OPTIMAL CONTROL OF THE LINEAR WAVE EQUATION BY
TIME-DEPENDING BV-CONTROLS: A SEMI-SMOOTH
NEWTON APPROACH

SEBASTIAN ENGEL∗
Institute for Mathematics and Scientific Computing
Karl-Franzens-Universität, Heinrichstr. 36
Graz, 8010, Austria

KARL KUNISCH
Radon Institute, Austrian Academy of Sciences
and Institute for Mathematics and Scientific Computing
Karl-Franzens-Universität, Heinrichstr. 36
Graz, 8010, Austria

(Communicated by Marius Tucsnak)

Abstract. An optimal control problem for the linear wave equation with con-
trol cost chosen as the BV semi-norm in time is analyzed. This formulation en-
hances piecewise constant optimal controls and penalizes the number of jumps.
Existence of optimal solutions and necessary optimality conditions are derived.
With numerical realisation in mind, the regularization by $H^1$ functionals is in-
vestigated, and the asymptotic behavior as this regularization tends to zero
is analyzed. For the $H^1$–regularized problems the semi-smooth Newton algo-
rithm can be used to solve the first order optimality conditions with super-linear
convergence rate. Examples are constructed which show that the distributional
derivative of an optimal control can be a mix of absolutely continuous measures
with respect to the Lebesgue measure, a countable linear combination of Dirac
measures, and Cantor measures. Numerical results illustrate and support the
analytical results.

1. Introduction. We investigate the following optimal control problem for the
wave equation with homogeneous Dirichlet boundary condition:

$$
\begin{align*}
(P) & \quad \min_{u \in BV(0,T)} \left\{ \frac{1}{2} \| y_u - y_d \|^2_{L^2(\Omega_T)} + \sum_{j=1}^m \alpha_j \| D_t u_j \|_{M(I)} \right\} \\
(W) & \quad \begin{cases}
\partial_{tt} y_u - \Delta y_u = \sum_{j=1}^m u_j g_j & \text{in } (0,T) \times \Omega \\
y_u = 0 & \text{on } (0,T) \times \partial \Omega \\
(y_u, \partial_t y_u) = (y_0, y_1) & \text{in } \{0\} \times \Omega,
\end{cases}
\end{align*}
$$

2010 Mathematics Subject Classification. Primary: 26A45, 35L05, 49J20, 49J52; Secondary:
35L10, 49K20.

Key words and phrases. Wave equation, optimal control problems, sparsity, non-smooth anal-
ysis, functions of bounded variation, semi-smooth Newton method.

∗ Corresponding author: Sebastian Engel.
where \( \Omega \subset \mathbb{R}^n \), with \( n \in \{1, 2, 3\} \), is a bounded open domain with Lipschitz boundary \( \Gamma := \partial \Omega \), \( T \in (0, \infty) \), and \( y_d \in L^2((0, T) \times \Omega) \). The temporally dependent controls \( u \) are chosen as \( u = (u_1, \ldots, u_m) \in BV(0, T)^m \), and \( BV(0, T)^m \) is endowed with the norm \( \| u \|_{BV(I)^m} = \sum_{j=1}^m \| u_j \|_{L^1(I)} + \| D_1 u_j \|_{M(I)} \), where \( I := (0, T) \).

Here \( M(I) \) denotes the space of Borel measures, endowed with the total variation norm \( \| \cdot \|_{M(I)} \). Further let \( (g_j)_j^m \subset \mathcal{L}^\infty(\Omega) \setminus \{0\} \) with pairwise disjoint supports \( w_j := \text{supp}(g_j) \), and \( \alpha_j > 0 \). The initial data are chosen as \( (y_0, y_1) \in H_0^1(\Omega) \times L^2(\Omega) \) and we abbreviate \( H := L^2(\Omega) \), \( V := H_0^1(\Omega) \), with \( V^* := H^{-1}(\Omega) \). Finally we set \( \Omega_T := (0, T) \times \Omega \).

In problem \((P)\), we focus our attention on sparse optimal controls in the sense that they are piecewise constant. In particular, using the total variation of a BV-function in the cost functional \( J \), enhances sparsity in the derivative of the optimal control. If the total variation norm of the derivative of the control is replaced with Hilbert-space control costs, a classical tracking type optimal control problem is obtained, c.f. [25, p. 295 et seq.], and also e.g. [14], [22].

For a piecewise constant optimal control of \((P)\) the jumps are located in the position of these Dirac measures, see for example [9]. This type of sparsity property is reflected in the necessary and sufficient first-order optimality condition. As far as the authors know, the \( L^1 \)-norm is one of the first discussed sparsity enhancing cost terms in the context of partial differential equations. A detailed discussion on the history of sparsity in optimal control of partial differential equations can be found in e.g. [2]. Furthermore, sparsity results for optimal control problems with linear partial differential equations are considered in several works. References were specified for example in [3] where the authors emphasize the papers [4], [5], [6], [7], [8], [11], [12], [19], and [20]. In image reconstruction, BV-functions are well investigated but modeling aspects are different compared to optimal control with partial differential equations. In mathematical image analysis the use of BV-functionals is motivated by their ability to preserve natural edges and corners in the image. An introduction to image reconstruction aspects can be found in [10].

For the purpose of numerical realization we rely on regularized problems by using the \( H^1 \) semi-norm. This enables us to approximate the \( BV \) optimal control of \((P)\) by the \( H^1 \) controls in the strict-\( BV \) sense. The main purpose of this regularization is to use the semi-smooth Newton algorithm for which we present super-linear convergence results. In particular, one is able to show that the regularized problem permits a point-wise formula for the derivative of the \( H^1 \) controls. This property is used for the well-posedness result of the Newton algorithm.

The choice of the control costs related to BV-norms or BV-seminorms has not yet received much attention in the literature. In [11] the effect of \( L^2, H^1 \)-, measure-valued and BV-valued control costs on the qualitative behavior of the optimal control is compared and a significant difference of the resulting optimal controls was pointed out. A systematic study of the use of controls which are BV-functions in time for optimal control related to semi-linear parabolic equation is given in [9]. In the numerical experiments of that paper it was noted that indeed the use of BV-control costs enhance the property that the optimal controls only exhibit a few jumps (switches). Here we aim for obtaining related results for the linear wave equation. Differently from [11], for the numerical realisation by means of a semi-smooth Newton method, we use an \( H^1 \) regularisation of the infinite dimensional problem and verify super-linear convergence of the method. While in [11] the numerical
realisation of the non-smooth part originating from the BV-function uses a duality formulation, it is based on a prox-operator approach, c.f. [26], in the present work.

Let us briefly outline the following sections. In section 2 we gather the necessary prerequisites on the wave equation and on one-dimensional BV-functions which will be needed later on in this paper. Section 3 is dedicated to the analysis of the optimal control problem and sparsity properties of the optimal controls. Section 4 is devoted to the regularized problem \((P_\gamma^1)\), the corresponding convergence results for the optimal controls of \((P_\gamma^1)\) as \(\gamma \to 0\), and the first-order optimality conditions of \((P_\gamma^1)\). Furthermore, the semi-smooth Newton algorithm and its super-linear convergence are presented. The algorithm is embedded into a path following algorithm to approximate the original unregularized problem. In section 5, we construct test cases for problem \((P)\) in such a manner that exact analytic solutions for \((P)\) can be found. The construction steps can be used to build all types of distributional realisations of the non-smooth part originating from the BV-function uses a duality formulation, it is based on a prox-operator approach, c.f. [26], in the present work.

\section{The wave equation and BV functions in time.}

\subsection{Preliminaries on the wave equation.}

Since in this work non-smooth data are used for the wave equation, we directly introduce the weak solution of the wave equation (see e.g. [28]). In particular \(y_u\) is understood as the weak solution of the wave equation \((W)\) in problem \((P)\). Furthermore, we present in this section standard regularity results, and an energy estimation for the weak solution of the wave equation.

\begin{definition}[(21, Chap.IV, Sec.4)]\label{def:wave}
We call \(y \in C([0, T]; V)\) with \(\partial_t y \in C([0, T]; H)\) a weak solution of \((W)\) with forcing \(f \in L^1(0, T; H)\), displacement \(y_0 \in V\), and velocity \(y_1 \in H\) if \(y|_{t=0} = y_0\)
\begin{equation}
\int_I \left(\langle \partial_t y, \partial_t \eta \rangle_H + \langle \nabla y, \nabla \eta \rangle_{L^2(\Omega_T)}\right) dt = \langle y_1, \eta|_{t=0} \rangle_H + \int_I \langle f, \eta \rangle_H dt
\end{equation}
for every \(\eta \in L^1(I; V)\) such that \(\partial_t \eta \in L^1(I; H), \eta|_{t=T} = 0\).
\end{definition}

\begin{theorem}[(28)]\label{thm:wave}
For each \((f, y_0, y_1) \in L^1(0, T; H) \times V \times H\) there exists a unique weak solution \(y = y(f, y_0, y_1) \in C([0, T]; V) \cap C^1([0, T]; H)\) of \((W)\).

The mapping \((f, y_0, y_1) \mapsto y(f, y_0, y_1)\) is linear and continuous from \(L^1(0, T; H) \times V \times H\) into \(C([0, T]; V) \cap C^1([0, T]; H)\).

In particular, there exists a constant \(c > 0\) such that for all \((f, y_0, y_1) \in L^1(I; H) \times V \times H\) the unique weak solution \(y = y(f, y_0, y_1)\) satisfies
\begin{equation}
\|y\|_{C(T; V)} + \|\partial_t y\|_{C(T; H)} \leq c (\|f\|_{L^1(I; H)} + \|y_0\|_V + \|y_1\|_{L^2})
\end{equation}
\end{theorem}

For the proof we refer to [28, Proposition 1.1].

\begin{definition}
Let us define the following continuous linear operators
\begin{align*}
L: \quad &L^2(\Omega_T) \to L^2(\Omega_T) &\quad Q: \quad V \times H \to L^2(\Omega_T)
\end{align*}
\begin{align*}
f &\quad \mapsto y(f, 0, 0) &\quad (y_0, y_1) &\quad \mapsto y(0, y_0, y_1)
\end{align*}
\end{definition}
Furthermore, we define the continuous affine linear solution operator \( \hat{S} \):
\[
\hat{S} : L^2(I)^m \rightarrow L^2(\Omega_T)
\]
\[
u \mapsto L(ug) + Q(y_0, y_1)
\]
with \( u = \sum_{j=1}^{m} u_j g_j \). (3)

**Lemma 2.4.** The action of the adjoint operator \( L^* \) is given by
\[
L^* : L^2(\Omega_T) \rightarrow L^2(\Omega_T)
\]
\[
w \mapsto p
\]

With \( p(t, x) = y(w(T - \cdot), 0, 0)(T - t, x) \).

2.2. **Preliminaries on BV functions in time.** Concerning BV-functions in one scalar variable we refer to [1]. In this section we only recall a few facts which we frequently refer to: A sequence \((u_k) \subset BV(I)\) is said to converge weakly* in \(BV(I)\) to \( u \) if \((u_k)\) converges to \( u \) in \(L^1(I)\), and the measures \((Du_k)\) converge weakly* in the measure space \(M(I)\) to \(Du\), i.e. \( \lim_{k \to \infty} \int_0^T \varphi Du_k = \int_0^T \varphi Du \) for every \( \varphi \in C_0(\Omega) \).

For all bounded sequences \((u_k)_k \subset BV(I)\) there exists a weakly* convergent sub-sequence with limit \( u \in BV(I) \).

A weakly*-converging sequence \((u_k)_k \) in \( BV(I) \) with limit \( u \) is also strongly converging in \( L^p(I) \) for \( 1 \leq p < \infty \) to \( u \).

The total variation functional \( \|Du \|_{M(I)} : L^1(I) \rightarrow \mathbb{R} := \mathbb{R} \cup \{ \infty \} \) is convex and lower semi-continuous with respect to \( L^1(I)\)-convergence [1, Proposition 3.6].

A sequence \((u_k) \subset BV(I)\) is said to converge strictly in \( BV(I) \) to \( u \) if \((u_k)\) converges strongly in \( L^1(I) \) and \( \|Du_k\|_{M(I)} \xrightarrow{k \to \infty} \|Du\|_{M(I)} \). Strictly converging sequences in \( BV(I) \) are also weakly* converging in \( BV(I) \), see [1, p. 126].

The following BV-Poincaré inequality holds:

**Lemma 2.5** ([1], page 152). There exists \( c > 0 \) such that for all \( u \in BV(I) \) and \( 1 \leq \sigma \leq \infty \) we have \( \|u - a\|_{L^\sigma(I)} \leq c \|Du\|_{M(I)} \), with \( a := \frac{1}{I} \int_I u \, dx \).

**Lemma 2.6.** For each \( m \in \mathbb{N}_{>0} \) the map \( u \rightarrow (Du, u(0)) \) is an isomorphism from \( BV(I)^m \) to \( M(I)^m \times \mathbb{R}^m \) with inverse \( (v, c) \rightarrow \int_0^T dv + c \). A similar result holds for \( H^1(I)^m \) and \( L^2(I)^m \times \mathbb{R}^m \).

3. **Analysis of the optimal control problem** \((P)\). In the following we show the existence of a unique solution of \((P)\). Furthermore, we will introduce a problem \((\hat{P})\) which is equivalent to \((P)\), for which the first-order optimality conditions are derived. These optimality conditions will be used to present sparsity results for the optimal control of \((P)\).

**Theorem 3.1.** Problem \((P)\) has a unique solution \( \pi \in BV(I)^m \).

**Proof.** Utilizing the fact that the forward mapping is continuous from \( L^2(I)^m \) to \( L^2(\Omega_T) \), the proof can be carried out along the lines of [9, Theorem 3.1]. \( \square \)

3.1. **Equivalent problem** \((\hat{P})\). Consider the following linear and continuous operator:
\[
B : M(I)^m \times \mathbb{R}^m \rightarrow L^2(\Omega_T)
\]
\[
(v, c) \mapsto \sum_{j=1}^{m} \left( \int_{[0,t]} dv_j + c_j \right) g_j
\]
(4)
The proof can be found in the appendix.

Proof.

where we have to modify the control to state operator \( \hat{S} \) to

\[
S : \quad M(I)^m \times \mathbb{R}^m \to L^2(\Omega_T)
\]

\[
(v, c) \to L(B(v, c)) + Q(y_0, y_1)
\]

3.2. First-order optimality condition for \((\hat{P})\). In this section the necessary and sufficient first-order optimality conditions for \((\hat{P})\) are presented. Furthermore, we show sparsity results for the optimal control of \((\hat{P})\) respectively \((P)\). Let us begin with the following theorem:

**Theorem 3.2.** The element \((\vec{v}, \vec{c}) \in M(I)^m \times \mathbb{R}^m\), is an optimal control for \((\hat{P})\) if

\[
-\left( \frac{p_1(s)}{p_1(0)} \right) := -\left( \int_0^T \int_{\Omega_T} \alpha_i \frac{\partial v_i}{\partial x_i} \, dx \, dt \right) \in \left( \alpha_i \|\vec{v}_i\|_{M(I)}^m, 0_{\mathbb{R}^m} \right)
\]

where \( p_1 \in C^2([0,T])^m \). This first-order optimality condition is equivalent to: For all \( i = 1, \ldots, m \) and \( v \in M(I) \) it holds that

\[
\langle v - \vec{v}_i, -p_{1,i}\rangle_{M(I),C_0(I)} \leq \alpha_i \|v\|_{M(I)} - \alpha_i \|\vec{v}_i\|_{M(I)} \text{ and } p_1(0) = 0_{\mathbb{R}^m}.
\]

**Proof.** The proof can be found in the appendix.

**Lemma 3.3.** Let \((\vec{v}, \vec{c}) \in M(I)^m \times \mathbb{R}^m\) be an optimal control for \((\hat{P})\). Then we have for all \( i = 1, \ldots, m \) and \( p_1 = (p_{1,i})_{i=1}^m \) given in \((6)\):

\[
\text{a) } \|p_{1,i}\|_{C_0(I)} \leq \alpha_i
\]

\[
\text{b) } \int_I \frac{p_{1,i}}{\alpha_i} \, dv_i = \int_I d|\vec{v}_i| = \|\vec{v}_i\|_{M(I)}
\]

\[
\text{c) } \text{supp}(\vec{v}_i^\pm) \subseteq \{ t \in I | p_{1,i}(t) = \mp \alpha_i \}
\]

The proof is analogous to the one of [9, Proposition 2.4]. The following corollary which is similar to a result in [9] exhibits an important structural property of the solution \( \pi_{\alpha,j} \) as a function of \( \alpha_j \).

**Corollary 1.** There exists \( M_j > 0 \) such that the \( j \)-th component \( \pi_{\alpha,j} \) of the optimal control \( \pi_{\alpha} \) of \((P)\) is constant in \( BV(I)^m \) for all \( \alpha_j > M_j \).

**Proof.** Let \( y^0, y_\alpha \) be the solutions of the state equation associated to the controls \( u = 0 \) respectively \( \pi_{\alpha} \). Furthermore, let us define \( \overline{p}_\alpha := L(y_\alpha - y_d) \). From the optimality of \( \pi_{\alpha} \) we get

\[
\frac{1}{2} ||\overline{y}_\alpha - y_d||^2_{L^2(\Omega_T)} \leq J(\pi_{\alpha}) \leq J(0) = \frac{1}{2} ||y^0 - y_d||^2_{L^2(\Omega_T)}.
\]
This implies that $\|\tilde{y}_t - y_d\|_{L^2(\Omega_T)} \leq \|y^0 - y_d\|_{L^2(\Omega_T)}$. From the adjoint state equation we obtain

$$\|\tilde{y}_t\|_{L^\infty(\mathcal{I} ; H)} = c_1 \|\tilde{y}_t\|_{C(\mathcal{I} ; V)} \leq c_1 c_2 \|\tilde{y}_t - y_d\|_{L^1(\mathcal{I} ; H)}$$

$$\leq c_1 c_2 c_3 \|\tilde{y}_t - y_d\|_{L^2(\Omega_T)} \leq c_1 c_2 c_3 \|y^0 - y_d\|_{L^2(\Omega_T)}.$$ 

The constant $c_1$ is defined with respect to the embedding $L^\infty(\mathcal{I} ; V) \hookrightarrow L^\infty(\mathcal{I} ; H)$, $c_2$ is depending on the embedding constant in (2), and $c_3$ is the embedding constant of $L^2(\Omega_T) = L^2(\mathcal{I} ; H) \hookrightarrow L^1(\mathcal{I} ; H)$. From the adjoint $p_1$, and the above estimation we get for all $t \in [0, T]$

$$|p_{1,j}(t)| \leq T \|\tilde{y}_t\|_{L^\infty(\mathcal{I} ; H)} \|g_j\|_{L^2(w_j)} \leq T c_1 c_2 c_3 \|g_j\|_{L^2(\omega_j)} \|y^0 - y_d\|_{L^2(\Omega_T)} =: M_j$$

where the first inequality follows from

$$|p_{1,j}(t)| \leq \int_T t \|\tilde{y}_t\|_{L^2(\omega_j)} \|g_j\|_{L^2(\omega_j)} ds \leq T \|\tilde{y}_t\|_{L^\infty(\mathcal{I} ; H)} \|g_j\|_{L^2(\omega_j)}.$$ 

The support relation in Lemma 3.3 now implies that $D_{i_t} \pi_{n,j} = 0$ if $\alpha_j > M_j$. 

**Corollary 2.** Let $\pi \in BV(\mathcal{I})^m$ be the optimal control of (P). Assume for some $i \in \{1, \cdots, m\}$ that the measures $D_{i_t} \pi^+_i$ and $D_{i_t} \pi^-_i$ are not trivial. Then we have

$$\text{dist}(\text{supp}(D_{i_t} \pi^+_i), \text{supp}(D_{i_t} \pi^-_i)) := \min_{x^\pm \in \text{supp}(D_{i_t} \pi^+_i)} |x^+ - x^-| > 0.$$ 

**Proof.** W.l.o.g. let us consider $m = 1$. Assume that $\text{dist}(\text{supp}(D_{i_t} \pi^+_i), \text{supp}(D_{i_t} \pi^-_i)) = 0$. Then there exists a sequence $(t_n)_n \in \text{supp}(D_{i_t} \pi^+_i) \subset I$ such that $p_1(t_n) = -\alpha$ and $\text{dist}(\{t_n\}, \text{supp}(D_{i_t} \pi^-_i)) \to 0$. Hence, there exists a subsequence $(t_{n_k})_k$ which converges to some $\hat{t}$ with $\text{dist}(\{\hat{t}\}, \text{supp}(D_{i_t} \pi^-_i)) = 0$. Furthermore, there exists a sequence $(\tau_n)_n \in \text{supp}(D_{i_t} \pi^-_i) \subset I$ such that $p_1(\tau_n) = \alpha$ and $\tau_n \to \hat{t}$. By the continuity of $p_1$ we have $-\alpha = \lim_{n \to \infty} p(t_{n_k}) = \lim_{n \to \infty} p(\tau_n) = \alpha$ which is a contradiction to $\alpha > 0$. 

**Remark 1.** If the set of points in which $p_{1,i}(t) \in \{\pm \alpha_i\}$, is finite, we have by Lemma 3.3 c) that $D_{i_t} \pi_j$ is a combination of Dirac measures centered at those points (not necessarily in all of these points). In particular, we obtain that the optimal control $\pi_j$ of (P) is piecewise constant in $[0, T]$ with jumps in supp$(D_{i_t} \pi_j)$. This remark can also be found in [9, Remark 3.5]. Later we will construct an analytically exactly solvable example for our problem (P), which allows us to show that the derivatives of the optimal controls can either be of Cantor or Dirac kind or alternatively absolutely continuous with respect to the Lebesgue measure. In particular, the derivatives of the optimal controls need not to be sparse. For further information about these characterizations of measures, see for example [1] on page 184.

4. **Regularization.** For numerical realization we aim at applying a semi-smooth Newton method. For this purpose we regularize problem (P). We then analyze the asymptotic behavior of the optimal controls of the regularized problem as well as the first-order optimality condition of the regularized problem. Finally, we will present convergence results for the semi-smooth Newton algorithm.
Lemma 4.3. The value function

\[ V(u) = \min_{u \in H^1(\Omega)^m} \left[ \frac{1}{2} \| y_u - y_d \|_{L^2(\Omega_T)}^2 + \sum_{j=1}^{m} \alpha_j \| \partial_t u_j \|_{L^1(I)} + \frac{\gamma}{2} \sum_{j=1}^{m} \| \partial_t u_j \|_{L^2(I)}^2 + \frac{\kappa(\gamma)}{2} \| u(0) \|_{\mathbb{R}^m}^2 \right] \]

Analogously, we have\[ \kappa(\gamma) = \epsilon_0 \cdot \tilde{f}(\gamma), \epsilon_0 \geq 0, \text{ monotonously increasing, } \tilde{f} \in C^1([0, \infty)), \tilde{f}(0) = 0, \text{ and supp}(\tilde{f}) = [0, \infty). \]
Note that for each \( u \in H^1(\Omega) \) the value \( u(0) \) is well defined, because \( H^1(\Omega) \) embeds continuously into \( C(\overline{\Omega}) \). The total variation cost term in \((P)\) can be identified with the cost term \( \sum_{j=1}^{m} \alpha_j \| \partial_t \cdot \|_{L^1(I)} \) for \( H^1(I)^m \)
functions in \((P)\) since now \( u \in H^1(\Omega) \). The symbol \( \partial_t \) represents the weak derivative.

4.1. Asymptotic behavior as \( \gamma \to 0^+ \). In this section we show that the unique solution of \((P)\) can be approximated by the unique solutions of the problems \((P)\) as \( \gamma \to 0 \).

In terms of the reduced costs \( J \), problem \((P)\) can be expressed as

\[ \left\{ \begin{array}{ll}
\min_{u \in BV(I)^m} & \frac{1}{2} \| \tilde{S}(u) - y_d \|_{L^2(\Omega_T)}^2 + \sum_{j=1}^{m} \alpha_j \| D_t u_j \|_{M(I)} =: J(u) \\
\end{array} \right. \]

Analogously, we have

\[ \left\{ \begin{array}{ll}
\min_{u \in H^1(\Omega)^m} & \frac{1}{2} \| \tilde{S}(u) - y_d \|_{L^2(\Omega_T)}^2 + \sum_{j=1}^{m} \alpha_j \| \partial_t u_j \|_{L^1(I)} + \frac{\gamma}{2} \sum_{j=1}^{m} \| \partial_t u_j \|_{L^2(I)}^2 + \frac{\kappa(\gamma)}{2} \| u(0) \|_{\mathbb{R}^m}^2 \\
\end{array} \right. =: J^1(\gamma)(u) \]

The following result follows with standard techniques.

**Theorem 4.1.** For every \( \gamma > 0 \) problem \((P)\) has a unique solution \( \pi_\gamma \in H^1(I)^m \).

Let us denote the unique optimal controls of \((P)\) and \((P)\) by \( \pi \) and \( \pi_\gamma \). To argue the BV-weak* and strict convergence of \( \pi_\gamma \) to \( \pi \) we use concepts from [26] and [9].

**Definition 4.2.** The value function is defined as

\[ \mathfrak{V} : [0, \infty) \to \mathbb{R}, \gamma \mapsto \mathfrak{V}(\gamma) := J^1(\gamma)(\pi_\gamma), \text{ where } \pi_\gamma = \arg \min_{u \in H^1(I)^m} J^1(\gamma)(u). \]

**Lemma 4.3.** The value function \( \mathfrak{V} \) maps \([0, \infty)\) into \([J(\overline{u}), \infty)\) and \( \mathfrak{V}(0) := J(\overline{u}) \).
It is (locally) Lipschitz-continuous, monotonically increasing, concave, and a.e. differentiable in \((0, \infty)\) with

\[ \mathfrak{V}'(\gamma) = \frac{1}{2} \sum_{i=1}^{m} \| \partial \pi_{\gamma,i} \|_{L^2(I)}^2 \quad \text{if } \kappa = 0, \text{ or} \]

\[ \mathfrak{V}'(\gamma) = \frac{1}{2} \sum_{i=1}^{m} \| \partial \pi_{\gamma,i} \|_{L^2(I)}^2 + \frac{\kappa'(\gamma)}{2} \| \pi_\gamma(0) \|_{\mathbb{R}^m}^2 \quad \text{if } \kappa \neq 0. \]

**Proof.** Utilizing the fact, that \( \kappa \in C^1([0, \infty)) \) and \( \kappa(0) = 0 \), the proof can be carried out along the line of [26, Proposition 2.26].

The following theorem can be found in the context of measure valued controls in [26, Proposition 2.27, Corollary 2.29], and for BV-controls in [15, Section 6].
Theorem 4.4. The value function $\mathfrak{V}$ is continuous in 0, and we have $0 \leq J_{\gamma}(\bar{u}_n) - J(\bar{u}) \leq \gamma \|\mathfrak{V}\|_{L^\infty(0,c_{\text{loc}})}$ for every $c_{\text{loc}} > 0$, and $\gamma \in (0,c_{\text{loc}})$.

Proof. Let $\epsilon > 0$. The space $C^\infty(I)^m$ is dense in $BV(I)^m$ with respect to the metric $d_{BV} : (\phi_1,\phi_2) \mapsto \|\phi_1 - \phi_2\|_{L^2(I)^m} + \|D_t\phi_1\|_{M(I)^m} - \|D_t\phi_2\|_{M(I)^m}$ in $BV(I)^m$. Hence, we can find a sequence $(u_n) \subset C^\infty(I)^m \subset H^1(I)^m$ such that $d_{BV}(u_n,\bar{u}) \to 0$ with $\bar{u}$ as the solution of $(P)$. Due to the continuity of $\mathcal{S}$, we have that $J$ is continuous with respect to the metric $d_{BV}$. The continuity of $J$ implies then, that there exists $N \in \mathbb{N}$ such that $|J(\bar{u}) - J(u_n)| \leq \epsilon$ for all $n \geq N$. Thus we have for all $\gamma > 0$:

$$
\begin{align*}
\mathfrak{V}(0) = & J(\bar{u}) - J(\bar{u}_n) + \frac{\gamma}{2} \sum_{i=1}^{m} \|\partial_t u_n\|^2_{L^2(I)^m} = J(\bar{u}_n) - J(\bar{u}) \\
= & J(\bar{u}) - \frac{\gamma}{2} \sum_{i=1}^{m} \|\partial_t u_n\|^2_{L^2(I)^m} + \frac{\kappa(\gamma)}{2} \|u_n(0)\|^2_{\mathbb{R}^m} \\
\leq & J(\bar{u}) + \epsilon + \frac{\gamma}{2} \sum_{i=1}^{m} \|\partial_t u_n\|^2_{L^2(I)^m} + \frac{\kappa(\gamma)}{2} \|u_n(0)\|^2_{\mathbb{R}^m} \\
= & \mathfrak{V}(0) + \epsilon + \frac{\gamma}{2} \sum_{i=1}^{m} \|\partial_t u_n\|^2_{L^2(I)^m} + \frac{\kappa(\gamma)}{2} \|u_n(0)\|^2_{\mathbb{R}^m}.
\end{align*}
$$

Because $\epsilon$ is arbitrary and $\sum_{i=1}^{m} \|\partial_t u_n\|^2_{L^2(I)^m}$, and $\|u_n(0)\|^2_{\mathbb{R}^m}$ are bounded, this implies that $\mathfrak{V}(0) = \liminf_{\gamma \to 0} \mathfrak{V}(\gamma) = \limsup_{\gamma \to 0} \mathfrak{V}(\gamma)$. Using that $\mathfrak{V}(\gamma) = \int_0^\gamma \mathfrak{W}(t)dt + \mathfrak{V}(0)$ holds, we have for all $c_{\text{loc}} > 0$, and $\gamma \leq c_{\text{loc}}$:

$$0 \leq J_{\gamma}(\bar{u}_n) - J(\bar{u}) = \mathfrak{V}(\gamma) - \mathfrak{V}(0) = \int_0^\gamma \mathfrak{W}(t)dt \leq \gamma \|\mathfrak{W}\|_{L^\infty(0,c_{\text{loc}})}$$

where we used that $\mathfrak{V}$ is (locally) Lipschitz-continuous, monotonously increasing (which implies that $\mathfrak{W} \geq 0$ a.e.), concave (which implies an a.e. decreasing derivative), and thus $\mathfrak{W}^r \in L^\infty(0,c_{\text{loc}})$.

Theorem 4.5. The unique optimal controls $\pi_n$ of $(P^1_\gamma)$ converge weakly* in $BV(I)^m$ to the optimal control $\pi$ of $(P)$.

Proof. Let $(\gamma_n)$ be an arbitrary null sequence in $\mathbb{R}^+$. In the following we show that the solutions $(\pi_{\gamma_n})_{n=1}^\infty$ of the problems $(P^1_\gamma)_{n=1}^\infty$ are bounded in $BV(I)^m$, with a proof which is similar to the one in [9]:

Because of the continuity of $\mathfrak{V}(\gamma)$ on $[0,\infty)$ we have that $(\mathfrak{V}(\gamma_n))_{n=1}^\infty$ is bounded in $\mathbb{R}$. Thus, we get that $\|\partial_t \pi_{\gamma_n}\|_{L^1(I)^m} = \|D_t \pi_{\gamma_n}\|_{M(I)^m}$ is bounded. Next, we have to prove that $\|\pi_{\gamma_n}\|_{L^1(I)^m}$ is bounded, which is required to show that $(\pi_{\gamma_n})_{n=1}^\infty$ is bounded in $BV(I)^m$. Consider the decomposition $\pi_{\gamma_n} = \pi_{\gamma_n} + \hat{\pi}_{\gamma_n}$ where

$$
\begin{align*}
\pi_{\gamma_n} &= (\pi_{\gamma_n,1},...,\pi_{\gamma_n,m}) , \\
\hat{\pi}_{\gamma_n} &= (\hat{\pi}_{\gamma_n,1},...,\hat{\pi}_{\gamma_n,m}) \\
\pi_{\gamma_n} &= \frac{1}{T} \int_0^T \pi_{\gamma_n}(t)dt \in \mathbb{R}^m , \\
\hat{\pi}_{\gamma_n} &= \pi_{\gamma_n} - \pi_{\gamma_n}.
\end{align*}
$$

At first we argue that $(\pi_{\gamma_n})_n$ is bounded in $BV(I)^m$. Note that $\|\hat{\mathcal{S}}(\pi_{\gamma_n}) - y_d\|^2_{L^2(I)^2}$ is bounded, because $(\nu(\gamma_n))_n$ is bounded. Thus, we get that $\hat{\mathcal{S}}(\pi_{\gamma_n})$ is bounded in
\[ L^2(\Omega_T). \] By (2), we have that \( \tilde{S}(\tilde{u}_{\gamma_n}) \) is bounded in \( L^2(\Omega_T) \) as well, in fact
\[ \| \tilde{S}(\tilde{u}_{\gamma_n}) \|_{L^2(\Omega_T)} \leq c \| \tilde{S}(\tilde{u}_{\gamma_n}) \|_{C(T,V)} \leq c_1 (\| u_{\gamma_n} \|_{L^1(I)^m} + c_2) \leq c_3 (\| \partial_t \pi_{\gamma_n} \|_{M(I)^m} + c_2) \]
where we used the BV-Poincaré inequality in the last estimate.

Now define \( z_n = y_n - \tilde{y}_n = L(\pi_{\gamma_n}, \tilde{y}) \) with \( y_n = \tilde{S}(\pi_{\gamma_n}) \) and \( \tilde{y}_n = \tilde{S}(\tilde{u}_{\gamma_n}) \). The sequence \( z_n \) is bounded in \( L^2(\Omega_T) \).

To argue that \( (a_{\gamma_n}) \) is bounded we argue by contradiction, and assume that (for a subsequence, denoted by the same index) \( \tilde{p}_n := \max_{1 \leq j \leq m} |a_{\gamma_n,j}| \xrightarrow{n \to \infty} \infty \). Let us introduce \( \xi_n = \frac{1}{\tilde{p}_n} z_n = L(\frac{1}{\tilde{p}_n} a_{\gamma_n}, \tilde{y}) \). Since
\[ \| \xi_n \|_{L^2(\Omega_T)} = \| \frac{1}{\tilde{p}_n} z_n \|_{L^2(\Omega_T)} = \frac{1}{\tilde{p}_n} \| z_n \|_{L^2(\Omega_T)} \xrightarrow{z_n \text{ bdd in } L^2(\Omega_T)} 0, \]
we have that \( \xi_n \xrightarrow{L^2(\Omega_T)} 0 \). Furthermore, we have
\[ \| \frac{1}{\tilde{p}_n} a_{\gamma_n} \tilde{y} \|_{L^2(\Omega)} = \sqrt{T} \| \frac{1}{\tilde{p}_n} a_{\gamma_n} \tilde{y} \|_{L^2(\Omega)} \xrightarrow{\text{Disj. supp}(g_i)} \sqrt{T} \sum_{i=1}^m |a_{\gamma_n,i}| \| g_i \|_{L^2(\omega_i)}, \]
which does not converge to 0 for \( n \to \infty \) since \( \tilde{p}_n \to \infty \). This is a contradiction to (10) by the injectivity of the \( L \) operator. Thus we get that \( (a_{\gamma_n}) \) is a bounded sequence in \( \mathbb{R}^m \) and hence \( (\pi_{\gamma_n})_{n=1}^\infty \) is bounded in \( BV(I)^m \). Here we use that there exists a constant \( C_T \) such that \( \| u \| := |u| + \| D_t u \|_{M(0,T)} \leq \max(1,T) \| u \|_{BV(0,T)} \leq C_T \| u \| \) for all \( u \in BV(0,T)^m \) and \( a = \frac{1}{T} \int_0^T u(t) dt \in \mathbb{R}^m \), use [1, Theorem 3.44].

Considering that bounded sequences in \( BV(I)^m \) are weak* compact, we obtain by [1, Theorem 3.23] that there exists a sub-sequence \( (\pi_{\gamma_n})_k \), which converges weakly* to a function \( \tilde{u} \in BV(I)^m \). The weak* convergence implies that \( \pi_{\gamma_n,k} \) converges in \( L^2(I)^m \) to \( \tilde{u} \), and \( \partial_t \pi_{\gamma_n,k} \) converges in the weak* topology of \( M(I)^m \) to \( \partial_t \tilde{u} \). Hence, by the weak* lower semi-continuity of \( \| \cdot \|_{M(I)} \) we get
\[ \liminf_{k \to \infty} \sum_{i=1}^m \alpha_i \| D_t \pi_{\gamma_n,k,i} \|_{M(I)} \geq \sum_{i=1}^m \alpha_i \| D_t \tilde{u}_i \|_{M(I)}. \]

Estimates (12) - (14) and Theorem 4.4.4 imply that
\[ \mathfrak{R}(0) = \lim_{\gamma_n \to 0^+} \left\{ \frac{1}{2} \| \tilde{S}(\pi_{\gamma_n}) - yd \|_{L^2(\Omega_T)}^2 + \sum_{i=1}^m \alpha_i \| \partial_t \pi_{\gamma_n,i} \|_{M(I)} \right\} + \frac{\gamma_n}{2} \sum_{i=1}^m \| \partial_t \pi_{\gamma_n,i} \|_{L^2(I)}^2 + \frac{\kappa(\gamma_n)}{2} \| \pi_{\gamma_n}(0) \|_{H^m}^2 \]
\[ \geq \frac{1}{2} \| \tilde{S}(\tilde{u}) - yd \|_{L^2(\Omega_T)}^2 + \sum_{i=1}^m \alpha_i \| D_t \tilde{u}_i \|_{M(I)} = J(\tilde{u}). \]
By uniqueness of the optimal control of (P) we get that \( \tilde{u} \) is equal to the optimal control \( \pi \) of (P). Thus, the unique solutions \( \pi_{\gamma_{nk}} \) of (\( P_{\gamma_{nk}}^1 \)) converge BV(I)^m-weak* to the optimal control \( \pi \) of (P).

**Corollary 3.** The unique optimal controls \( \pi_{\gamma} \) of (\( P_{\gamma}^1 \)) converge strictly in BV(I)^m to the optimal control \( \pi \) of (P).

**Proof.** Due to the weak* convergence by Theorem 4.5 we get that \( \pi_{\gamma} \) converges in \( L^1(I)^m \) to the optimal control \( \pi \). Using that \( \tilde{S}(\pi_{\gamma}) \to \tilde{S}(\pi) \) in \( L^2(\Omega_T) \), Theorem 4.4 implies that the total variations of \( \pi_{\gamma} \) converge to the total variation of \( \pi \).

4.2. **Equivalent regularized optimal control problem to (\( P_{\gamma}^1 \)).** In this section we introduce an equivalent problem (\( \hat{P}_{\gamma} \)) to (\( P_{\gamma}^1 \)). The latter will be solved by a semi-smooth Newton method. In the remaining part of the paper we restrict the operator \( B \) defined in (5) to \( L^2(I)^m \times \mathbb{R}^m \). Its adjoint has the form

\[
B^*: L^2(\Omega_T) \to L^2(I)^m \times \mathbb{R}^m, \quad \varphi \mapsto \left( \int_T \int_{\Omega_T} \varphi, \int_{\Omega_T} \varphi \right)
\]

Analogously we henceforth restrict \( S \) to \( L^2(I)^m \times \mathbb{R}^m \). The isomorphism in Lemma 2.6 translates (\( P_{\gamma}^1 \)) into the following equivalent form:

\[
(\hat{P}_{\gamma}) \left\{ \begin{array}{l}
\min_{(v,c) \in L^2(I)^m \times \mathbb{R}^m} \left\{ \frac{1}{2} \|S(v,c) - y_d\|_{L^2(\Omega_T)}^2 + \sum_{j=1}^{m} \alpha_j \|v_j\|_{L^1(I)} + \frac{1}{2} \|v\|_{L^2(I)}^2 + \frac{\kappa(\gamma)}{2} \|c\|_{\mathbb{R}^m}^2 \right\} \\
\end{array} \right. =: \hat{J}_\gamma(v,c)
\]

4.3. **Regularization - first-order optimality condition.** In this section we present the first-order optimality conditions for (\( \hat{P}_{\gamma} \)). We will use a prox-operator approach to represent implicitly the distributional derivative of the BV-optimal control of (P) with respect to the adjoint. This allows to replace the sub-differential in the first-order optimality conditions of (\( \hat{P}_{\gamma} \)). Finally we compare the sparsity results of (\( \hat{P}_{\gamma} \)) and (\( \hat{P} \)), and show the convergence of the adjoints of (\( \hat{P}_{\gamma} \)) to the adjoint of (\( \hat{P} \)) for \( \gamma \to \infty \).

**Lemma 4.6.** Let \((\tilde{v}, \overline{c}) \in L^2(I)^m \times \mathbb{R}^m \) be the optimal control of (\( \hat{P}_{\gamma} \)). We have the following necessary and sufficient optimality conditions for (\( \hat{P}_{\gamma} \)):

\[
(1) \quad \tilde{v}(s) = \max \left( 0, -\frac{1}{\gamma} \int_{\Omega} L^*(S(\tilde{v}, \overline{c}) - y_d)g \; dx - \frac{\alpha}{\gamma} \right) + \sum_{i=1}^{m} \left( \int_{\Omega} L^*(S(\tilde{v}, \overline{c}) - y_d)g \; dx - \alpha \overline{c}_i \right)
\]

\[
(2) -\int_{\Omega_T} L^*(S(\tilde{v}, \overline{c}) - y_d) \; dx - \kappa(\gamma) \overline{c} = 0 \in \mathbb{R}^m
\]

**Proof.** Since this proof is standard, we have deferred it to the appendix.

In the appendix it is also shown that

\[
\text{Prox}_{\sum_{i} \alpha_i \cdot \| \cdot \|_{L^1(I)}} \left( -\frac{1}{\gamma} \int_{\Omega} L^*(S(\tilde{v}, \overline{c}) - y_d)g \; dx \right)
\]
is equal to the right hand side of equation (1) in Lemma 4.6.

Due to the regularity of the adjoint wave equation, we have that the optimal control $\tilde{v}$ is at least Lipschitz continuous.

**Proposition 1.** Let $(\tilde{v}, \tilde{c}) \in L^2(I)^m \times \mathbb{R}^m$ be the optimal control of $(\tilde{P}_\gamma)$. Then we have for a.a. $s \in I$ and $i = 1, \ldots, m$:

$$\tilde{v}_{\gamma,i}(s) = \begin{cases} 0 & |\psi_{\gamma,i}(s)| < \alpha_i \\ -\frac{1}{\gamma} \psi_{\gamma,i}(s) + \frac{\alpha_i}{\gamma} & \psi_{\gamma,i}(s) \geq \alpha_i \\ -\frac{1}{\gamma} \psi_{\gamma,i}(s) - \frac{\alpha_i}{\gamma} & \psi_{\gamma,i}(s) \leq -\alpha_i \end{cases} \quad (16)$$

with $\psi_{\gamma}(s) = \int \int_{\Omega} L^*(S(\tilde{v}, \tilde{c}) - y_d) \tilde{g} \, dt \, dx$, and $\psi_{\gamma,i}(s) = \int \int_{\Omega} L^*(S(\tilde{v}, \tilde{c}) - y_d) \, g_i \, dt \, dx$.

One can compare the sparsity structure of the optimal controls associated to $(\tilde{P}_\gamma)$ to the sparsity for the optimal control of $(\tilde{P})$. The optimal measures $\tilde{v}_{i}$ in $(\tilde{P})$, see Lemma 3.3, are not supported, where $|p_{1,i}(t)| < \alpha_i$, while the optimal measures for $(\tilde{P}_\gamma)$ are not supported, where $|\psi_{\gamma,i}(t)| < \alpha_i$ holds.

We next address the convergence of the adjoints $\psi_{\gamma}$ of $(\tilde{P}_\gamma)$ to the adjoint $p_1$ of $(\tilde{P})$, which is defined in Theorem 3.2.

**Proposition 2.** For $0 < \gamma \rightarrow 0$ we find $\psi_{\gamma} \xrightarrow{H^2(I)^m} p_1$. Furthermore, we have for $\kappa = 0$,

$$\left\langle -\frac{\psi_{\gamma,i}}{\alpha_i}, \tilde{v}_{\gamma,i}\right\rangle_{C_0(I),M(I)} \xrightarrow{\gamma \rightarrow 0} \left\langle -\frac{p_{1,i}}{\alpha_i}, \tilde{v}_{i}\right\rangle_{C_0(I),M(I)} = \|\tilde{v}_{i}\|_{M(I)}, \quad (17)$$

and for $\kappa > 0$,

$$\left\langle -\frac{\psi_{\gamma,i}}{\alpha_i} d\tilde{v}_{\gamma,i}(s), \tilde{v}_{i}\right\rangle_{C_0(I),M(I)} \xrightarrow{\gamma \rightarrow 0} \left\langle -\frac{p_{1,i}}{\alpha_i}, \tilde{v}_{i}\right\rangle_{C_0(I),M(I)} = \|\tilde{v}_{i}\|_{M(I)}, \quad (18)$$

for $i = 1, \ldots, m$, where $(\tilde{v}, \tilde{c}) \in M(I)^m \times \mathbb{R}^m$ is the solution to $(\tilde{P})$, and $(\tilde{v}_{\gamma}, \tilde{c}_{\gamma}) \in L^2(I)^m \times \mathbb{R}^m$ is the solution to $(\tilde{P}_\gamma)$.

Due to Theorem 4.5 we know that $v_{\gamma}$, the derivative of the optimal control $\pi_{\gamma}$ of $(P_{\gamma})$, converges weakly* to $\tilde{v}$ in $M(I)^m$, the distributional derivative of the optimal control $\pi$ of $(P)$. Furthermore, recall that $\left\langle -\frac{p_{1,i}}{\alpha_i}, \tilde{v}_{i}\right\rangle_{C_0(I),M(I)} = \|\tilde{v}_{i}\|_{M(I)}$ holds due to Lemma 3.3.

**Proof.** We first show that $\|\psi_{\gamma,i} - p_{1,i}\|_{H^2(I)} \xrightarrow{\gamma \rightarrow 0} 0$, for $i = 1, \ldots, m$, holds. By regularity results of the wave equation we have that $p_{1,i}$ and $\psi_{\gamma,i}$ are elements of $H^2(I)$. Furthermore, using Theorem 4.5 in the last inequality of the following computation we find

$$\|\partial_t \psi_{\gamma,i} - \partial_t p_{1,i}\|_{L^2(I)}^2$$

$$= \left\| \int_{I} L^*(L((\bar{\pi}_{\gamma} - \bar{\pi}) \cdot \gamma g)) \, dt \right\|_{L^2(I)}^2$$

$$\leq \int_{I} \left( \int_{I} |L^*(L((\bar{\pi}_{\gamma} - \bar{\pi}) \cdot \gamma g))| \, |g| \, dt \right) \, dt$$
Because $\psi_{\gamma,i} \in H^1(I)$ with $(\psi_{\gamma,i} - p_{1,i})(T) = 0$ this implies that $\|\psi_{\gamma,i} - p_{1,i}\|_{H^1(I)} \xrightarrow{\gamma \to 0} 0$. Let us show that $\|\partial_t \psi_{\gamma,i} - \partial_t p_{1,i}\|_{L^2(I)} \xrightarrow{\gamma \to 0} 0$ holds as well. For this purpose, utilizing the dominated convergence theorem and Theorem 4.5 we obtain

$$
\leq \int_0^T \left( \|\partial_t L^*(L((\pi_{\gamma} - \pi) \cdot \overrightarrow{g} ))g_i \|_{L^2(I)} \right) dt
$$

$$
= c \|\partial_t L^*(L((\pi_{\gamma} - \pi) \cdot \overrightarrow{g} ))\|_{C(T,H)} \leq c \|\partial_t L^*(L((\pi_{\gamma} - \pi) \cdot \overrightarrow{g} ))\|_{C(T,H)} \leq c \|\pi_{\gamma} - \pi\|_{L^1(I)} \xrightarrow{\gamma \to 0} 0.
$$

Consider now the case $\gamma = 0$: To verify (17) let us note that $\psi_{\gamma,i} \in H^1_0(I)$ since $\gamma = 0$. Because $H^2(I)$ continuously embeds into $C^0(T)$, and $\psi_{\gamma,i}, p_{1,i} \in C^0(I)$ we have $\|\psi_{\gamma,i} - p_{1,i}\|_{C^0(I)} \xrightarrow{\gamma \to 0} 0$, for $i=1,\ldots,m$. Utilizing the weak* convergence of $\overrightarrow{\psi_{\gamma,i}} \xrightarrow{w^*,M(I)} \overrightarrow{\psi}$ and the strong convergence $\psi_{\gamma,i} \xrightarrow{C(I)} p_{1,i}$, for $i=1,\ldots,m$, we achieve the desired result.

We turn to verify (18). Since $\gamma \neq 0$ we do not have that $\psi_{\gamma,i} \in H^1_0(I)$.

Consider the following function $\varphi(t) = \cos(\frac{\pi}{T}t)1_{[0,\frac{T}{2}]}(t)$, which satisfies $\varphi(0) = 1$, $\varphi(t) = 0$, for $t \geq \frac{T}{2}$, $\varphi \in C(T)$ and thus

$$
\int_I \psi_{\gamma,i} d\overrightarrow{\psi_{\gamma,i}} = \langle \psi_{\gamma,i} + \kappa(\gamma)c_{\gamma,i} \varphi, \overrightarrow{\psi_{\gamma,i}} \rangle_{C_0(M(I))} - \int_I \kappa(\gamma) \overrightarrow{\psi_{\gamma,i}} \varphi d\overrightarrow{\psi_{\gamma,i}}. \tag{19}
$$

Furthermore, we have by the convergence of $\psi_{\gamma,i} \xrightarrow{H^2(I)} p_{1,i}$, for $i=1,\ldots,m$, and by the embedding $H^2(I) \hookrightarrow C(T)$ that $\psi_{\gamma,i} \xrightarrow{C(T)} p_{1,i}$. Due to weak* convergence $\pi_{\gamma} \xrightarrow{w^*,BV(I)^m} \pi$, we have that $(\pi_{\gamma})_\gamma$ is bounded in $BV(I)^m$ for $\gamma \to 0$. By the
isomorphism in Lemma 2.6 we get that \((D_l \overline{\alpha}_\gamma, \overline{\pi}_\gamma(0)) = (\overline{\theta}_\gamma, \overline{\gamma}_\gamma)\) is bounded in \(M(I)^m \times \mathbb{R}^m\) as well. Given the boundedness of \(|\overline{\theta}_\gamma, i|\) it holds that \(\|\kappa(\gamma) \overline{\theta}_\gamma, i \varphi\|_\infty \leq \kappa(\gamma) |\overline{\theta}_\gamma, i|\|\varphi\|_\infty \xrightarrow{\gamma \to 0} 0\). Summing up, we have \(\psi_{\gamma,i}\xrightarrow{\gamma \to 0} p_{1,i}\) and \(\kappa(\gamma) \overline{\theta}_\gamma, i \varphi \xrightarrow{\gamma \to 0} 0\), which implies \(\psi_{\gamma,i} + \kappa(\gamma) \overline{\theta}_\gamma, i \varphi \xrightarrow{\gamma \to 0} p_{1,i}\).

Together with weak* convergence in \(M(I)\) of \(\overline{\theta}_\gamma, i\) to \(\overline{\theta}_i\), we get

\[
\langle \psi_{\gamma,i} + \kappa(\gamma) \overline{\theta}_\gamma, i \varphi, \overline{\theta}_i \rangle_{C_0(M(I))} \xrightarrow{\gamma \to 0} \langle p_{1,i}, \overline{\theta}_i \rangle_{C_0(M(I)), M(I)}. \tag{20}
\]

Due to the boundedness of \((\overline{\theta}_\gamma)_\gamma\) in \(M(I)^m\) and \((\overline{\theta}_\gamma)_\gamma\) in \(\mathbb{R}^m\), for \(\gamma \to 0\), we get

\[
\int f \kappa(\gamma) \overline{\theta}_\gamma, i \varphi d \overline{\theta}_\gamma, i \leq \int f |\kappa(\gamma) \overline{\theta}_\gamma, i \varphi| d \overline{\theta}_\gamma, i \leq \kappa(\gamma) |\overline{\theta}_\gamma, i| \|\overline{\theta}_\gamma, i\|_{M(I)} \xrightarrow{\gamma \to 0} 0. \tag{21}
\]

Finally, we consider (20), and (21) in (19) and get (18).

4.4. Regularization - semi-smooth Newton method. In this section, we discuss the semi-smooth Newton method which is used to construct a sequence in \(L^2(\Omega)^m \times \mathbb{R}^m\) that solves the first-order condition (1), (2) in Lemma 4.6 in the limit. Later in section 5 a BV-path following algorithm is presented which uses these method, see Algorithm 1.

At first, let us introduce

\[
F_\gamma : L^2(I)^m \times \mathbb{R}^m \to L^2(I)^m \times \mathbb{R}^m, \quad F_\gamma(v, c) := \left( v - \text{Prox}_{\sum_i \alpha_i \|L^1_t(I)} \left( \frac{1}{\gamma} \tilde{\pi}(v, c) \right) \right)
\]

where

\[
\text{Prox}_{\sum_i \alpha_i \|L^1_t(I)} \left( \frac{1}{\gamma} \tilde{\pi}(v, c) \right) = \left( \max \left( 0, \frac{1}{\gamma} \tilde{\pi}_i(v, c) - \frac{\alpha_i}{\gamma} \right) + \min \left( 0, \frac{1}{\gamma} \tilde{\pi}_i(v, c) + \frac{\alpha_i}{\gamma} \right) \right)^m_{i=1} \in L^2(I)^m,
\]

\[
\tilde{\pi}(v, c)(s) = \int_\Omega \int_0^T L^t(S(v, c) - y_d) \overline{g} \, dt dx \in L^2(I)^m,
\]

and observe that \(F_\gamma(\overline{\theta}, \overline{\gamma}) = 0\) is equivalent to (1), (2) in Lemma 4.6.

Consider the following definition of [18, p. 120 et seq.]:

**Definition 4.7.** Let \(G : X \to Y\) be a continuous operator, between Banach spaces \(X\) and \(Y\). Further, let us consider a set-valued mapping \(\partial G : X \rightrightarrows L(X, Y)\) with non-empty images. We call \(\partial G\) a generalized differential. We define the operator \(G\) to be \(\partial G\)-semi-smooth or simply semi-smooth in \(x^*\), if

\[
\sup_{M \in \partial G(x^* + d)} \|G(x^* + d) - G(x^*) - Md\|_Y = o(\|d\|_X) \text{ for } \|d\|_X \to 0
\]

We recall the following theorem from [17, Theorem 1.1]:

**Theorem 4.8.** Suppose that \(x^*\) is a solution of the equation \(G(x^*) = 0\) and that \(G\) is \(\partial G\)-semi-smooth in a neighborhood \(U\) in \(X\) containing \(x^*\). If the set \(\partial G(x)\) contains only non-singular mapping and if \(\{\|M^{-1}\| \mid M \in \partial G(x)\}\) is bounded for all \(x \in U\), then the Newton iteration

\[
x^{k+1} = x^k - M^{-1}G(x^k), \text{ for any } M \in \partial G(x^k)
\]

(23)
converges super linearly to \( x^* \), provided that \( \|x^0 - x^*\| \) is sufficiently small.

**Remark 2.** Let us note that [17, Theorem 1.1] is actually more general. The authors are using the slant differentiability which is a weaker concept than the semi-smoothness, see [17, p. 868].

In the following, we prove that all conditions needed for Theorem 4.8 hold for \( G = F_\gamma \) and \( \kappa \neq 0 \) with \( x^* = (\overrightarrow{v}, \overrightarrow{c}) \) and \( X = Y = L^2(I)^m \times \mathbb{R}^m \). If the initial value \( x^0 \) is sufficiently close to \( x^* \) this guarantees that the sequence \((x^k)_{k \in \mathbb{N}}\) in (23) converges super linearly in \( L^2(I)^m \times \mathbb{R}^m \) to \( x^* \) with respect to \( F_\gamma \).

**Definition 4.9.** Define the following operators for \( (\overrightarrow{h}, \overrightarrow{k}) \in L^2(I)^m \times \mathbb{R}^m \):

\[
(B^* L^* \text{LB})_1(\cdot, \overrightarrow{k}) : L^2(I)^m \to L^2(I)^m, \quad \text{by} \quad \overrightarrow{h} \mapsto \frac{T}{\Omega} \int_\Omega L^*(L(B(\overrightarrow{h}, \overrightarrow{k}))) g \, dx \, dt,
\]

\[
(B^* L^* \text{LB})_2(\overrightarrow{h}, \cdot) : \mathbb{R}^m \to \mathbb{R}^m, \quad \text{by} \quad \overrightarrow{k} \mapsto \frac{T}{\Omega_T} \int_{\Omega_T} L^*(L(B(\overrightarrow{h}, \overrightarrow{k}))) g \, dx \, dt.
\]

Furthermore, we write for \( i = 1, \cdots, m \):

\[
(B^* L^* \text{LB})_1,i(\overrightarrow{h}, \overrightarrow{k}) = \frac{T}{\Omega} \int_\Omega L^*(L(B(\overrightarrow{h}, \overrightarrow{k}))) g_i \, dx \, dt \in L^2(I),
\]

\[
(B^* L^* \text{LB})_2,i(\overrightarrow{h}, \overrightarrow{k}) = \frac{T}{\Omega_T} \int_{\Omega_T} L^*(L(B(\overrightarrow{h}, \overrightarrow{k}))) g_i \, dx \, dt \in \mathbb{R}.
\]

For any function \( \Upsilon : X \to Y \), with \( X \) and \( Y \) Banach spaces, we denote by \( D\Upsilon(x)(z) \) the directional derivative of \( \Upsilon \) in \( x \) in direction \( z \).

Let us recall that the point-wise maximum and minimum operation from \( L^p \) to \( L^2 \) are semi-smooth if \( p > 2 \) (Norm gap), and a Newton derivative in \( f \in L^p(I) \) in direction \( h \in L^p(I) \) is given by

\[
\begin{cases}
  \mathbbm{1}_{(0,\infty)}(f)(h) & \text{for max,} \\
  \mathbbm{1}_{(-\infty,0)}(f)(h) & \text{for min.}
\end{cases}
\]

Hence, we get for \( \max/\min : L^p(I)^m \to L^2(I)^m \), \( \max(\overrightarrow{f}) := (\max(f_i))_{i=1,\ldots,m} \) respectively \( \min(\overrightarrow{f}) := (\min(f_i))_{i=1,\ldots,m} \), the following Newton derivative in \( \overrightarrow{f} \in L^p(I)^m \) in direction \( \overrightarrow{h} \in L^p(I)^m \):

\[
D\max/\min(\overrightarrow{f})(\overrightarrow{h}) = \begin{pmatrix}
  \mathbbm{1}_J(f_1) & 0 & \ldots & \ldots & 0 \\
  0 & \mathbbm{1}_J(f_2) & 0 & \ldots & 0 \\
  0 & 0 & \ldots & \ldots & \ldots \\
  \ldots & \ldots & \ldots & \ldots & \ldots \\
  0 & 0 & \ldots & 0 & \mathbbm{1}_J(f_m) \\
\end{pmatrix}
\begin{pmatrix}
  h_1 \\
  \cdot \\
  \cdot \\
  \cdot \\
  \cdot \\
\end{pmatrix} = \mathbbm{1}_J(\overrightarrow{f})(\overrightarrow{h}) := \text{diag}(\mathbbm{1}_J(f_i))\overrightarrow{h}_i
\]

for \( J = (0, \infty) \) in case of max and \( (-\infty,0) \) in case of min. The matrix \( \mathbbm{1}_J(\overrightarrow{f}) \) has only values equal to 1 or 0 on its diagonal.
Lemma 4.10. The first derivative of $F_\gamma$ in $(v, c) \in L^2(I)^m \times \mathbb{R}^m$ has the following form:

$$DF_\gamma(v, c) = \begin{pmatrix} (id_{L^2(I)^m}) & 0 \\ 0 & \frac{\kappa(c)}{\gamma} \text{id}_{\mathbb{R}^m} \end{pmatrix} + \frac{1}{\gamma} \left( \begin{array}{cc} \text{diag} & 0 \\ 0 & \text{id}_{\mathbb{R}^m} \end{array} \right) B^* L^* LB$$

$$= \begin{pmatrix} (id_{L^2(I)^m}) + \frac{1}{\gamma} \text{diag} \cdot (B^* L^* LB)_1(\cdot, 0) & \frac{1}{\gamma} \text{diag} \cdot (B^* L^* LB)_2(\cdot, 0) \\ \frac{1}{\gamma} (B^* L^* LB)_2(\cdot, 0) & \frac{\kappa(c)}{\gamma} \text{id}_{\mathbb{R}^m} + \frac{1}{\gamma} (B^* L^* LB)_2(\cdot, 0) \end{pmatrix}$$

with

$$\text{diag} := \begin{pmatrix} \mathbb{1}_{(0, \infty)}(-\frac{1}{\gamma} \bar{\pi}(v, c) - \frac{\mu}{\gamma}) + \mathbb{1}_{(-\infty, 0)}(-\frac{1}{\gamma} \bar{\pi}(v, c) + \frac{\mu}{\gamma}) \end{pmatrix}.$$  \hspace{1cm} (24)

In particular, we have for $(\bar{h}, \bar{k}) \in L^2(I)^m \times \mathbb{R}^m$:

$$DF_\gamma(v, c) \begin{pmatrix} \bar{h} \\ \bar{k} \end{pmatrix}(s) = \begin{pmatrix} \bar{h}(s) + \text{diag} \cdot \left( \frac{1}{\gamma} \int_0^T \int_{\Omega} L^*(L(B(\bar{h}, \bar{k}))) \bar{g} \, dt \, dx \right) \\ \frac{\kappa(c)}{\gamma} \bar{k} + \frac{1}{\gamma} \int_{\Omega_T} L^*(L(B(\bar{h}, \bar{k}))) \bar{g} \, dt \, dx \end{pmatrix}.$$  \hspace{1cm} (25)

Furthermore, the function $F_\gamma$ with $DF_\gamma(v, c)$ as generalized derivative is semi-smooth for all $(v, c) \in L^2(I)^m \times \mathbb{R}^m$.

Proof. Lemma 4.10 is a consequence of the semi-smoothness of $\max$ $\min$ from $L^p$ to $L^q$ with $p > q \geq 1$. \hfill $\Box$

Let us write $\text{diag}$ for $\text{diag}(s) := \text{diag}(X_1(s), \ldots, X_m(s))$ with $X_i : I \to \{0, 1\}$. The maps $X_i$ are $\mathcal{B}(I) = 2^{\{0, 1\}}$-measurable. Hence, we can define the measurable sets $I_{0,i} := \{s \in I | X_i(s) = 0\}, I_{1,i} := \{s \in I | X_i(s) = 1\}$.

Lemma 4.11. The linear continuous operator

$$B^* L^* LB : L^2(I)^m \times \mathbb{R}^m \to L^2(I)^m \times \mathbb{R}^m$$

is a self-adjoint non-negative and injective operator with spectrum inside the set $[0, \|B^* L^* LB\|]$.

Furthermore, $G := (\langle Lg_i, Lg_j \rangle_{L^2(\Omega_T)}{i,j=1}^m$ is invertible and we have for all $h \in L^2(I)^m$ that the continuous affine linear operator $(B^* L^* LB)_2(h, \cdot)$ is bijective with

$$\langle B^* L^* LB_2(h, k) = \phi \leftrightarrow k = G^{-1}(\phi - (B^* L^* LB)_2(h, 0)) \rangle.$$  \hspace{1cm} (26)

Proof. The non-negativity and injectivity can be seen by the strict inequality,

$$\langle B^* L^* LB_2(h, k) \rangle_{L^2(\Omega_T)} = \langle LB_2(h, LB_2) \rangle_{L^2(\Omega_T)} > 0 \text{ for } \phi \neq 0.$$  \hspace{1cm} (27)

The strictness is a consequence of the uniqueness of solutions of the wave equation defined by $L$.

The claim on the spectrum follows from selfadjointness and the fact that the spectral radius is $\|B^* L^* LB\|$, see [27, Theorem VI.6].

Let us now show that $G$ is invertible. Given the linear independence of $(g_i)_{i=1}^m$ in $L^2(\Omega_T)$ we get that $(Lg_i)_{i=1}^m$ is linearly independent in $L^2(\Omega_T)$ by the uniqueness of solutions of the wave equation. Further, introduce $\langle \lambda, \mu \rangle_L = \langle L(\lambda \cdot \bar{g}), L(\mu \cdot \bar{g}) \rangle_{L^2(\Omega_T)} = \langle \sum_{i=1}^m \lambda_i L(g_i), \sum_{j=1}^m \mu_j L(g_j) \rangle_{L^2(\Omega_T)}$. This is an inner product in $\mathbb{R}^m$.

Hence, the Gram-Schmidt Matrix $G = (\langle e_i, e_j \rangle_L)_{i,j=1}^m \in \mathbb{R}^{m \times m}$ is invertible.
To verify (26) let us derive that
\[(B^*L^*\text{LB})_2(0, k) = ((L^*L(k \cdot \vec{g}), g_i)_{L^2(\Omega_T)})_{i=1}^m = \left(\sum_{j=1}^m k_j L(g_j), L(g_i)\right)_{L^2(\Omega_T)} = G(c).\]

Then \(\phi = (B^*L^*\text{LB})_2(h, 0) + (B^*L^*\text{LB})_2(0, k)\) can be equivalently expressed by \(G(k) = (B^*L^*\text{LB})_2(0, k) = \phi - (B^*L^*\text{LB})_2(h, 0)\) and (26) follows.

In the following we present the injectivity results for the Newton derivative \(DF_\gamma(v, c)\). The final surjectivity results and uniform boundedness of \(DF_\gamma(v, c)^{-1}\) with respect to \(\gamma \to 0\) and \(\kappa > 0\) can be found in section 4.5. Combined, these results will allow us to conclude, that the super linear convergence of Theorem 4.8 holds for our control problem at least in the case \(\kappa \neq 0\).

**Theorem 4.12.** If \(\gamma > 0, \alpha_i > 0, i = 1, \cdots, m,\) and \((v, c) \in L^2(I)^m \times \mathbb{R}^m\), the Newton derivative \(DF_\gamma(v, c)\) is injective.

**Proof.** Case \(\kappa = 0\). Let \(0 \neq (h, k) \in L^2(I)^m \times \mathbb{R}^m\) and assume that \(DF_\gamma(v, c)(h, k) = 0\). By the first line of \(DF_\gamma(v, c)(h, k)\), see Lemma 4.10, we then have
\[h_i + X_i \left(\frac{1}{\gamma}(B^*L^*\text{LB})_\gamma(h, k)\right) = 0\]
for all \(i = 1, \cdots, m\). In the set \(I_{0,i}\) it holds that \(0 = h_i\) and in \(I_{1,i}\) we have \(-\gamma h_i = (B^*L^*\text{LB})_{\gamma 1}(h, k)\). Furthermore, by the second row of \(DF_\gamma(v, c)(h, k)\), see Lemma 4.10, we have \((B^*L^*\text{LB})_2(h, k) = 0\). Thus, we get by the positivity of \(B^*L^*\text{LB}\) and (27)
\[0 \leq \left\langle B^*L^*\text{LB}(h, k), \begin{pmatrix} h \\ k \end{pmatrix}\right\rangle_{L^2(I)^m \times \mathbb{R}^m} = -\gamma \sum_{i=1}^m \int_{I_{1,i}} h_i^2 dt.\]

This implies that \(h = 0\) for all \(i\). For \(h = 0\), we have that \(0 = (B^*L^*\text{LB})_2(0, k)\) based on the second row of \(DF_\gamma(v, c)(h, k)\). Because \((B^*L^*\text{LB})_2(0, \cdot)\) is invertible, the kernel is 0 and thus \(k = 0\), which is a contradiction. Hence \(DF_\gamma(v, c)\) is injective.

Case \(\kappa > 0\). Let \((h, k) \neq 0 \in L^2(I)^m \times \mathbb{R}^m\) and assume that \(DF_\gamma(v, c)(h, k) = 0\). By the first row of \(DF_\gamma(v, c)(h, k)\), see Lemma 4.10, we then have
\[h_i + X_i \left(\frac{1}{\gamma}(B^*L^*\text{LB})_{\gamma 1}(h, k)\right) = 0\]
for all \(i = 1, \cdots, m\). In the set \(I_{0,i}\) we have \(0 = h_i\), and in \(I_{1,i}\) we have \(-\gamma h_i = (B^*L^*\text{LB})_{\gamma 1}(h, k)\). By the second row of \(DF_\gamma(v, c)(h, k)\), see Lemma 4.10, we have
\[-\frac{\alpha_i}{\gamma} k = (B^*L^*\text{LB})_2(h, k)\]. Thus, we get by the positivity of \(B^*L^*\text{LB}\):
\[0 \leq \left\langle B^*L^*\text{LB}(h, k), \begin{pmatrix} h \\ k \end{pmatrix}\right\rangle_{L^2(I)^m \times \mathbb{R}^m} = -\gamma \sum_{i=1}^m \int_{I_{1,i}} h_i^2 dt - \frac{\alpha_i}{\gamma} \|k\|_{\mathbb{R}^m}^2 < 0,\]

which is a contradiction. Hence \(DF_\gamma(v, c)\) is injective. \(\square\)
4.5. Surjectivity results for the Newton derivative $DF_\gamma$. In this section we present surjectivity results for the Newton derivative $DF_{\gamma}(v,c)$ as well as uniform boundedness of the operator family $\{DF_{\gamma}(v,c)^{-1}\}_{(v,c) \in L^2(I)^m \times \mathbb{R}^m}$.

**Theorem 4.13.** For $\gamma, \kappa(\gamma)$, and $\alpha_i$, $i = 1, \cdots, m$, all positive, the Newton derivative $DF_{\gamma}(v,c)$ is surjective for each $(v,c) \in L^2(I)^m \times \mathbb{R}^m$. Furthermore, the operator family $\{DF_{\gamma}(v,c)^{-1}\}_{(v,c) \in L^2(I)^m \times \mathbb{R}^m}$ is uniformly bounded for each fixed $\kappa > 0$ in $\mathfrak{L}(L^2(I)^m \times \mathbb{R}^m)$.

**Proof.** (i) Surjectivity: In the following consider $(v,c)$ and $(\phi_1, \phi_2) \in L^2(I)^m \times \mathbb{R}^m$.

We have to show that there exists a $(h,k) \in L^2(I)^m \times \mathbb{R}^m$ such that

$$DF_{\gamma}(v,c)
\begin{pmatrix}
h \\
k
\end{pmatrix}
= \begin{pmatrix}
\phi_1 \\
\phi_2
\end{pmatrix}.
$$

(31)

In view of (31) and (25) we have $h_i = \phi_{1,i}$ for $i = 1, \cdots, m$ in $I_{0,i}$. This implies that $h_i = \phi_{1,i}1_{I_{0,i}} + h_i1_{I_{1,i}}$ for some $h_i$, $i = 1, \cdots, m$. If it holds that $|I_{1,i}| = 0$, for all $i = 1, \cdots, m$, we get that $h_i = \phi_{1,i}$ and by (31) we have

$$\phi_2 - \frac{1}{\gamma}(B^*L^*LB_2)(\phi_1,0) = \frac{\kappa(\gamma)}{\gamma} k + (B^*L^*LB_2)(0,k) =: \tilde{W}(k) \text{ with } \tilde{W} \in \mathbb{R}^m \times \mathbb{R}^m.$$

(32)

Since, $k \mapsto (B^*L^*LB_2)(0,k)$ is self-adjoint and positive definite from $\mathbb{R}^m$ to itself, there exists $k \in \mathbb{R}^m$ which solves (32).

Next, w.l.o.g. let $|I_{0,i}|, \cdots, |I_{n,i}| > 0$ and $|I_{1,i}+1|, \cdots, |I_{m,i}| = 0$ with $n > 0$. By (31), in $I_{1,i}$, $i = 1, \cdots, n$, we require

$$\phi_{1,i} = \tilde{h}_i1_{I_{1,i}} + \frac{1}{\gamma}(B^*L^*LB_2)_{1,i}(h,k),$$

$$\phi_11_{I_{1,i}} - \frac{1}{\gamma}(B^*L^*LB_2)_{1,i}(\phi_{1,i}1_{I_{0,i}})_m = \tilde{h}_i1_{I_{1,i}} + \frac{1}{\gamma}(B^*L^*LB_2)_{1,i}(\tilde{h}_i1_{I_{1,i}})_m =: \tilde{W}_2,$$

(33)

where in the last step we used $h_i = \phi_{1,i}1_{I_{0,i}} + \tilde{h}_i1_{I_{0,i}}$. Note that $\tilde{h}_i = 0$ for $i > n$ holds. By the second equation in (31) we have to fulfill the equation

$$\phi_2 - \frac{1}{\gamma}(B^*L^*LB_2)(\phi_{1,i}1_{I_{1,i}})_m = 0 = \frac{\kappa(\gamma)}{\gamma} k + \frac{1}{\gamma}(B^*L^*LB_2)((\tilde{h}_i1_{I_{1,i}}))_{i=1}^n, 0_{m-n}, k).$$

(34)

with $|I_{0,i}| = |I|$ for $i = n+1, \cdots, m$. We will use the solution of (33) and (34) to obtain the solution for (31). Let us consider in the following the linear continuous and self-adjoint operator

$$W_1 = \begin{pmatrix}
\tilde{id} & 0 \\
0 & \frac{\kappa(\gamma)}{\gamma} id_{\mathbb{R}^m}
\end{pmatrix} + \frac{1}{\gamma} \begin{pmatrix}
B^*L^*LB_1((\cdot1_{I_{1,i}})_{i=1}^n, 0_{m-n}, \cdot) \\
B^*L^*LB_2(((\cdot1_{I_{1,i}})_{i=1}^n, 0_{m-n}, \cdot)
\end{pmatrix}$$

$$\in \mathfrak{L}\left(\prod_{i=1}^n L^2(I_{1,i}) \times \mathbb{R}^m, \prod_{i=1}^n L^2(I_{1,i}) \times \mathbb{R}^m\right)$$

resulting in

$$W_1\begin{pmatrix}
a \\
b
\end{pmatrix} = \begin{pmatrix}
a \\
\frac{\kappa(\gamma)}{\gamma} b
\end{pmatrix} + \frac{1}{\gamma} \begin{pmatrix}
B^*L^*LB_1((a1_{I_{1,i}})_{i=1}^n, 0_{m-n}, b) \\
B^*L^*LB_2(((a1_{I_{1,i}})_{i=1}^n, 0_{m-n}, b)
\end{pmatrix}.$$
Since $W_2$ is non-negative, we conclude that $W_1$ is positive definite and hence invertible. This implies that there exists a $(\tilde{h}, k) \in \prod_{i=1}^{\tilde{n}} L^2(I_{1,i}) \times \mathbb{R}^m$ such that (33) and (34) holds true. Defining $h_i = \phi_{1,i} 1_{I_{0,i}} + \tilde{h}_i 1_{I_{1,i}}$, for $i = 1, \cdots, \tilde{n}$, and $h_i = \phi_{1,i}$ with $\tilde{h}_i = 0$, for $i = \tilde{n} + 1, \cdots, m$ provides the desired solution $(h, k) \in L^2(I)^m \times \mathbb{R}^m$ for (31): In fact, for $i = \tilde{n} + 1, \cdots, m$ we have $h_i = \phi_{1,i}$ a.e. in $I$. In $I_{0,i}$, $i = 1, \cdots, m$, holds $h_i = \phi_{1,i}$. Furthermore, we have in $I_{1,i}$

$$h_i + \frac{1}{\gamma} (B^* L^* LB)_{1,i} (h, k)$$

$$= \left\{ \begin{array}{l}
\tilde{h}_i 1_{I_{1,i}} + \frac{1}{\gamma} (B^* L^* LB)_{1,i} ((\phi_{1,i} 1_{I_{0,i}})_{i=1}^\tilde{n}, 0) 1_{I_{1,i}} \\
+ \frac{1}{\gamma} (B^* L^* LB)_{1,i} ((\tilde{h}_i 1_{I_{1,i}})_{i=1}^m, k) 1_{I_{1,i}}
\end{array} \right\}$$

$$= (\phi_{1,i} 1_{I_{1,i}},)$$

for $i = 1, \cdots, \tilde{n}$. With (34) we finally have (31). Hence, we have proved the surjectivity of $DF_\gamma(v, c)$.

(i) Boundedness: Let us now show that the operator family $\{DF_\gamma(v, c)^{-1}\}_{v, c}$ is uniformly bounded for fixed $\kappa > 0$. Let us consider $(\phi_1, \phi_2)^T \in L^2(I)^m \times \mathbb{R}^m$. By the surjectivity and injectivity of $DF_\gamma(v, c)$, there exists a unique $(h, k) \in L^2(I)^m \times \mathbb{R}^m$ with $h_i = \phi_{1,i} 1_{I_{1,i}} + \tilde{h}_i 1_{I_{1,i}}$, as we used above, such that

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} (\tilde{h}_i 1_{I_{1,i}})_{i=1}^m \\ \frac{\kappa(\gamma)}{\gamma} k \end{pmatrix} + \frac{1}{\gamma} \begin{pmatrix} \frac{B^* L^* LB_{1,i} ((\phi_{1,i} 1_{I_{0,i}})_{j=1}^m, 0) 1_{I_{1,i}})_{i=1}^m} {B^* L^* LB_{2,i} ((\tilde{h}_i 1_{I_{1,i}})_{j=1}^m, k)} \end{pmatrix}$$

with $\psi_1 := \left( \phi_{1,i} 1_{I_{1,i}} - \frac{1}{\gamma} B^* L^* LB_{1,i} ((\phi_{1,i} 1_{I_{0,i}})_{j=1}^m, 0) 1_{I_{1,i}})_{i=1}^m \right)$ and $\psi_2 := \phi_2 - \frac{1}{\gamma} B^* L^* LB_{2,i} ((\phi_{1,i} 1_{I_{0,i}})_{j=1}^m, 0)$, compare (33), and (34). Similarly as before, assume at first that $|I_{1,1}, \cdots, |I_{1,m}| = 0$. We have $h = \phi_1$ and

$$\psi_2 = \frac{\kappa(\gamma)}{\gamma} k - \frac{1}{\gamma} B^* L^* LB_{2} (0, k) = \tilde{W}(k)$$

by (31), (32). Recall that $\tilde{W}$ is a self-adjoint, and positive definite. Using $\tilde{W}^{-1}$ on both sides of (36) gives us the following:

$$\|k\|_{\mathbb{R}^m} = \|\tilde{W}^{-1}(\psi_2)\| \leq \|\tilde{W}^{-1}\| \left\| \psi_2 \right\|_{\mathbb{R}^m} + \frac{1}{\gamma} \|B^* L^* LB\| \|(\phi_1, 0)\|_{L^2(I)^m \times \mathbb{R}^m}$$

$$\leq 2\|\tilde{W}^{-1}\| \max\left(1, \frac{1}{\gamma} \|B^* L^* LB\|\right) \|(\phi_1, \phi_2)\|_{L^2(I)^m \times \mathbb{R}^m}$$

where in (*) we used that

$$\|(B^* L^* LB)_{2} ((\phi_{1,i} 1_{I_{0,i}})_{j=1}^m, 0)\|_{\mathbb{R}^m} \leq \|B^* L^* LB ((\phi_{1,i} 1_{I_{0,i}})_{j=1}^m, 0)\|_{L^2(I)^m \times \mathbb{R}^m}.$$
Next assume again that $\hat{\psi}_2 = \frac{\kappa(\gamma)}{\gamma} k - \frac{1}{\gamma} B^* L^* L B_2(\hat{h}, k)$. By (35) we have

$$\left\langle \left( \psi_1, \psi_2 \right) \right\rangle_{\mathbb{N} \times \mathbb{R}^m} \leq \left\langle \left( \hat{h}_N, k \right) \right\rangle_{\mathbb{N} \times \mathbb{R}^m}$$

This equation implies the following:

$$\| (\psi_1, \psi_2) \|_{L^2(I)^m \times \mathbb{R}^m} \| (\hat{h}_N, k) \|_{\mathbb{N} \times \mathbb{R}^m} \geq \left\langle (\psi_1)_{i \in \mathbb{N}}, \psi_2 \right\rangle_{\mathbb{N} \times \mathbb{R}^m} \| (\hat{h}_N, k) \|_{\mathbb{N} \times \mathbb{R}^m}$$

where (**) follows by the non-negativity of $B^* L^* L B$, i.e.

$$\frac{1}{\gamma} \left\langle \left( \frac{B^* L^* L B_1(\hat{h}, k) 1_{i_1}}{\mathbb{N}}, \frac{\hat{h}_N}{k} \right)_{i \in \mathbb{N}} \right\rangle_{\mathbb{N} \times \mathbb{R}^m} \geq \min \left( 1, \frac{\kappa(\gamma)}{\gamma} \right) \| (\hat{h}_N, k) \|_{\mathbb{N} \times \mathbb{R}^m}^2$$
Finally, we have by (40) and the definition of \( \psi_i \)
\[
\| (h, k) \|_{L^2(I)^m \times \mathbb{R}^m}^2 = \|(\phi_{ij1} 1_{I_{00}})_{i=1}^m + \tilde{h}, k) \|_{L^2(I)^m \times \mathbb{R}^m}^2
\]
\[
= \left\langle \left( (\phi_{ij1} 1_{I_{00}})_{i=1}^m + \tilde{h} \right), \left( (\phi_{ij1} 1_{I_{00}})_{i=1}^m + \tilde{h} \right) \right\rangle_{L^2(I)^m \times \mathbb{R}^m}
\]
where we used that \( \tilde{h} \) is the solution of the linear system in Algorithm 1 and \( \tilde{h} = \tilde{h}_{N/2} \) for \( j = 1, \cdots, \tilde{n} \) and \( \tilde{h} = 0 \) for \( j > \tilde{n} \).
Hence, we have
\[
\frac{1}{\min(1, \frac{\gamma}{\kappa(\gamma)})} \| (\psi_1, \psi_2) \|_{L^2(I)^m \times \mathbb{R}^m} \geq \| (\tilde{h}, k) \|_{\mathbb{P}\times \mathbb{R}^m}.
\] (40)

Finally, we have by (40) and the definition of \( \psi_i \)
\[
\| (h, k) \|_{L^2(I)^m \times \mathbb{R}^m}^2 = \|(\phi_{ij1} 1_{I_{00}})_{i=1}^m + \tilde{h}, k) \|_{L^2(I)^m \times \mathbb{R}^m}^2
\]
\[
= \left\langle \left( (\phi_{ij1} 1_{I_{00}})_{i=1}^m + \tilde{h} \right), \left( (\phi_{ij1} 1_{I_{00}})_{i=1}^m + \tilde{h} \right) \right\rangle_{L^2(I)^m \times \mathbb{R}^m}
\]
\[
\leq \| (\phi_1, \phi_2) \|_{L^2(I)^m \times \mathbb{R}^m}^2 + \| (\tilde{h}, k) \|_{\mathbb{P}\times \mathbb{R}^m}^2
\]
\[
\leq \| (\phi_1, \phi_2) \|_{L^2(I)^m \times \mathbb{R}^m}^2 + \frac{1}{\min(1, \frac{\gamma}{\kappa(\gamma)})} \| (\psi_1, \psi_2) \|_{L^2(I)^m \times \mathbb{R}^m}
\]
where \( \tilde{c} > 0 \) is some constant independent of \( (v, c) \) and \( (h, k) \). This finally concludes the boundedness of \( \| DF, (v, c)^{-1} \| \leq \tilde{c} \).

As a consequence of Theorem 4.8 - 4.13 we have the following result.

**Corollary 4.** If \( \gamma, \kappa(\gamma) \), and \( i \in \{1, \cdots, m\} \) are all positive, then the semi-smooth Newton algorithm
\[
(v^{k+1}, c^{k+1}) = (v^k, c^k) - DF, (v^k, c^k)^{-1}F, (v^k, c^k),
\] (41)
converges super-linearly to the optimal solution \( (\tilde{v}, \tilde{c}) \) of \( (P, \gamma) \), provided that \( \| (v^0, c^0) - (\tilde{v}, \tilde{c}) \|_{L^2(I)^m \times \mathbb{R}^m} \) is sufficiently small.

5. **Numerics and examples.** In the following sections we present numerical results which illustrate the effect of BV cost on the optimal controls. For the discretization of \( (W) \) we used the 3-level finite element method presented in [28]. In particular, we used the Crank-Nicholson method with linear continuous finite elements in time \( (S_T) \) and space \( (S_h) \). The resulting discrete solution of \( (W) \) is an element in the tensor space \( S_T \otimes S_h \).

We discretized the control \( (v, c) \in L^2(I)^m \times \mathbb{R}^m \) in \( (\tilde{P}, \gamma) \) by \( S_h \) elements. Furthermore, we used the trapezoidal rule to evaluate all time-depending integrals in problem \( (\tilde{P}, \gamma) \). The trapezoidal rule guarantees that the function inside the prox operator (see (15)) attains its maximum and minimum in the time nodes we considered for \( S_T \). We used the mass matrix for the space depending integral in \( (\tilde{P}, \gamma) \) with respect to the finite elements in \( S_h \). Further details can be found in [13].

In the following sections, we construct two test cases in such a manner that exact analytic solutions for \( (P) \), respectively \( (\tilde{P}) \), become available. We use Algorithm 1, which is a BV-path following algorithm, to approximate numerically the solutions of those examples. The solution of the linear system in Algorithm 1 is approximated by a Krylov iterative method.
A similar path following algorithm is used in the semi-linear parabolic case in [9]. A special aspect about the semi-smooth Newton method inside the BV-Path-

**Algorithm 1:** BV-path following algorithm.

Input: \((v_0, c_0) \in L^2(I)^m \times \mathbb{R}^m, \gamma_0 > 0, TOL_\gamma > 0, TOL_N > 0, k = 0\) and \(\nu \in (0, 1)\)

while \(\gamma_k > TOL_\gamma\) do

Set \(i = 0, (v_i^k, c_i^k) = (v_k, c_k)\)

while \(\|F_{\gamma_k}(v_i^k, c_i^k)\|_{L^2(I)^m \times \mathbb{R}^m} > TOL_N\) do

Solve \(DF_{\gamma_k}(v_i^k, c_i^k)(\delta v, \delta c) = -F_{\gamma_k}(v_k, c_k)\), set

\(\left(\begin{array}{c}
(\nu_i^{k+1}, c_i^{k+1}) = (v_i^k, c_i^k) + (\delta v, \delta c);
\end{array}\right) i = i + 1.\)

Define \((v_k, c_k+1) = (v_i^k, c_i^k)\) and \(\gamma_{k+1} = \nu \gamma_k\); set \(k = k + 1.\)

Following algorithm compared to the one in [9] is that we consider the derivative and an additional constant as control instead of a BV function. Besides, we have an additional term \(\frac{\alpha_{\gamma_0}}{\beta}\), which allows us to obtain super-linear convergence for the semi-smooth Newton algorithm for \(\kappa \neq 0\), see section 4.4 and 4.5.

5.1. Construction of test examples. This example is constructed in such a way that for the optimal adjoint state a wide range of different sets \(\{t \in I| p_i(t) = \alpha\}\) are possible. For the construction of test examples we consider \(\Omega \subset \mathbb{R}^d, d \in \{1, 2, 3\}, I = (0, T)\) with \(T < \infty, m \in \mathbb{N}_{>0}, \alpha_i > 0, \) and \(g_i \in L^\infty(\Omega) \setminus \{0\}\) with disjoint supports \(\text{supp}(g_i) = \omega_i\) for \(i = 1, \cdots, m.\)

**Adjoint State:** Choose a function \(f \in H^2(\Omega) \cap \Omega\) with \(\int f \cdot g_i dx \neq 0\) for all \(i = 1, \cdots, m.\) Furthermore, let \(h \in H^2(I)\) such that \(h(T) = 0, \partial_t h(T) = 0,\)

\[\int h(s)ds = 0, \bar{p}_i(t) := \int^T_{\Omega} h(s)ds \in [-\frac{\alpha_i}{\beta}, \frac{\alpha_i}{\beta}],\]

with \(\|p_i\|_\infty \leq \alpha_i,\) and \(p_i \in C_0(\Omega), \) for \(i = 1, \cdots, m.\)

**Control:** Let us consider now arbitrary positive measures \(\mu^+_i, \mu^-_i \in M(I)\) with support \(\text{supp}(\mu^+_i) \subset \{p_i = \alpha_i\}\) and define \(dD_t u_{i,t}(t) := -\frac{\alpha_i}{p_{i,t}(t)} (\mu^+_i + \mu^-_i).\) Due to the continuity of \(p_{i,t}\) the support of \(\mu^+_i\) is disjoint from the support of \(\mu^-_i.\) The measure \(dD_t u_{i,t}\) is a positive measure on \(\{p_i = -\alpha_i\}\) and a negative measure on \(\{p_i = +\alpha_i\}.\)

\[\left\|\frac{\alpha_i}{p_{i,t}(s)}(\mu^+_i + \mu^-_i)\right\|_{M(I)} = \|D_t u_{i,t}\|_{M(I)}.\]
Let us define $u_{1,i} := u_{1,i} + c_i$ with $u_{1,i}(t) = \int_0^t dD_t u_{1,i}(s)$ and $c_i \in \mathbb{R}$.

**State and Desired State:** Furthermore, let us fix the desired state according to $y_d = \hat{S}(\pi) - (x \cdot \partial_t h - h \cdot \Delta f)$, and some displacement and velocity functions $(y_0, y_1) \in V \times H$. For the resulting problem (P) the function $\pi$ is the optimal control.

5.2. Finitely many jumps example. This example is constructed in such a way that the set $\{ t \in I \mid |p_1(t)| = \alpha \}$ consists of finitely many active points. A similar construction steps can also be found in [9, Example 1]. Let $\beta > 0$, $l \in \mathbb{N}_{>0}$, $d \in \{1, 2, 3\}$, $\Omega = (-1, 1)^d$, $I = (0, 2)$, and define

$$g(x) = \mathbf{1}_{[-0.5,0.5]^d}(x) = \prod_{i=1}^d \mathbf{1}_{[-0.5,0.5]}(x_i).$$

Define the function $\varphi(t, x)$ by $\beta \sin(l \pi t) \sin(\frac{\pi l}{2} x) \prod_{i=1}^d \cos(\frac{\pi l}{2} x_i)$. Then $\varphi$ has the property $\varphi|_{\partial \Omega} = 0$, $\varphi(2) = 0$, and

$$\partial_t \varphi|_{t=2}(t, x) = \beta (l \pi \cos(2 \pi t) \sin(\frac{\pi l}{2} t) + \frac{l \pi}{2} \sin(l \pi t) \cos(\frac{\pi l}{2} t)) \prod_{i=1}^d \cos(\frac{\pi l}{2} x_i) = 0.$$ 

Using the wave operator $\partial_t - \Delta$ on $\varphi(t, x)$ gives us:

$$(\partial_t - \Delta)\varphi(t, x) = \left( \frac{d^2}{dt^2} - \frac{5d^2}{4} \right) \varphi + \beta (l \pi^2 \cos(l \pi t) \cos(\frac{\pi l}{2} t)) \prod_{i=1}^d \cos(\frac{\pi l}{2} x_i).$$

By an elementary computation we find

$$p_1(t) := \frac{2}{\Omega} \int \varphi(t, x)g(x)dxdt = \frac{4\alpha}{3\pi} \left( -\sin(\frac{l \pi}{2} t) \right)^3 \left( \frac{2\sqrt{\pi}}{\alpha} \right)^d.$$ 

It holds that $p_1(0) = p_1(2) = 0$, and $p_1 \in C_0(I)$ with $\|p_1\|_{C_0(I)} = \frac{4\alpha}{3\pi} \left( \frac{2\sqrt{\pi}}{\alpha} \right)^d$. To have equality $\|p_1\|_{C_0(I)} = \alpha$ at the optimum, we have to chose $\beta = \alpha \frac{3\pi l}{4} \left( \frac{2\sqrt{\pi}}{\alpha} \right)^{-d}$. Furthermore, we have $\{ t \in I \mid |p_1(t)| = \pm \alpha \} = \{ \frac{1+2n}{2} \mid n \in \{0, \ldots, l-1\} \} \subset I$.

The following equalities hold

$$-\alpha \sum_{n=0}^{l-1} c_n \left( -\sin\left(\frac{2n+1}{2}\right) \right)^3 = \int_0^2 -p_1(t) dD_t \pi_1 = \|p_1\|_{C_0(I)} \|D_t \pi_1\|_{M(I)} = \alpha \sum_{n=0}^{l-1} |c_n|$$

(42)

for $c_n = \text{sign}\left(\sin\left(\frac{2n+1}{2}\right)\right)$ or 0. Consider an arbitrary $\bar{c} \in \mathbb{R}$, and define $\pi := \pi_1 + \bar{c}$ with $\pi_1(t) = \int_0^t \sum_{n=0}^{l-1} c_n \mathbf{1}_{1+2n} \mathbf{1}_{\Omega}(t)$. Now determine the desired state as $y_d := \hat{S}(\pi) - (\partial_t - \Delta)\varphi(t, x)$ with arbitrary $(y_0, y_1) \in V \times H$. For the resulting problem (P) the function $\pi$ is the optimal control. The corresponding cost functional has the value

$$J(\pi) = \frac{1}{2} \|\hat{S}(\pi) - y_d\|_{L^2(\Omega_T)}^2 + \alpha \|D_t \pi\|_{M(I)}$$

$$= \frac{1}{2} \|(\partial_t - \Delta)\varphi\|_{L^2(\Omega_T)}^2 + \alpha \sum_{n=0}^{l-1} |c_n|$$

$$= \frac{\beta^2}{4} \left( \frac{d^2}{4} - \frac{5l^2\pi^2}{4} \right)^2 + \frac{\beta^2 l^4 \pi^4}{4} + \alpha \sum_{n=0}^{l-1} |c_n|$$

for $\pi_1(t) = \int_0^t \sum_{n=0}^{l-1} c_n \mathbf{1}_{1+2n} \mathbf{1}_{\Omega}(t)$.
where the last equality follows from an elementary computation.

We now turn to discuss numerical results. We considered dimension $d = 2$, and the number of Diracs $l = 3$. For the desired state $y_d := \tilde{S}(\pi) - (\partial_t - \Delta)\varphi(t, x)$ we used $(y_0, y_1) = (0, 0)$. The optimal constant is fixed by $\tau = 0$. The BV-path following algorithm starts with $\gamma_0 = 1$, $(v_0, c_0) = (0, 0)$, and we iterate according to $\gamma_{k+1} = 0.1\gamma_k$. We stopped the BV-path following algorithm when $\gamma_k = 10^{-8}$ was reached. The function $\kappa$ is defined as $\kappa(\gamma) = \gamma^4$.

In the Figures 1 and 2 we depict the optimal control for two different choices of d.o.f. On the right hand side of each Figure 1 and 2, we see the function $p_{1,\text{approx}} := \psi_1$ which appear in the prox operator (15). As suggested by (16) we
obtain $\partial_t \bar{\pi}_{\text{approx}} = 0$ whenever $|p_{1, \text{approx}}| < \alpha$ for the derivative of the approximated optimal control.

In the upper left sub-figure in Figure 1 - 2 the red curve depicts the approximated derivative of the approximated optimal control $\bar{\pi}_{\text{approx}}$. The blue pin line represents the exact Dirac measures approximated according to the mesh, i.e. for $a \in \mathbb{R}$ the Dirac measure $a \cdot \delta_t$ is approximated by a pin in the position $t$ with pin height of $\tau$ with $\tau$ the uniform distance between two time nodes. In the lower sub-figure in Figure 1 - 2 we see the exact optimal control $u$ in blue, $L^2$-projected on $V_h$, and the approximated optimal control $u_{\text{approx}}$ in red.

We stopped the semi-smooth Newton algorithm as soon as $\|F_{\gamma_k} (u_k)\|_{L^2(\Gamma)^m \times \mathbb{R}^m} \leq 10^{-6} =: \text{TOL}_N$. In Figure 3 we show the $\|F_{\gamma_k} (u_k)\|_{L^2(\Gamma)^m \times \mathbb{R}^m}$-error for different $\gamma$ values which were used in the in Algorithm 1. In Figure 3, we see the errors which correspond 2049 d.o.f. in time. In the last figure on the right we see the error corresponding to Figure 2. As expected, all figures show the super-linearity of the $\|F_{\gamma_k} (u_k)\|_{L^2(\Gamma)^m \times \mathbb{R}^m}$-error.

Figure 3.

5.3. Cantor function or Devil’s staircase example. Here we construct functions $p_{1,i} \in C_0(I)$ which enable us to use all three classes of measures for the distributional derivative of a $BV$ function in time. This means absolutely continuous measures with respect to the Lebesgue measure, countable linear combinations of Dirac measures, and Cantor measures. For further information about these measure characterisation see for example [1, p. 184]. Finally we will use $p_{1,i}$ to create a Cantor-like optimal control.

Let $0 < a_1 < b_1 < a_2 < b_2 < T$. Then for all closed non-trivial intervals $I_i \subseteq (a_i, b_i), i = 1, 2$, there exists $\tilde{p} \in C^\infty_c (I)$ such that $|\tilde{p}| \leq 1$ with

$$\tilde{p} = \begin{cases} 
0 & \text{in } (0, T) \setminus ((a_1, b_1) \cup (a_2, b_2)) \\
\geq 0 & \text{in } (a_1, b_1) \\
\leq 0 & \text{in } (a_2, b_2) \\
1 & \text{in } I_1 \\
-1 & \text{in } I_2 
\end{cases}.$$  \hspace{1cm} (43)

In the following we denote by $PC$ the set $\{ t \in I | \tilde{p}(t) = \pm 1 \} = I_1 \cup I_2$. Let us now fix $T, a_i, b_i$ such that the assumptions above (43) hold. We set $h = \partial_t \tilde{p}$, with $\tilde{p}$ as defined in (43). Then it holds that $h(T) = \partial_T h(T) = 0$ and $
\int_0^T h(t) dt = \tilde{p}(T) - \tilde{p}(0) = 0$

due to the compact support of $\tilde{p}$ inside $I$. Consider $\Omega, d, m, g_i, f$ such that the
assumptions in section 5.1 “Construction of Test Examples” are fulfilled, and define
\[ \varphi(t,x) := h(t) \cdot f(x). \] It holds that
\[ p^i_1(x) := \int_{\Omega} \varphi(t,x) \cdot g_i(x) \, dx = (\tilde{p}(t) - \tilde{p}(T)) \cdot \tilde{z}_i. \]
Under these circumstances, we define \( \alpha_i := |\tilde{z}_i|. \) Now, we consider positive measures \( \mu^\pm \in M(I) \) with support \( \text{supp}(\mu^\pm) \subset \{ p^i_1 = \mp \alpha_i \} \), and define \( d\mu_1(t) := -\frac{\alpha_i}{\text{supp}(\mu)}(\mu^+ + \mu^-). \) Following the instructions in section 5.1 “Construction of Test Examples” an optimal control \( \mathcal{U}. \) The measures \( \mu^\pm \) can be of the types described above.

Our next aim is to construct an optimal control which has a Cantor-like shape. Hence, denote by \( C(t) \) the Cantor function on \([0,1]\) (see [1, Example 3.34]). Define the function \( C^+(x) = C\left(\frac{x - PC_{left}^+}{2d^+}\right), \) with \( d^+ = |I_1|, \) \( PC_{left}^+ = \min_{x \in I_1} x, \) on the domain \((a_1, b_1)\). Additionally, let us define \( C^-(x) = C\left(\frac{PC_{right}^- - x}{2d^-}\right), \) with \( d^- = |I_2|, \) \( PC_{right}^- = \max_{x \in I_2} x, \) on the domain \((a_2, b_2)\). Accordingly we define the continuous function \( u_i(t) := \int_0^t d(\mu^+_1(s) + \mu^-_1) := \begin{cases} C^+(x), & \text{on } I_1 \\ \frac{1}{2}, & \text{on } [PC_{left}^+, PC_{right}^-] \setminus PC \\ C^-(x), & \text{on } I_2 \\ 0 & \text{else.} \end{cases} \)

Let us define \( \varpi_i = \tilde{c}_i \cdot u_i(t) + \tau_i \) with \( \tau_i \in \mathbb{R} \) and \( \tilde{c}_i > 0. \) The distributional derivative

\[ \frac{d}{dt}u_i(t) = \tilde{\varphi}(t,x) \cdot g_i(x) \]

**Figure 4.** In this figure we see one possible shape for \( \varpi_i. \)
In our numerical experiment we considered the following parameters:

\( a) \quad \Omega = [-2, 2]^2, \ T = 5, \ m = 1, \)

\( b) \quad g(x) := 10 \cdot 1_{[-\frac{1}{2}, \frac{1}{2}]^2}(x), \)

\( c) \quad \tilde{p}(t) := \varphi_\tau \ast (1_{[\frac{1}{2}, 2]} + 1_{[3.4, 5]})(t), \) with \( \varphi_\tau(x) := c_\tau e^{1 - (\tau^{-1} x)^2} \cdot 1_{[-\tau, \tau]}(x), \)

\( \tilde{\ell} = 0.28, \ c_\tau := \int_{\mathbb{R}} \varphi_\tau(x)dx, \)

\( d) \quad f(x, y) := \varphi_1(x) \cdot \varphi_2(y) with \ \varphi_1(x) := \exp(\frac{-1}{1-x^2}) 1_{[-1,1]}(x) \in C_\infty(O), \)

for \( i = 1, 2, \)

and the optimal control we want to approximate

\[ \overline{u}(t) := 10 \cdot C(\frac{1-0.8}{2, 14-0.8}) \cdot 1_{[0.8, 2.14]}(t) + 5 \cdot 1_{[2.14, 2.85]}(t) + 10 \cdot C(\frac{2-0.7}{2, 85-2.85}) \cdot 1_{[2.85, 4.2]}(t). \]

In Figures 5 and 6 we depict the numerical optimal control for two different choices of d.o.f. In the upper left sub-figure the red curve which is the approximated derivative of the approximated optimal control \( \overline{u}_{\text{approx}}. \) The blue curve represents an approximation to the derivative of \( \overline{u} \) by finite differences.

The BV-path following algorithm starts with \( \gamma_0 = 1, (v_0, c_0) = (0, 0), \) and we iterate according to \( \gamma_k+1 = 0.5 \gamma_k. \) We stopped the BV-path following algorithm when \( \gamma_k = 3.8 \cdot 10^{-6} \) was reached. The function \( \kappa \) is defined by \( \kappa(\gamma) = 0. \) We used \( \| F_\gamma(u_k) \|_{L^2(I)^m \times \mathbb{R}^m} \leq 0.5 \cdot 10^{-4} \) as the stopping criterion for the semi-smooth Newton algorithm.

**Figure 5.**

6. **Remarks on different cost functionals and boundary conditions.** The results presented in the previous sections can be extended in various directions. In particular the results presented in [13] include homogenous boundary conditions in \( (W) \) of the types:

\[ \frac{\partial y}{\partial \nu} = \phi_D \quad \text{on} \quad (0, T) \times \partial \Omega, \]

\[ a_N \frac{\partial y}{\partial \nu} + a_D y = \phi_N \quad \text{on} \quad (0, T) \times \partial \Omega, \]
with \( a_D \in \mathbb{R} \), and \( a_N \neq 0 \). More detailed discussion can be found in [13].

The work in [13] also includes more general cost functionals which in particular also involve velocity \( y_t \):

\[
(P^{II}) \quad \min_{u \in BV(0,T)^m} \begin{cases} \frac{\partial}{\partial t} u = v, \\ \Pi_1(y_u) - z_1 \big|_{\partial_1\Omega} + \frac{\beta_1}{2} \| \Pi_2(y_u(\overrightarrow{t})) - z_2 \|_{\partial_2\Omega}^2 \\ + \frac{\beta_2}{2} \| \Pi_3(\partial_t y_u(\overrightarrow{t})) - z_3 \|_{\partial_3\Omega}^2 + \sum_{j=1}^m \alpha_j \| D_j u_j \|_{M(I)} \end{cases} =: J^{II}(y,u)
\]

where \( y_u \) is again defined by \((W)\). The scalar values \( \overrightarrow{t} := (\overrightarrow{t}_1, \ldots, \overrightarrow{t}_r) \) represent measurement time points with \( 0 < \overrightarrow{t}_1 < \cdots < \overrightarrow{t}_r = T \), and \( r \) finite. Furthermore it is assumed that at least one \( \beta_1 > 0 \), that \( \partial_i \), \( i = 1, 2, 3 \), are separable Hilbert spaces, and that

\[
\Pi_1 : L^2(\Omega_T) \to \Omega_1, \quad \Pi_2 : L^2(\Omega)^r \to \Omega_2, \quad \Pi_3 : H^{-1}(\Omega)^r \to \Omega_3
\]

are linear, continuous operators with adjoints \( \Pi_1^*, \Pi_2^*, \text{ and } \Pi_3^* \). Let us denote by \( y_{g_j} \) the weak solution of \((W)\) with forcing \( g_j \), \( j = 1, \ldots, m \), and \( (y_0, y_1) = (0,0) \).

In [13] it is assumed that \((\Pi_1(y_{g_j}))_{j=1}^m\), \((\Pi_2(y_{g_j}(\overrightarrow{t})))_{j=1}^m\), and \((\Pi_3(\partial_t y_{g_j}(\overrightarrow{t})))_{j=1}^m\) are linear independent in \( \Omega_1 \), \( \Omega_2 \), respectively \( \Omega_3 \), for at least one of these sets of vectors where \( \beta_i > 0 \). Then the existence of solutions for \((P^{II})\) is shown in [13]. For additional technical assumptions we refer to [13, Section 2.4.1]. As in the case of distributed observations of \( y_u \) treated in the earlier sections, we again introduce a transformed problem in terms measures in \( M(0,T) \times \mathbb{R} \) instead of \( BV\)-functions in time, and thus arrive at an analogue of \((P)\) which we call \((\bar{P}^{II})\). Due to the more complex costs and observation operators in \((\bar{P}^{II})\), we present next the optimality conditions of \((\bar{P}^{II})\), c.f. [13, Theorem 2.10]. In this regard, let us first introduce the following time-space depending functions \( p_i \) for \( i = 1, \ldots, \overline{r} \) with \( \overline{r} \in \mathbb{N}_{>0} \), c.f. [13, Lemma 2.8].
Lemma 6.1. Consider the following system of wave equations:

\[
\begin{align*}
\rho \partial_t p_i - \Delta p_i &= g_i & \text{in } (\bar{t}_{i-1}, \bar{t}_i) \times \Omega := I_i \times \Omega, \\
p_i|_{\partial \Omega} &= 0 & \text{in } I_i, \\
p_i(\bar{t}_i) &= p_{i+1}(\bar{t}_i) + h_i & \text{in } \Omega, \\
\partial_t p_i(\bar{t}_i) &= \partial_t p_{i+1}(\bar{t}_i) + t_i & \text{in } \Omega,
\end{align*}
\]

with \( \bar{\tau} \in \mathbb{N}_{>0}, i = 1, \ldots, \bar{\tau} + 1, 0 < \bar{t}_1 < \cdots < \bar{t}_\bar{\tau} < T, \bar{t}_{\bar{\tau}+1} = T, \bar{t}_0 = 0, h_i \in H_0^1(\Omega), t_i \in L^2(\Omega), p_{\bar{\tau}+2}(T) := 0, \partial_t p_{\bar{\tau}+2}(T) := 0, \) and \( g_i \in L^2(I_i; L^2(\Omega)). \) Then the weak solution \( (p_i, t_i) \) are defined as in Definition 2.3 on the time-space cylinder \((0, \bar{t}_i - \bar{t}_{i-1}) \times \Omega.\)

Theorem 6.2. Define \( \hat{p}(\nu, c) := \sum_{i=1}^{\bar{\tau}} p_i(\nu, c) 1_{(\bar{t}_{i-1}, \bar{t}_i)} \) where the functions \( p_i(\nu, c) \) are the weak solutions of (44) with respect to the following parameters: Let \( p_{\bar{\tau}+1}(\nu, c)(\bar{t}_r) := 0 \) and \( \partial_t p_{\bar{\tau}+1}(\nu, c)(\bar{t}_r) := 0. \) Furthermore, for \( i = 1, \ldots, \bar{\tau} \), we set

\[
\begin{align*}
g_i := \beta_3(-\Delta)^{-1} \left[ \Pi_2(\Pi_3(\partial_t S\nu(\nu, c)(\bar{\nu})) - z_3) \right]_i & \in H_0^1(\Omega) \\
h_i := \beta_3(-\Delta)^{-1} \left[ \Pi_2(\Pi_3(\partial_t S\nu(\nu, c)(\bar{\nu})) - z_3) \right]_i \in H_0^1(\Omega)
\end{align*}
\]

with \( \bar{\nu} = (\bar{\nu}_1, \ldots, \bar{\nu}_\bar{\tau})^T. \) Thus, we have that \( (\bar{\nu}^H, \bar{\nu}^H) \in M(I)^m \times \mathbb{R}^m \) is an optimal control of \( (\hat{P}^H) \), if and only if

\[
- \frac{\partial \hat{p}^H(s)}{\partial \hat{p}^H(0)} := - \left( \int_{\Omega_T} \hat{p}(\bar{\nu}^H, \bar{\nu}^H) \nu \, dx \, dt \right) \in \left( \alpha, \partial \| \bar{\nu}^H \|_{M(I)} \right)_{m=1}^m, \quad (45)
\]

where \( (\hat{p}^H, \partial \hat{p}^H) \in (C_0(\bar{T}) \cap W^{1,\infty}(I))^m \times BV(I)^m. \)

Compared to \( p_1 \) in (6), we see that due to the possible discontinuities of \( \hat{p}(\bar{\nu}^H, \bar{\nu}^H) \) the solutions \( \hat{p}^H \) have lower regularity properties. Based on (45), similar results as for \( \hat{P} \), were shown for problem \( (\hat{P}^H) \), c.f. [13, Section 2.5-2.8] and Section 3 of this paper. For the numerical realization a semi-smooth Newton based on the same \( L^2 \)-regularization of \( (\hat{P}^H) \) as used in \( (\hat{P}^H) \) respectively \( (\hat{P}_\gamma) \) was employed. Let us call this regularized problem \( (\hat{P}^H) \) with focus on measure controls \( M(I)^m \times \mathbb{R}^m. \) These regularized problems are analysed [13, Section 3.1.1-3] to obtain results analogous to those in Sections 4.1 - 4.3. Due to lower regularity properties of \( \hat{p}^H \), the results of Proposition 2 holds in a more restrictive way for \( (\hat{P}^H) \), see [13, Corollary 3.5]. To solve \( (\hat{P}^H) \) by a semi-smooth Newton approach a prox operator ansatz is used and results as in Lemma 4.10 are obtained.

Appendix. The following proof of Theorem 3.2 is adapted from the proof of [9, Theorem 3.3].
Proof. Let us define the linear continuous operators

\[ P_i : M(I)^m \rightarrow M(I) \quad v \mapsto v_i, \quad \mathbb{D} : M(I)^m \times \mathbb{R}^m \rightarrow M(I)^m \quad (v, c) \mapsto v \]

for \( i = 1, \ldots, m \). By the convexity of \((\bar{P})\) we have that \( \bar{u} = (\bar{v}, \bar{c}) \in M(I)^m \times \mathbb{R}^m \) is an optimal control of \((\bar{P})\) if

\[
0 \in \partial \left( \frac{1}{2} \| S(\bar{u}) - y_d \|_{L^2(\Gamma_T)}^2 + \sum_{j=1}^{m} \alpha_j \| v_j \|_{M(I)} \right) \subseteq (M(I)^m \times \mathbb{R}^m)^*. \quad (46)
\]

Defining \( F(\bar{u}) := \frac{1}{2} \| S(\bar{u}) - y_d \|_{L^2(\Gamma_T)}^2 \) we have for \( 0 < \hat{\tau} < 1 \), and \( u = (v, c) \in M(I)^m \times \mathbb{R}^m \):

\[
0 \leq \frac{J(\hat{\tau}u + \hat{\tau}(u - \bar{u})) - J(\bar{u})}{\hat{\tau}} = \frac{F(\hat{\tau}u + \hat{\tau}(u - \bar{u})) - F(\bar{u})}{\hat{\tau}} + \sum_{j=1}^{m} \alpha_j \| P_j(\bar{v} + \hat{\tau}(v - \bar{v})) \|_{M(I)} - \| P_j(\bar{v}) \|_{M(I)} \frac{\hat{\tau}}{\hat{\tau}} \leq \frac{F(\hat{\tau}u + \hat{\tau}(u - \bar{u})) - F(\bar{u})}{\hat{\tau}} + \sum_{j=1}^{m} \frac{\alpha_j (1 - \hat{\tau}) \| P_j(\bar{v}) \|_{M(I)} + \hat{\tau} \| P_j(v) \|_{M(I)} - \| P_j(\bar{v}) \|_{M(I)}}{\hat{\tau}} \]

\[
\overset{\hat{\tau} \to 0}{\longrightarrow} DF_{\bar{u}}(u - \bar{u}) + \sum_{j=1}^{m} \alpha_j \| P_j \circ \mathbb{D}(u) \|_{M(I)} - \sum_{j=1}^{m} \alpha_j \| P_j \circ \mathbb{D}(\bar{u}) \|_{M(I)} \]

with \( DF_{\bar{u}} \) the Gateaux derivative of \( F \) in \( \bar{u} \). It has the following form:

\[
DF_{\bar{u}} := \left( \begin{array}{c} p_1(s) \\ p_1(0) \end{array} \right) := \left( \begin{array}{c} \int_{\Omega_T} L^* \left( S(\bar{u}) - y_d \right) \overline{g} \, dx \, dt \\ \int_{\Omega_T} L^* \left( S(\bar{u}) - y_d \right) \overline{g} \, dx \, dt \end{array} \right) \quad (48)
\]

Hence, \((47)\) implies that

\[
0 \in DF_{\bar{u}} + \partial \left( \sum_{j=1}^{m} \alpha_j \| P_j(\bar{v}) \|_{M(I)} \right) . \quad (49)
\]

Using standard techniques for the sub differential of a convex functional and \((49)\), this implies that:

\[
\left( p_1(s) \right) \\
\left( p_1(0) \right) \in \left( \begin{array}{c} \alpha_i \| v_i \|_{M(I)} \end{array} \right)_{i=1}^{m} \quad (50)
\]

which is equivalent to the following: For all \( i = 1, \ldots, m \) and \( v \in M(I) \) it holds that\( \langle v - v_i, -p_i, i \rangle_{M(I), M(I)} \leq \alpha_i \| v \|_{M(I)} - \alpha_i \| v_i \|_{M(I)} \) and we have \( p_1(0) = 0_{\mathbb{R}^m} \). \( \square \)

Note that the regularity of \( L^*(S(\bar{u}) - y_d) \in C(\bar{T}; L^2(\Omega)) \), \( p_1(0) = 0_{\mathbb{R}^m} \), and the definition of \( p_1 \) imply that \( p_1 \in C_0(I) \) and thus \( DF_{\bar{u}}(u - \bar{u}) \) is an element of \((M(I)^m \times \mathbb{R}^m)^*\).
In the following we prove Lemma 4.6:

Proof. Firstly, let us present the optimality conditions of \((\tilde{P}_\gamma)\): The control \((\tilde{v}, \tilde{c}) \in L^2(I)^m \times \mathbb{R}^m\) is optimal for \((\tilde{P}_\gamma)\) if

\[
\begin{pmatrix}
- \int_\Omega L^* (S(\tilde{v}, \tilde{c}) - y_d) \tilde{g} dtdx - \gamma \tilde{v} \\
- \int_\Omega L^* (S(\tilde{v}, \tilde{c}) - y_d) \tilde{g} dtdx - \kappa(\gamma) \tilde{c}
\end{pmatrix}
\in \left( \begin{pmatrix} (\alpha_i \partial (\|\tilde{v}\|_{L^1(I)})_{i=1}^m \end{pmatrix} \right)_{0 \in \mathbb{R}^m}.
\]

Consider now \(\tilde{u} := (\tilde{v}, \tilde{c}) \in L^2(I)^m \times \mathbb{R}^m\) and the following function

\[
\text{Prox}_{\gamma}^2 \frac{1}{2} \alpha_i \|\cdot\|_{L^1(I)} (\tilde{p}) = \arg \min_{v \in L^2(I)^m} \left( \sum_{i=1}^m \alpha_i \|v_i\|_{L^1(I)} + \frac{\gamma}{2} \sum_{i=1}^m \|v_i - p_i\|^2_{L^2(I)} \right)
\]

for \(\tilde{p} \in L^2(I)^m\), which we also call the Prox problem. Our aim is to calculate the first-order optimality conditions for this problem, and to find an explicit representation for \(\gamma(\tilde{v}, \tilde{c}) := \text{Prox}_{\gamma}^2 \frac{1}{2} \alpha_i \|\cdot\|_{L^1(I)} (\tilde{p})\). For the sub-differential of the non-smooth term we obtain \(\partial \left( \sum_{i=1}^m \alpha_i \|P_i(\cdot)\|_{L^1(I)} \right) = \sum_{i=1}^m \alpha_i P_i^* \partial \|P_i(\cdot)\|_{L^1(I)} \subseteq L^2(I)^m\) with the domain \(L^2(I)\) for the function \(\alpha_i \|\cdot\|_{L^1(I)}\). Thus, we have the following first-order optimality conditions for \(\text{Prox}_{\gamma}^2 \frac{1}{2} \alpha_i \|\cdot\|_{L^1(I)} (\tilde{p})\): \(\gamma(\tilde{v}, \tilde{c}) \in \partial \|v_i\|_{L^1(I)}\) for all \(i = 1, \ldots, m\), and \(\tilde{p} \in L^2(I)^m\).

Next, let us show that our optimal control can be explicitly written as

\[
\tilde{v} = \left( \max(0, p_i - \frac{\gamma}{\alpha_i}) + \min(0, p_i + \frac{\gamma}{\alpha_i}) \right)_{i=1}^m \subseteq L^2(I)^m
\]

We can proceed coordinate wise. By the definition of the sub-differential we have the equivalent inequality condition \(\langle \tilde{v}, v - \tau \rangle_{L^2(I)} \leq \|v\|_{L^1(I)} - \|\tau\|_{L^1(I)}\) for all \(v \in L^2(I)\). By a standard Lebesgue point argument \(\tau\) is an optimal control if and only if

\[
(v - \tau)_{\alpha_i} (p - \tau) \leq |v| - |\tau|\) holds for all \(v \in L^2(I)\) a.e. in \(I\)

This can also be expressed as

\[
\tilde{v}(x) = \begin{cases}
0, & |p(x)| < \frac{\gamma}{\alpha_i} \\
p(x) + \frac{\gamma}{\alpha_i}, & p(x) \leq -\frac{\gamma}{\alpha_i} \\
p(x) - \frac{\gamma}{\alpha_i}, & p(x) \geq \frac{\gamma}{\alpha_i}.
\end{cases}
\]

or equivalently as \(\tilde{v}(x) = \max \left( 0, p(x) - \frac{\gamma}{\alpha_i} \right) + \min \left( 0, p(x) + \frac{\gamma}{\alpha_i} \right)\). Now, we can state the following equivalent first-order optimality conditions for the Prox-problem: For \(\tilde{p} \in L^2(I)^m\), \(\tilde{v} \in L^2(I)^m\) is optimal if and only if

\[
\frac{\gamma}{\alpha_i} (p_i - v_i) \in \partial \|v_i\|_{L^1(I)}\) for all \(i = 1, \ldots, m\)

\[
\Leftrightarrow \tilde{v} = \left( \max(0, p_i - \frac{\gamma}{\alpha_i}) + \min(0, p_i + \frac{\gamma}{\alpha_i}) \right)_{i=1}^m \subseteq L^2(I)^m.
\]
Recall that \( \overline{u} = (\overline{v}, \overline{c}) \in L^2(I)^m \times \mathbb{R}^m \) is the optimal control for \((\tilde{P}_\gamma)\) if and only if for all \( i = 1, \ldots, m \) we have
\[
-\gamma v_i - \int_{\Omega} L^\ast(Su - y_d)g_i dtdx \in \alpha_i \partial \|v_i\|_{L^1(I)} \\
- \int_{\Omega} L^\ast(Su - y_d)\nabla g_i dtdx - \kappa(\gamma) c_i = 0_{\mathbb{R}^m}.
\]
(54)
(55)
Returning to \((\tilde{P}_\gamma)\), define \( \overline{p} \) as
\[
\overline{p} := \overline{v} - \frac{1}{\gamma} \left( \int_{s}^{T} L^\ast(S\overline{u} - y_d)g^\ast dtdx + \gamma \overline{v} \right) = -\frac{1}{\gamma} \int_{\Omega} L^\ast(S\overline{u} - y_d)g^\ast dtdx
\]
which implies the equation:
\[
\gamma (\overline{p} - \overline{v}) = -\gamma \overline{v} - \int_{\Omega} L^\ast(S\overline{u} - y_d)g^\ast dtdx = \text{ first line of the left hand side in (51)}.
\]
Thus, the first-order optimality system of the Prox problem is equal to the first line of the first-order optimality system of \((\tilde{P}_\gamma)\) in (51). Note that we have for \( \overline{v} \)
\[
\overline{v} = \text{Prox}_{\sum \alpha_i \cdot \| \cdot \|_{L^1(I)}} \left( \overline{p} \right) = \text{Prox}_{\sum \alpha_i \cdot \| \cdot \|_{L^1(I)}} \left( -\frac{1}{\gamma} \int_{s}^{T} L^\ast(S\overline{u} - y_d)g^\ast dtdx \right).
\]
Now, we can use the min-max formula in (53) for \( \text{Prox}_{\sum \alpha_i \cdot \| \cdot \|_{L^1(I)}} \left( \overline{p} \right) \) to rewrite (54). This implies, the equivalent first-order optimality system we had to prove. \( \square \)

Acknowledgments. We thank Philip Trautmann for the support he gave by his helpful hints and discussions.

All authors gratefully acknowledge support from the International Research Training Group IGDK 1754 “Optimization and Numerical Analysis for Partial Differential Equations with Nonsmooth Structures”, funded by the Austrian Science Fund (FWF) and the German Research Foundation (DFG): [W 1244-N18]. The second author also acknowledges partial support from the ERC advanced grant 668998 (OCLOC) under the EU H2020 research program.

REFERENCES

[1] L. Ambrosio, N. Fusco and D. Pallara, Functions of Bounded Variation and Free Discontinuity Problems, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 2000.

[2] E. Casas, A review on sparse solutions in optimal control of partial differential equations, *SEMA Journal*, **74** (2017), 319–344.

[3] E. Casas and K. Kunisch, Optimal control of the two-dimensional stationary Navier-Stokes equations with measure valued controls, *SIAM Journal on Control and Optimization*, **57** (2019), 1328–1354.

[4] E. Casas and K. Kunisch, Parabolic control problems in space-time measure space, *ESAIM Control Optim. Calc. Var.*, **22** (2016), 355–370.

[5] E. Casas and E. Zuazua, Spike controls for elliptic and parabolic PDEs, *Systems Control Lett.*, **62** (2013), 311–318.

[6] E. Casas, C. Clason and K. Kunisch, Approximation of elliptic control problems in measure spaces with sparse solutions, *SIAM J. Control Optimization*, **50** (2012), 1735–1752.

[7] E. Casas, C. Clason and K. Kunisch, Parabolic control problems in measure spaces with sparse solutions, *SIAM Journal Control Optimization*, **51** (2013), 28–63.

[8] E. Casas, B. Vexler and E. Zuazua, Sparse initial data identification for parabolic PDE and its finite element approximations, *Math. Control Relat. Fields*, **5** (2015), 377–399.
[9] E. Casas, F. Kruse and K. Kunisch, Optimal control of semilinear parabolic equations by BV-functions, SIAM Journal on Control and Optimization, 55 (2017), 1752–1788.

[10] A. Chambolle, V. Caselles, M. Novaga, D. Cremers and T. Pock, An introduction to total variation for image analysis, Theoretical Foundations and Numerical Methods for Sparse Recovery, Radon Ser. Comput. Appl. Math., Walter de Gruyter, Berlin, 9 (2009), 263–340.

[11] C. Clason and K. Kunisch, A duality-based approach to elliptic control problems in non-reflexive Banach spaces, ESIAM Control Optim. Calc. Var., 17 (2011), 243–266.

[12] C. Clason and K. Kunisch, A measure space approach to optimal source placement, Comput. Optim. Appl., 53 (2012), 155–171.

[13] S. Engel, Optimal Control and Bayesian Inversion for Linear Second-Order Hyperbolic Equations by BV Functions in Time, PhD thesis, Karl-Franzens-Universität Graz, 2018.

[14] M. Gugat, A. Keimer and G. Leugering, Optimal distributed control of the wave equation subject to state constraints, Z. angew. Math. Mech., 89 (2009), 420–444.

[15] D. Hafemeyer, Optimal Control of Differential Equations Using BV-Functions, thesis, Technical University of Munich, 2015.

[16] R. Herzog, G. Stadler and G. Wachsmuth, Directional sparsity in optimal control of partial differential equations, SIAM J. Control Optim., 50 (2012), 943–963.

[17] M. Hintermüller, K. Ito and K. Kunisch, The primal-dual active set strategy as a semismooth Newton method, SIAM J. Control Optim., 13 (2002), 865–888.

[18] M. Hinze, R. Pinnau, M. Ulbrich and S. Ulbrich, Optimization with PDE Constraints, Mathematical Modelling: Theory and Applications, 23. Springer, New York, 2009.

[19] K. Kunisch, K. Pieper and B. Vexler, Measure valued directional sparsity for parabolic optimal control problems, SIAM J. Control Optim., 52 (2014), 3078–3108.

[20] K. Kunisch, P. Trautmann and B. Vexler, Optimal control of the undamped linear wave equation with measure valued controls, SIAM J. Control Optim., 54 (2016), 1212–1244.

[21] O. A. Ladyzhenskaya, Boundary value problems of mathematical physics, Nauka, Moscow, (1973), 407 pp.

[22] I. Lasiecka, Control theory for partial differential equations volume 2: Abstract hyperbolic-like systems over a finite time horizon, Cambridge Univ. Press, (2000), 645–1067.

[23] I. Lasiecka and R. Triggiani, Regularity theory of hyperbolic equations with non-homogeneous Neumann boundary conditions. II. General boundary data, Journal of Differential Equations, 94 (1991), 112–164.

[24] I. Lasiecka, J.-L. Lions and R. Triggiani, Nonhomogeneous boundary value problems for second order hyperbolic operators, Journal de Mathématiques Pures et Appliquées, 65 (1986), 149–192.

[25] J.-L. Lions, Optimal Control of Systems Governed by Partial Differential Equations, Die Grundlehren der mathematischen Wissenschaften, Band 170 Springer-Verlag, New York-Berlin, 1971.

[26] K. Pieper, Finite Element Discretization and Efficient Numerical Solution of Elliptic and Parabolic Sparse Control Problems, PhD thesis, Technical University Munich, 2015.

[27] M. Reed and S. Barry, Functional Analysis. I, Academic Press, Inc., 1980.

[28] A. A. Zlotnik, Convergence Rate Estimates of Finite-Element Methods for Second-Order Hyperbolic Equations. Numerical Methods and Applications, Guri I. Marchuk, CRC Press, 1994.

Received October 2018; 1st revision July 2019; 2nd revision September 2019.

E-mail address: sebastian.engel@uni-graz.at
E-mail address: karl.kunisch@uni-graz.at