Non-Hermitian Yukawa interactions of fermions with axions: potential microscopic origin and dynamical mass generation

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Abstract. In this mini review, we discuss some recent developments regarding properties of (quantum) field-theory models containing anti-Hermitian Yukawa interactions between pseudoscalar fields (axions) and Dirac (or Majorana) fermions. Specifically, we first motivate physically such interactions, in the context of string-inspired low-energy effective field theories, involving right-handed neutrinos and axion fields. Then we proceed to discuss their formal consistency within the so-called Parity-Time-reversal(PT)-symmetry framework. Subsequently, we review dynamical mass generation, induced by the Yukawa interactions, for both fermions and axions. The Yukawa couplings are assumed weak, given that they are conjectured to have been generated by non-perturbative effects in the underlying microscopic string theory. The models under discussion contain, in addition to the Yukawa terms, also anti-Hermitian anomalous derivative couplings of the pseudoscalar fields to axial fermion currents, as well as interactions of the fermions with non-Hermitian axial backgrounds. We discuss the role of such additional couplings on the Yukawa-induced dynamically-generated masses. For the case where the fermions are right-handed neutrinos, we compare such masses with the radiative ones induced by both, the anti-Hermitian anomalous terms and the anti-Hermitian Yukawa interactions in phenomenologically relevant models.

1. Introduction

The Parity-Time-reversal(PT)-symmetry framework [1, 2, 3, 4] is an innovative approach to quantum theory, with a plethora of theoretical and experimental applications. These span various branches of physics, and are rapidly expanding to embrace new phenomena, even as this review is being written (for a partial but indicative list of such applications, the reader can consult the mini review [5].) PT symmetry guarantees the self consistency of quantum mechanical models with non-Hermitian Hamiltonians, characterised by real energy eigenvalues. The reality of the energy eigenvalues, despite the lack of Hermiticity of the Hamiltonian, can be understood [6, 7, 8, 9, 10] by means of the antilinear nature of PT symmetry. In fact, as argued in [10], PT symmetry constitutes only a special case of non-Hermitian Hamiltonians with real eigenvalues. If a quantum system is characterised by an antilinear symmetry, this is the most general condition that one can impose on a quantum theory for which one can have a time-independent inner product and a self-adjoint Hamiltonian with real energy eigenvalues. For each of the above properties Hermiticity is only a sufficient condition but not a necessary one.
Hermiticity is then a special case, in which the Hamiltonian of the system has both antilinearity and Hermiticity.

Extension of the methods of PT-symmetric non-Hermitian quantum-mechanics towards the formulation of non-Hermitian quantum field theories within the PT-symmetry framework is at present in its initial stages, but with very encouraging results [12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22]. There are studies in this framework associated with Dirac fermion field theories [23], neutrino models [24, 25], spontaneous breaking of global and local (gauge) symmetries in non-Hermitian field theories [26, 27, 28, 29, 30, 31, 32], discrete symmetries [33] and supersymmetry [34] in such models, chiral magnetic effect in non-Hermitian fermionic systems [35], (1+1)-dimensional time-like Liouville conformal field theories [36], a discussion on the potential rôle of non-Hermitian Hamiltonians for the stability of the Higgs vacuum and other field theories of interest to particle physics [37], as well as studies of non-Abelian magnetic monopole solutions [38], and more generally complex Bogomol'nyi-Prasad-Sommerfield (BPS) solitons with real energies [39].

All the above field-theoretic systems are relativistic, for which the reality of the energy eigenvalues can be understood by the extension of the quantum mechanical ideas of the existence of an underlying antilinear symmetry [8, 9, 10] to quantum field theories with non-Hermitian Hamiltonians. The antilinear symmetry is uniquely identified with CPT [10], where C is an appropriate definition of the charge conjugation, which may differ from the standard definition of the Dirac conjugation operator [21]. Indeed, as shown in [10], requiring the existence of time-independent inner products and invariance under complex Lorentz transformations, forces the antilinear symmetry to be uniquely CPT [10]. In this way, the standard CPT theorem (with C denoting the standard Dirac charge-conjugation operator), which is based on locality, unitarity and Lorentz invariance of the corresponding field-theoretic Lagrangian densities, can be extended to appropriate field-theoretic systems with non-Hermitian Hamiltonians. In this latter CPT-invariant framework, PT-symmetric systems are characterised by a separate invariance under charge conjugation C [10].

In the above examples, the non-Hermiticity of the Hamiltonians has been assumed, without an attempt to provide microscopic explanations. In [40, 41, 42] we have discussed possible microscopic explanations for a particular kind of non-Hermitian interactions, that of anti-Hermitian Yukawa interactions between pseudoscalar (axion-like) and fermion fields, within the framework of certain string-inspired models [43]. The underlying CPT symmetry that characterises the model is responsible for the reality of the energy eigenvalues, according to the general arguments of [6, 7, 8, 9, 10], mentioned above. Moreover, we have studied dynamical mass generation for the pseudoscalar and fermion fields in such systems, within a non-perturbative Schwinger-Dyson (SD) treatment. It is the purpose of this article, to briefly review these studies.

The structure of the article is as follows: in the next section, 2, we review the string-inspired model [43] and explain how the non-Hermitian Yukawa interactions emerge, along with non-Hermitian anomalous couplings of the pseudoscalar (axion-like) fields. We note at this point that the axion-like particles in this model are associated with stringy excitations and are in general different from the QCD axion. In the literature such fields are sometimes called axion-like particles (ALPs), a terminology used in [41, 42] where this review is partly based on. For notational brevity, though, in what follows, we shall refer to them simply as axions. We motivate the use of non-Hermitian Yukawa models (embedded in appropriately generalised-PT (CPT) frameworks) as providers of alternative ways of dynamical-mass generation for both axions and fermions. In the string-inspired model in which the fermions are right-handed neutrinos, both these fields and the axions in their massive phase could provide candidates for dark matter.

1 We also note that spontaneous breaking of PT symmetry, as for instance is the case in the (1+1)-dimensional quantum mechanical lattice system studied in [11], leads to the existence of an energy spectrum with complex branches.
In section 3, we discuss the SD mass generation for axions and fermions, in the absence of anomalous axion couplings. In section 4, we include such anomalous interactions of axions, and study their effects on the anti-Hermitian-Yukawa-interaction-induced mass generation. We also include external non-Hermitian axial backgrounds, which are motivated within the context of the string-inspired model of section 2. Conclusions and outlook are presented in section 5.

2. String-inspired Models and a potential origin of non-Hermitian Interactions

In this section we shall motivate the origin of non-Hermitian interactions of axions in low-energy effective actions inspired from string theory. To this end, we shall first review some basic features of the spectrum of string theories. We shall be dealing with perturbative effective actions of string theory [44, 45, 46, 47], restricting ourselves to at-most-quadratic order in space-time derivatives. This suffices when we discuss physics at energy scales much below the string mass scale, which serves as an ultraviolet (UV) momentum-cut-off scale of the point-like effective field theories stemming from strings [44, 45].

2.1. The Bosonic massless gravitational multiplet of strings and anomalies

In superstring theory [44, 45, 46, 47], after compactification to four space-time dimensions, the bosonic ground state of the closed-string sector consists of massless fields in the so-called gravitational multiplet, which contains a spin-0 (scalar) dilaton \( \Phi(x) \), a spin-2 traceless symmetric tensor field, \( g_{\mu\nu}(x) \), which is uniquely identified as the (3+1)-dimensional graviton, and a spin-1 antisymmetric tensor gauge field \( B_{\mu\nu}(x) = -B_{\nu\mu}(x) \), known as the Kalb-Ramond (KR) field.

In what follows, for brevity and concreteness, we shall set the four-dimensional dilaton field to a constant, \( \Phi(x) = \Phi_0 \). This will fix the string coupling

\[
g_s = \exp(\Phi) = \exp(\Phi_0) .
\]

There are always consistent solutions of the four-dimensional string theory with such a configuration, and this suffices for our purposes in this review.

There is a U(1) gauge symmetry of the closed-string (3+1)-dimensional target-space-time effective-field-theory action, associated with the KR \( B \)-field transformations

\[
B_{\mu\nu}(x) \rightarrow B_{\mu\nu}(x) + \partial_{[\mu}\theta_{\nu]}(x), \quad \mu, \nu = 0, \ldots, 3, \quad \theta_{\mu}(x) \in \mathbb{R},
\]

where Greek indices from now on denote space-time indices, taking on the values 0, \ldots, 3, and the symbol \([\ldots]\) denotes antisymmetrization of the respective indices.

The U(1) gauge symmetry of the closed-string sector implies that the corresponding effective action will be expressed only in terms of the field strength of the \( B \)-field:

\[
\mathcal{H}_{\mu\nu\rho}(x) = \partial_{\mu}B_{\nu\rho}(x).
\]

This is subject to the following Bianchi identity

\[
\mathcal{H}_{[\nu\rho;\mu]} = \partial_{[\mu}\mathcal{H}_{\nu\rho]} = 0 ,
\]

where, from now on, we omit the space-time-coordinate arguments of the fields \( (x) \) for brevity. The semicolon denotes covariant derivative with respect to the standard Christoffel connection \( \Gamma_{\mu\nu}^{\alpha} = \Gamma_{\mu\nu}^{\alpha} \) of the metric \( g_{\mu\nu} \). It should be stressed that, due to the total antisymmetry of \( \mathcal{H} \), the terms involving the gravitational Christoffel connection drop out from (4).
In superstring theory, anomaly cancellation requirements imply a modification of the KR field strength (3) by appropriate gauge (Yang-Mills ($Ω_{3Y}$)) and Lorentz ($Ω_{3L}$) (gravitational) Chern-Simons terms (Green-Schwarz (GS) mechanism) [45]

\[ \mathcal{H} = dB + \frac{\alpha'}{8\kappa} \left( \Omega_{3L} - \Omega_{3Y} \right), \]
\[ \Omega_{3L} = \omega^a \wedge d\omega_a + \frac{2}{3} \omega^a \wedge \omega^b \wedge \omega^c, \quad \Omega_{3Y} = A \wedge dA + A \wedge A \wedge A, \]

(5)

where $\alpha' = M_s^{-2}$, with $M_s$ the string mass scale, which is in general different from the four-dimensional Planck mass scale $M_P = 1.22 \times 10^{19}$ GeV $\equiv \sqrt{8\pi} \kappa^{-1}$. For notational brevity, in (5) we used differential-form language [48], with $d$ denoting the exterior-derivative one-form, $d = dx^\mu \partial_\mu$, and $\wedge$ the exterior (“wedge”) product among differential forms, such that $f^{(k)} \wedge g^{(\ell)} = (-1)^{k\ell} g^{(\ell)} \wedge f^{(k)}$, where $f^{(k)}$ and $g^{(\ell)}$ are $k-$ and $\ell-$ forms, respectively. Above, $A$ is the Yang-Mills gauge field one-form, and $\omega^a_b$ the spin-connection one-form (the Latin indices $a, b, c, d$ are $(3+1)$-dimensional tangent-space (i.e. Lorentz-group-SO(1,3)) indices, referring to the Minkowski manifold which is tangent to the space-time manifold at a coordinate point $x$).

The addition of the Chern-Simons terms in (5) leads to a modification of the Bianchi identity (4), which can now be written as [45]

\[ \varepsilon^{\mu\nu\rho\sigma} \mathcal{H}_{[\rho\sigma]:\mu} = \varepsilon_{abc} \mu \mathcal{H}^{abc}_{\mu} = \frac{\alpha'}{32\kappa} \sqrt{-g} \left( R_{\mu\nu\rho\sigma} \tilde{R}^{\mu\nu\rho\sigma} - F_{\mu\nu} \tilde{F}^{\mu\nu} \right) \]
\[ \equiv \sqrt{-g} G(A) = \partial_\mu K^\mu(A), \]

(6)

where the right-hand side denotes the mixed anomaly, due to chiral fermions in the theory circulating in the anomalous loop [49, 50], $g$ denotes the determinant of the metric tensor, $F = dA + A \wedge A$ is the two-form corresponding to the Yang-Mills field strength (we use form notation for brevity here), $R_{\mu\nu\rho\sigma}$ is the Riemann space-time curvature tensor\(^2\) and

\[ \varepsilon_{\mu\nu\rho\sigma} = \sqrt{-g} \varepsilon_{\mu\nu\rho\sigma}, \quad \varepsilon^{\mu\nu\rho\sigma} = \frac{\text{sgn}(g)}{\sqrt{-g}} \varepsilon^{\mu\nu\rho\sigma}, \]

(7)

with $\varepsilon^{0123} = +1$, etc., are the gravitationally covariant Levi-Civita tensor densities, totally antisymmetric in their indices. The symbol (…) over the curvature- or gauge-field-strength tensors denotes the corresponding dual, defined as

\[ \tilde{R}_{\mu\nu\rho\sigma} = \frac{1}{2} \varepsilon_{\mu\nu\lambda\rho} R^{\lambda\rho\sigma}, \quad \tilde{F}_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}. \]

(8)

The non-zero quantity on the right hand side of (6) is the “mixed (gauge and gravitational) quantum anomaly” [50, 49], which is known to be a total divergence of a function $K^\mu$ (containing the Chern-Simons forms in (5)).

To lowest order in the string Regge slope $\alpha'$, the $(3+1)$-dimensional effective action of the closed-string bosonic sector is then given by [44, 45, 46, 47]:

\[ S_B = - \int d^4 x \sqrt{-g} \left( \frac{1}{2\kappa^2} R + \frac{1}{6} \mathcal{H}_{\lambda\mu\nu} \mathcal{H}^{\lambda\mu\nu} + \ldots \right). \]

(9)

\(^2\) Our conventions and definitions used throughout this work are: signature of metric $(+, -, -, -)$, Riemann Curvature tensor $R^\lambda_{\mu\nu\sigma} = \partial_\lambda R^\lambda_{\mu\nu\sigma} + \Gamma^\lambda_{\mu\rho} \Gamma^\rho_{\nu\sigma} - (\nu \leftrightarrow \sigma)$, Ricci tensor $R_{\mu\nu} = R^\lambda_{\mu\lambda\nu}$, and Ricci scalar $R = R_{\mu\nu} g^{\mu\nu}$. 
The KR field strength terms $H^2$ in (9) can be absorbed up to an irrelevant total divergence into a contorted generalised curvature $\mathcal{R}(\Gamma)$, with a “torsional connection” \[\Gamma\] corresponding to a contorsion tensor proportional to the totally antisymmetric $H^\rho_{\mu
u}$ field strength,

$$\Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\nu\mu} + \frac{\kappa}{\sqrt{3}} H^\rho_{\mu
u} \neq \Gamma^\rho_{\nu\mu},$$

(10)

where $\Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\nu\mu}$ is the torsion-free (Riemannian) Christoffel symbol.

The reader should notice that the modifications (5) and the right-hand-side of (6) contain the torsion-free spin connection. In fact, it can be shown \[52, 53\] that any potential contributions from the (totally-antisymmetric) torsion $H$ three-form in the anomaly equation can be removed by adding to the string effective action appropriate counterterms order by order in perturbation theory.

Since the anomaly $G(\omega, A)$ is an exact one loop result, one may implement the Bianchi identity (6) as a $\delta$-functional constraint in the quantum path integral of the action (9) over the fields $H, A, g_{\mu\nu}$, and express the latter in terms of a Lagrange multiplier (pseudoscalar) field \[54, 55, 56, 57, 58\] $b(x)/\sqrt{3}$ (where the normalisation factor $\sqrt{3}$ is inserted so that the field $b(x)$ will acquire a canonical kinetic term, as we shall see below):

$$\Pi_x \delta\left(\varepsilon^{\mu\nu\rho\sigma} H_{\nu\rho\sigma}(x)_{,\mu} - G(\omega, A)\right) \Rightarrow$$

$$\int Db \exp \left[ i \int d^4x \sqrt{-g} \frac{1}{\sqrt{3}} b(x) \left( \varepsilon^{\mu\nu\rho\sigma} H_{\nu\rho\sigma}(x)_{,\mu} - G(\omega, A) \right) \right]$$

$$= \int Db \exp \left[ -i \int d^4x \sqrt{-g} \left( \partial^\mu b(x) \frac{1}{\sqrt{3}} \varepsilon^{\mu\nu\rho\sigma} H^{\nu\rho\sigma} + \frac{b(x)}{\sqrt{3}} G(\omega, A) \right) \right]$$

(11)

where the second equality has been obtained by partial integration, upon assuming that the KR field strength dies out at spatial infinity. Inserting (11) into the (Euclidean) path integral with respect to the action (9), and integrating over the $H$ field, one obtains \[54, 55\] an effective action in terms of the anomaly and a dynamical, massless, KR-axion field $b(x)$, with canonically-normalised kinetic terms.

### 2.2. Ambiguities in the KR-axion effective action and non-Hermitian interactions

There is a known ambiguity \[59\] in analytically continuing the resulting effective action back to Minkowski space time, which stems from the following fact: the $H$-field-strength Euclidean path integration results in the presence of a $b(x)$-dependent, quadratic in space-time derivatives, term in the bosonic (B) Euclidean (E) string effective action of the form:

$$S_{\text{eff}}^{(E)} B \ni \int d^4x \sqrt{g^{(E)}} \frac{1}{12} \varepsilon^{(E)}_{\mu\nu\rho\lambda} \varepsilon^{\mu\nu\rho\sigma} \partial^\lambda b \partial_\sigma b.$$  

(12)

where the covariant Levi-Civita tensor density $\varepsilon_{\mu\nu\rho\lambda}^{(E)}$ has been defined in (7), but here the notation (E) indicates that this quantity is evaluated in a Euclidean-signature metric (with the convention $(+,+,+,+)$). The ambiguity concerns the stage at which we analytically continue the quantity (12) back to Minkowski space-time.

If we first use the following property of the Levi-Civita tensor density in four space-time dimensions with Euclidean metric (“\textbf{Scheme I}”):

$$\varepsilon_{\mu\nu\rho\lambda}^{(E)} \varepsilon^{\mu\nu\rho\sigma} = +6 \delta^\sigma_\lambda,$$

(13)
where $\delta^\sigma_\lambda$ denotes the Kronecker delta, and then analytically continue to Minkowski space time, we obtain a real effective action for the dynamical field $b(x)$ (which plays the rôle of the KR gravitational axion): [54]

$$S_{\text{eff}(I)}^B = \int d^4x \sqrt{-g} \left[ -\frac{1}{2\kappa^2} R - \frac{1}{2} \partial_\mu b \partial^\mu b + \sqrt{\frac{2}{3}} \frac{\alpha' \kappa}{96} b(x) \left( R_{\mu\nu\rho\sigma} \tilde{R}^{\mu\nu\rho\sigma} - F_{\mu\nu} F^{\mu\nu} \right) + \ldots \right],$$

(14)

where the ellipsis denote gauge, as well as higher derivative, terms appearing in the string effective action, that we ignore for our discussion here. In this construction the KR axion appears as a standard pseudoscalar field, with a canonically-normalised kinetic term with the correct sign relative to the space-time curvature (Einstein-Hilbert) terms in (14). We also observe that, in view of the anomaly, the KR axion field couples to the gravitational and gauge fields. This latter interaction is P and T violating, and hence in view of the overall CPT invariance of the relativistic, local and unitary (quantum) field theory (14), also CP violating (we remind the reader that C denotes the standard Dirac charge-conjugation operator).

On the other hand, if one first analytically continues (12) to a Minkowski-signature space time, and then uses the Minkowski version of (13) ("Scheme II"):

$$\varepsilon_{\mu\nu\rho\lambda} \varepsilon^{\mu\nu\rho\sigma} = -6 \delta^\sigma_\lambda,$$

(15)

where the minus sign on the right-hand side is due to the dependence of the contravariant Levi-Civita tensor density (7) on the (Minkowski) signature of the metric tensor, then, one obtains an effective action in which the kinetic terms of the $b$ field have the wrong sign relative to the space-time-curvature terms in the effective action, and thus the KR axion would behave like a ghost field:

$$S_{\text{eff}(II)}^B = \int d^4x \sqrt{-g} \left[ -\frac{1}{2\kappa^2} R - \frac{1}{2} \partial_\mu b \partial^\mu b + i \sqrt{\frac{2}{3}} \frac{\alpha' \kappa}{96} b(x) \left( R_{\mu\nu\rho\sigma} \tilde{R}^{\mu\nu\rho\sigma} - F_{\mu\nu} F^{\mu\nu} \right) + \ldots \right],$$

(16)

Naively, one would ignore this second construction, because of the common perception that a ghost axion field does not carry any physical significance. However, in view of the PT-symmetric framework [1, 2, 3, 4, 8] and its field-theory extensions, one should reconsider this point of view. Indeed, by viewing the $b$ field in (16) as purely imaginary,

$$b(x) \rightarrow i b(x), \quad b(x) \in \mathbb{R},$$

(17)

one may write the corresponding effective action as

$$S_{\text{eff}(II)}^B = \int d^4x \sqrt{-g} \left[ -\frac{1}{2\kappa^2} R + \frac{1}{2} \partial_\mu b \partial^\mu b + i \sqrt{\frac{2}{3}} \frac{\alpha' \kappa}{96} b(x) \left( R_{\mu\nu\rho\sigma} \tilde{R}^{\mu\nu\rho\sigma} - F_{\mu\nu} F^{\mu\nu} \right) + \ldots \right],$$

(18)

which has a canonical kinetic term for the (redefined) axion field $b$, which no longer behaves as a ghost, but it is now characterised by non-Hermitian anomalous interactions. The latter make sense in a generalised PT framework, as has been discussed in [42] and will be reviewed below. Such non-Hermitian effective actions are characterised by a generalised antilinear CPT symmetry, with the charge conjugation operation $\mathcal{C}$ defined appropriately [10, 21], to be discussed below, which guarantees the reality of the energy eigenvalues of the system.
Although the parameters \( \alpha' \) and \( \kappa^2 \) are independent in generic string models [46, 47], especially in view of the possibility of large-extra-dimension compactifications, nonetheless for concreteness in what follows, we shall set [40, 42]

\[
\sqrt{\alpha'} = M_s^{-1} \sim \kappa = M_{\text{Pl}}^{-1} = \sqrt{8\pi M_P^{-1}} \sim (2.4 \times 10^{18})^{-1} \text{GeV}^{-1},
\]

where the reader should recall that \( M_s \) is the string mass scale, and \( M_{\text{Pl}} \) is the reduced Planck mass in (3+1)-dimensions. It goes without saying that this restriction on string parameters does not affect the qualitative conclusions of our analysis on mass generation, and one can straightforwardly extend it to include more general cases in which \( \sqrt{\alpha'} \neq \kappa \).

2.3. Inclusion of fermions

Upon the inclusion of (Dirac) fermions, the torsion interpretation (10) of the KR three-form \( \mathcal{H} \), implies that the curved-space Dirac terms read [54, 60, 56, 57]:

\[
S_{\text{Dirac}} = \int d^4x \sqrt{-g} \left[ \frac{i}{2} \left( \bar{\psi}_j \gamma^\mu \mathcal{D}(\mathcal{H})_{\mu} \psi_j - (\mathcal{D}(\mathcal{H})_{\mu} \bar{\psi}_j) \gamma^\mu \psi_j \right) - m(j) \bar{\psi}_j \psi_j \right],
\]

\[
= \int d^4x \sqrt{-g} \bar{\psi}_j \left( \frac{i}{2} \Gamma^a \mathcal{D}_a m(j) \right) \psi_j - \int d^4x \sqrt{-g} (\mathcal{F}_a + \mathcal{F}_0) \bar{\psi}_j \gamma^5 \Gamma^a \psi_j
\]

\[
\equiv S_{\text{Free Dirac}} + \int d^4x \sqrt{-g} (\mathcal{F}_a + \mathcal{F}_0) J^5 a,
\]

where \( \mathcal{A}_{abu} = \omega_{abu} + K_{abu}, \) \( K_{abc} = \frac{1}{2} (\mathcal{H}_{cab} - \mathcal{H}_{abc} - \mathcal{H}_{bca}) = \mp \frac{i}{2} \mathcal{H}_{abc}, \) is the generalised spin-connection with (totally-antisymmetric) torsion \( \mathcal{H} \). As before, Latin indices \( a, b, c, \ldots \) denote tangent-space indices, raised and lowered with the help of the Minkowski metric \( \eta^{ab} \) of the tangent space (at a point with coordinates \( x^\mu \)) of a space-time with metric \( g_{\mu\nu}(x) = \epsilon^a_\mu(x) \eta^{ab} \epsilon^b_\nu(x) \), with \( \epsilon^a_\mu(x) \) the vielbeins and \( \epsilon^a_\nu(x) \) their inverse. \( \Gamma^a \) is a tangent-space Dirac matrix, such that the space-time Dirac matrices \( \gamma^\mu(x) \) are given by \( \gamma^\mu(x) = \epsilon^a_\mu(x) \Gamma^a \), and we used the standard notation for \( \nabla \mathcal{D}_a \psi = \nabla \mathcal{D}_a \psi - \nabla \chi \psi \). The (gravitational) covariant derivative is given by \( \mathcal{D}_a = \partial_a - \frac{i}{4} \mathcal{A}_{bca} \sigma^{bc}, \) \( \sigma^{ab} = \frac{i}{2} [\Gamma^a, \Gamma^b] \).

In (20) we have defined the quantities \( \mathcal{F}^d = \epsilon^{abcd} \epsilon_{b\lambda} \partial_\lambda e^\gamma_c, \) \( B^d = \frac{1}{4} \epsilon_{abc} \mathcal{H}^{abc} \). In flat or Friedmann-Robertson-Walker space time backgrounds, of interest to us in this review, \( \mathcal{F}^a = 0 \), and thus it will not play any role in our analysis.

The axial fermionic current is given by : \( J^{5a} = \bar{\psi}_j \gamma^\mu \gamma^5 \psi_j \), and correspondingly \( J^{5a} = \bar{\psi}_j \Gamma^a \gamma^5 \psi_j \), with the repeated index \( j = 1, \ldots, N_f \) summed over the fermionic matter species \( \psi_j \) of the model. The matrix \( \gamma^5 = i \gamma^0 \gamma^1 \ldots \gamma^3 \) is the standard chirality matrix. The term involving the interactions of the b-field with the axial current in (20) can be partially integrated to give

\[
\int d^4x \sqrt{-g} \left[ \frac{\kappa}{2} \nabla_\mu b \right] J^{5a} = - \int d^4x \sqrt{-g} \left[ \frac{\kappa}{2} \partial_\mu b(x) J^{5a}_{\mu a} \right]
\]

where the covariant divergence of the axial current is non-zero in a theory of massless chiral fermions with chiral mixed anomalies, as in (20):

\[
J^{5a}_{\mu a} \propto \mathcal{G}(\omega, A) \neq 0
\]

where \( \mathcal{G}(\omega, A) \) is defined in (6). The proportionality factors involve the number of chiral fermionic degrees of freedom circulating in the anomalous loop [49], which is model dependent. In some models, e.g. the string-inspired cosmology models of [57, 58], one may encounter
a cancellation of the gravitational anomalies, during the radiation and matter epochs of the Universe, but chiral and QCD-anomalies (where the gauge field $A$ represents the gluon) survive.

It should be understood in what follows that one may extend the above considerations to include chiral as well as Majorana chiral fermions [41, 42], as is the case of the model of [43], which will be our main motivation for the non-Hermitian models discussed in this review.

Adding the fermionic action (20) to the bosonic one (9), implementing the constraint (11), performing the $H$-path integration (in Euclidean formalism), and analytically continuing back to Minkowski space time, we obtain the following effective action in the “Scheme I” (13):

$$S^{\text{eff}} (I) = \int d^4x \sqrt{-g} \left[ -\frac{1}{2\kappa^2} R + \frac{1}{2} \partial_{\mu} b \partial^{\mu} b - \sqrt{\frac{2}{3}} \frac{\kappa}{96} \partial_{\mu} b(x) K^{\mu} \right] + S_{\text{Dirac}} + \int d^4x \sqrt{-g} \left( F_{\mu} + \frac{\kappa}{2} \sqrt{\frac{3}{2}} \partial_{\mu} b \right) J^5_{\mu} - \frac{3\kappa^2}{16} \int d^4x \sqrt{-g} J^5_{\mu} J^5_{\mu} + \ldots + \ldots ,$$

where $K^{\mu}$ has been defined in (6), and is related to the mixed anomalies. The ellipsis in (23) indicate gauge field kinetic terms, as well as terms of higher order in derivatives, of no direct relevance to us here. The four-fermion axial-current-current repulsive term in (23), is characteristic of Einstein-Cartan fermionic theories on non-Riemannian spaces with torsion [51, 61], which in our case is provided by the field strength of the KR field, $H_{\mu\nu\rho}$ (10). The action (23) is Hermitian.

On the other hand, in the “Scheme II” (15), upon using (17), one arrives at the following non-Hermitian (complex) effective action:

$$S^{\text{eff}} (II) = \int d^4x \sqrt{-g} \left[ -\frac{1}{2\kappa^2} R + \frac{1}{2} \partial_{\mu} b \partial^{\mu} b - i \sqrt{\frac{2}{3}} \frac{\kappa}{96} b(x) K^{\mu} \right] + S_{\text{Dirac}} + \int d^4x \sqrt{-g} \left( F_{\mu} + i \frac{\kappa}{2} \sqrt{\frac{3}{2}} \partial_{\mu} b \right) J^5_{\mu} - \frac{3\kappa^2}{16} \int d^4x \sqrt{-g} J^5_{\mu} J^5_{\mu} + \ldots + \ldots ,$$

which is embeddable in a generalised PT-framework [42], and we shall discuss it below.

The reader should have noticed the invariance of the actions (23) and (24) under the shift symmetry of the KR axion field

$$b(x) \to b(x) + \text{constant},$$

which is characteristic of axion physics [62, 63]. In our string model this is associated [45, 54, 55] with the U(1) gauge invariance (2) of the closed-string actions (9) and (20).

2.4. Non-perturbative effects and breaking of axionic shift symmetry

In the presence of non-Abelian gauge fields in the anomalous couplings of the axions, e.g. QCD-type gluon fields, non-perturbative instantons are responsible for the generation of shift-symmetry-breaking axion potentials [62, 63]. In particular, for a generic axion field $a(x)$ with anomalous coupling to gluons (or in general non-Abelian gauge fields) of the (generic) form:

$$S_{\text{axion-QCD}} \ni \int d^4x \sqrt{-g} \frac{1}{f_a} a(x) F_{\mu\nu} \tilde{F}^{\mu\nu} ,$$

where $f_a$ denotes the pertinent axion coupling (with mass dimension +1), and $F_{\mu\nu}$ is the non-Abelian gauge-field strength, non-perturbative instanton effects are responsible for generating
an axion potential of the form

\[ \mathcal{U}(a) = \Lambda_{\text{inst}}^4 \left[ 1 - \cos \left( \frac{a(x)}{f_a} \right) \right], \quad (27) \]

where \( \Lambda_{\text{inst}} \) is the appropriate scale, e.g. the QCD scale (\( \Lambda_{\text{QCD}} \sim 200 \text{ MeV} \)), if the gauge field pertains to gluons. The potential (27), therefore, breaks the shift symmetry (25), which is now restricted to [62, 63]

\[ a(x) \rightarrow a(x) + 2\pi f_a. \quad (28) \]

In our string-inspired model, it is the KR field that may couple to such instanton effects, through the anomalous terms in (23). In such a case, the corresponding axion coupling \( f_a \) is given by

\[ f_b \equiv \sqrt{\frac{8}{3} \frac{\kappa}{\alpha'}} = \sqrt{\frac{8}{3} \left( \frac{M_s}{M_{\text{Pl}}} \right)^2 M_{\text{Pl}}}, \quad (29) \]

where \( M_{\text{Pl}} \) is defined in (19), and we have used the notation \( f_b \) to denote the axion coupling pertaining to the KR axion \( b(x) \). Here we keep the \( M_s \) and \( M_{\text{Pl}} \) different in magnitude, following the phenomenological study in [57, 58]. We shall come back to applying (19) later. We stress that the generation of a potential for the KR field implies that a torsion interpretation of this field is no longer possible in the massive phase.

From (29), we thus observe that the range of the axion coupling \( f_b \) depends on the allowed range of the string mass scale \( M_s \). For the model of, e.g., ref. [57, 58], the allowed range of the latter scale is

\[ M_{\text{Pl}} \gtrsim M_s \gtrsim 10^{-3} M_{\text{Pl}}, \quad (30) \]

where the upper bound of \( M_s \) corresponds to (19). This implies

\[ 3.9 \times 10^{12} \text{ GeV} \lesssim f_b \lesssim 3.9 \times 10^{18} \text{ GeV}. \quad (31) \]

On the other hand, the generic axion coupling constant \( f_a \) is constrained to lie in the range [63]

\[ 10^9 \text{ GeV} < f_a < 10^{12} \text{ GeV}. \quad (32) \]

We note, though, that astrophysical constraints [64, 65, 66, 67, 68] may extend the upper bound up to \( 10^{17} \) GeV. We thus observe that, even if the astrophysical constraints are ignored, there is still a marginal overlap (in order of magnitude) between the minimally allowed region of \( f_b \), (31), and the maximally allowed phenomenological region of the QCD axion coupling constant, (32).

From (27), for the case of the KR axion \( b(x) \), we infer that the instanton-induced KR-axion mass is [58]

\[ m_b = \left[ \frac{\partial^2 V_{\text{QCD}}^b}{\partial b^2} \right]_{b=0} = \frac{\Lambda_{\text{QCD}}^2}{f_b} = \sqrt{\frac{3}{8}} \left( \frac{\Lambda_{\text{QCD}}}{M_s} \right)^2 M_{\text{Pl}} = \sqrt{\frac{3}{8}} \left( \frac{\Lambda_{\text{QCD}}}{M_{\text{Pl}}} \right)^2 \left( \frac{M_{\text{Pl}}}{M_s} \right)^2 M_{\text{Pl}}, \quad (33) \]

which, in view of (30), (31), lies in the range

\[ 1.17 \times 10^{-11} \text{ eV} \lesssim m_b \lesssim 1.17 \times 10^{-5} \text{ eV}, \quad (34) \]
well within the range calculated in lattice QCD approaches (see, e.g. ref. [69] and references therein): $m_a \sim 5.7 \left( \frac{10^{12\text{ GeV}}}{f_a} \right) \times 10^{-6} \text{ eV}$. The above considerations pertain to the Hermitian effective action (23) in “Scheme I”. In this review we shall discuss the phenomenology of models with non-Hermitian anomalous couplings, (24), in “Scheme II”. Such models correspond to purely imaginary axion coupling constants for which the aforementioned mass generation through instantons is not clear (for, instance, for purely imaginary axion couplings in the potential (27), $f_a = i f_b$, $f_b \in \mathbb{R}$ (with $f_b$ defined in (29)), there are no mass terms generated, as the corresponding quadratic terms in the axion field $a(x)$ have the wrong sign). It is the point of this review to discuss alternative ways for mass generation for axions (and fermions) dynamically, which is done in sections 3 and 4. We use specific non-Hermitian effective actions which, as we shall discuss, complement (24) by appropriate anti-Hermitian Yukawa interactions of axions with the fermions [43]. The latter constitute the main source of dynamical-mass generation.

Before doing this, though, it is necessary to motivate the pertinent non-Hermitian models carefully within the microscopic string theory framework. This necessitates some discussion first on the various types of axions existing in string models, which we now proceed to review.

2.5. Stringy Model-independent (KR) and Model-dependent axions: kinetic mixing and Yukawa interactions

In ref. [43] it was postulated that non-perturbative physics might also be responsible for a further breaking of the shift symmetry by means of Yukawa interactions between axions and fermions, specifically right-handed neutrinos, in the context of the above-described Hermitian string-inspired model (23) in the “Scheme I”. To this end, one exploits the fact that in string theory [55, 70] there are many axion fields [55, 70], associated with the Kaluza-Klein (KK) zero modes of appropriate $p$-forms in the spectrum of strings compactified to four space-time dimensions, i.e. formulated on a target-space-time manifold of the form $M_{1,3} \times X_6$. Here, $M_{1,3}$ denotes the uncompactified (3+1)-dimensional space time, and $X_6$ the extra-dimensional space, assumed to be a smooth compact manifold. For instance, in heterotic string theory [45, 47], one has the (Neveu-Schwarz(NS)-type) two-form $B$ of the Kalb-Ramond field in ten dimensions, which, upon compactification on an appropriate (say Calabi-Yau [45]) six-dimensional compact space, $X_6$, can be written as:

$$B = B_{\mu\nu}(x) \, dx^\mu \wedge dx^\nu + \frac{1}{2\pi} \, b^I(x) \, \omega^{ij}_J(z, \bar{z}) \, dz^i \wedge d\bar{z}^j, \quad \mu, \nu = 0, \ldots, 3, \quad i, j = 1, 2, 3$$

where $z^i, i = 1, 2, 3$ are complex coordinates parametrising the compact manifold. The $B_{\mu\nu}(x)$ field yields, upon the dualisation procedure associated with the implementation of the corresponding Bianchi identity for its field strength (6) via a Lagrange multiplier, the KR axion $b^I(x)$, as discussed above. In the language of string theory this is the so-called model independent axion, as it is present in all string theories. The quantities $\omega^{ij}_J(z, \bar{z}), I = 1, \ldots, h^{1,1}_1$ in (35), represent harmonic (1,1) forms that depend only on the coordinates of the complex manifold, and are linked to the aforementioned KK zero modes. One uses the normalisation [55]

$$\int_{C^I} \omega^J = \delta^{IJ} \quad (36)$$

where $C^I$ is a 2-cycle in the compact manifold. In other words, the harmonic forms $\omega^I$ span the integer (1,1) cohomology group of the target space [48].

The quantities $b^I(x), I = 1, \ldots, h^{1,1}_1$ represent dimensionless pseudoscalar fields on the uncompactified space-time, and the factor $\frac{1}{2\pi}$ has been inserted so that the fields $b^I(x)$ have...
a period $2\pi$, as is conventional for axions. These four-dimensional fields correspond to the so-called model-dependent axions \cite{55}. The kinetic terms of the two-form yield the four-space-time-dimensional kinetic terms of the $b^I(x)$ fields. Indeed, to this end, and taking into account the structure and the space-time dependences of the field components in (35), we need only to consider the following components of the field strength $H_{MNP} = \partial_M B_{NP}$ (with capital Greek letters denoting indices referring to the ten-dimensional space-time of the superstring/brane theory, $M, N, P = 0, \ldots, 9$):

$$H_{\mu ij} = \partial_\mu B_{ij} = (\partial_\mu b^I(x)) \omega_{ij}^I(z, \bar{z}) \quad , \quad \mu = 0, \ldots, 3, \quad i, j = 1, 2, 3.$$ 

These components are the only ones associated with the $b^I$-model-dependent-axion terms on the right-hand side of (35), which are non-vanishing (the terms associated with the first, $B_{\mu
u}$-dependent terms yield, of course, the purely four-dimensional KR antisymmetric field strength (3)).

The pertinent kinetic terms for the axions $b^I(x)$ in $(3+1)$-dimensions stem from $H_{MNP} H^{MNP}$-structures in the ten-dimensional effective action, which, upon compactification down to $(3+1)$-dimensions, yield terms of the form

$$S_{10-\text{dim}} \cong \int \sqrt{-g} \, d^4x \int_{\mathcal{X}_6} \partial_\mu B_{ij} \partial^\mu B^{ij} = \int \sqrt{-g} \, d^4x \int \partial_\mu b^I(x) \partial^\mu b^I(x) \int_{\mathcal{X}_6} \omega_{ij}^I(z, \bar{z}) \omega^{J\bar{J}}(z, \bar{z}) ,$$

$$\equiv \int \sqrt{-g} \, d^4x \partial_\mu b^I(x) \partial^\mu b^I(x) \gamma^{I\bar{J}} \quad , \quad \mu = 0, \ldots, 3, \quad I, J = 1, \ldots, h^{1,1},$$

where, for brevity, we only indicated the structures, omitting numerical coefficients. The reader should observe the non-trivial kinetic mixing $\gamma^{I\bar{J}} \neq \delta^{I\bar{J}}$ of the model-dependent stringy axions $b(x)^I$.

It is important to stress that the kinetic terms of the model-dependent stringy axions do not suffer any ambiguities and are the same in both “Schemes I and II”, corresponding to standard axion kinetic terms, since they do not contain the Levi-Citiva tensor density $\varepsilon_{\mu\nu\rho\sigma}$ (7) (with $\mu, \nu, \rho, \sigma = 0, \ldots, 3$ denoting indices in the $(3+1)$-dimensional space-time manifold $M_{1,3}$).

The axion coupling for the fields $b^I(x)$ can be determined by looking \cite{55} at the (one-loop) counterterms required for the GS anomaly-cancellation mechanism in string theory \cite{45}. As a concrete example, we may consider the $E_8 \times E_8$ heterotic string, formulated on $M_{1,3} \times \mathcal{X}_6$, with the Standard Model gauge group $SU(3)_c \times SU(2) \times U_Y(1)$ embedded, say, in the first $E_8$ group factor. For brevity and concreteness, we take the uncompactified space-time manifold $M_{1,3}$ to be Minkowski flat. As shown in \cite{55}, then, one obtains the following effective-action anomaly terms for the axion-$b^I(x)$ fields:

$$S_{\text{anom string axion}} = - \left( \frac{1}{16 \pi^2} \int_{\mathcal{X}_6} \omega^I(z, \bar{z}) \wedge \left[ \text{Tr}_1 \mathbf{F} \wedge \mathbf{F} - \frac{1}{2} \mathbf{R} \wedge \mathbf{R} \right] \right) \int d^4x \, b^I(x) \frac{1}{16 \pi^2} \text{Tr}_1 \mathbf{F} \wedge \mathbf{F}$$

where, in order to arrive at (38) starting from the original form of the GS counterterms, the Bianchi identity (6) has been used. The term inside the parentheses on the right-hand side of the above relation expresses mixed anomalies in the compact manifold $\mathcal{X}_6$, with $\mathbf{F}$ ($\mathbf{R}$) the appropriate gauge-field (compact-space-$\mathcal{X}_6$ curvature) two-form over the compact space, and $\wedge$ the appropriate exterior product among differential forms \cite{48}, as mentioned previously; the trace $\text{Tr}_1$ pertains to the first $E_8$ gauge group.

---

3. In this formalism, the corresponding axion fields with mass dimension +1 and canonically-normalised kinetic terms in the effective action are given by $f_a \cdot b^i$, where $f_a$ are the corresponding axion couplings, of mass dimension +1, appearing in the (gauge and gravitational) anomalous, CP-violating, interactions of the axion moduli fields \cite{55}.

4. Such mixing has been exploited in \cite{71, 72} to discuss a widening of the allowed window of stringy-axion coupling constants, and also study large-field axion-induced inflation.
We also remark at this point that in generic string or D-brane models, compactified or projected to four space-time dimensions, model-dependent axion fields \( a^I(x) \) are also obtained as KK zero models of other appropriate \( p \)-form fields, \( C_p \), in the string spectrum on \( M_{1,3} \times X_6 \), for instance, the Ramond-Ramond(RR)-type \( p = 0, 2, 4 \)-forms of type IIB string theory, or the \( p = 1,3 \)-forms of type I\( A \) [55]. Similarly to the NS 2-form \( B(35) \), the corresponding model-dependent, dimensionless, (3+1)-dimensional axion fields are then given by

\[
a^I(x) = \frac{1}{2\pi} \int_{c^{(p)}_I} C_p, \quad I = 1, \ldots, M,
\]

where \( c^{(p)}_I \subset X_6 \) are appropriate homologically-non-equivalent \( p \)-cycles in the compact manifold, and we normalised again the axion so as to have period \( 2\pi \). We stress that qualitatively similar anomaly terms to (38) occur for the axions \( a^I(x) \).

In the model of [43], we may assume that there is an anomaly cancellation between gauge and gravitational anomaly terms in the compact manifold \( X_6 \) in (38), which means that such terms vanish. This is a special case, sufficient for our purposes here. In this case, the stringy model-dependent axions are characterised by their kinetic terms only and have no potential.

However, even in such a case, there is an induced coupling between the axions \( b^I(x) \) (or \( a^I(x) \)) and anomaly terms in the (3+1)-dimensional effective action, if there is kinetic mixing of such model-dependent axions with the model-independent KR \( b \)-axion field:

\[
S_{\text{kin. mix.}} = \gamma_J \int d^4x \sqrt{-g} \partial_\mu b \partial^\mu a_J, \quad J = 1, \ldots, M,
\]

where the repeated index \( J = 1, \ldots, M \), is summed over the stringy axion species. One could also include standard QCD axions in such a summation that may co-exist with stringy axions. Such a mixing has been assumed in the effective field theory of [43].

The term (40) could be added to (23), along with the following Hermitian shift-symmetry-breaking Yukawa interactions [43]

\[
S_{\text{Yukawa}} = i \sum_{j=1}^{N_f} \sum_{J=1}^{M} \lambda_{Jj} \int d^4x \sqrt{-g} \overline{\psi}_j \gamma_5 \gamma_\mu a_J \phi^\mu_j ,
\]

where \( j \) runs over fermion species \( N_f \) in the model.

Before proceeding in examining the consequences of (41) for mass generation, which is the main topic of our work here, we should make some important remarks, distinguishing the genuinely quantum Yukawa structures (41) from classical non-derivative Yukawa structures associated with derivative interactions of axions after the use of fermion equations of motion. The latter characterise any axion model, and are used, for instance, in experimental nuclear-physics searches of axions, as they describe interactions of axions to nucleons [63]. As such, they are also present in the classical effective action of the (string)model-independent KR \( b \)-axion, due to its torsion nature [54].

Let us explain now how such classical Yukawa interactions arise from derivative, shift-symmetry-respecting interactions. To this end, we first recall that a derivative coupling of a generic axion field \( a(x) \) to matter fermions has the form (cf. (20)):

\[
S^\text{eff}_{a-F} \ni \int d^4x \sqrt{-g} \overline{\psi}_j \gamma^\mu \gamma^5 \psi_j \partial_\mu a(x) = -\int d^4x \sqrt{-g} a(x) \nabla_\mu (\overline{\psi}_j \gamma^\mu \gamma^5 \psi_j),
\]

where in the right-hand side we performed partial integration, assuming as usual that the fields and their derivatives vanish in the boundaries of space-time. If the fermions are massless and
chiral, then there might be quantum anomalies that could spoil the conservation of the chiral current, \( \bar{\psi}_i \gamma^\mu \psi_j \), so this term yields non-trivial anomalous contributions to the effective action. If, however, the fermions \( \psi_j \) are massive, of mass \( m^{(j)} \), then the divergence of the chiral current is non-zero already classically, as follows from the equations of motion

\[
\nabla_\mu (\psi_j \gamma^\mu \gamma^5 \psi_j) = 2i m^{(j)} \bar{\psi}_j \gamma^5 \psi_j, \quad j = \ldots N_f.
\]

In an anomalous quantum theory, of course, the right-hand side of (43) also receives anomalous contributions of the form \( G(\omega, A) \) (defined in (6)), see (22).

Substituting (43) onto the effective action term (42), one then obtains non-derivative Yukawa coupling terms in the effective action of the form (41). But this constitutes a classical argument, since the fermion equations of motion have been used. In contrast, the structures (41) represent fully quantum interactions, for which the fermion equations of motion should not be used. This is why we cannot apply (41) to the model-independent KR axion, because the latter, as we have seen, arises from a dualisation procedure of the gauge-invariant (under the symmetry (2)) KR field strength \( \mathcal{H}_{\mu\nu\rho} \), and, as such, it should appear in the local low-energy effective Lagrangian of string theory through its derivatives.

At this point we remark that the reader might be tempted, following [54, 55], to redefine the field strength \( \mathcal{H}_{\mu\nu\rho} \) by appropriate counterterms involving fermions, so as to ensure the validity of the modified Bianchi identity

\[
\varepsilon^{\mu\nu\rho\sigma} \mathcal{H}_{\nu\rho\sigma} - \sqrt{-g} G(\omega, A) - i \lambda \sqrt{-g} \bar{\psi} \gamma^5 \psi = 0, \quad \lambda \in \mathbb{R},
\]

order-by-order in perturbation theory of the (Hermitian) string-inspired effective field theory described by the actions (9) and (20). As we have mentioned previously, classically (43) one may obtain the structures \( \bar{\psi} \gamma^5 \psi \) from the divergence of the axial fermion current, related to the anomalies \( G(\omega, A) \), upon use of the fermion equations of motion for massive fermions. By postulating the constraint (44) one ensures the appearance of such structures at a quantum level as independent operators in the effective action. This would formally ensure the presence of Yukawa interactions \( \lambda b(x) \bar{\psi}(x) \gamma^5 \psi(x) \) in the string-inspired field-theory action obtained after the dualisation procedure, by means of which one implements the constraint (44) in the path integral of the effective low-energy field theory via the Lagrange-multiplier field \( b(x) \). In such a case, under the “Scheme II”, one would obtain effective actions with anti-Hermitian anomaly and Yukawa terms for the model-independent KR axion \( b(x) \), embeddable directly in a generalised-PT (CPT) framework [17, 10, 40, 41, 42]. The problem with this approach, however, is that the constraint (44) would imply a redefinition of the field strength \( \mathcal{H}_{\mu\nu\rho} \), which, apart from the standard Chern-Simons local counterterms (5), required for the GS anomaly-cancellation mechanism [45], would involve additional non-local fermionic counterterms of the form (up to numerical coefficients, not relevant for our discussion): \( \propto i \lambda \varepsilon^{\mu\nu\rho\sigma} \nabla^\rho \frac{1}{\sqrt{g}} (\bar{\psi} \gamma^5 \psi) \), where \( \Box \) denotes the gravitationally-covariant D’Alembertian, \( \Box \equiv \nabla_\mu \nabla^\mu \), with \( \nabla_\mu \) the gravitational covariant derivative. Such non-local terms are not acceptable in the framework of local effective field theories obtained from strings, given that the perturbative-scattering-matrix approach that defines the perturbative-string-theory limit [44] would not be valid. Hence we shall not consider such a procedure here, but we felt like mentioning it, in case one is prepared to speculate on the existence of such non-local counterterms in a more general quantum-gravity framework.

2.6. Non-perturbative effects and various types of Majorana neutrino masses

In [43] we therefore considered the addition of the non-derivative interactions (41) in the sector of model-dependent axions, which are not related to the aforementioned dualisation procedure of \( \mathcal{H}_{\mu\nu\rho} \). In this respect, we have offered an alternative interpretation of the origin of such terms.
It has been conjectured in [43] that instanton or other non-perturbative effects in string/brane theories can generate such shift-breaking Yukawa couplings, which are therefore characterised by Yukawa couplings $\lambda_{ij}$ very small in magnitude (cf. see (47) below).

It is pointed out at this point that non-perturbatively generated masses for right-handed Majorana neutrinos in string/brane theories have been considered in the literature in the past [73, 74, 75, 76], which are associated with (effective (3+1)-dimensional) field operators of the form

$$M e^{-U \tilde{N}(x) N(x)},$$

(45)

where $M$ is the string mass scale, $N$ is the Majorana field of the neutrino, and $U$ denotes linear combinations of complex string moduli, whose imaginary parts correspond to axion-like particles (which thus are characterised by the standard restricted shift symmetry (28), corresponding to periodic shifts by $2\pi$ for the dimensionless fields; the Majorana-neutrino-mass operator (45) is also invariant under the standard-model gauge group and a $U(1)_{B-L}$ gauge symmetry, where $B(L)$ denotes Baryon (Lepton) quantum numbers). The string/D-brane instanton zero modes imply right-handed neutrino Majorana masses suppressed by $\propto \sum_r D_r e^{-S_r}$, where $D_r$ are coefficients depending on flavour couplings of the neutrinos, and $S_r$ is the (real part of the) instanton action, for the type-$r$ stringy instanton (we note that there are many types of instantons in string/brane theories [73, 74, 75, 76]).

The interaction (41) we are conjecturing in [43], and reviewing here, is rather different from (45), as it depends linearly on the model-dependent axion, and thus is not characterised by a periodic-shift symmetry, but breaks the shift symmetry completely. It remains to be seen whether microscopic mechanisms in concrete string/brane theories exist that yield (41). This falls beyond the scope of the current work, where the model with this extra interaction is considered phenomenologically [43] as a model beyond the standard model where dynamical right-handed neutrino masses can be generated in non-conventional ways. Moreover, the kinetic mixing (40) itself, could be the result of the same non-perturbative physics [40, 41], e.g. through one loop-effects involving heavy fermions, existing only in internal lines of graphs, coupled to axions $b$ and $a^I$ via the (non-perturbatively generated) Yukawa couplings (41).

In the model of [43], the non-perturbative Yukawa coupling couples to right-handed Majorana neutrinos $\psi_R$ only, hence one considers adding the following Hermitian terms in the Hermitian effective string action (23), within the “Scheme I” framework:

$$S_{a-b} = \int d^4x \sqrt{-g} \left\{ \frac{1}{2} (\partial_\mu a(x))^2 + \gamma^i \partial_\mu b \partial^\mu a + i \lambda a(x) \left( \bar{\psi}_R^C \psi_R - \bar{\psi}_{R'}^C \psi_{R'}^C \right) + \ldots \right\},$$

(46)

where $\psi_R^C = (\psi_R)^C$ is the (Dirac) charge-conjugate of the right-handed fermion $\psi_R$, which in [43] is considered to be a sterile neutrino $\nu_R$ (the reader should notice that, due to the chiral nature of the fermions $\psi_R$, the $\lambda$-dependent terms in (46) correspond to the axial form in (41)). In (46), for brevity, we did not write explicitly the kinetic terms of $\psi$. These are understood to be included in the . . . Due to the non-perturbative nature of $\lambda$ one may assume

$$\lambda \sim \exp(-S_{\text{instanton}}) \ll 1,$$

(47)

where $S_{\text{instanton}} \gg 1$ is the (real part of an appropriate) large and positive instanton action.$^5$

Redefining the $b$-field as $b \rightarrow b' = b + \gamma a$, one diagonalises the kinetic axion terms in (46), and can readily perform the $b'$ field integration in the path integral corresponding to the effective action obtained from the sum of (23) and (46). In this way one arrives at an effective

$^5$ In view of the above discussion on the existence of many types of instantons in string theory, one might consider $\lambda$ as proportional to the appropriate sum involving many exponential suppression factors of the (real parts of the) actions of individual instantons.
Figure 1. Feynman graph pertaining to the anomalously generated Majorana mass for the right-handed fermions $\psi_R$, identified with sterile neutrinos $\nu_R$ [43]. The dark blob denotes the operator $b(x) R_{\mu\nu\lambda\rho} \tilde{R}^{\mu\nu\lambda\rho}$ (up to numerical coefficients), which is the only part of the anomaly relevant for sterile neutrinos, which do not couple to gauge fields; $b(x)$ denotes the KR (gravitational) axion and $a(x)$ an axion-like particle. The wavy lines are gravitons $h_{\mu\nu}$, and dashed lines denote axions. The cross "×" indicates the $b(x)$-$a(x)$ kinetic mixing (40). Figure taken from [43].

(3+1)-dimensional action for the dynamics of the stringy axion field $a(x)$ [43]. Upon rescaling appropriately the $a(x)$ field, so as to have canonical kinetic terms after the above procedure, one arrives at the following effective action involving the $a$-field:

\[
S_{\text{eff}} \bigg|_{\text{Scheme I}} = \int d^4x \sqrt{-g} \left[ \frac{1}{2} (\partial_\mu a(x))^2 - \frac{\gamma a(x)}{f_b \sqrt{1 - \gamma^2}} G(\omega, \Lambda) \right] + \frac{i \lambda}{\sqrt{1 - \gamma^2}} a(x) \left( \bar{\psi}_R^C \psi_R - \bar{\psi}_R^C \psi_R^C \right) + \ldots ,
\]

where the KR-axion coupling $f_b$ is defined in (29). The ellipsis includes terms independent of $a(x)$, including the kinetic terms of the $\psi_R$ fermions, as well as anomaly-dependent terms stemming from the $b'(x)$-field path integration, of the form $K_\mu K^\mu$, with $K_\mu$ defined in (6). The reader should notice the induced anomalous interactions of the (model-dependent stringy) axion field $a(x)$ field in (48), proportional to the kinetic-mixing parameter $\gamma/\sqrt{1 - \gamma^2}$. It should be stressed that, to avoid the axion behaving as a ghost, one needs to impose the condition [43]

\[
0 < \gamma^2 < 1 .
\]

From the effective action (48), one may deduce novel ways of generating radiatively Majorana masses for the right-handed fermions (sterile neutrinos in the model of [43]), which are based on diagrams depicted in figure 1. The radiatively-generated Majorana sterile-neutrino mass is $M_R$ found to be [43]:

\[
M_R \sim \frac{\sqrt{3} \lambda \gamma \kappa^5 \Lambda^6}{49152 \sqrt{8} \pi^4 (1 - \gamma^2)} ,
\]

where $\Lambda$ is the Ultra-Violet (UV) momentum cutoff. The computation involved the use of graviton fluctuations $h_{\mu\nu}(x)$ about flat Minkowski space-time backgrounds. These yield non-trivial contributions to the gravitational-anomaly term $R_{\mu\nu\rho\sigma} \tilde{R}^{\mu\nu\rho\sigma}$, which are computed using perturbative (quantum) gravity methods developed in [77] (note that the gravitational-anomaly term vanishes for flat or cosmological Friedmann-Lemaître-Robertson-Walker background space-times). In the context of string theory, $\Lambda$ and $\kappa^{-1}$ are related [43] via the string mass scale and compactification radii of the extra-dimensional spatial manifold. For a generic quantum gravity
model, independent of string theory, one may use simply $\Lambda \sim \kappa^{-1}$. The sterile-fermion mass (50) is independent of the axion-$a$ potential [43], and thus its mass. As the reader can notice, the mass $M_R$ is generated for arbitrarily small $\gamma$ and $\lambda$ (real) couplings. This covers the case where both $\gamma$ and $\lambda$ are generated non-perturbatively, say by instanton effects, as discussed previously, and thus are very small, of order (47).

We now remark that, in the “Scheme II”, in which there are purely-imaginary anomalous couplings of the KR model-independent axions $b(x)$ in the effective action (24), a redefinition of the KR axion field $b \to b' = b + i \gamma a$, decouples the model-independent axion from the model-dependent one, $a(x)$, and, as the reader can easily verify, upon integrating out formally in the path integral $b'(x)$, one obtains the following effective action for the model-dependent axion field $a(x)$:

$$S_{\text{eff}}^a \bigg|_{\text{Scheme II}} = \int d^4x \sqrt{-g} \left[ \frac{1}{2} (\partial_\mu a(x))^2 + \frac{\gamma a(x)}{f_b \sqrt{1 + \gamma^2}} G(\omega, A) + \frac{i \lambda}{\sqrt{1 + \gamma^2}} a(x) \left( \bar{\psi}_R^C \psi_R - \bar{\psi}_R \psi_R^C \right) + \ldots \right],$$

(51)

where the ellipsis includes anomaly-dependent terms stemming from the $b'$ integration of the form $-K^\mu K^\mu$ (note the opposite sign as compared with the corresponding terms in the Hermitian case (48)). The reader should notice that there is no restriction in the range of $\gamma$ in this case. It should also be noticed that the action (51) is still Hermitian, despite the non-Hermiticity of the KR-$b$-axion Lagrangian (24), in “Scheme II”. Through the anomalous (radiative) graphs of fig. 1, one generates a right-handed-neutrino Majorana mass in this case of the form

$$M_R \sim \frac{\sqrt{3} \lambda \gamma \kappa^5 \Lambda^6}{49152 \sqrt{8} \pi^4 (1 + \gamma^2)},$$

(52)

for the entire range of $\gamma \in \mathbb{R}$ (in computing the graph of fig. 1 care should be taken so that the product $\frac{\lambda \gamma}{1 + \gamma^2}$ is not too big, otherwise our perturbative analysis [43] breaks down; this is guaranteed for sufficiently small $|\lambda \gamma| < 1$).

2.7. Motivation for embedding the model (51) in a generalised-PT (CPT) framework

We next remark that, although from a string point of view, the resulting effective actions involving model-dependent axions and non-derivative Yukawa interaction with fermions are real (cf. (48), (51)), nonetheless we may embed them in a non-Hermitian PT-symmetry framework by analytically continuing both coefficients $\gamma$ and $\lambda$ to purely imaginary values [40, 41]

$$\gamma \rightarrow i \gamma, \quad \lambda \rightarrow i \lambda, \quad \gamma, \lambda \in \mathbb{R}.$$

(53)

Formally, such a procedure will leave the radiatively generated masses (50), (52) real, but now the corresponding effective actions will have non-Hermitian interactions. This result, however, is only formal, given that the presence of non-Hermitian anomaly terms are yet to be understood. Hence, the methods developed for the Hermitian case to deal with graviton fluctuations of the gravitational anomaly terms [77] which lead [43] to the radiative mass (50) might not apply.

We therefore seek for alternative mass generation in such non-Hermitian cases embeddable in a generalised PT-symmetric framework [17], which is actually [40, 41, 42] CPT symmetric, under an appropriate definition of the charge-conjugation $C$ operator [21], as we shall discuss in the next section. As already mentioned, such an embedding guarantees the reality of the pertinent energy spectra [10] and, thus, potentially of the masses that can be generated dynamically.
The motivation for the analytic continuation (53), lies on the fact that the Hermitian Yukawa interactions cannot lead to dynamical generation of axion and fermion masses, unless there is a bare axion mass present [41, 42]. In contrast, in the non-Hermitian case (53) one can obtain truly dynamical masses for both axions and fermions, in the absence of bare masses for such fields, but in the presence of attractive four-fermion interactions, as we shall discuss below. The physical interest in having massive axions is linked with the potential rôle of such fields as dark matter.

It is the purpose of this article to review briefly the dynamical generation of masses for axions and fermions in this latter (non-Hermitian) set up. In the next section 3, we examine the issue of dynamical mass generation in theories with non-Hermitian (actually, anti-Hermitian) Yukawa couplings in the absence of anomalous terms, i.e. \( \gamma = 0, \lambda \neq 0 \). Motivated by our arguments about the potential generation of such Yukawa interactions through non-perturbative effects, we restrict ourselves to very small Yukawa coupling \( |\lambda| \ll 1 \). We incorporate a non zero \( \gamma \neq 0, |\gamma| < 1 \), as well as non-Hermitian axial backgrounds, in section 4. Despite the smallness of the couplings \( \lambda \) and \( \gamma \), we use non-perturbative Schwinger-Dyson (SD) analysis.

In what follows we shall use only prototype models, involving one Dirac fermion and one axion. This suffices to demonstrate the main conclusions. The extension to Majorana fermions is straightforward and has been done in [41, 42], where we refer the interested reader for details. These references also contain an analysis of SD dynamical mass generation for the Hermitian effective actions (48) (or (51)).

We remark at this point that, in addition to the \( \lambda \)- and \( \gamma \)-dependent terms in (48) (or (51)), which are analytically continued, as per (53), to yield the corresponding anti-Hermitian, CPT-invariant theories of interest to us here [40, 41, 42], we shall also include attractive Hermitian four-fermions interactions. Such interactions are present in any low-energy effective string theory with fermions, as a consequence of the exchange of heavy string states among fermions [78]. Integrating out such heavy states results in effective low-energy actions with contact four-fermion interactions of various types. It suffices for our purposes to consider the addition to our prototype models of four-fermion attractive interactions of the form

\[
-\frac{1}{2} f_4^2 \left( \bar{\psi} \gamma^5 \psi \right)^2 ,
\]

where \( f_4 \) is an appropriate coupling with dimensions of [mass]. As we shall discuss below, the rôle of such four-fermion interactions is crucial for fermion dynamical mass generation. Therefore, in view of the quantum-torsion-induced four-fermion repulsive interactions involving the covariant square of the axial fermion current \( J_5^\mu \) in (23) (or (24)), which upon use of Fierz identities, yield - among other terms - repulsive structures of the form (54), one must have sufficiently strong attractive interactions (54) to overcome the corresponding torsion-induced repulsions. This will be assumed in what follows, and from now on we ignore the repulsive four-fermion interactions due to torsion, as we assume that the dimensionful coupling \( f_4^2 \) of the interaction (54) describes an effective interaction where the effects of repulsion have been subtracted appropriately.

As we shall see in the following sections, the dynamically-generated masses will be much smaller than the ultraviolet (UV) momentum cut-off scale \( \Lambda \) in the effective, string-inspired, low-energy four-dimensional theory [43, 40, 41]. It is natural to consider \( \Lambda \lesssim M_s \), where \( M_s \) is the string scale, above which stringy effects appear, which would jeopardize the validity of any point-like string-inspired effective action. As we have demonstrated in [41, 42], and shall review below, the four-fermion dimensionful coupling \( f_4 \) and UV cut-off scale \( \Lambda \), will be determined self-consistently by the SD analysis in the massive phase of the models [41].
3. Dynamical Mass generation induced by non-Hermitian Yukawa interactions

We consider the following model consisting of a single pseudoscalar (axion) field \( \phi(x) \) and a single Dirac fermion, \( \psi(x) \), interacting with each other via a non-Hermitian Yukawa (specifically, anti-Hermitian) interaction with a real coupling \( \lambda \in \mathbb{R} \). Both axion and fermion fields have zero bare masses. We restrict ourselves to Minkowski space time for concreteness. The action reads

\[
S_{\text{Yuk, Herm}}^{\text{non}} = \int d^4x \left( \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \bar{\psi} i \gamma \psi + \lambda \phi \bar{\psi} \gamma^5 \psi - \frac{g^2}{2} f_4 (\bar{\psi} \gamma^5 \psi)^2 \right), \quad \lambda \in \mathbb{R},
\]

where we use the standard notation \( \gamma^\mu \partial_\mu \), with \( \gamma^\mu, \mu = 0, \ldots, 3 \), the four-dimensional Dirac matrices. Following the discussion after (39) in section 2.5, we include where we use the standard notation \( \partial_\mu \), with \( \lambda \in \mathbb{R} \):

\[
\begin{align*}
\text{parity} & : \quad \psi(t, \vec{r}) \rightarrow \gamma^0 \psi(t, -\vec{r}) \\
\text{time-reversal} & : \quad \psi(t, \vec{r}) \rightarrow \gamma^1 \gamma^3 \psi(-t, \vec{r}) \\
\text{charge conjugation} & : \quad \psi(t, \vec{r}) \rightarrow -i \gamma^2 \psi^*(t, \vec{r}),
\end{align*}
\]

such that the anti-Hermitian interaction \( \phi \bar{\psi} \gamma^5 \psi \) is \( \text{CPT}- \)even under the above transformations if \( \phi \) is a pseudo-scalar.\(^6\) It is the existence of this anti-linear \( \text{CPT} \) symmetry that guarantees the existence of real energy eigenvalues, and thus real dynamical masses \([40, 41]\), for this anti-Hermitian system, according to the general arguments of \([6, 7, 8, 9, 10]\).

In (55) the parameter \( g \) is set to take on the values \( g = 0, 1 \), depending on whether we consider the absence or the presence of attractive four-fermion interactions, respectively, and \( f_4 \) is an effective dimensionful coupling, with mass dimensions of \([\text{mass}]\), which is determined self-consistently in a SD analysis \([41]\), as we shall review below. As already mentioned, in the context of microscopic string-inspired models, the attractive four-fermion interactions in (55) express the net result, after appropriate subtraction of the corresponding repulsive interactions of the same form, obtained after applying the Fierz identities to the repulsive axial-current-current four-fermion interaction \( \frac{3\kappa^2}{16} J^5_\mu J^5_\mu \) in (24). Indeed, for a single fermion we are considering here, one has\(^7\)

\[
\frac{3\kappa^2}{16} J^5_\mu J^5_\mu = \frac{3\kappa^2}{16} (\bar{\psi} \gamma^5 \gamma^\mu \psi) (\bar{\psi} \gamma^5 \gamma_\mu \psi) = \frac{3\kappa^2}{16} (\bar{\psi} \gamma^\mu \psi) (\bar{\psi} \gamma_\mu \psi) + \frac{3\kappa^2}{8} (\bar{\psi} \psi)^2 - \frac{3\kappa^2}{8} (\bar{\psi} \gamma^5 \psi)^2. \tag{57}
\]

We shall come back to this point later on in the article, after we estimate the strength of \( f_4 \) necessary for a consistent SD dynamical mass generation \([41]\).

At the moment we remark that, motivated by the stringy axion models discussed above, we consider very small Yukawa couplings \( |\lambda| \ll 1 \) in (55). We are interested in dynamical mass generation, due to the coupling \( \lambda \), as a way of mass generation alternative to the radiative anomalous generation \((52)\), and independent of the anomalous coupling \( \gamma \), since in the model (55)

\(^6\) We note for completion that, in case \( \phi \) is a scalar, one can also guarantee the \( \text{CPT} \) invariance of the pertinent action, by defining appropriately the corresponding \( \mathcal{C} \) operator.

\(^7\) The reader should have noticed that, upon bringing the vector-current-current interaction onto the left-hand-side of (57), one would obtain a combination of four-fermion interactions which, if attractive, would coincide with the chiral-symmetry-invariant Nambu-Jona-Lasinio four-fermion interactions \([79]\). However, as we have seen above, the quantum torsion generates only the axial-current-current part of this and is repulsive. In string effective theories \([78]\), as already mentioned, one has various vector-current-current induced terms with coefficients inversely proportional to the square of the mass of the exchanged massive string states. In our work here we are not interested in a detailed study of such four-fermion interactions. In this review, we shall only use (57) to make some generic phenomenological observations on the magnitude of \( f_4 \), consistent with SD mass generation, which would allow us to ignore the repulsive torsion-induced interactions.
\( \gamma = 0 \). We shall follow a Schwinger-Dyson (SD) approach to study dynamical mass generation for both axions and fermion fields [40].

Before doing this, we would like to remark that in the absence of attractive four-fermion interactions, i.e. when \( g = 0 \), there is no dynamical mass generation for fermions, but there could be for axions (pseudoscalar) \( \phi(x) \) fields, as follows from energetics arguments [40] that we review in the next subsection.

### 3.1. Energetics arguments against dynamical generation of fermion mass in non-Hermitian Yukawa models without four-fermion self interactions

To this end [40], let us first assume a non-zero bare mass for the scalar, \( M_0 \neq 0 \). This will serve as a regulator for the definition of inverse propagators, as discussed in [40]. Our arguments on mass generation will not be affected by its presence, and thus \( M_0 \) can eventually be removed, as we shall see. To discuss dynamical mass generation for the fermions in non-Hermitian models a Euclidean path-integral quantisation is necessary (with a Euclidean-metric signature convention \((+,-,-,-)\), which is used throughout this work), in which the anti-Hermitian Yukawa interaction appears as a phase:

\[
Z_\lambda[j, \bar{\eta}, \eta] = \int \mathcal{D}[\phi, \bar{\psi}, \psi] \exp(-S_{\text{Herm}} - S_{\text{anti-Herm}} - S_{\text{sources}}),
\]

where

\[
S_{\text{Herm}} = \int d^4x \left( \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{M_0^2}{2} \phi^2 + \bar{\psi} i \gamma^5 \psi \right),
\]

\[
S_{\text{anti-Herm}} = -\lambda \int d^4x \, \phi \bar{\psi} \gamma^5 \psi = i\lambda \int d^4x \, \phi \Phi,
\]

with \( \Phi \equiv i\bar{\psi} \gamma^5 \psi \) a phase.

The source-\( J, \eta, \bar{\eta} \) terms in (59) have been inserted for the correct definition of the path integral. From a basic property of complex calculus we obtain the inequality [40]

\[
\exp(-S_{\text{eff}}) \leq \int \mathcal{D}[\phi] \left| \exp \left( -\int_x \bar{\psi} i \gamma^5 \psi + \bar{\eta} \psi + \psi \eta + \frac{1}{2} \phi G^{-1} \phi + i\lambda \Phi \right) \right|.
\]

We note that \( S_{\text{eff}} \) plays the role of the (Euclidean) vacuum energy functional, hence, as a consequence of (59), it appears that this quantity is larger in the non-Hermitian Yukawa model than in the free theory. This precludes energetically fermion dynamical mass generation [80], in contrast to the usual Hermitian case, where such a dynamical mass lowers the energy of the system [41]. This can also be seen explicitly by integrating out the massive scalar field \( \phi \). In such a case, one obtains an effective fermionic action \( S_{\text{eff}} \):

\[
\exp(-S_{\text{eff}}) \equiv \exp \left( -\int_x \bar{\psi} i \gamma^5 \psi + m \bar{\psi} \psi + \bar{\eta} \psi + \psi \eta \right) \int \mathcal{D}[\phi] \exp \left( -\int_x \frac{1}{2} \phi G^{-1} \phi + i\lambda \Phi \right)
= \exp \left( -\int_x \bar{\psi} i \gamma^5 \psi + m \bar{\psi} \psi + \bar{\eta} \psi + \psi \eta + \frac{\lambda^2}{2} \Phi G \Phi \right),
\]

where \( G^{-1} = -\Box + M_0^2 \) and \( \Phi \) is defined in (59) (here the reader recognises the importance of assuming a non-zero bare mass for the scalar, \( M_0 \neq 0 \), so as to have a well-defined operator \( G^{-1} \)). Ignoring higher-order derivatives, which are irrelevant for our analysis, we obtain

\[
S_{\text{eff}} \simeq \int_x \bar{\psi} i \gamma^5 \psi + m \bar{\psi} \psi + \frac{\lambda^2}{2 M_0^2} (\bar{\psi} \gamma^5 \psi)^2.
\]
The reader should then notice that this effective action includes a repulsive four-fermion interaction, for all values of the regulator $M_0$, which increases the energy of the system, in agreement with the generic argument (59), and thus dynamical mass generation for the fermions is impossible. Moreover, the form of the induced repulsive interaction implies the absence also of the chiral mass $\mu$, as the latter would be associated with a condensate of the form $\langle i\bar{\psi}\gamma^5\psi \rangle$, which is also not formed due to the above energetics reasons.

In constrast, one may have axion mass generation, which can be established upon considering the effective action $S^\text{scal}_{\text{eff}}$ for the pseudoscalar field $\phi$, obtained after integrating out massive fermions (of bare mass $m_0 \neq 0$)

$$\exp(-S^\text{scal}_{\text{eff}}) \equiv \exp\left( -\int x \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{M_0^2}{2} \phi^2 + J_\phi \right) \int \mathcal{D}[\psi, \bar{\psi}] \exp\left( -\int x \bar{\psi}(i\partial + m_0 - \lambda\phi\gamma^5)\psi \right).$$

For a constant scalar field configuration $\phi_0$, which suffices for demonstrating our main argument, the effective potential reads [40]

$$U_{\text{eff}}(\phi_0) = \frac{M_0^2}{2} \phi_0^2 - \text{Tr} \{\ln(p + m_0 - \lambda\phi_0\gamma^5)\}, \quad (62)$$

which implies

$$\frac{dU_{\text{eff}}}{d\phi_0} = M_0^2 \phi_0 + \frac{\lambda^2 \phi_0}{4\pi^2} \left( \Lambda^2 - (m_0^2 - \lambda^2 \phi_0^2) \ln \frac{\Lambda^2 + m_0^2 - \lambda^2 \phi_0^2}{m_0^2 - \lambda^2 \phi_0^2} \right), \quad (63)$$

where $\Lambda$ is the UV cut off. The energies are therefore real, and the non-Hermitian theory is self consistent, for $m^2 \geq \lambda^2 \phi_0^2$, as expected from the study in [17]. The possibility of real energies is attributed, of course, to the underlying CPT antilinear symmetry [10, 40]. In the limit $\lambda^2 \phi_0^2 \to m^2$, the effective potential becomes a mass term

$$U_{\text{eff}} \to \frac{1}{2}(M^{(1)})^2 \phi_0^2 \quad \text{with} \quad (M^{(1)})^2 = M_0^2 + \frac{\lambda^2}{4\pi^2} \Lambda^2. \quad (64)$$

From eq.(64) it is evident that dynamical axion-mass generation is possible in the anti-Hermitian-Yukawa-interaction model (55) (with $g = 0$), since on setting the bare mass to zero, $M_0 = 0$, one obtains a non-zero result from (64)

$$(M^{(1)})^2_0 = \frac{\lambda^2}{4\pi^2} \Lambda^2 > 0, \quad (65)$$

which is the same as the result of a one-loop calculation.

In [41] the same conclusions as above are reached by a detailed SD analysis, which avoids the use of regulator bare masses. We refer the interested reader to that work for details. Below we shall only describe the main results of such an analysis.

The terminology “chiral mass” $\mu$ (or, equivalently, non-Hermitian mass) is used here for the coefficient accompanying the (non-Hermitian) fermion bilinear term $\bar{\psi}\gamma^5\psi$ in the Lagrangian, extending appropriately the corresponding definition in the Hermitian case. The four-fermion interactions in (61) can be written in a Hartree-Fock (mean field) approximation, up to an irrelevant constant, as: $(\bar{\psi}\gamma^5\psi)^2 = -(i\bar{\psi}\gamma^5\psi)i\bar{\psi}\gamma^5\psi + \text{four-fermion interactions of pure quantum fluctuations}$. This implies the identification of $\mu$ with the chiral condensate $\langle i\bar{\psi}\gamma^5\psi \rangle$. In view of the repulsive nature of the four-fermion interactions in (61), such a condensate cannot form, and hence dynamical generation of $\mu \neq 0$ is not possible in the anti-Hermitian model.
3.2. Schwinger-Dyson Analysis of non-Hermitian Yukawa interactions, in the absence of four-fermion interactions

To formulate properly our quantum non-Hermitian prototype models, one necessarily uses, as already mentioned, a Euclidean ("E") path-integral formalism. For the model (55) with $g = 0$, of interest in this subsection, the appropriate Euclidean generating functional is given by

$$Z^E[J, \eta, \bar{\psi}] = \int \mathcal{D}[\phi \psi \bar{\psi}] \exp \left\{ - \int d^4 x \left[ \frac{1}{2} \partial_\nu \phi \partial^\nu \phi + \bar{\psi} \partial_\nu \psi - \lambda \phi \bar{\psi} \gamma^5 \psi \right] - \int d^4 x \left[ J \phi + \bar{\psi} \eta + \eta \psi \right] \right\}. \quad (66)$$

Taking into account that we are dealing with very small Yukawa couplings, $|\lambda| \ll 1$, possibly generated by non-perturbative effects in the underlying microscopic string/brane theories, and following standard SD methods, one arrives at the following SD equations for the pseudoscalar ($s$) and fermion ($f$) propagators in Fourier space [41]:

$$G_s^{-1}(k) - S_s^{-1}(k) = \lambda \Gamma^5 \int \frac{d^4 p}{(2\pi)^4} G_f(p) \Gamma^{(3)}(p, k) G_f(p - k),$$

$$G_f^{-1}(k) - S_f^{-1}(k) = -\lambda \Gamma^5 \int \frac{d^4 p}{(2\pi)^4} G_f(p) \Gamma^{(3)}(p, k) G_s(p - k), \quad (67)$$

where the dressed inverse propagators are given by: $G_f^{-1}(p) = p + m + \mu \gamma^5$, with $\mu$ the non-Hermitian (or "chiral") fermion mass, defined previously in footnote 8, $G_s^{-1}(p) = p^2 + M^2$, and we used the bare inverse propagators $S_f^{-1}(p) = p$, $S_s^{-1}(p) = p^2$. The vertex function is in the rainbow approximation [41], appropriate for weak Yukawa interactions, $|\lambda| \ll 1$, and reads:

$$\Gamma^{(3)}(p, k) \simeq \lambda \gamma^5. \quad (68)$$

In [41] and here, we restrict ourselves to the case of real $\mu \in \mathbb{R}$. This stems from the fact that we are interested in the (physically relevant) case where the energies of the system in the massive phase are real, for which one must have the following condition among the (dynamically generated) mass parameters [17, 40]

$$|\mu| \leq |m|. \quad (69)$$

The reality of the energy eigenspectrum in this case may be traced back [40, 41] to the underlying CPT (antilinear) symmetry of the field theory (55), with $C$ appropriately defined ((56)) [21], according to the general arguments of [6, 7, 8, 9, 10]. As discussed in [20, 21], the condition (69) guarantees unitarity, in the sense of the existence of a well-defined, conserved, probability density for the non-Hermitian fermionic system, which is less than unity.

Upon assuming vanishing external momenta $k = 0$ in the SD equations (67), and performing the Euclidean momentum integrations with an UV cutoff $\Lambda$, we arrive at [40]

$$M^2 = \frac{\lambda^2}{4\pi^2} \left[ \frac{(\Lambda^2 + m^2 - 3\mu^2)^2}{\Lambda^2 + m^2 - \mu^2} - (m^2 - 3\mu^2) \ln \left( 1 + \frac{\Lambda^2}{m^2 - \mu^2} \right) \right], \quad (70)$$

$$m + \mu \gamma^5 = -\frac{\lambda^2}{16\pi^2} \frac{(m - \mu \gamma^5)}{M^2 - m^2 + \mu^2} \left[ M^2 \ln \left( 1 + \frac{\Lambda^2}{M^2} \right) - (m^2 - \mu^2) \ln \left( 1 + \frac{\Lambda^2}{m^2 - \mu^2} \right) \right]. \quad (71)$$

Following our previous discussion in section 3.1 on the energetics of the action (61), implying non-formation of a dynamical chiral mass $\mu \neq 0$, we set from now on $\mu = 0$ [41].
On setting \( m = \mu = 0 \), which is consistent with (69), we observe from (70) that there is now a consistent solution for the dynamically generated axion mass \( M \), since (70) leads to

\[
M^2 = \frac{\lambda^2}{4\pi^2} \Lambda^2 \ll \Lambda^2, \quad \lambda^2 \ll 1,
\]  

(72)

which is the same as the expression (65). On account of (69), the case where \( m = 0 \) but \( \mu \neq 0 \) is not allowed, as it would lead to unphysical situations with complex energies.

For \( \mu = 0 \), and \( m M \neq 0 \), we have the following system of SD equations

\[
M^2 = \frac{\lambda^2}{4\pi^2} \left[ \Lambda^2 - m^2 \ln \left( 1 + \frac{\Lambda^2}{M^2} \right) \right]
\]

(73)

\[
1 = -\frac{\lambda^2}{16\pi^2} \frac{1}{M^2 - m^2} \left[ M^2 \ln \left( 1 + \frac{\Lambda^2}{M^2} \right) - m^2 \ln \left( 1 + \frac{\Lambda^2}{m^2} \right) \right]
\]

(74)

We will now consider partial solutions \( 0 \neq m \approx M \ll \Lambda \). From (74):

\[
-\frac{16\pi^2}{\lambda^2} = \ln \left( 1 + \frac{\Lambda^2}{M^2} \right),
\]

(75)

which is inconsistent. It is also readily seen that the case \( \mu = M = 0 \) also does not lead to fermion mass generation.

Hence dynamical fermion mass is not possible for pure anti-Hermitian Yukawa interactions, only scalar mass can be generated dynamically, in agreement with the generic energetics arguments \([40]\) provided in the previous subsection 3.1.

### 3.3. Non-Hermitian Yukawa Interactions in the presence of attractive four-fermion interactions

We next proceed to discuss explicitly the rôle on dynamical-mass generation played by additional Hermitian attractive four-fermion interactions in the non-Hermitian Yukawa model \([41]\), i.e. we consider the case \( g = 1 \) in (55). The pertinent Euclidean generating functional reads:

\[
Z^E[J, \eta, \overline{\eta}] = \int D[\phi \overline{\psi}] \exp \left\{- \int d^4x \left[ \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \overline{\psi} i \slashed{\partial} \psi - \lambda \phi \overline{\psi} \gamma^5 \psi + \frac{1}{2f_4^2} (\overline{\psi} \gamma^5 \psi)^2 \right] \right\}
\]

\[
- \int d^4x [J\phi + \overline{\psi} \eta + \eta \overline{\psi}] \right\}
\]

\[
= \int D[\sigma \phi \overline{\psi}] \exp \left\{- \int d^4x \left[ \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \overline{\psi} i \slashed{\partial} \psi - \lambda \phi \overline{\psi} \gamma^5 \psi + \frac{f_4^2}{2} \sigma^2 + i \sigma \overline{\psi} \gamma^5 \psi \right] \right\}
\]

\[
- \int d^4x [J\phi + \overline{\psi} \eta + \eta \overline{\psi} + K\sigma] \right\},
\]

(76)

where in the second equality we have linearised the four-fermion interactions using an auxiliary pseudoscalar field \( \sigma \). The additional ingredient with respect to the analysis in the previous subsection is the \( \sigma \) propagator, which in momentum space is given by \( G_\sigma(p) = 1/f_4^2 \).

As a result of the extra field \( \sigma(x) \), the SD equations with vanishing external momenta read now \([41]\):

\[
G_{f}^{-1}(0) - S_{f}^{-1}(0) = -\lambda^2 \gamma^5 \left( \int \frac{d^4p}{(2\pi)^4} G_f(p) \gamma^5 G_\sigma(p) \right) - \gamma^5 \left( \int \frac{d^4p}{(2\pi)^4} G_f(p) \gamma^5 G_\sigma(p) \right)
\]

(77)
\[ G_s^{-1}(0) - S_s^{-1}(0) = \lambda^2 \text{tr} \left[ \gamma^5 \int \frac{d^4p}{(2\pi)^4} G_f(p) \gamma^5 G_f(p) \right] \]  

(78)

Using the dressed inverse propagators, as in the previous subsection, and performing the integrals using an UV cutoff \( \Lambda \), we arrive at the following system of algebraic equations:

\[
m + \mu \gamma^5 = -\frac{\lambda^2}{16\pi^2} \frac{(m - \mu \gamma^5)}{M^2 - m^2 + \mu^2} \left[ M^2 \ln \left( 1 + \frac{\Lambda^2}{M^2} \right) - (m^2 - \mu^2) \ln \left( 1 + \frac{\Lambda^2}{m^2 - \mu^2} \right) \right] \\
+ \frac{(m - \mu \gamma^5)}{16\pi^2 f_4^2} \left[ \Lambda^2 - (m^2 - \mu^2) \ln \left( 1 + \frac{\Lambda^2}{m^2 - \mu^2} \right) \right]
\]

(79)

\[
M^2 = \frac{\lambda^2}{4\pi^2} \left[ \frac{(\Lambda^2 + m^2 - 3\mu^2)\Lambda^2}{\Lambda^2 + m^2 - \mu^2} - (m^2 - 3\mu^2) \ln \left( 1 + \frac{\Lambda^2}{m^2 - \mu^2} \right) \right]
\]

(80)

As the SD pseudoscalar mass equation is independent of \( f_4 \), it is straightforward to see from (80) that, for \( m = \mu = 0 \), one obtains the dynamically generated (pseudo)scalar mass (72).

If the case \( \mu = 0 \) but \( m \neq 0 \), the system of SD equations reads

\[
1 = -\frac{\lambda^2}{16\pi^2} \frac{1}{M^2 - m^2} \left[ M^2 \ln \left( 1 + \frac{\Lambda^2}{M^2} \right) - m^2 \ln \left( 1 + \frac{\Lambda^2}{m^2} \right) \right] + \frac{1}{16\pi^2 f_4^2} \left[ \Lambda^2 - m^2 \ln \left( 1 + \frac{\Lambda^2}{m^2} \right) \right]
\]

\[
M^2 = \frac{\lambda^2}{4\pi^2} \left[ \Lambda^2 - m^2 \ln \left( 1 + \frac{\Lambda^2}{m^2} \right) \right],
\]

(81)

which for the specific case

\[
\Lambda \gg m \simeq M \neq 0,
\]

(82)

leads to

\[
1 = -\frac{\lambda^2}{16\pi^2} \ln \left( 1 + \frac{\Lambda^2}{M^2} \right) + \frac{m^2}{4 f_4^2 \Lambda^2} \Rightarrow m^2 \simeq M^2 \simeq 4 \lambda^2 f_4^2 + \mathcal{O}(\lambda^4 \ln \lambda^2),
\]

\[
M^2 \simeq m^2 \simeq \frac{\lambda^2}{4\pi^2} \Lambda^2.
\]

(83)

For consistency we then obtain

\[
f_4 \simeq \frac{\Lambda}{4\pi} - \left| \mathcal{O}(\lambda^2 \ln \lambda^2) \right|, \quad \lambda^2 \ll 1.
\]

(84)

Thus, in non-Hermitian Yukawa interactions, upon the inclusion of sufficiently strong four-fermion attractive interactions, one can obtain dynamical mass for both fermions and axions.

Some important remarks are in order at this point. The reader should have noticed from (83) that there is SD dynamical mass for \textit{arbitrarily small values} of the Yukawa coupling \( \lambda \), which thus includes the case where such couplings are generated by stringy instanton effects, as discussed in section 2.5 (\textit{cf.} (47)). We stress that this is a consequence of the fact that the prototype model does not contain self-interactions of the axion-\( \phi \) field, which is motivated by the stringy considerations of section 2.5 (\textit{cf.} discussion following (39)). We note that, in the presence of such self-interactions, \textit{e.g.} a \( \phi^4 \)-term, thin constrast situation concerning fermion mass generation can be very different. For instance, in the specific (Hermitian) model of [81], where there is a common coupling between the Yukawa and \( \phi^4 \) self-interactions, there is a critical value of \( g \) above which dynamical fermion mass is possible. We do not consider axion self-interactions neither in this review nor in [40, 41, 42].

We also note that for our purposes it suffices that we considered the special case (82) for the dynamical masses of axions and fermions of approximately equal magnitude. More
general solutions of course may exist, but they are not relevant for our purposes, which are to demonstrate explicitly a non-trivial concrete solution of the SD system entailing dynamical masses for both axions and fermions.

We remark at this point that, although above we view the interaction (54) as an effective attractive interaction which incorporates the subtraction of repulsive terms of similar form stemming from the quantum-torsion-induced repulsive interactions (24) upon applying the Fierz identities (57), nonetheless one may also consider the case in which this interaction genuinely dominates the torsion terms, for an appropriate value of the cut-off scale $\Lambda$, which defines the momentum scale above which the validity of the low-energy effective field theory breaks down. In such a case one would justify ignoring the repulsive interactions altogether. To determine in this case the scale $\Lambda$, we should compare the coefficient $1/(2f_4^2)$ of the attractive terms (54), with $f_4$ determined by (84), with the corresponding coupling $\sqrt{3\kappa^2/8}$ of the repulsive quantum-torsion-induced terms (57). The reader observes that dominance of the attractive term (54) by, say, at least one order of magnitude, over the corresponding term in (57), would imply a cut-off momentum scale

$$\frac{1}{2 f_4^2} \gtrsim \mathcal{O}(10) \times \frac{3\kappa^2}{8} \overset{\text{cf. (84)}}{\Rightarrow} \Lambda \lesssim \mathcal{O}\left(\sqrt{\frac{4\pi}{15}}\right) M_P \sim 0.9 M_P,$$

(85)

which is phenomenologically reasonable for an effective theory embedded in an UV complete quantum gravity theory, such as strings. In fact, the range for $\Lambda$ in (85) is compatible with (19), which is satisfied for

$$\Lambda \sim M_s \approx 0.2 M_P,$$

(86)

implying that $M_s$ acts as an effective cut-off energy scale above which the validity of the point-like string-effective theory breaks down. Under (86), the dynamically generated mass (83) becomes

$$M \simeq m \approx \frac{|\lambda|}{2\pi} \Lambda \approx 0.03 \frac{|\lambda|}{M_P} \approx 0.04 \frac{|\lambda|}{M_P} \times 10^{19} \text{GeV}.$$  

(87)

If $\lambda$ is generated by string instantons (47), then, in order to have axion masses larger than $10^{-24}$ eV, which is a lower-bound for the mass of an axion in case the latter plays the rôle of a dominant dark-matter species [67], one needs the (real parts of the) stringy-instanton actions to be of order: $0 < S_{\text{instanton}} < 116.5$ (in natural units of $\hbar c = 1$). Smaller axion masses $> 10^{-32}$ eV are allowed if the axion constitutes only a small (e.g. < 5%) part of dark matter in the Universe, in which case one obtains for the instanton actions $0 < S_{\text{instanton}} < 134.9$. Such considerations allow for a rather wide variety of axion masses in our scenarios, which can thus play a rôle as dark matter candidates, including ultralight ones, with sensitivity to be falsified at immediate-future experimental facilities.

We also note that the presence of Yukawa interactions, coupling the axion to right-handed neutrinos in the model of [43], under the assumption (82), (83) for the dynamically generated masses, can be reconciled with phenomenologically realistic situations of particle physics by

9 Such instanton actions are smaller in magnitude than the ones required for axion quintessence in Hermitian stringy axion cosmologies, in which the contributions of axions to dark energy are non-negligible; in such cases one needs $S_{\text{instanton}} \sim 200 - 300$ [67]. We remark at this point, however, that, strictly speaking, one cannot directly compare the phenomenological constraints discussed in the context of our non-Hermitian models with the phenomenological constraints for generic Hermitian stringy axion models based on instanton-induced axion potentials [55], as a consequence of the anomalous couplings of the (stringy) axions. The dynamical axion masses (87) pertain to a model (55) without anomalous terms ($\gamma = 0$) and no potential for the pseudoscalar fields. Although non-Hermitian anomaly terms ($\gamma \neq 0$) are discussed in the next section 4, nonetheless for such non-Hermitian models the concept of instanton-induced potentials is not completely understood, as already mentioned. Hence, the considerations in [55] might not apply even to that case.
Figure 2. Resummation (dark elliptical blobs) of leading (quadratic) UV divergences due to four-fermion ‘bubble’ diagrams (ellipse without filling in the figures). Upper figure (a): effects on Yukawa interaction vertex. Lower figure (b): effects on four-fermion vertex. Continuous lines refer to fermions, dashed lines pertain to pseudoscalar fields. Figures taken from [41].

considering such ultralight right-handed neutrinos as being almost decoupled from the standard-model matter sector. In other words, the dominant coupling of such right-handed neutrinos with masses given by (83) to matter is the Yukawa one, while their mixing through Higgs-portal-type interactions with standard-model matter is negligible. Of course, such restrictions are lifted if one views the prototype models discussed here and in [40, 41, 42] as purely phenomenological, not necessarily connected to microscopic string models, in which case $\lambda$ is a free parameter, which need not be generated by string instanton effects, and hence need not be that small.

3.4. Resummation effects of strong four-fermion interactions

Before closing this section we would like to comment briefly on how resummation of the four-fermion interactions affects qualitatively the above conclusions. Indeed, this is an important aspect, given the strong coupling nature of the dimensionless four-fermion coupling (84)

$$\frac{\Lambda}{\sqrt{2} f_4} \sim 4\pi > 1,$$

(88)

which implies the necessity for a resummation of the loop “bubble” diagrams associated with quantum corrections of the four-fermion vertex, as indicated in fig. 2. This issue was discussed in detail in [41], where we refer the interested reader for details. For our purposes here, we only mention that the result of such a resummation, as far as the leading quadratic UV divergences are concerned, of interest to us here, is that the relevant quantum corrections can be absorbed in “renormalised” Yukawa and four-fermion couplings:

$$f_{4R} \sim \frac{1}{\sqrt{2}} f_4, \quad \lambda_R \sim 2 \lambda,$$

(89)
which are thus of the same order as the bare couplings. Therefore, the validity of the above SD analysis and the pertinent conclusions are guaranteed upon replacing the bare couplings \( \lambda, f_4 \) by the ‘renormalised’ ones (89) [41]. This is to be understood in all the previous formulae as well as the ones that follow. For notational convenience, we omit from now on the suffix \( R \).

4. Inclusion of non-Hermitian anomaly terms and non-Hermitian axial backgrounds

A final topic we would like to discuss in this mini-review is the rôle of non-Hermitian anomaly and axial background terms on the dynamical mass generation induced by the non-Hermitian Yukawa interactions in the model. The presence of the anomalous terms have been responsible, as we have discussed in section 2, for the radiative generation of Majorana masses for the right-handed neutrinos in the Hermitian model of [43]. Formally, when both Yukawa and anomalous interactions are characterised by purely imaginary couplings (53), i.e. when such interactions are anti-Hermitian, the expressions for the radiative fermion masses (52) retain their reality. However, the concept of anomalies, and the relevant studies, are not yet developed for non-Hermitian theories, except making formal analogies, and therefore it would be interesting to examine alternative ways for fermion (and axion) mass generation, through the anti-Hermitian Yukawa interaction, and scrutinise the rôle of the presence of anomalous terms on such a phenomenon. This has been done in detail in [42]. Below we shall only review the basic conclusions of that work.

In addition, we shall also examine in this section the rôle of non-Hermitian interactions of fermions with a constant axial background \( B_\mu \)

\[
S_{\text{axial-backgr}}^{\text{antiherm}} = \int d^4x \sqrt{-g} i B_\mu \bar{\psi} \gamma^\mu \gamma^5 \psi , \quad B_\mu \in \mathbb{R} .
\]  

(90)

In the context of microscopic string theories, discussed in section 2, the microscopic origin of the non-Hermitian background can be traced back to the non-Hermitian anomalous interactions of the KR gravitational axion of the massless bosonic string multiplet in some formulations of the theory (see “Scheme II”, (24)). To understand this, we mention that, in Hermitian theories, i.e. when \( iB_\mu \in \mathbb{R} \), such constant backgrounds (with temporal components only in a given Lorentz frame, specifically the cosmological Robertson-Walker (RW) frame) have been argued to characterise cosmological configurations of the KR gravitational axion field, \( b(x) \), exhibiting a linear dependence in the cosmic time, \( t \), in the RW frame [60, 57],

\[
b(x) = (\text{constant}) \times t .
\]  

(91)

Non-hermitian constant axial backgrounds (90), then, can arise in the “Scheme II” (24) of the effective string theory actions, in the way discussed in section 2. Moreover, in Hermitian models, constant axial backgrounds provide alternative ways for tree-level Leptogenesis [60, 56] (i.e. the generation of Lepton-number asymmetry between particles and anti-particles), in theories involving massive right-handed neutrinos, as a result of the asymmetric decays of the latter to standard-model leptons and antileptons in the presence of such backgrounds. It is yet to be seen whether non-Hermitian axial backgrounds (90) can lead to similar Leptogenesis phenomena in non-Hermitian theories.

Their contribution to mass generation, though, is one such aspect that can be settled already. In the work of [42], which will be reviewed briefly here, we shall study the rôle of such interactions on the masses of axions and fermions, generated dynamically by non-Hermitian Yukawa couplings. We also note that in [82] such non-Hermitian constant backgrounds have been examined from the point of view of their rôle in fermion mass generation in the context of
Hermitian Nambu-Jona-Lasinio four-fermion models [79], which however did not contain Yukawa interactions, in contrast to our case here, where the latter constitute the dominant source for mass generation.

Therefore, in this section we shall study the following prototype action:

$$S_{\text{anti-Herm-axial-back+anom}} = \int d^4x \sqrt{-g} \left( \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \bar{\psi} i D \psi - \frac{\gamma_f}{f_b} (\partial_\mu \phi) \bar{\psi} \gamma^\mu \psi + i B_\mu \bar{\psi} \gamma^\mu \gamma^5 \psi + \lambda \phi \bar{\psi} \gamma^5 \psi - \frac{g^2}{2f_b^2} (\bar{\psi} \gamma^5 \psi)^2 \right)$$

(92)

We remark at this stage that, in the context of the model [43], $f_b$ in (92) is given by (29), but here we consider it as an arbitrary mass scale to be determined self consistently in terms of the UV cutoff $\Lambda$ in the phase where dynamical mass generation occurs [42].

In (92) we have included non-trivial four-fermion attractive interactions, since, from arguments given previously (section 3.1), we know that their absence in anti-Hermitian Yukawa models disfavours dynamical mass generation for fermions. The coupling $g$ is inserted to demonstrate the difference (in general) of the strength of the four-fermion attractive interactions from that of the anomalous terms. The anomalies are encoded in the four-divergence of the axial current (cf. (22)), and appear after partial integration of the relevant (third) term on the right-hand-side of (92), with coefficient $-i \gamma_f/f_b$. In what follows, consistently with the discussion so far, we shall work in a model formulated in a Minkowski spacetime.

The action (92) is $CPT$ invariant, under the transformations (56) for the fermions, which guarantees the reality of the energy spectrum, and hence the dynamical masses [42], according to the generic analysis of [6, 7, 8, 9, 10]. We note that a constant anti-Hermitian axial background would break spontaneously the (complex) Lorentz symmetry, and thus the generalised CPT symmetry, as in the Hermitian case [60, 56]. Nonetheless, the reality of the dynamically-generated masses is guaranteed [42], as the condition (69) is maintained.

4.1. Models with non-Hermitian Yukawa and anomalous interactions - no axial background

We commence our study with the case of anti-Hermitian Yukawa interactions in the presence of anti-Hermitian anomalous couplings to fermions, in the absence of axial background, i.e. setting $B_\mu = 0$ in (92). In this case, the Euclidean partition function, under the linearization of the four-fermion interactions by means of the auxiliary pseudoscalar $\sigma$, reads (upon the standard inclusion of appropriate sources $\eta, \bar{\eta}, K, J$ for the fields $\psi, \bar{\psi}, \sigma$ and $\phi$, respectively):

$$Z[K, J, \eta, \bar{\eta}] = \int \mathcal{D}[\sigma \phi \bar{\psi} \psi] \exp \left( - \int d^4x \left[ \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \bar{\psi} i D \psi - \frac{\gamma_f}{f_b} (\partial_\mu \phi) \bar{\psi} \gamma^\mu \gamma^5 \psi - \gamma_f f_b (\bar{\psi} \gamma^5 \psi)^2 \right] ight)$$

In the way written, the anomalous interactions with coupling $\gamma$ in (92) imply that the Feynman rule for the (bare) axion-fermion vertex $\phi \bar{\psi} \psi$ is given by $\left( \lambda - \frac{\gamma_f}{f_b} \right) \gamma^5$, in the convention where all the momenta are incoming to the vertex.

The details of the SD analysis are given in [42] and will not be repeated here, where we only state the final result. We seek a partial solution of the SD equations for dynamical mass generation stemming from (93), for which there is an approximate equality of fermion ($m$) and axion ($M$) dynamical masses, $m \simeq M$, with the chiral mass $\mu = 0$, which suffices for our purposes, for reasons explained above. Using the rainbow approximation for the vertices,
appropriate for small Yukawa and anomalous couplings $|\gamma| \sim |\lambda| \ll 1$, assumed here, and taking into account the effects of resummation of the four-fermion interactions, as described in the previous section 3.4, we arrive at the following system of SD equations describing the dynamical generation of fermion and pseudoscalar masses [42]

\[
1 = -\frac{\lambda^2}{16\pi^2} \frac{1}{M^2 - m^2} \left[ M^2 \ln \left( 1 + \frac{\Lambda^2}{M^2} \right) - m^2 \ln \left( 1 + \frac{\Lambda^2}{m^2} \right) \right] \\
+ \frac{\gamma^2}{16\pi^2 f_b^2} \frac{1}{M^2 - m^2} \left[ -\lambda^2 (M^2 - m^2) + M^4 \ln \left( 1 + \frac{\Lambda^2}{M^2} \right) - m^4 \ln \left( 1 + \frac{\Lambda^2}{m^2} \right) \right] \\
+ \frac{g^2}{16\pi^2 f_b^2} \left( \Lambda^2 - m^2 \ln \left( 1 + \frac{\Lambda^2}{m^2} \right) \right)
\]

(94)

and

\[
M^2 = \frac{\lambda^2}{4\pi^2} \left[ \Lambda^2 - m^2 \ln \left( 1 + \frac{\Lambda^2}{m^2} \right) \right],
\]

(95)

where $\Lambda$ is the UV cut-off, as in previous sections. Seeking solutions with $m \simeq M \neq 0$ (cf. (82)) we obtain

\[
1 = -\frac{1}{16\pi^2} \left( \frac{\lambda^2 - \gamma^2 M^2}{f_b^2} \right) \ln \left( 1 + \frac{\Lambda^2}{M^2} \right) + \frac{g^2 - \gamma^2}{16\pi^2 f_b^2} \left[ \Lambda^2 - m^2 \ln \left( 1 + \frac{\Lambda^2}{M^2} \right) \right]
\]

and

\[
\frac{4\pi^2}{\lambda^2} M^2 = \left[ \Lambda^2 - m^2 \ln \left( 1 + \frac{\Lambda^2}{m^2} \right) \right]
\]

(96)

(97)

Considering $\Lambda^2 \gg M^2$, $|\lambda|, |\gamma| \ll 1$, with $g \neq \gamma$, and substituting (97) in (96), we arrive at [42]:

\[
M^2 \simeq m^2 \simeq \frac{4f_b^2 \lambda^2}{g^2 - \gamma^2} + |\mathcal{O}\left( (\lambda^4, \lambda^2 \gamma^2) \ln(\lambda^2) \right)|
\]

(98)

Hence, consistency between (98) and (97) requires

\[
\frac{f_b}{\sqrt{g^2 - \gamma^2}} \simeq \frac{\Lambda}{4\pi} - |\mathcal{O}(\lambda^2)| \quad \Rightarrow \quad M^2 \simeq m^2 \simeq \frac{\lambda^2}{4\pi^2} \Lambda^2 + |\mathcal{O}\left( (\lambda^4, \lambda^2 \gamma^2) \ln(\lambda^2) \right)|,
\]

(99)

for small $\gamma, \lambda$ couplings. In order to have dynamical mass generation one must have\(^{10}\)

\[
g^2 > \gamma^2,
\]

(100)

The reader should have noticed that bare axion and fermion masses are not necessary, which implies that dynamical mass generation is possible under the above restriction (100). Thus in this case, contrary to the Hermitian one (which is studied in [42]), it is a sufficiently strong attractive four-fermion interaction that catalyses the Yukawa-induced dynamical mass generation, since the non-Hermitian anomaly term resists mass generation.

In [40, 42] the case of Majorana fermions has also been studied, which are of relevance to the model of [43]. The case is qualitatively similar to the Dirasc case studied above, apart from some

\(^{10}\) For a discussion on the limiting case $g^2 = \gamma^2 + \epsilon$, $\epsilon \to 0^+$, we refer the reader to [42].
numerical factors in the expression for the dynamical mass and the one providing the connection of the four-fermion couplings $f_b$ to the cut-off $\Lambda$. In such a case, the analogue of (98) reads:

$$M_{\text{Dyn Maj}}^2 \simeq \frac{2 f_b^2 \lambda^2}{g^2 - \gamma^2} + \ldots, \quad \text{with} \quad \frac{f_b}{\sqrt{g^2 - \gamma^2}} \simeq \frac{\Lambda}{2 \sqrt{2\pi}},$$

(101)

implying:

$$M_{\text{Dyn Maj}}^\text{anti-herm} \sim \frac{\lambda}{2\pi} \Lambda, \quad |\lambda| \ll 1.$$  

(102)

Upon comparison of (102) with (52) (with $|\gamma|, |\lambda| \ll 1$, as required for consistency of our SD treatment, which also characterises the physically motivated case in which these interactions are generated by non-perturbative stringy effects, as discussed in section 2), one obtains:

$$M_{\text{Dyn Maj}}^\text{anti-herm} \sim \frac{\sqrt{3/2} \gamma (\kappa \Lambda)^{5/2}}{49152 \pi^3} M_{\text{Dyn Maj}}^\text{anti-herm} \simeq 8 \cdot 10^{-7} \gamma (\kappa \Lambda)^{5/2} M_{\text{Dyn Maj}}^\text{anti-herm} \ll M_{\text{Dyn Maj}}^\text{anti-herm},$$

(103)

for any value of a sub-planckian cutoff $\kappa \Lambda \lesssim 1$. Hence in the non-Hermitian case, the anomaly-and-Yukawa-induced dynamical mass dominates the radiative anomalously-induced mass, provided the existence of the latter is rigorously proven of course in this case, something which is currently pending.

4.2. Non-Hermitian axial background and Yukawa interactions - no anomalies

As a final topic for discussion in this review, we consider the case where an anti-Hermitian axial background (90) is present for an anomaly-free fermion model, i.e. upon setting $\gamma = 0$ in (92). The incorporation of both non-Hermitian axial background and anomalous interactions is straightforward and presents no conceptual difficulties, only algebraically more elaborate analysis. Hence it will not be discussed here.

For details of the SD study of models with both anti-Hermitian background and Yukawa interactions we refer the reader to [42]. Below we shall only give the final results. The pertinent solutions for SD mass generation for axions and fermions (with approximately equal masses $m \simeq M$ (cf. (82)) and chiral mass $\mu = 0$) read:

$$1 = -\frac{\lambda^2}{16\pi^2 M^2 B^2} \left(1 + 2(\tilde{M}^2 + \tilde{B}^2) - \sqrt{(1 + \tilde{M}^2 - \tilde{B}^2)^2 + 4\tilde{M}^2 \tilde{B}^2}ight)
- \left(\sqrt{(1 + \tilde{M}^2 - \tilde{B}^2)^2 + 4\tilde{M}^2 \tilde{B}^2} - (\tilde{M}^2 + \tilde{B}^2) \right)(\tilde{M}^2 + 7\tilde{B}^2)
+ 4\tilde{B}^2(\tilde{M}^2 - 2\tilde{B}^2) \log \left[1 + \tilde{M}^2 - \tilde{B}^2 + \sqrt{(1 + \tilde{M}^2 - \tilde{B}^2)^2 + 4\tilde{M}^2 \tilde{B}^2} \right] \frac{1}{2\tilde{M}^2}. $$

(104)
and

\[
1 = \frac{\lambda^2}{32\pi^2B^2} \left( 1 + \frac{B^2}{\sqrt{B^4 + 4M^2B^2}} \right) \log \left( 1 + \frac{1}{M^2} \right) - \left( \sqrt{(1 + M^2 - B^2)^2 + 4M^2B^2} - (M^2 + B^2) \right) \\
- 3B^2 \log \left[ 1 + \frac{1}{M^2} \right] + \frac{B^4}{\sqrt{B^4 + 4M^2B^2}} \log \left[ \frac{\left( \sqrt{B^4 + 4M^2B^2} + \sqrt{(1 + M^2 - B^2)^2 + 4M^2B^2} \right)^2 - (1 + M^2)^2}{\left( M^2 + B^2 \right) + \sqrt{B^4 + 4M^2B^2}} \right] \\
- \frac{\bar{g}^2}{64\pi^2B^2} \left( 1 + 2(M^2 + B^2) - \sqrt{(1 + M^2 - B^2)^2 + 4M^2B^2} \right) - \left( \sqrt{(1 + M^2 - B^2)^2 + 4M^2B^2} - (M^2 + B^2) \right) (M^2 + 7B^2) \\
+ 4B^2(M^2 - 2B^2) \log \left[ \frac{1 + (M^2 - B^2) + \sqrt{(1 + M^2 - B^2)^2 + 4M^2B^2}}{2M^2} \right],
\]  

(105)

where \( \bar{M} = M/\Lambda \) and \( \bar{B} = B/\Lambda \) and \( \bar{g} = (g\Lambda)/f_b \).

Figure 3 plots the right hand side of (104) as a function of the axion mass \( \bar{M} \) for different values of \( B \). All the curves intersect the solid line, corresponding to the fixed value 1, which represents the left hand side of (104). The reader should observe that, for a non-Hermitian axial background, the larger the background, the bigger the dynamical mass. This is to be contrasted with the case of a Hermitian axial background, for which, as we have discussed in [42], one faces the opposite situation, that is, the larger the background, the smaller the dynamical mass.

![Figure 3](image-url)  

**Figure 3.** Plot of the right-hand side of (104) versus the dynamical axion mass \( M \) in units of the cut-off \( \Lambda \), for the fixed value \( \lambda^2 = 0.02 \) for definiteness. All curves intersect the solid line, which represents the left hand side of (104), implying dynamical mass generation for the pseudoscalar field, without the need to introduce a bare mass for it. Figure taken from [42].

From (104) one obtains a value for \( M \), which, upon being inserted in (105), determines the value of the four-fermion coupling \( g/f_b \) for which there is a consistent dynamical mass for the fermion, of approximately equal magnitude to the pseudoscalar mass, (82). For concrete examples, we refer the reader to figure 4. We observe from the figure that the dashed line, corresponding to the right hand side of (104), intersects the constant dotted line at 1, representing the left-hand side of the equation. This implies the existence of a non-trivial
solution for $M/\Lambda = 0.159$. Using this value in (105), we then obtain a consistent solution for $\bar{g} = 12.58$, in the sense that for this value of $\bar{g}$ the three curves have a common intersection at $M/\Lambda = 0.159$. This demonstrates that there is dynamical mass generation with $m \simeq M < \Lambda$ in this case for strong (dimensionless) four-fermion couplings $\bar{g} > 1$.

![Figure 4](https://via.placeholder.com/150)

**Figure 4.** Plot of the right-hand-side of (104) (dashed line), for $\lambda^2 = 0.01$ and $\bar{B} = 0.0004$. The dotted-dashed line represents the right hand side of (105) for the above values of $\lambda^2$ and $\bar{B}$, and $\bar{g} = 12.58$. The constant dotted line at 1 represents the left hand side of the equations (104) and (105). The existence of a common intersection point for all three curves demonstrates the existence of dynamical mass generation, of approximately equal magnitude, for both fermions and axions. Figure taken from [42].

Although in our analysis above, and in [40, 42] we restricted ourselves to the partial solution $m \simeq M$, we remark that solutions beyond this restriction also exist. For instance, if we consider the case in which the Yukawa interactions are absent $\lambda^2 \to 0$, and only the (attractive) four-fermion interaction is present, $g \neq 0$, in (92), then we recover the result of [82] for dynamical mass generation only for fermions, for which the fermionic SD equation reads [42]

$$1 = -\frac{g^2}{64\pi^2f_b^2B^2} \left( \Lambda^4 + \Lambda^2 \left( 2(m^2 + B^2) - \sqrt{(\Lambda^2 + m^2 - B^2)^2 + 4m^2B^2} \right) 
+ (m^2 + 7B^2) \left( \sqrt{(\Lambda^2 + m^2 - B^2)^2 + 4m^2B^2} - (m^2 + B^2) \right) 
+ 4B^2(m^2 - 2B^2) \log \left[ \frac{(m^2 - B^2) + \Lambda^2 + \sqrt{(\Lambda^2 + m^2 - B^2)^2 + 4m^2B^2}}{2m^2} \right] \right). \quad (106)$$

This is essentially the same as equation (34) in ref. [82], where the Nambu Jona Lasinio four-fermion model was studied in the presence of a constant non-Hermitian axial background. The study of (106) indicates [82] that the inclusion of the background increases the dynamical fermion mass due to the strong four-fermion interactions alone.

### 5. Conclusions and Outlook

In this mini-review we discussed some aspects of non-Hermitian (specifically, anti-Hermitian) Yukawa models of interacting fermion and axion fields associated with their consistency as quantum field theoretic models, stemming, e.g., from microscopic models embedded in string theory [43]. Apart from the anti-Hermitian Yukawa interactions, the models included anti-Hermitian anomaly terms, as well as interactions of the fermions with constant anti-Hermitian
axial backgrounds. The underlying $CPT$ symmetry (56) of these models guarantee [10, 40, 41, 42] the reality of the energy spectra, and thus the dynamical masses.

We have addressed the issue of dynamical mass generation for both fermions and axions, with the conclusion that in the absence of attractive four-fermion (Hermitian) interactions, only an axion mass can be generated dynamically, due to energetics. In contrast, in the presence of sufficiently strong four-fermion attractive interactions, Yukawa-interaction-induced dynamical mass generation for both fermions and axions is possible. The role of the (sufficiently strong) four-fermion interactions, therefore, is that of a catalyst for the dynamical mass generation in this case. The anomaly terms, on the other hand, resist dynamical generation of fermion masses. The (constant) axial background assists the Yukawa-coupling-induced mass generation, in the presence of strong four-fermion interactions, in the sense that the larger the value of the background field, the larger the magnitude of the anti-Hermitian-Yukawa-interaction-induced dynamical mass.

In the context of microscopic string theories, the origin of the non-Hermitian background can be traced back to the non-Hermitian anomalous interactions of the KR gravitational axion of the massless bosonic string multiplet in some formulations of the theory (see “Scheme II”, (24), in section 2). It would be interesting to investigate further the consequences of such anti-Hermitian axial backgrounds for Leptogenesis, as per the scenario of [60, 56]. In general, non-Hermitian Yukawa interactions, of the type discussed here, may play a role in neutrino physics, including neutrino oscillations, as well as non-Hermitian extensions of the standard model.

We conclude, by mentioning the recent work of [83], according to which anti-Hermitian Yukawa interactions, of the type discussed in [40, 41] and reviewed here, may also arise as a result of linearising Hermitian four-fermion interactions, of Nambu-Jona-Lasinio type [79], using appropriate complex auxiliary (pseudo)scalar fields. According to such an analysis, one has spontaneous appearance of non-Hermiticity in these systems. In addition, four-fermion interactions of Nambu-Jona-Lasinio type with complex couplings, whose imaginary part represents dissipation, have been discussed in [84], with the conclusion that the imaginary coupling tends to enhance chiral-symmetry breaking up to a given threshold, above which the symmetry is restored. Such results could be testable in systems of ultracold atomic gases, or other materials that are characterised by Dirac-(Majorana-, or Weyl-)fermion-like excitations [85, 86, 87].

As an outlook, we mention that, from a pure field-theoretic viewpoint, the non-Hermitian models we have discussed here should be studied in a more rigorous way as far as their path-integral quantization is concerned. To this end, we should combine our SD treatment for mass generation with an appropriate extension to fermionic models of the methods developed in [22] for defining properly the path-integral measure of pseudoscalar fields in PT-symmetric pseudoscalar field theories, leading to their renormalization. Such a combined study might lead to a renormalization-group improved SD analysis for our models.

Moreover, an understanding of the concept of anomalies in such non-Hermitian quantum field theories is pressing. When, and if, this is accomplished, one could obtain a geometric understanding of the anti-Hermitian anomalous interactions that we focus our attention on in this review, and in [42]. A fully microscopic understanding of such terms in the context of UV-complete theories of quantum gravity, such as string/brane theory, is also desirable, but this might be a rather long shot.

Another potentially interesting topic is the application of the non-Hermitian anomalous Yukawa models in the presence of (approximately constant) non-Hermitian axial backgrounds to the generation of a Lepton-number asymmetry in the early Universe (Leptogenesis), by extending appropriately the corresponding scenarios of the Hermitian case [60, 56], provided, of course, that such an extension is possible. The dynamical fermion-mass generation in such non-Hermitian models, studied in [40, 42] and reviewed above, when applied to Majorana neutrinos, might lead
to Lepton-number asymmetries due to the asymmetric decays of such massive fields to standard-model leptons and antileptons in the presence of the non-Hermitian axial backgrounds. We hope to be able to come back to a study of some of these important issues in the near future.

We close this mini review with an optimistic note. We believe that, although several aspects of non-Hermitian quantum field theories embeddable in a generalised-PT(CPT) framework are yet to be understood, nonetheless the rapidly-growing interest in such theories is a strong indicator that soon they might found themselves at comparable levels of rigour and physical applicability with their Hermitian counterparts. In this respect, it would be interesting to see whether there are fundamental aspects of particle physics models, perhaps in the dark sector, that could be explained with such non-Hermitian interactions, which otherwise would remain a mystery. Such an example constituted the topic of this review, namely the novel mechanisms involving anti-Hermitian Yukawa interactions for dynamical generation of masses of right-handed neutrinos and axions, both of which can play the rôle of dark matter candidates in models beyond the standard model of particle physics.

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