Invariant minimal surfaces in the real special linear group of degree 2 *

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Dedicated to professor Koichi Ogiue on his 60th birthday

Abstract

Invariant minimal surfaces in the real special linear group $SL_2\mathbb{R}$ with canonical Riemannian and Lorentzian metrics are studied.

Constant mean curvature surfaces with vertically harmonic Gauß map are classified.

Introduction

In our previous works [15]–[16], we have investigated fundamental properties of the real special linear group $SL_2\mathbb{R}$ furnished with canonical left invariant Riemannian metric. It is known that $SL_2\mathbb{R}$ with canonical Riemannian metric admits a structure of naturally reductive homogeneous space and left invariant Sasaki structure. The isometry group of the canonical left invariant metric is 4-dimensional.

On the other hand, it is well known that the Killing form of $SL_2\mathbb{R}$ induces a biinvariant Lorentz metric of constant curvature on $SL_2\mathbb{R}$.

Thus $SL_2\mathbb{R}$ with biinvariant metric is identified with anti de Sitter 3-space $H^3_1$.

As we will see in Section 1, the canonical left invariant Riemannian metric and biinvariant Lorentzian metric (of constant curvature $-1$) belong to same one-parameter family of left invariant semi-Riemannian metrics. Based on this fact, in this paper, we shall give a unified approach to geometry of $H^3_1$ and $SL_2\mathbb{R}$ with canonical metric.

Since the canonical left invariant metric is of non-constant curvature, geometry of surfaces in $SL_2\mathbb{R}$ is complicated.

In fact, we have shown in [6], there are no extrinsic spheres (totally umbilical surfaces with constant mean curvature), especially no totally geodesic surfaces in $SL_2\mathbb{R}$.

In [18], Kokubu introduced the notions of rotational surface and conoid in $SL_2\mathbb{R}$ with canonical left invariant Riemannian metric. Further he classified constant mean curvature rotational surfaces and minimal conoids.

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Gorodski \cite{12} independently investigated constant mean curvature rotational surfaces.

In \cite{6}, Belkhelfa, Dillen and the author gave a characterisation of rotational surfaces with constant mean curvature. More precisely a surface in $\text{SL}_2\mathbb{R}$ is congruent to a rotational surface of constant mean curvature if and only if its second fundamental form is parallel.

In this paper we give some other characterisations of rotational surfaces (of constant mean curvature).

First we show that rotational surfaces in the sense of Kokubu coincide with Hopf cylinders (over curves in the hyperbolic 2-space $H^2$) in the sense of Pinkall \cite{22} and Barros–Ferrández–Lucas–Meroño \cite{4}. Based on this fact, we give a unified viewpoint for \cite{4} and \cite{18}.

Similarly we shall show that conoids in the sense of Kokubu coincide with Hopf cylinders over curves in Lorentz 2-sphere $S^2_1$.

When we identify the Lie algebra $\mathfrak{g}$ of $\text{SL}_2\mathbb{R}$ with (semi) Euclidean 3-space, both $H^2$ and $S^2_1$ are given by adjoint orbits in $\mathfrak{g}$. The adjoint orbits of $\text{SL}_2\mathbb{R}$ in $\mathfrak{g}$ are $H^2$, $S^2_1$ and lightcone $\Lambda$. Based on this fact, we shall introduce a new class of surfaces in $\text{SL}_2\mathbb{R}$. More precisely, in section 4, we shall investigate surfaces in $\text{SL}_2\mathbb{R}$ derived from curves in $\Lambda$.

For every surface in $\text{SL}_2\mathbb{R}$, we associate a map into the Grassmannian bundle $\text{Gr}_2(T\text{SL}_2\mathbb{R})$ of 2-planes—called the Gauß map of the surface. We shall give a characterisation of constant mean curvature rotational surfaces in terms of harmonicity for Gauß maps.

More precisely, in the final section, we shall prove that a constant mean curvature surface in $\text{SL}_2\mathbb{R}$ is congruent to a rotational surface with constant mean curvature if and only if its Gauß map is vertically harmonic.

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1 The special linear group

1.1 Let $G = \text{SL}_2\mathbb{R}$ be the real special linear group of degree 2:

$$\text{SL}_2\mathbb{R} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{R}, \ ad - bc = 1 \right\}.$$ 

By using the Iwasawa decomposition $G = NAK$ of $G$;

$$N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{R} \right\},$$ \hspace{1cm} \text{(Nilpotent part)}

$$A = \left\{ \begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix} \mid y > 0 \right\},$$ \hspace{1cm} \text{(Abelian part)}

$$K = \left\{ \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \mid 0 \leq \theta \leq 2\pi \right\},$$ \hspace{1cm} \text{(Maximal torus)}
we can introduce the following global coordinate system \((x, y, \theta)\) of \(G\):

\[
(x, y, \theta) \mapsto \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & \sqrt{y} \\ 0 & 0 & 1/\sqrt{y} \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}.
\]

We equip on \(G\) the following one-parameter family \(\{g[\nu]\}\) of semi-Riemannian metrics:

\[
g[\nu] = \frac{dx^2 + dy^2}{4y^2} + \nu \left( d\theta + \frac{dx}{2y} \right)^2, \quad \nu \in \mathbb{R}^*.
\]

Every metric \(g[\nu]\) is left invariant. Clearly \(g[\nu]\) is Riemannian for \(\nu > 0\) and Lorentzian for \(\nu < 0\).

Throughout this paper we restrict our attention to \(\nu = \pm 1\) for simplicity.

One can see that \(g[1]\) is only left invariant but \(g[-1]\) is a biinvariant Lorentz metric on \(G\).

We take the following orthonormal coframe field of \((G, g[\nu])\):

\[
\omega^1 = \frac{dx}{2y}, \quad \omega^2 = \frac{dy}{2y}, \quad \omega^3 = d\theta + \frac{dx}{2y}.
\]

The dual frame field of \(\{\omega^1, \omega^2, \omega^3\}\) is given by

\[
e_1 = 2y \frac{\partial}{\partial x} - \frac{\partial}{\partial \theta}, \quad e_2 = 2y \frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial \theta}.
\]

Note that this orthonormal frame field is not left invariant.

The Levi-Civita connection \(\nabla\) of \(g[\nu]\) is given by the following formulae:

\[
\nabla_{e_1} e_1 = 2e_2, \quad \nabla_{e_2} e_1 = -2e_1 - \nu e_3, \quad \nabla_{e_2} e_3 = \nu e_2,
\]

\[
\nabla_{e_2} e_2 = 0, \quad \nabla_{e_2} e_3 = -\nu e_1.
\]

The commutation relations of the basis are given by

\[
[e_1, e_2] = -2e_1 - 2e_3, \quad [e_1, e_3] = 0, \quad [e_2, e_3] = 0.
\]

The Riemannian curvature tensor \(R\) of the metric \(g\) defined by

\[
R(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \quad X, Y, Z \in \mathfrak{X}(G)
\]

is described by the following formulae:

\[
R(\epsilon_1, \epsilon_2)\epsilon_1 = (3\nu + 4)\epsilon_2, \quad R(\epsilon_1, \epsilon_2)\epsilon_2 = -(3\nu + 4)\epsilon_1,
\]

\[
R(\epsilon_1, \epsilon_3)\epsilon_1 = -\nu\epsilon_3, \quad R(\epsilon_1, \epsilon_3)\epsilon_3 = \nu \epsilon_1,
\]

\[
R(\epsilon_2, \epsilon_3)\epsilon_2 = -\nu \epsilon_3, \quad R(\epsilon_2, \epsilon_3)\epsilon_3 = \nu \epsilon_2.
\]

1.2 The one-form \(\eta = -d\theta - dx/(2y)\) is a contact form on \(G\), i.e., \(d\eta \wedge \eta \neq 0\).

Let us define an endomorphism field \(F\) by

\[
F \epsilon_1 = \epsilon_2, \quad F \epsilon_2 = -\epsilon_1, \quad F \epsilon_3 = 0.
\]

And put \(\xi = -\epsilon_3\). Then \((\eta, \xi, F, g[\nu])\) satisfies the following relations:

\[
F^2 = -I + \eta \otimes \xi, \quad d\eta(X, Y) = 2g(X, FY),
\]

3
\[ g(FX, FY) = g(X, Y) - \nu \eta(X)\eta(Y), \]
\[ \nabla_X \xi = -\nu FX, \]
\[ (\nabla_X F)Y = g(X, Y)\xi - \nu \eta(Y)X \]

for all \( X, Y \in \mathfrak{X}(G) \).

These formulae say that the structure \((\xi, F, g[\nu])\) is the associated almost contact structure of the contact manifold \((G, \eta)\) \[15\]. The resulting almost contact manifold \((G; \eta, \xi, F, g[\nu])\) is a homogeneous Sasaki manifold \[24\]. The structure \((\eta, \xi, F, g[\nu])\) is called the canonical Sasaki structure of \(G\). With respect to the canonical Sasaki structure, \((G, g[\nu])\) is a Sasaki manifold of constant holomorphic sectional curvature \(-3\nu + 4\). The vector field \(\xi\) is called the Reeb vector field of \(G\) associated to \(\eta\). In Lorentzian case, since \(\xi\) is a globally defined unit timelike vector field on \(G\), \(\xi\) time-orients \(G\).

**Remark 1.1** The Riemannian curvature tensor \(R\) of \((G, g[\nu])\) is given explicitly by

\[
R(X, Y)Z = -g(Y, Z)X + g(Z, X)Y - (1 + \nu) \left\{ \eta(Z)\eta(X)Y - \eta(Y)\eta(Z)X + g(Z, X)\eta(Y)\xi - g(Y, Z)\xi\eta(X) - g(Y, FZ)FX - g(Z, FX)FY + 2g(X, FY)FZ \right\}
\]

in terms of the canonical Sasaki structure. In particular, this explicit formula says \(g[-1]\) is a Lorentz metric of constant curvature \(-1\). As we will see later \((G, g[-1])\) is identified with the anti de Sitter space \(H^3\).

For more informations on the canonical Sasaki structure of \(G\), we refer to \[15\].

**1.3** The special linear group \(G\) acts transitively and isometrically on the upper half plane:

\[ H^2(1/2) = \left( \{(x, y) \in \mathbb{R}^2 \mid y > 0\}, \frac{dx^2 + dy^2}{4y^2} \right) \]

of constant curvature \(-4\). The isotropy subgroup of \(G\) at \((0, 1)\) is the rotation group \(K = SO(2)\). The natural projection \(\pi : (G, g[\nu]) \to G/K = H^2(1/2)\) is a semi-Riemannian submersion with totally geodesic fibres. Moreover \(\pi\) is given explicitly by

\[ \pi(x, y, \theta) = (x, y) \in H^2(1/2) \]

in terms of the global coordinate system \[1\].

The horizontal distribution of this semi-Riemannian submersion coincides with the contact distribution determined by \(\eta\). The submersion \(\pi : (G, g[-1]) \to H^2(1/2)\) is traditionally called the Hopf fibering of \(H^2(1/2)\). The Sasaki manifold \((G, \eta; \xi, F, g[-1])\) is an example of regular contact spacetime which is *not* globally hyperbolic.

**1.4** Let us denote by \(\mathfrak{g}\) the Lie algebra of \(G\), i.e., the tangent space of \(G\) at the identity matrix \(1\):

\[ \mathfrak{g} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{R}, \ a + d = 0 \right\}. \]
We take the following (split-quaternion) basis of $g$:

\[
i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad j' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad k' = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

Hereafter we identify $g$ with Cartesian 3-space $\mathbb{R}^3$ via the linear isomorphism:

\[
X = x_1 i + x_2 j' + x_3 k' \mapsto (x_1, x_2, x_3).
\]

Equivalently,

\[
X = \begin{pmatrix} -x_3 \\ x_1 + x_2 \\ x_3 \end{pmatrix} \mapsto (x_1, x_2, x_3).
\]

We denote the scalar product on $g$ induced by $g[1]$ and $g[-1]$ by $\langle \cdot, \cdot \rangle^{(\pm)}$ and $\langle \cdot, \cdot \rangle^{(-)}$ respectively.

The scalar products $\langle \cdot, \cdot \rangle^{(\pm)}$ are given explicitly by the following formulae:

\[
\langle X, Y \rangle^{(+)} = \frac{1}{2} \text{tr}(t^T XY), \ X, Y \in g,
\]

\[
\langle X, Y \rangle^{(-)} = \frac{1}{2} \text{tr}(XY), \ X, Y \in g.
\]

For $X \in g$,

\[
\langle X, X \rangle^{(\pm)} = \pm x_1^2 + x_2^2 + x_3^2.
\]

Thus we identify $(g, \langle \cdot, \cdot \rangle^{(+)})$ with Euclidean 3-space:

\[
\mathbb{E}^3 = (\mathbb{R}^3(x_1, x_2, x_3), dx_1^2 + dx_2^2 + dx_3^2).
\]

And $(g, \langle \cdot, \cdot \rangle^{(-)})$ is identified with Minkowski 3-space:

\[
\mathbb{E}^3_1 = (\mathbb{R}^3(x_1, x_2, x_3), -dx_1^2 + dx_2^2 + dx_3^2)
\]

respectively.

Moreover the semi-Euclidean 4-space

\[
\mathbb{E}^4 = (\mathbb{R}^4(x_0, x_1, x_2, x_3), \ -dx_0^2 - dx_1^2 + dx_2^2 + dx_3^2)
\]

is identified with the space $M_2 \mathbb{R}$ of all real 2 by 2 matrices:

\[
M_2 \mathbb{R} = \{ x_0 1 + x_1 i + x_2 j' + x_3 k' \}.
\]

The semi-Euclidean metric of $\mathbb{E}^4$ corresponds to the scalar product

\[
\langle X, Y \rangle = \frac{1}{2} \{ \text{tr}(XY) - \text{tr}(X)\text{tr}(Y) \}, \ X, Y \in M_2 \mathbb{R}.
\]

Since $\langle X, X \rangle = -\det X$ for all $X \in M_2 \mathbb{R}$, the special linear group $G$ with biinvariant Lorentz metric $g[-1]$ is identified with anti de Sitter 3-space:

\[
H_3^1 = \{ (x_0, x_1, x_2, x_3) \in \mathbb{E}^4 | \ -x_0^2 - x_1^2 + x_2^2 + x_3^2 = -1 \}.
\]
1.5 The Lie group $G$ acts on $\mathfrak{g}$ by the Ad-action:

$$\text{Ad} : G \times \mathfrak{g} \to \mathfrak{g}; \quad \text{Ad}(a)X = aXa^{-1}, \quad a \in G, \quad X \in \mathfrak{g}.$$ 

Since the determinant function $\det$ is Ad-invariant, the Ad-orbits in $\mathfrak{g}$ are parametrised in the following way:

$$\mathcal{O}_c = \{X \in \mathfrak{g} \mid \det X = c \}, \quad c \in \mathbb{R}.$$ 

For $c \geq 0$, put

$$\mathcal{O}_c^\pm = \{(x_1, x_2, x_3) \in \mathcal{O}_c \mid \pm x_1 > 0\}.$$ 

Then

$$\mathcal{O}_c = \mathcal{O}_c^+ \cup \mathcal{O}_c^-, \quad c > 0,$$

$$\mathcal{O}_0 = \mathcal{O}_0^+ \cup \{0\} \cup \mathcal{O}_0^- \quad \text{or} \quad \mathcal{O}_c, \quad c < 0.$$ 

**Proposition 1.1** The Ad-orbits of $G$ are $\mathcal{O}_c^\pm$, $(c > 0)$, $\mathcal{O}_0^\pm$, $(c = 0)$, $\{0\}$, or $\mathcal{O}_c$, $(c < 0)$.

With respect to the Lorentz scalar product $\langle \cdot, \cdot \rangle$, the non-trivial Ad-orbit $\mathcal{O}_c$ are classified as follows:

1. $c < 0$: The Ad-orbit $\mathcal{O}_c$ is the pseudo-2-sphere $S^2_1(\sqrt{-c})$ of radius $\sqrt{-c}$. In this case $\mathcal{O}_c = G/AZ^2$.

2. $c > 0$: The Ad-orbit $\mathcal{O}_c^\pm$ is the upper or lower imbedding of hyperbolic 2-space $H^2(\sqrt{c})$ with radius $\sqrt{c}$ in $E^3_1$. In this case $\mathcal{O}_c^\pm = G/K$.

3. $c = 0$: The Ad-orbit $\mathcal{O}_0^\pm$ is the future or past lightcone:

$$\Lambda_\pm = \{(x_1, x_2, x_3) \neq 0 \mid -x_1^2 + x_2^2 + x_3^2 = 0, \pm x_1 > 0\}.$$ 

The future lightcone $\Lambda_+$ is represented as $\Lambda_+ = G/N\mathbb{Z}_2$.

1.6 The Riemannian metric $g[1]$ is not only $G$-left invariant but also right $K$-invariant. Thus the product group $G \times K$ acts isometrically on $(G, g[1])$. Note that $(G, g[1])$ is represented by $(G \times K/K, g[1])$ as a naturally reductive (Riemannian) homogeneous space (See [25]).

On the other hand, since $g[-1]$ is bilinvariant, $G \times G$ acts isometrically on $(G, g[1])$. Moreover $(G, g[-1])$ is represented by $(G \times G/G, g[-1])$ as a Lorentzian symmetric space.

Hence every subgroup of $G \times K$ acts isometrically on both $(G, g[\nu])$. Kokubu introduced the notion of helicoidal motion for $(G, g[1])$. This notion can be naturally extended for $(G, g[\nu])$.

**Definition 1.1** Let $\{\sigma^\mu_t\}_{t \in \mathbb{R}}$ be a one parameter subgroup of $G \times K$ defined by

$$(6) \quad \sigma^\mu_t(X) = \begin{pmatrix} 1 & \mu t \\ 0 & 1 \end{pmatrix} X \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}, \quad \mu \in \mathbb{R}.$$ 

An element of $\{\sigma^\mu_t\}_{t \in \mathbb{R}}$ is called a helicoidal motion with pitch $\mu$.

Kokubu called surfaces in $(G, g[1])$ which are invariant under some helicoidal motion group $\{\sigma^\mu_t\}$ helicoidal surfaces.
2 Hopf cylinders

3.1 We recall two classes of surfaces in \((G, g[1])\) studied by Kokubu.

Definition 2.1 ([18]) An immersed surface in \(G\) is said to be a rotational surface if it is invariant under the right \(K\)-action.

A rotational surface can be parametrised as
\[
\varphi(u, v) = \begin{pmatrix} 1 & x(v) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y(v)} & 0 \\ 0 & 1/\sqrt{y(v)} \end{pmatrix} \begin{pmatrix} \cos u & \sin u \\ -\sin u & \cos u \end{pmatrix}.
\]

(7)

Obviously this definition is also valid for \(H^3_1\).

Next we recall the notion of Hopf cylinder introduced by Pinkall [22].

Let \(\pi: S^3 \rightarrow S^2(1/2)\) be the Hopf fibering of \(S^2(1/2)\). Take a curve \(\bar{\gamma}\) in the base space \(S^2(1/2)\). Then the inverse image \(M := \pi^{-1}\{\bar{\gamma}\}\) is a flat surface in \(S^3\) which is called a Hopf cylinder over \(\bar{\gamma}\) [22]. This construction is valid for other Hopf fiberings: \(H^3_1 \rightarrow H^2(1/2)\) and \(H^3_1 \rightarrow S^2(1/2)\).

In particular Hopf cylinders in \(H^3_1\) over curves in \(H^2(1/2)\) are timelike. Barros, Ferrández, Lucas and Meroño [4], [5], [10] developed detailed studies on Hopf cylinders in \(H^3_1\). It is easy to see that the notion of Hopf cylinder can be extended naturally to the fibering:
\[
\pi: (G, g[\nu]) \rightarrow H^2(1/2).
\]

By using \(\text{SL}_2\mathbb{R}\)-model of \(H^3_1\) and the coordinate system (1), we can see that Hopf cylinders over curves in \(H^2\) are nothing but surfaces in \(G\) invariant under the right action of \(K\).

Proposition 2.1 Let \(M\) be a surface in \((G, g[\nu])\). Then \(M\) is a Hopf cylinder over a curve in \(H^2(1/2)\) if and only if it is a rotational surface.

Thus we can unify two theories of “Hopf cylinders in \(H^3_1\)” and of “rotational surfaces in \((G, g[1])\)”.

Proposition 2.2 Let \(\varphi: I \times S^1 \rightarrow (G, g[\nu])\) be a Hopf cylinder over a curve \((x(v), y(v))\) in \(H^2(1/2)\) parametrised by arclength parameter \(v\). Then the induced metric of \(\varphi\) is
\[
I[\nu] = \nu \left( du + \frac{x'(v)}{2y} dv \right)^2 + dv^2.
\]

(8)

Hence the Hopf cylinder \((I \times S^1, \varphi)\) is flat.

Hopf cylinders of constant mean curvature are classified as follows (Compare [4–5] and Proposition 4.3 in [18]):

Proposition 2.3 (Classification of CMC Hopf cylinders)

Let \(c\) be a unit speed curve in \(H^2(1/2)\) with curvature \(\kappa\) and \(M_c\) the Hopf cylinder over \(c\) in \((G, g[\nu])\). Then \(M_c\) is of constant mean curvature if and only if \(c\) is a Riemannian circle in \(H^2(1/2)\). The mean curvature of \(M_c\) is \(H = \kappa/2\).
The Hopf cylinder $M_{c}$ is classified in the following way.

1. $M_{c}$ is a minimal complex circle if $\kappa = 0$,
2. $M_{c}$ is a non-minimal complex circle or a Hopf cylinder over a line segment $y = \pm (\sqrt{1 - 4\kappa^{2}/(2\kappa)} x)$ if $0 < \kappa^{2} < 4$,
3. $M_{c}$ is a Hopf cylinder over a horocycle or $y =$ constant if $\kappa^{2} = 4$,
4. $M_{c}$ is an embedded torus if $\kappa^{2} > 4$.

Note that, in $H_{1}^{3}$ case, $M_{c}$ is a $B$-scroll of the horizontal lift $\hat{c}$ of $c$. (See [8], [4]. Compare with Theorem 3.2).

**Remark 2.1** The notion of complex circle is introduced by Magid. (See [19], Example 1.12.) The non-minimal complex circle is an isometric immersion $\varphi : E_{1}^{2}(u,v) \rightarrow H_{1}^{3}$ of Minkowski plane into $H_{1}^{3}$ defined by

$$\varphi(u,v) = \begin{pmatrix} b \cosh v \cos u - a \sinh v \sin u \\ a \sinh v \cos u + b \cosh v \sin u \\ a \cosh v \cos u + b \sinh v \sin u \\ a \cosh v \sin u - b \sinh v \cos u \end{pmatrix},$$

where $a^{2} - b^{2} = -1$, $ab \neq 0$. The non-minimal complex circle $\varphi$ is a non-minimal flat timelike surface in $H_{1}^{3}$. (cf. Alías, Ferrández and Lucas [1], Example 3.3.)

If we interchange $+$ and $-$ in the third and fourth components of $\varphi$, then we obtain a timelike minimal surface in $H_{1}^{3}$. This timelike minimal surface has the following expression $\exp(u \hat{a}) \exp(v \hat{k}) \exp(t \hat{f})$. Here we put $b = \cosh t$ and $a = \sinh t$.

**Remark 2.2** It is straightforward to check that every rotational surface of constant mean curvature in $(G, g[1])$ has parallel second fundamental form (especially constant principal curvatures). Conversely, one can see that surfaces with parallel second fundamental form in $(G, g[1])$ are congruent to rotational surfaces of constant mean curvature. See [6]. Since rotational surfaces of constant mean curvature are not totally umbilical, there are no extrinsic spheres (totally umbilical surfaces with constant mean curvature) in $(G, g[1])$.

On the other hand, timelike isometric immersion of $E_{1}^{2}$ into $H_{1}^{3}$ with parallel second fundamental form are classified in p. 93, Corollary in [8]. See also [19].

**Remark 2.3** Let $c(t) = (x(t), y(t))$ be a curve in $H^{2}(1/2)$ parametrised by the arclength parameter $t$ and $M$ the Hopf cylinder over $c$. Then it is easy to see that $\xi$ is tangent to $M$. Moreover the horizontal lift $c'(t)^{\ast}$ of the tangent vector field $c'(t)$ of $c$ to $G$ also tangents to $M$. The tangent space of $M$ at $(x(t), y(t), \theta)$ is spanned by $c'(t)^{\ast}$ and $\xi$. Denote by $\mathcal{D}$ the distribution spanned by $c^{\ast}(t)$ and put $\mathcal{D} = \{0\}$. Then we have

$$TM = \mathcal{D} \oplus \mathcal{D}^{\perp} \oplus \langle \xi \rangle, \quad F(\mathcal{D}) \subset \mathcal{D}, \quad F(\mathcal{D}^{\perp}) = T^{\perp}M.$$ 

Here $\langle \xi \rangle$ is the distribution spanned by $\xi$. Thus the Hopf cylinder $M$ is an anti invariant submanifold of $G$ in the sense of [27].

3.2 Next we shall recall the notion of conoid introduced by Kokubu.
Definition 2.2 ([18]) An immersed surface in \((G, g[1])\) of the form:

\[
\varphi(u, v) = \begin{pmatrix}
1 & x(u) \\
0 & 1
\end{pmatrix} \begin{pmatrix}
\sqrt{v} & 0 \\
0 & 1/\sqrt{v}
\end{pmatrix} \begin{pmatrix}
\cos u & \sin u \\
-\sin u & \cos u
\end{pmatrix}
\]

is called a conoid in \(G\).

If we use the metric \(g[-1]\), then \((x, \theta) = (x(u), u)\) is a curve in the double covering manifold \(\tilde{S}^2_1\) of \(S^2_1\). Hence conoids in \((G, g[-1])\) may be regarded as Hopf cylinders over curves in \(S^2_1\).

Proposition 2.5 ([19]) The only (complete) minimal conoids in \((G, g[1])\) are helicoidal surfaces:

\[
\varphi(u, v) = \begin{pmatrix}
1 & \mu u + a \\
0 & 1
\end{pmatrix} \begin{pmatrix}
\sqrt{v} & 0 \\
0 & 1/\sqrt{v}
\end{pmatrix} \begin{pmatrix}
\cos u & \sin u \\
-\sin u & \cos u
\end{pmatrix}
\]

\[
= \sigma_{\mu}^a \left( \begin{pmatrix}
1 & a \\
0 & 1
\end{pmatrix} \begin{pmatrix}
\sqrt{v} & 0 \\
0 & 1/\sqrt{v}
\end{pmatrix} \right).
\]

Namely these minimal conoids are \(\{\sigma_{\mu}^a\}\)-orbits of a line \(\{(a, y, 0) \in H^2 \times S^1 \mid y > 0\}\). In particular \(\varphi\) is an imbedding.

The results in this section motivate us to study the class of surfaces which will be introduced in the next section.

3 Surfaces derived from curves in the lightcone

In this section, we shall introduce a new class of surfaces in \(G\). As we saw before, Ad-orbits of vectors in \(\mathfrak{g} = \mathfrak{sl}_2\mathbb{R}\) are classified in three types. The Ad-orbit of a spacelike [resp. timelike] vector is a hyperbolic 2-space [resp. Lorentz sphere]. The Ad-orbit of a null vector is the lightcone. In the preceding section, we saw that two kinds of surfaces, “rotational surfaces” and “conoids” coincide Hopf cylinders over curves in hyperbolic 2-space or Lorentz sphere. It seems to be
interesting to study surfaces obtained by curves in Ad-orbit of a null vector, i.e., the lightcone. This section is devoted to study such surfaces.

Let \( c \) be a curve in lightcone \( \Lambda \). Then its inverse image \( M \) in \( H^3_1 = (G, g[-1]) \) is given by

\[
\varphi(u, v) = \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y(u)} & 0 \\ 0 & 1/\sqrt{y(u)} \end{pmatrix} \begin{pmatrix} \cos u & \sin u \\ -\sin u & \cos u \end{pmatrix}.
\]

The partial derivatives of \( \varphi \) are

\[
\varphi_* \frac{\partial}{\partial u} = \frac{y'}{2y} \epsilon_2 + \epsilon_3, \quad \varphi_* \frac{\partial}{\partial v} = \frac{1}{2y} (\epsilon_1 + \epsilon_3).
\]

The induced metric \( I[\nu] \) of \( M \) is

\[
I[\nu] = \left\{ \nu + \left( \frac{y'(u)}{2y(u)} \right)^2 \right\} du^2 + \frac{\nu}{y(u)} dudv + \frac{1 + \nu}{4y(u)^2} dv^2.
\]

The determinant of \( I[\nu] \) is

\[
\det I[\nu] = \frac{1}{16y(u)^2} \left\{ \left( 1 + \nu \right)y'(u)^2 + 4\nu y(u)^2 \right\}.
\]

In particular \( \det I[-1] = -1/(4y^2) \), hence \( (M, \varphi) \) is timelike in \( H^3_1 \). Direct computations using (3) show that

\[
\nabla_{\partial_u} \varphi_* \frac{\partial}{\partial u} = -\nu \left( \frac{y'}{y} \right) \epsilon_1 + \left( \frac{y'}{2y} \right)' \epsilon_2,
\]

\[
\nabla_{\partial_v} \varphi_* \frac{\partial}{\partial v} = \frac{1}{4y^2} \left\{ -y'(\nu + 2) \epsilon_1 + 2\nu ye_2 - y' \epsilon_3 \right\},
\]

\[
\nabla_{\partial_v} \varphi_* \frac{\partial}{\partial v} = \frac{\nu + 1}{2y^2} \epsilon_2.
\]

The unit normal vector field \( n[\nu] \) is

\[
n[\nu] = \frac{1}{\sqrt{1 + (1 + \nu) \left( \frac{y'}{2y} \right)^2}} \left( \frac{y'}{2y} \epsilon_1 + \epsilon_2 - \frac{\nu y'}{2y} \epsilon_3 \right).
\]

Let us denote by \( \Pi = \Pi[\nu] \) the second fundamental form derived form \( n[\nu] \).

The second fundamental form \( \Pi \) is defined by the Gauß formula:

\[
\nabla_X \varphi_* Y = \varphi_* (\nabla^M_X Y) + \Pi(X,Y) n, \quad X, Y \in \mathfrak{X}(M).
\]

Here \( \nabla^M \) is the Levi-Civita connection of \( (M, I[\nu]) \).

Put \( \alpha = \sqrt{1 + (1 + \nu) \left( \frac{y'}{2y} \right)^2} \). Then \( \det I[\nu] = \nu \alpha^2/(4y^2) \).

The second fundamental form \( \Pi \) is described by the following formulae:

\[
\Pi \left( \frac{\partial}{\partial u}, \frac{\partial}{\partial u} \right) = -\frac{(1 + \nu)y'(u)^2 + y''(u)y(u)}{2\alpha y(u)^2},
\]

\[
\Pi \left( \frac{\partial}{\partial u}, \frac{\partial}{\partial v} \right) = -\frac{(1 + \nu)y'(u)^2 + 4\nu y(u)^2}{8\alpha y(u)^3},
\]

\[
\Pi \left( \frac{\partial}{\partial v}, \frac{\partial}{\partial v} \right) = 0.
\]
\[ \mathbb{I} \left( \frac{\partial}{\partial v} \frac{\partial}{\partial v} \right) = \frac{(1 + \nu)}{2\alpha y(u)^2}. \]

The mean curvature \( H[\nu] \) of \( \varphi \) is

\[ H[\nu] = \frac{1}{4\alpha^3 y(u)^2} \left\{ (1 + \nu) y''(u)y(u) + 4y(u)^2 \right\}. \]  

(11)

Here we used the formula:

\[ H[\nu] = \frac{1}{2} \text{tr}\{\mathbb{II}[\nu] \cdot \mathbb{I}[\nu]^{-1}\}. \]

Case 1: \( \nu = 1 \)

From (11), we have \( \varphi \) is minimal if and only if

\[ y'' = -2y. \]

Theorem 3.1 Let \( \varphi(u,v) \) an immersed surface in \((G,g[1])\) obtained by taking inverse image of a curve in \( \Lambda \) which is parametrised as (11). Then \( \varphi \) is minimal if and only if \( \varphi \) is the inverse image of

\[ (A\cos(\sqrt{2}u) + B\sin(\sqrt{2}u), u) \in \mathbb{R}^+ \times S^1. \]

Case 2: \( \nu = -1 \)

On the other hand, in \((G,g[-1])\), \( \varphi \) has constant mean curvature 1 and Gaussian curvature 0. Denote by \( D \) the discriminant of the characteristic equation:

\[ \det(tI - S) = 0 \]

for the shape operator \( S = \mathbb{II} \cdot \mathbb{I}^{-1} \). Then \( D \) is given by the following formula:

\[ D = H^2 - K - 1. \]

Thus \((M,\varphi)\) has real and repeated principal curvatures in \( H^3_1 \). Hence \( \varphi \) is a \( B \)-scroll of a null Frenet curve with constant torsion 1 in \( H^3_1 \). In particular \( M \) is flat totally umbilical timelike surface if and only if it is a \( B \)-scroll of a null geodesic with constant torsion 1. (See Theorem 3 in [8].)

Comparing the first and second fundamental forms we have the following

Proposition 3.1 Let \( \varphi(u,v) \) an immersed surface in \( H^3_1 \) obtained by taking inverse image of a curve in \( \Lambda \) which is parametrised as (11). Then \( \varphi \) is totally umbilical if and only if \( y \) is a solution to

\[ y'' - \frac{(y')^2}{2y} + 2y = 0. \]

(12)

The ordinary differential equation (12) with \( y > 0 \) can be solved explicitly. In fact let us introduce an auxiliary function \( \mathcal{I} \) by

\[ \mathcal{I}(u) := \frac{d}{du} \log y(u). \]

Then (12) is rewritten as

\[ \mathcal{I}' + \frac{1}{2} \mathcal{I}^2 + 2 = 0. \]
The general solutions of this ordinary equation are given explicitly by

\[ T(u) = -2 \tan(u + u_0), \quad u_0 \in \mathbb{R}. \]

Thus the solutions \( y \) to (12) are given by

\[ y(u) = A \cos^2(u + u_0), \quad A > 0. \]

**Theorem 3.2** Let \( \varphi(u, v) \) an immersed surface in \( H_1^3 \) obtained by taking inverse image of a curve in \( \Lambda \). Then \( \varphi \) is a B-scroll of a null Frenet curve with constant torsion 1 in \( H_1^3 \). In particular \( \varphi \) is totally umbilical if and only if \( \varphi \) is the inverse image of the curve

\[ (A \cos^2(u + u_0), u) \in \mathbb{R}^+ \times S^1, \quad A > 0. \]

**Remark 3.1** (Weierstrass-type representations for surfaces in \( H_1^3 \))

1. Hong [13] obtained a Bryant-type representation formula for timelike constant mean curvature 1 surfaces in \( H_1^3 \).
2. Balan and Dorfmeister [3] established a loop group theoretic Weierstrass-type representation (so-called DPW representation) for harmonic maps of Riemann surface into general Lie group with bi-invariant semi-Riemannian metric. Their general scheme is applicable to maximal (spacelike) surfaces in \( H_1^3 = (\text{SL}_2 \mathbb{R}, g[-1]) \).

**Remark 3.2** Hopf cylinders over curves in \( H^2 \) [resp. \( S^2 \)] are surfaces in \( G \) which are invariant under \( K \)-action [resp. \( AZ_2 \)-action]. Surfaces considered in this section are invariant under \( N \)-action. Thus all the surfaces investigated in preceding section and present section are invariant under 1-dimensional closed subgroup of the isometry group \( G \times K \). In [11], Figueroa, Mercuri and Pedrosa classified all constant mean curvature surfaces in the Heisenberg group which are invariant under 1-dimensional closed subgroups of the isometry group. Some results in [11] are independently obtained in [14]. Recently S. D. Pauls studied minimal surfaces in the Heisenberg group with Carnot-Carathéodory metric [21].

### 4 Tangential Gauß maps

**5.1** Let \( (N^n, g_N) \) be a Riemannian \( n \)-manifold and \( O(N) \) the orthonormal frame bundle of \( N \). As is well known, \( O(N) \) is a principal \( O(n) \)-bundle over \( N \).

Denote by \( \text{Gr}_\ell(T_p N) \) be the Grassmannian manifold of \( \ell \)-planes in the tangent space \( T_p N \) of \( N \) at \( p \in N \). The set \( \text{Gr}_\ell(TN) := \cup_{p \in N} \text{Gr}_\ell(T_p N) \) of all \( \ell \)-planes in the tangent bundle \( TN \) admits a structure of fibre bundle over \( N \). In fact, \( \text{Gr}_\ell(TN) \) is a fibre bundle associated to \( O(N) \):

\[ \text{Gr}_\ell(TN) = O(N) \times_{O(n)} \text{Gr}_\ell(\mathbb{E}^n) \]

whose standard fibre is the Grassmannian manifold \( \text{Gr}_\ell(\mathbb{E}^n) \) of \( \ell \)-planes in Euclidean \( n \)-space. This fibre bundle \( \text{Gr}_\ell(TN) \) is called the Grassmannian bundle of \( \ell \)-planes over \( N \).

The canonical 1-form of \( O(N) \) and the Levi-Civita connection 1-forms of \( g_N \) naturally induces an invariant Riemannian metric \( \langle \cdot, \cdot \rangle \) on \( \text{Gr}_\ell(TN) \). with
respect to this metric the projection \( pr : Gr_\ell(TN) \to N \) becomes a Riemannian submersion with totally geodesic fibres. For more details about the metric, see Jensen and Rigoli \cite{17} and Sanini \cite{23}.

**Definition 4.1** Let \( \varphi : M^m \to N^n \) be an immersed submanifold. Then the (tangential) Gauß map \( \psi : M \to Gr_m(TN) \) is defined by

\[
\psi(p) := \varphi_* p(T_p M) \in Gr_m(T_p N), \quad p \in M.
\]

**Remark 4.1** In case the ambient Riemannian \( n \)-manifold \( N \) is a Lie group with left invariant metric and \( M \) is a hypersurface, we can introduce another kind of Gauß map.

Let \( G \) be an \( n \)-dimensional Lie group with left invariant metric. For an immersed hypersurface \( \varphi : M \to G \) with unit normal \( n \), the **normal Gauß map** \( \Upsilon \) of \( M \) is a smooth map into the unit \((n - 1)\)-sphere in the Lie algebra \( g \) of \( G \) defined by

\[
\Upsilon(p) := L^{-1}_{\varphi(p)} n_p \in S^{n-1} \subset g.
\]

In our study for surfaces in \((G,g[1])\), to distinguish the Gauß maps into the \( Gr_2(TG) \) from the normal Gauß maps, we use the name “tangential Gauß maps” for the Gauß maps defined in Definition 4.1.

### 5.2 Here we recall and collect fundamental ingredients in the theory of harmonic maps from the lecture note \cite{9} by Eells and Lemaire.

Let \((M,g_M)\) and \((P,g_P)\) be Riemannian manifolds. And let \( f : M \to P \) be a smooth map of a manifold \( M \) into \( P \). The energy density \( e(f) \) of \( f \) is a smooth function on \( M \) defined by

\[
e(f) := \frac{|df|^2}{2}.
\]

It is obvious that \( e(f) = 0 \) if and only if \( f \) is constant.

The energy \( E(f) \) of \( f \) is

\[
E(f) := \int_M e(f) \, dV_M.
\]

Here \( dV_M \) is the volume element of \((M,g_M)\).

The tension field \( \tau(f) \) of \( f \) is a smooth section of \( f^*(TP) \) defined by

\[
\tau(f) := \text{tr} \nabla df.
\]

It is known that \( f \) is a critical point of the energy if and only if \( \tau(f) = 0 \).

A map \( f \) is said to be a **harmonic map** if \( \tau(f) = 0 \).

Baird and Eells introduced the notion of stress-energy tensor in \cite{4}. The stress-energy tensor \( S(f) \) of a map \( f \) is a symmetric \((0,2)\)-tensor field on \( M \) defined by

\[
S(f) := e(f) g_M - f^* g_P.
\]

In particular in case \( \dim M = 2 \) and \( f \) is nonconstant, \( f \) is conformal if and only if \( S(f) = 0 \).

Since \( S(f) \) is symmetric \((0,2)\)-tensor field, the divergence \( \text{div} \, S(f) \) of \( S(f) \) can be defined by the formula:

\[
\text{div} \, S(f) := C_{13}(\nabla S(f)).
\]
Here $C_{13}$ is the metric contraction operator in the 1st and 3rd entries. See p. 83 in [20]. The divergence of $S(f)$ is given explicitly by [2],

$$\text{div } S(f) = -g_P(\tau(f), df).$$

Thus if $f$ is a harmonic map then its stress-energy tensor is conservative.

5.3 Next we recall the notion of vertically harmonic map [20].

Let $(P, g_P)$ be a Riemannian manifold and $pr : (P, g_P) \to (N, g_N)$ a Riemannian submersion. With respect to the metric $g_P$, the tangent bundle $TP$ of $P$ is decomposed as:

$$T_uP = H_u \oplus V_u, \ u \in P.$$  

Here $V_u := \ker (pr_*)_u$ and $H_u = V_u^\bot$ are called the vertical subspace and horizontal subspace of $T_uP$ at $u$ respectively.

Now let $f : (M, g_M) \to (P, g_P)$ be a smooth map. With respect to the Riemannian submersion $pr$, $\tau(f)$ is decomposed into its horizontal and vertical components:

$$\tau(f) = \tau^H(f) + \tau^V(f).$$

The map $f$ is said to be a vertically harmonic map if the vertical component $\tau^V(f)$ vanishes.

In case $f : M = N \to P$ is a section of $P$, i.e., a smooth map satisfying $pr \circ f = \text{identity}$, C. M. Wood [20] showed that the vertical harmonicity for maps is equivalent to the criticality for the vertical energy under the vertical variations.

5.4 Now we investigate harmonicity of tangential Gauß maps for surfaces in $(G, g[1])$.

The following fundamental result is due to Sanini (See (3.2)-(3.3) in [23]).

**Lemma 4.1** Let $N$ be a Riemannian 3-manifold and $\varphi : M \to N$ an immersed surface with unit normal vector field $n$. Take a principal frame field $\{e_1, e_2, e_3 = n\}$, i.e., an orthonormal frame field such that $\{e_1, e_2\}$ diagonalise the shape operator. Put

$$R_{ijkl} = g_N(R(e_i, e_j) e_k, e_l)$$

and denote by $\psi$ the tangential Gauß map of $(M, \varphi)$. Then the following holds.

1. The tangential Gauß map $\psi$ is conformal if and only if $(M, \varphi)$ is totally umbilical or minimal.

2. Assume that $(M, \varphi)$ has constant mean curvature. Then $\psi$ is vertically harmonic if and only if $R_{1213} = R_{2123} = 0$. Moreover when $(M, \varphi)$ is minimal, $\psi$ is harmonic if and only if, in addition, $R_{3113} = R_{3223} = 0$.

3. Assume that the mean curvature is nonzero constant. Then the tangential Gauß map $\psi$ is vertically harmonic if and only if the stress energy tensor $S(\psi)$ of the tangential Gauß map $\psi$ is conservative (divergence free).

Sanini applied this Lemma to surfaces in 3-dimensional Heisenberg group with canonical left invariant metric [23].

Lemma 4.1 together with the nonexistence of extrinsic spheres (See Remark 2.2 and [6]) implies the following.
Corollary 4.1 Let $M$ be a constant mean curvature surface in $(G, g[1])$. Then $M$ is minimal if and only if its tangential Gauß map is conformal.

The following is the main result of this section.

Theorem 4.1 Let $M$ be a surface in $(G, g[1])$ with constant mean curvature. Then the tangential Gauß map of $M$ is vertically harmonic if and only if $M$ is a Hopf cylinder (rotational surface) of constant mean curvature. Hopf cylinders with nonzero constant mean curvature are (only) constant mean curvature surfaces whose tangential Gauß map are vertically harmonic but nonharmonic and have conservative stress-energies.

In particular the only minimal surface in $(G, g[1])$ with vertically harmonic tangential Gauß map is a Hopf cylinder over a geodesic. In this case the tangential Gauß map is a harmonic map.

Proof. Let $\varphi : M \to (G, g[1])$ be a surface with constant mean curvature and unit normal vector field $n$. Denote by $\theta^3$ the dual one-form of $n$. Express $\theta^3$ by

$$\theta^3 = a \omega^1 + b \omega^2 + c \omega^3, \quad a^2 + b^2 + c^2 = 1$$

in terms of the coframe field.

(1) Case 1 $c \neq 0$: In this case,

$$v_1 = -c \epsilon_2 + b \epsilon_3, \quad v_2 = (b^2 + c^2) \epsilon_1 - ab \epsilon_2 - ac \epsilon_3$$

gives a orthogonal frame field of $M$.

Direct computations show the following formulae:

$$g[1](R(v_1, v_2)v_1, n) = 8ac(b^2 + c^2), \quad g[1](R(v_1, v_2)v_2, n) = 8bc(b^2 + c^2).$$

Take a principal frame $\{e_1, e_2\}$. Then $\{e_1, e_2\}$ is expressed as

$$e_1 = \cos \mu \frac{v_1}{|v_1|} + \sin \mu \frac{v_2}{|v_2|}, \quad e_2 = -\sin \mu \frac{v_1}{|v_1|} + \cos \mu \frac{v_2}{|v_2|}.$$

Then we have

$$R_{1213} = \frac{8c(b^2 + c^2)}{|v_1||v_2|} \left( \frac{ac}{|v_1|} \cos \mu + \frac{b}{|v_2|} \sin \mu \right),$$

$$R_{2123} = \frac{8c(b^2 + c^2)}{|v_1||v_2|} \left( \frac{ac}{|v_1|} \sin \mu - \frac{b}{|v_2|} \cos \mu \right).$$

From these we have $\tau^V(\psi) = 0$ if and only if $a = b = 0$. Hence $\theta^3 = -\eta$. Namely $M$ is an integral surface of the distribution $\eta = 0$, but this is impossible, since $\eta$ is contact. (See p. 36, Theorem in [7]).

\footnote{This result is generalised to 3-dimensional Sasakian space forms by M. Tamura (Comment. Math. Univ. St. Pauli 52 (2003), no. 2, 117–123.)}
(2) Case 2 $c = 0$: Since $a^2 + b^2 = 1$, we may write $a = \cos \phi$, $b = \sin \phi$.

In this case $u_1 = \sin \phi \epsilon_1 - \cos \phi \epsilon_2$, $u_2 = \epsilon_3$ are orthonormal and tangent to $M$. The unit normal $n$ is given by $n = \cos \phi \epsilon_1 + \sin \phi \epsilon_2$. Then we have

\[
R(u_1, u_2)u_1 = -\sin^2 \phi \epsilon_3, \quad R(u_2, u_1)u_2 = -\sin \phi \epsilon_1 + \cos \phi \epsilon_2.
\]

Let us denote by $\mu$ the angle between the principal frame $\{e_1, e_2\}$ and $\{u_1, u_2\}$, i.e.,

\[
e_1 = \cos \mu u_1 + \sin \mu u_2, \quad e_2 = -\sin \mu u_1 + \cos \mu u_2.
\]

Using (14) and (15), we have $R_{1213} = R_{2123} = 0$. Thus $\tau^V(\psi) = 0$ is fulfilled automatically for $M$ with $c = 0$.

We have shown in [6] that constant mean curvature surfaces with $c = 0$ are Hopf cylinder of constant mean curvature. See the proof of Theorem in [6].

Furthermore, the second fundamental form $\mathbb{II}$ of $M$ relative to $n$ is given by (cf. (5) and (8) in [6])

\[
\mathbb{II}(u_1, u_1) = 2H, \quad \mathbb{II}(u_1, u_2) = 1, \quad \mathbb{II}(u_2, u_2) = 0.
\]

Next we see the case $\psi$ is harmonic. Using (14) and (15) again, we have

\[
R_{3113} = -7 \cos^2 \mu + \sin^2 \mu, \quad R_{3223} = -7 \sin^2 \mu + \cos^2 \mu.
\]

Thus $R_{3113} = R_{3223}$ if only if $\mu = \pm \pi/4$. Without loss of generality, we may assume $\mu = \pi/4$. In this case the principal frame $\{e_1, e_2\}$ is given by

\[
e_1 = \frac{1}{\sqrt{2}}(u_1 + u_2), \quad e_2 = \frac{1}{\sqrt{2}}(-u_1 + u_2).
\]

By definition, $\mathbb{II}(e_1, e_2) = 0$. On the other hand, direct computation using (16) shows $\mathbb{II}(e_1, e_2) = -H$. Thus a constant mean curvature surface $M$ with $c = 0$ satisfying $R_{3113} = R_{3223}$ is minimal.

Conversely one can check that every rotational surface of constant mean curvature has vertically harmonic tangential Gauß map and when $H \neq 0$, the tension field does not vanish by direct computations. It is also straightforward to check that every minimal Hopf cylinder has harmonic tangential Gauß map. \(\square\)

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