AN ASYMPTOTIC DESCRIPTION OF THE
NOETHER-LEFSCHETZ COMPONENTS
IN TORIC VARIETIES

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Abstract

We extend the definition of Noether-Lefschetz components to quasi-smooth hypersurfaces
in a projective simplicial toric variety $\mathbb{P}^{2k+1}_\Sigma$, and prove that asymptotically the components
whose codimension is upper bounded by a suitable effective constant correspond to hypersur-
faces containing a small degree $k$-dimensional subvariety. As a corollary we get an asymptotic
characterization of the components with small codimension, generalizing Otwinowska’s work
for $\mathbb{P}^{2k+1}_\Sigma = \mathbb{P}^{2k+1}$ and Green and Voisin’s for $\mathbb{P}^{2k+1}_\Sigma = \mathbb{P}^3$. Some tools that are developed in
this paper are a generalization of Macaulay theorem for projective irreducible varieties with
zero irregularity, and an extension of the notion of Gorenstein ideal to Cox rings of projective
simplicial toric varieties.
1 Introduction

The classical Noether-Lefschetz theorem states that a very general surface $X$ in $\mathbb{P}^3$ of degree $d \geq 4$ has Picard number 1. In recent years generalizations have been proved, using Hodge theory, for simplicial projective toric threefolds satisfying an explicit numerical condition [2], and more generally by Ravindra and Srinivas for normal projective threefolds using a purely algebraic approach [16].

The Noether-Lefschetz locus is the subscheme of the (hyper)surface parameter space where the Picard number is greater than the Picard number of the ambient variety. Green and Voisin proved that if $N_d$ is the Noether-Lefschetz locus for degree $d$ surfaces in $\mathbb{P}^3$, with $d \geq 4$, the codimension of every component of $N_d$ is bounded from below by $d - 3$, with equality exactly for the components corresponding to surfaces containing a line. Otwinowska gave an asymptotic generalization of Green and Voisin’s results to hypersurfaces in $\mathbb{P}^n$ [15].

In [3] (see also [11]) it was proved that for simplicial projective toric threefolds the codimension of the Noether-Lefschetz components are also bounded from below. There it was also proved that components corresponding to surfaces containing a “line,” defined as a curve which is minimal in a suitable sense, realize the lower bound. However the question whether these are exactly the components of smallest condimension was left open.

The purpose of the this paper is to extend and generalize Otwinowska’s ideas to odd dimensional simplicial projective toric varieties. In section 2 we present a generalization of Green’s restriction theorem (see [9] for Green’s result). In Section 3 we obtain an extension of the classical Macaulay theorem, while in section 4 we introduce a generalization of the notion of Gorenstein ideal, which we call a Cox-Gorenstein ideal; this will be one of key tools for the proof of our main result. In the rather technical Section 5 we give some applications of the generalized Macaulay theorem to Cox-Gorenstein ideals. In section 6 using Hodge theory we explicitly construct the tangent space at a point in the Noether-Lefschetz loci, which turns out to be a graded summand of a Cox-Gorenstein ideal. In section 7 using all the machinery so far developed we prove our main result.

We shall consider a projective simplicial toric variety $\mathbb{P}^{2k+1}_\Sigma$ of dimension $2k + 1$, whose fan is $\Sigma$, and an ample line bundle $L$ on $\mathbb{P}^{2k+1}_\Sigma$, with $\deg L = \beta \in \text{Pic}(\mathbb{P}^{2k+1}_\Sigma)$ satisfying for some $n \geq 0$ the condition

$$k\beta - \beta_0 = n\eta$$

where $\beta_0$ is the class of the anticanonical bundle and $\eta$ is the class of a primitive ample Cartier divisor (for $k = 1$ this reduces to the condition considered in [3]). $f \in H^0(\mathcal{O}_{\mathbb{P}^{2k+1}_\Sigma}(\beta))$ will be a section such that $X_f = \{ f = 0 \}$ is quasi-smooth hypersurface. Let $U_\beta \subset \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^{2k+1}_\Sigma}(\beta)))$ be the open subset parameterizing the quasi-smooth hypersurfaces and let $\pi : \chi_\beta \to U_\beta$ be its tautological family. Let $H^k_{\mathbb{Q}}$ be the local system $R^k_\pi, \mathbb{Q}$ and let $H^{2k}$ be the locally free sheaf $H^k_{\mathbb{Q}} \otimes \mathcal{O}_{U_\beta}$ over $U_\beta$. Let $0 \neq \lambda_f \in H^{k,k}(X_f, \mathbb{Q})/i^*(H^{k,k}(\mathbb{P}^{2k+1}_\Sigma))$ and let $U$ be a contractible open subset around $f$, so that $H^{2k}(U)$ is constant. Finally, Let $\lambda \in H^{2k}(U)$ be the section defined by

\footnote{Heuristically this means that $X_f$ has only singularities inherited from the ambient space, or more precisely, regarding $\mathbb{P}^{2k+1}_\Sigma$ as a smooth orbifold, that $X_f$ is a smooth suborbifold, see e.g. [10].}
λf and let \( \tilde{\lambda} \) be its image in \((H^{2k}/F^{k}H^{2k})(U)\), where \( F^{k}H^{2k} = H^{2k,0} \oplus H^{2k-1,1} \oplus \cdots \oplus H^{k,k} \).

**Definition 1.1** (Local Noether-Lefschetz Locus). \( N^{k,\beta}_{\lambda,U} := \{ G \in U | \lambda_G = 0 \} \).

The following is our main result. The degree of a subvariety \( Z \) of \( \mathbb{P}^{n+1} \) of codimension \( m \) is defined as

\[
\text{deg } Z = [Z] \cdot \eta^{2k+1-m}.
\]

**Theorem 1.2.** For every positive \( \epsilon \) there is positive \( \delta \) such that for every \( m \geq 1 \) and \( d \in [1, m\delta] \), if \( \text{codim } N^{k,\beta}_{\lambda,U} \leq d \), then every element of \( N^{k,\beta}_{\lambda,U} \) contains a \( k \)-dimensional subvariety whose degree is less than or equal to \((1 + \epsilon)d\).

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2 A restriction theorem

After fixing a positive integer \( n \), any other positive integer \( c \) can be written in the form

\[
c = \binom{k_n}{n} + \cdots + \binom{k_1}{1},
\]

with \( k_n > k_{n-1} > \cdots > k_1 \geq 0 \), see [9]. This is called the \( n \)-th Macaulay decomposition of \( c \). One adopts the convention that \( \binom{k}{\ell} = 0 \) if \( k < \ell \) so that one can also write

\[
c = \binom{k_n}{n} + \cdots + \binom{k_\delta}{\delta},
\]

where \( \delta = \min\{ m | k_m \geq m \} \).

Let \( c \) be the codimension of a linear subsystem \( W \subseteq H^0(\mathbb{P}^r, O_{\mathbb{P}^r}(n)) \), and let \( W_H \subseteq H^0(\mathcal{O}_H(n)) \) be the restriction of \( W \) to a general hyperplane \( H \) of codimension \( c_H \). Then the classical restriction theorem says that

\[
c_H \leq c_{<n>},
\]

where

\[
c_{<n>} := \binom{k_n-1}{n} + \cdots + \binom{k_1-1}{1}.
\]

We note two elementary properties of the function \( \phi : c \mapsto c_{<n>} \):

(A) If \( c' \leq c \), then \( c'_{<n>} \leq c_{<n>} \), i.e., the map \( \phi \) is non-decreasing;

(B) if \( k_\delta > \delta \) then \( (c-1)_{<n>} < c_{<n>} \), i.e., the map \( \phi \) is increasing when \( k_\delta > \delta \).
We generalize the inequality (2) to an irreducible projective variety $Y$ with zero irregularity, using induction under the degree and dimension. Given a divisor $D$ in $Y$ and a linear system $W \subseteq H^0(O_D(D))$, we will denote by $c_D$ the codimension of $W_D$ in $H^0(O_D(D))$. The next two lemmas are a preparation for the proof of the restriction theorem (Theorem 2.5).

**Lemma 2.1.** Let $D$ be a general effective ample Cartier divisor in $Y$, $W \subseteq H^0(Y, O_Y(D))$ a linear subsystem, and let $W_D \subseteq H^0(D, O_Y(D))$ be the restriction of $W$ to $D$. Then

$$c_D \leq c_{<1>} = \text{codim}(W, H^0(O_Y(D))) - 1.$$  

**Proof.** Taking cohomology in the fundamental short exact sequence of the divisor $D$, since $H^1(O_Y) = 0$ we obtain

\[ 0 \to H^0(O_Y) \to H^0(O_Y(D)) \to H^0(O_D(D)) \to 0 \]

so that

\[ h^0(O_Y(D)) = h^0(O_Y) + h^0(O_D(D)) = 1 + h^0(O_D(D)). \]  

(2)

Denoting by $r$ the projection $W \to W_D$ one has

$$\dim W = \dim \ker r + \dim W_D.$$  

(3)

so that subtracting (2) from (2) we have

$$\text{codim} W = \text{codim} W_D + 1 - \dim \ker r.$$  

Now, if $s_D$ is a section in $H^0(O_Y(D))$ such that $D = \text{div}_0(s_D)$, then

$$\ker r = \{ w \in W \mid \text{div}_0(w) = \lambda D, \; \lambda \in \mathbb{C} \},$$

and taking $s_D \notin W$, as is $D$ general, we conclude that $\ker r = \{0\}$. \qed

**Lemma 2.2.** Let $C$ be an irreducible smooth projective curve, $D = p_1 + \ldots + p_k$ a general effective divisor on $C$, let $W \subseteq H^0(O_C(nD))$ ($n \geq 1$) be a linear system, and let $W_D \subseteq H^0(O_D(nD))$ be its restriction. Then

$$c_D \leq c_{<n>}.$$  

**Proof.** If $L = O_C(nD)$, $L|D$ can be seen as a collection of $m$ lines $(L_1, \ldots, L_m)$, one for each point. Moreover given $s \in H^0(O_C(nD))$ its restriction provides a basis $(s_1, \ldots, s_k)$ of $\oplus_i H^0(O_{p_i}(np_i))$. Hence $\dim W_D = \dim H^0(O_D(nD))$, i.e., $c_D = 0$. \qed

Before proving our key theorem we recall the definition of Castelnuovo-Mumford regularity. We fix an ample line bundle $L$ on $Y$.
Definition 2.3. A coherent sheaf $\mathcal{F}$ on $Y$ is $m$-regular (with respect to $L$) if

$$H^i(\mathcal{F} \otimes L^{m-i}) = 0 \text{ for } i > 0.$$ 

$L$ itself is said to be $m$-regular it is so with respect to itself, i.e.,

$$H^i(L^{m-i+1}) = 0 \text{ for } i > 0.$$ 

Proposition 2.4 (Theorem 1.8.5 [12]). Let $\mathcal{F}$ be an $m$-regular sheaf on $Y$. Then for every $p \geq 0$, $\mathcal{F}$ is $(m+p)$-regular.

Theorem 2.5 (A Restriction Theorem). Let $Y$ a projective irreducible variety with zero irregularity. Let $W \subseteq H^0(Y, \mathcal{O}_Y(nD))$ ($n \geq 1$), be a linear system, and assume that $D$ be a 1-regular, general effective ample divisor; let $W_D \subseteq H^0(D, \mathcal{O}_D(nD))$ be its restriction. Then

$$c_D \leq c_{c_D^2}.$$ 

Proof. Let $l_n, \ldots, l_\delta$ be the coefficients of the $n$-th Macaulay decomposition of $c_D$. The inequality in the statement is equivalent to

$$\left(\binom{l_n + 1}{n}\right) + \left(\binom{l_{n-1} + 1}{n-1}\right) + \cdots + \left(\binom{l_\delta + 1}{\delta}\right) < c$$

By contradiction, and recalling that $\binom{l+1}{n} = \binom{l}{n} + \binom{l}{n-1}$, we have

$$c \leq \left(\binom{l_n}{n}\right) + \left(\binom{l_n}{n-1}\right) + \cdots + \left(\binom{l_\delta}{\delta}\right) + \left(\binom{l_\delta}{\delta-1}\right)$$

or equivalently

$$c - c_D \leq \left(\binom{l_n}{n-1}\right) + \cdots + \left(\binom{l_\delta}{\delta-1}\right). \quad (4)$$

From the exact sequence

$$0 \rightarrow W(-D) \rightarrow W \rightarrow W_D \rightarrow 0$$

one has

$$\dim W = \dim W_D + \dim W(-D). \quad (5)$$

Since $D$ is 1-regular, it is $n$-regular for every $n \geq 1$ and hence $H^1(Y, (n-1)D) = 0$, so that

$$0 \rightarrow H^0(\mathcal{O}_Y(n-1)D) \rightarrow H^0(\mathcal{O}_Y(nD)) \rightarrow H^0(\mathcal{O}_D(nD)) \rightarrow 0$$

and thus

$$h^0(\mathcal{O}_Y(nD)) = h^0(\mathcal{O}_Y(n-1)D) + h^0(\mathcal{O}_D(nD)). \quad (6)$$

Then (2) minus (2) yields

$$c = c_D + \text{codim } W(-D).$$

Taking $D' \in |D|$ general, we are within the same assumptions of the theorem but on $D$ as ambient variety, i.e., $D \cap D'$ is a 1-regular general divisor on $D$, and $D$ has zero irregularity.
The latter claim follows from the short exact sequence of the ample effective divisor $D$ and the Akizuki-Nakano vanishing for singular projective varieties [10, Thm. 5.1 p. 149].

Now, we have the short exact sequence

$$0 \to W_D(-(D \cap D')) \to W_D \to W_{D|D'} \to 0$$

which gives

$$c_D = \text{codim } W_{D|D'} + \text{codim } W_D(-(D \cap D'))$$

Note that $W(-(D'))_D \subset W_D(-(D \cap D'))$, hence

$$c_D \leq \text{codim } W_{D|D'} + \text{codim } W_D(-D')_D$$

By induction on $n$ and the dimension of the ambient variety $Y$, we may assume that the theorem holds true for $W_D$ and $W(-D)$; note that Lemmas 2.1 and 2.2 provide the induction base.

Applying the theorem to $W_D$ and $W(-D)$ we get

- $(c_D|D') \leq (c_D}_{<n>} = \binom{l_n - 1}{n} + \cdots + \binom{l_\delta - 1}{\delta}$

- $(c - c_D|D') \leq (c - c_D}_{<n-1>}$

Adding the two inequalities and keeping in mind that $D' \sim D$ we have

$$c_{D'} = c_D \leq (c_D}_{<n>} + (c - c_D}_{<n-1>}$$

and by $[2]$ and property (A)

$$(c - c_D}_{<n-1>} < \binom{l_n - 1}{n - 1} + \cdots + \binom{l_\delta - 1}{\delta - 1},$$

so that

$$c_D < \binom{l_n - 1}{n} + \cdots + \binom{l_\delta - 1}{\delta} + \binom{l_n - 1}{n - 1} + \cdots + \binom{l_\delta - 1}{\delta - 1} = c_D$$

which is a contradiction. 

\[\square\]

3 A theorem of Macaulay type

A generalization of another classical theorem by Macaulay (see e.g. [19]) can be obtained from the restriction Theorem 2.4. We are under the same hypotheses of the previous section: $Y$ is a projective irreducible variety with zero irregularity, and $D$ is a 1-regular, general effective ample divisor. Let $W \subset H^0(O_Y(nD))$ be a linear system and let $k_n, k_{n-1}, \ldots, k_\delta$ be the Macaulay coefficients of its codimension $c$; let $W_1$ be the image of the multiplication map $W \otimes H^0(O_Y(D)) \to H^0(O_Y(n + 1)D))$, and $c_1$ the codimension of its image. Let us denote

$$c^{<n>} := \binom{k_n + 1}{n + 1} + \cdots + \binom{k_\delta + 1}{\delta + 1}.$$  

This has the following elementary properties.
• If $c' \leq c$ then $c^{c_{n+}} \leq c^{c_{n+}}$, i.e., the map $c \mapsto c^{c_{n+}}$ is non-decreasing.

• $(c + 1)^{c_{n+}} = \begin{cases} 
  c^{c_{n+}} + k_1 + 1 & \text{if } \delta = 1 \\
  c^{c_{n+}} + 1 & \text{if } \delta > 1 
\end{cases}$

**Theorem 3.1.** (Generalized Macaulay’s Theorem) $c_1 \leq c^{c_{n+}}$.

**Proof.** This goes exactly as in the classical case. Let $l_{n+1}, l_n, \ldots, l_\delta$ be the $(n + 1)$-th Macaulay coefficients of $c_1$; then

$$(c_1)_D \leq c^{c_{n+}} = \left(\frac{l_{n+1} - 1}{n + 1}\right) + \cdots + \left(\frac{l_\delta - 1}{\delta}\right)$$

and by the sequence obtained by restriction it follows that

$$c_1 \leq c + (c_1)_D$$

so that

$$\left(\frac{l_{n+1} - 1}{n}\right) + \cdots + \left(\frac{l_\delta - 1}{\delta - 1}\right) \leq c$$

and then

$$\left(\frac{l_{n+1}}{n + 1}\right) + \cdots + \left(\frac{l_\delta}{\delta}\right) = c_1 \leq c^{c_{n+}}.$$

\[\square\]

## 4 Cox-Gorenstein ideals

We shall need a partial generalization of another theorem by Macaulay (see e.g. Thm. 6.19 in [19] for the classical result). This generalization is already basically contained in the work of Cox and Cattani-Cox-Dickenstein [7, 5] and was introduced in Section 4 of [4]. Here we only state the results, for the proofs see [4].

The Cox ring $S$ of a complete simplicial toric variety $\mathbb{P}_\Sigma$ is graded over the effective classes in the class group $\text{Cl}(\mathbb{P}_\Sigma)$

$$S = \sum_{\alpha \in \text{Cl}(\mathbb{P}_\Sigma)} S^\alpha, \quad S^\alpha = H^0(\mathbb{P}_\Sigma, \mathcal{O}_{\mathbb{P}_\Sigma}(\alpha))$$

(see e.g. [5]). As $\mathcal{O}_{\mathbb{P}_\Sigma}(\alpha)$ is coherent and $\mathbb{P}_\Sigma$ is complete, each $S^\alpha$ is finite-dimensional over $\mathbb{C}$; in particular, $S^0 \simeq \mathbb{C}$. We note that for every effective $N \in \text{Cl}(\mathbb{P}_\Sigma)$, the set of classes $\alpha \in \text{Cl}(\mathbb{P}_\Sigma)$ such that $N - \alpha$ is effective is finite.

We shall give a definition of Cox-Gorenstein ideal of the Cox rings which generalizes to toric varieties the definition given by Otwinowska in [15] for projective spaces. Let $B \subset S$ be the irrelevant ideal, and for a graded ideal $I \subset B$, denote by $V_T(I)$ the corresponding closed subscheme of $\mathbb{P}_\Sigma$.

**Definition 4.1.** A graded ideal $I$ of $S$ contained in $B$ is said to be a Cox-Gorenstein ideal of socle degree $N \in \text{Cl}(\mathbb{P}_\Sigma)$ if
1. there exists a $\mathbb{C}$-linear form $\Lambda \in (S^N)^\vee$ such that for all $\alpha \in \text{Cl}(\mathbb{P}_\Sigma)$

$$I^\alpha = \{ f \in S^\alpha | \Lambda(fg) = 0 \text{ for all } g \in S^{N-\alpha} \};$$

2. $V_T(I) = \emptyset$.

**Proposition 4.2.** Let $R = S/I$. If $I$ is Cox-Gorenstein then

1. $\dim \mathbb{C} R^N = 1$;
2. the natural bilinear morphism

$$R^\alpha \times R^{N-\alpha} \to R^N \cong \mathbb{C}$$

is nondegenerate whenever $\alpha$ and $N-\alpha$ are effective.

We shall need to use a **Euler form** on $\mathbb{P}_\Sigma$ [1, 7, 5]. We denote by $M$ the dual lattice of the lattice $N$ which contains the fan $\Sigma$, i.e., $\Sigma \subset N \otimes \mathbb{R}$. Let $d = \dim \mathbb{P}_\Sigma$.

**Definition 4.3.** Fix an integer basis $u_1, \ldots, u_d$ for the lattice $M$. Then given a subset $\iota = \{i_1, \ldots, i_d\} \subset \{1, \ldots, \# \Sigma(1)\}$, where $\# \Sigma(1)$ is the number of rays in $\Sigma$, we define

$$\det(e_{\iota}) := \det \left( < u_j, e_{i_h} >_{1 \leq j, h \leq d} \right);$$

moreover, $dx_i = dx_{i_1} \wedge \cdots \wedge dx_{i_d}$ and $\hat{x}_i = \Pi_{i \not\in \iota} x_i$.

**Definition 4.4.** A **Euler form** on $\mathbb{P}_\Sigma$ is a Zariski $d$-form $\Omega_0$ defined as

$$\Omega_0 := \sum_{|\iota|=d} \det(e_{\iota}) \hat{x}_i dx_{\iota}$$

where the sum is over all subsets $\iota \subset \{1, \ldots, \# \Sigma(1)\}$ with $d$ elements.

For more details about these definitions see [1].

Let $f_0, \ldots, f_d$ be homogeneous polynomials with $\deg(f_i) = \alpha_i$, where each $\alpha_i$ is ample, and let $N = \sum_i \alpha_i - \beta_0$; here $\beta_0$ is the anticanonical class of $\mathbb{P}_\Sigma$. Assume that the $f_i$ have no common zeroes in $\mathbb{P}_\Sigma$, i.e., $V_T(f_0, \ldots, f_d) = \emptyset$. For each $G \in S^N$ one can define a meromorphic $d$-form $\xi_G$ on $\mathbb{P}_\Sigma$ by letting

$$\xi_G = \frac{G \Omega_0}{f_0 \cdots f_d}$$

where $\Omega_0$ is a Euler form on $\mathbb{P}_\Sigma$. The form $\xi_G$ determines a class in $H^d(\mathbb{P}_\Sigma, \omega)$, where $\omega$ is the canonical sheaf of $\mathbb{P}_\Sigma$, i.e., the sheaf of Zariski $d$-forms on $\mathbb{P}_\Sigma$, and the trace morphism $\text{Tr}_{\mathbb{P}_\Sigma} : H^d(\mathbb{P}_\Sigma, \omega) \to \mathbb{C}$ associates a complex number to $G$. We define $\Lambda \in (S^N)^\vee$ as

$$\Lambda(G) = \text{Tr}_{\mathbb{P}_\Sigma}([\xi_G]) \in \mathbb{C}. \quad (7)$$

We can now state a toric version of Macaulay’s theorem.

**Theorem 4.5.** The linear map defined in Eq. [4] satisfies the condition in Definition [4, 7]. Therefore, the ideal $I = (f_0, \ldots, f_d)$ is a Cox-Gorenstein ideal of socle degree $N = \sum_i \deg(f_i) - \beta_0$.  

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Another example of a Cox-Gorenstein ideal is given in terms of toric Jacobian ideals. For every ray $\rho \in \Sigma(1)$ we shall denote by $v_\rho$ its rational generator, and by $x_\rho$ the corresponding variable in the Cox ring. Recall that $d$ is the dimension of the toric variety $\mathbb{P}_\Sigma$, while we denote by $r = \# \Sigma(1)$ the number of rays. Given $f \in S^3$ one defines its toric Jacobian ideal as

$$J_0(f) = \left( x_{\rho_1} \frac{\partial f}{\partial x_{\rho_1}}, \ldots, x_{\rho_r} \frac{\partial f}{\partial x_{\rho_r}} \right).$$

We recall from [1] the definition of nondegenerate hypersurface and some properties (Def. 4.13 and Prop. 4.15).

**Definition 4.6.** Let $f \in S^3$, with $\beta$ an ample Cartier class. The associated hypersurface $X_f$ is nondegenerate if for all $\sigma \in \Sigma$ the affine hypersurface $X_f \cap O(\sigma)$ is a smooth codimension one subvariety of the orbit $O(\sigma)$ of the action of the torus $\mathbb{T}^d$.

**Proposition 4.7.**
1. Every nondegenerate hypersurface is quasi-smooth.
2. If $f$ is generic then $X_f$ is nondegenerate.

We collect here, with some changes in the terminology, some results that are already contained in Prop. 5.3 of [2].

**Proposition 4.8.** Let $f \in S^3$, and let $\{\rho_1, \ldots, \rho_d\} \subset \Sigma(1)$ be such that $v_{\rho_1}, \ldots, v_{\rho_d}$ are linearly independent.

1. The toric Jacobian ideal of $f$ coincides with the ideal

$$\left( f, x_{\rho_1} \frac{\partial f}{\partial x_{\rho_1}}, \ldots, x_{\rho_d} \frac{\partial f}{\partial x_{\rho_d}} \right).$$

2. The following conditions are equivalent:
   (a) $f$ is nondegenerate;
   (b) the polynomials $x_{\rho_i} \frac{\partial f}{\partial x_{\rho_i}}$, $i = 1, \ldots, r$, do not vanish simultaneously on $X_f$;
   (c) the polynomials $f$ and $x_{\rho_i} \frac{\partial f}{\partial x_{\rho_i}}$, $i = 1, \ldots, d$, do not vanish simultaneously on $X_f$.

3. If $\beta$ is ample and $f$ is nondegenerate, then $J_0(f)$ is a Cox-Gorenstein ideal of socle degree $N = (d + 1)\beta - \beta_0$, where $\beta_0$ is the anticanonical class of $\mathbb{P}_\Sigma$.

## 5 Some applications of Macaulay’s theorem

In this section we prove some applications of Theorem 3.1 to Cox-Gorenstein ideals. This generalizes some of the results in [14, 15] to the more general setting toric varieties, as opposed to classical projective spaces, which is the case considered there. We assume that $\mathbb{P}_\Sigma$ is a simplicial toric variety of dimension $d$, $D$ an primitive divisor in $\mathbb{P}_\Sigma$ and we denote $\deg D = \eta \in \text{Pic}(\mathbb{P}_\Sigma)$.

**Lemma 5.1.** Let $W \subseteq H^0(\mathcal{O}_{\mathbb{P}_\Sigma}(n\eta))$ be a linear subspace whose base locus has dimension $t$ and degree $b$. Then

$$\text{codim}(W) \geq \binom{n + t + 1}{t + 1} - \binom{n - b + t + 1}{t + 1}.$$
Proof. Let $Z$ be the base-locus of $W$ and $I_Z$ its ideal. To easy the notation we write $S^n$ for the piece of degree $nη$ of the Cox ring $S$, and $I^n_Z$ for $I^{nη}_Z$ (no confusion with a power of the ideal should arise). Since $I_W ⊂ I_Z$, codim $W ≥$ codim $I^n_Z$ so it is enough to prove that the result holds for codim $I^n_Z$. We will prove that by induction over $n$ and $t$. For $n = 0$ the result is trivial. For $t = 0$ and $n > 0$ we need to show that codim $I^n_Z ≥ b$. Taking, for some $r > 0$, cohomology in the exact sequence

$$0 → I_Z(rD) → O_{P^n}(rD) → O_Z(rD) → 0$$

we have

$$0 → H^0(I_Z(rD)) → H^0(O_{P^n}(rD)) → H^0(O_Z(rD)) → H^1(I_Z(rD)) → \cdots$$

where by Serre’s vanishing theorem $H^1(I_Z(rD)) = 0$ for $r >> 0$. Thus

$$c := \text{codim } I^n_Z = h^0(O_{P^n}(rD)) - h^0(I_Z(rD)) = h^0(O_Z(rD)) = b$$

as $Z$ has degree $b$ (which in this is the length of the 0-cycle $Z$). Taking $n > b$ and reasoning by contradiction we have $c < b < n$, so that

$$b = \left(\frac{n}{n}\right) + \cdots + \left(\frac{n - (b - 1)}{n - (b - 1)}\right) = 1 + \cdots + 1.$$  

By applying the generalized Macaulay theorem and using the fact that the map $c → c^{<n>}$ is increasing, we have

$$c_1 < c^{<n>} < b$$

where $c_1 = \text{codim } I^{(n+1)η}_Z$;

repeating the same argument after replacing $c$ with $c_1$ we have

$$c_2 < c^{<n+1>} < (c^{<n>})^{<n+1>} < b$$

where $c_2 = \text{codim } I^{(n+2)η}_Z$, so that

$$c_r < (c^{<n>})^{<n+1> \cdots <n+r-1>} < b$$

which implies $c_r ≤ b - 1$. This is a contradiction as $c_r = b$.

Now we assume that the result is true for the pairs $(n-1, t)$ and $(n, t-1)$ and prove that it holds for $(n, t)$.

Claim: Since $D$ is general, the multiplication for a global section $s_D$ of $O_Y(D)$

$$\mu_D : S^n/I^n_Z → S^n/I^n_Z$$

is injective.

In principle the base locus $D$ may contain $Z$, but since $D$ is general we may assume by Bertini’s theorem that $Z \cap D ≠ Z$, i.e., $\mu_D ≠ 0$. Now, if $\mu(f) = 0$ then $f \cdot s_D = 0$ and since $s_D ≠ 0$ then $f = 0$.

We have a well defined surjective restriction map (again, $D$ is general), $S^n/I^n_Z → S^n/I^n_{Z \cap D}$. There is a short exact sequence

$$0 → \text{ker } r → S^n/I^n_Z → S^n/I^n_{Z \cap D} → 0.$$
ker \ r \ contains \ S^{n-1}/I^n_Z, \ so \ that

\dim(S^n/I^n_Z) = \dim(S^n/I^n_{Z \cap D}) + \dim \ ker \ r \geq \dim(S^n/I^n_{Z \cap D}) + \dim(S^{n-1}/I^{n-1}_Z).

By the induction hypothesis we have

\text{codim} \ I^n_{Z \cap D} \geq \left( \frac{n}{t} \right)^t - \left( \frac{n-b+t}{t} \right)^t \quad (8)

and

\text{codim} \ I^n_Z \geq \left( \frac{n+t}{t} \right)^t - \left( \frac{n-b+t}{t} \right)^t \quad (9)

thus adding (8) and (9) we get the result.

\textbf{Corollary 5.2.} Let \ W \subseteq H^0(\mathcal{O}_{\text{PS}}(n \eta)) a linear system whose base locus has dimension \ t \ and degree greater than or equal to \ b. Then for every \ x \leq \min\{b,n\}, one has

\text{codim} \ W \geq x \left( \frac{n-x}{t!} \right)^t.

\textbf{Proof.} Since \ \left( \frac{n+t+1}{t+1} \right)^t - \left( \frac{n-b+t+1}{t+1} \right)^t = \sum_{j=1}^b \binom{t+1+n-j}{n-j+1}^t \ \text{applying the above lemma we get}

\sum_{j=1}^b \left( \frac{t+1+n-j}{n-j+1} \right)^t \geq \sum_{j=1}^b \left( \frac{t+1+n-j}{t!} \right)^t \left( 2+n-j \right)^t \geq \sum_{j=1}^b \frac{(n-j)^t}{t!} \geq x \left( \frac{n-x}{t!} \right)^t.

We establish a preorder in \ N^1(\text{PS}) = \text{Pic}(\text{PS}) \otimes \mathbb{Q} \ by letting \ N < N' \ when \ N' - N \ is effective.

\textbf{Proposition 5.3.} Let \ t \ be a positive integer. For every \ \epsilon_1 > 0 \ there exists \ \delta_1 > 0 \ such that for every \ m \geq \frac{1}{\delta_1} \ and every real number \ b \in [1, \delta_1 m], \ if a Cox-Gorenstein ideal \ I \ of socle degree \ N \ satisfies:

- \ \beta - \beta_0 \leq \ N - \beta = n \eta \ with \ n \geq 0;
- \ \text{codim} \ I^\beta \leq b \frac{m^t}{t!} \ where \ m = \max\{i \in \mathbb{N}^+ \mid i \eta \leq \beta\},

then

1. For every integer \ i \in \{0, \ldots, \lceil \delta_1 m \rceil\} \ one has

\text{codim} \ I^{\beta-i} \eta \leq (1 + \epsilon_1)b \frac{m^t}{t!}.

2. For every \ i \in \{0, \ldots m\} \ one has

\text{codim} \ I^{\beta-i} \eta \leq 4t b \frac{m^t}{t!}.
Proof. First note that since \( I \) is Cox-Gorenstein of socle degree \( N \),
\[
\operatorname{codim} I^{\beta - in} = \operatorname{codim} I^{N - (\beta - in)} = \operatorname{codim} I^{(n+1)\eta}.
\]
So by the generalized Macaulay Theorem 3.1
\[
\operatorname{codim} I^{\beta - in} \leq (\operatorname{codim} I^{\eta})^{c < r < c + n + i - 1},
\]
and since for a fixed \( c \) the map \( c^{<r} \) is decreasing, and for a fixed \( n \) the map \( c^{<\eta} \) is increasing, for every natural number \( x \leq n \)
\[
\operatorname{codim} I^{\beta - in} \leq (\operatorname{codim} I^{\eta})^{c < r < c + n + i - 1}.
\] (10)
Also note that if
\[
\operatorname{codim} I^{\beta} \leq \left( \frac{T + x}{x} \right) + \cdots + \left( \frac{T + x + i - 1}{x + i} \right) \quad \text{where}\ t, v \in \mathbb{N}, \tag{11}
\]
as the map \( c \mapsto c^{<\eta} \) is increasing, (3) and (4) imply
\[
\operatorname{codim} I^{\beta - in} \leq \left( \frac{T + x + i}{x + i} \right) + \cdots + \left( \frac{T + x + i - 1}{x + i} \right) \tag{12}
\]
Take \( \delta_1 \) small enough such that \( b \leq \frac{m - 2r}{2 + \gamma} \) for \( r = \min\{i \mid \beta_0 \leq in\} \). By assumption \( \beta - \beta_0 \leq n\eta \), i.e., \( m - r \leq n \), so that
\[
\left\lfloor \frac{m}{2} \right\rfloor + 2b \leq \left\lfloor \frac{m}{2} \right\rfloor + \frac{m - 2r}{2} \leq m - r \leq n.
\]
Let \( \gamma \) be the smallest positive real number such that \( (2 + \gamma)^t b \) is an integer and
\[
\left\lfloor \frac{m}{2 + \gamma} \right\rfloor + (2 + \gamma)^t b \leq n;
\]
then the inequality (3) holds for \( x = \left\lfloor \frac{m}{2 + \gamma} \right\rfloor + (2 + \gamma)^t b \). On the other hand,
\[
m^t \leq (\gamma + 2 + m)^t = (1 + \frac{m}{2 + \gamma})^t \leq (1 + \left\lfloor \frac{m}{2 + \gamma} \right\rfloor)^t = (2 + \left\lfloor \frac{m}{2 + \gamma} \right\rfloor)^t (t + \left\lfloor \frac{m}{2 + \gamma} \right\rfloor) \cdots (2 + \left\lfloor \frac{m}{2 + \gamma} \right\rfloor) = \frac{(t + \left\lfloor \frac{m}{2 + \gamma} \right\rfloor)!}{(\frac{m}{2 + \gamma} + 1)!}
\]
so that
\[
\frac{m^t}{t!} \leq \left( \frac{t + \left\lfloor \frac{m}{2 + \gamma} \right\rfloor}{\left\lfloor \frac{m}{2 + \gamma} \right\rfloor + 1} \right)
\]
and
\[
b \frac{m^t}{t!} \leq \left( \frac{t + \left\lfloor \frac{m}{2 + \gamma} \right\rfloor}{\left\lfloor \frac{m}{2 + \gamma} \right\rfloor + (2 + \gamma)^t b - 1} \right) + \cdots + \left( t + \left\lfloor \frac{m}{2 + \gamma} \right\rfloor \right) \tag{2 + \gamma)^t \text{d terms}
\]
Then by the second assumption the inequality (3) holds for
\begin{itemize}
  \item \( x = \left\lfloor \frac{m}{2 + \gamma} \right\rfloor + (2 + \gamma)^t b \),
  \item \( \tau = t - 1 \),
\end{itemize}
Lemma 5.6. Let $t \in \{1, \ldots, r-1\}$. For every $\epsilon_2 > 0$ there exists $\delta_2 > 0$ such that for every $m \geq \frac{1}{\delta_2}$ and $b \in [1, \delta_2 m]$, if a Cox-Gorenstein ideal $I \subset B$ with socle degree $N$ satisfies:

- $N - \beta = \eta \eta$
- $\text{codim } I^\beta \leq b m^t$, where $m = \max\{i \in \mathbb{N}^+ \mid i \eta \leq \beta\}$,

then

\[ l_i(I) \leq \epsilon_2 (m - 1) \quad \forall i \in \{0, \ldots, d - t - 1\} . \]
Proof. Note that for the above second remark, it is enough to prove the Lemma for \( i = d - t - 1 \), so we take \( \epsilon_1 = 1 \) and \( x = 1 \). Then, for \( l = \min(l_{d-t-1}(I) - 1, m) \), we have

\[
\frac{(l-1)^{t+1}}{(t+1)!} \leq \text{codim } I^n \leq 4^t b^m t!
\]

so that

\[
l \leq 1 + (4^t b m^t (t + 1))^\frac{1}{m} \leq \left( \frac{1}{m} + (4^t t + 1) \frac{b}{m} \right) m \leq (\delta_2 + (4^t (t+1) \delta_2)) m
\]

and since \( m \delta_2 \geq 1 \),

\[
l \leq (3 \delta_2 + (4^t (t+1) \delta_2)) m - 1.
\]

So, given \( \epsilon_2 > 0 \), we take \( \delta_2 \) small enough to have \( 3 \delta_2 + (4^t (t+1) \delta_2) < \min\{1, \epsilon_2\} \); then \( l < m \) i.e., \( l = l_{2k-t-1}(I) - 1 \) or, in other words, \( l_{d-t-1}(I) < \epsilon_2 m - 1 \), and taking \( \epsilon_2 \leq 1 \), we get that \( l_{d-t-1}(I) < \epsilon_2 (m - 1) \) as desired. \( \square \)

**Proposition 5.7.** Let \( t \in \{1, \ldots, d - 1\} \). For every \( \epsilon > 0 \) there exists \( \delta > 0 \) such that for every integer \( m > \frac{1}{\delta} \) and for every \( b \in [1, \delta m] \), if a Cox-Gorenstein ideal \( I \) with socle degree \( N \) satisfies:

i) \( N = (t+1)\beta - \beta_0 \) and \( N - \beta = n \eta \);

ii) \( I \) contains \( d+1 \) homogenous polynomials in quasi-smooth intersection \( \{f_i\}_{i=0}^d \) with \( \deg f_i = \beta \) and whose associated ideal is base point free;

iii) \( \text{codim } I^3 \leq b^m t! \) where \( m = \max\{i \in \mathbb{N}^+ \mid i \eta \leq \beta\} \),

then \( I \) contains the ideal \( I_V \) of a closed scheme \( V \subset \mathbb{P}_\Sigma^d \) of pure dimension \( t \) and degree less than or equal to \( (1 + \epsilon)b \). Moreover, \( I \) and \( I_V \) coincide in degree less than or equal to

\[
\beta - (d-t)(\deg V) \eta \in \text{Pic}(\mathbb{P}_\Sigma)
\]

where \( \deg V \) is the number \([V] \cdot \eta^t\).

**Proof.** The ideal \( I \) satisfies the assumptions of Lemma \( \text{[5.6]} \), so there are \( f_0, f_1, \ldots, f_{d-t-1} \in I^{\varepsilon_2(m-1)} \) in quasi-smooth intersection and by Corollary 1.5 in \( \text{[13]} \) with \( \dim V^n(< f_1, \ldots, f_{d-t-1} >) = t \). By the second assumption, it is possible to find \( t+1 \) polynomials \( f_{d-t}, \ldots, f_d \), where \( \deg(f_i) = \beta \) ( \( i > d-t-1 \) ), so that, the ideal \( < f_0, \ldots, f_d > \) is a Cox-Gorenstein ideal of with socle degree

\[
\sum_{i=0}^{d-t-1} \deg(f_i) + \sum_{i=d-t}^{d} \deg(f_i) - \beta_0 \leq (d-t)((m-1)\epsilon_2)\eta + (t+1)\beta - \beta_0.
\]

Now, by inclusion of Cox-Gorenstein ideals, there exists a homogeneous polynomial \( P \) not contained in \( < f_0, \ldots, f_d > \), with

\[
\deg P \leq (d-t)((m-1)\epsilon_2)\eta + (t+1)\beta - \beta_0 - N = (d-t)((m-1)\epsilon_2)\eta
\]

such that \( I = ((f_0, \ldots, f_d) : P) \). Moreover, \( I \) and \( J = ((f_0, \ldots, f_{d-t-1}) : P) \) coincide in degree less than
\[ \beta - \deg P > \beta - (d - t)(m - 1)\epsilon_2 \eta \leq (m - (d - t)(m - 1)\epsilon_2)\eta; \]

Now, let us set
\[ l = [m - (d - t)(m - 1)\epsilon_2] \]
and apply the previous results to \( I^n \), which is equivalent to find \( i \) such that \( \text{codim} I^{\beta - i} = \text{codim} I^{N - \beta + i} = \text{codim} I_1^{l} (i = l - n) \). Note that \( i \leq m\delta_1 \) and \( l - n \leq m\delta_1 \) for every \( m \geq \frac{\delta_2}{\eta_1} \), with \( r = \min \{ i \in \mathbb{N} \mid \beta_0 \leq i \eta \} \).

Then, for every \( x \leq \text{min}(\deg V, l) \)

one has
\[ x(x - 1)! \leq \text{codim} I^l \leq (1 + \epsilon_1) b \]

and
\[ x(1 - \frac{(d - t)(m - 1)\epsilon_2)}{m} + x)^l \leq (1 + \epsilon_1) b \]

so that, taking \( x = [(d - t)m\epsilon_2 + 2] \), we get
\[ x \leq \frac{(1 + \epsilon_1) b}{(1 - 2(d - t)\epsilon_2)^l} \]

Then for \( 0 < \epsilon \leq 1 \) and taking \( \epsilon_1 \) and \( \epsilon_2 \) such that
\[ \frac{(1 + \epsilon_1) b}{(1 - 2(d - t)\epsilon_2)^l} \leq (1 + \epsilon) b, \]

one has \( x \leq (1 + \epsilon) b < 2b \), but on the other hand for \( \delta \leq \min(\frac{\delta_1}{\eta_1}, \delta_2, \frac{\delta_2}{\eta_1}) \)
\[ 2b \leq 2\eta m < |m\epsilon_2| \leq |(d - t)m\epsilon_2| + 2 \leq x, \]

which is a contradiction. Thus, \( \deg(V) = x \) and \( \deg(V) \leq (1 + \epsilon) \).

Now, since \( V \) has pure dimension \( t \), there exist \( d - t \) polynomials with degree less than or equal to \( \deg V \). The way to get those polynomials is as follows. By the construction of \( V \), we can find \( t + 1 \) hyperplanes in quasi-smooth intersection \( \{ H_i \}_{i=t}^d \) such that \( V \cap V_t(< H_{d-t}, \ldots, H_d>) = \emptyset \).

Now, since \( \text{dim} V_t(< H_{d-t}, \ldots, H_d>) = d - t - 1 \) we can choose \( d - t - 1 \) points \( \{ p_i \}_{i=t}^{d-t-1} \) in general position so that we get cone with base \( V \) and vertices \( \{ p_i \}_{i=t}^{d-t-1} \), which is a divisor. So we get a polynomial which has degree (intersection with \( \eta^{d-1} \)) less than or equal to \( \deg V \). Varying the choice of the points we get the other required polynomials. Thus, we have \( f_0, \ldots, f_{d-t-1} \in I^{\delta_{(\deg V)^n}} \) and \( f_{d-t}, \ldots, f_d \) with degree \( \beta \) and reasoning as the beginning of the proof we get that \( I \) and \( I_V \) coincide in degree less than \( \beta - (d - t)(\deg V)\eta \).
6 The tangent space to the Noether-Lefschetz loci

Since $\mathbb{P}^{2k+1}_\Sigma$ has a pure Hodge structure [17, [20], there is a well defined residue map for it, and we can use it to construct the tangent space at a point of the Noether-Lefschetz locus. This is again basically done as in [15], however we provide more details, and use the properties of the residue map as developed in [1] for simplicial toric varieties.

Let $X = \{ f = 0 \}$ be a quasi-smooth hypersurface in $\mathbb{P}_\Sigma$, with deg $f = \beta$. Denote by $i : X \to \mathbb{P}_\Sigma$ the embedding, and by $i^* : H^\bullet(\mathbb{P}^{2k+1}_\Sigma, \mathbb{Q}) \to H^\bullet(X, \mathbb{Q})$ the associated morphism in cohomology; $i^* : H^{2k}(\mathbb{P}^{2k+1}_\Sigma, \mathbb{Q}) \to H^{2k}(X, \mathbb{Q})$ is injective by a suitable version of the Lefschetz hyperplane theorem, see Proposition 10.8 in [1].

Definition 6.1. The primitive cohomology group $H^{2k}_{\text{prim}}(X)$ is the quotient

$$H^{2k}(X, \mathbb{Q})/i^*(H^{2k}(\mathbb{P}^{2k+1}_\Sigma, \mathbb{Q})).$$

Both $H^{2k}(\mathbb{P}^{2k+1}_\Sigma, \mathbb{Q})$ and $H^{2k}(X, \mathbb{Q})$ have pure Hodge structures, and the morphism $i^*$ is compatible with them, so that $H^{2k}_{\text{prim}}$ inherits a pure Hodge structure [1].

Theorem 6.2. $T_{[f]}(NL^{k, \beta}_{\lambda(U)}) \cong E^\beta$, where

$$E = \{ K \in B^\bullet | \sum_{i=1}^b \lambda_i \int_{\text{Tub}} K R \Omega_0 \frac{f}{k+1} = 0 \text{ for all } R \in S^{N-\bullet} \},$$

and $\text{Tub}(-)$ is the adjoint of the residue map.

Proof. By [2, Prop. 2.10] the $p$-th residue map

$$r_p : H^0(\mathbb{P}_\Sigma, \Omega^{2k+1}_{\Sigma}(2k+1-p)X) \to H^{p,2k-p}_{\text{prim}}(X) \text{ for } 0 \leq p \leq 2k$$

exists; it is surjective and has kernel $H^0(\mathbb{P}_\Sigma, \Omega_{\Sigma}^{2k+1}(2k-p)X) + dH^0(\mathbb{P}_\Sigma, \Omega_{\Sigma}^{2k}(2k-p)X)$. So

$$\text{res} \ H^0(\Omega^{2k+1}_{\Sigma}(2k+1)X) = r_{2k} H^0(\Omega^{2k+1}(X)) \oplus \cdots \oplus r_0 H^0(\Omega^{2k+1}(2k+1)X))$$

by definition of $H^0(\Omega^{2k+1}_{\Sigma}(2k+1)X)$. Or, equivalently,

$$\text{res} \ H^0(\Omega^{2k+1}_{\Sigma}(2k+1)X) = H^{2k,0}_{\text{prim}}(X) \oplus \cdots \oplus H^{0,2k}_{\text{prim}}(X) = H^{2k}_{\text{prim}}(X).$$

Similarly

$$\text{res} \ H^0(\Omega^{2k+1}(kX)) = F^{k+1}H^{2k}_{\text{prim}}(X).$$

On the other hand by [1, Thm 9.7] we have

$$H^0(\Omega^{2k+1}_{\Sigma}(kX)) = \left\{ \frac{K \Omega_0}{f^k} | K \in S^{k\beta-\beta_0} \right\} = \left\{ \frac{K \Omega_0}{f^k} | K \in B^{k\beta-\beta_0} \right\};$$

the last equality holds true because we are assuming that $k\beta - \beta_0$ is ample and hence $B^{k\beta-\beta_0} = S^{k\beta-\beta_0}$ by Lemma 9.15 in [1] and $\Omega_0$ as in Definition 4.3.
Now fixing a basis \( \{ \gamma_i \}_{i=1}^b \) for \( H_{2k}(X, \mathbb{Q}) \), we have that the components of any element in \( F^{k+1}H^2_{\text{prim}}(X) \) are
\[
\left( \int_{\gamma_1} \text{res} \frac{K \Omega_0}{f^k}, \ldots, \int_{\gamma_b} \text{res} \frac{K \Omega_0}{f^k} \right),
\]
or, equivalently,
\[
\left( \int_{\text{Tub}(\gamma_1)} \frac{K \Omega_0}{f^k}, \ldots, \int_{\text{Tub}(\gamma_b)} \frac{K \Omega_0}{f^k} \right).
\]
Taking \( 0 \neq \lambda_f \in H^{k,k}(X, \mathbb{Q}) \) one has \( \lambda_f \perp F^{k+1}H^2_{\text{prim}}(X) \) (see [13]) and since the sheaf \( \mathcal{H}^2 \) is constant on \( U \) we have
\[
NL^{k,\beta}_{\lambda,U}(\gamma) = \{ G \in U \mid \lambda_G \in F^kH^2_{\text{prim}}(X_G) \} = \{ G \in U \mid \lambda_f \perp F^{k+1}H^2_{\text{prim}}(X_G) \}.
\]
More explicitly, if \( \{ \lambda_1, \ldots, \lambda_b \} \) are the components of \( \lambda_f \), one gets
\[
\lambda_f \perp F^{k+1}H^2_{\text{prim}}(X) \iff \sum_{i=1}^b \lambda_i \int_{\text{Tub}(\gamma_i)} \frac{K \Omega_0}{f^k} = 0 \quad \forall K \in S^{N-\beta}
\]
where \( N \) is equal to \( (k+1)\beta - \beta_0 \). Thus we can characterize the local Noether-Lefschetz locus in the following way.

Let us consider the differentiable map \( \psi \) which assigns to every homogeneous polynomial \( G \in B^\beta \) a linear map \( \psi_G : (B^{N-\beta})^\gamma \to (B^{N-\beta})^\gamma \) sends \( G \) to
\[
\psi_G : B^{N-\beta} \to \mathbb{C}, \quad K \mapsto \sum_i \lambda_i \int_{\text{Tub}(\gamma_i)} \frac{K \Omega_0}{G^k};
\]
then \( NL^{k,\beta}_{\lambda,U} = \psi^{-1}_G(0) \), hence the tangent space at \( f \) is the kernel of \( d\psi_f \). Now \( T_{[f]}U \cong S^3 \) and since \( \beta \) is ample, \( S^\beta = B^\beta \). Thus we can identified canonically \( T_{[f]}(NL^{k,\beta}_{\lambda,U}) \) with the subspace \( E^\beta \subset B^\beta \), which is the \( \beta \)-summand of the Cox-Gorenstein ideal
\[
E = \{ K \in B^* \mid \forall R \in S^{N-\bullet}, \sum_{i=1}^b \lambda_i \int_{\text{Tub}(\gamma_i)} \frac{KR \Omega_0}{f^{k+1}} = 0 \}
\]
whose socle degree is \( N = (k+1)\beta - \beta_0 \).

**Remark 6.3.** Note that \( E \) contains the toric Jacobian ideal \( J(f) \) which is Cox-Gorenstein. △

We also consider the Cox-Gorenstein ideals
\[
E_s := \{ K \in B^* \mid \forall R \in S^{N+r\beta-\bullet}, \sum_{i=1}^b \lambda_i \int_{\text{Tub}(\gamma_i)} \frac{KR \Omega_0}{f^{k+r+1}} = 0 \}
\]
with \( s \in \mathbb{N}^+ \), which have socle degree \( N+r\beta \). For a fixed \( s \), the ideal \( E_s \) describes the deformation of order \( s+1 \) of \( NL^{k,\beta}_{\lambda,U} \) in a neighborhood of \( f \).

**Proposition 6.4.** The Cox-Gorenstein ideals \( E_s \) have the following properties:

i. \( E_s = (E_{s+1} : f) \);

ii. If \( f \) is a generic point of \( NL^{\text{red}}_{\lambda,U} \) then \( (E_r)^2 \Theta \subset E_{s+1} \), where \( \Theta \subset S^\beta = B^\beta \) is the image of the tangent space \( T_f(NL_{\lambda,U})^{\text{red}} \);
iii. for all \( K \in E_s \) and all \( j \in \{1, \ldots, r\} \) one has \( \frac{\partial K}{\partial x_j} f - (k + s + 1)K \frac{\partial f}{\partial x_j} \in E_{s+1} \).

Proof. 1. Clear.

2. For every \( G \in NL_{k,\beta}^{k,\beta} \) and for every \( i \in \mathbb{N}^+ \) such that \( N + \beta - i \eta \) is ample, consider the bilinear map

\[
\mathcal{Q}_i(G) : B^{in} \times B^{N+r-\beta+i} \to \mathbb{C}
\]

\[
(K, R) \mapsto \sum_{i=1}^{b} \lambda_i \int_{\text{Tub}_{\gamma_i}} \frac{KR\Omega}{G^{s+1}}
\]

For a fixed \( R \) we have ker \( \mathcal{Q}_i(G) = E^i_s(G) \), and for a fixed \( K \) we have

\[
\ker \mathcal{Q}_i(G) = E_s(G)^{N+r-L-iD},
\]

where \( E_s(G) \) is the Cox-Gorenstein ideal associated to the class \( \lambda_G \). Since \( f \) is a quasi-smooth point of \( (NL_{k,\beta}^{k,\beta})^{\text{red}} \), the map \( G \to \mathcal{Q}_i(G) \) has constant rank for every \( G \) close to \( f \). So for each \( \tilde{v} \in T_f(NL_{k,\beta}^{k,\beta})^{\text{red}} \) associated to \( M \in \Theta \) the differential of the bilinear map

\[
d\mathcal{Q}_i(f)(\tilde{v}) : S^{in} \times S^{N+r-\beta-i} \to \mathbb{C}
\]

\[
(K, R) \mapsto -(k + s + 2) \sum_{i=1}^{b} \lambda_i \int_{\text{Tub}_{\gamma_i}} \frac{KR\Omega}{G^{s+1}}
\]

is zero on \( E^i_s \times E^i_s^{N+r-\beta-i} \), or, in other words, \( E^i_s E_s^{N+r-\beta-i} \Theta \subset E^{N+(s+1)\beta}_{s+1} \).

3. Given \( K \in E_s \), for every \( R \in S^{N+s+\eta-\deg(K)} \) we have

\[
R \left( \frac{\partial K}{\partial x_i} f - (k + s + 1)K \frac{\partial f}{\partial x_i} \right) = 
\left( \frac{\partial (KR)}{\partial x_i} f - (k + r + 1)KR \frac{\partial f}{\partial x_i} \right)_A - KR \frac{\partial R}{\partial x_i} _B.
\]

Note that \( \frac{A\Omega_0}{f^{s+1}} \) is an exact form in the kernel of the residue map, so that \( A \in E_{s+1} \). By assumption \( K \frac{\partial R}{\partial x_i} \in E_s \) so \( B \in E_{s+1} \) by the first property. Thus \( R \left( \frac{\partial K}{\partial x_i} f - (k + r + 1)K \frac{\partial f}{\partial x_i} \right) \in E_{s+1} \) and since \( R \) is arbitrary we get the result. \( \square \)

7 The Main Theorem

Now we have all the machinery necessary to prove our main result.

**Theorem 7.1.** For every \( \epsilon > 0 \) there exists \( \delta > 0 \) such that for all \( m \geq \frac{1}{\delta} \) and for all \( b \in [1, m\delta] \), if codim \( X_{k,\beta}^{k,\beta,\lambda,U} \leq \frac{b}{m} \) where \( m = \max\{i \mid \eta \leq \beta\} \) and if \( G \in N_{k,\beta}^{k,\beta} \), then there exists a \( k \)-dimensional subvariety \( V \subset X_G \) with degree less than or equal to \( 1 + \epsilon \).

**Proof.** If \( f \) is a generic point in \( (NL_{k,\beta}^{k,\beta})^{\text{red}} \), by Proposition 5.7 there exists a subscheme \( V \subset \mathbb{P}_\Sigma \) of pure dimension \( k \) and degree \( b' \leq (1 + \epsilon)b \leq 2\delta m \) such that \( I_V \subset E \) and the two ideals coincide in degree less or equal to \( \beta - (d - k)(\deg V) \eta \), so it is enough to prove that \( f \in \sqrt{I_V} \). Firstly, let us fix:

- \( 0 < \epsilon \leq 1 \)
- \( \delta \) satisfying Proposition 5.7 and \( \delta < \frac{1}{4(k+1)} \).
\[ \epsilon_2 = \frac{1}{2(k+1)} \] in Lemma 5.6 and \( \delta \) small enough such that \( \delta_2 = \delta \).

Secondly, let us prove the following

Claim 1. \( (I_{V'}^{\beta/\eta})^2 \subset E_1 \).

Let \( L \in (I_{V'}^{\beta/\eta})^2 \); by Proposition 6.4 one has \( \Theta \subset (E_1 : R) \). By assumption \( \text{codim}(E_1 : R) \leq b \), and hence we are under the assumptions of Lemma 5.6, so that for \( \epsilon_2 = \frac{1}{2(k+1)} \) there exists a \( \delta_2 \) which we take to be equal to \( \delta \). Hence,

\[ l_i(E_1 : L) \leq \frac{m-1}{2(k+1)} \quad \forall i \in \{0, \ldots, k\} . \]

On the other hand, by the form of \( L \) its derivatives belong to \( E \) and by the second part of Proposition 6.4 its derivatives belong to \( (E_1 : L) \) and hence to the toric jacobian as well.

Now, by contradiction if \( L \notin E_1 \) then \( (E_1 : L) \) is a Cox-Gorenstein ideal with socle degree \( N + \beta - \deg L \geq N + \beta - 2b' \geq N + (1 - 4\delta)m\eta \) but by the above construction \( (E_1 : L) \) contains a Cox-Gorenstein sub-ideal with socle degree

\[ (k+1) \frac{m-1}{2(k+1)} + (k+1)\beta - \beta_0 \leq N + \frac{m}{2} \eta \]

and since \( \delta < \frac{1}{2} \), we get a contradiction.

Claim 2: \( f \in \sqrt{I_V} \).

It is enough to prove that \( f \in \sqrt{I_W} \) for every irreducible subscheme \( W \subset V \) associated to the primary decomposition of \( I_V \).

Let \( W' \subset V \) the smallest subscheme such that \( I_V = I_W \cap I_{W'} \) and let \( P \) be a general linear \((k-1)\)-dimensional projective subspace with Cox ring \( S_P = \mathbb{C}[x_0, \ldots, x_{k+1}] \) in such way that \( I_W \cap S_P \subset S_P \) and \( I_{W'} \cap S_P \subset S_P \) correspond to the divisor in \( P \). Let \( K_{P,W} \) be their equations in \( S \). Note that \( K_{P,W} \) and \( K_{P,W'} \) are cones with bases \( W \) and \( W' \) and vertices on \( P \). Thus, \( \tau = \deg K_{P,W} \leq \deg W \) and \( \tau' = \deg K_{P,W'} \leq \deg W' \). Considering \( K_P = K_{P,W}K_{P,W'}^2 \) by its form \( K_P \in I_V \subset E \). Since no derivatives of \( K_P \) belong to \( I_W \) \((\dim V_T(I_W) = k)\), they do not all belong to \( I_V \) either, and since \( I_V \) coincides with \( E \) in degree strictly less than \( \tau + 2\tau' \), not all the partial derivatives of \( K_P \) belong to \( E \). Now, by the previous Proposition, we have \( K_P \notin E_1 \) and thus \( (E_1 : K_P) \) is a Cox-Gorenstein ideal with socle degree \( N + \beta - (\tau + 2\tau') \). Note that, \( (E_1 : K_P) \) contains the derivatives \( \frac{\partial f}{\partial x_{k+2}}, \ldots, \frac{\partial f}{\partial x_{2k+1}} \) and moreover since \( I_W^{\deg W} K_P \subset (I_{V'}^{\beta/\eta})^2 \) and by claim 1 \( (I_{V'}^{\beta/\eta})^2 \subset E_1^{2b'} \), we have \( I_W \subset (E_1 : K_P)^{\deg W} \).

So, the ideal \( (E_1 : K_P) \) contains the ideal

\[ J_P^0 = (f, I_W^{\deg W}, x_{k+2} \frac{\partial f}{\partial x_{k+2}}, \ldots, x_{2k+1} \frac{\partial f}{\partial x_{2k+1}}) \]

By contradiction, if \( f \notin \sqrt{I_W} \), since \( W \) is irreducible, the base locus of the ideal \( (f, I_W^{\deg W}) \) has dimension less or equal to \( k-1 \). The ideal \( J_P \) contains a Cox-Gorenstein ideal with socle degree \((k+1)(\deg W) \eta + (k+1)\beta - \beta_0 \leq (k+1)b' + N \). On the other hand, since \( J_P^0 \subset (E_1 : K_P) \) then

\[ N + \beta - (\tau + 2\tau') \geq N + \beta - 3b' \eta \]
which is a contradiction since $\delta < \frac{1}{k+4}$. 

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