POWERS OF COMPLEX PERSYMMETRIC ANTI-TRIDIAGONAL MATRICES WITH CONSTANT ANTI-DIAGONALS

Wang Yusun*, Qin Mei, Wang Haibo & Pan Xue
College of Science, University of Shanghai for Science and Technology, Shanghai, China, 200093
*Email: wangyusun2012@126.com

ABSTRACT
In this paper, we derive a general expression for the $p$th power ($p \in \mathbb{N}$) of any complex persymmetric anti-tridiagonal Hankel (constant anti-diagonals) matrices, in terms of the Chebyshev polynomials of second kind. Numerical examples are presented, which show that our results generalize the results in [4], [5], [7].

Keywords: Anti-tridiagonal matrices; Eigenvalues; Eigenvectors; Chebyshev polynomials.
MSC2010: 15B48.

1. INTRODUCTION
From a practical point of view, Antidiagonal matrices arise frequently in many areas of mathematics and engineering, such as numerical analysis, solution of the boundary value problems, high order harmonic filtering theory[1][2]. In many of such problems, we need to calculate some matrix functions such the powers, inverse or the exponential.

There is a lot of work dealing with the inverse of a Anti-tridiagonal matrix and solving the resulting linear system can be done in an efficient way. However, computing the integer powers of Anti-tridiagonal matrices has been a very popular problem in last few years. There have been several papers on computing the positive integer powers of various kinds of square matrices by J.Rimas, Jesús Gutiérrez, Yin, etc [3]-[7]. J.Rimas [4] gave the general expression of the $p$th power for this type of symmetric odd order anti-tridiagonal matrices ($antitridia g_n(a,b,a)$) in 2008. In [5]-[6] a similar problem was solved for anti-tridiagonal matrices having zeros in main skew diagonal and units in the neighbouring diagonals. In 2010, the general expression for the entries of the power of odd order anti-tridiagonal matrices with zeros in main skew diagonal and elements $1,1,\cdots;1,-1,-1,\cdots,-1$ in neighbouring diagonals was derived by J.Rams [7]. In 2011, the general expression for the entries of the power of complex persymmetric or skew-persymmetric anti-tridiagonal matrices with constant anti-diagonals are presented by Jesús Gutiérrez [3]. In 2013, J.Rimas [10] gave the eigenvalue decomposition for real odd order skew-persymmetric anti-tridiagonal matrices with constant anti-diagonals ($antitridia g_n(a,b,a)$) and derived the general expression for integer powers of such matrices.

In the present paper, we derive a general expression for the $p$th power ($p \in \mathbb{N}$) of any complex persymmetric anti-tridiagonal matrices with constant anti-diagonals ($antitridia g_n(b,a,b)$), in terms of the Chebyshev polynomials of the second kind.

This paper is organized as follows:
- In Section 2, we give the derivation of general expression.
- In Section 3, Numerical examples are presented.
- In Section 3, we summarize the paper.

2. DERIVATION OF GENERAL EXPRESSION
In this paper, we study the entries of positive integer power of $n \times n$ complex persymmetric anti-tridiagonal matrices with constant anti-diagonals
Consider the following $n \times n$ complex Toeplitz tridiagonal matrices

\[ B_n = \text{antitridiag}_n(b, a, b) = \begin{pmatrix}
    b & a & & \\
    b & a & b & \\
    & b & a & b \\
    a & b & & \\
\end{pmatrix} \tag{2.1} \]

and

\[ C_n = \text{antitridiag}_n(b, 0, -b) = \begin{pmatrix}
    b & 0 & & \\
    b & 0 & -b & \\
    & b & 0 & -b \\
    & & b & 0 & -b \\
\end{pmatrix} \tag{2.2} \]

where $a \in C, b \in C \setminus \{0\}$.

The next trivial result relates the matrix $B_n$ with $A_n$ (or matrix $C_n$ with $D_n$) and with the $n \times n$ backward identity [3]
Lemma 2.1 Let \( a \in C, b \in C \setminus \{0\} \) and \( n \in N \). Then
\[
B_n = J_n A_n, \tag{2.6}
\]
\[
C_n = J_n D_n, \tag{2.7}
\]
where \( B_n = \text{antitridia}_n(b,a,b) \), \( A_n = \text{tridiag}_n(b,a,b) \), \( C_n = \text{antitridia}_n(b,0,-b) \), \( D_n = \text{tridiag}_n(b,0,-b) \).

Proof. We have
\[
[J_n A_n]_{j,k} = \sum_{h=1}^{n} [J_n]_{j,h} [A_n]_{h,k} = [A_n]_{n+1-j,k} = \begin{cases} b, & \text{if } n + 1 - (j + k) = -1, \\ a, & \text{if } n + 1 - (j + k) = 0, \\ b, & \text{if } n + 1 - (j + k) = 1, \\ 0, & \text{other}. \end{cases}
\]
and
\[
[J_n D_n]_{j,k} = \sum_{h=1}^{n} [J_n]_{j,h} [D_n]_{h,k} = [D_n]_{n+1-j,k} = \begin{cases} -b, & \text{if } n + 1 - (j + k) = -1, \\ 0, & \text{if } n + 1 - (j + k) = 0, \\ b, & \text{if } n + 1 - (j + k) = 1, \\ 0, & \text{other}. \end{cases}
\]
This completes the proof.

We shall find the \( q \) th power \(( q \in N )\) of the matrices (2.1) and (2.2). Theorem 2.2 relates all positive integer powers of the matrix \( B_n \) with \( A_n \) and \( J_n \) (or \( C_n \) with \( D_n \) and \( J_n \)).

Theorem 2.2 If \( a \in C, b \in C \setminus \{0\} \), \( n \in N \), \( B_n = \text{antitridia}_n(b,a,b) \) and \( C_n = \text{antitridia}_n(b,0,-b) \), then
\[
B_n^q = \begin{cases} A_n^q, & \text{if } q \text{ is even}, \\ J_n A_n^q, & \text{if } q \text{ is odd}. \end{cases} \tag{2.8}
\]
\[ C_n^q = \begin{cases} (-1)^{\frac{q}{2}} D_n^q, & \text{if } q \text{ is even}, \\ (-1)^{\frac{q-1}{2}} J_n D_n^q, & \text{if } q \text{ is odd}. \end{cases} \quad (2.9) \]

where \( A_n = \text{tridiag}_n(b, a, b) \), \( D_n = \text{tridiag}_n(b, 0, -b) \).

**Proof.** We will proceed by induction on \( q \). The case \( q = 1 \) is obvious.

Suppose that the result is true for \( q \geq 1 \), and consider that case \( q + 1 \).

By the induction hypothesis we have

\[ B_n^{q+1} = B_n^q B_n^q = \begin{cases} B_n J_n A_n^q, & \text{if } q + 1 \text{ is even}, \\ B_n A_n^q, & \text{if } q + 1 \text{ is odd}. \end{cases} \]

Since \( B_n = J_n A_n \), we obtain that

\[ B_n^{q+1} = \begin{cases} J_n A_n J_n A_n^q, & \text{if } q + 1 \text{ is even}, \\ J_n A_n (A_n^q), & \text{if } q + 1 \text{ is odd}. \end{cases} \]

Since \( A_n \) is symmetric and \( J_n^{-1} = J_n \), we have

\[ B_n^q = \begin{cases} A_n^q, & \text{if } q \text{ is even}, \\ J_n A_n^q, & \text{if } q \text{ is odd}. \end{cases} \]

Similarly, we have

\[ C_n^q = \begin{cases} (-1)^{\frac{q}{2}} D_n^q, & \text{if } q \text{ is even}, \\ (-1)^{\frac{q-1}{2}} J_n D_n^q, & \text{if } q \text{ is odd}. \end{cases} \]

This completes the proof.

Next, we have to solve \( A_n^q \) and \( D_n^q \). We begin this work by reviewing a theorem regarding the Hermitian Toeplitz tridiagonal matrices \( A_n \) and \( D_n \).

**Theorem 2.3** Let \( a \in C, b \in C \setminus \{0\} \) and \( n \in \mathbb{N} \). Then \( A_n \) has eigenvalues 

\[ \lambda_j = a + 2|b| \cos \frac{i\pi}{n+1}, 1 \leq j \leq n, \]

and \( D_n \) has eigenvalues 

\[ \lambda_j = 2|b| \cos \frac{i\pi}{n+1}, 1 \leq j \leq n. \]

**Proof.** See [6].

With the tridiagonal matrix \( A_n \), we associate the polynomial sequence \( \{P_i\}_{1 \leq i \leq n} \) characterized by a three-term recurrence relation:

\[ xP_i(x) = bP_{i+1}(x) + aP_i(x) + bP_{i-1}(x), \quad i = 0, 1, \ldots, n-1. \quad (2.10) \]

With the tridiagonal matrix \( D_n \), we associate the polynomial sequence \( \{M_i\}_{1 \leq i \leq n} \) characterized by a three-term recurrence relation:

\[ xM_i(x) = bM_{i+1}(x) - bM_{i-1}(x), \quad i = 0, 1, \ldots, n-1. \quad (2.11) \]

With initial conditions \( P_0(x) = 0 \), \( P_1(x) = 1 \), \( M_0(x) = 0 \), and \( M_1(x) = 1 \) we can write the relations (2.10) and (2.11) in matrix form:
\[ xQ_{n-1}(x) = A_n Q_{n-1}(x) + P_n(x) E_n, \quad (2.12) \]

\[ xN_{n-1}(x) = D_n N_{n-1}(x) + M_n(x) E_n, \quad (2.13) \]

where \[ Q_{n-1}(x) = \begin{bmatrix} P_0(x), P_1(x), \ldots, P_{n-1}(x) \end{bmatrix}^T, \quad N_{n-1}(x) = \begin{bmatrix} M_0(x), M_1(x), \ldots, M_{n-1}(x) \end{bmatrix}^T \]

and \[ E_n = [0, 0, \ldots, 1]^T \in \mathbb{R}^n. \]

**Lemma 2.4** For \( i \geq 0 \), the degree of the polynomial \( P_i \) is \( i \) and \( P_i \) and \( P_{i+1} \) has no common root. The degree of the polynomial \( M_i \) is \( i \) and \( M_i \) and \( M_{i+1} \) has no common root.

**Proof.** See[8].

One can show that the characteristic polynomial of \( A_n \) is precisely \( nbP_n(x) \) and the characteristic polynomial of \( D_n \) is precisely \( -nbM_n(x) \). Hence the eigenvalues of \( A_n \) are exactly the roots of \( P_n \). Similarly we have the eigenvalues of \( D_n \) are exactly the roots of \( M_n \).

If \( \{\lambda_j\}_{0 \leq j \leq n-1} \) are the roots of the polynomial \( P_n \), then it follows from (2.7) that each \( \lambda_j \) is an eigenvalue of the matrix \( A_n \) and \( Q_{n-1}(x) = \begin{bmatrix} P_0(x), P_1(x), \ldots, P_{n-1}(x) \end{bmatrix}^T \) is a corresponding eigenvector [7]-[9]. For \( D_n \), we have same results. This observation should be taken into account elsewhere in the manuscript.

The polynomials \( \{P_i\}_{0 \leq i \leq n} \) and \( \{M_i\}_{0 \leq i \leq n} \) verify the well-known Christoffel-Darboux Identity:

**Lemma 2.5** We have:

\[ \sum_{i=0}^{n-1} P_i(x) P_i(y) = \frac{P_n(y)P_{n-1}(x) - P_n(x)P_{n-1}(y)}{y-x}, \quad \text{for} \quad x \neq y. \quad (2.14) \]

\[ \sum_{i=0}^{n-1} M_i(x) M_i(y) = \frac{M_n(y)M_{n-1}(x) - M_n(x)M_{n-1}(y)}{y-x}, \quad \text{for} \quad x \neq y. \quad (2.15) \]

**Proof.** See[8].

Tending \( y \) to \( x \) in formulas (2.14) and (2.15), we get:

\[ \sum_{i=0}^{n-1} P_i(x) P_i(x) = P'_n(x)P_{n-1}(x) - P_n(x)P'_{n-1}(x). \quad (2.16) \]

\[ \sum_{i=0}^{n-1} M_i(x) M_i(x) = M'_n(x)M_{n-1}(x) - M_n(x)M'_{n-1}(x). \quad (2.17) \]

Since the matrix \( A_n \) has distinct eigenvalues \( \lambda_0, \lambda_1, \ldots, \lambda_{n-1} \). Thus, the eigendecomposition of the matrix \( A_n \) is

\( A_n = TET^{-1} \),

where \( E = \text{diag}(\lambda_0, \lambda_1, \ldots, \lambda_{n-1}) \) and \( T \) is the transforming matrix formed by the eigenvectors of \( A_n \). Namely, \( T = (t_{ij} = P_{i-1}(\lambda_j))_{1 \leq i, j \leq n} \), where \( \{P_i\}_{1 \leq i \leq n} \) are defined as above. For \( D_n \), we have same results: the eigendecomposition of the matrix \( D_n \) is

\( D_n = SFS^{-1} \),

where \( F = \text{diag}(\lambda_0, \lambda_1, \ldots, \lambda_{n-1}) \) and \( S \) is the transforming matrix formed by the eigenvectors of \( D_n \). Namely, \( S = (s_{ij} = M_{i-1}(\lambda_j))_{1 \leq i, j \leq n} \), where \( \{M_i\}_{1 \leq i \leq n} \) are defined as above.
Lemma 2.6 If \( T^{-1} = (x_{i,j})_{i\leq j, j \leq n} \) and \( S^{-1} = (y_{i,j})_{i\leq j, j \leq n} \), then
\[
x_{i,j} = \frac{P_{j-1}(\lambda_{j-1})}{P_n'(\lambda_{j-1})P_n(\lambda_{j-1})},
\]
\[
y_{i,j} = \frac{M_{j-1}(\lambda_{j-1})}{M_n'(\lambda_{j-1})M_n(\lambda_{j-1})}.
\] (2.18), (2.19)

Proof. By using the relations (2.14) and (2.18) (or relations (2.15) and (2.19)), we obtain:
\[
\sum_{k=1}^{n} x_{i,k} t_{k,j} = \sum_{k=1}^{n} P_{k-1}(\lambda_{k-1})P_{k-1}(\lambda_{j-1}) = \delta_{i,j},
\]
\[
\sum_{k=1}^{n} y_{i,k} s_{k,j} = \sum_{k=1}^{n} P_{k-1}(\lambda_{k-1})P_{k-1}(\lambda_{j-1}) = \delta_{i,j},
\]
where \( \delta_{i,j} = 1 \) if \( i = j \) and \( \delta_{i,j} = 0 \) if \( i \neq j \).

This completes the proof.

For \( q \in \mathbb{N} \), we have \( A_q^T = TE^qT^{-1} \) and \( D_q^T = TF^qT^{-1} \).

We get immediately:

Theorem 2.7 Assume that \( q \in \mathbb{N} \) and \( A_q = (\alpha^{(q)}_{i,j})_{i\leq j, j \leq n} \), \( D_q = (\mu^{(q)}_{i,j})_{i\leq j, j \leq n} \).

Then:
\[
\alpha^{(q)}_{i,j} = \sum_{k=1}^{n} \frac{P_{k-1}(\lambda_{k-1})P_{k-1}(\lambda_{j-1})}{P_n'(\lambda_{k-1})P_n(\lambda_{k-1})},
\]
\[
\mu^{(q)}_{i,j} = \sum_{k=1}^{n} \frac{M_{k-1}(\lambda_{k-1})M_{k-1}(\lambda_{j-1})}{M_n'(\lambda_{k-1})M_n(\lambda_{k-1})}.
\]

By using the Cauchy Integral Formula, we can give another expressions of the coefficients \( \alpha^{(q)}_{i,j} \);
\[
\alpha^{(q)}_{i,j} = \int_{X} z^q \frac{P_{i-1}(z)P_{j-1}(z)}{P_n(z)P_{n-1}(z)} dz,
\] (2.20)
\[
\mu^{(q)}_{i,j} = \int_{Y} z^q \frac{M_{i-1}(z)M_{j-1}(z)}{M_n(z)M_{n-1}(z)} dz,
\] (2.21)

where \( X \) is a closed curve containing the roots of \( P_n \) and no roots of \( P_{n-1} \), \( Y \) is a closed curve containing the roots of \( M_n \) and no roots of \( M_{n-1} \).

Proof: Obviously this theorem holds.

Corollary 2.8 If the matrix \( A_n \) is nonsingular with \( A^{-1} = (\alpha^{(-1)}_{i,j})_{i\leq j, j \leq n} \), the matrix \( D_n \) is nonsingular with \( E^{-1} = (\mu^{(-1)}_{i,j})_{i\leq j, j \leq n} \), then
\[
A_n^{-1} = TE^{-1}T^{-1}
\] (2.22)
\[
D_n^{-1} = SF^{-1}S^{-1}
\] (2.23)
\[ \alpha_{i,j}^{(-1)} = \sum_{k=1}^{n} \frac{1}{\lambda_{k-1}} \frac{P_{i-1}(\lambda_{k-1}) P_{j-1}(\lambda_{k-1})}{P'_{n}(\lambda_{k-1}) P_{n-1}(\lambda_{k-1})} . \]  

(2.24)

\[ \mu_{i,j}^{(-1)} = \sum_{k=1}^{n} \frac{1}{\lambda_{k-1}} \frac{M_{i-1}(\lambda_{k-1}) M_{j-1}(\lambda_{k-1})}{M'_{n}(\lambda_{k-1}) M_{n-1}(\lambda_{k-1})} . \]  

(2.25)

By using the Cauchy Integral Formula, we can give another expressions of the coefficients \( \alpha_{i,j}^{(-1)} \) and \( \mu_{i,j}^{(-1)} \):

\[ \alpha_{i,j}^{(-1)} = \int_{X} z^{-1} \frac{P_{i-1}(z) P_{j-1}(z)}{P_{n}(z) P_{n-1}(z)} \, dz, \]  

(2.26)

\[ \mu_{i,j}^{(-1)} = \int_{Y} z^{-1} \frac{M_{i-1}(z) M_{j-1}(z)}{M_{n}(z) M_{n-1}(z)} \, dz, \]  

(2.27)

where \( X \) is a closed curve containing the roots of \( P_{n} \) and no roots of \( P_{n-1} \), \( Y \) is a closed curve containing the roots of \( M_{n} \) and no roots of \( M_{n-1} \).

**Theorem 2.9** Assume that \( q \in \mathbb{N}, B_{n}^{q} = (\beta_{i,j}^{(q)})_{1 \leq i,j \leq n} \) and \( C_{n}^{q} = (\omega_{i,j}^{(q)})_{1 \leq i,j \leq n} \). Then:

\[ \beta_{i,j}^{(q)} = \begin{cases} \sum_{k=1}^{n} \frac{\lambda_{k}^{q} P_{i-1}(\lambda_{k-1}) P_{j-1}(\lambda_{k-1})}{P'_{n}(\lambda_{k-1}) P_{n-1}(\lambda_{k-1})}, & \text{if } q \text{ is even}, \\ \sum_{k=1}^{n} \frac{\lambda_{k}^{q} P_{n-1}(\lambda_{k-1}) P_{j-1}(\lambda_{k-1})}{P'_{n}(\lambda_{k-1}) P_{n-1}(\lambda_{k-1})}, & \text{if } q \text{ is odd}. \end{cases} \]  

(2.28)

\[ \omega_{i,j}^{(q)} = \begin{cases} (-1)^{\frac{q}{2}} \sum_{k=1}^{n} \lambda_{k}^{q} M_{i-1}(\lambda_{k-1}) M_{j-1}(\lambda_{k-1}), & \text{if } q \text{ is even}, \\ (-1)^{\frac{q-1}{2}} \sum_{k=1}^{n} \lambda_{k}^{q} M_{n-1}(\lambda_{k-1}) M_{j-1}(\lambda_{k-1}), & \text{if } q \text{ is odd}. \end{cases} \]  

(2.29)

By using the Cauchy Integral Formula, we can give another expressions of the coefficients \( \beta_{i,j}^{(q)} \) and \( \omega_{i,j}^{(q)} \):

\[ \beta_{i,j}^{(q)} = \begin{cases} \int_{X} z^{q} \frac{P_{i-1}(z) P_{j-1}(z)}{P_{n}(z) P_{n-1}(z)} \, dz, & \text{if } q \text{ is even}, \\ \int_{X} z^{q} \frac{P_{n-1}(z) P_{j-1}(z)}{P_{n}(z) P_{n-1}(z)} \, dz, & \text{if } q \text{ is odd}. \end{cases} \]  

(2.30)

\[ \omega_{i,j}^{(q)} = \begin{cases} (-1)^{\frac{q}{2}} \int_{X} z^{q} \frac{M_{i-1}(z) M_{j-1}(z)}{M_{n}(z) M_{n-1}(z)} \, dz, & \text{if } q \text{ is even}, \\ (-1)^{\frac{q-1}{2}} \int_{X} z^{q} \frac{M_{n-1}(z) M_{j-1}(z)}{M_{n}(z) M_{n-1}(z)} \, dz, & \text{if } q \text{ is odd}. \end{cases} \]  

(2.31)

where \( X \) is a closed curve containing the roots of \( P_{n} \) and no roots of \( P_{n-1} \), \( Y \) is a closed curve containing the roots of \( M_{n} \) and no roots of \( M_{n-1} \).
Proof. From Theorem 2.2 we get:

\[
[B^q_n]_{i,j} = \begin{cases} 
[A^q_n]_{i,j}, & \text{if } q \text{ is even,} \\
\sum_{n=1}^{k} [J_n A^q_n]_{i,j} & \text{if } q \text{ is odd.}
\end{cases}
\]

Namely,

\[
\beta^{(q)}_{i,j} = \begin{cases} 
\alpha^{(q)}_{i,j}, & \text{if } q \text{ is even,} \\
\alpha^{(q)}_{n+1-i,j} & \text{if } q \text{ is odd.}
\end{cases}
\]

From Theorem 2.7 it follows:

\[
\beta^{(q)}_{i,j} = \begin{cases} 
\sum_{k=1}^{n} \lambda_{k-1}^{p} P_{i-1}(\lambda_{k-1}) P_{j-1}(\lambda_{k-1}), & \text{if } q \text{ is even,} \\
\sum_{k=1}^{n} \lambda_{k-1}^{p} P_{n-1}(\lambda_{k-1}) P_{j-1}(\lambda_{k-1}), & \text{if } q \text{ is odd.}
\end{cases}
\]

By using the Cauchy Integral Formula, we can give another expressions of the coefficients \( \beta^{(q)}_{i,j} \):

\[
\beta^{(q)}_{i,j} = \begin{cases} 
\int_{X} z^{q} \frac{P_{i-1}(z) P_{j-1}(z)}{P_{n}(z) P_{n-1}(z)} dz, & \text{if } q \text{ is even,} \\
\int_{X} z^{q} \frac{P_{n-1}(z) P_{j-1}(z)}{P_{n}(z) P_{n-1}(z)} dz, & \text{if } q \text{ is odd.}
\end{cases}
\]

Similarly, we have

\[
\omega^{(q)}_{i,j} = \begin{cases} 
(-1)^{\frac{q}{2}} \int_{Y} z^{q} \frac{M_{i-1}(z) M_{j-1}(z)}{M_{n}(z) M_{n-1}(z)} dz, & \text{if } q \text{ is even,} \\
(-1)^{\frac{q-1}{2}} \int_{Y} z^{q} \frac{M_{n-1}(z) M_{j-1}(z)}{M_{n}(z) M_{n-1}(z)} dz, & \text{if } q \text{ is odd.}
\end{cases}
\]

This completes the proof.

Corollary 2.10 Assume that \( q \in N \), \( B^{-1}_n = (\beta^{(-1)}_{i,j})_{i\leq j, j=1} \) and \( C^{-1}_n = (\omega^{(-1)}_{i,j})_{i\leq j, j=1} \). Then:

\[
\beta^{(-1)}_{i,j} = \sum_{k=1}^{n} \frac{1}{\lambda_{k-1}} \frac{P_{i-1}(\lambda_{k-1}) P_{j-1}(\lambda_{k-1})}{P_{n}(\lambda_{k-1}) P_{n-1}(\lambda_{k-1})},
\]

\[
\omega^{(-1)}_{i,j} = \sum_{k=1}^{n} \frac{-1}{\lambda_{k-1}} \frac{M_{i-1}(\lambda_{k-1}) M_{j-1}(\lambda_{k-1})}{M_{n}(\lambda_{k-1}) M_{n-1}(\lambda_{k-1})}.
\]

By using the Cauchy Integral Formula, we can give another expressions of the coefficients \( \beta^{(-1)}_{i,j} \) and \( \omega^{(-1)}_{i,j} \):

\[
\beta^{(-1)}_{i,j} = \int_{X} z^{(-1)} \frac{P_{i-1}(z) P_{j-1}(z)}{P_{n}(z) P_{n-1}(z)} dz,
\]

(2.32)

\[
\omega^{(-1)}_{i,j} = \int_{Y} z^{(-1)} \frac{M_{i-1}(z) M_{j-1}(z)}{M_{n}(z) M_{n-1}(z)} dz,
\]

(2.33)

where \( X \) is a closed curve containing the roots of \( P_{n} \) and no roots of \( P_{n-1} \), \( Y \) is a closed curve containing the
roots of $M_n$ and no roots of $M_{n-1}$.

3. NUMERICAL EXAMPLES

(a). The Persymmetric Case

Consider the $n$ order anti-tridiagonal matrix $B_n$ of the following type:

$$B_n = \begin{pmatrix}
1 & a & 1 & & & \\
1 & a & 1 & & & \\
& & & \ddots & & \\
& & & & & \\
a & & & & & 1
\end{pmatrix}$$

Assume that

$$A_n = \begin{pmatrix}
a & 1 & & & \\
& a & 1 & & \\
& & & \ddots & & \\
& & & & & a \\
& & & & & 1
\end{pmatrix},$$

and

$$J_n = \begin{pmatrix}
1 & & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & \\
& & & & & 1
\end{pmatrix},$$

where $A_n$ and $J_n$ are $n \times n$ matrix. The polynomial sequence $\{P_i\}_{i=0}^{n-1}$ verifies

$$xP_i(x) = P_{i+1}(x) + aP_i(x) + P_{i-1}(x), \quad i = 0, 1, \ldots, n-1.$$ 

With initial conditions $P_{-1}(x) = 0$ and $P_0(x) = 1$.

By simple calculation we can show that:

$$P_i(x) = U_i(\frac{x-a}{2}), \quad i = 0, \ldots, n,$$

where $U_i$ are the Chebyshev polynomials [9] of the second kind which satisfies the three-term recurrence relations:

$$2xU_i(x) = U_{i+1}(x) + U_{i-1}(x),$$

with initial conditions $U_0(x) = 1$ and $U_1(x) = 2x$.

Each $U_n$ satisfies
\[ U_n(x) = \frac{\sin((n+1)\arccos x)}{\sin(\arccos x)}, \]
and thus the roots of \( U_n(x) \) are \( z_k = \cos \frac{k\pi}{n+1}, k = 1, \ldots, n \). Then, the eigenvalues of \( A \) are
\[ \lambda_k = a + 2\cos \frac{k\pi}{n+1}, k = 1, \ldots, n. \]

By Theorem 2.7 We get: assume that \( q \in N \) and \( A^q = (\alpha_{i,j}^{(q)})_{|\leq s, j \leq n} \), then:
\[ \alpha_{i,j}^{(q)} = 2 \sum_{k=1}^{n} (a + 2\cos \frac{k\pi}{n+1})^q \frac{U_{i-1}(\cos \frac{k\pi}{n+1})U_{j-1}(\cos \frac{k\pi}{n+1})}{U_n'(\cos \frac{k\pi}{n+1})U_{n-1}(\cos \frac{k\pi}{n+1})}. \]

If the matrix \( A \) is nonsingular, and \( A^{-1} = (\alpha_{i,j}^{(-1)})_{|\leq s, j \leq n} \). Then:
\[ l\alpha_{i,j}^{(-1)} = 2 \sum_{k=1}^{n} \frac{1}{a + 2\cos \frac{k\pi}{n+1}} \frac{U_{i-1}(\cos \frac{k\pi}{n+1})U_{j-1}(\cos \frac{k\pi}{n+1})}{U_n'(\cos \frac{k\pi}{n+1})U_{n-1}(\cos \frac{k\pi}{n+1})}. \]

We can obtain:
\[ \beta_{i,j}^{(q)} = \begin{cases} 
2 \sum_{k=1}^{n} (a + 2\cos \frac{k\pi}{n+1})^q \frac{U_{i-1}(\cos \frac{k\pi}{n+1})U_{j-1}(\cos \frac{k\pi}{n+1})}{U_n'(\cos \frac{k\pi}{n+1})U_{n-1}(\cos \frac{k\pi}{n+1})}, & \text{if } q \text{ is even,} \\
2 \sum_{k=1}^{n} (a + 2\cos \frac{k\pi}{n+1})^q \frac{U_{n-1}(\cos \frac{k\pi}{n+1})U_{j-1}(\cos \frac{k\pi}{n+1})}{U_n'(\cos \frac{k\pi}{n+1})U_{n-1}(\cos \frac{k\pi}{n+1})}, & \text{if } q \text{ is odd.}
\end{cases} \]

and
\[ \beta_{i,j}^{(-1)} = 2 \sum_{k=1}^{n} (a + 2\cos \frac{k\pi}{n+1})^{-1} \frac{U_{i-1}(\cos \frac{k\pi}{n+1})U_{j-1}(\cos \frac{k\pi}{n+1})}{U_n'(\cos \frac{k\pi}{n+1})U_{n-1}(\cos \frac{k\pi}{n+1})}. \]

If \( a = 0 \), we have the following theorems.

**Theorem 3.1** Consider an odd natural number \( n = 2m + 1, m \in N \). Let \( B_n = \text{antitridia}_n(1,0,1) \) and \( \lambda_k = -2\cos \frac{k\pi}{n+1} \) for every \( 1 \leq k \leq n \). Then
\[ \left[ B_n^* \right]_{ij} = \frac{1 + (-1)^{q+i+j}}{2n+2} \sum_{k=1}^{n-1} \beta_{i,j}(k) \lambda_k^2 (4 - \lambda_k^2) U_{n-i} \left( \frac{\lambda_k}{2} \right) U_{n-j} \left( \frac{\lambda_k}{2} \right), \]
for all \( q \in N \) and \( 1 \leq i, j \leq n \), where
\[
\beta_{i,j}(k) = \begin{cases} 
1, & \text{if } i + j \text{ is even}, \\
(-1)^{k-1}, & \text{if } i + j \text{ is odd}.
\end{cases}
\]
\( \lambda_k (k = 1, 2, \cdots, n) \) are the eigenvalues of the matrix \( B_n \), \( U_n(x) \) is the \( k \) th degree Chebyshev polynomial of the second kind.

**Theorem 3.2** Consider an even natural number \( n = 2m \), \( m \in N \). Let \( B_n = antitridia g_n(1,0,1) \) and \( \lambda_k = -2 \cos \frac{k\pi}{n+1} \) for every \( 1 \leq k \leq n \). Then
\[
[B^q_n]_{i,j} = \frac{1}{n+1} \gamma_{i,j} \sum_{k=1}^{n} \lambda_k^q (4-\lambda_k^2) U_{n-1}(\frac{\lambda_k^2}{2}) U_{n-j}(\frac{\lambda_k^2}{2})
\]
for all \( q \in N \) and \( 1 \leq i, j \leq n \), where \( \gamma_{i,j} = \begin{cases} 
1, & \text{if } i + j \text{ is even}, \\
0, & \text{if } i + j \text{ is odd}.
\end{cases} \)

For even order matrix \( B \) the following condition is fulfilled: \( \lambda_k \neq 0(k = 2, 4, \cdots, n) \). This means, that even order matrix \( B_n = antitridia g_n(1,0,1) \) is nonsingular (its determinant is not equal to zero) and derived expression of \( B^q_n \) can be applied for computing negative integer powers, as well. Taking \( q = -1 \), we get the following expression for elements of the inverse matrix \( B_n^{-1} \)
\[
[B_n^{-1}]_{i,j} = \frac{1}{n+1} \gamma_{i,j} \sum_{k=1}^{n} \lambda_k^q (4-\lambda_k^2) U_{n-1}(\frac{\lambda_k^2}{2}) U_{n-j}(\frac{\lambda_k^2}{2}), i, j = 0, 1, \cdots, n.
\]

If \( a = 1 \), we have the following theorems.

**Theorem 3.3** Consider an even natural number \( n = 2m \), \( m \in N \). Let \( B_n = antitridia g_n(1,1,1) \) and \( \lambda_k = (-1)^{k-1}(1+2 \cos \frac{k\pi}{n+1}), (k = 1, 2, \cdots, n) \). Then
\[
[B^q_n]_{i,j} = \frac{2}{n+1} \sum_{k=1}^{n} \lambda_k^q \sin \frac{k\pi}{n+1} \sin \frac{k\pi}{n+1}
\]
\[
= \frac{2}{n+1} \sum_{k=1}^{n} [\lambda_k^q + (-1)^{i+j}(\lambda_k + 2(-1)^q)] \sin \frac{k\pi}{n+1} \sin \frac{k\pi}{n+1}.
\]

**b. The Skew-persymmetric Case**

For \( C_n = antitridia g_n(b,0,-b) \), if \( b = 1 \), we have the following theorem by Theorem 2.9.

**Theorem 3.4** Consider \( n \in N \). Let \( C_n = antitridia g_n(1,0,-1) \) and \( \lambda_k = -2 \cos \frac{k\pi}{n+1}, (k = 1, 2, \cdots, n) \). If \( q \equiv i + j \), we have
\[
[C_n]_{i+j}^{q} = \frac{(-1)^{i+j}}{n+1} \sum_{k=1}^{n} \lambda_k^q (4-\lambda_k^2) U_{i-1}(\frac{\lambda_k^2}{2}) U_{j-1}(\frac{\lambda_k^2}{2}).
\]
In other case, we get
\[
[C_n]_{i,j}^{q} = 0,
\]
where $\tilde{i} = \begin{cases} 
 i & \text{if } q \text{ is even}, \\
 n+1-i & \text{if } q \text{ is odd}. 
\end{cases}$

From Theorem 3.4, we can find any of these positive integer power of $C_n = antitridia g_n(1,0,-1)$. As an example, we consider the cases $n = 3$.

$$C_3^q = \begin{cases} 
 \left( \begin{array}{ccc}
 \gamma_1 & \gamma_2 & \gamma_3 \\
 \gamma_2 & \gamma_4 & \gamma_2 \\
 \gamma_3 & \gamma_2 & 0 \\
 \gamma_2 & 0 & -\gamma_2 \\
 0 & -\gamma_2 & -\gamma_3 
\end{array} \right) & \text{if } q \text{ is even,} \\
 \left( \begin{array}{ccc}
 \gamma_1 & \gamma_2 & \gamma_3 \\
 \gamma_2 & \gamma_4 & \gamma_2 \\
 \gamma_3 & \gamma_2 & 0 \\
 \gamma_2 & 0 & -\gamma_2 \\
 0 & -\gamma_2 & -\gamma_3 
\end{array} \right) & \text{if } q \text{ is odd.} 
\end{cases}$$

where $\gamma_1 = c^{q-3}[1 + (-1)^q]$, $\gamma_2 = c^{q-3}[1 + (-1)^q]$, $\gamma_3 = -c^{q-4}[1 + (-1)^q]$, $\gamma_4 = c^{q-2}[1 + (-1)^q]$ with $c = \sqrt{2}$.

4. CONCLUSION

In this paper, we derive a general expression for the $p$th power ($p \in N$) of any complex persymmetric anti-tridiagonal Hankel (constant anti-diagonals) matrices with constant anti-diagonals ($antitridia g_n(b,a,b)$ and $antitridia g_n(b,0,-b)$). Numerical examples are presented. This novel expression is both an extension of the one obtained by Rimas for the powers of the matrix $antitridia g_n(1,0,1)$ with $n \in N$ (see [4] for the odd case and [7] for the even case) and an extension of the one obtained by Honglin Wu for the powers of the matrix $antitridia g_n(1,1,1)$ with $n \in N$ (see [5] for the even case). We may safely draw the conclusion: our results generalize the results in [4],[5],[7].

REFERENCES

[1]. R.M. Gray, Toeplitz and circulant matrices: a review, Foundations and Trends in Communications and Information Theory 2 (3) (2006) 155–239.
[2]. P.M. Crespo, J. Gutiérrez-Gutiérrez, On the elementwise convergence of continuous functions of Hermitian banded Toeplitz matrices, IEEE Transactions on Information Theory 53 (3) (2007) 1168–1176.
[3]. J.Gutierrez, Powers of complex persymmetric or skew-persymmetric anti-tridiagonal matrices with constant anti-diagonals, Appl.Math.Comput. 217 (2011) 6125-6132.
[4]. J. Rimas, On computing of arbitrary positive integer powers for one type of symmetric anti-tridiagonal matrices of odd order, Applied Mathematics and Computation 203 (2008) 573-581.
[5]. Honglin Wu, On computing of arbitrary positive powers for one type of anti-tridiagonal matrices of even order, Appl. Math. Comput. 217 (2010) 2750-2756.
[6]. Qingxiang Yin, On computing of arbitrary positive powers for anti-tridiagonal matrices of even order, Appl. Math. Comput. 203 (2008) 252-257.
[7]. J.Rimas, on computing of arbitrary positive integer power of odd order anti-tridiagonal matrices with zeros in main skew diagonal and elements 1,1,...,1;1,1,...,1 in neighbouring diagonals. Appl. Math. Comput.210(2009)64-71.
[8]. R.A.Horn, C.R. Johnson, Matrix analysis, Cambridge University press. New York. 1990.
[9]. L.Fox, J.B.parke, Chebyshev Polynomials in Numerical Analysis, Oxford University Press. London. 1968.
[10]. J.Rimas, Integer powers of real odd order skew-persymmetric anti-tridiagonal matrices with constant anti-diagonals, Appl. Math. Comput.219(2013)7075-7088.