Pion Form Factor in Chiral Limit of Hard-Wall AdS/QCD Model

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We develop a formalism to calculate form factor and charge density distribution of pion in
the chiral limit using the holographic dual model of QCD with hard-wall cutoff. We introduce
two conjugate pion wave functions and present analytic expressions for these functions and
for the pion form factor. They allow to relate such observables as the pion decay constant
and the pion charge electric radius to the values of chiral condensate and hard-wall cutoff
scale. The evolution of the pion form factor to large values of the momentum transfer is
discussed, and results are compared to existing experimental data.

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I. INTRODUCTION

During the last few years applications of gauge/gravity duality [1] to hadronic physics attracted
a lot of attention, and various holographic dual models of QCD were proposed in the literature
(see, e.g., [2–14]). These models were able to incorporate such essential properties of QCD as
confinement and chiral symmetry breaking, and also to reproduce many of the static hadronic
observables, (e.g. decay constant, masses), with values sufficiently close to experimental ones.
Amongst these dual models, a special class is the so called “bottom-up” approaches, (see, e.g.,
[6–9]), the goal of which is to reproduce known properties of QCD by choosing an appropriate
theory in the 5-dimensional (5D) AdS bulk. In these AdS/QCD models, by modifying the theory
in the bulk one can approach sufficiently close to the experimental results in different sectors of
QCD.

Here, we will be interested in the AdS/QCD model with hard-wall cutoff [6–8] where, as the
name suggests, the confinement is modeled by hard-wall cutting off the AdS space along the extra
fifth dimension at some finite value $z = z_0$. In the framework of this hard wall model, it was possible
to find the form factors and wave functions of vector mesons, (see e.g. [17, 18]). In particular, the
vector sector of QCD in the model [6, 7] works rather well for the lowest states. To reproduce the
general features of the spectrum for the higher states (“linear confinement”), a soft-wall model was
proposed in [9].

In general, the vector sector is less sensitive to the infrared (IR) effects, since this symmetry is not broken in QCD. However, the axial-vector sector appears to be very sensitive to the way the chiral symmetry is broken or in other words to the bulk content and the shape of IR wall [9].

In this respect, one of the interesting objects to study in the holographic dual models of QCD is the pion. The properties of the pion were studied in various holographic approaches, (see e.g. Refs. [5, 6, 8, 12–16]). In particular, the approach of Ref. [6] managed to reproduce the (Gell-Mann–Oakes–Renner) relation $m_{\pi}^2 \sim m_q$ between the quark mass $m_q$ and mass of the pion $m_{\pi}$ and also the $g_{\rho\pi\pi}$ coupling (the coupling between $\rho$ meson and two pions). In Ref. [8] the solution of the pion wave-function equation was explicitly written.

In this paper, working in the framework of the model proposed in [6] (hard-wall model), we develop a formalism to calculate the form factor and wave functions (and also the density function) of the pion in the chiral limit of two flavor QCD.

In particular, we extract and analyze the behavior of the pion’s electric radius in various regions of the holographic parameters space and conclude that the radius of the pion is smaller than what is known from experiment. However, we suggest that, as in case of the radius of the rho meson, the smoothing of the IR wall may improve the results.

In the derivation process, we introduce two types of holographic “wave functions” which are basically similar to the analogous objects introduced in Refs. [17, 18], where we studied vector mesons.

We start with recalling, in Section II, the basics of the hard-wall model and some results obtained in Ref. [6], in particular, the form of the relevant action, the eigenvalue equations for bound states and their solutions. In Section III, we develop a formalism to calculate the pion form factor and express it in terms of the two wave functions mentioned above. In Section IV, we discuss the relation of our AdS/QCD results with experimental data. We express the values of pion’s decay constant and the pion charge radius in terms of the fundamental parameters of the theory and study their behavior in different regions of the parametric space. At the end, we study the behavior of the pion form factor at large momentum transfer. Finally, we summarize the paper.

II. PRELIMINARIES

In the holographic model of hadrons, QCD resonances correspond to Kaluza-Klein (KK) excitations in the sliced AdS$_5$ background. In particular, vector mesons correspond to the KK modes of
transverse vector gauge field in this background. Since the gauge symmetry in the vector sector of the H-model is not broken, the longitudinal component of the vector gauge field is unphysical, and only transverse components correspond to physical mesons. Similarly, the axial-vector mesons are the modes of the transverse part of the axial-vector gauge field. However, because the axial-vector gauge symmetry is broken in the 5D background, the longitudinal components have physical meaning and are related to the pion field. This should be taken into account if we want to incorporate pions in a consistent way.

A. Action and Equations of Motion

The standard prescription of the holographic model is that there is a correspondence between the 4D vector and axial-vector currents and the corresponding 5D gauge fields:

\[ J^a_V(\mu(x)) = \bar{q}(x)\gamma^\mu t^a q(x) \rightarrow V^a_\mu(x,z) \]
\[ J^a_A(\mu(x)) = \bar{q}(x)\gamma^\mu\gamma^5 t^a q(x) \rightarrow A^a_\mu(x,z) , \]

where \( t^a = \frac{1}{2} \sigma^a \), \( a = 1, 2, 3 \) and \( \sigma^a \) are usual Pauli matrices).

In general, one can write \( A = A_\perp + A_\parallel \), where \( A_\perp \) and \( A_\parallel \) are transverse and longitudinal components of the axial-vector field. The spontaneous symmetry breaking causes \( A_\parallel \) to be physical and associated with the Goldstone boson, pion in this case. The longitudinal component may be written in the form: \( A^a_\parallel(x,z) = \partial_M \psi^a(x,z) \). Then \( \psi^a(x,z) \) corresponds to the pion field. Physics of the axial-vector and pseudoscalar sector is described by the action

\[ S_{AdS}^A = \text{Tr} \int d^4x \int dz \left[ \frac{1}{z^3} (D^M X)^\dagger (D_M X) + \frac{3}{z^5} X^\dagger X - \frac{1}{4g_5^2 z} A^{MN} A_{MN} \right] , \]

where \( DX = \partial X - iA_L X + i X A_R \) (\( A_{L(R)} = V \pm A \)) and \( X(x,z) = v(z) U(x,z) \) is taken as a product of the chiral field \( U(x,z) = \exp (2it^a \pi^a (x,z)) \) and the function \( v(z) = m_q z + \sigma z^3 \) containing the chiral symmetry breaking parameters \( m_q \) and \( \sigma \), with \( m_q \) playing the role of the quark mass and \( \sigma \) that of the quark condensate. Expanding \( U(x,z) \) in powers of \( \pi^a \) gives the relevant piece of the action

\[ S_{AdS}^{A(2)} = \text{Tr} \int d^4x \int dz \left[ -\frac{1}{4g_5^2 z} A^{MN} A_{MN} + \frac{v^2(z)}{2z^3} \left( A^a_M - \partial_M \pi^a \right)^2 \right] . \]

This Higgs-like mechanism breaks the axial-vector gauge symmetry by bringing a \( z \)-dependent mass term in the \( A \)-part of the lagrangian. Varying the action with respect to the transverse part of the axial-vector gauge field \( A^a_\perp(x,z) \) and representing the Fourier image of \( A^a_\perp(x,z) \) as \( \tilde{A}^a_\perp(p,z) \) we
will get the following equation of motion

$$\left[ z^3 \partial_z \left( \frac{1}{z} \partial_z \tilde{A}_\mu \right) + p^2 z^2 \tilde{A}_\mu - g_5^2 v^2 \tilde{A}_\mu \right] = 0,$$

that determines physics of the axial-vector mesons, like $A_1$. The axial-vector bulk-to-boundary propagator $\mathcal{A}(p, z)$ is introduced by the relation $\tilde{A}_\mu^a(p, z) = \mathcal{A}(p, z) A_{\mu}^a(p)$. It satisfies Eq. (4) with boundary conditions (b.c.) $\mathcal{A}(p, 0) = 1$ and $\mathcal{A}'(p, z_0) = 0$. Similarly, variation with respect to the longitudinal component $\partial_\mu \psi^a$ gives

$$z^3 \partial_z \left( \frac{1}{z} \partial_z \psi^a \right) - g_5^2 v^2 (\psi^a - \pi^a) = 0.$$  

Finally, varying with respect to $A_z$ produces

$$p^2 z^2 \partial_z \psi^a - g_5^2 v^2 \partial_z \pi^a = 0.$$  

The pion wave function is determined from Eqs. (5) and (6) with b.c. $\partial_z \psi(z_0) = 0$, $\psi(\epsilon) = 0$ and $\pi(\epsilon) = 0$.

Within the framework of the model of Ref. [6], it is possible to derive the Gell-Mann–Oakes–Renner relation $m_\pi^2 \sim m_q$ producing massless pion in the $m_q = 0$ limit. Taking $p^2 = m_\pi^2$ in Eq. (6) gives

$$\partial_z \pi = \frac{m_\pi^2 z^2}{g_5^2 v^2} \partial_z \psi.$$  

A perturbative solution in the form of $m_\pi^2$ expansion was proposed in Ref [6], with $\psi(z) = \mathcal{A}(0, z) - 1$ in the lowest order. Then it was shown that, in the $m_q \to 0$ limit, $\pi(z)$ tends to $-\theta(z - z_0)$ or, roughly speaking, $\pi = -1$ in this limit. Since our goal is to calculate the pion form factor in the chiral limit, this approximation will be sufficient for us.

**B. Two-Point Function**

The spectrum in the axial-current channel consists of the pseudoscalar pion $\langle 0 | J_A^a | \pi(p) \rangle = i f_\pi p^a$ and axial-vector mesons $\langle 0 | J_A^a | A_n(p, s) \rangle = F_{A,n} \epsilon_n^a(p, s)$, where $F_{A,n}$ correspond to the $n^{th}$ axial-vector meson decay constant (and we ignored the flavor indexes). Thus, the two-point function for the axial-vector currents has the form:

$$\langle J_A^\alpha(p) J_A^\beta(-p) \rangle = p^\alpha p^\beta \frac{f_\pi^2}{p^2} + \sum_n \Pi_n^{\alpha\beta}(p) \frac{F_{A,n}^2}{p^2 - M_{A,n}^2}.$$  

(8)
where the meson polarization tensor is given by
\[ \Pi_{\alpha\beta}^n(p) = \sum_\sigma \epsilon_\alpha^n(p,\sigma)\epsilon_\beta^n(p,\sigma) = -\eta^{\alpha\beta} + \frac{p^\alpha p^\beta}{M_{A,n}^2}. \] (9)

The representation for the two-point function can be also written as
\[ \langle J_\alpha^A(p)J_\beta^A(-p) \rangle = p^\alpha p^\beta \frac{f_{\pi}^2}{p^2} \left( -\eta^{\alpha\beta} + \frac{p^\alpha p^\beta}{p^2} \right) \sum_n \frac{F_{A,n}^2}{p^2 - M_{A,n}^2} + \text{(nonpole terms)}, \] (10)
in which the second term on the rhs is explicitly transverse to \( p \).

As noted in Ref. [6], using holographic correspondence one can relate the two-point function to
\[ \left[ \partial_z A(p,z)/z \right]_{z=0} \] and derive that
\[ f_{\pi}^2 = -\frac{1}{g_5^2} \left( \frac{1}{z} \partial_z A(0,z) \right)_{z=\epsilon\to0}. \] (11)

For large spacelike \( p^2 \), Eq. (4) gives the same solution as in case of vector mesons, and the same asymptotic logarithmic behavior, just as expected from QCD.

C. Pion Wave Functions

The longitudinal component of the axial-vector gauge field was defined as \( A_{\parallel} = \partial_\psi \). In the chiral limit, when \( p^2 = m_\pi^2 = 0 \), we have \( \partial_z \pi = 0 \), and the basic equation for \( \psi \), Eq. (5) can be rewritten as the equation
\[ z^3 \partial_z \left( \frac{1}{z} \partial_z \Psi \right) - g_5^2 v^2 \Psi = 0 \] (12)
for the function \( \Psi \equiv \psi - \pi \). In the chiral limit, when \( \pi(z) \to -1 \), the value of \( \Psi(\epsilon) \) tends to 1 as \( \epsilon \to 0 \). This value and the b.c. \( \Psi'(z_0) = 0 \) are the same as those for \( A(p,z) \) and, furthermore, Eq. (12) coincides with the \( p^2 = 0 \) version of equation (4) for \( A(p,z) \). Hence, the solution for \( \Psi(z) \) coincides with \( A(0,z) \):
\[ \Psi(z) = A(0,z), \] (13)
and we may write
\[ f_{\pi}^2 = -\frac{1}{g_5^2} \left( \frac{1}{z} \partial_z \Psi(z) \right)_{z=\epsilon\to0}. \] (14)

In our analysis of \( \rho \)-meson wave functions in Refs. [17, 18], we emphasized that it makes sense to consider also the conjugate functions \( \Phi(z) \sim \Psi'(z)/z \) of the corresponding Sturm-Liouville equation. As we observed, they are closer in their structure to the usual quantum mechanical
bound state wave functions than the solutions of the original equation. In the pion case, it is
convenient to define the Φ function as

$$\Phi(z) = -\frac{1}{g_5^2 f_\pi^2} \left( -\frac{1}{z} \partial_z \Psi(z) \right). \quad (15)$$

It vanishes at the IR boundary $z = z_0$ and, according to Eq. (11), is normalized as

$$\Phi(0) = 1 \quad (16)$$
at the origin. Note also that using Eq. (12) we can express Ψ as derivative of Φ:

$$\Psi(z) = -\frac{f_\pi^2 z^3 v}{u^2} \partial_z \Phi(z). \quad (17)$$

### III. EXTRACTING PION ELECTROMAGNETIC FORM FACTOR

#### A. Three-point function

To obtain the pion form factor, we need to consider three-point correlation functions. The
correlator should include the external EM current $J_\mu^el(0)$ and currents having nonzero projection
onto the pion states, e.g. the axial currents $J_{\alpha 5}(x_1), J_{\alpha 5}^\dagger(x_2)$

$$T_{\mu\alpha\beta}(p_1, p_2) = \int d^4x_1 \int d^4x_2 \ e^{ip_1 x_1 - ip_2 x_2} \langle 0 | T J_{\alpha 5}^\dagger(x_2) J_\mu^el(0) J_{\alpha 5}(x_1) | 0 \rangle, \quad (18)$$

where $p_1, p_2$ are the corresponding momenta, with the momentum transfer carried by the EM
source being $q = p_2 - p_1$ (as usual, we denote $q^2 = -Q^2, Q^2 > 0$). The spectral representation for
the three-point function is a two-dimensional generalization of Eq. (8)

$$T^{\mu\alpha\beta}(p_1, p_2) = p_1^\alpha p_2^\beta (p_1 + p_2)^\mu \frac{f_\pi^2 F_\pi(Q^2)}{p_1^2 p_2^2} + \sum_{n, m} (\text{transverse terms}) + (\text{nonpole terms}), \quad (19)$$

where the first term, longitudinal both with respect to $p_1^\alpha$ and $p_2^\beta$ contains the pion electromagnetic
form factor $F_\pi(Q^2)$

$$\langle \pi(p_1) | J_\mu^el(0) | \pi(p_2) \rangle = F_\pi(q^2) (p_1 + p_2)_\mu, \quad (20)$$
normalized by $F_\pi(0) = 1$, while other pole terms contain the contributions involving axial-vector
mesons and are transverse either with respect to $p_1^\alpha$ or $p_2^\beta$, or both. Hence, the pion form factor
can be extracted from the three-point function using

$$p_1^\alpha p_2^\beta T^{\mu\alpha\beta}(p_1, p_2)|_{p_1^2=0, p_2^2=0} = (p_1 + p_2)^\mu \frac{f_\pi^2}{F_\pi(Q^2)}. \quad (21)$$
B. Trilinear Terms in $F^2$ Part of Action

To obtain form factor from the holographic model, we need the action at the third order in the fields. There are two types of terms contributing to the pion electromagnetic form factor: $|DX|^2$ term and $F^2$ terms. Let us consider first the contribution from $F^2$ terms. They contain $VVV$, $VAA$ and $AVA$ interactions and may be written as

$$S_{\text{AdS}}^{F^2} = \frac{i}{g_5^2} \text{Tr} \int d^4x \, dz \, \frac{1}{z} \left( V_{MN}[V^M, V^N] + V_{MN}[A^M, A^N] + A_{MN}[V^M, A^N] \right) ,$$

(22)

where $V_{MN} = \partial_M V_N - \partial_N V_M$ and $A_{MN} = \partial_M A_N - \partial_N A_M$. Taking $V_z = A_z = 0$ gauge, we pick out the part of the action which is contributing to the 3-point function $\langle J^5_\alpha J^\mu J^5_\beta \rangle$:

$$W_3 = \frac{i}{g_5^2} \text{Tr} \int d^4x \, dz \, \frac{1}{z} \left( V_{\mu\nu}[A^\mu, A^\nu] + A_{\mu\nu}[V^\mu, A^\nu] \right) .$$

(23)

Introducing Fourier transforms of fields, we define, as usual, $V_\mu(q, z) = \tilde{V}_\mu(q)\mathcal{V}(q, z)$ for the vector field, where $\tilde{V}_\mu(q)$ is the Fourier transform of the 4-dimensional field $V_\mu(x)$ and $\mathcal{V}(q, z)$ is the bulk-to-boundary propagator satisfying the equation

$$z \partial_z \left( \frac{1}{z} \partial_z \mathcal{V}(q, z) \right) + q^2 \mathcal{V}(q, z) = 0$$

(24)

with b.c. $\mathcal{V}(q, 0) = 1$ and $\partial_z \mathcal{V}(q, z_0) = 0$. It can be written as the sum

$$\mathcal{V}(q, z) = g_5 \sum_{m=1}^{\infty} \frac{f_m \psi_m^V(z)}{-q^2 + M_m^2}$$

(25)

involving all the bound states in the $q$-channel, with $M_m$ being the mass of the $m$th bound state and $\psi_m^V(z)$ its wave function given by a solution of the basic equation of motion in the vector sector.

The projection (21) picks out only the longitudinal part $A_{||\mu}(p, z)$ of the axial-vector field. Taking into account that $A_{||\mu}(x, z) = \partial_\mu \psi(x, z)$, we may write

$$A_{||\mu}^a(p, z) = ip_\mu \psi^a(p, z) .$$

(26)

Furthermore, there is only one particle in the expansion over bound state in this case, namely, the massless pion. Thus, we have $A_{||\mu}^a(p, z) = A_{||\mu}^0(p) \psi(z)$ and, therefore,

$$\psi^a(p, z) = -\frac{ip^a}{p^2} \tilde{A}_{||\alpha}^a(p) \psi(z) .$$

(27)

This allows us to rewrite $A_{||\mu}^a(p, z)$ in the form

$$A_{||\mu}^a(p, z) = \frac{p^\mu p_\mu}{p^2} \tilde{A}_{||\alpha}^a(p) \psi(z) .$$

(28)
involving the longitudinal projector $p^a p_\mu / p^2$ and the pion wave function $\psi(z)$, which is the solution of the basic equation (5). Using this representation and making Fourier transform of $W_3$ gives

$$W_3 = -\frac{1}{2g_5^2} \epsilon_{abc} \int \frac{d^4u d^4v d^4w}{(2\pi)^{12}} \frac{i(2\pi)^4 \delta^{(4)}(u + v + w)}{u^2 v^2} \frac{u^\mu v^\nu u^\alpha v^\beta}{u v} \left( w_\mu \tilde{\nabla}_\nu (w) - w_\nu \tilde{\nabla}_\mu (w) \right) \int_\epsilon^{z_0} dz \frac{1}{z} \mathcal{V}(w, z) \psi^2(z).$$

(29)

Varying this functional with respect to sources produces the following 3-point function:

$$\langle J_{V,a}^\mu (q) J_{A,b}^\alpha (p_1) J_{A,c}^\beta (-p_2) \rangle = -i(2\pi)^4 \delta^{(4)}(q + p_1 - p_2) \epsilon_{abc} \frac{p_1^\alpha p_2^\beta}{p_1^2 p_2^2} (p_1 + p_2)^\mu \left( \frac{1}{2g_5^2} q^2 \int_\epsilon^{z_0} dz \frac{1}{z} \mathcal{V}(q, z) \psi^2(z) \right),$$

(30)

where, anticipating the limit $p_1^2 \to 0, p_2^2 \to 0$, we took $(p_1 q) = -(p_2 q) = -q^2/2$ in the numerator factors. Now, representing $\langle J_{V,a}^\mu (q) J_{A,b}^\alpha (p_1) J_{A,c}^\beta (-p_2) \rangle$ and applying the projection suggested by Eq. (21), we will have

$$\lim_{p_1^2 \to 0} \lim_{p_2^2 \to 0} p_{1\alpha} p_{2\beta} T^{\mu \alpha \beta} (p_1, p_2) = \frac{1}{2g_5^2} (p_1 + p_2)^\mu Q^2 J(Q),$$

(31)

where $J(Q)$ is the dynamic factor given by the convolution

$$J(Q) = \int_\epsilon^{z_0} dz \frac{1}{z} \mathcal{J}(Q, z) \psi^2(z).$$

(32)

C. Dynamic Factor and Wave Functions

The vector bulk-to-boundary propagator $\mathcal{J}(Q, z) \equiv \mathcal{V}(iQ, z)$ for spacelike momenta, entering into the dynamic factor $J(Q)$, satisfies the equation

$$z \partial_z \left( \frac{1}{z} \partial_z \mathcal{J}(Q, z) \right) = Q^2 \mathcal{J}(Q, z)$$

(33)

with b.c. $\mathcal{J}(Q, 0) = 1$ and $\partial_z \mathcal{J}(Q, z_0) = 0$. Its explicit form is given by

$$\mathcal{J}(Q, z) = Q^2 \left[ K_1(Qz) + I_1(Qz) \frac{K_0(Qz_0)}{I_0(Qz_0)} \right].$$

(34)

One can easily see that $\mathcal{J}(0, z) = 1$. Combining all the factors, we get

$$f_\pi^2 F_\pi^{(F^2)}(Q^2) = \frac{1}{2g_5^2} \frac{Q^2}{2} \int_0^{z_0} dz \frac{1}{z} \mathcal{J}(Q, z) \psi^2(z).$$

(35)

Integrating by parts and using equations of motion both for $\mathcal{J}$ and $\psi$ gives

$$F_\pi^{(F^2)}(Q^2) = \frac{1}{g_5^2 f_\pi^2} \int_0^{z_0} dz \frac{1}{z} \mathcal{J}(Q, z) \left[ \frac{(\partial_z \psi)^2}{z^2} + \frac{q_5^2 v^2}{z^4} \psi (\psi - \pi) \right].$$

(36)
We need also to add the $V\pi\pi$ contribution from the $|DX|^2$ term of the AdS action (2). It is generated by

$$S_{AdS}^{\pi\pi} = \epsilon_{abc} \int d^4x \, dz \left[ \frac{v^2(z)}{z^3} \left( A^a_M - \partial_M \pi^a \right) \pi^b \pi^c M \right],$$

(37)

and its inclusion changes $\psi(\psi - \pi)$ into $(\psi - \pi)^2$ in Eq. (36). The total result may be now conveniently expressed in terms of the $\Psi(z) \equiv \psi - \pi$ wave function

$$F_\pi(Q^2) = \frac{1}{g^2_f f^2_\pi} \int_0^{z_0} dz \, z \, J(Q,z) \left[ \left( \frac{\partial_z \Psi}{z} \right)^2 + \frac{g^2_5 v^2}{z^4} \Psi^2(z) \right].$$

(38)

Using equation of motion for $\Psi(z)$, one can see that the expression in square brackets coincides with

$$\frac{1}{z} \partial_z \left( \Psi(z) \frac{1}{z} \partial_z \Psi(z) \right) = -g^2_5 f^2_\pi \frac{1}{z} \partial_z \left( \Psi(z) \Phi(z) \right),$$

and write the form factor as

$$F_\pi(Q^2) = - \int_0^{z_0} dz \, J(Q,z) \partial_z \left( \Psi(z) \Phi(z) \right).$$

(39)

This representation allows one to easily check the normalization

$$F_\pi(0) = - \int_0^{z_0} dz \, \partial_z \left( \Psi(0) \Phi(z) \right) = \Psi(0) \Phi(0) = 1,$$

(40)

where we took into account that $J(0, z) = 1$ and $\Phi(z_0) = 0$. We can also represent our result for the pion form factor as

$$F_\pi(Q^2) = \int_0^{z_0} dz \, J(Q,z) \left[ g^2_5 f^2_\pi \Phi^2(z) + \frac{\sigma^2}{f^2_\pi} z^2 \Psi^2(z) \right] \equiv \int_0^{z_0} dz \, J(Q,z) \rho(z),$$

(41)

and interpret the function $\rho(z)$ as the radial distribution density, as it was done in Refs. [17, 18]. Note that keeping only the first term in square brackets gives an expression similar to our result [17] for the $\rho$-meson form factor

$$\mathcal{F}_{11}(Q^2) = \int_0^{z_0} dz \, J(Q,z) |\phi_1(z)|^2$$

(42)

in terms of the function $\phi_1$ conjugate to the solution of the basic equation of motion. The value of $\phi_1(z)$ at the origin is proportional to the $\rho$-meson decay constant $f_\rho/m_\rho \equiv g_\rho$ (experimentally, $g^\exp_\rho \approx 207$ MeV), namely, $\phi_1(0) = g_5 g_\rho$. Thus, the pion wave function $g_5 f_\pi \Phi(z) \equiv \phi_\pi(z)$ is a direct analog of the $\rho$-meson wave function $\phi_1(z)$. Main difference is that, in the pion case, there is also the second term in the form factor expression. The latter, in fact, is necessary to secure correct normalization of the form factor at $Q^2 = 0$. In Eq. (38), this term is written in terms of the $\Psi(z)$ wave function, but using Eq. (17) we can rewrite it also in terms of $\Phi(z)$ or $\phi_\pi(z)$:

$$\rho(z) = \phi^2_\pi(z) + \frac{1}{g^2_5 \sigma^2} \left( \frac{1}{z} \partial_z \phi_\pi(z) \right)^2.$$

(43)
IV. WAVE FUNCTIONS AND FORM FACTOR

A. Structure of Pion Wave Functions

Explicit form of the $\Psi$ wave function follows from the solution of Eq. (12):

$$
\Psi(z) = z \Gamma[2/3] \left( \frac{\alpha}{2} \right)^{1/3} \left[ I_{-1/3} (\alpha z^3) - I_{1/3} (\alpha z^3) \frac{I_{2/3} (\alpha z^3)}{I_{-2/3} (\alpha z^3)} \right],
$$

where $\alpha = g_5 \sigma / 3 \approx 1.481 \sigma$ (recall that $g_5 = \sqrt{2\pi}$, see e.g. Ref.[18]). As a result, $\Phi(z)$ is given by

$$
\Phi(z) = -\frac{1}{g_5^2 f_\pi^2} \left( \frac{1}{z} \partial_z \Psi(z) \right) = \frac{3 z^2}{g_5^2 f_\pi^2} \Gamma[2/3] \left( \frac{\alpha}{2} \right)^{1/3} \left[ -I_{2/3} (\alpha z^3) + \alpha I_{-2/3} (\alpha z^3) \frac{I_{2/3} (\alpha z^3)}{I_{-2/3} (\alpha z^3)} \right].
$$

This formula, combined with Eq. (16), establishes the relation

$$
f_\pi^2 = 3 \cdot 2^{1/3} \frac{\Gamma[2/3]}{\Gamma[1/3]} \frac{I_{2/3} (\alpha z_0^3)}{I_{-2/3} (\alpha z_0^3)} \frac{\alpha^{2/3}}{g_5^2}.
$$

(46)

for $f_\pi$ in terms of the condensate parameter $\alpha$ and the confinement radius $z_0$. Since $\sigma$ appears in the solutions only through $\alpha$, we will use $\alpha$ in what follows. Note also that $\alpha^{1/3} \approx 1.14 \cdot \alpha^{1/3}$.

Realizing that the equations of motion for the vector sector in this holographic model are not affected by the chiral symmetry-breaking effects expressed through the function $v(z)$, it is natural to set the value of $z_0$ from the vector sector spectrum, i.e., by the $\rho$-meson mass. The numerical value of $z_0$ (call it $z_0^0$) is then $z_0^0 \approx 1/323\text{MeV}$. As given by Eq. (46), $f_\pi$ looks like a rather complicated function of two scales, $z_0$ and $\alpha$. Note, however, that the ratio $I_{2/3}(a)/I_{-2/3}(a)$ is very close to 1 for $a \gtrsim 2$ and practically indistinguishable from 1 for $a \gtrsim 3$. Hence, for sufficiently large values of the confinement radius, $z_0 \gtrsim 1/\alpha^{1/3}$, the value of $f_\pi$ is determined by the value of $\alpha$ alone. This limiting value of $f_\pi$ is given by

$$
f_\pi|_{z_0 \to \infty} = 2^{1/6} \frac{\alpha^{1/3}}{g_5} \sqrt{\frac{3 \Gamma[2/3]}{\Gamma[1/3]}} = \frac{3^{1/2}}{2^{1/3} \pi} \frac{\Gamma[2/3]}{\Gamma[1/3]} \alpha^{1/3} \approx \frac{\alpha^{1/3}}{3.21}.
$$

(47)

Requiring that $f_\pi|_{z_0 \to \infty}$ coincides with the experimental value, $f_\pi \approx 131\text{MeV}$, one should take $\alpha^{1/3} \approx 420 \text{MeV}$. For such $\alpha$, the value of $1/\alpha^{1/3}$ is close to $z_0^0$, i.e., we are in the region $\alpha z_0^3 \sim 1$ and we may expect that, even if we use exact formula (46) with $z_0 = z_0^0$, the value of $f_\pi$ would not change much. Indeed, to get $f_\pi \approx 131\text{MeV}$ from Eq. (46) for $1/z_0 = 323\text{MeV}$, we should take $\alpha^{1/3} \approx 424\text{MeV} \equiv \alpha_0^{1/3}$. Thus, in this range of parameters, the value of $f_\pi$ is practically in one-to-one correspondence with the value of $\alpha$. It is convenient to introduce a dimensionless
variable
\[ a \equiv \alpha z_0^3 = \frac{1}{3} g_\sigma z_0^3. \] (48)

Then the values \( \alpha_0^{1/3} = 424 \text{ MeV} \) and \( 1/z_0^6 = 323 \text{ MeV} \) correspond to \( a = 2.26 \equiv a_0 \). As one can see from Fig.(1), the dependence of \( f_\pi \) is practically flat for \( a \gtrsim 2 \).

![Graph](image)

**FIG. 1:** Left: Pion decay constant \( f_\pi \) as a function of \( a \) for fixed \( \alpha^{1/3} = 424 \text{ MeV} \). Right: Function \( n(a) \)

The confinement radius \( z_0 \) presents a natural scale to measure length, so it makes sense to rewrite the form factor formula (38) as an integral over the dimensionless variable \( \zeta \equiv z/z_0 \):

\[
F_\pi(Q^2) = 3 \int_0^1 d\zeta \zeta \mathcal{J}(Q, \zeta, z_0) \left[ n(a) \varphi^2(\zeta, a) + \frac{a^2 \zeta^2}{n(a)} \psi^2(\zeta, a) \right] \equiv \int_0^1 d\zeta \zeta \mathcal{J}(Q, \zeta, z_0) \rho(\zeta, a),
\] (49)

where the mass scale \( \alpha \) is reflected by the dimensionless parameter \( a \). The factor \( n(a) \) takes care of the correct normalization of the form factor. It is given by

\[
n(a) = 2^{1/3} a^{2/3} \frac{\Gamma[2/3]}{\Gamma[1/3]} \frac{I_{2/3}(a)}{I_{-2/3}(a)}. \] (50)

For small \( a \), it may be approximated by \( \frac{3}{4} a^2 \). For large \( a \), using the fact that \( I_{2/3}(a)/I_{-2/3}(a) \) is very close to 1 for \( a \gtrsim 2 \), we may approximate \( n(a) \approx 0.637 a^{2/3} \) in this region. In terms of \( n(a) \), the pion decay constant can be written as

\[
f_\pi = \frac{1}{\pi a^{1/3}} \sqrt{\frac{3}{2} n(a)} \alpha^{1/3}. \] (51)

For large \( a \), this gives

\[
f_\pi \big|_{a \gtrsim 2} \approx 0.311 \alpha^{1/3}. \] (52)

For small \( a \), we have

\[
f_\pi \big|_{a \lesssim 1} = \frac{3 a^{2/3}}{2 \sqrt{2 \pi}} \alpha^{1/3} + \ldots \approx 0.338 \alpha z_0^2 = 0.338 \frac{a}{z_0}. \] (53)
The functions $\varphi(\zeta, a), \psi(\zeta, a)$ are just the $\Phi$ and $\Psi$ wave functions written in $\zeta$ and $a$ variables. For $a = 0$, the limiting forms are $\varphi(\zeta, 0) = 1 - \zeta^4$ and $\psi(\zeta, 0) = 1$. As $a$ increases, both functions become more and more narrow. For density, we have $\rho(z, a = 0) = 4z^2$ in the $a \to 0$ limit, a function that vanishes at the origin (see Fig.(3)). For nonzero $a$, the value of $\rho(z = 0, a)$ monotonically increases with $a$, and the function itself narrows. The increase of $\rho(\zeta = 0, a)$ with $a$ is generated by the monotonically increasing function $n(a)$. It is interesting to compare the pion density $\rho(\zeta, 2.26)$ (taken at the “experimental” value” $a = 2.26$) with the $\rho$-meson density $\rho_\rho(\zeta)$. These densities are rather close for $\zeta > 0.5$, but strongly differ for small $\zeta$. In particular, the $\rho$-meson density is more than two times larger for $\zeta = 0$, which corresponds to the hard-wall model result that $g_\rho$ is essentially larger than $f_\pi$.

**B. Pion Charge Radius**

It is interesting to investigate how well these values $z_0 = 1/323\text{MeV}$ and $\alpha = (424\text{MeV})^3$ describe another important low-energy characteristics of the pion – its charge radius. Using the
$Q^2$-expansion of the vector source [17]

$$J(Q, \zeta, z_0) = 1 - \frac{Q^2}{4} z_0^2 \zeta^2 \left[ 1 - 2 \ln \zeta \right] + \ldots$$

(54)

and explicit form of the density

$$\rho(\zeta, a) = \frac{3}{2} \Gamma(1/3) \Gamma(2/3) a^2 \zeta^4 \left[ \left( \frac{\nu(a) I_{-2/3}(a\zeta^3)}{I_{2/3}(a\zeta^3)} - \frac{I_{2/3}(a\zeta^3)}{I_{2/3}(a\zeta^3)} \right) + \left( \frac{I_{-1/3}(a\zeta^3)}{I_{1/3}(a\zeta^3)} - \nu(a) I_{1/3}(a\zeta^3) \right)^2 \right],$$

(55)

where $\nu(a) \equiv \sqrt{I_{2/3}(a)/I_{-2/3}(a)}$, we obtain for the pion charge radius:

$$\langle r_{\pi}^2 \rangle = \frac{3}{2} \frac{z_0^2}{z_0^2} \int_0^1 d\zeta \zeta^3 \left[ 1 - 2 \ln \zeta \right] \rho(\zeta, a) = \frac{4}{3} \frac{z_0^2}{z_0^2} \left\{ 1 - \frac{a^2}{4} + O(a^4) \right\}.$$

(56)

![FIG. 4: $\langle r_{\pi}^2 \rangle$ in fm$^2$ for $z_0 = z_0^\rho$ as a function of $a$.](image)

Hence, for fixed $z_0$ and small $a$, when $\alpha \ll 1/z_0^3$, the pion radius is basically determined by the confinement scale $z_0$. In particular, $\langle r_{\pi}^2 \rangle = \frac{4}{3} \frac{z_0^2}{z_0^2}$ for $\alpha = 0$. Numerically, taking $z_0 = z_0^\rho \approx 1/323$ MeV = 0.619 fm, we obtain $\langle r_{\pi}^2 \rangle = 0.51$ fm$^2$. This result is very close to the value $\langle r_{\rho}^2 \rangle_C \approx 0.53$ fm$^2$ that we obtained in the hard-wall model for the $\rho$-meson electric radius determined in [17] from the slope of the $G_C(Q^2)$ form factor. However, since $G_C(Q^2)$ involves kinematic-type terms $Q^2/m_{\rho}^2$, it seems more appropriate to compare $F_{\pi}(Q^2)$ with the $F_{11}(Q^2)$ form factor (42) given directly by a wave function overlap integral. The slope of $F_{11}(Q^2)$ is smaller than that of $G_C(Q^2)$, and the corresponding radius is also smaller: $\langle r_{\rho}^2 \rangle_F = 0.27$ fm$^2$. Thus, for $\alpha = 0$, the pion r.m.s. radius is about 1.4 times larger than the $\rho$-meson size determined by $\langle r_{\rho}^2 \rangle_F^{1/2}$.

With the increase of $\alpha$, the pion becomes smaller. The experimental value of 0.45 fm$^2$ [19] is reached for $a \sim 0.9$. However, the corresponding value $f_{\pi} \approx 80$ MeV is too small. If we take $a = a_0 = 2.26$, then $\langle r_{\pi}^2 \rangle = 0.34$ fm$^2$. Thus, if we insist on using $z_0 = z_0^\rho$ dictated by the hard-wall
model calculation of the \( \rho \)-meson mass, and the value of \( \alpha \) producing the experimental \( f_\pi \) (note that then \( \alpha^{-2/3} \approx 0.222 \text{ fm}^2 \)), the pion radius is smaller than the experimental value. In linear units, the difference, in fact, does not look very drastic: just 0.58 fm instead of 0.66 fm. Given that the hard-wall model for confinement is rather crude, the agreement may be considered as encouraging. Furthermore, one may expect that, in a more realistic softer model of confinement, the size of the pion will be larger. Such an expectation is supported by our soft-wall model calculation of the \( \rho \)-meson electric radius, for which we obtained \( \langle r_{\pi}^2 \rangle_C = 0.66 \text{ fm}^2 \) (0.40 fm\(^2\) for \( \langle r_{\pi}^2 \rangle_F \)), i.e., the result by 0.13 fm\(^2\) larger than in the hard-wall model. If \( \langle r_{\pi}^2 \rangle \) would increase by a similar amount, the result will be very close to the quoted experimental value.

To find \( \langle r_{\pi}^2 \rangle \) for large \( a \) (i.e., when \( \alpha > z_0^{-3} \) for fixed \( z_0 \), or when \( z_0 > \alpha^{-1/3} \) for fixed \( \alpha \)), we use first the observation that, in the region \( a > 2 \), we may approximate \( \nu(a) \approx 1 \). Then the factor in square brackets in Eq. (55) becomes a function of the combination \( a \zeta^3 \equiv \mu \) (call it \( R(\mu) \)), and we can write

\[
\langle r_{\pi}^2 \rangle \bigg|_{a \gtrsim 2} \approx \frac{3}{4} \Gamma(1/3) \Gamma(2/3) \left( \frac{1}{\alpha} \right)^{2/3} \int_0^a d\mu \mu^{5/3} R(\mu) \left[ 1 - \frac{2}{3} \ln \frac{\mu}{a} \right].
\] (57)

For \( a \gtrsim 2 \), the upper limit of integration in this expression may be safely substituted by infinity producing

\[
\int_0^\infty d\mu \mu^{5/3} R(\mu) \approx 0.289 , \quad \int_0^\infty d\mu \mu^{5/3} R(\mu) \ln \mu \approx -0.164 .
\] (58)

As a result, when the confinement radius \( z_0 \) is larger than the scale \( \alpha^{-1/3} \) set by the condensate parameter, the pion charge radius is primarily determined by \( \alpha^{-1/3} \), with a slow logarithmic increase of \( \langle r_{\pi}^2 \rangle \) as \( z_0 \) raises:

\[
\langle r_{\pi}^2 \rangle \bigg|_{a \gtrsim 2} \approx \left( \frac{1}{\alpha} \right)^{2/3} \left[ 1.511 + 0.524 \ln \left( \frac{\alpha z_0^3}{2.26} \right) \right].
\] (59)

Using Eq. (52), we can also write

\[
\langle r_{\pi}^2 \rangle \bigg|_{a \gtrsim 2} \approx \frac{3}{2\pi^2 f_\pi^2} \left[ 0.962 + 0.333 \ln \left( \frac{\alpha z_0^3}{2.26} \right) \right].
\] (60)

This formula is remarkably close to an old prediction [22] that the slope of the pion form factor at \( Q^2 = 0 \) is given by \( 1/4\pi^2 f_\pi^2 \) (numerically, though, this prediction, as we have seen, is not supported by recent experimental data).

### C. Form Factor at Large \( Q^2 \)

In the large-\( Q^2 \) limit, the source \( J(Q, z) \) is given by its free-field version \( zQK_1(Qz) \) that behaves asymptotically like \( e^{-Qz} \). As a result, only small values \( z \sim 1/Q \) are important in the form factor.
integral, and the large-$Q^2$ asymptotic behavior of the form factor is determined by the value of $ho(z)$ at the origin [2, 4, 18], namely,

$$F_\pi(Q^2) \to \frac{2 \rho(0)}{Q^2} = \frac{2 \phi_\pi^2(0)}{Q^2} = \frac{4\pi^2 f_\pi^2}{Q^2} \equiv \frac{s_0}{Q^2}. \quad (61)$$

It is worth mentioning that the combination $s_0 \approx 0.68 \text{GeV}^2$ is the basic scale in the local duality model [23], the “pion duality interval”.

The leading contribution comes entirely from the $\Phi^2$ term of the form factor integral (41) while the $\Psi^2$ term contribution behaves asymptotically like $1/Q^4$ since it is accompanied by extra $z^2$ factor. Note, however, that it is quite visible in the experimentally interesting region $Q^2 \lesssim 10 \text{GeV}^2$: it is responsible for more than 20% of the form factor value in this region (moreover, at $Q^2 = 0$, the $\Psi^2$ term contributes about 40% into the normalization of the form factor).

From a phenomenological point of view, different AdS/QCD-like models for the pion form factor differ in the shape of the density $\rho(\zeta)$ that they produce. If we require, that the density $\rho(z)$ equals $2\pi^2 f_\pi^2$ at the origin, the asymptotic behavior is $F_\pi(Q^2) \to s_0/Q^2$ in any such model. For $Q^2 = 0$, the form factor is normalized to one, so basically the models would differ in how they interpolate between these two limits. In particular, the simplest interpolation is provided by the monopole formula

$$F_\pi^{\text{mono}}(Q^2) = \frac{1}{1 + Q^2/s_0}, \quad (62)$$

while our hard-wall calculation gives a curve that goes above $F_\pi^{\text{mono}}(Q^2)$: the ratio $F_\pi(Q^2)/F_\pi^{\text{mono}}(Q^2)$ is larger than 1 for all $Q^2 > 0$, slowly approaching unity as $Q^2 \to \infty$.

In fact, a purely monopole form factor was obtained in our paper [18], where we studied the $\rho$-meson form factors in the soft-wall holographic model, in which confinement is generated by $\sim z^2$
oscillator-type potential. It was shown in [18] that the form factor integral

$$\mathcal{F}(Q^2, \kappa) = \int_0^\infty dz z \mathcal{J}^O(Q, z) |\Phi(z, \kappa)|^2,$$

in which $$\Phi(z, \kappa) = \sqrt{2} \kappa e^{-z^2\kappa^2/2}$$ is the lowest bound state wave function, and

$$\mathcal{J}^O(Q, z) = z^2 \kappa^2 \int_0^1 \frac{dx}{(1-x)^2} x^{Q^2/4\kappa^2} \exp \left[ -\frac{x}{1-x} z^2 \kappa^2 \right]$$

is the bulk-to-boundary propagator of this oscillator-type model, is exactly equal to $$1/(1 + Q^2/(4\kappa^2))$$. The magnitude of the oscillator scale $$\kappa$$ was fixed in our paper [18] by the value of the $$\rho$$-meson mass: $$\kappa = \kappa_\rho \equiv m_\rho/2$$. As a result, the form factor $$\mathcal{F}(Q^2, \kappa = m_\rho/2)$$ had the $$\rho$$-dominance behavior $$1/(1 + Q^2/m_\rho^2)$$. If we take $$\kappa = \kappa_\pi \equiv \pi f_\pi$$, the integral (63) gives $$1/(1 + Q^2/s_0)$$. The relevant wave function $$\Phi(z, \kappa_\pi)$$ has the expected correct normalization $$\Phi(0, \kappa_\pi) = \sqrt{2\pi} f_\pi$$, however, the slope $$1/s_0$$ of $$1/(1 + Q^2/s_0)$$ at $$Q^2 = 0$$ (corresponding to 0.35 fm$^2$ for the radius squared) is smaller than that of the experimental pion form factor. Furthermore, $$Q^2 F_\pi^{\text{mono}}(Q^2)$$ tends to $$s_0 \approx 0.68 \text{GeV}^2$$ for large $$Q^2$$, achieving values about 0.5 GeV$^2$ for $$Q^2 \sim 2 \text{GeV}^2$$, and thus exceeding by more than 25% the experimental JLab values [20] measured for $$Q^2 = 1.6$$ and 2.45 GeV$^2$. The authors of Ref. [21] proposed to use Eqs. (63),(64) as an AdS/QCD model for the pion form factor, with $$\kappa = 375 \text{MeV}$$ chosen so as to fit these high-$$Q^2$$ data. However, such a choice underestimates the value of $$f_\pi^2$$ by almost 30%. Our opinion is that the AdS/QCD models describe first the low-energy properties of hadrons, and one may use basic low-energy characteristics, such as $$m_\rho$$ and $$f_\pi$$, to fix the model parameters. On the other hand, if the form factor calculations based on these parameters disagree with the large-$$Q^2$$ data, this may be just an indication that one is using the model beyond its applicability limits. Furthermore, as we have seen in the hard-wall model, to correctly describe the pion one needs to include the chiral symmetry breaking effects, and there are no reasons to
expect that the pion density in a soft-wall model should have the same shape as the $\rho$-meson one.

Below, we give an example of a density $\rho^{\text{mod}}(z)$ that is normalized at the origin $\rho^{\text{mod}}(0) = 2\kappa_\pi^2$ by the experimental value of $f_\pi$, i.e., $\rho^{\text{mod}}(0) = 2\kappa_\pi^2$, but which reproduces, in addition, the value of the pion charge radius. The latter is a low-energy characteristics that is visibly underpredicted both by the hard-wall model calculation and the simplest interpolation formula $F^{\text{mono}}_\pi(Q^2)$.

Evidently, to increase the radius we should take a density which is larger for large $z$ than $\Phi^2(z, \kappa = \pi f_\pi)$. Since the overall integral normalization of the density is kept fixed, this can be achieved only by decreasing the density for small $z$ values. To give an example, consider a simple ansatz

$$\rho^{\text{mod}}(z) = 2\kappa_\pi^2 e^{-z^2\kappa_\pi^2} \left[ 1 - Az^2 \kappa_\rho^2 + Bz^4 \kappa_\rho^4 \right], \quad (65)$$

with $A = 1 - \kappa_\rho^2/\kappa_\pi^2 + 2B$. It has both the desired value for $z = 0$ and satisfies the normalization condition

$$\int_0^\infty dz \, z \rho^{\text{mod}}(z) = 1. \quad (66)$$

Integrating it with $J^0(Q, z)$ taken at $\kappa = \kappa_\rho$ produces the model form factor given by the following sum of contributions of the three lowest vector states:

$$F^{\text{mod}}_\pi(Q^2) = \frac{2 - (1 - 2B)s_0/m_\rho^2}{1 + Q^2/m_\rho^2} - \frac{1 - (1 - 4B)s_0/m_\rho^2}{1 + Q^2/2m_\rho^2} + \frac{2Bs_0/m_\rho^2}{1 + Q^2/3m_\rho^2}. \quad (67)$$

The slope of $F^{\text{mod}}_\pi(Q^2)$ at $Q^2 = 0$ is given by

$$\frac{dF^{\text{mod}}_\pi(Q^2)}{dQ^2} = -\frac{1}{m_\rho^2} \left[ \frac{3}{2} - \left( \frac{1}{2} - \frac{2}{3}B \right) \frac{s_0}{m_\rho^2} \right]. \quad (68)$$

Taking $B = 1/4$, one obtains the experimental value 0.45 fm$^2$ for $\langle r_\pi^2 \rangle$. It is interesting to note that the model density providing this value, has an enhancement for larger values of $z$, just like

![Graph](image-url)
the pion densities in the hard-wall model (see Fig.(3)). Due to larger slope, $F_{\pi}^{\text{mod}}(Q^2)$ decreases faster than the simple monopole interpolation $F_{\pi}^{\text{mono}}(Q^2)$ and, as a result, is in better agreement with the data. In fact, it goes very close to $Q^2 \lesssim 1 \text{ GeV}^2$ data, but exceeds the values of the JLab $Q^2 = 1.6$ and $2.45 \text{ GeV}^2$ points by roughly 10% and 20%, respectively.

This discrepancy has a general reason. The asymptotic AdS/QCD prediction is $Q^2 F_\pi(Q^2) \to 4\pi^2 f_\pi^2$. If one takes the experimental value of $f_\pi$, then $4\pi^2 f_\pi^2 \approx 0.68 \text{ GeV}^2$. On the other hand, JLab experimental points correspond to $Q^2 F_{\pi}^{\text{exp}}(Q^2) \approx 0.4 \text{ GeV}^2$, which is much smaller than the theoretical value quoted above. The pre-asymptotic effects, as we have seen, reduce the discrepancy, but there still remains a gap. As we already stated, such a disagreement may be just a signal that we are reaching a region where AdS/QCD models should not be expected to work. In particular, AdS/QCD describes the pion as a whole, in terms of an effective field or current, without specifying whether the current is built from spin-1/2 fields, or from scalar fields, etc. For $Q^2$ above $1 \text{ GeV}^2$, the quark substructure of the pion may be resolved by the electromagnetic probe (which is a wide-spread belief), and the description of the pion “as a whole” may be insufficient.

V. CONCLUSION

In this paper we study the pion in the chiral limit of two flavor QCD. For that we develop a formalism that allows to extract pion form factor in the framework of the holographic dual model of QCD with hard wall cutoff. We identify the pion with the longitudinal component of the axial-vector gauge field and define two “wave functions” which determine the charge density distribution of pion along the parametric coordinate $z$. It is essential that one of this wave functions is similar to the wave function defined for vector meson in [17], however the second wave function is what makes pion principally different from the vector meson in the holographic picture. In some sense the second wave function determines the deformation in radial charge distribution in the presence of chiral symmetry breaking. Besides, it is required for normalization at $Q^2 = 0$. These wave functions provide a very convenient framework to study the holographic physics of pion.

As a result we found the analytic expression for the pion decay constant in terms of two parameters of the model: $\sigma$ and $z_0$, similar to [8]. Analyzing the results, it is convenient to work with the combinations of these parameters such as: $\alpha = g_5 \sigma/3$ and $a = \alpha z_0^3$. The value of parameter $a$ corresponding to the experimental $\rho$ meson mass and pion decay constant is $a_0 = 2.26$. This parameter determined the regions of model where pion is either governed with confinement effects or with the effects from chiral symmetry breaking. For example, for $a > 2$ the pion’s decay con-
stant depends on $\sigma$ only. However, for $a < 1$ the decay constant is proportional to the ratio $a/z_0$. Besides, for small $a \ll 1$ the radius of the pion is given by $\langle r^2_\pi \rangle = \frac{4}{3} z_0^2$. However, for $a > 2$, the combination $f^2_\pi \langle r^2_\pi \rangle$ depends only on $\ln a/a_0$.

We also found that the electric radius of pion from this model is smaller than experimentally predicted. However, we claim that by “softening” the IR wall, in the same way as we did in [18], one can increase the size of the pion by sufficient amount. However, notice, that this idea can’t be directly applied for model [9], since this model, as described in their paper, doesn’t have the appropriate chiral symmetry breaking pattern. This is because in the chiral limit the condensate vanishes with the quark mass. What we did to soften the IR is choose an ansatz for pion density function and used the vector current source from the model with linear confinement [18]. As a result we achieved some improvements.

Finishing the write-up of this paper, we have learned that the paper [24] studying the similar problem was posted into the arxive. We did not observe there any essential overlaps with our ideas and results.

VI. ACKNOWLEDGMENTS

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