BOUNDARY TRANSFER MATRICES AND BOUNDARY QUANTUM KZ EQUATIONS

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Abstract. A simple relation between inhomogeneous transfer matrices and boundary quantum KZ equations is exhibited for quantum integrable systems with reflecting boundary conditions, analogous to an observation by Gaudin for periodic systems. Thus the boundary quantum KZ equations receive a new motivation. We also derive the commutativity of Sklyanin’s boundary transfer matrices by merely imposing appropriate reflection equations, i.e. without using the conditions of crossing symmetry and unitarity of the R-matrix.

1. Introduction

Many interesting objects associated to integrable models and representation theory are known to satisfy difference or differential equations. The Knizhnik-Zamolodchikov (KZ) equations are differential equations defining conformal blocks in Wess-Zumino-Witten conformal field theory [15] and describing intertwiners between certain representations of affine Kac-Moody algebras [26]. Quantum deformations of these equations yield difference equations, known as quantum Knizhnik-Zamolodchikov (qKZ) equations; they arise independently as equations satisfied by correlation functions and form factors of quantum integrable models [13, 24] and by matrix elements of intertwiners for representations of quantum affine algebras [7].

More recently, connections with combinatorics have been extensively investigated when the (multiplicative) shift parameter in the equations assumes root-of-unity values, in conjunction with the Razumov-Stroganov conjectures for loop models, e.g. in [5, 11, 22, 28].

Cherednik [3, 4] constructed generalizations of the qKZ equations in terms of a so-called R-matrix datum associated to arbitrary affine root systems. In this framework the aforementioned “original” qKZ equations correspond to the case where the affine root system is of A-type and are related to integrable systems with periodic boundary conditions (or rather twisted-periodic, but we will nevertheless use “periodic” in our terminology). If the affine root system is of a different classical type (i.e. of B/C/D-type) we arrive at the boundary qKZ equations (bqKZ), relating to integrable systems with up to two reflecting boundaries. In the current work we will focus on the case when the affine root system is of C-type (since the affine Weyl groups of types B/D naturally appear as normal subgroups of the affine Weyl group of C-type, many of our results can be modified to results for types B/D).

The bqKZ equations were first studied and motivated in their own right in [14], where they describe correlation functions of semi-infinite spin chains with integrable boundary conditions. It would be interesting to find other motivations, analogous to the ones for the A-type qKZ equations. This paper provides such a motivation by showing that in the limit that the shift parameter goes to 1, the bqKZ equations are...
related, through Yang’s notion of scattering matrices, to interpolants of commuting
transfer matrices, which is entirely parallel to an observation by Gaudin [10 Ch. 10]
for the A-type case (also cf. [19]). It is beneficial to review some basic terminology
in more detail at this stage.

1.1. Quantum Knizhnik-Zamolodchikov equations and scattering matrices. Consider a collection of complex vector spaces $V_1, \ldots, V_N$, called local state spaces, and construct the global state space $W = V_1 \otimes \cdots \otimes V_N$. In this paper we will deal with the case $V_i = \ldots = V_N = V$ only. For $p \in \mathbb{C}^\times := \mathbb{C} \setminus \{0\}$ and $i = 1, \ldots, N$, write $p^{x_i} : (\mathbb{C}^\times)^N \to (\mathbb{C}^\times)^N$ for multiplication by $p$ in the $i$-th entry:

$$p^{x_i} z = (z_1, \ldots, z_{i-1}, p z_i, z_{i+1}, \ldots, z_N), \quad \text{for } z = (z_1, \ldots, z_N) \in (\mathbb{C}^\times)^N.$$ 

The qKZ equations are the following system of $p$-difference equations for meromorphic functions $f : (\mathbb{C}^\times)^N \to V \otimes N$:

$$f(p^{x_i} z) = A_i(z;p)f(z), \quad \text{for } i = 1, \ldots, N,$$

for certain qKZ transport matrices $A_i(z;p) \in \text{End}(V \otimes N)$ depending meromorphically on the $z_i$. They are consistent if the two possible ways of resolving $f(p^{x_i} p^{y_i} z)$ amount to the same; this happens precisely if

$$A_i(p^{x_i} z; p)A_j(z;p) = A_j(p^{y_i} z; p)A_i(z;p), \quad \text{for } i, j = 1, \ldots, N.$$

Thus, (1.2) can be viewed as a “flatness” condition for a discrete connection defined by (1.1). We assume the $A_i$ are composed of “local” operators (i.e. acting on one or two tensorands $V$). The main such ingredient is an R-matrix $R(x) \in \text{End}(V \otimes V)$, i.e. a meromorphic solution of the (quantum) Yang-Baxter equation (YBE):

$$R_{12}(x/y)R_{13}(x)R_{23}(y) = R_{23}(y)R_{13}(x)R_{12}(x/y) \in \text{End}(V \otimes V \otimes V)$$

for generic $x, y$. The subscripts indicate in which tensor factors the operator in question acts nontrivially, also cf. Appendix A. (1.3) can be seen as a condition expressing the equivalence of the two possible factorizations of three-particle interactions into three two-particle interactions. Then (1.2), a global condition, follows from the YBE (1.3) and other local conditions.

In the B/C/D-type case the R-matrix datum consists of the usual R-matrices, i.e. solutions of the YBE, and two operators $K^+(x), K^-(x) \in \text{End}(V)$, meromorphically depending on $x$, the so-called K-matrices. They are required to satisfy reflection equations (REs), also known as boundary Yang-Baxter equations, one corresponding to either of two boundaries present in the system. The left and right reflection equations (LRE, RRE) are the relations

$$R_{12}(x/y)K^+_1(x)R_{21}(xy)K^+_2(y) = K^+_2(y)R_{12}(xy)K^+_1(x)R_{21}(x/y), \quad \text{(1.4)}$$

$$R_{21}(x/y)K^-_1(x)R_{12}(xy)K^-_2(y) = K^-_2(y)R_{21}(xy)K^-_1(x)R_{12}(x/y), \quad \text{(1.5)}$$

both acting on $V \otimes 2$; here $R_{12}(x) = PR(x)P$ with the “flip” $P \in \text{GL}(V \otimes 2)$ defined by $P(v \otimes v') = v' \otimes v$ for $v, v' \in V$. Taking the same viewpoint as with the YBE (1.3), (1.4) express the equivalence of the two possible factorizations of two-particle-and-wall interactions into two one-particle-and-wall interactions. We will review the precise expression of the qKZ transport matrices in terms of the R- and K-matrices in Section B.

1 It is possible that this restriction may be lifted for some of the theory under consideration; we will return to this question in Subsection B.
In the special case that \( p = 1 \) the consistency condition (1.2) simplifies to the statement that the matrices \( A_i(z; 1) \) \((1 \leq i \leq N)\) mutually commute. The \( A_i(z; 1) \) correspond to the *scattering matrices* introduced by Yang in his investigations into the delta Bose gas [27]. Scattering matrices predate the qKZ equations and play an important role in quantum integrability (cf. [10, Ch. 10] and [19]). Hence we may view (1.2) as a \( p \)-deformed criterion for integrability. Moreover, the qKZ equations (1.1) then naturally appear in this story as being deformations of the eigenvector equation for scattering matrices with eigenvalue 1 (see e.g. the introductory remarks in [28]), thus motivating the study of (solutions of) qKZ equations.

1.2. Transfer matrices. Another main criterion of quantum integrability is Baxter’s notion of commuting *transfer matrices* which is at the basis of the quantum inverse scattering method (algebraic Bethe ansatz) as developed by the Faddeev school from the 1980s; for textbook accounts see [1, 16]. The transfer matrix, originally associated specifically to vertex models from statistical mechanics, is a parameter-dependent linear operator \( T(x) \) acting on a state space \( W \). As before, if \( W = V_1 \otimes \cdots \otimes V_N \), transfer matrices can be built up from local operators in such a way that the integrability criterion (commutativity) can be derived from local integrability conditions like the YBE. Also, it is possible to introduce an additional dependence on an \( N \)-tuple of complex numbers (so-called inhomogeneities) into the transfer matrix, yielding inhomogeneous transfer matrices. In Section 2 we study transfer matrices in more detail. The method of commuting transfer matrices is best understood in the case of systems with periodic or “closed” boundary conditions, but for systems with reflecting or “open” boundary conditions there is a more elaborate version due to Sklyanin [23], who constructed commuting *boundary transfer matrices* from R- and K-matrices and derived the algebraic Bethe ansatz for special types of R- and K-matrices. Similar to the setup for the qKZ transport matrices, two K-matrices are required. One of these can be taken equal to one of the K-matrices featuring in the qKZ transport matrix, say \( K^- \); thus it satisfies the RRE (1.5). In the present context it is crucial that with that choice, the other necessary K-matrix \( K' \) is in general not a solution of the LRE (1.4), but a third reflection equation, see (2.8).

1.3. The connection between qKZ equations and transfer matrices. Gaudin [10, Ch. 10] has highlighted that scattering matrices are proportional to interpolants of inhomogeneous transfer matrices (those with the spectral parameter running through the set of inhomogeneities), in case the underlying affine root system is of A-type. Thus the problem of finding eigenvectors of transfer matrices is related to the problem of finding eigenvectors of scattering matrices, adding to the relevance of the qKZ equations. For the bqKZ equations, such a connection with inhomogeneous boundary transfer matrices has heretofore been unclear, as observed by Pasquier [19]; this owes mainly to the fact that the K-matrices for the left boundary appearing in the formulae satisfy different reflection equations. We address this question in Thm. 3.13 thus providing a new motivation for the bqKZ equations. For the special case where the R-matrix is of \( U_q(\hat{sl}_2) \)-type this connection was already made in [25].

1.4. Outline. Each of the main three sections of this paper is split up in a part about periodic systems and a part on reflecting systems, the former of which consists
mainly of a review of existing results which is provided as a background for the new results proposed in the latter.

In Section 2 we will discuss transfer matrices for periodic and reflecting systems for general state spaces $W$ (not necessarily tensor products of local state spaces $V_i$). The new result here addresses an unsatisfactory aspect of the state-of-affairs of quantum integrability for reflecting systems, namely the large number of conditions on the R-matrix datum required to establish the commutativity of the boundary transfer matrices in [23], both compared to the analogon in the periodic case and to the requirements for the consistency of the bqKZ equations. This problem has already been reduced significantly in [18] and [8] and the main improvement in the current work (cf. Thm. 2.2) is to show it is unnecessary to assume unitarity or crossing symmetry of the R-matrix, as in the periodic case, essentially leaving the appropriate REs at both boundaries as the only conditions.

From here onwards we will focus on the case $W = V^\otimes N$. In Section 3 we will discuss the connection between inhomogeneous transfer matrices and qKZ transport matrices. For reflecting systems this leads to the main theorem Thm. 3.13 of this paper, which settles the aforementioned problem identified by Pasquier: it establishes a simple relation between bqKZ transport matrices [8, 4, 14] and the inhomogeneous boundary transfer matrices [23, 18, 8]. This relies on a careful analysis of the relation between the K-matrices $K^+$ and $K'$, q.v. Lemma 3.12.

As an application of these relations, in Section 4 we will derive the commutativity of transfer matrices from the qKZ consistency conditions for special classes of integrable systems, both periodic and reflecting, yielding a generalization of a result by Razumov, Stroganov and Zinn-Justin [22].

Finally, in Section 5 we will outline future work and possible generalizations.

1.5. Some notational conventions. Let $\text{Mer}$ denote the associative algebra of meromorphic functions: $\mathbb{C} \to \mathbb{C}$. Given a complex vector space $V$, let $\text{Mer}(V)$ denote the associative algebra of meromorphic functions: $\mathbb{C} \to \text{End}(V)$, i.e. the associative algebras of linear operators on $V$ meromorphically depending on one complex variable. We write $\text{Mer}^\times$ for the complement in $\text{Mer}$ of the constant function zero, and similarly by $\text{Mer}(V)^\times$ we denote the subset of $\text{Mer}(V)$ consisting of meromorphic functions ranging in $\text{End}(V)$ whose images are generically invertible (i.e. whose images have generically nonzero determinants, if $V$ is finite-dimensional). If $X \in \text{Mer}(V)^\times$ then by $X^{-1}$ we denote the element of $\text{Mer}(V)^\times$ defined by $X^{-1}(x) = (X(x))^{-1}$ for generic values of $x$.

For $X, Y \in \text{End}(V)$ the notation $X \propto Y$ denotes the equivalence relation $X = mY$ for some $m \in \mathbb{C}^\times$. Moreover, if $X, Y \in \text{Mer}(V)$ then $X \propto_x Y$ means $X(x) = m(x)Y(x)$ for generic values of $x$ and some $m \in \text{Mer}^\times$.

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2. Commuting transfer matrices

In this section we compare the requirements for the existence of boundary commuting transfer matrices with the periodic case and formulate an improved commutativity statement. First we summarize the method of commuting transfer matrices.

Let $W$ be a complex vector space (state space). The transfer matrix $T$ is a distinguished element of $\text{Mer}(W)$; the variable on which it depends as an element of $\text{Mer}$ is called the spectral parameter. The condition of integrability for systems modelled in this way is that transfer matrices with different spectral parameters should commute: $[T(x), T(y)] = 0$. The transfer matrix relates directly to the notion of partition function of statistical mechanical models; for quantum mechanical models the importance of the transfer matrix is that the quantum Hamiltonian of the model can be expressed in terms of it, typically as the logarithmic derivative of the transfer matrix for a special value of the spectral parameter.

The transfer matrix $t$ is usually constructed in terms of a partial trace over another (finite-dimensional) linear space $V$ of a certain element of $\text{End}(V \otimes W)$, satisfying a quadratic relation (“exchange relation”) in $\text{End}(V \otimes V \otimes W)$ involving an invertible solution of the Yang-Baxter equation; for this reason $V$ is called auxiliary space. From this quadratic relation one derives $[T(x), T(y)] = 0$.

2.1. Commuting transfer matrices in the periodic setting. For periodic integrable systems one is given operators $U \in \text{Mer}(V \otimes W)$ and $D \in \text{End}(V)$, where the monodromy matrix $U$ represents the interactions in the bulk and $D$ encodes the twist in the periodicity (with $D = \text{Id}_V$ pertaining to the special case of untwisted periodic boundary conditions). One may construct the transfer matrix $T \in \text{Mer}(W)$ in terms of a partial trace over $V$ (cf. Appendix A):

$$(2.1) \quad T(x) := \text{Tr}_V(D \otimes \text{Id}_W)U(x).$$

We have the following standard result (cf. e.g. [1,15]).

**Theorem 2.1.** Let $U \in \text{Mer}(V \otimes W)$ and $D \in \text{End}(V)$. Suppose there exists $R \in \text{Mer}(V \otimes V) \times$ such that the pair $(R, U)$ satisfies the following version of the Yang-Baxter equation, viz.

$$(2.2) \quad R_{12}(x/y)U_{13}(x)U_{23}(y) = U_{23}(y)U_{13}(x)R_{12}(x/y) \in \text{End}(V \otimes V \otimes W)$$

for generic $x$ and $y$, and $(R, D)$ satisfies the compatibility condition

$$(2.3) \quad [R(x), D \otimes D] = 0 \in \text{End}(V \otimes V)$$

for generic $x$. Then the transfer matrices defined by (2.1) form a commuting family:

$$(2.4) \quad [T(x), T(y)] = 0, \quad \text{for all } x, y \in \text{dom}(T).$$

**Proof.** The argument below holds for generic $x, y$. Applying (A3) we have

$$T_1(x)T_1(y) = \text{Tr}_0 D_0U_{01}(x)\text{Tr}_{0'}D_{0'}U'_{01}(y) = \text{Tr}_{0,0'}D_0D_{0'}U_{01}(x)U_{0'1}(y).$$

Inserting $R_{00'}(x/y)^{-1}R_{00'}(x/y) = \text{Id}_{V \otimes V}$ and applying (2.2)(2.3) we obtain

$$T_1(x)T_1(y) = \text{Tr}_{0,0'}D_0D_{0'}R_{00'}(x/y)^{-1}R_{00'}(x/y)U_{01}(x)U_{0'1}(y)$$
\[ T_1(x)T_1(y) = \text{Tr}_{0,0'} D_0 D_0 U_{01}(y) U_{01}(x), \]

which equals \( T_1(y)T_1(x) \) by virtue of (2.2). Because the meromorphic function \([T(x), T(y)]\) is zero for generic \( x, y \), it must be zero for all \( x, y \in \text{dom}(T) \). \( \square \)

### 2.2. Commuting boundary transfer matrices

The notion of commutative transfer matrices in the reflecting setting was first investigated by Sklyanin in the seminal paper \[23\]. It involves the boundary monodromy matrix \( U \in \text{Mer}(V \otimes W) \), which represents the interactions in the bulk and the interactions at the right boundary, and a reflection matrix \( K' \in \text{Mer}(V) \), corresponding to the interactions at the left boundary (it is equally possible to treat the right boundary separately and combine the bulk interactions with those at the left boundary\(^3\)). The boundary transfer matrix is given by

\[ (2.5) \quad \mathcal{T}(x) = \text{Tr}_V (K'(x) \otimes \text{Id}_W) U(x). \]

Conditions on \( K' \) and \( U \) need to be imposed (cf. \[23\]) to derive the commutativity of the boundary transfer matrices. As in the periodic setting these conditions involve an \( R \)-matrix \( R \in \text{Mer}(V^{\otimes 2})^\times \).

The RRE for the pair \((R, U)\) in \( \text{Mer}(V \otimes V \otimes W) \), i.e. the identity

\[ (2.6) \quad R_{21}(x/y) U_{13}(x) R_{12}(x y) U_{23}(y) = U_{23}(y) R_{21}(x y) U_{13}(x) R_{12}(x/y) \]

in \( \text{End}(V \otimes V \otimes W) \), for generic \( x \) and \( y \). It plays a similar role as (2.2), the YBE for \((R, U)\), does in the periodic case.

Assume that \( R^{t_1} \in \text{Mer}(V^{\otimes 2})^\times \) (this rules out the constant solution \( R(x) = P \) of (1.3)) and write

\[ (2.7) \quad \tilde{R}(x) := ((R_{12}(x))^{t_1})^{-1} = ((R_{12}(x)^{t_2} = (1)_{t_2}. \]

Here, \( X^{t_i} \) denotes the partial transpose of an operator \( X \in \text{End}(V^{\otimes 2}) \) with respect to the \( i \)-th tensor factor \( V \) (cf. Appendix [A]). The dual reflection equation (DRE) for the datum \((R, K')\) is the relation

\[ (2.8) \quad R_{12}(x/y)^{-1} K'_1(x) \tilde{R}_{21}(xy) K'_2(y) = K'_2(y) \tilde{R}_{12}(xy) K'_1(x) R_{21}(x/y)^{-1} \]

in \( \text{End}(V \otimes V) \), for generic \( x \) and \( y \).

In the existing derivations of the commutativity of the boundary transfer matrices \([\mathcal{T}(x), \mathcal{T}(y)] = 0\) further conditions have been used.

- Sklyanin \[23\] imposed the conditions \( PR(x)P = R(x) \) (P-symmetry) and \( R(x)^t = R(x) \) (T-symmetry). These conditions were replaced by the single condition \( PR(x)P = R(x)^t \) (PT-symmetry) at the hands of Mezincescu

\(^2\)Where an object in the periodic case has a direct analogon in the reflecting case, the symbol representing the latter will be the same as the former, but typeset in calligraphic typeface.

\(^3\)In \[23\] the more general case is considered where the bulk interactions are split up into two parts, according to some factorization \( W^+ \equiv W_+ \otimes W_- \) and these parts of the bulk interactions are combined with the interactions at the respective adjacent boundaries to yield a left and a right boundary monodromy matrix acting in \( V \otimes W^+ \) and \( V \otimes W^- \), respectively. We will not consider it here but cf. Section [B].
and Nepomechie [15]. Fan, Shi, Hou and Yang [8] presented a modification of Sklyanin’s argument which did not rely on any such conditions.

- We emphasize that in all of [23, 18, 8] the condition of unitarity is assumed:

\[(2.9)\quad R(x)PR(x^{-1})P \propto_x \text{Id}_{V \otimes 2}.\]

- Crossing symmetry is the condition that there exists \(M \in \text{GL}(V)\) and \(r \in \mathbb{C}^\times\) such that

\[(2.10)\quad M_2^{-1}R_{12}(r^2x)^{-1}M_2 \propto_x \tilde{R}_{12}(x) \text{ for generic } x,\]

i.e.

\[\left((\tilde{R}_{12}(r^2x)^{-1})^{t_1}\right)^{-1} \propto_x M_2R_{12}(x)M_2^{-1}.\]

One usually also imposes the following compatibility condition:

\[(2.11)\quad [R(x), M \otimes M] = 0 \in \text{End}(V \otimes V).\]

The conditions (2.10), in the guise of crossing unitarity, the result of combining (2.11) with (2.9), and (2.11) are assumed in [23, 18, 8] to derive the commutativity of the boundary transfer matrices.

Conditions (2.9), (2.11) are not very stringent, as opposed to P- or T-symmetry. They are guaranteed in the context of representations of quantum affine algebras \(U_q(\hat{g})\), see e.g. [6, 7], In particular, if \(R\) is an intertwiner of a tensor product of finite-dimensional \(U_q(\hat{g})\)-modules, then (2.10), (2.11) are satisfied with \(M = \text{Id}(q^{2\rho})\) and \(r = q^{h^-}\), with \(\rho\) the half-sum of positive roots of \(\hat{g}\) and \(h^-\) the dual Coxeter number of \(\hat{g}\). Furthermore, if the \(U_q(\hat{g})\)-modules are irreducible, (2.9) holds.

It would nevertheless be pleasing theoretically if the commutativity of the \(T(x)\) can be derived without them, as can be done in the periodic case for the \(T(x)\), cf. Thm. 2.1. This is possible by a natural generalization of the proof given in [8].

**Theorem 2.2.** Suppose we have \(K' \in \text{Mer}(V)\) and \(U \in \text{Mer}(V \otimes W)\). If there exists \(R \in \text{Mer}(V \otimes 2)^\times\) with \(R^\dagger \in \text{Mer}(V \otimes 2)^\times\) such that (2.6) and (2.8) are satisfied, then the boundary transfer matrices \(T(x)\) defined by (2.5) form a commuting family of operators:

\[\{T(x), T(y)\} = 0, \quad \text{for all } x, y \in \text{dom}(T).\]

**Proof.** The following argument holds for generic values of \(x, y\). We have \(T(x) = \text{Tr}_0 K'_0(x)U_0(x)^{t_0}\) due to (A.3). We have

\[T_1(x)T_1(y) = \left(\text{Tr}_0 K'_0(x)U_0(x)^{t_0}\right)\left(\text{Tr}_0 K'_0(y)U_0(y)^{t_0}\right)\]

\[= \text{Tr}_0 K'_0(y)K'_0(x)U_0(x)^{t_0}U_0(y)^{t_0}\]

\[= \text{Tr}_0 K'_0(y)K'_0(x)^{t_0}\tilde{R}_{00'}(xy)^{t_0}U_0(x)^{t_0}U_0(y)^{t_0}\]

\[= \text{Tr}_0 (K'_0(y)\tilde{R}_{00'}(xy)K'_0(x))^{t_0}(U_0(x)R_{00'}(xy)U_0(y))^{t_0},\]

by subsequently applying (A.3), inserting the identity \(\tilde{R}_{00'}(xy)^{t_0}R_{00'}(xy)^{t_0} = \text{Id}_{V \otimes 2}\) and applying (A.1). Hence, by (A.4),

\[(2.12)\quad T_1(x)T_1(y) = \text{Tr}_0 K'_0(y)\tilde{R}_{00'}(xy)K'_0(x)U_0(x)R_{00'}(xy)U_0(y).\]

\[\text{In [23, 8] the conditions (2.10) and (2.11) were assumed with the special choice } M = \pm \text{Id}.\]
Inserting the identity $R_{00}(\frac{\pi}{2})^{-1}R_{00}(\frac{\pi}{2}) = \text{Id}_{V \otimes 2}$ between $K'_0(x)$ and $U_0(x)$, applying the reflection equations (2.6) and (2.8), and applying (A.2) we obtain

$$T_1(x)T_1(y) = \text{Tr}_{0,0} R_{00}(\frac{\pi}{2})^{-1} K'_0(x) R_{00}(\pi) K'_0(y) U_{01}(y) R_{00}(\pi) U_{01}(x) R_{00}(\frac{\pi}{2})$$

$$= \text{Tr} K'_0(x) R_{00}(\pi) K'_0(y) U_{01}(y) R_{00}(\pi) U_{01}(x).$$

This equals $T_1(y)T_1(x)$ by virtue of (2.12). Because the meromorphic function $[T(x), T(y)]$ is zero for generic $x, y$, it must be zero for all $x, y \in \text{dom}(T)$.

Remark 2.3. Thm. 2.2 shows that the conditions required for commutativity of transfer matrices for reflecting integrable models are no more stringent than those for periodic integrable models, cf. Thm 2.1. The RRE (2.6) for $(R, U)$ corresponds to the YBE (2.2) for $(R, U)$ and the DRE (2.8) for $(R, K')$ corresponds to the compatibility condition (2.3) for $(R, D)$.

2.2.1. Double-row boundary monodromy matrices. The following proposition gives a canonical way of constructing boundary monodromy matrices $U \in \text{Mer}(V \otimes W)$ from “ordinary” monodromy matrices $U \in \text{Mer}(V \otimes W)$ and “local” solutions $K^- \in \text{Mer}(V)$ to the RRE.

Proposition 2.4 (E.g. [23]). Let $K^- \in \text{Mer}(V)$ and $U \in \text{Mer}(V \otimes W)^\times$ and define $U \in \text{Mer}(V \otimes W)$ by means of

$$U(x) := U(x^{-1})^{-1}(K^-(x) \otimes \text{Id}_W)U(x).$$

If there exists $R \in \text{Mer}(V^\otimes 2)^\times$ such that (2.2) and (1.5) are satisfied, then (2.6) holds true. In other words, if $U$ satisfies the (global) YBE in $\text{End}(V \otimes V \otimes W)$ and $K^-$ satisfies the (local) RRE in $\text{End}(V \otimes V)$, then $U$ satisfies the (global) RRE in $\text{End}(V \otimes V \otimes W)$.

Combining Thm. 2.2 and Prop. 2.4 we obtain

Corollary 2.5 (E.g. [23]). Suppose we have $K', K^- \in \text{Mer}(V)$ and $U \in \text{Mer}(V \otimes W)^\times$. If there exists $R \in \text{Mer}(V^\otimes 2)^\times$ such that $R_{12}^t \in \text{Mer}(V^\otimes 2)^\times$ and (1.5), (2.2) and (2.8) are satisfied, then the associated boundary transfer matrices

$$T(x) = \text{Tr}_V (K'(x) \otimes \text{Id}_W)U(x^{-1})^{-1}(K^-(x) \otimes \text{Id}_W)U(x)$$

form a commuting family: $[T(x), T(y)] = 0$.

Let $K \in \text{Mer}(V)$. Regularity for $K$ is the condition

$$K(\pm 1) \propto \text{Id}_V.$$

From (2.13) we immediately derive

Lemma 2.6. Let $K', K^- \in \text{Mer}(V)$ and $U \in \text{Mer}(V \otimes W)$, and let $T$ be given by (2.14). If $K^-$ satisfies (2.15) and if $\pm 1 \in \text{dom}(K')$, then

$$T(\pm 1) \propto (\text{Tr} K'(\pm 1))\text{Id}_W.$$
3. INHOMOGENEOUS TRANSFER MATRICES AND SCATTERING MATRICES

We will now review Gaudin’s observation that the operator appearing in the qKZ equations of A-type and inhomogeneous transfer matrices for periodic systems are related and prove our main theorem, which is the analogon of this for the reflecting case. The various statements are most naturally expressed in the language of Heisenberg spin chains. Integrable inhomogeneous Heisenberg spin chains are obtained when the state space is a tensor product of “local” state spaces \( V_i \) of the remaining tensor factors. Then

\begin{equation}
W \otimes V \otimes V_i \ldots \otimes V_i = R(z_1) \cdots R(z_N)
\end{equation}

where we have labelled the first tensor factor \( V \) by 0 and suppressed the labels 1, \ldots, \( N \) of the remaining tensor factors. Then \( (2.2) \) holds.

Starting from the datum \((R, D, z)\), where \( R \in \text{Mer}(V^\otimes 2) \), \( D \in \text{End}(V) \) and \( z \in (\mathbb{C}^\times)^N \), assume \( (1.3) \) and \( (2.3) \). For \( W = V^\otimes N \), with \( U \) given as in Prop. 3.1, the transfer matrix of an inhomogeneous periodic spin chain is given by, cf. \( (2.1) \),

\begin{equation}
T(x) = T(x; z) = T_0 D_0 R_0 (\frac{z}{z_N}) \cdots R_{01}(\frac{z}{z_1}).
\end{equation}

From Theorem 2.1 and Proposition 3.1 we derive

**Corollary 3.2.** Let \( R \in \text{Mer}(V^\otimes 2) \), \( D \in \text{End}(V) \) and \( z = (z_1, \ldots, z_N) \in (\mathbb{C}^\times)^N \). Construct the inhomogeneous transfer matrices according to \( (3.1) \). Suppose the YBE \( (1.3) \) and the compatibility condition \( (2.3) \) are satisfied. Then we have

\[ [T(x; z), T(y; z)] = 0 \quad \text{for generic } x, y, z. \]

To complete the description of the qKZ equations given in the Introduction for the A-type case we define the qKZ transport matrices \( A_i(z; p) \) in terms of the R-matrices \( R \in \text{Mer}(V)^\times \) and the constant linear operator \( D \in \text{GL}(V) \) as follows:

\begin{equation}
A_i(z; p) := R_{i-1}(\frac{p_{i+1}}{z_{i-1}}) \cdots R_i(\frac{p}{z_1}) D R_{Ni}(\frac{z}{z_i})^{-1} \cdots R_{i+1}(\frac{z_{i+1}}{z_i})^{-1}.
\end{equation}

For these operators we have (see, e.g. \([7\text{ Thm. 5.4}]\)) the following statement.

**Proposition 3.3.** Suppose we have \( R \in \text{Mer}(V)^\times \) and \( D \in \text{End}(V) \) satisfying \( (1.3) \) and \( (2.3) \) and let \( A_i(z; p) \) be given by \( (3.2) \). Then the consistency conditions \( (1.2) \) hold true.

**Remark 3.4.** Note that the local conditions imposed in Cor. 3.2 and Prop. 3.3 are manifestly the same. As we will see in Subsect. 3.2 for reflecting integrable systems this is more subtle.

Let \( R \in \text{End}(V \otimes V) \) and note that the flip \( P \) is a constant solution to the YBE \( (1.3) \). The “initial condition”

\begin{equation}
R(1) \propto P
\end{equation}
is called regularity for $R$. If (3.3) holds true, then by setting $x = 1$ in (1.3) one derives the unitarity condition (2.9).

**Remark 3.5.** The condition (3.3) is required to express the Hamiltonian of the homogeneous XXZ spin chain in terms of $\frac{d}{dx} \log T(x; (1, \ldots, 1))_{x=1}$; this relation was first noted by Lieb [17].

The following result involving Yang’s scattering matrices appears to have been observed for the first time by Gaudin [10, Ch. 10].

**Theorem 3.6.** Assume that $R$ satisfies (2.9) and (3.3). Then the interpolants of the transfer matrix satisfy

$$T(z;\, z) \propto A_i(z; 1) \quad \text{for } i = 1, \ldots, N,$$

for generic values of $z$ and hence $[T(x; z), T(y; z)] = 0$ for $x, y \in \{z_1, \ldots, z_N\}$.

**Proof.** Using (2.9) and (3.3) the statement is easily obtained by straightforwardly moving the permutation operator $P$ past various $R$-matrices:

$$T(z_i; z) \propto \text{Tr}_0 R_0 N(z_i) \cdots R_0 i+1(z_i)\, P_0 R_0 i-1(z_i) \cdots R_0 1(z_i) = \text{Tr}_0 P_0 D_i R_i N(z_i) \cdots R_i i+1(z_i) R_0 i-1(z_i) \cdots R_0 1(z_i)$$

$$= \left(\text{Tr}_0 P_0 R_0 i-1(z_i) \cdots R_0 1(z_i)\right) D_i R_i N(z_i) \cdots R_i i+1(z_i)$$

$$= R_i i-1(z_i) \cdots R_i 1(z_i) \left(\text{Tr}_0 P_0\right) D_i R_i N(z_i) \cdots R_i i+1(z_i)$$

$$= A_i(z; 1).$$

**Remark 3.7.** For periodic systems, the unitarity condition (2.9) is not necessary to derive the qKZ consistency or the commutativity of the transfer matrices, cf. Corollaries 3.2 and 3.3 However, because of the inverted appearance of certain $R$-matrices in $A_i(z; p)$ cf. (3.2), it is required for Thm. 3.6. Alternatively, we could have chosen to replace the inverted $R$-matrices $R_{ij}(\frac{\pi}{2})^{-1}$ in the definition of $A_i(z; p)$ by their “unitarity-counterparts” $R_{ij}(\frac{\pi}{2})$. With this choice, (2.9) would be necessary for the consistency condition of the $A_i$ instead of the proof of $T(z_i; z) \propto A_i(z; 1)$. We argue that with a view of extending Thm. 3.6 to the reflecting case it is more natural to use our choice of $A_i(z; p)$: it will turn out that for the proof of the version of this theorem for reflecting systems both the YBE and regularity for $R$ are needed (so that unitarity for $R$ is guaranteed, as well).

3.2. **Reflecting systems.** Whereas the connection between transfer matrix and qKZ transport matrix has long been understood in the periodic case, in the reflecting case it has been less clear and it is the main purpose of this paper to show that this very connection holds true. Given the datum $(R, K', K^{-}, z)$, with $R \in \text{Mer}(V \otimes \mathbb{C}^N)^\times$, $K', K^{-} \in \text{Mer}(V)$ and $z \in \mathbb{C}^N$ such that (1.3), (1.5) and (2.8) are satisfied, cf. Prop. 2.4 the corresponding boundary monodromy matrix $U \in \text{Mer}(V \otimes V^N)$ satisfying (2.6) is given by

$$U_0(x; z) := R_0 1(\frac{\pi}{2z^i})^{-1} \cdots R_0 N(\frac{\pi}{2z^i})^{-1} K_0^-(x) R_0 N(\frac{\pi}{2z^i}) \cdots R_0 1(\frac{\pi}{2z^i});$$

again we have suppressed the labels 1, ..., $N$ of the factors representing state space. Next, according to (2.5) we define the transfer matrix $T \in \text{Mer}(V^N)$ of the inhomogeneous Heisenberg spin chain with reflecting boundary conditions as

$$T(x; z) := \text{Tr}_0 K_0^+(x) R_0 1(\frac{\pi}{2z^i})^{-1} \cdots R_0 N(\frac{\pi}{2z^i})^{-1} K_0^+(x) R_0 N(\frac{\pi}{2z^i}) \cdots R_0 1(\frac{\pi}{2z^i}).$$
By virtue of Theorem 2.2 and Prop. 3.1, we have

**Corollary 3.8.** Let \( R \in \text{Mer}(V \otimes 2)^{\otimes} \), \( K^+, K^- \in \text{Mer}(V) \) and \( z \in (\mathbb{C}^\times)^N \). Suppose that \( R^+ \in \text{Mer}(V \otimes 2)^{\otimes} \) and \( (1.3), (1.5) \) and \( (2.8) \) are satisfied. Let \( T \) be defined by \( (3.4) \). Then for generic \( x, y, z \) we have

\[
[T(x; z), T(y; z)] = 0.
\]

Here we provide the explicit formula for the bqKZ transport matrix corresponding to the C-type case of Cherednik’s works \([3, 4]\) (or, equivalently, following \([14]\)). The ingredients are \( R \in \text{Mer}(V \otimes 2)^{\otimes} \) and \( K^\pm \in \text{Mer}(V) \). Then the bqKZ transport matrices are the operators \( A_i(z; p) \in \text{End}(V^{\otimes N}) \) are given by

\[
A_i(z; p) = R_{i-1} \cdots R_{i+1} K^+(p^{1/2} z_i) R_{i+2} \cdots R_{i-1} \cdot R_{i+1} (z_{i+1} z_i)^{-1}. \tag{3.5}
\]

The bqKZ equations are given by the system

\[
f(p^{\epsilon_i} z) A_i(z; p) f(z) = 1 \quad \text{for } i = 1, \ldots, N,
\]

for meromorphic functions \( f : (\mathbb{C}^\times)^N \to V^{\otimes N} \). We have the following statement

**Proposition 3.9** \((\text{E.g. } [3, 4])\). Suppose the datum \((R, K^+, K^-)\) satisfies \((1.3) (1.5) \) and construct the \( A_i \) as per \((3.5) \). Then the system \((3.6) \) is consistent, viz. the \( A_i \) satisfy the conditions

\[
A_i(p^{\epsilon_i} z; p) A_j(z; p) = A_j(p^{\epsilon_i} z; p) A_i(z; p) \quad \text{for } i, j = 1, \ldots, N. \tag{3.7}
\]

**Remark 3.10.** Note that Cor. 3.8 and Prop. 3.9 require the same type of conditions; both require the YBE \((1.3)\), the RRE \((1.5)\) and one additional reflection equation (the DRE \((2.8)\) and the LRE \((1.4)\), respectively). This is a consequence from the new result Thm. 2.2 and ties these two notions of integrability more closely together. In Lemma 3.12 we will see that the DRE and the LRE are in fact equivalent (for essentially all R-matrices) and then complete the connection in Thm. 3.13. All of these statements show that there are clear parallels between systems with reflecting boundary conditions and periodic systems.

**3.2.1. Relations between solutions of reflection equations.** In order to connect the boundary transfer matrices \( T(x; z) \) to the bqKZ transport matrices \( A_i(z; p) \), we need to relate solutions of the DRE \((2.8)\), which appear in \( T(x; z) \), to solutions of the LRE, which are used in \( A_i(z; p) \). Fix \( R \in \text{Mer}(V \otimes 2)^{\otimes} \). First we introduce subsets of \( \text{Mer}(V) \) defined by the various reflection equations:

\[
\text{Refl}^+(R) := \{ K^+ \in \text{Mer}(V) \mid \text{the LRE } (1.4) \text{ holds} \},
\]

\[
\text{Refl}^-(R) := \{ K^- \in \text{Mer}(V) \mid \text{the RRE } (1.5) \text{ holds} \},
\]

\[
\text{Refl}'(R) := \{ K' \in \text{Mer}(V) \mid \text{the DRE } (2.8) \text{ holds} \}.
\]

There are several noteworthy relations between these three sets.

Given \( \sigma \in \text{GL}(V) \), define the following bijection \( \chi_\sigma : \text{Mer}(V) \to \text{Mer}(V) \):

\[
\chi_\sigma(Y)(x) = \sigma^{-1} Y(x) \sigma
\]

for \( Y \in \text{Mer}(V) \) and generic \( x \in \mathbb{C} \); evidently \( \chi_\sigma^{-1} = \chi_{\sigma^{-1}} \). If for generic \( x \) we have

\[
R_{12}(x) \sigma_1 \sigma_2 = \sigma_1 \sigma_2 R_{21}(x), \tag{3.8}
\]

we have

\[
\chi_\sigma \circ \text{Refl}^+(R) \subseteq \text{Refl}'(R), \quad \chi_\sigma \circ \text{Refl}^-(R) \subseteq \text{Refl}'(R),
\]

\[
\chi_\sigma \circ \text{Refl}'(R) \subseteq \text{Refl}^+(R), \quad \chi_\sigma \circ \text{Refl}'(R) \subseteq \text{Refl}^-(R).\]
Lemma 3.12. Suppose we have

Now note that by applying (A.5) twice we obtain

Suppose we have

φ

i.e.

Mer(V)κ → Mer(V)κ:

\[ \psi_{M,r}(Y)(x) = Y(rx)^{-1}M, \quad \text{with} \quad \psi_{M,r}^{-1}(Y)(x) = MY(x/r)^{-1}. \]

Provided crossing symmetry (2.10-2.11) are satisfied, ψ_{M,r} restricts to a bijection:

Refl^−(R) ∩ Mer(V)^κ → Refl^+r(R) ∩ Mer(V)^κ, as observed by [23] [18] [8].

There is also a (more elaborate) bijection\(^5\) from Refl^r(R) to Refl^+(R). More precisely, we will show that there exists an invertible C-linear map φ : Mer(V) → Mer(V) which restricts to a bijection from Refl^r(R) to Refl^+(R). Namely, for Y ∈ Mer(V) define φ_R(Y) ∈ Mer(V) by

\[ \phi_R(Y)(x) = \text{Tr}_0 Y_0(x)P_{01}R_{01}(x^2) \quad \text{for generic} \ x. \]

Recall the notation \( \tilde{R} \) defined in (2.7) relevant to the DRE (2.8).

Lemma 3.11. Suppose we have R ∈ Mer(V⊗2) such that \( R^{t_1} \in Mer(V⊗2)^\times \). Then \( \phi_R \) is bijective; in fact \( \phi_R^{-1} = \phi_{\tilde{R}}. \)

Proof. Let Y ∈ Mer(V) and x ∈ C generic. Owing to (A.3) we have

\[ (\phi_R \circ \phi_{\tilde{R}})(Y)(1) = \text{Tr}_0 \phi_R(Y)_0(x)P_{01}R_{01}(x^2) = \text{Tr}_0 Y_0(x)P_{00}^{00}\tilde{R}_{00}^{00}(x^2)P_{01}R_{01}(x^2) = \text{Tr}_0 P_{00}^0 Y_1(x)\text{Tr}_0 P_{00}^0 \tilde{R}_{00}^{00}(x^2)R_{01}(x^2). \]

Now note that by applying (A.5) twice we obtain

\[ \left( \text{Tr}_0 P_{00}^{00} \tilde{R}_{00}^{00}(x^2)R_{01}(x^2) \right)^{t_{0'}} = \text{Tr}_0 \tilde{R}_{01}(x^2)^{t_{0'}}(P_{00}^{00}R_{01}(x^2))^{t_{0'}} = \tilde{R}_{01}(x^2)^{t_{0'}}\left( \text{Tr}_0 R_{01}(x^2)P_{00}^{00} \right)^{t_{0'}} = \tilde{R}_{01}(x^2)^{t_{0'}}R_{01}(x^2)^{t_{0'}} = \text{Id}_{01}, \]

so that \( \text{Tr}_0 P_{00}^{00} \tilde{R}_{00}^{00}(x^2)R_{01}(x^2) = \text{Id}_{01} \). It follows that

\[ (\phi_R \circ \phi_{\tilde{R}})(Y)(1) = \text{Tr}_0 P_{00}^0 Y_1(x) = Y_1(x), \]

i.e. \( \phi_R \circ \phi_{\tilde{R}} = \text{Id}_V \) since \( \tilde{R} = R \) and \( R^{t_1} \in Mer(V⊗2)^\times \) precisely if \( \tilde{R}^{t_1} \in Mer(V⊗2)^\times \), we immediately obtain \( \phi_R \circ \phi_{\tilde{R}} = \text{Id}_V \), which completes the proof. □

Lemma 3.12. Suppose we have R ∈ Mer(V⊗2)^\times such that \( R^{t_1} \in Mer(V⊗2)^\times \) and the YBE (1.3) is satisfied. Then \( \phi_R \) restricts to a bijection: Refl^r(R) → Refl^+(R).

The long and technical proof of this lemma is given in Appendix [B].

Lemmas 3.11 and 3.12 will play a key role in the main Theorem 3.13 of this paper, linking interpolants of inhomogeneous boundary transfer matrices to bqKZ transport matrices.

The bijectons \( \phi_R, \chi_\sigma \) and \( \psi_{M,r} \) are presented diagrammatically in Fig. [I]. Evidently

\(^5\)This bijection is related to Sklyanin’s “less obvious isomorphism” [23] Remark 2 for P-symmetric R-matrices.
Such a relation between solutions of a RE is known as a boundary crossing symmetry. 

3.2.2. The main theorem. We have

\[
T(3.11)
\]

If in addition the REs \((1.4-1.5)\) are satisfied we have

\[
T(3.10)
\]

by \((3.4)\). Let \(\sigma_i, \sigma_{i+1} \in M, r \in M\). Assume \(K' = \phi_R(K^+)\). Then, for generic values of \(z\),

\[
T(3.12)\]

Furthermore, if \(K^+, K^-\) satisfy \((3.9)\) then, for generic values of \(z\),

\[
T(3.13)\]

If in addition the REs \((3.3)\) are satisfied we have \([T(z_i^{\pm 1} \mid z), T(z_j^{\pm 1} \mid z)] = 0\) for all \(1 \leq i, j \leq N\) and all sign choices.

Proof. We have, owing to \((3.3)\),

\[
T(z_i \mid z) \propto \text{Tr} K'_0(z_i) R_{01} \cdot R_{0 \cdot 1} \cdots R_{0 \cdot (i-1)} \cdot R_{0 \cdot i}^{-1} \cdot R_{0 \cdot (i+1)}^{-1} \cdots R_{0 \cdot N}^{-1} \cdot K_0^{-1} (z_i) \cdot R_{0 \cdot (i+1)} \cdots R_{0 \cdot N} \cdot P_{0i} R_{0 \cdot (i+1)} \cdots R_{01} (z_i).
\]

Theorem 3.13. Let \(R \in \text{Mer}(V^{\otimes 2})\) and \(K^+ \in \text{Mer}(V)\) and \(z \in (\mathbb{C}^\times)^N\). Assume the YBE \((1.3)\) and the regularity condition \((3.3)\). Let \(A_i\) be given by \((3.4)\) and \(T\) by \((3.1)\), where we have written \(K' = \phi_R(K^+)\). Then, for generic values of \(z\),

\[
T(3.14)\]

Unitarity for \(K\) is the condition

\[
K(x)K(x^{-1}) \propto x Id_V.
\]

We have

\[
\text{Figure 1. The bijections } \phi_R, \chi, \text{ and } \psi_{M, r} \text{ relating the solutions of the three reflection equations with the necessary assumptions on } R. \text{ Strictly speaking, } \psi_{M, r} \text{ maps between subsets consisting of generically invertible solutions of appropriate reflection equations.}
\]
Moving the factors $P_0$ and $R_{0+i-1}(\frac{z_i}{z_{i-1}}) \cdots R_{01}(\frac{z_1}{z_0})$ to the left and applying (2.9) (a consequence of (3.3) and (3.4)) we have
\[
\mathcal{T}(z_i; z) \propto \prod_0^i K_0'(z_i)P_0R_{01}(z_i) \cdots R_{i-11}(z_iz_{i-1})R_{0i}(z_i^2)R_{0+i-1}(\frac{z_i}{z_{i-1}}) \cdots R_{01}(\frac{z_1}{z_0}).
\]
Now applying the YBE (1.3) repeatedly in the first line of this expression and moving various factors through $P$, we obtain
\[
\mathcal{T}(z_i; z) \propto R_{i+11}(\frac{z_i}{z_{i-1}}) \cdots R_{i1}(\frac{z_i}{z_{i-1}})\left(\prod_0^i K_0'(z_i)P_0R_{0i}(z_i^2)\right) \cdot R_{i1}(z_iz_{i+1}) \cdots R_{i-11}(z_{i-1}z_{i+1})R_{i+11}(z_iz_{i+1}) \cdots R_{i1}(z_iz_{i+1})^{-1}.
\]
Using Lemma 3.11 we recognize the partial trace as $\phi_R(K_k)_{1}(z_i) = K_i^+(z_i)$ and we obtain (3.4). (3.4) is obtained in a similar fashion, but in this case we also need the unitarity condition (3.4) to match the inverted $K$-matrices in $A_i(z; 1)^{-1}$ to the non-inverted ones in $\mathcal{T}(z^{-1}; z)$. For the final commutativity statement we note that the conditions for Prop. 3.3 and Lemma 3.12 are satisfied.

Both the qKZ consistency conditions and the commutativity of transfer matrices can be seen as quantum integrability conditions, and we have highlighted how to go from one to the other in Thm. 3.13 in the reflecting setting. In Fig. 2 we have summarized the main results. The key new results from this paper are Cor. 3.3 and Thm. 3.13 and clearly there is now greater similarity in the required conditions on the R-matrix datum, both comparing between the type of boundary conditions (periodic vs. reflecting) and the type of integrability criterion (commuting transfer matrices vs. qZK consistency condition).

Remark 3.14. Note that, as in the periodic case, the condition (2.4) is not necessary to derive the consistency condition $A_i(p^+; z; p)A_j(z; p) = A_j(p^+; z; p)A_i(z; p)$ or the commutativity $[\mathcal{T}(x; z), \mathcal{T}(y; z)] = 0$; however it is required for the relation $\mathcal{T}(z_i; z) \propto A_i(z; 1)$, where it is a consequence of the explicitly assumed YBE (1.3) and the regularity condition (3.3).

4. The commutativity of transfer matrices revisited

Thms. 3.6 and 3.13 can be wielded to recover the commutativity statements Thms. 2.1 and 2.2 for inhomogeneous transfer matrices built up out of special classes of solutions of the local integrability conditions. It is then possible (cf. Prop. 4) to deduce information about the ground state of certain quantum and statistical mechanics models if the parameter $p$ assumes special values.

The class of solutions we will have in mind is based on the image of the universal R-matrix of $U_q(sl_n)$ in the tensor square of its fundamental (i.e. $n$-dimensional) representation. In particular, let $V = \mathbb{C}^n$ for $n \in \mathbb{Z}_{>1}$ and consider the standard ordered orthonormal basis $(v_\alpha)_{\alpha=1}^n$ of $\mathbb{C}^N$, i.e. the $\alpha$-th entry of $v_\alpha$ is 1 and all other entries of $v_\alpha$ are 0. Fix $q \in \mathbb{C}^\times$ and define $R \in \text{Mer}(V^\otimes 2)^\times$ by
\[
R(x)(v_\alpha \otimes v_\beta) = \begin{cases} 
q(1-x)v_\alpha \otimes v_\beta + (1-q^2)xv_\beta \otimes v_\alpha, & \alpha < \beta, \\
(1-q^2)xv_\alpha \otimes v_\alpha, & \alpha = \beta, \\
(1-q^2)v_\beta \otimes v_\alpha + q(1-x)v_\alpha \otimes v_\beta, & \alpha > \beta.
\end{cases}
\]
We immediately see that $R$ is a polynomial element of $\text{Mer}(V^\otimes 2)$ of degree 1 satisfying (3.3). Using this we can directly check the YBE (1.3); it is of course an existing result, cf. e.g. [2].

4.1. Periodic systems. As an application of Thm. 3.6 we will now present a novel proof of the commutativity of the inhomogeneous transfer matrices $t$ for special classes of $R$-matrices and associated solutions $D$ of (2.3).

Let $R$ be as above and let $D \in \text{GL}(V)$ be diagonal w.r.t. $(v_\alpha)^n_{\alpha=1}$; the compatibility condition (2.3) is satisfied. The special case $n = 2$ of this example corresponds to the datum $(R, D)$ for the periodic inhomogeneous Heisenberg XXZ spin-$\frac{1}{2}$ chain.

In addition to the conditions (1.3), (2.3) and (3.3) necessary for Prop. 3.9 and Thm. 3.6 we need further technical conditions to recover the commutativity of the transfer matrices. Note that we have, for all $\alpha, \beta \in \{1, \ldots, n\},$

\begin{align}
(1.2) & \quad R(x)(v_\alpha \otimes v_\beta) \in \mathbb{C}v_\alpha \otimes v_\beta + \mathbb{C}v_\beta \otimes v_\alpha, \quad \text{for all } x \in \text{dom}(R), \\
(2.3) & \quad R(0)(v_\alpha \otimes v_\beta) \in \sum_{\gamma \leq \alpha, \delta \geq \beta} \mathbb{C}v_\gamma \otimes v_\delta, \\
(3.3) & \quad Dv_\alpha \in \mathbb{C}v_\alpha.
\end{align}

For this datum $(R, D)$ we can derive $[T(x; z), T(y; z)] = 0$ from the consistency condition (1.2).

**Theorem 4.1.** Let $R \in \text{Mer}(V)$ and $D \in \text{End}(V)$. Assume that (1.3), (2.3) and (3.3) hold, that $R$ is a polynomial in $x$ of degree 1, and that, with respect to a certain ordered basis $(v_\alpha)^n_{\alpha=1}$ of $V$ we have (1.2). Then for all $x, y \in \mathbb{C}$ we have $[T(x; z), T(y; z)] = 0$.

**Remark 4.2.** Razumov, Stroganov and Zinn-Justin essentially established this statement for $n = 2$ in [22, Prop. 4]. There it is used to deduce information about the ground state of XXZ spin chains and Temperley-Lieb loop models in case the parameter $p$ assumes root-of-unity values.
For the proof of Thm. 4.1 it is helpful to write \( v_\beta = v_{\beta_1} \otimes \cdots \otimes v_{\beta_N} \) for an \( N \)-tuple \( \beta = (\beta_1, \ldots, \beta_N) \in \{1, \ldots, n\}^N \) and consider the decomposition
\[
V^N = \bigoplus_{\alpha \in S_N} W_\alpha^N, \quad \text{where} \quad W_\alpha^N := \bigoplus_{\beta \in S_N(\alpha)} v_\beta.
\]
Here, \( S_N(\alpha) = \{ (\alpha_{w1}, \ldots, \alpha_{wN}) \mid w \in S_N \} \) for \( \alpha = (\alpha_1, \ldots, \alpha_N) \) denotes the orbit of \( \alpha \) under the standard action of the symmetric group.

**Proof of Thm. 4.1.** From (1.3), (2.3) and (3.3) we deduce that (1.2) holds true. Also, Theorem 3.13 applies; in particular we deduce that \([T(z; z), T(z; z)] = 0\) for \( 1 \leq i, j \leq N \).

Because of the conditions on \( R(x) \) and \( D \), each \( A_i(z; p) \) preserves each subspace \( W_\alpha^N \). On the other hand, Lemma 3.14 applies, yielding that \( T(0; z) \) acts trivially on each \( W_\alpha^N \). Hence, \([T(0; z), A_i(z; 1)] = 0\).

Combining these facts we see that \([T(x; z), T(y; z)] = 0\) where \( x \) and \( y \) assume values in a collection of \( N + 1 \) interpolation points
\[
Z := \{ z_1, \ldots, z_N, 0 \}.
\]
Since \( T(x; z) \) is a polynomial of degree \( N \), we may conclude that, given \( x \in Z \), the polynomials \([T(x; z), T(y; z)]\), of degree \( N \) in \( y \), vanish for \( N + 1 \) values of \( y \); hence these polynomials are zero for all values of \( y \), provided that \( x \in Z \). Therefore for all \( y \in \mathbb{C} \) we can draw the following conclusion: the polynomials \([T(x; z), T(y; z)]\), of degree \( N \) in \( x \), vanish for \( N + 1 \) values of \( x \), so that these polynomials are zero for all values of \( x \). The desired conclusion follows. \( \square \)

### 4.2. Reflecting systems.

The goal of this subsection is to give an alternative proof of the commutativity property \([T(x; z), T(y; z)] = 0\) using the bqKZ consistency conditions for \( p = 1 \), viz. \([A_i(z; 1), A_j(z; 1)] = 0\), for \( R\)- and \( K\)-matrix datum satisfying further conditions, in analogy with the periodic case. In particular, we will rely on the main Thm. 3.13. The main example we will have in mind is the following datum \((R, K^+, K^-)\). With \( V = (v_\alpha)_{\alpha = 1}^n \), let \( R \) be as at the start of this section. For \( \alpha \in \{1, \ldots, n\} \), write \( \bar{\alpha} := n + 1 - \alpha \in \{1, \ldots, n\} \). Then \( R \) satisfies (3.3) with \( \sigma \in \text{GL}(V) \) given by \( \sigma(v_\alpha) = v_{\bar{\alpha}} \).

Also, crossing symmetry (2.10) and the associated condition (2.11) are satisfied, with \( r = q^{h^\vee} = q^n \) and \( M = \text{diag}(q^{2\rho}) = \text{diag}(q^{-1}, q^{-3}, \ldots, q^{-(n-1)}) \), i.e.
\[
M(v_\alpha) = q^{\bar{\alpha}} v_\alpha, \quad \alpha = 1, \ldots, n.
\]

Given \( \theta, \kappa \in \mathbb{C} \) and \( \mu = (\mu_1, \ldots, \mu_{[n/2]}) \in (\mathbb{C}^\times)^{[n/2]} \) we define \( K_{\theta, \kappa, \mu} \in \text{Mer}(V)^\times \) by
\[
K_{\theta, \kappa, \mu}(x)(v_\alpha) = \theta x v_\alpha + \begin{cases} 
(k-1)v_\alpha + \frac{x}{\mu_\alpha} (1-x^2) v_{\bar{\alpha}}, & \alpha < \frac{n+1}{2}, \\
(kx^2 - 1)v_\alpha, & \alpha = \frac{n+1}{2}, \text{ n odd}, \\
\mu_\alpha (1-x^2) v_{\bar{\alpha}} + (k-1)x^2 v_\alpha, & \alpha > \frac{n+1}{2}.
\end{cases}
\]
Evidently, \( K_{\theta, \kappa, \mu} \) is a polynomial element of \( \text{Mer}(V) \) of degree 2 satisfying (2.15). Using these and properties satisfied by \( R \), a straightforward argument shows that \( K_{\theta, \kappa, \mu} \) satisfies the LRE (1.4) and \( K\)-matrix unitarity (3.9). Fix parameters \( \theta^\pm, \kappa^\pm \in \mathbb{C} \).
For all \( R \) and the property \( K \) which follows from the definitions of Prop. 3.9 and Thm. 3.13, we note that the datum \((R, K)\) polynomial in \( x \) satisfies the DRE (2.8). Explicitly it can be checked that, for generic conditions. We have \( T \) were polynomial in the spectral parameter; however the boundary transfer matrix in the periodic case. In the periodic case we relied on the fact that the \( R \)-matrices and assume that \( \pm \) and \( \mu \) satisfy regularity (2.15) and unitarity (3.9).

Moreover, define \( K' \in \text{Mer}(V)^\times \) as in Thm. 3.13 by \( K' = \phi_R(K^+) \), so that \( K' \) satisfies the DRE (2.8). Explicitly it can be checked that, for generic \( x \),

\[
K'(x) = \begin{cases} 
q^{-1}K_{\theta^+,\kappa^+,\mu^+}(q^n x)M, & n \text{ even} \\
K_{q^{-1}\theta^+,-q^{-1}\kappa^+,\Delta_\mu^+}(q^n x)M, & n \text{ odd},
\end{cases}
\]

where \( \Delta_q = \text{diag}(q^{2(n/2)-1}, q^{2(n/2)-3}, \ldots, q) \in \text{GL}(\mathbb{C}^{n/2}) \).

In addition to the identities just discussed, which are necessary for Lemma 2.6, (4.4) \( K'(\pm q^{-n}) \propto M \).

For all \( \alpha \in \{1, \ldots, n\} \) we have

\[
K^-(0)(v_{\alpha}) = \sum_{\gamma \leq \alpha} \mathbb{C}v_{\gamma}, \quad \text{and} \quad K'(0)(v_{\alpha}) = \sum_{\gamma \leq \alpha} \mathbb{C}v_{\gamma},
\]

which follows from the definitions of \( K^- \) and \( K' \) in terms of \( K_{\theta, \kappa, \mu} \), respectively, and the property \( K_{\theta, \kappa, \mu}(0)(v_{\alpha}) \in \sum_{\gamma \geq \alpha} \mathbb{C}v_{\gamma} \). Finally, note that \( K^- \) and \( K' \) are polynomial in \( x \) of degree 2.

Before we state the Theorem, we need to address a subtlety which is absent from the periodic case. In the periodic case we relied on the fact that the \( R \)-matrices were polynomial in the spectral parameter; however the boundary transfer matrix \( T(x; z) \) contains inverses of \( R \)-matrices so that it cannot be polynomial and the analogon of the argument in the proof of Thm. 4.1 will not apply. Therefore we will work with the modified boundary transfer matrix \( \tilde{T} \) defined by

\[
\tilde{T}(x; z) := \text{Tr}_0 K'_0(x)R_{10}(xz_1) \cdots R_{N0}(xz_N)K_0^-(x)R_{0N}(z_{N1}) \cdots R_{01}(\overline{z_1})
\]

which is polynomial in \( x \), provided the \( R^- \) and \( K^- \)-matrices are. We have \( \tilde{T}(x; z) \propto_x T(x; z) \) for generic \( z \) provided unitarity \( (2.4) \) is satisfied (the hidden factor in this proportionality relation only depends on the products \( xz_i \)). In Thm. 3.13 we have seen that, under suitable assumptions on the \( R^- \) and \( K^- \)-datum, the interpolants \( \tilde{T}(z_i^{\pm 1}; z) \), and hence also the modified interpolants \( \tilde{T}(z_i^{\pm 1}; z) \), are proportional to (inverted) bqKZ transport matrices for \( p = 1 \):

\[
\tilde{T}(z_i; z) \propto A(z; 1), \quad \tilde{T}(z_i^{-1}; z) \propto A(z; 1)^{-1}, \quad i = 1, \ldots, N.
\]

Using these modified \( \tilde{T} \) we can derive \([T(x; z), \tilde{T}(y; z)] = 0\) from the bqKZ consistency conditions \( (5.7) \), analogously to Thm. 4.1.

**Theorem 4.3.** Let \( R \in \text{Mer}(V^\otimes 2)^\times \) such that \( R^{t_i} \in \text{Mer}(V^\otimes 2)^\times \), \( K^+, K^- \in \text{Mer}(V)^\times \), \( M \in \text{GL}(V) \), \( r \in \mathbb{C}^x \) and \( z \in (\mathbb{C}^x)^N \). Assume the conditions \( (1.3) \), \( (2.10) \), \( (2.11) \), \( (2.13) \), \( (3.8) \), \( (3.9) \) and \( \pm r^{-1} \in \text{dom}(K^-) \). Write \( K' = \phi_R(K^+) \) and assume that \( \pm 1 \in \text{dom}(K') \) and \( (4.3) \) are satisfied. Assume that \( R, K' \) and...
\(K^-\) are polynomial in \(x\) of degree 1, 2 and 2, respectively, and that, with respect to a certain ordered basis \((v_i)_{i=1}^n\) of \(V\) we have (4.1) and (4.2). Then for all \(x, y \in \text{dom}(T)\) we have \([T(x; z), T(y; z)] = 0\).

Before we give the proof, we consider another decomposition of \(V^\otimes N\). The hyperoctahedral group \(S_N\) acts on \(\{1, \ldots, n\}^N\) by permutations and inversions of entries: when writing \(S_N = S_N \times \langle e_1, \ldots, e_N \rangle\) we let elements from \(S_N\) act by permutations and the \(e_i\) by inversions:

\[e_i(\alpha_1, \ldots, \alpha_N) = (\alpha_1, \ldots, \alpha_{i-1}, \bar{\alpha}_i, \alpha_{i+1}, \ldots, \alpha_N)\]

for \(i = 1, \ldots, N\) and \(\alpha_1, \ldots, \alpha_N \in \{1, \ldots, n\}\). Given an \(N\)-tuple \(\alpha = (\alpha_1, \ldots, \alpha_N) \in \{1, \ldots, n\}^N\), the orbit of \(\alpha\) under \(S_N\) is denoted

\[S_N(\alpha) = \{w(\alpha) \mid w \in S_N\} \subset \{1, \ldots, n\}^N\]

We have the decomposition

\[V^\otimes N = \bigoplus_{\alpha_1 \leq \ldots \leq \alpha_N \leq n+1 \over 2} W^S_\alpha, \quad \text{where} \quad W^S_\alpha := \bigoplus_{\beta \in S_N(\alpha)} v_\beta.\]

**Proof of Thm. 4.3.** Because (1.3) and (1.8) are satisfied, we have (2.4). Hence, \(\bar{T}(x; z) \propto T(x; z)\) for generic values of \(z\). Since (1.3) hold true, the bqKZ consistency conditions (3.7) are satisfied. Hence, \([A_i(z; 1), A_j(z; 1)] = [A_i(z; 1), A_j(z; 1)^{-1}] = [A_i(z; 1)^{-1}, A_j(z; 1)^{-1}] = 0\) for all \(i, j \in \{1, \ldots, N\}\). Consider the modified transfer matrix \(\bar{T}(x; z)\), which is a polynomial in \(x\) of degree \(2N + 4\). Theorem 3.13 combined with Lemmas 2.6 and C.3 (note that \(K^-\) is polynomial so that \(\pm r^{-1} \in \text{dom}(K^-)\)), yields \([\bar{T}(x; z), \bar{T}(y; z)] = 0\) where \(x\) and \(y\) assume values in a collection of \(2N + 5\) interpolation points

\[Z := \{z_1, \ldots, z_N, z_1^{-1}, \ldots, z_N^{-1}, 1, -1, r^{-1}, -r^{-1}, 0\}.
\]

From a similar argument as the one concluding the proof of Thm. 1.4 it follows that the polynomials \([\bar{T}(x; z), \bar{T}(y; z)]\) are zero for all values of \(x\) and \(y\). Hence also the original boundary transfer matrices generically commute, viz. \([\bar{T}(x; z), \bar{T}(y; z)] = 0\) for generic values of \(x\) and \(y\). Because \(\bar{T}(x; z)\) depends meromorphically on \(x\), the desired statement follows. \(\square\)

### 5. Outlook

In Section 2 we have focused on the case where one of the factors in the partial trace that defines the transfer matrices is an element of \(\text{End}(V)\) (\(D\) in the periodic case and \(K'(x)\) in the reflecting case). It is very natural to generalize this to the situation where the transfer matrix is an element of \(\text{Mer}(W^+ \otimes W^-)\) given by the partial trace over a finite-dimensional \(V\) of \(U^+(x)U^-(x)\), with \(U_{\pm} \in \text{Mer}(V \otimes W^\pm)\) respectively. In the periodic case, one easily obtains the analogon of Thm. 2.1 (cf. e.g. [23]). In the reflecting case it is less straightforward, in particular if both \(W^\pm\) are infinite-dimensional. Sklyanin [23] deals with this in the context of a P- and T-symmetric (and unitary and crossing-symmetric) R-matrix. To provide an analogon of Thm. 2.2 for more general \(R\) is work in progress.
With respect to the results of Section 3 one may consider the generalization of Thms. 3.6 and 3.13 to the case where in the state space \( V_1 \otimes \cdots \otimes V_N \) not all \( V_i \) are isomorphic. Then some \( R \)-matrices making up the \((b)qKZ\) transport matrix act in a tensor product of different spaces, and for them no direct analogon of regularity exists. It would be interesting to see in how far the argument can be salvaged.

It should be possible to generalize the analysis in Section 4 in the following ways.

1. There are also polynomial solutions \( K \) of the DRE or RRE of degree 1 (for \( R \) of \( U_q(\mathfrak{sl}_n) \)-type). In this case it seems that only one of \( K(1), K(-1) \) is a multiple of the identity. However, since the degree of \( T \) is reduced by one, we need one fewer interpolation point. Moreover, there are also constant (off-diagonal) solutions \( K \) to the REs, which do not satisfy regularity; however, in this case the degree of \( T \) is reduced by two, and we do not need these interpolation points at all. Hence, conjecturally, the argument of Section 4 can be modified to deal with these cases, as well. A special case of this would be the \( qKZ \) equations associated to affine Weyl groups of B- and D-type (in Cherednik’s framework), which correspond to special choices of constant \( K \)-matrices (for \( K^+ \) in B-type and for both \( K^\pm \) in D-type); note that \( K' \) is constant (up to a scalar factor) if and only if \( K' = \phi_R(K') \) is.

2. We can also look at non-fundamental finite-dimensional representations of \( U_q(\mathfrak{sl}_n) \), in which case \( R \)- and \( K \)-matrices can be obtained from the ones discussed here through fusion (cf. [21] and references therein for the case \( n = 2 \)). Then the degree of \( R \) will be higher, so more interpolation points are needed. For the periodic case, this problem was considered in [9], where additional interpolation points of the transfer matrix were found, related to the \( q \)-dependent shift in the spectral parameter associated with fusion. To extend this to the reflecting case is work in progress.

3. The \( R \)- and \( K \)-matrices from Section 4 are gauge-equivalent to the trigonometric solutions of the additive YBE and REs. It is equally natural to do a similar analysis with rational or elliptic \( R \)- and \( K \)-matrices.

4. It would be interesting to consider \( R \)- and \( K \)-matrices associated to \( U_q(\mathfrak{g}) \) for the orthogonal and symplectic Lie-algebras \( \mathfrak{g} \).

**Appendix A. Some Linear Algebra**

Consider a tensor product \( \otimes_{i \in I} V_i =: V_I \) of complex vector spaces \( V_i \), where \( I \) is a finite ordered set (typically, \{1, \ldots, N\} or \{0, 1, \ldots, N\}). We use standard subscript “tensor leg notation” to turn a local operator (i.e. an operator acting on one or two of the \( V_i \)) into a global operator (one acting on \( V_I \)) by stipulating that it acts nontrivially only in those \( V_i \) specified by the subscript. Thus, for \( i \in I \) and \( Y \in \text{End}(V_i) \) we write \( Y_i \in \text{End}(V_I) \) for the operator that acts trivially in all \( V_j \) where \( j \neq i \) and as \( Y \) in \( V_i \). Similarly, for \( i, j \in I, i \neq j \) and \( X \in \text{End}(V_i \otimes V_j) \), we have the operator \( X_{ij} \in \text{End}(V_I) \) which acts trivially in all \( V_k \) where \( k \neq i, j \) and as \( X \) in \( V_i \otimes V_j \). Furthermore, given parameter-dependent local operators \( X \in \text{Mer}(V_{ij}), Y \in \text{Mer}(V_i) \) we define \( Y_i, X_{ij} \in \text{Mer}(V_I) \) by \( X_{ij}(x) = (X(x))_{ij} \) for \( x \in \text{dom}(X) \) and \( Y_i(x) = (Y(x))_i \) for \( x \in \text{dom}(Y) \).

For finite-dimensional \( V \), we will saliently identify \( V \) and \( V^* \), so that transposition in \( V \) becomes an algebra-antiautomorphism of \( \text{End}(V) \). Partial transposition, i.e. transposing \( X \in \text{End}(V_I) \) with respect to a finite-dimensional \( V_i \) (\( i \in I \)) is
denoted by \( X^{t_i} \in \text{End}(V_j) \). More precisely, the partial transpose in \( V_t \) with respect to \( V_i \) is the unique linear operator \( t_i : \text{End}(V_j) \to \text{End}(V_t) : Z \mapsto Z^{t_i} \) such that

\[
X_i^{t_i} = \text{Id}_{V_i} \otimes X^t \otimes \text{Id}_{V_n}, \quad \text{for } X \in \text{End}(V_i),
\]

where \( V_{<i} = \otimes_{j \in I, j < i} V_j \) and \( V_{>i} = \otimes_{j \in I, j > i} V_j \). From now on, whenever there is a partial transposition with respect to a vector space, it is assumed to be finite-dimensional.

Let \( X \in \text{End}(V_i \otimes V_j) \) and \( \tilde{X} \in \text{End}(V_i \otimes V_k) \). Then in \( \text{End}(V_t) \) we have

\[
\text{(A.1) } (X_{ij} \tilde{X}_{ik})^{t_i} = \tilde{X}_{ik}^{t_i} X_{ij}^{t_i}, \quad (X_{ij} \tilde{X}_{ik})^{t_j} = X_{ij}^{t_j} \tilde{X}_{ik}^{t_j}, \quad (X_{ij} \tilde{X}_{ik})^{t_k} = X_{ij}^{t_k} \tilde{X}_{ik}^{t_k}
\]

Furthermore, the notion of taking the trace of \( X \in \text{End}(V_t) \) with respect to \( V_i \) (“partial trace”) is denoted \( \text{Tr}_i(X) \) or \( \text{Tr}_{V_i}(X) \). More precisely, if \( V_i \) for some \( i \in I \) is finite-dimensional, then the partial trace in \( V_t \) with respect to \( V_i \), denoted \( \text{Tr}_i \) or \( \text{Tr}_{V_i} \), is the unique linear operator: \( \text{End}(V_j) \to \text{End}(V_{t\setminus\{i\}}) \) such that

\[
\text{Tr}_i(X \otimes Y \otimes \tilde{X}) = \text{Tr}(Y^t)X \otimes \tilde{X}, \quad \text{for } X \in \text{End}(V_{<i}), \ Y \in V_i, \ \tilde{X} \in \text{End}(V_{>i}).
\]

Again, whenever a space is traced out from now on we will assume it is finite-dimensional. In particular, we have \( \text{Tr}_i P_{ij} = \text{Id}_{V_i} \) if \( V_i = V_j \), and \( \text{Tr}_i Y_i = (\text{Tr}_i Y^{t})\text{Id}_{V_{t\setminus\{i\}}} \) for any \( Y \in \text{End}(V_i) \). Also, we have the useful relation

\[
\text{(A.2) } \text{Tr}_i Z Y_i = \text{Tr}_i Y_i^t Z \in \text{End}(V_{t\setminus\{i\}}), \quad \text{for } Y \in \text{End}(V_i), \ Z \in \text{End}(V_t).
\]

If \( i, j \in I \) such that \( i \neq j \) then we can consecutively take partial traces with respect to \( V_i \) and \( V_j \); the order of this does not matter and we employ the notation

\[
\text{Tr}_{i,j} := \text{Tr}_i \text{Tr}_j = \text{Tr}_j \text{Tr}_i : \text{End}(V_t) \to \text{End}(V_{t\setminus\{i,j\}}).
\]

We have the identity

\[
\text{(A.3) } \text{Tr}_{i,j} X_{ik} \tilde{X}_{jk} = (\text{Tr}_i X_{ik})(\text{Tr}_j \tilde{X}_{jk}) \in \text{End}(V_{t\setminus\{i,j\}}),
\]

where \( X \in \text{End}(V_i \otimes V_k) \), \( \tilde{X} \in \text{End}(V_j \otimes V_k) \); note that \( \text{Tr}_i X_{ik} \) acts trivially in \( V_j \) so that we may view it as an element of \( \text{End}(V_{t\setminus\{i,j,k\}}) \).

The interplay between partial traces and partial transposes is captured by the following identities in \( \text{End}(V_{t\setminus\{i\}}) \):

\[
\text{(A.4) } \text{Tr}_{i} Z^{t_i} \tilde{Z}^{t_i} = \text{Tr}_{i} Z \tilde{Z}, \quad \text{for } Z, \tilde{Z} \in \text{End}(V_t),
\]

\[
\text{(A.5) } \text{Tr}_{i} (Z^{t_i}) = (\text{Tr}_{i} Z)^{t_i}, \quad \text{for } Z \in \text{End}(V_i), \ j \in I \setminus \{i\}.
\]

**Appendix B. Proof of Lemma 3.12**

**Proof.** First we will show that \( \phi_{R}(\text{Refl}'(R)) \subset \text{Refl}^{+}(R) \), in other words derive

\[
R_{12}(x/y)\phi_{R}(K')_1(x)R_{21}(xy)\phi_{R}(K')_2(y) = \phi_{R}(K')_2(y)R_{12}(xy)\phi_{R}(K')_1(x)R_{21}(x/y)
\]

from the DRE \([2.8]\) for generic values of \( x, y \). Owing to \([A.4]\) we have

\[
\phi_{R}(K')_1(x) = \text{Tr}_{0}(K'_{0}(x)P_{01})^{t_{0}} R_{01}(x^{2})^{t_{0}}.
\]

Owing to \([A.3]\), the identity \( \hat{R}_{00'}(xy)^{t_{0}}R_{00'}(xy)^{t_{0}} = \text{Id}_{V^{02}} \) and \([A.1]\) we have

\[
\phi_{R}(K')_1(x)R_{21}(xy)\phi_{R}(K')_2(y) = \text{Tr}_{0}(K'_{0}(x)P_{01})^{t_{0}} R_{01}(x^{2})^{t_{0}} R_{21}(xy) \text{Tr}_{0'}(K'_{0}(y)P_{0'}R_{0'})(y^{2})
\]
It follows that (B.4) leads to
\[
\phi_R(K')_1(x) R_{21}(x) y R_{12}(x/y) = R_{21}(x) y R_{12}(x/y)
\]
(B.1)

It follows that
\[
R_{12}(\tilde{x}_y) \phi_R(K')_1(x) R_{21}(x) y R_{12}(x/y) = R_{12}(x/y) R_{12}(x/y)
\]
where we have applied (B.3) twice and (A.2). Now applying the DRE (2.8) followed by (A.3) (twice) we obtain
\[
R_{12}(\tilde{x}_y) \phi_R(K')_1(x) R_{21}(x) y R_{12}(x/y) = \phi_R(K')_1(x) R_{21}(x) y R_{12}(x/y)
\]
(B.2)

which equals \( \phi_R(K')_1(x) R_{21}(x/y) \) as desired by virtue of (B.1).

It remains to show that \( \phi_R(\text{Ref}^+(R)) \subset \text{Ref}^+(R) \) which can be done in an analogous way as before, with the following modifications. Instead of inserting \( R_{00'}(x/y) R_{00'}(x/y) = \text{Id}_{V \otimes V} \) we insert \( R_{00'}(x/y) R_{00'}(x/y) = \text{Id}_{V \otimes V} \). Initially it leads to
\[
\phi_R(K')_1(x) R_{21}(x/y) \phi_R(K')_1(x) R_{21}(x/y) = \phi_R(K')_1(x) R_{21}(x/y) \phi_R(K')_1(x) R_{21}(x/y)
\]
(B.3)

Now we claim that, since \( R \) satisfies the YBE (1.3), for generic values \( x, y \) we have
\[
R_{12}(\tilde{x}_y) R_{12}(x/y) = R_{23}(y) R_{13}(x) R_{12}(y)_y^{-1}
\]
which can be straightforwardly checked using (A.1). Repeated use of (B.4) instead of (1.3), as well as applying (A.2) and (A.4) now allows us to continue along the same lines as before:
\[
R_{12}(\tilde{x}_y) R_{12}(x/y) R_{12}(x/y) \phi_R(K')_1(x) R_{21}(x/y) \phi_R(K')_1(x) R_{21}(x/y) = R_{12}(x/y) R_{12}(x/y) R_{12}(x/y) R_{12}(x/y)
\]
Recall from the proof of Lemma C.1 that \((4.2)\) implies \((C.1)\). In the same way we can derive
\[
\begin{align*}
\bar{R}(v_\alpha \otimes v_\beta) &= \sum_{\gamma, \delta} r_{\alpha \beta}^{\gamma \delta} v_\gamma \otimes v_\delta, \\
\bar{K}(v_\alpha) &= \sum_{\gamma} k_\alpha^\gamma v_\gamma, \\
\bar{L}(v_\alpha) &= \sum_{\gamma} l_\alpha^\gamma v_\gamma.
\end{align*}
\]

**Appendix C. Properties of R- and K-matrices associated to the fundamental representation of \(U_q(\hat{\mathfrak{sl}}_n)\)**

For the next two lemmas, denote the matrix entries of \(\hat{R} := R(0), D, \hat{K} := K^-(0)\) and \(\hat{L} := K'(0)\) by \(r_{\alpha \beta}^{\gamma \delta},d_\alpha^\gamma,k_\alpha^\gamma,l_\alpha^\gamma \in \mathbb{C}\), respectively, for \(\alpha, \beta, \gamma, \delta \in \{1, \ldots, n\}\), viz.

\[
\begin{align*}
\hat{R}(v_\alpha \otimes v_\beta) &= \sum_{\gamma, \delta} r_{\alpha \beta}^{\gamma \delta} v_\gamma \otimes v_\delta, & D(v_\alpha) &= \sum_{\gamma} d_\alpha^\gamma v_\gamma, \\
\hat{K}(v_\alpha) &= \sum_{\gamma} k_\alpha^\gamma v_\gamma, & \hat{L}(v_\alpha) &= \sum_{\gamma} l_\alpha^\gamma v_\gamma.
\end{align*}
\]

**Lemma C.1.** Let \(R \in \text{Mer}(V)\) and \(D \in \text{End}(V)\). Assume that \(0 \in \text{dom}(R)\) and that, with respect to a certain ordered basis \((v_\alpha)_\alpha=1^N\) of \(V\), we have \((4.2)\) and \((4.3)\).

Then for all \(N\)-tuples \(\alpha\) satisfying \(1 \leq \alpha_1 \leq \cdots \leq \alpha_N \leq n\), \(T(0; z)\) acts trivially on each subspace \(W_\alpha^S\), i.e. there exists \(C_\alpha \in \mathbb{C}\) such that

\[
T(0; z)(v_\beta) = C_\alpha v_\beta, \quad \text{for } v_\beta \in W_\alpha^S.
\]

**Proof.** The condition \((4.2)\) can be generalized by induction with respect to \(N\) to

\[
\bar{R}_{0N} \cdots \bar{R}_{01}(v_\alpha \otimes v_\beta) = \left( \prod_{i=1}^{N} r_{\alpha \beta_i}^{\alpha_i \gamma_i} \right) v_\alpha \otimes v_\beta + \sum_{\gamma < \alpha} v_\gamma \otimes V^{\otimes N}
\]

for \(\beta \in \{1, \ldots, n\}^N\). Hence

\[
D_0 \bar{R}_{0N} \cdots \bar{R}_{01}(v_\alpha \otimes v_\beta) \in d_\alpha \left( \prod_{i=1}^{N} r_{\alpha \beta_i}^{\alpha_i \gamma_i} \right) v_\alpha \otimes v_\beta + \sum_{\gamma < \alpha} v_\gamma \otimes V^{\otimes N}
\]

and

\[
T(0; z)(v_\beta) = \left( \sum_{\alpha_1=0}^{n} d_\alpha \prod_{i=1}^{N} r_{\alpha \beta_i}^{\alpha_i \gamma_i} \right) v_\beta.
\]

Evidently the coefficient in front of \(v_\beta\) is unchanged if we permute the \(\beta_i\). \(\square\)

**Lemma C.2.** Let \(R \in \text{Mer}(V^{\otimes 2})^\times, K', K^- \in \text{Mer}(V)^\times\) and \(z \in (\mathbb{C}^\times)^N\). Suppose that there exists \(\sigma \in \text{GL}(V)\) such that \((3.3)\) holds. Assume that \(0\) is in the domains of \(R, K'\) and \(K^-\) and that, with respect to a certain ordered basis \((v_\alpha)_\alpha=1^N\) of \(V\), we have \((4.2)\) and \((4.3)\). Then for all \(N\)-tuples \(\alpha\) satisfying \(1 \leq \alpha_1 \leq \cdots \leq \alpha_N \leq n+1\), \(\hat{T}(0; z)\) acts trivially on each subspace \(W_\alpha^S\), i.e. there exists \(C_\alpha \in \mathbb{C}\) such that

\[
\hat{T}(0; z)(v_\beta) = C_\alpha v_\beta, \quad \text{if } v_\beta \in W_\alpha^S.
\]

**Proof.** Recall from the proof of Lemma C.1 that \((4.2)\) implies \((4.4)\). In the same way we can derive

\[
\bar{R}_{10} \cdots \bar{R}_{N0}(v_\alpha \otimes v_\beta) = \left( \prod_{i=1}^{N} r_{\beta_i \alpha}^{\beta_i \alpha_i} \right) v_\alpha \otimes v_\beta + \sum_{\gamma < \alpha} v_\gamma \otimes V^{\otimes N},
\]

By virtue of \((3.3)\) the right-hand side of the last expression can be seen to equal \(\phi_R(K'^+)(y)\bar{R}_{12}(x)\phi_R(K^+)(x)R_{21}(x)^{-1}\), yielding the desired conclusion. \(\square\)
where owing to (4.3) we may re-write each $r_{i,\bar{a}}^{\alpha,\beta}$ as $r_{\bar{a},\beta}^{\alpha,\beta}$. Combining the above remarks with (4.4) we obtain

$$
\bar{K}_0\bar{R}_0\cdots \bar{R}_1(v_\alpha \otimes v_\beta) \in k_0^{\bar{\alpha}} \left( \prod_{i=1}^{N} r_{\bar{a},\beta_i}^{\alpha,\beta_i} \right) v_\alpha \otimes v_\beta + \sum_{\gamma < \alpha} v_\gamma \otimes V^{\otimes N},
$$

$$
\bar{L}_0\bar{R}_{10}\cdots \bar{R}_{N0}(v_\alpha \otimes v_\beta) \in l_0^{\alpha} \left( \prod_{i=1}^{N} r_{\beta_i,\bar{\alpha_i}}^{\alpha,\beta_i} \right) v_\alpha \otimes v_\beta + \sum_{\gamma < \alpha} v_\gamma \otimes V^{\otimes N},
$$

so that

$$
\bar{L}_0\bar{R}_{10}\cdots \bar{R}_{N0}\bar{K}_0\bar{R}_0\cdots \bar{R}_1(v_\alpha \otimes v_\beta) \in
k_0^{\bar{\alpha}}l_0^{\alpha} \left( \prod_{i=1}^{N} r_{\alpha_i,\bar{\beta}_i}^{\alpha,\beta_i} \right) v_\alpha \otimes v_\beta + \sum_{\gamma < \alpha} v_\gamma \otimes V^{\otimes N}.
$$

Hence any $v_\beta$ is an eigenfunction of $\bar{T}(0; z)$ with the eigenvalue invariant under the action of $\mathcal{S}_N$:

$$
\bar{T}(0; z)(v_\beta) = \left( \sum_{\alpha=1}^{N} k_0^{\bar{\alpha}}l_0^{\alpha} \left( \prod_{i=1}^{N} r_{\alpha_i,\bar{\beta}_i}^{\alpha,\beta_i} \right) \right) v_\beta.
$$

Lemma C.3. Let $R \in \text{Mer}(V^{\otimes 2})$, $K^{-}, K' \in \text{Mer}(V)^*$, $M \in \text{GL}(V)$, $r \in \mathbb{C}^{\times}$ and $z \in (\mathbb{C}^{\times})^N$. Let $\mathcal{T}$ be given by (3.4). If (2.10), (2.11), (4.3) and $\pm r^{-1} \in \text{dom}(K^{-})$ are satisfied then

$$
\mathcal{T}(\pm r^{-1}; z) \propto (\text{Tr } K^{-}(\pm r^{-1})M)\text{Id}_{V^{\otimes N}}.
$$

Proof. Using (4.3) we have

$$
\mathcal{T}(\pm r^{-1}; z) \propto \text{Tr } M_0 R_{01}(\pm r z_1^{-1})^{-1} \cdots R_{0N}(\pm r z_N^{-1})^{-1}.
$$

Hence, applying (A.4) and (A.11) we have

$$
\mathcal{T}(\pm r^{-1}; z) = \text{Tr } M_0 R_{01}(\pm r z_1^{-1})^{-1} \cdots R_{0N}(\pm r z_N^{-1})^{-1} K_0^{-}(\pm r^{-1})^{t_0} \cdot (R_{0N}(\pm r z_N^{-1})^{-1} \cdots R_{01}(\pm r z_1^{-1})^{-1})^{t_0} M_0^t.
$$

Combining (2.10) with (2.11) yields, for generic values of $z$,

$$
M_0^t(R_0(\pm r^{-1} z_1^{-1})^{t_0} \propto (R_0(\pm r z_1^{-1})^{t_0} M_0^t.
$$

Repeatedly applying this we obtain $\mathcal{T}(\pm r^{-1}; z) \propto \text{Tr } M_0(\pm r^{-1})^{t} M_0^t$, from which the Lemma follows after applying (A.4)

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