Uncertainty about Uncertainty: Near-Optimal Adaptive Algorithms for Estimating Binary Mixtures of Unknown Coins

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Abstract
Given a mixture between two populations of coins, “positive” coins that have (unknown and potentially different) probabilities of heads $\geq \frac{1}{2} + \Delta$ and negative coins with probabilities $\leq \frac{1}{2} - \Delta$, we consider the task of estimating the fraction $\rho$ of coins of each type to within additive error $\epsilon$. We introduce new techniques to show a fully-adaptive lower bound of $\Omega(\frac{\rho \epsilon^2}{\Delta^2})$ samples (for constant probability of success). We achieve almost-matching algorithmic performance of $O\left(\frac{\rho \epsilon^2}{\Delta^2} (1 + \rho \log \frac{1}{\epsilon})\right)$ samples, which matches the lower bound except in the regime where $\rho = \omega\left(\frac{\log (1/\epsilon)}{\log (1/\epsilon)}\right)$. The fine-grained adaptive flavor of both our algorithm and lower bound contrasts with much previous work in distributional testing and learning.

1 Introduction
We consider a natural statistical estimation task, motivated by a practical setting, with an intriguing adaptive flavor. We provide both algorithms and (fully adaptive) lower bounds for this problem that match each other in most parameter regimes. We conjecture that, in the remaining parameter regime, our lower bound is tight and our algorithm can be improved to match it.

In our problem setting, there is a finite universe of coins of two types: positive coins each have a (potentially different) probability of heads that lies in the interval $[\frac{1}{2} + \Delta, 1]$, while negative coins lie in the interval $[0, \frac{1}{2} - \Delta]$, where $\Delta \in (0, \frac{1}{2}]$ parameterizes the “quality” of the coins. Our only access to the coins is by choosing a coin and then flipping it, without knowing the true biases of the coins. An algorithm in this setting may employ arbitrary adaptivity—for example, flipping three different coins in sequence and then flipping the first coin 5 more times if and only if the results of the first 3 flips were heads, tails, heads. The challenge is to estimate the fraction $\rho$ of coins that are of positive type, to within a given additive error $\epsilon$, using as few coin flips (samples) as possible. We assume because of the symmetry of the problem (between positive and negative coins) that $\rho \leq \frac{1}{2}$.

This model was inspired by a practical problem that has been studied by other authors in the context of data science. Given a set of data and a predicate on the data, the task is to estimate what fraction of the data satisfies the predicate. One emerging tool that potentially offers sophisticated capabilities but at the cost of unreliability is crowdsourcing. Namely, for each data item, ask many users/workers online whether they think the item satisfies the predicate. In the case that the workers have no ability to distinguish the predicate, we cannot hope to succeed; however, if the histograms of detection probabilities $p(x)$ differ significantly on the positive and negative items
then the challenge is to estimate $\rho$ as accurately as possible from a limited budget of queries to workers \[12\].

A key feature that makes this estimation problem distinct from many others studied in the literature is the richness of adaptivity available to the algorithm. Much of the previous work in the area of statistical estimation is focused on non-adaptive algorithms and lower bounds; however see [5], in particular, Sections 4.1 and 4.2 for a survey of several distribution testing models that allow for adaptivity. In our setting there are two distinct kinds of adaptivity that an algorithm can leverage: 1) adaptivity for a single coin, deciding how many times a particular coin should be flipped in terms of the results of its previous flips, and 2) adaptivity between coins, deciding which coin to flip next in terms of the results of previous flips across all coins. Our results demonstrate, perhaps surprisingly, that only adaptivity of the first kind is useful: 1) our proposed algorithm (Section 3) crucially leverages adaptivity of the first kind, but makes use of no adaptivity of the second kind, and furthermore, 2) our almost-tight lower bound analysis (Section 4) gives a reduction (Section 4.1) from fully-adaptive algorithms that leverage both kinds of adaptivity to “single-coin adaptive” algorithms that process each coin independently.

The main algorithmic challenge in this problem is our uncertainty about the uncertainty: we make no assumptions about the quality of the coins beyond the existence of a gap $2\Delta$ between biases of the coins of different types (centered at $\frac{1}{2}$). If we relaxed the problem, and assumed (perhaps unrealistically) that we know 1) the conditional distribution of biases of positive coins, and 2) the same for negative coins, and 3) some guess of the mixture parameter $\rho$ between the two distributions, then we show that it is easy to construct—using mathematical programming techniques in Section 5.1—an estimation algorithm with sample complexity that is optimal by construction up to a multiplicative constant (see Section 5.2). On the other hand, our algorithm for the original setting has to return estimates with small bias, and be sample efficient at the same time, regardless of the bias of the coins, be they all deterministic, or all maximally noisy as allowed by the $\Delta$ parameter, or some quality in between. While intuitively, information theoretically, the hardest settings to distinguish involve coins with biases as close to each other as possible (and indeed our lower bound relies on mixtures of only $\frac{1}{2} \pm \Delta$ coins), settings with biases near but not equal to $\frac{1}{2} \pm \Delta$ introduce “uncertainty about uncertainty” challenges. While our upper bounds show that we cover all possibilities with almost-optimal sample complexity, we suspect and conjecture that this uncertainty in fact does not increase the asymptotic sample complexity of our estimation task, and instead requires algorithmic developments.

For our lower bounds, the main difficulty lies in characterizing the potential benefits of the two kinds of adaptivity mentioned above: adaptively choosing how many samples to draw from a single coin based on results of its own prior flips; and adaptively switching between coins based on the prior flips across all coins. The bulk of the analysis in the paper focuses on bounding the efficiency of single-coin adaptive algorithms. Intuitively, for the small-$\rho$ case, if an algorithm discovers a (rare) coin with probability $\geq \frac{1}{2} + \Delta$, this gives significantly more information about the fraction $\rho$ of such coins than if the algorithm discovers a common coin with probability $\leq \frac{1}{2} + \Delta$. Thus the algorithmic challenge for the single-coin regime is essentially to “give up quickly” when given a (common) $\leq \frac{1}{2} - \Delta$ coin, while investing samples into exploring and confirming the (rare) $\geq \frac{1}{2} + \Delta$ coins. Namely, among the space of all adaptive “stopping rules”, we analyze in Section 4.2 the trade off between 1) the expected number of samples taken, and 2) the posterior probability of sampling a $\geq \frac{1}{2} + \Delta$ coin, to prove a lower bound showing that our “triangular walk” stopping rule
is essentially optimal.

Finally, we analyze the much larger space of fully-adaptive algorithms, and derive in Section 4.1 a general inequality showing that the information gained by any fully-adaptive algorithm (measured via the square of the Hellinger distance between the views of the algorithm in the cases of the two hypotheses we aim to distinguish) can be upper bounded by a sum over all the coins of single-coin algorithms run independently on each coin. This decomposition preserves expected sample complexity, and thus, we show that the effectiveness of any algorithm—when measured via the ratio of squared Hellinger distance per expected sample taken—is subadditive when adaptively combining information across many coins. Through this proof technique, we show that our above characterization of single-coin performance in fact bounds all fully-adaptive algorithms too.

1.1 Our Approaches and Results

To motivate the new algorithms of this paper, we start by describing the straightforward analysis of perhaps the most natural subsampling approach to the problem.

Recall that it takes $\Omega(\frac{1}{\Delta^2})$ samples to distinguish a coin of bias $\frac{1}{2} - \Delta$ from a coin of bias $\frac{1}{2} + \Delta$. We can therefore imagine an algorithm that chooses a random subset of the coins, and flips each coin $\Omega(\frac{1}{\Delta^2})$ many times. Asking for $\Theta(\frac{1}{\Delta^2} \log \frac{1}{\epsilon})$ flips from each coin guarantees that all but $\epsilon$ fraction of the coins in the subset will be accurately classified. Given an accurate classification of $m$ randomly chosen coins, we use the fraction of these that appear positive as an estimate on the overall mixture parameter $\rho$. To estimate $\rho$ to within error $\epsilon$, with probability at least $1 - \delta$, the median-of-means method requires $m = O(\rho \log \frac{1}{\epsilon})$ randomly chosen coins. Overall, taking $\Theta(\frac{1}{\Delta^2} \log \frac{1}{\epsilon})$ samples from each of $m = \Theta(\frac{\rho}{\epsilon^2} \log \frac{1}{\epsilon})$ coins uses $\Theta(\frac{\rho}{\epsilon^2 \Delta^2} \log \frac{1}{\epsilon^2} \log \frac{1}{\delta})$ samples.

The above straightforward algorithm is wasteful in samples, since it makes $\Theta(\frac{1}{\Delta^2} \log \frac{1}{\epsilon})$ flips for every single coin. If $\rho$ is small, then most of the coin flips will be “wasted” on the negative coins (the common coins, with probabilities $\leq \frac{1}{2} - \Delta$), even though after $\Theta(\frac{1}{\Delta^2})$ flips of a single coin we have already gained some information as to whether the coin is positive or negative. This suggests the need for an algorithm that is at least single-coin adaptive, as introduced earlier at the beginning of the introduction.

Consider on the other hand a “wishful thinking” algorithm, where for negative coins, we somehow notice that the coin is negative and do not invest in coin flips for this coin, while we invest a substantial number of coin flips only for positive coins. The main algorithm of this paper imitates and almost achieves such idealistic behavior.

We view single-coin adaptive algorithms generically as a random walk on the Pascal triangle with specified stopping conditions (explained in more detail in Section 2), and, using techniques from random walk theory, we design a particular stopping rule such that: 1) for a negative coin, the algorithm will terminate in expectation within $O(\frac{1}{\epsilon^2 \Delta^2})$ coin flips and 2) for a positive coin, the algorithm will terminate within $O(\frac{1}{\Delta})$ coin flips with constant probability, and terminate after $O(\frac{1}{\Delta^2} \log \frac{1}{\epsilon})$ coin flips with complementary constant probability. We design a linear estimator based on this behavior, and using the median-of-means method to estimate the expectation, our algorithm provides us our main upper bound, Theorem 1, in Section 3.

**Theorem 1.** There is an algorithm that estimates $\rho$ to within an additive error of $\epsilon$ with probability at least $1 - \delta$, with an expected sample complexity of $O(\frac{\rho}{\epsilon^2 \Delta^2} (1 + \rho \log \frac{1}{\epsilon}) \log \frac{1}{\delta})$.

Complementary to our algorithm, we show a lower bound of $\Omega(\frac{\rho}{\epsilon^2 \Delta^2})$ samples for a success probability of $2/3$ for the problem. This lower bound matches our upper bound in most parameter regimes, except when $\rho = \omega(1/\log \frac{1}{\epsilon})$. To show the lower bound, we use the following setup: consider a scenario where all positive coins have bias exactly $\frac{1}{2} + \Delta$ and all negative coins have
bias exactly $\frac{1}{2} - \Delta$. We show that no fully-adaptive algorithm can distinguish the following two scenarios with probability at least 2/3, using $o(\frac{\epsilon}{\epsilon^2 \Delta^2})$ samples: 1) when a $\rho$ fraction of the coins are positive, and 2) when a $\rho + \epsilon$ fraction of the coins are positive. This is formalized as the following theorem (Theorem 2), and proved in Section 4.

**Theorem 2.** For $\rho \in [0, \frac{1}{2})$ and $\epsilon \in (0, 1 - 2\rho]$, the following two situations are impossible to distinguish with at least $\frac{2}{3}$ probability in $o(\frac{\rho \epsilon^2}{\epsilon^2 \Delta^2})$ samples: A) $\rho$ fraction of the coins have probability $\frac{1}{2} + \Delta$ of landing heads and $1 - \rho$ fraction of the coins have probability $\frac{1}{2} - \Delta$ of landing heads, versus B) $\rho + \epsilon$ fraction of the coins have probability $\frac{1}{2} + \Delta$ of landing heads and $1 - (\rho + \epsilon)$ fraction of the coins have probability $\frac{1}{2} - \Delta$ of landing heads. This impossibility crucially includes fully-adaptive algorithms.

By contrast with the above results that analyze the “uncertainty about uncertainty” regime with unknown populations of coins, we shed light on this phenomenon by providing a tight analysis in the case where knowledge about the populations of coins can be leveraged by the algorithm. Explicitly, consider the regime where we know 1) the conditional distribution of the biases of positive coins, 2) the conditional distribution of the biases of negative coins and 3) a reasonable guess of $\rho$. In Section 5.1 we use quadratic and linear programming techniques to find a single-coin adaptive algorithm that gives a linear estimator with minimum variance possible given the information. We further show in Section 5.2 that such linear estimator is in fact optimal in sample complexity, up to constant multiplicative factors, in the regime of constant probability success, culminating in the following theorem (Theorem 3).

**Theorem 3.** The linear estimator produced from solving the linear program in Figure 3, as described in Section 5.1, has (total) expected sample complexity that is within a constant factor of the optimal single-coin adaptive algorithm (with $\geq \frac{2}{3}$ probability of success) subject to the same maximum depth constraint.

Finally, motivated by practical applications, we consider the setting where we repeatedly run single-coin adaptive algorithms under a fixed budget $T$ of total coin flips, and must return an estimate of the mixture ratio $\rho$ within this budget. The issue is that, cutting off execution when our budget is met might introduce bias, as the adaptive execution of the current coin will typically not be independent of the “type” of the current coin, and thus adaptive sampling of positive versus negative coins may be cut off with rather different probabilities, leading to bias in the overall estimator. We show how to analyze and address this in Section 6.

### 1.2 Related Work

Most related to our problem is a line of work concerning the learning of distributions of (e.g. coin) parameters over a population, which arises in various scientific domains [18, 19, 20, 21, 13, 3]. In particular, the works of Lord [18], and Kong et al. [26, 31] consider a model similar to ours, with the crucial difference that each coin is sampled a fixed number $t$ many times—instead of allowing adaptive sampling as in the current work—with the objective of learning the distribution of biases of the coins in the universe.

Our problem also sits in the context of estimation and learning tasks with noisy or uncalibrated queries. The noiseless version of our problem would be when $\Delta = \frac{1}{2}$ and thus $\frac{1}{2} \pm \Delta$ equals either 0 or 1. That is, all coins are either deterministically heads or deterministically tails, and thus estimating the mixture parameter $\rho$ is equivalent to estimating the parameter of a single coin with bias $\rho$, which has a standard analysis. Prior works have considered noisy versions of well-studied computational problems, such as (approximate) sorting and maximum selection under noisy access.
to pairwise comparisons \[16, 14\] and maximum selection under access to uncalibrated numerical scores that are consistent with some global ranking \[32\].

While many models in distributional testing, estimation and learning use inherently non-adaptive sampling, there are also access models that allows for more sophisticated adaptive sampling techniques. For example, in testing contexts, conditional sampling oracles have been considered \[10, 8, 15, 1\], where a subset of the domain is given as input to the oracle, which in turn outputs a sample from the underlying unknown distribution conditioned on the subset. Evaluation oracles have also been considered \[22, 2, 17, 7\], where the testing algorithm has access to an oracle that evaluates the probability mass function or the cumulative mass function of the underlying distribution. See the survey by Canonne \[5\] for detailed comparisons between the different standard and specialized access models, along with a discussion of recent results.

Adaptive lower bounds of problems related to testing monotonicity of high-dimensional Boolean functions have a somewhat similar setup to ours, where binary decisions adaptively descend a decision tree according to probabilities that depend both on the algorithm and its (unknown) input that it seeks to categorize \[4, 11\]. Lower bounds in these works rely on showing that the probabilities of reaching any leaf in the decision tree under the two scenarios that they seek to distinguish are either exponentially small or within a constant factor of each other. This proof technique does not work in our setting, as certain outcomes of the adaptive process can yield significant insight into which of the two scenarios we are in; by contrast, in our work we show that if such leaves are reached with high probability, then we must pay a correspondingly high sample complexity cost somewhere else in the adaptive tree.

As described at the beginning of the introduction, results of this work have practical applications in crowdsourcing algorithms in the context of data science and beyond. Related theoretical studies have been undertaken on handling potentially noisy answers from crowdsourced workers due to lack in expertise \[25, 24\] (including this work), as well as incentivizing workers to answer truthfully \[23\]. Our work also addresses directly the practical problem proposed by Chung et al. \[12\], to issue queries to potentially unreliable crowdsourced workers in order to estimate the fraction of records containing “wrong” data within a database; here adaptive queries are a natural capability of the model.

## 2 Algorithmic Framework

We will describe the algorithms in this paper in a particular form, that we may refer to as a “triangular walk” with output coefficients. In Sections \[1\] and \[5.2\] we show that our algorithms are in fact close to optimal.

The algorithms we consider are of the form: 1) independently sample a set of coins from the universe; 2) for each coin, keep flipping the coin, deciding at each point only in terms of the previous number of heads and tails for that single coin whether to ask for another flip; 3) output some statistic for the coin that is a function of its observed number of heads and tails; and 4) aggregate the per-coin statistics into an estimate for \(\rho\).

The most general form of an adaptive algorithm for Step 2) can be modelled as a random walk on the structure of the Pascal Triangle, in which the states are represented by pairs \((n, k)\), where \(n\) is the total number of flips on the coin so far, and \(k \leq n\) is the number of “heads” responses. At each state \((n, k)\), the algorithm terminates with probability \(\gamma_{n,k}\) (the set \(\{\gamma_{n,k}\}\) we call a stopping rule), or requests a coin flip and continues the walk. This is a slight generalization of Algorithm \[1\] that we ultimately adopt. The statistic produced for the coin would be the path from the \((0,0)\) state to its terminating state. However, conditioned on a stopping rule and a terminating state, it is
possible to show that the distribution over paths that ends at the terminating state is independent of the coin’s bias. This shows that we can just use the terminating state as the statistic, instead of the entire path of the random walk.

After processing all the individual coins, the algorithm now has the number of coins that terminated at each state \((n, k)\). There are still a multitude of methods to aggregate such information into one single estimate for the mixture parameter \(\rho\). On one extreme, one can attempt to fit this empirical histogram to some model before using the model to produce an estimate, such as in the work of Valiant and Valiant [27, 29, 30]. However, in this scenario, the terminating state for a coin’s random walk only tells us the bias of the coin, instead of the mixture parameter that we want to estimate. A model fit would essentially give us the same (not useful) kind of information. We therefore use linear estimators instead, a simple and natural estimator form that is shown to be optimal in certain classes of estimation tasks [28]. That is, for each state \((n, k)\) there is an output coefficient \(v_{n,k}\), and we simply take the average of these output coefficients over the sampled coins.

We give this final algorithmic form as Algorithm 1 which we call the Triangular Walk, parameterized by its stopping rule \(\{\gamma_{n,k}\}\) and its output coefficients \(\{v_{n,k}\}\).

\textbf{Algorithm 1 Triangular Walk}

1. Input: a coin of bias \(p\)
2. Initialize state \((n, k)\) to \((0, 0)\).
3. Repeat until termination:
   (a) With probability \(\gamma_{n,k}\), terminate and output coefficient \(v_{n,k}\).
   (b) Otherwise, sample one more coin flip. Increment \(n\), and increment \(k\) by the result of the flip (0 or 1).

To limit the depth of the triangle to a maximum of \(n_{\text{max}}\), one can set \(\gamma_{n_{\text{max}},k}\) to be 1 for all \(k\), which is a strategy we use in the algorithms we present in this work for controlling the sample complexity and for error bounding. This restriction is mild: with a sufficiently large \(n_{\text{max}}\), by concentration guarantees a finite depth triangle can always well-approximate the performance of an infinite-depth triangle.

Before we move onto the next sections describing our instantiations of the triangular walk, we make an important structural observation on the distribution induced by these walks. Given a stopping rule and a coin with bias \(p\), the distribution over states \((n, k)\) that the triangular walk terminates at can be easily characterized by the form \(\alpha_{n,k}p^{k}(1-p)^{n-k}\) (noting that \(\alpha_{n,k}\) is independent of \(p\)), where \(\{\alpha_{n,k}\}\) satisfy the following system of recurrence relations that relates them to \(\{\beta_{n,k}\}\), which have the semantics that \(\beta_{n,k}p^{k}(1-p)^{n-k}\) is the probability that the walk visits the state \((n, k)\) but not necessarily terminating at it:

\[
\begin{align*}
\beta_{0,0} & = 1 \\
\beta_{n+1,k+1} & = \beta_{n,k+1}(1-\gamma_{n,k+1}) + \beta_{n,k}(1-\gamma_{n,k}) \\
\alpha_{n,k} & = \beta_{n,k}\gamma_{n,k}
\end{align*}
\] (1)

This formulation of triangular walks will be useful in both the derivation and analysis of instantiations that we present later in the paper, as well as for the lower bound analyses. For ease of notation, we also define \(\eta_{n,k} = \beta_{n,k} - \alpha_{n,k} = \beta_{n,k}(1-\gamma_{n,k})\), which is interpreted as the coefficient for the probability the walk visits and then leaves the state \((n, k)\).
3 Algorithm for Unknown Coins

In this section, we present and analyze our main algorithm. We first present an algorithm intended for the regime of constant $\Delta$, resulting in Proposition [5], and then augment this with a standard majority-voting technique to yield our main result, Theorem [1], which upper bounds the sample complexity by $O\left(\frac{\rho}{\Delta^2} (1 + \rho \log \frac{1}{\epsilon}) \log \frac{1}{\delta}\right)$ and has optimal dependence on $\Delta$.

In the next section (Section [4]), we show that the sample complexity achieved in Theorem [1] is almost tight. In particular, we show a lower bound of $\Omega(\frac{\rho}{\Delta^2})$ in the regime where the error probability $\delta$ is constant. The upper and lower bounds are matching as long as $\rho = O(1/ \log \frac{1}{\epsilon})$.

We describe our algorithm (Algorithm 2), now, as a particular stopping rule on a triangular walk (an instantiation of Algorithm [1]), along with coefficients to be combined to produce the final output. For a given coin, our algorithm stops if and only if it reaches the “negative” half of the triangle or if it reaches a maximum depth.

Algorithm 2. Instantiate Algorithm [1] setting the stopping probabilities to be: 1) $\gamma_{n,k} = 1$ for $n \geq 1$ and $2k \leq n$; 2) $\gamma_{n,k} = 1$ again for $n = n_{\text{max}}$ where $n_{\text{max}}$ is the bound we give in Lemma [6] and 3) $\gamma_{n,k} = 0$ otherwise. As for the output coefficients, we set $v_{n,k}$ to be 0 for all $n < n_{\text{max}}$, and for $n = n_{\text{max}}$ to be $v_{n_{\text{max}},k} = \min(1/\Delta, 1/(2k_{\text{max}} - 1))$.

The analysis of this algorithm consists of two parts. Using random walk techniques, we show that for sufficiently large $n_{\text{max}}$, under our choice of $\gamma_{n,k}$ and $v_{n,k}$, the triangular walk algorithm can be abstracted (Lemma [5]) to a simple random process (Process [4]). Furthermore, we show that the simple random process is an estimator with small bias and good concentration guarantees (Lemma [5]). We introduce the abstract process now.

Process 4 (A simple abstraction to our triangular walk algorithm).

1. Randomly choose a coin from the universe. Let the bias of this coin be $p$.

2. If $p \leq \frac{1}{2} - \Delta$, output a random variable $X_0$ such that $E[X_0] \in [\pm \epsilon/2]$, $\text{Var}[X_0] \leq \epsilon^3$ and $|X_0| \leq \frac{1}{\Delta}$.

3. If $p \geq \frac{1}{2} + \Delta$, output a random variable $X_1$ such that $E[X_1] \in [1 \pm \epsilon/2]$, $\text{Var}[X_1] \leq O(\frac{1}{\Delta})$ and $|X_1| \leq \frac{1}{\Delta}$.

Across the random choice of coin in Step 1, the output of Process [4] has an expectation in $[p \pm \frac{\epsilon}{2}]$ by construction. We show its concentration in the following lemma.

Lemma 5. To approximate $\rho$ to within additive error $\epsilon$ with probability at least $1 - \delta$, it suffices to take the median-of-means of $O(\log \frac{1}{\delta})$ groups of $m = O(\frac{\rho}{\Delta^2})$ instances of Process [4].

Proof. Process [4] is the mixture $\rho X_1 + (1 - \rho)X_0$. Its expectation is in $[p \pm \epsilon/2]$, and therefore we just need an estimate of its mean to an additive error of $\epsilon/2$. Since we estimate the expectation of Process [4] by taking the median-of-means, it suffices to bound the variance of $\rho X_1 + (1 - \rho)X_0$ by $O(\frac{\rho}{\Delta})$.

$$\text{Var}[\rho X_1 + (1 - \rho)X_0] = \rho \text{Var}[X_1] + (1 - \rho) \text{Var}[X_0] + \sum_{i \sim \text{Bernoulli}(\rho)} \text{Var} [E[X_i]]$$

$$\leq O\left(\frac{\rho}{\Delta}\right) + (1 - \rho) \epsilon^3 + \rho (E[X_1])^2 + (1 - \rho) (E[X_0])^2$$

$$\leq O\left(\frac{\rho}{\Delta}\right) + \epsilon^3 + O(\rho) + O(\epsilon^2)$$
\[ \leq O\left(\frac{\rho}{\delta}\right) + O(\epsilon^2) \]

from which the stated sample size bound follows. \hfill \square

The following lemma shows that our triangular walk is indeed an instance of Process \ref{process}.  

**Lemma 6.** If the maximum depth of the triangular walk satisfies \( n_{\text{max}} \geq c \cdot \frac{1}{\Delta^2} \log \frac{1}{\delta} \) for some universal constant \( c \), then the following statements hold.

1. Given an arbitrary negative coin (which has a bias \( \leq \frac{1}{2} - \Delta \)), the output of the walk has expectation in \( \pm \frac{\epsilon}{2} \) and variance upper bounded by \( \epsilon^3 \).

2. Given an arbitrary positive coin (that is \( \geq \frac{1}{2} + \Delta \)), the output of the walk has expectation in \( \pm \frac{\epsilon}{2} \) and variance upper bounded by \( \frac{1}{\Delta} \).

Furthermore, by construction, the triangular walk has output absolutely bounded by \( \frac{1}{\Delta} \).

We defer the proof of Lemma 6 to the appendix. The main idea is to use random walk techniques to calculate \( \alpha_{n,k} \) for the triangular walk; from this the results follow from applications of Chernoff bounds. The walk only stops at locations that satisfy \( n = n_{\text{max}} \) and \( k > n_{\text{max}} \), as well as for \( n \leq n_{\text{max}} \) that are even numbers and \( k = n \), and thus these are the coefficients of interest. In order to calculate these coefficients, we use the following two standard facts on 1-D random walks.

**Fact 7** (The Ballot Theorem). Consider a 1-D walk that starts at the origin, and moves one step in either the positive or negative direction at each time. The number of paths from the origin that end at \( v \) at time \( n_{\text{max}} \), which do not revisit the origin, is a \( \frac{|v|}{n_{\text{max}}} \) fraction of the total number of paths from the origin to \( v \) at time \( n_{\text{max}} \).

We see immediately that \( \alpha_{n_{\text{max}},k} = 2k - n_{\text{max}} \binom{n_{\text{max}}}{k} \) for \( k > n_{\text{max}} \), by applying a linear transformation to the triangular walk so that it becomes the 1-D walk in the Ballot Theorem.

**Fact 8.** Consider a 1-D walk that starts at the origin, and moves one step in either the positive or negative direction at each time. The number of paths from the origin that revisit the origin for the first time at \( 2n \), is a \( \frac{1}{2n-1} \) fraction of the number of paths from the origin at time 0 that return to the origin at time \( n \) (and also possibly at other intermediate times).

From Fact 8 we see that \( \alpha_{2n,2n} = \frac{1}{2n-1} \binom{2n}{n} \) by applying the same linear transformation to our triangular walk so that it becomes the 1-D random walk.

With Lemma 6 we are equipped to state and prove the correctness and sample complexity result for the triangular walk algorithm.

**Proposition 9.** Randomly choosing \( m \) coins and taking the median-of-means of their triangular walk estimates will return an estimate for \( \rho \) to within an additive error of \( \epsilon \) with probability at least \( 1 - \delta \). Furthermore, the expected sample complexity is \( O\left(m\left(\frac{1}{\Delta} + \rho n_{\text{max}}\right)\right) = O\left(\frac{\rho}{\Delta^2} \log \frac{1}{\delta}\right) \), where \( m \) and \( n_{\text{max}} \) are taken as the bounds in Lemmas 5 and 6 respectively.

The accuracy guarantees follow directly from Lemmas 5 and 6. As for the expected sample complexity, 1) using Chernoff bounds we can bound the expected sample complexity of any negative coin to be \( O(1/\Delta) \) and 2) positive coins always have sample complexity bounded by \( n_{\text{max}} \), thus giving the stated result. We again defer the actual calculations to the appendix.

As remarked above, this proposition is only intended for the constant \( \Delta \) regime and does not give an optimal dependence on the \( \Delta \) parameter. In order to yield the correct \( \Delta \) dependence, we
adapt the algorithm in the following way: instantiate our triangular walk algorithm (Algorithm 2) with $\Delta$ being a sufficiently small constant (say $\Delta = \frac{1}{4}$), and for each coin flip the triangular walk asks for, emulate it with the majority vote of $O(1/\Delta^2)$ actual coin flips from the universe; majority vote boosts the quality of a $\frac{1}{2} \pm \Delta$ coin flip to now have a constant (say $\frac{1}{4}$) separation from $\frac{1}{2}$. This gives our main algorithmic theorem.

**Theorem** 1. There is an algorithm that estimates $\rho$ to within an additive error of $\epsilon$ with probability at least $1 - \delta$, with an expected sample complexity of $O((\rho^2 \epsilon^2 \Delta^2 (1 + \frac{1}{\rho} \log \frac{1}{\delta} \log \frac{1}{\Delta})).$

### 4 Fully-adaptive lower bounds

We show in this section that the triangular walk algorithm, augmented with majority voting, is in fact close to optimal, even when compared to all fully-adaptive algorithms that are adaptive across different coins. In particular, we show the following theorem (Theorem 2).

**Theorem** 2. For $\rho \in [0, \frac{1}{2})$ and $\epsilon \in (0, 1 - 2\rho]$, the following two situations are impossible to distinguish with at least $\frac{2}{3}$ probability in $o(\rho^2 \epsilon^2 \Delta^2)$ samples:  

A) $\rho$ fraction of the coins have probability $\frac{1}{2} + \Delta$ of landing heads and $1 - \rho$ fraction of the coins have probability $\frac{1}{2} - \Delta$ of landing heads, versus  

B) $\rho + \epsilon$ fraction of the coins have probability $\frac{1}{2} + \Delta$ of landing heads and $1 - (\rho + \epsilon)$ fraction of the coins have probability $\frac{1}{2} - \Delta$ of landing heads. This impossibility crucially includes fully-adaptive algorithms.

With the algorithmic result in Theorem 1 this lower bound is therefore tight as long as $\rho = O(1/\log \frac{1}{\epsilon})$. We note that the restrictions $\rho < \frac{1}{2}$ and $\epsilon \leq 1 - 2\rho$ reflect the symmetry of the problem, where the pair $\rho, \rho + \epsilon$ is exactly as hard to distinguish as the pair $1 - \rho - \epsilon, 1 - \rho$, yielding analogous results for the symmetric parameter regime.

The phrase “$o(\rho^2 \epsilon^2 \Delta^2)$ samples” in the theorem can be interpreted either as a deterministic bound or as an expected bound on the number of samples. We show the impossibility for the deterministic bound, but a straightforward Markov inequality argument extends this to the expected case.

The $\Omega(\rho^2 \epsilon^2 \Delta^2)$ lower bound requires significant analysis, forming the bulk of the remainder of this paper, but two special cases have direct proofs. When $\Delta = \Theta(1)$ we can prove a $\Omega(\frac{\rho^2 \epsilon^2}{\Delta^2})$ lower bound without the $\Delta$ dependence: consider the case where all coins are unbiased and perfect, meaning that the only source of randomness is from the mixture of coins, which is itself a Bernoulli distribution of bias either $\rho$ or $\rho + \epsilon$. We quote the standard fact that, in order to estimate a Bernoulli coin flip of bias $\rho$ to up to additive $\epsilon$, we need $\Omega(\frac{\rho^2}{\epsilon^2})$ samples to succeed with constant probability; this can be proven by a standard (squared) Hellinger distance argument. On the other hand, it is also straightforward to prove a $\frac{1}{\Delta^2}$ lower bound (covering the regime where $\rho$ and $\epsilon$ are constant): consider easiest regime for $\rho$ and $\epsilon$, where $\rho = 0$ and $\epsilon = 1$; thus coins either all have $\frac{1}{2} + \Delta$ bias or all have $\frac{1}{2} - \Delta$ bias. To distinguish whether we have access to positive coins or negative coins requires $\Omega(\frac{1}{\Delta^2})$ samples.

In order to show the indistinguishability result, we use the notion of Hellinger distance between probability distributions.

**Definition 10** (Hellinger Distance). Given two discrete distributions $P$ and $Q$, the Hellinger distance $H(P,Q)$ between them is

$$
\frac{1}{\sqrt{2}} \sqrt{\sum_i (\sqrt{p_i} - \sqrt{q_i})^2} = \sqrt{1 - \sum_i \sqrt{p_i}q_i}
$$
There are two properties of the Hellinger distance that are particularly relevant to proving lower bounds. First, the Hellinger distance upper bounds $\ell_1$ distance except for a multiplicative factor:

$$\ell_1(P, Q) \leq \sqrt{2} H(P, Q)$$

Thus a sufficiently small upper bound to Hellinger distance gives an indistinguishability result. Second, for a pair of product distributions $P_1 \otimes P_2$ and $Q_1 \otimes Q_2$, we have that the squared Hellinger distance is subadditive:

$$H^2(P_1 \otimes P_2, Q_1 \otimes Q_2) \leq H^2(P_1, Q_1) + H^2(P_2, Q_2)$$

which extends inductively to products of any finite number of distributions.

The main challenge in proving a general lower bound for our problem, is that there are two kinds of adaptivity that algorithms may employ that were both absent in the special cases of the previous paragraph. Explicitly, when taking samples from a given coin, we can choose whether to ask for another sample based on A) previous results of this coin, and also B) previous results of the other coins. This first kind of adaptivity, “single-coin adaptivity”, is crucially used in the algorithms presented in the rest of the paper (e.g., the “shape” of the stopping rule for our triangular-walk algorithms); in Proposition 12 we analyze the best possible performance of such triangular stopping rules. The main part of the proof of Theorem 2 consists of showing, in general, that the second kind of adaptivity (“B”, in this paragraph) cannot help.

4.1 Reduction to Single-Coin Adaptive Algorithms

In the next subsection, we show Corollary 11, which upper bounds the squared Hellinger distance for single-coin adaptive algorithms by a quantity that is proportional to its sample complexity. Using the corollary, we now complete the proof of the indistinguishability result (Theorem 2) for fully-adaptive algorithms. The bulk of the proof consists of showing that adaptivity between coins does not help.

Corollary 11. Consider an arbitrary single-coin adaptive algorithm. Let $H^2$ be the squared Hellinger distance between a single run of the algorithm where 1) a coin with bias $\frac{1}{2} + \Delta$ is used with probability $\rho$ and a coin with bias $\frac{1}{2} - \Delta$ is used otherwise, versus a run of the algorithm where 2) a coin with bias $\frac{1}{2} + \Delta$ is used with probability $\rho + \epsilon$ and a coin with bias $\frac{1}{2} - \Delta$ is used otherwise. Furthermore, let $E_{\rho+\frac{\epsilon}{2}}[n]$ be the expected number of coin flips on this random walk, where we use a $\frac{1}{2} + \Delta$ coin with probability $\rho + \frac{\epsilon}{2}$ (instead of $\rho$ or $\rho + \epsilon$), and a $\frac{1}{2} - \Delta$ coin otherwise. If all of $\rho$, $\epsilon$, $\Delta$, and $\epsilon/\rho$ are smaller than some universal absolute constant, then

$$\frac{H^2}{E_{\rho+\frac{\epsilon}{2}}[n]} = O\left(\frac{\epsilon^2 \Delta^2}{\rho}\right)$$

Proof of Theorem 2. We leverage the upper bound on Hellinger distance for single-coin adaptive algorithms to in turn upper bound Hellinger distance for fully-adaptive algorithms, thereby proving our indistinguishability result. We consider only deterministic adaptive algorithms without loss of generality, by standard derandomization/averaging arguments.

A deterministic fully-adaptive algorithm is a decision tree, where each node is labelled by the identity of a coin, and each edge out of a node is labelled by a heads or tails result for the coin at the node. Here, we are considering decision trees of height $O(\frac{1}{\epsilon^2 \Delta^2})$. We can view a run of the algorithm as follows: 1) first draw $O(\frac{1}{\epsilon^2 \Delta^2})$ random coins that are either $(\frac{1}{2} + \Delta)$-biased or $(\frac{1}{2} - \Delta)$-biased, and then 2) flip these coins according to this fully-adaptive algorithm. After step
1, fixing whether each coin has bias $\frac{1}{2} + \Delta$ or $\frac{1}{2} - \Delta$, the probability of ending up at the $i^{th}$ leaf of the decision tree is simply the probability that every edge along the path from the root to that leaf is followed. Note that each edge is a probabilistic event depending on only one coin. Therefore, this probability can be factored into a product of probabilities, one term for each of the $O(\frac{1}{\epsilon^2})$ different coins. Explicitly, if we use a $\frac{1}{2} + \Delta$ coin for coin $j$, we denote its term in the probability product (for leaf $i$) by $q_{j,i}^+$, and similarly we have the notation $q_{j,i}^-$ for $\frac{1}{2} - \Delta$ coins. As an example, if for every coin, we used a $\frac{1}{2} + \Delta$ coin with probability $\rho$ and a $\frac{1}{2} - \Delta$ coin otherwise, the probability of reaching the $i^{th}$ leaf would be $\prod_{j\text{ coin}} (\rho q_{j,i}^+ + (1 - \rho)q_{j,i}^-)$. An immediate corollary is that, letting $\rho_j$ be an arbitrary vector whose $j^{th}$ entry denotes the probability that the $j^{th}$ coin has type $\frac{1}{2} + \Delta$, then summing the above product over all leaves $i$ will give a total probability of 1, since it covers all the possibilities. That is, we have

$$\sum_{\text{leaf } i} \prod_{\text{coin } j} (\rho_j q_{j,i}^+ + (1 - \rho_j)q_{j,i}^-) = 1$$

To work with the results of Corollary 11, we first derive some properties of Hellinger distance. Given two distributions $a$ and $b$, 1 minus their squared Hellinger distance can be rewritten as $\sum_i \sqrt{a_i b_i}$. For the two scenarios ($\rho$ versus $\rho + \epsilon$ mixture probability) we are considering, the $i^{th}$ leaf contributes the following to 1 minus their squared Hellinger distance:

$$\sqrt{\prod_{\text{coin } j} (\rho q_{j,i}^+ + (1 - \rho)q_{j,i}^-) \prod_{\text{coin } j} ((\rho + \epsilon)q_{j,i}^+ + (1 - \rho - \epsilon)q_{j,i}^-)}$$

For any two non-negative (and finite) sequences $\{c_{j,i}\}_j$ and $\{d_{j,i}\}_j$, we have

$$\sqrt{\prod_{\text{coin } j} c_{j,i} \prod_{\text{coin } j} d_{j,i}} = \left(\prod_{\text{coin } j} \frac{c_{j,i} + d_{j,i}}{2}\right) \left(\prod_{\text{coin } j} \frac{2\sqrt{c_{j,i} d_{j,i}}}{c_{j,i} + d_{j,i}}\right) \geq \left(\prod_{j} \frac{c_{j,i} + d_{j,i}}{2}\right) \left[1 - \sum_{j} \left(1 - \frac{2\sqrt{c_{j,i} d_{j,i}}}{c_{j,i} + d_{j,i}}\right)\right]$$

where the last inequality holds because each $\frac{2\sqrt{c_{j,i} d_{j,i}}}{c_{j,i} + d_{j,i}}$ is less than 1 by the AM-GM inequality, and therefore we can apply a union bound. We apply this inequality to the sequences $c_{j,i} = (\rho q_{j,i}^+ + (1 - \rho)q_{j,i}^-)$ and $d_{j,i} = ((\rho + \epsilon)q_{j,i}^+ + (1 - \rho - \epsilon)q_{j,i}^-)$. Note that the average of $c_{j,i}$ and $d_{j,i}$ is simply $((\rho + \frac{\epsilon}{2})q_{j,i}^+ + (1 - \rho - \frac{\epsilon}{2})q_{j,i}^-)$ which we shall call $e_{j,i}$.

Summing the inequality over all leaves $i$, on the left hand side we get 1 minus the squared Hellinger distance of the fully-adaptive algorithm we are considering, and on the right hand side we get (by simple calculations)

$$\left(\sum_{\text{leaf } i} \prod_{\text{coin } j} e_{j,i}\right) - \sum_{\text{coin } k} \left(\sum_{\text{leaf } i} \prod_{\text{coin } j} e_{j,i}\right) - \left(\sum_{\text{leaf } i} \prod_{\text{coin } j \neq k} e_{j,i}\right) \sqrt{c_{k,i} d_{k,i}}$$

Since each $e_{j,i}$ is a mixture of $q_{j,i}^+$ with $\rho + \frac{\epsilon}{2}$ probability and $q_{j,i}^-$ otherwise, it follows from Equation 2 that $\sum_{\text{leaf } i} \prod_{\text{coin } j} e_{j,i} = 1$, and therefore for the right hand side we have

$$1 - \sum_{\text{coin } k} \left(1 - \sum_{\text{leaf } i} \prod_{\text{coin } j \neq k} e_{j,i}\right) \sqrt{c_{k,i} d_{k,i}}$$

which is exactly 1 minus the sum of the following squared Hellinger distances: for each coin $k$, we run the fully-adaptive algorithm simulating all coins $j \neq k$ such that (corresponding to the $e_{j,i}$
terms) coin \( j \) is a \( \frac{1}{2} + \Delta \) coin with probability \( \rho + \frac{\epsilon}{2} \) and a \( \frac{1}{2} - \Delta \) coin otherwise. Coin \( k \) is a “real” (not simulated) coin, and therefore depending on which of the two scenarios we are in, either it is a \( \frac{1}{2} + \Delta \) coin with \( \rho \) probability or it is that with \( \rho + \epsilon \) probability.

Summarizing, 1 minus the squared Hellinger distance of the fully-adaptive algorithm is at least 1 minus the sum of squared Hellinger distances for each coin \( k \), where we simulate the other coins with a \( \rho + \frac{\epsilon}{2} \) mixture. Thus the squared Hellinger distance of the fully-adaptive algorithm is upper bounded by the sum of squared Hellinger distances of the individual coin’s simulated runs. Let us denote this inequality by

\[
H_{\text{full}}^2 \leq \sum_{\text{coin } k} H_k^2 = \sum_{\text{coin } k} \left[ 1 - \sum_{\text{leaf } i} \left( \prod_{\text{coin } j \neq k} e_{j,i} \right) \sqrt{c_{k,i} d_{k,i}} \right]
\]

where \( H_{\text{full}}^2 \) is the squared Hellinger distance of the fully-adaptive algorithm, and each \( H_k^2 \) is the squared Hellinger distance when we run the fully-adaptive algorithm for coin \( k \), simulating all other coins by a \( \rho + \frac{\epsilon}{2} \) mixture. In other words, for any fully-adaptive algorithm (intuitively seeking to maximize the discrepancy of the squared Hellinger distance between the two possibilities we seek to distinguish) we have decomposed its performance into a sum of performances of single-coin adaptive algorithms; we will now show that the sum of the sample complexities of these single-coin algorithms is essentially the same as the original fully-adaptive algorithm.

Observe that each individual coin’s simulated run is now simply an instance of a single-coin adaptive algorithm (which may now be randomized again, even if we derandomized the fully-adaptive algorithm at the beginning, since the simulated runs contain simulations of coin flips of other coins). Thus we can apply Corollary 11 to upper bound each \( H_k^2 \) by \( O\left(\frac{1}{\epsilon \Delta^2}\right) \) times the expected sample complexity of coin \( k \), where coin \( k \) is a “real” (not simulated) coin that is randomly drawn from a \( \rho + \frac{\epsilon}{2} \) mixture. Recall that the simulated coins (all coins \( j \neq k \)) are also randomly drawn from a \( \rho + \frac{\epsilon}{2} \) mixture. Therefore, across all the different runs, every coin is independently drawn from the same \( \rho + \frac{\epsilon}{2} \) mixture, whether it is simulated or not. It follows that the sum of expected sample complexities across every coin \( k \) in its own simulated run is equal to the expected sample complexity of one run of the fully-adaptive algorithm with all the coins non-simulated, where each coin \( k \) is drawn independently from a \( \rho + \frac{\epsilon}{2} \) mixture. Without loss of generality we already assumed that the fully-adaptive algorithm in consideration is a decision tree with a height bounded by \( O\left(\frac{1}{\epsilon \Delta^2}\right) \), and thus the aforementioned total sample complexity is also bounded by this quantity. Thus \( \sum_k H_k^2 = O\left(\frac{1}{\epsilon \Delta^2}\right) \cdot O\left(\frac{1}{\rho \Delta^2}\right) = O(1) \), where the quantity in the first big-O expression is the total number of samples of our fully-adaptive algorithm. If we set the multiplicative constant in this bound to be sufficiently small, then we have that the squared Hellinger distance of the fully-adaptive algorithm is smaller than a constant of our choice. Since Hellinger distance bounds total variation distance, we conclude that for any fully-adaptive algorithm, the total variation distance between its views in the two settings is smaller than a constant of our choice; and therefore one can never distinguish the different mixture scenarios with \( 2/3 \) probability with \( o\left(\frac{1}{\epsilon \Delta^2}\right) \) samples.

4.2 Upper Bounding the Squared Hellinger Distance for Single-Coin Adaptive Algorithms

In this section, we prove Corollary 11. Explicitly, we prove Proposition 12, which is a simpler-to-analyze special case. We discuss now why each of the simplifying assumptions does not give up generality.

First, given a single-coin algorithm, by standard symmetrization arguments, it can always be implemented such that decisions only depend on the number of flips for a coin as well as the
number of observed "heads"", as opposed to the explicit sequence of heads/tails observations. Thus we restrict our attention to stopping rules in the sense of Algorithm 1 (specified by stopping coefficients \{γ_{n,k}\}, but where the output \(v_{n,k}\) is irrelevant).

Second, there is in some sense a “phase change” once an algorithm has received \(\Omega(\frac{1}{\Delta^2})\) samples from a single coin, where after this point, the algorithm might have good information about whether the coin is of type \(\frac{1}{2} + \Delta\) or type \(\frac{1}{2} - \Delta\), and might productively make subtle adaptive decisions after this point. We restrict our analysis to the regime where no coin is flipped more than \(10^{-8}/\Delta^2\) times: formally, we show an impossibility result in the following stronger setting, where we assume that whenever a single coin is flipped \(10^{-8}/\Delta^2\) times, then then the coin’s true bias (either \(\frac{1}{2} + \Delta\) or \(\frac{1}{2} - \Delta\)) is revealed to the algorithm. Thus any coin flips beyond \(10^{-8}/\Delta^2\) that an algorithm desires can instead be simulated at no cost.

Formally, an impossibility result in this setting with “advice” (Proposition 12) implies the analogous result in the original setting (Corollary 11) by the data processing inequality for Hellinger distance (since Hellinger distance is an \(f\)-divergence): simulating additional coin flips in terms of “advice” is itself “data processing”, and thus can only decrease the Hellinger distance. Thus the setting without advice has smaller-or-equal Hellinger distance, and uses greater-or-equal number of samples, and hence the bound on their ratio in Proposition 12 implies the corresponding bound in Corollary 11.

Third and finally, we restrict our attention to stopping rules that only stop at rows (i.e. number of coin flips) that are powers of 2. We can justify this analogously to the previous paragraph, by noting that any stopping rule \(S\) that potentially stops at rows that are not powers of 2 could be converted into a rule \(S'\) with better performance by rounding each potential stopping row in \(S\) up to the next power of 2. This will sacrifice at most a factor of 2 in sample complexity, and can only improve our Hellinger distance (via taking more samples, by the data processing inequality, as above).

**Proposition 12.** Consider an arbitrary stopping rule \(\{γ_{n,k}\}\) that 1) is non-zero only for \(n\) that are powers of 2, and 2) \(γ_{10^{-8}/\Delta^2,k} = 1\) for all \(k\), that is the random walk always stops after \(10^{-8}/\Delta^2\) coin flips. Suppose that given a coin, after a random walk on the Pascal triangle according to the stopping rule, the position \((n,k)\) that the walked ended at is always revealed, and furthermore, if \(n = 10^{-8}/\Delta^2\), then the bias of the coin is also revealed. Let \(H^2\) be the squared Hellinger distance between a single run of the above process where 1) a coin with bias \(\frac{1}{2} + \Delta\) is used with probability \(ρ\) and a coin with bias \(\frac{1}{2} - \Delta\) is used otherwise versus 2) a coin with bias \(\frac{1}{2} + \Delta\) is used with probability \(ρ + \epsilon\) and a coin with bias \(\frac{1}{2} - \Delta\) is used otherwise. Furthermore, let \(\mathbb{E}_{\rho + \frac{\epsilon}{\rho}}[n]\) be the expected number of coin flips on this random walk, where we use a \(\frac{1}{2} + \Delta\) coin with probability \(\rho + \frac{\epsilon}{\rho}\) (instead of \(\rho\) or \(\rho + \epsilon\)), and a \(\frac{1}{2} - \Delta\) coin otherwise. If all of \(ρ, \epsilon, \Delta\) and \(\epsilon/ρ\) are smaller than some universal absolute constant, then

\[
\frac{H^2}{\mathbb{E}_{\rho + \frac{\epsilon}{\rho}}[n]} = O \left( \frac{\epsilon^2 \Delta^2}{\rho} \right)
\]

It remains to prove Proposition 12. For the rest of the section, we shall use the notation \(h_{n,k}^+ = (\frac{1}{2} + \Delta)^k(\frac{1}{2} - \Delta)^{n-k}\) and \(h_{n,k}^- = (\frac{1}{2} - \Delta)^k(\frac{1}{2} + \Delta)^{n-k}\) for convenience.

The first step in the proof is the following lemma that writes out the squared Hellinger distance induced by a given stopping rule \(\{γ_{n,k}\}\), whose proof can be found in the appendix. The two lines in the expression below capture the different forms of the Hellinger distance for stopping before the last row versus at the last row—recall that we prove impossibility under the stronger model where, upon reaching the last row the algorithm receives the true bias of the coin (as “advice”). Thus the squared Hellinger distance coefficients from elements of the last row are typically much larger than for other rows, capturing the cases when this advice is valuable.
Lemma 13. Consider the two probability distributions in Proposition 12 over locations \((n, k)\) in the Pascal triangle of depth \(10^{-8}/\Delta^2\) and bias \(p \in \{\frac{1}{2} \pm \Delta\}\), generated by the given stopping rule \(\{\gamma_{n, k}\}\) in the two cases of 1) a coin with bias \(\frac{1}{2} + \Delta\) is used with probability \(\rho\) and a coin with bias \(\frac{1}{2} - \Delta\) is used otherwise versus 2) a coin with bias \(\frac{1}{2} + \Delta\) is used with probability \(\rho + \epsilon\) and a coin with bias \(\frac{1}{2} - \Delta\) is used otherwise. If \(\epsilon/\rho\) is smaller than some universal constant, then the squared Hellinger distance between these two distributions can be written as

\[
\Theta(\epsilon^2) \left[ \sum_{n = \frac{10^{-8}}{\Delta^2} \text{ s.t. } h_{n,k}^+ < \frac{1}{\rho^2}} \alpha_{n,k}((\rho + \frac{\epsilon}{2})h_{n,k}^+ + (1 - \rho - \frac{\epsilon}{2})h_{n,k}^-) \frac{(h_{n,k}^+ - h_{n,k}^-)^2}{(\rho h_{n,k} + (1 - \rho)h_{n,k})^2} \right] \\
\sum_{n = \frac{10^{-8}}{\Delta^2} \text{ s.t. } h_{n,k}^+ < \frac{1}{\rho^2}} \alpha_{n,k}((\rho + \frac{\epsilon}{2})h_{n,k}^+ + (1 - \rho - \frac{\epsilon}{2})h_{n,k}^-) \frac{h_{n,k}^+ + h_{n,k}^-}{\rho h_{n,k} + (1 - \rho)h_{n,k}} \right]
\]

We shall upper bound the squared Hellinger distance by splitting the sum into three components, depending on their location in the Pascal triangle: 1) the last row \(n = \frac{10^{-8}}{\Delta^2}\), 2) a “high discrepancy region” where \(h_{n,k}^+/h_{n,k}^- \geq 1/\rho^{0.1}\) which is towards the right of the triangle, potentially contributing large amounts to the squared Hellinger distance and 3) a “central” region that is the rest of the triangle. Propositions 22 and 15 assert that for each of the respective regions, their contribution to the squared Hellinger distance, divided by the expected sample complexity, is at most \(O(\epsilon^2\Delta^2/\rho)\). Summing up the three terms is an upper bound on the total squared Hellinger distance per expected sample, and loses only by a factor of \(3\) and hence is still \(O(\epsilon^2\Delta^2/\rho)\), which would complete the proof of Proposition 12. We state formally and prove the three propositions in the next three sections, in the order of “central” region, “high discrepancy” region and the last row.

4.2.1 “Central” Region

For the purposes of this section, define \(b_{n,k,\rho+\frac{\epsilon}{2}}\) to equal \(((\rho + \frac{\epsilon}{2})h_{n,k}^+ + (1 - \rho - \frac{\epsilon}{2})h_{n,k}^-)\), so that \(\alpha_{n,k}b_{n,k,\rho+\frac{\epsilon}{2}}\) is the probability of reaching and stopping at location \((n, k)\) under a \(\rho + \frac{\epsilon}{2}\) mixture of the two coin types. Further, let \(R_{n,k,\rho}\) be defined to equal \(\frac{(h_{n,k}^+ - h_{n,k}^-)^2}{(\rho h_{n,k} + (1 - \rho)h_{n,k})^2}\), which is the contribution of location \((n, k)\) to the squared Hellinger distance per unit of probability mass that stops there.

By Lemma 13, the contribution to the squared Hellinger distance from the central region of the triangle is bounded by the sum, over this region, of \(\epsilon^2\alpha_{n,k}b_{n,k,\rho+\frac{\epsilon}{2}}R_{n,k,\rho}\).

Proposition 14. For an arbitrary stopping rule, the contribution of the central region to the squared Hellinger distance, divided by the (total) expected sample complexity \(E_{\rho+\frac{\epsilon}{2}}[n]\) of the walk using a \(\rho + \frac{\epsilon}{2}\) mixture of \(\frac{1}{2} \pm \Delta\) coins, is at most \(O(\epsilon^2\Delta^2/\rho)\). Explicitly, with notation for \(b\) and \(R\) defined in the previous paragraphs, we have

\[
\epsilon^2 \sum_{n = \frac{10^{-8}}{\Delta^2} \text{ s.t. } h_{n,k}^+ < \frac{1}{\rho^2}} \alpha_{n,k}b_{n,k,\rho+\frac{\epsilon}{2}}R_{n,k,\rho} = O\left(\frac{\epsilon^2\Delta^2}{\rho}\right) E_{\rho+\frac{\epsilon}{2}}[n]
\]

Proof. We upper bound this quantity here by instead 1) replacing \(R_{n,k,\rho}\) by a similar quantity \(\hat{R}_{n,k,\rho}\) that is an upper bound on \(R\) in the central region, and 2) summing over the entire triangle instead of just the central region. Let \(\hat{R}_{n,k,\rho} = 2\left(\min\left(h_{n,k}^+, \frac{1}{\rho^2}\right) - 1\right)^2\). This bounds \(R\) in the
region where \( h_{n,k}^+ / h_{n,k}^- \leq 1 / \rho^{0.1} \) since the numerator of \( R \) is at most \( \frac{1}{2} \hat{R} \), and the denominator of \( R \) is at least \( \frac{1}{2} \).

Thus we instead prove the related fact that

\[
\epsilon^2 \sum_{n \leq 10^{-8} \Delta^{-2}, k \in [0..n]} \alpha_{n,k} b_{n,k,\rho + \frac{\Delta}{2}} \hat{R}_{n,k,\rho} = O \left( \frac{\epsilon^2 \Delta^2}{\rho} \right) \mathbb{E}_{\rho + \frac{\Delta}{2}}[n] \tag{3}
\]

We prove this by induction on a row \( i \), where we define \( A_{n,k}^i \) to be the stopping probabilities (corresponding to the product \( \alpha_{n,k} b_{n,k,\rho + \frac{\Delta}{2}} \)) for the variant of the given stopping rule where we force the rule to stop at row \( i \) if it reaches this row; analogously define \( \mathbb{E}_{\rho + \frac{\Delta}{2}}^i[n] \) to be the expected number of samples taken by this rule. We consider how both the left hand side and \( \mathbb{E}_{\rho + \frac{\Delta}{2}}^i[n] \) change as we increase \( i \) by 1, and show that the ratio of their change is \( O \left( \frac{\epsilon^2 \Delta^2}{\rho} \right) \).

See Lemma 27 in the appendix for a complete proof of this fact.

\[ \square \]

As a proof sketch of the ground covered by Lemma 27, if for some location \((i,k)\) some amount of probability mass \( m \) continues down to row \( i + 1 \) instead of stopping here, then the expected number of samples increases by exactly \( m \). Meanwhile, this probability mass \( m \) will end up split between locations \((i+1,k)\) and \((i+1,k+1)\), where for a coin of bias \( p \) (that will be \( \frac{1}{2} \pm \Delta \)), we will have \( m(1-p) \) mass going left and \( mp \) mass going right, contributing to \( A_{i+1,k}^{i+1} \) and \( A_{i+1,k+1}^{i+1} \) entries respectively. The change in the left hand side of Equation 3 induced by sending mass \( m \) down to level \( i + 1 \) is thus expressed as a linear combination of 3 evaluations of the function \( \hat{R}_{n,k,\rho} \). Since \( \hat{R}_{n,k,\rho} \) is essentially a quadratic function of the ratio \( h_{n,k}^+ / h_{n,k}^- \), this linear combination evaluates to the difference between a quadratic evaluated at 1 point, versus the weighted average of the quadratic at 2 surrounding points, and is bounded by \( m \cdot O(\frac{\Delta^2}{\rho^2}) \) essentially because of the second derivative of the quadratic in the central region.

### 4.2.2 “High Discrepancy” Region

**Proposition 15.** Consider an arbitrary stopping rule \( \{\gamma_{n,k}\} \) that 1) is non-zero only for \( n \) that are powers of 2, and 2) \( \gamma_{10^{-8}/\Delta^2,k} = 1 \) for all \( k \), that is the random walk always stops after \( 10^{-8}/\Delta^2 \) coin flips. Let

\[
H_{\text{disc}}^2 = \Theta(\epsilon^2) \sum_{n < 10^{-8}/\Delta^2, k \text{ s.t. } \frac{h_{n,k}^+}{h_{n,k}^-} > \frac{1}{\rho^{0.1}}} \alpha_{n,k} \left( \left( \rho + \frac{\epsilon}{2} \right) h_{n,k}^+ + \left( 1 - \rho - \frac{\epsilon}{2} \right) h_{n,k}^- \right) \frac{(h_{n,k}^+ - h_{n,k}^-)^2}{(\rho h_{n,k}^+ + (1-\rho) h_{n,k}^-)^2}
\]

be the contribution to the squared Hellinger distance by the “high discrepancy” region. Furthermore, again let \( \mathbb{E}_{\rho + \frac{\Delta}{2}}[n] \) be the expected number of coin flips on this random walk, where we use a \( \frac{1}{2} + \Delta \) coin with probability \( \rho + \frac{\epsilon}{2} \) (instead of \( \rho \) or \( \rho + \epsilon \)), and a \( \frac{1}{2} - \Delta \) coin otherwise. If all of \( \rho, \epsilon, \Delta \) and \( \epsilon/\rho \) are smaller than some universal absolute constant, then

\[
\frac{H_{\text{disc}}^2}{\mathbb{E}_{\rho + \frac{\Delta}{2}}[n]} = O \left( \frac{\epsilon^2 \Delta^2}{\rho} \right)
\]

\[ 15 \]
The key observation for this section is that the “high discrepancy” region is in fact at least \( \Omega(\log \frac{1}{\rho}) \) standard deviations away from where a random walk on the triangle (without a stopping rule) would concentrate; and thus it is very unlikely for the random walk to enter the region. However, the existence of a stopping rule could potentially skew the distribution of the random walk on each row towards the “high discrepancy” side of the triangle, while saving on sample complexity by stopping early whenever the walk enters the other side of the triangle. In this section, we essentially show that this cannot happen.

The analysis in this section relies on our assumption that the stopping rule only stops at rows that are powers of 2 (unlike the analysis of the previous section). Intuitively, if the random walk ends up very far to the right, then there must be a single region of rows \([2^i, 2^{i+1}]\) where, without any stopping rule on intermediate rows to guide it, the walk still somehow makes unlikely progress to the right. More explicitly, if the distribution of reaching-and-not-stopping-at row \(2^{i+1}\) is skewed significantly far to the right of the distribution of reaching-and-not-stopping-at row \(2^i\) (despite the intervening process being strictly a binomially distributed random walk), then the only way this could have occurred is if an overwhelming fraction of the probability mass reaching row \(2^{i+1}\) stops there. Namely, if probability mass \(m\) emerges below row \(2^{i+1}\) and skewed far to the right, the potential Hellinger distance gains this induces will be more than counterbalanced by the huge addition to sample complexity induced by the overwhelming (relative to \(m\)) probability of stopping at row \(2^{i+1}\).

We utilize the following fact, essentially a consequence of a Binomial distribution being upper bounded by a corresponding Gaussian.

**Fact 16.** Let \(\text{Bin}(n, p, k)\) denote the probability that a Binomial distribution with \(n\) trials and bias \(p\) has value \(k\). If \(\Delta\) is sufficiently small, then there exists some absolute constant \(C\) such that for all \(n \geq 1\), and for both \(\frac{1}{2} + \Delta\) and \(\frac{1}{2} - \Delta\) substituted in the expression “\(\frac{1}{2} \pm \Delta\)” below,

\[
\sum_{k \in [0..n]} e^{\frac{(k-(\frac{1}{2} \pm \Delta)n)^2}{n}} \text{Bin}(n, \frac{1}{2} \pm \Delta, k) \leq C
\]

The sum of the pointwise products of the Binomial pmf and the inverse Gaussian can instead be re-expressed as the evaluation of a convolution between corresponding functions evaluated at a single point. We express this straightforward corollary below, and use it crucially in this section and the next.

**Fact 17.** Consider the sequences \(f_{n,k}^+(m) = e^{\frac{(k-(\frac{1}{2} + \Delta)n-m)^2}{n}}\) for \(m \in \mathbb{Z}\), and \(f_{n,k}^-(m) = e^{\frac{(k-(\frac{1}{2} - \Delta)n-m)^2}{n}}\) for \(m \in \mathbb{Z}\). Let \(\text{Bin}(n, p)\) be the pmf of the Binomial distribution with \(n\) trials and bias \(p\). If \(\Delta\) is sufficiently small, then there exists some absolute constant \(C\) such that for all \(n \geq 1\) and all \(k\),

\[
(f_{n,k}^+ * \text{Bin}(n, \frac{1}{2} + \Delta))(k) \leq C
\]

and

\[
(f_{n,k}^- * \text{Bin}(n, \frac{1}{2} - \Delta))(k) \leq C
\]

To start lower bounding the expected sample complexity of the random walk, we start with the following two lemmas stating that if there is probability \(c\) of reaching a right tail on a particular power-of-2 row, then there must be a tail on the previous power-of-2 row that the walk has high probability reaching. These are formalized as Lemma 18 and 19 for \(\frac{1}{2} + \Delta\) coins and \(\frac{1}{2} - \Delta\) coins respectively. The crux of the arguments are (weighted) averaging arguments based on Fact 17.
Lemma 18. Consider an arbitrary stopping rule \( \{\gamma_{n,k}\} \) that is non-zero only for \( n \) that are powers of 2. For a coin with bias \( \frac{1}{2} + \Delta \), suppose at row \( 2^j \) there is some position \( k \in [(\frac{1}{2} + \Delta)2^j..2^j] \) such that the total probability mass of the random walk reaching positions \( \geq k \) at row \( 2^j \) is at least \( c \). Then, there must be some position \( k' \in [0..2^{j-1}] \) at row \( 2^{j-1} \) such that the probability of reaching positions \( \geq k' \) at that row is at least \( \xi \cdot f_{2j-1,k}^+(k') = \xi \cdot e^{-\frac{(k-(\frac{1}{2} + \Delta)2^{j-1}-k')^2}{2^{j-1}}} \), where the constant \( C \) and the function \( f_{n,k}^+ \) are defined in Fact 17.

Proof. Let us denote by \( D_1 \) the vector (over \( k \in [0..n] \)) of probabilities that the random walk using a coin of bias \( \frac{1}{2} + \Delta \) reaches but does not stop at the location \((n,k)\). Similarly, let us denote by \( D_n \) the vector (over \( k' \in [0..n] \)) of probabilities that the random walk using a \( \frac{1}{2} + \Delta \) coin reaches the location \((n,k)\) (and can either stop at or leave the location).

Consider the vector \( I \) that is 1 for all coordinates \( \leq 0 \), and 0 otherwise. Then for any vector \( v \), \( (v * I)(k) = \sum_{i \leq k} v(k) \), using “∗” to denote convolution.

Assume for the sake of contradiction that the statement is false, namely that for all \( k' \in [0..2^{j-1}] \), \((D_{2j-1} * I)(k') < \xi \cdot f_{2j-1,k}^+(k') \). Then, since \( D_{2j-1} \leq D_{2j-1} \) pointwise, we have for all \( k' \in [0..2^{j-1}] \), \((D_{2j-1} * I)(k') < \xi \cdot f_{2j-1,k}^+(k') \). Observe that \( D_{2j-1} * I \) is constant for all coordinates \( \leq 0 \), and that \( f_{2j-1,k}^+ \) is a decreasing function in the same region if \( k \in [(\frac{1}{2} + \Delta)2^j..2^j] \) (as in the lemma assumption), and therefore \( D_{2j-1} * I < f_{2j-1,k}^+ \) also for that region since the inequality holds at coordinate 0. As for coordinates \( > 2^j \), \( D_{2j-1} * I = 0 \), whilst \( f_{2j-1,k}^+ \) is strictly positive. It follows that the inequality also holds for coordinates \( > 2^j \), and thus it holds everywhere.

From this, using the commutativity of convolution, we have
\[
D_{2j} * I = \left( D_{2j-1} * \text{Bin}(2^{j-1}, \frac{1}{2} + \Delta) \right) * I \\
= (D_{2j-1} * I) * \text{Bin}(2^{j-1}, \frac{1}{2} + \Delta) \\
< f_{2j-1,k}^+ * \text{Bin}(2^{j-1}, \frac{1}{2} + \Delta)
\]
which holds pointwise, in particular at coordinate \( k \). However, \((D_{2j} * I)(k) = c \) by assumption, but \( f_{2j-1,k}^+ * \text{Bin}(2^{j-1}, \frac{1}{2} + \Delta)(k) \leq c \) by Fact 17 which is a contradiction. \( \square \)

Lemma 19. Consider an arbitrary stopping rule \( \{\gamma_{n,k}\} \) that is non-zero only for \( n \) that are powers of 2. For a coin with bias \( \frac{1}{2} - \Delta \), suppose at row \( 2^j \) there is some position \( k \in [(\frac{1}{2} - \Delta)2^j..2^j] \) such that the total probability mass of the random walk reaching positions \( \geq k \) at row \( 2^j \) is at least \( c \). Then, there must be some position \( k' \in [0..2^{j-1}] \) at row \( 2^{j-1} \) such that the probability of reaching positions \( \geq k' \) at that row is at least \( \xi \cdot f_{2j-1,k}^-(k') = \xi \cdot e^{-\frac{(k-(\frac{1}{2} - \Delta)2^{j-1}-k')^2}{2^{j-1}}} \), where the constant \( C \) and the function \( f_{n,k}^- \) are defined in Fact 17.

Proof. The proof is completely analogous to that of Lemma 18. \( \square \)

In order to conclude the sample complexity lower bound corresponding to a particular row, we need the following lemma saying that, if we repeatedly apply Lemma 18 (or Lemma 19), then some row \( 2^j \) will have a large probability of stopping at that row, which will contribute a large amount to the overall sample complexity. Further, when \( 2^j \) is smaller (corresponding to fewer samples taken before stopping), the probability bound induced by the following lemma will be correspondingly higher, so that the product of the row and its stopping probability (i.e., a lower bound on total sample complexity) will be high for the \( j \) produced by the lemma.
Lemma 20. Consider an arbitrary sequence of numbers \( \{g_j\}_{j \in [0..J]} \) such that \( \sum_j g_j = K \). Let \( r_j \) be chosen arbitrarily such that \( r_j \geq \frac{1}{2} e^{g_j/2^j} \), where \( C \) is the constant in Fact 17 and let \( \pi_j = \prod_{i=j}^J r_i \). Furthermore suppose that \( K^2 \geq 100 \log(2C) \cdot 2^J \). Then there exists \( j \in [0..J] \) such that \( \pi_j 2^{j-j} \geq e^{0.01 K^2/2^j} \).

Proof. Taking logarithms and rearranging, we see that it suffices to show the existence of \( j \) such that \( (J - J - 1) \log(2C) + \sum_{i=j}^J g_j \geq 0.01 K^2 \).

The sequence \( \frac{K^2}{2} \cdot 0.8^{J-j} \) for \( j \in [0..J] \) sums up to less than \( K \). Since \( \sum_j g_j = K \), there must exist a \( j \) such that \( g_j \geq \frac{K^2}{2} 0.8^{J-j} \). Therefore, \( \frac{g_j}{2^j} \geq \frac{K^2}{25} 0.64^{J-j} = \frac{K^2}{25} 2^{2J-j} \).

It suffices to show that \( \frac{K^2}{25} 1.28^{J-j} \geq 0.01 K^2 + (J - j + 1) \log(2C) \). It is easy to check that a sufficient condition is \( K^2/2^j \geq 100 \log(2C) \), as assumed in the lemma statement; thus we conclude the above inequality for all \( j \in [0..J] \).

Now we use Lemmas 18, 19 and 20 to prove the sample complexity lower bound corresponding to a particular row (Lemma 21). Afterwards we shall combine these bounds across all possible power-of-2 rows to prove Proposition 15.

Lemma 21. Consider an arbitrary stopping rule \( \{\gamma_{n,k}\} \) that is non-zero only for \( n \) that are powers of 2. For a mixture coin that has bias \( \frac{1}{2} + \Delta \) with probability \( \rho + \frac{1}{2} \) and bias \( \frac{1}{2} - \Delta \) otherwise, suppose at row \( 2^{J+1} \) there is some position \( k \in [(\frac{1}{2} + \Delta)2^{J+1}, 2^{J+1}] \) such that the probability mass of the random walk reaching positions \( \geq k \) at row \( 2^{J+1} \) is \( c \). If \( k \geq (\frac{1}{2} + \Delta)2^{J+1} + \sqrt{100 \log(2C)2^J} + 1 \), then the expected sample complexity of a single random walk using the above mixture coin is at least \( 2^{J-1} \cdot c \cdot e^{0.01(k-(\frac{1}{2}+\Delta)2^{J+1})^2/2^J} \).

We point out that the restriction on \( k \) (that it lies at least a constant number of standard deviations to the right of its mean) includes the entire high discrepancy region, as analyzed in this section, and further includes all of the larger yet analogous region for the analysis of the last row in the next section.

Proof of Lemma 21. The probability of the random walk reaching positions \( \geq k \) at row \( 2^{J+1} \) using a mixture coin is the sum of \( \frac{\rho}{2} + \frac{1}{2} \) times such probability of the random walk using a \( \frac{1}{2} + \Delta \) coin and \( 1 - \rho - \frac{1}{2} \) times such probability of the random walk using a \( \frac{1}{2} - \Delta \) coin. Since the total probability of this walk reaching positions \( \geq k \) equals \( c \), at least half this probability must come from one of the two coin types. Explicitly, at least one of the following two statements has to be true: 1) the probability that the random walk using a coin with \( \frac{1}{2} + \Delta \) bias reaches positions \( \geq k \) at row \( 2^{J+1} \) is at least \( c/(2\rho + \epsilon) \), or 2) the same probability but using a \( \frac{1}{2} - \Delta \) coin instead is at least \( c/(2-2\rho - \epsilon) \).

For case 1, we repeatedly apply Lemma 18 to generate a sequence of \( \{k_j\} \) from \( j = J \) backwards (and \( k_{J+1} = k \)), until \( k_J < (\frac{1}{2} + \Delta)2^{J^*} \) or \( J^* = 0 \). By induction, the probability of reaching positions \( \geq k_j \) at row \( 2^{J^*} \) is at least \( \frac{c}{2^p+\epsilon} \cdot \prod_{i=j}^J \frac{1}{C} e^{(k_{i+1} - (\frac{1}{2}+\Delta)2^i)} \). We would now apply Lemma 20 with \( g_i = k_{i+1} - k_i - (\frac{1}{2} + \Delta)2^i \) for \( i \geq J^* \), and \( g_i = 0 \) for \( i < J^* \), noting that \( K \) in that lemma that we get is \( K = \sum_{i=j}^J k_{i+1} - k_i - (\frac{1}{2} + \Delta)2^i \geq k_{J+1} - k_J - \sum_{i=j}^J (\frac{1}{2} + \Delta)2^i > k_{J+1} - (\frac{1}{2} + \Delta)2^{J+1} \) since \( k_J < (\frac{1}{2} + \Delta)2^{J^*} \) if \( J^* > 0 \) and \( k_0 \leq 1 \) when \( J^* = 0 \). Since we assumed in the lemma statement that \( k \geq (\frac{1}{2} + \Delta)2^{J+1} + \sqrt{100 \log(2C)2^J} + 1 \), we have \( K^2/2^J \geq 100 \log(2C) \).

Therefore, as a result of applying Lemma 20, we know that there exists \( j \) such that

\[
2^{J-j} \prod_{i=j}^J \frac{1}{C} e^{(k_{i+1} - (\frac{1}{2}+\Delta)2^i)} \geq e^{0.01(k-(\frac{1}{2}+\Delta)2^{J+1})^2/2^J}
\]
Thus in case 1, we multiply the left hand side by \( c/(2\rho + \epsilon)2^J \) to give a lower bound on the expected sample complexity of the random walk, using a \( \frac{1}{2} + \Delta \) coin. We thus use the above inequality to conclude a lower bound of \( 2^J \frac{c}{2\rho + \epsilon}e^{0.01(k-(\frac{1}{2}+\Delta)2^{J+1})^2/2^J} \) for the expected sample complexity conditioned on a \( \frac{1}{2} + \Delta \) coin. Since the mixture coin has probability \( \rho + \frac{\Delta}{2} \) of being a \( \frac{1}{2} + \Delta \) coin, the lemma statement follows.

The proof for case 2 is completely analogous, using Lemma 19 instead of Lemma 18 and noting that \( k - (\frac{1}{2} - \Delta)2^{J+1} \geq k - (\frac{1}{2} + \Delta)2^{J+1} \geq 0 \).

Equipped with Lemma 21, we prove Proposition 15.

Proof of Proposition 15. The general strategy is to show using Lemma 21 that, for each row (from 1 to \( 10^{-8}/\Delta^2 \)), if there is some probability \( c_J \) for the random walk to reaching the high discrepancy region, then: 1) the total expected sample complexity must be large, and 2) by Lemma 13, if there is probability \( c_J \) of reaching the high discrepancy region at row \( 2^J \), then the contribution to the squared Hellinger distance by the high discrepancy region at row \( 2^J \) is upper bounded by \( \Theta(c_J2^2/\rho^2) \). Thus the squared Hellinger distance per sample complexity for the high discrepancy region of each row is small, and our bounds are in fact strong enough for us to simply take a union bound over the rows and lose by no more than a constant factor. We now formalize the above argument.

Consider the rows \( 2^{J+1} \) for \( J \in [-1,(\log_2 \frac{10^{-8}}{\Delta^2})] - 1 \). Recall that the high discrepancy region consists of coordinates \( k \in [0,2^{J+1}] \) such that \( h_{2^{J+1},k}^+/h_{2^{J+1},k}^- \geq 1/\rho^{0.1} \). Observe that

\[
\frac{h_{2^{J+1},k}^+}{h_{2^{J+1},k}^-} = \left( \frac{1 + 2\Delta}{1 - 2\Delta} \right)^{2k - 2^{J+1}}
\]

and therefore the high discrepancy region consists of \( k \) such that \( 2k - 2^{J+1} \geq \frac{.1 \log \frac{1}{\rho}}{\log \frac{1}{1+2\Delta}} \), implying that

\[
k \geq 2^J + \frac{.1 \log \frac{1}{\rho}}{\log \frac{1}{1+2\Delta}} \geq \frac{1}{2} 2^{J+1} + \frac{.099 \log \frac{1}{\rho}}{4\Delta}
\]

Furthermore, since \( J \leq (\log_2 \frac{10^{-8}}{\Delta^2}) - 1 \), we have \( 2^J \leq \frac{0.01}{2\Delta} \), which for sufficiently small \( \rho \) and \( \Delta \) (both smaller than some absolute constant, with no requirements on how they depend on each other) means that \( \frac{.099 \log \frac{1}{\rho}}{4\Delta} \geq \Delta^2 2^{J+1} + \sqrt{100 \log(2C)2^J} + 1 \). Thus the coordinates \( k \) in the high discrepancy region always satisfy the precondition of Lemma 21

Now note that for sufficiently small \( \rho \) (smaller than some absolute constant),

\[
\left( k - \left( \frac{1}{2} + \Delta \right)2^{J+1} \right)^2 \geq \left( \frac{.098 \log \frac{1}{\rho}}{4\Delta} \right)^2 \geq \frac{10^{-8}(\log \frac{1}{\rho})^2}{\Delta^2 2^J}
\]

Therefore, if the probability of the random walk using a random coin reaches the high discrepancy region at row \( 2^{J+1} \) is \( c_{J+1} \), then by Lemma 21 the total expected sample complexity of the random walk must be at least \( 2^{J-1} \cdot c_{J+1} \cdot e^{0.01\log(2C)2^J} \).

We can now upper bound the ratio between the high discrepancy region contribution to the squared Hellinger distance and the total expected sample complexity of the random walk by

\[
\frac{\sum_{J \in [-1,(\log_2 \frac{10^{-8}}{\Delta^2})] - 1} \Theta \left( \frac{c_J^2}{\rho^2} \right) c_{J+1}}{E_{\rho + \frac{\Delta}{2}}[\frac{\epsilon}{\log \frac{1}{\rho}}]}
\]

19
\[ H_{\text{last}}^2 = \Theta(\epsilon^2) \sum_{n=\log_2 \frac{10^{-8}}{\Delta^2}, k \in [0..n]} \alpha_{n,k} \left( (\rho + \frac{\epsilon}{2})h^+_{n,k} + (1 - \rho - \frac{\epsilon}{2})h^-_{n,k} \right) \left( \frac{h^+_{n,k} + h^-_{n,k}}{\rho h^+_{n,k} + (1 - \rho)h^-_{n,k}} \right) \]

be the contribution of the squared Hellinger distance from the last row of the triangle, namely row \(10^{-8}/\Delta^2\). Furthermore, again let \(E_{\rho + \frac{\epsilon}{2}}[n]\) be the expected number of coin flips on this random walk, where we use a \(\frac{1}{2} + \Delta\) coin with probability \(\rho + \frac{\epsilon}{2}\) (instead of \(\rho\) or \(\rho + \epsilon\)), and a \(\frac{1}{2} - \Delta\) coin otherwise. If all of \(\rho, \epsilon, \Delta\) and \(\epsilon/\rho\) are smaller than some universal absolute constant, then

\[ \frac{H_{\text{last}}^2}{E_{\rho + \frac{\epsilon}{2}}[n]} = O\left( \frac{\epsilon^2 \Delta^2}{\rho} \right) \]

The squared Hellinger distance contribution from the last row has a different form from the rest of the triangle, and can be large even outside the previously “high discrepancy” region. While the term \((h^+_n h^-_n)/(\rho h^+_n + (1 - \rho)h^-_n)\) is still upper bounded by \(1/\rho^2\) everywhere, it may be as large as \(\Theta(1/\rho^2)\) even when \(h^+_n/h^-_n = \Theta(1)\). The intuition for this section is again that despite having a stopping rule that may have subtle effects on the distribution, it is impossible to skew the distribution of the random walk so much that it appears mostly in the “high discrepancy” side of the triangle. We shall use Lemma 21 again along with a case analysis and a weighted averaging argument to show sample complexity lower bounds, which lets us upper bound the squared Hellinger distance contribution per expected sample, as required.

4.2.3 The Last Row

We lastly analyze the squared Hellinger distance contribution from the last row of the triangle.

**Proposition 22.** Consider an arbitrary stopping rule \(\{\gamma_{n,k}\}\) that 1) is non-zero only for \(n\) that are powers of \(2\), and 2) \(\gamma_{10^{-8}/\Delta^2,k} = 1\) for all \(k\), that is the random walk always stops after \(10^{-8}/\Delta^2\) coin flips. Let

\[ H_{\text{last}}^2 = \Theta(\epsilon^2) \sum_{n=\log_2 \frac{10^{-8}}{\Delta^2}, k \in [0..n]} \alpha_{n,k} \left( (\rho + \frac{\epsilon}{2})h^+_{n,k} + (1 - \rho - \frac{\epsilon}{2})h^-_{n,k} \right) \left( \frac{h^+_{n,k} + h^-_{n,k}}{\rho h^+_{n,k} + (1 - \rho)h^-_{n,k}} \right) \]

be the contribution of the squared Hellinger distance from the last row of the triangle, namely row \(10^{-8}/\Delta^2\). Furthermore, again let \(E_{\rho + \frac{\epsilon}{2}}[n]\) be the expected number of coin flips on this random walk, where we use a \(\frac{1}{2} + \Delta\) coin with probability \(\rho + \frac{\epsilon}{2}\) (instead of \(\rho\) or \(\rho + \epsilon\)), and a \(\frac{1}{2} - \Delta\) coin otherwise. If all of \(\rho, \epsilon, \Delta\) and \(\epsilon/\rho\) are smaller than some universal absolute constant, then

\[ \frac{H_{\text{last}}^2}{E_{\rho + \frac{\epsilon}{2}}[n]} = O\left( \frac{\epsilon^2 \Delta^2}{\rho} \right) \]
Proof. We separate the last row again into a “high discrepancy” region and a “central” region, but with a different criterion: whether
\[
\frac{h_{n,k}^+ + h_{n,k}^-}{\rho h_{n,k}^+ + (1 - \rho)h_{n,k}^-} \geq C
\]
where \( C \) is the constant specified in Fact \( \lfloor \frac{C}{\rho} \rfloor \). The criterion can be equivalently stated as whether \( h_{n,k}^+/h_{n,k}^- \geq r \) for some \( r = \Theta(1) \).

For the “central” region, suppose there is probability \( c_{n,k} \) of reaching position \((n,k)\) in that region for the random walk that uses a \( \rho + \frac{\epsilon}{2} \) mixture random coin. Consider an alternate form of the squared Hellinger distance contribution that is within a constant factor of that presented in the proposition statement, assuming that \( \epsilon/\rho \) is small:

\[
\Theta(\epsilon^2) \sum_{n=10^{-8}, k \in [0..n]} \alpha_{n,k} \left( \frac{h_{n,k}^+}{\rho} + h_{n,k}^- \right)
\]

This approximation holds since \( \rho h_{n,k}^+ + (1 - \rho)h_{n,k}^- \) and \( (\rho + \frac{\epsilon}{2})h_{n,k}^+ + (1 - \rho - \frac{\epsilon}{2})h_{n,k}^- \) are within constant factors of each other. Note that \( c_{n,k} = \alpha_{n,k}((\rho + \frac{\epsilon}{2})h_{n,k}^+ + (1 - \rho - \frac{\epsilon}{2})h_{n,k}^-) \), and so when \( h_{n,k}^+/h_{n,k}^- \leq r = \Theta(1) \), we have both \( \alpha_{n,k}h_{n,k}^+ = O(c) \) and \( \alpha_{n,k}h_{n,k}^- = O(c) \). Thus the squared Hellinger contribution of this location is upper bounded by \( O(\epsilon^2/\rho) \), yet the total sample complexity is lower bounded by \( \Omega(cn) = \Omega(\epsilon/\Delta^2) \), giving a fraction that is \( O(\epsilon^2\Delta^2/\rho) \).

For the “central” region, recall that it consists of the locations where

\[
\frac{h_{n,k}^+}{\rho} + h_{n,k}^- \leq \frac{\rho h_{n,k}^+ + (1 - \rho)h_{n,k}^-}{\rho h_{n,k}^+ + (1 - \rho)h_{n,k}^-}
\]

ranges from \( \frac{C}{\rho} \) to \( \frac{1}{\rho^2} \). We separate this region into \( O(\log \frac{1}{\rho}) \) buckets delimited by consecutive powers of 2.

Suppose there is probability \( c_{\text{disc}} \) of the \( \rho + \frac{\epsilon}{2} \) mixture coin random walk entering the “high discrepancy” region in the last row. Note that the geometric sequence 1, 0.8, 0.64, … converges to 5, and therefore the sequence \( \{ \frac{C}{\rho} \cdot 0.8^i \} \) sums to \( c_{\text{disc}} \). If we took only the first \( O(\log \frac{1}{\rho}) \) many terms, they sum to strictly less than \( c_{\text{disc}} \). By a standard averaging argument, there must exist a bucket such that the probability of reaching locations in that bucket \( i \) is greater than \( \frac{\log 5}{\rho} \cdot 0.8^i \).

Note that the values (Equation 4) inside bucket \( i \) range from \( C \cdot 2^i/\rho \) to \( 2 \cdot C \cdot 2^i/\rho \). It is possible to calculate that the locations \( k \) within bucket \( i \) satisfy \( k \geq \frac{n}{2} + \frac{1}{16} \cdot \frac{\log C}{\Delta} \), where \( n \) again is \( 10^{-8}/\Delta^2 \).

For a sufficiently small \( \Delta \), this lower bound in location is at least \( \frac{n}{2} + \Delta \), and therefore we can apply Lemma 21.

Furthermore, the locations are also at least \( (i \cdot 10^{-2} \log C)/\Delta \) away from \( \frac{n}{2} + \Delta \), and so Lemma 21 guarantees a sample complexity of at least \( \Theta(1/\Delta^2) \cdot \frac{c_{\text{disc}}}{\rho} \cdot 0.8^i \cdot e^{0.01(i \cdot 10^{-2} \log C)/\Delta^2} \cdot \Delta^2 \cdot 10^{8} \geq \Theta(1/\Delta^2) \cdot \frac{c_{\text{disc}}}{\rho} \cdot 0.8^i \cdot e^{100(i \cdot \log C)^2} \geq \Omega(2^i c_{\text{disc}}/\Delta^2) \), where the last inequality is true because the large exponential term has a logarithm that is quadratic in \( i \) and with a base that is a lot greater than 1/0.8.

The squared Hellinger distance contribution from the “high discrepancy” region in the last row is upper bounded by \( O(c_{\text{disc}}^2/\rho) \), and we have shown a sample complexity lower bound of \( \Omega(2^i c_{\text{disc}}/\Delta^2) \). We therefore conclude a fraction of \( O(\epsilon^2\Delta^2/\rho) \) for the squared Hellinger distance per expected sample, for contributions from the “high discrepancy” region in the last row.
Summarizing, both the “high discrepancy” and “central” region contribute no more than \(O(\epsilon^2 \Delta^2 / \rho)\) times \(\mathbb{E}_{\rho} [\eta]\) to \(H^2_{\text{last}}\). Therefore, the proposition follows from summing the two contributions.

\[\square\]

5 Algorithm for Known Conditional Distributions of Coins

We now present our algorithms for the scenario where we know 1) the conditional distribution \(h^+\) of the biases of positive coins, 2) analogously the distribution \(h^-\) for the negative coins, as well as 3) a good guess \(\rho\) for the mixture parameter itself, with the goal being to refine the estimate of the mixture parameter. We assume that our knowledge of the two conditional distributions as well as the mixture parameter are perfect (even if in practice it is only a guess), and derive a method for computing the triangular walk linear estimator (an instantiation of Algorithm \[\text{Algorithm 1}\] in Section 2), with the minimum variance subject to the constraint that the estimator has expected output 0 when given a randomly chosen negative coin, and expected output 1 for a randomly chosen positive coin. That is, we enforce that the estimator is unbiased no matter what the true mixture parameter is, but we optimize its variance given our knowledge of the mixture parameter.

5.1 A Quadratic+Linear Programming Approach

In order to derive the minimum variance unbiased estimator (in the form of Algorithm \[\text{Algorithm 1}\]), as described at the beginning of the section, we first assume we are given a stopping rule, and derive output coefficients for the corresponding linear estimator that has minimum variance. We formulate a quadratic program with the output coefficients \(\{v_{n,k}\}\) as the variables, fixing \(\alpha_{n,k}\) as constants. The quadratic program can be solved analytically, which allows us to derive for \(\{v_{n,k}\}\) closed form expressions that makes an unbiased estimator with minimum variance assuming the given stopping rule, as well as perfect knowledge of the conditional distribution of biases and the mixture parameter. Furthermore, the objective value of the quadratic program turns out (Lemma \[23\]) to be the reciprocal of a linear function in terms of \(\alpha_{n,k}\). We can therefore use the structural observations in Section 2 to formulate a linear program that solves for the optimum stopping rule given the histograms and mixture parameter. In practice, the linear program is first solved to give the stopping rule, then the output coefficients can be calculated from the first step in the analysis.

To simplify notation, let \(h_{n,k}^-\) be shorthand for \(\mathbb{E}_{p \leftarrow h^-} (p^k (1-p)^{n-k})\) (a generalization of the notation from Section 2), and similarly for \(h_{n,k}^+\). Thus \(\alpha_{n,k} h_{n,k}^-\) is the probability that if we randomly choose a negative coin, executing the triangular walk with that coin will stop at state \((n,k)\). Similarly, \(\alpha_{n,k} h_{n,k}^+\) is the analogous probability using a randomly chosen positive coin instead.

The quadratic program mentioned above is given in Figure 5.1. We use variables \(\{\tilde{v}_{n,k}\}\), constraining them such that the expected output over a randomly chosen positive coin (from distribution \(h^+)\) has value 1 greater than that over a randomly chosen negative coin (from distribution \(h^-\)). Under this constraint, we minimize the second moment of the output when items are drawn from the mixture \(\rho h^++(1-\rho)h^-\). Any optimal solution to this optimization will choose the variables \(\{\tilde{v}_{n,k}\}\) such that the expected output of an item drawn from the universe is 0, implying that \(\sum_{n,k} \alpha_{n,k} h_{n,k}^+ \tilde{v}_{n,k} = 1 - \rho\) and \(\sum_{n,k} \alpha_{n,k} h_{n,k}^- \tilde{v}_{n,k} = -\rho\). Therefore, we can compute \(\{v_{n,k}\}\) using \(\{\tilde{v}_{n,k}\}\) by setting \(v_{n,k} = \tilde{v}_{n,k} + \rho\). As a consequence, \(\sum_{n,k} \alpha_{n,k} h_{n,k}^+ v_{n,k} = 1\) and \(\sum_{n,k} \alpha_{n,k} h_{n,k}^- v_{n,k} = 0\), satisfying the unbiasedness requirement as desired.

The quadratic program in Figure 5.1 can be solved analytically using Langrange multipliers. We give the results as Lemma \[23\] and defer the calculations to the appendix.
minimize $\sum_{n,k} \alpha_{n,k} \left( \rho h_{n,k}^+ + (1 - \rho) h_{n,k}^- \right) \tilde{v}_{n,k}^2$
subject to $\sum_{n,k} \alpha_{n,k} h_{n,k}^- \tilde{v}_{n,k} = 1 + \sum_{n,k} \alpha_{n,k} h_{n,k}^- \tilde{v}_{n,k}$

Figure 1: A QP formulation for computing the output coefficients in terms of the stopping rule

maximize $\frac{1}{n_0} \sum_{n,k} \frac{(h_{n,k}^+ - h_{n,k}^-)^2}{\rho h_{n,k}^+ + (1 - \rho) h_{n,k}^-} \alpha_{n,k}$
subject to $\beta_{0,0} = 1$
$\beta_{n+1,k+1} = \beta_{n,k+1} - \alpha_{n,k+1} + \beta_{n,k} - \alpha_{n,k}$
$\alpha_{n,k} \leq \beta_{n,k}$
$\alpha_{n_{\text{max}},k} = 1$ for all $k$ (Max depth constraint)
$\sum_{n,k} n \cdot \alpha_{n,k} (\rho h_{n,k}^+ + (1 - \rho) h_{n,k}^-) \leq n_0$ (Bounding expected sample complexity)

where $\alpha_{n,k}, \beta_{n,k} \geq 0$

Figure 2: An LP formulation for finding the best stopping rule given an expected sample complexity

Lemma 23. For the quadratic program in Figure 5.1, the optimal assignments to $\{\tilde{v}_{n,k}\}$ are

$$\tilde{v}_{n,k} = \frac{h_{n,k}^+ - h_{n,k}^-}{\rho h_{n,k}^+ + (1 - \rho) h_{n,k}^-} \sum_{m,j} \alpha_{m,j} \frac{(h_{m,j}^+ - h_{m,j}^-)^2}{\rho h_{m,j}^+ + (1 - \rho) h_{m,j}}$$

(and we choose $v_{n,k} = \tilde{v}_{n,k} + \rho$), giving an objective value of

$$\frac{1}{\sum_{n,k} \alpha_{n,k} (h_{n,k}^+ - h_{n,k}^-)^2 / (\rho h_{n,k}^+ + (1 - \rho) h_{n,k})}$$

As mentioned at the beginning of the section, the optimal objective value of the quadratic program, namely the minimum variance achievable given a stopping rule, is the reciprocal of a linear function in $\{\alpha_{n,k}\}$. Note that the total sample complexity of the linear estimator, if we use the median-of-means method to estimate its expectation, is proportional to product of the variance of the linear estimator and the expected sample complexity of one run of the random walk. Therefore, if we fix the expected sample complexity of one run to be $n_0$, we can in fact optimize the total sample complexity by minimizing the variance over all possible stopping rules with the expected sample complexity of $n_0$. Observe that the reciprocal of the variance, divided by $n_0$, is simply the reciprocal of the total sample complexity of the stopping rule, that we would therefore like to maximize. Moreover, such function is a linear function in $\{\alpha_{n,k}\}$. Thus, we can write the optimization problem as the linear program in Figure 2 by taking the objective to maximize the reciprocal of the quadratic program solution, divided by $n_0$. The program includes (slightly adapted versions of) the recurrence relations introduced in Equation 1 as constraints. Moreover, in order to control the sample complexity of the algorithm, the program also contains constraints enforcing that 1) the expected number of responses solicited for a random item is bounded by $n_0$ and 2) the maximum depth of the triangle is bounded by some parameter $n_{\text{max}}$.

Since, ultimately, we wish to optimize over all possible values in $n_0$, such a linear program formulation (in Figure 2) cannot be used directly. However, consider the following rewriting of the program. We can always divide the $\{\alpha_{n,k}, \beta_{n,k}\}$ variables by $n_0$ and not change the meaning of
 maximize \[ \sum_{n,k} \frac{(h^+_n - h^-_n)^2}{\rho h^+_n + (1-\rho)h^-_n} \alpha_{n,k} \]

subject to 
\[ \beta_{n+1,k+1} = \beta_{n,k+1} - \alpha_{n,k+1} + \beta_{n,k} - \alpha_{n,k} \]
\[ \alpha_{n,k} \leq \beta_{n,k} \]
\[ \alpha_{n_{\text{max}},k} = \beta_{0,0} \quad \text{for all } k \quad \text{(Max depth constraint)} \]
\[ \sum_{n,k} n \cdot \alpha_{n,k} (\rho h^+_n + (1-\rho)h^-_n) \leq 1 \]
where 
\[ \alpha_{n,k}, \beta_{n,k} \geq 0 \]

Figure 3: An LP formulation for finding the best stopping rule independent of the expected sample complexity for a single coin.

To obtain the optimal stopping rule \( \{\gamma_{n,k}\} \), we solve the linear program in Figure 3, rescale every variable such that \( \beta_{0,0} = 1 \), and calculate \( \gamma_{n,k} = \alpha_{n,k}/\beta_{n,k} \). If the solution to the linear program (in Figure 3) is \( 1/S \), then the expected sample complexity is \( O\left(\frac{S}{\epsilon^2} \log \frac{1}{\delta}\right) \) to estimate \( \rho \) to within an additive \( \epsilon \) with probability at least \( 1 - \delta \). This can be achieved by taking the median-of-means of \( O(\log \frac{1}{\delta}) \) groups of samples of size \( O(\frac{S}{\epsilon^2}) \), each of which has a constant probability concentration to within additive \( \epsilon \) by Chebyshev’s inequality.

5.2 Optimality of such linear estimators

In this section, we show that in fact, the linear estimators produced by the linear program in Figure 2 are optimal compared with any single-coin adaptive but possibly non-linear estimators, subject to the same maximum depth constraints.

Our approach for lower bounding the sample complexity is to fix the distributions \( h^+ \) and \( h^- \), and show that with a small number of samples, it is impossible to distinguish the case between A) a \( \rho \) and \( (1-\rho) \) mixture of positive and negative coins and B) a \( (\rho + \epsilon) \) and \( (1-\rho - \epsilon) \) mixture. To show indistinguishability, we again use the notion of Hellinger distance. Since each stopping rule induces different distribution on the Pascal triangle, under randomly chosen coins from each of the A and B scenarios, we will upper bound the (squared) Hellinger distance between the scenarios.

Lemma 24 shows that the squared Hellinger distance is in fact a linear function in \( \{\alpha_{n,k}\} \) and furthermore, in the regime where \( \epsilon \ll \rho \), is within a constant factor of the objective in the linear program in Figure 3. The coincidence will allow us to show matching lower bounds.

Lemma 24. Consider an arbitrary stopping rule \( \{\gamma_{n,k}\} \) giving coefficients \( \{\alpha_{n,k}\} \). If \( \epsilon/\rho \) is smaller than some universal constant, then the squared Hellinger distance between 1) a coin randomly chosen as in case A (described in the paragraphs above) inducing a distribution on the Pascal triangle given
the stopping rule and 2) a coin randomly chosen as in case B instead, is

$$\Theta(\epsilon^2) \sum_{n,k} \frac{(h^+_{n,k} - h^-_{n,k})^2}{\rho h^+_{n,k} + (1 - \rho)h^-_{n,k}} \alpha_{n,k}$$

We defer the proof and calculations to Appendix B, but it is completely analogous to that of Lemma 13.

With Lemma 24, we now prove Theorem 3.

**Theorem 3.** The linear estimator produced from solving the linear program in Figure 3, as described in Section 5.1, has (total) expected sample complexity that is within a constant factor of the optimal single-coin adaptive algorithm (with $\geq \frac{2}{3}$ probability of success) subject to the same maximum depth constraint.

**Proof.** Given an arbitrary stopping rule, if it induces a squared Hellinger distance of $H^2$ between the two cases with a single random walk, then we can lower bound the number of random walks needed in the single-coin adaptive algorithm in order to solve the distinguishing task with constant probability of success, by $\Theta(1/H^2)$, using the subadditivity of squared Hellinger distance, and that the total Hellinger distance needs to be at least constant to solve the distinguishing task. Thus, if $n_0$ is the expected number of coin flips for a random walk, the overall expected sample complexity is lower bounded by $\Omega(n_0/H^2)$. Since we need to find a lower bound that applies to all single-coin adaptive algorithms, we need to find the smallest $n_0/H^2$ over all the possible stopping rules (subject to the same max-depth constraint), or equivalently, maximize $H^2/n_0$ (which can alternatively be interpreted as the squared Hellinger distance per expected sample). Lemma 24 tells us that we can replace $H^2$ with the expression in the lemma and lose no more than multiplicative constants. Thus, if we fix $n_0$, finding the best lower bound up to multiplicative constants is equivalent to solving the optimization problem that is exactly the one in Figure 3 except for an extra factor of $\Theta(\epsilon^2)$ in the objective. We again wish to maximize the $H^2/n_0$ over all possible choices of $n_0$ as well, and therefore, following the same reasoning as before, we arrive at the linear program that is essentially the one in Figure 3 again except for the factor of $\Theta(\epsilon^2)$ in the objective. This linear program has no $n_0$ dependency, and has objective that is $\Theta(\epsilon^2)$ times that of the one in Figure 3 which is the reciprocal of the (expected) total sample complexity of the optimal linear estimator produced as described in Section 5.1. Summarizing, if the solution to the linear program in Figure 3 is $1/S$, then the maximum $H^2/n_0$ over all possible stopping rules would be within a constant factor of $\epsilon^2/S$, giving a lower bound of $\Omega(S/\epsilon^2)$ on the expected sample complexity (under case A) for a constant probability of success in the task of distinguishing between case A and case B. This lower bound matches the upper bound of $O(S/\epsilon^2)$ on the total sample complexity of the linear estimator we produce according to Section 5.1.

6 Stopping Given a Fixed Budget

In the previous sections, the concentration results were phrased in terms of the number of coins that need to be sampled, which are then input to the triangular walk algorithm (which flips each coin an *a priori* unknown number of times) to produce estimates. Since the sample complexity of a single triangular walk is random, the concentration results only give an expected overall sample complexity for the algorithm. On the other hand, in practice one may wish to impose a fixed budget for sample complexity and simply use the entire budget. Such an approach introduces the issue that the triangular walk started last will probably not have finished by the time the budget...
is exhausted. How then can we aggregate the estimates obtained from the completed triangular walks without introducing bias in subtle ways?

Here we show that the most natural algorithm does in fact work as is: that is, we take the average estimate of all the completed walks, ignoring the incomplete walk in progress. The analysis is not completely trivial. To show that this estimator is unbiased, we view it as the following two-stage estimator: 1) We estimate without bias the distribution over states \((n, k)\) that the triangular walk terminates at, when given a randomly chosen coin from the universe, namely the numbers \(\{\alpha_{n,k}(\rho h_{n,k}^+ + (1-\rho)h_{n,k}^-)\}\). 2) We simply take the dot product of this distribution with the corresponding output values \(\{v_{n,k}\}\).

In order to perform step 1), that is to estimate the distribution of termination over the states, we use the estimator \(\frac{n_{n,k}}{\sum_{m,j} i_{(m,j)}}\) where \(n_{n,k}\) is the number of walks terminating at \((n, k)\), ignoring the incomplete walks. The following proposition shows that the estimation in step 1) is unbiased, from which it follows that the entire estimator is indeed also unbiased.

**Proposition 25.** Suppose there is a fixed limit \(T\) on the number of coin flips. Given a set of outcomes indexed by \(k \in \{1, \ldots, K\}\), let \(p_k\) be a probability distribution over outcomes, and let \(t_k\) denote the number of coin flips taken to reach this outcome. When an outcome using \(t\) coin flips is drawn, if \(t\) is less than or equal to the remaining budget, then \(t\) is subtracted from the remaining budget; and otherwise the most recent outcome is discarded as “over budget” and the algorithm terminates. Let \(i_k\) be the number of times that outcome \(k\) is drawn. Then \(\frac{i_k}{\sum_j i_j}\) is an unbiased estimator of \(p_k\).

**Proof.** Given the coin budget \(T\), the possible sequences of samples can be classified into the following cases. Either 1) the sequence ends exactly at time \(T\), or 2) the sequence ends with a time interval of length \(t_m\) for some \(m\), which in turn ends after time \(T\). For a vector \(i\), whose \(k^{th}\) index denotes the number of times outcome \(k\) occurs, the dot product with vector \(t\) counts the total number of coin flips used by this sequence. Thus, if \(i \cdot t = T\), then the probability of \(i\) occurring equals

\[
\binom{i_1 + \cdots + i_K}{i_1; \cdots; i_K} p_{i_1}^{i_1} \cdots p_{i_K}^{i_K}
\]

This expression captures all cases where we use exactly our budget \(T\). In the remaining cases, there is a final (discarded) outcome \(m\) that goes “over budget”. In this case, \(i \cdot t \in [T-t_m+1, T-1]\), and the probability of observing \(i\) and discarding \(m\) equals

\[
\binom{i_1 + \cdots + i_K}{i_1; \cdots; i_K} p_{i_1}^{i_1} \cdots p_{i_K}^{i_K} \cdot p_m \cdot \frac{i_k}{i_1 + \cdots + i_K}
\]

Therefore, the expectation of \(\frac{i_k}{\sum_j i_j}\) can be written as

\[
\sum_{\text{vector } i} \binom{i_1 + \cdots + i_K}{i_1; \cdots; i_K} p_{i_1}^{i_1} \cdots p_{i_K}^{i_K} \cdot \frac{i_k}{i_1 + \cdots + i_K}
\]

\[+
\sum_m \sum_{\text{vector } i} \binom{i_1 + \cdots + i_K}{i_1; \cdots; i_K} p_{i_1}^{i_1} \cdots p_{i_K}^{i_K} \cdot p_m \cdot \frac{i_k}{i_1 + \cdots + i_K}
\]

Now observe that

\[
\binom{i_1 + \cdots + i_K}{i_1; \cdots; i_K} \frac{i_k}{i_1 + \cdots + i_K} = \binom{i_1 + \cdots + (i_k - 1) + \cdots + i_K}{i_1; \cdots; i_k - 1; \cdots; i_K}
\]
meaning that, letting the vector $i'$ equal the vector $i$ with its $k^{th}$ entry decreased by 1, the expectation can be rewritten and simplified as

$$p_k \left[ \sum_{\text{s.t. } i' \cdot t = T'} \left( \frac{i'_1 + \cdots + i'_K}{i'_1; \cdots; i'_K} \right) p_{i'_1} \cdots p_{i'_K} + \sum_{m} \sum_{\text{s.t. } i' \cdot t \in [T' - t_m + 1, T' - 1]} \left( \frac{i'_1 + \cdots + i'_K}{i'_1; \cdots; i'_K} \right) p_{i'_1} \cdots p_{i'_K} \cdot p_m \right]$$

where $T' = T - t_k$. The term inside the square brackets sums to 1, as we observed at the beginning of the proof, but substituting $T'$ for $T$. Thus the expectation is $p_k$, as desired.

A further subtlety arises if (for practical reasons) one runs multiple instances of the same algorithm, meaning that at every moment in time there are a number of triangular walks being run concurrently, but under a combined budget on the total number of coin flips. One natural attempt at an overall estimate is to “run $c$ instances of triangular walks in parallel; stop when we run out of budget, and return the average estimate of all completed triangles.” However, this algorithm is not unbiased. The correct way to apply the results of this section is instead a subtle reweighting of the above average: for each of the $c$ parallel tracks, average the outputs of completed triangles, and then average these $c$ averages to produce a final unbiased estimate. Each of the $c$ “track” averages will be unbiased by Proposition 25 and thus the overall average will be unbiased too.

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A Full Proof of Results of Section 4.2.1

In this appendix, we shall be using the notations \( \{\gamma_{n,k}\}, \{\alpha_{n,k}\}, \{\beta_{n,k}\} \) and \( \{\eta_{n,k}\} \) defined in Section 2.

**Proposition 26.** Consider an arbitrary stopping rule \( \{\gamma_{n,k}\} \) that 1) is non-zero only for \( n \) that are powers of 2, and 2) \( \gamma_{10^{-8}/\Delta^2,k} = 1 \) for all \( k \), that is the random walk always stops after \( 10^{-8}/\Delta^2 \) coin flips. Let

\[
H_{central}^2 = \Theta(\epsilon^2) \sum_{n<10^{-8}/\Delta^2, k \in [0..n]} \alpha_{n,k}((\rho + \frac{\epsilon}{2})h_{n,k}^+ + (1 - \rho - \frac{\epsilon}{2})h_{n,k}^-) \frac{(h_{n,k}^+ - h_{n,k}^-)^2}{(\rho h_{n,k}^+ + (1 - \rho)h_{n,k}^-)^2}
\]

be the contribution of the squared Hellinger distance by the “central” region. Furthermore, again let \( E_{\rho + \frac{\epsilon}{2}}[n] \) be the expected number of coin flips on this random walk, where we use a \( \frac{1}{2} + \Delta \) coin with probability \( \rho + \frac{\epsilon}{2} \) (instead of \( \rho \) or \( \rho + \epsilon \)), and a \( \frac{1}{2} - \Delta \) coin otherwise. If all of \( \rho, \epsilon, \Delta \) and \( \epsilon/\rho \) are smaller than some universal absolute constant, then

\[
\frac{H_{central}^2}{E_{\rho + \frac{\epsilon}{2}}[n]} = O \left( \frac{\epsilon^2 \Delta^2}{\rho} \right)
\]

Note that in the central region, where \( h_{n,k}^+/h_{n,k}^- \leq 1/\rho^{0.1} \), we have that

\[
\frac{(h_{n,k}^+/h_{n,k}^- - 1)^2}{(\rho(h_{n,k}^+/h_{n,k}^- - 1) + 1)^2} \leq 2 \left( \min \left( \frac{h_{n,k}^+}{h_{n,k}^-}, \frac{1}{\rho^{0.1}} \right) - 1 \right)^2
\]

and furthermore, 0 is also upper bounded by this quantity. Therefore, it is sufficient to upper bound the fraction of

\[
\Theta(\epsilon^2) \sum_{n<10^{-8}/\Delta^2, k \in [0..n]} \alpha_{n,k}((\rho + \frac{\epsilon}{2})h_{n,k}^+ + (1 - \rho - \frac{\epsilon}{2})h_{n,k}^-) \left( \min \left( \frac{h_{n,k}^+}{h_{n,k}^-}, \frac{1}{\rho^{0.1}} \right) - 1 \right)^2
\]

divided by \( E_{\rho + \frac{\epsilon}{2}}[n] \).

The key lemma we need is Lemma 27 which informally states that, for a given stopping rule and a location \( (n, k) \) in the central region, any probability mass not stopping at \( (n, k) \) contributes Hellinger distance proportional to \( O\left(\frac{\epsilon^2 \Delta^2}{\rho^{0.1}}\right) \) times the mass. In the lemma statement, the two terms on the left hand side are the contributions of the squared Hellinger distance of mass leaving \( (n, k) \) and ending up in \( (n + 1, k + 1) \) and \( (n, k) \) respectively. The right hand side states that such sum of contributions is upper bounded by the contribution of squared Hellinger distance at \( (n, k) \), except for an excess of \( O\left(\frac{\epsilon^2 \Delta^2}{\rho^{0.1}}\right) \) times the probability mass. Intuitively, this implies that for every unit of probability mass, every time it leaves a location \( (n, k) \), it incurs a contribution of \( O\left(\frac{\epsilon^2 \Delta^2}{\rho^{0.1}}\right) \) to the squared Hellinger distance in the “central” region. For each unit of probability mass, the number of times it leaves a location is equal to the sample complexity it contributes to, and therefore summing such quantity over each unit of probability mass gives the expected sample complexity. Therefore, we can use this lemma to bound the contribution of squared Hellinger distance by the expected sample complexity times \( O\left(\frac{\epsilon^2 \Delta^2}{\rho^{0.1}}\right) \), thereby proving Proposition 26.
Lemma 27. For any \((n, k)\),

\[
e^2 \eta_{n,k} \left[ \left( (\rho + \frac{\epsilon}{2})h^+_{n+1,k+1} + (1 - \rho - \frac{\epsilon}{2})h^-_{n+1,k+1} \right) \times 2 \left( \frac{h^+_{n+1,k+1}}{h^-_{n+1,k+1}} \right)^2 \right]
\]

\[
+ \left( (\rho + \frac{\epsilon}{2})h^+_{n+1,k} + (1 - \rho - \frac{\epsilon}{2})h^-_{n+1,k} \right) \times 2 \left( \frac{h^+_{n+1,k}}{h^-_{n+1,k}} - 1 \right)^2 \]

\[
\leq e^2 \eta_{n,k} \left( (\rho + \frac{\epsilon}{2})h^+_{n,k} + (1 - \rho - \frac{\epsilon}{2})h^-_{n,k} \right) \left[ 2 \left( \frac{h^+_{n,k}}{h^-_{n,k}} - 1 \right)^2 + O\left( \frac{\Delta^2}{\rho^{0.2}} \right) \right]
\]

Proof. It suffices to show that the left hand side of the inequality is upper bounded by the right hand side, substituting in both options for the minimum. For the \(1/\rho^{0.1}\) case, since both summands on the left hand side are upper bounded by the \(1/\rho^{0.1}\) case of their expressions, the inequality follows trivially and in fact without the excess term of \(O(\Delta^2/\rho^{0.2})\).

We now prove the other case, for which it is sufficient to show that

\[
e^2 \eta_{n,k} \left[ \left( (\rho + \frac{\epsilon}{2})h^+_{n+1,k+1} + (1 - \rho - \frac{\epsilon}{2})h^-_{n+1,k+1} \right) \times 2 \left( \frac{h^+_{n+1,k+1}}{h^-_{n+1,k+1}} - 1 \right)^2 \right]
\]

\[
+ \left( (\rho + \frac{\epsilon}{2})h^+_{n+1,k} + (1 - \rho - \frac{\epsilon}{2})h^-_{n+1,k} \right) \times 2 \left( \frac{h^+_{n+1,k}}{h^-_{n+1,k}} - 1 \right)^2 \]

\[
\leq e^2 \eta_{n,k} \left( (\rho + \frac{\epsilon}{2})h^+_{n,k} + (1 - \rho - \frac{\epsilon}{2})h^-_{n,k} \right) \left[ 2 \left( \frac{h^+_{n,k}}{h^-_{n,k}} - 1 \right)^2 + O\left( \frac{\Delta^2}{\rho^{0.2}} \right) \right]
\]

when \(h^+_{n,k}/h^-_{n,k} \leq 1/\rho^{0.1}\).

In turn, we can break this inequality into a conjunction of two inequalities, that

\[
h^+_{n+1,k+1} \left( \frac{h^+_{n+1,k+1}}{h^-_{n+1,k+1}} - 1 \right)^2 + h^+_{n+1,k} \left( \frac{h^+_{n+1,k}}{h^-_{n+1,k}} - 1 \right)^2 \leq h^+_{n,k} \left( \frac{h^+_{n,k}}{h^-_{n,k}} - 1 \right)^2 + O\left( \frac{\Delta^2}{\rho^{0.2}} \right)
\]

and

\[
h^-_{n+1,k+1} \left( \frac{h^+_{n+1,k+1}}{h^-_{n+1,k+1}} - 1 \right)^2 + h^-_{n+1,k} \left( \frac{h^+_{n+1,k}}{h^-_{n+1,k}} - 1 \right)^2 \leq h^-_{n,k} \left( \frac{h^+_{n,k}}{h^-_{n,k}} - 1 \right)^2 + O\left( \frac{\Delta^2}{\rho^{0.2}} \right)
\]

again assuming that \(h^+_{n,k}/h^-_{n,k} \leq 1/\rho^{0.1}\).

For the first inequality, observe that

\[
\frac{h^+_{n+1,k+1}}{h^-_{n+1,k+1}} = \frac{\frac{1}{2} + \Delta}{\frac{1}{2} - \Delta} \text{ and } \frac{h^+_{n+1,k}}{h^-_{n+1,k}} = \frac{\frac{1}{2} - \Delta}{\frac{1}{2} + \Delta}
\]

and also \(h^+_{n+1,k+1} = h^+_{n,k} \left( \frac{1}{2} + \Delta \right)\) and \(h^-_{n+1,k} = h^-_{n,k} \left( \frac{1}{2} - \Delta \right)\). We therefore factor out and drop the \(h^+_{n,k}\) on both sides, simplify, and reduce to showing that

\[
\left( \frac{1}{2} + \Delta \right) \left( \frac{1}{2} + \Delta \frac{h^+_{n,k}}{h^-_{n,k}} - 1 \right)^2 + \left( \frac{1}{2} - \Delta \right) \left( \frac{1}{2} - \Delta \frac{h^+_{n,k}}{h^-_{n,k}} - 1 \right)^2 \leq \left( \frac{h^+_{n,k}}{h^-_{n,k}} - 1 \right)^2 + O\left( \frac{\Delta^2}{\rho^{0.2}} \right)
\]

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The left hand side is
\[
\left(\frac{1}{2} + \Delta\right) \left(\frac{1}{2} + \Delta h_{n,k}^+ - 1\right)^2 + \left(\frac{1}{2} - \Delta\right) \left(\frac{1}{2} - \Delta h_{n,k}^+ - 1\right)^2
\]
\[
= \left(\frac{h_{n,k}^+}{h_{n,k}^-}\right)^2 \left(\frac{1}{2} + \Delta\right)^2 \left(\frac{1}{2} - \Delta\right)^2 + \left(\frac{1}{2} - \Delta\right) \left(\frac{1}{2} + \Delta\right)^2 + 1
\]
\[
= \left(\frac{h_{n,k}^+}{h_{n,k}^-}\right)^2 \left(1 + O(\Delta^2)\right) - 2\frac{h_{n,k}^+}{h_{n,k}^-} \left(\frac{1}{2} - \Delta\right)^2 + \left(\frac{1}{2} + \Delta\right)^2 + 1
\]
\[
\leq \left(\frac{h_{n,k}^+}{h_{n,k}^-}\right)^2 \left(1 + O(\Delta^2)\right) - 2\frac{h_{n,k}^+}{h_{n,k}^-} + 1
\]
\[
= \left(\frac{h_{n,k}^+}{h_{n,k}^-} - 1\right)^2 + O\left(\frac{\Delta^2}{\rho^{0.2}}\right)
\]
where the last inequality holds again because we have \(h_{n,k}^+/h_{n,k}^- \leq 1/\rho^{0.1}\) by our case analysis.

For the second inequality, via similar reasoning as above, we only need to show that
\[
\left(\frac{1}{2} - \Delta\right) \left(\frac{1}{2} + \Delta h_{n,k}^+ - 1\right)^2 + \left(\frac{1}{2} + \Delta\right) \left(\frac{1}{2} - \Delta h_{n,k}^+ - 1\right)^2 \lesssim \left(\frac{h_{n,k}^+}{h_{n,k}^-} - 1\right)^2 + O\left(\frac{\Delta^2}{\rho^{0.2}}\right)
\]

The left hand side is
\[
\left(\frac{1}{2} - \Delta\right) \left(\frac{1}{2} + \Delta h_{n,k}^- - 1\right)^2 + \left(\frac{1}{2} + \Delta\right) \left(\frac{1}{2} - \Delta h_{n,k}^- - 1\right)^2
\]
\[
= \left(\frac{h_{n,k}^-}{h_{n,k}^+}\right)^2 \left(\frac{1}{2} + \Delta\right)^2 \left(\frac{1}{2} - \Delta\right)^2 + \left(\frac{1}{2} - \Delta\right) \left(\frac{1}{2} + \Delta\right)^2 + 1
\]
\[
\leq \left(\frac{h_{n,k}^-}{h_{n,k}^+}\right)^2 \left(1 + O(\Delta^2)\right) - 2\frac{h_{n,k}^-}{h_{n,k}^+} + 1
\]
\[
= \left(\frac{h_{n,k}^-}{h_{n,k}^+} - 1\right)^2 + O\left(\frac{\Delta^2}{\rho^{0.2}}\right)
\]
with reasoning as in the previous inequality, thus completing the proof of the lemma.

Before we move onto the proof of Proposition 26, we need another lemma that formally rephrases the following simple fact, that for any depth \(n\), the probability of leaving \((n, k)\) for any \(k\) is equal to the sum of probabilities of stopping at \((N, K)\) over \(N > n\).

Lemma 28. For any \(n\),
\[
\sum_{k \in [0..n]} \eta_{n,k}(\rho h_{n,k}^+ + (1 - \rho)h_{n,k}^-) = \sum_{N > n, K \in [0..N]} \alpha_{N,K}(\rho h_{N,K}^+ + (1 - \rho)h_{N,K}^-)
\]
Proof. Since the stopping rules have bounded maximum depth and therefore stops with probability 1, the lemma amounts to the conservation of probability mass.

We can now use the two lemmas to prove Proposition 26.

Proof of Proposition 26. The proof strategy is to inductively apply Lemma 27 from the bottommost row of the triangle, namely the $10^{-8}/\Delta^2$ row, back up to the tip of the triangle, keeping track of the excess squared Hellinger distance for rows below the current row. More formally, we show the following statement by induction, from the bottommost depth up to the depth 0: for any $m \in [0..10^{-8}/\Delta^2)$,

$$
\varepsilon^2 \sum_{n,k \in [0..n]} \alpha_{n,k} ((\rho + \frac{\varepsilon}{2})h_{n,k}^+ + (1 - \rho - \frac{\varepsilon}{2})h_{n,k}^-) \times 2 \left( \min \left( \frac{h_{n,k}^+}{h_{n,k}^-}, \frac{1}{\rho^{0.1}} \right) - 1 \right)^2
$$

$$
\leq \varepsilon^2 \sum_{n \leq m, k \in [0..n]} \alpha_{n,k} ((\rho + \frac{\varepsilon}{2})h_{n,k}^+ + (1 - \rho - \frac{\varepsilon}{2})h_{n,k}^-) \times 2 \left( \min \left( \frac{h_{n,k}^+}{h_{n,k}^-}, \frac{1}{\rho^{0.1}} \right) - 1 \right)^2
$$

$$
+ \varepsilon^2 \sum_{k' \in [0..m]} \beta_{m,k'} ((\rho + \frac{\varepsilon}{2})h_{m,k'}^+ + (1 - \rho - \frac{\varepsilon}{2})h_{m,k'}^-) \times 2 \left( \min \left( \frac{h_{m,k'}^+}{h_{m,k'}^-}, \frac{1}{\rho^{0.1}} \right) - 1 \right)^2
$$

$$
+ O \left( \frac{\varepsilon^2 \Delta^2}{\rho^{0.2}} \right) \sum_{N > m, k' \in [0..N]} \alpha_{N,k'} ((\rho + \frac{\varepsilon}{2})h_{N,k'}^+ + (1 - \rho - \frac{\varepsilon}{2})h_{N,k'}^-) \times (N - m - 1)
$$

The first term on the right hand side corresponds to the squared Hellinger distance for rows we have not “processed”, the second term corresponding to the current row, and the last term is the excess squared Hellinger distance we have collected so far using Lemma 27.

Since the stopping rule has a bounded maximum depth of $10^{-8}/\Delta^2$, the case of $m = 10^{-8}/\Delta^2$ is true by the fact that $\beta_{10^{-8}/\Delta^2,k} = \alpha_{10^{-8}/\Delta^2,k}$.

For the inductive step, for $m < 10^{-8}/\Delta^2$,

$$
\varepsilon^2 \sum_{n,k \in [0..n]} \alpha_{n,k} ((\rho + \frac{\varepsilon}{2})h_{n,k}^+ + (1 - \rho - \frac{\varepsilon}{2})h_{n,k}^-) \times 2 \left( \min \left( \frac{h_{n,k}^+}{h_{n,k}^-}, \frac{1}{\rho^{0.1}} \right) - 1 \right)^2
$$

$$
\leq \varepsilon^2 \sum_{n \leq m, k \in [0..n]} \alpha_{n,k} ((\rho + \frac{\varepsilon}{2})h_{n,k}^+ + (1 - \rho - \frac{\varepsilon}{2})h_{n,k}^-) \times 2 \left( \min \left( \frac{h_{n,k}^+}{h_{n,k}^-}, \frac{1}{\rho^{0.1}} \right) - 1 \right)^2
$$

$$
+ \varepsilon^2 \sum_{k' \in [0..m+1]} \beta_{m+1,k'} ((\rho + \frac{\varepsilon}{2})h_{m+1,k'}^+ + (1 - \rho - \frac{\varepsilon}{2})h_{m+1,k'}^-) \times 2 \left( \min \left( \frac{h_{m+1,k'}^+}{h_{m+1,k'}^-}, \frac{1}{\rho^{0.1}} \right) - 1 \right)^2
$$

$$
+ O \left( \frac{\varepsilon^2 \Delta^2}{\rho^{0.2}} \right) \sum_{N > m+1, k' \in [0..N]} \alpha_{N,k'} ((\rho + \frac{\varepsilon}{2})h_{N,k'}^+ + (1 - \rho - \frac{\varepsilon}{2})h_{N,k'}^-) \times (N - m - 1)
$$

by the induction hypothesis.

$$
= \varepsilon^2 \sum_{n \leq m, k \in [0..n]} \alpha_{n,k} ((\rho + \frac{\varepsilon}{2})h_{n,k}^+ + (1 - \rho - \frac{\varepsilon}{2})h_{n,k}^-) \times 2 \left( \min \left( \frac{h_{n,k}^+}{h_{n,k}^-}, \frac{1}{\rho^{0.1}} \right) - 1 \right)^2
$$

$$
+ \varepsilon^2 \sum_{k' \in [0..m+1]} (\eta_{m,k'} + \eta_{m,k'-1}) ((\rho + \frac{\varepsilon}{2})h_{m+1,k'}^+ + (1 - \rho - \frac{\varepsilon}{2})h_{m+1,k'}^-) \times 2 \left( \min \left( \frac{h_{m+1,k'}^+}{h_{m+1,k'}^-}, \frac{1}{\rho^{0.1}} \right) - 1 \right)^2
$$
\[ + \mathcal{O}\left( \frac{\varepsilon^2 \Delta^2}{\rho^{0.2}} \right) \sum_{N > m+1, k', \rho' \in [0..N]} \alpha_{N, k'}((\rho + \varepsilon/2)h_{N, k'}^+ + (1 - \rho - \varepsilon/2)h_{N, k'}^-) \times (N - m - 1) \]

since \( \beta_{m+1, k'} = \eta_{m, k'} + \eta_{m, k'-1} \) (if we define \( \eta_{m, -1} = 0 \))

\[ \leq \varepsilon^2 \sum_{n \leq m, k \in [0..m]} \alpha_{n, k}((\rho + \varepsilon/2)h_{n, k}^+ + (1 - \rho - \varepsilon/2)h_{n, k}^-) \times 2 \left( \min \left( \frac{h_{n, k}^+}{h_{n, k}^+}, \frac{1}{\rho^{0.1}} \right) - 1 \right)^2 \]

\[ + \varepsilon^2 \sum_{k' \in [0..m+1]} \eta_{m, k'}((\rho + \varepsilon/2)h_{m, k'}^+ + (1 - \rho - \varepsilon/2)h_{m, k'}^-) \times \left[ 2 \left( \min \left( \frac{h_{m, k'}^+}{h_{m, k'}^+}, \frac{1}{\rho^{0.1}} \right) - 1 \right)^2 + \mathcal{O}\left( \frac{\varepsilon^2 \Delta^2}{\rho^{0.2}} \right) \right] \]

by Lemma 27

\[ = \varepsilon^2 \sum_{n \leq m, k \in [0..m]} \alpha_{n, k}((\rho + \varepsilon/2)h_{n, k}^+ + (1 - \rho - \varepsilon/2)h_{n, k}^-) \times 2 \left( \min \left( \frac{h_{n, k}^+}{h_{n, k}^+}, \frac{1}{\rho^{0.1}} \right) - 1 \right)^2 \]

by Lemma 28

\[ \leq \varepsilon^2 \sum_{n \leq m-1, k \in [0..m]} \alpha_{n, k}((\rho + \varepsilon/2)h_{n, k}^+ + (1 - \rho - \varepsilon/2)h_{n, k}^-) \times 2 \left( \min \left( \frac{h_{n, k}^+}{h_{n, k}^+}, \frac{1}{\rho^{0.1}} \right) - 1 \right)^2 \]

\[ + \varepsilon^2 \sum_{k' \in [0..m]} \beta_{m, k'}((\rho + \varepsilon/2)h_{m, k'}^+ + (1 - \rho - \varepsilon/2)h_{m, k'}^-) \times 2 \left( \min \left( \frac{h_{m, k'}^+}{h_{m, k'}^+}, \frac{1}{\rho^{0.001}} \right) - 1 \right)^2 \]

\[ + \mathcal{O}\left( \frac{\varepsilon^2 \Delta^2}{\rho^{0.2}} \right) \sum_{N > m, k' \in [0..N]} \alpha_{N, k'}((\rho + \varepsilon/2)h_{N, k'}^+ + (1 - \rho - \varepsilon/2)h_{N, k'}^-) \times (N - m) \]

since \( \beta_{m, k} = \alpha_{m, k} + \eta_{m, k} \)

proving the induction step.

The \( m = 0 \) case gives a right hand side of

\[ \mathcal{O}\left( \frac{\varepsilon^2 \Delta^2}{\rho^{0.2}} \right) \sum_{N > 0, k' \in [0..N]} \alpha_{N, k'}((\rho + \varepsilon/2)h_{N, k'}^+ + (1 - \rho - \varepsilon/2)h_{N, k'}^-) \times N \]

(since the first two term are 0), which is equal to \( \mathcal{O}\left( \frac{\varepsilon^2 \Delta^2}{\rho^{0.2}} \right) E_{\rho + \varepsilon/2}[n] \), proving the proposition. \( \square \)

**B Remaining Proofs/Calculations of Results**

**Lemma 6.** If the maximum depth of the triangular walk satisfies \( n_{\max} \geq c \cdot \frac{1}{\Delta^2} \log \frac{1}{\Delta \varepsilon} \) for some universal constant \( c \), then the following statements hold.
1. Given an arbitrary negative coin (which has a bias \( \leq \frac{1}{2} - \Delta \)), the output of the walk has expectation in \([\pm 2\varepsilon]\) and variance upper bounded by \(\varepsilon^3\).

2. Given an arbitrary positive coin (that is \( \geq \frac{1}{2} + \Delta \)), the output of the walk has expectation in \([1 \pm \frac{2\varepsilon}{3}]\) and variance upper bounded by \(\frac{1}{3}\).

Furthermore, by construction, the triangular walk has output absolutely bounded by \(\frac{1}{3}\).

**Proof.** Recall by Fact 7 (the Ballot theorem) that \(\alpha_{n, k} = \frac{2k - n_{\text{max}}}{n_{\text{max}}} \binom{n_{\text{max}}}{k}\) for \(k > \frac{n_{\text{max}}}{2}\).

For a coin with bias \(p\), the expected output of the walk is

\[
\sum_{\frac{n_{\text{max}}}{2} < k < \frac{1+\Delta}{2} n_{\text{max}}} \frac{2k - n_{\text{max}}}{\Delta n_{\text{max}}} \binom{n_{\text{max}}}{k} p^k (1-p)^{n_{\text{max}}-k} + \sum_{k > \frac{1+\Delta}{2} n_{\text{max}}} \binom{n_{\text{max}}}{k} p^k (1-p)^{n_{\text{max}}-k}
\]

which is clearly always in \([0, 1]\).

If \(p \leq \frac{1}{2} - \Delta\), then the expectation is upper bounded by \(\frac{1}{3} \mathbb{P}(\text{Bin}(n_{\text{max}}, p) > n_{\text{max}}/2)\). Thus, by choosing a sufficiently large constant \(c\) in the lemma statement, the expectation is upper bounded by \(O(\Delta^2 \varepsilon^3) \leq \varepsilon^3\) using Chernoff bounds. Similarly, if \(p \geq \frac{1}{2} + \Delta\), then the expectation is lower bounded by \(\mathbb{P}(\text{Bin}(n_{\text{max}}, p) > \frac{1+\Delta}{2} n_{\text{max}})\), which in turn is equal to \(1 - \mathbb{P}(\text{Bin}(n_{\text{max}}, p) \leq \frac{1+\Delta}{2} n_{\text{max}})\).

By choosing a sufficiently large constant \(c\), the expectation is then lower bounded by \(1 - \varepsilon^3\).

As for the variance, it is upper bounded by the second moment, which is in turn upper bounded by

\[
\sum_{k > \frac{n_{\text{max}}}{2}} \frac{1}{\Delta} \binom{n_{\text{max}}}{k} p^k (1-p)^{n_{\text{max}}-k}
\]

For \(p \leq \frac{1}{2} - \Delta\), we again use Chernoff bounds and choose a sufficiently large constant \(c\) such that the variance is upper bounded by \(\varepsilon^3\). For \(p \geq \frac{1}{2} + \Delta\), this is clearly upper bounded by \(\frac{1}{3}\). \(\square\)

**Proposition 9.** Randomly choosing \(m\) coins and taking the median-of-means of their triangular walk estimates will return an estimate for \(\rho\) to within an additive error of \(\varepsilon\) with probability at least \(1 - \delta\). Furthermore, the expected sample complexity is \(O(m(\frac{1}{2} - \rho n_{\text{max}}) = \frac{1}{2\sigma^2}(1 + \frac{1}{\Delta} \log \frac{1}{2\varepsilon}) \log \frac{1}{\delta})\), where \(m\) and \(n_{\text{max}}\) are taken as the bounds in Lemmas 5 and 6 respectively.

**Proof.** The accuracy guarantee follows directly from Lemmas 5 and 6.

We now calculate the expected sample complexity. Since we sample \(m\) many coins to run the triangular walk algorithm on, and each coin is drawn from the universe, it suffices to bound the (conditional) expected number of steps in the walk by 1) \(\frac{1}{\Delta}\) in the case where \(p \leq \frac{1}{2} - \Delta\) and 2) \(n_{\text{max}}\) otherwise. The second case is trivial, since \(n_{\text{max}}\) is the maximum depth in the triangle before we terminate.

For the first case, the expected number of coin clips is upper bounded by the triangular walk with \(n_{\text{max}} = \infty\). Since \(p \leq \frac{1}{2} - \Delta\), the walk will almost surely revisit the boundary of \(k = \frac{n}{2}\). Recalling that the Triangular Walk stops at \((n, k)\) with probability \(\alpha_{n, k} p^k (1-p)^{n-k}\) where \(\alpha_{2n, n} = \frac{1}{2n-1} \binom{2n}{n}\), the conditional expectation is

\[
\sum_{n \geq 1} (2n) \frac{1}{2n-1} \binom{2n}{n} p^n (1-p)^n = 4p (1-p) \sum_{n \geq 0} \binom{2n}{n} p^n (1-p)^n = \frac{4p (1-p)}{1 - 2p}
\]
\[
\leq \frac{e^{-4\Delta^2}}{2\Delta} \quad \text{for} \quad p \leq \frac{1}{2} - \Delta \\
= O\left(\frac{1}{\Delta}\right)
\]

where the second line is by inspecting the Taylor series of \( \frac{1}{\sqrt{1-4\epsilon}} \) and substituting \( z = p(1-p) \). □

**Lemma 13.** Consider the two probability distributions in Proposition 12 over locations \((n, k)\) in the Pascal triangle of depth \(10^{-8}/\Delta^2\) and bias \( p \in \{\frac{1}{2} \pm \Delta\} \), generated by the given stopping rule \( \{\gamma_{n,k}\} \) in the two cases of 1) a coin with bias \( \frac{1}{2} + \Delta \) is used with probability \( p \) and a coin with bias \( \frac{1}{2} - \Delta \) is used otherwise versus 2) a coin with bias \( \frac{1}{2} + \Delta \) is used with probability \( p + \epsilon \) and a coin with bias \( \frac{1}{2} - \Delta \) is used otherwise. If \( \epsilon / \rho \) is smaller than some universal constant, then the squared Hellinger distance between these two distributions can be written as

\[
\Theta(\epsilon^2) \left[ \sum_{n < \frac{10^{-8}}{\Delta^2}, k \in [0..n]} \alpha_{n,k}((\rho + \frac{\epsilon}{2})h_{n,k}^+ + (1 - \rho - \frac{\epsilon}{2})h_{n,k}^-) \frac{(h_{n,k}^+ - h_{n,k}^-)^2}{(\rho h_{n,k}^+ + (1 - \rho)h_{n,k}^-)^2} + \sum_{n = \frac{10^{-8}}{\Delta^2}, k \in [0..n]} \alpha_{n,k}((\rho + \frac{\epsilon}{2})h_{n,k}^+ + (1 - \rho - \frac{\epsilon}{2})h_{n,k}^-) \frac{h_{n,k}^+}{\rho h_{n,k}^+ + (1 - \rho)h_{n,k}^-} \right]
\]

**Proof.** For \( n < \frac{10^{-8}}{\Delta^2} \), the probability of that the location \((n, k)\) is revealed, for the two distributions we consider in Proposition 12 are \( \alpha_{n,k}(\rho h_{n,k}^+ + (1 - \rho)h_{n,k}^-) \) and \( \alpha_{n,k}(\rho + \epsilon)h_{n,k}^+ + (1 - \rho - \epsilon)h_{n,k}^- \) respectively. Thus, the contribution by these locations to the squared Hellinger distance is proportional to:

\[
\sum_{n,k} \left( \sqrt{\alpha_{n,k}(\rho h_{n,k}^+ + (1 - \rho)h_{n,k}^-)} - \sqrt{\alpha_{n,k}((\rho + \epsilon)h_{n,k}^+ + (1 - \rho - \epsilon)h_{n,k}^-)} \right)^2
\]

\[
= \sum_{n,k} \alpha_{n,k}(\rho h_{n,k}^+ + (1 - \rho)h_{n,k}^-) \left( 1 - \sqrt{\frac{(\rho + \epsilon)h_{n,k}^+ + (1 - \rho - \epsilon)h_{n,k}^-}{\rho h_{n,k}^+ + (1 - \rho)h_{n,k}^-}} \right)^2
\]

\[
= \sum_{n,k} \alpha_{n,k}(\rho h_{n,k}^+ + (1 - \rho)h_{n,k}^-) \left( 1 - \sqrt{1 + \epsilon \frac{h_{n,k}^+ - h_{n,k}^-}{\rho h_{n,k}^+ + (1 - \rho)h_{n,k}^-}} \right)^2
\]

\[
= \sum_{n,k} \alpha_{n,k}(\rho h_{n,k}^+ + (1 - \rho)h_{n,k}^-) \left( 1 - \left(1 + \frac{1}{2} \epsilon \frac{h_{n,k}^+ - h_{n,k}^-}{\rho h_{n,k}^+ + (1 - \rho)h_{n,k}^-} + \Theta\left(\frac{h_{n,k}^+ - h_{n,k}^-}{\rho h_{n,k}^+ + (1 - \rho)h_{n,k}^-}\right)^2\right) \right)^2
\]

Note that the multiplier to \( \epsilon \) is upper bounded by \( 1/\rho \), and therefore if \( \epsilon / \rho \) is sufficiently small, we have the last line being equal to

\[
\sum_{n,k} \alpha_{n,k}(\rho h_{n,k}^+ + (1 - \rho)h_{n,k}^-) \left( \Theta\left(\frac{h_{n,k}^+ - h_{n,k}^-}{\rho h_{n,k}^+ + (1 - \rho)h_{n,k}^-}\right)^2 \right)
\]

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Proof. We use the method of Lagrangian multiplier to find the optimal assignment to \( \rho \) of \( \rho \) factor of \( \rho \) with algebraic manipulation and approximations as in the contribution of \( \alpha \) for the two scenarios respectively. A similar calculation above gives a squared Hellinger distance

For the quadratic program in Figure 5.1, the optimal assignments to \( \rho \) are

Finally, note that is \( \epsilon/\rho \) is a small constant, then \( \rho \) and \( 1-\rho \) are respectively within a small constant factor of \( \rho/2 \) and \( 1-\rho - \frac{\epsilon}{\rho} \), meaning that \( (\rho h_{n,k}^+ + (1-\rho) h_{n,k}^-) \) is within a constant factor of \( (\rho + \frac{\epsilon}{\rho}) h_{n,k}^+ + (1-\rho - \frac{\epsilon}{\rho}) h_{n,k}^- \).

For \( n = 10^{-8}/\Delta^2 \), the probability that \( ((n,k), \frac{1}{2} + \Delta) \) is revealed is \( \alpha_{n,k} \rho h_{n,k}^+ \) and \( \alpha_{n,k} (\rho + \epsilon) h_{n,k}^+ \) for the two scenarios respectively. A similar calculation above gives a squared Hellinger distance contribution of

As for the contribution from the revealing of \( ((n,k), \frac{1}{2} - \Delta) \), the respective probabilities are \( \alpha_{n,k} (1-\rho) h_{n,k}^- \) and \( \alpha_{n,k} (1-\rho - \epsilon) h_{n,k}^- \), and similar calculations give a squared Hellinger distance contribution of

which with algebraic manipulation and approximations as in the \( n < 10^{-8}/\Delta^2 \) case completes the proof of the lemma.

\[ \Theta(\epsilon^2) \alpha_{n,k} h_{n,k}^- \]

Lemma 23. For the quadratic program in Figure 5.1, the optimal assignments to \( \{\tilde{v}_{n,k}\} \) are

\[ \tilde{v}_{n,k} = \frac{h_{n,k}^+ - h_{n,k}^-}{\rho h_{n,k}^+ + (1-\rho) h_{n,k}^-} \sum_{m,j} \alpha_{m,j} \frac{(h_{m,j}^+ - h_{m,j}^-)^2}{\rho h_{m,j}^+ + (1-\rho) h_{m,j}} \]

(and we choose \( v_{n,k} = \tilde{v}_{n,k} + \rho \), giving an objective value of

\[ \frac{1}{\sum_{n,k} \alpha_{n,k} \frac{(h_{n,k}^+ - h_{n,k}^-)^2}{\rho h_{n,k}^+ + (1-\rho) h_{n,k}}} \]

Proof. We use the method of Lagrangian multiplier to find the optimal assignment to \( \{\tilde{v}_{n,k}\} \).

The Lagrangian of the program is

\[ L = \sum_{m,j} \alpha_{m,j} (\rho h_{m,j}^+ + (1-\rho) h_{m,j}^-) \tilde{v}_{m,j}^2 + \lambda \left( \sum_{m,j} \alpha_{m,j} (h_{m,j}^+ - h_{m,j}^-) \tilde{v}_{m,j} \right) - 1 \]

where \( \lambda \) is the Lagrange multiplier.

We need to find assignments to \( \{\tilde{v}_{n,k}\} \) and \( \lambda \) such that \( \nabla_{\{\tilde{v}_{n,k}\},\lambda} L = 0 \). Computing the partial derivatives gives the following system of equations:

\[ 2\alpha_{n,k} (\rho h_{n,k}^+ + (1-\rho) h_{n,k}^-) \tilde{v}_{n,k} + \lambda \alpha_{n,k} (h_{n,k}^+ - h_{n,k}^-) = 0 \text{ for all } n, k \]  

(5)

\[ \sum_{m,j} \alpha_{m,j} (h_{m,j}^+ - h_{m,j}^-) \tilde{v}_{m,j} = 1 \]

(6)

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Rearranging Equation 5 gives
\[
\tilde{v}_{n,k} = \frac{-\lambda (h^+_{n,k} - h^-_{n,k})}{2(\rho h^+_{n,k} + (1 - \rho)h^-_{n,k})}
\] (7)
and substituting this into Equation 6 gives
\[
-\lambda \sum_{m,j} \frac{\alpha_{m,j} (h^+_{m,j} - h^-_{m,j})^2}{2(\rho h^+_{m,j} + (1 - \rho)h^-_{m,j})} = 1
\]
which lets us solve for \(\lambda\)
\[
\lambda = -1/\sum_{m,j} \frac{\alpha_{m,j} (h^+_{m,j} - h^-_{m,j})^2}{2(\rho h^+_{m,j} + (1 - \rho)h^-_{m,j})}
\]
which when substituted back into Equation 7 gives
\[
\tilde{v}_{n,k} = \frac{h^+_{n,k} - h^-_{n,k}}{\rho h^+_{n,k} + (1 - \rho)h^-_{n,k}} \frac{\sum_{m,j} \frac{\alpha_{m,j} (h^+_{m,j} - h^-_{m,j})^2}{\rho h^+_{m,j} + (1 - \rho)h^-_{m,j}}}{\sum_{m,j} \frac{\alpha_{m,j} (h^+_{m,j} - h^-_{m,j})^2}{\rho h^+_{m,j} + (1 - \rho)h^-_{m,j}}}
\]
as desired.

The optimal value of the program can be calculated by substituting the assignment to the objective function.

Lemma 24. Consider an arbitrary stopping rule \(\{\gamma_{n,k}\}\) giving coefficients \(\{\alpha_{n,k}\}\). The squared Hellinger distance between 1) a random coin drawn in case A inducing a distribution on the Pascal triangle given the stopping rule and 2) a random coin drawn in case B instead, is
\[
\Theta(\epsilon^2) \sum_{n,k} \frac{(h^+_{n,k} - h^-_{n,k})^2}{\rho h^+_{n,k} + (1 - \rho)h^-_{n,k}} \alpha_{n,k}
\]
assuming that \(\epsilon/\rho\) is smaller than some universal constant.

Proof. In scenario 1, the distribution induced by a random coin on the Pascal triangle is
\[
\alpha_{n,k} (\rho h^+_{n,k} + (1 - \rho)h^-_{n,k})
\]
and similarly for scenario 2,
\[
\alpha_{n,k} ((\rho + \epsilon)h^+_{n,k} + (1 - \rho - \epsilon)h^-_{n,k})
\]
The squared Hellinger distance is therefore proportional to
\[
\sum_{n,k} \sqrt{\alpha_{n,k} (\rho h^+_{n,k} + (1 - \rho)h^-_{n,k})} - \sqrt{\alpha_{n,k} ((\rho + \epsilon)h^+_{n,k} + (1 - \rho - \epsilon)h^-_{n,k})}^2
\]
\[
= \sum_{n,k} \alpha_{n,k} (\rho h^+_{n,k} + (1 - \rho)h^-_{n,k}) \left( 1 - \sqrt{\frac{(\rho + \epsilon)h^+_{n,k} + (1 - \rho - \epsilon)h^-_{n,k}}{\rho h^+_{n,k} + (1 - \rho)h^-_{n,k}}} \right)^2
\]
\[
= \sum_{n,k} \alpha_{n,k} (\rho h^+_{n,k} + (1 - \rho)h^-_{n,k}) \left( 1 - \sqrt{1 + \frac{\epsilon h^-_{n,k}}{\rho h^+_{n,k} + (1 - \rho)h^-_{n,k}}} \right)^2
\]
\[
\sum_{n,k} \alpha_{n,k} \left( \rho h_{n,k}^+ + (1 - \rho) h_{n,k}^- \right) \left( 1 - \left( 1 + \frac{1}{2} \epsilon \frac{h_{n,k}^+ - h_{n,k}^-}{\rho h_{n,k}^+ + (1 - \rho) h_{n,k}^-} + \Theta \left( \frac{\left( \epsilon \frac{h_{n,k}^+ - h_{n,k}^-}{\rho h_{n,k}^+ + (1 - \rho) h_{n,k}^-} \right)^2}{\rho h_{n,k}^+ + (1 - \rho) h_{n,k}^-} \right) \right)^2 \right)
\]

\[
= \sum_{n,k} \alpha_{n,k} \left( \rho h_{n,k}^+ + (1 - \rho) h_{n,k}^- \right) \left( \frac{1}{2} \epsilon \frac{h_{n,k}^+ - h_{n,k}^-}{\rho h_{n,k}^+ + (1 - \rho) h_{n,k}^-} + \Theta \left( \frac{\left( \epsilon \frac{h_{n,k}^+ - h_{n,k}^-}{\rho h_{n,k}^+ + (1 - \rho) h_{n,k}^-} \right)^2}{\rho h_{n,k}^+ + (1 - \rho) h_{n,k}^-} \right) \right)^2
\]

Note that the multiplier to \( \epsilon \) is upper bounded by \( 1/\rho \), and therefore if \( \epsilon / \rho \) is sufficiently small, we have the last line being equal to

\[
\sum_{n,k} \alpha_{n,k} \left( \rho h_{n,k}^+ + (1 - \rho) h_{n,k}^- \right) \Theta \left( \frac{\left( \epsilon \frac{h_{n,k}^+ - h_{n,k}^-}{\rho h_{n,k}^+ + (1 - \rho) h_{n,k}^-} \right)^2}{\rho h_{n,k}^+ + (1 - \rho) h_{n,k}^-} \right)^2
\]

\[
= \sum_{n,k} \alpha_{n,k} \left( \rho h_{n,k}^+ + (1 - \rho) h_{n,k}^- \right) \Theta \left( \frac{\left( h_{n,k}^+ - h_{n,k}^- \right)^2}{\rho h_{n,k}^+ + (1 - \rho) h_{n,k}^-} \right)^2
\]

\[
= \Theta(\epsilon^2) \sum_{n,k} \frac{(h_{n,k}^+ - h_{n,k}^-)^2}{\rho h_{n,k}^+ + (1 - \rho) h_{n,k}^-} \alpha_{n,k}
\]