ON THE FOUR-COLOR-MAP THEOREM

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Abstract

Coloring planar Feynman diagrams in spinor quantum electrodynamics, is a non trivial model soluble without computer. Four colors are necessary and sufficient.
INTRODUCTION

The representation of fundamental processes in the theory of elementary particles proceeds mainly by means of the so-called Feynman diagrams. This is the case, for instance, for the gauge theories (QED, Weak interactions, QCD), Higgs scalars, and others. These diagrams, as is well-known, are drawn on a plane, on which different kinds of lines represent the various elementary particles entering the interactions. Therefore (see Appendix), they are similar to the maps of different countries which have given rise to the well-known four-color map theorem (K. Appel & W. Haken, 1977) [1].

We shall say that “normal maps”, as defined by Kempe [2], summarize the set of rules in terms of vertices, planarity, boundaries of adjacent countries, number of neighbors and so on. Coloring those countries, demands that two countries with an adjacent border must have different colors. And the fact that four colors are always enough to color any normal map is the Appel-Haken four-color theorem.

In the present paper we shall consider the case of countries that represent a planar self-energy of an electron, in QED with any number of photons emitted and then reabsorbed, forming the photon cloud of the physical electron. This is a spinor quantum electro-dynamics problem. The rules are, for the most important: energy conservation, electric charge conservation and Furry’s theorem (which stems from the properties of Clifford algebra. In other words from the algebra of a Dirac gamma-matrices in four dimensions). The grammar is more restrictive, and therefore simplifies the problem of coloring the countries, the borders of which are both photon and electron lines (resp. wavy and full lines).

In a first section, we shall examine the case when the electron line only emits and reabsorbs photons in a planar way. This fundamental case will be called the “rainbow case”. The various colors will refer to the numbers: 1, 2, 3, 4. It will be shown that such a configuration needs only four colors for the represented countries. Then in a second section, we consider the inclusions of fermion loops: vacuum polarization, scattering of light by light, generally electron–positron boxes with interaction of any even number of photons. It will be shown that those inclusions will not destroy the result of the first section, the rainbow case. We conclude therefore that the four-color map theorem for coloring the regions of an electron self-energy planar diagram in spinor quantum electrodynamics is confirmed without the use of a computer.
Addendum to the Introduction:
As a warm up exercise, the interested reader can try to color diagrams of the planar \( \varphi^4 \)-theory, and find that only two colors are enough in order to color the regions of any diagrams.

1 RAINBOW CASE

We start by dealing with the electron self-energy case in QED planar theory. That is, the electron emits \( n \) photons which are all reabsorbed when reaching the final state. Planarity means that photons-lines do not cross and we are therefore in the so-called ‘rainbow’ case. We shall not enter in this note into the technicalities of rainbow diagrams. Several exhaustive investigations have been made in various cases [3].

We draw an horizontal fermion line (full line) and cross this line vertically by a cut in its middle. If a total of \( n \) ‘photons’ lines (wavy lines), are exchanged from one side of the cut to the other side and if the cut is ‘maximal’ (in other words if all emitted photons cross the cut before being reabsorbed), there will be \( \binom{n}{n-\kappa} \) combinations for the emission of \( n-\kappa \) lines in the upperhalf plane and \( \kappa \) lines in the lower half plane. Similarly, \( \binom{n}{n-\kappa} \) ways of absorbing these photons lines.

We call this the ‘maximal rainbows’ at order \( n \). As it is well known in the theory of Feynman diagrams, there are also rainbow of order \( n \) (\( n \) wavy lines) which are not maximal. They have, along the electron line, between the emission or absorption of two photons lines of the main rainbow, on each side of the cut, some so called ‘subrainbows’ with \( \ell \) photons lines. These subrainbows have the same properties as the maximals but are of the order \( \ell \). To analyse them, if it is not evident to the reader, one can also define subcuts for them and the results follows with \( \ell \) instead of \( n \) lines. The combinatorics, in those case, is trivial and we leave the alert reader to verify it.\(^1\)

The above consideration allow ourselves to formulate the following theorem:

**Theorem:** In the pure rainbow case (no electron loops), for every planar

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\(^1\)The case of a subrainbow connected to another subrainbow only by the electron line and no photon line is not considered. By cutting the electron line between the rainbows one is led to two disconnected diagrams.
configuration, at any $n$, the partition of colors can be the following: colors 1 and 3 for the upper half-plane; 2 and 4 for the lower half plane.

If follows that four colors are always enough to color the rainbow configurations of an electron self-energy. Two adjacent regions will always have borders with different colors on each side. The trick is to start with color 1 upside the most upper photon line. Then alternating with color 3, once the photon boundary of this upper region is crossed. Since photon lines do not cross (planarity), one always ends with colors 1 or 3 along the electron line boundary.

2 THE INSERTIONS OF PLANAR FERMION BOXES IN THE RAINBOW DIAGRAMS

The rainbows diagrams dealt with in Section 1 are subject of corrections due to the insertions of planar electron boxes (ex. vacuum polarization, scattering of light by light, hexagons ...). The electron lines of these boxes are in turn, corrected by rainbows themselves containing boxes and so on until the maximum order is reached, order which can tend to infinity. Finding a rule of coloration for all kind of boxes inserted in a rainbow diagram, say the maximal rainbow at order $s < n$, such that this corrected rainbow is of order $n$, will give the rule for the corrections (photons propagators, boxes of all sizes) to the boxes insertions themselves by recursion.

This rule of coloration stems fundamentally from a property of Clifford algebra in four-dimensions: the algebra of Dirac $\gamma$-matrices. For each electron box, corresponds a Trace of the product of $\gamma$-matrices. It turns out to be identically zero, if the number of $\gamma$’s is odd. Therefore, only boxes with an even number of sides survive. This is all we need in order to show that boxes’ insertions in any simple rainbow diagrams do not need more then four colors. Indeed, suppose we are dealing with a rainbow in the lower half-plane. The boxes having only an even number of neighbors, if we alternate for this rainbow the colors 2 and 4, a single box insertion, with color 1 and inside photon corrections with colors 1 and 3 alternating, will provide, trivially, configurations with no two adjacent neighbors with the same color. The box internal photon rainbows corrections, are of colors 1 and 3, whereas the external photon rainbows corrections to the box will be of colors 2 and 4 (the
same alternating colors as the lower half-plane rainbows). Finally, the insertion of further (or several) box(es) inside the box-corrections themselves, will alternate with the other pair of colors 2 and 4 and so on and so forth\textsuperscript{2}. All the previous properties are due to the main fact that no boxes with odd neighbors will ever appear. This fact spots the crucial difference between the general geographic case of Appel and Haken, and the planar Feynman diagrams for quantum electrodynamics. It explains that the constraints of an additional grammar-rule (no odd neighbors countries to boxes) to the general ‘normal case’ for drawing countries, makes the problem so considerably simpler, that it can be dealt with, without recourse to the computer’s aid. The way used to color any planar spinor Q.E.D. self-energy diagrams uses a systematic alternance of pairs of colors (1, 3) and (2, 4) and provides the rule of coloring we were looking for. That is:

**Theorem**: four colors are sufficient in order to color any planar electron self-energy diagram in spinor quantum electrodynamics.

### 3 CONCLUSIONS AND REMARKS

Why only planar graphs have been considered in this paper is self-evident. Non-planar diagrams have indeed parts in common in the plane and are therefore meaning less for the color-problem we have investigated. The restriction to planar diagrams (here the self-energy) does nevertheless have a link with physical situations. An example is discussed in \cite{4} for a $(\bar{\phi}^2)^2$ field theory with O(N) symmetry. One sees that, in the calculations of the critical exponents, the leading and subleading terms in a $1/N$ expansion are all planar. The same happens when considering a field theory\cite{6}, involving one scalar field an N massless Dirac fermion-fields $\psi^i$ and $\bar{\psi}^i$ coupled via a Yukawa type interaction. And then performing a $1/N$ expansion.

Here, however, our aim was simply to figure out a non trivial example where maps (here Feynman diagrams) need four colors and are sufficient to avoid two countries with the same color on each side of the border which separates them. As emphasized, the grammar for the drawing of countries is more restrictive:

\textsuperscript{2}It is easy to notice the self-similarity of the corrections at all scales, from which is derived the recursive procedure. By all scales, we mean dominant, sub-dominant, sub-sub-dominant and so on (see also Section 3).
– No odd-sized fermion polygons;

– Two kinds of borders: fermions (full lines) and photons (wavy lines);

– All vertices of order three, with two fermion-lines and one photon lines ending at these vertices.

The proof proceeds by recursion, since the same patterns of diagrams, reduced in scale, always appear at a lower level, showing a patent self-similarity. Physically this can be translated in the following way: each particle has a virtual accompanying cloud which surrounds it as it propagates. And each of the virtual particles in the cloud also drags along, its own virtual cloud and so on ad infinitum. Paraphrasing R. Péter [7], recursion is based on the same thing happening on several different levels at once.
Appendix

As we said in the Introduction, in physics (especially in field theory) it is well known that the representation of interactions between elementary particles can be visualized by means of the so-called Feynman diagrams. For a given process, described by Green functions, the usual way of calculating its contribution is to use perturbation expansions in terms of a given interaction density Lagrangian, which characterizes the theory one is dealing with. One of the most popular examples is spinor Quantum Electrodynamics (QED for short) for which the Lagrangian is $eu^*Au$ with $u$ the electron field and $A$ the photon field vector product with Dirac matrices (4-dimensional).

If one wants to draw all conceivable Feynman diagrams corresponding to a given process to the perturbative order $n$, the standard method is the derivation of all these diagrams from a generating functional $W(J_i)$ ($J_i$ being the sources of the various particles)(Ref. [8]). $W(J_i)$ is obtained as the logarithm of the partition functional $Z(J_i)$. This method is described with full details in all quantum field theory textbooks, so we shall not develop it here. Our main interest in this note is coloring planar Feynman diagrams according to Tait’s method of coloring edges of a diagram. This means that at each cubic vertex corresponding to a given process to the perturbative order $n$, the three edges segments starting from this vertex are of different colors, three colors having been fixed once for all (Ref.[9]). The generating functional, when expanded in powers of the coupling ($e$ in the case of QED), will give rise, as already said, to all possible cubic diagrams at each order $n$. But here, we are, for evident reasons, only interested in planar diagrams of which we want to color the facets (countries). So we must first proceed to a filtering and eliminate a plethora of non-planar and improper diagrams which are of no interest for our scope. The residue of this filtering will provide all conceivable diagrams and can be shown to be topologically equivalent to all conceivable geographical cubic maps. By Wick ordering and contractions (contractions give rise to propagators (=edges)), and since at each vertex three different colors meet, all the edges are 3-colored according to Tait’s coloring; and from this, the Petersen conjecture on hamiltonian paths and disjoint even subcircuits (Ref.[10]) is also demonstrated. This conjecture is equivalent to the 4-color problem. This proof has already been done in the case of QED, using straightforward methods. But according to the present
note, is valid for any cubic geographical map. In conclusion, the Feynman diagrams of QED have the crucial advantage, over other methods of coloring maps, that one can always follow the path of the electric charge, which is a conserved quantity. Thus, deleting all wavy lines (photons), one is dealing with the fermionic motion of the charge, which is a hamiltonian circuit or a collection of mutually disjoint even subcircuits. And this is the Petersen condition for the 4-color map theorem to hold.

References

[1] K. Appel and W. Haken, Illinois Journal of Mathematics (1977).
[2] For Kempe’s work, see K. Appel and W. Haken, Scientific American, Oct. 1977, p. 108.
[3] G.’t Hooft, Nucl. Phys. B75 (1974) 461; G.’t Hooft, NATO, Adv. Inst. Series Plenum (1984).
[4] E. Brezin, J. Le Guillou and J. Zinn-Justin in ‘Phase Transitions and Critical Phenomena’, Eds. Domb and Green, Academic Press (1976) p. 127.
[5] S. K. Ma, Ibid., p. 250.
[6] K. G. Wilson, Phys. Rev. D7 (1973) 2911; D. J. Gross and A. Neveu. Phys. Rev. D10 (1974) 3235
[7] R. P´eter ‘Recursive Functions’ Academic Press (1977).
[8] J. Schwinger, Proc. Natl. Acad. Sci. 37 (1951)p. 452.
[9] P.G. Tait, Trans. Roy. Soc. Edinburgh 29 (1880)pp. 657-660.
[10] Petersen, (1891). See also A.S. Calude, “The Journey of the Four Colour Theorem Through Time”. Preprint Dept. of Mathematics, University of Auckland (June 2001).