Thermalization and its Breakdown for a Large Nonlinear Spin

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By developing a semi-classical analysis based on the Eigenstate Thermalization Hypothesis, we determine the long time behavior of a large spin evolving with a nonlinear Hamiltonian. Despite integrable classical dynamics, we find the Eigenstate Thermalization Hypothesis is satisfied in the majority of eigenstates and thermalization is generic. The exception is a novel mechanism for the breakdown of thermalization based on an unstable fixed point in the classical dynamics. Using the semi-classical analysis we derive how the equilibrium values of observables encode properties of the initial state. We conclude with a discussion of relevant experiments and the potential generality of this mechanism for the breakdown of thermalization.

In recent years, experiments on ultra cold atoms and trapped ions[1–4] have succeeded in producing quantum systems that, on relevant time scales, are completely isolated from an environment. Surprisingly, many of these experiments find long time behavior that mimic a system coupled to an environment. These experiments prompt the question of thermalization: Given an initial state \(|\psi(t = 0)\rangle\), a Hamiltonian \(H = \sum_n E_n |n\rangle \langle n|\), and an observable \(O\), when and why does the long time average of \(O\):

\[
O(t,T) = \frac{1}{T} \int_t^{t+T} \text{d}\tau \langle \psi(\tau) | O | \psi(\tau) \rangle
\]

lose memory of its initial state? In other words, when does \(O(t,T)\), at long time \(t\), depend only on the energy of the initial state?

The eigenstate thermalization hypothesis (ETH)[5–10] attempts to answer this question. Briefly, it states that if \(A1\) \(|n\rangle \langle n|O\rangle \langle n|n\rangle\) changes very little between eigenstates with similar energy; \(A2\) the level spacings, \(E_n - E_{n+1}\), are sufficiently small; and \(A3\) the energy uncertainty of the initial state is sufficiently small, then an eigenstate, randomly selected from a micro-canonical ensemble at the energy of the initial state, will describe the long time average observable (LTO):

\[
O(t,T) \approx \langle n|O|n\rangle \text{for large } t \text{ and } T.
\]

ETH is often discussed in extended systems where interactions produce chaotic behavior[9], thermalization is described as a process in which a larger subsystem acts as a thermal bath for a smaller one, and the reduced density matrix of the smaller system is approximated as a Gibbs ensemble[7]. In these systems, the standard mechanism for the breakdown of thermalization is the emergence of an extensive set of conserved charged due to underlying integrability[11–13] or a random disorder potential[14–16].

In few mode bosonic systems, micro-canonical thermalization has been investigated using a semi-classical analysis[17–23]. In these works, the loss of initial state memory is explained by the appearance of chaos in the corresponding classical system as opposed to a direct analysis of eigenstates.

In this letter, we explore a similar phenomenon for the long time behavior of a quantum evolution, but for a system which is not extended nor classically chaotic. The model we study is that of an SU(2) spin with large fixed size \(|J| > 50\) and evolving with respect to the Hamiltonian \(H = -J_x + \frac{\Lambda}{2J_z} J_z^2\), where \(J_x, J_z\) and \(J_y\) are the canonical SU(2) spin operators, and we assume \(\Lambda > 1\).

We formulate the question of thermalization for this system by asking: 1) for which initial states do LTOs thermalize and approach a micro-canonical ensemble, and 2) for states that do not thermalize, what is the mechanism that maintains information about the initial state. We focus our analysis on the observables \(O = J_x\) and \(O = J_z\), and refer to the Supplemental Material for other observables.

This spin Hamiltonian is expected to describe boson tunneling experiments[4, 24], and the theory community has explored its dynamics[25–34]. Particularly relevant results are those relating to a semi-classical analysis[25–28, 31–33, 35], and describe the classical trajectories shown in Fig. 1. These trajectories, and corresponding eigenstates, have two distinct behaviors known as Josephson oscillation and self trapping, and are separated by a separatrix at \(E = 1\). Unlike the few-mode boson models, these trajectories are not chaotic and relaxation occurs through quantum effects[31]. Thus, to consider the question of thermalization, we use the correspondence between periodic classical trajectories and eigenstates[36] to access the assumptions required by ETH and answer
the questions posed above. We find that 1) for initial states with energy sufficiently different from the energy of the separatrix, $E = 1$, the assumptions of ETH are obeyed and observables come to an equilibrium described by a micro-canonical ensemble; and 2) for initial states with energy on the separatrix, the assumptions of ETH do not hold and LTO do not thermalize. We find that the breakdown of thermalization is due to the $E \approx 1$ eigenstates becoming localized in classical phase space. The semi-classical analysis finds that the localization is due to the asymptotically slow classical dynamics near the unstable fixed point shown in Fig. 1. The ubiquity of this semi-classical feature suggests that this mechanism for the breakdown of thermalization is a general phenomenon present in other models.

Semi-Classical Picture and ETH: We first consider the case when the assumptions of ETH are valid and the large spin thermalizes. To do so it will be useful to first consider why assumptions of ETH generally imply thermalization. First consider the eigenstate decomposition of the initial state density matrix, $\sum_{nm} c_n^* c_m \langle n | m \rangle$. At long times, $t$ and for sufficiently large $T$, we can expect that only the diagonal terms of the density matrix to contribute to observables[8, 31]:

$$O(t, T) \approx \sum_{m} |c_m|^2 \langle m|O|m \rangle.$$  

If A3) of ETH is true, then $|c_m|^2$ is non zero only in a small energy window. Furthermore, if A1) of ETH holds, then $\langle m|O|m \rangle$ is approximately constant over the eigenstates with significant probability $|c_m|^2$. Finally A2) ensures there are multiple eigenstates in the micro-canonical ensemble which can be sampled, and we can conclude that a representative eigenstate $\langle n|O|n \rangle$ can be chosen to factor out of the average in Eq. 2 yielding: $O(t, T) \approx \langle n|O|n \rangle$.

We now use a semi-classical analysis to determine when these three assumptions of ETH hold for the nonlinear spin Hamiltonian. The semi-classical analysis is based on a Wigner-function formalism in which states and operators are represented as functions, $W(z, \phi)$ and $O(z, \phi)$, of $z$, the eigenvalue of $J_z/|J|$, and its conjugate momentum $\phi$. In this formalism, the observables $J_z$ and $J_x$ are given by $|J|z$ and $|J|\sqrt{1 - z^2} \cos(\phi)$ respectively, and the Hamiltonian is written as:

$$\frac{H(z, \phi)}{|J|} = \frac{\Lambda}{2} z^2 - \sqrt{1 - z^2} \cos(\phi)$$  

The expectation values of a state $W(z, \phi)$ with an observable $O(z, \phi)$ is computed with:

$$\langle \psi | O | \psi \rangle = \frac{1}{4\pi} \int_{-\pi}^{\pi} dz d\phi W(z, \phi) O(z, \phi).$$  

We use the set of spin coherent states as our initial states because they are regularly created in experiments[4, 24]. In the Wigner-function formalism these states are represented by Gaussian distributions (see [37] for specific form) that become more localized around a mean $z'$ and a mean $\phi'$ as the size of the spin, $|J|$, is increased. Since a state which is more local around a specific $z'$ and $\phi'$ has smaller energy uncertainty, assumption A3) of ETH is satisfied when $|J|$ is sufficiently large.

We now consider when assumptions A1) and A2) hold by constructing the Wigner functions of the eigenstates via a semi-classical analysis. The zeroth order classical analysis treats Eq. 3 as a classical Hamiltonian which yields the periodic trajectories depicted in Fig. 1. Fig. 1 shows two distinct types of periodic trajectories depending on the energy: for $E < 1$, the trajectories known as Josephson oscillation[25] occur in which $z$ and $\phi$ periodically oscillates around a stable fixed point at $(z, \phi) = (0, 0)$, while for $E > 1$ trajectories called self trapping[38] occur in which $z$ does not change sign, and $\phi$ monotonically increases ($z < 0$) or decreases ($z > 0$) depending on the sign of $z$. At $E = 1$, there is a separatrix separating the two dynamical behaviors.

Using the correspondence between classical periodic trajectories and eigenstates[36], the eigenstate Wigner-functions (EWF) with energy $E$ can be written as $W_E(z, \phi) = w(E) \delta(H(z, \phi) - E)$, where $w(E)$ is the normalization of the eigenstate with energy $E$. The quantized energy levels, $E = E_n$, are then determined by the rule[28] stating that the area swept out by the eigenstate trajectories is quantized to $2\pi/|J|$. Thus, the energy difference between the eigenstate trajectories goes to 0 as $|J|$ is increased, and assumption A2) of ETH holds. true.
Considering assumption A1), we first identify that the Hamiltonian in Eq. 3 has two distinct types of eigenstates corresponding to the Josephson oscillation and the self trapping trajectories. The self trapping eigenstates are further structured because, for a given energy $E > 1$, there are two disconnected trajectories depending on the initial sign of $z$. These two trajectories will be identified with the sign of $z$ and their associated EWFs are calculated by selecting the correct trajectory when inverting $H(z, \phi)$:

$$\rho_{E \pm}(z, \phi) = w(E) \frac{dH(z, \phi)}{dz}^{-1} \delta(z \pm |H^{-1}(E, \phi)|)\chi(5)$$

At lowest order in a semi-classical expansion, these two trajectories correspond to two degenerate eigenstates, while at higher order the degeneracy is lifted with splitting exponentially decreasing with $|J|$. Since this splitting is exponentially small, we will ignore it and assume all measurements occur before its dynamics are realized ($t < T_1 \approx e^{-|J|}$).

With EWFs identified, we are now able to assess the validity of assumption A1) of ETH. For the Josephson oscillation eigenstates, the difference between two neighboring eigenstate trajectories decreases to 0 as $|J|$ is increased. Thus, we expect the eigenstate observables will be smooth in energy for $-1 < E < 1$. The same is also true in each of the sgn($z$) = ±1 branches of the self trapping eigenstates with $E > 1$. On the separatrix, the self trapping trajectories meet the free oscillating ones, a discontinuity emerges, and non analytic behavior of the eigenstate observables is expected. The smooth behavior away from $E = 1$ along with the non-analytic behavior at $E = 1$ has been identified previously and we confirm for $J_x$ in Fig. 1.

Thus, we find that away from $E = 1$ and for large enough $|J|$, the assumptions of ETH hold, and we expect the LTOs to be described by a micro-canonical ensemble. While for eigenstates with energy $E \approx 1$, assumption A1) of ETH does not hold, and additional consideration is required to understand the long time behavior.

**Numerical Analysis of the Diagonal Ensemble:** From the analysis of the previous section we expect initial coherent states with $z'$ and $\phi'$ away from the separatrix to show thermal behavior at long times. That is, we expect the LTO to 1) depend solely on conserved quantities (the energy and initial sign of $z$ for $E > 1$), 2) match a micro-canonical ensemble and 3), for large enough spin, to match a characteristic eigenstate selected from the micro-canonical ensemble. To confirm these expectations, we focus on initial coherent states and compute LTOs using exact diagonalization and Eq. 2. In the left column of Fig. 2, we show the long time expectation of $J_x$ and $J_z$ for a set of initial coherent states with different $\phi'$, but with energy fixed to $E = 0.5$ in the Josephson Oscillation regime. It demonstrates that the LTO cannot distinguish between initial states with different $\phi'$, and thus memory of the initial state is lost.

We next identify if a micro-canonical ensemble, and a characteristic eigenstate describe the LTOs. The micro-canonical ensemble is constructed as a Gaussian distribution in the energy eigenstates with a projection onto one of the self trapping branches (see SM Sec.2 for numerical details). We take the eigenstate with maximum amplitude as the characteristic eigenstate to be compared with the LTO. Fig. 1 depicts how the expectation value of $J_x$ for the micro-canonical ensemble, the characteristic eigenstate, and the diagonal ensemble depend on energy, and confirms that the micro-canonical ensemble and the characteristic eigenstate describe the LTO computed by the diagonal ensemble for $E \neq 1$.

Close the $E = 1$, the micro-canonical ensemble and the characteristic eigenstate no longer match LTOs. Failure of the initial states at $E = 1$ to thermalize is further demonstrated in the right column of Fig. 2, which shows a dramatic dependence of the diagonal ensemble expectation value on the initial phase, $\phi'$. This does not invalidate ETH because assumption A1) of ETH does not hold for these eigenstates.

**Semi-Classical Analysis of the Breakdown of Thermalization:** To better understand this breakdown of thermalization we investigate, using the semi-classical analysis, how the $E \approx 1$ eigenstates affect the LTO of the initial coherent states with $E \approx 1$. We begin by calculating the diagonal ensemble and its expectation values for the initial coherent states used above. Semi-classically, this
diagonal ensemble is given as:

$$\rho_{\text{diag}} = \frac{1}{4\pi} \int_{-\pi}^{\pi} dz d\phi W_c(z, \phi, z', \phi') \rho_{E,s}(z, \phi)$$  \hfill (6)

where \(W_c\) is the initial coherent state Gaussian distribution centered around \(z'\) and \(\phi'\) with variance \(\frac{1}{2}\), and the EWF, \(\rho_{E,s}\), is given by Eq. 5. This integral can be computed when \((z', \phi') \neq (0, \pi)\) since the Jacobian \(\left| \frac{dH(z, \phi)}{dz} \right|^{-1} \) is approximately constant in the vicinity of \(z'\) and \(\phi'\). Performing the Dirac delta and Gaussian integrations yields:

$$\rho_{\text{diag}}(\phi', z', E, s) \sim e^{-\frac{E-H(\phi', z')}{2\sigma} + \ln(w(E))}$$  \hfill (7)

where \(w(E)\) is the eigenstate normalization, and the energy variance \(\sigma(\phi', z')\) scales with \(|J|\) as \(\sim \frac{1}{\sqrt{|J|}}\) with proportionality dependent on \(\phi'\) and \(z'\).

To calculate the LTOs, one must convolve the diagonal ensemble with the eigenstate expectation values:

$$O_{\text{diag}}(\phi', z') = \int_{-1}^{z'} dE \sum_s \rho_{\text{diag}}(\phi', z', E, s) O(E, s)$$  \hfill (8)

where the sum over \(s\) is the sum over self trapping states when \(E > 1\) and a fixed \(s = 0\) for \(E < 1\), and \(O(E, s)\) is the eigenstate expectation value calculated using Eq. 4 with \(W(z, \phi) = \rho_{E,s}(z, \phi)\).

Understanding this integral, and consequently why the LTOs encode information about the initial phase \(\phi'\), requires understanding the structure of the eigenstates and their EWFs. While an EWF is constrained to an equal energy surface, the shape of the energy surface affects how the EWF is distributed within the energy surface. This is captured by the Jacobian \(\left| \frac{dH(z, \phi)}{dz} \right|\), which appears in Eq. 5 due to the transformation of the energy delta function to phase space coordinates. Take the \(s = 1\) self trapping eigenstate for example. If one integrates over \(z\) using the delta function, the Jacobian \(\frac{dH}{dz}(E, \phi) = \left| \frac{dH}{dz}(E, \phi) \right|\) weights the EWF. Therefore, the EWF will have more weight in regions where \(\phi\) is changing slower in time.

On the separatrix, \(E = 1\), the classical spin comes to a complete stop on the unstable fixed point: the Jacobian limits to 0, \(\lim_{E \to 1} \lim_{\phi \to \pi} \left| \frac{dH}{dz}(E, \phi) \right| = 0\); and the EWFs with \(E \to 1\) become localized on the unstable fixed point: \(\rho_{E \to 1}(z, \phi) \approx \delta(z)\delta(\phi - \pi)\). The singularity of this localization result in the non-analytic behavior of the eigenstate expectation values near \(E = 1\) (see for example \(J_z\) in Fig. 1).

This singular localization also produces a non-analyticity in the eigenstate overlaps for the set of initial coherent states with \(E \approx 1\), but \(\phi' \neq \pi\). Since these initial states have Wigner functions localized around \(\phi'\) and \(z' = H^{-1}(E = 1, \phi')\) and the EWFs for \(E \approx 1\) are localized around \(\phi = \pi \neq \phi'\) and \(z = 0 \neq z'\), their overlap integrals in Eq. 8 will vanish.

These two non-analyticities are integrated over in Eq. 8 and result in the memory effects depicted in Fig. 2. In one limit, an initial coherent state with \(\phi' \approx \pi\) will overlap the unstable fixed point eigenstate at \(E = 1\), and the LTOs will closely match the observables of that same eigenstate\((J_z = 0, J_y = -1)\). In the other limit, when the initial \(\phi'\) is away from \(\pi\), the initial coherent state will have negligible overlap with the \(E = 1\) eigenstate, the LTOs will depart from the observables of the \(E = 1\) eigenstate. This is depicted in Fig. 2, in which the closer \(\phi'\) to \(\pi\), the closer \(J_z = J_z/J\) and \(J_x = J_x/J\) approach 0 and -1 respectively.

To capture this behavior analytically, we perform a saddle point expansion for the integral Eq. 8, and evaluate the integrand at the peaks of \(\rho_{\text{diag}}\), given in Eq. 7. To calculate the integrands that give the eigenstate normalization, \(w(E)\), and the eigenstate observables, \(O(E, s)\), we perform an expansion of the trajectories around the unstable fixed point to second order in \((z, \phi)\). The phase, \(\phi'\), and size, \(|J|\), dependence of diagonal ensemble observables are then derived as[42]:

$$j_{x,z,\text{diag}}(|J|, E = 1, \phi') = \frac{4\pi \sqrt{(\Lambda - 1)}}{3\Lambda \text{Ln}[F(\phi') \text{Ln}[|J|]]} \left[ 3 + \Lambda - \frac{1}{F(\phi') \text{Ln}[|J|]} \right]$$  \hfill (9)

where the factor \(F(\phi') = [2\sigma(z', \phi')^2 |J|]^{-1}\), \(j_{x,z}(z) = J_z/J\), and \(z'\) is fixed by energy \(z' = H^{-1}(E = 1, \phi')\). The factor \(F(\phi')\) is constant in \(|J|\) but has a non trivial dependence on the initial phase \(\phi'\) via \(\sigma(z', \phi')\), the energy variance of the coherent state. This non trivial dependence in \(\phi'\) describe the memory effects shown in
Fig. 2 for the initial states with $\phi' \neq \pi$. For the initial states with $\phi' \approx \pi$ the approximation for $\rho_{\text{diag}}$ given in Eq. 7 breaks down, and the results in Eq. 9 become invalid. For these states, a different approximation is required and results in LTO closer to the unstable fix point [28, 33]. While the exact diagonal ensemble for $J_z$ becomes numerically unstable for large $|J|$, we can still compare exact results for $J_x$ with Eq. 9. This comparison is shown in Fig. 3, where the $\sqrt{|J/L_n|} |J|$ scaling is confirmed. Note that the breakdown of thermalization is a quantum effect which is lost in the $J \to \infty$ classical limit.

Discussion and Possible Experimental Realizations: Above we discussed how, for the large non-linear spin with energy $E \neq 1$, the assumptions of ETH hold and the spin thermalizes, while for $E = 1$ the spin does not thermalize. We found that spin fails to thermalize for $E \approx 1$ because the slow dynamics of an unstable fixed point produces eigenstates localized in the classical phase space. These results are valid when $J$ is finite and quantum effects are present, but large enough ($J > 200$) that assumption A2) and A3) hold.

The appearance of unstable fixed points in semiclassical dynamics is ubiquitous, and we expect this mechanism for the breakdown of thermalization to be general. Quantum systems with integrable classical dynamics offer the simplest generalization, because the correspondence between classical trajectories and eigenstates still holds and the analysis above can be easily repeated. For classically chaotic systems, the same correspondence does not hold, but the eigenstates must still capture the slow dynamics in the semi-classical limit. Thus, it is reasonable to still expect the existence of a set of eigenstates that are localized on the unstable classical fixed points and responsible for the slow dynamics.

This mechanism for the breakdown of thermalization may be observable in ultra cold BECs [4, 43] in which the bosons can be condensed into one of two modes such as two different hyperfine states. A spin boson mapping then yields the non-linear spin Hamiltonian, where the parameter $\Lambda$ is a ratio between the bosonic interaction energy and the energy associated with the tunneling between the two modes. This mapping hides additional energy and the energy associated with the tunneling between the two modes.

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Supplemental Materials

I) Large-J Scaling At Different Energies and for Different Observables

Thermalization in the Self-Trapping Limit

In Fig. S1, we demonstrate the validity of the ETH for $E = 3$.

![Graphs demonstrating thermalization in the self-trapping limit](image)

**FIG. S1.** (Color online) These plots demonstrate the validity of ETH in the self trapping limit for a fixed energy of $E = 3$. The different colors corresponds to different values of $\phi'$: blue (darker) $\phi = 0$, pink (lighter) $\phi = \pi$. The dashed line is the for the micro-canonical ensemble.

Ensemble Averages v.s. Energy

In this section we repeat a similar analysis done in Fig.1 in the main text for the observables $J_z$, $J_x^2$, $J_y^2$ and $J_z^2$. The results are shown in Figure S2. Each observable shows a non analyticity in the eigenstate observable as a function of energy. $J_z$ shows a discontinuity while, the square observables show a cusp approaching the fixed point value.

II) Numerical Details of Exact Construction of Micro-Canonical Ensembles

In the main text we discuss the use of exact diagonalization to construct the diagonal ensemble and micro-canonical ensemble. We use standard algorithms BLAS and LAPACK algorithms. The one subtlety is the identification of the two degenerate self trapping trajectories. We find that these algorithms naturally find a basis in which there is no tunneling between the two self trapping eigenstates. We then identify $s = \pm 1$ by the sign of $\langle J_z \rangle$ of the produced eigenstate. To confirm a lack of tunneling in the produced degenerate basis, we compute the overlap $\sum_{j \neq 0} |\langle s | j \rangle|^2$.

If the overlap is close to 1 then we know the there is very little tunneling between the two self trapping trajectories. Note for the diagonal ensemble $\sum_n \rho_n |n\rangle \langle n|$, we use the non-tunneling degenerate basis, and thus treat the sign of $J_z$ as a quasi conserved integral of motion.

III) Semi-Classical Eigenstates and Eigenstate normalization

In the main text we defined the semi-classical eigenstate Wigner function (EWF) as:

$$\rho_E(z, \phi) = \omega(E) \delta(H(z, \phi) - E)$$  \hspace{1cm} (S1)

where the Hamiltonian is given as:

$$H = \frac{\Lambda z^2}{2} - \sqrt{1 - z^2} \cos(\phi),$$  \hspace{1cm} (S2)

and the normalization $\omega(E)$ is given as:

$$\omega(E)^{-1} = \int \int dz d\phi \delta(E - H(z, \phi)).$$  \hspace{1cm} (S3)
FIG. S2. These plots demonstrate the validity of ETH in the self trapping and free oscillating limit, but it fails on the separatrix ($E=1$).

To compute this integral, we focus on the energy close to the separatrix, $E = 1 \pm |\delta|$, and expand the Hamiltonian around $E = 1$:

$$H - 1 = \frac{\Lambda - 1}{2} z^2 - \frac{\phi - \pi)^2}{2}$$  \hspace{1cm} (S4)

Close to the unstable fixed point the trajectories trace out a hyperbola:

$$z = \pm \sqrt{\frac{2}{\Lambda - 1}} \sqrt{\frac{|\delta|^2}{2} + (E - 1)}$$  \hspace{1cm} (S5)

$$\phi = \sqrt{2(1 - E) + (\Lambda - 1)z^2}$$

The Jacobian for both these trajectories are:

$$\left| \frac{dH}{dz} \right| = (\Lambda - 1)z = \sqrt{(\Lambda - 1)\sqrt{2(E - 1) + \phi^2}}$$  \hspace{1cm} (S6)

$$\left| \frac{dH}{d\phi} \right| = \phi = \sqrt{(\Lambda - 1)\sqrt{2(1 - E) + \Lambda - 1} + z^2}.$$  \hspace{1cm} (S7)

Since the the inverse Jacobians, $\left| \frac{dH}{dz} \right|^{-1}$ and $\left| \frac{dH}{d\phi} \right|^{-1}$, contribute the most near the unstable fixed point and we can expand the integrand for $\omega(E)^{-1}$ near them and write:

$$\omega(1 + |\delta|)^{-1} = \int_{r_e}^{r_t} \left| \frac{dH}{dz} \right|^{-1} (\phi, \delta) + C_+$$

$$\omega(1 - |\delta|)^{-1} = \int_{r_e}^{r_t} \left| \frac{dH}{d\phi} \right|^{-1} (z, \delta) + C_-$$
where \( r_\pm \) denotes the limits where the hyperbolic expansion is valid and \( C_\pm \) are small and approximately constant for \( \delta \) small. Defining \( a \) as:

\[
a_+ = 2(E - 1) \\
a_- = \frac{2(1 - E)}{\Lambda - 1},
\]

these integrals can be expressed as:

\[
\frac{1}{\sqrt{a(\Lambda - 1)}} \int_{-r}^{r} \frac{1}{\sqrt{1 - a^{-1}x^2}} \, dx = \frac{1}{\sqrt{(\Lambda - 1)}} \left[ \sinh^{-1} \left( \frac{r}{\sqrt{a}} \right) \right]
\]

and for \( E \approx 1 \), this approximates to as:

\[
\omega(1 + |\delta|)^{-1} = -\frac{\text{Ln}(|\delta|)}{2\sqrt{(\Lambda - 1)}} \\
\omega(1 - |\delta|)^{-1} = -\frac{\text{Ln}(|\delta|)}{\sqrt{(\Lambda - 1)}}
\]

**IV) Eigenstate Overlaps and Saddle Point Approximation**

To approximate the eigenstate overlap for initial states on the separatrix but away from the fixed points, we expand the energy to linear order in \( z \) and \( \phi \):

\[
H = \kappa_1 \phi + \gamma_1 z + E_0
\]

We first write the coherent state with initial imbalance \( z' \) and phase \( \phi' \) as:

\[
\rho(N, z', \phi', z, \phi) = \frac{\alpha_z(N, z')\alpha_\phi(N, z')}{\pi} \text{Exp} \left[-\alpha_z(N, z')(z - z')^2 - \alpha_\phi(N, z')(\phi - \phi')^2\right]
\]

where the inverse variances are:

\[
\alpha_\phi(J, z') = \frac{1}{2} J \left( 1 - z^2 \right) \\
\alpha_z(J, z') = \frac{2J}{1 - z^2}
\]

The eigenstate overlap is then given as:

\[
\rho_{\text{diag}}(z', \phi', E, s) = \frac{\omega(E)}{\gamma_1} \int d\phi \left( N, z', \phi', \frac{\delta_0 - \kappa_1 \phi}{\gamma}, \phi \right)
\]

where \( \delta_0 = E - E_0 \). This integrates by parts as:

\[
\frac{\omega(E)}{\gamma_1} \frac{\sqrt{\alpha_z(N, z')\alpha_\phi(N, z')}}{\sqrt{\alpha_\phi + \frac{\kappa_1^2 \alpha_z}{\gamma_1}}} \exp \left( -\frac{\delta_0^2 \alpha_\phi \alpha_z}{\gamma_1^2 \alpha_\phi + \kappa_1^2 \alpha_z} \right)
\]

Here we note that the energy uncertainty depends on the coherent state via the uncertainties \( \alpha_z \) and \( \alpha_\phi \). To find long term expectation values, we compute the diagonal ensemble average,

\[
O = \int \rho_{\text{diag}}(E)O(E),
\]

by performing a saddle point integration in the function \( \rho_{\text{diag}} \).
Saddles

Analytic solutions for the saddle point only exist if $E_0 = 1$ so we focus on coherent states on this line. To find the saddle points we rewrite $\rho_{diag}$ as:

$$\rho_{diag}(E = 1 \pm |\delta|) = \frac{K_\pm}{(1 - G_\pm \ln|\delta|)} \exp[-2JF\delta^2]$$  \hspace{1cm} (S17)

where $K_\pm$ and $G_\pm$ are constants in $\delta$, depend on $C_\pm$, and with $\pm$ depending on the sign of $\delta$. This function has a saddle at:

$$|\delta| = \frac{i}{\sqrt{(2JF)W_{-1}(\frac{-e^{-2/\gamma_\pm}}{\pi})}}$$  \hspace{1cm} (S18)

Where the product log, $W_{-1}(X)$, is the inverse of $e^X$: $W_{-1}(e^x) = x$ and the $-1$ says to take the negative branch. For small $x$ we get:

$$\lim_{x \to 0^-} W_{-1}(x) \ln(x) = 1$$  \hspace{1cm} (S19)

and we know $W_{-1}(x) \approx \ln(-x) - \ln(-\ln(-x)) + \ldots$. We can then write this as:

$$|\delta| = \frac{i}{\sqrt{2JFLn(\frac{-e^{-2/\gamma_\pm}}{\pi})}}$$  \hspace{1cm} (S20)

Which in the large-$J$ limit goes as:

$$\frac{1}{\sqrt{2JFLn(J)}}$$  \hspace{1cm} (S21)

and

$$2JF = \frac{\alpha_\phi \alpha_z}{\gamma_1^2 \alpha_\phi + \kappa_1^2 \alpha_z}$$  \hspace{1cm} (S22)

Thus the difference in initial states on the separatrix again shows up in the scaling to the large $J$ limit. Also note $G$ comes from $W(E)$ which depends on which side of the separatrix we are on (sign of $\delta$). In the large-$J$ limit the points become symmetric as indicated by the lack of dependence on $G$.

Eigenstate observables close to the separatrix

Next we compute the eigenstate observables, $O(E)$, which are given as

$$W(E) \int O(z, \phi) \delta[H(z, \phi) - E].$$  \hspace{1cm} (S23)

$J_z$ for $\Lambda$ large has a amazingly simple solution. For $E < 1$, $J_z(E) = 0$ for $E > 1$ we integrate:

$$\int dz \delta[H(z, \phi) - E] = \int_{-\pi}^{\pi} d\phi \zeta(\phi) \left|\frac{dH}{dz}\right|^{-1}$$  \hspace{1cm} (S24)

and for $\Lambda >> 1$, $\left|\frac{dH}{dz}\right|^{-1} \approx \Lambda z$, the $z$’s cancel and we get

$$J_z(E) = \frac{W(E)2\pi}{\Lambda}.$$  \hspace{1cm} (S25)

$J_x$ is more involved. We will take the same approach as the integral for $\omega(E)$. We assume the integral is dominated by the contribution near the unstable fixed point. Doing so allows us to expand $J_x$ near the unstable fixed point: $J_x \approx -1 + \phi^2/2$. Solving for $\phi$, we find that it is written as: $J_x \approx \frac{\Lambda - 1}{2} z^2 - E$. 
\[ J_x(1 + |\delta|)^{-1} = \omega(E)(\Lambda - 1) \int_{-r_\nu}^{r_\nu} \left( -\frac{E}{\Lambda - 1} + z'^2/2 \right) \left| \frac{dH}{dz} \right|^{-1}(\phi, \delta) + K_+ \]
\[ J_x(1 - |\delta|)^{-1} = \omega(E) \int_{-r_\nu}^{r_\nu} (-1 + \phi^2/2) \left| \frac{dH}{d\phi} \right|^{-1}(z, \delta) + K_- \]

Similarly to the integral for \( \omega(E) \), these can be computed and in the limit of small \( \delta \) we get:
\[ J_x(1 + |\delta|)^{-1} = -1 + \omega(|\delta|) \left( K_+ - \frac{|\delta| Ln(|\delta|)}{(\Lambda - 1)^{3/2}} \right) \]
\[ J_x(1 - |\delta|)^{-1} = -1 + \omega(|\delta|) \left( K_- - \frac{|\delta| Ln(|\delta|)}{\sqrt{\Lambda - 1}} \right) \]

\( \omega(|\delta|) \) goes to 0 faster than \( \omega(|\delta|) |\delta| Ln(|\delta|) \) and we get:
\[ J_x(1 + |\delta|)^{-1} = -1 - \omega(|\delta|) \frac{|\delta| Ln(|\delta|)}{(\Lambda - 1)^{3/2}} \]
\[ J_x(1 - |\delta|)^{-1} = -1 - \omega(|\delta|) \frac{|\delta| Ln(|\delta|)}{\sqrt{\Lambda - 1}} \]

Substituting \( \omega \):
\[ J_x(1 + |\delta|)^{-1} = -1 + \frac{2|\delta|}{\Lambda - 1} \]
\[ J_x(1 - |\delta|)^{-1} = -1 + |\delta| \]
\[ J_x(1 + |\delta|)^{-1} = \frac{4\pi \sqrt{(\Lambda - 1)}}{ALn(|\delta|)} \]
\[ J_x(1 - |\delta|) = 0 \]

V) Semi-Classical Equilibrium Observables

Using the saddle point values to evaluate the diagonal ensemble observables we get:
\[ O_{diag}(z', \phi') = \int \rho_{diag}(E, z', \phi') O(E) = \frac{1}{3} [2O(1 + |\delta|) + O(1 - |\delta|)] \]

There is a factor of 2 for +|\delta| because \( \rho_{diag}(E, z', \phi') \) in the small |\delta| limit is twice as large on the self trapping side. (The integral in \( \omega(E) \) only integrates over one of the hyperbole on one side of the fixed point while it integrates both sides in the free oscillating limit). This gives the diagonal ensemble observables as:
\[ J_z(z', \phi') = -\frac{8\pi \sqrt{(\Lambda - 1)}}{3ALn[|\delta|(|z', \phi'|)]} \]
\[ J_x(z', \phi') = -1 + \frac{3 + \Lambda}{3(\Lambda - 1)} \]

By substituting the saddle point location, we get the expressions given in the main text:
\[ J_z(z', \phi') = \frac{4\pi \sqrt{(\Lambda - 1)}}{3ALn[F(z_0, \phi_0)JLn|J|]} \]
\[ J_x(z', \phi') = -1 + \frac{1}{\sqrt{F(z_0, \phi_0)JLn|J|}} \frac{3 + \Lambda}{3(\Lambda - 1)}. \]