Bose-Einstein Condensation in the presence of an artificial spin-orbit interaction

S.-K. Yip

Institute of Physics, Academia Sinica, Nankang, Taipei 115, Taiwan

(Dated: January 12, 2011)

Bose-Einstein condensation in the presence of a synthetic spin-momentum interaction is considered, focusing on the case where a Dirac or Rashba potential is generated via a tripod scheme. We found that the ground states can be either plane wave states or superpositions of them, each characterized by their unique density distributions.

PACS numbers: 03.75.-b, 67.85.Bc, 03.75.Mn

The possibility of producing an artificial gauge field \[ \text{[1]} \] opens up a new era in cold atom physics. The concept of gauge fields is ubiquitous in many branches of physics, from coupling of electromagnetic fields to charged particles \[ \text{[2]} \] relevant to ordinary laboratory settings to fundamental forces between elementary particles \[ \text{[3]} \]. In addition to the abelian gauge field already realized in \[ \text{[1]} \], there are numerous proposals to generate other artificial abelian and non-abelian gauge fields, with and without optical lattices, with the prediction of some exotic properties \[ \text{[4–12]} \].

While Dirac and Rashba interaction are often investigated in fermionic electronic systems mentioned above, in this paper we investigate the consequences of Bose-Einstein condensation (BEC) of Bosons in such artificial gauge fields. We investigate the possible ground states of this system, and show that one can have BEC into plane-wave states or states corresponding to a superposition of plane waves, depending on the physical parameters. We further investigate the observable consequences, assuming that the gauge fields are produced via the tripod scheme proposed in \[ \text{[4, 5, 7, 8]} \]. We shall see how the single particle terms due to the laser fields or external potentials and the interparticle interactions determine which ground state would be realized. Each ground state is associated with a characteristic density distribution of the physical states involved in the tripod scheme, enabling one to distinguish these states experimentally.

We mention here the closely related works of \[ \text{[6, 13, 14]} \]. \[ \text{[13]} \] investigates BEC in the artificial gauge field in the experiment cited in \[ \text{[12]} \]. \[ \text{[14]} \] considers also the Dirac/Rashba interaction as in the present work, but without reference to a particular production scheme and so did not discuss the potentials arising from the laser fields, nor the density distributions in the original atomic states. They also limit themselves to a less general form of the interparticle interaction included in this paper. \[ \text{[6]} \] did not include the single-particle and interaction energies to be discussed in detail below. They obtained a ground state qualitatively different from \[ \text{[13, 14]} \] and ours.

For definiteness, we consider the tripod scheme proposed in \[ \text{[5]} \]. For convenience we shall first review it briefly. We consider a cloud which is either three-dimensional or confined by y-dependent optical potentials so that one only needs to consider motion along x and z. Three (almost) degenerate states \[ | 1 >, | 2 >, | 3 > \] are coupled to another state \[ | 0 > \] with light fields with strengths \[ \Omega_{1,2,3} \]. That is, we have the coupling term \[ H_{1} = -|0 > (\Omega_{1} < 1 | + \Omega_{2} < 2 | + \Omega_{3} < 3 | ) + c.c. \]. \[ \Omega_{1,2} \] are chosen to be plane waves of equal strengths propagating in opposite directions, while \[ \Omega_{3} \] involves a plane wave propagating perpendicular to the previous two with the same wavelength: we have \[ \Omega_{1} = \frac{1}{\sqrt{2}} \Omega \sin \theta e^{-ik_{0}x}, \Omega_{2} = \frac{1}{\sqrt{2}} \Omega \sin \theta e^{+ik_{0}x}, \Omega_{3} = \Omega \cos \theta e^{ik_{0}z} \]., with \[ \Omega \] and \[ \theta \] parameters specifying the coupling strengths. There are two linear combinations among the states \[ | 1 >, | 2 >, | 3 > \] which are not affected by \[ H_{1} \]. They are referred to as "dark states" which we shall choose to be

\[
| D_{1} > \equiv \frac{1}{\sqrt{2}} [ | 1 > e^{+ik_{0}x} - | 2 > e^{-ik_{0}x} ] e^{ik_{0} \cos \theta' z} \quad (1)
\]

\[
| D_{2} > \equiv \frac{1}{\sqrt{2}} \cos \theta [ | 1 > e^{+ik_{0}x} + 2 | 2 > e^{-ik_{0}x} ] e^{ik_{0} \cos \theta' z} - \sin \theta | 3 > e^{-ik_{0} (1 - \cos \theta) z} \quad (2)
\]

Here we have generalized slightly \[ \text{[13]} \] and include a yet undetermined parameter \[ \theta' \] in the z-dependent phase factors in eq (1) and (2). The value for \[ \theta' \] does not affect the actual physics, but we shall choose a special value for it for convenience below. The "bright" state \[ | B > \], orthogonal to both \[ | D_{1,2} > \], couples to \[ | 0 > \] with strength \[ \Omega \]. We shall assume that the energy splitting of these two resulting states from \[ | D_{1,2} > \] are large compared with all other energy scales considered below, hence \[ | D_{1,2} > \] are the only states physically relevant.

Within the \[ | D_{1,2} > \] subspace, the single particle Hamiltonian is given by
Here the matrix $\tilde{A}$ has elements $\tilde{A}_{\mu\nu} = i < D_\mu | \nabla D_\nu > (\mu, \nu = 1 \text{ or } 2)$ is the resulting (non-Abelian) gauge field, and $\Phi_{\mu\nu} = \frac{1}{i m} < \nabla D_\mu | B > \cdot < B | \nabla D_\nu >$ [4]. $\mathbf{V}$ includes the terms that may arise if the internal energies or the potentials acting on the states $|1 - 3\rangle$ are not identical. We shall provide more details below. $\tilde{A}$ has components $\tilde{A}_{11} = - k_0 \cos \theta \hat{z}, \tilde{A}_{12} = - k_0 \sin \theta \hat{x}, \tilde{A}_{22} = - k_0 (\cos \theta - \sin \theta \hat{z}) \hat{z}$. With the choice $\cos \theta = \frac{\sin^2 \theta}{2}$, we then have

$$\tilde{A} = - k_0 \cos \theta \sigma_x \hat{x} - \frac{k_0}{2} \sin^2 \theta \sigma_z \hat{z}$$

where $\sigma_{x,y,z}$ are Pauli (pseudospin) matrices acting within the $|D_1 >, |D_2 >$ space. With this $\tilde{A}$, we see that we have an anisotropic Dirac-like term in the Hamiltonian $\frac{\hbar}{m} \left( \cos \theta \sigma_x p_x + \frac{\sin^2 \theta}{2} \sigma_z p_z \right)$, with the special case $\cos \theta = \sqrt{2} - 1 = \frac{\sin^2 \theta}{2}$ where this term becomes isotropic in the $x$-$z$ plane [7]. We shall however not assume such a special value for $\theta$ below.

In the above vector potential $\tilde{A}$, $\hat{x}$ and $\hat{z}$ result from the $x$ and $z$ dependence of $\Omega_{12}$ and $\Omega_3$ respectively. One can thus produce a Rashba-like term $\frac{\hbar}{m} \left( \cos \theta \sigma_x p_x - \frac{\sin^2 \theta}{2} \sigma_z p_z \right)$ if one replaces $x$ by $z$ in $\Omega_{12}$ and $z$ by $-x$ in $\Omega_3$, that is, changing the directions of the lasers or relabeling the coordinates. Alternatively, one can also use a different choice for $|D_{1,2} >$ corresponding to a “spin-rotation” [8]. For definitiveness, we shall continue to deal with the Dirac-like term in our Hamiltonian.

$\Phi$ can be easily found to be $\frac{\hbar^2}{2m} \sin^2 \theta \left( \frac{1}{2} \cos \theta + \frac{\sin^2 \theta}{2} \sigma_z \right)$. The first term is a momentum independent scalar which we shall drop. In the presence of a potential (or internal energies) $V_{1,2,3}$ on the states $|1 - 3\rangle$, $\mathbf{V}$ is given by, apart from a constant which we shall again drop, $\mathbf{V} = \frac{\hbar}{m} \left( (V_1 - V_2) \cos \sigma_x + (V_1 + V_2 - 2V_3) \sin \theta \sigma_z \right)$. $\Phi$ and $\mathbf{V}$ act like Zeeman fields in the space $|D_{1,2} >$.

We shall first consider BEC in the presence of the gauge potential $\tilde{A}$, but ignoring $\Phi, \mathbf{V}$, and interparticle interactions for the moment. These would be included later. The Hamiltonian is then simply the kinetic energy term $K = \frac{\hbar^2}{2m} \left( \hat{p}^2 - \frac{\hbar^2}{m} \hat{A} \cdot \hat{p} + \frac{\hbar^2}{2m} A^2 \right)$. $A^2$ is simply given by the constant $\hbar^2 \left( \cos^2 \theta + \frac{\sin^2 \theta}{2} \right)$, which we shall drop below. $K$ can be easily diagonalized giving the energies $E = \frac{\hbar^2}{2m} \pm \frac{\hbar^2}{m} \left[ \cos^2 \theta p_x^2 + \frac{\sin^2 \theta}{2} p_z^2 \right]^{1/2}$. BEC should occur in the state of the lowest energy, hence we make the negative sign and find the $\tilde{p}$ where $\frac{\partial E}{\partial p_z} = 0$ and $\tilde{p}_z = 0$. Obviously $p_y$ is zero and for simplicity we shall not write this component explicitly. The possible minima are $\tilde{p} = (0, \pm k_0 \sin \theta) = (0, \pm p^0_x)$, with energy $E = \frac{k_0^2}{2m} \left( \frac{\sin^4 \theta}{4} \right)$, and $\tilde{p} = (\pm k_0 \cos \theta, 0) = (\pm p^0_x, 0)$, with energy $E = - \frac{k_0^2}{2m} \cos^2 \theta$. Hence if $\cos \theta < \sqrt{2} - 1$, the minima are at $\tilde{p} = \pm p^0_x \hat{x}$, whereas if $\cos \theta > \sqrt{2} - 1$, the minima are at $\tilde{p} = \pm p^0_x \hat{x}$. If $\cos \theta = \sqrt{2} - 1$, we have the very special case that all momenta given by $(p^2 + p_z^2)^{1/2} = k_0 (\sqrt{2} - 1)$ are degenerate. We shall not deal with this very special circumstance in the present paper.

If we have condensation in one of these four minima, the corresponding wavefunctions in $|D_{1,2} >$ space are

$$\Psi_1(\tilde{r}) = \Phi_1 e^{i k_0 \sin^2 \theta} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\Psi_2(\tilde{r}) = \Phi_2 e^{-i k_0 \sin^2 \theta} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\Psi_3(\tilde{r}) = \frac{\Phi_3}{\sqrt{2}} \left( e^{-i k_0 \cos \theta} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right)$$

$$\Psi_4(\tilde{r}) = \frac{\Phi_4}{\sqrt{2}} \left( -e^{i k_0 \cos \theta} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right)$$

where $\Phi_{1-4}$ are complex numbers. We shall call them states 1-4 (not be too confused with those which enter the tripod scheme). We note here that even though these are plane wave states, they carry no current since we have demanded $\frac{\partial E}{\partial p_{x,z}} = 0$. Alternatively, we should note that the velocity operator $\mathbf{v}$ is given by $\frac{\partial H}{\partial \tilde{p}}$, hence $\mathbf{v} = \frac{1}{m} \left( \tilde{p} - \tilde{A} \right) = \frac{\hbar}{m} \left( \cos \theta \sigma_x \hat{x} + \frac{\sin^2 \theta}{2} \sigma_z \hat{z} \right)$. One can easily verify that the expectation values of the velocity is zero in each of the plane wave states [6,3] above.

Since in each case we still have two degenerate minima, we must consider other terms in the Hamiltonian to determine which BEC would occur. We first consider the other single particle terms in the Hamiltonian, that is $\mathbf{V} + \phi \equiv H_0 = \tilde{H} = \frac{\hbar}{m} \sigma z \sigma x \hat{z}$, assuming for the moment that they are dominant over the interparticle interactions. Due to the above discussion, we shall consider the case where $H_0$ has only $\sigma_{x,z}$ terms, With finite $\hbar$, the relevant branch of the spectrum is

$$E = \frac{\hbar^2}{2m} \left( \cos \theta p_x - \frac{\hbar \sigma z}{m} \right)^2 + \left( \frac{\sin^2 \theta p_x - \hbar \sigma z}{2m} \right)^2 \left( \frac{m}{2m} \right)$$

We now need to find the value of $\tilde{p}$ where $E$ is a minimum. In general this is very complicated for general $\hbar$. In the limit of first order in $\hbar$, one can easily verify that the energies of the above four states become $E = - \frac{k_0^2}{2m} \left( \frac{\sin^4 \theta}{4} \right) \pm \hbar z$ and $E = - \frac{k_0^2}{2m} \cos^2 \theta \pm \hbar x$. Hence if $\cos \theta < \sqrt{2} - 1$ and $\hbar z < (>)0$, then state 1 (2) should be realized, whereas if $\cos \theta > \sqrt{2} - 1$ and $\hbar z < (>)0$,
states 3 (4) should be realized. The above statements assumed that \( \hbar \) is sufficiently small so that the ordering of energies are not changed except the lifting of degeneracies. Also, strictly speaking, one also needs to consider the modification of the wavefunctions in eq (10) due to \( \hbar \), as both the momentum and the "spin" wavefunction at which the minimum occurs are modified. We however would not give these rather lengthy formulas here. (See however near the end of this paper below).

Let us at this point give some physical properties related to these states, ignoring for simplicity the modifications of wavefunctions just mentioned. If \( \cos \theta < \sqrt{2} - 1 \) and \( h_z < 0 \), then (5) is the wavefunction in \( |D_{1,2} > \) space. Using (1) and (2), one easily finds the wavefunctions in the original \( |1 - 3 > \) basis:

\[
\psi_{1,2}(\vec{r}) = \frac{\Phi_1}{\sqrt{2}} \cos \theta e^{\pm ik_0 x} e^{i k_0 \sin^2 \theta} (10)
\]

\[
\psi_3(\vec{r}) = -\Phi_1 \sin \theta e^{-ik_0 \cos^2 \theta} (11)
\]

and so the corresponding particle densities are

\[
|\psi_{1,2}(\vec{r})|^2 = \frac{|\Phi_1|^2}{2} \cos^2 \theta (12)
\]

\[
|\psi_3(\vec{r})|^2 = |\Phi_1|^2 \sin^2 \theta (13)
\]

Hence \( |\psi_1(\vec{r})|^2 = |\psi_2(\vec{r})|^2 \) (again ignoring corrections due to \( \hbar \), a statement which we shall not repeat).

If \( \cos \theta < \sqrt{2} - 1 \) and \( h_z > 0 \), the system condenses into state 2 (eq (6)) , we have instead

\[
\psi_{1,2}(\vec{r}) = \pm \frac{\Phi_2}{\sqrt{2}} e^{\pm ik_0 x} (14)
\]

\[
\psi_3(\vec{r}) = 0 (15)
\]

and with the corresponding particle densities

\[
|\psi_{1,2}(\vec{r})|^2 = \frac{|\Phi_2|^2}{2} (16)
\]

\[
|\psi_3(\vec{r})|^2 = 0 (17)
\]

which is very different from the state 1.

For \( \cos \theta > \sqrt{2} - 1 \) and \( h_x < 0 \) we have condensation into state 3, where

\[
\psi_{1,2}(\vec{r}) = \frac{\Phi_3}{2} [-1 + \cos \theta] e^{\pm ik_0 (1 + \cos \theta) x} e^{i k_0 \sin^2 \theta} (18)
\]

\[
\psi_3(\vec{r}) = -\frac{\Phi_3}{\sqrt{2}} \sin \theta e^{ik_0 \cos x} e^{-ik_0 (1 - \cos^2 \theta) z} (19)
\]

The densities are

\[
|\psi_{1,2}(\vec{r})|^2 = |\Phi_3|^2 \frac{(1 + \cos \theta)^2}{4} (20)
\]

\[
|\psi_3(\vec{r})|^2 = |\Phi_3|^2 \frac{\sin^2 \theta}{2} (21)
\]

For \( \cos \theta > \sqrt{2} - 1 \) and \( h_x > 0 \) we have condensation into state 4. We have

\[
\psi_{1,2}(\vec{r}) = \frac{\Phi_4}{2} [1 + \cos \theta] e^{\pm ik_0 (1 + \cos \theta) x} e^{i k_0 \sin^2 \theta} (22)
\]

\[
\psi_3(\vec{r}) = -\frac{\Phi_4}{\sqrt{2}} \sin \theta e^{-ik_0 \cos x} e^{-ik_0 (1 - \cos^2 \theta) z} (23)
\]

with densities

\[
|\psi_{1,2}(\vec{r})|^2 = |\Phi_4|^2 \frac{(1 + \cos \theta)^2}{4} (24)
\]

\[
|\psi_3(\vec{r})|^2 = |\Phi_4|^2 \frac{\sin^2 \theta}{2} (25)
\]

which are those of state 3 with \( |1 > \) and \( |2 > \) interchanged. In passing, we remark here also that, as indicated by the wavefunctions \( \psi_{1,2,3}(\vec{r}) \) above, the atomic states \( |1 >, |2 >, |3 > \) are each associated with their characteristic wavevector, which can also be measured by time-of-flight experiments as in 3.

The above is for when the single particle terms dominate and determine the ground state. Generally, if the energy differences between the above plane wave states are sufficiently small, we can have condensation into superposition of plane wave states. The energetics is analogous to that of superfluid mixtures 10 as we shall see. To be specific, let us consider the case where we have \( \cos \theta < \sqrt{2} - 1 \), with energies of states 1 and 2 sufficiently lower than 3 and 4 so that the later two need not be considered. We are left with considering condensation into states described by 1 and 2 in eq (11) and (22). (We are now considering normalized single particle states, that is, without the \( \Phi_{1,2} \) coefficient). Consider the scattering between two particles, each one in the state 1. Since the total momentum is zero and the relevant low energy sector must correspond to that that the outgoing particles are also in state 1. Similarly, if there is one particle from each state 1 and 2, then the total momentum is zero and the relevant low energy process also correspond to outgoing particles with one each in state 1 and 2. Let \( d_{\vec{r}_0,1} \) and \( d_{\vec{r}_0,2} \) be the operators corresponding to the (normalized) plane wave states in states 1 and 2. The general form of the Hamiltonian in the low energy subspace is

\[
\frac{1}{2} g_{ij1} d_{\vec{r}_0,1}^\dagger d_{\vec{r}_0,1} d_{\vec{r}_0,1} + g_{ij2} d_{\vec{r}_0,2}^\dagger d_{\vec{r}_0,2} d_{\vec{r}_0,2} + \frac{1}{2} g_{ij3} d_{\vec{r}_0,1}^\dagger d_{\vec{r}_0,2}^\dagger d_{\vec{r}_0,1} d_{\vec{r}_0,2} + \frac{1}{2} g_{ij4} d_{\vec{r}_0,1}^\dagger d_{\vec{r}_0,2}^\dagger d_{\vec{r}_0,2} d_{\vec{r}_0,1},
\]

where \( g_{ij} \) are coefficients. Other terms in the Hamiltonian are off-shell and can be dropped when we eventually take the mean-field approximation below. \( g_{ij} \) can be evaluated once the interaction Hamiltonian in terms of the states \( |1 - 3 > \) in the tripod scheme is known. There are two many possibilities so we do not provide the formulas here. We simply remark here that if the interaction among the particles \( |1 - 3 > \) are all identical, then \( g_{11} = g_{12} = g_{22} \).

Within the mean-field approximation, we replace the operators \( d_{\vec{r}_0,1(2)} \) by the amplitudes \( \Phi_{1,2} \), obtaining the
interaction energy $E_{int} = \frac{1}{2} g_{11} |\Phi_1|^4 + \frac{1}{2} g_{12} |\Phi_1|^2 |\Phi_2|^2 + \frac{1}{2} g_{22} |\Phi_2|^4$. Also, in the presence of the bias field $H_0$, we have energy $E_0 = h c (|\Phi_1|^2 - |\Phi_2|^2)$ to linear order in $\hbar$ corresponding to the discussion below eq (10). Note that there is no such term such as $-h_\perp (\Phi_1^* \Phi_2 + \Phi_2^* \Phi_1)$. There cannot be a term in the form of $-h_z (d_{\perp 0,1}^* d_{\perp 0,2} + d_{\perp 0,2}^* d_{\perp 0,1})$ due to momentum conservation.

Including also a chemical potential $\mu$ gives us then the total energy $E_{tot} = -(\mu - h_z) |\Phi_1|^2 - (\mu + h_z) |\Phi_2|^2 + E_{int}$ as in the case of a mixture of two species 1 and 2, with effective chemical potential for component 1 (2) being $\mu_{1,2} = \mu \mp h_z$. Note that therefore there are no terms which would depend on the phase difference between $\Phi_{1,2}$. We shall have another point of view of this below. The phase diagram of a two component mixture is well-known [19], and we shall not repeat those results here. The state corresponding to a "mixture" between 1 and 2 has a wavefunction in the $|D_{1,2}| >$ space as a the superposition of $\Phi_1$ and $\Phi_2$, that is,

$$\Psi (\vec{r}) = \Phi_1 e^{ik_0 \frac{\sin \theta}{2} z} (0 1) + \Phi_2 e^{-ik_0 \frac{\sin \theta}{2} z} (1 0)$$

(26)

and likewise for their projection into the states $|1 \sim 3 >$. We can understand easily why the energy is independent of the relative phase between $\Phi_{1,2}$. Changing the relative phase between these two complex numbers can simply be reabsorbed by a shift of the origin for $z$. Below, we shall therefore assume that both $\Phi_{1,2}$ are real.

The densities for this "mixture" of 1 and 2 are

|$\psi_{1,2}(\vec{r})|^2 = \frac{1}{2} \left[ \cos^2 \theta |\Phi_1|^2 + |\Phi_2|^2 \pm 2 \Phi_1 \Phi_2 \cos \theta \cos (k_0 \sin^2 \theta z) \right]$ (27)

|$\psi_{3}(\vec{r})|^2 = |\Phi_1|^2 \sin^2 \theta $ (28)

The coefficients of $|\Phi_{1,2}|^2$ are the same as eq (12)-(17). An interference term $\propto \Phi_1 \Phi_2$ arises from the wavevector difference $k_0 \sin^2 \theta \vec{z}$ between the $\Phi_{1,2}$ components in eq (26). Similar discussion holds for a "mixture" of states 3 and 4. The densities are

|$\psi_{1,2}(\vec{r})|^2 = \frac{1}{4} \left[ (1 - \cos \theta)^2 |\Phi_3|^2 + (1 + \cos \theta)^2 |\Phi_4|^2 - 2 \Phi_3 \Phi_4 \sin^2 \theta \cos (2 k_0 \cos \theta x) \right]$ (29)

|$\psi_{3}(\vec{r})|^2 = \frac{1}{2} \sin^2 \theta \left[ |\Phi_3|^2 + |\Phi_4|^2 + 2 \Phi_3 \Phi_4 \cos (2 k_0 \cos \theta x) \right]$ (30)

The densities thus in general oscillate in space, as has been discussed in [13, 14].

Next we would like to consider in more detail the case of $H_0 = -h_z \sigma_z$, without restricting to small $h_z$. Recall that $\Phi$ contains such a component even in the special case that $V = 0$, and also that $h_z < 0$ in that case. Due to the lack of space we would mainly state the results. First we ignore interparticle interactions. If $\cos \theta < \sqrt{2} - 1$, the state 1 (2) with $\vec{p} = (0, \pm k_0 \sin \theta \vec{z})$ is the energy minima if $h_z < (>) 0$ with energy $E = -\frac{k_0^2}{2m} \left( \frac{\sin \theta}{2} \right) - |h_z|$. For $\cos \theta > \sqrt{2} - 1$, and if $|h_z| < h_c$ where $h_c = \frac{k_0^2}{m} D(\theta)$ with $D(\theta) \equiv \cos^2 \theta - \left( \frac{\sin \theta}{2} \right)$ (note $D > 0$ in the above range for $\theta$), then the energy minima occurs at $(\pm p_x, p_y)$ where $p_x^0 = k_0 \cos \theta [1 - (\frac{\theta}{2\pi})^2]^{1/2}$ and $p_y^0 = -h_z \frac{\sin \theta}{2\pi}$. Here $h_z \equiv h_z/(k_0^2/m)$ is a dimensionless measure of the energy $h_z$. Note there are two degenerate minima of opposite $p_x$, and $\text{sgn}(p_x^0) = -\text{sgn}(h_z)$. We shall call these states 5 and 6. The energy is given by $E = -\frac{k_0^2}{2m} \left[ \cos^2 \theta + \frac{\theta^2}{4\pi^2} \right]$. At $\mp h_c$, $\vec{p}$ becomes $(0, \pm k_0 \sin \theta \vec{z})$ and thus merge with the other minima stated before. For $|h_z| > h_c$, the minima is at $\vec{p} = (0, \pm k_0 \sin \theta \vec{z})$ according to $h_z < (>0)$. It is convenient to define the quantity $\beta \equiv \text{tan}^{-1} \left( \frac{-h_z/\sqrt{1 - h_z^2/D(\theta)}}{1 - h_z^2/D(\theta)} \right)$. $0$ if $h_z = 0$ and $\beta = -\text{sgn}(h_z) \frac{\pi}{2}$ at $|h_z| = h_c$. The wavefunction in $|D_{1,2}| >$ space for state 5, with wavevector $(p_x^0, p_y^0)$, is

$$\Psi_5(\vec{r}) = \frac{\Phi_5}{\sqrt{2}} e^{i(p_x^0 x + p_y^0 z)} \left( \frac{-\cos \frac{\beta}{2} - \sin \frac{\beta}{2}}{\cos \frac{\beta}{2} + \sin \frac{\beta}{2}} \right)$$

(31)

whereas that for state 6 with wavevector $(-p_x^0, p_y^0)$ is

$$\Psi_6(\vec{r}) = \frac{\Phi_6}{\sqrt{2}} e^{-i(p_x^0 x + p_y^0 z)} \left( \frac{\cos \frac{\beta}{2} - \sin \frac{\beta}{2}}{\cos \frac{\beta}{2} + \sin \frac{\beta}{2}} \right)$$

(32)

These two states are orthogonal due to their different wavevectors $(\pm p_x^0, p_y^0)$, but their "spinor" part has finite $(\sin \beta)$ overlap. At $h_z = 0$, $\beta = 0$, $p_x^0 = k_0 \cos \theta$, $p_y^0 = 0$, states 5 and 6 are identical respectively with state 3 and 4. At $h_z = h_c$, $\beta = \frac{\pi}{2}$, and as stated, $p_x^0 = 0$, $p_y^0 = k_0 \sin \theta$, they become identical with each other and with state 1.

The densities corresponding to state 5 are
\[ |\psi_{1,2}(\vec{r})|^2 = \frac{|\Phi_5|^2}{4} \left[ 1 + \cos^2 \theta - \sin \beta \sin^2 \theta \mp 2 \cos \theta \cos \beta \right] \]  
\[ |\psi_3(\vec{r})|^2 = \frac{|\Phi_5|^2}{2} \sin^2 \theta (1 + \sin \beta) \]  
(33)  
(34)

The densities of state 6 are those of 5 with states \(|1\rangle\) and \(|2\rangle\) interchanged.

For a superposition between states 5 and 6, the densities have three contributions, the terms proportional to \(|\Phi_5,6|^2\) are the same as those in eq (33) and (34). In addition, there are the interference terms

\[ -\frac{\Phi_5 \Phi_6}{2} \left[ \sin^2 \theta - \sin \beta (1 + \cos^2 \theta) \right] \cos(2p_x x) \]

for \(|\psi_{1,2}(\vec{r})|^2\), and

\[ \Phi_5 \Phi_6 \sin^2 \theta (1 + \sin \beta) \cos(2p_x x) \]

for \(|\psi_3(\vec{r})|^2\). Note that they reduce to the appropriate limits eq (29) (30) if \(\beta = 0\). At \(\beta = \frac{\pi}{2}\), we get eq (12) and (13) with \(\Phi_1 \rightarrow \Phi_5 + \Phi_6\).

If \(V = 0\), we have \(h_z = -\frac{k^2}{m} \left( \frac{\sin^2 \theta}{2} \right)^2 \). Hence \(|\hat{h}_z| = h_c\) at \(\cos \theta = \frac{\sqrt{\pi - \sqrt{\pi}}}{2} \approx 0.518\), corresponding to \(\theta \approx 0.326\pi\). Hence, due to this finite \(h_z\), we have states 5, 6 or their mixtures if \(\theta < 0.326\pi\), and state 1 if otherwise (assuming that interaction does not change the ordering of the energies).

In [8], standing waves for \(\Omega_{1,2}\) are considered instead of plane waves. This situation can be treated in a similar manner as in this paper. The corresponding results can be obtained by a simple unitary transformation among the atomic states \(|1\rangle\) and \(|2\rangle\).

In conclusion, we have considered Bose-Einstein condensation in an artificial spin-orbit field. In general BEC occurs in a state with finite wavevector, or their superpositions. Due to the momentum-pseudospin coupling, each state is characterized by their unique density distributions.

This research was supported by the National Science Council of Taiwan. The author would also like to thank the Aspen Center for Physics where this study was motivated.

[1] Y.-J. Lin et al, Phys. Rev. Lett. 102, 130401 (2009); Nature, 462, 628 (2009)
[2] J. J. Sakurai, Modern Quantum Mechanics, Addison-Wesley, 1994, revised addition.
[3] C. Itzykson and J.-B. Zuber, Quantum Field Theory, MaGraw-Hill, 1980.
[4] J. Ruseckas et al, Phys. Rev. Lett. 95, 010404 (2005)
[5] T. D. Stanescu, C. Zhang and V. M. Galitski, Phys. Rev. Lett. 99, 110403 (2007)
[6] T. D. Stanescu, B. Anderson and V. M. Galitski, Phys. Rev. A 78, 023616 (2008)
[7] G. Juzeliunas et al, Phys. Rev. Lett. 100, 200405 (2008)
[8] J. Y. Vaishnav and C. W. Clark, Phys. Rev. Lett. 100, 153002 (2008)
[9] N. Goldman et al, Phys. Rev. Lett. 103, 035301 (2009)
[10] J. Larson and E. Sjöqvist, Phys. Rev. A 79, 043627 (2009)
[11] L.-K. Lim et al, Phys. Rev. A 81, 023404 (2010)
[12] see also the new experiment of Spielman referred to in ref [13, 14]
[13] T.-L. Ho and S. Zhang, arXiv:1007.0650
[14] C. J. Wang et al, arXiv:1006.5148
[15] A. H. Castro Neto et al, Rev. Mod. Phys. 81, 109 (2009)
[16] M. Z. Hasan and C. L. Kane, arXiv:1002.3895
[17] N. Nagaosa, J. Phys. Soc. Jpn, 77, 031010 (2008)
[18] V. P. Mineev and M. Sigrist, arXiv:0904.2962
[19] T.-L. Ho and V. B. Shenoy, Phys. Rev. Lett. 77, 3276 (1996); P. Ao and S. T. Chui, Phys. Rev. A 58, 4836 (1998); D. M. Stamper-Kurn et al, Phys. Rev. Lett. 83, 661 (1999)