REMARKS ON SCHRÖDINGER OPERATORS WITH SINGULAR MATRIX POTENTIALS

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Abstract. In this paper the asymmetric generalization of the Glazman–Povzner–Wienholtz theorem is proved for one-dimensional Schrödinger operators with strongly singular matrix potentials from the space $H^{-1}_{loc}(\mathbb{R}, \mathbb{C}^{m \times m})$. This result is new in the scalar case as well.

1. Introduction and main results

Let us consider in the complex separable Hilbert space of vector functions $L^2(\mathbb{R}, \mathbb{C}^m)$, $m \in \mathbb{N}$ the operators generated by the formal differential expression:

$$l[u] := -u'' + qu, \quad u = (u_1, \ldots, u_m),$$

where the matrix potential $q = \{q_{ij}\}_{i,j=1}^m$ belongs to the Sobolev negative class $H^{-1}_{loc}(\mathbb{R}, \mathbb{C}^{m \times m})$. Without loss of generality, we assume that the potential $q$ in (1) may be presented in the form

$$q = Q' + s, \quad Q \in L^2_{loc}(\mathbb{R}, \mathbb{C}^{m \times m}), \quad s \in L^1_{loc}(\mathbb{R}, \mathbb{C}^{m \times m}),$$

where the derivative is understood in the sense of the distributions. Then the block Shin–Zettl matrices are defined:

$$A(x) := \begin{pmatrix} Q & I_m \\ -Q^2 + s & -Q \end{pmatrix} \in L^1_{loc}(\mathbb{R}, \mathbb{C}^{2m \times 2m}),$$

where $I_m$ is a unit $(m \times m)$-matrix. Similarly to the scalar case [7,15] Shin–Zettl matrices define quasiderivatives [13]:

$$u[0] := u, \quad u[1] := u' - Qu, \quad u[2] := (u[1])' + Qu[1] + (Q^2 - s)u.$$

Then formal differential equation (1) is a quasidifferential one:

$$l[u] := -u[2], \quad \text{Dom}(l) := \left\{ u \mid u, u[1] \in AC_{loc}(\mathbb{R}, \mathbb{C}^{m}) \right\},$$

where by $AC_{loc}(\mathbb{R}, \mathbb{C}^{m})$ we denote the class of locally absolutely continuous vector functions. This definition is motivated by the fact that

$$-u[2] = -u'' + qu$$

in the sense of distributions, i. e.,

$$\langle -u[2], \varphi \rangle = \langle -u'' + qu, \varphi \rangle, \quad u \in \text{Dom}(l), \varphi \in C^\infty_0(\mathbb{R}, \mathbb{C}^{m}).$$

We say that function $u$ solves the Cauchy problem

(3) $l[u] = f, \quad f \in L^1_{loc}(\mathbb{R}, \mathbb{C}^m),$  
(4) $u(x_0) = c_0, \quad u[1](x_0) = c_1, \quad x_0 \in \mathbb{R}, \quad c_0, c_1 \in \mathbb{C}^{m},$

if $u$ is the first coordinate of the vector function solving the Cauchy problem for the associated Cauchy problem with initial conditions [4]

(5) $\frac{d}{dx} \begin{pmatrix} u \\ u[1] \end{pmatrix} = A(x) \begin{pmatrix} u \\ u[1] \end{pmatrix} + \begin{pmatrix} 0 \\ -f \end{pmatrix}.$

The existence and uniqueness theorem implies that the Cauchy problem for system (5) has a unique solution (see [13 Theorem 16.1] and [17 Theorem 2.1]). Therefore our definition of a solution of the equation (3) is correct.

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Differential expression (1) gives rise to the associated maximal and preminimal operators $L$ and $L_{00}$ in the Hilbert space $L^2(\mathbb{R}, \mathbb{C}^m)$:

$$Lu := l[u], \quad \text{Dom}(L) := \left\{ u \in L^2(\mathbb{R}, \mathbb{C}^m) \bigg| u, u^{(1)} \in AC_{loc}(\mathbb{R}, \mathbb{C}^m), \ l[u] \in L^2(\mathbb{R}, \mathbb{C}^m) \right\},$$

and

$$L_{00}u := l[u], \quad \text{Dom}(L_{00}) := \{ u \in \text{Dom}(L) \mid \text{supp} \ u \subset \mathbb{R} \}.$$

The block Shin–Zettl matrix (2) defines the Lagrange adjoint quasidifferential expression $l^+$ in the following way:

$$v^{(0)} := v, \quad v^{(1)} := v' - Q^*v, \quad v^{(2)} := \left(v^{(1)}\right)' + Q^*v^{(1)} + ((Q^*)^2 - s^2)v,$$

$$l^+[v] := -v^{(2)}, \quad \text{Dom}(l^+) := \left\{ v \bigg| v, v^{(1)} \in AC_{loc}(\mathbb{R}, \mathbb{C}^m) \right\},$$

where the matrix $Q^* := \overline{Q^T}$ is Hermitian conjugate to $Q$. The matrix $s^*$ has a similar meaning.

Quasidifferential expression $1^+$ gives rise to the associated maximal and preminimal operators $L^+$ and $L_{00}^+$:

$$L^+v := l^+[v], \quad \text{Dom}(L^+) := \left\{ v \in L^2(\mathbb{R}, \mathbb{C}^m) \bigg| v, v^{(1)} \in AC_{loc}(\mathbb{R}, \mathbb{C}^m), \ 1^+[v] \in L^2(\mathbb{R}, \mathbb{C}^m) \right\},$$

and

$$L_{00}^+v := l^+[v], \quad \text{Dom}(L_{00}^+) := \{ v \in \text{Dom}(L^+) \mid \text{supp} \ v \subset \mathbb{R} \}.$$

Below we prove (Proposition 7) that preminimal operators $L_{00}$, $L^+_{00}$ are densely defined in the space $L^2(\mathbb{R}, \mathbb{C}^m)$ and have closures $L_0$ and $L^+_0$ which are called minimal operators. Maximal operators $L$ and $L^+$ are closed.

For the case of potential $q$ being a real-valued symmetric matrix such operators were considered earlier in [13]. Matrix Schrödinger operators with strongly singular self-adjoint potentials of Miura class were investigated in detail in [2]. There one may find a more detailed review and a more extensive bibliography. For the scalar case of quasidifferential operators generated by Shin–Zettl matrices of general form one may find a review of results in [4], see also [8, 18].

Recall that an operator $A$ in the Hilbert space $H$ is called accretive if

$$\text{Re} \ (Au, u)_H \geq 0, \quad u \in \text{Dom}(A).$$

If in addition the left half-plane \( \{ \lambda \in \mathbb{C} \mid \text{Re} \lambda < 0 \} \) belongs to the resolvent set of the operator $A$ then operator $A$ is called $m$-accretive [10, 11]. This operator is also maximal accretive in the sense that it has no accretive extensions in the space $H$. If operator $A$ is $m$-accretive then operator $-A$ generates a semigroup of contractions in the space $H$. Converse is also true.

The main result of this paper is a non-symmetric generalization of the Glazman–Povzner–Wienholtz theorem for operators generated by differential expression (1).

**Theorem 1.** The operator $L_0$ is $m$-accretive if and only if preminimal operators $L_{00}$ and $L^+_{00}$ are accretive. In this case $L_0 = L$.

Note that in this theorem we assume both preminimal operators $L_{00}$ and $L^+_{00}$ to be accretive. In the scalar case one of these operators being accretive implies that other is also accretive.

**Corollary 2** (Cf. [5]). If matrix potential $q$ is self-adjoint: $Q = Q^*$ and $s = s^*$, then operator $L_0$ is symmetric. Moreover if operator $L_0$ is bounded below then it is self-adjoint and $L_0 = L$.

For $m = 1$ this is known [11, Remark III.2], see also [9, 11].

**Remark 3.** If the complex matrices $Q$ and $s$ are symmetric, i. e., $Q = Q^T$, $s = s^T$, then Theorem 1 can be strengthened. As operator $L_{00}$ is accretive, the operator $L_0$ is maximal accretive and its residual spectrum is empty.

In particular, this condition is satisfied in the scalar case, when $m = 1$. In this case, the operators $L_{00}$ and $L^+_{00}$ obviously are accretive if the real part of the potential $q$ is positive in the sense of distributions. This condition is equivalent to

$$q = \mu + i\nu,$$

where $\mu$ is a nonnegative Radon measure on a locally compact space $\mathbb{R}$ and $\nu$ is a real-valued distribution from $H^{loc}_{-1}(\mathbb{R}, \mathbb{C}^{m \times m})$. 
The paper is organized as follows. In Section 2 we state a list of the symbols used in the paper and thoroughly investigate the properties of the operators \( L, L_0 \) and \( L_0^+ \). Section 3 contains proofs of the main Theorem 4 Corollary 5 and Remark 6.

2. Properties of the minimal and maximal operators

In this paper, we use the following notation. We denote by \( (\cdot, \cdot)_{C^m} \) the inner product in the space \( C^m \):

\[
(u, v)_{C^m} := \sum_{i=1}^{m} u_i \overline{v}_i, \quad u = (u_1, \ldots, u_m), \quad v = (v_1, \ldots, v_m) \in C^m.
\]

We denote by \( (\cdot, \cdot)_{L^2(\mathbb{R}, C^m)} \) the inner product in the Hilbert space of square-integrable vector functions \( L^2(\mathbb{R}, C^m) \):

\[
(u, v)_{L^2(\mathbb{R}, C^m)} := \int_{\mathbb{R}} (u, v)_{C^m} \, dx
\]

For an arbitrary matrix \( A = (a_{ij})_{i,j=1}^{m} \in C^{m \times m} \) we denote the transposed matrix by \( A^T = (a_{ij})_{i,j=1}^{m} \) and Hermitian conjugate matrix by \( A^* = (a_{ij}^*)_{i,j=1}^{m} : a_{ij}^* = \overline{a_{ji}} \). For an arbitrary complex number \( a \in \mathbb{C} \) we denote the corresponding complex conjugate number by \( \overline{a} \).

We say that matrix function \( A(x) = (a_{ij}(x))_{i,j=1}^{m} \) belongs to the space \( L^p_{loc}(\mathbb{R}, C^{m \times m}) \), if each element of this matrix \( a_{ij}(x) \) belongs to the space \( L^p_{loc}(\mathbb{R}, \mathbb{C}) \), \( p \in (1, \infty) \).

J. Weidmann [17] previously studied in detail the quasidifferential matrix-valued Sturm-Liouville operators generated by quasidifferential expressions \( \tau \),

\[
\tau[u] := -(u' - Qu)' - Q^*(u' - Qu) - (Q^*Q - s)u,
\]

\( Q \in L^2_{loc}(\mathbb{R}, C^{m \times m}), \quad s \in L^1_{loc}(\mathbb{R}, C^{m \times m}), \quad s = s^* \).

In this case preminimal operators generated by quasidifferential expressions \( \tau \) are symmetric [17, Theorem 3.1].

Obviously, if matrices \( Q = Q^* \) and \( s = s^* \) are self-adjoint then operators generated by quasidifferential expressions \( \tau \) and operators generated by quasidifferential expressions \( l \) and \( l^+ \) coincide.

The following properties of the operators \( L, L_0, L_0 \) and \( L_0^+, L_0^+ \) we state without proof, because they are proved in the same way as the properties of operators generated by quasidifferential expressions \( \tau \) [17].

Lemma 4. For arbitrary vector functions \( u \in \text{Dom}(L), \ v \in \text{Dom}(L^+) \) and arbitrary finite interval \([a, b]\) we have

\[
\int_a^b (l[u], v)_{C^m} \, dx - \int_a^b (u, l^+[v])_{C^m} \, dx = [u, v]_{a}^{b},
\]

where

\[
[u, v](t) \equiv [u, v] := \begin{pmatrix} u, v(1) \end{pmatrix}_{C^m} - \begin{pmatrix} u(1), v \end{pmatrix}_{C^m},
\]

\[
[u, v]_{a}^{b} := [u, v](b) - [u, v](a), \quad -\infty \leq a \leq b \leq \infty.
\]

Lemma 5. For arbitrary vector functions \( u \in \text{Dom}(L) \) and \( v \in \text{Dom}(L^+) \) the following limits exist and are finite:

\[
[u, v](-\infty) := \lim_{t \to -\infty} [u, v](t), \quad [u, v](\infty) := \lim_{t \to \infty} [u, v](t).
\]

Lemma 6 (Generalized Lagrange identity). For arbitrary vector functions \( u \in \text{Dom}(L) \) and \( v \in \text{Dom}(L^+) \) the following relation holds:

\[
\int_{-\infty}^{\infty} (l[u], v)_{C^m} \, dx - \int_{-\infty}^{\infty} (l[u], v)_{C^m} \, dx = [u, v]_{-\infty}^{\infty}.
\]

Proposition 7. The operators \( L, L_0 \) and \( L^+, L_0^+ \) have the following properties:

1°. Operators \( L_0 \) and \( L_0^+ \) are densely defined in the Hilbert space \( L^2(\mathbb{R}, C^m) \).

2°. The equalities

\[
(L_0)^* = L^+ = L,
\]

hold. In particular, operators \( L, L^+ \) are closed and operators \( L_0, L_0^+ \) are closable.

3°. Domains of operators \( L_0, L_0^+ \) may be described in the following way:

\[
\text{Dom}(L_0) = \{ u \in \text{Dom}(L) \mid [u, v]_{-\infty}^{\infty} = 0, \forall v \in \text{Dom}(L^+) \},
\]

\[
\text{Dom}(L_0^+) = \{ v \in \text{Dom}(L^+) \mid [u, v]_{-\infty}^{\infty} = 0, \forall u \in \text{Dom}(L) \}.
\]
The following inclusions take place:

$$\text{Dom}(L) \subset H^1_{\text{loc}}(\mathbb{R}, \mathbb{C}^m), \quad \text{Dom}(L^+) \subset H^1_{\text{loc}}(\mathbb{R}, \mathbb{C}^m).$$

For the case $m = 1$ the results of this section are established in [42].

3. PROOFS

The following lemma is proved by direct calculation.

**Lemma 8.** For arbitrary vector functions $u \in \text{Dom}(L)$, $v \in \text{Dom}(L^+)$ and functions $\varphi \in C_0^\infty(\mathbb{R}, \mathbb{C})$ we have

i) $\|\varphi I_m u\| = \varphi I_m[u] - \varphi'' I_m u - 2\varphi' I_m u', \quad \varphi I_m u \in \text{Dom}(L_{00});$

ii) $\varphi^+ [\varphi I_m v] = \varphi I_m l^+ [v] - \varphi'' I_m v - 2\varphi' I_m v', \quad \varphi I_m v \in \text{Dom}(L_{00}).$

**Proof of Theorem 1. Sufficiency.** Due to the assumptions of theorem the minimal operators $L_0$ and $L_0^+$ are accretive. Without loss of generality we assume that the following inequalities hold:

$$\text{Re} \langle L_0 u, u \rangle_{L^2(\mathbb{R},\mathbb{C}^m)} \geq \langle u, u \rangle_{L^2(\mathbb{R},\mathbb{C}^m)}, \quad u \in \text{Dom}(L_0),$$

and

$$\text{Re} \langle L_0^+ v, v \rangle_{L^2(\mathbb{R},\mathbb{C}^m)} \geq \langle v, v \rangle_{L^2(\mathbb{R},\mathbb{C}^m)}, \quad v \in \text{Dom}(L_0^+).$$

To prove the minimal operator $L_0$ to be $m$-accretive one suffices to show that the kernel of operator $L^+$ contains only the zero element.

Let $v$ be a solution to the equation

$$L^+ v = 0.$$

We will show that $v \equiv 0$.

For an arbitrary function $\varphi \in C_0^\infty(\mathbb{R}, \mathbb{R})$ due to Lemma 8 we have $\varphi I_m v \in \text{Dom}(L_{00})$. Therefore, taking into account that $1^+[v] = 0$, after some simple calculations we obtain:

$$\langle L_0^+ \varphi I_m v, \varphi I_m v \rangle_{L^2(\mathbb{R},\mathbb{C}^m)} = \int_{\mathbb{R}} (\varphi')^2(v, v)_{\mathbb{C}^m} d x + \int_{\mathbb{R}} \varphi \varphi' ((v, v')_{\mathbb{C}^m} - (v', v)_{\mathbb{C}^m}) d x.$$

As

$$\text{Re} \int_{\mathbb{R}} \varphi \varphi' ((v, v')_{\mathbb{C}^m} - (v', v)_{\mathbb{C}^m}) d x = 0,$$

from (7) taking into account (8) we receive:

$$\int_{\mathbb{R}} (\varphi')^2(v, v)_{\mathbb{C}^m} d x \geq \int_{\mathbb{R}} (\varphi)^2(v, v)_{\mathbb{C}^m} d x \quad \forall \varphi \in C_0^\infty(\mathbb{R}, \mathbb{R}).$$

Furthermore, let us take a sequence of functions $\{\varphi_n\}_{n \in \mathbb{N}}$ which has the following properties:

i) $\varphi_n \in C_0^\infty(\mathbb{R}, \mathbb{R});$

ii) $\text{supp } \varphi_n \subset [-n - 1, n + 1];$

iii) $\varphi_n(x) = 1, \quad x \in [-n, n];$

iv) $|\varphi_n(x)| \leq C$ where $C > 0$ is an absolute constant.

Substituting in (8) we get

$$\int_{-n}^n (v, v)_{\mathbb{C}^m} d x \leq \int_{\mathbb{R}} (\varphi_n')^2(v, v)_{\mathbb{C}^m} d x \leq C^2 \int_{n \leq |x| \leq n + 1} (v, v)_{\mathbb{C}^m} d x,$$

i. e.

$$\int_{-n}^n (v, v)_{\mathbb{C}^m} d x \leq C^2 \int_{n \leq |x| \leq n + 1} (v, v)_{\mathbb{C}^m} d x.$$

As $v \in L^2(\mathbb{R}, \mathbb{C}^m)$ passing in (9) to the limit as $n \to \infty$, we receive $v \equiv 0$.

Thus we have proved that operator $L_0$ is $m$-accretive.

In a similar way one may prove that operator $L_0^+$ is $m$-accretive. Then taking into account that an adjoint operator to the $m$-accretive operator is $m$-accretive [16 Proposition 3.20] from the property $2^o$ of Proposition 7 we get that the maximal operator $L$ is also $m$-accretive. By the definition of the maximal accretivity and [16] Примерложение 3.24 we have $L_0 = L$ as $L_0 \subset L$. Sufficiency is proved.
Necessity. Let us suppose that the operator $L_0$ is $m$-accretive. Then taking into account that an adjoint operator to the $m$-accretive operator is $m$-accretive [16 Proposition 3.20] from the property 20 of Proposition [7] we get that the operator $L_{00}^+$ is $m$-accretive. Therefore the operators $L_{00}$ and $L_{00}^+$ are accretive. Necessity is proved.

Theorem is proved completely. \hfill \Box

Proof of Corollary \[^{[8]}\] One only needs to note that in the case of self-adjoint potential $q$ preminimal operators $L_{00}$ and $L_{00}^+$ coincide and due to property 20 of Proposition \[^{[7]}\] (see also \[^{[17]}\] Theorem 3.1) are symmetric. \hfill \Box

Proof of Remark \[^{[9]}\] Note that in the case of complex symmetric matrix potentials, we have:

\[ Q^* = \overline{Q} = \{\overline{Q}_{ij}\}_{i,j=1}^m, \quad s^* = \overline{s} = \{\overline{s}_{ij}\}_{i,j=1}^m. \]

Thus domains of preminimal operators $L_{00}$ and $L_{00}^+$ are related by

\[ u \in \text{Dom}(L_{00}) \iff \overline{u} \in \text{Dom}(L_{00}^+). \]

Therefore the accretivity of the operator $L_{00}$ implies the $m$-accretivity of the operator $L_{00}^+$ and vice versa.

Moreover, let $J$ be an antilinear operator of complex conjugation. Then one may easily verify that the following inclusion takes place:

\[ JL_{00} = L_{00}^+ \subset L_{00} = L_{00}^+ J, \]

that is, the operator $L_0$ is $J$-symmetric \[^{[6]}\]. If operators $L_{00}$ are accretive, then due to Theorem \[^{[1]}\] and property 20 of Proposition \[^{[4]}\] the operator $L_0$ is $J$-self-adjoint:

\[ JL_{00} = L_0. \]

Therefore its residual spectrum is empty. \hfill \Box

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