On special limits of the Mixed Painlevé $P_{III-V}$ Model

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Abstract. The paper discusses $P_{III-V}$ equation for special values of its parameters for which this equation reduces to $P_{III}$, as well as, to some special cases of $I_{38}$ and $I_{49}$ equations from the Ince’s list of 50 second order differential equations possessing Painlevé property.

These reductions also yield symmetries governing the reduced models obtained from the $P_{III-V}$ equation. We point out that the solvable equations on Ince’s list emerge in this reduction scheme when the underlying reflections of the Weyl symmetry group no longer include an affine reflection through the hyperplane orthogonal to the highest root and therefore do not give rise to an affine Weyl group. We hypothesize that on the level of the underlying algebra and geometry this might be a fundamental feature that distinguishes the six Painlevé equations from the remaining 44 solvable equations on the Ince’s list.

1. Introduction

Painlevé equations emerged in a study of ordinary second order differential equations with solutions that have no movable critical points other than poles. Equations with such characteristic referred to as Painlevé Property [6] can be identified with one of 50 canonical types listed by Ince [9]. Forty four of these equations can be either linearized or are solvable in terms of known transcendental functions. The relevant, for this paper, examples are equations $I_{12}$, $I_{38}$ and $I_{49}$ listed in Appendix A. The remaining six equations are known as Painlevé $P_I$, $P_{II}, \ldots, P_{VI}$ equations, see Appendix A for explicit expressions of equations $P_{III}$ and $P_{V}$.

One of the most fundamental developments in the study of integrable models has been Ablowitz, Ramani and Segur [1] conjecture that partial differential evolution equations of integrable hierarchies reduce in self similarity limit to differential equations with Painlevé Property. In particular the 2M-Boson integrable model [5] obtained as reductions of KP integrable models connected to Toda lattice hierarchy gives rise to Painlevé equations invariant under extended affine Weyl groups. It was shown in reference [4] that the 4-Boson integrable model ($M=2$), can be reduced after elimination of a pair of degrees of freedom by Dirac reduction in a self-similarity limit to a mixed $P_{III-V}$ equation, namely,
\[ q_{zz} = -\frac{1}{z} q_z + \left( \frac{1}{2q} + \frac{1}{2(q - r_1)} \right) \left( q_z^2 - \frac{c_0^2 r_0^2 z^{-2} - 2J}{2} - (2\alpha_2 + \alpha_1 + \alpha_3 - 1) \frac{(q - r_1) q r_0}{z} \right) \\
\]  
\[ + \frac{r_0^2}{2} q (q - r_1) (2q - r_1) + \frac{\alpha_1^2 r_1 (q - r_1)}{2 z^2 q} - \frac{\alpha_3 r_1 q}{2 z^2 (q - r_1)} \]  
\[ + \frac{c_0 r_0 z^{-J-2}}{q(q - r_1)} \left[ (\alpha_1 + \alpha_3 - J) q^2 + qr_1 (-2\alpha_1 + J) + \alpha_1 r_1^2 \right] - 2\epsilon_1 r_1 z^{J-1} q (q - r_1). \]  
\[ (1) \]

This equation fulfills the necessary condition for having the Painlevé Property and further reduces to $P_{III}$ and $P_V$ equations for special values of its parameters. Here $J, \epsilon_0, \epsilon_1, r_0, r_1$ together with $\alpha_j, j = 0, 1, 2, 3$ (with $\sum_{j=0}^{3} \alpha_j = 1$) define the extended parameter space of mixed $P_{III-V}$ model.

In this paper we systematically study submodels and their symmetries that are obtained from $P_{III-V}$ model for special values of its parameters. For the purpose of this study it is convenient to alternatively define $P_{III-V}$ equation in terms of symmetric equations:

\[ z \frac{f_{i+1}}{f_i} = f_{i+1} (f_{i+1} - f_{i+3}) + (-1)^i f_i \left( \alpha_1 + \alpha_3 - (1 + J) / 2 \right) + \alpha_i (f_i + f_{i+2}) \\
- (-1)^{i/2} \epsilon_{i+1} (f_{i+1} + f_{i+3}), \quad i = 0, 1, 2, 3 \]
\[ (2) \]

for $\epsilon_0 = \epsilon_2, \epsilon_1 = \epsilon_3$ and with symbol $[i/2]$ that is $i/2$, if $i$ is even or $(i + 1)/2$, if $i$ is odd.

For equations (2) the constraints:

\[ f_1 + f_3 = r_1 z^{(1+J)/2}, \quad f_0 + f_2 = r_0 z^{(1-J)/2}, \]
\[ (3) \]

are automatically satisfied with $r_0, r_1$ being integration constants. Equations (2) are obtained when the $P_V$ Hamiltonian (see e.g. [10, 11, 12]):

\[ h_0 = f_0 f_1 f_2 f_3 + \frac{1}{4} (\alpha_1 + 2\alpha_2 - 3\alpha_3) f_0 f_1 + \frac{1}{4} (\alpha_1 + 2\alpha_2 + 3\alpha_3) f_1 f_2 \\
- \frac{1}{4} (3\alpha_1 + 2\alpha_2 + \alpha_3) f_2 f_3 + \frac{1}{4} (\alpha_1 - 2\alpha_2 - 3\alpha_3) f_0 f_3 + \frac{1}{4} (\alpha_1 + \alpha_3)^2 \]

is augmented by two symmetry breaking terms:

\[ \tilde{h}_0 = h_0 + \frac{\epsilon_0}{2} (f_0^2 - f_2^2) + \frac{\epsilon_1}{2} (f_1^2 - f_3^2). \]
\[ (4) \]

These terms break the $A_3^{(1)}$ symmetry of $P_V$ equation down to invariance under one single automorphism operation:

\[ \pi(\alpha_i) = \alpha_{i+1}, \quad \pi(f_i) = f_{i+1}, \quad i = 0, 1, 2, 3 \]
\[ \pi(\epsilon_0) = \epsilon_1, \quad \pi(\epsilon_1) = -\epsilon_0, \quad \pi(J) = -J, \quad \pi(r_0) = r_1, \quad \pi(r_1) = r_0, \]
\[ (5) \]

such that $\pi^4 = 1$. Defining canonical variables $q, p$ as:

\[ q = f_1 z^{-(1+J)/2}, \quad p = -f_2 z^{(1+J)/2}, \]
\[ (7) \]

one finds that equation (2) is equivalent to the two first-order Hamilton equations:

\[ z q_z = q (q - r_1) (2p + r_0 z) - (\alpha_1 + \alpha_3) q + \alpha_1 r_1 + \epsilon_0 r_0 z^{-J} \]
\[ z p_z = p (p + r_0 z) (r_1 - 2q) + (\alpha_1 + \alpha_3) p - \alpha_2 r_0 z - \epsilon_1 r_1 z^{J+1} \]
\[ (8) \]
that lead back to $P_{III-V}$ equation (1) upon elimination of $p$. Equations (8) follow from the Hamiltonian:

$$zh = q(q - r_1)p(p + r_0z) - (\alpha_1 + \alpha_3)qp + (\alpha_1r_1 + \epsilon_0r_0z^{-J})p + (\alpha_2r_0z + \epsilon_1r_1z^{J+1})q,$$

which agrees with the Hamiltonian (4) up to a constant. The above automorphism $\pi$ from relation (5) can be rewritten in terms of canonical variables as

$$\pi(q) = -p/z, \quad \pi(p) = (q - r_1)z, \quad \pi(\alpha_i) = \alpha_{i+1},$$

and $\pi$ as defined above and in relation (6) keeps equations (8) invariant.

The $P_{III}$ or $P_V$ Painlevé models emerge from $P_{III-V}$ for different values of the underlying parameters. See below the list $i) - v)$ [4, 2] for a complete summary of models that can be obtained from $P_{III-V}$, their symmetries and the corresponding values of parameters. The notation $W[s_1, s_3, \pi^2]$ used below denotes the symmetry group generated by $s_1, s_3, \pi^2$.

i) $P_{III-V}$ defined for $r_0 \neq 0$ and $r_1 \neq 0, J \neq 0$ is invariant under automorphism $\pi$ for $\epsilon_0 \neq 0$ and $\epsilon_1 \neq 0$.

ii) $P_{III-V}$ defined for $r_0 \neq 0$ and $r_1 \neq 0, J \neq 0$ with only one of the parameters $\epsilon_0$ (or $\epsilon_1$) being $\neq 0$ is invariant under the extended affine Weyl group $W[s_0, s_2, \pi^2]$ (or $W[s_1, s_3, \pi^2]$).

Note that $\pi^2$ remains a symmetry even with one of the $\epsilon_i$ parameters being set to zero.

iii) $P_V$ (see equation (A.5)) is obtained for $r_0 \neq 0$ and $r_1 \neq 0$, and either $J = 0$ or parameters $\epsilon_i = 0$ for $i = 0, 1$ and is invariant under the $A_3^{(1)}$ symmetry $W[s_0, s_1, s_2, s_3, \pi]$.

iv) $P_{III}$ (see equation (A.4)) is obtained in a limit when either $r_0 \rightarrow 0$ and $J \neq -1$ or $r_1 \rightarrow 0$ and $J \neq 1$ and is invariant under the extended affine Weyl group $W[s_0, s_2, \pi_0, \pi_2, \pi^2]$ (or $W[s_1, s_3, \pi_1, \pi_3, \pi^2]$) . It is possible to realize this symmetry as $C_2^{(1)}$ extended affine Weyl group [4].

v) Ince’s equations XII ($I_{12}$), (incomplete) XXXVIII ($I_{38}$) and XLIX ($I_{49}$) are obtained as a limit when either $r_0 \rightarrow 0$ and $J = -1$ or $r_1 \rightarrow 0$ and $J = 1$. The symmetry is still $W[s_0, s_2, \pi_0, \pi_2, \pi^2]$ (or $W[s_1, s_3, \pi_1, \pi_3, \pi^2]$) but actions of $\pi_i$ on $\alpha_j$ become identical to these of $s_i$ and consequently the $C_2^{(1)}$ realization can no longer be established.

In the next two sub-sections we will give more detailed discussion of limits $r_i \rightarrow 0$, $i = 0, 1$ discussed in cases iv) and v) with special attention to symmetries valid at these limits for various values of the parameter $J$.

2. The $r_1 \rightarrow 0$ limit of $P_{III-V}$ model

Setting $r_1 = 0$ in (1) yields

$$q_{zz} = -\frac{q_z}{z} + \frac{(q^2 - \epsilon_0r_0z^{-2-2J})}{q} - (2\alpha_2 + \alpha_1 + \alpha_3 - 1)\frac{q^2r_0}{z} + r_0q^3 + \epsilon_0r_0z^{-J-2}(\alpha_1 + \alpha_3 - J)$$

(11)

For the special value of $J = -1$ this equation takes form of the conventional Painlevé III equation (A.4) [13] invariant under $W[s_0, s_2, \pi_0, \pi_2, \pi^2$] [4].

However for arbitrary values of $J$ equation (11) remains invariant under

$$\pi_0(q) = -\frac{\epsilon_0z^{-(1+J)}}{q}, \quad \pi_0(p) = \frac{z^{(1+J)}}{\epsilon_0}(q^2p + \alpha_2q)$$

$$\pi_0(\alpha_1 + \alpha_3) = J + 1 - 2\alpha_2 - \alpha_1 - \alpha_3, \quad \pi_0(\alpha_2) = \alpha_2, \quad \pi_0(\alpha_0) = 1 - J - \alpha_0$$

(12)
and

\[ \pi_2(q) = \frac{\epsilon_0 z^{-1-J}}{q}, \quad \pi_2(p) = -\frac{\epsilon_0}{z^{1+J}} (q^2(p + r_0 z) + (1 - \alpha_2 - \alpha_1 - \alpha_3)q) - r_0 z \]

\[ \pi_2(\alpha_1 + \alpha_3) = J - 1 + 2\alpha_2 + \alpha_1 + \alpha_3, \quad \pi_2(\alpha_2) = 1 - J - \alpha_2, \quad \pi_2(\alpha_0) = \alpha_0, \]

which formally generalize to all values of \( J \) the transformations that kept \( P_{III} \) invariant for \( J = -1 \) [4].

In addition to (12) and (13) the system is also invariant under \( s_0, s_2 \) transformations:

\[ s_2(q) = q + \frac{\alpha_2}{p}, \quad s_2(p) = p, \quad s_2(\alpha_1 + \alpha_3) = 2\alpha_2 + \alpha_1 + \alpha_3, \quad s_2(\alpha_2) = -\alpha_2 \]

and

\[ s_0(q) = q + \frac{1 - \alpha_2 - \alpha_1 - \alpha_3}{p + r_0 z}, \quad s_0(p) = p, \quad s_0(\alpha_1 + \alpha_3) = 2 - 2\alpha_2 - \alpha_1 - \alpha_3, \quad s_0(\alpha_2) = \alpha_2. \]

Together, these transformations satisfy the following relations:

\[ s_i^2 = 1 = \pi_i^2, \quad \pi_i \pi_i = \pi_{i+2}, \quad \pi_i \pi_i = s_{i+2}, \quad i = 0, 2, \]

for

\[ \pi^2(q) = -q, \quad \pi^2(p) = -p - r_0 z, \quad \pi^2(\alpha_i) = \alpha_{i+2}, \quad \pi^2(\epsilon_0) = -\epsilon_0 \]

as well as the commutation relations:

\[ s_is_{i+2} = s_{i+2}s_i, \quad \pi_i \pi_i = \pi_{i+2}, \quad \pi_i s_{i+2} = s_{i+2} \pi_i, \quad i = 0, 2, \]

that define the extended affine Weyl group \( W[s_0, s_2, \pi_0, \pi_2, \pi^2] \) as established previously in [4] (see equations (5.9) and (5.10) there).

One expects that this extended affine Weyl symmetry should define the model uniquely. The question is therefore if all these models labeled by \( J \) are really not equivalent to each other. To explore this question we will cast the above transformations in a more standard form by first performing a canonical transformation:

\[ q \rightarrow \tilde{q} = q/z^{-(1+J)/2}, \quad p \rightarrow \tilde{p} = pz^{-(1+J)/2}, \]

with the Hamiltonian system of equations

\[ z\tilde{q}_z = \tilde{q}^2 \left( 2\tilde{p} + r_0 z^{-(1+J)/2} \right) - ((J - 1)/2 + \alpha_1 + \alpha_3) \tilde{q} + \epsilon_0 r_0 z^{1-(1-J)/2} \]

\[ z\tilde{p}_z = \tilde{p} \left( \tilde{p} + r_0 z^{-(1+J)/2} \right) (-2\tilde{q}) + ((J - 1)/2 + \alpha_1 + \alpha_3) \tilde{p} - \alpha_2 r_0 z^{1-(1-J)/2} \]

that leads to simplified symmetry transformations by absorbing factors like \( z^{-(1+J)/2} \) appearing in e.g. (12):

\[ \pi_0(\tilde{q}) = -\frac{\epsilon_0}{\tilde{q}}, \quad \pi_0(\tilde{p}) = \frac{1}{\epsilon_0} \left( \tilde{q}^2 \tilde{p} \right) + \alpha_2 \tilde{q} \]

Furthermore for \( J \neq 1 \) we are able to define new variables \( W, F \) as

\[ W = \tilde{q}/\sqrt{(1-J)/2}; \quad F = \tilde{p}/\sqrt{(1-J)/2}. \]
The above transformation is not canonical, however introducing
\[ \xi = z^{(1-J)/2}, \quad \text{for } J \neq 1 \]
we can rewrite the corresponding equations as a Hamiltonian system
\[
\begin{align*}
\xi W_\xi &= \frac{\partial H_1}{\partial F} = W^2 (2F + \hat{r}_0 \xi) - (\hat{\alpha}_1 + \hat{\alpha}_3) W + \hat{\epsilon}_0 \hat{r}_0 \xi \\
\xi F_\xi &= -\frac{\partial H_1}{\partial W} = F (F + \hat{r}_0 \xi) (-2W) + (\hat{\alpha}_1 + \hat{\alpha}_3) F - \hat{\alpha}_2 \hat{r}_0 \xi
\end{align*}
\]
(19)
with new parameters:
\[ \hat{r}_0 = r_0/\sqrt{(1-J)/2}, \quad \hat{\epsilon}_0 = \epsilon_0/((1-J)/2), \quad \hat{\alpha}_2 = \alpha_2/((1-J)/2), \quad \hat{\alpha}_1 + \hat{\alpha}_3 = (C + \alpha_1 + \alpha_3)/((1-J)/2), \]
with respect to the new Hamiltonian:
\[ H_1 = W^2 F^2 + W^2 F \hat{r}_0 \xi - (\hat{\alpha}_1 + \hat{\alpha}_3) W F + \hat{\epsilon}_0 \hat{r}_0 F W + \hat{\alpha}_2 \hat{r}_0 \xi W. \]
We note that with this association the following relation holds
\[ 1 - \hat{\alpha}_1 - \hat{\alpha}_2 - \hat{\alpha}_3 = (1 - \alpha_1 - \alpha_2 - \alpha_3)/((1-J)/2) \]
that shows that the system is properly normalized for \( J \neq 1 \) with \( \hat{\alpha}_0 = \alpha_0/(1-J)/2 \).
Therefore as long as \( J \neq 1 \) we were able to cast the system for \( r_1 = 0 \) and general \( J \neq 1 \) into Hamilton equations (8) previously obtained for \( J = -1, r_1 = 0 \) with \( \alpha_i, i = 0, 1, 2, 3 \) replaced by \( \hat{\alpha}_i, i = 0, 1, 2, 3 \). Thus, as long as \( J \neq 1 \) the model obtained in \( r_1 \to 0 \) limit is equivalent to \( P_{III} \) model with an extended affine Weyl group acting according to relations (12), (13), (15) and (14) with \( J = -1 \). In particular, we find by substituting \( J = -1 \) in (12), (13), (15) and (14) that
\[ v_1 = \frac{1}{2} (\alpha_0 + \alpha_2), \quad v_2 = \frac{1}{2} (\alpha_0 - \alpha_2), \]
transform under \( s_2, \pi^2, \pi_0 \) as
\[
\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \mapsto \begin{pmatrix} \pi^2 \\ v_2 \end{pmatrix}, \quad \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \mapsto \begin{pmatrix} -1 - v_2 \\ -1 \end{pmatrix}
\]
(20)
One sees that actions of \( \pi_0, s_2, \pi^2 \) on parameters \( (v_1, v_2) \) realize a representation of the extended affine Weyl group for the root system \( C_2^{(1)} \) [7, 4]. Consider namely a 2-dimensional vector space \( \mathbf{V} \) consisting of vectors \( \mathbf{v} = v_1 \mathbf{e}_1 + v_2 \mathbf{e}_2 \), with \( \mathbf{e}_1, \mathbf{e}_2 \) being a canonical basis of \( \mathbf{V} \). Define next a symmetric bilinear form \( \langle \cdot , \cdot \rangle \) such that \( \langle \mathbf{e}_i | \mathbf{e}_j \rangle = \delta_{ij} \). Then according to [13] vectors
\[ \mathbf{a}_1 = \mathbf{e}_1 - \mathbf{e}_2, \quad \mathbf{a}_2 = \mathbf{e}_2 \]
(21)
are the fundamental roots of the \( C_2 \) root system and
\[ \mathbf{a}_0 = \mathbf{e}_1 + \mathbf{e}_2 \]
(22)
is its highest root. Geometrically, the transformations \( s_2, \pi^2 \) are reflections in the hyperplane perpendicular to vectors \( \mathbf{a}_i, i = 1, 2 \) and the transformation \( \pi_0 \) corresponds to reflections in the hyperplane \( \{ v : \langle \mathbf{a}_0 | v \rangle = -1 \} \) [4].
As one can see from (12), (13) the transformation \( \pi_0 \) for the special value of \( J = 1 \) transforms \( \alpha_i \) exactly as \( s_0 \) and \( \pi_0(v_1) = -v_2, \pi_0(v_2) = -v_1 \) no longer involves reflection in the hyperplane perpendicular to the highest root. Thus actions of these transformations do not coincide in this case with an extended affine Weyl symmetry within this geometric interpretation.
We now turn our attention to the remaining case of reduction of the \( P_{III-V} \) model for \( r_1 = 0 \) and \( J = 1 \).
2.1. The \( r_1 \to 0 \) limit when \( J = 1 \)

We now consider separately the case \( J = 1 \) when \( r_1 = 0 \). Inserting \( J = 1 \) into equations (18) and defining

\[
x = \ln z, \quad w = \dot{q}, \quad f = \dot{p}
\]

we obtain

\[
\begin{align*}
w_x &= w^2 (2f + r_0) - (\alpha_1 + \alpha_3 - 1) w + \epsilon_0 r_0 \\
f_x &= f( f + r_0)(-2w) + (\alpha_1 + \alpha_3 - 1)f - \alpha_2 r_0
\end{align*}
\]

that originate from a Hamiltonian

\[
H = f^2 w^2 + w^2 q r_0 + (\alpha_1 + \alpha_3 - 1) f w + \epsilon_0 r_0 f + \alpha_2 r_0 w
\]

The second order equation for \( w \) is given by:

\[
w_{xx} = \frac{w^2}{w} + w^3 r_0^2 + w^2 r_0 (\alpha_2 - \alpha_0) - \epsilon_0 r_0 (\alpha_0 + \alpha_2) - \frac{\epsilon_0^2 r_0^2}{w}
\]

and agrees with equation I_{12} of Ince as reproduced in (A.1) in Appendix A.

The second order equation for \( f \) written in terms of \( y \) such that

\[
f = -\frac{r_0 y}{y - 1}
\]

leads to Ince’s equation XXXVIII (A.2) with \( A = (1 - \alpha_1 - \alpha_2 - \alpha_3)^2 / 2, B = -\alpha_2^2 / 2, C = -2\epsilon_0 r_0^2 \) and \( D = 0 \) and thus the equation obtained in this limit is only an incomplete version of Ince’s 38-th equation (A.2).

3. The \( r_0 \to 0 \) limit of \( P_{\text{III–V}} \) model

Setting \( r_0 \to 0 \) in equation (1) yields

\[
q_{zz} = -\frac{q_z}{z} + \left( \frac{1}{2q} + \frac{1}{2(q - r_1)} \right) q^2 + \frac{\alpha_1^2 r_1 (q - r_1)}{2z^2 q} - \frac{\alpha_3^2 r_1 q}{2z^2 (q - r_1)} - 2\epsilon_1 r_1 z^{J-1} q(q - r_1)
\]

For \( J = 0 \) one recognizes in the above equation for \( y = (q - r_1)/q \) the Painlevé V equation (A.5) with the parameter \( D = 0 \). For \( D = 0 \) the Painlevé V equation is known to be equivalent to the Painlevé III equation [8].

Applying automorphism \( \pi \) (10) one transforms the symmetry transformations \( \pi_i, s_i, i = 0, 2 \) to symmetry transformations \( \pi_i i = 1, 3 : \)

\[
\begin{align*}
\pi_1(p) &= \epsilon_1 z^{J-1} p, \quad \pi_1(q) = \frac{z^{-J-1}}{\epsilon_1} (-p^2 q + \alpha_1 p), \quad \pi_1(\alpha_1) = \alpha_1, \quad \pi_1(\alpha_3) = J + 1 - \alpha_3 \\
\pi_3(p) &= -\epsilon_1 z^{J-1} p, \quad \pi_3(q) = \frac{z^{-J-1}}{\epsilon_1} (p^2 (q - r_1) - \alpha_3 p) + r_1, \quad \pi_3(\alpha_1) = J + 1 - \alpha_1, \quad \pi_3(\alpha_3) = \alpha_3
\end{align*}
\]

that together with transformations \( s_1 = \pi s_0 \pi \) and \( s_3 = \pi s_2 \pi \) keep invariant equations

\[
\begin{align*}
zh &= q (q - r_1) 2p - (\alpha_1 + \alpha_3) q + \alpha_1 r_1 \\
zp &= p^2 (r_1 - 2q) + (\alpha_1 + \alpha_3)p - \epsilon_1 r_1 z^{J+1}
\end{align*}
\]
obtained from (8) in the limit $r_0 \to 0$. For $J \neq -1$ the transformation

$$q \to q/z^{-(1+J)/2}/\sqrt{-{(1+J)/2}} = F, \quad p \to = pz^{-(1+J)/2}/\sqrt{-{(1+J)/2}} = W,$$

followed by a change of variable $z \to \xi = z^{(1+J)/2}$ leads to equations:

$$\xi W_\xi = W^2(2F - \dot{r}_1) - (\dot{\alpha}_1 + \dot{\alpha}_3)W + \ddot{r}_1\xi$$

$$\xi F_\xi = F(F - \dot{r}_1)(-2W) + (\dot{\alpha}_1 + \dot{\alpha}_3) F - \dot{\alpha}_1\ddot{r}_1\xi$$

(29)

where $\ddot{r}_1 = r_1/\sqrt{-{(1+J)/2}}, \dot{\epsilon}_1 = \epsilon_1/\sqrt{-{(1+J)/2}}, \dot{\alpha}_i = \alpha_i/\sqrt{-{(1+J)/2}}$. One obtains from (29) the following second order equation for $W$:

$$W_\xi = \frac{W^2_\xi}{W} - \frac{W_\xi}{\xi} + W^3\dot{r}_1^2 + W^2\dot{r}_1\frac{\dot{\alpha}_1 - \dot{\alpha}_3 + 1}{\xi} + \frac{\dot{\epsilon}_1^2\dot{r}_1^2}{W} - \frac{\dot{\epsilon}_1\dot{r}_1}{\xi}(\dot{\alpha}_1 - \dot{\alpha}_3) + 1),$$

which is Painlevé III equation (A.4). Thus we have obtained Painlevé III equation in $r_0 \to 0$ limit for any $J \neq -1$. This establishes another way to understand an equivalence between Painlevé V equation (A.5) with $D = 0$ and Painlevé III equation (A.4) realized in a setting of Hamilton equations.

3.1. The $r_0 \to 0$ limit for $J = -1$

The transformations $\pi_i, i = 1, 3$ in (26), (27) for the special value of $J = -1$ transform $\alpha_i$ in the same way as $s_1, s_3$ and it does not look in such case that actions of these transformations on roots will form an extended affine Weyl symmetry group. To investigate this further we set $J = -1$ directly in (28) to obtain (for $x = \ln z$):

$$q_x = q(q - r_1)2p - (\alpha_1 + \alpha_3)q + \alpha_1 r_1,$$

$$p_x = p^2(r_1 - 2q) + (\alpha_1 + \alpha_3)p - \epsilon_1 r_1.$$

Let us set as before $q = w, p = f$ and note that the Hamiltonian that reproduces the above equation is given by:

$$H = f^2w^2 - wf^2r_1 - (\alpha_1 + \alpha_3)fw + \epsilon_1 r_1 w + \alpha_1 r_1 f.$$

(30)

Note that the major difference from (24) is the term $wf^2$ instead for $w^2f$.

For the quantity $f = p$ we find from the above equations a second order equation:

$$f_{xx} = \frac{f^2}{f} + f^3r_1^2 + f^2r_1(-\alpha_1 + \alpha_3) + \epsilon_1 r_1(\alpha_1 + \alpha_3) - \frac{\epsilon_1^2 r_1^2}{f}$$

in which we again recognize the XII-th equation of Ince (A.1). Furthermore we derive:

$$w_{xx} = \frac{w^2}{2} \left( \frac{1}{w} + \frac{1}{w - r_1} \right) - 2r_1\epsilon_1 w^2 + \alpha_1^2 r_1$$

$$+ \frac{2r_1^2\epsilon_1 w^2}{w - r_1} - w r_1 \frac{\alpha_1^2 + \alpha_3^2 + 4\epsilon_1^2 r_1}{2(w - r_1)} + \frac{r_1^3\alpha_1^2}{2(w - r_1)}$$

Defining $y$ in terms of $w$ as

$$y = \frac{w}{w - r_1} \quad \text{or} \quad w = \frac{r_1 y}{y - 1}$$

one obtains a special case of Ince’s equation $I_{19}$ (A.3) listed in Appendix A with the parameters $A = 1, B = \alpha_1^2/2, C = -\alpha_1^2/2$ and $D + E = 2r_1^2\epsilon_1$.

Note that Ince’s equation $I_{38}$ (A.2) with $D = 0$ can be rewritten as Ince’s equation 49 (A.3) with $A = 1$ and vice versa.
4. Discussion

One of main lessons derived from the above exercises of reducing $P_{III-V}$ is that the Hamilton functions of the type

$$H = f^2 w^2 + \kappa w f^2 + \beta f w + \gamma f + \delta w,$$

or

$$H = f^2 w^2 + \kappa w^2 f + \beta f w + \gamma f + \delta w$$

(31)

will lead to $I_{12}$ equation and will be invariant under the symmetry generators that satisfy the Coxeter group relations (16), (17). Let us illustrate this using the first of Hamiltonians in (31). The corresponding Hamilton equations:

$$w_x = 2w^2 f + 2\kappa w f + \beta w + \gamma,$$

$$f_x = -2f^2 w - \kappa f^2 - \beta f - \delta.$$  (32)

lead to a second order equation for $f$:

$$f_{xx} = f^2 f + \delta \beta + f^2 (\kappa \beta - 2\gamma) + f^2 \kappa^2 - \delta^2 / f$$

which is $I_{12}$ equation (A.1) from Ince’s list.

Eqs. (32) are invariant under $s_2$ transformations:

$$s_2(f) = f + \frac{\gamma}{\kappa w},$$

$$s_2(w) = w, s_2(\beta) = \beta - 2\gamma / \kappa, s_2(\kappa) = \kappa, s_2(\delta) = \delta, s_2(\gamma) = -\gamma,$$  (33)

and $\pi_0$ transformations:

$$\pi_0(f) = -\frac{\delta}{\kappa f},$$

$$\pi_0(w) = \frac{\kappa}{\delta} (f^2 w + \gamma f / \kappa),$$

$$\pi_0(\beta) = -\beta + 2\frac{\gamma}{\kappa},$$

$$\pi_0(\delta) = \delta,$$

$$\pi_0(\gamma) = \gamma,$$  (34)

with $\pi_0(\kappa) = \kappa$ as well as a version of $\pi^2$:

$$\pi^2(f) = -f,$$

$$\pi^2(w) = -w - \kappa,$$

$$\pi^2(\gamma) = -\gamma + \kappa \beta,$$

$$\pi^2(\delta) = -\delta,$$

$$\pi^2(\kappa) = \kappa,$$

$$\pi^2(\beta) = \beta.$$  

As in (16) we can now define $s_0, \pi_2$ as $\pi^2 \pi_0 \pi^2 = \pi_2,$ $\pi^2 s_0 \pi^2 = s_0$ and obtain the Coxeter relations (17). As observed above the resulting symmetry $W[s_0, s_2, \pi_0, \pi_2, \pi^2]$ can not be given the extended affine Weyl group interpretation that holds for $W[s_0, s_2, \pi_0, \pi_2, \pi^2]$ structure in the setting of $P_{III}$ equation. Comparison of actions of $s_2$ and $\pi_0$ on parameters $\beta, \gamma, \delta, \kappa$ in equations (33) and (34) indeed reveals identical behavior (up to the sign) of those two transformations, which in the discussion below (22) was recognized as a reason for why the geometric interpretation of $W[s_0, s_2, \pi_0, \pi_2, \pi^2]$ as an extended affine Weyl group did not extend to the case of symmetry of $I_{12}$ equation.

The remaining questions of how to complete Hamiltonian structures seen in this paper in such a way as to obtain full equations $I_{38}, I_{49}$ and what are the symmetries governing $I_{38}, I_{49}$ models will be addressed in a paper in preparation [3].

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Appendix A. Selected Equations from Ince’s List

Here we list the three equations, $I_{12}, I_{38}$ and $I_{49}$, from Ince’s list, and two Painlevé equations that are subject of our discussion:

$$I_{12} : y_{xx} = \frac{y_x^2}{y} + Ay^3 + By^2 + C + \frac{D}{y} \tag{A.1}$$

$$I_{38} : y_{xx} = \left(\frac{1}{2y} + \frac{1}{y - 1}\right) y_x^2 + y(y - 1) \left(A(y - 1) + B\frac{y - 1}{y^2} + \frac{C}{y - 1} + \frac{D}{(y - 1)^2}\right) \tag{A.2}$$

$$I_{49} : y_{xx} = \left(\frac{1}{y} + \frac{1}{y - 1} + \frac{1}{y - A}\right) \frac{y_x^2}{2} + y(y - 1)(y - A) \left(B + \frac{C}{y^2} + \frac{D}{(y - 1)^2} + \frac{E}{(y - A)^2}\right) \tag{A.3}$$

$$P_{III} : y_{zz} = -\frac{1}{z} y_z + \frac{y_z^2}{y} + Ay^2 z + Cy^3 + \frac{B}{z} + \frac{D}{y} \tag{A.4}$$

$$P_{V} : y_{zz}(z) = \left(\frac{1}{y - 1} + \frac{1}{2y}\right) y_z^2 - \frac{y_z}{z} + \frac{(y - 1)^2 \left(Ay + \frac{B}{y}\right)}{z^2} + \frac{Cy}{z} + \frac{Dy(1 + y)}{y - 1} \tag{A.5}$$

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