Energy equality for the isentropic compressible Navier-Stokes equations without upper bound of the density

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Abstract

In this paper, we are concerned with the minimal regularity of both the density and the velocity for the weak solutions keeping energy equality in the isentropic compressible Navier-Stokes equations. The energy equality criteria without upper bound of the density are established. Almost all previous corresponding results requires $\rho \in L^\infty(0,T;L^\infty(T^d))$.

MSC(2000): 35Q30, 35Q35, 76D03, 76D05

Keywords: compressible Navier-Stokes equations; energy equality; vacuum

1 Introduction

The classical isentropic compressible Navier-Stokes equations

$$\begin{aligned} 
\rho_t + \nabla \cdot (\rho v) &= 0, \\
(\rho v)_t + \text{div} (\rho v \otimes v) + \nabla P(\rho) - \text{div} (\mu \mathbb{D}v) - \nabla (\lambda \text{div} v) &= 0, 
\end{aligned} \tag{1.1}$$

where $\rho$ stands for the density of the flow, $v$ represents the flow velocity field and $P(\rho) = \rho^\gamma$ is the scalar pressure; The viscosity coefficients $\mu$ and $\lambda$ satisfy $\mu \geq 0$ and $2\mu + d\lambda > 0$. $\mathbb{D}v = \frac{1}{2}(\nabla v \otimes \nabla v^T)$ is the strain tensor; We complement equations (1.1) with initial data

$$\begin{aligned} \rho(0,x) = \rho_0(x), \quad (\rho v)(0,x) = (\rho_0 v_0)(x), \quad x \in \Omega, 
\end{aligned} \tag{1.2}$$

where we define $v_0 = 0$ on the sets $\{x \in \Omega : \rho_0 = 0\}$. In the present paper, we consider the periodic case, that is $\Omega = T^d$ with dimension $d \geq 2$.

One of the celebrated results of the isentropic compressible Navier-Stokes equations (1.1) is the global existence of the finite energy weak solutions due to Lions [18] with $\gamma \geq \frac{3d}{d+2}$ for $d = 2$ or 3, where $d$ is the spatial dimension. Subsequently, in [10], Feireisl-Novotny-Petzeltová further extended the Lions’ work to $\gamma > \frac{d}{2}$ for $d = 3$. In [13], Jiang and Zhang

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considered the global existence of weak solutions for the case \( \gamma > 1 \) with the spherical symmetric initial data. For the convenience of the reader, we recall the definition of the finite energy weak solutions:

**Definition 1.1.** A pair \((\varrho, v)\) is called a weak solution to (1.1) with initial data (1.2) if \((\varrho, v)\) satisfies

(i) equation (1.1) holds in \( D'(0,T;\Omega) \) and

\[
P(\varrho), \varrho|v|^2 \in L^\infty(0,T;L^1(\Omega)), \quad \nabla v \in L^2(0,T;L^2(\Omega)),
\]

(ii) the density \( \varrho \) is a renormalized solution of (1.1) in the sense of \([3]\).

(iii) the energy inequality holds

\[
E(t) + \int_0^T \int_\Omega \left( \mu |\nabla v|^2 + (\mu + \lambda)|\text{div } v|^2 \right) \, dx \, dt \leq E(0),
\]

where \( E(t) = \int_\Omega \left( \frac{1}{2} \varrho |v|^2 + \frac{\varrho^\gamma}{\gamma - 1} \right) \, dx \).

Note that the finite energy weak solutions constructed in \([10, 18]\) satisfy the energy inequality (1.4). A natural question is how much regularity of weak solutions in these system are required to ensure energy equality. In \([32]\), Yu opened the door of research of the Lions-Shinbrot type energy equality criterion to the weak solutions of the compressible Navier-Stokes equations with both non vacuum and vacuum cases. The so-called Lions-Shinbrot type criterion is that if the Leray-Hopf weak solutions \( v \) of the 3D incompressible Navier-Stokes equations satisfy \( v \in L^p(0,T;L^q(\mathbb{T}^d)) \), with \( \frac{2}{p} + \frac{2}{q} = 1 \) and \( q \geq 4 \), then the following corresponding energy equality holds

\[
\|v\|^2_{L^2(\Omega)} + 2 \int_0^T \|\nabla v\|^2_{L^2(\Omega)} \, ds = \|v_0\|^2_{L^2(\Omega)}.
\]

Yu’s results in \([32]\) without vacuum can be formulated as: if a weak solution \((\varrho, v)\) of the compressible Navier-Stokes equations (1.1) with \( \mu = 2\varrho \) and \( \lambda = 0 \) satisfies

\[
\sqrt{\varrho} v \in L^\infty(0,T;L^2(\Omega)), \quad \sqrt{\varrho} \nabla v \in L^2(0,T;L^2(\Omega)),
\]

\[
0 < c_1 \leq \varrho \leq c_2 < \infty, \quad \nabla \sqrt{\varrho} \in L^\infty(0,T;L^2(\Omega)),
\]

\[
v \in L^p(0,T;L^q(\Omega)) \quad \frac{1}{p} + \frac{1}{q} \leq \frac{5}{12} \quad \text{and} \quad q \geq 6, \sqrt{\varrho_0} v_0 \in L^4(\Omega),
\]

then such a weak solution \((\varrho, v)\) for any \( t \in [0,T] \) satisfies

\[
\int_{\mathbb{T}^d} \left( \frac{1}{2} |\varrho|^2 + \frac{\varrho^\gamma}{\gamma - 1} \right) \, dx + \int_0^t \int_{\mathbb{T}^d} \varrho |v|^2 \, dx \, dt = \int_{\mathbb{T}^d} \left( \frac{1}{2} \varrho_0 |v_0|^2 + \frac{\varrho_0^\gamma}{\gamma - 1} \right) \, dx.
\]

Subsequently, Nguyen-Nguyen-Tang \([21]\) obtained the following Lions-Shinbrot type criterion for the weak solutions of the compressible Navier-Stokes equations (1.1) with \( \mu = \mu(\varrho) \), \( \lambda = \lambda(\varrho) \) and general pressure law \( P(\varrho) \in C^2(0,\infty) \),

\[
0 < c_1 \leq \varrho \leq c_2 < \infty, \quad v \in L^\infty(0,T;L^2(\mathbb{T}^d)), \quad \nabla v \in L^2(0,T;L^2(\mathbb{T}^d)),
\]

\[
\sup_{t \in (0,T)} \sup_{|h| \leq \varepsilon} \||\varrho(\cdot + h, t) - \varrho(\cdot, t)||_{L^2(\mathbb{T}^d)} < \infty,
\]

\[
v \in L^4(0,T;L^4(\mathbb{T}^d)).
\]
ensures that the corresponding energy equality is valid. Very recently, in [21], the authors refined (1.7) to
\begin{equation}
0 < c_1 \leq \varrho \leq c_2 < \infty, v \in L^\infty(0, T; L^2(\mathbb{R}^d)), \nabla v \in L^2(0, T; L^2(\mathbb{R}^d)),
\end{equation}
with \( k > \max\{\frac{4p}{p-2}, \frac{8-2p}{2}\}, l > \max\{\frac{4p}{q-2}, \frac{8-2p}{2}\} \) and \( k(4-p) + 7p \geq 0, l(4-q) + 7q \geq 0 \).

It is worth pointing out that a special case \( p = q = 2 \) in (1.8) is an improvement of (1.7). Other kind Lions-Shinbrot type criterion for the compressible Navier-Stokes system (1.1) can be found in [16, 27].

**Theorem 1.1.** Let \((\varrho, v)\) be a weak solutions in the sense of definition (1.1). Assume
\begin{equation}
0 < c \leq \varrho \leq L^k(0, T; L^I(\mathbb{R}^d)),
v \in L^p(0, T; L^q(\mathbb{R}^d)), \nabla v \in L^{\frac{kp}{p-2}}(0, T; L^{\frac{lp}{l-2}}(\mathbb{R}^d)),
\end{equation}
with \( k > \max\{\frac{4p}{p-2}, \frac{8-2p}{2}\}, l > \max\{\frac{4p}{q-2}, \frac{8-2p}{2}\} \) and \( k(4-p) + 7p \geq 0, l(4-q) + 7q \geq 0 \).

The energy equality below is valid, for any \( t \in [0, T] \),
\begin{align}
\int_{\mathbb{T}^d} \left( \frac{1}{2}|v|^2 + \frac{\varrho^\gamma}{\gamma - 1} \right) dx + \int_0^t \int_{\mathbb{T}^d} \left[ \mu |\nabla v|^2 + (\mu + \lambda) |\text{div} v|^2 \right] dx dt
&= \int_{\mathbb{T}^d} \left( \frac{1}{2}|\varrho_0|^2 + \frac{\varrho_0^\gamma}{\gamma - 1} \right) dx.
\end{align}

**Remark 1.1.** This theorem is a generalization of recent works [21, 27]. It seems that this is the first energy equality criterion for the compressible Navier-Stokes equations without the upper bound of the density.

**Remark 1.2.** A special case of (1.9) is that \( v \in L^4(0, T; L^4(\mathbb{R}^d)), \nabla v \in L^{\frac{8k}{k-8}}(0, T; L^{\frac{8l}{l-8}}) \) and \( \varrho \in L^k(0, T; L^I(\mathbb{R}^d)) \) with \( k > \max\{8, 4(\gamma - 2)\} \) and \( l > \max\{8, 4(\gamma - 2)\} \) guarantee the energy conservation of the weak solutions. This shows that lower integrability of the density \( \varrho \) means that more integrability of gradient of the velocity \( \nabla v \) is necessary in energy conservation of the isentropic compressible fluid with non-vacuum and the inverse is also true.

Next, we turn our attentions to the energy equality of the isentropic Navier-Stokes equations allowing vacuum. In aforementioned work [32], Yu also dealt with the energy conservation of system (1.1) in the present of vacuum and proved that if a weak solution \((\varrho, v)\) in the sense of Definition (1.1) satisfies
\begin{equation}
0 \leq \varrho \leq c < \infty, \nabla \sqrt{\varrho} \in L^\infty(0, T; L^2(\mathbb{R}^d)),
v \in L^p(0, T; L^q(\mathbb{R}^d)) \quad p \geq 4 \text{ and } q \geq 6, \text{ and } v_0 \in L^{q_0}, \quad q_0 \geq 3,
\end{equation}
then the energy equality (1.10) is valid for \( t \in [0, T] \).

Developing the technique used in [32], the authors showed Lions’s \( L^4(0, T; L^4(\Omega)) \) condition for energy balance is also valid for the weak solutions of the isentropic compressible
Navier-Stokes equations allowing vacuum via the following energy equality criterion: if a weak solution \((\rho, v)\) satisfies, for any \(p, q \geq 4\) and \(dp < 2q + 3d\) with \(d \geq 2\),
\[
0 \leq \varrho < c < \infty, \quad \nabla \sqrt{\varrho} \in L^{\frac{2p}{p-2}}(0, T; L^{\frac{2q}{q-2}}(T^d)),
\]
\[
v \in L^{\frac{2p}{p-2}}(0, T; L^{\frac{2q}{q-2}}(T^d)), \quad \nabla v \in L^p(0, T; L^q(T^d)), \quad v_0 \in L^{\frac{q}{q-1}}(\mathbb{T}^d).
\]
(1.12)

then there holds (1.10). Now, we state our second result for the compressible Navier-Stokes equations (1.1) in the presence of vacuum as follows:

**Theorem 1.2.** For any dimension \(d \geq 2\), the energy equality (1.10) of weak solutions \((\varrho, v)\) to the compressible Navier-Stokes equation (1.1) with vacuum is valid for \(t \in [0, T]\) provided
\[
0 \leq \varrho \in L^k(0, T; L^1(T^d)), \quad \nabla \sqrt{\varrho} \in L^{\frac{2kp}{(p-3)p}}(0, T; L^{\frac{2kq}{(q-3)q}}(T^d)),
\]
\[
v \in L^p(0, T; L^q(T^d)), \quad \nabla v \in L^p(0, T; L^{\frac{q \gamma}{q-\gamma}}(T^d)) \quad \text{and} \quad v_0 \in L^{\frac{q}{q-1}}(\mathbb{T}^d),
\]
(1.13)

where \(k > \max\{\frac{p}{2(p-3)}, \frac{p}{2(p-3)}, \frac{(\gamma-1)p}{2q-2\gamma p} \}\), \(l \geq \max\{\frac{q}{2(q-3)}, \frac{(\gamma-1)q}{2q-2\gamma q} \}\), \(p > 3\) and \(q \geq \max\{3, \frac{d(p-3)}{2} \}\).

**Remark 1.3.** If let \(k = l \to +\infty\), then Theorem 1.2 will reduce to the result obtained in [30], hence, this result can be seen as a generalization of recent works in [30, 32]. To the best of our knowledge, this seems to be the first result of the energy conservation criteria for the weak solutions of compressible Navier-Stokes equations without upper bound of density in the presence of vacuum.

**Corollary 1.3.** The energy equality (1.10) of weak solutions \((\varrho, v)\) to the compressible Navier-Stokes equation (1.1) allowing vacuum holds for \(t \in [0, T]\) provided

(1) \(\nabla v \in L^2(0, T; L^2(T^d))\),
\[
0 \leq \varrho \in L^k(0, T; L^1(T^d)), \quad \nabla \sqrt{\varrho} \in L^{\frac{4k}{k+4}}(0, T; L^{\frac{4l}{l+4}}(T^d)),
\]
\[
v \in L^{\frac{4k}{k+4}}(0, T; L^{\frac{4l}{l+4}}(T^d)), \quad v_0 \in L^{\frac{4l}{l+4}}(\mathbb{T}^d),
\]
with \(k, l > 2\gamma\) and \(k > \frac{2(2d+4\gamma+4l-4d\gamma)}{(8-d)d+2d} \);  

(2) \(v \in L^4(0, T; L^4(T^d))\),
\[
0 \leq \varrho \in L^k(0, T; L^1(T^d)), \quad \nabla \sqrt{\varrho} \in L^{\frac{4k}{k+4}}(0, T; L^{\frac{4l}{l+4}}(T^d)),
\]
\[
\nabla v \in L^{\frac{4k}{k+4}}(0, T; L^{\frac{4l}{l+4}}(T^d)) \quad \text{and} \quad v_0 \in L^{\frac{4l}{l+4}}(\mathbb{T}^d),
\]
with \(k > \max\{2, \frac{4(\gamma-1)(d+4)}{8-d} \}\) and \(l \geq \max\{2, 2(\gamma - 1)\};

**Remark 1.4.** This shows that lower integrability of the density \(\varrho\) means that more integrability of the velocity \(v\) or the gradient of the velocity \(\nabla v\) are necessary in energy conservation of the isentropic compressible fluid with vacuum and the inverse is also true.
The minimum regularity of weak solution keeping energy conservation in the fluid equations originated from Onsager’s work [23]. The recent progress of Onsager’s conjecture can be found in [1, 5, 8, 22]. We refer the readers to [28] for the Onsager conjecture on the energy conservation for the incompressible Navier-Stokes equation via establishing the energy conservation criterion involving the density \( \rho \in L^k(0, T; L^1(\mathbb{T}^d)) \). There are very significant recent developments on energy equality of the Leray-Hopf weak solutions of the 3D incompressible Navier-Stokes equations in [2, 3, 7, 29, 31, 33, 34].

Moreover, if \( q \) belongs to \( [1, \infty] \) with \( \frac{1}{p} + \frac{1}{q} = \frac{1}{p_1} + \frac{1}{q_2} \), then we have the following standard properties of the mollifier kernel:

**Lemma 2.2.** Let \( p, q, p_1, p_2, q_1, q_2 \in [1, \infty) \) with \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} \). Assume \( f \in L^p(0, T; L^{q_1}(\mathbb{T}^d)) \) and \( g \in L^p(0, T; L^{q_2}(\mathbb{T}^d)) \). Then for any \( \varepsilon > 0 \), there holds

\[
\|(fg)^\varepsilon - f^\varepsilon g^\varepsilon\|_{L^p(0, T; L^q(\mathbb{T}^d))} \to 0, \quad \text{as } \varepsilon \to 0.
\] (2.1)

We also recall the general Constantin-E-Titi type and Lions type commutators on mollifying kernel proved in [21, 27].

**Lemma 2.2.** Let \( 1 \leq p, q, p_1, p_2, q_1, q_2 \leq \infty \) with \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \) and \( \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} \). Assume \( f \in L^p(0, T; W^{1,q_1}(\mathbb{T}^d)) \) and \( g \in L^p(0, T; L^{q_2}(\mathbb{T}^d)) \). Then for any \( \varepsilon > 0 \), there holds

\[
\|(fg)^\varepsilon - f^\varepsilon g^\varepsilon\|_{L^p(0, T; L^q(\mathbb{T}^d))} \leq C\varepsilon \|
abla f\|_{L^p(0, T; L^{q_1}(\mathbb{T}^d))} \|g\|_{L^p(0, T; L^{q_2}(\mathbb{T}^d))},
\] (2.2)

Moreover, if \( q_1, q_2 < \infty \) then

\[
\limsup_{\varepsilon \to 0} \varepsilon^{-1} \|(fg)^\varepsilon - f^\varepsilon g^\varepsilon\|_{L^p(0, T; L^q(\mathbb{T}^d))} = 0.
\] (2.3)
The following two lemmas plays an important role in the proof of Theorem 1.1. We refer the reader to [16, 21] for their original version.

**Lemma 2.3.** Suppose that $f \in L^p(0, T; L^q(\mathbb{T}^d))$. Then for any $\varepsilon > 0$, there holds

$$\|\nabla f^\varepsilon\|_{L^p(0, T; L^q(\mathbb{T}^d))} \leq C\varepsilon^{-1}\|f\|_{L^p(0, T; L^q(\mathbb{T}^d))},$$

(2.4)

and, if $p, q < \infty$

$$\limsup_{\varepsilon \to 0} \|\nabla f^\varepsilon\|_{L^p(0, T; L^q(\mathbb{T}^d))} = 0.$$  

(2.5)

Moreover, if $0 < c_1 \leq g \in L^k(0, T; L^l(\mathbb{T}^d))$, then there holds, for any $\varepsilon > 0$,

$$\left\|\nabla f^\varepsilon \right\|_{L^p(0, T; L^q(\mathbb{T}^d))} \leq C\varepsilon^{-1}\|f\|_{L^p(0, T; L^q(\mathbb{T}^d))}\|g\|_{L^k(0, T; L^l(\mathbb{T}^d))},$$

(2.6)

and if $1 \leq q, l < \infty$

$$\limsup_{\varepsilon \to 0} \left\|\nabla f^\varepsilon \right\|_{L^p(0, T; L^q(\mathbb{T}^d))} = 0.$$  

(2.7)

**Proof.** The proof of (2.4) and (2.5) was given in [21], hence we only focus on the proof of (2.6) and (2.7) here.

By direct calculation and using the lower bound of $g$, we have

$$\left\|\nabla \left( \frac{f^\varepsilon}{g^\varepsilon} \right) \right\|_{L^\frac{ql}{p+ql}} \leq C \left( \left\| \frac{g^\varepsilon}{(g^\varepsilon)^2} \nabla f^\varepsilon \right\|_{L^\frac{ql}{p+ql}} + \left\| \frac{f^\varepsilon}{(g^\varepsilon)^2} \nabla g^\varepsilon \right\|_{L^\frac{ql}{p+ql}} \right)$$

$$\leq C \left( \|g\| \|\nabla f^\varepsilon\|_{L^{\frac{ql}{p+ql}}} + \|f\| \|\nabla g^\varepsilon\|_{L^{\frac{ql}{p+ql}}} \right).$$

(2.8)

Then we need to deal with the two terms on the right-hand side of the inequality (2.8), since

$$g^\varepsilon \nabla f^\varepsilon = \int \frac{1}{\varepsilon^d} g(y) \eta\left( \frac{x-y}{\varepsilon} \right) dy \int \frac{1}{\varepsilon^d} f(y) \nabla \eta\left( \frac{x-y}{\varepsilon} \right) dy$$

$$\leq C\varepsilon^{-1} \int_{B(x, \varepsilon)} \frac{1}{\varepsilon^d} |g(y)| dy \int_{B(x, \varepsilon)} \frac{1}{\varepsilon^d} |f(y)| dy$$

$$\leq C\varepsilon^{-1} \left( \int_{B(0, \varepsilon)} |g(x-z)| \frac{1}{\varepsilon^d} |B(0, \varepsilon)\eta(z)| \frac{1}{\varepsilon^d} dz \right) \left( \int_{B(0, \varepsilon)} |f(x-z)| \frac{1}{\varepsilon^d} |B(0, \varepsilon)\eta(z)| \frac{1}{\varepsilon^d} dz \right)$$

$$\leq C\varepsilon^{-1} (|g| \ast J_{1\varepsilon}(x)) (|f| \ast J_{1\varepsilon}(x)),$$

(2.9)

where $J_{1\varepsilon} = \frac{1}{\varepsilon^d} |B(0, \varepsilon)| \geq 0$ and $\int g dz = \text{measure of } B(0, 1)$. Then it follows from the Hölder’s inequality and Minkowski’s inequality, one has

$$\|g^\varepsilon \nabla f^\varepsilon\|_{L^{\frac{ql}{p+ql}}} \leq C\varepsilon^{-1} \|g\| \|J_{1\varepsilon}(x)\| \|f\| \|J_{1\varepsilon}(x)\|_{L^q}$$

$$\leq C\varepsilon^{-1} \|g\| \|J_{1\varepsilon}\|_{L^q} \|f\| \|J_{1\varepsilon}\|_{L^q}$$

$$\leq C\varepsilon^{-1} \|g\|_{L^q} \|f\|_{L^q}.$$  

(2.10)

Similarly, we also have

$$\|f^\varepsilon \nabla g^\varepsilon\|_{L^{\frac{ql}{p+ql}}} \leq C\varepsilon^{-1} \|f\|_{L^q} \|g\|_{L^l}.$$  

(2.11)
Substituting (2.10) and (2.11) into (2.8), one can obtain
\[
\left\| \nabla \left( \frac{f^\varepsilon}{g^\varepsilon} \right) \right\|_{L^{\frac{q}{p'q}}(0,T;L^q)} \leq C\varepsilon^{-1} \|f\|_{L^p} \|g\|_{L^q}.
\] (2.12)

It follows from Hölder's inequality that
\[
\left\| \nabla \left( \frac{f^\varepsilon}{g^\varepsilon} \right) \right\|_{L^{\frac{q}{p'q}}(0,T;L^q)} \leq C\varepsilon^{-1} \|f\|_{L^p(0,T;L^q)} \|g\|_{L^q(0,T;L^q)}.
\] (2.13)

Furthermore, if \(1 \leq q, l < \infty\), let \(\{f_n\}, \{g_n\} \in C_0^\infty(\mathbb{T}^d)\) with \(f_n \to f\), \(g_n \to g\) strongly in \(L^q(\mathbb{T}^d)\) and \(L^l(\mathbb{T}^d)\), respectively. Thus, by the density arguments, we find that
\[
\varepsilon \left\| \nabla \left( \frac{f^\varepsilon}{g^\varepsilon} \right) \right\|_{L^{\frac{q}{p'q}}} \leq C\varepsilon \left( \left\| \nabla \left( \frac{(f - f_n)^\varepsilon}{g^\varepsilon} \right) \right\|_{L^{\frac{q}{p'q}}} + \left\| \nabla \left( \frac{f_n^\varepsilon}{g^\varepsilon} \right) \right\|_{L^{\frac{q}{p'q}}} \right)
\leq C\varepsilon \left( \varepsilon^{-1} \|f - f_n\|_{L^q} \|g\|_{L^l} + \|g^\varepsilon \nabla f_n^\varepsilon\|_{L^{\frac{q}{p'q}}} + \|f_n^\varepsilon \nabla (g - g_n)^\varepsilon\|_{L^{\frac{q}{p'q}}} \right)
\leq C \left( \|f - f_n\|_{L^q} \|g\|_{L^l} + \varepsilon \|g\|_{L^l} \|\nabla f_n\|_{L^q} + \|f_n\|_{L^q} \|g - g_n\|_{L^l} + \varepsilon \|f_n\|_{L^q} \|\nabla g_n\|_{L^l} \right),
\] (2.14)

which gives
\[
\limsup_{\varepsilon \to 0} \varepsilon \left\| \nabla \left( \frac{f^\varepsilon}{g^\varepsilon} \right) \right\|_{L^{\frac{q}{p'q}}}
\leq C \left( \|f - f_n\|_{L^q} \|g\|_{L^l} + \|f_n\|_{L^q} \|g - g_n\|_{L^l} \right) \to 0, \text{ as } n \to +\infty.
\] (2.15)

Then taking \(L^{\frac{q}{p'q}}\) norm with respect to \(t\) on (2.15) and using the Hölder’s inequality, it leads to (2.17). □

**Lemma 2.4.** Assume that \(0 < c \leq q(x,t) \in L^l(\mathbb{T}^d)\) and \(\nabla v \in L^q(\mathbb{T}^d)\) with \(l \geq \frac{2q}{q-1}\), \(1 \leq q \leq \infty\). Then
\[
\left\| \partial \left( \frac{(qv)^\varepsilon}{\varrho^\varepsilon} \right) \right\|_{L^{\frac{q}{p'q}}(\mathbb{T}^d)} \leq C\|\nabla v\|_{L^q(\mathbb{T}^d)} \left( \|\varrho\|_{L^l(\mathbb{T}^d)} + \|\varrho\|_{L^l(\mathbb{T}^d)}^2 \right).
\] (2.16)

**Proof.** By direct computation, one has
\[
\partial \left( \frac{(qv)^\varepsilon}{\varrho^\varepsilon} \right) = \frac{\partial(qv)^\varepsilon - v\varrho^\varepsilon}{\varrho^\varepsilon} - \frac{(qv)^\varepsilon - q^\varepsilon v\varrho^\varepsilon}{(q^\varepsilon)^2} \partial\varrho^\varepsilon := I_1 + I_2.
\] (2.17)

Let \(B(x,\varepsilon) = \{y \in \mathbb{T}^d, |x - y| < \varepsilon\}\), then Using the Hölder’s inequality, we have
\[
|I_1| \leq C \left| \int_{\mathbb{T}^d} \varrho(y) (v(y) - v(x)) \nabla_x \eta_{\varepsilon}(x - y) dy \right|
\leq C \int_{\mathbb{T}^d} \varrho(y) \frac{v(y) - v(x)}{\varepsilon} \frac{1}{\varepsilon^{d-1}} \nabla \eta \left( \frac{x - y}{\varepsilon} \right) dy,
\] (2.18)

\[
\leq C \left( \frac{1}{\varepsilon^d} \int_{B(x,\varepsilon)} |v(y) - v(x)|_{s_1} dy \right)^{\frac{1}{s_1}} \left( \frac{1}{\varepsilon^{d} \int_{B(x,\varepsilon)} |\varrho(y)|_{s_2} dy \right)^{\frac{1}{s_2}},
\]

where \(s_1 \leq q\), \(2s_2 \leq l\) and \(\frac{1}{s_1} + \frac{1}{s_2} = 1\).
Using the mean value theorem, one has
\[
\frac{1}{\varepsilon^d} \int_{B(x,\varepsilon)} \frac{|v(y) - v(x)|^{s_1}}{\varepsilon^{s_1}} dy \leq C \frac{1}{\varepsilon^d} \int_{B(x,\varepsilon)} \int_0^1 |\nabla v(x + (y - x)s)|^{s_1} \frac{|y - x|^{s_1}}{\varepsilon^{s_1}} dsdy \\
\leq C \int_0^1 \int_{B(0,1)} |\nabla v(x + s\varepsilon \omega)|^{s_1} d\omega ds \\
\leq C \int_{\mathbb{R}^d} |\nabla v(x - z)|^{s_1} \int_0^1 \frac{1}{(\varepsilon^d)} d\sigma_d z \\
= |\nabla v|^{s_1} * J_\varepsilon(x),
\]
where \(J_\varepsilon(z) = \int_0^1 \frac{1}{(\varepsilon^d)} d\sigma_d z \geq 0\) and it's easy to check that \(\int_{\mathbb{R}^d} J_\varepsilon dz = \text{measure of } (B(0,1))\). Similarly, we also have
\[
\frac{1}{\varepsilon^d} \int_{B(x,\varepsilon)} |\phi(y)|^{s_2} dy \leq C \int_{B(0,\varepsilon)} |\phi(x - z)|^{s_2} dz \\
\leq C \int_{\mathbb{R}^d} |\phi(x - z)|^{s_2} \frac{1}{(\varepsilon^d)} d\sigma_d z \\
\leq C |\phi|^{s_2} * J_{1\varepsilon}(x),
\]
where \(J_{1\varepsilon} = \frac{1}{(\varepsilon^d)} \geq 0\) and \(\int_{\mathbb{R}^d} J_{1\varepsilon} dz = \text{measure of } (B(0,1))\).

Next, to estimate \(I_2\), due to the Hölder’s inequality, one deduces
\[
|I_2| = |\int \phi(y) (v(y) - v(x)) \eta_\varepsilon(x - y) dy | \int \phi(y) \nabla_x \eta_\varepsilon(x - y) dy | \\
\leq C \int_{B(x,\varepsilon)} \phi(y) |v(y) - v(x)| \frac{1}{\varepsilon^d} dy \int_{B(x,\varepsilon)} \frac{1}{\varepsilon^d} \phi(y) |\nabla \eta_\varepsilon(x - y)| \frac{1}{\varepsilon^d} dy \\
\leq C \left( \frac{1}{\varepsilon^d} \int_{B(x,\varepsilon)} \frac{|v(y) - v(x)|^{s_1}}{\varepsilon^{s_1}} dy \right)^{\frac{1}{2}} \left( \frac{1}{\varepsilon^d} \int_{B(x,\varepsilon)} |\phi(y)|^{s_2} dy \right)^{\frac{1}{2}}.
\]

Then from the Young’s inequality, we arrive at
\[
\left\| \frac{\partial (\phi u)^\varepsilon}{\phi^\varepsilon} \right\|_{L^q(\mathbb{R}^d)} \leq C \left( (\|\nabla v|^{s_1} * J_\varepsilon(z)) \right)^{\frac{1}{2}} \left( \|\|\phi(y)|^{s_2} * J_{1\varepsilon} \|_{L^q} \right)^{\frac{1}{2}} \left( \|\phi(y)|^{s_2} * J_{1\varepsilon} \|_{L^q} \right)^{\frac{1}{2}} (2.23)
\]
\[
\leq C \|\nabla v\|_{L^q} \left( \|\|J_\varepsilon\|_{L^1} \right)^{\frac{1}{2}} \left( \|\|\phi\|_{L^q} \|\|J_{1\varepsilon}\|_{L^q} \right)^{\frac{1}{2}} \left( \|\|\phi\|_{L^q} \|\|J_{1\varepsilon}\|_{L^q} \right)^{\frac{1}{2}}
\]
\[
\leq C \|\nabla v\|_{L^q} \left( \|\|\phi\|_{L^q} \right)^{\frac{1}{2}} \left( \|\|\phi\|_{L^q} \right)^{\frac{1}{2}}.
\]

Then we have completed the proof of lemma 2.4.
On the other hand, if we let \( \eta \) be non-negative smooth function supported in the space-time ball of radius 1 and its integral equals to 1, we define the rescaled space-time mollifier 
\[
\eta_\varepsilon(t,x) = \frac{1}{\varepsilon^{d+1}} \eta(\frac{t}{\varepsilon}, \frac{x}{\varepsilon})
\]
and
\[
f_\varepsilon(t,x) = \int_0^T \int_{\mathbb{R}^d} f(\tau,y) \eta_\varepsilon(t-\tau, x-y) dy d\tau.
\]

We list two lemmas for the proof of Theorem (1.2). The fist one is the Lions type commutators on space-time mollifying kernel. The second one is the generalized Aubin-Lions Lemma, which helps us to extend the energy equality up to the initial time.

**Lemma 2.5** ([2, 14, 30]) Let \( 1 \leq p, q, p_1, q_1, p_2, q_2 \leq \infty \), with \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \) and \( \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} \). Let \( \partial \) be a partial derivative in space or time, in addition, let \( \partial_t f, \nabla f \in L^{p_1}(0,T; L^{q_1}(\Omega)) \), \( g \in L^{p_2}(0,T; L^{q_2}(\Omega)) \). Then, there holds
\[
\| \partial (fg) - \partial (fg_\varepsilon) \|_{L^p(0,T; L^q(\Omega))} \leq C (\| \partial_t f \|_{L^{p_1}(0,T; L^{q_1}(\Omega))} + \| \nabla f \|_{L^{p_1}(0,T; L^{q_1}(\Omega))}) \| g \|_{L^{p_2}(0,T; L^{q_2}(\Omega))},
\]
for some constant \( C > 0 \) independent of \( \varepsilon, f \) and \( g \). Moreover,
\[
\partial (fg) - \partial (fg_\varepsilon) \to 0 \quad \text{in} \quad L^p(0,T; L^q(\Omega)),
\]
as \( \varepsilon \to 0 \) if \( p_2, q_2 < \infty \).

**Lemma 2.6** ([23]). Let \( X \hookrightarrow B \hookrightarrow Y \) be three Banach spaces with compact imbedding \( X \hookrightarrow Y \). Further, let there exist \( 0 < \theta < 1 \) and \( M > 0 \) such that
\[
\| v \|_B \leq M \| v \|_X^{1-\theta} \| v \|_Y^\theta \quad \text{for all} \quad v \in X \cap Y.
\]
(2.24)

Denote for \( T > 0 \),
\[
W(0,T) := W^{s_0,r_0}((0,T), X) \cap W^{s_1,r_1}((0,T), Y)
\]
(2.25)

with
\[
s_0, s_1 \in \mathbb{R}; \quad r_0, r_1 \in [1, \infty],
\]
\[
s_\theta := (1-\theta)s_0 + \theta s_1, \quad \frac{1}{r_\theta} := \frac{1-\theta}{r_0} + \frac{\theta}{r_1}, \quad s^* := s_\theta - \frac{1}{r_\theta}.
\]
(2.26)

Assume that \( s_\theta > 0 \) and \( F \) is a bounded set in \( W(0,T) \). Then, we have

If \( s_* \leq 0 \), then \( F \) is relatively compact in \( L^p((0,T), B) \) for all \( 1 \leq p < p^* := -\frac{1}{s_*} \).

If \( s_* > 0 \), then \( F \) is relatively compact in \( C((0,T), B) \).

3 Energy equality in compressible Navier-Stokes equations without vacuum

When we consider the compressible Navier-Stokes equations with the density containing non-vacuum, due to the momentum equation (1.1), just spatially regularized velocity \( v_\varepsilon \) can be used as a test function to generate the global energy equality. However, to avoid a commutator estimate involving time \( t \), we choose \( \frac{(\omega v_\varepsilon)}{\varepsilon} \) instead of \( v_\varepsilon \) as the test function, which was introduced in [15, 21]. Hence, in this section, we let \( \eta \) be non-negative smooth
function only supported in the space ball of radius 1 and its integral equals to 1. We define the rescaled space mollifier \( \eta_{\varepsilon}(x) = \frac{1}{\varepsilon^n} \eta\left(\frac{x}{\varepsilon}\right) \) and

\[
f^{\varepsilon}(x) = \int_{\Omega} f(y) \eta_{\varepsilon}(x - y) dy ds.
\]

**Proof of Theorem 4.4**. Multiplying (4.1) by \( \left(\frac{(qv)^{\varepsilon}}{\varepsilon}\right) \), then integrating over \((s,t) \times \mathbb{T}^d\) with \(0 < s < t < T\), we have

\[
\int_s^t \int \left(\frac{(qv)^{\varepsilon}}{\varepsilon}\right) \partial_t (qv)^{\varepsilon} + \text{div} \left( (qv \otimes v)^{\varepsilon} \right) + \nabla P(\varrho)^{\varepsilon} - \mu \Delta v^{\varepsilon} - (\mu + \lambda) \nabla (\text{div} v)^{\varepsilon} \right] = 0. \tag{3.1}
\]

We will rewrite every term of the last equality to pass the limit of \( \varepsilon \). For the first term, a straightforward calculation and (4.1) yields that

\[
\int_s^t \int \left(\frac{(qv)^{\varepsilon}}{\varepsilon}\right) \partial_t (qv)^{\varepsilon} = \int_s^t \int \frac{1}{2} \partial_t \left( \frac{|(qv)^{\varepsilon}|^2}{\varrho^{\varepsilon}} \right) + \frac{1}{2} \partial_t \varrho^{\varepsilon} \frac{|(qv)^{\varepsilon}|^2}{(\varrho^{\varepsilon})^2} \tag{3.2}
\]

Integration by parts means that

\[
\int_s^t \int \left(\frac{(qv)^{\varepsilon}}{\varepsilon}\right) \text{div} (qv \otimes v)^{\varepsilon} = - \int_s^t \int \nabla \left(\frac{(qv)^{\varepsilon}}{\varepsilon}\right) [(qv \otimes v)^{\varepsilon} - (qv)^{\varepsilon} \otimes v^{\varepsilon}] - \int_s^t \int \nabla \left(\frac{(qv)^{\varepsilon}}{\varepsilon}\right) (qv)^{\varepsilon} \otimes v^{\varepsilon}. \tag{3.3}
\]

Making use of integration by parts once again, we infer that

\[
- \int_s^t \int \nabla \left(\frac{(qv)^{\varepsilon}}{\varepsilon}\right) (qv)^{\varepsilon} \otimes v^{\varepsilon} = \int_s^t \int \text{div} v^{\varepsilon} \frac{|(qv)^{\varepsilon}|^2}{\varrho^{\varepsilon}} + \frac{1}{2} v^{\varepsilon} \nabla |(qv)^{\varepsilon}|^2 \tag{3.4}
\]

\[
- \int_s^t \int \frac{1}{2} \text{div} v^{\varepsilon} \frac{|(qv)^{\varepsilon}|^2}{\varrho^{\varepsilon}} - \frac{1}{2} v^{\varepsilon} \nabla (\frac{1}{\varrho^{\varepsilon}}) |(qv)^{\varepsilon}|^2 \\
= \frac{1}{2} \int_s^t \int \text{div} \left(\frac{q^{\varepsilon}}{\varrho^{\varepsilon}}\right)^{\varepsilon} |(qv)^{\varepsilon}|^2 \left(\frac{q^{\varepsilon}}{\varrho^{\varepsilon}}\right)^2 \\
= \frac{1}{2} \int_s^t \int \text{div} \left[ q^{\varepsilon} v^{\varepsilon} - (qv)^{\varepsilon}\right] \frac{|(qv)^{\varepsilon}|^2}{(q^{\varepsilon})^2} + \frac{1}{2} \int_s^t \int \text{div} (qv)^{\varepsilon} \frac{|(qv)^{\varepsilon}|^2}{(q^{\varepsilon})^2} \\
= - \int_s^t \int \left[ q^{\varepsilon} v^{\varepsilon} - (qv)^{\varepsilon}\right] \frac{(qv)^{\varepsilon}}{q^{\varepsilon}} \nabla (\frac{(qv)^{\varepsilon}}{q^{\varepsilon}}) + \frac{1}{2} \int_s^t \int \text{div} (qv)^{\varepsilon} \frac{|(qv)^{\varepsilon}|^2}{(q^{\varepsilon})^2}.
\]

Inserting (3.4) into (3.3), we arrive at

\[
\int_s^t \int \left(\frac{(qv)^{\varepsilon}}{\varepsilon}\right) \text{div} (qv \otimes v)^{\varepsilon} \tag{3.5}
\]

\[
= - \int_s^t \int \nabla \left(\frac{(qv)^{\varepsilon}}{\varepsilon}\right) [(qv \otimes v)^{\varepsilon} - (qv)^{\varepsilon} \otimes v^{\varepsilon}] - \int_s^t \int \left[ q^{\varepsilon} v^{\varepsilon} - (qv)^{\varepsilon}\right] \frac{(qv)^{\varepsilon}}{q^{\varepsilon}} \nabla (\frac{(qv)^{\varepsilon}}{q^{\varepsilon}}) + \frac{1}{2} \int_s^t \int \text{div} (qv)^{\varepsilon} \frac{|(qv)^{\varepsilon}|^2}{(q^{\varepsilon})^2}.
\]
For the pressure term, by the integration by parts, one has
\[
\int_{s}^{t} \int \frac{(\rho v)^{\varepsilon}}{\rho^{\varepsilon}} \nabla (P(\varrho)^{\varepsilon}) = \int_{s}^{t} \int \frac{(\rho v)^{\varepsilon}}{\rho^{\varepsilon}} \nabla \left[ (P(\varrho)^{\varepsilon} - P(\varrho^{\varepsilon})) \right] + \int_{s}^{t} \int \frac{(\rho v)^{\varepsilon}}{\rho^{\varepsilon}} \nabla P(\varrho^{\varepsilon}) \\
= - \int_{s}^{t} \int \text{div} \left[ \frac{(\rho v)^{\varepsilon}}{\rho^{\varepsilon}} \right] [(P(\varrho)^{\varepsilon} - P(\varrho^{\varepsilon}))] + \int_{s}^{t} \int \frac{(\rho v)^{\varepsilon}}{\rho^{\varepsilon}} \nabla P(\varrho^{\varepsilon}).
\]

Using the mass equation (1.1)\textsuperscript{1}, the second term on the right hand-side of (3.6) can be rewritten as
\[
\int_{s}^{t} \int \frac{(\rho v)^{\varepsilon}}{\rho^{\varepsilon}} \nabla P(\varrho^{\varepsilon}) = \int_{s}^{t} \int (\rho v)^{\varepsilon} \gamma (\varrho^{\varepsilon})^{-2} \nabla \varrho^{\varepsilon} = \int_{s}^{t} \int (\rho v)^{\varepsilon} \frac{\gamma}{\gamma - 1} \nabla (\varrho^{\varepsilon})^{\gamma - 1} = \int_{s}^{t} \int \partial_{t} \varrho^{\varepsilon} \frac{\gamma}{\gamma - 1} (\varrho^{\varepsilon})^{\gamma - 1} dx d\tau = \int_{s}^{t} \int \frac{1}{\gamma - 1} \partial_{t} P(\varrho^{\varepsilon}).
\]

It is clear that
\[
- \mu \int_{s}^{t} \int \frac{(\rho v)^{\varepsilon}}{\rho^{\varepsilon}} \Delta v^{\varepsilon} = \mu \int_{s}^{t} \int - \Delta v^{\varepsilon} v^{\varepsilon} - \Delta v^{\varepsilon} \frac{(\rho v)^{\varepsilon} - \varrho^{\varepsilon} v^{\varepsilon}}{\rho^{\varepsilon}},
\]
\[
- (\mu + \lambda) \int_{s}^{t} \int \frac{(\rho v)^{\varepsilon}}{\rho^{\varepsilon}} \nabla (\text{div } v)^{\varepsilon} = (\mu + \lambda) \int_{s}^{t} \int - \nabla (\text{div } v)^{\varepsilon} v^{\varepsilon} - \nabla (\text{div } v)^{\varepsilon} \frac{(\rho v)^{\varepsilon} - \varrho^{\varepsilon} v^{\varepsilon}}{\rho^{\varepsilon}}.
\]

Substituting (3.2), (3.5) - (3.8) into (3.1), we see that
\[
\int_{s}^{t} \int \partial_{t} \left( \frac{1}{2} |(\rho v)^{\varepsilon}|^{2} + \frac{1}{\gamma - 1} P(\varrho^{\varepsilon}) \right) + \mu \int_{s}^{t} \int |\nabla v^{\varepsilon}|^{2} + (\mu + \lambda) \int_{s}^{t} \int |\nabla v^{\varepsilon}|^{2} = \int_{s}^{t} \int \mu \Delta v^{\varepsilon} \frac{(\rho v)^{\varepsilon} - \varrho^{\varepsilon} v^{\varepsilon}}{\rho^{\varepsilon}} + (\mu + \lambda) \int_{s}^{t} \int \nabla (\text{div } v)^{\varepsilon} \frac{(\rho v)^{\varepsilon} - \varrho^{\varepsilon} v^{\varepsilon}}{\rho^{\varepsilon}}
\]
\[
+ \int_{s}^{t} \int \text{div} \left[ \frac{(\rho v)^{\varepsilon}}{\rho^{\varepsilon}} \right] [(P(\varrho)^{\varepsilon} - P(\varrho^{\varepsilon}))]
\]
\[
+ \int_{s}^{t} \int \nabla (\frac{(\rho v)^{\varepsilon}}{\rho^{\varepsilon}}) [(\rho v \otimes v)^{\varepsilon} - (\rho v)^{\varepsilon} \otimes v^{\varepsilon}] + \int_{s}^{t} \int \frac{(\rho v)^{\varepsilon} - \rho^{\varepsilon} v^{\varepsilon}}{\rho^{\varepsilon}} \nabla (\frac{(\rho v)^{\varepsilon}}{\rho^{\varepsilon}}).
\]

Next, we need to prove that the terms on the right hand-side of (3.9) tend to zero as \( \varepsilon \to 0 \).

Under the hypothesis
\[
0 < c \leq \varrho \in L^{k}(0, T; L^{l}(\mathbb{T}^{d})),
\]
\[
v \in L^{p}(0, T; L^{q}(\mathbb{T}^{d})), \quad \nabla v \in L^{\frac{kp}{k - 2}}(0, T; L^{\frac{1q}{1 - 2}}(\mathbb{T}^{d})),
\]
with \( k > \max \{ \frac{4p}{p - 2}, \frac{(\gamma - 2)p}{2} \}, \)
\[
l > \max \{ \frac{4q}{q - 2}, \frac{(\gamma - 2)q}{2} \}
\]
and \( k(4 - p) + 7p \geq 0, \ l(4 - q) + 7q \geq 0 \).

It follows from Lemma 2.3 that
\[
\| \Delta v^{\varepsilon} \|_{L^{\frac{kp}{k - 2}}(0, T; L^{\frac{1q}{1 - 2}})} \leq C \| \nabla v^{\varepsilon} \|_{L^{\frac{kp}{k - 2}}(0, T; L^{\frac{1q}{1 - 2}})} \leq C \varepsilon \| \nabla v \|_{L^{\frac{kp}{k - 2}}(0, T; L^{\frac{1q}{1 - 2}})}
\]
\[
\| \Delta v^{\varepsilon} \|_{L^{\frac{kp}{k - 2}}(0, T; L^{\frac{1q}{1 - 2}})} \leq C \varepsilon \| \nabla v \|_{L^{\frac{kp}{k - 2}}(0, T; L^{\frac{1q}{1 - 2}})},
\]
and
\[
\limsup_{\varepsilon \to 0} \varepsilon \| \Delta v^{\varepsilon} \|_{L^{\frac{kp}{k - 2}}(0, T; L^{\frac{1q}{1 - 2}})} = 0.
\]
For some $\theta$ which follows from that, by Lemma 2.1, if
\[ \leq C\|\varepsilon\|_{L^{k(p-2)-4p}(L^{(q-2)-2q})} \leq \|\nabla v\|_{L^{k(p-2)-4p}(L^{(q-2)-2q})}, \]
we know that
\[ \| (pv)^\varepsilon - g^\varepsilon v^\varepsilon \|_{L^{k+4p}(L^{2+4q})} \leq C\varepsilon \| (pv)^\varepsilon - g^\varepsilon v^\varepsilon \|_{L^{k+4p}(L^{2+4q})}. \]

Combining the the Hölder’s inequality and (3.11)-(3.13), we arrive at
\[ \int_s^t \int \mu \Delta v^\varepsilon \frac{(pv)^\varepsilon - g^\varepsilon v^\varepsilon}{\varepsilon} \]
\[ \leq C\|\Delta v^\varepsilon\|_{L^{k(p-2)-4p}(L^{(q-2)-2q})} \leq \|\nabla v\|_{L^{k(p-2)-4p}(L^{(q-2)-2q})}. \]

Using the Constantin-E-Titi type commutators on mollifying kernel Lemma 2.2, we know that
\[ \| (pv)^\varepsilon - g^\varepsilon v^\varepsilon \|_{L^{k+4p}(L^{2+4q})} \leq C\varepsilon \| (pv)^\varepsilon - g^\varepsilon v^\varepsilon \|_{L^{k+4p}(L^{2+4q})}. \]

Combining the the Hölder’s inequality and (3.11)-(3.13), we arrive at
\[ \int_s^t \int \mu \Delta v^\varepsilon \frac{(pv)^\varepsilon - g^\varepsilon v^\varepsilon}{\varepsilon} \]
\[ \leq C\|\Delta v^\varepsilon\|_{L^{k(p-2)-4p}(L^{(q-2)-2q})} \leq \|\nabla v\|_{L^{k(p-2)-4p}(L^{(q-2)-2q})}. \]

where we need $k > \frac{4p}{p-2}$, $k(4-p) + 7p \geq 0$ and $l > \frac{4q}{q-2}$, $l(4-q) + 7q \geq 0$.

As a consequence, we get
\[ \limsup_{\varepsilon \to 0} \int_s^t \int \Delta v^\varepsilon \frac{(pv)^\varepsilon - g^\varepsilon v^\varepsilon}{\varepsilon} = 0. \]

Likewise, there holds
\[ \limsup_{\varepsilon \to 0} \int_s^t \int \nabla (\nabla v)^\varepsilon \frac{(pv)^\varepsilon - g^\varepsilon v^\varepsilon}{\varepsilon} = 0. \]

Applying Lemma 2.4 we get
\[ \| \nabla (\nabla v)^\varepsilon \|_{L^{k(p-2)-4p}(L^{(q-2)-2q})} \leq C\|\nabla v\|_{L^{k(p-2)-4p}(L^{(q-2)-2q})}. \]

For some $\theta \in [0,1]$, the mean value theorem, the Hölder’s inequality and the triangle inequality ensure that
\[ \|g^\gamma - (\theta^\varepsilon)^\gamma\|_{L^{\frac{ql}{2(\varepsilon+q)}}} = \|g + \theta(\varphi - g^\varepsilon)^\gamma-1(\varphi - g^\varepsilon)\|_{L^{\frac{ql}{2(\varepsilon+q)}}} \leq C\|\varphi\|_{L^{\frac{ql}{2(\varepsilon+q)}}} \|g - \theta^{\varepsilon}\|_{L^{\frac{ql}{2(\varepsilon+q)}}}, \]

which follows from that, by Lemma 2.1 if $g \in L^{\frac{ql}{2(\varepsilon+q)}}$, as $\varepsilon \to 0$,
\[ \|g^\gamma - (\theta^\varepsilon)^\gamma\|_{L^{\frac{ql}{2(\varepsilon+q)}}} \to 0. \]

With the help of the triangle inequality, the Hölder’s inequality and (3.16)-(3.18), we obtain
\[ \int_s^t \int \nabla v^\varepsilon \frac{(pv)^\varepsilon - g^\varepsilon v^\varepsilon}{\varepsilon} \]
\[ \leq C\|\nabla v\|_{L^{k(p-2)-4p}(L^{(q-2)-2q})} \leq \|\nabla v\|_{L^{k(p-2)-4p}(L^{(q-2)-2q})}. \]
which means that
\[
\limsup_{\varepsilon \to 0} \int_s^t \int \text{div} \left( \frac{(qv)^\varepsilon}{q^\varepsilon} \right) ((P(q))^\varepsilon - P(q^\varepsilon)) = 0.
\]

At this stage, it is enough to show
\[
\limsup_{\varepsilon \to 0} \int_s^t \int \nabla \left( \frac{(qv)^\varepsilon}{q^\varepsilon} \right) [((qv \otimes v)^\varepsilon - (qv)^\varepsilon \otimes v^\varepsilon) + \int_s^t \int \left[ \frac{(q^\varepsilon v^\varepsilon - (qv)^\varepsilon)}{q^\varepsilon} \right] \nabla \left( \frac{(qv)^\varepsilon}{q^\varepsilon} \right)] = 0,
\]
which in turn implies
\[
\parallel \nabla \left( \frac{(qv)^\varepsilon}{q^\varepsilon} \right) \parallel_{L^k(L^t)} \leq C \varepsilon^{-1} \parallel v \parallel_{L^k(L^t)}^2,
\]
and
\[
\limsup_{\varepsilon \to 0} \varepsilon \parallel \nabla \left( \frac{(qv)^\varepsilon}{q^\varepsilon} \right) \parallel_{L^k(L^t)} = 0.
\]

In view of (2.6) and Lemma 2.3
\[
\parallel \nabla \left( \frac{(qv)^\varepsilon}{q^\varepsilon} \right) \parallel_{L^q(L^t)} \leq C \varepsilon^{-1} \parallel v \parallel_{L^k(L^t)}.
\]

Taking advantage of (2.2) in Lemma 2.2
\[
\parallel (qv \otimes v)^\varepsilon - (qv)^\varepsilon \otimes v^\varepsilon \parallel_{L^q(L^t)} \leq C \varepsilon \parallel \nabla v \parallel_{L^k(L^t)} \parallel v \parallel_{L^k(L^t)}.
\]

Thanks to the Hölder’s inequality and (3.21)–(3.23), we find
\[
\left| \int_s^t \int \nabla \left( \frac{(qv)^\varepsilon}{q^\varepsilon} \right) [((qv \otimes v)^\varepsilon - (qv)^\varepsilon \otimes v^\varepsilon) + \int_s^t \int \left[ \frac{(q^\varepsilon v^\varepsilon - (qv)^\varepsilon)}{q^\varepsilon} \right] \nabla \left( \frac{(qv)^\varepsilon}{q^\varepsilon} \right)] \right| = 0.
\]

We turn our attentions to the term \( \int_s^t \int \frac{(qv)^\varepsilon}{q^\varepsilon} \nabla \left( \frac{(qv)^\varepsilon}{q^\varepsilon} \right) \). We conclude from Lemma 2.2 that
\[
\parallel q^\varepsilon v^\varepsilon - (qv)^\varepsilon \parallel_{L^k(L^t)} \leq C \varepsilon \parallel \nabla v \parallel_{L^k(L^t)} \parallel v \parallel_{L^k(L^t)}.
\]

Using the Höder’s inequality and (3.25), we find,
\[
\left| \int_s^t \int \left[ q^\varepsilon v^\varepsilon - (qv)^\varepsilon \right] \frac{(qv)^\varepsilon}{q^\varepsilon} \nabla \left( \frac{(qv)^\varepsilon}{q^\varepsilon} \right) \right| \leq C \parallel q^\varepsilon v^\varepsilon - (qv)^\varepsilon \parallel_{L^k(L^t)} \parallel v \parallel_{L^k(L^t)} \parallel \nabla v \parallel_{L^k(L^t)}.
\]

Together this with (3.21) and (3.22) yield that
\[
\limsup_{\varepsilon \to 0} \parallel \nabla \left( \frac{(qv)^\varepsilon}{q^\varepsilon} \right) \parallel_{L^k(L^t)} = 0.
\]

Collecting all the above estimates, using the weak continuity of \( \rho \) and \( qv \), we complete the proof of Theorem 1.1. \( \square \)
4 Energy equality in compressible Navier-Stokes equations allowing vacuum

Unlike the case with non-vacuum, when we consider the compressible Navier-Stokes equations with the density containing vacuum, only spatially regularized velocity \( v^\varepsilon \) fails to possess enough temporal regularity to qualify for a test function. To over this difficulty, we need to mollify the velocity both in space and time. Hence, in this section, we let \( \eta \) be non-negative smooth function supported in the space-time ball of radius 1 and its integral equals to 1. We define the rescaled space mollifier \( \eta_\varepsilon(t, x) = \frac{1}{\varepsilon t+\tau} \eta(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}) \) and

\[
f^\varepsilon(t, x) = \int_0^T \int_{\mathbb{T}^d} f(\tau, y) \eta_\varepsilon(t - \tau, x - y) dyd\tau.
\]

**Proof of Theorem 4.1.** Let \( \phi(t) \) be a smooth function compactly supported in \((0, +\infty)\). Multiplying (4.1) by \((\phi v^\varepsilon)^\varepsilon\), then integrating over \((0, T) \times \mathbb{T}^d\), we infer that

\[
\int_0^T \int_0^T \phi(t)v^\varepsilon \left[ \partial_t (\varepsilon g v^\varepsilon) + \div (\varepsilon g \otimes v) + \nabla P(\varepsilon g) - \mu \Delta v^\varepsilon - (\mu + \lambda) \nabla (\div v^\varepsilon) \right] = 0. \tag{4.1}
\]

To pass the limit of \( \varepsilon \), we reformulate every term of the last equation. A straightforward computation leads to

\[
\int_0^T \int_0^T \phi(t)v^\varepsilon \partial_t (\varepsilon g v^\varepsilon) = \int_0^T \int_0^T \phi(t)v^\varepsilon \left[ \partial_t (\varepsilon g v^\varepsilon) - \partial_t (\varepsilon g v^\varepsilon) \right] + \int_0^T \int_0^T \phi(t)v^\varepsilon \partial_t (\varepsilon g v^\varepsilon)
\]

\[= \int_0^T \int_0^T \phi(t)v^\varepsilon \left[ \partial_t (\varepsilon g v^\varepsilon) - \partial_t (\varepsilon g v^\varepsilon) \right] + \int_0^T \int_0^T \phi(t) \varepsilon \partial_t |v^\varepsilon|^2 + 1 + \frac{1}{2} \varepsilon g \nabla |v^\varepsilon|^2. \tag{4.2}
\]

It follows from integration by parts and the mass equation (1.1) that

\[
\int_0^T \int_0^T \phi(t)v^\varepsilon \div (\varepsilon g \otimes v) = \int_0^T \int_0^T \phi(t)v^\varepsilon (\div g \otimes v^\varepsilon) - (\div v^\varepsilon) + \int_0^T \int_0^T \phi(t) \varepsilon \div (\varepsilon g \otimes v^\varepsilon)
\]

\[= - \int_0^T \int_0^T \phi(t) \div (\varepsilon g \otimes v^\varepsilon) - (\div v^\varepsilon) \div (\varepsilon g \otimes v^\varepsilon) + \frac{1}{2} \int_0^T \int_0^T \phi(t) \varepsilon \div (\varepsilon g \otimes v^\varepsilon)
\]

\[= - \int_0^T \int_0^T \phi(t) \div (\varepsilon g \otimes v^\varepsilon) - (\div v^\varepsilon) \div (\varepsilon g \otimes v^\varepsilon) + \frac{1}{2} \int_0^T \int \phi(t) \varepsilon \div (\varepsilon g \otimes v^\varepsilon)
\]

\[= - \int_0^T \int_0^T \phi(t) \div (\varepsilon g \otimes v^\varepsilon) - (\div v^\varepsilon) - \frac{1}{2} \int_0^T \int \phi(t) \varepsilon \div (\varepsilon g \otimes v^\varepsilon). \tag{4.3}
\]

We rewrite the pressure term as

\[
\int_0^T \int_0^T \phi(t)v^\varepsilon \nabla (g^\varepsilon) = \int_0^T \int_0^T \phi(t)v^\varepsilon \nabla (g^\varepsilon) - v \nabla (g^\varepsilon) + \int_0^T \int \phi(t) v \nabla (g^\varepsilon). \tag{4.4}
\]
Then using the integration by parts and mass equation (1.1) again, we find

\[ \int_0^T \int \phi(t) v \cdot \nabla (\varrho^\gamma) = - \int_0^T \int \phi(t) \varrho^{\gamma-1} \varrho \text{div} v \]
\[ = \int_0^T \int \phi(t) \varrho^{\gamma-1} (\partial_t \varrho + v \cdot \nabla \varrho) \]
\[ = \frac{1}{\gamma} \int_0^T \int \phi(t) \partial_t \varrho^\gamma + \frac{1}{\gamma} \int_0^T \int \phi(t) v \cdot \nabla \varrho^\gamma, \]

which in turn means that

\[ \int_0^T \int \phi(t) v \cdot \nabla (\varrho^\gamma) = \frac{1}{\gamma - 1} \int_0^T \int \phi(t) \partial_t \varrho^\gamma. \quad (4.5) \]

Thanks to integration by parts, we arrive at

\[ - \mu \int_0^T \int \phi(t) v^\varepsilon \Delta v^\varepsilon = \mu \int_0^T \int \phi(t) |\nabla v^\varepsilon|^2, \]
\[ - (\mu + \lambda) \int_0^T \int \phi(t) v^\varepsilon \nabla \text{div} v^\varepsilon = (\mu + \lambda) \int_0^T \int \phi(t) |\text{div} v^\varepsilon|^2. \quad (4.6) \]

Plugging (4.2)-(4.6) into (4.1) and using the integration by parts, we conclude that

\[ - \int_0^T \int \phi(t) \left( \frac{|v^\varepsilon|^2}{2} + \frac{1}{\gamma - 1} \varrho^\gamma \right) + \int_0^T \int (\mu |\nabla v^\varepsilon|^2 + (\mu + \lambda) |\text{div} v^\varepsilon|^2) \]
\[ = - \int_0^T \int \phi(t) v^\varepsilon \left[ \partial_t (\varrho v^\varepsilon) - \partial_t (\varrho v^\varepsilon) \right] + \int_0^T \int \phi(t) \nabla v^\varepsilon \left[ (\varrho v \otimes v)^\varepsilon - (\varrho v) \otimes v^\varepsilon \right] \]
\[ - \int_0^T \int \phi(t) [v^\varepsilon \nabla (\varrho^\gamma) - v \nabla (\varrho^\gamma)]. \quad (4.7) \]

It is enough to prove that the terms on the right hand-side of (4.7) tend to zero as \( \varepsilon \to 0. \)

Under the hypothesis

\[ 0 \leq \varrho \in L^k(0,T; L^1(\mathbb{T}^d)), \nabla \sqrt{\varrho} \in L^{\frac{2k}{k-1}}(0,T; L^{\frac{2g}{q-\frac{q}{2}}}(\mathbb{T}^d)) \]
\[ v \in L^p(0,T; L^q(\mathbb{T}^d)), \nabla v \in L^{\frac{nk}{p-3q}}(0,T; L^{\frac{ng}{q-\frac{q}{2}}}(\mathbb{T}^d)) \text{ and } v_0 \in L^{\max\{\frac{2k}{q-\frac{q}{2}}, \frac{2}{2}\}}(\mathbb{T}^d), \quad (4.8) \]

with \( k > \max\{\frac{p}{2(p-3)}, \frac{p}{p-2}, \frac{(\gamma-1)p}{2}, \frac{(\gamma-1)(d+q)}{2g-d(p-3)}\}, \ l > \max\{\frac{q}{2(q-3)}, \frac{q}{q-2}, \frac{(\gamma-1)q}{2}\}, \ p > 3 \) and \( q > \max\{3, \frac{d(p-3)}{2}\}. \)

In view of Hölder’s inequality and Lemma (2.5), we know that

\[ \int_0^t \int \phi(t) v^\varepsilon \left[ \partial_t (\varrho v^\varepsilon) - \partial_t (\varrho v^\varepsilon) \right] \leq C \|v^\varepsilon\|_{L^p(L^q)} \|\partial_t (\varrho v^\varepsilon) - \partial_t (\varrho v^\varepsilon)\|_{L^\frac{p}{p-1}(L^\frac{q}{q-1})} \]
\[ \leq C \|v\|_{L^p(L^q)}^2 \left( \|\varrho_t\|_{L_{p-2}{q-2}} + \|\nabla \varrho\|_{L_{p-2}{q-2}} \right). \quad (4.9) \]

To bound \( \varrho_t \) and \( \nabla \varrho, \) we employ mass equation (1.1) to obtain

\[ \varrho_t = -2\sqrt{\varrho v} \cdot \nabla \sqrt{\varrho} - g \text{div} v, \text{ and } \nabla \varrho = 2\sqrt{\varrho} \nabla \sqrt{\varrho}. \]
As a consequence, the triangle inequality and Hölder’s inequality guarantee that

\[
\|q_t\|_{L^p Q(L^{\infty Q})} \leq C \left( 2^{\beta_0} \|q_t\|_{L^{p_0 Q}(L^Q)} + \|\text{div} \ v\|_{L^{p_0 Q}(L^Q)} \right)
\]

\[
\|q_t\|_{L^{p_0 Q}(L^Q)} \leq C \left( \|v\|_{L^{p_0 Q}(L^Q)} \frac{2^{\beta_0}}{L^{p_0 Q}(L^{Q_0 - Q_0})} \|q\|_{1/2}^{1/2} \right) + \|q\|_{1/2}^{1/2} \|\text{div} \ v\|_{L^{p_0 Q}(L^{Q_0 - Q_0})} \|q\|_{L^{1/2}(L^Q)} \right),
\]

(4.10)

and

\[
\|\nabla q\|_{L^{p_0 Q}(L^{Q_0 - Q_0})} \leq C \left( \|\text{div} \ q\|_{L^{p_0 Q}(L^{Q_0 - Q_0})} \|q\|_{1/2}^{1/2} \right) + \|\text{div} \ v\|_{L^{p_0 Q}(L^{Q_0 - Q_0})} \|q\|_{L^{1/2}(L^Q)} \right).
\]

(4.11)

Plugging (4.10) and (4.11) into (4.9), we get

\[
\int_0^T \int \phi(t) v^\varepsilon \left[ \partial_t (q v)^\varepsilon - \partial_t (q v^\varepsilon) \right] dtdt \leq C \|v\|^2_{L^p Q(L^Q)} \left( \|v\|_{L^p Q(L^Q)} + 1 \right) \|\nabla q\|_{L^{p_0 Q}(L^{Q_0 - Q_0})} \|q\|_{1/2}^{1/2} \|\text{div} \ v\|_{L^{p_0 Q}(L^{Q_0 - Q_0})} \|q\|_{L^{1/2}(L^Q)} \right),
\]  

(4.12)

From Lemma (2.5), we end up with, as \( \varepsilon \to 0 \),

\[
\int_0^T \int \phi(t) v^\varepsilon \left[ \partial_t (q v)^\varepsilon - \partial_t (q v^\varepsilon) \right] dtdt \to 0.
\]

In the light of the Hölder’s inequality, we obtain

\[
\|q v \otimes v\|_{L^{p_0 Q}(L^{Q_0 - Q_0})} \leq \|v\|^2_{L^p Q(L^Q)} \|q\|_{L^{1/2}(L^Q)} \quad (4.13)
\]

Using the integration by parts, we observe that

\[
\left| \int_0^T \int \phi(t) \nabla v^\varepsilon [((q v) \otimes v)^\varepsilon - (q v) \otimes v]^\varepsilon \right|
\]

\[
\leq C \|\nabla v\|_{L^{p_0 Q}(L^{Q_0 - Q_0})} \left( \|\text{div} \ v\|_{L^{p_0 Q}(L^{Q_0 - Q_0})} + \|q v \otimes v\|_{L^{p_0 Q}(L^{Q_0 - Q_0})} \right),
\]

Hence, by the standard properties of the mollification, we have

\[
\int_0^T \int \phi(t) \nabla v^\varepsilon [((q v) \otimes v)^\varepsilon - (q v) \otimes v]^\varepsilon \to 0 \quad \text{as} \ \varepsilon \to 0.
\]

According to Hölder’s inequality on bound domain, we observe that

\[
\|\nabla (q^\gamma)\|_{L^{p_0 Q}(L^{Q_0 - Q_0})} \leq C \|q\|_{L^{p_0 Q}(L^{Q_0 - Q_0})} \|\text{div} \ q\|_{L^{p_0 Q}(L^{Q_0 - Q_0})} \|\text{div} \ v\|_{L^{p_0 Q}(L^{Q_0 - Q_0})} \|q\|_{L^{1/2}(L^Q)} \right),
\]

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which in turn implies that
\[ \int_0^T \int \phi(t)[v^\varepsilon \nabla (\varphi \gamma)^\varepsilon - v \nabla (\varphi \gamma)] \rightarrow 0, \]  
(4.15)
where we have used lemma 2.1 and the condition \( k \geq \frac{(\gamma - 1)p}{2}, l \geq \frac{(\gamma - 1)q}{2} \).

Then together with (4.12) - (4.15), passing to the limits as \( \varepsilon \to 0 \), we know that
\[ - \int_0^T \int \phi_t \left( \frac{1}{2} |v|^2 + \frac{\varphi \gamma}{\gamma - 1} \right) + \int_0^T \int \phi(t) \left( \mu |\nabla v|^2 + (\mu + \lambda) |\text{div} v|^2 \right) = 0. \]  
(4.16)

The next objective is to get the energy equality up to the initial time \( t = 0 \) by the similar method in [6] and [32], for the convenience of the reader and the integrity of the paper, we give the details. First we prove the continuity of \( \sqrt{\varphi}v(t) \) in the strong topology as \( t \to 0^+ \).

To do this, we define the function \( f \) on \([0,T]\) as
\[ f(t) = \int_{\mathbb{T}^d} (\varphi v)(t,x) \cdot \phi(x) dx, \]  
for any \( \phi(x) \in \mathcal{D}(\mathbb{T}^d) \),
which is a continuous function with respect to \( t \in [0,T] \). Moreover, since
\[ \varrho \in L^\infty(0,T; L^\gamma(\mathbb{T}^d)) \text{ and } \sqrt{\varrho}v \in L^\infty(0,T; L^2(\mathbb{T}^d)), \]
we can obtain \( \varrho v \in L^\infty(0,T; L^\frac{2\gamma}{\gamma+1}(\mathbb{T}^d)) \).

From the moment equation, we have
\[ \frac{d}{dt} \int_{\mathbb{T}^d} (\varrho v)(t,x) \cdot \phi(x) dx = \int_{\mathbb{T}^d} \varrho v \otimes v : \nabla \phi(x) - P \text{div} \phi(x) - \mu \nabla v \nabla \phi(x) - (\mu + \lambda) \text{div} v \text{div} \phi(x) dx, \]
which is bounded for any function \( \phi \in \mathcal{D}(\mathbb{T}^d) \). Then it follows from the Corollary 2.1 in [11] that
\[ \varrho v \in C([0,T]; L^\frac{2\gamma}{\gamma+1}_{\text{weak}}(\mathbb{T}^d)). \]  
(4.17)

On the other hand, we derive from the mass equation (1.1) that
\[ \partial_t (\varphi \gamma) = -\gamma \varrho \gamma \text{div} v - 2\gamma \varrho \gamma^{-\frac{1}{2}} v \cdot \nabla \sqrt{\varrho}, \]  
(4.18)
and
\[ \partial_t (\sqrt{\varrho}) = \frac{\sqrt{\varrho}}{2} \text{div} v + v \cdot \nabla \sqrt{\varrho}, \]  
(4.19)
which together with (1.12) gives
\[ \partial_t \varrho \gamma \in L^{\frac{k}{p}\gamma(p-2)\gamma(p-1)\gamma} \left( L^{\frac{l}{q}\gamma(q-2)\gamma(q-1)\gamma} \right), \]  
\[ \nabla \varrho \gamma \in L^{\frac{k}{p}\gamma(p-3)\gamma(p-1)\gamma} \left( L^{\frac{l}{q}\gamma(q-3)\gamma(q-1)\gamma} \right), \]
and
\[ \partial_t \sqrt{\varrho} \in L^{\frac{k}{2p}\gamma(p-2)\gamma} \left( L^{\frac{2l}{q}\gamma(q-2)\gamma} \right), \]  
\[ \nabla \sqrt{\varrho} \in L^{\frac{2k}{2p}\gamma(p-3)\gamma} \left( L^{\frac{2l}{q}\gamma(q-3)\gamma} \right), \]
Hence, using the Aubin-Lions Lemma 2.6, we can obtain
\[ \varrho \gamma \in C([0,T]; L^{\frac{l}{q}\gamma(q-3)\gamma}((\mathbb{T}^d)) \text{ and } \sqrt{\varrho} \in C([0,T]; L^{\frac{2l}{q}\gamma(q-3)\gamma}((\mathbb{T}^d))), \]  
(4.20)
for \( k \geq \frac{(\gamma - 1)(d+q)p}{2q-d(p-3)}, p > 3 \) and \( q > \max\{3, \frac{d(p-3)}{2}\} \).
Meanwhile, using the natural energy (4.11), (4.17) and (4.20), we have
\[ 0 \leq \lim_{t \to 0} \int |\varphi v - \sqrt{\varrho_0} v_0|^2 dx \]
\[ = 2 \lim_{t \to 0} \left( \int \left( \frac{1}{2} \varrho v^2 + \frac{1}{\gamma - 1} \varrho^\gamma \right) dx - \int \left( \frac{1}{2} \varrho_0 v_0^2 + \frac{1}{\gamma - 1} \varrho_0^\gamma \right) dx \right) \]
\[ + 2 \lim_{t \to 0} \left( \int \sqrt{\varrho_0} v_0 (\sqrt{\varrho_0} v_0 - \sqrt{\varrho v}) dx + \frac{1}{\gamma - 1} \int (\varrho_0^\gamma - \varrho^\gamma) dx \right) \] (4.21)
\[ \leq 2 \lim_{t \to 0} \int \sqrt{\varrho_0} v_0 (\sqrt{\varrho_0} v_0 - \sqrt{\varrho v}) dx \]
\[ = 2 \lim_{t \to 0} \int v_0 (\varrho_0 v_0 - \varrho v) dx + \lim_{t \to 0} \int v_0 \sqrt{\varrho} (\sqrt{\varrho} - \sqrt{\varrho_0}) dx = 0, \]
from which it follows
\[ \sqrt{\varrho v}(t) \to \sqrt{\varrho v}(0) \text{ strongly in } L^2(\Omega) \text{ as } t \to 0^+. \] (4.22)

Similarly, one has the right temporal continuity of \( \sqrt{\varrho v} \) in \( L^2(\Omega) \), hence, for any \( t_0 \geq 0 \), we infer that
\[ \sqrt{\varrho v}(t) \to \sqrt{\varrho v}(t_0) \text{ strongly in } L^2(\Omega) \text{ as } t \to t_0^+. \] (4.23)

Before we go any further, it should be noted that (4.16) remains valid for function \( \phi \) belonging to \( W^{1,\infty} \) rather than \( C^1 \), then for any \( t_0 > 0 \), we redefine the test function \( \phi \) as \( \phi_t \) for some positive \( \tau \) and \( \alpha \) such that \( \tau + \alpha < t_0 \), that is
\[
\phi_t(t) = \begin{cases} 
0, & 0 \leq t \leq \tau, \\
\frac{t - \tau}{\alpha}, & \tau \leq t \leq \tau + \alpha, \\
1, & \tau + \alpha \leq t \leq t_0, \\
\frac{t_0 - t}{\alpha}, & t_0 \leq t \leq t_0 + \alpha, \\
0, & t_0 + \alpha \leq t.
\end{cases}
\] (4.24)

Then substituting this test function into (4.16), we arrive at
\[ - \int_\tau^{t + \alpha} \int \frac{1}{\alpha} \left( \frac{1}{2} \varrho v^2 + \frac{1}{\gamma - 1} \varrho^\gamma \right) + \frac{1}{\alpha} \int_{t_0}^{t + \alpha} \int \left( \frac{1}{2} \varrho v^2 + \frac{1}{\gamma - 1} \varrho^\gamma \right) \]
\[ + \int_\tau^{t_0 + \alpha} \int \phi_t (\mu |\nabla v|^2 + (\mu + \lambda) |\text{div } v|^2) = 0. \] (4.25)

Taking \( \alpha \to 0 \) and using the fact that \( \int_0^t \int (\mu |\nabla v|^2 + (\mu + \lambda) |\text{div } v|^2) \) is continuous with respect to \( t \) and the Lebesgue point Theorem, we deduce that
\[ - \int \left( \frac{1}{2} \varrho v^2 + \frac{1}{\gamma - 1} \varrho^\gamma \right) (\tau)dx + \int \left( \frac{1}{2} \varrho v^2 + \frac{1}{\gamma - 1} \varrho^\gamma \right) (t_0)dx \]
\[ + \int_\tau^{t_0} \int (\mu |\nabla v|^2 + (\mu + \lambda) |\text{div } v|^2) = 0. \] (4.26)

Finally, letting \( \tau \to 0 \), using the continuity of \( \int_0^t \int (\mu |\nabla v|^2 + (\mu + \lambda) |\text{div } v|^2) \), (4.17) and (4.22), we can obtain
\[ \int \left( \frac{1}{2} \varrho v^2 + \frac{1}{\gamma - 1} \varrho^\gamma \right) (t_0)dx + \int_0^{t_0} \int (\mu |\nabla v|^2 + (\mu + \lambda) |\text{div } v|^2) dxds \]
\[ = \int \left( \frac{1}{2} \varrho_0 v_0^2 + \frac{1}{\gamma - 1} \varrho_0^\gamma \right) dx. \] (4.27)

Then we complete the proof of Theorem 1.2. \( \square \)
Acknowledgement

The authors would like to express their sincere gratitude to Prof. Quansen Jiu for pointing out this problem to us. Ye was partially supported by the National Natural Science Foundation of China under grant (No.11701145) and China Postdoctoral Science Foundation (No. 2020M672196). Wang was partially supported by the National Natural Science Foundation of China under grant (No. 11971446, No. 12071113 and No. 11601492). Yu was partially supported by the National Natural Science Foundation of China (NNSFC) (No. 11901040), Beijing Natural Science Foundation (BNSF) (No. 1204030) and Beijing Municipal Education Commission (KM202011232020).

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