Taking a critical look at holographic critical matter

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Despite a recent flurry of applications of the broadly defined (‘non-AdS/non-CFT’) holographic correspondence to a variety of condensed matter problems, the status of this intriguing, yet speculative, approach remains largely undermined. This note exposes a number of potential inconsistencies between the previously made holographic predictions and advocates for a compelling need to systematically contrast the latter against the results of alternate, more conventional, approaches as well as experimental data. It is also proposed to extend the list of computed observables and utilise the general relations between them as a further means of bringing the formal holographic approach into a closer contact with the physical realm.

Introduction

Quantum many-body theory has long been seeking to expand its toolbox of computational techniques, thus allowing one to describe and classify a broad variety of non-Fermi liquid (NFL) states of strongly correlated fermions.

In generic 1d fermion systems, the conventional Fermi liquid behavior gets (marginally) destroyed by an arbitrarily weak short-ranged repulsive interaction, thereby giving way to the so-called Luttinger behavior. As one of the hallmarks of the Luttinger regime, the electron propagator exhibits an algebraic decay with distance/time, $G(x) \propto 1/x^\Delta$, governed by an anomalous dimension $\Delta > 1$. Moreover, long-range interactions, such as Coulomb, modify the Fermi liquid behavior even more drastically, resulting in, e.g., the 1d Wigner crystal state where the fermion propagator decays faster than any power law, $G(x) \sim \exp(-\# \ln^{3/2} x)$.

In higher dimensions, the Fermi liquid is generally believed to be more robust, although it is not expected to remain absolutely stable. While in the case of short-ranged repulsive interactions any departures from the Fermi liquid are likely to be limited to the strong-coupling regime, long-ranged couplings can possibly result in the NFL types of behavior without any threshold.

Of a particular interest are the spectroscopic and transport properties of such emergent critical behaviors as incipient $s^-, p^-$, and $d$-wave charge/spin density waves and orbital current-type instabilities in itinerant (anti)ferromagnets, quantum spin liquids, compressible (‘composite fermion’) Quantum Hall states, etc. Recently, the focus has also been on the $d > 1$-dimensional zero density (‘neutral’) Dirac/Weyl systems characterized by the presence of isolated points (‘nodes’) or lines (‘arcs’) of vanishing quasiparticle energy.

The intrinsic complexity of these systems has long been recognized, prompting the use of such sophisticated techniques as renormalization group, $1/N$- and $\epsilon$-expansions, Keldysh functional integral and quantum kinetic equation, supersymmetric diffusive and ballistic $\sigma$-models, multi-dimensional bosonization, etc. In spite of all the effort, however, the overall progress towards a systematic classification of various ‘strange’ metallic (compressible) states that are often indiscriminately referred to as ‘higher dimensional Luttinger liquids’ has been rather slow.

In that regard, the recent idea of a (broadly defined) holographic correspondence could provide a sought-after powerful alternative technique. Specifically, its widely used ‘bottom-up’ version could potentially offer an advanced phenomenological framework for discovering new and classifying the already known types of NFL behavior.

Although in much of the pertinent literature the validity of the generalized (‘non-AdS/non-CFT’) holographic conjecture appears to be taken for granted, it might be worth reminding that the actual status of the entire holographic approach remains anything but firmly established.

Indeed, in most of its applications this bold adaptation of the original ‘bona-fide’ string-holographic correspondence does not seem to be subject to much (or, for that matter, any) of the former’s stringent symmetry conditions, as the pertinent non-relativistic systems at finite density and temperature, in general, tend to be neither non-Abelian/multicomponent/supersymmetric, nor even Lorentz invariant.

The precious few examples of a quantitative agreement between the holographic approach and other (e.g., Monte Carlo) techniques involve some carefully tailored gravity duals (whose physical nature still remains rather obscure, though)$^{24,5}$.

In other cases, under a closer inspection the purported agreement appears to be largely limited to an apparent similarity between the results of some (for the most part, numerical) calculations and certain selected sets of the available experimental data.

For one, in Ref.$^4$ the holographically computed optical conductivity was claimed to agree (over less than half of a decade, $2 < \omega \tau < 8$) with the enigmatic power-law decay, $\sigma(\omega) \sim \omega^{-2/3}$, observed in the normal state of the superconducting cuprates ($BSCYO$), pnictides, and certain heavy fermion materials, often up to the energies of order $eV$. Notably, the original claim was not corroborated by the later analysis of Ref.$^4$ and was also argued to be intermittent with the ‘more universal’ $\sim 1/\omega$ behavior$^5$.

Also, while being customarily wordy and profuse on technical details, most of the works on holography end up with rather simple scaling relations as their final answers, thereby suggesting that there might be more eco-
nomic and physically illuminating ways of obtaining such results.

Thus, in order for its status to be definitively ascertained, the holographic approach needs to be assessed critically and applied to those systems where a preliminary insight can be (or has already been) gained by some alternative means, so that a systematic comparison with the holographic results can be made. Also, in order to gain a predictive power the holographic calculations would have to be made for as many observables as possible and then applied to the host of experimental data on the documented NFL materials.

The present communication takes another step towards filling in this gap.

**Practical holography of condensed matter systems**

In its original formulation, the holographic principle postulates that certain $d+1$-dimensional 'boundary' field theories allow for a dual description involving, alongside other 'bulk' fields, $d+2$-dimensional gravity. Moreover, when the boundary theory is strongly coupled, the higher-dimensional gravity appears to allow for a semi-classical treatment, thus facilitating a powerful new approach to the problem of strong interactions.

So far, the holographic approach has opportunistically applied to a variety of systems which includes 'strange' Fermi and Bose metals describing quantum-critical $U(1)$ and $Z_2$ spin liquids, itinerant (anti)ferromagnets, quantum nematics, Mott transitions in lattice and cold atom systems, Hall effect, graphene, etc.

On the gravity side, the system in question would be characterized by a (weakly) fluctuating background metric $g_{\mu\nu} = g_{\mu\nu}^{(0)}(r) + \delta g_{\mu\nu}(t, \vec{x}, r)$ determining the interval

$$ds^2 = g_{tt}dt^2 + g_{rr}dr^2 + \sum_{ij} g_{ij}dx^idx^j$$

The early applications of the holographic approach revolved around a handful of the classic 'black brane' solutions, such as the Reissner-Nordstrom AdS (anti-de-Sitter) black hole with the metric

$$g_{tt} \sim -\frac{1}{r^{2\alpha}}, \quad g_{rr} \sim \frac{1}{r^{2\beta}}, \quad g_{ii} \sim \frac{1}{r^{2\gamma}}$$

where the emblackening factor $f(r)$ vanishes at the horizon of radius $r_h$ which is inversely proportional (ostensibly, similar to the case of the Schwarzschild black hole in the asymptotically Minkowski space-time, despite the variable's $r$ being the inverse of the actual radius in the $d+1$-dimensional bulk space) to the Hawking temperature $T$ shared by the bulk and boundary degrees of freedom. The explicit form of this function depends on how the gravito-electro-magnetic background is described.

In the black brane geometry of the minimal Einstein-Maxwell theory, one has $f_{EM}(r) = 1-(r/r_h)^{d+1}$, whereas in the Dirac-Born-Infeld (DBI) theory with the Lagrangian

$$L_{DBI} = \sqrt{-\det[g_{\mu\nu} + F_{\mu\nu}]}$$

geared to the strong-field limit $f_{DBI}(r) = \sqrt{1+en|2r^2|}$, $en$ being the boundary density of electric charge.

In the early works on the subject, the metric (2) was claimed to provide a potential gravity dual to the class of strongly correlated condensed matter problems - most notably, heavy fermion materials and cuprates - which are believed to manifest a certain 'semi-locally critical' behavior. However, soon thereafter it was realized that the corresponding physical scenario appears to be much too limited to encompass more general types of the real-life NFLs, so the focus shifted towards a broader class of geometries.

Further attempts of 'reverse engineering' have brought out such Lorentz non-invariant metrics as the Shroedinger, Lifshitz, helical Bianchi, etc. Amongst them, a particular attention has been paid to the static, diagonal, and isotropic metrics with algebraic radial dependence

$$g_{tt} \sim -\frac{1}{r^{2\alpha}}, \quad g_{rr} \sim \frac{1}{r^{2\beta}}, \quad g_{ii} \sim \frac{1}{r^{2\gamma}}$$

These exponents are defined modulo a change of the radial variable $r \rightarrow r^\delta$ resulting in the substitution

$$\alpha \rightarrow \alpha \delta, \quad \beta \rightarrow \beta \delta - \delta + 1, \quad \gamma \rightarrow \gamma \delta$$

Unless $\gamma = 0$, the metric (4) is conformally equivalent to the one

$$g_{tt} \sim -r^{2(\theta/d-z)}, \quad g_{rr} = g_{ii} \sim r^{2(\theta/d-1)}$$

characterized by only two parameters

$$\theta = d - \frac{1 - \beta}{1 - \beta + \gamma}, \quad z = \frac{1 + \alpha - \beta}{1 - \beta + \gamma}$$

which describe a family of 'hyperscaling-violating' (HV) backgrounds where the dynamical exponent $z$ controls the boundary excitation spectrum $\omega \propto q^z$, while $\theta$ quantifies a non-trivial scaling of the interval $ds \rightarrow \lambda^{\ell/d}ds$, the scaling-(albeit not Lorentz-) invariant ('Lifshitz') case corresponding to $\theta = 0$.

The finite-$T$ version of the HV metric can be constructed by decorating (6) with the additional factor $f_{HV}(r) = 1 - (r/r_h)^{d+z-\theta}$, akin to Eq.(2), which introduces the black brane’s horizon located at $r_h \sim T^{-1/z}$.

The physically sensible values of $z$ and $\theta$ are expected to satisfy the all-important ‘null energy conditions’

$$(d-\theta)(d(z-1) - \theta) \geq 0, \quad (z-1)(d+z-\theta) \geq 0 \quad (8)$$

signifying a thermodynamic stability of the corresponding geometry.

The HV metrics have been extensively discussed in the content of various generalized gravity theories, including those with massive vector fields as well as the Einstein-Maxwell-dilaton (EMD) theory which includes an additional scalar field, alongside the cosmological constant
In its minimal version, both, the dilaton potential \( U(\phi) \) and the effective gauge coupling \( V(\phi) \) are given by some exponential functions of \( \phi \).

At the (semi)classical level, gravitating matter added to the EMD Lagrangian (9) can be described in terms of its energy-momentum tensor and electric current

\[
T_{\mu \nu} = (E + P) u_{\mu} u_{\nu} + P g_{\mu \nu}, \quad J_{\mu} = e n u_{\mu}
\]

where \( e, n, E, P, u_{\mu} \) are the charge and energy densities, pressure, and (covariant) velocity, respectively. The 1st law of thermodynamics then relates the above quantities as follows

\[
E + P = ST + \mu n
\]

where \( S \) is entropy density and \( \mu = e A_1(r)|_{r=0} \) is the chemical potential. In the particle-hole symmetric (‘neutral’) system \( \mu = 0 \) and the equation of state reads \( E = (d - \theta)P/z \).

The HV solutions (6) have also been obtained by taking into account a back-reaction of the matter on the background geometry.\(^{2}\) Such analyses might typically use the Fermi distribution when summing over the occupied fermion states, thereby achieving a partial account of the (Hartree-type) effects of the Fermi statistics, while leaving out more subtle (exchange and correlation) ones.

However, for an already chosen gravitational background the customary way of introducing a finite charge density into the holographic scheme is by embedding a D-brane into such geometry and treating it in the probe approximation (no back-reaction). The pertinent dynamics is then described by the DBI action (3) with the background electric field

\[
F_{\tau t} = \partial_\tau A_t = \frac{|g_{tt} g_{rr} (en)^2|}{\prod_i g_{ii} + (en)^2}
\]

The DBI approach has been used to study thermodynamics of the HV theory. In that regard, in Refs.\(^{10,11}\) the specific heat of a finite density (‘charged’) system was found to scale with temperature as

\[
C_{\text{DBI, charged}} \sim T^{-(d-\theta)/z}
\]

Being primarily interested in the limit \( z \to \infty, -\theta/z \to \text{const} \), the authors did not seem to be particularly concerned with the implications of this result, including its apparent inapplicability in the potentially physically relevant case of \( \theta = d - 1 > 0 \) (see below).

Moreover, even for \( \theta = 0 \) Eq. (12) differs from the expression obtained in the earlier work of Refs.\(^{12,13}\) where the standard (‘black-body’) leading term \( \sim T^{d/z} \) was deliberately discarded in favor of the subdominant (yet, charge density dependent) one, \( \sim T^{2d/z/en} \).

Nevertheless, Eq.(12) can be rationalized by comparing it to the result of a direct calculation for the HV metric (6) and \( \mu = 0 \)

\[
C_{\text{DBI, neutral}} \sim \frac{\partial}{\partial T} \int_0^{r_h} dr \sqrt{-\det g_{\mu \nu}} \\
\sim T^{(d-\theta-2\theta/d)/z}
\]

Physically, this expression can also be recognized as the (number, rather than charge) density \( n(T) \) of thermally excited carriers (of either sign).

One subtle point is that in the charged case it is not the latter but the charge density that gets replaced with a finite value at not too high temperatures. Based on that insight, the entire temperature dependence in Eq.(12) should then be attributed to the effective (\( T \)-dependent) charge

\[
e \sim T^{2\theta/dz}
\]

while the charge density itself scales as the ratio between Eqs.(12) and (13)

\[
en = \frac{C_{\text{DBI, neutral}}}{C_{\text{DBI, charged}}} \sim T^{(d-\theta)/z}
\]

The temperature dependence (13) would also be shared by the concomitant thermal entropy

\[
S_{\text{DBI}} \sim n \sim T^{(d-\theta-2\theta/d)/z}
\]

Notably, this result of a straightforward thermodynamic calculation appears to be at odds with both, the naive estimate

\[
S_{BH} = \frac{1}{4} A \sim r_h^{-d} \sim T^{d/z}
\]

for the Bekenstein-Hawking entropy a black \( d \)-brane of radius \( r_h \) as well as the entanglement entropy

\[
S_{\text{ent}} \sim T^{(d-\theta)/z}
\]

which relation would often be quoted \textit{ad hoc} with regard to the empiric interpretation of the parameter \( \theta \) as a ‘dimensional defect’ that gives rise to the effective dimension \( d_{eff} = d - \theta \) in the above expression for \( S_{BH} \).

The proper choice of \( \theta \) has been extensively discussed in the context of fermionic entanglement entropy which points to the value \( \theta = d - 1 \), consistent with the notion of the Fermi surface as a \( d - 1 \)-dimensional manifold spanning the tangential directions in the reciprocal (momentum) space of the boundary theory.\(^{12}\)

In fact, by adhering to the above value one chooses to treat the HV system in question as fermionic and, therefore, must use the fermion quasiparticle dispersion \( \omega \sim (k - k_F)^2 \) to determine the value of the dynamical index. In what follows, the metric with the parameters \( z = z_f, \theta = d - 1 \) will be called ‘Model I’.

Alternatively, one could treat the system as bosonic and use the value \( z_b \) deduced from the dispersion of the bosonic mode, \( \omega \sim k^{2z_b} \). A straightforward choice for the
HV parameter would then be $\theta_h = 0$, thereby reducing the corresponding metric back to the Lifshitz one (hereafter, ‘Model II’).

Yet another possibility corresponds to choosing $z = z_f$ and $\theta = \theta (1 - z_f / z_b)$, which choice will be called ‘Model III’. In fact, the two latter metrics are related by a conformal transformation, so some of the results turn out to be the same in both cases. And, lastly, there also exists a choice $z = z_b$ and $\theta = d - z_b / z_f$ which will be referred to as ‘Model IV’.

**Scaling properties of hyperscaling-violating systems**

The above exposition of the physically incomprehensible holographically computed specific heat suggests that any substantive comparison with the calculations performed by alternate techniques has to involve more than one quantity.

Indeed, a power-law behavior of the specific heat is just one of the many scaling laws which describe quantum-critical systems. The list of other observables includes (tunneling) density of states, charge, current, and spin susceptibilities, electrical, thermal, and spin conductivities, shear and bulk viscosities, etc.

In the quantum-critical regime of a massless ($m = 0$) and particle-hole symmetric (or neutral, $\mu = 0$) system, the single most important scale is set by temperature $T$ or frequency $\omega$, whichever is greater.

The anticipated algebraic behavior of a physical observable $A$ is then fully characterized by its scaling dimension $[A]$, namely:

$$A(\omega, T) \sim \max |T, \omega|^{\Delta A}, \quad \Delta A = [A] / z$$

Although in the following discussion no distinction is made between the exponents controlling the frequency and temperature dependencies, this point will be addressed later.

Once a new scale enters the game, the pure algebraic dependencies would only hold at high enough frequencies and/or temperatures, while at smaller $\omega$ or $T$ any pure power-law gets complemented by a universal function of the ratios between $T$ and all the competing scales ($m, \mu$, etc.).

Also, in the case of a vanishing exponent one can encounter a logarithmic behavior $\sim \log \max |T, \omega|$ stemming from, e.g., quantum localization or the well-known classical ‘long-tail’ phenomenon.

The scaling analysis begins with a proper assignment of the scaling dimensions under transformation of the space-time coordinates in the boundary theory.

In accordance with the underlying dispersion relation ($\omega \sim k^2$), the dimensions of the space-time coordinates and their conjugate energies/momenta take the values

$$[x_i] = -[\xi] = -1, \quad [t] = -[\varepsilon] = -[\mu] = -[T] = -2$$

whereas those of the gauge potential differ from the above values by the dimension of the effective charge (14)

$$[A_i] = [k_i / \varepsilon] = 1 - 2\theta / d, \quad [A_0] = [\mu / \varepsilon] = z - 2\theta / d$$

From that one can also obtain the scaling relations

$$[\nu_i] = [x_i / t] = z - 1, \quad [\nabla_i T] = z + 1$$

$$[\mathcal{E}_i] = [A_i / t] = [A_0 / x_i] = z + 1 - 2\theta / d$$

$$[B_i] = [A_j / x_k] = 2 - 2\theta / d$$

(20)

where $\nu_i, A_i, \mathcal{E}_i, B_i$ are the velocity, vector potential, electric and magnetic fields, respectively.

The dimensions of the energy and number densities can be read off directly from Eq.(13)

$$E = [P] = [T_{tt}] = [T_{ij}] = [\mu n] = z + 1 - 2\theta / d$$

(21)

Then, in order for the boundary action to maintain scale invariance, a spatial integration must be thought of as contributing the extra dimension

$$[dx] = -[n] = -d + \theta + 2\theta / d,$$  

(22)

under which convention the total (quasi)particle number $\int ndx$ is dimensionless and, therefore, conserved. In contrast, the total charge $\int \mu ndx$ appears to scale with the non-vanishing dimension (14) imposed by the ‘running’ dilaton-dependent gauge coupling. For a more systematic derivation of such assignment the scaling properties of the entire bulk theory (9) have to be considered.

Likewise, the dimensions of the electrical current and the remaining components of the stress-energy tensor can be readily deduced from the conservation laws

$$\frac{\partial n}{\partial t} + \frac{\partial J_i}{\partial x_i} = 0, \quad \frac{\partial T_{tt}}{\partial t} + \frac{\partial T_{tj}}{\partial x_i} = 0, \quad \frac{\partial T_{ti}}{\partial t} + \frac{\partial T_{ij}}{\partial x_j} = 0$$

In this way, one obtains

$$[J_i] = d + z - 1 - \theta,$$

$$[T_{ti}] = d - \theta + 1 - 2\theta / d,$$

$$[Q_i] = [T_{it}] = d + 2z - 1 - \theta - 2\theta / d$$

(23)

where $Q_i$ is the thermal current.

It is also worth pointing out that in the absence of the Lorentzian symmetry the stress-energy tensor becomes non-symmetrical (cf. with Refs. which dealt exclusively with the case of $\theta = 0$, though). The density susceptibility $\chi$ (related to the charge one by the factor $\varepsilon^2$), electrical $\sigma$ and thermal $\kappa$ conductivities are then characterized by the following dimensions

$$[\chi] = [E/\mu^2] = d - z - \theta - 2\theta / d,$$

(24)

$$[\sigma] = [J_i / E_i] = d - 2 - \theta + 2\theta / d,$$

(25)

$$[\kappa] = [Q_i / \nabla_i T] = d + 2 - \theta - 2\theta / d$$

(26)
Curiously enough, prior to the release of the original version of this note such simple scaling relations do not seem to have appeared in the holographic analyses for general values of \(z\) and \(\theta\).

In the recent Ref.\(^{16}\), an attempt was made to further generalize these scaling relations to the situation where the spatial dimension of the gauge degrees of freedom \(d_s\) can be different from that of the gravitational ones.

In this approach, instead of attributing the 'subleading' term \(\phi = 2\theta / d\) appearing in the above expressions (cf. Eq.(3.6) in Ref.\(^{16}\)) to the dimension of the electric charge (14), thus distinguishing between the number and charge densities (or, for that matter, \(\mu\) and \(A_0 = \mu / e\) which Ref.\(^{16}\) makes no distinction between), the gauge sector was assigned its own HV parameter \(\theta_m\).

Moreover, it was argued in Ref.\(^{16}\) that introducing an extra parameter (either \(\theta_m\) or \(\phi\)) in addition to \(z\) and \(\theta\) is necessary for the proper description of a charged system with the HV geometry. However, even after such a modification the approach of Ref.\(^{16}\) still struggles to reproduce the thermodynamics of the DBI system self-consistently.

Specifically, while it asserts that the dimensions of the energy densities in the gravitational and gauge sectors can generally be different, it still seeks to make the two equal, provided that the following condition is satisfied

\[
\theta = d - d_s + \theta_m + \phi \quad (27)
\]

(cf. Eq.(3.16) in Ref.\(^{16}\) where the implicit assumption \(d_s = d\) seems to have been made).

In fact, for \(d = d_s\) and the original DBI action (3) this condition would be impossible to meet for any \(\theta \neq 0\), given that the pertinent value of the gauge HV parameter is \(\theta_m,DBI = d_s \theta / d\) (see Eq.(3.8) in Ref.\(^{16}\)).

Among other things, such inference implies that the dimension of the gauge sector’s entropy is \([S] = [E/T] = d_s - \theta_m - \phi\) (see Eq.(3.11) in Ref.\(^{16}\)) which agrees with Eq.(16) for \(d_s = d\), yet differs from the much-anticipated empirical dependence (17) (in Ref.\(^{16}\), the latter was instead postulated for the entropy of the gravitational sector). In any case, though, this extended scheme does not provide a suitable framework for assessing the status of the earlier DBI studies pertaining to the original (i.e., 2-parameter) HV systems.

In contrast, the discussion presented in this Section (specifically, for the case of \(d_s = d\)) involves just one type of entropy (16) and requires no additional parameters or some other sleight of hand.

It is, therefore, quite remarkable that despite such discrepancies both variants of the dimensional assignments result in the same Eq.(25) for the conductivity, thereby attesting to the intrinsic robustness of this and similar expressions (moreover, in the original version of this note the very same result was obtained in still another couple of different ways).

Furthermore, even robuster than the individual thermodynamic and kinetic coefficients are their universal ratios which are dimensionless and, therefore, scale invariant (hence, constant for \(\mu \ll T\)). Amongst those are the standard Wilson and Wiedemann-Franz ratios whose vanishing dimensions follow from the above Eqs.(13,14) and (24-26)

\[
\frac{e^2 \kappa}{\sigma T} = \text{const} \quad (28)
\]

Should, however, a new scale emerge, these ratios, albeit remaining dimensionless, would no longer remain constant. In fact, they may deviate strongly from their Fermi liquid values, thereby signalling, e.g., the formation of a strongly correlated (hydrodynamic) quantum-critical state.

In that regard, albeit being irrelevant for the general scaling properties, a practically important distinction has to be made between the formally defined thermal conductivity (see Eq.(30) below) and that computed under the condition of vanishing electric current (which setup more faithfully represents the actual measurement).

As a means of lending further support to the above scaling relations, one can also reproduce the dimensions (25) and (26) of the electrical and thermal conductivities from the Kubo formulæ

\[
\sigma = \frac{1}{\omega} Im \int dx \int_0^\infty dt e^{i \omega t} < [J(t,x),J(0,0)] > \quad (29)
\]

and

\[
\kappa = \frac{1}{\omega} Im \int dx \int_0^\infty dt e^{i \omega t} < [Q(t,x),Q(0,0)] > \quad (30)
\]

where the use the scaling rule (22) is instrumental.

As yet another independent check, the shear viscosity

\[
\eta = \frac{1}{\omega} Im \int dx \int_0^\infty dt e^{i \omega t} < [T_{xy}(t,x),T_{xy}(0,0)] > \quad (31)
\]

features the dimension

\[
|\eta| = d - \theta - 2 \theta / d,
\]

which, together with Eq.(16), guarantees that the celebrated viscosity-to-entropy ratio \(\eta / S\) is indeed dimensionless.

It can also be easily seen that the dimensions are consistent with the classical (Einstein’s) relations

\[
\sigma = e^2 \chi D, \quad \eta = D(E + P) / v^2 \quad (31)
\]

where the diffusion coefficient \(D = v^2 / d\) contains a scattering rate \(\Gamma\) expected to assume the universal linear form

\[
\Gamma \sim T \quad (32)
\]

in the quantum-critical regime, thereby allowing one to link the kinetic and thermodynamic coefficients together.\(^{17}\)
Also, observe that the ratio between Eqs. (22)

$$\frac{\eta e^2 v^2}{\sigma T^2} = \text{const}$$  \( (33) \)

is dimensionless and, therefore, constant in the neutral massless case. As such, it should be contrasted against the proposal

$$\frac{\eta}{\sigma T^{2/2}} = \text{const}$$

which was put forward for \( \theta = 0 \) in Ref. 18. Their obvious discrepancy (even in this limit) stems from the improper account of the velocity’s dimension in Ref. 19.

In the charged case \( (\mu \gg T) \), by using \( \chi_c = dn/d\mu \) and Eq. (10) one can cast the conductivity in the form

$$\sigma \sim \frac{e^2 \gg^2 m}{\mu \Gamma}$$

Should the rate \( \Gamma \) then happen to be linear, as in Eq. (27), the conductivity would exhibit the ubiquitous in strongly correlated systems \( \sim 1/T \) behavior20, seemingly in agreement with various scenarios of the cuprates and other ‘strange metals’ that emphasize their proximity to one or another putative quantum-critical point.

The above scaling relations can also be generalized to include anisotropic spatial geometries. In the simplifying case of a unidirectional rotationally anisotropic metric21

$$g_{tt} \sim -r^{2\theta/d-2z}, \ g_{\|\|} \sim r^{2\theta/d-2w}, \ g_{rr} = g_{\perp\perp} \sim r^{2\theta/d-2}$$

the scaling dimensions read

$$[t] = -z, \ [x_\|] = -1, \ [x_\perp] = -w \]$$

$$[A_\|] = w - 2\theta/d, \ [A_\perp] = 1 - 2\theta/d,$$

$$[E_\|] = z + w - 2\theta/d, \ [E_\perp] = 1 + z - 2\theta/d,$$

$$[J_\|] = d + z - 1 - \theta, \ [J_\perp] = d + z + w - 2 - \theta,$$

$$[n] = d + w - 1 - \theta - 2\theta/d, \ [B] = 1 + w - 2\theta/d$$  \( (35) \)

where \( B \) is a magnetic field perpendicular to both \( \vec{E} \) and \( \vec{J} \).

Choosing the axes \( x \) and \( y \) along the \( \| \) and one of the \( d - 1 \perp \) directions, respectively, one obtains

$$[\sigma_{xx}] = d - 1 - w - \theta + 2\theta/d,$$

$$[\sigma_{xy, yx}] = d - 2 - \theta + 2\theta/d,$$

$$[\sigma_{yy}] = d + w - 3 - \theta + 2\theta/d$$  \( (38) \)

In the charged case, all the components are expected to be proportional to the density, as the Hall response of a particle-hole symmetric system vanishes identically. However, the common density factors cancel out in the Hall angle

$$\cot \theta_H = \frac{\sigma_{xx}}{\sigma_{xy}} \sim \frac{en}{B\sigma_{xx}} \sim B^{-1}T^{2(2/z)(1-\theta/d)}$$  \( (39) \)

where the linear proportionality of \( \sigma_{xy} \) to a weak magnetic field has also been taken into account.

Eqs. (36)-(39) agree with the results of Ref. 20, where only the case of \( \theta = 0 \) was considered. A further generalization to the fully anisotropic case would be quite straightforward, too.

It was concluded in Refs. 20 that both goals of reproducing the linear resistivity and quadratic Hall angle characteristic of, e.g., the behavior found in the superconducting cuprates can not be achieved simultaneously, regardless of the choice of \( z \) and \( w \).

For instance, by choosing \( z = 1, w = 1/2, \theta = 0 \) one does obtain \( \sigma_{xx} \sim 1/T, \cot \theta_H \sim T^2 \) in the charged case, although it can only come at the expense of acquiring a strong spatial anisotropy (now \( \sigma_{yy} \sim 1/T^2 \), independent of \( w \) or \( d \)).

One can also check that having yet another available parameter \( \theta \) does not change the above conclusions. For instance, in the isotropic charged system \( (z = w = 1) \) one can get \( \sigma_{xx, yy} \sim 1/T \) by simply choosing \( \theta = d/2 \), but then the concomitant Hall angle is \( \cot \theta_H \sim T \).

It is also instructive to compare the above scaling dimensions to the predictions of the holographic ‘membrane paradigm’ which offers simple integral expressions for such important thermodynamic characteristics as charge susceptibility21

$$e^2 \chi = \frac{\int_{r_H}^{\infty} dr \sqrt{g_{rr}(g_{ii})^{-1}}}{\sqrt{\prod g_{i\i}}}$$  \( (40) \)

or enthalpy density

$$E + P = \left( \frac{\int_{r_H}^{\infty} dr \sqrt{g_{rr}(g_{ii})^{-1}}}{\sqrt{\prod g_{i\i}}} \right)^{-1}$$  \( (41) \)

For the HV geometry (6) Eq. (40) yields

$$e^2 \chi \sim T^{(d-\theta-z+2\theta/d)/z}$$  \( (42) \)

which fully agrees with (14) and (24).

In contrast, the result of computing Eq. (41)

$$E + P \sim T^{(d-\theta-z+2)/z}$$  \( (43) \)

is clearly at odds with Eq. (21) even for \( \theta = 0 \), as long as \( z \neq 1 \).

One can readily check that for any \( \theta \) such discrepancy can not be fixed by adding any powers of the velocity and, if taken at its face value, questions the validity of Eq. (41).
Moreover, the ‘membrane paradigm’ approach offers a closed expression for the WF ratio
\[
\frac{e^2 \kappa}{T \sigma} = \left( \frac{E + P}{Tn} \right)^2
\]  
(44)

Physically, the WF ratio provides a measure of the energy dependence of the dominant scattering rate. The classic WF law stating a constancy of this ratio would be expected to hold in any regime dominated by (quasi)elastic scattering, including, e.g., the case of electron-phonon scattering either well below or well above the Debye temperature.

In the important case of a zero-density ‘relativistic’ system with \( z = 1 \), the thermal conductivity appears to be formally proportional to the momentum and, therefore, becomes infinite in the absence of momentum relaxation, thus making the WF ratio diverge, in accordance with Eq.(44) for \( n = 0 \), and signalling an extreme form of the WF law’s violation.

However, if the density appearing in Eq.(44) were interpreted as its zero-temperature value \( n(T = 0) \), then in the neutral case the WF ratio would be divergent regardless of the value of \( z \). Conversely, if it were treated as the equilibrium \( T \)-dependent density of particles/holes with the dimension given by Eq.(13), then in the neutral system the r.h.s. of Eq.(44) would always be finite, including the case of \( z = 1 \). Clearly, a further clarification on the conditions under which Eq.(44) holds is warranted here.

The scaling analysis of the quantum-critical regime can also be extended to spin dynamics. A small field expansion of the free energy yields the dimension of the spin susceptibility
\[
\chi_s = \frac{[E/B^2]}{d - \theta + z - 4 + 2\theta/d}
\]  
(45)

However, there seems to be neither a solid holographic result to compare with, nor even a commonly accepted recipe for computing this quantity.

The few previous attempts range from using a radial equation for the variation of the vector potential \( \delta A_r \) similar to Eq.(46) in the next Section
\[
\partial_r^2 \delta A_r + \partial_r \left( \frac{\Pi_i g_{ii} g_{tt}^{1/2}}{g_{r\tau} g_{xx}} \right) \partial_r \delta A_r
\]  
(44)

\[
- \left( \omega^2 g_{rr} \frac{g_{tt}}{g_{tt}} + k^2 \frac{g_{rr}}{g_{yy}} \right) \delta A_\perp = 0
\]

or for the magnetic field itself \( \delta B = \partial_\tau \delta A_r \) to that for the spin connection \( \delta \omega^{ix}_y \sim \partial_\tau \delta g_{xy} \) from which one can evaluate \( \chi_s \), as if it was just another response function of the Kubo type.

In particular, in the 2d case and for \( \omega/\tau \gg 1 \) the thus-obtained results \( \chi_s \sim \omega^{2/3} \), was claimed to compare favorably with that experimentally observed in the conjectured spin-liquid state of the quasi-2d materials YbRh$_2$(Si$_{1-x}$Ge$_x$)$_2$ and ZnCu$_3$(OH)$_6$Cl$_2$. It can be easily checked, however, that the scaling dimension (45) does not appear to support the above estimate for any relevant values of \( z \) and \( \theta \) (see below).

### Nifty shades of holographic conductivity

Electrical conductivity has been computed for a variety of holographic models and in a number of different ways. However, establishing a consistency between the results of different calculations (or a lack thereof) does not seem to have always been particularly high of the agenda.

The most frequently employed calculation of the electrical conductivity and other kinetic coefficients is based on the holographic adaptation of the Kubo formula. It proceeds by solving linearized equations for small variations of the electromagnetic potential \( \delta A_\mu \) and (possibly) such coupled component(s) of the metric as \( \delta g_{\mu
u} \), and/or other degrees of freedom, depending on the field content of the bulk theory in question.

In the case of a generic electro-magneto-gravitational background treated in the customary probe limit by virtue of the DBI action (3), the relevant quasi-normal mode obeys the equation
\[
\partial_r^2 \delta A_i + \partial_r \log \left( \frac{\Pi_i g_{ii} g_{tt}^{1/2}}{g_{r\tau} g_{xx}} \right) \partial_r \delta A_i - \frac{g_{rr} g_{tt}}{g_{tt}} \left( \omega^2 + g_{rr} k^2 h^2 - \omega^2 \right) \delta A_i = 0
\]  
(46)

where
\[
h = \sqrt{\frac{|g_{tt}| g_{rr} - F_{rr}^2}{g_{rr} g_{xx}}} = \sqrt{\frac{|g_{tt}|}{g_{xx}(1 + (en)^2/\Pi_i g_{ii})}}
\]

As per the standard holographic prescription, the optical conductivity is then defined as the reflection coefficient of a radial in-falling wave
\[
\sigma = Im \frac{\partial_r \delta A_i}{\delta A_i} \big|_{r \to 0}
\]  
(47)

where the Fourier-transformed function \( \delta A_i(r, \tau, k) \) and its derivative are evaluated at the boundary (a.k.a., the UV limit).

As one technicality, in order to compute Eq.(47) one first solves Eq.(46) in the opposite, IR or \( r \to \infty \) limit (which is also formally attainable by putting \( n = 0 \) where it reads
\[
\partial_r^2 \delta A_i + (2\gamma - \alpha + \beta) \frac{\partial_r \delta A_i}{r} + \omega^2 r^{2(\alpha - \beta)} \delta A_i = 0
\]  
(48)

the coefficients being those of the metric (4).

Next, imposing the in-falling boundary condition at \( r \to \infty \) one obtains the solution
\[
\delta A_i \sim u^\nu H^{(1)}_\nu(u)
\]

where \( \nu = 1/2 - \gamma/(1 + \alpha - \beta) \) and \( u = \omega r^2/\tau \), which then has to be matched with that of the equation obtained from (46) in the \( r \to 0 \) limit by comparing the
two in the region \( u \sim 1 \) where they overlap (which, in turn, requires one to take the small \( \omega \) limit).

Skipping the algebra (unlike much-needed physical discussions of the results of this calculation, such formal manipulations, complete with all the auxiliary technical details, can be readily found in many of the pertinent papers), one obtains

\[
\sigma_{\text{Kubo}} \sim \omega^{-2/(1+\alpha-\beta)} \tag{49}
\]

Somewhat surprisingly, instead of deriving this general result once and for all, in much of the holographic literature this calculation would be performed anew for every equation of the type (48).

Also, observe that Eq.(49) is invariant under the transformation (5), in agreement with the aforementioned conformal equivalence of the corresponding metrics.

Barring the fact that under the aforementioned matching condition the power-law dependence (49) is derived for low frequencies, this asymptotic behavior and its analogues (see below) have been contrasted against the experimental data taken at energies up to \( eV \) (e.g., in the case of the cuprate superconductors)\(^{25}\).

Applying Eq.(49) to the HV metric (6) paired with the radial electric field (11), one obtains the optical conductivity of a charged \((n \neq 0)\) holographic system

\[
\sigma_{\text{Kubo,charged}} \sim \omega^{-(2/z)(1-\theta/d)} \tag{50}
\]

for small \( \omega \) and \( z > 2(1-\theta/d) \), while in the opposite case one gets \( \sigma \sim 1/\omega \). This result was reported in Refs.\(^{19,20}\) (the first two of these references addressed only the limit \( z \to \infty, -\theta/z = \text{const.} \), though).

In the neutral case, the conductivity can be obtained by expanding and solving Eq.(46) directly at the boundary \((r \to 0)\) where the electric field is negligible, thereby yielding

\[
\sigma_{\text{Kubo,neutral}} \sim \omega^{(d-2)(1-\theta/d)/z} \tag{51}
\]

which estimate is in a perfect agreement with the scaling dimension (25).

In turn, Eq.(50) can then be readily rationalized by observing that the ratio \( \sigma_{\text{Kubo,neutral}}/\sigma_{\text{Kubo,charged}} \) scales as the charge density (15), which in the neutral system is played by the density of thermally activated quasiparticles (of either charge sign).

Notably, in the \( 2d \) case Eq.(51) allows for no faster than logarithmic dependence. Besides, there seems to be little difference between the general HV and the Lifshitz \((\theta = 0)\) geometries. In that regard, it is worth mentioning that the experimentally observed optical conductivity of a neutral (undoped) \( 2d \) Dirac metal, such as graphene, indeed appears to be nearly constant \((\sim e^2/h)\).

Specifically, in Ref.\(^{22}\) the latter was found to behave as

\[
\sigma_{\text{graphene}} \sim \text{Re} \frac{T}{i\omega + g^2(T)T} \tag{52}
\]

where the logarithmically running effective charge \( g(T) \sim 1/\log T \) represents the effect of the Coulomb interactions.

In the higher dimensions \( d > 2 \), according to Eq.(51) the conductivity of a neutral system generally vanishes for \( \omega \to 0 \), regardless of the value of \( z \), which behavior is consistent with the intrinsically semi-metallic nature of such systems. In the pertinent example of the \( 3d \) ‘Weyl metal’ where \( z = 1 \) it was recently found that \( \Delta_0 = 3 - 4M \) with \(|M| < 1/\sqrt{32} \).

It is again instructive to compare Eqs.(50,51) to the predictions of the ‘membrane paradigm’ which also provides a simple algebraic expression for the low-\( \omega \) value of the conductivity. The latter is cast solely in terms of the geometry at the (necessarily, non-degenerate) horizon (thus, such results would not be applicable to the extremal black branes) without the need of solving any differential equations.

Furthermore, this approach can also be extended to include a magnetic field. To first order in the weak field \( B \), both the diagonal and off-diagonal components of the DC conductivity tensor take the following closed form\(^{20}\)

\[
\sigma_{xx} \sim e^{-2\phi_0} \sqrt{(en)^2 + 4\sqrt{en} |g_{ii}|_{r \to r_h}} \tag{53}
\]

where \( \phi_0 \) is the fixed point value of the dilaton.

Quite remarkably, for \( \phi_0 = 0 \) and \( n = 0 \), \( \phi_0 = 0 \) the first of Eqs.(53) exactly reproduces Eq.(50) and (51) for \( n \neq 0 \) and \( n = 0 \), respectively, whereas the second one is fully consistent with the scaling dimension (37).

The problem, however, is that, as opposed to Eqs.(50) and (51), Eqs.(53) are supposed to be evaluated at the horizon, rather than the boundary.

In general, the local conductivity defined according to Eq.(49) at an arbitrary \( r \), is expected to be independent of \( r \), unless there is a ‘running’ field, such as an electrical scalar potential or dilaton which can bring about a non-trivial \( r \)-dependence.

In the neutral case and in the absence of a dilaton, no radial evolution should indeed occur, and so the conductivity could be equally well evaluated either at the boundary or the horizon \((\sigma_{\text{Kubo,neutral}}^{(B)} = \sigma_{\text{Kubo,neutral}}^{(H)})\).

However, in the charged case one would expect the local conductivity to vary with the radial variable, whether or not a non-trivial dilaton field is present.

To that end, in Ref.\(^{22}\) a general relation was proposed

\[
\frac{\sigma^{(B)}}{\sigma^{(H)}} = \left( \frac{ST}{E + P} \right)^2 \tag{54}
\]

which ratio becomes unity at zero density, as per the equation of state (10).

In contrast, at finite density Eq.(54) implies a non-trivial radial (hence, temperature) dependence of the local conductivity, thereby predicting the low-\( T \) behavior

\[
\sigma^{(H)}_{\text{Kubo,charged}} \approx \sigma^{(B)}_{\text{Kubo,charged}} \left( \frac{M_{\text{H}}}{ST} \right)^2 \sim T^{(2/z)(-d+\theta-z-1+3\theta/d)}
\]
which clearly contradicts Eq.(50).

Conversely, if one chooses to treat Eq.(50) as the horizon value $\sigma_{Kubo,charged}^{(H)}$ then Eq.(56) implies

$$\sigma_{Kubo,charged}^{(B)} \sim T^{2/3}(d-\theta + z - 1 - \theta/d)$$

which is again different from the predictions of the scaling analysis.

This adds to the argument that a better understanding of the applicability of such formulae as Eqs.(44) and (54) for $n \neq 0$ is definitely called for.

However, in spite of some confusion with their terms of use, Eqs.(53) can still capture such intrinsic properties of the conductivity tensor as, e.g., the relative scaling of its components with temperature.

Namely, the scaling dimensions (36-38) would seemingly imply that the following relation

$$\sigma_{xy} \sim B\sigma_{xx}^2/(en)$$

sets in, as the system approaches the neutral regime at high temperatures. In fact, such a relation does hold - but only for the partial (particle and hole) contributions towards the total Hall conductivity. Should both components happen to have equal mobilities (as, e.g., in the case of a particle-hole symmetric spectrum), the overall $\sigma_{xy}$ would only be proportional to the charge imbalance given by $en(T = 0)$, thereby resulting in the different relative scaling rule

$$\sigma_{xy} \sim B\sigma_{xx}^2/(en)^2$$

The naive relation (55) could still hold, though, if the spectrum were lacking particle-hole symmetry (as, e.g., in the case of topological insulators where the Dirac spectrum emerges as a result of the bulk gap inversion).

In that regard, the recent Ref.18 claimed that in the extended class of the 3-parameter HV systems and at sufficiently high temperatures the cuprate-like behaviors of, both, $\sigma_{xx}$ and $\sigma_{xy}$ could be recovered even in the spatially isotropic case.

Indeed, by using Eq.(56), one can see that in the original, 2-parameter HV system with $\theta \neq 0$ and $d \neq 0$, this does not happen, as the desired dependencies $\sigma_{xx} \sim T^{-1}$ and $\sigma_{xy} \sim T^{-3}$ would only occur in the unphysical dimension $d_s = 2/3$.

However, in the 3-parameter family of the generalized HV systems, such dependencies could emerge under the choice of parameters19

$$\phi = 2 - 3z/2, \quad \theta_m = d_s - z/2$$

which conditions would only be consistent with Eq.(27), provided that

$$d + 2 - 2z - \theta = 0$$

(cf. Eq.(5.5) in Ref.20 where $d$ is replaced with $d_s$ - obviously, in error, as its derivation utilizes Eq.(3.16) where the identification $d_s = d$ has already been made).

The physical tangibility of such a scenario remains to be discerned, as does the whole notion of different dimensions for the gravitational and gauge (matter) degrees of freedom. In the known examples of layered strongly correlated systems, the emergent (both, gauge and matter) fields always tend to be confined to the supporting lower-dimensional subspace. On the other hand, in, e.g., graphene the Coulomb interactions do permeate the surrounding 3d space, but then their dynamics becomes affected by the charges outside the graphene plane, thus hindering the possibility of finding a closed holographic description.

One more remark is in order here. According to Eqs.(53), a finite longitudinal (Ohm’s) conductivity can arise due to, both, current relaxation as well as (Schwinger’s) pair production. While the latter mechanism operates even at zero charge density (unlike the current, the system’s momentum can then remain conserved), the former one requires a finite density of carriers immersed in a dissipative medium composed of some neutral modes.

Moreover, in the framework of the ‘membrane paradigm’ the two sources of finite conductivity combine together in a rather peculiar manner

$$\sigma = \sqrt{\sigma_{mr}^2 + \sigma_{pc}^2}$$

where $\sigma_{mr}$ and $\sigma_{pc}$ stand for the contributions due to momentum relaxation and pair creation, respectively.

For comparison, yet another recent application of the Kubo approach yields25

$$\sigma = \sigma_{mr} + \sigma_{pc}$$

Notably, both Eqs.(57) and (58) violate the standard Matthiessen’s rule, according to which it is the inverse of the partial conductivities, rather than the conductivities themselves, that tend to add up. Instead, Eq.(58) is reminiscent of the combination rule for those scenarios where more than one type of current carriers (as opposed to more than one mechanism of scattering for the same type of carriers) is present, and the best conducting one short-circuits the rest of the system.

Moreover, the two contributions entering Eqs.(57),(58) were found to behave as

$$\sigma_{mr,1}(\omega) \sim \omega^{1+(d-2-\theta)/z-1}$$

or

$$\sigma_{mr,2}(\omega) \sim \omega^{1+(\theta-2-d)/z-1}$$

whereas

$$\sigma_{pc}(\omega) \sim \omega^{(1-\theta)/z}$$
28,29 where a new (‘conduction’) exponent $\zeta$ was introduced to describe the scalar potential $A_0 \sim \varphi^{-\zeta}$. Conceptually, it can be related to the aforementioned $\phi$-factor.

Given that the exponents appearing in Eqs.(59)-(61) differ from those discussed earlier in this Section, a better understanding of their physical nature as well as the origin of the combination rules (57) and (58), would once again be warranted.

Interestingly, though, the momentum relaxation exponent (60) found in Ref.28 agrees with the results of still another recent work of Refs.28 where an important effort was made to include elastic scattering, alongside the inelastic one.

In fact, the analysis of Refs.28 represents a 'holography-augmented' transport theory, rather than a systematic all-holographic calculation. Conceivably, though, such a hybrid approach might be better equipped for capturing the underlying physics of the relevant transport phenomena.

Specifically, these works employed the so-called memory function formalism28 which does not explicitly rely on the existence of well-defined quasiparticles and presents, e.g., the electrical conductivity in the form

$$\sigma_{\text{memory}} = \chi_{JP,PP}(T) = \frac{\int dk d^2 \mathbf{k} D(\omega, k) - i\omega \chi_{PP}}{\omega}^{-1} \tag{62}$$

where $\chi_{JP,PP}(T)$ are the current-momentum and momentum-momentum susceptibilities.

The formula (62) assumes that momentum is the only (nearly) conserved physical quantity and relates the conductivity to the spectral density of the operator that breaks momentum conservation. It is expected to work best in the hydrodynamic regime where the rate of momentum relaxation due to a breaking of translational invariance by elastic impurity or lattice-assisted inelastic Umklapp scattering is smaller than the inelastic rate which controls a formation of the hydrodynamic state itself. For instance, in the case of $\mu \gg T$ the rate of the Umklapp scattering is of order $\sim T^2/\mu$, whereas the latter one is given by the universal quantum-critical rate (32).

In general, the onset of hydrodynamics is a distinct property of strong correlations which would be routinely absent in the Fermi liquid regime. Such a regime would also be absent in 1d, thanks to the peculiar 1d kinematics facilitating the emergence of infinitely many (almost) conserved currents.

In the absence of any (nearly) conserved quantities Eq.(62) ceases to be applicable. Although the corresponding 'incoherent' metals do not allow for any simple description, they have been eloquently argued28 to conform to the ubiquitous $\sigma \sim 1/T$ dependence stemming from the universal scattering rate (32).

In many cases, though, strong interactions often go hand-in-glove with (and enhance the effects of) strong disorder. The combined effects of the two can hardly be accounted for by means of the perturbative Altshuler-Aronov theory and are likely to require some intrinsically non-perturbative approaches, such as the Efros-Shklovskii one, thus allowing for other, essentially non-linear, $T$-dependencies, $\sigma \sim \exp(-\#/T^n)$.

It is also worth noting that, unlike Eqs.(57) and (58), the applications of Eq.(62) would have a good chance to be in compliance with the Matthiessen’s rule, as the different scattering mechanisms tend to correspond to separate contributions to the integral kernel $D(\omega, k)$, thereby producing additive terms in the expression for the inverse conductivity.

The main inference from Eq.(62) is a transfer of the spectral weight from the coherent Drude peak to the incoherent high-frequency tail. It is worth noting, though, that in the previous applications of Eq.(62) a possible quasiparticle renormalization was not, de facto, considered, as the behavior of $\chi_{JP,PP}$ was believed to be non-singular and, at most, only weakly $T$-dependent (such an assumption notwithstanding, e.g., at the onset of the Mott transition one expects $\chi_{JP} = 0$).

As mentioned above, the behavior found in Ref.30

$$\sigma_{\text{memory}} \sim T^{(\theta - 2 - d)/z} \tag{63}$$

coincides with that reported in Ref.28. However, in Ref.28 it was shown to emerge only in the strong coupling regime, whereas the lowest (second) order perturbative result was found to be non-universal $\sigma_{\text{memory}} \sim T^{(z - d - 3)/z}$, $\delta$ being the anomalous dimension of the operator that breaks momentum conservation.

Taking into account the HV scaling relations (16) one observes that in the neutral case Eq.(63) appears to be inversely proportional to entropy (equivalently, specific heat or viscosity), as conjectured earlier in Ref.29. However, the dependence $\sigma \sim 1/S_{\text{ent}} \sim T^{(\theta - d)/z}$ advocated in29 can only occur in the limit $z \to \infty$. Otherwise, Eq.(63) features an additional factor that, incidentally, behaves as the inverse square of a $T$-dependent 'graviton mass', $m \sim T^{1/z}$.

In a series of works28 it was indeed proposed to incorporate the effects of static disorder by introducing a graviton mass $m$ which is weakly (if at all) $T$-dependent. Although under such an assumption the desired dependence $\sigma \sim 1/S_{\text{ent}}$ does indeed set in, it remains to be seen whether such a scenario can be justified beyond the ad hoc level.

As yet another effort towards marrying the formal holographic manipulations with the more traditional transport theory, it was also proposed to mimic the momentum-relaxing Umklapp processes brought about by the presence of a regular crystal lattice with expressly anisotropic geometries and periodic scalar and/or dilaton potentials.

To that end, in the previously quoted Ref.28 the crystal lattice was modelled by a periodic electric potential, resulting in $\Delta \varphi = -2/3, -\sqrt{3}/2$ in 2d and 3d, respectively.

Also, noteworthy is the proposal34 to use the helical
Bianchi-$VII_0$ metric with a pitch in the $x$-direction

\[ g_{tt} = -g_{rr} \sim -1/r^2, \quad g_{xx} \sim r^{2/3}, \]
\[ g_{yy} \sim 1/r^{4/3}, g_{zz} \sim 1/r^{2/3} \quad (64) \]
as a holographic description of the anisotropic 3d periodic structure which gives rise to the interaction-induced Mott-type state with the 'bad-metallic' conductivity $\sigma_{xx} \sim T^{4/3}$ in the direction of the pitch, alongside a gapless behavior of the entropy, $S_{\text{ent}} \sim T^{2/3}$.

Although the Bianchi geometry (64) does have its intellectual appeal, it should be noted that the above choice is not unique. As follows from Eq.(51), the same behavior of, both, $\sigma_{xx}$ and entanglement entropy can be found for an entire family of the uniaxially anisotropic 3d metrics

\[ g_{tt} \sim -1/r^{2\alpha}, \quad g_{rr} \sim 1/r^{2\beta}, \quad g_{zz} \sim 1/r^{2\gamma} \quad (65) \]

which satisfy the conditions $2\gamma/(1 + \alpha - \beta) = -2/3$, $\sum\gamma_i = 2/3$. For instance, by choosing $\alpha = \beta = 1$, $\gamma_x = -\gamma_y = -\gamma_z = -2/3$ one finds that fully spatially anisotropic geometries, as in (64), may not be necessary for constructing a holographic dual of the 'bad metal', after all.

Continuing with the list of the previously obtained holographic results it might be worth mentioning a few more examples whose physical interpretation (as well as mutual consistency) is yet to be ascertained.

For one, there has been a variety of predictions for the dimension of electrical conductivity. In the neutral Lifshitz case ($\theta = 0$), Ref.\cite{34} found $\Delta_\sigma = (d + 2 - 2z)/z$, while Ref.\cite{35} reported $\Delta_\sigma = (3 - d - z)/z$, and Ref.\cite{36} arrived at the exponent $\Delta_\sigma = (d + 2z - 4)/z$.

None of these values appears to be consistent with the above scaling predictions and the universal quantum-critical scattering rate (32).

Going beyond the Lifshitz case, Ref.\cite{34} found $\Delta_\sigma = 3$ for $d = 3$ and $\Delta_\sigma = (2z - 3)/z$ for $\theta = d - 1$, whereas Ref.\cite{35} reported $\Delta_\sigma = (d - 2)/z$, but only for $z = d - 2$, $\Delta_\sigma = 1/z$ for $z = (d-4)/3$, and $\Delta_\sigma = (d - \theta)/z$, also in conflict with the above scaling results.

Also, for $d = 2$ Ref.\cite{36} delivered $\Delta_\sigma = 2$, while Ref.\cite{34} obtained $\Delta_\sigma = 7/2$. Moreover, Ref.\cite{37} presented an even greater variety of values, $\Delta_\sigma = 1, 2, 3, d, d - 2, d - 4$ for $z = 1$ and $(2z + d - 2)/z$ for $z \neq 1$, as well as a whole discrete series $(1 + 3p)/(3 + p)$, whereas other works featured the entire plethora of non-universal exponents as functions of one or even two continuous parameters appearing in the holographic Lagrangian\cite{38}.

On the other hand, Ref.\cite{34} utilized the metrics (4) with $\beta = 2 - \alpha$ obtaining the results $\Delta_\sigma = -2\gamma/(2\alpha - 1)$ and $\cot \theta_H \sim T^{2\gamma/(2\alpha - 1)}$, in agreement with (49).

Still other available methods of computing conductivity include extracting it from the hydrodynamic expansion or computing a drag force for massive charge carriers. Although some of those results may seem more plausible than others, they are still awaiting for their physical interpretation and a systematic comparison with the predictions made by the alternative techniques.

In that regard, the general universal relations, such as Eq.(28) or (33), provide an important consistency test, while reinforcing the notion that the dynamic properties of quantum-critical systems are closely related to their thermodynamics. Technically, such a relationship implies that, apart from the relaxation rate (32), the kinetic coefficients can be found in terms of the thermodynamic ones.

Yet another important test would be provided by the sum rules for the optical conductivity and other kinetic coefficients, akin those extensively employed in Ref.\cite{39}. Obviously, no monotonic low-frequency asymptotic obtained by solving the differential equation (46) in the Kubo formula approach can be up for this test. However, the frequency-dependent counterparts of the purely algebraic Eqs.(53) could indeed be used to that effect, once their closed expressions are obtained in a wide range of frequencies.

**Mother of all non-Fermi liquids**

One concrete context for a comparative discussion of the different holographic models is provided the theories of fermions coupled to gapless over-damped bosonic modes. This 'mother of all NFLs' has long been at the forefront of theoretical research, since the singular interactions mediated by soft gauge field-like bosons are often associated with incipient ground state instabilities and concomitant NFL types of behavior.

Such effective long-range and strongly retarded interactions may occur even in microscopic systems with purely short-ranged couplings. In the close proximity to a quantum-critical point, the role of the corresponding modes is then played by (nearly) gapless excitations of an emergent order parameter.

Important examples include such problems as ordinary electromagnetic fluctuations in metals and plasmas, spin and charge ordering transitions in itinerant (anti)ferromagnets, compressible Quantum Hall Effect, Pomeranchuk instabilities resulting in rotationally anisotropic 'quantum nematic' states, etc.

Despite the differences in their physical nature, all these systems conform to the general problem of a finite density fermion gas coupled to an overdamped bosonic mode whose own dynamics is governed by the (transverse) gauge field-like propagator

\[ D(\omega, q) = \frac{1}{|\omega|^2 + q^2} \quad (66) \]

In the context of electrodynamics of conducting media, the first and second terms account for the phenomena of Landau damping and diamagnetism, respectively.

Over the past two decades this problem has been repeatedly attacked with a variety of techniques.

At the early stage, it was believed that the functional form of the one-loop fermion self-energy

\[ \Sigma(\omega) = \int d\epsilon d^d q \frac{D(\omega, q)}{(\epsilon + i\gamma - q\mathbf{v})} \sim \omega^{(d-1)/d} \quad (67) \]
survives in the higher orders of perturbation theory, akin to the situation in the Eliashberg theory of electron-phonon interactions. However, the more recent analyses demonstrated an inapplicability of the naive weak coupling and $1/N$-expansions, thus calling the earlier results into question.

There have been also attempts to study this theory without introducing the Landau damping from the outset, which analysis yields a self-energy $\Sigma \sim \omega^{1-\epsilon/4}$ in $d = 3 - \epsilon$ dimensions and for $\rho = 2$, $\xi = 1$ that is markedly different from the counterpart of (67), $\Sigma \sim \omega^{1-\epsilon/3}$. In still other approaches, the problem was attacked by expanding in $z_b - 2\frac{d}{\xi}$ or $d = 5/2 - \frac{3}{4}\frac{d}{\xi}$.

Despite somewhat conflicting results, Eqs.(66) and (67) would often be used for evaluating the boson and fermion dynamical exponents

$$z_b = \rho + \xi, \quad z_f = \frac{\xi + \rho}{d - 1 + \xi} \quad (68)$$

Conceivably, a hypothetical holographic dual (if any) of the boundary theory with the interaction (66) might involve such bulk degrees of freedom as gauge potential, metric and scalar fields and, therefore, it could be envisioned amongst the solutions of the EMD Lagrangian (9).

Along these lines, in Ref.30 a comparison was made between the two-point correlation function computed holographically in a yet-to-be-specified HV geometry and those obtained directly in the boundary theory with the use of the eikonal technique.

The agreement was found, provided that the $\theta$-parameter of the HV metric (6) was chosen as

$$\theta = d\frac{\rho + 1 - d}{\xi + d - 1} \quad (69)$$

thereby satisfying the relation $z_f = 1 + \theta/d$ and, incidentally, turning the first of the conditions (8) into an exact equality. In particular, for $d = 2$, $\rho = 2$, $\xi = 1$ one obtains $z_f = 3/2$ and $\theta = 1$ which values have also been independently singled out on the basis of analysing the entanglement entropy.

In Table I, we compare the exponents governing a power-law decay of the conductivity computed holographically with the use of Eqs.(50), (51), and (62). These values pertain to the aforementioned models I-IV and are complemented by those for the new model V which is characterized by the exponents $z_f$ and $\theta$ given by Eqs.(68) and (69), respectively.

The first two columns contain the exponents $\Delta_{\omega}^{(\infty)}$ governing the $\omega$-dependence for $\omega \gg T$ and given by Eqs.(50) and (51), whereas the third column contains the values of $\Delta_{\omega}^{(0)}$ pertinent to the $T$-dependence for $\omega \ll T$ and given by Eq.(62) (a potentially strong $\omega$-dependence of the functions $\chi_{JF,PP}$ complicates the analysis of $\sigma(\omega)$ in the framework of the memory function method).

| $\Delta_\omega$ | $Kubo_{charged}$ | $Kubo_{neutral}$ | Memory function |
|-----------------|------------------|-----------------|-----------------|
| Model I        | $\Delta$         | $\Delta$        | $\Delta$        |
| Model II       | $\Delta$         | $\Delta$        | $\Delta$        |
| Model III      | $\Delta$         | $\Delta$        | $\Delta$        |
| Model IV       | $\Delta$         | $\Delta$        | $\Delta$        |
| Model V        | $\Delta$         | $\Delta$        | $\Delta$        |

A few comments are in order:

Firstly, despite being spurious as far as its physical implications are concerned (see below), the much-desired exponent $-2/3$ is quite robust and can be obtained in any of the models I-IV for both $d = 2$ and $d = 3$, as long as $\xi = 1$. Moreover, for $\xi = 1$ all the results for the models I and III as well as those for the models II and IV are identical.

Secondly, the model V with its conjectured boundary dual represented by the gauge-fermion model can also be amenable to the application of the standard Drude theory. The latter (nominal) assumes the existence of a quasiparticle description and yields the conductivity

$$\sigma_{Drude} \sim \Re \frac{1}{\omega} \frac{1}{Z + 1/T} \quad (70)$$

where the (possibly strong) quasiparticle renormalization is accounted for via the Green function's residue, $Z = 1/(1 - \partial\Sigma/\partial\omega)$.

Potentially, Eq.(70) can take rather different forms at small and large $\omega$, as compared to $T$, depending on whether or not the transport scattering rate $\Gamma_T$ behaves differently from that of quasiparticle decay.

Estimating the latter as $\Sigma(\omega) \sim \omega^{3/2}$ one then finds the former (as well as the entire $T$-dependent DC conductivity) to be governed by the modified exponent

$$\sigma_{Drude}^{(0)} \sim 1/\Gamma_T \sim T^{-1/\xi_f + 2/z_b} \quad (71)$$

In contrast, at high $\omega$ one obtains

$$\sigma_{Drude}^{(\infty)} \sim \frac{Z^2\Gamma_T}{\omega^2} \sim \omega^{1/z_f + 2/z_b} \quad (72)$$

Thus, the DC and AC Drude conductivities exhibit the same exponent only in the limit $z_b \to \infty$ which corresponds to a particular case of the generic short-ranged (and, therefore, only weakly momentum-dependent) scattering mechanism.

By contrast, for any finite $z_b$ there will be a disparity between the transport and quasiparticle decay rates and, as a result, different values of the exponents controlling the $\omega$ and $T$ dependencies. This fairly mundane observation should be contrasted with such exotic proposals as a 'wrong' sign of the expression under the absolute value in Eqs.(59)-(61) or a parameter-dependent dominance of one term in Eq.(58) over the other which were put forward in Ref.28.

By applying Eq.(70) to the Model V one observes that the low-$\omega$ Drude conductivity (specifically, its exponent $\Delta_{\omega}^{(0)}$) agrees with that of the memory function approach.
for, at least, one of the models I-IV, while in the high-$\omega$ regime (i.e., for $\Delta_{\sigma}^{(\infty)}$) this is generally not the case.

Specifically, for $d = 2$, $\rho = 2$, $\xi = 1$ the high-$\omega$ Drude formula yields $\Delta_{\sigma}^{(\infty)} = 0$, while in $3d$ one gets $\Delta_{\sigma}^{(\infty)} = -1/3$. By contrast, in the DC limit one gets $\Delta_{\sigma}^{(0)} = -4/3$ and $-5/3$ (up to a power of logarithm) in $2d$ and $3d$, respectively.

In the case of $\rho = 2, \xi = 0$, the counterparts of the above values read $\Delta_{\sigma}^{(\infty)} = 1/2$ ($2d$) and 0 in ($3d$), while in the DC limit one gets $\Delta_{\sigma}^{(0)} = -3/2$ and $-2$ in $2d$ and $3d$, correspondingly.

The latter estimate should not be compared directly with the prediction $\Delta_{\sigma}^{(\infty)} = -1/3$ made in Ref.29 for the scenario of an incipient $2d$ spin density wave instability with the large momentum. The cuprate-like shape of the Fermi surface and the dominant scattering involving its opposite regions modify the above results obtained under the assumption of a spherical Fermi surface by a missing factor of $\omega^{2/z_b}$ due to the scattering between the conjugate pairs of hot spots and an additional factor of $\omega^{1/2}$ due to a finite span of the region around each hot spot. Together, the two effect conspire to result in a somewhat accidental cancellation, thereby producing $\Delta_{\sigma}^{(\infty)} = 0$.

As to the quoted exponent $-1/3$, it was obtained in Ref.29 by going well beyond the Drude approximation and focusing on certain 'energy transfer' processes which involve pairs of soft bosons with small total momenta.

The story does not seem to end there, though. The recent calculation for the 'Ising nematic' model produced yet another term that dominates over all the other ones at low $T$,

$$\sigma_{IN} \sim (T \log T)^{1/2} \quad (73)$$

which was associated with the dominant process of scattering off of a (quasi)static 'random Ising magnetic field'.

Under a closer inspection, the behavior (73) turns out to be indicative of the IR divergence of the momentum integral in Eq.62 from which it is rescued by introducing a cut-off at energies of order the mass of the bosonic mode $m \sim (T \log T)^{1/2}$.

However, should such a mass happen to be prohibited on the grounds of, e.g., unbroken gauge invariance, the problem in question would turn out to be intrinsically strongly-coupled and possibly resulting in quite different, yet to be determined, conductivity behavior (in the case of the Ising nematic, this possibility was claimed to be conveniently pre-empted by the onset of a superconducting instability22).

Besides, albeit being seemingly innocuous to first order29, the T-dependent corrections to the susceptibilities $\chi_{PP}^{(\infty)}$ may get promoted to the exponent in the higher orders, thus altering the overall power counting.

In that regard, a particularly interesting would be the actual gauge field problem where, unlike the longitudinal, the transverse gauge boson does not develop any (thermal) mass, except in the case of a symmetry-breaking phase transition.

Real-life non-Fermi liquids

The list of documented NFLs is extensive and includes ferromagnetic metals (e.g., MnSi, ZrZn) and superconductors (UCeGe, URhGe, UCoGe), heavy fermions (e.g., YbRh$_2$Si$_2$, CeCoIn$_5$ or URu$_2$Si$_2$), unconventional superconductors such as cuprates and iron pnictides, electronic nematics (e.g., $S_{\gamma}R_{\alpha}Ru_{\gamma}O_{\alpha}$), insulating magnets (e.g., $CoNb_{2}O_{6}$ and TlCuCl$_3$), quasi-one dimensional Mott insulators (e.g., (TMTSF)$_2$PF$_6$ or (TMTSF)$_2$ClO$_4$, etc.

Given that in most cases the dynamical exponent $z > 1$, one might naively expect all the 3d systems to show the classical mean-field scaling behavior, since the effective dimension of spatial fluctuations, which equals $d + z$, appears to exceed the upper critical dimension $d_{uc} = 4$.

Moreover, the FM systems with a conserved order parameter and $z_b = \rho + \xi = 3$ would be anticipated to follow the classical scenario for any $d > 1$, whereas for the AFM ones (where the order parameter is not conserved and $z_b = \rho + \xi = 2$) it would then happen for all $d > 2$.

However, this argument can be invalidated by dangerously irrelevant variables presenting a potential source of hyperscaling violation and resulting in the breakdown of the corresponding relation between the specific heat ($\alpha$) and correlation length ($\nu$) exponents, $d\nu = 2 - \alpha$.

For one, the 3d helical ferromagnet MnSi demonstrates a NFL behavior for $\sigma \sim T^{-3/2}$ (which reverts to $\sim T^2$ in a field of $3T$). The itinerant ferromagnet ZrZn$_2$ shows somewhat similar properties. Such prototypical NFL materials have long been viewed as potential candidates to the application of the 3d gauge-fermion theory discussed in the previous Section, although the thus-obtained conductivity would behave as $\sim T^{-5/3}$, in disagreement with the above dependencies.

In turn, the 3d AFM heavy-fermion compound YbRh$_2$Si$_2$ exhibits a quantum-critical point at a finite field $H_c$, featuring $\chi_s \sim T^{1/4}, C \sim T^{3/4}, \sigma \sim T^{-3/4}$, which behavior is suggestive of the critical exponents $z = 4, \alpha = 1/4, \nu = 1/3$.

Its doped cousin YbRh$_2$Si$_2(\text{Si}_{1-x}\text{Ge}_x)_2$ shows a power-law behavior of the low-$\omega$ spin susceptibility $Re\chi_s \sim T^{-0.6}, \text{Im}\chi_s \sim \omega/T^{1.3}$ for $x \approx 0.05$.

Another example of the 3d AFM materials, UC$_{\alpha}u_{\alpha}P_{\alpha}d_{\alpha}$, manifests $C \sim T^2, \sigma \sim T^{-1/3}$ and $\chi_s \sim T^{\gamma}$ where $\gamma$ ranges between 0 for $x = 1$ (i.e., $\chi_s \sim \ln T$ and $\chi_s \sim T^{-1/3}$ for $x = 1.5$).

The list of the 3d AFM also includes CeIn$_3$ with $\sigma \sim T^{-3/2}$ (under near-critical pressure), CePd$_2$Si$_2$ with $\sigma \sim T^{-5/4}$, CeRu$_5$Sn with anisotropic resitivity: $\sigma_{ab} \sim T^{-3/2}, \sigma_{c} \sim 1/T$, and magnetic susceptibility: $\chi_{s,ab} \sim T^{-1/3}, \chi_{s,c} \sim T^{-1.5}$.

Another (this time, quasi-2d) AFM material, CeCu$_6\text{Cu}_{x}\text{Au}_{1-x}$, shows $C \sim T^{7/8}, \sigma \sim T^{-7/8}, \chi_s \sim T^{1/8}$ for $x \approx 0.15$, which data hint at the exponents
\[ z = 8/3, \alpha = 1/8, \nu = 3/7. \]

It was also argued in Ref.\textsuperscript{55} that such data could be explained in terms of the anisotropic dynamical susceptibility \( \chi_s(\omega, k) = (k_1^2 + k_\parallel^2 + |\omega|^\gamma)^{-1} \) where \( \gamma = 4/5 \), yielding \( \rho \sim T, \chi_s \sim T^{-3/2} \).

As regards the quasi-2d AFM materials, the Kagome AFM ZnCu\textsubscript{3}(OH)\textsubscript{6}Cl\textsubscript{2} (a.k.a. Herbstsmithite) shows a power-law behavior of the bulk susceptibility, \( \chi_s \sim T^{-2/3} \), and the spin relaxation rate \( 1/T_1 \sim T^{0.7} \), although the issue of its possibly non-analytical \( \omega \)-dependence at small \( \omega \) has not been completely settled yet\textsuperscript{56}.

Also, the in-plane optical conductivity of this material \( \sigma \sim \omega^{7/5} \) was argued to be consistent with the picture of a spin gapless (since \( C \sim T \)) but charge gapped 2d Dirac spin liquid state\textsuperscript{55}.

The gauge theory of the \( U(1) \) spin-liquid states is also expected to reproduce such observed metal-like properties as \( C/T, \chi_s, \kappa/T \to \text{const} \) (despite \( 1/T_1 \sim T^2 \) which might indicate a soft nodal gap) in the organic compound EtM\textsubscript{3}Sb(Ph(dmit))\textsubscript{2}2 which shows the conductivity exponent \( \Delta_s \) varying between 3/4 to 3/2\textsuperscript{55}. A similar spin-liquid state (although, possibly, with a small spinon gap) characterized by the conductivity exponent \( \Delta_s \) ranging between 0.8 and 1.5 occurs in \( \kappa \)-BEET-Cu\textsubscript{2}CN\textsubscript{3}\textsuperscript{55}.

While the complete theory is still being developed and perfected, its viable variant was proposed in the framework of the phenomenological analysis of Ref.\textsuperscript{55} where the NFL self-energy was assumed to be independent of momentum, \( \Sigma(\omega) \sim \omega^{1-\alpha} \). The latter was found as a self-consistent solution of the self-consistent equations

\[
\Sigma(\omega, q) = \int d\epsilon d^d p \Lambda^2(\epsilon) \chi_E(\omega + \epsilon, p + q) G(\epsilon, p) \quad (74)
\]

where \( G(\omega, q) = (i\omega/Z - \mathbf{v}q)^{-1} \) and \( \chi_E(\omega, q) = \int d\epsilon d^d p G_{\text{GDD}} \) is the effective propagator of soft bosonic pair-exchange processes. The interaction vertices \( \Lambda \) are decorated with the renormalization factor \( Z \), enforcing the corresponding Ward identity.

The exponents obtained by solving Eq. (74) turn out to be solely determined by the spatial dimension

\[
\alpha = 1/2 - 1/z_b, \quad \nu = \frac{1}{2 + z_b \alpha}, \quad z_b = 4d/3, \quad z_f = \frac{1}{1 - \alpha} \quad (75)
\]

The resulting observables

\[
C \sim T^{1-\alpha}, \quad \sigma \sim \omega^{\alpha-1}, \quad \chi_s \sim T^n \quad (76)
\]

turn out to describe quite well the aforementioned data on YbRh\textsubscript{2}Si\textsubscript{2} and CeCu\textsubscript{6-x}Au\textsubscript{x} for \( d = 3 \) and 2, respectively.

It would be a real challenge (and an impressive achievement in the case of success) for the holographic approach to reproduce more than one of the above exponents (for the same material).

**Summary**

The holographic approach aspires to provide a potential framework for treating those strongly coupled systems that do not fit into the conventional quasiparticle picture but could be still amenable to a description in terms of certain one- and two-particle Green functions.

In fact, had this implicit assumption failed as well, it would make any comparison with the experimental data (deduced by means of the available one- and two-particle probes) rather problematic.

To that end, a comparison with the results obtained by other, more traditional, techniques might be helpful for setting up a proper holographic model. Besides, in order to become a viable practical tool, the holographic approach would have to be able to reproduce the behavior of not just one, but a whole variety of observables, such as specific heat, compressibility, magnetic susceptibility, electrical, thermal, and spin conductivities, etc. A host of such data on the documented NFL materials is available and, for the most part, is still awaiting its interpretation.

In the present communication, a number of the existing holographic predictions for thermodynamic and kinetic coefficients in the theories dual to the HV geometries were analysed in the framework of the scaling theory and with an eye on the general universal relations. In the course of such analysis, a number of contradictions between the predictions for, e.g., the conductivity obtained by virtue of the Kubo vs 'membrane paradigm' techniques were exposed and the related subtleties emphasized.

Providing a solid physical interpretation for the holographic results is instrumental for ascertaining their true status. In the absence of such physical input, the only (obviously, unwanted) alternative for the holographic predictions would be to get stuck in the situation where any formal result would seem to be (almost) as good as any other one. Only after having proven to be more than tenuously related to the actual materials, will the holographic approach become a genuine breakthrough in the field of strongly correlated systems.

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1 S. A. Hartnoll, Class. Quant. Grav. \textbf{26}, 224002 (2009); C. P. Herzog, J.Phys. \textbf{A42} 343001 (2009); J. McGreevy, Adv. High Energy Phys. \textbf{2010}, 723105 (2010). S.Sachdev, Annual Review of Cond. Matt. Phys. \textbf{3}, 9 (2012).
K. Hashimoto, N. Iizuka, T. Kimura, arXiv:1304.3126

D. N. Basov et al, Rev.Mod.Phys. 83, 471 (2011).

M. Mueller, J. Schmalian, L. Fritz, Phys. Rev. Lett. 103, 025301 (2009); M. Mueller et al, arXiv:0910.5680

V. P. J. Jacobs, S. J. G. Vandoren, and H. T. C. Stoof, arXiv:1403.3608

B. Goutreaux, arXiv:1401.5430

B. Goutreaux, JHEP 01 (2014) 080.

A. Lucas, S. Sachdev, K. Schalm, arXiv:1401.7993; S. A. Hartnoll et al, arXiv:1401.7012.

S. A. Hartnoll, D. M. Hofman, Phys.Rev.Lett.108, 241601 (2012); K. Balasubramanian, C. P. Herzog, arXiv:1312.4953; S. A. Hartnoll, J. E. Santos, arXiv:1402.0872.

S. A. Hartnoll, arXiv:1405.3651

R. A. Davison, Phys. Rev. D 88, 086003 (2013); M. Blake, D. Tong, Phys. Rev. D 88, 106004 (2013); T. Andrade, B. Withers, arXiv:1311.5157, M. Blake, D. Tong, D. Vegh, Phys. Rev. Lett. 112, 071602 (2014).

A. Donos, S. A. Hartnoll, Nature Phys.9, 649 (2013).

P. M. Hogan, A. G. Green, Phys. Rev. B 78, 195104 (2008).

J.-R. Sun, S.-Y. Wu, H.-Q. Zhang, Physics Lett. B 729 (2014) 177.

M. Alishahiha, M. R. M. Mozaffar, A. Mollabashi, arXiv:1208.2535.

M. Alishahiha, E. O. Colgin, H. Yavartanoo, JHEP11(2012)137.

K. Goldstein et al, JHEP 1008:078, 2010.

S. S. Gubser, F. D. Rocha, Phys. Rev. Lett. 102, 061601 (2009).

B. Goutreaux, E. Kiritsis, JHEP 1112, 036 (2011).

S. A. Hartnoll et al, arXiv:1401.7012.

A. Donos and J. P. Gauntlett, arXiv:1401.5077.

B.-H. Lee, D.-W. Pang, C. Park, arXiv:1006.1719.

1006.0779; Int.J.Mod.Phys.A26, 2279 (2011).

W. W. Witczak-Krempa and S. Sachdev, Phys. Rev. B 86, 235115 (2012); ibid 87, 155149 (2013).

B. L. Altshuler, L. B. Ioffe, and A. J. Millis, Phys. Rev. B 50, 14048 (1994); A. V. Chubukov, Phys. Rev. B 70, 2010.

S.-S. Lee, Phys. Rev. B 80, 165102 (2009); M. A. Metlitski and S. Sachdev, ibid B82, 075127 (2010); ibid B82, 075128 (2010).

A. L. Fitzpatrick et al, arXiv:1307.0004 arXiv:1312.3321.

D. F. Mross et al, Phys. Rev. B 82, 045121 (2010).

D. Dalidovich, S.-S. Lee, Phys. Rev. B 88, 245106 (2013).

D. V. Khveschenko, Phys. Rev. B 80, 115115 (2012).

S. A. Hartnoll et al, Phys. Rev. B 84, 125115 (2011); A. V. Chubukov, D. L. Maslov, V. I. Yudson, arXiv:1401.1461.

M. Metlitski et al, arXiv:1403.3694.

C. M. Varma, Z. Nussinov, W. van Saarloos, Phys. Rep. 361, 267 (2002).

P. Gegenwart et al, PRL 94, 076402 (2005).

A. Schoeder et al, Nature 407, 351 (2000).

J. S. Helton et al, PRL 104, 147201 (2010).

D. Pilon et al, Phys.Rev.Lett. 111, 127401 (2013).

M. Yamashita et al, Science 328, 1246 (2010).

S. Elssasser, et al, Phys.Rev.B86, 155150 (2012).

E. Abrahams, J. Schmalian, and P. Woelfle, arXiv:1303.3926.