Spanning tree-connected subgraphs with small degrees

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Dedicated to Hisako Tsunoda on the occasion of her 67th birthday

Abstract

Let $G$ be a graph with a spanning subgraph $F$, let $m$ be a positive integer, and let $f$ be a positive integer-valued function on $V(G)$. In this paper, we show that if for all $S \subseteq V(G)$,

$$\Omega_m(G \setminus S) \leq \sum_{v \in S} (f(v) - 2m) + m + \Omega_m(G[S]),$$

then $G$ has a spanning $m$-tree-connected subgraph $H$ containing $F$ such that for each vertex $v$, $d_H(v) \leq f(v) + \max\{0, d_F(v) - m\}$, where $G[S]$ denotes the induced subgraph of $G$ with the vertex set $S$ and $\Omega_m(G_0)$ is a parameter to measure $m$-tree-connectivity of a given graph $G_0$.

By applying this result, we show that every $k$-edge-connected graph $G$ with $k \geq 2m$ has a spanning $m$-tree-connected subgraph $H$ such that for each $v \in V(H)$,

$$d_H(v) \leq \left\lceil \frac{m}{2} (d_G(v) - 2m) \right\rceil + 2m,$$

where $G$ is $k$-tree-connected and $k \geq m$, then $G$ has a spanning $m$-tree-connected subgraph $H$ such that $d_H(v) \leq \left\lceil \frac{m}{2} (d_G(v) - m) \right\rceil + m$ for each $v \in V(H)$. As a consequence, we conclude that every $(r - 2m)$-edge-connected graph with $r \geq 4m$ admits a spanning $m$-tree-connected subgraph with maximum degree at most $3m$.

Next, we prove that a graph $G$ admits a spanning $m$-tree-connected subgraph $H$ satisfying $\Delta(H) \leq 2m + 1$, if for all $S \subseteq V(G)$,

$$\omega(G \setminus S) + \frac{m + 1}{2} \text{iso}(G \setminus S) \leq \frac{1}{m} |S| + 1,$$

where $\omega(G \setminus S)$ and $\text{iso}(G \setminus S)$ denote the number of components and the number of isolated vertices of $G \setminus S$, respectively. As a consequence, we conclude that every $m(n - 1)$-connected $K_{1,n}$-free simple graph with a sufficiently large minimum degree and $n \geq 3$ admits a spanning $m$-tree-connected subgraph with maximum degree at most $2m + 1$.

Keywords:
Spanning tree; connected factor; toughness; tree-connected graph; strongly tough graph.
1 Introduction

In this article, graphs have no loops, but multiple edges are allowed and a simple graph is a graph without multiple edges. Let $G$ be a graph. The vertex set, the edge set, the minimum degree, the maximum degree, and the number of components of $G$ are denoted by $V(G)$, $E(G)$, $\delta(G)$, $\Delta(G)$, and $\omega(G)$, respectively. We also denote by $iso(G)$ (resp. $I(G)$) the number (resp. the set) of isolated vertices of $G$. For a set $X \subseteq V(G)$, we denote by $G[X]$ the induced subgraph of $G$ with the vertex set $X$ containing precisely those edges of $G$ whose ends lie in $X$. The degree $d_G(v)$ of a vertex $v$ is the number of edges of $G$ incident to $v$. We also denote by $d_G(X)$ the number of edges of $G$ with exactly one end in $X$. For a spanning subgraph $H$ with a given integer-valued function $h$ on $V(H)$, the total excess of $H$ from $h$ is defined as follows:

$$te(H, h) = \sum_{v \in V(H)} \max\{0, d_H(v) - h(v)\}.$$ 

According to this definition, $te(H, h) = 0$ if and only if for each vertex $v$, $d_H(v) \leq h(v)$. A spanning subgraph of a graph $G$ is called a factor. For a set $A$ of integers, an $A$-factor is a spanning subgraph with vertex degrees in $A$. Let $F$ be a factor of $G$. For an edge set $E$, we denote by $F - E$ the graph obtained from $F$ by removing the edges of $E$ from $F$. Likewise, we denote by $F + E$ the graph obtained from $F$ by inserting the edges of $E$ into $F$. For convenience, we use $e$ instead of $E$ when $E = \{e\}$. For two edge sets $E_1$ and $E_2$, we also use the notation $E_1 + E_2$ for the union of them. A component of $F$ is said to be trivial, if it consists of only one vertex. Likewise, $F$ is said to be trivial, if it has no edge. A vertex set $S$ of a graph $G$ is called independent, if there is no edge of $G$ connecting vertices in $S$. Let $S \subseteq V(G)$. The graph obtained from $G$ by removing all vertices of $S$ is denoted by $G \setminus S$. Denote by $e_G(S)$ the number of edges of $G$ with both ends in $S$. Let $P$ be a partition of $V(G)$. Denote by $e_G(P)$ the number of edges of $G$ whose ends lie in different parts of $P$. The graph obtained from $G$ by contracting all vertex sets of $P$ is denoted by $G/P$. A graph $G$ is called $m$-tree-connected, if it has $m$ edge-disjoint spanning trees. (Note that the trivial graph having one vertex and no edge is also $m$-tree-connected). In addition, an $m$-tree-connected graph $G$ is called minimally $m$-tree-connected, if for any edge $e$ of $G$, the graph $G - e$ is not $m$-tree-connected. Hence, an $m$-tree-connected graph is minimally $m$-tree-connected if and only if $|E(G)| = m(|V(G)| - 1)$.

(We should note that somewhere we consider a minimal $m$-tree-connected subgraph of $G$ that contains a fixed vertex subset $X$ of $V(G)$; in this case, our minimality is with respect to inclusion and the desired subgraph is not necessarily minimally $m$-tree-connected). The vertex set of any graph $G$ can be expressed uniquely as a disjoint union of vertex sets of maximal $m$-tree-connected subgraphs. (More precisely, we can define the relation on $V(G)$ as follows: $x \sim y$ if $x$ and $y$ are contained in an $m$-tree-connected subgraph of $G$. This is an equivalence relation which partitions $V(G)$ into equivalence classes; see Observation 2.2). These subgraphs are called the $m$-tree-connected components of $G$. For a graph $G$, we define the parameter $\Omega_m(G) = m|P| - e_G(P)$ to measure tree-connectivity, where $P$ is the unique partition of $V(G)$ obtained from the $m$-tree-connected components of $G$. Note that $\Omega_1(G)$ is the same number of components of $G$, while $\frac{1}{m}\Omega_m(G)$ is less than or equal to the number of $m$-tree-connected components of $G$. The definition implies that the null graph $K_0$ with no vertices is not $m$-tree-connected and $\Omega_m(K_0) = 0$. (Note that every nonnull graph $G$ satisfies $\Omega_m(G) \geq m$ and the equality holds if and only if $G$ is $m$-tree-connected; see
Theorem 2.16). In this paper, we assume that all graphs are nonnull, except for the graphs that are obtained by removing vertices. We say that a graph $G$ is $m^{+}$-tree-connected, if it is non-trivial and $G - e$ remains $m$-tree-connected for every edge $e$. A graph $F$ is $m$-sparse, if $e_F(S) \leq m|S| - m$ for all nonempty subsets $S \subseteq V(F)$. Clearly, 1-sparse graphs are forests. It is not hard to show that a graph $F$ is $m$-sparse if and only if all $m$-tree-connected components of $F$ are minimally $m$-tree-connected; see Corollary 2.7. We will show that every $m$-tree-connected graph $H$ with the minimum number of edges containing a given $m$-sparse subgraph $F$ is also minimally $m$-tree-connected. Let $t$ be a positive real number. A graph $G$ is said to be $t$-tough, if $\omega (G \setminus S) \leq \max \{1, \frac{1}{t}|S|\}$ for all $S \subseteq V(G)$. Likewise, $G$ is said to be $m$-strongly $t$-tough, if $\frac{1}{m}\Omega_m(G \setminus S) \leq \max \{1, \frac{1}{t}|S|\}$ for all $S \subseteq V(G)$. We will show that every $m$-strongly $t$-tough graph must be $t$-tough; see Theorem 2.19. In addition, we show that tough enough graphs with sufficiently large order are also $m$-strongly tough enough; see Corollary 6.4. We say that a simple graph $G$ is $K_{1,f}$-free, if $G$ does not have an induced star of size $f(v)$ with center $v$ for all $v \in V(G)$, where $f$ is a positive integer-valued function on $V(G)$. Throughout this article, all variables $k$ and $m$ are positive integers.

Recently, the present author [13] investigated spanning trees with small degrees and established the following theorem. This result is an improvement of several results due to Win (1989) [28], Ellingham and Zha (2000) [10], and Ellingham, Nam, and Voss (2002) [9].

**Theorem 1.1.** ([13]) Let $G$ be a graph with a factor $F$. Let $f$ be a positive integer-valued function on $V(G)$.

If for all $S \subseteq V(G)$,

$$\omega(G \setminus S) \leq \sum_{v \in S} (f(v) - 2) + 1 + \omega(G[S]),$$

then $G$ has a connected factor $H$ containing $F$ such that for each vertex $v$, $d_H(v) \leq f(v) + \max \{0, d_F(v) - 1\}$.

We derived the following result from Theorem 1.1 which is an improvement of some results due to Liu and Xu (1998) [16], Ellingham, Nam, and Voss (2002) [9], and the present author (2015) [12].

**Theorem 1.2.** ([13]) Let $G$ be a connected graph. Then every matching of $G$ can be extended to a connected factor $H$ such that for each vertex $v$,

$$d_H(v) \leq \begin{cases} \left\lceil \frac{d_G(v) - 2}{k} \right\rceil + 2, & \text{if } G \text{ is } k\text{-edge-connected;} \\ \left\lceil \frac{d_G(v) - 1}{k} \right\rceil + 1, & \text{if } G \text{ is } k\text{-tree-connected.} \end{cases}$$

In this paper, we investigate tree-connected factors with small degrees and generalize Theorem 1.1 toward this concept by proving the following theorem.

**Theorem 1.3.** Let $G$ be a graph with a factor $F$ and let $f$ be a positive integer-valued function on $V(G)$.

If for all $S \subseteq V(G)$,

$$\Omega_m(G \setminus S) \leq \sum_{v \in S} (f(v) - 2m) + m + \Omega_m(G[S]),$$

then $G$ has an $m$-tree-connected factor $H$ containing $F$ such that for each vertex $v$, $d_H(v) \leq f(v) + \max \{0, d_F(v) - m\}$. 

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In Section 5, we derive the following result from Theorem 1.3 on highly edge-connected graphs. As a consequence, we conclude that every \((r - 2m)\)-edge-connected \(r\)-regular graph with \(r \geq 4m\) must have an \(m\)-tree-connected factor with maximum degree at most \(3m\).

**Theorem 1.4.** Let \(G\) be a graph. Then every factor of \(G\) with maximum degree at most \(m\) can be extended to an \(m\)-tree-connected factor \(H\) such that for each vertex \(v\),

\[
d_H(v) \leq \begin{cases} \left\lceil \frac{m(d_G(v) - 2m)}{k} \right\rceil + 2m, & \text{if } G \text{ is } k \text{-edge-connected and } k \geq 2m; \\ \left\lfloor \frac{m(d_G(v) - m)}{k} \right\rfloor + m, & \text{if } G \text{ is } k \text{-tree-connected and } k \geq m. \end{cases}
\]

In 1973 Chvátal [6] conjectured that tough enough graphs of order at least three admit a Hamiltonian cycle. In Section 6, we establish a sufficient toughness-type condition for the existence of tree-connected factors with a bounded maximum degree as the following theorem.

**Theorem 1.5.** Let \(G\) be a graph. Then every factor of \(G\) with maximum degree at most \(m\) can be extended to an \(m\)-tree-connected factor \(H\) with \(\Delta(H) \leq 2m + 1\), if for all \(S \subseteq V(G)\),

\[
\omega(G \setminus S) + \frac{m + 1}{2} \text{iso}(G \setminus S) \leq \frac{1}{m} |S| + 1.
\]

In addition, we show that this result is sharp in the sense that the coefficient of \(|S|\) cannot be increased according to the following theorem.

**Theorem 1.6.** Let \(m\) be an integer with \(m \geq 2\). For every real number \(\varepsilon \in (0, 1)\), there are infinitely many \(2m\)-connected graphs \(G\) having no \(m\)-tree-connected factors \(H\) with \(\Delta(H) \leq 2m + 1\), while for all \(S \subseteq V(G)\) with \(|S| \geq 2m\),

\[
\omega(G \setminus S) + \frac{m + 1}{2} \text{iso}(G \setminus S) \leq \left(\frac{1}{m} + \varepsilon\right) |S|.
\]

It remains to be decided whether higher toughness can guarantee the existence of an \(m\)-tree-connected factor with maximum degree at most \(2m\). The special case \(m = 1\) of this question investigates the existence of Hamiltonian paths in tough enough graphs. We put forward the following conjecture for this purpose.

**Conjecture 1.7.** For every positive integer \(m\), there is a positive real number \(t_m\) such that every \(t_m\)-tough graph \(G\) of order at least \(2m\) admits an \(m\)-tree-connected factor \(H\) with \(\Delta(H) \leq 2m\).

## 2 Basic tools

In this section, we present some basic tools for working with tree-connected graphs. We begin with the following well-known result which gives a criterion for a graph to have \(m\) edge-disjoint spanning trees.
Theorem 2.1. (Nash-Williams [20] and Tutte [27]) A graph $G$ is $m$-tree-connected if and only if for every partition $P$ of $V(G)$, $e_G(P) \geq m(|P| - 1)$.

For every vertex $v$ of a graph $G$, consider an induced $m$-tree-connected subgraph of $G$ containing $v$ with the maximal order. It is known that these subgraphs are unique and partition the vertex set of $G$; see [4], and [24, Lemma 4.7]. In fact, these subgraphs are the $m$-tree-connected components of $G$ that were introduced in the Introduction. The following observation simply shows that these subgraphs are well-defined.

Observation 2.2. Let $G$ be a graph and let $X \subseteq V(G)$ and $Y \subseteq V(G)$. If $G[X]$ and $G[Y]$ are $m$-tree-connected and $X \cap Y \neq \emptyset$, then $G[X \cup Y]$ is also $m$-tree-connected.

Proof. Let $P$ be a partition of $X \cup Y$. Define $P'$ and $P_0$ to be the partitions of $X$ and $Y$ with

$$P' = \{ A \cap X : A \in P \text{ and } A \cap X \neq \emptyset \}$$

and

$$P_0 = \{ A \in P : A \cap X = \emptyset \} \cup \{ Y \cap \bigcup_{A \in P, A \cap X \neq \emptyset} A \}.$$ 

Since $|P'| + (|P_0| - 1) = |P|$, by Theorem 2.1, we have

$$e_{G[X \cup Y]}(P) \geq e_{G[X]}(P') + e_{G[Y]}(P_0) \geq m(|P'| - 1) + m(|P_0| - 1) = m(|P| - 1).$$

Again, by applying Theorem 2.1, the graph $G[X \cup Y]$ must be $m$-tree-connected. □

The next observation presents a simple way for deducing tree-connectivity of a graph.

Observation 2.3. Let $G$ be a graph and let $X \subseteq V(G)$. If $G[X]$ and $G/X$ are $m$-tree-connected, then $G$ itself is $m$-tree-connected.

Proof. It is enough to apply the same argument as in the proof of Observation 2.2 by setting $Y = V(G)$. Note that, by Theorem 2.1, we again have $e_{G[Y]}(P_0) \geq m(|P_0| - 1)$, since $G/X$ is $m$-tree-connected. □

2.1 Edge-density and the existence of non-trivial $m$-tree-connected components

The following theorem shows that graphs with high edge-density must have $m^+$-tree-connected subgraphs.

Theorem 2.4. Every graph $G$ of order at least two containing at least $m(|V(G)| - 1) + 1$ edges has an $m^+$-tree-connected subgraph with at least two vertices.

Proof. The proof is by induction on $|V(G)|$. For $|V(G)| = 2$, the proof is clear. Assume $|V(G)| \geq 3$. Suppose the theorem is false. Since $G$ is not $m^+$-tree-connected, there exits (by Theorem 2.1) an edge $e$ and a partition $P$ of $V(G)$ such that $e_{G_0}(P) < m(|P| - 1)$, where $G_0 = G - e$. Note that $e_G(P) \leq m(|P| - 1)$. 5

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By induction hypothesis, for every $A \in P$ (since no part $A \in P$ has an $m^+$-tree-connected subgraph), we have $e_G(A) \leq m(|A| - 1)$ regardless of $|A| = 1$ or not. Therefore,

$$m(|V(G)| - 1) < |E(G)| = e_G(P) + \sum_{A \in P} e_G(A) \leq m(|P| - 1) + m \sum_{A \in P} (|A| - 1) \leq m(|V(G)| - 1).$$

This result is a contradiction, as desired. □

The following corollary gives a sufficient condition for the existence of a non-trivial $m$-tree-connected subgraph.

**Corollary 2.5.** ([29]) Every graph $G$ of order at least two containing at least $m(|V(G)| - 1)$ edges has an $m$-tree-connected subgraph with at least two vertices.

This condition can be improved a little when the existence of a non-trivial $m$-edge-connected subgraph is considered.

**Corollary 2.6.** ([26, Section 2]) Every graph $G$ of order at least two containing at least $(m-1)(|V(G)|-1)+1$ edges has an $m$-edge-connected subgraph with at least two vertices.

**Corollary 2.7.** A graph $G$ is $m$-sparse if and only if its $m$-tree-connected components are $m$-sparse.

**Proof.** Suppose, to the contrary, that $G$ is not $m$-sparse. Thus by Theorem 2.4, the graph $G$ has a non-trivial $m^+$-tree-connected subgraph $G[A]$. Since $G[A]$ is not $m$-sparse, the $m$-tree-connected component of $G$ containing it is not $m$-sparse, which is a contradiction. □

**Corollary 2.8.** Every graph $M$ with maximum degree at most $m$ must be $m$-sparse.

**Proof.** Suppose, to the contrary, that $M$ is not $m$-sparse. Thus by Theorem 2.4, the graph $M$ has a non-trivial $m^+$-tree-connected subgraph $M[A]$. Since $M[A]$ is $(m+1)$-edge-connected, $M$ has minimum degree at least $m+1$ which is a contradiction. (A simpler proof can be obtained from the definition, because every non-trivial subgraph $M[A]$ has size at most $\frac{1}{2}m|A|$ which is less than or equal to $m(|A| - 1)$.) □

**Corollary 2.9.** Let $G$ be an $m$-tree-connected graph with an $m$-sparse factor $M$. If $G - e$ is not $m$-tree-connected for every $e \in E(G) \setminus E(M)$, then $G$ must be minimally $m$-tree-connected.

**Proof.** Suppose, to the contrary, that $G$ is not $m$-sparse. Thus by Theorem 2.4, the graph $G$ has a non-trivial $m^+$-tree-connected subgraph $G[A]$. Since $M[A]$ is $m$-sparse, there must be an edge $e \in E(G) \setminus E(M)$ with both ends in $A$. Therefore, $G[A] - e$ must be $m$-tree-connected and so is $G - e$ which is a contradiction. □
We shall here prove the following similar version of Corollary 2.9 based on Theorem 2.4. This result was proved in [18, 21] when $M$ is the trivial factor.

**Corollary 2.10.** Let $k_0$ and $k$ be two nonnegative integers, and let $G$ be a $(k_0 + k)$-edge-connected graph having a $k_0$-edge-connected factor $G_0$ and a $k$-sparse factor $M$. If $G - e$ is not $(k + k_0)$-edge-connected for every $e \in E(G) \setminus E(G_0 \cup M)$, then the supergraph $G \setminus E(G_0)$ of $M$ must be $k$-sparse.

**Proof.** Suppose, to the contrary, that $H$ is not $k$-sparse, where $H = G \setminus E(G_0)$. Thus by Theorem 2.4, the graph $H$ has a non-trivial $k^+$-tree-connected subgraph $H[A]$. Since $M[A]$ is $k$-sparse, there must be an edge $e \in E(H) \setminus E(M)$ with both ends in $A$. Since $G - e$ is not $(k + k_0)$-edge-connected, there is a vertex set $X$ having exactly one end of $e$ such that $d_G(X) = k_0 + k$. Note that $A$ is not a subset of $X$ and $X \neq V(G_0)$. Since $H[A]$ is $(k + 1)$-edge-connected, we must have $d_G(X) \geq d_{G_0}(X) + d_{H[A]}(X \cap A) \geq k_0 + (k + 1)$, which is a contradiction. \hfill \Box

### 2.2 Sparsity of minimally $m$-tree-connected graphs

Let $G$ be a graph satisfying $|E(G)| = m(|V(G)| - 1)$. It is known that $G$ is $m$-sparse if and only if $G$ is $m$-tree-connected. We shall below form a similar stronger version.

**Theorem 2.11.** If $G$ is an $m$-sparse graph satisfying $|E(G)| = m(|V(G)| - 1)$, then for every partition $P$ of $V(G)$, $e_G(P) \geq m(|P| - 1) + p_0$, where $p_0$ is the number of vertex sets $X \in P$ such that $G[X]$ is not $m$-tree-connected.

**Proof.** Since $G$ is $m$-sparse, for every nonempty subset $A$ of $V(G)$, $e_G(A) \leq m(|A| - 1)$. Let $P_0$ be the set of all $A \in P$ such that $G[A]$ is not $m$-tree-connected. Let $A \in P_0$. By Theorem 2.1, there exists a partition $P'$ of $A$ such that $e_{G[A]}(P') < m(|P'| - 1)$. Thus $e_G(A) = e_{G[A]}(P') + \sum_{X \in P} e_G(X) < m(|P'| - 1) + \sum_{X \in P} m(|X| - 1) = m(|A| - 1)$. Therefore,

$$e_G(P) = |E(G)| - \sum_{X \in P} e_G(X) \geq m(|V(G)| - 1) - (\sum_{X \in P} m(|X| - 1) - |P_0|) = m(|P| - 1) + |P_0|.$$

This completes the proof. \hfill \Box

The following corollary shows an application of Theorem 2.11 that plays an essential role in this paper.

**Corollary 2.12.** Let $F$ be an $m$-sparse graph and let $x, y \in V(G)$. Let $Q$ be a minimal $m$-tree-connected subgraph of $F$ containing $x$ and $y$. If $z \in V(Q) \setminus \{x, y\}$, then $d_Q(z) \geq m + 1$.

**Proof.** Let $A = V(Q) \setminus \{z\}$. By minimality of $Q$, the graph $Q[A]$ is not $m$-tree-connected. In addition, $Q$ is $m$-sparse and $|E(Q)| = m(|V(Q)| - 1)$. Thus by Theorem 2.11, we must have $d_Q(A) = e_Q(P) \geq m(|P| - 1) + 1 = m + 1$, where $P = \{A, \{z\}\}$. Hence the proof is completed. \hfill \Box
The next corollary is a very useful tool for finding a pair of edges such that replacing them preserves sparsity of a given \( m \)-sparse factor. This tool for working with sparse graphs can be obtained using matroid theory; see [8].

**Corollary 2.13.** Let \( F \) be an \( m \)-sparse graph, let \( x, y \in V(F) \), and let \( e_{xy} \) be an edge joining \( x \) and \( y \) satisfying \( e_{xy} \notin E(F) \). If \( Q \) is a minimal \( m \)-tree-connected subgraph of \( F \) containing \( x \) and \( y \), then for every \( e \in E(Q) \), the graph \( F - e + e_{xy} \) remains \( m \)-sparse.

**Proof.** Suppose, to the contrary, that \( F - e + e_{xy} \) is not \( m \)-sparse. Thus there is a vertex subset \( A \) of \( V(F) \) containing \( x \) and \( y \) such that \( e_F(A) \geq m(|A| - 1) \) and \( e \notin E(F[A]) \). This implies that \( A \notin V(Q) \).

Since \( F \) is \( m \)-sparse, we must have \( e_F(A) = m(|A| - 1) \). Let \( B = V(Q) \). Since \( Q \) is \( m \)-tree-connected, \( e_F(A \cap B) \geq e_F(A) + e_F(B) - e_F(A \cup B) \geq m(|A| - 1) + m(|B| - 1) - m(|A \cup B| - 1) = m(|A \cap B| - 1) \).

Therefore, \( e_F(A \cap B) = m(|A \cap B| - 1) \). According to Theorem 2.11, the graph \( F[A \cap B] \) must be \( m \)-tree-connected, which contradicts minimality of \( Q \). Hence the proof is completed. \( \square \)

**Corollary 2.14.** Let \( H \) be an \( m \)-tree-connected graph, let \( x, y \in V(H) \), and let \( e_{xy} \) be an edge joining \( x \) and \( y \) satisfying \( e_{xy} \notin E(H) \). If \( Q \) is a minimal \( m \)-tree-connected subgraph of \( H \) containing \( x \) and \( y \), then for every \( e \in E(Q) \), the graph \( H - e + e_{xy} \) remains \( m \)-tree-connected.

**Proof.** By the assumption, \( Q \) is minimally \( m \)-tree-connected and so it is \( m \)-sparse. Let \( H' \) be a minimally \( m \)-tree-connected factor of \( H \) containing \( E(Q) \). By Corollary 2.9, the graph \( H' \) is minimally \( m \)-tree connected and so it is \( m \)-sparse. Thus by Corollary 2.13, \( H' - e + e_{xy} \) remains \( m \)-sparse. Since the size of \( H' - e + e_{xy} \) is \( m(|V(H')| - 1) \), this graph must be \( m \)-tree-connected and so is \( H - e + e_{xy} \). \( \square \)

**Corollary 2.15.** Let \( F \) be an \( m \)-sparse graph, let \( x, y \in V(F) \), and let \( e_{xy} \) be an edge joining \( x \) and \( y \) satisfying \( e_{xy} \notin E(F) \). If \( x \) and \( y \) are in different \( m \)-tree-connected components of \( F \), then \( F + e_{xy} \) is still \( m \)-sparse.

**Proof.** Suppose, to the contrary, that \( F + e_{xy} \) is not \( m \)-sparse. Thus there is a vertex set \( A \) containing both ends of \( e_{xy} \) such that \( e_F(A) = m(|A| - 1) \). From Theorem 2.11, we infer that \( e_{F[A]}(P) \geq m(|P| - 1) \) for all partitions \( P \) of \( A \), and so by Theorem 2.1 we find that \( F[A] \) is \( m \)-tree-connected. Hence both ends of \( e_{xy} \) are in the same \( m \)-tree-connected component of \( G \) which is a contradiction. \( \square \)

### 2.3 Some properties of tree-connectivity measures

The following theorem introduces an interesting property of tree-connectivity measures.

**Theorem 2.16.** For every graph \( G \), we have

\[
\Omega_m(G) = \max_{P \in A} \{ m|P| - e_G(P) \} \geq m,
\]
where \( \mathcal{A} \) is the set of all partitions \( P \) of \( V(G) \). In addition, the equality holds if and only if \( G \) is \( m \)-tree-connected.

**Proof.** Let \( P \) be a partition of \( V(G) \) with the maximum \( m|P| - e_G(P) \). We may assume that \( P \) has the minimum size. We first claim that for every \( X \in P \), \( G[X] \) is \( m \)-tree-connected. Otherwise, by Theorem 2.1, there is a partition \( P_X \) of \( X \) such that \( e_{G[X]}(P_X) < m(|P_X| - 1) \). Let \( P' \) be the partition of \( V(G) \) consisting of all parts of \( P \setminus \{ X \} \) and \( P_X \). Since \( e_{G'}(P') = e_G(P) + e_{G[X]}(P_X) \), we must have \( m|P'| - e_G(P') = m(|P| - 1 + |P_X|) - e_G(P) - e_{G[X]}(P_X) > m|P| - e_G(P) \), which is a contradiction. For every \( X \in P \), let \( X_0 \) be the vertex set of the maximal \( m \)-tree-connected subgraph of \( G \) containing \( X \). We claim that \( X = X_0 \). Otherwise, by applying the first claim, there exists a partition \( P_0 \) of \( X_0 \) obtained from some parts of \( P \) satisfying \( |P_0| \geq 2 \). Let \( P' = (P \setminus P_0) \cup \{ X_0 \} \). Note that \( P' \) is a partition of \( V(G) \) and \( |P'| < |P| \). Since \( e_{G'}(P') = e_G(P) - e_{G[X_0]}(P_0) \), we must have \( m|P'| - e_G(P') = m(|P| - |P_0| + 1) - e_G(P) + e_{G[X_0]}(P_0) \geq m|P| - e_G(P) \), which is a contradiction to the minimality of \( |P| \). Therefore, \( P \) is the partition of \( V(G) \) obtained from the \( m \)-tree-connected components of \( G \). Thus \( \Omega_m(G) = m|P| - e_G(P) \). By applying Corollary 2.5 to the contracted graph \( G/P \), we must have \( e_G(P) < m(|P| - 1) \) provided that \( |P| \geq 2 \). In addition, \( e_G(P) = m(|P| - 1) = 0 \) when \( |P| = 1 \). Thus \( \Omega_m(G) = m \) if and only if \( G \) is \( m \)-tree-connected. Hence the proof is completed. \( \square \)

**Corollary 2.17.** For every graph \( G \), we have \( \Omega_m(G) \geq m|V(G)| - |E(G)| \). In addition, the equality holds if and only if \( G \) is \( m \)-sparse.

**Proof.** According to Theorem 2.16, \( \Omega_m(G) \geq m|P| - e_G(P) = m|V(G)| - |E(G)| \), where \( P \) is the trivial partition of \( V(G) \) obtained from vertices of \( G \). Now, we prove the second assertion. Let \( P \) be the partition of \( V(G) \) obtained from \( m \)-tree-connected components of \( G \). Obviously, \( e_G(A) \geq m(|A| - 1) \) for all \( A \in P \). In addition, by Corollary 2.7, the equalities hold simultaneously if and only if \( G \) is \( m \)-sparse. Thus \( |E(G)| = \sum_{A \in P} e_G(A) + e_G(P) \geq m|V(G)| - (m|P| - e_G(P)) = m|V(G)| - \Omega_m(G) \). Moreover, the equality holds if and only if \( G \) is \( m \)-sparse. Hence the assertion holds. \( \square \)

**Corollary 2.18.** Let \( G \) be a graph. If \( H \) is a factor of \( G \), then \( \Omega_m(G) \leq \Omega_m(H) \).

**Proof.** If we set \( P \) to be the partition of \( V(G) \) obtained from the \( m \)-tree-connected components of \( G \), then by Theorem 2.16, we must have \( \Omega_m(G) = m|P| - e_G(P) \leq m|P| - e_H(P) \leq \Omega_m(H) \). \( \square \)

The following theorem describes a relationship between tree-connectivity measures of a graph.

**Theorem 2.19.** For every graph \( G \), we have
\[
\omega(G) = \Omega_1(G) \leq \frac{1}{2} \Omega_2(G) \leq \frac{1}{3} \Omega_3(G) \leq \cdots \leq |V(G)|.
\]
Therefore, $G_e$.

Obviously, $e_{G[A]}(P_A) = (m + 1)(|P_A| - 1) = 0$ when $|P_A| = 1$. In addition, by applying Corollary 2.5 to the graph $G[A]$, we must have $e_{G[A]}(P_A) < (m + 1)(|P_A| - 1)$ when $|P_A| \geq 2$. Thus

$$e_G(P') - e_G(P) \leq \sum_{A \in P} e_{G[A]}(P_A) \leq (m + 1) \sum_{A \in P} (|P_A| - 1) = (m + 1)(|P'| - |P|).$$

Therefore,

$$\frac{1}{m} \Omega_m(G) = |P| - \frac{1}{m} e_G(P) \leq |P| - \frac{1}{m + 1} e_G(P) \leq |P'| - \frac{1}{m + 1} e_G(P') = \frac{1}{m + 1} \Omega_{m+1}(G).$$

This inequality completes the proof. □

3 Structures of tree-connected factors with the minimum total excess

Here, we state the following fundamental theorem, which gives much information about $m$-tree-connected factors with the minimum total excess. This result develops some results in [9, 11, 13, 28] and it can also develop the main result of this paper in terms of total excess as the paper [11].

Theorem 3.1. Let $G$ be an $m$-tree-connected graph and let $h$ be an integer-valued function on $V(G)$. Let $M$ be a factor of $G$ satisfying $\Delta(M) \leq m$. If $H$ is a minimally $m$-tree-connected factor of $G$ containing $M$ with the minimum total excess from $h$, then there exists a subset $S$ of $V(G)$ with the following properties:

1. $\Omega_m(G \setminus S) = \Omega_m(H \setminus S)$.
2. $S \supseteq \{v \in V(G) : d_H(v) > h(v)\}$.
3. For each vertex $v$ of $S$, $d_H(v) \geq h(v)$.

Proof. Note that $M$ is $m$-sparse (by Corollary 2.8) and so $H$ is well-defined (by Corollary 2.9). Define $V_0 = \emptyset$ and $V_1 = \{v \in V(H) : d_H(v) > h(v)\}$. For any $S \subseteq V(G)$ and $u \in V(G \setminus S$, let $C(S, u)$ be the vertex set of the $m$-tree-connected component of $H \setminus S$ containing $u$. Let $\mathcal{A}(S, u)$ be the set of factors $H'$ of $G$ containing $M$ such that

(a) $H'$ is minimally $m$-tree-connected.

(b) $d_{H'}(v) \leq h(v)$ for all $v \in V(G \setminus V_1$.

(c) if $e \in (E(H) \cup E(H')) \setminus (E(H) \cap E(H'))$, then both ends of $e$ are contained in $C(S, u)$.
According to item (c), since $|E(H')| = |E(H)| = m(|V(H)| - 1)$, we must have $e_{H'}(X) = e_H(X)$, where $X = C(S, u)$. On the other hand, since $H'[X]$ is $m$-sparse and $m$-tree-connected, $e_H(X) = m(|X| - 1)$ which implies that $e_{H'}(X) = m(|X| - 1)$. Thus the $m$-sparse graph $H'[X]$ must be minimally $m$-tree-connected with respect to Theorems 2.11 and 2.1. Now, for each integer $n$ with $n \geq 2$, recursively define $V_n$ as follows:

$$V_n = V_{n-1} \cup \{ v \in V(G) \setminus V_{n-1}: d_{H'}(v) \geq h(v) \text{ for all } H' \in A(V_{n-1}, v) \}.$$ 

Now, we prove the following claim.

**Claim.** Let $x$ and $y$ be two vertices in different $m$-tree-connected components of $H \setminus V_{n-1}$. If $xy \in E(G) \setminus E(H)$, then $x \in V_n$ or $y \in V_n$.

**Proof of Claim.** By induction on $n$. For $n = 1$, the proof is clear. Assume that the claim is true for $n - 1$. Now we prove it for $n$. Suppose, to the contrary, that vertices $x$ and $y$ are in different $m$-tree-connected components of $H \setminus V_{n-1}$, $xy \in E(G) \setminus E(H)$, and $x, y \notin V_n$. Let $X$ and $Y$ be the vertex sets of the $m$-tree-connected components of $H \setminus V_{n-1}$ containing $x$ and $y$, respectively. Since $x, y \notin V_n$, there exist $H_x \in A(V_{n-1}, x)$ and $H_y \in A(V_{n-1}, y)$ with $d_{H_x}(x) < h(x)$ and $d_{H_y}(y) < h(y)$. By the induction hypothesis, $x$ and $y$ are in the same $m$-tree-connected component of $H \setminus V_{n-2}$ with the vertex set $Z$ so that $\{x, y\} \subseteq X \cup Y \subseteq Z$. Obviously, $Z \cap V_{n-2} = \emptyset$. Let $Q$ be a minimal $m$-tree-connected subgraph of $H[Z]$ containing $x$ and $y$.

Notice that $Q$ includes at least a vertex $z \in V_{n-1} \setminus V_{n-2}$ so that $d_Q(z) \geq h(z)$. By Corollary 2.12, we have $d_Q(z) > m \geq d_M(z)$ and so there is an edge $zz' \in E(Q) \setminus E(M)$. By Corollary 2.13, the graph $H - zz' + xy$ is $m$-sparse. Since this graph contains $m(|V(H)| - 1)$ edges, by Theorem 2.11, it must be $m$-tree-connected. Now, let $H'$ be the factor of $G$ with

$$E(H') = E(H) - zz' + xy - E(H[X]) + E(H_x[X]) - E(H[Y]) + E(H_y[Y]).$$

Recall that $H_x[X]$ and $H_y[Y]$ are minimally $m$-tree-connected. Thus $H'$ and $H$ have the same size $m(|V(H)| - 1)$. By applying Observation 2.3 twice, one can easily check that $H'$ is minimally $m$-tree-connected. Since both of graphs $H_x$ and $H_y$ contain $E(M)$ and $zz' \notin E(M)$, the graph $H'$ contains $E(M)$. For each $v \in V(H') \setminus \{z\}$, we have

$$d_{H'}(v) = \begin{cases} d_{H_x}(v) - 1, & \text{if } v = z' \in X \setminus \{x\}, \\ d_{H_y}(v) - 1, & \text{if } v = z' \in Y \setminus \{y\}, \\ d_H(v) - 1, & \text{if } v = z' \notin (X \cup Y), \\ d_{H_z}(v), & \text{if } v = z' \in \{x, y\}. \end{cases}$$

and

$$d_{H'}(v) = \begin{cases} d_{H_x}(v), & \text{if } v \in X \setminus \{x, z'\}; \\ d_{H_y}(v), & \text{if } v \in Y \setminus \{y, z'\}; \\ d_H(v), & \text{if } v \notin X \cup Y \cup \{z, z'\}; \\ d_{H_z}(v) + 1, & \text{if } v \in \{x, y\} \setminus \{z'\}. \end{cases}$$

If $n \geq 3$, then it is not hard to see that $d_{H'}(z) = d_H(z) - 1 < h(z)$ and $H'$ lies in $A(V_{n-2}, z)$. Since $z \in V_{n-1} \setminus V_{n-2}$, we arrive at a contradiction. For the case $n = 2$, since $z \in V_1$, it is easy to see that $d_{H'}(z) = d_H(z) - 1 \geq h(z)$ and $te(H', h) < te(H, h)$, which is again a contradiction. Hence the claim holds.

Obviously, there exists a positive integer $n$ such that and $V_1 \subseteq \cdots \subseteq V_{n-1} = V_n$. Put $S = V_n$. Since $S \supseteq V_1$, Condition 2 clearly holds. For each $v \in V_i \setminus V_{i-1}$ with $i \geq 2$, we have $H \in A(V_{i-1}, v)$ and so
such that for each vertex

\[ d_H(v) \geq h(v). \]

This establishes Condition 3. Because \( S = V_n \), the previous claim implies Condition 1 and completes the proof. \( \square \)

### 4 Sufficient conditions depending on tree-connectivity measures

Our aim in this subsection is to prove Theorem 1.3. We begin with the following lemma that allows us to make the proof shorter. This lemma is an extension of Lemma 4.1 in [13].

**Lemma 4.1.** Let \( G \) be a graph with a factor \( F \). If a maximal factor \( M \) of \( F \) satisfying \( \Delta(M) \leq m \) can be extended to an \( m \)-tree-connected factor \( T \), then \( F \) itself can be extended to an \( m \)-tree-connected factor \( H \) such that for each vertex \( v \),

\[ d_H(v) \leq d_T(v) + \max\{0, d_F(v) - m\}. \]

**Proof.** Let \( A = \{v \in V(G) : d_M(v) < m\} \). According to the maximality of \( M \), the vertex set \( A \) must be an independent set of \( F \setminus E(M) \). Otherwise, we can insert a new edge of \( F \setminus E(M) \) into \( M \) to expand it to a larger factor with maximum degree at most \( m \) which is a contradiction. Define \( G_0 = T \cup F \) and let \( A \) be the set of all \( m \)-tree-connected factors \( T_0' \) of \( G_0 \) containing \( M \) such that \( d_{T_0'}(u) \leq d_T(u) \) for all \( u \in A \).

Note that \( A \) is nonempty, because \( T \in A \). Consider \( T_0 \in A \) with the maximum \( |E(T_0) \cap E(F)| \) and let \( H = T_0 \cup F \). We claim that \( H \) is the desired factor that we are looking for. Let \( v \in V(H) \). If \( d_M(v) = m \), then

\[ d_H(v) \leq d_{G_0}(v) \leq d_T(v) + d_F(v) - d_M(v) = d_T(v) + d_F(v) - m. \]

So, suppose \( v \in A \). Define \( F_0 \) to be the factor of \( F \) with \( E(F_0) = E(F) \cap E(T_0) \). To complete the proof, we are going to show that \( d_{F_0}(v) \geq \min\{m, d_F(v)\} \). More precisely, this implies that

\[ d_H(v) = d_{T_0'}(v) + d_F(v) - d_{F_0}(v) \leq d_T(v) + d_F(v) - \min\{m, d_F(v)\} = d_T(v) + \max\{0, d_F(v) - m\}. \]

Suppose, to the contrary, that \( d_{F_0}(v) < \min\{m, d_F(v)\} \). Pick \( vx \in E(F) \setminus E(F_0) \) so that \( vx \notin E(T_0) \cup E(M) \). Let \( Q \) be a minimal \( m \)-tree-connected subgraph of \( T_0 \) containing \( v \) and \( x \). Since \( Q \) is \( m \)-edge-connected, \( d_Q(v) \geq m \). On the other hand, \( d_{F_0}(v) < m \). Thus there exists an edge \( vy \in E(Q) \setminus E(F_0) \) so that \( vy \notin E(T_0) \setminus E(F) \) (we might have \( x = y \)). Define \( T_0' = T_0 - vy + vx \). By Corollary 2.14, the graph \( T_0' \) is still \( m \)-tree-connected. Note that \( T_0' \) contains the edges of \( M \), because \( vy \notin E(M) \). According to this

![Figure 1: An example showing all possibilities of the vertices y, v, and x.](image-url)
construction, \(d_{T_0}(u) = d_{T_0}(x)\) for all \(u \in V(G) \setminus \{x, y\}\). Moreover, \(d_{T_0}(x) = d_{T_0}(x)+1\) and \(d_{T_0}(y) = d_{T_0}(y)-1\) when \(x \neq y\), and \(d_{T_0}(x) = d_{T_0}(x)\) when \(x = y\). Since \(xy \in E(F) \setminus E(M)\) and \(v \in A\), we must have \(x \not\in A\). Therefore, \(d_{T_0}(u) \leq d_{T_0}(u) \leq d_T(u)\) for all \(u \in A\). Since \(|E(T_0') \cap E(F)| > |E(T_0) \cap E(F)|\), we derive a contradiction to the maximality of \(T_0\), as desired. \(\Box\)

The following lemma establishes a simple but important property of minimally \(m\)-tree-connected graphs.

**Lemma 4.2.** Let \(H\) be a minimally \(m\)-tree-connected graph. If \(S \subseteq V(H)\), then

\[
\Omega_m(H \setminus S) = \sum_{v \in S}(d_H(v) - m) + m - e_H(S).
\]

**Proof.** Let \(P\) be the partition of \(V(H) \setminus S\) obtained from the \(m\)-tree-connected components of \(H \setminus S\). Obviously, \(e_H(P \cup \{\{v\} : v \in S\}) = \sum_{v \in S}d_H(v) - e_H(S) + e_{H \setminus S}(P)\). On the other hand, \(e_H(P \cup \{\{v\} : v \in S\}) = m(|P| + |S| - 1)\), since \(|E(H)| = m(|V(H)| - 1)\) and for any \(A \subseteq P\), \(e_H(A) = m(|A| - 1)\). Therefore, we must have \(\Omega_m(H \setminus S) = m|P| - e_{H \setminus S}(P) = \sum_{v \in S}(d_H(v) - m) + m - e_H(S)\). Hence the lemma holds. \(\Box\)

The following theorem is essential in this section.

**Theorem 4.3.** Let \(G\) be a graph with \(X \subseteq V(G)\) and with a factor \(F\). Let \(f\) be a positive integer-valued function on \(X\). If for all \(S \subseteq X\),

\[
\Omega_m(G \setminus S) \leq \sum_{v \in S}(f(v) - 2m) + m + \Omega_m(G[S]),
\]

then \(G\) has an \(m\)-tree-connected factor \(H\) containing \(F\) such that for each \(v \in X\), \(d_H(v) \leq f(v) + \max\{0, d_F(v) - m\}\).

**Proof.** Note that \(G\) must automatically be \(m\)-tree-connected, because of \(\Omega_m(G \setminus \emptyset) \leq m\). For each \(v \in V(G) \setminus X\), define \(f(v) = d_G(v) + 1\). Choose a maximal factor \(M\) of \(F\) satisfying \(\Delta(M) \leq m\). Let \(H\) be a minimally \(m\)-tree-connected factor of \(G\) containing \(M\) with the minimum total excess from \(f\). Define \(S\) to be a subset of \(V(G)\) with the properties described in Theorem 3.1. If \(v \in V(G) \setminus X\), then \(d_H(v) \leq d_G(v) < f(v)\). This implies that \(S \subseteq X\). By Lemma 4.2 and Theorem 3.1,

\[
\sum_{v \in S}f(v) + te(H, f) = \sum_{v \in S}d_H(v) = \Omega_m(H \setminus S) + m|S| - m + e_H(S),
\]

and hence

\[
\sum_{v \in S}f(v) + te(H, f) = \Omega_m(G \setminus S) + m|S| - m + e_H(S).
\]

Since \(H[S]\) is \(m\)-sparse, by Corollaries 2.17 and 2.18, one can conclude that \(e_H(S) = m|S| - \Omega_m(H[S]) \leq m|S| - \Omega_m(G[S])\). By the assumption, we therefore have

\[
te(H, f) \leq \Omega_m(G \setminus S) - \sum_{v \in S}(f(v) - 2m) - m - \Omega_m(G[S]) \leq 0.
\]
Hence $te(H, f) = 0$. By Lemma 4.1, the factor $F$ itself can be extended to an $m$-tree-connected connected factor $H$ such that for each vertex $v$, $d_H(v) \leq f(v) + \max\{0, d_F(v) - m\}$. Hence the theorem is proved. □

The next corollary gives a sufficient condition, similar to toughness condition, that guarantees the existence of a highly tree-connected factor with bounded maximum degree.

**Corollary 4.4.** Let $G$ be an $m$-tree-connected graph with a factor $F$. Let $f$ be a positive integer-valued function on $V(G)$. If for all $S \subseteq V(G)$,

$$
\Omega_m(G \setminus S) \leq \sum_{v \in S} (f(v) - 2m) + 2m,
$$

then $G$ has an $m$-tree-connected factor $H$ containing $F$ such that for each vertex $v$, $d_H(v) \leq f(v) + \max\{0, d_F(v) - m\}$.

**Proof.** Since $G$ is $m$-tree-connected, it is obvious that $\Omega_m(G \setminus \emptyset) = m$. Let $S$ be a nonempty subset of $V(G)$. Since $\Omega_m(G[S]) \geq m$, we must have $\Omega_m(G \setminus S) \leq \sum_{v \in S} (f(v) - 2m) + 2m \leq \sum_{v \in S} (f(v) - 2m) + m + \Omega_m(G[S])$. Now, it is enough to apply Theorem 4.3. □

**Corollary 4.5.** Let $G$ be a graph with an independent set $X \subseteq V(G)$ and with a factor $F$. Let $f$ be a positive integer-valued function on $X$. If for all $S \subseteq X$,

$$
\Omega_m(G \setminus S) \leq \sum_{v \in S} (f(v) - m) + m,
$$

then every factor $F$ can be extended to an $m$-tree-connected factor $H$ such that for each $v \in X$, $d_H(v) \leq f(v) + \max\{0, d_F(v) - m\}$.

**Proof.** Let $S$ be a subset of $X$. Since $X$ is an independent set, we must have $\Omega_m(G[S]) = m|S|$ which implies that $\Omega_m(G \setminus S) \leq \sum_{v \in S} (f(v) - m) + m = \sum_{v \in S} (f(v) - 2m) + m + \Omega_m(G[S])$. Now, it is enough to apply Theorem 4.3. □

## 5 Highly edge-connected graphs

### 5.1 Edge-connected and tree-connected graphs

Highly edge-connected graphs are natural candidates for graphs satisfying the assumptions of Theorem 4.3. We examine them in this subsection, beginning with the following extended version of Lemma 4.9 in [13].

**Lemma 5.1.** Let $G$ be a graph with $S \subseteq V(G)$. Then

$$
\Omega_m(G \setminus S) \leq \begin{cases}
\sum_{v \in S} \frac{m(d_G(v) - 2m)}{k} + \frac{2m}{k} \Omega_m(G[S]), & \text{if } G \text{ is } k\text{-edge-connected, } k \geq 2m, \text{ and } S \neq \emptyset; \\
\sum_{v \in S} \left( \frac{m(d_G(v) - m)}{k} - m \right) + m + \frac{m}{k} \Omega_m(G[S]), & \text{if } G \text{ is } k\text{-tree-connected and } k \geq m.
\end{cases}
$$
Theorem 5.2. Let 

Now, we are ready to generalize Theorems 1.2 as mentioned in the abstract. 

Proof. Let \( P \) be the partition of \( V(G) \setminus S \) obtained from the \( m \)-tree-connected components of \( G \setminus S \). Obviously, we have 

If \( G \) is \( k \)-edge-connected and \( S \neq \emptyset \), then there are at least \( k \) edges of \( G \) with exactly one end in \( C \), for any \( C \in P \). Thus \( e_G(P \cup \{v : v \in S\}) \geq k|P| - e_{G \setminus S}(P) + e_G(S) \) and so if \( k \geq 2m \), then 

When \( G \) is \( k \)-tree-connected, we have \( e_G(P \cup \{v : v \in S\}) \geq k(|P| + |S| - 1) \) and so if \( k \geq m \), then 

By Corollary 2.17, since \( e_G(S) \geq m|S| - \Omega_m(G[S]) \), these inequalities complete the proof. \( \square \)

Now, we are ready to generalize Theorems 1.2 as mentioned in the abstract.

Theorem 5.2. Let \( G \) be a graph with \( X \subseteq V(G) \). Then every factor \( F \) can be extended to an \( m \)-tree-connected factor \( H \) such that for each \( v \in X \),

Furthermore, for an arbitrary given vertex \( u \), the upper bound can be reduced to \( \lceil \frac{m}{k} d_G(u) \rceil + \max\{0, d_F(u) - m\} \).

Proof. Let \( S \) be a subset of \( X \). If \( S \) is empty, then \( \Omega_m(G \setminus S) \leq m \), since \( G \) is \( m \)-tree-connected. Assume that \( S \) is not empty. Hence \( \Omega_m(G[S]) \geq m \) by Theorem 2.16. If \( G \) is \( k \)-edge-connected and \( k \geq 2m \), then by Lemma 5.1, we have 

where \( f(u) = \lfloor \frac{m}{k} d_G(u) \rfloor \) and \( f(v) = \lceil \frac{m(d_G(v) - 2m)}{k} \rceil + 2m \) for all \( v \in V(G) \setminus \{u\} \). If \( G \) is \( k \)-tree-connected and \( k \geq m \), then by Lemma 5.1, we also have 

where \( f(u) = \lfloor \frac{m}{k} d_G(u) \rfloor \) and \( f(v) = \lceil \frac{m(d_G(v) - m)}{k} \rceil + m \) for all \( v \in V(G) \setminus \{u\} \). Thus the first two assertions follow from Theorem 4.3. Now, suppose that \( X \) is an independent set. If \( G \) is \( k \)-edge-connected and \( k \geq 2m \),

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then by Lemma 5.1, we have
\[ \Omega_m(G \setminus S) \leq \sum_{v \in S} m k d_G(v) < 1 + \sum_{v \in S} (f(v) - m) + m, \]
where \( f(u) = \lceil \frac{m}{k} d_G(u) \rceil \) and \( f(v) = \lceil \frac{md_G(v)}{k} \rceil + m \) for all \( v \in X \setminus \{u\} \). If \( G \) is \( k \)-tree-connected and \( k \geq m \), then by Lemma 5.1, we also have
\[ \Omega_m(G \setminus S) \leq \sum_{v \in S} m k d_G(v) - m + m < 1 + \sum_{v \in S} (f(v) - m) + m, \]
where \( f(u) = \lceil \frac{m}{k} d_G(u) \rceil \) and \( f(v) = \lceil \frac{md_G(v)}{k} \rceil \) for all \( v \in X \setminus \{u\} \). Thus the second two assertions follow from Corollary 4.5.

A generalization of Corollary 1 in [12] is given in the following corollary.

**Corollary 5.3.** Every \((r - 2m)\)-edge-connected \( r \)-regular graph \( G \) with \( r \geq 4m \) admits an \( m \)-tree-connected \{\( m, m + 1, \ldots, 3m \}\}-factor.

**Proof.** Apply Theorem 5.2 with \( k = r - 2m \).

**Corollary 5.4.** Every \( 2m \)-edge-connected graph \( G \) has an \( m \)-tree-connected factor \( H \) such that for each vertex \( v \),
\[ d_H(v) \leq \lceil \frac{d_G(v)}{2} \rceil + m. \]
Furthermore, for an arbitrary given vertex \( u \), the upper bound can be reduced to \( \lceil \frac{d_G(u)}{2} \rceil \).

**Proof.** Apply Theorem 5.2 with \( k = 2m \).

### 5.2 Alternative proofs for \( 2m \)-edge-connected graphs

In the following, we shall give two simpler proofs for Corollary 5.4 inspired by the proofs that introduced in [2, 16, 25] for the special case \( m = 1 \). For this purpose, we need some well-known results. Note that the second one implicitly appeared in [3] for \( m = 2, 6 \). Moreover, two interesting developments of this corollary are given in [1].

**Theorem 5.5.** (Mader [17], see Section 3 in [23]) Let \( G \) be a \( 2m \)-edge-connected graph with \( z \in V(G) \). If \( d_G(z) \geq 2m + 2 \), then there are two edges \( xz \) and \( yz \) incident to \( z \) such that after removing them and inserting a new edge \( xy \) for the case \( x \neq y \), the resulting graph is still \( 2m \)-edge-connected.

**The second proof of Corollary 5.4.** By induction on \( \sum_{v \in V(G)} \max\{0, d_G(v) - 2m - 1\} \). First, suppose that this summation is zero. This means that \( \Delta(G) \leq 2m + 1 \). Since \( G \) is \( 2m \)-edge-connected, every
Lemma 5.6. Let \( M \subseteq E_G(u) \) be an edge set of size \( m \) or \( m+1 \) with respect to \( d_G(u) = 2m \) or \( d_G(u) = 2m+1 \), where \( E_G(u) \) denotes the set of edges of \( G \) that are incident to \( u \). We claim that \( G \setminus M \) is \( m \)-tree-connected and so the theorem obviously holds by setting \( H = G \setminus M \). Otherwise, Theorem 2.1 implies that there is a partition \( P \) of \( V(G) \) such that \( m|\{P|−1\}| > e_{G,M}(P) \geq e_G(P) − |M| = \sum_{X \in P} d_G(X)/2 − |M| \geq m|P| − |M| \). This implies that the edges of \( M \) join different parts of \( P \), \( |M| = m+1 \), \( d_G(u) = 2m+1 \), and \( d_G(X) = 2m \) for all \( X \in P \). It is not hard to check that \( (E_G(U, \overline{U}) \cup E_G(u)) \setminus M \) forms an edge cut of size \( 2m − 1 \) for \( G \), which is contradiction, where \( u \in U \in P \) and \( E_G(U, \overline{U}) \) denotes the set of edges of \( G \) with exactly one end in \( U \).

Now, suppose that there is a vertex \( z \) with \( d_G(z) \geq 2m+2 \). By Theorem 5.5, there are two edges \( xz \) and \( yz \) incident to \( z \) such that after removing them, and inserting a new edge \( xy \) for the case \( x \neq y \), the resulting graph \( G' \) is still \( 2m \)-edge-connected. By the induction hypothesis, the graph \( G' \) has a factor \( H' \) containing \( m \) edge-disjoint spanning trees \( T_1, \ldots, T_m \) such that \( d_{H'}(u) \leq \lfloor d_{G'}(u)/2 \rfloor \) and for each vertex \( v \) with \( v \neq u \), \( d_{H'}(v) \leq \lfloor d_{G'}(v)/2 \rfloor + m \). If \( xy \notin E(T_1 \cup \cdots \cup T_m) \), then the theorem clearly holds. Thus we may assume that \( xy \in E(T_1) \) and \( z \) and \( x \) lie in the same component of \( T_1 − xy \). Define \( T'_1 = T_1 − xy + yz \). It is easy to see that \( T'_1 \) is connected and \( T'_1 \cup T_2 \cup \cdots \cup T_m \) is the desired factor of \( G \) that we are looking for.

Before stating the third proof, let us establish the following lemma and state the next two well-known results.

Lemma 5.6. Let \( G \) be a graph with a factor \( M \) satisfying \( \Delta(M) \leq m \). If \( G \) can be decomposed into an \( m \)-tree-connected factor \( H \) and a factor \( F \) having an orientation such that for each vertex \( v \), \( d^+_F(v) \geq l_0(v) \), then \( G \) can also be decomposed into an \( m \)-tree-connected factor \( H' \) containing \( M \) and a factor \( F' \) having an orientation such that for each vertex \( v \), \( d^+_{F'}(v) \geq l_0(v) \), where \( l_0 \) is a nonnegative integer-valued function on \( V(G) \).

Proof. Decompose \( G \) into a minimally \( m \)-tree-connected factor \( H \) and a factor \( F \) having an orientation such that for each vertex \( v \), \( d^+_F(v) \geq l_0(v) \). Consider the pair \( (H, F) \) with the maximum \( |E(H) \cap E(M)| \). We claim that \( H \) contains the edges of \( M \). Suppose, to the contrary, that there is an edge \( vx \in E(M) \cap E(F) \). We may assume that \( vx \) is directed from \( v \) to \( x \) in \( F \). Let \( Q \) be a minimal \( m \)-tree-connected factor of \( G \) containing \( v \) and \( x \). Since \( d_Q(v) \geq m > d_{H \cap M}(v) \), there is an edge \( vy \in E(Q) \setminus E(M) \). We define \( H_0 = H + vx − vy \) and \( F_0 = F − vx + vy \), and we orient the edge \( vy \) from \( v \) to \( y \) in \( F_0 \). Obviously, for each vertex \( u \), we still have \( d^+_{F_0}(u) = d^+_{F}(u) \). Moreover, by Corollary 2.14, the graph \( H_0 \) is still \( m \)-tree-connected. Thus the new pair \( (H_0, F_0) \) has the desired properties while \( |E(H_0) \cap E(M)| > |E(H) \cap E(M)| \), which is a contradiction. Hence the proof is completed.

Theorem 5.7. (Nash-Williams [19], see Theorem 2.1 in [2]) Every \( 2m \)-edge-connected graph \( G \) has an \( m \)-arc-strong orientation such that for each vertex \( v \), \( \lfloor d_G(v)/2 \rfloor \leq d^+_G(v) \leq \lceil d_G(v)/2 \rceil \).
Theorem 5.8. (Edmonds [7]) Let $G$ be a directed graph with $u \in V(G)$. If $d_G^+(X) \geq m$ for all $X \subseteq V(G)$ with $u \in X$, then $G$ has a spanning subdigraph $H$ such that its underlying graph is $m$-tree-connected, $d_H^+(u) = 0$, and $d_H^+(v) = m$ for all $v \in V(G) \setminus \{u\}$, where $d_G^+(X)$ denotes the number of incoming edges to $X$ in $G$.

Now, we are in a position to provide the third proof of Corollary 5.4 by proving the following stronger version.

Theorem 5.9. Let $G$ be a graph with a factor $M$ satisfying $\Delta(M) \leq m$. If $G$ is $2m$-edge-connected, then it can be decomposed into an $m$-tree-connected factor $H$ containing $M$ and a factor $F$ having an orientation such that for each vertex $v$,

$$d_F^+(v) \geq \left\lfloor \frac{d_G(v)}{2} \right\rfloor - m.$$ 

Furthermore, for an arbitrary given vertex $u$ the lower bound can be increased to $\left\lceil \frac{d_G(u)}{2} \right\rceil$.

Proof. Consider an $m$-arc-strong orientation for $G$ with the properties stated in Theorem 5.7. We may assume that the out-degree of $u$ is equal to $\left\lfloor d_G(u)/2 \right\rfloor$; otherwise, we reverse the orientation of $G$. Take $H$ to be a spanning subdigraph of $G$ with the properties stated in Theorem 5.8. For each vertex $v$, we have $d_H(v) = d_H^+(v) + d_H^-(v) \leq d_G^+(v) + d_H^+(v) \leq \left\lfloor d_G(v)/2 \right\rfloor + m$. In particular, $d_H(u) \leq d_G^+(u) + d_H^+(u) \leq \left\lfloor d_G(u)/2 \right\rfloor$. These imply that for each vertex $v$, $d_F^+(v) \geq \left\lfloor \frac{d_G(v)}{2} \right\rfloor - m$, and $d_F^+(u) \geq \left\lceil \frac{d_G(u)}{2} \right\rceil$, where $F$ is the complement of $H$ in $G$. Now, it is enough to apply Lemma 5.6 to complete the proof. \qed

6 Tough enough graphs

6.1 The existence of $m$-tree-connected $[m, 2m + 1]$-factors

As we have already shown in Theorem 2.19, $m$-strongly tough enough graphs are tough enough. In this subsection, we shall prove the converse statement and examine tough enough graphs for Corollary 4.4. For this purpose, we need the following two lemmas.

Lemma 6.1. Let $G$ be a graph and let $\varepsilon$ be a real number with $0 \leq \varepsilon \leq 1$. If $S$ is a subset of $V(G)$ with the maximum $\Omega_m(G \setminus S) - \varepsilon |S|$ and with the maximal $|S|$, then every component of $G \setminus S$ is $m$-tree-connected or has maximum degree at most $m$.

Proof. Let $v$ be an arbitrary vertex of $G \setminus S$ and define $S' = S \cup \{v\}$. If $v$ is contained in a non-trivial $m$-tree-connected component of $G \setminus S$ with vertex set $X$, then it is not difficult to check that $\Omega_m(G \setminus S') = \Omega_m(G \setminus S) - m + \Omega_m(G[X \setminus v]) + d$, where $d$ denotes the number of edges incident to $v$ having one end in $V(G) \setminus (X \cup S)$. Thus by Theorem 2.16, $\Omega_m(G \setminus S') - \varepsilon |S'| \geq \Omega_m(G \setminus S) - \varepsilon |S| - \varepsilon + d$. According to the assumption, we must have $d < \varepsilon$ and so $d = 0$. Therefore, every non-trivial $m$-tree-connected component of $G \setminus S$ is also a component of it. If $v$ is a trivial $m$-tree-connected component of $G \setminus S$, then it is not difficult to
Theorem 6.3. Let

Now, we are ready to prove the main result of this section.

Lemma 6.2.\(\) Let \(I\) be an independent subset of \(G\). According to the assumption, we must have \(d_G(v) < m + \varepsilon\) and so \(d_G(v) \leq m\). This implies that if a component of \(G \setminus S\) is not \(m\)-tree-connected, then it has maximum degree at most \(m\). This completes the proof. \(\square\)

\textbf{Lemma 6.2.} \textit{(14)} Let \(H\) be a graph. If \(\varphi\) is a nonnegative real function on \(V(H)\), then there is an independent subset \(I\) of \(V(H)\) such that

\[
\sum_{v \in V(H)} \varphi(v) \leq \sum_{v \in I} \varphi(v)(d_H(v) + 1).
\]

Now, we are ready to prove the main result of this section.

\textbf{Theorem 6.3.} Let \(G\) be a graph and let \(\varepsilon\) and \(c\) be two real numbers satisfying \(0 \leq \varepsilon \leq 1/m\) and \(1 \leq c\). If for all \(S \subseteq V(G)\),

\[
\omega(G \setminus S) + \frac{m + 1}{2} + \text{iso}(G \setminus S) \leq \varepsilon|S| + c,
\]

then for all \(S \subseteq V(G)\),

\[
\frac{1}{m} \Omega_m(G \setminus S) \leq \varepsilon|S| + c.
\]

\textbf{Proof.} Let \(S \subseteq V(G)\) with the properties described in Lemma 6.1. Denote by \(\sigma\) the number of non-trivial components of \(G \setminus S\) which are \(m\)-tree-connected. Let \(C\) be the induced subgraph of \(G\) consisting of the vertices of trivial \(m\)-tree-connected components of \(G \setminus S\). By Lemma 6.2, there is an independent set \(I\) of \(C\) such that

\[
\sum_{v \in V(C)} (1 + \varepsilon - \frac{d_C(v)}{2m}) \leq \sum_{v \in I} (1 + \varepsilon - \frac{d_C(v)}{2m})(d_C(v) + 1).
\]

If \(V(C)\) is empty, then we must automatically have \(\frac{1}{m} \Omega_m(G \setminus S) = \omega(G \setminus S) \leq \varepsilon|S| + c\). We may therefore assume that \(V(C)\) is nonempty and so is \(I\). Since \(\Delta(C) \leq m\) and \(0 \leq \varepsilon \leq 1/m\), we must have

\[
\sum_{v \in I} (1 + \varepsilon - \frac{d_C(v)}{2m})(d_C(v) + 1) \leq \frac{1}{2m} \sum_{v \in I} (2m + 2m\varepsilon - m)(m + 1) \leq \left(\frac{m + 3}{2}\right) + \varepsilon|I|,
\]

which implies that

\[
\sum_{v \in V(C)} (1 - \frac{d_C(v)}{2m}) \leq \left(\frac{m + 3}{2}\right) + \varepsilon|I| - \varepsilon|V(C)|.
\] (1)

Let \(S' = S \cup (V(C) \setminus I)\) so that \(\omega(G \setminus S') = \sigma + |I|\) and \(\text{iso}(G \setminus S') = |I|\). Thus by the assumption,

\[
\sigma + \frac{m + 3}{2}|I| = \omega(G \setminus S') + \frac{m + 1}{2} + \text{iso}(G \setminus S') \leq \varepsilon|S'| + c = \varepsilon(|S| + |V(C)| - |I|) + c,
\]

which implies that

\[
\sigma + \left(\frac{m + 3}{2} + \varepsilon\right)|I| - \varepsilon|V(C)| \leq \varepsilon|S| + c.\] (2)

Therefore, Inequations (1) and (2) can conclude that

\[
\frac{1}{m} \Omega_m(G \setminus S) = \sigma + \sum_{v \in V(C)} (1 - \frac{d_C(v)}{2m}) \leq \varepsilon|S| + c.
\]

Hence the theorem holds. \(\square\)
The following corollary says that tough enough graphs with sufficiently large order are also \emph{m}-strongly tough enough.

\textbf{Corollary 6.4.} Let \( G \) be a graph and let \( t \) be a real number with \( t \geq 1 \). If \( G \) is \( 2m^2t \)-tough and \( |V(G)| \geq 2m^2t \), then \( G \) is also \( m \)-strongly \( t \)-tough.

\textbf{Proof.} We may assume that \( m \geq 2 \). Since \( G \) is \( \frac{m+3}{2}(m+1)t \)-tough, for every \( S \subseteq V(G) \), \( \omega(G \setminus S) + \frac{m+1}{2} \text{iso}(G \setminus S) \leq \frac{1}{(m+1)t}|S| + 1 \) provided that \( \omega(G \setminus S) \geq 2 \) or \( \text{iso}(G \setminus S) = 0 \). In addition, if \( \omega(G \setminus S) = \text{iso}(G \setminus S) = 1 \), then \( |S| = |V(G)| - 1 \), and so \( \omega(G \setminus S) + \frac{m+1}{2} \text{iso}(G \setminus S) = \frac{m+3}{2} \leq \frac{1}{(m+1)t}|S| + 1 \). Thus by Theorem 6.3, for every \( S \subseteq V(G) \), \( \frac{1}{m}\Omega_m(G \setminus S) \leq \frac{1}{(m+1)t}|S| + 1 \). Therefore, if \( \Omega_m(G \setminus S) > m \), then we must have \( |S| \geq \frac{(m+1)m}{m} \) which implies that \( \frac{1}{m}\Omega_m(G \setminus S) \leq \frac{1}{(m+1)t}|S| + 1 \leq \frac{1}{t}|S| \). This means that \( G \) is \( m \)-strongly \( t \)-tough. \( \square \)

The following result gives a sufficient toughness-type condition for a graph to have an \( m \)-tree-connected factor with maximum degree at most \( 2m + 1 \).

\textbf{Corollary 6.5.} Let \( G \) be a graph with a factor \( M \) satisfying \( \Delta(M) \leq m \). If for all \( S \subseteq V(G) \),

\[ \omega(G \setminus S) + \frac{m+1}{2} \text{iso}(G \setminus S) \leq \frac{1}{m}|S| + 1, \]

then \( G \) admits an \( m \)-tree-connected factor \( H \) containing \( M \) such that for each vertex \( v \), \( d_H(v) \leq 2m + 1 \), and also \( d_H(u) \leq m + 1 \) for an arbitrary given vertex \( u \).

\textbf{Proof.} By applying Theorem 6.3 with setting \( \varepsilon = 1/m \) and \( c = 1 \), we must have \( \Omega_m(G \setminus S) \leq |S| + m \) for all \( S \subseteq V(G) \). Hence the assertion follows from Corollary 4.4 with setting \( f(u) = m + 1 \) and \( f(v) = 2m + 1 \) for each \( v \in V(G) \setminus \{u\} \). Note that \( G \) must automatically be \( m \)-tree-connected, because of \( \Omega_m(G \setminus \emptyset) \leq m \). \( \square \)

\textbf{Corollary 6.6.} Every \( \frac{1}{2}m(m+3) \)-tough graph \( G \) of order at least \( 2m \) has an \( m \)-tree-connected factor \( H \) satisfying \( \Delta(H) \leq 2m + 1 \).

\textbf{Proof.} If \( |V(G)| \leq \frac{1}{2}m(m+3) \), then \( G \) must be complete and the proof is straightforward. Assume that \( |V(G)| > \frac{1}{2}m(m+3) \). Let \( S \) be a subset of \( V(G) \). If \( \omega(G \setminus S) \geq 2 \), then \( \omega(G \setminus S) + \frac{m+1}{2} \text{iso}(G \setminus S) \leq \frac{m+3}{m} \omega(G \setminus S) \leq \frac{1}{m}|S| \). If \( \omega(G \setminus S) \leq 1 \) and \( \text{iso}(G \setminus S) = 0 \), then \( \omega(G \setminus S) + \frac{m+1}{2} \text{iso}(G \setminus S) \leq 1 \). If \( \omega(G \setminus S) = \text{iso}(G \setminus S) = 1 \), then \( S = |V(G)| - 1 \geq \frac{1}{2}m(m+3) - 1 \) and so \( \omega(G \setminus S) + \frac{m+1}{2} \text{iso}(G \setminus S) = \frac{m+3}{2} \leq \frac{1}{m}|S| + 1 \). Now, it is enough to apply Corollary 6.5. \( \square \)

\subsection{Sharpness: \((m-\varepsilon)\)-tough graphs with no \( m \)-tree-connected \([m, 2m+1]\)-factors}

Our aim in this subsection is to present a family of tough graphs with no \( m \)-tree-connected \([m, 2m+1]\)-factors and show that the coefficient \( 1/m \) of Corollary 6.5 is sharp. For our purpose, we first need to establish the following simple lemma.
Lemma 6.7. Let $m$ be an integer with $m \geq 2$. For every $\varepsilon \in (0,1)$, there is a simple graph $G$ satisfying $e_G(P) \geq (m - \varepsilon)(|P| - 1)$ for every partition $P$ of $V(G)$, while $G$ is not $m$-tree-connected.

Proof. Let $G_0$ be an essentially $(4m - 2)$-edge-connected $2m$-regular simple graph $G_0$ of order at least $1/\varepsilon + 1$; a graph is called essentially $\lambda$-edge-connected, if all edges of any edge cut of size strictly less than $\lambda$ are incident to a common vertex. (For example, the graph with vertices $v_1, \ldots, v_n$ and edges $v_i v_{i+j}$ (modulo $n$) where $j \in \{1, \ldots, m\}$ and $n$ is a sufficiently large integer). Since $m \geq 2$, the graph $G_0$ is essentially $(2m + 2)$-edge-connected. Take $M$ to be a set of $m + 1$ edges of $G_0$ which are not incident to a common vertex. We claim that $G = G_0 \setminus M$ is the desired graph. Since $|E(G)| = m(|V(G)| - 1) - 1$, the graph $G$ is not $m$-tree-connected. Let $P$ be a partition of $V(G)$ of size at least two. First, assume that $|P| = 2$. If there exists a set $X \in P$ satisfying $|X| = 1$, then by the assumption, $e_G(P) \geq 2m - (|M| - 1) \geq m$. Otherwise, $e_G(P) \geq 2m + 2 - |M| > m$. Thus, in both cases, $e_G(P) \geq m(|P| - 1)$. Now, assume that $|P| \geq 3$. If there exists a set $X \in P$ satisfying $|X| = 2$, then $d_G(X) \geq 2m + 2$ and hence $e_G(P) \geq e_G(P) - |M| \geq \sum_{A \in P} d_G(A)/2 - |M| \geq m|P| + 1 - (m + 1) = m(|P| - 1)$. Otherwise, $|P| = |V(G)|$ and hence $e_G(P) = |E(G)| = m(|V(G)| - 1) - 1 \geq (m - \varepsilon)(|P| - 1)$. This completes the proof. □

The following theorem shows that the coefficient $1/m$ in Corollary 6.5 is sharp and cannot be improved even by arbitrarily increasing the coefficient of iso$(G \setminus S)$ or the upper bound on the maximum degree.

Theorem 6.8. Let $m$ be an integer with $m \geq 2$, and let $\Delta$, $k$, and $n$ be arbitrary positive integers. If $\varepsilon \in (0,1)$, then there exist infinitely many $k$-connected simple graphs $G$ having no $m$-tree-connected factor with maximum degree at most $\Delta$, while for all $S \subseteq V(G)$ satisfying $|S| \geq k$,

$$\omega(G \setminus S) + n \text{iso}(G \setminus S) \leq \left(\frac{1}{m} + \varepsilon\right)(|S| - k) + 1.$$

Proof. Let $s$ and $p$ be two arbitrary integers satisfying $s \geq k$ and $p > \Delta s$. By Lemma 6.7, there exists a non-$m$-tree-connected simple graph $H'$ satisfying $e_{H'}(P) \geq m'(|P| - 1)$ for every partition $P$ of $V(G)$, where $m' = 1/(1/m + \varepsilon) < m$. We replace each vertex of $H'$ by a copy of the complete graph $K_{n_0}$ such that every vertex in the new graph is adjacent to at most one edge of $H'$, where $n_0$ is a fixed positive integer satisfying $n_0 \geq m'(n + 1)|V(H')| + 1$. We call the resulting graph $H$. For every integer $i$ with $1 \leq i \leq p$, let $H_i$ be a copy of the graph $H$ and let $U_i$ be the set of all vertices of a complete subgraph of $H_i$ corresponding to $K_{n_0}$. First, add all possible edges between all vertices of $U_i$ and $U_j$, where $i, j \in \{1, \ldots, p\}$. Next, add a new complete graph $R_s$ of order $s$ and join all its vertices to all vertices of every graph $H_i$. We call the resulting graph $G$; see Figure 2.

Obviously, $G$ must be $s$-connected. Suppose, to the contrary, that $G$ has an $m$-tree-connected factor $T$ with maximum degree at most $\Delta$. According to the construction of $G$, since $H_i$ is not $m$-tree-connected, there must be at least one edge of $T$ having one end in $V(R_s)$ the other one in $V(H_i) \setminus U_i$. Thus there are at least $p$ edges of $T$ with exactly one end in $V(R_s)$. Therefore, $T$ contains a vertex in $V(R_s)$ with degree at least $p/s > \Delta$ which is a contradiction.
Let $S \subseteq V(G)$ with $|S| \geq k$. If $\omega(G \setminus S) \leq 1$ and $iso(G) = 0$, then $\omega(G \setminus S) + n iso(G \setminus S) = 1 \leq \frac{1}{m'}(|S| - k) + 1$. If $\omega(G \setminus S) = iso(G) = 1$, then $|S| = |V(G)| - 1 \geq n_0 + k$ and so $\omega(G \setminus S) + n iso(G \setminus S) = (n + 1) \leq \frac{1}{m'}(|S| - k)$. We may assume that $\omega(G \setminus S) \geq 2$. Obviously, $S$ must contain all vertices of $R_s$.

For every integer $i$ with $1 \leq i \leq p$, let $S_i = S \cap V(H_i)$. Thus $|S| = s + \sum_{1 \leq i \leq p} |S_i|$. Define $\omega_i$ to be the number of components of $H_i \setminus S_i$ having no vertices of $U_i$. According to the construction of $G$, we must have

$$|S_i| \geq e_{H'}(P_i) \geq m'(|P_i| - 1) = m'\omega_i(G(H_i \setminus S_i) - 1) = m'\omega_i,$$

which implies that $\omega_i + n iso(H_i \setminus S_i) = \omega_i \leq \frac{1}{m'}|S_i|$. If $|S_i| \geq n_0 - 1$, then we must have $|S_i| \geq m'(n + 1)|V(H_i)|$, which again implies that $\omega_i + n iso(H_i \setminus S_i) \leq (n + 1)|V(H_i)| \leq \frac{1}{m'}|S_i|$. Therefore, one can conclude that

$$\frac{|S| - k}{\omega(G \setminus S) - 1 + n iso(G \setminus S)} \geq \frac{\sum_{1 \leq i \leq p} |S_i|}{\sum_{1 \leq i \leq p} (\omega_i + n iso(H_i \setminus S_i))} \geq m'.$$

These inequalities complete the proof. \hfill \square

### 6.3 An application to highly connected star-free simple graphs

Matthews and Summer (1984) \cite{22} showed that every $k$-connected $K_{1,3}$-free simple graph is $\frac{k}{2}$-tough. This result is generalized to $k$-connected $K_{1,n}$-free simple graphs by Jackson and Wormald (1990) \cite{15} and Chen and Schelp (1995) \cite{5} independently. In the following lemma, we provide a stronger version for their result for $K_{1,f}$-free simple graphs.
Lemma 6.9. Let $G$ be a simple graph and let $f$ be an integer-valued function on $V(G)$ with $f \geq 2$. If $G$ is $k$-connected and $K_{1,f}$-free, then for every $S \subseteq V(G)$,
\[
\omega(G \setminus S) + \sum_{v \in I(G \setminus S)} \left(\frac{1}{k}d_G(v) - 1\right) \leq \max\{1, \frac{1}{k} \sum_{v \in S} (f(v) - 1)\}.
\]

Proof. We repeat the proof of Theorem 4.2 in [15] with some modifications. Let $S$ be a subset of $V(G)$. Since $G$ is $k$-connected, every component of $G \setminus S$ is joined to at least $k$ vertices in $S$ when $\omega(G \setminus S) \geq 2$. In addition, every component of $G \setminus S$ consisting of a single vertex $v$ is joined to at least $d_G(v)$ vertices in $S$. Since $G$ is $K_{1,f}$-free, every vertex of $S$ is joined to at most $f(v) - 1$ components of $G \setminus S$. Thus
\[
k(\omega(G \setminus S) - iso(G \setminus S)) + \sum_{v \in I(G \setminus S)} d_G(v) \leq \sum_{v \in S} (f(v) - 1)
\]
provided that $\omega(G \setminus S) \geq 2$ or $\omega(G \setminus S) = iso(G \setminus S) = 1$. Therefore, for all vertex sets $S$, $\omega(G \setminus S) + \sum_{v \in I(G \setminus S)} \left(\frac{1}{k}d_G(v) - 1\right) \leq \max\{1, \frac{1}{k} \sum_{v \in S} (f(v) - 1)\}$ regardless of $\omega(G \setminus S) \leq 1$ and $iso(G \setminus S) = 0$ or not. Hence the assertion holds. □

The following result is an application of Lemma 6.9 and Corollary 6.5.

Theorem 6.10. Every $m(n-1)$-connected $K_{1,n}$-free simple graph $G$ with $n \geq 3$ admits an $m$-tree-connected factor $H$ satisfying $\Delta(H) \leq 2m + 1$, provided that $\delta(G) \geq \frac{1}{2}m(m+3)(n-1)$.

Proof. By Lemma 6.9, for every $S \subseteq V(G)$, we must have
\[
\omega(G \setminus S) + \left(\frac{\delta(G)}{m(n-1)} - 1\right)iso(G \setminus S) \leq \omega(G \setminus S) + \frac{m+1}{2}iso(G \setminus S) \leq \max\{1, \frac{n-1}{m(n-1)} |S|\} \leq \frac{1}{m} |S| + 1.
\]
Thus by Corollary 6.5, the graph $G$ has an $m$-tree-connected factor $H$ satisfying the theorem. □

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