On the Parallel Undecided-State Dynamics with Two Colors

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July 18, 2017

Abstract

The Undecided-State Dynamics is a well-known protocol that achieves Consensus in distributed systems formed by a set of \( n \) anonymous nodes interacting via a communication network. We consider this dynamics in the parallel \texttt{PULL} communication model on the complete graph for the binary case, i.e., when every node can either support one of two possible colors (say, Alpha and Beta) or stay in the undecided state. Previous work in this setting only considers initial color configurations with no undecided nodes and a large bias (i.e., \( \Theta(n) \)) towards the majority color. A major open question here is whether this dynamics reaches consensus quickly, i.e. within a polylogarithmic number of rounds.

In this paper we present an unconditional analysis of the Undecided-State Dynamics which answers to the above question in the affirmative. Our analysis shows that, starting from any initial configuration, the Undecided-State Dynamics reaches a monochromatic configuration within \( O(\log^2 n) \) rounds, with high probability (w.h.p.). Moreover, we prove that if the initial configuration has bias \( \Omega(\sqrt{n \log n}) \), then the dynamics converges toward the initial majority color within \( O(\log n) \) round, w.h.p.

At the heart of our approach there is a new analysis of the symmetry-breaking phase that the process must perform in order to escape from (almost-)unbiased configurations. Previous symmetry-breaking analysis of consensus dynamics essentially concern sequential communication models (such as Population Protocols) and/or symmetric updated rules (such as majority rules).
1 Introduction

Strong research interest has been recently focussed on the study of simple, local mechanisms for Consensus problems in distributed systems [3, 2, 16, 17, 22, 23]. In one of the basic versions of the consensus problem, the system consists of a finite set of \( n \) anonymous entities (nodes) that run elementary operations and interact by exchanging messages. Every node initially supports a value (i.e. a color) chosen from a finite alphabet \( \Sigma \) and a Consensus Protocol is a local procedure that, starting from any color configuration, let the system converge in finite time to a monochromatic configuration where every node supports the same color. The consensus is valid if the winning color is a valid one: It is one among those initially supported by at least one node. Moreover, the consensus configurations must result equilibria of the protocol process: Once the system reaches a consensus configuration, it will stay there forever unless some external event takes place.

We study the consensus problem in the PULL model [12, 15, 19] in which, at every round, each active node of a communication network contacts one neighbor uniformly at random to pull information. A well-studied natural consensus protocol in this model is the Undecided-State Dynamics (for short, the U-Dynamics) in which the state of a node can be either a color or the undecided state. When a node is activated, it pulls the state of a random neighbors and updates its state according to the following updating rule (see Table 1): If a colored node pulls a different color from its current one, then it becomes undecided, while in all other cases it keeps its color; moreover, if the node is in the undecided state then it will take the state of the pulled neighbor.

The U-Dynamics has been studied in both sequential and parallel models: Informally, in the former, at every round, only one random node is activated, while in the latter, at every round, all nodes are activated synchronously.

As for the sequential model\[^2\] provides an unconditional analysis showing (among other results) that the U-Dynamics solves the binary consensus problem (i.e. when \( |\Sigma| = 2 \)) in the complete graph within \( O(n \log n) \) activations (and, thus in \( O(\log n) \) “parallel” time), with high probability\[^3\].

As for the parallel PULL model, even though it is easy to verify that the U-Dynamics achieves consensus in the complete graph (with high probability), the convergence time of this dynamics is still a major open issue, even in the binary case. We remark that the stochastic process yielded by the parallel dynamics significantly departs from the process yielded by the sequential one. To get just one immediate evidence of this difference, observe that, in the former model, the system can converge to the (non-valid) configuration where all nodes are undecided even if starting from a “fully-colored” configuration (where all nodes are not undecided). On the other hand, it is easy to see that this evolution cannot happen in the sequential setting. A deeper, crucial difference lies in the random number of nodes that may change color at every round: In the sequential model, this is at most one\[^4\] while in the parallel one, all nodes may change state

\[^{1}\text{In some previous papers [23] on the binary case (}\mid\Sigma\mid = 2\text{), this protocol has been also called the Third-State Dynamics. We here prefer the term “undecided” since it also holds for the non-binary case and, moreover, the term well captures the role of this additional state.}\]

\[^{2}\text{[3] in fact considers the Population-Protocol model which is, in our specific context, equivalent to the sequential PULL model.}\]

\[^{3}\text{As usual, we say that an event } E_n \text{ holds w.h.p. if } P(\ E_n\ ) \geq 1 - n^{-\Theta(1)}.}\]

\[^{4}\text{This number becomes 2 if the sequential communication model activate a random edge}\]
in one shot and indeed, for most phases of the process, the expected number of changes is linear in $n$. It thus turns out that the probabilistic arguments used in the analysis of $[3]$ appear not useful in the parallel setting. In $[5]$, the author analyze the U-Dynamics in the parallel $\textsc{Pull}$ model on the complete graph when the alphabet $\Sigma$ has size $k$, where $k = o(n^{1/3})$. The analysis in $[5]$ considers this dynamics as a protocol for Plurality Consensus $[2, 3, 21]$, a variant of Consensus, where the goal is to reach consensus on the color that was initially supported by the plurality of the nodes: Their analysis requires that the initial configuration must have a relatively-large bias $s = c_1 - c_2$ between the size $c_1$ of the (unique) initial plurality and the size $c_2$ of the second-largest color. More in details, in $[5]$ it is assumed that $c_1 \geq \alpha c_2$, for some absolute constant $\alpha > 1$ and, thus, this condition for the binary case would result into requiring a very-large initial bias, i.e., $s = \Theta(n)$. This analysis clearly does not show that the U-Dynamics efficiently solves the binary consensus problem, mainly because it does not manage balanced initial configurations.

The main contribution of this paper is an unconditional analysis showing the U-Dynamics solves the binary consensus problem in the parallel $\textsc{Pull}$ model for the complete graph within $O(\log^2 n)$ rounds, with high probability.

Our results

We prove that, starting from any color configuration on the complete graph, the U-Dynamics reaches a monochromatic configuration (thus consensus) within $O(\log^2 n)$ rounds, with high probability. This bound is almost tight since, for some (in fact, a large number of) initial configurations, the process requires $\Omega(\log n)$ rounds to converge.

Not assuming a large initial bias of the majority color significantly complicates the analysis. Indeed, the major challenges arise from (almost) balanced initial configurations where the system needs to break symmetry (see Section 4). So far, this issue for the parallel U-Dynamics has never been addressed. A key ingredient of our analysis is a suitable application of the martingale optional stopping theorem (see, e.g., $[20, 24]$). While the use of that theorem is standard in the analysis of sequential processes of interacting particles that can be modeled as birth-and-death chains, our new approach allows us to analyze the process yielded by running the U-Dynamics in synchronous parallel rounds, that is a somewhat “wild” process where an unbounded number of particles may change state at every round.

The symmetry-breaking phase terminates when the U-Process reaches some configuration having a bias $s = \Omega(\sqrt{n \log n})$. Then (see Section 5) we prove that, starting from any configuration having that bias, the process reaches consensus within $O(\log n)$ rounds, with high probability. Even though our analysis of this “majority” part of the process is based on standard concentration arguments, it must cope with some non-monotone behaviour of the key random variables (such as the bias and the number of undecided nodes at the next round): Again, this is due to the non-symmetric role played by the undecided nodes. A good intuition about this “non-monotone” process can be gained by looking at the mutually-related formulas giving the expectation of such key random variables.

Our refined analysis shows that, during this majority phase, the winning color per round, rather than one single node $[3]$.

Our analysis also considers initial configurations with undecided nodes.
never changes and, thus, the U-Dynamics also ensures Plurality Consensus in logarithmic time whenever the initial bias is $s = \Omega(\sqrt{n \log n})$.

Interestingly enough, we also show that configurations with $s = \mathcal{O}(\sqrt{n})$ exist so that the system may converge toward the minority color with non-negligible probability.

Further motivation and related work

On the U-Dynamics. The interest in the U-Dynamics arises in fields beyond the borders of Computer Science and it seems to have a key-role in important biological processes modelled as so-called chemical reaction networks \cite{11,17}. For such reasons, the convergence time of this dynamics has been analyzed on different communication models \cite{1,3,4,8,13,16,18,21,23}.

As previously mentioned, the U-Dynamics has been analysed in the parallel $\text{PULL}$ model in \cite{5} and their results concern the evolution of the process for the multi-color case when there is a significant initial bias (as a protocol for plurality consensus).

As for the sequential model, the U-Dynamics has been introduced and analyzed in \cite{3} in the complete graph. They prove that this dynamics, with high probability, converges to a valid consensus within $\mathcal{O}(n \log n)$ activations and, moreover, it converges to the majority whenever the initial bias is $\omega(\sqrt{n \log n})$.

Still concerning the sequential model, \cite{21} recently analyzes, besides other protocols, the U-Dynamics in arbitrary graphs when the initial configuration is sampled uniformly at random between the two colors. In this (average-case) setting, they prove that the system converges to the initial majority color with higher probability than the initial minority one. They also give results for special classes of graphs where the minority can win with large probability if the initial configuration is chosen in a suitable way. Their proof for this last result relies on an exponentially-small upper bound on the probability that a certain minority can win in the complete graph (see \cite{21} for more details).

In \cite{4,8,15,23}, the same dynamics for the binary case has been analyzed in further sequential communication models.

On some other consensus dynamics. Recently, further simple consensus protocols have been deeply analyzed in several papers, thus witnessing the high interest of the scientific community on such processes \cite{3,7,10,11,13,14,16,23}. Due to the lack of space, we here mention only those results that are closer to ours.

The parallel 3-$\text{MAJORITY}$ is a protocol where at every round, each node picks the colors of three random neighbors and updates its color according to the majority rule (taking the first one or a random one to break ties). All theoretical results for 3-$\text{MAJORITY}$ consider the complete graph. The authors of \cite{7} assume that the bias is $\Omega(\min\{\sqrt{2k}, (n/\log n)^{1/6}\} \cdot \sqrt{n \log n})$. Under this assumption, they prove that consensus is reached with high probability in $\mathcal{O}(\min\{k, (n/\log n)^{1/3}\} \cdot \log n)$ rounds, and that this is tight if $k \leq (n/\log n)^{1/4}$. The first result without bias \cite{6} restricts the number of initial colors to $k = \mathcal{O}(n^{1/3})$. Under this assumption, they prove that 3-$\text{MAJORITY}$ reaches consensus with high probability in $\mathcal{O}(k^2(\log n)^{1/2} + k \log n \cdot (k + \log n))$ rounds. Very recently, such result has been generalized to the whole range of $k$ in \cite{9}.

It is important to emphasize the 3-$\text{MAJORITY}$ rule is “fully-anonymous”, i.e., it does not depend on the current state of the node and it is symmetric w.r.t all
colors: A crucial consequence is that the variance of the 3-Majority process is essentially the same along all the symmetry-breaking phase. Such properties make the analysis of this process significantly different from that of the U-Process we provide here in this work.

In [13, 14], the authors consider the 2-Choices dynamics, a non-symmetric variant of 3-Majority for plurality consensus in the binary case (i.e. \( k = 2 \)). For random \( d \)-regular graphs, [13] proves that all nodes agree on the majority color in \( O(\log n) \) rounds, provided that the bias is \( \omega(n \cdot \sqrt{1/d + d/n}) \). The same holds for arbitrary \( d \)-regular graphs if the bias is \( \Omega(\lambda_2 \cdot n) \), where \( \lambda_2 \) is the second largest eigenvalue of the transition matrix. In [14], these results are extended to general expander graphs.

2 Preliminaries

We analyze the parallel version of the dynamics called U-Dynamics in the (uniform) PULL model on a complete graph: Starting from an initial configuration where every node supports a color, i.e. a value from a set \( \Sigma \) of possible colors, at every round, each agent \( u \) pulls the color of a randomly-selected neighbor \( v \). If the color of agent \( v \) differs from its own color, then agent \( u \) enters in an undecided state (an extra state with no color). When an agent is in the undecided state and pulls a color, it gets that color. Finally, an agent that pulls either an undecided agent or an agent with its own color remains in its current state.

| \( i \setminus j \) | undecided | color \( i \) | color \( j \) |
|-----------------|-----------|-----------|-----------|
| undecided       | undecided | \( i \)    | \( j \)    |
| \( i \)         | \( i \)    | undecided |           |
| \( j \)         | \( j \)    | undecided |           |

Table 1: The update rule of the U-Dynamics where \( i, j \in [k] \) and \( i \neq j \).

In this paper we consider the case in which there are two possible colors (say color Alpha and color Beta). Let us name \( C \) the space of all possible configurations and observe that, since we are on the complete graph, a configuration \( x \in C \) is completely determined by the number of agents with color Alpha and the number of agents with color Beta, say \( a(x) \) and \( b(x) \), respectively.

It is convenient to give names also to two other quantities that will appear often in the analysis: the number \( q(x) = n - a(x) - b(x) \) of undecided agents and the difference \( s(x) = a(x) - b(x) \) between the numbers of Alpha-colored and Beta-colored agents. We will call \( s(x) \) the bias of configuration \( x \). Notice that any two of the quantities \( a(x), b(x), q(x), \) and \( s(x) \) uniquely determine the configuration. When it will be clear from the context, we will omit \( x \) and write \( a, b, q, \) and \( s \) instead of \( a(x), b(x), q(x), \) and \( s(x) \).

Observe that the U-Dynamics defines a finite-state Markov chain \( \{ X_t \}_{t \geq 0} \) with state space \( C \) and three absorbing states, namely, \( q = n \), \( a = n \), and \( b = n \). We call U-Process the random process obtained by applying the U-Dynamics starting at a given state. Once we fix the configuration \( x \) at round \( t \) of the process, i.e. \( X_t = x \), we use the capital letters \( A, B, Q, \) and \( S \) to refer to the random variables \( a(X_{t+1}), b(X_{t+1}), q(X_{t+1}), s(X_{t+1}) \).
From the definition of U-Dynamics it is easy to compute the following expected values (see also Section 3 in [5])

\[ E[A | X_t = x] = a \left( \frac{a + 2q}{n} \right) \]  
\[ E[Q | X_t = x] = \frac{q^2 + 2ab}{n} \]  
\[ E[S | X_t = x] = \frac{a(a + 2q) - b(b + 2q)}{n} = s \left( 1 + \frac{q}{n} \right) \]

2.1 The expected evolution of the U-Dynamics

Equations (1)-(3) can be used to have a preliminary intuitive idea on the expected evolution of the U-Dynamics. From (3) it follows that the bias \( s \) increases exponentially, in expectation, as long as the number \( q \) of undecided agents is a constant fraction of \( n \) (say, \( q \geq \delta n \), for some positive constant \( \delta \)). By rewriting (2) in terms of \( q \) and \( s \) we have that

\[ E[Q | X_t = x] = \frac{q^2 + 2ab}{n} = \frac{2q^2 + (n - q)^2 - s^2}{2n} \geq \frac{n}{3} - \frac{s^2}{2n} \]

where in the inequality we used the fact that the minimum of \( 2q^2 + (n - q)^2 \) is achieved at \( q = n/3 \) and its value is \( 2n^2/3 \). From (1) it thus follows that, as long as the magnitude of the bias is smaller than a constant fraction of \( n \) (say \( |s| < 2n/3 \)), the expected number of undecided nodes will be larger than a constant fraction of \( n \) at the next round (say, \( E[Q | X_t = x] \geq n/9 \)).

When the magnitude of the bias \( |s| \) reaches \( 2n/3 \), it is easy to see that the expected number of nodes with the minority color decreases exponentially. Indeed, suppose wlog that \( B \) is the minority color and let us rewrite (1) for \( B \) and in terms of \( b \) and \( s \), we get

\[ E[B | X_t = x] = b \left( \frac{b + 2q}{n} \right) = b \left( 1 - \frac{2s + 3b - n}{n} \right). \]

Hence, when \( s > 2n/3 \) we have that \( E[B | X_t = x] \leq (1 - 2/3)b \).

The above sketch of the analysis in expectation would suggest that the process should end up in a monochromatic configuration within \( O(\log n) \) rounds. Indeed, in Theorem 2 we prove that this is what happens with high probability (w.h.p., from now on) when the process starts from a configuration that already has some bias, namely \( s = \Omega(\sqrt{\log n}) \).

When the process starts from a configuration with a smaller bias, the analysis in expectation looses its predictive power. As an extremal example, observe that when \( a = b = n/3 \) the system is “in equilibrium” according to (1)-(3). However, the equilibrium is “unstable” and the symmetry is broken by the variance of the process. As mentioned in the Introduction, the analysis of this symmetry-breaking phase is the key technical contribution of the paper and it will be described in Section 4. This analysis will show that, starting from any initial configuration, the system reaches a configuration where the magnitude of the bias is \( \Omega(\sqrt{\log n}) \) within \( O(\log^2 n) \) rounds, w.h.p.
Figure 1: \(\{H_1, \ldots, H_7\}\) is the considered partitioning the configuration space \(C\). Missing arrows are transitions that have negligible probabilities.

3 Main results and the digraph of the U-Process’ phases

As informally discussed in the introduction, we prove the two following results characterizing the evolution of the U-Dynamics on the synchronous PLULL model in the complete graph.

**Theorem 1** (Consensus). Let the U-Process start from any configuration in \(C\). Then the process converges to a (valid) monochromatic configuration within \(O(\log^2 n)\) rounds, w.h.p. Furthermore, if the initial configuration has at least one colored node (i.e. \(q \leq n - 1\)), then the process converges to a configuration such that \(|s| = n\), w.h.p.

**Theorem 2** (Plurality consensus). Let \(\gamma\) be any positive constant and assume that the U-Process starts from any biased configuration such that \(|s| \geq \gamma \sqrt{n \log n}\) and assume w.l.o.g. the majority color is \(\text{Alpha}\). Then the process converges to the monochromatic configuration with \(a = n\) within \(O(\log n)\) rounds, w.h.p. Furthermore, the result is almost tight in a twofold sense: (i) An initial configuration exists, with \(|s| = \Omega(\sqrt{n \log n})\), such that the process requires \(\Omega(\log n)\) rounds to converge w.h.p., and (ii) there is an initial configuration with \(|s| = \Theta(\sqrt{n})\) such that the process converges to the minority color with constant probability.

Outline of the two proofs. The two theorems above are consequences of our refined analysis of the evolution of the U-Process. The analysis is organized into a set of possible process phases, each of them is defined by specific ranges
of parameters \( q \) and \( s \). A high-level description of this structure is shown in Fig. 1, where every rectangular region represents a subset of configurations with specific ranges of \( s \) and \( q \) and it is associated to a specific phase. In details, let \( \gamma \) be any positive constant, then the regions are defined as follows: \( H_1 \) is the set of configurations such that \( s \leq \frac{\gamma \sqrt{n \log n}}{\log n} \) and \( q \geq \frac{1}{\log n} \); \( H_2 \) is the set of configurations such that \( s \leq \frac{\gamma \sqrt{n \log n}}{\log n} \) and \( \frac{1}{18} \leq q \leq \frac{1}{\log n} \); \( H_3 \) is the set of configurations such that \( s \leq \frac{\gamma \sqrt{n \log n}}{\log n} \) and \( q \leq \frac{1}{\log n} \). \( H_4 \) is the set of configurations such that \( \frac{\gamma \sqrt{n \log n}}{\log n} \leq s \leq \frac{\gamma n}{16} \) and \( q \geq \frac{1}{\log n} \). \( H_5 \) is the set of configurations such that \( \gamma \sqrt{n \log n} \leq s \leq \frac{\gamma n}{3} \) and \( q \leq \frac{1}{\log n} \); \( H_6 \) is the set of configurations such that \( \frac{\gamma n}{3} \leq s \leq n - 5\sqrt{n \log n} \) and \( q \leq \sqrt{n \log n} \). \( H_6 \) is the set of configurations such that \( s \geq \frac{2}{3} n \) minus \( H_7 \).

For each region, Fig. 1 specifies our upper bound on the exit time from the corresponding phase, while black arrows represent all possible phase transitions which may happen with non-negligible probability.

As a first, important remark, we point out that the scheme of Fig. 1 can be seen as a directed acyclic graph \( G \) with a single sink \( H_6 \), which is reachable from any other region. We also remark that, starting from certain configurations, the monochromatic state may be reached via different paths in \( G \). This departs from previous analysis of consensus processes [5, 7, 16] in which the phase transition graph is essentially a path.

We now outline the proofs of the two main results of this paper.

Outline of the Proof of Theorem 2. Consider an initial configuration \( x \) such that \( s(x) \geq \frac{\gamma n \log n}{\log n} \), for some positive constant \( \gamma \), and assume w.l.o.g. that the majority color in \( x \) is Alpha. In Section 5, we first show (see Lemma 5.1) that if the process lies in \( H_4 \) the bias grows exponentially fast and thus the process enters in \( H_6 \) within \( O(\log n) \) rounds. Then we prove Lemma 5.2 stating that, starting from any configuration in \( H_6 \), the process ends in the monochromatic configuration where \( a = n \) in \( O(\log n) \) rounds. Next, we show that, starting from any configuration in \( H_6 \), the process falls into \( H_4 \) or in \( H_6 \) in one round (Lemma 5.3) and that, starting from any configuration in \( H_7 \) the process falls into \( H_4, H_5 \) or \( H_6 \) in one round (Lemma 5.4). Concerning the tightness of the result stated in the second part of the theorem, we have that the lower bound on the convergence time is an immediate consequence of Claim (ii) of Lemma 5.1. While, Claim (ii), concerning the lower bound on the initial bias, will be proved in Claim 14, which is provided in Appendix C.

Outline of the Proof of Theorem 1. We first observe that the configuration where all nodes are undecided (i.e. \( q = n \)) is an absorbing state of the U-Process and thus, for this initial configuration, Theorem 1 trivially holds. In Section 2, we will show that, starting from any balanced configuration, i.e. with \( |s| = o(\sqrt{n \log n}) \), the U-Process “breaks symmetry” reaching a configuration \( y \) with \( |s(y)| = \Omega(\sqrt{n \log n}) \) within \( O(\log^2 n) \) rounds, w.h.p. Then, the thesis easily follows by applying Theorem 2 with initial configuration \( y \). As for the symmetry-breaking phase, in Lemma 4.1 we prove that, if the process starts from a configuration in \( H_1 \) or \( H_3 \) (see Figure 1), then after \( O(\log n) \) rounds either the bias between the two colors becomes \( \Omega(\sqrt{n \log n}) \) or the system reaches some configuration in \( H_2 \), w.h.p. In Lemma 4.3 we then prove that, if the process is in a configuration in \( H_2 \), then the bias between the two colors will become \( \Omega(\sqrt{n \log n}) \) within \( O(\log^2 n) \) rounds, w.h.p.
4 Symmetry breaking

In this section we show that, starting from any (almost-)balanced configuration, i.e. with \(|s| = o(\sqrt{n \log n})\), the U-Process “breaks symmetry” reaching a configuration with \(|s| = \Omega(\sqrt{n \log n})\) within \(O(\log^2 n)\) rounds, w.h.p. This part of our analysis is structured as follows.

In Lemma 4.1 we prove that, if the process starts in a configuration in \(H_1\) or \(H_3\) (see Figure 1), i.e., when the number of undecided nodes is either smaller than \(n/18\) or larger than \(n/2\), then, after \(O(\log n)\) rounds, either the bias between the two colors already gets order of \(\Omega(\sqrt{n \log n})\) or the system reaches some configuration in \(H_2\) (i.e., a configuration where the number of undecided nodes is between \(n/18\) and \(n/2\)). In Lemma 4.4 we then prove that, if the process is in a configuration in \(H_2\), then the bias between the two colors will be \(\Omega(\sqrt{n \log n})\) within \(O(\log^2 n)\) rounds w.h.p.

Lemma 4.1 is a simple consequence of the three following claims. Claims 1 and 2 follow from Chernoff bound applied to \((\ref{1})\) and \((\ref{1})\), respectively. Claim 3 follows from the fact that, when the number of colored nodes is very small, the U-Process behaves essentially like a pull process (the detailed proofs can be found in Appendix A).

Claim 1. Let \(x \in \mathcal{C}\) be any configuration with \(s(x) \leq (2/3)n\). Then, at the next round, the number of undecided nodes of the U-Process is \(Q \geq n/18\), w.h.p.

Claim 2. Let \(x \in \mathcal{C}\) be any configuration with \(q(x) \geq n/2\) and \(a(x) \geq \log n\). Then, at the next round, the number of Alpha-colored nodes of the U-Process is \(A \geq (1 + 3/4)a(x)\), w.h.p.

Claim 3. Starting from any configuration \(x \in \mathcal{C}\) with \(1 \leq a(x) + b(x) < 2 \log n\), the U-Process reaches a configuration \(X'\) with \(a(X') + b(X') \geq 2 \log n\) within \(O(\log n)\) rounds, w.h.p.

Lemma 4.1 (Phases \(H_1\) and \(H_3\): Starters). Starting from any configuration \(x \in H_1\), the U-Process reaches a configuration \(X' \in (H_2 \cup H_4)\) within \(O(\log n)\) rounds, w.h.p.

Starting from any configuration \(x \in H_3\), the U-Process reaches a configuration \(X' \in (H_1 \cup H_2 \cup H_4)\) in one round, w.h.p.

If the system lies in a configuration of \(H_2\), we need more complex probabilistic arguments to prove that the bias between the two colors reaches \(\Omega(\sqrt{n \log n})\) within \(O(\log^2 n)\) rounds w.h.p. Indeed, we will make use of the following “symmetry-breaking lemma” for discrete-time processes. It’s proof is deferred to Appendix A and it is essentially a simple application of the martingale Optional-Stopping Theorem (see e.g. [23, 24]).

Lemma 4.2 (Symmetry breaking). Let \(n\) be a positive integer, \(\{X_t\}_{t \in \mathbb{N}}\) be a discrete-time stochastic process with a finite state space \(\Omega\), let \(f : \Omega \to [-n, n]\) be a function mapping the state of the process in a range of integer numbers, and consider the stochastic process \(\{f(X_t)\}_{t \in \mathbb{N}}\). Let \(m \in [0, n]\) be a “target” value and let \(\tau\) be the random variable indicating the first time \(|f(X_t)|\) reaches or exceeds \(m\), i.e., \(\tau = \inf\{t \in \mathbb{N} : |f(X_t)| \geq m\}\). Assume that for any \(x \in \Omega\) such that \(|f(x)| < m\) the following conditions hold:

\[\int_0^\tau f(X_s) \, ds \geq \frac{m}{2} \cdot \mathbb{E}[\tau] \quad \text{and} \quad \int_0^\tau (f(X_s))^2 \, ds \leq M \cdot \mathbb{E}[\tau] \]
1. (Non-negative expected drift) $E[|f(X_{t+1})| \mid X_t = x]^2 \geq f(x)^2$.

2. (Non-zero variance) $\text{Var}(f(X_{t+1}) \mid X_t = x) \geq \delta^2 > 0$.

Then, for every starting state $x \in \Omega$ it holds that $E_x[\tau] \leq \frac{E_x[f(X_0)^2]}{\delta^2}$.

The basic idea would be to apply the above lemma to the U-Process $\{X_t\}$ with $f(X_t) = s(X_t)$ and target value $m = \Theta(\sqrt{n \log n})$ in order to get an upper bound on $\tau$, for any starting configuration $x_0 \in H_2$. Observe that Condition 1 in Lemma 4.2 holds and it is an immediate consequence of (3). However, in order to get a meaningful bound on the expectation of $\tau$ out of the lemma, we would need a lower bound $\delta^2$ on the conditional variance $\text{Var}(s(X_{t+1}) \mid X_t = x)$ holding for all $x \in \mathcal{C}$ with $|s(x)| < \sqrt{n \log n}$ and large enough as compared to $E_{x_0}[s(X_0)^2]$. Since our target value is $\Theta(\sqrt{n \log n})$, the expectation $E_{x_0}[s(X_0)^2]$ is $\Omega(n \log n)$ (it would not be difficult to prove that it is also $O(\sqrt{n \log n})$, see Claim 5), so a lower bound on the conditional variance $\delta^2 = \Omega(n)$ is necessary in order to get a logarithmic upper bound on the expectation of $\tau$. Unfortunately, while for any configuration $x \in H_2$ it is true that $\text{Var}(s(X_{t+1}) \mid X_t = x) = \Theta(n)$ (see Claim 6), there are other configurations $x \in \mathcal{C}$ with $|s(x)| < \sqrt{n \log n}$ such that the conditional variance is much smaller (namely, all configurations with $q = n - o(\sqrt{n})$). Hence, if we applied directly Lemma 4.2 to the U-Process we would only get a useless upper bound on $E_{x_0}[s(X_0)^2]$.

However, the helpful, key point is that, starting from any configuration $x \in H_2$, the probability that the process goes in one of those configurations with $|s| < \sqrt{n \log n}$ and $q = n - o(\sqrt{n})$ is negligible (see Claim 3). Thus, intuitively speaking, all the configurations actually visited by the process before exiting $H_2$ do satisfy Condition 2 in Lemma 4.2. In order to make this intuitive argument rigorous, in what follows, we first define a suitably pruned process by removing from $H_2$ all the unwanted transitions that have negligible probability (see (6)) and, then, thanks to Lemma 4.2 we prove that the time the pruned process takes to reach a configuration with sufficiently large bias is $O(\log^2 n)$, w.h.p. Finally, using a coupling argument between the two processes, we show that the time the U-Process needs to reach a configuration with a sufficiently large bias is $O(\log n)$, w.h.p. (see Lemma 4.3).

The following three claims will be used in the proof of Lemma 4.3 (the first two will also turn out useful in the next section). Their proofs can be found in Appendix C.

**Claim 4.** For every configuration $x \in H_2$, the probability that the number of undecided nodes in the next round of the U-Process is not between $n/18$ and $n/2$ is $P(q(X_{t+1}) \notin (n/18, n/2) \mid X_t = x) \leq e^{-\Theta(n)}$.

**Claim 5.** For every configuration $x \in \mathcal{C}$ with $|s(x)| \leq \sqrt{n \log n}$ and for every constant $\alpha > 2$, the probability that the bias in the next round of the U-Process is larger than $\alpha \sqrt{n \log n}$ is $P(|s(X_{t+1})| > \alpha \sqrt{n \log n} \mid X_t = x) \leq 4/n^{(\alpha - 2)^2}$.

**Claim 6.** For every configuration $x \in H_2$, the conditional variance of the bias at the next round of the U-Process is $\text{Var}(s(X_{t+1}) \mid X_t = x) \geq \beta n$ for some constant $\beta > 0$. 

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The pruned process. Let \( \mathcal{A} \) be the set of all configurations \( \mathbf{x} \) such that the magnitude of the bias is smaller than \( 4\sqrt{n \log n} \) and the number of undecided nodes is between \( n/18 \) and \( n/2 \).

\[ \mathcal{A} = \{ \mathbf{x} \in \mathcal{C} : |s(\mathbf{x})| \leq 4\sqrt{n \log n} \text{ and } \frac{1}{18} n \leq q(\mathbf{x}) \leq \frac{1}{2} n \}. \]

We remark that \( H_2 \subset \mathcal{A} \) and that \( \mathcal{A} \setminus H_2 \) is the set of all configurations \( \mathbf{x} \in \mathcal{C} \) with \( \sqrt{n \log n} \leq s(\mathbf{x}) \leq 4\sqrt{n \log n} \) and \( n/18 \leq q(\mathbf{x}) \leq n/2 \). Let us name \( \mathbf{\bar{x}} \in H_2 \) the perfectly-balanced configuration with \( s(\mathbf{\bar{x}}) = 0 \) and \( q(\mathbf{\bar{x}}) = n/3 \). We define a new Markov chain \( \{\mathbf{Y}_t\} \) with state space \( \mathcal{A} \) by pruning all transitions of chain \( \{\mathbf{X}_t\} \) from \( \mathcal{A} \) to \( \mathcal{C} \setminus \mathcal{A} \) and by redirecting all the pruned probability to state \( \mathbf{\bar{x}} \). More formally, for any \( (\mathbf{x}, \mathbf{y}) \in \mathcal{C} \times \mathcal{C} \), let us name \( p_{\mathbf{x},\mathbf{y}} = \mathbf{P} (\mathbf{X}_{t+1} = \mathbf{y} | \mathbf{X}_t = \mathbf{x}) \) the transition probability from \( \mathbf{x} \) to \( \mathbf{y} \) in the original U-Process. For every \( (\mathbf{x}, \mathbf{y}) \in \mathcal{A} \times \mathcal{A} \) we define the transition probability \( p_{\mathbf{x},\mathbf{y}}^* = \mathbf{P} (\mathbf{Y}_{t+1} = \mathbf{y} | \mathbf{Y}_t = \mathbf{x}) \) in the new process as follows

\[
p_{\mathbf{x},\mathbf{y}}^* = \begin{cases} 
  p_{\mathbf{x},\mathbf{y}} & \text{if } \mathbf{y} \neq \mathbf{\bar{x}} \\
  p_{\mathbf{x},\mathbf{y}} + r_\mathbf{x} & \text{if } \mathbf{y} = \mathbf{\bar{x}} 
\end{cases}
\]

(6)

where for each configuration \( \mathbf{x} \in \mathcal{A} \) we named \( r_\mathbf{x} \) the pruned probability

\[ r_\mathbf{x} = \mathbf{P} (\mathbf{X}_{t+1} \notin \mathcal{A} | \mathbf{X}_t = \mathbf{x}) = \sum_{\mathbf{y} \in \mathcal{C} \setminus \mathcal{A}} p_{\mathbf{x},\mathbf{y}}. \]

From Claims 4 and 5 it follows that the pruned probability \( r_\mathbf{x} \) is negligible for every configuration \( \mathbf{x} \in H_2 \). Recall that all configurations \( \mathbf{x} \in \mathcal{A} \) with \( s(\mathbf{x}) \leq \sqrt{n \log n} \) are in \( H_2 \). In the next lemma we prove that the expected exit time from \( H_2 \) in the pruned process is \( O(\log n^2) \), w.h.p.

Lemma 4.3 (The pruned process). Let \( \{\mathbf{Y}_t\} \) be the pruned process and let \( \tau^* \) be the first time it exits from \( H_2 \), \( \tau^* = \inf \{t \in \mathbb{N} : \mathbf{Y}_t \notin \mathcal{A} \setminus H_2 \} = \inf \{t \in \mathbb{N} : |s(\mathbf{Y}_t)| \geq \sqrt{n \log n} \} \). Then, for every \( \mathbf{x} \in \mathcal{A} \) it holds that \( \mathbf{P}_\mathbf{x} (\tau^* > c \log^2 n) \leq 1/n, \) for a suitable constant \( c \).

Proof. We show that the bias of Markov chain \( \{\mathbf{Y}_t\} \) satisfies the hypothesis of Lemma 4.2

Non-negative expected drift. First observe that for every \( \mathbf{x} \in H_2 \)

\[
\mathbf{E} [s(\mathbf{Y}_{t+1}) | \mathbf{Y}_t = \mathbf{x}] = \sum_{\mathbf{y} \in \mathcal{A}} s(\mathbf{y}) p_{\mathbf{x},\mathbf{y}}^* = \sum_{\mathbf{y} \in \mathcal{A} \setminus \{\mathbf{x}\}} s(\mathbf{y}) p_{\mathbf{x},\mathbf{y}}^* = \sum_{\mathbf{y} \in \mathcal{A} \setminus \{\mathbf{x}\}} s(\mathbf{y}) p_{\mathbf{x},\mathbf{y}} = \mathbf{E} [s(\mathbf{X}_{t+1}) | \mathbf{X}_t = \mathbf{x}] - \sum_{\mathbf{y} \in \mathcal{C} \setminus \mathcal{A}} s(\mathbf{y}) p_{\mathbf{x},\mathbf{y}},
\]

(7)

where we used the fact that \( s(\mathbf{x}) = 0 \) and the fact that \( p_{\mathbf{x},\mathbf{y}}^* = p_{\mathbf{x},\mathbf{y}} \) for all \( \mathbf{y} \neq \mathbf{x} \).

Hence, for every \( \mathbf{x} \in H_2 \) we have that

\[
|\mathbf{E} [s(\mathbf{Y}_{t+1}) | \mathbf{Y}_t = \mathbf{x}] - \mathbf{E} [s(\mathbf{X}_{t+1}) | \mathbf{X}_t = \mathbf{x}]| = \left| \sum_{\mathbf{y} \in \mathcal{C} \setminus \mathcal{A}} s(\mathbf{y}) p_{\mathbf{x},\mathbf{y}} \right| \leq \sum_{\mathbf{y} \in \mathcal{C} \setminus \mathcal{A}} |s(\mathbf{y})| p_{\mathbf{x},\mathbf{y}} \leq n \sum_{\mathbf{y} \in \mathcal{C} \setminus \mathcal{A}} p_{\mathbf{x},\mathbf{y}} \leq 1/n^2,
\]

(8)

where we used triangle inequality, the fact that \( |s(\mathbf{y})| \leq n \) and, in the last inequality, Claims 4 and 5. Hence,
\[
\mathbb{E} [s(Y_{t+1}) \mid Y_t = x]^2 \geq \mathbb{E} [s(X_{t+1}) \mid X_t = x] - \frac{1}{n^2} \]
\[
\geq \mathbb{E} [s(X_{t+1}) \mid X_t = x]^2 - \frac{1}{n^2} \mathbb{E} [s(X_{t+1}) \mid X_t = x] \geq \mathbb{E} [s(X_{t+1}) \mid X_t = x]^2 - \frac{1}{n}
\]
\[
= s(x)^2 \left( 1 + \frac{q}{n} \right)^2 - \frac{1}{n} \geq s(x)^2 \left( 1 + \frac{1}{18} \right) - \frac{1}{n} \geq s(x)^2
\]
(9)

where from the third to the fourth line we used (3) and then we used the fact that \( q(x) \geq n/18 \) for \( x \in H_2 \) and the fact that \( s(x) \geq 1 \) (notice that for \( s(x) = 0 \) the non-negative expected drift is trivially satisfied).

Lower bound on the conditional variance. As in (7) we easily obtain that for every \( x \in H_2 \)
\[
\mathbb{E} [s(Y_{t+1})^2 \mid Y_t = x] = \mathbb{E} [s(X_{t+1})^2 \mid X_t = x] - \sum_{y \in \mathcal{C} \setminus A} s(y)^2 \mathbb{P}_{x,y},
\]
and as in (6), from triangle inequality, the fact that \( |s(y)| \leq n \) and Claims 4 and 5 it follows that \( \mathbb{E} [s(Y_{t+1})^2 \mid Y_t = x] - \mathbb{E} [s(X_{t+1})^2 \mid X_t = x] \leq 1/n \). From (7) it also easily follows that
\[
\mathbb{E} [s(Y_{t+1}) \mid Y_t = x]^2 - \mathbb{E} [s(X_{t+1}) \mid X_t = x]^2 \leq 1/n. \quad \text{We then get that for every } x \in H_2
\]
\[
|\text{Var} (s(Y_{t+1}) \mid Y_t = x) - \text{Var} (s(X_{t+1}) \mid X_t = x)| \leq 1/n \quad (10)
\]

From (10) and Claim 6 we can conclude that a positive constant \( \alpha > 0 \) exists such that, for any \( x \in H_2 \) \( \text{Var} (s(Y_{t+1}) \mid Y_t = x) \geq \alpha n \). Thus, since \( s(Y_t) \) satisfies the hypothesis of Lemma 1.2 with \( \delta^2 = \alpha n \), we get that for every starting state \( x \in H_2 \)
\[
\mathbb{E}_x [\tau^*] \leq \mathbb{E}_x [\alpha n] \leq 16n \log n \leq c \log n, \quad \text{for a suitable constant } c.
\]
From Markov inequality we then have that \( \mathbb{P}_x (\tau^* \geq 2c \log n) \leq 1/2 \) and thus \( \mathbb{P}_x (\tau^* \geq 2c \log^2 n) \leq 2^{-c \log n} \).

Back to the original process. We can now prove the main lemma of this section by using a coupling argument.

**Lemma 4.4 (Phase H2).** Starting from any configuration \( x \in H_2 \), the U-Process reaches a configuration \( X' \in H_4 \) within \( O(\log^2 n) \) rounds, w.h.p.

**Proof.** We define a coupling between the original process \( \{X_t\} \) and the pruned one \( \{Y_t\} \) in the obvious way: If the two processes lie in different configurations then the two processes move independently. Otherwise, let \( x \in A \) be the current state, i.e. \( X_t = Y_t = x \), we set
\[
Y_{t+1} = \begin{cases} 
X_{t+1} & \text{if } X_{t+1} \in A \\
\times & \text{otherwise}
\end{cases}
\]
Recall that \( \tau = \inf \{t \in \mathbb{N} : |s(X_t)| \geq \sqrt{n \log n} \} \) and
\[
\tau^* = \inf \{t \in \mathbb{N} : |s(Y_t)| \geq \sqrt{n \log n} \} = \inf \{t \in \mathbb{N} : Y_t \in A \setminus H_2 \}.
\]
Now observe that, if the two coupled processes start in the same configuration \( x_0 \in H_2 \) and \( \tau > c \log^2 n \), then either \( \tau^* > c \log^2 n \) as well, or a round \( t \leq c \log^2 n \) exists such that chains \( \{X_t\} \) and \( \{Y_t\} \) separated at round \( t+1 \), i.e. \( X_t = Y_t = x \) (for some \( x \in H_2 \)) and \( X_{t+1} \notin \mathcal{A} \) and \( Y_{t+1} = x \). Hence,

\[
P_{x_0,x_0}(\tau > c \log^2 n) \leq P_{x_0,x_0}(\tau^* > c \log^2 n) \cup \left\{ \exists x \in H_2 : (X_t = Y_t = x) \land (X_{t+1} \neq Y_{t+1}) \right\}
\]

\[
\leq P_{x_0,x_0}(\tau^* > c \log^2 n) + P_{x_0,x_0} \left( \exists x \in H_2 : (X_t = Y_t = x) \land (X_{t+1} \neq Y_{t+1}) \right).
\]

(11)

As for the first term in (11), from the analysis of the pruned process (Lemma 4.3) we have that it is upper bounded by \( 1/n \). As for the second term, we have that

\[
P_{x_0,x_0} \left( \exists x \in H_2 : (X_t = Y_t = x) \land (X_{t+1} \neq Y_{t+1}) \right)
\]

\[
\leq \sum_{t=1}^{c \log^2 n} P_{x_0,x_0} \left( \exists x \in H_2 : (X_t = Y_t = x) \land (X_{t+1} \neq Y_{t+1}) \right)
\]

\[
= \sum_{t=1}^{c \log^2 n} \sum_{x \in H_2} P_{x_0,x_0} \left( (X_t = Y_t = x) \land (X_{t+1} \neq Y_{t+1}) \right)
\]

\[
= \sum_{t=1}^{c \log^2 n} \sum_{x \in H_2} P_{x_0,x_0} \left( (X_{t+1} \neq Y_{t+1}) \mid (X_t = Y_t = x) \right) P_{x_0,x_0}(X_t = Y_t = x)
\]

\[
\leq \frac{1}{n} \sum_{t=1}^{c \log^2 n} \sum_{x \in H_2} P_{x_0,x_0}(X_t = Y_t = x) = \frac{c \log^2 n}{n}.
\]

(12)

The thesis then follows by using in (11) the bound in Lemma 4.3 for the pruned process and (12).

\[\square\]

## 5 Convergence to the majority

In this section we provide the arguments needed to prove our second main result, namely Theorem 3 which essentially states that starting from any sufficiently biased configuration, the U-Process converges to the monochromatic configuration where all nodes have the majority color. Remind that the outline of the proof is given in Section 3. Here, we formalize the arguments of the provided high-level description. Due to space limitation the proofs of the technical claims are moved to Appendix C.

**Phase \( H_4 \): The age of the undecideds** We first show that under some parameter ranges including \( H_4 \) (and hence when the number of the undecideds are large enough) the growth of the bias is exponential.

**Claim 7.** Let \( \gamma \) be any positive constant and \( x \in \mathcal{C} \) be any configuration such that \( s \geq \gamma \sqrt{n \log n} \) and \( q \geq \frac{1}{10} n \). Then, it holds that \( s(1 + \frac{1}{50}) < S < 2s \), w.h.p.
Lemma 5.1 (Phase $H_4$). Let $x \in H_4$ be a configuration with $a > b$. Then, (i) starting from $x$, the U-Process reaches a configuration $X' \in H_4$ with $a > b$ within $O(\log n)$ rounds, w.h.p. Moreover, (ii) an initial configuration $y \in H_4$ exists such that the U-Process stays in $H_4$ for $\Omega(\log n)$ rounds, w.h.p.

Proof. We iteratively apply Claim 7 and Claim 1 and after $t = \Theta(\log n)$ rounds we have that either there is a round $t' < t$ such that $s(X_{t'}) > \frac{2}{3}n$, or $s(X_t) > (1 + 1/36)^t s(x) \geq (1 + 1/36)^t$. In both cases, the process has reached a configuration $X'$ such that $s(X') > \frac{2}{3}n$ and $q(X') \geq \frac{n}{18}$: So $X'$ belongs to $H_0$. Since each step of the iteration holds w.h.p. and the number of steps is $O(\log n)$, we easily obtain that the result holds w.h.p. by a simple application of the Union Bound.

Concerning the second part of the lemma, consider an initial configuration $y$ such that $s(y) = n^{2/3}$. By iteratively applying (the upper bound of) Claim 7 and Claim 1 for $t = \frac{1}{4} \log n$ rounds, we have that $s(X_t) < 2^{t} s(y) = 2^{t/3} = n^{1/3} n^{2/3} = o(n)$. $\square$

Phase $H_3$: The victory of the majority This is the phase in which a large bias let the nodes converge to the majority color within a logarithmic number of rounds. We first prove that the number of nodes that support the minority color decreases exponentially fast (Claim 8) and that the bias is preserved round by round (Claims 9 and 10). Then, when $b \leq 2\sqrt{n \log n}$, the undecided nodes start to decrease exponentially fast as well (Claim 11). At the very end, when there are only few nodes (i.e., $O(\sqrt{n \log n})$) that do not still support the majority color, the minority color disappears in few steps and thus the U-Process converges to majority within $O(\log n)$ rounds (Claim 12).

Claim 8. Let $x \in C$ be any configuration such that $s \geq \frac{2}{3}n$ and $b \geq \log n$ then it holds that $B \leq b(1 - \frac{1}{2})$, w.h.p.

In order to iteratively apply the above claim we now show that, if there are enough undecided nodes, the bias is preserved round by round until the number of Beta-colored nodes decreases below $2\sqrt{n \log n}$.

Claim 9. Let $x \in C$ be any configuration such that $s \geq \frac{2}{3}n$ and $q \geq \sqrt{n \log n}$. Then it holds that $S \geq \frac{2}{3}n$, w.h.p.

Claim 10. Let $x \in C$ be any configuration such that $s \geq \frac{2}{3}n$ and $b \geq 2\sqrt{n \log n}$. Then it holds that $Q > \sqrt{n \log n}$, w.h.p.

The three above claims imply that, after $O(\log n)$ rounds, the process reaches a configuration such that $s \geq \frac{2}{3}n$, $q \geq \sqrt{n \log n}$ and $b \leq 2\sqrt{n \log n}$. The next claim shows that starting from any such configuration the number of undecided nodes decrease exponentially fast. Next, we show that if the process reaches a configuration such that $q \leq 12\sqrt{n \log n}$ and $b \leq 2\sqrt{n \log n}$ then within few rounds the U-Process converges to the configuration where all nodes support Alpha.

Claim 11. Let $x \in C$ be any configuration such that $12\sqrt{n \log n} \leq q \leq \frac{1}{2}n$ and $b \leq 2\sqrt{n \log n}$ it holds that $Q \leq q(1 - \frac{1}{2})$, w.h.p.

Claim 12. Let $\gamma$ be any positive constant and let $x \in C$ be any configuration such that $q \leq \gamma \sqrt{n \log n}$ and $b \leq 2\sqrt{n \log n}$ then the U-Process reaches a configuration $X'$ with $a(X') = n$ within $O(\log n)$ rounds, w.h.p.
We are now ready to show the following

**Lemma 5.2 (Phase $H_6$).** Starting from any configuration $x \in H_6$ with $a > b$, the U-Process ends in the monochromatic configuration where $a = n$ within $O(\log n)$ rounds, w.h.p.

*Proof.* Let us first assume that $s(x) \geq n - 5\sqrt{n \log n}$ and $q(x) \leq \sqrt{n \log n}$. This implies that $b(x) \leq 2\sqrt{n \log n}$ and thanks to Claim 12 we get that the process end in the configuration such that $a = n$ within $O(\log n)$ rounds. Otherwise $s(x) \geq \frac{n}{2}$ and $q \geq \sqrt{n \log n}$. Then, starting from $x$, we iteratively apply Claim 8 together with Claim 9 and Claim 10 and we get that the process reaches a configuration $X'$ such that $s(X') \geq \frac{n}{2}, q(X') \geq \sqrt{n \log n}$ and $b(X') \leq 2\sqrt{n \log n}$ in $O(\log n)$ rounds. Then we iteratively apply Claim 11 together with Claim 8 (if $b < \log n$ we cannot apply Claim 8 in order to show that $B$ does not overtake $2\sqrt{n \log n}$ but we can get the claim with a simple application of the Markov inequality) and Claim 9 and we get that the process reaches a configuration $X''$ such that $q(X'') \leq 12\sqrt{n \log n}$ and $b(X'') \leq 2\sqrt{n \log n}$ in $O(\log n)$ rounds and now we apply Claim 12 and the process reaches the monochromatic configuration w.h.p. Since every step of the iterations holds w.h.p. and the number of steps is $O(\log n)$, we easily obtain the thesis by a simple application of the Union Bound.

**Phases $H_5$ and $H_7$: Starters** We show that if the process is in a configuration where the number of the undecided nodes is relatively small with respect to the bias, then in the next round the number of the undecided nodes becomes large while the bias does not decrease too much, w.h.p. This essentially implies that if the process starts in $H_5$ then in the next round the process moves to a configuration belonging to $H_5$ or $H_6$ (Lemma 5.3), while if it starts in $H_7$ then in the next round it moves to $H_4$ or $H_5$ or $H_6$ (Lemma 5.4).

**Claim 13.** Let $\gamma, \varepsilon$ be any two positive constants and $x \in C$ any configuration such that $s \geq \gamma \sqrt{n \log n}$ then it holds that $S \geq (\gamma - \varepsilon) \sqrt{n \log n}$, w.h.p.

The above claim together with Claim 11 immediately implies the following

**Lemma 5.3 (Phase $H_5$).** Starting from any configuration $x \in H_5$ with $a > b$, the U-Process reaches a configuration $X' \in (H_4 \cup H_6)$ with $a > b$ in one round, w.h.p.

Concerning phase $H_7$, we have

**Lemma 5.4 (Phase $H_7$).** Starting from any configuration $x \in H_7$ with $a > b$, the U-Process reaches a configuration $X' \in (H_4 \cup H_5 \cup H_6)$ with $a > b$ in one round, w.h.p.

*Proof.* Note that Claim 13 implies that in the next round the process does not enter in $H_1, H_2$ or $H_3$ w.h.p. The hypothesis that $s \leq n - 5\sqrt{n \log n}$ and $q \leq \sqrt{n \log n}$ implies that $b \geq 2\sqrt{n \log n}$ and thus we can apply the Claim 10 and get that the process leaves $H_7$ because of the grown of the undecided nodes.
6 Conclusions

We provided the full analysis of the U-Dynamics in the parallel PULL model for the binary case showing that the resulting process converges quickly, regardless of the initial configuration. Besides giving almost-tight bounds on the convergence time, our set of results well-clarifies the main aspects of the process evolution and the crucial role of the undecided nodes in each phase of this evolution.

We believe that our probabilistic argument for the symmetry-breaking analysis may result useful in the study of other similar parallel processes. A possible further step could be that of considering the same process in the multi-color case and to derive bounds on the time required to break symmetry from balanced configurations, as well. A more specific open question is to establish whether our time bound $O(\log^2 n)$ for symmetry breaking is optimal or it can be improved.

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Appendix

A Proof of the Symmetry Breaking Lemma (Lemma 4.2)

For the sake of brevity we define \( Y_t = f(X_t) \). We define a stochastic process \( \{Z_t\}_t \) such that \( Z_t = Y_t^2 - t\delta^2 \). We first show this new stochastic process is a supermartingale:

\[
E[Z_{t+1}|X_t = x] = E[Y_{t+1}^2|X_t = x] - (t + 1)\delta^2 \\
= E[Y_{t+1}|X_t = x]^2 + \text{Var}(Y_{t+1}|X_t = x) - (t + 1)\delta^2 \\
\geq E[Y_{t+1}|X_t = x]^2 + \delta^2 - (t + 1)\delta^2 \\
\geq E[Y_{t+1}|X_t = x]^2 - t\delta^2 \\
\geq f(X_t)^2 - t\delta^2 \\
= Y_t^2 - t\delta^2 \\
= Z_t
\]

Where in (13) we used the Assumption 2. It is clear that \( \tau \) is a stopping time, and we show it meets the conditions of the Optional Stopping Theorem [20] Corollary 17.7. Indeed:

\[
E[Y_{t+1}^2|X_t = x] = \text{Var}(Y_{t+1}|X_t = x) + E[Y_{t+1}|X_t = x]^2 \geq \delta^2 + f(x)^2
\]

The expectation \( E[Y_{t+1}^2|X_t = x] \) can we rewritten in the following way:

\[
E[Y_{t+1}^2|X_t = x] = E[Y_{t+1}^2|X_t = x \land Y_{t+1} \leq i] P(Y_{t+1} \leq i|X_t = x) \\
+ E[Y_{t+1}^2|X_t = x \land Y_{t+1} \geq i] P(Y_{t+1} \geq i + 1|X_t = x) \\
\geq \delta^2 + f(x)^2
\]

Now we focus on the probability of reaching a configuration \( x' \) such that \(|f(x')| \geq i + 1:\)

\[
P(Y_{t+1} \geq i + 1|X_t = x) \geq \frac{\delta^2 + f(x)^2 - E[Y_{t+1}^2|X_t = x \land Y_{t+1} \leq i] P(Y_{t+1} \leq i)}{E[Y_{t+1}^2|X_t = x \land Y_{t+1} \geq i]} \\
\geq \frac{\delta^2 + f(x)^2 - E[Y_{t+1}^2|X_t = x \land Y_{t+1} \leq i]}{\delta^2 + f(x)^2 - i^2} \\
\geq \frac{E[Y_{t+1}^2|X_t = x \land Y_{t+1} \geq i]}{E[Y_{t+1}^2|X_t = x \land Y_{t+1} \geq i]}
\]

Then, for any \( x \) such that \( f(x) = i \geq 0: \)

\[
P(Y_{t+1} \geq i + 1|X_t = x \land f(x) = i) \geq \frac{\delta^2 + i^2 - i^2}{E[Y_{t+1}^2|X_t = x \land Y_{t+1} \geq i]} \geq \frac{\delta^2}{n^2} = \varepsilon
\]

Where \( \varepsilon \) is a positive constant (it does not depend on \( t \)). This means that the r.v. representing the unbalance have always a constant positive probability of
increase of one step, round by round. Thus starting from any state \( x \in A \) such that \(|f(x)| \leq m - 1\) it easy to see after \( m \) rounds the probability \( P_x (\tau \leq m) \geq \varepsilon^m \).

Hence:

\[
E_x [\tau] = \sum_{i=0}^{\infty} P_x (\tau \geq i) \leq \sum_{i=0}^{\infty} mP_x (\tau \geq im) \leq \sum_{i=0}^{\infty} m(1 - \varepsilon^m)^i < \infty
\]

As for the second condition of the Optional Stopping Theorem, it is clear that for any \( x \in C \) we have \( E [|Y_{t+1} - Y_t| \mid X_t = x] \leq 2n \) (the stochastic process \( \{Y_t\}_{t \in \mathbb{N}} \) defined in \([-n,n]\)), thus we can apply the bound on the expectation of the supermartingale \( \{Z_t\}_t \) to the stopping time \( \tau \) and we get

\[
E[Z_\tau] \geq E[Y_\tau^2] - E[\tau] \delta^2 \geq E[Z_0] \geq 0
\]

\[
E[\tau] \leq E[Y_\tau^2] / \delta^2
\]

\( \square \)

B Useful inequalities

B.1 Chernoff Bound multiplicative form

Let \( X_1, \ldots, X_n \) be independent 0-1 random variables. Let \( X = \sum_{i=1}^{n} X_i \) and \( \mu \leq E[X] \leq \mu' \). Then, for any \( 0 < \delta < 1 \) the following Chernoff bounds hold:

\[
P(X \geq (1 + \delta)\mu) \leq e^{-\mu \delta^2/3}
\]

(14)

\[
P(X \leq (1 - \delta)\mu) \leq e^{-\mu' \delta^2/2}
\]

(15)

B.2 Chernoff Bound additive form

Let \( X_1, \ldots, X_n \) be independent 0-1 random variables. Let \( X = \sum_{i=1}^{n} X_i \) and \( \mu = E[X] \). Then the following Chernoff bounds hold:

for any \( 0 < \lambda < n - \mu \)

\[
P(X \leq \mu - \lambda) \leq e^{-2\lambda^2/n}
\]

(16)

for any \( 0 < \lambda < \mu \)

\[
P(X \geq \mu + \lambda) \leq e^{-2\lambda^2/n}
\]

(17)

B.3 Reverse Chernoff Bound

Let \( X_1, \ldots, X_n \) be independent 0-1 random variables, \( X = \sum_{i=1}^{n} X_i \), \( \mu = E[X] \) and \( \delta \in (0, 1/2] \). Assuming that \( \mu \leq \frac{1}{2}n \) and \( \delta^2 \mu \geq 3 \) then the following bounds hold:

\[
P(X \geq (1 + \delta)\mu) \geq e^{-9\delta^2 \mu}
\]

(18)

\[
P(X \leq (1 - \delta)\mu) \geq e^{-9\delta^2 \mu}
\]

(19)
C.1 Proof of Claim 1

From (4) we get

\[ E[Q|X_t = x] \geq \frac{n}{3} - \frac{s^2}{2n} \geq \frac{1}{3} n - \frac{2}{9} n = \frac{1}{9} n. \]

By applying the additive form of the Chernoff Bound (see (16)) to the random variable \( Q \) we easily get the claim, i.e.,

\[ P(Q \leq \frac{1}{18} n) = P(Q \leq \frac{1}{18} n + \frac{1}{18} n - \frac{1}{18} n) \leq P(Q \leq E[Q] - \frac{1}{18} n) \leq e^{-2n^2/18^2 n} = e^{-\Theta(n)}. \]

C.2 Proof of Claim 2

Proof. We recall (1) and we get

\[ P(A|X_t = x) = a\left(\frac{a + 2q}{n}\right) \]
\[ \geq a\left(\frac{n - q}{2n} + 2q\right) \]
\[ = a\left(\frac{1}{2} + \frac{3q}{2n}\right) \geq a\left(\frac{1}{2} + 3\right) = a\frac{7}{2}. \]

where in (20) we used that \( a = \frac{n - q + s}{2n} \geq \frac{n - q}{2}. \) We apply the multiplicative form of the Chernoff Bound ((15) in Appendix B.1) with \( \delta = \frac{1}{2} \)

\[ P\left(A \leq a\frac{7}{2}(1 - \frac{1}{2})\right) \leq \exp^{-a\frac{7}{2}/3} \leq \exp^{-\log n \frac{7}{2}} \frac{1}{n^{\Theta(1)}}. \]

Thus w.h.p.

\[ A > a\frac{7}{4} = a(1 + \frac{3}{4}). \]

C.3 Proof of Claim 3

Proof. Let’s consider the random variable counting the number of coloured nodes \( A + B \) and his evolution during the process. The probability that in one round a Alpha node picks a Beta node (or vice versa) is less than \( \frac{(2\log n)^2}{n} \). Applying the Union Bound for \( \mathcal{O}(\log n) \) rounds we get that the probability that this event happens at least one time is negligible. If this kind of event does not happen, the coloured nodes will remain coloured, and a undecided node becomes coloured if it picks a coloured node. Thus it immediate see that we can couple this process with a standard rumour spreading process via \textsc{Pull} messages (a coloured node is now called an informed node). Starting from the same state the two process will stay coupled for \( \Omega(\log n) \) rounds w.h.p. and it is known that the spreading process reaches \( 2\log n \) nodes within \( \mathcal{O}(\log n) \) rounds w.h.p.
C.4 Proof of Lemma 4.1

Thanks to Claim 3 we can say that \( a \geq \log n \) within \( \mathcal{O}(\log n) \). Then we can iteratively apply the Claim 2 in order to say that within \( \mathcal{O}(\log n) \) rounds the number of undecided nodes has to drop below \( \frac{1}{2}n \) and the process enters in \( H_2 \). Note that can also happen that the process directly enters in \( H_4 \) because in these rounds the bias is increased. Thanks to Claim 1 the process does not enter into \( H_3 \) or \( H_5 \). Since every step of the iterations holds w.h.p. and the number of steps is \( \mathcal{O}(\log n) \), we easily obtain the thesis by a simple application of the Union Bound.

\( \square \)

C.5 Proof of Claim 4

The upper bound is a consequence of Claim 1. In order to show that \( Q \leq \frac{1}{2}n \) w.h.p., from (4) we get

\[
\mathbb{E}[Q | X_t = x] = \frac{2q^2 + (n - q)^2 - s^2}{2n} \leq \frac{2q^2 + (n - q)^2}{2n}.
\]

Note that if \( \frac{1}{12} n \leq q \leq \frac{1}{2} n \) then the maximum of \( 2q^2 + (n - q)^2 \) is in \( q = \frac{1}{18} n \).

\[
\frac{2q^2 + (n - q)^2}{2n} \leq \frac{2}{12^2} n^2 + (n - \frac{1}{18} n)^2 = \frac{n^2(\frac{2}{12^2} + \frac{17^2}{18^2})}{2n} = \frac{2 + 17^2}{2 \cdot 18^2} n = \frac{291}{648} n = (\frac{1}{2} - \frac{66}{648}) n.
\]

By using the additive form of the Chernoff bound (see (17) in Appendix B.2) with \( \lambda = \frac{66}{648} n \), we obtain

\[
P(Q \geq \frac{1}{2} n) \leq P(Q \geq \mathbb{E}[Q | X_t = x] - \lambda) \leq e^{-2 \cdot \frac{66^2 n^2}{648^2} n} = e^{-\Theta(n)}.
\]

\( \square \)

C.6 Proof of Claim 5

By using the additive form of the Chernoff bound (16) and (17) in Appendix B.2 with \( \lambda = c \sqrt{n \log n} \), for a suitable positive constant \( c \), we get that

\[
P(A \geq \mathbb{E}[A | X_t = x] + c \sqrt{n \log n}) \leq e^{-2c^2 \log n} = \frac{1}{n^{2c^2}}
\]

\[
P(B \leq \mathbb{E}[B | X_t = x] - c \sqrt{n \log n}) \leq e^{-2c^2 \log n} = \frac{1}{n^{2c^2}}
\]

Then, the following holds w.h.p.

\[
S \leq \mathbb{E}[A | X_t = x] + c \sqrt{n \log n} - \mathbb{E}[B | X_t = x] + c \sqrt{n \log n}
\]

\[
= \mathbb{E}[S | X_t = x] + 2c \sqrt{n \log n}
\]

\[
= s(1 + \frac{q}{n}) + 2c \sqrt{n \log n}
\]

\[
\leq 2s + 2c \sqrt{n \log n}
\]

\[
\leq 2 \sqrt{n \log n} + 2c \sqrt{n \log n}.
\]

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Thus, for \( c = \frac{\alpha - 2}{2} > 0 \), we can apply the union bound and get
\[
P \left( S > \alpha \sqrt{n \log n} \mid X_t = x \right) \leq \frac{2}{n^{(\alpha-2)^2}}. \tag{21}
\]

Using simple symmetry argument (i.e. by renaming \( a \) as \( b \) and vice versa), we can also prove that
\[
P \left( S < -\alpha \sqrt{n \log n} \mid X_t = x \right) \leq \frac{2}{n^{(\alpha-2)^2}}.
\]

Finally, by applying the union bound we get the claim, i.e.,
\[
P \left( |S| > \alpha \sqrt{n \log n} \mid X_t = x \right) \leq \frac{4}{n^{(\alpha-2)^2}}.
\]

\( \square \)

### C.7 Proof of Claim 6

Now we show that the variance of the U-Process is big in all the configurations in \( H_2 \). W.l.o.g. by a simple symmetry argument, we can assume that \( a \geq b \). Fix \( x \in C \) with parameters \( a, b, q \). Let \( V_a, V_b, V_q \) be the sets of nodes that are coloured \( \text{Alpha}, \text{Beta} \) and Undecided, respectively. Notice that \( |V_a| = a \), \( |V_b| = b \) and \( |V_q| = q \). We define the following random variables: for each \( i \in V_a, j \in V_b, k \in V_q \):

\[
X_i = \begin{cases} 
1 \text{ with prob. } \frac{a+q}{b} \\
0 \text{ with prob. } \frac{b}{n}
\end{cases} \\
Y_j = \begin{cases} 
0 \text{ with prob. } \frac{a}{n} \\
-1 \text{ with prob. } \frac{b+q}{n}
\end{cases} \\
Z_k = \begin{cases} 
1 \text{ with prob. } \frac{a}{n} \\
0 \text{ with prob. } \frac{b}{n} \\
-1 \text{ with prob. } \frac{q}{n}
\end{cases}
\]

We remark that these r.v.s. are independent. We define \( X = \sum_{i \in V_a} X_i, Y = \sum_{j \in V_b} Y_j \) and \( \sum_{k \in V_q} Z_k \). Note that we can write the bias \( S \) in the next step:

\[
S = X + Y + Z
\]

It holds that:

\[
\text{Var} (X) = a \frac{b(a+q)}{n^2}
\]
\[
\text{Var} (Y) = b \frac{a(b+q)}{n^2} \tag{22}
\]
\[
\text{Var} (Z) = q \frac{n(a+b) - (a-b)^2}{n^2}. \tag{23}
\]

Moreover note that, for any state \( x \in H_2, a = \Theta(n), b = \Theta(n) \) and \( q = \Theta(n) \) then, for some constant \( \beta > 0 \)
\[
\text{Var} (S \mid X_t = x) = a \frac{b(a+q)}{n^2} + b \frac{a(b+q)}{n^2} + q \frac{n(a+b) - (a-b)^2}{n^2} \geq \beta n \tag{24}
\]

\( \square \)
C.8 Proof of Claim \[\text{[7]}\]

Proof. Recall that \(S = A - B\). In order to show that \(S > s(1 + \frac{1}{36})\) w.h.p., we provide two independent bounds to the values of \(A\) and \(B\), respectively. We use the additive form of the Chernoff bound ((16) and (17) in Appendix B.2) with \(\lambda = \frac{\gamma \sqrt{n \log n}}{12}\). Hence, we have

\[P(A \leq \mathbb{E}[A|X_t = x] - \lambda) \leq e^{-2\lambda^2/n} = e^{-2\gamma^2 \log n/72} = \frac{1}{n^{\Theta(1)}},\]

and

\[P(B \geq \mathbb{E}[B|X_t = x] + \lambda) \leq e^{-2\lambda^2/n} = e^{-2\gamma^2 \log n/72} = \frac{1}{n^{\Theta(1)}}.\]

Then w.h.p.

\[
S > \mathbb{E}[A|X_t = x] - \lambda - \mathbb{E}[B|X_t = x] - \lambda \\
= \mathbb{E}[A - B|X_t = x] - 2\lambda \\
= \mathbb{E}[S|X_t = x] - 2\lambda \\
= \mathbb{E}[S|X_t = x] - 2\lambda \\
\geq s(1 + \frac{q}{n} - \frac{\gamma \sqrt{n \log n}}{36}) \\
\geq s(1 + \frac{1}{18} - \frac{1}{36}) \\
= s(1 + \frac{1}{36}).
\]

We now show that \(S < 2s\) w.h.p. using similar arguments as above. Once again, we use the additive form of the Chernoff bound with \(\lambda = \frac{\gamma \sqrt{n \log n}}{4}\). We have

\[P(A \geq \mathbb{E}[A|X_t = x] + \lambda) \leq e^{-2\lambda^2/n} = e^{-2\gamma^2 \log n/16} = \frac{1}{n^{\Theta(1)}},\]

and

\[P(B \leq \mathbb{E}[B|X_t = x] - \lambda) \leq e^{-2\lambda^2/n} = e^{-2\gamma^2 \log n/16} = \frac{1}{n^{\Theta(1)}}.\]

As a consequence, we have that w.h.p.

\[
S < \mathbb{E}[A|X_t = x] + \lambda - \mathbb{E}[B|X_t = x] + \lambda \\
= \mathbb{E}[A - B|X_t = x] + 2\lambda \\
= \mathbb{E}[S|X_t = x] + 2\lambda \\
= \mathbb{E}[S|X_t = x] + 2\lambda \\
< s(1 + \frac{q}{n} + \frac{\gamma}{2} \sqrt{n \log n}) \\
< s(1 + \frac{1}{2}) + \frac{1}{2}s \\
= 2s.
\]

\[\Box\]
C.9 Proof of Claim 8

Proof. From (5), since \( s \geq \frac{2}{3} n \), we have that

\[
E \left[ B \mid X_t = x \right] = b \left( 1 - \frac{2s + 3b - n}{n} \right) \leq b \left( 1 - \frac{2s - n}{n} \right) \leq b \left( 1 - \frac{\frac{2}{3} n - n}{n} \right) = b \left( 1 - \frac{1}{3} \right).
\]

Thus, we apply the multiplicative form of the Chernoff Bound ((14) in Appendix B.1) with \( \delta = \frac{1}{3} \), and we obtain

\[
P \left( B \geq (1 + \delta) \left( 1 - \frac{1}{3} \right) b \right) \leq e^{-b(1-\frac{1}{3})\delta^2/3} \leq e^{-\log n(1-\frac{1}{3})\delta^2/3} = \frac{1}{n^{\Theta(1)}}.
\]

As a consequence, we have that w.h.p.

\[
B \leq b(1 + \delta) \left( 1 - \frac{1}{3} \right) = b \left( 1 + \frac{1}{3} \right) \left( 1 - \frac{1}{3} \right) = b \left( 1 - \frac{1}{9} \right).
\]

\[\square\]

C.10 Proof of Claim 9

Proof. We recall that \( S = A - B \), thus we provide two independent bounds to the values of \( A \) and \( B \) respectively. We use the additive form of the Chernoff bound ((16) and (17) in Appendix B.2) with \( \lambda = \varepsilon \sqrt{n \log n} \). We have

\[
P (A \leq E [A | X_t = x] - \lambda) \leq e^{-2\lambda^2/n} = e^{\varepsilon^2 \log n/2} = \frac{1}{n^{\Theta(1)}},
\]

and

\[
P (B \geq E [B | X_t = x] + \lambda) \leq e^{-2\lambda^2/n} = e^{\varepsilon^2 \log n/2} = \frac{1}{n^{\Theta(1)}}.
\]

Then it holds that, w.h.p.

\[
S \geq E [A | X_t = x] - \lambda - E [B | X_t = x] - \lambda = E [A - B | X_t = x] - 2\lambda = E [S | X_t = x] - 2\lambda = s(1 + \frac{q}{n}) - \varepsilon \sqrt{n \log n} \geq s + \frac{2 \sqrt{n \log n}}{3} - \varepsilon \sqrt{n \log n} > s
\]

\[\square\]

C.11 Proof of Claim 10

Proof. The number of \texttt{Beta}-colored nodes is at least \( 2 \sqrt{n \log n} \) and each node has probability at least \( 2/3 \) to pick a \texttt{Alpha}-colored node. Thus \( E [Q] > \frac{1}{3} \sqrt{n \log n} \) and we get the claim by a simple application of the additive form of the Chernoff bound.
C.12 Proof of Claim 11

Proof. From (2), we have:

\[
E[Q|X_t = x] = \frac{q^2 + 2ab}{n} \leq \frac{q^2 + 4n\sqrt{n \log n}}{n} = q\left(\frac{q + 4\sqrt{n \log n}}{q}\right) \leq q\left(\frac{1}{3} + \frac{1}{3}\right) = q\left(1 - \frac{1}{3}\right).
\]

Thus we apply the multiplicative form of the Chernoff Bound (14 in Appendix B.1) with \(\delta = \frac{1}{3}\):

\[
P\left(Q \geq (1 + \delta)\left(1 - \frac{1}{3}\right)q\right) \leq e^{-q(1-\frac{1}{3})\delta^2/3} \leq e^{-\log n(1-\frac{1}{3})\delta^2/3} = \frac{1}{n^{3\delta(1)}}.
\]

Thus we get that, w.h.p.

\[
Q \leq \left(1 + \frac{1}{3}\right)\left(1 - \frac{1}{3}\right)q = q\left(1 - \frac{1}{9}\right).
\]

\(\square\)

C.13 Proof of Claim 12

Proof. We first show that in one round the number nodes that support the color Beta becomes logarithmic and the number of undecided nodes does not increase.

\[
E[B|X_t = x] = b\left(\frac{b + 2q}{n}\right) \leq 2\sqrt{n \log n}\left(\frac{2\sqrt{n \log n} + 2\gamma \sqrt{n \log n}}{n}\right) = 4(\gamma + 1) \log n.
\]

It is immediate concentrate using the multiplicative form of the Chernoff bound and get that \(B < 8(\gamma + 1) \log n\) w.h.p. We now show that the number of the undecided nodes is still \(O(\sqrt{n \log n})\). Indeed

\[
E[Q|X_t = x] = \frac{q^2}{n} + \frac{2ab}{n} \leq \gamma^2 \log n + 4\sqrt{n \log n}.
\]

Then using the additive form of the bernoulli bound we get that \(Q \leq 5\sqrt{n \log n}\) w.h.p. In the next round, w.h.p., no undecided node picks a node colored of Beta or viceversa so we can conclude that there are no nodes supporting Beta left (and it easy to show that there is at least one supporter of Alpha w.h.p.). From now on, the stochastic process is equivalent to a classic spreading process via Pull operations, and thus, in the \(O(\log n)\) rounds, all the nodes will support Alpha w.h.p. \(\square\)

C.14 Proof of Claim 13

Proof. We recall that \(S = A - B\), thus we provide two independent bounds to the values of \(A\) and \(B\) respectively. We use the additive form of the Chernoff bound (16 and 17 in Appendix B.2) with \(\lambda = \frac{\sqrt{n \log n}}{2}\). We have

\[
P(A \leq E[A|X_t = x] - \lambda) \leq e^{-2\lambda^2/n} = e^{2\log n/2} = \frac{1}{n^{3\delta(1)}},
\]

and
\[ P \left( B \geq E[B|X_t = x] + \lambda \right) \leq e^{-2\lambda^2/n} = e^{\varepsilon^2 \log n/2} = \frac{1}{n^{\Theta(1)}}. \]

Then it holds that, w.h.p.

\[
S \geq E[A|X_t = x] - \lambda - E[B|X_t = x] - \lambda \\
= E[A - B|X_t = x] - 2\lambda \\
= E[S|X_t = x] - 2\lambda \\
= s(1 + \frac{q}{n}) - \varepsilon \sqrt{n \log n} \\
\geq s - \varepsilon \sqrt{n \log n} \\
\geq \gamma \sqrt{n \log n} - \varepsilon \sqrt{n \log n} \\
= (\gamma - \varepsilon) \sqrt{n \log n}.
\]

\[\Box\]

C.15 Tightness of Theorem 2

Claim 14. An initial configuration exists with \(|s| = \Theta(\sqrt{n})\) such that the process converges to the minority color with constant probability

Proof. Let us consider the configuration \(x\) such that \(q(x) = n/3, a(x) = n/3 + \sqrt{n}\) and \(b(x) = n/3 - \sqrt{n}\). We prove that in one round there is constant probability that the bias becomes zero or negative. After that, by simple symmetry argument, we get the claim.

We define \(A^q, B^q, Q^q\) the random variables counting the nodes that was undecided in the configuration \(x\), and in the next round are respectively colored of Alpha, Beta or undecided. Similarly \(A^a (B^b)\) counts the nodes that was supporting the color Alpha (Beta) in the configuration \(x\) and that are still support the same color in the next round.

Since it is impossible that a node supporting a color in one round could support the other color in the next round, it holds that \(A = A^q + A^a\) and \(B = B^q + B^b\). Note that, among these random variables, only \(A^q\) and \(B^q\) are not independents. Thus, for any positive constant \(\delta\), it holds that

\[ P( S \leq 0 ) = P( B \geq A ) \\
= P( B^q + B^b \geq A^q + A^a ) \\
\geq P( B^q \geq A^q, B^b \geq n/3 + \delta \sqrt{n}, A^a \leq n/3 + \delta \sqrt{n} ) \\
= P( B^q \geq A^q ) \cdot P( B^b \geq n/3 + \delta \sqrt{n} ) \cdot P( A^a \leq n/3 + \delta \sqrt{n} ) \\
\]

With a simple application of the Reverse Chernoff bound (see [18]) we get that \( P( B^q \geq n/3 + \delta \sqrt{n} ) \) is atleast constant, whereas the fact that \( P( A^a \leq n/3 + \delta \sqrt{n} ) \) is atleast constant is an immediate consequence of the additive form of the Chernoff Bound (see [17]).
Thus we need to show that also $P(B^q \geq A^q)$ is at least constant. Note that the distribution $B^q$ conditioned to the event $Q^q = k$ is a binomial distribution with parameters $(\frac{n}{3} - k, \frac{b(x)}{a(x) + b(x)})$ and with expectation $E[B^q | Q^q = k, X_t = x] = (\frac{n}{3} - k)/2 - (\frac{n}{4} - k)/(6\sqrt{n})$. Thus we get

$$P(B^q \geq A^q) = \sum_{k=1}^{n/3} P(B^q \geq A^q | Q^q = k) P(Q^q = k)$$

$$> \sum_{k=n/4}^{n/2} P(B^q \geq A^q | Q^q = k) P(Q^q = k)$$

$$= \sum_{k=n/4}^{n/2} P\left(B^q \geq \left(\frac{n}{3} - k\right)/2 \mid Q^q = k\right) P(Q^q = k)$$

$$= \sum_{k=n/4}^{n/2} P\left(B^q \geq E[B^q | Q^q = k] + (\frac{n}{3} - k)/(6\sqrt{n}) \mid Q^q = k\right) P(Q^q = k)$$

$$= \sum_{k=n/4}^{n/2} P\left(B^q \geq E[B^q | Q^q = k] + (\frac{n}{3} - k)/(6\sqrt{n}) \mid Q^q = k\right) P(Q^q = k)$$

$$\geq \sum_{k=n/4}^{n/2} \exp\left(-9 \cdot \frac{1}{(3\sqrt{n}-1)^2} \cdot n \cdot E[B^q | Q^q = k]\right) P(Q^q = k)$$

$$\geq \sum_{k=n/4}^{n/2} \exp\left(-9 \cdot \frac{1}{(3\sqrt{n}-1)^2} \cdot n\right) P(Q^q = k)$$

$$= \exp\left(-9 \cdot \frac{1}{(3\sqrt{n}-1)^2} \cdot n\right) \sum_{k=n/4}^{n/2} P(Q^q = k)$$

$$= O(1) \sum_{k=n/4}^{n/2} P(Q^q = k)$$

$$= O(1)(1 - e^{O(n)})$$

(25)

(26)

Where in (25) we used the reverse Chernoff bound (see [18]) and in (26) we used that $E[Q^q] \approx \frac{n}{4}$ and the additive form of the Chernoff bound.