Smooth statistics for a point process on the unit circle with reflection-type interactions across the real line

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Abstract
We consider the point process
\[ \frac{1}{Z_n} \prod_{1 \leq j < k \leq n} |e^{\theta_j} - e^{-i\theta_k}|^\beta \prod_{j=1}^{n} d\theta_j, \quad \theta_1, \ldots, \theta_n \in (-\pi, \pi], \quad \beta > 0, \]
where \( Z_n \) is the normalization constant. The feature of this process is that the points \( e^{i\theta_1}, \ldots, e^{i\theta_n} \) only interact with the mirror points reflected over the real line \( e^{-i\theta_1}, \ldots, e^{-i\theta_n} \).

We study smooth linear statistics of the form
\[ \sum_{j=1}^{n} g(\theta_j) \] as \( n \to \infty \), where \( g \) is \( 2\pi \)-periodic.

We prove that a wide range of asymptotic scenarios can occur: depending on \( g \), the leading order fluctuations around the mean can (i) be of order \( n \) and purely Bernoulli, (ii) be of order 1 and purely Gaussian, (iii) be of order 1 and purely Bernoulli, or (iv) be of order 1 and of the form \( B N_1 + (1 - B) N_2 \), where \( N_1, N_2 \) are two independent Gaussians and \( B \) is a Bernoulli that is independent of \( N_1 \) and \( N_2 \). The above list is not exhaustive: the fluctuations can be of order \( n \), of order 1 or \( o(1) \), and other random variables can also emerge in the limit.

We also obtain large \( n \) asymptotics for \( Z_n \) (and some generalizations), up to and including the term of order 1.

Our proof is inspired from a method developed by McKay and Wormald [15] to estimate related \( n \)-fold integrals.

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1 Introduction

We consider the joint probability density
\[ \frac{1}{Z_n} \prod_{1 \leq j < k \leq n} |e^{i\theta_j} - e^{-i\theta_k}|^\beta \prod_{j=1}^{n} d\theta_j, \quad \theta_1, \ldots, \theta_n \in (-\pi, \pi], \tag{1.1} \]
where \( \beta > 0 \) and \( Z_n \) is the normalization constant. The points \( e^{i\theta_1}, \ldots, e^{i\theta_n} \) do not directly interact with each other, but instead interact with the image points \( e^{-i\theta_1}, \ldots, e^{-i\theta_n} \) obtained by reflection over the real line. This interaction has the following consequence on the global behavior of the points: for \( n \geq 3 \), only two configurations maximize (1.1), namely \((e^{i\theta_1}, \ldots, e^{i\theta_n}) = (i, \ldots, i)\) and \((e^{i\theta_1}, \ldots, e^{i\theta_n}) = (-i, \ldots, -i)\). Our first result makes precise the idea that for large \( n \) only the point configurations that are close to either \((i, \ldots, i)\) or \((-i, \ldots, -i)\) are likely to occur.

Theorem 1.1. Fix \( \beta > 0 \). For any \( \epsilon \in (0, \frac{1}{15}] \), there exists \( c > 0 \) such that, for all large enough \( n \),
\[ P \left( \left\{ |e^{i\theta_j} - i| \leq n^{-\frac{1}{15} + \epsilon} \text{ for all } j \in \{1, \ldots, n\} \right\} \cup \left\{ |e^{i\theta_j} + i| \leq n^{-\frac{1}{15} + \epsilon} \text{ for all } j \in \{1, \ldots, n\} \right\} \right) \geq 1 - e^{-cn^2\epsilon}. \]
Point processes with only reflection-type interactions across a line such as (1.1) have not been considered before to our knowledge. The main goal of this paper is to investigate the asymptotic fluctuations as \( n \to \infty \) of linear statistics of the form \( \sum_{j=1}^{n} g(\theta_{j}) \), for fixed \( \beta \) and where \( g: \mathbb{R} \to \mathbb{R} \) is \( 2\pi \)-periodic and smooth enough in neighborhoods of \( \frac{\pi}{2} \) and \( -\frac{\pi}{2} \) (it is already clear from Theorem 1.1 that the regularity of \( g \) outside neighborhoods of \( \frac{\pi}{2} \) and \( -\frac{\pi}{2} \) does not matter). One of the interesting properties of (1.1) is that, as shown in Theorem 1.5 below, many different types of scenarios can occur.

It is natural to expect from Theorem 1.1 some oscillations in the large \( n \) behavior of \( \sum_{j=1}^{n} g(\theta_{j}) \). Indeed, for large \( n \), the average point configuration contains \( \frac{\pi}{2} \) points near \( i \) and \( \frac{\pi}{2} \) points near \( -i \), but with overwhelming probability a random point configuration contains either all \( n \) points near \( i \), or all \( n \) points near \( -i \), and is thus “very far” from the average. As a consequence, the empirical measure \( \mu_{n} := \frac{1}{n} \sum_{j=1}^{n} \delta_{\theta_{j}} \) has no deterministic limit as \( n \to \infty \). In fact, we prove in Theorem 1.5 that \( \mu_{n} \) converges weakly in distribution to the random measure \( \mu := B\delta_{\frac{\pi}{2}} + (1-B)\delta_{-\frac{\pi}{2}} \) where \( B \sim \text{Bernoulli}(\frac{1}{2}) \), namely

\[
\int_{(-\pi,\pi]} g(x) d\mu_{n}(x) \xrightarrow{\text{law as } n \to \infty} \int_{(-\pi,\pi]} g(x) d\mu(x).
\]  

(1.2)

We will first prove a general result about the large \( n \) asymptotics of \( n \)-fold integrals of the form

\[
I(f) = \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \prod_{1 \leq j < k \leq n} |e^{i\theta_{j}} - e^{-i\theta_{k}}|^{\beta} \prod_{j=1}^{n} e^{f(\theta_{j})} d\theta_{j},
\]

where \( f: \mathbb{R} \to \mathbb{C} \) is regular enough and \( 2\pi \)-periodic, see Theorem 1.2. As corollaries, we will obtain large \( n \) asymptotics for the characteristic function of \( \sum_{j=1}^{n} g(\theta_{j}) \) simply by considering the ratio \( \mathbb{E} \left[ \exp \left( it \sum_{j=1}^{n} g(\theta_{j}) \right) \right] = \frac{I(f(t))}{I(0)} \), \( t \in \mathbb{R} \). The large \( n \) asymptotics of \( Z_{n} = I(0) \) are also obtained as the special case \( f = 0 \) of Theorem 1.2.

The work \cite{15} has been the main inspiration for the present paper. In \cite{15}, motivated by a combinatorial problem about the enumeration of regular graphs, McKay and Wormald developed a method to obtain large \( n \) asymptotics of \( n \)-fold integrals of the form

\[
\frac{1}{(2\pi i)^{n}} \int_{\gamma} \cdots \int_{\gamma} \prod_{1 \leq j < k \leq n} \frac{(1 + z_{j}z_{k})}{z_{j}^{d+1} \cdots z_{k}^{d+1}} dz_{1} \cdots dz_{n},
\]

where each integral is taken along a circle centered at 0, and \( d = d(n) \in \mathbb{N} \) grows with \( n \) at a suitable speed. Remarkably, the method of \cite{15} does not rely at all on the fact that the integrand in (1.4) is analytic, and can actually be adapted with little effort to handle integrals of the form (1.3).

Let \( M(f) := \sup_{\theta \in (-\pi,\pi]} \text{Re } f(\theta) \). We now state our first main result.

**Theorem 1.2.** Fix \( \beta > 0 \). Let \( f: \mathbb{R} \to \mathbb{C} \) be \( 2\pi \)-periodic, bounded, and \( C^{2,\tilde{q}} \) in neighborhoods of \( \frac{\pi}{2} \) and \( -\frac{\pi}{2} \), with \( 0 < q \leq 1 \). Assume also that

\[
M(f) + \beta \log \cos(\frac{\pi \theta}{2}) < \min\{\text{Re } f(\frac{\pi}{2}), \text{Re } f(-\frac{\pi}{2})\}.
\]

(1.5)

Then, for any fixed \( 0 < \zeta < \min\{\frac{1}{2}, \frac{\pi}{2}\} \), as \( n \to \infty \) we have

\[
I(f) = 2^{\frac{n(n-1)}{2}} \left( \frac{8\pi}{(2\pi)^{n}} \right)^{\frac{n}{2}} e^{-\frac{n\pi}{4}} \left( e^{\frac{f'(\frac{\pi}{2})}{\beta} - \frac{f''(\frac{\pi}{2})}{\beta}} + O(n^{-\zeta}) \right)
\]
Furthermore, if \( \text{Re } f \equiv 0 \) and \( t \in \mathbb{R} \), then the large \( n \) asymptotics of \( I(tf) \), which are given by (1.6) with \( f \) replaced by \( tf \), hold uniformly for \( t \) in compact subsets of \( \mathbb{R} \).

**Remark 1.3.** Condition (1.5) comes from some technicalities in our analysis and can probably be weakened. For the applications on the linear statistics below, we will only use Theorem 1.2 with \( f : \mathbb{R} \rightarrow i\mathbb{R} \), i.e. \( \text{Re } f \equiv 0 \), for which (1.5) is automatically verified.

The following result on the characteristic function of \( \sum_{j=1}^{n} g(\theta_j) \) is a direct consequence of Theorem 1.2.

**Theorem 1.4.** Fix \( \beta > 0 \). Let \( t \in \mathbb{R} \) and let \( g : \mathbb{R} \rightarrow \mathbb{R} \) be \( 2\pi \)-periodic, bounded, and \( C^2 \) in neighborhoods of \( \frac{x}{2} \) and \( -\frac{x}{2} \), with \( 0 < q \leq 1 \). For any fixed \( 0 < \zeta < \min\{\frac{1}{2}, q\} \), as \( n \rightarrow \infty \) we have

\[
E\left[e^{it\sum_{j=1}^{n} g(\theta_j)}\right] = \frac{I(itg)\{itg(\frac{x}{2})\}}{I(0)} \exp\left(-\frac{g(\frac{x}{2})^2}{\beta} t^2 + \frac{2g''(\frac{x}{2})}{\beta} it + O(n^{-\zeta})\right) + \frac{e^{itg(-\frac{x}{2})}}{2} \exp\left(-\frac{g(-\frac{x}{2})^2}{\beta} t^2 + \frac{2g''(-\frac{x}{2})}{\beta} it + O(n^{-\zeta})\right).
\]

Furthermore, the above asymptotics hold uniformly for \( t \) in compact subsets of \( \mathbb{R} \).

Let us define

\[
\nu_1 = \frac{g(\frac{x}{2}) + g(-\frac{x}{2})}{2}, \quad \nu_2 = \frac{g''(\frac{x}{2}) + g''(-\frac{x}{2})}{\beta}.
\]

Our next theorem establishes convergence in distribution of the smooth linear statistics.

**Theorem 1.5.** Fix \( \beta > 0 \). Let \( g : \mathbb{R} \rightarrow \mathbb{R} \) be \( 2\pi \)-periodic, bounded, and \( C^2 \) in neighborhoods of \( \frac{x}{2} \) and \( -\frac{x}{2} \), with \( 0 < q \leq 1 \).

(a) Define \( \mu_n = \frac{1}{n} \sum_{j=1}^{n} \delta_{\theta_j} \). We have

\[
\int_{[-\pi, \pi]} g(x) d\mu_n(x) \xrightarrow{\text{law } n \rightarrow \infty} \int_{[-\pi, \pi]} g(x) d\mu(x),
\]

where \( \mu := B\delta_{\frac{x}{2}} + (1-B)\delta_{-\frac{x}{2}} \) and \( B \sim \text{Bernoulli}(\frac{1}{2}) \). Equivalently,

\[
n^{-1}\left(\sum_{j=1}^{n} g(\theta_j) - \nu_1 n\right) \xrightarrow{\text{law } n \rightarrow \infty} \frac{g(\frac{x}{2}) - g(-\frac{x}{2})}{2}(2B - 1),
\]

where \( B \sim \text{Bernoulli}(\frac{1}{2}) \).

(b) If \( g(\frac{x}{2}) = g(-\frac{x}{2}) \) and \( g'(\frac{x}{2}) \neq 0 \neq g'(-\frac{x}{2}) \), then

\[
\sum_{j=1}^{n} g(\theta_j) - \nu_1 n \xrightarrow{\text{law } n \rightarrow \infty} BN_1 + (1-B)N_2,
\]

where \( N_1, N_2, B \) are random variables independent of each other and distributed as

\[
N_1 \sim N_2\left(\frac{2g''(\frac{x}{2})}{\beta}, \frac{2g''(-\frac{x}{2})}{\beta}\right), \quad N_2 \sim N_2\left(\frac{2g''(\frac{x}{2})}{\beta}, \frac{2g''(-\frac{x}{2})}{\beta}\right), \quad B \sim \text{Bernoulli}\left(\frac{1}{2}\right).
\]
In particular, if \( g(\frac{1}{2}) = g(-\frac{1}{2}) \), \( g'(\frac{1}{2}) = g'(-\frac{1}{2}) \neq 0 \) and \( g''(\frac{1}{2}) = g''(-\frac{1}{2}) \), then
\[
\sum_{j=1}^{n} g(\theta_j) - \nu_1 n \xrightarrow{\text{law}} N_1,
\]
where \( N_1 \sim N_\mathbb{R}(\frac{2g''(\frac{1}{2})}{\beta}, \frac{2g'(\frac{1}{2})^2}{\beta}) \).

(c) If \( g(\frac{1}{2}) = g(-\frac{1}{2}) \), \( g'(\frac{1}{2}) \neq 0 \), and \( g'(-\frac{1}{2}) = 0 \), then
\[
\sum_{j=1}^{n} g(\theta_j) - \nu_1 n \xrightarrow{\text{law}} BN_1 + (1 - B) \frac{2g''(-\frac{1}{2})}{\beta},
\]
where \( N_1 \sim N_\mathbb{R}(\frac{2g''(-\frac{1}{2})}{\beta}, \frac{2g'(\frac{1}{2})^2}{\beta}) \) and \( B \sim \text{Bernoulli}(\frac{1}{2}) \).

Similarly, if \( g(\frac{1}{2}) = g(-\frac{1}{2}) \), \( g'(-\frac{1}{2}) \neq 0 \), and \( g'(\frac{1}{2}) = 0 \), then
\[
\sum_{j=1}^{n} g(\theta_j) - \nu_1 n \xrightarrow{\text{law}} BN_2 + (1 - B)N_2,
\]
where \( N_2 \sim N_\mathbb{R}(\frac{2g''(-\frac{1}{2})}{\beta}, \frac{2g'(\frac{1}{2})^2}{\beta}) \) and \( B \sim \text{Bernoulli}(\frac{1}{2}) \).

(d) If \( g(\frac{1}{2}) = g(-\frac{1}{2}) \) and \( g'(-\frac{1}{2}) = 0 = g'(\frac{1}{2}) \), then
\[
\sum_{j=1}^{n} g(\theta_j) - (\nu_1 n + \nu_2) \xrightarrow{\text{law}} \frac{g''(\frac{1}{2}) - g''(-\frac{1}{2})}{\beta} (2B - 1),
\]
where \( B \sim \text{Bernoulli}(\frac{1}{2}) \).

**Remark 1.6.** Each random variable in Theorem 1.5 has a variance that increases as \( \beta^{-1} \) increases. This is consistent with the expectation that as \( \beta^{-1} \) increases, the random point configurations of (1.1) should become less localized around \((i, \ldots, i)\) and \((-i, \ldots, -i)\).

**Remark 1.7.** If \( g(\frac{1}{2}) = g(-\frac{1}{2}) \), \( g'(\frac{1}{2}) = 0 = g'(-\frac{1}{2}) \) and \( g''(\frac{1}{2}) = g''(-\frac{1}{2}) \), then \( \sum_{j=1}^{n} g(\theta_j) - (\nu_1 n + \nu_2) \) is typically of order \( O(n^{-\xi}) \) and our result (1.7) is not precise enough to understand the fluctuations in this situation. We believe this case involves other random variables that just Bernoulli and Gaussian random variables (because we expect the error term in (1.7) to involve higher powers of \( t \) than \( t^2 \)).

**Proof of Theorem 1.5.** Recall that (1.7) holds uniformly for \( t \) in compact subsets of \( \mathbb{R} \). Hence, using (1.7) but with \( t \) replaced by \( s/n \), \( s \in \mathbb{R} \) fixed, we obtain
\[
\mathbb{E} \left[ e^{isn^{-1} \sum_{j=1}^{n} g(\theta_j) - \nu_1 n} \right] = \frac{e^{isn^{-1} g(\frac{1}{2})} \exp \left( \mathcal{O}(n^{-\xi}) \right) + e^{isn^{-1} g(-\frac{1}{2})} \exp \left( \mathcal{O}(n^{-\xi}) \right)}{2} + \mathcal{O}(n^{-\xi}),
\]
as \( n \to \infty \). Claim (a) now directly follows from Lévy’s continuity theorem. Let us now consider the case \( g(\frac{1}{2}) = g(-\frac{1}{2}) \). Using again (1.7), but now with \( t \) \( \in \mathbb{R} \) fixed, we obtain
\[
\mathbb{E} \left[ e^{itn^{-1} \sum_{j=1}^{n} g(\theta_j) - \nu_1 n} \right] = \frac{1}{2} e^{-n(t^2 / 2 + 2g''(\frac{1}{2})t^2 + 2g''(\frac{1}{2})t + \mathcal{O}(n^{-\xi}))} + \frac{1}{2} e^{-n(t^2 / 2 + 2g''(-\frac{1}{2})t^2 + 2g''(-\frac{1}{2})t + \mathcal{O}(n^{-\xi}))}
\]
as \( n \to \infty \). Since

\[
\mathbb{E}[e^{it(BN_1 + (1-B)N_2)}] = \frac{1}{2} \mathbb{E}[e^{itN_1}] + \frac{1}{2} \mathbb{E}[e^{itN_2}] = \frac{1}{2} e^{\frac{g'(-N)^2}{2} + 2\Re(\xi^{\prime\prime})} e^{2it\Re(\xi^{\prime\prime})} + \frac{1}{2} e^{\frac{g'(-N)^2}{2} + 2\Re(\xi^{\prime\prime})} e^{2it\Re(\xi^{\prime\prime})},
\]

claims (b), (c) and (d) now also directly follows from Lévy’s continuity theorem.

\[\square\]

**Comparison with other point processes.** We are not aware of an earlier work on a point process with only reflection-type interactions over a line such as (1.1). There is however a vast literature on point processes with different types of interactions, such as

(a) the circular \( \beta \)-ensemble (C\( \beta \)E), distributed as \( \sim \prod_{j<k} |e^{i\theta_j} - e^{i\theta_k}|^\beta \prod_{j=1}^n d\theta_j \),

(b) point processes on the unit circle with Riesz pairwise interactions,

(c) the point process on the unit circle with density \( \sim \prod_{j<k} |e^{i\theta_j} - e^{i\theta_k}|^\beta \prod_{j=1}^n d\theta_j \),

(d) the two-dimensional point process with density \( \sim \prod_{j<k}|z_j - z_k|^2 |z_j - \overline{z}_k|^2 \prod_{j=1}^n |z_j - \overline{z}_j|^2 e^{-N|z|^2} d^2 z_j \),

e) the point process on the unit circle with density \( \sim \prod_{j<k} |e^{i\theta_j} + e^{i\theta_k}|^\beta \prod_{j=1}^n d\theta_j \).

Other examples of point processes can be found in e.g. [10, 14]. The four examples listed above at least one common feature with (1.1): (a), (b), (c) and (e) are point processes defined on the unit circle, and (c) and (d) are point processes involving the image points reflected across the real line. In sharp contrast with (1.1), the C\( \beta \)E and the circular Riesz gas favor the configurations with equispaced points on the unit circle and the associated smooth linear statistics have always Gaussian fluctuations, see e.g. [12] for the C\( \beta \)E and [3] for the Riesz gas. Example (c) is discussed in [10, Section 2.9] (see also [13]) for its connection to random matrix theory. Here the reflection-type interactions \( \prod_{j<k} |e^{i\theta_j} - e^{-i\theta_k}|^\beta \) are damped by the pairwise logarithmic interactions \( \prod_{j<k} |e^{i\theta_j} - e^{-i\theta_k}|^\beta \), and just like (a) and (b), the limiting empirical measure of point process (c) is the uniform measure on \((-\pi, \pi]\), see [10, Proposition 3.6.3 and Exercise 4.1.1]. Example (d) is an integrable Pfaffian point process on the plane introduced by Ginibre [17] and later generalized in several works, see e.g. [1, 4, 6] and the review [5]. Here too, the reflection-type interactions \( \prod_{j<k} |z_j - \overline{z}_k|^2 \prod_{j=1}^n |z_j - \overline{z}_j|^2 \) are damped by the pairwise logarithmic interactions \( \prod_{j<k} |z_j - z_k|^2 \). The overall effect is that the repulsion between the points and the real axis is only visible on local scales, see e.g. [1, Figure 1(b)] (this is in sharp contrast with (1.1), see Theorem 1.1).

The point process (e) involves reflection-type interactions across a point (here the origin) and was considered in [7]. In contrast with (a), (b), (c) and (d), and just like (1.1), the empirical measure \( \mu_N := \frac{1}{N} \sum_{j=1}^n \delta_{\theta_j} \) of (e) has no deterministic limit as \( n \to \infty \). In fact, it was proved in [7] that if \( g : \mathbb{R} \to \mathbb{R} \) is \( 2\pi \)-periodic and H"older continuous, then

\[
\int_{(-\pi, \pi]} g(x) d\mu_N(x) \xrightarrow{\text{law}} \int_{(-\pi, \pi]} g(x) d\mu^\varepsilon(x), \tag{1.8}
\]

where \( \mu^\varepsilon = \delta_U \), where \( U \sim \text{Uniform}(-\pi, \pi] \).
It is also interesting to make a comparison between (1.1) and β-ensembles in the “multi-cut regime”. For such ensembles, the empirical measure converges to a deterministic limiting measure supported on several intervals, and the oscillations of various statistics are typically of order 1 and described by θ-functions and discrete Gaussian random variables, see e.g. [9, 16, 2, 8]. The point process (1.1) is very different: even if one views the point masses \( \theta = \frac{2}{z} \) and \( \theta = -\frac{2}{z} \) in (1.2) as degenerated cuts, (1.1) is not in the two-cut regime, but in another kind of regime which we can call a “Bernoulli-one-cut regime” (and similarly, the point process (e) above is in a “Uniform-one-cut regime”).

## 2 Preliminaries

In this section, we introduce some notation and record some results from [15]. These results will be used in Section 3 to obtain large n asymptotics for \( I(f) \).

**Lemma 2.1.** (Special case of [15, Lemma 1].) For all \( x \in \mathbb{R} \),

\[
|\frac{1 + \cos x}{2}| = (\frac{1 + \cos x}{2})^2 = |\cos \frac{x}{2}| \leq \exp(-\frac{x^2}{2} + \frac{3}{2} x^4).
\]

**Lemma 2.2.** ([15, Eq (3.3)]) Let \( \ell \in \mathbb{N}_{>0} \). For all \( x_1, \ldots, x_\ell \in \mathbb{R} \),

\[
\sum_{1 \leq j < k \leq \ell} (x_j + x_k)^2 \geq (\ell - 2) \sum_{j=1}^\ell x_j^2, \quad \sum_{1 \leq j < k \leq \ell} (x_j + x_k)^4 \leq 8(\ell - 1) \sum_{j=1}^\ell x_j^4.
\]

Following [15, Section 2], we also introduce the following quantities:

\[
\gamma = 1 - \sqrt{\frac{n - 2}{2(n - 1)}}, \quad J_n = \text{the } n \times n \text{ matrix of all ones},
\]

\[
T = I_n - \gamma J_n/n, \quad y, \eta \in \mathbb{R}^n, \quad \eta = Ty, \quad \mu_k = \sum_{j=1}^n y_j^k \text{ for } k \geq 0,
\]

\[
U_n(t) = \{ x \in \mathbb{R}^n : |x_i| \leq t, i = 1, \ldots, n \} \text{ for } t \geq 0.
\]

**Lemma 2.3.** ([15, Lemma 2])

(a) \( \sum_{j=1}^n \eta_j = (1 - \gamma) \mu_1 \), \( \sum_{j=1}^n \eta_j^2 = \mu_2 - \gamma(2 - \gamma) \mu_4/n \), \( \sum_{1 \leq j < k \leq n} (\eta_j + \eta_k)^2 = (n - 2) \mu_2 \),

\[
\sum_{1 \leq j < k \leq n} (\eta_j + \eta_k)^4 = (n - 8) \mu_4 + 3 \mu_2^2 + (4(1 - 2\gamma) + 32\gamma/n) \mu_1 \mu_3 - (24\gamma(1 - \gamma)/n + 48\gamma^2/n^2) \mu_4 \mu_2 + (8\gamma^2(1 - \gamma)(3 - \gamma)/n^2 + 8\gamma^3(4 - \gamma)/n^3) \mu_4^2.
\]

(b) \( \det(I_n - s J_n/n) = 1 - s \) for any \( s \).

(c) For any \( t \geq 0 \), \( T U_n(t) \subset (1 + \gamma) U_n(t) \) and \( T^{-1} U_n(t) \subset (1 - \gamma)^{-1} U_n(t) \).

The following lemma is a minor extension of [15, Lemma 3].

**Lemma 2.4.** Let \( a = 9 \), \( \epsilon = \epsilon(n) \) and \( \epsilon' = \epsilon'(n) \) be such that \( 0 < \epsilon' < 2\epsilon < \frac{1}{n} \). Let \( A = A(n) \) be a bounded complex-valued function such that \( \Im A = O(n^{-1}) \) and \( \Re A \geq n^{-\epsilon'} \) for sufficiently large \( n \). Let \( B = B(n), C = C(n), \ldots, K = K(n) \) be complex valued functions such that the ratios \( B/A, C/A, \ldots, K/A \) are bounded. Suppose that \( \delta > 0 \), \( 0 < \Delta < \frac{1}{4} - \frac{1}{2} \epsilon' \), and that

\[
h(y) = \exp \left( -An \mu_2 + Bn \mu_3 + C \mu_1 \mu_2 + D \mu_4^3/n + En \mu_4 + F \mu_2^2 \right)
\]
The proof closely follows the ideas of [15, Proof of Lemma 3]. The case of non-zero $K$ only requires to modify $\psi_n(y)$ in [15, Proof of Lemma 3] into

$$
\psi_n(y) = \exp (-A\eta_1 + En\beta + B\eta_2 + C\theta_1 + D\eta_3) + G\eta_4
+ H\eta_5 + I\eta_6 + J\eta_7 + K\eta_8 + L\eta_9
+ M\eta_{10} + N\eta_{11} + O(\eta_{12})
$$

Also, in [15] this lemma is stated for real-valued $A$, but the extension to complex-valued $A$ with $\text{Im } A = O(n^{-1})$ is straightforward.

**Lemma 2.5.** ([15, Top of p. 572]) If $t > 0$ and $0 < \alpha < \frac{3}{2}$ are fixed, then as $m \to \infty$,

$$
\int_{-2t}^{2t} \exp \left( -mx^2 + \frac{3}{2}m(1 + o(1))x^4 \right) dx \leq \sqrt{\frac{2}{m} (1 + O(m^{-1+\delta}))}.
$$

## 3 Proof of Theorems 1.1 and 1.2

We divide the proof into two parts: we will first prove (1.6) and Theorem 1.1, and the text written below (1.6) about the dependence of the error terms in $t$ will be proved afterwards.

### 3.1 Proof of (1.6) and of Theorem 1.1

The proof closely follows the ideas of [15, Proof of Theorem 1]. For convenience, we first make the change of variables $\eta_j = \theta_j - \frac{\pi}{2}$ in (1.3); this yields

$$
I(f) = \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \prod_{1 \leq j < k \leq n} |e^{i\eta_j} + e^{-i\eta_k}|^\beta \prod_{j=1}^{n} e^{f(\eta_j)} \eta_{ij},
$$

where $f(\eta) := f(\eta + \frac{\pi}{2})$. We first show that the main contribution to $I(f)$ comes from the point configurations for which either all the $e^{i\eta_j}$ are close to 1, or all the $e^{i\eta_j}$ are close to $-1$. Let $\tau = \frac{\pi}{8}$ and fix $\epsilon \in (0, \frac{\pi}{16})$. Consider the partition $(-\pi, \pi)^n = \mathcal{J}_1 \cup \mathcal{J}_2$, with

$$
\mathcal{J}_1 := \left\{ \eta = (\eta_1, \ldots, \eta_n) \in (-\pi, \pi)^n : n_0 n_2 \geq n^{1+\epsilon} \text{ or } \frac{n_1}{2} \geq n^{1+\epsilon} \text{ or } \frac{n_3}{2} \geq n^{1+\epsilon} \right\},
$$

Even for $K \equiv 0$ the statement only holds for $a = 9$ (or $a = 7$ if $D(n) \equiv 0$). This lemma is stated in [15] for $K \equiv 0$ and $a = 6$. However, it seems to us that there is a small typo in [15] and that [15, Eq (2.2)] only holds for $\eta = \frac{\pi}{2} - \alpha \epsilon$. 


and where \( n_0 = n_0(\eta), n_1 = n_1(\eta), n_2 = n_2(\eta), n_3 = n_3(\eta) \) are the numbers of \( \eta \) in the regions \([-\tau, \tau], (\tau, \pi - \tau), (\pi - \tau, \pi) \cup (-\pi, -\pi + \tau) \) and \((-\pi + \tau, -\tau)\), respectively. Define

\[
J_1 = \int_{\mathcal{S}} \prod_{1 \leq j < k \leq n} |e^{\imath \eta_j} + e^{-\imath \eta_k}|^2 \prod_{j=1}^{n} e^{(e^{\imath \eta_j})} d\eta_j.
\]

Using \( |e^{\imath \eta_j} + e^{-\imath \eta_k}|^2 = 2^2 \cos \frac{\eta_j + \eta_k}{2} \), we get

\[
|J_1| \leq e^{nM(f)} (2\pi)^n 2^\beta \frac{n(n-1)}{2} \left[ \left( \cos \tau \right)^{\beta n_1(n_1-1)} + \left( \cos \tau \right)^{\beta n_2(n_2-1)} + \left( \cos \tau \right)^{\beta n_3(n_3-1)} \right] 
\leq 3 e^{nM(f)} (2\pi)^n 2^\beta \frac{n(n-1)}{2} \left( \cos \tau \right)^{\beta n}
\]

(3.2)

It remains to estimate the integral over \( \mathcal{S}_1 \), for which we have \( n_1 = O(n^{\frac{1+\epsilon}{2}}), n_3 = O(n^{\frac{1+\epsilon}{2}}) \), and either \( n_0 = O(n^\epsilon) \) or \( n_2 = O(n^\epsilon) \). For sufficiently large \( n \), we can write \( \mathcal{S}_1 = \mathcal{S}_2 \cup \mathcal{S}_3 \), where

\[
\mathcal{S}_2 = \{ \eta \in \mathcal{S}_1 : n_1 \leq 2n^{\frac{1+\epsilon}{2}} \text{ and } n_3 \leq 2n^{\frac{1+\epsilon}{2}} \text{ and } n_2 \leq n/2 \}.
\]

\[
\mathcal{S}_3 = \{ \eta \in \mathcal{S}_1 : n_1 \leq 2n^{\frac{1+\epsilon}{2}} \text{ and } n_3 \leq 2n^{\frac{1+\epsilon}{2}} \text{ and } n_0 \leq n/2 \}.
\]

We first consider the \( n \)-fold integral over \( \mathcal{S}_2 \). Define \( S_0 = S_0(\eta), S_1 = S_1(\eta) \) and \( S_2 = S_2(\eta) \) by

\[
S_0 = \{ j : |\eta_j| \leq \tau \}, \quad S_1 = \{ j : \tau < |\eta_j| \leq 2\tau \}, \quad S_2 = \{ j : 2\tau < |\eta_j| \leq \pi \},
\]

and let \( s_0 = \#S_0, s_1 = \#S_1 \) and \( s_2 = \#S_2 \). For \( \eta \in \mathcal{S}_2 \), we note that \( s_0 = n_0 \geq n - 5n^{\frac{1+\epsilon}{2}} \) and \( s_1 + s_2 = n_1 + n_2 + n_3 \leq 5n^{\frac{1+\epsilon}{2}} \). Moreover, we have

\[
|\cos \frac{\eta_j + \eta_k}{2}|^2 \leq \begin{cases} 
\exp(-\frac{\beta}{8} \eta_j \eta_k)^2 + \frac{\beta}{96} \eta_j^2 \eta_k^2 + \frac{\beta}{12} (n - s_2 - 1) \eta_3^2, & \text{if } j, k \in S_0 \cup S_1, \\
(\cos \frac{\beta}{2} \eta_j)^2, & \text{if } j \in S_0, \ k \in S_2,
\end{cases}
\]

(3.3)

where for the top inequality we have used Lemma 2.1. Let \( \alpha = -\beta \log \cos \frac{\beta}{2} \). Using (3.3) and Lemma 2.2, we infer that the modulus of the integrand in (3.1) is bounded above by

\[
2^\beta \frac{n(n-1)}{2} \exp \left( -\frac{\beta}{8} \sum_{1 \leq j < k \leq n} (n - s_2 - 2) \sum_{j \in S_0 \cup S_1} \eta_j^2 + \frac{\beta}{12} (n - s_2 - 1) \sum_{j \in S_0 \cup S_1} \eta_j^3 \right) 
\leq e^{nM(f)} 2^\beta \frac{n(n-1)}{2} \exp \left( -\frac{\beta}{8} \sum_{j \in S_0 \cup S_1} \eta_j^2 + \frac{\beta}{12} (n - s_2 - 1) \sum_{j \in S_0 \cup S_1} \eta_j^3 \right).
\]

Let \( J_2(m_2) \) be the contribution to (3.1) from \( \eta \in \mathcal{S}_2 \). Using the above inequality and Lemma 2.5 (with \( \delta = \frac{\beta}{2} \)), for all sufficiently large \( n \) and \( m_2 \leq 5n^{\frac{1+\epsilon}{2}} \) we obtain

\[
|J_2(m_2)| \leq \left| \int_{m_2}^{n} \prod_{j=1}^{m_2} \left| e^{\imath \eta_j} + e^{-\imath \eta_k} \right|^2 \prod_{j=1}^{n} e^{(e^{\imath \eta_j})} d\eta_j \right|
\leq e^{nM(f)} 2^\beta \frac{n(n-1)}{2} (2\pi - 4\pi m_2) e^{-\alpha m_2 (n - 5n^{\frac{1+\epsilon}{2}})} \left( \frac{n}{m_2} \right) \left( \frac{\pi}{2} \right)^n \left( \frac{\pi}{2} \right)^{n-m_2} \left( 1 + O(n^{-\frac{1}{4}}) \right)^{n-m_2}
\]

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We now turn to the analysis of $J_2(m_2)$ and let $	ilde{m}_1 = \tilde{m}_1(\eta)$ by

$$
\tilde{m}_0 = \{j : |\eta_j| \leq n^{-\frac{1}{2} + \epsilon} \},
\tilde{m}_1 = \{j : n^{-\frac{1}{2} + \epsilon} < |\eta_j| \leq 2\tau \},
$$

and let $\tilde{s}_0 = \# \tilde{m}_0$ and $\tilde{s}_1 = \# \tilde{m}_1$. Define also $J_2(\tilde{m}_1)$ to be the contribution to (3.1) from $\{\eta \in (-\pi, \pi)^n: \tilde{s}_0(\eta) = n - \tilde{m}_1$ and $\tilde{s}_1(\eta) = \tilde{m}_1 \}$, and note that $J_2(0) = \sum_{\tilde{m}_1=1}^n J_2(\tilde{m}_1)$. In the same way as we proved (3.4) (but with $s_2 = 0$), we note that the modulus of the integrand in (3.1) is bounded above by

$$
2^{\frac{n(n-1)}{2}} e^{\frac{1}{2} M(f)} \exp \left( -\frac{\beta}{8} (n-2) \sum_{j=1}^n \eta_j^2 + \frac{\beta}{12} (n-1) \sum_{j=1}^n \eta_j^4 + \sum_{j=1}^n \text{Re}(e^{i\eta_j}) \right).
$$

Using (3.6) and Lemma 2.5 with $\delta = \frac{1}{4}$, we find

$$
|J_3(\tilde{m}_1)| = \left| \left( \frac{n}{\tilde{m}_1} \right) \int_{|\eta_1|, \ldots, |\eta_{\tilde{m}_1}| \leq n^{-\frac{1}{2} + \epsilon}} \prod_{1 \leq j < k \leq \tilde{m}_1} |e^{i\eta_j} + e^{-i\eta_k}|^\beta \prod_{j=1}^n e^{f(\eta_j)} d\eta \right|
$$

$$
\leq 2^{\frac{n(n-1)}{2}} n^{-1} \tilde{m}_1 \left( \int_{-n^{-\frac{1}{2} + \epsilon}}^{n^{-\frac{1}{2} + \epsilon}} e^{\text{Re}(f(z))} \exp \left( -\frac{\beta}{8} (n-2) x^2 + \frac{\beta}{12} (n-1) x^4 \right) dx \right)^{n-\tilde{m}_1}
$$

$$
\times e^{\frac{1}{2} M(f)} \left( 2 \int_{-n^{-\frac{1}{2} + \epsilon}}^{n^{-\frac{1}{2} + \epsilon}} \exp \left( -\frac{\beta}{8} (n-2) x^2 + \frac{\beta}{12} (n-1) x^4 \right) dx \right)^{\tilde{m}_1}
$$

$$
\leq 2^{\frac{n(n-1)}{2}} n^{-1} \tilde{m}_1 e^{n(n-1)\text{Re}(f(0)) + O(\sqrt{\tilde{m}_1})} \left( \frac{8\pi}{\beta n - 2} \right)^{\frac{n-\tilde{m}_1}{2}} \exp \left( -\frac{\beta}{16} n^{2\epsilon} \tilde{m}_1 + O(n^4) + O(\tilde{m}_1 \log n) \right).
$$

Hence, for some $c_3 > 0$,

$$
\sum_{\tilde{m}_1=1}^n |J_3(\tilde{m}_1)| \leq e^{n \text{Re}(f(0))} 2^{\frac{n(n-1)}{2}} \left( \frac{8\pi}{\beta n} \right)^{\frac{n-\tilde{m}_1}{2}} \exp \left( -\frac{\beta}{8} n^{2\epsilon} \tilde{m}_1 + O(n^4) + O(\tilde{m}_1 \log n) \right).
$$

Finally, we turn to the analysis of $J_3(0)$. Since $f$ is $C^2, \beta$ is a neighborhood of 0, as $x \to 0$ we have

$$
\log \left[ 2^\beta \cos \frac{\beta}{2} x \right] = \beta \log 2 - \frac{\beta}{8} x^2 - \frac{\beta}{192} x^4 + O(x^6),
$$

$$
f(x) = f(0) + f'(0)x + \frac{1}{2} f''(0)x^2 + O(x^{2+\epsilon}),
$$

and thus

$$
J_3(0) = 2^{\frac{n(n-1)}{2}} e^{n f(0)} \int_{U_n(n^{-\frac{1}{2} + \epsilon})} \exp \left( -\frac{\beta}{8} \sum_{j \leq k} (\eta_j + \eta_k) - \frac{\beta}{192} \sum_{j \leq k} (\eta_j + \eta_k)^4 + O\left( \sum_{j \leq k} (\eta_j + \eta_k)^6 \right) \right)
$$
\[
+f'(0) \sum_{j=1}^{n} \eta_j + \frac{f''(0)}{2} \sum_{j=1}^{n} \eta_j^2 + O\left( \sum_{j=1}^{n} \eta_j^{2+\epsilon} \right) \prod_{j=1}^{n} d\eta_j.
\]

For \( \eta \in U_n(n^{-\frac{1}{2}+\epsilon}) \),

\[
O\left( \sum_{j<k} (\eta_j + \eta_k)^6 \right) = O(n^{-1+6\epsilon}), \quad O\left( \sum_{j=1}^{n} \eta_j^{2+\epsilon} \right) = O(n^{-\frac{1}{2}+(2+\epsilon)\epsilon}).
\]

Since \( \epsilon \in (0, \frac{1}{18}] \) is fixed, we have \( n^{-\frac{1}{2}+(2+\epsilon)\epsilon} + n^{-1+6\epsilon} = O(n^{-\frac{1}{2}+(2+\epsilon)\epsilon}) \). Hence, applying the transformation \( \eta = Ty \) of Section 2, and using Lemma 2.3 (a) and (b) (using in particular that \( \det T = 1 - \gamma = \frac{1}{14} + O(n^{-1}) \)), we obtain

\[
J_3(0) = 2^{\frac{a+n-1}{2}} \frac{e^{f(0)}}{\sqrt{2}} \int_{T^{-1}U_n(n^{-\frac{1}{2}+\epsilon})} \exp \left\{ -\frac{\beta}{8} (n-2) \mu_2 - \frac{\beta}{192} (n-8) \mu_4 + \left( 4(1-2\gamma) + \frac{32\gamma}{n} \right) \mu_1 \mu_3 \right\} \\
+ 3\mu_2^2 \left( \frac{24\gamma(1-\gamma)}{n} + \frac{48\gamma^2}{n^2} \right) \mu_1^2 \mu_2 + \left( 8\gamma^2(1-\gamma)(3-\gamma) + \frac{8\gamma^3(4-\gamma)}{n^2} \right) \mu_1^4 \\
+ f'(0)(1-\gamma)\mu_1 + \frac{f''(0)}{2} \left( \mu_2 - \gamma(2-\gamma) \frac{\mu_1^2}{n} \right) + O(n^{-\frac{1}{2}+(2+\epsilon)\epsilon}) \prod_{j=1}^{n} d\eta_j.
\]

(3.8)

By Lemma 2.3 (c), \( U_n(n^{-\frac{1}{2}+\epsilon}) \subseteq T^{-1}U_n(n^{-\frac{1}{2}+\epsilon}) \subseteq U_n(n^{-\frac{1}{2}+\epsilon}) \). Hence, using [11, Theorem 4.3] with \( \phi_1, \phi_2 = n^{-\frac{1}{2}+\epsilon} \), we infer that we can replace \( T^{-1}U_n(n^{-\frac{1}{2}+\epsilon}) \) by \( U_n(n^{-\frac{1}{2}+\epsilon}) \) in (3.8) at the cost of a multiplicative error \( \times (1 + O(n^{-1+8\epsilon})) \). Since \( \epsilon \in (0, \frac{1}{18}] \), we can then apply Lemma 2.4 with \( \delta = \frac{\beta}{2} - (2+\epsilon)\epsilon > 0 \) and

\[
A = \frac{\beta}{8} \frac{n-2}{n} - \frac{f''(0)}{2n}, \quad B = 0, \quad C = 0, \quad D = 0, \quad E = -\frac{\beta}{192} \frac{n-8}{n}, \quad F = -\frac{\beta}{64}, \\
G = -\frac{\beta}{48} \left( 1 - 2\gamma + \frac{8\gamma}{n} \right), \quad H = \frac{\beta}{8} \left( \gamma(1-\gamma) + \frac{2\gamma^2}{n} \right), \quad I = -\frac{\beta}{24} \left( \gamma^2(1-\gamma)(3-\gamma) + \frac{\gamma^3(4-\gamma)}{n} \right), \\
J = f'(0)(1-\gamma), \quad K = -\frac{f''(0)}{2} \gamma(2-\gamma),
\]

to get

\[
J_3(0) = 2^{\frac{a+n-1}{2}} \frac{e^{f(0)}}{\sqrt{2}} \left( \frac{8\pi}{\beta n} \right)^{\frac{3}{2}} \exp \left( 1 - \frac{1}{2\beta} + \frac{f'(0)^2}{\beta} + \frac{2f''(0)}{\beta} + O(n^{-\zeta}) \right),
\]

(3.9)

for any fixed \( \zeta < \min\{ \frac{\beta}{2} - (2+\epsilon)\epsilon, \frac{\beta}{2} - (2+\epsilon)\epsilon \} \). By combining (3.5), (3.7), (3.9) and (1.5), we conclude that

\[
\int \ldots \int \mathcal{J}_2 |e^{in_j} + e^{-in_k}|^\beta \prod_{j=1}^{n} e^{f(n_j)} d\eta_j = J_3(0)(1 + O(e^{-cn^2})),
\]

for some \( c > 0 \). Similarly, reducing \( c > 0 \) if necessary, we find

\[
\int \ldots \int \mathcal{J}_2 |e^{in_j} + e^{-in_k}|^\beta \prod_{j=1}^{n} e^{f(n_j)} d\eta_j = \tilde{J}_3(0)(1 + O(e^{-cn^2})),
\]

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where \( \tilde{J}_3(0) \) satisfies
\[
\tilde{J}_3(0) = 2^\beta \sigma^{2(n-1)/2} \frac{e^{nf(\pi)}}{\sqrt{2}} \left( \frac{8\pi}{\beta n} \right)^{\frac{3}{2}} \exp \left( 1 - \frac{1}{2\beta} + \frac{f'(\pi)^2}{\beta} + \frac{2f''(\pi)}{\beta} + O(n^{-\zeta}) \right).
\]  
(3.10)

Hence, by (3.2), (3.5), (3.7), (3.9) and (3.10), we have
\[
I(f) = (J_3(0) + \tilde{J}_3(0))(1 + O(e^{-cn^2}))
\]
\[
= 2^\beta \sigma^{2(n-1)/2} \left( \frac{8\pi}{\beta n} \right)^{\frac{3}{2}} \left[ e^{nf(0)} \exp \left( 1 - \frac{1}{2\beta} + \frac{f'(0)^2}{\beta} + \frac{2f''(0)}{\beta} + O(n^{-\zeta}) \right) + e^{nf(\pi)} \exp \left( 1 - \frac{1}{2\beta} + \frac{f'(\pi)^2}{\beta} + \frac{2f''(\pi)}{\beta} + O(n^{-\zeta}) \right) \right],
\]
which is (1.6).

Note that for \( f \equiv 0, (J_3(0) + \tilde{J}_3(0))/I(0) \) is equal to
\[
\mathbb{P}\left( \left| e^{i\theta_j} - i \right| \leq n^{\frac{1}{\beta} + \epsilon} \text{ for all } j \in \{1, \ldots, n\} \right) \text{ or } \left( \left| e^{i\theta_j} + i \right| \leq n^{\frac{1}{\beta} + \epsilon} \text{ for all } j \in \{1, \ldots, n\} \right).
\]

Hence Theorem 1.1 directly follows from (3.11).

### 3.2 The case of \( I(tf) \), \( t \in \mathbb{R} \) and \( \text{Re } f \equiv 0 \)

If \( f \) is replaced by \( tf \) in Subsection 3.1, and if \( \text{Re } f \equiv 0 \), then the estimates (3.2), (3.5), (3.7) become
\[
|J_1| \leq 3(2\pi)^n 2^\beta \sigma^{2(n-1)/2} (\cos \tau)^{\beta n^{1+\epsilon}},
\]
\[
O(n^{1+\epsilon}) \sum_{m_1=1}^n |J_2(m_2)| \leq e^{-n\alpha 2^{\beta(n-1)/2}} \left( \frac{8\pi}{\beta n} \right)^{\frac{3}{2}} e^{-c_s n^{2\epsilon}},
\]
\[
O(n^{1+\epsilon}) \sum_{n_1=1}^n |J_3(\tilde{m}_1)| \leq 2^\beta \sigma^{2(n-1)/2} \left( \frac{8\pi}{\beta n} \right)^{\frac{3}{2}} e^{-c_s n^{2\epsilon}},
\]
which are independent of \( t \). Furthermore, it directly follows from Lemma 2.4 that the \( O(n^{-\zeta}) \)-terms in (3.9) and (3.10) are uniform for \( t \) in compact subsets of \( \mathbb{R} \). This proves that the \( O(n^{-\zeta}) \)-terms in (1.6) are uniform for \( t \) in compact subsets of \( \mathbb{R} \).

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**References**

[1] G. Akemann, S.-S. Byun and N.-G. Kang, Scaling limits of planar symplectic ensembles, *SIGMA Symmetry Integrability Geom. Methods Appl.* **18** (2022), 40 pp.

[2] G. Borot and A. Guionnet, Asymptotic expansion of beta matrix models in the multi-cut regime, arXiv:1303.1045.

[3] J. Boursier, Optimal local laws and CLT for 1D long-range Riesz gases, arXiv:2112.05881.

[4] S.-S. Byun and C. Charlier, On the almost-circular symplectic induced Ginibre ensemble, to appear in *Stud. Appl. Math.*, arXiv:2206.06021

[5] S.-S. Byun and P.J. Forrester, Progress on the study of the Ginibre ensembles II: GinOE and GinSE, preprint.

[6] S.-S. Byun, N.-G. Kang, S.-M. Seo, Partition functions of determinantal and Pfaffian Coulomb gases with radially symmetric potentials, arXiv:2210.02799.
[7] C. Charlier, Smooth statistics for a point process on the unit circle with reflection-type interactions across the origin, preprint.
[8] C. Charlier, B. Faha, C. Webb and M.D. Wong, Asymptotics of Hankel determinants with a multi-cut regular potential and Fisher-Hartwig singularities, arXiv:2111.08395.
[9] P. Deift, A. Its and X. Zhou, A Riemann-Hilbert approach to asymptotic problems arising in the theory of random matrix models, and also in the theory of integrable statistical mechanics, Ann. of Math. (2) 146 (1997), no. 1, 149–235.
[10] P.J. Forrester, Log-gases and random matrices, London Mathematical Society Monographs Series, 34. Princeton University Press, Princeton, NJ, 2010.
[11] M. Isaev and B.D. McKay, Complex martingales and asymptotic enumeration, Random Structures Algorithms 52 (2018), no. 4, 617–661.
[12] K. Johansson, On Szegő’s asymptotic formula for Toeplitz determinants and generalizations, Bull. Sci. Math. (2) 112 (1988), no. 3, 257–304.
[13] R. Killip and I. Nenciu, Matrix models for circular ensembles, Int. Math. Res. Not. 2004, no. 50, 2665–2701.
[14] T. Leblé and S. Serfaty, Large deviation principle for empirical fields of log and Riesz gases, Invent. Math. 210 (2017), no. 3, 645–757.
[15] B.D. McKay and N.C. Wormald, Asymptotic enumeration by degree sequence of graphs of high degree, European J. Combin. 11 (1990), no. 6, 565–580.
[16] M. Shcherbina, Fluctuations of Linear Eigenvalue Statistics of $\beta$ Matrix Models in the Multi-cut Regime, J. Stat. Phys. 151 (2013), no. 6, 1004–1034.
[17] J. Ginibre, Statistical ensembles of complex, quaternion, and real matrices, J. Mathematical Phys. 6 (1965), 440–449.