Generalization of the Darboux transformation and generalized harmonic oscillators

Dae-Yup Song†‡ and John R. Klauder§
† Department of Physics, University of Florida, Gainesville, FL 32611, USA
‡ Department of Physics, Sunchon National University, Suncheon 540-742, Korea
§ Department of Physics and Mathematics, University of Florida, Gainesville, FL 32611, USA
E-mail: dsong@sunchon.ac.kr, klauder@phys.ufl.edu

Abstract. The Darboux transformation is generalized for time-dependent Hamiltonian systems which include a term linear in momentum and a time-dependent mass. The formalism for the $N$-fold application of the transformation is also established, and these formalisms are applied for a general quadratic system (a generalized harmonic oscillator) and a quadratic system with an inverse-square interaction up to $N = 2$. Among the new features found, it is shown, for the general quadratic system, that the shape of potential difference between the original system and the transformed system could oscillate according to a classical solution, which is related to the existence of coherent states in the system.

PACS numbers: 03.65.Ge 03.65.Ca

1. Introduction

There has been great interest in using the Darboux transformation \[1,2\] for the analysis of physical systems and for finding new solvable systems. It has been shown that the transformation method is useful in finding soliton solutions of the integrable systems \[3\] and constructing supersymmetric quantum mechanical systems \[4\]. An excellent survey for developments and some applications of the transformation method is given in \[5\].

If a Darboux transformation is applied to a time-independent Schrödinger equation of confining potential, one of the immediate consequences is that the energy eigenvalues of the transformed system will be almost identical with those of the original system except for a finite number of addition(s) and/or deletion(s) to the spectrum \[6\]. On the other hand, Abraham and Moses (AM) have developed an integral equation algorithm \[7\], which can also be used to find a new solvable system based on a known one. When applied to a harmonic oscillator, the AM algorithm gives a transformed system whose energy spectrum coincides with that of the harmonic oscillator except that the lowest eigenvalue has been removed. It has been shown that the result given by AM can also be derived through the factorization method \[8\] and the factorization method has been
Darboux transformation and generalized harmonic oscillators

applied for various systems \[9\]. After all, the AM method and the factorization method are intimately related to the Darboux transformation \[6\,\,10\,\,11\].

In order to obtain a new solvable system by implementing the Darboux transformation, it is necessary to choose the auxiliary function of the transformation judiciously \[6\]. After the transformation being extended to include a time-dependent potential, the application of the Darboux transformation and the two-fold application of the transformation have been explicitly carried out for the harmonic oscillator model (with time-dependent frequency), to obtain new solvable systems \[11\,\,12\]. For the simple harmonic oscillator, unphysical negative-energy eigenstates, which can be obtained by invoking the symmetry of the Hamiltonian, have been used as the auxiliary functions of the Darboux transformation. For the two-fold application of the transformation, a theorem has already been established in \[13\] for the choice of the auxiliary functions.

In this article, we will generalize the Darboux transformation to be applicable to time-dependent Hamiltonian systems which include a term linear in momentum and a time-dependent mass. The formalism for the \(N\)-fold application of the transformation will also be established. The transformation method will then be applied for a general quadratic system (a generalized harmonic oscillator) and a quadratic system with an inverse-square interaction up to \(N = 2\). The general quadratic system \[14\] is known to be related to a simple harmonic oscillator through unitary transformations, which may be a reason for the existence of coherent states in the simple harmonic oscillator \[15\]. Even for the cases already considered \[11\,\,12\], new features of the Darboux transformed systems will be given. In particular, it will be shown that, the shape of the potential difference between the original Hamiltonian and the transformed one could oscillate according to the classical solution of the quadratic system, which is related to the existence of coherent states in the system.

2. A generalization of the Darboux transformation

For smooth functions \(P(x), Q(x)\), we assume a function \(u(x)\) satisfies

\[
\frac{d^2u}{dx^2} + P(x)\frac{du}{dx} + (Q(x) + C)u = 0, \tag{1}
\]

where \(C\) is a constant. If \(\phi(x)\) satisfies the linear differential equation

\[
\frac{d^2\phi}{dx^2} + P(x)\frac{d\phi}{dx} + Q(x)\phi = 0, \tag{2}
\]

Darboux shows that the following equation is true \[14\]

\[
\left[\frac{d^2}{dx^2} + P(x)\frac{d}{dx} + \frac{dP}{dx} + Q(x) + 2\left(\frac{d^2\ln u}{dx^2}\right)\right] \left(\frac{d\phi}{dx} - \phi\frac{du}{dx}\right) = 0. \tag{3}
\]

This Darboux transformation has immediate consequences for a time-independent Schrödinger equation \[6\].
For the extension of the transformation to be applicable for the time-dependent system described by the Hamiltonian

\[ H = \frac{p^2}{2M(t)} + (R(x,t)p + pR(x,t)) + V(x,t), \]

we consider the operator

\[ O(t,x) = -i\hbar \frac{\partial}{\partial t} + H = -i\hbar \frac{\partial}{\partial t} - \frac{\hbar^2}{2M(t)} \frac{\partial^2}{\partial x^2} - i\hbar \{ 2R(x,t) \frac{\partial}{\partial x} + R'(x,t) \} + V(x,t). \]

In \( H \), \( M(t) \) denotes time-dependent mass and the term proportional to \( R(x,t) \) is included to resemble a three dimensional model which is under a vector potential. A wave function of the system of \( H \) should satisfy the time-dependent Schrödinger equation

\[ O(t,x)\psi(t,x) = 0. \]

We also introduce the auxiliary functions \( u_k \) satisfying

\[ O(t,x)u_k(t,x) = 0 \quad (k = 1, 2, \cdots, N). \]

We assume that \( u_k \) is a smooth function for any finite \( x \), but \( u_k \) need not be square-integrable as it is just an auxiliary function.

From (6) and (7), it is straightforward to show that

\[ \left( O(t,x) - 2i\hbar R'(x,t) - \frac{\hbar^2}{M(t)} \frac{\partial^2 \ln u_1}{\partial x^2} \right) \psi^{1d}(x,t) = 0, \]

where

\[ \psi^{1d}(x,t) = \left( \psi' - \psi \frac{u_1'}{u_1} \right), \]

and \( ' \) denotes the partial derivative with respect to \( x \). Equation (8) is a generalization of the Darboux transformation.

The Darboux transformation can be applied repeatedly. For the description of \( k \)-fold transformations, we define the Wronskian determinants as

\[ W_k = \begin{vmatrix} u_1 & u_2 & \cdots & u_k \\ u_1' & u_2' & \cdots & u_k' \\ \vdots & \vdots & \ddots & \vdots \\ u_1^{(k-1)} & u_2^{(k-1)} & \cdots & u_k^{(k-1)} \end{vmatrix}, \]

\[ W_{k,\psi} = \begin{vmatrix} u_1 & u_2 & \cdots & u_k & \psi \\ u_1' & u_2' & \cdots & u_k' & \psi' \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ u_1^{(k)} & u_2^{(k)} & \cdots & u_k^{(k)} & \psi^{(k)} \end{vmatrix}. \]
Making use of Crum’s formula \[2\]

\[ W_{k,\psi} W_{k-1} = W_{k} \frac{\partial}{\partial x} W_{k-1,\psi} - W_{k-1,\psi} \frac{\partial}{\partial x} W_{k}, \]  

(11)

one can find that \( \psi^{kd}(x, t) \) satisfying

\[ \left( O(t, x) - 2i\hbar R'(x, t) - \frac{\hbar^2}{M(t)} \frac{\partial^2 \ln W_k}{\partial x^2} \right) \psi^{kd}(x, t) = 0 \]  

(12)

is given by

\[ \psi^{kd}(x, t) = \frac{W_{k,\psi}}{W_k}. \]  

(13)

There is a difficulty in interpreting equation (12) as a Shrödinger equation, since the associated Hamiltonian is, in general, not Hermitian. As discussed in special cases [11], this difficulty can be resolved if there exists a purely time-dependent function \( \alpha_k(t) \) satisfying

\[ \frac{d}{dt} \ln \alpha_k = 2kR' - i\frac{\hbar}{2M(t)} \frac{\partial^2 \ln W_k}{\partial x^2}, \]  

(14)

where \( \bar{W}_k \) denotes the complex conjugate of \( W_k \). In this case, equation (12) can be rewritten as

\[ \left( O(t, x) - \frac{\hbar^2}{2M(t)} \frac{\partial^2 (W_k \bar{W}_k)}{\partial x^2} \right) \psi^{kD}(x, t) = 0, \]  

(15)

where

\[ \psi^{kD}(x, t) = \alpha_k(t) \psi^{kd}(x, t). \]  

(16)

If \( V(x, t) \) is a real function, equation (15) is the Schrödinger equation of a system described by the Hermitian Hamiltonian

\[ H_k = \frac{p^2}{2M(t)} + (R(x, t)p + pR(x, t)) + V(x, t) - \frac{\hbar^2}{2M(t)} \frac{\partial^2 \ln (W_k \bar{W}_k)}{\partial x^2}. \]  

(17)

One of the crucial conditions for \( \psi^{kD}(x, t) \) being square-integrable is that \( W_k \) should not have any zero in the entire space of \( x \). Indeed, the derivations of (12) and (13) are not valid for the zeros of \( W_1, W_2, \cdots, W_k \), while, if \( W_k \) has no zero, the formulas would still be useful even for the cases that \( W_1, W_2, \cdots, W_{k-1} \) have zeros, as examples will show later.

In addition to \( \psi^{kD}(x, t) \), other solutions of the Schrödinger equation of \( H_k \) would exist [6]. For the \( H_1 \) system, from the fact that

\[ i\hbar \frac{\partial}{\partial t} u = \frac{\hbar^2}{2M} \left( \frac{\partial^2}{\partial x^2} \frac{1}{u} \right) - 2i\hbar R \left( \frac{\partial}{\partial x} \frac{1}{u} \right) + (-V + i\hbar R') \frac{1}{u} + \frac{1}{u} \frac{\hbar^2}{M} \left( \frac{\partial^2 \ln u}{\partial x^2} \right), \]  

(18)

a solution \( \psi^{1D}_u \) satisfying \([ -i\hbar(\partial/\partial t) + H_1 ] \psi^{1D}_u = 0 \) is given as [6, 11]

\[ \psi^{1D}_u(x, t) = \frac{1}{\alpha u_1} = \frac{1}{\alpha W_1}. \]  

(19)
3. A general quadratic system

For an application of the transformation of the previous section, it may be essential to
find $u_1$ or $W_k$ which does not vanish over the whole coordinate space. Moreover, for an
Hermitian Hamiltonian, such auxiliary solutions are required to support the existence
of a purely time-dependent function, $a_k(t)$, defined in (14). For the non-vanishing
$u_1$ of the harmonic oscillator, unphysical negative-energy eigenstates can be used as
the auxiliary functions for the transformation. For the two-fold application, the sign
theorem established in [13] may be used to find non-vanishing $W_2$. Nevertheless, it has
been shown that the application and two-fold application of the Darboux transformation
are possible for the harmonic oscillator (of time-dependent frequency) with and without
an inverse-square potential [11, 12].

In this section, we will show that the application and two-fold application of the
Darboux transformation are possible for a general quadratic system. Even for the cases
already considered [11, 12], new features of the applications will be found.

3.1. Darboux transformation

A general quadratic system can be conveniently described by the Lagrangian [14]

$$L_Q = \frac{1}{2}M(t)\dot{x}^2 - \frac{1}{2}M(t)w^2(t)x^2 + F(t)x + \frac{d}{dt}\left(M(t)a(t)x^2\right) + \frac{d}{dt}(b(t)x) + f(t),$$

(20)

where the overdot denotes the derivative with respect to $t$. This Lagrangian gives the
classical equation of motion

$$\frac{d}{dt}(M(t)\dot{x}) + M(t)w^2(t)x = F(t)$$

(21)

which is that of a generalized harmonic oscillator of mass $M(t)$ and frequency $w(t)$ in
an external force $F(t)$. The corresponding Hamiltonian is written as

$$H_Q = \frac{p^2}{2M(t)} - a(t)[px + xp] + \frac{1}{2}M(t)c(t)x^2 - \frac{b(t)}{M(t)}p + d(t)x + \left(\frac{b^2(t)}{2M(t)} - f(t)\right),$$

(22)

where

$$c(t) = w^2 + 4a^2 - 2\dot{a} - 2\frac{\dot{M}}{M}a, \quad d(t) = 2ab - \dot{b} - F.$$ (23)

We assume that $H_Q$ is Hermitian. The general solution of equation (21) is a linear
combination of a particular solution $x_p(t)$ and two linearly independent homogeneous
solutions $u(t), v(t)$. We assume $x_p(t), u(t), v(t)$ are real, and define $\rho(t)$ and a time-
constant $\Omega$, for later use, as

$$\rho = \sqrt{u^2(t) + v^2(t)}, \quad \Omega = M(t)[\dot{v}(t)u(t) - \dot{u}(t)v(t)].$$

(24)

Since the general quadratic system can be obtained from a simple harmonic
oscillator through unitary transformations, we first consider the simple harmonic
oscillator of unit mass and frequency. The Schrödinger equation for the simple harmonic
oscillator, whose time is $\tau$, is written as

$$i\hbar \frac{\partial}{\partial \tau} \psi_s = -\frac{\hbar^2}{2} \frac{\partial^2}{\partial x^2} \psi_s + \frac{x^2}{2} \psi_s = H_s \psi_s.$$ (25)
Darboux transformation and generalized harmonic oscillators

Making use of the invariance of the Schrödinger equation under the exchange of \( \tau \leftrightarrow -\tau \) and \( x \leftrightarrow ix \), from the well-known wave functions

\[
\psi_n^s(\tau, x) = \frac{e^{-i(n+1/2)\tau}}{(2\pi \hbar)^{1/2}} \exp \left[ -\frac{x^2}{2\hbar} \right] H_n \left( \frac{x}{\sqrt{\hbar}} \right)
\]

one can find auxiliary functions satisfying the same Schrödinger equation as

\[
v_n^s(\tau, x) = e^{i(n+1/2)\tau} \exp \left[ \frac{x^2}{2\hbar} \right] H_n \left( \frac{ix}{\sqrt{\hbar}} \right), \quad n = 0, 1, 2, \ldots.
\]

Indeed, \( v_n^s(\tau, x) \) for even \( n \) has no zero over the whole coordinate space, and it can be used as an auxiliary function of the Darboux transformation. In this subsection, we restrict our attention on the cases that \( n = 0, 2, 4, \ldots \).

By defining

\[
O_s(\tau, x) = -i\hbar \frac{\partial}{\partial \tau} + H_s, \quad O_Q(t, x) = -i\hbar \frac{\partial}{\partial t} + H_Q,
\]

and if \( \tau \) and \( t \) are related by

\[
d\tau = \frac{\Omega}{M(t)\rho^2(t)} dt,
\]

then from the results given in [15] one can find that

\[
O_Q(t, x) = U_G U_F U_S O_s(\tau, x) U_S^\dagger U_F^\dagger U_G^\dagger |_{\tau = \tau(t)},
\]

where

\[
U_G = \exp[\frac{i}{\hbar}(M(t)a(t)x^2 + b(t)x + \int f(z)dz)]
\]

\[
U_F(x_p) = \exp[\frac{i}{\hbar}(M\dot{x}_p x + \delta(x_p))] \exp(-\frac{i}{\hbar}x_pp)
\]

\[
U_S(\rho, \Omega) = \exp[\frac{i}{2\hbar}\frac{\rho^2}{\rho} x^2] \exp[-\frac{i}{4\hbar}\ln(\frac{\rho^2}{\Omega}) (xp + px)]
\]

with \( \delta \) defined through the relation

\[
\delta(x_p) = \frac{1}{2}Mw^2x_p^2 - \frac{1}{2}M\dot{x}_p^2.
\]

A point that should be mentioned is the unitary operators \( U_F \) and \( U_S \) are not unique since they depend on the choice of classical solutions. Instead of \( \{u, v\} \) and \( x_p \), one can take another set of two linearly independent homogeneous solutions \( \{\tilde{u}, \tilde{v}\} \) and a particular solution \( \tilde{x}_p \) of equation (21). After defining \( \tilde{\rho}, \tilde{\Omega} \) and \( \tilde{\delta} \) from \( \{\tilde{u}, \tilde{v}\} \) and \( \tilde{x}_p \) as \( \rho, \Omega \) and \( \delta \) are defined from \( \{u, v\} \) and \( x_p \), one can find that the unitary relation (30) is also valid with the unitary operators \( U_F(\tilde{x}_p), U_S(\tilde{\rho}, \tilde{\Omega}) \).

Making use of the unitary relation, the normalized wave functions of the system of \( H_Q \) have been given in [15] as

\[
\psi_m^Q(t, x) = \frac{1}{\sqrt{2mm!}} \left( \frac{\Omega}{\pi \hbar} \right)^{\frac{m}{2}} \frac{1}{\sqrt{\rho(t)}} \left[ \frac{u(t) - iv(t)}{\rho(t)} \right]^{m+\frac{1}{2}} \exp \left[ \frac{i}{\hbar} \left( \delta(t) + \int f(z)dz \right) \right]
\]

\[
\times \exp \left[ \frac{i}{\hbar} \left[ M(t)a(t)x^2 + (M(t)\dot{x}_p(t) + b(t))x \right] \right]
\]
The unitary relation can also be used to find a solution $v_n^Q(t, x)$ of the Schrödinger equation

$$O(t, x)v_n^Q(t, x) = 0,$$

as

$$v_n^Q(t, x) = U_G U_F(x_p(t)) U_S(\bar{\rho}, \tilde{\Omega}) v_n^s(\tau, x)|_{\tau = \tau(t)}$$

$$= \sqrt{\frac{\Omega}{\bar{\rho}(t)}} \left[ \frac{\hat{u}(t) + i\hat{v}(t)}{\bar{\rho}(t)} \right]^{n+\frac{1}{2}} \exp \left[ \frac{i}{\hbar} \left( \hat{\delta}(t) + \int^t f(z)dz \right) \right]$$

$$\times \exp \left[ \frac{i}{\hbar} \left[ M(t)a(t)x^2 + \left( M(t)\hat{x}_p(t) + b(t) \right)x \right] \right]$$

$$\times \exp \left[ \frac{(x - \hat{x}_p(t))^2}{2\hbar} \left( \frac{\tilde{\Omega}}{\bar{\rho}^2(t)} + iM(t)\frac{\tilde{\rho}(t)}{\bar{\rho}(t)} \right) \right] H_n \left( i\sqrt{\frac{\tilde{\Omega} - x - \hat{x}_p(t)}{\bar{\rho}(t)} \right) . \quad (37)$$

From the properties of the unitary transformations, it is manifest that $v_n^Q(t, x)$ for even $n$ does not have a zero over whole coordinate space.

Since, for even $n$, $H_n \left( i\sqrt{\frac{\tilde{\Omega} - x - \hat{x}_p(t)}{\bar{\rho}(t)} \right)$ is a real function, it is easy to find that

$$-\frac{i}{2M} \frac{\partial^2}{\partial x^2} \ln \frac{v_n^Q}{v_n^Q} = 2a + \frac{\hat{\rho}}{\bar{\rho}}, \quad (38)$$

which shows, for the choice of $u_1^Q = W_1^Q = v_n^Q$, that $\alpha_1^Q(t)$ defined in (14) can be found as

$$\alpha_1^Q(t) = \bar{\rho}(t) \quad (39)$$

up to a normalization constant. Therefore, the transformation formalism developed in the previous section can be applied with $u_1^Q(t, x) = v_n^Q(t, x)$ to give the solvable model described by the Hamiltonian

$$H_1^Q(t, x, p) = H_Q - \frac{\hbar^2}{M\bar{\rho}^2} + \frac{\hbar^2}{M\bar{\rho}^2} \left[ (n - 1) \frac{H_{n-2}(z)}{H_n(z)} - n \left( \frac{H_{n-1}(z)}{H_n(z)} \right)^2 \right]. \quad (40)$$

where $z = i\sqrt{\frac{\Omega - x - \hat{x}_p(t)}{\bar{\rho}(t)}}$. If we adopt the notation that $H_{-2}(z) = H_{-1}(z) = 0$, equation (40) is valid for $n = 0, 2, 4, \cdots$. The magnitude of $\Delta V = (\equiv H_1^Q - H_Q + \frac{\hbar^2}{M\bar{\rho}^2})$ is $O(\hbar)$. Since $\Delta V$ vanishes in the limit of $|z| \to \infty$, $\Delta V/\hbar$ approaches 0 except for the region in the vicinity of $x_p$ where the width of the region is $O(\sqrt{\hbar})$.

From equations (10), (13) and (16), the unnormalized wave functions $\psi_n^Q(t, x)$ satisfying the Schrödinger equation

$$\left( -i\hbar \frac{\partial}{\partial t} + H_1^Q \right) \psi_n^Q(t, x) = 0,$$

\( (41) \)
can be given as
\[\psi_m^{nQ}(t, x) = \tilde{\rho}\psi_m \left[ i\hbar M(\dot{x}_p - \dot{\bar{x}}_p) + \frac{x - x_p}{\hbar} \left( -\frac{\Omega}{\rho^2} + iM\frac{\dot{\rho}}{\rho} \right) - \frac{x - \bar{x}_p}{\hbar} \left( \frac{\bar{\Omega}}{\rho^2} + iM\frac{\dot{\rho}}{\rho} \right) \right] + 2m\sqrt{\frac{\Omega}{\hbar} \frac{1}{H_{m-1}(w)}} \frac{1}{H_m(w)} - 2in\sqrt{\frac{\Omega}{\hbar} \frac{1}{H_{n-1}(z)}} \frac{1}{H_n(z)},\]
(42)
where \(w = \sqrt{\frac{\Omega}{\hbar} \frac{x - x_p}{\rho^2}}\). As in the general quadratic system [14, 15], a different choice of the classical solutions \(u(t), v(t), x_p(t)\) gives different wave functions. When we choose \(u(t) = \tilde{u}(t), v(t) = \tilde{v}(t), \) and \(x_p(t) = \tilde{x}_p(t)\), for even integer \(n\), an unnormalized \(\psi_m^{nQ}(t, x)\) is written as
\[\psi_m^{nQ}(t, x) = -\sqrt{\frac{2\Omega}{\hbar}} \left[ \sqrt{m+1} \left( \frac{u + iv}{\rho} \right) \psi_{m+1} + \sqrt{2m} \frac{H_{n-1}(iw)}{H_n(iw)} \psi_m \right],\]
(43)
for \(m = 0, 1, 2, \ldots\). For the case of \(M(t) = 1\) and \(a(t) = 0\), \(\psi_m^{nQ}(t, x)\) in equation (43) reproduces the wave functions given in [12], up to normalization. From equation (19), another wave function satisfying \((-i\hbar \frac{\partial}{\partial t} + H_1^{nQ}) \psi_{-n}^{nQ} = 0\) is given as
\[\psi_{-n}^{nQ} = \frac{1}{\rho \nu_n^Q}.\]
(44)
For non-negative even integer \(n\), it is clear that \(\psi_m^{nQ} (m = -n, 0, 1, 2 \ldots)\) are square-integrable.

Even for the simple harmonic oscillator of unit mass and frequency, since the shape of the probability density of a wave function could breathe and oscillate [14, 15], the region, where \(\Delta V/\hbar\) is different from 0 by a certain amount, could breathe and oscillate. If we take the classical solutions as \(\tilde{u}(t) = u(t) = \cos t, \tilde{v}(t) = v(t) = c \sin t (c \neq 0),\) and \(x_p(t) = d \cos t,\) the breathing and/or oscillating behavior of the non-vanishing region appears when \(c \neq 1\) and/or \(d \neq 0,\) respectively. For the choice \(c = 1\) and \(d = 0,\) \(H_1^{nQ}\) becomes
\[H_1^{ns} = \frac{1}{2} (p^2 + x^2) - \hbar - \hbar^2 \frac{\partial^2}{\partial x^2} \ln H_n(i \frac{x}{\sqrt{\hbar}}).\]
(45)
Through the same choice of classical solutions, for non-negative even integer \(n,\) one can also find the eigenfunctions of \(H_1^{ns}\) as
\[\psi_{m}^{s, n} = -\sqrt{\frac{2}{\hbar}} \left[ \sqrt{m+1} e^{it} \psi_{m+1} + \sqrt{2m} \frac{H_{n-1}(i \frac{x}{\sqrt{\hbar}})}{H_n(i \frac{x}{\sqrt{\hbar}})} \psi_m \right],\]
\[\psi_{-n}^{s, n} = \frac{1}{\tilde{u}_n^s},\]
whose eigenvalues are \((m + \frac{1}{2})\hbar\) and \(-(n + \frac{1}{2})\hbar,\) respectively, as expected in [6].

3.2. A two-fold transformation

In this subsection, we will show that the two-fold application of the Darboux transformation is also possible for a general quadratic system. It will also be manifested
that, in the two-fold transformation, the breathing and oscillating behavior would still appear in the transformed systems of a simple harmonic oscillator.

By taking
\[ u_1(t, x) = \psi_n(t, x), \quad u_2(t, x) = \psi_{n+1}(t, x), \]  
(47)
one can find that
\[ W_2 = \psi_n \psi_{n+1}' - \psi_{n+1} \psi_n' \]
\[ = \frac{1}{2^n n! \sqrt{2(n + 1)}} \left( \frac{u - iv}{\rho} \right)^{2n+1} \frac{1}{\rho} \sqrt{\frac{\Omega}{\hbar}} \psi_0^2(t, x) J_n(w), \]
where
\[ J_n(w) = H_n(w) \frac{d}{dw} H_{n+1}(w) - H_{n+1}(w) \frac{d}{dw} H_n(w) \]
(49)
with \( w = \sqrt{\frac{x-x_0}{\rho}} \). Making use of a recursive relation among the Hermite polynomials [16], one can show that \( J_n(w) = 2H_2^2(w) + 2nJ_{n-1}(w) \). This relation, with the fact \( J_0 = 2 \), shows that \( J_n(w) \) is positive definite for all \( x \), which in fact has been implied in [13]. One can also find that
\[ -i \hbar \frac{\partial^2}{\partial x^2} \ln \frac{W_2}{W_2} = 4a + 2 \dot{\rho} / \rho. \]
(50)
Equation (50) shows that \( \alpha_2^Q \) can be given in accordance with equation (14) as
\[ \alpha_2^Q(t) = \rho^2(t), \]
(51)
up to a multiplicative constant.

With the \( W_2 \), the two-fold transformation can, therefore, be applied to the quadratic system to give the transformed Hamiltonian
\[ H_{nQ}^2 = H_Q + 2 \frac{\hbar \Omega}{M \rho^2} - \frac{\hbar^2}{M} \frac{\partial}{\partial x^2} \ln J_n(w). \]
(52)
For the case of \( M(t) = 1 \) and \( a(t) = 0 \), \( H_{nQ}^2 \) reduces to the Hamiltonian found in [12]. Making use of equations (10), (13) and (16), it is also possible to find the wave functions of the system of \( H_{nQ}^2 \). For non-vanishing \( W_2 \), it turns out that the same set of classical solutions must be used for both \( u_1 \) and \( u_2 \) in equation (47). However, in obtaining wave functions, a different set of classical solution could be used for \( \psi \) in equation (10), as in the Darboux transformation of the model.

4. A quadratic system with an inverse-square interaction

In this section, we will consider the application of the Darboux transformation for the quadratic system with an inverse-square interaction described by the Hamiltonian
\[ H_{in} = \frac{p^2}{2M(t)} - a(t)(xp + px) + \frac{1}{2} M(t)c(t)x^2 + \frac{g}{M(t)} \frac{1}{x^2}, \]
(53)
Darboux transformation and generalized harmonic oscillators

defined on the half line \( x > 0 \), where \( g \) is a constant. The system of \( H_{in} \) is related to the system described by the Hamiltonian

\[
H_{in}^s = \frac{\hat{p}^2}{2} + \frac{1}{2} x^2 + \frac{g}{x^2},
\]

through a unitary relation [15]. If \( u(t), v(t) \) denote the homogeneous solutions of equation (21) as in the previous section, and \( \rho(t) \) and \( \Omega \) are defined by equation (24), a wave function of the system of \( H_{in} \) is given, for non-negative integer \( n \), as

\[
\psi_{n,in}^i(t,x) = \left( \frac{4\Omega}{\hbar \rho^2} \right)^{1/4} \left( \frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)} \right)^{1/2} \left( \frac{u-iv}{\rho} \right)^{(2n+\alpha+1)} \left( \frac{\Omega x^2}{\hbar \rho^2} \right)^{(2\alpha+1)/4} \times \exp \left[ -\frac{x^2}{2\hbar} \left( \frac{\Omega_0 x^2}{\rho^2} - iM\frac{\dot{\rho}}{\rho} + 2iMa \right) \right] L_n^\alpha \left( \frac{\Omega x^2}{\hbar \rho^2} \right),
\]

where \( \alpha \) is defined through the relation

\[
g = \frac{1}{2} (\alpha + \frac{1}{2})(\alpha - \frac{1}{2})\hbar^2,
\]

and \( L_n^\alpha \) is the Laguerre polynomial [16]. For \( \alpha > -1 \), \( \psi_{n,in}^i(t,x) \) is square-integrable on the half line.

Through similar procedures used in the previous section, one can find the auxiliary function

\[
v_{n,in} = \frac{1}{\sqrt{\rho}} \left( \frac{u+iv}{\rho} \right)^{(2n+\alpha+1)} \left( \frac{\Omega_0 x^2}{\hbar \rho^2} \right)^{(2\alpha+1)/4} \times \exp \left[ \frac{x^2}{2\hbar} \left( \frac{\Omega_0 x^2}{\rho^2} + iM\frac{\dot{\rho}}{\rho} + 2iMa \right) \right] L_n^\alpha \left( \frac{\Omega_0 x^2}{\hbar \rho^2} \right),
\]

satisfying

\[
\left( -i\hbar \frac{\partial}{\partial t} + H_{in} \right) v_{n,in} = 0.
\]

From the properties of the Laguerre polynomial, \( v_{n,in} \) has no zero on the half line, and it is easy to see that

\[
-\frac{i}{2M} \frac{\partial^2}{\partial x^2} \ln v_{n,in}^i = 2\alpha + \frac{\dot{\rho}}{\rho}.
\]

Equations (14) and (57) show that \( \alpha_{1,in}^i(t) \) can be given as

\[
\alpha_{1,in}^i(t) = \rho(t)
\]

up to a normalization constant. The Darboux transformation can therefore be carried out, with \( u_1 = v_{n,in}^i(t,x) \), to find the Hamiltonian of a solvable model as

\[
H_{1,in} = H_{in} - \frac{\hbar^2}{2M} \frac{\partial^2 (v_{n,in}^i v_{n,in}^i)}{\partial x^2}.
\]

In implementing the formulas (10), (13) and (16), different sets of homogeneous solutions of equation (21) can be used for \( v_{n}^i = u_1 \) and \( \psi_{m}^i = \psi \) to find the general expression of the wave functions of the \( H_{1,in} \) system, as in the general quadratic system of the previous section. For simplicity, however, we only consider the case that the same set \( \{u(t), v(t)\} \) of the homogeneous solutions is used in \( v_{n,in}^i \) and \( \psi_{m,in}^i \). The wave functions \( \psi_{m,in}^i \) of the \( H_{1,in} \) system, if we adopt the notation \( L_{-1}(y) = 0 \), is given as

\[
\psi_{m,in}^i = -2\frac{\rho}{x} \sqrt{m(m+\alpha)} \left( \frac{u-iv}{\rho} \right)^2 \psi_{m-1}^i.
\]
Darboux transformation and generalized harmonic oscillators

\[ + 2\rho \frac{\partial}{x} \left[ (m - n - 1)y + (n + \alpha) \frac{L_{n-1}^\alpha(-y)}{L_n^\alpha(-y)} \right] \psi_m^{in}, \]  

(60)

for \( m = 1, 2, 3, \ldots \), where \( y = \frac{\Omega x^2}{\hbar \rho^2} \).

For \( m = 0 \), the wave function of the \( H_1^{\text{in}} \) system is given as

\[ \psi_0^{in} = 2\rho \frac{\partial}{x} \left[ -(n + 1)y + (n + \alpha) \frac{L_{n-1}^\alpha(-y)}{L_n^\alpha(-y)} \right] \psi_0. \]  

(62)

It is clear that \( \psi_m^{in}(t, x) \) is square-integrable on the half line for \( \alpha > 0 \). Though another formal solution of the Schrödinger equation of the \( H_1^{\text{in}} \) system can be found through equation (19), that solution turns out to be not square-integrable for \( \alpha > 0 \).

As implied in [13], for the two-fold application of the Darboux transformation, the auxiliary functions can be chosen as

\[ u_1(t, x) = \psi_n^{in}(t, x), \quad u_2(t, x) = \psi_{n+1}^{in}(t, x), \]  

(63)

which gives

\[ W_2^{in} = 2\frac{n!}{(n + \alpha)!} \sqrt{\frac{n + 1}{n + 1 + \alpha}} \left( \frac{u - i\nu}{\rho} \right)^{4n+2} \left( \psi_0^{in} \right)^2 \frac{\Omega x}{\hbar \rho^2} K_n(y), \]  

(64)

where

\[ K_n(y) = L_n^\alpha(y) \frac{d}{dy} L_{n+1}^\alpha(y) - L_{n+1}^\alpha(y) \frac{d}{dy} L_n^\alpha(y). \]  

(65)

Making use of a recurrence relation among the Laguerre polynomials, one can easily find that

\[ K_n(y) = -\frac{1}{n+1} (L_n^\alpha(y))^2 + \frac{n + \alpha}{n + 1} K_{n-1}(y). \]  

(66)

With the fact that \( K_0 = -1 \), (66) proves that \( K_n(y) \) is negative-definite for all \( n \), so that \( W_2^{in} \) does not vanish for all \( 0 < x < \infty \). The fact

\[ -i \frac{\hbar}{2M} \frac{\partial^2}{\partial x^2} \ln \frac{W_2^{in}}{W_0^{in}} = 4a + 2\frac{\dot{\rho}}{\rho}, \]  

(67)

shows that the Hermiticity condition (14) of the Hamiltonian can be satisfied with

\[ \alpha_2^{in}(t) = \rho^2(t). \]  

(68)

With \( W_2^{in} \), from (17), the Hamiltonian of the transformed system can be found as

\[ H_2^{in} = H_\text{in} + \frac{2\hbar^2(\alpha + 1)}{M x^2} + \frac{2 \hbar\Omega}{M \rho^2} - \frac{\hbar^2}{M} \frac{\partial^2}{\partial x^2} \ln K_n(y). \]  

(69)

It may be possible, through the formulas (10), (13) and (16), to find the wave functions of the system of \( H_2^{in} \).
5. Discussion

We have generalized the Darboux transformation to be applicable to a general one-dimensional time-dependent Hamiltonian system. The formalism for an $N$-fold application of the transformation has also been established. It has been shown that an Hermitian system can be found from the transformed system, if the auxiliary function(s) of the transformation could support the existence of a purely time-dependent function satisfying a certain condition (equation (14)). The formalisms have been applied to a general quadratic system and a quadratic system with an inverse-square interaction. As the potential difference between the original and the transformed systems is calculated from a (formal) solution of the Schrödinger equation, due to unitary relations responsible for the existence of coherent states in the system, the shape of the potential difference could oscillate according to a classical solution for the general quadratic system.

Since the potential difference is calculated from a solution of the Schrödinger equation, the difference depends on the Planck constant $\hbar$. In the cases considered, the magnitude of the potential difference is $O(\hbar)$. The range within which the potential difference effectively depends on the space coordinate would also be described in terms of $\hbar$, as has been shown explicitly in the examples considered. These features are not usual in the standard text book of quantum mechanics, and the potential difference appears as if it is a quantum correction which depends on the position.

It should also be mentioned that, for the systems considered in this paper, there may exist other choices of auxiliary functions which lead to Hermitian systems. Only for limited cases, however, it may be possible to find exact solutions of the Schrödinger equation of a time-dependent system. Even if the solutions are found, the auxiliary function should be chosen judiciously so that the transformed system could be Hermitian. As an example, $v_1^Q + v_2^Q$ is a solution of the Schrödinger equation of the general quadratic system, so that the Darboux transformation can be formally applied to it. However, with this solution, one can not find $\alpha(t)$ satisfying (14).

Acknowledgments

One of us (DYS) is grateful to Prof. J.H. Park for her help on Crum’s formula. This work was supported in part by the Korea Research Foundation Grant (KRF-2002-013-D00025) and by NSF Grant 1614503-12.

References

[1] Darboux 1882 G, C.R. Acad. Sci. (Paris) 94 1456
[2] Crum M M 1955 Quat. J. Math. 6 121
[3] Matveev V B and Salle M A 1991 Darboux Transformations and Solitons (Berlin: Springer)
   Ablowitz M J and Segur H 1981 Solitons and the Inverse Scattering Transform (Philadelphia: SIAM)
   Park Q-H and Shin H J 2001 Physica D 157 1
[4] Witten E 1981 Nucl. Phys. B 185 513
   Cooper F, Khare A, and Sukhatme U 1995 Phys. Rep. 251 267
Darboux transformation and generalized harmonic oscillators

[5] Rosu H C 1998 Preprint quant-ph/9809056
[6] Luban M and Pursey D L 1986 Phys. Rev. D 33 431
[7] Abraham P B and Moses H E 1980 Phys. Rev. A 22 1333
[8] Mielnik B 1984 J. Math. Phys. 25 3387
[9] Fernandez C D J 1984 Lett. Math. Phys. 8 337
    Fernandez C D J, Hussin V, and Nieto L M 1994 J. Phys. A: Math. Gen. 27 3547
    Fernandez C D J, Hussin V, and Mielnik B 1998 Phys. Lett. A 244 309
[10] Schnizer W A and Leeb H 1993 J. Phys. A: Math. Gen. 26 5145
    Samsonov B F 1995 J. Phys. A: Math. Gen. 28 6989
[11] Bagrov V G and Samsonov B F, 1996 J. Phys. A: Math. Gen. 29 1011
    Bagrov V G and Samsonov B F, 1997 Phys. Part. Nucl. 28 374
[12] Samsonov B F and Shekoyan L A 2000 Physics of Atomic Nuclei 63 657
    Samsonov B F, Glasser M L, and Nieto L M 2003 Preprint quant-ph/0304144
[13] Adler V E 1994 Theor. Math. Phys. 101 1381
[14] Song D-Y 1999 Phys. Rev. A 59 2616
[15] Song D-Y 2000 Phys. Rev. A 62 014103
    Song D-Y 2000 Phys. Rev. Lett. A 85 1141
    Song D-Y 1999 J. Phys. A: Math. Gen. 32 3449
    Song D-Y 2001 Phys. Rev. A 63 032104
[16] Gradshteyn I S and Ryzhik I M 1994 Tables of Integrals, Series, and Products (Boston: Accademic Press)