Peakons Arising as Particle Paths Beneath Small-Amplitude Water Waves in Constant Vorticity Flows

Delia Ionescu-Kruse

To cite this article: Delia Ionescu-Kruse (2010) Peakons Arising as Particle Paths Beneath Small-Amplitude Water Waves in Constant Vorticity Flows, Journal of Nonlinear Mathematical Physics 17:4, 415–422, DOI: https://doi.org/10.1142/S140292511000101X

To link to this article: https://doi.org/10.1142/S140292511000101X

Published online: 04 January 2021
We present a new kind of particle path in constant vorticity water of finite depth, within the framework of small-amplitude waves.

Keywords: Water-waves; small-amplitude; vorticity; particle paths; peakon.

1. Introduction

A peakon is a soliton with discontinuous first derivative [17]. The concept was introduced in 1993 by Camassa and Holm in the paper [4], where they derived the CH shallow water equation

$$u_t + 2\kappa u u_x + 3uu_x = 2u u_{xx} + uu_{xxx}, \quad (\text{CH})$$  \hspace{1cm} (1.1)

$$(x, t) \in \mathbb{R} \times (0, \infty), \quad \kappa \text{ being a real constant. Alternative derivations of CH equation are provided in the papers [22, 9, 19]. The peakons arise as solution of this equation for } \kappa = 0.\text{ The CH peakons are given by}$$

$$u(x, t) = c \exp(-|x - ct|), \quad c \in \mathbb{R}. \quad \hspace{1cm} (1.2)$$

Since peakon solutions are only piecewise differentiable, they must be interpreted in a suitable weak sense. The derivative

$$u_x = -c \text{ sgn}(x - ct) \exp(-|x - ct|)$$  \hspace{1cm} (1.3)

has a jump discontinuity at the peak. The second derivative $u_{xx}$ must be taken in the sense of distributions and will contain a Dirac delta function

$$u_{xx} = c \exp(-|x - ct|) - 2c \delta(x - ct). \quad \hspace{1cm} (1.4)$$

415
The function $m$ is defined by

$$m(x, t) := u - u_{xx} = 2c\delta(x - ct).$$  

(1.5)

Physically $m$ has the interpretation of momentum [4, 18].

The peakon (1.2) has amplitude $c$ and travels at speed $c$. At $x = ct$ the momentum (1.5) blows up at $+\infty$.

A small perturbation of a CH peakon yields another one which remains close to some translate of the initial one at all later times. In this sense the CH peakons are orbitally stable [10]. Of particular interest is the description of peakon dynamics in terms of a system of completely integrable Hamiltonian equations for the locations of the peaks of the solution. Thus, each peakon solution can be associated with a mechanical system of moving particles. Being solitons, they retain their shape and speed after interacting with other peakons [1].

The peakon interaction plays an important role in the general dynamics of the solutions to the equation (see the discussion in [16]) and provided the framework for the construction of global weak solutions both in the conservative case [2] as well as in the dissipative case [5]. One of the main interests in CH equation was that, in contrast to other standard shallow water equations, as for example the KdV equation, it models breaking waves: smooth solutions that develop singularities in finite time, the solution being bounded but its slope becoming unbounded. This fact was already noted in [4] and subsequently proved in [6].

Another completely integrable CH-type equation which has peakon solutions [13] is the Degasperis–Procesi equation [14]

$$u_t + 4uu_x - u_{txx} + 3u_xu_{xx} + uu_{xxx}, \quad (\text{DP})$$  

(1.6)

$(x, t) \in \mathbb{R} \times (0, \infty)$. The DP equation possesses not only peaked solitons (1.2) but also discontinuous solitons, so-called shock-peakons [25] of the form

$$u(x, t) = c\exp(-|x|) - \frac{1}{t + k} \text{sgn}(x) \exp(-|x|), \quad k > 0.$$  

(1.7)

At the peak they have a finite jump in the function $u$ itself. The shock-peakon solutions must be interpreted in a proper weak formulation. The derivative $u_x$ will contain $\delta$ and the function $m := u - u_{xx}$ will be a linear combination of $\delta$ and $\delta'$ distributions. It is not known to the author if the function $m$ can be in this case interpreted as momentum. We point out that the CH equation with $\kappa = 0, \kappa \neq 0$, is a geodesic equation on the diffeomorphism group of the circle [8], respectively on the Bott–Virasoro group [26, 7], while the DP equation is a non-metric equation [15].

The shock-peakon (1.7) moves at constant speed $c$ (in particular, does not move if $c = 0$ [25]) which is equal to the average amplitude at the jump. The shock “dissipates away” like $1/t$ as $t \rightarrow +\infty$.

The peakons of the DP equation are also true solitons that interact via elastic collisions under the DP dynamics [24], and are also orbitally stable [23]. In what follows we will see that in the study of particle motion beneath small-amplitude water waves in constant vorticity flows a peakon trajectory comes up. This solution contains arctanh() function, having a vertical asymptote in the positive direction.
2. Particle Path Beneath Small-Amplitude Water Waves in Constant Vorticity Flows

We consider two-dimensional gravity waves on constant vorticity water of finite depth. They are described, in non-dimensional scaled variables, by the following boundary value problem (see, for example [21]):

\[
\begin{align*}
  &u_t + \epsilon (u u_x + v u_z) = -p_x, \\
  &\delta^2 [v_t + \epsilon (u v_x + v v_z)] = -p_z, \\
  &u_x + v_z = 0 \\
  &u_z = \delta^2 v_x + \frac{\sqrt{g h_0}}{g} \omega_0, \\
  &v = \eta_t + \epsilon u \eta_x, \quad \text{on } z = 1 + \epsilon \eta(x, t) \\
  &p = \eta, \quad \text{on } z = 1 + \epsilon \eta(x, t) \\
  &v = 0, \quad \text{on } z = 0,
\end{align*}
\]  

(2.1)

The system (2.2) has the solution

\[
\begin{align*}
  &\eta(x, t) = \cos(2\pi(x - ct)) \\
  &u(x, z, t) = \frac{2\pi \delta c}{\sinh(2\pi \delta)} \cosh(2\pi \delta z) \cos(2\pi(x - ct)) + \frac{\omega_0 \sqrt{g h_0}}{g} z + c_0 \\
  &v(x, z, t) = \frac{2\pi c}{\sinh(2\pi \delta)} \sinh(2\pi \delta z) \sin(2\pi(x - ct)) \\
  &p(x, z, t) = \frac{2\pi \delta c^2}{\sinh(2\pi \delta)} \cosh(2\pi \delta z) \cos(2\pi(x - ct))
\end{align*}
\]  

(2.3)
with the non-dimensional speed of the linear wave given by

\[ c^2 = \frac{\tanh(2\pi \delta)}{2\pi \delta}. \] (2.4)

Let \((x(t), z(t))\) be the path of a particle in the fluid domain, with location \((x(0), z(0)) := (x_0, z_0)\) at time \(t = 0\). The motion of the particles below the small-amplitude gravity water waves given by (2.3), is described by the following system of differential equations

\[
\begin{aligned}
\frac{dx}{dt} &= u(x, z, t) = 2\pi \delta c \frac{\cosh(2\pi \delta z) \cos(2\pi(x - ct)) + \omega_0 \sqrt{gh_0}}{g} + c_0 \\
\frac{dz}{dt} &= v(x, z, t) = 2\pi c \frac{\sinh(2\pi \delta z) \sin(2\pi(x - ct))}{g}.
\end{aligned}
\] (2.5)

Notice that the constant \(c_0\) is the average of the horizontal fluid velocity on the bottom over any horizontal segment of length 1, that is,

\[ c_0 = \frac{1}{1} \int_0^{\epsilon+1} u(s, 0, t) ds. \] (2.6)

This is accordance with Stokes’ definition of the wave speed for irrotational flows (see the discussion in [12]).

To study the exact solution of the system (2.5) it is more convenient to rewrite it in the following moving frame

\[ X = 2\pi(x - ct), \quad Z = 2\pi \delta z. \] (2.7)

This transformation yields

\[
\begin{aligned}
\frac{dX}{dt} &= 4\pi \delta c \frac{\cosh(Z) \cos(X) + \omega_0 \sqrt{gh_0}}{g} + 2\pi(c_0 - c) \\
\frac{dZ}{dt} &= 4\pi \delta c \frac{\sinh(Z) \sin(X)}{g}.
\end{aligned}
\] (2.8)

We denote by

\[ A := \frac{4\pi \delta c}{\sinh(2\pi \delta)} \quad \text{and} \quad \Omega_0 := \frac{\omega_0 \sqrt{gh_0}}{g}. \] (2.9)

With the notations (2.9), the system (2.8) becomes:

\[
\begin{aligned}
\frac{dX}{dt} &= A \cosh(Z) \cos(X) + \Omega_0 Z + 2\pi(c_0 - c) \\
\frac{dZ}{dt} &= A \sinh(Z) \sin(X).
\end{aligned}
\] (2.10)

We write the second equation of this system in the form

\[ \frac{dZ}{\sinh(Z)} = A \sin X(t) dt. \] (2.11)
Integrating, we get

$$\log \left[ \tanh \left( \frac{Z}{2} \right) \right] = \int A \sin X(t) \, dt.$$  \hspace{1cm} (2.12)

If

$$\int A \sin X(t) \, dt < 0$$  \hspace{1cm} (2.13)

then

$$Z(t) = 2 \arctanh \left[ \exp \left( \int A \sin X(t) \, dt \right) \right].$$  \hspace{1cm} (2.14)

Taking into account the formula:

$$\cosh(2x) = \frac{1 + \tanh^2(x)}{1 - \tanh^2(x)},$$  \hspace{1cm} (2.15)

and the expression (2.14) of $Z(t)$, the first equation of the system (2.10) becomes

$$\frac{dX}{dt} = A \frac{1 + w^2}{1 - w^2} \cos(X) + 2\Omega_0 \arctanh (w) + 2\pi(c_0 - c),$$  \hspace{1cm} (2.16)

where we have denoted by

$$w = w(t) := \exp \left( \int A \sin X(t) \, dt \right).$$  \hspace{1cm} (2.17)

With (2.13) in view, we have

$$0 < w < 1.$$  \hspace{1cm} (2.18)

From (2.17) we get

$$A \sin X(t) = \frac{1}{w(t)} \frac{dw}{dt}.$$  \hspace{1cm} (2.19)

Differentiating with respect to $t$ this relation, we obtain

$$A \cos(X) \frac{d^2X}{dt^2} = \frac{1}{w^3} \left[ \frac{d^2w}{dt^2} w - \left( \frac{dw}{dt} \right)^2 \right].$$  \hspace{1cm} (2.20)

From (2.19) we have furthermore

$$A^2 \cos^2(X) = A^2 - \frac{1}{w^2} \left( \frac{dw}{dt} \right)^2.$$  \hspace{1cm} (2.21)

Thus, taking into account (2.20), (2.21), Eq. (2.16) becomes

$$\frac{d^2w}{dt^2} + \frac{2w}{1 - w^2} \left( \frac{dw}{dt} \right)^2 - A^2 \frac{1 + w^2}{1 - w^2}$$

$$\sqrt{A^2w^2 - \left( \frac{dw}{dt} \right)^2} [2\Omega_0 \arctanh (w) + 2\pi(c_0 - c)] = 0.$$  \hspace{1cm} (2.22)
We make the following substitution
\[ \xi^2(w) := A^2 w^2 - \left( \frac{dw}{dt} \right)^2. \]  
(2.23)

Differentiating with respect to \( t \) this relation, we get
\[ \xi \frac{d \xi}{dw} = A^2 w - \frac{d^2 w}{dt^2}. \]  
(2.24)

We replace (2.23), (2.24) into Eq. (2.22) and we obtain the equation
\[ \xi \frac{d \xi}{dw} + \frac{2w}{1 - w^2} \xi^2 + [\Omega_0 \arctanh(w) + 2\pi (c_0 - c)] \xi = 0. \]  
(2.25)

A solution of Eq. (2.25) is
\[ \xi = 0 \]  
(2.26)

which, in view of (2.23) and (2.19) implies
\[ \sin X(t) = \pm 1. \]  
(2.27)

Therefore, from (2.14) with the condition (2.13), and further from (2.7), a solution of the system (2.5) is
\[ x(t) = ct + k_1 \]
\[ z(t) = \frac{1}{\pi \delta} \arctanh[\exp(-A|t|)]. \]  
(2.28)

\( k_1 \) being a constant. We observe that
\[ \lim_{t \to 0^+} x(t) = k_1, \quad \lim_{t < 0} z(t) = \lim_{t < 0} z(t) = +\infty \]  
(2.29)

and
\[ \lim_{t \to \pm \infty} x(t) = \pm \infty, \quad \lim_{t \to \pm \infty} z(t) = 0. \]  
(2.30)

Therefore, \( x = k_1 \) will be a vertical asymptote and \( z = 0 \) will be a horizontal asymptote for the curve (2.28). The graph of the parametric curve (2.28) is drawn in Fig. 1.

Fig. 1.
Notice that within the setting of irrotational flows with no underlying current (see [5]) there are no such paths but in the context of irrotational flows with an underlying (uniform) current, the possibility of such paths was already noticed; see [12] for the exact solutions, where this shape can be thought of as a limiting case of the situation depicted in Fig. 4.4(ii), as well as [20] for the linearized problem, where somewhat similar particle paths are encountered.

References
[1] R. Beals, D. Sattinger and J. Szmigielski, Multi-peakons and a theorem of Stieltjes, *Inverse Problems* 15 (1999) L1–L4.
[2] A. Bressan and A. Constantin, Global conservative solutions of the Camassa–Holm equation, *Arch. Ration. Mech. Anal.* 183 (2007) 215–239.
[3] A. Bressan and A. Constantin, Global dissipative solutions of the Camassa–Holm equation, *Analysis and Applications* 5 (2007) 1–27.
[4] R. Camassa and D. D. Holm, An integrable shallow water equation with peaked solitons, *Phys. Rev. Letters* 71 (1993) 1661–1664.
[5] A. Constantin, The trajectories of particles in Stokes waves, *Invent. Math.* 166 (2006) 523–535.
[6] A. Constantin and J. Escher, Wave breaking for nonlinear nonlocal shallow water equations, *Acta Mathematicae* 181 (1998) 229–243.
[7] A. Constantin, T. Kappeler, B. Kolev and P. Topalov, On geodesic exponential maps of the Virasoro group, *Ann. Glob. Anal. Geom.* 31 (2007).
[8] A. Constantin and B. Kolev, Geodesic flow on the diffeomorphism group of the circle, *Comment. Math. Helv.* 78(4) (2003) 787–804.
[9] A. Constantin and D. Ionescu-Kruse, Variational derivation of the Camassa–Holm shallow water equation, *J. Nonlinear Math. Phys.* 14 (2007) 303–312.
[10] A. Constantin and W. Strauss, Stability of peakons, *Comm. Pure Appl. Math.* 53 (2000) 603–610.
[11] A. Constantin and W. Strauss, Exact steady periodic water waves with vorticity, *Comm. Pure Appl. Math.* 57 (2004) 481–527.
[12] A. Constantin and W. Strauss, Pressure beneath a Stokes wave, *Comm. Pure Appl. Math.* 63 (2010) 533–557.
[13] A. Degasperis, D. D. Holm and A. N. I. Hone, A new integrable equation with peakon solutions, *Teoret. Mat. Fiz.* 133 (2002) 170–183.
[14] A. Degasperis and M. Procesi, Asymptotic integrability, in *Symmetry and Perturbation Theory* eds. A. Degasperis and G. Gaeta (World Scientific, Singapore, 1999), pp. 23–37.
[15] J. Escher and B. Kolev, The Degasperis–Procesi equation as a non-metric Euler equation, arxiv.org/abs/0908.0508 (2009).
[16] H. Holden and X. Raynaud, A convergent numerical scheme for the Camassa–Holm equation based on multiperiodic, *Discrete Contin. Dyn. Syst.* 14 (2006) 505–523.
[17] D. D. Holm, Peakons, *Encyclopedia of Mathematical Physics* 4, eds. J.-P. Francoise, G. L. Naber, S. T. Tsou (Elsevier, Oxford, 2006), pp. 12–20.
[18] D. D. Holm, J. E. Marsden and T. Ratiu, The Euler–Poincare equations and semidirect products with applications to continuum theories, *Adv. Math.* 137 (1998) 1–81.
[19] D. Ionescu-Kruse, Variational derivation of the Camassa–Holm shallow water equation, *J. Nonlinear Math. Phys.* 14 (2007) 303–312.
[20] D. Ionescu-Kruse, Small-amplitude capillary-gravity water waves: Exact solutions and particle motion beneath such waves, *Nonlinear Analysis: Real World Applications* 11 (2010) 2988–3000.
[21] R. S. Johnson, *A Modern Introduction to the Mathematical Theory of Water Waves* (Cambridge University Press, Cambridge, 1997).
[22] R. S. Johnson, Camassa–Holm, Korteweg-de Vries and related models for water waves, *J. Fluid Mech.* **455** (2002) 63–82.
[23] Z. W. Lin and Y. Liu, Stability of peakons for the Degasperis–Procesi equation, *Comm. Pure Appl. Math.* **62** (2008) 1–22.
[24] H. Lundmark and J. Szmigielski, Multi-peakon solutions of the Degasperis–Procesi equation, *Inverse Problems* **19** (2003) 1241–1245.
[25] H. Lundmark, Formation and dynamics of shock waves in the Degasperis–Procesi equation, *J. Nonlinear Sci.* **17** (2007) 169–198.
[26] G. Misiolek, A shallow water equation as a geodesic flow on the Bott–Virasoro group, *J. Geom. Phys.* **24** (1998) 203–208.