MORDELL-LANG PLUS BOGOMOLOV II: THE DIVISION GROUP

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1. INTRODUCTION

In [Po], we formulated a conjecture for semiabelian varieties which would include both the
Mordell-Lang conjecture and the Bogomolov conjecture, and we proved this conjecture in
the case where the semiabelian variety was almost split (i.e., isogenous to the product of an
abelian variety and a torus), and where we had only a finitely generated subgroup instead
of its division group.

In this paper we prove the conjecture for the full division group. The “almost split”
hypothesis remains, but this is only because it is under this hypothesis that the Bogomolov
conjecture and an equidistribution theorem have been proved so far [CL]. In fact, if we
assume the Bogomolov conjecture and an equidistribution theorem for general semiabelian
varieties, then we can prove our conjecture entirely.

Our proof requires also the Mordell-Lang conjecture (which is completely proven). But we
do not need its full strength: we use only the “Mordellic” (finitely generated) part. Hence
our proof gives a new reduction of the full Mordell-Lang conjecture to the Mordellic part, at
least for almost split semiabelian varieties.

We postpone the precise statement of our result until Section 5. First we discuss the notion
of “sequence of small points” on semiabelian varieties, and describe the major ingredients
from which our result will follow.

2. SMALL POINTS ON SEMIABELIAN VARIETIES

For the rest of this paper, $k$ denotes a number field. Let $A$ be a semiabelian variety over
$k$. When $A$ is almost split, then after enlarging $k$ we have an isogeny $\phi = (\phi_1, \phi_2) : A \to
A_0 \times \mathbb{G}_m^r$, and we can define a canonical height $h : A(\overline{k}) \to \mathbb{R}$ by $h(x) = h_1(\phi_1(x)) + h_2(\phi_2(x))$, where $h_1 : A_0(\overline{k}) \to \mathbb{R}$ is the Néron-Tate canonical height associated to a symmetric ample
line bundle on $A_0$, and $h_2 : \mathbb{G}_m^r(\overline{k}) \to \mathbb{R}$ is the sum of the naive heights of the coordinates.
Then a sequence of points $x_i \in A(\overline{k})$ is said to be a sequence of small points if $h(x_i) \to 0$.

But for more general semiabelian varieties over $k$, there seems to be no natural canonical
height with reasonable properties. Therefore we instead use the definition of sequence of
small points given in [Po], which was based on a suggestion of Hrushovski. For the details
of the following, see [Po].

Embed $A$ as a quasi-projective variety in some $\mathbb{P}^n$, and let $h$ be a naive Weil height. Fix
an integer $m \geq 2$, and let $[m] : A \to A$ denote multiplication by $m$. Then there exists
an integer $r \geq 1$ and real constants $M > 0$ and $c > 1$ such that for $z \in A(\overline{k})$, $h(z) > M$
implies $h([m^r]z) > ch(z)$. If $z \in A(\overline{k})$, let $N(z)$ be the smallest integer $N \geq 1$ such that $h([m^N]z) > M$, or $\infty$ if no such $N$ exists. We say that $\{z_i\}_{i \geq 1}$ is a sequence of small points if $N(z_i) \to \infty$ in $\mathbb{P}^1(\mathbb{R})$.

It is shown in [Po] that this notion is independent of choices. Moreover, when $A$ is almost split, this notion agrees with the one defined using canonical heights.

3. The Mordell-Lang conjecture

By the “Mordell-Lang conjecture” we mean the theorem below, which was proved by McQuillan [McQ], following work by Faltings, Vojta, Hindry, Raynaud, and many others. The division group $\Gamma'$ of a subgroup $\Gamma$ of a group $G$ is defined as

$$\Gamma' := \{ x \in G \mid \text{there exists } n \geq 1 \text{ such that } nx \in \Gamma \}.$$ 

**Theorem 1** (Mordell-Lang conjecture). Let $A$ be a semiabelian variety over a number field $k$. Let $\Gamma$ be a finitely generated subgroup of $A(\overline{k})$, and let $\Gamma'$ be the division group. Let $X$ be a geometrically integral closed subvariety of $A$ that is not equal to the translate of a sub-semiabelian variety (over $\overline{k}$). Then $X(\overline{k}) \cap \Gamma'$ is not Zariski dense in $X$.

**Remarks.**

1. We obtain an equivalent statement if “translate of a sub-semiabelian variety” is replaced by “translate of a sub-semiabelian variety by a point in $\Gamma'$.”
2. We obtain an equivalent statement if “not Zariski dense in $X$” is replaced by “contained in a finite union $\bigcup Z_j$ where each $Z_j$ is a translate of sub-semiabelian variety of $A_{\overline{k}}$, and $Z_j \subseteq X_{\overline{k}}$.” (Here $A_{\overline{k}}$ denotes $A \times_k \overline{k}$, and so on.)
3. The theorem is true for any field $k$ of characteristic 0: specialization arguments let one reduce to the number field case.
4. There are function field analogues: see [Hi].

We will need only the weaker statement obtained by replacing $\Gamma'$ by $\Gamma$ in Theorem 1. This is sometimes called the “Mordellic part,” because the special case where $X \subset A$ is a curve of genus $\geq 2$ in its Jacobian is equivalent to Mordell’s conjecture about the finiteness of $X(k)$.

4. The Bogomolov and equidistribution conjectures

If $X$ is a geometrically integral variety, then a sequence of points $z_i$ is said to be generic in $X$ if $z_i \in X(\overline{k})$ for all $i$, and the $z_i$ converge to the generic point of $X_{\overline{k}}$. The latter condition means that each subvariety $Y$ of $X_{\overline{k}}$ other than $X_{\overline{k}}$ itself contains only finitely many points in the sequence.

**Conjecture 2** (Bogomolov conjecture for semiabelian varieties). Let $A$ be a semiabelian variety over a number field $k$. Let $X$ be a geometrically integral closed subvariety of $A$. Let $\{z_i\}$ be a sequence of small points of $A(\overline{k})$, generic in $X$. Then $X_{\overline{k}}$ is the translate of a sub-semiabelian variety of $A_{\overline{k}}$ by a torsion point.

Bogomolov’s original conjecture was for a curve of genus $\geq 2$ in its Jacobian. This case was proved by Ullmo [Ul]. For $A$ an abelian variety or a torus, Conjecture 2 was proved by Zhang in [Zh3] and [Zh2], respectively. Recently, a proof for the case where $A$ is almost split was announced by Chambert-Loir [CL].
Recall that if $\mu_i$ for $i \geq 1$ and $\mu$ are probability measures on a metric space $X$, then one says that the $\mu_i$ converge weakly to $\mu$ if for every bounded continuous function $f$ on $X$, $\lim_{i \to \infty} \int f \mu_i = \int f \mu$. For the following, we fix an embedding $\sigma : \bar{k} \hookrightarrow \mathbb{C}$, and let $A_\sigma$ denote the semiabelian variety over $\mathbb{C}$ obtained by base extension by $\sigma$.

**Conjecture 3** (Equidistribution conjecture). Let $A$ be a semiabelian variety over a number field $k$. Let $\{z_i\}$ be a sequence of small points of $A(\bar{k})$, generic in $A$. Let $\mu_i$ be the uniform probability measure on the finite set $\sigma(G_k(z_i)) \subset A_\sigma(\mathbb{C})$. Then the $\mu_i$ converge weakly to the normalized Haar measure $\mu$ on the maximal compact subgroup $A_0$ of $A(\mathbb{C})$.

**Remark.** The subgroup $A_0$ also equals the closure of $A_\sigma(\mathbb{C})_{\text{tors}}$ in the complex topology. It is all of $A(\mathbb{C})$ is $A$ is abelian, and it is a “polydisc” if $A$ is a torus.

The cases where $A$ is an abelian variety or a torus are proved in [SUZ] and [Bi], respectively. A proof for the case where $A$ is an almost split semiabelian variety has been announced by Chambert-Loir [CL]. Bilu suggested in [Bi] that Haar measure on $A_0$ should be the limit measure, but he could not formulate Conjecture 3 precisely, because at the time there was no notion of “sequence of small points” for general semiabelian varieties.

### 5. Statement of the result

The following was conjectured in the final section of [Po]:

**Conjecture 4** (Mordell-Lang + Bogomolov). Let $A$ be a semiabelian variety over a number field $k$. Let $\Gamma$ be a finitely generated subgroup of $A(\bar{k})$, and let $\Gamma'$ be its division group. Let $X$ be a geometrically integral closed subvariety of $A$. For $i \geq 1$, suppose that $x_i = \gamma_i + z_i \in X(\bar{k})$ where $\gamma_i \in \Gamma'$ and $\{z_i\}_{i \geq 1}$ is a sequence of small points in $A(\bar{k})$. If $X_{\bar{k}}$ is not a translate of a sub-semiabelian variety of $A_{\bar{k}}$ by an element of $\Gamma'$, then the $x_i$ are not Zariski dense in $X$.

**Remark.**

1. Equivalently, we could let $B_\epsilon := \{ z \in A(\bar{k}) \mid N(z) > 1/\epsilon \}$ and conjecture that for some $\epsilon > 0$, $X(\bar{k}) \cap (\Gamma' + B_\epsilon)$ is not Zariski dense in $X$. (Here $N$ is as in Section 2.)
2. It is equivalent if we remove the restriction that $X_{\bar{k}}$ not be the translate of a sub-semiabelian variety by an element of $\Gamma'$ and replace the conclusion “not Zariski dense in $X$” by “contained in a finite union $\bigcup Z_j$ where each $Z_j$ is a translate of a sub-semiabelian variety of $A_{\bar{k}}$ by an element of $\Gamma'$, and $Z_j \subseteq X_{\bar{k}}$.”

Let $S(A)$ be the set of semiabelian varieties over number fields that can be obtained from $A$ by taking algebraic subgroups, quotients, and products, and changing the field of definition.

**Theorem 5.** Let $A$ be a semiabelian variety over a number field $k$. Assume that Conjectures 2 and 3 hold for all $B \in S(A)$. Then Conjecture 4 holds for subvarieties $X$ in $A$.

**Corollary 6.** Conjecture 4 holds when $A$ is almost split.

**Proof.** One checks that all $B \in S(A)$ are almost split. Chambert-Loir [CL] proved Conjectures 2 and 3 for almost split semiabelian varieties. \qed
6. Properties of small points

In preparation for the proof of Theorem 5, we derive a few more properties of sequences of small points. Throughout this section, $A$ denotes a semiabelian variety over a number field $k$. Let $\beta_n : A^n \to A^{n-1}$ be the map sending $(x_1, \ldots, x_n)$ to $(x_2 - x_1, x_3 - x_1, \ldots, x_n - x_1)$. Up to an automorphism of $A^{n-1}$, this is the same as the map $\alpha_n$ used in [ZiH3].

Let $G_k = \text{Gal}(\overline{k}/k)$. For $x \in A(\overline{k})$, let $G_k(x)$ denote the $G_k$-orbit of $x$, and let $\text{Diff}_n(x)$ be the (arbitrarily ordered) list of $[k(x) : k]^n$ elements of $A^{n-1}$ obtained by applying $\beta_n$ to the elements of $G_k(x)^n$. Given any sequence $x_1, x_2, \ldots$ of points in $A(\overline{k})$, let $\mathcal{D}_n = \mathcal{D}_n(\{x_i\})$ denote the infinite sequence obtained by concatenating $\text{Diff}_n(x_1), \text{Diff}_n(x_2), \ldots$.

**Lemma 8.** Let $A$ and $B$ be semiabelian varieties over a number field $k$. Let $\{x_i\}$ be a sequence of small points in $A$, and let $\{y_i\}$ be a sequence of small points in $B$.

1. If $\{x_j\}$ is an infinite subsequence of $\{x_i\}$, then $\{x_j\}$ is a sequence of small points in $A$.
2. If $\{x_j\}$ is an infinite sequence obtained from $\{x_i\}$ by replacing each $x_i$ with a finite number of copies of $x_i$, then $\{x_j\}$ is a sequence of small points in $A$.
3. If $\sigma \in G_k$, then $\{\sigma x_i\}$ is a sequence of small points in $A$.
4. The sequence $\{(x_i, y_i)\}$ is a sequence of small points in $A \times B$.
5. If $f : A \to B$ is a homomorphism, then $\{f(x_i)\}$ is a sequence of small points in $B$.
6. If $A = B$, then $\{x_i + y_i\}$ and $\{x_i - y_i\}$ are sequences of small points in $A$.
7. For any $n \geq 2$, $\mathcal{D}_n = \mathcal{D}_n(\{x_i\})$ is a sequence of small points in $A$.
8. If $[k(x_i) : k]$ is bounded, then there is a finite subset $T \subset A(\overline{k})_{\text{tors}}$ containing all but finitely many of the $x_i$.
9. There exists a sequence of positive integers $\{n_i\}$ with $n_i \to \infty$, such that if $1 \leq m_i \leq n_i$, then the sequence $\{[m_i, x_i]\}$ is still a sequence of small points.

**Proof.** Properties (1) through (4) are immediate from the definition, and (5) is proved in [Pe]. Property (6) follows from (4) and (5), and (7) follows from (2), (3), (4), and (6).

Next we prove (8). Let $S = \{x \in A(\overline{k}) : h(x) \leq M\}$. By Northcott’s Theorem, $s := |S|$ is finite. By definition of $N(x_i)$, $[m^j]x_i \in S$ for $j < N(x_i)$. We have $N(x_i) > 1 + s$ for all but finitely many $i$. For these $i$, the pigeonhole principle yields $[m^j]x_i = [m^{j'}]x_i$ for some $j < j' \leq 1 + s$. In particular $x_i \in T$, where $T := \bigcup_{q \leq m_i + s} A(\overline{k})[q]$.

Finally, we prove (9). Define $a_{ij} := N([j], x_i)$. By part (5) applied to $[j] : A \to A$, we have $\lim_{i \to \infty} a_{ij} = \infty$ for each $j$. It follows formally that there is a sequence of positive integers $\{n_i\}$ tending to infinity, such that $a_{i, m_i} \to \infty$ whenever $1 \leq m_i \leq n_i$.

Next we have a sequence of lemmas leading up to Lemma 11, which is the only other result from this section that will be used later. Recall from [Pe] that one possible choice of $h$ is as follows. We have an exact sequence $0 \to T \to A \to A_0 \to 0$ where $T$ is a torus and $A_0$ is an abelian variety. Enlarging $k$, we may assume that $T \cong G_m$. There is a compactification $\overline{A}$ to which $\alpha$ extends, equipped with effective line bundles $\mathcal{L}_0, \mathcal{L}_1$ with $\mathcal{L} := \mathcal{L}_0 \otimes \mathcal{L}_1$ ample, such that $\mathcal{L} \cong \mathcal{L}_0^\otimes 2$ and $\mathcal{L} \cong \mathcal{L}_1^\otimes 4$. In fact, $\overline{A}$ is also equipped with a map $\overline{\pi} : \overline{A} \to A_0$ extending $A \to A_0$, and $\mathcal{L}_1 = \overline{\pi}^*\mathcal{M}$, where $\mathcal{M}$ is a symmetric ample line bundle on $A_0$. (See Section 1.1 of [McO], for example.) Define $h = h_0 + h_1$ where $h_0$ is a Weil height associated to $\mathcal{L}_0$, and $h_1$ is the canonical height associated to $\mathcal{L}_1$ (i.e., the pullback of the canonical height on $A_0$ associated to $\mathcal{M}$).

**Lemma 8.** Given $a \in A(\overline{k})$, there exists $M_a > 0$ such that if $x \in A(\overline{k})$ and $h(x) > M_a$, then $h([2]x + a) > (3/2)h(x)$.
Proof. Translation-by-\(a\) extends to a morphism \(\tau_a : \overline{A} \to \overline{A}\), and \(\tau_0^* \mathcal{L}_0 = \mathcal{L}_0\). It follows that \(h_0(x + a) = h_0(x) + O(1)\), where the \(O(1)\) depends on \(a\). Also \(\mathcal{L}_0^* = \mathcal{L}_0^\otimes 2\), so \(h_0(2x) = 2h_0(x) + O(1)\). On the other hand, since \(h_1\) is a quadratic function, \(h_1(x + a) = h_1(x) + O(h_1(x)^{1/2}) + O(1)\). Hence

\[
\begin{align*}
h(2x + a) &= h_0(2x + a) + h_1(2x + a) \\
&= [2h_0(x) + O(1)] + [4h_1(x) + O(h_1(x)^{1/2}) + O(1)] \\
&= 2h(x) + [2h_1(x) + O(h_1(x)^{1/2}) + O(1)] \\
&\geq 2h(x) + O(1).
\end{align*}
\]

\[\square\]

Lemma 9. Let \(\Gamma\) be a finitely generated subgroup of \(A(\overline{k})\), and let \(\{x_i\}\) be a sequence in \(\Gamma\). If the image of \(\{x_i\}\) in \(\Gamma' \otimes \mathbb{R} = \Gamma / \Gamma_{\text{tors}}\) converges to zero in the usual real vector space topology, then \(\{x_i\}\) is a sequence of small points.

Proof. Let \(S := \{\gamma_1, \gamma_2, \ldots, \gamma_n\} \subset \Gamma\) be a \(Z\)-basis for \(\Gamma / \Gamma_{\text{tors}}\). Let \(U = \{\sum \epsilon_i \gamma_i : \epsilon_i \in \{-1, 0, 1\}\}\). Let \(f_1, f_2, \ldots, f_u\) be the maps \(A \to A\) of the form \(x \mapsto [2]x + a\) for \(a \in U\).

By repeated application of Lemma 8, we can find \(M > 0\) such that \(h(x) > M\) implies \(h(f_i(x)) > (3/2)h(x)\) for all \(i\).

Let \(B\) be the subset of elements of \(\Gamma'\) whose image in \(\Gamma \otimes \mathbb{R}\) have coordinates (with respect to the basis \(S\)) bounded by 1 in absolute value. For every \(b_0 \in B\), there exists \(i\) such that \(b_1 := f_i(b_0) \in B\), and then there exists \(j\) such that \(b_2 := f_j(b_1) \in B\), and so on. The intersection \(I\) of \(B\) with the finitely generated subgroup generated by \(b_0\) and \(S\) is finite, and \(b_i \in I\) for all \(i\). But if \(h(b_0) > M\), then \(h(b_{m+1}) > (3/2)h(b_m)\) for all \(m\), and in particular, the \(b_i\) would be all distinct. Hence \(h(b_0) \leq M\) for all \(b_0 \in B\). The lemma now follows from the definition of sequence of small points.

\[\square\]

Remark. The converse to Lemma 9 is true, but we do not need it.

Lemma 10. Let \(\Gamma\) be a finitely generated subgroup of \(A(\overline{k})\). Then \(A(k) \cap \Gamma'\) is a finitely generated group.

Proof. By the Mordell-Weil theorem, \(A_0(k)\) is finitely generated. Since we have assumed that the toric part \(T\) of \(A\) is split, Hilbert’s Theorem 90 guarantees that there is no obstruction to lifting generators of \(A_0(k)\) to elements of \(A(k)\). Without loss of generality, enlarge \(\Gamma\) to contain these lifts. Then \(A(k) \subseteq T(k) + \Gamma\), and \(A(k) \cap \Gamma' \subseteq (T(k) \cap \Gamma') + \Gamma\), so we reduce to the case where \(A = T = G_m^r\).

Let \(\pi_i : G_m^r \to G_m\) be the \(i\)-th projection. Enlarging \(\Gamma\) by replacing it by \(\prod_{i=1}^r \pi_i(\Gamma)\), we reduce to the case \(r = 1\); i.e., \(A = G_m\). The finite generation of the unit group and the finiteness of the class group of \(k\) imply that \(k^*\) is isomorphic as abstract group to the direct sum of a finite torsion group with a free abelian group of countable rank. Any finite rank subgroup of such a group is finitely generated.

\[\square\]

Lemma 11. Let \(\Gamma\) be a finitely generated subgroup of \(A(\overline{k})\). Suppose \(\{x_i\}\) is a sequence in \(A(k)\), and \(x_i = \gamma_i + z_i\) where \(\gamma_i \in \Gamma'\), and \(\{z_i\}\) is a sequence of small points in \(A(\overline{k})\). Then there is a finitely generated subgroup of \(\Gamma'\) containing all but finitely many of the \(x_i\).

\[\square\]
Proof. We may enlarge $k$ to assume $\Gamma \subset A(k)$. Choose $\{n_i\}$ for $\{z_i\}$ as in part (9) of Lemma 7. By elementary diophantine approximation (the pigeonhole principle), there exist integers $m_i$ with $1 \leq m_i \leq n_i$, and $\nu_i \in \Gamma$ such that the images of $m_i\gamma_i - \nu_i \in \Gamma \otimes \mathbb{R}$ approach zero as $i \to \infty$. Then $\{m_i\gamma_i - \nu_i\}$ is a sequence of small points by Lemma 9 but $\{m_i\gamma_i\}$ also is a sequence of small points, so by part (6) of Lemma 7, $\{m_i\gamma_i - \nu_i\}$ is a sequence of small points. On the other hand, $m_i\gamma_i - \nu_i \in A(k)$, so by part (8) of Lemma 4, $x_i \in \Gamma'$ for all but finitely many $i$. Finally, Lemma 10 implies that there is a finitely generated subgroup of $\Gamma'$ containing all but finitely many $x_i$. □

7. Measure-theoretic lemmas

We recall the following lemma from [Po]:

**Lemma 12.** Let $V$ be a projective variety over $\mathbb{C}$. Let $S$ be a connected quasi-projective variety over $\mathbb{C}$. Let $Y \to V \times S$ be a closed immersion of $S$-varieties, where $Y \to S$ is flat with $d$-dimensional fibers. For $i \geq 1$, let $s_i \in S(\mathbb{C})$ and let $Y_i \subset V$ be the fiber of $Y \to S$ above $s_i$. Let $\mu_i$ be a probability measure supported on $Y_i(\mathbb{C})$. If the $\mu_i$ converge weakly to a probability measure $\mu$ on $V(\mathbb{C})$, then the support of $\mu$ is contained in a $d$-dimensional Zariski closed subvariety of $V$.

**Proof.** Choose an embedding $V \hookrightarrow \mathbb{P}^m$. Without loss of generality, we may replace $V$ and $Y$ by their closures in $\mathbb{P}^m$ and $\mathbb{P}^m \times S$, respectively. Replacing $S$ by a dense open subset $U$ and $Y$ by $\pi^{-1}(U)$, and passing to a subsequence, we may assume that $Y \to S$ is flat. The result now follows from Lemma 12. □

**Lemma 13.** Let $V$ and $S$ be quasi-projective varieties over $\mathbb{C}$, with $S$ integral. Let $Y$ be a subvariety of $V \times S$. Let $s_1, s_2, \ldots$ be a sequence in $S(\mathbb{C})$, Zariski dense in $S$. Let $\mu_i$ be a probability measure with support contained in the fiber of $\pi : Y \to S$ above $s_i$, considered as subvariety of $V$. Suppose the $\mu_i$ converge weakly to a probability measure $\mu$ on $V(\mathbb{C})$. Then the support of $\mu$ is contained in a subvariety of $V$ of dimension $\dim Y - \dim S$.

**Proof.** We have $\dim Y - \dim S < \dim V$, so by the previous lemma, $\mu$ is supported on a subvariety of $V$ of positive codimension. But $V^0$ is Zariski dense in $V$. □

**Lemma 14.** Retain the assumptions of the previous lemma, but assume in addition that $V$ is a semiabelian variety over $\mathbb{C}$, and that $Y$ is not Zariski dense in $V \times S$. Then $\mu$ does not equal the normalized Haar measure on the maximal compact subgroup $V^0$ of $V(\mathbb{C})$.

**Proof.** We have $\dim Y - \dim S < \dim V$, so by the previous lemma, $\mu$ is supported on a subvariety of $V$ of positive codimension. But $V^0$ is Zariski dense in $V$. □

8. Proof of Theorem 5

The proof will proceed through various reductions; to aid the reader, we box cumulative assumptions and other partial results to be used later in proof. Let $G$ be the group of translations preserving $X$; i.e., the largest algebraic subgroup of $A$ such that $X + G = X$. We may assume $\dim G = 0$, since otherwise we consider $X/G \hookrightarrow A/G$ and use part (5) of Lemma 7. We may also enlarge $k$ to assume that $\Gamma \subset A(k)$.

If the theorem is false, then there exists a sequence $x_i = \gamma_i + z_i \in X(\bar{k})$, generic in $X$, with $\gamma_i \in \Gamma'$ and with $\{z_i\}$ a sequence of small points in $A(\bar{k})$. For $\sigma, \tau \in G_k$,

$$\sigma x_i - \tau x_i = (\sigma \gamma_i - \tau \gamma_i) + (\sigma z_i - \tau z_i).$$
Some multiple of $\gamma_i$ is in $\Gamma \subset A(k)$, so $\sigma \gamma_i - \tau \gamma_i$ is torsion. Applying part (7) of Lemma 7 to $D_2(\{z_i\})$, and then applying parts (2) and (6), we find that $D_2 := D_2(\{x_i\})$ is a sequence of small points in $A$. Then by parts (2) and (4) of Lemma 6, $D_n$ is a sequence of small points for each $n \geq 2$. Fix $n > \dim A$.

By repeated application of Conjecture 2, we may discard finitely many of the $x_i$ in order to assume that the Zariski closure $\overline{D}_n$ of $D_n$ in $A_\mathbb{F}$ is a finite union $\bigcup_{j=1}^s (B_j + t_j)$ where $B_j$ is a sub-semiabelian variety of $A_\mathbb{F}$, and $t_j \in A(\overline{k})$ is a torsion point. If we replace $X$ by the image of $X$ under multiplication by a positive integer $N$, and replace each $x_i$ by $N x_i$, then $\bigcup_{j=1}^s B_j$ is unchanged. If we pass to a subsequence of the $x_i$ or enlarge $k$, then the new $\bigcup_{j=1}^s B_j$ can only be smaller. Since $A_\mathbb{F}$ is noetherian, we may assume without loss of generality that these operations are done so as to make $\bigcup_{j=1}^s B_j$ minimal. Moreover, by multiplying by a further integer $N$ we may assume that $t_j = 0$ for all $j$. Now, any further operations of the types above will leave $\overline{D}_n$ unchanged, equal to $\bigcup_{j=1}^s B_j$. Enlarging $k$, we may assume that each $B_j$ is defined over $k$.

Repeating the same procedure with 2 instead of $n$, we may minimize $\overline{D}_2 = \bigcup_{j=1}^a C_j$, and assume that each $C_j$ is a sub-semiabelian variety of $A$. We now show that there is only one $C_j$. By the pigeonhole principle, there is some $C_j$, say $C := C_1$, such that for infinitely many $i$, at least a fraction $1/u$ of the elements of $G_k(x_i^2)$ are mapped by $\beta_2$ into $C$. Passing to a subsequence of the $x_i$, we may suppose that this holds for all $i$. Let $\tau : A \to A/C$ be the projection, and let $y_i = \pi(x_i)$. If $\ell$ is a finite Galois extension of $k$ containing $k(x_i)$, then it follows that $\sigma y_i - \tau y_i = 0$ for at least a fraction $1/u$ of the pairs $(\sigma, \tau) \in \text{Gal}(\ell/k)^2$. Thus the subgroup of $\text{Gal}(\ell/k)$ stabilizing $y_i$ must have index at most $u$. Hence $\deg_{\mathbb{F}}(y_i) = \#G_k(y_i) \leq u$. Then $D_2(\{y_i\})$ consists of points of degree bounded by $u^2$. On the other hand, $D_2(\{y_i\})$ is a sequence of small points in $A/C$, by parts (5) and (7) of Lemma 6. By part (8) of Lemma 6, there is a finite subset $T$ of torsion points of $A/C$ such that $\text{Diff}_2(y_i) \subseteq T$ for all sufficiently large $i$. Passing to a subsequence of the $x_i$, and multiplying everything by an integer $N$ to kill $T$, we may assume that $\text{Diff}_2(y_i) = \{0\}$ for all $i$. Then $\text{Diff}_2(x_i) \subseteq C$, so $\overline{D}_2 \subseteq C$, and hence $\overline{D}_2 = C$ by definition of $C$.

We next show that there is only one $B_j$, and that it equals $C^{n-1}$. By definition of $D_2$ and $D_n$, we have $B_j \subseteq C^{n-1}$ for each $j$. By the pigeonhole principle, there is some $B_j$, say $B := B_1$, such that for infinitely many $i$, at least a fraction $1/s$ of the elements of $G_k(x_i^2)$ are mapped by $\beta_n$ into $B$. Passing to a subsequence, we may suppose that this holds for all $i$. For $1 \leq q \leq n-1$, define the “coordinate axis” $C_{(q)}(x) = 0 \times \cdots \times 0 \times C \times 0 \times \cdots \times 0$, with $C$ in the $q$-th place. Let $B_0 := B \cap C_{(q)}$. By the pigeonhole principle again, given $i$, there exist $w_1, w_2, \ldots, w_q, w_{q+2}, \ldots, w_n \in G_k(x_i)$ such that $\beta_0(w_1, w_2, \ldots, w_q, \zeta, w_{q+2}, \ldots, w_n) \in B$ for at least a fraction $1/s$ of the pairs of elements $\zeta$ of $G_k(x_i)$. Subtracting, we find that $\beta_0(0, 0, \ldots, 0, \zeta - \zeta', 0, \ldots, 0) \in B(q)$ for at least a fraction $1/s^2$ of the pairs of elements $\zeta, \zeta'$ of $G_k(x_i)$. As before, this implies (after passing to a subsequence and multiplying by a positive integer again) that the image $y_i$ of $x_i$ in $C_B(q)$ satisfies $\text{Diff}_2(y_i) = \{0\}$. Then $\overline{D}_2 \subseteq B(q) \subseteq C$, so $B(q) = C$. This holds for all $q$, so $\overline{D}_n = B = C^{n-1}$.

If $C = \{0\}$, then $D_2 = \{0\}$, and then by definition of $D_2$, $x_i \in X(k)$ for all $i$. Lemma 11 implies that all but finitely many $x_i$ are contained in a finitely generated subgroup $\tilde{\Gamma}$ of $\Gamma'$. The Mordellic part of Theorem 1 applied to $\tilde{\Gamma}$ implies that $X_{\mathbb{F}}$ is a translate of a sub-semiabelian variety. But $\dim G = 0$, so $X$ is a point. Moreover this point is in $\Gamma'$ (by
our application of Lemma [1], so we contradict the hypotheses. Therefore we may assume
\[ \dim C \geq 1. \]

Let \( S = \pi(X) \subseteq A/C \). Note that \( S \) is integral. Consider the fibered power \( X^n_S := X \times_S X \times_S \cdots \times_S X \) as a subvariety of \( X^n \).

Let \( m = \dim C \). Note that \( 1 \leq m \leq \dim A < n \). Let \( \dim(X/S) \) denote the relative dimension; i.e., the dimension of the generic fiber of \( X \to S \). Then \( \dim(X/S) < m \), since otherwise \( X \) (being closed) would equal the entire inverse image of \( S \) under \( A \to A/C \), and then \( C \subseteq G \), contradicting \( \dim G = 0 \). Hence
\[
\dim(X^n_S/S) = n \dim(X/S) \leq n(m - 1) < m(n - 1) = \dim(C^{n-1} \times S/S).
\]

The homomorphism \( \beta_n \) restricts to a morphism \( X^n_S \to C^{n-1} \). We also have the obvious morphism \( X^n_S \to S \). Let \( \mathcal{Y} \) denote the image of the product morphism \( X^n_S \to C^{n-1} \times S \).

Then \( \dim \mathcal{Y} < \dim C^{n-1} \times S \), by (1).

Since \( \sigma x_i - \tau x_i \in C(\overline{k}) \) for all \( \sigma, \tau \in G_k \) and all \( i \geq 1 \), concatenating the finite subsets \( \beta(G_k(x^n)) \times \{ \pi(x_i) \} \) of \( C^{n-1} \times S \) yields a sequence of points \( y_j = (c_j, s_j) \) in \( \mathcal{Y} \). The \( c \)-sequence is simply \( D_n \), and each \( s_j \) equals \( \pi(x_i) \) for some \( i \). Now fix an embedding \( \sigma : \overline{k} \hookrightarrow C \), and let \( \mu_j \) be the uniform probability measure on the finite subset \( \sigma(G_k(c_j)) \subset C^{n-1}_\sigma(C) \).

Conjecture \( [3] \) implies that the \( \mu_j \) converge to the normalized Haar measure \( \mu \) on the maximal compact subgroup of \( C^{n-1}_\sigma(C) \).

On the other hand, \( \mu_j \) is supported on \( \sigma:\beta(G_k(x^n)) \), which is contained in the fiber of \( \mathcal{Y} \to S \) above \( s_j \), when we consider the fiber as a subvariety of \( V := C^{n-1}_\sigma \). Lemma [4] implies that the \( \mu_j \) cannot converge to \( \mu \). This contradiction completes the proof of Theorem [5].

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