Virtual fundamental classes for moduli spaces of sheaves on Calabi–Yau four-folds

Dennis Borisov and Dominic Joyce

Abstract

Let \((X, \omega_X)\) be a separated, \(-2\)-shifted symplectic derived \(\mathbb{C}\)-scheme, in the sense of Pantev, Toën, Vezzosi and Vaquié \cite{PTVV}, of complex virtual dimension \(\text{vdim}_\mathbb{C} X = n \in \mathbb{Z}\), and \(X_{an}\) the underlying complex analytic topological space. We prove that \(X_{an}\) can be given the structure of a derived smooth manifold \(X_{dm}\), of real virtual dimension \(\text{vdim}_\mathbb{R} X_{dm} = n\). This \(X_{dm}\) is not canonical, but is independent of choices up to bordisms fixing the underlying topological space \(X_{an}\). There is a 1-1 correspondence between orientations on \((X, \omega_X)\) and orientations on \(X_{dm}\).

Because compact, oriented derived manifolds have virtual classes, this means that proper, oriented \(-2\)-shifted symplectic derived \(\mathbb{C}\)-schemes have virtual classes, in either homology or bordism. This is surprising, as conventional algebro-geometric virtual cycle methods fail in this case. Our virtual classes have half the expected dimension, and from purely complex algebraic input, can yield a virtual class of odd real dimension.

Now derived moduli schemes of coherent sheaves on a Calabi–Yau 4-fold are expected to be \(-2\)-shifted symplectic (this holds for stacks). We propose to use our virtual classes to define new Donaldson–Thomas style invariants ‘counting’ (semi)stable coherent sheaves on Calabi–Yau 4-folds \(Y\) over \(\mathbb{C}\), which should be unchanged under deformations of \(Y\).

Contents

1 Introduction 2
2 Background material 5
   2.1 Commutative differential graded algebras 5
   2.2 Derived algebraic geometry and derived schemes 7
   2.3 PTVV’s shifted symplectic geometry 8
   2.4 Orientations on \(k\)-shifted symplectic derived schemes 10
   2.5 Derived smooth manifolds 12
   2.6 Orientations on derived manifolds 16
   2.7 Bordism and virtual classes 17
1 Introduction

This paper will relate two apparently rather different classes of ‘derived’ geometric spaces. The first class is derived \( \mathbb{C} \)-schemes \( X \), in the Derived Algebraic Geometry of Toën and Vezzosi \([34, 36]\), equipped with a \(-2\)-shifted symplectic structure \( \omega_X^* \) in the sense of Pantev, Toën, Vaquié and Vezzosi \([31]\). Such \((X, \omega_X^*)\) are the expected structure on 4-Calabi–Yau derived moduli \( \mathbb{C} \)-schemes.

The second class is derived smooth manifolds \( X_{dm} \), in Derived Differential Geometry. There are several different models available: the derived manifolds of Spivak \([32]\) and Borisov–Noël \([7, 8]\) (which form \( \infty \)-categories \( \text{DerMan}_{\text{Spi}} \) and \( \text{DerMan}_{\text{BoNo}} \)), and the second author’s \( d \)-manifolds \([21, 23]\) (a strict 2-category \( \text{dMan} \)), \( M \)-Kuranishi spaces \([25] \) \( \S 2 \) (an ordinary category \( \text{MKur} \)) and \( \text{Kuranishi spaces with trivial orbifold groups} \([25] \) \( \S 4 \) (a weak 2-category \( \text{Kur}_\text{trG} \)).

As it is known that equivalence / isomorphism classes of all of these (higher) categories are in natural bijection, these five models are interchangeable for our purposes. But we use theorems proved for \( d \)-manifolds or \( (M-) \)Kuranishi spaces.

Here is a summary of our main results, taken from Theorems 3.18, 3.20 and 3.25 and Proposition 3.19 below.
Theorem 1.1. Let $(X, \omega^*_X)$ be a $-2$-shifted symplectic derived $\mathbb{C}$-scheme, in the sense of Panet et al. [31], with complex virtual dimension $\text{vdim}_C X = n$ in $\mathbb{Z}$, and write $X_{\text{an}}$ for the set of $\mathbb{C}$-points of $X = t_0(X)$, with the complex analytic topology. Suppose that $X$ is separated, and $X_{\text{an}}$ is second countable. Then we can make the topological space $X_{\text{an}}$ into a derived manifold $X_{\text{dm}}$ of real virtual dimension $\text{vdim}_R X_{\text{dm}} = n$, in the sense of any of [7,8,21,23,25,32].

There is a natural $1$-$1$ correspondence between orientations on $(X, \omega^*_X)$, in the sense of [2,4] and orientations on $X_{\text{dm}}$, in the sense of [2,6].

The (oriented) derived manifold $X_{\text{dm}}$ above depends on arbitrary choices made in its construction. However, $X_{\text{dm}}$ is independent of choices up to (oriented) bordisms of derived manifolds which fix the underlying topological space.

All the above extends to (oriented) $-2$-shifted symplectic derived schemes $(\pi : X \to Z, \omega^*_X/\mathbb{Z})$ over a base $Z$ which is a smooth, connected, affine $\mathbb{C}$-scheme, yielding an (oriented) derived manifold $\pi_{\text{dm}} : X_{\text{dm}} \to Z_{\text{an}}$ over the complex manifold $Z_{\text{an}}$ associated to $Z$, regarded as an (oriented) real manifold.

We prove Theorem 1.1 using a ‘Darboux Theorem’ for $k$-shifted symplectic derived schemes by Bussi, Brav and the second author [5]. This paper is related to the series [5,6,24], mostly concerning the $-1$-shifted (3-Calabi–Yau) case.

An important motivation for proving Theorem 1.1 is that compact, oriented derived manifolds have virtual classes. Virtual classes are used in several areas of geometry to define enumerative invariants (e.g. algebraic and symplectic Gromov–Witten invariants, and Donaldson–Thomas invariants [20,33]). Given a compact, oriented derived manifold $X_{\text{dm}}$ with $\text{vdim}_R X_{\text{dm}} = n$, one can define a virtual class $[X_{\text{dm}}]_{\text{virt}}$ in the (Steenrod) homology $H_n(X_{\text{dm}}, \mathbb{Z})$, or in the homology $H_n(Y; \mathbb{Z})$ of a manifold $Y$ with a morphism $X_{\text{dm}} \to Y$.

A more elementary approach to virtual classes is via bordism. As we explain in §2.7, one can define bordism groups $\text{dB}_n(*)$ of compact, oriented derived manifolds $X_{\text{dm}}$ with $\text{vdim}_R X_{\text{dm}} = n$, and these are isomorphic to the classical bordism groups $B_n(*)$ of compact, oriented manifolds, which are well understood. As in §3.6–3.7 from Theorem 1.1 we may deduce:

Corollary 1.2. Let $(X, \omega^*_X)$ be a proper, oriented $-2$-shifted symplectic derived $\mathbb{C}$-scheme, with $\text{vdim}_C X = n$. Theorem 1.1 gives a compact, oriented derived manifold $X_{\text{dm}}$ with $\text{vdim}_R X_{\text{dm}} = n$, depending on choices.

The $d$-bordism class $[X_{\text{dm}}]$ in $\text{dB}_n(*) \cong B_n(*)$ from §2.7 is independent of choices, and depends only on $(X, \omega^*_X)$ and its orientation. We regard $[X_{\text{dm}}]$ as a virtual class for $(X, \omega^*_X)$.

If $n = 0$ then $\text{dB}_n(*) \cong \mathbb{Z}$, and $[X_{\text{dm}}]$ is a virtual count of $(X, \omega^*_X)$.

If $(\pi : X \to Z, \omega^*_X/\mathbb{Z})$ is an oriented $-2$-shifted symplectic derived scheme proper over a connected, affine $\mathbb{C}$-scheme $Z$, then for any $z_1, z_2 \in Z_{\text{an}}$, the bordism classes $[X_{\text{dm}}^{z_1}], [X_{\text{dm}}^{z_2}]$ from the fibres $X_{\text{dm}}^{z_1}, X_{\text{dm}}^{z_2}$ satisfy $[X_{\text{dm}}^{z_1}] = [X_{\text{dm}}^{z_2}]$.

So, proper, oriented $-2$-shifted symplectic derived $\mathbb{C}$-schemes $(X, \omega^*_X)$ have virtual classes. This is not obvious, in fact it is rather surprising. Firstly, if $(X, \omega^*_X)$ is $-2$-shifted symplectic then $X = t_0(X)$ has a natural obstruction theory $L_X |_X \to L_X$ in the sense of Behrend and Fantechi [2], which is perfect in
the interval $[-2, 0]$. But the Behrend–Fantechi construction of virtual cycles [2] works only for obstruction theories perfect in $[-1, 0]$, and does not apply here.

Secondly, our virtual cycle has real dimension $\text{vdim}_C X = \frac{1}{2} \text{vdim}_R X$, which is half what we might have expected. A heuristic explanation is that one should be able to make $X$ into a ‘derived $C^\infty$-scheme’ $X_{C^\infty}$ (not a derived manifold), in some sense similar to Lurie [28 §4.5] or Spivak [32], and $(X_{C^\infty}, \text{Im } \omega^*_X)$ should be a ‘real $-2$-shifted symplectic derived $C^\infty$-scheme’, with $\text{Im } \omega^*_X$ the imaginary part of $\omega^*_X$. There should be a morphism $X_{C^\infty} \to X_{\text{dm}}$ which is a ‘Lagrangian fibration’ of $(X_{C^\infty}, \text{Im } \omega^*_X)$. So $\text{vdim}_R X_{\text{dm}} = \frac{1}{2} \text{vdim}_R X_{C^\infty} = \frac{1}{2} \text{vdim}_R X$, as for Lagrangian fibrations $\pi : (S, \omega) \to B$ we have $\dim B = \frac{1}{2} \dim S$.

Note that $n = \text{vdim}_C X$ can be odd. So starting with the complex algebraic data $(X, \omega^*_X)$, we may construct a virtual class of odd real dimension, which seems strange, and unlike conventional Behrend–Fantechi virtual cycles [2].

The main application we intend for these results, motivated by Donaldson and Thomas [15] and explained in §3.8–§3.9, is to define new invariants ‘counting’ (semi)stable coherent sheaves on Calabi–Yau 4-folds $Y$ over $\mathbb{C}$, which should be unchanged under deformations of $Y$. These are similar to Donaldson–Thomas invariants [26, 27, 33] ‘counting’ (semi)stable coherent sheaves of Calabi–Yau 3-folds, and could be called ‘holomorphic Donaldson invariants’, as they are complex analogues of Donaldson invariants of 4-manifolds [14].

Pantev, Toën, Vaquié and Vezzosi [31, §2.1] show that any derived moduli stack $\mathcal{M}$ of coherent sheaves (or complexes of coherent sheaves) on a Calabi–Yau $m$-fold has a $(2 - m)$-shifted symplectic structure $\omega^*_\mathcal{M}$, so in particular 4-Calabi–Yau moduli stacks are $-2$-shifted symplectic. Given an analogue of this for derived moduli schemes, and a way to define orientations upon them, Corollary 1.2 would give virtual classes for moduli schemes of (semi)stable coherent sheaves on Calabi–Yau 4-folds, and so enable us to define invariants.

It is well known that there is a great deal of interesting and special geometry, related to String Theory, concerning Calabi–Yau 3-folds and 3-Calabi–Yau categories — Mirror Symmetry, Donaldson–Thomas theory, and so on. One message of this paper is that there should also be special geometry concerning Calabi–Yau 4-folds and 4-Calabi–Yau categories, which is not yet understood.

During the writing of this paper, Cao and Leung [9–12] also proposed a theory of invariants counting coherent sheaves on Calabi–Yau 4-folds, based on gauge theory rather than derived geometry. We discuss their work in §3.9.

Section 2 provides background material on derived schemes, shifted symplectic structures upon them, and derived manifolds. The heart of the paper is §3 with the definitions, main results, shorter proofs, and discussion. Longer proofs of results in §3 are deferred to sections §4–7.

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2 Background material

We begin with some background material and notation needed later. Some references are Toën and Vezzosi [34–36] for §2.1–§2.2, Pantev, Toën, Vezzosi and Vaquié [31] and Brav, Bussi and Joyce [5] for §2.3, and Spivak [32], Borisov and Noël [7, 8] and Joyce [21–23, 25] for §2.5–§2.7.

2.1 Commutative differential graded algebras

Definition 2.1. Write $\mathsf{cdga}_C$ for the category of commutative differential graded $C$-algebras in nonpositive degrees, and $\mathsf{cdga}_C^{\text{op}}$ for its opposite category. In fact $\mathsf{cdga}_C$ has the additional structure of a model category (a kind of $\infty$-category), but we only use this in the proof of Theorem 3.2 in §4. In the rest of the paper we treat $\mathsf{cdga}_C, \mathsf{cdga}_C^{\text{op}}$ just as ordinary categories.

Objects of $\mathsf{cdga}_C$ are of the form $\cdots \xrightarrow{d} A^{-2} \xrightarrow{d} A^{-1} \xrightarrow{d} A^0$. Here $A^k$ for $k = 0, -1, -2, \ldots$ is the $C$-vector space of degree $k$ elements of $A$, and we have a $C$-bilinear, associative, supercommutative multiplication $\cdot : A^k \times A^l \to A^{k+l}$ for $k, l \leq 0$, an identity $1 \in A^0$, and differentials $d : A^k \to A^{k+1}$ for $k < 0$ satisfying

$$d(a \cdot b) = (da) \cdot b + (-1)^k a \cdot (db)$$

for all $a \in A^k$, $b \in A^l$. We write such objects as $A^\bullet$ or $(A^\bullet, d)$.

Here and throughout we will use the superscript $^\bullet$ to denote graded objects (e.g. graded algebras or vector spaces), where $\bullet$ stands for an index in $\mathbb{Z}$, so that $A^\bullet$ means $(A^k, k \in \mathbb{Z})$. We will use the superscript $^\ast$ to denote differential graded objects (e.g. differential graded algebras or complexes), so that $A^\ast$ means $(A^k, d)$, the graded object $A^\bullet$ together with the differential $d$.

Morphisms $\alpha : A^\bullet \to B^\bullet$ in $\mathsf{cdga}_C$ are $C$-linear maps $\alpha^k : A^k \to B^k$ for all $k \leq 0$ commuting with all the structures on $A^\bullet, B^\bullet$.

A morphism $\alpha : A^\ast \to B^\ast$ is a quasi-isomorphism if $H^k(\alpha) : H^k(A^\ast) \to H^k(B^\ast)$ is an isomorphism on cohomology groups for all $k \leq 0$. A fundamental principle of derived algebraic geometry is that $\mathsf{cdga}_C$ is not really the right category to work in, but instead one wants to define a new category (or better, $\infty$-category) by inverting (localizing) quasi-isomorphisms in $\mathsf{cdga}_C$.

We will call $A^\bullet \in \mathsf{cdga}_C$ of standard form if $A^0$ is a smooth finitely generated $C$-algebra with Spec $A^0$ connected, and the graded $C$-algebra $A^\bullet$ is freely generated over $A^0$ by finitely many generators in each degree $i = -1, -2, \ldots$.

Remark 2.2. Brav, Bussi and Joyce [5, Def. 2.9] work with a stronger notion of standard form cdgas than us, as they require $A^\ast$ to be freely generated over $A^0$ by finitely many generators, all in negative degrees. In contrast, we allow infinitely many generators, but only finitely many in each degree $i = -1, -2, \ldots$.

The important thing for us is that since standard form cdgas in the sense of [5] are also standard form in the (slightly weaker) sense of this paper, we can apply some of their results [5, Th.s 4.1, 4.2, 5.18] on the existence and properties of nice standard form cdga local models for derived schemes.
Definition 2.3. Let $A^\bullet \in \text{cdga}_C$, and write $D(\text{mod } A)$ for the derived category of dg-modules over $A^\bullet$. Define a **derivation of degree $k$** from $A^\bullet$ to an $A^\bullet$-module $M^\bullet$ to be a $C$-linear map $\delta : A^\bullet \to M^\bullet$ that is homogeneous of degree $k$ with

$$\delta(fg) = \delta(f)g + (-1)^{kf}f\delta(g).$$

Just as for ordinary commutative algebras, there is a universal derivation into an $A^\bullet$-module of **Kähler differentials** $\Omega^1_{A^\bullet}$, which can be constructed as $I/I^2$ for $I = \text{Ker}(m : A^\bullet \otimes A^\bullet \to A^\bullet)$. The universal derivation $\delta : A^\bullet \to \Omega^1_{A^\bullet}$ is $\delta(a) = a \otimes 1 - 1 \otimes a \in I/I^2$. One checks that $\delta$ is a universal degree 0 derivation, so that $\delta : \text{Hom}_{A^\bullet}(\Omega^1_{A^\bullet}, M^\bullet) \to \text{Der}^0(A^\bullet, M^\bullet)$ is an isomorphism of dg-modules.

Note that $\Omega^1_{A^\bullet} = (\Omega^1_{A^\bullet})^\circ$, is canonical up to strict isomorphism, not just up to quasi-isomorphism of complexes, or up to equivalence in $D(\text{mod } A)$. Also, the underlying graded vector space $(\Omega^1_{A^\bullet})^\circ$, as a module over the graded algebra $A^\bullet$, depends only on $A^\bullet$ and not on the differential $d$ in $A^\bullet = (A^\bullet, d)$.

Similarly, given a morphism of cdgas $\Phi : A^\bullet \to B^\bullet$, we can define the **relative Kähler differentials** $\Omega^1_{B^\bullet/A^\bullet}$.

The cotangent complex $L_{A^\bullet}$ of $A^\bullet$ is related to the Kähler differentials $\Omega^1_{A^\bullet}$, but is not quite the same. If $\Phi : A^\bullet \to B^\bullet$ is a quasi-isomorphism of cdgas over $C$, then $\Phi^\circ : (\Omega^1_{A^\bullet})^\circ \otimes_{A^\bullet} B^\bullet \to \Omega^1_{B^\bullet}$ may not be a quasi-isomorphism of $B^\bullet$-modules. So Kähler differentials are not well-behaved under localizing quasi-isomorphisms of cdgas, which is bad for doing derived algebraic geometry.

The cotangent complex $L_{A^\bullet}$ is a substitute for $\Omega^1_{A^\bullet}$ which is well-behaved under localizing quasi-isomorphisms. It is an object in $D(\text{mod } A)$, canonical up to equivalence. We can define it by replacing $A^\bullet$ by a quasi-isomorphic, cofibrant (in the sense of model categories) cdga $B^\bullet$, and then setting $L_{A^\bullet} = (\Omega^1_{B^\bullet})^\circ \otimes_{B^\bullet} A^\bullet$. We will be interested in the $p^{th}$ exterior power $L^p_{A^\bullet}$, and the dual $(L_{A^\bullet})^\vee$, which is called the tangent complex, and written $T_{A^\bullet} = (L_{A^\bullet})^\vee$.

There is a de Rham differential $d_{\text{dR}} : L^p_{A^\bullet} \to L^{p+1}_{A^\bullet}$, a morphism of complexes, with $d_{\text{dR}}^2 = 0 : L^p_{A^\bullet} \to L^{p+2}_{A^\bullet}$. Note that each $L^p_{A^\bullet}$ is also a complex with its own internal differential $d : (L^p_{A^\bullet})^k \to (L^p_{A^\bullet})^{k+1}$, and $d_{\text{dR}}$ being a morphism of complexes means that $d \circ d_{\text{dR}} = d_{\text{dR}} \circ d$.

Similarly, given a morphism of cdgas $\Phi : A^\bullet \to B^\bullet$, we can define the relative cotangent complex $L_{B^\bullet/A^\bullet}$.

As in [5 §2.3], an important property of our standard form cdgas $A^\bullet$ in Definition 2.3 is that they are sufficiently cofibrant that the Kähler differentials $\Omega^1_{A^\bullet}$ provide a model for the cotangent complex $L_{A^\bullet}$, so we can take $\Omega^1_{A^\bullet} = L_{A^\bullet}$, without having to replace $A^\bullet$ by an unknown cdga $B^\bullet$. Thus standard form cdgas are convenient for doing explicit computations with cotangent complexes.

A morphism $\Phi : A^\bullet \to B^\bullet$ of standard form cdgas will be called a **submersion** if the corresponding morphism $\Phi^\circ : (\Omega^1_{A^\bullet})^\circ \otimes_{A^\bullet} B^\bullet \to \Omega^1_{B^\bullet}$ is injective in every degree. (By analogy, a smooth map of manifolds $f : X \to Y$ is a submersion if $df : f^*(T^*Y) \to T^*X$ is an injective morphism of vector bundles.)

If $\Phi : A^\bullet \to B^\bullet$ is a submersion of standard form cdgas then the relative Kähler differentials $\Omega^1_{B^\bullet/A^\bullet}$ are a model for the relative cotangent complex.
\[ \Omega^1_{B^*/A^*} \], so we can take \( \Omega^1_{B^*/A^*} = L_{B^*/A^*} \). Thus submersions are a convenient class of morphisms for doing explicit computations with cotangent complexes.

### 2.2 Derived algebraic geometry and derived schemes

**Definition 2.4.** Write \( \mathsf{dSt}_C \) for the \( \infty \)-category of derived \( C \)-stacks (or \( D^- \)-stacks) defined by Toën and Vezzosi \[36\] Def. 2.2.2.14, \[34\] Def. 4.2. Objects \( X \) in \( \mathsf{dSt}_C \) are \( \infty \)-functors

\[ X : \{ \text{simplicial commutative } C\text{-algebras} \} \to \{ \text{simplicial sets} \} \]

satisfying sheaf-type conditions. There is a \emph{spectrum functor}

\[ \text{Spec} : \text{cdga}_{C}^{\op} \to \mathsf{dSt}_C. \]

A derived \( C \)-stack \( X \) is called an \emph{affine derived }\( C \)-scheme if \( X \) is equivalent in \( \mathsf{dSt}_C \) to \( \text{Spec } A^* \) for some cdga \( A^* \) over \( C \). As in \[34\] §4.2, a derived \( C \)-stack \( X \) is called a \emph{derived }\( C \)-scheme if it may be covered by Zariski open \( Y \subseteq X \) with \( Y \) an affine derived \( C \)-scheme. Write \( \mathsf{dSch}_C \) for the full \( \infty \)-subcategory of derived \( C \)-schemes in \( \mathsf{dSt}_C \), and \( \mathsf{dSch}_C^{\text{aff}} \subset \mathsf{dSch}_C \) for the full \( \infty \)-subcategory of affine derived \( C \)-schemes.

We shall assume throughout this paper that all derived \( C \)-schemes \( X \) are \emph{locally finitely presented} in the sense of Toën and Vezzosi \[36\] Def. 1.3.6.4.

There is a \emph{classical truncation functor} \( t_0 : \mathsf{dSch}_C \to \mathsf{Sch}_C \) taking a derived \( C \)-scheme \( X \) to the underlying classical \( C \)-scheme \( X = t_0(X) \). On affine derived schemes \( \mathsf{dSch}_C^{\text{aff}} \) this maps \( t_0 : \text{Spec } A^* \to \text{Spec } H^0(A^*) = \text{Spec}(A^0/d(A^{-1})) \).

Toën and Vezzosi show that a derived \( C \)-scheme \( X \) has a \emph{cotangent complex} \( L_X \) \[36\] §1.4, \[34\] §4.2.4–§4.2.5 in a stable \( \infty \)-category \( L_{qcoh}(X) \) defined in \[34\] §3.1.7, §4.2.4. We will be interested in the \( p \)-th exterior power \( \Lambda^p L_X \), and the dual \( (L_X)^\vee \), which is called the \emph{tangent complex} \( \mathbb{T}_X \). There is a \emph{de Rham differential} \( d_{\text{dr}} : \Lambda^p L_X \to \Lambda^{p+1} L_X \).

Restricted to the classical scheme \( X = t_0(X) \), the cotangent complex \( L_X |_X \) may Zariski locally be modelled as a finite complex of vector bundles \( [F^{-m} \to F^{-m-1} \to \cdots \to F^0] \) on \( X \) in degrees \([-m, 0]\) for some \( m \geq 0 \). The \emph{(complex) virtual dimension} \( \text{vdim}_C X \) is \( \text{vdim}_C X = \sum_{i=0}^m (-1)^i \text{rank } F^{-i} \). It is a locally constant function \( \text{vdim}_C X : X \to \mathbb{Z} \), so is constant on each connected component of \( X \). We say that \( X \) has \emph{(complex) virtual dimension} \( n \in \mathbb{Z} \) if \( \text{vdim}_C X = n \).

When \( X = X \) is a classical scheme, the homotopy category of \( L_{qcoh}(X) \) is the triangulated category \( D_{qcoh}(X) \) of complexes of quasicoherent sheaves. These have the usual properties of (co)tangent complexes. For instance, if \( f : X \to Y \) is a morphism in \( \mathsf{dSch}_C \) there is a distinguished triangle

\[ f^*(L_Y) \xrightarrow{L_{\mathbb{L}f}} L_X \xrightarrow{L_{L_X/Y}} f^*(L_Y)[1], \]

where \( L_{L_X/Y} \) is the \emph{relative cotangent complex} of \( f \).

Now suppose \( A^* \) is a cdga over \( C \), and \( X \) a derived \( C \)-scheme with \( X \simeq \text{Spec } A \) in \( \mathsf{dSch}_C \). Then we have an equivalence of triangulated categories \( L_{qcoh}(X) \simeq D(\mod A) \), which identifies cotangent complexes \( L_X \simeq L_{A^*} \). If also \( A^* \) is of standard form then \( L_{A^*} \simeq \Omega^1_{A^*} \), so \( L_X \simeq \Omega^1_{A^*} \).
finite presentation), and $x \in X$. Then there exists a standard form cdga $A^\bullet$ over $\mathbb{C}$ and a Zariski open inclusion $\alpha : \text{Spec} A^\bullet \to X$ with $x \in \text{Im} \alpha$.

See Remark 2.2 on the difference in definitions of ‘standard form’. They also explain [5, Th. 4.2] how to compare two such standard form charts $\text{Spec} A^\bullet \hookrightarrow X$, $\text{Spec} B^\bullet \hookrightarrow X$ on their overlap in $X$, using a third chart.

### 2.3 PTVV’s shifted symplectic geometry

Next we summarize parts of the theory of shifted symplectic geometry, as developed by Pantev, Toën, Vaquié, and Vezzosi in [31]. We explain them for derived $\mathbb{C}$-schemes $X$, although Pantev et al. work more generally with derived stacks.

Given a (locally finitely presented) derived $\mathbb{C}$-scheme $X$ and $p \geq 0$, $k \in \mathbb{Z}$, Pantev et al. [31, §1.2] define complexes of $k$-shifted p-forms $\mathcal{A}_C^k(X, k)$ and $k$-shifted closed p-forms $\mathcal{A}_C^{k,\mathrm{cl}}(X, k)$. These are defined first for affine derived $\mathbb{C}$-schemes $Y = \text{Spec} A^\bullet$ for $A^\bullet$ a cdga over $\mathbb{C}$. Then for general $X$, $k$-shifted (closed) p-forms are defined by smooth descent techniques; basically, a $k$-shifted (closed) p-form $\omega$ on $X$ is the functorial choice for all $Y$, $f$ of a $k$-shifted (closed) p-form $f^*(\omega)$ on $Y$ whenever $Y = \text{Spec} A^\bullet$ is affine and $f : Y \to X$ is smooth.

**Definition 2.6.** Let $Y \simeq \text{Spec} A^\bullet$ be an affine derived $\mathbb{C}$-scheme, for $A^\bullet$ a cdga over $\mathbb{C}$. A $k$-shifted p-form on $Y$ for $k \in \mathbb{Z}$ is an element $\omega_A^\bullet \in (\wedge^p \Lambda^L A^\bullet)^k$ with $d \omega_A^\bullet = 0$ in $(\wedge^p \Lambda^L A^\bullet)^{k+1}$, so that $\omega_A^\bullet$ defines a cohomology class $[\omega_A^\bullet] \in H^k(\Lambda^p \Lambda^L A^\bullet)$. When $p = 0$, we call $\omega_A^\bullet$ nondegenerate, or a $k$-shifted presymplectic form, if the induced morphism $\omega_A^{\bullet,\bullet} : \Lambda^{\bullet,\bullet} \to \Lambda^{\bullet,\bullet}[k]$ is a quasi-isomorphism.

A $k$-shifted closed $p$-form on $Y$ is a sequence $\omega_A^{\bullet,\bullet} = (\omega_A^0, \omega_A^1, \omega_A^2, \ldots)$ such that $\omega_A^m \in (\wedge^{p+m} \Lambda^L A^\bullet)^{k-m}$ for $m \geq 0$, with $d \omega_A^m = 0$ and $d \omega_A^{1+p+m} + d \omega_A^m = 0$ in $(\wedge^{p+m+1} \Lambda^L A^\bullet)^{k-m}$ for all $m \geq 0$. Note that if $\omega_A^{\bullet,\bullet} = (\omega_A^0, \omega_A^1, \omega_A^2, \ldots)$ is a $k$-shifted closed $p$-form then $\omega_A^0$ is a $k$-shifted $p$-form.

When $p = 2$, we call a $k$-shifted closed 2-form $\omega_A^{\bullet,\bullet}$ a $k$-shifted symplectic form if the associated 2-form $\omega_A^2$ is nondegenerate (presymplectic).

If $X$ is a general derived $\mathbb{C}$-scheme, then Pantev et al. [31, §1.2] define $k$-shifted 2-forms $\omega_X$, which may be nondegenerate (presymplectic), and $k$-shifted closed 2-forms $\omega_X^{\bullet,\bullet}$, which have an associated $k$-shifted 2-form $\omega_X^0$, and where $\omega_X^{\bullet,\bullet}$ is called a $k$-shifted symplectic form if $\omega_X^0$ is nondegenerate (presymplectic).

We will not go into the details of this definition for general $X$.

The important thing for us is this: if $Y \subseteq X$ is a Zariski open affine derived $\mathbb{C}$-subscheme with $Y \simeq \text{Spec} A^\bullet$ then a $k$-shifted 2-form $\omega_X$ (or a $k$-shifted closed 2-form $\omega_X^{\bullet,\bullet}$) on $X$ induces a $k$-shifted 2-form $\omega_A^\bullet$ (or a $k$-shifted closed 2-form $\omega_A^{\bullet,\bullet}$) on $Y$ in the sense above, where $\omega_A^\bullet$ is unique up to cohomology in the complex $(\Lambda^2 \Lambda^L A^\bullet)^\bullet$, $d$ (or $\omega_A^{\bullet,\bullet}$ is unique up to cohomology in the complex $(\prod_{m \geq 0} (\Lambda^{2+m} \Lambda^L A^\bullet)^{\bullet-m}, d + d \text{dR})$, and $\omega_X$ nondegenerate/presymplectic (or $\omega_X^{\bullet,\bullet}$ symplectic) implies $\omega_A^\bullet$ nondegenerate/presymplectic (or $\omega_A^{\bullet,\bullet}$ symplectic).
It is easy to show that if $X$ is a derived $\mathbb{C}$-scheme with a $k$-shifted symplectic or presymplectic form, then $k \leq 0$, and the complex virtual dimension $\text{vdim}_C X$ satisfies $\text{vdim}_C X = 0$ if $k$ is odd, and $\text{vdim}_C X$ is even if $k \equiv 0 \pmod{4}$ (which includes classical complex symplectic schemes when $k = 0$), and $\text{vdim}_C X \in \mathbb{Z}$ if $k \equiv 2 \pmod{4}$. In particular, in the case $k = -2$ of interest in this paper, $\text{vdim}_C X$ can take any value in $\mathbb{Z}$.

The main examples we have in mind come from Pan et al. [31, §2.1]:

**Theorem 2.7.** Suppose $Y$ is a Calabi–Yau $m$-fold over $\mathbb{C}$, and $\mathcal{M}$ a derived moduli stack of coherent sheaves (or complexes of coherent sheaves) on $Y$. Then $\mathcal{M}$ has a natural $(2 - m)$-shifted symplectic form $\omega_\mathcal{M}$.

In particular, derived moduli schemes and stacks on a Calabi–Yau 4-fold $Y$ are $-2$-shifted symplectic.

Bussi, Brav and Joyce [5] prove ‘Darboux Theorems’ for $k$-shifted symplectic derived $\mathbb{C}$-schemes $(X, \omega_X)$ for $k < 0$, which give explicit Zariski local models for $(X, \omega_X)$. We will explain their main result for $k = -2$. The next definition is taken from [5, Ex. 5.16] (with notation changed, $2q_js_j$ in place of $s_j$).

**Definition 2.8.** A pair $(A^*, \omega_{A^*})$ is called in $-2$-Darboux form if $A^*$ is a standard form cdga over $\mathbb{C}$, and $\omega_{A^*} \in (\Lambda^2 L A^*)^{-2} = (\Lambda^2 \Omega^1_{A^*})^{-2}$ with $d \omega_{A^*} = 0$ in $(\Lambda^2 L A^*)^{-1}$ and $d_{\text{dR}} \omega_{A^*} = 0$ in $(\Lambda^3 L A^*)^{-2}$, so that $\omega_{A^*} := (\omega_{A^*}, 0, 0, \ldots)$ is a $-2$-shifted closed 2-form on $A^*$ (which is symplectic by (iii)), such that:

(i) $A^0$ is a smooth $\mathbb{C}$-algebra of dimension $m^0$, and there exist $x_1^0, \ldots, x_{m^0}^0$ in $A^0$ forming an étale coordinate system on $V = \text{Spec} A^0$.

(ii) The commutative graded algebra $A^*$ is freely generated over $A^0$ by elements $y_1, \ldots, y_{m^1}$ of degree $-1$ and $z_1, \ldots, z_{m^2}$ of degree $-2$.

(iii) There are invertible functions $q_1, \ldots, q_{m^1}$ in $A^0$ such that

\[
\omega_{A^*} = d_{\text{dR}} z_1^{-2} d_{\text{dR}} x_1^0 + \cdots + d_{\text{dR}} z_{m^2}^{-2} d_{\text{dR}} x_{m^0}^0 + d_{\text{dR}} (q_1 y_1^1) d_{\text{dR}} y_1^{-1} + \cdots + d_{\text{dR}} (q_{m^1} y_{m^1}^{-1}) d_{\text{dR}} y_{m^1}^{-1}.
\]

(iv) There are functions $s_1, \ldots, s_{m^1} \in A^0$ satisfying

\[
q_1 (s_1)^2 + \cdots + q_{m^1} (s_{m^1})^2 = 0 \quad \text{in } A^0,
\]

such that the differential $d$ on $A^* = (A^*, d)$ is given by

\[
dx_i^0 = 0, \quad dy_j^{-1} = s_j, \quad dz_i^{-2} = \sum_{j=1}^{m^1} y_j^{-1} \left( 2q_j \frac{\partial s_j}{\partial x_i^0} + s_j \frac{\partial q_j}{\partial z_i^{-2}} \right),
\]

and $d \circ dz_i^{-2} = 0$ follows from applying $\frac{\partial}{\partial z_i^{-2}}$ to (2.2).

The main result of Bussi, Brav and Joyce [5, Th. 5.18] when $k = -2$ yields:
Theorem 2.9. Suppose \((X, \omega_X^\bullet)\) is a \(-2\)-shifted symplectic derived \(\mathbb{C}\)-scheme. Then for each \(x \in X = t_0(X)\) there exists a pair \((A^\bullet, \omega_A^\bullet)\) in \(-2\)-Darboux form and a Zariski open inclusion \(\alpha : \text{Spec} A^\bullet \hookrightarrow X\) such that \(x \in \text{Im} \alpha\) and \(\alpha^*(\omega_X^\bullet) \cong \omega_A^\bullet\) in \(\mathcal{A}^{2,\text{cl}}_k(\text{Spec} A^\bullet, -2)\). Furthermore, we can choose \(A^\bullet\) minimal at \(x\), in the sense that \(m^t = \dim H^i(T_X|_x)\) in Definition 2.8 for \(i = 0, 1\).

### 2.4 Orientations on \(k\)-shifted symplectic derived schemes

Let \(Y\) be a classical \(\mathbb{C}\)-scheme. A complex \(E^\bullet\) in the derived category of coherent sheaves \(D^b\text{coh}(Y)\) is called perfect if Zariski locally on \(Y\), \(E^\bullet\) is quasi-isomorphic to a complex of vector bundles \(F^\bullet = [0 \to F^a \to F^{a+1} \to \cdots \to F^b \to 0]\) in a bounded range of degrees \(a, a+1, \ldots, b\) in \(\mathbb{Z}\). Then \(E^\bullet\) has a determinant line bundle \(\det(E^\bullet)\), a line bundle on \(Y\), natural up to canonical isomorphism, where if \(E^\bullet \cong F^\bullet\) as above then \(\det(E^\bullet) \cong \bigotimes_{i=0}^b (L^\text{top} F^i)^{(-1)^i}\).

If \(X\) is a derived \(\mathbb{C}\)-scheme (always assumed locally finitely presented), with classical \(\mathbb{C}\)-scheme \(X = t_0(X)\), the cotangent complex \(L_X|_X\) restricted to \(X\) is a perfect complex, so \(\det(L_X|_X)\) is a line bundle on \(X\).

The following notion is important for \(-1\)-shifted symplectic derived schemes, 3-Calabi–Yau moduli spaces, and generalizations of Donaldson–Thomas theory:

**Definition 2.10.** Let \((X, \omega_X^\bullet)\) be a \(-1\)-shifted symplectic derived \(\mathbb{C}\)-scheme (or more generally \(k\)-shifted symplectic, for \(k < 0\) odd). An orientation for \((X, \omega_X^\bullet)\) is a choice of square root line bundle \(\det(L_X|_X)^{1/2}\) for \(\det(L_X|_X)\).

Writing \(X_{\text{an}}\) for the complex analytic topological space of \(X\), the obstruction to existence of orientations for \((X, \omega_X^\bullet)\) lies in \(H^2(X_{\text{an}}; \mathbb{Z}_2)\), and if the obstruction vanishes, the set of orientations is a torsor for \(H^1(X_{\text{an}}; \mathbb{Z}_2)\).

This notion of orientation, and its analogue for ‘d-critical loci’, are used by Ben-Bassat, Brav, Dupont, Joyce, Meinhardt, and Szczech in a series of papers [3][6][24]. They use orientations on \((X, \omega_X^\bullet)\) to define natural perverse sheaves, D-modules, mixed Hodge modules, and motives on \(X\). A very similar idea appeared first in Kontsevich and Soibelman [27] §5 as ‘orientation data’ needed to define motivic Donaldson–Thomas invariants of Calabi–Yau 3-folds.

This paper concerns \(-2\)-shifted symplectic derived schemes, and 4-Calabi–Yau moduli spaces. It turns out that there is a parallel notion of orientation in the \(-2\)-shifted case, needed to construct virtual cycles.

To define this, note that determinant line bundles \(\det(E^\bullet)\) have the properties that \(\det([E^\bullet]^\vee) \cong [\det(E^\bullet)]^{-1}\), and \(\det(E^\bullet[k]) \cong [\det(E^\bullet)]^{(-1)^k}\). If \((X, \omega_X^\bullet)\) is a \(k\)-shifted symplectic derived \(\mathbb{C}\)-scheme, then \(T_X \cong L_X[k]\), where \(T_X \cong (L_X)^\vee\). Restricting to \(X\) and taking determinant line bundles gives \(\det(L_X|_X)^{-1} \cong \det(L_X|_X)^{(-1)^k}\). If \(k\) is odd this is trivial, but for \(k\) even, this gives a canonical isomorphism of line bundles on \(X\):

\[
\iota_X \omega_X^\bullet : [\det(L_X|_X)]^\otimes \to \mathcal{O}_X \cong \mathcal{O}_X^{\otimes 2}.
\]

The next definition is new, so far as the authors know.

**Theorem 2.9.** Suppose \((X, \omega_X^\bullet)\) is a \(-2\)-shifted symplectic derived \(\mathbb{C}\)-scheme. Then for each \(x \in X = t_0(X)\) there exists a pair \((A^\bullet, \omega_A^\bullet)\) in \(-2\)-Darboux form and a Zariski open inclusion \(\alpha : \text{Spec} A^\bullet \hookrightarrow X\) such that \(x \in \text{Im} \alpha\) and \(\alpha^*(\omega_X^\bullet) \cong \omega_A^\bullet\) in \(\mathcal{A}^{2,\text{cl}}_k(\text{Spec} A^\bullet, -2)\). Furthermore, we can choose \(A^\bullet\) minimal at \(x\), in the sense that \(m^t = \dim H^i(T_X|_x)\) in Definition 2.8 for \(i = 0, 1\).
Definition 2.11. Let \((X, \omega^X_X)\) be a \(-2\)-shifted symplectic derived \(\mathbb{C}\)-scheme (or more generally \(k\)-shifted symplectic, for \(k < 0\) with \(k \equiv 2 \mod 4\)). An orientation for \((X, \omega^X_X)\) is a choice of isomorphism \(o : \det(L_X|_{\mathbb{C}}) \to O_X\) such that \(o \otimes o = \iota_{X, \omega^X_X}\) for \(\iota_{X, \omega^X_X}\) as in (2.3).

Writing \(X_{an}\) for the complex analytic topological space of \(X\), the obstruction to existence of orientations for \((X, \omega^X_X)\) lies in \(H^1(X_{an}; \mathbb{Z}_2)\), and if the obstruction vanishes, the set of orientations is a torsor for \(H^0(X_{an}; \mathbb{Z}_2)\).

This definition makes sense for \(k\)-shifted symplectic derived \(\mathbb{C}\)-schemes with \(k\) even, but when \(k \equiv 0 \mod 4\) (including the classical symplectic case \(k = 0\)) there is a natural choice of orientation \(o\), so we restrict to \(k \equiv 2 \mod 4\).

So for \(k\) odd, orientations are square roots of objects (line bundles), but for \(k\) even, orientations are square roots of morphisms (isomorphisms \(\iota_{X, \omega^X_X}\)).

We will find it helpful to rewrite orientations on \((X, \omega^X_X)\) in terms of a principal \(\mathbb{Z}_2\)-bundle \(O_{X, \omega^X_X}\) on \(X_{an}\).

Definition 2.12. Let \((X, \omega^X_X)\) be a \(-2\)-shifted symplectic derived \(\mathbb{C}\)-scheme, and write \(X_{an}\) for the set of \(\mathbb{C}\)-points of \(X = t_0(X)\) with the complex analytic topology. For \(\iota_{X, \omega^X_X}\) as in (2.3), define the orientation bundle

\[
O_{X, \omega^X_X} := \{ (x, o_x) : x \in X_{an}, \ o_x : \det(L_X|_{x}) \xrightarrow{\otimes} \mathbb{C}, \ o_x \otimes o_x = \iota_{X, \omega^X_X}|_{x} \}, \tag{2.5}
\]

and define a projection \(\pi : O_{X, \omega^X_X} \to X_{an}\) by \(\pi : (x, o_x) \mapsto x\), and a free action of \(\mathbb{Z}_2 = \{1, -1\}\) on \(O_{X, \omega^X_X}\) by \(-1 : (x, o_x) \mapsto (x, -o_x)\).

Then \(O_{X, \omega^X_X}\) has a natural topology making \(\pi : O_{X, \omega^X_X} \to X_{an}\) into a topological principal \(\mathbb{Z}_2\)-bundle. Orientations of \((X, \omega^X_X)\) correspond naturally to continuous sections of \(O_{X, \omega^X_X}\). More generally, if \(U \subseteq X\) is Zariski open, then orientations of \((U, \omega^X_X|_{U})\) correspond to continuous sections of \(O_{X, \omega^X_X}|_{U_{an}}\).

For \(x \in X_{an}\), we have a canonical isomorphism

\[
\det(L_X|_{x}) \cong \Lambda^{top} H^0(L_X|_{x}) \otimes [\Lambda^{top} H^{-1}(L_X|_{x})]^* \otimes \Lambda^{top} H^{-2}(L_X|_{x}).
\]

Now \(H^{-1}(L_X|_{x}) \cong H^{1}(T_X|_{x})^*, \) and \(\omega^X_X|_{x}\) gives \(H^0(L_X|_{x}) \cong H^{-2}(L_X|_{x})^*, \) so \(\Lambda^{top} H^0(L_X|_{x}) \cong [\Lambda^{top} H^{-2}(L_X|_{x})]^*\). Thus we have a canonical isomorphism

\[
\det(L_X|_{x}) \cong \Lambda^{top} H^1(T_X|_{x}). \tag{2.6}
\]

Write \(Q_x\) for the nondegenerate \(\mathbb{C}\)-bilinear pairing

\[
Q_x := \omega^0_X|_{x} : H^1(T_X|_{x}) \times H^1(T_X|_{x}) \to \mathbb{C}. \tag{2.7}
\]

Considering degrees shows that \(\Lambda^{top}\) is symmetric, that is, \(Q_x\) is a nondegenerate complex quadratic form \(Q_x\) on \(H^1(T_X|_{x})\). The determinant \(\det Q_x\) is then an isomorphism \(\Lambda^{top} H^1(T_X|_{x})|^{\otimes 2} \to \mathbb{C}\), and one can show that \(\det Q_x\) corresponds to \(\iota_{X, \omega^X_X}|_{x}\) under the isomorphism (2.6).

From this it follows that there is a canonical identification

\[
O_{X, \omega^X_X}|_{x} \cong \{ \text{orientations of } (H^1(T_X|_{x}), Q_x) \} \tag{2.8}
\]
where orientations of \((H^1(T_{X|_x}), Q_x)\) are equivalence classes of bases \((v_1, \ldots, v_k)\) of \(H^1(T_{X|_x})\) orthonormal w.r.t. \(Q_x\), and two bases \((v_1, \ldots, v_k), (v'_1, \ldots, v'_k)\) are equivalent if they differ by a matrix in \(\text{SO}(k, \mathbb{C}) \subset O(k, \mathbb{C})\).

Note that as \((H^1(T_{X|_x}), Q_x)\) does not vary continuously with \(x \in X_{an}\) (in general \(x \mapsto \dim H^1(T_{X|_x})\) is only upper semicontinuous), the topology on \(O_X, \omega^*_X\) is not obvious from the description (2.8).

2.5 Derived smooth manifolds

Derived algebraic geometry \([28, 34–36]\) is the study of ‘derived’ schemes and stacks. There is also a less well-known theory of derived differential geometry, which is the study of ‘derived’ smooth manifolds and ‘derived’ smooth orbifolds. Here are some of the main milestones in this theory so far:

(i) The earliest reference to derived differential geometry we are aware of (though see (v) below) is a short final paragraph by Jacob Lurie \([28, \S 4.5]\), where he explains how to use his machinery to define an \(\infty\)-category of objects one could call ‘derived \(C^\infty\)-stacks’, and notes that ‘derived smooth manifolds’ will appear as an \(\infty\)-subcategory of these.

(ii) Broadly following \([28, \S 4.5]\), Lurie’s student David Spivak \([32]\) constructed an \(\infty\)-category \(\text{DerMan}_{\text{Spi}}\) of ‘derived manifolds’. He also defined ‘derived manifolds with boundary’, and proved that bordism groups of derived manifolds are isomorphic to bordism groups of ordinary manifolds.

Spivak’s constructions are complicated and difficult to work with.

(iii) Borisov and Noël \([7]\) defined an \(\infty\)-category \(\text{DerMan}_{\text{BoNo}}\) of derived manifolds, quite a lot simpler than Spivak’s, in which the objects are special simplicial \(C^\infty\)-rings, and showed that \(\text{DerMan}_{\text{Spi}} \simeq \text{DerMan}_{\text{BoNo}}\).

(iv) Joyce \([21–23]\) defined strict 2-categories \(\text{dMan}\) of ‘\(d\)-manifolds’ (a kind of derived manifold), and \(\text{dOrb}\) of ‘\(d\)-orbifolds’ (a kind of derived orbifold), and also strict 2-categories of \(d\)-manifolds and \(d\)-orbifolds with boundary \(\text{dMan}^b, \text{dOrb}^b\) and with corners \(\text{dMan}^c, \text{dOrb}^c\), and developed their differential geometry in a lot of detail.

Borisov \([8]\) studied the relation between \(\text{DerMan}_{\text{Spi}}, \text{DerMan}_{\text{BoNo}}\) and \(\text{dMan}\). He constructed a 2-functor \(F : \pi_2(\text{DerMan}_{\text{BoNo}}) \rightarrow \text{dMan}\), where \(\pi_2(\text{DerMan}_{\text{BoNo}})\) is the 2-category truncation of \(\text{DerMan}_{\text{BoNo}}\), and proved that \(F\) is close to being an equivalence of 2-categories; in particular, \(F\) induces a 1-1 correspondence on equivalence classes of objects in \(\pi_2(\text{DerMan}_{\text{BoNo}})\) and \(\text{dMan}\).

(v) Joyce \([25]\) defined an ordinary category \(\text{MKur}\) of ‘\(M\)-Kuranishi spaces’ (yet another kind of derived manifold), and a weak 2-category \(\text{Kur}\) of ‘Kuranishi spaces’ (another kind of derived orbifold), with an equivalence \(\text{MKur} \simeq \text{Ho}(\text{Kur}_{\text{trG}})\), where \(\text{Kur}_{\text{trG}} \subset \text{Kur}\) is the full 2-subcategory
of Kuranishi spaces with trivial orbifold groups, and \( \text{Ho(Kur} \text{trG}) \) its homotopy category. He also defined (2-)categories of (M-)Kuranishi spaces with boundary \( \text{MKur}^b, \text{Kur}^b \) and with corners \( \text{MKur}^c, \text{Kur}^c \).

Amorim and Joyce \[1\] study the differential geometry of (M-)Kuranishi spaces, and prove equivalences \( \text{MKur} \simeq \text{Ho(dMan)}, \text{Kur} \simeq \text{dOrb} \).

(v) In fact Kuranishi spaces (though with a different definition) have been used since before the invention of derived algebraic geometry, in the work of Fukaya, Oh, Ohta and Ono \[16,18\], as the geometric structure on moduli spaces of \( J \)-holomorphic curves in symplectic geometry.

There were some problems with the theory, stemming from the lack of ‘derived’ ideas. The contribution of \[25\] was to repair the definition of Kuranishi space, and make (M-)Kuranishi spaces into a well-behaved (2-)category (there are no morphisms of Kuranishi spaces in \[16,18\]).

There is a difference in character between (0)–(iii), based on derived algebraic geometry of \( \mathbb{C}^\infty \)-rings, and (iv)–(v), which use an ‘atlas of charts’ approach.

One goal of this paper to show that if \((X,\omega_X)\) is a \(-2\)-shifted symplectic derived \(\mathbb{C}\)-scheme (with some extra conditions), then the underlying topological space \(X\) can be given the structure of a derived manifold \(X_{\text{dm}}\), which is canonical up to bordisms fixing the underlying topological space.

For this purpose, we could work with derived manifolds in any of the five (higher) categories \(\text{DerMan}_{\text{Spi}}, \text{DerMan}_{\text{BoNo}}, \text{dMan}, \text{MKur}, \text{Kur}_{\text{trG}}\) above, as the results we quoted show that there are canonical 1-1 correspondences between equivalence/isomorphism classes of objects in each category.

However, the theory of d-manifolds (and to a lesser extent (M-)Kuranishi spaces) has been developed much further than for \(\text{DerMan}_{\text{Spi}}, \text{DerMan}_{\text{BoNo}}\). We will need a theorem telling us that we can build a derived manifold out of a certain collection of differential-geometric data, and this has been written down in the form we want only for d-manifolds \[21\] Th. 4.16, \[22\] Th. 4.17, \[23\] §3.6, and for (M-)Kuranishi spaces \[25\] Th. 4.66.

We now explain a little about how to define (iv) (M-)Kuranishi spaces in \[24\], (ii) derived manifolds in \[7,8\], and (iii) d-manifolds in \[21,23\]. The next two definitions are adapted from \[25\], and we discuss them in Remark 2.15.

**Definition 2.13.** Let \(X\) be a topological space. A **Kuranishi neighbourhood** on \(X\) is a quadruple \((V, E, s, \psi)\) such that:

(a) \(V\) is a smooth manifold.

(b) \(\pi : E \to V\) is a real vector bundle over \(V\), called the **obstruction bundle**.

(c) \(s : V \to E\) is a smooth section of \(E\), called the **Kuranishi section**.

(d) \(\psi\) is a homeomorphism from \(s^{-1}(0)\) to an open subset \(R = \text{Im} \psi\) in \(X\), where \(\text{Im} \psi = \{\psi(x) : x \in s^{-1}(0)\}\) is the image of \(\psi\).

If \(S \subseteq X\) is open, by a **Kuranishi neighbourhood over \(S\)**, we mean a Kuranishi neighbourhood \((V, E, s, \psi)\) on \(X\) with \(S \subseteq \text{Im} \psi \subseteq X\).
**Definition 2.14.** Let \((V_J, E_J, s_J, \psi_J), (V_K, E_K, s_K, \psi_K)\) be Kuranishi neighbourhoods on a topological space \(X\), and \(S \subseteq \text{Im} \psi_J \cap \text{Im} \psi_K \subseteq X\) be open. A coordinate change \((V_{JK}, \phi_{JK}, \hat{\phi}_{JK}) : (V_J, E_J, s_J, \psi_J) \to (V_K, E_K, s_K, \psi_K)\) over \(S\) is a triple \((V_{JK}, \phi_{JK}, \hat{\phi}_{JK})\) satisfying:

(a) \(V_{JK}\) is an open neighbourhood of \(\psi_J^{-1}(S)\) in \(V_J\).

(b) \(\phi_{JK} : V_{JK} \to V_K\) is a smooth map.

(c) \(\hat{\phi}_{JK} : E_J|_{V_{JK}} \to \phi_J^*(E_K)\) is a morphism of vector bundles on \(V_{JK}\).

(d) \(\hat{\phi}_{JK}(s_J|_{V_{JK}}) = \phi_J^*(s_K)\).

(e) \(\psi_J = \psi_K \circ \phi_{JK}\) on \(s_J^{-1}(0) \cap V_{JK}\).

(f) Let \(x \in S\), and set \(v_J = \psi_J^{-1}(x) \in V_J\) and \(v_K = \psi_K^{-1}(x) \in V_K\). Then the following is an exact sequence of real vector spaces:

\[
\begin{align*}
0 &\longrightarrow T_{v_J}V_J \xrightarrow{ds_J|_{v_J} \oplus d\phi_{JK}|_{v_J}} E_J|_{v_J} \oplus T_{v_K}V_K \xrightarrow{-\hat{\phi}_{JK}|_{v_J} \oplus ds_K|_{v_K}} E_K|_{v_K} \longrightarrow 0.
\end{align*}
\]  

(2.9)

There is a good notion of composition of coordinate changes: if \((V_{JK}, \phi_{JK}, \hat{\phi}_{JK}) : (V_J, E_J, s_J, \psi_J) \to (V_K, E_K, s_K, \psi_K)\) and \((V_{KL}, \phi_{KL}, \hat{\phi}_{KL}) : (V_K, E_K, s_K, \psi_K) \to (V_L, E_L, s_L, \psi_L)\) are coordinate changes over \(S_{JK}, S_{KL}\), then

\[
(V_{JK} \cap \phi_J^{-1}(V_{KL}), \phi_K \circ \phi_{JK} |_{\phi_J^{-1}(V_{KL})} \circ \hat{\phi}_{KL} \circ \hat{\phi}_{JK} |_{\phi_J^{-1}(V_{KL})} : (V_J, E_J, s_J, \psi_J) \longrightarrow (V_L, E_L, s_L, \psi_L)
\]

is a coordinate change over \(S_{JK} \cap S_{KL}\).

**Remark 2.15.** (a) Definition 2.14 coincides with ‘M-Kuranishi neighbourhoods’ in [25](§2.1). ‘Kuranishi neighbourhoods’ \((V, E, \Gamma, s, \psi)\) in [25](§4.1) are an orbifold version of this with a finite group \(\Gamma\) acting on \(V, E\), but we do not need this, so set \(\Gamma = \{1\}\).

By [25](Ths 2.13 & 4.14), Definition 2.14 is essentially equivalent to ‘M-coordinate changes’ of M-Kuranishi neighbourhoods in [25](§2.1), and to ‘coordinate changes’ of Kuranishi neighbourhoods in [25](§4.1) with finite groups \(\Gamma = \{1\}\). There are small differences: in [25] we allow a more general version of Definition 2.14(d), and M-coordinate changes in [25] are equivalence classes \([V_{JK}, \phi_{JK}, \hat{\phi}_{JK}]\) of the triples \((V_{JK}, \phi_{JK}, \hat{\phi}_{JK})\) in Definition 2.14.

(b) An \(M\)-Kuranishi space \(X\) in [25](§2.3) is a Hausdorff, second countable topological space \(X\) with a family \(\{V_J, E_J, s_J, \psi_J : J \in A\}\) of Kuranishi neighbourhoods with \(X = \bigcup_{J \in A} \text{Im} \psi_J\), and a family \(\{[V_{JK}, \phi_{JK}, \hat{\phi}_{JK}] : J, K \in A\}\) of M-coordinate changes \([V_{JK}, \phi_{JK}, \hat{\phi}_{JK}] : (V_{JK}, E_{JK}, s_{JK}, \psi_{JK}) \to (V_K, E_K, s_K, \psi_K)\) over \(\text{Im} \psi_J \cap \text{Im} \psi_K\) satisfying conditions over triple overlaps \(\text{Im} \psi_J \cap \text{Im} \psi_K \cap \text{Im} \psi_L\).

A Kuranishi space \(X\) in [25](§4.3) is similar, but including finite groups \(\Gamma\), and with extra ‘2-morphism’ data on triple overlaps \(\text{Im} \psi_J \cap \text{Im} \psi_K \cap \text{Im} \psi_L\) relating the coordinate changes \((V_{JK}, \phi_{JK}, \hat{\phi}_{JK}), (V_{KL}, \phi_{KL}, \hat{\phi}_{KL}), (V_{JL}, \phi_{JL}, \hat{\phi}_{JL})\).

(c) All this is heavily based on the work of Fukaya, Oh, Ohta and Ono [16][18], and also draws on McDuff and Wehrheim [29][30], and other authors. Our
Kuranishi neighbourhoods above are essentially the same as those in [16][18][29][30], omitting finite groups, and our coordinate changes above are a common generalization of the different notions used in [16][18] and [29][30].

(d) In fact Kuranishi neighbourhoods \((V,E,s,\psi)\) are the local models for all the forms of derived manifolds discussed in (i)–(iv) above, as we will see in Definitions 2.16 and 2.17. Coordinate changes \((V_J,K,\phi_J,k)\) also induce local equivalences of the derived manifolds associated to \((V_J,E_J,s_J,\psi_J)\) and \((V_K,E_K,s_K,\psi_K)\) in all of (i)–(iv).

**Definition 2.16.** Let \((V,E,s,\psi)\) be a Kuranishi neighbourhood on a topological space \(X\). Define a simplicial \(C^\infty\)-ring \(\mathcal{C}_{V,E,s}\) by

\[ \mathcal{C}_{V,E,s} = \{ C^\infty(V)/m^9, C^\infty(E)/\pi^*(m^9), C^\infty(E^\oplus^2)/\pi^*(m^9), \ldots \} , \]  

where \(m^9 \subseteq C^\infty(V)\) is the ideal consisting of functions having 0-germ at \(s^{-1}(0) \subseteq V\). The simplicial structure maps are defined as follows: the degeneracy \(s_0 : C^\infty(V)/m^9 \to C^\infty(E)/\pi^*(m^9)\) is given by \(\pi^*\), and the two face maps \(d_0, d_1 : C^\infty(E)/\pi^*(m^9) \to C^\infty(V)/m^9\) are \(0^*, s^*\) respectively, with \(0 : V \to E\) being the 0-section. The rest of the simplicial structure maps are freely generated by this triple \((s_0,d_0,d_1)\).

To describe them explicitly we index copies of \(E\) in \(E^\oplus^n\) by surjective weakly order preserving maps \(\{\alpha_n^i : \underline{n} \to \underline{1}\}_{0 \leq i \leq n-1}\), where for a non-negative integer \(n\) we write \(\underline{n}\) to mean the ordinal \((0, \ldots, n)\). Given a surjective order preserving map \(c : \underline{n}+1 \to \underline{n}\), we have \(c^* : \{\alpha_n^i\} \to \{\alpha_n^j\}\), and then the copy of \(C^\infty(E)/\pi^*(m^9)\) indexed by \(\alpha_n^i\) is mapped identically to the copy indexed by \(c^*(\alpha_n^i)\). Similarly we use \(id\) to define the action of any injective weakly order preserving \(d : \underline{n} \to \underline{n}+1\), when we have that \(d^*(\alpha_n^i) : \underline{n} \to \underline{1}\) is surjective.

There are two cases when \(d^*(\alpha_n^i)\) is not surjective: if \(j=0\) and \(d\) misses \(0\), and if \(j=n\) and \(d\) misses \(n+1\). In the former case we define the action of \(d^*\) to be \(0^*\), and in the latter \(s^*\). For example, the simplicial structure maps between \(C^\infty(E)/\pi^*(m^9)\) and \(C^\infty(E \oplus E)/\pi^*(m^9)\) are given by

\[
\begin{array}{ccc}
\mathbb{id} \oplus \beta_0 & \pi_1 \\
\Delta & E & E \oplus E, \\
\beta_1 \oplus \mathbb{id} & \pi_2 & \mathbb{id}
\end{array}
\]

where \(\beta_0, \beta_1\) are the composite maps

\[ \beta_0 : E \xrightarrow{\pi} V \xrightarrow{0} E, \quad \beta_1 : E \xrightarrow{\pi} V \xrightarrow{s} E. \]

A (Borisov–Noël) derived manifold \(X\) of virtual dimension \(\text{vdim} X = n\) in \(\mathbb{Z}\), as in [7][8], is a pair \(X = (X, \mathcal{O}_{\bullet,X})\), where \(X\) is a Hausdorff, second countable topological space, and \(\mathcal{O}_{\bullet,X}\) is a sheaf of simplicial \(C^\infty\)-rings on \(X\), satisfying:

(i) the sheaf \(\mathcal{O}_{0,X}\) is soft, and its stalks are local \(C^\infty\)-rings with residue field \(\mathbb{R}\),
(ii) for any $x \in X$ there is a Kuranishi neighbourhood $(V, E, s, \psi)$ on $X$ with $x \in R = \text{Im} \psi \subseteq X$ and $\dim V - \text{rank} E = n$, such that $\Gamma(R, \mathcal{O}_{\bullet, R})$ is weakly equivalent to the simplicial $C^\infty$-ring $\mathcal{E}_{V, E, s}$ in (2.10).

A morphism of derived manifolds $f = (f, f^\sharp) : (X, \mathcal{O}_{\bullet, X}) \to (Y, \mathcal{O}_{\bullet, Y})$ is a continuous map $f : X \to Y$ and a morphism of sheaves of simplicial $C^\infty$-rings $f^\sharp : f^{-1}(\mathcal{O}_{\bullet, X}) \to \mathcal{O}_{\bullet, Y}$. A morphism $(f, f^\sharp)$ is a weak equivalence if $f$ is a homeomorphism and $f^\sharp$ is a weak equivalence of sheaves of simplicial $C^\infty$-rings.

**Definition 2.17.** A d-manifold $X$ of virtual dimension $\text{vdim} X = n \in \mathbb{Z}$, as in [21, 23], is a pair $X = (X, \mathcal{O}^\bullet_X)$, where $X$ is a Hausdorff, second countable topological space, and $\mathcal{O}^\bullet_X$ is a sheaf of $dg$ $C^\infty$-rings, satisfying:

(i) $\mathcal{O}^0_X = 0$ for $n \neq 0, -1$, and $(\text{d}(\mathcal{O}^{-1}_X)) \cdot \mathcal{O}^{-1}_X = 0$,
(ii) the sheaf $\mathcal{O}^0_X / \text{d}(\mathcal{O}^{-1}_X)$ is soft, and its stalks are local $C^\infty$-rings with residue field $\mathbb{R}$,
(iii) for any $x \in X$, there is a Kuranishi neighbourhood $(V, E, s, \psi)$ on $X$ with $x \in R = \text{Im} \psi \subseteq X$ and $\dim V - \text{rank} E = n$, and two quasi-isomorphisms of $dg$ $C^\infty$-rings

\[
\left( \frac{C^\infty(E^\ast)}{I_s} \right) \xrightarrow{s} \left( \frac{C^\infty(V)}{I_s} \right) \xrightarrow{(\Gamma(R, \mathcal{O}^{-1}_X) \xrightarrow{d} \Gamma(R, \mathcal{O}^0_X))}
\]

with both compositions being homotopic to the identities, where $I_s = s \cdot C^\infty(E^\ast)$ is the ideal generated by $s$.

A 1-morphism of d-manifolds $f = (f, f^\sharp) : (X, \mathcal{O}^\bullet_X) \to (Y, \mathcal{O}^\bullet_Y)$ is given by a continuous map $f : X \to Y$ and a morphism of sheaves of $dg$ $C^\infty$-rings $f^\sharp : f^{-1}(\mathcal{O}^\bullet_Y) \to \mathcal{O}^\bullet_X$. A 2-morphism between 1-morphisms of d-manifolds $(f, f^\sharp), (g, g^\sharp) : (X, \mathcal{O}^\bullet_X) \to (Y, \mathcal{O}^\bullet_Y)$ with $f = g : X \to Y$ is given by a sheaf of $C^\infty$-derivations $\eta : f^{-1}(\mathcal{O}^0_Y) \to \mathcal{O}^{-1}_X$, such that $g^\sharp = f^\sharp + [d, \eta]$.

D-manifolds and their 1- and 2-morphisms form a strict 2-category $\text{dMan}$. By [23, Th. 3.39], a coordinate change $(V_{JK}, \phi_{JK}, \phi_{JK}) : (V_J, E_{J, s, J, \psi_J}) \to (V_{K}, E_{K, s, K, \psi_K})$ in Definition [2.14] induces an equivalence in $\text{dMan}$ of open subsets of the d-manifolds associated to $(V_J, E_{J, s, J, \psi_J}), (V_{K}, E_{K, s, K, \psi_K})$.

### 2.6 Orientations on derived manifolds

Derived manifolds have a good notion of orientation, which behaves much like orientations on ordinary manifolds. Some references are Joyce [21 §4.8], [22 §4.8], [23 §4.6] for d-manifolds, Amorim and Joyce [11] for (M-)Kuranishi spaces in the sense of Joyce [25], and Fukaya, Oh, Ohta and Ono [15 §5], [16 §A1.1] for Kuranishi spaces in their sense.

For d-manifolds $X$, in a similar way to [23, 25] defines a virtual cotangent bundle $L_X$, a 2-term complex of quasicoherent sheaves on the classical $C^\infty$-scheme $\mathcal{X} = t_0(X)$, and a (real) determinant line bundle $\text{det } L_X$ on $\mathcal{X}$. Orientations on $X$ are then orientations on this real line bundle $\text{det } L_X$.
Also in a similar way to §2.4, for any $d$-manifold (or other kind of derived manifold) $X$, we can define a topological principal $\mathbb{Z}_2$-bundle $\pi : P_X \to X$ called the orientation bundle, where $X$ is the underlying topological space of $X$, such that (local) continuous sections of $P_X$ correspond to (local) orientations on $X$.

We can describe $P_X$ explicitly as follows. If $X$ is a derived manifold, at each $x \in X$ we can define a tangent space $T_x X$ and obstruction space $O_x X$ (see [25 §2.5] for the definition for M-Kuranishi spaces). We have

$$\det L_{|x} \cong (\Lambda^{\text{top}} T_x X)^* \otimes_{\mathbb{R}} \Lambda^{\text{top}} O_x X.$$  \hfill (2.11)

So in a similar way to (2.8), we may write

$$P_X|_x \cong \{\text{orientations on the real vector space } T_x X \oplus O_x X\}. \hfill (2.12)$$

If $(V, E, s, \psi)$ is a Kuranishi neighbourhood on $X$ and $v \in s^{-1}(0) \subseteq V$ with $\psi(v) = x \in X$, then there is a natural exact sequence

$$0 \longrightarrow T_x X \longrightarrow T_v V \longrightarrow E|_v \longrightarrow O_x X \longrightarrow 0. \hfill (2.13)$$

Taking top exterior powers in (2.13) gives an isomorphism

$$(\Lambda^{\text{top}} T_x X)^* \otimes_{\mathbb{R}} \Lambda^{\text{top}} O_x X \cong (\Lambda^{\text{top}} T_v V)^* \otimes_{\mathbb{R}} \Lambda^{\text{top}} E|_v,$$  \hfill (2.14)

and thus, with a suitable orientation convention, bijections

$$P_X|_x \cong \{\text{orientations on } T_x X \oplus O_x X\} \cong \{\text{orientations on } T_v V \oplus E|_v\}. \hfill (2.15)$$

Note that the right hand sides of (2.11)–(2.12) do not vary continuously with $x \in X$ (the dimensions of $T_x X, O_x X$ are only upper semicontinuous in $x$). So, as for (2.8), the topology on $P_X$ is not obvious from the description (2.12). However, the right hand sides of (2.14)–(2.15) do vary continuously with $v \in s^{-1}(0)$, and give the correct topology on $P_X$ over the open set $\text{Im} \psi \subseteq X$.

## 2.7 Bordism and virtual classes

We now discuss bordism for manifolds and d-manifolds, following [21 §4.10], [22 §15] and [23 §13].

**Definition 2.18.** Let $Y$ be a manifold, and $k \in \mathbb{N}$. Consider pairs $(X, f)$, where $X$ is a compact, oriented manifold with $\dim X = k$, and $f : X \to Y$ is a smooth map. Define an equivalence relation $\sim$ on such pairs by $(X, f) \sim (X', f')$ if there exists a compact, oriented $(k + 1)$-manifold with boundary $W$, a smooth map $e : W \to Y$, and a diffeomorphism of oriented manifolds $j : -X \sqcup X' \to \partial W$, such that $f \sqcup f' = e \circ i_W \circ j$, where $-X$ is $X$ with the opposite orientation, and $i_W : \partial W \to W$ is the inclusion map.

Write $[X, f]$ for the $\sim$-equivalence class (bordism class) of a pair $(X, f)$. For each $k \in \mathbb{Z}$, define the $k^{\text{th}}$ bordism group $B_k(Y)$ of $Y$ to be the set of all such bordism classes $[X, f]$ with $\dim X = k$. We give $B_k(Y)$ the structure of an
Theorem 2.20. For any manifold $Y$, we have $dB_k(Y) = 0$ for $k < 0$, and $\Pi^\text{dbo}_{k}: B_k(Y) \to dB_k(Y)$ is an isomorphism for $k \geq 0$.

The main idea of the proof of Theorem 2.20 is that (compact, oriented) d-manifolds $X$ can be turned into (compact, oriented) manifolds $\hat{X}$ by a small perturbation. By Theorem 2.20, we may define a projection $\Pi^\text{hom}_{\text{dbo}}: dB_k(Y) \to H_k(Y;\mathbb{Z})$ for $k \geq 0$ by $\Pi^\text{hom}_{\text{dbo}} = \Pi^\text{hom}_{\text{bo}} \circ (\Pi^\text{dbo}_{k})^{-1}$. We think of $\Pi^\text{hom}_{\text{dbo}}$ as a virtual class map. Virtual classes (or virtual cycles, or virtual chains) are used in several areas of geometry to construct enumerative invariants using moduli spaces, for example in [16] §A1, [18] §6 for Fukaya–Oh–Ohta–Ono’s Kuranishi spaces, and in Behrend and Fantechi [2] in algebraic geometry.

In particular, if $X$ is a compact, oriented d-manifold with vdim $X = 0$, then the bordism class $[X] \in B_0(*) = \mathbb{Z}$ is a ‘virtual count’ of $X$.
Remark 2.21. One can also define virtual classes $[X]_{\text{virt}}$ for compact, oriented derived manifolds $X$ directly using homology, rather than going via bordism, along the lines of Fukaya et al. [16,18] for their Kuranishi spaces, and with some extra work one can even define them in the (Steenrod) homology $H_*(X;\mathbb{Z})$ of the underlying topological space. We chose to explain virtual classes via bordism in this paper, because it is elementary and easy to understand.

3 The main results

We now give our main results. We begin in §3.1 with a general existence result for a special kind of atlas for $\pi : X \to Z$, where $X$ is a separated derived $\mathbb{C}$-scheme and $Z$ a smooth affine classical $\mathbb{C}$-scheme, an atlas in which the charts are spectra of standard form cdgas, the coordinate changes are submersions, and composition of coordinate changes is strictly associative. Sections 3.2–3.5 build up to our primary goal, Theorem 3.18 in §3.5, that to a separated, $-2$-shifted symplectic derived $\mathbb{C}$-scheme $(X,\omega_X^*)$ with $\text{vdim}_\mathbb{C} X = n$, we can associate a derived manifold $X_{dm}$ with the same topological space, with $\text{vdim}_\mathbb{R} X_{dm} = n$. In §3.6 we show that orientations on $(X,\omega_X^*)$ and on $X_{dm}$ correspond, and prove that for $(X,\omega_X^*)$ proper and oriented, the bordism class $[X_{dm}] \in dB_n(*)$ is a 'virtual cycle' independent of choices. Section 3.7 extends §3.2–§3.6 to families $(\pi : X \to Z, [\omega_{X/Z}])$ over a connected base $\mathbb{C}$-scheme $Z$, and shows that the bordism class $[X_{dm,z}] \in dB_n(*)$ associated to a fibre $\pi^{-1}(z)$ is independent of $z \in Z_{an}$. Finally, §3.8–§3.9 discuss applying our results to define Donaldson–Thomas style invariants ‘counting’ coherent sheaves on Calabi–Yau 4-folds, and motivation from gauge theory.

3.1 Zariski homotopy atlases on derived schemes

Derived schemes and stacks, discussed in §2.2, are very abstract objects, and difficult to do computations with. But standard form cdgas $A^*, B^*$ and their submersions $\Phi : A^* \to B^*$ in §2.1 are easy to write down and work with explicitly. Our first main result constructs well-behaved homotopy atlases for a derived scheme $X$, built from standard form cdgas and submersions. First we need to define when a derived scheme is separated. Following the classical characterization of separated schemes [19, Prop. I.5.3.6], we use affine intersections.

Definition 3.1. Let $X$ be a derived $\mathbb{C}$-scheme. We say that $X$ is separated, if for any two Zariski open inclusions $U_1, U_2 \hookrightarrow X$, such that $U_1, U_2$ are affine, the homotopy pullback $U_1 \times^h_X U_2 \hookrightarrow X$ in $\text{dSch}_\mathbb{C}$ is a Zariski open inclusion, and $U_1 \times^h_X U_2$ is affine.

The next theorem will be proved in §4.

Theorem 3.2. Let $X$ be a separated derived $\mathbb{C}$-scheme, $Z = \text{Spec} B$ be a smooth, connected, classical affine $\mathbb{C}$-scheme for $B$ a smooth $\mathbb{C}$-algebra, and $\pi : X \to Z$ be a morphism. Suppose we are given data $\{(A^*_i, \alpha_i, \beta_i) : i \in I\}$,
where $I$ is an indexing set and for each $i \in I$, $A^*_i \in \text{cdga}_C$ is a standard form cdga, and $\alpha_i : \text{Spec} \, A^*_i \hookrightarrow X$ is a Zariski open inclusion in $\text{dSch}_C$, and $\beta_i : B \to A^*_0$ is a smooth morphism of classical $\mathbb{C}$-algebras such that the following diagram homotopy commutes in $\text{dSch}_C$:

$$
\begin{array}{ccc}
\text{Spec} \, A^*_i & \xrightarrow{\alpha_i} & X \\
\downarrow & & \downarrow \pi \\
\text{Spec} \, B = Z
\end{array}
$$

regarding $\beta_i$ as a morphism $B \to A^*_i$. Then we can construct the following data:

(i) For all finite subsets $\emptyset \neq J \subseteq I$, a standard form cdga $A^*_J \in \text{cdga}_C$, a Zariski open inclusion $\alpha_J : \text{Spec} \, A^*_J \hookrightarrow X$, with image $\text{Im} \alpha_J = \bigcap_{i \in J} \text{Im} \alpha_i$ as a Zariski open subset of the topological space of $X = t_0(X)$, and a smooth morphism of classical $\mathbb{C}$-algebras $\beta_J : B \to A^*_0$, such that the following diagram homotopy commutes in $\text{dSch}_C$:

$$
\begin{array}{ccc}
\text{Spec} \, A^*_J & \xrightarrow{\alpha_J} & X \\
\downarrow & \xleftarrow{\beta_J} & \downarrow \pi \\
\text{Spec} \, B = Z
\end{array}
$$

and when $J = \{i\}$ for $i \in I$ we have $A^*_J = A^*_i$, $\alpha_J = \alpha_i$, and $\beta_J = \beta_i$.

(ii) For all inclusions of finite subsets $\emptyset \neq K \subseteq J \subseteq I$, a submersion of standard form cdgas $\Phi_{JK} : A^*_K \to A^*_J$ with $\beta_J = \Phi_{JK} \circ \beta_K : B \to A^*_0$, such that the following diagram homotopy commutes in $\text{dSch}_C$:

$$
\begin{array}{ccc}
\text{Spec} \, A^*_J & \xleftarrow{\Phi_{JK}} & \text{Spec} \, A^*_K \\
\downarrow & & \downarrow \alpha_K \\
\text{Spec} \, B = Z & \xrightarrow{\alpha_J} & X
\end{array}
$$

and if $\emptyset \neq L \subseteq K \subseteq J \subseteq I$ then $\Phi_{JL} = \Phi_{JK} \circ \Phi_{KL} : A^*_L \to A^*_J$.

Remark 3.3. (a) For greater generality, we work over a base $Z = \text{Spec} \, B$ which is a smooth classical affine $\mathbb{C}$-scheme. But we are mostly interested in the case $B = \mathbb{C}$, so that $Z = *$ is the point, and the data $\beta_i, \beta_J$ in Theorem 3.2 is trivial and can be ignored. In §3.3 §3.6 when we construct a derived manifold $X_{\text{dm}}$ from a $-2$-shifted symplectic derived $\mathbb{C}$-scheme $(X, \omega_X^*)$, we restrict to $B = \mathbb{C}$. But in §3.7 we work over a base $Z$, to show that if $(X, \omega_X^*)$ deforms algebraically over $Z$, then $X_{\text{dm}}$ also deforms smoothly over $Z_{\text{an}}$ by bordism. For this purpose it is sufficient to take $Z$ to be smooth, connected, and affine, rather than a more general classical or derived $\mathbb{C}$-scheme.

(b) The theorem is intended to be used when $\{\text{Im} \alpha_i : i \in I\}$ is an open cover of $X$. Then $\text{Spec} \, A^*_i$ gives a useful local description of $X$ in the open set $\text{Im} \alpha_i$, for each $i \in I$. To be able to work globally on $X$, we need to relate these
descriptions on multiple overlaps $\bigcap_{i \in J} \text{Im} \alpha_i$ in $X$ for $\emptyset \neq J \subseteq I$ finite. The theorem provides a good way to do this. It is important that the morphisms $\Phi_{JK} : A^*_K \to A^*_J$ are submersions (arguably the nicest kind of morphism of standard form cdgas), and that in (ii) we have $\beta_J = \Phi_{JK} \circ \beta_K$, rather than just a homotopy $\beta_J \simeq \Phi_{JK} \circ \beta_K$, and $\Phi_{JL} = \Phi_{JK} \circ \Phi_{KL}$ rather than just a homotopy $\Phi_{JL} \simeq \Phi_{JK} \circ \Phi_{KL}$.

(c) We need $X$ to be separated as then if $X_i \subseteq X$ for $i \in J$ are affine open derived $\mathbb{C}$-subschemes, with $J$ a finite set, then $\bigcap_{i \in J} X_i$ is also affine. This is necessary for $\alpha_J : \text{Spec} A^*_J \hookrightarrow X$ in (i) with image $\bigcap_{i \in J} \text{Im} \alpha_i$ to exist.

### 3.2 Interpreting Zariski atlases using complex geometry

Given a $-2$-shifted symplectic derived $\mathbb{C}$-scheme $(X, \omega_X^*)$ satisfying conditions, we will construct a $d$-manifold structure $X_{dm}$ on the complex analytic topological space $X_{an}$ underlying $X$. To do this, we need a change of language: we have to pass from talking about derived schemes $X$, cdgas $A^*$, etc., to talking about smooth manifolds $V$, vector bundles $E \to V$, smooth sections $s : V \to E$, as $X_{dm}$ will be built by gluing together such local Kuranishi models $(V, E, s)$.

Therefore we now rewrite part of the output $A^*_J, \beta_J : B \to A^*_J, \Phi_{JK} : A^*_L \to A^*_J$ of Theorem 3.2 in terms of complex manifolds $V$, holomorphic vector bundles $E \to V$, and holomorphic sections $s : V \to E$. In §3.5 we will pass to certain real vector bundles $E^+ = E/E^-$ to define $X_{dm}$.

First we interpret standard form cdgas $A^* \in \text{cdga}_C$ using holomorphic data. We discuss only data from degrees $0, -1, -2$ in $A^*$, as this is all we need, but one could also define vector bundles $G, H, \ldots$ over $V$ corresponding to $M^{-3}, M^{-4}, \ldots$, and many vector bundle morphisms, satisfying equations.

**Definition 3.4.** Let $A^* = (\cdots \to A^{-2} \xrightarrow{d} A^{-1} \xrightarrow{d} A^0)$ be a standard form cdga over $\mathbb{C}$, as in [2.1]. Then $A^0$ is a finitely generated smooth $\mathbb{C}$-algebra, so $V_{\text{alg}} := \text{Spec} A^0$ is a smooth affine $\mathbb{C}$-scheme, assumed connected, as in [2.1].

Now any $\mathbb{C}$-scheme $S$ has an underlying complex analytic space $S_{an}$, which is a complex manifold if $S$ is smooth and connected. (We assume $S$ connected for a trivial reason: otherwise $S$ could have connected components of different dimensions, and then $S_{an}$ would not count as a complex manifold.)

Write $V$ for the complex manifold $(V_{\text{alg}})_{an}$ associated to $V_{\text{alg}} = \text{Spec} A^0$.

As $A^*$ is of standard form, the graded $\mathbb{C}$-algebra $A^*$ is freely generated over $A^0$ by a series of finitely generated free $A^0$-modules $M^{-1} \subseteq A^{-1}, M^{-2} \subseteq A^{-2}, \ldots$ Thus $A^{-1} \cong M^{-1}, A^{-2} \cong M^{-2} \oplus \Lambda^2_{A^0}M^{-1}$, and so on, giving

$$M^{-1} = A^{-1}, \quad M^{-2} \cong A^{-2}/\Lambda^2_{A^0}A^{-1}, \ldots$$

Hence, the $M^i$ are determined by $A^*$ as $A^0$-modules up to canonical isomorphism, although for $i \leq -2$ the inclusions $M^i \hookrightarrow A^i$ involve an arbitrary choice.

Now finitely generated free $A^0$-modules $M$ are those of the form $M \cong H^0(C^\text{alg})$ for $C^\text{alg} \to V_{\text{alg}} = \text{Spec} A^0$ a trivial algebraic vector bundle. Write $E_{\text{alg}} \to V_{\text{alg}}, F_{\text{alg}} \to V_{\text{alg}}$ for the trivial algebraic vector bundles (unique up
to canonical isomorphism) with $M^{-1} \cong H^0((E^{\text{alg}})^*)$, $M^{-2} \cong H^0((F^{\text{alg}})^*)$. We do not choose trivializations $E^{\text{alg}} \cong \mathcal{O}_{V^{\text{alg}}}^m$, $F^{\text{alg}} \cong \mathcal{O}_{V^{\text{alg}}}^m$ for $E^{\text{alg}}, F^{\text{alg}}$. Write $E \to V$, $F \to V$ for the holomorphic vector bundles corresponding to $E^{\text{alg}}, F^{\text{alg}}$.

We now have isomorphisms

$$
A^0 \cong H^0(\mathcal{O}_{V^{\text{alg}}}), \quad A^{-1} \cong H^0((E^{\text{alg}})^*),
$$

$$
A^{-2} \cong H^0((F^{\text{alg}})^*) \oplus H^0(\Lambda^2(E^{\text{alg}})^*). \quad \tag{3.5}
$$

Thus $d : A^{-1} \to A^0$ is identified with an $A^0$-module morphism $H^0((E^{\text{alg}})^*) \to H^0(\mathcal{O}_{V^{\text{alg}}})$, that is, a morphism $(E^{\text{alg}})^* \to \mathcal{O}_{V^{\text{alg}}}$ of algebraic vector bundles, which is dual to a morphism $\mathcal{O}_{V^{\text{alg}}} \cong \mathcal{O}_{V^{\text{alg}}} \to E^{\text{alg}}$, i.e. a section $s^{\text{alg}} \in H^0(E^{\text{alg}})$ of $E^{\text{alg}}$. Write $s \in H^0(E)$ for the corresponding holomorphic section.

Similarly, write $t^{\text{alg}} : E^{\text{alg}} \to F^{\text{alg}}$ for the algebraic vector bundle morphism dual to the component of $d : A^{-2} \to A^{-1}$ mapping $H^0((F^{\text{alg}})^*) \to H^0((E^{\text{alg}})^*)$ under (3.5), and write $t : E \to F$ for the corresponding morphism of holomorphic vector bundles. Then $d \circ d = 0$ implies that

$$
t \circ s = 0 : \mathcal{O}_V \to F.
$$

We should also consider how this data $E, F, s, t$ depends on the choice of inclusion $M^{-2} \hookrightarrow A^{-2}$. Here $E, F$ are independent of choices up to canonical isomorphism, and $s$ is independent of choices. Changing the inclusion $M^{-2} \hookrightarrow A^{-2}$ is equivalent to choosing an algebraic vector bundle morphism $\gamma^{\text{alg}} : \Lambda^2 E^{\text{alg}} \to F^{\text{alg}}$ and identifying $M^{-2}$ with the image of $\text{id} \oplus (\gamma^{\text{alg}})^* : H^0((F^{\text{alg}})^*) \hookrightarrow H^0((\Lambda^2 E^{\text{alg}})^*) \oplus H^0(\Lambda^2 E^{\text{alg}})^*)$. Writing $\gamma : \Lambda^2 E \to F$ for the corresponding holomorphic morphism, this changes $t$ to $t$, where

$$
\tilde{t} = t + \gamma \circ (- \wedge s). \quad \tag{3.6}
$$

Notice that $t|_v : E|_v \to F|_v$ is independent of choices at $v \in V$ with $s(v) = 0$.

Next suppose $X$ is a derived C-scheme and $\alpha : \text{Spec } A^* \hookrightarrow X$ a Zariski open inclusion. Write $X = t_0(X)$ for the classical C-scheme, and $X_{\text{an}}$ for the set of C-points of $X$ equipped with the complex analytic topology. (One can give $X_{\text{an}}$ the structure of a complex analytic space, but we will not use this.) Then $t_0(\text{Spec } A^*)$ is the C-subscheme $(s^{\text{alg}})^{-1}(0) \subseteq V^{\text{alg}}$, so $\alpha = t_0(\alpha)$ is a Zariski open inclusion $(s^{\text{alg}})^{-1}(0) \hookrightarrow X$. Write $\psi : s^{-1}(0) \hookrightarrow X_{\text{an}}$ for the corresponding map of C-points. Then $\psi$ is a homeomorphism with an open set $R = \text{Im } \psi \subseteq X_{\text{an}}$. Note that $(V, E, s, \psi)$ is a Kuranishi neighbourhood on $X_{\text{an}}$, in the sense of [2.5].

As we explained in [2.1–2.2] if $A^*$ is a standard form cdga then it is easy to compute the cotangent complex $\mathbb{L}_{A^*} \simeq \Omega^1_{A^*}$, and this also can be identified with the cotangent complex $\mathbb{L}_{\text{Spec } A^*}$ of the derived scheme $\text{Spec } A^*$. Let $v \in s^{-1}(0) \subseteq V$ with $\psi(v) = x \in X_{\text{an}}$. Then $v$ is a C-point of $\text{Spec } A^*$ and $x$ a C-point of $X$ with $\alpha(v) = x$, so $L_{\alpha}|_v : \mathbb{L}_{X}|_x \to \mathbb{L}_{\text{Spec } A^*}|_v$ is a quasi-isomorphism, and induces an isomorphism on cohomology. One can show that $\mathbb{L}_{\text{Spec } A^*}|_v$ is represented by the complex of C-vector spaces

$$
\cdots \longrightarrow F|_v \longrightarrow E|_v \longrightarrow T|_v^* \longrightarrow 0, \quad \tag{3.7}
$$
with \( T^*_v V \) in degree 0. Dualizing to tangent complexes and taking cohomology, we get canonical isomorphisms

\[
H^0(T_{\alpha|v}) : \text{Ker}(ds|_v : T_v V \to E|_v) \longrightarrow H^0(T_X|_x), \quad (3.8)
\]

\[
H^1(T_{\alpha|v}) : \frac{\text{Ker}(t|_v : E|_v \to F|_v)}{\text{Im}(ds|_v : T_v V \to E|_v)} \longrightarrow H^1(T_X|_x). \quad (3.9)
\]

Now suppose that \( Z = \text{Spec } B \) is a smooth, connected, classical affine \( \mathbb{C} \)-scheme, \( \pi : X \to Z \) is a morphism, and \( \beta : B \to A^0 \) is a smooth morphism of \( \mathbb{C} \)-algebras, such that as for (3.1)–(3.2) the following homotopy commutes

\[
\begin{array}{ccc}
\text{Spec } A^* & \xrightarrow{\alpha} & X \\
\downarrow \text{Spec } \beta & & \downarrow \pi \\
\text{Spec } B & = & Z.
\end{array}
\quad (3.10)
\]

Then \( Z_{an} \) is a complex manifold, and \( \tau_{\text{alg}} := \text{Spec } \beta : V_{\text{alg}} \to Z \) is a smooth morphism of \( \mathbb{C} \)-schemes, and \( \tau := (\tau_{\text{alg}})_{an} : V \to Z_{an} \) is a holomorphic submersion of complex manifolds. We can form the relative cotangent complexes \( L \), \( T \), and \( \tau \) as \( \mathbb{C} \)-vector spaces, and dual relative tangent complexes \( \tau^* \), \( \tau^* \), and dual relative tangent bundles \( \tau^* \), \( \tau^* \), and dual relative tangent complexes \( \tau^* \), \( \tau^* \). As for (3.7), \( L_{\text{Spec } A^*/Z} \) is represented by the complex of \( \mathbb{C} \)-vector spaces

\[
\cdots \longrightarrow F|_v^* \xrightarrow{t_v^*} E|_v^* \xrightarrow{ds|_v^*} T_v^*(V/Z_{an}) \longrightarrow 0,
\]

with \( T_v^*(V/Z_{an}) \) in degree 0. As for (3.8)–(3.9) we get canonical isomorphisms

\[
H^0(T_{\alpha|v}) : \text{Ker}(ds|_v : T_v V \to E|_v) \longrightarrow H^0(T_X|_x), \quad (3.11)
\]

\[
H^1(T_{\alpha|v}) : \frac{\text{Ker}(t|_v : E|_v \to F|_v)}{\text{Im}(ds|_v : T_v V \to E|_v)} \longrightarrow H^1(T_X|_x). \quad (3.12)
\]

**Example 3.5.** Suppose \( (A^*, \omega_{A^*}) \) is in \(-2\)-Darboux form, in the sense of Definition 3.3, with coordinates \( x_0, \ldots, x_{m_0}, y^{-1}_1, \ldots, y^{-1}_{m_1}, z^{-2}_1, \ldots, z^{-2}_{m_2} \), and 2-form \( \omega_{A^*} \) in (2.1), depending on invertible functions \( q_1, \ldots, q_{m_1} \in A^0 \).

Let \( V, E, F, s, t \) be as in Definition 3.3. Then \( V \) is a smooth \( \mathbb{C} \)-scheme of dimension \( m_0 \), with \( \text{étale} \) coordinates \( (x_0, \ldots, x_{m_0}) \), so that \( TV \) is a trivial vector bundle with basis of sections \( \left. \frac{\partial}{\partial x_i} \right|_{m_0}, \ldots, \left. \frac{\partial}{\partial x_{m_0}} \right|_{m_0}. \) Also \( E \) is a trivial vector bundle of rank \( m_1 \), with basis \( e_1 := \left. \frac{\partial}{\partial y_1} \right|_{m_1}, \ldots, e_{m_1} := \left. \frac{\partial}{\partial y_{m_1}} \right|_{m_1} \), and \( F \) is trivial of rank \( m_0 \), with basis \( \left. \frac{\partial}{\partial z_1} \right|_{m_0}, \ldots, \left. \frac{\partial}{\partial z_{m_0}} \right|_{m_0}. \) Using the first line of \( \omega_{A^*} \) in (2.1), it is natural to identify \( F \cong TV \) by identifying \( \left. \frac{\partial}{\partial z_i} \right|_{m_0} \cong d_{dR}x_i^0 \) for \( i = 1, \ldots, m_0 \).
The natural section \( s \in H^0(E) \) is \( s = s_1e_1 + \cdots + s_me_m \). Write \( e_1, \ldots, e_m \) for the basis of sections of \( E^* \) dual to \( e_1, \ldots, e_m \), so that \( e_1 \cong d_{\text{dR}}y_1^{-1} \). Motivated by the second line of \( \omega_{AK} \) in (2.1), define \( Q = q_1 e_1 \otimes e_1 + \cdots + q_m e_m \otimes e_m \) in \( H^0(S^2E^*) \). Then \( Q \) is a natural nondegenerate quadratic form on the fibres of \( E \), and (2.2) implies that \( Q(s,s) = 0 \).

Identifying \( F = T^*V \), from (2.3) we see that \( t : E \rightarrow F \) is given by

\[
t(e_j) = \sum_{i=1}^{m^0} \left( 2q_j \frac{\partial s_j}{\partial x_i} + s_j \frac{\partial q_j}{\partial x_i} \right) d_{\text{dR}}x_i^0 = 2q_j d_{\text{dR}}s_j + s_j d_{\text{dR}}q_j,
\]

for \( j = 1, \ldots, m^1 \). Then \( t \circ s = 0 \) follows from applying \( d_{\text{dR}} \) to \( Q(s,s) = 0 \).

What will matter later is that we have a complex manifold \( V \), a holomorphic vector bundle \( E \rightarrow V \), a section \( s \in H^0(E) \), and a nondegenerate holomorphic quadratic form \( Q \in H^0(S^2E^*) \) with \( Q(s,s) = 0 \), such that the classical complex analytic topological space \( (\text{Spec } H^0(A^*))_{an} \) is \( s^{-1}(0) \subseteq V \).

Next we interpret submersions of standard form cdgas \( \Phi_{JK} : A^*_{JK} \rightarrow A^*_J \), as in Theorem 3.2(a)(ii), in terms of complex geometry.

**Definition 3.6.** Let \( \Phi_{JK} : A^*_{JK} \rightarrow A^*_J \) be a submersion of standard form cdgas over \( \mathbb{C} \), as in (2.1). Let \( V^\text{alg}_J, E^\text{alg}_J, s_j^\text{alg}, t_j^\text{alg}, V_J, E_J, s_J, t_J \) be as in Definition 3.4 for \( A^*_J \), and \( \phi^\text{alg}_{JK} = E^\text{alg}_J, F^\text{alg}_J, s^\text{alg}_K, t^\text{alg}_K, V_K, E_K, F_K, s_K, t_K \) for \( A^*_{JK} \).

Then \( \phi^\text{alg}_{JK} := \text{Spec } \Phi^\text{alg}_{JK} = V^\text{alg}_J = \text{Spec } A^*_J \) is a \( \mathbb{C} \)-scheme morphism. Write \( \phi_{JK} : V_J \rightarrow V_K \) for the corresponding holomorphic map. The submersion condition on \( \Phi_{JK} \) implies that \( d \circ \Phi^\text{alg}_{JK} : (\phi^\text{alg}_{JK})^*(T^*V^\text{alg}_K) \rightarrow T^*V^\text{alg}_J \) is injective, so \( d \circ \phi_{JK} : \phi^\text{alg}_{JK}(T^*V_K) \rightarrow T^*V_J \) is injective, that is, \( \phi_{JK} : V_J \rightarrow V_K \) is a submersion of complex manifolds.

Now \( \Phi_{JK}^{-1}_J : A^{-1}_K \rightarrow A^{-1}_J \) induces an \( A^0_J \)-linear map \( (\Phi_{JK}^{-1}_J)_* : A^{-1}_K \otimes A^0_J ightarrow A^{-1}_J \), which under (3.3) corresponds to an algebraic vector bundle morphism \( (\phi^\text{alg}_{JK})^*(E^\text{alg}_J) \rightarrow (F^\text{alg}_K)^* \), \( (\phi^\text{alg}_{JK})^*(E^\text{alg}_J) \rightarrow (F^\text{alg}_K)^* \) for the dual morphism, and \( \chi_{JK} : E_J \rightarrow \phi_{JK}^*(E_K) \) for the corresponding morphism of holomorphic vector bundles. It is surjective, as \( \Phi_{JK} \) is a submersion. Then \( d \circ \Phi_{JK}^{-1}_J = \Phi^\text{alg}_{JK} \circ d \) implies that

\[
\chi_{JK}(s_J) = \phi_{JK}^*(s_K) \in H^0(\phi^*_{JK}(E_K)).
\]

By (3.3) we have a natural composition of morphisms

\[
H^0((F^\text{alg})^*) \cong M^2_K \cong A^2_K / A^1_K, A^{1}_K(\Phi_{JK}^{-1}_J)_* A^{-2}_J / A^{1}_J A^{-1}_J \cong M^{-2}_J \cong H^0((F^\text{alg})^*). \]

The induced \( A^0_J \)-linear map corresponds to a natural algebraic vector bundle morphism \( (\phi^\text{alg}_{JK})^*(E^\text{alg}_K) \rightarrow (F^\text{alg}_J)^* \). Write \( \xi^\text{alg}_{JK} : F^\text{alg}_J \rightarrow (\phi^\text{alg}_{JK})^*(F^\text{alg}_K) \) for the dual morphism, and \( \xi_{JK} : F_J \rightarrow \phi_{JK}^*(F_K) \) for the corresponding morphism of holomorphic vector bundles. It is surjective, as \( \Phi_{JK} \) is a submersion.
These $\xi_{JK}^{\text{alg}}$, $\xi_{JK}$ are independent of choices, as they depend on the canonical isomorphism $M^{-2} \cong A^{-2}/\Lambda_{A^2}^2 A^{-1}$ rather than on the non-canonical inclusion $M^{-2} \to A^{-2}$ in Definition 3.4. However, $\Phi_{JK}^2$ need not map $M_{K}^{-2} \subseteq A_{K}^{-2}$ to $M_{T}^{-2} \subseteq A_{T}^{-2}$, and so under the isomorphisms (3.4) need not map $H^0((F_{K}^{\text{alg}})^*) \to H^0((F_{T}^{\text{alg}})^*)$. Write $\delta_{JK}^{\text{alg}} : \Lambda^2 E_{J}^{\text{alg}} \to (\phi_{JK})^{*}(F_{K}^{\text{alg}})$ for the algebraic vector bundle morphism dual to the component of $\Phi_{JK}^2$ mapping $H^0((F_{K}^{\text{alg}})^*) \to H^0(\Lambda^2(E_{J}^{\text{alg}})^*)$, and $\delta_{JK} : \Lambda^2 E_{J} \to \phi_{JK}(F_{K})$ for the corresponding morphism of vector bundles. Then $d \circ \Phi_{JK}^2 = \Phi_{JK}^2 \circ d$ implies that

$$\xi_{JK} \circ t_J + \delta_{JK} \circ (- \wedge s_J) = \phi_{JK}^*(t_{K}) \circ \chi_{JK} : E_J \to \phi_{JK}^{*}(F_{K}) \quad (3.15)$$

Thus $\chi_{JK}, \xi_{JK}$ do not strictly commute with $t_J, t_K$, which is not surprising, since $t_J, t_K$ depend on arbitrary choices as in (3.6). But notice that $\xi_{JK}|_{t_J|v} = t_K|_{\phi_{JK}(v)} \circ \chi_{JK}|_v$ at $v \in V_J$ with $s_J(v) = 0$.

Next suppose we are given Zariski open inclusions $\alpha_J : \text{Spec} A_J^* \to X$, $\alpha_K : \text{Spec} A_K^* \to X$ into a derived $\mathbb{C}$-scheme $X$, such that (3.3) homotopy commutes, and let $\psi_J : s_{J}^{-1}(0) \to X_{\text{an}}$. $\psi_K : s_{K}^{-1}(0) \to X_{\text{an}}$ be as in Definition 3.3. As the classical truncation of (3.3) commutes, we see that

$$\psi_J = \psi_K \circ \phi_{JK}|_{s_{J}^{-1}(0)} : s_{J}^{-1}(0) \to X_{\text{an}} \quad (3.16)$$

Suppose $v_j \in s_{J}^{-1}(0) \subseteq V_J$ with $\phi_{JK}(v_j) = v_K \in s_{K}^{-1}(0) \subseteq V_K$ and $\psi_J(v_j) = \psi_K(v_K) = x \in X_{\text{an}}$. As (3.3) homotopy commutes, the corresponding morphisms of tangent complexes $\mathbb{T}_{\text{Spec} A_J^*}$, $\mathbb{T}_{\text{Spec} A_K^*}$, $\mathbb{T}X$ commute up to homotopy, so restricting to $v_j, v_K, x$ and taking homology gives strictly commuting diagrams. Thus using (3.3)–(3.9), we see that the following diagrams commute:

$$\begin{array}{cccc}
\text{Ker}(d_{s_{J}}|_{v_{j}} : T_{v_{j}} V_{J} \to E_{J}|_{v_{j}}) & \text{H}^{0}(\mathbb{T}_{\alpha_{J}}|_{v_{j}}) \\
\text{Ker}(d_{s_{K}}|_{v_{K}} : T_{v_{K}} V_{K} \to E_{K}|_{v_{K}}) & \text{H}^{0}(\mathbb{T}_{\alpha_{K}}|_{v_{K}}) \\
\text{Im}(d_{s_{J}}|_{v_{j}} : T_{v_{j}} V_{J} \to E_{J}|_{v_{j}}) & \text{H}^{1}(\mathbb{T}_{\alpha_{J}}|_{v_{j}}) \\
\text{Im}(d_{s_{K}}|_{v_{K}} : T_{v_{K}} V_{K} \to E_{K}|_{v_{K}}) & \text{H}^{1}(\mathbb{T}_{\alpha_{K}}|_{v_{K}}) \\
\end{array} \quad (3.17)$$

$$\begin{array}{cccc}
\text{Ker}(t_{J}|_{v_{j}} : E_{J}|_{v_{j}} \to F_{J}|_{v_{j}}) & \text{H}^{0}(\mathbb{T} X|_{v_{j}}) \\
\text{Ker}(t_{K}|_{v_{K}} : E_{K}|_{v_{K}} \to F_{K}|_{v_{K}}) & \text{H}^{1}(\mathbb{T} X|_{v_{j}}) \\
\end{array} \quad (3.18)$$

Now suppose that $Z = \text{Spec} B$ is a smooth, connected, classical affine $\mathbb{C}$-scheme, $\pi : X \to Z$ is a morphism, and $\beta_J : B \to A_J^*$, $\beta_K : B \to A_K^*$ are smooth morphisms of $\mathbb{C}$-algebras, such that (3.2) homotopy commutes for $J, K$, and $\beta_J = \Phi_{JK} \circ \beta_K$. As in Definition 3.4 we have holomorphic submersions $\tau_J : V_J \to Z_{\text{an}}$, $\tau_K : V_K \to Z_{\text{an}}$, with $\tau_J = \tau_K \circ \phi_{JK} : V_J \to Z_{\text{an}}$ as $\beta_J = \Phi_{JK} \circ \beta_K$. Let $v_j \in s_{J}^{-1}(0) \subseteq V_J$ with $\phi_{JK}(v_j) = v_K \in s_{K}^{-1}(0) \subseteq V_K$, and $\psi_J(v_j) = \psi_K(v_K) = x \in X_{\text{an}}$, and $\tau_J(v_j) = \tau_K(v_K) = \pi(x) = z \in Z_{\text{an}}$. Then
Corollary 3.7. In the situation of Theorem 3.2 write $X_{an}$ for the set of $C$-points of $X = t_0(X)$, regarded as a topological space with the complex analytic topology. Then we obtain the following data in complex geometry:

\begin{enumerate}[(i)]
    \item For all inclusions of finite subsets $\emptyset \neq J \subseteq I$, a complex manifold $V_J$, a holomorphic submersion $\tau_J : V_J \to X_{an}$, holomorphic vector bundles $E_J, F_J \to V_J$, a holomorphic section $s_J : V_J \to E_J$, and a homeomorphism $\psi_J : s_J^{-1}(0) \to R_J \subseteq X_{an}$, where $R_J \subseteq X_{an}$ is open, with $\pi \circ \psi_J = \tau_J|_{s_J^{-1}(0)} : s_J^{-1}(0) \to Z_{an}$. These image subsets satisfy $R_J = \bigcap_{i \in J} R_{\{i\}}$.

By making an additional arbitrary choice we also obtain a morphism of holomorphic vector bundles $t_J : E_J \to F_J$, with $t_J \circ s_J = 0$. Different choices $t_J, \tilde{t}_J$ are related by (3.19). The restrictions $t_J|_{v_J}, \tilde{t}_J|_{v_J} : E_J|_{v_J} \to F_J|_{v_J}$ for $v_J \in s_J^{-1}(0)$ are independent of choices. For each $v_J \in s_J^{-1}(0)$ with $\psi_J(v_J) = x \in X_{an}$, there are canonical isomorphisms $\kappa_{s_J} \circ \kappa_{V_J}(x)$ writing $H^i(\mathbb{T}_{X_{an}})$ for $i = 0, 1$ and (3.19), (3.20) writing $H^i(\mathbb{T}_{X_{an}})$ for $i = 0, 1$ in terms of $V_J, E_J, F_J, s_J, t_J, \tau_J$ at $v_J$.

    \item For all inclusions of finite subsets $\emptyset \neq K \subseteq J \subseteq I$, a holomorphic submersion $\phi_{JK} : V_J \to V_K$, and surjective morphisms of holomorphic vector bundles $\chi_{JK} : E_J \to \phi_{JK}^*(E_K)$ and $\xi_{JK} : F_J \to \phi_{JK}^*(F_K)$. These satisfy $\tau_J = \tau_K \circ \phi_{JK} : V_J \to Z_{an}$, and $\chi_{JK}(s_J) = \phi_{JK}(s_K)$, and $\psi_J = \psi_K \circ \phi_{JK}|_{s_J^{-1}(0)} : s_J^{-1}(0) \to X_{an}$.

If $t_J, t_K$ are possible choices in (i) then $\chi_{JK}, \xi_{JK}, t_J, t_K$ are related as in (3.15). If $v_J \in s_J^{-1}(0)$ with $\phi_{JK}(v_J) = v_K \in s_K^{-1}(0)$, this implies that

$$\xi_{JK}|_{\{v_J\}} \circ t_J|_{\{v_J\}} = t_K|_{\{v_K\}} \circ \chi_{JK}|_{\{v_J\}} : E_J|_{v_J} \to F_K|_{v_K}.$$
3.3 Subbundles $E^- \subseteq E$ and Kuranishi neighbourhoods

Throughout \([3.3]–[3.6]\) when we apply Theorem \([3.2]\) we take $B = \mathbb{C}$, so that $Z$ is the point $\ast = \text{Spec} \mathbb{C}$, and the data $\pi, \beta_{ij}, \beta_j, \tau_{ij}$ is trivial, so we omit it.

Suppose $(X, \omega_X)$ is a $-2$-shifted symplectic derived $\mathbb{C}$-scheme, $A^\bullet$ a standard form cdga over $\mathbb{C}$, and $\alpha : \text{Spec} \ A^\bullet \to X$ a Zariski open inclusion. Then Definition \([3.4]\) defines complex geometric data $V, E, F, s, t, \psi, R$, such that $(V, E, s, \psi)$ is a Kuranishi neighbourhood on the topological space $X_{an}$ of $X$.

However these are not the Kuranishi neighbourhoods we want: they depend only on $X$, not on $\omega_X$, and in general two such neighbourhoods $(V_j, E_j, s_j, \psi_j)$ and $(V_K, E_K, s_K, \psi_K)$ are not compatible over their intersection $R_J \cap R_K$ in $X_{an}$ (e.g. the virtual dimensions $\dim \mathbb{R}$ $\forall V_j - \text{rank} \mathbb{R} E_j$ and $\dim \mathbb{R} V_K - \text{rank} \mathbb{R} E_K$ may be different), so we cannot glue them to make $X_{an}$ into a derived manifold.

The basic problem is that the rank of $E$ may be too large -- for instance, we can modify $A^\bullet$ to replace $E, F, s, t$ by $\tilde{E} = E \oplus G$, $\tilde{F} = F \oplus G$, $\tilde{s} = s \oplus 0$, $\tilde{t} = t \oplus id_G$ for some holomorphic vector bundle $G \to V$. Our solution is to choose a real vector subbundle $E^- \subseteq E$ satisfying some conditions involving $\omega_X$, and set $E^+ = E/E^-$ to be the quotient bundle and $s^+ = s + E^-$ in $C^\infty(E^+)$ to be the quotient section. The conditions on $E^-$ imply that $s^{-1}(0) = (s^+)^{-1}(0)$, so $(V, E^+, s^+, \psi^+)$ is also a Kuranishi neighbourhood on $X_{an}$. Under good conditions we can make two such $(V_j, E_j, s_j^+, \psi_j^+)$ and $(V_K, E_K, s_K^+, \psi_K^+)$ compatible over $R_J \cap R_K$, and glue these local models to make $X_{an}$ into a derived manifold.

We define the class of subbundles $E^- \subseteq E$ we are interested in:

**Definition 3.8.** Let $(X, \omega_X)$ be a $-2$-shifted symplectic derived $\mathbb{C}$-scheme with $\text{vdim}_\mathbb{C} X = n$, and suppose $A^\bullet \in \text{cdga}_\mathbb{C}$ is of standard form and $\alpha : A^\bullet \hookrightarrow X$ is a Zariski open inclusion. Define complex geometric data $V, E, F, s, t$ and $\psi : s^{-1}(0) \to R \subseteq X_{an}$ as in Definition \([3.4]\) and suppose $R \neq \emptyset$. Then for each $v \in s^{-1}(0)$ with $\psi(v) = x \in X_{an}$, equation \([3.21]\) gives an isomorphism from a vector space depending on $V, E, F, s, t$ at $v$ to $H^1(T_{X|_x})$.

Equation \([2.7]\) defined a quadratic form $Q_x$ on $H^1(T_{X|_x})$. Define

$$
\tilde{Q}_v : \frac{\text{Ker}(t|_v : E|_v \to F|_v)}{\text{Im}(ds|_v : T_v V \to E|_v)} \times \frac{\text{Ker}(t|_v : E|_v \to F|_v)}{\text{Im}(ds|_v : T_v V \to E|_v)} \to \mathbb{C}
$$

(3.21)

to be the nondegenerate complex quadratic form identified with $Q_x$ in \([2.7]\) by the isomorphism $H^1(T_{X|_x})$ in \([3.9]\).

Consider pairs $(U, E^-)$, where $U \subseteq V$ is open and $E^-$ is a real vector subbundle of $E|_U$. Given such $(U, E^-)$, we write $E^+ = E|_U/E^- |_U$ for the quotient vector bundle over $U$, and $s^+ \in C^\infty(E^+)$ for the image of $s|_U$ under the projection $E|_U \to E^+$, and $\psi^+ := \psi|_{s^{-1}(0) \cap U} : s^{-1}(0) \cap U \to X_{an}$. We say that $(U, E^-)$ satisfies condition $(\ast)$ if:

$(\ast)$ For each $v \in s^{-1}(0) \cap U$, we have

$$
\text{Im}(ds|_v : T_v V \to E|_v) \cap E^-|_v = \{0\} \quad \text{in } E|_v,
$$

$$
t|_v(E^-|_v) = t|_v(E|_v) \quad \text{in } F|_v.
$$

(3.22)

(3.23)
and the natural real linear map
\[ \Pi_v : E|_v \cap \ker(t|_v : E|_v \to F|_v) \to \frac{\ker \left( t|_v : E|_v \to F|_v \right)}{\im (ds|_v : T_v V \to E|_v)}, \tag{3.24} \]

which is injective by \(\text{[3.22]}\), has image \(\im \Pi_v\) a real vector subspace of dimension exactly half the real dimension of \(\ker(t|_v)/\im (ds|_v)\), and the real quadratic form \(\real \im \left( \tilde{Q}_v \right)\) on \(\ker(t|_v)/\im (ds|_v)\) from \(\text{[3.21]}\) restricts to a negative definite real quadratic form on \(\im \Pi_v\).

We say \((U, E^-)\) satisfies condition \((\dagger)\) if:

\((\dagger)\) \((U, E^-)\) satisfies condition \((*)\) and \(s^{-1}(0) \cap U = (s^+)^{-1}(0) \subseteq U\).

Then \((U, E^+, s^+, \psi^+)\) is a Kuranishi neighbourhood on \(X_{\text{an}}\).

Observe that if \(v \in s^{-1}(0) \cap U\) with \(\psi(v) = x \in X_{\text{an}}\) then using \(\text{[3.8]} - \text{[3.9]}\) and \(\text{[3.22]} - \text{[3.24]}\) we find there is an exact sequence
\[ 0 \longrightarrow H^0(T_X|_x) \longrightarrow T_v U \longrightarrow E^+_v \longrightarrow H^1(T_X|_x)/\im \Pi_v \longrightarrow 0. \tag{3.25} \]

Hence
\[
\dim_U U - \rank_E E^+ = \dim_{\real} H^0(T_X|_x) - \dim_{\real} H^1(T_X|_x) + \dim_{\real} \im \Pi_v
= 2 \dim_{\complex} H^0(T_X|_x) - \dim_{\complex} H^1(T_X|_x)
= \dim_{\complex} H^0(T_X|_x) - \dim_{\complex} H^1(T_X|_x) + \dim_{\complex} H^2(T_X|_x)
= \vdim_{\complex} X = n. \tag{3.26}
\]

Here in the second step we use \(\dim_{\real} \Pi_v = \frac{1}{2} \dim_{\real} H^1(T_X|_x)\) by \((*)\) and \(\text{[3.9]}\), in the third that \(H^0(T_X|_x) \cong H^2(T_X|_x)^*\) as \((X, \omega^*_X)\) is \(-2\)-shifted symplectic (or \(-2\)-shifted presymplectic will do), and in the fourth that \(T_X\) is perfect in the interval \([0, 2]\) as \((X, \omega^*_X)\) is \(-2\)-shifted symplectic (or presymplectic).

Equation \(\text{[3.20]}\) says that the Kuranishi neighbourhood \((U, E^+, s^+, \psi^+)\) has real virtual dimension \(\dim U - \rank E^+ = n = \vdim_{\complex} X = \frac{1}{2} \vdim_{\real} X\). Note that this is half the virtual dimension we might have expected, and the real virtual dimension can be odd, even though \(X, V, E, s, \ldots\) are all complex.

Note that we have now moved from complex geometry to real geometry: the real vector bundles \(E^\pm\) need not be holomorphic, and can have odd real rank. Here are some important properties of such \(U, E^-, E^+, s^+\), proved in \(\text{[3]}\)

**Theorem 3.9.** In the situation of Definition \(\text{[3.8]}\) with \(X, \omega^*_X, A^*, \alpha, V, E, F, s, t, \psi\) fixed, we have:

(a) If the conditions in \((*)\) hold at some \(v \in s^{-1}(0) \cap U\), then they also hold for all \(v'\) in an open neighbourhood of \(v\) in \(s^{-1}(0) \cap U\).

(b) Suppose \(C \subseteq V\) is closed, and \((U, E^-)\) satisfies condition \((*)\) with \(C \subseteq U \subseteq V\). (We allow \(C = U = \emptyset\).) Then there exists \((\tilde{U}, \tilde{E}^-)\) satisfying \((*)\) with \(C \cup s^{-1}(0) \subseteq \tilde{U} \subseteq V\), and an open neighbourhood \(U'\) of \(C\) in \(U \cap \tilde{U}\) such that \(E^-|_{U'} = \tilde{E}^-|_{U'}\).
(c) If \((U, E^-)\) satisfies \((\ast)\), the closed subsets \(s^{-1}(0) \cap U\) and \((s^+)^{-1}(0)\) in 
\(U \subseteq V\) coincide in an open neighbourhood \(U'\) of \(s^{-1}(0) \cap U\) in \(U\). Hence
\((U', E^-|_{U'})\) satisfies condition \((\dagger)\), and \((U', E^+|_{U'}, s^+|_{U'}, \psi^+)\) is a Kuranishi
 neighbourhood on \(X_{an}\). Thus, we can make \((U, E^-)\) satisfying \((\ast)\) also satisfy
\((\dagger)\) by shrinking \(U\), without changing \(R = \text{Im} \psi\) in \(X_{an}\).

The next example proves Theorem 3.9(c) near \(v \in s^{-1}(0) \cap U\) in a special
case, when \((A^*, \omega_{A^*})\) is in \(-2\)-Darboux form and minimal at \(v\), as in Definition
3.8 and Theorem 2.9. The general case in 3.10 is proved by reducing to Example
3.10.

**Example 3.10.** Suppose \((X, \omega_X)\) is a \(-2\)-shifted symplectic derived \(\mathbb{C}\)-scheme,
and \(x \in X_{an}\). Then Theorem 2.9 gives a pair \((A^*, \omega_{A^*})\) in \(-2\)-Darboux form
and a Zariski open inclusion \(\alpha : \text{Spec} A^* \to X\) which is minimal at \(x \in \text{Im} \alpha\),
with \(\alpha^*(\omega_X^*) \simeq \omega_{A^*}\) in \(A^*_c(\text{Spec} A^*, -2)\).

Example 3.5 describes the data \(V, E, F, s, t\) associated to \(A^*\) in 3.2, and
defines a nondegenerate quadratic form \(Q \in H^0(S^2E^*)\) with \(Q(s, s) = 0\) using
\(\omega_{A^*}\). As \(x \in \text{Im} \alpha\) there is \(v \in s^{-1}(0) \subseteq V\) with \(\alpha(v) = x\), and \((A^*, \alpha)\) minimal
at \(x\) means that \(ds|_v = 0\), so that \(t|_v = 0\) by (3.13). Thus in (3.9) we have
\(\text{Ker}(t|_v)/\text{Im}(ds|_v) = E|_v\), identified with \(H^1(T_X|_v)\). Since
\(\alpha^*(\omega_X^*) \simeq \omega_{A^*}\), the quadratic form \(Q_v\) on \(\text{Ker}(t|_v)/\text{Im}(ds|_v) = E|_v\) in (3.21)
is \(Q|_v\).

Given a pair \((U, E^-)\) as in Definition 3.8 with \(v \in U\), the map \(\Pi_v\) in (3.23)
is just the inclusion \(E^-|_v \to E|_v\). So \((\ast)\) at \(v\) says that \(E^-|_v\) is a real vector
subspace of \(E|_v\) with \(\dim_{\mathbb{R}} E^-|_v = \frac{1}{2} \dim_{\mathbb{R}} E|_v = \dim_{\mathbb{C}} E|_v\), such that \(\text{Re} Q|_v\) is
negative definite on \(E^-|_v\).

As this is an open condition, there exists an open neighbourhood \(U'\) of \(v\) in \(U\)
such that \(\text{Re} Q|_{U'}\) is negative definite on \(E^-|_{U'}\). Define a real vector
subbundle \(\bar{E}^+\) of \(E|_{U'}\) to be the orthogonal subbundle of \(E^-|_{U'}\) w.r.t. the non-
degenerate real quadratic form \(\text{Re} Q|_{U'}\). Then \(E|_{U'} = \bar{E}^+ \oplus E^{-}|_{U'}\), so we can
write \(s|_{U'} = \bar{s}^+ \oplus s^-\), for \(\bar{s}^+ \in C^\infty(\bar{E}^+)\) and \(s^- \in C^\infty(E^-|_{U'})\). The projection
\(E|_{U'} \to E^+|_{U'} = E|_{U'}/E^-|_{U'}\) restricts to an isomorphism \(\bar{E}^+ \to E^+|_{U'}\), which
maps \(\bar{s}^+ \mapsto s^+|_{U'}\).

Because \(\text{Re} Q\) is the real part of a complex form, it has the same number of
positive as negative eigenvalues. Thus \(\text{Re} Q|_{U'}\) is positive definite on \(\bar{E}^+\). Now
\[
0 = \text{Re} Q(s, s)|_{U'} = \text{Re} Q(\bar{s}^+ + s^-) = \text{Re} Q(\bar{s}^+, \bar{s}^+) + \text{Re} Q(s^-, s^-),
\]
using \(\text{Re} Q(s^+, s^-) = 0\) as \(\bar{E}^+, E^-|_{U'}\) are orthogonal w.r.t. \(\text{Re} Q|_{U'}\).

For each \(u \in U'\), we now have
\[
s^+(u) = 0 \iff \bar{s}^+(u) = 0 \iff \text{Re} Q(\bar{s}^+, \bar{s}^+)|_u = 0 \iff \text{Re} Q(s^-, s^-)|_u = 0 \iff \bar{s}^+(u) = s^-(u) = 0 \iff s(u) = 0,
\]
using \(\bar{E}^+ \to E^+|_{U'}\) an isomorphism mapping \(\bar{s}^+ \mapsto s^+|_{U'}\) in the first step, \(\text{Re} Q\)
positive definite on \(\bar{E}^+\) in the second, (3.27) in the third, \(\text{Re} Q \) negative definite
on \(E^-|_{U'}\) in the fourth, and \(s|_{U'} = \bar{s}^+ \oplus s^-\) in the fifth.
This proves there exists an open neighbourhood $U'$ of $v$ in $U$ such that $s^{-1}(0) \cap U' = (s^+)^{-1}(0) \cap U'$, which is Theorem 3.9(c), except that $U'$ is a neighbourhood of $v$ rather than of $s^{-1}(0) \cap U$.

Remark 3.11. Pairs $(U, E^-)$ satisfying $(\dagger)$ will be used to prove our main result, constructing a derived manifold structure $X_{dm}$ on the complex analytic topological space $X_{an}$ of a $-2$-shifted symplectic derived $\mathbb{C}$-scheme $(X, \omega_X^*)$.

Our construction apparently uses less than the full $-2$-shifted symplectic structure $\omega_X^*$ on $X$. In particular, conditions $(\ast), (\dagger)$ only involve the non-degenerate pairings $\omega_X^0|_{x}$ on $H^1(T_{X|x})$ in (2.7), which depend only on the presymplectic structure $\omega_X^0$, not the symplectic structure $\omega_X^* = (\omega_X^0, \omega_X^1, \ldots)$. The proofs of Theorem 3.9(a), (b) in §5.1–5.2 also use only $\omega_X^0$ rather than $\omega_X^*$.

However, the proof of Theorem 3.9(c) in §5.3 involves $\omega_X^*$, as it uses the existence of a minimal $-2$-Darboux form presentation for $(X, \omega_X^*)$ near each $x \in X_{an}$, as in Definition 3.8 and Theorem 2.4. The authors do not know whether Theorem 3.9(c) holds for $-2$-shifted presymplectic $(X, \omega_X^0)$ which are not symplectic.

For $-1$-shifted symplectic derived schemes, the second author [23] defined a classical truncation called ‘d-critical loci’. In subsequent papers with Ben-Bassat, Brav, Bussi, Dupont, Meinhardt, and Szendrői [3–6] which constructed natural perverse sheaves, D-modules, mixed Hodge modules, and motives on $-1$-shifted symplectic derived schemes, we showed the constructions actually factored through this classical truncation.

In the same way, the authors expect that there should exist a new geometric structure which is a classical truncation of $-2$-shifted symplectic derived $\mathbb{C}$-schemes, similar in spirit to [23], and that our construction of $X_{dm}$ factors through this classical truncation. We hope to address this in a future paper.

### 3.4 Comparing $(U_J, E^-_J), (U_K, E^-_K)$ under a submersion $\phi_{JK}$

Section 3.3 discussed how to use standard form charts $\alpha : \text{Spec} A^* \to X$ on $(X, \omega_X^*)$ to choose pairs $(U, E^-)$, and so define Kuranishi neighbourhoods $(U, E^+, s^+, \psi^+)$ on $X_{an}$. We now explain how to pull back such pairs $(U_K, E^-_K)$ along a submersion $\Phi_{JK} : A^*_K \to A^*_J$, and construct coordinate changes between the Kuranishi neighbourhoods $(U_J, E^+_J, s^+_J, \psi^+_J), (U_K, E^+_K, s^+_K, \psi^+_K)$.

**Definition 3.12.** Let $(X, \omega_X^*)$ be a $-2$-shifted symplectic derived $\mathbb{C}$-scheme with $\text{vdim}_{\mathbb{C}} X = n$, and suppose $\Phi_{JK} : A^*_K \to A^*_J$ is a submersion of standard form cdgas over $\mathbb{C}$ and $\alpha_J : \text{Spec} A^*_J \hookrightarrow X$, $\alpha_K : \text{Spec} A^*_K \to X$ are Zariski open inclusions such that (3.3) homotopy commutes. Define complex geometric data $V_J, E_J, F_J, s_J, t_J, \psi_J, R_J, V_K, E_K, F_K, s_K, t_K, \psi_K, R_K, \phi_{JK}, \chi_{JK}, \xi_{JK}$ in Definitions 3.4 and 3.9 and suppose $R_J \neq \emptyset$, so $R_K \neq \emptyset$ as $R_J \subseteq R_K \subseteq X_{an}$.

Consider pairs $(U_J, E^-_J)$ for $A^*_J$ and $(U_K, E^-_K)$ for $A^*_K$ satisfying condition $(\ast)$ in Definition 3.8. We say that $(U_J, E^-_J)$ and $(U_K, E^-_K)$ are compatible if $\phi_{JK}(U_J) \subseteq U_K$ and $\chi_{JK}|_{U_J}(E^-_J) \subseteq \phi_{JK}|_{U_J}(E^-_K) \subseteq \phi_{JK}|_{U_J}(E_K)$.
For $(U_J, E_J^+)$, $(U_K, E_K^-)$ compatible, define a vector bundle morphism $\chi^+_J : E_J^+ \to \phi_{JK} |_{U_J}(E_K^-)$ on $U_J$ by the commutative diagram with exact rows:

$$
\begin{array}{cccccc}
0 & \to & E_J^+ & \to & E_J^+ |_{U_J} & \to & E_J^+ |_{v_J} \oplus \chi^+_J |_{v_J} & \to & 0 \\
& & \downarrow{\chi_{JK}|_{E_J}} & & \downarrow{\chi_{JK}|_{v_J}} & & \downarrow{\chi^+_J} & & \\
0 & \to & \phi_{JK} |_{U_J}(E_K^-) & \to & \phi_{JK} |_{U_J}(E_K^-) & \to & \phi_{JK} |_{U_J}(E_K^-) & \to & 0.
\end{array}
$$

Let $v_J \in s^{-1}_J(0) \subseteq U_J \subseteq V_J$ with $\phi_{JK}(v_J) = v_K \in s^{-1}_K(0) \subseteq U_K \subseteq V_K$ and $\psi_J(v_J) = \psi_K(v_K) = x \in X_\text{an}$. Consider the diagram, with rows (3.28) for $(U_J, E_J^+), v_J$ and $(U_K, E_K^-), v_K$:

$$
\begin{align*}
0 & \to H^0(T_X|_x) \to T_{v_J}U_J \xrightarrow{\partial s_J |_{v_J}} E_J^+ |_{v_J} \to H^1(T_X|_x)/\text{Im } \Pi_{v_J} & \to 0 \\
& \quad \downarrow{\text{id}} \quad \downarrow{\partial s_J |_{v_J}} \quad \downarrow{\chi^+_J |_{v_J}} \quad \downarrow{\text{id}} \\
0 & \to H^0(T_X|_x) \to T_{v_K}U_K \xrightarrow{\partial s_K |_{v_K}} E_K^- |_{v_K} \to H^1(T_X|_x)/\text{Im } \Pi_{v_K} & \to 0.
\end{align*}
$$

Here if we regard $\text{Im } \Pi_{v_J}, \text{Im } \Pi_{v_K}$ from (3.24) as subspaces of $H^1(T_X|_x)$ using (3.29), compatibility $\chi_{JK}(E_J |_{v_J}) \subseteq E_K^{-} |_{v_K}$ and (3.18) imply that $\text{Im } \Pi_{v_J} \subseteq \text{Im } \Pi_{v_K}$, so $\text{Im } \Pi_{v_J} = \text{Im } \Pi_{v_K}$ as they have the same dimension by (*), and the right hand column of (3.28) makes sense. From (3.14), (3.17) and (3.18) we see that (3.28) commutes. Elementary linear algebra then gives an exact sequence:

$$
\begin{align*}
0 & \to T_{v_J}U_J \xrightarrow{\partial s_J |_{v_J} \oplus \partial \phi_{JK} |_{v_J}} E_J^+ |_{v_J} \oplus \chi^+_J |_{v_J} \oplus \partial s_K |_{v_K} & \to E_K^- |_{v_K} & \to 0.
\end{align*}
$$

From (3.29) and Definition 2.14 we deduce:

**Corollary 3.13.** In the situation of Definition 3.12, if $(U_J, E_J^+)$ and $(U_K, E_K^-)$ are compatible and satisfy ($\dagger$) then in the sense of (3.25)

$$(U_J, \phi_{JK} |_{U_J}, \chi^+_J) : (U_J, E_J^+, s^+_J, \psi_J) \to (U_K, E_K^+, s^+_K, \psi_K)$$

is a coordinate change of Kuranishi neighbourhoods on $X_\text{an}$.

The next lemma is easy to prove. We deduce $\nabla s_J$ is injective on $\text{Ker } \partial \phi_{JK}$ at $v_J \in s^{-1}_J(0)$ using (3.17), check that ($*$) for $U_J, E_J^+$ is equivalent to $E_J = E_J^+ \oplus E_J''$ at each $v_J \in s^{-1}_J(0)$, and note that both are open conditions.

**Lemma 3.14.** In the situation of Definition 3.12 fix $(U_K, E_K^-)$ satisfying ($*$) for $A^*_K, \alpha_K$. Set $U_J'' = \phi_{JK}^{-1}(U_K) \subseteq V_J$. Then $E_J'' := \chi_{JK}^{-1} |_{U_J''}(E_K^-)$ is a vector subbundle of $E_J |_{U_J''}$, as $\chi_{JK}$ is surjective. Choose a complementary real vector subbundle $E_J''$, so that $E_J |_{U_J''} = E_J' \oplus E_J''$.

Choose a connection $\nabla$ on $E_J$, so that $\nabla s_J : TV_J \to E_J$ is a vector bundle morphism. Now $\text{Ker } (\phi_{JK} : TV_J \to \phi_{JK}^*(TV_K))$ is a vector subbundle of $TV_J$, as $\phi_{JK}$ is surjective, and $\nabla s_J$ is injective on $\text{Ker } \phi_{JK} |_{s^{-1}_J(0)}$, so $E_J'' := (\nabla s_J)[\text{Ker } \phi_{JK}]$ is a vector subbundle of $E_J |_{s^{-1}_J(0)}$ in $V_J$.  

31
Then \((U_J, E_J^{-})\) satisfies \((*)\) for \(A_j^*\), \(\alpha_j\) and is compatible with \((U_K, E_K^{-})\) if and only if \(U_J\) is open in \(U_J^\prime_k\), and \(E_J^\prime_k\) is a vector subbundle of \(E_J^\prime_k\) satisfying \(E_J^\prime_k|U_J = E_J^\prime_k - E_J^\prime_k U_J + E_J^\prime_k U_J\) near \(s^{-1}_J(0) \cap U_J\) in \(U_J\). Alternatively, identifying \(E_J^\prime_k\) with \(E_J^\prime_k U_J|/E_J^\prime_k\), this condition may be written as \(E_J^\prime_k|U_J = E_J^\prime_k - E_J^\prime_k U_J + E_J^\prime_k U_J\) near \(s^{-1}_J(0) \cap U_J\).

Lemma 3.14 shows we can always pullback \((U_K, E_K^{-})\) satisfying \((*)\) along submersions \(\phi_{JK} : V_J \to V_K\); we just have to choose a complement \(E_J^{-}\) to \((E_J^\prime_k \cap E_J^\prime_k)/E_J^\prime_k\) in \(E_J^\prime_k\) on some small open neighbourhood \(U_J\) of \(s^{-1}_J(0)\) in \(U_J^\prime_k\), for instance, the orthogonal complement w.r.t. any metric on \(E_J^\prime_k\). By Theorem 3.9(c), making \(U_J\) smaller, we can suppose \((U_J, E_J^{-})\) satisfies \((\dagger)\).

### 3.5 Derived manifolds from \(-2\)-shifted symplectic derived \(\mathbb{C}\)-schemes

Let \((X, \omega_X^\bullet)\) be a \(-2\)-shifted symplectic derived \(\mathbb{C}\)-scheme with \(\text{vdim}_\mathbb{C} X = n\) in \(\mathbb{Z}\), and write \(X_\text{an}\) for the set of \(\mathbb{C}\)-points of \(X = t_0(X)\) with the complex analytic topology. Suppose \(X\) is separated, so that \(X_\text{an}\) is Hausdorff, and also that \(X_\text{an}\) is a second countable topological space (a necessary and sufficient condition for this is that \(X\) admits a Zariski open cover \(\{X_c : c \in C\}\) with \(C\) countable and \(X\), a finite type \(\mathbb{C}\)-scheme). If \(X\) is proper (equivalently, \(X_\text{an}\) is compact) then \(X_\text{an}\) is automatically second countable.

This section will prove one of our main results, Theorem 3.18 below, that we can give \(X_\text{an}\), the structure of a derived manifold \(X_{\text{dm}}\) (in any of senses (i)–(iv)) in \([2, 3]\) with \(\text{vdim}_\mathbb{C} X_{\text{dm}} = n\).

First choose a family \(\{(A_i^*, \alpha_i) : i \in I\}\), where \(A_i^* \in \text{cdga}_\mathbb{C}\) is a standard form cdga, and \(\alpha_i : \text{Spec} A_i^* \to X\) a Zariski open inclusion in \(\text{dSch}_\mathbb{C}\) for each \(i \in I\), an indexing set, such that \(\{R_i := (\text{Im} \alpha_i)_{\text{an}} : i \in I\}\) is an open cover of the complex analytic topological space \(X_\text{an}\). This is possible by Theorem 2.3.

Apply Theorem 3.2 to get data \(A_i^* \in \text{cdga}_\mathbb{C}\), \(\alpha_j : \text{Spec} A_i^* \to X\) for finite \(\emptyset \neq J \subseteq I\) and submersions \(\Phi_{JK} : A_J^* \to A_K^*\) for all finite \(\emptyset \neq K \subseteq J \subseteq I\).

Use the notation of 3.2 to rewrite \(A_j^*\), \(\Phi_{JK}\) in terms of complex geometry. As in Corollary 3.7, this gives data \(V_J, E_J, E_J, s_J, t_J, \psi_J, R_J\) for all finite \(\emptyset \neq J \subseteq I\), and \(\phi_{JK}, \chi_{JK}, \xi_{JK}\) for all finite \(\emptyset \neq K \subseteq J \subseteq I\).

For brevity we write \(A = \{J : \emptyset \neq J \subseteq I, J \text{ is finite}\}\).

Recall that a topological space \(Z\) is paracompact if every open cover of \(Z\) admits a locally finite refinement, and normal if whenever \(C, D \subseteq Z\) are closed with \(C \cap D = \emptyset\) there exist open \(T, U \subseteq Z\) with \(C \subseteq T, D \subseteq U\) and \(T \cap U \neq \emptyset\). We will need the following result, proved in [6, 1].

**Proposition 3.15.** Suppose \(Z\) is a paracompact, normal topological space and \(\{R_i : i \in I\}\) an open cover of \(Z\). Then we can choose closed subsets \(C_J \subseteq Z\) for all \(\emptyset \neq J \subseteq I\), satisfying:

(i) \(C_J \subseteq \bigcap_{i \in J} R_i\) for all \(J\).

(ii) Each \(z \in Z\) has an open neighbourhood \(U_z \subseteq Z\) with \(U_z \cap C_J \neq \emptyset\) for only finitely many \(J\).
(iii) \( C_J \cap C_K \neq \emptyset \) only if \( J \subseteq K \) or \( K \subseteq J \).

(iv) \( \bigcup_{\emptyset \neq J \subseteq I} C_J = Z \).

**Remark 3.16.** Results similar to Proposition 3.15 have been used for a long time in the construction of virtual cycles for moduli spaces of \( J \)-holomorphic curves in symplectic geometry. See in particular McDuff and Wehrheim [30, Lem. 7.1.7], on which our proof in §6 is based, and Fukaya et al. [17, Th. 2.7].

In our case, \( X_{an} \) is Hausdorff and second countable. It is also locally compact and regular, as it is locally homeomorphic to closed subsets \( s_J^{-1}(0) \) of complex manifolds \( V_J \). But locally compact and second countable imply paracompact, and paracompact and regular imply normal, so \( X_{an} \) is paracompact and normal. Thus Proposition 3.15 applies to \( X = X_{an} \) with the open cover \( \{ R_i : i \in I \} \), and we can choose closed subsets \( C_J \subseteq R_J = \bigcap_{i \in J} R_i \subseteq X_{an} \) for all \( J \in A \) satisfying conditions (i)–(iv).

The next proposition, proved in §6.2 using Theorem 3.9 and Lemma 3.14, chooses pairs \((U_J, E^+_J)\) satisfying (†), as in §3.3 with \((U_J, E^-_J), (U_K, E^-_K)\) compatible near \( C_J \cap C_K \) under the submersions \( \Phi_{JK} : A^*_K \to A^*_J \).

**Proposition 3.17.** In the situation above, we can choose \((U_J, E^+_J)\) satisfying condition (†) for \( V_J, E_J, \ldots \) for each \( J \in A \), such that \( \psi_J^{-1}(C_J) \subseteq U_J \subseteq V_J \), and setting \( S_J = \psi_J(s_J^{-1}(0) \cap U_J) \) so that \( S_J \) is an open neighbourhood of \( C_J \) in \( X_{an} \), then for all \( J, K \in A \), we have \( S_J \cap S_K \neq \emptyset \) only if \( J \subseteq K \) or \( K \subseteq J \), and if \( K \subseteq J \) then there exists open \( U_{JK} \subseteq U_J \) with \( s_J^{-1}(0) \cap U_{JK} = \psi_J^{-1}(S_J \cap S_K) \) such that \((U_{JK}, E^+_J |_{U_{JK}})\) is compatible with \((U_K, E^-_K)\), in the sense of §3.3.

We can now prove a central result of this paper. We will discuss the dependence of \( X_{dm} \) on choices made in the construction in §3.6.

**Theorem 3.18.** Let \((X, \omega_X^e)\) be a \(-2\)-shifted symplectic derived \( \mathbb{C} \)-scheme with complex virtual dimension \( \text{vdim}_C X = n \in \mathbb{Z} \), and write \( X_{an} \) for the set of \( \mathbb{C} \)-points of \( X = t_0(X) \) with the complex analytic topology. Suppose that \( X \) is separated, so that \( X_{an} \) is Hausdorff, and also that \( X_{an} \) is a second countable topological space, which holds if and only if \( X \) admits a Zariski open cover \( \{ X_c : c \in C \} \) with \( C \) countable and each \( X_c \) a finite type \( \mathbb{C} \)-scheme.

Then we can make the topological space \( X_{an} \) into a derived manifold \( X_{dm} \) with real virtual dimension \( \text{vdim}_C X_{dm} = n \), in any of the senses (i) Spi-vak’s derived manifolds \( \text{DerMan}_{\mathbb{Sp}} \) [32], (ii) Borisov–Noël’s derived manifolds \( \text{DerMan}_{\mathbb{BN}} \) [8], (iii) Joyce’s \( d \)-manifolds \( \text{dMan} \) [21, 22], or (iv) Joyce’s \( M \)-Kuranishi spaces \( \text{MKur} \) [23, §2], and Joyce’s Kuranishi spaces with trivial orbifold groups \( \text{Kur}_{\text{trG}} \) [23, §4], all discussed in §2.3.

**Proof.** In the discussion from the beginning of §3.5 up to Proposition 3.17 we have constructed the following data:

(a) A Hausdorff, second countable topological space \( X_{an} \).

(b) An indexing set \( I \), where we write \( A = \{ J : \emptyset \neq J \subseteq I, J \text{ is finite} \} \).
(c) An open cover \( \{ S_J : J \in A \} \) of \( X_{\text{an}} \), such that \( S_J \cap S_K \neq \emptyset \) for \( J, K \in A \) only if \( J \subseteq K \) or \( K \subseteq J \).

(d) For each \( J \in A \), a Kuranishi neighbourhood \( (U_J, E^+_J, s_J^+, \psi^+_J) \) on \( X_{\text{an}} \) with \( \dim U_J - \text{rank } E^+_J = n \), constructed as in §3.3 from \( (U_J, E^-_J) \) satisfying (1), with \( \text{Im } \psi^+_J = S_J \subseteq X_{\text{an}} \).

(e) For all \( J, K \in A \) with \( K \subseteq J \), a coordinate change of Kuranishi neighbourhoods over \( S_J \cap S_K \), as in Corollary 3.13

\[
(U_{JK}, \phi_{JK}|_{U_{JK}, \chi^+_{JK}}) : (U_J, E^+_J, s_J^+, \psi^+_J) \rightarrow (U_K, E^+_K, s_K^+, \psi^+_K),
\]

since \( (U_{JK}, E^-_J|_{U_{JK}}) \) is compatible with \( (U_K, E^-_K) \).

(f) For all \( J, K, L \in A \) with \( L \subseteq K \subseteq J \), Corollary 3.7 implies that \( \phi_{JL} = \phi_{KL} \circ \phi_{JK} \) and \( \chi^+_{JL} = \phi^*_{JK} \chi^+_{KL} \circ \chi^+_{JK} \) on \( U_{JK} \cap U_{JL} \cap U_{KL} \).

All this is sufficient to build a derived manifold \( X_{dm} \).

For d-manifolds, [21 Th. 4.16] or [22 Th. 4.17] or [23 Th. 3.42] imply that there is a d-manifold \( X_{dm} \) with topological space \( X_{an} \) and \( \text{vdim}_R X_{dm} = n \), unique up to equivalence in the 2-category \( \text{dMan} \), such that \( X_{dm} \) has an open cover \( \{ S_J : J \in A \} \) corresponding to \( \{ S_J : J \in A \} \) for \( X_{an} \), and for all \( J \in A \) there are equivalences \( \psi^+_J : S_{U_J, E^+_J, s_J^+} \rightarrow S_J \) in \( \text{dMan} \) with underlying continuous map \( \psi^+_J \), where \( S_{U_J, E^+_J, s_J^+} \) is a ‘standard model’ d-manifold, which for \( J, K \in A \) with \( K \subseteq J \) are compatible over \( S_J \cap S_K \) with a ‘standard model’ 1-morphism \( S_{U_{JK}, E^+_J, s_J^+} \rightarrow S_{U_K, E^+_K, s^+_K} \).

For Borisov–Noël derived manifolds, Borisov [8] produces a 1-1 correspondence between equivalence classes of objects in \( \text{dMan} \) and \( \text{DerMan}_{\text{BoNo}} \). So the d-manifold \( X_{dm} \) corresponds to a Borisov–Noël derived manifold \( X'_{dm} \), unique up to equivalence in \( \text{DerMan}_{\text{BoNo}} \).

For Spivak’s derived manifolds, Borisov and Noël [7] defined an equivalence of \( \infty \)-categories \( \text{DerMan}_{\text{Spi}} \simeq \text{DerMan}_{\text{BoNo}} \). So \( X'_{dm} \) corresponds to \( X''_{dm} \in \text{DerMan}_{\text{Spi}} \), uniquely up to equivalence.

There is also an independent construction of \( X_{dm} \) as an (M-)Kuranishi space. It follows from [24 §2.1–§2.3] that there is a unique M-Kuranishi structure \( \mathcal{K} = (\mathcal{K}, (U_J, E^+_J, s_J^+, \psi^+_J), J \in A, \Psi_{JKL} : \mathcal{K}_{JK} \rightarrow \mathcal{K}_{KL}) \) on \( X_{an} \) such that if \( J, K \in A \) with \( K \subseteq J \) then \( \Psi_{JKL} = [U_{JK}, \phi_{JK}|_{U_{JK}, \chi^+_{JK}}, \chi^+_{JK}] \), where \( [U_{JK}, \phi_{JK}|_{U_{JK}, \chi^+_{JK}}, \chi^+_{JK}] \) is the \( \sim \)-equivalence class of \( (U_{JK}, \phi_{JK}|_{U_{JK}, \chi^+_{JK}}, \chi^+_{JK}) \), in the sense of [24 §2.1].

Then \( X'''_{dm} = (X_{an}, \mathcal{K}) \) is an M-Kuranishi space in \( \text{MKur} \), with \( \text{vdim}_R X_{dm} = n \).

Then [25 Th. 4.60] implies that from \( X'''_{dm} \) we can obtain a Kuranishi space \( X''''_{dm} \) with trivial orbifold groups, unique up to equivalence in the 2-category \( \text{Kur}_{tr,G} \). Alternatively, we can build \( X''''_{dm} \) directly from the data (a)–(f) above by defining a ‘fair coordinate system’ on \( X_{an} \), as in [25 §4.8], and then applying [25 Th. 4.66] to get \( X''''_{dm} \), unique up to equivalence in \( \text{Kur}_{tr,G} \).

Under the equivalence \( \text{MKur} \simeq \text{Ho}(\text{dMan}) \) in Amorim and Joyce [1], \( X''''_{dm} \) is identified with \( X_{dm} \) up to isomorphism.

Note that \( X_{dm} \) in Theorem 3.18 has dimension \( \text{vdim}_R X_{dm} = \text{vdim}_C X = \frac{1}{2} \text{vdim}_R X \), which is exactly half what we might have expected. A heuristic
explanation, which we will not attempt to make precise, is that one should be able to make the derived \( \mathbb{C} \)-scheme \( X \) into a ‘derived \( C^\infty \)-scheme’ \( X^\infty \) (not a derived manifold), in some sense similar to Lurie [28, §4.5] or Spivak [32], and then \( (X^\infty, \Im \omega_X) \) should become a ‘real \(-2\)-shifted symplectic derived \( C^\infty \)-scheme’, where \( \Im \omega_X \) is the ‘imaginary part’ of \( \omega_X \).

Then there should be a morphism \( X^\infty \to X_{\text{dm}} \) which is a ‘Lagrangian fibration’ of \( (X^\infty, \Im \omega_X) \). So \( v\dim_r X_{\text{dm}} = \frac{1}{2} v\dim_r X^\infty = \frac{1}{2} v\dim_r X \), as for Lagrangian fibrations \( \pi : (S, \omega) \to B \) we have \( \dim B = \frac{1}{2} \dim S \).

### 3.6 Orientations and bordism classes

Work in the situation of Theorem 3.18 so that we have a \(-2\)-shifted symplectic derived \( \mathbb{C} \)-scheme \( (X, \omega_X) \) with complex analytic topological space \( X_{\text{an}} \), and a derived manifold \( X_{\text{dm}} \) with the same topological space \( X_{\text{an}} \).

Now \( [24] \) defined a notion of orientation of \( (X, \omega_X) \), corresponding to continuous sections of a topological principal \( \mathbb{Z}_2 \)-bundle \( \pi : O(X, \omega_X) \to X_{\text{an}} \). Also \( [26] \) defined a notion of orientation of \( X_{\text{dm}} \), corresponding to continuous sections of a topological principal \( \mathbb{Z}_2 \)-bundle \( \pi : P_{X_{\text{dm}}} \to X_{\text{an}} \). The next proposition, proved in \([3.6]\) justifies the definition of orientations of \( (X, \omega_X) \) in \([24]\).

**Proposition 3.19.** In the situation of Theorem 3.18 there is a canonical isomorphism of topological principal \( \mathbb{Z}_2 \)-bundles \( O(X, \omega_X) \cong P_{X_{\text{dm}}} \) over \( X_{\text{an}} \). Thus, there is a natural 1-1 correspondence between orientations on \( (X, \omega_X) \) in the sense of \([24]\) and orientations on \( X_{\text{dm}} \) in the sense of \([26]\).

Next we consider how the derived manifold \( X_{\text{dm}} \) in Theorem 3.18 depends on choices made in the construction. We explained in the proof of Theorem 3.18 that once we have chosen the data (a)–(f) listed there, then \( X_{\text{dm}} \) is determined up to equivalence / isomorphism in its (higher) category. However, (a)–(f) still involve many arbitrary choices, and the next theorem, proved in \([3.7]\), explains how \( X_{\text{dm}} \) depends on these. We use ideas on bordism from \([2.7]\).

**Theorem 3.20.** In the situation of Theorem 3.18, for \( (X, \omega_X) \), \( n \) fixed, the derived manifold \( X_{\text{dm}} \) depends on choices made in the construction only up to bordisms of derived manifolds which fix the underlying topological space \( X_{\text{an}} \).

That is, if \( X_{\text{dm}}, X'_{\text{dm}} \) are possible derived manifolds in Theorem 3.18, then we can construct a derived manifold with boundary \( W_{\text{dm}} \) with topological space \( X_{\text{an}} \times [0, 1] \) and \( \text{vdim}_r W_{\text{dm}} = n + 1 \), and an equivalence of derived manifolds \( \partial W_{\text{dm}} \cong X_{\text{dm}} \coprod X'_{\text{dm}} \), topologically identifying \( X_{\text{dm}} \) with \( X_{\text{an}} \times \{0\} \) and \( X'_{\text{dm}} \) with \( X_{\text{an}} \times \{1\} \). We regard \( W_{\text{dm}} \) as a bordism from \( X_{\text{dm}} \) to \( X'_{\text{dm}} \).

This bordism \( W_{\text{dm}} \) is compatible with orientations in Proposition 3.19. That is, there is a canonical isomorphism \( \pi^*_X (O_X, \omega_X) \cong P_{W_{\text{dm}}} \) of topological principal \( \mathbb{Z}_2 \)-bundles over \( X_{\text{an}} \times [0, 1] \), where \( \pi^*_X : X_{\text{an}} \times [0, 1] \to X_{\text{an}} \) is the projection. Under the equivalence \( \partial W_{\text{dm}} \cong X_{\text{dm}} \coprod X'_{\text{dm}} \), this restricts on \( X_{\text{an}} \times \{1\} \) to the isomorphism \( O_X, \omega_X \cong P_{X'_{\text{dm}}} \) for \( X'_{\text{dm}} \) in Proposition 3.19 and on \( X_{\text{an}} \times \{0\} \) to the isomorphism \( O_X, \omega_X \cong P_{X_{\text{dm}}} \) for \( X_{\text{dm}} \), multiplied by \(-1\).
Thus, given an orientation on \((X, \omega_X^*)\), we get natural orientations on \(X_{\mathrm{dm}}\), \(X'_{\mathrm{dm}}\), \(W_{\mathrm{dm}}\), and an equivalence of oriented derived manifolds \(\partial W_{\mathrm{dm}} \simeq -X_{\mathrm{dm}}\) II \(X'_{\mathrm{dm}}\), where \(-X_{\mathrm{dm}}\) is \(X_{\mathrm{dm}}\) with the opposite orientation.

Combining this with material in \([27]\) yields:

**Corollary 3.21.** Suppose \((X, \omega_X^*)\) is a proper \(-2\)-shifted symplectic derived \(\mathbb{C}\)-scheme, with \(\vdim X = n\), and with an orientation in the sense of \([27]\). Then Theorem 3.18 constructs a compact derived manifold \(X_{\mathrm{dm}}\) with \(\vdim X_{\mathrm{dm}} = n\), and Proposition 3.19 defines an orientation on \(X_{\mathrm{dm}}\).

Although \(X_{\mathrm{dm}}\) depends on arbitrary choices, the \(d\)-bordism class \([X_{\mathrm{dm}}]\) in \(dB_n(*) \cong B_n(*)\) from \([27]\) is independent of these, and depends only on \((X, \omega_X^*)\) and its orientation. We regard \([X_{\mathrm{dm}}]\) as a virtual class for \((X, \omega_X^*)\).

If \(n = 0\) then \(dB_n(*) \cong \mathbb{Z}\), and \([X_{\mathrm{dm}}]\) is a ‘virtual count’ of \((X, \omega_X^*)\).

Here the assumption that \(X\) is proper by definition includes \(X\) separated, so that \(X_{\mathrm{an}}\) is compact and Hausdorff, and as \(X = t_0(X)\) is locally of finite type, also implies that \(X_{\mathrm{an}}\) is second countable.

Note that \(n\) can be odd. So starting with the strictly complex algebraic input of \((X, \omega_X^*)\) and its orientation, we may construct a virtual class of odd real dimension, which seems rather strange.

### 3.7 Working relative to a smooth base \(\mathbb{C}\)-scheme \(Z\)

Let \(Z = \text{Spec } B\) be a smooth, connected, classical affine \(\mathbb{C}\)-scheme. Then the set \(Z_{\mathrm{an}}\) of \(\mathbb{C}\)-points of \(Z\) is a complex manifold, and hence a real manifold. In this section we will show that all of \([3.1\text{-}3.6]\) also works relatively over the base \(Z\).

To do this, we will need a notion of a family \((\pi : X \to Z, \omega_X^*)\) of \(-2\)-shifted symplectic derived \(\mathbb{C}\)-schemes over the base \(Z\).

To understand the next definition, recall from Remark 3.11 that if \((X, \omega_X^*)\) is \(-2\)-shifted symplectic, then the derived manifold \(X_{\mathrm{dm}}\) in constructed \((3.5)\) does not depend on the whole sequence \(\omega_X^* = (\omega_0^*, \omega_1^*, \ldots)\), but only on the nondegenerate pairings \(\omega_0^*|_x \in H^1(TX|_x)\) for \(x \in X_{\mathrm{an}}\), and therefore only on the cohomology class \([\omega_X^*] \in H^{-2}(L_X)\). We require that choices of \(\omega_X^*, \omega_{X,1}^*, \ldots\) should exist (they are required to apply Theorem 2.9 which is used in the proof of Theorem 3.19(c)), but \(X_{\mathrm{dm}}\) does not depend on them.

**Definition 3.22.** Let \(X\) be a derived \(\mathbb{C}\)-scheme, \(Z = \text{Spec } B\) a smooth, connected, classical affine \(\mathbb{C}\)-scheme, and \(\pi : X \to Z\) a morphism. A family of \(-2\)-shifted symplectic structures on \(X/Z\) is \([\omega_X^*] \in H^{-2}(L_X)\), such that for each \(z \in Z_{\mathrm{an}}\), writing \(X_z = \pi^{-1}(z) = X \times_{Z_{\mathrm{an}}, z} \text{Spec } \mathbb{C}\) for the fibre of \(\pi\) over \(z\) and \([\omega_X^*]|_{X_z} \in H^{-2}(L_{X_z})\) for the restriction of \([\omega_X^*] \to X_z\), then there should exist a \(-2\)-shifted symplectic structure \(\omega_{X_z}^* = (\omega_{X_z,1}^*, \omega_{X_z,2}^*, \ldots)\) on \(X_z\) such that \([\omega_X^*]|_{X_z} = [\omega_{X_z}^*] \in H^{-2}(L_{X_z})\).

That is, a family of \(-2\)-shifted symplectic structures on \(X/Z\) is a \(-2\)-shifted relative 2-form \([\omega_X^*] \to X/Z\), which on each fibre \(X_z\) extends to a closed 2-form which is \(-2\)-shifted symplectic. We will explain how to extend the arguments of \([3.3\text{-}3.6]\) to the relative case. Here is the analogue of Definition 3.8.
**Definition 3.23.** Let $X$ be a separated derived $\mathbb{C}$-scheme, $Z = \text{Spec} B$ a smooth, connected, classical affine $\mathbb{C}$-scheme, $\pi : X \to Z$ a morphism, and $[\omega_{X/Z}] \in H^{-2}(L_X/Z)$ a family of $-2$-shifted symplectic structures on $X/Z$. Write $\dim_\mathbb{C} Z = k$ and $\dim_\mathbb{C} X = n + k$. Suppose $A^* \in \text{cdga}_\mathbb{C}$ is of standard form, $\alpha : A^* \to X$ is a Zariski open inclusion, and $\beta : B \to A^0$ is a smooth morphism of $\mathbb{C}$-algebras, such that \ref{3.10} homotopy commutes. Define complex geometric data $V, \tau, E, F, s, t$ and $\psi : s^{-1}(0) \xrightarrow{\sim} R \subseteq X_{an}$ as in Definition \ref{3.4} and suppose $\mathcal{R} \neq \emptyset$. Then for each $v \in s^{-1}(0)$ with $\psi(v) = x \in X_{an}$ and $\tau(v) = \pi(x) = z \in Z_{an}$, equation \ref{3.12} gives an isomorphism from a vector space depending on $V, \tau, Z_{an}, E, F, s, t, \tau$ at $v$ to $H^1(T_{X/Z}|_x)$. As in \ref{2.7}, the relative 2-form $[\omega_{X/Z}]$ induces a pairing

$$Q_x := \omega_{X/Z|_x}^0 : H^1(T_{X/Z}|_x) \times H^1(T_{X/Z}|_x) \to \mathbb{C}, \quad (3.30)$$

which is nondegenerate as under the equivalence $T_{X/Z}|_x \simeq T_{x|_x}$, $Q_x$ is identified with the pairing induced by a $-2$-shifted symplectic form $\omega^*_{X_z}$ on $X_z$, as in Definition \ref{3.22}. Define

$$\bar{Q}_v : \frac{\ker(t|_v : E|_v \to F|_v)}{\text{Im}(ds|_v : T_v(V/Z_{an}) \to E|_v)} \times \frac{\ker(t|_v : E|_v \to F|_v)}{\text{Im}(ds|_v : T_v(V/Z_{an}) \to E|_v)} \to \mathbb{C} \quad (3.31)$$

to be the nondegenerate complex quadratic form identified with $Q_x$ in \ref{3.30} by the isomorphism $H^1(T_{\alpha|_v})$ in \ref{3.12}.

Consider pairs $(U, E^-)$, where $U \subseteq V$ is open and $E^-$ is a real vector subbundle of $E|_U$. Given such $(U, E^-)$, we write $E^+ = E|_U/E^-$ for the quotient vector bundle over $U$, and $s^+ \in C^\infty(E^+)$ for the image of $s|_U$ under the projection $E|_U \to E^+$, and $\psi^+ := \psi|_{s^+(0) \cap U} : s^+(0) \cap U \to X_{an}$. We say that $(U, E^-)$ satisfies condition $(\ast)$ if:

$(\ast)$ For each $v \in s^{-1}(0) \cap U$, we have

$$\text{Im}(ds|_v : T_v(V/Z_{an}) \to E|_v) \cap E^-|_v = \{0\} \quad \text{in } E|_v, \quad (3.32)$$

$$t|_v(E^-|_v) = t|_v(E|_v) \quad \text{in } F|_v, \quad (3.33)$$

and the natural real linear map

$$\Pi_v : E^-|_v \cap \ker(t|_v : E|_v \to F|_v) \to \frac{\ker(t|_v : E|_v \to F|_v)}{\text{Im}(ds|_v : T_v(V/Z_{an}) \to E|_v)} \quad (3.34)$$

which is injective by \ref{3.32}, has image $\text{Im} \Pi_v$ a real vector subspace of dimension exactly half the real dimension of $\ker(t|_v)/\text{Im}(ds|_v)$, and the real quadratic form $\text{Re} \bar{Q}_v$ on $\ker(t|_v)/\text{Im}(ds|_v)$ from \ref{3.31} restricts to a negative definite real quadratic form on $\text{Im} \Pi_v$.

We say $(U, E^-)$ satisfies condition $(\dagger)$ if:

$(\dagger)$ $(U, E^-)$ satisfies condition $(\ast)$ and $s^{-1}(0) \cap U = (s^+)^{-1}(0) \subseteq U$. 

37
Then $(U, E^+, s^+, \psi^+)$ is a Kuranishi neighbourhood on $X_{an}$.

Observe that if $v \in s^{-1}(0) \cap U$ with $\psi(v) = x \in X_{an}$ then using (3.11), (3.12) and (3.32) we find as for (3.25) that there is an exact sequence

$$0 \rightarrow H^0(T_{X/Z}|_x) \rightarrow T_v(V/Z_{an}) \rightarrow E^+_v \rightarrow H^1(T_{X/Z}|_x) / \text{Im} \Pi_v \rightarrow 0. \quad (3.35)$$

Hence as for (3.26) we have

$$\dim \mathbb{R} U - \dim \mathbb{R} Z_{an} - \text{rank}_{\mathbb{R}} E^+ = \dim \mathbb{R} H^0(T_{X/Z}|_x) - \dim \mathbb{R} H^1(T_{X/Z}|_x) + \dim \mathbb{R} \text{Im} \Pi_v$$

$$= 2 \dim_{\mathbb{C}} H^0(T_{X/Z}|_x) - \dim_{\mathbb{C}} H^1(T_{X/Z}|_x)$$

$$= \dim_{\mathbb{C}} H^0(T_{X/Z}|_x) - \dim_{\mathbb{C}} H^1(T_{X/Z}|_x) + \dim_{\mathbb{C}} H^2(T_{X/Z}|_x)$$

$$= v \dim_{\mathbb{C}} X - \dim_{\mathbb{C}} Z = n.$$

Thus the Kuranishi neighbourhood $(U, E^+, s^+, \psi^+)$ has virtual dimension

$$\dim U - \text{rank} E^+ = n + 2k = \frac{1}{2}(v \dim_{\mathbb{R}} X - \dim \mathbb{R} Z_{an}) + \dim \mathbb{R} Z_{an}, \quad (3.36)$$

which is the real dimension of the base $Z_{an}$, plus half the real virtual dimension of the fibres $X_z$. The real virtual dimension can be odd, even though $X, Z_{an}, V, E, s, \ldots$ are all complex.

Note that essentially the only important difference between Definitions 3.8 and 3.23 is that $T_v V$ in equations (3.21), (3.22), (3.24) is replaced by $T_v(V/Z_{an})$ in equations (3.31), (3.32), and (3.34).

**Theorem 3.24.** Theorem 3.9 holds with Definition 3.23 in place of Definition 3.8.

**Proof.** In the proofs of Theorem 3.9(a), (b) in §3.11–3.12 we replace $ds|_v : T_v V \rightarrow E|_v$ by $ds|_v : T_v(V/Z_{an}) \rightarrow E|_v$ throughout, and no other changes are needed.

For part (c), fix $z \in Z_{an}$, so that Definition 3.22 gives a $-2$-shifted symplectic derived $\mathbb{C}$-scheme $(X_z, \omega_{X_z})$ with $[\omega_{X/Z}|_{X_z}] = [\omega_{X/\mathbb{C}}]_x$ in $H^{-2}(L_{X_z})$. Consider the complex submanifolds $V_z = \tau^{-1}(z)$ in $V$ and $U_z = U \cap V_z$ in $U$, and write $E_z, F_z, s_z, t_z$ for the restrictions of $E, F, s, t$ to $V_z$, and $E^+_z, E^+_z, s^+_z, \psi^+_z$ for the restrictions of $E^-, E^+, s^+, \psi^+$ to $U_z$. Then $(X_z, \omega_{X_z}), V_z, E_z, \ldots$ satisfy Definition 3.8, so Theorem 3.9(c) shows that $s^{-1}_z(0) \cap U_z$ and $(s^+_z)^{-1}(0)$ coincide near $s^{-1}_z(0) \cap U_z$. Hence $(s^{-1}_z(0) \cap U) \cap \tau^{-1}(z)$ and $((s^+_z)^{-1}(0)) \cap \tau^{-1}(z)$ coincide near $(s^{-1}_z(0) \cap U) \cap \tau^{-1}(z)$ in $U$. As this holds for all $z \in Z_{an}$, $s^{-1}_z(0) \cap U$ and $(s^+_z)^{-1}(0)$ coincide near $s^{-1}(0) \cap U$ in $U$, and the theorem follows. \qed

When we extend 3.11 to the relative case, in the analogue of Definition 3.12 we also include data $\pi : X \rightarrow Z = \text{Spec} B$ and smooth $\beta_j : B \rightarrow A^0_j$, $\beta_K : B \rightarrow A^0_K$ with $\beta_j = \Phi_{JK} \circ \beta_K$ and (3.2) homotopy commuting for $J, K$. We obtain an analogue of (3.28) with rows (3.35) rather than (3.27), and so as for (3.29) we get an exact sequence

$$0 \rightarrow T_{v,J}(U_J/Z_{an}) \xrightarrow{\partial_s \psi_J \oplus \partial \phi_{JK} \mid_{v,J}} E^+_{v,J} \rightarrow T_{v,K}(U_K/Z_{an}) \xrightarrow{-\chi_{JK} \psi_J \oplus \partial s_{JK} \mid_{v,K}} E^+_{v,K} \rightarrow 0.$$
But by taking the direct sum of this with $\text{id} : T_2 Z_{\text{an}} \to T_2 Z_{\text{an}}$ in the second and third positions, we see that this implies (3.24) is exact, and the analogue of Corollary 3.13 follows. The relative analogue of Lemma 3.14 in which we replace $TV_J, TV_K$ by $T(V_J/Z_{\text{an}}), T(V_K/Z_{\text{an}})$, is immediate.

For (3.5) we prove the following relative analogue of Theorem 3.18.

**Theorem 3.25.** (a) Let $X$ be a separated derived $\mathbb{C}$-scheme, $Z = \text{Spec} B$ a smooth, connected, classical affine $\mathbb{C}$-scheme, $\pi : X \to Z$ a morphism, and $[\omega_{X/Z}]$ a family of $-2$-shifted symplectic structures on $X/Z$, with $\dim_{\mathbb{C}} Z = k$ and $\text{vdim}_C X = n + k$. Write $X_{\text{an}}$ for the set of $\mathbb{C}$-points of $X = t_0(X)$ with the complex analytic topology, and suppose $X_{\text{an}}$ is second countable.

Then we can make the topological space $X_{\text{an}}$ into a derived manifold $X_{\text{dm}}$ with real virtual dimension $\text{vdim}_B X_{\text{dm}} = n + 2k$, in any of the senses (i) Spivak’s derived manifolds $\text{DerMans}_\text{spi}$, (ii) Borisov–Noël’s derived manifolds $\text{DerMans}_{\text{BN}}$, (iii) Joyce’s $d$-manifolds $\text{dMan}$, or (iv) Joyce’s $M$-Kuranishi spaces $\text{MKur}$, all discussed in [25], §4, and Joyce’s Kuranishi spaces with trivial orbifold groups $\text{Kur}_{\text{trG}}$ [28] §4, all discussed in [25].

(b) We can also define a morphism of derived manifolds $\pi_{\text{dm}} : X_{\text{dm}} \to Z_{\text{an}}$ to the complex manifold $Z_{\text{an}}$ regarded as a real manifold, and hence a derived manifold, with underlying continuous map $\pi_{\text{an}} : X_{\text{an}} \to Z_{\text{an}}$.

(c) For each $z \in Z_{\text{an}}$, the fibre $X_{\text{dm}, z} = \pi_{\text{dm}}^{-1}(z) = X_{\text{dm}} \times_{\pi_{\text{dm}}, Z_{\text{an}}, z} *$ is a derived manifold with $\text{vdim}_B X_{\text{dm}, z} = n$. From Definition 3.22, $X_{z} = \pi_{\text{dm}}^{-1}(z)$ has a $-2$-shifted symplectic structure $\omega_{X_{z}}^\bullet$, and both $X_{\text{dm}, z}, X_{z}$ have (complex analytic) topological space $\pi_{\text{dm}}^{-1}(z) \subseteq X_{\text{an}}$. Then $X_{\text{dm}, z}$ is up to equivalence a possible choice for the derived manifold associated to $(X_{z}, \omega_{X_{z}}^\bullet)$ in Theorem 3.18.

**Proof.** First choose a family $\{ (A_i^*, \alpha_i, \beta_i) : i \in I \}$, where $A_i^* \in \text{cdga}_C$ is a standard form cdga, and $\alpha_i : \text{Spec} A_i^* \to X$ is a Zariski open inclusion in $\text{dSch}_C$ for each $i$ in $I$, an indexing set, and $\beta_i : B \to A_i^0$ is a smooth morphism of classical $\mathbb{C}$-algebras such that (3.31) homotopy commutes, with $\{ R_i := (\text{Im} \alpha_i)_{\text{an}} : i \in I \}$ an open cover of the complex analytic topological space $X_{\text{an}}$. This is possible by a relative version of Theorem 2.23, easily proved by modifying the proof of [1] Th. 4.1] to work over the base $Z = \text{Spec} B$.

Apply Theorem 3.2 to get data $\pi_{\text{dm}} \in \text{cdga}_C$, $\alpha_i : \text{Spec} A_i^* \to X$, $\beta_i : B \to A_i^0$ for finite $\emptyset \neq J \subseteq I$ and submersions $\Phi_{JK} : A_K^* \to A_J^*$, for all finite $\emptyset \neq K \subseteq J \subseteq I$.

Use the notation of (3.22) to rewrite $A_i^*, \beta_i, \Phi_{JK}$ in terms of complex geometry. As in Corollary 3.7, this gives data $V_J, \tau_J, E_J, F_J, s_J, t_J, \psi_J, R_J$ for all finite $\emptyset \neq J \subseteq I$, and $\phi_{JK}, \chi_{JK}, \xi_{JK}$ for all finite $\emptyset \neq K \subseteq J \subseteq I$.

Proposition 3.17 now also holds in our relative situation. Its proof in [6.2] uses Theorem 3.4 and Lemma 3.14 which as above hold in the relative situation with Definition 3.22 and $T(V_J/Z_{\text{an}})$ in place of Definition 3.3 and $TV_J$.

The proof of Theorem 3.18 in [3.3] now builds the derived manifold $X_{\text{dm}}$ using the data $I, A, U_J, E_J^+, s_J^+, \psi_J^+, \ldots$. The only change is that in part (d) of the proof we have $\dim U_J - \text{rank} E_J^+ = n + 2k$ by (3.36), so that $\text{vdim}_B X_{\text{dm}} = n + 2k$. This proves Theorem 3.25 (a).
For part (b), note that by Corollary 3.7 we have holomorphic submersions
\( \tau_j : V_j \to Z_{\text{an}} \) for \( J \in A \) with \( \tau_K = \tau_j \circ \phi_{JK} \) for \( K \subseteq J \) in \( A \). The restrictions \( \tau|_{U_j} : U_j \to Z_{\text{an}} \) for \( J \in A \) are the data we need to build the morphism of derived manifolds \( \pi_{\text{dm}} : X_{\text{dm}} \to Z_{\text{an}} \), as in [21 Th. 4.16] or [22 Th. 4.17] or [23 Th. 3.42] for d-manifolds, or just by the definition of (1-)morphisms for M-Kuranishi spaces [25 §2.3] or Kuranishi spaces [25 §4.3].

For part (c), if \( z \in Z_{\text{an}} \) then as in (2.11). However, because \( \tau_j : V_j \to Z_{\text{an}} \) is a holomorphic submersion for \( J \in A \), the fibre \( V_{J,z} := \tau_j^{-1}(z) \) is a complex submanifold of \( V_j \). Setting \( U_{J,z} = U_j \cap V_{J,z} \) and writing \( E_{J,z}, F_{J,z}, s_{J,z}, t_{J,z} \) for the restrictions of \( E_j, F_j, s_j, t_j \) to \( V_{J,z} \), and \( E_{+J,z}, E_{-J,z}, s_{+J,z}, s_{-J,z}, \psi_{+J,z}, \psi_{-J,z} \) for the restrictions of \( E_{+j}, E_{-j}, s_{+j}, s_{-j}, \psi_{+j}, \psi_{-j} \) to \( U_{J,z} \), we see that \( I, A, V_{J,z}, E_{J,z}, F_{J,z}, s_{J,z}, t_{J,z}, U_{J,z}, E_{+J,z}, E_{-J,z}, s_{+J,z}, s_{-J,z}, \psi_{+J,z}, \psi_{-J,z} \) are a possible choice for the data \( I, A, V_j, E_j \), ... in the application of Theorem 3.18 to \((X_z, \omega^*_X)\). But from facts about fibre products of derived manifolds in [21 §4.6], [22 §4.6], [23 §4.3], [1] we see that the derived manifold \( X_{\text{dm},z} = X_{\text{dm}} \times_{\pi_{\text{dm}}, Z_{\text{an}}, z} \ast \) may be constructed as above from the data \( I, A, U_{J,z}, E_{J,z}, s_{J,z}, t_{J,z}, \psi_{J,z}, \ast \). The theorem follows.

Next we discuss orientations, generalizing [24 §4] and [36] to the relative case. Here is the analogue of Definition 2.1.

**Definition 3.26.** Let \( X \) be a derived C-scheme, \( Z = \text{Spec} B \) a smooth, connected, classical affine C-scheme, \( \pi : X \to Z \) a morphism, and \( [\omega_X/Z] \in H^{-2}(\mathcal{L}_X/Z) \) a family of \(-2\)-shifted symplectic structures on \( X/Z \). Then as in [24], \([\omega_X/Z]\) induces a canonical isomorphism of line bundles on \( X = t_0(X) \):

\[ \iota_{X/Z, \omega_X/Z} : [\det(\mathcal{L}_X/Z)|x]|^\otimes 2 \longrightarrow \mathcal{O}_X \cong \mathcal{O}_X^\otimes 2. \]

An orientation for \((\pi : X \to Z, [\omega_X/Z])\) is an isomorphism \( o : \det(\mathcal{L}_X/Z)|x| \to \mathcal{O}_X \) such that \( o \otimes o = \iota_{X/Z, \omega_X/Z} \).

The relative analogue of Proposition 3.19 holds. The proof in [40,3] should be modified in a straightforward way, by replacing \( T_X \) by \( T_{X/Z} \) and \( T_{V_j} \) by \( T_{V_{J,z}}(V_j/Z_{\text{an}}) \) throughout, and working with relative versions \( T_{\pi}(X_{\text{dm}}/Z_{\text{an}}) \), \( O_x(X_{\text{dm}}/Z_{\text{an}}) \) of \( T_{\pi}(X_{\text{an}}) \), \( O_x(X_{\text{an}}) \) in the exact sequence as in (2.13)

\[ 0 \longrightarrow T_{x}(X_{\text{dm}}/Z_{\text{an}}) \longrightarrow T_{V_j}(V_j/Z_{\text{an}}) \overset{dx^*_j|_{V_j}}{\longrightarrow} E^+_j|_{V_j} \longrightarrow O_x(X_{\text{dm}}/Z_{\text{an}}) \longrightarrow 0. \]

Actually, what corresponds to an orientation for \((\pi : X \to Z, [\omega_X/Z])\) is most naturally not an orientation for \( X_{\text{dm}} \), but a coorientation (relative orientation) for \( \pi_{\text{dm}} : X_{\text{dm}} \to Z_{\text{an}} \), which may be defined as in [20] as orientations on a real line bundle \( \det \mathcal{L}_{X_{\text{dm}}/Z_{\text{an}}} \) on the classical \( C^\infty \)-scheme \( X_{\text{dm}} \), with fibres

\[ \det \mathcal{L}_{X_{\text{dm}}/Z_{\text{an}}}|_x \cong (\Lambda^\top T_x(X_{\text{dm}}/Z_{\text{an}}))^* \otimes \Lambda^\top O_x(X_{\text{dm}}/Z_{\text{an}}), \]

as in (2.11). However, because \( Z_{\text{an}} \) is a complex manifold it has a canonical orientation, so orientations on \( X_{\text{dm}} \) are in natural 1-1 correspondence with coorientations for \( \pi_{\text{dm}} : X_{\text{dm}} \to Z_{\text{an}} \). Thus we obtain:
Proposition 3.27. In the situation of Theorem 3.25 there is a natural 1-1 correspondence between orientations on \((\pi : X \to Z, [\omega_{X/Z}])\) in the sense of Definition 3.26, and orientations on \(X_{dm}\) in the sense of 2.6.

The relative analogue of Theorem 3.20 does hold, but we will not prove it, as we do not need it. The next theorem says that the ‘virtual class’ \([X_{dm}] \in dBn(\ast)\) of a proper oriented \(-2\)-shifted symplectic derived \(C\)-scheme \((X, \omega_X)\) defined in Corollary 3.21 is unchanged under deformation in families. Note that it is essential that the base \(C\)-scheme \(Z\) be connected in Theorem 3.28.

Theorem 3.28. Let \(X\) be a separated derived \(C\)-scheme, \(Z = \text{Spec } B\) a smooth, connected, classical affine \(C\)-scheme, \(\pi : X \to Z\) a proper morphism, and \([\omega_{X/Z}]\) a family of \(-2\)-shifted symplectic structures on \(X/Z\), equipped with an orientation, with \(\text{dim}_C Z = k\) and \(\text{vdim}_C X = n + k\).

For each \(z \in Z_{an}\), as in Definition 3.22 we have a proper, oriented \(-2\)-shifted symplectic \(C\)-scheme \((X_z, \omega_{X_z})\) with \(\text{vdim} X_z = n\), so Corollary 3.21 defines a \(d\)-bordism class \([X_{dm,z}]\) in \(dBn(\ast) \cong Bn(\ast)\) which depends on \((X_z, \omega_{X_z})\) and not on any additional choices. Then \([X_{dm,z_1}] = [X_{dm,z_2}]\) for all \(z_1, z_2 \in Z_{an}\).

Proof. Theorem 3.25 constructs a derived manifold \(X_{dm}\) with \(\text{vdim} X_{dm} = n + 2k\) and a morphism \(\pi_{dm} : X_{dm} \to Z_{an}\), which is proper as \(\pi\) is proper, and Proposition 3.27 gives an orientation on \(X_{dm}\).

Let \(z_1, z_2 \in Z_{an}\). As \(Z\) is connected we can choose a smooth map \(\gamma : [0, 1] \to Z_{an}\) with \(\gamma(0) = z_1\) and \(\gamma(1) = z_2\). The fibre product

\[ W_{dm} = X_{dm} \times_{\pi_{dm}, Z_{an}, \gamma} [0, 1] \]

exists as a derived manifold with boundary by [22 §7.5], [23 §7.6], \([1]\), with \(\text{vdim} W_{dm} = n + 1\), and \(W_{dm}\) is compact as \([0, 1]\) is and \(\pi_{dm}\) is proper, and oriented since \(X_{dm}, Z_{an}, [0, 1]\) are. As \(\partial X_{dm} = \partial Z_{an} = \emptyset\), the boundary is

\[ \partial W_{dm} = X_{dm} \times_{\pi_{dm}, Z_{an}, \gamma} \emptyset [0, 1] = X_{dm,z_1} \amalg X_{dm,z_2}, \]

where \(X_{dm,z_1}, X_{dm,z_2}\) are the fibres of \(\pi_{dm} : X_{dm} \to Z_{an}\) at \(z_1, z_2\).

Since \(\partial [0, 1] = -\{0\} \amalg \{1\}\) in oriented 0-manifolds, we have \(\partial W_{dm} = -X_{dm,z_1} \amalg X_{dm,z_2}\) in oriented derived manifolds. Therefore Definition 2.19 gives \([X_{dm,z_1}] = [X_{dm,z_2}]\) in \(dBn(\ast)\). By Theorem 3.21(c) \(X_{dm,z_1}, X_{dm,z_2}\) are possible outcomes of Theorem 3.18 applied to \((X_{z_1}, \omega_{X_{z_1}}), (X_{z_2}, \omega_{X_{z_2}})\), so \([X_{dm,z_1}]\), \([X_{dm,z_2}]\) are the \(d\)-bordism classes associated to \((X_{z_1}, \omega_{X_{z_1}}), (X_{z_2}, \omega_{X_{z_2}})\) in Corollary 3.21. This completes the proof.

Remark 3.29. The assumptions that \(Z\) is smooth, classical, and affine in Theorem 3.28 are easily removed; we can work over a base \(Z\) which is a general classical or derived \(C\)-scheme, provided it is connected.

To see this, suppose \(\pi : X \to Z\) is a proper morphism of derived \(C\)-schemes with \(Z\) connected, and \([\omega_{X/Z}] \in H^{-2}(L_X/Z)\) is a family of \(-2\)-shifted symplectic structures on \(X/Z\) equipped with an orientation, generalizing Definitions 3.22 and 3.26 to general \(Z\) in the obvious way.
Suppose $z_1, z_2 \in \mathbb{Z}_{an}$. As $Z$ is connected we can find a sequence $z_1 = z_0^1, z_1^1, \ldots, z_N^1$ of points in $\mathbb{Z}_{an}$, and a sequence of smooth affine curves $C_i^1, \ldots, C_N^1$ over $\mathbb{C}$ with morphisms $\pi^i : C^i \to Z$, such that $\pi^i(C^i)$ contains $z_i^{i-1}, z_i^i$ for $i = 1, \ldots, N$. Then $X^i = X \times_{\mathbb{Z}_{an}, \pi^i} C^i$ is a derived $\mathbb{C}$-scheme, and $[\omega_{X/Z}]$ pulls back to a family $[\omega_{X^i/C^i}]$ of oriented $-2$-shifted symplectic structures on $X^i/C^i$. Applying Theorem 3.28 to $(X^i \to C^i, [\omega_{X^i/C^i}])$ we see that $[X_{dm,z_1}] = [X_{dm,z^0}] = [X_{dm,z^1}] = \cdots = [X_{dm,z^N}] = [X_{dm,z_2}]$.

We took $Z$ to be smooth above to avoid defining families $\pi_{dm} : X_{dm} \to Z$ of derived manifolds over a base $Z$ which is not a (derived) manifold.

3.8 ‘Holomorphic Donaldson invariants’ of C–Y 4-folds

We now outline how the results of 3.3-3.7 can be used to define new enumerative invariants of (semi?)stable coherent sheaves on Calabi–Yau 4-folds $Y$, which we could call ‘holomorphic Donaldson invariants’, and which should be unchanged under deformations of $Y$. A closely related programme using gauge theory has recently been proposed by Cao and Leung [9–12], and we discuss their work in §3.9. Please note that the material of this section is provisional, sketchy, and incomplete, and we will prove no theorems.

We begin by discussing Donaldson–Thomas invariants $DT^\alpha(\tau)$ of Calabi–Yau 3-folds, introduced by Thomas [33]. Suppose $Z$ is a Calabi–Yau 3-fold over $\mathbb{C}$ with an ample line bundle $O_Z(1)$, which defines a Gieseker stability condition $\tau$ on coherent sheaves on $Z$, and $\alpha \in H^{even}(Z; \mathbb{Q})$. Then one can form coarse moduli $\mathbb{C}$-schemes $\mathcal{M}_\text{st}(\tau), \mathcal{M}_\text{ss}(\tau)$ of $\tau$-(semi)stable coherent sheaves on $Z$ of Chern character $\alpha$, with $\mathcal{M}_\text{st}(\tau) \subseteq \mathcal{M}_\text{ss}(\tau)$ Zariski open, and $\mathcal{M}_\text{ss}(\tau)$ proper.

Thomas [33] showed that $\mathcal{M}_\text{st}(\tau)$ carries an ‘obstruction theory’ $\varphi : E^\bullet \to L_{\mathcal{M}_\text{st}(\tau)}$ of virtual dimension 0, in the sense of Behrend and Fantechi [2]. Thus, if there are no strictly $\tau$-semistable sheaves in class $\alpha$, so that $\mathcal{M}_\text{st}(\tau) = \mathcal{M}_\text{ss}(\tau)$ and $\mathcal{M}_\text{ss}(\tau)$ is proper, then [2] gives a virtual count $DT^\alpha(\tau) = [\mathcal{M}_\text{ss}(\tau)]_{\text{virt}} \in \mathbb{Z}$. Thomas proved that $DT^\alpha(\tau)$ is unchanged under continuous deformations of $Z$.

Later, Joyce and Song [26] extended the definition of $DT^\alpha(\tau)$ to invariants $DT^\alpha(\tau) \in \mathbb{Q}$ for all $\alpha \in H^{even}(Z; \mathbb{Q})$, dropping the condition that there are no strictly $\tau$-semistable sheaves in class $\alpha$, and proved a wall-crossing formula for $DT^\alpha(\tau)$ under change of stability condition $\tau$. At about the same time, Kontsevich and Soibelman [27] defined a motivic generalization of Donaldson–Thomas invariants (assuming existence of ‘orientation data’ as in [2, 4]), and proved their own wall-crossing formula under change of $\tau$.

Thomas [33] called his invariants $DT^\alpha(\tau)$ ‘holomorphic Casson invariants’, though they are now generally known as Donaldson–Thomas invariants. Here Casson invariants are integer invariants of oriented real 3-manifolds $\mathbb{Z}_R$ which are homology 3-spheres, which ‘count’ flat connections on $\mathbb{Z}_R$.

This followed a programme of Donaldson and Thomas [15], which starting with some well-known geometry in real dimensions 2, 3 and 4, aimed to find
analyses in complex dimensions 2,3 and 4; so the complex analogues of homology 3-spheres, and flat connections upon them, are Calabi–Yau 3-folds, and holomorphic vector bundles (or coherent sheaves) upon them.

Donaldson invariants \[14\] are invariants of compact, oriented 4-manifolds \(Y_4\), defined by ‘counting’ moduli spaces \(M_{\alpha}^{\text{inst}}\) of \(SU(2)\)-instantons \(E\) on \(Y_4\) with \(c_2(E) = \alpha \in \mathbb{Z}\). In contrast to Casson and Donaldson–Thomas invariants, the (virtual) dimension \(d^\alpha\) of \(M_{\alpha}^{\text{inst}}\) need not be zero. Oversimplifying / lying a bit, one first constructs an orientation on \(M_{\alpha}^{\text{inst}}\), \[14\] \S 5.4. Then we have a virtual class \([M_{\alpha}^{\text{inst}}]|_{\text{virt}} \in H_{d^\alpha}(M_{\alpha}^{\text{inst}}; \mathbb{Z})\). For each \(\beta \in H_2(Y_4; \mathbb{Z})\) we construct a natural cohomology class \(\mu(\beta) \in H^2(M_{\alpha}^{\text{inst}}; \mathbb{Z})\), with \(\mu(\beta_1 + \beta_2) = \mu(\beta_1) + \mu(\beta_2)\). Then if \(d^\alpha = 2k\), we define Donaldson invariants \(D^\alpha(\beta_1, \ldots, \beta_k) = (\mu(\beta_1) \cup \cdots \cup \mu(\beta_k)) \cdot [M_{\alpha}^{\text{inst}}]|_{\text{virt}} \in \mathbb{Z}\) for all \(\beta_1, \ldots, \beta_k \in H_2(Y_4; \mathbb{Z})\). We can think of \(D^\alpha\) as a \(\mathbb{Z}\)-valued homogeneous degree \(k\) polynomial on \(H_2(Y_4; \mathbb{Z})\).

We propose, following \[15\], to define ‘holomorphic Donaldson invariants’ of Calabi–Yau 4-folds. The gauge theory ideas which were the primary focus of \[15\] will be discussed in \S 3.8 here we work in the world of (derived) algebraic geometry. Suppose \(Y\) is a Calabi–Yau 4-fold over \(\mathbb{C}\) (i.e. \(Y\) is smooth and projective with \(H^i(O_Y) = \mathbb{C}\) if \(i = 0,4\) and \(H^i(O_Y) = 0\) otherwise), and \(\alpha = (\alpha^0, \alpha^2, \alpha^4, \alpha^6) \in H^2(\mathbb{Y}; \mathbb{Q})\). As above we can form coarse moduli \(\mathbb{C}\)-schemes \(M_{\alpha}^{\text{ss}}(\tau) \subseteq M_{\alpha}^{\text{inst}}(\tau)\) of Gieseker (semi)stable coherent sheaves on \(Y\) of Chern character \(\alpha\), with \(M_{\alpha}^{\text{ss}}(\tau)\) proper.

To make contact with the work of \[3.1–3.7\] we need to show:

**Claim 3.30.** There is a \(-2\)-shifted symplectic derived \(\mathbb{C}\)-scheme \((M_{\alpha}^{\text{ss}}(\tau), \omega^*)\), natural up to equivalence, with classical truncation \(t_0(M_{\alpha}^{\text{ss}}(\tau)) = M_{\alpha}^{\text{ss}}(\tau)\), of virtual dimension \(\text{vdim}_{\mathbb{C}} M_{\alpha}^{\text{ss}}(\tau) = d^\alpha := 2 - \deg(\alpha \cup \bar{\alpha} \cup \text{td}(TY))_s\), where \(\bar{\alpha} = (\alpha^0, -\alpha^2, \alpha^4, -\alpha^6)\), and \(\text{td}(\cdot)\) is the Todd class.

Pantev et al. \[31\], \S 2.1] prove the analogue of Claim 3.30 in the context of (derived) Artin stacks, but we want to reduce to (derived) schemes. Roughly this means factoring out the \(\mathbb{C}^*\) stabilizer groups at each point of the \(\tau\)-stable derived moduli stack. Actually, it should not be difficult to extend \[3.1–3.7\] to derived algebraic \(\mathbb{C}\)-spaces rather than derived \(\mathbb{C}\)-schemes, and then it would be enough to construct \(M_{\alpha}^{\text{ss}}(\tau)\) as a derived algebraic \(\mathbb{C}\)-space. Also, using the ideas of Remark 3.11, we could probably do without Claim 3.30.

Next we would need to answer the:

**Question 3.31.** Does \((M_{\alpha}^{\text{ss}}(\tau), \omega^*)\) in Claim 3.30 have a natural orientation, in the sense of \[2.4\] possibly depending on some choice of data on \(Y\)?

Following the argument of Donaldson \[14\] \S 5.4, Cao and Leung \[11\] Th. 2.2] prove an orientability result, which should translate to the statement that if the Calabi–Yau 4-fold \(Y\) has holonomy \(SU(4)\) with \(H_*(Y; \mathbb{Z})\) torsion-free, and \(M_{\alpha}^{\text{ss}}(\tau)\) is a derived moduli scheme of coherent sheaves on \(Y\), then orientations on \(M_{\alpha}^{\text{ss}}(\tau)\) exist, though they do not construct a natural choice.

If both these problems are solved, then Theorem 5.18 makes \(M_{\alpha}^{\text{ss}}(\tau)\) into a derived manifold \(M_{\alpha}^{\text{ss}}(\tau)_{\text{dim}}\) of real virtual dimension \(d^\alpha\), which is oriented by
If there are no strictly $\tau$-semistable sheaves in class $\alpha$ then $\mathcal{M}^{\alpha}_{\text{st}}(\tau)_{\text{an}}$ is also compact, and has a d-bordism class $[\mathcal{M}^{\alpha}_{\text{st}}(\tau)_{\text{an}}] \in dB_{d^a}(\ast) \cong B_{d^a}(\ast)$, which is independent of choices by Corollary 3.21.

If $d^a = 0$ then $[\mathcal{M}^{\alpha}_{\text{st}}(\tau)_{\text{an}}] \in \mathbb{Z}$ is the virtual count we want. But if $d^a > 0$, then rather than taking bordism classes we would prefer to find cohomology classes on $\mathcal{M}^{\alpha}_{\text{st}}(\tau)_{\text{an}}$ with total dimension $d^a$ and ‘integrate’ them over $[\mathcal{M}^{\alpha}_{\text{st}}(\tau)_{\text{an}}]$, as for the $\mu(\beta_i)$ for Donaldson invariants above.

**Claim 3.32.** One can define natural cohomology classes $\mu(\beta)$ on $\mathcal{M}^{\alpha}_{\text{st}}(\tau)_{\text{an}}$ depending on homology classes $\beta$ on $Y$, which can be combined with $[\mathcal{M}^{\alpha}_{\text{st}}(\tau)_{\text{an}}]$ to give integer invariants, in a similar way to Donaldson invariants.

If $\mathcal{M}^{\alpha}_{\text{st}}(\tau)$ is a fine moduli space, there is a universal sheaf $\mathcal{E}$ on $\mathcal{M}^{\alpha}_{\text{st}}(\tau) \times Y$, with Chern classes $c_i(\mathcal{E}) \in H^{2i}(\mathcal{M}^{\alpha}_{\text{st}}(\tau) \times Y) \cong \bigoplus_k H^{2i-k}(\mathcal{M}^{\alpha}_{\text{st}}(\tau)_{\text{an}}) \otimes H^k(Y)$, and we can make $\mu_i(\beta) \in H^{2i-k}(\mathcal{M}^{\alpha}_{\text{st}}(\tau)_{\text{an}})$ by contracting $c_i(\mathcal{E})$ with $\beta \in H_k(Y)$. Using the results of 3.7, we should be able to prove that the resulting invariants are unchanged under continuous deformations of $Y$.

This would take us to the same point as Thomas [33] in the Calabi–Yau 3-fold case: we could ‘count’ moduli spaces $\mathcal{M}^{\alpha}_{\text{st}}(\tau)$ for those classes $\alpha$ containing no strictly $\tau$-semistable sheaves, and get a deformation-invariant answer. Many questions would remain, for instance, how to count strictly $\tau$-semistables, wall-crossing formulae as in [26, 27], computation in examples, and so on.

We hope to return to these issues in future work.

**3.9 Motivation from gauge theory, and ‘SU(4) instantons’**

Finally we discuss some ideas of Donaldson and Thomas [15], which were part of the motivation for this paper, and the work of Cao and Leung [9,12].

Let $Y$ be a Calabi–Yau 4-fold over $\mathbb{C}$, regarded as a compact real 8-manifold $Y$ with complex structure $J$, Ricci-flat Kähler metric $g$, Kähler form $\omega$, and holomorphic volume form $\Omega$. Fix a complex vector bundle $E \rightarrow Y$ of rank $r > 0$ with Hermitian metric $h$ and Chern character $\text{ch}(E) = \alpha$, and as in [9,10] assume for simplicity that $c_1(E) = 0$. Consider connections $\nabla$ on $E$ preserving $h$ with curvature $F \in C^\infty(\text{End}(E) \otimes_{\mathbb{C}} (\Lambda^2 T^*Y \otimes_{\mathbb{C}} \mathbb{C}))$. The splitting

$$\Lambda^2 T^*Y \otimes_{\mathbb{R}} \mathbb{C} = \langle \omega \rangle_\mathbb{C} \oplus \Lambda^{1,1}_{0}T^*Y \oplus \Lambda^{2,0}T^*Y \oplus \Lambda^{0,2}T^*Y$$

induces a corresponding decomposition $F = F^\omega \oplus F^{1,1}_{0} \oplus F^{2,0} \oplus F^{0,2}$.

We call $\nabla$ a Hermitian–Einstein connection if $F^\omega = F^{2,0} = F^{0,2} = 0$. We can split $\nabla = \partial_E \oplus \bar{\partial}_E$, where $\bar{\partial}_E$ gives $E$ the structure of a holomorphic vector bundle on $(Y,J)$, as $F^{0,2} = 0$. The Hitchin–Kobayashi correspondence says that if $(E, \bar{\partial}_E)$ is a holomorphic vector bundle and is slope-stable, then $\bar{\partial}_E$ extends to a unique Hermitian–Einstein connection $\nabla = \partial_E \oplus \bar{\partial}_E$ preserving $h$. Also, holomorphic vector bundles on $Y$ are algebraic. Thus, studying moduli spaces $\mathcal{M}_{\text{alg-vb}}$ of stable algebraic vector bundles is roughly equivalent to studying moduli spaces $\mathcal{M}_{\text{HE}}$ of Hermitian–Einstein connections, modulo gauge.

44
As a system of p.d.e.s, the Hermitian–Einstein equations are overdetermined: there are $8r^2$ unknowns, $13r^2$ equations and $r^2$ gauge equivalences, with $8r^2 - 13r^2 - r^2 < 0$. Algebraically, this corresponds to the fact that the natural obstruction theory on $\mathcal{M}_{\text{alg,vb}}$ is not perfect, so we cannot form virtual classes.

Using $\Omega, g$ we can define real splittings $\Lambda^{2,0}T^*Y = \Lambda^{2,0}_+T^*Y \oplus \Lambda^{2,0}_-T^*Y,$ $\Lambda^{0,2}T^*Y = \Lambda^{0,2}_+T^*Y \oplus \Lambda^{0,2}_-T^*Y$ and corresponding decompositions $F^{2,0} = F_+^{2,0} \oplus F_0^{2,0}, F^{0,2} = F_+^{0,2} \oplus F_0^{0,2}$. Following Donaldson and Thomas [15, §3], we call $\nabla$ an SU(4)-instanton if $F^\omega = F_+^{2,0} = F_0^{0,2} = 0$. This gives $8r^2$ unknowns, $7r^2$ equations and $r^2$ gauge equivalences, with $8r^2 - 7r^2 - r^2 = 0$. It is a determined elliptic system, so that we can hope to define virtual classes. This is special to Calabi–Yau 4-folds, a complex analogue of instantons on real 4-manifolds.

Writing $\mathcal{M}_{\text{SU}(4)}$ for the moduli space of SU(4)-instantons, we have $\mathcal{M}_{\text{HE}}^{\alpha} \subseteq \mathcal{M}_{\text{SU}(4)}^{\alpha}$, as the SU(4) instanton equations are weaker than the Hermitian–Einstein equations. Now $\alpha = \text{ch}(E) = \bigoplus_{p=0}^{4} H^{p,p}(Y)$ if $E$ admits Hermitian–Einstein connections. Conversely, as in [15, p. 36], if $\alpha \in \bigoplus_{p} H^{p,p}(Y)$ then one can use $L^2$-norms of components of $F$ to show that any SU(4)-instanton is Hermitian–Einstein. Thus, either $\mathcal{M}_{\text{HE}}^{\alpha} = \mathcal{M}_{\text{SU}(4)}^{\alpha}$, or $\mathcal{M}_{\text{HE}}^{\alpha} = \emptyset$.

However, the equality $\mathcal{M}_{\text{HE}}^{\alpha} = \mathcal{M}_{\text{SU}(4)}^{\alpha}$ holds only at the level of sets, or topological spaces. Since $\mathcal{M}_{\text{HE}}^{\alpha}$ is defined by more equations, if we regard $\mathcal{M}_{\text{SU}(4)}^{\alpha}$ as (derived) $C^\infty$-schemes, for instance, then $\mathcal{M}_{\text{HE}}^{\alpha} \subseteq \mathcal{M}_{\text{SU}(4)}^{\alpha}$.

In the setting of [5][6][8][9] we should compare $\mathcal{M}_{\text{HE}}^{\alpha}$ (a Calabi–Yau 4-fold moduli space, without a virtual class, equivalent to an algebraic moduli scheme $\mathcal{M}_{\text{alg,vb}}^{\alpha}$) with the $-2$-shifted symplectic derived $C$-scheme $(\mathcal{X}, \omega^*_\mathcal{X})$, and $\mathcal{M}_{\text{SU}(4)}^{\alpha}$ (an elliptic moduli space, hopefully with a virtual class, equal to $\mathcal{M}_{\text{HE}}^{\alpha}$ on the level of topological spaces) with the derived manifold $\mathcal{X}_{\text{dim}}$. It was these ideas from Donaldson and Thomas [15] that led the authors to believe that one could modify a $-2$-shifted symplectic derived $C$-scheme to get a derived manifold with the same topological space, and so define a virtual class.

Donaldson and Thomas [15] envisaged using gauge theory to define invariants of Calabi–Yau 4-folds ‘counting’ moduli spaces $\mathcal{M}_{\text{SU}(4)}^{\alpha}$, and also invariants of compact Spin(7)-manifolds ‘counting’ moduli spaces of ‘Spin(7)-instantons’.

This would require finding suitable compactifications $\overline{\mathcal{M}}_{\text{SU}(4)}^{\alpha}$ of the moduli spaces $\mathcal{M}_{\text{SU}(4)}^{\alpha}$, and giving them a nice enough geometric structure to define virtual classes, which is a formidably difficult problem in gauge theory in dimensions $> 4$. A huge advantage of our approach is that, working in algebraic geometry, with moduli spaces of coherent sheaves rather than vector bundles, we often get compactness of moduli spaces for free, without doing any work.

Cao and Leung [9][12] also aim to define enumerative invariants of Calabi–Yau 4-folds $Y$, which they call ‘$DT_1$-invariants’, and their ideas overlap with ours. As for our outline in [3][8] their general theory is still rather incomplete, but they prove many partial results, going much further than we do, and do computations in examples, which we have not done.

Given a vector bundle moduli space $\mathcal{M}_{\text{alg,vb}}^{\alpha} \cong \mathcal{M}_{\text{HE}}^{\alpha} \cong \mathcal{M}_{\text{SU}(4)}^{\alpha}$ in topological spaces, assuming it is compact, and with a choice of orientation (com-
pare Question 3.31, which they prove exists under topological assumptions on \( Y \), Cao and Leung \([10]\) §5 define a virtual class \( \mathcal{M}^{\text{virt}}_{\text{SU}(4)} \) for \( \mathcal{M}^\alpha_{\text{SU}(4)} \), and contract this with some cohomology classes \( \mu(\beta) \) (compare Claim 3.32) to get integer invariants, which they prove are unchanged under deformations of \( Y \).

All this involves fairly standard material from gauge theory. Rather than making \( \mathcal{M}^{\text{virt}}_{\text{SU}(4)} \) into a derived manifold / Kuranishi space as we do, they write it globally as the zeroes of a Fredholm section \( s \) of a Banach vector bundle \( E \) over a Banach manifold \( B \) of connections modulo gauge. Their virtual class \( \mathcal{M}^{\text{virt}}_{\text{SU}(4)} \) lies in \( H_0^* (B; \mathbb{Z}) \), and their cohomology classes \( \mu(\beta) \) in \( H^*(B; \mathbb{Z}) \).

They also discuss the case in which one has a compact moduli space of coherent sheaves \( \mathcal{M}^\alpha_{\text{coh-sh}} \) which contains the vector bundle moduli space \( \mathcal{M}^\alpha_{\text{alg-vb}} \) as a (possibly empty) open subset. They want to define a virtual class for \( \mathcal{M}^\alpha_{\text{coh-sh}} \), as we want to, and they can do this under the assumptions that either \( \mathcal{M}^\alpha_{\text{coh-sh}} \) is smooth, or (roughly speaking, in our language) that the \(-2\)-shifted symplectic derived scheme \( (\mathcal{M}^\alpha_{\text{coh-sh}}, \omega) \) is locally of the form \( T^* X[2] \) for \( X \) a quasi-smooth derived \( C \)-scheme.

To compare our work with theirs: given \( \mathcal{M}^\alpha_{\text{alg-vb}} \subset \mathcal{M}^\alpha_{\text{coh-sh}} \) as above, assuming Claim 3.30, our Theorem 3.18 gives \( \mathcal{M}^\alpha_{\text{coh-sh}} \) the structure of a derived manifold, but one depending on arbitrary choices. By topologically identifying \( \mathcal{M}^\alpha_{\text{alg-vb}} \cong \mathcal{M}^\alpha_{\text{SU}(4)} \), in effect Cao and Leung make \( \mathcal{M}^\alpha_{\text{alg-vb}} \) into a derived manifold, canonically up to equivalence (though depending on the Kähler metric \( g \) and holomorphic volume form \( \Omega \)). However, there seems no reason why their derived manifold structure on \( \mathcal{M}^\alpha_{\text{alg-vb}} \subset \mathcal{M}^\alpha_{\text{coh-sh}} \) should extend smoothly to \( \mathcal{M}^\alpha_{\text{coh-sh}} \). This is a reason why our approach may in the end be more effective.

### 4 Proof of Theorem 3.2

In this proof we write \( \text{cdga}_C \) for the ordinary category of cdgas over \( C \), and \( \text{cdga}_C^\infty \) for the \( \infty \)-category of cdgas over \( C \), defined using the model structure on \( \text{cdga}_C \). All objects in \( \text{cdga}_C \) are fibrant, but only free cdgas are cofibrant. If \( \phi: A \to B \) is a morphism in \( \text{cdga}_C \) then \( \phi: A \to B \) is also a morphism in \( \text{cdga}_C^\infty \). However, morphisms \( \phi: A \to B \) in \( \text{cdga}_C^\infty \) may not correspond to morphisms \( A \to B \) in \( \text{cdga}_C \) unless \( A \) is cofibrant.

The spectrum functor \( \text{Spec} \) maps \( (\text{cdga}_C)^{\text{op}} \to \text{dSch}_C \) and \( (\text{cdga}_C^\infty)^{\text{op}} \to \text{dSch}_C \), and \( (\text{cdga}_C)^{\text{op}} \to \text{dSch}_C \) is an equivalence with the full \( \infty \)-subcategory of \( \text{dSch}_C \) with affine objects. So, morphisms \( \phi: A \to B \) in \( \text{cdga}_C^\infty \) are essentially the same thing as morphisms \( \text{Spec} B \to \text{Spec} A \) in \( \text{dSch}_C \).

Let \( \pi: X \to Z = \text{Spec} B \) and \( \{ (A^*_i, \alpha_i, \beta_i) : i \in I \} \) be as in Theorem 3.2 Our task is to construct a standard form cdga \( A^*_j = (A^*_j, d) \), a Zariski open inclusion \( \alpha_j : \text{Spec} A^*_j \hookrightarrow X \), and a morphism \( \beta_j: B \to A^*_j \) for all finite \( \emptyset \neq J \subseteq I \), and a submersion \( \Phi_{JK}: A^*_K \to A^*_J \) for all finite \( \emptyset \neq K \subseteq J \subseteq I \), satisfying conditions. We will do this by induction on increasing \( k = |J| \). Here is our inductive hypothesis:

**Hypothesis 4.1.** Let \( k = 1, 2, \ldots \) be given. Then
(a) We are given finite subsets $S^*_J$ for all $\emptyset \neq J \subseteq I$ with $|J| \leq k$ and all $n = -1, -2, \ldots$.

(b) For all $\emptyset \neq J \subseteq I$ with $|J| \leq k$ we have $A^0_J = \bigotimes_{i \in J} B A^0_i$ as a smooth $C$-algebra, where the tensor products are over $B$ using $\beta_i : B \to A^0_i$ to make $A^0_i$ into a $B$-algebra, so that if $J = \{i_1, \ldots, i_j\}$ then

\[ A^0_J = A_{i_1} \otimes_B A_{i_2} \otimes_B \cdots \otimes_B A_{i_j}. \tag{4.1} \]

The morphism $\beta_J : B \to A^0_J$ is induced by (4.1) and the $\beta_i : B \to A^0_i$ for $i \in J$, and is smooth as the $\beta_i$ are.

(c) For all $\emptyset \neq J \subseteq I$ with $|J| \leq k$, as graded $C$-algebra, $A^*_J$ is freely generated over $A^0_J$ by generators $\bigcup_{\emptyset \neq L \subseteq J} S^*_L$ in degree $n$ for $n = -1, -2, \ldots$.

(d) For all $\emptyset \neq K \subseteq J \subseteq I$ with $|J| \leq k$, the morphism $\Phi^0_{JK} : A^0_K \to A^0_J$ in degree 0 is the morphism

\[ A^0_K = \bigotimes_{i \in K} B A^0_i = (\bigotimes_{i \in K} B A^0_i) \otimes_B \bigotimes_{i \in J \setminus K} B \to \bigotimes_{i \in J} B A^0_i = A^0_J \]

induced by the morphisms $id : A^0_i \to A^0_i$ for $i \in K$ and $\beta_i : B \to A^0_i$ for $i \in J \setminus K$. Then $\Phi^0_{JK} : A^*_K \to A^*_J$ is the unique morphism of graded $C$-algebras acting by $\Phi^0_{JK}$ in degree zero, and mapping $\Phi^0_{JK} : \gamma \to \gamma$ for each $\gamma \in S^*_L$ for $\emptyset \neq L \subseteq K \subseteq J \subseteq I$ and $n = -1, -2, \ldots$, so that $\gamma$ is a free generator of both $A^*_K$ over $A^*_K$ and $A^*_J$ over $A^*_J$.

Note that $\Phi^0_{JK} : A^0_K \to A^0_J$ is a smooth morphism of $C$-algebras, since $id : A^0_i \to A^0_i$ and $\beta_i : B \to A^0_i$ are. Also $\Phi^0_{JK}$ maps independent generators $\bigcup_{\emptyset \neq L \subseteq K} S^*_L$ of $A^*_K$ over $A^0_K$ to independent generators of $A^*_J$ over $A^0_J$.

Hence $\Phi_{JK} : A^*_K \to A^*_J$ is a submersion.

Clearly $\beta_J = \Phi^0_{JK} \circ \beta_K = \Phi_{JK} \circ \beta_K : B \to A^0_J$.

Also, if $\emptyset \neq L \subseteq K \subseteq J \subseteq I$ with $|J| \leq K$ then clearly $\Phi^0_{JK} = \Phi^0_{JK} \circ \Phi^0_{KL} : A^0_L \to A^0_J$, as $\Phi_{KL} = \Phi_{JK} \circ \Phi_{KL} : A^*_L \to A^*_J$.

(e) For all $\emptyset \neq J \subseteq I$ with $|J| \leq k$ and all $n = -1, -2, \ldots$, we are given maps $\delta^*_J : S^*_J \to A^*_J$.

(f) Let $\emptyset \neq J \subseteq I$ with $|J| \leq k$. Define $d : A^*_J \to A^*_J$ uniquely by the conditions that $d$ satisfies the Leibnitz rule, and

\[ d\gamma = \Phi_{JK} \circ \delta^*_K (\gamma) \quad \text{for all } \emptyset \neq K \subseteq J, \ n \leq -1 \text{ and } \gamma \in S^*_K. \tag{4.2} \]

We require that $d \circ d = 0 : A^*_J \to A^*_J$, so that $A^*_J = (A^*_J, d)$ is a cdga.

This defines $A^*_J = (A^*_J, d)$, as a standard form cdga over $C$. Observe that if $\emptyset \neq K \subseteq J \subseteq I$ with $|J| \leq k$ then as $\Phi_{JK} : A^*_K \to A^*_J$ is a morphism of graded $C$-algebras with $\Phi_{JK} \circ d\gamma = d \circ \Phi_{JK}(\gamma)$ for all $\gamma$ in the generating sets $\bigcup_{\emptyset \neq L \subseteq K} S^*_L$ for $A^*_K$ over $A^*_K$, we have $\Phi_{JK} \circ d = d \circ \Phi_{JK} : A^*_K \to A^*_J$; and so $\Phi_{JK} : A^*_K \to A^*_J$ is a morphism of cdgas.
(g) For all $\emptyset \neq J \subseteq I$ with $|J| \leq k$, we are given a Zariski open inclusion $\alpha_j : \text{Spec} A^*_j \hookrightarrow X$, with image $\text{Im} \alpha_j = \bigcap_{i \in J} \text{Im} \alpha_i$, such that (3.2) homotopy commutes.

If $\emptyset \neq K \subseteq J \subseteq I$ with $|J| \leq k$ then (3.2) homotopy commutes.

**Remark 4.2.** (i) In Hypothesis 4.1 the only actual data required are the finite sets $S^n_j$ in (a), the maps $\delta^n_j : S^n_j \to A^n_{j+1}$ in (e), and the morphisms $\alpha_j : \text{Spec} A^*_j \hookrightarrow X$ in (g).

Also, the only statements requiring proof are that $d \circ d = 0$ in (f), and that $\alpha_j$ is a Zariski open inclusion with image $\bigcap_{i \in J} \text{Im} \alpha_i$, and that (3.2) and (3.3) homotopy commute in (g). All of (b),(c),(d) are definitions and deductions.

(ii) Most of the conclusions of Theorem 3.2 are immediate from the definitions in (a)–(g): that $A^*_{j+1}$ is a standard form cdga, and $\beta_j : B \to A^n_{j+1}$ is smooth, and $\Phi_{J,K} : A^n_{K+1} \to A^n_{j+1}$ is a submersion, and $\beta_j = \Phi_{J,K} \circ \beta_K$, and $\Phi_{J,L} = \Phi_{J,K} \circ \Phi_{K,L}$.

For the first step in the induction, we prove Hypothesis 4.1 when $k = 1$. Then the only subsets $\emptyset \neq J \subseteq I$ with $|J| \leq k$ are $J = \{i\}$ for $i \in I$, and the only subsets $\emptyset \neq K \subseteq J \subseteq I$ with $|J| \leq k$ are $J = K = \{i\}$ for $i \in I$.

As in Theorem 3.2 we are given data $\{(A^*_i, \alpha_i, \beta_i) : i \in I\}$, where $A^*_i$ is a standard form cdga, so that $A^*_i$ is freely generated over $A^0_i$ by finitely many generators in each degree $n = -1, -2, \ldots$, as in Definition 2.1. For each $i \in I$ and each $n = -1, -2, \ldots$, choose a subset $S^n_{\{i\}} \subseteq A^0_i$, as in part (a) for $J = \{i\}$, such that $A^*_i$ is freely generated over $A^0_i$ by $\prod_{n \leq -1} S^n_{\{i\}}$. Set $A^*_i = A^*_i$ and $\beta_{\{i\}} = \beta_i$, so that parts (b),(c) hold for $J = \{i\}$.

Part (d) is a definition, and when $k = 1$ only says that when $J = K = \{i\}$ we have $\Phi_{\{i\}} = \text{id} : A^*_i \to A^*_i$. For (e), define $\delta^n_{\{i\}} : S^n_{\{i\}} \to A^{n+1}_i$ by $\delta^n(\gamma) = d\gamma$, using $d$ in the cdga $A^*_i = (A^*_i, d)$. Given (e), part (f) says that the differentials $d$ in $A^*_i = (A^*_i, \{i\})$ and $A^*_i = (A^*_i, d)$ agree, consistent with setting $A^*_i = A^*_i$, so $d \circ d = 0$ in $A^*_i$, as $A^*_i$ is a cdga.

For (g), if $i \in I$ define $\alpha_i : A^*_i = A^*_i \to X$. Then the assumptions on $\{(A^*_i, \alpha_i, \beta_i) : i \in I\}$ in Theorem 3.2 imply that $\alpha_i$ is a Zariski open inclusion, with image $\text{Im} \alpha_i = \text{Im} \alpha_i$, and (3.2) homotopy commutes for $J = \{i\}$ as (3.1) does. The only $\emptyset \neq K \subseteq J \subseteq I$ with $|J| \leq k = 1$ are $J = K = \{i\}$, and then (3.3) homotopy commutes as $\alpha_K = \alpha_K = \alpha_i$ and $\Phi_{J,K} = \text{id}$. This completes Hypothesis 4.1 when $k = 1$. Note that our definitions $A^*_i = A^*_i$, $\alpha_i = \alpha_i$, and $\beta_i = \beta_i$ for $i \in I$ are as required in Theorem 3.2.

Next we prove the inductive step. Let $l \geq 1$ be given, and suppose Hypothesis 4.1 holds with $k = l$. Keeping all the data in (a),(c),(g) for $|J| \leq l$ the same, we will prove Hypothesis 4.1 with $k = l + 1$. To do this, for each $J \subseteq I$ with $|J| = l + 1$, we have to construct the data of finite sets $S^n_{\{i\}}$ for $n = -1, -2, \ldots$, in (a), and maps $\delta^n_{\{i\}} : S^n_{\{i\}} \to A^{n+1}_i$ in (e), and the morphism $\alpha_J : \text{Spec} A^*_J \hookrightarrow X$ in (g), and then prove the claims in (f) that $d \circ d = 0$, and in (g) that $\alpha_J$ is a Zariski open inclusion with image $\bigcap_{i \in J} \text{Im} \alpha_i$, and that (3.2) and (3.3) homotopy commute.
Note that as Hypothesis 4.1 involves no compatibility conditions between data for distinct $J, J' \subseteq I$ with $|J| = |J'| = k$, we can do this independently for each $J \subseteq I$ with $|J| = l + 1$, that is, it is enough to give the proof for a single such $J$. So fix a subset $J \subseteq I$ with $|J| = l + 1$.

We first define a standard form cdga $\tilde{A}_J^*$ which is an approximation to the cdga $A_J^*$ that we want, and morphisms $\tilde{\beta}_J : B \to \tilde{A}_J^0, \tilde{\Phi}_{JK} : A_K^* \to \tilde{A}_J^*$ for all $\emptyset \neq K \subseteq J$, so that $|K| \leq l$ and $A_K^*$ is already defined:

- Define $\tilde{A}_J^0 = A_j^0$ and $\tilde{\beta}_J = \beta_j : B \to \tilde{A}_J^0 = A_j^0$ as in Hypothesis 4.1(b).
- Define $\tilde{A}_J^*$ to be the graded $\mathbb{C}$-algebra freely generated over $A_j^0$ by generators $\prod_{s \not\in K \subseteq J} S^K_n$ in degree $n$ for $n = -1, -2, \ldots$. This is the same as for $A_J^*$ in Hypothesis 4.1(c), except that we do not include generators $S_J^n$, since $S_J^n$ is not yet defined.
- If $\emptyset \neq K \subseteq J$, so that $A_K^*$ is defined, define $\Phi^0_{JK} : A_K^0 \to A_J^0 = \tilde{A}_J^0$ as in Hypothesis 4.1(d), and define $\Phi_{JK} : A_K^* \to \tilde{A}_J^*$ to be the unique morphism of graded $\mathbb{C}$-algebras acting by $\Phi^0_{JK}$ in degree zero, and mapping $\Phi_{JK} : \gamma \mapsto \gamma$ for each $\gamma \in S_K^n$ for $\emptyset \neq L \subseteq K$ and $n = -1, -2, \ldots$.
- The differential $d : \tilde{A}_J^* \to \tilde{A}_J^{*+1}$ in the cdga $\tilde{A}_J^* = (\tilde{A}_J^*, d)$ is determined uniquely as in (4.2) by

$$d \gamma = \tilde{\Phi}_{JK} \circ \delta_K^n(\gamma) \quad \text{for all } \emptyset \neq K \subseteq J, n \leq -1 \text{ and } \gamma \in S_K^n.$$ 

Then $\tilde{\Phi}_{JK} : A_K^* \to \tilde{A}_J^*$ is a cdga morphism for all $\emptyset \neq K \subseteq J$, as in Hypothesis 4.1(e) for $\Phi_{JK}$.

Consider now two diagrams $\Gamma \subseteq \tilde{\Gamma}$ of objects and morphisms in cdga$_c$, where $\Gamma$ has vertices the objects $B$ and $A_K^*$ for all $K$ with $\emptyset \neq K \subseteq J$, and edges the morphisms $\beta_K : B \to A_K^*$ and $\Phi_{K,K_2} : A_{K_2}^* \to A_K^*$ for $\emptyset \neq K_2 \subseteq K_1 \subseteq J$, and $\tilde{\Gamma}$ is $\Gamma$ with an additional vertex $A^*_J$, and additional edges $\tilde{\beta}_J : B \to \tilde{A}_J^*$ and $\tilde{\Phi}_{JK} : A_K^* \to \tilde{A}_J^*$ for all $K$ with $\emptyset \neq K \subseteq J$. The proof that $\beta_J = \Phi_{JK} \circ \beta_K$ and $\Phi_{JL} = \Phi_{JK} \circ \Phi_{KL}$ in Hypothesis 4.1 also works for $\tilde{\beta}_J, \tilde{\Phi}_{JK}$, so $\Gamma, \tilde{\Gamma}$ commute. We illustrate these in Figure 4.1 when $J = \{1, 2, 3\}$.

We claim that $A_J^*$ is the colimit of the commutative diagram $\Gamma$ in the ordinary category cdga$_c$, where the commutative diagram $\Gamma$ is $\Gamma$ together with its colimit $A_J^*$ and the projections to the colimit. To see this note that if $C^*$ is any graded $\mathbb{C}$-algebra and $\emptyset \neq K \subseteq J$ then morphisms $\Psi : A_K^* \to C^*$ of graded $\mathbb{C}$-algebras are equivalent to choices of the data:

- a morphism of $\mathbb{C}$-algebras $\beta : B \to C^0$;
- morphisms of $\mathbb{C}$-algebras $\psi_i : A_i^0 \to C^0$ for all $i \in K$ with $\psi_i \circ \beta_i = \beta$; and
- elements $\Psi \circ \Phi_{KL}(\gamma) \in C^n$ for all $\emptyset \neq L \subseteq K, n \leq -1$ and $\gamma \in S_L^n$.

Similarly, morphisms $\tilde{\Psi} : \tilde{A}_J^* \to C^*$ are equivalent to choices of the data:

- a morphism of $\mathbb{C}$-algebras $\beta : B \to C^0$;
- morphisms of $\mathbb{C}$-algebras $\psi_i : A_i^0 \to C^0$ for all $i \in J$ with $\psi_i \circ \beta_i = \beta$; and
Figure 4.1: Graph $\Gamma$ when $J = \{1, 2, 3\}$. For $\Gamma$, delete vertex $A^*_j$ and its edges

- elements $\Psi \circ \Phi_{JK}(\gamma) \in C^n$ for all $\emptyset \neq K \subseteq J$, $n \leq -1$ and $\gamma \in S^n_K$.

Using these, we see that morphisms from the diagram $\Gamma$ of graded $\mathbb{C}$-algebras to $C^*$ correspond to data $\beta, \psi_i : i \in J$ and elements of $C^n$ for all $\emptyset \neq K \subseteq J$, $n \leq -1$ and $\gamma \in S^n_K$, and hence to morphisms $\tilde{\Psi} : \tilde{A}^*_j \to C^*$, and $\tilde{A}^*_j$ is the colimit of $\Gamma$ in graded $\mathbb{C}$-algebras, with colimit diagram $\tilde{\Gamma}$. Since $\Gamma$ is also a commutative diagram of cdgas, it is a colimit diagram of cdgas, as the universal property for graded $\mathbb{C}$-algebras implies the universal property for cdgas.

Next we claim that $A^*_j$ is also the homotopy colimit of the commutative diagram $\Gamma$ in the $\infty$-category $\text{cdga}^\infty$. To see this, we will build the colimit of $\Gamma$ up as an iterated fibre product of standard form cdgas by submersions.

Call a subdiagram $\Delta$ of $\Gamma$ convex if $B$ is a vertex of $\Delta$, and if $\emptyset \neq L \subseteq J$ is a vertex of $\Delta$ and $\emptyset \neq L \subseteq K$ then $L$ is also a vertex of $\Delta$, and the edges of $\Delta$ are all edges of $\Gamma$ joining two vertices of $\Delta$. Then we can define a standard form cdga $A^*_\Delta$ in a similar way to $A^*_j$, where $A^*_\emptyset = A^*_K$ for $K$ the union of all $\emptyset \neq L \subseteq J$ which are vertices of $\Delta$, and $A^*_\Delta$ is freely generated over $A^*_\emptyset$ by $S^n_K$ in degree $n$ for all $n = -1, -2, \ldots$ and vertices $K$ of $\Delta$. Then a similar proof to the above shows that $A^*_\Delta$ is the colimit of $\Delta$ in the ordinary category of cdgas, with $A^*_\Gamma = A^*_j$.

If $\Delta_1, \Delta_2$ are convex subdiagrams of $\Gamma$ then so are $\Delta_1 \cap \Delta_2$ and $\Delta_1 \cup \Delta_2$, and properties of colimits mean we have a cocartesian diagram of cdgas:

$$
\begin{array}{ccc}
A^*_\Delta_1 \cap \Delta_2 & \longrightarrow & A^*_\Delta_1 \\
\downarrow & & \downarrow \\
A^*_\Delta_2 & \longrightarrow & A^*_\Delta_1 \cup \Delta_2.
\end{array}
$$

The morphisms in (4.3) are easy to write down — they act as the identity on generators in $S^n_K$ — and are submersions. Now a cocartesian diagram of standard form cdgas in the ordinary category $\text{cdga}^\text{st}$ with morphisms submersions is also a homotopy cocartesian diagram in the $\infty$-category $\text{cdga}^\infty$, because of cofibrancy properties of submersions. Therefore by induction on increasing number
of vertices of $\Delta$, we find that $A_{\Delta}$ is the homotopy colimit of $\Delta$ in the $\infty$-category cdga$_C^\infty$ for all convex $\Delta \subseteq \Gamma$, and so $A_{\gamma} = A_{\Gamma}$ is the homotopy colimit of $\Gamma$.

Applying Spec to $\Gamma$ and $\Gamma$ gives homotopy commutative diagrams in $dSch_C$, with directions of arrows reversed. Since the full $\infty$-subcategory of affine derived $\mathbb{C}$-schemes in $dSch_C$ is equivalent to the opposite $\infty$-category of cdga$_C^\infty$, we see that Spec $A_{\gamma}$ is the homotopy limit of Spec $\Gamma$ in $dSch_C$.

For $\emptyset \neq K \subseteq J$, consider $\bigcap_{i \in K} \text{Im} \alpha_i$, as an open derived $\mathbb{C}$-scheme of $X$. Then by Hypothesis 4.1(g), $\alpha_K : \text{Spec} A_{\gamma} \rightarrow \bigcap_{i \in K} \text{Im} \alpha_i$ is an equivalence in $dSch_C$. We also have the open derived $\mathbb{C}$-scheme $\bigcap_{i \in J} \text{Im} \alpha_i$ in $X$, which is affine by Definition 3.1 as $X$ is separated and $\text{Im} \alpha_i \simeq \text{Spec} A_i$ is affine for $i \in J$. Thus we may choose a standard form cdga $A_{\gamma}$ and an equivalence $\hat{\alpha}_J : \text{Spec} A_{\gamma} \rightarrow \bigcap_{i \in K} \text{Im} \alpha_i$.

Define morphisms $\hat{\beta}_J : \text{Spec} A_{\gamma} \rightarrow Z = \text{Spec} B$ by $\beta_J = \pi \circ \hat{\alpha}_J$, and $\hat{\phi}_{JK} : \text{Spec} A_{\gamma} \rightarrow \text{Spec} A_{K}$ for $\emptyset \neq K \subseteq J$ to be the composition

$$\text{Spec} A_{\gamma} \xrightarrow{\hat{\alpha}_J} \bigcap_{i \in J} \text{Im} \alpha_i \xrightarrow{\bigcap_{i \in K} \alpha_i} \bigcap_{i \in K} \text{Im} \alpha_i \xrightarrow{\alpha_K^{-1}} \text{Spec} A_{K},$$

where $\alpha_K^{-1}$ is a quasi-inverse for the equivalence $\alpha_K : \text{Spec} A_{\gamma} \rightarrow \bigcap_{i \in K} \text{Im} \alpha_i$.

Now as Spec $A_{\gamma}$ is the homotopy limit of Spec $\Gamma$ in $dSch_C$, and Spec $A_{\gamma}$ has morphisms $\hat{\beta}_J, \hat{\phi}_{JK}$ to the vertices of Spec $\Gamma$ making a homotopy commuting diagram, there exists a morphism $\psi : \text{Spec} A_{\gamma} \rightarrow \text{Spec} A_{\gamma}$ in $dSch_C$ making the whole diagram of Spec $\Gamma$ plus Spec $A_{\gamma}$ and morphisms $\hat{\beta}_J, \hat{\phi}_{JK}, \psi$ homotopy commute. We can then write $\psi \simeq \text{Spec} \Psi$ for $\Psi : A_{\gamma} \rightarrow A_{\gamma}$ a morphism in cdga$_C^\infty$, unique up to homotopy. However, we do not yet know that $\Psi$ descends to a morphism in cdga$_C$.

The homotopy limit property of Spec $\Gamma$ implies that we have homotopies $\hat{\beta}_J \simeq \text{Spec} \beta_J \circ \psi$ and $\hat{\phi}_{JK} \simeq \text{Spec} \Phi_{JK} \circ \psi$ for $\emptyset \neq K \subseteq J$. So the definitions of $\hat{\beta}_J, \hat{\phi}_{JK}$ and $\psi \simeq \text{Spec} \Psi$ give homotopies

$$\pi \circ \hat{\alpha}_J \simeq \text{Spec} \hat{\beta}_J \circ \text{Spec} \Psi : \text{Spec} A_{\gamma} \rightarrow Z,$$

$$\hat{\alpha}_J \simeq \alpha_K \circ \text{Spec} \hat{\phi}_{JK} \circ \text{Spec} \Psi : \text{Spec} A_{\gamma} \rightarrow X, \emptyset \neq K \subseteq J. \quad (4.4)$$

Consider the composition of morphisms of classical $\mathbb{C}$-algebras

$$A^0_j \xrightarrow{\gamma} \hat{A}^0_j \xrightarrow{H^0(\gamma)} \hat{H}^0(\hat{A}^0_j) \xrightarrow{H^0(\Psi)} H^0(\hat{A}^0_j). \quad (4.5)$$

As the morphism $A_j^0 \rightarrow H^0(A_j^0)$ is a closed immersion for $i \in J$, and (4.5) is the tensor product of these morphisms for $i \in J$ over $B$, equation (4.5) is a closed immersion. Therefore we can replace $\hat{A}_j^0$ by an equivalent object in cdga$_C^\infty$, such that $\hat{A}_j^0 = A_j^0$, and the following homotopy commutes in cdga$_C^\infty$:
Now $\Psi : \hat{A}_J^* \rightarrow \hat{A}_J^*$ is a morphism in $cdga_C^\infty$. For this to descend to a morphism in $cdga_C$, the simplest condition is that $\hat{A}_J^*$ should be cofibrant and $\hat{A}_J^*$ fibrant in the model category $cdga_C$. Here $\hat{A}_J^*$ is fibrant, as all objects are, but $\hat{A}_J^*$ may not be cofibrant, i.e. a free cdga. However, $\hat{A}_J^*$ is almost cofibrant, as it is free in negative degrees, and (1.10) says that $\Psi$ does descend to a morphism $\Psi : \hat{A}_J^* \rightarrow \hat{A}_J^*$ in $cdga_C$.

Let us summarize our progress so far:

- We have defined an explicit cdga $\hat{A}_J^*$, which is very similar to the cdga $A_J^*$ we want eventually to construct, but lacks the generators $S_n^0, n = -1, -2, \ldots$, which we have not yet chosen.
- We have defined an explicit smooth morphism $\hat{\beta}_J : B \rightarrow \hat{A}_J^\theta$, and explicit submersions $\Phi_{JK} : A_K^* \rightarrow A_J^*$ in $cdga_C$ for all $\emptyset \neq K \subset J$, all part of a commutative diagram $\Gamma$ in $cdga_C$.
- We have constructed a non-explicit cdga $\hat{A}_J^*$, with morphisms $\Psi : \hat{A}_J^* \rightarrow \hat{A}_J^*$ in $cdga_C$ and $\hat{\alpha}_J : \text{Spec} \hat{A}_J^* \rightarrow X$ in $dSch_C$, which is a Zariski open inclusion with image $\bigcap_{i \in J} \text{Im} \alpha_i$, satisfying the homotopies (1.1).

![Figure 4.2: Graph $\text{Spec} \hat{\Gamma}$ when $J = \{1, 2, 3\}$, plus $X$ and $\text{Spec} \hat{A}_J^*$](image-url)

We illustrate the diagram $\text{Spec} \hat{\Gamma}$ plus vertices $X, \text{Spec} \hat{A}_J^*$ and edges $\pi, \alpha_K, \hat{\alpha}_J, \hat{\beta}_J, \hat{\phi}_{JK}, \psi$ in Figure 4.2 when $J = \{1, 2, 3\}$. Note that this diagram is not completely homotopy commutative, as the morphisms $\alpha_K \circ \text{Spec} \hat{\phi}_{JK} : \text{Spec} \hat{A}_J^* \rightarrow X$ may depend on $\emptyset \neq K \subset J$, even up to homotopy.

Next, by induction on decreasing $n = -1, -2, \ldots$ we will choose the data $S_n^\theta, \delta_n^\theta$ in Hypothesis (1.1)(a),(e). Here is our inductive hypothesis:
Hypothesis 4.3. Let \( N = 0, -1, -2, \ldots \) be given. Then:

(a) We are given finite subsets \( S^n_j \) for \( n = -1, -2, \ldots, N \). Write \( A^*_j, N = \tilde{A}^*_j[S^1_j, \ldots, S^n_j] \) for the graded \( C \)-algebra freely generated over \( \tilde{A}^*_j \) by the sets of extra generators \( S^n_j \) in degree \( n \) for all \( n = -1, -2, \ldots, N \).

(b) We are given maps \( \delta^n_j : S^n_j \to A^*_j, N + 1 \) for \( n = -1, -2, \ldots, N \). Define \( d : A^*_j, N \to A^*_j, N + 1 \) uniquely by the conditions that \( d \) satisfies the Leibnitz rule, and \( d \) is as in \( \tilde{A}^*_j(d) \) on \( \tilde{A}^*_j \subseteq A^*_j, N \), and on the extra generators \( \gamma \in S^n_j \) for \( n = -1, -2, \ldots, N \), we have \( d\gamma = \delta^n_j(\gamma) \in A^*_j, N + 1 \). We require that \( d \circ d = 0 : A^*_j, N \to A^*_j, N + 2 \), so that \( A^*_j, N = (A^*_j, N, d) \) is a cdga.

(c) We are given maps \( \xi^n_j : S^n_j \to \tilde{A}^*_j \) for \( n = -1, -2, \ldots, N \).

Define \( \Xi_N : A^*_j, N \to \tilde{A}^*_j \) to be the morphism of graded \( C \)-algebras such that \( \Xi_N = \Psi \) on \( \tilde{A}^*_j \subseteq A^*_j, N \), and on the extra generators \( \gamma \in S^n_j \) for \( n = -1, -2, \ldots, N \), we have \( \Xi_N(\gamma) = \xi^n_j(\gamma) \in \tilde{A}^*_j \).

We require that \( \Xi_N \circ d = d \circ \Xi_N : A^*_j, N \to A^*_j, N + 1 \), so that \( \Xi_N : A^*_j, N \to \tilde{A}^*_j \) is a cdga morphism.

We also require that \( H^n(\Xi_N) : H^n(A^*_j, N) \to H^n(\tilde{A}^*_j) \) should be an isomorphism for \( n = 0, -1, -2, \ldots, N + 1 \), and surjective for \( n = N \).

For the first step \( N = 0 \), there is no data \( S^0_j, \delta^0_j, \xi^0_j \), and \( A^*_j, 0 = \tilde{A}^*_j \), and \( \Xi_0 = \Psi \), and the only thing to prove is that \( H^0(\Psi) : H^0(\tilde{A}^*_j) \to H^0(\tilde{A}^*_j) \) is surjective, which holds as \( \Psi^0 = \text{id} : \tilde{A}^*_j \to \tilde{A}^*_j \) from above. So Hypothesis 4.3 holds for \( N = 0 \).

For the inductive step, let \( m = 0, -1, -2, \ldots \) be given, and suppose Hypothesis 4.3 holds with \( N = m \). Keeping all the data \( S^m_j, \delta^m_j, \xi^m_j \) for \( n = -1, \ldots, m \) the same, we will prove Hypothesis 4.3 with \( N = m + 1 \). Note that with \( S^m_j, \ldots, S^n_j \) the same, the graded \( C \)-algebras \( A^*_j, m, A^*_j, m - 1 \) agree in degrees \( 0, -1, \ldots, m \), so it makes sense to say that \( \delta^m_j : S^m_j \to A^*_j, m + 1 \) and \( \delta^n_j : S^n_j \to A^*_j, m - 1 \) are equal for \( n = -1, -2, \ldots, m \). We must choose data \( S^m_j, \delta^m_j, \xi^m_j \) such that \( \delta^m_j : S^m_j \to A^*_j, m + 1 \) and \( \xi^m_j : S^m_j \to A^*_j, m - 1 \) and verify the last two conditions of Hypothesis 4.3.

Choose a finite subset \( S^{m-1}_j \) of \( \text{Ker}(H^m(\Xi_m) : H^m(A^*_j, m) \to H^m(A^*_j)) \) which generates \( \ker(\cdots) \) as an \( H^0(A^*_j, m) \)-module, and a finite subset \( S^{m-1}_j \) of \( H^{m-1}(A^*_j) \) such that \( \delta^{m-1}_j \) and \( \text{Im}(H^{m-1}(\Xi_m) : H^{m-1}(A^*_j, m) \to H^{m-1}(A^*_j)) \) generate \( H^{m-1}(A^*_j) \) as an \( H^0(A^*_j) \)-module. Finite subsets suffice in each case since \( A^*_j, m, \tilde{A}^*_j \) are of standard form, so that \( H^n(A^*_j, m), H^n(\tilde{A}^*_j) \) are finitely generated over \( H^0(A^*_j, m), H^0(\tilde{A}^*_j) \) for all \( n \). Set \( S^{m-1}_j = \tilde{S}^{m-1}_j \sqcup \tilde{S}^{m-1}_j \).

Then Hypothesis 4.3(a) defines \( A^*_j, m - 1 \) as a graded \( C \)-algebra, with \( A^*_j, m - 1 = A^*_j, m \) in degrees \( n \geq m \). For all \( \gamma \in S^{m-1}_j \), choose a representative \( \delta^{m-1}_j(\gamma) \) in \( A^*_j, m - 1 = A^*_j, m \) for the cohomology class \( \gamma \) in \( H^{m-1}(A^*_j, m) \), so that \( d(\delta^{m-1}_j(\gamma)) = 0 \).
in $A_{J,m+1}$. Define $\delta_j^{-1}(\gamma) = 0$ in $A_{J,m-1}^\circ$ for all $\gamma \in \tilde{S}_j^{m-1}$. This defines $\delta_j^{m-1} : S_j^{m-1} \to A_{J,m-1}^\circ$ in Hypothesis (c), and hence $d : A_{J,m-1}^\circ \to A_{J,m-1}^\circ$.

To see that $d \circ d = 0 : A_{J,m-1}^\circ \to A_{J,m-1}^{\circ+2}$, note that $A_{J,m-1}^\circ = A_{J,m}^\circ[S_j^{m-1}]$, so $d$ on $A_{J,m-1}^\circ$ is determined by $d$ on $A_{J,m}^\circ$, which already satisfies $d \circ d = 0$ by induction, and $d$ on the extra generators $S_j^{m-1}$, which satisfy $d \circ d = 0$ as for $\gamma \in S_j^{m-1}$ we have $d \circ d\gamma = d(\delta_j^{-1}(\gamma)) = 0$, and for $\gamma \in S_j^{m-1}$ we have $d\gamma = 0$ so $d \circ d\gamma = 0$. Hence $A_{J,m-1}^\circ = (A_{J,m-1}^\circ, d)$ is a cdga, as we have to prove.

For all $\gamma \in \tilde{S}_j^{m-1}$, as $\delta_j^{m-1}(\gamma) \in A_{J,m}^\circ$ represents a cohomology class in $	ext{Ker}(H^m(\Xi_m) : H^m(A_{J,m}^\circ) \to H^m(\hat{A}_j^\circ))$, we see that $\Xi_m \circ \delta_j^{m-1}(\gamma)$ is exact in $\hat{A}_j^\circ$, so we can choose an element $\xi_j^{m-1}(\gamma) \in \hat{A}_j^{m-1}$ with $d \circ \xi_j^{m-1}(\gamma) = \Xi_m \circ \delta_j^{m-1}(\gamma)$. For all $\gamma \in \tilde{S}_j^{m-1} \subset H^{m-1}(\hat{A}_j^\circ)$, choose an element $\xi_j^{m-1}(\gamma) \in \hat{A}_j^{m-1}$ representing $\gamma$, so that $d \circ \xi_j^{m-1}(\gamma) = 0$. This defines $\xi_j^{m-1} : S_j^{m-1} \to \hat{A}_j^{m-1}$.

Hypothesis (c) now defines $\Xi_m = A_{J,m-1}^\circ \to \hat{A}_j^\circ$. To prove that $\Xi_m = d \circ \Xi_m$, note that $A_{J,m-1}^\circ = A_{J,m}^\circ[S_j^{m-1}]$, and on $A_{J,m-1}^\circ \subset A_{J,m-1}^\circ$ we have $\Xi_m = \Xi_m$, and $\Xi_m = d \circ \Xi_m$ by induction. So it is enough to prove that $\Xi_m \circ d(\gamma) = d \circ \Xi_m(\gamma)$ for all $\gamma \in S_j^{m-1}$. If $\gamma \in S_j^{m-1}$ then

$$\Xi_m \circ d(\gamma) = \Xi_m \circ \delta_j^{m-1}(\gamma) = \Xi_m \circ \delta_j^{m-1}(\gamma) = d \circ \Xi_j^{m-1}(\gamma) = d \circ \Xi_m(\gamma),$$

as we want. Similarly, if $\gamma \in \tilde{S}_j^{m-1}$ then

$$\Xi_m \circ d(\gamma) = \Xi_m \circ \delta_j^{m-1}(\gamma) = d \circ \Xi_j^{m-1}(\gamma) = d \circ \Xi_m(\gamma).$$

Therefore $\Xi_m = d \circ \Xi_m$, and $\Xi_m : A_{J,m-1}^\circ \to \hat{A}_j^\circ$ is a cdga morphism.

Finally we have to show that $H^n(\Xi_m) : H^n(A_{J,m-1}^\circ) \to H^n(\hat{A}_j^\circ)$ is an isomorphism for $n = -1, -2, \ldots, m$, and surjective for $n = m - 1$. Since $\Xi_m : A_{J,m}^\circ \to \hat{A}_j^\circ$ and $\Xi_m : A_{J,m-1}^\circ \to \hat{A}_j^\circ$ coincide in degrees $0, -1, \ldots, m$, in cohomology they coincide in degrees $0, -1, \ldots, m + 1$, so $H^n(\Xi_m)$ is an isomorphism for $n = 0, -1, \ldots, m + 1$ as $H^n(\Xi_m)$ is, by induction.

As $H^n(\Xi_m) : H^n(A_{J,m}^\circ) \to H^n(\hat{A}_j^\circ)$ is surjective, and the added generators $\hat{S}_j^{m-1}$ in $A_{J,m-1}^\circ$ span Ker$(H^n(\Xi_m))$, adding $\hat{S}_j^{m-1}$ makes $H^n(\Xi_m)$ an isomorphism. Also, since the added generators $\hat{S}_j^{m-1}$ together with Im$(H^{m-1}(\Xi_m))$ generate $H^{m-1}(\hat{A}_j^\circ)$, adding $\hat{S}_j^{m-1}$ makes $H^{m-1}(\Xi_m) : H^{m-1}(A_{J,m-1}^\circ) \to H^{m-1}(\hat{A}_j^\circ)$ surjective.

This proves Hypothesis (c) for $N = m - 1$, so by induction Hypothesis (c) holds for all $N = 0, -1, -2, \ldots$. Taking the limit $\lim_{N \to -\infty} A_{J,N}^\circ$ gives the cdga $\hat{A}_j^\circ$ defined in Hypothesis (g) using the data $S_j^\circ, \delta_j^\circ$ for all $n = -1, -2, \ldots$ from Hypothesis (c), as $N \to -\infty$. The data $\delta_j^\circ$ for $n = -1, -2, \ldots$ from part (c) defines a morphism $\Xi = \lim_{N \to -\infty} \Xi_N : A_j^\circ \to \hat{A}_j^\circ$, where $\Xi, A_j^\circ$ agree with $\Xi_N, A_j^\circ$ in degrees $0, -1, \ldots, N$ for all $N \leq 0$.

Hence $H^n(\Xi) : H^n(A_j^\circ) \to H^n(\hat{A}_j^\circ)$ agrees with $H^n(\Xi_N) : H^n(A_j^\circ) \to H^n(\hat{A}_j^\circ)$ for all $n = 0, -1, \ldots, N + 1$, so $H^n(\Xi)$ is an isomorphism for all $n \leq 0$ by Hypothesis (c), and $\Xi : A_j^\circ \to \hat{A}_j^\circ$ is a quasi-isomorphism in cdga, hence
an equivalence in $\text{cdga}^\infty$. Thus $\text{Spec} \Xi : \text{Spec} A^*_J \to \text{Spec} A^*_J$ is an equivalence in $d\text{Sch}_C$. So we can choose a quasi-inverse $\chi : \text{Spec} A^*_J \to \text{Spec} A^*_J$ in $d\text{Sch}_C$.

Write $\iota : A^*_J \to A^*_J$ for the inclusion. Then $\Psi = \Xi \circ \iota : A^*_J \to A^*_J$, since $\Xi|_{A^*_J} = \Psi$, so taking the limit $N \to -\infty$ gives $\Xi|_{A^*_J} = \Psi$. Also the definitions of $\beta_J : B \to A^*_J$ and $\Phi_{JK} : A^*_K \to A^*_J$ for $\emptyset \neq K \subseteq J$ in Hypothesis 4.1(b),(d) satisfy $\beta_J = \iota \circ \beta_J$ and $\Phi_{JK} = \iota \circ \Phi_{JK}$.

Define $\alpha_J = \tilde{\alpha}_J \circ \chi : \text{Spec} A^*_J \to X$. Since $\tilde{\alpha}_J$ is a Zariski open inclusion with image $\bigcap_{i \in J} \text{Im} \alpha_i$, and $\chi$ is an equivalence, $\alpha_J : \text{Spec} A^*_J \to X$ is a Zariski open inclusion with image $\bigcap_{i \in J} \text{Im} \alpha_i$, as in Hypothesis 4.1(g). Then we have

\[
\pi \circ \alpha_J = \pi \circ \tilde{\alpha}_J \circ \chi \simeq \text{Spec} \tilde{\beta}_J \circ \text{Spec} \Psi \circ \chi
\]

\[
\simeq \text{Spec} \beta_J \circ \text{Spec} \iota \circ \text{Spec} \Xi \circ \chi \simeq \text{Spec} \beta_J \circ \text{Spec} \iota = \text{Spec} \beta_J,
\]

using (4.4) in the second step, $\Psi = \Xi \circ \iota$ in the third, $\text{Spec} \Xi, \chi$ quasi-inverse in the fourth, and $\beta_J = \iota \circ \beta_J$ in the fifth. Thus (3.2) homotopy commutes.

Similarly, if $\emptyset \neq K \subseteq J$ then

\[
\alpha_J = \tilde{\alpha}_J \circ \chi \simeq \text{Spec} \tilde{\Phi}_{JK} \circ \text{Spec} \Psi \circ \chi
\]

\[
\simeq \alpha_K \circ \text{Spec} \tilde{\Phi}_{JK} \circ \text{Spec} \iota \circ \text{Spec} \Xi \circ \chi \simeq \alpha_K \circ \text{Spec} \Phi_{JK}
\]

using (4.4) in the second step, $\Psi = \Xi \circ \iota$ in the third, and $\Phi_{JK} = \iota \circ \tilde{\Phi}_{JK}$ and $\text{Spec} \Xi, \chi$ quasi-inverse in the fourth. Hence (3.3) homotopy commutes.

This proves that Hypothesis 4.1 holds with $k = l + 1$, and completes the inductive step begun shortly after Remark 4.2. Hence by induction, Hypothesis 4.1 holds for all $k = 1, 2, \ldots$, so Hypothesis 4.1 holds for $k = \infty$. Theorem 3.9 follows, since all the conclusions of Theorem 3.2(i),(ii) are either part of Hypothesis 4.1 or for $A^*_i = A^*_i, \alpha_i = \alpha_i, \beta_i = \beta_i$ in part (i) were included in the first step of the induction. This completes the proof.

5 Proof of Theorem 3.9

5.1 Theorem 3.9(a): (*) is an open condition

Suppose $X, \omega_X, A^*, \alpha, V, E, F, s, t, \psi$ are as in Definition 3.8 and suppose that $U \subseteq V$ is open, $E^-$ is a real vector subbundle of $E|_U$, and $v \in s^{-1}(0) \cap U$, such that the assumptions on $E^-|_v$ in condition (*) hold at $v$. We must show that these assumptions also hold for all $v'$ in an open neighbourhood of $v$ in $s^{-1}(0) \cap U$. Suppose for a contradiction that this is false. Then we can choose a sequence $(v_i)_{i=1}^\infty$ in $s^{-1}(0) \cap U$ such that $v_i \to v$ as $i \to \infty$, and the assumptions on $E^-|_{v_i}$ in (*) do not hold for any $i = 1, 2, \ldots$.

By passing to a subsequence of $(v_i)_{i=1}^\infty$, we can assume that $\dim \text{Im} d_s|_{v_i}$ and $\dim \text{Ker} t|_{v_i}$ are independent of $i = 1, 2, \ldots$. By trivializing $E$ near $v$, we can regard $(\text{Im} d_s|_{v_i})_{i=1}^\infty$ and $(\text{Ker} t|_{v_i})_{i=1}^\infty$ as sequences in complex Grassmannians, which are compact. Thus, passing to a subsequence of $(v_i)_{i=1}^\infty$, we can assume
they converge, and there are complex vector subspaces \( I_v, K_v \subseteq E|_v \) such that \( \Im ds|_{v,i} \to I_v \) and \( \Ker t|_{v,i} \to K_v \) as \( i \to \infty \).

As \( i \to \infty \), we have \( \Im ds|_{v,i} \subseteq \Ker t|_{v,i} \), and so \( I_v \subseteq K_v \). Also \( \Im ds|_{v,i} \subseteq I_v \), since if \( w \in T_v V \) we can find \( w_i \in T_{v,i} V \) with \( w_i \to w \) as \( i \to \infty \), and then \( ds|_{v,i}(w_i) \to ds|_{v}(w) \) as \( i \to \infty \). Similarly \( K_v \subseteq \Ker t|_{v,i} \).

We now have a quotient vector space \((\Ker t|_v)/(\Im ds|_v)\), which as in (5.21) carries a nondegenerate quadratic form \( Q_v \). There are subspaces \( I_v/(\Im ds|_v) \subseteq K_v/(\Im ds|_v) \subseteq (\Ker t|_v)/(\Im ds|_v) \). Also, for each \( i = 1, 2, \ldots \) we have quotient space \((\Ker t|_{v,i})/(\Im ds|_{v,i})\) with quadratic forms \( Q_{v,i} \). As \( i \to \infty \) we have

\[
(\Ker t|_v)/(\Im ds|_v) \rightarrow K_v/I_v \cong [K_v/(\Im ds|_v)]/\left[I_v/(\Im ds|_v)\right].
\] (5.1)

One can prove using a representative \( \omega_{A^*} \) for \( \omega_{\bar{\omega}^*} \) that

\[
I_v/(\Im ds|_v) = \{ e \in (\Ker t|_v)/(\Im ds|_v) : \tilde{Q}_v(e,k) = 0 \ \forall k \in K_v/(\Im ds|_v) \},
\]

that is, \( I_v/(\Im ds|_v) \) and \( K_v/(\Im ds|_v) \) are orthogonal subspaces w.r.t. \( \tilde{Q}_v \). Hence the restriction of \( \tilde{Q}_v \) to \( K_v/(\Im ds|_v) \) is null along \( I_v/(\Im ds|_v) \), and descends to a nondegenerate quadratic form \( Q_v \) on \( [K_v/(\Im ds|_v)]/\left[I_v/(\Im ds|_v)\right] \equiv K_v/I_v \). Then under the limit (5.21), we have \( \tilde{Q}_v \to Q_v \) as \( i \to \infty \).

By (\*) for \((U,E^-)\) at \( v \), we have \( \Im ds|_v \cap E^-|_v = \{0\} \), and the map \( \Pi_v : E^-|_v \cap \Ker(t|_v) \to (\Ker t|_v)/(\Im ds|_v) \) in (3.21) has image \( \Im \Pi_v \) of half the total dimension, with \( \Re Q_v \) negative definite on \( \Im \Pi_v \). Since \( Q_v \) is zero on \( I_v/(\Im ds|_v) \), it follows that \( \Im \Pi_v \cap (I_v/(\Im ds|_v)) = \{0\} \), and thus

\[
E^-|_v \cap I_v = \{0\}.
\] (5.2)

Condition 3.23 that \( t|_v(E^-|_v) = t|_v(E|_v) \), is equivalent to the equation \( E^-|_v + \Ker(t|_v) = E|_v \), in subspaces of \( E|_v \). As \( \Im \Pi_v \) is a maximal negative definite subspace for \( \Re Q_v \) in \( (\Ker t|_v)/(\Im ds|_v) \), and \( K_v/(\Im ds|_v) \) is the orthogonal to a null subspace \( I_v/(\Im ds|_v) \) w.r.t. \( \Re Q_v \), it follows that \( \Im \Pi_v + K_v/(\Im ds|_v) = (\Ker t|_v)/(\Im ds|_v) \). Lifting to \( \Ker t|_v \) gives \( [E^-|_v \cap (\Ker t|_v)] + K_v = \Ker t|_v \). Thus the subspace \( E^-|_v + K_v \) in \( E|_v \) contains \( E^-|_v \) and \( \Ker t|_v \), so as \( E^-|_v + \Ker(t|_v) = E|_v \), we see that

\[
E^-|_v + K_v = E|_v.
\] (5.3)

Write \( \tilde{\Pi}_v : E^-|_v \cap K_v \to K_v/I_v \) for the natural projection. It is injective by (5.2). Using (5.21) and the facts that \( \Im \Pi_v \) has half the dimension of \( (\Ker t|_v)/(\Im ds|_v) \), and \( \dim[I_v/(\Im ds|_v)] + \dim[K_v/(\Im ds|_v)] = \dim[\Ker t|_v/(\Im ds|_v), K_v/(\Im ds|_v)] \) are orthogonal subspaces, by a dimension count we find that \( \Im \Pi_v \) has half the total dimension of \( K_v/I_v \). Also, since the quadratic form on \( \tilde{Q}_v \) on \( K_v/I_v \cong [K_v/(\Im ds|_v)]/\left[I_v/(\Im ds|_v)\right] \) descends from the restriction of \( Q_v \) to \( K_v/(\Im ds|_v) \), and \( \Im \Pi_v \) descends from \( \Im \Pi_v \cap [K_v/(\Im ds|_v)] \), and \( \Re Q_v \) is negative definite on \( \Im \Pi_v \), we see that \( \Re Q_v \) is negative definite on \( \Im \Pi_v \).

As \( E^-|_{v,i} \to E^-|_v \) and \( \Im ds|_{v,i} \to I_v \) as \( i \to \infty \), we see from (5.2) that

\[
E^-|_{v,i} \cap (\Im ds|_{v,i}) = \{0\} \quad \text{for } i > 0.
\] (5.4)
Since $E^{-}|_{v_i} \to E^{-}|_{v_i}$ and $\text{Ker} t|_{v_i} \to K_v$ as $i \to \infty$, we see from (5.3) that $E^{-}|_{v_i} + \text{Ker} t|_{v_i} = E|_{v_i}$ for $i \gg 0$. But this is equivalent to
\[ t|_{v_i}(E^{-}|_{v_i}) = t|_{v_i}(E|_{v_i}) \quad \text{in } F|_{v_i} \text{ for } i \gg 0. \] (5.5)

Using (5.4)–(5.5), the same dimension count as above implies that $\text{Im} \tilde{\Pi}_{v_i}$ has half the dimension of $(\text{Ker} \ t|_{v_i})/\text{Im} \ ds|_{v_i}$ for $i \gg 0$. Under the limit (5.1), we have $\tilde{Q}_{v_i} \to \tilde{Q}$ and $\text{Im} \tilde{\Pi}_{v_i} \to \text{Im} \Pi$. Thus, as $\text{Re} \tilde{Q}_v$ is negative definite on $\text{Im} \tilde{\Pi}_{v_i}$, we see that $\text{Re} \tilde{Q}_{v_i}$ is negative definite on $\text{Im} \tilde{\Pi}_{v_i}$ for $i \gg 0$. Together with (5.4)–(5.5), this shows that the assumptions on $E^{-}|_{v_i}$ in (5) hold for $i \gg 0$, which contradicts the choice of sequence $(v_i)_{i=1}^{\infty}$. This proves Theorem 3.9(a).

### 5.2 Theorem 3.9(b): extending pairs $(U, E^-)$ satisfying (5)

Suppose $X, \omega_X, A, \alpha, V, E, F, s, t, \psi$ are as in Definition 3.8 and $(U, E^-)$ satisfying (5) is as in Definition 3.8 and $C \subseteq V$ is closed with $C \subseteq U$. Our goal is to construct $(\tilde{U}, \tilde{E}^-)$ satisfying (5) for $V, E, \ldots$ with $C \cup s^{-1}(0) \subseteq \tilde{U} \subseteq V$, such that $E^-|_{\tilde{U}} = \tilde{E}^-|_{\tilde{U}}$ for an open neighbourhood of $C$ in $U \cap \tilde{U}$.

Using the notation of (3.2), $s^{-1}(0) \text{alg}$ is a finite type closed $C$-subscheme of $V \text{alg}$, and the maps $v \mapsto \dim \text{Ker} \ ds|_v$, $v \mapsto \dim \text{Ker} t|_v$ are upper semicontinuous, algebraically constructible functions $s^{-1}(0) \text{alg} \to \mathbb{N}$, noting that $t|_v$ is independent of choices for $v \in s^{-1}(0) \text{alg}$. Therefore by some standard facts about constructible sets in algebraic geometry, we can choose a stratification of Zariski topological spaces $s^{-1}(0) \text{alg} = \coprod_{a \in A} W^a \text{alg}$, where $A$ is a finite indexing set, and $W^a \text{alg}$ is a smooth, connected, locally closed $C$-subscheme of $s^{-1}(0) \text{alg} \subseteq V \text{alg}$ for each $a \in A$, with closure $\overline{W^a} \text{alg}$ in $s^{-1}(0) \text{alg}$ a finite union of strata $W_b$, such that $v \mapsto \dim \text{Ker} \ ds|_v$ and $v \mapsto \dim \text{Ker} t|_v$ are both constant functions on $W^a \text{alg}$.

Writing $W_a \subseteq s^{-1}(0) \subseteq V$ for the set of $C$-points of $W^a \text{alg}$, each $W_a$ is a connected, locally closed complex submanifold of $V$ lying in $s^{-1}(0)$, with closure $\overline{W}_a$ a finite union of submanifolds $W_b$, such that $s^{-1}(0) = \bigcup_{a \in A} W_a$. On each $W_a$, the maps $v \mapsto \dim \text{Ker} \ ds|_v$ and $v \mapsto \dim \text{Ker} t|_v$ are constant. This implies that $\text{Ker} \ ds|_{W_a}$ is a holomorphic vector subbundle of $TV|_{W_a}$, and $\text{Im} \ ds|_{W_a}$ a holomorphic vector subbundle of $E|_{W_a}$, and $\text{Ker} t|_{W_a}$ a holomorphic vector subbundle of $E|_{W_a}$, and $\text{Im} t|_{W_a}$ a holomorphic vector subbundle of $F|_{W_a}$. We have $\text{Im} \ ds|_{W_a} \subseteq \text{Ker} t|_{W_a} \subseteq E|_{W_a}$ as $t \circ ds = 0$ on $s^{-1}(0)$.

Thus we have a holomorphic vector bundle $(\text{Ker} t|_{W_a})/(\text{Im} \ ds|_{W_a})$ over $W_a$, whose fibre at $v \in W_a$ is identified with $H^1(T_X|_x)$ for $x = \psi(v)$ by (3.9). As in (3.2), we have a quadratic form $Q_v$ on $H^1(T_X|_x)$, and as in (3.24), $Q_v$ is the quadratic form on $(\text{Ker} t|_{W_a})/(\text{Im} \ ds|_{W_a})|_v$ identified with $Q_x$ by (3.9). One can prove using a representative $\omega_a$ for $\alpha^*(\omega_X)$ that $\tilde{Q}_v$ depends holomorphically on $v \in W_a$. Hence $\tilde{Q}_v = \tilde{Q}_a|_v$ for $\tilde{Q}_a \in H^0(S^2((\text{Ker} t|_{W_a})/(\text{Im} ds|_{W_a}))^*)$, a non-degenerate holomorphic quadratic form on the fibres of $(\text{Ker} t|_{W_a})/(\text{Im} ds|_{W_a})$.

The idea of the proof is to choose $\tilde{E}^-$ near $W_a$ by induction on increasing $\dim W_a$, starting with $a \in A$ with $\dim W_a = 0$, then $\dim W_a = 1$, and so on. Since $\dim(\overline{W}_a \setminus W_a) < \dim W_a$, we see that $\overline{W}_a \setminus W_a$ is a finite union of $W_b$ with $\dim W_b < \dim W_a$, so when we choose $\tilde{E}^-$ near $W_a$ we will already have chosen $\tilde{E}^-$ near $\overline{W}_a \setminus W_a$, and the extension over $W_a$ should be compatible with this.
Our inductive hypothesis \((\dagger)_m\) for \(m = 0, 1, 2, \ldots\) is:

\((\dagger)_m\) For all \(a \in A\) with \(\dim W_a \leq m\) we have chosen a pair \((\tilde{U}_a, \tilde{E}_a^-)\) satisfying 

\((*)\) for \(V, E, s, t, \ldots\) with \(W_a \subseteq \tilde{U}_a \subseteq V\), such that there is an open neighbourhood  \(\tilde{U}_a\) of \(C \cap \tilde{U}_a\) in \(U \cap \tilde{U}_a\) with \(E^-|_{\tilde{U}_a} = \tilde{E}_a^-|_{\tilde{U}_a}\), and if \(b \in A\) with \(W_b \subseteq \overline{W}_a \setminus W_a\) (which implies that \(\dim W_b < \dim W_a \leq m\), so \((\tilde{U}_b, \tilde{E}_b^-)\) is defined), then there is an open neighbourhood \(\tilde{U}_{ab}\) of \(W_b\) in \(\tilde{U}_b\) such that \(\tilde{E}_a^-|_{\tilde{U}_a \cap \tilde{U}_{ab}} = \tilde{E}_b^-|_{\tilde{U}_a \cap \tilde{U}_{ab}}\).

First consider how to choose \((\tilde{U}_a, \tilde{E}_a^-)\) satisfying \((*)\) with \(W_a \subseteq \tilde{U}_a \subseteq V\) for \(a \in A\) with no compatibility conditions, either with \((U, E^-)\) near \(C\), or with \((\tilde{U}_b, \tilde{E}_b^-)\) for \(W_b \subseteq \overline{W}_a \setminus W_a\). We can do this as follows:

(i) Choose a real vector subbundle \(\tilde{E}_a\) of \((\text{Ker} t|_{W_a})/(\text{Im} ds|_{W_a})\), whose real rank is half the real rank of \((\text{Ker} t|_{W_a})/(\text{Im} ds|_{W_a})\), such that \(\text{Re} \tilde{Q}_a\) is negative definite on \(\tilde{E}_a\).

(ii) Lift \(\tilde{E}_a\) to a real vector subbundle \(\hat{E}_a\) of \(t|_{W_a}\). That is, the projection \(t|_{W_a} \to (\text{Ker} t|_{W_a})/(\text{Im} ds|_{W_a})\) induces an isomorphism \(\hat{E}_a \to \tilde{E}_a\).

(iii) Choose a real vector subbundle \(\tilde{E}_a\) of \(E|_{W_a}\) with \(E|_{W_a} = \tilde{E}_a \oplus \text{Ker} t|_{W_a}\).

(iv) Set \(\tilde{E}_a^-|_{W_a} = \tilde{E}_a \oplus \hat{E}_a\). Then \(\tilde{E}_a^-|_{W_a}\) is a real vector subbundle of \(E|_{W_a}\), and the assumptions on \(\tilde{E}_a^-|_v\) in condition \((*)\) in \(\S 3.3\) hold for all \(v \in W_a\).

(v) Choose any real vector subbundle \(\tilde{E}_a^-\) of \(E|_{\tilde{U}_a}\) on a small open neighbourhood \(\tilde{U}_a\) of \(W_a\) in \(V\), extending the given \(\tilde{E}_a^-|_{W_a} = \tilde{E}_a \oplus \hat{E}_a\) on \(W_a\).

Observe that by Theorem \(\S 3.3\) \((a)\), proved in \(\S 3.1\) condition \((*)\) holds for \(\tilde{E}_a^-\) on an open neighbourhood of \(W_a\). So by making \(\tilde{U}_a\) smaller, we can suppose \((\tilde{U}_a, \tilde{E}_a^-)\) satisfies \((*)\). All of these are possible. Any \((\tilde{U}_a, \tilde{E}_a^-)\) satisfying \((*)\) with \(W_a \subseteq \tilde{U}_a \subseteq V\) arises from steps \((i)-(v)\) (though \(\hat{E}_a\) in \((iii)\) is not uniquely determined by \(\tilde{E}_a^-\)). Furthermore (taking germs in \((v)\)), the space of choices in each step is contractible.

Now suppose \(m = 0, 1, \ldots\), and \((\dagger)_{m-1}\) holds if \(m > 0\), and \(a \in A\) with \(\dim W_a = m\). To choose \((\tilde{U}_a, \tilde{E}_a^-)\) with the compatibility conditions required in \((\dagger)_m\), we follow \((i)-(v)\), but modified as follows. In step \((i)\), we choose \(\tilde{E}_a\) with

\[
\tilde{E}_a|_{W_a \cap \tilde{U}_a} = \left[(E^- \setminus \text{Ker} t)|_{W_a \cap \tilde{U}_a} + (\text{Im} ds|_{W_a \cap \tilde{U}_a})\right]/(\text{Im} ds|_{W_a \cap \tilde{U}_a}), \tag{5.6}
\]

for some small open neighbourhood \(\tilde{U}_a\) of \(C \cap \tilde{U}_a\) in \(U\), and if \(b \in A\) with \(W_b \subseteq \overline{W}_a \setminus W_a\) then

\[
\tilde{E}_a|_{W_a \cap \tilde{U}_{ab}} = \left[\tilde{E}_b^- \setminus \text{Ker} t|_{W_a \cap \tilde{U}_{ab}}\right]/(\text{Im} ds|_{W_a \cap \tilde{U}_{ab}}) \tag{5.7}
\]

for some small open neighbourhood \(\tilde{U}_{ab}\) of \(W_b\) in \(\tilde{U}_b\).

To see this is possible, first note that the first part of \((\dagger)_{m-1}\) with \(b\) in place of \(a\) implies that equations \(\tag{5.6}\) and \(\tag{5.7}\) are compatible, that is they prescribe the same value for \(\tilde{E}_a\) on \(W_a \cap \tilde{U}_a \cap \tilde{U}_{ab}\), provided the open neighbourhoods \(\tilde{U}_a, \tilde{U}_{ab}\) are small enough. Also given distinct \(b, b' \in A\) with \(W_{b}, W_{b'} \subseteq \overline{W}_a \setminus W_a\), either
This is possible provided \( \mathcal{U}_{ab} \) may not extend continuously to the closure \( \mathcal{W}_a \). Kuranishi neighbourhoods, as in \( \S \) (3.37) we explained how to pullback pairs \((U_K, E_K^-)\) satisfying (\ast\) along a submersion \( \Phi_{JK} : A_K^* \to A_J^* \). We can also pushforward \((U_J, E_J^-)\) along \( \Phi_{JK} \).

**Definition 5.1.** Let \( X, \omega_X, n, \Phi_{JK} : A_K^* \to A_J^* \) and \( V_J, E_J, V_J, E_J \) be as in Definition (3.12) and suppose \((U_J, E_J^-)\) satisfies (\ast\) for \( A_J^* \). Our goal is to construct \((U_K, E_K^-)\) satisfying (\ast\) for \( A_K^* \), with \( \psi_J(s_J^{-1}(0) \cap U_J) = \psi_K(s_K^{-1}(0) \cap U_K) \subseteq X_{an} \), and if \((U_J, E_J^-), (U_K, E_K^-)\) also satisfy (\dagger\), a coordinate change of Kuranishi neighbourhoods, as in (2.5).

\[
(U_K, \theta_{KJ}, \eta_{KJ}) : (U_K, E_K^+, s_K^+, \psi_K) \to (U_J, E_J^+, s_J^+, \psi_J). \tag{5.8}
\]

Let \( v_J \in s_J^{-1}(0) \cap U_J \) with \( \phi_{JK}(v_J) = v_K \in s_K^{-1}(0) \subseteq V_K \) and \( \psi_J(v_J) = \psi_K(v_K) = x \in X_{an} \). We claim that we can choose splittings of real vector spaces

\[
T_{v_J} V_J = \tilde{T}_{v_J} V_J \oplus T_{v_J}^r V_J, \quad E_J|_{v_J} = \tilde{E}_{J|_{v_J}} \oplus E_J^r|_{v_J} \oplus E''_J|_{v_J},
\]

\[
E_J^r|_{v_J} = \tilde{E}_J^r|_{v_J} \oplus E''_J^r|_{v_J}, \quad F_J|_{v_J} = \tilde{F}_J|_{v_J} \oplus F''_J|_{v_J} \oplus F'''_J|_{v_J}. \tag{5.9}
\]

5.3 **Theorem (3.9)**: \( s^{-1}(0) = (s^+)^{-1}(0) \) locally in \( U \)

In (3.3) we explained how to pull back pairs \((U_K, E_K^-)\) satisfying (\ast\) along a submersion \( \Phi_{JK} : A_K^* \to A_J^* \). We can also pushforward \((U_J, E_J^-)\) along \( \Phi_{JK} \).

**Definition 5.1.** Let \( X, \omega_X, n, \Phi_{JK} : A_K^* \to A_J^* \) and \( V_J, E_J, V_J, E_J \) be as in Definition (3.12) and suppose \((U_J, E_J^-)\) satisfies (\ast\) for \( A_J^* \). Our goal is to construct \((U_K, E_K^-)\) satisfying (\ast\) for \( A_K^* \), with \( \psi_J(s_J^{-1}(0) \cap U_J) = \psi_K(s_K^{-1}(0) \cap U_K) \subseteq X_{an} \), and if \((U_J, E_J^-), (U_K, E_K^-)\) also satisfy (\dagger\), a coordinate change of Kuranishi neighbourhoods, as in (2.5).

\[
(U_K, \theta_{KJ}, \eta_{KJ}) : (U_K, E_K^+, s_K^+, \psi_K) \to (U_J, E_J^+, s_J^+, \psi_J). \tag{5.8}
\]

Let \( v_J \in s_J^{-1}(0) \cap U_J \) with \( \phi_{JK}(v_J) = v_K \in s_K^{-1}(0) \subseteq V_K \) and \( \psi_J(v_J) = \psi_K(v_K) = x \in X_{an} \). We claim that we can choose splittings of real vector spaces

\[
T_{v_J} V_J = \tilde{T}_{v_J} V_J \oplus T_{v_J}^r V_J, \quad E_J|_{v_J} = \tilde{E}_{J|_{v_J}} \oplus E_J^r|_{v_J} \oplus E''_J|_{v_J},
\]

\[
E_J^r|_{v_J} = \tilde{E}_J^r|_{v_J} \oplus E''_J^r|_{v_J}, \quad F_J|_{v_J} = \tilde{F}_J|_{v_J} \oplus F''_J|_{v_J} \oplus F'''_J|_{v_J}. \tag{5.9}
\]

59
fitting into a commutative diagram of the form

\[
\begin{array}{cccccc} 
0 & \to & T_{v_j}V_J & \to & E_j^{(i)}_{\mid v_j} & \to & F_j^{(i)}_{\mid v_j} & \to & 0 \\
0 & \to & T_{v_j}V_K & \to & E_K^{(i)}_{\mid v_j} & \to & F_K^{(i)}_{\mid v_j} & \to & 0 \\
\end{array}
\]

\[
\begin{bmatrix}
\ast & 0 \\
0 & \ast & 0 \\
0 & \ast & 0 \\
\ast & 0 & \ast & 0 & \ast & 0 & \ast & 0
\end{bmatrix}
\]

(5.10)

To prove this, note that the rows of (5.10) are $\mathbb{A}^n_{\text{Spec } A_j^{\ast}, \text{Spec } A_j^{\ast}} \mid v_j$, and are complexes, and the lower columns are induced by $\Phi_{jk}$, so injective and induce isomorphisms on cohomology as in (3.32). Then:

(i) Define $T_{v_j}V_J = \ker d^J_{\mid v_j}$.

(ii) Choose arbitrary $T_{v_j}V_J$ with $T_{v_j}V_J \cong T_{v_j}V_J \oplus T'_vV_J$. Then $T_{v_j}V_J \cong T_{v_j}V_J$ as $d^J_{\mid v_j}$ is surjective.

(iii) Define $E'_j_{\mid v_j} = d^J_{\mid v_j}[T'_vV_J]$. Then $E'_j_{\mid v_j} \cong T'_vV_J$ as the columns of (5.10) are isomorphisms in cohomology, and $E'_j_{\mid v_j} \subseteq \ker(\chi_{jk})_{\mid v_j}$ as the left hand square of (5.10) commutes.

(iv) Choose $E''_j_{\mid v_j}$ with $\ker(\chi_{jk})_{\mid v_j} = E'_j_{\mid v_j} \oplus E''_j_{\mid v_j}$.

(v) Since the columns of (5.10) are isomorphisms on cohomology, $t^J_{\mid v_j}$ is injective on $E''_j_{\mid v_j}$. Define $F''_j_{\mid v_j} = t^J_{\mid v_j}[E''_j_{\mid v_j}]$. Then $F''_j_{\mid v_j} \cong E''_j_{\mid v_j}$. Also $F''_j_{\mid v_j} \subseteq \ker \xi_{jk} \mid v_j$, as the right hand square of (5.10) commutes.

(vi) Choose $F''_j_{\mid v_j}$ with $Ker \xi_{jk} \mid v_j = F''_j_{\mid v_j} \oplus F''_j_{\mid v_j}$.

(vii) Since the columns of (5.10) are isomorphisms on cohomology, we have

\[
F'_j_{\mid v_j} = \ker(\chi_{jk})_{\mid v_j} \cap \ker t^J_{\mid v_j} \cap \ker F''_j_{\mid v_j} = \ker(\chi_{jk})_{\mid v_j} \cap \ker t^J_{\mid v_j} \cap \ker F''_j_{\mid v_j}.
\]

Thus we may choose $F''_j_{\mid v_j}$ with $F'_j_{\mid v_j} = F''_j_{\mid v_j}$ and $\ker t^J_{\mid v_j} \subseteq F'_j_{\mid v_j} \oplus F''_j_{\mid v_j}$ and $\ker F''_j_{\mid v_j} \subseteq F'_j_{\mid v_j} \oplus F''_j_{\mid v_j}$. So the third row of $t_{\mid v_j}$ in (5.10) is zero. Also $F''_j_{\mid v_j}$ is surjective.

(viii) Set $E''_j_{\mid v_j} = E'_j_{\mid v_j} \cap t^J_{\mid v_j}[F''_j_{\mid v_j}]$. We claim $\chi_{jk} \mid v_j$ is injective on $E''_j_{\mid v_j}$.

To see this, note that we have an exact sequence

\[
0 \to E''_j_{\mid v_j} \cap \ker t^J_{\mid v_j} \to E''_j_{\mid v_j} \to t^J_{\mid v_j}[E''_j_{\mid v_j}] \cap F''_j_{\mid v_j} \to 0,
\]

as $\ker t^J_{\mid v_j} \subseteq t^J_{\mid v_j}[F''_j_{\mid v_j}]$. The last part of (5) implies that $\chi_{jk} \mid v_j$ maps $E''_j_{\mid v_j} \cap \ker t^J_{\mid v_j}$ injectively into $\ker t^J_{\mid v_j}$. Also $\xi_{jk} \mid v_j$ is injective on $F''_j_{\mid v_j}$, and the right square of (5.10) commutes, so the claim follows.

60
(ix) Choose $\tilde{E}_J|_{v_J} \subseteq E_J|_{v_J}$ such that $\tilde{E}_J|_{v_J} \subseteq \tilde{E}_J|_{v_J}$ and $E_J|_{v_J} = E_J|_{v_J} \oplus \text{Ker}(\chi_{JK}|_{v_J}) = E_J|_{v_J} \oplus E_J'|_{v_J} \oplus E_J''|_{v_J}$ by (iv) and $t_J|_{v_J}[\tilde{E}_J|_{v_J}] \subseteq F_J|_{v_J}$. This is possible as $\chi_{JK}|_{v_J}$ is injective on $\tilde{E}_J|_{v_J}$, and using (v), (vii) and (viii). Then $\tilde{E}_J|_{v_J} \cong E_K|_{v_K}$ as $\chi_{JK}$ is surjective.

(x) Choose $\tilde{E}'_J|_{v_J}$ such that $E_J'|_{v_J} = \tilde{E}_J|_{v_J} \oplus \tilde{E}'_J|_{v_J}$ and $t_J|_{v_J}[\tilde{E}'_J|_{v_J}] \subseteq F'_J|_{v_J}$.

This is possible by (viii) and $\text{Im } t_J|_{v_J} \subseteq F_J|_{v_J} \oplus F'_J|_{v_J}$.

As $t_J|_{v_J}(E_J|_{v_J}) = t_J|_{v_J}(E_J|_{v_J})$ by (5.23) and $F''_J|_{v_J} = t_J|_{v_J}[E'_J|_{v_J}]$, we see that $t_J|_{v_J}[\tilde{E}'_J|_{v_J}] = F''_J|_{v_J}$. Also $t_J|_{v_J}: E_J|_{v_J} \to F''_J|_{v_J}$ is injective, as $\text{Ker } t_J|_{E_J|_{v_J}} \subseteq E_J|_{v_J}$ by (viii). Hence $E''_J|_{v_J} \cong F''_J|_{v_J}$.

We can do all this, not just at one $v_J \in s^{-1}(0) \cap U_J$, but in an open neighbourhood $U'_J$ of $s^{-1}(0) \cap U_J$. That is, we can choose $U'_J$, and splittings

$$TV_J|_{U'_J} = TV_J \oplus T' V_J, \quad E_J|_{U'_J} = \tilde{E}_J \oplus E'_J \oplus E''_J|_{v_J},$$

$$E'_J|_{U'_J} = \tilde{E}'_J \oplus \tilde{E}''_J, \quad F'_J|_{U'_J} = F''_J \oplus F''_J \oplus F''_J,$$

with $\tilde{E}_J \subseteq \tilde{E}_J$, such that $\text{(5.10)}$ holds at each $v_J \in s^{-1}(0) \cap U_J$. To see this, note that the argument above can be carried out on $s^{-1}(0) \cap U_J$ regarded as a $C^\infty$-subscheme of $U_J$, in the sense of $C^\infty$-algebraic geometry in [20], and the splittings $\text{(5.11)}$ with $\tilde{E}_J \subseteq \tilde{E}_J$ can then be extended from $s^{-1}(0) \cap U_J$ to an open neighbourhood $U'_J$. Making $U'_J$ smaller, we can suppose the component of $\chi_{JK}$ mapping $\tilde{E}_J \to \phi_{JK}|_{U'_J}(E_K)$ is an isomorphism. We can also choose the splittings so that away from $s^{-1}(0) \cap U_J$, $t_J|_{U'_J}$ has the form

$$t_J|_{U'_J} = \begin{pmatrix} * & * & 0 \\ * & * & \cong \\ 0 & 0 & 0 \end{pmatrix}: E_J|_{v_J} \oplus E'_J \oplus E''_J \to \tilde{F}_J \oplus F''_J \oplus F''_J. \quad \text{(5.12)}$$

Write $s_J|_{U'_J} = \tilde{s}_J \oplus s'_J \oplus s''_J$, for $\tilde{s}_J \in C^\infty(\tilde{E}_J)$, $s'_J \in C^\infty(E'_J)$ and $s''_J \in C^\infty(E''_J)$. Then $\text{(5.12)}$ and $t_J|_{v_J} = 0$ imply that $s''_J = 0$. From $\text{(5.10)}$ we see that at each $v_J \in s^{-1}(0) \cap U_J$, $\text{ds}'_J|_{v_J} \circ T_{\tilde{E}_J} V_J \to E'_J|_{v_J}$ is injective, and $\text{ds}_{JK}|_{v_J} : \text{Ker } (\text{ds}'_J|_{v_J}) \to T_{\tilde{E}_J} V_K$ is an isomorphism. Hence $s'_J$ is transverse near $v_J$ so that $(s'_J)^{-1}(0)$ is an embedded submanifold of $V_J$ near $v_J$ with tangent space $\text{Ker } (\text{ds}'_J|_{v_J})$ at $v_J$, and $\phi_{JK}|_{(s'_J)^{-1}(0)}: (s'_J)^{-1}(0) \to V_K$ is a local diffeomorphism near $v_J$. Thus, making $U'_J$ smaller, we can suppose that $s'_J$ is transverse on $U'_J$, so that $(s'_J)^{-1}(0)$ is an embedded submanifold of $U'_J$, and $\phi_{JK}|_{(s'_J)^{-1}(0)}: (s'_J)^{-1}(0) \to V_K$ is a local diffeomorphism. But $\phi_{JK}$ is injective on $s^{-1}(0) \cap U_J$, so making $U'_J$ smaller, we can also suppose that $\phi_{JK}|_{(s'_J)^{-1}(0)}$ is a diffeomorphism with an open set $U_K \subset V_K$, with inverse $\theta_{JK}: U_K \xrightarrow{\cong} (s'_J)^{-1}(0) \subseteq U'_J \subseteq U_J$.

We now have a vector bundle $\theta_{JK}(E_J)$ of $U_K$, and vector subbundles

$$\theta_{JK}(\tilde{E}_J), E'_J, E''_J, E''_J, \tilde{E}_J, E''_J, (\text{with } \theta_{JK}(E_J) = \theta_{JK}(\tilde{E}_J) = \theta_{JK}(E'_J) = \theta_{JK}(E''_J), \theta_{JK}(\tilde{E}_J) = \theta_{JK}(E'_J) \oplus \theta_{JK}(E''_J) \oplus \theta_{JK}(E''_J), \theta_{JK}(E'_J) \subseteq \theta_{JK}(\tilde{E}_J)).$$

Since $\phi_{JK} \circ \theta_{JK} = \text{id}_{U_K}$, pulling back $\chi_{JK}: E_J \to \phi_{JK}(E_K)$ by $\theta_{JK}$ gives a surjective vector
bundle morphism \( \theta^*_{K,J}(\chi_{JK}) : \theta^*_{K,J}(E_j) \to E_K|_{U_K} \), where \( \theta^*_{K,J}(\chi_{JK}) \) restricts to an isomorphism \( \theta^*_{K,J}(E_j) \to E_K \). We also have a section \( \theta^*_{K,J}(s_j) \) of \( \theta^*_{K,J}(E_j) \), whose components in \( \theta^*_{K,J}(\tilde{E}_j), \theta^*_{K,J}(E'_j), \theta^*_{K,J}(E''_j) \) are \( \theta^*_{K,J}(s_j), 0, 0 \). Applying \( \theta^*_{K,J} \) to \( (5.4) \) and using \( E''_j \subseteq \text{Ker} \chi_{JK} \) shows that

\[
\theta^*_{K,J}(\chi_{JK})[\theta^*_{K,J}(s_j)] = \theta^*_{K,J}(\chi_{JK})[\theta^*_{K,J}(\tilde{s}_j)] = s_K|_{U_K}. \tag{5.13}
\]

Define a vector subbundle \( E_K^- \subseteq E_K|_{U_K} \) by \( E_K^- = \theta^*_{K,J}(\chi_{JK})[\theta^*_{K,J}(\tilde{E}_j)] \). This is valid as \( \theta^*_{K,J}(\tilde{E}_j) \subseteq \theta^*_{K,J}(E_j) \), and \( \theta^*_{K,J}(\chi_{JK}) \) is an isomorphism on \( \theta^*_{K,J}(E_j) \). We claim that \( (U_K, E_K^-) \) satisfies condition \((*)\). To see this, let \( v_K \in s_K^{-1}(0) \cap U_K \), and set \( v_J = \theta^*_{K,J}(v_K) \). Then \( v_J \in s_J^{-1}(0) \cap U'_j \) with \( \phi_{JK}(v_J) = v_K \), so \((5.9) \sim (5.10)\) hold, with the columns of \((5.10)\) isomorphisms on cohomology. From this and \((*)\) for \((U_J, E'_j) \) at \( v_J \), we can deduce \((*)\) for \((U_K, E_K^-) \) at \( v_K \).

Writing \( E^+_j = E_j|_{U_j}/E'_j, s^+_j = s_j + E'_j \in C^\infty(E^+_j) \), and similarly for \( E_K^+, s_K^+ \), define a vector bundle morphism \( \eta_{K,J} : E_K^+ \to \theta^*_{K,J}(E^+_j) \) by

\[
\eta_{K,J} : e_K + E_K^- \mapsto \theta^*_{K,J}(\chi_{JK})[\theta^*_{K,J}(\tilde{E}_j)]|_{e_K} + \theta^*_{K,J}(E'_j).
\]

This is well-defined as \( \theta^*_{K,J}(\chi_{JK})[\theta^*_{K,J}(\tilde{E}_j) : \theta^*_{K,J}(E_j) \to E_K \) is an isomorphism, with inverse \( \theta^*_{K,J}(\chi_{JK})[\theta^*_{K,J}(\tilde{E}_j)] : E_K \to \theta^*_{K,J}(E_j) \), which maps \( E_K^- \to \theta^*_{K,J}(E'_j) \subseteq \theta^*_{K,J}(E_j) \) by definition of \( E_K^- \). Also \((5.13)\) implies that \( \eta_{K,J}(s_K^+) = \theta^*_{K,J}(s_j) \).

Using \((5.10)\) we can also show that the analogue of \((2.9)\) for \( \eta_{K,J}, v_K \) at \( v_K \) is exact. Therefore, if \((U_J, E'_j), (U_K, E_K^-)\) also satisfy \((*)\), then \((U_K, \eta_{K,J}, v_K) \) in \((5.8)\) is a coordinate change of Kuranishi neighbourhoods. This completes Definition \((5.1)\).

We now prove Theorem \((3.9)\)(c). Suppose \( X, \omega_X, A^*, \alpha, V, E, F, s, t, \psi \) and \((U, E^-)\) satisfying \((*)\) are as in Definition \((3.8)\). Then \( X' := \alpha(\text{Spec} A^*) \subseteq X \) is an affine derived \( \mathcal{C}\)-subscheme of \( X \), and so is separated. Let \( v \in s^{-1}(0) \cap U \), and set \( x = \psi(v) \in X_{\text{an}} \). Write \((A^*_X, \alpha_X) = (A^*, \alpha) \), \( V_1 = V, E_1 = E, v_1 = v \) and so on. Applying Theorem \((2.9)\) to \((X', \omega_X|_{X'}) \) at \( x \) gives a pair \((A^*_X, \omega_X) \) in \(-2\)-Darboux form and a Zariski open inclusion \( \alpha_2 : \text{Spec} A^*_X \hookrightarrow X' \subseteq X \) which is minimal at \( x \in \text{Im} \alpha_2 \) with \( \alpha_2^*(\omega_X) \simeq \omega_{A^*_X} \). Section \((3.3)\) applied to \( A^*_X, \alpha_2 \) gives \( V_2, E_2, s_2, \ldots \). Set \( v_2 = \psi_2^{-1}(x) \in s_2^{-1}(0) \subseteq V_2 \).

Applying Theorem \((3.2)\) to the separated derived \( \mathcal{C}\)-scheme \( X' \) with \( I = \{1, 2\} \) and initial data \((A^*_X, \alpha_1, A^*_X, \alpha_2)\) gives \((A^*_{12}, \alpha_{12})\) with image \( \text{Im} \alpha_{12} = \text{Im} \alpha_1 \cap \text{Im} \alpha_2 \) and submersions \( \Phi_{12,1} : A^*_1 \to A^*_{12}, \Phi_{12,2} : A^*_2 \to A^*_{12} \) such that \((3.3)\) homotopy commutes in \( \text{dSch}_\mathcal{C} \). Section \((3.2)\) applied to \( A^*_{12} \) gives \( V_{12}, E_{12}, s_{12}, \ldots\), and to \( \Phi_{12,1}, \Phi_{12,2} \) gives \( \phi_{12,1} : V_{12} \to V_1 = V, \chi_{12,1}, \xi_{12,1} \) and \( \phi_{12,2} : V_{12} \to V_2, \chi_{12,2}, \xi_{12,2} \), simplifying notation a little. Set \( v_{12} = \psi_{12}^{-1}(x) \in s_{12}^{-1}(0) \subseteq V_{12} \), so that \( \phi_{12,1}(v_{12}) = v_1 \) and \( \phi_{12,2}(v_{12}) = v_2 \).

We have \((U, E^-)\) satisfying \((*)\) for \( A^*_X, \alpha_1, V_1, E_1, s_1, \ldots \). Thus by Lemma \((3.14)\) we can choose \((U_{12}, E_{12})\) satisfying \((*)\) for \( V_{12}, E_{12}, s_{12}, \ldots\) and compatible with \((U, E^-)\) under \( \phi_{12,1}, \chi_{12,1} \) in the sense of \((3.4)\) such that \( v_{12} \in s_{12}^{-1}(0) \cap \)}
\[ \phi_{12,1}^{-1}(U) \subseteq U_{12} \subseteq V_{12} \]. Also \(\chi_{12,1}^+\) defines \(\chi_{12,1}^+\) such that if \((U, E^-)\) and \((U_{12}, E_{12}^-)\) satisfy (\(\dagger\)) (we do not assume this), then

\[ (U_{12}, \phi_{12,1}(U_{12}), \chi_{12,1}^+) : (U_{12}, E_{12}^+, s_{12}^+, \psi_{12}^+) \rightarrow (U, E^+, s^+, \psi^+) \]  \hspace{1cm} (5.14)

is a coordinate change of Kuranishi neighbourhoods, as in Corollary 5.13.

Now apply Definition 5.1 to pushforward \((U_{12}, E_{12}^-)\) in \(V_{12}, E_{12}, s_{12}, \ldots\) along \(\phi_{12,2}, \chi_{12,2}, \xi_{12,2}\). This yields \((U_2, E_2^-)\) satisfying (\(\ast\)) for \(V_2, E_2, s_2, \ldots\) with \(\phi_{12,2}(s_{12}^{-1}(0) \cap U_{12}) \subseteq U_2 \subseteq V_2\), so in particular \(v_2 \in U_2\), and data \(\theta_{2,12}, \eta_{2,12}\) such that if \((U_2, E_2^-)\) and \((U_{12}, E_{12}^-)\) satisfy (\(\dagger\)) (we do not assume this), then

\[ (U_2, \theta_{2,12}, \eta_{2,12}) : (U_2, E_2^+, s_2^+, \psi_2^+) \rightarrow (U_{12}, E_{12}^+, s_{12}^+, \psi_{12}^+) \]  \hspace{1cm} (5.15)

is a coordinate change of Kuranishi neighbourhoods, as in (5.8).

Since \((A_2^*, \omega_2^*)\) is in \(-2-\)Darboux form and minimal at \(x\), Example 3.10 proves that there exists an open neighbourhood \(U'_2\) of \(v_2\) in \(U_2\) such that \(s_2^{-1}(0) \cap U'_2 = (s_2^+)^{-1}(0) \cap U'_2\). Then \((U'_2, E_2', U'_2)\) satisfies (\(\dagger\)). The construction in Definition 5.1 implies that \(\theta_{2,12}\) identifies \(s_{12}^{-1}(0)\) near \(v_2\) with \(s_{12}^{-1}(0)\) near \(v_{12}\), and identifies \((s_2^+)^{-1}(0)\) near \(v_2\) with \(s_{12}^+)^{-1}(0)\) near \(v_{12}\) (the second follows from the fact that the analogue of (2.4) for \(\theta_{2,12}, \eta_{2,12}\) at \(v_2, v_{12}\) is exact, so (5.12) is a coordinate change of Kuranishi neighbourhoods near \(v_2, v_{12}\)).

Similarly, \(\phi_{12,1}\) identifies \(s_{12}^{-1}(0)\) near \(v_{12}\) with \(s^{-1}(0)\) near \(v\), and identifies \((s_{12}^+)^{-1}(0)\) near \(v_{12}\) with \((s^+)^{-1}(0)\) near \(v\), so there exists an open neighbourhood \(U'_2\) of \(v\) in \(U\) such that \(s^{-1}(0) \cap U'_v = (s^+)^{-1}(0) \cap U'_v\). This holds for all \(v \in s^{-1}(0) \cap U\). Define \(U' = \bigcup_{v \in s^{-1}(0)} U'_v\). Then \(U'\) is an open neighbourhood of \(s^{-1}(0) \cap U\) in \(U\), and \(s^{-1}(0) \cap U' = (s^+)^{-1}(0) \cap U'\). Theorem 3.3(c) follows.

6 Proofs of some auxiliary results

Next we prove Propositions 3.15, 5.17 and 5.19.

6.1 Proof of Proposition 3.15

Let \(Z\) be a paracompact, normal topological space and \(\{R_i : i \in I\}\) an open cover of \(Z\). By paracompactness we can choose a locally finite refinement \(\{S_i : i \in I\}\) of \(\{R_i : i \in I\}\). That is, \(S_i \subseteq R_i \subseteq Z\) is open with \(\bigcup_{i \in I} S_i = Z\), and each \(z \in Z\) has an open \(z \in U_z \subseteq Z\) with \(U_z \cap S_i \neq \emptyset\) for only finitely many \(i \in I\).

We claim we can choose open sets \(T_i^1 \subseteq Z\) with closures \(\overline{T_i^1} \subseteq Z\) for \(i \in I\) such that \(T_i^1 \subseteq \overline{T_i^1} \subseteq S_i\) for \(i \in I\) and \(\bigcup_{i \in I} T_i^1 = Z\). To see this, fix a well-ordered total order \(\prec\) on \(I\). We will choose \(T_i^1\) in the order \(\prec\) by transfinite induction (that is, when we choose \(T_i^1\), we may assume we have already chosen
\(T_i^1\) for all \(i' \in I\) with \(i' < i\) satisfying the conditions that

\[
\begin{align*}
\bigcup_{j \in I \setminus i} T_j^1 \cup \bigcup_{j \in I : j \leq i} S_j &= Z, \\
\bigcup_{j \in I \setminus i} T_j^1 \cup \bigcup_{j \in I : j < i} S_j &= Z, \quad (6.1) \\
Z \setminus \left[ \bigcup_{j \in I \setminus i} T_j^1 \cup \bigcup_{j \in I : j < i} S_j \right] &\subseteq T_i^1 \subseteq T_i^1 \subseteq S_i. \quad (6.3)
\end{align*}
\]

Suppose \(i \in I\), and we have chosen \(T_i^1\) satisfying \((6.1)-(6.3)\) for all \(i' \in I\) with \(i' < i\). Then \((6.2)\) for all \(i' \in I\) with \(i' < i\) implies that \((6.1)\) holds for \(i\). (Since \((6.1)\) does not involve \(T_i^1\), this makes sense.) If there are no \(i' \in I\) with \(i' < i\) (the first step in the induction), then \((6.2)\) becomes \(\bigcup_{i \in I} S_i = Z\), which is true. Apply the normal condition on \(Z\) with closed subsets \(C = Z \setminus \left[ \bigcup_{j \in I \setminus i} T_j^1 \cup \bigcup_{j \in I : j < i} S_j \right]\) and \(D = Z \setminus S_i\). These are disjoint by \((6.1)\) for \(i\). Now for each finite \(\emptyset \neq J \subseteq I\), define a closed subset \(C_J \subseteq Z\) by

\[
C_J = \bigcap_{j \in J} T_j^1 \setminus \bigcup_{i \in I \setminus J} T_i^{\|J\|+1}. \quad (6.5)
\]

Then part (i) of the proposition follows from \(T_j^{\|J\|} \subseteq S_j \subseteq R_j\) for \(j \in J\) by \((6.4)\), and (ii) from \(\{S_i : i \in I\}\) locally finite with \(C_J \subseteq \bigcap_{i \in J} S_i\). For (iii), suppose \(\emptyset \neq J, K \subseteq I\) are finite with \(J \not\subseteq K\) and \(K \not\subseteq J\). Without loss of generality, suppose \(|J| \leq |K|\). Then there exists \(j \in J \setminus K\), and \((6.5)\) gives \(C_J \subseteq T_j^{\|J\|}\) and \(C_K \subseteq Z \setminus T_j^{\|K\|+1}\), which forces \(C_J \cap C_K = \emptyset\) as \(T_j^{\|J\|} \subseteq T_j^{\|K\|+1}\) by \((6.4)\).

For part (iv), if \(z \in Z\), define

\[
J_z = \bigcup_{J \subseteq I\text{ finite}, \, z \in \bigcap_{j \in J} T_j^{\|J\|}} J.
\]

Then \(J_z\) is finite as \(\{S_i : i \in I\}\) is locally finite, so \(z \in S_j\) for only finitely many \(j \in I\), and \(J_z\) is nonempty as \(\{T_i^1 : i \in I\}\) covers \(Z\), so \(z \in T_i^1 \subseteq T_i^1\) for some \(i \in I\), and \(J = \{i\}\) is a possible set in the union \((6.6)\). If \(j \in J_z\) then \(j \in J\) for some \(J\) in the union \((6.6)\), so that \(z \in T_j^{\|J\|} \subseteq T_j^{\|J\|}\) as \(|J| \leq |J_z|\). If \(i \in I \setminus J_z\) then \(z \notin \bigcap_{j \in J_z} T_j^{\|J_z\|+1}\), as \(J_z \cup \{i\}\) is not one of the sets \(J\) in \((6.6)\), but \(z \in \bigcap_{j \in J_z} T_j^{\|J_z\|+1}\), so \(z \notin T_i^{\|J_i\|+1}\). Hence \(z \in C_J\), by \((6.5)\), and part (iv) follows. This completes the proof of Proposition 6.15.
6.2 Proof of Proposition 3.17

We work in the situation of §5.5 just after Remark 3.16 so that we have data $X_n, I, V_j, E_j, s_j, \psi_j$ and $C_j \subseteq R_j = \bigcap_{i \in J} R_i \subseteq X_n$ for all $J \subseteq A$, and $\phi_{JK}, \chi_{JK}$ for all $J, K \subseteq A$ with $K \subseteq J$. We will first prove the following inductive hypothesis $(+)_m$ by induction on $m = 1, 2, \ldots$:

$(+)_m$ For all $J \in A$ with $|J| \leq m$, we can choose $(\tilde{U}_J, \tilde{E}_J)$ satisfying condition $(*)$ for $A^j, V_j, E_j, F_j, s_j, t_j, \psi_j, \ldots$, such that $\psi_j^{-1}(C_j) \subseteq \tilde{U}_J \subseteq V_j$, and if $J, K \subseteq A$ with $K \subseteq J$ and $0 < |K| < |J| \leq m$ then there exists open $\tilde{U}_{JK} \subseteq \tilde{U}_J$ with $\psi_j^{-1}(C_j \cap C_K) \subseteq \tilde{U}_{JK}$ such that $(\tilde{U}_{JK}, \tilde{E}_{JK})$ is compatible with $(\tilde{U}_K, \tilde{E}_K)$, in the sense of §5.4. That is, $\phi_{JK}(\tilde{U}_{JK}) \subseteq \tilde{U}_K \subseteq V_K$ and $\chi_{JK}|_{\tilde{U}_{JK}}(\tilde{E}_{JK} \mid_{\tilde{U}_{JK}}) \subseteq \phi_{JK}|_{\tilde{U}_{JK}}(\tilde{E}_K) \subseteq \phi_{JK}|_{\tilde{U}_{JK}}(E_K)$.

For the first step, to prove $(+)_1$, for all $J = \{i\}$ with $i \in I$ we choose $(\tilde{U}_J, \tilde{E}_J)$ for $A^i, V_i, E_i, \ldots$, satisfying $(*)$ with $s_i^{-1}(0) \subseteq \tilde{U}_J$, so that $\psi_j^{-1}(C_j) \subseteq \tilde{U}_J$, by applying Theorem 3.9(b) with $C = U = 0$. The second part of $(+)_1$ is trivial, as there are no $J, K \in A$ with $0 < |K| < |J| < 1$.

For the inductive step, suppose $(+)_m$ holds for some $m > 1$. We will prove $(+)_m$. Using the existing choices of $(\tilde{U}_J, \tilde{E}_J)$ and $\tilde{U}_{JK}$ for $J, K \subseteq A$ with $|J|, |K| < m$ from $(+)_m$, it remains to choose $(\tilde{U}_J, \tilde{E}_J)$ when $|J| = m$, and $\tilde{U}_{JK}$ when $0 < |K| < |J| = m$. So fix $J \subseteq I$ with $|J| = m$.

Then $(+)_m$ gives $(\tilde{U}_K, \tilde{E}_K)$ satisfying $(*)$ for all $\emptyset \neq K \subseteq J$. Using the notation of Lemma 3.14 set $\tilde{U}'_{JK} = \phi_{JK}^{-1}(\tilde{U}_K) \subseteq V_J$, and define $\tilde{E}_{JK} = \chi_{JK}|_{\tilde{U}'_{JK}}(\tilde{E}_K)$, a vector subbundle of $E_J|_{\tilde{U}'_{JK}}$. Then $\tilde{U}'_{JK}$ is an open neighbourhood of $\psi_j^{-1}(C_K \cap C_L)$ in $V_J$, by (3.18).

If $\emptyset \neq L \subseteq K \subseteq J$ then by $(+)_m$ there exists open $\tilde{U}_{KL} \subseteq \tilde{U}_K$ with $\psi_K^{-1}(C_K \cap C_L) \subseteq \tilde{U}_{KL}$ such that $\phi_{KL}(\tilde{U}_{KL}) \subseteq \tilde{U}_L$ and $\chi_{KL}|_{\tilde{U}_{KL}}(\tilde{E}_L) \subseteq \phi_{KL}|_{\tilde{U}_{KL}}(\tilde{E}_L)$. Pulling back by $\phi_{JK}$, applying $\chi_{JK}$, and using the last part of Corollary 3.7(ii) then shows that we have an open neighbourhood $\tilde{U}_{JKL} = \phi_{JK}^{-1}(\tilde{U}_{KL})$ of $\psi_j^{-1}(C_K \cap C_L)$ in $\tilde{U}_{JK}' \cap \tilde{U}_{JL}' \subseteq V_J$, such that

$$\tilde{E}_{JKL}'|_{\tilde{U}'_{JKL}} \subseteq \tilde{E}_{JL}'|_{\tilde{U}'_{JKL}} \subseteq E_J|_{\tilde{U}'_{JKL}}. \quad (6.7)$$

As in Lemma 3.14 choose vector subbundles $\tilde{E}_{JKL}'' \subseteq E_J|_{\tilde{U}'_{JKL}}$ with $E_J|_{\tilde{U}'_{JKL}} = \tilde{E}_{JK}' \oplus \tilde{E}_{JKL}''$ on $\tilde{U}'_{JKL}$ for all $\emptyset \neq K \subseteq J$. Choose a connection $\nabla$ on $E_J$. As in Lemma 3.14 for all $\emptyset \neq K \subseteq J$, $\tilde{E}_{JKL}'' := (\nabla s_j) [ \text{Ker } d \phi_{JK} ]$ is a vector subbundle of $E_J$ near $s_j^{-1}(0)$ in $V_J$. Making the open neighbourhoods $\tilde{U}_{JK}' \cap \tilde{U}_{JL}'$ smaller, we can suppose $\tilde{E}_{JKL}''$ is a vector subbundle of $E_J|_{\tilde{U}'_{JKL}}$. If $\emptyset \neq L \subseteq K \subseteq J \subseteq I$ then $\text{Ker } d \phi_{JK} \subseteq \text{Ker } d \phi_{KL} \circ \phi_{JK}$, as $\phi_{KL} = \phi_{KL} \circ \phi_{JK}$, and so

$$\tilde{E}_{JKL}''|_{\tilde{U}'_{JKL}} \subseteq \tilde{E}_{ JK ''|_{\tilde{U}'_{JKL}}} \subseteq E_J|_{\tilde{U}'_{JKL}}. \quad (6.8)$$

Next, by reverse induction on $l = m - 1, m - 2, \ldots, 1$, we will prove the following inductive hypothesis $(\times)_{JI}$:
For all $\emptyset \neq L \subseteq J$ with $l \leq |L|$ we can choose an open neighbourhood $\hat{U}_{JL}$ of $\psi^{-1}_J(C_J \cap C_L)$ in $\hat{U}_{JL}$ and a vector subbundle $E_{JL}^{-1}$ of $E'_{JL}|\hat{U}_{JL}$ such that

$$E_{JL}|\hat{U}_{JL} = E_{JL}^{-1} \oplus E''_{JL}|\hat{U}_{JL} \oplus E'''_{JL}|\hat{U}_{JL},$$

or equivalently, identifying $E'_{JL}$ with $E_{JL}/E''_{JL}$ on $\hat{U}_{JL}$,

$$E'_{JL}|\hat{U}_{JL} = \hat{E}_{JL}^{-1} \oplus \left([E''_{JL} \oplus E'''_{JL}] / E''_{JL}\right)|\hat{U}_{JL},$$

and such that if $\emptyset \neq L \subseteq K \subseteq J$ with $l \leq |L| < |K|$ then there exists an open neighbourhood $\hat{U}_{JKL}$ of $\psi^{-1}_J(C_J \cap C_K \cap C_L)$ in $\hat{U}_{JK} \cap \hat{U}_{JL}$ with $\hat{E}_{JKL}^{-1}|\hat{U}_{JKL} = \hat{E}_{JK}|\hat{U}_{JKL}$.

For the first step $l = m - 1$, for each $L \subseteq J$ with $|L| = m - 1$ we take $\hat{U}_{JL} = U_{JL}$ and take $\hat{E}_{JL}$ to be an arbitrary complement to $[E''_{JL} \oplus E'''_{JL}] / E''_{JL}$ in $E'_{JL}|\hat{U}_{JL}$, as in (6.10), which implies (6.9). The second part of $(\times)_{J_{m-1}}$ is trivial as there are no $\emptyset \neq L \subseteq K \subseteq J$ with $|L| < |K| < |J| = m$.

For the inductive step, suppose $(\times)_{J_{l+1}}$ holds for some $1 \leq l < m-1$, and fix $L \subseteq J$ with $|L| = l$. Choose open neighbourhoods $\hat{U}_{JKL}$ of $\psi^{-1}_J(C_J \cap C_K \cap C_L)$ in $V_j$ for all $\emptyset \subseteq K \subseteq J$ with the properties that:

(a) $\hat{U}_{JKL} \subseteq \hat{U}_{JK} \cap \hat{U}_{JL}$, where $\hat{U}_{JK}$ is already chosen by $(\times)_{J_{l+1}}$.

(b) If $L \subseteq K_1, K_2 \subseteq J$ with $K_1 \subseteq K_2$ and $K_2 \subseteq K_1$ then $\hat{U}_{JK_1L} \cap \hat{U}_{JK_2L} = \emptyset$.

(c) If $L \subseteq K_2 \subseteq K_1 \subseteq J$ then $\hat{U}_{JK_1L} \cap \hat{U}_{JK_2L} \subseteq \hat{U}_{JK_1K_2}$, where $\hat{U}_{JK_1K_2}$ is already chosen by $(\times)_{J_{l+1}}$.

This is possible, using Proposition 3.15(iii) to ensure (b).

Next, we have to choose an open neighbourhood $\hat{U}_{JL}$ of $\psi^{-1}_J(C_J \cap C_L)$ in $\hat{U}_{JL}$ and a vector subbundle $\hat{E}_{JL}$ of $E'_{JL}|\hat{U}_{JL}$ satisfying (6.9)–(6.10), such that for all $K$ with $L \subseteq K \subseteq J$ we have $\hat{U}_{JKL} \subseteq \hat{U}_{JL}$ and $\hat{E}_{JKL}^{-1}|\hat{U}_{JKL} = \hat{E}_{JK}|\hat{U}_{JKL}$.

First note from Lemma 3.14 that (6.9)–(6.10) near $\psi^{-1}_J(C_J \cap C_L)$ are equivalent to $(\hat{U}_{JL}, \hat{E}_{JL})$ near $\psi^{-1}_J(C_J \cap C_L)$ satisfying $(\ast)$ and being compatible with $(\hat{U}_{L}, \hat{E}_{L})$. By $(\times)_{J_{l+1}}$ we already known that $\hat{E}_{JKL}^{-1}|\hat{U}_{JKL}$ near $\psi^{-1}_J(C_J \cap C_L)$ satisfies $(\ast)$ and is compatible with $(\hat{U}_K, \hat{E}_K)$, which implies that $\hat{E}_{JKL}^{-1}|\hat{U}_{JKL}$ is compatible with $(\hat{U}_L, \hat{E}_L)$ near $\psi^{-1}_J(C_J \cap C_L)$ as $(\hat{U}_K, \hat{E}_K)$ is compatible with $(\hat{U}_L, \hat{E}_L)$ by $(\ast)_{m-1}$. Therefore the prescribed value $\hat{E}_{JKL}^{-1}|\hat{U}_{JKL}$ for $\hat{E}_{JL}$ on $\hat{U}_{JKL}$ satisfies (6.9)–(6.10) near $\psi^{-1}_J(C_J \cap C_L)$, and making $\hat{U}_{JKL}$ smaller, we can suppose $\hat{E}_{JKL}^{-1}|\hat{U}_{JKL}$ satisfies (6.9)–(6.10) on $\hat{U}_{JK}$.

This proves that (6.9)–(6.10) are compatible with the conditions $\hat{E}_{JKL}^{-1}|\hat{U}_{JKL} = \hat{E}_{JK}|\hat{U}_{JKL}$ for all $\emptyset \neq L \subseteq K \subseteq J$.

Next, observe that the prescribed values $\hat{E}_{JKL}^{-1}|\hat{U}_{JKL}$ for $\hat{E}_{JL}$ on $\hat{U}_{JKL}$ for different $K_1, K_2$ with $L \subseteq K_1, K_2 \subseteq J$ agree on overlaps $\hat{U}_{JK_1L} \cap \hat{U}_{JK_2L}$. This follows from (b), (c) above and $\hat{E}_{JK_1L}|\hat{U}_{JK_1K_2} = \hat{E}_{JK_2L}|\hat{U}_{JK_1K_2}$, which holds by $(\times)_{J_{l+1}}$. Therefore the last part of $(\times)_{J_{l+1}}$ can be rewritten to say that we have
one prescribed value for $\hat{E}_{JL}$ on the subset $\hat{U}_{JL} := \bigcup_{K:L \subseteq K \subseteq J} \hat{U}_{JKL}$, which satisfies \([6.9]-[6.10]\) on $\hat{U}_{JL}$.

So, we are given a prescribed value of $\hat{E}_{JL}$ on an open set $\hat{U}_{JL} \subseteq V_J$ satisfying \([6.10]\), and we have to extend it to a larger open set $\hat{U}_{JL} \subseteq V_J$ containing both $U_{JL}$ and $\psi^{-1}_J(C_J \cap C_K \cap C_L)$. This may not be possible: if we have chosen previous $\hat{E}_{JK}'$s badly near the ‘edge’ of $\hat{U}_{JL}$ in $V_J$, then the prescribed values of $\hat{E}_{JL}$ may not extend continuously to the closure $\bar{\hat{U}}_{JL}$ of $\hat{U}_{JL}$ in $V_J$, and in particular, may not extend continuously over points in $[\bar{\psi}^{-1}_J(C_J \cap C_K \cap C_L)] \cap [\bar{U}_{JL} \setminus \hat{U}_{JL}]$. However, we can deal with this problem by shrinking all the $\hat{U}_{JKL}'$s, such that the closure $\bar{\hat{U}}_{JL}$ of the new $\hat{U}_{JL}$ lies inside the old $\hat{U}_{JL}$. Then it is guaranteed that the prescribed value of $\hat{E}_{JL}$ on $\hat{U}_{JL}$ extends smoothly to an open neighbourhood of $\bar{\hat{U}}_{JL}$ in $V_J$, so we can choose $(\hat{U}_{JL}, \bar{\hat{E}}_{JL})$ satisfying all the required conditions. As this holds for all $L \subseteq J$ with $|L| = l$, this completes the inductive step, and $(\times)_{J,l}$ holds for all $l = m - 1, m - 2, \ldots, 1$.

Fix data $\hat{U}_{JL}, \bar{\hat{E}}_{JL}, \bar{\hat{U}}_{JKL}$ as in $(\times)_{J,l}$. For all $\emptyset \neq K \subseteq J$, choose open neighbourhoods $\hat{U}_{JK}$ of $\psi^{-1}_J(C_J \cap C_K)$ in $\hat{U}_{JK}$ such that if $K_1 \subseteq K_2$ and $K \subseteq K_1 \subseteq K_2$ then $\hat{U}_{JK_1} \cap \hat{U}_{JK_2} = \emptyset$, and if $\emptyset \neq L \subseteq K \subseteq J$ then $\hat{U}_{JK} \cap \hat{U}_{JL} \subseteq \hat{U}_{JKL}$. This is possible provided the $\hat{U}_{JK}$ are small enough, using Proposition \(3.13\)ii) to ensure $\hat{U}_{JK_1} \cap \hat{U}_{JK_2} = \emptyset$.

Define $\hat{U}_J = \bigcup_{K: \emptyset \neq K \subseteq J} \hat{U}_{JK}$. It is an open neighbourhood of the closed set $\bar{C}_J$ in $V_J$, where $\bar{C}_J = \bigcup_{K: \emptyset \neq K \subseteq J} \psi^{-1}_J(C_J \cap C_K)$ in $V_J$. Define a vector subbundle $\hat{E}_J$ of $E_J|_{\hat{U}_J}$ by $\hat{E}_J|_{\hat{U}_{JK}} = \bar{\hat{E}}_{JK} |_{\hat{U}_{JK}}$ for all $\emptyset \neq K \subseteq J$. These prescribed values for different $K_1$ and $K_2$ are compatible on the overlap $\hat{U}_{JK_1} \cap \hat{U}_{JK_2}$ by construction, so $\hat{E}_J$ is well-defined.

Now apply Theorem \(3.2\)ii) to $A^*_J, V_J, E_J, s_J, \ldots$, with closed set $\bar{C}_J \subseteq \hat{U}_J$, and pair $(\hat{U}_J, \hat{E}_J)$ satisfying $(\ast)$ with $\bar{C}_J \subseteq \hat{U}_J$. This shows that there exists $(\hat{U}_J, \hat{E}_J)$ satisfying $(\ast)$ for $A^*_J, V_J, E_J, s_J, \ldots$, and an open neighbourhood $\bar{U}_J'$ of $\bar{C}_J$ in $\bar{U}_J \cap \hat{U}_J$ such that $\hat{E}_J|_{\bar{U}_J'} = \bar{E}_J|_{\bar{U}_J'}$. For all $\emptyset \neq K \subseteq J$, set $\bar{U}_{JK} = \hat{U}_J \cap \hat{U}_{JK}$. Then $\bar{U}_{JK}$ is an open neighbourhood of $\psi^{-1}_J(C_J \cap C_K)$ in $V_J$, and $\bar{E}_J|_{\bar{U}_{JK}} = \bar{E}_J|_{\hat{U}_{JK}}$, which is compatible with $(\bar{U}_J, \bar{E}_J)$ by definition. This completes the proof of the inductive step of $(\ast)_m$. So by induction, $(\ast)_m$ holds for all $m = 1, 2, \ldots$.

Fix data $(\hat{U}_J, \bar{E}_J)$ for all $J \in A$ and $\hat{U}_{JK}$ for all $J, K \in A$ with $K \subseteq J$ as in $(\ast)_m$ as $m \to \infty$ (or $m = |I|$ if $I$ is finite). For all $J \in A$, choose open neighbourhoods $U_J$ of $\psi^{-1}_J(C_J)$ in $\hat{U}_J$, such that setting $\bar{E}_J = \bar{E}_J|_{U_J}$ and $S_J = \psi_J(s^{-1}_J(0) \cap U_J)$, so that $S_J$ is an open neighbourhood of $C_J$ in $X_m$, then $(\hat{U}_J, \bar{E}_J)$ satisfies condition $(\dagger)$, and for all $J, K \in A$, if $J \not\subseteq K$ and $K \not\subseteq J$ then $S_J \cap S_K = \emptyset$, and if $K \not\subseteq J$ then $\psi^{-1}_J(S_J \cap S_K) \subseteq \hat{U}_{JK}$. If $K \not\subseteq J$, we define $U_{JK} = \hat{U}_{JK} \cap U_J \circ \phi^{-1}_{JK}(U_K)$. Then $s^{-1}_J(0) \cap U_{JK} = \psi^{-1}_J(S_J \cap S_K)$, and $(U_{JK}, \bar{E}_J|_{\hat{U}_{JK}})$ is compatible with $(\bar{U}_J, \bar{E}_J)$.

To see that we can choose $U_J$ for all $J \in A$ satisfying all these conditions, note that by Theorem \(3.9\)c), if $U_J$ is small enough then $(U_J, E_J)$ satisfies $(\dagger)$,
as $(\tilde{U}_J, \tilde{E}_J)$ satisfies $(\ast)$. If $J \not\subseteq K$ and $K \not\subseteq J$ then Proposition 3.15 iii) implies that $S_J \cap S_K = \emptyset$ provided both $U_J, U_K$ are sufficiently small. Similarly, if $K \subseteq J$ then $\psi_J^{-1}(S_J \cap S_K) \subseteq \tilde{U}_{JK}$ holds provided both $U_J, U_K$ are sufficiently small. Now if $I$ is infinite, it is possible that an individual set $U_I$ may have to satisfy infinitely many smallness conditions, for compatibility with infinitely many sets $\emptyset \not= K \subseteq I$. However, the local finiteness condition Proposition 3.15 ii) means that in an open neighbourhood of any $v_J \in \psi_J^{-1}(C_J)$, only finitely many smallness conditions on $U_J$ are relevant, so we can solve them. This completes the proof of Proposition 3.17.

6.3 Proof of Proposition 3.19

Let $(X, \omega_X)$, $X_{an}$ and $X_{dm}$ be as in Theorem 3.18 and use the notation of 3.18. Then 2.4 and 3.10 define topological principal $\mathbb{Z}_2$-bundles $O_{X, \omega_X}$ and $P_{X, \omega}$ over $X_{an}$, and we have to construct a canonical isomorphism $O_{X, \omega_X} \cong P_{X, \omega}$.

We first do this over a single point $x \in X_{an}$.

As $X_{an} = \bigcup_{J \in A} C_J$ by Proposition 3.15 iv), there exists $J \in A$ with $x \in C_J \subseteq S_J = \text{Im} \psi_J$, so there exists unique $v_J \in s_J^{-1}(0) \cap U_J = (s_J^{-1})^{-1}(0) \subseteq U_J \subseteq V_J$ with $\psi_J(v_J) = x$. Equation (2.3) gives

$$O_{X, \omega_X} |_x \cong \{\text{complex orientations of } (H^1(T_X |_x), Q_x)\},$$

where $Q_x = \omega_X^0_x$ is the nondegenerate complex quadratic form on $H^1(T_X |_x)$ in 2.7. Equation 3.9 gives an isomorphism of complex vector spaces

$$H^1(T_X |_{\alpha_J \cdot v_J}) : \frac{\text{Ker}(t_J |_{v_J} : E_J |_{v_J} \to F_J |_{v_J})}{\text{Im}(d_J |_{v_J} : T_{v_J} V_J \to E_J |_{v_J})} \to H^1(T_X |_x). \quad (6.11)$$

Write $\tilde{Q}_{v_J}$ for the complex quadratic form on $\text{Ker}(t_J |_{v_J})/\text{Im}(d_J |_{v_J})$ identified with $Q_x$ by (6.11), as in Definition 3.8. Then we have

$$O_{X, \omega_X} |_x \cong \{\text{complex orientations of } (\text{Ker}(t_J |_{v_J})/\text{Im}(d_J |_{v_J}), \tilde{Q}_{v_J})\}. \quad (6.12)$$

Condition $(\ast)$ for $(U_J, E_J)$ at $v_J$ requires that

$$\Pi_{v_J} : E_J |_{v_J} \cap \text{Ker}(t_J |_{v_J} : E_J |_{v_J} \to F_J |_{v_J}) \to \frac{\text{Ker}(t_J |_{v_J} : E_J |_{v_J} \to F_J |_{v_J})}{\text{Im}(d_J |_{v_J} : T_{v_J} V_J \to E_J |_{v_J})}$$

should be injective, with image $\Pi_{v_J}$ a real vector subspace of half the real dimension of $\text{Ker}(t_J |_{v_J})/\text{Im}(d_J |_{v_J})$, on which the real quadratic form $\text{Re} \tilde{Q}_{v_J}$ is negative definite. As $(U_J, E_J, s_J^J, \psi_J |_{s_J^{-1}(0) \cap U_J})$ is a Kuranishi neighbourhood on $X_{dm}$ by the proof of Theorem 3.18 equation (2.13) gives an exact sequence

$$0 \to T_x X_{dm} \to T_{v_J} V_J \to d_{s_J}^J |_{v_J} E_J |_{v_J} \to O_x X_{dm} \to 0.$$

Condition $(\ast)$ implies that $\text{Ker}(d_J |_{v_J}) = \text{Ker}(d_s^J |_{v_J})$, so we have

$$T_x X_{dm} \cong \text{Ker}(d_J |_{v_J} : T_{v_J} V_J \to E_J |_{v_J}). \quad (6.13)$$
Also from (*) we see there is a canonical isomorphism

\[ O_x X_{dm} \cong \frac{\text{Ker}(t_J|_{v_J})}{\text{Im}(d_J|_{v_J})} \]  \quad (6.14)

By (6.13), \( T_x X_{dm} \) is a complex vector space, so \( T_x X_{dm} \) and \( T_x^* X_{dm} \) have natural orientations as real vector spaces. Thus (2.12) gives bijections

\[ P_{X_{dm}}|_x \cong \{ \text{real orientations of } T_x^* X_{dm} \oplus O_x X_{dm} \} \]

\[ \cong \{ \text{real orientations of } O_x X_{dm} \}. \]  \quad (6.15)

Suppose we are given a complex basis \( e_1, \ldots, e_k \) of \( \text{Ker}(t_J|_{v_J})/\text{Im}(d_J|_{v_J}) \cong \mathbb{C}^k \) which is orthonormal w.r.t. \( \hat{Q}_{v_J} \). As \( e_1, \ldots, e_k \) are orthonormal w.r.t. \( Q_{v_J} \), the real quadratic form \( \text{Re} \hat{Q}_{v_J} \) is positive definite on the real span \( \langle e_1, \ldots, e_k \rangle \mathbb{R} \), and \( \text{Re} Q_{v_J} \) is negative definite on \( \text{Im} \Pi_{v_J} \), so \( \langle e_1, \ldots, e_k \rangle \cap \text{Im} \Pi_{v_J} = \{0\} \). Therefore \( e_1 + \text{Im} \Pi_{v_J}, \ldots, e_k + \text{Im} \Pi_{v_J} \) are linearly independent in the real vector space \( \langle \text{Ker}(t_J|_{v_J})/\text{Im}(d_J|_{v_J}) \rangle \cap \text{Im} \Pi_{v_J} \cong \mathbb{R}^k \), so they are a basis as \( \text{Im} \Pi_{v_J} \) has half the real dimension of \( \text{Ker}(t_J|_{v_J})/\text{Im}(d_J|_{v_J}) \). Define an identification

\[ \{ \text{complex orientations of } \langle \text{Ker}(t_J|_{v_J})/\text{Im}(d_J|_{v_J}) \rangle \cap \text{Im} \Pi_{v_J} \} \]

\[ \cong \{ \text{real orientations of } \langle \text{Ker}(t_J|_{v_J})/\text{Im}(d_J|_{v_J}) \rangle \cap \text{Im} \Pi_{v_J} \}, \]  \quad (6.16)

such that orientations on both sides are identified if, whenever \( e_1, \ldots, e_k \) is an oriented orthonormal complex basis for \( \langle \text{Ker}(t_J|_{v_J})/\text{Im}(d_J|_{v_J}) \rangle \), then \( e_1 + \text{Im} \Pi_{v_J}, \ldots, e_k + \text{Im} \Pi_{v_J} \) is an oriented basis for \( \langle \text{Ker}(t_J|_{v_J})/\text{Im}(d_J|_{v_J}) \rangle \cap \text{Im} \Pi_{v_J} \).

Combining equations (6.12), (6.14), (6.15) and (6.16) gives an identification

\[ O_{X, \omega_X}|_x \cong P_{X_{dm}}|_x. \]  \quad (6.17)

To show (6.17) is independent of the choice of \( J \in A \) with \( x \in C_J \), let \( K \subseteq A \) with \( x \in C_K \). Proposition 3.15(iii) gives \( J \subseteq K \) or \( K \subseteq J \), so suppose \( K \subseteq J \). Then we have potentially different bijections (6.17) \( J \) defined using \( v_J, V_J, \ldots \) and (6.17) \( K \) defined using \( v_K, V_K, \ldots \). We also have \( \phi_{JK} : V_J \to V_K \) with \( \phi_{JK}(v_J) = v_K \) and \( \chi_{JK} : \xi_K \). Equations (6.12)–(6.13) give isomorphisms

\[ \text{Ker}(d_J|_{v_J}) : T_{v_J} V_J \to E_{J|_{v_J}} \cong \text{Ker}(d_K|_{v_K}) : T_{v_K} V_K \to E_{K|_{v_K}} \]  \quad (6.18)

\[ \text{Ker}(t_J|_{v_J}) : E_{J|_{v_J}} \to F_{J|_{v_J}} \cong \text{Ker}(t_K|_{v_K}) : E_{K|_{v_K}} \to F_{K|_{v_K}} \]  \quad (6.19)

where (6.19) identifies \( \hat{Q}_{v_J} \cong \hat{Q}_{v_K} \). Since \( (U_J, E_J), (U_K, E_K) \) are compatible, as in (3.4) we have \( \chi_{JK|v_J}(E_J|_{v_J}) \subseteq E_K|_{v_K} \), which implies that (6.19) identifies \( \text{Im} \Pi_{v_J} \) with \( \text{Im} \Pi_{v_K} \). Thus all the ingredients used to define (6.17) \( J \) and (6.17) \( K \) are canonically identified, so (6.17) is independent of choices.

Taken over all \( x \in X_{an} \), equation (6.17) gives a bijection \( O_{X, \omega_X}|_x \cong P_{X_{dm}}|_x \), which is \( \mathbb{Z}_2 \)-equivariant and commutes with the projections \( O_{X, \omega_X} \to X_{an} \) and \( P_{X_{dm}} \to X_{an} \). By considering the local models we can show that (6.17) varies continuously with \( v_J \in s_J^{-1}(0) \cap U_J \), and so with \( x \in X_{an} \), and thus \( O_{X, \omega_X} \cong P_{X_{dm}} \) is a homeomorphism, and a canonical isomorphism of topological principal \( \mathbb{Z}_2 \)-bundles. This completes the proof of Proposition 3.19.
7 Proof of Theorem 3.20

Suppose \((X, \omega^*_X)\) is a separated, \(-2\)-shifted symplectic derived \(\mathbb{C}\)-scheme with vdim\(_C\) \(X = n\), whose complex analytic topological space \(X_{an}\) is second countable. Let \(X_{dm}, X'_{dm}\) be different possible outcomes in Theorem 3.18 so that \(X_{dm}, X'_{dm}\) are derived manifolds with topological space \(X_{an}\) and vdim\(_C\) \(X_{dm} = n\), defined using different arbitrary choices in the programme of 3.6.

We must build a derived manifold with boundary \(W_{dm}\) with topological space \(X_{an} \times [0, 1]\) and vdim\(_C\) \(W_{dm} = n+1\), and an equivalence \(\partial W_{dm} \simeq X_{dm}\) of \(X_{dm}\) topologically identifying \(X_{dm}\) with \(X_{an} \times \{0\}\) and \(X'_{dm}\) with \(X_{an} \times \{1\}\).

As in 3.6 let \(X_{dm}\) be constructed using the family \(\{(A^*_J, \alpha_J) : i \in I\}\), and data \(A^*_J, \alpha_J\) for \(J \in A, \Phi_{JK}\) for \(K \subseteq J\) in \(A\) from Theorem 3.2 where \(A = \{J : \emptyset \neq J \subseteq I, J\ finite\}\), as in 3.2 use notation \(V_J, E_J, F_J, s_J, t_J, \xi, \psi_J\) and \(R_J = \bigcap_{i \in J} R_i \subseteq X_{an}\) from \(A^*_J, \alpha_J\) and \(\phi_{JK}, \chi_{JK}, \xi_{JK}\) from \(\Phi_{JK}\). Let \(X_{dm}\) be defined using closed subsets \(C_J \subseteq R_J \subseteq X_{an}\) for \(J \in A\) in Proposition 3.15 and pairs \((U_J, E_J)\) and open subsets \(U_{JK} \subseteq U_J\) in Proposition 3.17.

Similarly, let \(X'_{dm}\) be constructed using the family \(\{(A'^*_J, \alpha'_J) : i' \in I'\}\) and data \(A'^*_J, \alpha'_J\) for \(J' \in A', \Phi_{JK'}\) for \(K' \subseteq J'\) in \(A'\) from Theorem 3.2 where \(A' = \{J' : \emptyset \neq J' \subseteq I', J'\ finite\}\), as in 3.2 use notation \(V_{J'}, E_{J'}, F_{J'}, s_{J'}, t_{J'}, \xi_{J'}, \psi_{J'}\) and \(R_{J'} = \bigcap_{i' \in J'} R'_{i'} \subseteq X_{an}\) from \(A'^*_J, \alpha'_J\) and \(\phi_{JK'}, \chi_{JK'}, \xi_{JK'}\) from \(\Phi_{JK'}\). Let \(X'_{dm}\) be defined using closed subsets \(C'_{J'} \subseteq R'_{J'} \subseteq X_{an}\) for \(J' \in A'\) in Proposition 3.15 and pairs \((U'_{J'}, E'_{J'})\) and open subsets \(U'_{JK'} \subseteq U'_{J'}\) in Proposition 3.17.

Set \(I = I \amalg I'\), the disjoint union of \(I\) and \(I'\). Define a family \(\{(A^*_J, \alpha_J) : i \in I\}\) by \((A^*_J, \alpha_J) = (A^*_i, \alpha_i)\) for \(i \in I\) and \((A'^*_J, \alpha'_J) = (A'^*_i, \alpha'_i)\) for \(i' \in I'\). Now apply Theorem 3.2 to the family \(\{(A^*_J, \alpha_J) : i \in I\}\) on \(X\), to construct data \(A^*_J, \alpha_J\) for \(J \in A, \Phi_{JK}\) for \(K \subseteq J\) in \(A\), where \(A = \{J : \emptyset \neq J \subseteq I, J\ finite\}\).

The proof of Theorem 3.2 chooses \(A^*_J, \alpha_J\) for \(J \in A\) and \(\Phi_{JK}\) for \(K \subseteq J\) by induction on increasing \(|J|\), where the choice of \(A^*_J, \alpha_J, \Phi_{JK}\) depends only on previous choices of \(A^*_L, \alpha_L\) for \(\emptyset \neq L \subseteq J\) and \(\Phi_{LM}\) for \(\emptyset \neq M \subseteq L \subseteq J\). Therefore we can choose the \(A^*_J, \alpha_J, \Phi_{JK}\) such that \((A^*_J, \alpha_J) = (A'^*_J, \alpha'_J)\) if \(J \in A \subseteq \tilde{A}\), and \(\Phi_{JK} = \Phi_{JK}\) if \(K \subseteq J \in A \subseteq \tilde{A}\) and \((A'^*_J, \alpha'_J) = (A'^*_J, \alpha'_J)\) if \(J' \in A' \subseteq \tilde{A}\), and \(\Phi_{JK'} = \Phi_{JK'}\), for \(K' \subseteq J' \in A' \subseteq \tilde{A}\). Note that as \(\tilde{A} \neq A\), this does not determine the \(A^*_J, \alpha_J, \Phi_{JK}\) for all \(J, K\), we have to make new choices for \(J, K\) with \(J \cap K \neq \emptyset \neq \tilde{J} \cap \tilde{K}\).

As in 3.2 use notation \(\tilde{V}_J, \tilde{E}_J, \tilde{F}_J, \tilde{s}_J, \tilde{t}_J, \tilde{\psi}_J\) and \(\tilde{R}_J\) from \(A^*_J, \alpha_J\) and \(\tilde{\Phi}_{JK}, \tilde{\chi}_{JK}, \tilde{\xi}_{JK}\) from \(\tilde{\Phi}_{JK}\). Then \(\tilde{V}_J = V_J, \tilde{E}_J = E_J, \ldots\) for \(J \in A \subseteq \tilde{A}\), and \(\tilde{\Phi}_{JK} = \phi_{JK}, \tilde{\chi}_{JK} = \chi_{JK}, \ldots\) for \(K \subseteq J \in A \subseteq \tilde{A}\), and \(\tilde{V}_{J'} = V_{J'}, \tilde{E}_{J'} = E_{J'}, \ldots\) for \(J' \in A' \subseteq \tilde{A}\), and \(\tilde{\phi}_{JK'} = \phi_{JK'}, \tilde{\chi}_{JK'} = \chi_{JK'}, \ldots\) for \(K' \subseteq J' \in A' \subseteq \tilde{A}\).

At this point we pass from working over \(X_{an}\) to working over \(X_{an} \times [0, 1]\). The rule is this:

(i) For data associated to \(J \in A \subseteq \tilde{A}\) or \(K \subseteq J \in A \subseteq \tilde{A}\), we take the product with \([0, \frac{1}{2}]\), and we use accents \(\tilde{\cdot}\) to denote the corresponding product data. So, we consider the open set \(\tilde{R}_J := R_J \times [0, \frac{1}{2}] \subseteq X_{an} \times [0, 1]\), the real manifold with boundary \(\tilde{V}_J := V_J \times [0, \frac{1}{2}]\), the vector bundle \(\tilde{E}_J \to \tilde{V}_J\).
with \( E_j = E_j \times [0, \frac{1}{3}] = \pi_{V_j}(E_j) \), the section \( s_j = \pi_{V_j}^{-1}(s_j) : V_j \to \hat{E}_j \), and so on, and the smooth map \( \phi_{JK} : \hat{V}_j \to \hat{V}_k \) with \( \phi_{JK} = \phi_{JK} \times \id_{[0, \frac{1}{3}]} : V_j \times [0, \frac{1}{3}] \to V_k \times [0, \frac{1}{3}] \), and so on.

(ii) For data associated to \( J' \in A' \subseteq A \) or \( K' \subseteq J' \in A' \subseteq A \), we take the product with \( [\frac{1}{3}, 1] \), and use accents ‘ \( \acute{\cdot} \)’. So, we consider the open set \( \hat{R}_{J'} := \hat{R}_{J'} \times [\frac{1}{3}, 1] \subseteq X_{\text{an}} \times [0, 1] \), the real manifold with boundary \( \hat{V}_{J'} := V_{J'} \times [\frac{1}{3}, 1] \), the vector bundle \( \hat{E}_{J'} \to \hat{V}_{J'} \) with \( \hat{E}_{J'} = E_{J'} \times [\frac{1}{3}, 1] = \pi_{V_j}^{-1}(E_{J'}) \), the smooth map \( \phi_{J'K'} = \phi_{J'K'} \times \id_{[\frac{1}{3}, 1]} : \hat{V}_{J'} \to \hat{V}_{K'} \), etc.

(iii) For data associated to \( \hat{J} \in \hat{A} \) with \( \hat{J} \notin A, A' \), we take the product with \( [\frac{1}{3}, \frac{2}{3}] \), and use accents ‘ \( \check{\cdot} \)’. So, we consider the open set \( \hat{R}_j := \hat{R}_j \times (\frac{1}{3}, \frac{2}{3}) \subseteq X_{\text{an}} \times [0, 1] \), the real manifold \( \hat{V}_j := \hat{V}_j \times (\frac{1}{3}, \frac{2}{3}) \), and so on.

For data associated to \( \hat{K} \subseteq \hat{J} \in \hat{A} \) with \( \hat{K} \notin A, A' \), allowing all three possibilities \( \hat{K} \in A, \hat{K} \notin A, A' \), we take the product with \( (\frac{1}{3}, \frac{2}{3}) \), and use accents ‘ \( \check{\cdot} \)’. So we define \( \hat{\phi}_{JK} = \hat{\phi}_{JK} \times \id_{[\frac{1}{3}, \frac{2}{3}]} : \hat{V}_j \to \hat{V}_k \), where \( \hat{V}_j := \hat{V}_j \times (\frac{1}{3}, \frac{2}{3}) \), but \( \hat{V}_k \) may be \( V_k \times [0, \frac{2}{3}] \) or \( V_k \times (\frac{1}{3}, \frac{2}{3}) \) or \( \hat{V}_k \times (\frac{1}{3}, \frac{2}{3}) \); and so on.

Next we follow the programme of [3.5] from Proposition 3.15 onwards, but working over \( X_{\text{an}} \times [0, 1] \) rather than \( X_{\text{an}} \). The rule is that any choices we make for \( J \in A \subseteq \hat{A} \) or \( K \subseteq J \in A \subseteq \hat{A} \) should coincide over \( X_{\text{an}} \times [0, \frac{1}{3}] \) with the corresponding choices made in the definition of \( X_{\text{dm}} \), after taking products with \( [0, \frac{1}{3}] \), and any choices we make for \( J' \in A' \subseteq A \) or \( K' \subseteq J' \in A' \subseteq \hat{A} \) should coincide over \( X_{\text{an}} \times [\frac{2}{3}, 1] \) with the corresponding choices made in the definition of \( X_{\text{dm}} \), after taking products with \( [\frac{2}{3}, 1] \). The choices we make over \( X_{\text{an}} \times (\frac{1}{3}, \frac{2}{3}) \) will interpolate smoothly between these two.

First we apply Proposition 3.15 to the topological space \( X_{\text{an}} \times [0, 1] \) with open cover \( \{ \hat{R}_i : i \in I \} \), where \( \hat{R}_i = R_i \times [0, \frac{1}{3}] \) if \( i \in I \subseteq \hat{I} \) and \( \hat{R}_i = \hat{R}_{i'} \times [\frac{1}{3}, 1] \) if \( i' \in J' \subseteq \hat{I} \). This gives closed subsets \( \hat{C}_j \subseteq X_{\text{an}} \times [0, 1] \) for all \( \hat{J} \in \hat{A} \). From the proof of Proposition 3.15 we see that we can choose these with \( \hat{C}_j \cap (X_{\text{an}} \times [0, \frac{1}{3}]) = C_j \times [0, \frac{1}{3}] \) for all \( J \in A \subseteq \hat{A} \), and \( \hat{C}_{J'} \cap (X_{\text{an}} \times [\frac{2}{3}, 1]) = C_{J'} \times [\frac{2}{3}, 1] \) for all \( J' \in A' \subseteq \hat{A} \).

Next we apply the proof of Proposition 3.17 in [6.2] which depends on Theorem 3.19 to choose pairs \((\hat{U}_j, \hat{E}_j^-)\) for \( \hat{J} \in \hat{A} \) satisfying the analogue of condition (†) for \( \hat{V}_j, \hat{E}_j, s_j, \ldots \), with open \( \hat{U}_{JK} \subseteq \hat{U}_j \) satisfying compatibility under the \( \hat{\phi}_{JK} \). Working over \( X_{\text{an}} \times [0, 1], V_j \times [0, 1] \) rather than \( X_{\text{an}}, V_j \) causes no new nontrivial problems.

We can choose these \( \hat{U}_j, \hat{E}_j^-\) such that \( \hat{U}_j \cap (V_j \times [0, \frac{1}{3}]) = U_j \times [0, \frac{1}{3}] \) and \( \hat{E}_j^- \mid_{\hat{U}_j \cap (V_j \times [0, \frac{1}{3}])] = \pi_{V_j}^{-1}(E_j^-) \) for all \( J \in A \subseteq \hat{A} \), and \( \hat{U}_{JK} \cap (V_j \times [0, \frac{1}{3}]) = U_{JK} \times [0, \frac{1}{3}] \) for all \( K \subseteq J \in A \subseteq \hat{A} \), and \( \hat{U}_{J'} \cap (V_{J'} \times [\frac{2}{3}, 1]) = U_{J'} \times [\frac{2}{3}, 1] \) and \( \hat{E}_{J'} \mid_{\hat{U}_{J'} \cap (V_{J'} \times [\frac{2}{3}, 1])} = \pi_{V_j}^{-1}(E_{J'}^-) \) for all \( J' \in A' \subseteq \hat{A} \), and \( \hat{U}_{J'K'} \cap (V_{J'} \times [\frac{2}{3}, 1]) = U_{J'K'} \times [\frac{2}{3}, 1] \) for all \( K' \subseteq J' \in A' \subseteq \hat{A} \). As in the proof of Theorem 3.18 in
we have now constructed the following data:

(a) A Hausdorff, second countable topological space $X_{\text{an}} \times [0,1]$.
(b) An indexing set $\mathcal{I}$, where we write $\mathcal{A} = \{ J : \emptyset \neq J \subseteq \mathcal{I}, \, J \text{ is finite} \}$.
(c) An open cover $\{ \tilde{S}_j : j \in \mathcal{A} \}$ of $X_{\text{an}} \times [0,1]$, such that $\tilde{S}_j \cap \tilde{S}_K \neq \emptyset$ for $J, K \in \mathcal{A}$ only if $J \subseteq K$ or $K \subseteq J$.
(d) A Kuranishi neighbourhood with boundary $(\tilde{U}_J, E^+_J, s^+_J, \psi^+_J)$ on $X_{\text{an}} \times [0,1]$ with $\dim \tilde{U}_J - \text{rank } E^+_J = n + 1$ for each $J \in \mathcal{A}$, constructed from $(\tilde{U}_J, E^-_J)$ satisfying (i), with $\text{Im } \psi^+_J = \tilde{S}_j \subseteq X_{\text{an}} \times [0,1]$.

3.5 This is straightforward. In place of (6.13)–(6.14), for

\[ T_{(x,t)}W_{\text{dm}} \cong \text{Ker}((\partial s_j|_{v_J}) : T_{v_J}\bar{V}_j \to \bar{E}_j|_{v_J}), \]

\[ O_{(x,t)}W_{\text{dm}} \cong \frac{\text{Ker}(\bar{f}_J|_{v_J})/\text{Im}(\partial s_j|_{v_J})}{\text{Im } H_{v_J}}. \]

Here $\bar{V}_j$ is of the form $V_j \times [0, \frac{3}{4}]$ or $V_j \times [\frac{1}{4}, 1]$ or $\bar{V}_j \times (\frac{1}{4}, \frac{3}{4})$ with $V_j, V'_j, \bar{V}_j$ complex manifolds, so $T_{v_J}\bar{V}_j = (\text{complex vector space}) \times \mathbb{R}$, and $\partial s_j|_{v_J}$ is a complex linear map on the first factor, so $T_{(x,t)}W_{\text{dm}} = (\text{complex vector space}) \times \mathbb{R}$.
Thus $T_{(x,t)} W_{dm}$ has a natural orientation, coming from the complex structure on the first factor, and the standard orientation on $[0,1]$ for the second factor. Hence as in (6.17) we have

$$P W_{dm} |_{(x,t)} \cong \{ \text{real orientations of } O_{(x,t)} W_{dm} \}.$$ 

We can then argue as for (6.17) in §6.3 to get a canonical identification

$$O_{X, \omega^*_{\mathcal{X}}} |_x \cong P W_{dm} |_{(x,t)}$$

and thus a natural isomorphism $\pi^*_{\mathcal{X}} (O_{X, \omega^*_{\mathcal{X}}}) \cong P W_{dm}$ of topological principal $\mathbb{Z}_2$-bundles over $X_{an} \times [0,1]$, as we have to prove.

Under the equivalence $\partial W_{dm} \cong X_{dm} \sqcup X'_{dm}$, this restricts on $X_{an} \times \{1\}$ to the isomorphism $O_{X, \omega^*_{\mathcal{X}}} \cong P_{X_{dm}}$ for $X'_{dm}$ in Proposition 3.19, but on $X_{an} \times \{0\}$ to the isomorphism $O_{X, \omega^*_{\mathcal{X}}} \cong P_{X_{dm}}$ for $X_{dm}$ multiplied by $-1$, because with the standard orientation on $[0,1]$ used to define the orientation on $T_{(x,t)} W_{dm}$ we have $\partial [0,1] = -\{0\} \sqcup \{1\}$, so the identification $\partial U_J = U_J \times \{0\} \cong U_J$ for $J \in A \subseteq \tilde{A}$ above reverses orientations. This completes the proof of Theorem 3.20.

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Dennis Borisov, Mathematisches Institut, Georg-August Universität Göttingen, Bunsenstrasse 3–5, D-37073 Göttingen, Germany.
E-mail: dennis.borisov@gmail.com.

Dominic Joyce, The Mathematical Institute, Radcliffe Observatory Quarter, Woodstock Road, Oxford, OX2 6GG, U.K.
E-mail: joyce@maths.ox.ac.uk.