A large Hilbert space QRPA and RQRPA calculation of neutrinoless double beta decay *

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Abstract

A large Hilbert space is used for the calculation of the nuclear matrix elements governing the light neutrino mass mediated mode of neutrinoless double beta decay (0νββ-decay) of 76Ge, 100Mo, 116Cd, 128Te and 136Xe within the proton-neutron quasiparticle random phase approximation (pn-QRPA) and the renormalized QRPA with proton-neutron pairing (full-RQRPA) methods. We have found that the nuclear matrix elements obtained with the standard pn-QRPA for several nuclear transitions are extremely sensitive to the renormalization of the particle-particle component of the residual interaction of the nuclear hamiltonian. Therefore the standard pn-QRPA does not guarantee the necessary accuracy to allow us to extract a reliable limit on the

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effective neutrino mass. This behaviour, already known from the calculation of the two-neutrino double beta decay matrix elements, manifests itself in the neutrinoless double-beta decay but only if a large model space is used. The full-RQRPA, which takes into account proton-neutron pairing and considers the Pauli principle in an approximate way, offers a stable solution in the physically acceptable region of the particle-particle strength. In this way more accurate values on the effective neutrino mass have been deduced from the experimental lower limits of the half-lifes of neutrinoless double beta decay.
I. INTRODUCTION

The neutrinoless double beta decay ($0\nu\beta\beta$-decay) continues to attract the attention of both experimentalists and theoreticians for a long period. In this process the nucleus ($A,Z$) undergoes the transition to nucleus ($A,Z+2$) with the emission of two electrons:

$$(A,Z) \rightarrow (A,Z+2) + 2e^-.$$  \hspace{1cm} (1)

It is obvious that this process violates the lepton number $L_e$ by two units and is forbidden in the Standard Model. The background considerations imply that the $0\nu\beta\beta$-decay is measured on nuclei for which the ordinary single beta decay is either forbidden by energy law conservation or strongly inhibited by large spin changes. The $0\nu\beta\beta$-decay has not been seen in the experiment till now. A possible detection would be undoubtedly a signal of new physics beyond the standard model. The experimental lower limits provide us e.g. with the most stringent limits on the effective neutrino mass, and the parameters of right-handed currents and coupling constants of the supersymmetric particles. If neutrinos turn out to be massive, the $0\nu\beta\beta$-decay experiment is considered to be the most sensitive to the existence of Majorana neutrinos coupled to electron. It is worthwhile to notice that there is another two-neutrino mode of double beta ($2\nu\beta\beta$-decay) with two antineutrinos and electrons in the final state, which is allowed within the Standard Model. This mode being independent of the unknown particle physics parameters serves as a sensitive test of nuclear structure calculations. There exist extensive reviews of the theory and phenomenology of the double beta decay and we refer the interested reader e.g., to Refs. [1]-[5] for details. New contributions to the R-parity violating ($R_p$) supersymmetry (SUSY) mechanisms in $0\nu\beta\beta$-decay are discussed in Ref. [6].

In order to deduce the limits on the parameters from the particle physics point of view, it is necessary to calculate the corresponding nuclear matrix elements. The proton-neutron Quasiparticle Random Phase Approximation (pn-QRPA) has been considered the most practical method for nuclear structure calculations of nuclear systems which are far away from
closed shells [7]-[14]. However, the extreme sensitivity of the calculated $2\nu\beta\beta$-decay matrix elements in the physically acceptable region of the particle-particle strength of the nuclear Hamiltonian renders it difficult to make definite rate predictions [12]-[14]. This quenching behavior of the $2\nu\beta\beta$-decay matrix elements is a puzzle and has attracted the attention of many theoreticians. Other shortcomings of the pn-QRPA (e.g., particle number non-conservation, the question of the proton-neutron pairing, the violation of the Pauli exclusion principle etc.) indicate that we have to go beyond the pn-QRPA in the evaluation of the double beta decay nuclear matrix elements. Toivanen and Suhonen have proposed a proton-neutron renormalized QRPA (pn-RQRPA) to study the double beta decay [15]. The pn-QRPA is based on the renormalized quasiboson approximation, which considers the Pauli exclusion principle in an approximate way. Schwieger, Šimkovic and Faessler have extended the pn-RQRPA to include proton-neutron pairing (full-RQRPA) [16]. The $2\nu\beta\beta$-decay matrix elements calculated via pn-RQRPA and full-RQRPA have been found significantly less sensitive to the increasing strength of the particle-particle interaction [15]-[17]. This fact indicated that both the Pauli exclusion principle and proton-neutron pairing play an important role in the evaluation of the many-body Green functions for the double beta decay.

In the meanwhile, some critical studies have shown that the renormalized QRPA has a few shortcomings, e.g., the violation of the Ikeda sum rule [17,18]. J. Hirsch et al. [18,19] and J. Engel et al. [20] studied the validity of the renormalized QRPA within schematic exactly solvable models. Their studies confirmed that the renormalized QRPA offers advantages over the QRPA. But, they found some discrepancies between the exact and the RQRPA solutions after the point of collapse of the QRPA. However it is not clear whether their results would also hold for realistic calculations with large model spaces and realistic effective NN-interactions. In fact we do not know of any exactly solvable realistic model. The exact solution for the intermediate nuclear state discussed in Refs. [18,19] within a schematic one level model space, after the point of collapse of QRPA, is below the initial ground state. This is not however the case of nuclei which undergo double beta decay. We note also that the violation of the Ikeda sum rule within the full-RQRPA with a large model space and a
realistic effective NN-interaction is rather small (about 10-20 %) \[21\].

We believe that the full-RQRPA is the most reliable method to deduce the desired interesting lepton number non-conserving parameters from the experimental lower limits on the half-lifes of neutrinoless double beta decay of heavier nuclei at present. To our knowledge the full-RQRPA has been applied for the first time by Šimkovic et al. \[21\] in calculations of $0\nu\beta\beta$-decay. It has been found there by calculating the $0\nu\beta\beta$-decay of $^{76}\text{Ge}$ via the pn-QRPA and full-RQRPA that by increasing the model space the pn-QRPA results are extremely sensitive to the renormalization of the particle-particle interaction in the physically acceptable region of the nuclear strength. This behaviour was similar to the one known from $2\nu\beta\beta$-decay calculations. On the other hand the full-RQRPA results show an increased stability in respect to the strength of the particle-particle interaction with increasing model space. This then sugest that the quenching of the $2\nu\beta\beta$-decay matrix elements by the pn-QRPA is not a special phenomenon for the $2\nu\beta\beta$-decay process but is a common feature to all many-body Green functions defining second order processes. As this quenching seems to have its origin in an inaccuracy of the calculation steming from the quasiboson approximation scheme it is necessary to recalculate the $0\nu\beta\beta$-decay matrix elements with the full-RQRPA, which takes into account the Pauli exclusion principle and therefore one can expect to deduce more reliable limits on the lepton number non-conserving parameters.

In this article, we shall study the nuclear matrix elements of light neutrino mass mediated mode of $0\nu\beta\beta$-decay of $^{76}\text{Ge}$, $^{100}\text{Mo}$, $^{116}\text{Cd}$, $^{128}\text{Te}$ and $^{136}\text{Xe}$. The most stringent experimental lower limit of the life-time and large phase-space factors favour especially these processes for extracting an upper limit on the effective neutrino mass. Our first consideration is to find out whether the strong sensitivity of the $0\nu\beta\beta$-decay matrix elements within the pn-QRPA is a general feature to all double beta decay transitions. For that purpose we perform our calculations with a considerably large Hilbert model space. Our second task is to calculate the $0\nu\beta\beta$-decay matrix elements within the full-RQRPA and investigate in this way whether one might be able to extract more accurate values on the effective neutrino
mass.

II. THEORY

In the case of the light neutrino mass mediated mode of the $0\nu\beta\beta$-decay the weak beta decay Hamiltonian acquires the form:

$$\mathcal{H}^\beta(x) = \frac{G_F}{\sqrt{2}} 2 [\bar{e}_L(x) \gamma_\alpha \nu_{e L}(x)] j_\alpha(x) + \text{h.c.},$$  

(2)

where $j_\alpha(x)$ is the strangeness conserving charged hadron currents and $e_L(x)$ and $\nu_{e L}(x)$ are operators of the left components of fields of the electron and electron neutrino, respectively. We suppose that neutrino mixing does take place according to,

$$\nu_{e L} = \sum_k U_{e k}^L \chi_{k L},$$  

(3)

where, $\chi_{k L}$ is the field of the Majorana neutrinos with mass $m_k$ and $U_{e k}^L$ is unitary mixing matrix.

If we consider the usual approximations, i.e., non-relativistic momentum approximation for hadron currents and long-wave approximation, and replace the energies of the outgoing electrons in the denominators with the half of the available energy for this process, we get the following matrix element:

$$<f | S^{(2)} | i> = \frac{i}{2(2\pi)^3} \left( \frac{G_F}{\sqrt{2}} \right)^2 \frac{1}{\sqrt{p_{10}p_{20}}} <m_\nu > \frac{g_A^2}{R} \times$$

$$\bar{u}(p_1)(1 - \gamma_5) C \bar{u}^T(p_2)$$

(4)

Here, $p_1, p_2$ are the four-vector momenta of the electrons and $E^i, E^f$ are respectively the energies of the initial and final nuclear states. The effective neutrino mass is given as follows:

$$<m_\nu> = \sum_j |U_{ej}|^2 m_j e^{i\alpha_j},$$  

(5)

where $\exp(i\alpha_j)$ is the CP eigenvalue of the neutrino mass eigenstate $|\chi_j>$. The nuclear matrix element of the process takes the form:
\[ M_{\text{mass}}^0 = M_{GT}^0 - \left( \frac{g_V}{g_A} \right)^2 M_F^0, \]  

(6)

The Gamow-Teller and Fermi nuclear matrix elements take the form

\[ M_{GT}^0 = R \sum_{mJ,L} \int_0^{\infty} q^2 dq < 0^+_f | \mathcal{O}_{GT}^0(qr; L, J) | Jm > < Jm | \mathcal{O}_{GT}^0(qr; L, J) | 0^+_i > \left[ q_0 + E^m_j - (E^i + E^f)/2 \right], \]  

(7)

\[ M_F^0 = R \sum_{mJ} \int_0^{\infty} q^2 dq < 0^+_f | \mathcal{O}_F^0(qr, J) | Jm > < Jm | \mathcal{O}_F^0(qr, J) | 0^+_i > \left[ q_0 + E^m_j - (E^i + E^f)/2 \right], \]  

(8)

with

\[ \mathcal{O}_{GT}^0(qr, J) = \sum_k \tau^+_k 2\sqrt{2} i^L j_L(qr_k) \{ Y_L \otimes \sigma(k)_1 \}_J, \]  

(9)

\[ \mathcal{O}_F^0(qr, J) = \sum_k \tau^+_k 2\sqrt{2} i^L j_L(qr_k) Y_J, \]  

(10)

Here, \( R = r_0 A^{1/3} \) is the nuclear radius (\( r_0 = 1.1 \) fm), \( g_V = 1.00, g_A = 1.25 \) and \( q_0 \approx q \) for light neutrino. In the formulae of the \( M_{GT}^0 \) and \( M_F^0 \) in Eqs. (7) and (8) it is somehow difficult to include the correlation function of the two interacting nucleons. Some attempts have been made by Krmpotić and Sharma [11]. However, the operators \( M_{GT}^0 \) and \( M_F^0 \) in (7) and (8) containing two one-body matrix elements are usually transformed to ones containing two-body matrix elements in relative coordinates by using the second quantization formalism.

We then obtain:

\[ \begin{align*}  
M_{GT}^0 & = \left\langle H(r_{12}) \sigma_1 \cdot \sigma_2 \right\rangle,  
M_F^0 & = \left\langle H(r_{12}) \right\rangle,  
\end{align*} \]  

(11)

\[ < O_{12} > = \sum_{k\ell i j k' \ell' i' J, J^*} (-)^{j_i + j_{k'} + J + J^*} (2J + 1) \begin{pmatrix} j_k & j_i & J \\ j_{k'} & j_{i'} & J^* \end{pmatrix} \times \left[ c_{pk} c_{\ell r} \right] J | f(r_{12}) \tau^+_2 \tau^+_k \mathcal{O}_{12} f(r_{12})| n l, n l'; J > \times < 0^+_f | J^* m_f > < J^* m_f | J^* m_i > < J^* m_i | [c_{pk} c_{\ell r}] J | 0^+_i > . \]  

(12)

The short-range correlations between the two interacting nucleons are now taken into account by a correlation function [3][4].
\[ f(r_{12}) = 1 - e^{-ar_{12}^2} (1 - br_{12}^2) \] (13)

with \( a = 1.1 \, \text{fm}^{-2} \), \( b = 0.68 \, \text{fm}^{-2} \). The neutrino-potential \( H(r) \) takes the form

\[
H(r) = \frac{2R}{\pi r} \int_0^\infty \frac{\sin(qr)}{q + (\Omega_{j\pi}^{m_i} + \Omega_{j\pi}^{m_i})/2} \frac{1}{(1 + q^2/\Lambda^2)^4} dq.
\] (14)

The parameter \( \Lambda \) of the dipole shape nucleon form factor is chosen to be 0.85 GeV \[5,10\]. \( \Omega_{j\pi}^{m_i} = E_{j\pi}^{m_i} - E_{0^+} \) and \( \Omega_{j\pi}^{m_i} = E_{j\pi}^{m_i} - E_{0^+} \).

For the half-life of the 0\( \nu \beta \beta \)-decay we obtain:

\[
[T_{1/2}^{0\nu}]^{-1} = G_{01} (M_{mass}^{0\nu})^2 \left( \frac{< m_\nu >}{m_e} \right)^2.
\] (15)

\( G_{01} \) is the integrated kinematical factor for the 0\( \nu \rightarrow 0_f \) transition \[1,3,23\].

In order to calculate the nuclear matrix element \( M_{mass}^{0\nu} \) the full set of the intermediate nuclear states has to be constructed e.g., by the QRPA or RQRPA diagonalization. The full-RQRPA, which describes the excited states of the even-even nucleus, has been studied in Ref. \[16\] and the pn-QRPA, which is a special case of the full-RQRPA in Refs. \[12–14\]. Therefore, here we shall present only the formulae relevant to this work.

The quasiparticle creation and annihilation operators \( a_{\mu \alpha m}^+ \) and \( a_{\mu \alpha m} \), \( \mu = 1, 2 \) are defined through the Hartree- Fock- Bogoliubov (HFB) transformation, which includes proton-proton, neutron-neutron and proton-neutron pairing \[22,23\]:

\[
\begin{pmatrix}
  c_{pkm_k}^+ \\
  c_{nkkm_k}^+ \\
  \tilde{c}_{pkm_k} \\
  \tilde{c}_{nk\tilde{m}_k}
\end{pmatrix} =
\begin{pmatrix}
  u_{k1p} & u_{k2p} & -v_{k1p} & -v_{k2p} \\
  u_{k1n} & u_{k2n} & -v_{k1n} & -v_{k2n} \\
  v_{k1p} & v_{k2p} & u_{k1p} & u_{k2p} \\
  v_{k1n} & v_{k2n} & u_{k1n} & u_{k2n}
\end{pmatrix}
\begin{pmatrix}
  a_{1km_k}^+ \\
  a_{2km_k}^+ \\
  \tilde{a}_{1km_k} \\
  \tilde{a}_{2km_k}
\end{pmatrix}.
\] (16)

Here, \( c_{\tau m}^+ \) \( (c_{\tau m}) \) denotes the particle creation (annihilation) operator acting on a single particle level \( k \) with isospin \( \tau = p, n \). The tilde \( \sim \) indicates the time reversed states \( \tilde{c}_{\tau km_k} = (-1)^{j_k - m_k} c_{\tau k - m_k} \) etc.

In the full-RQRPA the commutator of two bifermion operators is replaced with its expectation value in the correlated QRPA ground state \( |0_{QRPA}^+ \rangle \) (renormalized quasiboson approximation). We have
\[
\langle 0^+_\text{QRP A} | [A_{\mu \nu}^+(k, l, J, M), A_{\mu' \nu'}^+(k', l', J, M)] | 0^+_\text{QRP A} \rangle
= n(k\mu, l\nu) n(k'\mu', l'\nu') \left( \delta_{kk'} \delta_{\mu \mu'} \delta_{ll'} \delta_{\nu \nu'} - \delta_{lk'} \delta_{\nu' \mu} \delta_{kl} \delta_{\mu' \nu} (-1)^{j_k + j_{l'} - J} \right)
\times \left\{ 1 - \frac{1}{j_l} < 0^+_\text{QRP A} | \{ a_{\nu l} \tilde{a}_{\nu l} \} | 0 | 0^+_\text{QRP A} > \right\},
\]

with \( \hat{j}_k = \sqrt{2j_k + 1} \). The operator \( A_{\mu \nu}^+(k, l, J, M) \) creates a pair of quasiparticles coupled to angular momentum \( J \) with projection \( M \).

\[
A_{\mu \nu}^+(k, l, J, M) = n(k\mu, l\nu) \sum_{m_k, m_l} C_{Jk, j_m, j_{m_l}}^{JM} a_{\mu, k}^+ a_{\nu, l}^+,
\]

\[n(k\mu, l\nu) = (1 + (-1)^j \delta_{kk'} \delta_{\mu \mu'})/(1 + \delta_{kk'} \delta_{\mu \mu'})^{3/2} \] (18)

If we replace \( |0^+_\text{QRP A} > \) in Eq. (17) with the uncorrelated HFB ground state, we obtain the quasiboson approximation (i.e. \( D_{\mu \nu, k, l; J, \pi} = 1 \)), which violates the Pauli exclusion principle by neglecting the terms coming from the commutator. The full-RQRPA takes into account the Pauli exclusion principle more carefully. The coefficients \( D_{\mu \nu, k, l; J, \pi} \), which renormalize the particle-hole and particle-particle interaction entering the \( A \) and \( B \) matrices of the full-RQRPA equation

\[
\begin{pmatrix}
A & B \\
B & A
\end{pmatrix}_{J^*} \begin{pmatrix}
\bar{X}^m \\
\bar{Y}^m
\end{pmatrix}_{J^*} = \Omega^m_{J^*} \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix} \begin{pmatrix}
\bar{X}^m \\
\bar{Y}^m
\end{pmatrix}_{J^*},
\]

are determined by solving numerically the system of non-linear equations (14) (15):

\[
D_{k\mu \nu, J^*} = 1 - \frac{1}{j_k^2} \sum_{k', \mu', l', \nu'} D_{k \mu \nu, k', \mu', l', \nu'} J^{J^*} |\sum_{m} \bar{Y}^m_{\mu \nu} (k, k', J^*)|^2
- \frac{1}{j_l^2} \sum_{l', \nu', J'^*} D_{l \nu \nu', l', \nu', J'^*} J^{J^*} |\sum_{m} \bar{X}^m_{\nu \nu'} (l, l', J'^*)|^2.
\]

The selfconsistent scheme of the calculation of forward- (backward-) going free variational amplitude \( \bar{X}^m \) (\( \bar{Y}^m \)), energies of the excited states \( \Omega^m_{J^*} \) and coefficients \( D_{\mu \nu, k, l; J, \pi} \) is a double iterative problem which requires the solution of coupled-non-linear equations. We note that in the limit \( D_{\mu \nu, k, l; J, \pi} = 1 \) and in the case proton-neutron pairing is switched off, the solution of the full-RQRPA coincides with the solution of the pn-QRPA.
For the one-body densities in the full-RQRPA we can write:

\[ < J^\pi_{mi} || \{ c_{pk}^i c_{nl}^i \} || J^\pi_{0i} > = \sqrt{2J+1} \sum_{\mu,\nu=1,2} m(\mu k, \nu l) \left[ u_{k\mu}^{(i)} v_{l\nu}^{(i)} X_{\mu\nu}^{mi}(k, l, J^\pi) \right. \\
+ \left. v_{k\mu}^{(i)} u_{l\nu}^{(i)} X_{\nu\mu}^{ni}(k, l, J^\pi) \right] \sqrt{D_{k\mu l\nu J^\pi}}, \]  

(21)

\[ < J^\pi_{mf} || \{ c_{pk'}^f c_{nl'}^f \} || J^\pi_{mf} > = \sqrt{2J+1} \sum_{\mu,\nu=1,2} m(\mu k', \nu l') \left[ v_{k'\mu}^{(f)} u_{l'\nu}^{(f)} X_{\nu\mu}^{nf}(k', l', J^\pi) \right. \\
+ \left. u_{k'\mu}^{(f)} v_{l'\nu}^{(f)} X_{\mu\nu}^{nf}(k', l', J^\pi) \right] \sqrt{D_{k'\mu l'\nu J^\pi}}, \]  

(22)

with \( m(\mu a, \nu b) = \frac{1+(-1)^J \delta_{\mu a} \delta_{\nu b}}{1+\delta_{\mu a} \delta_{\nu b}} \). We note that the \( X_{\mu\nu}^{ni}(k, l, J^\pi) \) and \( Y_{\mu\nu}^{ni}(k, l, J^\pi) \) amplitudes are calculated by the renormalized QRPA equation only for the configurations \( \mu a \leq \nu b \) (i.e., \( \mu = \nu \) and the orbitals are ordered \( a \leq b \) and \( \mu = 1 \), \( \nu = 2 \) and the orbitals are not ordered) [23]. For different configurations \( X_{\mu\nu}^{ni}(k, l, J^\pi) \) and \( Y_{\mu\nu}^{ni}(k, l, J^\pi) \) in Eqs. (21) and (22) are given as follows:

\[ X_{\mu\nu}^{ni}(k, l, J^\pi) = -(-1)^j k + j l - J X_{\nu\mu}^{ni}(l, k, J^\pi), \]  

(23)

\[ Y_{\mu\nu}^{ni}(k, l, J^\pi) = -(-1)^j k + j l - J Y_{\nu\mu}^{ni}(l, k, J^\pi). \]  

(24)

The index \( i \) (\( f \)) indicates that the quasiparticles and the excited states of the nucleus are defined with respect to the initial (final) nuclear ground state \( |0_i^+ > \) (\( |0_f^+ > \)). We note that for \( D_{k\mu l\nu J^\pi} = 1 \) (i.e. there is no renormalization) and \( u_{2p} = v_{2p} = u_{1n} = v_{1n} = 0 \) (i.e. there is no proton-neutron pairing), Eqs. (21) and (22) reduce to the expressions of the pn-QRPA [7]-[9]. The overlap between two intermediate nuclear states belonging to two different sets is given by:

\[ < J^\pi_{mf} | J^\pi_{mi} > = \sum_{\mu k \leq \nu l} \left[ X_{\mu\nu}^{nf}(kl, J^\pi) X_{\mu\nu}^{ni}(kl, J^\pi) - Y_{\mu\nu}^{nf}(kl, J^\pi) Y_{\mu\nu}^{ni}(kl, J^\pi) \right]. \]  

(25)

**III. CALCULATION AND DISCUSSION**

We applied both the pn-QRPA and the full-RQRPA methods to calculate the \( 0\nu\beta\beta \)-decay of the \( A = 76, 100, 116, 128, 136 \) systems. In our calculations we tried to use as large
as possible Hilbert model spaces limited only by the power of the available computers. We have taken the following single particle model spaces for both, protons and neutrons:

(i) For $^{76}Ge \rightarrow ^{76}Se$, $^{100}Mo \rightarrow ^{100}Ru$ and $^{116}Cd \rightarrow ^{116}Sn$ the model space comprises 21 levels: $0s_{1/2}$, $0p_{1/2}$, $0p_{3/2}$, $0d_{3/2}$, $1s_{1/2}$, $1p_{1/2}$, $1p_{3/2}$, $0f_{5/2}$, $0f_{7/2}$, $2s_{1/2}$, $1d_{3/2}$, $1d_{5/2}$, $0g_{7/2}$, $0g_{9/2}$, $2p_{1/2}$, $2p_{3/2}$, $1f_{5/2}$, $1f_{7/2}$, $0h_{9/2}$, $0h_{11/2}$.

(ii) For $^{128}Te \rightarrow ^{128}Xe$ and $^{136}Xe \rightarrow ^{136}Ru$ we used 20 levels: $1s_{1/2}$, $0d_{3/2}$, $0d_{5/2}$, $1p_{1/2}$, $1p_{3/2}$, $0f_{5/2}$, $0f_{7/2}$, $2s_{1/2}$, $1d_{3/2}$, $1d_{5/2}$, $0g_{7/2}$, $0g_{9/2}$, $2p_{1/2}$, $2p_{3/2}$, $1f_{5/2}$, $1f_{7/2}$, $0h_{9/2}$, $0h_{11/2}$, $0i_{11/2}$, $0i_{13/2}$.

These model spaces are considerably larger as those used in any previous pn-QRPA calculations [7] - [14]. The single particle energies have been calculated with a Coulomb-corrected Woods-Saxon potential. The nucleon-nucleon interaction used to calculate the nuclear wave functions is based on the Brueckner G-matrix derived from the Bonn one-boson-exchange potential, which in principle is a more consistent and better description of the NN-interaction in nuclei. The Brueckner reaction matrix is obtained by solving the Bethe-Goldstone equation [24]. The pn-QRPA is based on a Bardeen-Cooper-Schrieffer (BCS) transformation including proton-proton and neutron-neutron pairing correlations. In the case of the full-RQRPA the single quasiparticle energies and occupation amplitudes have been found by solving the HFB equation with p-n pairing in the above mentioned space [22]. Since our model spaces are finite, all pairing potential gaps are renormalized to the empirical gaps by the strength parameters $d_{pp}$, $d_{nn}$ and $d_{pn}$. Technically this is achieved by performing a BCS and HFB calculation and comparing the obtained pairing gaps with the ones extracted from the empirical separation energies in a manner described in Ref. [24]. The renormalization parameters $d_{pp}$, $d_{nn}$ and $d_{pn}$ together with the experimental proton ($\Delta_p^{exp}$), neutron ($\Delta_n^{exp}$) and proton - neutron ($\delta_{pn}^{exp}$) pairing gaps, for all studied nuclear systems, are listed in Table I. The experimental pairing gaps are defined by Moeller and Nix [25,26].

By glancing at the Table I we see that the $d_{pp}$ and $d_{nn}$ values are close to unity and $d_{pn}$ is higher than these values. It is because for spherical nuclei the HFB-transformation can only describe correlations for pairs with $J=0$ and $T=1$ and not for pairs with $J=0$ and $T=0$. 
The T=0 pairing is effectively taken into account by the renormalization of the T=1 J=0 n-p interaction leading to a higher value of \( d_{pn} \). However, we do not want to focus our attention to the problem of proton-neutron pairing, which seems to play a significant role in the QRPA calculation of the double beta decay process \([16,23]\) but it is less important in the case of the renormalized QRPA calculations \([21]\). The proton-neutron pairing problem is extensively discussed in \([22,27]\) and references cited there. We note that in the BCS and HFB calculations we neglected the mixing of different ”n” but the same ”ljm” orbitals. We suppose that shell mixing is not significantly affecting the BCS and HFB solutions because their off-diagonal pairing matrix elements are quite small.

After settling the values of the pairing parameters, the parameters which remain to be fixed are the particle- particle and the particle- hole strengths. The particle - particle and particle - hole channels of the G-matrix interaction are renormalized by introducing the two parameters \( g_{pp} \) and \( g_{ph} \), which, in principle, should be close to unity. Our adopted values were \( g_{ph} = 0.8 \), as in our previous calculations \([21,23]\) and \( g_{pp} \) is varied in the interval 0.70 - 1.30 which can be regarded as physical.

The nuclear matrix elements \( M_{0\nu}^{\text{mass}} \) for the most interesting nuclei obtained within the pn-QRPA are shown in Fig. 1 (a). We see that \( M_{0\nu}^{\text{mass}} \) for the A=76, 100, 116, 128 and 136 systems, becomes singular with increasing strength of the particle-particle interaction and even crosses zero in the physically acceptable region of the parameter \( g_{pp} \). This behavior has not been found in the previous calculations \([7]-[10]\) because the model spaces used there, were too small. To our knowledge only Krmpotić and Sharma \([11]\) have studied the model space dependence of the 0\( \nu\beta\beta \)-decay matrix elements. For a relatively large model space they found for the 0\( \nu\beta\beta \)-decay of \(^{48}\text{Ca}\) a similar behavior. However this was further out of the physical region of the strength renormalization parameters. We note that there is a principal difference between our calculations and those of Krmpotić and Sharma. They used a zero range \( \delta \)-force interaction and a different treatment to the two-nucleon correlation function, which were incorporated in the formalism in Eqs. (6-10). To our opinion their type of two-nucleon correlations influences the result unsignificantly. The two-nucleon correlations
presented in this work give an effect of about 30-40%. We should note further that $^{48}\text{Ca}$ is a closed shell nucleus and therefore it is not very suitable for the QRPA calculation.

The QRPA quenching mechanism for $M^{0\nu}_{\text{mass}}$ in Fig. 1 (a) has its origin, exactly in the same way as the quenching mechanism of $2\nu\beta\beta$-decay matrix elements, i.e., in the generation of too much ground state correlations with increasing $g_{pp}$ near the collapse of the QRPA. A larger model space means more ground state correlations, i.e., a collapse of the QRPA solution for smaller $g_{pp}$. As a consequence the validity of the quasiboson approximation in the evaluation of the $0\nu\beta\beta$-decay matrix elements is questioned because of the generation of too much ground state correlations. For that reason it is necessary to perform the calculation of $M^{0\nu}_{\text{mass}}$ in the framework of the renormalized QRPA, which takes into account the Pauli exclusion principle in an approximate way and also considers proton-neutron pairing correlations. In Fig. 1 (b) we present our results with the full-RQRPA method. From the comparison with the Fig. 1 (a) it follows that the inclusion of ground state correlations beyond the QRPA in the calculation of $M^{0\nu}_{\text{mass}}$ removes the difficulties associated with the extreme sensitivity of $M^{0\nu}_{\text{mass}}$ on the particle-particle strength. The strong differences between the results of both methods indicate that the Pauli exclusion principle plays an important role in the evaluation of the $0\nu\beta\beta$-decay.

A weakly dependence of the $M^{0\nu}_{\text{mass}}$ on $g_{pp}$ inside the physical range $0.8 \leq g_{pp} \leq 1.2$ allows us to have more confidence for deducing the effective neutrino mass $< m_\nu >$ from the available experimental lower limits on the half-lives of the $0\nu\beta\beta$-decays $T^{0\nu-\exp}_{1/2}$. The nuclear matrix elements $M^{0\nu}_{\text{mass}}$ obtained within the full-RQRPA for $g_{pp} = 1.0$, the integrated kinematical factors $G_{01}$, $T^{0\nu-\exp}_{1/2}$ and the limits on the effective neutrino mass $| < m_\nu > |$ deduced from $M^{0\nu}_{\text{mass}} (g_{pp} = 1.0)$ and $T^{0\nu-\exp}_{1/2}$ are listed in Table [I]. A more stringent upper limit on $< m_\nu >$ is favored by large values of $M^{0\nu}_{\text{mass}}$, $G_{01}$ and $T^{0\nu-\exp}_{1/2}$. We see that the most stringent upper limit $< m_\nu > \leq 1.1$ eV is deduced for the $A=76$ system mainly because of the unbelievable high upper limit on $T^{0\nu-\exp}_{1/2}$ given by the Heidelberg-Moscow collaboration [28]. Also of interest is the value $T^{0\nu-1\text{eV}}_{1/2}$ (see Table [I]), which is calculated by assuming $| < m_\nu > | = 1$ eV. This value indicates that for further experimental measurements the
most perspective candidate is $^{100}Mo$.

**IV. SUMMARY AND CONCLUSION**

In summary, we have studied the nuclear matrix elements entering the light neutrino mediated mode of the neutrinoless double beta decay of some experimentally interesting nuclear systems, $A=76, 100, 116, 128$ and $136$. The calculations have been performed within both, the pn-QRPA and the full-RQRPA methods with a large Hilbert model space. We have found that in the framework of the pn-QRPA for a large enough model space the $0\nu\beta\beta$-decay matrix elements demonstrate an instability with respect to the renormalization of the particle-particle strength, similar to the one known from the $2\nu\beta\beta$-decay mode. The value of the matrix element crosses zero and it is then difficult to make definite rate predictions. We believe that the common quenching phenomenon which is independent of the studied nucleus and double beta decay process could have its origin only in the approximation scheme. The full-RQRPA which includes proton-neutron pairing and the Pauli effect of fermion pairs goes beyond the quasi-boson approximation. The inclusion of the Pauli principle eliminates the instabilities that plague the pn-QRPA. The $0\nu\beta\beta$-decay matrix elements calculated via the full-RQRPA are stable in respect to the changes of the particle-particle force and it allows us to deduce more accurate limits on the effective neutrino mass. The largest $0\nu\beta\beta$-decay matrix elements are 4.22 and 3.28 associated with the $A=100$ and 128 systems, respectively. A large value of the matrix element together with a large value of the kinematical factor favour especially $^{100}Mo$ for further experimental study of the $0\nu\beta\beta$-decay. At present the $A=76$ and 128 systems provide us with the most stringent limit on the effective neutrino mass i.e., 1.1 - 1.2 eV.
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### TABLE I

Experimental proton ($\Delta_{p}^{exp}$), neutron ($\Delta_{n}^{exp}$) and proton - neutron ($\delta_{pn}^{exp}$) pairing gaps and renormalization constants of the proton - proton ($d_{pp}$), neutron - neutron ($d_{nn}$) and proton - neutron ($d_{pn}$) pairing interactions for all nuclei studied here.

| Nucleus  | ($\Delta_{p}^{exp}$, $\Delta_{n}^{exp}$, $\delta_{pn}^{exp}$) | Model | $d_{pp}$ | $d_{nn}$ | $d_{pn}$ |
|----------|-------------------------------------------------|-------|---------|---------|---------|
| $^{76}_{32}$Ge$_{44}$ | (1.561, 1.535, 0.388) | 21 level | 0.899 | 1.028 | 1.506 |
| $^{76}_{34}$Se$_{42}$ | (1.751, 1.710, 0.459) | 21 level | 0.934 | 1.059 | 1.325 |
| $^{100}_{42}$Mo$_{58}$ | (1.612, 1.358, 0.635) | 21 level | 0.980 | 0.923 | 1.766 |
| $^{100}_{44}$Ru$_{56}$ | (1.548, 1.296, 0.277) | 21 level | 1.002 | 0.945 | 1.568 |
| $^{116}_{48}$Cd$_{68}$ | (1.493, 1.377, 0.371) | 21 level | 0.953 | 0.922 | 1.822 |
| $^{116}_{50}$Cd$_{66}$ | (1.763, 1.204, 0.128) | 21 level | 1.00 | 0.873 | 1.460 |
| $^{128}_{52}$Te$_{76}$ | (1.127, 1.177, 0.149) | 20 level | 0.873 | 0.942 | 1.780 |
| $^{128}_{54}$Xe$_{74}$ | (1.307, 1.266, 0.199) | 20 level | 0.920 | 0.972 | 1.530 |
| $^{136}_{54}$Xe$_{82}$ | (0.971, 1.408, 0.0) | 20 level | 0.771 | 0.803 | 0.0 |
| $^{136}_{56}$Ba$_{80}$ | (1.245, 1.032, 0.165) | 20 level | 0.875 | 0.899 | 1.716 |
TABLE II. The nuclear matrix elements $M_{\text{mass}}^{0\nu}$ (see Eqs. (6) and (11-12)) obtained within the full-RQRPA for $g_{pp} = 1.0$, the integrated kinematical factors $G_{01}$, the limits on the effective neutrino mass $|<m_\nu>|$ deduced from the experimental limit of the $0\nu\beta\beta$-decay lifetime $T_{1/2}^{0\nu-\text{exp}}$ for the nuclei studied in this work. $T_{1/2}^{0\nu-1\text{eV}}$ is the calculated $0\nu\beta\beta$-decay half-life times assuming $|<m_\nu>| = 1\text{eV}$.

| nucleus | $M_{\text{mass}}^{0\nu}$ | $G_{01}$ | $T_{1/2}^{0\nu-1\text{eV}}$ | $T_{1/2}^{0\nu-\text{exp}}$ | $|<m_\nu>|$ |
|---------|-----------------|-------|-----------------|-----------------|--------|
|         | [$10^{-14}$ years$^{-1}$] | years | [years] ref. | [eV] |
| $^{76}\text{Ge}$ | 1.86 | 0.7928 | $9.5 \times 10^{24}$ | $\geq 7.4 \times 10^{24}$ (90% C.L.) | $\leq 1.1$ |
| $^{100}\text{Mo}$ | 4.22 | 5.731 | $2.6 \times 10^{23}$ | $\geq 4.4 \times 10^{22}$ (68% C.L.) | $\leq 2.4$ |
| $^{116}\text{Cd}$ | 2.47 | 6.237 | $6.9 \times 10^{23}$ | $\geq 2.9 \times 10^{22}$ (90% C.L.) | $\leq 4.9$ |
| $^{128}\text{Te}$ | 3.28 | 0.2207 | $1.1 \times 10^{25}$ | $\geq 7.3 \times 10^{24}$ (68% C.L.) | $\leq 1.2$ |
| $^{136}\text{Xe}$ | 0.96 | 5.914 | $3.4 \times 10^{23}$ | $\geq 6.4 \times 10^{23}$ (90% C.L.) | $\leq 3.7$ |
FIGURES

FIG. 1. The calculated nuclear matrix element $M_{\text{mass}}^{0\nu}$ for the $0\nu\beta\beta$-decay of $^{76}\text{Ge}$, $^{100}\text{Mo}$, $^{116}\text{Cd}$, $^{128}\text{Te}$ and $^{136}\text{Xe}$ as a function of the particle-particle interaction strength $g_{pp}$. In (a) $M_{\text{mass}}^{0\nu}$ has been calculated with the pn-QRPA method, in (b) with the full-RQRPA.
