Beyond the Equivalence Principle: Gravitational Magnetic Monopoles

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Received April 21, 2021; revised April 21, 2021; accepted May 11, 2021

Abstract—We review the hypothesis on the existence of gravitational magnetic monopoles (H-poles for short) defined by analogy with Dirac’s hypothesis on magnetic monopoles in electrodynamics. These hypothetic dual particles violate the equivalence principle and are accelerated by a gravitational field. We propose an expression for the gravitational force exerted upon an H-pole. According to GR, ordinary matter (which we call E-poles) follows geodesics in a background metric $g_{\mu\nu}$. The dual H-poles follows geodesics in an effective metric $\tilde{g}_{\mu\nu}$.

DOI: 10.1134/S0202289321030117

1. INTRODUCTION

Einstein’s General Relativity describes the effect of gravity on any particle as the modification of the background geometry. Particles, free of any other interaction, follow geodesics in this modified metric. In other words, gravity does not accelerate the bodies. Nevertheless, we will analyze here a suggestion [1, 2] according to which there may exist particles (as yet not observed) such that gravity could be responsible for their acceleration. How is this possible? In order to examine such a hypothesis, we follow the original idea of the existence of magnetic monopoles made by Dirac long time ago [3].

The gravitational correspondence comes from the observation that the connecting vector of neighboring geodesics (which we call a E-pole) is controlled by the electric part $E_{\mu\nu}$ of the Weyl tensor in a space free of matter. This led us by analogy with Dirac’s procedure to suggest that there could be possible to consider the existence of paths followed by real particles (which we call H-poles) such that their connecting vector is controlled by the magnetic part of the Weyl tensor ($H_{\mu\nu}$) (see the appendix for a short compilation of Weyl’s tensor and the notations).

In the next sections we shortly describe the original idea of the electromagnetic magnetic monopole and the analogous idea for the gravitational magnetic monopole, and also suggest a formula for the force on an H-pole.

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2. DIRAC’S MAGNETIC MONOPOLE

The equation of motion of a charged particle with charge $e$ and mass $m$ in a given electromagnetic field is given by

$$
\frac{d^2 x_\mu}{ds^2} = \frac{e}{m} F_{\mu\nu} \frac{dx_\nu}{ds} = \frac{e}{m} E_\mu,
$$

(1)

that is, an electric monopole couples only to the electric part of the electromagnetic field.

Dirac [3] put forward the hypothesis, based on dual properties, that there could exist a magnetic monopole that obeys the dual equation

$$
\frac{d^2 z_\mu}{ds^2} = \frac{g}{m} F^*_{\mu\nu} \frac{dz_\nu}{ds} = \frac{g}{m} H_\mu.
$$

(2)

Let us consider now the motion of particles in a gravitational field. Before, let us make a brief comment on the Jacobi field.

2.1. Jacobi Field

A vector $Z^\alpha$ which connects points on two infinitesimally neighboring geodesics $\Gamma$ of a congruence $x^\alpha(s)$ with equal values of the parameter $s$ is called a connecting vector. For a Ricci-flat geometry ($R_{\mu\nu} = 0$) we call a Jacobi field along $\Gamma$ any connecting vector $Z^\alpha$ that satisfies the equation

$$
\frac{D^2}{Ds^2} Z^\alpha + W^\alpha_{\beta\gamma} v^\beta v^\gamma Z^\gamma = 0,
$$

(3)

which, in terms of the electric part of the Weyl tensor, yields (see the Appendix for the definitions)

$$
\frac{D^2}{Ds^2} Z^\alpha = -E^\alpha_\beta Z^\beta.
$$

(4)
Let us note that in the case of spin one, the electric (magnetic) part of the field is obtained by a single projection of the field on the velocity vector of an observer. For the case of spin two, the corresponding electric (magnetic) part needs a double projection, as we shall see.

For accelerated curves, the generalized Jacobi field is defined in terms of a polynomial function \( N_\alpha^\beta \) of the curvature tensor satisfying the equation
\[
\frac{D^2}{Ds^2}Z^\alpha = N_\alpha^\beta Z^\beta.
\] (5)

3. GRAVITATIONAL MAGNETIC MONOPOLE

In the general relativity theory, a test body follows a geodesic in a given metric. The important point of a contact with the dual framework pointed out by Dirac comes from the remark that the evolution of the connecting vector of two neighboring particles, with the same value of the affine parameter, call it \( \eta^\alpha \), satisfies the Jacobi equation (4). This is a motivation to call such ordinary matter E-poles.

Following the dual approach, let us put forward the hypothesis that there exist particles whose path is not a geodesic but which are accelerated by gravity. This will mimic the procedure that led to the suggestion of magnetic monopoles [3].

The dual operation led us to construct a congruence of accelerated curves in a gravitational field such that its corresponding connecting vector \( \Pi_\alpha \) satisfies the dual equation that generalizes the Jacobi equation:
\[
\frac{D^2\Pi_\alpha}{Ds^2} = -\ast W_\mu^\alpha_\beta_\nu_\beta_\pi_\mu_\pi_\beta = -H^\alpha_\mu_\beta_\Pi_\beta.
\] (6)

The gravitational force produces an accelerated path of an H-pole (see [1] for details),
\[
\frac{d^2}{ds^2}z^\alpha(s) + \Gamma^\alpha_{\mu\nu} \frac{dz^\mu}{ds} \frac{dz^\nu}{ds} = F^\alpha,
\] (7)
where \( F^\alpha \) satisfies the condition
\[
\left(F_\alpha^\mu + R^\alpha_{\beta\mu\nu} v^\beta v^\nu + H_\alpha^\mu \right) \Pi^\alpha = 0.
\] (8)
In the particular case in which
\[
F_\alpha^\mu = -H_\alpha^\mu - R^\alpha_{\beta\mu\nu} v^\beta v^\nu,
\] (9)
it satisfies the two conditions:

- \( F_\alpha \) is a gradient, \( F_\alpha = d\Phi/dx^\alpha \).
- The scalar field \( \Phi \) obeys the wave equation \( \Box \Phi = -R_{\mu\nu} v^\mu v^\nu \).

In the case where the Ricci tensor \( R_{\mu\nu} \) does not vanish, the expression (6) must be modified once there are two possibilities of taking the dual of the Riemann curvature tensor:
\[
\ast R^\alpha_{\beta\mu\nu} := \frac{1}{2}\eta^{\alpha\beta\rho\sigma} R_{\rho\sigma}^{\mu\nu},
\] and
\[
R^\alpha_{\beta\mu\nu} := \frac{1}{2}\eta^{\mu\nu\rho\sigma} R_{\alpha\beta\rho\sigma}.
\]
Thus we set
\[
\frac{D^2\Pi_\alpha}{Ds^2} + \frac{1}{2} \left( \ast R_{\mu_\beta\nu} + R_{\alpha_\beta\mu_\nu} \right) v^\mu v^\nu \pi_\beta = 0.
\]
This modification does not changes the properties of the force, that still satisfies the conditions that \( F_\alpha = d\Phi/dx^\alpha \) and the scalar field \( \Phi \) obeys the wave equation \( \Box \Phi = -R_{\mu\nu} v^\mu v^\nu \).

3.1. The Gravitational Force

The very important result that it is possible to annihilate the local gravitational force was the true basis of the program of geometrization done by Einstein in the theory of general relativity (GR). It implies that the net effect of the gravitational field on a body is to produce the geodesics that become the natural paths that all bodies free from any other force must follow. In Maxwell’s electrodynamics, the acceleration of a charged body of mass \( m \), electric charge \( q \) and velocity \( v^\mu \) induced by an electromagnetic field is given by the Lorentz formula
\[
a_\mu = \frac{q}{m} F_{\mu\nu} v^\nu.
\] (10)

The hypothesis on the existence of a gravitational magnetic pole (H-pole) implies that it does not follow geodesics. Thus the acceleration \( a^\mu \) of an H-pole in the geometric framework must depend only on the geometric properties.

Besides, once the natural motion of E-poles does not contain any specific constant related to the body (that is, the equivalence between inertial and gravitational masses), it is natural to make the hypothesis that the same must occur for an H-pole. The question is to find an expression for \( a^\mu \) that has such a property. There is no better way than to make an appeal to the potential of Weyl’s conformal tensor, that is, the Lanczos tensor. In other words, we will make the assumption that in the case in which the Ricci tensor vanishes, the acceleration of an H-pole is given in terms of the Lanczos tensor.
3.2. The Lanczos Tensor

Let us consider the third-order tensor \( L_{\alpha \beta \mu} \) introduced by Lanczos [5], that obeys the following relations:

\[
L_{\alpha \beta \mu} + L_{\beta \alpha \mu} = 0, \tag{11}
\]
\[
L_{\alpha \beta \mu} + L_{\alpha \mu \beta} + L_{\mu \alpha \beta} = 0. \tag{12}
\]

The conditions (11) and (12) imply that such a tensor has only 20 independent components. Lanczos showed [5] (see also [6]) that the Weyl conformal tensor has only 20 independent components. Lanczos showed [5] (see also [6]) that the Weyl conformal tensor can be written in terms of \( L_{\alpha \beta \mu} \) as follows:

\[
W_{\alpha \beta \mu \nu} = L_{\alpha \beta [\mu ; \nu]} + L_{\mu [\nu ; \alpha ; \beta]} + \frac{2}{3} \mathcal{L}_{\alpha \lambda} \mathcal{L}_{\beta \lambda} g_{\alpha \beta \mu \nu}, \tag{13}
\]
where

\[
S_{\alpha \beta \mu \nu} := \frac{1}{2} \bigg( L_{(\alpha \nu)\beta} g_{\mu \beta} + L_{(\beta \mu)\alpha} g_{\alpha \nu} - L_{(\alpha \mu)\beta} g_{\nu \beta} - L_{(\beta \nu)\alpha} g_{\mu \beta} \bigg),
\]
\[
L_{\alpha \beta} := L_{\alpha \sigma ; \mu ; \sigma} - L_{\alpha \sigma ; \sigma ; \mu},
\]
\[
g_{\alpha \beta \mu \nu} := g_{\alpha \mu} g_{\beta \nu} - g_{\alpha \nu} g_{\beta \mu}.
\]

We are using \((ab) := ab + ba\) and \([ab] := ab - ba\).

A very important result [7] comes from the remark that it is possible to use the irreducible components associated to a congruence of curves to write the Lanczos tensor [8]. We then set

\[
L_{\alpha \beta \mu} = \sigma_{\mu [\alpha ; \beta]} + \omega_{\alpha \beta} v_{\mu} - \frac{1}{2} \omega_{[\alpha ; \beta]} v_{\mu} + \frac{3}{2} a_{[\alpha ; \beta]} v_{\mu} - \frac{1}{2} (a_{\alpha} g_{\beta \mu} - a_{\beta} g_{\alpha \mu}), \tag{14}
\]
where \(\sigma_{\mu \nu}\) denotes the shear, and \(\omega_{\alpha \beta}\) the vorticity, as defined in the Appendix. It is immediate to show that such a tensor fulfills the conditions (11) and (12) for a Lanczos tensor. Besides, it satisfies the Lanczos gauge

\[
L_{\alpha \beta \mu} g^{\beta \mu} = 0.
\]

By analogy with the Lorentz force (10), the decomposition (14) suggests a unique form for the acceleration, that is,

\[
a_{\alpha} = L_{\alpha \beta \mu} v^{\beta} v^{\mu}. \tag{15}
\]

Note that this expression does not depend on the mass of the body. In other words, the gravitational field acts on H-poles in a universal way, regardless of its mass. This is a direct consequence of the dimension (length)\(^{-1}\) of the Lanczos tensor. We note that in all these expressions we are setting the velocity of light \(c = 1\). In the next section we present an explicit example of this formula.

4. H-POLES IN THE SCHWARZSCHILD METRIC

Let us consider the geometry of a static, spherically symmetric configuration given by

\[
ds^2 = e^{\nu(r)} dt^2 - e^{\nu(r)} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2,
\]

where \(e^{\nu} = 1 - 2m/r\). In this case, the gravitational force that drives the H-poles is controlled by the potential that obeys the equation

\[
\square \Phi = 0.
\]

The acceleration vector is given by

\[
a_{\mu} = \left(0, \frac{r_H}{2r^2(1 - r_H/r)}, 0, 0\right), \tag{16}
\]

that is, the acceleration is a gradient of \(\Phi\), \(a_{\mu} = \partial_{\mu} \Phi\), whose solution is

\[
\Phi = \frac{1}{2} \ln \left(1 - \frac{r_H}{r}\right). \tag{17}
\]

4.1. The Effective Metric

According to GR, ordinary matter (which we denote as E-poles) follows geodesics in the gravitational metric \(g_{\mu \nu}\). Let us now show that the gravitational magnetic poles (H-poles) that are accelerated by the gravitational field follow geodesics in an associated effective metric. To do that, let us recall the following Lemma [9]:

**Lemma.** Consider an accelerated curve with the velocity \(v^\mu\) in an arbitrary background metric \(g_{\mu \nu}\) such that its acceleration is a gradient \(a_{\mu} = \partial_{\mu} \Psi\). It is always possible to construct an effective metric

\[
\hat{g}^{\mu \nu} = g^{\mu \nu} + \beta v^\mu v^\nu, \tag{18}
\]

such that this curve is mapped into a geodesics, where

\[
1 + \beta = e^{-2\Psi}. \tag{19}
\]

The inverse covariant form of the effective metric is given by

\[
\hat{g}_{\mu \nu} = g_{\mu \nu} - \frac{\beta}{1 + \beta} v^\mu v^\nu. \tag{20}
\]

In the case of an H-pole, Eq. (9) which implies that the acceleration of an H-pole is a gradient, fulfills the applicability condition of the above Lemma. Thus, it follows that H-poles follow geodesics in the metric (18), where \(\beta\) is related to the acceleration through the condition (19).
4.2. The Equation of Motion of an H-Pole

Let \( x^\alpha(s) \) be the path of an H-pole, and its corresponding velocity is \((i, \dot{r}, \dot{\theta}, \phi)\). Its evolution is provided by

\[
\frac{d}{ds}(g_{\alpha\lambda} \dot{x}^\alpha) - \frac{1}{2} g_{\mu\nu,\lambda} \dot{x}^\mu \dot{x}^\nu = \Phi_{,\lambda}.
\]  

(21)

The comma means a simple derivative: \( \Phi_{,\lambda} = \partial_\lambda \Phi \), and \( \Phi \) is given by (17). From the equations for \( x^2 = \theta \) and \( \dot{x}^3 = \phi \) it follows that the angle \( \theta \) is a constant of motion. We choose \( \theta = \pi/2 \). For the angle \( \varphi \) we have

\[
\dot{\varphi} = \frac{h}{r^2}
\]

for \( h = \text{const} \). For the variable \( t \) it follows

\[
i \left(1 - \frac{r_H}{r}\right) = l,
\]

with \( l \) constant. From the auxiliary condition \( v^\mu v^\nu g_{\mu\nu} = 1 \),

we have

\[
\left(1 - \frac{r_H}{r}\right) i^2 - \frac{i^2}{1 - r_H/r} - \frac{h^2}{r^2} = 1. 
\]

(22)

Once the acceleration has the form \( a_\mu = (0, a_1, 0, 0) \), it follows that \( dr/ds = 0 \). Thus the orbits of H-poles are only circular. That is,

\[
v^\mu = \left(\frac{l}{1 - r_H/R_o}, 0, 0, h/R_o^2\right).
\]

The norm of \( v^\mu \) implies

\[
l^2 = \left(1 + \frac{h^2}{R_o^2}\right) (1 - r_H/R_o),
\]

where \( R_o \) is the constant radius of the circular orbit. We note that \( R_o > r_H \).

It then follows that the angular velocity of the H-pole is given by

\[
\omega^2 = \frac{r_H}{R_o^2} (R_o - r_H).
\]

Let us note that the acceleration can be written according to Eq. (15). Indeed, the velocity of the H-pole takes the form

\[
v^\mu = (i, 0, 0, \dot{\phi}).
\]

The nonzero components of the Lanczos tensor in the Schwarzschild background reduces to

\[
L_{100} = \frac{1}{2} r_H r^2, \quad L_{122} = L_{133} = -\frac{1}{6} r_H (1 - r_H/r).
\]

Thus it follows that we can write the acceleration (16) in the form of Eq. (15).

Just as it is extremely difficult to detect Dirac’s magnetic monopole, possibly the same could occur with gravitational magnetic monopoles. Nevertheless, a further analysis on these hypothetical particles should be important to eventually elucidate the reason why Nature did not find it necessary to create them.

Appendix

MATHEMATICAL COMPRENDIUM

The Riemann curvature tensor can be decomposed into its irreducible parts by the relation

\[
R_{\alpha\beta\mu\nu} = W_{\alpha\beta\mu\nu} + M_{\alpha\beta\mu\nu} - \frac{1}{6} R g_{\alpha\beta\mu\nu},
\]

where \( W_{\alpha\beta\mu\nu} \) is the Weyl conformal tensor,

\[
2M_{\alpha\beta\mu\nu} = R_{\alpha\mu} g_{\beta\nu} + R_{\beta\nu} g_{\alpha\mu} - R_{\alpha\nu} g_{\beta\mu} - R_{\alpha\mu} g_{\beta\nu},
\]

and \( g_{\alpha\beta\mu\nu} = g_{\alpha\mu} g_{\beta\nu} - g_{\alpha\nu} g_{\beta\mu} \). The duality operation for an arbitrary antisymmetric tensor \( F_{\mu\nu} \) is defined by

\[
F^*_{\mu\nu} \equiv \frac{1}{2} \eta_{\mu\nu\alpha\beta} F^{\alpha\beta},
\]

with

\[
\eta_{\alpha\beta\mu\nu} = \sqrt{-g} \varepsilon_{\alpha\beta\mu\nu},
\]

\( g \) being the determinant of \( g_{\mu\nu} \), and \( \varepsilon_{\alpha\beta\mu\nu} \) is the Levi-Civita totally antisymmetric quantity. We define the electric vector \( E^\mu \) and magnetic vector \( H^\mu \) by setting

\[
E_\alpha = F_{\alpha\mu} v^\mu,
\]

\[
H_\alpha = F^{\alpha\beta} v^\beta.
\]

The Weyl tensor has ten independent components and can also be separated by an arbitrary observer endowed with the four velocity \( v^\mu \) into its electric \((E_\alpha)\) and magnetic \((H_\alpha)\) tensor parts, that is,

\[
E_\alpha = W_{\alpha\beta\mu\nu} v^\mu v^\nu,
\]

\[
H_\alpha = W^{\alpha\beta} v_\beta v^\nu.
\]

Thus the electric and magnetic tensors are symmetric, traceless and orthogonal to the observer’s velocity:

\[
E_{\mu\nu} = E_{\nu\mu}, \quad E_{\mu\nu} v^\mu = 0, \quad \text{and} \quad E_{\mu\nu} g^{\mu\nu} = 0,
\]

\[
H_{\mu\nu} = H_{\nu\mu}, \quad H_{\mu\nu} v^\mu = 0, \quad \text{and} \quad H_{\mu\nu} g^{\mu\nu} = 0.
\]

The kinematic parameters

Shear: \( \sigma_{\mu\nu} = \frac{1}{2} h_{\nu}^{\alpha} (\mu h^{\beta}_{\nu}) - \frac{1}{3} \theta h_{\mu\nu} \);

Vorticity: \( \omega_{\mu\nu} = \frac{1}{2} h_{\nu}^{\alpha} [\mu h^{\beta}_{\nu}] \);

Expansion factor: \( \theta = v^\alpha_{,\alpha} \);

Projection: \( h_{\mu\nu} = g_{\mu\nu} - v_{\mu} v_{\nu} \).
FUNDING

MN would like to thank the support from Brazilian agencies FAPERJ, CNPq and FINEP.

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