Locating overlap information in quantum systems

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Abstract

When discussing the black hole information problem the term “information flow” is frequently used in a rather loose fashion. In this article I attempt to make this notion more concrete. I consider a Hilbert space which is constructed as a tensor product of two subspaces (representing for example inside and outside the black hole). I discuss how the system has the capacity to contain information which is in neither of the subspaces. I attempt to quantify the amount of information located in each of the two subspaces, and elsewhere, and analyze the extent to which unitary evolution can correspond to “information flow”. I define the notion of “overlap information” which appears to be well suited to the problem.

1 Introduction

The “black hole information problem” (BHIP) is one of the most interesting topics in theoretical physics. At the root of this problem lie such fundamental questions as “is quantum gravity unitary?” and “can black holes decay completely?” (for an excellent review see [1]). The discussion often is phrased in terms of “the flow of information” in and out of a black hole.
This is the first in a series of articles with the purpose of answering the question “what does information flow mean in quantum systems?”. Is information some locally conserved quantity which can be followed as it “flows” from place to place? If not, is it possible to sensibly define information flow at all?

The motivation for this work is to understand what constraints on information flow are implied by unitary dynamics. Because the preservation of all inner products is the defining feature of unitary dynamics, I define “overlap information”, which allows one to reproduce the inner product of two states. It is this type of information which seems to be most relevant to the problems considered here.

Aside from motivational aspects, this first paper is not directly concerned with dynamics. I consider a Hilbert space which is constructed as a tensor product of two subspaces. I ask to what extent information can be said to be “located in” one or the other of the spaces, and try to provide quantitative answers to this question. The constraints on this quantified information implied by unitary evolution are then examined. A companion paper will build on this work and discuss the dynamics of information flow in more detail.

The BHIP is the motivating force behind this work. In section 2 I use the BHIP to specify just what I mean by “information”. However, the BHIP is quite complex and requires more than a definition of information flow in order to be resolved. At this stage I cannot claim that my efforts to quantify information flow offer any specific new insights into the BHIP. None the less, I do conclude that the connections between unitarity and information flow are not simple ones. It may prove productive to understand how the subtleties discussed here apply to the BHIP.

This paper is organized as follows: Section 2 provides the basic framework for the discussion which follows. Among other things, I define what I mean by information. The definition I chose is very narrowly motivated by the BHIP and I call this information “overlap information”. There purposely is no explicit reference to “information theory”, or any other of the various ways the term “information” shows up in the discussion of physical problems.

Section 3 illustrates how it is possible to hide information from both of the subspaces. I explicitly construct a complete basis for the whole tensor product space for which the information is completely hidden from the subsystems. That is, different basis vectors are indistinguishable from the
point of view of either subsystem. I point out that many such bases can be constructed.

Section 4 develops a way of quantifying the information in $a$ and the information in $b$, and discusses the implications of unitary evolution on these quantities. In Section 5 I quantify the overlap information which is located elsewhere.

Section 6 follows up on some technical issues, and Section 7 contains my conclusions. Appendix A provides proofs for the inequalities which I quote in the paper, and Appendix B provides a concise list of the main definitions and inequalities from this paper.

2 Preliminaries

2.1 What is information?

At the center of the BHIP is the question: “Does a decaying black hole evolve in an entirely unitary fashion?” The hallmark of unitary evolution is that the inner product between any two vectors in the Hilbert space is preserved.

Hawking [2, 3] has argued that the process of black hole decay can evolve two orthogonal initial states into overlapping final states, and thus that the decay process is non-unitary. This flies in the face of convictions which many physicists hold dear, and Hawking’s work has generated much heated debate. Those who wish to argue the case for a unitary decay process must show how information about the initial state can be encoded in the decay products (the “Hawking radiation”) so that orthogonal initial states remain so over time. If such a process can be identified, it would be natural to say that information flows from inside to outside the black hole.

In this work I take the point of view that the whole purpose of information is to determine the inner product between two given states in the full Hilbert space. The question then is: Given any two states, where does one have to look to acquire enough information to correctly determine their inner product? The answer, of course, will depend on which two states one is working with.

Because one starts with two states, and uses the information to discuss the relationship between these two state, I call the information discussed here “overlap information”. The notion appears to be quite different than other
ways the term information is used. I stress that I wish to remain as focused as possible on issues motivated by the BHIP. Although I will sometimes drop the word “overlap”, the term “information” will always refer to overlap information in what follows.

2.2 The framework
I will consider a “world” Hilbert space (denoted by $w$) which is constructed as a tensor product of subspaces $a$ and $b$. The sizes of $a$ and $b$ are $N_a$ and $N_b$ respectively, and I will take $N_a \leq N_b$. I also take all spaces to be finite.

Given two bases
\[ \{ |i\rangle_a |i = 1...N_a\} \]
and
\[ \{ |j\rangle_b |j = 1...N_b\} \]
which span subspaces $a$ and $b$ respectively, a tensor product basis which spans the $w$ space is given by
\[ \{ |i\rangle_a |j\rangle_b |i = 1...N_a, j = 1...N_b\}. \]

Any state $|\psi\rangle_w$ in $w$ can be written as
\[ |\psi\rangle_w = \sum_{i,j} \alpha_{ij} |i\rangle_a |j\rangle_b. \]

where $i$ and $j$ run over their complete ranges.

The results of any measurements of subsystem $a$ only can be predicted using the density matrix given by
\[ \rho_a \equiv \text{tr}_b |\psi\rangle_w \langle \psi| = \sum_k \alpha_k^* \alpha_{jk} |j\rangle_a \langle i|. \]

The subsystem $a$ is only in a pure state if $\rho_a$ has but a single non-zero eigenvalue. Similarly, measurements on $b$ only can be predicted using $\rho_b$ produced by tracing over the $a$ subspace. As long as $w$ is in a pure state the non-zero eigenvalues of $\rho_a$ and $\rho_b$ are always identical, and thus $a$ is in a pure state if and only if $b$ is in a pure state.

Given two states $|1\rangle_w$ and $|2\rangle_w$, one can construct the corresponding $\rho_a^{(1)}$ and $\rho_a^{(2)}$. If the value of $w \langle 1 |2\rangle_w$ can be correctly calculated using only $\rho_a^{(1)}$
and $\rho^{(2)}_a$ I will say that the information resides in subsystem $a$. Likewise one can construct $\rho^{(1)}_b$ and $\rho^{(2)}_b$, and if the value of $w\langle 1 \mid 2 \rangle_w$ can be calculated from these I will say that the information lies in subsystem $b$.

I am not concerned in this paper with the question of what measurements one actually has to do to extract the necessary information from $a$ or $b$. In many situation these may be impossible for human beings to perform. Despite such limitations, a discussion of the information content of the $\rho_a$’s and $\rho_b$’s seems a natural starting point for this work. One must bear in mind that in the end a discussion of the BHIP will deal with whether the fundamental Hamiltonian (not a human!) is able to move this information about.

### 2.3 The Schmidt expansion

It is often illuminating to expand a state $|\psi\rangle_w$ in the “Schmidt” basis $|1\rangle_a$, $|2\rangle_b$, $|3\rangle_a$, $|4\rangle_b$. This is the tensor product basis formed by using the eigenstates of $\rho_a$ and $\rho_b$ to span their respective subspaces. Denoting the Schmidt basis by $S$:

$$|\psi\rangle_w = \sum_{j=1}^{N_a} \sqrt{p_j} e^{i\phi_j} |j\rangle^S_a |j\rangle^S_b$$

(6)

where $p_i$ are the non-zero eigenvalues which are all shared by the two density matrices. Note that the number of terms in the sum is just the size of the smaller space ($N_a$) as opposed to $N_a \times N_b$ in Eq 4. Each eigenstate of $\rho_a$ is correlated with a unique eigenstate of $\rho_b$ with which it shares the eigenvalue.

For a discussion of the Schmidt result see Appendix A in [7]. Normally the phase $\phi_j$ in Eq 4 is absorbed into the definition of the Schmidt basis states. It is written explicitly here because everything in Eq 6 can almost always be determined from $\rho_a$ and $\rho_b$ except for this phase. The eigenstates can be determined up to a phase, and the correlations (that is which eigenstates of $\rho_a$ and $\rho_b$ are paired together in the Schmidt expansion) can be determined because the correlated eigenstates share the same eigenvalue.

The exception occurs when two or more $p_i$’s are equal. In this case the “eigenvalue matching” just described does not work in the subspace corresponding to the degeneracy. One is thus unable to identify the correlations within this subspace simply by knowing $\rho_a$ and $\rho_b$. In the next section I shall take advantage of this uncertainty to hide information from the subsystems.
3 The hidden basis

3.1 Defining a hidden basis

I will now construct a special “hidden” basis which will serve to illustrate some interesting places where information can be located. The hidden basis is comprised of a complete set of orthonormal states. For each element of the hidden basis $|i\rangle^H$ the corresponding $\rho_a$’s are all given by

$$\rho_a = \frac{1}{N_a}I_a$$ (7)

where $I_a$ is the unit matrix in $a$.

Likewise, for each hidden basis state

$$\rho_b = \frac{1}{N_b}I_b.$$ (8)

Complete hidden bases can only be constructed when $N_a = N_b$. Since all the $\rho_a$’s and the $\rho_b$’s are identical for all hidden basis states, one can not distinguish among these states by examining the subsystems.

3.2 The $N_a = N_b = 2$ case

First I will take $N_a = N_b = 2$. Using any bases denoted by $\{|1\rangle_a, |2\rangle_a\}$ and $\{|1\rangle_b, |2\rangle_b\}$ to span their respective subspaces the tensor product basis for $w$ is

$$\{|1\rangle_a|1\rangle_b, |1\rangle_a|2\rangle_b, |2\rangle_a|1\rangle_b, |2\rangle_a|2\rangle_b\}.$$ (9)

Now consider the first hidden basis state

$$|1\rangle_w^H \equiv \frac{|1\rangle_a|1\rangle_b + |2\rangle_a|2\rangle_b}{\sqrt{2}}.$$ (10)

This state is written in Schmidt form, so one can immediately see that $\rho_a = \frac{1}{2}I_a$ and $\rho_b = \frac{1}{2}I_b$ as promised.

Now consider

$$|2\rangle_w^H \equiv \frac{|1\rangle_a|1\rangle_b - |2\rangle_a|2\rangle_b}{\sqrt{2}}.$$ (11)

This state is also in Schmidt form and the subsystem density matrices are identical to those corresponding to $|1\rangle_w^H$. Even though $w^H\langle 1|2\rangle_w^H = 0$, one can
not possibly deduce this by inspecting \( \rho_a \) and \( \rho_b \). This information is simply not located in \( a \) or \( b \). One can say that the information is located in the relative sign of the two terms. The remaining members of the hidden basis are

\[
|3\rangle_w^H \equiv \frac{|1\rangle_a|2\rangle_b + |2\rangle_a|1\rangle_b}{\sqrt{2}} \quad (12)
\]

and

\[
|4\rangle_w^H \equiv \frac{|1\rangle_a|2\rangle_b - |2\rangle_a|1\rangle_b}{\sqrt{2}}. \quad (13)
\]

These are constructed by interchanging the correlations present in \( |1\rangle_w^H \) and \( |2\rangle_w^H \) between the two subsystems. The correlations can be changed without changing the \( \rho \)'s only because the \( p_i \)'s are degenerate.

If one wants to reconstruct \( H_w\langle 3|1\rangle_w^H = 0 \) one could say that the information was “in the correlations” (whereas for \( H_w\langle 4|3\rangle_w^H = 0 \) the information is again in the relative sign).

### 3.3 Generalizing

This construction can easily be generalized to arbitrary \( N_a = N_b \equiv N \). In this case “information in the relative sign” generalizes to “information in the relative phases”. One would have the first \( N_a \) hidden basis states being:

\[
|n\rangle_w^H \equiv \sum_{k=1}^{N_a} e^{ink2\pi/N} |k\rangle_a|k\rangle_b. \quad (14)
\]

The remaining \( N^2 - N \) hidden basis states can be constructed from these by making all possible exchanges of correlations.

In the case where \( N_b/N_a \) is an integer greater than unity one can make a similar construction. Since \( \rho_b \) can have at most \( N_a \) non-zero eigenvalues, the information can not be completely hidden from the \( b \) subsystem for all possible pairs of hidden basis elements. Some of the information will be contained in which eigenvalues of \( \rho_b \) are zero.

Note that many hidden bases can be constructed. For every possible choice of bases in the \( a \) and \( b \) subspaces one can construct a different hidden basis in the manner illustrated above. Still, the hidden basis states correspond to the special case where \( \rho_a \) and \( \rho_b \) have completely degenerate eigenvalues. (It is interesting to note that when \( 1 << N_a << N_b \) randomly chosen \( |\psi\rangle_w \)'s correspond to \( \rho_a \approx I_a/N_a \) [3, 5, 10].)
4 Quantifying the information in $a$ and in $b$

4.1 The Schmidt perspective

Consider now two arbitrary states $|1\rangle_w$ and $|2\rangle_w$. They each can be Schmidt expanded giving

$$|1\rangle_w = \sum_j \sqrt{p_j^{(1)}} \exp(i\phi_j^{(1)}) |j\rangle_a^{(1)} |j\rangle_b^{(1)}$$

$$|2\rangle_w = \sum_k \sqrt{p_k^{(2)}} \exp(i\phi_k^{(2)}) |k\rangle_a^{(2)} |k\rangle_b^{(2)}.$$  (15)  (16)

I have dropped the explicit $S$ superscript for Schmidt states. Note that in general $|1\rangle_w$ and $|2\rangle_w$ will generate different Schmidt bases. As discussed in section 2.3, as long as the $p_i$’s are not degenerate one can construct Eqs 15 and 16 except for the $\phi$’s just from knowing $\rho_a$ and $\rho_b$. For our purposes, the utility of the Schmidt form lies in the ease with which it allows one to identify what information about a state can be accessed from the $\rho$’s.

4.2 Measuring the overlap of two $\rho$’s

It will be useful to define the quantity

$$M_a \equiv \text{tr}_a \left( \sqrt{\rho_a^{(1)} \rho_a^{(2)}} \right) \equiv \sum_{jk} \sqrt{p_j^{(1)} p_k^{(2)}} |\langle j|k\rangle_a^{(1)} \rangle_{a} |\langle j|k\rangle_a^{(2)} \rangle_{b}|^2.$$  (17)

This quantity only depends on $\rho_a^{(1)}$ and $\rho_a^{(2)}$. I show in Appendix A that $M_a$ is bounded by

$$0 \leq M_a \leq 1.$$  (18)

The quantity $M_a$ has the following properties:

$$M_a = 1 \quad \text{if} \quad \rho_a^{(1)} = \rho_a^{(2)}$$

$$M_a = 0 \quad \text{if} \quad \rho_a^{(1)} \perp \rho_a^{(2)}.$$  (19)  (20)

By $\rho_a^{(1)} \perp \rho_a^{(2)}$ I mean that $\rho_a^{(1)}$ and $\rho_a^{(2)}$ assign non-zero probabilities only to orthogonal subspaces. The properties stated in Eqs 19 and 20 are evident from inspection of Eq 17.
Equations 19 and 20 suggest that the value of $M_a$ gives a good measure of the “overlap” of $\rho^{(1)}_a$ with $\rho^{(2)}_a$. This is how I will use $M_a$. (Note that using other powers of $\rho_a$ in Eq 17 would not work. Check Eq 19 on $\rho^{(1)}_a = \rho^{(2)}_a = I_a/N_a$ to see this.)

Similarly one can describe the overlap between $\rho^{(1)}_b$ and $\rho^{(2)}_b$ using

$$M_b \equiv \text{tr}_b \left( \sqrt{\rho^{(1)}_b \rho^{(2)}_b} \right) \equiv \sum_{jk} \sqrt{p^{(1)}_j p^{(2)}_k} |j_k\rangle_b \langle j_k|_b|^2.$$ (21)

4.3 The information in $a$

In Appendix A I show that $M_a$ actually obeys

$$|w\langle 1 |2\rangle_w| \leq \sqrt{M_a} \leq 1$$ (22)

(which is more restrictive than Eq 18). This means that if $M_a = 0$ one knows for sure that $w\langle 1 |2\rangle_w = 0$. When $M_a \neq 0$ Eq 22 simply constrains $|w\langle 1 |2\rangle_w|$ to a range of possible values. In general, one is not able to constrain $|w\langle 1 |2\rangle_w|$ any more tightly than

$$0 \leq |w\langle 1 |2\rangle_w| \leq \sqrt{M_a}$$ (23)

if one only knows $\rho_a$.

As an illustration, suppose $\rho^{(1)}_a = \rho^{(2)}_a = I/N_a$. Two possibilities are $|1\rangle_w = |1\rangle_H$ and $|2\rangle_w = |2\rangle_H$, or $|2\rangle_w = |1\rangle_w = |1\rangle_H$ (where I am using the hidden basis vectors defined in Section 3.2). These give values of $|w\langle 1 |2\rangle_w|$ at opposite ends of the allowed range.

I offer here a conjecture (without proof) that for any $\rho^{(1)}_a$ and $\rho^{(2)}_a$ one can choose corresponding $|1\rangle_w$’s and $|2\rangle_w$’s so that $w\langle 1 |2\rangle_w$ lies anywhere in the range given by Eq 23.

Recalling that for present purposes “information” corresponds to our ability to determine $w\langle 1 |2\rangle_w$, I define $O_a$, the “overlap information in $a$” as

$$O_a \equiv 1 - \sqrt{M_a}$$ (24)

When $O_a = 1$ one is certain that $w\langle 1 |2\rangle_w = 0$ and one can say that all the information is in $a$. When $O_a \neq 1$ one is less certain of the value of
w\langle 1 \mid 2 \rangle_w$, and there is incomplete information in $a$. When $O_a = 0$ there is no information in $a$ and $w\langle 1 \mid 2 \rangle_w$ can take on any value ($0 \leq |w\langle 1 \mid 2 \rangle_w| \leq 1$).

Similarly, the information in $b$ is

$$O_b \equiv 1 - M_b.$$  \hfill (25)

### 4.4 Information flow between $a$ and $b$

It is important to note that knowing $O_a$ alone tells us nothing about $O_b$. The information may be duplicated in both $a$ and $b$, or there could be less in one than the other. As is illustrated by the hidden basis, it is quite possible that there is no information in $a$ or $b$. Table 1 gives some illustrations.\[1]

Does unitarity alone constrain the evolution of $O_a$ and $O_b$? Not at all! The only requirement unitarity imposes is that orthogonal states must remain so over time. Given two orthogonal initial states, one can choose any pair of orthogonal final states and construct a unitary time evolution which relates the two. Thus one can construct unitary processes whereby $O_a$ and $O_b$ rise and fall independently. (Explicit examples of such constructions will appear in \[1\].)

Of course particular types of unitary evolution may produce a relationship between $O_a$ and $O_b$, but the point I wish to make here is that unitary evolution alone does not imply “information flow” between $a$ and $b$. One can for example construct unitary evolution where $O_a$ evolves from unity to zero (so information “flows out” of $a$), while $O_b$ holds steady at zero (so no information “flows in” to $b$). For example, evolution from columns $a)$ to column $c)$ in table \[1\] can do this (if $|a\langle X\mid Y \rangle_a| = 0$).

I have defined overlap information as that which allows you to evaluate the inner product between two states. Since unitary evolution preserves inner products, surely there must be some kind of “conservation of overlap information” associated with unitary evolution. This is in fact the case. The reason that the conservation of information does not force information to

\[1\]The density matrices $\rho_a^{(1)}$ and $\rho_a^{(2)}$ contain more information than just $O_a$. Under certain circumstances it is possible to learn something about $O_b$ if $\rho_a^{(1)}$ and $\rho_a^{(2)}$ are known. For example, because $\rho_a$ and $\rho_b$ share the same eigenvalues, when $\rho_a \propto I_a$ one can be sure that $\rho_b \propto I_b$ (as long as $N_a = N_b$). Thus (taking $N_a = N_b$) if $\rho_a^{(1)} = \rho_a^{(2)} \propto I_b$ one can be certain that $O_b = 0$. This point does not affect the discussion in this section because with other forms for $\rho_a^{(1)}$ and $\rho_a^{(2)}$, $O_a$ and $O_b$ can be truly independent.
flow between \(a\) and \(b\) is that there are other places where the information might be located. In what follows I will try to quantify this information.

5 Quantifying information which is neither in \(a\) nor \(b\)

5.1 Preliminaries

As is amply illustrated by the hidden basis, it is possible for overlap information to be located in neither \(a\) nor \(b\). Quantifying this information is not as straightforward as the construction of \(O_a\) and \(O_b\).

My starting point is to use Eqs \((15)\) and \((16)\) to write

\[
\langle 1 | 2 \rangle_w = \sum_{jk} \sqrt{p_j^{(1)} p_k^{(2)}} \exp \left\{ i \left( \phi_k^{(2)} - \phi_j^{(1)} \right) \right\} \langle j | k \rangle_a^{(1)} \langle k | j \rangle_b^{(2)} \tag{26}
\]

\[
\equiv \sum_{jk} M_{jk} e^{i\theta_{jk}}. \tag{27}
\]

where

\[
M_{jk} \geq 0. \tag{28}
\]

Equation \((26)\) is a sum over \(N_a^2\) complex numbers. Equation \((27)\) expresses each of these complex numbers in terms of its magnitude \(M_{jk}\) and complex phase \(\theta_{jk}\). Note that there in general are fewer \(\phi\)’s \((2N_a)\) than \(\theta_{jk}\)’s \((\text{which total } N_a^2)\). The values of the \(\theta\)’s are in part determined by the complex phases of the inner product of density matrix eigenstates. Thus, some information about the \(\theta\)’s can be deduced from the \(\rho\)’s. Barring eigenvalue degeneracy, the values of the \(M_{jk}\)’s can be completely determined from the \(\rho\)’s.

One can visualize Eq \((26)\) as a “chain” in the complex plane. Each term on the right side of Eq \((26)\) is represented by a link in the chain with length \(M_{jk}\). The angle of orientation of each link is given by the complex phase \(\theta_{jk}\). The separation of the ends of the chain equals \(|\langle 1 | 2 \rangle_w|\).

In general there are \(N_a^2\) links in the chain, but if \(\langle 1 | 2 \rangle_w = 1\) then

\[
M_{jk} = p_j \delta_{jk} \tag{29}
\]

and

\[
\theta_{jj} = 0 \quad \forall j. \tag{30}
\]
Equation 30 means that the chain is fully extended in the case of unit norm (all the links are parallel). Equation 29 shows that in this case the chain has at most $N_a$ links of non-zero length.

5.2 The length of the chain: $\bar{M}$

Below I shall define some quantities for which no simple analytic expression appears to exist. For this reason it is helpful to start by defining something which does have a simple form. It will help us get oriented, even though in most cases it is not the most interesting quantity\footnote{The quantity $\bar{M}$ is actually useful in one of the proofs in Appendix A.}

I defined $\bar{M}$ by

$$\bar{M} \equiv \sum_{jk} M_{jk}. \quad (31)$$

This is just the length of the chain. In Appendix A I show that $\bar{M}$ obeys

$$\bar{M} \leq \sqrt{M_a M_b}. \quad (32)$$

Also, the fact that the chain can not extend farther than its length gives

$$\bar{M} \geq |\langle 1 | 2 \rangle_w|. \quad (33)$$

Taking Eqns 32, 33, and 18 together gives

$$|\langle 1 | 2 \rangle_w| \leq \bar{M} \leq \sqrt{M_a M_b}. \quad (34)$$

Except in the case of eigenvalue degeneracy, The quantity $\bar{M}$ can be calculated if one knows both $\rho_a$’s and $\rho_b$’s, since then the only uncertainty lies in orientations of the links, not their lengths. In fact, since $\bar{M}$ is completely independent of the link orientations, it actually fails to make use of some information which is available if one knows both the $\rho_a$’s and $\rho_b$’s. (Remember, since in general there are more $\theta$’s than $\phi$’s, the link orientations are not entirely independent of the $\rho_a$’s and $\rho_b$’s.) This is why $\bar{M}$ is not exactly the quantity I would most like to evaluate. When some eigenvalues are degenerate, knowing all the $\rho$’s will not be enough to determine $\bar{M}$. (For an illustration of this point, compare columns c) and d) in table 1.)
5.3 The hidden information: $O_H$

Having defined $\tilde{M}$ as a “warm up”, I will now define quantities which identify how much information is hidden from both $a$ and $b$.

First define $M^\star$ as the norm of the right hand side of Eq 26:

$$M^\star \equiv \left| \sum_{jk} \sqrt{p_j^{(1)} p_k^{(2)}} \exp \left\{ i(\phi_k^{(2)} - \phi_j^{(1)}) \right\} \langle a|j\rangle_a \langle 1|k\rangle_1 \langle 2|k\rangle_2 \right|$$

(35)

Now define:

$M^\uparrow$: The maximum value $M^\star$ can achieve when the $\phi$’s are allowed to vary arbitrarily. When eigenvalues of a $\rho$ are degenerate, the corresponding ambiguity in choice of eigenstates (which appear in Eq 35) must also be fully explored as well, and the maximum value of $M^\star$ assigned to $M^\uparrow$.

and

$M^\downarrow$: The minimum value of $M^\star$ achieved by making the same variations as described in the definition of $M^\uparrow$.

The hidden information $O_H$ is then

$$O_H \equiv M^\uparrow - M^\downarrow.$$  

(36)

Which represents the remaining uncertainty in $|w\langle 1|2\rangle_w|$ once one utilizes both the $\rho_a$’s and the $\rho_b$’s to constrain $|w\langle 1|2\rangle_w|$ as tightly as possible.

5.4 Putting it all together

The quantity $|w\langle 1|2\rangle_w|$ can lie anywhere between zero and unity. To the extent that one is able to constrain $|w\langle 1|2\rangle_w|$ to lie in a more narrow range $\Delta$ I will say that one has an amount of overlap information given by $1 - \Delta$. This definition makes sense in the extreme limits of $\Delta = 0$ and $\Delta = 1$, and implies a certain measure for intermediate values (which I discuss in Section 6.2). My convention is that when the overlap information equals unity one has complete information.
Knowing $\mathcal{M}_a$ (the overlap of the two $\rho_a$’s) allows one to bound $|w\langle 1|2\rangle_w|$ according to Eq.22. Thus I arrived at the “overlap information in a” defined by $O_a \equiv 1 - \sqrt{\mathcal{M}_a}$.

Knowing both $\mathcal{M}_a$ and $\mathcal{M}_b$ produces the constraint

$$0 \leq |w\langle 1|2\rangle_w| \leq \sqrt{\mathcal{M}_a\mathcal{M}_b}$$

(37)

so it is natural to define

$$O_{ab} \equiv 1 - \sqrt{\mathcal{M}_a\mathcal{M}_b}.$$  

(38)

But actually, if one knows the $\rho_a$’s and the $\rho_b$’s one knows a lot more than just $\mathcal{M}_a$ and $\mathcal{M}_b$. One can employ the eigenvalue matching described in Section 2.3 to determine the correlations. Above I defined $\mathcal{O}_H$ to be the information which remains hidden after the $\rho_a$’s and $\rho_b$’s are used to maximal effect. Thus one can define

$$O_{ab} \equiv 1 - \mathcal{O}_H$$

(39)

which is all the information one can possibly extract from the $\rho_a$’s and the $\rho_b$’s. When there is no eigenvalue degeneracy one can say that $O_{ab} - O_{ab}$ is the information in the correlations, and $\mathcal{O}_H$ is the “information in the phases”. As we have seen in the extreme case of the hidden basis, when there is eigenvalue degeneracy some (or all) of the information in the correlations can be hidden (and contribute to $\mathcal{O}_H$ instead of $O_{ab} - iab$).

I show in Appendix A that (naturally enough)

$$O_{ab} \geq O_a$$

(40)

$$O_{ab} \geq O_b.$$  

(41)

Table II gives some illustrations, where the $\mathcal{M}$’s and $\mathcal{O}$’s are given for a variety of $|1\rangle_w$’s and $|2\rangle_w$’s.

6 Further discussion

6.1 Relation to the Von Neumann entropy

Given a state $|\psi\rangle_w$ and the corresponding $\rho_a$ and $\rho_b$, the Von Neumann entropy $S$ (relative to the $a \otimes b$ partition) is given by

$$S \equiv - \text{tr}_a (\rho_a \ln \rho_a) = - \text{tr}_b (\rho_b \ln \rho_b).$$  

(42)
If $|\psi\rangle_w$ has the pure form

$$|\psi\rangle_w = |X\rangle_a \otimes |Y\rangle_b$$

then $S = 0$. When

$$\rho_a \propto \mathbb{I}_a / N_a$$

the entropy is maximal (so $S = \ln(N_a)$) \footnote{I am still taking (without loss of generality) $N_a \leq N_b$.}

Given any two pure ($S = 0$) states, no overlap information can be hidden and $\mathcal{O}_H = 0$. The overlap information in $a$ and in $b$ ($\mathcal{O}_a$ and $\mathcal{O}_b$) can each take on any value between zero and unity (see columns $a$ and $b$) in table \footnote{Table}. In the limit of two $S = 0$ states equality holds in Eq \footnote{Equation} giving

$$|w\langle 1|2\rangle_w| = \mathcal{M} = \sqrt{\mathcal{M}_a \mathcal{M}_b}.$$ \footnote{Equation}

In the other extreme, given any two states with maximal $S$, no overlap information can be found in $a$, and if $N_b = N_a$ the same holds for $b$ giving $\mathcal{O}_a = \mathcal{O}_b = 0$ and $\mathcal{O}_H = 1$. The hidden basis states are an example of this.

Clearly the Von Neumann entropy has something to do with overlap information. Is it possible that the entire discussion can be re-phrased in terms of $S$, allowing one to avoid defining a new notion, “overlap information”, as I have done? For example, Page \footnote{Page} considers a world partitioned in two, and uses $S$ to discuss the information in each of the two subsystems, as well as the information in the correlations.

I believe it is clear that a discussion involving the Von Neumann entropy of single states can not completely replace the notion of overlap information. For example if $N_b \geq 2N_a$ one can consider pairs of orthogonal states with maximal entropy for which $\mathcal{O}_b$ can take on values anywhere between zero and unity (depending on the overlap of the $\rho_b$’s). Any discussion involving simply the Von Neumann entropy could not distinguish among these cases. Still, it may be productive to try and understand more carefully the relationship between Von Neumann entropy and overlap information.

### 6.2 Measures

The $\mathcal{M}$’s (and thus the $\mathcal{O}$’s) which I have defined place bounds on $|w\langle 1|2\rangle_w|$. I could just as well placed bounds on $|w\langle 1|2\rangle_w|^\alpha$ for any real value of $\alpha$. For
example, raising Eq 23 to the $\alpha$ power could result in

$$0 \leq |w\langle 1 | 2\rangle_w|^\alpha \leq (\mathcal{M}_a)^{\alpha/2} \equiv \sqrt{\mathcal{M}} \equiv 1 - \tilde{O}_a. \quad (46)$$

Choosing a value of $\alpha$ other than unity would imply a different measure of overlap information. At this point I do not have a concrete basis on which to distinguish among these different measures. Things would be different if there were equalities relating sums of $\mathcal{O}$'s (other than the rather trivial $\mathcal{O}_{ab} + \mathcal{O}_H \equiv (1 - \mathcal{O}_H) + \mathcal{O}_H = 1$) which would not hold for the $\tilde{\mathcal{O}}$'s I might construct along the lines of Eq 46. I believe the fact that such equalities do not exist is related to the lack of a clear notion of information flow. This is a result of the many places overlap information can be located, and the possibility that information can copied rather than being required to flow out of one place as it flows into another.

It is interesting to note that if one defines

$$m_a \equiv \frac{1}{2} \ln(\mathcal{M}_a) \quad (47)$$

(and similarly $m_b$) then Eq 34 becomes

$$\ln |w\langle 1 | 2\rangle_w| \leq m_a + m_b \quad (48)$$

which has an interesting additive form. At this stage however, it is not clear that there are any real advantages to taking this path.

Another point is that if the conjecture in Section 4.3 turns out to be wrong, one probably would want to modify the choice of measure.

### 7 Conclusions

I have defined the notion of “overlap information” which allows one to try and relate the “conservation of inner products” corresponding to unitarity with the notion of information flow.

I studied a space $w$ which is a tensor product space of two subspaces: $w = a \otimes b$. There are four different places where the overlap information can be located: In $a$, in $b$, “in the phases”, and “in the correlations”. Given two pure states in the $w$ space I have quantified these four types of information in a way which seems useful, but is not unique. Unitarity alone does not require
information to execute a “conserved flow”, in the sense that information can flow into one location without being required to flow out of another.

The existence of many complete “hidden bases” (for which no information can be found in a or b) means it is possible to start with a complete set of initial states for which all the overlap information is jointly held between a and b, and unitarily evolve this complete set into one in which the information is completely hidden from a and b. Those who wish a unitary resolution of the black hole information problem might want to look in all possible locations for the necessary information. In \cite{12} Wilczek takes a very interesting step in this direction.

This article has not discussed what type of Hamiltonians are required to move information among these different locations. Some work along these lines is currently underway \cite{11}.

8 Acknowledgments

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A Proofs of inequalities

In this appendix I prove the various inequalities quoted in the paper. They all derive from the Schwarz inequality applied to suitably identified real vector spaces with suitably defined inner products. Everything I do will be symmetrical under the exchange a ↔ b so, in particular, everything I prove for \( M_a \) also applies to \( M_b \).

A.1 The Schwarz inequality

If \( V \) and \( W \) are two vectors, any real quantity \( \langle V, W \rangle \) is an inner product if it obeys:

\[
\langle V, (\mu_1 W_1 + \mu_2 W_2) \rangle = \mu_1 \langle V, W_1 \rangle + \mu_2 \langle V, W_2 \rangle
\]
where $\mu_1$ and $\mu_2$ are real numbers.

2. \[ \langle V, W \rangle = \langle W, V \rangle \] (50)

3. \[ \langle V, V \rangle \geq 0 \] (51)

with \[ \langle V, V \rangle = 0 \Rightarrow V = 0. \] (52)

For any inner product, and any vectors $V$ and $W$ the Schwarz inequality holds:
\[ |\langle V, W \rangle| \leq \langle V, V \rangle^{1/2} \langle W, W \rangle^{1/2}. \] (53)

Equality occurs when the $W$ and $V$ are linearly dependent.

Given any scalar product the norm $||V||$ of any vector can be defined as $||V|| \equiv \langle V, V \rangle^{1/2}$. When working with vectors with unit norm, the Schwarz inequality has unity on the right hand side.

A.2 Bounds on $M_a$

Equation 17 defines
\[ M_a \equiv \text{tr}_a \left( \sqrt{\rho_a^{(1)}} \sqrt{\rho_a^{(2)}} \right) \equiv \sum_{jk} \sqrt{p_j^{(1)} p_k^{(2)}} |(j | 1 a) \rangle \langle (k | 2 a) |^2 . \] (54)

One can think of $\sqrt{\rho}$ as a vector in a vector space with inner product
\[ \langle \sqrt{\rho}_V, \sqrt{\rho}_W \rangle \equiv \text{tr} \left( \sqrt{\rho}_V : \sqrt{\rho}_W \right). \] (55)

(One can check that this meets all the requirements to be an inner product)

Properly normalized density matrices (with unit trace) have unit norm, and so
\[ 0 \leq M_a \leq 1. \] (56)

(Equation 18) follows directly from the Schwarz inequality and Eq 51. I derive the relationship between $M_a$ and $|w\langle 1 | 2 \rangle_w|$ in section A.4.
A.3 Bounds on $\bar{\mathcal{M}}$

The quantity $\bar{\mathcal{M}}$ is defined by Eq (31) which, taken with Eqs (26) and (27) gives

$$\bar{\mathcal{M}} \equiv \sum_{j,k} \sqrt{p_j(1) \langle j | k \rangle_a(2)} \sqrt{p_k(2) \langle b | j \rangle_b(1)}.$$  

(57)

It will be useful to think of Eq (57) in terms of a single sum by assigning to each unique ordered pair $(j, k)$ a unique integer $l(j, k)$. The relation $l(j, k)$ can be inverted to give $j(l)$ and $k(l)$. Thus one can write

$$\bar{\mathcal{M}} \equiv \sum_l \sqrt{P_j(l) P_k(l)} \langle a | j(l) \rangle_{a(1)} \langle k(l) | k \rangle_{a(2)} \langle b | j(l) \rangle_{b(1)} \langle k(l) | b \rangle_{b(2)}.$$  

(58)

Now one can define

$$A_l \equiv \frac{4}{P_j(l) P_k(l)} \langle a | j(l) \rangle_{a(1)} \langle k(l) | k \rangle_{a(2)} \langle b | j(l) \rangle_{b(1)} \langle k(l) | b \rangle_{b(2)}$$  

(59)

and

$$B_l \equiv \frac{4}{P_j(l) P_k(l)} \langle a | j(l) \rangle_{a(1)} \langle k(l) | k \rangle_{a(2)} \langle b | j(l) \rangle_{b(1)} \langle k(l) | b \rangle_{b(2)}$$  

(60)

so $\bar{\mathcal{M}}$ can be written as

$$\bar{\mathcal{M}} = \sum_l A_l B_l \equiv \vec{A} \cdot \vec{B}.$$  

(61)

Using this notation one finds that

$$\mathcal{M}_a = \vec{A} \cdot \vec{A}$$  

(62)

and

$$\mathcal{M}_b = \vec{B} \cdot \vec{B}.$$  

(63)

One can identify $\vec{A}$ and $\vec{B}$ as vectors, and “·” as a legitimate inner product. The Schwarz inequality then gives

$$\bar{\mathcal{M}} \leq \sqrt{\mathcal{M}_a \mathcal{M}_b}$$  

(64)
A.4 Relating $\mathcal{M}_a$ and $|w\langle 1 \mid 2 \rangle_w|$ 

Equation (64) implies a relation between $\mathcal{M}_a$ and $|w\langle 1 \mid 2 \rangle_w|$, even if $\mathcal{M}_b$ is unknown. One can simply use the fact that $\mathcal{M}_b$ is bounded above by unity (from Eq (56)) to get

$$|w\langle 1 \mid 2 \rangle_w| \leq \sqrt{\mathcal{M}_a}. \quad (65)$$

Equality in Eq (64) occurs when $\vec{A} \propto \vec{B}$. Since the derivation of Eq (65) is somewhat indirect, one might wonder when (or even if) equality is achieved. An example where Eq (65) becomes an equality appears in column a) of Table I.

A.5 Relating $\mathcal{O}_{\bar{a}b}$, $\mathcal{O}_a$ and $\mathcal{O}_b$

The quantities $\mathcal{O}_{\bar{a}b}$, $\mathcal{M}^\dagger$, and $\mathcal{M}^\downarrow$ are defined in Section 5.3 and Eq (39). If there is no eigenvalue degeneracy, then

$$\mathcal{M}^\dagger \leq \bar{\mathcal{M}}. \quad (66)$$

This is because the maximization process which gives $\mathcal{M}^\dagger$ can not do better than straightening out the chain, (giving $\mathcal{M}^\dagger = \bar{\mathcal{M}}$) and it could do worse (since the maximization process can not rotate all links arbitrarily). One thus has

$$\mathcal{O}_H \equiv \mathcal{M}^\dagger - \mathcal{M}^\downarrow \leq \bar{\mathcal{M}} \leq \sqrt{\mathcal{M}_a \mathcal{M}_b} \leq \sqrt{\mathcal{M}_a}. \quad (67)$$

Multiplying Eq (67) by $-1$, adding unity and using

$$\mathcal{O}_{\bar{a}b} \equiv 1 - \mathcal{O}_H \quad (68)$$

and

$$\mathcal{O}_a \equiv 1 - \sqrt{\mathcal{M}_a} \quad (69)$$

gives

$$\mathcal{O}_{\bar{a}b} \geq \mathcal{O}_a \quad (70)$$

Similarly one can show that

$$\mathcal{O}_{\bar{a}b} \geq \mathcal{O}_b. \quad (71)$$

If there are eigenvalue degeneracies it is possible to have $\mathcal{M}^\dagger > \bar{\mathcal{M}}$. However, it is possible to construct an $\bar{\mathcal{M}}'$ for which $\mathcal{M}^\dagger \leq \bar{\mathcal{M}}'$ and follow the
above steps to arrive at Eqs 70 and 71. This $\mathcal{M}'$ is constructed by constructing new states $|1\rangle'_w$ and/or $|2\rangle'_w$ by changing the correlations present in the subspace corresponding to the degenerate eigenvalues in order to maximize $\mathcal{M}$. This maximum value of $\mathcal{M}$ is $\mathcal{M}'$. Since this maximization procedure does not change $\mathcal{M}_a$, $\mathcal{M}_b$, or $\mathcal{M}^\dagger$ (which are the relevant quantities), the above proof can be used with $\mathcal{M}'$. Thus Eqs 70 and 71 apply even in the case of eigenvalue degeneracy.

B Compilation of definitions and results

Here is a concise compilation of definitions and results which appear in this paper.

B.1 The framework

Two states, $|1\rangle_w$ and $|2\rangle_w$, in a space $w$ partitioned according to $w = a \otimes b$ may be Schmidt expanded (Eqs 15 and 16) to give

$$
|1\rangle_w = \sum_j \sqrt{p_j^{(1)}} \exp(i\phi_j^{(1)}) |j\rangle_a^{(1)} |j\rangle_b^{(1)}
$$

$$
|2\rangle_w = \sum_k \sqrt{p_k^{(2)}} \exp(i\phi_k^{(2)}) |k\rangle_a^{(2)} |k\rangle_b^{(2)}.
$$

Each of $|1\rangle_w$ and $|2\rangle_w$ generates its own $\rho_a$ and $\rho_b$, resulting in $\rho_a^{(1)}$, $\rho_a^{(2)}$, $\rho_b^{(1)}$, and $\rho_b^{(2)}$.

B.2 Overlap and information in $a$

The overlap of $\rho_a^{(1)}$ and $\rho_a^{(2)}$ is given by Eq 17:

$$
\mathcal{M}_a \equiv \text{tr}_a \left( \sqrt{\rho_a^{(1)}} \sqrt{\rho_a^{(2)}} \right) \equiv \sum_{jk} \sqrt{p_j^{(1)} p_k^{(2)}} |\langle j|k\rangle_a^{(1)} |k\rangle_a^{(2)}|^2
$$

which obeys (Eq 22)

$$
|\langle w|1\rangle_w |^2 \leq \mathcal{M}_a \leq 1.
$$
This leads to the definition of $O_a$ the “overlap information in $a$”:

$$O_a \equiv 1 - \sqrt{M_a}$$  \hspace{1cm} (76)

(Eq 24). The equivalent quantities can be defined for $b$ (Eqs 21 and 23).

### B.3 The quantity $\bar{M}$

The quantity $\bar{M}$ is defined by Eqs 31, 26, and 27:

$$\bar{M} \equiv \sum_{j,k} \sqrt{p_j^{(1)} p_k^{(2)}} |\langle j|k\rangle^{(1)}_a| |\langle j|k\rangle^{(2)}_b|$$  \hspace{1cm} (77)

which obeys

$$|_{w\langle 1|2\rangle_w} | \leq \bar{M} \leq \sqrt{M_a M_b}$$  \hspace{1cm} (Eq 34).

### B.4 The hidden information

The quantity $\mathcal{M}^\bullet$ is given by (Eq 35)

$$\mathcal{M}^\bullet \equiv \sum_{j,k} \sqrt{p_j^{(1)} p_k^{(2)}} \exp\left\{i(\phi_k^{(2)} - \phi_j^{(1)})\right\} |\langle j|k\rangle^{(1)}_a| |\langle j|k\rangle^{(2)}_b|$$  \hspace{1cm} (79)

The quantities $\mathcal{M}^\uparrow$ and $\mathcal{M}^\downarrow$ are:

$\mathcal{M}^\uparrow$: The maximum value $\mathcal{M}^\bullet$ can achieve when the $\phi$’s are allowed to vary arbitrarily. When eigenvalues of a $\rho$ are degenerate, the corresponding ambiguity in choice of eigenstates (which appear in Eq 35) must also be fully explored as well, and the maximum value of $\mathcal{M}^\bullet$ assigned to $\mathcal{M}^\uparrow$.

and

$\mathcal{M}^\downarrow$: The minimum value of $\mathcal{M}^\bullet$ achieved by making the same variations as described in the definition of $\mathcal{M}^\uparrow$.

The hidden information $O_H$ (Eq 36) is then

$$O_H \equiv \mathcal{M}^\uparrow - \mathcal{M}^\downarrow$$  \hspace{1cm} (80)

which represents the remaining uncertainty in $|_{w\langle 1|2\rangle_w}|$ once one utilizes both the $\rho_a$’s and the $\rho_b$’s to constrain $|_{w\langle 1|2\rangle_w}$ as tightly as possible.
B.5 The information in both $a$ and $b$

The most overlap information one can extract from the $\rho_a$'s and $\rho_b$'s is

$$O_{ab} \equiv 1 - O_H$$

(Eq 24). Naturally,

$$O_{ab} \geq O_a, O_b.$$  \hspace{1cm} (82)

The overlap information in both $M_a$ and $M_b$ is

$$O_{ab} \equiv 1 - \sqrt{M_a M_b}.$$  \hspace{1cm} (83)

When there is no eigenvalue degeneracy one can say that $O_{\bar{ab}} - O_{ab}$ is the information in the correlations, and $O_H$ is the “information in the phases”. When there is eigenvalue degeneracy some (or all) of the information in the correlations can be hidden (and contributes to $O_H$ instead of $O_{\bar{ab}} - O_{ab}$).

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|   | a) | b) | c) | d) | e) |
|---|----|----|----|----|----|
| $|1\rangle_w$ | $|X\rangle_a|U\rangle_b$ | $|X\rangle_a|U\rangle_b$ | $|1\rangle^H_w$ | $|1\rangle^H_w$ | $|1\rangle^H_w$ |
| $|2\rangle_w$ | $|Y\rangle_a|U\rangle_b$ | $|Y\rangle_a|V\rangle_b$ | $|2\rangle^H_w$ | $|3\rangle^H_w$ | $|1\rangle^H_w$ |
| $|w\langle 1 \ 2\rangle_w|$ | $|_a\langle X|Y\rangle_a|$ | $|_a\langle X|Y\rangle_a||_a\langle U|V\rangle_a|$ | 0 | 0 | 1 |
| $\sqrt{\mathcal{M}_a}$ | $|_a\langle X|Y\rangle_a|$ | $|_a\langle X|Y\rangle_a|$ | 1 | 1 | 1 |
| $\sqrt{\mathcal{M}_b}$ | 1 | $|_a\langle U|V\rangle_a|$ | 1 | 1 | 1 |
| $\sqrt{\mathcal{M}_a\mathcal{M}_b}$ | $|_a\langle X|Y\rangle_a|$ | $|_a\langle X|Y\rangle_a||_a\langle U|V\rangle_a|$ | 1 | 1 | 1 |
| $\bar{\mathcal{M}}$ | $|_a\langle X|Y\rangle_a|$ | $|_a\langle X|Y\rangle_a||_a\langle U|V\rangle_a|$ | 1 | 0 | 1 |
| $\mathcal{M}^i$ | $|_a\langle X|Y\rangle_a|$ | $|_a\langle X|Y\rangle_a||_a\langle U|V\rangle_a|$ | 1 | 1 | 1 |
| $\mathcal{M}^s$ | $|_a\langle X|Y\rangle_a|$ | $|_a\langle X|Y\rangle_a||_a\langle U|V\rangle_a|$ | 0 | 0 | 0 |
| $\mathcal{O}_a$ | $1 - |_a\langle X|Y\rangle_a|$ | $1 - |_a\langle X|Y\rangle_a|$ | 0 | 0 | 0 |
| $\mathcal{O}_b$ | 0 | $1 - |_a\langle U|V\rangle_a|$ | 0 | 0 | 0 |
| $\mathcal{O}_{ab}$ | $1 - |_a\langle X|Y\rangle_a|$ | $1 - |_a\langle X|Y\rangle_a||_a\langle U|V\rangle_a|$ | 0 | 0 | 0 |
| $\mathcal{O}_{ab}$ | 1 | $1 - |_a\langle X|Y\rangle_a||_a\langle U|V\rangle_a|$ | 0 | 0 | 0 |
| $\mathcal{O}_H$ | 0 | 1 | 1 | 1 | 1 |

Table 1: Some examples. Each column represents a particular choice of $|1\rangle_w$ and $|2\rangle_w$. The values of the $\mathcal{M}$'s and $\mathcal{O}$'s are given for each choice. Note that here the hidden basis states (with superscript $H$) refer to the $N_a = N_b = 2$ case discussed in Section 3.2.