A TOPOLOGICAL CENTRAL POINT THEOREM

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ABSTRACT. In this paper a generalized topological central point theorem is proved for maps of a simplex to finite-dimensional metric spaces. Similar generalizations of the Tverberg theorem are considered.

1. INTRODUCTION

Let us state the discrete version of the Neumann–Rado theorem [9, 11, 5] (see also the reviews [4] and [3]):

**Theorem** (The discrete central point theorem). Suppose \( X \subset \mathbb{R}^d \) is a finite set with \(|X| = (d + 1)(r - 1) + 1\). Then there exists \( x \in \mathbb{R}^d \) such that for any halfspace \( H \ni x \)

\[ |H \cap X| \geq r. \]

In this theorem a halfspace is a set \( \{ x \in \mathbb{R}^d : \lambda(x) \geq 0 \} \) for a (possibly not homogeneous) linear function \( \lambda : \mathbb{R}^d \to \mathbb{R} \). Using the Hahn–Banach theorem [12] we restate the conclusion of this theorem as follows: the point \( x \) is contained in the convex hull of any subset \( F \subseteq X \) of at least \( d(r - 1) + 1 \) points.

When stated in terms of convex hulls, the central point theorem has an important and nontrivial generalization proved in [15]:

**Theorem** (Tverberg’s theorem). Consider a finite set \( X \subset \mathbb{R}^d \) with \(|X| = (d+1)(r-1)+1\). Then \( X \) can be partitioned into \( r \) subsets \( X_1, \ldots, X_r \) so that

\[ \bigcap_{i=1}^{r} \text{conv } X_i \neq \emptyset. \]

In [2, 16] a topological generalization of the Tverberg theorem was established. Instead of taking a finite point set in \( \mathbb{R}^d \) and the convex hulls of its subsets, we take the continuous image of a simplex in \( \mathbb{R}^d \) and the images of its faces (faces of the simplex viewed as a simplicial complex):

**Theorem** (The topological Tverberg theorem). Let \( m = (d + 1)(r - 1), r \) be a prime power, and let \( \Delta^m \) be the \( m \)-dimensional simplex. Suppose \( f : \Delta^m \to Y \) is a continuous map to a \( d \)-dimensional manifold \( Y \). Then there exist \( r \) disjoint faces \( F_1, \ldots, F_r \subset \Delta^m \) such that

\[ \bigcap_{i=1}^{r} f(F_i) \neq \emptyset. \]

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It is still unknown whether such a theorem holds for \( r \) not equal to a prime power. But if we return to the central point theorem, we see that the following topological version holds without restrictions on \( r \). Moreover, the target space can be any \( d \)-dimensional metric space, not necessarily a manifold. So the main result of this paper is:

**Theorem 1.1.** Let \( m = (d + 1)(r - 1) \), let \( \Delta^m \) be the \( m \)-dimensional simplex, and let \( W \) be a \( d \)-dimensional metric space. Suppose \( f : \Delta^m \to W \) is a continuous map. Then

\[
\bigcap_{F \subseteq \Delta^m, \dim F = d(r-1)} f(F) \neq \emptyset,
\]

where the intersection is taken over all faces of dimension \( d(r-1) \).

Note that for \( W = \mathbb{R}^d \) this theorem can also be deduced from the topological Tverberg theorem (see Section 4 for details). The goal of this paper is to give another proof of Theorem 1.1, valid for any \( d \)-dimensional \( W \). In Section 5 we show that a similar generalization of the Tverberg theorem for maps into finite-dimensional spaces essentially needs larger values of \( m \).

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### 2. Index of \( \mathbb{Z}_2 \)-spaces

Let us recall some basic facts on the homological index of \( \mathbb{Z}_2 \)-actions (\( \mathbb{Z}_2 \) is a group with two elements); the reader may consult the book [8] for more details. Denote \( G = \mathbb{Z}_2 \), if we consider \( \mathbb{Z}_2 \) as a transformation group. The algebra \( H^*(BG; \mathbb{F}_2) \) is a polynomial ring \( \mathbb{F}_2[c] \) with the one-dimensional generator \( c \).

In this section we consider the cohomology with \( \mathbb{F}_2 \) coefficients, the coefficients being omitted from the notation. Define the equivariant cohomology for a space \( X \) with continuous action of \( G \) (a \( G \)-space) by

\[
H^*_G(X) = H^*(X \times_G EG) = H^*((X \times EG)/G),
\]

thus we have \( H^*_G(pt) = H^*(BG) \) for a one-point space with trivial action of \( G \) and \( H^*_G(X) = H^*(X/G) \) for a free \( G \)-space. For \( G = \mathbb{Z}_2 \) we may take \( EG \) to be the infinite-dimensional sphere \( S^\infty \) with the antipodal action of \( G \), and \( BG = \mathbb{R}P^\infty \). For any \( G \)-space \( X \) the natural map \( X \to pt \) induces the natural cohomology map

\[
\pi_X : H^*_G(pt) = H^*(BG) \to H^*_G(X).
\]

**Definition 2.1.** For a \( G \)-space \( X \) define \( \text{ind}_G X \) to be the maximal \( n \) such that \( \pi_X^*(c^n) \neq 0 \in H^*_G(X) \).

Note that if \( X \) has a \( G \)-fixed point then the map \( \pi_X^* \) is necessarily injective and the index is infinite. The following property of index is obvious by definition:

**Lemma 2.2.** If \( X \) is a topological disjoint union of \( G \)-invariant subspaces \( X_1, \ldots, X_k \), then

\[
\text{ind}_G X = \max_i \text{ind}_G X_i.
\]

The next property is the generalized Borsuk–Ulam theorem (see [8] for example):

**Lemma 2.3.** Let \( \text{ind}_G X \geq n \) and let \( V \) be an \( n \)-dimensional vector space with antipodal \( G \)-action. Then for every continuous \( G \)-equivariant map \( f : X \to V \)

\[
f^{-1}(0) \neq \emptyset.
\]
The following lemma is proved in [20], see also [6]:

**Lemma 2.4.** Let \( X \) be a compact metric \( G \)-space, \( \text{ind}_G X \geq (d+1)k \), and let \( W \) be a \( d \)-dimensional metric space with trivial \( G \)-action. Then for every continuous \( G \)-equivariant map \( f : X \to W \) there exists \( x \in W \) such that

\[
\text{ind}_G f^{-1}(x) \geq k.
\]

In this lemma it is important to use the Čech cohomology, which is assumed in the sequel.

**3. Proof of Theorem 1.1**

Consider a continuous map \( f : \Delta^m \to W \). Let us map the \( m \)-dimensional sphere \( S^m \) to \( \Delta^m \) by the formula:

\[
g(x_1, \ldots, x_{m+1}) = (x_1^2, \ldots, x_{m+1}^2).
\]

Apply Lemma 2.4 to the composition \( f \circ g \), which is possible because \( g(x) = g(-x) \). We obtain a point \( x \in W \) such that for \( Z = (f \circ g)^{-1}(x) \) we have \( \text{ind}_G Z \geq r - 1 \).

We are going to show that \( x \) is the required intersection point. Assume the contrary: a face \( F \subseteq \Delta^m \) of dimension \( d(r - 1) \) does not intersect \( g(Z) \). Without loss of generality, let \( g^{-1}(F) \) be defined by the equations

\[
x_1 = \cdots = x_{r-1} = 0.
\]

Note that the \( r-1 \) coordinates \( x_{r-1}, \ldots, x_m \) give a continuous \( G \)-equivariant map \( h : S^m \to \mathbb{R}^{r-1} \), where \( G \) acts on \( \mathbb{R}^{r-1} \) antipodally. By Lemma 2.3 the intersection \( g^{-1}(F) \cap Z = h^{-1}(0) \cap Z = h|_Z^{-1}(0) \) should be nonempty. The proof is complete.

**4. Remark on the case \( W = \mathbb{R}^d \) of Theorem 1.1**

Recall the known fact: The case \( W = \mathbb{R}^d \) of Theorem 1.1 follows from the topological Tverberg theorem (only the case of prime \( r \) is needed). For the reader’s convenience we present a proof here (see also [7] Section 6).

Consider a simplicial map \( \varphi : \Delta^M \to \Delta^m \), where \( R = k(r-1)+1 \) is a prime (for some \( k \) this is so by the Dirichlet theorem on arithmetic progressions), \( M = (R-1)(d+1)+k-1 \), and there are \( k \) vertices of \( \Delta^M \) in the preimage of every vertex of \( \Delta^m \). For \( \Delta^M \) the topological Tverberg theorem holds (since \( M \geq (R-1)(d+1) \)), and there exist \( R \) disjoint faces \( \tilde{F}_1, \ldots, \tilde{F}_R \) of \( \Delta^M \) such that

\[
\bigcap_{i=1}^R f(\varphi(\tilde{F}_i)) \ni x.
\]

Consider a face \( F \subseteq \Delta^m \) of dimension \( d(r - 1) \) and assume that \( \varphi^{-1}(F) \) does not contain any \( \tilde{F}_i \), then \( M + 1 \) must be at least the number of vertices in \( \varphi^{-1}(F) \) plus \( R \), that is

\[
M + 1 \geq k(r-1)d + k + R = (R-1)d + k + R = M + 2,
\]

which is a contradiction. So \( \varphi^{-1}(F) \) contains some \( \tilde{F}_i \), and \( f(F) \ni x \).

**5. Tverberg type theorems for maps to finite-dimensional spaces**

It is natural to ask whether the corresponding version of the Tverberg theorem holds for maps from \( \Delta^m \) to a \( d \)-dimensional metric space, at least for \( r \) a prime power. In fact, the number \( m = (d+1)(r-1) \) must be increased, as claimed by the following:
Theorem 5.1. Let $m = (d + 1)r - 2$. Then there exists a $d$-dimensional polyhedron $W$ and a continuous map $f : \Delta^m \to W$ with the following property. For any pairwise disjoint faces $F_1, \ldots, F_r \subseteq \Delta^m$ there exists $i$ such that

$$f(F_i) \cap f(F_j) = \emptyset$$

for all $j \neq i$.

This theorem also shows that our approach used to prove Theorem 5.1 cannot be applied to the topological Tverberg theorem. Indeed, this proof does not distinguish between $\mathbb{R}^d$ and any metric $d$-dimensional space, but the topological Tverberg theorem does not hold for maps to $d$-dimensional metric spaces.

**Proof of Theorem 5.1.** The construction in the proof is taken from [19]. Let $\Delta^m$ be a regular simplex in $\mathbb{R}^m$, centered at the origin. Denote by $\Delta_{d-1}^m$ its $(d-1)$-skeleton, and $W = C\Delta_{d-1}^m$ the cone (geometrically centered at the origin) on this skeleton. Define the PL-map (of the barycentric subdivision to the barycentric subdivision) $f : \Delta^m \to W$ as follows. For every face $F \subseteq \Delta^m$ of dimension $\leq d - 1$ its barycenter is mapped to itself, for every face $F \subseteq \Delta^m$ of dimension $\geq d$ its barycenter is mapped to the origin.

Let $F_1, \ldots, F_r \subseteq \Delta^m$ be a set of $r$ pairwise disjoint faces. For some $i$ the dimension $\dim F_i$ is at most $d - 1$ by the pigeonhole principle. For such a face we have $f(F_i) = F_i$, and

$$f(F_i) \cap f(F_j) \subseteq F_i \cap f(F_j) \subseteq \partial \Delta^m.$$

Since $f(F_i) \cap \partial \Delta^m \subseteq F_j$ we obtain

$$f(F_i) \cap f(F_j) \subseteq F_i \cap F_j = \emptyset.$$

The following positive result for larger $m$ is a direct consequence of the reasoning in [18]:

**Theorem 5.2.** Let $m = (d + 1)r - 1$ and let $r$ be a prime power. Suppose $f : \Delta^m \to W$ is a continuous map to a $d$-dimensional metric space $W$. Then there exist $r$ disjoint faces $F_1, \ldots, F_r \subset \Delta^m$ such that

$$\bigcap_{i=1}^r f(F_i) \neq \emptyset.$$

**Proof.** Without loss of generality we may assume $W$ to be a finite $d$-dimensional polyhedron. Assume the contrary and denote $\Delta^m$ by $K$ for brevity. Then there exists a map

$$\tilde{f} : K_{\Delta(2)}^{*r} \to W_{\Delta(r)}^{*r}$$

from the $r$-fold pairwise deleted join $K_{\Delta(2)}^{*r}$ in the simplicial sense to the $r$-fold $r$-wise deleted join $W_{\Delta(r)}^{*r}$ in the topological sense (see the definitions of the deleted joins in [8]). Following [16], put $r = p^a$ and consider the group $G = (\mathbb{Z}_p)^a$ and let $G$ act on the factors of the deleted join transitively. The rest of the reasoning is based on the following facts from [17, 18]:

Let $X$ be a connected $G$-space. Consider the Leray–Serre spectral sequence with

$$E_2^{*,*} = H^*(BG; H^*(X; \mathbb{F}_p))$$

converging to $H^*_G(X; \mathbb{F}_p)$. Here $G$ may act on $H^*(X; \mathbb{F}_p)$ so the cohomology $H^*(BG; \cdot)$ may be with twisted coefficients.

**Definition 5.3.** Denote by $i_G(X)$ the minimum $r$ such that the differential $d_r$ of this spectral sequence has nontrivial image in the bottom row.
The index $i_G$ has the following properties, if $G$ is a $p$-torus $G = (\mathbb{Z}_p)^n$:

1. (Monotonicity) If there is a $G$-map $f : X \to Y$, then $i_G(X) \leq i_G(Y)$. If in addition $i_G(X) = i_G(Y) = n + 1$ then the map $f^* : H^n(Y; \mathbb{F}_p) \to H^n(X; \mathbb{F}_p)$ is nontrivial.

2. (Dimension upper bound) $i_G(X) \leq \text{hdim}_p X + 1$.

3. (Cohomology lower bound) If $X$ is connected and acyclic over $\mathbb{F}_p$ in degrees $\leq N - 1$, then $i_G(X) \geq N + 1$.

Now note that from the cohomology lower bound it follows that $i_G(K^r_{\Delta(2)}) \geq m + 1$, from the dimension upper bound it follows that $i_G(W^r_{\Delta(r)}) \leq m + 1$, and from (1) the map

$$\tilde{f}^* : H^m(W^r_{\Delta(r)}; \mathbb{F}_p) \to H^m(K^r_{\Delta(2)}; \mathbb{F}_p)$$

must be nontrivial. From the cohomology exact sequence of a pair it follows that the natural map

$$g^* : H^m(W^r; \mathbb{F}_p) \to H^m(W^r_{\Delta(r)}; \mathbb{F}_p)$$

is surjective because $H^{m+1}(W^r, W^r_{\Delta(r)}; \mathbb{F}_p) = 0$ by dimensional considerations. Now it follows that the map

$$(g \circ \tilde{f})^* : H^m(W^r; \mathbb{F}_p) \to H^m(K^r_{\Delta(2)}; \mathbb{F}_p)$$

is nontrivial. But the map $g \circ \tilde{f}$ is a composition of the natural inclusion

$$h : K^r_{\Delta(2)} \to K^r$$

with the map

$$f^r : K^r \to W^r.$$

The latter map has contractible domain, and therefore induces a zero map on cohomology $H^m(\cdot; \mathbb{F}_p)$. We obtain a contradiction. \qed

6. The case $r = 2$ of Theorem 1.1 and the Alexandrov width

Let us give a definition, generalizing the definition in [14]. The reader may also consult the book [10] in English. Throughout this section we use the notation

$$\delta A = \{\delta a : a \in A\} \quad \text{and} \quad A + B = \{a + b : a \in A, b \in B\}.$$

**Definition 6.1.** Let $K \subseteq \mathbb{R}^n$ be a convex body. Denote by $b_k(K)$ the maximal number such that for any map $K \to Y$ to a $k$-dimensional polyhedron there exists $y \in Y$ such that for any $\delta < b_k(K)$ the set $f^{-1}(y)$ cannot be covered by a translate of $\delta K$.

We use $k$-dimensional polyhedra $Y$ following [14], but we may also use $k$-dimensional metric spaces as above.

The definition of the *Alexandrov width* (in [14]) is a bit different: A subset $X$ of some normed space $E$ is considered and $a_k(X)$ denotes the maximal number such that for any map $X \to Y$ to a $k$-dimensional polyhedron there exists $y \in Y$ such that for any $\delta < a_k(X)$ the set $f^{-1}(y)$ cannot be covered by a ball (in the given norm of $E$) of radius $\delta$.

In [14] Theorem 1, p. 268] the results of K. Sitnikov and A.M. Abramov [11, 13] are cited, which assert that $a_k(X) = 1$ for any $k \leq n - 1$, if $X$ is the unit ball of a norm in $\mathbb{R}^n$. In terms of Definition 6.1 this means that $b_k(K) = 1$ for centrally symmetric convex bodies in $\mathbb{R}^n$ if $k \leq n - 1$ and obviously $b_k(K) = 0$ for $k \geq n$.

Note that Theorem 1.1 for $r = 2$ actually asserts that $b_k(\Delta^n) = 1$ if $k \leq n - 1$. Indeed, if $f^{-1}(y)$ intersects all facets of $\Delta^n$ then it cannot be contained in a smaller homothetic copy of $\Delta^n$. Now it makes sense to extend the result of K. Sitnikov and A.M. Abramov to (possibly not symmetric) convex bodies:
Theorem 6.2. If $K$ is a convex body in $\mathbb{R}^n$ and $k \leq n-1$, then $b_k(K) = 1$.

Proof. The proof in [14, Proposition 1, pp. 84–85] actually works in this case. Assume the contrary: the map $f : K \to Y$ is such that every preimage $f^{-1}(y)$ can be covered by a smaller copy of $K$ and $\dim Y \leq n-1$. For a fine enough finite closed covering of $Y$ its pullback covering $\mathcal{U}$ of $K$ has the following properties: the multiplicity of $\mathcal{U}$ is at most $n$ and any $X \in \mathcal{U}$ can be covered by a translate of $\delta K$ for some fixed $0 < \delta < 1$.

Assume $0 \in \text{int} \ K$ and call the point $t$ the center of a translate $\delta K + t$. Assign to any $X \in \mathcal{U}$ the center $t_X$ of $\delta K + t_X \subseteq X$. Using the partition of unity subordinate to $\mathcal{U}$ we map $K$ to the nerve of $\mathcal{U}$, and then map this nerve to at most $(n-1)$-dimensional subcomplex of $\mathbb{R}^n$ by assigning $t_X$ to $X$. Finally we obtain a continuous map $\varphi : K \to \mathbb{R}^n$ such that for any $x \in K$ we have $x \in \varphi(x) + \delta K$ and the image $\varphi(K)$ has dimension $\leq n-1$.

Under the above condition the image $\varphi(\partial K)$ cannot intersect $\varepsilon K$ if $\varepsilon < 1-\delta$, because $\varepsilon K + \delta K$ is in the interior of $K$. If we compose $\varphi|_{\partial K}$ with the central projection of $K \setminus \{0\}$ onto $\partial K$, we obtain a map homotopic to the identity map of $\partial K$. Therefore the map of pairs $\varphi : (K, \partial K) \to (K, K \setminus \varepsilon K)$ has degree 1, and $\varphi(K) \supseteq \varepsilon K$. Therefore $\varphi(K)$ is $n$-dimensional, which is a contradiction. □

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