Anti-DeSitter Spaces and Nonextreme Black Holes

Finn Larsen
Department of Physics and Astronomy
University of Pennsylvania
Philadelphia, PA 19104

At low energy the near horizon geometry of nonextreme black holes in four dimensions exhibits an effective $SL(2,\mathbb{R})_L \times SL(2,\mathbb{R})_R$ symmetry. The parameters of the corresponding induced conformal field theory gives the correct expression for the black hole entropy. The resulting spectrum of the Schwarzschild black hole is compared with another proposal.

1 Introduction

There has been many recent attempts at deriving the entropy of Schwarzschild black holes in four dimensions from string theory, but a fully satisfying understanding has not yet been achieved. This paper elaborates on the non-extreme black hole counting in and exploits the symmetries of the near-horizon region to learn about general non-extreme black holes, including Schwarzschild black holes. The result provides an appealing generalization of the computations for the dilute-gas black holes that are standard by now. However, just as other computations of the Schwarzschild entropy in string theory, this work is presently without secure foundation in the microscopic theory, and thus speculative.

2 The Black Hole Background

The starting point is a large class of four-dimensional black holes specified by their mass $M$ and four $U(1)$ charges $Q_i$ or, more conveniently, parametrized by the non-extremality parameter $m$ and four boosts $\delta_i$:

$$G_4 M = \frac{1}{4} m \sum_{i=0}^{3} \cosh 2\delta_i ; \quad G_4 Q_i = \frac{1}{4} m \sinh 2\delta_i \quad (i = 0, 1, 2, 3) , \quad (1)$$

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where $G_4$ is the four-dimensional Newton’s constant. The metric is:

$$ds^2 = -\frac{1}{\sqrt{H_0H_1H_2H_3}}(1-\frac{2m}{r})dt^2 + \sqrt{H_0H_1H_2H_3}(\frac{1}{1-\frac{2m}{r}}dr^2 + r^2d\Omega_2^2),$$

where $H_i = 1 + 2m \sinh^2 \delta_i / r$. The Reissner-Nordström black hole corresponds to the case where the four $U(1)$ charges are identical $Q_0 = Q_1 = Q_2 = Q_3 \equiv \frac{1}{4}Q_{RN}$; the Schwarzschild black hole is the special case $Q_{RN} = 0$. From the metric eq. (2) one finds the black hole entropy:

$$S \equiv \frac{A_4}{4G_4} = \frac{4\pi m^2}{G_4} \prod_{i=0}^3 \cosh \delta_i,$$

where $A_4$ is the area of the outer horizon.

The dilute gas black holes satisfy $\delta_{1,2,3} \gg 1$, and their near horizon geometry is $\text{BTZ} \times S^2$. The BTZ black hole is locally $\text{AdS}_3$, so in this case the CFT/AdS correspondence applies directly. However, the dilute gas conditions imply a near extreme limit that does not include the Reissner-Nordström black holes. In this paper the CFT/AdS correspondence is applied without any conditions on the black hole parameters.

### 3 The Wave Equation

The black hole geometry can be analyzed by considering test fields that propagate in the background, e.g., a minimally coupled scalar field satisfying the massless Klein-Gordon equation:

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\Phi) = 0.$$

The variables are separated by writing the wave function in spherical coordinates as:

$$\Phi \equiv \Phi_r(r) \chi(\theta)e^{-i\omega t + im\phi},$$

and we introduce the dimensionless radial coordinate:

$$x \equiv \frac{r - \frac{1}{2}(r_+ + r_-)}{r_+ - r_-},$$

where $r_+ = 2m$ and $r_- = 0$. Then the outer and inner event horizons are located at $x = \pm \frac{1}{2}$, respectively; and the asymptotic space is at large $x$. The radial wave equation becomes:

$$\frac{\partial}{\partial x}(x^2 - \frac{1}{4})\frac{\partial}{\partial x}\Phi_r + \left[\frac{1}{x - \frac{1}{2}}\frac{\omega^2}{4\kappa_+} - \frac{1}{x + \frac{1}{2}}\frac{\omega^2}{4\kappa_-} + V(x)\right]\Phi_r = l(l+1)\Phi_r,$$
where the surface accelerations $\kappa_{\pm}$ at the outer and inner horizons are:

$$\frac{1}{\kappa_+} = 4m \prod_i \cosh \delta_i \quad ; \quad \frac{1}{\kappa_-} = 4m \prod_i \sinh \delta_i ,$$

(8)

respectively; and the effective potential is:

$$V(x) = 4x^2 m^2 \omega^2 + x \, 8G_4 M m \omega^2 + \left(1 + \sum_{i<j} \cosh 2\delta_i \cosh 2\delta_j \right) m^2 \omega^2 .$$

(9)

The radial wave equation has pole terms at the horizons $x = \pm \frac{1}{2}$ that are characteristic of black holes; they encode the distinctive features of the causal structure. In contrast, the potential $V(x)$ contains the boundary condition that space is asymptotically flat; and that gravitational potentials at large distances are governed by Newton’s $1/r$ law. Thus the potential $V(x)$ contains the “unimportant” part of the black hole geometry that is unlikely to reveal much about its internal structure.

There are several circumstances where it can be justified to neglect the potential $V(x)$. Here we exploit that the potential $V(x)$ is negligible when the frequency of the probe is small. In contrast, the standard strategy for computations of greybody factors in string theory imposes conditions on both the black hole parameters and the probe energy. The precise low-energy condition $M \omega \ll 1$ needed here ensures the universal low-energy cross-section $\sigma_{\text{abs}}(\omega \to 0) = A_4$ as well; so it is reasonable to expect that this condition also suffices to reveal the model-independent aspects of black hole entropy.

The radial wave equation exhibits an interesting structure when the potential $V(x)$ is omitted: it realizes the group $SL(2, \mathbb{R})_L \times SL(2, \mathbb{R})_R$ that is also the world-sheet conformal symmetry group of string theory. This feature of the background geometry can be exploited to shed light on the underlying effective string theory. According to the previous paragraph the $SL(2, \mathbb{R})_L \times SL(2, \mathbb{R})_R$ symmetry applies to all black holes at very low energy.

It is customary to interpret the four dimensional dilute gas black hole as a five dimensional black string. This effective string in turn arises as the intersection of three M5-branes that are mutually orthogonal and wrapped on a small six-torus. From the five dimensional point of view the near horizon geometry of the M5-branes is $AdS_3 \times S^2$ and the isometries of the $AdS_3$ accounts for the $SL(2, \mathbb{R})_L \times SL(2, \mathbb{R})_R$ symmetry. However, this interpretation is not mandatory. Probes that are massless in four dimensions are independent of the fifth direction so, for such probes, the “extra” coordinate is a redundant variable whose role is to realize the $SL(2, \mathbb{R})_L \times SL(2, \mathbb{R})_R$ linearly.
interpret the $SL(2, \mathbb{R})_L \times SL(2, \mathbb{R})_R$ symmetry of the near-horizon geometry in this way it may persist even when the higher dimensional interpretation has no preferred direction which, together with the $t$ and $r$ coordinates, could form the $AdS_3$. From this abstract point of view, the symmetry makes sense for black holes that do not satisfy the dilute gas condition.

4 Counting Black Hole Microstates

The working hypothesis is that the underlying microscopic theory is similar to the one governing the BTZ black hole; i.e. a two dimensional conformal field theory with the central charges:

$$c_L = c_R = \frac{3\lambda}{2G_3},$$

where the effective cosmological constant in three dimensions is $\Lambda = -\lambda^2$.

The wave equation for a probe in the BTZ background agrees precisely with eq. 7, for $V(x) = 0$, if we make the identifications:

$$\beta_{L,R} = \frac{2\pi}{\kappa_+} + \frac{2\pi}{\kappa_-} = 8\pi m \left( \prod_{i=0}^{3} \cosh \delta_i \mp \prod_{i=0}^{3} \sinh \delta_i \right) = \frac{2\pi \lambda R_{11}}{\sqrt{M_3 \lambda^2 + 8\lambda G_3 J_3}},$$

where $M_3$ and $J_3$ are the parameters of the BTZ black hole and the auxiliary length $R_{11}$ relates the four dimensional Schwarzschild time and the BTZ-time through $\lambda t^4 = t_{BTZ} R_{11}$. Then the conformal weights of the underlying quantum states:

$$h_{L,R} = \frac{M_3 \lambda^2 \mp 8\lambda G_3 J_3}{16\lambda G_3},$$

determine the entropy of the left- and right-movers as:

$$S_{L,R} = 2\pi \sqrt{\frac{c_{L,R}}{6}} h_{L,R} = \frac{\lambda}{32G_3 m \left( \prod_{i=0}^{3} \cosh \delta_i \mp \prod_{i=0}^{3} \sinh \delta_i \right) 2\pi R_{11}}.$$

Alternatively, this result follows from the standard expression for entropy of gasses in one spatial dimension of length $L = 2\pi R_{11}$:

$$S_{L,R} = \frac{c_{L,R}}{6} \frac{\pi}{\beta_{L,R}} L,$$

with the parameters given in eqs. [10] and [11]².

A low energy wave that is incident on the four-dimensional black hole experiences a $BTZ \times S^2$ geometry. It has no dependence on the angular BTZ
variable, the “extra coordinate”, and the radius of the sphere is related to the cosmological constant as $R_{sph} = \frac{1}{2} \lambda$, by the equations of motion. Comparing the dimensional reductions of this five-dimensional geometry to four physical dimensions, and to three BTZ-dimensions, we find:

$$\frac{1}{G_3} = \frac{1}{G_4} \frac{\lambda^2}{2 R_{11}}.$$  
(15)

Thus the $S_{L,R}$ in eq. [3] are independent of $R_{11}$, as they should be.

The thermodynamic argument that leads to eq. [14] for free gasses implies that the $N_{L,R} = \frac{c L}{6} h_{L,R}$ are quantized with integer spacings. On the other hand, quantization rules on the string theory charges give the relation $64 G_2^3 \prod_{i=0}^3 Q_i = \prod_{i=0}^3 n_i = \text{integer}$, for general non-extreme black holes[^4]. Consistency then determines the cosmological constant as:

$$\lambda = 4 m \left( \prod_{i=0}^3 \cosh \delta_i - \prod_{i=0}^3 \sinh \delta_i \right)^{\frac{1}{2}},$$  
(16)

after a short computation of $N_L - N_R$. This generalizes the dilute gas result $\lambda = 8 G_4 (Q_1 Q_2 Q_3)^{\frac{1}{2}}$, in a way that treats all four charges symmetrically.

We now find:

$$S = 2\pi \left( \sqrt{\frac{c_L}{6} h_L} + \sqrt{\frac{c_R}{6} h_R} \right) = \frac{4\pi m^2}{G_4} \prod_{i=0}^3 \cosh \delta_i,$$  
(17)

in perfect agreement with eq. [3]. In particular, the full functional dependence on all charges is reproduced correctly.

Angular momentum was not included in the above. However, this can be easily remedied and the ensuing computation is much tighter, giving supporting evidence for the argument. The general semiclassical quantization rule now states that:

$$N_L = \frac{c_L}{6} h_L = \left[ \frac{m^2}{G_4} \left( \prod_{i=0}^3 \cosh \delta_i + \prod_{i=0}^3 \sinh \delta_i \right) \right]^2,$$  
(18)

$$N_R = \frac{c_R}{6} h_R = \left[ \frac{m^2}{G_4} \left( \prod_{i=0}^3 \cosh \delta_i - \prod_{i=0}^3 \sinh \delta_i \right) \right]^2 - J^2,$$  
(19)

have integer spacings.

The above presentation relied on the computation of BTZ entropy by Strominger[^3]. Albeit elegant, this calculation is not above criticism, as recently
emphasized in [2]. Thus the ultimate justification for entropy calculations, in
the dilute gas regime, remains our understanding of $D$-branes. Similarly, the
non-extreme entropy must eventually be derived from the complete string the-
ory spectrum, far from BPS. The form of the non-extreme black hole entropy
suggests that, far from being complicated, this spectrum is structured in the
same way as the near-extreme one.

As it stands, the computation of non-extreme entropy has several weak-
nesses, beyond those present in the dilute gas case. In particular, the $\text{BTZ} \times S^2$
geometry is an auxiliary concept that is not realized explicitly; the area $\pi \lambda^2$
of the effective sphere is in general unrelated to the area of the outer horizon.
On the other hand, this also ascertains that the final result was not “put in”
by hand. Another concern is that the low energy condition $M \omega \ll 1$ imposed
on the test field implies $\beta_{L,R} \omega \ll 1$, whereas the modes responsible for the
black hole entropy appears to satisfy $\beta_{L,R} \omega \sim 1$. This mismatch of scales is
related to the absence of a “decoupling limit” ensuring that the worldvolume
field theory does not couple to gravity. A possible interpretation is that near-
extreme and non-extreme black holes are separated by a phase transition, and
the present work assumes that this does not happen.

5 Comparison with Area Quantization

It is instructive to compare the quantization rule given in eqs. [18-19] with
the “area quantization” of Bekenstein-Mazur-Mukhanov (BMM) [26,27,28] (see
also [29]). Consider for definiteness the Kerr-Newman entropy:

$$S = 2\pi [(G_4 M^2 - J^2) + \sqrt{G_4^2 M^4 - G_4^2 M^2 Q_{KN}^2 - J^2}].$$  (20)

Eqs. [18-19] imply that the expression under the square root and the square
of the expression in round brackets are both quantized with integer spacings.
The BMM prescription states that only the square bracket is so quantized. In
particular, eqs. [18-19] gives $G_4^2 M^4$ = integer for Schwarzchild black holes; in
contrast, BMM advocate $G_4 M^2$ = integer.

These differences can be elucidated by considering various extreme rotating
black holes. For neutral Kerr black holes the expression under the square root
vanishes in the extreme limit, and so $G_4 M^2 = J = \text{integer}$. If this is the
complete spectrum also in the Schwarzchild case, we find the BMM result.
However, in the dilute gas limit we trust eqs. [18-19] in particular:

$$N_L - N_R = q + J^2,$$  (21)

where the quantized charge $q$ has integer spacings. In the extreme limit $N_R \to 0$; so, for exactly neutral black holes ($q = 0$), the quantum number $N_L$ is the
square of an integer. Since the extremal entropy $S = 2\pi \sqrt{N_L}$, this is again the BMM result. However, for generic (charged) black holes the $N_L$ can be any integer, even in the extreme limit; so the allowed masses are generically spaced much closer than the BMM argument indicates. If this is the complete spectrum also in the Schwarzschild case, we find the quantization rule advocated here.

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