ON A CLASS OF ONE-SIDED MARKOV SHIFTS

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Abstract. We study one-sided Markov shifts, corresponding to positively recurrent Markov chains with countable (finite or infinite) state spaces. The following classification problem is considered: when two one-sided Markov shifts are isomorphic up to a measure preserving isomorphism? In this paper we solve the problem for the class of \( \rho \)-uniform (or finitely \( \rho \)-Bernoulli) one-sided Markov shifts considered in [Ru].

We show that every ergodic \( \rho \)-uniform Markov shift \( T \) can be represented in a canonical form \( T = T_\Phi \) by means of a canonical (uniquely determined by \( T \)) stochastic graph \( \Phi \). In the canonical form, two such shifts \( T_\Phi \) and \( T_{\Phi'} \) are isomorphic if and only if their canonical stochastic graphs \( \Phi \) and \( \Phi' \) are isomorphic.

1. Introduction

In this paper we consider the classification problem for one-sided Markov shifts with respect to measure preserving isomorphism. Let \( G \) be a finite or countable stochastic graph, i.e. a directed graph, whose edges \( g \in G \) are equipped with positive weights \( p(g) \). The weights \( p(g) \) determine transition probabilities of a Markov chain on the discrete state space \( G \). The corresponding one-sided Markov shift \( T_G \) acts on the space \( (X_G, m_G) \), where \( X_G = \mathbb{Z}^G \) and \( m_G \) is a stationary (probability) Markov measure on \( X_G \). We deal only with irreducible positively recurrent Markov chains, so that such a Markov measure exists and the shift \( T_G \) is an ergodic endomorphism of the Lebesgue space \( (X_G, m_G) \). The problem under consideration is: When for given two stochastic graphs \( G_1 \) and \( G_2 \), does there exist an isomorphism \( \Phi : X_{G_1} \rightarrow X_{G_2} \) such that \( m_{G_2} = m_{G_1} \circ \Phi^{-1} \) and \( \Phi \circ T_{G_1} = T_{G_2} \circ \Phi \)?

It is obvious, that any (weight preserving) graph isomorphism \( \phi : G_1 \rightarrow G_2 \) generates such an isomorphism \( \Phi = \Phi_\phi \), but nonisomorphic graphs can generate the same shift \( T_G \).

Recently J. Ashley, B. Marcus and S. Tuncel [AsMaTu] solved the classification problem for one-sided Markov shifts corresponding to finite Markov chains. They used an approach which is based on the following important fact: Two one-sided Markov shifts \( T_{G_1} \) and \( T_{G_2} \) (on finite state spaces) are isomorphic iff there exists a common extension \( G \) of \( G_1 \) and \( G_2 \) by right resolving graph homomorphisms of degree 1. The result was proved implicitly in [BoTu], and regular isomorphisms and right closing maps for two-sided Markov shifts were studied (See also [As], [KiMaTr], [Tu], [Ki] and references cited there)

It should be noted that the classification problem for two-sided shifts is quite different from the one-sided case. Namely, any mixing two-sided Markov shift is isomorphic to the Bernoulli shift with the same entropy \([FrOr]\) and two-sided Bernoulli shifts are isomorphic iff they have the same entropy by the Sinai-Ornstein theorem \([Si]-[Or]\).

On the other hand, let \( T_\rho \) be the one-sided Bernoulli shift with a discrete state space \( (I, \rho) \), where \( I \) is a finite or countable set, \( 1 < |I| \leq \infty \), and \( \rho = \{ \rho_i \}_{i \in I} \), \( \sum \rho_i = 1 \), \( \rho_i > 0 \). The endomorphism \( T_\rho \) acts as the one-sided shift on the product space \( (X_\rho, m_\rho) = \prod_{n=1}^\infty (I, \rho) \). Consider the measurable partition \( T_\rho^{-1} = \{ T_\rho^{-1} x , x \in X_\rho \} \) generated by \( T_\rho \) on \( X_\rho \). The partition admits an independent complement \( \delta \), which is unique in general,
but necessarily has the distribution $\rho$. This implies that one-sided Bernoulli shifts $T_\rho_1$ and $T_\rho_2$ are isomorphic iff the distributions $\rho_1$ and $\rho_2$ coincide.

This simple observation motivates the following definition. An endomorphism $T$ of a Lebesgue space $(X, m)$ is called $\rho$-uniform (or finitely $\rho$-Bernoulli according to [Ru$_6$]) if the measurable partition $T^{-n}\varepsilon = \{T^{-1}x, \ x \in X\}$ admits an independent complement $\delta$ with distr $\delta = \rho$. We denote by $\mathcal{UE}(\rho)$ the class of all $\rho$-uniform endomorphisms.

Recall that the cofiltration $\xi(T)$ generated by an endomorphism $T$ is the decreasing sequence $\{\xi_n\}_{n=1}^{\infty}$ of the measurable partitions $\xi_n = T^{-n}\varepsilon$ of the space $X$ onto inverse images $T^{-n}x$. If two endomorphisms $T_1$ and $T_2$ are isomorphic, i.e. there exists an isomorphism $\Phi$ such that $\Phi \circ T_1 = T_2 \circ \Phi$, then $\Phi(T_1^{-n}x) = T_2^{-n}(\Phi x)$ for almost all $x \in X$, i.e. $\Phi(T_1^{-n}\varepsilon) = T_2^{-n}\varepsilon$ for all $n$. This means that the cofiltrations $\xi(T_1)$ and $\xi(T_2)$ are isomorphic.

If $T \in \mathcal{UE}(\rho)$, the cofiltration $\xi(T)$ is not necessarily isomorphic to the standard cofiltration $\xi(T_x)$, generated by the Bernoulli shift $T_x$. However, it is finitely isomorphic to $\xi(T_x)$, i.e. for every $n \in \mathbb{N}$ there exists an isomorphism $\Phi_n$ such that $\Phi_n(T^{-k}\varepsilon) = T_x^{-k}\varepsilon$ for all $1 \leq k \leq n$.

The isomorphism problem for $\rho$-uniform endomorphisms is decomposed into the following two parts: When are the cofiltrations $\xi(T_1)$ and $\xi(T_2)$ isomorphic? When are $T_1$ and $T_2$ isomorphic provided that $\xi(T_1) = \xi(T_2)$?

In particular, for given $T \in \mathcal{UE}(\rho)$: When is the cofiltration $\xi(T)$ standard, i.e. isomorphic to $\xi(T_x)$? When are $T_1$ and $T_2$ isomorphic provided that $\xi(T_1) = \xi(T_2)$?

All these problems are quite nontrivial even in the dyadic case $\rho = (\frac{1}{2}, \frac{1}{2})$. Various classes of decreasing sequences of measurable partitions were considered by A.M. Vershik [Ve$_1$, Ve$_2$, V.G. Vinokurov [V$_1$, A.M. Stepin [St] and by author [Ru$_1$, Ru$_2$, Ru$_4$, Ru$_6$]. A new remarkable progress in the theory is due to J. Feldman, D.J. Rudolph, D. Heicklen and Ch. Hoffman (See [FeR], [HeHo], [HeHoR], [Ho], [HoR]). Note also that, as it was shown in [Ru$_6$] Corollary 4.4, a $\rho$-uniform one-sided Markov shift $T_G$ is isomorphic to the Bernoulli shift $T_{\rho}$ iff the cofiltration $\xi(T_G)$ is isomorphic to standard cofiltration $\xi(T_{\rho})$.

The purpose of this paper is to classify the $\rho$-uniform one-sided Markov shifts. We show that every ergodic $\rho$-uniform Markov shift $T$ can be represented in a canonical form $T = T_G$ by means of a canonical (uniquely determined by $T$) stochastic graph $G$. In the canonical form, two such shifts $T_{G_1}$ and $T_{G_2}$ are isomorphic if and only if their canonical stochastic graphs $G_1$ and $G_2$ are isomorphic.

First we consider (Section 2) general $\rho$-uniform endomorphisms and use the following results from [Ru$_6$]. Any ergodic $T \in \mathcal{UE}(\rho)$ can be represented as a skew product over $T_\rho$ on the space $X_\rho \times Y_d$, $d \in \mathbb{N} \cup \{\infty\}$, where $Y_d$ consists of $d$ atoms of equal measure $\frac{1}{d}$ for $d < \infty$ and $Y_\infty$ is a Lebesgue space with no atoms, (see Section 2.2 below). According to [Ru$_6$] we introduce the minimal index $d(T)$ of $T \in \mathcal{UE}(\rho)$ as the minimal possible $d$ in the above skew product representation of $T$. The index $d(T)$ is an invariant of the endomorphism $T$ and $d(T) = 1$ iff $T$ is isomorphic to the Bernoulli shift $T_\rho$.

Other important invariants of $T \in \mathcal{UE}(\rho)$ (introduced also in [Ru$_6$]) are the partitions $\gamma(T)$, $\beta(T)$ and the index $d_{\gamma, \beta}(T)$. The partition $\gamma(T)$ is the smallest (i.e. having the most coarsely measurable) measurable partition of $X$ such that almost all elements of the partition $\beta_n := \gamma(T) \vee T^{-n}\varepsilon$ have homogeneous conditional measures for all $n$. The corresponding tail partition is defined by $\beta(T) = \bigwedge_{n=1}^{\infty} \beta_n \geq \gamma(T)$, and the index $d_{\gamma, \beta}(T)$ is the number of elements of $\beta(T)$ in typical elements of the partition $\gamma(T)$ (Proposition 2.4).

It was proved in [Ru$_6$] that $d(T) = d_{\gamma, \beta}(T) < \infty$ for any $\rho$-uniform one-sided Markov shift $T = T_G$. This result implies, in particular, that $T_G$ is simple in the sense of Definition
The classification of general simple $\rho$-uniform endomorphisms is reduced to a description of equivalent $d$-extensions of the Bernoulli shift $T_{\rho}$ (Theorem 2.10).

Next we turn to $\rho$-uniform one-sided Markov shifts.

It is easy to see that a Markov shift $T_G$ is $\rho$-uniform iff the graph $G$ satisfies the following condition: For any vertex $u$ the set $G_u$ of all edges starting in $u$, equipped with the corresponding weights $p(g), g \in G_u$, is isomorphic to $(I, \rho)$. This means that the transition probabilities of the Markov chain (starting from any fixed state) coincide with $\rho(i), i \in I$, up to a permutation. We call these graphs and Markov chains $\rho$-uniform. In particular, $(I, \rho)$ itself is considered as a $\rho$-uniform graph having a single vertex. The corresponding Markov shift is the Bernoulli shift $T_\rho$.

Following [AsMaTu] we use in the sequel graph homomorphisms of the form $\phi : G_1 \to G_2$, which are assumed to be weight preserving and deterministic, i.e. right resolving in the terminology of [AsMaTu], (see Definition 3.3 for details). Thus a stochastic graph $G$ is $\rho$-uniform iff there exists a homomorphisms $\phi : G \to I$.

Two particular kinds of homomorphisms are of special interest in our explanation, they are homomorphisms of degree 1 and $d$-extensions. A homomorphism $\phi : G_1 \to G_2$ has degree 1, $d(\phi) = 1$, if the corresponding factor map $\Phi_\phi : X_{G_1} \to X_{G_2}$ is an isomorphism. So that $\Phi_\phi \circ T_{G_1} = T_{G_2} \circ \Phi_\phi$, i.e. $T_{G_1}$ and $T_{G_2}$ are isomorphic.

The $d$-extensions homomorphism are defined in Section 3.2 by the condition: $|\phi^{-1}(g)| = d, g \in G$. They can be described (up to equivalence) by the graph skew products, (see Example 3.7 and Definition 3.8 in Section 3.2).

As the first step to the construction of the canonical graph we show (Theorem 3.24) that any homomorphism $\phi : G \to I$ can be extended to a $d$-extension $\hat{\phi}$ by homomorphisms of degree 1 (See Diagram 3.39). To this end we consider a $d$-contractive semigroup $S(\phi)$, associated with the homomorphism $\phi$, and the corresponding persistent sets (Section 4.1). Thus we reduce the classification problem to the study of diagrams of the form

\[
(\pi, \psi) : \hat{H} \xrightarrow{\pi} H \xrightarrow{\psi} I
\]

where $\hat{H}$ is a $d$-extension, $\psi$ is a degree 1 homomorphism and the shift $T_{\hat{H}}$ is isomorphic to the shift $T_G$.

The second step is to minimize $d$ in the above Diagram 1.1. We show (Theorem 3.25) that, passing possibibly to a "n-stringing" graph $G^{(n)}$, one can choose the minimal $d = d(T)$. Note that the result is based on [Ru] Theorem 4.2 and 4.3.

The third final step is to reduce the homomorphism $\psi$ in Diagram 1.1 as much as possible. Let $\text{Ext}^d(I, \rho)$ denotes the set of all $d$-extensions of the Bernoulli graph $(I, \rho)$ of the form (1.1). We show that $\text{Ext}^d(I, \rho)$ can be equipped with a natural partial order $\preceq$ and equivalence relation $\sim$ (Definition 4.10). The minimal elements of $\text{Ext}^d(I, \rho)$ with respect to the order are called irreducible (Definition 4.11). We describe these irreducible $(\pi, \psi)$-extensions by means of the persistent $d$-partitions, associated with elements of $\text{Ext}^d(I, \rho)$ (Theorem 4.13).

Now we can formulate the main result of the paper (Theorems 5.1 and 5.3).

- Let $T_G$ be a $\rho$-uniform ergodic one-sided Markov shift. A stochastic graph $\hat{H} = \hat{H}(G)$ is said to be a canonical graph for the shift $T$ if there exists an irreducible $(\pi, \psi)$-extension (1.1) from $\text{Ext}^d(I, \rho)$ with $d = d(T)$ such that the shift $T_{\hat{H}}$ is isomorphic to $T_G$.

- Any $\rho$-uniform ergodic one-sided Markov shift can be represented in the canonical form $T = T_{\hat{H}}$ by a canonic graph $\hat{H} = \hat{H}(G)$. 
In this canonical form, two shifts $T_{H_1}$ and $T_{H_2}$ are isomorphic iff the canonical graphs $H_1$ and $H_2$ are isomorphic, and iff the corresponding irreducible $(\pi, \psi)$-extensions are equivalent.

The paper is organized as follows. In Section 2 we study general $\rho$-uniform endomorphisms (class $UE(\rho)$) and simple $\rho$-uniform endomorphisms (subclass $SUE(\rho)$). Following [Ru], we introduce the partitions $\gamma(T), \beta(T)$ and the index $d_{\gamma, \beta}(T)$. Two main conclusions of the section are Theorem 2.10 (classification of simple $\rho$-uniform endomorphisms) and Theorem 2.14 which states that every ergodic $\rho$-uniform one-sided Markov shift $T_G$ is simple and $d(T_G) = d_{\gamma, \beta}(T_G) < \infty$. In Section 3 we consider general properties of stochastic graphs and their homomorphisms. In particular, we define $\rho$-uniform graphs corresponding to $\rho$-uniform Markov shifts. We prove that the index $d(T_G)$ of any ergodic $\rho$-uniform Markov shift $T_G$ is finite (Theorem 3.18). This follows from the finiteness of the degree $d(\phi)$ of any homomorphism $\phi: G \to I$ from any $\rho$-uniform graph $G$ onto the standard Bernoulli graph $(I, \rho)$. The degree $d(\phi)$, in turn, can be computed by means of a special $d$-contractive semigroup $S(\phi)$, induced by $\phi$ (Theorem 3.21).

Section 4 contains some essential stages of the proof of Main Theorems 5.1 and 5.3. Homomorphisms of degree 1 and extensions of the Bernoulli graph are considered in Sections 4.1 and 4.2. Theorem 4.5 (Section 4.3) reduces the classification of skew product over Markov shifts $T_H$ to the classification of the corresponding graph skew product over $H$. In Sections 4.4 and 4.5 we study the set $Ext_d(I, \rho)$ of all $(\pi, \psi)$-pairs of the form (1.1). The main result of Section 4 is Theorem 4.12 which claims the existence and uniqueness of the irreducible $(\pi, \psi)$-pair $(\pi_*, \psi_*)$, majorized by a given $(\pi, \psi) \in Ext_d(I, \rho)$.

In Section 5 we prove Main Theorems 5.1 and 5.3 and give some consequences and examples. As a consequence we prove also (Theorem 5.4) that two shifts $T_{G_1}$ and $T_{G_2}$ are isomorphic iff the graphs $G_1$ and $G_2$ have a common extension of degree 1.

We do not study here the classification problem for general, not necessarily $\rho$-uniform, one-sided Markov shifts as well as the classification problem of the cofiltrations, generated by the shifts. Our approach seems to be a good tool to this end and we hope to deal with these two problems in another paper.

We do not also consider the classification problem of one-sided Markov shifts with infinite invariant measure, in particular, of null-recurrent one-sided Markov shifts. One can find a good introduction to the topic and more references in [Aar, Chapters 4 and 5].

2. Class of $\rho$-uniform endomorphisms

2.1. Lebesgue spaces and their measurable partitions. We use terminology and results of the Rokhlin’s theory of Lebesgue spaces and their measurable partitions (See [Rok], [Rok]). An improved and more detailed explanation can be found in [ViRuFe]. We fix the terms ”homomorphism , isomorphism, endomorphism” only for measure preserving maps of Lebesgue spaces.

Let $(X, \mathcal{F}, m)$ be a Lebesgue space with $mX = 1$. The space $X$ is called homogeneous if it is non-atomic or if it consists of $d$ points of measure $\frac{1}{d}$, $d \in \mathbb{N}$.

Let $\zeta$ be a partition of $X$ onto mutually disjoint sets $C \in \zeta$. The element of $\zeta$ containing a point $x$ is denoted by $C_\zeta(x)$. The partition $\zeta$ is measurable iff there exists a measurable
function \( f : X \to \mathbb{R} \) such that
\[
x \sim y \iff C_\zeta(x) = C_\zeta(y) \iff f(x) = f(y) , \ x, y \in X
\]

Elements of \( \zeta \) are considered as Lebesgue spaces \((C, \mathcal{F}^C, m^C)\), \( C \in \zeta \), with canonical system of conditional measures \( m^C \), \( C \in \zeta \). We shall denote also by \( m(A|C) \) the conditional measures \( m^C(A \cap C) \) of a measurable set \( A \in \mathcal{F} \) in the element \( C \) of \( \zeta \).

Two measurable partitions \( \zeta_1 \) and \( \zeta_2 \) are said to be independent \((\zeta_1 \perp \zeta_2)\) if the corresponding \( \sigma \)-algebras \( \mathcal{F}(\zeta_1) \) and \( \mathcal{F}(\zeta_2) \) are independent, where \( \mathcal{F}(\zeta) \) denotes the \( m \)-completion of the \( \sigma \)-algebra of all measurable \( \zeta \)-sets. We shall write also \( \zeta_1 \perp \zeta_2 \) (mod \( \zeta \)) if the partitions \( \zeta_1 \) and \( \zeta_2 \) are conditionally independent with respect to the third measurable partition \( \zeta \). This means that
\[
m(A \cap B \mid C_\zeta(x)) = m(A\mid C_\zeta(x)) \cdot m(B \mid C_\zeta(x))
\]
for all \( A \in \mathcal{F}(\zeta_1) \), \( B \in \mathcal{F}(\zeta_2) \) and a.a. \( x \in X \).

We denote by \( \varepsilon = \varepsilon_X \) the partition of \( X \) onto separate points and by \( \nu = \nu_X \) the trivial partition of \( X \).

An independent complement of \( \delta \) is a measurable partition \( \eta \) such that \( \zeta \perp \eta \) and \( \zeta \cup \eta = \varepsilon \). The partition \( \zeta \) admits an independent complement iff almost all elements \((C, m^C)\) of \( \zeta \) are mutually isomorphic. The collection of all independent complements of \( \zeta \) is denoted by \( IC(\zeta) \).

We shall use induced endomorphisms, which are defined as follows. Let \( A \in \mathcal{F} \), \( mA > 0 \) and \( T \) be an endomorphism of \((X, m)\). Then the return function
\[
(2.2) \quad \varphi_A(x) := \min \{ n \geq 1 : T^n x \in A \} , \ x \in A
\]
is finite a.e. on \( A \). The induced endomorphism \( T_A \) on \( A \) is defined now by \( T_A x = T^{\varphi_A(x)} x \). It is an endomorphism of \((A, \mathcal{F} \cap A, m\mid_A)\) and it is ergodic if \( T \) is ergodic.

\( |E| \) denotes the cardinality of the set \( E \).

2.2. Classes \( UE(\rho) \) and index \( d(T) \). Let \((I, \rho)\) be a finite or countable state space
\[
\rho = \{ \rho(i) , \ i \in I \} , \ \rho(i) > 0 , \ \sum_{i \in I} \rho(i) = 1.
\]

**Definition 2.1.** An endomorphism \( T \) of a Lebesgue space \((X, m)\) is said to be \( \rho \)-uniform or finitely \( \rho \)-Bernoulli endomorphism \((T \in UE(\rho))\), if there exists a discrete measurable partition \( \delta \) of \( X \), which satisfies the following condition:

(i) \( distr \ \delta = \rho \), i.e. \( \delta = \{ B(i) \}_{i \in I} \) with \( m(B(i)) = \rho(i), \ i \in I \),

(ii) \( \delta \in IC(T^{-1}\varepsilon) \), i.e. \( \delta \perp T^{-1}\varepsilon \) and \( \delta \cup T^{-1}\varepsilon = \varepsilon \).

So \( UE(\rho) \) denotes the class of all \( \rho \)-uniform endomorphisms. Denote by \( \Delta_\rho(T) \) the set of all partitions \( \delta \) satisfying the condition (i) and (ii). Then \( T \in UE(\rho) \) means \( \Delta_\rho(T) \neq \emptyset \).

For \( T \in UE(\rho) \) and \( \delta \in \Delta_\rho(T) \) define
\[
(2.3) \quad \delta^{(n)} = T^{-n+1}\delta , \ \delta^{(n)} = \{ T^{-n+1}B(i) \}_{i \in I} , \ n \in \mathbb{N}
\]
Then \( distr \ \delta_n = \rho \) and the partitions \( \delta_1 , \delta_2 , \delta_3 , \ldots \) are independent.

The partitions
\[
(2.4) \quad \delta^{(n)} = \bigvee_{k=1}^{n} \delta_k , \ \delta^{(\infty)} = \bigvee_{k=1}^{\infty} \delta_k
\]
satisfy for all $n$ the conditions
\[ \delta^{(n)} \in IC(T^{-n}\varepsilon) \, , \, \delta^{(\infty)} \perp T^{-n}\varepsilon \pmod{\delta^{(\infty)} \wedge T^{-n}\varepsilon} \]
and
\[ \delta^{(\infty)} \lor T^{-n}\varepsilon = \varepsilon \, , \, \delta^{(\infty)} \land T^{-n}\varepsilon = T^{-n}\delta^{(\infty)}. \]
In particular, let $T = T_\rho$ be a Bernoulli endomorphism, which acts on the space
\[ (X_\rho, m_\rho) = \prod_{n=1}^{\infty} (I, \rho) \]
as the one-sided shift
\[ T_\rho x = \{x_{n+1}\}_{n=1}^{\infty} \, , \, x = \{x_n\}_{n=1}^{\infty} \in X_\rho. \]
We can set
\[ \delta_\rho = \{B_\rho(i)\}_{i \in I} \, , \, B_\rho(i) = \{x = \{x_n\}_{n=1}^{\infty} \in X_\rho : x_1 = i\}. \]
Then $\delta_\rho \in \Delta_\rho(T_\rho)$ and $\delta_\rho$ is an one-sided Bernoulli generator of $T_\rho$, that is
\[ \delta_\rho^{(\infty)} = \bigvee_{n=1}^{\infty} T^{-n+1}\delta_\rho = \varepsilon X_\rho. \]

In general case, for $T \in \mathcal{UE}(\rho)$ and $\delta \in \Delta_\rho(T)$, the partition $\delta^{(\infty)}$ does not equal $\varepsilon$, but we can define the canonical factor map
\[ \Phi_\delta : X \ni x \to \Phi_\delta(x) = \{i_n(x)\}_{n=1}^{\infty} \in X_\rho, \]
where $i_n(x) \in I$ is uniquely defined by the inclusion $T^n x \in B(i_n(x)) \in \delta$.

The homomorphism $\Phi_\delta$ satisfies $\Phi_\delta \circ T = T_\rho \circ \Phi_\delta$, and it determines the following representation of $T$ by a skew product over $T_\rho$ (see [Rub Proposition 2.2]).

**Proposition 2.2.** Let $T \in \mathcal{UE}(\rho)$ be an endomorphism of $(X, m)$ and $\delta \in \Delta_\rho(T)$. Then

(i) There exists an independent complement $\sigma$ of the partition $\delta^{(\infty)}$.

(ii) The pair $(\delta^{(\infty)}, \sigma)$ induces decomposition of the space $(X, m)$ into the direct product $(X_\rho \times Y, m_{X_\rho} \times m_Y)$ such that the factor map $\Phi_\delta$ coincides under the decomposition with the canonical projection
\[ \pi : X_\rho \times Y \ni (x, y) \to x \in X_\rho \]
and
\[ \delta = \pi^{-1}\delta_\rho \, , \, \delta^{(\infty)} = \pi^{-1}\varepsilon_{X_\rho} \times \varepsilon_Y \, , \, \sigma = \nu_{X_\rho} \times \varepsilon_Y \]

(iii) The endomorphism $T$ is identified with the following skew product over $T_\rho$
\[ T(x, y) = (T_\rho x, A(x)y), \quad (x, y) \in X_\rho \times Y \]
where $\{A(x), x \in X_\rho\}$ is a measurable family of automorphisms of $Y$.

(iv) If $T$ is ergodic, $Y$ is a homogeneous Lebesgue space.

Every homogeneous Lebesgue space $Y$ is isomorphic to $Y_d \cdot d \in \mathbb{N} \cup \{\infty\}$, where $Y_\infty$ is the Lebesgue space with a continuous measure and $Y_d$, $d \in \mathbb{N}$, consists of $d$ points of measure $\frac{1}{d}$. Thus for any ergodic $T$ endomorphism $T \in \mathcal{UE}(\rho)$ and $\delta \in \Delta_\rho(T)$ there exists $d = d(T, \delta) \in \mathbb{N} \cup \{\infty\}$ such that
\[ u_{\delta^{(\infty)}}(x) := m^{C_{\delta^{(\infty)}}}(\{x\}) = \frac{1}{d} \]
for a.a. $x \in X$.  

Definition 2.3.  
(i) The number \( d(T, \delta) \) will be called the **index** of \( T \in \mathcal{UE}(\rho) \) with respect to \( \delta \in \Delta_\rho(T) \).

(ii) The **minimal index** \( d(T) \) of \( T \) is defined as
\[
(2.7) \quad d(T) = \min \{ d(T, \delta) : \delta \in \Delta_\rho(T) \}
\]
Note that an ergodic endomorphism \( T \) is isomorphic to the Bernoulli shift \( T_\rho \) iff \( T \in \mathcal{UE}(\rho) \), and \( d(T) = 1 \), that is, there exists \( \delta \in \Delta_\rho(T) \) such that \( d(T, \delta) = 1 \), i.e. \( \delta^{(\infty)} = \varepsilon \).

2.3. Partitions \( \alpha(T), \beta(T), \gamma(T) \) and indices \( d_\alpha(T), d_{\gamma;\beta}(T) \). Let \( T \) be an endomorphism of \((X, m)\) and let \( \{\xi_n\}_{n=1}^{\infty} \) be the decreasing sequence of measurable partitions \( \xi_n := T^{-n} \varepsilon \), generated by \( T \). The element of \( \xi_n \), containing a point \( x \in X \), has the form
\[
C_{\xi_n}(x) = T^{-n} T^m(x),
\]
In order to introduce the partitions \( \gamma(T) \) and \( \beta(T) \), consider the measurable functions
\[
u_n(x) = m_{C_{\xi_n}(x)}(C_{\xi_{n-1}}(x)) , \quad n \in \mathbb{N} , \quad x \in X ,
\]
where \( \xi_0 := \varepsilon \). With these \( u_n : X \to [0, 1] \) we can consider the measurable partitions
\[
\gamma_n = \bigvee_{k=1}^n u_k^{-1} \varepsilon_{[0,1]} , \quad n \in \mathbb{N} ,
\]
generated by \( u_k, \quad k \leq n \), and also
\[
(2.8) \quad \gamma = \bigvee_{n=1}^{\infty} \gamma_n , \quad \beta_n = \gamma \lor T^{-n} \varepsilon , \quad \beta = \bigwedge_{n=1}^{\infty} \beta_n
\]
We shall write \( \gamma_n(T), \gamma(T), \beta_n(T), \beta(T) \) to indicate \( T \), if it will be necessary.

**Proposition 2.4.** Suppose that \( T \in \mathcal{UE}(\rho) \) and \( T \) is ergodic. Then there exists \( d \in \mathbb{N} \cup \{\infty\} \) such that
\[
m_{C_{\gamma}(x)}(C_{\beta}(x)) = \frac{1}{d}
\]
for a.a. \( x \in X \).

We may define now the index \( d_{\gamma;\beta}(T) \) of an ergodic endomorphism \( T \in \mathcal{UE}(\rho) \) as the number \( d \) constructed in Proposition 2.4, i.e.
\[
d_{\gamma;\beta}(T) := (m_{C_{\gamma}(x)}(C_{\beta}(x)))^{-1}
\]
for a.e. \( x \in X \).

We shall use the following properties of the partitions (2.8).

**Proposition 2.5.** Suppose that \( T \in \mathcal{UE}(\rho) \), let \( \delta \in \Delta_\rho(T) \) and the partitions \( \delta_n, \delta^{(n)}, \delta^{(\infty)} \) defined by (2.3) and (2.4). Then
\begin{align*}
(i) & \quad \gamma^{(n)} \leq \delta_n, \quad \beta_n \perp \delta^{(n)} \pmod{\gamma_n}, \quad n \in \mathbb{N} . \\
(ii) & \quad \gamma \leq \delta^{(\infty)}, \quad \beta \perp \delta^{(\infty)} \pmod{\gamma} . \\
(iii) & \quad d_{\gamma;\beta}(T) \leq d(T)
\end{align*}
We shall also use the **tail** measurable partition \( \alpha(T) := \bigwedge_{n=1}^{\infty} T^{-n} \varepsilon \). An endomorphism \( T \) is called **exact** if \( \alpha(T) = \nu \). The **tail index** \( d_\alpha(T) \) (which is, in fact, the **period** of \( T \)) is defined as follows: \( d_\alpha(T) = \infty \) if \( X/\alpha(T) \) is a continuous Lebesgue space and \( d_\alpha(T) = d \) if \( X/\alpha(T) \) consists of \( d \) atoms of measure \( \frac{1}{d} \). So that \( d_\alpha(T) \in \mathbb{N} \cup \{\infty\} \).

It is easily to see, that
\[
(2.9) \quad T^{-1} \alpha = \alpha , \quad \alpha \lor \gamma \leq \beta , \quad T^{-1} \gamma \leq \gamma , \quad T^{-1} \beta \leq \beta
\]
and \( \alpha \perp \delta^{(\infty)} \) for any \( \delta \in \Delta_\rho(T) \).
Proposition 2.6. \( (i) \) \( \alpha(T_\rho) = \nu \) , \( \beta(T_\rho) = \gamma(T_\rho) \).
(ii) \( \gamma_n(T) = \Phi^1_\delta \gamma_n(T_\rho) \) , \( \gamma(T) = \Phi^1_\delta \gamma(T_\rho) \).

The stated above propositions were proved in [Ru5] Propositions 2.5 - 2.9.

2.4. Simple \( \rho \)-uniform endomorphisms. We use now the partitions \( \gamma(T) \) and \( \beta(T) \) to introduce an important subclass of the class \( \mathcal{UE}(\rho) \).

Definition 2.7. An endomorphism \( T \in \mathcal{UE}(\rho) \) of a Lebesgue space \( (X,m) \) is said to be a simple \( \rho \)-uniform endomorphism \( (T \in \mathcal{SUE}(\rho)) \), if there exists partition \( \delta \in \Delta_\rho(T) = IC(T^{-1}\varepsilon) \) such that
\[
\delta^{(\infty)} \lor \beta(T) = \varepsilon
\]

(2.10)

We denote by \( \mathcal{SUE}(\rho) \) the class of all simple \( \rho \)-uniform endomorphisms.

The Bernoulli endomorphism (one-sided Bernoulli shift) \( T = T_\rho \) belongs to \( \mathcal{SUE}(\rho) \). In this case there exists a partition \( \delta = \delta_\rho \in \Delta_\rho(T) \) such that \( \delta^{(\infty)} = \varepsilon \) and hence \( \beta(T) \lor \delta^{(\infty)} = \varepsilon \)

Remark 2.8. It is easily to show that the condition (2.10) holds iff there exists an independent complement \( \sigma \in IC(\delta^{(\infty)}) \) of \( \delta^{(\infty)} \) that satisfies
\[
\sigma \lor \gamma(T) = \beta(T) , \sigma \in IC(\delta^{(\infty)}) , \delta \in \Delta_\rho(T) = IC(T^{-1}\varepsilon)
\]
(2.11)

Proposition 2.9. Suppose \( T \in \mathcal{UE}(\rho) \) is ergodic and \( d(T) < \infty \). Then \( T \) is simple iff \( d(T) = d_{\gamma;\beta}(T) \).

Proof. Since \( d(T) < \infty \) we have, by Proposition 2.5 \((iii)\), that \( d_{\gamma;\beta}(T) \leq d(T) < \infty \).

Definition of the index \( d_{\gamma;\beta}(T) \) (Proposition 2.4) means that \( m^{C_\gamma(x)}(C_\beta(x)) = d^{-1} \) for a.a. \( x \in X \) and \( d = d_{\gamma;\beta}(T) \in \mathbb{N} \). Almost every element of \( \gamma(T) \) consists precisely of \( d \) elements of the partition \( \beta(T) \). On the other hand there exists \( \delta \in \Delta_\rho(T) \) such that almost every element of the corresponding partition \( \delta^{(\infty)} \) consists precisely of \( d(T) \) points, \( d \leq d(T) \).

By Proposition 2.5 \((ii)\) we have
\[
\beta(T) \perp \delta^{(\infty)} \mod \gamma(T)) , \beta(T) \land \delta^{(\infty)} = \gamma(T)
\]

Whence, the condition (2.10) holds iff \( d(T) = d \).

Let \( T \in \mathcal{SUE}(\rho) \). By Proposition 2.2 any choice of the partition \( \sigma \) in the equality (2.11) determines a skew product representation (2.6) of \( T = T_\rho \) over \( T_\rho \). Herewith, all statements of Proposition 2.2 hold and we have also by (2.11) and Proposition 2.6
\[
\beta(T_\rho) = \gamma(T_\rho) , \gamma(T) = \gamma(T_\rho) \times \nu_Y , \beta(T) = \gamma(T_\rho) \times \varepsilon_Y
\]
(2.12)

These arguments imply the following

Theorem 2.10. Let \( T \) be a \( \rho \)-uniform simple endomorphism, \( T \in \mathcal{SUE}(\rho) \).

\( i) \) \( T \) can be represented in the skew product form (2.6) \( T = \tilde{T} \) over \( T_\rho \)
\[
\tilde{T}(x,y) = (T_\rho x, A(x)y) , (x,y) \in X_\rho \times Y
\]
(2.13)

with a measurable family \( \{ A(x) , x \in X_\rho \} \) of automorphisms of \( Y \) such that \( \beta(\tilde{T}) = \gamma(T_\rho) \times \varepsilon_Y \).
(ii) Two such skew product endomorphisms $T_k$, $k = 1, 2$,

$$T_k(x, y) = (T_\rho x, A_k(x)y) \quad (x, y) \in X_\rho \times Y$$

are isomorphic iff the corresponding families $A_1(x)$ and $A_2(x)$ are cohomologous, i.e.

$$A_2(x)W(x) = W(T_\rho x)A_1(x) \quad x \in X_\rho$$

for a measurable family of $\{W(x), x \in X_\rho\}$ of automorphisms of $Y$.

Proof. Part (i) follows from Proposition 2.2 with (2.12).

Let $T_1$ and $T_2$ be two skew product endomorphisms of the form (2.14). Denote $\bar{W}(x, y) := (x, W(y))$. Then (2.15) implies $\bar{T}_2 \circ S = S \circ \bar{T}_1$ if we use the automorphism $S = \bar{W}$.

Conversely, suppose there exists an automorphism $S$ such that $\bar{T}_2 \circ S = S \circ \bar{T}_1$. Then the partitions

$$\bar{\gamma} := \gamma(\bar{T}_1) = \gamma(\bar{T}_2) = \gamma(T_\rho) \times \nu_Y$$

and

$$\bar{\beta} := \beta(\bar{T}_1) = \beta(\bar{T}_2) = \gamma(T_\rho) \times \nu_Y$$

are invariant with respect to $S$. Moreover, $\bar{\gamma}$ is element-wise invariant with respect to $S$. Hence, $S|_{C(\bar{\beta}|C)} = \bar{\beta}|_C$ for almost every element $C \in \bar{\gamma}$. The restriction $S|_C$ induces a factor automorphism $W_C$ on the factor space $C/\bar{\beta}|_C \cong Y$. We obtain a measurable family $W(x) := W_{C(x)}$, $x \in X_\rho$, of automorphisms of $Y$. Since the partition $\bar{\gamma} = T_\rho \times \nu_Y$ is $\bar{T}_1$- and $\bar{T}_2$-invariant, the functions $A_1(x)$ and $A_2(x)$ (as well as $W(x)$) are constant on elements of $\gamma(T_\rho)$. Therefore the equality $\bar{T}_2 \circ S = S \circ \bar{T}_1$ implies $\bar{T}_2 \circ W = \bar{W} \circ \bar{T}_1$ and (2.15) holds.

Consider two special cases.

Absolutely non-homogeneous $\rho$. The distribution $\rho = \{\rho(i)\}_{i \in I}$ is called absolutely non-homogeneous if $\rho(i) \neq \rho(j)$ for all $i \neq j$.

In this case we have $\gamma_1(T) \lor T^{-1}\varepsilon = \varepsilon$. On the other hand $\gamma_1(T) \land T^{-1}\varepsilon$. Thus $\Delta_\rho(T)$ consists of the only partition, which is $\delta = \gamma_1(T)$. Hence

$$\delta^{(\infty)} = \gamma(T), \quad \beta_n(T) = \gamma_n(T) \lor T^{-n}\varepsilon = \varepsilon,$$

$$\beta(T) = \bigwedge_{n=1}^{\infty} \beta_n(T) = \varepsilon, \quad \beta(T) \lor \delta^{(\infty)} = \varepsilon$$

Thus we have

Proposition 2.11. Every $\rho$-uniform endomorphisms with absolutely non-homogeneous $\rho$ is simple.

Homogeneous $\rho$. We have another extremal case if $\rho$ is homogeneous, i.e. if for some $l \in \mathbb{N}$, $I = \{1, 2, \ldots, l\}$ and $\rho(i) = l^{-1}$ for all $i \in I$.

All the functions $u_n$, which generate the partitions $\gamma_n(T)$, are constant,

$$u_n(x) = m_{C^{(n)}_\epsilon(x)}(C^{(n-1)}_\epsilon(x)) = l^{-1}, \quad n \in \mathbb{N}, \quad x \in X$$

We have $\gamma(T) = \gamma_n(T) = \nu$, and $\beta_n(T) = T^{-n}\varepsilon$, whence, $\beta(T) = \bigwedge_{n=1}^{\infty} T^{-n}\varepsilon = \alpha(T)$. Therefore, for any $\delta \in \Delta_\rho(T)$ the equality $2.10$ is equivalent to $\delta^{(\infty)} \lor \alpha(T) = \varepsilon$. On the other hand $\delta^{(\infty)} \land \alpha(T)$ for every $\delta \in \Delta_\rho(T)$.

Thus we have for $T \in \mathcal{UE}(\rho)$ with homogeneous $\rho$

Proposition 2.12. Let $T \in \mathcal{UE}(\rho)$ with homogeneous $\rho$. Then
(i) $T$ is simple iff there exists $\delta \in \Delta_\rho(T)$ such that $\delta^{(\infty)} \in IC(\alpha(T))$.
(ii) The skew product decomposition in Theorem 2.10 is a direct product $T_\rho \times S$ with $S = T/\alpha(T)$.
(iii) Two such direct products $T_\rho \times S_1$ and $T_\rho \times S_2$ are isomorphic iff the automorphisms $S_1$ and $S_2$ are isomorphic.
(iv) If, in addition, $T$ is exact, i.e. $\alpha(T) = \nu$, then $T$ is simple iff $T$ is isomorphic to $T_\rho$.

It is easily to construct a skew product $T$ over $T_\rho$, which is exact and has entropy $h(T) > h(T_\rho) = \log l$. Every such endomorphism is $\rho$-uniform, $T \in \mathcal{UE}(\rho)$, but it is not isomorphic to $T_\rho$, whence, it is not simple. See also [FeR], [HeHo], [HeHoR], [Ho], [HoR], for more interesting examples of such kind of endomorphisms.

Remark 2.13. It can be shown that there exist non-simple exact endomorphisms in each class $\mathcal{UE}(\rho)$ in the case, when $\rho$ is not absolutely non-homogeneous, i.e. $\rho(i) = \rho(j)$ for some $i, j \in I$.

The following result plays an important role in present paper.

Theorem 2.14. Every ergodic $\rho$-uniform one-sided Markov shift $T_G$, corresponding to a positively recurrent Markov chain on a finite or countable state space, is simple and

$$(2.16) \quad d(T_G) = d_{\gamma, \beta}(T_G) < \infty.$$\[\textit{Proof.}\] The last statement was proved in [Ru], Theorem 4.3]. It implies that $T_G$ is simple by Proposition 2.9. \[\square\]

3. Stochastic graphs and their homomorphisms.

3.1. Stochastic graphs and Markov shifts. We need some terminology concerning stochastic graphs and their homomorphisms.

Consider a directed graph with countable (finite or infinite) set $G$ of edges. Denote by $G^{(0)}$ the set of all vertices of the graph. We also denote by $s(g)$ the starting vertex and by $t(g)$ the terminal vertex of an edge $g \in G$

$$t(g) \overset{g}{\longrightarrow} s(g)$$

The maps

$$s : G \ni g \rightarrow s(g) \in G^{(0)} \quad , \quad t : G \ni g \rightarrow t(g) \in G^{(0)}$$

completely determine the structure of the graph $G$.

In the sequel we assume that both the sets

$$\gamma G = \{ g \in G : t(g) = v \} \quad , \quad \gamma u = \{ g \in G : s(g) = u \}$$

are not empty for all vertices $u, v \in G^{(0)}$.

Denote by $G^{(n)}$ the set of all paths of length $n$ in $G$, i.e.

$$(3.17) \quad G^{(n)} = \{ g_1 g_2 \ldots g_n \in G^n : s(g_1) = t(g_2), \ldots, s(g_{n-1}) = t(g_n) \}$$

A graph $G$ is said to be irreducible if for every pair of vertices $u, v \in G^{(0)}$ there exists a finite $G$-path $g_1 g_2 \ldots g_n \in G^{(n)}$ such that $u = s(g_n)$ and $v = t(g_1)$.

Take into account that we use here and in the sequel the notation $g_1 g_2 \ldots g_n$ for backward paths

$$t(g_1) \overset{g_1}{\longleftarrow} s(g_1) = t(g_2) \overset{g_2}{\longleftarrow} s(g_2) = t(g_3) \overset{g_3}{\longleftarrow} \ldots \overset{g_n}{\longleftarrow} s(g_n)$$
A graph $G$ is called **stochastic** if its edges $g$ are equipped with positive numbers $p(g)$ such that $\sum_{g \in G_u} p(g) = 1$ for all $u \in G^{(0)}$. The weights $p(g)$, $g \in G$, determine the backward transition probabilities of the Markov chain induced by $G$.

We shall assume in the sequel that there exist stationary probabilities $p^{(0)}(u) > 0$ on $G^{(0)}$ such that

$$\sum_{u \in G^{(0)}} p^{(0)}(u) = 1, \quad \sum_{g \in v} p(g) p^{(0)}(s(g)) = p^{(0)}(v)$$

for all vertices $u, v \in G^{(0)}$.

It is known that the stationary probabilities on $G^{(0)}$ exist iff the corresponding to $G$ Markov chain is positively recurrent. If, in addition, the irreducibility condition hold, the stationary probabilities $p^{(0)}(u)$, $u \in G^{(0)}$ on the vertices are uniquely determined by the transition probabilities $p(g)$, $g \in G$ on the edges.

Thus any stochastic graph $(G, p)$ induces a Markov chain on the state space $G$ with the transition probabilities matrix $P = (P(g, h))_{g \in G, h \in G}$, where

$$P(g, h) = \begin{cases} p(h), & \text{if } t(g) = s(h) \\ 0, & \text{otherwise.} \end{cases}$$

In the sequel we mainly deal with stochastic graphs, which induce irreducible positively recurrent Markov chains.

The one-sided Markov shift $T_G$, induced by the stochastic graph $G$, is defined as follows. Let

$$X_G = \{ x = \{ g_n \}_{n=1}^\infty \in G^{\mathbb{N}} : s(g_1) = t(g_2), s(g_2) = t(g_3), \ldots \}$$

and the Markov measure $m_G$ on $X_G$ is given by

$$p(g_1) p(g_2) \ldots p(g_n) p^{(0)}(s(g_n))$$

on the cylindrinc sets of the form

$$A(g_1 g_2 \ldots g_n) := \{ x = \{ x_k \}_{k=1}^\infty \in X_G : x_1 = g_1, \ldots, x_n = g_n \}$$

where $g_1 g_2 \ldots g_n \in G^{(n)}$ is a $G$-path of length $n$ in $G$.

The one-sided shift $T_G$ acts on the probability space $(X_G, m_G)$ by

$$T_G(\{ x_n \}_{n=1}^\infty) = \{ x_{n+1} \}_{n=1}^\infty$$

and $T_G$ preserves the Markov measure $m_G$. The shift $T_G$ is ergodic iff the graph $G$ is irreducible. Under the irreducibility condition, the stationary probabilities $p^{(0)}$ on $G^{(0)}$ and, hence, the $T_G$-invariant Markov measure $m_G$ are uniquely determined by the stochastic graph $(G, p)$.

The coordinate functions

$$Z_n : X_G \ni x = \{ x_k \}_{k=1}^\infty \rightarrow x_n \in G, \ n \in \mathbb{N}$$

form a stationary Markov chain on $(X_G, m_G)$ with the backward transition probabilities

$$P(g, h) = m_G( \{ Z_n = h \ | \ Z_{n+1} = g \} ) = p(h), \ n \in \mathbb{N}$$

for all $(h, g) \in G^{(2)}$.

Consider now the partitions

$$\zeta_n = Z_n^{-1} \varepsilon_G = T_G^{-n+1} \zeta_1 = \{ T_G^{-n+1} A(g) \}_{g \in G}, \ n \in \mathbb{N},$$

generated by $Z_n$ on $X_G$, where

$$A(g) = \{ x = \{ x_k \}_{k=1}^\infty \in X_G : x_1 = g \}$$
Setting $\zeta = \zeta_1$ and $T = T_G$, we have

$$(3.19) \quad \bigvee_{n=1}^{\infty} T^{-n+1}\zeta = \varepsilon$$

$$(3.20) \quad \zeta \perp \bigvee_{n=1}^{\infty} T^{-n}\zeta \quad (\text{mod } T^{-1}\zeta)$$

Recall that a measurable partition $\zeta$ of $(X, m)$ is said to be a one-sided Markov generator or one-sided Markov generating partition for an endomorphism $T$ of $(X, m)$, if the above conditions (3.19) and (3.20) hold.

The partition $\zeta_G$ will be called the standard one-sided Markov generator of the one-sided Markov shift $T_G$ on $X_G$.

**Example 3.1 (Standard Bernoulli Graph).** Let $(I, \rho)$ be a finite or countable alphabet and

$$\rho = \{\rho(i), i \in I\}, \quad \rho(i) > 0, \quad \sum_{i \in I} \rho(i) = 1.$$  

be a probability on $I$. We shall consider $(I, \rho)$ as a stochastic graph, which has the set of edges $i \in I$ with weights $\rho(i)$ and a single vertex, denoted by $"o"$. So $G^{(0)} = \{o\}$ is a singleton and $s(i) = t(i) = o$ for all $i \in I$. We shall say that $(I, \rho)$ is the standard Bernoulli graph.

For instance, $p \bigcup o \bigcap q$ if $|I| = 2$ and $\rho = (p, q)$.

The corresponding to $(I, \rho)$ one-sided Markov shift $T_I$ coincides with the Bernoulli shift $T_I = T_\rho$. The generating partition $\zeta_I$ coincides with the standard Bernoulli generator $\delta_\rho = \{B_\rho(i)\}_{i \in I}$, defined by (2.2).

**Induced shift $T_u$.** For any $u \in G^{(0)}$, denote

$$D(u) := \{x = \{x_k\}_{k=1}^{\infty} \in X_G : t(x_1) = u\}, \quad u \in G^{(0)}.$$  

and consider the partition $\zeta^{(0)} := \{D(u)\}_{u \in G^{(0)}}$ on the space $X_G$. The partition $\zeta^{(0)}$ is a Markov partition with respect to shift $T_G$, i.e.

$$(3.21) \quad \zeta^{(0)} \perp T_G^{-1}\varepsilon X_G \quad (\text{mod } T_G^{-1}\zeta^{(0)}),$$

but it is not a one-sided generator for $T_G$, in general.

We shall use in the sequel the endomorphisms $T_u := (T_G)_{D(u)}$, induced by the shift $T_G$ on elements $D(u)$ of $\zeta^{(0)}$, $u \in G^{(0)}$. The Markov property (3.21) provides that for every $u \in G^{(0)}$ the induced endomorphism $T_u$ is a Bernoulli shift. More exactly, in accordance with the general definition of return functions (2.2) we have

$$\varphi_{u}(x) = \varphi_{D(u)}(x) := \min\{n \geq 1 : T_G^n x \in D(u)\}, \quad x \in D(u)$$

and

$$T_u x = T_G^{\varphi_{u}(x)} x, \quad x \in X_G.$$  

Take $I_u = \bigcup_{n=1}^{\infty} I_{u,n}$, where $I_{u,n}$ be the set of all $g_1 g_2 \ldots g_n \in G^{(n)}$ such that

$$(3.22) \quad t(g_k) = s(g_n) = u, \quad s(g_k) = t(g_{k+1}) \neq u, \quad k = 1, 2, \ldots, n - 1.$$  

Define also $\rho_u = \{\rho_u(i)\}_{i \in I_u}$ by

$$(3.23) \quad \rho_u(i) = p(g_1)p(g_2) \ldots p(g_n), \quad i = g_1 g_2 \ldots g_n \in I_{u,n}, \quad n \in \mathbb{N}.$$
For any \( i = g_1 g_2 \ldots g_n \in I_{u,n} \) we set \( B_u(i) := A(g_1 g_2 \ldots g_n) \) and consider the partition \( \zeta_u = \{ B_u(i) \}_{i \in I_u} \), whose elements are enumerated by the alphabet \( I_u \). The Markov property \( \Phi \) implies that the partitions \( T_u^{-n} \zeta_u \), \( n \in \mathbb{N} \) are independent. Thus

**Proposition 3.2.** The induced endomorphism \( T_u \) is isomorphic to the Bernoulli shift \( T_{\rho_u} \) and \( \zeta_u \) is a one-sided Bernoulli generator of \( \Gamma_u \).

### 3.2. Graph homomorphisms and skew products

Now we want to establish the class of graph homomorphisms that we shall use.

**Definition 3.3.** Let \( G \) and \( H \) be two stochastic graphs.

(i) A map \( \phi : G \to H \) is a **graph homomorphism** if there exists a map \( \phi^{(0)} : G^{(0)} \to H^{(0)} \) such that

\[
 s(\phi(g)) = \phi^{(0)}(s(g)) \quad t(\phi(g)) = \phi^{(0)}(t(g))
\]

for all \( g \in G \). (Note that, if the map \( \phi^{(0)} \) exists it is unique.)

(ii) A graph homomorphism \( \phi : G \to H \) is **deterministic** if \( \phi^{(0)}(G^{(0)}) = H^{(0)} \) and for every \( u \in G^{(0)} \) the restriction of \( \phi \) on \( G_u \)

\[
 \phi|_{G_u} : G_u \to H_{\phi^{(0)}(u)}
\]

is a bijection of this set onto \( H_{\phi^{(0)}(u)} \).

(iii) A graph homomorphism is **weight preserving** or **p-preserving** if \( p(\phi(g)) = p(g) \) for all \( g \in G \).

Two edges \( g_1 \) and \( g_2 \) are said to be **congruent** if

\[
 s(g_1) = s(g_2) \quad t(g_1) = t(g_2) \quad p(g_1) = p(g_2).
\]

The map \( \phi^{(0)} \) in the above definition is uniquely determined by \( \phi \), but \( \phi^{(0)} \) does not determines \( \phi \) if \( G \) has congruent edges.

Anyway one can use a more explicit notation

\[
 (\phi, \phi^{(0)}) : (G, G^{(0)}) \to (H, H^{(0)})
\]

for the homomorphism \( \phi : G \to H \).

We shall denote by \( \text{Hom}(G, H) \) the set of all weight preserving deterministic graph homomorphisms \( \phi : G \to H \). In the sequel the term ”homomorphism” always means just weight preserving deterministic graph homomorphism.

**Proposition 3.4.** Let \( \phi : G \to H \) be a map.

(i) If \( \phi \) is a graph homomorphism, it induces a factor map

\[
 \Phi_\phi : X_G \to X_H \quad \Phi_\phi(\{x_n\}_{n=1}^\infty) = \{\phi(x_n)\}_{n=1}^\infty
\]

such that \( \Phi_\phi \circ T_G = T_H \circ \Phi_\phi \).

(ii) If, in addition, \( \phi \) is weight preserving, the factor map \( \Phi_\phi \) is measure preserving, \( (m_H = m_G \circ \Phi_\phi^{-1}) \).

(iii) If \( \phi \) is also deterministic, the shift \( T_G \) can be represented as a skew product

\[
 T(x, y) = \left(T_H x, A(x)y \right) \quad (x, y) \in X_H \times Y
\]

where \( \{A(x), x \in X_H\} \) is a measurable family of automorphisms of \( Y \).

(iv) If \( T_G \) is ergodic, \( Y \) is a homogeneous Lebesgue space.
Proof. Parts (i) and (ii) follow directly from Definition 3.3; Part (iii) and (iv) can be proved by analogy with Proposition 2.2. □

Moreover

Theorem 3.5. Let $\phi \in \mathcal{H}om(G,H)$ and suppose that the shift $T_G$ is ergodic. Then there exists $d \in \mathbb{N}$ such that $|\Phi_\phi^{-1}(x)| = d$ for almost all $x \in X_H$. That is, in the skew product (3.24) the space $Y$ is finite, $|Y| = d$.

Note that Theorem 3.5 claims the finiteness of $d$ even in the case, when the graph $G$ is not finite, i.e. $|G| = \infty$. This is a consequence of positive recurrence of the corresponding $G$ Markov chain. The skew product decomposition (3.24) of $T_G$ over $T_H$ is a $d$-extension.

Theorem 3.5 was proved earlier in a particular case, when $H$ is a Bernoulli graph, i.e. when $H^{(0)} = \{\emptyset\}$ is a singleton (See Theorem 3.3 and Corollary 3.4 from [Ru, and also Theorem 3.18 below).

We omit the proof of Theorem 3.5 in general case, since only the pointed out particular case is considered in this paper.

Definition 3.6. The integer $d$ in Theorem 3.5, i.e. the degree of the factor map $\Phi_\phi$ will be called the degree of the homomorphism $\phi$.

Denoting the degree by $d(\phi)$, we have $d(\phi) = |\Phi_\phi^{-1}(x)|$ for a.a. $x \in X_H$.

The following construction plays a central role in our explanation.

Example 3.7 (Graph Skew Product). Let $d \in \mathbb{N}$ and let $Y_d = \{1,2,\ldots,d\}$ consists of $d$ points of measure $\frac{1}{d}$. Denote by $A_d = A(Y_d)$ the full group of all permutations of $Y_d$.

Given a stochastic graph $H$, equipped with a function $a : H \ni h \rightarrow a(h) \in A_d$, we construct a stochastic graph $\tilde{H}_a$ and a homomorphism $\pi_H : \tilde{H}_a \rightarrow H$ by

$$\tilde{H}_a = H \times Y_d, \quad \tilde{H}_a^{(0)} = H^{(0)} \times Y_d,$$

with

$$s(\tilde{h}) = (s(h),y), \quad t(\tilde{h}) = (t(h),a(h)y), \quad p(\tilde{h}) = p(h)$$

for $\tilde{h} = (h,y) \in \tilde{H}_a = H \times Y_d$ and also

$$p^{(0)}(\tilde{u}) = p^{(0)}(u), \quad \tilde{u} = (u,y) \in \tilde{H}_a^{(0)} = H^{(0)} \times Y_d$$

The natural projection

$$\pi_H : \tilde{H}_a = H \times Y_d \twoheadrightarrow H, \quad \pi_H^{(0)} : \tilde{H}_a^{(0)} = H^{(0)} \times Y_d \twoheadrightarrow H^{(0)}$$

is a homomorphism.

Definition 3.8. We shall say that the graph $\tilde{H}_a$ is a skew product over $H$ and the homomorphism $\pi_H : \tilde{H}_a \rightarrow H$ is a graph skew product (or GSP) $d$-extension of $H$.

In the above construction we have $|\pi_H^{-1}(h)| = d$ for all $h \in H$ and this is, in fact, a characteristic property of the graph skew product $d$-extension in the following sense

Definition 3.9. Two homomorphisms $\phi_k : G_k \rightarrow H$, $k = 1,2$ are said to be equivalent if $\phi_2 = \kappa \circ \phi_1$ for an appropriate isomorphism $\kappa : G_1 \rightarrow G_2$.

Definition 3.10. Let $d \in \mathbb{N}$. A homomorphism $\phi \in \mathcal{H}om(G,H)$ is called a $d$-extension if

$$|\phi^{-1}(h)| = d, \quad h \in H.$$  

Proposition 3.11. Any $d$-extension $\phi : G \rightarrow H$ is equivalent to a GSP $d$-extension $\pi_H : \tilde{H}_a \rightarrow H$. 

Proof. Let $\phi \in \mathcal{H}\text{om}(G, H)$ be a $d$-extension. Since $\phi$ is deterministic the restrictions $\phi|_{G_u}$ are bijections between $G_u$ and $H_{\phi(0)}(u)$ for all $u \in H^{(0)}$. Hence the condition (3.25) is equivalent to

$$|\phi^{(0)}(u)| = d, \ u \in H^{(0)}.$$  

For each $u \in H^{(0)}$ we can choose a bijection $w_u$ of $\phi^{(0)}(u)$ onto $Y_d$. With any fixed choice of these bijections we set

$$H \ni h \rightarrow a(h) = w_t(h) \circ w_s(h)^{-1} \in A_d,$$

and consider the corresponding skew product graph $\bar{H}_a$. The bijections $w_u$ uniquely determine an isomorphism $\kappa : G \rightarrow \bar{H}_a$ such that $\phi = \pi_H \circ \kappa$. \hfill $\Box$

Remark 3.12. The Markov shift $T_{\bar{H}_a}$ corresponding to a graph skew product $\bar{H}_a$ can be identified with the skew product endomorphism $T_{H,a}$, defined by

$$\bar{T}_{H,a}(x,y) = (T_{H}x, a(a(x))^{-1}y), \ x = \{x_n\}_{n=1}^{\infty} \in X_H, \ y \in Y_d.$$

and thus any $d$-extension is a homomorphism of degree $d$.

Indeed, the shift $T_{\bar{H}_a}$ acts on the space

$$X_{\bar{H}_a} = \{(x_n, y_n)\}_{n=1}^{\infty} : x = \{x_n\}_{n=1}^{\infty} \in X_H, \ y_n = a(x_n) y_{n+1} \in Y_d \}$$

and the map

$$\Psi : X_{\bar{H}_a} \ni \{x_n\}_{n=1}^{\infty} \rightarrow (\{x_n\}_{n=1}^{\infty}, y_n) \in X_H \times Y_d$$

realizes the identification, that is, $m_H \otimes m_{Y_d} = m_{\bar{H}_a} \circ \Psi^{-1}$ and $\bar{T}_{H,a} \circ \Psi = \Psi \circ T_{H,a}$. Note also that

$$(3.26) \quad \Psi(\zeta_{\bar{H}_a}) = \zeta_H \times \varepsilon_{Y_d}, \quad \Psi(\zeta^{(0)}_{\bar{H}_a}) = \zeta^{(0)}_H \times \varepsilon_{Y_d}.$$  

Consider now two skew product endomorphisms $T_{H,a_k}$, corresponding to graph skew products $\bar{H}_a_k$ with two functions $a_k : H \rightarrow A_d, \ k = 1, 2$.

Definition 3.13. (i) Two functions $a_k : H \rightarrow A_d, \ k = 1, 2$, are said to be cohomologous with respect to $H$ if there exists a map $w : H^{(0)} \rightarrow A_d$ such that

$$(3.27) \quad a_2(h)w(s(h)) = w(t(h))a_1(h), \ h \in H$$

(ii) Two measurable functions $A_k : X_H \rightarrow A_d, \ k = 1, 2$ are said to be cohomologous with respect to $T_H$ if there exists a measurable map $W : X_H \rightarrow A_d$ such that

$$(3.28) \quad A_2(x)W(x) = W(T_Hx)A_1(x), \ x \in X_H.$$  

In accordance with Definitions 3.9 and 3.13 we can say now that the homomorphisms $\pi_H : \bar{H}_a_k \rightarrow H$ are equivalent iff the functions $a_k : H \rightarrow A_d, \ k = 1, 2$ are cohomologous with respect to $H$.

The equality (3.27) is equivalent to (3.28) if we take

$$A_k(x) = a_k(x_1)^{-1}, \ k = 1, 2, \ W(x) = w(t(x_1))$$

for $x = \{x_n\}_{n=1}^{\infty} \in X_H$ and given $a_k$ and $w$. Hence if $a_1$ and $a_2$ are cohomologous with respect to $H$, then $A_1$ and $A_2$ cohomologous with respect to $T_H$.

We shall show in Section 4.3 that the inverse is also true.
Remark 3.14. Let $\chi : H \to H_1$ be a homomorphism and $\pi_1 : H_1 \to H_1$ be a $d$-extension of $H_1$ generated by a function $a_1 : H_1 \to \mathcal{A}_a$. Setting $a(h) := a_1(\chi(h))$ we obtain a $d$-extension $\pi : H := H_a \to H$ of $H$. The map $\tilde{\chi}(h,y) := (\kappa(h), y)$, $(h,y) \in H$ is a homomorphism and the diagram

$$
\begin{array}{ccc}
\tilde{H} & \xrightarrow{\tilde{\chi}} & \tilde{H}_1 \\
\downarrow \pi & & \downarrow \pi_1 \\
H & \xrightarrow{\chi} & H_1
\end{array}
$$

commutes. The homomorphism $\tilde{\chi}$ is called a trivial extension of $\chi$. If, in addition, $d(\chi) = 1$, then $d(\tilde{\chi}) = 1$ and hence the corresponding endomorphisms $\tilde{T}_{H,a}$ and $\tilde{T}_{H_1,a_1}$ are isomorphic.

3.3. Stochastic $\rho$-uniform graphs. We continue to consider $(I, \rho)$ as the standard Bernoulli stochastic graph, (Example 3.1).

Definition 3.15. A stochastic graph $(G, p)$ is called $\rho$-uniform if there exists a homomorphism $\phi \in \text{Hom}(G, I)$.

For any such homomorphism $\phi$ and for every $u \in G^{(0)}$

$$
\phi |_{G_u} : (G_u, p|_{G_u}) \to (I, \rho)
$$

is a weight preserving bijection. Thus the atomic probability spaces $(G_u, p|_{G_u})$ are isomorphic to $(I, \rho)$ for every $u \in G^{(0)}$.

Proposition 3.16. $T_G \in \mathcal{UE}(\rho)$ iff $G$ is $\rho$-uniform.

Proof. Consider the partition $\xi_1 := T_G^{-1} \varepsilon_{X_G}$ generated by the shift $T_G$. The Markov property of the measure $m_G$ on $X_G$ implies

$$
m_{C_{\xi_1}}(\{x\}) = m_G\{Z_1 = x_1 \mid Z_2 = x_2\} = p(x_1)
$$

for a.a. $x = \{x_n\}_{n=1}^{\infty} \in X_G$ . Here $m_{C_{\xi_1}}(\{x\})$ is the conditional measure of the point $x$ in the element $C_{\xi_1}(x) = T_{G^{-1}} T_G x$ of the partition $\xi_1$. Hence for every $u \in G^{(0)}$ almost all elements $(C, m_C)$ of the partition $\xi_1$ are isomorphic to $(G_u, p|_{G_u})$ on the set

$$
\{x = \{x_n\}_{n=1}^{\infty} \in X_G : s(x_1) = u\} .
$$

But $T_G \in \mathcal{UE}(\rho)$ iff a.a. elements $(C, m_C)$ are isomorphic to $(I, \rho)$. Hence $T_G \in \mathcal{UE}(\rho)$ iff $(G_u, p|_{G_u})$ are isomorphic to $(I, \rho)$ for every $u \in G^{(0)}$.

Let $G$ be a $\rho$-uniform graph and $\phi \in \text{Hom}(G, I)$ . Consider the partition $\phi^{-1} \varepsilon_I = \{\phi^{-1}(i), i \in I\}$ of $G$. The first coordinate function

$$
Z_1 : X_G \ni x = \{x_k\}_{k=1}^{\infty} \to x_1 \in G
$$

generates the following partition

$$
\delta_{\phi} = Z_1^{-1}(\phi^{-1} \varepsilon_I)
$$

do the space $X_G$. Elements of $\delta_{\phi}$ have the form

$$
B(i) = Z_1^{-1}(\phi^{-1}(i)) = \{x = \{x_k\}_{k=1}^{\infty} \in X_G : \phi(g) = i\} , i \in I
$$

Using the standard Markov generator

$$
\zeta_G = Z_1^{-1} \varepsilon_G = \{A(g)\}_{g \in G} , A(g) = Z_1^{-1}(g)
$$
of $T_G$, we have

$$
B(i) = \bigcup_{g \in \phi^{-1}(i)} A(g)
$$
and
\[ m_G(B(i)) = \sum_{g \in \phi^{-1}(i)} p(g)p^{(0)}(s(g)) = \rho(i) \sum_{u \in G^{(0)}} p^{(0)}(u) = \rho(i) \]
for \( i \in I \). Hence for \( \delta = \delta_{\phi} \) we have
\[ (3.30) \quad \delta \in IC(T_G^{-1} \varepsilon_{X_G}) = \Delta_{\rho}(T_G) \quad \delta \leq \zeta_G \]
Denoting by \( \Delta_{\rho}(T_G, \zeta_G) \) the set of all \( \delta \) that satisfy (3.30), we have also

**Proposition 3.17.** \( \Delta_{\rho}(T_G, \zeta_G) \) is precisely the set of all \( \delta \) of the form \( \delta = \delta_{\phi} \).

Now we introduce a semigroup \( S(\phi) \) of maps \( f : G^{(0)} \to G^{(0)} \) induced by the homomorphism \( \phi \).

Let \( i \in I \). Since \( \phi \) is deterministic the restriction \( \phi|_{G_u} : G_u \to I \) is a bijection of \( G_u \) onto \( I \) for every \( u \in G^{(0)} \). Hence for any pair \((i, u)\) there exists an unique \( g_{i,u} \) such that \( \phi(g_{i,u}) = i \) and \( s(g_{u,i}) = u \). Putting \( f_iu = g_{i,u} \), we get a map \( f_i : G^{(0)} \to G^{(0)} \). Let \( S(\phi) \) be the semigroup generated by the maps \( \{f_i, i \in I\} \).

Let \( FS(I) \) be the set of all finite words \( i_1i_2 \ldots i_n \) in the alphabet \( I \). We shall consider \( FS(I) \) as a free semigroup with the generating set \( I \) and with juxtaposition multiplication:
\[ i_1i_2 \ldots i_m \cdot j_1j_2 \ldots j_n = i_1i_2 \ldots i_mj_1j_2 \ldots j_n \]
and set
\[ f_{i_1i_2 \ldots i_n} = f_{i_1} \circ f_{i_2} \circ \ldots \circ f_{i_n} \quad i_1i_2 \ldots i_n \in I^n \]
Then \( i_1i_2 \ldots i_n \rightarrow f_{i_1i_2 \ldots i_n} \) is a homomorphism from the semigroup \( FS(I) \) onto the semigroup
\[ S(\phi) = \{f_{i_1i_2 \ldots i_n}, i_1i_2 \ldots i_n \in FS(I)\} \]
genrated by \( \{f_i, i \in I\} \).

Now we can describe the partitions
\[ \delta_{\phi} = \{B(i)\}_{i \in I} \quad \delta_{\phi}^{(n)} = \bigvee_{k=1}^{n} T_G^{-k+1} \delta_{\phi}, n \in \mathbb{N} \]
as follows.

First recall that the partition \( \zeta^{(0)} \) consists of the atoms \( D(u) = Z^{-1}(uG) \), \( u \in G^{(0)} \) and rename the elements \( A(g) \), \( g \in G \) of the partition \( \zeta_G \) by
\[ D(i, u) := A(g_{i,u}) \quad u \in G^{(0)}, i \in I. \]
Then for all \( i \in I \) and \( u \in G^{(0)} \) we have
\[ D(i, u) = B(i) \cap T_G^{-1}D(u), \]
\[ D(i, u) = \{x = \{x_n\}_{n=1}^{\infty} \in X_G : t(x_1) = u, \phi(x_1) = i\} \]
and
\[ (3.31) \quad B(i) = \bigcup_{u \in G^{(0)}} D(i, u), D(u) = \bigcup_{v : f_i(v) = u} D(i, v) \]
Further for any \( g_1g_2 \ldots g_n \in G^{(n)} \) there exists a unique pair \( (i_1i_2 \ldots i_n, u) \in I^n \times G^{(0)} \) such that
\[ (3.32) \quad u = s(g_n), i_k = \phi(g_k), t(g_k) = f_i(s(g_k)), k = 1, 2, \ldots, n \]
Hence any atom \( A(g_1g_2 \ldots g_n) \) of the partition \( \zeta_G^{(n)} = \bigvee_{k=1}^n T^{-k+1}\zeta_G \) can be renamed by
\[
D(i_1i_2 \ldots i_n, u) = A(g_1g_2 \ldots g_n),
\]
where the pair \( (i_1i_2 \ldots i_n, u) \) satisfies \( (3.32) \). By \( (3.31) \), any atom \( B(i_1i_2 \ldots i_n) \) of the partition \( \delta^{(n)}_\phi = \bigvee_{k=1}^n T^{-k+1}\delta_\phi \) has the form
\[
B(i_1i_2 \ldots i_n) = \bigcup_{u \in G^{(0)}} D(i_1i_2 \ldots i_n, u)
\]
and since
\[
D(i_1i_2 \ldots i_n, u) = B(i_1i_2 \ldots i_n) \cap T^{-n}D(u)
\]
we have
\[
m_G(B(i_1i_2 \ldots i_n)) = \rho(i_1)\rho(i_2) \ldots \rho(i_n),
\]
\[
m_G(D(i_1i_2 \ldots i_n, u)) = \rho(i_1)\rho(i_2) \ldots \rho(i_n)p^{(0)}(u).
\]
Any \( \zeta_G^{(0)} \)-set has the form
\[
D(E) = \{ x = \{x_n\}_{n=1}^\infty \in X_G : t(x_1) \in E \}.
\]
for a subset \( E \subseteq G^{(0)} \). Then for any \( i_1i_2 \ldots i_n \in I^n \)
\[
D(E) \cap B(i_1i_2 \ldots i_n) = \bigcup_{u: f_{i_1i_2 \ldots i_n}(u) \in E} D(i_1i_2 \ldots i_n, u).
\]
Hence
\[
(3.33) \quad m_G(D(E) | B(i_1i_2 \ldots i_n)) = p^{(0)}(f_{i_1i_2 \ldots i_n}(E)).
\]
Next theorem is basic for our explanation. Let
\[
(\delta_\phi^{(\infty)}) = \bigvee_{n=1}^\infty \delta_\phi^{(n)} = \bigvee_{n=1}^\infty T^{-n+1}\delta_\phi
\]

**Theorem 3.18.** Let \( \phi \in \text{Hom}(G, I) \). Then

(i) \( \zeta_G \vee \delta^{(\infty)}_\phi = \varepsilon_{X_G} \).

(ii) \( d(T_G) \leq d(T_G, \delta_\phi) = d(\phi) < \infty \)

**Proof.** It was proved in [Ru3, Theorem 3.3] that if \( \zeta \) is a one-sided Markov generator of \( T \)
and
\[
T \in \mathcal{U}(\rho), \ \delta \in \Delta_\rho(T), \ \delta \leq \zeta,
\]
then \( \zeta \vee \delta^{(\infty)} = \varepsilon \). Hence (i) follows by putting \( \delta = \delta_\phi \) and \( \zeta = \zeta_G \).

Since the partition \( \zeta_G \) is finite or countable the equality (i) implies that almost all elements \( (C, m_G) \) of the partition \( \delta^{(\infty)}_\phi \) are atomic. Taking into account the ergodicity of \( T_G \), we see that almost all elements of \( \delta^{(\infty)}_\phi \) consist of \( d \) atoms of measure \( \frac{1}{d} \) for an natural \( d \). Herewith by Definitions 2.3 and 3.6 we have \( d = d(T_G, \delta_\phi) = d(\phi) \) and, whence, (ii) follows.

We need the following sharp version of Part (i) of Theorem 3.18.

**Lemma 3.19.** \( \zeta_G^{(0)} \vee \delta^{(\infty)}_\phi = \varepsilon_{X_G} \)

**Proof.** Choose an increasing sequence of positive numbers \( c_n > 0 \) and an increasing sequence of finite subsets \( E_n \) of \( G^{(0)} \) such that
\[
(3.34) \quad \sum_{n=1}^\infty E_n = G^{(0)} \quad \sum_{n=1}^\infty (1-c_n) < \infty \quad p^{(0)}(E_n) > c_n.
\]
Since $|E_n| < \infty$ there exist $i_1^{(n)} i_2^{(n)} \ldots i_k^{(n)} \in I^k$ and $f_n := f_{i_1^{(n)} i_2^{(n)} \ldots i_k^{(n)}} \in \mathcal{S}(\phi)$ such that $|f_n(E_n)| = \min\{|f(E)| : f \in \mathcal{S}(\phi)|$.

The choice of $f_n$ provides that all restrictions $f|_{f_n(E_n)}$, $f \in \mathcal{S}(\phi)$ are bijections.

Consider the sets

$$B_n := B(i_1^{(n)} i_2^{(n)} \ldots i_k^{(n)}) ,$$

$$B'_n := B_n \cap T^{-k_n} D(E_n) = \bigcup_{u \in E_n} D(i_1^{(n)} i_2^{(n)} \ldots i_k^{(n)}, u)$$

and also

$$F_n := \{ x \in X_G : T^\omega_n(x) + n x \in B'_n \} ,$$

where

$$\omega_n(x) := \min\{ k \geq 0 : T^{n+k} x \in B_n \}$$

Then it is not hard to see that

$$m_G(F_n) = m_G(B'_n \mid B_n) = p^{(0)}(E_n) > c_n .$$

Set $F := \lim \inf_{n \to \infty} F_n$. Then we have $m_G(F) = 1$, since $\sum (1 - c_n) < \infty$. By constructing, the set $F$ has the following property. Suppose $x = \{ x_k \}^\infty_{k=1}$ and $y = \{ y_k \}^\infty_{k=1}$ belong to $F$ and

$$\Phi_{\delta_\phi}(x) = \Phi_{\delta_\phi}(y) = \{ i_k \}^\infty_{k=1} \in X_\rho .$$

If $s(x_m) \neq s(y_m)$ for some $m \geq 1$, then

$$t(x_k) = f_{i_k} s(x_k) \neq t(y_k) = f_{i_k} s(y_k) , \quad k = 1, 2, \ldots m .$$

In other words, if $t(x_1) = t(y_1)$ and $\Phi_{\delta_\phi}(x) = \Phi_{\delta_\phi}(y)$ then $x = y$. Thus $\zeta_G^{(0)} \vee \delta_\phi^{(\infty)} = \varepsilon_{X_G}$ on the set $F$ of measure 1.

3.4. Semigroup $\mathcal{S}(\phi)$ and persistent $d$-sets. Let $U$ be a finite or countable set.

**Definition 3.20.** Let $\mathcal{S}$ be a semigroup of maps $f : U \to U$ on $U$ and let $d \in \mathbb{N}$.

Call the semigroup $\mathcal{S}$ **d-contractive** if there exists a subset $L \subseteq U$ such that

(i) $|f(L)| = |L| = d$ for all $f \in \mathcal{S}$.

(ii) For every finite subset $E \subseteq U$ there exists $f \in \mathcal{S}$ with $f(E) \subseteq L$.

The sets $L$, satisfying (i) and (ii), will be called **persistent d-sets** with respect to $\mathcal{S}$.

Denote by $\mathcal{L}(\mathcal{S})$ the set of all such $L$. We have directly from the definition:

- For $L \in \mathcal{L}(\mathcal{S})$ and $f \in \mathcal{S}$ the restriction $f|_{L} : L \to f(L)$ is a bijection and $f(L) \in \mathcal{L}(\mathcal{S})$.

- The semigroup $\mathcal{S}$ acts transitively on $\mathcal{L}(\mathcal{S})$, i.e. for every pair $L_1, L_2 \in \mathcal{L}(\mathcal{S})$ there exists $f \in \mathcal{S}$ such that $f(L_1) = L_2$.

- The integer $d$ is equal to

$$d(\mathcal{S}) := \sup_{E \subseteq U : |E| < \infty} \min_{f \in \mathcal{S}} |f(E)| ,$$

and $d(\mathcal{S}) = \min_{f \in \mathcal{S}} |f(U)|$ if $|U| < \infty$.

Let $G$ be a $\rho$-uniform stochastic graph and $\phi \in \mathcal{H}(G, I)$ be a homomorphism $\phi : G \to I$.

Return to the semigroup $\mathcal{S}(\phi)$ which acts on $U = G^{(0)}$.
Theorem 3.21. Let $T_G$ be an ergodic one-sided Markov shift corresponding to a $p$-uniform stochastic graph $G$ and let $\phi \in \text{Hom}(G, I)$. Then the semigroup $S(\phi)$ is $d$-contractive on $G^{(0)}$ and

$$d = d(S(\phi)) = d(T_G, \delta_\phi) = d(\phi)$$

Proof. To prove the theorem we shall use the partition $\zeta_G^{(0)}$ on $G^{(0)}$. Recall that $\zeta_G^{(0)}$ consists of all atoms of the form $D(u) = Z_1^{-1}(uG)$, $u \in G^{(0)}$. For any subset $E$ of $G^{(0)}$ we denote

$$D(E) = \{x = \{x_n\}_{n=1}^\infty \in X_G : t(x_1) \in E\} = \bigcup_{u \in E} D(u),$$

i.e. $D(E)$ is a $\zeta_G^{(0)}$-set corresponding to $E$ in the space $X_G$.

It follows from Theorem 3.18 Part (ii) that almost all elements $(C, m_C)$ of the partition $\delta_\phi^{(\infty)}$ are isomorphic to $Y_d$, where $d = d(T_G, \delta_\phi) \in \mathbb{N}$. Hence

$$m(\{x\} | C_{\delta_\phi^{(\infty)}}(x)) = \frac{1}{d}$$

for a.e. $x \in X_G$. Then Lemma 3.17 implies that there exists a measurable family $\{l(x), x \in X\}$ of subsets $l(x) \subseteq G^{(0)}$ such that

$$m(D(l(x)) | C_{\delta_\phi^{(\infty)}}(x)) = 1, \ |l(x)| = d$$

almost everywhere on $X_G$.

For any $L \subseteq G^{(0)}$ denote

$$\bar{L} := \{x \in X_G : l(x) = L\}, \ \mathcal{L} := \{L \subseteq G^{(0)} : m_G(\bar{L}) > 0\},$$

i.e. $\mathcal{L}$ is the (finite or countable) set of all essential values of the function $x \rightarrow l(x)$.

We show that $\mathcal{L} \subseteq \mathcal{L}(S(\phi))$, i.e. that every $L \in \mathcal{L}$ satisfies the conditions (i) and (ii) of Definition 3.20.

Take any finite subset $E \subseteq G^{(0)}$ and choose $c > 0$ such that $c < \min_{u \in E} p_0^{(0)}(u)$.

For $L \in \mathcal{L}$ and almost all $x = \{x_n\}_{n=1}^\infty \in \bar{L}$ we have by (3.33)

$$\lim_{n \to \infty} m(D(L) | C_{\delta_\phi^{(\infty)}}(x)) = m_G(D(l(x)) | C_{\delta_\phi^{(\infty)}}(x)) = 1$$

and by (3.33)

$$m(D(L) | C_{\delta_\phi^{(\infty)}}(x)) = p_0^{(0)}(f_{x_1x_2 \ldots x_n}^{-1}(L)).$$

Hence we can choose $n$ and $(x_1x_2 \ldots x_n) \in G^{(n)}$ such that

$$m(B(x_1x_2 \ldots x_n) \cap \bar{L}) > 0$$

and then

$$p_0^{(0)}(f_{x_1x_2 \ldots x_n}^{-1}(L)) > 1 - c$$

The choice of $c$ provides $f_{x_1x_2 \ldots x_n}(E) \supseteq L$ and thus Part (ii) of Definition 3.20 holds. Part (i) follows from the equalities

$$f_{x_1x_2 \ldots x_n} l(T_G^n x) = l(x), \ |l(x)| = d, \ x \in X_G$$

by the definition of $l(x)$. We have proved the inclusion $\mathcal{L} \subseteq \mathcal{L}(S(\phi))$, which implies that the semigroup $S$ is $d$-contractive with $d = d(T_G, \delta_\phi)$.

\begin{flushright}$\square$\end{flushright}

Corollary 3.22. $\mathcal{L} = \mathcal{L}(S(\phi))$. 
Proof. It was proved above that $\mathcal{L} \subseteq \mathcal{L}(S(\phi))$. Take $M \in \mathcal{L}(S(\phi))$ and $L \in \mathcal{L}$. Since also $L \in \mathcal{L}(S(\phi))$, there exists $i_1 i_2 \ldots i_n \in I^n$ such that

$$f_{i_1 i_2 \ldots i_n}(M) = L = I(x), \ x \in L$$

Then the equality (3.38) implies $M = l(T^n x)$ on a set of positive measure in $X$ and hence $M \in \mathcal{L}$. Thus $\mathcal{L} \supseteq \mathcal{L}(S(\phi))$. \hfill $\square$

Remark 3.23. Note that the notion of $d$-contractive semigroup was introduced in [Ru4], where an analog of Theorem 3.21 was also proved. Definition 3.20 is a generalization of what is called ”point collapsing” by M. Rosenblatt [Ro], [Ro2] in the case $|U| < \infty$ and $d = 1$. The case, when $|U| = \infty$ and $d = 1$, was considered in [KuMuTo].

3.5. Graph skew product representation. From now on let $G$ be a $\rho$-uniform stochastic graph, which is irreducible and satisfies the positive recurrence condition.

Theorem 3.24. Let $\phi \in \text{Hom}(G, I)$ be a homomorphism of degree $d = d(\phi)$. Then there exists a commutative diagram

$$\begin{array}{ccc}
\hat{H} & \xrightarrow{\psi} & G \\
\downarrow \phi & & \downarrow \phi \\
H & \xrightarrow{\psi} & I
\end{array}$$

where the graph $\hat{H} = \hat{H}_a$ is a graph skew product over $H$, generated by a function $a : H \ni h \rightarrow a(h) \in A_i$, the homomorphism $\hat{\phi}$ coincides with the natural projection $\pi_H$, and both the homomorphisms $\hat{\psi} \in \text{Hom}(\hat{H}, G)$ and $\psi \in \text{Hom}(H, I)$ are of degree 1. In particular, $(\hat{\phi}, \psi) \in \text{Ext}^d(I, \rho)$

Proof. We construct a commutative diagram

$$\begin{array}{ccc}
\hat{H} & \xrightarrow{\psi} & G \\
\downarrow \hat{\phi} & & \downarrow \phi \\
H & \xrightarrow{\psi} & I
\end{array}$$

such that the homomorphism $\hat{\phi} \in \text{Hom}(\hat{H}, H)$ is a $d$-extension (See Definition 3.10) and both homomorphisms $\hat{\psi} \in \text{Hom}(\hat{H}, G)$ and $\psi \in \text{Hom}(H, I)$ are of degree 1.

We shall use the persistent $d$-sets $\mathcal{L} = \mathcal{L}(S(\phi))$ of the semigroup $S(\phi)$, described in Theorem 3.21 (Section 3.5). Since $\mathcal{L}$ is finite or countable we can enumerate the set by an alphabet $J$, setting $\mathcal{L} = \{L_j : j \in J\}$.

Recall that the semigroup $S(\phi)$ is $d$-contractive with $d = d(S(\phi))$ (Theorem 3.24). For any pair $i \in I$, $j \in J$ we have $|f_i(L_j)| = |L_j| = d$ and the restrictions $f_i|_{L_j}$ is a bijection of $L_j$ onto $f_i(L_j)$, whence, $f_i(L_j) \in \mathcal{L}$ for all $i$ and $j$. For any $i \in I$ denote by $f_i^j$ the map $J \rightarrow J$, which is defined by $f_i^j(j) = j'$, where $f_i(L_j) = L_{j'}$.

To construct Diagram 3.40 define first $\psi : H \rightarrow I$ with $H := I \times J$, $H^{(0)} := J$, where

$$s(i, j) = j, \ t(i, j) = f_i^j(j), \ p(i, j) = \rho(i), \ \psi(i, j) = i$$

Next set

$$\hat{H}^{(0)} := \{(j, u) \in J \times G^{(0)} : j \in J, \ u \in L_j\}, \ \hat{H} := I \times \hat{H}^{(0)}$$

with

$$s(i, j, u) = (j, u), \ t(i, j, u) = (f_i^j(j), f_i u), \ p(i, j, u) = \rho(i).$$
Finally, we define the maps \( \hat{\psi} \) and \( \hat{\phi} \) by

\[
\hat{\phi} : \hat{H} \ni (i, j, u) \rightarrow (i, j) \in H , \quad \hat{\psi} : \hat{H} \ni (i, j, u) \rightarrow g_{i,u} \in G .
\]

where \( g_{i,u} \) is uniquely determined by the conditions \( s(g) = u \) and \( \phi(g) = i \).

It follows directly from this constructing that \( H \) and \( \hat{H} \) are stochastic graphs, and that \( \hat{\phi} \), \( \hat{\psi} \) and \( \psi \) are homomorphisms, and that Diagram 3.40 commutes. Point out only that \( \hat{\phi} \) is a \( d \)-extension, since \(|L_j| = d\) for all \( j \) and hence \( \hat{\phi} \) is of degree \( d \). This implies that \( \hat{\psi} \) and \( \psi \) are of degree 1, since \( \phi \) is of degree \( d \).

It remains to apply Proposition 3.11 to the homomorphism \( \hat{\phi} : \hat{H} \rightarrow H \). \( \Box \)

Next we construct \( d \)-extensions with a minimal possible \( d \). Let \( G \) as above be a \( \rho \)-uniform stochastic graph. Recall that \( \Hom(G^{(n)}) \) denotes the set of all \( n \)-paths in \( G \), see (3.17). We shall consider \( G^{(n)} \) as a stochastic graph with the set of vertices \( G^{(n-1)} \), where for any \( g^{(n)} = g_1 \, g_2 \, \ldots \, g_n \),

\[
s(g^{(n)}) = g_2 \, g_3 \, \ldots \, g_n , \quad t(g^{(n)}) = g_1 \, g_2 \, \ldots \, g_{n-1} \]

and \( p(g^{(n)}) = p(g_1) \, p(g_2) \, \ldots \, p(g_n) \). If \( G \) is \( \rho \)-uniform, the ”\( n \)-stringing” graph \( G^{(n)} \) is also \( \rho \)-uniform. The natural projection

\[
\pi^{(n)} : G^{(n)} \ni g^{(n)} = (g_1 \, g_2 \, \ldots \, g_n) \rightarrow g_1 \in G
\]

is a homomorphism and \( \phi \circ \pi^{(n)} \in \Hom(G^{(n)}, I) \) for any \( \phi \in \Hom(G, I) \). However, if \( (I, \rho) \) has congruent edges there exist \( \phi_1 \in \Hom(G^{(n)}, I) \), which are not of the above form \( \phi_1 = \phi \circ \pi^{(n)} \). It is an obvious fact, that \( d(\pi^{(n)}) = 1 \), i.e. \( \Phi^{(n)} : X_{G^{(n)}} \rightarrow X_G \) is an isomorphism. We use the index \( d(T, \delta) \) and the minimal index \( d(T) \), which were defined by Definition 2.3.

Theorem 3.25. Let \( G \) be a \( \rho \)-uniform stochastic graph, which is irreducible and satisfies the positive recurrence condition. Then there exist an integer \( n \in \mathbb{N} \), a homomorphism \( \phi \in \Hom(G^{(n)}, I) \) and a commutative diagram

\[
\begin{array}{ccc}
\hat{H} & \xrightarrow{\hat{\psi}} & G^{(n)} \\
\phi \downarrow & & \downarrow \phi \\
H & \xrightarrow{\psi} & I
\end{array}
\]

such that

(i) The graph \( H = \tilde{H}_a \) is a skew product over a graph \( H \), generated by a function \( a : H \ni h \rightarrow a(h) \in \mathcal{A}_d \), \( d \in \mathbb{N} \), and the homomorphism \( \tilde{\phi} \) coincides with the natural projection \( \pi_H \) of \( H \) onto \( H \);

(ii) \( d = d(\phi) = d(T_G) \),

(iii) The homomorphisms \( \tilde{\psi} \in \Hom(\hat{H}, G) \) and \( \psi \in \Hom(H, I) \) are of degree 1 .

Proof. Let \( \zeta = \zeta_G \) be the standard Markov generator of the shift \( T_G \). It was proved in [Ru6] Theorem 4.2 and 4.3] that there exist \( n \in \mathbb{N} \) and \( \delta \in \Delta_\rho(T_G) \) such that

\[
d \leq \zeta^{(n)} := \bigvee_{k=1}^{n} T^{-k+1} \zeta , \quad d(T, \delta) = d(T) .
\]

Take \( \zeta^{(n)} \) and \( G^{(n)} \) instead \( \zeta \) and \( G \) in Proposition 3.17. Then we obtain \( \phi \in \Hom(G^{(n)}, I) \) with \( \delta = \delta_\phi \) and thus, by using Corollary 3.24, we complete the proof. \( \Box \)
4. Homomorphisms and finite extensions.

4.1. Homomorphisms of degree 1. Let $H$ be a $\rho$-uniform graph and consider a homomorphism $\psi : H \to I$. Suppose that $\psi$ is of degree 1. By Theorem 3.21 the semigroup $S(\psi)$, generated by $f_i = f_i^\psi$, $i \in I$, is 1-contractive and all its persistent sets are singletons. Using $\psi$ we can identify the graph $H$ with $I \times J$, where $J = H^{(0)}$ and for any $h = (i, j) \in H$

$$\psi(h) = i, \ s(h) = j, \ t(h) = f_i j, \ \rho(h) = \rho(i)$$

Since $d(\psi) = 1$ the partition $\delta_\psi$ is a one-sided Bernoulli generator for the Markov shift $T_H$. The factor map $\Phi_\delta : X_H \to X_\rho$ is an isomorphism, $\Phi_\delta \circ T_H = T_\rho \circ \Phi_\delta$ and we can consider the Markov partitions $\zeta_\rho := \Phi_\delta(\zeta_H)$ and $\zeta_\rho^{(0)} := \Phi_\delta(\zeta_H^{(0)})$ for $T_\rho$ on $X_\rho$, which correspond to the Markov partitions $\zeta_H$ and $\zeta_H^{(0)}$ for $T_H$ on $X_H$. The partition $\delta_\rho = \Phi_\delta(\delta_H)$ coincides with the standard Bernoulli generator of the Bernoulli shift $T_\rho$.

Thus we have, with the notations from Section 3.3

$$\delta_\rho = \{B_\rho(i)\}_{i \in I}, \ \zeta_\rho = \{D_\rho(i, j)\}_{(i, j) \in I \times J}, \ \zeta_\rho^{(0)} = \{D_\rho(j)\}_{j \in J}$$

where

$$D_\rho(i, f_i(j)) = B_\rho(i) \cap T_\rho^{-1}D_\rho(j), \ (i, j) \in I \times J$$

Hence the homomorphism $\psi$ is determined by the partitions $\delta_\rho$, $\zeta_\rho$, $\zeta_\rho^{(0)}$ uniquely up to equivalence (see Definition 3.9).

Our aim now is to construct a common extension of degree 1 for two homomorphisms of degree 1.

**Theorem 4.1.** Let $\psi_1 : H_1 \to I$, $\psi_2 : H_2 \to I$ be two homomorphisms of $\rho$-uniform graphs $H_1$ and $H_2$ onto the Bernoulli graph $(I, \rho)$ and suppose that $\psi_1$ and $\psi_2$ are of degree 1. Then there exist a $\rho$-uniform graph $H$ and homomorphisms $\psi, \chi_1$ and $\chi_2$ of degree 1, for which the following diagram commutes:

$$\begin{array}{c}
H_2 & \xrightarrow{\psi_2} & I \\
\downarrow{\chi_2} & & \downarrow{\psi} \\
H & \xrightarrow{\psi_1} & H_1
\end{array}$$

The homomorphism $\psi$ will be called a common extension of $\psi_1$ and $\psi_2$ of degree 1.

**Proof.** Denote by $(\zeta_1, \zeta_1^{(0)})$ and $(\zeta_2, \zeta_2^{(0)})$ the pairs of Markov partitions of the space $X_\rho$, which correspond to the homomorphisms $\psi_1$ and $\psi_2$. Here we omit the subscript ”$\rho$” and mark the partitions and their elements by subscripts ”$1$” and ”$2$”, respectively.

We have to construct the desired $H$ and $\psi : H \to I$ by means of the partitions

$$\zeta := \zeta_1 \lor \zeta_2, \ \zeta^{(0)} := \zeta_1^{(0)} \lor \zeta_2^{(0)}.$$ 

By the identification $H_1 = I \times J_1$ and $H_2 = I \times J_2$, we have

$$\zeta_1 = \{D_1(i, j_1)\}_{(i, j_1) \in I \times J_1}, \ \zeta_1^{(0)} = \{D_1(j_1)\}_{j_1 \in J_1},$$

$$\zeta_2 = \{D_2(i, j_2)\}_{(i, j_2) \in I \times J_2}, \ \zeta_2^{(0)} = \{D_2(j_2)\}_{j_2 \in J_2},$$

and then the partition $\zeta^{(0)}$ consists of all elements

$$D(j) = D_1(j_1) \cap D_2(j_2), \ j = (j_1, j_2) \in J.$$
where the set $J$ is defined by

$$J := \{ j = (j_1, j_2) : p^{(0)}(j) := m_\rho(D_1(j_1) \cap D_2(j_2)) > 0 \} \subset J_1 \times J_2 .$$

For any $i \in I$ and $j = (j_1, j_2) \in J$ we set $f_{i,j} := (f_{i,j_1}, f_{i,j_2})$. Then

$$D(f_{i,j}) := D_1(f_{i,j_1}) \cap D_2(f_{i,j_2}) \supseteq D_1(i, j_1) \cap D_2(i, j_2) = B(i) \cap T_\rho^{-1}(D_1(j_1) \cap D_2(j_2)) = B(i) \cap T_\rho^{-1}D(j) .$$

Since $\delta$ and $T_\rho^{-1} \in$ are independent, this implies

$$\rho^{(0)}(f_{i,j}) = m_\rho(D(f_{i,j})) \geq m_\rho(B(i) \cap T_\rho^{-1}D(j)) = \rho(i)p^{(0)}(j) .$$

Hence $f_{i,j} \in J$ for all $j \in J$ and $i \in I$.

Thus we are able to define a stochastic graph $H := I \times J$ with $H^{(0)} := J$ such that for any $j \in H^{(0)}$ and $h = (i, j) \in H$

$$s(h) := j , \ t(h) := f_{i,j} , \ p(h) := \rho(i) , \ \psi(h) := i .$$

The construction provides that $H$ is a $\rho$-uniform graph, $p^{(0)}$ is a stationary probability on $H^{(0)}$ and $\psi : H \to I$ is a homomorphism of index 1. Moreover, if we set

$$\chi_1(h) := (i, j_1) , \ \chi_2(h) := (i, j_2) , \ h = (i, j_1, j_2) \in H = I \times J_1 \times J_2 ,$$

then $\chi_1 : H \to H_1$ and $\chi_2 : H \to H_2$ are homomorphisms and Diagram (4.32) commutes. \qed

We shall use also the following sharpening of the previous theorem, which can be proved in a similar way.

**Theorem 4.2.** Let

$$\kappa_1 : H_1 \to H_0 , \ \kappa_2 : H_2 \to H_0 , \ \psi_0 : H_0 \to I$$

be homomorphisms of $\rho$-uniform graphs $H_1$, $H_2$ and $H_0$ and suppose they are of degree 1. Then there exist a $\rho$-uniform graph $H$ and homomorphisms $\chi$, $\chi_1$ and $\chi_2$ of degree 1, for which the following diagram commutes

$$\begin{array}{ccc}
H_2 & \xrightarrow{\kappa_2} & H_0 \\
\chi_2 \downarrow & & \psi_0 \downarrow \\
H & \xrightarrow{\chi_1} & H_1 \\
\chi \uparrow & & \kappa_1 \uparrow \\
\end{array}$$

(4.44)

Note that this theorem holds without adding of homomorphism $\psi_0$ i.e. for graphs, which are not necessary $\rho$-uniform, but we do not use the fact in this paper.

### 4.2. Extensions of Bernoulli graphs.

Consider a very special case of the graph skew product construction $H_a$ (see Example [3.7]), when the graph $H$ is the standard Bernoulli graph $(I, \rho)$. Let $\kappa \in \mathbb{N}$ and let $a : I \to A_\kappa$, be a function on $I$ with the values $a(i), i \in I$, in the group $A_\kappa$ of all permutations of $Y_\kappa = \{1, 2, \ldots , \kappa\}$ . According to the general GSP-construction we have $\tilde{I}_a = I \times Y_\kappa$ , $\tilde{I}_a^{(0)} = Y_\kappa$ and $\pi : \tilde{I}_a \to I$ , where for any $\tilde{h} = (i, y) \in \tilde{I}_a$

$$s(\tilde{h}) = y , \ t(\tilde{h}) = a(i)y , \ \pi(\tilde{h}) = i , \ p(\tilde{h}) = \rho(i) , \ p^{(0)}(y) = d^{-1} .$$

The stochastic graph $\tilde{I}_a$ is $\rho$-uniform and it is irreducible iff the group $\Gamma(a)$, generated by \{a(i), $i \in I$\} $\subseteq A_\kappa$ , is transitive on $Y_\kappa$.

As it was noted in Section [3.2] (see Remark 3.12) the Markov shift $T_{\tilde{I}_a}$ is isomorphic to the skew product $T_{\rho,a}$ , which acts on $X_{\rho} \times Y_\kappa$ by

$$T_{\rho,a}(x, y) = (T_{\rho x}, a(x_1)^{-1}) , \ x = \{x_n\}_{n=1}^{\infty} \in X_{\rho} , \ y \in Y_\kappa .$$
Theorem 4.3. Let $\pi_k : I_{ak} \to I$, $k = 1, 2$, be two $d$-extensions of the Bernoulli graph $(I, \rho)$, generated by functions $a_k : I \to \mathcal{A}_d$, respectively. Let the functions $A_k : X_{\rho} \to \mathcal{A}_d$, $k = 1, 2$, are defined by

$$A_k(x) := a_k(x)^{-1} , \quad x \in \{x_n\}_{n=1}^{\infty} \in X_{\rho} .$$

If there exists a measurable function $W : X_{\rho} \to \mathcal{A}_d$ such that

$$A_2(x) \cdot W(x) = W(T_\rho x) \cdot A_1(x) , \quad x \in X_{\rho}$$

then $W(x)$ does not depend on $x$, i.e. $W(x) = w_0 \in \mathcal{A}_d$ a.e. on $X_{\rho}$. Thus $A_1$ and $A_2$ are cohomologous with respect to $T_\rho$ iff $a_1$ and $a_2$ are conjugate in $\mathcal{A}_d$, i.e. $a_2(i) \cdot w_0 = w_0 \cdot a_1(i)$, $i \in I$.

Note that the last equality means the equivalence of $a_1$ and $a_2$ in the sense of Definition 3.13, since $I^{(0)} = \{o\}$.

To prove the theorem we need the following simple lemma.

Lemma 4.4. Let $\Gamma$ be a finite group with the identity element $e$. For any $b : I \to \Gamma$ denote

$$B^{(n)}(x) := b(x_1) \cdot b(x_2) \cdot \ldots \cdot b(x_n) , \quad x \in \{x_n\}_{n=1}^{\infty} \in X_{\rho}$$

and

$$\omega_b(x) := \min\{n \in \mathbb{N} : B^{(n)}(x) = e\} , \quad x \in X_{\rho} .$$

Then the transformation $T^{\omega_b}_\rho$, defined by

$$X_{\rho} \ni x \to T^{\omega_b}_\rho x := T^{\omega_b(x)}_\rho x \in X_{\rho} ,$$

is an ergodic endomorphism of $X_{\rho}$, which is in fact a one-sided Bernoulli shift.

Proof. Consider the $\Gamma$-extension of the graph $(I, \rho)$ generated by $b$.

Namely, set $\tilde{I}_b := I \times \Gamma$ and $\tilde{I}^{(0)}_b := \Gamma$ with

$$s(\tilde{i}) = (s(i), \gamma) , \quad t(\tilde{i}) = (t(i), b(i) \cdot \gamma) , \quad p(\tilde{i}) = \rho(i) , \quad p^{(0)}(\gamma) = |\Gamma|^{-1}$$

The skew product endomorphism $\tilde{T}_{\rho,b}$ corresponding to the stochastic graph $\tilde{I}_b$, acts on the space $X_{\rho} \times \Gamma$ by

$$\tilde{T}_{\rho,b}(x, \gamma) := (T^{\omega_b}_\rho x , B(x) \cdot \gamma) , \quad x \in \{x_n\}_{n=1}^{\infty} \in X_{\rho} , \quad \gamma \in \Gamma .$$

where $B(x) := b(x^{-1})$. The skew product $\tilde{T}_{\rho,b}$ can be identified with the Markov shift $T^{\omega}_\tilde{I}_b$ (see Remark 3.12). Under this identification the partition $\zeta^{(0)}_{\tilde{I}_b}$ coincides with the partition

$$\zeta^{(0)} = \nu_{X_{\rho}} \times \varepsilon_{\Gamma} = \{\tilde{E}(\gamma)\}_{\gamma \in \Gamma} ,$$

where

$$\tilde{E}(\gamma) := X_{\rho} \times \{\gamma\} \subseteq X_{\rho} \times \Gamma . \quad \gamma \in \Gamma .$$

For any $\gamma \in \Gamma$ consider the endomorphism $(\tilde{T}_{\rho,b})_{\tilde{E}(\gamma)}$ induced by $\tilde{T}_{\rho,b}$ on the set $\tilde{E}(\gamma)$. Let

$$\varphi_{\tilde{E}(\gamma)} : \tilde{E}(\gamma) \ni (x, \gamma) \to \varphi_{\tilde{E}(\gamma)}(x, \gamma) \in \mathbb{N}$$

be the corresponding return functions (2.2).

Since we use the left shifts on $\Gamma$ in the definition of the skew product $\tilde{T}_b$ and they commute with the right shifts, we have

$$\varphi_{\tilde{E}(\gamma)}(x, \gamma) = \varphi_{\tilde{E}(\gamma) \cdot \beta}(x, \gamma \cdot \beta) , \quad \gamma, \beta \in \Gamma , \quad x \in X_{\rho} .$$

Hence with (4.49) and (4.48) we have

$$\omega^b(x) = \varphi_{\tilde{E}(\gamma)}(x, \gamma) ,$$

where

$$0 \leq \omega^b(x) < \log |\Gamma| .$$
and
\[ \bar{T}_b^{\omega_b}(x, \gamma) = (T^{\omega_b}(x), \gamma) \cdot \gamma \in \Gamma, \quad x \in X_\rho, \]
Thus \( T^{\omega_b} \) is isomorphic to the endomorphisms \( (T^{\omega_b})_{D(\gamma)} \) induced by the Markov shift \( T_{\bar{h}} \) on elements \( D(\gamma) \) of the partition \( \zeta^{(0)} \). So that \( T^{\omega_b} \) is a Bernoulli shift by Proposition 4.2 \( \square \)

**Proof of Theorem 4.3** For given two functions \( a_1 \) and \( a_2 \) put
\[ b : I \ni i \to b(i) := (a_1(i), a_2(i)) \in \Gamma := \mathcal{A}_d \times \mathcal{A}_d. \]
and denote for \( k = 1, 2 \)
\[ A_k^{\omega_b}(x) := A_k(T^{\omega_b(x)}_\rho - 1 \cdot x) \cdot \ldots \cdot A_k(T^{\omega_b(x)}_\rho x) \cdot A_k(x), \quad k = 1, 2 \]
with \( A_1 \) and \( A_2 \) defined by (4.46). Then by definition of \( b \) and \( \omega_b \) we have
\[ A_2^{\omega_b}(x) := A_1^{\omega_b}(x) = e, \quad x \in X_\rho, \]
where \( e \) is the identity of \( \mathcal{A}_d \). The equality (4.47) implies
\[ A_2^{\omega_b}(x) \cdot W(x) = W(T^{\omega_b(x)}_\rho x) \cdot A_1^{\omega_b}(x) \]
and then \( W(T^{\omega_b}(x) = W(x) \) a.e. on \( X_\rho \). By Lemma 4.3 \( T^{\omega_b}_\rho \) is ergodic and hence \( W(x) \) is constant a.e. on \( X_\rho \). \( \square \)

4.3. **Equivalent extensions.** Let \( d \in \mathbb{N} \), and \( H \) be an irreducible positively recurrent stochastic graph. Given a function \( a : H \ni h \to a(h) \in \mathcal{A}_d \) consider the graph skew product \( d \)-extension \( \bar{H}_a \) of \( H \) generated by the function \( a \) (See Example 3.7). Recall that the skew product endomorphism \( \bar{T}_{H,a} \), corresponding to \( H_a \), acts on the space \( X_H \times Y_d \) by
\[ \bar{T}_{H,a}(x, y) = (T_H x, A(x)y), \quad x = \{x_n\}_{n=1}^\infty \in X_H, \quad y \in Y_d, \]
where \( A(x) := a(x_1)^{-1} \). We shall use Definition 3.13

**Theorem 4.5.** Let \( \pi_k : \bar{H}_{a_k} \to H \), \( k = 1, 2 \), be two \( d \)-extensions of \( H \) generated by functions \( a_1 \) and \( a_2 \), respectively, and let the functions \( A_k : X_H \to \mathcal{A}_d \), \( k = 1, 2 \) are defined by
\[ A_k(x) := a_k(x_1)^{-1}, \quad x = \{x_n\}_{n=1}^\infty \in X_H. \]
Then the following two conditions are equivalent
(i) \( A_1 \) and \( A_2 \) cohomologous with respect to \( T_H \), i.e. there exists a measurable map \( W : X_H \to \mathcal{A}_d \) such that
\[ A_2(x) \cdot W(x) = W(T x) \cdot A_1(x), \quad x \in X_H. \]
(ii) \( a_1 \) and \( a_2 \) cohomologous with respect to \( H \), i.e. there exists a map \( w : H^{(0)} \to \mathcal{A}_d \) such that
\[ a_2(h) \cdot w(s(h)) = w(t(h)) \cdot a_1(h), \quad h \in H \]
**Proof.** It is obvious that (4.52) implies (4.51) with
\[ W(x) = w(t(x_1)), \quad x = \{x_n\}_{n=1}^\infty \in X_H \]
That is (ii) implies (i).
To prove the converse, suppose that (4.51) holds with a suitable measurable function
\( W : X_H \to \mathcal{A}_d \).
We have to show that the function \( W(x) \) necessarily has the form (4.53), i.e. \( W(x) \) is constant on each element \( D(u) = Z_1^{-1}(u)H \) of the partition \( \zeta^{(0)}_H = \{D(u), \; u \in H^{(0)}\} \).
To this purpose we shall use induced endomorphisms, which are defined as follows.
Fix an atom $D(u)$ of the partition $\zeta^{(0)}_H$ and consider the endomorphism $T_u := (T_H)^{D(u)}$, induced by the shift $T_H$ on $D(u)$, see Section 3.1. In accordance with the general definition (2.2), the return function
\[
\varphi_u(x) = \varphi_{D(u)}(x) := \min\{n \geq 1 : T_H^n x \in D(u)\}, \quad x \in D(u)
\]
induces $T_u$ by $T_u x = T_H^{\varphi_u(x)} x$. By Proposition 3.2 the induced endomorphism $T_u$ is isomorphic to the Bernoulli shift $T_{\rho_u}$, where $I_u = \bigcup_{n=1}^{\infty} I_{u,n}$, and $\rho_u = \{\rho_u(i)\}_{i \in I_u}$ is defined by (3.22) and (3.23). That is, $I_{u,n}$ consists of all $h_1 h_2 \ldots h_n \in H^{(n)}$ such that
\[
t(h_1) = s(h_n) = u, \quad s(h_m) = t(h_{m+1}) \neq u, \quad m = 1, 2, \ldots, n - 1
\]
and
\[
\rho_u(i) = p(h_1)p(h_2)\ldots p(h_n), \quad i = h_1 h_2 \ldots h_n \in I_{u,n}, \quad n \in \mathbb{N}.
\]
For any $u \in H^{(0)}$ and $k = 1, 2$ set
\[
A^{\varphi_u}_{k}(x) := A_k(T_H^{\varphi_u(x)-1} x) \cdot \ldots \cdot A_k(T x) \cdot A_k(x), \quad x \in D(u)
\]
and
\[
a^u_k(i) := a_k(h_1) \cdot a_k(h_2) \cdot \ldots \cdot a_k(h_n), \quad i = h_1 h_2 \ldots h_n \in I_{u,n}.
\]
Then
\[
A^{\varphi_u}_{k}(x) = a^u_k(i)^{-1}, \quad x \in B_u(i) \subseteq D(u), \quad i \in I_u,
\]
where
\[
B_u(i) := \{x = \{x_n\}_{n=1}^{\infty} \in D(u) : (x_1 x_2 \ldots x_n) = i \in I_{u,n}\}.
\]
Then the equality (4.5.1) implies
\[
A^{\varphi_u}_{2}(x) \cdot W(x) = W(T_u x) \cdot A^{\varphi_u}_{1}(x), \quad x \in D(u),
\]
i.e. $A^{\varphi_u}_{1}$ and $A^{\varphi_u}_{2}$ are cohomologous on $D(u)$ with respect to $T_u = T_H^{\varphi_u}$.

Since for any fix $u \in D(u)$ the partition $\zeta_u = \{B_u(i)\}_{i \in I_u}$ is a one-sided Bernoulli generator for $T_u$, we may apply Theorem 4.3 with the Bernoulli shift $T_u = T^{\rho_u}$ and the functions $A^u_k$, $k = 1, 2$. Therefore, it follows from (4.5.1) that there exists $w(u) \in A_d$ such that $W(x) = w(u) \text{ a.e. on } D(u)$.

For every $u$ we have now an element $w(u)$ such that $W(x) = w(u) = w(t(x_1))$ for a.e. $x \in D(u)$. Hence $W(x)$ is of the form (4.5.3), $W(T_H x) = w(s(x_1))$. Thus (4.5.1) implies (4.5.2).

As a consequence we obtain

**Theorem 4.6.** Let $\pi_k : \tilde{H}_{a_k} \to H$, $k = 1, 2$, be two $d$-extensions of $H$ generated by functions $a_1$ and $a_2$, respectively. Let also $\psi : H \to I$ be an homomorphism of degree 1. Suppose $d = d(T_{H,a_1}) = d(T_{H,a_2})$. Then the endomorphisms $\tilde{T}_{H,a_1}$ and $\tilde{T}_{H,a_2}$ are isomorphic iff $a_1$ and $a_2$ cohomologous with respect to $H$.

**Proof.** Since $d(\psi) = 1$ the factor map $\Psi : \Phi_\psi : X_H \to X_I$ is an isomorphism. Consider two skew products over the Bernoulli shift $T_\rho$
\[
\tilde{T}_k(x, y) = (T_\rho x, B_k(x) y), \quad (x, y) \in X_\rho \times Y_d, \quad k = 1, 2
\]
where $B_k(x) := A_k(\Psi^{-1} x)$ and $A_k$ induced by $a_k$ as above (4.5.0). Each of the shifts $T_{H,a_k}$ is a simple $\rho$-uniform endomorphism by Theorem 2.1. The skew products $\tilde{T}_{H,a_k}$ as well as the shifts $T_{H,a_k}$, are isomorphic to $\tilde{T}_k$. They are $\rho$-uniform endomorphisms and $d = d(\tilde{T}_1) = d(\tilde{T}_2)$. By Theorem 2.10 $\tilde{T}_1$ and $\tilde{T}_2$ are isomorphic iff the functions $B_1$ and $B_2$ are cohomologous with respect to $T_\rho$. This means that the functions $A_1$ and $A_2$ are cohomologous with respect to $T_H$. Finally, by Theorem 4.5 the last condition holds iff $a_1$ and $a_2$ cohomologous with respect to $H$. \qed
4.4. **GSP-extensions and persistent d-partitions.** Let $H$ be a stochastic graph and $(I, \rho)$ a standard Bernoulli graph. In this section we study extensions of the form

$$\Psi \Psi \Psi \Psi \Psi \Psi \Psi \Psi$$

where the graph $H$ be an extension of the Bernoulli graph $(I, \rho)$ by a homomorphism $\psi$ of degree $d(\psi) = 1$ and $\bar{H} = \bar{H}_a$ be a graph skew product $d$-extension of $H$ generated by a function $a : H \to A_d$ (See Example 3.7). The diagrams of the above form (4.55) will be referred to as (π, ψ)-extensions. We assume that the graph $\bar{H}$ is irreducible, i.e. the corresponding Markov shift $T_{\bar{H}}$ and skew product $T_{\bar{H}, a}$ are ergodic.

Fixing an extension (4.55) and setting $J = H(0)$, we identify $H$ with $I \times J$ such that

$$\psi(h) = i, \ s(h) = j, \ t(h) = f_i(j), \ p(h) = \rho(i)$$

for any $h = (i, j) \in H = I \times J$. Here the maps $f_i : J \to J$ are uniquely determined by

$$f_{i,j} = t(i,j), \ (i,j) \in I \times J$$

and the semigroup $S(\psi)$, generated by $\{f_i, i \in I\}$ is 1-contractive, since $d(\psi) = 1$ (Theorem 3.21).

The $d$-extension $\bar{H} = \bar{H}_a$ is described now as follows:

$$\bar{H} = I \times J \times Y_d, \quad \bar{H}(0) = H(0) \times Y_d = J \times Y_d,$$

where for any $\bar{h} = (i, j, y)$

$$s(\bar{h}) = (j, y), \ t(\bar{h}) = (f_{i,j}, a(i,j)y), \ p(\bar{h}) = \rho(i), \ a(\bar{h}) = a(i,j)$$

The homomorphisms $\psi, \pi$ and $\phi = \psi \circ \pi$ are defined by

$$\pi(\bar{h}) = h = (i, j), \quad \pi(0)(j, y) = j, \quad \phi(\bar{h}) = \psi(h) = i,$$

where $d(\phi) = d(\pi) = d$ and the diagram

$$\begin{array}{ccc}
I \times J \times Y_d & \xrightarrow{\phi} & I \\
\downarrow{\pi} & & \downarrow{\psi} \\
I \times J & \xrightarrow{\psi} & I
\end{array}$$

commutes. The semigroup $S(\phi)$ can be described now as a $d$-extension $\bar{S} = \bar{S}(\pi, \psi)$ of the semigroup $S(\psi)$.

Set

$$\bar{f}(i,j) := t(i,j,y) = (f_{i,j}, a(i,j)y), \ (j,y) \in J \times Y_d, \ i \in I.$$

The maps $\bar{f}$ act on $J \times Y_d$.

The semigroup $\bar{S}$, generated by $\bar{f}, \ i \in I$, consists of all maps of the form:

$$\{\bar{f}_{i_1i_2...i_n}, i_1i_2...i_n \in I^n, n \in \mathbb{N}\},$$

where

$$\bar{f}_{i_1i_2...i_n}(j,y) = (f_{i_1i_2...i_n,j}, a(i_1i_2...i_n,j)y)$$

and

$$a(i_1i_2...i_n,j) := a(i_1, f_{i_1i_2...i_n,j})...a(i_{n-1}, f_{i_n,j})a(i_n,j).$$

Note that $\bar{S}(\psi) \ni f \to \bar{f} \in \bar{S}$ is an isomorphism between the semigroups.

**Proposition 4.7.** The semigroup $\bar{S}$ is $d$-contractive and its persistent $d$-sets are of the form

$$\mathcal{L}(\bar{S}) = \{L_j, j \in J\}, \quad L_j := \{j\} \times Y_d.$$
Definition 4.8. Let the semigroup $\bar{S}$ be as above.

(i) A subset $S$ of $\bar{S}$ is called transversal with respect to $\phi$ if $\phi(S) = \bar{S}$ and the restriction $\phi|_S : S \to \bar{S}$ is a bijection. A partition $\pi = \{R_1, R_2, \ldots, R_d\}$ will be called transversal with respect to $\phi$ if all the set $R_1, R_2, \ldots, R_d$ are transversal.

(ii) A transversal partition $\pi$ will be called persistent with respect to semigroup $\bar{S}$, if for every transversal partition $\pi$ and every finite subset $E \subseteq J$ there exists $\bar{f} \in \bar{S}$ such that $\bar{f}^{-1}r_1 |_E = r |_E$.

Denote by $\mathcal{R}$ the set of all transversal partitions and by $\mathcal{R}(\bar{S})$ the set of all persistent partitions for the semigroup $\bar{S}$.

For any finite set $F \subseteq J \times Y_d$ the set $E := \pi(0)(F) \subseteq J$ is also finite. Since the semigroup $\bar{S}$ is 1-contractive there exist $i_1 \leq i_2 \leq \ldots \leq i_n \in \mathbb{N}$ and $j \in J$ such that $f_{i_1i_2\ldots i_n}(E) = \{j\}$ and hence $\bar{f}_{i_1i_2\ldots i_n}(F) \subseteq L_j$. On the other hand $d = |L_j| = |f_{i}(L_j)|$ for all $i \in I, j \in J$. Thus the sets $L_j$ and only they are persistent sets for the semigroup $\bar{S}$.

For every $E \subseteq J$ set $\bar{E} := \pi(0)^{-1}E = E \times Y_d$.

Lemma 4.9. (i) The set $\pi(0) \subseteq \mathcal{R}(\bar{S})$ is not empty.

(ii) $\bar{f}^{-1} \mathcal{R}(\bar{S}) \subseteq \mathcal{R}(\bar{S})$ for all $\bar{f} \in \bar{S}$.

(iii) $\mathcal{R}(\bar{S})$ is the least subset of $\mathcal{R}$ with the property (ii).

Proof. Consider a subset $\mathcal{R}_0(\bar{S})$ of $\mathcal{R}$ consisting of all $r \in \mathcal{R}$ having the following property:

For any finite subset $E \subseteq J$ there exists $f \in \mathcal{S}(\psi)$ such that $f(E)$ is a singleton, i.e. $|f(E)| = 1$, and $r |_{\bar{E}} = \bar{f}^{-1} |_{\bar{E}}$, where $\bar{E} := E \times Y_d$ and $\bar{r} = \varepsilon_{J \times Y_d}$.

We show that $\mathcal{R}(\bar{S}) = \mathcal{R}_0(\bar{S}) \neq \emptyset$.

Take a sequence $E_n \not\owns J$, $|E_n| < \infty$. Since $d(\mathcal{S}(\psi)) = 1$ we can find a sequence $g_n \in \mathcal{S}(\psi)$ such that for all $n \in \mathbb{N}$ and $f_n := g_1 \cdot g_2 \cdot g_1$, the set $f_n(E_n)$ is single-point, i.e. $|f_n(E_n)| = 1$. Using the decreasing sequence of partitions

$$\varepsilon \geq \bar{f}_1^{-1} \varepsilon \geq \bar{f}_2^{-1} \varepsilon \geq \ldots \geq \bar{f}_n^{-1} \varepsilon \geq \ldots$$

set $r_0 := \bigwedge_{n=1}^{\infty} \bar{f}_n^{-1} \varepsilon$. Since $|f_n(E_n)| = 1$ the restriction $r_0 |_{E_n}$ consists of $d$ sets, whose projections on $J$ are $E_n$. Hence $r_0 \in \mathcal{R}$ and $r_0 |_{E_n} = \bar{f}_n^{-1} \varepsilon |_{E_n}$. We see that $r_0 \in \mathcal{R}_0(\bar{S})$, i.e. $\mathcal{R}_0(\bar{S})$ is not empty.

Let $r \in \mathcal{R}_0(\bar{S})$ and $r_1 \in \mathcal{R}$. For any finite subset $E \subseteq J$ there exists $\bar{f} \in \mathcal{S}(\bar{S})$ such that $r |_{\bar{E}} = \bar{f}^{-1} \varepsilon |_{\bar{E}}$. Then

$$\bar{f}^{-1}r_1 |_{\bar{E}} \leq \bar{f}^{-1} \varepsilon |_{\bar{E}} = r |_{\bar{E}}.$$
Conversely, let \( r \in \mathcal{R}(\mathcal{S}) \) and \( E \) be a finite subset of \( J \). There are exist \( \bar{f} \in \mathcal{S} \) and \( r_1 \in \mathcal{R} \) such that \( r_1 \mid_E = \bar{f}^{-1}\varepsilon \mid_E \). On the other hand, since \( r \in \mathcal{R}(\mathcal{S}) \), we can choose \( \bar{f}_1 \in \mathcal{S} \) for which \( \bar{f}_1^{-1}r_1 \mid_E = r \mid_E \). Hence
\[
\bar{f}^{-1}r_1 = \bar{f}_1^{-1}\bar{f}^{-1}\varepsilon \mid_E = (\bar{f}_1^{-1}\bar{f})^{-1}\varepsilon \mid_E.
\]
We see that \( r \in \mathcal{R}_0(\mathcal{S}) \) and thus \( \mathcal{R}(\mathcal{S}) = \mathcal{R}_0 \) and Part (i) follows.

Parts (ii) and (iii) follow in the same manner by the definition of \( \mathcal{R}(\mathcal{S}) \) and by the equality \( \mathcal{R}(\mathcal{S}) = \mathcal{R}_0(\mathcal{S}) \).

\[\square\]

4.5. **Irreducible \( d \)-extensions.** In this section we continue to study \((\pi, \psi)\)-extensions of the form \((4.55)\)
\[
(\pi, \psi) : \begin{array}{ccc}
\bar{H} & \xrightarrow{\pi} & H \\
\downarrow{\kappa} & & \downarrow{\psi} \\
H_1 & \xrightarrow{\psi_1} & I
\end{array},
\]
where the graph \( H \) is an extension of the Bernoulli graph \((I, \rho)\) by a homomorphism \( \psi \) of degree 1 and \( \bar{H} = \bar{H}_a \) be a GSP \( d \)-extension of \( H \), generated by a function \( a : H \to A_d \).

Fix \( d \) and \((I, \rho)\) and consider the set \( \mathcal{E}xt^d(I, \rho) \) of all \((\pi, \psi)\)-extensions of the form \((4.55)\). This set is equipped with a natural partial order and with an equivalence relation as follows

**Definition 4.10.** Let \((\pi, \psi) : \begin{array}{ccc}
\bar{H} & \xrightarrow{\pi} & H \\
\downarrow{\kappa} & & \downarrow{\psi} \\
H_1 & \xrightarrow{\psi_1} & I
\end{array} \quad \text{and} \quad (\pi_1, \psi_1) : \begin{array}{ccc}
\bar{H}_1 & \xrightarrow{\pi_1} & H_1 \\
\downarrow{\kappa} & & \downarrow{\psi_1} \\
H_1 & \xrightarrow{\psi_1} & I
\end{array} \)
be two \((\pi, \psi)\)-extensions from \( \mathcal{E}xt^d(I, \rho) \).

Let \( a \) and \( a_1 \) be the functions, which generate the extensions \( \bar{H} \) and \( \bar{H}_1 \), respectively.

(i) A homomorphism \( \bar{\kappa} : \bar{H} \to \bar{H}_1 \) is said to be a **trivializable** \( d \)-extension of a homomorphism \( \kappa : H \to H_1 \), if the square part of Diagram \((4.60)\) (below) commutes and the functions
\[a_1 \circ \chi \quad \text{and} \quad a \] are cohomologous with respect to \( H \).

(ii) We shall say that \((\pi_1, \psi_1) \preceq (\pi, \psi)\) if there is a commutative diagram
\[
\begin{array}{ccc}
\bar{H} & \xrightarrow{\pi} & H \\
\downarrow{\bar{\kappa}} & & \downarrow{\psi} \\
\bar{H}_1 & \xrightarrow{\psi_1} & I
\end{array},
\]
where \( \bar{\kappa} \in \text{Hom}(\bar{H}, \bar{H}_1) \) is a trivializable \( d \)-extension of \( \kappa \in \text{Hom}(H, H_1) \).

(iii) We shall say that \((\pi_1, \psi_1) \sim (\pi, \psi)\) if there is commutative Diagram \((4.60)\) where both \( \kappa : H \to H_1 \) and its \( d \)-extension \( \bar{\kappa} : \bar{H} \to \bar{H}_1 \) are isomorphisms.

In connection with Part (i) of the definition, note that an extension \( \bar{\kappa} : \bar{H} \to \bar{H}_1 \) is trivializable iff it is equivalent to a trivial extension of \( \kappa : H \to H_1 \) (see Remark \((3.14)\)).

It can be checked also that \((\pi_1, \psi_1) \preceq (\pi, \psi)\) and \((\pi, \psi) \preceq (\pi_1, \psi_1)\) imply \((\pi_1, \psi_1) \sim (\pi, \psi)\), but we do not use the fact in this paper.

Our aim now is to describe "minimal" elements of \( \mathcal{E}xt^d(I, \rho), \preceq \).

**Definition 4.11.** An extension \((\pi, \psi) \in \mathcal{E}xt^d(I, \rho)\) is called **irreducible** if \((\pi_1, \psi_1) \sim (\pi, \psi)\) as soon as \((\pi_1, \psi_1) \in \mathcal{E}xt^d(I, \rho)\) and \((\pi_1, \psi_1) \preceq (\pi, \psi)\).
Theorem 4.12. For any \((\pi, \psi) \in \mathcal{E}xt^d(I, \rho)\) there exists a unique up to equivalence irreducible \((\pi, \psi)\)-extension \((\pi_s, \psi_s) \in \mathcal{E}xt^d(I, \rho)\) such that \((\pi_s, \psi_s) \preceq (\pi, \psi)\).

To prove the theorem we fix a pair \((\pi, \psi) \in \mathcal{E}xt^d(I, \rho)\) and again use the identification \((4.56)\). Namely,

\[
(\pi, \psi) : \tilde{H} = I \times J \times Y_d \xrightarrow{\pi} H = I \times J \xrightarrow{\psi} I
\]

where \(H^{(0)} = J\) and \(\tilde{H}^{(0)} = H^{(0)} \times Y_d = J \times Y_d\) as in Section 4.4.

We construct the desired irreducible \((\pi_s, \psi_s)\)-extension and a corresponding commutative diagram

\[
\begin{array}{ccc}
\tilde{H} & \xrightarrow{\pi} & H \\
\downarrow & & \downarrow \\
\tilde{H}_s & \xrightarrow{\pi_s} & H_s
\end{array}
\]

\[
\begin{array}{ccc}
& & \psi \\
\downarrow & & \downarrow \\
& & \psi_s \\
\tilde{H}_s & \xrightarrow{\pi_s} & H_s
\end{array}
\]

by means of the semigroup \(S = \tilde{S}(\pi, \psi)\) and its persistent partitions \(\mathcal{R}(\tilde{S})\).

Definition 4.13. A partition \(\xi\) of \(J = H^{(0)}\) is called \textbf{reducing} partition if the following two conditions hold

(i) \(f^{-1}\xi \subseteq \xi\) for all \(f \in \mathcal{S}(\psi)\), i.e. \(\xi\) is \(\mathcal{S}(\psi)\)-invariant

(ii) For any element \(C \in \xi\) denote \(\bar{C} := \pi^{-1}C\) and let \(r|_{\bar{C}}\) be the restriction of the partition \(r \in \mathcal{R}(\mathcal{S})\) on the set \(\bar{C}\). Then all the partitions \(r|_{\bar{C}}, r \in \mathcal{R}(\mathcal{S})\) coincide with each other.

Consider the set \(\Xi\) of all reducing partitions \(\xi\) on \(H^{(0)}\).

For any \(\xi \in \Xi\) we have \(\pi^{(0)}|_{\bar{C}}^{-1}\xi = \xi \times \nu_d\) and the partition \(\pi^{(0)}|_{\bar{C}}^{-1}\xi \vee r\) does not depend on the choice of \(r \in \mathcal{R}(\mathcal{S})\). So that we may set

\[
\bar{\xi} := \pi^{(0)}|_{\bar{C}}^{-1}\xi \vee r, \ \xi \in \Xi
\]

and \(\bar{\Xi} := \{\bar{\xi} : \xi \in \Xi\}\) on \(\tilde{H}^{(0)}\).

Since \(\xi\) is \(\mathcal{S}(\psi)\)-invariant and \(\mathcal{R}(\mathcal{S})\) is \(\tilde{S}\)-invariant by Lemma 4.9 the partition \(\bar{\xi}\) is also \(\tilde{S}\)-invariant.

Therefore we may introduce the \textbf{factor pair}

\[
\begin{array}{ccc}
\tilde{H}/\xi & \xrightarrow{\pi/\xi} & H/\xi \\
\downarrow & & \downarrow \\
\tilde{H}/\xi & \xrightarrow{\psi/\xi} & I
\end{array}
\]

Namely, we set

\[
H/\xi := I \times J/\xi, \ \tilde{H}/\xi := I \times J/\xi \times Y_d
\]

Any element of \(\bar{\xi}\) consists of \(d\) elements of the form \(R^C_y, \ y \in Y_d\), where \(C \in \xi\) and \(\pi^{(0)}(R^C_y) = C\). Hence, by possibly passing to an equivalent extension, we may assume that \(R^C_y = C \times \{y\}\), i.e. \(\bar{\xi} = \xi \times \nu_d\). This means that the function \(a = a(i, j)\), generating the extension \(\tilde{H} = H_d\), does not depend on \(j\) on the elements of \(\xi\). Hence the equalities \((4.57)\) and \((4.59)\) well define \(a/\xi\) and \(\tilde{H}/\xi := (H/\xi)_{a/\xi}\). Thus we have shown

Proposition 4.14. For any \(\xi \in \Xi\) the natural projections

\[
\pi^{(0)}_\xi : \tilde{H}^{(0)} \to \tilde{H}^{(0)}/\xi, \ \pi^{(0)} : H^{(0)} \to H^{(0)}/\xi
\]
uniquely determine \((\pi/\xi, \psi/\xi) \in \mathcal{E}_{xt^d}(I, \rho)\) such that \((\pi/\xi, \psi/\xi) \preceq (\pi, \psi)\) with the corresponding commutative diagram

\[
\begin{array}{ccc}
\hat{H} & \xrightarrow{\pi} & H \\
\downarrow \pi\xi & \downarrow \pi\xi & \downarrow \pi\xi \\
\hat{H}/\xi & \xrightarrow{\pi/\xi} & H/\xi \\
& \psi/\xi \nearrow & 1
\end{array}
\]

Conversely

**Proposition 4.15.** For any \((\pi_1, \psi_1) \preceq (\pi, \psi)\) there exists \(\xi \in \Xi\) such that \((\pi/\xi, \psi/\xi) \sim (\pi_1, \psi_1)\)

**Proof.** Take the map \(\kappa^{(0)} : H^{(0)} \to H_1^{(0)}\) induced by homomorphism \(\kappa : H \to H_1\) from Diagram 3.29 and set \(\xi := \kappa^{(0)} \cdot \varepsilon_{H_1^{(0)}}\). Then \(\xi \in \Xi\) and it is desired \(\square\)

**Proof of Theorem 4.12.** It is easily to see that \(\Xi\) is a lattice, i.e. \(\xi_1 \lor \xi_2 \in \Xi\) and \(\xi_1 \land \xi_2 \in \Xi\) for all \(\xi_1, \xi_2 \in \Xi\). Herewith, \(\Xi\) has the least element. Denote the least element by \(\xi_*\) and let \(\xi_* := (\xi_*)\) be the corresponding partition of \(\hat{H}^{(0)}\). Note that \(\xi_*\) is the least element of \(\Xi\). Herewith \(\xi_*\) is the least partition of \(\hat{H}^{(0)}\) such that for all \(r \in \mathcal{R}(\hat{S})\) and every \(C \in \xi\) the restriction \(r\mid_C\) consists precisely of \(d\) elements.

Putting \(\xi = \xi_*\) in Diagram 4.65 (Proposition 4.14) we obtain Diagram 4.62 with

\[
H_* = H/\xi_* , \quad \hat{H}_* = \hat{H}/\xi_* , \quad \pi_* = \pi/\xi_* , \quad \psi_* = \psi/\xi_* .
\]

and \((\pi_*, \psi_*) \preceq (\pi, \psi)\).

Using by the above propositions and Lemma 4.9 we see that the pair \((\pi_*, \psi_*)\) is irreducible and that it is the only (up to equivalence) irreducible pair majorized by \((\pi, \psi)\). \(\square\)

**Remark 4.16.** The above arguments show that a pair \((\pi, \psi) \in \mathcal{E}_{xt^d}(I, \rho)\) is irreducible iff \((\pi_*, \psi_*)\) is irreducible, i.e. \(\Leftrightarrow \varepsilon_{H^{(0)}}\). The last equality means that the persistent partitions \(\mathcal{R}(\hat{S})\) separate the points of \(H^{(0)}\) in the following sense: for every pair \(u_1, u_2 \in H^{(0)}\) there exist \(R_1 \in r_1 \in \mathcal{R}(\hat{S})\) and \(R_2 \in r_2 \in \mathcal{R}(\hat{S})\) such that

\[
\pi^{(0)}(u_1) \cap R_1 \cap R_2 \neq \emptyset , \quad \pi^{(0)}(u_2) \cap R_1 \cap R_2 = \emptyset .
\]

5. Canonical form and classification.

5.1. Main Theorems. The following two theorems claim the existence and uniqueness of the canonical form of \(\rho\)-uniform one-sided Markov shifts.

**Theorem 5.1.** Let \(G\) be a \(\rho\)-uniform stochastic graph, which is irreducible and positively recurrent. Then there exists a \((\pi, \psi)\)-extension

\[
(\pi, \psi) : \quad \hat{H} \xrightarrow{\pi} H \xrightarrow{\psi} I
\]

such that

(i) The shifts \(T_G\) and \(T_H\) are isomorphic,

(ii) \((\pi, \psi) \in \mathcal{E}_{xt^d}(I, \rho),\) where \(d = d(T_G)\) is the minimal index of the shift \(T_G\),

(iii) The extension \((\pi, \psi)\) is irreducible.
Proof. Combining the results of Theorems 4.12 and 3.25 we obtain from Diagrams 4.62 and 3.41 the following commuting diagram

\[
\begin{array}{ccc}
\tilde{H} & \xrightarrow{\psi} & G^{(n)} & \xrightarrow{\pi^{(n)}} & G \\
\downarrow & & \downarrow & & \downarrow \\
H & \xrightarrow{\phi} & I \\
\downarrow & & \downarrow & & \downarrow \\
\bar{H}_s & \xrightarrow{\pi_s} & H & \xrightarrow{\psi} & I \\
\end{array}
\]

Here, \(\pi, \pi_s\) and \(\phi\) are homomorphisms of degree \(d = d(T_G)\), all other homomorphisms are of degree 1, and the extension \((\pi_s, \psi) \in \mathcal{E}xt^d(I, \rho)\) is irreducible.

Since \(G\) and \(\bar{H}_s\) have a common extension \(\tilde{H}\) of degree 1, the shifts \(T_G\) and \(T_{\bar{H}_s}\) are isomorphic. Thus the the extension \((\pi_s, \psi)\) is desired. \(\square\)

**Definition 5.2.** We shall say that \(T_{\bar{H}}\) is a canonical form of the shift \(T_G\), if there exists an extension \(5.66\) satisfying the conditions of Theorem 5.1. Herewith the graph \(\tilde{H}\) is said to be the canonical graph for \(T_G\).

Theorem 5.1 states the existence of the canonical form. Turn to the uniqueness.

**Theorem 5.3.** Let \(G_1\) and \(G_2\) be two \(\rho\)-uniform stochastic graphs, which are irreducible and satisfy the positive recurrence condition. Suppose the shifts \(T_{G_1}\) and \(T_{G_2}\) are represented in the canonical form \(T_{\bar{H}_1}\) and \(T_{\bar{H}_2}\), respectively, and let

\[
(\pi_k, \psi_k) : \tilde{H}_k \xrightarrow{\pi_k} H \xrightarrow{\psi_k} I, \quad k = 1, 2
\]

be corresponding canonical \((\pi, \psi)\)-extensions.

Then the following conditions are equivalent

(i) The shifts \(T_{G_1}\) and \(T_{G_2}\) are isomorphic, \((T_{G_1} \sim T_{G_2})\).

(ii) The graphs \(\tilde{H}_1\) and \(\tilde{H}_2\) are isomorphic, \((\tilde{H}_1 \sim \tilde{H}_2)\).

(iii) The extensions \((\pi_1, \psi_1)\) and \((\pi_1, \psi_2)\) are equivalent, \(((\pi_1, \psi_1) \sim (\pi_2, \psi_2))\).

Proof. By the definition we have \(T_{G_1} \sim T_{\bar{H}_1}, T_{G_2} \sim T_{\bar{H}_2}\) and

\[
(\pi_1, \psi_1) \sim (\pi_2, \psi_2) \implies \tilde{H}_1 \sim \tilde{H}_2 \implies T_{\bar{H}_1} \sim T_{\bar{H}_2}
\]

Thus we need to prove only

\[
T_{\bar{H}_1} \sim T_{\bar{H}_2} \implies (\pi_1, \psi_1) \sim (\pi_2, \psi_2)
\]

Suppose \(T_{\bar{H}_1} \sim T_{\bar{H}_2}\) and let \(a_k : H_k \rightarrow A_d\), \(k = 1, 2\), be the functions generating \(\tilde{H}_k\), where \(d = d(T_{\bar{H}_1}) = d(T_{\bar{H}_2})\).

Since both of \(\psi_1 : H_1 \rightarrow I\) and \(\psi_2 : H_2 \rightarrow I\) are of degree 1, we can apply Theorem 4.1 and to construct a common extension \(H\) of \(H_1\) and \(H_2\). Herewith, the corresponding Diagram 4.42 commutes and the homomorphisms \(\psi : H \rightarrow I\), \(\chi_1 : H \rightarrow H_1\) and \(\chi_2 : H \rightarrow H_2\) are of degree 1.
By Remark 3.14 each of homomorphisms $\chi_k : H_{b_k} \to H_k$, $k = 1, 2$ admits the trivial extension $\bar{\chi}_k : H_{b_k} \to \bar{H}_k$ with the commuting diagram

\[
\begin{array}{ccc}
H_{b_k} & \xrightarrow{\bar{\chi}_k} & \bar{H}_k \\
\downarrow{\pi_k} & & \downarrow{\pi_k} \\
H & \xrightarrow{\chi_k} & H_k
\end{array}
\]

Here $\bar{\chi}_k$ is of degree 1 and $b_k := a_k \circ \chi_k$ for $k = 1, 2$. Since $d(\bar{\chi}_1) = d(\bar{\chi}_2) = 1$ we have $T_{H_1} \sim T_{H_{b_1}}$ and $T_{H_2} \sim T_{H_{b_2}}$. Therefore $T_{H_1} \sim T_{H_2}$ implies that the skew products $\bar{T}_{H_{b_1}}$ and $\bar{T}_{H_{b_2}}$ are isomorphic.

Thus we have two GSP $d$-extensions $\pi_{b_k} : \bar{H}_{b_k} \to H$, $k = 1, 2$, of $H$ and a homomorphism $\psi : H \to I$ of degree 1. Herewith, the number $d$ is the minimal index of $\bar{T}_{H_{b_1}}$ and $\bar{T}_{H_{b_2}}$. By Theorem 4.6 the functions $b_1$ and $b_2$ are cohomologous with respect to $H$. Hence two constructed $(\pi, \psi)$-extensions

\[
(\pi_{b_k}, \psi) : \bar{H}_{b_k} \xrightarrow{\pi} H \xrightarrow{\psi} I, \ k = 1, 2
\]

are equivalent, $(\pi_{b_1}, \psi) \sim (\pi_{b_2}, \psi)$.

On the other hand by constructing both two diagrams

\[
\begin{array}{ccc}
\bar{H}_{b_k} & \xrightarrow{\bar{\pi}_{b_k}} & H \\
\downarrow{\bar{\pi}_k} & & \downarrow{\pi_k} \\
\bar{H}_k & \xrightarrow{\pi_k} & H_k
\end{array}
\]

\[
\begin{array}{ccc}
\bar{H}_{b_k} & \xrightarrow{\bar{\psi}_k} & I \\
\downarrow{\bar{\pi}_k} & & \downarrow{\pi_k} \\
\bar{H}_k & \xrightarrow{\pi_k} & H_k
\end{array}
\]

commute. This means that $(\pi_1, \psi_1) \preceq (\pi_{b_1}, \psi)$ and $(\pi_2, \psi_2) \preceq (\pi_{b_2}, \psi)$.

The pairs $(\pi_1, \psi_1)$ and $(\pi_2, \psi_2)$ are irreducible and they are majorized by equivalent pairs. Hence they are equivalent.

We have shown (5.69).

As a consequence we have also

**Theorem 5.4.** Under conditions of Theorem 5.3 the shifts $T_{G_1}$ and $T_{G_2}$ are isomorphic iff the graphs $G_1$ and $G_2$ have a common extension of degree 1, i.e. there exists a diagram

\[
G_1 \xleftarrow{\phi_1} G \xrightarrow{\phi_2} G_2
\]

where homomorphisms $\phi_1$ and $\phi_2$ are of degree 1.

**Proof.** By Theorem 3.25 we have two diagram of homomorphisms

\[
G_k \xleftarrow{\pi(n)} G_{k}^{(n)} \xrightarrow{\psi_k} H_k \xrightarrow{\pi_k} H_k \xrightarrow{\psi_k} I, \ k = 1, 2
\]

where $d(\pi(n)) = d(\psi_k) = d(\psi_k) = 1$ and $\pi_k$ is a $d$-extension. So that $(\pi_k, \psi_k) \in E x t^d(I, \rho)$.

By Theorem 4.12 each pair $(\pi_k, \psi_k)$, $k = 1, 2$ majorizes an irreducible pair from $E x t^d(I, \rho)$. If the the shifts $T_{G_1}$ and $T_{G_2}$ are isomorphic the irreducible pairs are equivalent (Theorem 5.3) and we may assume without loss of generality that they coincide with each other.
Thus there exists \((\pi_0, \psi_0) \in \mathcal{E}xt^d(I, \rho)\) with two commuting diagrams

\[
\begin{array}{c}
\tilde{H}_k \xrightarrow{\pi_k} H_k \\
\downarrow \tilde{\kappa}_k \quad \downarrow \kappa_k \\
\tilde{H}_0 \xrightarrow{\pi_0} H_0 \xrightarrow{\psi_0} I
\end{array}
\]

\hspace{1cm} k = 1, 2

Passing possibly to equivalent extensions we may also assume that \(\tilde{\kappa}_1\) and \(\tilde{\kappa}_2\) are trivial extensions of \(\kappa_1\) and \(\kappa_2\).

By Theorem 5.2 and Remark 3.14 we find a common extension of degree 1

\[
H_1 \xleftarrow{\chi_1} H \xrightarrow{\chi_2} H_2
\]

of \(H_1\) and \(H_2\) with the trivial extensions

\[
\tilde{H}_1 \xleftarrow{\tilde{\chi}_1} \tilde{H} \xrightarrow{\tilde{\chi}_2} \tilde{H}_2
\]

of \(\chi_1\) and \(\chi_2\) such that the corresponding diagram

\[
\begin{array}{c}
\tilde{H}_2 \xrightarrow{\tilde{\kappa}_2} \tilde{H}_0 \\
\downarrow \tilde{\chi}_2 \quad \downarrow \chi \\
\tilde{H} \xrightarrow{\pi} \tilde{H}_1 \xrightarrow{\pi_1} \tilde{H}_0 \\
\downarrow \chi \quad \downarrow \pi_2 \\
H_2 \xrightarrow{\kappa_2} H_0 \xrightarrow{\psi} I
\end{array}
\]

commutes. Therefore we have

\[
G_1 \xleftarrow{\pi^{(n)}_1} G_1^{(n)} \xleftarrow{\tilde{\psi}_1} \tilde{H}_1 \xleftarrow{\tilde{\kappa}_1} \tilde{H} \xrightarrow{\tilde{\chi}_2} \tilde{H}_2 \xrightarrow{\chi_2} H_2 \xrightarrow{\kappa_2} H_0 \xrightarrow{\psi_2} G_2^{(n)} \xrightarrow{\pi^{(n)}} G_2
\]

Putting \(G := \tilde{H}\) and \(\phi_k := \tilde{\kappa}_k \circ \tilde{\psi}_k \circ \pi^{(n)}\) for \(k = 1, 2\), we obtain the desired common extension of degree 1 \((5.73)\).

\[\square\]

5.2. **Consequences and examples.** Consider some particular cases.

**Extensions of Bernoulli graphs.** Let \((I, \rho)\) be a standard Bernoulli graph and let \(d \in \mathbb{N}\). Let \(a : I \to A_d\) be a function \(a : I \to A_d\) on \(I\) with the values \(a(i)\), \(i \in I\), in the group \(A_d\) of all permutations of \(Y_d = \{1, 2, \ldots, d\}\). Consider a \(d\)-extension \(\tilde{I}_a\) generated by the function \(a\) (See Section 4.2). We assume that the group \(\Gamma(a)\), generated by \(a(i), i \in I\), acts transitively on \(Y_d\). This provides that the shift \(T_{\tilde{I}_a}\) and the skew product \(\tilde{T}_{I,a}\) are ergodic.

We want to clarify: when is \(\tilde{I}_a\) the canonical graph for the corresponding Markov shift \(T_{\tilde{I}_a}\) (Definition 5.2). Let \(\pi : \tilde{I}_a \to I\) be the projection and \((\pi, \psi) \in \mathcal{E}xt^d(I, \rho)\). Since every homomorphism \(\psi : I \to I\) is an automorphism, the pair \((\pi, \psi)\) is irreducible. Therefore \(\tilde{I}_a\) is a the canonical graph iff \(d(T_{\tilde{I}_a}) = d\).

**Proposition 5.5.** If the function \(a\) satisfies the following condition

\[
(5.80) \quad \rho(i) = \rho(i') \implies a(i) = a(i') , \ i, i' \in I
\]

then \(d(T_{\tilde{I}_a}) = d\).
Proof. Suppose the condition (5.80) holds. The Markov shift $T_{I_a}$ is isomorphic to the skew product $T = T_{\rho,a}$, which acts on $X_{\rho} \times Y_d$ by (4.45). So that we have $d(T_{I_a}) = d(T)$ and by Theorem 2.14 $d(T) = d_{\gamma,\beta}(T)$.

A direct computation, using (5.80), the definition of $\gamma(T)$ and $\beta(T)$ and Proposition 2.6, shows that $\beta(T) = \gamma(T_{\rho}) \times \varepsilon_{Y_d}$.

This means that any element of $\gamma(T)$ consists precisely of $d$ elements of the partition $\beta(T)$. By the definition of the index $d_{\gamma,\beta}$ we have $d_{\gamma,\beta} = d$. Thus $d(T_{I_a}) = d$.

Taking into account Theorem 4.3 we have

Corollary 5.6. Let $\pi_k : \bar{I}_{a_k} \to I$, $k = 1, 2$, be two $d$-extensions of the Bernoulli graph $(I, \rho)$, generated by functions $a_k : I \to \mathcal{A}_d$, respectively, and suppose both the functions $a_k$, $k = 1, 2$ satisfy the condition 5.80. Then the Markov shifts $T_{I_{a_1}}$ and $T_{I_{a_2}}$ are isomorphic iff $a_1$ and $a_2$ are conjugate in $\mathcal{A}_d$, i.e. there exists $w_0 \in \mathcal{A}_d$ such that $a_2(i) \cdot w_0 = w_0 \cdot a_1(i)$, $i \in I$.

Remark 5.7. It can be proved that for $d$-extension $\bar{I}_a$, the condition 5.80 is equivalent to $d(T_{I_a}) = d$.

Absolutely non-homogeneous $\rho$. Consider the case, when $\rho$ is absolutely non-homogeneous (see Section 2.4). This means that $\rho(i) \neq \rho(i')$ for all $i \neq i'$ from $I$, i.e. the Bernoulli graph $(I, \rho)$ has no congruent edges.

In this case for any $\rho$-uniform graph $G$ there exists a unique homomorphism $\phi : G \to I$ . Therefore Theorem 3.25 can be sharpened as follows

Theorem 5.8. Let $G$ be a $\rho$-uniform stochastic graph, which is irreducible and satisfies the positive recurrence condition. Suppose that $\rho$ is absolutely non-homogeneous. Then there exist a unique homomorphism $\phi \in \text{Hom}(G, I)$ and a commutative diagram

\[
\begin{array}{ccc}
H & \xrightarrow{\psi} & G \\
\downarrow{\pi} & & \downarrow{\phi} \\
H & \xrightarrow{\psi} & I
\end{array}
\]

such that

(i) The pair $(\pi, \psi) \in \text{Ext}^d(I, \rho)$ is a $(\pi, \psi)$-extension.

(ii) $d = d(\phi) = d(T_G)$ ,

A natural question, which is arisen in connection with the previous theorem is:

Question 5.9 (Generalized Road Coloring Problem). Does Theorem 3.25 hold with $n = 1$ in general case, when $\rho$ is not necessarily absolutely non-homogeneous, i.e. when $(I, \rho)$ has congruent edges ?

As we know, the problem is open even in the case, when the graph $G$ is finite (See [AsMaTu] and references therein.)

Homogeneous $\rho$ and Road Problem Consider a special case, when $\rho$ is homogeneous, i.e. $\rho(i) = l^{-1}$, $i \in I$ with an integer $l = |I| \in \mathbb{N}$. Theorem 2.14 and arguments adduced in Section 2.4 imply
**Theorem 5.10.** Suppose $\rho$ is homogeneous. Then every ergodic $\rho$-uniform Markov shift $T_G$ is isomorphic to a direct product $T_\rho \times \sigma_d$ of the Bernoulli shift $T_\rho$ and a cyclic permutation $\sigma_d$ of $Y_d$, where $d$ is the period of $T_G$. If, in addition, $T_G$ is exact, then it is isomorphic to the Bernoulli shift $T_\rho$, herewith, there exists $n \in \mathbb{N}$ and a homomorphism $\phi : G^{(n)} \to I$ of degree 1.

The result was proved earlier in [Ru3] for finite $G$ and in [Ru4] for general case.

If $G$ is finite and $\rho$ is homogeneous Question 5.9 is a reformulation of well-known Road Coloring Problem (See [Fr], [O'B], [AdGoWe], [Ki]). As we know, the problem is still open.

### 5.3. Some $(p,q)$-uniform graphs.

We construct some simple examples to illustrate the case, when the $\psi$-part in the canonical pair $(\pi, \psi)$ is not trivial.

Let $I = \{0, 1\}$ and $\rho = (p,q)$, where $0 < p < 1$ and $q = 1 - p$. Given $n \in \mathbb{N}$ consider the following random walk on $J_n := \{1, 2, \ldots, n\}$

\begin{equation}
(5.82) \quad \begin{array}{cccccc}
q & 1 & q & 2 & q & \cdots & q & n \end{array}^p,
\end{equation}

which is known as a **Finite Drunkard Ruin**. We set here: $H := I \times J_n$, $H^{(0)} := J_n$ and

$s(h) = j$, $t(h) = f_{ij}$, $\psi(h) = i$, $h = (i, j) \in H$,

where the maps $f_i : J_n/ \to J_n$, $i = 0, 1$, are defined by

$f_{ij} = \min(j + 1, n)$, $f_{0j} = \max(j - 1, 1)$, $j \in J_n$

and the weights of edges $p(h)$, $h \in H$ are given according to (5.82) by $p(1, j) = p$, $p(0, j) = q$.

Then the finite stochastic graph $H$ is irreducible and $\rho$-uniform, $\psi \in \mathcal{H}(H, I)$. The semigroup $S(\psi)$, generated by $\{f_0, f_1\}$, is 1-contractive, since $(f_0)^n(J_n) = \{1\}$. Whence, $d(\psi) = 1$ and the Markov shift $T_H$ is isomorphic to the Bernoulli shift $T_\psi$.

Given $p$ and $n$ we construct a $\mathbb{Z}_2$-extension $\tilde{H}_a$ of the graph $H$, where $a : H \to \mathbb{Z}_2$ and $\mathbb{Z}_2 := \{0, 1\}$ be the cyclic group of order 2.

Define $a : H = I \times J_n \ni h = (i, j) \to a(h) \in \mathbb{Z}_2$ by

\begin{equation}
(5.83) \quad a(i, j) = \begin{cases} 
1, & \text{if } (i, j) = (1, 1) \\
0, & \text{if } (i, j) \neq (1, 1).
\end{cases}
\end{equation}

Then the corresponding graph $\tilde{H}_a$ has the form

\begin{equation}
(5.84) \quad \begin{array}{cccccc}
q & 11 & q & 21 & q & \cdots & q & n1 \end{array}^p \quad z = 1 \\
q & 10 & q & 20 & q & \cdots & q & n0 \end{array}^p \quad z = 0
\end{equation}

for $n > 2$. and

\begin{equation}
(5.85) \quad \begin{array}{cccccc}
q & 11 & q & 21 & p & \cdots & p & \cdots & p \end{array}^p \quad z = 1 \\
q & 10 & q & 20 & p & \cdots & p & \cdots & p \end{array}^p \quad z = 0
\end{equation}

for two special cases $n = 1, 2$.
Suppose \( p \neq q \). We claim in this case that for all \( n \in \mathbb{N} \) the graphs (5.85) and (5.84) are canonical. Indeed, \( d(\pi_H) = d(T_{H_a}) = 2 \), since \( \rho = (p,q) \) is absolutely non-homogeneous. In order to check the irreducibility of the 2-extension \((\pi_H, \psi) : \tilde{T}_a \to H \to I\) consider the semigroup \( \tilde{S} = \tilde{S}(\pi, \psi) \) and its persistent partitions \( \mathcal{R}(\tilde{S}) \).

The semigroup \( \tilde{S} \) is generated by \( \{ \tilde{f}_i, i \in I \} \), where

\[
\tilde{f}_i (j, z) = (f_i j, z + a(i, j)) \pmod{2}, \quad (j, z) \in J_n \times \mathbb{Z}_2.
\]

A direct computation shows that for \( n = 1, 2 \) any transversal partition of \( J_n \times \mathbb{Z}_2 \) is persistent in the sense of Definition 1.12. Moreover, this is the only transversal partition, which is not persistent. This implies that for every \( n \in \mathbb{N} \) the persistent partitions \( \mathcal{R}(\tilde{S}) \) separate points of \( J_n \times \mathbb{Z}_2 \) in the sense of Remark 1.16 and the 2-extension \((\pi_H, \psi)\) is irreducible. Thus

- For all \( n \in \mathbb{N} \) and \( p \neq q \) the graphs \( \tilde{H}_a \) are canonical graphs for the corresponding shifts \( T_{H_a} \).

Just in the same way we can consider the following **Infinite Drunkard Ruin**

\[
(5.86) \quad \begin{array}{cccccccc}
\text{1} & & \text{2} & & & & \cdots & \text{n} & \text{...}
\end{array}
\]

where \( H := I \times \mathbb{N} , \ H^{(0)} := \mathbb{N} \).

Suppose \( p < q \). Then the corresponding Markov chain is positively recurrent and the Markov shift \( T_H \) is isomorphic to the Bernoulli shift \( T_{\rho} \).

Again define the functions \( a : H = I \times \mathbb{N} \ni h = (i, j) \to a(h) \in \mathbb{Z}_2 \) by (5.83). Then \( \mathbb{Z}_2 \)-extension \( \tilde{H}_a \) of the graph \( H \) (3.84) has the form

\[
(5.87) \quad \begin{array}{cccccccc}
\text{1} & & \text{21} & & & & \cdots & \text{n1} & \text{...}
\end{array}
\]

\[
\begin{array}{cccccccc}
\text{10} & & \text{20} & & & & \cdots & \text{n0} & \text{...}
\end{array}
\]

It can be shown in this case that any transversal set is persistent. Thus

- If \( p < q \) the graph \( \tilde{H}_a \) (5.87) is the canonical graph for the shift \( T_{H_a} \).

Note that the shift \( T_{H_a} \) is a \( \mathbb{Z}_2 \)-extension of the Bernoulli shift \( T_{p,q} \); therefore, \( T_{H_a} \) has a 4-element one-sided generator. On the other hand the shift is not isomorphic to Markov shifts on finite state spaces. Thus

- If \( p < q \) the one-sided Markov shift \( T_{H_a} \) has no finite one-sided Markov generator.

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