Spin dynamics in semiconductor nanostructures has recently become a topic of great interest due to the possibility of using the spin degree of freedom instead of charge in electronic circuits and equally important due to the proposal of using electron spin in a semiconductor quantum dot as a fundamental building block of the quantum computing device. GaAs quantum dots are the main candidates in practical realizations of these proposals due to the well developed manufacturing technology. However, unavoidable inhomogeneous hyperfine interaction of electron spin with many nuclear spins of the host crystallite acts as a noisy environment that is the main source of dephasing for the electron spin at low temperature when relaxation due to the phonons is ineffective.

The limit of fully polarized nuclear spin bath was analyzed exactly in \cite{3}, including spectral properties. However, it is rather hard to achieve a significant polarization dynamically, and thermodynamic polarization, requiring sub-milli Kelvin temperatures, is still out of reach for semiconductors. Currently, a more promising route is to actively reduce the distribution width of the nuclear Overhauser field by projective measurements \cite{5–7}. This has been partially achieved in experiments leading to significantly longer decoherence times \cite{8–10}. To further optimize projective measurement techniques it is essential to gain a better understanding of the spectral properties of the unpolarized system which, so far, have only been understood qualitatively.

In this paper we diagonalize the central spin Hamiltonian for a quantum dot in the high magnetic field limit using a 1/B-expansion. Projecting the eigenstates on an unpolarized state of the nuclear spin bath we find that their density has Gaussian tails. The level spacing of the nuclear sublevels is exponentially small in the middle of each of the two electron Zeeman levels but increases super-exponentially away from the center. This suggests to select states from the wings of the distribution when the system is projected on a single eigenstate by a measurement to reduce the noise of the nuclear spin bath. This theory is valid when the external magnetic field is larger than a typical Overhauser field at high nuclear spin temperature.

The main source of decoherence for an electron spin confined to a quantum dot is the hyperfine interaction with nuclear spins. To analyze this process theoretically we diagonalize the central spin Hamiltonian in the high magnetic field limit. Then we project the eigenstates onto an unpolarized state of the nuclear bath and find that the resulting density of states has Gaussian tails. The level spacing of the nuclear sublevels is exponentially small in the middle of each of the two electron Zeeman levels but increases super-exponentially away from the center. This suggests to select states from the wings of the distribution when the system is projected on a single eigenstate by a measurement to reduce the noise of the nuclear spin bath. This theory is valid when the external magnetic field is larger than a typical Overhauser field at high nuclear spin temperature.

Our theory is applicable when the external magnetic field \( B \) is larger than a typical Overhauser field at high nuclear spin temperature due to fluctuations \( B_{\text{fluc}} = A\sqrt{S/N}/\mu \), where \( A/\mu \) is the maximum Overhauser field, \( N \) is the number of nuclei under the electron envelope wave function, and \( S \) is a number of degenerate hyperfine couplings. At low field \( B < B_{\text{fluc}} \) the spectrum can be obtained by a numerical solution of the Richardson equations \cite{2} where the 1/B-expansion of the present paper can be used as a benchmark for complex numerical procedures.

The spin of an electron in a quantum dot couples to nuclear spins in the presence of an external \( B \)-field as

\[
H = \mu B S_0^z + \sum_{j=1}^{N} A_j S_0^z \cdot S_j, \tag{1}
\]
where \( \mu = g\mu_B \) is the electron magneton (in the following we neglect the nuclear Zeeman splitting). \( S_0^+ , S_0^- = S_0^0 = 0 \) are electron spin-1/2 operators and \( S_j \) (\( j \geq 1 \)) are spin operators of nuclear shell \( j \) with the maximum angular momentum \( S_j \geq 1/2 \) constructed out of 2\( S_j \) nuclei of spin-1/2 which have the same hyperfine coupling to the electron spin, \( \mathbf{S}_j = \sum_{i=1}^{2S_j} \mathbf{I}_{ji} \), where \( i \) labels individual nuclei within the shell, \( \mathbf{I}_{ji} \) are nuclear spin-1/2 operators, and \( N \) is the number of nuclear shells. Assuming harmonic confinement of the electron in all spatial directions the couplings are \( A_j = A_0 \exp (-r_j^2/r_0^2) \), where \( A_0 \) is the coupling in the middle of the quantum dot, and \( r_0 \) and \( r_j \) are spatial size of the quantum dot and radius of \( j \)th shell in units of the lattice parameter.

In 1D only two nuclei have the same coupling ignoring the isotope effects and assuming equidistant lattice sites \( r_j = j \), thus the maximum total angular momentum is \( S_j = S = 1 \). In 2D degeneracy of the couplings gives \( S_j = S = 4 \) but the radii of the sequential shells are not equidistant because the number of nuclei grows linearly away from the center. We thus model the system as a set of concentric nuclear shells, \( r_j = r + 4m/(\pi \nu) \) and also change the summation indices in Eq. (1). \( \sum_{j=1}^{N} \rightarrow \sum_{r=1,m=1}^{N,\pi/4} \). In 3D the degeneracy is larger than in 2D, \( S_j = S = 12 \), and the number of the nuclei grows quadratically away from the center, \( r_j = r + 6m/(\pi \nu^2) \), \( \sum_{j=1}^{N} \rightarrow \sum_{r=1,m=1}^{N,\pi^2/16}. \)

This model conserves the number of excitations \( [H, J_z] = 0 \), where \( J_z = \sum_{j=0}^{N} S_j^z \), and the total angular momentum of each nuclear shell \( [H, \mathbf{S}_j^2] = 0 \). All of them also commute with each other, \( [J_z, \mathbf{S}_j^2] = 0 \) and \( [\mathbf{S}_j^2, \mathbf{S}_j^z] = 0 \). Thus the Hilbert space is partitioned into a set of disconnected subspaces labeled by the following quantum numbers: \( n \) is an eigenvalue of \( J_z \) and \( l_j \) [13] correspond to \( \mathbf{S}_j^2, \mathbf{S}_j^z | \Psi \rangle = l_j(l_j+1) | \Psi \rangle \). The latter becomes trivial when all of the nuclear spins have different couplings as for spin-1/2 operators \( \mathbf{S}_j^2 = 3/4 \) is a number but is nontrivial when \( S_j > 1/2 \).

The diagonalization in each subspace can be performed using degenerate perturbation theory when the \( B \)-field is large. Splitting the Hamiltonian into the unperturbed part \( H_0 = \mu B S_0^z \) and a perturbation \( V = \sum_j A_j \mathbf{S}_j \). \( \mathbf{S}_j \) defines two electron Zeeman levels, \( E = \pm \mu B/2 \) but leaves the nuclear spin sublevels hugely degenerate in the zeroth-order approximation. The latter degeneracy has to be lifted via a diagonalization of the perturbation \( V \).

In the basis of eigenstates of \( J_z \), \( | \Psi \rangle = | \pm, \{ l_j, k_j \} \rangle \), \( \langle \Psi | \Psi \rangle = 1 \), \( V \) is a diagonal matrix within both of the electron spin subspaces where the spin-flip part of \( V \) that couples opposite electron levels can be neglected when the external field is very large. Here \( \pm \) refers to the “up” and “down” electron Zeeman levels and \( k_j \) are the numbers of nuclear spin excitations on each shell such that the quantum number \( n = (1 \pm 1)/2 + \sum_{j=1}^{N} k_j \). The second order correction to the eigenenergies are due to the spin-flip part of \( V \). Using the matrix elements of \( V \) in the basis of eigenstates of \( J_z \) we obtain

\[
E = \pm \frac{\mu B}{2} \pm \sum_{j=1}^{N} \left[ A_j (-l_j + k_j) \right] + \frac{A_j^2 (2l_j - k_j + \frac{1}{2}) (k_j + \frac{1}{2})}{4 \mu B},
\]

where the energy denominator in the last term was also expanded up to the leading order in \( 1/\mu B \). Including the first order corrections to the eigenfunctions we get

\[
| \Psi \rangle = | \pm, \{ l_j, k_j \} \rangle \pm \sum_{m=1}^{N} \frac{A_m}{\mu B} S_m^+ | \mp, \{ l_j, k_j \} \rangle.
\]

The large magnetic field expansion has different conditions of applicability for the eigenenergies Eq. (2) and the eigenstates Eq. (3) in the subspaces of unpolarized nuclear spins \( k_j \approx \ell_j \). The subleading terms in Eq. (2) are small in all subspaces when \( B > B_{\text{fluc}} \) where \( B_{\text{fluc}} = \sqrt{\sum_j A_j^2 S_j^2}/\mu \). But the next (second) subleading correction to Eq. (3) is small only when \( B > B_{\text{max}} \) where \( B_{\text{max}} = r_0^2 A_0/2 \mu \) in 1D and 2D \( (B_{\text{max}} = r_0^3 A_0/\sqrt{8} \mu \) in 3D) [13] is a much larger field than \( B_{\text{fluc}} \). The latter signals that the choice of the eigenfunctions, \( | \Psi \rangle = | \pm, \{ l_j, k_j \} \rangle \), is a poor zeroth order approximation in the intermediate field regime, \( B_{\text{fluc}} \ll B < B_{\text{max}} \). The correct approximation can be identified by merging the inner nuclear shells with different couplings up to the radius \( \tilde{r} = r_0 \sqrt{\ln ((B_{\text{max}}/B)/\text{units of the lattice parameter})} \) in 1D and 2D \( (\tilde{r} = r_0 (1 + \sqrt{\ln (B_{\text{max}}/B)}) / \sqrt{2} \) in 3D) [13] into a single shell with the same coupling \( A_0 \). Then, diagonalizing \( H + V' \), where \( V' = \sum_{j,r \leq \tilde{r}} (A_j - A_0) \mathbf{S}_j \cdot \mathbf{S}_0 \), when \( B_{\text{fluc}} < B < B_{\text{max}} \) instead of the original model \( H \) we obtain the same result as in Eqs. (2) [3] but a different definition of nuclear shells \( \tilde{S}_j \), where the first element is \( \tilde{S}_1 = \sum_{j,r \leq \tilde{r}} S_j \), the middle elements are \( S_j = 0 \) for \( 1 < r_j \leq \tilde{r} \), and the outer elements, \( r_j > \tilde{r} \), are \( S_j = S_j \).

In 2D and 3D the parameter \( B_{\text{max}} \) is proportional to the measurable maximum Overhauser field \( A = \sum_{j=1}^{N} S_j A_j \) of the order of a few Tesla [14] with the numerical factor \( \pi^{-1} \) and \( (2\pi^3 e)^{-1} \). In 1D, \( B_{\text{max}} = \tilde{N} A/(\pi \mu) \) is much larger than \( A \), here \( \tilde{N} = \sum_{j=1}^{N} 2S_j A_j / A_0 \). The parameter \( B_{\text{fluc}} = A \sqrt{S/N}/\mu \) scales with the number of nuclei under the electron envelope function in all dimensions.

In terms of density of states the bare electron level acquires a finite smearing due to coupling to many degrees of freedom of unpolarized nuclear spins. When the quantum dot is empty the nuclei at different lattice sites
are uncorrelated. After an electron, say with spin “up”, populates the quantum dot, the state of the combined system \(|\Psi_0\rangle = S_j^z \prod_i \langle i,j \rangle I_j^z |\downarrow\rangle\) is not an eigenstate of the Hamiltonian Eq. (13), where \(\{j, i\}\) labels a subset of nuclear lattice sites and \(|\downarrow\rangle\) is the all spins down (including the central spin) state. We analyze the distribution of the eigenenergies Eq. (2) using a projected density of states \(\nu(E) = \sum_{\{i,j\}} P(\{i,j\}) \delta(E - E(\{i,j\}))\), where \(P(\{i,j\}) = 1\) when \(|\Psi_0\rangle \{i,j\}\) is not 0 and \(P(\{i,j\}) = 0\) when \(|\Psi_0\rangle \{i,j\}\) is 0. Here the \(\sum_{\{i,j\}}\) runs over all subspaces and all eigenstates within each subspace. Note that for any shell with \(S_j > 1\) the complete set of the eigenstates includes \(I_j\) with multiplicities greater than one \([13]\). Only one of each \(I_j\) is kept since these multiplicities do not change \(P(\{i,j\})\). We calculate the overlaps matrix elements only in the leading 1/\(\mu B\)-order as the probability of measuring other eigenstates coming from subleading orders is at least as small as \(A_j/\mu B\).

By representing the delta function as \(\delta(x) = \int d\lambda e^{ix\lambda}/(2\pi)\), the Fourier transform of \(\nu(E)\) can be written as a product of sums over each nuclear spin shell

\[
\nu(\lambda) = \sum_{\{i,j\}} P(\{i,j\}) e^{-i\lambda E(\{i,j\})}
\]

\[
= \prod_{j=1}^N e^{-i\lambda(p_jA_j - \mu B)} \sum_{k = p_j(1+sgn p_j)}^{p_j + \hat{S}_j} e^{-i\lambda A_j^2 (k-2p_j)(k+1)} /
\]

\[
\sqrt{4\pi} \sigma^2 , \tag{4}
\]

where \(p_j = \langle \Psi_0|S_j^z|\Psi_0\rangle\), \(|p_j| \leq I_j\), are polarizations of the shells given by the state of the system \(|\Psi_0\rangle\).

Assuming that each shell is unpolarized \(p_j \ll \hat{S}_j\) and \(\hat{S}_j \gg 1\), the sum over each shell can be calculated as an integral, \(I_j(\lambda) = \int_{0}^{\hat{S}_j} \frac{dk}{\sqrt{\pi}} e^{-k^2/4} = 1/\sqrt{\pi} e^{-\lambda^2/4} \left[ \text{erf} \left( (1+2\hat{S}_j) \sqrt{x/2} \right) - \text{erf} \left( (\sqrt{x/2}) \right) \right] / (2\sqrt{x})\), \(x = \lambda^2 A_j^2 / (4\mu B)\), which is an oscillating function of \(\lambda\). Then the product of the oscillating functions can be approximated in the large-\(N\) limit by turning it into an exponential of a sum of logarithms, \(\prod_{j=1}^N I_j(\lambda) = I_1(\lambda) \exp \left( \sum_{j;r>j} \log I_j(\lambda) \right)\), and by expanding the exponent in \(\lambda\), \(\sum_{j;r>j} \log I_j(\lambda) \approx \sum_{j;r>j} \log S_j - i \{ S_j/2 + S_j^2/3 \} \lambda A_j^2 / (4\mu B) - \{ S_j^2/24 + S_j^3/12 + 2S_j^4/45 \} \lambda^2 A_j^4 / (16\mu B^2)\).

In 1D \(I_j(\lambda)\) can not be calculated as an integral since the degeneracy of the hyperfine couplings is two but the explicit evaluation of the sum of only two terms within each shell and the small-\(\lambda\) expansion yields a similar expression, \(\sum_{j; r>j} \log I_j(\lambda) \approx \sum_{j; r>j} i \{ 2 - i \lambda A_j^2 / (4\mu B) - \lambda^2 A_j^4 / (24\mu B^2) \}\). Strictly speaking, the small-\(\lambda\) expansion is good when \(\lambda \ll 16\mu B A_j^2\) but the resulting Gaussian is also quite a good approximation for a large \(\lambda\) since the original product of many oscillating functions is zero due to random phases of \(I_j(\lambda)\) when \(\lambda \geq 4\mu B/A_j^2\), provided that the couplings \(A_j\) have a non regular distribution.

By evaluating the inverse Fourier transform \(\nu(E) = \int d\lambda \nu(\lambda) \exp(-i\lambda E)\) in the limit \(B \gg B_{\text{max}}\) we obtain

\[
\nu(E) = \frac{\hat{S}_1 \prod_{j; r>j} S_j}{\sqrt{\pi} \sigma} \exp \left[ \frac{-(E - E_0)^2}{\sigma^2} \right] , \tag{5}
\]

where \(E_0 = \sum_{j=1}^N p_j A_j/2 - \mu B/2\) is a shift of the bare electron level that depends on the momentary state of the nuclei and a finite linewidth \(\sigma = \sqrt{\sum_{j; r>j} (S_j^2/96 + S_j^4/48 + S_j^4/90) A_j^4 / (\mu B)} \approx \mu B^2_{\text{flu}} / (\sqrt{\pi}N^2 B)\) that is common for all unpolarized nuclear states. In the intermediate regime \(B_{\text{flu}} \ll B \ll B_{\text{max}}\) Eq. (5) is valid when \(E \geq S_1^2 A_1^2 / (4\mu B)\). The contribution of the inner shells can be approximated as \(I_1(\lambda) = \hat{S}_1\) when, due to the fast oscillating exponential, the main contribution to the inverse Fourier transform comes from \(\lambda \leq 4\mu B / (\hat{S}_1 A_1^2)\).

In 1D, the Gaussian result agrees precisely with the spectroscopically measurable lineshape when \(B \gg B_{\text{max}}\). As the degeneracy of hyperfine couplings is 2 for all shells, all projections \([13]\) are the overlap of the singlet (or triplet) and two nuclear spin states which give \(1/\sqrt{\pi}\) and the calculation of the lineshape gives Eq. (13). When the degeneracy is larger than 2 the two calculations are different. It is also worth noting that the state \(|\Psi(0)\rangle\) is an eigenstate of the model Eq. (1) with \(S_j = 1/2\) in the high \(B\)-field \(B \gg B_{\text{max}}\).

Rediscretization of Eq. (5) recovers the average level spacing of the nuclear spin levels. From the definition of the density of states, \(d = 1/\nu(E)\) is an energy range that contains only one state. But, as the prefactor in \(\nu(E)\) increases to infinity when more and more outer shells are taken into account, the level spacing becomes zero. On the other hand the coupling strengths of the outer shells become super-exponentially small which makes the splitting of the inner shells’ levels into sublevels due to the outer shells very narrow. Thus, by selecting an effective number of the significantly coupled nuclear shells \(r_j < 4r_0\), we find

\[
d(E) = d(E_0) \exp \left[ (E - E_0)^2 / \sigma^2 \right] , \tag{6}
\]

where \(d(E_0) = \sqrt{\pi} \sigma / (\hat{S}_1 \prod_{j; r<j} S_j)\) is exponentially small, \(d(E_0) \approx \hat{S}_1 \exp(\hat{N}/\hat{S})\). Thereby, \(d(E_0)\) is a tiny level spacing in the middle of the upper electron Zeeman line but \(d(E)\) increases super-exponentially at a finite detuning \(E \neq E_0\) on a characteristic energy scale \(\sigma\) when \(B \gg B_{\text{max}}\) and \(S_1^2 A_1^2 / (4\mu B)\) when \(B_{\text{flu}} \ll B \ll B_{\text{max}}\).

There is also a finite temperature smearing. To average the hyperfine shift \(E_0\) over all possible nuclear spin configurations at a high temperature,
\( \nu_0(E) = \sum_{\{p_j\}} \delta(E - E_0), \) we use the same approach as in the calculation of \( \nu(E) \) and obtain the Gaussian distribution of levels with a width \( \sigma_0 = \sqrt{\sum_{j=1}^{N} S_j^2 A_j^2 / 6} \) and an average level spacing \( d_0 = \exp \left[ (E - \mu B/2)^2 / \sigma_0^2 \right] S_1 \prod_{j;x<r_j<4a_0} 2S_j / (\sqrt{\pi} \sigma_0) \).

This implies that if the nuclear spin state is not prepared in a specific way but is a thermal state, there are two energy scales in a projective measurement to narrow the nuclear spin bath \( 3 \frac{1}{10} \) in order to suppress fluctuations of the Overhauser field \( 3 \frac{7}{10} \). A measurement in the coarse resolution of \( d_0 \) will select a single specific nuclear spin configuration suppressing only thermal fluctuations and a measurement in the fine resolution of \( d \) will project the system on an eigenstate within a given nuclear bath state.

Using the eigenstates and the spectrum in Eqs. (2) one can evaluate the time-dependent density matrix of the electron with an unpolarized state of the nuclear bath \( 3 \frac{1}{10} \frac{1}{10} \) in order to suppress fluctuations of the Overhauser field \( 3 \frac{7}{10} \). A measurement in the coarse resolution of \( d_0 \) will select a single specific nuclear spin configuration suppressing only thermal fluctuations and a measurement in the fine resolution of \( d \) will project the system on an eigenstate within a given nuclear bath state.

In conclusion we have diagonalized the central spin Hamiltonian in the high \( B \)-field limit. Projecting the eigenstates on an unpolarized state of the nuclear bath we have shown that the level spacing of the nuclear sub-levels, which is exponentially small in the middle of the bare electron level, becomes super-exponentially large with detuning away from the middle. This suggests to select states from the wings of the distribution when one attempts to eliminate the decohering effect of the nuclei by projective measurement techniques. This theory is valid when the external \( B \)-field is larger than typical Overhauser fields.

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**Supplementary materials**

Here we provide more details on the calculations and approximations of the main text. Figs. 1 and 2 illustrate construction of the shells in the Hamiltonian Eq. (1) of the main text. Section A contains a description of the zeroth order approximation to the eigenfunctions in the intermediate field regime. In section B we solve explicitly the Richardson equations in the large magnetic $B$-field limit.

**Intermediate field regime $B_{\text{fluc}} \ll B \ll B_{\text{max}}$**

The large magnetic field expansion has different conditions of applicability for the eigenenergies,

$$ E = \pm \frac{\mu B}{2} \pm \sum_{j=1}^{N} \left[ \frac{A_j (-l_j + k_j)}{2} + \frac{A_j^2 (2l_j - k_j + \frac{1}{2}) (k_j + \frac{1}{2})}{4\mu B} \right], $$

and the eigenstates,

$$ |\Psi\rangle = |\pm, \{l_j, k_j\}\rangle \pm \sum_{m=1}^{N} \frac{A_m^2 S_m^x}{\mu} |\mp, \{l_j, k_j\}\rangle, $$

in the subspaces of unpolarized nuclear spins $k_j \approx l_j$. Comparison of the first and the third terms in Eq. (8) set the limitation on external magnetic field as $B_{\text{fluc}}^2 / B^2 \ll 1$ in all subspaces where

$$ B_{\text{fluc}} = \sqrt{\sum_{j=1}^{N} A_j^2 S_j^2}.$$  

The next order correction in Eq. (8) is small as $B_{\text{fluc}}^3 / B^3$ compared to the leading term.

The second term of Eq. (8) is small when $B \gg B_{\text{fluc}}$ but the next order correction is not. The second order term in the flip-flop part of $V$ contains a denominator which is a difference between unperturbed eigenenergies that belong to the same electron spin level. This sets the limitation on the magnetic field as

$$ \frac{A_j^2}{[2\mu B (A_j - A_i)]} \ll 1 $$

for all pairs $i \neq j$. Assuming that $r_0 \gg 1$, the most restrictive condition comes from a pair of sequential shells at $r_1 = 1$ in 1D and 2D as $B / B_{\text{max}} \ll 1$, where

$$ B_{\text{max}} = \frac{r_0^2 A_0}{2\mu}, $$

In 3D the most restrictive condition comes from a pair of sequential shell at an intermediate radius $r_j = r_0 / \sqrt{2}$ with

$$ B_{\text{max}} = \frac{r_0^2 A_0}{\sqrt{8e\mu}}, $$

where $e$ is the base of the natural logarithm. In all cases $B_{\text{max}}$ is a much larger field than $B_{\text{fluc}}$.

The divergence of perturbation series for the eigenstates Eq. (8) when $B_{\text{fluc}} \ll B \ll B_{\text{max}}$ does not mean that a perturbation theory in $1 / B$ is inapplicable at all but only signals that the choice of the eigenfunctions, $|\Psi\rangle = (S_0^+)^{(1 \pm 1)/2} \prod_{j=1}^{N} (S_j^+)^{k_j} |\Psi\rangle$, is a poor zeroth order approximation to split the nuclear spin sublevels within one of the electron Zeeman levels. The correct approximation can be identified by merging the inner nuclear shells (in 1D and 2D) with different couplings up to the radius $r = r_0 \sqrt{\ln (B_{\text{max}} / B)}$ into a single shell with the same coupling $A_0$ as the most restrictive limitation originates from the shells at the middle of the electron envelope function. In 3D the correct approximation can be identified by merging the shells with intermediate radii between $r = r_0 (1 + \sqrt{\ln (B_{\text{max}} / B)}) / \sqrt{2}$. The inner radius vanishes fast, when $B = B_{\text{max}} / e$, thus we neglect it and merge all of the inner shells up to the radius $\tilde{r} = r_0 \left(1 + \sqrt{\ln (B_{\text{max}} / B)}\right) / \sqrt{2}$ in 3D as well as in 1D and 2D.

Then we diagonalize $H + V'$, where $V' = \sum_{j,r_j < \tilde{r}} (A_i - A_j) S_i \cdot S_j$, instead of the original model $H$ from Eq. (1) of the main text by repeating the same calculation as for $H$ and obtain Eqs. (9) with a different definition of nuclear shells $\tilde{S}_j$, where the first element is $\tilde{S}_1 = \sum_{j,r_j < \tilde{r}} S_j$, the middle elements are $\tilde{S}_j = 0$ for $1 < r_j \lesssim \tilde{r}$, and the outer elements, $r_j > \tilde{r}$, are $\tilde{S}_j = S_j$. Corrections to this result due to $V'$ are small for $|r_j - \tilde{r}| \gg 1$. The specific form of the zeroth order eigenstates for $r_j \approx \tilde{r}$ can only be found numerically and we neglect this crossover region assuming a sharp transition between the two types of eigenstates.

**A solution for the Richardson equations**

The eigenenergies of the central spin Hamiltonian,

$$ H = \mu B S_0^z + \sum_{j=1}^{N} A_j S_j \cdot S_j, $$

at arbitrary external magnetic field in each subspace with a given set of the quantum numbers $n, l_j$ can be found by solving a set of Richardson equations [1],

$$ \sum_{j=1}^{N} \frac{2l_j A_j / 2}{E_{\nu} + A_j / 2} + 1 - \frac{\mu B}{E_{\nu}} + \sum_{k=1}^{n} \frac{2E_k}{E_{\nu} - E_k} = 0, $$

where $E_{\nu} = E_{\nu}(\{n, l_j\})$.
as \[ E = \sum_{\nu=1}^{n} E_{\nu} + \sum_{j=1}^{N} l_j A_j/2 - \mu B/2. \] (16)

All sets of \( E_{\nu} \), s, which are the solution of the above set of equations, also uniquely define the Gaudin eigenfunctions as \[ |\{ E_{\nu} \}\rangle = \frac{1}{\sqrt{\prod_{\nu=1}^{n} \prod_{j=1}^{N} (E_{\nu} + l_j A_j/2 + E_j S_j^+ + S_0^+)}} |\psi\rangle. \] (17)

that correspond to these eigenenergies. The unexcited state \(|\psi\rangle\) is all spins down, including the central spin, state. The normalization factor is determinant of an \( n \times n \) matrix \([2]\), \( z = \sqrt{\det M} \), which diagonal and the off-diagonal matrix elements are

\[
M_{kk} = 1 + \frac{2 l_j (A_j/2)^2}{(E_k + A_j/2)^2} - \frac{2 E_p}{\prod_{p=1 \neq k} (E_k - E_p)^2},
\]

\[
M_{kk'} = \frac{2 E_p^2}{(E_k - E_{k'})^2}. \quad (18)
\]

This diagonalization procedure can be constructed in an easy way by solving a complementary bosonic model instead of Eq. (14) to obtain the ansatz for the eigenstate, Eq. (17). Then, the Richardson equations emerge as the requirement for the states in Eq. (17) to be the eigenstates of the model Eq. (14). In this way the exact form of Eq. (15) can be found by using the spin commutation relations only. This approach was developed in [3] in the context of the BCS model and in [4] in the context of the Dicke model.

When magnetic field is very large the last term in the Richardson equations, Eq. (15), can be neglected in leading \( 1/\mu B \) order therefore all of the roots \( E_{\nu} \) are close to either \(-A_{\nu}/2\) or \( \mu B \). There is up to one root close to \( \mu B \) and, we consider here 1D case only, there are up to two roots, \( k_j \leq 2 \), close to \(-A_{\nu}/2\). Collection of all possible sets of \( k_j \), such that \( n = \sum_{j=1}^{N} k_j + (1 \pm 1)/2 \) equal to the number of excitations, explores the complete set of the eigenstates in a given subspace of the Hamiltonian. We will use \(-A_{\nu}/2\) and \( \mu B \) to label an eigenstate instead of \( E_{\nu} \).

Corrections to these limiting values of the roots at a finite \( B \), \( E_{\nu} = -A_{\nu}/2 + \delta_{\nu} \) and \( E_{\nu} = \mu B + \delta_{\nu} \), can be found from the \( 1/\mu B \) expansion of the Richardson equations, Eq. (15), as

\[
\delta_{\nu} = \begin{cases} 
\frac{-A_{\nu}^2}{2 \mu B}, & \text{if one } E_{\nu} = -A_{\nu}/2, \\
\frac{(1 \pm 1) A_{\nu}^2}{4 \mu B}, & \text{if two } E_{\nu} = -A_{\nu}/2, \\
\sum_{k=1}^{n-1} A_k + \left( \frac{k-k_j}{\mu B} \right) A_{j}^2, & \text{if } E_{\nu} = \mu B.
\end{cases}
\] (19)

The sum \( \sum_{k=1}^{n-1} A_k + \left( \frac{k-k_j}{\mu B} \right) A_{j}^2 \) in the third case is over the remaining \( n - 1 \) roots \(-A_{\nu}/2\). Note that the corrections to the roots \(-A_{\nu}/2\) were obtained by linearizing the system of equations, Eq. (15), and the correction to the root \( \mu B \) was obtained by expanding Eq. (15) up to the second order in \( 1/\mu B \) with the first order corrections to the roots \(-A_{\nu}/2\) obtained from the linearized equations.

The normalization factor \( z \) can also be expanded in a \( 1/\mu B \) series and we find that the product of the diagonal matrix elements of \( M \) gives in leading order

\[
z = \begin{cases} 
(2\alpha)^n \prod_{k=1}^{n} \frac{1}{A_k}, & \text{if all } E_{\nu} = -A_{\nu}/2, \\
(2\alpha)^{n-1} \prod_{k=1}^{n-1} \frac{1}{A_k}, & \text{if one } E_{\nu} = \mu B,
\end{cases}
\] (20)

where the product in the second case is over the remaining \( n - 1 \) roots \(-A_{\nu}/2\). Upper bound of the first subleading correction to \( z \) can be estimated as \( r_j^2 \sqrt{n!} \), assuming that a few off-diagonal matrix elements \([18]\) are of the order of \( r_j^2 \) and they all contribute with the same sign to the determinant, which is smaller than the leading terms if \( \mu B \gg A_0 \sqrt{n!} \) when number of the excitations is large \( n \gg 1 \) (\( A_0 \) is the maximal coupling strength). We used the Stirling’s formula to approximate \( n! \) for a large \( n \). At high external magnetic field, \( B \gg B_{\text{Duc}} \), the first subleading correction is small.

Performing summation over \( E_{\nu} \) with the accuracy of Eq. (19) we obtain the eigenenergies of the Hamiltonian Eq. (14) at high magnetic field,

\[
E = \pm \frac{\mu B}{2} \pm \sum_{j=1}^{N} \left[ \frac{A_j (-l_j + k_j)}{2} + \frac{A_j^2 (2l_j - k_j + \frac{1+1}{2}) (k_j + \frac{1+1}{2})}{4\mu B} \right], \quad (21)
\]

where \( \pm \) refers to the “up” and “down” electron Zeeman levels and \( k_j \) labels the number of roots of Eq. (15) that are close to \(-A_{\nu}/2\). Substituting \( E_{\nu} \) with the corrections from Eq. (19) into Eq. (17) we obtain the eigenfunctions that correspond to the eigenenergies \( E \),

\[
|\{ E_{\nu} \}\rangle = (S_0^+) \prod_{j=1}^{N} (S_j^+)^{k_j} |\psi\rangle, \quad (22)
\]

in leading \( 1/\mu B \) order. Corrections to this wave function contains admixture of states from the opposite electron level with single nuclear spin flip but each of them is proportional to a small factor \( A_j/\mu B \).

The \( 1/B \)-expansion of the Richardson equations in Eqs. (21) (22) coincides with the result in Eqs. (8) obtained using the \( 1/B \)-expansion of the main text.

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Figure 2: Shells of constant couplings for a 2D square lattice. The majority of the shells has degeneracy 8. Two circles marked by dashed lines select a subgroup of shells between $r = 5$ and $r = 6$. 
Figure 3: Distribution of shell spacings for a 2D square lattice, $100 < r < 101$. The dashed line marks the uniform spacing we use in the main text.