Multifractality of eigenstates in the delocalized non-ergodic phase of some random matrix models: Wigner–Weisskopf approach

Cécile Monthus

Institut de Physique Théorique, Université Paris Saclay, CNRS, CEA,
91191 Gif-sur-Yvette, France

E-mail: cecile.monthus@cea.fr

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Abstract

The delocalized non-ergodic phase existing in some random $N \times N$ matrix models is analyzed via the Wigner–Weisskopf approximation for the dynamics from an initial site $j_0$. The main output of this approach is the inverse $\Gamma_{j_0}(N)$ of the characteristic time to leave the state $j_0$ that provides some broadening $\Gamma_{j_0}(N)$ for the weights of the eigenvectors. In this framework, the localized phase corresponds to the region where the broadening $\Gamma_{j_0}(N)$ is smaller in scaling than the level spacing $\Delta_{j_0}(N) \propto \frac{1}{N}$, while the delocalized non-ergodic phase corresponds to the region where the broadening $\Gamma_{j_0}(N)$ decays with $N$ but is bigger in scaling than the level spacing $\Delta_{j_0}(N)$. Then the number $\frac{\Gamma_{j_0}(N)}{\Delta_{j_0}(N)}$ of resonances grows only sub-extensively in $N$. This approach allows to recover the multifractal spectrum of the Generalized–Rosenzweig–Potter (GRP) Matrix model (Kravtsov et al 2015 New. J. Phys. 17 122 002). We then consider the Lévy generalization of the GRP Matrix model, where the off-diagonal matrix elements are drawn with an heavy-tailed distribution of Lévy index $1 < \mu < 2$: the dynamics is then governed by a stretched exponential of exponent $\beta = \frac{2(\mu-1)}{\mu}$ and the multifractal properties of eigenstates are explicitly computed.

Keywords: multifractality, random matrix, Lévy statistics
1. Introduction

The recent huge activity in the field of many-body-localization (MBL) (see the recent reviews [1–4] and references therein) has renewed the interest into some subtle properties of various Anderson localization models. In particular, the MBL-delocalized phase, which is usually expected to be ergodic and to follow the eigenstate thermalization hypothesis [5–9], has been found to display anomalously slow dynamical properties [10–12] and to be nontrivial (see the recent review [13, 14] and references therein). Another possibility that has been raised is the existence of a delocalized non-ergodic phase, as discussed in [15–19]. This delocalized non-ergodic scenario is usually explained within the point of view that the MBL transition is somewhat similar to an Anderson localization transition in the Hilbert space of ‘infinite dimensionality’ where the size of the Hilbert space grows exponentially with the volume [20–24]. As a consequence, this issue has motivated many recent works to confirm or to rule out the existence of a delocalized non-ergodic phase in the short-ranged Anderson model on the Bethe lattice, either with boundaries [25, 26] or without boundaries, where this issue remains extremely controversial, since many recent papers contain completely opposite conclusions [27–32].

Within random matrix models, the question of the existence of a non-ergodic delocalized phase has been actually raised more than twenty years ago by Cizeau and Bouchaud [33] in their pioneering work on Random Lévy Matrices, that has attracted a lot of interest among physicists [34–40] and mathematicians [41–48]. More recently, the Generalized–Rosenzweig–Porter model has been proposed as the simplest matrix model exhibiting a delocalized non-ergodic phase with an explicit multifractal spectrum for eigenvectors in [49]. It has been then revisited from various points of view, namely via the statistics of the local resolvent [50], via the super-symmetry approach [51] and via the self-consistent cavity equations [32]. In this paper, our goal is to propose still another point of view based on the Wigner–Weisskopf approximation for the dynamics: this approach is applied to the usual Generalized–Rosenzweig–Porter (GRP) model, as well as to some Lévy generalization of the GRP model that we introduce (it should be stressed that it is different from the usual Lévy matrix model of Cizeau and Bouchaud [33] mentioned above).

The paper is organized as follows. Section 2 contains the definition of the models. In section 3, we describe how the multifractal properties of the localized phase and of the critical point can be obtained by the strong disorder perturbative expansion. In section 4, the dynamics from an initial site \( j_0 \) is analyzed via the Wigner–Weisskopf approximation in order to obtain the weights of the eigenvectors in the delocalized non-ergodic phase. This general framework is then applied to the Generalized–Rosenzweig–Porter model in section 5 and to its Lévy generalization in section 6. Our conclusions are summarized in section 7.

2. Models and notations

In this paper, we focus on \( N \times N \) symmetric matrix models, where the diagonal matrix elements \( H_{ii} \) are \( O(1) \) random variables drawn with some distribution \( P_{\text{diag}}(H_{ii}) \), while the off-diagonal matrix elements \( H_{ij} \) are rescaled with respect to the system size \( N \) with some exponent \( a \)

\[
H_{ij} = \frac{v_{ij}}{N^a}
\]  

(1)

where \( v_{ij} \) are \( O(1) \) random variables drawn with some symmetric probability distribution \( p_{\text{off}}(v) = p_{\text{off}}(-v) \).
2.1. Generalized–Rosenzweig–Porter (GRP) model

The Generalized–Rosenzweig–Porter model introduced in [49] and revisited from various points of view [32, 50, 51] corresponds to the case where the variance is finite and can be chosen to be unity

\[ \overline{v_{ij}^2} = \int_{-\infty}^{+\infty} dv v^2 p_{\text{off}}(v) = 1. \]  

(2)

Then the eigenvalues of the matrix remain finite \( O(1) \) in the region

\[ a \geq \frac{1}{2}. \]  

(3)

2.2. Lévy version of the Generalized–Rosenzweig–Porter (Lévy-GRP) model

We will also consider the case where \( v_{ij} \) is drawn with some heavy-tailed distribution with \( 0 < \mu < 2 \)

\[ p_{\text{off}}(v_{ij}) = \frac{\mu}{2|v_{ij}|^{1+\mu}} \theta(|v_{ij}| \geq 1) \]  

(4)

so that the variance does not exist in contrast to the case of equation (2).

The probability distribution of off-diagonal elements reads (equations (1) and (4))

\[ P_{\text{off}}(H_{ij}) = \frac{\mu}{2N^{-\mu} |H_{ij}|^{1+\mu}} \theta(|H_{ij}| \geq N^{-a}). \]  

(5)

The typical value scales as expected as

\[ H_{ij}^{\text{typ}} \propto N^{-a} \]  

(6)

but the maximum value seen by some given site \( j_0 \) is much bigger and scales as

\[ \max_{j \neq j_0} (H_{ij}) \propto N^{-(a-\frac{1}{\mu})}. \]  

(7)

As a consequence, the eigenvalues of the matrix remain finite \( O(1) \) in the region

\[ a \geq \frac{1}{\mu} \]  

(8)

that replaces equation (3).

3. Multifractal properties in the localized phase and at criticality

3.1. Strong disorder perturbative expansion

In the Strong disorder perturbative expansion, one considers the perturbation theory in the off-diagonal terms [35, 49, 52]. At order 0, the eigenvectors are completely localized on a single site

\[ |\phi_j^{(0)}\rangle = |j\rangle \]  

(9)

and the eigenvalues are given by the \( O(1) \) diagonal matrix elements

\[ E_j^{(0)} = H_{jj}. \]  

(10)
At first order in the off-diagonal elements that decay with the size \(N\), the eigenvalues remain unchanged
\[
E_{j}^{(0+1)} = H_{jj}
\]
while the eigenstates become
\[
|\phi_{j}^{(0+1)}\rangle = |j\rangle + \sum_{k \neq j} \frac{H_{kj}}{H_{jj} - H_{kk}} |k\rangle.
\]

The idea is that this expression makes sense as long as the number of resonances defined by 
\[|H_{kj}| > |H_{jj} - H_{kk}|\] does not grow with the system size \(N\), and this corresponds to the localized phase. The multifractal properties of the eigenstates can be then derived from the weights of equation (12)
\[
w_{j}^{\text{loc}}(j_{0}) \equiv |<j_{0}|\phi_{j}^{(0+1)}|> |^{2} \simeq \frac{H_{jj}^{2}}{(H_{jj} - H_{j_{0}j_{0}})^{2}}.
\]

3.2. Multifractality in the localized phase of the Generalized–Rosenzweig–Porter (GRP) model

The typical value of the weights of equation (13) corresponds to finite energy differences
\[
H_{jj} - H_{j_{0}j_{0}} = O(1)
\]
\[
[w_{j}^{\text{loc}}]_{\text{typ}} \propto N^{-2a}
\]
while the maximal weight occurs for nearby states separated by level spacing
\[
\Delta_{j_{0}}(N) \equiv |H_{j_{0}j_{0}} - H_{\text{next}}| = \frac{1}{N\rho(H_{j_{0}j_{0}})}
\]
and scales as
\[
[w_{j}^{\text{loc}}]_{\text{max}} \propto \frac{N^{-2a}}{\Delta_{j_{0}}^{2}(N)} \propto N^{-2(a-1)}.
\]

This shows that the localized phase corresponds to the region [49]
\[
a_{\text{loc}} > a_{c} = 1.
\]

The probability distribution of the weight of equation (13)
\[
\mathcal{P}_{N}^{\text{loc}}(w) = \int dH_{jj} P_{\text{diag}}(H_{jj}) \int dv P_{\text{off}}(v) \delta \left( w - \frac{N^{-2a}v^{2}}{(H_{jj} - H_{j_{0}j_{0}})^{2}} \right)
\]
\[
= \int dv P_{\text{off}}(v) |v| \left[ P_{\text{diag}} \left( H_{j_{0}j_{0}} + v\sqrt{\frac{N^{-a}}{\pi}} \right) + P_{\text{diag}} \left( H_{j_{0}j_{0}} - v\sqrt{\frac{N^{-a}}{\pi}} \right) \right]
\]
\[
2N^{a}w^{2}
\]
displays the power-law tail
\[
\mathcal{P}_{N}^{\text{loc}}(w) \sim \frac{P_{\text{diag}}(H_{j_{0}j_{0}}) \int dv P_{\text{off}}(v) |v|}{N^{a}w^{2}}.
\]
For the exponent
\[ \alpha \equiv -\frac{\ln w}{\ln N}. \] (20)

Equation (19) translates into the multifractal spectrum for the probability \( \Pi^{\text{loc}}(\alpha) \) of \( \alpha \)
\[ \Pi^{\text{loc}}(\alpha) \simeq N^{2^{-\alpha}}. \] (21)

The typical exponent corresponding to a finite probability \( \Pi^{\text{loc}}(\alpha_{\text{typ}}) = O(1) \) is \( \alpha_{\text{typ}} = 2a \) in agreement with equation (14), while the exponent associated to the maximal weight of equation (16) is \( \alpha_{\text{min}} = 2(a - 1) \) and corresponds to a probability of order \( \Pi^{\text{loc}}(\alpha_{\text{min}}) \sim \frac{1}{N} \). So the number \( N^{\text{loc}}(\alpha) \) of weights scaling as \( w \propto N^{-\alpha} \) involves the linear multifractal spectrum [49]
\[ N^{\text{loc}}_{\alpha \geq 1}(\alpha) \simeq N^{2-(a-1)\theta}(2(a-1) \leq \alpha \leq 2a). \] (22)

3.3. Multifractality in the localized phase of the Lévy-GRP model

The above calculation, in particular equation (19), shows that the multifractal properties remain the same as long as the average of the absolute value of \( v \) converges
\[ \int dv p_{\text{eff}}(v)|v| < +\infty \] (23)
i.e. in the region \( 1 < \mu < 2 \) of the Lévy case of equation (4), leading to
\[ N^{\text{loc}}_{1<\mu<2a>1}(\alpha) \simeq N^{2-(a-1)\theta}(2(a-1) \leq \alpha \leq 2a). \] (24)

For \( 0 < \mu < 1 \) where equation (23) diverges, the probability distribution of the weight of equation (13) reads using equation (4)
\[ P^{\text{loc}}_{0<\mu<1}(w) = \int dh_{j} P_{\text{diag}}(H_{j}) \int dv \frac{\mu}{2|v|^{1+\mu}} \theta(|v| \geq 1) \delta \left( w - \frac{N^{-2a}v^{2}}{(H_{j} - H_{j\text{th}})^{2}} \right) \]
\[ = \frac{\mu}{2N^{a+1/2}w^{1+\frac{\mu}{2}}} \int dh_{j} P_{\text{diag}}(H_{j}) |H_{j} - H_{j\text{th}}|^{-\mu} \theta \left( |H_{j} - H_{j\text{th}}| \geq \frac{N^{-a}}{\sqrt{w}} \right). \] (25)

In particular it displays the power-law tail
\[ P^{\text{loc}}_{0<\mu<1}(w) \sim \frac{\mu}{w^{1+\frac{\mu}{2}}} \frac{\int dh_{j} P_{\text{diag}}(H_{j}) |H_{j} - H_{j\text{th}}|^{-\mu}}{2N^{a+1/2}w^{1+\frac{\mu}{2}}} \] (26)
that translates for the exponent \( \alpha \equiv -\frac{\ln w}{\ln N} \) into the multifractal spectrum for the probability \( \Pi^{\text{loc}}(\alpha) \) of \( \alpha \)
\[ \Pi^{\text{loc}}_{0<\mu<1}(\alpha) \simeq N^{\alpha_{\mu} - a\mu}. \] (27)

The typical exponent corresponding to a finite probability \( \Pi^{\text{loc}}(\alpha_{\text{typ}}) = O(1) \) is \( \alpha_{\text{typ}} = 2a \), while the exponent associated to the maximal weight corresponding to a probability of order \( \Pi^{\text{loc}}(\alpha_{\text{min}}) \sim \frac{1}{N} \) is \( \alpha_{\text{min}} = 2(a - \frac{1}{\mu}) \). So the number \( N^{\text{loc}}(\alpha) \) of weights scaling as \( w \propto N^{-\alpha} \)
\[ N^{\text{loc}}_{0<\mu<1,a>\frac{1}{\mu}}(\alpha) \simeq N^{\alpha_{\mu} - (a-1)\theta}(2(a - \frac{1}{\mu}) \leq \alpha \leq 2a). \] (28)
3.4. Critical point

For the GRP model and (equation (22)) the Lévy-GRP model for $1 < \mu < 2$ (equation (24)), the critical point $a_c = 1$ can be obtained as the limit $a \to a_c = 1$ of the localized phase, and corresponds to the well known ‘Strong multifractality spectrum’ [53, 54]

$$N^\text{crit}(\alpha) \simeq N^{2/3} \theta \ (0 \leq \alpha \leq 2)$$

(29)

that appear in various Anderson localization models (see the review [71]) and that has been studied by various methods [55–70]. It also appears in many-body-localization models [17, 18].

3.5. Discussion

In summary, the perturbative expression of eigenvectors (equation (12)) is sufficient to derive the multifractal properties of the eigenstates in the localized phase and at criticality, but does not allow to go beyond the critical point $a_c = 1$. The goal of the present paper is thus to describe how the Wigner–Weisskopf approximation for the dynamics yields a self-consistent perturbative expression for the eigenstates containing some broadening with respect to equation (12), in order to study the multifractal properties in the delocalized phase $a < a_c = 1$.

4. Dynamics within the Wigner–Weisskopf approximation

4.1. Physical picture of the delocalized non-ergodic phase

In this section, we describe the Wigner–Weisskopf approximation for the quantum dynamics from an initial site $j_0$ in order to obtain the inverse $\Gamma_{j_0}(N)$ of the characteristic time to leave the state $j_0$

$$| < j_0 | e^{-iHt} | j_0 > |^2 \simeq e^{-\Gamma_{j_0}t}$$

(30)

(here we have written the simplest exponential case, but we will also find the stretched exponential behavior in the Lévy case). The corresponding weights for the eigenvectors $|\phi_j>$ on the site $j_0$

$$| < j_0 | \phi_j > |^2 \simeq \frac{|H_{j_0 j}|^2}{(H_{j_0 j} - H_{j_0 j_0})^2 + \left(\frac{\Gamma_{j_0}(N)}{2}\right)^2}$$

(31)

then display the additional broadening $\Gamma_{j_0}(N)$ with respect to equation (13). When the broadening $\Gamma_{j_0}(N)$ is smaller in scaling than the level spacing $\Delta_{j_0}(N) \sim |H_{j_0 j} - H_{j_0 j_0}|$, one recovers the localized phase with the weights of equation (13). When the broadening $\Gamma_{j_0}(N)$ is bigger in scaling than the level spacing $\Delta_{j_0}(N) \sim |H_{j_0 j} - H_{j_0 j_0}|$, but remains smaller than the typical difference $|H_{j_0 j} - H_{j_0 j_0}|_{\text{typ}}$, one obtains that the delocalization is only partial since it involves the sub-extensive number $\frac{\Gamma_{j_0}(N)}{\Delta_{j_0}(N)}$ of states in the energy range $\Delta_{j_0}(N) \leq |H_{j_0 j} - H_{j_0 j_0}| \leq \Gamma_{j_0}(N)$.

The multifractal spectrum of eigenvectors can be then obtained from equation (31). Note that here the broadening $\Gamma_{j_0}(N)$ as determined by the dynamics (equation (30)) has a well-defined scaling in $N$ for each model as a function of its parameters. So this dynamical point of view is somewhat different from the closely recent studies based of the Green function $G(z)$ as a function of the complex variable $z = E + i\eta$, where the imaginary part $\eta$ introduced as a formal
regularization can be chosen with various scalings with respect to the system size $N$ in order to probe various regimes [26, 32, 50, 72].

4.2. Dynamics from an initial site $j_0$

In terms of the components in the spatial basis

$$|\psi(t)\rangle = \sum_{j=1}^{N} \psi_j(t) |j\rangle$$  \hspace{1cm} (32)

the Schrödinger equation reads

$$i \frac{d\psi_j(t)}{dt} = \sum_{k=1}^{N} H_{jk} \psi_k(t) = H_{jj} \psi_j(t) + \sum_{k \neq j} H_{jk} \psi_k(t)$$  \hspace{1cm} (33)

with the initial condition

$$\psi_j(t=0) = \delta_{j_0}.$$  \hspace{1cm} (34)

It is convenient to work in the interaction picture, i.e. to make the change of variables

$$\psi_j(t) = b_j(t) e^{-iH_0 t}$$  \hspace{1cm} (35)

so that equation (33) becomes

$$i \frac{db_j(t)}{dt} = \sum_{k \neq j} H_{jk} e^{i(H_0 - H_{00}) t} b_k(t)$$  \hspace{1cm} (36)

with the initial condition

$$b_j(t=0) = \delta_{j_0}.$$  \hspace{1cm} (37)

4.3. First-order perturbation theory in the off-diagonal matrix elements

At order zero in the off-diagonal matrix elements, the solution is of course that the system remains forever in its initial condition $j_0$

$$b_j^{(0)}(t) = \delta_{j_0}.$$  \hspace{1cm} (38)

At first order, the amplitudes on the other sites $j \neq j_0$ satisfy (equation (36))

$$i \frac{db_j^{(1)}(t)}{dt} = H_{j_0} e^{i(H_0 - H_{00}) t}$$  \hspace{1cm} (39)

and thus read

$$b_j^{(1)}(t) = -iH_{j_0} \int_0^t d\tau e^{i(H_0 - H_{00}) \tau} = iH_{j_0} \frac{2 \sin \left( \frac{(H_0 - H_{00}) \tau}{2} \right)}{H_{j_0} \tau} e^{i\frac{(H_0 - H_{00}) \tau}{2}}.$$  \hspace{1cm} (40)

At lowest order, the probability to be still on the initial site $j_0$ at time $t$ will thus display the decay

$$|b_{j_0}(t)|^2 \equiv 1 - \gamma_{j_0}(t)$$  \hspace{1cm} (41)
where the probability to be elsewhere reads
\[
\gamma_{j_0}(t) = \sum_{j \neq j_0} |b_j(t)|^2 = \sum_{j \neq j_0} |H_{j_0}| f_j(H_{j_0} - H_{j_0})
\]  
(42)
in terms of the well-known auxiliary function
\[
f_j(\omega) = \left( \frac{\sin \left( \frac{\omega t}{2} \right)}{\frac{\omega}{2}} \right)^2.
\]
(43)
For large time \( t \), this function becomes peaked around the origin
\[
f_j(\omega = 0) = r^2
\]
(44)
on the interval \([-\frac{2\pi}{T}, \frac{2\pi}{T}]\).

In the standard study of the decay into a continuum of states, this function is replaced by the delta function
\[
f_j(\omega) \simeq \frac{2\pi t}{\pi T} \delta(\omega)
\]
(45)
and one obtains the famous Fermi golden Rule
\[
\gamma_{j_0}^{GR}(t) \simeq \frac{2\pi}{T} \theta \left( H_{j_0} - \frac{2\pi}{T} \leq H_j \leq H_{j_0} + \frac{2\pi}{T} \right)
\]
(46)
with the rate
\[
\Gamma_{j_0}^{GR} = 2\pi \sum_{j=1}^{N} |H_{j_0}|^2 \delta(H_j - H_{j_0}).
\]
(47)

Here since the states are discrete, one needs to keep the finite regularization of the delta function on the interval \([-\frac{2\pi}{T}, \frac{2\pi}{T}]\) leading to
\[
\gamma_{j_0}(t) \simeq \frac{t^2}{2} \sum_{j=1}^{N} |H_{j_0}|^2 \theta \left( H_{j_0} - \frac{2\pi}{T} \leq H_j \leq H_{j_0} + \frac{2\pi}{T} \right)
\]
(48)
It is thus useful to introduce the number of resonances the interval \([-\frac{2\pi}{T}, \frac{2\pi}{T}]\)
\[
N_t \equiv \sum_{j=1}^{N} \theta \left( H_{j_0} - \frac{2\pi}{T} \leq H_j \leq H_{j_0} + \frac{2\pi}{T} \right).
\]
(49)
As long as this number remains large \( N_t \gg 1 \), it will concentrate around its averaged value
\[
N_t \simeq N \int_{H_{j_0} - \frac{2\pi}{T}}^{H_{j_0} + \frac{2\pi}{T}} dH_j P_{\text{diag}}(H_j) \simeq NP_{\text{diag}}(H_{j_0}) \frac{4\pi}{T}
\]
(50)
involving the probability density \( P_{\text{diag}}(H_{j_0}) \) of the diagonal element \( H_{j_0} \).

4.4. Wigner–Weisskopf approximation

As explained in quantum mechanics textbooks, the Wigner–Weisskopf approximation allows to promote the linear perturbative decay of equation (41) into an exponential decay as follows.
Equation (36) is written exactly for the initial site $j_0$

$$i \frac{d b_{j_0}(t)}{dt} = \sum_{j \neq j_0} H_{j_0 j} e^{i(H_{j_0 j} - H_{j_0})t} b_j(t) \tag{51}$$

while for the other sites $j \neq j_0$, one keeps only the dominant term $k = j_0$ on the right-hand-side

$$i \frac{d b_j(t)}{dt} = H_{j_0 j} e^{i(H_{j_0} - H_{j_0})t} b_j(t). \tag{52}$$

The integration

$$b_j(t) = -i H_{j_0} \int_0^t d\tau e^{i(H_{j_0} - H_{j_0})\tau} b_{j_0}(\tau) \tag{53}$$

is plugged into equation (51) to obtain a closed equation for the amplitude at $j_0$

$$i \frac{d b_{j_0}(t)}{dt} \approx -i \sum_{j \neq j_0} |H_{j_0 j}|^2 \int_0^t d\tau e^{i(H_{j_0 j} - H_{j_0})\tau} b_j(t-x) \tag{54}$$

To simplify further, one makes the Markovian approximation $b_0(t-x) \approx b_0(t)$ to obtain

$$\frac{1}{b_{j_0}(t)} \frac{d b_{j_0}(t)}{dt} \approx -\sum_{j \neq j_0} |H_{j_0 j}|^2 \int_0^t dx e^{i(H_{j_0 j} - H_{j_0})x}. \tag{55}$$

The integration with the initial condition $b_{j_0}(t = 0) = 1$ yields

$$\ln b_{j_0}(t) \approx -\sum_{j \neq j_0} |H_{j_0 j}|^2 \int_0^t d\tau \int_0^\tau dx e^{i(H_{j_0 j} - H_{j_0})x}$$

$$\approx -\sum_{j \neq j_0} |H_{j_0 j}|^2 \frac{1 + i(H_{j_0 j} - H_{j_0})t - e^{i(H_{j_0 j} - H_{j_0})t}}{(H_{j_0 j} - H_{j_0})^2}$$

$$\approx -\frac{1}{2} \sum_{j \neq j_0} |H_{j_0 j}|^2 \sin^2 \left( \frac{(H_{j_0 j} - H_{j_0})t}{2} \right) - i \sum_{j \neq j_0} |H_{j_0 j}|^2 \frac{(H_{j_0 j} - H_{j_0}) t - \sin[(H_{j_0 j} - H_{j_0})t]}{(H_{j_0 j} - H_{j_0})^2}. \tag{56}$$

The first real term involves the function $\gamma_{j_0}(t)$ already introduced in equation (42), while the second imaginary term is dominated by the contribution which is linear in time, where the coefficient

$$\delta_{j_0} = \sum_{j \neq j_0} \frac{|H_{j_0 j}|^2}{H_{j_0 j} - H_{j_0}} \tag{57}$$

is well-known as the second-order perturbation correction to the eigenvalue $H_{j_0 j}$.

In summary, the amplitude on the initial site $j_0$ of equation (56) follows the exponential form

$$b_{j_0}(t) \approx e^{-\gamma_{j_0}(t) - i\delta_{j_0} t}. \tag{58}$$
while the amplitudes on the other sites $j \neq j_0$ become (equation (53))

$$b_j(t) \simeq -iH_{jj} \int_0^t dt'e^{-\gamma_j(t')} - i(H_{jj_0} - H_{jj_0} + \delta_{j_0}) \tau.$$  (59)

In particular, this Wigner–Weisskopf approximation yields the final probabilities of the other states $j \neq j_0$ in the limit $t \to +\infty$

$$|\psi_j(t \to +\infty)|^2 = |b_j(t \to +\infty)|^2 \simeq \left| -iH_{jj_0} \int_0^{+\infty} dt'e^{-\gamma_j(t')} - i(H_{jj_0} - H_{jj_0} + \delta_{j_0}) \tau \right|^2.$$  (60)

4.5. Interpretation from the point of view of the eigenstates

For the amplitudes $\psi_j(t)$ of equation (32), the solution of equation (59) yields via equation (35)

$$\psi_j(t) \simeq \frac{b_j(\infty)}{e^{-iH_{jj}t} b_j(\infty)}.$$  (61)

The comparison with the spectral decomposition into eigenstates

$$|\psi(t)|^2 = \sum_{n=1}^{N} e^{-i\mathbf{E}_n t} |\phi_n> <\phi_n|j_0>$$  (62)

means that at this approximation, the eigenvalues are $E_j = H_{jj} + ...$, the corresponding eigenstates are $|\phi_j> = |j > + ..., so that the amplitudes of these eigenstates at $j_0$ can be identified to

$$<\phi_j|j_0> = b_j(\infty) \simeq -iH_{jj_0} \int_0^{+\infty} dt'e^{-\gamma_j(t')} - i(H_{jj_0} - H_{jj_0} + \delta_{j_0}) \tau.$$  (63)

4.6. Example with the exponential decay $\gamma_j(t) = \Gamma_{j0} t$

The exponential decay $\gamma_j(t) = \Gamma_{j0} t$ corresponds to the standard Golden-Rule form (equation (46)) and to the standard Wigner–Weisskopf approximation, where the amplitudes of equation (63)

$$<\phi_j|j_0> \simeq \frac{H_{jj_0}}{(H_{jj} - H_{jj_0} - \delta_{j_0})} + i\frac{\Gamma_{j0}}{2}$$  (64)

lead to the well-known Lorentzian shape for the weights

$$|<\phi_j|j_0>|^2 \simeq \frac{|H_{jj_0}|^2}{(H_{jj} - H_{jj_0} - \delta_{j_0})^2 + \left(\frac{\Gamma_{j0}}{2}\right)^2}.$$  (65)

4.7. Example with the stretched exponential decay $\gamma_j(t) = (\Gamma_{j0} t)^\beta$ with $0 < \beta < 1$

For the stretched exponential decay $\gamma_j(t) = (\Gamma_{j0} t)^\beta$ with $0 < \beta < 1$, the weights

$$|<\phi_j|j_0>|^2 \simeq \left|H_{jj_0} \int_0^{+\infty} dt'e^{-\gamma_j(t')} - i(H_{jj_0} - H_{jj_0} + \delta_{j_0}) \tau \right|^2 = |H_{jj_0} \beta_j(\Gamma_{j0} - H_{jj_0} + \delta_{j_0} \beta_j)|^2$$  (66)
involve the half-Fourier of a stretched exponential

\[ I_\beta(\omega; \Gamma) \equiv \int_0^{+\infty} d\tau e^{-i\omega \tau} \]  

which does not seem to have a simple explicit expression (while the full Fourier corresponds to the Lévy symmetric stable law of index \( \beta \)). However the stretched exponential can be rewritten as the Laplace transform of the fully asymmetric Lévy stable law \( L_\beta(x) \) of index \( \beta \)

\[ e^{-i\omega x} = \int_0^{+\infty} dx L_\beta(x) e^{-\left(\frac{\Gamma}{2} - \frac{1}{2}\right) x} \]  

and one obtains the weights of equation (66) in terms of these integrals.

However, in the following we will only need the two simple limits:

(i) for \( \Gamma \ll |\omega| \), we may approximate by the value for \( \Gamma \to 0 \)

\[ |I_\beta(\omega; \Gamma \to 0)|^2 = \frac{1}{\omega^2} \]  

so that the weights of equation (66) become

\[ |\langle \phi_j | \phi_{j_0} \rangle| \sim \frac{\text{constant}}{|H_{j_0} - H_j|} \]  

(iii) for \( \Gamma \gg |\omega| \), we may approximate by the value for \( \omega = 0 \)

\[ |I_\beta(\omega = 0; \Gamma)|^2 = \frac{2 \int_0^{+\infty} du u^\frac{1}{2} e^{-u}}{\Gamma^2} \]  

so that the weights of equation (66) reads

\[ |\langle \phi_j | \phi_{j_0} \rangle| \sim \frac{\text{constant}}{|H_{j_0} - H_j|} \left\{ \frac{H_{j_0}}{|\Gamma_j|} \right\}^2 \]  

i.e. apart from numerical constants, the energy difference \( |H_{j_0} - H_j| \) of equation (71) is simply replaced by the broadening \( \Gamma_{j_0} \), exactly as in the Lorentzian simpler case of equation (65).

5. Generalized–Rosenzweig–Porter matrix model

As recalled in the Introduction, the Generalized–Rosenzweig–Porter model is the simplest matrix model exhibiting a delocalized non-ergodic phase with an explicit multifractal spectrum for eigenvectors in [49], and has been analyzed recently from various points of view [32, 50, 51]. In this section, our goal is to show how the present dynamical approach is able to recover the multifractal spectrum obtained in [49].
5.1. Dynamics within the Wigner–Weisskopf approximation

Here the number of resonances of equation (49) scales as equation (50)

\[ N_t \approx N P_{\text{diag}}(H_{j0j0}) \frac{4\pi}{t}. \]  

(74)

Since all off-diagonal matrix elements have the same scaling (equations (1) and (2)), equation (48) becomes

\[ \gamma_j(t) \approx \frac{t^2}{2N^{2a}} N_t \approx 2\pi P_{\text{diag}}(H_{j0j0}) N^{1-2a} t. \]  

(75)

It is thus linear in the time \( t \) as the case discussed in section (4.6), leading to the Lorentzian weights (equation (65))

\[ w_j \equiv | \langle \phi_j \rangle | f_0 > |^2 \approx \frac{|H_{j0}|^2}{(H_{jj} - H_{j0j0} - \delta_{j0})^2 + \left( \frac{\Gamma_{j0}}{2} \right)^2}. \]  

(76)

with the broadening

\[ \Gamma_{j0}(N) = 2\pi P_{\text{diag}}(H_{j0j0}) N^{1-2a} \]  

(77)

that should be compared with the level spacing \( \Delta_{j0}(N) \) of equation (15). For \( a > \alpha_c = 1 \), the broadening \( \Gamma_{j0}(N) \) is smaller in scaling than the level spacing \( \Delta_{j0}(N) \) and one recovers the localized phase discussed in section 3.3.

5.2. Multifractality in the delocalized non-ergodic phase \( \frac{1}{2} < a < \alpha_c = 1 \)

For \( \frac{1}{2} < a < \alpha_c = 1 \), the broadening \( \Gamma_{j0}(N) \) of equation (77) decays with \( N \) but is bigger in scaling than the level spacing \( \Delta_{j0}(N) \) of equation (15), so here we need to analyze the Lorentzian weights

\[ w_j \approx \frac{|H_{j0}|^2}{(H_{jj} - H_{j0j0})^2 + \left( \frac{\Gamma_{j0}}{2} \right)^2}. \]  

(78)

The typical value remains the same as in equation (14),

\[ w_j^{\text{typ}} \propto N^{-2a} \]  

(79)

while the maximal weight is not equation (16) anymore but is given instead by

\[ w_{\text{max}}(N) \equiv 4 \frac{N^{-2a}}{\Gamma_{j0}^2} \propto N^{-2(1-a)}. \]  

(80)

In terms of this maximal value \( w_{\text{max}}(N) \), the probability distribution reads

\[
P(w) = \int \text{d}H_{jj} P_{\text{diag}}(H_{jj}) \delta \left( w - \frac{N^{-2a}}{(H_{jj} - H_{j0j0})^2 + \left( \frac{\Gamma_{j0}}{2} \right)^2} \right) \\
\quad = \theta(w \leq w_{\text{max}}(N)) \frac{P_{\text{diag}}(H_{j0j0} + N^{-a} \sqrt{\frac{1}{2} - \frac{1}{w w_{\text{max}}(N)}}) + P_{\text{diag}}(H_{j0j0} - N^{-a} \sqrt{\frac{1}{2} - \frac{1}{w w_{\text{max}}(N)}})}{2 N w^{\frac{1}{2}} \sqrt{1 - \frac{w}{w_{\text{max}}(N)}}}.
\]  

(81)
For the exponent $\alpha = -\frac{\ln w}{m}$, this translates into the multifractal spectrum for the number $N(\alpha)$

$$N_{\text{energo}}(\alpha) \simeq N^{\frac{1}{2} + 1 - \alpha} \theta(2(1 - \alpha) \leq \alpha \leq 2\alpha).$$

(82)

The physical meaning of this delocalized non-ergodic phase is thus as follows: the delocalization is limited to the energies inside the broadening scale $|H_j - H_{0bj}| < \Gamma_j(N) \propto N^{1 - 2\alpha}$ containing the sub-extensive $\sum_j \frac{\Gamma_j(N)}{\Delta(\Gamma_j(N))} \propto N^{2(1 - \alpha)}$ number of states that have weights scaling as $w_{\text{max}}(N) \propto N^{-2(1 - \alpha)}$ (equation (80)). The other exponents $\alpha > 2(1 - \alpha)$ arising in the linear spectrum of equation (82) corresponds to energies outside the broadening scale $|H_j - H_{0bj}| < \Gamma_j(N) \propto N^{1 - 2\alpha}$. For the generalized fractal dimensions $D(q)$ that govern the generalized moments of arbitrary index $q > 0$

$$N < w^q >_N = \int d\alpha N(\alpha) N^{-\alpha q} \simeq N^{\left(1 - q\right) D(q)}.$$ 

(83)

Equation (82) translates into

$$D_{\frac{1}{2} < \alpha < 1}^{\text{energo}}(q) = 2(1 - \alpha) \quad \text{for} \quad q \geq \frac{1}{2},$$

$$D_{\frac{1}{2} < \alpha < 1}^{\text{energo}}(q) = \frac{1 - 2aq}{1 - q} \quad \text{for} \quad 0 \leq q \leq \frac{1}{2}.$$ 

(84)

So the region inside the broadening scale $|H_j - H_{0bj}| < \Gamma_j(N)$ govern all the fractal dimensions $D(q)$ for $q > \frac{1}{2}$, while the region outside the broadening scale $|H_j - H_{0bj}| < \Gamma_j(N)$ dominates for $q < \frac{1}{2}$.

It is interesting to consider the two boundaries of the delocalized non-ergodic region $\frac{1}{2} < a < a_c = 1$. For $a \rightarrow a_c = 1$, one recovers the critical spectrum of equation (29) as it should. For $a \rightarrow \frac{1}{2}$, one reaches the monofractal spectrum of the ergodic phase

$$N_{\text{ergo}}(\alpha) \simeq N^\delta(\alpha - 1).$$

(85)

Note that for this case $a = \frac{1}{2}$ where the broadening $\Gamma_j$ does not decay with $N$ anymore, the Lorentzian distribution of equation (78) is nevertheless a non-perturbative exact result as a consequence of the free probability theory as applied to eigenvectors (see [73, 74] and references therein).

6. Lévy version of the Generalized–Rosenzweig–Porter matrix model

6.1. Dynamics within the Wigner–Weisskopf approximation in the region $1 < \mu < 2$

The sum of equation (48) that we have to evaluate

$$\gamma_j(t) = \frac{1}{2} \sum_{j=1}^{N} |H_{jb}|^2 \theta \left( H_{hbj} - \frac{2\pi}{T} \leq H_j \leq H_{hbj} + \frac{2\pi}{T} \right)$$

(86)

involves the number (equations (49) and (50))

$$N_t \equiv \sum_{j=1}^{N} \theta \left( H_{hbj} - \frac{2\pi}{T} \leq H_j \leq H_{hbj} + \frac{2\pi}{T} \right) \simeq NP_{\text{diag}}(H_{hbj}) \frac{4\pi}{T}$$

(87)
of random positive variables \( y_j \equiv |H_{jj}|^2 \), whose distribution is obtained from equation (5)

\[
P(y_j) = \frac{\mu}{2N^\mu y_j^{1+\frac{2}{\mu}}} \theta(y_j \geq N^{-2\mu}).
\] (88)

As a consequence, the sum \( S_{N_t} \) of \( N_t \) variables \( y_j \) is distributed with the asymmetric Lévy stable distribution of index \( \frac{2}{\mu} \). In particular displays the tail

\[
P(S_{N_t}) \approx \frac{\mu N_t}{2N^\mu S_{N_t}^{1+\frac{2}{\mu}}}
\] (89)

so that its typical scaling reads

\[
S_{N_t}^{typ} \approx \left( \frac{\mu N_t}{2N^\mu} \right)^{\frac{2}{\mu}}.
\] (90)

Putting everything together, equation (86) scales as

\[
\gamma_j(\tau) \approx \frac{\tau^2}{2} S_{N_t}^{typ} \approx \frac{\tau^2}{2} \left( \frac{\mu P_{diag}(H_{jj})2\pi}{N^\mu N_{N+1}} \right)^{\frac{2}{\mu}} \approx \frac{(\Gamma_j \tau)^{\beta}}{2}.
\] (91)

This corresponds to the stretched exponential case discussed in section (4.7) with the exponent

\[
\beta = \frac{2}{\mu}(\mu - 1)
\] (92)

varying in the interval \( 0 < \beta < 1 \) for \( 1 < \mu < 2 \). The inverse time scale in equation (91)

\[
\Gamma_j = \frac{[\mu P_{diag}(H_{jj})2\pi N_{N+1}]}{N^{\mu N_{N+1}}}
\] (93)

decays as a function of the system size \( N \) for \( a > \frac{1}{\mu} \). The comparison with the level spacing \( \Delta_{\text{del}}(N) \propto N^{1/\mu(N_{1/\mu})} \) shows that the delocalized non-ergodic phase corresponds to the region

\[
\frac{1}{\mu} < a_{\text{nonergo}} < a_c = 1.
\] (94)

6.2. Multifractal properties in the delocalized non-ergodic region \( \frac{1}{\mu} < a < a_c = 1 \) for \( 1 < \mu < 2 \)

In the region of equation (94), the broadening \( \Gamma_j(N) \) of equation (93) decays with \( N \) but is bigger in scaling than the level spacing of equation (15). As explained in section 4.7, the weights of the eigenstates are more complicated than Lorentzian, but to obtain the multifractal spectrum, we only need to take into account the two simple limits of equations (71) and (73) as follows:

(i) In the region outside the broadening scale \( |H_{jj} - H_{jj0}| > \Gamma_j \), the weights still follow equation (71)

\[
w^{\text{outside}}_j \approx \frac{H_{jj0}^2}{(H_{jj0} - H_{jj})^2} = \frac{N^{-2\mu}2}{(H_{jj0} - H_{jj})^2}.
\] (95)
Using equation (4), the probability distribution of these weights reads

\[
P_{\text{outside}}(w) = \int \frac{dH_{jj}}{2Nw^2} \int \frac{dv}{2|v|^{1+\mu}} \theta(|v| \geq 1) \delta(w - \frac{N^{-2a}v^2}{\Gamma_j^2})
\]

\[
= \frac{1}{2Nw^2} \int \frac{dv}{2|v|^{1+\mu}} \theta(|v| \geq 1) \delta(w \leq \frac{N^{-2a}v^2}{\Gamma_j^2}) \left[ P_{\text{diag}}(H_{jj} - v^{N^{-\mu}}) + P_{\text{diag}}(H_{jj} + v^{N^{-\mu}}) \right].
\]

This translates into the multifractal spectrum for probability distribution of the exponent \( \alpha = -\frac{\ln w}{\ln N} \)

\[
\Pi_{\text{outside}}(\alpha) \simeq N^{\frac{2}{2} - a} \theta \left( \frac{2(1-a)}{\mu - 1} \leq \alpha \leq 2a \right). \tag{96}
\]

Since there are \( O(N) \) weights outside, the corresponding number of weights decaying with the exponent \( \alpha \) reads

\[
N_{\text{outside}}(\alpha) \simeq N^{\frac{2}{2} + a} \theta \left( \frac{2(1-a)}{\mu - 1} \leq \alpha \leq 2a \right). \tag{97}
\]

(ii) In the region inside the broadening scale \(|H_{jj} - H_{jj0}| \leq \Gamma_j\), the number of states scales as

\[
\Gamma_j \propto N^{1 - \frac{\mu - 1}{2} - \frac{1}{a}} = N^{\frac{1}{2} - \frac{\mu - 1}{2} - \frac{1}{a}} \tag{98}
\]

and the weights follow equation (73)

\[
w_{\text{inside},i} \simeq \frac{H_{jj0}}{\Gamma_j^2} = \frac{N^{-2a}v^2}{\Gamma_j^2} \propto N^{-2\frac{\mu - 1}{2} - \frac{1}{a}}. \tag{99}
\]

Using equation (4), the probability distribution of these weights reads

\[
P_{\text{inside}}(w) = \int \frac{dv}{2|v|^{1+\mu}} \theta(|v| \geq 1) \delta \left( w - \frac{N^{-2a}v^2}{\Gamma_j^2} \right)
\]

\[
= \frac{\mu}{2a^{1+\frac{1}{2}}} N^{-a - \frac{1}{2}} \Gamma_j^{-\mu} \theta \left( w \geq \frac{N^{-2a}}{\Gamma_j^2} \right). \tag{100}
\]

Using that \( \Gamma_j \) decays with \( N \) as equation (93), this translates into the multifractal spectrum for probability distribution of the exponent \( \alpha = -\frac{\ln w}{\ln N} \)

\[
\Pi_{\text{inside}}(\alpha) \simeq N^{\frac{2}{2} - \frac{1}{2} - \frac{1}{2}} \theta \left( 0 \leq \alpha \leq 2 \frac{1-a}{\mu - 1} \right). \tag{101}
\]

Since the number of states inside the broadening scales as equation (98), the corresponding number of exponents \( \alpha \)

\[
N_{\text{inside}}(\alpha) \simeq N^{\frac{2}{2} \theta} \left( 0 \leq \alpha \leq 2 \frac{1-a}{\mu - 1} \right). \tag{102}
\]

Putting together the two contributions of equation (96) and equation (101), one obtains that the total multifractal spectrum is the sum of two linear spectra of slopes \( \frac{2}{2} \) and \( \frac{1}{2} \)
For the generalized fractal dimensions $D(q)$ that govern the generalized moments of arbitrary index $q > 0$ (equations (83), and (103)) translates into the three domains

\[
D_{\frac{1}{\mu} < 2; \frac{1}{a} < \alpha < a}^\text{nonergo}(q) = 0 \quad \text{for } q > \frac{\mu}{2},
\]

\[
D_{\frac{1}{\mu} < 2; \frac{1}{a} < \alpha < a}^\text{nonergo}(q) = \frac{\mu(1-a)}{(\mu-1)} \left( \frac{1 - \frac{q}{\mu}}{1-q} \right) \quad \text{for } \frac{1}{2} \leq q < \frac{\mu}{2},
\]

\[
D_{\frac{1}{\mu} < 2; \frac{1}{a} < \alpha < a}^\text{nonergo}(q) = \frac{1 - 2aq}{1-q} \quad \text{for } 0 \leq q < \frac{1}{2}.
\]

(104)

The results are thus very different from the spectrum of equation (84) concerning the Generalized–Rosenzweig–Potter. The ‘delocalization’ for the energies inside the broadening scale $|H_j - H_{hh}| < \Gamma_{hh}(N)$ containing the sub-extensive number of states is not homogeneous as in the Generalized–Rosenzweig–Potter, but is instead strongly inhomogeneous as a consequence of the Lévy distribution of the off-diagonal matrix elements. So this ‘delocalization’ is actually not so effective. In particular the generalized dimensions $D(q)$ vanish in the whole region $q > \frac{\mu}{2}$ including the information dimension $D(q = 1)$ and the return dimension $D(q = 2)$, while the transmission dimension does not vanish $D(q = \frac{1}{2}) > 0$. Our conclusion is thus that the possibility proposed by Cizeau and Bouchaud [33] to have at the same time $D(q = \frac{1}{2}) > 0$ ($Y = \infty$ in the notation of [33]) and $D(q = 2) = 0$ ($Y > 0$ in the notation of [33]) indeed comes true for the present Lévy-GRP model.

At the critical point $a_c = 1$, equation (103) yields the critical spectrum of equation (29) corresponding to the contribution $N_{\text{outside}}(\alpha)$ only

\[
N_{\frac{1}{\mu} < 2; \frac{1}{a} < \alpha < a = 1}^\text{crit}(\alpha) = N^\frac{\alpha}{2} \theta (0 \leq \alpha < 2a).
\]

(105)

At the other boundary $a = \frac{1}{\mu}$ on the contrary, only the contribution $N_{\text{inside}}(\alpha)$ survives and gives

\[
N_{\frac{1}{\mu} < 2; \frac{1}{a} < \alpha < a = \frac{1}{\mu}}^\text{nonergo}(\alpha) = N^\alpha \frac{\alpha}{2} \theta \left( 0 \leq \alpha < \frac{2}{\mu} \right).
\]

(106)

7. Conclusions

In this paper, we have proposed to analyze the delocalized non-ergodic phase of some random matrix models via the Wigner–Weiskopf approximation for the dynamics from an initial site $j_0$. The main output of this approach is the inverse $\Gamma_{hh}(N)$ of the characteristic time to leave the state $j_0$ that provides some broadening $\Gamma_{hh}(N)$ for the weights of the eigenvectors. In this framework, the localized phase is recovered as the region where the broadening $\Gamma_{hh}(N)$ is smaller in scaling than the level spacing $\Delta_{hh}(N)$. Here we have focused on the delocalized non-ergodic phase existing in the region of parameters where the broadening $\Gamma_{hh}(N)$ decays with $N$ but is bigger in scaling than the level spacing $\Delta_{hh}(N)$. Then the number of resonances
grows only sub-extensively in $N$ as $\sum_{\Delta N} \Delta < N$. For the Generalized–Rosenzweig–Potter (GRP) Matrix model, we have shown how the present approach allows to recover the results obtained previously via other methods [32, 49–51]. For the Lévy generalization of the GRP model with $1 < \mu < 2$, we have obtained that the dynamics is governed by a stretched exponential and we have computed the multifractal properties of eigenstates.

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