A TABLEAU FORMULA OF DOUBLE GROTHENDIECK POLYNOMIALS FOR 321-AVOIDING PERMUTATIONS

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Abstract. In this article, we prove a tableau formula for the double Grothendieck polynomials associated to 321-avoiding permutations. The proof is based on the compatibility of the formula with the $K$-theoretic divided difference operators.

1. Introduction

Let $S_n$ be the permutation group of $\{1, \ldots, n\}$ and a permutation $w \in S_n$ is called 321-avoiding if there are no numbers $i < j < k$ such that $w(i) > w(j) > w(k)$. The Grassmannian permutations are also examples of such permutations. The goal of this paper is to prove a tableau formula of the double Grothendieck polynomials $G_w(x, b)$ associated to those 321-avoiding permutations $w$.

Lascoux and Schützenberger ([14], [13]) introduced the (double) Grothendieck polynomials which are polynomial representatives of the (equivariant) $K$-theory classes of structure sheaves of Schubert varieties in a full flag variety. Fomin and Kirillov ([9], [8]) further studied them in the terms of $\beta$-Grothendieck polynomials and Yang-Baxter equations, obtaining their combinatorial formula. For the case of Grassmannian permutations, or more generally vexillary permutations (i.e. 2143-avoiding), the corresponding double Grothendieck polynomials are expressed in terms of set-valued tableaux by the work of Buch [4], McNamara [16], and Knutson–Miller–Yong [12]. On the other hand, the author jointly with Hudson, Ikeda and Naruse obtained a determinant formula of double Grothendieck polynomials for Grassmannian permutations ([10], see [1] and [11] for vexillary case), in the context of degeneracy loci formulas in $K$-theory. Recently, Anderson–Chen–Tarasca [2] extended the method in [10] to the case of 321-avoiding permutations in the study of Brill-Noether loci in $K$-theory, and obtained a determinant formula of the corresponding double Grothendieck polynomials. Combined with the work [15] by the author on skew Grothendieck polynomials, it implies a tableaux formula of (single) Grothendieck polynomials associated to 321 avoiding permutations. See also the most recent related work [5] by Chan–Pflueger. This new development motivated our work in this paper.

Let $w \in S_n$ be a 321-avoiding permutation. Following the work of Billey–Jockusch–Stanley [3] (cf. Chen–Yan–Yang [6] and Anderson–Chen–Tarasca [2]), one can find a skew partition...
\[ \sigma(w) = \lambda/\mu \] and a flagging \( f(w) \) which is a subsequence of \( (1, \ldots, n) \). Namely, let \( f(w) = (f_1, \ldots, f_r) \) be the increasing sequence of numbers \( i \) such that \( w(i) > i \). We define the partitions \( \lambda \) and \( \mu \) by

\[ \lambda_i = w(f_r) - r - f_i + i, \quad \mu_i = w(f_r) - r - w(f_i) + i, \quad i = 1, \ldots, r. \]

As mentioned above, it is known that the single Grothendieck polynomials \( \mathfrak{G}_w(x) \) has the following tableau formula

\[ \mathfrak{G}_w(x) = \sum_{T \in \text{SVT}(\sigma(w), f(w))} \beta^{|T| - |\sigma(w)|} \prod_{e \in T} x_{\text{val}(e)}, \]

where \( \text{SVT}(\sigma(w), f(w)) \) is the set of set-valued tableaux of shape \( \sigma(w) \) with flagging \( f(w) \). On the other hand, Chen–Yan–Yang \cite{6} showed that the double Schubert polynomials \( \mathfrak{S}_w(x, b) \) of Lascoux and Schützenberger associated to \( w \) has the formula

\[ \mathfrak{S}_w(x, b) = \sum_{T} \prod_{e \in T} (x_{\text{val}(e)} + b_{\lambda_r(e)} + f_{r(e)} - c(e) - \text{val}(e) + 1), \]

where \( T \) runs over the set of all semistandard tableaux of shape \( \sigma(w) \) with flagging \( f(w) \). Note that both formulas specialize to the formula of the single Schubert polynomial \( \mathfrak{G}_w(x) \) found earlier by Billey–Jockusch–Stanley \cite{3}.

Our main theorem (Theorem 3.1) unifies the two formulas above. We show that the double Grothendieck polynomial \( \mathfrak{G}_w(x, b) \) associated to \( w \) is given by

\[ \mathfrak{G}_w(x, b) = \sum_{T \in \text{SVT}(\sigma(w), f(w))} \beta^{|T| - |\sigma(w)|} \prod_{e \in T} (x_{\text{val}(e)} \oplus b_{\lambda_r(e)} + f_{r(e)} - c(e) - \text{val}(e) + 1), \]

where \( u \oplus v := u + v + \beta uv \) for any variables \( u \) and \( v \).

The main ingredient for the proof is Proposition 4.3, the compatibility of the tableau formula with the \( K \)-theoretic divided difference operators which are used to define Grothendieck polynomials. In proving this compatibility, we closely follow the argument used by Wachs in \cite{17} Lemma 1.1.

The paper is organized as follows. In §2 we recall the definitions of the divided difference operators and the double Grothendieck polynomials, as well as a few basic formulas. In §3.1 we recall the definitions of set-valued tableaux of skew shape with flagging and describe our main theorem (Theorem 3.1). Grassmannian permutations are basic examples of 321-avoiding permutations and we also prove the theorem in this case (Lemma 3.4). In §4 we show a key proposition and then prove our main theorem by induction.

2. Divided difference and double Grothendieck polynomials

Let \( x = (x_1, x_2, \cdots, x_n) \) and \( b = (b_1, b_2, \ldots, b_n) \) be two sets of variables. Let \( \mathbb{Z}[\beta][x, b] \) be the polynomial ring in variable \( x \) and \( b \) over the coefficient ring \( \mathbb{Z}[\beta] \) where \( \beta \) is a formal variable of degree \(-1\). Let \( S_n \) be the permutation group of \( \{1, \ldots, n\} \) and \( s_i = (i, i + 1) \), \( i = 1, \ldots, n-1 \), denote the transpositions that generates \( S_n \). The length of a permutation \( w \), denoted by \( \ell(w) \),
is the number of transpositions in the reduced word decomposition of \( w \). Let the action of \( S_n \) on \( \mathbb{Z}[\beta][x, b] \) be defined by
\[
\sigma(f(x_1, \ldots, x_n)) := f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}).
\]
For each \( i \in \mathbb{Z}_{>0} \), we define the divided difference operator \( \pi_i \) on \( \mathbb{Z}[\beta][x, b] \) by
\[
\pi_i(f) = \frac{(1 + \beta x_{i+1})f - (1 + \beta x_i)s_i f}{x_i - x_{i+1}}
\]
for each polynomial \( f \in \mathbb{Z}[\beta][x, b] \).

The double Grothendieck polynomial \( G_w = G_w(x, b) \in \mathbb{Z}[\beta][x, b] \) associated to a permutation \( w \in S_n \) is defined inductively as follows. Our convention coincides with the one in [4] after setting \( \beta = -1 \). For any variable \( u \) and \( v \), we use the notation \( u \oplus v := u + v + \beta uv \). For the longest element \( w_0 \) in \( S_n \), we set
\[
G_{w_0} = \prod_{i+j \leq n} (x_i \oplus b_j).
\]
If \( w \) is not the longest, we can find a positive integer \( i \) such that \( \ell(ws_i) = \ell(w) + 1 \) and then define
\[
G_w := \pi_i(G_{ws_i}).
\]
This definition is independent of the choice of \( s_i \) because the operators \( \pi_i \) satisfy the Coxeter relations.

To conclude this section, we recall a few basic formulas for \( \pi_i \) (cf. [15, §2.1]). Let \( f, g \in \mathbb{Z}[\beta][[x, b]] \). First, we have the following Leibniz rule for \( \pi_i \):
\[
\pi_i(fg) = \pi_i(f)g + s_i(f)\pi_i(g) + \beta s_i(f)g.
\]
(2.1)
If \( f \) is symmetric in \( x_i \) and \( x_{i+1} \), then we have
\[
\pi_i(f) = -\beta f,
\]
(2.2)
\[
\pi_i(fg) = f\pi_i(g),
\]
(2.3)
and moreover we have
\[
\pi_i(x_i^k) = \begin{cases} 
-\beta f & (k = 0), \\
\sum_{s=0}^{k-1} x_i^s x_{i+1}^{k-1-s} + \beta \sum_{s=1}^{k-1} x_i^s x_{i+1}^{k-s} & (k > 0).
\end{cases}
\]
(2.4)

3. FLAGGED SKEW PARTITIONS AND 321-AVOIDING PERMUTATIONS

3.1. Definitions and the main theorem. Let \( \lambda = (\lambda_1 \geq \cdots \geq \lambda_r > 0) \) and \( \mu = (\mu_1 \geq \cdots \geq \mu_r \geq 0) \) be partitions such that \( \mu_i \leq \lambda_i \) for all \( i = 1, \ldots, r \). We identify a partition with its Young diagram. A skew partition \( \lambda/\mu \) is given by the pair of \( \lambda \) and \( \mu \). We identify a skew partition \( \lambda/\mu \) with its skew diagram which is the collection of boxes in \( \lambda \) that are not in \( \mu \). More precisely, we denote \( \lambda/\mu = \{(i,j) \mid \mu_i < j \leq \lambda_i, k = 1, \ldots, r\} \). When \( \mu \) is empty, the
skew partition $\lambda/\mu$ is regarded as the partition $\lambda$. Let $|\lambda/\mu|$ be the numbers of boxes in the corresponding skew diagram.

For finite subsets $a$ and $b$ of positive integers, we define $a < b$ if $\max(a) < \min(b)$, and $a \leq b$ if $\max(a) \leq \min(b)$. A set-valued tableau $T$ of shape $\lambda/\mu$ is a labeling by which each box of the skew diagram $\lambda/\mu$ is assigned a finite subset of positive integers, called a filling, in such a way that the rows are weakly increasing from left to right and the columns strictly increasing from top to bottom. An element $e$ of a filling of $T$ is called an entry and denoted by $e \in T$. The numerical value of an entry $e \in T$ is denoted by $\text{val}(e)$, and the row and column indices of the box to which $e$ belongs are denoted by $r(e)$ and $c(e)$ respectively. Let $|T|$ be the total number of entries of $T$.

A flagging of $\lambda/\mu$ is a sequence of positive integers $f = (f_1, \ldots, f_r)$ such that $f_1 \leq \cdots \leq f_r$. A set-valued tableau of skew shape $\lambda/\mu$ with a flagging $f$ is a set-valued tableau of skew shape $\lambda/\mu$ such that each filling in the $i$-th row is a subset of $\{1, \ldots, f_i\}$ for all $i$. Let $\text{SVT}(\lambda/\mu, f)$ denote the set of all skew tableaux of shape $\lambda/\mu$ with a flagging $f$. If $f_1 = \cdots = f_r$, then the associated set-valued tableaux are nothing but the set-valued tableaux of skew shape $\lambda/\mu$ considered by Buch in [4]. Note that in [17] and [15], one considers more general flagged skew partitions and their set-valued tableaux.

A permutation $w \in S_n$ is called 321-avoiding if there are no numbers $i < j < k$ such that $w(i) > w(j) > w(k)$. Such permutation $w$ is completely characterized by a pair of increasing subsequence of $(1, \ldots, n)$ (cf. [7, §2]). Namely, let $f(w) = (f_1, \ldots, f_r)$ be the increasing sequence of indices $i$ such that $i < w(i)$ and then $h(w) := (w(f_1), \ldots, w(f_r))$ is also an increasing sequence. If $f^c(w) = (f^c_1, \ldots, f^c_{n-r})$ is the increasing sequence of indices $i$ such that $i \geq w(i)$, then $h^c(w) = (w(f^c_1), \ldots, w(f^c_{n-r}))$ is also an increasing sequence. Therefore one can see that $f(w)$ and $h(w)$ determines $w$ uniquely.

To $w$ one assigns a skew partition $\sigma'(w) = \lambda'/\mu'$ used by Anderson–Chen–Tarasca [2]:

$$\lambda'_i := w(f_r+i) - (r+1-i), \quad \mu'_i := f_r+i - (r+1-i).$$

The skew partition $\sigma(w) = \lambda/\mu$ defined by Billey–Jockusch–Stanley [3] can be obtained from rotating $\sigma'(w)$ by 180 degree and it relates to $\sigma'(w)$ by

$$\lambda_i := \lambda'_1 - \mu'_{r+1-i} = w(f_r) - r - (f_i - i), \quad (3.1)$$

$$\mu_i := \lambda'_1 - \lambda'_{r+1-i} = w(f_r) - r - (w(f_i) - i). \quad (3.2)$$

We have $\ell(w) = |\sigma(w)|$. Note also that we have $\lambda'_i = w(f_r) - r = \lambda_i + f_i - i$ for all $i = 1, \ldots, r$.

The following is the main theorem of this article.

**Theorem 3.1.** Let $w \in S_n$ be a 321-avoiding permutation and $\sigma(w)$ its associated skew partition with the flagging $f(w)$. Then we have

$$\mathfrak{G}_w(x,b) = \sum_{T \in \text{SVT}(\sigma(w), f(w))} b^{|T|-|\sigma(w)|} \prod_{e \in T} x_{\text{val}(e)} \oplus b^{\lambda_{r(e)}+f_r(e)-c(e)-\text{val}(e)+1}. \quad (3.3)$$
Example 3.2. Let $w = (31254)$. Then $f(w) = (1, 4)$, $h(w) = (3, 5)$, $f^c(w) = (2, 3, 5)$ and $h^c(w) = (1, 2, 4)$. We find that $\lambda' = (3, 2)$, $\mu' = (2, 0)$, $\lambda = (3, 1)$ and $\mu = (1, 0)$. Below is a few examples of set-valued tableaux in $SVT(\sigma(w), f(w))$.

\[
\begin{array}{cccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 3 & 4 & 12 & 13 & 14 & 15 & \\
\end{array}
\ldots
\]

If $T$ is the 5th one above, the corresponding term in the summation of (3.3) is

$$\beta(x_1 \oplus b_2)(x_1 \oplus b_1)(x_1 \oplus b_1)(x_2 \oplus b_3).$$

3.2. Grassmannian case. In this section, we prove Theorem 3.1 in the Grassmannian case. A permutation $w \in S_n$ is called Grassmannian with descent at $d$, i.e. $w(1) < \cdots < w(d)$ and $w(d + 1) < \cdots < w(n)$. By definition, a Grassmannian permutation is 321-avoiding. In this case, $f(w) = (s + 1, \ldots, s + r)$ where $s + 1$ is the smallest index $i$ such that $i < w(i)$ and $f_r = s + r = d$. Hence, for each $i = 1, \ldots, r$, we find that $\lambda'_i = w(d + 1 - i) - (r + 1 - i)$ and $\mu'_i = s$ so that $\lambda(w) := \lambda' - \mu'$ is a partition. We can also find that $\lambda_i = w(d) - d$ and $\mu_i = w(d) - r - (w(s + i) - i)$.

Example 3.3. Consider $w = (13524)$ which is a Grassmannian permutation with descent at $d = 3$. Then we have $f(w) = (2, 3), h(w) = (3, 5), \lambda' = (3, 2), \mu' = (1, 1), \lambda = (2, 2), \mu = (1, 0), \lambda(w) = (2, 1)$ and $\bar{f}(w) = (3, 3)$.

The double Grothendieck polynomial associated to a Grassmannian permutation $w$ is known to be a symmetric polynomial in $x_1, \ldots, x_d$ with coefficients in $Z[\beta][b]$ and it can be expressed by the following formula (see [4] and [16]): let $\lambda(w)$ and $\bar{f}(w)$ be Grassmannian with descent at $d = 3$, then we have

$$\mathcal{G}_w(x, b) = \sum_{T \in SVT(\lambda(w), \bar{f})} \beta^{(T)} \beta(\bar{\lambda}(w)) \prod_{e \in T} (x_{val(e)} \oplus b_{val(e)} + c_{val(e)} - r(e)).$$

(3.4)

Lemma 3.4. If $w \in S_n$ is Grassmannian with descent at $d$, the equation (3.4) holds.

Proof. First observe that there is a bijection

$$SVT(\sigma(w), f(w)) \to SVT(\lambda(w), \bar{f})$$

sending $T$ to $T'$ which is obtained by changing the value $i$ of each entry in $T$ to $d + 1 - i$ and rotating it by 180 degree. The inverse map can be defined by the same way. Although $T$ is a tableau with flagging $(1, 2, \ldots, r)$ and the numbers used in the $i$-th row of the tableau $T'$ seems bounded below by $i$, this would not affect the bijection because of the column strictness of tableaux of the partition $\lambda(w)$.

If $e \in T$ corresponds to $e' \in T'$ under this bijection, we have

$$val(e) = d + 1 - val(e'), \quad r(e) = r + 1 - r(e'), \quad c(e) = w(d) - d + 1 - c(e').$$
Thus, we have

\[
\sum_{T \in SVT(\sigma(w),f(w))} \beta^{|T|-|\sigma(w)|} \prod_{e \in T} x_{\mathrm{val}(e)} \oplus b_{\lambda_e(w) + f_e(w) - c(e) - \mathrm{val}(e) + 1}
\]

\[
= \sum_{T \in SVT(\sigma(w),f(w))} \beta^{|T|-|\sigma(w)|} \prod_{e \in T} x_{\mathrm{val}(e)} \oplus b_{w(d) - r(e) - c(e) - \mathrm{val}(e) + 1}
\]

\[
= \sum_{T' \in SVT(\lambda(w),f')} \prod_{e' \in T'} x_{d+1-\mathrm{val}(e')} \oplus b_{\mathrm{val}(e') - r(e') + c(e')}
\]

\[
= \mathcal{G}_w(x,b),
\]

where the third equality follows from the fact that \( \mathcal{G}_w(x,b) \) is a symmetric polynomials in \( x_1, \ldots, x_d \).

\[\Box\]

4. Proof of the main theorem

4.1. Preparation. The following two lemmas will be used in the proof of Proposition 4.3 which allows us to prove the main theorem by induction.

**Lemma 4.1.** For an arbitrary finite sequence of positive integers \( \ell = (\ell_1, \ldots, \ell_k) \), we have

\[
\pi_i((x_1 \oplus \ell_1) \cdots (x_i \oplus \ell_i)) = \sum_{j=1}^k \left( \prod_{v=1}^{j-1} (x_i \oplus \ell_v) \right) \prod_{v=j+1}^k (x_{i+1} \oplus \ell_v) \]

\[
+ \beta \sum_{j=1}^{k-1} \left( \prod_{v=1}^{j} (x_i \oplus \ell_v) \right) \prod_{v=j+1}^k (x_{i+1} \oplus \ell_v). \quad (4.1)
\]

In particular, \( \pi_i(x_i \oplus \ell_i) = 1 \). Furthermore, the expression on the right hand side of (4.1) is symmetric in \( x_i \) and \( x_{i+1} \).

**Proof.** It suffices to show the claim for \( i = 1 \) and \( \ell = (1, \ldots, k) \):

\[
\pi_1((x_1 \oplus b_1) \cdots (x_1 \oplus b_k)) = \sum_{j=1}^k \left( \prod_{v=1}^{j-1} (x_1 \oplus b_v) \right) \prod_{v=j+1}^k (x_2 \oplus b_v) \]

\[
+ \beta \sum_{j=1}^{k-1} \left( \prod_{v=1}^{j} (x_1 \oplus b_v) \right) \prod_{v=j+1}^k (x_2 \oplus b_v). \quad (4.2)
\]

Choose an integer \( n > k \) and consider the Grassmannian permutation \( w = (k+1,1,2,\ldots,k,k+2,\ldots,n) \) with descent at 1. We have \( \lambda(w) = (k) \) and \( f(w) = 1 \). There is only one set-valued tableau of shape \( \lambda(w) \) with flagging \( f(w) \), which assigns \( \{1\} \) to each box. Therefore it follows from (4.2) that

\[
\mathcal{G}_w(x,b) = (x_1 \oplus b_1) \cdots (x_1 \oplus b_k).
\]
Now consider the element $w_1 = (1, k + 1, 2, 3, \ldots, k, k + 2, \ldots, n)$. We have $\lambda(w_1) = (k - 1)$ and $f(w) = (2)$. Since $\ell(w) = \ell(w_1) + 1$, we find that

$$\pi_1 \mathfrak{G}_w(x, b) = \mathfrak{G}_{w_1}(x, b).$$

We can also see that $\mathfrak{G}_{w_1}(x, b)$ coincides with the right hand side of (4.2) by (3.4). Therefore, (4.2) holds. The last claim follows from the fact that $\mathfrak{G}_{w_1}(x, b)$ is symmetric in $x_1$ and $x_2$. □

**Lemma 4.2.** Let $w$ be a 321-avoiding permutation. Let $f(w) = (f_1, \ldots, f_r)$ and $\sigma(w) = λ/µ$. Suppose that $f_i + 1 < f_{i+1}$ and $w(f_i) > f_i + 1$ for some $i$. Then $w_{f_i}$ is a 321-avoiding permutation and we have

$$f(w_{f_i}) = (f_1, \ldots, f_{i-1}, f_i + 1, f_{i+1}, \ldots, f_r), \quad h(w_{f_i}) = h(w),$$

$$\lambda(w_{f_i}) = (\lambda_1, \ldots, \lambda_{i-1}, \lambda_i - 1, \lambda_{i+1}, \ldots, \lambda_r), \quad \mu(w_{f_i}) = \mu(w).$$

In particular, $\ell(w) = \ell(w_{f_i}) + 1$.

**Proof.** First observe that $f_i + 1 < f_{i+1}$ implies that $w(f_i + 1) < f_i + 1$. Indeed, if $w(k) = k$ for some $k$, we find that $w' = (w(1), \ldots, w(k - 1))$ is a permutation in $S_{k-1}$ since $w(f_j) > f_j$ for all $j$ and $h'(w)$ is an increasing sequence. Therefore $w(k - 1) \leq k - 1$. It follows that $w_{f_i}(f_i) = w(f_i + 1) \leq f_i$ and $w_{f_i}(f_i + 1) = w(f_i) > f_i + 1$. From this, we find that $f(w_{f_i})$ and $h(w_{f_i})$ are as given in the claim. The rest can be checked by computing $\lambda(w_{f_i})$ and $\mu(w_{f_i})$ from the definition (3.1) and (3.2). □

The following proposition is the main ingredient of the proof of Theorem 3.1.

**Proposition 4.3.** Let $w$ be a 321-avoiding permutation. Let $f(w) = (f_1, \ldots, f_r)$ and $\sigma(w) = λ/µ$. Suppose that $f_i + 1 < f_{i+1}$ and $w(f_i) > f_i + 1$ for some $i$. Let $t := f_i$. Then we have

$$\partial_t \mathbb{T}_w(x, b) = \mathbb{T}_{w_{f_i}}(x, b),$$

where $\mathbb{T}_w(x, b)$ is the right hand side of (3.3).

**Proof.** We define an equivalence relation in $\text{SVT}(\sigma(w), f(w))$ as follows: $T_1 \sim T_2$ if the collection of boxes containing $t$ and $t' := t + 1$ is the same for $T_1$ and $T_2$. We can write

$$\mathbb{T}_w = \mathbb{T}_w(x, b) = \sum_{\mathcal{A} \in \text{SVT}(\sigma(w), f(w))/\sim} \left( \sum_{T \in \mathcal{A}} \beta^{[T]}(x) \left| \sigma(w) \right| (x|b)^T \right),$$

where we denote

$$(x|b)^T = \prod_{e \in T} x_{\text{val}(e)} \oplus b_{\lambda(e) + f_r(e) - c(e) - \text{val}(e) + 1} = \prod_{e \in T} x_{\text{val}(e)} \oplus b_{\lambda(e) + f_r(e) - c(e) - \text{val}(e) + 1}.$$

By (3.1), we see that the condition $f_i + 1 < f_{i+1}$ implies $\lambda_i > \lambda_{i+1}$. Let $\mathcal{A}$ be the equivalence class for $\text{SVT}(\sigma(w), f(w))$ whose tableaux have the same configuration of $t$ and $t'$ as shown in Figure 1.
In Figure 1, the rightmost one-row rectangle with entries $t$ has $m_1$ boxes and is denoted by $A_1$. For $s \geq 2$, the $s$-th one-row rectangle with *, denoted by $A_s$, has $m_s$ boxes each of which contains $t$, $t'$, or both so that the total number of entries $t$ and $t'$ in $A_s$ is $m_s$ or $m_s + 1$. The $s$-th rectangle with two rows where the first row contains $t$ and the second row contains $t'$ has $r_s$ columns and is denoted by $B_s$. Note that $m_s$ and $r_s$ may be 0 and hence the rectangles in Figure 1 may not be connected. Also the leftmost box of $A_s$ ($s \geq 1$) may contain a number less than $t$, the rightmost box of $A_s$ ($s \geq 2$) may contain a number greater than $t'$, and so on. Let $a_s$ be the column index of the leftmost box in $A_s$. Similarly, let $b_s$ be the column index of the leftmost column in $B_s$.

We can write

$$
\sum_{T \in \mathcal{A}} \beta^{|T| - |\sigma(w)|} (x|b)^T = R(A_1) \cdot \prod_{s=2}^{k} R(A_s) \cdot \prod_{s=1}^{k} R(B_s) \cdot R(\mathcal{A})
$$

where $R(A_s)$ and $R(B_s)$ are the polynomials contributed from $A_s$ and $B_s$ respectively, and $R(\mathcal{A})$ is the polynomial contributed from the entries other than $t$ and $t'$. More precisely we have

$$
R(A_1) = \prod_{\ell=a_1}^{a_1+m_1-1} (x_\ell \oplus b_{\lambda'_s+i-(a_1+v-1)\ell-t+1}) = \prod_{\ell=1}^{m_1} (x_\ell \oplus b_{\lambda'_s+i-(a_1+v-1)\ell-t+1})
$$

$$
R(A_s) = \sum_{j=0}^{m_s} \left( \prod_{\ell=a_s}^{a_s+j-1} (x_\ell \oplus b_{\lambda'_s+i+(i+s-1)\ell-t+1}) \prod_{\ell=a_s+j+1}^{a_s+m_s} (x_\ell' \oplus b_{\lambda'_s+i+(i+s-1)\ell-t+1}) \right)

+ \beta \sum_{j=1}^{m_s} \left( \prod_{\ell=a_s}^{a_s+j-1} (x_\ell \oplus b_{\lambda'_s+i+(i+s-1)\ell-t+1}) \prod_{\ell=a_s+j+1}^{a_s+m_s} (x_\ell' \oplus b_{\lambda'_s+i+(i+s-1)\ell-t+1}) \right) \quad (2 \leq s \leq k)
$$

$$
R(B_s) = \prod_{\ell=b_s}^{b_s+r_s-1} (x_\ell \oplus b_{\lambda'_s+i+(i+s-1)\ell-t-t'+1})(x_\ell' \oplus b_{\lambda'_s+i+(i+s)\ell-t-t'+1}) \quad (1 \leq s \leq k).
$$
Observe that $R(B_s)$ for $s \geq 1$ is symmetric in $x_t$ and $x_{t'}$, and so is $R(A_s)$ for $s \geq 2$ by Lemma 4.1 Therefore, if $m_1 = 0$, then $r_1 = 0$, $R(A_1) = 0$ and by (2.2) we have

$$
\pi_t \left( \sum_{T \in \mathcal{A}} \beta^{|T| - |\sigma(w)|} (x|b)^T \right) = -\beta \prod_{s=2}^{k} R(A_s) \cdot \prod_{s=2}^{k} R(B_s) \cdot R(\mathcal{A}). \quad (4.3)
$$

If $m_1 = 1$, we have, by (2.3) and Lemma 4.1

$$
\pi_t \left( \sum_{T \in \mathcal{A}} \beta^{|T| - |\sigma(w)|} (x|b)^T \right) = \prod_{s=2}^{k} R(A_s) \cdot \prod_{s=1}^{k} R(B_s) \cdot R(\mathcal{A}). \quad (4.4)
$$

If $m_1 \geq 2$, we have, also by (2.3) and Lemma 4.1

$$
\pi_t \left( \sum_{T \in \mathcal{A}} \beta^{|T| - |\sigma(w)|} (x|b)^T \right) = \left( \prod_{j=1}^{m_1} \left( \prod_{v=1}^{j-1} (x_t \oplus b_{\ell_v}) \prod_{s=j+1}^{m_1} (x_{t'} \oplus b_{\ell_v}) \right) \right)

+ \beta \sum_{j=1}^{m_1-1} \left( \prod_{v=1}^{j} (x_t \oplus b_{\ell_v}) \prod_{v=j+1}^{m_1} (x_{t'} \oplus b_{\ell_v}) \right)

\times \prod_{s=2}^{k} R(A_s) \cdot \prod_{s=1}^{k} R(B_s) \cdot R(\mathcal{A}), \quad (4.5)
$$

where $\ell_v = \lambda'_1 + i - (a_1 + v - 1) - t + 1$.

We consider the decomposition

$$
SVT(\sigma(w), f(w))/\sim = F_1 \sqcup F_2 \sqcup F_3 \sqcup F_4
$$

where $F_1, \ldots, F_4$ are the sets of equivalence classes whose configurations of the boxes containing $t$ or $t'$ satisfy the following conditions respectively:

1. $m_1 = 0$ (so that $r_1 = 0$),
2. $m_1 = 1$ and the box at $(i, \lambda_i)$ in $\lambda$ contains more than one entry (so that $r_1 = 0$),
3. $m_1 = 1$ and the box at $(i, \lambda_i)$ in $\lambda$ contains contains only $t$,
4. $m_1 \geq 2$.

Observe that there is a bijection from $F_2$ to $F_1$ sending $\mathcal{A}_2$ to $\mathcal{A}_1$ by deleting $t$ in the rectangle $A_1$ and that $R(\mathcal{A}_2) = \beta R(\mathcal{A}_1)$. Therefore, by the expressions (4.3) and (4.4), we have

$$
\sum_{\mathcal{A} \in F_1 \sqcup F_2} \pi_t \left( \sum_{T \in \mathcal{A}} \beta^{|T| - |\sigma(w)|} (x|b)^T \right) = 0.
$$

Thus we obtain

$$
\pi_t(T_w(x, b)) = \sum_{\mathcal{A} \in F_3 \sqcup F_4} \pi_t \left( \sum_{T \in \mathcal{A}} \beta^{|T| - |\sigma(w)|} (x|b)^T \right).
$$

On the other hand, by Lemma 4.2 the skew partition $\sigma(ws_t)$ is obtained from $\sigma(w)$ by deleting the rightmost box of the $i$-th row and the flagging $f(ws_t)$ is obtained from $f(w)$ by adding 1 to $f_i$. Thus for each equivalence class $\mathcal{A}'$ for $SVT(\sigma(ws_t), f(ws_t))$, the corresponding tableaux have the following configuration of $t$ and $t'$ in Figure 2 where $m_1 \geq 1$. 


Consider the decomposition

\[
SVT(\sigma(ws), f(ws))/\sim = \mathcal{F}_3^r \sqcup \mathcal{F}_4^r
\]

where \( \mathcal{F}_3^r \) is the set of equivalence classes such that \( m_1 = 1 \) and \( \mathcal{F}_4^r \) is the set of equivalence classes such that \( m_1 \geq 2 \). Obviously there are bijections \( \mathcal{F}_3 \rightarrow \mathcal{F}_3^r \) and \( \mathcal{F}_4 \rightarrow \mathcal{F}_4^r \). Namely, \( \mathcal{A} \in \mathcal{F}_3 \) corresponds to \( \mathcal{A}^r \in \mathcal{F}_3^r \) by removing the single \( t \) in the rectangle \( A_1 \), and \( \mathcal{A} \in \mathcal{F}_4 \) corresponds to \( \mathcal{A}^r \in \mathcal{F}_4^r \) by removing the last box of \( A_1 \) with \( t \) and replace all other \( t \)'s by \(*\), without changing all other entries of the tableaux.

Under these bijections, \( \sum_{T' \in \mathcal{A}^r} \beta^{T'}|\sigma(ws)t|(x|b)T' \) is exactly the right hand side of (4.4) if \( \mathcal{A} \in \mathcal{F}_3 \) and (4.5) if \( \mathcal{A} \in \mathcal{F}_4 \). Thus we have

\[
\pi_t(\sum_{T \in \mathcal{A}} \beta^{T}|\sigma(w)t|(x|b)T) = \sum_{T' \in \mathcal{A}^r} \beta^{T'}|\sigma(ws)|t|(x|b)T'.
\]

Therefore

\[
\pi_t(T_w) = \sum_{\mathcal{A} \in \mathcal{F}_3} \pi_t(\sum_{T \in \mathcal{A}} \beta^{T}|\sigma(w)|t|(x|b)T) + \sum_{\mathcal{A} \in \mathcal{F}_4} \pi_t(\sum_{T \in \mathcal{A}} \beta^{T}|\sigma(w)t|(x|b)T)
= \sum_{\mathcal{A} \in \mathcal{F}_3} \sum_{T' \in \mathcal{A}^r} \beta^{T'}|\sigma(ws)|t|(x|b)T' + \sum_{\mathcal{A} \in \mathcal{F}_4} \sum_{T' \in \mathcal{A}^r} \beta^{T'}|\sigma(ws)t|(x|b)T'
= T_{wst}.
\]

This completes the proof. \( \square \)

4.2. Proof of Theorem 3.1 Let \( h = (h_1, \ldots, h_r) \) be an increasing finite sequence of positive integers such that \( h_1 \geq 2 \) and \( h_r \leq n \). Let \( C_h \) be the set of the 321-avoiding permutations \( v \) in \( S_n \) such that \( h(v) = h \). The union of \( C_h \)'s where \( h \) runs over the set of all such increasing sequences coincides with the set of all 321-avoiding permutations in \( S_n \). We can define a total order on \( C_h \) by \( v < v' \) if \( f(v) < f(v') \) in the lexicographic order. We show that \( \mathcal{G}_v = T_v \) for
each \( v \in C_h \) by induction on this order. The minimum element in \( C_h \) is the Grassmannian permutation \( v^{(0)} \) where \( f(v^{(0)}) = (1, \ldots, r) \) and \( h(v^{(0)}) = h \), and we have \( \mathfrak{G}_{v^{(0)}}(x, b) = T_{v^{(0)}}(x, b) \) by Lemma 3.4. Let \( v \in C_h \) and suppose that \( \mathfrak{G}_w = T_w \) for all \( w < v \). Let \( f(v) = (f_1, \ldots, f_r) \). If \( v^{(0)} < v \), then there is an index \( i \) such that \( f_{i-1} < f_i - 1 \). Then \( v = w s_t \) with \( t := f_i - 1 \) where \( w \in C_h \) is defined by \( f(w) = (f_1, \ldots, f_{i-1}, f_i - 1, f_{i+1}, \ldots, f_r) \). Since \( w < v \), we have \( \mathfrak{G}_w = T_w \). Furthermore, \( \ell(w) = \ell(v) + 1 \) and \( w \) satisfies the conditions in Proposition 4.3. Now it follows from Proposition 4.3 that

\[
\mathfrak{G}_v = \pi_t(\mathfrak{G}_w) = \pi_t(T_w) = T_v.
\]

This completes the proof of Theorem 3.1. \( \Box \)

**Example 4.4.** Let us demonstrate how Proposition 4.3 implies Theorem 3.1 in examples. Let \( h = (3, 5) \). Then the minimum element in \( C_h \) is the Grassmannian permutation \( v^{(0)} = (35124) \) where \( f(v^{(0)}) = (1, 2) \). Consider \( v = (31254) \in C_h \) where \( f(v) = (1, 4) \). We have \( v s_3 s_2 = v^{(0)} \), \( \ell(v^{(0)}) = \ell(v) + 2 \), and hence \( \mathfrak{G}_v = \pi_3 \pi_2(\mathfrak{G}_{v^{(0)}}) \). Let \( w = v^{(0)} s_2 = (31524) \) so that \( f(w) = (1, 3) \). The corresponding skew partitions are

\[
\begin{array}{ccc}
| & | & | \\
\hline
| & | & |
\end{array}
\quad
\begin{array}{ccc}
| & | & | \\
\hline
| & | & |
\end{array}
\quad
\begin{array}{ccc}
| & | & | \\
\hline
| & | & |
\end{array}
\]

Since we know \( \mathfrak{S}_{v^{(0)}} = T_{v^{(0)}} \) from Lemma 3.4, Proposition 4.3 implies

\[
\mathfrak{G}_v = \pi_3 \pi_2(\mathfrak{G}_{v^{(0)}}) = \pi_3 \pi_2(T_{v^{(0)}}) = \pi_3(T_w) = T_v.
\]

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