Robust structural damage detection and localization based on joint approximate diagonalization technique in frequency domain

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Abstract

The structural characteristic deflection shapes (CDS’s) such as mode shapes and operational deflection shapes are highly sensitive to structural damage in beam- or plate-type structures. Nevertheless, they are vulnerable to measurement noise and could result in unacceptable identification errors. In order to increase the accuracy and noise robustness of damage identification based on CDS’s using vibration responses of random excitation, joint approximate diagonalization (JAD) technique and gapped smoothing method (GSM) are combined to form a sensitive and robust damage index (DI), which can simultaneously detect the existence of damage and localize its position. In addition, it is possible to apply this approach to damage identification of structures under ambient excitation. First, JAD method which is an essential technique of blind source separation is investigated to simultaneously diagonalize a set of power spectral density matrices corresponding to frequencies near a certain natural frequency to estimate a joint unitary diagonalizer. The columns of this joint diagonalizer contain dominant CDS’s. With the identified dominant CDS’s around different natural frequencies, GSM is used to extract damage features and a robust damage identification index is then proposed. Numerical and experimental examples of beams with cracks are used to verify the validity and noise robustness of JAD based CDS estimation and the proposed DI. Furthermore, damage identification using dominant CDS’s estimated by JAD method is demonstrated to be more accurate and noise robust than by the commonly used singular value decomposition method.

Keywords: damage identification, characteristic deflection shape, joint approximate diagonalization, power spectral density, gapped smoothing method, output only, ambient vibration

(Some figures may appear in colour only in the online journal)

1. Introduction

Structural health monitoring and damage identification are essentially important in maintaining the safety and reliability of civil, mechanical and aeronautical engineering structures, which monitor structural healthy states in order to avoid catastrophic structural failure (Doebling et al 1996, Montalvao et al 2006). Vibration-based structural health monitoring and damage identification have the advantage of low cost and can be applied continuously under various operational and environmental conditions (Maia et al 2011). The basic principle of this kind of methods is that damage induced changes in structural physical characteristics (mass, damping
or stiffness) can alter structural modal parameters such as natural frequencies and mode shapes, which can be used to extract damage sensitive features for damage identification (Husain et al. 2009, Fan and Qiao 2011).

Here, structural damage identification methods are discussed as model-based and non-model-based approaches (Doebbling et al. 1998, Carden and Fanning 2004, Sohn et al. 2004). For model-based damage identification approaches, a well-correlated theoretical structural model and the accurate initial state of the structure are required (Friswell and Penny 2002, Gao et al. 2013, Yang et al. 2014). Moreover, the large number of updating parameters and the non-uniqueness of updated models increase the difficulties of model updating based damage identification methods (Gola et al. 2001, Mottershead et al. 2011). On the other hand, non-model based damage identification methods can accomplish damage detection and localization without a theoretical structural model. The majority of this kind of methods is based on differences in structural modal parameters between damaged structures and healthy structures (Mottershead et al. 1999, Reynders et al. 2014). Some other methods take advantage of the damage induced sudden shape changes in frequency response function or operational deflection shape to detect and localize damage without baseline data of undamaged structures (Yoon et al. 2010, Yang et al. 2012, Amanashari and Sinha 2014).

For practical engineering structures, structural damage identification utilizing output-only measurements is attractive and promising, which does not require the knowledge of input excitation and a theoretical structural model (Deraemaeker et al. 2008, Rainieri and Fabbrocino 2014). The identified characteristic deflection shapes (CDS’s) containing spatial knowledge of structures are highly sensitive to detect and localize the damage (Pai and Young 2001, Xu et al. 2015). Here, CDS’s refer to vector features that contain information about mode shapes such as operational deflection shapes or mode shapes (Surace 2013). The purpose of this study is to apply the identified CDS’s (the demand of their convergence to mode shapes is not required) using output-only vibration responses without baseline information of healthy structures for damage detection and localization.

There are two approaches to increasing the accuracy and noise robustness of damage identification when using CDS’s. One is to apply signal processing techniques such as fractal dimension method, wavelet transform method and gapped smoothing method (GSM) to CDS’s (Qiao and Cao 2008, Bai et al. 2014, Cao et al. 2016), which are able to reduce the influence of noise and provide better damage identification results. The other way is to improve the noise robustness of CDS’s during the estimation procedures. In this study, before applying signal processing techniques, a new approach is proposed to estimate noise robust CDS’s by applying joint approximate diagonalization (JAD) technique to diagonalize a set of power spectral density (PSD) matrices in frequency domain. JAD (also called simultaneous diagonalization or jointly diagonalization) was originally studied for simultaneously diagonalization of commuting normal matrices (Bunse-Gerstner et al. 1992, 1993) and then was applied successfully to blind source separation (Belouchrani et al. 1997, Cichocki and Amari 2002). Recently the possibility of applying blind source separation to operational modal analysis and structural damage identification has been studied by many authors, for example Zhou and Chelidze (2007), Antoni and Chauhan (2013) and Musafere et al. (2016).

In this study, JAD technique is studied in frequency domain based on the similar theoretical background as frequency domain decomposition method. Frequency domain decomposition method proposed by Brincker et al. (2000, 2001) is an important tool in operational modal analysis and able to reveal and calculate the dominant modes at a frequency by singular value decomposition (SVD) of its PSD matrix. As an improvement, JAD estimates an integrated joint diagonalizer using the information contained in a couple of PSD matrices corresponding to frequencies around a natural frequency rather than exactly diagonalizing the PSD matrix at some frequency using SVD. The advantage of doing this is that JAD method provides a kind of ‘average Eigen-structure’ shared by the PSD matrices of a narrow frequency band, which statistically deals better with the measurement noise than SVD method. Therefore, the CDS’s at a frequency estimated by JAD method tend to be more accurate and noise robust than by SVD method. Moreover, unlike traditional JAD method in blind source separation, the pre-whitening procedure is not required in this proposed JAD based CDS estimate. As it is well known, the pre-whitening procedure introduces errors or biases during the processing procedure, which can degrade the estimate accuracy of CDS’s (Cichocki and Amari 2002). Another merit of applying JAD method in frequency domain rather than in time domain is that it overcomes the limitations of closely spaced modes or even repeated frequencies.

At a frequency near natural frequency, the CDS corresponding to the largest dominant mode is called dominant CDS or DCDS. With the identified DCDS’s around different natural frequencies, damage detection and localization are accomplished through identifying jump discontinuities or sudden shape distortions in DCDS’s without prior information about healthy structures. For some severe damage cases, the damage locations can be roughly determined by observing sudden shape distortions in DCDS curvature plot. However, in order to obtain more accurate and reliable damage detection and localization results, a robust damage index (DI) is proposed based on GSM using DCDS’s around different natural frequencies.

The structure of this paper is outlined as follows. In section 2, JAD is proposed (as an original contribution) and demonstrated to estimate CDS’s in frequency domain. And a comprehensive theoretical background of frequency domain decomposition is presented. In section 3, a damage detection and localization index is defined using DCDS’s based on GSM. In section 4, a numerical example of a cantilever beam with two cracks is studied under random excitation to verify the feasibility and effectiveness of the proposed damage identification index. Moreover, DCDS’s and DI computed by JAD method are statistically demonstrated to be more noise
robust than by SVD method when white noise is present. However, in this numerical example an ideal excitation is used and white noise is added to the output data, in order to test the feasibility and robustness of proposed DI against measurement noise. Then in section 5, two experimental examples are used to validate the damage identification index and demonstrate the higher damage identification accuracy of JAD method when comparing with SVD method. Furthermore, a comparison study of damage identification performance using real part, imaginary part and absolute value of DCDS’s is conducted based on the experimental data. Finally, conclusions are given to summarize the findings of this JAD based damage detection and localization method.

2. Applying JAD technique in frequency domain

Estimating DCDS’s such as mode shapes or operational deflection shapes requires a set of spatial measurement points along the structure. And the localization accuracy is largely depended on the density of measurement points. With the advanced measurement techniques such as embedded sensors or non-contact laser scanning sensors, experimental data of many measurement points are readily acquired now. Consider the equation of motion of an n degree-of-freedom system subjected to general excitation forces:

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{C}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{f}(t),$$

(1)

where $\mathbf{q}(t)$ is the displacement vector, $\mathbf{f}(t)$ is the excitation force vector, and $\mathbf{M} \in \mathbb{R}^{n \times n}$, $\mathbf{C} \in \mathbb{R}^{n \times n}$ and $\mathbf{K} \in \mathbb{R}^{n \times n}$ represent the mass, damping matrix and stiffness matrix respectively. According to the modal expansion theorem of structural response, vibration responses can be expressed in terms of mode shapes and modal coordinates.

$$\mathbf{x}(t) = \Phi\mathbf{q}(t),$$

(2)

where $\Phi \in \mathbb{R}^{n \times n}$ represents the modal matrix (the rth column represents the rth mode shape), $\mathbf{q}(t)$ is the vector of modal coordinates and $\xi = 0, 1, 2$ indicates the corresponding responses are displacement, velocity or acceleration respectively. In order to link with the experimental velocity measurements from a scanning laser Doppler vibrometer, the following equations are derived in terms of velocity. The measured velocity vector $\dot{\mathbf{y}}(t)$ in $\mathbb{R}^m$ is obtained at $m$ sensors with $N$ data samples and is expressed as

$$\dot{\mathbf{y}}(t) = \Phi_\gamma \mathbf{q}(t) + \nu(t), \quad j = 1, 2, \cdots, N,$$

(3)

where $\Phi_\gamma \in \mathbb{R}^{m \times n}$ represents the matrix of mode shape vectors at the measured degrees of freedom and $\nu(t)$ is a vector of measurement noise. Before computing the covariance matrix of vibration responses, the zero mean procedure is required and a bar is added on variables such as $\bar{y}(t)$, $\bar{q}(t)$ and $\bar{\nu}(t)$ to indicate the mean value has been taken. Provided that $\mathbf{u}(t)$ and $\Phi_\gamma \mathbf{q}(t)$ are uncorrelated, covariance matrix should be computed as

$$\mathbf{R}_{\gamma \gamma}(\tau) = \Phi_\gamma \mathbf{R}_{\mathbf{q}\mathbf{q}}(\tau)\Phi_\gamma^T + \mathbf{R}_{\nu\nu}(\tau),$$

(4)

where $\tau (= 0, 1, 2 \cdots, N - 1)$ denotes time delay (which is the sampling period). $\mathbf{R}_{\mathbf{q}\mathbf{q}} \in \mathbb{R}^{n \times n}$ and $\mathbf{R}_{\nu\nu} \in \mathbb{R}^{m \times m}$ are covariance matrices of modal coordinates and noise respectively. Taking discrete Fourier transform of equation (4), the PSD matrix should be obtained as

$$\mathbf{S}_{\gamma\gamma}(\omega) = \Phi_\gamma \mathbf{S}_{\mathbf{q}\mathbf{q}}(\omega)\Phi_\gamma^H + \mathbf{S}_{\nu\nu}(\omega),$$

(5)

where $\omega$ indicates the discrete frequency of excitation and superscript $H$ denotes Hermitian transpose. In equation (5), PSD matrix $\mathbf{S}_{\gamma\gamma}(\omega) \in \mathbb{R}^{m \times m}$ is a Hermitian (positive definite) matrix having the common mode shape matrix $\Phi_\gamma$ at the measurement degrees of freedom. When the number of measured points is greater than the number of excited modes $n_1$ of the structure ($m > n_1$), the factorization of $\mathbf{S}_{\gamma\gamma}(\omega)$ by SVD is

$$\mathbf{S}_{\gamma\gamma}(\omega) = \mathbf{UDD}^H = \begin{bmatrix} \mathbf{U}_1 & \mathbf{U}_2 \end{bmatrix} \begin{bmatrix} \mathbf{D}_1 & 0 \\ 0 & \mathbf{D}_2 \end{bmatrix} \begin{bmatrix} \mathbf{U}_1 \mathbf{U}_2 \end{bmatrix}^H,$$

(6)

where $p$ indicates the frequency of a peak in singular value spectrum plot (such as point 1, 2 and 3 in figure 1), $\mathbf{U} \in \mathbb{R}^{m \times m}$ is an estimate of DCDS’s corresponding to the columns of $\mathbf{U}$ related to mode shapes at this frequency and $\mathbf{U}_1 \in \mathbb{R}^{m \times n_1}$ consists of the singular vectors corresponding to noise. $\mathbf{D}$ is a real diagonal matrix and its diagonal entries are singular values in descending order indicating the contribution magnitude of singular vectors in $\mathbf{U}$ to vibration responses at this frequency.

Figure 1 shows the singular value spectrum calculated using the random vibration responses of the numerical example in section 4. At each frequency, the singular values are simply obtained from its PSD matrix using equation (6). The first five singular values at each frequency are plotted to indicate the number of peaks in figure 1. Under white noise excitation, the number of peaks in figure 1 can be used to identify the number of excited modes $n_1$ present in the experimental data. And at a given frequency, the number of dominant modes or DCDS’s is determined by the number of singular values that are obviously greater than the remaining ones using singular value plot shown in figure 2. From figure 2, it indicates that only one mode plays the dominant role at point 1. If more than one dominant mode exists at a frequency, only the DCDS (dominant CDS) estimated from
that minimize the following cost function

\[ J(U_p, D_{p+k}) = \sum_{k=-K}^{K} \left\| S_{yy}(\omega_{p+k}) - U_p D_{p+k} U_p^H \right\|. \]  

The non-diagonalization of \( S_{yy}(\omega_{p+k}) \) caused by damping and noise effects mainly deteriorates the singular vectors in \( U_p \) corresponding to the smaller singular values of \( D_p \) while the singular vector corresponding to the largest singular value is robust to this effect. As a consequence, only the estimated DCDS’s are used for structural damage identification in this paper. This is the advantage of applying JAD in frequency domain rather than time domain. Fortunately, there is a numerically effective and efficient algorithm for solving JAD through the extended Jacobi technique (Bunse-Gerstner et al. 1993, Cardoso and Souloumiac 1996). The main steps of applying JAD to estimate the joint unitary diagonalizer are shown as follows.

**Step 1:** Select a set of PSD matrices of frequencies around a frequency \( \omega_p \) (which is one singular value peak in singular value spectrum plot), \( S = \{ S_{yy}(\omega_{p+k}) \} \) \( k = [-K, -K+1, \ldots, K] \) \( (S \in \mathbb{R}^{m \times (2K+1)m}) \), set the initial joint unitary diagonalizer \( U_0 = I(U_p \in \mathbb{R}^{m \times m}) \) and the threshold \( \epsilon \) (see Step 3 for its use).

**Step 2:** For a given pair of \((i_1, j_1)\) \( 1 \leq i_1 \leq j_1 \leq m \), elementary Givens rotation matrix \( V(i_1, j_1, a, b) \in \mathbb{R}^{m \times m} \) is an identity matrix except the following entries

\[
\begin{bmatrix}
v_{i_1 i_1} & v_{i_1 j_1} \\
v_{j_1 i_1} & v_{j_1 j_1}
\end{bmatrix} = \begin{bmatrix} a & b^H \\ -b & a^H \end{bmatrix},
\]

where \( H \) indicates Hermitian transpose. Now, the minimization of equation (8) is equivalent to find \( a \) and \( b \) for each pair of \((i_1, j_1)\) that minimize the following cost function

\[
J_2(a, b) = \sum_{k=-K}^{K} \text{off}(V(i_1, j_1, a, b)S_{yy}(\omega_{p+k}))V^H(i_1, j_1, a, b)),
\]

where ‘off’ denotes the sum of squares of all off-diagonal terms.

**Step 3:** If all the Givens rotations in a sweep \( 1 \leq i_1, j_1 \leq m \), \( i_1 = j_1 \) result in \( |b| < \epsilon \), JAD ends. If not, go to Step 4.

**Step 4:** Sweep \( i_1 = 1, 2, \ldots, m-1 \) and \( j_1 = i_1 + 1, i_1 + 2, \ldots, m \) to iteratively calculate \( a \) and \( b \) based on the approach proposed by Cardoso and

Figure 2. Singular value plot of point 1 in figure 1.
Souloumiac (1996). Then, update $U_p$ and $S_{yy}^k(\omega_{p+k})$:

$$U_p = U_p \chi^{H}(i_1, j_1, a, b),$$

$$S_{yy}^k(\omega_{p+k}) = V(i_1, j_1, a, b) S_{yy}^k(\omega_{p+k}) \chi^{H} (i_1, j_1, a, b),$$

$k = [-K, -K + 1, \cdots, K]$.

**Step 5:** Go to Step 3 and repeat the process till the criterion is satisfied.

The number $2K + 1$ of PSD matrices and stopping threshold value $\epsilon$ need to be set appropriately to ensure the accuracy of JAD method. Around the peaks in singular value spectrum plot, only several PSD matrices are enough to promise an accurate estimate of DCDS’s. $K = 4$ and $K = 10$ are used for the numerical example in section 4 and experimental examples in section 5 respectively. For the stopping threshold $\epsilon$, the machine precision is not necessary and $\epsilon = 10^{-8}$ is chosen in this study (Rainieri and Fabrocinò 2015). In addition, the computational time of JAD technique will increase when increasing $K$ or decreasing $\epsilon$.

The DCDS at frequency $\omega_p$ is a complex vector in $U_p$ which corresponds to the largest diagonal element in $D_p$. (If the diagonal element is complex, the absolute value is used.) This technique only uses Givens rotations to compute the joint unitary diagonalizer $U_p$ without a pre-whitening procedure, which avoids the bias or error in pre-processing stage (Cichocki and Amari 2002). The most prominent feature of JAD is that it diagonalizes a set of PSD matrices simultaneously rather than one by one. From a statistical point of view, the identified DCDS’s by JAD method are more accurate and noise-robust for structural damage identification than by SVD method. The demonstration of this will be given in sections 4 and 5. In the following sections, the real part of DCDS’s will be firstly studied for damage identification and the DCDS’s refer to the real part of DCDS’s except in section 5.3, in which the damage identification performance of the imaginary part and the absolute value of DCDS’s will be investigated.

### 3. Damage detection and localization index

With the estimated DCDS’s by JAD method and SVD method from damaged structures, damage can be identified through the difference with DCDS’s of intact structures. However, the drawback of this approach is that it requires the DCDS’s of healthy structures and also the identified DCDS’s of damaged structures need to match the baseline DCDS’s of intact structures, which is difficult and unreliable due to the various operational and environmental conditions. In fact, for beam- and plate- type structures, damage detection and localization can be achieved without the prior knowledge of healthy structures under the basic principle that their DCDS’s of healthy structures are smooth. Generally, it is not effective and robust to identify damage using DCDS at single frequency, because the DCDS at certain frequency will be sensitive to damage at some locations while not sensitive to damage at other locations. Hence, a combination of identified DCDS’s around different natural frequencies is promising to define a robust DI as expressed in equation (13):

$$DI_l = \mu_l \gamma_l,$$

$$\mu_l = \sum_{p=1}^{L} (\Delta \varphi_{p,l}/ \max_{n=1,2,\cdots,m} \Delta \varphi_{p,n}),$$

$$\gamma_l = \sum_{p=1}^{L} (\Delta \varphi_{p,l}^2 / \max_{n=1,2,\cdots,m} \Delta \varphi_{p,n}^2),$$

$$\Delta \varphi_{p,l} = |\hat{\varphi}_{p,l} - \varphi_{p,l}|,$$

where $\varphi_{p,l}$ denotes DCDS value at measurement point $l (= 1, 2, \cdots, m)$ of frequency $\omega_{p}$. $p (= 1, 2, \cdots, L)$ indicates the selected singular value peaks in singular value spectrum plot such as figure 1. $\Delta \varphi_{p,l}$ is the absolute DCDS value difference between smoothed $\hat{\varphi}_{p,l}$ and original estimated DCDS $\varphi_{p,l}$ at measurement point $l$. In equation (13), $\mu_l$ and $\gamma_l$ are the sum of normalized first order and second order moments of $\Delta \varphi_{p,l}$ at measurement point $l$ for all the DCDS’s. Normally, the absolute DCDS difference $\Delta \varphi_{p}$ has a big variation in magnitude for different $p$. If they are simply added together, the contribution of each $\Delta \varphi_{p}$ to the DI is not even. In order to equally use the damage information in every $\Delta \varphi_{p}$, a normalization is done to remove the influences of magnitudes of different $\Delta \varphi_{p}$. In addition, the second order moments of $\Delta \varphi_{p,l}$ are used as they enhance the difference of the DI value. The equation of GSM using cubic polynomial is

$$\hat{\varphi}_{pl} = c_3 x^3 + c_2 x^2 + c_1 x + c_0,$$

where $x l$ indicates the location of measurement point $l$ and $c = [c_0, c_1, c_2, c_3]$ are coefficients of the above cubic polynomial. The two DCDS values $\varphi_{pl(-2)}$ and $\varphi_{pl(-1)}$ before measurement point $l$ and the two DCDS values $\varphi_{pl(+1)}$ and $\varphi_{pl(+2)}$ after measurement point $l$ are used to estimate $e$ and predict DCDS value $\hat{\varphi}_{pl}$ at measurement point $l$. GSM is sensitive to the shape irregular in DCDS’s of higher frequencies and can result in inaccuracy for damage detection and localization. However, the DCDS’s of higher frequencies are normally more sensitive to local damage.

### Table 1. Material properties of steel beam.

| Property                  | Value   |
|---------------------------|---------|
| Length (mm)               | 700     |
| Cross-section dimensions (mm × mm) | 20 × 20 |
| Young’s modulus (GPa)     | 210     |
| mass density (kg m⁻³)     | 7861    |
| Poisson ratio             | 0.33    |

### Table 2. Configurations of cracks.

| Cracks | Location | Measurement points | Depth | Width |
|--------|----------|--------------------|-------|-------|
| Crack 1 | 200 mm   | 5–6                | 4 mm  | 1 mm  |
| Crack 2 | 400 mm   | 10–11              | 4 mm  | 1 mm  |

* The measurement points are numbered from left to right.

### Cross-section dimensions (Length × Width × Height) (mm × mm × mm)

| Property                  | Value   |
|---------------------------|---------|
| Young’s modulus (GPa)     | 210     |
| mass density (kg m⁻³)     | 7861    |
| Poisson ratio             | 0.33    |
Hence, a compromise between damage identification accuracy and sensitivity should be made when using GSM.

4. Numerical example

A cantilever beam with two open cracks is simulated to demonstrate the validity of the proposed damage identification index. Moreover, DCDS’s and DI estimated by JAD method are proved to be more noise robust than by SVD method. This damaged beam is modeled with Rayleigh damping, $C = \alpha M + \beta K$ with $\alpha = 0.3693$ and $\beta = 1.2889 \times 10^{-6}$, using 670 CPS8R elements in ABAQUS. The geometry and material properties of the beam are tabulated in table 1. And the configurations of the cracks are presented in table 2. Pseudo-random excitation in frequency...
range of 0–800 Hz is applied to the free end and velocity time series are ‘measured’ at the prescribed 18 red points along the beam with an equal distance of 0.04 m as illustrated in figure 3. Finally, the responses of the finite element model are solved via the dynamic-implicit approach.

In order to compare the noise robustness of computed DCDS’s and DI between JAD method and SVD method, Gaussian white noise is added to contaminate the simulation velocity responses $\mathbf{Y}_{\text{m}}$ in the form of

$$ s_l = \mathbf{d}_{\text{nl}} \mathbf{Y}_l + n_{\text{level}} \sigma(\mathbf{Y}_l), \quad l = 1, 2, \ldots, m, $$

where $\mathbf{d} \in \mathbb{R}^{2 \times N}$ contains normally distributed random values with a zero mean and variance being 1, $n_{\text{level}}$ is the noise level range of [0 1] and $\sigma(\mathbf{Y}_l)$ denotes standard deviation of vibration responses at $l$th measurement point (Mao et al 2010, Gao et al 2013). The noise is added to $\mathbf{Y} \in \mathbb{R}^{m \times N}$ 1000 times with the same noise level 3%. With each noise realization of $\mathbf{Y} \in \mathbb{R}^{m \times N}$, the DCDS’s of peak singular value points as shown in figure 1 are estimated by JAD and SVD individually. The mean and averaged relative errors of DCDS at each peak singular value over 1000 independent noise realizations are presented in figure 4. The averaged relative error is calculated according to equation (16)

$$ E_{\text{rel, } l} = \frac{1}{1000} \sum_{k=1}^{1000} \frac{|\varphi_{p,l} - \varphi_{p,l}^*|}{|\varphi_{p,l}|}, \quad (16) $$

where $\varphi_{p,l}^*$ and $\varphi_{p,l}$ denote the DCDS values estimated at measurement point $l$ without noise and with noise respectively.

From figures 4(a), (c) and (e), the mean DCDS’s of JAD method are almost identical with these of SVD method. But the identified DCDS’s by JAD method are demonstrated to be

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**Figure 5.** DCDS curvatures and DCDS differences $\Delta \varphi$. (a) DCDS curvature of peak point 1. (b) DCDS difference $\Delta \varphi$ of peak point 1. (c) DCDS curvature of peak point 2. (d) DCDS difference $\Delta \varphi$ of peak point 2. (e) DCDS curvature of peak point 3. (f) DCDS difference $\Delta \varphi$ of peak point 3.
more noise robust than by SVD method due to their smaller relative errors as shown in figures 4 (b), (d) and (f) (table A1 in the appendix tabulates the relative error values). The curvatures of the mean DCDS’s and the differences between experimental DCDS’s and GSM prediction DCDS’s are plotted in figure 5.

Some crack locations can be identified roughly from the DCDS curvature plots in figures 5(a), (c) and (e), while it is impossible to detect and localize the cracks from DCDS’s directly in figures 4(a), (c) and (e). This clearly shows that the DCDS curvature is more sensitive to damage than DCDS’s. When the number of measurement points is small, the DCDS curvatures calculated from DCDS’s using central difference approximation cannot localize the damage positions as precisely as DCDS’s. When there are many measurement points, the damage localization accuracy of using DCDS curvatures or DCDS’s will not have a big difference. Thus, in order to identify the damage more accurately when using less measurement points, the proposed DI is computed using the DCDS’s and the estimated DI values are presented in figure 6. Figures 6(a)–(c) show the DI values using mean DCDS’s of peak points 1, 2 and 3 in figure 1 respectively and figure 6(d) gives the integrated DI values using mean DCDS’s of all the three peaks.

Figure 6 shows that each DCDS of a frequency is sensitive to damage at some locations while not sensitive to damage at other locations. For instance, the DCDS of peak point 1 is sensitive to the first crack whilst the DCDS of peak point 2 is sensitive to the second crack. This also indicates that for DCDS of peak point 3 (higher frequency), several peaks show up in figure 5(f) not only around the damage locations but also at other positions (false alarms). Moreover, comparing figures 6(a)–c with figure 6(d) shows that the DI estimated using DCDS’s at several frequencies is more sensitive and reliable than using individual DCDS. In order to demonstrate the better damage identification performance of JAD method, the standard deviations of DI computed form 1000 independent noise realizations is presented in figure 7. It shows that these of JAD method are smaller than those of SVD method, which indicates DI calculated by JAD method is more noise robust than by SVD method.

![Figure 6. Damage identification results of the numerical example. (a) DI of peak point 1. (b) DI of peak point 2. (c) DI of peak point 3. (d) Integrated DI.](image1)

![Figure 7. Standard deviations of damage index.](image2)
5. Experimental study

The purposes of this section are twofold. The proposed damage identification index based on identified DCDS’s around different natural frequencies will be validated experimentally. And the damage identification results from JAD method will be compared with SVD method to demonstrate the better damage identification accuracy of JAD method. Two steel beams having dimensions of $0.7 \times 0.02 \times 0.02 \text{m}^3$ with two open cracks are tested. Experimental set-up is presented in figures 8 and 9. A PSV-500 Scanning Laser Vibrometer is used to acquire velocity responses of prescribed 21 points along the beams as shown in figure 9. Pseudo-random excitation of frequency range

\[
\begin{array}{cccccc}
\text{Examples} & \text{Cracks} & \text{Location} & \text{Measurement points} & \text{Depth} & \text{Width} \\
1 & \text{Crack 1} & 200 \text{ mm} & 6–7 & 6 \text{ mm} & 1 \text{ mm} \\
1 & \text{Crack 2} & 400 \text{ mm} & 12–13 & 6 \text{ mm} & 1 \text{ mm} \\
2 & \text{Crack 1} & 200 \text{ mm} & 6–7 & 4 \text{ mm} & 1 \text{ mm} \\
2 & \text{Crack 2} & 400 \text{ mm} & 12–13 & 4 \text{ mm} & 1 \text{ mm} \\
\end{array}
\]

\(a\) The measurement points are numbered from left to right.
0–800 Hz is generated by the PSV-500 system and applied to the free end of cantilever beam by a shaker (LDS V406). Damage is simulated by cutting small slots at different locations and depths. The information of cracks in the two experimental examples is listed in Table 3. In addition, the cracks are on the back surface and marked by the blue lines in the front view as illustrated in Figure 9.

Figure 10 shows the PSD comparisons of the output excitation signal from PSV 500 and real input force to the steel beam. Obviously, there are decreases in the force spectrum occurring in the vicinity of resonant frequencies due to the exciter-structure interactions (Maia and Silva 1997, Worden and Tomlinson 2000). Under the effects of exciter-structure interactions, the frequencies of singular peak values are not the exact structure resonant frequencies. But this issue does not affect the application of JAD technique to identify the DCDS’s at the frequencies of peak singular values and the proposed damage identification index is still feasible and effective in this case.

5.1. Example 1

In this example, a beam with two open cracks (both 30% of the beam depth) is tested under pseudo-random excitation. Figure 11 presents the singular value spectrum calculated by SVD of PSD matrices. The three peak points are selected to estimate their DCDS’s through SVD and JAD methods respectively, which are shown in Figure 12.

From figures 12(b), (d) and (f), it is obvious that DCDS’s estimated by JAD method are smoother than by SVD method, which indicates the better noise robustness of JAD method during DCDS estimation. With the smoother DCDS’s, JAD method is able to enhance the accuracy of damage detection and localization as shown in Figure 13. Figures 13(a)–(c) shows the DI values calculated using individual DCDS of the three peak points in Figures 11 and 13(d) presents the DI values computed using DCDS’s of all the three peak points. Apparently, the combination of several DCDS’s around different natural frequencies can obtain a better noise robust and sensitive damage identification index.

In order to demonstrate that the proposed damage identification index is reliable and sensitive to small damage and further show that damage identification results by JAD method tend to be more noise robust and accurate than by SVD method, a second beam with two cracks is tested.

5.2. Example 2

Example 2 is tested under the same excitation level, boundary conditions and measurement points as shown in Figure 9 except that the depth of the two cracks are decreased from 30% to 20% of the beam depth. Figure 14 shows the singular value spectrum computed by SVD of PSD matrices.

Similar to example 1, the DCDS’s and their curvatures of the three selected peak points are presented in Figure 15 for damage detection and localization. The positions of the two cracks are hard to observe from Figure 15 but it shows that the identified DCDS’s by JAD method are smoother than by SVD method.

Figures 16(a)–(c) shows the proposed DI values calculated using DCDS’s of the three peak points individually. Figure 16(d) shows the DI calculated using DCDS’s of all the three peak points, which is better in identifying the damage than figures 16(a)–(c). Another conclusion from Figure 16 is that DI values calculated using JAD method are much more accurate and noise robust to detect and localize damage than SVD method.
In the above study, damage identification is studied using the real part amplitude changes of DCDS’s without considering the imaginary part or the absolute value of DCDS’s. However, the imaginary part of DCDS’s corresponds to the out-of-phase vibration and should be more sensitive to phase changes and the absolute value of DCDS’s includes the total vibration information and thus can be also used to detect the damage-induced local stiffness reduction (Masciotta et al. 2016, 2017). Thus, a damage identification performance comparison of the real part, imaginary part and the absolute value of DCDS’s should be conducted. Before investigating the imaginary part and the absolute value of DCDS’s, the approach to improve damage identification accuracy using the real part of DCDS’s is presented firstly.

### 5.3. Imaginary part and absolute value of DCDS’s

In the above study, damage identification is studied using the real part amplitude changes of DCDS’s without considering the imaginary part or the absolute value of DCDS’s. However, the imaginary part of DCDS’s corresponds to the out-of-phase vibration and should be more sensitive to phase changes and the absolute value of DCDS’s includes the total vibration information and thus can be also used to detect the damage-induced local stiffness reduction (Masciotta et al. 2016, 2017). Thus, a damage identification performance comparison of the real part, imaginary part and the absolute value of DCDS’s should be conducted. Before investigating the imaginary part and the absolute value of DCDS’s, the approach to improve damage identification accuracy using the real part of DCDS’s is presented firstly.

#### 5.3.1. Improvements of real part of DCDS’s for damage identification

In order to measure the relative vibration magnitude between the real part and imaginary part of DCDS’s, a criterion is defined in equation (17):

$$E_{rp} = \frac{||\text{real}(\varphi_p)||}{||\text{imag}(\varphi_p)||},$$

where ‘real’ implies the real part of DCDS’s and ‘imag’ denotes the imaginary part of DCDS’s. Then the relative vibration magnitude results in experimental examples 1 and 2 are listed in table 4. It is obvious from table 4 that the relative vibration magnitude measure $E_{rp}$ of JAD method is much larger than SVD method. In order to validate that maximizing $E_{rp}$ is possible to improve the damage identification accuracy of the real part of DCDS’s, an improved SVD method is
proposed to increase $E_{\gamma}$ of DCDS’s provided by SVD method through a plane rotation. In this method, the best straight line fit to all the elements of a DCDS plotted on a complex plane is determined and then the whole DCDS vector is rotated to make this best line to align with the horizontal (real) axis, which will boost $E_{\gamma}$ of DCDS’s from SVD method. $E_{\gamma}$ of the improved SVD method is also tabulated in table 4 and $E_{\gamma}$ increase is remarkable when compared with the plain SVD method. In addition, figure 17 presents the damage identification comparison between JAD method and the improved SVD method using the real part of DCDS’s.

It is clear in figure 17 that the improved SVD method pinpoints the damage locations whereas the SVD method provides inaccurate damage identification results in figures 13(d) and 16(d), which confirms that maximizing $E_{\gamma}$ is a feasible way to enhance damage identification accuracy using real part of DCDS’s. Nevertheless, the damage identification results of JAD method is still better than the improved SVD method as the improved SVD method provides false damage alarms around measurement point 19 in figure 17(a) and measurement point 10 in figure 17(b).

5.3.2. Imaginary part of DCDS’s for damage identification.

The imaginary part of DCDS’s contains the out-of-phase vibration due to the phase differences between vibrations of measurement points of the original structure, and the phase differences between vibrations of measurement points of the damaged structure induced by the local damage. For practical engineering structures, the phase differences should be considered a sensitive damage feature since the local damage tends to cause unsynchronized movements (Masciotta et al 2017). In this study, the imaginary part curvature of DCDS’s in experimental example 2 is shown in figure 18.

It can be seen that the imaginary part does not provide reliable damage detection due to the uncertainties caused by measurement noise. The other reason for this is that the damage in the beams under study is linear damage (open cracks), which mainly affects the vibration amplitudes by reducing the local stiffness and the resultant local unsynchronized vibrations are not obvious. However, when using the CDS’s of PSD matrix for damage identification, the
imaginary part of DCDS’s should be always investigated due to its sensitivity to phase changes.

5.3.3. Absolute value of DCDS’s for damage identification. It is worth noting that the improved SVD method is merely a plane rotation of DCDS’s from SVD method and has the same absolute value. Thus, the improved SVD method will not be presented here. The absolute DCDS results of experimental example 2 are given in figure 19.

The absolute DCDS value plots show irregular shape features in DCDS’s caused by taking absolute value (figures 19(b) and (c)), which lead to false damage alarms and as a result GSM is difficult to be applied. But in this case, damage could be identified by comparing absolute DCDS value of undamaged structures with that of the damaged one. In order to apply GSM to the absolute value of DCDS’s for damage identification without knowledge of the healthy structures, the signed absolute value of DCDS’s (having the same sign as the real part of DCDS’s) is adopted. Then the damage identification index values are calculated and presented in figure 20.

Figure 20 illustrates that the absolute value of DCDS’s can improve the damage identification results of SVD method but the improvement for JAD method is not obvious compared with the results in figure 17 using the real part of DCDS’s. In addition, it is also demonstrated that JAD method performs better than SVD method using the signed absolute value of DCDS’s, as SVD method gives false damage detection around measurement point 19 in figure 20(a) and measurement point 10 in figure 20(b).
6. Conclusions

This paper proposed a new damage detection and localization index using the identified CDS’s, which is able to simultaneously detect the existence of damage and localize its position. JAD method and GSM are investigated to enhance the damage identification accuracy. Numerical simulation contaminated by Gaussian white noise demonstrates that JAD

Table 4. Relative vibration magnitude measure $E_{ij}$.

| Methods      | Experimental example 1 | Experimental example 2 |
|--------------|------------------------|------------------------|
|              | $E_{ij1}$   | $E_{ij2}$   | $E_{ij3}$   | $E_{ij1}$   | $E_{ij2}$   | $E_{ij3}$   |
| JAD          | 572.2       | 111.4       | 28.72       | 966.4       | 62.43       | 32.41       |
| SVD          | 12.28       | 5.491       | 3.372       | 8.647       | 2.919       | 4.535       |
| Improved SVD | 586.5       | 97.24       | 39.07       | 871.6       | 89.05       | 46.39       |

Figure 16. Damage identification results of experimental example 2. (a) DI of peak point 1. (b) DI of peak point 2. (c) DI of peak point 3. (d) Integrated DI.

Figure 17. Damage identification results using real part of DCDS’s. (a) Integrated DI of experimental example 1. (b) Integrated DI of experimental example 2.
method performs better in estimation of dominant CDS’s and DI than SVD method. Under white noise excitation, the identified dominant CDS’s are an estimate of mode shapes and this indicates that JAD method has the potential to improve the accuracy of operational modal analysis when compared with SVD method (also known as frequency domain decomposition method). Form two experimental examples, the dominant CDS’s estimated by JAD method are illustrated to be more accurate and noise robust for damage identification than those estimated by SVD method. Consequently, JAD method that diagonalizes several PSD matrices can enhance the noise robustness of CDS’s and augment damage identification accuracy. In addition, GSM is efficient and effective to extract shape distortion features and achieve accurate damage identification.

Another conclusion is that the proposed damage identification index is attractive and promising to be applied in practical engineering applications with ambient excitations. It is readily calculated from the output-only vibration responses and can be used for multiple damage detection and localization under various operational and environmental conditions without the theoretical model and prior knowledge of healthy structures.

Figure 18. The imaginary part curvature of DCDS’s in experimental example 2. (a) DCDS of peak point 1. (b) DCDS of peak point 2. (c) DCDS of peak point 3.

Figure 19. Absolute value of DCDS’s in experimental example 2. (a) DCDS of peak point 1. (b) DCDS of peak point 2. (c) DCDS of peak point 3.

Figure 20. Damage identification results using signed absolute value of DCDS’s. (a) Integrated DI of experimental example 1. (b) Integrated DI of experimental example 2.
Table A1. Relative error values ($\times 10^{-4}$) of figures 4(b), (d), (f).

| Measurement points | Figure 4(b) SVD | JAD | Figure 4(d) SVD | JAD | Figure 4(f) SVD | JAD |
|--------------------|-----------------|-----|-----------------|-----|-----------------|-----|
| 1                  | 5.6690          | 5.4571 | 1.5204          | 1.4788 | 3.9613          | 2.9371 |
| 2                  | 4.9980          | 4.8671 | 1.6112          | 1.5552 | 4.0183          | 2.9898 |
| 3                  | 4.4311          | 4.3934 | 1.4657          | 1.4393 | 4.0461          | 2.9440 |
| 4                  | 4.0392          | 3.9472 | 1.3733          | 1.3400 | 3.9917          | 2.9990 |
| 5                  | 3.3435          | 3.2421 | 1.4884          | 1.4553 | 4.4358          | 3.2978 |
| 6                  | 2.9359          | 2.8347 | 1.3468          | 1.3038 | 5.0023          | 3.7802 |
| 7                  | 2.5118          | 2.4270 | 1.2417          | 1.2036 | 6.1090          | 4.5298 |
| 8                  | 2.2470          | 2.1915 | 1.2926          | 1.2447 | 10.0368         | 7.4051 |
| 9                  | 2.0283          | 1.9849 | 1.3972          | 1.3278 | 29.5903         | 22.4556 |
| 10                 | 1.7648          | 1.7289 | 1.4096          | 1.3693 | 34.5500         | 25.3922 |
| 11                 | 1.4334          | 1.4102 | 1.7040          | 1.6348 | 12.5561         | 9.3751 |
| 12                 | 1.3437          | 1.3261 | 2.4589          | 2.4208 | 9.5378          | 7.2169 |
| 13                 | 1.2326          | 1.2129 | 3.7533          | 3.6723 | 9.8054          | 7.3874 |
| 14                 | 1.1721          | 1.1600 | 12.3267         | 12.1111 | 12.2744      | 9.1271 |
| 15                 | 1.1146          | 1.0892 | 11.2403         | 10.7392 | 23.3646     | 18.1075 |
| 16                 | 1.1233          | 1.1024 | 4.5486          | 4.4181 | 276.9493       | 206.8615 |
| 17                 | 1.1079          | 1.1060 | 2.6944          | 2.5879 | 22.0259        | 16.3751 |
| 18                 | 1.1265          | 1.1025 | 1.9807          | 1.9259 | 11.0240        | 8.4034 |

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Appendix

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