Duality and Moduli Spaces for Time-Like Reductions

C.M. Hull

Physics Department, Queen Mary and Westfield College,
Mile End Road, London E1 4NS, U.K.

and

B. Julia

Laboratoire de Physique Théorique, Ecole Normale Supérieure, 24 Rue Lhomond, 75231 Paris Cedex 05, France.

ABSTRACT

We consider the dimensional reduction/compactification of supergravity, string and M-theories on tori with one time-like circle. We find the coset spaces in which the massless scalars take their values, and identify the discrete duality groups.
The standard dimensional reduction of 11-dimensional supergravity [1,2] on a torus $T^d$ gives a supergravity theory in $11 - d$ dimensions which is invariant under a rigid duality symmetry $G_d$ and a local symmetry $H_d$; the groups $G_d, H_d$ are listed in table 1. $H_d$ is the maximal compact subgroup of $G_d$ and the theory has scalars taking values in the coset $G_d/H_d$. If in $D$ dimensions some of the $p$-form gauge fields for various values of $p$ are dualised to $(D - p - 2)$-form gauge fields, then Noether and topological charges are interchanged, and internal symmetries are traded for (generalized) gauge symmetries [3]. The reduction to two dimensions is particularly involved but follows the higher dimensional pattern, see for instance [4-8]. In that case the duality groups are infinite dimensional, with affine Kac-Moody symmetries and the associated Virasoro (Witt) symmetries. In one dimension, it is conjectured that there is an $E_{10(10)}$ symmetry [9].

The duality $G_d$ of the dimensionally reduced supergravity theory was first discovered as a symmetry of the compactified theory after discarding the nonzero Kaluza-Klein modes. In any quantum theory for which the supergravity theory is a low-energy effective description, the rigid $G_d$ invariance is broken to (at most) a discrete subgroup $G_d(\mathbb{Z})$ [10]. It is conjectured that 11-dimensional supergravity is a low-energy effective description of a consistent quantum M-theory, and that $G_d(\mathbb{Z})$ is a symmetry of the toroidally compactified theory, the U-duality symmetry [10]. (We will distinguish between dimensional reduction, in which the Kaluza-Klein modes are truncated, and compactification, in which they are kept.)
In three dimensions, the bosonic sector consists of gravity coupled to an $E_8/SO(16)$ coset space, and reduction to $D=2$ gives scalars in $E_8/SO(16) \times \mathbb{R}$, coupled to gravity and a vector field. This has an infinite dimensional symmetry [2] $E_9(9)$ (which is an affine $E_8(8)$), and the quantum U-duality symmetry is conjectured to be a discrete subgroup of this [10]. The natural subgroup $H_2$ to consider in this context is an infinite dimensional subgroup of $E_9(9)$ containing $SO(16)$, which can be characterised as a fixed set of a Cartan involution of $E_9(9)$, as in the case of symmetric spaces. It is also the subspace of the adjoint representation on which the Killing form, or rather the invariant bilinear form, is negative definite [4] and deserves the name of maximal compact subalgebra by analogy with the finite dimensional semi-simple situation with the Cartan compactness criterion. $H_2$, as acting on solutions in the Geroch [11] group fashion, does not contain the central charge of $E_9$. A different subalgebra arises in the dressing method, see [8].

Instead of reducing on a space-like torus to obtain a theory with Lorentzian signature in $11-d$ dimensions, one can instead compactify time along with $d-1$ spatial directions and reduce on the Lorentzian-signature torus $T^{d-1,1}$ to obtain a supergravity theory in $11-d$ dimensions with Euclidean-signature. Truncating

| $D=11-d$ | $G_d$ | $H_d$ | U-Duality |
|----------|-------|-------|-----------|
| 10       | $SO(1,1)$ | 1     | -         |
| 9        | $SL(2,\mathbb{R}) \times SO(1,1)$ | $SO(2)$ | $SL(2,\mathbb{Z})$ |
| 8        | $SL(3,\mathbb{R}) \times SL(2,\mathbb{R})$ | $SO(3) \times SO(2)$ | $SL(3,\mathbb{Z}) \times SL(2,\mathbb{Z})$ |
| 7        | $SL(5,\mathbb{R})$ | $SO(5)$ | $SL(5,\mathbb{Z})$ |
| 6        | $SO(5,5)$ | $SO(5) \times SO(5)$ | $SO(5,5;\mathbb{Z})$ |
| 5        | $E_{6(6)}$ | $USp(8)$ | $E_{6(6)}(\mathbb{Z})$ |
| 4        | $E_{7(7)}$ | $SU(8)$ | $E_{7(7)}(\mathbb{Z})$ |
| 3        | $E_{8(8)}$ | $SO(16)$ | $E_{8(8)}(\mathbb{Z})$ |
| 2        | $E_{9(9)}$ | $H_2$ | $E_{9(9)}(\mathbb{Z})$ |
the dependence on the internal dimensions leaves a dimensionally reduced theory in which some of the fields have kinetic terms of the wrong sign, but in the full compactified theory such ghosts can be gauged away. This is readily seen in the example of dimensional reduction of (super) Yang-Mills theory from $D + d$ dimensions to $D$ dimensions. Reducing on a Euclidean torus $T^d$ gives Yang-Mills plus $d$ scalars in Minkowski space, acted on by an $SO(d)$ ‘R-symmetry’, while reducing on a Lorentzian torus $T^{d-1,1}$ gives Yang-Mills plus $d$ scalars in Euclidean space, acted on by an $SO(d-1,1)$ ‘R-symmetry’. In particular, the scalar field coming from $A_0$, the time component of the gauge field, has a kinetic term of the wrong sign. However, in the full compactified theory, one can eliminate $A_0$ by a gauge choice (e.g. $A_0 = 0$) leaving a theory without ghosts. Such dimensional reductions of super Yang-Mills are useful in constructing topological field theories [12,13].

Timelike dimensional reductions of supergravity and string theories were investigated in [14, 9, 15, 16], and can be useful for constructing stationary solutions in $12 - d$ dimensions. For example, in [15] the case $d = 8$ was considered and the resulting Euclidean 3-dimensional theory was used to construct stationary solutions of $N = 8$ supergravity in $3 + 1$ dimensions. In [15], it was found that the Euclidean maximal supergravity in 3 dimensions still had $G_d = E_{8(+8)}$ but that $H_d$, which had been $SO(16)$ in $2 + 1$ dimensions, had become the non-compact group $SO^*(16)$; the noncompact $SO^*(2k)$ had been discussed already in [9]. In [16], the bosonic string was compactified to zero dimensions on $T^{25,1}$, allowing an algebraic study of the theory.

The remaining possibility is to consider compactification on a torus which includes a light-like circle; such null reductions were studied in [17]. This could be viewed as the result of compactifying on a spacelike torus, boosting in one of the toroidal directions and taking the limit of infinite boost. M-theory compactifications on such null tori have recently been the focus of considerable interest, as taking a suitable limit defines the matrix theory for toroidal compactifications of M-theory [18-22]. It is of particular interest to study the duality symmetries for such null reductions and compactifications, and their implications for the U-duality
of M-theory. Such reductions would also be useful in constructing solutions with null Killing vectors.

Our aim here is to study the duality symmetries and scalar coset spaces of gravity and supergravity theories compactified on tori including a time-like circle, and to return to the null case in a future publication.

For compactification of gravity on a space $X$, the dimensionally reduced theory has massless scalars taking values in the moduli space of Ricci-flat metrics on $X$, $M_X$. For supergravity theories, or for gravity coupled to anti-symmetric tensor gauge fields, there are moduli for both gravity and the anti-symmetric tensor gauge fields. For compactification on a Ricci-flat $X$, with all other fields vanishing, there are, in the linear approximation, massless scalars corresponding to flat anti-symmetric tensor gauge fields on $X$ that extend the moduli space of Ricci-flat metrics $M_X$ into some moduli space $N_X$ in which all massless scalars take their values. There will be massless scalars taking values in $N_X$ independent of whether one keeps the full Kaluza-Klein theory (compactification) or whether one performs a consistent truncation to the massless sector (dimensional reduction). For compactification of string or M-theories, the massless fields can be studied using the appropriate (super)-gravity theory, and they again take values in the appropriate $N_X$.

In this paper we will discuss compactifications in which $X$ is a torus and the moduli space takes the form $M_G = G(Z)\backslash G/H$ for some groups $G, H$ where $G(Z)$ is a discrete subgroup of $G$. For compactification on Euclidean tori, $H$ is the maximal compact subgroup of $G$ so that $G/H$ is a Riemannian manifold, a symmetric space of the noncompact type, and $M_G$ is a Hausdorff space with cusps in which the massless scalars take their values. For compactification on a torus with a time-like direction, the situation is more complicated [16]. In this case, $H$ is non-compact and $G/H$ has indefinite signature. The action of $H$ on $G(Z)\backslash G$ is typically ergodic, with generic orbits being space-filling. This leads to non-commutative geometry [16, 23]. (See also [24].) In the following we will not
discuss such points further, but restrict ourselves to finding the groups $G, H, G(\mathbb{Z})$ that arise for various compactifications.

**Toroidal Compactifications of Gravity**

Consider the dimensional reduction of $D + d$-dimensional gravity on $T^d$ to $D$ dimensions. The $(D + d)$-dimensional metric $G_{MN}$ gives a $D$-dimensional metric $g_{\mu\nu}$, $d$ vector fields $g_{i\mu}$ and $d(d + 1)/2$ scalars, represented by a metric $g_{ij}$ on $T^d$. For a Euclidean torus, the reduced theory has a rigid $GL(d, \mathbb{R})$ symmetry and the scalars can be viewed as taking values in $GL(d, \mathbb{R})/SO(d)$, which is the moduli space of flat metrics on $T^d$. There is a subgroup $GL(d, \mathbb{Z})$ which preserves the periodicities of the toroidal directions and which is a discrete gauge symmetry of the compactified theory, so that configurations related by $GL(d, \mathbb{Z})$ transformations should be identified.

If instead one reduces on a torus $T^{d-1,1}$ with $d - 1$ space-like circles and one time-like circle, a similar theory results, with rigid $GL(d, \mathbb{R})$ symmetry but with scalars taking values in $GL(d, \mathbb{R})/SO(d - 1, 1)$. As $GL(d, \mathbb{R})$ is a remnant of the $(D + d)$-dimensional diffeomorphisms $[1, 2]$, it remains a symmetry whatever the metric on the torus, and the discrete subgroup $GL(d, \mathbb{Z})$ remains as the discrete gauge symmetry. However, the scalar coset space is now the moduli space of Lorentzian metrics on $T^d$, which is $GL(d, \mathbb{R})/SO(d - 1, 1)$ (as $SO(d - 1, 1)$ is the subgroup of $GL(d, \mathbb{R})$ preserving a Lorentzian metric). The scalar coset space is no longer a Riemannian space.

For reductions to $D = 3$ dimensions, the $d$ vector fields $g_{i\mu}$ can be dualised to give additional scalars, and the $GL(d, \mathbb{R})$ symmetry is enlarged by Ehlers-type transformations. For example, reduction from $3 + 1$ Lorentzian dimensions on a circle gives a scalar and a vector, which can be dualised to give a second scalar. For reductions on a spacelike circle, the two scalars take values in $SL(2, \mathbb{R})/SO(2)$ and the Ehlers symmetry is $SL(2, \mathbb{R})$ $[25, 11]$. For reduction on a timelike circle, there are two minus signs which conspire to leave the structure intact, so that the
scalar coset is again $SL(2, \mathbb{R})/SO(2)$, but 4-dimensional euclidean gravity reduced to 3 dimensions leads to the coset space $SL(2, \mathbb{R})/SO(1, 1)$ [14].

**Bosonic and Heterotic String Compactifications**

Consider first the bosonic string compactified on $T^d$ (with $D + d = 26$), or a theory of $(D + d)$-dimensional gravity coupled to a 2-form gauge field and dilaton with action

$$\int d^{D+d}x \sqrt{-g} e^{-2\Phi} \left( R + \frac{1}{12} H^2 + 4(\nabla \Phi)^2 \right)$$

(1)

We will first consider the case of reduction to more than 4 dimensions, with $D > 4$. For a Euclidean torus, this has a moduli space

$$\mathbb{R} \times \frac{O(d,d)}{O(d) \times O(d)}$$

(2)

with $O(d,d; \mathbb{Z})$ being the T-duality symmetry of the theory, and the $\mathbb{R}$ factor representing the dilaton. The coset space $O(d,d)/O(d) \times O(d)$ has dimension $d^2$ and is parameterised by constant metric $g_{ij}$ and anti-symmetric tensor $b_{ij}$ on $T^d$.

For a Lorentzian torus $T^{d-1,1}$, we expect a coset space $G_d/H_d$ with the following properties. (i) As $T^{d-1,1}$ contains both $T^{d-1,0}$ and $T^{d-2,1}$, we must have

$$\frac{O(d-1,d-1)}{O(d-1) \times O(d-1)} \subset G_d/H_d, \quad G_{d-1}/H_{d-1} \subset G_d/H_d$$

(ii) As $GL(d, \mathbb{R})/SO(d-1,1)$ is the moduli space of metrics on $T^{d-1,1}$, we must have

$$\frac{GL(d, \mathbb{R})}{SO(d-1,1)} \subset G_d/H_d$$

(iii) The dimension of $G_d/H_d$ is $d^2$, and is parameterised by constant Lorentzian metrics $g_{ij}$ taking values in $GL(d, \mathbb{R})/SO(d-1,1)$ and an anti-symmetric tensor $b_{ij}$ on $T^{d-1,1}$. Splitting the torus coordinates $x^i$ into a time coordinate $x^0$ and $d-1$ space coordinates $x^a$, the $d(d-1)/2$ degrees of freedom corresponding to $b_{ij}$ split into $s = (d-1)(d-2)/2$ degrees of freedom $b_{ab}$ with space-like norm and $d$ degrees of freedom $b_{a0}$ with time-like norm, so that
The simplest coset space with these properties is \( G_d = O(d, d) \) and \( H_d = O(d - 1, 1) \times O(d - 1, 1) \), so that the full moduli space is

\[
\frac{O(d, d)}{O(d - 1, 1) \times O(d - 1, 1)} \times \mathbb{R}
\]

and this is indeed the correct moduli space [9, 26, 16]. The effective action has a rigid \( O(d, d) \times \mathbb{R} \) symmetry, and the discrete subgroup \( O(d, d; \mathbb{Z}) \) is again the T-duality symmetry of the perturbative string theory.

This is easily generalised to the heterotic (or type I) string toroidally compactified from 10 dimensions. Compactification on \( T^d \) gives the moduli space [27, 9, 28, 29]

\[
\mathbb{R} \times \frac{O(d, d + n)}{O(d) \times O(d + n)}
\]

with \( n = 16 \), while similar arguments to the above give the moduli spaces

\[
\frac{O(d, d + n)}{O(d - 1, 1) \times O(d + n - 1, 1)} \times \mathbb{R}
\]

for compactification on \( T^{d-1,1} \). The same spaces emerge for toroidal compactification of half-maximal supergravity coupled to \( n \) vector multiplets, for any \( n \) [30].

**Compactifications to 4 Dimensions and Less**

For reduction of the bosonic or heterotic string, or of the related supergravity theories to 4 dimensions or less gives scalars in the moduli spaces (2) or (3), as before. However, in 4 dimensions the 2-form gauge field \( b_{\mu \nu} \) can be dualised to a pseudo-scalar, which combines with the dilaton in the \( \mathbb{R} \) factor to take values in a two-dimensional coset space. For the usual compactification on a spacelike
torus this is $SL(2, \mathbb{R})/SO(2)$, while for compactification on a Lorentzian torus it is $SL(2, \mathbb{R})/SO(1, 1)$ (the difference in sign for the pseudoscalar kinetic term coming from the duality transformation). Thus the coset space for the heterotic string compactified on a Lorentzian torus $T^{5,1}$ to Euclidean 4-space is

$$\frac{O(6, 22)}{O(5, 1) \times O(11, 1)} \times \frac{SL(2, \mathbb{R})}{SO(1, 1)} \quad (6)$$

In both the Lorentzian and the Euclidean cases, there is a non-perturbative $SL(2, \mathbb{Z})$ S-duality symmetry.

Note that a similar structure can occur in $N = 4$ super Yang-Mills in 4 dimensions. Starting from the $(2,0)$ supersymmetric self-dual tensor multiplet theory in 6 dimensions, reducing on $T^2$ gives the usual Lorentzian $N = 4$ super Yang-Mills with coupling constants in $SL(2, \mathbb{R})/SO(2)$, while compactifying on $T^{1,1}$ gives a Euclidean $N = 4$ super Yang-Mills with coupling constants in $SL(2, \mathbb{R})/SO(1, 1)$. In both cases, the moduli of the torus become the coupling constants on reduction and the $SL(2, \mathbb{Z})$ torus symmetry becomes the S-duality of the 4-dimensional theory. This Euclidean theory can be twisted to give a topological field theory, as in [12, 13].

Toroidal compactification of gravity from $d+3$ dimensions to 3 dimensions gives scalars in $GL(d, \mathbb{R})/SO(d)$ for a space-like torus $T^d$ or $GL(d, \mathbb{R})/SO(d - 1, 1)$ for a Lorentzian torus $T^{d-1,1}$, together with $d$ vector fields from the reduction of the metric. In 3 dimensions, these can be dualised to give an extra $d$ scalars. The dimensional reduction of gravity from 4 to 3 dimensions gives rise to an $SL(2, \mathbb{R})$ Ehlers symmetry [25], and there are $d$ such $SL(2, \mathbb{R})$ symmetries, corresponding to the ways of first reducing on a $d-1$ torus to four dimensions, and then reducing from four to three dimensions. There are then these $d$ $SL(2, \mathbb{R})$ symmetries together with the geometrical $GL(d, \mathbb{R})$ symmetry, which do not commute but fit together to generate the group $SL(d+1, \mathbb{R})$ of symmetries of the 3-dimensional field theory. For reduction on a spacelike torus, the scalars in $GL(d, \mathbb{R})/SO(d)$ and the extra $d$
scalars from dualising the vectors combine to take values in the coset space

\[ \frac{SL(d + 1, \mathbb{R})}{SO(d + 1)} \]

For a Lorentzian torus, the extra scalars transform as a \( d \) of \( SO(d - 1, 1) \) and so, after dualisation \( (d - 1) \) of those also have a kinetic term of the wrong sign. These \( d \) scalars combine with those in \( GL(d, \mathbb{R})/SO(d - 1, 1) \) to parameterise the coset space or

\[ \frac{SL(d + 1, \mathbb{R})}{SO(d - 1, 2)} \]

In both cases, the expected discrete symmetry is \( SL(d + 1, \mathbb{Z}) \).

For the bosonic string, the dimensional reduction on a \( d \) torus gives moduli in

\[ \frac{O(d, d)}{O(d) \times O(d)} \times \mathbb{R} \] \hspace{1cm} (7)

or

\[ \frac{O(d, d)}{O(d - 1, 1) \times O(d - 1, 1)} \times \mathbb{R} \] \hspace{1cm} (8)

together with \( 2d \) vectors from the reduction of the metric and 2-form gauge field. In four dimensions S duality appears due to the 2-form, and for compactifications to 3 dimensions, the vectors are dual to scalars, and can be combined with the other scalars to take values in the coset spaces

\[ \frac{O(d + 1, d + 1)}{O(d + 1) \times O(d + 1)} \] \hspace{1cm} (9)

or

\[ \frac{O(d + 1, d + 1)}{O(d - 1, 2) \times O(d - 1, 2)} \] \hspace{1cm} (10)

Thus the dilaton is combined with the other scalars and are acted on by an \( O(d + 1, d + 1; \mathbb{Z}) \) U-duality combining the \( O(d, d; \mathbb{Z}) \) T-duality with discrete \( SL(2, \mathbb{Z}) \)
Ehlers-type symmetries, and the $SL(2,\mathbb{Z})$ S-dualities inherited from 4-dimensions, as in [31]. For compactifications to 3 dimensions of the heterotic string (with $d = 7$, $n = 16$), or for any theory of half-maximal supergravity coupled to $n$ vector multiplets, a similar analysis gives the coset spaces
\[ \frac{O(d + 1, d + n + 1)}{O(d + 1) \times O(d + n + 1)} \quad (11) \]
for compactification on a spacelike torus [31], or
\[ \frac{O(d + 1, d + n + 1)}{O(d - 1, 2) \times O(d + n - 1, 2)} \quad (12) \]
for compactification on a Lorentzian torus [31]. For the heterotic string, the coset space is
\[ \frac{O(8, 24)}{O(6, 2) \times O(22, 2)} \quad (13) \]
and there is an $O(8, 24; \mathbb{Z})$ U-duality symmetry [31,10].

For reductions to 2 dimensions, the situation is more complicated [4, 5, 7, 8, 6]. In the reduction of the heterotic string to 1+1 dimensions, one obtains scalars in
\[ \frac{O(8, 24)}{O(8) \times O(24)} \times \mathbb{R} \]
coupled to 2-dimensional gravity (which includes a coupling $\int d^2x \sqrt{g} R e^{-\phi}$ where $\phi$ is the dilaton) [6]. The classical supergravity theory has an affine $O(8, 24)$ symmetry [6] in which the central charge acts through scale transformations, and a discrete subgroup of this is conjectured to be the U-duality symmetry of the full string theory [10, 6].

For the reduction on a Lorentzian torus the situation is similar, with scalars taking values in
\[ \frac{O(8, 24)}{O(7, 1) \times O(23, 1)} \times \mathbb{R} \quad (14) \]
coupled to gravity, with an affine $O(8, 24)$ symmetry. However, if one first reduces on a Lorentzian torus to three dimensions, dualises the vectors to scalars so that
the scalar target space is (13), then reduces to two dimensions, one obtains a coset space

\[
\frac{O(8, 24)}{O(6, 2) \times O(22, 2)} \times \mathbb{R}
\]

(15)

which is different from (14). However, in two Euclidean dimensions, the dual of a scalar \( \phi \) with kinetic term \((D\phi)^2\) is a scalar \( \tilde{\phi} \) with kinetic term \(- (D\tilde{\phi})^2\), and some of the scalars in the coset space (15) can be dualised so that the resulting coset space is (14). Thus in two Euclidean dimensions, duality can be used to analytically continue the coset space to another real form, as T-dualities for Euclidean world-sheets can change the target space signature.

One can also consider reduction to 0 dimensions. For the bosonic string, this was considered in [16]. The metric, anti-symmetric tensor and dilaton moduli take values in

\[
\mathbb{R} \times \frac{O(26, 26)}{O(25, 1) \times O(25, 1)}
\]

(16)

However, if one first compactifies to 3 Euclidean dimensions on \(T^{22,1}\), dualises the vectors to obtain scalars in \(O(23, 23)/O(21, 2) \times O(21, 2)\), then compactifies the remaining three dimensions, one obtains a moduli space which is a different real form of (16), and going via 2 dimensions, duality transformations could give further real forms of (16). Similarly, for the heterotic string, the moduli take values in

\[
\mathbb{R} \times \frac{O(10, 26)}{O(9, 1) \times O(25, 1)}
\]

(17)

and other real forms of this can be obtained by going via 2 or 3 dimensions and performing duality transformations.

**COMPACTIFICATIONS OF 11-DIMENSIONAL SUPERGRAVITY AND M-THEORY**

We now turn to the toroidal compactification of M-theory. Compactification on the Euclidean torus \(T^d\) gave the coset spaces \(G_d/H_d\) in table 1, and the U-duality group \(G_d(\mathbb{Z})\). Compactification on \(T^{d-1,1}\) gives a moduli space which we will
assume to be a coset space $\tilde{G}_d/\tilde{H}_d$. This must have the following properties. (i) As $T^{d-1,1}$ contains both $T^{d-1,0}$ and $T^{d-2,1}$, $G_{d-1}/H_{d-1} \subset \tilde{G}_d/\tilde{H}_d$ and $\tilde{G}_{d-1}/\tilde{H}_{d-1} \subset \tilde{G}_d/\tilde{H}_d$. (ii) Since $GL(d, \mathbb{R})/SO(d-1,1)$ is the moduli space of metrics on $T^{d-1,1}$, 

$$\frac{GL(d, \mathbb{R})}{SO(d-1,1)} \subset \tilde{G}_d/\tilde{H}_d$$

(iii) The same theory emerges from the type IIB theory compactified on $T^{d-2,1}$, and this has $O(d, d)$ T-duality and a commuting $SL(2)$ from the type IIB S-duality. Thus, 

$$\frac{O(d, d)}{O(d-1,1) \times O(d-1,1)} \times \frac{SL(2, \mathbb{R})}{U(1)} \subset \tilde{G}_d/\tilde{H}_d$$

(iv) The dimension of $\tilde{G}_d/\tilde{H}_d$ is the same as that of $G_d/H_d$. It is parameterised by constant Lorentzian metrics $g_{ij}$ taking values in $GL(d, \mathbb{R})/SO(d-1,1)$ and an antisymmetric tensor $A_{ijk}$ on $T^{d-1,1}$, together with extra scalars from dualising antisymmetric tensor gauge fields. For the Euclidean torus, $g_{ij}$ and $A_{ijk}$ parameterise

$$\frac{GL(d, \mathbb{R})}{SO(d)} \times \mathbb{R}^{c_d}$$

where $c_d = d!/6(d-3)!$, while for $T^{d-1,1}$, the $A_{ijk}$ split into $A_{abc}$ with space-like norm and $A_{ab0}$ with time-like norm, so that $g_{ij}$ and $A_{ijk}$ parameterise

$$\frac{GL(d, \mathbb{R})}{SO(d-1,1)} \times \mathbb{R}^{c_{d-1,t}}$$

where $t = (d-1)(d-2)/2$ (recall that $\mathbb{R}^{c_{d,t}}$ has $c_d$ spacelike dimensions and $t$ timelike ones). Thus

$$\frac{GL(d, \mathbb{R})}{SO(d-1,1)} \times \mathbb{R}^{c_{d-1,t}} \subset \tilde{G}_d/\tilde{H}_d$$

In $D = 5$, there is an extra scalar that arises from dualising $A_{\mu
u p}$, in $D = 4$ there are 7 scalars from dualising $A_{\mu
u i}$ and in $D = 3$ there are $28 + 8$ scalars from dualising $A_{\mu ij}$ and $g_{\mu i}$. 

13
The unique coset spaces $\tilde{G}_d/\tilde{H}_d$ satisfying these conditions have $\tilde{G}_d = G_d$ and $\tilde{H}_d$ is a non-compact form of $H_d$. These non-compact forms are given in table 2.

| D=11-d | $G_d$          | $H_d$          | $\tilde{H}_d$     | $K_d$          |
|--------|---------------|---------------|-------------------|----------------|
| 10     | $SO(1,1)$     | 1             | $SO(1,1)$         | 1              |
| 9      | $SL(2,\mathbb{R}) \times SO(1,1)$ | $SO(2)$      | $SO(2,1) \times SO(2)$ | 1              |
| 8      | $SL(3,\mathbb{R}) \times SL(2,\mathbb{R})$ | $SO(3) \times SO(2)$ | $SO(3) \times SO(2)$ | 1              |
| 7      | $SL(5,\mathbb{R})$ | $SO(5)$      | $SO(3,2)$         | 1              |
| 6      | $SO(5,5)$     | $SO(5) \times SO(5)$ | $SO(5, C)$        | $SO(5) \times SO(5)$ |
| 5      | $E_{6(6)}$    | $USp(8)$      | $USp(4,4)$        | $SO(5) \times SO(5)$ |
| 4      | $E_{7(7)}$    | $SU(8)$       | $SU^*(8)$         | $USp(8)$       |
| 3      | $E_{8(8)}$    | $SO(16)$      | $SO^*(16)$        | $U(8)$         |
| 2      | $E_{9(9)}$    | $H_2$         | $H_2^*$           | $K_2$          |

**Table 2** Toroidal reductions of M-theory to $D = 11 - d$. Reduction on $T^d$ gives the scalar coset space $G_d/H_d$ while reduction on $T^{d-1,1}$ gives the scalar coset space $G_d/\tilde{H}_d$. $\tilde{H}_d$ is a non-compact form of $H_d$ with maximal compact subgroup $K_d$.

$\tilde{H}_d$ is a non-compact form of $H_d$ with maximal compact subgroup $K_d$, so that the generators of $H_d$ can be decomposed into generators $k$ of $K_d$ and generators $h$ of $H_d \setminus K_d$, and the Lie algebra takes the form

$$[k, k] \sim k, \quad [k, h] \sim h, \quad [h, h] \sim k$$  \hspace{1cm} (18)

and $h, k$ can be taken to be anti-hermitian. The algebra $\tilde{H}_d$ is obtained from this by substituting $h \rightarrow \tilde{h} = ih$, so that the generators $\tilde{h}$ in $\tilde{H}_d \setminus K_d$ are hermitian and so are non-compact generators.

The bosonic part of the supergravity action in 3 dimensions is gravity coupled to an $E_{8(8)}/SO^*(16)$ sigma-model and reducing to 2 dimensions gives scalars in $E_{8(8)}/SO^*(16) \times \mathbb{R}$ coupled to gravity, with an $E_{9(9)}$ symmetry. In two dimensions, $H_2^*$ is an infinite dimensional subgroup of $E_{9(9)}$ containing $SO^*(16)$, which is a fixed set of an involution of $E_{9(9)}$. 

14
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