3α description of $^{12}$C with microscopic nonlocal potentials

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Abstract. The well known αα local potentials Buck-Friedrich-Wheatley and Ali-Bodmer which reproduce very well the αα phase shifts are not successful for describing the $^{12}$C nucleus as a 3α system. The nonlocal nature of the αα interaction seems to be indispensable.

1. Introduction

A microscopic cluster model is useful for understanding the structure of light nuclei. It is successful for obtaining the low-lying spectrum of such nuclei and it has the advantages that it needs only a nucleon-nucleon (NN) interaction and that the antisymmetry requirement is exactly taken into account. However, it is difficult to extend its application to many cluster systems due to the heavy computational work. This motivates the use of a macroscopic model where the clusters are treated as point particles with local potentials for the cluster-cluster interactions usually set phenomenologically to fit some data for the binary cluster systems. However, applying a macroscopic model for the 3α system with the local αα potentials of Buck-Friedrich-Wheatley (BFW) $[7]$ or of Ali-Bodner (AB) $[6]$ gives poor results in comparison with experiment and the microscopic results $[1]$. It was found that the energy of the 3α system depends on how accurately the redundant states of the BFW potential are eliminated from the configuration space $[1, 2]$. A good step to resolve this dilemma has recently been developed by Fujiwara et al. $[3]$, where the energy-dependent nonlocal kernel of the resonating group method (RGM) was proposed as the intercluster potential. With this semi-microscopic model ($V^{\text{eK}}$) the difficulty mentioned above disappears $[3, 4]$. But we face another problem that physical quantities other than the ground state energy are not always predicted very well. By eliminating the energy dependence, we arrive at a new nonlocal potential ($V^{\text{RGM}}$) $[5]$, which allows us to calculate simultaneously the ground and higher states of $^{12}$C having the same symmetry. In this work we show the characteristics of this renormalized energy-independent nonlocal potential $V^{\text{RGM}}$ and compare its results with $V^{\text{eK}}$ and those obtained from phase-equivalent local potentials as well as a fully microscopic calculation.
2. Local versus nonlocal potentials

2.1. Local potentials

In the beginning we calculate the energy of the $3\alpha$ system using the local potential BFW. The main motivation was to investigate the strange behaviour of the $3\alpha$ ground state energy since it was noticed that it depends on how accurately the redundant states of the BFW potential are eliminated from the configuration space and also how accurately the forbidden states are defined [2]. So we tried to investigate this enigmatic behaviour in Ref. [1], and in that paper we defined the forbidden states in two ways, one as the bound states (BS) of the BFW potential and the other as harmonic oscillator (HO) wave functions. For the case of HO it was found that the dimension of the Pauli allowed space (PAS) is larger by two states than that of the BS, see Fig. 1. The curve denoted 'BFW' corresponds to the definition that the forbidden states are the bound states of the BFW potential, while the curve denoted 'HO' is the one with the forbidden states defined as the HO wave functions. $\gamma$ in the figure is the eigenvalue of the projection operator $\Gamma$

$$\Gamma = \sum_{i<j=1}^{3} \Gamma_{ij}, \quad \Gamma_{ij} = \sum_{f} |\chi_{ij}^{f}\rangle \langle \chi_{ij}^{f}|,$$

where $\chi_{ij}^{f}$ is the forbidden state wave function for the pair of $\alpha$ particles $ij$. Of course $\gamma = 0$ corresponds to a Pauli allowed state. For more details about this figure see Ref. [1]. In the case of the $2\alpha$ system, both BS and HO reproduce the resonance and the phase shifts very well, as one can see in Fig. 2. But when we tried to calculate the $3\alpha$ energy we found that BS and HO give very different results, and actually both of them are in disagreement with the experimental value and the theoretical calculation, as shown in Table 1. The result of the Ali-Bodmer $d$ potential (ABd) is also shown for comparison.

![Figure 1](image1.png)

**Figure 1.** Dimension of the Pauli allowed space for the $3\alpha$ system depending on the definition of the forbidden states.

![Figure 2](image2.png)

**Figure 2.** BFW $\alpha\alpha$ phase shifts with the two definitions of the forbidden states. The dots are the experimental data.

2.2. Nonlocal potentials

Next we replace the local potential with a nonlocal one. This semi-microscopic method (semi-micro) is a way to perform more realistic calculations and get more realistic results while avoiding at the same time the laborious work of a microscopic scheme. A first attempt of a $3\alpha$ model with energy-dependent nonlocal $\alpha\alpha$ RGM kernels [8, 9] has been developed in Ref. [3]. This interaction contains all antisymmetrization effects in the $\alpha\alpha$ system. It accurately describes the $^8\text{Be}$ ground-state resonance and the $\alpha\alpha$ phase shifts. The only model assumptions are: a
simple \((0s)^4\) description of the \(\alpha\) clusters within the translation-invariant HO model and the use of an effective nucleon-nucleon interaction \([10]\). Employing the resulting nonlocal \(\alpha\alpha\) interaction is rendered complicated by the existence of an energy dependence and of Pauli-forbidden states. Within this semi-microscopic \(3\alpha\) model, the difficulties associated with the accuracy of the forbidden-state elimination disappear \([11, 4]\). Despite a reasonable success for the ground state, the problem now is that the \(0^+_2\) excited state is not obtained simultaneously. Moreover, the energy dependence of the \(\alpha\alpha\) nonlocal interaction raises another difficulty. This energy is well defined in a two-body system but not in a three-body system. Its choice raises another ambiguity in the model \([4]\).

The principle of the elimination of the energy dependence from the RGM equation is known for a long time \([9, 12, 13, 14]\). For two-body systems, its interest is mainly academic. The fact that \(3\alpha\) calculations should be performed with the complicated nonlocal potential resulting from this transformation has been suggested several times \([13, 14]\) but its use has never thoroughly been examined in view of the many difficulties involved. Indeed, not only must these nonlocal potentials be derived, but their use and evaluation of the validity simultaneously require mastering accurate calculations of macroscopic three-body systems with nonlocal forces and of microscopic three-cluster systems for comparison. This know-how has been developed in recent years by our groups \([3, 11, 15, 16, 17, 18, 19]\).

The three-body equation including clusters 1, 2 and 3 reads
\[
(T_1 + T_2 + T_3 - T_{cm} + V_{12} + V_{13} + V_{23})\Psi = E\Psi, \tag{2}
\]
where \(T_i\) is the kinetic energy of the cluster \(i\) and the center-of-mass kinetic energy \(T_{cm}\) is subtracted. The intercluster potential \(V_{ij}\) between the \(ij\) pair can be either local in the macroscopic model or nonlocal in the semi-microscopic model. Except for shallow potentials, Eq.(2) implicitly assumes that \(\Psi\) belongs to the Pauli-allowed space of the three \(\alpha\alpha\) pairs.

When \(V_{ij}\) is a nonlocal RGM potential derived from an effective nucleon-nucleon interaction, the three-body equation does not contain any free parameter. It is a semi-microscopic model in the sense that exchanges involving two clusters are taken into account in the nonlocal part of \(V_{ij}\), but exchanges involving all three clusters are neglected. This model is quite different from the orthogonality-condition model (OCM) \([9]\) where the full antisymmetrization is approximately taken into account through the orthogonality with respect to the forbidden states, but where the \(\alpha\alpha\) interactions are phenomenological.

2.3. RGM potentials
Before solving Eq.(2), we briefly recall the characteristics of two variants of nonlocal RGM potential. The RGM equation for the intercluster relative motion function \(\chi\) reads
\[
(T + V + \varepsilon K)\chi = \varepsilon \chi, \quad V = V_D + V_{EX}, \tag{3}
\]
where \(T\) is the intercluster kinetic energy, \(V_D\) is local (direct) potential, and the nonlocal potential \(V_{EX} = K_T + K_V\), where \(K_T\) and \(K_V\) are the exchange nonlocal kernels for the kinetic and potential energies (including the Coulomb term), respectively. The energy \(\varepsilon\) is defined with respect to the two-cluster threshold. Equation (3) suggests that the microscopically founded intercluster potential is
\[
V_{\varepsilon K} = V + \varepsilon K, \tag{4}
\]
which is energy-dependent. The norm kernel \(\mathcal{N}\) is defined by
\[
\mathcal{N} = 1 - K, \tag{5}
\]
where $K$ is the overlap kernel. When applying the RGM potential Eq. (4) for $V_{ij}$, one must note that a particular function $\chi_f$, called a Pauli-forbidden state (PFS), exists in general. It is defined as a solution of the equation, $N\chi_f = 0$ or $K\chi_f = \chi_f$. The three-body wave function $\Psi$ of Eq. (2) is constrained to be orthogonal to all the two-cluster PFSs which exist for the three intercluster pairs,

$$< \chi_f | \Psi > = 0. \quad (6)$$

A potential which has no explicit energy-dependence and still maintains basic properties such as the phase shifts can be constructed for a macroscopic (or renormalized) relative motion function $g = \sqrt{N} \chi$, by requiring the condition [12]

$$(T+V_{RGM})g = \varepsilon g. \quad (7)$$

Comparing Eqs. (3) and (7) gives the nonlocal potential $V_{RGM}$ as

$$V_{RGM} = N^{-1/2}(T+V)N^{-1/2} - T = V + W, \quad (8)$$

where the new nonlocal operator $W$ is the difference between the renormalized RGM potential $V_{RGM}$ and the bare RGM potential $V$.

The relative motion function $g$ has the nice property of preserving the orthonormality of microscopic two-cluster wave function. In addition, the asymptotic of $g$ is the same as that of $\chi$ because $N$ approaches unity at large distances. The phase shift determined from $g$ using the potential $V_{RGM}$ is exactly equal to that determined from the RGM equation (3) for $\chi$.

The potential $V_{RGM}$ is defined in the PFS-free space. To make this point more transparent, we consider the eigenvalue problem $K\psi_\kappa = \kappa \psi_\kappa$. The PFSs are nothing but the eigenfunctions with eigenvalue $\kappa = 1$. For cluster wave functions described with HO functions with a common size parameter, this eigenvalue problem can be solved [20]. The eigenfunctions are HO wave functions $\psi_{nlt}$, and the corresponding eigenvalues are given by $\kappa_{nl} = 4(1/2)^{2n+\ell} - 3\delta_{2n+\ell,0}$ for $\alpha + \alpha$. There are thus three PFSs, $\psi_{000}$, $\psi_{100}$ and $\psi_{02m}$ for even $\ell$, and all states are PFSs for odd $\ell$.

The nonlocal kernel $V_{RGM}$ can be expanded in terms of the eigenfunctions of $K$. For example $W$ is expressed by

$$W(r, r') = \sum_{nlt} W_{nn't} \psi_{nlm}(r) \psi_{ntl}(r'),$$

with

$$W_{nn't} = \left[ \frac{1}{\sqrt{(1-\kappa_{nl})(1-\kappa_{ntl})}} - 1 \right] \langle \psi_{nlm}|T + V|\psi_{ntl} \rangle,$$

where the prime over the sum indicates that the PFSs are excluded from the summation. Here $V$ is assumed to be rotation-invariant, which makes its matrix element independent of $m$. In practice, we include $n$ and $n'$ up to 100. The potential $V_{RGM}$ is expanded into partial waves as

$$V_{\ell}^{RGM}(r, r') = 2\pi \rho r' \int_{-1}^{1} V_{RGM}(r, r') P_\ell(t) dt = V_D(r) \delta(r-r') + V_{EX}\ell(r, r') + W_\ell(r, r'), \quad (9)$$

with $t = r \cdot r'/rr'$ and $P_\ell$ is a Legendre polynomial. The potential $V_{\ell}^{RGM}$ acts on $g_\ell$, the macroscopic wave function in partial wave $\ell$. Fig. 3 displays $V_{EX}\ell(r, r')$ and $W_\ell(r, r')$ for the $\alpha \alpha$ system. The parameters of the RGM kernels, $\nu = 0.257$ fm$^{-2}$ and $\alpha = 0.94687$, taken from Ref. [17], reproduce the $\alpha \alpha$ phase shifts. In the figure one can notice that they are having different patterns and that the $W$ potential is smaller than $V_{EX}$ by almost a factor 10. Also one can see the rapid oscillating behaviour for both of them in the $S$ and $D$ waves, which is attributed to the existence of forbidden states in these partial waves.
3. Results and discussion

The values $h^2/m_N = 41.47$ MeV fm$^2$ and $m_{\alpha} = 4m_N$ are used everywhere for consistency between Micro and non-Micro calculations. The NN potential (Minnesota) used in Micro is the same as the one used to derive the RGM kernels, and thus the comparison between Micro and Semi-micro indicates how closely the various potentials simulate the microscopic 3$\alpha$ calculation. We stress that all models, except $V$ of Semi-micro, reproduce the $\alpha\alpha$ phase shifts in an essentially identical way. The rms radius of the point $\alpha$-particle distribution is obtained for Micro by subtracting $r_{\alpha}^2$ ($r_{\alpha} = 1.479$ fm) from the $h^2 r_{\alpha}^2$ value calculated at the nucleon level. As explained before, the results using local potentials are not satisfactory. They cannot reproduce the microscopic results. The shallow AB potential gives a weakly bound energy for the ground state, while the deep BFW potential gives either a weakly bound or overbound energy depending on the definition of the forbidden states. The bare RGM potential $V$ cannot reproduce the microscopic results either. These results are summarized in Table 1 which compares the energies and root-mean-square radii of the 3$\alpha$ system obtained using local potentials (Local), the nonlocal $V^{eK}$ and $V^{RGM}$ potentials (Semi-micro) as well as a fully microscopic calculation (Micro) [16, 19]. The three-body equation (2) is solved under different conditions depending on the potential model. In the Local model, the orthogonality constraint (6) is imposed in the case of BFW potential. In the Semi-micro model, the HO PFSs are used for $\chi_f$ in all the cases, and in addition the self-consistency condition described in Refs. [3, 4] is put in the case of $V^{eK}$. In this model the $0^+_1$ and $0^+_2$ states cannot be obtained simultaneously without loosing their mutual orthogonality by changing the value of $\varepsilon$. As the $0^+_2$ state calculated with the Micro and $V^{RGM}$ potentials is above the 3$\alpha$ threshold, the value of its rms radius depends on the basis choice. Our calculations here suggest that it is very large (around 3.5 fm).
Table 1. Comparison of the energies (MeV) from the 3α threshold and the rms radii (fm) of point α-particle distribution for the 3α system with different models and potentials. Experimental energies are –7.27 and 0.38 MeV for the $0^+_1$ and $0^+_2$ states, respectively.

| Model   | Potential | $L^*$ | $E$  | $\sqrt{\langle r^2 \rangle}$ |
|---------|-----------|------|-----|------------------------------|
| Local   | ABd [6]   | $0^+_1$ | –1.52 | 2.34                      |
|         | BFW (BS) [7] | $0^+_1$ | –0.66 | 2.31                      |
|         | BFW (HO) [7] | $0^+_1$ | –20.62 | 1.29                      |
| Semi-micro | $V$     | $0^+_1$ | –19.50 | 1.42                      |
|         | $V^\pm K$ | $0^+_1$ | –9.60  | 1.45                      |
|         | $V^{RGM}$ | $0^+_1$ | –9.44  | 1.62                      |
|         |           | $0^+_2$ | 0.597  |                          |
| Micro   | $0^+_1$   | –11.37 | 1.64  |                            |
|         | $0^+_2$   | 0.598   |        |                            |

4. Conclusion
The important thing that we want to highlighten is that the results of the Semi-micro model are much better than those of local potentials. One advantage for $V^{RGM}$ over $V^\pm K$ is that we can now calculate all the $0^+$ states at the same time. Also one can notice that the rms radius of the $V^{RGM}$ is in better agreement with the microscopic result than that of the $V^\pm K$. Though the Semi-micro model with $V^{RGM}$ is qualitatively successful, its ground state energy is about 1.9 MeV higher than Micro. The former takes account of binary exchanges between α clusters but not of simultaneous exchanges among the three α clusters. Such three-cluster exchanges produce attraction as observed here. This effect is rather significant in the compact ground state but negligible in the spatially extended $0^+_2$ state. The success here of the potential $V^{RGM}$ encourages us to try it with more cluster systems e.g. 4α and 5α.

References
[1] Matsumura H, Orabi M, Suzuki Y and Fujiwara Y 2006 Nucl. Phys. A 776 1
[2] Tursunov E M 2001 J. Phys. G 27 1381
[3] Fujiwara Y, Nemura H, Suzuki Y, Miyagawa K and Kohno M 2002 Prog. Theor. Phys. 107 745
[4] Theeten M, Baye D and Descouvemont P 2006 Phys. Rev. C 74 044304
[5] Suzuki Y, Matsumura H, Orabi M, Fujiwara Y, Descouvemont P, Theeten M and Baye D accepted for publication in Physics Letters B
[6] Ali S and Bodmer A R 1966 Nucl. Phys. 80 99
[7] Buck B, Friedrich H and Wheatley C 1977 Nucl. Phys. A 275 246
[8] Wildermuth K and Tang Y C 1977 A Unified Theory of the Nucleus (Vieweg Braunschweig)
[9] Saito S 1977 Prog. Theor. Phys. Suppl. 62 11
[10] Thompson D R, LeMere M and Tang Y C 1977 Nucl. Phys. A 286 53
[11] Fujiwara Y, Suzuki Y and Kohno M 2004 Phys. Rev. C 69 037002; 2004 Few-Body Systems 34 237
[12] Timm W, Fiebig H R and Friedrich H 1982 Phys. Rev. C 25 79
[13] Suzuki Y, Lovas R G, Yabana K and Varga K 2003 Structure and Reactions of Light Exotic Nuclei (Taylor & Francis, London)
[14] Schmid E W 1984 Nucl. Phys. A 416 347c
[15] Varga K and Suzuki Y 1995 Phys. Rev. C 52 2885
[16] Matsumura H and Suzuki Y 2004 Nucl. Phys. A 739 238
[17] Fujiwara Y, Miyagawa K, Kohno M, Suzuki Y, Baye D and Sparenberg J M 2004 Phys. Rev. C 70 024002
[18] Theeten M, Baye D and Descouvemont P 2005 Nucl. Phys. A 753 233
[19] Koreniov S and Descouvemont P 2004 Nucl. Phys. A 740 249
[20] Horiuchi H 1977 Prog. Theor. Phys. Suppl. 62 90