Abstract

This paper investigates stochastic and adversarial combinatorial multi-armed bandit problems. In the stochastic setting, we first derive problem-specific regret lower bounds, and analyze how these bounds scale with the dimension of the decision space. We then propose COMBUCB, algorithms that efficiently exploit the combinatorial structure of the problem, and derive finite-time upper bound on their regrets. These bounds improve over regret upper bounds of existing algorithms, and we show numerically that COMBUCB significantly outperforms any other algorithm. In the adversarial setting, we propose two simple algorithms, namely COMBEXP-1 and COMBEXP-2 for semi-bandit and bandit feedback, respectively. Their regrets have similar scaling as state-of-the-art algorithms, in spite of the simplicity of their implementation.

1. Introduction

Multi-Armed Bandits (MAB) problems (Robbins, 1985) constitute the most fundamental sequential decision problems with an exploration vs. exploitation trade-off. In such problems, the decision maker selects an arm in each round, and observes a realization of the corresponding unknown reward distribution. Each decision is based on past decisions and observed rewards. The objective is to maximize the expected cumulative reward over some time horizon by balancing exploitation (arms with higher observed rewards should be selected often) and exploration (all arms should be explored to learn their average rewards). Equivalently, the performance of a decision rule or algorithm can be measured through its expected regret, defined as the gap between the expected reward achieved by the algorithm and that achieved by an oracle algorithm always selecting the best arm. MAB problems have found applications in many fields, including sequential clinical trials, communication systems, economics, see e.g., (Bubeck & Cesa-Bianchi, 2012; Cesa-Bianchi et al., 2006).

In this paper, we investigate generic combinatorial MAB problems with linear reward function, as introduced in (Cesa-Bianchi & Lugosi, 2012). In each round $n \geq 1$, a decision maker selects an arm from a finite set $\mathcal{M} \subset \{0, 1\}^d$ and receives a reward $\mathcal{M}^\top X(n) = \sum_{i=1}^d M_i X_i(n)$ if $M \in \mathcal{M}$ is selected, and where the reward vector $X(n) \in \mathbb{R}^d_+$ is unknown. We focus here on the case where all arms consist of the same number of basic actions in the sense that $\|M\|_1 = m, \forall M \in \mathcal{M}$. After selecting an arm $M$ in round $n$, the decision maker receives some feedback. We consider both (i) semi-bandit feedback under which after round $n$, for all $i \in \{1, \ldots, d\}$, the component $X_i(n)$ of the reward vector is revealed if and only if $M_i = 1$; (ii) bandit feedback under which only the reward $\mathcal{M}^\top X(n)$ is revealed. Based on the feedback received up to round $n-1$, the decision maker selects an arm for the next round $n$, and her objective is to maximize her cumulative reward over a given time horizon consisting of $T$ rounds. The challenge in these problems resides in the very large number of arms, i.e., in its combinatorial structure: the size of $\mathcal{M}$ could well grow as $d^m$. Fortunately, one may hope to exploit the problem structure to speed up the exploration of sub-optimal arms.

We consider two instances of combinatorial bandit prob-
lems, depending on how the sequence of reward vectors is generated. We first analyze the case of stochastic rewards, where for all \( i \in \{1, \ldots, d\} \), \( (X_i(n), n \geq 1) \) are i.i.d. with Bernoulli distribution of unknown mean. The reward sequences are also independent across \( i \). We then address the problem in the adversarial setting where the sequence of vectors \( X(n) \) is arbitrary and selected by an adversary at the beginning of the experiment. In the stochastic setting, we provide sequential arm selection algorithms whose performance exceeds that of existing algorithms, whereas in the adversarial setting, we devise simple algorithms whose regret have the same scaling as that of state-of-the-art algorithms.

2. Results and Related Work

2.1. Stochastic Setting

For stochastic combinatorial bandits, we restrict our attention to the case of semi-bandit feedback, and our contributions are as follows.

(a) Our first contribution is to derive an asymptotic (as the time horizon \( T \) grows large) regret lower bound satisfied by any arm selection algorithm (Theorem 1). Our derivation leverages the theory of optimal control of Markov chains with unknown transition rates. To our knowledge, this is the first time such fundamental performance limits are presented for stochastic combinatorial bandits. The dependency in \( m \) and \( d \) of the lower bound is unfortunately not explicit. However, we further provide a simplified lower bound (Theorem 2) that indicates its scaling in \( m \) and \( d \) in specific cases.

(b) We then propose \( \text{COMBUCB} \), an algorithm whose regret scales at most as \( O\left(\sqrt{m d T \log \mu_{\text{min}}^{-1}}\right) \) (Theorem 5), where \( \Delta_{\text{min}} \) denotes the smallest gap between the average rewards of the best arm and of a sub-optimal arm. \( \text{COMBUCB} \) relies on assigning an index to each arm. The index of given arm can be interpreted as performing likelihood tests with vanishing risk on its average reward. In fact, our indexes can be seen as the natural extension of KL-UCB indexes defined for unstructured bandits (Garivier & Cappé, 2011). Numerical experiments for some specific combinatorial problems are presented in the supplementary material, and show that \( \text{COMBUCB} \) significantly outperforms existing algorithms.

Related work. Stochastic combinatorial bandits have not received a lot of attention in the literature. Some research contributions concern problems where the set of arms exhibits very specific structures, such as \( m \)-set (Anantharam et al., 1987), matroid (Kveton et al., 2014a), or matching in bi-partite graphs (Gai et al., 2010). Generic combinatorial problems have been investigated in (Gai et al., 2012), (Chen et al., 2013), and (Kveton et al., 2014b). The proposed algorithms, LLR and CUCB, respectively, are variants of UCB algorithm, and their performance guarantees are presented in Table 1. Our algorithms improve the regret under LLR and CUCB by at least a (multiplicative) factor \( \sqrt{m} \).

2.2. Adversarial Setting

We study adversarial combinatorial bandits under both semi-bandit and bandit feedback. For semi-bandit feedback, we present \( \text{COMBEXP-1} \), an algorithm whose regret scales as at most as \( O\left(\sqrt{m d T \log \mu_{\text{min}}^{-1}}\right) \) (Theorem 7), where \( \mu_{\text{min}} = \min_{i \in [d]} \frac{1}{|M_i|} \sum_{M \in M} M_i \). When \( \mu_{\text{min}}^{-1} = O(\text{poly}(d)) \), which happens in several specific problems of interest, the regret under \( \text{COMBEXP-1} \) matches the regret minimax lower bound \( \sqrt{mdT} \) up to a logarithmic factor (Audibert et al., 2013). The case of bandit feedback, we design \( \text{COMBEXP-2} \), an algorithm with regret \( O\left(\sqrt{m^3 T \left( d + \frac{m^{1/2}}{2} \right) \log \mu_{\text{min}}^{-1}}\right) \), where \( \lambda \) is the smallest nonzero eigenvalue of the matrix \( \mathbb{E}[M M^\top] \) when \( M \) is uniformly distributed over \( M \) (Theorem 8). Again when \( \mu_{\text{min}}^{-1} = O(\text{poly}(d)) \) and when the size of \( M \) is growing as \( d^n \), taking into account that for most problems of interest \( \frac{\mu_{\text{min}}^{-1}}{d} = O(1) \) (Cesa-Bianchi & Lugosi, 2012), we deduce that \( \text{COMBEXP-2} \) provides a regret upper-bounded by \( O(m^{3/2} \sqrt{dT \log \mu_{\text{min}}^{-1}}) \). Note that a known regret minimax lower bound is \( \Omega(m \sqrt{dT}) \) (Audibert et al., 2013), and hence the regret gap between \( \text{COMBEXP-2} \) and this lower bounds scales at most as \( m^{1/2} \) up to a logarithmic factor.

Related work. Adversarial combinatorial bandits have been extensively investigated recently, see (Audibert et al., 2013) and references therein. Some papers consider specific instances of these problems, e.g. shortest-path routing (György et al., 2006; 2007) or \( m \)-sets (Kale et al., 2010). For generic combinatorial problems, known regret lower bounds scale as \( \Omega\left(\sqrt{mdT}\right) \) and \( \Omega\left(m \sqrt{dT}\right) \) (if \( d \geq 2m \)) in the case of semi-bandit and bandit feedback, respectively (Audibert et al., 2013). In the case of semi-bandit feedback, (Audibert et al., 2013) proposes OSMD, an algorithm whose regret upper bound matches the lower

| Algorithm         | Regret                                      |
|-------------------|---------------------------------------------|
| LLR (Gai et al., 2012) | \( O\left(\frac{m^{1/2}}{d} \log(T)\right) \) |
| CUCB (Chen et al., 2013) | \( O(d n^{3/2}) \log(T) \) |
| CUCB (Kveton et al., 2014b) | \( O\left(\frac{m^{1/2}}{d} \log(T)\right) \) |
| COMBUCB (Theorem 6)          | \( O\left(\frac{m^{1/2}}{d} \log(T)\right) \) |

Table 1. Regret upper bounds for stochastic combinatorial bandits under semi-bandit feedback.
bound. COMBEXP-1 is also order-optimal in most cases, and has a simpler implementation than OSMD. It is worth mentioning FPL WITH GR (Neu & Bartók, 2013), an algorithm with regret scaling as $O(m\sqrt{dT\log(d)})$, but that runs in polynomial time. For problems with bandit feedback, (Cesa-Bianchi & Lugosi, 2012) proposes COMBAND and derives a regret upper bound which depends on the structure of action set $\mathcal{M}$. For most problems of interest, the regret under COMBAND is upper-bounded by $O(m\sqrt{dT\log(|\mathcal{M}|)}).$ (Bubeck et al., 2012) addresses generic linear optimization with bandit feedback, and has a simpler implementation than OSMD. It is worth mentioning FPL WITH GR and EXP2 AND EXP-1 is also order-optimal in most cases, as for EXP2. For most problems of interest, the regret under COMBAND is upper-bounded by $O(m\sqrt{dT\log(|\mathcal{M}|)}).$ (Bubeck et al., 2012) addresses generic linear optimization with bandit feedback, and has a simpler implementation than OSMD. It is worth noting that FPL WITH GR and EXP2 AND EXP-1 is also order-optimal in most cases, as for EXP2.

### Examples

#### $m$-sets. Here, $\mathcal{M}$ is the set of all $d$-dimensional binary vectors with $m$ non-zero coordinates. We have $\mu_{\min} = \frac{m}{d}$ and $N = \min\{m, d\}$ (refer to the supplementary material for details). Hence when $m = o(d)$, the regret upper bound of COMBEXP-2 becomes $O(\sqrt{m^3dT\log(d/m)})$, which is the same as that of COMBAND and EXP2 with John's Exploration.

#### Matching. In this example, the set of arms $\mathcal{M}$ is the set of perfect matchings in $K_{m,m}$. $d = m^2$ and $|\mathcal{M}| = m!$. We have $\mu_{\min} = \frac{d}{m^2}$ and $\lambda = \frac{1}{m-1}$. Hence the regret upper bound of COMBEXP-2 is $O(\sqrt{m^3dT\log(m)})$, as for COMBAND and EXP2 with John's Exploration.

#### Spanning Trees. Assume now that $\mathcal{M}$ is the set of spanning trees in the complete graph $K_N$. In this case, $d = \binom{N}{2}$, $m = N - 1$, and by Cayley's formula $\mathcal{M}$ has $N^{N-2}$ arms. $\log \mu_{\min} \leq 2N$ for $N \geq 2$ and $\frac{m}{\lambda} < 7$ when $N \geq 6$. The regret upper bound of COMBAND and EXP2 with John's Exploration becomes $O(\sqrt{N^5T\log(N)})$. As for COMBEXP-2, we get the same regret upper bound $O(\sqrt{N^3T\log(N)})$.

### 3. Models and Objectives

We consider MAB problems where each arm $M$ is a subset of $d$ basic actions taken from $[d] = \{1, \ldots, d\}$. For $i \in [1, \ldots, d]$, $X_i(n)$ denotes the reward of basic action $i$ if this action is present in the selected arm in round $n$. In the stochastic setting, for each $i$, the sequence of rewards $(X_i(n))_{n \geq 1}$ is i.i.d. with Bernoulli distribution with mean $\theta_i$. Rewards are assumed to be independent across actions. We denote by $\theta = (\theta_1, \ldots, \theta_d)^\top \in [0, 1]^d$ the vector of expected rewards of the various basic actions, which is assumed to be unknown. In the adversarial setting, the (column) reward vector $X(n) = (X_1(n), \ldots, X_d(n))^\top \in [0, 1]^d$ is arbitrary, and the sequence $(X(n))_{n \geq 1}$ is decided (but unknown) at the beginning of the experiment.

The set of arms $\mathcal{M}$ is an arbitrary subset of $\{0, 1\}^d$, such that each of its elements $M$ has $m$ basic actions. Arm $M$ is identified with a binary column vector $(M_1, \ldots, M_d)^\top$, and we have $\|M\|_1 = m$, $\forall M \in \mathcal{M}$. At the beginning of each round $n$, an algorithm or policy $\pi$, selects an arm $M^\pi(n) \in \mathcal{M}$ based on the arms chosen in previous rounds and their observed rewards. The reward of arm $M^\pi(n)$ selected in round $n$ is

$$X^M(n) = \sum_{i \in [d]} M_i^\pi(n) X_i(n) = M^\pi(n)^\top X(n).$$

We consider both semi-bandit and bandit feedback. Under semi-bandit feedback and policy $\pi$, at the end of round $n$, the outcome of basic actions $X_i(n)$ for all $i \in M^\pi(n)$ are revealed to the decision maker, whereas under bandit feedback, $M^\pi(n)^\top X(n)$ only can be observed.

Let $\Pi$ be the set of all feasible policies. The objective is to identify a policy in $\Pi$ maximizing the cumulative expected reward over a finite time horizon $T$. The expectation is here taken with respect to possible randomness in the rewards (in the stochastic setting) and the possible randomization in the policy. Equivalently, we aim at designing a policy that minimizes regret, where the regret of policy $\pi \in \Pi$ is
Finally, for the stochastic setting, we denote by $\Delta M$ or distinguishing from $B$ which arm $M$ is unique. We further define: $\mu^*(\theta) = M^* \theta$, $\Delta_{\min} = \min_{M \neq M^*} (\mu^*(\theta) - \mu^*(\theta))$, and $\Delta_{\max} = \max_{M} (\mu^*(\theta) - \mu^*(\theta))$. Finally, define for all $M \in \mathcal{M}$, $\Delta^M = \mu^*(\theta) - \mu^*(\theta)$.

4. Stochastic Combinatorial Bandits

4.1. Regret Lower Bound

Given $\theta$, define the set of parameters that cannot be distinguished from $\theta$ when selecting action $M^*(\theta)$, and for which arm $M^*(\theta)$ is suboptimal:

$$B(\theta) = \{ \lambda \in \Theta : M^*(\theta) \neq \lambda \}$$

We define $\mathcal{X}$ as the set of positive vectors indexed by $\mathcal{M}$ and $\text{kl}(u, v)$ the Kullback-Leibler divergence between Bernoulli distributions of respective means $u$ and $v$, i.e., $\text{kl}(u, v) = u \log(u/v) + (1 - u) \log((1 - u)/(1 - v))$. Finally, for $(\theta, \lambda) \in \Theta^2$, we define the vector $\text{kl}(\theta, \lambda) = (\text{kl}(\theta_i, \lambda_i))_{i \in [d]}$.

We are now ready to derive a regret lower bound that applies to any uniformly good algorithm. An algorithm $\pi$ is uniformly good iff $R^\pi(T) = O(T^{\alpha})$ for all $\alpha > 0$ and all parameters $\theta \in \Theta$. The proof of this result relies on a general result on controlled Markov chains (Graves & Lai, 1997).

**Theorem 1** For all $\theta \in \Theta$, for any uniformly good policy $\pi \in \Pi$,

$$\lim_{T \to \infty} \inf_{\pi} \frac{R^\pi(T)}{\log(T)} \geq C(\theta),$$

where $C(\theta)$ is the optimal value of the optimization problem:

$$\inf_{\theta \in \mathcal{X}} \sum_{M \in \mathcal{M}} x_M (M^*(\theta) - M)^\top \theta$$

s.t. $\left( \sum_{M \in \mathcal{M}} x_M \right)^\top \text{kl}(\theta, \lambda) \geq 1$, $\forall \lambda \in B(\theta)$. (3)

Observe first that optimization problem (5) is a linear program which can be solved for any fixed $\theta$, but its optimal value is not fully explicit. Determining how $C(\theta)$ scales as a function of the problem dimensions $d$ and $m$ is not obvious. Also note that (5) has the following interpretation: assume that (5) has a unique solution $x^*$. Then any uniformly good algorithm must select action $M$ at least $x^*_M \log(T)$ times over the $T$ first rounds. From (Graves & Lai, 1997), we know that there exists an algorithm which is asymptotically optimal, so that its regret matches the lower bound of Theorem 1. However, this algorithm suffers from two problems: it is computationally infeasible for large problems since it involves solving (5) $T$ times, furthermore the algorithm has no finite time performance guarantees, and numerical experiments suggest that its finite time performance on typical problems is rather poor. Further remark that if $\mathcal{M}$ is the set of singleton (classical bandit), Theorem 1 reduces to the Lai-Robbins bound (Lai & Robbins, 1985) and if $\mathcal{M}$ is the set of m-sets (bandit with multiple plays), Theorem 1 reduces to the lower bound of (Anantharam et al., 1987). Finally, Theorem 1 can be generalized in a straightforward manner for when rewards belong to a one-parameter family of distributions (say Gaussian, Exponential, Gamma etc.) by replacing $\text{kl}$ by the appropriate divergence measure.

**A Simplified Lower Bound**

We now study how the regret $C(\theta)$ scales as a function of the problem dimensions $d$ and $m$. To this aim, we present a simplified regret lower bound. Given $\theta$, we say that a set $\mathcal{H} \subset \mathcal{M} \setminus M^*$ has property $P(\theta)$ iff, for all $(M, M') \in \mathcal{H}^2$, $M \neq M'$ we have $M_i M_i'(1 - M_i^*(\theta)) = 0$ for all $i$. We say that $\mathcal{M}$ is $a$-flippable iff and for all $M \in \mathcal{M}$ we have

| Algorithm | Regret |
|-----------|--------|
| Lower Bound (Audibert et al., 2013) | $\Omega(m \sqrt{dT})$, if $d \geq 2m$ |
| COMBAND (Cesa-Bianchi & Lugosi, 2012) | $O(m \sqrt{dT \log |\mathcal{M}| (1 + \frac{2n}{d \min})}$ |
| EXP2 with John’s Exploration (Bubeck et al., 2012) | $O(\sqrt{m^3dT \log (\frac{1}{\Delta})}$ |
| COMBEXP-2 (Theorem 8) | $O(\sqrt{m^3dT \log \mu_{\min}^{-1}(1 + \frac{m^{1/2}d}{2})}$ |

Table 3. Regret of various algorithms for adversarial combinatorial bandits with bandit feedback.
Elements of 

\[ \beta \]

of \( M \)

\[ \begin{align*}
\text{(a) } M^* \\
\text{(b) } (c) \\
\text{(d) } (e) \\
\text{(f) } (g)
\end{align*} \]

Figure 1. Matchings in \( K_{4,4} \): (a) The optimal matching \( M^* \), (b)-

(g) Elements of \( H \).

Remark 1

Corollary 1

Consider \( \beta \) of \( M \)

of \( M \)

\[ \begin{align*}
\text{Theorem 2 Let } H & \text{ be a maximal (inclusion-wise) subset of } M \text{ with property } P(\theta). \text{ Then:} \\
C(\theta) & \geq \sum_{M \in H} \max_{i \in M \setminus M^*} \frac{\beta(\theta)}{\max_{j \in M \setminus M^*} \theta_j} \cdot \frac{1}{\theta_i}, \\
\theta & = \min_{M \neq M^*} \Delta(\theta), \quad \Delta(\theta) = \frac{\Delta M}{\max_{M \neq M^*} \Delta(\theta)}.
\end{align*} \]

Corollary 1 Consider \( 0 \leq x < y \leq 1 \) such that \( \theta = y M^* + x(1 - M^*) \). Assume that \( M \) is a-flippable. Then:

\[ C(\theta) \geq (d - m)/a \cdot \frac{y(1 - y)}{\Delta_{\min}}. \]

Remark 1 (a) For \( m \)-sets, \( M \) is 1-flippable, (b) for matchings in a bipartite graph \( M \) is 2-flippable, (c) if all elements of \( M \) are disjoint \( M \) is \( m \)-flippable. (see Figure 1 and supplement).

Theorem 2 and Corollary 1 provide explicit regret lower bounds. In the worse case, one cannot obtain a regret smaller than \( O(d - m) \Delta_{\min}^{-1} \log(T) \), which is intuitive since \( d - m \) is the number of parameters not observed when selecting the optimal arm. The algorithms proposed below have a regret of \( O(d \Delta_{\min}^{-1} \log(T)) \), which is acceptable since typically, we consider \( m \) much smaller than \( d \). Note that Corollary 1 cannot be improved without further assumptions on \( M \). Indeed, in the case of \( m \)-sets, there exists an algorithm with \( O(d \Delta_{\min}^{-1} \log(T)) \) regret (Anantharam et al., 1987).

4.2. Indexes

CombUCB relies on arm indexes. In general, an index of arm \( M \) in round \( n \), say \( b_M(n) \), should be defined so that \( b_M(n) \geq M^T \theta \) with high probability. Then as for UCB1 and KL-UCB, applying the principle of optimism against uncertainty, a natural way to devise algorithms based on indexes is to select in each round the arm with the highest index.

Under a given algorithm, at time \( n \), define \( t_i(n) = \sum_{n'=1}^{n} X_i(n') M_i(n') \) the number of times basic action \( i \) has been sampled. The empirical mean reward of action \( i \) is then defined as \( \hat{\theta}_i(n) = (1/t_i(n)) \sum_{n'=1}^{n} X_i(n') M_i(n') \) if \( t_i(n) > 0 \) and \( \hat{\theta}_i(n) = 0 \) otherwise. We define the corresponding vectors \( t(n) = (t_i(n))_{i \in [d]} \) and \( \hat{\theta}(n) = (\hat{\theta}_i(n))_{i \in [d]} \).

The indexes we propose are functions of the round \( n \) and of \( \theta(n) \). Our first index for arm \( M \), referred to as \( b_M(n, \hat{\theta}(n)) \) or \( b_M(n) \) for short, is an extension of KL-UCB index. Let \( f(n) = \log(n) + 4m \log(\log(n)) \), \( b_M(n, \hat{\theta}(n)) \) is the optimal value of the following optimization problem:

\[ \max_{q \in \Theta} M^T q \]

s.t. \( (M(n)) \log(\hat{\theta}(n), q) \leq f(n) \).

where we use the convention that for \( v, u \in \mathbb{R}^d \), \( vu = (v_i u_i)_{i \in [d]} \). Index \( b_M(n) \) can be interpreted as performing likelihood tests with vanishing risk (more precisely \( O(n^{-1}) \)) on the average reward of arm \( M \). As we show later, \( b_M(n) \) may be computed efficiently using a line search procedure similar to that used to determine KL-UCB index.

Our second index \( c_M(n, \hat{\theta}(n)) \) or \( c_M(n) \) for short is a generalization of the UCB1 and UCB-tuned indexes:

\[ c_M(n) = M^T \hat{\theta}(n) + \sqrt{\frac{f(n)}{2} \sum_{i=1}^{d} M_i(n) t_i(n)} \]

Note that, in the classical bandit problems with independent arms, i.e., when \( m = 1 \), \( b_M(n) \) reduces to the KL-UCB index (which yields and asymptotically optimal algorithm) and \( c_M(n) \) reduces to the UCB-tuned index.

The next theorem provides generic properties on our indexes. An important consequence of these properties is that the expected number of times where \( b_M(n, \hat{\theta}(n)) \) or \( c_M(n, \hat{\theta}(n)) \) underestimate \( M^* \) is finite, as stated in the corollary below.

Theorem 3 (i) For all \( n \geq 1 \), \( M \in \mathcal{M} \) and \( \tau \in [0, 1]^d \), we have \( b_M(n, \tau) \leq c_M(n, \tau) \).

(ii) There exists \( C_m > 0 \) depending on \( m \) only such that,
for all \( M \in \mathcal{M} \) and \( n \geq 1 \):
\[
\mathbb{P}[b_M(n, \hat{\theta}(n)) \leq M^1 \theta] \leq C_m n^{-1}(\log(n))^{-2}.
\]

**Corollary 2**
\[
\sum_{n \geq 0} \mathbb{P}[b_M^*(n, \hat{\theta}(n)) \leq \mu^*] \leq C_m \sum_{n \geq 0} n^{-1}(\log(n))^{-2} < \infty
\]

Statement (i) in the above theorem is obtained combining Pinsker and Cauchy–Schwarz inequalities. The proof of statement (ii) is based on a concentration inequality on sums of empirical KL divergences proven in (Magureanu et al., 2014). It enables to control the fluctuations of multivariate empirical distributions for exponential families. It should also be observed that indexes \( b_M(n) \) and \( c_M(n) \) can be extended in a straightforward manner to the case of continuous linear bandit problems, where the set of arms is the unit sphere and one wants to maximize the dot product between the arm and an unknown vector. \( b_M(n) \) can also be extended to the case where reward distributions are not Bernoulli but lie in an exponential family (e.g., Gaussian, Exponential, Gamma, etc.), replacing kl by a suitably chosen divergence measure.

A close look at \( c_M(n) \) reveals that indexes proposed in (Chen et al., 2013), (Kveton et al., 2014b), and (Gai et al., 2013) are too conservative to be optimal: there the “confidence bonus” \( \sum_{i=1}^{d} \frac{M_i}{t_i(n)} \) was replaced by (at least) \( m \sum_{i=1}^{d} \frac{M_i}{t_i(n)} \).

**4.2.2. INDEX COMPUTATION**

While index \( c_M(n) \) is explicit, index \( b_M(n) \) is defined as the solution to an optimization problem. We show that in fact, it may be computed by a simple line search. For \( \lambda \geq 0, m \in [0, 1] \) and \( v \in \mathbb{N} \), define:
\[
g(\lambda, m, v) = \frac{(1 - \lambda v + \sqrt{(1 - \lambda v)^2 + 4mv\lambda})}{2}.
\]

Fix \( n, M, \hat{\theta}(n) \) and \( t(n) \). Define \( I' = \{ i : M_i = 1, \hat{\theta}_i(n) \neq 1 \} \), and for \( \lambda > 0 \), define:
\[
F(\lambda) = \sum_{i \in I'} t_i(n) kl(\hat{\theta}_i(n), g(\lambda, \hat{\theta}_i(n), t_i(n))).
\]

**Theorem 4** If \( I' = \emptyset, b_M(n) = ||M||_1 \). Otherwise:

(i) \( \lambda \mapsto F(\lambda) \) is strictly increasing, and \( F(\mathbb{R}^+) = \mathbb{R}^+ \).

(ii) Define \( \lambda^* \) as the unique solution to \( F(\lambda) = f(n) \). Then \( b_M(n) = ||M||_1 - |I'| + \sum_{i \in I'} g(\lambda^*, \hat{\theta}_i(n), t_i(n)) \).

Theorem 4 shows that \( b_M(n) \) can be computed using a line search procedure such as bisection, as this computation amounts to solving the non-linear equation \( F(\lambda) = f(n) \), where \( F \) is a strictly increasing function. The proof of Theorem 4 follows from the KKT conditions and the convexity of the KL-divergence.

**4.2.3. COMBUCB ALGORITHM**

The pseudo-code of COMBUCB is presented in Algorithm 1.

**Algorithm 1 COMBUCB**

for \( n \geq 1 \) do

Select action \( M(n) \in \arg \max_{M \in \mathcal{M}} \xi_M(n) \).

Observe the rewards, and update \( t_i(n) \) and \( \hat{\theta}_i(n), \forall i \).

end for

In the following theorem, we provide a finite time analysis of our two COMBUCB algorithms.

**Theorem 5** The regret under algorithms \( \pi \in \{\text{COMBUCB-1, COMBUCB-2}\} \) satisfies for any time horizon \( T \):
\[
R^\pi(T) \leq dm \Delta_{\min}^{-1} \left(3f(T) + 4m^2 \Delta_{\min}^{-1}\right) + C_m',
\]
where \( C_m' \geq 0 \) does not depend on \( \theta, d \) and \( T \). As a consequence \( R^\pi(T) = O(dm \Delta_{\min}^{-1} \log(T)) \) when \( T \to \infty \).

We also prove the following result:

**Theorem 6** The regret under algorithms \( \pi \in \{\text{COMBUCB-1, COMBUCB-2}\} \) satisfies for any time horizon \( T \):
\[
R^\pi(T) \leq 16d \sqrt{m \Delta_{\min}^{-1}} f(T) + 4dm^3 \Delta_{\min}^{-2} + C_m',
\]
where \( C_m' \geq 0 \) does not depend on \( \theta, d \) and \( T \). As a consequence \( R^\pi(T) = O(d \sqrt{m} \Delta_{\min}^{-1} \log(T)) \) when \( T \to \infty \).

**4.2.4. A NUMERICAL EXAMPLE**

We briefly illustrate the performance under the various algorithms in the case where the set of arms is that of perfect matchings in \( K_{m,m} \) as described at the end of Section 2. We set \( m = 5 \), in which case there are 120 matchings or arms, and \( d = 25 \) basic actions. We also set \( \theta \) such that \( \theta_i = a \) if \( i \in M^* \), and \( \theta_i = b \) otherwise, with \( 0 < b < a < 1 \). Figure 5 presents the regret vs. time horizon under COMBUCB and other existing algorithms. Note that COMBUCB-2 significantly outperforms other al-
Figure 2. Regret of various algorithms for perfect matchings averaged over 50 runs; 95% confidence intervals – $a = 0.7$ and $b = 0.5$.

algorithms. These observations are confirmed for other combinatorial bandit problems, as shown in the supplementary material.

5. Adversarial Combinatorial Bandits

In this section, we study combinatorial bandit problems in the adversarial setting under both semi-bandit and bandit feedbacks. In (Auer et al., 2002b), a regret bound of $O(\sqrt{T})$ is derived for adversarial bandits, where the constant scales as the square root of the number of arms (up to logarithmic factors). In our case, the number of arms grows very rapidly with $d$ and $m$ even in simple cases. We propose algorithms with the same dependence in time as in (Auer et al., 2002b) but with much smaller constants.

We start with the following observation:

$$\max_{M \in \mathcal{M}} X^M = \max_{M \in \mathcal{M}} M^T X = \max_{\mu(M) \geq 0, \sum_{M \in \mathcal{M}} \mu(M) = 1} \sum_{M \in \mathcal{M}} \mu(M) M^T X = \max_{\mu \in Co(\mathcal{M})} \mu^T X,$$

where $Co(\mathcal{M})$ is the convex hull of the set $\mathcal{M}$. We can embed $\mathcal{M}$ in the simplex of distributions in $\mathbb{R}^d$ by multiplying all the entries by $1/m$. Let $\mathcal{P}$ be this scaled version of $Co(\mathcal{M})$. We also define the vector $\mu^0 \in \mathbb{R}^d$ with

$$\mu_i^0 = \frac{1}{m|\mathcal{M}|} \sum_{M \in \mathcal{M}} M_i, \quad \forall i \in [d].$$

Clearly $\mu^0 \in \mathcal{P}$. Furthermore, we define

$$\mu_{\min} = \min_i \mu_i^0 \geq \frac{1}{|\mathcal{M}|}.$$

Inspired by the algorithm of (Helmbold & Warmuth, 2009) for full information setting, we propose two algorithms COMBEXP-1 and COMBEXP-2 for semi-bandit and bandit feedbacks, respectively. The main key difference is the use of the projection onto the simplex of distributions using the KL divergence to determine the distribution at each round (for a thorough description of projection using KL divergence, see Chapter 3, I-projections in (Csiszar & Shields, 2004)). We denote the KL divergence between distributions $q$ and $p$ in $\mathcal{P}$ (or more generally in the simplex of distribution in $\mathbb{R}^d$) by:

$$KL(q, p) = \sum_{i \in [d]} q(i) \log \frac{q(i)}{p(i)},$$

with the usual convention where $p \log \frac{p}{q}$ is defined to be 0 if $p = 0$ and $+\infty$ if $p > q = 0$. By definition, the projection of a distribution $q$ onto a closed convex set $\Xi$ of distributions is the $p^* \in \Xi$ such that

$$KL(p^*, q) = \min_{p \in \Xi} KL(p, q).$$

COMBEXP-1 and COMBEXP-2 algorithms are respectively described in Algorithm 3 and Algorithm 4. We note that the KL projection in these algorithms ensures that $m \mu_{\min} - 1 \in Co(\mathcal{M})$. As a result, there must exist a probability vector $\mu$ such that $m \mu_{\min} - 1 = \sum_{M} \mu(M) M$. This guarantees that the step for computing distribution $p$ always has a solution. In Section 3.4 of the supplementary document, we give an iterative algorithm for computing the projection of distribution $q$ onto $\mathcal{P}$ using KL divergence. Theorem 7 and Theorem 8 respectively give the regret bound for COMBEXP-1 and COMBEXP-2.

It is easy to verify that $\mu^0$ over $[d]$ induces uniform distribution over $\mathcal{M}$. Thus, COMBEXP-2 uses uniform sampling for exploration.

**Theorem 7** We have

$$R^{\text{COMBEXP-1}}(T) \leq \sqrt{2mdT \log \mu_{\min}^{-1}}.$$

Recalling that $\mu_{\min} \geq 1/|\mathcal{M}|$, COMBEXP-1 has a regret of $O(\sqrt{mdT \log |\mathcal{M}|}) = O(m \sqrt{dT \log (d/m)})$ in the generic case. However, when $\mu_{\min}^{-1} = O(\text{poly}(d))$, its regret is $O(\sqrt{mdT \log d})$, which is optimal up to a logarithmic factor.

**Theorem 8** Let $\lambda$ be the smallest nonzero eigenvalue of $\mathbb{E}[MM^T]$, where $M$ is uniformly distributed over $\mathcal{M}$. We have

$$R^{\text{COMBEXP-2}}(T) \leq 2 \sqrt{m^3dT \left(d + m^{1/2}\right) \log \mu_{\min}^{-1}} + \frac{m^{5/2} \log \mu_{\min}^{-1}}{\lambda}.$$
Algorithm 2 COMBEXP-1

**Initialization:** Start with the distribution \( q_0 = \mu^0 \) and \( \eta = \sqrt{\frac{2m \log \mu_{\min}^{-1}}{dT}} \).

for all \( n \geq 1 \) do

Select a distribution \( p_{n-1} \) over \( \mathcal{M} \) such that \( \sum M p_{n-1}(M) = m q_{n-1} \).

Select a random action \( M(n) \) with distribution \( p_{n-1} \).

Observe the reward vector: \( X_i(n) \) for all \( i \in M(n) \).

Construct the vector: \( \tilde{X}_i(n) = \frac{1-X_i(n)}{mq_{n-1}(i)} \) for all \( i \) with \( M_i(n) = 1 \) and all other entries are 0.

Update \( \tilde{q}_n(i) \propto q_{n-1}(i) \exp \left(-\eta \tilde{X}_i(n)\right) \).

Set \( q_n \) to be the projection of \( \tilde{q}_n \) onto the set \( \mathcal{P} \) using the KL divergence.

end for

Algorithm 3 COMBEXP-2

**Initialization:** Start with the distribution \( q_0 = \mu^0 \), and set \( \gamma = \sqrt{\frac{m \log \mu_{\min}^{-1}}{C(M^2d + m)T}} \) and \( \eta = \gamma C \), with \( C = \frac{1}{m^{3/2}} \).

for all \( n \geq 1 \) do

Let \( q_{n-1} = (1-\gamma)q_{n-1} + \gamma \mu^0 \).

Select a distribution \( p_{n-1} \) over \( \mathcal{M} \) such that \( \sum M p_{n-1}(M) = m q_{n-1} \).

Select a random action \( M(n) \) with distribution \( p_{n-1} \).

Observe a reward \( Y_n = \sum_i X_i(n) M_i(n) \).

Let \( \Sigma_{n-1} = \mathbb{E} \left[M M^\top\right] \), where \( M \) has law \( p_{n-1} \). Set \( \tilde{X}(n) = Y_n \Sigma_{n-1}^{-1} M(n) \), where \( \Sigma_{n-1}^{-1} \) is the pseudo-inverse of \( \Sigma_{n-1} \).

Update \( \tilde{q}_n(i) \propto q_{n-1}(i) \exp \left(\eta \tilde{X}_i(n)\right) \).

Set \( q_n \) to be the projection of \( \tilde{q}_n \) onto the set \( \mathcal{P} \) using the KL divergence.

end for

As described in (Cesa-Bianchi & Lugosi, 2012), for most classes of \( \mathcal{M} \), we have \( \frac{m}{\Delta^2} = O(1) \). For these classes, COMBEXP-2 has a regret of \( O(m^{3/2} \sqrt{dT \log \mu_{\min}^{-1}}) \). Moreover, when \( \mu_{\min}^{-1} = O(\text{poly}(d)) \), it yields a regret of \( O(m^{3/2} \sqrt{dT \log d}) \), which is a factor \( m^{1/2} \sqrt{\log d} \) off the lower bound (see Table 3).

6. Conclusion

This paper investigated stochastic and adversarial multiarmed bandit problems where the set of arm exhibits a combinatorial structure. In the stochastic setting, we have provided a generic regret lower bound, that in most cases, scales at least as \( (d-m)\Delta_{\min}^{-1} \log(T) \). Our proposed algorithm for this setting, COMBUCB-2, has a regret bounded by \( O(d \sqrt{\Delta_{\min}^{-1} \log(T)}) \). It might be interesting to try to reduce the gap between this regret guarantee and the regret lower bound. In the stochastic setting, it would be also interesting to investigate the performance of Thompson sampling algorithm. In the adversarial setting, we proposed two algorithms, COMBEXP-1 and COMBEXP-2 for semibandit and bandit feedbacks, respectively. There is a gap between the regret under COMBEXP-2, and the existing regret lower bound in this setting, and we plan to reduce it as much as possible.

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Stochastic and Adversarial Combinatorial Bandits
Supplementary Material

1. Stochastic Combinatorial Bandits: Regret Lower Bounds

1.1. Proof of Theorem 1

To derive regret lower bounds, we apply the techniques used by Graves and Lai (Graves & Lai, 1997) to investigate efficient adaptive decision rules in controlled Markov chains. First we give an overview of their general framework.

Consider a controlled Markov chain $(X_n)_{n \geq 0}$ on a finite state space $S$ with a control set $U$. The transition probabilities given control $u \in U$ are parameterized by $\theta$ taking values in a compact metric space $\Theta$: the probability to move from state $x$ to state $y$ given the control $u$ and the parameter $\theta$ is $p(x, y; u, \theta)$. The parameter $\theta$ is not known. The decision maker is provided with a finite set of stationary control laws $G = \{g_1, \ldots, g_K\}$, where each control law $g_j$ is a mapping from $S$ to $U$: when control law $g_j$ is applied in state $x$, the applied control is $u = g_j(x)$. It is assumed that if the decision maker always selects the same control law $g$, the Markov chain is then irreducible with stationary distribution $\pi_\theta^g$. Now the reward obtained when applying control $u$ in state $x$ is denoted by $r(x, u)$, so that the expected reward achieved under control law $g$ is: $\mu_\theta(g) = \sum_x r(x, g(x)) \pi_\theta^g(x)$. There is an optimal control law given $\theta$ whose expected reward is denoted by $\mu^*_\theta = \max_{g \in G} \mu_\theta(g)$. Now the objective of the decision maker is to sequentially select control laws so as to maximize the expected reward up to a given time horizon $T$. As for MAB problems, the performance of a decision scheme can be quantified through the notion of regret which compares the expected reward to that obtained by always applying the optimal control law.

**Proof.** The parameter $\theta$ takes values in $[0, 1]^d$. The Markov chain has values in $S = \{0, 1\}^d$. The set of controls corresponds to the set of feasible actions $M$, and the set of control laws is also $M$. These laws are constant, in the sense that the control applied by control law $M \in M$ does not depend on the state of the Markov chain, and corresponds to selecting action $M$. The transition probabilities are given as follows: for all $x, y \in S$,

$$p(x, y; M, \theta) = p(y; M, \theta) = \prod_{i \in [d]} p_i(y_i; M, \theta),$$

where for all $i \in [d]$, if $M_i = 0$, $p_i(0; M, \theta) = 1$, and if $M_i = 1$, $p_i(1; M, \theta) = \theta_i^p(1 - \theta_i)^{1 - y_i}$. Finally, the reward $r(y, M)$ is defined by $r(y, M) = M^T y$. Note that the state space of the Markov chain is here finite, and so, we do not need to impose any cost associated with switching control laws (see the discussion on page 718 in (Graves & Lai, 1997)).

We can now apply Theorem 1 in (Graves & Lai, 1997). Note that the KL number under action $M$ is

$$kl^M(\theta, \lambda) = \sum_{i \in [d]} M_i kl(\theta_i, \lambda_i).$$

From Theorem 1 in (Graves & Lai, 1997), we conclude that for any uniformly good rule $\pi$,

$$\lim_{T \to \infty} \inf_{\pi} \frac{R_{\pi}(T)}{\log(T)} \geq C(\theta),$$

where $C(\theta)$ is the optimal value of the following optimization problem:

$$\inf_{x_{M \geq 0, M \in M}} \sum_{M \neq M^*} x_M (\mu^* - \mu_M(\theta)), \quad (5)$$

$$\text{s.t. } \inf_{\lambda \in B(\theta)} \sum_{Q \neq M^*} x_Q kl^Q(\theta, \lambda) \geq 1. \quad (6)$$
The result is obtained by observing that \( B(\theta) = \bigcup_{M \neq M^*} B_M(\theta) \), where
\[
B_M(\theta) = \{ \lambda \in \Theta : M^*_i(\theta)(\theta_i - \lambda_i) = 0, \forall i, \mu^*(\theta) < \mu_M(\lambda) \}.
\]

\[\square\]

1.2. Proof of Theorem 2

The proof proceeds in three steps. In the subsequent analysis, given the optimization problem \( P \), we use \( \text{val}(P) \) to denote its optimal value.

**Step 1.** In this step, first we introduce an equivalent formulation for problem (5) above by simplifying its constraints. We show that constraint (6) is equivalent to:
\[
\text{inf}_{\lambda \in B_M(\theta)} \sum_{M \neq M^*} \text{kl}(\theta, \lambda) \sum_{Q \in \mathcal{M}} Q_i x_Q \geq 1, \ \forall M \neq M^*.
\]

Observe that:
\[
\sum_{Q \neq M^*} x_Q \text{kl}(\theta, \lambda) = \sum_{Q \neq M^*} x_Q \sum_{i \in [d]} Q_i \text{kl}(\theta_i, \lambda_i) = \sum_{i \in [d]} \text{kl}(\theta_i, \lambda_i) \sum_{Q \neq M^*} Q_i x_Q.
\]

Fix \( M \neq M^* \). In view of the definition of \( B_M(\theta) \), we can find \( \lambda \in B_M(\theta) \) such that \( \lambda_i = \theta_i, \forall i \in ([d] \setminus M) \cup M^* \). Thus, for the r.h.s. of the \( M \)-th constraint in (6), we get:
\[
\text{inf}_{\lambda \in B_M(\theta)} \sum_{Q \neq M^*} x_Q \text{kl}(\theta, \lambda) = \text{inf}_{\lambda \in B_M(\theta)} \sum_{i \in [d]} \text{kl}(\theta_i, \lambda_i) \sum_{Q \neq M^*} Q_i x_Q
\]
\[
= \text{inf}_{\lambda \in B_M(\theta)} \sum_{i \in M \setminus M^*} \text{kl}(\theta_i, \lambda_i) \sum_{Q} Q_i x_Q,
\]
and therefore problem (5) can be equivalently written as:
\[
C(\theta) = \inf_{x_M \geq 0, M \in \mathcal{M}} \sum_{M \neq M^*} x_M (\mu^* - \mu_M(\theta)), \quad \text{s.t.} \quad \text{inf}_{\lambda \in B_M(\theta)} \sum_{i \in M \setminus M^*} \text{kl}(\theta_i, \lambda_i) \sum_{Q} Q_i x_Q \geq 1, \ \forall M \neq M^*.
\]

Next, we formulate an LP whose value gives a lower bound for \( C(\theta) \). Define \( \hat{\lambda}(M) = (\hat{\lambda}_i(M), i \in [d]) \) with

\[
\hat{\lambda}_i(M) = \left\{ \begin{array}{ll}
\frac{1}{|M \setminus M^*|} \sum_{j \in M \setminus M^*} \theta_j & \text{if } i \in M \setminus M^* \\
\theta_i & \text{otherwise.}
\end{array} \right.
\]

Clearly \( \hat{\lambda}(M) \in B_M(\theta) \), and therefore:
\[
\text{inf}_{\lambda \in B_M(\theta)} \sum_{i \in M \setminus M^*} \text{kl}(\theta_i, \lambda_i) \sum_{Q} Q_i x_Q \leq \sum_{i \in M \setminus M^*} \text{kl}(\theta_i, \hat{\lambda}_i(M)) \sum_{Q} Q_i x_Q.
\]

Then, we can write:
\[
C(\theta) \geq \inf_{x_M \geq 0} \sum_{M \neq M^*} \Delta_M x_M \quad \text{s.t.} \quad \sum_{i \in M \setminus M^*} \text{kl}(\theta_i, \hat{\lambda}_i(M)) \sum_{Q} Q_i x_Q \geq 1, \ \forall M \neq M^*.
\]
For any $M \neq M^*$ introduce: $g_M = \max_{i \in M \setminus M^*} \text{kl}(\theta_i, \hat{\lambda}_i(M))$. Now we form $P_1$ as follows:

$$P_1: \inf_{x \geq 0} \sum_{M \neq M^*} \Delta^M x_M \tag{11}$$

subject to

$$\sum_{i \in M \setminus M^*} \sum_{Q} Q_i x_Q \geq \frac{1}{g_M}, \quad \forall M \neq M^*. \tag{12}$$

Observe that $C(\theta) \geq \text{val}(P_1)$ since the feasible set of problem (9) is contained in that of $P_1$.

**Step 2.** In this step, we formulate an LP to give a lower bound for $\text{val}(P_1)$. To this end, for any suboptimal edge $i \in [d]$, we define $z_i = \sum_M M_i x_M$. Further, we let $z = [z_i, i \in [d]]$. Next, we represent the objective of $P_1$ in terms of $z$, and give a lower bound for it as follows:

$$\sum_{M \neq M^*} \Delta^M x_M = \sum_{M \neq M^*} x_M \sum_{i \in M \setminus M^*} \Delta^M \frac{M_i}{|M \setminus M^*|} = \sum_{M \neq M^*} x_M \sum_{i \in M \setminus M^*} \Delta^M M_i \geq \min_{M \neq M^*} \frac{\Delta^M}{|M \setminus M^*|} \sum_{i \in [d] \setminus M^*} \sum_{M' \neq M^*} M' x_M = \min_{M \neq M^*} \frac{\Delta^M}{|M \setminus M^*|} \sum_{i \in [d] \setminus M^*} z_i = \beta(\theta) \sum_{i \in [d] \setminus M^*} z_i.$$

Then, defining

$$P_2: \inf_{z \geq 0} \beta(\theta) \sum_{i \in [d] \setminus M^*} z_i \quad \text{s.t.} \quad \sum_{i \in M \setminus M^*} z_i \geq \frac{1}{g_M}, \quad \forall M \neq M^*,$$

yields: $\text{val}(P_1) \geq \text{val}(P_2)$.

**Step 3.** Introduce set $\mathcal{H}$ satisfying property $P(\theta)$ as stated in Section 4. Now define

$$\mathcal{Z} = \left\{ z \in \mathbb{R}_+^d : \sum_{i \in M \setminus M^*} z_i \geq \frac{1}{g_M}, \quad \forall M \in \mathcal{H} \right\},$$

and

$$P_3: \inf_{z \in \mathcal{Z}} \beta(\theta) \sum_{i \in [d] \setminus M^*} z_i.$$

Observe that $\text{val}(P_2) \geq \text{val}(P_3)$ since the feasible set of $P_2$ is contained in $\mathcal{Z}$. 
It can be easily seen that:

\[ \text{val(P3)} = \sum_{M \in \mathcal{H}} \frac{\beta(\theta)}{g_M} \geq \sum_{M \in \mathcal{H}} \frac{\beta(\theta)}{\max_{i \in M \setminus M^*} \text{kl}(\theta_i, \lambda_i(M))} = \sum_{M \in \mathcal{H}} \frac{\beta(\theta)}{\max_{i \in M \setminus M^*} \text{kl}(\theta_i, \sum_{j \in M \setminus M^*} \theta_j)} . \]

The proof is completed by observing that: \( C(\theta) \geq \text{val(P1)} \geq \text{val(P2)} \geq \text{val(P3)} \).

1.3. Proof of Corollary 1

Assume that \( \theta = yM^* + x(1 - M^*) \). Hence \( \mu_* = my \) and for all \( M; M^T \theta = y(m - |M^*/M|) + x|M^*/M| \). Hence \( \mu_* - M^T \theta = |M^*/M|(y - x) \) so that \( \beta(\theta) = y - x \). Also \( \Delta_{\min} = \min_{M \in \mathcal{M}} |M^*/M|(y - x) \geq a(y - x) \) since \( \mathcal{M} \) is \( a \)-flippable. Hence \( (y - x) \leq \Delta_{\min}/a \). Consider \( i \) such that \( M_i^* = 0 \), and \( M \neq M^* \). Then:

\[ \text{kl} \left( \theta_i, |M \setminus M^*|^{-1} \sum_{j \in M \setminus M^*} \theta_j \right) = \text{kl}(x, y) \leq \frac{(x - y)^2}{y(1 - y)} . \]

We have used the fact that for all \((p, q) \in [0, 1]^2\), \( \text{kl}(p, q) \leq \frac{(p - q)^2}{q(1 - q)} \). This inequality comes from the fact that \( \log(u) \leq u - 1 \), \( u \geq 0 \). Define \( I = \{i : M_i^* = 0\} \). We have \( |I| = d - m \geq a[(d - m)/a] \). Define \( K = [(d - m)/a] \) and define \( I_1, \ldots, I_K \) disjoint subsets of \( I \) with \( |I_k| = a \) for all \( k \). We have assumed that \( \mathcal{M} \) is \( a \)-flippable, so for each \( k \), there exists \( M^k \in \mathcal{M} \) such that \( I_k \subset M^k \) and \( (M^k \setminus A) \subset M^* \). Now define \( \mathcal{H} = \{M^1, \ldots, M^K\} \). Applying the first statement of the theorem and using the above we get:

\[ C(\theta) \geq |\mathcal{H}| \beta(\theta) y(1 - y)^2 = |(d - m)/a| (y - x) \frac{y(1 - y)^2}{(x - y)^2} \geq a[(d - m)/a] \frac{y(1 - y)}{\Delta_{\min}} , \]

which is the announced result.

1.4. Examples of Scaling of the Lower Bound

1.4.1. Spanning trees

Consider the problem of finding the minimum spanning tree in a complete graph \( K_N \). This corresponds to letting \( \mathcal{M} \) be the set of all spanning trees in \( K_N \), where \( |\mathcal{M}| = N^{N-2} \) (Cayley’s formula). In this case, we have \( d = \binom{N}{2} = \frac{N(N-1)}{2} \), which is the number of edges of \( K_N \), and \( m = N - 1 \). A maximal subset \( \mathcal{H} \) of \( \mathcal{M} \) satisfying property \( P(\theta) \) can be constructed by composing all spanning trees that differ from the optimal tree by one edge only, see Figure 3. In this case, \( \mathcal{H} \) has \( d - m = \frac{(N-1)(N-2)}{2} \) elements.

1.4.2. Routing in a grid

Now we give an example, in which \( |\mathcal{H}| \) is not scaling as \( \Omega(d) \). Consider routing in an \( N \)-by-\( N \) directed grid, whose topology is shown in Figure 4(a) where the source (resp. destination) node is shown in red (resp. blue). Here \( \mathcal{M} \) is the

Figure 3. Spanning trees in \( K_5 \): (a) The optimal spanning tree \( M^* \). (b)-(g) Elements of \( \mathcal{H} \).
Figure 4. Routing in a grid: (a) Grid topology with source (red) and destination (blue) nodes, (b) Optimal path $M^*$, (c)-(d) Elements of $H$.

set of all $\binom{2N-2}{N-1}$ paths with $m = 2(N - 1)$ edges. We further have $d = 2N(N - 1)$. In this example, elements of any maximal set $H$ satisfying $P(\theta)$ do not cover all basic actions. For instance, for the grid shown in Figure 4(a), the two edges incident to the right lower corner do not appear in any arm in $H$. It can be easily verified that in this case, $|H|$ scales as $N$ rather than $N^2 = d$. 
2. Stochastic Combinatorial Bandits: Regret Analysis of COMBUCB

We use the convention that for $v, u \in \mathbb{R}^d$, $vu = (v_i u_i)_{i \in [d]}$.

2.1. A concentration inequality

We first recall Lemma 1, a concentration inequality derived in (Magureanu et al., 2014)[Theorem 2].

Lemma 1 There exists a number $C_m > 0$ depending only on $m$ such that, for all $M$ and all $n$:

$$\mathbb{P}[(Mt(n))^\top kl(\hat{\theta}(n), \theta) \geq f(n)] \leq C_m n^{-1}(\log(n))^{-2}$$

2.2. Proof of Theorem 3

First statement

Consider $q \in \Theta$, and apply the Cauchy-Schwartz inequality:

$$M^\top (q - \hat{\theta}(n)) = \sum_{i=1}^{d} \sqrt{t_i(n)}(q_i - \hat{\theta}_i(n)) \leq \sqrt{\sum_{i=1}^{d} M_i t_i(n)(q_i - \hat{\theta}_i(n))^2} \leq \sqrt{\sum_{i=1}^{d} M_i t_i(n)}$$

By Pinsker’s inequality, for all $(p, q) \in [0, 1]^2$ we have $2(p - q)^2 \leq kl(p, q)$ so that:

$$M^\top (q - \hat{\theta}(n)) \leq \sqrt{(Mt(n))^\top kl(\hat{\theta}(n), q)} \leq \sqrt{\sum_{i=1}^{d} M_i t_i(n)}$$

Hence, $(Mt(n))^\top kl(\hat{\theta}(n), q) \leq f(n)$ implies:

$$M^\top q = M^\top \hat{\theta}(n) + M^\top (q - \hat{\theta}(n)) \leq M^\top \hat{\theta}(n) + \sqrt{\frac{f(n)}{2}} \sum_{i=1}^{d} \frac{M_i}{t_i(n)} = c_M(n),$$

so that, by definition of $b_M(n)$, we have $b_M(n) \leq c_M(n)$.

Second statement If $(Mt(n))^\top kl(\hat{\theta}(n), \theta) \leq f(n)$ then, by definition of $b_M(n)$ we have $b_M(n) \geq M^\top \theta$. Therefore, using Lemma 1, there exists $C_m$ such that for all $n$ we have:

$$\mathbb{P}[b_M(n) < M^\top \theta] \leq \mathbb{P}[(Mt(n))^\top kl(\hat{\theta}(n), \theta) \geq f(n)] \leq C_m n^{-1}(\log(n))^{-2},$$

which concludes the proof.

2.3. Proof of Theorem 4

We recall the following facts about the KL divergence $kl$, for all $p \in [0, 1]$:

(i) $q \mapsto kl(p, q)$ is strictly convex on $[0, 1]$ and attains its minimum at $p$, with $kl(p, p) = 0$.

(ii) Its derivative with respect to the second parameter $q \mapsto kl'(p, q) = \frac{q-p}{q(1-q)}$ is strictly increasing on $(p, 1)$.

(iii) For $p < 1$, we have $kl(p, q) \xrightarrow{q \to 1^-} \infty$ and $kl'(p, q) \xrightarrow{q \to 1^-} \infty$.

Consider $M$ and $n$ fixed throughout the proof. Define $I = \{i \in [d] : M_i = 1\}$ and $I' = \{i \in I : \hat{\theta}_i(n) \neq 1\}$. Consider $q^* \in \Theta$ the optimal solution of optimization problem:

$$\max_{q \in \Theta} M^\top q$$

s.t. $(Mt(n))^\top kl(\hat{\theta}(n), q) \leq f(n)$.
so that $b_M(n) = M^T q^*$. Consider $i \notin I$, then $M^T q$ does not depend on $q_i$ and from (i) we get $q_i = \hat{\theta}_i(n)$. Now consider $i \in I$. From (i) we get that $1 \geq q^*_i \geq \hat{\theta}_i(n)$. Hence $q^*_i = 1$ if $\hat{\theta}_i(n) = 1$. If $I'$ is empty, then $q^*_i = 1$ for all $i \in I$, so that $b_M(n) = \|M\|_1$.

Consider the case where $I' \neq \emptyset$. From (iii) and the fact that $t(n)^T \text{kl}(\hat{\theta}(n), q^*) < \infty$ we get $\hat{\theta}_i(n) \leq q^*_i < 1$. From the Karush-Kuhn-Tucker conditions, there exists $\lambda^* > 0$ such that for all $i \in I'$:

$$1 = \lambda^* t_i(n) \text{kl}'(\hat{\theta}_i(n), q^*_i).$$

For $\lambda > 0$ define $\bar{\theta}_i(n) = \min\{l^i : l^i < 1\}$ a solution to the equation:

$$1 = \lambda t_i(n) \text{kl}'(\hat{\theta}_i(n), \bar{\theta}_i(n)).$$

From (i) we have that $\lambda \mapsto \bar{\theta}_i(n)$ is uniquely defined, is strictly decreasing and $\bar{\theta}_i(n) < \bar{\theta}_i(n) < 1$. From (iii) we get that $\bar{\theta}_i(n) = [\hat{\theta}_i(n), 1]$. Define the function:

$$F(\lambda) = \sum_{i \in I'} t_i(n) \text{kl}'(\hat{\theta}_i(n), \bar{\theta}_i(n)).$$

From the reasoning below, $F$ is well defined, strictly increasing and $F(\mathbb{R}^+) = \mathbb{R}^+$. Therefore, $\lambda^*$ is the unique solution to $F(\lambda^*) = f(n)$, and $q^*_i = \bar{\theta}_i(\lambda^*)$. Furthermore, replacing $\text{kl}'$ by its expression we obtain the quadratic equation:

$$\bar{\theta}_i(\lambda^2 + \bar{\theta}_i(n)(\lambda t_i(n) - 1) - \lambda t_i(n) \hat{\theta}_i(n) = 0$$

and solving for $\bar{\theta}_i(\lambda)$ we obtain that $\bar{\theta}_i(\lambda) = g(\lambda, \hat{\theta}_i(n), t_i(n))$ which concludes the proof.

\[\square\]

2.4. Proof of Theorem 5

Define the following sets of instants:

$$A = \{n \geq 1 : \xi_M(n) < \mu^*\},$$

$$B_i = \{n \in [T] : M_i(n) = 1, |\hat{\theta}_i(n) - \theta_i| \geq m^{-1} \Delta_{\min}/2\}, B = \bigcup_{i=1}^d B_i,$$

$$C_i = \{n \in [T] \setminus (A \cup B) : M(n) \neq M^*, M_i(n) = 1, t_i(n) = \min_{j : M_j(n) = 1} t_j(n)\}.$$

Consider $n$ such that $M(n) \neq M^*$. Then $n \in (A \cup B \cup C)$. Indeed, either $n \in A$, or $n \in B$, or $n \in C_i$ for all $i \in \arg\min_{j : M_j(n) = 1} t_j(n)$. Since $\|M\|_1 \leq m \forall M$, the regret is upper bounded by:

$$R^\pi(T) \leq m E[|A|] + m E[|B|] + \sum_{i=1}^d E\left[\sum_{n \in C_i} (\mu^* - M(n)\theta)\right].$$

We will prove the following inequalities: (i) $E[|A|] \leq m^{-1} C'_m$, with $C'_m \geq 0$ independent of $\theta$, $d$ and $T$ (ii) $E[|B|] \leq 4dm^2 \Delta_{\min}^{-2}$ and (iii) $\sum_{n \in C_i} (\mu^* - M(n)\theta) \leq 3f(T)m \Delta_{\min}^{-1} \text{a.s.}$ for all $i \in [d]$. Hence, as announced:

$$R^\pi(T) \leq \sum_{i=1}^d E[|B_i|] \leq 4dm^2 \Delta_{\min}^{-2} + C'_m.$$

Inequality (i): From Theorem 3, there exists $C_m$ (independent of $\theta$, $d$ and $T$) such that

$$E[|A|] = \sum_{n \geq 1} \mathbb{P}(|\xi_M(n) < M^T \theta|) \leq \sum_{n \geq 1} C_m n^{-1}(\log(n))^{-2} \equiv m^{-1} C'_m < \infty.$$

Inequality (ii): Consider $n \in B_i$ fixed and define $s = \sum_{n' \in B_i} \mathbb{1}\{n' \in B_i\}$. We have that $n' \in B_i$ implies $M_i(n') = 1$, hence $t_i(n) \geq s$. Therefore, applying (Combes & Proutiere, 2014)[Lemma B.1], we have that $E[|B_i|] \leq 4m^2 \Delta_{\min}^{-2}$. Using a union bound: $E[|B|] \leq \sum_{i=1}^d E[|B_i|] \leq 4dm^2 \Delta_{\min}^{-2}$.

Inequality (iii): Consider $n \in C_i$ fixed and define $s = \sum_{n' \in C_i} \mathbb{1}\{n' \in C_i\}$. We have the following facts:
• $c_{M(n)}(n) \geq \xi_{M(n)}(n) \geq \xi_{M^*}(n) \geq \mu^*$ using Theorem 3, and since $M(n)$ is selected and $n \notin A$.
• $t_j(n) \geq s$ for all $j$ such that $M_j(n) = 1$. Since $n' \in C_i$ implies $M_i(n') = 1$, we have $t_i(n) \geq s$. Also $n \in C_i$ implies that $t_i(n) = \min_{j:M_i(n)=1} t_j(n)$.
• $M(n) \geq \theta(n) \leq M(n)^T \theta + \Delta_{\min}/2$ since $n \notin B_i$ implies $\theta_j(n) \leq \theta_j + \Delta_{\min}/(2n)$ and $\|M(n)\|_1 \leq m$.

Putting these together we obtain:

$$
\mu^* \leq c_{M(n)}(n) = M(n)^T \theta(n) + \sqrt{\frac{f(n)}{2} \sum_{j=1}^{d} \frac{M_j}{t_j(n)}} \leq M(n)^T \theta + \frac{\Delta_{\min}}{2} + \sqrt{\frac{f(T)m}{2s}}.
$$

Since $M(n) \neq M^*$ we have

$$
\Delta_{\min} \leq \mu^* - M(n)^T \theta \leq \frac{\Delta_{\min}}{2} + \sqrt{\frac{f(T)m}{2s}}.
$$

We deduce that $s \leq \bar{s} = 2f(T)m\Delta_{\min}^{-2}$, so $|C_i| \leq \bar{s}$. Furthermore:

$$
\sum_{n \in C_i} (\mu^* - M(n)^T \theta) \leq \bar{s} \frac{\Delta_{\min}}{2} + \sqrt{\frac{f(T)m}{2s}} \leq \bar{s} \frac{\Delta_{\min}}{2} + 2\sqrt{s} \frac{f(T)m}{\Delta_{\min}} = \frac{3f(T)m}{\Delta_{\min}}.
$$

where we used the inequality $\sum_{x=1}^{\bar{s}} x^{-1/2} \leq \int_{0}^{\bar{s}} x^{-1/2}dx = 2\sqrt{\bar{s}}$. This concludes the proof. \qed

2.5. Proof of Theorem 6

To prove Theorem 6, we borrow some ideas from proof of (Kveton et al., 2014b)[Theorem 3].

For any $n \in \mathbb{N}$, $s \in \mathbb{R}^d$, and $M \in \mathcal{M}$ define $h_{n,s,M} = \sqrt{\frac{f(n)}{2} \sum_{i=1}^{d} \frac{M_i}{s_i}}$, and introduce the following events:

$$
G_n = \{(M^*t(n))^T \text{kl}(\hat{\theta}(n), \theta) > f(n)\},
$$

$$
H_{i,n} = \{M_i(n) = 1, |\hat{\theta}_i(n) - \theta_i| \geq m^{-1}\Delta_{\min}/2\}, \quad H_n = \cup_{i=1}^{d} H_{i,n},
$$

$$
F_n = \{\Delta_{M(n)} \leq 2h_{T,t(n),M(n)}\}.
$$

Then the regret can be bounded as:

$$
R^s(T) = \mathbb{E}[\sum_{n=1}^{T} \Delta_{M(n)}] \leq \mathbb{E}[\sum_{n=1}^{T} \Delta_{M(n)}\mathbb{1}{G_n}] + \mathbb{E}[\sum_{n=1}^{T} \Delta_{M(n)}\mathbb{1}{H_n}] + \mathbb{E}[\sum_{n=1}^{T} \Delta_{M(n)}\mathbb{1}{\overline{G_n}}\mathbb{1}{\overline{H_n}}]
$$

$$
\leq m\mathbb{E}[\sum_{n=1}^{T} \mathbb{1}{G_n}] + m\mathbb{E}[\sum_{n=1}^{T} \mathbb{1}{H_n}] + \mathbb{E}[\sum_{n=1}^{T} \Delta_{M(n)}\mathbb{1}{\overline{G_n}}\mathbb{1}{\overline{H_n}}],
$$

since $\Delta_{M(n)} \leq m$.

Next we show that for any $n$, $\Delta_{M(n)}\mathbb{1}{\overline{G_n}}\mathbb{1}{\overline{H_n}} \leq \Delta_{M(n)}\mathbb{1}{F_n}$. Recall that $c_{M(n)} \geq b_{M(n)}$ for any $M$ and $n$ (Theorem 3). Moreover, if $\overline{G_n}$ holds, we have $(M^*t(n))^T \text{kl}(\hat{\theta}(n), \theta) \leq f(n)$, which by definition of $b_M$ implies: $b_{M^*}(n) \geq M^*^T \theta$. Hence we have:

$$
\Delta_{M(n)}\mathbb{1}{\overline{G_n}}\mathbb{1}{\overline{H_n}} = \Delta_{M(n)}\mathbb{1}{\overline{G_n},\overline{H_n},\xi_{M(n)}(n) \geq \xi_{M^*}(n)}
$$

$$
\leq \Delta_{M(n)}\mathbb{1}{\overline{H_n},c_{M(n)}(n) \geq M^*^T \theta}
$$

$$
= \Delta_{M(n)}\mathbb{1}{\overline{H_n},M(n)^T \theta(n) + h_{n,t(n),M(n)} \geq M^*^T \theta}
$$

$$
\leq \Delta_{M(n)}\mathbb{1}{M(n)^T \theta + \Delta_{M(n)}/2 + h_{n,t(n),M(n)} \geq M^*^T \theta}
$$

$$
= \Delta_{M(n)}\mathbb{1}{2h_{t(n),M(n)} \geq \Delta_{M(n)}}
$$

$$
= \Delta_{M(n)}\mathbb{1}{F_n}.
$$
where the second inequality follows from the fact that event $\overline{c_n}$ implies: $M(n)^T \tilde{\theta}(n) \leq M(n)^T \theta + \Delta_{\min}/2 \leq M(n)^T \theta + \Delta_{M(n)}/2$.

Hence, the regret is upper bounded by:

$$R^*(T) \leq mE[\sum_{n=1}^{T} \mathbb{1}\{G_n\}] + mE[\sum_{n=1}^{T} \mathbb{1}\{H_n\}] + \mathbb{E}[\sum_{n=1}^{T} \Delta_{M(n)} \mathbb{1}\{F_n\}].$$

We will prove the following inequalities: (i) $E[\sum_{n=1}^{T} \mathbb{1}\{G_n\}] \leq m^{-1}C'_m$, with $C'_m \geq 0$ independent of $\theta$, $d$, and $T$, (ii) $E[\sum_{n=1}^{T} \mathbb{1}\{H_n\}] \leq 4dm^2 \Delta_{\min}^{-2}$, and (iii) $E[\sum_{n=1}^{T} \Delta_{M(n)} \mathbb{1}\{F_n\}] \leq 16d\sqrt{m} \Delta_{\min}^{-1} f(T)$.

Hence as announced:

$$R^*(T) \leq 16d\sqrt{m} \Delta_{\min}^{-1} f(T) + 4dm^2 \Delta_{\min}^{-2} + C'_m.$$

**Inequality (i):** An application of Lemma 1 gives

$$E[\sum_{n=1}^{T} \mathbb{1}\{G_n\}] = \sum_{n=1}^{T} \mathbb{P}[((M^* t(n))^T kl(\tilde{\theta}(n), \theta) > f(n))] \leq \sum_{n \geq 1} C_m n^{-1} (\log(n))^{-2} = m^{-1} C'_m < \infty.$$

**Inequality (ii):** (The same as the proof of “Inequality (ii)” in the proof of Theorem 5.)

**Inequality (iii):** Let $\ell > 0$. For any $n$ introduce the following events:

- $S_n := \{i \in M(n) : t_i(n) \leq 4mf(T) \Delta_{M(n)}^{-2}\}$,
- $A_n := \{|S_n| \geq \ell\}$,
- $B_n := \{|S_n| < \ell, \exists i \in M(n) : t_i(n) \leq 4\ell f(T) \Delta_{M(n)}^{-2}\}$.

We claim that for any $n$ such that $M(n) \neq M^*$, we have $F_n \subset (A_n \cup B_n)$. To prove this, we show that when $F_n$ holds and $M(n) \neq M^*$, the event $\overline{A_n \cup B_n}$ cannot happen. Let $n$ be a time instant such that $A_n \neq M^*$ and $F_n$ holds, and assume that $\overline{A_n \cup B_n} = \{|S_n| < \ell, \forall i \in M(n) : t_i(n) > 4\ell f(T) \Delta_{M(n)}^{-2}\}$ happens. Then $F_n$ implies:

$$\Delta_{M(n)} \leq 2h_{T_t(n), M(n)} = 2\sqrt{\frac{f(T)}{2}} \sqrt{\sum_{i \in [d] \setminus S_n} \frac{M_i}{t_i(n)} + \sum_{i \in S_n} \frac{M_i}{t_i(n)}}$$

$$< 2\sqrt{\frac{f(T)}{2}} \sqrt{m \Delta_{M(n)}^2 \frac{\Delta_{M(n)}}{4mf(T)}} + |S_n| \Delta_{M(n)}^2 \frac{4mf(T)}{4mf(T)} < \Delta_{M(n)}^2,$$

where the last inequality uses the observation that $\overline{A_n \cup B_n}$ implies $|S_n| < \ell$. Clearly, (13) is a contradiction. Thus $F_n \subset (A_n \cup B_n)$ and consequently:

$$\sum_{n=1}^{T} \Delta_{M(n)} \mathbb{1}\{F_n\} \leq \sum_{n=1}^{T} \Delta_{M(n)} \mathbb{1}\{A_n\} + \sum_{n=1}^{T} \Delta_{M(n)} \mathbb{1}\{B_n\}.$$

To further bound the r.h.s. of the above, we introduce the following events for any $i$:

- $A_{i,n} := A_n \cap \{i \in M(n), t_i(n) \leq 4mf(T) \Delta_{M(n)}^{-2}\}$,
- $B_{i,n} := B_n \cap \{i \in M(n), t_i(n) \leq 4\ell f(T) \Delta_{M(n)}^{-2}\}$.

It is noted that:

$$\sum_{i \in [d]} \mathbb{1}\{A_{i,n}\} = \mathbb{1}\{A_n\} \sum_{i \in [d]} \mathbb{1}\{i \in S_n\} = |S_n| \mathbb{1}\{A_n\} \geq \mathbb{1}\{A_n\},$$
and hence: $\mathbb{1}\{A_{n}\} \leq \frac{1}{T} \sum_{i \in [d]} \mathbb{1}\{A_{i,n}\}$. Moreover $\mathbb{1}\{B_{n}\} \leq \sum_{i \in [d]} \mathbb{1}\{B_{i,n}\}$. Let each basic action $i$ belong to $K_i$ suboptimal arms, ordered based on their gaps as: $\Delta^{i,1} \geq \cdots \geq \Delta^{i,K_i} > 0$. Also define $\Delta^{i,0} = \infty$. Plugging the above inequalities into (14), we have

\[
\sum_{n=1}^{T} \Delta_{M(n)} \mathbb{1}\{F_n\} \leq \sum_{n=1}^{T} \sum_{i=1}^{d} \frac{\Delta_{M(n)}}{\ell} \mathbb{1}\{A_{i,n}\} + \sum_{n=1}^{T} \sum_{i=1}^{d} \Delta_{M(n)} \mathbb{1}\{B_{i,n}\}
\]

\[
= \sum_{n=1}^{T} \sum_{i=1}^{d} \frac{\Delta_{M(n)}}{\ell} \mathbb{1}\{A_{i,n}, M(n) \neq M^*\} + \sum_{n=1}^{T} \sum_{i=1}^{d} \Delta_{M(n)} \mathbb{1}\{B_{i,n}, M(n) \neq M^*\}
\]

\[
\leq \sum_{i=1}^{d} \sum_{n=1}^{T} \sum_{k \in [K_i]} \frac{\Delta^{i,k}}{\ell} \mathbb{1}\{A_{i,n}, M(n) = k\} + \sum_{n=1}^{T} \sum_{i=1}^{d} \sum_{k \in [K_i]} \Delta^{i,k} \mathbb{1}\{B_{i,n}, M(n) = k\}
\]

\[
\leq \sum_{i=1}^{d} \sum_{n=1}^{T} \sum_{k \in [K_i]} \frac{\Delta^{i,k}}{\ell} \mathbb{1}\{i \in M(n), t_i(n) \leq 4mf(T)(\Delta^{i,k})^{-2}, M(n) = k\}
\]

\[
+ \sum_{i=1}^{d} \sum_{n=1}^{T} \sum_{k \in [K_i]} \Delta^{i,k} \mathbb{1}\{i \in M(n), t_i(n) \leq 4\ell f(T)(\Delta^{i,k})^{-2}, M(n) = k\}
\]

\[
\leq \frac{8df(T)}{\Delta_{\min}} \left( \frac{m}{\ell} + \ell \right),
\]

where in the last inequality we used the fact for any $i$ with $K_i \geq 1$ and $C > 0$ that does not depend on $n$, we have:

\[
\sum_{n=1}^{T} \sum_{k=1}^{K_i} \mathbb{1}\{i \in M(n), t_i(n) \leq C(\Delta^{i,k})^{-2}, M(n) = k\} \Delta^{i,k}
\]

\[
= \sum_{n=1}^{T} \sum_{k=1}^{K_i} \mathbb{1}\{i \in M(n), t_i(n) \in (C(\Delta^{i,j-1})^{-2}, C(\Delta^{i,j})^{-2}], M(n) = k\} \Delta^{i,k}
\]

\[
\leq \sum_{n=1}^{T} \sum_{k=1}^{K_i} \mathbb{1}\{i \in M(n), t_i(n) \in (C(\Delta^{i,j-1})^{-2}, C(\Delta^{i,j})^{-2}], M(n) = k\} \Delta^{i,j}
\]

\[
\leq \sum_{n=1}^{T} \sum_{k=1}^{K_i} \mathbb{1}\{i \in M(n), t_i(n) \in (C(\Delta^{i,j-1})^{-2}, C(\Delta^{i,j})^{-2}], M(n) \neq M^* \} \Delta^{i,j}
\]

\[
\leq \frac{C}{\Delta_{\min}} + \sum_{j=2}^{K_i} (C(\Delta^{i,j})^{-2} - (\Delta^{i,j-1})^{-2}) \Delta^{i,j} \leq \frac{C}{\Delta_{\max}} + \int_{\Delta^{i,j} \geq \Delta^{i,j-1}} C x^{-2} dx \leq \frac{2C}{\Delta_{\max}},
\]

The proof is completed by choosing $\ell = \sqrt{m}$. \qed

### 2.6. Numerical Experiments

In this section, we examine the performance of COMBUCB against existing ones through numerical experiments for some classes of $\mathcal{M}$. For COMBUCB we have ignored the term $\log(\log(n))$.

#### 2.6.1. Experiment 1: Matching

In our first experiment, we consider matching problem described in Section 5.3.1(a) with $N_1 = N_2 = 5$, which corresponds to $d = 5^2 = 25$ and $m = 5$. We also set $\theta$ such that $\theta_i = a$ if $i \in M^*$, and $\theta_i = b$ otherwise, with $0 < b < a < 1$.

Figure 5(a)-(b) depict the regret of various algorithms for the case of $a = 0.7$ and $b = 0.5$. The curves in Figure 5(a) are shown with % 95 confidence interval. We observe that COMBUCB-2 attains the best regret amongst the others. Moreover,
CUBUCB-1 achieves a regret that is significantly better than that of CUCB and LLR, and slightly worse than that of CUBUCB-2.

Figures 6(a)-(b) portray the regret of various algorithms for the case of \( a = 0.95 \) and \( b = 0.3 \). The difference compared to the former case is that CUBUCB-2 significantly outperforms CUBUCB-1. The reason is in the former case, mean rewards of the most of the basic actions were close to 0.5, for which UCB-type algorithms give close to optimal performance. On the other hand, when mean rewards are not close to 0.5, as for the previous case, we expect an increased gap between performance of CUBUCB-1 and CUBUCB-2.

### 2.6.2. EXPERIMENT 2: SPANNING TREES

In the second experiment, we consider spanning trees problem described in Section 1.4.1 for the case of \( N = 5 \). In this case, we have \( d = \binom{5}{2} = 10 \), \( m = 4 \), and \( |\mathcal{M}| = 5^3 = 125 \).

Figure 7 portrays the regret of various algorithms with 95% confidence intervals, with \( \Delta_{\text{min}} = 0.54 \). Our algorithms significantly outperform CUCB and LLR. In line with the discussion in Section 5.3.2 in the main document, in this case the regret of CUCB takes the form \( O\left( \frac{d}{\Delta_{\text{min}}} \log(T) \right) = O\left( \frac{N^2}{\Delta_{\text{min}}} \log(T) \right) \) on the account of (Kveton et al., 2014a). This experiment supports our conjecture that when \( \mathcal{M} \) is a matroid, CUBUCB admits a regret bound of \( O\left( \frac{d}{\Delta_{\text{min}}} \log(T) \right) \).
3. Supplementary Materials for Adversarial Combinatorial Bandits

3.1. Proof of Theorem 6

We first prove the following result:

**Lemma 2** We have for any $q \in \mathcal{P}$,

$$\sum_{n=1}^{T} q_{n-1}^\top \tilde{X}(n) - \sum_{n=1}^{T} q_{n}^\top \tilde{X}(n) \leq \frac{\eta}{2} \sum_{n=1}^{T} q_{n-1}^\top \tilde{X}^2(n) + \frac{\text{KL}(q,q_0)}{\eta},$$

where $\tilde{X}^2(n)$ is the vector that is the coordinate-wise square of $\tilde{X}(n)$.

**Proof:** We have

$$\text{KL}(q, \tilde{q}_n) - \text{KL}(q, q_{n-1}) = \sum_{i \in [d]} q(i) \log \frac{q_{n-1}(i)}{\tilde{q}_n(i)} = \eta \sum_{i \in [d]} q(i) \tilde{X}_i(n) + \log Z_n,$$

with

$$\log Z_n = \log \sum_{i \in [d]} q_{n-1}(i) \exp\left(-\eta \tilde{X}_i(n)\right) \leq \log \sum_{i \in [d]} q_{n-1}(i) \left(1 - \eta \tilde{X}_i(n) + \frac{\eta^2}{2} \tilde{X}_i^2(n)\right) \leq -\eta q_{n-1}^\top \tilde{X}(n) + \frac{\eta^2}{2} q_{n-1}^\top \tilde{X}^2(n),$$

where we used $\exp(-z) \leq 1 - z + z^2/2$ for $z \geq 0$ in the first inequality and $\log(1 + z) \leq z$ for all $z > -1$ in the second inequality. We note that the use of the latter inequality is allowed, i.e. $q_{n-1}^\top \left(-\eta \tilde{X}(n) + \frac{\eta^2}{2} \tilde{X}^2(n)\right) > -1$, since we have

$$1 + q_{n-1}^\top \left(-\eta \tilde{X}(n) + \frac{\eta^2}{2} \tilde{X}^2(n)\right) \geq \sum_{i \in [d]} q_{n-1}(i) \exp\left(-\eta \tilde{X}_i(n)\right) > 0.$$

Hence, we have

$$\text{KL}(q, \tilde{q}_n) - \text{KL}(q, q_{n-1}) \leq \eta q_{n-1}^\top \tilde{X}(n) - \eta q_{n-1}^\top \tilde{X}(n) + \frac{\eta^2}{2} q_{n-1}^\top \tilde{X}^2(n).$$

Generalized Pythagorean inequality (see Theorem 3.1 in (Csiszár & Shields, 2004)) gives

$$\text{KL}(q,q_n) + \text{KL}(q_n, \tilde{q}_n) \leq \text{KL}(q, \tilde{q}_n).$$
Since $\text{KL}(q_n, \tilde{q}_n) \geq 0$, we get

$$
\text{KL}(q, q_n) - \text{KL}(q, q_{n-1}) \leq \eta q^\top \tilde{X}(n) - \eta q_{n-1}^\top \tilde{X}(n) + \frac{\eta^2}{2} q_{n-1}^\top \tilde{X}^2(n).
$$

Finally, summing over $n$ gives

$$
\sum_{n=1}^T \left( q_{n-1}^\top \tilde{X}(n) - q_{n-1}^\top \tilde{X}(n) \right) \leq \frac{\eta}{2} \sum_{n=1}^T q_{n-1}^\top \tilde{X}^2(n) + \frac{\text{KL}(q, q_0)}{\eta}.
$$

Let $E_n$ be the expectation conditioned on all the randomness chosen by the algorithm up to time $n$. For any $q \in \mathcal{P}$, we have

$$
E_n \left[ q^\top \tilde{X}(n) \right] = \sum_{i \in [d]} q(i) E_n[\tilde{X}_i(n)] = \sum_{i \in [d]} q(i)(1 - X_i(n)) = 1 - q^\top X(n),
$$

and hence $E_n \left[ q_{n-1}^\top \tilde{X}(n) - q_{n-1}^\top \tilde{X}(n) \right] = q^\top X(n) - q_{n-1}^\top X(n)$.

Moreover, we have

$$
E_n \left[ q_{n-1}^\top \tilde{X}^2(n) \right] = \sum_{i \in [d]} q_{n-1}(i) E_n \left[ \tilde{X}_i^2(n) \right] = \sum_{i \in [d]} q_{n-1}(i) \frac{(1 - X_i(n))^2}{m^2 q_{n-1}(i)} m q_{n-1}(i)
$$

$$
= \sum_{i \in [d]} \frac{(1 - X_i(n))^2}{m} \leq \frac{d}{m},
$$

since $X_i(n) \in [0, 1]$.

Using Lemma 2 and the above bound, we get with $mq^*$ the optimal arm, i.e. $q^*(i) = \frac{1}{m}$ iff $M_i^* = 1$,

$$
R^{\text{COMBEXP-1}}(T) = E \left[ \sum_{n=1}^T m q_{n-1}^\top \tilde{X}(n) - \sum_{n=1}^T m q^*^\top \tilde{X}(n) \right] \leq \frac{\eta d T}{2} + \frac{m \log \mu_{\min}^{-1}}{\eta},
$$

since

$$
\text{KL}(q^*, q_0) = -\frac{1}{m} \sum_{i \in M^*} \log m \mu_i^0 \leq \log \mu_{\min}^{-1}.
$$

The proof is completed by setting $\eta = \sqrt{\frac{2m \log \mu_{\min}^{-1}}{dT}}$.

### 3.2. Proof of Theorem 7

We first prove a simple result:

**Lemma 3** For all $x \in \mathbb{R}^d$, we have $\Sigma_{n-1}^+ \Sigma_{n-1} x = \bar{x}$, where $\bar{x}$ is the orthogonal projection of $x$ onto $\text{span}(\mathcal{M})$, the linear space spanned by $\mathcal{M}$.

**Proof:** Note that for all $y \in \mathbb{R}^d$, if $\Sigma_{n-1} y = 0$, then we have

$$
y^\top \Sigma_{n-1} y = E \left[ y^\top MM^\top y + (y^\top M)^2 \right] = 0,
$$

where $M$ has law $p_{n-1}$ such that $\sum_M M_i p_{n-1}(M) = q'_{n-1}(i), \forall i \in [d]$ and $q'_{n-1} = (1 - \gamma) q_{n-1} + \gamma \mu^0$. By definition of $\mu^0$, each $M \in \mathcal{M}$ has a positive probability. Hence, by (15), $y^\top M = 0$ for all $M \in \mathcal{M}$. In particular, we see that the linear application $\Sigma_{n-1}$ restricted to $\text{span}(\mathcal{M})$ is invertible and is zero on $\text{span}(\mathcal{M})^\perp$, hence we have $\Sigma_{n-1}^+ \Sigma_{n-1} x = \bar{x}$. \hfill \square
Lemma 4 We have for any $\eta \leq \frac{2\lambda}{m\eta z}$ and any $q \in \mathcal{P}$,

$$
\sum_{n=1}^{T} q^T \hat{X}(n) - \sum_{n=1}^{T} q_{n-1}^T \hat{X}(n) \leq \frac{\eta}{2} \sum_{n=1}^{T} q_{n-1}^T \hat{X}^2(n) + \frac{\text{KL}(q, q_0)}{\eta},
$$

where $\hat{X}^2(n)$ is the vector that is the coordinate-wise square of $\hat{X}(n)$.

Proof: We have

$$
\text{KL}(q, \hat{q}_n) - \text{KL}(q, q_{n-1}) = \sum_{i \in [d]} q(i) \log \frac{q_{n-1}(i)}{\hat{q}_n(i)} = -\eta \sum_{i \in [d]} q(i) \hat{X}_i(n) + \log Z_n,
$$

with

$$
\log Z_n = \log \sum_{i \in [d]} q_{n-1}(i) \exp \left( \eta \hat{X}_i(n) \right)
\leq \log \sum_{i \in [d]} q_{n-1}(i) \left( 1 + \eta \hat{X}_i(n) + \eta^2 \hat{X}^2_i(n) \right)
\leq \eta q_{n-1}^T \hat{X}(n) + \eta^2 q_{n-1}^T \hat{X}^2(n),
$$

where we used $\exp(z) \leq 1 + z + z^2$ for all $|z| \leq 1$ in (16) and $\log(1 + z) \leq z$ for all $z > -1$ in (17). Later we verify the condition for the former inequality.

Hence we have

$$
\text{KL}(q, \hat{q}_n) - \text{KL}(q, q_{n-1}) \leq \eta q_{n-1}^T \hat{X}(n) - \eta q_{n-1}^T \hat{X}(n) + \eta^2 q_{n-1}^T \hat{X}^2(n).
$$

Generalized Pythagorean inequality (see Theorem 3.1 in (Csiszar & Shields, 2004)) gives

$$
\text{KL}(q, q_n) + \text{KL}(q_n, \hat{q}_n) \leq \text{KL}(q, \hat{q}_n).
$$

Since $\text{KL}(q_n, \hat{q}_n) \geq 0$, we get

$$
\text{KL}(q, q_n) - \text{KL}(q, q_{n-1}) \leq \eta q_{n-1}^T \hat{X}(n) - \eta q_{n-1}^T \hat{X}(n) + \eta^2 q_{n-1}^T \hat{X}^2(n).
$$

Finally, summing over $n$ gives

$$
\sum_{n=1}^{T} \left( q^T \hat{X}(n) - q_{n-1}^T \hat{X}(n) \right) \leq \eta \sum_{n=1}^{T} q_{n-1}^T \hat{X}^2(n) + \frac{\text{KL}(q, q_0)}{\eta}.
$$

To satisfy the condition for the inequality (16), i.e., $\eta |\hat{X}_i(n)| \leq 1$, $\forall i \in [d]$, we find the upper bound for $\max_{i \in [d]} |\hat{X}_i(n)|$ as follows:

$$
\max_{i \in [d]} |\hat{X}_i(n)| \leq \|\hat{X}(n)\|_2
= \|\Sigma_{n-1}^+ M(n) Y_n\|_2
\leq m \|\Sigma_{n-1}^+ M(n)\|_2
\leq m \sqrt{M(n)^T \Sigma_{n-1}^+ \Sigma_{n-1}^+ M(n)}
\leq m \|M(n)\|_2 \sqrt{\lambda_{\max} \left( \Sigma_{n-1}^+ \Sigma_{n-1}^+ \right)}
= m^{3/2} \sqrt{\lambda_{\max} \left( \Sigma_{n-1}^+ \Sigma_{n-1}^+ \right)}
= m^{3/2} \lambda_{\max} \left( \Sigma_{n-1}^+ \right)
= m^{3/2} \lambda_{\min} \left( \Sigma_{n-1} \right),
$$

$\lambda_{\min}$.
where $\lambda_{\text{max}}(A)$ and $\lambda_{\text{min}}(A)$ respectively denote the maximum and the minimum nonzero eigenvalue of matrix $A$. Note that $\mu^0$ induces uniform distribution over $\mathcal{M}$. Thus by $q_{n-1}' = (1-\gamma)q_{n-1} + \gamma\mu^0$ we see that $p_{n-1}$ is a mixture of uniform distribution and the distribution induced by $q_{n-1}$. Note that we have:

$$\lambda_{\text{min}}(\Sigma_{n-1}) = \min_{\|x\|_2 = 1, x \in \text{span}(\mathcal{M})} x^\top \Sigma_{n-1} x.$$  

Moreover, we have

$$x^\top \Sigma_{n-1} x = \mathbb{E} [x^\top M(n)M(n)^\top x] = \mathbb{E} [(M(n)^\top x)^2] \geq \gamma \mathbb{E} [(M^\top x)^2],$$

where in the last inequality $M$ has law $\mu^0$. By definition, we have for any $x \in \text{span}(\mathcal{M})$ with $\|x\|_2 = 1$,

$$\mathbb{E} [(M^\top x)^2] \geq \lambda,$$

so that in the end, we get $\lambda_{\text{min}}(\Sigma_{n-1}) \geq \gamma \lambda$, and hence $\eta|\tilde{X}_i(n)| \leq \frac{mn^{3/2}}{\gamma \lambda}$, $\forall i \in [d]$. Finally, we choose $\eta \leq \frac{2\Lambda}{mn^{3/2}}$ to satisfy the condition for the inequality we used in (16).

We have

$$\mathbb{E}_n \left[ \tilde{X}(n) \right] = \mathbb{E}_n \left[ Y_n \Sigma_{n-1}^+ M(n) \right] = \mathbb{E}_n \left[ \Sigma_{n-1}^+ M(n)M(n)^\top X(n) \right] = \Sigma_{n-1}^+ \Sigma_{n-1} X(n) = \overline{X(n)},$$

where the last equality follows from Lemma 3 and $\overline{X(n)}$ is the orthogonal projection of $X(n)$ onto $\text{span}(\mathcal{M})$. In particular, for any $mq' \in \text{Co}(\mathcal{M})$, we have

$$\mathbb{E}_n \left[ mq'^\top \tilde{X}(n) \right] = mq'^\top \overline{X(n)} = mq'^\top X(n).$$

Moreover, we have:

$$\mathbb{E}_n \left[ q_{n-1}' \tilde{X}^2(n) \right] = \sum_{i \in [d]} q_{n-1}(i)\mathbb{E}_n \left[ \tilde{X}_i^2(n) \right]$$

$$= \sum_{i \in [d]} q_{n-1}'(i) - \gamma \mu^0(i) = \mathbb{E}_n \left[ \tilde{X}_i^2(n) \right]$$

$$\leq \frac{1}{m(1-\gamma)} \sum_{i \in [d]} mq_{n-1}'(i)\mathbb{E}_n \left[ \tilde{X}_i^2(n) \right]$$

$$= \frac{1}{m(1-\gamma)} \mathbb{E}_n \left[ \sum_{i \in [d]} \tilde{M}_i(n)\tilde{X}_i^2(n) \right],$$

where $\tilde{M}(n)$ is a random arm with the same law as $M(n)$ and independent of $M(n)$. Note that $\tilde{M}_i^2(n) = \tilde{M}_i(n)$, so that we have

$$\mathbb{E}_n \left[ \sum_{i \in [d]} \tilde{M}_i(n)\tilde{X}_i^2(n) \right] = \mathbb{E}_n \left[ X(n)^\top M(n)M(n)^\top \Sigma_{n-1}^+ \tilde{M}(n)\tilde{M}(n)^\top \Sigma_{n-1}^+ M(n)M(n)^\top X(n) \right]$$

$$\leq m^2 \mathbb{E}_n [M(n)^\top \Sigma_{n-1}^+ M(n)],$$

where we used the bound $M(n)^\top X(n) \leq m$. By Lemma 15 in (Cesa-Bianchi & Lugosi, 2012), $\mathbb{E}_n [M(n)^\top \Sigma_{n-1}^+ M(n)] \leq d$, so that we have:

$$\mathbb{E}_n \left[ q_{n-1}' \tilde{X}^2(n) \right] \leq \frac{md}{1-\gamma}.$$
Observe that
\[
\mathbb{E}_n \left[ q^{*\top} \tilde{X}(n) - q_{n-1}^{\top} \tilde{X}(n) \right] = \mathbb{E}_n \left[ q^{*\top} \tilde{X}(n) - (1 - \gamma)q_{n-1}^{\top} \tilde{X}(n) - \gamma \mu_0^{\top} \tilde{X}(n) \right]
\]
\[
= \mathbb{E}_n \left[ q^{*\top} \tilde{X}(n) - q_{n-1}^{\top} \tilde{X}(n) \right] + \gamma q_{n-1}^{\top} X(n) - \gamma \mu_0^{\top} X(n)
\]
\[
\leq \mathbb{E}_n \left[ q^{*\top} \tilde{X}(n) - q_{n-1}^{\top} \tilde{X}(n) \right] + \gamma q_{n-1}^{\top} X(n)
\]
\[
\leq \mathbb{E}_n \left[ q^{*\top} \tilde{X}(n) - q_{n-1}^{\top} \tilde{X}(n) \right] + \gamma.
\]

Using Lemma 4 and the above bounds, we get with \(mq^*\) the optimal arm, i.e. \(q^*(i) = \frac{1}{m}\) iff \(M^*_i = 1\),

\[
R^{\text{COMBEXP-2}}(T) = \mathbb{E} \left[ \sum_{n=1}^{T} mq^*^{\top} \tilde{X}(n) - \sum_{n=1}^{T} mq_{n-1}^{\top} \tilde{X}(n) \right]
\]
\[
\leq \mathbb{E} \left[ \sum_{n=1}^{T} mq^*^{\top} \tilde{X}(n) - \sum_{n=1}^{T} mq_{n-1}^{\top} \tilde{X}(n) \right] + m\gamma T
\]
\[
\leq \frac{\eta m^2 T}{1 - \gamma} + \frac{m \log \mu_{0\min}^{-1}}{\eta} + m\gamma T,
\]

since

\[
\text{KL}(q^*, q_0) = -\frac{1}{m} \sum_{i \in M^*} \log m \mu_i^0 \leq \log \mu_{0\min}^{-1}.
\]

Choosing \(\eta = \gamma C\) with \(C = \frac{\Lambda}{m^{1/2}}\) gives

\[
R^{\text{COMBEXP-2}}(T) \leq \frac{\gamma C m^2 d T}{1 - \gamma} + \frac{m \log \mu_{0\min}^{-1}}{\gamma C} + m\gamma T
\]
\[
= \frac{C m^2 d + m \gamma}{1 - \gamma} T + \frac{m \log \mu_{0\min}^{-1}}{\gamma C}
\]
\[
\leq \frac{(C m^2 d + m) \gamma T}{1 - \gamma} + \frac{m \log \mu_{0\min}^{-1}}{\gamma C}.
\]

The proof is completed by setting \(\gamma = \frac{\sqrt{m \log \mu_{0\min}^{-1}}}{\sqrt{m \log \mu_{0\min}^{-1} + \sqrt{C (C m^2 d + m) T}}} \). \(\square\)

### 3.3. Implementation: The Case of Graph Coloring

In this subsection, we present an iterative algorithm for projection step of algorithms COMBEXP-1 and COMBEXP-2, for the graph coloring problem described next.

Consider a graph \(G = (V, E)\) consisting of \(m\) nodes indexed by \(i \in [m]\). Each node can use one of the \(c \geq m\) available colors indexed by \(j \in [c]\). A feasible coloring is represented by a matrix \(M \in \{0, 1\}^{m \times c}\), where \(M_{ij} = 1\) if and only if node \(i\) is assigned color \(j\). Coloring \(M\) is feasible if (i) for all \(i\), node \(i\) uses at most one color, i.e., \(\sum_{j \in [c]} M_{ij} \in \{0, 1\}\); (ii) neighboring nodes are assigned different colors, i.e., for all \(i, i' \in [m]\), \((i, i') \in E\) implies for all \(j \in [c]\), \(M_{ij} M_{i'j} = 0\). In the following we denote by \(K = \{K_{i\ell}, \ell \in [k]\}\) the set of maximal cliques of the graph \(G\). We also introduce \(K_{i\ell} \in \{0, 1\}\) such that \(K_{i\ell} = 1\) if and only if node \(i\) belongs to the maximal clique \(K_{i\ell}\).

There is a specific case where our algorithm can be efficiently implementable: when the convex hull \(\text{Co}(M)\) can be captured by polynomial in \(m\) many constraints. Note that this cannot be ensured unless restrictive assumptions are made on the graph \(G\) since there are up to \(3^m/m^3\) maximal cliques in a graph with \(m\) vertices (Moon & Moser, 1965). There are families of graphs in which the number of cliques is polynomially bounded. These families include chordal graphs, complete graphs, triangle-free graphs, interval graphs, and planar graphs. Note however, that a limited number of cliques does not ensure a priori that \(\text{Co}(M)\) can be captured by a limited number of constraints. To the best of our knowledge,
this problem is open and only particular cases have been solved as for the stable set polytope (corresponding to the case \( c = 2, X_{11} = 1 \) and \( X_{12} = 0 \) with our notation) (Schrijver, 2003).

For the coloring problem described above we have

\[
\text{Co}(\mathcal{M}) = \text{Co}\{ \forall i, \sum_{j \in [c]} M_{ij} \leq 1, \ \forall \ell, j, \sum_{i \in [m]} K_{\ell i} M_{ij} \leq 1 \}.
\]  

(18)

Note that in the special case where \( G \) is the complete graph, such a representation becomes

\[
\text{Co}(\mathcal{M}) = \text{Co}\{ \sum_{j \in [c]} M_{ij} \leq 1, \ \forall i, \ \sum_{i \in [m]} M_{ij} \leq 1, \ \forall j \}.
\]

We now give an algorithm for the projection a distribution \( p \) onto \( \mathcal{P} \) using KL divergence. Since \( \mathcal{P} \) is a scaled version of \( \text{Co}(\mathcal{M}) \), we give an algorithm for the projection of \( mp \) onto \( \text{Co}(\mathcal{M}) \) given by (18).

Set \( \lambda_i(0) = \mu_j(0) = 0 \) for all \( i, j \) and then define for \( t \geq 0 \),

\[
\forall i \in [m], \ \lambda_i(t + 1) = \log \left( \sum_{j} mp_{ij} e^{-\mu_j(t)} \right)
\]

(19)

\[
\forall j \in [c], \ \mu_j(t + 1) = \max_{\ell} \log \left( \sum_{i} K_{\ell i} mp_{ij} e^{-\lambda_i(t+1)} \right).
\]

(20)

We can show that

**Proposition 1** Let \( p^*_t = \lim_{t \to \infty} p_{ij} e^{-\lambda_i(t)-\mu_j(t)} \). Then \( mp^* \) is the projection of \( mp \) onto \( \text{Co}(\mathcal{M}) \) using the KL divergence.

Although this algorithm is shown to converge, we must stress that the step (20) might be expensive as the number of distinct values of \( \ell \) might be exponential in \( m \). When \( G \) is a complete graph, this step is easy and our algorithm reduces to Sinkhorn’s algorithm (see (Helmbold & Warmuth, 2009) for a discussion).

**Proof:** First note that the definition of projection can be extended to non-negative vectors thanks to the relation

\[
\text{KL}(p^*, q) = \min_{p \in \Xi} \text{KL}(p, q).
\]

More precisely, given an alphabet \( A \) and a vector \( q \in \mathbb{R}_+^A \), we have for any probability vector \( p \in \mathbb{R}_+^A \)

\[
\sum_{a \in A} p(a) \log \frac{p(a)}{q(a)} \geq \sum_{a} p(a) \log \frac{\sum_{a} p(a)}{\sum_{a} q(a)} = \log \frac{1}{\|q\|_1},
\]

thanks to the log-sum inequality. Hence we see that \( p^*(a) = \frac{q(a)}{\|q\|_1} \) is the projection of \( q \) onto the simplex of \( \mathbb{R}_+^A \).

Now define \( \mathcal{A}_i = \text{Co}\{ M_{ij}, \sum_j M_{ij} \leq 1 \} \) and \( \mathcal{B}_{lj} = \text{Co}\{ M_{ij}, \sum_i K_{\ell i} M_{ij} \leq 1 \} \). Hence \( \bigcap_i \mathcal{A}_i \cap \bigcap_{lj} \mathcal{B}_{lj} = \text{Co}(\mathcal{M}) \). By the argument described above, iteration (19) (resp. (20)) corresponds to the projection onto \( \mathcal{A}_i \) (resp. \( \bigcap_{lj} \mathcal{B}_{lj} \)) and the proposition follows from Theorem 5.1 in (Csiszár & Shields, 2004).

\[ \square \]

### 3.4. Examples

In this subsection, we compare the performance of \textsc{CombExp-1} and \textsc{CombExp-2} against state-of-the-art algorithms. The regret of these algorithms for semi-bandit and bandit cases are summarised respectively in Table 2 and Table 3.

#### 3.4.1. \( m \)-Sets

In this case, \( \mathcal{M} \) is the set of all \( d \)-dimensional binary vectors with \( m \) ones. We have

\[
\mu_{\text{min}} = \min_i \frac{1}{(m)_i} \sum_{\mathcal{M}} M_i = \frac{(d-1)}{(m-1)} = \frac{m}{d}.
\]
and hence \( \text{COMBEXP-1} \) has a regret of \( O(\sqrt{mdT \log(d/m)}) \), which matches that of OSMD up to a logarithmic factor. Moreover, according to (Cesa-Bianchi & Lugosi, 2012)[Proposition 12], we have \( \Delta = \frac{m(d-m)}{d(d-1)} \). When \( m = o(d) \), the regret of \( \text{COMBEXP-2} \) becomes \( O(\sqrt{m^3d^2T \log(d/m)}) \), namely it has the same performance as \( \text{COMBAND} \) and \( \text{EXP2} \) with John’s Exploration.

### 3.4.2. Matching

Let \( M \) be the set of perfect matchings in \( K_{m,m} \), where we have \( d = m^2 \) and \( |M| = m! \). We have

\[
\mu_{\text{min}} = \min_{i} \frac{1}{m!} \sum_{M} M_i = \frac{(m-1)!}{m!} = \frac{1}{m},
\]

which results in a regret of \( O(\sqrt{m^3T \log(m)}) \) for \( \text{COMBEXP-1} \). Hence, in this case OSMD and \( \text{COMBEXP-1} \) attain the same regrets up to a factor \( \sqrt{\log(m)} \). Furthermore, from (Cesa-Bianchi & Lugosi, 2012)[Proposition 4] we have that \( \Delta = \frac{1}{m-1} \), thus giving \( R_{\text{COMBEXP-2}}(T) = O(\sqrt{m^3T \log(m)}) \), which is the same as the regret of \( \text{COMBAND} \) and \( \text{EXP2} \) with John’s Exploration in this case.

### 3.4.3. Spanning Trees

In our next example, we assume that \( M \) is the set of spanning trees in the complete graph \( K_N \). In this case, we have \( d = \binom{N}{2} \), \( m = N - 1 \), and by Cayley’s formula \( M \) has \( N^{N-2} \) elements. Observe that

\[
\mu_{\text{min}} = \min_{i} \frac{1}{N^{N-2}} \sum_{M} M_i = \frac{(N-1)^{N-3}}{N^{N-2}},
\]

which gives for \( N \geq 2 \)

\[
\log \mu_{\text{min}}^{-1} = \log \left( \frac{N^{N-2}}{(N-1)^{N-3}} \right) = (N-3) \log \left( \frac{N}{N-1} \right) + \log N \\
\leq (N-3) \log 2 + \log(N) \leq 2N.
\]

As a result, \( R_{\text{COMBEXP-1}}(T) = O(\sqrt{N^3T \log(N)}) \), which matches the regret of OSMD up to \( \sqrt{\log(N)} \) factor. From (Cesa-Bianchi & Lugosi, 2012)[Corollary 7], we also get \( \Delta \geq \frac{1}{N} - \frac{12}{2N^2} \). For \( N \geq 6 \), the regret of \( \text{COMBAND} \) takes the form \( O(\sqrt{N^3T \log(N)}) \) since \( \frac{m}{d} \Delta < 7 \) when \( N \geq 6 \). Further, \( \text{EXP2} \) with John’s Exploration attains the same regret. On the other hand, we get

\[
R_{\text{COMBEXP-2}}(T) = O(\sqrt{N^5T \log(N)}), \quad N \geq 6,
\]

and therefore it gives the same regret as \( \text{COMBAND} \) and \( \text{EXP2} \) with John’s Exploration.

### 3.4.4. Cut sets

Consider the case where \( M \) is the set of balanced cuts of the complete graph \( K_{2N} \), where a balanced cut is defined as the set of edges between a set of \( N \) vertices and its complement. It is easy to verify that \( d = \binom{2N}{2} \) and \( m = N^2 \). Moreover, \( M \) has \( \binom{2N}{N} \) balanced cuts and hence

\[
\mu_{\text{min}} = \min_{i} \frac{1}{\binom{2N}{N}} \sum_{M} M_i = \frac{\binom{2N-2}{N-1}}{\binom{2N}{N}} = \frac{N}{4N-2},
\]

which gives \( O(N^2\sqrt{T}) \) for the regret of \( \text{COMBEXP-1} \), that is the same as the regret of OSMD. Moreover, by (Cesa-Bianchi & Lugosi, 2012)[Proposition 9], we have

\[
\Delta = \frac{1}{4} + \frac{8N - 7}{4(2N-1)(2N-3)}, \quad N \geq 2,
\]

And consequently, the regret of \( \text{COMBEXP-2} \) becomes \( O(N^4\sqrt{T}) \) for \( N \geq 2 \), which is the same as that of \( \text{COMBAND} \) and \( \text{EXP2} \) with John’s Exploration.
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