.

Let $\Psi = (H_n)_{n \geq 1}$ be a family of graphs $H_n$ of order $\varphi(n)$ for which $\lim_{n \to \infty} \varphi(n) = \infty$. If there does not exist a constant number $c > 0$ such that $F_{xt}(H_n) \leq c$ for every $n \geq 1$, then we say that $\Psi$ has unbounded fixatic number.

**Example 3.9.** Let $G = \text{Cay}(Z_n; S)$ be an undirected Cayley graph with vertex set $V(G) = \{v_0, v_1, \ldots, v_{n-1}\}$ and $S$ is its connection set. Let $n = 2p + 1$ is prime. We know that, $\text{fix}(\text{Cay}(Z_n; S)) = 2$ (every fixing set of $\text{Cay}(Z_n; S)$ must contain at least two vertices from $V(G)$). So, to form a fixatic partition of maximum cardinality, we divide the vertex set $V(G)$ into $p - 1$ classes of order two, and one class of order three, otherwise all the classes may not be fixing. So, there will be $(p - 1) + 1$ classes, which form a fixatic partition. Hence, $F_{xt}(\text{Cay}(Z_n; S)) = p$, which implies that $G = \text{Cay}(Z_n; S)$ is a family of graphs with unbounded fixatic number.

A family $\Omega = (G_n)_{n \geq 1}$ of connected graphs is a family with constant fixing number if there exists a constant $0 < c < \infty$ such that $\text{fix}(G_n) = c$, for all $n$. The family of paths $P_n, n \geq 2$ is a family with constant fixing number, because $\text{fix}(P_n) = 1$.

**Theorem 3.10.** A family of connected graphs with constant fixing number is a family of graphs with unbounded fixatic number.

**Proof.** Let $\Omega = (G_n)_{n \geq 1}$ be a family of graphs with constant fixing number $\text{fix}(G_n)$ and fixatic number $F_{xt}(G_n)$, then there exists a constant $c > 0$ such that $\text{fix}(G_n) = c$ for all $n$. In order to form a fixatic partition $\Pi = \{F_1, F_2, \ldots, F_{F_{xt}}\}$ of $V(G_n)$, each $F_j$ has $|F_j| \geq c$. So, there are at least $\left\lceil \frac{n}{c} \right\rceil$ classes in $\Pi$, which implies that there does not exist a constant $c > 0$ such that $F_{xt}(G_n) \leq c$ for all $n$. □

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