Proper incorporation of self-adjoint extension method to Green’s function formalism: one-dimensional $\delta'$-function potential case

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Abstract

One-dimensional $\delta'$-function potential is discussed in the framework of Green’s function formalism without invoking perturbation expansion. It is shown that the energy-dependent Green’s function for this case is crucially dependent on the boundary conditions which are provided by self-adjoint extension method. The most general Green’s function which contains four real self-adjoint extension parameters is constructed. Also the relation between the bare coupling constant and self-adjoint extension parameter is derived.
1 Introduction

Since Kronig-Penny model[1] has been successful for the description of energy band in solid state physics, the point interaction problem has been applied in the various branches of physics for a long time. Recently the two-dimensional $\delta$-function potential has been of interests in the context of the Aharonov-Bohm(AB) effect of spin-1/2 particles[2, 3] in which the delta function occurs as the mathematical description of the Zeeman interaction of the spin with a magnetic flux tube. In Ref.[4] two different approaches, renormalization and self-adjoint extension methods[5, 6], are presented for this subject. More recently same problem is re-examined in the framework of Green’s function formalism[7, 8]. In Ref.[8] present author showed how to incorporate the self-adjoint extension method within the Green’s function formalism without invoking the perturbation expansion.

Unlike two- and three-dimensional cases, one-dimensional point interaction provides a four-parameter family solution, characterized by the boundary conditions at $x = 0$ :

$$
\varphi(\epsilon) = \omega a \varphi(-\epsilon) + \omega b \varphi'(-\epsilon), \quad (1)
$$

$$
\varphi'(\epsilon) = \omega c \varphi(-\epsilon) + \omega d \varphi'(-\epsilon),
$$

where $\epsilon$ is infinitesimal positive parameter and $\omega \in \mathbb{C}$; $a, b, c, d \in \mathbb{R}$, satisfying $|\omega| = 1$ and $ad - bc = 1[6, 9]$. Recently path-integral for the one-dimensional $\delta'$-function potential is calculated by incorporating Neumann boundary conditions within the usual perturbation theory of one-dimensional Dirac particle in order for the coupling constant to be infinitely repulsive[10].

In this paper we will discuss the one-dimensional $\delta'$-function potential in
the framework of Green’s function formalism without using a perturbation expansion like Ref.[8]. It will be shown that the energy-dependent Green’s function is crucially dependent on the boundary conditions which are provided by self-adjoint extension method in the present formalism. Choosing the boundary condition

\[ \varphi^\prime(\epsilon) = \varphi^\prime(-\epsilon) = \varphi^\prime(0), \]
\[ \varphi(\epsilon) - \varphi(-\epsilon) = \beta \varphi^\prime(0), \]

which is easily obtained from Eq.(1) by requiring \( c = 0, \omega = a = d = 1, \) and \( b = \beta, \) one can derive a similar result with that of Ref.[10].

However, the advantage of this formalism presented here is that it is free to choose boundary conditions. This means that one can get more general Green’s function by choosing more general boundary conditions. If one chooses the most general boundary conditions (1) of one-dimensional point interaction, the most general Green’s function, in which four real self-adjoint extension parameters are contained, can be derived. It is worthwhile to note that this formalism does not use the complicated perturbation expansion. Therefore, calculation is very simple and clear.

This paper is organized as follows. In Sec.2 we will discuss why the derivation of Green’s function for one-dimensional \( \delta^\prime \)-function potential case is difficult. In this section we will show that the representation of \( \delta^\prime \)-function as two usual \( \delta \)-function with infinitesimal distance does generate the physically irrelevant Green’s function. In Sec.3 we will derive energy-dependent Green’s function by incorporating the self-adjoint method in the Green’s function formalism. In this section we will show that the most general Green’s function for one-dimensional point interaction can be constructed
by using only $\delta'$-function potential. In Sec.4 a brief conclusion is given.

2 Calculational Difficulty of Green’s function for one-dimensional $\delta'$-function potential

In this section we will calculate the energy-dependent Brownian motion Green’s function $\hat{G}[x, y; E]$ for one-dimensional $\delta'$-function potential by using a same method presented in Ref.[11], in which the one-dimensional $\delta$-function potential case is calculated. For this purpose consider a one-dimensional system whose Hamiltonian is

$$H = H_0 + v\delta'(x),$$

where $v$ is bare coupling constant. Although $H_0$ can involve an arbitrary potential, in this paper we will only consider the free particle case for simplicity:

$$H_0 = \frac{p^2}{2}.$$  \(4\)

It is well-known that the time-dependent Brownian motion propagator for the Hamiltonian (3) obeys integral equation[12, 13]

$$G[x, y; t] = G_0[x, y; t] - v \int_0^t ds \int dz G_0[x, z; t - s] \delta'(z) G[z, y; s].$$  \(5\)

In order to follow the same method used in Ref.[11] for $\delta$-function potential case, we regard $\delta'$-function as

$$\delta'(z) = \lim_{\epsilon \to 0^+} \frac{\delta(z + \epsilon) - \delta(z - \epsilon)}{2\epsilon}. \quad (6)$$

After inserting Eq.(6) into (5), one can derive easily

$$\hat{G}[x, y; E] = \hat{G}_0[x, y; E]$$

$$+ \frac{v}{2\epsilon} \left[ \hat{G}_0[x, \epsilon; E] \hat{G}[\epsilon, y; E] - \hat{G}_0[x, -\epsilon; E] \hat{G}[-\epsilon, y; E] \right]$$

5
by taking Laplace transform
\[ \hat{f}(E) \equiv \int_0^\infty dt e^{-Et} f(t) \] (8)
of both sides of Eq.(5). At this stage the limit of \( \epsilon \) is omitted for brevity.
We will take this limit after calculation. After inserting \( x = \pm \epsilon \) in Eq.(7),
one can obtain with much ease
\[
\hat{G}[\epsilon, y; E] = \left(1 + \frac{v}{2\sqrt{2E}}\right) \hat{G}_0[\epsilon, y; E] - \frac{ve^{-\sqrt{2E}\epsilon}}{2\sqrt{2E}} \hat{G}_0[-\epsilon, y; E] \\
1 - \frac{v^2}{8E\epsilon^2} \left(1 - e^{-\sqrt{2E}\epsilon}\right)
\]
(9)
When deriving Eq.(9) we used the explicit result of energy-dependent Green’s function for one-dimensional free particle
\[ \hat{G}_0[x, y; E] = \frac{1}{\sqrt{2E}} e^{-\sqrt{2E}|x-y|}. \] (10)
By inserting Eq.(9) into Eq.(7) \( \hat{G}[x, y; E] \) becomes
\[
\hat{G}[x, y; E] = \hat{G}_0[x, y; E] + \frac{v}{4\pi\epsilon} \frac{1}{1 - \frac{v^2}{8E\epsilon^2} \left(1 - e^{-\sqrt{2E}\epsilon}\right)}
\times \left[ \left(1 + \frac{v}{2\sqrt{2E}}\right) e^{-\sqrt{2E}(|x-\epsilon|+|y-\epsilon|)} - \frac{ve^{-\sqrt{2E}2\epsilon}}{2\sqrt{2E}} e^{-\sqrt{2E}(|x-\epsilon|+|y+\epsilon|)} \right.

\left. - \frac{ve^{-\sqrt{2E}2\epsilon}}{2\sqrt{2E}} e^{-\sqrt{2E}(|x+\epsilon|+|y-\epsilon|)} - \left(1 - \frac{v}{2\epsilon \sqrt{2E}}\right) e^{-\sqrt{2E}(|x+\epsilon|+|y+\epsilon|)} \right].
\] (11)
Now let us assume that the coupling constant \( v \) is \( \epsilon \)-independent and finite.
If one calculates \( \hat{G}[x, y; E] \) by taking \( \epsilon \to 0^+ \) limit in Eq.(11) at the following four regions
\[ x > \epsilon \quad y > \epsilon \] (12)
\[ x > \epsilon \quad y < -\epsilon, \]
\[ x < -\epsilon \quad y > \epsilon, \]
\[ x < -\epsilon \quad y < -\epsilon, \]

one can show that the infinity terms \( O(\epsilon^{-1}) \) cancel, leaving the same finite results at these four regions

\[
\hat{G}[x, y; E] = \hat{G}_0[x, y; E] - \frac{1}{\sqrt{2E}} e^{-\sqrt{2E}|x|+|y|}. \tag{13}
\]

At this stage the regions \(-\epsilon < x < \epsilon \) and \(-\epsilon < y < \epsilon \) are not considered since both shrink infinitesimally if \( \epsilon \to 0^+ \) limit is taken.

The result (13) is physically irrelevant since it is independent of coupling constant \( v \). There is another reason which makes Eq.(13) physically unacceptable. From Ref.[11] the energy-dependent Brownian motion Green’s function for one-dimensional \( \delta \)-function potential case is

\[
\hat{G}_\delta[x, y; E] = \hat{G}_0[x, y; E] - \frac{c}{\sqrt{2E}(\sqrt{2E} + c)} e^{-\sqrt{2E}|x|+|y|} \tag{14}
\]

where \( c \) is coupling constant of \( \delta \)-function potential. Then one can easily show

\[
\hat{G}[x, y; E] = \lim_{c \to \infty} \hat{G}_\delta[x, y; E] \tag{15}
\]

which results in the physically irrelevant deduction

\[
v\delta'(x) \overset{?}{=} \lim_{c \to \infty} c \delta(x)
\]

at quantum level. Thus the description of \( \delta' \)-function as two usual \( \delta \)-function with infinitesimal distance does not make sense physically if \( v \) is \( \epsilon \)-independent and finite. Maybe some relations between \( v \) and \( \epsilon \) can give physically relevant solutions. Upon my knowledge it is not clear how to derive the relations systematically.
In next section we will present the correct procedure for the calculation of energy-dependent Green’s function when the potential is one-dimensional \( \delta' \)-function.

3 Green’s function approach to one-dimensional \( \delta' \)-function potential

In this section we will show how to incorporate the self-adjoint extension method into Green’s function formalism by using one-dimensional \( \delta' \)-function potential without invoking a perturbation expansion. The two- and three-dimensional cases are already discussed in Ref.[8].

Now let us start with Eq.(5). After performing integration with respect to \( z \) in Eq.(5), one can show easily

\[
\hat{G}[x, y; E] = \hat{G}_0[x, y; E] + v \left( \frac{\partial \hat{G}_0[x, z; E]}{\partial z} \right)_{z=0} \hat{G}_0[0, y; E] + v \hat{G}_0[x, 0; E] \left( \frac{\partial \hat{G}[z, y; E]}{\partial z} \right)_{z=0}.
\]

Eq.(16) is purely formal. This is easily deduced from fact that \( \hat{G}[0, y; E] \) is not well-defined because of the factor \( |x| \) which is contained in \( (\partial \hat{G}_0[x, z; E]/\partial z)_{z=0} \). Therefore, at this stage one has to conjecture the modification of Eq.(16).

Our conjecture for the modification of Eq.(16) is simply to extract the problematic zero point at \( \hat{G}[x, y; E] \) as follows:

\[
\hat{G}[x, y; E] = \hat{G}_0[x, y; E] + v \left( \frac{\partial \hat{G}_0[x, z; E]}{\partial z} \right)_{z=0} \hat{G}[\epsilon, y; E] \quad \text{for} \quad x > 0
\]
\[
\hat{G}[x, y; E] = \hat{G}_0[x, y; E] + v \left( \frac{\partial \hat{G}_0[x, z; E]}{\partial z} \right)_{z=0} \hat{G}[-\epsilon, y; E] \quad \text{for } x < 0
\]
\[
+ v \hat{G}_0[x, 0; E] \left( \frac{\partial \hat{G}[z, y; E]}{\partial z} \right)_{z=-\epsilon}
\]

which might be a natural modification of Eq.(16).

In Eq.(17) the infinitesimal positive parameter \( \epsilon \) is introduced again. By inserting \( x = \pm \epsilon \) in the first and second equations of Eq.(17) respectively, one can derive

\[
\left( \frac{\partial \hat{G}[z, y; E]}{\partial z} \right)_{z=\epsilon} = \frac{\sqrt{2E}}{v} \left[ (1 - v)\hat{G}[\epsilon, y; E] - \hat{G}_0[0, y; E] \right] \quad (18)
\]
\[
\left( \frac{\partial \hat{G}[z, y; E]}{\partial z} \right)_{z=-\epsilon} = \frac{\sqrt{2E}}{v} \left[ (1 + v)\hat{G}[-\epsilon, y; E] - \hat{G}_0[0, y; E] \right].
\]

By inserting Eq.(18) into Eq.(17) \( \hat{G}[x, y; E] \) becomes

\[
\hat{G}[x, y; E] = \hat{G}_0[x, y; E] - \frac{1}{\sqrt{2E}} e^{-\sqrt{2E}|x|+|y|} \hat{G}[\epsilon, y; E] \quad \text{for } x > 0
\]
\[
\hat{G}[x, y; E] = \hat{G}_0[x, y; E] - \frac{1}{\sqrt{2E}} e^{-\sqrt{2E}|x|+|y|} \hat{G}[-\epsilon, y; E] \quad \text{for } x < 0.
\]

Note that in Eq.(19) the \( v \)-dependence of \( \hat{G}[x, y; E] \) is hidden in \( \hat{G}[\pm \epsilon, y; E] \).

Now it is time to incorporate the self-adjoint extension method into Green’s function formalism. Firstly let us consider the simple boundary conditions given in Eq.(2). By applying these two boundary conditions to
\( \hat{G}[x, y; E] \), one can show that the boundary conditions generate two independent equations

\[
\hat{G}[\epsilon, y; E] + \hat{G}[-\epsilon, y; E] = \frac{2}{\sqrt{2E}} e^{-\sqrt{2E}|y|} \tag{20}
\]

\[
\hat{G}[\epsilon, y; E] - \hat{G}[-\epsilon, y; E] = \beta \left[(\epsilon(y) + 1)e^{-\sqrt{2E}|y|} - \sqrt{2E}\hat{G}[\epsilon, y; E]\right]
\]

where \( \epsilon(y) \) is usual alternating function. Therefore, by solving Eq.(20) the solutions

\[
\hat{G}[\epsilon, y; E] = \frac{1}{\sqrt{2E}} e^{-\sqrt{2E}|y|} \left[1 + \frac{\sqrt{2E}}{\sqrt{2E} + \frac{2}{\beta}} \epsilon(y)\right], \tag{21}
\]

\[
\hat{G}[-\epsilon, y; E] = \frac{1}{\sqrt{2E}} e^{-\sqrt{2E}|y|} \left[1 - \frac{\sqrt{2E}}{\sqrt{2E} + \frac{2}{\beta}} \epsilon(y)\right]
\]

are easily obtained. By combining Eqs.(19) and (21) we get a final result

\[
\hat{G}[x, y; E] = \hat{G}_0[x, y; E] +\frac{\epsilon(x)\epsilon(y)}{\sqrt{2E} + \frac{2}{\beta}} e^{-\sqrt{2E}(|x|+|y|)}. \tag{22}
\]

Also the relation between the bare coupling constant \( v \) and self-adjoint extension parameter \( \beta \) is obtained by inserting Eq.(21) into Eq.(18) and using the continuity of \( \partial \hat{G}[z, y; E]/\partial z \) at \( z = 0 \). Unlike two- and three-dimensional cases the relation is dependent on the space:

\[
\frac{1}{v} = \left(1 + \sqrt{\frac{2}{E} \frac{1}{\beta}}\right) \quad \text{for} \quad y > 0, \tag{23}
\]

\[
\frac{1}{v} = -\left(1 + \sqrt{\frac{2}{E} \frac{1}{\beta}}\right) \quad \text{for} \quad y < 0.
\]

After taking inverse Laplace transform of Eq.(22) and using analytic continuation in time, one can obtain Feynman propagator(or Kernel) \( K[x, y; t] \):

\[
K[x, y; t] = \frac{1}{\sqrt{2\pi it}} \exp\left(\frac{i}{2t} |x - y|^2\right) \tag{24}
\]
\[ + \frac{1}{\sqrt{2\pi}it} \exp\left(\frac{i}{2t}(|x| + |y|)^2\right) \epsilon(x)\epsilon(y) \]
\[ - \frac{1}{\beta} \exp\left(\frac{2}{\beta}(|x| + |y|) + \frac{2it}{\beta}\right) \times \text{erfc}\left[\frac{1}{\sqrt{2it}(|x| + |y|) + \frac{2it}{\beta}}\right] \epsilon(x)\epsilon(y), \]

where \( \text{erfc}(z) \) is usual error function. Eq.(24) coincides with Eq.(14) of Ref.[10] if one changes \( \beta \) of Ref.[10] into \(-\beta/2\). Therefore we derived a similar result with that of Ref.[10] without invoking perturbation expansion. Furthermore, in this formalism one can derive more general Green’s function (or propagator) by using more general boundary conditions. Therefore, let us use the most general boundary condition (1) of one dimensional point interaction. Like same way as before these two boundary conditions provide two independent equations

\[ \begin{align*}
\left(\frac{d}{b} + \sqrt{2E}\right)\hat{G}[\epsilon, y; E] - \frac{\omega}{b} \hat{G}[-\epsilon, y; E] &= (\epsilon(y) + 1)e^{-\sqrt{2E}|y|} \\
\frac{\omega^*}{b} \hat{G}[\epsilon, y; E] - (\sqrt{2E} + \frac{a}{b})\hat{G}[-\epsilon, y; E] &= (\epsilon(y) - 1)e^{-\sqrt{2E}|y|}
\end{align*} \]

where \( \omega^* \) is complex conjugate of \( \omega \). By inserting the solutions of Eq.(25)

\[ \begin{align*}
\hat{G}[\epsilon, y; E] &= -\frac{e^{-\sqrt{2E}|y|}}{\frac{d}{b} + \sqrt{2E}\frac{a+d}{b} + 2E} \left[ \epsilon(y) \left(\frac{\omega}{b} - \sqrt{2E} - \frac{a}{b}\right) - \left(\frac{\omega^*}{b} + \sqrt{2E} + \frac{a}{b}\right) \right] \\
\hat{G}[-\epsilon, y; E] &= -\frac{e^{-\sqrt{2E}|y|}}{\frac{d}{b} + \sqrt{2E}\frac{a+d}{b} + 2E} \left[ \epsilon(y) \left(\frac{d}{b} + \sqrt{2E} - \frac{\omega^*}{b}\right) - \left(\frac{d}{b} + \sqrt{2E} + \frac{\omega^*}{b}\right) \right]
\end{align*} \]

to Eq.(19) it is straightforward to derive the energy-dependent Green’s function corresponding to the most general boundary conditions

\[ \hat{G}[x, y; E] = \hat{G}_0[x, y; E] + \frac{\sqrt{2E}b}{D(E)} e^{-\sqrt{2E}(|x|+|y|)} \epsilon(x)\epsilon(y) \]
\[- \frac{e^{-\sqrt{2E}|x|+|y|}}{D(E)} \left[ \frac{c}{\sqrt{2E}} + \frac{1}{2}(a + d - \omega - \omega^*) \right. \\
+ \frac{1}{2}(d - a + \omega^* - \omega)\epsilon(x) + \frac{1}{2}(d - a + \omega - \omega^*)\epsilon(y) \left. - \frac{1}{2}(a + d - \omega - \omega^*)\epsilon(x)\epsilon(y) \right] \]

where

\[D(E) = c + (a + d)\sqrt{2E} + 2Eb.\]  \hspace{1cm} (28)

Note that Eq.(27) coincides with Eq.(22) at \(c = 0, \omega = a = d = 1,\) and \(b = \beta.\)

The energy-dependent Green’s function for the one-dimensional point interaction is calculated in Ref.[14]. The result (27) is exactly same with that of Ref.[14] although the authors of Ref.[14] clamed that their result is a consequence of appropriate mixture of one-dimensional \(\delta-\) and \(\delta'-\)potentials. In this paper same result can be derived by using only one-dimensional \(\delta'\)–function potential. Of course by following the procedure presented in Ref.[14] one can also obtain the time-dependent Brownian motion propagator and Feynman Kernel straightforwardly. Also one can derive the relation between bare coupling constant and self-adjoint parameters as before.

4 Conclusion

The one-dimensional \(\delta'-\)function potential is analyzed in the framework of Green’s function formalism. It is shown that the energy-dependent Brownian motion Green’s function for one-dimensional \(\delta'-\)function potential is crucially dependent on the boundary conditions. By choosing the most general boundary condition of one-dimensional point interaction the most general Green’s function which contains four real self-adjoint extension pa-
rameters is constructed. Grosch’s result which is obtained recently by using
the perturbation expansion of one-dimensional Dirac particle is special case
of ours. Also the relation between the bare coupling constant and self-adjoint
extension parameters is derived. Unlike two- and three-dimensional cases
the relation is dependent on the space.

References

[1] R. de L. Kronig and W. G. Penny, Proc. Roy. Soc. 130A, 499(1931).

[2] Ph. de Sousa Gerbert, Phys. Rev. D40, 1346(1989).

[3] C.R.Hagen, Phys. Rev. Lett. 64, 503(1990); Int. J. Mod. Phys. A6, 3119(1991).

[4] R. Jackiw, in M. A. Bég Memorial Volume, edited by A. Ali and P. Hoodbhoy(World Scientific, Singapore, 1991).

[5] M. Reed and B. Simon, Methods of Modern Mathematical Physics(Academic, New York, 1975).

[6] S. Albeverio, F. Gesztesy, R. Hoegh-Krohn, and H. Holden, Solvable Models in Quantum Mechanics(Springer, Berlin, 1988).

[7] C. Grosche, Ann. Physik 3, 283(1994).

[8] D. K. Park, J. Math. Phys. 36, 5453(1995).

[9] P. R. Chernoff and R. J. Hughes, J. Funct. Anal. 111, 97(1993).

[10] C. Grosche, J. Math. Phys. 28, L99(1995).
[11] B. Gaveau and L. S. Schulman, J. Phys. A19, 1833(1986).

[12] R. P. Feynman and A. R. Hibbs, Quantum Mechanics and Path Integrals(McGraw-Hill, New York, 1965).

[13] L. S. Schulman, Techniques and Applications of Path Integrals(Wiley, New York, 1981).

[14] S. Albeverio, Z. Brzezniak, and L. Dabrowski, J. Math. Phys. A27, 4933(1994).