THE ZARISKI-LIPMAN CONJECTURE FOR LOG CANONICAL SPACES

STEFAN HEUVER

Abstract. In this paper we give an elementary proof of the Zariski-Lipman conjecture for log canonical spaces.

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1. Introduction

The Zariski-Lipman conjecture claims that an n-dimensional complex variety with locally free tangent sheaf of rank n is smooth. Although not proven in general, this conjecture holds in special cases (see [Dru13]). In 2013 Druel has proven this conjecture for log canonical spaces by using foliations and the Camacho-Sad formula. In 2014 Graf-Kovács obtained the result of Druel by strengthening the extension theorem (see [GKRPT11 Theorem 1.5]) for 1-forms on log canonical pairs (see [GK14]). For more information on log canonical spaces, see [Rei87] or [KM98]. The goal of this paper is to give a more elementary proof for log canonical surfaces in the style of [GKK10] and to conclude the Zariski-Lipman conjecture for log canonical spaces by using the reduction technique of Druel (see [Dru13 Theorem 5.2]).

Theorem 1.1 (The Zariski-Lipman conjecture for log canonical spaces). Let \( X \) be a log canonical variety of dimension \( n \) such that the tangent sheaf \( T_X \) is locally free of rank \( n \). Then \( X \) is smooth.

To prove the 2-dimensional case, we will use an argument of [vSS85]. The idea is that under the given properties, a smooth 1-form on the variety will extent to a smooth 1-form on the resolution, which leads to a contraction, unless the variety has already been smooth. After this we will reduce the n-dimensional case to the surface case by using hyperplane sections.

Conventions and basics. In this paper a variety is a integral, separated scheme of finite type over an algebraic closed field. Varieties are also assumed to be reduced and irreducible. In [Lip65] Lipman has proven that his conjecture fails to be true if \( X \) is a variety over a field with positive characteristic and that a variety with locally free tangent sheaf is necessarily normal. That is why we will work over the field of complex numbers and assume that \( X \) is normal. The sheaf \( \Omega^1_X \) of Kähler differentials behaves badly near a singular point. Therefore it is more useful to work with the reflexive hull \( \Omega^1_X \), which is the double dual of \( \Omega^1_X \) (see [Rei87] (1.5),(1.7)). For the definition of logarithmic differential forms and logarithmic vector fields we recommend [Sai80] and [EV92].

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2. The 2-dimensional case

To prove the Zariski-Lipman conjecture for log canonical surfaces, we use the minimal resolution \( \pi : Y \to X \) and show that a reflexive 1-form on \( X \) lifts to a regular 1-form on \( Y \) under \( \pi \). An example of [Rei87] (1.8), (1.9)] shows that this is wrong if \( X \) is not log canonical. In general a reflexive 1-form
only lifts to a 1-form that is regular outside the exceptional locus. We will have to show that it extends regular over the exceptional locus.

**Proposition 2.1** (Logarithmic extension). Let $X$ be a log canonical surface with locally free tangent sheaf $T_X$ of rank 2. Let $\omega \in H^0(X, \Omega_X^{[1]} \mathbb{Q})$ be a reflexive 1-form, $\pi : Y \to X$ the minimal resolution and $E$ the largest reduced divisor included in $\text{excep}(\pi)$. Then $\omega$ extends to a logarithmic 1-form $\tilde{\omega} := \pi^* \omega \in H^0(Y, \Omega_Y^{[1]}(\log E))$ on $Y$.

**Proof.** (cf. 

Let $\omega \in H^0(X, \Omega_X^{[1]} \mathbb{Q})$. Since $T_X$ is locally free of rank 2 we can assume without loss of generality that $\mathcal{O}_X(K_X) \cong \mathcal{O}_X$. Since $\Omega_X^{[1]}$ is the dual of $T_X$ there exists a unique vector field $\xi \in H^0(Y, \mathcal{O}_Y \mathbb{Q}(\log E))$ corresponding to $\omega$ via the perfect pairing

$$\Omega_X^{[1]} \times \Omega_Y^{[1]} \to \mathcal{O}_X(K_X) \cong \mathcal{O}_X.$$

The minimal resolution $\pi$ is functorial and we can lift the vector field $\xi$ to a vector field $\tilde{\xi} \in H^0(Y, T_Y(\log E))$. As $X$ is assumed to be log canonical we have $\mathcal{O}_Y(K_Y + E) \cong \mathcal{O}_Y(D)$ for some effective divisor $D$ on $Y$. Hence $\tilde{\xi}$ corresponds to an element $\tilde{\omega} \in H^0(Y, \Omega_Y^{[1]}(\log E) \otimes \mathcal{O}_Y(-D))$ via the pairing

$$\Omega_Y^{[1]}(\log E) \times \Omega_Y^{[1]}(\log E) \to \mathcal{O}_Y(K_Y + E) \cong \mathcal{O}_Y(D).$$

This yields the extension of $\omega$. \hfill $\Box$

The following result is a consequence of the negative definiteness of the self-intersection form in $E$.

**Proposition 2.2.** Let $X$ be a normal surface and $\pi$ and $E$ as in Proposition 2.1. Then the inclusion

$$H^0(Y, \Omega_Y^{[1]}) \hookrightarrow H^0(Y, \Omega_Y^{[1]}(\log E))$$

is an isomorphism.

**Proof.** See [Wah85, Lemma 1.3.b]. \hfill $\Box$

**Theorem 2.3** (The Zariski-Lipman conjecture for log canonical surfaces). Let $X$ be a log canonical surface such that the tangent sheaf $T_X$ is locally free of rank 2. Then $X$ is smooth.

**Proof.** Let $\pi$, $Y$, and $E$ be as defined above. Using Proposition 2.1 and 2.2 we see that a reflexive 1-form on $X$ lifts to a regular 1-form on $Y$ under $\pi$. With $\pi_!(T_Y(\log E)) \cong T_X$ Theorem 2.3 is a consequence of a classical argument presented in [NSSS55, (1.6)]. \hfill $\Box$

### 3. The n-dimensional case

In this section $X$ is an $n$-dimensional log canonical variety with locally free tangent sheaf and $\pi : Y \to X$ a functorial resolution (see [Kol07, 3.45]). We will use $n-2$ hyperplane sections $G_1, \ldots, G_{n-2} \subset X$ to cut $X$ down to a surface $S := X \cap G_1 \cap \cdots \cap G_{n-2}$ and show that $S$ is a log canonical surface with locally free tangent sheaf and therefore already smooth. Using the fact that a singularity of $X$ necessarily is a singularity of $S$, we conclude that $X$ must have been smooth. For more information on hyperplane sections used in this paper see [GKKP11, 2.E.]. With $H := \pi^{-1}(G)$ we will denote the preimage of $G$ under $\pi$. Please note the following facts.

**Fact 3.1** (see [GKKP11, 2.E.]). Let $X, G, H$ and $\pi$ be as defined above and $E$ be the largest reduced divisor contained in $\text{excep}(\pi)$ then $G$ is affine and normal, $H$ is smooth and $\pi|_H$ is a functorial resolution and the largest reduced divisor $E|_H = E \cap H$ contained in $\text{excep}(\pi|_H)$ is a simple normal crossing (snc) divisor (this uses Bertini’s Theorem).

Define $T := Y \cap H_1 \cap \cdots \cap H_{n-2}$, then by induction we get

1. $S$ is affine and normal and $T$ is smooth,
2. $\pi|_T : T \to S$ is a functorial resolution and the largest reduced divisor $C := E|_T$ contained in the exceptional locus $\text{excep}(\pi|_T)$ of $\pi|_T$ is a snc divisor.

We will now show that $S$ has the right properties.

**The behavior of the singularities under reduction.** The following lemma gives us a connection between $\text{Sing}(X)$ and $\text{Sing}(G)$.
Lemma 3.2. Let $X$ be a normal variety and $G \subset X$ an effective, ample Cartier divisor. If $G$ is smooth, then $\text{Sing}(X) \cap G = \emptyset$ and the singularities of $X$ are isolated.

Proof. This is a consequence of [Che96 Lemma 1] and the Jacobian criteria.

Lemma 3.3. The surface $S$ constructed above is log canonical.

Proof. Since $X$ is log canonical and normal, $|G|$ is a basepoint free system of Cartier divisors and $G_1 \in |G|$ is a hyperplane section, Theorem 1.13 in [Rei80] shows that $G$ is log canonical, too. Using Fact 3.1 we get Lemma 3.3 by induction.

The tangent sheaf of $S$ is locally free. To show that $T_S$ is locally free we first need to prove the following Lemma.

Lemma 3.4. Let $Y$ be a smooth variety of dimension $n \geq 2$ and $E \subset Y$ a snc divisor. Let $H \subset Y$ be a smooth hyperplane such that $E \cap H$ is a snc divisor. Then the sequence

$$0 \to \mathcal{N}_{H|Y}^* \to Y^*(\log E)|_{H} \to H^1(Y^*(\log E)|_{H}) \to 0$$

is exact.

Proof. Since $E$ and $E|_H$ are snc divisors we can use the sequence of [EV92 2.3 a]. Using the Snake Lemma we then get Lemma 3.4.

Theorem 3.5. Let $X$ be a log canonical variety of dimension $n$ with locally free tangent sheaf of rank $n$. Then the tangent sheaf $T_S$ of the surface $S$ defined above is locally free of rank $2$.

Proof. The following proof is basically the cutting-down procedure of Druel (see the proof of [Dru13 Theorem 5.2]) supplemented with additional steps for the convenience of the reader. For $n = 2$ the theorem is clear and we can assume that $n \geq 3$. Suppose that $\text{Sing}(X) \neq \emptyset$. Since $X$ is normal $\text{codim}_X(\text{Sing}(X)) \geq 2$ and with [Fle88 p.318] we get that $\text{codim}_X(\text{Sing}(X)) = 2$. Replacing $X$ with an affine open dense subset we may assume that $X$ is affine, $\text{Sing}(X)$ is irreducible of codimension 2 and $T_X \cong \mathcal{O}_X^{\oplus n}$.

Let $\pi : Y \to X$ be a functorial resolution and $E$ the largest reduced divisor contained in $\text{excep}(\pi)$. Note that $E \neq \emptyset$. We consider the morphism of vector bundles

$$F : \pi^* T_X \to T_Y(-\log E).$$

Since $T_X \cong \pi_* T_Y(-\log E)$ the morphism $F$ is induced by the evaluation map

$$\pi^* (\pi_* T_Y(-\log E)) \hookrightarrow T_Y(-\log E)$$

and induces an injective map of sheaves

$$\pi^* (\mathcal{O}_X(-K_X)) \cong \pi^* \text{det}(T_X) \xrightarrow{\pi^* \text{det}(T_Y(-\log E))} \mathcal{O}_Y(-K_Y - E).$$

Using the ramification formula $K_Y := \pi^* K_X + \sum a_i E_i$ this yields $a_i \leq -1$. Since $X$ is log canonical we get $a_i = -1$. Thus $F$ is an isomorphism. Since $T_X$ is free we deduce that $T_Y(-\log E)$ is free and that $\Omega_{H}^1(\log E)|_{H_1} \cong \mathcal{O}_{H_1}^{\oplus n}$.

Let $G_1 \subset X$ be a general hyperplane section and $H_1 = \pi^{-1}(G_1) \subset Y$. By Lemma 3.4 we have the exact sequence

$$0 \to \mathcal{N}_{H_1|X}^* \xrightarrow{\Phi} \Omega^1_Y(\log E)|_{H_1} \xrightarrow{\Psi} \Omega^1_{H_1}(\log E)|_{H_1} \to 0.$$ 

We want to prove

$$\Omega^1_{H_1}(\log E)|_{H_1} \cong \mathcal{O}_{H_1}^{\oplus \text{dim}(H_1)}.$$

Since $X$ is affine and $G_1 \in |G|$ is an effective Cartier divisor and $|G|$ basepoint free, we can define $G_1$ by a global function $g$. Due to this the ideal sheaf $\mathcal{O}_X(-G_1) = \mathcal{N}_{G_1|X}$ is free. Since $H_1$ is the total transform of $G_1$ we get $\mathcal{N}_{H_1|X} = \pi^* \mathcal{N}_{G_1|X} \cong \mathcal{O}_{H_1}$. Thus we can represent the map

$$\Phi : \mathcal{O}_{H_1} \cong \mathcal{N}_{H_1|X} \to \Omega^1_Y(\log E)|_{H_1} \cong \mathcal{O}_{H_1}^{\oplus n}$$

by regular functions $f_1, \ldots, f_n$ on $H_1$ and since $\pi_*(\mathcal{O}_{H_1}) \cong \mathcal{O}_{G_1}$ by regular functions $g_1, \ldots, g_n$ in $G_1$ with $f_i = g_i \circ \pi|_{H_1}$. If $g_j|_{G_1 \cap \text{Sing}(X)} = 0$ for all $i$ then $\Phi$ would vanish at every point of $\pi^{-1}(G_1 \cap \text{Sing}(X))$. Since $E_1|_{H_1}$ is snc divisor, the sheaf $\Omega^1_{H_1}(\log E)|_{H_1}$ is locally free, which yields in a contradiction. Take an $i \in \{1, \ldots, n\}$, so that $g_i|_{G_1 \cap \text{Sing}(X)} \neq 0$ then, by replacing $X$ with
we can ensure that $\pi_{|H_1}(x) \neq 0$ for all $x \in H_1$ and thus assume that $\Phi$ has full rank 1 in every fiber. We obtain the following exact sequence:

$$0 \to \mathcal{O}_{H_1} \xrightarrow{\Phi} \mathcal{O}_{H_1}^{\oplus n} \xrightarrow{\Psi} \mathcal{O}_{H_1}^{\oplus (n-1)} \to 0.$$ 

Thus $\Omega_{H_1}(\log E|H_1) \cong \mathcal{O}_{H_1}^{\oplus \dim(H_1)}$ and $E_{H_1} \neq \emptyset$. By replacing $X$ with an appropriate open subset we may assume that $T_{H_1}(-\log C|H_1) \cong \mathcal{O}_{H_1}^{\oplus 2}$ and thus that $T_S$ is locally free of rank 2.

Proof of Theorem 1.1. Using the notation of Theorem 3.5 we assume that $\text{Sing}(X) \neq \emptyset$. With Lemma 3.2 (by induction) we can deduce that the surface $S$ constructed above necessarily has an isolated singularity. However $S$ is a log canonical surface with locally free tangent sheaf and thus smooth by Theorem 2.3. This contradicts the assumption. □

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