KOVALEVSKAYA EXPONENTS AND POISSON STRUCTURES

A. V. BORISOV, S. L. DUDOLADOV
Faculty of Mechanics and Mathematics,
Department of Theoretical Mechanics Moscow State University
Vorob’iev gory, 119899 Moscow, Russia
E-mail: borisov@uni.udm.ru, dsl@online.ru

Abstract
We consider generalizations of pairing relations for Kovalevskaya exponents in quasihomogeneous systems with quasihomogeneous tensor invariants. The case of presence of a Poisson structure in the system is investigated in more detail. We give some examples which illustrate general theorems.

1 Quasihomogeneous systems. Kovalevskaya exponents

A system of $n$ differential equations

$$\dot{x}^i = v^i(x^1, \ldots, x^n), \quad i = 1, \ldots, n,$$  \hspace{1cm} (1)

is called quasihomogeneous with quasihomogeneity exponents $g_1, \ldots, g_n$, if

$$v^i(\alpha^{g_i}x^1, \ldots, \alpha^{g_n}x^n) = \alpha^{g_i+1}v^i(x^1, \ldots, x^n)$$  \hspace{1cm} (2)

for all values of $\mathbf{x}$ and $\alpha > 0$. Thus, the equations (1) are invariant under substitution $x^i \mapsto \alpha^{g_i}x^i, \ t \mapsto \frac{t}{\alpha^{g_i}}$ [15].

Remark 1. A more general definition of quasihomogeneity of degree $m$ is the invariance of the system (1) under transformation $x^i \mapsto \alpha^{g_i}x^i, \ t \mapsto \frac{t}{\alpha^{m-1}}$ [14]. All further results hold for this case as well.

An important example of the equations (1), (2) is a system with quasihomogeneous quadratic right-hand sides; in this case $g_1 = \ldots = g_n = 1$. Motion equations of many important problems of dynamics (Euler-Poisson equations, Kirchhoff equations, Euler-Poincaré equations on Lie algebras, Toda lattices, etc.) are of the quasihomogeneous form.

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Differentiating (2) with respect to $\alpha$ and setting $\alpha = 1$, we obtain the Euler formula for quasihomogeneous functions:

$$\sum_{k=1}^{n} g_k x^k \frac{\partial v^i}{\partial x^k} = (g_i + 1)v^i, \quad i = 1, \ldots, n.$$  \hfill (3)

The equations (1) possess partial solutions

$$x^i = C_i t^{-g_i}, \quad i = 1, \ldots, n,$$  \hfill (4)

where the complex constants $C_1, \ldots, C_n$ should satisfy the algebraic system of equations

$$v^i(C_1, \ldots, C_n) = -g_i C_i, \quad i = 1, \ldots, n.$$  \hfill (5)

Let us write variational equations for the partial solution (5) as

$$\dot{y}^i = \sum_{k=1}^{n} \frac{\partial v^i}{\partial x^k} (C_1 t^{-g_1}, \ldots, C_n t^{-g_n}) y^k.$$  \hfill (6)

The linear system (6) possesses partial solutions of the form

$$y^1 = \varphi^1 t^{\rho-g_1}, \ldots, y^n = \varphi^n t^{\rho-g_n},$$

where $\rho$ is an eigenvalue and $\varphi$ is an eigenvector of a matrix $K = \|K^i_j\|$, $K^i_j = \left(\frac{\partial v^i}{\partial x^j}(C) + g_i \delta^i_j\right)$, $\delta^i_j$ is the Kronecker symbol. The matrix $K$ is called Kovalevskaya matrix, its eigenvalues are called Kovalevskaya exponents (see [18]). One of the Kovalevskaya exponents always equals $-1$ [18].

If the general solution of the system (1) is expressed in terms of single-valued (meromorphic) functions of complex time, then Kovalevskaya exponents, except for $-1$, are integer (nonnegative integer, respectively).

Relations between Kovalevskaya exponents are pointed out in [12]. They occur due to the presence of an invariant tensor field in the system (1).

Recall, that a tensor field $T$ of $(p, q)$ type is called quasihomogeneous of degree $m$ with quasihomogeneity exponents $g_1, \ldots, g_n$, if

$$T_{j_1 \ldots j_q}^{i_1 \ldots i_p}(\alpha^{g_1} x^1, \ldots, \alpha^{g_n} x^n) = \alpha^{m-g_1-\ldots-g_q+g_{i_1}+\ldots+g_{i_p}} T_{j_1 \ldots j_q}^{i_1 \ldots i_p}(x^1, \ldots, x^n).$$

This tensor field is invariant for the system (1), if its Lie derivative along the vectorfield $v$ equals zero.
2 Hamilton equations

Let us consider quasihomogeneous equations of the form:

\[ \dot{x}^i = \sum_k J^{ik} \frac{\partial H}{\partial x^k}, \quad i = 1, \ldots, n, \quad (7) \]

where \( J = \|J^{ik}\| \) is a constant skew-symmetric tensor of type (2, 0), \( H \) is a quasihomogeneous function of degree \( m + 1 \):

\[ H(\alpha^{g_1}x^1, \ldots, \alpha^{g_n}x^n) = \alpha^{m+1} H(x^1, \ldots, x^n). \quad (8) \]

Checking the fulfillment of the condition (2) and using (8), we obtain

\[ \sum_{k=1}^n J^{ik} \alpha^{m+1-g_k} \frac{\partial H(x)}{\partial x^k} = \alpha^{g_i+1} \sum_{k=1}^n J^{ik} \frac{\partial H(x)}{\partial x^k}. \]

Let \( \Gamma = \text{diag}(g_1, \ldots, g_n) \). Differentiating the latter identity with respect to \( \alpha \) and setting \( \alpha = 1 \), we obtain the following conditions in matrix form, to which the quasihomogeneity exponents should satisfy:

\[ J\Gamma + \Gamma J = mJ. \quad (9) \]

Let us note, that the equations (7) are Hamilton equations with the Hamiltonian \( H \) in (possibly) noncanonical variables. If \( J \) is a symplectic matrix, the conditions (9) have a simple form:

\[ g_k + g_{k+n_2} = m. \]

It is shown in [16, 12], that in the case of diagonalizable Kovalevskaya matrix its exponents satisfy the analogous relations

\[ \rho_k + \rho_{k+n_2} = m, \]

moreover, there are always \(-1\) and \( m + 1 \) among them.

The following statement extends the corresponding results [16, 12] in general case of nondiagonalizable matrix \( K \). Let us note that one should not speak about single-valuedness and meromorphy of the general solution in the nondiagonalizable situation.

**Theorem 1.** Let \( J \) be a nondegenerate skew-symmetric matrix for the equations (7). Then the Kovalevskaya exponents decompose into pairs, which satisfy the relations

\[ \rho_k + \rho_{k+n_2} = m, \quad k = 1, \ldots, \frac{n}{2}, \]

and the structure of Jordan cells, associated with the exponents \( \rho_* \) and \( (m - \rho_*) \), is the same.
Proof.
Let us present the Kovalevskaya matrix in the form $K = JB + \Gamma$, where

$$B = \frac{\partial^2 H}{\partial x^i \partial x^k}(C)$$

is a symmetric matrix. Therefore, all conclusions of Theorem 1 follow from the chain of equivalent statements:

$$\det\|K - \rho E\| = 0 \iff \det\|K - \rho E\|^T = 0 \iff \det\|(K - \rho E)J^{-1}\| = 0 \iff$$

$$\det\|(-BJ + \Gamma - \rho E)J^{-1}\| = 0 \iff \det\|J^{-1}(JB + \Gamma + (h - 1 - \rho)E)\| = 0.$$  

In the last link of the chain we used the fact, that it is possible to substitute $J^{-1}$ for $J$ in (9).

Let us generalize the above arguments on the case when the matrix $J^{ik}$ is not certainly nondegenerate and constant. It corresponds to the considerations of quasi-homogeneous systems which admit the Poisson structure of a more general form (Lie-Poisson structures, quadratic structures, etc.) Let us preliminary prove the main theorem.

**Theorem 2.** Let us assume that the equations (11) admit a quasihomogeneous tensor invariant $T$ of degree $m$ and type $(2,0)$. Then natural numbers form 1 to $n$ can be grouped in a set $(k_1, \ldots, k_n)$ so that $\rho_1, \ldots, \rho_n$ satisfy at least $r = \text{rank } T(C)$ relations:

$$\rho_i + \rho_{k_i} = -m, \quad i = 1, \ldots, n.$$  

**Proof.**

Using the expression of Lie derivative for an invariant tensor field, it is easy to show (see [12] for details) that the tensors $T$ and $K$ are connected by the following relations:

$$-mT^{ij} = K_s^iT^sj + T^isK_j^s,$$

which we shall write in matrix form

$$-mT = KT + TK^T,$$  \hspace{1cm} (10)

where $T = \|T^{ij}\|, K = \|K^i_j\|$.

Let the matrix $A$ be formed of column-vectors $e_1, \ldots, e_n$, which are Jordan vectors of $K$:

$$KA = AK_s,$$

where $K_s$ for definiteness has the following form: there are the Kovalevskaya exponents $(\rho_1, \ldots, \rho_n)$ on the principal diagonal, and there can be units above the principal diagonal. The analogous relation holds for the transposed matrix $K^T$:

$$K^T(A^{-1})^T = (A^{-1})^TK^T_s.$$
Let us denote column-vectors, which form the matrix \((A^{-1})^T\), by \(f_1, \ldots, f_n\). On account of (10), we obtain
\[
KT(A^{-1})^T = -mT(A^{-1})^T - TK^T(A^{-1})^T = T(A^{-1})T(-mE - (K_s)^T).
\]

The matrix \((-mE - (K_s)^T)\) is also of Jordan form, but now there can be \(-1\) under the principal diagonal. Thus, under transformation \(A \mapsto T(A^{-1})^T\) to Jordan vectors \((e_1, \ldots, e_n)\) there correspond \(r = \text{rank} T(C)\) independent vectors \((Tf_1, \ldots, Tf_n)\), which are also Jordan vectors of \(K\) with the eigenvalues \((-m - \rho_1), \ldots, (-m - \rho_n)\).

**Remark 2.** It is possible that \(i = k_i\) in general case. Then \(\rho_i = -\frac{m}{2}\). The following Corollary specifies the theorem in the case of skew-symmetric tensor invariant.

**Corollary.** Let the tensor \(T\) be skew-symmetric. Then among natural numbers from 1 to \(n\) one can extract two subsets with distinct numbers \((i_1, \ldots, i_l)\) and \((k_1, \ldots, k_l)\), \(l = \frac{1}{2} \text{rank} T(C)\), such that Kovalevskaya exponents satisfy \(l\) relations
\[
\rho_{i_s} + \rho_{k_s} = -m, \quad s = 1, \ldots, l.
\]

Indeed, a nonzero vector \(Tf_i\) can not be proportional to \(e_i\) in the case of skew-symmetric \(T\). It follows from the fact that \((e_i, f_j) = \delta_{ij}\), where \((\cdot, \cdot)\) is the standard scalar product in \(\mathbb{R}^n\) \((AA^{-1} = E)\), and the skew-symmetric property of \(T\) implies \((Tf_i, f_i) = 0\).

Since the skew-symmetric structural tensor \(J^{ij}\) is a tensor invariant of motion equations for general Hamiltonian systems (Section 2), then it follows from the corollary that the Kovalevskaya exponents are coupled, and the number of pairs equals \(\frac{1}{2} \text{rank} J(C)\).

### 3 Invariant measure

As a rule, quasihomogeneous equations of dynamics (the Euler-Poisson equations, the Kirchhoff equations, etc.) possess an invariant measure besides the degenerate Poisson structure (determined by algebra \(e(3)\)). The existence of invariant measure imposes an additional condition on Kovalevskaya exponents.

Indeed, let us assume that the system \(\Pi\) admits a quasihomogeneous tensor invariant of type \((n,0)\)
\[
\Omega = \Omega(x)dx^1 \wedge \ldots \wedge dx^n, \quad \Omega(C) \neq 0.
\]
Then \(\sum_{i=1}^n \rho_i = m\), where \(m\) is the quasihomogeneity degree of \(\Omega\). If \(\Omega\) is the standard measure, then \(\sum_{i=1}^n \rho_i = \sum_{i=1}^n g_i\); in particular, the sum of Kovalevskaya
exponents equals the system dimension $n$ for systems with homogeneous quadratic right-hand sides. This result follows from the main theorem of [12].

As it was pointed out in [7], in the homogeneous case ($g_i = 1$) Kovalevskaya exponents are connected with multipliers of periodic solutions, and their pairing for Hamiltonian systems follows from the Poincaré-Lyapunov theorem on recurrence of roots of characteristic polynomial of variational equations.

4 Examples

a) Let us consider a variant of the system of Lotka-Volterra type [5, 17], which can be written as

$$
\dot{x}_i = x_i(\alpha_{i+1}x_{i+1} - \beta_{i-1}x_{i-1}), \quad i = 1, \ldots, n,
$$

$$
x_0 = x_n, \quad x_{n+1} = x_1,
$$

(11)

where $\alpha_i, \beta_i$ are constants. The equations (11) are generalizations of the integrable periodic Volterra system, for which $\alpha_i = \beta_i = \text{const}$ [2].

A straightforward calculation of Kovalevskaya exponents for the system (11) shows, that they satisfy the pairing conditions $\rho_i + \rho_j = 0$; it corresponds to occurrence of a quadratic tensor invariant $T_{ij}$ (by definition of quasihomogeneity degree). However, the fulfillment of the theorem condition is not sufficient for the presence of the Poisson structure. If the relation $\prod_{i=1}^n \alpha_i = \prod_{i=1}^n \beta_i$ holds, the equations (11) possess an additional linear integral $F = (l, x)$, $l \in \mathbb{R}^n$, and under condition $\alpha_i = \beta_i$ the system (11) is indeed Hamiltonian with the quadratic Poisson bracket $J_{ij} = C_{ij}x_ix_j$ and linear Hamiltonian.

b) Let us consider a generalized Suslov problem as another example. It describes a rigid body motion around a fixed point with the nonholonomic constraint $\omega_3 = 0$. If the center of mass is on the principal axis, along which $\omega_3 = 0$, then motion equations of the system have the form:

$$
\begin{align*}
I_1 \dot{\omega}_1 &= \varepsilon \gamma_2, \\
I_2 \dot{\omega}_2 &= -\varepsilon \gamma_1, \\
\dot{\gamma}_1 &= -\omega_2 \gamma_3, \\
\dot{\gamma}_2 &= \omega_1 \gamma_3, \\
\dot{\gamma}_3 &= \omega_2 \gamma_1 - \omega_1 \gamma_2,
\end{align*}
$$

(12)

where $I_1, I_2$ are components of the inertia tensor, $\varepsilon$ is the distance from the fixed point to the center of mass. Calculation of Kovalevskaya exponents gives the following values:

1. $\rho_1 = -1, \rho_2 = 2, \rho_3 = 4, \rho_{4,5} = \frac{1}{2} \left( 3 \pm \sqrt{1 + \frac{8}{I_1}} \right)$,

2. $\rho_1 = -1, \rho_2 = 2, \rho_3 = 4, \rho_{4,5} = \frac{1}{2} \left( 3 \pm \sqrt{1 + \frac{8}{I_2}} \right)$.

Similar in structure, but more complicated expressions for the Kovalevskaya exponents $\rho_{4,5}$ can be obtained in general case, when the position of the center of
mass and the nonintegrable constraint in the body are not related anyhow \[13\].

These Kovalevskaya exponents are coupled: \( \rho_1 + \rho_3 = \rho_4 + \rho_5 = 3 \). Therefore, it is natural to expect the presence of a tensor invariant in the system \([12]\) and possibility of its representation in the Hamiltonian form \([7]\) with some in general nonconstant structural tensor \( J^{ik} \). It is valid indeed, if one chooses the geometrical integral \( F = \frac{1}{2}(\gamma_1^2 + \gamma_2^2 + \gamma_3^2) \) as the Hamiltonian.

Let \( x = (x^1, x^2, x^3, x^4, x^5) = (\omega_1, \omega_2, \gamma_1, \gamma_2, \gamma_3) \) and rewrite \([12]\) as

\[ \dot{x}^i = J^{ij}(x) \frac{\partial F}{\partial x^j}, \]

where

\[
J = \|J^{ij}\| = \begin{pmatrix}
0 & 0 & 0 & \varepsilon & 0 \\
0 & 0 & -\frac{\varepsilon}{I_1} & 0 & 0 \\
0 & \frac{\varepsilon}{I_2} & 0 & 0 & -\omega_2 \\
-\frac{\varepsilon}{I_1} & 0 & 0 & 0 & \omega_1 \\
0 & 0 & \omega_2 & -\omega_1 & 0
\end{pmatrix}.
\]

For \([12]\) \( J \) is a tensor invariant of degree 3, which satisfies the Jacobi identity, what one verifies by straightforward calculations. The Casimir function of the Poisson structure \( J \) is the energy integral (of the Suslov problem)

\[ \frac{1}{2}(I_1\dot{\omega}_1^2 + I_2\dot{\omega}_2^2) + \varepsilon\gamma_2 = h. \quad (13) \]

The reduction on a symplectic sheet can be carried out explicitly, if one notes, that the equations \([12]\) can be rewritten as Lagrange equations \([8]\) after excluding \( \gamma_3 \) from \([13]\)

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{\omega}_i} = \frac{\partial L}{\partial \omega}, \quad L = T - V, \]

where

\[ T = \frac{1}{2}(I_1^2\dot{\omega}_1^2 + I_2^2\dot{\omega}_2^2). \]

These equations are not integrable for \( I_1 \neq I_2 \). The Hamiltonian property for the Suslov problem is rather unexpectable, since motion equations of nonholonomic dynamics do not preserve the invariant measure in general case \([11]\).

c) Let us consider a restricted problem in rigid body dynamics.

\[ \dot{\omega}_1 = \omega_2\omega_3 + z\gamma_2, \quad \dot{\omega}_2 = -\omega_1\omega_3 - z\gamma_1, \quad \dot{\omega}_3 = -\gamma_2, \quad \dot{\gamma} = \gamma \times \omega. \quad (14) \]

The equations \([14]\) at \( z = 0 \) were studied in \([13]\), where their nonintegrability was shown with the help of the separatrix splitting method; pictures of stochastic behavior are presented in \([8]\).
In general case when \( z \neq 0 \), the system (14) is quasihomogeneous and possesses the following sets of partial solutions \( \omega_i = \frac{\omega_i^0}{t}, \gamma_i = \frac{\gamma_i^0}{t^2} \), where \( \omega_0^0_1 = 0 \), \( \omega_0^0_2 = 0 \), \( \omega_0^0_3 = 2^i \), \( \gamma_0^0_1 = 2^i \), \( \gamma_0^0_2 = 2^i \), \( \gamma_0^0_3 = 0 \);

and

\[ \omega_0^0_1 = 0, \quad \omega_0^0_2 = 2^i, \quad \omega_0^0_3 = 0, \quad \gamma_0^0_1 = \frac{2^i}{z}, \quad \gamma_0^0_2 = 0, \quad \gamma_0^0_3 = \frac{2}{z}. \]

The Kovalevskaya exponents, corresponding to the chosen partial solutions, are of the form:

\[ \rho = (-1, 0, 1, \rho_1, \rho_2, \rho_3), \]

where \( (\rho_1, \rho_2, \rho_3) \) are roots of the cubic equation \( \rho^3 - 9\rho^2 + 26\rho - (24 + 8z) = 0 \), the solutions of which for any \( z \) have a complicated algebraic form. Apparently, it prevents from the existence of algebraic integrals of motion for the system (14). The occurrence of a Poisson structure for the equations (14) has not been also investigated.

d) Motion of a ferromagnet with the Barnett-London effect

The essence of the quantum-mechanic effect of Barnett is that a neutral ferromagnet magnetizes along the axis of rotation. In this case the magnetic moment \( B \) is connected with its angular velocity \( \omega \) by relation \( B = \Lambda_1 \omega \), where \( \Lambda_1 \) is a symmetric linear operator. The analogous moment occurs under rotation of a superconducting rigid body under the London effect. If a body rotates in homogeneous magnetic field with the intensity \( H \), then it is under magnetic forces with the moment \( B \times H \).

Let us denote \( \gamma = H \), then equations of motion can be written as:

\[
\begin{align*}
\dot{M} &= M \times AM + AM \times \gamma, \\
\dot{\gamma} &= \gamma \times AM, \\
\Lambda &= \Lambda_1 A, \\
A &= I^{-1} = \text{diag} (a_1, a_2, a_3).
\end{align*}
\]

As it is shown in [9], the equations are Hamiltonian at \( \Lambda = A \) (they are reduced to the Kirchhoff equations, i.e. to equations on the algebra \( e(3) \)), and also at \( \Lambda = \text{diag} (\lambda_1, \lambda_2, \lambda_3), \ A = E \). In the last case they are integrable and reduced to the Clebsh case on the algebra \( e(3) \) by a linear coordinate transformation [3].

The equations (15) possess two integrals \( F_1 = (M, \gamma), \ F_2 = (\gamma, \gamma) \) and the standard invariant measure. There is the lack of two integrals for their integrability in general case.

These integrals are \( F_3 = (M, M), \ F_4 = (M, AM) \) at \( \Lambda = 0 \). At \( \Lambda = \text{diag} (\lambda_1, \lambda_2, \lambda_3) \) using the method of separatrix splitting, one can show that for \( a_1 \neq a_2 \neq a_3 \neq a_1 \) the existence conditions for at least one of additional motion integrals, generated by \( F_3 \) or \( F_4 \), have the form

\[
\sum_{\leftrightarrow} \frac{\lambda_2 - \lambda_3}{a_1} = 0, \quad \sum_{\leftrightarrow} a_1^{-1}[a_2\lambda_3 - a_3\lambda_2 + \lambda_1(a_2 - a_3)] = 0.
\]
It is obvious from (16), that one more integral can actually exist at \( \Lambda = E \). It is the integral of moment \( F_3 = (M, M) \). The system (15) is completely integrable at \( a_1 = a_2 = a, \Lambda = E \), and its additional integral is

\[ F_4 = aM_3 + \gamma_3. \]

The question concerning the Hamiltonian property of the equations (15) was arisen in [10], however it has not been solved yet. As it is noted in [1], the matrix \( \Lambda \) should be diagonal \( \Lambda = \text{diag} (\Lambda_1, \Lambda_2, \Lambda_3) \) for the Hamiltonian property in the case of \( a_1 \neq a_2 \neq a_3 \neq a_1 \).

Calculation of Kovalevskaya exponents at \( a_2 = a_3 = B, a_1 = 1 \) for the solution

\[
(c_1, \ldots, c_6) = \left(0, \frac{1}{B} \sqrt{\frac{\lambda_3}{\lambda_2 - \lambda_3}}, \frac{1}{B} \sqrt{\frac{\lambda_2}{\lambda_3 - \lambda_2}}, \frac{c_3}{\lambda_3}, -\frac{Bc_2}{\lambda_3}, -\frac{Bc_3}{\lambda_2}\right)
\]

gives the set \((-1, 1, 2, 1 + \sqrt{B^2 - 2B}, 1 - \sqrt{B^2 - 2B})\), and for the solution

\[
(c_1, \ldots, c_6) = \left(i, \sqrt{\frac{\lambda_1}{B(\lambda_2 - \lambda_3)}}, \sqrt{\frac{\lambda_1}{B(\lambda_3 - \lambda_2)}}, 0, -\frac{Bc_2}{\lambda_1}, -\frac{Bc_3}{\lambda_1}\right)
\]

the set \((-1, 2, 2, 2, B, 1 - B)\).

The pairing condition does not hold in this case, what is typical (in general situation of nondegeneracy of the structural tensor at the point \((c_1, \ldots, c_6)\)) for Hamiltonian systems. However, this observation can not be considered to be a strong evidence of the absence of the Hamiltonian property for the system (16).

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