On quantum groups and Lie bialgebras related to $sl(n)$

A Stolin and I Pop
Department of Mathematical Sciences, University of Gothenburg, SE-405 30 Gothenburg, Sweden

Abstract. Given an arbitrary field $\mathbb{F}$ of characteristic 0, we study Lie bialgebra structures on $sl(n, \mathbb{F})$, based on the description of the corresponding classical double. For any Lie bialgebra structure $\delta$, the classical double $D(sl(n, \mathbb{F}), \delta)$ is isomorphic to $sl(n, \mathbb{F}) \otimes_{\mathbb{F}} A$, where $A$ is either $\mathbb{F}[\varepsilon]$, with $\varepsilon^2 = 0$, or $\mathbb{F} \oplus \mathbb{F}$ or a quadratic field extension of $\mathbb{F}$. In the first case, the classification leads to quasi-Frobenius Lie subalgebras of $sl(n, \mathbb{F})$. In the second and third cases, a Belavin-Drinfeld cohomology can be introduced which enables one to classify Lie bialgebras on $sl(n, \mathbb{F})$, up to gauge equivalence. The Belavin-Drinfeld untwisted and twisted cohomology sets associated to an $r$-matrix are computed.

2010 Mathematics Subject Classification: 17B37, 17B62.
Keywords: Quantum group, Lie bialgebra, classical double, $r$-matrix, admissible triple.

1. Introduction
Following [3], we recall that a quantized universal enveloping algebra (or a quantum group) over a field $k$ of characteristic zero is a topologically free topological Hopf algebra $H$ over the formal power series ring $k[[\hbar]]$ such that $H/\hbar H$ is isomorphic to the universal enveloping algebra of a Lie algebra $g$ over $k$.

The quasi-classical limit of a quantum group is a Lie bialgebra. A Lie bialgebra is a Lie algebra $g$ together with a cobracket $\delta$ which is compatible with the Lie bracket. Given a quantum group $H$, with comultiplication $\Delta$, the quasi-classical limit of $H$ is the Lie bialgebra $g$ of primitive elements of $H/\hbar H$ and the cobracket is the restriction of the map $(\Delta - \Delta^{21})/\hbar (\text{mod} \hbar)$ to $g$.

The operation of taking the semiclassical limit is a functor $SC : QUE \to LBA$ between categories of quantum groups and Lie bialgebras over $k$. The existence of universal quantization functors was proved by Etingof and Kazhdan [4, 5]. They used Drinfeld’s theory of associators to construct quantization functors for any field $k$ of characteristic zero. More precisely, let $(g, \delta)$ be a Lie bialgebra over $k$. Then one can associate a Lie bialgebra $g_\hbar$ over $k[[\hbar]]$ defined as $(g \otimes_k k[[\hbar]], \hbar \delta)$. According to Theorem 2.1 of [5] there exists an equivalence $\hat{Q}$ between the category $LBA_0(k[[\hbar]])$ of topologically free over $k[[\hbar]]$ Lie bialgebras with $\delta \equiv 0 (\text{mod} \hbar)$ and the category $H_{A_0}(k[[\hbar]])$ of topologically free Hopf algebras cocommutative modulo $\hbar$. Moreover, for any $(g, \delta)$ over $k$, one has the following: $\hat{Q}(g_\hbar) = U_\hbar(g)$.

Due to this equivalence, the classification of quantum groups whose quasi-classical limit is $g$ is equivalent to the classification of Lie bialgebra structures on $g \otimes_{\mathbb{C}} \mathbb{C}[[\hbar]]$. Since any cobracket over $\mathbb{C}[[\hbar]]$ can be extended to one over $\mathbb{C}((\hbar))$ and conversely, any cobracket over $\mathbb{C}((\hbar))$, multiplied by an appropriate power of $\hbar$, can be restricted to a cobracket over $\mathbb{C}[[\hbar]]$, this in turn reduces to the problem of finding Lie bialgebras on $g \otimes_{\mathbb{C}} \mathbb{C}((\hbar))$. Denote, for the sake of simplicity, $\mathbb{K} := \mathbb{C}((\hbar))$ and $g(\mathbb{K}) := g \otimes_{\mathbb{C}} \mathbb{K}$.

As a first step towards classification, following ideas of [6], we proved in [8] that for any Lie bialgebra structure on $g(\mathbb{K})$, the associated classical double is of the form $g(\mathbb{K}) \otimes_{\mathbb{K}} A$, where $A$
is one of the following associative algebras: $K[e]$, where $e^2 = 0$, $K \oplus K$ or $K[j]$, where $j^2 = h$.

As it was shown in [8], the classification of Lie bialgebras with classical double $g(K[e])$ leads to the classification of quasi-Frobenius Lie algebras over $K$, which is a complicated and still open problem.

Unlike this case, the classification of Lie bialgebras with classical double $g(K) \oplus g(K)$ can be achieved by cohomological and combinatorial methods. In [8], we introduced a Belavin-Drinfeld cohomology theory which proved to be useful for the study of Lie bialgebra structures. To any non-skewsymmetric $r$-matrix $r_{BD}$ from the Belavin-Drinfeld list [1], we associated a cohomology set $H^1_{BD}(g, r_{BD})$. We proved the existence of a one-to-one correspondence between any Belavin-Drinfeld cohomology and gauge equivalence classes of Lie bialgebra structures on $g(K)$. In case $g = sl(n)$, we showed that for any non-skewsymmetric $r$-matrix $r_{BD}$, the cohomology set $H^1_{BD}(sl(n), r_{BD})$ has only one class, which is represented by the identity.

Regarding the classification of Lie bialgebras whose classical double is isomorphic to $g(K[j])$, with $j^2 = h$, a cohomology theory can be introduced too. Our result states that there exists a one-to-one correspondence between Belavin-Drinfeld twisted cohomology and gauge equivalence classes of Lie bialgebra structures on $g(K)$ whose classical double is isomorphic to $g(K[j])$. In [8], we proved that the twisted cohomology corresponding to the Drinfeld-Jimbo $r$-matrix has only one class, represented by a certain matrix $J$ (not the identity). A deeper investigation was done in the subsequent article [9] where twisted cohomologies for $sl(n)$ associated to generalized Cremmer-Gervais $r$-matrices were studied.

The aim of the present article is the study of Lie bialgebra structures on $sl(n, F)$, for an arbitrary field $F$ of characteristic zero. Again the idea is to use the description of the classical double. We will show that for any Lie bialgebra structure $\delta$, the classical double $D(sl(n, F), \delta)$ is isomorphic to $sl(n, F) \otimes_F A$, where $A$ is either $F[e]$, with $e^2 = 0$, or $F \oplus F$ or a quadratic extension of $F$. In the first case, the classification leads to quasi-Frobenius Lie subalgebras of $sl(n, F)$. In the second and third cases, we will introduce a Belavin-Drinfeld cohomology which enables one to classify Lie bialgebras on $sl(n, F)$, up to gauge equivalence. In the particular case $F = C((h))$ we recover the classification of quantum groups whose classical limit is $sl(n, C)$ obtained in [8,9].

2. Description of the classical double

From the general theory of Lie bialgebras it is known that for each Lie bialgebra structure $\delta$ on a fixed Lie algebra $L$ one can construct the corresponding classical double $D(L, \delta)$. As a vector space, $D(L, \delta) = L \oplus L^*$. Moreover, since the cobracket of $L$ induces a Lie bracket on $L^*$, there exists a Lie algebra structure on $L \oplus L^*$, induced by the bracket and cobracket of $L$, and such that the canonical symmetric nondegenerate bilinear form $Q$ on this space is invariant.

Let $F$ be an arbitrary field of zero characteristic. Let us assume that $\delta$ is a Lie bialgebra structure on $sl(n, F)$. Then one can construct the corresponding classical double $D(sl(n, F), \delta)$.

Similarly to Lemma 2.1 from [6], one can prove that $D(sl(n, F), \delta)$ is a direct sum of regular adjoint $sl(n)$-modules. Combining this result with Prop. 2.2 from [2], one obtains the following

**Theorem 2.1.** There exists one associative, unital, commutative algebra $A$ of dimension 2 over $F$, such that $D(sl(n, F), \delta) \cong sl(n, F) \otimes_F A$.

The symmetric invariant nondegenerate bilinear form $Q$ on $sl(n, F) \otimes_F A$ is given in the following way. For arbitrary elements $f_1, f_2 \in sl(n, F)$ and $a, b \in A$ we have

$$Q(f_1 \otimes a, f_2 \otimes b) = K(f_1, f_2) \cdot t(ab)$$

where $K$ denotes the Killing form on $sl(n, F)$ and $t : A \rightarrow F$ is a trace function.

Let us now investigate the algebra $A$. Since $A$ is unital and of dimension 2 over $F$, one can choose a basis $\{e, 1\}$, where 1 denotes the unit. Moreover, there exist $p$ and $q$ in $F$ such that $e^2 + pe + q = 0$. Let $\Delta = p^2 - 4q \in F$. We distinguish the following cases:
(i) Assume $\Delta = 0$. Let $\varepsilon := (e + p)/2$. Then $\varepsilon^2 = 0$ and $A = F\varepsilon \oplus F = F[\varepsilon]$.

(ii) Assume $\Delta$ is the square of a nonzero element of $F$. In this case, one can choose $e' \in F^*$ such that $e'^2 = \Delta$. Then $A = F \oplus e'F = F \oplus F$.

(iii) Assume $\Delta$ is not a square of an element of $F$. Then $A = F + e'F$, where $e' = (e + p)/2$ and $e'^2 = \Delta/4 \in F$. Thus $A$ is a quadratic field extension of $F$.

Summing up the above observations, we get

**Theorem 2.2.** Let $\delta$ be an arbitrary Lie bialgebra structure on $sl(n, F)$. Then $D(sl(n, F), \delta) \cong sl(n, F) \otimes F A$, where $A = F[\varepsilon]$ and $\varepsilon^2 = 0$, $A = F \oplus F$ or $A$ is a quadratic field extension of $F$.

The classification of Lie bialgebras with classical double $sl(n, F[\varepsilon])$ leads to the classification of quasi-Frobenius Lie algebras over $F$. More precisely, due to the correspondence between Lie bialgebras and Manin triples (see [3]), the following result holds:

**Proposition 2.3.** There exists a one-to-one correspondence between Lie bialgebra structures on $sl(n, F)$ whose corresponding double is isomorphic to $sl(n, F[\varepsilon])$ and Lagrangian subalgebras $W$ of $sl(n, F[\varepsilon])$ complementary to $sl(n, F)$.

Similarly to Theorem 3.2 from [7], one can prove

**Proposition 2.4.** Any Lagrangian subalgebra $W$ of $sl(n, F[\varepsilon])$ complementary to $sl(n, F)$ is uniquely defined by a subalgebra $L$ of $sl(n, F)$ together with a nondegenerate 2-cocycle $B$ on $L$.

We recall that a Lie algebra is called quasi-Frobenius if there exists a nondegenerate 2-cocycle $B$ on it. The complete classification of quasi-Frobenius Lie subalgebras of $sl(n, F)$ is not generally known for large $n$.

### 3. Belavin-Drinfeld untwisted cohomologies

Unlike the previous case, the classification of Lie bialgebras with classical double $sl(n, F) \oplus sl(n, F)$ can be achieved by cohomological and combinatorial methods.

**Lemma 3.1.** Any Lie bialgebra structure $\delta$ on $sl(n, F)$ for which the associated classical double is isomorphic to $sl(n, F) \oplus sl(n, F)$ is a coboundary $\delta = dr$ given by an $r$-matrix satisfying $r + r^{21} = f\Omega$, where $f \in F^*$ and CYB($r$) = 0.

We may suppose that $f = 1$. Naturally we want to classify all such $r$ up to $GL(n, F)$-equivalence. Let $\overline{F}$ denote the algebraic closure of $F$. Any Lie bialgebra structure $\delta$ over $F$ can be extended to a Lie bialgebra structure $\overline{\delta}$ over $\overline{F}$.

According to [1], the Lie bialgebra structures on a simple Lie algebra $g$ over an algebraically closed field are coboundaries given by non-skewsymmetric $r$-matrices. Suppose we have fixed a Cartan subalgebra $h$ and the corresponding root system. Any $r$-matrix depends on a discrete and a continuous parameter. The discrete parameter is an admissible triple $(\Gamma_1, \Gamma_2, \tau)$, i.e. an isometry $\tau : \Gamma_1 \rightarrow \Gamma_2$ where $\Gamma_1, \Gamma_2 \subset \Gamma$ such that for any $\alpha \in \Gamma_1$ there exists $k \in \mathbb{N}$ satisfying $\tau^k(\alpha) \notin \Gamma_1$. The continuous parameter is a tensor $r_0 \in h \otimes h$ satisfying

$$r_0 + r_0^{21} = \Omega_0, \quad (\tau(\alpha) \otimes 1 + 1 \otimes \alpha)(r_0) = 0, \quad \forall \alpha \in \Gamma_1$$

Here $\Omega_0$ denotes the Cartan part of the quadratic Casimir element $\Omega$. Then the associated $r$-matrix is given by the following formula

$$r = r_0 + \sum_{\alpha > 0} e_\alpha \otimes e_{-\alpha} + \sum_{\alpha \in (\text{Span}\Gamma_1)^+} \sum_{k \in \mathbb{N}} e_\alpha \wedge e_{-\tau^k(\alpha)}$$
Now, let us assume that $\delta$ is a Lie bialgebra structure on $\mathfrak{sl}(n, F)$. Then its extension $\widetilde{\delta}$ has a corresponding $r$-matrix. Up to $GL(n, \overline{F})$-equivalence, we have the Belavin-Drinfeld classification. We may therefore assume that our $r$-matrix is of the form $r_{\chi} = (\text{Ad}_{\chi} \otimes \text{Ad}_{\chi})(r)$, where $X \in GL(n, \overline{F})$ and $r$ satisfies the system $r + r^{21} = \Omega$ and CYB($r$) = 0.

Let $\sigma \in \text{Gal}(\overline{F}/F)$. Since

$$\delta(a) = [r_{\chi}, a \otimes 1 + 1 \otimes a]$$

for any $a \in \mathfrak{sl}(n, F)$ we have

$$(\sigma \otimes \sigma)(\delta(a)) = [\sigma(r_{\chi}), a \otimes 1 + 1 \otimes a]$$

and $$(\sigma \otimes \sigma)(\delta(a)) = \delta(a).$$ Consequently, $\sigma(r_{\chi}) = r_{\chi} + \lambda \Omega$, for some $\lambda \in \overline{F}$. Let us show that $\lambda = 0$. Really,

$$\Omega = \sigma(\Omega) = \sigma(r_{\chi}) + \sigma(r_{\chi}^{21}) = r_{\chi} + r_{\chi}^{21} + 2\lambda \Omega.$$ Thus $\lambda = 0$ and $\sigma(r_{\chi}) = r_{\chi}$. Consequently,

$$(\text{Ad}_{\chi}^{-1}\sigma(r_{\chi}) \otimes \text{Ad}_{\chi}^{-1}\sigma(r_{\chi}))\sigma(r) = r.$$

We recall the following.

**Definition 3.2.** Let $r$ be an $r$-matrix. The centralizer $C(r)$ of $r$ is the set of all $X \in GL(n, \overline{F})$ satisfying $(\text{Ad}_{\chi} \otimes \text{Ad}_{\chi})(r) = r$.

Using the same arguments as in the proof of Theorem 4.3 [8], it follows that $\sigma(r) = r$ and $X^{-1}\sigma(X) \in C(r)$, for any $\sigma \in \text{Gal}(\overline{F}/F)$.

**Definition 3.3.** Let $r$ be a non-skewsymmetric $r$-matrix from the Belavin-Drinfeld list and $C(r)$ its centralizer. We say that $X \in GL(n, \overline{F})$ is a Belavin-Drinfeld cocycle associated to $r$ if $X^{-1}\sigma(X) \in C(r)$, for any $\sigma \in \text{Gal}(\overline{F}/F)$.

The set of Belavin-Drinfeld cocycles associated to $r$ will be denoted by $Z_{BD}(\mathfrak{sl}(n, F), r)$. Note that this set contains the identity.

**Definition 3.4.** Two cocycles $X_1$ and $X_2$ in $Z_{BD}(\mathfrak{sl}(n, F), r)$ are called equivalent if there exists $Q \in GL(n, \overline{F})$ and $C \in C(r)$ such that $X_1 = QX_2C$.

**Definition 3.5.** Let $H^1_{BD}(\mathfrak{sl}(n, F), r)$ denote the set of equivalence classes of cocycles from $Z_{BD}(\mathfrak{sl}(n, F), r)$. We call this set the Belavin-Drinfeld cohomology associated to the $r$-matrix $r$. The Belavin-Drinfeld cohomology is said to be trivial if all cocycles are equivalent to the identity, and non-trivial otherwise.

Combining the above definitions with the preceding discussion, we obtain

**Proposition 3.6.** For any non-skewsymmetric $r$-matrix $r$, there exists a one-to-one correspondence between $H^1_{BD}(\mathfrak{sl}(n, F), r)$ and gauge equivalence classes of Lie bialgebra structures on $\mathfrak{sl}(n, F)$ with classical double isomorphic to $\mathfrak{sl}(n, F) \oplus \mathfrak{sl}(n, F)$ and $\overline{F}$-isomorphic to $\delta = dr$.

The Belavin-Drinfeld cohomology set can be computed as in [8] and the following result holds.

**Theorem 3.7.** For any non-skewsymmetric $r$-matrix $r$, $H^1_{BD}(\mathfrak{sl}(n, F), r)$ is trivial. Any Lie bialgebra structure on $\mathfrak{sl}(n, F)$ with classical double $\mathfrak{sl}(n, F) \oplus \mathfrak{sl}(n, F)$ is of the form $\delta = dr$, where $r$ is an $r$-matrix which is, up to a multiple from $F^*$, $GL(n, F)$-equivalent to a non-skewsymmetric $r$-matrix from the Belavin-Drinfeld list.
4. Belavin-Drinfeld twisted cohomologies

We focus on the study of Lie bialgebra structures on $\mathfrak{sl}(n, \mathbb{F})$ whose classical double is isomorphic to $\mathfrak{sl}(n, \mathbb{F}) \otimes \mathbb{F} A$, where $A$ is a quadratic extension of $\mathbb{F}$. We may suppose that $A = \mathbb{F}(\sqrt{d})$, where $d$ is not a square in $\mathbb{F}$. We will show that Lie bialgebras of this type can also be classified by means of certain cohomology sets.

Twisted cohomologies associated to $r$-matrices for $\mathfrak{sl}(n, \mathbb{F})$ can be defined as in [8], where we studied the particular case $\mathbb{F} = \mathbb{C}(\hbar)$. First, similarly to Prop. 5.3 of [8], one can prove the following

**Proposition 4.1.** Any Lie bialgebra structure on $\mathfrak{sl}(n, \mathbb{F})$ with classical double isomorphic to $\mathfrak{sl}(n, \mathbb{F}[\sqrt{d}])$ is given by an $r$-matrix $r'$ which satisfies CYB$(r') = 0$ and $r' + r'_{21} = \sqrt{d}\Omega$.

Over $\mathbb{F}$, all $r$-matrices are gauge equivalent to the ones from Belavin-Drinfeld list. It follows that there exists a non-skewsymmetric $r$-matrix $r$ and $X \in GL(n, \mathbb{F})$ such that $r' = \sqrt{d}(\text{Ad}X \otimes \text{Ad}X)(r)$.

The field $\mathbb{F}[\sqrt{d}]$ is endowed with a conjugation $\overline{a + b\sqrt{d}} = a - b\sqrt{d}$. Denote by $\sigma$ its lift to $\text{Gal}(\mathbb{F}[\sqrt{d}]/\mathbb{F})$. If $X \in GL(n, \mathbb{F}[\sqrt{d}])$, then $\sigma(X) = \overline{X}$. Now let us consider the action of $\sigma$ on $r'$. We have $\sigma(r') = r' + \lambda\Omega$, for some $\lambda \in \mathbb{F}$. Let us show that $\lambda = -\sqrt{d}$. Indeed, since $r' + r'_{21} = \sqrt{d}\Omega$, we also have $\sigma(r') + \sigma(r'_{21}) = -\sqrt{d}\Omega$. Combining these relations with $\sigma(r') = r' + \lambda\Omega$, we get $\lambda = -\sqrt{d}$ and therefore $\sigma(r') = r' - \sqrt{d}\Omega = -r'_{21}$.

Recall now that $r' = \sqrt{d}(\text{Ad}X \otimes \text{Ad}X)(r)$. Then condition $\sigma(r') = -r'_{21}$ implies

$$
(\text{Ad}_{X^{-1}\sigma(X)}(X) \otimes \text{Ad}_{X^{-1}\sigma(X)}(X))(\sigma(r)) = r^{21}
$$

For any $\sigma \in \text{Gal}(\mathbb{F}/\mathbb{F}[\sqrt{d}])$, $\sigma(r') = r'$, which in turn implies

$$
(\text{Ad}_{X^{-1}\sigma(X)}(X) \otimes \text{Ad}_{X^{-1}\sigma(X)}(X))(\sigma(r)) = r
$$

Now, using the same type of arguments as in the proof of Theorem 4.3 [8], one can deduce that $\sigma(r) = r$ and therefore the following result holds.

**Proposition 4.2.** Any Lie bialgebra structure on $\mathfrak{sl}(n, \mathbb{F})$ with classical double isomorphic to $\mathfrak{sl}(n, \mathbb{F}[\sqrt{d}])$ is given by $r' = \sqrt{d}(\text{Ad}X \otimes \text{Ad}X)(r)$, where $r$ is, up to a multiple from $\mathbb{F}^*$, a non-skewsymmetric $r$-matrix from the Belavin-Drinfeld list and $X \in GL(n, \mathbb{F})$ satisfies $(\text{Ad}_{X^{-1}\sigma(X)}(X) \otimes \text{Ad}_{X^{-1}\sigma(X)}(X))(r) = r^{21}$ and, for any $\sigma \in \text{Gal}(\mathbb{F}/\mathbb{F}[\sqrt{d}])$, $(\text{Ad}_{X^{-1}\sigma(X)}(X) \otimes \text{Ad}_{X^{-1}\sigma(X)}(X))(r) = r$.

**Definition 4.3.** Let $r$ be a non-skewsymmetric $r$-matrix from the Belavin-Drinfeld list. We say that $X \in GL(n, \mathbb{F})$ is a Belavin-Drinfeld twisted cocycle associated to $r$ if

$$
(\text{Ad}_{X^{-1}\sigma(X)}(X) \otimes \text{Ad}_{X^{-1}\sigma(X)}(X))(r) = r^{21}
$$

and for any $\sigma \in \text{Gal}(\mathbb{F}/\mathbb{F}[\sqrt{d}])$,

$$
(\text{Ad}_{X^{-1}\sigma(X)}(X) \otimes \text{Ad}_{X^{-1}\sigma(X)}(X))(r) = r
$$

The set of Belavin-Drinfeld twisted cocycle associated to $r$ will be denoted by $\mathcal{Z}_{BD}(\mathfrak{sl}(n, \mathbb{F}), r)$. Let us analyse for which admissible triples this set is non-empty.

Let $S \in GL(n, \mathbb{F})$ be the matrix with 1 on the second diagonal and 0 elsewhere. Let us denote by $s$ the automorphism of the Dynkin diagram given by $s(\alpha_i) = \alpha_{n-i}$ for all $i \leq n - 1$.

**Proposition 4.4.** Let $r$ be a non-skewsymmetric $r$-matrix associated to an admissible triple $(\Gamma_1, \Gamma_2, r)$. If $\mathcal{Z}_{BD}(\mathfrak{sl}(n, \mathbb{F}), r) \neq \emptyset$, then $s(\Gamma_1) = \Gamma_2$ and $s\tau = \tau^{-1}s$. 

5
Definition 4.5. Let $X_1$ and $X_2$ be two Belavin-Drinfeld twisted cocycles associated to $r$. We say that they are equivalent if there exist $Q \in GL(n, \mathbb{F})$ and $C \in C(r)$ such that $X_2 = QX_1C$.

The set of equivalence classes of twisted cocycles corresponding to a non-skewsymmetric $r$-matrix $r$ will be denoted by $\mathcal{H}_{BD}^1(sl(n, \mathbb{F}), r)$.

Remark 4.6. If two twisted cocycles $X_1$ and $X_2$ are equivalent, then the corresponding $r$-matrices $\sqrt{d}(Ad_{X_1} \otimes Ad_{X_1})(r)$ and $\sqrt{d}(Ad_{X_2} \otimes Ad_{X_2})(r)$ are gauge equivalent via $Q$.

Remark 4.7. In fact, by obvious reasons it is better to denote $\mathcal{H}_{BD}^1(sl(n, \mathbb{F}), r)$ by $\mathcal{H}_{BD}^1(sl(n, \mathbb{F}), r, d)$. However, we fix $d$ and the notation $\mathcal{H}_{BD}^1(sl(n, \mathbb{F}), r)$ is not misleading.

Proposition 4.8. There exists a one-to-one correspondence between the twisted cohomology set $\mathcal{H}_{BD}^1(sl(n, \mathbb{F}), r)$ and gauge equivalence classes of Lie bialgebra structures on $sl(n, \mathbb{F})$ with classical double isomorphic to $sl(n, \mathbb{F}[\sqrt{d}])$ and $\mathbb{F}$-isomorphic to $\delta = dr$.

Let $r_{DJ}$ be the Drinfeld-Jimbo $r$-matrix. Having fixed a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and the associated root system, we choose a system of generators $e_\alpha$, $e_{-\alpha}$, $h_\alpha$ where $\alpha > 0$ such that $K(e_\alpha, e_{-\alpha}) = 1$. Denote by $\Omega_0$ the Cartan part of $\Omega$. Then

$$r_{DJ} = \sum_{\alpha > 0} e_\alpha \otimes e_{-\alpha} + \frac{1}{2} \Omega_0$$

The twisted cohomology corresponding to $r_{DJ}$ can be studied in the same manner as was done in [8] (see Prop. 7.15). Let $J \in GL(n, \mathbb{F}[\sqrt{d}])$ denote the matrix with entries $a_{kk} = 1$ for $k \leq m$, $a_{kk} = -\sqrt{d}$ for $k \geq m + 1$, $a_{k,n+1-k} = 1$ for $k \leq m$, $a_{k,n+1-k} = \sqrt{d}$ for $k \geq m + 1$, where $m = [(n + 1)/2]$.

Theorem 4.9. The Belavin-Drinfeld twisted cohomology $\mathcal{H}_{BD}^1(sl(n), r_{DJ})$ is non-empty and consists of one element, the class of $J$.

Proof. Let $X$ be a twisted cocycle associated to $r_{DJ}$. Then $X$ is equivalent to a twisted cocycle $P \in GL(n, \mathbb{F}[\sqrt{d}])$, associated to $r_{DJ}$. We may therefore assume from the beginning that $X \in GL(n, \mathbb{F}[\sqrt{d}])$ and it remains to prove that all such cocycles are equivalent. The proof will be done by induction.

For $n = 2$, consider

$$J = \begin{pmatrix} 1 & 1 \\ \sqrt{d} & -\sqrt{d} \end{pmatrix}$$

and let $X = (a_{ij}) \in GL(2, \mathbb{F}[\sqrt{d}])$ satisfy $X = XSD$ with

$$D = \text{diag}(d_1, d_2) \in GL(2, \mathbb{F}[\sqrt{d}])$$

The identity is equivalent to the following system:

$$\begin{align*}
a_{11} & = a_{12}d_1, & a_{12} & = a_{11}d_2, & a_{21} & = a_{22}d_1, & a_{22} & = a_{21}d_2
\end{align*}$$

Assume that $a_{21}a_{22} \neq 0$. Let $a_{11}/a_{21} = a' + b'\sqrt{d}$. Then $a_{12}/a_{22} = a' - b'\sqrt{d}$. One can immediately check that $X = QJD'$, where

$$Q = \begin{pmatrix} a' & b' \\ 1 & 0 \end{pmatrix} \in GL(2, \mathbb{F}), \quad D' = \text{diag}(a_{21}, a_{22}) \in \text{diag}(2, \mathbb{F}[\sqrt{d}])$$
For \( n = 3 \), set

\[
J = \begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
\sqrt{d} & 0 & -\sqrt{d}
\end{pmatrix}
\]

and let \( X = (a_{ij}) \in GL(3, \mathbb{F}[\sqrt{d}]) \) satisfy \( \overline{X} = XSD \) with \( D = \text{diag}(d_1, d_2, d_3) \in GL(3, \mathbb{K}[\sqrt{d}]) \).

The identity is equivalent to the following system:

\[
\frac{a_{11}}{a_{21}} = d_1 a_{13}, \quad \frac{a_{12}}{a_{22}} = d_1 a_{23}, \quad \frac{a_{13}}{a_{23}} = d_1 a_{33},
\]

\[
\frac{a_{12}}{a_{22}} = d_2 a_{11}, \quad \frac{a_{13}}{a_{23}} = d_2 a_{12}, \quad \frac{a_{23}}{a_{33}} = d_2 a_{21}, \quad \frac{a_{33}}{a_{33}} = d_2 a_{32},
\]

Assume that \( a_{21}a_{22}a_{23} \neq 0 \). Let

\[
a_{11}/a_{21} = b_{11} + b_{13}\sqrt{d}, \quad a_{31}/a_{21} = b_{31} + b_{33}\sqrt{d}
\]

Then

\[
a_{13}/a_{23} = b_{31} - b_{33}\sqrt{d}, \quad a_{33}/a_{23} = b_{31} - b_{33}\sqrt{d}
\]

On the other hand, let \( b_{12} := a_{12}/a_{22} \) and \( b_{32} := a_{32}/a_{22} \). Note that \( b_{12} \in \mathbb{F}, b_{32} \in \mathbb{F} \). One can immediately check that \( X = QJD' \), where

\[
Q = \begin{pmatrix}
b_{11} & b_{12} & b_{13} \\
1 & 1 & 0 \\
b_{31} & b_{32} & b_{33}
\end{pmatrix} \in GL(3, \mathbb{F}), \quad D' = \text{diag}(a_{21}, a_{22}, a_{23}) \in \text{diag}(3, \mathbb{F}[\sqrt{d}])
\]

Assume \( n > 3 \). Denote the constructed above \( J \in GL(n, \mathbb{F}[\sqrt{d}]) \) by \( J_n \). We are going to prove that if \( X \in GL(n, \mathbb{F}[\sqrt{d}]) \) satisfies \( \overline{X} = XSD \), then using elementary row operations with entries from \( \mathbb{F} \) and multiplying columns by proper elements from \( \mathbb{F}[\sqrt{d}] \) we can bring \( X \) to \( J_n \).

We will need the following operations on a matrix \( M = \{m_{pq}\} \in \text{Mat}(n) \):

1. \( u_n(M) = \{m_{pq}, \ p, q = 2, 3, \ldots, n-1\} \in \text{Mat}(n-2) \);
2. \( g_n(M) = \{m_{pq}\} \in \text{Mat}(n+2) \), where \( m_{pq} \) are already defined for \( p, q = 1, 2, \ldots, n \), \( m_{00} = m_{n+1,n+1} = 1 \) and the rest \( m_{0,a} = m_{n,0} = m_{n+1,a} = m_{a,n+1} = 0 \).

It is clear that \( u_n(X) \) satisfies the twisted cocycle condition. However, its determinant might vanish. To avoid this complication, we note that columns 2, 3, \ldots, \( n-1 \) of \( X \) are linearly independent. Applying elementary row operations (in fact, they are permutations) we obtain a new cocycle \( X_1 \), which is equivalent to \( X \) and such that \( u_n(X_1) \) is a cocycle in \( GL(n-2, \mathbb{F}[\sqrt{d}]) \). Then, by induction, there exist \( Q_{n-2} \in GL(n-2, \mathbb{F}) \) and a diagonal matrix \( D_{n-2} \) such that

\[
Q_{n-2} \cdot u_n(X_1) \cdot D_{n-2} = J_{n-2}
\]

Consider

\[
X_n = g_{n-2}(Q_{n-2}) \cdot X_1 \cdot g_{n-2}(D_{n-2})
\]

Clearly, \( X_n \) is a twisted cocycle equivalent to \( X \) and \( u_n(X_n) = J_{n-2} \).

Applying elementary row operations with entries from \( \mathbb{F} \) and multiplying by a proper diagonal matrix we can obtain a new cocycle \( Y_n = (y_{pq}) \) equivalent to \( X \) with the following properties:
Proof. According to Lemma 4.10, \( Y = Y_n \cdot S \cdot \text{diag}(h_1, \ldots, h_n) \) that \( h_1 = h_n = 1 \) and hence, \( y_{n1} = \frac{1}{y_{nn}} \).

It follows from the cocycle condition \( Y^n = Y_n \cdot S \cdot \text{diag}(h_1, \ldots, h_n) \) and after that, we use the first and the last rows to “kill” \( \{ y_{k1}, k = 2, \ldots, n-1 \} \). Then the set \( \{ y_{kn}, k = 2, \ldots, n-1 \} \) will be “killed” automatically. We have obtained \( J_n \) from \( X \) and thus, have proved that \( X \) is equivalent to \( J_n \).

Now investigate twisted cohomologies associated to arbitrary non-skewsymmetric \( r \)-matrices. The following two results will prove to be useful for our study.

Lemma 4.10. Assume \( X \in Z_{BD}(sl(n), r) \). Then there exists a twisted cocycle \( Y \in GL(n, \mathbb{F}[\sqrt{d}]) \), associated to \( r \), and equivalent to \( X \).

Proof. We have \( X \in GL(n, \mathbb{F}) \) and for any \( \sigma \in \text{Gal}(\mathbb{F}/\mathbb{F}[\sqrt{d}]), \quad X^{-1} \sigma(X) \in C(r) \)

On the other hand, the Belavin-Drinfeld cohomology for \( sl(n) \) associated to \( r \) is trivial. This implies that \( X \) is equivalent to the identity, where in the equivalence relation we consider \( \mathbb{F}[\sqrt{d}] \) instead of \( \mathbb{F} \). So there exists \( Y \in GL(n, \mathbb{F}[\sqrt{d}]) \) and \( C \in C(r) \) such that \( X = Y C \). On the other hand, \( Y \in Z_{BD}(sl(n), r) \) since

\[
(\text{Ad}_{X^{-1}\sigma_2(X)} \otimes \text{Ad}_{X^{-1}\sigma_2(X)})(r) = r^{21} \implies (\text{Ad}_{Y^{-1}\sigma_2(Y)} \otimes \text{Ad}_{Y^{-1}\sigma_2(Y)})(r) = r^{21}
\]

Proposition 4.11. Let \( r \) be a non-skewsymmetric \( r \)-matrix associated to an admissible triple \((\Gamma_1, \Gamma_2, \tau)\) satisfying \( s(\Gamma_1) = \Gamma_2 \) and \( s\tau = \tau^{-1}s \). If \( X \in Z_{BD}(sl(n, \mathbb{F}), r) \), then there exist \( R \in GL(n, \mathbb{F}) \) and \( D \in \text{diag}(n, \mathbb{F}) \) such that \( X = RJD \).

Proof. According to Lemma 4.10, \( X = Y C \), where \( Y \in GL(n, \mathbb{F}[\sqrt{d}]) \) and \( C \in C(r) \). Since

\[
(\text{Ad}_{Y^{-1}\sigma_2(Y)} \otimes \text{Ad}_{Y^{-1}\sigma_2(Y)})(r) = r^{21}, \quad (\text{Ad}_S \otimes \text{Ad}_S)(r) = r^{21}
\]

it follows that \( S^{-1}Y^{-1}\sigma_2(Y) \in C(r) \). On the other hand, by Lemma 4.11 from [8], \( C(r) \subset \text{diag}(n, \mathbb{F}) \). We get \( S^{-1}Y^{-1}\sigma_2(Y) \in \text{diag}(n, \mathbb{F}) \). Now Theorem 4.9 implies that \( Y = RJD_0 \), where \( R \in GL(n, \mathbb{F}) \) and \( D_0 \in \text{diag}(n, \mathbb{F}) \). Consequently, \( X = RJD_0 C = RJD \) with \( D = D_0 C \in \text{diag}(n, \mathbb{F}) \).

Let \( T \) denote the automorphism of \( \text{diag}(n, \mathbb{F}) \) defined by \( T(D) = SD^{-1}T \).

Lemma 4.12. Let \( r \) be a non-skewsymmetric \( r \)-matrix with centralizer \( C(r) \). Let \( X = RJD \), with \( R \in GL(n, \mathbb{F}) \) and \( D \in \text{diag}(n, \mathbb{F}[\sqrt{d}]) \). Then \( X \in Z_{BD}(sl(n, \mathbb{F}), r) \) if and only if \( D \in T^{-1}(C(r)) \).

Proof. Let us first note that \( X \in Z_{BD}(sl(n, \mathbb{F}), r) \) if and only if for any \( \sigma \in \text{Gal}(\mathbb{F}/\mathbb{F}[\sqrt{d}]) \), \( X^{-1} \sigma(X) \in C(r) \) and \( SX^{-1}X \in C(r) \). We have \( X = RJD \) which implies

\[
X = RJD = RJS = RJJD^{-1}S = XD^{-1}S = XST(D)
\]

We immediately get that \( SX^{-1}X \in C(r) \) if and only if \( T(D) \in C(r) \).

Lemma 4.13. Let \( X_1 = R_1JD_1 \) and \( X_2 = R_2JD_2 \) be two Belavin-Drinfeld twisted cocycles associated to \( r \). Then \( X_1 \) and \( X_2 \) are equivalent if and only if \( D_2D_1^{-1} \in C(r) \cdot \text{Ker}(T) \).
\[ Q = R_2 J D_2 C^{-1} D_1^{-1} J^{-1} R_1^{-1} \]

Since \( Q = \overline{Q} \) and \( J = JS \), we get

\[ D_2 C^{-1} D_1^{-1} = S D_2 C^{-1} D_1^{-1} S \]

Thus \( D_2 C^{-1} D_1^{-1} \in \text{Ker}(T) \). On the other hand, \( C \in C(r) \subset \text{diag}(n, F) \), so \( D_2 C^{-1} D_1^{-1} = D_2 D_1^{-1} C^{-1} \). We have obtained that \( D_2 D_1^{-1} \in C(r) \cdot \text{Ker}(T) \). Conversely, if this condition is satisfied, then write \( D_2 D_1^{-1} = D_0 C \), where \( C \in C(r) \) and \( D_0 \in \text{Ker}(T) \). Denote \( Q := R_2 J D_0 J^{-1} R_1^{-1} \). Then, by construction, \( Q = \overline{Q} \) and \( X_2 = Q X_1 C \).

By lemmas 4.12 and 4.13, we get

**Proposition 4.14.** Let \( r \) be a non-skewsymmetric \( r \)-matrix associated to an admissible triple \((\Gamma_1, \Gamma_2, \tau)\) satisfying \( s(\Gamma_1) = \Gamma_2 \) and \( s\tau = \tau^{-1}s \). Then

\[ \overline{H}_{BD}^1(\mathfrak{sl}(n, F), r) = \frac{T^{-1}(C(r))}{C(r) \cdot \text{Ker}(T)} \]

At this point, one needs the explicit description of the centralizer and its preimage under \( T \).

**Lemma 4.15.** Let \( r \) be a non-skewsymmetric \( r \)-matrix associated to an admissible triple \((\Gamma_1, \Gamma_2, \tau)\). Then the following hold:

(a) \( C(r) \) consists of all diagonal matrices \( D = \text{diag}(d_1, \ldots, d_n) \) such that \( d_i = s_i s_{i+1} \ldots s_n \), where \( s_i \in F \) satisfy the condition: \( s_i = s_j \) if \( \alpha_i \in \Gamma_1 \) and \( \tau(\alpha_i) = \alpha_j \).

(b) \( T^{-1}(C(r)) \) consists of all diagonal matrices \( D = \text{diag}(d_1, \ldots, d_n) \) such that \( d_i = s_i s_{i+1} \ldots s_n \), where \( s_i \in F \) satisfy the condition: \( s_i s_{n-i} = s_j s_{n-j} \) if \( \alpha_i \in \Gamma_1 \) and \( \tau(\alpha_i) = \alpha_j \).

**Proof.** Part (a) can be proved in the same way as Lemma 5.5 from [8] and (b) follows immediately from (a).

Let us make the following remark. Any admissible triple \((\Gamma_1, \Gamma_2, \tau)\) can be viewed as a union of strings

\[ \alpha_{i_1} \tau \alpha_{i_2} \tau \ldots \tau \alpha_{i_k}, \quad \tau(\alpha_{i_k}) \notin \Gamma_1 \]

The above lemma implies that elements of \( C(r) \) have the property that \( s_{i_1} = s_{i_2} = \ldots = s_{i_k} \), i.e. \( s_i \) is constant on each string. In turn, elements of \( T^{-1}(C(r)) \) satisfy

\[ s_1 s_{n-1} = s_2 s_{n-2} = \ldots = s_k s_{n-k} \]

i.e. \( s_i s_{n-i} \) is constant on each string.

**Theorem 4.16.** Suppose \( r \) is a non-skewsymmetric \( r \)-matrix with admissible triple \((\Gamma_1, \Gamma_2, \tau)\) satisfying \( s\tau = \tau^{-1}s \). Let \( \text{str}(\Gamma_1, \Gamma_2, \tau) \) denote the number of symmetric strings not containing the middlepoint. Then

\[ \overline{H}_{BD}^1(\mathfrak{sl}(n, F), r) = \left( \frac{F^*}{N_F(\sqrt{d})/F(\sqrt{d})^*} \right)^{\text{str}(\Gamma_1, \Gamma_2, \tau)} \]
Proof. Let \( \varphi : (\mathbb{F}^n)^n \to \text{diag}(n, \mathbb{F}) \) be the map

\[
\varphi(s_1, \ldots, s_{n-1}, s_n) = \text{diag}(s_1 \ldots s_n, s_2 \ldots s_n, \ldots, s_{n-1}s_n, s_n)
\]

Consider \( \tilde{T} = \varphi^{-1}T\varphi \). Since \( \text{Ker}(T) = \varphi\text{Ker}(\tilde{T}) \), we have

\[
\frac{T^{-1}(C(r))}{\text{Ker}(T) \cdot C(r)} = \frac{\tilde{T}^{-1}\varphi^{-1}(C(r))}{\text{Ker}(\tilde{T}) \cdot \varphi^{-1}(C(r))}
\]

We make the following remarks:

(i) \((s_1, \ldots, s_n) \in \text{Ker}(\tilde{T})\) if and only if \(\bar{s}_is_{n-i} = 1\) for all \(i \leq n-1\) and \(\bar{s}_n = s_1 \ldots s_n\).

(ii) \((s_1, \ldots, s_n) \in \varphi^{-1}(C(r))\) is equivalent to \(s_i\) is constant on each string of the given triple.

(iii) \((s_1, \ldots, s_n) \in \tilde{T}^{-1}\varphi^{-1}(C(r))\) implies that \(\bar{s}_is_{n-i}\) is constant on each string.

Step 1.
Suppose that the admissible triple is the disjoint union of two symmetric strings

\[
\alpha_{i_1} \to \alpha_{i_2} \to \cdots \to \alpha_{i_k}, \quad \alpha_{n-i_k} \to \alpha_{n-i_{k-1}} \to \cdots \to \alpha_{n-i_1}
\]

Here we recall that \(\tau\) has the property that \(\tau(\alpha_{n-j}) = \alpha_{n-i}\) if \(\tau(\alpha_i) = \alpha_j\).

Let \((s_1, \ldots, s_n) \in \tilde{T}^{-1}\varphi^{-1}(C(r))\). Then

\[
\bar{s}_{i_1}s_{n-i_1} = \cdots = \bar{s}_{i_k}s_{n-i_k} = \bar{t}, \quad \bar{s}_{n-i_1}s_{i_1} = \cdots = \bar{s}_{n-i_k}s_{i_k} = \bar{t}
\]

One can check that \((s_1, \ldots, s_n) \in \text{Ker}(\tilde{T}) \cdot \varphi^{-1}(C(r))\). Indeed, let us assume first that \(n = 2m+1\). Then \((s_1, \ldots, s_n)\) is the product of the following elements:

\[
(s_1, \ldots, s_m, (\bar{s}_m)^{-1}, \ldots, (\bar{s}_1)^{-1}, \bar{s}_1 \ldots \bar{s}_m), \quad (1, \ldots, 1, s_{m+1}\bar{s}_m, \ldots, s_{n-1}\bar{s}_1, s_n(\bar{s}_1 \ldots \bar{s}_m)^{-1})
\]

The first factor belongs to \(\text{Ker}(\tilde{T})\) and the second is in \(\varphi^{-1}(C(r))\) since the \(n - i_1, \ldots, n - i_k\) coordinates have the constant value \(t\).

Suppose that \(n = 2m\). Consider

\[
(s_1, \ldots, s_{m-1}, r_m, (\bar{s}_{m+1})^{-1}, \ldots, (\bar{s}_1)^{-1}, \bar{s}_n), \quad (1, \ldots, 1, s_m/r_m, s_{m+1}\bar{s}_{m-1}, \ldots, s_{n-1}\bar{s}_1)
\]

where

\[
r_m = \frac{s_1 \ldots s_{m-1}s_n}{\bar{s}_1 \ldots \bar{s}_{m-1}\bar{s}_n}
\]

The first factor is in \(\text{Ker}(\tilde{T})\) since \(r_mr_m = 1\) and the second is in \(\varphi^{-1}(C(r))\) since neither \(n - i_1, \ldots, n - i_k\) can be \(m\), and the corresponding coordinates all equal \(t\).

Step 2.
Let us assume that the admissible triple includes a symmetric string

\[
\alpha_{i_1} \to \alpha_{i_2} \to \cdots \to \alpha_{i_k} \to \cdots \to \alpha_{n-i_k} \to \alpha_{n-i_{k-1}} \to \cdots \to \alpha_{n-i_1}
\]

not containing the midpoint. Let \((s_1, \ldots, s_n) \in \tilde{T}^{-1}\varphi^{-1}(C(r))\). Then

\[
\bar{s}_{i_1}s_{n-i_1} = \cdots = \bar{s}_{i_k}s_{n-i_k} = \bar{s}_{n-i_1}s_{i_1} = \cdots = \bar{s}_{n-i_k}s_{i_k} = \bar{t}
\]

We note that \(t \in \mathbb{F}\) since \(\bar{t} = t\).
Case 1. Assume there exists \( q \in \mathbb{F}(\sqrt{d}) \) such that \( t = q\bar{q} \). Then \((s_1, \ldots, s_n) \in \text{Ker}(\bar{T}) \cdot \varphi^{-1}(C(r))\). Indeed, one can make the same construction as in Step 1, except for the positions \( i_1, \ldots, i_k, n - i_1, \ldots, n - i_k \) where we consider instead the decomposition

\[
(\ldots, s_{i_1}, \ldots, s_{n-i_1}, \ldots) = (\ldots, s_{i_1}/q, \ldots, s_{n-i_1}/q, \ldots) \cdot (\ldots, q, \ldots, q, \ldots)
\]

Case 2. Assume for any \( q \in \mathbb{F}(\sqrt{d}) \), \( t \neq q\bar{q} \). Then it follows that \((s_1, \ldots, s_n) \notin \text{Ker}(\bar{T}) \cdot \varphi^{-1}(C(r))\). Indeed, let us assume the contrary, i.e. we may write \( s_i = p_i r_i \), where \( p_i p_{n-i} = 1 \) for all \( i \leq n - 1 \), \( p_n = p_1 \ldots p_n \) and

\[
r_{i_1} = \ldots = r_{i_k} = r_{n-i_1} = \ldots = r_{n-i_k}
\]

It follows that

\[
t = s_{i_1} s_{n-i_1} = r_{i_1} r_{n-i_1} = r_{i_1} r_{i_1}
\]

which is a contradiction.

Step 3.
Let us suppose that the admissible triple includes a symmetric string

\[
\alpha_{i_1} \xrightarrow{r} \alpha_{i_2} \xrightarrow{r} \ldots \xrightarrow{r} \alpha_{i_k} \xrightarrow{r} \ldots \xrightarrow{r} \alpha_{n-i_k} \xrightarrow{r} \alpha_{n-i_{k-1}} \xrightarrow{r} \ldots \xrightarrow{r} \alpha_{n-i_1}
\]

containing the middlepoint. In this case

\[
\overline{s}_{i_1} s_{n-i_1} = \ldots = \overline{s}_{i_k} s_{n-i_k} = \overline{s}_{n-i_1} s_{i_1} = \ldots = \overline{s}_{n-i_k} s_{i_k} = t
\]

Moreover, \( t = s_m s_m \), where \( s_m \) is the coordinate corresponding to the middlepoint \( \alpha_m \). Then again \((s_1, \ldots, s_n) \in \text{Ker}(\bar{T}) \cdot \varphi^{-1}(C(r))\) since we may proceed as in Step 2, case 1 by taking \( q = s_m \).

Example 4.17. For \( \mathbb{F} = \mathbb{R} \) and \( d = -1 \), it follows that given an \( r \)-matrix \( r \) with admissible triple \((\Gamma_1, \Gamma_2, \tau)\) we have

\[
\overline{H}^1_{BD}(sl(n, \mathbb{R}), r) = (\mathbb{Z}_2)^{\text{str}((\Gamma_1, \Gamma_2, \tau)}
\]

Example 4.18. Let us consider \( \mathbb{F} = \mathbb{C}((\hbar)) \) and \( d = \hbar \). Then \( N(\mathbb{F}(\sqrt{d})) = \mathbb{F} \) and Theorem 4.16 implies that \( \overline{H}^1_{BD}(sl(n, \mathbb{C}((\hbar))), r) \) is trivial (consists of one element) for any \( r \)-matrix \( r \) satisfying the condition of Proposition 4.4 and empty otherwise. We have thus generalized our previous results [9], where we proved that twisted cohomologies for \( sl(n) \) associated to generalized Cremmer-Gervais \( r \)-matrices are trivial.

This result completes classification of quantum groups which have \( sl(n, \mathbb{C}) \) as the classical limit. Summarizing, we have the following picture:

1. According to [4, 5], there exists an equivalence between the category \( HA_0(\mathbb{C}[[:h]]) \) of topologically free Hopf algebras cocommutative modulo \( \hbar \) and the category \( LBA_0(\mathbb{C}[[:h]]) \) of topologically free over \( \mathbb{C}[[:h]] \) Lie bialgebras with \( \delta \equiv 0 \mod \hbar \).
2. To describe the category \( LBA_0(\mathbb{C}[[:h]]) \), it is sufficient (multiplying by a proper power of \( \hbar^N \)) to classify Lie bialgebra structures on the Lie algebra \( \mathfrak{g} \otimes \mathbb{C}((\hbar)) \).
3. Following [6], only three classical Drinfeld doubles are possible, namely
\[ D(\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}((h))) = \mathfrak{g} \otimes_{\mathbb{C}} A_k, \quad k = 1, 2, 3 \]

Here
\[ A_1 = \mathbb{K}[\varepsilon], \quad \varepsilon^2 = 0, \quad A_2 = \mathbb{K} \oplus \mathbb{K}, \quad A_3 = \mathbb{K}(\sqrt{h}) \quad \text{with} \quad \mathbb{K} = \mathbb{C}((h)) \]

4. Lie bialgebra structures related to the case \( A_1 \) are in a one-to-one correspondence with quasi-Frobenius subalgebras of \( \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}((h)) \).

5. Now we turn to the case \( D(\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}((h))) = \mathfrak{g} \otimes_{\mathbb{C}} A_2 \) with \( \mathfrak{g} = \mathfrak{sl}(n) \). Up to multiplication by \( h^N \) and conjugation by an element of \( GL(n, \mathbb{K}) \), the related Lie bialgebra structures are defined by the Belavin-Drinfeld data (see [1] and Section 2, the main ingredient is the triple \( \tau : \Gamma_1 \to \Gamma_2 \) and an additional data called a Belavin-Drinfeld cohomology. In the case \( \mathfrak{g} = \mathfrak{sl}(n) \), the cohomology consists of one element independently of the Belavin-Drinfeld data. As a representative of this cohomology class one can choose the identity matrix.

6. Finally, in the case \( A_3 \) and \( \mathfrak{g} = \mathfrak{sl}(n) \) the description is as follows. Up to multiplication by \( h^N \) and conjugation by an element of \( GL(n, \mathbb{K}) \), the related Lie bialgebra structures are defined by the Belavin-Drinfeld data and an additional data called a twisted Belavin-Drinfeld cohomology. In this case the twisted cohomology consists of one element if \( \tau : \Gamma_1 \to \Gamma_2 \) satisfies the condition of Proposition 4.4 and is empty otherwise (no Lie bialgebra structures of the type \( A_3 \) if \( \tau \) does not satisfy the condition of Proposition 4.4). If the cohomology class is non-empty, it can be represented by the matrix \( J \) introduced before Theorem 4.9.

Appendix A.

Throughout the paper we use the following convenient notations for the arXiv references:

- [8] Stolin A and Pop I 2013 arXiv:1303.4046
- [9] Stolin A and Pop I 2013 arXiv:1309.7133

References

[1] Belavin A and Drinfeld V 1984 Math. Phys. Rev. 4 93
[2] Benkart G and Zelmanov E 1996 Invent. Math. 126 1
[3] Drinfeld V G 1997 Quantum groups Proc. ICM, Berkeley 1996 (Providence, RI: AMS) 1 798
[4] Etingof P and Kazhdan D 1996 Sel. Math. (NS) 2 1
[5] Etingof P and Kazhdan D 1998 Sel. Math. (NS) 4 213
[6] Montaner F, Stolin A and Zelmanov E 2010 Sel. Math. (NS) 16 935
[7] Stolin A 1999 Comm. Algebra 27 4289
[8] Bibliographic description is given in Appendix A
[9] Bibliographic description is given in Appendix A