Distributing a multiparticle state by entanglement swapping

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Using entanglement swapping, we construct a scheme to distribute an arbitrary multiparticle state to remote receivers. Only Bell states and two-qubit collective measurements are required.

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The superposition principle is the most novel and least understood feature of the quantum theory. Quantum entanglement arises from the superposition of multiparticle states. For particles located far apart from one another, quantum entanglement gives rise to the mysterious phenomenon of non-local correlation to which there exists no local classical explanation [1, 2, 3, 4]. With the advent of quantum information science in recent years, instead of remaining merely a mystery to be solved, quantum entanglement has become a valuable and often indispensable resource in every branch of the field.

A bipartite state $|\psi_{1,2}\rangle$ is entangled if it cannot be written in a product form:

$$|\psi_{1,2}\rangle = |\psi_1\rangle \otimes |\psi_2\rangle.$$ (1)

The two-qubit Bell (or EPR) states,

$$|\phi^\pm_{1,2}\rangle = \frac{1}{\sqrt{2}} (|0_1\rangle|1_2\rangle \pm |0_1\rangle|1_2\rangle),$$ (2)

$$|\psi^\pm_{1,2}\rangle = \frac{1}{\sqrt{2}} (|0_1\rangle|0_2\rangle \pm |1_1\rangle|1_2\rangle),$$ (3)

are the most common entangled states employed in quantum information science. (The $\otimes$ sign will be understood hereafter.) Usually two separated particles are entangled because they were once in contact and interacted with each other. However, by entanglement swapping [5, 6, 7], we can entangle two remotely separated particles which do not have a history of mutual interaction. This is a vivid demonstration of nonlocal correlations in the quantum theory. The idea of entanglement swapping is mathematically very simple. Consider two Bell states, say $|\psi^+_1\rangle$ and $|\phi^-_{1,4}\rangle$, where particles 1 and 2 have never been in contact with particles 3 and 4 before. The product of these two states can be rewritten as

$$|\varphi_{1,2}\rangle|\phi_{3,4}^-\rangle = |\varphi^+_1\rangle|\phi^+_{2,4}\rangle + |\varphi^-_{1,3}\rangle|\phi^+_{2,4}\rangle - |\varphi^+_{1,3}\rangle|\phi^-_{2,4}\rangle - |\varphi^-_{1,3}\rangle|\phi^+_{2,4}\rangle.$$ (4)

Therefore if we make a Bell measurement on the pair (1,3), then depending on the outcome the (2,4) pair would collapse into one of the corresponding Bell states: $|\phi^+_{2,4}\rangle$, $|\phi^-_{2,4}\rangle$, $|\varphi^+_{2,4}\rangle$, and $|\varphi^-_{2,4}\rangle$. In other words, particles 2 and 4 would become entangled even though they are far apart and have never interacted with each other in the past.

In this paper, we shall use entanglement swapping to distribute an arbitrary $N$-particle state to $M \leq N$ remote parties. Distribution of quantum information is essential in quantum secret sharing (QSS) and other applications in quantum information science, such as quantum communication network and distributed quantum computation [8]. QSS is the quantum counterpart of classical secret sharing first discussed by Blakely [9] and Shamir [10] in 1979. The idea is as follows. Suppose Alice wants to send a secret message to a remote location, and she can send it to either of her two agents, Bob and Charlie. As a precaution against information leakage and misuse, it is safer for her to split the message into two pieces and send them separately to Bob and Charlie, such that anyone alone has absolutely no knowledge of the message. Bob and Charlie can reconstruct

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the original secret message only if they collaborate with each other. Clearly the above consideration can be
generalized to secret sharing by \( N \) parties.

QSS refers to the implementation of the secret sharing task outlined above using quantum mechanical
resources. Hillary et al. \(^{11}\) and Karlsson et al. \(^{12}\) were the first to propose QSS protocols using respectively
three-particle Greenberger-Horne-Zeilinger (GHZ) states and two-particle Bell states. Since then, a wide
variety of other QSS protocols have been proposed. Although the goal of most QSS protocols is to protect a
classical secret message, the notion of QSS has also been generalized to the sharing of a secret quantum state
\(^{11,12,13,14}\), which is also referred to as “quantum state sharing”.

Whether the goal is to share a classical or a quantum secret, in almost all cases, it is necessary to distribute
a \( N \)-qubit entangled state among \( M \) parties, where \( N \geq M \geq 2 \). For example, in the QSS protocol of Hillery
et al. \(^{11}\), Alice uses GHZ states to split a quantum key into two shares such that each of the two agents, Bob
and Charlie, gets only one share. To do so, Alice must be able to safely distribute two of the particles in each
tripartite GHZ state separately to Bob and Charlie. In the quantum state sharing protocol proposed by Cleve
et al. \(^{13}\), the secret message to be shared is a single qutrit state. The protocol is similar to an error-correcting
code, in which a three-qutrit entangled state is generated from the secret qutrit. The “dealer” then distributes
the resulting three qutrits to three different parties. More generally, in a so-called \( (k, n) \) threshold scheme,
the secret (quantum or classical) is divided into \( n \) shares, such that any \( k \) of those shares can be used to
reconstruct the secret, while any set of less than \( k \) shares contains absolutely no information about the secret
at all. So in general one needs to distribute an arbitrary entangled state to \( n \) different parties.

Of course Alice could send the particles involved directly over quantum channels to their respective desti-
nations. This practice is however both inefficient and unsafe. First of all, since noise is always present in any
available channel, quantum information may get distorted; more seriously decoherence effects may even cause
the particle to collapse. The loss of any one particle in an unknown multiparticle state due to decoherence or
dissipation will require the whole state to be regenerated again. Hence direct distribution of the particles is not
an efficient way to proceed. Furthermore the state to be shared is itself the carrier of information, therefore
for security reasons, it is not safe to send them directly over long-distance quantum channels which may be
monitored by eavesdroppers. Of course eavesdropping activities can be detected by security testing methods,
but they inevitably involve measuring some of the particles going through the channels, which is obviously not
acceptable if they are part of the multiparticle state being distributed.

It is therefore more desirable to distribute the particles using quantum entanglement plus local operations
and classical communications only. Entanglement resources are usually supplied by two-qubit Bell states or
sometimes three-qubit GHZ states shared between the sender and the intended receivers. The advantage of
this approach is that the required entanglement can be established and tested independent of the state to be
distributed. Once it is securely established, quantum channels are no longer needed, and transmission noise is
no longer a problem.

In the following, we show how to faithfully distribute an arbitrary \( N \)-qubit state using entanglement swap-
ping; our scheme generalizes a multipartite QSS protocol discussed in Ref. \(^{12}\). Schemes of distributing
\( N \)-qubit states by teleportation have also been proposed \(^{13,16}\). However, these protocols have the undesirable
features that the complexity of the required collective measurements increases with \( N \). In our scheme,
only Bell states are used and only two-qubit collective measurements are required.

To proceed, we first show that entanglement swapping between two Bell states can be generalized to that
between an arbitrary \( N \)-qubit state and a Bell state. The setting is that Alice owns an arbitrary \( N \)-qubit state
\(|\Psi_{1,...,N}\rangle\), and she shares a Bell state \(|\phi^\mu_\nu\rangle\) with Bob who is somewhere far away; Alice holds qubit-\( \mu \) and Bob
qubit-\( \nu \). From
\[
|\Psi_{1,...,N}\rangle = (|0_i\rangle + |1_i\rangle) |\Psi_{1,...,N}\rangle,
\]

we see that \(|\Psi_{1,...,N}\rangle\) can always be rewritten as
\[
|\Psi_{1,...,N}\rangle = a|0_i\rangle |\Phi_{1,...,i−1,i+1,...,N}\rangle + b|1_i\rangle |\Phi'_{1,...,i−1,i+1,...,N}\rangle,
\]

where \(|a|^2 + |b|^2 = 1\), and \(|\Phi_{1,...,i−1,i+1,...,N}\rangle\) and \(|\Phi'_{1,...,i−1,i+1,...,N}\rangle\) are normalized states of \((N − 1)\) qubits.
Note that unless \( a \) or \( b \) vanishes, or \(|\Phi_{1,...,i−1,i+1,...,N}\rangle\) and \(|\Phi'_{1,...,i−1,i+1,...,N}\rangle\) differ only by a phase factor,
otherwise qubit-\( i \) is entangled with the rest of the group.
similar to Eq. (4), we can rewrite the product of $|\Psi_{1,...,N}\rangle$ and $|\phi_{\mu,\nu}^+\rangle$ as

$$
|\Psi_{1,...,N}\rangle|\phi_{\mu,\nu}^+\rangle = \frac{1}{2} \left[ \begin{array}{c}
|\varphi_{i,\mu}^+\rangle(a|1_\nu\rangle|\Phi_{1,...,i-1,i+1,...,N}\rangle - b|0_\nu\rangle|\Phi'_{1,...,i-1,i+1,...,N}\rangle) \\
+ |\varphi_{i,\mu}^-\rangle(a|1_\nu\rangle|\Phi_{1,...,i-1,i+1,...,N}\rangle + b|0_\nu\rangle|\Phi'_{1,...,i-1,i+1,...,N}\rangle) \\
- |\phi_{i,\mu}^+\rangle(a|0_\nu\rangle|\Phi_{1,...,i-1,i+1,...,N}\rangle - b|1_\nu\rangle|\Phi'_{1,...,i-1,i+1,...,N}\rangle) \\
+ |\phi_{i,\mu}^-\rangle(a|0_\nu\rangle|\Phi_{1,...,i-1,i+1,...,N}\rangle + b|1_\nu\rangle|\Phi'_{1,...,i-1,i+1,...,N}\rangle) \end{array} \right].
$$

(7)

Hence a Bell measurement by Alice on the $(i, \mu)$ pair will entangle the remote qubit-$\nu$ to the local group of $(N-1)$ qubits $(1, ..., i-1, i+1, ..., N)$. This process is depicted diagrammatically in Fig. 1.

The resulting $N$-qubit state depends on the outcome of the Bell measurement:

1. $|\varphi_{i,\mu}^+\rangle \rightarrow a|1_\nu\rangle|\Phi_{1,...,i-1,i+1,...,N}\rangle - b|0_\nu\rangle|\Phi'_{1,...,i-1,i+1,...,N}\rangle$,
2. $|\varphi_{i,\mu}^-\rangle \rightarrow a|1_\nu\rangle|\Phi_{1,...,i-1,i+1,...,N}\rangle + b|0_\nu\rangle|\Phi'_{1,...,i-1,i+1,...,N}\rangle$,
3. $|\phi_{i,\mu}^+\rangle \rightarrow a|0_\nu\rangle|\Phi_{1,...,i-1,i+1,...,N}\rangle - b|1_\nu\rangle|\Phi'_{1,...,i-1,i+1,...,N}\rangle$,
4. $|\phi_{i,\mu}^-\rangle \rightarrow a|0_\nu\rangle|\Phi_{1,...,i-1,i+1,...,N}\rangle + b|1_\nu\rangle|\Phi'_{1,...,i-1,i+1,...,N}\rangle$.

(8) (9) (10) (11)

Comparing with Eqs. (5), we see that if Alice tells Bob which of the four Bell states she has obtained, then by a local unitary transformation on qubit-$\nu$ Bob can rotate the state of the $N$-qubit group $(1, ..., i-1, \nu, i+1, ..., N)$ back to the original state $|\Psi_{1,...,N}\rangle$ (with qubit-$i$ → qubit-$\nu$). The required unitary operators are $(\sigma_x, \sigma_x, \sigma_z, I)$ respectively for the four possible outcomes listed in Eqs. (10-14). It is easily seen that the information cost for the whole operation is one e-bit plus two c-bits (classical bits) from Alice to Bob. If qubit-$i$ is not entangled with the group $(1, ..., i-1, i+1, ..., N)$, that is, if Eq. (5) can be reduced to

$$
|\Psi_{1,...,N}\rangle = (a|0_i\rangle + b|1_i\rangle)|\Phi_{1,...,i-1,i+1,...,N}\rangle,
$$

(12)

then what we have done is simply the teleportation [17] of a single qubit $(a|0_i\rangle + b|1_i\rangle)$ from Alice to Bob. For a general $|\Psi_{1,...,N}\rangle$ our procedure effectively teleports the entanglement to Bob.

Notice that the above procedure is entirely general, in the sense that it is independent of the state of the inactive qubits $(1, ..., i-1, i+1, ..., N)$. Therefore it can be repeated until all the qubits are distributed to their respective remote locations as desired. We note in passing that, if Alice leaves some of the qubits undistributed, then the distributed qubits form a mixed state.

It is interesting to note that each round of the operation (Bell measurement plus unitary rotation) described above can be summarized by the action of an operator $U_{\nu\nu}$ which interchanges the states of qubit-$i$ and qubit-$\nu$:

$$
U_{\nu\nu}(|\Psi_{1,...,i-1,i+1,...,N}\rangle|\phi_{\mu,\nu}^+\rangle) = |\Psi_{1,...,i-1,\nu,i+1,...,N}\rangle|\phi_{\mu,\nu}^-\rangle,
$$

(13)

where without loss of generality we have assumed that the measured Bell state has been rotated back to the singlet Bell state $|\phi_{\mu,\nu}^-\rangle$, which Alice can easily do by a local unitary transformation. The effect of $U_{\nu\nu}$ is
identical to that of a regular two-qubit swap gate \[18\] except for the fact that here the qubits involved are remotely separated; hence \(U_{i\nu}\) is a kind of nonlocal swap operator. The whole distribution process can be accomplished by simply repeating this swapping operation \(N\) times:

\[
\prod_{i=1}^{N} U_{i\nu} \left( |\Psi_{1,\ldots,N}\rangle \prod_{i=1}^{N} |\phi_{\mu_{i},\nu_{i}}\rangle \right) = |\Psi_{\nu_{1},\ldots,\nu_{N}}\rangle \prod_{i=1}^{N} |\phi_{\mu_{i},i}\rangle ,
\]

where the \(N\) qubits \((\nu_{1},\ldots,\nu_{N})\) are located at their respective destinations.

From this perspective, it is clear that we could also use the general nonlocal swap operation \[19, 20\] in place of \(U_{i\nu}\), although it would be rather inefficient to do so. It is known that the implementation of a general nonlocal swap operation requires at least two e-bits plus two c-bits from Alice to Bob plus another two c-bits from Bob to Alice \[19, 20\]. However in our scheme, only one e-bit and two c-bits from Alice to Bob are needed per swap. This difference in resources consumption is due to the fact that \(U_{i\nu}\) is actually not an universal swap operator, because it cannot be used to swap two arbitrary remote qubits. Our scheme is only good for the special purpose of qubit distribution, where each round of swapping is analogous to the teleportation of a single qubit, hence relatively less resources are required.

In summary, we have constructed a scheme to distribute an arbitrary \(N\)-qubit state (pure or mixed) to \(M \leq N\) remote parties. The basic operation used is entanglement swapping which involves only Bell states and two-qubit collective measurements. Our scheme is experimentally feasible with currently available technologies \[7, 14\].