Relativistic Mechanics and a Special Role for the Coulomb Potential

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Abstract

It is shown that a nonrelativistic mechanical system involving a general nonrelativistic potential \( V(|\mathbf{r}_1 - \mathbf{r}_2|) \) between point particles at positions \( \mathbf{r}_1 \) and \( \mathbf{r}_2 \) can be extended to a Lagrangian system which is invariant under Lorentz transformation through order \( v^2/c^2 \). However, this invariance requires the introduction of velocity-dependent and acceleration-dependent forces between particles. The textbook treatments of "relativistic mechanics" can be misleading; the discussions usually deal with only one particle experiencing prescribed forces and so make no mention of these additional velocity- and acceleration-dependent forces. A simple example for a situation analogous to a parallel-plate capacitor is analyzed for all the conservation laws of Galilean invariance or Lorentz invariance. For this system, Galilean invariance requires that the mechanical momentum is given by \( \mathbf{p}_{\text{mech}} = m\mathbf{v} \) but places no restriction on the position-dependent potential function. On the other hand, Lorentz invariance requires that the mechanical momentum is given by \( \mathbf{p}_{\text{mech}} = m\mathbf{v}(1 - v^2/c^2)^{-1/2} \), and in addition requires that the potential function is exactly the Coulomb potential \( V(|\mathbf{r}_1 - \mathbf{r}_2|) = k/|\mathbf{r}_1 - \mathbf{r}_2| \). It is also noted that the transmission of the interparticle-force signal at the speed of light again suggests a special role for the Coulomb potential. A nonrelativistic particle system interacting through the Coulomb potential becomes the Darwin Lagrangian when extended to a system relativistic through order \( v^2/c^2 \), and then allows extension to classical electrodynamics as a fully Lorentz-invariant theory of interacting particles.
A. Introduction

Contemporary physics regards special relativity as a metatheory to which (locally) all theories describing nature should conform. Thus in nonrelativistic classical mechanics, there is the unspoken implication that the nonrelativistic interaction between point particles at positions \( r_1 \) and \( r_2 \) under a general potential \( V(|r_1 - r_2|) \) is the small-velocity limit of some fully relativistic theory of interacting point particles which might occur in nature. However, the use of a general potential can be misleading for both students and researchers. Here we demonstrate that an arbitrary nonrelativistic potential function can indeed be extended to a Lagrangian which is Lorentz-invariant through order \( v^2/c^2 \); however, the extension requires the introduction of velocity-dependent and acceleration-dependent forces which go unmentioned in the mechanics textbooks. Also, we present a simple example showing that the \( 1/r \) potential between point particles is singled out as the only nonrelativistic force law which will lead to appropriate Lorentz-invariant behavior (\textit{without} the appearance of these additional forces) for groups of particles arranged in a fashion analogous to parallel capacitor plates. Finally, we note that the Coulomb potential is suggested when the potential satisfies the wave equation for signal transmission at the speed of light \( c \). All these results emphasize both the sometimes misleading nature of current textbook treatments of "relativistic mechanics" and also the special role played by the Coulomb potential.

B. Textbook Discussion of the "Relativistic Lagrangian"

Current textbooks of classical mechanics encourage the common misconception among physicists that a \textit{relativistic} classical system can be obtained from a \textit{nonrelativistic} mechanical system involving an arbitrary nonrelativistic potential \( V(|r_1 - r_2|) \) between particles simply by introducing the relativistic expressions for mechanical linear momentum and mechanical energy for the particles. Thus, for example, standard classical mechanics textbooks suggest\([1,2]\) that the "relativistic Lagrangian" is obtained by using the relativistic Lagrangian for a free particle and adding an arbitrary nonrelativistic potential. One textbook\([3]\) indeed has a section on "The relativistic one-dimensional harmonic oscillator."

The usual discussion of relativistic particle motion in classical mechanics texts considers only a single particle \( m \) and involves the replacement of the Lagrangian \( L(r, \dot{r}) \) for the
nonrelativistic motion in a time-independent potential $V(r)$

$$L(r, \dot{r}) = \frac{1}{2} m \dot{r}^2 - V(r) \quad (1)$$

giving the nonrelativistic equation of motion

$$\frac{d}{dt}(m\dot{r}) = -\nabla V(r) \quad (2)$$

by the "relativistic Lagrangian"

$$L(r, \dot{r}) = -mc^2(1 - \dot{r}^2/c^2)^{1/2} - V(r) \quad (3)$$

with the equation of motion

$$\frac{d}{dt} \left( \frac{m\dot{r}}{(1 - \dot{r}^2/c^2)^{1/2}} \right) = -\nabla V(r) \quad (4)$$

Thus in the equation of motion, the nonrelativistic particle momentum $p_{\text{nonrel}} = m\dot{r}$ is replaced by the relativistic particle momentum $p_{\text{rel}} = m\dot{r}(1 - \dot{r}^2/c^2)^{-1/2}$, and the time rate of change of the momentum is given by the same force $-\nabla V(r)$ in both relativistic and nonrelativistic cases. Of course, these one-particle systems take this simple form in only one inertial frame. In other inertial frames, there are velocity-dependent and acceleration-dependent forces.

Some authors go one step further and insist that the Lagrangian itself should be written in manifestly covariant form despite the fact that the forces on the particle may take a simple form in only one inertial frame. Such one-particle systems (other than the free particle) exhibit neither conservation of linear momentum nor constant motion of the center of energy, both of which are expected in a Lorentz-invariant system. These one-particle systems may provide mathematical exercises for students; however, with the sole exception of electromagnetic forces, they are largely irrelevant to physics as a description of nature, and indeed are misleading to students, instructors, and researchers. Insistence upon a covariant appearance is a mere distraction, with no connection to nature. Indeed, as was pointed out long ago by Kretchmann, any expression can be written in manifestly Lorentz-covariant notation, indeed in general covariant notation.
C. Nonrelativistic Lagrangians for Particles and Lorentz-Invariant Extension to Order $v^2/c^2$

When describing nature, we regard the fundamental interactions as those between point particles. Thus here we turn to the question as to what potential functions $V(|r_1 - r_2|)$ between two point particles can be regarded as describing the nonrelativistic limit arising from a fully Lorentz-invariant interaction between particles. We first try to solve this problem by working backwards, trying to construct a relativistic theory which produces a specific potential in the nonrelativistic limit.

The nonrelativistic mechanical behavior of two point particles interacting through a potential $V(|r_1 - r_2|)$ can be written in terms of a Lagrangian

$$L(r_1, r_2, \dot{r}_1, \dot{r}_2) = \frac{1}{2} m_1 \dot{r}_1^2 + \frac{1}{2} m_2 \dot{r}_2^2 - V(|r_1 - r_2|)$$

The invariance of this Lagrangian under spacetime translations and spatial rotations leads to the conservation laws for energy, linear momentum, and angular momentum. The system is also invariant under Galilean transformations where the generator of proper Galilean transformations is given by the system total mass times the system center of mass. In order to extend this system to a Lorentz-invariant system, we must preserve the invariance of the Lagrangian under spacetime translations and spatial rotations while changing the system invariance under Galilean transformations over to invariance under Lorentz transformations. The generator of Lorentz transformations is the system total energy times the system center of energy. The first step in this transformation is the replacement of the nonrelativistic expression for particle kinetic energy by the Lagrangian for a relativistic free particle

$$\frac{1}{2} m \dot{r}^2 \to -mc^2(1 - \dot{r}^2/c^2)^{1/2}$$

just as was done in moving from Eq. (1) to Eq. (3) above. With this replacement, the nonrelativistic Lagrangian of Eq. (5) becomes now

$$L(r_1, r_2, \dot{r}_1, \dot{r}_2) = -m_1 c^2(1 - \dot{r}_1^2/c^2)^{1/2} - m_2 c^2(1 - \dot{r}_2^2/c^2)^{1/2} - V(|r_1 - r_2|)$$

This Lagrangian, which is of the sort given in the mechanics textbooks, will lead to relativistic expressions for particle kinetic energy and particle linear momentum. It is invariant under spacetime translations and spatial rotations. However, this system is not Lorentz invariant.
Since the energy and momentum of an isolated system form a Lorentz four-vector, we expect the potential energy $V(|\mathbf{r}_1 - \mathbf{r}_2|)$ to be related to momentum in a different inertial frame. Let us label as $S$ the inertial frame in which the potential function $V(|\mathbf{r}_1 - \mathbf{r}_2|)$ gives the nonrelativistic interaction of the particles. Then when viewed from any inertial frame $S'$ moving with constant velocity with respect to the frame $S$, we expect to find velocity-dependent forces between the particles in addition to the position-dependent forces found in the frame $S$. If we require Lorentz invariance through order $v^2/c^2$, then the velocity-dependent terms must appear in the Lagrangian in any inertial frame. By working backwards from the requirement of Lorentz invariance through order $v^2/c^2$, we find that the Lagrangian extended from the nonrelativistic expression (5) can be written as

$$L(\mathbf{r}_1, \mathbf{r}_2, \dot{\mathbf{r}}_1, \dot{\mathbf{r}}_2) = -m_1c^2(1 - \dot{\mathbf{r}}_1^2/c^2)^{1/2} - m_2c^2(1 - \dot{\mathbf{r}}_2^2/c^2)^{1/2} - V(|\mathbf{r}_1 - \mathbf{r}_2|)$$

$$+ \frac{1}{2}V(|\mathbf{r}_1 - \mathbf{r}_2|) \frac{\dot{\mathbf{r}}_1 \cdot \dot{\mathbf{r}}_2}{c^2} - \frac{1}{2}V'(|\mathbf{r}_1 - \mathbf{r}_2|) \frac{\dot{\mathbf{r}}_1 \cdot (\mathbf{r}_1 - \mathbf{r}_2) \dot{\mathbf{r}}_2 \cdot (\mathbf{r}_1 - \mathbf{r}_2)}{c^2 |\mathbf{r}_1 - \mathbf{r}_2|}$$

(8)

where $V'(|\mathbf{r}_1 - \mathbf{r}_2|)$ refers to the derivative of the potential function with respect to its argument.

We can check the Lorentz invariance of this Lagrangian through order $v^2/c^2$ by showing that the system center of energy moves with constant velocity through order $v^2/c^2$. Indeed, we expect

$$\frac{d}{dt}(U \vec{X}) = c^2 \mathbf{P}$$

(9)

where $U$ is the system energy, $\vec{X}$ is the system center of energy, and $\mathbf{P}$ is the system linear momentum. The system energy $U$ times the center of energy of the system $\vec{X}$ through zero-order in $v/c$ is given by

$$U \vec{X} = m_1(c^2 + \frac{1}{2} \dot{\mathbf{r}}_1^2)\mathbf{r}_1 + m_2(c^2 + \frac{1}{2} \dot{\mathbf{r}}_2^2)\mathbf{r}_2 + V(|\mathbf{r}_1 - \mathbf{r}_2|) \frac{(\mathbf{r}_1 + \mathbf{r}_2)}{2}$$

(10)

corresponding to the restmass energy and kinetic energy of the two particles located at their respective positions $\mathbf{r}_1$ and $\mathbf{r}_2$ plus the interaction potential energy located half way between the positions of the two particles. Since the Lagrangian in Eq. (8) has no explicit time dependence, the system energy $U$ is constant in time. Taking the time derivative of Eq. (10), we find

$$\frac{d}{dt}(U \vec{X}) = U \frac{d\vec{X}}{dt} = m_1(c^2 + \frac{1}{2} \dot{\mathbf{r}}_1^2)\dot{\mathbf{r}}_1 + m_2(c^2 + \frac{1}{2} \dot{\mathbf{r}}_2^2)\dot{\mathbf{r}}_2 + (m_1 \dot{\mathbf{r}}_1 \cdot \mathbf{r}_1 + m_2 \dot{\mathbf{r}}_2 \cdot \mathbf{r}_2)\mathbf{r}_2$$

$$+ \frac{1}{2}V(|\mathbf{r}_1 - \mathbf{r}_2|)(\dot{\mathbf{r}}_1 \cdot \dot{\mathbf{r}}_2) + \frac{1}{2}V'(|\mathbf{r}_1 - \mathbf{r}_2|) \frac{(\dot{\mathbf{r}}_1 - \dot{\mathbf{r}}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|} \cdot (\mathbf{r}_1 - \mathbf{r}_2)(\mathbf{r}_1 + \mathbf{r}_2)$$

(11)
It is sufficient to use the nonrelativistic equations of motion,
\[ m_1 \ddot{r}_1 = -V'(|r_1 - r_2|) \frac{(r_1 - r_2)}{|r_1 - r_2|} \]  
\[ m_2 \ddot{r}_2 = V'(|r_1 - r_2|) \frac{(r_1 - r_2)}{|r_1 - r_2|} \]  
(12)  
(13)
to transform Eq. (11) into the form
\[ \frac{d}{dt}(UX) = U \frac{dX}{dt} = m_1(c^2 + \frac{1}{2} \dot{r}_1^2) \dot{r}_1 + m_2(c^2 + \frac{1}{2} \dot{r}_2^2) \dot{r}_2 \]
\[ - \left( V'(|r_1 - r_2|) \frac{(r_1 - r_2)}{|r_1 - r_2|} \dot{r}_1 \right) r_1 + \left( V'(|r_1 - r_2|) \frac{(r_1 - r_2)}{|r_1 - r_2|} \dot{r}_2 \right) r_2 \]
\[ + \frac{1}{2} V(|r_1 - r_2|)(\dot{r}_1 + \dot{r}_2) + \frac{1}{2} V'(|r_1 - r_2|) \left( \frac{\dot{r}_1 - \dot{r}_2}{|r_1 - r_2|} \cdot (r_1 - r_2)(r_1 + r_2) \right) \]
(14)
The momenta can be obtained from the Lagrangian in Eq. (8) as
\[ p_1 = \frac{\partial L}{\partial \dot{r}_1} = m_1 \dot{r}_1 \left( 1 - \frac{\dot{r}_1^2}{c^2} \right)^{-1/2} + \frac{1}{2} V(|r_1 - r_2|) \frac{\dot{r}_1}{c^2} \]
\[ - \frac{1}{2} V'(|r_1 - r_2|) \frac{(r_1 - r_2)}{|r_1 - r_2|} \cdot (r_1 - r_2) \]  
(15)
\[ p_2 = \frac{\partial L}{\partial \dot{r}_2} = m_2 \dot{r}_2 \left( 1 - \frac{\dot{r}_2^2}{c^2} \right)^{-1/2} + \frac{1}{2} V(|r_1 - r_2|) \frac{\dot{r}_2}{c^2} \]
\[ - \frac{1}{2} V'(|r_1 - r_2|) \frac{(r_1 - r_2)}{|r_1 - r_2|} \cdot (r_1 - r_2) \]  
(16)
giving total linear momentum
\[ P = m_1 \dot{r}_1 \left( 1 - \frac{\dot{r}_1^2}{c^2} \right)^{-1/2} + m_2 \dot{r}_2 \left( 1 - \frac{\dot{r}_2^2}{c^2} \right)^{-1/2} \]
\[ + \frac{1}{2} V(|r_1 - r_2|) \frac{\dot{r}_1}{c^2} + \frac{1}{2} V(|r_1 - r_2|) \frac{\dot{r}_2}{c^2} \]
\[ - \frac{1}{2} V'(|r_1 - r_2|) \frac{(r_1 - r_2)}{|r_1 - r_2|} \cdot (r_1 - r_2) + \frac{1}{2} V'(|r_1 - r_2|) \frac{(r_1 - r_2)}{|r_1 - r_2|} \cdot (r_1 - r_2) \]
(17)
Comparing Eqs. (14) and (17) after reorganizing a few terms, we find that indeed Eq. (9) holds. The system of Eq. (8) is indeed Lorentz invariant through order \( v^2/c^2 \).
D. Velocity-Dependent and Acceleration-Dependent Forces in Lorentz-Invariant Systems

The Lagrange equations of motion follow from the Lagrangian in Eq. (8); for the particle at \( \mathbf{r}_1 \), the equation takes the form

\[
0 = \frac{d}{dt} \left( \frac{m_1 \ddot{\mathbf{r}}_1}{(1 - \dot{\mathbf{r}}_1^2/c^2)^{1/2}} + \frac{1}{2} V(|\mathbf{r}_1 - \mathbf{r}_2|) \frac{\ddot{\mathbf{r}}_2}{c^2} - \frac{1}{2} V'(|\mathbf{r}_1 - \mathbf{r}_2|) \frac{(\mathbf{r}_1 - \mathbf{r}_2) \cdot \ddot{\mathbf{r}}_2}{c^2} \cdot (\mathbf{r}_1 - \mathbf{r}_2) \right)
- \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|} V'(|\mathbf{r}_1 - \mathbf{r}_2|) \left( -1 + \frac{\ddot{\mathbf{r}}_1 \cdot \ddot{\mathbf{r}}_2}{2c^2} + \frac{\ddot{\mathbf{r}}_1 \cdot (\mathbf{r}_1 - \mathbf{r}_2) \ddot{\mathbf{r}}_2 \cdot (\mathbf{r}_1 - \mathbf{r}_2)}{2c^2 |\mathbf{r}_1 - \mathbf{r}_2|^2} \right)
+ \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|} V''(|\mathbf{r}_1 - \mathbf{r}_2|) \left( \frac{\ddot{\mathbf{r}}_1 \cdot (\mathbf{r}_1 - \mathbf{r}_2) \ddot{\mathbf{r}}_2 + \ddot{\mathbf{r}}_2 \cdot (\mathbf{r}_1 - \mathbf{r}_2) \ddot{\mathbf{r}}_1}{2c^2 |\mathbf{r}_1 - \mathbf{r}_2|} \right)
+ V'(|\mathbf{r}_1 - \mathbf{r}_2|) \left( \frac{\ddot{\mathbf{r}}_1 \cdot (\mathbf{r}_1 - \mathbf{r}_2) \ddot{\mathbf{r}}_2 + \ddot{\mathbf{r}}_2 \cdot (\mathbf{r}_1 - \mathbf{r}_2) \ddot{\mathbf{r}}_1}{2c^2 |\mathbf{r}_1 - \mathbf{r}_2|} \right) \tag{18}
\]

The equations of motion can be rewritten as forces acting on the particles to change the mechanical momentum. For the particle at \( \mathbf{r}_1 \), this becomes

\[
\frac{d}{dt} \left( \frac{m \ddot{\mathbf{r}}_1}{(1 - \dot{\mathbf{r}}_1^2/c^2)^{1/2}} \right) = -V'(|\mathbf{r}_1 - \mathbf{r}_2|) \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|} \left[ 1 + \frac{1}{2} \left( \frac{\ddot{\mathbf{r}}_2}{c} \right)^2 \right]
- \frac{1}{2c^2 |\mathbf{r}_1 - \mathbf{r}_2|} \left( V''(|\mathbf{r}_1 - \mathbf{r}_2|) \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|} - \frac{V'(|\mathbf{r}_1 - \mathbf{r}_2|)}{|\mathbf{r}_1 - \mathbf{r}_2|^2} \right) \ddot{\mathbf{r}}_2 \cdot (\mathbf{r}_1 - \mathbf{r}_2)^2
- \frac{1}{2c^2} \left( V(|\mathbf{r}_1 - \mathbf{r}_2|) \ddot{\mathbf{r}}_2 - V'(|\mathbf{r}_1 - \mathbf{r}_2|) \frac{\ddot{\mathbf{r}}_2 \cdot (\mathbf{r}_1 - \mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|} \right) \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^2}
- \frac{\ddot{\mathbf{r}}_1}{c} \times \left( \frac{\ddot{\mathbf{r}}_2}{c} \times \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|} V'(|\mathbf{r}_1 - \mathbf{r}_2|) \right) \tag{19}
\]

We notice that the force on the first particle involves not only the force arising from the original nonrelativistic potential function, but also forces depending upon the velocities of both particles and upon the acceleration of the other particle. These forces were not part of the original nonrelativistic theory. Such forces are absent from the accounts in the mechanics textbooks and from the articles which treat ”relativistic” motion for a single article. The single particle appearing in the Lagrangian of these treatments produces velocity-dependent and acceleration-dependent forces back on the prescribed sources whose momentum and energy are never discussed.

The most famous Lagrangian which is Lorentz invariant through \( v^2/c^2 \) is that obtained from the Coulomb potential \( V(|\mathbf{r}_1 - \mathbf{r}_2|) = q_1 q_2/|\mathbf{r}_1 - \mathbf{r}_2| \). In this case the Lagrangian of
Eq. (8) becomes

\[
L(\mathbf{r}_1, \mathbf{r}_2, \dot{\mathbf{r}}_1, \dot{\mathbf{r}}_2) = -m_1 c^2 \left(1 - \frac{\dot{\mathbf{r}}_1^2}{c^2}\right)^{1/2} - m_2 c^2 \left(1 - \frac{\dot{\mathbf{r}}_2^2}{c^2}\right)^{1/2} - \frac{q_1 q_2}{|\mathbf{r}_1 - \mathbf{r}_2|} \\
+ \frac{1}{2} \frac{q_1 q_2}{|\mathbf{r}_1 - \mathbf{r}_2|} \frac{\dot{\mathbf{r}}_1 \cdot \dot{\mathbf{r}}_2}{c^2} + \frac{1}{2} \frac{q_1 q_2}{|\mathbf{r}_1 - \mathbf{r}_2|} \frac{\dot{\mathbf{r}}_1 \cdot (\mathbf{r}_1 - \mathbf{r}_2) \dot{\mathbf{r}}_2 \cdot (\mathbf{r}_1 - \mathbf{r}_2)}{c^2 |\mathbf{r}_1 - \mathbf{r}_2|^2}
\]

(20)

If in Eq. (20) we expand the free-particle expressions \(-mc^2(1 - \dot{r}^2/c^2)^{1/2}\) through second order in \(v/c\), then this becomes the Darwin Lagrangian which sometime appears in electromagnetism textbooks as an approximation to the interaction of charged particles. The approximation is an accurate description of the classical electromagnetic interaction between charged particles through second order in \(v/c\) for small separations between the particles.

The Lagrangian equation of motion following from Eq.(20) becomes (for the particle at position \(\mathbf{r}_1\))

\[
\frac{d}{dt} \left( \frac{m_1 \mathbf{\dot{r}}_1}{(1 - \dot{\mathbf{r}}_1^2/c^2)^{1/2}} \right) = q_1 q_2 \left( \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|} \right)^3 \left\{ 1 + \frac{1}{2} \left( \frac{\mathbf{\dot{r}}_2}{c} \right)^2 - \frac{3}{2} \left( \frac{(\mathbf{r}_1 - \mathbf{r}_2) \cdot \mathbf{\dot{r}}_2}{|\mathbf{r}_1 - \mathbf{r}_2|} \right)^2 \right\} \\
- \frac{q_2}{2c} \left( \mathbf{\ddot{r}}_2 + \frac{[\mathbf{\dot{r}}_2 \cdot (\mathbf{r}_1 - \mathbf{r}_2)](\mathbf{r}_1 - \mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|^2} \right) \\
+ q_1 \left[ \frac{\mathbf{\ddot{r}}_1}{c} \times \frac{\mathbf{\dot{r}}_2}{c} \times \frac{(\mathbf{r}_1 - \mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|^3} \right]
\]

(21)

where we have rewritten the Lagrangian equation in the form \(d\mathbf{p}_1/dt = q_1 \mathbf{E} + q_1 (\mathbf{\dot{r}}_1/c) \times \mathbf{B}\) with \(\mathbf{p}_1\) the mechanical particle momentum. The velocity- and acceleration-dependent forces in Eq.(21) correspond to fields arising from electromagnetic induction. In the textbooks, electromagnetic induction is always treated without reference to any charged particles which may be producing the induction fields, a very different point of view from that which follows from the Darwin Lagrangian.

E. Special Role for the Coulomb Potential for a Parallel-Disk System

Since the potential energy \(V(|\mathbf{r}_1 - \mathbf{r}_2|)\) and associated relativistic momentum depend upon pairs of particles whereas the mechanical energy and momentum depend upon individual particles, the quantities associated with the potential can be made arbitrarily large compared to the mechanical quantities by considering a group of particles held together by forces of constraint. This means that for a Lorentz-invariant multiparticle system, the velocity-dependent and acceleration-dependent forces might dominate any consideration of particle
mechanics. And indeed this does occur in connection with the self inductance and mutual
inductance of electromagnetic inductors where the mass of the charge carriers plays so small
a role that it is never mentioned. However, there is at least one case where the additional
velocity-dependent and acceleration-dependent forces do not dominate the multi-particle
system. In an earlier article [12] providing illustrations of the center-of-energy motion in
relativistic systems, the example of a parallel-plate capacitor was used. Here we point out
that a Coulomb potential and only a Coulomb potential allows relativistic behavior for a
parallel-plate system without any consideration of the velocity-dependent or acceleration-
dependent forces generally required for relativistic behavior.

The forces of constraint holding together a group of particles can introduce energy and
momentum into the system unless they are perpendicular to the direction of motion of the
group of particles. Because of this consideration, we will consider two groups of interacting
particles which are arranged in a disk fashion analogous to those of a parallel-plate capacitor,
and we will consider the motion of the disks along a single axis perpendicular to the plates.
In this case, the forces of constraint holding each plate together are perpendicular to the
direction of motion and so introduce neither energy nor momentum into the system.

In order to calculate the forces between the plates, we sum the forces between particles
assuming superposition holds. For definiteness, we assume a potential of the form $1/r^n$, so
that the force between two particles A and B in the nonrelativistic limit takes the form

$$\mathbf{F}_{onA} = -\nabla_A V(|\mathbf{r}_A - \mathbf{r}_B|) = \frac{-nk(\mathbf{r}_A - \mathbf{r}_B)}{|\mathbf{r}_A - \mathbf{r}_B|^{n+2}} \quad (22)$$

where

$$V(|\mathbf{r}_A - \mathbf{r}_B|) = \frac{-k}{|\mathbf{r}_A - \mathbf{r}_B|^n} \quad (23)$$

and we assume $n > 0$ so that the potential decreases as the separation between the particles
increases. Next we consider a uniform disk of particles of type B in the $yz$-plane with $\sigma$
particles per unit area. We obtain the force on particle A at a small distance $L$ above the
center of the disk of large radius $R$, $R >> L$, by summing the contributions of the particles
in the disk. Taking account of the cylindrical symmetry, the force on $A$ is given by

$$F_{onA} = \int_{0}^{R} (2\pi r dr) \sigma_B nk \frac{-L}{(L^2 + r^2)^{n/2+1}}$$

$$= 2\pi \sigma_B kL \left( \frac{1}{(L^2 + r^2)^{n/2}} \right)_{r=0}^{r=R}$$

$$= -2\pi \sigma_B k \left( \frac{1}{L^{n-1}} \right)$$

(24)

where we have used the assumptions $n > 0$ and $R >> L$. From this result, we can obtain the attractive forces between a pair of parallel disks of radius $R$, one made up of particles of type $A$ and the other of type $B$, separated by a distance $L$,

$$F = -2\pi \sigma_B kL^{1-n}(\sigma_A \pi R^2) = -2\pi^2 \sigma_A \sigma_B kR^2 L^{1-n}$$

(25)

The potential energy function associated with this force is

$$V(L) = 2\pi^2 \sigma_A \sigma_B kR^2 \frac{L^{2-n}}{2-n}$$

(26)

We now go over to the mechanical motion. We imagine that the two disks are allowed to accelerate toward each other due to the force between them. We wish to consider the conservation laws for the system of these two disks. We imagine the two disks as being oriented parallel to the $yz$-plane with the $x$-axis running through the centers of the disks. The disk of particles of type $A$ has mass $m$ and is located at $x$ and while the other disk has mass $M$ and is located at coordinate $X$ with $x < X$. If we write the constants appearing in Eqs. (25) and (26) as

$$C = 2\pi^2 \sigma_A \sigma_B kR^2$$

(27)

and assume that this force is the only force acting on the disks, then Newton’s equations of motion for the disks give the momentum changes along the $x$-axis as

$$\frac{dp_m}{dt} = C(X - x)^{1-n} = -\frac{dp_M}{dt}$$

(28)

If we assume that the system momentum is entirely mechanical, then the total momentum $P$ is given by

$$P = \hat{p}_m + \hat{p}_M$$

(29)
By symmetry, the angular momentum taken about the origin vanishes

$$L = 0.$$  (30)

The total energy $U$ of the system includes both the mechanical energies $U_m$ and $U_M$ of the disks and the potential energy $V$ between the disks as given in Eq. (26)

$$U = U_m + U_M + \frac{C}{2 - n}(X - x)^{2-n}$$  (31)

The center of rest-mass $\vec{X}_{mass}$ of the system is given by

$$(m + M)\vec{X}_{mass} = m\vec{i}x + M\vec{i}X$$  (32)

while the center of energy $\vec{X}_{energy}$ of the system is given by

$$U\vec{X}_{energy} = U_m\vec{i}x + U_M\vec{i}X + \frac{C}{2 - n}(X - x)^{2-n}\left(\vec{i}x + \vec{i}X\right)$$  (33)

where the center-of-energy location for the potential energy has been taken as half-way between the disks.

We now wish to consider the conservation laws for this system. The conservation of linear momentum associated with space-translation invariance in the $x$-direction follows as

$$\frac{dP}{dt} = \vec{i}\frac{dp_m}{dt} + \vec{i}\frac{dp_M}{dt} = 0$$  (34)

from Newton’s equations of motion in Eq. (28). The conservation of energy associated with time-translation invariance follows as

$$\frac{dU}{dt} = \frac{dU_m}{dt} + \frac{dU_M}{dt} + \frac{dV}{dt}$$

$$= \left(\frac{dp_m}{dt} - C(X - x)^{1-n}\right)\frac{dx}{dt} + \left(\frac{dp_M}{dt} + C(X - x)^{1-n}\right)\frac{dX}{dt} = 0$$  (35)

where we have used $dV/dt = C(X - x)^{1-n}(dx/dt + dX/dt)$ together with the equations of motion appearing in Eq. (28) and the basic relation

$$\frac{dU_{mech}}{dt} = \frac{dP_{mech}}{dt} \cdot \frac{dx}{dt}$$  (36)

which holds for each disk.

When considering the conservation of linear momentum, angular momentum, and energy, we have not had to specify whether our system was invariant under Galilean transformations.
or under Lorentz transformations. However, we now wish to apply the last conservation law associated with change from one inertial frame to another. The generator of Galilean transformations is the total rest-mass times the center of rest mass as in Eq. (32). The conservation law associated with this generator is related to the relativistic conservation law in Eq. (9) when we divide Eq. (9) by \( c^2 \) and take the limit \( c^2 \to \infty \); this leaves only the restmass contributions to the energy and the linear momentum

\[
\frac{d}{dt} \left( \sum_i m_i \vec{X}_{\text{mass}} \right) = \sum_i \left( m_i \frac{dr_i}{dt} \right) = P
\] (37)

For our example involving Eq. (32), this gives the conservation law

\[
\frac{d}{dt} (m + M) \vec{X}_{\text{mass}} = \frac{d}{dt} (\hat{m}x + \hat{M}X) \\
= \hat{m} \frac{dx}{dt} + \hat{M} \frac{dX}{dt} = P
\] (38)

If we compare this Eq. (38) with Eq. (29) for the total momentum, we see that we must identify

\[
P_m = \hat{m} \frac{dx}{dt} \quad P_M = \hat{M} \frac{dX}{dt}
\] (39)

as is indeed appropriate for nonrelativistic physics. This result in Eq. (39) combined with the basic expression for the rate of change of mechanical energy in Eq. (36) then forces us to choose the nonrelativistic expression for mechanical energy

\[
U_m = \frac{1}{2} m \left( \frac{dx}{dt} \right)^2 \quad U_M = \frac{1}{2} M \left( \frac{dX}{dt} \right)^2
\] (40)

With these familiar nonrelativistic identifications, we find that the conservation laws for linear momentum, angular momentum, energy, and constant motion of the center of mass are all satisfied and the system is Galilean invariant. There is no restriction on the force between the disks which is given in Eq. (25).

Suppose now that we were to demand that our system of accelerating disks was invariant under Lorentz transformation. This requires the result of Eq. (9) which, from Eq. (33),
becomes here
\[
\frac{d}{dt} \left( U \hat{X}_{\text{energy}} \right) = \gamma \frac{d}{dt} \left( U_m x + U_M X + \frac{C}{2 - n} (X - x)^{2-n} \left(\frac{x + X}{2}\right) \right) \\
= \gamma \left[ \frac{d}{dt} \left( \frac{dp_m}{dt} \right) x + U_m \frac{dx}{dt} + \frac{d}{dt} \left( \frac{dp_M}{dt} + X + U_m \frac{dX}{dt} \right) \right] \\
+ \gamma \left[ C(X - x)^{1-n} \left( \frac{dX}{dt} - \frac{dx}{dt} \right) \frac{(x + X)}{2} + \frac{C}{2 - n} (X - x)^{2-n} \frac{1}{2} \left( \frac{dx}{dt} + \frac{dX}{dt} \right) \right] \\
= \gamma \left( \frac{dp_m}{dt} - C(X - x)^{1-n} \right) \frac{dx}{dt} + \gamma \left( \frac{dp_M}{dt} + C(X - x)^{1-n} \right) \frac{dX}{dt} + \gamma \left[ U_m \frac{dx}{dt} + U_m \frac{dX}{dt} \right] \\
+ \gamma \left[ \frac{C(X - x)^{1-n}}{2} \left( 1 - \frac{1}{2 - n} \right) \left( \frac{dx}{dt} - \frac{dX}{dt} \right) X + \frac{dx}{dt} \frac{dX}{dt} - \frac{dx}{dt} \frac{dX}{dt} \right] \\
= c^2 P 
\]

where we have used the equations of motion in Eq. (28) to simplify the expression. Thus Lorentz invariance for our system requires

\[
c^2 P = c^2 (\hat{p}_m + \hat{p}_M) \\
= \gamma U_m \frac{dx}{dt} + \gamma U_M \frac{dX}{dt} \\
+ \gamma \left[ \frac{C(X - x)^{1-n}}{2} \left( 1 - \frac{1}{2 - n} \right) \left( \frac{dx}{dt} - \frac{dX}{dt} \right) X + \frac{dx}{dt} \frac{dX}{dt} - \frac{dx}{dt} \frac{dX}{dt} \right] 
\]

Since the velocities \( \frac{dx}{dt}, \frac{dX}{dt} \), and the positions \( x, X \), are arbitrary, the only way for this requirement to be met is for the mechanical momentum \( c^2 P_{\text{mech}} \) to be given by \( U_{\text{mech}} \frac{dr}{dt} \)

\[
c^2 P_{\text{mech}} = U_{\text{mech}} \frac{dr}{dt} 
\]

and for the second line to vanish, implying

\[
\left( 1 - \frac{1}{2 - n} \right) = 0 \quad \text{or} \quad n = 1 
\]

Combining Eqs. (36) and (43), so as to eliminate the velocity \( \mathbf{v} = \frac{dr}{dt} \), we find \( U_{\text{mech}} \frac{dU_{\text{mech}}}{dt} = c^2 P_{\text{mech}} \cdot \frac{dP_{\text{mech}}}{dt} \) so that \( U_{\text{mech}}^2 = c^2 P_{\text{mech}}^2 + \text{const.} \). Denoting this constant of integration by \( \text{const} = m^2 c^4 \), we have precisely the requirements of relativistic mechanical momentum and energy related as

\[
U_{\text{mech}} = \left( c^2 P_{\text{mech}}^2 + m^2 c^4 \right)^{1/2} 
\]
and combining this with Eq. (43), we find the familiar relativistic mechanical momentum

$$p_{\text{mech}} = \frac{mv}{(1 - v^2/c^2)^{1/2}}$$  \hspace{1cm} (46)

The requirement in Eq. (44) that the exponent $n = 1$ corresponds exactly to the Coulomb potential in Eq. (23). Thus our disks provide a Lorentz-invariant system only in the case where they can be reinterpreted within classical electrodynamics as accelerating plates of a parallel-plate capacitor. We should note that classical electrodynamics does indeed involve velocity-dependent and acceleration dependent forces such as are mentioned in Section D, but these forces do not enter the disk example within the approximations $L \ll R \ll c^2/a$ where $a$ is the maximum acceleration of the plates.

The analysis here suggests three important aspects. First the conservation laws of energy, linear momentum, and angular momentum can hold independent of whether relativistic or nonrelativistic physics (or some combination of both) is employed in the analysis. Second, Galilean invariance requires that nonrelativistic expressions are used for mechanical energy and momentum but makes no restrictions upon a potential function $V(|r_1 - r_2|)$. Third, relativistic invariance requires not only that relativistic expressions are used for the mechanical energy and momentum but also places restrictions on the form of the interactions between the particles. In the disk example, the Coulomb potential is the unique potential associated with Lorentz invariance.

F. Requirements for a Fully Relativistic Extension

Although the calculations in Section C show that a system of two point particles interacting through a potential function $V(|r_1 - r_2|)$ can indeed be extended to a Lagrangian system which is Lorentz invariant through order $v^2/c^2$, this, in general, is as far as we can go. Already in equations (15) and (16) above, we have seen that a relativistic system requires that the interaction between particles involves not only energy $V(|r_1 - r_2|)$ but also additional velocity-dependent terms associated with the potential energy. Also, the Lagrangian equations of motion involve an additional time derivative and so require that there are velocity- and acceleration-dependent forces, as seen in Eq. (19). Presumably a fully Lorentz-invariant interaction requires a full field theory, and not every mechanical potential can be extended to a field theory.
For the Lorentz-invariant interaction of point particles, we expect the forces to be transmitted at the speed of light $c$. This speed is the only one which is the same in every inertial frame. Thus we expect the forces to be associated with a wave equation involving wave speed $c$. This wave-equation assumption has strong implications.

Let us consider the situation where the potential function $V$ arises from the interaction of a very massive point particle at the origin of the $S$ frame and a much lighter point particle $m$ at position $r$, so that the potential function can be regarded as given by $V(r)$ where $r = |r| = (x^2 + y^2 + z^2)^{1/2}$ is the distance from the origin in $S$. This same situation can be observed from the $S'$ frame moving with constant velocity $u = \hat{i}u$ relative to the $S$ frame. Then if $V(r)$ satisfies tensor behavior, we expect that in $S'$ the potential function $V'(x', y', z', t')$ moves rigidly with velocity $-u = -\hat{i}u$ becoming a function of $x' + ut'$, and so satisfies the wave equation

$$\frac{1}{u^2} \frac{\partial^2 V'}{\partial t'^2} - \frac{\partial^2 V'}{\partial x'^2} = (\delta - \text{function singularity at } r' = -ut') \quad (47)$$

Furthermore, the relativistic behavior requires that the potential function $V'(x', y', z', t')$ acting on the particle $m$ arises from a signal traveling with velocity $c$, and so the potential function satisfies the wave equation

$$\frac{1}{c^2} \frac{\partial^2 V'}{\partial t'^2} - \frac{\partial^2 V'}{\partial x'^2} - \frac{\partial^2 V'}{\partial y'^2} - \frac{\partial^2 V'}{\partial z'^2} = (\delta - \text{function singularity at } r' = -ut') \quad (48)$$

Subtracting Eq. (47) from Eq. (48) so as to eliminate the time derivatives, we find

$$c^2 \left(1 - \frac{u^2}{c^2}\right) \frac{\partial^2 V'}{\partial x'^2} + c^2 \frac{\partial^2 V'}{\partial y'^2} + c^2 \frac{\partial^2 V'}{\partial z'^2} = (\delta - \text{function singularity at } r' = -ut') \quad (49)$$

This suggests Lorentz contraction in the $x'$-direction. Also, if we take the limit as $u$ goes to zero so that we are back in the $S$ frame where the potential function is time independent, then we find Eq. (49) becomes

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = (\delta - \text{function singularity at } r = 0) \quad (50)$$

Thus the potential $V(r)$ which allows both the natural rigid behavior in another inertial frame and a natural relativistic extension to wave behavior at the relativistic speed $c$ must necessarily satisfy Laplace’s equation. But if the potential is rotationally symmetric in the $S$ frame where the massive particle is at rest, then the potential satisfying Laplace’s equation must be the Coulomb potential

$$V(r) = \frac{k}{r} \quad (51)$$
We conclude that a nonrelativistic potential function arising from a point source which allows a natural extension to a relativistic theory involving the wave equation must necessarily be the Coulomb potential.

The calculations given here suggest the possibility that the nonrelativistic Lagrangian for the interaction of two point particles given in Eq. (8) may be an approximation to nature only in the case of the Coulomb/Kepler potential.[15] In the case of interacting electric charges, the extension to a relativistic interaction through order $v^2/c^2$ gives the Darwin Lagrangian. And the Darwin Lagrangian is known to be a valid approximation to fully relativistic classical electrodynamics.

G. Implications for Classical Physics

Classical electromagnetism is a relativistic theory which was developed during the nineteenth century before the ideas of special relativity. Indeed, special relativity arose at the beginning of the twentieth century as a response to the conflict of electromagnetism with nonrelativistic mechanics. Around the same time, quantum mechanics was introduced in response to the mismatch between electromagnetic radiation equilibrium (blackbody radiation) and classical statistical mechanics (which is based on nonrelativistic mechanics). Although quantum theory and special relativity have gone on to enormous successes, they have left behind a number of unresolved questions within classical physics. For example, the blackbody radiation problem has never been solved within relativistic classical physics. [16] There have been discussions of classical radiation equilibrium using nonrelativistic mechanical scatterers and even one calculation of a scattering particle using relativistic mechanical momentum in a general class of non-Coulomb potentials. [5] However, there has never been a treatment of scattering by a relativistic particle in a Coulomb potential, despite the fact that the Coulomb potential has all the qualitative aspects which might allow classical radiation equilibrium at a spectrum with finite thermal energy.

We conclude that the misconceptions regarding potentials which allow extensions to relativistic systems is relevant for treatments in mechanics textbooks and perhaps also for the description of nature within classical theory.

Acknowledgement

The argument given here in Section E was adapted from the work of Dr.
Hans de Vries appearing as the answer to an unrelated query on the internet, http://www.physicsforums.com/showthread.php?t=114620

[1] H. Goldstein, C. Poole, and J. Safko, Classical Mechanics 3rd ed. (Addison-Wesley, New York 2002), p. 313.

[2] J. V. Jose and E.J. Saletan, Classical Dynamics: A Contemporary Approach (Cambridge University Press 1998), p. 210. The text includes some remarks indicating discomfort with the "relativistic Lagrangian."

[3] See ref. 1, Section 7.9, p. 316.

[4] One referee for an earlier version of this article wrote, "My understanding of what is meant by the 'relativistic Lagrangian' is a scalar functional of spacetime coordinates and their proper time derivatives, \( L = L(x^\mu, dx^\mu/d\tau) \), that is invariant under all coordinate transformations in a Minkowski spacetime. This is the definition used by A. O. Barut in his monograph, Electrodynamics and Classical Theory of Fields and Particles, MacMillan, 1964." This is the definition used by R.W. Brehme, "The Relativistic Lagrangian," Am. J. Phys. 39, 275-280 (1971).

[5] This mistaken idea of Lorentz-invariant behavior as a description of nature involving only the use of relativistic mechanical momentum appears in the classical theoretical analysis of black-body radiation by R. Blanco, L. Pesquera, and E. Santos, "Equilibrium between radiation and matter for classical relativistic multiperiodic systems. Derivation of Maxwell-Boltzmann distribution from Rayleigh-Jeans spectrum," Phys. Rev. D 27, 1254-1287 (1983), and "Equilibrium between radiation and matter for classical relativistic multiperiodic systems. II. Study of radiative equilibrium with Rayleigh-Jeans radiation," ibid. 29, 2240-2254 (1984).

[6] E. Kretchmann, "Über den physikalischen Sinn der Relativitätspostulate: A. Einsteins neue und seine ursprüngliche Relativitätstheorie," Annalen der Physik 53, 575-614 (1917).

[7] S. Coleman and J.H. Van Vleck, "Origin of 'hidden momentum forces' on magnets," Phys. Rev. 171, 1370-1375 (1968).

[8] See, for example, ref. 1, p. 313, Eq. (7.136) or ref. 2, p. 210, Eq. (5.27).

[9] See, for example, ref. 7 or T. H. Boyer, "Illustrations of the relativistic conservation law for the center of energy," Am. J. Phys. 73, 953-961 (2005).
[10] I am not aware of a derivation of the Darwin Lagrangian which takes the form given here. A different derivation is given by J. D. Jackson, *Classical Electrodynamics 3rd ed* (Wiley, New York 1999), p. 596-598; the Darwin Lagrangian is given in Eq. (12.82).

[11] The expressions for the electric and magnetic fields agree with those given by L. Page and N.I. Adams, "Action and Reaction Between Moving Charges," Am. J. Phys. 13, 141-147 (1945).

[12] See the article by Boyer in reference 9.

[13] See, for example, ref. 2, pp. 211-212.

[14] See, for example, ref. 10, p. 582, Eq. (12.12).

[15] Some people have suggested that particle interactions through the Yukawa potential provide a relativistic interaction; however, I am not aware of any *classical* relativistic theory of such forces.

[16] See, for example, the discussions by T. H. Boyer, "Blackbody radiation, conformal symmetry, and the mismatch between classical mechanics and electromagnetism," J. Phys. A: Math. Gen. 38, 1807-1821 (2005); "Connecting blackbody radiation, relativity, and discrete charge in classical electrodynamics," Found. Phys. 37, 999-1026 (2007).