Adaptive Federated Minimax Optimization with Lower Complexities

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Abstract

Federated learning is a popular distributed and privacy-preserving learning paradigm in machine learning. Recently, some federated learning algorithms have been proposed to solve the distributed minimax problems. However, these federated minimax algorithms still suffer from high gradient or communication complexity. Meanwhile, few algorithms focuses on using adaptive learning rate to accelerate these algorithms. To fill this gap, in the paper, we study a class of nonconvex distributed minimax optimization problems based on the data distributed in multiple clients (such as mobile devices, institutions, organizations, etc.), defined as

\[
\min_{x \in \mathbb{R}^d} \max_{y \in \mathbb{R}^p} f(x, y) \equiv \frac{1}{K} \sum_{k=1}^{K} f^k(x, y),
\]

where \( f^k(x, y) = E_{\xi \sim D^k} [f^k(x, y; \xi^k)] \) denotes the local objective function at \( k \)-th client for any \( k \in [K] = \{1, 2, \cdots, K\} \). The global objective function \( f(x, y) \) is possibly nonconvex on the variable \( x \in \mathbb{R}^d \), while it is Strongly-Concave (SC) on the variable \( y \in \mathbb{R}^p \), or is still nonconvex on the variable \( y \in \mathbb{R}^p \) but satisfies Polyak-Lojasiewicz (PL) condition [40]. Here \( \xi^k \) for any \( k \in [K] \) are independent random variables following unknown distributions \( D^k \), and for any \( k, j \in [K] \) possibly \( D^k \neq D^j \). Let \( y^*(x) = \arg\max_{y \in \mathbb{R}^p} f(x, y) \) and \( F(x) = f(x, y^*(x)) \). In solving the above minimax problem (1), our goal is to search for an \( \epsilon \)-stationary solution, i.e., \( \|\nabla F(x)\| \leq \epsilon \) \( (\epsilon \geq 0) \) as in [10, 45].

When \( K = 1 \) in Problem (1), i.e., non-distributed minimax optimization, [31, 32] proposed the stochastic gradient descent ascent (SGDA) method, which is a simple generalization of stochastic gradient descent (SGD) [3]. Specifically, it alternately conducts SGD for updating the variable \( x \) and stochastic gradient ascent (SGA) for updating the variable \( y \). Subsequently, some accelerated SGD methods [34, 59, 19, 21, 58] have been developed to solve the NonConvex-Strongly-Convex (NC-SC) minimax optimization at a single client. Meanwhile, [39, 57, 4, 17] studied the NonConvex-PL (NC-PL) minimax optimization. For example, [34] proposed an acceler-
Table 1: Gradient (i.e., SFO) and Communication complexities comparison of the representative federated minimax optimization algorithms in searching for an $\epsilon$-stationary point of the NC-SC or NC-PL minimax problem (1), i.e., $\mathbb{E}\|\nabla F(x)\| \leq \epsilon$ or its equivalent variants. ALR denotes adaptive learning rate.

| Algorithm          | Reference | Gradient | Communication | NC-SC | NC-PL | ALR |
|--------------------|-----------|----------|---------------|-------|-------|-----|
| Local-SGDA         | [10]      | $O(\epsilon^{-3})$ | $O(\epsilon^{-4})$ | √     | √     |     |
| FEDNEST            | [49]      | $O(\epsilon^{-4})$ | $O(\epsilon^{-4})$ | √     | √     |     |
| Momentum-Local-SGDA| [45]      | $O(\epsilon^{-4})$ | $O(\epsilon^{-4})$ | √     | √     |     |
| SAGDA              | [56]      | $O(\epsilon^{-2})$ | $O(\epsilon^{-2})$ | √     | √     |     |
| CDMA               | [55]      | $O(\epsilon^{-2})$ | $O(\epsilon^{-2})$ | √     | √     |     |
| FGDA               | Ours      | $O(\epsilon^{-3})$ | $O(\epsilon^{-2})$ | √     | √     | √   |
| AdaFGDA            | Ours      | $O(\epsilon^{-3})$ | $O(\epsilon^{-2})$ | √     | √     | √   |

Federated Learning (FL) [36] is an effective distributed and privacy-preserving learning paradigm in machine learning. In the paper, thus, we focus on the federated learning algorithms for minimax optimization. From Table 1, the existing FL methods for the NC-SC and NC-PL minimax problem (1) still suffer high gradient (i.e., stochastic first-order oracle, SFO) or communication complexities in searching for an $\epsilon$-stationary point of the minimax problem (1) (i.e., $\mathbb{E}\|\nabla F(x)\| \leq \epsilon$). Thus there exists an open question:

**Could we develop federated algorithms with lower gradient and communication complexities simultaneously in finding an $\epsilon$-stationary point of Problem (1)?**

In the paper, we affirmatively answer to the above question and propose a class of accelerated federated minimax optimization methods (i.e., FGDA and AdaFGDA) to solve the NC-SC or NC-PL minimax problem (1), which build on the momentum-based variance reduced [7] and local-SGD [47] techniques. In particular, our adaptive algorithm (i.e., AdaFGDA) can flexibly incorporate various adaptive learning rates by using the unified adaptive matrices. Moreover, our FL methods obtain lower sample and communication complexities simultaneously. In summary, our main contributions are:

1. We propose a class of accelerated federated minimax optimization methods (i.e., FGDA and AdaFGDA) to solve the minimax Problem (1). In particular, our AdaFGDA can use various adaptive learning rates.

2. We provide a solid convergence analysis framework for our algorithms, and prove that they obtain lower gradient complexity of $O(\epsilon^{-3})$ with lower communication complexity of $O(\epsilon^{-2})$ in finding an $\epsilon$-stationary point of Problem (1). From [1], the optimal gradient complexity is $O(\epsilon^{-3})$ in finding an $\epsilon$-stationary point of non-
convex smooth problem \( \min_{x \in \mathbb{R}^d} f(x) \). Thus, our algorithms obtain the optimal gradient complexity with lower communication complexity.

(3) Experimental results demonstrate efficiency of our algorithms on the deep AUC maximization and robust neural network training tasks.

2 Related Works

In this section, we overview some representative federated learning algorithms and distributed minimax optimization, respectively.

2.1 Federated Learning Algorithms

Federated Learning (FL) \cite{FedAvg} is an effective distributed and privacy-preserving learning paradigm, which learns a global model from a set of located clients under the coordination of a server. In FL, the edge clients do not send their data to the server to improve the privacy afforded to the clients. Meanwhile, FL applies the local-SGD technique to reduce the cost of communication. FedAvg \cite{FedAvg}/Local-SGD \cite{Local-SGD} algorithm is one of the earliest FL algorithms, where each client takes multiple steps of SGD with its local data and then sends the learned parameter to the server for averaging. Recently, the convergence properties of local-SGD and FedAvg algorithms have been studied in \cite{FedAvg, Local-SGD, Decentralized}. For example, \cite{FedAvg} provided the convergence analysis of FedAvg/local-SGD algorithms for strongly-convex optimization. \cite{Local-SGD} studied the convergence rates of local-SGD for both convex and nonconvex optimizations. Due to lacking of solution personalization, the basic FL methods often show poor performances in the presence of local data heterogeneity deteriorating the performance of the global FL model on individual clients. Thus, some personalized FL methods \cite{Personalized, Personalized2} have been studied. Meanwhile, to accelerate the basic local-SGD and FedAvg, some accelerated FL algorithms \cite{Accelerated, Accelerated2, Accelerated3, Distributed} have been developed. For example, \cite{Accelerated} proposed a faster FL algorithm for nonconvex optimization with simultaneously near-optimal sample and communication complexities. More recently, \cite{Distributed} proposed a faster federated learning for nonconvex optimization via global and local momentum. In parallel, some adaptive FL methods \cite{AdaptiveFL, AdaptiveFL2} have been developed to accelerate the basic local-SGD and FedAvg algorithms. For example, \cite{AdaptiveFL} proposed a class of adaptive FL algorithms via using adaptive learning rates at the server side. Meanwhile, an efficient local-AMSGrad algorithm \cite{AMSGrad} has been proposed, where clients locally update variables by using adaptive learning rates shared with all clients.

2.2 Distributed Minimax Optimization

Minimax optimization is widely applied in many machine learning problems such as robust learning, fair learning and reinforcement learning. For the big data applications, recently, there exists an increasing interest in distributed minimax optimization, e.g., training robust Deep Neural Networks (DNNs) over multiple clients and policy evaluation over multi-agents. Recently, decentralized optimization methods \cite{Decentralized, Decentralized2, Decentralized3, Decentralized4, Decentralized5, Decentralized6} for distributed minimax optimization have been developed. For example, \cite{Decentralized6} studied the decentralized optimization methods for the nonconvex-(strongly)-concave minimax optimization. Subsequently, \cite{Decentralized5} proposed a faster decentralized minimax optimization method for NC-SC minimax optimization. In parallel, some federated minimax optimization methods \cite{Federated, Federated2, Federated3, Federated4, Federated5, Federated6} have been developed to solve the distributed minimax problems. For example, \cite{Federated6} studied the federated learning methods for NC-PL minimax optimization. \cite{Federated7} proposed a class of effective Local-SGDA methods for minimax optimization, and provide the convergence analysis for the general minimax optimization. \cite{Federated8, Federated9} proposed some communication-efficient federated algorithms for NC-SC/NC-PL minimax optimization. Subsequently, \cite{Federated10, Federated11, Federated12} proposed some accelerated Local-SGDA methods based on the variance reduced techniques.

3 Preliminaries

3.1 Notations

\([K]\) denotes the set \( \{1, 2, \cdots, K\} \). \( \| \cdot \| \) denotes the \( \ell_2 \) norm for vectors and spectral norm for matrices. \((x, y)\) denotes the inner product of two vectors \( x \) and \( y \). For vectors \( x \) and \( y \), \( x' = (r > 0) \) denotes the element-wise power operation, \( x/y \) denotes the element-wise division and \( \max(x, y) \) denotes the element-wise maximum. \( I_d \) denotes a \( d \)-dimensional identity matrix. Matrix \( A \succ 0 \) is positive definite. Given function \( f(x, y) \), \( f(x, \cdot) \) denotes function w.r.t. the second variable with fixing \( x \), and \( f(\cdot, y) \) denotes function w.r.t. the first variable with fixing \( y \). \( a_m = O(b_m) \) denotes that \( a_m \leq c b_m \) for some constant \( c > 0 \). The notation \( O(\cdot) \) hides logarithmic terms.

3.2 Some Assumptions

Assumption 1. For any \( k \in [K] \), the local function \( f^k(x, y; \xi^k) \) has a \( L_f \)-Lipschitz gradient, e.g., for all \( x, x_1, x_2 \in \mathbb{R}^d \) and \( y, y_1, y_2 \in \mathbb{R}^p \), we have

\[
\| \nabla_x f^k(x_1, y; \xi^k) - \nabla_x f^k(x_2, y; \xi^k) \| \leq L_f \| x_1 - x_2 \|, \\
\| \nabla_y f^k(x, y_1; \xi^k) - \nabla_y f^k(x, y_2; \xi^k) \| \leq L_f \| y_1 - y_2 \|.
\]

Assumption 1 imposes the smoothness of stochastic
function \( f(x, y; \xi) \) as in the variance reduced federated algorithm [27].

**Assumption 2.** For \( x \in \mathbb{R}^d \), the global function \( f(x, y) = \frac{1}{K} \sum_{k=1}^{K} f^k(x, y) \) is \( \mu \)-strongly concave on variable \( y \in \mathbb{R}^p \), i.e., for all \( x \in \mathbb{R}^d \) and \( y, y' \in \mathbb{R}^p \), we have

\[
f(x, y) \leq f(x, y') + \langle \nabla_y f(x, y'), y - y' \rangle - \frac{\mu}{2} \| y - y' \|^2. \tag{2}\]

**Assumption 3.** For \( x \in \mathbb{R}^d \), the global function \( f(x, y) = \frac{1}{K} \sum_{k=1}^{K} f^k(x, y) \) satisfies \( \mu \)-PL condition in variable \( y \in \mathbb{R}^p \) for some \( \mu > 0 \) if for any given \( x \in \mathcal{X} \), it holds that

\[
\| \nabla_y f(x, y') \|^2 \geq 2\mu \left( \max_y f(x, y) - f(x, y') \right), \quad \forall y' \in \mathbb{R}^p. \tag{3}\]

By maximizing the inequality (2) with respect to \( y \), we have

\[
\max_y f(x, y) \leq \max_y \left\{ f(x, y') + \langle \nabla_y f(x, y'), y - y' \rangle - \frac{\mu}{2} \| y - y' \|^2 \right\}. \tag{4}\]

For its right hand side, we have

\[
\nabla f(x, y') - \mu(y - y') = 0 \Rightarrow y = y' + \frac{1}{\mu} \nabla f(x, y'). \tag{5}\]

Then putting \( y = y' + \frac{1}{\mu} \nabla f(x, y') \) into the right hand side of the above inequality (4), we have

\[
\max_y f(x, y) \leq f(x, y') + \frac{1}{2\mu} \| \nabla f(x, y') \|^2. \tag{6}\]

Then we can get

\[
\| \nabla_y f(x, y') \|^2 \geq 2\mu \left( \max_y f(x, y) - f(x, y') \right), \quad \forall y \in \mathbb{R}^p. \tag{7}\]

Thus, the strong concavity implies Polyak-Lojasiewicz inequality is satisfied. In other words, Assumption 3 implies that Assumption 2 holds. In the following our convergence analysis, thus, we only use the above Assumption 3, i.e., satisfying PL condition.

### 3.3 Distributed Minimax Optimization

In this subsection, we review the first-order method to solve the following distributed minimax optimization problem,

\[
\min_{x \in \mathbb{R}^d} \max_{y \in \mathbb{R}^p} f(x, y) = \frac{1}{K} \sum_{k=1}^{K} f^k(x, y). \tag{8}\]

For solving Problem (8), we can iteratively conduct the gradient descent for the variables \( x \) and the gradient ascent for the variables \( y \) at the \( t \)-th step

\[
x_{t+1} = x_t - \gamma \nabla_x f(x_t, y_t), \quad y_{t+1} = y_t + \lambda \nabla_y f(x_t, y_t),
\]

where \( \lambda > 0 \) and \( \gamma > 0 \) denote the learning rates. Based on the above Assumption 3, the function \( f(x, y) = \frac{1}{K} \sum_{k=1}^{K} f^k(x, y) \) satisfies PL condition in \( y \in \mathbb{R}^p \). Thus, there exists a unique solution to the problem \( \max_{y \in \mathbb{R}^p} f(x, y) \) for any \( x \). Here we let \( y^*(x) = \arg \max_{y \in \mathbb{R}^p} f(x, y) = \arg \max_{y \in \mathbb{R}^p} \frac{1}{K} \sum_{k=1}^{K} f^k(x, y) \), and \( F(x) = f(x, y^*(x)) = \max_{y \in \mathbb{R}^p} \frac{1}{K} \sum_{k=1}^{K} f^k(x, y) \).

In the paper, we mainly focus on the distributed stochastic minimax problem (1). For any \( k \in [K] \), \( f^k(x, y) = E_{\xi^k} \left[ f^k(x, y; \xi^k) \right] \). Next, we review a useful lemma in [39].
Algorithm 1 FGDA and AdaFGDA Algorithms

1: Input: $T, q$, tuning parameters $\{\gamma, \lambda, \eta, \alpha_t, \beta_t\}$, initial inputs $x_1 \in \mathbb{R}^d, y_1 \in \mathbb{R}^p$;
2: initialize: Set $x^k_1 = x_1$ and $y^k_1 = y_1$ for $k \in [K]$, and draw $q$ samples $\{\xi^k_{ij}\}_{j=1}^q$, and then compute $v^k_t = \frac{1}{q} \sum_{j=1}^q \nabla_x f^k(x_t^1, y_t^1, \xi^k_{ij}),$ and $w^k_t = \frac{1}{q} \sum_{j=1}^q \nabla_y f^k(x_t^1, y_t^1, \xi^k_{ij}),$ for all $k \in [K]$, and generate adaptive matrices $A_t \in \mathbb{R}^{d \times d}$ and $B_t \in \mathbb{R}^{p \times p}$;
3: for $t = 1 \to T$ do
4: if mod $(t, q) = 0$ then
5: $\bar{v}_t = \frac{1}{K} \sum_{k=1}^K v^k_t, \bar{y}_t = \frac{1}{K} \sum_{k=1}^K y^k_t, \bar{x}_t = \frac{1}{K} \sum_{k=1}^K x^k_t$;
6: Generate the adaptive matrices $A_t \in \mathbb{R}^{d \times d}$ and $B_t \in \mathbb{R}^{p \times p}$;
7: Compute $a_t = \gamma \alpha_t - 1 + (1 - \varrho)\bar{y}_t^2, A_t = \operatorname{diag}(\sqrt{\bar{y}_t} + \rho)$;
8: Compute $b_t = \gamma \beta_t - 1 + (1 - \varrho)\bar{v}_t^2, B_t = \operatorname{diag}(\sqrt{\bar{v}_t} + \rho)$;
9: else
10: for each $k \in [K]$ (in parallel) do
11: $\bar{x}_{t+1} = x_t^k - \gamma A_t^{-1} w^k_t$;
12: $\bar{y}_{t+1} = y_t^k + \eta_t (\bar{y}_{t+1} - \bar{y}_t)$, $x_{t+1} = x_t^k + \eta_t (\bar{x}_{t+1} - x_t)$;
13: $A_{t+1} = A_t, B_{t+1} = B_t$;
14: Draw one sample $\xi_{t+1}$ for any $k \in [K]$;
15: $v^k_{t+1} = \nabla_x f^k(x_{t+1}^k, y_{t+1}^k; \xi_{t+1}^k) + (1 - \alpha_{t+1}) \frac{\nabla_y f^k(x_{t+1}^k, y_{t+1}^k; \xi_{t+1}^k)}{\kappa} + (1 - \beta_{t+1}) \frac{\nabla_y f^k(x_{t+1}^k, y_{t+1}^k; \xi_{t+1}^k)}{\kappa}$;
16: $w^k_{t+1} = \nabla_x f^k(x_{t+1}^k, y_{t+1}^k; \xi_{t+1}^k)$;
17: end for
18: end if
19: end for
20: Output: Chosen uniformly random from $\{\bar{x}_t, \bar{y}_t\}_{t=1}^T$.

Lemma 1. (Lemma A.5 of [39]) Let $F(x) = f(x, y^*(x))$ with $y^*(x) \in \arg \max x \in \operatorname{arg} \max x, f(x, y)$. Under the above Assumptions 1, 3, we have $\nabla F(x) = \nabla_x f(x, y^*(x))$ and $F(x)$ is L-smooth, i.e.,

$$||\nabla F(x_1) - \nabla F(x_2)|| \leq L ||x_1 - x_2||, \quad \forall x_1, x_2 \quad (9)$$

where $L = L_f(1 + \frac{\kappa}{\rho})$ with $\kappa = \frac{L_f}{\rho}$.

4 Faster Federated Minimax Optimization Algorithms

In this section, we propose a class of accelerated federated minimax optimization methods (i.e., FGDA and AdaFGDA) to solve Problem (1), based on the momentum-based variance reduced and local-SGD techniques. In particular, our AdaFGDA algorithm uses the unified adaptive matrices to flexibly incorporate various adaptive learning rates to update variables $x$ and $y$. Figure 1 shows the basic idea of our federated minimax optimization algorithms. Meanwhile, Algorithm 1 shows a procedure framework of our FGDA and AdaFGDA algorithms.

In Algorithm 1, when mod $(t, q) = 0$ (i.e., synchronization step), the server receives the updated variables $\{x^k_t, y^k_t\}_{k=1}^K$ from the clients, and then averages them to obtain the averaged variables $\{\bar{x}_t, \bar{y}_t\}$ and averaged gradients $\{\bar{v}_t, \bar{v}_t\}$. Based on these averaged gradients, we can generate some adaptive matrices (i.e., adaptive learning rates). Besides one example given at the line 6 of Algorithm 1, we can also generate many other adaptive matrices. For example, we can generate adaptive matrices as in AdaBelief [63] algorithm, defined as

$$a_t = \varrho \alpha_{t-1} + (1 - \varrho) (\bar{y}_t - \bar{y}_{t-1})^2, A_t = \operatorname{diag}(\sqrt{\bar{y}_t} + \rho),$$
$$b_t = \varrho \beta_{t-1} + (1 - \varrho) (\bar{v}_t - \bar{v}_{t-1})^2, B_t = \operatorname{diag}(\sqrt{\bar{v}_t} + \rho),$$

where $t_0 = t - q$. We update the variables $x$ and $y$ in the server by using these adaptive matrices, then sent the updated variables to each client.

When mod $(t, q) \neq 0$ (i.e., asynchronization step), the clients receive the updated variables $\{\bar{x}_t, \bar{y}_t\}$ and the generated adaptive matrices $\{A_t, B_t\}$ from the server. Then the clients use the momentum-based variance reduced technique of STORM [7]/ ProxHSGD [51] to update the stochastic gradients based on local data: for $k \in [K]$

$$v^k_{t+1} = \nabla_y f^k(x^k_{t+1}, y^k_{t+1}; \xi^k_{t+1}) + (1 - \alpha_{t+1}) \frac{\nabla_y f^k(x^k_{t+1}, y^k_{t+1}; \xi^k_{t+1})}{\kappa} + (1 - \beta_{t+1}) \frac{\nabla_y f^k(x^k_{t+1}, y^k_{t+1}; \xi^k_{t+1})}{\kappa},$$

where $\alpha_{t+1} \in (0, 1)$, and it is similar for $w^k_{t+1}$. Based
on the estimated stochastic gradients and adaptive matrices, the clients update the variables \( \{x^k, y^k\} \), defined as

\[
\begin{align*}
\hat{y}^k_{t+1} &= y^k_t - \lambda B^{-1}_{t} x^k_t, \\
\hat{x}^k_{t+1} &= x^k_t - \gamma A_t^{-1} u^k_t, \\
\tilde{x}^k_{t+1} &= x^k_t + \eta_t (\hat{x}^k_{t+1} - x^k_t).
\end{align*}
\]

In our algorithms, all clients use the same adaptive matrices generated from the server to avoid model divergence. \textbf{Note that} for our non-adaptive FGDA algorithm, we only set \( A_t = I_d \) and \( B_t = I_p \) for all \( t \geq 1 \) in Algorithm 1.

5 Convergence Analysis

In this section, we study the convergence properties of our FGDA and AdaFGDA algorithms under some mild assumptions. All related proofs are provided in the Appendix. We first review some useful lemmas and assumptions.

\textbf{Assumption 4.} For any \( k \in [K] \), each component function \( f^k(x, y; \xi^k) \) has an unbiased stochastic gradient with bounded variance \( \sigma^2 \), i.e., for all \( \xi^k \sim \mathcal{D}^k \), \( x \in \mathbb{R}^d, y \in \mathbb{R}^p \)

\[
\begin{align*}
\mathbb{E}[\nabla f^k(x, y; \xi^k)] &= \nabla f^k(x, y), \\
\mathbb{E}[\|\nabla f^k(x, y) - \nabla f^k(x, y; \xi^k)\|^2] &\leq \sigma^2.
\end{align*}
\]

\textbf{Assumption 5.} For any \( k, j \in [K] \), \( x \in \mathbb{R}^d \) and \( y \in \mathbb{R}^p \), we have \( \|\nabla_x f^k(x, y) - \nabla_x f^j(x, y)\| \leq \delta_x \), \( \|\nabla_y f^k(x, y) - \nabla_y f^j(x, y)\| \leq \delta_y \), where \( \delta_x > 0 \) and \( \delta_y > 0 \) are constants.

\textbf{Assumption 6.} The function \( F(x) = \arg \max_{y \in \mathbb{R}^p} f(x, y) \) is bounded below, i.e., \( F^* = \inf_{x \in \mathbb{R}^d} F(x) > -\infty \).

\textbf{Assumption 7.} In our AdaFGDA algorithm, the adaptive matrices \( A_t \) and \( B_t \) for all \( t \geq 1 \) satisfy \( A_t \succeq \rho I_d > 0 \) and \( \rho_u I_p \succeq B_t \succeq \rho I_p > 0 \), where \( \rho_u \geq \rho > 0 \) is an appropriate positive number.

Assumption 4 shows that the stochastic gradients in each client are unbiased, and their variances are bounded, which is very common in the stochastic optimization [13, 12, 7]. Assumption 5 shows that under non-i.i.d. setting, the data heterogeneity is bounded, which is very common in the federated optimization [27, 45]. Assumption 6 guarantees the feasibility of Problem (1). Assumption 7 ensures that the adaptive matrices \( A_t \) for all \( t \geq 1 \) are positive definite as in [20, 17].

Next, based on the above assumptions, we give the convergence properties of our FGDA and AdaFGDA algorithms.

5.1 Convergence Properties of AdaFGDA Algorithm

\textbf{Theorem 1.} Assume the sequence \( \{\tilde{x}_t, y^t\}_{t=1}^T \) be generated from Algorithm 1. Under the above Assumptions 1, 3, 7, and let \( \eta_t = \frac{nK^{1/3}}{(m+1)^{1/3}} \) for all \( t \geq 0 \), \( \alpha_{t+1} = \beta_{t+1} = \gamma_t^2 \), \( m \geq \max(2, n^3, (c_n^1)^3 K, (c_n^3)^3 K, \frac{(12\sqrt{2}\pi n\alpha t \lambda)}{\rho^3}) \), \( n > 0 \), \( c_1 + c_2 \leq \frac{12\lambda^2 \rho^2 \sigma^2 t^2}{\rho^2} + \gamma^2 t^2 + \frac{9\rho^2 L^2}{\rho^2} \), \( c_2 \geq \frac{2}{3nK} + \frac{9}{2}, \gamma \tau \lambda \leq \min\left(\frac{\sqrt{nK}}{4\sqrt{2}}, 1\right) \), \( \gamma \leq \min\left(\frac{n^{1/3}}{4\sqrt{2}}, \frac{\lambda \mu}{16\pi \rho_1^2} \right) \), \( \rho_1^2 \leq \frac{\lambda \mu}{16\pi \rho_1^2} \), \( \rho_u^2 \geq \frac{\lambda \mu}{16\pi \rho_1^2} \), \( \Delta = \left(\frac{1}{\rho_u^2} + \frac{16\lambda^2 \rho^2}{K^2}\right) \), \( \delta \geq \delta_x + \delta_y \).

\textbf{Remark 1.} Assume the bounded stochastic gradient \( \|\nabla_x f^k(x^k_t, y^k_t; \xi^k_t)\| \leq C_{fx} \) for all \( k \in [K], \) we have \( \frac{1}{t} \sum_{k=1}^{K} \mathbb{E}[\|A_t\|^2] \leq 2\left(C_{fx}^2 + \rho^2\right) \). Similarly, assume the bounded stochastic gradient \( \|\nabla_y f^k(x^k_t, y^k_t; \xi^k_t)\| \leq C_{fy} \) for all \( k \in [K] \), we can obtain \( \rho_u = O(1) \).

\textbf{Remark 2.} Without loss of generality, let \( k = O(1), \rho = \rho_u = O(1), \) \( c_1 = O(1), \) \( c_2 = O(1) \) and \( m = O(q^3) \), we have \( G = O(1) \) and \( \sqrt{\frac{1}{T}} \sum_{t=1}^{T} \mathbb{E}[\|A_t\|^2] = O(1) \). Based on the above Theorem 1, let \( q = T^{1/3} \), we have

\[
\frac{1}{T} \sum_{t=1}^{T} \mathbb{E}[\|\nabla F(\tilde{x}_t)\|] \leq O\left(\sqrt{\frac{q}{T}} + \frac{1}{T^{1/3}}\right) = O\left(\frac{1}{T^{1/3}}\right) \leq \epsilon,
\]

then we can obtain \( T = O(\epsilon^{-3}) \). Our AdaFGDA algorithm needs to compute two stochastic gradients at each iteration except for the first iteration requires \( 2q \) stochastic gradients, so it has a gradient (i.e., SFO) complexity of \( 2q + 2T = O(\epsilon^{-3}) \). Thus, our AdaFGDA algorithm requires \( O(\epsilon^{-3}) \) gradient complexity and \( \frac{T}{q} = T^{2/3} = O(\epsilon^{-2}) \) communication complexity in searching for an \( \epsilon \)-stationary point of Problem (1), which improves the existing federated mini
maximization methods by a factor of $O(\epsilon^{-1})$ in gradient or communication complexities (Please see Table 1).

5.2 Convergence Properties of FGDA Algorithm

**Theorem 2.** Assume the sequence $\{\tilde{x}_t, \tilde{y}_t\}_{t=1}^T$ be generated from Algorithm 1 when $A_t = I_d$ and $B_t = I_p$ for all $t \geq 1$. Under the above Assumptions 1,3-6, and let $\eta_t = \frac{nK^{1/3}}{(m+1)^{1/3}}$ for all $t \geq 0$, $\alpha_{t+1} = c_1\eta_t^2$, $\beta_{t+1} = c_2\eta_t^2$, $m \geq \max\left(2, n^3, (c_1n)^3K, (c_2n)^3K, (12\sqrt{2}n\lambda qL_f)^3\right)$, $n > 0$, $c_1^2 + c_2^2 \leq 12^4\lambda^4q^2L_f^2$, $c_1 \geq \frac{2}{3n^2K} + \frac{9\lambda q}{2\eta^2}$, $c_2 \geq \frac{2}{3n^2K} + \frac{g}{2}$, $\gamma = \tau\lambda$, $\tau \leq \min\left(\frac{\sqrt{K}}{4\sqrt{2}K}, 1\right)$, $\gamma \leq \min\left(\frac{\sqrt{K}}{4\sqrt{2}K}, \frac{\lambda t}{\tau\Delta - \frac{2\lambda}{15K^2L_f^2}}, \frac{\mu}{\tau\Delta - \frac{2\lambda}{15K^2L_f^2}}, \frac{2m^2}{8\sqrt{3}L_f}, \frac{\sqrt{G}}{K}\right)$, and $\lambda \leq \min\left(\frac{\mu}{\tau\Delta - \frac{2\lambda}{15K^2L_f^2}}, \frac{3\sqrt{G}}{\sqrt{2}\sqrt{2}\sqrt{2}}\right)$. We have

$$\frac{1}{T} \sum_{t=1}^{T} \mathbb{E}[\|\nabla F(\tilde{x}_t)\|] \leq \frac{\sqrt{3Gm^{1/6}}}{K^{1/6}T^{1/2}} + \frac{\sqrt{3G}}{K^{1/6}T^{1/3}},$$

where $g$ is defined as:

$$g = \frac{4m^{1/6}L_f^2}{K^{1/6}T^{1/2}} + \frac{4m^{1/6}L_f^2}{K^{1/6}T^{1/2}} + 8n^2\left(\frac{c_1^2 + c_2^2}{K}\right)^{3/2} + \frac{4m^{1/6}L_f^2}{K^{1/6}T^{1/2}} \ln(m + t), \Delta = c_2^2\sigma^2 + \frac{c_1^2\sigma^2}{K} + 3c_2^2\frac{\delta_y^2}{K} + 3c_2^2\frac{\delta_y^2}{K}$$

and $\lambda = \frac{1}{16} + \frac{L_f^2}{4\sigma^2} + \frac{16\lambda qL_f^2}{K}$.

**Remark 3.** The proof of Theorem 2 can follow the proofs of the above Theorem 1 with $A_t = I_d$ and $B_t = I_p$ for all $t \geq 1$, and $\rho = \rho_n = \rho_t = 1$. Since the conditions of Theorem 2 are similar to these of Theorem 1, clearly, our FGDA algorithm still can obtain a lower gradient complexity of $O(\epsilon^{-3})$ and lower communication complexity of $O(\epsilon^{-2})$ for finding an $\epsilon$-stationary point of Problem (1).

6 Numerical Experiments

In this section, we perform numerical experiments on some federated minimax optimization problems to demonstrate the efficiency of our FGDA and AdaFGDA algorithms. We compare our FGDA and AdaFGDA algorithms with state-of-the-art federated minimax optimization algorithms, including Local-SGD [10], Momentum-Local-SGD [45], CDMA [55] and FEDNEST [49]. All experiments are run over machine with Intel(R) Xeon(R) W-2255 CPU and Nvidia RTX2080ti(s).

6.1 Synthetic Federated Minimax Problem

In the subsection, we conduct a synthetic federated minimax optimization problem as in [49] formulated as:

$$\min_{x \in \mathbb{R}^d} \max_{y \in \mathbb{R}^p} \frac{1}{K} \sum_{k=1}^{K} f_k(x, y),$$

where $f_k(x, y) = \frac{1}{2} \|x\|^2 - \frac{1}{2} \|y\|^2 - b_k^T y + c_k^T x$. In fact, this minimax problem (12) is expected to find a saddle point of the following problem:

$$\min_{x \in \mathbb{R}^d} \frac{1}{K} \sum_{k=1}^{K} A_k x - b_k^2,$$

s.t. $b_k = b_k - \frac{1}{K} \sum_{k=1}^{K} b_k', A_k = t_k I_d$.

Here we set $\tau$ to 10 and sampled $b_k$ and $t_k$ from $b_k \sim \mathcal{N}(0, s^2 I_d)$ and $t_k \sim \mathcal{U}(0, 0.1)$, respectively. In the experiment, we train the model for 200 epochs as [49].

Figure 2 shows that our FGDA and AdaFGDA converge linearly despite the significant heterogeneity, which is a significant improvement over Local-SGD and optimized Momentum-Local-SGD. Our methods also achieve a faster and more stable convergence rate than FEDNEST for varying heterogeneities $s = 1$ and $s = 10$, where a larger $s$ represents a larger heterogeneity. We can witness a mutation on FEDNEST in $||y - y^*||^2$ in the early training stage. Although the synthetic data simulation experiment is relatively simple and ideal compared to general federated minimax optimization, it provides a more detailed and specific comparison than simulation experiments on real datasets since the optimal solution is available.

6.2 Deep AUC Maximization

Data imbalance, where the number of samples from different classes is skewed, is a fundamental issue that can lead to model bias. Although Federated Learning (FL) offers an effective framework for leveraging multiple data sources, most existing FL methods still do not have the ability to address model bias caused by data imbalance, especially when such imbalance varies...
CIFAR100, CheXpert [23] and ImageNet datasets, we  
split them further into different groups to increase  
balanced heterogeneous dataset, we manually select  
and use a LeNet5 for training. To construct the im-
  
For the CIFAR10 experiment, we set the local itera-
  
We evaluate the efficiency of our proposed FGDA and  
AdaFGDA on five benchmark datasets, i.e., MNIST,  
CIFAR10, CIFAR100, ImageNet, CheXpert, and com-
  
As shown in Figures 3 and 4, our proposed FGDA and  
AdaFGDA algorithms achieve state-of-the-art perfor-
  
6.3 Robust Neural Network Training  
In this subsection, we will address the problem of train-
  
For the CIFAR10 experiment, we set the local itera-
  
employed ResNet50 as the backbone model and set the  
local iterations to \( q = 20 \) for all methods. To estab-
  
In the experiment, we use a grid search approach to de-
  
Adaptive Federated Minimax Optimization with Lower Complexities
Figure 4: AUC Scores on ImageNet (Left), CIFAR100 (Middle) and CheXpert (Right).

Figure 5: Test Accuracy for the robust NN training problem on the MNIST dataset, with 3-layer MLP. A comparison of different $q$ is also provided.

Figure 6: TestAccuracy for the robust NN training problem on the FashionMNIST dataset, with 3 layer MLP. A comparison of different $q$ is also provided.

7 Conclusion

In the paper, we studied a class of distributed non-convex minimax optimization under non-i.i.d. setting, and proposed a class of efficient adaptive federated minimax optimization methods (i.e., AdaFGDA and FGDA) based on momentum-based variance reduced and local-SGD techniques. Moreover, we provided a convergence analysis framework for our methods and proved that they obtain lower gradient and communication complexities simultaneously.

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Checklist

1. For all models and algorithms presented, check if you include:
   (a) A clear description of the mathematical setting, assumptions, algorithm, and/or model. [Yes]
   (b) An analysis of the properties and complexity (time, space, sample size) of any algorithm. [Yes]
   (c) (Optional) Anonymized source code, with specification of all dependencies, including external libraries. [No]

2. For any theoretical claim, check if you include:
   (a) Statements of the full set of assumptions of all theoretical results. [Yes]
   (b) Complete proofs of all theoretical results. [Yes]
   (c) Clear explanations of any assumptions. [Yes]

3. For all figures and tables that present empirical results, check if you include:
   (a) The code, data, and instructions needed to reproduce the main experimental results (either in the supplemental material or as a URL). [Yes]
   (b) All the training details (e.g., data splits, hyperparameters, how they were chosen). [Yes]
   (c) A clear definition of the specific measure or statistics and error bars (e.g., with respect to the random seed after running experiments multiple times). [Yes]
   (d) A description of the computing infrastructure used. (e.g., type of GPUs, internal cluster, or cloud provider). [Yes]

4. If you are using existing assets (e.g., code, data, models) or curating/releasing new assets, check if you include:
   (a) Citations of the creator If your work uses existing assets. [Yes]
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5. If you used crowdsourcing or conducted research with human subjects, check if you include:
   (a) The full text of instructions given to participants and screenshots. [Not Applicable]
   (b) Descriptions of potential participant risks, with links to Institutional Review Board (IRB) approvals if applicable. [Not Applicable]
   (c) The estimated hourly wage paid to participants and the total amount spent on participant compensation. [Not Applicable]
A Appendix

In this section, we provide the detailed convergence analysis of our algorithms. We first introduce some useful notations: $\bar{v}_t = \frac{1}{K} \sum_{k=1}^{K} v^k_t$, $\bar{w}_t = \frac{1}{K} \sum_{k=1}^{K} w^k_t$, $\bar{x}_t = \frac{1}{K} \sum_{k=1}^{K} x^k_t$, $\bar{\lambda}_t = \frac{1}{K} \sum_{k=1}^{K} \lambda^k_t$, $\bar{y}_t = \frac{1}{K} \sum_{k=1}^{K} y^k_t$, $\bar{\hat{y}}_t = \frac{1}{K} \sum_{k=1}^{K} \hat{y}^k_t$,

\[
F(x) = \frac{1}{K} \sum_{k=1}^{K} f^k(x, y^k(x)), \quad \nabla_x f(x, y) = \frac{1}{K} \sum_{k=1}^{K} \nabla_x f^k(x, y), \quad \nabla_y f(x, y) = \frac{1}{K} \sum_{k=1}^{K} \nabla_y f^k(x, y),
\]

\[
\nabla_x f(x_t, y_t) = \frac{1}{K} \sum_{k=1}^{K} \nabla_x f^k(x^k_t, y^k_t), \quad \nabla_y f(x_t, y_t) = \frac{1}{K} \sum_{k=1}^{K} \nabla_y f^k(x^k_t, y^k_t), \quad \forall t \geq 1.
\]

Next, we review and provide some useful lemmas.

**Lemma 2.** ([27]) Function $f(x) : \mathbb{R}^d \to \mathbb{R}$ is $L$-smooth and satisfies PL condition with constant $\mu > 0$, then it also satisfies error bound (EB) condition with $\mu$, i.e., for all $x \in \mathbb{R}^d$

\[
\|\nabla f(x)\| \geq \mu \|x^* - x\|,
\]

where $x^* \in \text{arg\,min}_x f(x)$. It also satisfies quadratic growth (QG) condition with $\mu$, i.e.,

\[
f(x) - \min_x f(x) \geq \frac{\mu}{2} \|x^* - x\|^2.
\]

From the above lemma 2, when consider the problem $\max_x f(x)$ that is equivalent to the problem $\min_x -f(x)$, we have

\[
\|\nabla f(x)\| \geq \mu \|x^* - x\|,
\]

\[
\max_x f(x) - f(x) \geq \frac{\mu}{2} \|x^* - x\|^2.
\]

**Lemma 3.** ([37]) Assume function $f(x)$ is convex and $\mathcal{X}$ is a convex set. $x^* \in \mathcal{X}$ is the solution of the constrained problem $\min_{x \in \mathcal{X}} f(x)$, if

\[
\nabla f(x^*), x - x^* \geq 0, \; \forall x \in \mathcal{X}.
\]

where $\nabla f(x^*)$ denote the (sub-)gradient of function $f(x)$ at $x^*$.

**Lemma 4.** Given $K$ vectors $\{v^k\}_{k=1}^{K}$, the following inequalities satisfy: $\|v^k + v^j\|^2 \leq (1 + \alpha)\|v^k\|^2 + (1 + \frac{1}{\alpha})\|v^j\|^2$ for any $\alpha > 0$, and $\|\sum_{k=1}^{K} v^k\|^2 \leq K \sum_{k=1}^{K} \|v^k\|^2$.

**Lemma 5.** Given a finite sequence $\{w^k\}_{k=1}^{K}$, and $\bar{w} = \frac{1}{K} \sum_{k=1}^{K} w^k$, the following inequality satisfies $\sum_{k=1}^{K} \|w^k - \bar{w}\|^2 \leq \sum_{k=1}^{K} \|w^k\|^2$.

**Lemma 6.** Suppose the sequence $\{\bar{x}_t, \bar{y}_t\}_{t=1}^{T}$ be generated from Algorithm 1. Under the Assumptions 1,3, given $0 < \gamma \leq \min\left(\frac{\lambda \mu}{16 \rho u}, \frac{\gamma \mu}{16 \rho u}\right)$ and $0 < \lambda \leq \frac{1}{2L_f \rho u \gamma^2}$ for all $t \geq 1$, we have

\[
F(\bar{x}_{t+1}) - f(\bar{x}_{t+1}, \bar{y}_{t+1}) \leq (1 - \eta \frac{\beta^2}{2\rho u})(F(\bar{x}_t) - f(\bar{x}_t, \bar{y}_t)) + \frac{\eta \beta}{8\gamma^2} \|\bar{x}_{t+1} - \bar{x}_t\|^2 - \frac{\eta \gamma}{4L_f \rho u} \|\bar{y}_{t+1} - \bar{y}_t\|^2 + \frac{\eta \beta}{\rho u} \|\nabla y f(\bar{x}_t, \bar{y}_t) - \bar{v}_t\|^2,
\]

where $F(\bar{x}_t) = f(\bar{x}_t, y^*(\bar{x}_t))$ with $y^*(\bar{x}_t) \in \text{arg\,max}_y f(\bar{x}_t, y)$ for all $t \geq 1$.

**Proof.** Using $L_f$-smoothness of $f(x, \cdot)$, such that

\[
f(\bar{x}_{t+1}, \bar{y}_t) + \langle \nabla y f(\bar{x}_{t+1}, \bar{y}_t), \bar{y}_{t+1} - \bar{y}_t \rangle - \frac{L_f}{2} \|\bar{y}_{t+1} - \bar{y}_t\|^2 \leq f(\bar{x}_{t+1}, \bar{y}_{t+1}),
\]

then we have

\[
f(\bar{x}_{t+1}, \bar{y}_t) \leq f(\bar{x}_{t+1}, \bar{y}_{t+1}) - \langle \nabla y f(\bar{x}_{t+1}, \hat{y}_t), \hat{y}_{t+1} - \hat{y}_t \rangle + \frac{L_f}{2} \|\hat{y}_{t+1} - \hat{y}_t\|^2
\]

\[
= f(\bar{x}_{t+1}, \bar{y}_{t+1}) - \eta_{t}\langle \nabla y f(\bar{x}_{t+1}, \bar{y}_t), \hat{y}_{t+1} - \hat{y}_t \rangle + \frac{L_f \eta_t^2}{2} \|\bar{y}_{t+1} - \bar{y}_t\|^2.
\]
Since $\rho_u I_p \succeq B_1 \succeq \rho I_p > 0$ for any $t \geq 1$ is positive definite, we set $B_t = L_t(L_t)^T$, where $\sqrt{\rho u} I_p \succeq L_t \succeq \sqrt{\rho} I_p > 0$. Thus, we have $B_t^{-1} = (L_t^{-1})^2 L_t^{-1}$, where $\frac{1}{\sqrt{\rho u}} I_p \succeq L_t^{-1} \succeq \frac{1}{\sqrt{\rho}} I_p > 0$.

When $t = s_t = q[t/q] + 1$, according to the line 7 of Algorithm 1, we have $\hat{y}_{t+1} = \hat{y}_t + \lambda B_{t-1}^{-1} \hat{v}_t$. When $t \in (s_t, s_t+1)$, according to the line 11 of Algorithm 1, we have for all $k \in [K]$, $\hat{y}_{t+1} = y^k + \lambda B_{t-1}^{-1} e^k_t$, and then we also have $\hat{y}_{t+1} = \frac{1}{K} \sum_{k=1}^K y^k_{t+1} = \frac{1}{K} \sum_{k=1}^K (y^k + \lambda e^k_t) = \hat{y}_t + \lambda B_{t-1}^{-1} \hat{v}_t$.

Next, we bound the inner product in (22). According to $\hat{y}_{t+1} = \hat{y}_t + \lambda B_{t-1}^{-1} \hat{v}_t$, we have

\[-\eta_t \langle \nabla_y f(\bar{x}_{t+1}, \bar{y}_t), y_{t+1} - \bar{y}_t \rangle = -\eta_t \lambda \eta_t \langle \nabla_y f(\bar{x}_{t+1}, \bar{y}_t), B_{t-1}^{-1} \hat{v}_t \rangle = -\eta_t \lambda \eta_t \langle L_t^{-1} \nabla_y f(\bar{x}_{t+1}, \bar{y}_t), L_t^{-1} \hat{v}_t \rangle \]
\[\leq -\frac{\eta_t \lambda}{2} \eta_t \langle |L_t^{-1} \nabla_y f(\bar{x}_{t+1}, \bar{y}_t)|^2 + |L_t^{-1} \nabla_y f(\bar{x}_{t+1}, \bar{y}_t) - L_t^{-1} \nabla_y f(\bar{x}_t, \bar{y}_t) - L_t^{-1} \nabla_y f(\bar{x}_t, \bar{y}_t) - L_t^{-1} \hat{v}_t|^2 \rangle \]
\[\leq -\frac{\eta_t \lambda}{2} \eta_t \langle |L_t^{-1} \nabla_y f(\bar{x}_{t+1}, \bar{y}_t)|^2 + |L_t^{-1} \nabla_y f(\bar{x}_{t+1}, \bar{y}_t) - L_t^{-1} \nabla_y f(\bar{x}_t, \bar{y}_t) - L_t^{-1} \hat{v}_t|^2 \rangle \]
\[\leq -\frac{\eta_t \lambda}{2} \eta_t \langle F(\bar{x}_{t+1}) - f(\bar{x}_{t+1}, \bar{y}_t) - F(\bar{x}_{t+1}, \bar{y}_t) - \hat{v}_t|^2 \rangle, \tag{23} \]

where the last inequality is due to the quadratic growth condition of $\mu$-PL functions, i.e.,

\[\| \nabla_y f(\bar{x}_{t+1}, \bar{y}_t) \|^2 \geq 2\mu \bigg( \max_{y'} f(\bar{x}_{t+1}, y') - f(\bar{x}_{t+1}, \bar{y}_t) \bigg) = 2\mu \bigg( F(\bar{x}_{t+1}) - f(\bar{x}_{t+1}, \bar{y}_t) \bigg). \tag{24} \]

Substituting (23) in (22), we have

\[f(\bar{x}_{t+1}, \bar{y}_t) = f(\bar{x}_{t+1}, \bar{y}_{t+1}) - \frac{\eta_t \lambda}{\rho_u} \bigg( F(\bar{x}_{t+1}) - f(\bar{x}_{t+1}, \bar{y}_t) \bigg) - \frac{\eta_t \lambda^2}{2\rho_u} \| y_{t+1} - \bar{y}_t \|^2 + \eta_t \lambda \langle |L_t^{-1} \nabla_y f(\bar{x}_{t+1}, \bar{y}_t)|^2 \rangle \]
\[+ \eta_t \lambda \langle |L_t^{-1} \nabla_y f(\bar{x}_{t+1}, \bar{y}_t) - \hat{v}_t|^2 \rangle + \frac{L_t \eta_t^2}{2} \| y_{t+1} - \bar{y}_t \|^2, \tag{25} \]

then rearranging the terms, we can obtain

\[F(\bar{x}_{t+1}) - f(\bar{x}_{t+1}, \bar{y}_{t+1}) = \bigg( 1 - \frac{\eta_t \lambda}{\rho_u} \bigg) \bigg( F(\bar{x}_{t+1}) - f(\bar{x}_{t+1}, \bar{y}_t) \bigg) - \frac{\eta_t \lambda^2}{2\rho_u} \| y_{t+1} - \bar{y}_t \|^2 + \eta_t \lambda \langle |L_t^{-1} \nabla_y f(\bar{x}_{t+1}, \bar{y}_t)|^2 \rangle \]
\[+ \eta_t \lambda \langle |L_t^{-1} \nabla_y f(\bar{x}_{t+1}, \bar{y}_t) - \hat{v}_t|^2 \rangle + \frac{L_t \eta_t^2}{2} \| y_{t+1} - \bar{y}_t \|^2. \tag{26} \]

Next, using $L_f$-smoothness of function $f(\cdot, \bar{y}_t)$, such that

\[f(\bar{x}_t, \bar{y}_t) + \langle \nabla_x f(\bar{x}_t, \bar{y}_t), \bar{x}_{t+1} - \bar{x}_t \rangle - \frac{L_f}{2} \| \bar{x}_{t+1} - \bar{x}_t \|^2 \leq f(\bar{x}_{t+1}, \bar{y}_t), \tag{27} \]

then we have

\[f(\bar{x}_t, \bar{y}_t) - f(\bar{x}_{t+1}, \bar{y}_t) \leq -\langle \nabla_x f(\bar{x}_t, \bar{y}_t), \bar{x}_{t+1} - \bar{x}_t \rangle + \frac{L_f}{2} \| \bar{x}_{t+1} - \bar{x}_t \|^2 \]
\[= -\eta_t \langle \nabla_x f(\bar{x}_t, \bar{y}_t) - \nabla F(\bar{x}_t), \bar{x}_{t+1} - \bar{x}_t \rangle - \eta_t \langle \nabla F(\bar{x}_t), \bar{x}_{t+1} - \bar{x}_t \rangle + \frac{L_f \eta_t^2}{2} \| \bar{x}_{t+1} - \bar{x}_t \|^2 \]
\[\leq \frac{\eta_t}{8} \| \bar{x}_{t+1} - \bar{x}_t \|^2 + 2\eta_t \langle \nabla_x f(\bar{x}_t, \bar{y}_t) - \nabla F(\bar{x}_t), \bar{x}_{t+1} - \bar{x}_t \rangle + \frac{L_f \eta_t^2}{2} \| \bar{x}_{t+1} - \bar{x}_t \|^2 \]
\[\leq \frac{\eta_t}{8} \| \bar{x}_{t+1} - \bar{x}_t \|^2 + \frac{2L_f \eta_t}{8} \| \bar{x}_{t+1} - \bar{y}_t \|^2 + \frac{L_f \eta_t^2}{2} \| \bar{x}_{t+1} - \bar{x}_t \|^2 \]
\[+ \eta_t^2 \| \bar{x}_{t+1} - \bar{x}_t \|^2 + \frac{\eta_t^2 L_f}{2} \| \bar{x}_{t+1} - \bar{x}_t \|^2 \]
\[\leq \frac{4L_f \eta_t}{\mu} \bigg( F(\bar{x}_t) - f(\bar{x}_t, \bar{y}_t) \bigg) + F(\bar{x}_t) - F(\bar{x}_{t+1}) + \eta_t \bigg( \frac{1}{8} + \eta_t L \bigg) \| \bar{x}_{t+1} - \bar{x}_t \|^2, \tag{28} \]
where the second last inequality is due to Lemma 1, i.e., $L$-smoothness of function $F(x)$, and the last inequality holds by Lemma 2 and $L_f \leq L$. Then we have

$$F(\tilde{x}_{t+1}) - f(\tilde{x}_{t+1}, \tilde{y}_t) = F(\tilde{x}_{t+1}) - F(\tilde{x}_t) + F(\tilde{x}_t) - f(\tilde{x}_t, \tilde{y}_t) + f(\tilde{x}_t, \tilde{y}_t) - f(\tilde{x}_{t+1}, \tilde{y}_t) \leq (1 + \frac{4L_f^2 \eta \gamma}{\mu})(F(\tilde{x}_t) - f(\tilde{x}_t, \tilde{y}_t)) + \eta(\frac{1}{8\gamma} + \eta L)\|\tilde{x}_{t+1} - \tilde{x}_t\|^2. \tag{29}$$

Substituting (29) in (26), we get

$$F(\tilde{x}_{t+1}) - f(\tilde{x}_{t+1}, \tilde{y}_{t+1}) \leq (1 - \frac{\eta \lambda \mu}{\rho u})(1 + 4L_f^2 \eta \gamma)(F(\tilde{x}_t) - f(\tilde{x}_t, \tilde{y}_t)) + \eta(\frac{1}{8\gamma} + \eta L)(1 - \frac{\eta \lambda \mu}{\rho u})\|\tilde{x}_{t+1} - \tilde{x}_t\|^2$$

$$- \frac{\eta}{2}\|\tilde{y}_{t+1} - \tilde{y}_t\|^2 - \frac{\eta \lambda L_f^2}{\rho u}\|\tilde{x}_{t+1} - \tilde{x}_t\|^2 + \eta \lambda \|\nabla_y f(\tilde{x}_t, \tilde{y}_t) - \tilde{v}_t\|^2 + \frac{L_f \eta L_f^2}{2}\|\tilde{y}_{t+1} - \tilde{y}_t\|^2$$

$$= (1 - \frac{\eta \lambda \mu}{\rho u})(1 + 4L_f^2 \eta \gamma)(F(\tilde{x}_t) - f(\tilde{x}_t, \tilde{y}_t)) + \eta(\frac{1}{8\gamma} + \eta L) - \frac{\eta \lambda \mu}{\rho u}\|\nabla_y f(\tilde{x}_t, \tilde{y}_t) - \tilde{v}_t\|^2$$

$$- \frac{\eta}{2}\|\tilde{y}_{t+1} - \tilde{y}_t\|^2 + \frac{\eta \lambda \|\nabla_y f(\tilde{x}_t, \tilde{y}_t) - \tilde{v}_t\|^2}{\rho u}$$

$$\leq (1 - \frac{\eta \lambda \mu}{\rho u})(F(\tilde{x}_t) - f(\tilde{x}_t, \tilde{y}_t)) + \eta(\frac{1}{8\gamma} + \eta L)\|\tilde{x}_{t+1} - \tilde{x}_t\|^2 - \frac{\eta \lambda \mu}{\rho u}\|\tilde{y}_{t+1} - \tilde{y}_t\|^2 + \frac{\eta \lambda \|\nabla_y f(\tilde{x}_t, \tilde{y}_t) - \tilde{v}_t\|^2}{\rho u}, \tag{30}$$

where the last inequality holds by $\gamma \leq \min\left(\frac{\lambda \mu}{16\rho u L}, \frac{\rho \mu}{16\rho u L_f^2}\right)$ and $\lambda \leq \frac{1}{2\gamma L_f \rho u}$ for all $t \geq 1$, i.e.,

$$\gamma \leq \frac{\lambda \mu}{16\rho u L} \Rightarrow \lambda \geq \frac{16\rho u L \gamma}{\mu} = 16\rho u \gamma + \frac{\kappa^2}{2} \geq \frac{8\rho u \kappa^2 \gamma}{2} \Rightarrow \frac{\eta \lambda \mu}{\rho u} \geq \frac{4L_f^2 \eta \gamma}{\mu}$$

$$\gamma \leq \min\left(\frac{\lambda \mu}{16\rho u L}, \frac{\rho \mu}{16\rho u L_f^2}\right) \Rightarrow \frac{\eta \lambda \mu}{8\rho u} \geq \eta L + \frac{\eta L_f^2 \lambda}{\rho u}$$

$$\lambda \leq \frac{1}{2\gamma L_f \rho u} \Rightarrow \frac{1}{2\lambda \rho u} \geq \eta L_f, \forall t \geq 1. \tag{31}$$

**Lemma 7.** Assume the sequences $\{\tilde{x}_t, \tilde{y}_t\}_{t=1}^T$ generated from Algorithm 1. Under the Assumptions 1,3, given $0 < \gamma \leq \frac{2\gamma}{\pi \eta}$, we have

$$F(\tilde{x}_{t+1}) \leq F(\tilde{x}_t) + \frac{4L_f^2 \eta \gamma}{\rho u}(F(\tilde{x}_t) - f(\tilde{x}_t, \tilde{y}_t)) + \frac{2\gamma \eta}{\rho u}\|\nabla_y f(\tilde{x}_t, \tilde{y}_t) - \tilde{v}_t\|^2 - \frac{\rho \mu}{2\gamma} \|\tilde{x}_{t+1} - \tilde{x}_t\|^2. \tag{32}$$

**Proof.** For notational simplicity, let $\tilde{x}_t = \frac{1}{K} \sum_{k=1}^K x^k, \tilde{y}_t = \frac{1}{K} \sum_{k=1}^K y^k, \tilde{w}_t = \frac{1}{K} \sum_{k=1}^K w^k$ and $\tilde{v}_t = \frac{1}{K} \sum_{k=1}^K v^k$. When $t = s_t = q[t/q] + 1$, according to the lines 7 and 8 of Algorithm 1, we have $\tilde{y}_{t+1} = \tilde{y}_t + \lambda B_t^{-1} \tilde{v}_t, \tilde{y}_{t+1} = \tilde{y}_t + \eta (\tilde{y}_{t+1} - \tilde{y}_t), \tilde{x}_{t+1} = \tilde{x}_t - \gamma A_t^{-1} \tilde{w}_t$ and $\tilde{x}_{t+1} = \tilde{x}_t + \eta (\tilde{x}_{t+1} - \tilde{x}_t)$. Thus, we have for all $t \geq 1$,

$$\tilde{x}_{t+1} = \tilde{x}_t + \gamma A_t^{-1} \tilde{v}_t = \arg \min_x \left\{\langle \tilde{w}_t, x - \tilde{x}_t \rangle + \frac{1}{2\gamma} (x - \tilde{x}_t)^T A_t (x - \tilde{x}_t)\right\}. \tag{33}$$

By using the optimal condition of the above problem (33), we have, for all $x \in \mathbb{R}^d$

$$\langle \tilde{w}_t + \frac{1}{\gamma} A_t (\tilde{x}_{t+1} - \tilde{x}_t), x - \tilde{x}_{t+1} \rangle \geq 0. \tag{34}$$

Let $x = x_t$, we can obtain

$$\langle \tilde{w}_t + \frac{1}{\gamma} A_t (\tilde{x}_{t+1} - \tilde{x}_t), x - \tilde{x}_{t+1} \rangle \leq -\frac{\rho}{\gamma} \|\tilde{x}_{t+1} - \tilde{x}_t\|^2. \tag{35}$$
Adaptive Federated Minimax Optimization with Lower Complexities

According to Lemma 1, i.e., function $F(x)$ is $L$-smooth, we have

$$F(\bar{x}_{t+1}) \leq F(\bar{x}_t) + \langle \nabla F(\bar{x}_t), \bar{x}_{t+1} - \bar{x}_t \rangle + \frac{L}{2} \| \bar{x}_{t+1} - \bar{x}_t \|^2$$

$$= F(\bar{x}_t) + \langle \nabla F(\bar{x}_t), \eta_t (\bar{x}_{t+1} - \bar{x}_t) \rangle + \frac{L}{2} \| \eta_t (\bar{x}_{t+1} - \bar{x}_t) \|^2$$

$$= F(\bar{x}_t) + \eta_t (\bar{\omega}_t, \bar{x}_{t+1} - \bar{x}_t) + \eta_t \langle \nabla F(\bar{x}_t) - \bar{\omega}_t, \bar{x}_{t+1} - \bar{x}_t \rangle + \frac{L\eta_t^2}{2} \| \bar{x}_{t+1} - \bar{x}_t \|^2$$

$$\leq F(\bar{x}_t) - \frac{\eta_t}{\gamma} \| \bar{x}_{t+1} - \bar{x}_t \|^2 + \eta_t \langle \nabla F(\bar{x}_t) - \bar{\omega}_t, \bar{x}_{t+1} - \bar{x}_t \rangle + \frac{L\eta_t^2}{2} \| \bar{x}_{t+1} - \bar{x}_t \|^2,$$  (36)

where the second equality is due to $\bar{x}_{t+1} = \bar{x}_t + \eta_t (\bar{x}_{t+1} - \bar{x}_t)$, and the last inequality holds by the above inequality (35). Meanwhile, we have

$$\langle \nabla F(\bar{x}_t) - \bar{\omega}_t, \bar{x}_{t+1} - \bar{x}_t \rangle \leq \| \nabla F(\bar{x}_t) - \bar{\omega}_t \| \cdot \| \bar{x}_{t+1} - \bar{x}_t \|$$

$$\leq \frac{\gamma}{\rho} \| \nabla F(\bar{x}_t) - \bar{\omega}_t \|^2 + \frac{\rho}{4\gamma} \| \bar{x}_{t+1} - \bar{x}_t \|^2$$

$$= \frac{\gamma}{\rho} \| \nabla_x f(\bar{x}_t, y^*(\bar{x}_t)) - \nabla_x f(\bar{x}_t, \bar{y}_t) + \nabla_x f(\bar{x}_t, \bar{y}_t) - \bar{\omega}_t \|^2 + \frac{\rho}{4\gamma} \| \bar{x}_{t+1} - \bar{x}_t \|^2$$

$$\leq \frac{2\gamma}{\rho} \| \nabla_x f(\bar{x}_t, y^*(\bar{x}_t)) - \nabla_x f(\bar{x}_t, \bar{y}_t) \|^2 + \frac{2\gamma}{\rho} \| \nabla_x f(\bar{x}_t, \bar{x}_t) - \bar{\omega}_t \|^2 + \frac{\rho}{4\gamma} \| \bar{x}_{t+1} - \bar{x}_t \|^2$$

$$\leq \frac{2\gamma L^2}{\rho} \| y^*(\bar{x}_t) - \bar{y}_t \|^2 + \frac{2\gamma}{\rho} \| \nabla_x f(\bar{x}_t, \bar{x}_t) - \bar{\omega}_t \|^2 + \frac{\rho}{4\gamma} \| \bar{x}_{t+1} - \bar{x}_t \|^2,$$  (37)

where the first inequality is due to the Cauchy-Schwarz inequality and the second is due to Young’s inequality. By plugging the above inequalities (37) into (36), we obtain

$$F(\bar{x}_{t+1}) \leq F(\bar{x}_t) + \eta_t \langle \nabla F(\bar{x}_t) - \bar{\omega}_t, \bar{x}_{t+1} - \bar{x}_t \rangle + \eta_t \langle \bar{\omega}_t, \bar{x}_{t+1} - \bar{x}_t \rangle + \frac{L\eta_t^2}{2} \| \bar{x}_{t+1} - \bar{x}_t \|^2$$

$$\leq F(\bar{x}_t) + \frac{2\gamma L^2 \eta_t}{\rho} \| y^*(\bar{x}_t) - \bar{y}_t \|^2 + \frac{2\gamma}{\rho} \| \nabla_x f(\bar{x}_t, \bar{x}_t) - \bar{\omega}_t \|^2 + \frac{\rho}{4\gamma} \| \bar{x}_{t+1} - \bar{x}_t \|^2$$

$$\leq \frac{\rho}{4\gamma} \| \bar{x}_{t+1} - \bar{x}_t \|^2$$

$$F(\bar{x}_{t+1}) \leq F(\bar{x}_t) + \frac{2\gamma L^2 \eta_t}{\rho} \| y^*(\bar{x}_t) - \bar{y}_t \|^2 + \frac{2\gamma}{\rho} \| \nabla_x f(\bar{x}_t, \bar{x}_t) - \bar{\omega}_t \|^2 + \frac{\rho}{4\gamma} \| \bar{x}_{t+1} - \bar{x}_t \|^2$$

$$\leq \frac{\rho}{4\gamma} \| \bar{x}_{t+1} - \bar{x}_t \|^2$$

$$\leq F(\bar{x}_t) + \frac{2\gamma L^2 \eta_t}{\rho} \| y^*(\bar{x}_t) - \bar{y}_t \|^2 + \frac{2\gamma}{\rho} \| \nabla_x f(\bar{x}_t, \bar{x}_t) - \bar{\omega}_t \|^2 + \frac{\rho}{4\gamma} \| \bar{x}_{t+1} - \bar{x}_t \|^2$$

$$\leq \frac{\rho}{4\gamma} \| \bar{x}_{t+1} - \bar{x}_t \|^2,$$  (38)

where the second last inequality is due to $0 < \gamma \leq \frac{\rho}{4\gamma}$, and the last inequality holds by the above Lemma 2 using in $F(\bar{x}_t) = f(\bar{x}_t, y^*(\bar{x}_t))$ with $y^*(\bar{x}_t) \in \arg \max f(\bar{x}_t, y)$.

\[\square\]

**Lemma 8.** Under the above assumptions, and assume the stochastic gradient estimators $\{\hat{\omega}_t, \hat{\omega}_t\}^T_{t=1}$ be generated from Algorithm 1, we have

$$\mathbb{E}[\| \hat{\omega}_{t+1} - \nabla_y f(x_{t+1}, y_{t+1}) \|^2] \leq (1 - \alpha_{t+1}) \mathbb{E}[\| \hat{\omega}_t - \nabla_y f(x_t, y_t) \|^2] + \frac{2\alpha_{t+1} \sigma^2}{K}$$

$$+ \frac{4L^2 \eta_t^2}{K^2} \sum_{k=1}^{K} \mathbb{E}[\| \bar{x}_{t+1} - \bar{x}_t \|^2 + \| \bar{y}_{t+1} - y_t \|^2],$$  (39)

$$\mathbb{E}[\| \hat{\omega}_{t+1} - \nabla_x f(x_{t+1}, y_{t+1}) \|^2] \leq (1 - \beta_{t+1}) \mathbb{E}[\| \hat{\omega}_t - \nabla_x f(x_t, y_t) \|^2] + \frac{2\beta_{t+1} \sigma^2}{K}$$

$$+ \frac{4L^2 \eta_t^2}{K^2} \sum_{k=1}^{K} \mathbb{E}[\| \bar{x}_{t+1} - \bar{x}_t \|^2 + \| \bar{y}_{t+1} - y_t \|^2].$$  (40)
Proof. Without loss of generality, we only prove the above inequality (40), and it is similar to (39). Since \( \tilde{w}_{t+1} = \frac{1}{K} \sum_{k=1}^{K} \left( \nabla_x f^k(x_{t+1}^k, y_{t+1}^k; \xi_{t+1}^k) + (1 - \beta_{t+1})(w_t^k - \nabla_x f^k(x_t^k, y_t^k; \xi_{t+1}^k)) \right) \), we have

\[
\mathbb{E}\|\tilde{w}_{t+1} - \nabla_x f(x_{t+1}, y_{t+1})\|^2 = \frac{1}{K^2} \sum_{k=1}^{K} \left( \nabla_x f^k(x_{t+1}^k, y_{t+1}^k; \xi_{t+1}^k) + (1 - \beta_{t+1})(w_t^k - \nabla_x f^k(x_t^k, y_t^k; \xi_{t+1}^k)) - \nabla_x f^k(x_t^k, y_t^k) \right)^2
\]

\[
+ (1 - \beta_{t+1}) \frac{1}{K} \sum_{k=1}^{K} (w_t^k - \nabla_x f^k(x_t^k, y_t^k))^2
\]

\[
\leq 2 \left(1 - \beta_{t+1}\right)^2 \frac{K}{2} \sum_{k=1}^{K} \mathbb{E}\|\nabla_x f^k(x_{t+1}^k, y_{t+1}^k; \xi_{t+1}^k) + (1 - \beta_{t+1})(w_t^k - \nabla_x f^k(x_t^k, y_t^k; \xi_{t+1}^k)) - \nabla_x f^k(x_t^k, y_t^k)\|^2
\]

\[
+ (1 - \beta_{t+1})^2 \|\tilde{w}_t - \nabla_x f(x_t, y_t)\|^2
\]

\[
\leq (1 - \beta_{t+1})^2 \mathbb{E}\|\tilde{w}_t - \nabla_x f(x_t, y_t)\|^2 + \frac{2\beta_{t+1}^2 \sigma^2}{K} + \frac{4(1 - \beta_{t+1})^2 L^2}{2} \sum_{k=1}^{K} \mathbb{E}(\|x_{t+1}^k - x_t^k\|^2 + \|y_{t+1}^k - y_t^k\|^2)
\]

where the forth equality holds by, for any \( k \in [K] \),

\[
\mathbb{E}_{\xi_{t+1}^k} \left[ \nabla_x f(x_{t+1}^k, y_{t+1}^k; \xi_{t+1}^k) - \nabla_x f(x_t, y_t) \right] = 0, \quad \mathbb{E}_{\xi_{t+1}^k} \left[ \nabla_x f(x_t, y_{t+1}^k; \xi_{t+1}^k) - \nabla_x f(x_t, y_t) \right] = 0.
\]

and for any \( k \neq j \in [K] \), \( \xi_{t+1}^k \) and \( \xi_{t+1}^j \) are independent, i.e.,

\[
\mathbb{E}_{\xi_{t+1}^k} \left[ \nabla_x f^k(x_{t+1}^k, y_{t+1}^k; \xi_{t+1}^k) - \nabla_x f^k(x_{t+1}^k, y_{t+1}^k) \right] = 0;
\]

the second inequality holds by the inequality \( \mathbb{E}\|\mathcal{C} - \mathbb{E}\mathcal{C}\|_F^2 \leq \mathbb{E}\|\mathcal{C}\|_F^2 \) and Assumption 4; the second last inequality is due to Assumption 1; the last inequality holds by \( 0 < \beta_{t+1} \leq 1 \) and \( x_{t+1}^k = x_t^k + \eta_t(x_{t+1}^k - x_t^k) \), \( y_{t+1}^k = y_t^k + \eta_t(y_{t+1}^k - y_t^k) \).

Lemma 9. Based on the above Assumptions 1 and 5, we have

\[
\sum_{k=1}^{K} \mathbb{E}\|\nabla_x f^k(x_t^k, y_t^k) - \frac{1}{K} \sum_{j=1}^{K} \nabla_y f^j(x_t^j, y_t^j)\|^2 \leq 12 L^2 \sum_{k=1}^{K} \mathbb{E}(\|x_t^k - \bar{x}_t\|^2 + \|y_t^k - \bar{y}_t\|^2) + 3K\delta_t^2,
\]

\[
\sum_{k=1}^{K} \mathbb{E}\|\nabla_y f^k(x_t^k, y_t^k) - \frac{1}{K} \sum_{j=1}^{K} \nabla_y f^j(x_t^j, y_t^j)\|^2 \leq 12 L^2 \sum_{k=1}^{K} \mathbb{E}(\|x_t^k - \bar{x}_t\|^2 + \|y_t^k - \bar{y}_t\|^2) + 3K\delta_t^2.
\]
Lemma 10. Suppose the iterates \( \{x^k_t, y^k_t\}_{t=1}^T \), for all \( k \in [K] \) generated from Algorithm 1 satisfy:

\[
\sum_{k=1}^{K} \mathbb{E}[\|x^k_t - \bar{x}_t\|^2] \leq (q - 1) \sum_{l=s_t+1}^{t-1} \gamma^2 \eta^2 \sum_{k=1}^{K} \mathbb{E}[\|A^{-1}_k(u^k_l - \bar{w}_l)\|^2],
\]
\[
\sum_{k=1}^{K} \mathbb{E}[\|y^k_t - \bar{y}_t\|^2] \leq (q - 1) \sum_{l=s_t+1}^{t-1} \lambda^2 \eta^2 \sum_{k=1}^{K} \mathbb{E}[\|B^{-1}_k(v^k_l - \bar{v}_l)\|^2].
\]

Proof. From Algorithm 1, when \( s_t = q[t/q] \), we have \( t = s_t + 1 \) and \( x^k_t = \bar{x}_t \), the above inequality holds trivially. When \( t \in (s_t + 1, s_t + q] \), we have

\[
x^k_t = x^k_{s_t+1} - \sum_{l=s_t+1}^{t-1} \gamma \eta A^{-1}_l w^k_l, \quad \text{and} \quad \bar{x}_t = \bar{x}_{s_t+1} - \sum_{l=s_t+1}^{t-1} \gamma \eta A^{-1}_l \bar{w}_l.
\]

Thus we have

\[
\sum_{k=1}^{K} \mathbb{E}[\|x^k_t - \bar{x}_t\|^2] = \sum_{k=1}^{K} \mathbb{E} \left[ \|x^k_{s_t+1} - \bar{x}_{s_t+1} - \left( \sum_{l=s_t+1}^{t-1} \gamma \eta A^{-1}_l w^k_l - \sum_{l=s_t+1}^{t-1} \gamma \eta A^{-1}_l \bar{w}_l \right) \|^2 \right]
\]
\[
= \sum_{k=1}^{K} \mathbb{E} \left[ \sum_{l=s_t+1}^{t-1} \left( \gamma \eta A^{-1} w^k_l - \gamma \eta A^{-1} \bar{w}_l \right) \|^2 \right] \leq (q - 1) \sum_{l=s_t+1}^{t-1} \gamma^2 \eta^2 \sum_{k=1}^{K} \mathbb{E}[\|A^{-1}_k(u^k_l - \bar{w}_l)\|^2],
\]

where the above inequality is due to \( t - s_t - 1 \leq q - 1 \). Similarly, we can obtain the above inequality (46).
\[ c_1^2 + c_2^2 \leq \frac{12t^4 L^2 q^2 L_1^2}{\rho^2} \]. Let \( s_t = [t/q] \) and \( t \in [s_t, s_t + q - 1] \), we have

\[
\sum_{t=s_t}^{s_t+q-1} \eta_t \mathbb{E}(\|A_{t-1}^{-1}(w_t^k - \bar{w}_t^k)\|^2 + \|B_t^{-1}(v_t^k - \bar{v}_t^k)\|^2) \\
\leq 8K \sum_{t=s_t}^{s_t+q-1} \eta_t \mathbb{E}(\|\bar{w}_t^k\|^2 + \|\bar{v}_t^k\|^2) + \frac{2K \Delta}{15L^2 L_1^2} \sum_{t=s_t}^{s_t+q-1} \eta_t.
\]

where \( \Delta = c_1^2 \sigma^2 + c_2^2 \sigma^2 + 3c_3^2 \sigma^2 + 3c_3^2 \sigma^2 \).

**Proof.** When \( t = s_t = q[t/q] \), we have \( w_{t+1}^k = \bar{w}_{t+1} \) for all \( k \in [K] \), and then we have \( \sum_{k=1}^{K} \mathbb{E}\|A_{t+1}^{-1}(w_{t+1}^k - \bar{w}_{t+1})\| = 0 \). When \( t \in (s_t, s_t + q) \), we have

\[
\sum_{k=1}^{K} \mathbb{E}\|A_{t+1}^{-1}(w_{t+1}^k - \bar{w}_{t+1})\|^2 \\
= \sum_{k=1}^{K} \mathbb{E}\|A_{t+1}^{-1}\left(\nabla_x f^k(x_{t+1}^k, y_{t+1}^k; \xi_{t+1}^k) + (1 - \beta_{t+1})(w_t^k - \nabla_x f^k(x_t^k, y_t^k; \xi_t^k) - 1 - \beta_{t+1})(w_t^k - \nabla_x f^k(x_t^k, y_t^k; \xi_t^k))\right)\|^2 \\
= \sum_{k=1}^{K} \mathbb{E}\|A_{t+1}^{-1}\left(1 - \beta_{t+1})(w_t^k - \bar{w}_t^k) + (\nabla_x f^k(x_{t+1}^k, y_{t+1}^k; \xi_{t+1}^k) - \frac{1}{K} \sum_{k=1}^{K} \nabla_x f^k(x_{t+1}^k, y_{t+1}^k; \xi_{t+1}^k)\| - 1 - \beta_{t+1})(\nabla_x f^k(x_{t+1}^k, y_{t+1}^k; \xi_{t+1}^k) - \frac{1}{K} \sum_{k=1}^{K} \nabla_x f^k(x_{t+1}^k, y_{t+1}^k; \xi_{t+1}^k)\|\|^2 \\
\leq (1 + \nu)(1 - \beta_{t+1})\sum_{k=1}^{K} \mathbb{E}\|A_{t+1}^{-1}(w_t^k - \bar{w}_t^k)\|^2 + (1 + \frac{1}{\nu}) \frac{1}{\rho^2} \sum_{k=1}^{K} \mathbb{E}\|\nabla_x f^k(x_{t+1}^k, y_{t+1}^k; \xi_{t+1}^k) - \frac{1}{K} \sum_{k=1}^{K} \nabla_x f^k(x_{t+1}^k, y_{t+1}^k; \xi_{t+1}^k)\|^2 \\
\leq 2 \sum_{k=1}^{K} \mathbb{E}\|\nabla_x f^k(x_{t+1}^k, y_{t+1}^k; \xi_{t+1}^k) - \frac{1}{K} \sum_{k=1}^{K} \nabla_x f^k(x_{t+1}^k, y_{t+1}^k; \xi_{t+1}^k)\|^2 \\
+ 2\beta_{t+1}^2 \sum_{k=1}^{K} \mathbb{E}\|\nabla_x f^k(x_{t+1}^k, \xi_{t+1}^k) - \frac{1}{K} \sum_{k=1}^{K} \nabla_x f^k(x_{t+1}^k, \xi_{t+1}^k)\|^2 \\
\leq 4L_1^2 \sum_{k=1}^{K} \mathbb{E}(\|x_t^k - x_{t+1}^k\|^2 + \|y_t^k - y_{t+1}^k\|^2) + 2\beta_{t+1}^2 \sum_{k=1}^{K} \mathbb{E}\|\nabla_x f^k(x_{t+1}^k, \xi_{t+1}^k) - \frac{1}{K} \sum_{k=1}^{K} \nabla_x f^k(x_{t+1}^k, \xi_{t+1}^k)\|^2,
\]

where the last inequality holds by Assumption 3.
Consider the term \( \sum_{k=1}^{K} \| \nabla_x f^k(x^k_t, y^k_t; \xi^k_t) - \frac{1}{K} \sum_{k=1}^{K} \nabla_x f^k(x^k_t, y^k_t; \xi^k_t) \| \), we have

\[
\sum_{k=1}^{K} \| \nabla_x f^k(x^k_t, y^k_t; \xi^k_t) - \frac{1}{K} \sum_{k=1}^{K} \nabla_x f^k(x^k_t, y^k_t; \xi^k_t) \|^2 \\
= \sum_{k=1}^{K} \| \nabla_x f^k(x^k_t, y^k_t; \xi^k_t) - \nabla_x f^k(x^k_t, y^k_t) - \frac{1}{K} \sum_{k=1}^{K} (\nabla_x f^k(x^k_t, y^k_t; \xi^k_t) - \nabla_x f^k(x^k_t, y^k_t)) \|
\]

where the last inequality holds by Assumption 1 and the above Lemma 9.

By combining the above inequalities (47), (48) and (49), we have

\[
\sum_{k=1}^{K} \| \nabla_x f^k(x^k_t, y^k_t; \xi^k_t) - \frac{1}{K} \sum_{k=1}^{K} \nabla_x f^k(x^k_t, y^k_t; \xi^k_t) \|^2 \\
\leq 2 \sum_{k=1}^{K} \| \nabla_x f^k(x^k_t, y^k_t; \xi^k_t) - \nabla_x f^k(x^k_t, y^k_t) - \frac{1}{K} \sum_{k=1}^{K} (\nabla_x f^k(x^k_t, y^k_t; \xi^k_t) - \nabla_x f^k(x^k_t, y^k_t)) \|
\]

where the last inequality holds by Assumption 1 and the above Lemma 9.
Similarly, we can also obtain

\[
\sum_{k=1}^{K} \mathbb{E}\|B_{t+1}^{-1}(v_{t+1}^k - \bar{v}_{t+1})\|^2 \\
\leq (1 + \nu)(1 - \alpha_{t+1})^2 \sum_{k=1}^{K} \mathbb{E}\|B_{t}^{-1}(v_{t}^k - \bar{v}_{t})\|^2 + (1 + \frac{1}{\nu})^2 \rho^2 \left( 4L_f^2 \sum_{k=1}^{K} \mathbb{E}\|x_{t+1}^k - x_t^k\|^2 + \mathbb{E}\|y_{t+1}^k - y_t^k\|^2 \right) \\
+ 4K\alpha_{t+1}^2 \sigma^2 + 48\alpha_{t+1}^2 L_f^2 \sum_{k=1}^{K} \left( \mathbb{E}\|x_{t}^k - \bar{x}_t\|^2 + \mathbb{E}\|y_{t}^k - \bar{y}_t\|^2 + 12\alpha_{t+1}^2 K\delta_g^2 \right) \\
\leq (1 + \nu)(1 - \alpha_{t+1})^2 \sum_{k=1}^{K} \mathbb{E}\|B_{t}^{-1}(v_{t}^k - \bar{v}_{t})\|^2 + (1 + \frac{1}{\nu})^2 \rho^2 \left( 8L_f^2 \sum_{k=1}^{K} \mathbb{E}\left( \gamma^2 \eta_t^2 \|A_{t}^{-1}B_{t}^{-1}(v_{t}^k - \bar{v}_{t})\|^2 + \lambda^2 \eta_t^2 \|B_{t}^{-1}(v_{t}^k - \bar{v}_{t})\|^2 \right) \\
+ 8L_f^2 \sum_{k=1}^{K} \mathbb{E}\left( \gamma^2 \eta_t^2 \|A_{t}^{-1}\bar{w}_t\|^2 + \lambda^2 \eta_t^2 \|B_{t}^{-1}\bar{v}_t\|^2 \right) + 4\beta_{t+1}^2 K\sigma^2 \\
+ 48\beta_{t+1}^2 L_f^2 \left( (q - 1) \sum_{k=1}^{K} \gamma^2 \eta_t^2 \|A_{t}^{-1}\bar{w}_t\|^2 + \lambda^2 \eta_t^2 \|B_{t}^{-1}\bar{v}_t\|^2 \right) + 4\beta_{t+1}^2 K\sigma^2 \\
+ (1 + \nu)(1 - \alpha_{t+1})^2 \sum_{k=1}^{K} \mathbb{E}\|B_{t}^{-1}(v_{t}^k - \bar{v}_{t})\|^2 + (1 + \frac{1}{\nu})^2 \rho^2 \left( 8L_f^2 \sum_{k=1}^{K} \mathbb{E}\left( \gamma^2 \eta_t^2 \|A_{t}^{-1}(w_{t}^k - \bar{w}_{t})\|^2 + \lambda^2 \eta_t^2 \|B_{t}^{-1}(w_{t}^k - \bar{w}_{t})\|^2 \right) \\
+ 8L_f^2 \sum_{k=1}^{K} \mathbb{E}\left( \gamma^2 \eta_t^2 \|A_{t}^{-1}\bar{w}_t\|^2 + \lambda^2 \eta_t^2 \|B_{t}^{-1}\bar{v}_t\|^2 \right) + 4K\alpha_{t+1}^2 \sigma^2 \\
+ 48\alpha_{t+1}^2 L_f^2 \left( (q - 1) \sum_{k=1}^{K} \gamma^2 \eta_t^2 \|A_{t}^{-1}(w_{t}^k - \bar{w}_{t})\|^2 + \lambda^2 \eta_t^2 \|B_{t}^{-1}(w_{t}^k - \bar{w}_{t})\|^2 \right) \right) \\
\leq \max \left( (1 + \nu)(1 - \beta_{t+1})^2 + 16\gamma^2 \eta_t^2 (1 + \frac{1}{\nu}) \rho^2 L_f^2 (1 + \nu)(1 - \alpha_{t+1})^2 + 16\lambda^2 \eta_t^2 (1 + \frac{1}{\nu}) \rho^2 L_f^2 \right) \\
\cdot \sum_{k=1}^{K} \mathbb{E}\|A_{t}^{-1}(w_{t}^k - \bar{w}_{t})\|^2 + \|B_{t}^{-1}(v_{t}^k - \bar{v}_{t})\|^2 \\
+ 16\eta_t^2 L_f^2 (1 + \frac{1}{\nu}) \rho^2 \sum_{k=1}^{K} \mathbb{E}\left( \gamma^2 \|A_{t}^{-1}\bar{w}_t\|^2 + \lambda^2 \|B_{t}^{-1}\bar{v}_t\|^2 \right) \\
+ (1 + \frac{1}{\nu})^2 \rho^2 \left( 4K\beta_{t+1}^2 \sigma^2 + 4K\alpha_{t+1}^2 \sigma^2 + 12\beta_{t+1}^2 K\delta_g^2 + 12\alpha_{t+1}^2 K\delta_g^2 \right) \\
+ 48(q - 1)(1 + \frac{1}{\nu}) \rho^2 \sum_{i=s+1}^{K} \sum_{k=1}^{K} \mathbb{E}\left( \|A_{t}^{-1}(w_{t}^k - \bar{w}_{t})\|^2 + \|B_{t}^{-1}(v_{t}^k - \bar{v}_{t})\|^2 \right) \right). \quad (51)
\]
Let $\gamma = \gamma \tau$ ($0 < \tau \leq 1$), $\nu = \frac{1}{q}$ and $\eta_t \leq \frac{\nu^2}{\sqrt{2} \lambda \alpha_t \rho L_f}$ for all $t \geq 1$. Since $\alpha_{t+1} \in (0, 1)$ and $\beta_{t+1} \in (0, 1)$ for all $t \geq 0$, we have

\[
(1 + \nu)(1 - \beta_{t+1})^2 + 16\gamma^2 \eta^2_t (1 + \frac{1}{\nu}) \frac{1}{\rho^2} L_f^2 \\
\leq 1 + \frac{1}{q} + 16 (1 + q) \frac{\gamma^2}{\rho^2} L_f^2 \frac{\rho^2}{288 \lambda^2 q^2 L_f^2} \\
\leq 1 + \frac{1}{q} + \gamma^2 \frac{1 + q}{\lambda^2 18 q^2} \leq 1 + \frac{10}{9q}
\] (53)

Similarly, we can also obtain $(1 + \nu)(1 - \alpha_{t+1})^2 + 16\lambda^2 \eta^2_t (1 + \frac{1}{\nu}) \frac{1}{\rho^2} L_f^2 \leq 1 + \frac{10}{9q}$. Thus, we have

\[
\sum_{k=1}^{K} \mathbb{E} \left( \left\| A_{t+1}^{-1}(w_{t+1}^k - \bar{w}_{t+1}) \right\|^2 + \left\| B_{t+1}^{-1}(v_{t+1}^k - \bar{v}_{t+1}) \right\|^2 \right) \\
\leq (1 + \frac{10}{9q}) \sum_{k=1}^{K} \mathbb{E} \left( \left\| A_{t+1}^{-1} (w_t^k - \bar{w}_t) \right\|^2 + \left\| B_{t+1}^{-1} (v_t^k - \bar{v}_t) \right\|^2 \right) \\
+ 16 \eta^2_t L_f^2 (1 + q) \frac{\lambda^2}{\rho^2} \sum_{k=1}^{K} \mathbb{E} \left( \tau^2 \left\| A_{t+1}^{-1} \bar{w}_t \right\|^2 + \left\| B_{t+1}^{-1} \bar{v}_t \right\|^2 \right) \\
+ (1 + q) \frac{1}{\rho^2} (4K \beta^2 \delta^2 + 4K \alpha^2 \sigma^2 + 24 \lambda^2 K \delta^2 + 24 \lambda^2 K \delta^2) \\
+ 48 \eta^2_t \frac{\lambda^2}{\rho^2} (\beta_{t+1} + \alpha_{t+1}) L_f^2 \sum_{l=\chi_{t+1}}^{K-1} \eta_l^2 \sum_{k=1}^{K} \mathbb{E} \left( \left\| A_{t+1}^{-1} (w_l^k - \bar{w}_l) \right\|^2 + \left\| B_{t+1}^{-1} (v_l^k - \bar{v}_l) \right\|^2 \right) \\
\leq (1 + \frac{10}{9q}) \sum_{k=1}^{K} \mathbb{E} \left( \left\| A_{t+1}^{-1} (w_t^k - \bar{w}_t) \right\|^2 + \left\| B_{t+1}^{-1} (v_t^k - \bar{v}_t) \right\|^2 \right) \\
+ \frac{\sqrt{2} K \eta^3}{3 \rho L_f} (c_2 \sigma^2 + c_3 \sigma^2 + 3 c_2^2 \delta^2 + 3 c_3^2 \delta^2) \\
+ (c_2^2 + c_3^2) \frac{\eta^2_t L_f^2}{6} \sum_{l=\chi_{t+1}}^{K-1} \eta_l^2 \sum_{k=1}^{K} \mathbb{E} \left( \left\| A_{t+1}^{-1} (w_l^k - \bar{w}_l) \right\|^2 + \left\| B_{t+1}^{-1} (v_l^k - \bar{v}_l) \right\|^2 \right), \tag{54}
\]
\( \bar{w}_{t+1} \| = 0 \) and \( \sum_{k=1}^K E \| A_{t+1}^{-1}(w_t^k - \bar{w}_t^{k+1}) \| + E \| B_{t+1}^{-1}(v_t^k - \bar{v}_t^{k+1}) \| = 0 \). When \( t \in (s_t, s_t + q) \), we have

\[
\begin{align*}
\sum_{k=1}^K E \| A_{t+1}^{-1}(w_t^k - \bar{w}_{t+1}) \|^2 + E \| B_{t+1}^{-1}(v_t^k - \bar{v}_{t+1}) \|^2 \\
\leq \frac{1}{9q} \sum_{s = s_t}^t \left( 1 + \frac{10}{9q} \right)^{t-s} \sum_{k=1}^K E(\tau^2 \| A_{s+1}^{-1} \|_{\infty}^2 + E \| B_{s+1}^{-1} \|_{\infty}) \right) \\
+ \frac{\sqrt{2K}}{3\rho L_f} (c_3^2 \sigma^2 + c_1^3 \sigma^2 + 3c_2^2 \delta_x^2 + 3c_2^2 \delta_y^2) \sum_{s = s_t}^t \left( 1 + \frac{10}{9q} \rho^{s-4} \right) \\
+ \frac{(c_3^2 + c_1^3) L_f^2}{6} \sum_{s = s_t}^t \left( 1 + \frac{10}{9q} \right)^{t-s} \rho^{s-4} \sum_{k=1}^K E \left( \| A_{s+1}^{-1}(w_t^k - \bar{w}_t^k) \|^2 + E \| B_{s+1}^{-1}(v_t^k - \bar{v}_t^k) \|^2 \right) \\
\leq 4K \sum_{s = s_t}^t \left( 1 + \frac{10}{9q} \right)^{t-s} \sum_{k=1}^K E \left( \tau^2 \| A_{s+1}^{-1} \|_{\infty}^2 + E \| B_{s+1}^{-1} \|_{\infty} \right) \\
+ \frac{4\sqrt{2K}}{3\rho L_f} (c_3^2 \sigma^2 + c_1^3 \sigma^2 + 3c_2^2 \delta_x^2 + 3c_2^2 \delta_y^2) \sum_{s = s_t}^t \rho^{s-4} \\
+ \frac{\rho^3 (c_3^2 + c_1^3)}{36 \sqrt{2}(12)^2 \lambda^4 \eta^2 L_f} \sum_{s = s_t}^t \rho^{s-4} \sum_{k=1}^K E \left( \| A_{s+1}^{-1}(w_t^k - \bar{w}_t^k) \|^2 + E \| B_{s+1}^{-1}(v_t^k - \bar{v}_t^k) \|^2 \right),
\end{align*}
\]

where the last inequality holds by \( (1 + \frac{10}{9q})^9 \leq e^{10/9} \leq 4 \).

By multiplying both sides of (55) by \( \eta_{s+1} \) and summing over \( t = s_t - 1 \) to \( s_t + q - 2 \), we have

\[
\begin{align*}
\sum_{s = s_t}^{s_t + q - 1} \eta_{s+1} K E \left( \| A_{s+1}^{-1}(w_t^k - \bar{w}_t^k) \|^2 + E \| B_{s+1}^{-1}(v_t^k - \bar{v}_t^k) \|^2 \right) \\
\leq 4K \sum_{s = s_t}^{s_t + q - 1} \eta_{s+1} \left( \tau^2 \| A_{s+1}^{-1} \|_{\infty}^2 + E \| B_{s+1}^{-1} \|_{\infty} \right) \\
+ \frac{K}{\lambda^2 L_f^2} (c_3^2 \sigma^2 + c_1^3 \sigma^2 + 3c_2^2 \delta_x^2 + 3c_2^2 \delta_y^2) \sum_{s = s_t}^{s_t + q - 1} \eta_{s+1} \\
+ \frac{\rho^3 (c_3^2 + c_1^3)}{72 \sqrt{2}(12)^2 \lambda^4 \eta^2 L_f} \sum_{s = s_t}^{s_t + q - 1} \rho^{s-4} \sum_{k=1}^K E \left( \| A_{s+1}^{-1}(w_t^k - \bar{w}_t^k) \|^2 + E \| B_{s+1}^{-1}(v_t^k - \bar{v}_t^k) \|^2 \right),
\end{align*}
\]

Let \( c_3^2 + c_1^3 \leq \frac{12 \lambda^4 \eta^2 L_f^2}{\rho^3} \), we have \( \eta_{s+1} \leq 1 - \frac{\rho^3 (c_3^2 + c_1^3)}{72 \sqrt{2}(12)^2 \lambda^4 \eta^2 L_f} \). we have

\[
\begin{align*}
\sum_{s = s_t}^{s_t + q - 1} \eta_{s+1} K E \left( \| A_{s+1}^{-1}(w_t^k - \bar{w}_t^k) \|^2 + E \| B_{s+1}^{-1}(v_t^k - \bar{v}_t^k) \|^2 \right) \\
\leq \frac{8K}{15} \sum_{s = s_t}^{s_t + q - 1} \eta_{s+1} \left( \tau^2 \| A_{s+1}^{-1} \|_{\infty}^2 + E \| B_{s+1}^{-1} \|_{\infty} \right) \\
+ \frac{2K}{15 \lambda^2 L_f^2} (c_3^2 \sigma^2 + c_1^3 \sigma^2 + 3c_2^2 \delta_x^2 + 3c_2^2 \delta_y^2) \sum_{s = s_t}^{s_t + q - 1} \eta_{s+1}.
\end{align*}
\]
Theorem 3. (Restatement of Theorem 1) Assume the sequence \( \{\bar{x}_t, \bar{y}_t\}_{t=1}^T \) be generated by Algorithm 1. Under the above Assumptions 1,3-7, and let \( \eta_t = \frac{nK^{1/3}}{(m+1)\sqrt{T}} \) for all \( t \geq 0 \), \( \alpha_{t+1} = c_1 \eta_t^2 \), \( \beta_{t+1} = c_2 \eta_t^2 \), \( m \geq \max\left(2, n^3, (c_1 n)^3 K, (c_2 n)^3 K, K^{12/7}, mK^{1/2}L_f \right) \), \( n > 0 \), \( c_1^2 + c_2^2 \leq \frac{12\lambda^6 L_f^2}{\rho^n} \), \( c_1 \geq \frac{2}{\sqrt{m}K^{1/2}} + \frac{g_0}{4L_f \rho^n} \), \( c_2 \geq \frac{2}{\sqrt{m}K^{1/2}} + \frac{g}{\sqrt{m}K^{1/2}} \), \( \gamma = \tau \lambda \), \( \tau \leq \min\left(\frac{2\lambda}{\sqrt{m}K^{1/2}}, 1\right) \), \( \gamma \leq \min\left(\frac{m^{1/3}}{4L_f \rho^n}, 1\right) \), \( \lambda \leq \min\left(\frac{m^{1/3}}{4L_f \rho^n}, \frac{3\sqrt{m}K^{1/2}}{\lambda L_f} \right) \), \( 0 < \rho \leq 1 \) and \( 0 < \rho_u \leq \frac{1}{\sqrt{m}K^{1/2}} \), we have

\[
\frac{1}{T} \sum_{t=1}^{T} \mathbb{E}\|\nabla F(\bar{x}_t)\| \leq \sqrt{\frac{1}{T} \sum_{t=1}^{T} \mathbb{E}\|\nabla A_t\|^2 \left( \frac{\sqrt{3Gm^{1/6}}}{K^{1/6}\sqrt{T}^{1/2}} + \sqrt{\frac{3G}{K^{1/6}T^{3/2}}} \right)},
\]

where \( G = \frac{4\rho L_f}{\rho^n} F(\bar{x}_1) + \frac{36\rho_n L_f^2}{\rho^n \xi} \left( F(\bar{x}_1) - f(\bar{x}_1, \bar{y}_1) + \frac{8m^{1/3}a_n}{QK^{1/2}n^2} + 8K^{1/2} \left( \frac{c_1^2 + c_2^2}{\rho^n} \right) \right) \ln(m+1) \), \( \Delta = \frac{c_1^2}{2} + \frac{c_2^2}{2} + 3c_1^2 \delta^2 + 3c_2^2 \delta^2 \) and \( \lambda = \frac{1}{16} L_f^2 \rho^{n/2} + \frac{16c_1^2 L_f^2}{K^{1/2}} \).

Proof. According to Lemma 7, we have

\[
F(\bar{x}_{t+1}) \leq F(\bar{x}_t) + \frac{4\gamma L_f^2 \eta_t}{\rho \mu} \left( F(\bar{x}_t) - f(\bar{x}_t, \bar{y}_t) \right) + \frac{2\gamma \eta_t}{\rho} \sum_{k=1}^{K} \left( \frac{\gamma^2 \eta_t^2}{\rho^2} \sum_{k=1}^{K} \sum_{k=1}^{K} \mathbb{E}\|A_t^{-1}(w_t^k - \bar{w}_t)\|^2 \right) + \frac{8\gamma \eta_t}{\rho} \left( \frac{q - 1}{2} \right) \frac{L_f^2}{K} \left( \sum_{l=1}^{t-1} \frac{\lambda^2 \eta_t^2}{\rho^2} \sum_{k=1}^{K} \sum_{k=1}^{K} \mathbb{E}\|B_t^{-1}(v_t^k - \bar{v}_t)\|^2 \right).
\]

By combining the above inequalities (59) with (60), we have

\[
F(\bar{x}_{t+1}) \leq F(\bar{x}_t) + \frac{4\gamma L_f^2 \eta_t}{\rho \mu} \left( F(\bar{x}_t) - f(\bar{x}_t, \bar{y}_t) \right) + \frac{2\gamma \eta_t}{\rho} \sum_{k=1}^{K} \left( \frac{\gamma^2 \eta_t^2}{\rho^2} \sum_{k=1}^{K} \sum_{k=1}^{K} \mathbb{E}\|A_t^{-1}(w_t^k - \bar{w}_t)\|^2 \right) + \frac{8\gamma \eta_t}{\rho} \left( \frac{q - 1}{2} \right) \frac{L_f^2}{K} \left( \sum_{l=1}^{t-1} \frac{\lambda^2 \eta_t^2}{\rho^2} \sum_{k=1}^{K} \sum_{k=1}^{K} \mathbb{E}\|B_t^{-1}(v_t^k - \bar{v}_t)\|^2 \right).
\]

Since \( \eta_t = \frac{nK^{1/3}}{(m+1)\sqrt{T}} \) on \( t \) is decreasing and \( m \geq Kn^3 \), we have \( \eta_t \leq \eta_0 = \frac{nK^{1/3}}{(m+1)\sqrt{T}} \leq 1 \) and \( \gamma \leq \min\left(\frac{m^{1/3}}{4L_f \rho^n}, 1\right) \), \( \lambda \leq \min\left(\frac{m^{1/3}}{4L_f \rho^n}, \frac{3\sqrt{m}K^{1/2}}{\lambda L_f} \right) \), \( 0 < \rho \leq 1 \) and \( 0 < \rho_u \leq \frac{1}{\sqrt{m}K^{1/2}} \), for any \( t \geq 0 \). Similarly, we have \( 0 < \eta \leq \frac{m^{1/3}}{4L_f \rho^n} \leq \frac{1}{2L_f \eta_0 \rho_u} \). Since \( \eta_t \leq \frac{\rho}{12\lambda^2 L_f} \) for all \( t \geq 0 \), we have \( nK^{1/3} \leq n \leq \frac{\rho}{12\lambda^2 L_f} \), then we have \( m \geq \frac{K^{12/7} \lambda^2 L_f^3}{\rho^3} \). Due to \( 0 < \eta \leq 1 \) and \( m \geq (c_1 n)^3 K \), we have \( \alpha_{t+1} = c_1 \eta_t^2 \leq c_1 \eta_t \leq c_1 \eta_0 = \frac{c_1 nK^{1/3}}{m^{1/3}} \leq 1 \). Similarly, due to \( m \geq (c_1 n)^3 K \), we have \( \beta_{t+1} \leq 1 \).
According to Lemma 8, we have
\[
\begin{align*}
\frac{1}{\eta_t}E[\|\tilde{v}_{t+1} - \nabla_y f(x_{t+1}, y_{t+1})\|^2] &- \frac{1}{\eta_{t-1}}E[\|\tilde{v}_t - \nabla_y f(x_t, y_t)\|^2] \\
\leq (1 - \frac{\alpha_{t+1}}{\eta_t} - \frac{1}{\eta_{t-1}})E[\|\tilde{v}_t - \nabla_y f(x_t, y_t)\|^2] + \frac{4L_t^2}{K^2} \eta_t \sum_{k=1}^{K} (\|\tilde{x}_{t+1}^k - x_t^k\|^2 + \|\tilde{y}_{t+1}^k - y_t^k\|^2) + \frac{2\alpha_{t+1}^2 \sigma^2}{K \eta_t} \\
= (1 - \frac{1}{\eta_t} - c_1 \eta_t)E[\|\tilde{v}_t - \nabla_y f(x_t, y_t)\|^2] + \frac{4L_t^2}{K^2} \eta_t \sum_{k=1}^{K} (\|\tilde{x}_{t+1}^k - x_t^k\|^2 + \|\tilde{y}_{t+1}^k - y_t^k\|^2) + \frac{2\alpha_{t+1}^2 \sigma^2}{K}.
\end{align*}
\]
where the second equality is due to \(\alpha_{t+1} = c_1 \eta_t^2\). Similarly, we have
\[
\begin{align*}
\frac{1}{\eta_t}E[\|\tilde{w}_{t+1} - \nabla_x f(x_{t+1}, y_{t+1})\|^2] &- \frac{1}{\eta_{t-1}}E[\|\tilde{w}_t - \nabla_x f(x_t, y_t)\|^2] \\
\leq (1 - \frac{\beta_{t+1}}{\eta_t} - \frac{1}{\eta_{t-1}})E[\|\tilde{w}_t - \nabla_x f(x_t, y_t)\|^2] + \frac{4L_t^2}{K^2} \eta_t \sum_{k=1}^{K} (\|\tilde{x}_{t+1}^k - x_t^k\|^2 + \|\tilde{y}_{t+1}^k - y_t^k\|^2) + \frac{2\beta_{t+1}^2 \sigma^2}{K \eta_t} \\
= (1 - \frac{1}{\eta_t} - c_2 \eta_t)E[\|\tilde{w}_t - \nabla_x f(x_t, y_t)\|^2] + \frac{4L_t^2}{K^2} \eta_t \sum_{k=1}^{K} (\|\tilde{x}_{t+1}^k - x_t^k\|^2 + \|\tilde{y}_{t+1}^k - y_t^k\|^2) + \frac{2\beta_{t+1}^2 \sigma^2}{K}.
\end{align*}
\]
By \(\eta_t = \frac{K_{1/3}}{(m + t)^{1/3}}\), we have
\[
\begin{align*}
\frac{1}{\eta_t} &- \frac{1}{\eta_{t-1}} = \frac{1}{nK^{1/3}}((m + t)^{1/3} - (m + t - 1)^{1/3}) \leq \frac{1}{3nK^{1/3}(m + t - 1)^{2/3}} \leq \frac{1}{3nK^{1/3}(m/2 + t)^{2/3}} \\
&= \frac{2^{2/3}}{3nK^{1/3}(m + t)^{2/3}} \leq \frac{2^{2/3}}{3nK^{1/3}K^2/3} = \frac{2^{2/3}}{3nK^{1/3}} \eta_t \leq \frac{2}{3nK^{1/3} \eta_t},
\end{align*}
\]
where the first inequality holds by the concavity of function \(f(x) = x^{1/3}\), i.e., \((x + y)^{1/3} \leq x^{1/3} + \frac{y}{m + t}\); the second inequality is due to \(m \geq 2\), and the last inequality is due to \(0 < \eta_t \leq 1\).

Let \(c_1 \geq \frac{2}{3nK^{1/3}} + \frac{9L_t^2}{2\mu_0 n^2}\), we have
\[
\begin{align*}
\frac{1}{\eta_t}E[\|\tilde{v}_{t+1} - \nabla_y f(x_{t+1}, y_{t+1})\|^2] &- \frac{1}{\eta_{t-1}}E[\|\tilde{v}_t - \nabla_y f(x_t, y_t)\|^2] \\
\leq - \frac{9L_t^2}{2\mu_0 n^2}E[\|\tilde{v}_t - \nabla_y f(x_t, y_t)\|^2] + \frac{4L_t^2}{K^2} \eta_t \sum_{k=1}^{K} (\|\tilde{x}_{t+1}^k - x_t^k\|^2 + \|\tilde{y}_{t+1}^k - y_t^k\|^2) + \frac{2\alpha_{t+1}^2 \sigma^2}{K} \\
\leq - \frac{9L_t^2}{2\mu_0 n^2}E[\|\tilde{v}_t - \nabla_y f(x_t, y_t)\|^2] + \frac{4L_t^2}{K^2} \eta_t \sum_{k=1}^{K} (\|\tilde{x}_{t+1}^k - x_t^k\|^2 + \|\tilde{y}_{t+1}^k - y_t^k\|^2) + \frac{2\alpha_{t+1}^2 \sigma^2}{K} \\
+ \lambda^2 \|B_t^{-1}(v_t^k - \tilde{v}_t)\|^2 + \lambda^2 \|B_t^{-1}\tilde{v}_t\|^2 + \frac{2\sigma^2}{K}.
\end{align*}
\]
Let \(c_2 \geq \frac{2}{3nK^{1/3}} + \frac{9}{2}\), we have
\[
\begin{align*}
\frac{1}{\eta_t}E[\|\tilde{w}_{t+1} - \nabla_x f(x_{t+1}, y_{t+1})\|^2] &- \frac{1}{\eta_{t-1}}E[\|\tilde{w}_t - \nabla_x f(x_t, y_t)\|^2] \\
\leq - \frac{9L_t^2}{2\mu_0 n^2}E[\|\tilde{w}_t - \nabla_x f(x_t, y_t)\|^2] + \frac{4L_t^2}{K^2} \eta_t \sum_{k=1}^{K} (\|\tilde{x}_{t+1}^k - x_t^k\|^2 + \|\tilde{y}_{t+1}^k - y_t^k\|^2) + \frac{2\alpha_{t+1}^2 \sigma^2}{K} \\
\leq - \frac{9L_t^2}{2\mu_0 n^2}E[\|\tilde{w}_t - \nabla_x f(x_t, y_t)\|^2] + \frac{4L_t^2}{K^2} \eta_t \sum_{k=1}^{K} (\|\tilde{x}_{t+1}^k - x_t^k\|^2 + \|\tilde{y}_{t+1}^k - y_t^k\|^2) + \frac{2\alpha_{t+1}^2 \sigma^2}{K} \\
\leq - \frac{9L_t^2}{2\mu_0 n^2}E[\|\tilde{w}_t - \nabla_x f(x_t, y_t)\|^2] + \frac{8L_t^2}{K^2} \eta_t \sum_{k=1}^{K} (\gamma^2 \|A_t^{-1}(w_t^k - \tilde{w}_t)\|^2 + \gamma^2 \|A_t^{-1}\tilde{w}_t\|^2) + \frac{c_2 \sigma^2}{K}.
\end{align*}
\]
According to Lemma 6, we have

$$F(\bar{x}_{t+1}) - f(\bar{x}_{t+1}, \hat{y}_{t+1}) \leq (1 - \frac{\eta_t \lambda_t}{2\rho u})(F(\bar{x}_t) - f(\bar{x}_t, \hat{y}_t)) + \frac{\eta_t}{8} \|\bar{x}_{t+1} - \bar{x}_t\|^2 - \frac{\eta_t}{4\lambda u} \|\hat{y}_{t+1} - \hat{y}_t\|^2$$

$$+ \frac{\eta_t \lambda_t}{\rho u} \|\nabla_y f(\bar{x}_t, \hat{y}_t) - \hat{v}_t\|^2$$

$$= (1 - \frac{\eta_t \lambda_t}{2\rho u})(F(\bar{x}_t) - f(\bar{x}_t, \hat{y}_t)) + \frac{\gamma \eta_t}{8} \|A_t^{-1} \hat{w}_t\|^2 - \frac{\lambda u}{4\rho u} \|B_t^{-1} \hat{v}_t\|^2$$

$$+ \frac{\eta_t \lambda_t}{\rho u} \|\nabla_y f(\bar{x}_t, \hat{y}_t) - \nabla_y f(x_t, y_t)\| \|\hat{v}_t\|^2$$

$$\leq (1 - \frac{\eta_t \lambda_t}{2\rho u})(F(\bar{x}_t) - f(\bar{x}_t, \hat{y}_t)) + \frac{\gamma \eta_t}{8} \|A_t^{-1} \hat{w}_t\|^2 - \frac{\lambda u}{4\rho u} \|B_t^{-1} \hat{v}_t\|^2$$

$$+ 2\frac{\eta_t \lambda_t}{\rho u} \|\nabla_y f(\bar{x}_t, \hat{y}_t) - \nabla_y f(x_t, y_t)\| \|\hat{v}_t\|^2$$

$$\leq (1 - \frac{\eta_t \lambda_t}{2\rho u})(F(\bar{x}_t) - f(\bar{x}_t, \hat{y}_t)) + \frac{\gamma \eta_t}{8} \|A_t^{-1} \hat{w}_t\|^2 - \frac{\lambda u}{4\rho u} \|B_t^{-1} \hat{v}_t\|^2$$

$$+ \frac{4\eta_t \lambda_t L_t^2}{\rho u K} \sum_{k=1}^{K} \left(\|x_k - \bar{x}_t\|^2 + \|y_k - \hat{y}_t\|^2\right) + \frac{2\eta_t \lambda_t}{\rho u} \|\nabla_y f(x_t, y_t) - \hat{v}_t\|^2$$

$$\leq (1 - \frac{\eta_t \lambda_t}{2\rho u})(F(\bar{x}_t) - f(\bar{x}_t, \hat{y}_t)) + \frac{\gamma \eta_t}{8} \|A_t^{-1} \hat{w}_t\|^2 - \frac{\lambda u}{4\rho u} \|B_t^{-1} \hat{v}_t\|^2$$

$$+ \frac{4(q - 1) \eta_t \lambda_t L_t^2}{\rho u K} \left(\sum_{l=1}^{t-1} \gamma^2 \eta_l^2 \sum_{k=1}^{K} \mathbb{E}(\|A_t^{-1} (v_l^k - \hat{w}_l)\|^2) + \sum_{l=1}^{t-1} \lambda^2 \eta_l^2 \sum_{k=1}^{K} \mathbb{E}(\|B_t^{-1} (v_l^k - \hat{v}_l)\|^2)\right)$$

$$+ \frac{2\eta_t \lambda_t}{\rho u} \|\nabla_y f(x_t, y_t) - \hat{v}_t\|^2,$$  \hspace{1cm} (67)

Next, we define a potential function, for any $t \geq 1$

$$\Omega_t = \mathbb{E} \left[ F(\bar{x}_t) + \frac{9\rho u \gamma L_t^2}{\rho \lambda u^2} (F(\bar{x}_t) - f(\bar{x}_t, \hat{y}_t)) + \frac{\gamma}{\rho u \eta_t} (\|\hat{v}_t - \nabla_y f(x_t, y_t)\|^2 + \|\hat{w}_t - \nabla_y f(x_t, y_t)\|^2) \right].$$
Then we have

$$\Omega_{t+1} - \Omega_t = F(\bar{x}_{t+1}) - F(x_t) + \frac{9\rho u \gamma L_f^2}{\rho \lambda L} \left(F(\bar{x}_{t+1}) - f(\bar{x}_{t+1}, \bar{y}_{t+1}) - (F(x_t) - f(x_t, y_t))\right) + \frac{\gamma}{\rho} \left(\frac{1}{\eta} E\|\bar{v}_{t+1} - \nabla_y f(x_{t+1}, y_{t+1})\|^2 - \frac{1}{\eta_{t+1}} E\|\bar{v}_t - \nabla_y f(x_t, y_t)\|^2\right)$$

$$\leq (F(x_t) + \frac{4\gamma L_f^2}{\rho} \left(F(\bar{x}_{t+1}) - f(\bar{x}_{t+1}, \bar{y}_{t+1})\right) + \frac{4\gamma \eta}{\rho} E\|\bar{v}_t - \nabla_y f(x_t, y_t)\|^2 - \frac{\rho \gamma \eta}{2} \|A_1^{-1}\bar{v}_t\|^2$$

$$+ \frac{8\gamma \eta \frac{(q - 1) L_f^2}{\rho K}}{\|x_t\|^2} \left(\sum_{l = s_{t+1}}^{t-1} \frac{\gamma^2}{\eta^2} \sum_{k=1}^K E\|A_1^{-1}(w_k - \bar{w}_t)\|^2 + \sum_{l = s_{t+1}}^{t-1} \frac{\lambda^2}{\eta^2} \sum_{k=1}^K E\|B_1^{-1}(v_k - \bar{v}_t)\|^2\right)$$

$$+ \frac{9u \gamma L_f^2}{\rho \lambda L} \left(F(x_t) - f(x_t, y_t)\right) + \frac{\lambda \eta}{4\rho u} \|A_1^{-1}\bar{v}_t\|^2$$

$$+ \frac{4(q - 1) \eta \gamma L_f^2}{\rho K} \left(\sum_{l = s_{t+1}}^{t-1} \frac{\gamma^2}{\eta^2} \sum_{k=1}^K E\|A_1^{-1}(w_k - \bar{w}_t)\|^2 + \sum_{l = s_{t+1}}^{t-1} \frac{\lambda^2}{\eta^2} \sum_{k=1}^K E\|B_1^{-1}(v_k - \bar{v}_t)\|^2\right)$$

$$+ \frac{\gamma}{\rho} \left(- \frac{9\rho u \gamma L_f^2}{2\rho K} E\|\bar{v}_t - \nabla_y f(x_t, y_t)\|^2 + \frac{8\gamma L_f^2}{K^2 \eta} \sum_{k=1}^K (\gamma \|A_1^{-1}(w_k - \bar{w}_t)\|^2 + 2\gamma \|A_1^{-1}\bar{w}_t\|^2$$

$$+ \lambda^2 \|B_1^{-1}(v_k - \bar{v}_t)\|^2 + \frac{2\gamma \eta^2 \sigma^2}{K} E\|B_1^{-1}(v_k - \bar{v}_t)\|^2$$

$$+ \frac{2\gamma \eta^2 \sigma^2}{K} E\|B_1^{-1}(v_k - \bar{v}_t)\|^2 + \frac{2\gamma \eta^2 \sigma^2}{K} E\|B_1^{-1}(v_k - \bar{v}_t)\|^2\right)$$

$$= -\frac{\gamma L_f^2 \eta}{2\rho \gamma} \left(F(x_t) - f(x_t, y_t)\right) - \frac{\gamma L_f^2 \eta}{2\rho K} \left(\sum_{l = s_{t+1}}^{t-1} \frac{\gamma^2}{\eta^2} \sum_{k=1}^K E\|A_1^{-1}(w_k - \bar{w}_t)\|^2 + \frac{8\gamma L_f^2}{K^2 \eta} \sum_{k=1}^K (\gamma \|A_1^{-1}(w_k - \bar{w}_t)\|^2 + 2\gamma \|A_1^{-1}\bar{w}_t\|^2$$

$$+ \lambda^2 \|B_1^{-1}(v_k - \bar{v}_t)\|^2 + \frac{2\gamma \eta^2 \sigma^2}{K} E\|B_1^{-1}(v_k - \bar{v}_t)\|^2$$

$$+ \frac{2\gamma \eta^2 \sigma^2}{K} E\|B_1^{-1}(v_k - \bar{v}_t)\|^2\right)$$

where the above inequality holds by the above inequalities (61), (65), (66) and (67).

Here considering the term $\|\bar{v}_t - \nabla_y f(x_t, y_t)\|^2$, we have

$$\|\bar{v}_t - \nabla_y f(x_t, y_t)\|^2$$

$$= \|\bar{w}_t - \nabla_y f(x_t, y_t)\|^2 + \nabla_y f(x_t, y_t) - \nabla_y f(x_t, y_t)\|^2$$

$$\leq 2\|\bar{w}_t - \nabla_y f(x_t, y_t)\|^2 + \nabla_y f(x_t, y_t) - \nabla_y f(x_t, y_t)\|^2$$

$$\leq 2\|\bar{w}_t - \nabla_y f(x_t, y_t)\|^2 + \frac{1}{K} \sum_{k=1}^K \nabla_x f^k(x_t, y_t) - \frac{1}{K} \sum_{k=1}^K \nabla_x f^k(x_t, y_t)\|^2$$

$$\leq 2\|\bar{w}_t - \nabla_y f(x_t, y_t)\|^2 + \frac{4L_f^2}{K} \sum_{k=1}^K (\|x_k - \bar{x}_t\|^2 + \|y_k - \bar{y}_t\|^2)$$

$$\leq 2\|\bar{w}_t - \nabla_y f(x_t, y_t)\|^2 + \frac{4(q - 1) L_f^2}{K} \left(\sum_{l = s_{t+1}}^{t-1} \frac{\gamma^2}{\eta^2} \sum_{k=1}^K E\|A_1^{-1}(w_k - \bar{w}_t)\|^2 + \frac{8\gamma L_f^2}{K^2 \eta} \sum_{k=1}^K (\gamma \|A_1^{-1}(w_k - \bar{w}_t)\|^2 + 2\gamma \|A_1^{-1}\bar{w}_t\|^2$$

$$+ \lambda^2 \|B_1^{-1}(v_k - \bar{v}_t)\|^2 + \frac{2\gamma \eta^2 \sigma^2}{K} E\|B_1^{-1}(v_k - \bar{v}_t)\|^2\right), \quad (69)
then we obtain
\[-\|\bar{w}_t - \nabla_x f(x_t, y_t)\|^2\]
\[\leq -\frac{1}{2}\|\bar{w}_t - \nabla_x f(\bar{x}_t, \bar{y}_t)\|^2 + \frac{2(q-1)L_f^2}{K}\left(\sum_{l=\tilde{s}+1}^{t-1} \gamma_l^2 \eta_l^2 \sum_{k=1}^{\tilde{K}} \mathbb{E}\|A_t^{-1}(w_k^\ell - \bar{w}_t)\|^2 + \sum_{l=\tilde{s}+1}^{t-1} \lambda_l^2 \eta_l^2 \sum_{k=1}^{\tilde{K}} \mathbb{E}\|B_t^{-1}(v_k^\ell - \bar{v}_t)\|^2\right), \quad (70)\]

By combining the above inequalities 68 with 70, we can obtain
\[\Omega_{t+1} - \Omega_t \leq \frac{\gamma L_f^2 \eta}{2\rho K} (F(\bar{x}_t) - f(\bar{x}_t, \bar{y}_t)) - \frac{\gamma L_f^2 \eta}{2\rho K} \|\bar{w}_t - \nabla_x f(\bar{x}_t, \bar{y}_t)\|^2 - \left(\frac{\gamma L_f^2 \eta}{2\rho K} + \frac{9\rho \gamma L_f^2 \eta}{8\rho \lambda^2 \mu^2} - \frac{16\gamma^3 L_f^2 \eta}{\rho K}\right)\|A_t^{-1}\bar{w}_t\|^2\]
\[+ \left(\frac{\gamma L_f^2 \eta}{2\rho K} + \frac{36(q-1)\eta L_f^2 \gamma \rho_u}{\mu^2 K \rho}\right) \sum_{l=\tilde{s}+1}^{t-1} \left(\gamma_l^2 \eta_l^2 \sum_{k=1}^{\tilde{K}} \mathbb{E}\|A_t^{-1}(w_k^\ell - \bar{w}_t)\|^2 + \sum_{l=\tilde{s}+1}^{t-1} \lambda_l^2 \eta_l^2 \sum_{k=1}^{\tilde{K}} \mathbb{E}\|B_t^{-1}(v_k^\ell - \bar{v}_t)\|^2\right)\]
\[- \left(\frac{3\gamma L_f^2 \eta}{4\rho^2 K} + \frac{16\gamma L_f^2 \eta}{K^2 \rho}\right) \sum_{k=1}^{\tilde{K}} \|A_t^{-1}(w_k^\ell - \bar{w}_t)\|^2 + \frac{16\gamma L_f^2 \eta}{K^2 \rho} \sum_{k=1}^{\tilde{K}} \|B_t^{-1}(v_k^\ell - \bar{v}_t)\|^2. \quad (71)\]

Let \(s_t = q|l/q|\), summing the above inequality (71) over \(t = s_t\) to \(t = s_t + q - 1\), we have
\[\sum_{t=s_t}^{s_t+q-1} \left(\Omega_{t+1} - \Omega_t\right) \leq \sum_{t=s_t}^{s_t+q-1} \left(\frac{\gamma L_f^2 \eta}{2\rho K} (F(\bar{x}_t) - f(\bar{x}_t, \bar{y}_t)) - \frac{\gamma L_f^2 \eta}{2\rho K} \|\bar{w}_t - \nabla_x f(\bar{x}_t, \bar{y}_t)\|^2 - \left(\frac{\gamma L_f^2 \eta}{2\rho K} + \frac{9\rho \gamma L_f^2 \eta}{8\rho \lambda^2 \mu^2} - \frac{16\gamma^3 L_f^2 \eta}{\rho K}\right)\|A_t^{-1}\bar{w}_t\|^2\]
\[+ \left(\frac{\gamma L_f^2 \eta}{2\rho K} + \frac{36(q-1)\eta L_f^2 \gamma \rho_u}{\mu^2 K \rho}\right) \sum_{l=\tilde{s}+1}^{t-1} \left(\gamma_l^2 \eta_l^2 \sum_{k=1}^{\tilde{K}} \mathbb{E}\|A_t^{-1}(w_k^\ell - \bar{w}_t)\|^2 + \sum_{l=\tilde{s}+1}^{t-1} \lambda_l^2 \eta_l^2 \sum_{k=1}^{\tilde{K}} \mathbb{E}\|B_t^{-1}(v_k^\ell - \bar{v}_t)\|^2\right)\]
\[- \left(\frac{3\gamma L_f^2 \eta}{4\rho^2 K} + \frac{16\gamma L_f^2 \eta}{K^2 \rho}\right) \sum_{k=1}^{\tilde{K}} \|A_t^{-1}(w_k^\ell - \bar{w}_t)\|^2 + \frac{16\gamma L_f^2 \eta}{K^2 \rho} \sum_{k=1}^{\tilde{K}} \|B_t^{-1}(v_k^\ell - \bar{v}_t)\|^2. \quad (72)\]
where the second inequality is due to $\lambda \geq \gamma > 0$ and $\eta_t \leq \frac{\rho}{\gamma \eta_t}$ for all $t \geq 1$.

According to the above inequality (72), we have

\[
\sum_{t=s_1}^{s_1+q-1} (\Omega_{t+1} - \Omega_t) \leq \sum_{t=s_1}^{s_1+q-1} \left( -\frac{L_7^2\eta_t}{2\mu \rho} (f(x_t) - f(x, y_t)) - \frac{\gamma \eta_t}{A} \|w_t - \nabla_x f(x_t, x_t)\|^2 \right. - \left( \frac{\eta_t \gamma \rho}{2} - \frac{9\rho_u \gamma^2 L_7^2 \eta_t}{8\rho \lambda \mu^2} - \frac{16\gamma^3 L_7^2 \eta_t}{\rho K} \right) \|A^{-1} w_t\|^2 + \frac{9\rho_u \gamma^2 L_7^2 \eta_t}{8\rho \lambda \mu^2} - \frac{16\gamma^3 L_7^2 \eta_t}{\rho K} \|A^{-1} w_t\|^2 \\
+ \frac{\rho}{16} + \frac{L_7^2 \rho \rho_u}{4\mu^2} + \frac{16\gamma^3 L_7^2 \eta_t}{8\rho \lambda \mu^2} - \frac{9\rho_u \gamma^2 L_7^2 \eta_t}{8\rho \lambda \mu^2} - \frac{16\gamma^3 L_7^2 \eta_t}{\rho K} \|A^{-1} w_t\|^2 \\
+ \left( \frac{\rho}{16} + \frac{L_7^2 \rho \rho_u}{4\mu^2} + \frac{16\gamma^3 L_7^2 \eta_t}{8\rho \lambda \mu^2} - \frac{9\rho_u \gamma^2 L_7^2 \eta_t}{8\rho \lambda \mu^2} - \frac{16\gamma^3 L_7^2 \eta_t}{\rho K} \right) \|A^{-1} w_t\|^2 \\
\leq \sum_{t=s_1}^{s_1+q-1} \left( -\frac{L_7^2\eta_t}{2\mu \rho} (f(x_t) - f(x, y_t)) - \frac{\gamma \eta_t}{A} \|w_t - \nabla_x f(x_t, x_t)\|^2 \right. - \left( \frac{\eta_t \gamma \rho}{2} - \frac{9\rho_u \gamma^2 L_7^2 \eta_t}{8\rho \lambda \mu^2} - \frac{16\gamma^3 L_7^2 \eta_t}{\rho K} \right) \|A^{-1} w_t\|^2 + \frac{9\rho_u \gamma^2 L_7^2 \eta_t}{8\rho \lambda \mu^2} - \frac{16\gamma^3 L_7^2 \eta_t}{\rho K} \|A^{-1} w_t\|^2 \\
+ \frac{\rho}{16} + \frac{L_7^2 \rho \rho_u}{4\mu^2} + \frac{16\gamma^3 L_7^2 \eta_t}{8\rho \lambda \mu^2} - \frac{9\rho_u \gamma^2 L_7^2 \eta_t}{8\rho \lambda \mu^2} - \frac{16\gamma^3 L_7^2 \eta_t}{\rho K} \|A^{-1} w_t\|^2 \\
+ \left( \frac{\rho}{16} + \frac{L_7^2 \rho \rho_u}{4\mu^2} + \frac{16\gamma^3 L_7^2 \eta_t}{8\rho \lambda \mu^2} - \frac{9\rho_u \gamma^2 L_7^2 \eta_t}{8\rho \lambda \mu^2} - \frac{16\gamma^3 L_7^2 \eta_t}{\rho K} \right) \|A^{-1} w_t\|^2 \\
= \sum_{t=s_1}^{s_1+q-1} \left( \gamma \eta_t \rho^2 - \frac{9\rho_u \gamma^2 L_7^2 \eta_t}{8\rho \lambda \mu^2} - \frac{16\gamma^3 L_7^2 \eta_t}{\rho K} \right) \|A^{-1} w_t\|^2 \\
\geq \frac{\gamma}{2} \eta_t \rho^2 - \frac{9\rho_u \gamma^2 L_7^2 \eta_t}{8\rho \lambda \mu^2} - \frac{16\gamma^3 L_7^2 \eta_t}{\rho K} \|A^{-1} w_t\|^2.
\]

where the second inequality holds by Lemma 11.
Summing the above inequality (76) from $t = 1$ to $T$, then we have

\[
\sum_{t=1}^{T} (\Omega_{t+1} - \Omega_t)
\leq -\frac{\gamma L^2}{2\rho \mu} \sum_{t=1}^{T} \eta_t (F(\bar{x}_t) - f(\bar{x}_t, \bar{y}_t)) - \frac{\gamma}{4\rho} \sum_{t=1}^{T} \eta_t \|\bar{w}_t - \nabla_x f(\bar{x}_t, \bar{x}_t)\|^2 - \frac{7\rho^2}{4} \sum_{t=1}^{T} \eta_t \|\bar{A}_t^{-1} \bar{w}_t\|^2
\]

\[+ \frac{1}{K\rho} \sum_{t=1}^{T} \eta_t^3 + \frac{2\rho \gamma \Delta}{15KL^2} \sum_{t=1}^{T} \eta_t^3. \tag{77}\]

Since $v^k_t = \frac{1}{q} \sum_{j=1}^{q} \nabla_y f^k(x^k_t, y^k_t; \xi^k_{t,j})$, and $w^k_t = \frac{1}{q} \sum_{j=1}^{q} \nabla_x f^k(x^k_t, y^k_t; \xi^k_{t,j})$, we have

\[
\Omega_t = \mathbb{E}\left[F(\bar{x}_1) + \frac{9\rho \gamma L^2}{\rho \mu^2} (F(\bar{x}_1) - f(\bar{x}_1, \bar{y}_1)) + \frac{\gamma}{\rho \mu^2} \left(\|\bar{v}_t - \nabla_y f(\bar{x}_1, \bar{y}_1)\|^2 + \|\bar{w}_t - \nabla_x f(\bar{x}_1, \bar{y}_1)\|^2\right)\right]
\leq F(\bar{x}_1) + \frac{9\rho \gamma L^2}{\rho \mu^2} (F(\bar{x}_1) - f(\bar{x}_1, \bar{y}_1)) + \frac{2\gamma \sigma^2}{qK\rho \mu^2}, \tag{78}\]

where the last inequality holds by Assumption 4.

Since $\eta_t = \frac{n^{K^{1/3}}}{(m+t)^{1/3}}$ is decreasing, i.e., $\eta_t^{-1} \geq \eta_{t+1}^{-1}$ for any $0 \leq t \leq T$, we have

\[
\frac{1}{T} \sum_{t=1}^{T} \mathbb{E}\left[\frac{2L^2}{\rho \mu^2} (F(\bar{x}_t) - f(\bar{x}_t, \bar{y}_t)) + \frac{1}{\rho^2} \|\bar{w}_t - \nabla_x f(\bar{x}_t, \bar{x}_t)\|^2 + \|\bar{A}_t^{-1} \bar{w}_t\|^2\right]
\leq \frac{4}{T \rho \eta^3} \left(\Omega_t - \Omega_{t+1}\right) + \left(\frac{c_1^2 + c_2^2}{\rho^2 K} + \frac{\Lambda \Delta}{15KL^2} \right) \frac{8}{T \eta^3} \sum_{t=1}^{T} \eta^3
\leq \frac{4}{T \rho \eta^3} \left(F(\bar{x}_1) - F^* + \frac{9\rho \gamma L^2}{\rho \mu^2} (F(\bar{x}_1) - f(\bar{x}_1, \bar{y}_1)) + \frac{2\gamma \sigma^2}{qK\rho \mu^2} + \frac{(c_1^2 + c_2^2)\sigma^2}{\rho^2 K} + \frac{\Lambda \Delta}{15KL^2} \right) \frac{8}{T \eta^3} \sum_{t=1}^{T} \eta^3
\leq \frac{4}{T \rho \eta^3} \left(F(\bar{x}_1) - F^* + \frac{9\rho \gamma L^2}{\rho \mu^2} (F(\bar{x}_1) - f(\bar{x}_1, \bar{y}_1)) + \frac{2\gamma \sigma^2}{qK\rho \mu^2} + \frac{(c_1^2 + c_2^2)\sigma^2}{\rho^2 K} + \frac{\Lambda \Delta}{15KL^2} \right) \frac{8K^3}{T \eta^3} \ln(m+t)
\leq \frac{4}{T \rho \eta^3} \left(F(\bar{x}_1) - F^* + \frac{9\rho \gamma L^2}{\rho \mu^2} (F(\bar{x}_1) - f(\bar{x}_1, \bar{y}_1)) + \frac{2\gamma \sigma^2}{qK\rho \mu^2} + \frac{(c_1^2 + c_2^2)\sigma^2}{\rho^2 K} + \frac{\Lambda \Delta}{15KL^2} \right) \frac{8K^3}{T \eta^3} \ln(m+t)
= \frac{4F(\bar{x}_1) - F^* + 36\rho \mu L^2}{\rho \mu^2 n} \cdot \frac{1}{m+t} \cdot \frac{1}{K^{1/3}T^{1/3}},
\]

where the second inequality holds by the above inequality (78).

Let $G = \frac{4F(\bar{x}_1) - F^* + 36\rho \mu L^2}{\rho \mu^2 n} \cdot \frac{1}{m+t} \cdot \frac{1}{K^{1/3}T^{1/3}} \ln(m+t)$, we have

\[
\frac{1}{T} \sum_{t=1}^{T} \mathbb{E}\left[\frac{2L^2}{\rho \mu^2} (F(\bar{x}_t) - f(\bar{x}_t, \bar{y}_t)) + \frac{1}{\rho^2} \|\bar{w}_t - \nabla_x f(\bar{x}_t, \bar{x}_t)\|^2 + \|\bar{A}_t^{-1} \bar{w}_t\|^2\right] \leq \frac{G}{K^{1/3}T} (m + T)^{1/3}. \tag{80}\]

We define a useful metric

\[
\mathcal{M}_1 = \frac{1}{\rho} \left(\frac{\|\nabla \bar{x}_t\|}{\sqrt{T}} \right) \cdot \frac{\sqrt{F(\bar{x}_t) - f(\bar{x}_t, \bar{y}_t) + \|\nabla_x f(\bar{x}_t, \bar{x}_t) - \bar{w}_t\|}}{/K^{1/3}T} + \|\bar{A}_t^{-1} \bar{w}_t\|}. \tag{81}\]

According to the above inequality 80, we have

\[
\frac{1}{T} \sum_{t=1}^{T} \mathbb{E}[\mathcal{M}_1^2] \leq \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}\left[\frac{6L^2}{\rho^2 \mu} (F(\bar{x}_t) - f(\bar{x}_t, \bar{y}_t)) + \frac{3}{\rho^2} \|\bar{w}_t - \nabla_x f(\bar{x}_t, \bar{x}_t)\|^2 + 3\|\bar{A}_t^{-1} \bar{w}_t\|^2\right] \leq \frac{3G}{K^{1/3}T} (m + T)^{1/3}. \tag{82}\]
Let $F(\bar{x}_t) = f(\bar{x}_t, y^*(\bar{x}_t)) = \max_y f(\bar{x}_t, y)$. According to the Lemma 2, i.e., $\nabla F(\bar{x}_t) = \nabla_x f(\bar{x}_t, y^*(\bar{x}_t))$, we have
\[
\|\nabla F(\bar{x}_t) - \bar{w}_t\| = \|\nabla_x f(\bar{x}_t, y^*(\bar{x}_t)) - \bar{w}_t\| = \|\nabla_x f(\bar{x}_t, y^*(\bar{x}_t)) - \nabla_x f(\bar{x}_t, y) + \nabla_x f(\bar{x}_t, y) - \bar{w}_t\|
\leq \|\nabla_x f(\bar{x}_t, y^*(\bar{x}_t)) - \nabla_x f(\bar{x}_t, y)\| + \|\nabla_x f(\bar{x}_t, y) - \bar{w}_t\|
\leq L_f \|y^*(\bar{x}_t) - y\| + \|\nabla_x f(\bar{x}_t, y) - \bar{w}_t\|.
\]
(83)

Meanwhile, according to the Lemma 2, we have
\[
F(\bar{x}_t) - f(\bar{x}_t, y_t) = f(\bar{x}_t, y^*(\bar{x}_t)) - f(\bar{x}_t, y_t) = \max_y f(\bar{x}_t, y) - f(\bar{x}_t, y_t) \geq \frac{\mu}{2} \|y^*(\bar{x}_t) - y_t\|^2,
\]
then we can obtain
\[
\frac{\sqrt{2}}{\sqrt{\mu}} \sqrt{F(\bar{x}_t) - f(\bar{x}_t, y_t)} \geq \|y^*(\bar{x}_t) - y_t\|.
\]
(84)

Thus we have
\[
\mathcal{M}_t = \|A_t^{-1} \bar{w}_t\| + \frac{1}{\rho} \left( \frac{\sqrt{2} L_f}{\sqrt{\mu}} \sqrt{F(\bar{x}_t) - f(\bar{x}_t, y_t)} + \|\nabla_x f(\bar{x}_t, y) - \bar{w}_t\| \right)
\geq \|A_t^{-1} \bar{w}_t\| + \frac{1}{\rho} \left( L_f \|y^*(\bar{x}_t) - y_t\| + \|\nabla_x f(\bar{x}_t, y) - \bar{w}_t\| \right)
\geq \|A_t^{-1} \bar{w}_t\| + \frac{1}{\rho} \|\nabla F(\bar{x}_t) - \bar{w}_t\|
= \frac{1}{\|A_t\|} \|A_t\| \|A_t^{-1} \bar{w}_t\| + \frac{1}{\rho} \|\nabla F(\bar{x}_t) - \bar{w}_t\|
\geq \frac{1}{\|A_t\|} \|\bar{w}_t\| + \frac{1}{\rho} \|\nabla F(\bar{x}_t) - \bar{w}_t\|
\geq \left(\frac{1}{\|A_t\|} \|\bar{w}_t\| + \frac{1}{\rho} \|\nabla F(\bar{x}_t) - \bar{w}_t\| \right)
\geq \frac{1}{\|A_t\|} \|\nabla F(\bar{x}_t)\|,
\]
(85)

where the above inequality (i) holds by $\|A_t\| \geq \rho$ for all $t \geq 1$ due to Assumption 7. Then we have
\[
\|\nabla F(\bar{x}_t)\| \leq \mathcal{M}_t \|A_t\|.
\]
(86)

According to Cauchy-Schwarz inequality, we have
\[
\frac{1}{T} \sum_{t=1}^{T} \mathbb{E}[\|\nabla F(\bar{x}_t)\|] \leq \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}[\mathcal{M}_t \|A_t\|] \leq \left( \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}[\mathcal{M}_t^2] \right)^{1/2} \left( \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}[\|A_t\|^2] \right)^{1/2}. \quad (87)
\]

By plugging the above inequalities (82) into (87), we can obtain
\[
\frac{1}{T} \sum_{t=1}^{T} \mathbb{E}[\|\nabla F(\bar{x}_t)\|] \leq \sqrt{\frac{1}{T} \sum_{t=1}^{T} \mathbb{E}[\|A_t\|^2 \left( \frac{\sqrt{3G^{1/6}T^{1/2}}}{K^{1/6}T^{1/2}} + \frac{\sqrt{3G^{1/6}T^{1/2}}}{K^{1/6}T^{1/2}} \right)}. \quad (88)
\]