YAMABE FLOW AND THE MYERS-TYPE THEOREM ON COMPLETE MANIFOLDS

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Abstract. In this paper, we prove the following Myers-type theorem: if \((M^n, g), n \geq 3,\) is an n-dimensional complete locally conformally flat Riemannian manifold with bounded Ricci curvature satisfying the Ricci pinching condition \(Rc \geq \epsilon Rg > 0,\) where \(\epsilon > 0\) is a uniform constant, then \(M^n\) must be compact.

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1. Introduction

The Yamabe flow has been proposed by R.Hamilton [18] in the early 1980s as a tool for constructing metrics of constant scalar curvature in a given conformal class of Riemannian metrics on a manifold of dimension no less than three. There are some interesting results of the Yamabe flow on closed manifolds. For the case when the initial metric is locally conformally flat and has positive Ricci curvature, B.Chow [10] has proved that the normalized Yamabe flow converges to a metric of constant curvature on closed manifolds. Then R.Ye [29] has improved B.Chow’s result, by assuming only that the initial metric is locally conformally flat, that the normalized Yamabe flow converges to a metric of constant scalar curvature on closed manifolds in this case. Recently, S.Brendle [3] has proved that the normalized Yamabe flow converges to a metric of constant scalar curvature on closed manifolds if the initial metric is locally conformally flat or \(3 \leq n \leq 5,\) where \(n\) is the dimension of the manifold. For other recent works of Yamabe flow on closed manifolds, one may see [4] and [28].

Precisely, in this paper, we consider the Yamabe flow \((M^n, g(t)), t \in (0, T),\) satisfying

\[
\frac{\partial g}{\partial t} = -Rg
\]
on a complete locally conformally flat Riemannian manifold \((M^n, g(0))\), 
\(n \geq 3\), where \(g(t)\) evolves in the conformal class of the given metric \(g(0)\), i.e. \(g(t) = u(t)^{\frac{4}{n-2}} g(0)\) for \(u(t)\) being a positive smooth function on \(M^n\), and \(R\) is the scalar curvature of \(g(t)\).

By using the Ricci flow and the Yamabe flow as tools, there are some interesting Myers-type results, which state that if a complete manifold with bounded curvature satisfies a pinching condition, then this manifold must be compact. B.L.Chen and X.P.Zhu [9], by using Ricci flow, have firstly proved that if \(M^n\) is a complete \(n\)-dimensional Riemannian manifold with positive and bounded scalar curvature and satisfies the pinching condition

\[ |W|^2 + |V|^2 \leq \delta_n(1 - \epsilon)|U|^2, \]

where \(W, V, U\) are the Weyl part, scalar curvature part and traceless Ricci part of the Riemannian curvature tensor respectively and \(\delta_4 = \frac{4}{5}, \delta_5 = \frac{1}{10}, \delta_n = \frac{2}{(n-2)(n+1)}\), \(n \geq 6, \epsilon > 0\) is a constant. When \(n = 3\), they have also proved if \(M^3\) is 3-dimensional complete Riemannian manifold with bounded and nonnegative sectional curvature satisfying the Ricci pinching condition

\[ Rc \geq \epsilon Rg > 0, \]

where \(\epsilon > 0\) is a constant, then \(M^n\) must be compact. Then L.Ni and B.Q.Wu [23] have improved B.L.Chen and X.P.Zhu's result for \(n \geq 4\) and have proved that if \(M^n\) is an \(n\)-dimensional complete Riemannian manifold with bounded curvature operator satisfying the pinching condition

\[ Rm \geq \delta R_I > 0, \]

where \(\delta > 0\) is a constant and \(R_I\) is the scalar part of the curvature operator \(Rm\), then \(M^n\) must be compact. In the recent interesting work [6], S.Brendle and R. Schoen have proved if \(M^n, n \geq 4\), is an \(n\)-dimensional complete Riemannian manifold with bounded curvature operator satisfying the pinching condition

\[ Rm(e_1, e_3, e_1, e_3) + \lambda^2 Rm(e_1, e_4, e_1, e_4) + \mu^2 Rm(e_2, e_3, e_2, e_3) + \lambda^2 \mu^2 Rm(e_2, e_4, e_2, e_4) - 2\lambda \mu Rm(e_1, e_2, e_3, e_4) \geq \delta R > 0 \]

for all orthonormal four frames \(\{e_1, e_2, e_3, e_4\}\) and all \(\lambda, \mu \in [-1, 1]\), where \(\delta > 0\) is a constant and \(R\) is the scalar curvature, then \(M^n\) must be compact. In [16], H.L.Gu has proved that, by using the Yamabe flow, if \((M^n, g), n \geq 3\), is an \(n\)-dimensional complete locally conformally flat Riemannian manifold with bounded Ricci curvature and nonnegative sectional curvature satisfying the Ricci pinching condition

\[ Rc \geq \epsilon Rg > 0, \]
where $\epsilon > 0$ is an uniform constant, then $M^n$ must be compact.

We remark that all the results above need the condition that $M^n$ has nonnegative sectional curvature (or positive curvature operator). This assumption gives a nature injectivity bound needed in the convergence theorem, which also needed to apply dimension reduction techniques (see [20]). Without this assumption, we meet the difficult conjecture of R.Hamilton, which states that if $M^3$ is a 3-dimensional complete Riemannian manifold satisfying the Ricci pinching condition

$$Rc \geq \epsilon Rg > 0,$$

where $\epsilon > 0$ is an uniform constant, then $M^3$ must be compact (see [2]).

In this paper, we prove the following result which partially confirms the conjecture of R.Hamilton on 3-dimensional locally conformally flat manifolds.

**Theorem 1.1.** If $(M^n, g)$, $n \geq 3$, is an n-dimensional complete locally conformally flat Riemannian manifold with bounded Ricci curvature satisfying the pinching condition

$$Rc \geq \epsilon Rg > 0,$$

where $\epsilon > 0$ is an uniform constant, then $M^n$ must be compact.

**Remark 1.2.** Note that by the strong maximum principle, Theorem 1.1 is equivalent to say that if $(M^n, g)$, $n \geq 3$, is an n-dimensional complete noncompact locally conformally flat Riemannian manifold with bounded Ricci curvature satisfying the Ricci pinching condition $Rc \geq \epsilon Rg$ and $R \geq 0$, where $\epsilon > 0$ is an uniform constant, then $M^n$ must be flat.

2. **Preliminaries**

We shall recall some basic formulae and convergence results concerning with the Yamabe flows. Similar results for the Ricci flow are well-known.

We first recall some formulae from the fundamental paper [10] (see Lemmas 2.2 and 2.4).

**Lemma 2.1.** If $(M^n, g(t))$, $n \geq 3$, is the solution to the Yamabe flow (1.1) on an n-dimensional complete locally conformally flat Riemannian manifold, then

$$R_t = (n - 1)\Delta R + R^2.$$
and
\[ \partial_t R_{ij} = (n - 1) \Delta R_{ij} + \frac{1}{n - 2} B_{ij}, \]
where
\[ B_{ij} = (n - 1) |Ric|^2 g_{ij} + n R R_{ij} - n(n - 1) R^2_{ij} - R^2 g_{ij}. \]

As pointed out in [12], we can rewrite the equation (2.2) for \( Rc \) as
\[ \partial_t Rc = (n - 1) \Delta Rc + Rc \ast Rc, \]
where \( Rc \ast Rc \) stands for any linear combination of tensors formed by contraction on \( R_{ij} \cdot R_{kl} \). Notice that the evolution for \( Rc \) along the Yamabe flow has the same form as the evolution for \( Rm \) along the Ricci flow. Techniques similar to Shi’s in [27] can be applied to the Yamabe flow as well, and we can show that all the covariant derivatives of \( Rc \) are locally uniformly bounded on \((0, T)\) if \( |Rc| \) is bounded on \([0, T)\). Hence, all the covariant derivatives of the Riemannian curvature \( Rm \) are uniformly bounded on \((0, T)\) if \( |Rc| \) is bounded on \([0, T)\) in the locally conformally flat Riemannian manifolds under the Yamabe flow. Similarly, techniques similar to Shi’s in [26] can also be applied to the Yamabe flow, and we have that there exists a solution to the Yamabe flow on noncompact locally conformally flat Riemannian manifolds with bounded Ricci curvature in some time interval \([0, T)\) (see [1]).

It is well known that the singularity analysis plays the important role in the study of geometric flows. To study the singularities of geometric flows, one often dilates about a singularity based on the blow up rate of the curvature. Note that the curvature bound is immediately satisfied for the blow up about a singularity, but the injectivity radius bound is not. Especially, it seems hard to get the injectivity radius bound without extra conditions along the geometric flows in noncompact manifolds, since the injectivity radius may not have uniformly lower bound at the initial time. In order to handle this problem, we shall use the method firstly proposed by K.Fukaya [13] in metric geometry and later by D.Glickenstein [15] to the Ricci flow. The latter work gives a kind of precompactness theorem ([15], Theorem 3) of the Ricci flow without injectivity radius estimates. We note that Fukaya-Glickenstein’s theorem also holds for the Yamabe flow. The reasons are below.

First we need an elementary fact in Riemannian geometry and one can find a proof in [21]: Let \((M^n, g)\) be a complete Riemannian manifold with bounded sectional curvature \(|sec| \leq 1\). Given a point \( p \in M^n \). Denote the exponential map by \( exp_p : T_p M^n \to M^n \) with \( B(o, \pi) \subset T_p M^n \) equipped with metric \( exp_p^* g \). Then the injectivity radius at \( o \) in \( B(o, \pi) \) has its lower bound that \( \text{inj}(o) > \frac{\pi}{2} \).
So we have the following precompactness theorem for the Yamabe flow.

**Lemma 2.2.** Let \( \{(M^n_i, g_i(t), x_i)\}_{i=1}^{\infty}, t \in [0, T] \), be a sequence of the Yamabe flows on the complete locally conformally flat Riemannian manifolds such that
\[
\sup_{M^n_i \times [0,T]} |Rm(g_i(t))|_{g_i(t)} \leq 1.
\]

Let \( \phi_i = \exp_{x_i, g_i(0)} \) be the exponential map with respect to metric \( g_i(0) \) and \( B(o_i, \frac{\pi}{2}) \subset T_{x_i}M_i \) equipped with metric \( \tilde{g}_i(t) \equiv \phi_i^*g(t) \). Then \( (B(o_i, \frac{\pi}{2}), \tilde{g}_i(t), o_i) \) subconverges to a Yamabe flow \( (B(o, \frac{\pi}{2}), \tilde{g}(t), o) \) in \( C^\infty \) sense, where \( B(o, \frac{\pi}{2}) \subset \mathbb{R}^n \) equipped with metric \( \tilde{g}(t) \).

**Proof.** Since \( \tilde{g}_i(t) \equiv \phi_i^*g(t) \), we have
\[
\sup_{B(o_i, \frac{\pi}{2}) \times [0,T]} |Rm(\tilde{g}_i(t))|_{\tilde{g}_i(t)} \leq 1.
\]

As we mentioned before, Shi’s local derivative estimates of curvature operator also hold for the Yamabe flow on complete locally conformally flat manifolds. Note that we also have \( inj(o_i, \tilde{g}_i(0)) > \frac{\pi}{2} \). Then the result follows from the proof of Hamilton’s precompactness theorem for the Ricci flow (see [19]). \( \square \)

Next we recall the following two definitions in [15].

**Definition 2.3.** [15] A sequence of pointed n-dimensional Riemannian manifolds \( \{(M^n_i, g_i, x_i)\}_{i=1}^{\infty} \) locally converges to a pointed metric space \( (X, d, x) \) in the sense of \( C^\infty \)-local submersions at \( x \) if there is a Riemannian metric \( \overline{g} \) on an open neighborhood \( V \subset \mathbb{R}^n \) of \( o \), a pseudogroup \( \Gamma \) such that the quotient is well defined, an open set \( U \subset X \), and maps \( \varphi_i : (V, o) \rightarrow (M_i, x_i) \) such that

1. \( \{(M^n_i, g_i, x_i)\}_{i=1}^{\infty} \) converges to \( (X, d, x) \) in the pointed Gromov-Hausdorff distance,
2. the identity component of \( \Gamma \) is a Lie group germ,
3. \( (V/T, d_{\overline{g}}) \) is isometric to \( (U, d) \), where \( d_{\overline{g}} \) is the induced distance in the quotient,
4. \( (\varphi_i)_* \) is nonsingular on \( V \) for all \( i \in N \), and
5. \( \overline{g} \) is the \( C^\infty \) limit of \( \varphi_i^*g_i \) (uniform convergence on compact sets together with all derivatives).

**Definition 2.4.** [15] A sequence of pointed n-dimensional Riemannian manifolds \( \{(M^n_i, g_i, x_i)\}_{i=1}^{\infty} \) converges to a pointed metric space \( (X, d, x) \) in the sense of \( C^\infty \)-local submersions if for every \( y \in X \) there exist \( y_i \in M_i \) such that \( \{(M^n_i, g_i, y_i)\}_{i=1}^{\infty} \) locally converges to \( (X, d, y) \) in the sense of \( C^\infty \)-local submersions at \( y \).
Note that there exists subsequence \( \{(M_k, g_k(t), x_k)\}_{k=1}^{\infty} \) converges to \((X, d(t), x)\) for each \( t \in [0, T] \) in Gromov-Hausdorff distance by \(|Rm(g(t))|_{g(t)} \leq 1 \) and Theorem 19 in [15]. In fact \( \phi_i = \exp_{x_i, g_i(0)} \) in Lemma 2.2 defines a 'locally' covering map between \( B(o_i, \frac{\pi}{2}) \subset T_{x_i} M_i \) and \( B(x_i, \frac{\pi}{2}) \subset M_i \). This defines pseudogroups \( \Gamma_i \) acts isometrically on \( B(o, \frac{1}{4}) \) (see [13], P.9 or [15], §5). Furthermore, \( \Gamma_i \) converge to a limit pseudogroup \( \Gamma \) (see see [13], P.9) such that \( (B(o, \frac{1}{4}), \Gamma_i) \), where \( B(o, \frac{1}{4}) \subset B(o, \frac{\pi}{2}) \), converges to \( (B(o, \frac{1}{4}), \Gamma) \) in the equivariant Gromov-Hausdorff distance (see [13], Definition 1.9), and hence \( B(o_i, \frac{1}{4})/\Gamma_i \) converges to \( B(o, \frac{1}{4})/\Gamma \) in the Gromov-Hausdorff distance (see [13], Lemma 1.11). Since \( B(o_i, \frac{1}{4})/\Gamma_i \) is isometric to a neighborhood of \( x_i \), \( B(o, \frac{1}{4})/\Gamma \) is isometric to a neighborhood of \( x \). Note that \( B(o_i, \frac{1}{4}) \) converges to \( B(o, \frac{1}{4}) \) in \( C^\infty \) sense by Lemma 2.2. So we have proved that \( (M_i, d_{g_i(t)}, x_i) \) converges to \((X, d(t), x)\) in the sense of \( C^\infty \)-local submersions at \( x \). If we identify \( B(o_i, \frac{1}{4}) \) with \( B(o, \frac{1}{4}) \) by map \( id \), then \( \varphi_i = \phi_i \circ id \), where \( \varphi_i \) is defined in Definition 2.3. Note that \( \Gamma \) is a Lie group germ by [13], §3. Just notice that if \((M_i, d_{g_i(t)}, x_i) \) converges to \((X, d(t), x)\) in the pointed Gromov-Hausdorff distance, then for every \( y \in X \) there exist \( y_i \in X_i \) such that \((M_i, d_{g_i(t)}, y_i) \) converges to \((X, d(t), x)\) in the pointed Gromov-Hausdorff distance (see [12], Proposition 12). Then \((M_i, d_{g_i(t)}, x_i) \) converges to \((X, d(t), x)\) in the sense of \( C^\infty \)-local submersions.

**Theorem 2.5.** Let \( \{(M^n_i, g_i(t), x_i)\}_{i=1}^{\infty} \), where \( t \in [0, T] \), be a sequence of pointed solutions to the Yamabe flows on the locally conformally flat Riemannian manifolds such that

\[
\sup_{M^n_i \times [0, T]} |Rm(g_i(t))|_{g_i(t)} \leq 1,
\]

and for all \( i \in \mathbb{N} \) and \( t \in [0, T] \).

Then there is a subsequence which still denote by \( \{(M_i, g_i(t), x_i)\}_{i=1}^{\infty} \) and a one parameter family of complete pointed metric spaces \((X, d(t), x)\) such that for each \( t \in [0, T] \), \((M_i, d_{g_i(t)}, x_i) \) converges to \((X, d(t), x)\) in the sense of \( C^\infty \)-local submersions and the metric \( \overline{g}(t) \) in Definition 2.3 is solution to the Yamabe flow.

**Remark 2.6.** In fact, Fukaya-Glickenstein’s theorem holds for any sequence of geometric flows \( \{(M^n_i, g_i(t), x_i)\}_{i=1}^{\infty} \), \( \frac{\partial \overline{g}}{\partial t} = h(g) \) satisfying \( h(g_i(t)) < C \), \( |Rm(g_i(t))|_{g_i(t)} \leq C \) on \([0, T]\) and Shi’s local derivative estimates of curvature operators hold.

Note that the limit space \((X, d(t))\) is an Alexandrov space, since the sectional curvature of \( M_i \) has a uniformly lower bound. Finally, we need the following theorem which can be found in [7], Theorem 3.6.
Theorem 2.7. Let $M$ be a complete Alexandrov space with curvature $K > 0$. Then $\text{diam}(M) \leq \frac{\pi}{\sqrt{K}}$.

3. Singularity model of the Yamabe flow

Recall that R.Hamilton has proposed the singularity models which classify all the maximal solutions to Ricci flow into three types. We note that the same classification can be applied to the Yamabe flow; every maximal solution to the Yamabe flow on locally conformally flat manifolds with nonnegative Ricci curvature is of only one of the following three types:

**Definition 3.1.** Suppose that $(M^n, g(t))$ is a solution to the Yamabe flow on a locally conformally flat manifold with nonnegative Ricci curvature. If $T < \infty$, we say that the solution forms a

1. Type I singularity if $\sup_{M \times [0,T]} (T-t)R < \infty$,
2. Type IIa singularity if $\sup_{M \times [0,T]} (T-t)R = \infty$.

Similarly, if $T = \infty$, we say that the solution forms a

1. Type IIb singularity if $\sup_{M \times [0,\infty)} tR = \infty$,
2. Type III singularity if $\sup_{M \times [0,\infty)} tR < \infty$.

For any maximum solution to the Yamabe flow on a locally conformally flat manifold with nonnegative Ricci curvature, we have that if the infimum of injectivity radius $\rho(t)$ at all points satisfies $\rho(t) \geq \frac{c}{\sqrt{M(t)}}$, where $c > 0$ is a uniform constant and $M(t)$ denotes the supremum of the curvature at time $t$, then there exists a sequence of dilations of the solution which converges in the limit to one of the following singularity model of the corresponding type (see [20]) in the sense of Definition 3.2 below.

**Definition 3.2.** Suppose that $(M^n, g(t))$ is a limit solution to the Yamabe flow on a locally conformally flat manifold with nonnegative Ricci curvature. We say that the limit solution is

1. Type I limit solution if it exists for $-\infty < t < \Omega$ for some constant $\Omega$ with $0 < \Omega < +\infty$ and $R \leq \frac{\Omega}{\Omega - t}$ everywhere with equality holds somewhere at $t = 0$,
2. Type II limit solution if it exists for $-\infty < t < +\infty$ and $R \leq 1$ everywhere with equality holds somewhere at $t = 0$. 


(3) Type III limit solution if it exists for $-A < t < +\infty$ for some constant $A$ with $0 < A < +\infty$ and $R \leq \frac{A}{A+t}$ with equality holds somewhere at $t = 0$.

As we mentioned before, the injectivity radius lower bound may not be available in the sequence of dilations of the maximal solution to the Yamabe flow on complete and noncompact manifolds. Hence the singularity model in Definition 3.2 may not suitable for our original Yamabe flow. However, we shall show how to use Theorem 2.5 to avoid the assumption of uniform injectivity radius bound in the next section.

In order to prove Theorem 1.1, we need a local version of a result proved by H.L.Gu [16], which is based on the B.Chow’s Harnack inequality [10].

**Theorem 3.3.** Let $D \subset M^n$ be a simply connected open domain of a complete $n$-dimensional locally conformally flat Riemannian manifold such that B.Chow’s Harnack inequality and the strong maximum principle for the Harnack quantity $Z$ of the Yamabe flow (see (3.1)) hold true on $D$. Then any Type III limit solution with positive Ricci curvature to the Yamabe flow on $D \subset M^n$ is necessarily a homothetically expanding gradient soliton.

**Proof.** We follow the argument in [16] and assume that $D = M$ without loss of generality. We may assume that, after a shift of the time variable, the Type III limit solution of the Yamabe flow on locally conformally flat manifolds is defined for $0 < t < +\infty$, where $tR$ achieves its maximum in space-time. Recall that B.Chow [10] has proved the following Harnack inequality

$$Z = \frac{\partial R}{\partial t} + <\nabla R, X> + \frac{1}{2(n-1)} R_{ij} X^i X^j + \frac{R}{t} \geq 0,$$  

(3.1) for the Yamabe flow on the closed locally conformally flat manifolds with positive Ricci curvature. We remark that by the same proof and by using the maximum principle, this Harnack inequality clearly holds for the Yamabe flow on the complete locally conformally flat manifolds with nonnegative and bounded Ricci curvature.

Since $tR$ achieves its maximum at some $(x_0, t_0)$, (3.1) vanishes in the direction $X = 0$ at $(x_0, t_0)$. By the strong maximum principle (see [12], Lemma 3.2), we know that at any $t < t_0$ and any point $x \in M^n$, there is a vector $X \in T_x M^n$ such that $Z = 0$. Take the first variation of $Z$ in $X$, we get

$$\nabla_i R + \frac{1}{n-1} R_{ij} X^j = 0.$$  

(3.2)
We remark that for \((R_{ij}) > 0\), the equation above uniquely determines a vector field \(X\).

Substituting (3.2) into \(Z = 0\), we have

\[
\frac{\partial R}{\partial t} + \frac{R}{t} + \frac{1}{2} \nabla_i R \cdot X^i = 0. \tag{3.3}
\]

We now denote \(\partial_t - (n - 1)\Delta\) by \(\Box\). Applying \(\frac{1}{2} X^i \Box\) to (3.2), \(\Box\) to (3.3) and then take the sum, we have

\[
X^i \Box(\nabla_i R) + \frac{1}{2(n - 1)} X^i X^j \Box R_{ij} - \nabla_k R_{ij} (\nabla_k X^j) X^i \\
- (n - 1) \nabla_i \nabla_k R \cdot \nabla_k X^i + \Box(\frac{\partial R}{\partial t} + \frac{R}{t}) = 0. \tag{3.4}
\]

We also have

\[
\Box(\nabla_i R) = \nabla_i (\Box R) - (n - 1) R_{il} \nabla_l R \\
= \nabla_i (R^2) - (n - 1) R_{il} \nabla_l R. \tag{3.5}
\]

By Lemma 3.8 in [10], we get

\[
\Box(\frac{\partial R}{\partial t} + \frac{R}{t}) = 3(n - 1) R \Delta R + \frac{1}{2} (n - 1)(2 - n) |\nabla R|^2 \\
+ 2 R^3 + \frac{R^2}{t} - \frac{R}{t^2}. \tag{3.6}
\]

Substituting (2.2), (3.5) and (3.6) into (3.4), we get

\[
X^i (\nabla_i (R^2) - (n - 1) R_{il} \nabla_l R) + \frac{1}{2(n - 1)(n - 2)} X^i X^j B_{ij} \\
- \nabla_k R_{ij} (\nabla_k X^j) X^i - (n - 1) \nabla_k \nabla_i R \cdot \nabla_k X^i \\
+ 3(n - 1) R \Delta R + \frac{1}{2} (n - 1)(2 - n) |\nabla R|^2 \\
+ 2 R^3 + \frac{R^2}{t} - \frac{R}{t^2} = 0. \tag{3.7}
\]

It follows from (3.2) that

\[
\nabla_k \nabla_l R + \frac{1}{n - 1} (\nabla_k R_{ij}) X^i = -\frac{1}{n - 1} R_{ij} \nabla_k X^i, \tag{3.8}
\]

and

\[
X^i R_{il} \nabla_l R + \frac{1}{n - 1} R_{il} R_{lj} X^i X^j = 0. \tag{3.9}
\]
We also have
\begin{equation}
Z = (n-1)\Delta R^+ <\nabla R, X> + \frac{1}{2(n-1)} R_{ij} X^i X^j + R^2 + \frac{R}{t} = 0.
\end{equation}

Substituting (3.8), (3.9) and (3.11) into (3.7), we get
\begin{equation}
- R(R + \frac{1}{t})^2 + \frac{1}{2(n-1)(n-2)} B_{ij} X^i X^j - \frac{1}{2(n-1)} R R_{ij} X^i X^j
\end{equation}
\begin{equation}
+ \frac{n}{2(n-1)} R_{il} R_{jl} + R_{ij} \nabla_k X^i \nabla_k X^j = 0.
\end{equation}

By (3.2), we have
\[ \nabla_k \nabla_i R = - \frac{1}{n-1} (X^j \nabla_k R_{ij} + R_{ij} \nabla_k X^j), \]
and then by taking the trace and using the evolution equation of scalar curvature, we get
\begin{equation}
R_{ij} ((R + \frac{1}{t})g_{ij} - \nabla_i X^j) = 0.
\end{equation}

By (3.11) and (3.12), we conclude that
\begin{equation}
R_{ij} (\nabla_k X^i - (R + \frac{1}{t}) g_{ik}) (\nabla_k X^j - (R + \frac{1}{t}) g_{jk}) + A_{ij} X^i X^j = 0,
\end{equation}
where
\[ A_{ij} = \frac{1}{2(n+1)(n+2)} B_{ij} + \frac{1}{2(n-2)} (n R_{il} R_{jl} - R R_{ij}). \]

Then in local coordinates where \( g_{ij} = \delta_{ij} \) and the Ricci tensor \( (R_{ij}) \) is diagonal, we have
\[ \sum_i \lambda_i (\nabla_k X^i - (R + \frac{1}{t}) g_{ik})^2 + A_{ij} X^i X^j = 0. \]

By [10](3.13), we have \( \nu_i = \frac{1}{2(n+1)(n-2)} \sum_{k,l \neq i,k > l} (\lambda_k - \lambda_l)^2 \), where \( \nu_i \) is the eigenvalue of \( A_{ij} \). Since \( \lambda_i > 0 \), the theorem holds immediately. \( \square \)

Finally, we need the following

**Theorem 3.4.** [16] There exists no noncompact locally conformally flat Type III limit solution of the Yamabe flow which satisfies the Ricci pinching condition
\[ Rc \geq \epsilon Rg > 0, \]
for some constant \( \epsilon > 0 \).
4. Pinching estimates

In [10], B. Chow proved the inequality $R_{ij} \geq \epsilon R_{ij} > 0$ is preserved under the Yamabe flow on compact locally conformally flat manifolds. Clearly his proof also works in the complete setting all curvature operators are uniformly bounded in space, at each time-slice, which can apply the maximum principle for complete manifolds. B. Chow [10] also gets the pinching estimate that

$$Rc_{\max} - Rc_{\min} \leq CR^1 - \epsilon$$

where $C$ is a constant only depending on $g(0)$. But this pinching estimate may not strong enough for our purpose. In this section, we calculate the term $|Rc|^2 - \frac{1}{n} R^2$ directly and get an improved pinching estimate.

**Lemma 4.1.** If $(M^n, g(0))$, $n \geq 3$, is an $n$-dimensional locally conformally flat complete Riemannian manifold and bounded Ricci curvature, then the following equality holds for any constant $\delta$ under the Yamabe flow (1.1),

$$\partial_t \frac{|Rc|^2 - \frac{1}{n} R^2}{R^2 - \delta} = 2(1 - \delta)(n - 1) \frac{\langle \nabla f, \nabla R \rangle}{R}$$

$$- \frac{2(n - 1)}{R^{4-\delta}} |R \nabla Rc - \nabla RRc|^2$$

$$- \frac{(1 - \delta)(n - 1)}{R^{4-\delta}} (|Rc|^2 - \frac{1}{n} R^2) |\nabla R|^2$$

$$+ \frac{1}{R^{2-\delta}} (\delta R (|Rc|^2 - \frac{1}{n} R^2) - J),$$

where $f = \frac{|Rc|^2 - \frac{1}{n} R^2}{R^2 - \delta}$ and $J = \frac{2}{n-2} \left( n(n-1) tr(Rc^3) + R^3 - (2n-1)R|Rc|^2 \right)$. 

**Proof.** By (2.2) and $|Rc|^2 = g^{ik} g^{jl} R_{ij} R_{kl}$, we have

$$\partial_t |Rc|^2 = 2g^{ik} g^{jl} (\partial_t R_{ij}) R_{kl} + 2g^{ik} g^{jl} R_{ij} R_{kl}$$

$$= (n - 1) \Delta |Rc|^2 - 2(n - 1) |\nabla Rc|^2 + \frac{n - 1}{n - 2} R |Rc|^2$$

$$- \frac{2}{n - 2} R^3 - \frac{2(n - 1)}{n - 2} tr(Rc^3).$$

From (2.1), we get

$$\partial_t R^2 = (n - 1) \Delta R^2 - 2(n - 1) |\nabla R|^2 + 2R^3.$$
Hence
\[ \partial_t(|Rc|^2 - \frac{1}{n}R^2) = (n-1)\Delta(|Rc|^2 - \frac{1}{n}R^2) - 2(n-1)(\nabla Rc\cdot \nabla R) \]
\[ +6\frac{n-1}{n-2}R|Rc|^2 - (\frac{2}{n-2} + \frac{2}{n})R^3 - \frac{2n(n-1)}{n-2}tr(Rc^3). \]

Now we denote \( \partial_t - (n-1)\Delta \) by \( \square \). So we have
\[
\Box f = \frac{\Box(|Rc|^2 - \frac{1}{n}R^2)}{R^{2-\delta}} - (2-\delta)(\frac{|Rc|^2 - \frac{1}{n}R^2}{R^{4-\delta}})^{\frac{1}{2}}|\nabla R|^2
+ \frac{2(2-\delta)(n-1)}{R^{3-\delta}} < \nabla R, \nabla (|Rc|^2 - \frac{1}{n}R^2) >
\]
\[ = A + B, \]

where
\[ A = -\frac{2(n-1)}{R^{2-\delta}}(|\nabla Rc|^2 - \frac{1}{n}|\nabla R|^2)
- (2-\delta)(3-\delta)(n-1)\frac{|Rc|^2 - \frac{1}{n}R^2}{R^{4-\delta}}|\nabla R|^2
+ \frac{2(2-\delta)(n-1)}{R^{3-\delta}} < \nabla R, \nabla (|Rc|^2 - \frac{1}{n}R^2) > \]

contains the gradient terms and
\[ B = \frac{1}{R^{2-\delta}}(6\frac{n-1}{n-2}R|Rc|^2 - (\frac{2}{n-2} + \frac{2}{n})R^3 - \frac{2n(n-1)}{n-2}tr(Rc^3))
- (2-\delta)(\frac{|Rc|^2 - \frac{1}{n}R^2}{R^{4-\delta}})^{\frac{1}{2}}R^2
\]
\[ = \frac{1}{R^{2-\delta}}(\delta(|Rc|^2 - \frac{1}{n}R^2)R - J) \]

contains the curvature terms. We rewrite \( A \) as
\[ \frac{A}{n-1} = -\frac{2}{R^{2-\delta}}(|\nabla Rc|^2 - \frac{1}{n}|\nabla R|^2) - (2-\delta)(3-\delta)\frac{|Rc|^2 - \frac{1}{n}R^2}{R^{4-\delta}}|\nabla R|^2
+ \frac{2(2-\delta)}{R^{3-\delta}} < \nabla R, \nabla (|Rc|^2 - \frac{1}{n}R^2) >
\]
\[ = -\frac{2}{R^{2-\delta}}(|\nabla Rc|^2 - \frac{1}{n}|\nabla R|^2) - (2-\delta)(3-\delta)\frac{|Rc|^2 - \frac{1}{n}R^2}{R^{4-\delta}}|\nabla R|^2
+ \frac{2(1-\delta)}{R^{3-\delta}} < \nabla R, \nabla (|Rc|^2 - \frac{1}{n}R^2) >
+ \frac{2}{R^{3-\delta}} < \nabla R, \nabla (|Rc|^2 - \frac{1}{n}R^2) > \]
Since
\[ \nabla \left( \frac{|Rc|^2 - \frac{1}{n}R^2}{R^{2-\delta}} \right) = \nabla \left( \frac{|Rc|^2 - \frac{1}{n}R^2}{R^{2-\delta}} \right) - (2 - \delta) \frac{|Rc|^2 - \frac{1}{n}R^2}{R^{3-\delta}} \nabla R, \]
we get
\[ A_{n-1} = -\frac{2}{R^{2-\delta}} (|\nabla Rc|^2 - \frac{1}{n}|
abla R|^2) - (2 - \delta)(1 + \delta) \frac{|Rc|^2 - \frac{1}{n}R^2}{R^{4-\delta}} |
abla R|^2 \\
+ \frac{2(1 - \delta)}{R} < \nabla R, \nabla \left( \frac{|Rc|^2 - \frac{1}{n}R^2}{R^{2-\delta}} \right) > \\
+ \frac{2}{R^{3-\delta}} < \nabla R, \nabla (|Rc|^2 - \frac{1}{n}R^2) >. \]

Note that
\[-\frac{2}{R^{2-\delta}} |\nabla Rc|^2 - 2\frac{|Rc|^2}{R^{4-\delta}} |
abla R|^2 + \frac{2}{R^{3-\delta}} < \nabla R, \nabla |Rc|^2 > = -\frac{2}{R^{4-\delta}} |R \nabla Rc - \nabla RRc|^2, \]
so we have
\[ A_{n-1} = \frac{2(1 - \delta)}{R} < \nabla \left( \frac{|Rc|^2 - \frac{1}{n}R^2}{R^{2-\delta}} \right), \nabla R > -\frac{2}{R^{4-\delta}} |R \nabla Rc - \nabla RRc|^2 \\
(4.2) \\
- \frac{(1 - \delta)\delta}{R^{4-\delta}} (|Rc|^2 - \frac{1}{n}R^2) |
abla R|^2. \]
Combining with (4.2) and (4.1), we conclude Lemma 4.1. \(\square\)

Next we need the following lemma to control the term \(J\) defined in Lemma 4.1.

**Lemma 4.2.** If \((M^n, g), n \geq 3\), is an \(n\)-dimensional complete locally conformally flat Riemannian manifold and bounded Ricci curvature satisfying \(Rc \geq \epsilon Rg > 0\), then we have the following inequality holds
\[ J \geq \frac{4}{3} n \epsilon R (|Rc|^2 - \frac{1}{n}R^2), \]
where \(J\) is defined in Lemma 4.1.
Proof. Let $\lambda_i$ be the eigenvalues of $Rc$ and assume $\lambda_n \geq \lambda_{n-1} \geq \cdots \geq \lambda_1$. Then we compute

$$\frac{n-2}{2} J = n(n-1) \sum \lambda_i^3 + (\sum \lambda_i)((\sum \lambda_i)^2 - (2n-1) \sum \lambda_i^2)$$

$$= n(n-1) \sum \lambda_i^3 + (\sum \lambda_i)(-2(n-1) \sum \lambda_i^2 + 2 \sum \lambda_i \lambda_j)$$

$$= n(n-1) \sum \lambda_i^3 - 2(n-1)(\sum \lambda_i^3 + \sum \lambda_i^2 \lambda_j + \sum \lambda_i \lambda_j^2)$$

$$+ 2(\sum \lambda_i^2 \lambda_j + \sum \lambda_i \lambda_j^2 + 3 \sum \lambda_i \lambda_j \lambda_k)$$

$$= (n-1)(n-2) \sum \lambda_i^3 - 2(n-2)(\sum \lambda_i^2 \lambda_j + \sum \lambda_i \lambda_j^2)$$

$$+ 6 \sum \lambda_i \lambda_j \lambda_k$$

$$= 2 \sum_{i<j<k} (\lambda_k(\lambda_k - \lambda_i)(\lambda_k - \lambda_j) + \lambda_j(\lambda_j - \lambda_i)(\lambda_j - \lambda_k)$$

$$+ \lambda_i(\lambda_i - \lambda_k)(\lambda_i - \lambda_j))$$

Note that $\lambda_j \leq \lambda_k$ for $j \leq k$. We have

$$\frac{n-2}{2} J = 2 \sum_{i<j<k} (\lambda_k(\lambda_k - \lambda_i)(\lambda_k - \lambda_j) - \lambda_j(\lambda_j - \lambda_i)(\lambda_k - \lambda_j)$$

$$+ \lambda_i(\lambda_k - \lambda_i)(\lambda_j - \lambda_i))$$

$$\geq 2 \sum_{i<j<k} (\lambda_k(\lambda_k - \lambda_i)(\lambda_k - \lambda_j) - \lambda_k(\lambda_j - \lambda_i)(\lambda_k - \lambda_j)$$

$$+ \lambda_i(\lambda_k - \lambda_i)(\lambda_j - \lambda_i))$$

$$\geq 2 \sum_{i<j<k} (\lambda_k(\lambda_k - \lambda_j)^2 + \lambda_i(\lambda_j - \lambda_i)^2).$$
Since $\lambda_i \geq \epsilon R$ for any $i$ and $(\lambda_k - \lambda_j)^2 + (\lambda_j - \lambda_i)^2 \geq \frac{1}{3}((\lambda_k - \lambda_j)^2 + (\lambda_k - \lambda_i)^2 + (\lambda_j - \lambda_i)^2)$, we get
\[
\frac{n-2}{2} J \geq \frac{2}{3} \epsilon R \sum_{i<j<k} ((\lambda_k - \lambda_j)^2 + (\lambda_k - \lambda_i)^2 + (\lambda_j - \lambda_i)^2)
= \frac{2}{3} \epsilon (n-2) n R \sum_{i<j} \frac{(\lambda_i - \lambda_j)^2}{n}
= \frac{2}{3} \epsilon (n-2) n R (|Rc|^2 - \frac{1}{n} R^2).
\]
Hence Lemma 4.2 holds immediately. 

Finally, we get the following improved pinching estimate.

**Theorem 4.3.** If $(M^n, g(0))$, $n \geq 3$, is a $n$-dimensional complete locally conformally flat Riemannian manifold and bounded Ricci curvature satisfying $Rc \geq \epsilon Rg > 0$, then the following inequality holds under the Yamabe flow (1.1)
\[
f(t) \leq \left( \frac{1}{3t} \right)^{\frac{n}{3}},
\]
where $f$ is defined in Lemma 4.1 and $\delta = \frac{n\epsilon}{3}$.

**Proof.** The assertion is trivial if $(M^n, g(t))$ is Einstein at some time. Now by Lemma 4.1 and 4.2, we have
\[
\partial_t f(t) \leq (n-1) \Delta f + \frac{2(1 - \frac{n\epsilon}{3})(n-1)}{R} < \nabla f, \nabla R > -n \epsilon f.
\]
Since $Rc > 0$, clearly $f \leq R^\delta$. So we get
\[
\partial_t f(t) \leq (n-1) \Delta f + \frac{2(1 - \delta)(n-1)}{R} < \nabla f, \nabla R > -n \epsilon f^{1+\frac{2}{n}}.
\]
Hence Theorem 4.3 follows from maximum principle immediately. 

5. **Proof of Theorem 1.1**

Before presenting the proofs Theorem 1.1 we give some remarks.

First, as we mentioned before, it is hard to control the injectivity radius uniformly in the sequence of dilations of the maximal solution to the Yamabe flow on noncompact manifolds without extra conditions. Note that the curvature bound is satisfied for the sequence of dilations in all the singularity models. Hence, by Theorem 2.5 we have $(M_i, g_i(t), x_i)$ with positive Ricci curvature subconverges to metric space $(X, d(t), x)$ in the sense of $C^\infty$-local submersions for the sequence of dilations in all the singularity models. We have a neighborhood $V \subset \mathbb{R}^n$ of $0$ with metric $g(t)$ is the solution to the Yamabe flow, and
(V, \overline{g_\nu}(t))$ modulo an isometric pseudogroup action $\Gamma_V$ is isometric to a neighborhood $V'$ of $x$ in the limit metric space $(X, d(t))$. Moreover, there are maps $\varphi_i : (V, o) \to (M_i, x_i)$ such that $\overline{g_\nu}$ is the $C^\infty$ limit of $\varphi_i^* g_i$.

Note that one difficulty in applying the methods in the proof of Theorem 1.1 is that we may not apply the weak maximum principle directly on $V$. Fortunately, we can apply the weak maximum principle on $(M, g_i(t))$ and get the Harnack inequality $Z(g_i) \geq 0$ and then $Z(\varphi_i^* g_i) \geq 0$. Since $\overline{g_\nu}$ is the $C^\infty$ limit of $\varphi_i^* g_i$, we still have Harnack inequality holds on $(V, \overline{g_\nu}(t))$. Similarly, Theorem 4.3 also holds on $(V, \overline{g_\nu}(t))$.

Second, we need to establish the strong maximum principle for the Harnack quantity $Z$ in $V$. Suppose that $Z$ is positive for all $Y \in T_{x_0} V$ at $t = t_0$, for any given point $y \in V$. Let $\Omega \subset V$ be a connected open set such that $\Omega$ is a compact manifold with smooth boundary and $\Omega$ contains both $x_0$ and $y$. We can find a nonnegative function $f$ on $V$ with support on $\Omega$ so that $f(x_0) > 0$ and $Z \geq \frac{f}{t_0}$ for all $Y \in T_x V$ for all $x \in \Omega$ at $t_0$. Let $f$ evolves as

$$\begin{cases} 
\partial_t f = (n - 1) \Delta f & \text{in } \Omega \times [t_0, T], \\
 f(x, t) = 0 & \text{on } \partial \Omega \times [t_0, T]. 
\end{cases}$$

By the scalar strong maximum principle, we conclude that $f > 0$ on $\Omega \times (t_0, T]$. Since $(\partial_t - (n - 1) \Delta) Z \geq -\frac{2}{t} Z$ (see [10], (3.14)), we get $(\partial_t - (n - 1) \Delta)(Z - \frac{f}{t_0}) \geq -\frac{2}{t} (Z - \frac{f}{t_0})$ on $\Omega \times [t_0, T]$. Moreover, since $Z \geq 0$, we have $Z \geq \frac{f}{t_0}$ on $\Omega \times \{t_0\} \cup \partial \Omega \times [t_0, T]$. By the weak maximum principle, we have $Z \geq \frac{f}{t_0}$. So $Z$ is positive for all $Y \in T_x V$ for $x \in \Omega$ for any $t > t_0$.

Third, the arguments below show the relation between the Riemannian neighborhood above the different points in limit space $X$, i.e. we show that they always have the subset locally isometric to each other if the intersection of their projection is not empty. By Theorem 2.5 we know that for all $y \in X$ there exist $y_i \in M_i$ such that $(M_i^n, g_i(t), y_i)$ locally converges to $(X, d(t), y)$ in the sense of $C^\infty$-local submersions at $y$. Again, we emphasize that the above conclusion holds because of Proposition 12 in [15]. Then we have a neighborhood $U \subset \mathbb{R}^n$ of $o$ with metric $\overline{g}(t)$ being the solution to the Yamabe flow, and $(U, \overline{g}(t))$ modulo an isometric pseudogroup action $\Gamma_U$ is isometric to a neighborhood $U'$ of $y$ in the limit metric space $(X, d(t))$. Now we assume $W' = U' \cap V' \neq \emptyset$ and define $\pi_U : U \to U'$, $\pi_V : V \to V'$. By the definition of the Gromov-Hausdorff distance, there is a Gromov-Hausdorff approximation map $\psi_i : X \to M_i$ such that $\psi_i(W')$ converges to $W'$. 


Clearly \( \overline{g_U} \upharpoonright_{\pi^{-1}_U(W')} \) and \( \overline{g_V} \upharpoonright_{\pi^{-1}_V(W')} \) are the \( C^\infty \) limits of \( (\varphi_U)_t^* g_t \upharpoonright_{\psi(W')} \) and \( (\varphi_V)_t^* g_t \upharpoonright_{\psi(W')} \). Hence, \( \pi^{-1}_U(W') \) is locally isometric to \( \pi^{-1}_V(W') \).

With the preparations above we now give the proof of Theorem 1.1.

**Proof of Theorem 1.1.** Since the sectional curvature is bounded at the initial time \( t = 0 \), the Yamabe flow has a solution on the complete non-compact manifold \( M^n \) in some time interval \([0, T)\).

If the singularity is of Type I, Type IIa, Type IIb. Just as [20], we can take a sequence \((x_i, t_i)\) and define the pointed rescaled solutions \((M^n, g_i(t), x_i), t \in [\alpha_i, 0]\) by letting \( g_i(t) = Q_i g(t_i + Q_i^{-1} t) \), where \( Q_i = R(x_i, t_i) \) and \( \alpha_i = -t_i Q_i \), such that

\[
R_{g_i}(x, t) \leq C,
\]

for all \( x \in M^n, t \in (\alpha_i, 0] \),

\[
R_{g_i}(x_i, 0) = 1,
\]

and

\[
t_i Q_i \to \infty.
\]

Since the Weyl tensor of \( M^n \) is vanishing and \( Rc > 0 \), then we get

\[
\sup_{M \times (-\epsilon, 0]} |Rm|_{g_i}(x, t) \leq C.
\]

By theorem 2.5, we have \((M_i, g_i(t), x_i)\) subconverges to metric space \((X, d(t), x)\) in the sense of \( C^\infty\)-local submersions. Hence we have a neighborhood \( V \subset \mathbb{R}^n \) of \( o \) with metric \( \overline{g_V}(t) \) is the ancient solution to the Yamabe flow, and \((V, \overline{g_V}(t))\) modulo an isometric pseudogroup action \( \Gamma_V \) is isometric to a neighborhood \( V' \) of \( x \) in the limit metric space \((X, d(t))\). Moreover, there are maps \( \varphi_i : (V, o) \to (M_i, x_i) \) such that \( \overline{g_V} \) is the \( C^\infty \) limit of \( \varphi_i^* g_i \). Hence \( |R_{\overline{g_V}}(o, 0)| = 1 \). Applying Theorem 4.1 on time interval \([-\alpha, 0]\), we get

\[
(|Rc|^2 - \frac{1}{n} R^2)(\overline{g_V}(0)) \leq \frac{R^2 - \frac{n}{2} \overline{\nabla}^2(\overline{g_V}(0))}{(3\alpha)^{n+1}}.
\]

Since \( \overline{g_V}(t) \) is an ancient solution, letting \( \alpha \to \infty \), we get \( (|Rc|^2 - \frac{1}{n} R^2)(\overline{g_V}(0)) \equiv 0 \). Then this implies \( Rc(\overline{g_V}(0)) \equiv c_1 > 0 \) in \( V \). Since the Weyl tensor of \( \overline{g_V} \) is vanishing, we conclude that \( sec(\overline{g_V}(0)) \equiv c_2 > 0 \) in \( V \).

By Theorem 2.5, we know that for all \( y \in X \) there exists \( y_i \in M^n_i \) for each \( M^n_i \) such that \((M^n_i, g_i(t), y_i)\) locally converges to \((X, d(t), y)\) in the sense of \( C^\infty\)-local submersions at \( y \). Then we have a neighborhood \( U \subset \mathbb{R}^n \) of \( o \) with the metric \( \overline{g_V}(t) \) being the ancient solution to the Yamabe flow, and \((U, \overline{g_V}(t))\) modulo an isometric pseudogroup action \( \Gamma_U \) is isometric to a neighborhood \( U' \) of \( y \) in the limit metric space.
ing soliton of Yamabe flow, i.e. there is smooth vector field satisfying 
\[ \text{sec}_{\pi_U^{-1}(W')} = \text{sec}_{\pi_V^{-1}(W')} \equiv c_2 > 0. \]
Now repeat the same arguments before, we can conclude \( \text{sec}_{\tilde{g}_t(0)} = \text{sec}_{\tilde{g}_t(0)} \equiv c_2 > 0. \)

Hence clearly we have for all the point in \( X \) there exists a neighborhood isometric to a Riemannian neighborhood, which has constant curvature, modula a pseudo-group action. Furthermore, the curvature Riemannian neighborhood in different point has the same value \( c_2 > 0. \)

Then \( X \) is an Alexandrov space with curvature \( \geq c_2 > 0 \) by the Corollary in [7] (see §4.6, in Page 16). Then \( X \) must be compact by Theorem [2.7] which is a contradiction.

If the singularity is of Type III, i.e. \( \sup_{M \times [0, \infty)} tR < \infty. \) Set \( A = \limsup_{t \to \infty} tM(t) \), where \( M(t) = \sup_{M} R(x, t) \).

By B.Chow's Harnack inequality [3.1] and taking \( X = 0 \), we get \( \frac{\partial R}{\partial t}(tR) \geq 0. \) Hence we have \( A > 0. \) So we can take a sequence \( (x_i, t_i) \) such that \( t_i \to \infty \) and \( A_i = t_iR(x_i, t_i) \to A. \) Define the pointed rescaled solutions \( (M^n, g_i(t, x_i), t \in (-t_iR_i, \infty), g_i(t) = Q_iQ_i^{-1}t \), where \( Q_i = R(x_i, t_i) \). For any \( \epsilon > 0 \) we can find a time \( \tau < \infty \) such that for \( t \geq \tau \) and any \( x \in M^n \)
\[
tR(x, t) \leq A + \epsilon.
\]

Then we have
\[
R_{g_i}(x, t) \leq \frac{A + \epsilon}{A_i + t},
\]
for all \( x \in M^n, t \in [-\frac{A_i(t_i - \tau)}{t_i}, \infty) \) and
\[
R_{g_i}(x_i, 0) = 1.
\]

Set \( \phi_i = exp_{x_i, g_i(0)} \) and \( B(o_i, \frac{\pi}{2}) \subset T_{x_i} M \) equipped with metric \( \tilde{g}_i(t) \triangleq \phi_i^*g_i(t) \). By Lemma [2.2] we get \( (B(o_i, \frac{\pi}{2}), \tilde{g}_i(t), o_i) \) subconverges to a Yamabe flow \( (B(o, \frac{\pi}{2}), \tilde{g}(t), o) \) in \( C^\infty \) sense.

Hence
\[
R_{\tilde{g}(t)}(x, t) \leq \frac{A}{A + t}
\]
for all \( x \in B(o, \frac{\pi}{2}), t \in (-A, \infty) \) and
\[
R_{\tilde{g}(t)}(o, 0) = 1.
\]

Then by Theorem [3.3] we conclude that \( (B(o, \frac{\pi}{2}), \tilde{g}(t)) \) is an expanding soliton of Yamabe flow, i.e. there is smooth vector field satisfying 
\[
\nabla_k X^i - (R + \frac{1}{2})g_{ik} = 0.
\]
Moreover, \( X \) is the unique solution of the
equation

\[ \nabla_i R + \frac{1}{n-1} R_{ij} X^j = 0. \]

We use the arguments due to A. Chau and L. F. Tam \[8\] (see Theorem 2.1) to show the injectivity radius of \( x_i \) have the uniformly lower bound with respect to \( g_i(0) \). By our assumptions on the positivity of Ricci curvature, we may then let \( W(i) \in TM \) be the unique solutions to (5.2) on \((M^n, g_i(t))\) for any \( i \). Set \( V(i) = \phi_i^* W(i) \). Then \( V(i) \) converges to \( X \) in the \( C^\infty \) sense.

In some coordinates \( x^\alpha \) of \( B(o, \pi/2) \), the integral curves of \( -X(\cdot, 0) \) (i.e. the vector field \( X \) at time \( t = 0 \)) are given by the following

\[ x'_\alpha = -\lambda_\alpha x_\alpha + F_\alpha(x) \]

where \( \lambda_\alpha \geq c > 0 \) are the positive eigenvalues of \((R + \frac{1}{t}) g(\cdot, 0), |F(x)| = O(|x|^2) \) and \( |dF(x)| = O(|x|) \). For any \( \epsilon > 0 \), there exists sufficient larger \( i \) and \( 0 < r_1 < \pi/2 \) such that the integral curves of \( -V(i)(\cdot, 0) \) is given by

\[ x'_\alpha = -\lambda_\alpha x_\alpha + G^\alpha_i(x), \]

with \( |G^i - F| + |dG^\alpha - dF| \leq \epsilon \) in \( B_{g_i}(o, r_1) \).

Let \( x(\tau) \) be an integral curve of \( -V(i)(\cdot, 0) \) in \( B_{g_i}(o, r_1) \). Set \( |x| \leq r_2 < r_1 \), where \( r_2 \) is to be determined later. We calculate

\[ \frac{d}{d\tau} |x|^2 \leq -2c|x|^2 + |G^i||x| \]

\[ \leq -2c|x|^2 + \epsilon |x| + C_1 |x|^2 \]

\[ \leq -\frac{3c}{2} |x|^2 + \epsilon |x|, \]

where \( C_1 > 0 \) is a constant only depending on \( F \) and \( r_2 < \frac{\epsilon}{2C_1} \). Then if \( \frac{3c}{2} \leq |x| \leq r_2 \), we have \( \frac{d}{d\tau} |x|^2 < 0 \) if \( \epsilon \) is sufficient small. Hence for \( i \) large enough any integral curve of \( -V(i)(\cdot, 0) \) starting in \( B_{g_i}(o, r_2) \) will stay inside \( B_{g_i}(o, r_2) \).

Now let \( x(\tau) \) and \( y(\tau) \) be two integral curves of \( -V(i)(\cdot, 0) \) inside \( B_{g_i}(o, r_2) \). Then we calculate

\[ \frac{d}{d\tau} |x - y|^2 \leq -2c|x - y|^2 + ||dG^i|||x - y|^2 \]

\[ \leq -c|x - y|^2. \]

Hence

\[ |x - y|(\tau) \leq \exp(-c\tau)|x - y|^2(0) \leq 4r_2^2 \exp(-c\tau). \]
Set $y(\tau) = x(\tau_2 - \tau_1 + \tau)$. Then we have $|x(\tau_1) - y(\tau_1)| = |x(\tau_1) - x(\tau_2)| \leq 4r^2 \exp(-c\tau_1)$. Hence $x(\tau)$ converges to a point $x_0 \in B_{g_i}(o, r_2)$. By (5.3), we conclude that $y(\tau)$ also converges to $x_0$.

Next we show $\phi_i$ is injective on $B_{g_i}(o, r_2)$ for $i$ sufficient large, which imply that $\text{inj}(g_i(0), x_i) \geq r_2$. Otherwise, there exist two points $p_1 \neq p_2 \in B_{g_i}(o, r_2)$ such that $\phi_i(p_1) = \phi_i(p_2) = q \in M^n$. Let $\gamma_1$ and $\gamma_2$ be two integral curves for $-V(i)(\cdot, 0)$ starting at $p_1$ and $p_2$ respectively. Hence $\phi_i(\gamma_1)$ and $\phi_i(\gamma_2)$ be two integral curves for $-W(i)(\cdot, 0)$ starting at $q$. By uniqueness of the integral curves, we have $\phi_i(\gamma_1(\tau)) = \phi_i(\gamma_2(\tau))$ for all $\tau$. On the other hand, for all $\tau$, we also have $\gamma_1(\tau) \neq \gamma_2(\tau)$ by uniqueness of integral curves. But $\gamma_1$ and $\gamma_2$ both converge to the point $x_0 \in B_{g_i}(o, r_2)$. It contradicts the fact that $\phi$ is the diffeomorphism in some neighborhood of $x_0$.

Then we have $(M^n, g_i(t), x_i)$ subconverges to a noncompact Type III limit solution to Yamabe flow (an expanding soliton) with $Rc \geq \epsilon Rg > 0$ which contradicts to Theorem 3.4.

This completes the proof of Theorem 1.1. □

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