Superfluidity in dipolar binary Bose mixtures

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We study the superfluidity and the coherence in dipolar binary Bose mixtures using the hydrodynamic approach. Useful analytical formula for the excitations spectrum, the correlation function and the superfluid fraction are derived. We find that in the case of highly imbalanced mixture, the superfluidity can occur in the dilute component only at extremely low temperatures. The behavior of the first-order correlation function for both dipolar and nondipolar Bose mixtures is deeply analyzed. Then we face the two-dimensional case which encodes a non-trivial physics due to the roton modes.

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I. INTRODUCTION

Superfluidity of either fermionic or bosonic ultracold gases is one of the most exciting phenomena in condensed matter systems. In the last years, superfluidity in ultracold atomic gas mixtures have attracted a great interest from both the experimental and theoretical sides owing of their potential applications in different areas of physics.

Mixtures of superfluid liquids which originally focused on helium atoms, were first addressed by Khalatnikov in 1957 using a hydrodynamic approach [1]. Bassichis [2] and Nepomnyashchii [3] employed the Bogoliubov method to study the superfluidity in Bose-Bose mixtures at zero temperature. Later on, Colson and Fetter [4] generalized the Bogoliubov method to finite temperature. Superfluidity in Bose-Fermi mixtures has been investigated in Refs (see e.g. [5, 6]). Recently, observation of spin superfluidity in a Bose gas mixture has been reported in Ref [7].

On the other hand, dipolar Bose-Bose mixtures has attracted tremendous attention due to the intriguing role played by their anisotropic nature and long-range character. More recently, Er-Dy mixture has been experimentally achieved in two-species magneto-optical trap [8]. Among the theoretical activity in binary mixtures with dipole-dipole interactions (DDIs), one can quote: rotonization [9], solitons [10], supersolidity (see e.g. [11, 12]), quantum fluctuations [13, 14] and quantum droplets [15].

The purpose of the present paper is therefore to extend the earlier works of Refs [2, 3] and check the existence of superfluidity in a dipolar Bose mixture. To this end, we use the hydrodynamic approach which has been successfully applied to study the properties of both three (3D) and low-dimensional single component Bose-Einstein condensate (BEC) (see for book [15, 19]). At zero temperature, the hydrodynamic equations can be derived from the standard Gross-Pitaevskii equation and thus, the hydrodynamic approach is valid only in the low-wavelength excitations regime. At finite temperatures, the validity of the theory requires fast thermalization times due to collisional relaxation [18−20]. Furthermore, due to the semi-classical nature of the hydrodynamic theory, its applicability necessitates that the mixture should be confined by a weakly-varying trap potential.

The hydrodynamic theory allows us to derive useful relations for the excitations energy and the superfluid fraction. General formulas for the first-order correlation function (one-body density matrix) in both dipolar and nondipolar mixtures are also derived. We show that such a quantity is tending to a constant at large distances signaling the existence of the long-range order and thus, the formation of true condensates. The role of the interspecies DDI and temperature in the one-body density matrix is discussed. Furthermore, it is found that the intra- and interspecies DDI may crucially reduce the superfluid fraction of the whole mixture and render it anisotropic. For a highly imbalanced mixture, the superfluid behavior disappears in the dilute component. Whereas, in the case of a balanced mixture, the superfluid fraction becomes identical to that of the one-component Bose liquid. In quasi-2D case, we derive useful dispersion relations which exhibit the roton-maxon structure. Effect of the rotonization on the coherence of the mixture is highlighted. We show that at low temperature, the superfluid density coincides with the result of a nondipolar mixture.

The rest of the paper is organized as follows. In Sec II, we introduce the main features of the hydrodynamic approach for dipolar Bose-Bose mixture. We derive general formulas for the excitations spectrum which we employ to analyze the coherence and the superfluidity. We discuss the 3D homogeneous case in Sec III and the quasi-2D homogeneous mixture in Sec IV. Finally, we summarize our results in Sec V.
II. HYDRODYNAMIC APPROACH

Consider two-component dipolar Bose-Bose mixture labeled \((j = 1, 2)\) in the weakly interacting regime. The dynamics of such a system is governed by the following nonlocal coupled GP equations:

\[ i\hbar \dot{\Phi}_j(r, t) = h_j \Phi_j(r, t) + \int dr' V_j(r - r') n_j(r', t) \Phi_j(r, t) + \int dr' V_{12}(r - r') n_{3-j}(r', t) \Phi_j(r, t), \]

where \(h_j = -(h^2/2m_j) \Delta + U_j(r)\) with \(m_j\) being the atomic mass and \(U_j(r)\) external traps, \(V_j(r)\) and \(V_{12}(r)\) are respectively, the intraspecies and interspecies two-body interactions potentials. The condensed density is defined as \(n_j(r) = |\Phi_j(r)|^2\), and \(\Phi_j(r)\) is the condensate wavefunction.

Writing the condensate wavefunction for each component in the form:

\[ \Phi_j(r, t) = \sqrt{n_j(r, t)} \exp[i\phi_j(r, t)], \]

where \(\phi\) is the phase which is real and related to the superfluid velocity as \(v_j = (\hbar/m) \nabla \phi_j\). After having substituting (2) in the coupled nonlocal GP equations (1) and separating real and imaginary parts, one obtaines the continuity and the Euler-like equations, respectively

\[ \frac{\partial n_j}{\partial t} + \nabla \cdot (n_j v_j) = 0, \]

and

\[ m \frac{\partial v_j}{\partial t} = -\nabla \left[ -\frac{\hbar^2}{2m} \frac{\Delta n_j}{n_j} + \frac{1}{2} m v_j^2 + U_j(r) \right] + \int dr' V_j(r - r') n_j(r') + \int dr' V_{12}(r - r') n_{3-j}(r'), \]

where \(\Delta \frac{n_j}{\sqrt{n_j}}\) represents the so-called quantum pressures.

We consider small fluctuations of the condensed density \(\delta n_j = n_j(r) + \delta n_j\) where \(\delta n_j/n_j \ll 1\). We then linearize Eqs. (3) and (4) with respect to \(\delta n_j\) and \(\nabla \phi_j\) around the stationary solution. The zero-order terms yield:

\[ \mu_j = -\frac{\hbar^2}{2m} \frac{\Delta n_j}{n_j} + U_j(r) + \int dr' V_j(r - r') n_j(r') + \int dr' V_{12}(r - r') n_{3-j}(r'), \]

where \(\mu_j\) is the chemical potential for each component. The first order terms provide equations for the density and phase fluctuations:

\[ \hbar \frac{\partial}{\partial t} \frac{\delta n_j}{\sqrt{n_j}} = \left[ \mathcal{L}_j + \int dr' V_j(r - r') n_j(r') \right] \]

\[ + \int dr' V_{12}(r - r') n_{3-j}(r') \]

\[ + \int dr' V_{12}(r - r') n_{3-j}(r') \]

\[ - 2 \int dr' V_j(r - r') \sqrt{n_j(r)} \delta \phi_j(r') \]

\[ + 2 \int dr' V_{12}(r - r') \sqrt{n_j(r)} \delta n_{3-j}(r'), \]

where \(\mathcal{L}_j = -(\hbar^2/2m_j) \Delta + U_j(r) - \mu_j\).

A. Elementary excitations

Writing the phase and the density in the basis of the excitations \(\phi_j(r) = (-i/2\sqrt{n_j(r)}) \sum_k [f_{jk}(r) \exp(-i\varepsilon_k t/\hbar) \hat{b}_{jk} - H.c.]\) and \(\delta n_j(r) = \sqrt{n_j(r)} \sum_k [f_{jk}(r) \exp(-i\varepsilon_k t/\hbar) \hat{b}_{jk} + H.c.]\), where \(\int df_{jk} f_{jk}^* + f_{jk}^* f_{jk} = 2\delta_{kk'}\). After some algebra, we obtain the nonlocal Bogoliubov-de Gennes equations (BdGE):

\[ \varepsilon_k f_{jk}^-(r) = \left[ \mathcal{L}_j + \int dr' V_j(r - r') n_j(r') \right] f_{jk}^+(r), \]

and

\[ \varepsilon_k f_{jk}^+(r) = \left[ \mathcal{L}_j + \int dr' V_j(r - r') n_j(r') \right] f_{jk}^-(r), \]

Equations (8) and (9) form a complete set to calculate the ground state and collective modes of dipolar Bose binary mixtures. They reduce to the usual BdG equations for vanishing inter- and intra-species DDI.

In the homogeneous mixture where \(U_j(r) = 0\) and \(f_{jk} = (\varepsilon_{1,2k}/E_k)^{1/2}\), the spectrum corresponding to the BdG equations (8) and (9) is found to be composed of upper \((\varepsilon_{1k})\) and lower \((\varepsilon_{2k})\) branches:

\[ \varepsilon_{1,2k} = \sqrt{E_k^2 + 2E_k \nu_{\pm}(k)}, \]
where \( n_{\pm} \) = \( V_1(k)n_1 F_\pm(k)/2 \), \( F_\pm(k) = 1 + \alpha \pm \sqrt{(1-\alpha)^2 + 4\Delta^{-1}\alpha} \), \( \alpha(k) = V_2(k)n_2/V_1(k)n_1 \), and
\[
\Delta(k) = \frac{V_1(k)V_2(k)}{V_1^2(k)}, \tag{11}
\]
is the miscibility parameter, where \( V_j(k) \) is the Fourier transform of the two-body interaction potential. For \( \Delta(k) > 1 \), the mixture is miscible while it is immiscible for \( \Delta(k) < 1 \). The spectrum \( \omega(\mathbf{k}) \) is identical to that derived earlier in \([26, 27]\). It is worth noticing that \( \beta \) is the miscibility parameter of a nondipolar mixture.

At low momenta (\( \mathbf{k} \to 0 \)), the Bogoliubov excitations \( \epsilon_j(k) = \sqrt{V_j(|\mathbf{k}|)} = \hbar c_j(0)k \) is the sound velocity of a single condensate. Therefore, the total dispersion reads
\[
\epsilon_{12k} = \hbar c_{12}(0)k, \tag{18}
\]
where the sound velocities \( c_{ij} \) are
\[
c_{12}^2(\theta) = \frac{1}{2} \left[ c_1^2 + c_2^2 \pm \sqrt{(c_1^2 - c_2^2)^2 + 4\Delta^{-1}c_1^2c_2^2} \right], \tag{19}
\]
and
\[
\Delta(k) = \frac{V_1(k)V_2(k)}{V_1^2(k)}, \tag{17}
\]
for \( \Delta(\theta) > 1 \), \( c_2 \) tends to zero indicating that the system becomes unstable and thus, the two condensates spatially separate. We see that the sound velocity is angular dependence due to the anisotropy of the DDI \([13]\).

For a nondipolar mixture \( c_{12}^2 = c_{12}^2 = 0 \) and at zero temperature, the one-body density matrix turns out to be given as
\[
g_j^{(1)}(s) = n_j + \int_0^\infty \frac{d^3k}{(2\pi)^3} \left[ v_j^2(k) + (u_j^2(k) + v_j^2(k))N_j \right] e^{ik \cdot s}. \tag{15}
\]
The one-body density matrix is important for describing the coherence of the mixture.

### III. THREE-DIMENSIONAL CASE

In 3D geometry, the two-body interactions potential reads
\[
V_{ji}(r) = g_{ji} \delta(r) + d_j d_i \frac{1 - 3 \cos^2 \theta}{r^3}, \tag{16}
\]
where \( g_{ji} = g_{ij} = 2\pi \hbar^2 (m_j^{-1} + m_i^{-1})a_{ji} \) correspond to the interspecies short-range part of the interaction, they characterized by the interspecies \( a_{ji} = a_{ij} \) s-wave scattering lengths, \( d_j \) stands for the magnitude of the dipole moment of component \( j \), and \( \theta \) is the angle between the polarization axis- \( z \) and the relative separation of the two dipoles. In the Fourier space, the interaction potential \( \epsilon_{dd}^d \) can be written as \( \epsilon_{dd}^d = 1 + \epsilon_{dd}^d|\mathbf{k}|^2 \text{exp}(-2\pi\hbar h^2 |\mathbf{k}|) \) \([13]\), where \( \epsilon_{dd}^d = r_{xji}/3a_{ji} \) with \( r_{xji} = 2d_i d_j / [\hbar^2 (m_j^{-1} + m_i^{-1})] \) being the characteristic dipole-dipole distance. Therefore, the above parameters turn out to be given as
\[
\alpha(k) = \beta \frac{1 + \epsilon_{dd}^d(3\cos^2 \theta - 1)}{1 + \epsilon_{dd}^d(3\cos^2 \theta - 1)}, \tag{10}
\]
and
\[
\Delta(k) = \frac{V_1(k)V_2(k)}{V_1^2(k)}, \tag{17}
\]
where \( \beta = n_{2}g_{2}/n_{1}g_{1} \) and \( \Delta = g_{1}g_{2}/g_{12}^2 \) is the miscibility parameter of a nondipolar mixture.

The expression of the superfluid fraction can be given in the frame of the hydrodynamic approximation \([22-24]\). Assuming that the two condensates are now moving with the same particle flow velocity \( \mathbf{v} = \hbar \mathbf{p}/m \) \([2, 3]\). As in the case of a single dipolar Bose gas, the total momentum in each component is represented by a tensor as \([22, 23]\) \( P_j = -\psi_j (\int dN_{jk}/d\epsilon_j) \mathbf{p}_j \otimes \mathbf{p}_j (2\pi\hbar)^3 \).

The normal fraction of the dipolar Bose mixture liquid turns out to be given by
\[
n_{\pm} = \int \frac{d\mathbf{k}}{(2\pi)^3} m \hbar^2 k \otimes k \). \tag{12}
\]
This clearly indicates the superfluidity depends on the direction of the superfluid motion with respect to the orientation of the dipoles. In the absence of the DDI and interspecies interaction, the expression \([12]\) reduces to that derived earlier in \([26, 27]\). It is worth noticing that the zero-temperature spectrum \([10]\) remains valid for the calculation of the finite-temperature superfluid density \([3]\).

### C. First order correlation function

At equal times, the first order correlation function is defined as
\[
g_j^{(1)}(s, 0) = n_j + \langle \psi_j(0, \mathbf{0}) \psi_j(0, \mathbf{0}) \rangle, \tag{13}
\]
where \( s = |\mathbf{r} - \mathbf{r}'| \). For the second term in the right hand side of this equation, one has to use the transformation
\[
\psi_j(\mathbf{r}) = \left[ u_{jk}(r) \hat{b}_{jk} e^{i\mathbf{k} \cdot \mathbf{r}/\hbar} - v_{jk}(r) \hat{b}_{jk}^\dagger e^{-i\mathbf{k} \cdot \mathbf{r}/\hbar} \right], \tag{14}
\]
where \( u_{jk}, v_{jk} = (f_{jk}^\dagger \pm f_{jk})/2 \). Thus,
\[
g_j^{(1)}(s, 0) = n_j + \int_0^\infty \frac{d^3k}{(2\pi)^3} \left[ v_j^2 + (u_j^2 + v_j^2)N_j \right] e^{ik \cdot s}. \tag{15}
\]
The one-body density matrix is important for describing the coherence of the mixture.
In the parallel direction, the superfluid fraction is given by

\[ g^{(1)}_{\|}(s) = n_j + \frac{\Lambda^2}{2s}; \quad s \to \infty, \]

where \( \Lambda = \sqrt{2\pi\hbar^2/mT} \) is the thermal de Broglie wavelength. We see that the thermal contribution decays as \( 1/s \) at \( s \to \infty \) and it exceeds the zero temperature contribution to \( g^{(1)}_{\|}(s) \) since this latter decays as \( 1/s^2 \). Furthermore, Eq. (21) shows clearly the presence of the long-range order even at finite temperatures.

For a dipolar mixture, the behavior of the one-body density matrix is also a decaying function at \( s \to \infty \) and diverging near the center as is seen in Fig. 1 showing the existence of the long-range order (i.e., existence of the two condensates). We observe also from the same figure that \( g^{(1)}_{\|}(s) \) is decreasing with increasing temperature.

As we have stated above, the tensorial superfluid fraction splits into a parallel and a perpendicular components. In the parallel direction, the superfluid fraction is given as

\[ \frac{n^{\|}_{\|}}{n} = 1 - \frac{4\hbar^2}{n_j T m} \int \frac{dk d\theta}{(2\pi)^2 4 \sinh^2(\varepsilon_{jk}/2T)} \frac{k^4 \sin \theta \cos^2 \theta}{8\pi^2 4 \sinh^2(\varepsilon_{jk}/2T)} \]

\[ = 1 - \frac{2\pi^2 T^4}{45 mn\hbar^3} \left[ \frac{\mathcal{J}^{\|}_{1}(\varepsilon_{dd})}{c_1^2} + \frac{\mathcal{J}^{\|}_{2}(\varepsilon_{dd})}{c_2^2} \right]. \]

In the perpendicular direction, the superfluid fraction can be computed as

\[ \frac{n^{\perp}_{\|}}{n} = 1 - \frac{4\hbar^2}{n_j T m} \int \frac{dk d\theta}{(2\pi)^2 4 \sinh^2(\varepsilon_{jk}/2T)} \frac{k^4 \sin \theta \sin^2 \theta}{8\pi^2 4 \sinh^2(\varepsilon_{jk}/2T)} \]

\[ = 1 - \frac{\pi^2 T^4}{45 mn\hbar^3} \left[ \frac{\mathcal{J}^{\|}_{3}(\varepsilon_{dd})}{c_1^5} + \frac{\mathcal{J}^{\|}_{4}(\varepsilon_{dd})}{c_2^5} \right]. \]

Note that expressions (22a) and (22b) are calculated at low temperature where the main contribution to the above integrals comes from the phonon branch. The functions \( \mathcal{J}_{\|}(\varepsilon_{dd}) \) are defined as

\[ \mathcal{J}_{\|}^{\parallel}(\varepsilon_{dd}) = \int_0^\infty \sin \theta \cos^2 \theta \left[ 1 + \varepsilon^{dd}_1(3\cos^2 \theta - 1) \right]^{\ell/2} F_j^{\ell/2} d\theta, \]

\[ \mathcal{J}_{\perp}^{\parallel}(\varepsilon_{dd}) = \int_0^\infty \sin^3 \theta \left[ 1 + \varepsilon^{dd}_1(3\cos^2 \theta - 1) \right]^{\ell/2} F_j^{\ell/2} d\theta, \]

their behavior is displayed in Fig. 2.

Figure 2 (top panel) shows that the interspecies DDIs may strongly reduce the superfluid fraction associated with the upper branch in both directions. We observe also that \( n^{\perp}_{\|} \) is smaller than \( n^{\|}_{\|} \) for any value of \( \varepsilon_{dd} \) since \( \mathcal{J}_{\perp}^{\|}(\varepsilon_{dd}) \approx \mathcal{J}_{\parallel}^{\|}(\varepsilon_{dd}) \). For large \( \beta \), i.e., the second component (\( n_2 \)) is highly concentrated, the superfluid fraction rises in both directions and the whole mixture liquid becomes practically superfluid. On the other hand, the superfluidity can occur in the dilute component (\( n_1 \)) only at ultralow temperatures. A similar behavior arises in \( ^4\text{He} - ^{3}\text{He} \) mixture superfluid [3, 4]. The superfluidity disappears in the second component related to the lower branch as is seen in Fig. 2 (bottom panel) due to the instability of \( c_2 \). The normal part of the mixture superfluid increases with \( T \) whatever the values of \( \varepsilon^{dd}_1 \), \( \beta \) and \( \sqrt{n_{c1} n_{c2}} \). Close to the transition, it coincides with the noncondensed density of a noninteracting Bose gas.

For a balanced mixture where \( n_{c1} = n_{c2} \) and \( \tilde{V}_1(k) = \tilde{V}_2(k) = \tilde{V}_{12}(k) = V(k) \), one has \( F_1 = 4 \) and \( F_2 = 0 \), thus, the spectrum of the upper branch is identical to the spectrum of a single component dipolar BEC, \( \varepsilon_{kk} = \sqrt{F_1^2 + 8 E_k n_{c1} V(k)} \), while the spectrum associated with the lower branch becomes identical to that of free particles, \( \varepsilon_{2k} = E_k \). In such a case the superfluid fractions in the parallel direction reduce to

\[ \frac{n^{\perp}_{\|}}{n} = 1 - \frac{\sqrt{2}}{8} \frac{2\pi^2 T^4}{45 mn\hbar^3} Q^{\perp}_{5}(\varepsilon_{dd}) + \frac{\Lambda^3 \zeta(3/2)}{n}, \]

where \( Q^{\perp}_{\ell}(x) = \int_0^1 dy y^\ell (1 - x + 3xy^2)^{\ell/2} \), have the properties \( Q^{\perp}_{\ell}(0) = 1/3 \) and become imaginary for \( x > 1 \).

In the perpendicular direction

\[ \frac{n^{\perp}_{\perp}}{n} = 1 - \frac{\sqrt{2}}{8} \frac{2\pi^2 T^4}{45 mn\hbar^3} Q^{\perp}_{5}(\varepsilon_{dd}) + \frac{\Lambda^3 \zeta(3/2)}{n}, \]

where \( Q^{\perp}_{\ell}(x) = \int_0^1 dy (1 - y^2)(1 - x + 3xy^2)^{\ell/2} \). Importantly, the superfluid fraction of a balanced mixture in

![Fig. 1. First-order correlation function for 3D dipolar two-component Bose gas for various temperatures. Parameters are: \( \beta = 0.2 \), \( \delta = 1.5 \), \( \varepsilon^{dd}_1 = 0.9 \), \( \varepsilon^{dd}_2 = 0.5 \), and \( \varepsilon^{dd}_3 = 0.3 \).](image-url)
both parallel and perpendicular directions is larger than that of one Bose liquid. This can be attributed to the competition between the intra- and inter-component interactions. We see also from Eqs. (25) and (26) that the normal part of the superfluid mixture in both directions coincides with the normal density of a dipolar one-component Bose fluid obtained at higher temperatures and with the noncondensed density of an ideal gas. The difference between both quantities might be found by calculating higher order fluctuations corrections.

Near the phase separation and at low temperature, the lower branch has the free-particle dispersion law $\varepsilon_{k\perp} = \varepsilon_{\parallel}$ while the upper branch is phonon-like $\varepsilon_{k\parallel} = \hbar c_{\parallel}(1 + \alpha)^{1/2}$. This results indicates that the total normal density in the parallel direction has two different temperature dependence form: $n_{\parallel} = p^\parallel T^4 + q T^{3/2}$, where $p^\parallel = (\sqrt{2\pi^2/360mn\hbar^3 c^5}) T^\parallel$, the function $T^\parallel = \int_0^\pi d\theta \sin \theta \cos^2 \theta/(1 + \alpha)^{3/2}$, and $p = (m/2\pi\hbar^2)^{3/2} \xi(3/2)$. Whereas, in the perpendicular direction: $n_{\perp} = p^\perp T^4 + q T^{3/2}$, where $p^\perp = (\sqrt{2\pi^2/360mn\hbar^3 c^5}) T^\perp$, and the function $T^\perp = \int_0^\pi d\theta \sin^3 \theta/(1 + \alpha)^{5/2}$. One can conclude that the component related to the lower branch is extremely dilute.

**IV. QUASI-TWO-DIMENSIONAL CASE**

IV. QUASI-TWO-DIMENSIONAL CASE

We focus our attention now on homogeneous quasi-2D Bose-Bose mixture. It is well known that in such systems, thermal fluctuations can destroy the long-range order associated with BEC at low temperatures, but cannot suppress superfluidity in an interacting system [27–32].

Let us consider dipolar bosonic mixture confined in quasi-2D geometry by means of an external harmonic potential in the direction perpendicular to the motion (pancake trap) and all dipoles are aligned perpendicularly to the plane of their translational motion, by means of a strong electric (or magnetic) field. In momentum space, the two-body interaction potential reads [28]

$$\hat{V}_{ji}(k) = g_{ji}^{2D} (1 - C_{ji}^{dd} k),$$

where the 2D short-range coupling constant is $g_{ji}^{2D} = g_{ji}/\sqrt{l_0}$ with $l_0 = \sqrt{\hbar/m\omega}$, $\omega$ is the confinement frequency, and $C_{ji}^{dd} = 2\pi d_{ji}^2/g_{ji}^{2D}$ are dimensionless relative strengths describing the interplay between the DDI and short-range interactions.

The miscibility parameter becomes

$$\Delta(k) = \Delta^{2D} \left(1 - C_{1}^{dd} k \right) \left(1 - C_{2}^{dd} k \right),$$

where $\Delta^{2D} = g_1^{2D} g_2^{2D} / (g_{12}^{2D})^2$ is the miscibility parameter for 2D nondipolar mixture. One should stress that in addition to the usual (long-wavelength) immiscibility-miscibility transition, the system is also sensitive to a supplementary transition originates due to the roton modes [33]. In our case, such a transition could be controlled by the parameter $C_{ji}^{dd}$.

The behavior of the Bogoliubov dispersion relation is presented in Fig. 3. The upper branch of the two-component dispersion [34] has a roton-maxon structure. The height of the roton depends on the intra- and interspecies DDIs as is seen in Fig. 3(a). Whereas the lower branch is unstable (see Fig. 3(b)).

![FIG. 2. Functions $J_{A}^{-5}$, $J_{A}^{-5/2}$, $J_{B}^{-5/2}$ and $J_{B}^{-5}$, which describes the dependence of the superfluid fractions in the parallel and perpendicular directions on the relative inter-component dipolar interaction strength $\varepsilon_{dd}$. Parameters are: $\beta = 0.2$, $\delta = 1.5$, $\varepsilon_{1}^{dd} = 0.9$ and $\varepsilon_{2}^{dd} = 0.5$.](image1)

![FIG. 3. Upper (a) and lower (b) branches of excitation energy of the quasi-2D dipolar Bose mixture as a function of momentum $k$. Dashed lines: $\beta = 5.2$, $\Delta^{2D} = 1.1$, $c_1/\xi_1 = 0.3$, $c_2/\xi_1 = 0.6$, and $c_1/\xi_1 = 0.5$. Solid lines: $\beta = 5.2$, $\Delta^{2D} = 1.1$, $c_1/\xi_1 = 0.3$, $c_2/\xi_1 = 0.1$, and $c_1/\xi_1 = 0.4$. Here $\xi_1 = \hbar/\sqrt{mn_1g_1}$ is the standard healing length.](image2)
The one-body density matrix shows that at $T=0$, correlations and tend to its asymptotic value at large distances. Therefore, the long-range order is destroyed by the phase fluctuations prohibiting the formation of a true BEC in 2D geometry.

The superfluid fraction can be obtained easily from Eq. (31)

$$n_{sj} = 1 - \frac{3\zeta(3)}{2\pi\hbar^2 mn_j c_j^3} T^3,$$

which is similar to that obtained for a single component BEC in Refs [27, 34]. It is clear that in the roton regime, the superfluid density is strongly reduced yielding the transition to supersolid state similarly to the 2D single dipolar BEC [28, 33, 35].

V. CONCLUSION

We theoretically investigated the superfluid properties of weakly interacting homogeneous dipolar binary Bose condensates. Within the realm of the hydrodynamic approach, we derived useful expressions for the excitations spectrum, the first-order correlation function, and the superfluid density. We examined in addition the impact of temperature and interspecies DDI on the correlation function and on the superfluid density in both 3D and 2D geometries. We showed that the first-order correlation function decays at larger distances at any temperature in the 3D case indicating the existence of the long-range order. In 2D case, it goes to its asymptotic value at large distances. Our analysis revealed also that in the roton regime, the correlation function develops a peak near the center signaling the destruction of the long-range order associated with the BEC and hence, the transition to a new state of matter. Moreover, we found that $c_{12}^2$ may crucially reduce the superfluid fraction of the whole mixture. For a highly imbalanced mixture, the superfluid behavior disappears in the dilute component. Whereas, in the case of a balanced mixture, the superfluid fraction becomes identical to that of the one-component Bose liquid. To the best of our knowledge, this is the first study of the superfluidity in dipolar binary condensates. Our findings will pave the way for studying the superfluidity of liquid dipolar mixture droplets [13].

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