SEMISTABILITY AND RESTRICTIONS OF TANGENT BUNDLE TO CURVES

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ABSTRACT. We consider all complex projective manifolds $X$ that satisfy at least one of the following three conditions:

(1) There exists a pair $(C, \varphi)$, where $C$ is a compact connected Riemann surface and $\varphi : C \longrightarrow X$ a holomorphic map, such that the pull back $\varphi^*TX$ is not semistable.

(2) The variety $X$ admits an étale covering by an abelian variety.

(3) The dimension $\dim X \leq 1$.

We prove that the following classes are among those that are of the above type.

- All $X$ with a finite fundamental group.
- All $X$ such that there is a nonconstant morphism from $\mathbb{CP}^1$ to $X$.
- All $X$ such that the canonical line bundle $K_X$ is either positive or negative or $c_1(K_X) \in H^2(X, \mathbb{Q})$ vanishes.
- All $X$ with $\dim \mathbb{C}X = 2$.

1. INTRODUCTION

The tangent bundle of a complex projective manifold equipped with a polarization is often semistable. For example, if $X$ is a complex projective manifold such that the canonical line bundle $K_X$ is ample, then the tangent bundle $TX$ is semistable with respect to the polarization defined by $K_X$. More generally, if $X$ admits a Kähler–Einstein metric then $TX$ is semistable. Let $V$ be a holomorphic vector bundle on a complex projective manifold $X$ equipped with a very ample line bundle $\zeta$. If $V$ is semistable, then the restriction of $V$ to any smooth complete intersection curve in $X$, obtained by intersecting hyperplanes from the linear systems of sufficiently large powers of $\zeta$, remains semistable (see [10, Ch. 7]).

Here we consider all connected complex projective manifolds $X$ with the property that for every pair of the form $(C, \varphi)$, where $C$ is a compact connected Riemann surface and $\varphi : C \longrightarrow X$ a holomorphic map, the pull back $\varphi^*TX$ is a semistable vector bundle over $C$. If $\dim \mathbb{C}X \leq 1$, or $X$ admits an étale covering by an abelian variety, then $X$ satisfies this condition (if $A \longrightarrow X$ is an étale covering with $A$ an abelian variety, then the pull back of $\varphi^*TX$ to the fiber product $C \times _X A$ is trivial; hence $\varphi^*TX$ is semistable). We conjecture that these are all.

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For convenience, let us define that a connected complex projective manifold $M$ satisfies Condition $C$ if at least one of the following three statements holds:

1. There exists a pair $(Y, \varphi)$, where $Y$ is a compact connected Riemann surface and $\varphi : Y \to M$ a holomorphic map, such that $\varphi^*TM$ is not semistable.
2. The variety $M$ admits an étale covering by an abelian variety.
3. The variety $M$ is a curve or a point.

The above conjecture says that all connected complex projective manifolds satisfy Condition $C$. We prove that the following classes satisfy Condition $C$:

- All $M$ with a finite fundamental group (Theorem 2.3).
- All $M$ such that there is a nonconstant morphism from $\mathbb{CP}^1$ to $M$ (Proposition 3.1).
- All $M$ such that either the canonical line bundle $K_M$ is ample or $K_M^{-1}$ is ample or $c_1(K_M) \in H^2(M, \mathbb{Q})$ vanishes (Corollary 4.2).
- All $M$ with $\dim_{\mathbb{C}} M = 2$ (Proposition 5.1).

2. Flat connection and fundamental group

Let $M$ be an irreducible smooth complex projective variety. The complex dimension of $M$ will be denoted by $d$. Let $\mathbb{P}(TM)$ denote the projectivized tangent bundle that parametrizes all lines in the tangent spaces of $M$. Let $FPGL(d, \mathbb{C})$ be the holomorphic principal $PGL(d, \mathbb{C})$ over $M$ defined by $\mathbb{P}(TM)$.

We recall that a holomorphic connection on $\mathbb{P}(TM)$ is a holomorphic splitting of the Atiyah exact sequence for the $PGL(d, \mathbb{C})$–bundle $FPGL(d, \mathbb{C})$ (see [1, page 188, Definition]). The projective bundle $\mathbb{P}(TM)$ admits a flat holomorphic connection if and only if it admits local holomorphic trivializations such that all the transition functions are locally constant.

**Proposition 2.1.** Assume that $M$ has the property that for every pair of the form $(Y, \varphi)$, where $Y$ is a compact connected Riemann surface and $\varphi : Y \to M$ a holomorphic map, the pull back $\varphi^*TM$ is a semistable vector bundle over $Y$. Then the projective bundle $\mathbb{P}(TM)$ admits a flat holomorphic connection.

**Proof.** Consider the adjoint action of $PGL(d, \mathbb{C})$ on the Lie algebra $M(d, \mathbb{C})$ of $GL(d, \mathbb{C})$. Let

$$\rho : PGL(d, \mathbb{C}) \to GL(M(d, \mathbb{C})) =: G$$

be the corresponding homomorphism. This homomorphism $\rho$ is injective.

The Lie algebra of $G$ in Eq. (2.1) will be denoted by $\mathfrak{g}$. The Lie algebra of $PGL(d, \mathbb{C})$ is the subalgebra of $M(d, \mathbb{C})$ defined by the trace zero matrices. We will denote the Lie algebra of $PGL(d, \mathbb{C})$ by $M_0(d, \mathbb{C})$ Let

$$\hat{\rho} : M_0(d, \mathbb{C}) \to \mathfrak{g}$$
be the injective homomorphism of Lie algebras associated to $\rho$ in Eq. (2.1). The group $\text{PGL}(d, \mathbb{C})$ has the adjoint action $M_0(d, \mathbb{C})$, and $\text{PGL}(d, \mathbb{C})$ acts on $\mathfrak{g}$ by combining the homomorphism $\rho$ and the adjoint action of $G$ on $\mathfrak{g}$. The injective homomorphism $\hat{\rho}$ in Eq. (2.2) is a homomorphism of $\text{PGL}(d, \mathbb{C})$–modules. The group $\text{PGL}(d, \mathbb{C})$ is reductive. Hence any short exact sequence of $\text{PGL}(d, \mathbb{C})$–modules splits. Therefore, there is a homomorphism of $\text{PGL}(d, \mathbb{C})$–modules $(2.3)$

$$\eta : \mathfrak{g} \longrightarrow M_0(d, \mathbb{C})$$

such that $\eta \circ \hat{\rho} = \text{Id}_{M_0(d, \mathbb{C})}$. Fix such a homomorphism $\eta$.

Let

$$F_G := F_{\text{PGL}(d, \mathbb{C})}(G)$$

be the holomorphic principal $G$–bundle over $M$ obtained by extending the structure group of $F_{\text{PGL}(d, \mathbb{C})}$ using the homomorphism $\rho$ in Eq. (2.1). We recall that $F_G$ is a quotient of $F_{\text{PGL}(d, \mathbb{C})} \times G$. Two points $(z_1, g_1)$ and $(z_2, g_2)$ of $F_{\text{PGL}(d, \mathbb{C})} \times G$ are identified in $F_G$ if and only if there is an element $g \in \text{PGL}(d, \mathbb{C})$ such that

$$(z_2, g_2) = (z_1 g^{-1}, \rho(g) g_1).$$

We have a holomorphic map

$$\alpha : F_{\text{PGL}(d, \mathbb{C})} \longrightarrow F_G$$

(2.5)

that sends any $z \in F_{\text{PGL}(d, \mathbb{C})}$ to the element in $F_G$ defined by $(z, e)$. Since $\rho$ is injective, the map $\alpha$ in Eq. (2.5) is an embedding.

We note that the vector bundle over $M$ associated to $F_G$ for the standard action of $G = \text{GL}(M(d, \mathbb{C}))$ on $M(d, \mathbb{C})$ is the endomorphism bundle

$$\mathcal{E}nd(TM) = TM \otimes \Omega^1_M.$$
the restriction of $\nabla^G$ to any fiber of the projection $F_G \to M$ coincides with the Maurer–Cartan form, and

- the form $\nabla^G$ is equivariant for the action of $G$ on $F_G$ and the adjoint action of $G$ on $\mathfrak{g}$.

Consider the $\mathfrak{g}$–valued holomorphic one–form $\alpha^*\nabla^G$ on $F_{\text{PGL}(d, \mathbb{C})}$, where $\alpha$ is the embedding constructed in Eq. (2.5). The composition $\eta \circ (\alpha^*\nabla^G)$, where $\eta$ is the projection in Eq. (2.3), is a $M_0(d, \mathbb{C})$–valued holomorphic one–form on $F_{\text{PGL}(d, \mathbb{C})}$.

Since $\eta$ in Eq. (2.3) is a homomorphism of $\text{PGL}(d, \mathbb{C})$–modules satisfying the condition that $\eta \circ \hat{\rho} = \text{Id}_{M_0(d, \mathbb{C})}$, it follows that $\eta \circ (\alpha^*\nabla^G)$ defines a holomorphic connection on the principal $\text{PGL}(d, \mathbb{C})$–bundle $F_{\text{PGL}(d, \mathbb{C})}$.

The curvature of this holomorphic connection on $F_{\text{PGL}(d, \mathbb{C})}$ defined by $\eta \circ (\alpha^*\nabla^G)$ clearly coincides with $\eta \circ \mathcal{K}(\nabla^G)$, where $\mathcal{K}(\nabla^G)$ is the curvature of the connection $\nabla^G$ on $F_G$. But $\nabla^G$ is flat. Hence the holomorphic connection on $F_{\text{PGL}(d, \mathbb{C})}$ defined by $\eta \circ (\alpha^*\nabla^G)$ is flat. This completes the proof of the proposition.

But we put it down the following lemma for later reference.

**Lemma 2.2.** Take $M$ as in Proposition 2.1. Then

$$(d - 1)c_1(TM)^2 = 2d \cdot c_2(TM).$$

**Proof.** From [4, Theorem 1.2] we have $c_2(\text{End}(TM)) = 0$. Now the lemma follows from the fact that $c_2(\text{End}(TM)) = 2d \cdot c_2(TM) - (d - 1)c_1(TM)^2$. □

**Theorem 2.3.** Let $M$ be a connected complex projective manifold of complex dimension $d$, with $d \geq 2$. Assume that $M$ has the property that for every pair of the form $(Y, \varphi)$, where $Y$ is a compact connected Riemann surface and

$$\varphi : Y \to M$$

a holomorphic map, the pull back $\varphi^*TM$ is a semistable vector bundle over $Y$. Then the cardinality of the fundamental group of $M$ is infinite.

**Proof.** Assume that the fundamental group of $M$ is finite. Fix a universal cover

$$\gamma : \widetilde{M} \to M$$

of $M$. Since the fundamental group of $M$ is finite, this $\widetilde{M}$ is also a connected complex projective manifold of complex dimension $d$.

Let $Y$ be a compact connected Riemann surface, and let

$$\phi : Y \to \widetilde{M}$$

be a holomorphic map. Set

$$\varphi := \gamma \circ \phi,$$
where $\gamma$ is the map in Eq. (2.6). Since $\gamma$ is an étale covering, we have

$$\varphi^*TM = \phi^*\tilde{T}\tilde{M}.$$ 

Therefore, using the given condition on $M$ it follows that the vector bundle $\phi^*\tilde{T}\tilde{M}$ is semistable.

Hence from Proposition [2.1] we know that the projective bundle $\mathbb{P}(T\tilde{M})$ admits a flat holomorphic connection. On the other hand, $\tilde{M}$ is simply connected. Hence the projective bundle $\mathbb{P}(T\tilde{M})$ is trivial. This immediately implies that the tangent bundle $T\tilde{M}$ splits into a direct sum of holomorphic line bundles.

Since $T\tilde{M}$ splits into a direct sum of holomorphic line bundles, and $\tilde{M}$ is a compact connected Kähler manifold, using [6, page 242, Theorem 1.2] we know that $\tilde{M}$ is biholomorphic to the Cartesian product $(\mathbb{C}\mathbb{P}^1)^d$.

Fix a point $x_0 \in \mathbb{C}\mathbb{P}^1$. Consider the map

$$\phi : \mathbb{C}\mathbb{P}^1 \longrightarrow (\mathbb{C}\mathbb{P}^1)^d = \tilde{M}$$

defined by $x \longmapsto (x, x_0, \cdot \cdot \cdot, x_0)$. We have

$$\phi^*T\tilde{M} = \mathcal{O}_{\mathbb{C}\mathbb{P}^1}(2) \bigoplus (\mathcal{O}_{\mathbb{C}\mathbb{P}^1})^\oplus(d-1).$$

Since $d \geq 2$, it follows immediately from this decomposition that the vector bundle $\phi^*T\tilde{M}$ is not semistable.

This contradicts the earlier observation that $\phi^*T\tilde{M}$ is semistable. Hence the fundamental group of $M$ is infinite. This completes the proof of the theorem. \hfill \Box

3. Maps from the projective line

Let $M$ be a connected complex projective manifold of complex dimension $d$, with $d \geq 2$. Assume that $M$ has the property that for every pair of the form $(Y, \varphi)$, where $Y$ is a compact connected Riemann surface and

$$\varphi : Y \longrightarrow M$$
a holomorphic map, the pull back $\varphi^*TM$ is a semistable vector bundle over $Y$.

**Proposition 3.1.** There is no nonconstant morphism from $\mathbb{C}\mathbb{P}^1$ to $M$.

**Proof.** To prove by contradiction, let

$$f : \mathbb{C}\mathbb{P}^1 \longrightarrow M$$

be a nonconstant morphism. The given condition on $M$ says that $f^*TM$ is a semistable vector bundle over $\mathbb{C}\mathbb{P}^1$. Any holomorphic vector bundle over $\mathbb{C}\mathbb{P}^1$ splits into a direct sum of holomorphic of line bundles [9, page 122, Théorème 1.1]. Therefore, we have

$$f^*TM = (\mathcal{O}_{\mathbb{C}\mathbb{P}^1}(n))^{\oplus d}$$

for some integer $n$. 

The differential \( df : T\mathbb{CP}^1 \rightarrow f^*TM \) of the map \( f \) in Eq. (3.1) does not vanish identically because \( f \) is nonconstant. Since there is a nonzero homomorphism from \( T\mathbb{CP}^1 = \mathcal{O}_{\mathbb{CP}^1}(2) \) to \( f^*TM \), it follows that
\[
n \geq 2,
\]
where \( n \) is the integer in Eq. (3.2). Consequently, the pull back \( f^*TM \) is an ample vector bundle.

Since \( f^*TM \) is ample, the variety \( M \) is rationally connected (see [11, page 433, Theorem 2.1] and [11, page 434, Definition–Remark 2.2]). This in turn implies that \( M \) is simply connected [7, p. 545, Theorem 3.5], [13, page 362, Proposition 2.3]. But this contradicts Theorem 2.3.

Therefore, there is no nonconstant morphism from \( \mathbb{CP}^1 \) to \( M \). This completes the proof of the proposition. \( \square \)

4. The case of Kähler–Einstein manifolds

As before, \( M \) is a complex projective manifold of complex dimension \( d \), with \( d \geq 2 \). Assume that there exists a Kähler form \( \omega \) on \( M \) with the following property:

There is a non–positive real number \( \lambda \in \mathbb{C} \) such that the cohomology class of \( \lambda \cdot \omega \) coincides with the Chern class \( c_1(TM) \in H^2(M, \mathbb{R}) \).

A theorem due to Yau, [14], says that there is a Kähler metric \( \tilde{\omega} \) on \( M \) satisfying the following two conditions:

(1) the Kähler metric \( \tilde{\omega} \) is Kähler–Einstein, and

(2) the cohomology class \( [\tilde{\omega}] \in H^2(M, \mathbb{R}) \) coincides with that of \( \omega \).

(In [2], this was proved under the assumption that \( c_1(TM) \) is positive.)

**Theorem 4.1.** Assume that for every compact connected Riemann surface \( Y \), and for every holomorphic map
\[
\varphi : Y \rightarrow M,
\]
the pull back \( \varphi^*TM \) is a semistable vector bundle over \( Y \). Then \( M \) admits a flat Kähler metric.

**Proof.** Let \( \tilde{\omega} \) be the Kähler–Einstein metric on \( M \). We will show that \( \tilde{\omega} \) is projectively flat.

Consider the Hermitian structure \( \tilde{\omega}' \) on the vector bundle \( \text{End}(TM) \) induced by the Hermitian metric \( \tilde{\omega} \) on \( TM \). Since \( \tilde{\omega} \) is a Kähler–Einstein metric, it follows that \( \tilde{\omega}' \) is a Hermitian–Einstein metric. From Lemma 2.2 we know that
\[
c_2(\text{End}(TM)) = 2d \cdot c_2(TM) - (d - 1)c_1(TM)^2 \in H^2(M, \mathbb{Q})
\]
vanishes. In view of this, the condition that \( \tilde{\omega}' \) is a Hermitian–Einstein metric implies that \( \tilde{\omega}' \) is flat (see [13] Ch. IV, page 115, Theorem 4.11). Therefore, \( \tilde{\omega} \) is projectively flat.

Let \( \gamma : \tilde{M} \rightarrow M \)
be a universal cover $M$. The pulled back Kähler metric $\gamma^* \tilde{\omega}$ on $\tilde{M}$ is projectively flat because $\tilde{\omega}$ is projectively flat. Consequently, the holonomy of $\gamma^* \tilde{\omega}$ is contained in the center $U(1) \subset U(d)$, where $d = \dim_{\mathbb{C}} M$.

Since the holonomy of $\gamma^* \tilde{\omega}$ is contained in the center $U(1) \subset U(d)$, and $\tilde{M}$ is simply connected, we conclude the following. There are connected Riemann surfaces $C_i$, $1 \leq i \leq d$, equipped with Kähler forms $\omega_i$, such that the product Kähler manifold

$$\prod_{i=1}^{d} (C_i, \omega_i) = (\prod_{i=1}^{d} C_i, \bigoplus_{i=1}^{d} \omega_i)$$

is holomorphically isometric to $\tilde{M}$ equipped the Kähler form $\gamma^* \tilde{\omega}$ [8, page 49, Theorem 3.2.7].

Fix a point $(x_1, \cdots, x_{d-1}) \in \prod_{i=1}^{d-1} C_i$. Consider the holomorphic Hermitian vector bundle of rank $d$ on $C_d$ obtained by restricting to

$$(x_1, \cdots, x_{d-1}, C_d) \subset \prod_{i=1}^{d} C_i$$

the tangent bundle $T(\prod_{i=1}^{d} C_i)$ equipped with the Hermitian metric $\bigoplus_{i=1}^{d} \omega_i$. We note that this holomorphic Hermitian vector bundle on $C_d$ decomposes into a direct sum of $(TC_d, \omega_d)$ with the trivial holomorphic Hermitian vector bundle of rank $d-1$ with fiber $\bigoplus_{i=1}^{d-1} T_{x_i} C_i$ equipped with the Hermitian structure $\bigoplus_{i=1}^{d-1} \omega_i(x_i)$.

We observed earlier that $\gamma^* \tilde{\omega}$ is projectively flat. Since the restriction of the holomorphic Hermitian vector bundle $(T(\prod_{i=1}^{d} C_i), \bigoplus_{i=1}^{d} \omega_i)$ to $C_d$ decomposes into the direct sum of $(TC_d, \omega_d)$ with a trivial holomorphic Hermitian vector bundle of positive rank, it follows that $(TC_d, \omega_d)$ is a flat line bundle. Indeed, the condition that the restriction of $(T(\prod_{i=1}^{d} C_i), \bigoplus_{i=1}^{d} \omega_i)$ to $C_d$ is projectively flat implies that curvatures on $C_d$ of all the direct summands coincide.

Interchanging $(TC_d, \omega_d)$ with $(TC_i, \omega_i)$, $i \in [1, d-1]$, we conclude that

$$\prod_{i=1}^{d} (C_i, \omega_i)$$

is a flat Kähler manifold. Therefore, $\gamma^* \tilde{\omega}$ is a flat metric. This completes the proof of the theorem. \hfill \square

**Corollary 4.2.** Let $M$ be a connected complex projective manifold of complex dimension $d$, with $d \geq 2$. Assume that either the canonical line bundle $K_M$ is ample or $K_M^{-1}$ is ample or $c_1(K_M^{-1}) \in H^2(M, \mathbb{Q})$ vanishes. Then exactly one of the following two statements is valid:

1. There is a pair $(Y, \varphi)$, where $Y$ is a compact connected Riemann surface and

$$\varphi : Y \to M$$

a holomorphic map, such that the pull back $\varphi^* TM$ is not semistable.

2. There is an étale covering map $A \to M$, where $A$ is an abelian variety.
Proof. If $A \to M$ is an étale covering, where $A$ is an abelian variety, then the fiber product $Y \times_M A$ is an étale covering of $Y$, and furthermore, the pull back of $\varphi^*TM$ to $Y \times_M A$ is a trivial vector bundle. This implies that $\varphi^*TM$ is a semistable vector bundle. Hence the two statements in the corollary cannot be simultaneously valid.

If $K_M^{-1}$ is ample, then $M$ is rationally connected [12, p. 766, Theorem 0.1]. Hence in this case Proposition 3.1 implies that the first statement holds.

Now assume that either $K_M$ is ample or $c_1(K_M) \in H^2(M, \mathbb{Q})$ vanishes. Then we know that $M$ admits a flat Kähler metric (see Theorem 4.1). If $M$ admits a flat Kähler metric, then the universal cover of $M$ is $\mathbb{C}^d$, and the deck transformations are contained in the automorphisms of $\mathbb{C}^d$ that preserve the constant metric on $\mathbb{C}^d$. Consequently, the second statement in the corollary holds. This completes the proof of the corollary. □

Corollary 4.3. Let $M$ be a compact connected Kähler manifold such that the rank of the Néron–Severi group $\text{NS}(M) = H^2(M, \mathbb{Z}) \cap H^{1,1}(M)$ is one. Then exactly one of the following two statements is valid:

1. There is a pair $(Y, \varphi)$, where $Y$ is a compact connected Riemann surface and
   \[ \varphi: Y \to M \]
   a holomorphic map, such that the pull back $\varphi^*TM$ is not semistable.

2. The surface $M$ admits an étale covering map $A \to M$, where $A$ is an abelian variety.

Proof. If $\text{rank} (\text{NS}(M)) = 1$, then either $K_M$ is ample or $K_M^{-1}$ is ample or $c_1(K_M) \in H^2(M, \mathbb{Q})$ vanishes. Therefore, the corollary follows from Corollary 4.2. □

5. THE CASE OF A SURFACE

Let $M$ be an irreducible smooth complex projective surface.

Proposition 5.1. Exactly one of the following two statements is valid:

1. There is a pair $(Y, \varphi)$, where $Y$ is a compact connected Riemann surface and
   \[ \varphi: Y \to M \]
   a holomorphic map, such that the pull back $\varphi^*TM$ is not semistable.

2. The surface $M$ admits an étale covering by an abelian variety.

Proof. It was shown in the proof of Corollary 4.2 that the two statements cannot be simultaneously valid.

Assume that the first statement does not hold. So for every pair of the form $(Y, \varphi)$, where $Y$ is a compact connected Riemann surface and

\[ \varphi: Y \to M \]
a holomorphic map, the pull back $\varphi^*TM$ is a semistable vector bundle over $Y$. 
From Proposition 3.1 we know that $M$ is a minimal surface. If $M$ is of general type, then $c_2(TM) > 0$, and also the Miyaoka inequality
\[ c_1(TM)^2 \leq 3c_2(TM) \]
holds (see [3, page 207, Theorem (1.1)]). Hence
\[ c_1(TM)^2 - 4c_2(TM) < 0. \]
This contradicts Lemma 2.2. Hence $M$ is not of general type.

From Proposition 3.1 we know that $M$ is not a ruled surface.

Hence from the list of minimal projective surfaces (see [3, page 188, Table 10]) we know $c_1(TM)^2 = 0$. Therefore, from Lemma 2.2 we know that $c_2(TM) = 0$. Hence, from the list of minimal projective surfaces we conclude that exactly one of the following two statements holds:

1. The surface $M$ admits an étale covering by an abelian surface.
2. There is an elliptic fibration $M \to C$ with genus($C$) $\geq 2$.

The proof of the proposition will be completed by showing that the second statement does not hold. To prove this by contradiction, let
\[
\beta : M \to C
\]
be an elliptic fibration such that $C$ is a smooth projective curve of genus at least two.

Since there is no nonconstant map from $\mathbb{CP}^1$ to $M$, all the singular fibers of $\beta$ in Eq. (5.1) must be multiples of smooth elliptic curves. From this it follows that there is a finite covering
\[
\alpha : \widetilde{C} \to C
\]
with possible ramifications such that the normalization $\widetilde{M}$ of the fiber product $M \times_C \widetilde{C}$ is a smooth elliptic fibration over $\widetilde{C}$, and furthermore, the resulting morphism
\[
\gamma : \widetilde{M} \to M
\]
is an étale covering map. Note that since
\[ c_1(TM)^2 = 0 = c_2(TM), \]
using the Hirzebruch–Riemann–Roch theorem it follows that the Euler characteristic of $\mathcal{O}_M$ vanishes. Hence the above assertion can be deduced using [3, page 162, Remark] and [3, page 162, Proposition (12.2)].

Since $\widetilde{M} \to \widetilde{C}$ is a smooth elliptic fibration, the $j$–invariant map, that associates to each point $x \in \widetilde{C}$ the $j$–invariant of the fiber $\widetilde{M}_x$ over $x$, is in fact a constant map. Therefore, there is a finite étale Galois covering
\[
\alpha' : \widetilde{C'} \to \widetilde{C}
\]
such that
\[
\widetilde{M}' := \widetilde{M} \times_{\widetilde{C}} \widetilde{C}' = Z \times \widetilde{C'},
\]
where $Z$ is a smooth elliptic curve.
Let
(5.4) $\gamma' : \tilde{M}' \to M$
be the composition of the natural projection $\tilde{M}' \to \tilde{M}$ with the morphism $\gamma$ in Eq. (5.2). Let
(5.5) $\tilde{\gamma} : Z \times \tilde{C}' \to M$
be the composition of the identification $\tilde{M}' = Z \times \tilde{C}'$ in Eq. (5.3) with $\gamma'$ in Eq. (5.4).

Fix a point $z_0 \in Z$. Let
(5.6) $\varphi_0 : \tilde{C}' \to \tilde{M}' = Z \times \tilde{C}'$
be the map defined by $c \mapsto (z_0, c)$, where $\tilde{\gamma}$ is constructed in Eq. (5.5). Define
$$\varphi := \tilde{\gamma} \circ \varphi_0 = \tilde{\gamma}|_{z_0 \times \tilde{C}'} : \tilde{C}' \to M,$$
where $\tilde{\gamma}$ is constructed in Eq. (5.5). Since the map $\gamma'$ in Eq. (5.4) is étale, we have
(5.7) $\varphi^*TM = \varphi_0^*T\tilde{M}' = \mathcal{O}_{\tilde{C}'} \bigoplus T\tilde{C}'$.

Since genus($\tilde{C}'$) $\geq$ genus($\tilde{C}$) $\geq$ genus($C$) $\geq$ 2, we have degree($T\tilde{C}'$) $\neq$ 0. Hence from Eq. (5.7) it follows that $\varphi^*TM$ is not semistable.

But this contradicts the initial assumption that the first statement in the proposition does not hold. Therefore, there is no elliptic fibration $M \to C$ with genus($C$) $\geq$ 2. This completes the proof of the proposition. $\square$

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