Two stable modifications of the finite section method

MARKO LINDNER

February 24, 2010

Abstract. In this article we demonstrate and compare two modified versions of the classical finite section method for band-dominated operators in case the latter is not stable. For both methods we give explicit criteria for their applicability.

Mathematics subject classification (2000): 65J10; 47N40, 47L40.

Keywords and phrases: finite section method, projection methods, stability.

1 Introduction

Infinite Matrices. In this paper, we look at truncation methods for the approximate solution of certain operator equations \( Au = b \) on the space \( E := \ell^p(\mathbb{Z}^N, X) \) of functions \( u : \mathbb{Z}^N \to X \) with

\[
\|u\| = \begin{cases} 
\sqrt[p]{\sum_{k \in \mathbb{Z}^N} |u(k)|^p}, & p \in [1, \infty) \\
\sup_{k \in \mathbb{Z}^N} |u(k)|, & p = \infty \end{cases} < \infty,
\]

where \( N \in \mathbb{N}, p \in [1, \infty] \) and \( X \) is an arbitrary complex Banach space. The operators \( A \) that we have in mind are bounded linear operators \( E \to E \) which are induced, via

\[
(Au)(i) = \sum_{j \in \mathbb{Z}^N} a_{ij} u(j), \quad i \in \mathbb{Z}^N,
\]

by a matrix \( (a_{ij})_{i,j \in \mathbb{Z}^N} \) with operator entries \( a_{ij} : X \to X \). Among those operators we call \( A \) a band operator if it is induced by a banded matrix, i.e. \( a_{ij} = 0 \) if \( |i - j| \) is large enough, and we call \( A \) a band-dominated operator and write \( A \in \text{BDO}(E) \) if \( A \) is the limit, with respect to the operator norm induced by the norm on \( E \), of a sequence of band operators. Also for \( A \in \text{BDO}(E) \), there is a unique (see [18, §2.1.2] or [10, §1.3.5]) matrix \( (a_{ij})_{i,j \in \mathbb{Z}^N} \) which induces \( A \) via (1); we denote it by \( [A] \).

Finite Sections. If \( A \in \text{BDO}(E) \) is invertible then \( Au = b \) has a unique solution \( u \in E \) for every right-hand side \( b \in E \). An exact computation of \( u \), however, is in general not possible which is why one uses approximation methods. One of the most popular
approximation methods is as follows: Choose a sequence \( \Omega_1 \subset \Omega_2 \subset \cdots \) of finite subsets of \( \mathbb{Z}^N \) that eventually covers every point of \( \mathbb{Z}^N \) and replace the infinite system

\[
Au = b \quad \text{i.e.} \quad \sum_{j \in \mathbb{Z}^N} a_{ij} u(j) = b(i), \quad i \in \mathbb{Z}^N
\]

by the sequence of finite systems

\[
\sum_{j \in \Omega_n} a_{ij} \tilde{u}_n(j) = b(i), \quad i \in \Omega_n
\]  \( (2) \)

for \( n = 1, 2, \ldots \). This procedure is called the finite section method (FSM). The FSM is called applicable if there exists an \( n_0 \in \mathbb{N} \) such that, for every \( b \in E \), \( (2) \) is uniquely solvable for all \( n \geq n_0 \) and if the sequence \( (\tilde{u}_n) \) of solutions is bounded in \( E \) and converges componentwise to the exact solution \( u \) of \( Au = b \) as \( n \to \infty \).

Here is how we will choose the finite sets \( \Omega_1, \Omega_2, \ldots \) in \( (2) \):

**Definition 1.1** We will say that \( \Omega \subset \mathbb{R}^N \) is a valid starlike set if \( \Omega \) is bounded, nonempty and has the property that, for every \( x \in \Omega \) and \( \alpha \in [0, 1) \), \( \alpha x \) is an interior point of \( \Omega \).

So in particular, 0 is an interior point of every valid starlike set. Moreover, all bounded convex sets \( \Omega \subset \mathbb{R}^N \) with interior point 0 are valid starlike sets. Now, for every \( n \in \mathbb{N} \), put

\[
\Omega_n := n\Omega \cap \mathbb{Z}^N \quad \text{and} \quad P_n := P_{\Omega_n},
\]

where, for a set \( U \subseteq \mathbb{Z}^N \), by \( P_U : E \to E \) we denote the operator of multiplication by the characteristic function \( \chi_U \) of \( U \). Then we can abbreviate \( (2) \) as

\[
P_n A P_n \tilde{u}_n = P_n b, \quad n = 1, 2, \ldots \]  \( (4) \)

This truncation procedure is a very natural idea and the fact that it can be performed on all infinite matrices creates the temptation to simply use it and keep fingers crossed it will work. A positive outcome, however, i.e. applicability as defined above, is in general far from guaranteed. Here is the probably most elementary example for which the FSM fails to apply:

**Example 1.2** Consider the shift operator \( A = V_c : u \mapsto v \) on \( E \) with \( u(k) = v(k+c) \) for every \( k \in \mathbb{Z}^N \) and a fixed nonzero vector \( c \in \mathbb{Z}^N \). Then \( V_c \) is invertible on \( E \) but since \( V_c \) maps functions with support in \( \Omega_n \) to functions supported in \( \Omega_n + c \), the truncated equation \( (2) \) alias \( (4) \) is not solvable for general right-hand sides (and even if it is solvable, the solution is not unique) – no matter how big \( n \) is and how \( \Omega \) is chosen. \( \Box \)

Here is a slightly more sophisticated example:

**Example 1.3** Let \( N = 1 \) and consider the operator \( A \) induced by the block diagonal matrix

\[
\text{diag} \left( \cdots, \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right), \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right), \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right), \cdots \right)
\]
with the single 1 entry at position zero. Then \( A = A^{-1} \) is invertible and, for \( \Omega = [-1, 1] \), its truncations \( P_nAP_n \) correspond to the finite \((2n + 1) \times (2n + 1)\) matrices
\[
\text{diag}\left(\left(\begin{array}{cc}0 & 1 \\ 1 & 0\end{array}\right), \ldots, \left(\begin{array}{cc}0 & 1 \\ 1 & 0\end{array}\right), 1, \left(\begin{array}{cc}0 & 1 \\ 1 & 0\end{array}\right), \ldots, \left(\begin{array}{cc}0 & 1 \\ 1 & 0\end{array}\right)\right)
\]
if \( n \) is even and to
\[
\text{diag}\left(0, \left(\begin{array}{cc}0 & 1 \\ 1 & 0\end{array}\right), \ldots, \left(\begin{array}{cc}0 & 1 \\ 1 & 0\end{array}\right), 1, \left(\begin{array}{cc}0 & 1 \\ 1 & 0\end{array}\right), \ldots, \left(\begin{array}{cc}0 & 1 \\ 1 & 0\end{array}\right), 0\right)
\]
if \( n \) is odd. So the FSM (4) is not applicable since all operators \( P_nAP_n|_{\text{im } P_n} \) with an odd \( n \) are non-invertible. □

By [10, Corollary 1.77] (which is a consequence of [18, Theorem 6.1.3]) one has that the FSM (4) is applicable iff \( A \) is invertible and the sequence
\[
(P_nAP_n + Q_n)_{n \in \mathbb{N}}
\]
(5)
is stable. Here we have put \( Q_n := I - P_n \) and we call a sequence \((A_n)_{n \in \mathbb{N}}\) of operators \( A_n : E \to E \) stable if there exists an \( n_0 \in \mathbb{N} \) such that all operators \( A_n \) with \( n \geq n_0 \) are invertible and \( \sup_{n \geq n_0} \|A_n^{-1}\| \) is finite. Also note that \( P_nAP_n + Q_n \) is invertible on \( E \) iff \( P_nAP_n \) is invertible on the image of \( P_n \) and that \( \|(P_nAP_n + Q_n)^{-1}\| = \max(1, \|(P_nAP_n|_{\text{im } P_n})^{-1}\|) \).

The aim of this paper is to demonstrate two strategies, originally developed in [20, 12] and [5], that can be used if the FSM (4) is not applicable:

**Strategy 1: Pass to a subsequence.** As we have just seen, the finite section method cannot be expected to work for every operator \( A \). But in some cases it is possible to “adjust” the method to the operator at hand by choosing the right geometry \( \Omega \) and an appropriate subsequence of (5). The philosophy here is to give the operator \( A \) the chance to impose some of its “personality” on the (otherwise too “impersonal”) method of finite sections. In the previous example, for instance, one simply has to remove all elements from the sequence (5) that correspond to an odd value of \( n \) to get a stable approximation method for \( A \) (or alternatively, one could replace \( \Omega = [-1, 1] \) by \([-2, 2] \) and work with the whole sequence (5)). We believe that, for a given operator \( A \), finding the right geometry \( \Omega \) and an appropriate sequence \( n_1, n_2, \ldots \) of natural numbers such that the corresponding subsequence of finite sections \( P_{n_i}AP_{n_i} \) is stable (meaning that (4) is only expected to be uniquely solvable, with solutions \( \tilde{u}_n \) convergent to \( u \), for a particular sequence \( n = n_1, n_2, \ldots \)) is a major task in the numerical analysis of the equation \( Au = b \).

We will show that, under an additional condition on the operator \( A \), the finite section subsequence \((P_{n_i}AP_{n_i})_{i=1}^{\infty}\) is stable iff \( A \) and every element from an associated set of operators is invertible and the inverses are uniformly bounded. We give a description of this associated set that depends on \( A, \Omega \) and the sequence \((n_i)_{i=1}^{\infty}\).

**Strategy 2: Use rectangular instead of square systems.** As an alternative approach to the FSM (2), we discuss the slightly modified truncation scheme \( P_mAP_n \tilde{u}_{m,n} \approx P_mb \), i.e.
\[
\sum_{j \in \Omega_n} a_{ij} \tilde{u}_{m,n}(j) \approx b(i), \quad i \in \Omega_m,
\]
(6)
leading to rectangular instead of quadratic finite subsystems of \(Au = b\) that are now to be solved approximately instead of exactly.

We prove that if \(A\) is induced by a matrix \((a_{ij})\) with \(\|a_{ij}\| \to 0\) as \(|i| \to \infty\) for every \(j\) and if \(A\) is invertible then the modified method (6) is applicable. By the latter we mean that, for every \(\varepsilon > 0\) and every \(b \in E\), there exist \(m_0, n_0 \in \mathbb{N}\) and a precision \(\delta > 0\) such that all (approximate) solutions of the rectangular system \(\|P_m AP_n u_{m,n} - P_m b\| < \delta\) with \(m > m_0\) and \(n > n_0\) are in the \(\varepsilon\)-neighbourhood of the exact solution \(u\) of \(Au = b\).

We also discuss how the two truncation parameters \(m\) and \(n\) are to be coupled.

**Short History.** The idea of the FSM is so natural that it is difficult to give a historical starting point. First rigorous treatments are from Baxter [1] and Gohberg & Feldman [3] on Wiener-Hopf and convolution operators in dimension \(N = 1\) in the early 1960's. For convolution equations in higher dimensions \(N \geq 2\), the FSM goes back to Kozak & Simonenko [7, 8], and for general band-dominated operators with scalar [15] and operator-valued [16, 17] coefficients, most results are due to Rabinovich, Roch & Silbermann. For the state of the art in the scalar case for \(p = 2\), see [23].

The quest for stable subsequences if the FSM itself is unstable is getting more attention recently [19, 20, 24, 25, 12]. In [20], the stability theorem for subsequences is used to remove the uniform boundedness condition in dimension \(N = 1\). Also the consideration of rectangular finite sections, although not new in the numerical community, is now gaining more focus in the numerical functional analysis literature (see [6, 26] for Toeplitz operators, [24, 25] for band-dominated operators and [5] for even more general operators).

## 2 Strategy One: Stable Subsequences of the FSM

### 2.1 Preliminaries

Let \(E = \ell^p(\mathbb{Z}^N, X)\), \(A \in \text{BDO}(E)\) and \(\Omega \subset \mathbb{R}^N\) be a valid starlike set as in Definition 1.1. For an infinite index set \(I = \{n_1, n_2, \ldots\} \subseteq \mathbb{N}\), we study the stability of the operator sequence

\[
(P_n AP_n + Q_n)_{n \in I} = (P_n, AP_n + Q_n)_{i=1}^\infty,
\]

where we suppose that \(n_1, n_2, \ldots\) is a strictly monotonous enumeration of \(I\). For the study of this sequence as one item, we will assemble it to a single operator. To do this, let

\[
A_i := \begin{cases} 
P_n, AP_n + Q_n, & i \in \mathbb{N}, \\
I, & i \in \mathbb{Z} \setminus \mathbb{N},
\end{cases}
\]

put \(E' := \ell^p(\mathbb{Z}^{N+1}, X)\), thought of as \(\ell^p(\mathbb{Z}, E)\), and write \(\oplus A_i\) for the map \(u \mapsto v\) on \(E'\) with

\[
v(j, i) = (A_i u(\cdot, i))(j), \quad j \in \mathbb{Z}^N, i \in \mathbb{Z}.
\]

In other words, we think of \(u \in E'\) as decomposed into layers \(u(\cdot, i) \in E, i \in \mathbb{Z}\), and let each \(A_i\) act on the \(i\)-th layer of \(u\). We will therefore refer to \(A_i\) as the \(i\)-th layer of \(\oplus A_i\). One can show that then \(\oplus A_i \in \text{BDO}(E')\).
A key argument in [17], refined later in [9, 10, 18, 23], is that the stability of (7) is equivalent to \( \oplus A_i \) being invertible at infinity. Here we say that an operator \( B \in \text{BDO}(E') \) is invertible at infinity if there exist \( C, D \in \text{BDO}(E') \) and \( m \in \mathbb{N} \) such that \( CB\Theta_m = \Theta_mBD \) holds, where \( \Theta_m \) is the operator of multiplication by the characteristic function of \( \mathbb{Z}^{N+1} \setminus \{-m, \ldots, m\}^{N+1} \).

So it remains to study invertibility at infinity of \( \oplus A_i \). This is done in terms of so-called limit operators \([18, 2, 10]\). The idea is to reflect the behaviour of an operator \( B \in \text{BDO}(E') \) at infinity by a family of operators on \( E' \) and to evaluate this family. To do this, we need two notations. Firstly, for \( B, B_1, B_2, \ldots \in \text{BDO}(E') \), we write \( B = \mathcal{P}^\alpha\text{-lim} B_n \) if \( B_n \) converges entrywise (in the norm of \( L(X) \)) to \( B \) as \( n \to \infty \) and if \( \sup_n \|B_n\| < \infty \). Secondly, for \( \alpha \in \mathbb{Z}^{N+1} \), let \( V_\alpha' : E' \to E' \) denote the shift operator with \( (V_\alpha'u)(k) = u(k - \alpha) \) for all \( k \in \mathbb{Z}^{N+1} \) and \( u \in E' \).

If \( B \in \text{BDO}(E') \), \( h = (h(1), h(2), \ldots) \subseteq \mathbb{Z}^{N+1} \) is a sequence with \( |h(n)| \to \infty \) and the operator sequence \( V'_{-h(n)}BV'_{h(n)} \) is \( \mathcal{P}^\alpha \)-convergent as \( n \to \infty \) then its limit will be denoted by \( B_h \) and is called limit operator of \( B \) w.r.t. the sequence \( h \). In an analogous fashion, one defines limit operators in \( \text{BDO}(E) \). To distinguish between operators on \( E' \) and on \( E \) we write \( \mathcal{P}\text{-lim} \) and \( V_\alpha \) with \( \alpha \in \mathbb{Z}^N \) if we are in the \( E \) setting. Different sequences \( h \) generally lead to different limit operators and often the sequence \( V_{-h(n)}BV_{h(n)} \) does not \( \mathcal{P} \)-converge at all. We will call \( B \in \text{BDO}(E) \) a rich operator if every sequence \( h = (h(1), h(2), \ldots) \subseteq \mathbb{Z}^N \) with \( |h(n)| \to \infty \) has a subsequence \( g \) such that the limit operator \( B_g \) exists.

As a final preparation, we turn our attention to the geometry of \( \Omega \). Let \( \Gamma := \partial \Omega \) be the boundary of \( \Omega \) and, for every \( n \in \mathbb{N} \), put

\[
\Gamma_n := (n\Gamma + H) \cap \mathbb{Z}^N \quad \text{with} \quad H = (-1/2, 1/2)^N
\]

and then let

\[
\Gamma_\mathcal{I} := \bigcup_{n \in \mathcal{I}} \Gamma_n.
\]

For a sequence \( h = (h(1), h(2), \ldots) \subseteq \Gamma_\mathcal{I} \), say \( h(k) \in \Gamma_{m_k} \) for some \( m_k \in \mathcal{I} \), and a set \( S \subseteq \mathbb{Z}^N \), we call \( S \) the geometric limit of \( \Omega \) w.r.t. \( h \) and write \( S = \Omega_h \) if, for every \( m \in \mathbb{N} \), there exists a \( k_0 \in \mathbb{N} \) such that

\[
(\Omega_{m_k} - h(k)) \cap \{-m, \ldots, m\}^N = S \cap \{-m, \ldots, m\}^N, \quad k \geq k_0.
\]

Note that in this case \( V_{-h(k)} P_{m_k} V_{h(k)} \) is \( \mathcal{P} \)-convergent to \( P_S \) as \( k \to \infty \). For a polytope \( \Omega \), the only candidates for the geometric limit \( S \) w.r.t. a sequence \( h \subseteq \Gamma_\mathcal{I} \) are intersections of finitely many half spaces and \( \mathbb{Z}^N \) (discrete half spaces, edges, corners, etc.).

### 2.2 The Stability Theorem for Subsequences

Given a rich operator \( A \in \text{BDO}(E) \) on \( E = l^p(\mathbb{Z}^N, X) \) with \( p \in [1, \infty] \), \( N \in \mathbb{N} \) and a complex Banach space \( X \), a valid starlike set \( \Omega \in \mathbb{R}^N \), and an index set \( \mathcal{I} = \{n_1, n_2, \ldots\} \subseteq \mathbb{N} \) with \( n_1 < n_2 < \cdots \), we put

\[
\mathcal{H}_{\Omega, \mathcal{I}}(A) := \{ h = (h(1), h(2), \ldots) : h(k) \in \Gamma_\mathcal{I} \forall k, |h(k)| \to \infty, A_h \text{ exists}, \Omega_h \text{ exists} \}
\]
and

\[ \sigma_{\Omega,\mathcal{I}}^{\text{stab}}(A) := \{ A \} \cup \{ P_{\Omega} A_{h} P_{\Omega} + Q_{\Omega} : h \in \mathcal{H}_{\Omega,\mathcal{I}}(A) \}. \] (10)

Then the following theorem holds.

**Theorem 2.1** Under the conditions mentioned above, the following are equivalent.

(i) The sequence \((P_{n} A P_{n} + Q_{n})_{n=1}^{\infty}\) is stable.

(ii) The operator \(\oplus A_{i}\), with \(A_{i}\) as in (8), is invertible at infinity.

(iii) All operators in \(\sigma_{\Omega,\mathcal{I}}^{\text{stab}}(A)\) are invertible with their inverses uniformly bounded.

**Proof.** See Theorems 3.1 and 3.5 in [12].

For dimension \(N = 1\), our statement coincides with a two-sided version of [20, Theorem 3]. As such it generalizes [17, Theorem 3] (also see [18, Theorem 6.2.2], [10, Theorem 4.2] and [23, Theorem 2.7]) from the full sequence \(\mathcal{I} = \mathbb{N}\) to arbitrary infinite subsequences with index set \(\mathcal{I} \subseteq \mathbb{N}\). For \(N = 2\) and \(\Omega\) a convex polygon with integer vertices, our Theorem 2.1, together with (10), corrects another version of the stability spectrum (see (11) and Example 2.2 below) that was previously suggested in the literature (see [17, 18]) for \(\mathcal{I} = \mathbb{N}\). Moreover, our result demonstrates how to deal with subsequences \(\mathcal{I} \subseteq \mathbb{N}\) by restricting consideration to sequences \(h = (h(1), h(2), \ldots)\) with values in the set \(\Gamma_{\mathcal{I}} = \bigcup_{n \in \mathcal{I}} \Gamma_{n}\). For dimensions \(N > 2\), to our knowledge, the result is new – even in cases like \(\mathcal{I} = \mathbb{N}\) or \(\Omega\) a convex polytope.

### 2.3 Examples

As a particularly illustrative and not too difficult class of examples, we will look at operators that are induced by an adjacency matrix. Therefore, put \(X = \mathbb{C}\), let \(\mathcal{E}\) denote a set of pairwise disjoint doubletons \(\{i, j\}\) (i.e. sets \(\{i, j\} = \{j, i\}\) with exactly two elements) with \(i, j \in \mathbb{Z}^{N}\), \(i \neq j\), and put

\[ a_{ij} := \begin{cases} 1, & \text{if } \{i, j\} \in \mathcal{E} \text{ or } i = j \not\in \bigcup_{e \in \mathcal{E}} e, \\ 0, & \text{otherwise}, \end{cases} \]

for all \(i, j \in \mathbb{Z}^{N}\). Then \((a_{ij})_{i,j \in \mathbb{Z}^{N}}\) is the extended adjacency matrix of the undirected graph \(\mathcal{G} = (\mathbb{Z}^{N}, \mathcal{E})\) with vertex set \(\mathbb{Z}^{N}\) and edges \(\mathcal{E}\). We write \(\text{Adj}(\mathcal{G})\) for the operator that is induced by this matrix \((a_{ij})\) and note that \(\text{Adj}(\mathcal{G})\) is band-dominated iff \(b := \sup_{(i,j) \in \mathcal{E}} |i - j|\) is finite, in which case \(\text{Adj}(\mathcal{G})\) is even a band operator with band-width \(b\).

If applied to an element \(u \in E = l^{p}(\mathbb{Z}^{N}, X)\), the operator \(\text{Adj}(\mathcal{G})\) “swaps” the values \(u(i)\) and \(u(j)\) around if \(\{i, j\}\) is an edge of \(\mathcal{G}\), and it leaves all values \(u(k)\) untouched for which \(k \in \mathbb{Z}^{N}\) is not part of an edge of \(\mathcal{G}\). From this it is obvious that \(\|\text{Adj}(\mathcal{G})\| = 1\) and that \(\text{Adj}(\mathcal{G})\) is invertible and coincides with its inverse. Moreover, it is clear that, for \(n \in \mathbb{N}\), the \(n\)-th finite section \(P_{n}\text{Adj}(\mathcal{G})P_{n} + Q_{n}\) is invertible iff each edge \(e \in \mathcal{E}\) has either both or no vertices in \(\Omega_{n} = n\Omega \cap \mathbb{Z}^{N}\). In the latter case, \(P_{n}\text{Adj}(\mathcal{G})P_{n} + Q_{n}\) equals \(\text{Adj}(\mathcal{G}_{n})\), where \(\mathcal{G}_{n} = (\mathbb{Z}^{N}, \mathcal{E} \cap \Omega_{n}^{2})\), is again its own inverse and has norm 1. So we get that, for \(A = \text{Adj}(\mathcal{G})\), the sequence \((7)\) is stable iff, for all sufficiently large \(n \in \mathcal{I}\), each edge \(e \in \mathcal{E}\) has either both or no vertices in \(\Omega_{n}\).
Note that Example 1.3 was already of the form $A = \text{Adj}(\mathcal{G})$, namely with $N = 1$ and

$$E = \left\{ \ldots, \{-4, -3\}, \{-2, -1\}, \{1, 2\}, \{3, 4\}, \ldots \right\}.$$  

Here $\Omega_n$ separates the vertices of the edge $\{-n - 1, -n\}$ and also of $\{n, n + 1\}$ if $n$ is odd.

We continue with two examples demonstrating that two particular sets of operators that are closely related to $\sigma^\text{stab}(A)$ — and that have, in the past, been suggested to replace (10) in the $N = 2$, $\mathcal{I} = \mathbb{N}$ version of Theorem 2.1 — are actually not stability spectra (meaning that Theorem 2.1 is incorrect for $\mathcal{I} = \mathbb{N}$ with $\sigma^\text{stab}(A)$ replaced by any of them) if $N > 1$. These two “non-replacements” for $\sigma^\text{stab}(A)$ are

$$\{A\} \cup \bigcup_{x \in \Gamma} \{ P_{\Omega_x} B \Gamma \Omega_x + Q_{\Omega_x} : B \in \sigma^\text{op}_x(A) \} \quad (11)$$

and

$$\{A\} \cup \bigcup_{x \in \Gamma} \{ P_{\Omega_x} B \Gamma \Omega_x + Q_{\Omega_x} : B \in \sigma^\text{op}_{x, \text{ray}}(A) \}, \quad (12)$$

where $\Gamma = \partial \Omega$ and, for every $x \in \Gamma$, $\Omega_x \subseteq \mathbb{Z}^N$ is the limit of $n(\Omega - x) \cap \mathbb{Z}^N$ as $n \to \infty$ in the sense that, for each $m \in \mathbb{N}$,

$$n(\Omega - x) \cap \{-m, \ldots, m\}^N = \Omega_x \cap \{-m, \ldots, m\}^N$$

for all sufficiently large $n \in \mathbb{N}$. Finally, $\sigma^\text{op}_x(A)$ is the set of all limit operators $A_k$ of $A$ with respect to sequences $h = (h(1), h(2), \ldots) \subseteq \mathbb{Z}^N$ going to infinity in the direction $x$, i.e. $h(n)/|h(n)| \to x/|x|$, and $\sigma^\text{op}_{x, \text{ray}}(A)$ is the set of all limit operators $A_k$ with respect to sequences of the form $h = ([m_1 x], [m_2 x], \ldots) \subseteq \mathbb{Z}^N$ where $(m_n)$ is an unbounded monotonously increasing sequence of positive reals and $[\cdot]$ means componentwise rounding to the nearest integer.

**Example 2.2** Take $N = 2$, $\Omega = [-1, 1]^2$ and let $A = \text{Adj}(\mathcal{G})$ with $\mathcal{G} = (\mathbb{Z}^2, E)$ and

$$E = \left\{ \{(k^2 - k - 1, k^2)\}, \{(k^2 - k, k^2)\} : k = 1, 2, \ldots \right\}.$$  

Then, with respect to $h = (h(1), h(2), \ldots)$ with $h(k) = (k^2 - k - 1, k^2) \in \mathbb{Z}^2$, the limit operator of $A$ exists and is equal to $B = \text{Adj}(\mathcal{G}')$, where $\mathcal{G}' = (\mathbb{Z}^2, \{(0, 0), (1, 0)\})$. Since $h(k)/|h(k)| \to x/|x|$ with $x = (1, 1)$, we have that $B \in \sigma^\text{op}_x(A)$. But $\Omega_x = \{-1, 0\}^2$ separates $(0, 0)$ from $(1, 0)$ so that $P_{\Omega_x} B \Gamma \Omega_x + Q_{\Omega_x} \in (11)$ is not invertible. However, the whole finite section sequence (5) is stable since all edges $e \in E$ have either both or no points in $\Omega_n$, so that $P_n A P_n + Q_n = \text{Adj}(G_n)$ with $G_n = (\mathbb{Z}^2, E \cap \Omega_n^2)$ for every $n \in \mathbb{N}$. So (11) is not a valid replacement of (10) as stability spectrum.

Note that the element of (10) that corresponds to the limit operator $B = A_k$ of $A$ is $P_{\Omega_k} B \Gamma \Omega_k + Q_{\Omega_k}$ with $\Omega_k = \mathbb{Z} \times \{-1, 0\}$ instead of $\{-1, 0\}^2$, which is again equal to $B$ (since both $(0, 0)$ and $(1, 0)$ are in $\Omega_k$) and hence invertible. $\Box$

Similarly, we can rule out (12) as stability spectrum by the following example:

**Example 2.3** Again take $N = 2$, $\Omega = [-1, 1]^2$ and let $A = \text{Adj}(\mathcal{G})$ with $\mathcal{G} = (\mathbb{Z}^2, E)$ and

$$E = \left\{ \{(k^2 - k, k^2)\}, \{(k^2 - k, k^2 + 1)\} : k = 1, 2, \ldots \right\}.$$
Then, with respect to \( h = (h(1), h(2), \ldots) \) with \( h(k) = (k^2 - k, k^2) \in \mathbb{Z}^2 \), the limit operator of \( A \) exists and is equal to \( B = \text{Adj}(\mathcal{G}') \), where \( \mathcal{G}' = \left( \mathbb{Z}^N, \left\{ (0, 0), (0, 1) \right\} \right) \).

Again \( B \in \sigma_{x^p}^\text{op}(A) \) with \( x = (1, 1) \). But \( B \not\in \sigma_{x^p, \text{ray}}^\text{op}(A) \) neither is \( B \in \sigma_{y^p, \text{ray}}^\text{op}(A) \) for any other \( y \in \Gamma \!). In fact, it holds that \( \sigma_{y^p, \text{ray}}^\text{op}(A) = \{ I \} \) for all \( y \in \Gamma \), whence (12) is elementwise invertible with uniformly bounded inverses. However, the finite section sequence (5) is not stable since \( \Omega_n \) separates \((k^2 - k, k^2)\) from \((k^2 - k, k^2 + 1)\) if \( n = k^2 \). So also (12) is not a valid replacement of (10) as stability spectrum.

Note that, for \( I = \mathbb{N} \), (10) contains the operator \( P_{\Omega}B P_{\Omega} + Q_{\Omega} \) with \( \Omega = \mathbb{Z} \times \{ \ldots, -1, 0 \} \), which is non-invertible since \( \Omega \) separates \((0, 0)\) from \((0, 1)\). This operator is however removed from (10) if we remove all (sufficiently large) square numbers from \( I \), which matches our observation that \( P_n \text{Adj} P_n + Q_n \) is non-invertible iff \( n \) is a square. □

It is clear that Examples 2.2 and 2.3 can easily be heaved to dimensions \( N > 2 \). Let us look at another example, for simplicity also in dimension \( N = 2 \).

**Example 2.4** We look at \( A = \text{Adj}(\mathcal{G}) \) for \( \mathcal{G} = (\mathbb{Z}^2, \mathcal{E}) \), where

\[
\mathcal{E} = \left\{ (k, 1), (k + 1, 0) : k = 1, 2, \ldots \right\}.
\]

It is not hard to see that every limit operator of \( A \) is either the identity operator \( I \) or the operator \( B = \text{Adj}(\mathcal{G}') \) for \( \mathcal{G}' = (\mathbb{Z}^2, \mathcal{E}') \), where

\[
\mathcal{E}' = \left\{ (k, 1), (k + 1, 0) : k \in \mathbb{Z} \right\},
\]

or it is a translate of \( B \). Looking at \( B \) and noting that \( B = A_h \) for all sequences \( h = (h(1), h(2), \ldots) \) with \( h(k) = (m_h, 0) \) and \( m_h \to +\infty \), we can say how \( \Omega \) has to look locally at the intersection \( z \) of its boundary \( \Gamma \) with the positive \( x \)-axis in order for the finite section method to be stable. Here the upward tangent of \( \Gamma \) at \( z \) has to enclose an angle \( \alpha \in (90^\circ, 135^\circ) \) with the positively directed \( x \)-axis. So, for example, the finite section sequence is stable if \( \Omega \) is the square \( \text{conv}\{(1, 0), (0, 1), (-1, 0), (0, -1)\} \) or the triangle \( \text{conv}\{(0, 2), (2, -2), (-2, -2)\} \), whereas it does not even have a stable subsequence if \( \Omega \) is the square \([-1, 1]^2\]. □

The next example is closely related to Example 1.3.

**Example 2.5** a) Let \( A = \text{Adj}(\mathcal{G}) \) where \( \mathcal{G} = (\mathbb{Z}, \mathcal{E}) \) is the following infinite graph:

![Graph](image)

Then, no matter how we choose \( \Omega = [a, b] \) with integers \( a < 0 < b \), the finite section method does not even have a stable subsequence. A workaround would be to take \( \Omega = [-1, 1] \) or to increase the dimension to \( N = 2 \), where we place the edges \( \mathcal{E} \) along the \( x \)-axis and put \( \Omega = \text{conv}\{(-1, 0), (1, 1), (0, -1)\} \), for example. In the latter case, the finite section subsequence corresponding to \( I = 4N + 1 \) turns out to be stable.

b) In contrast to a), there is no workaround whatsoever if \( A = \text{Adj}(\mathcal{G}) \) with the following graph \( \mathcal{G} \) (embedded in dimension \( N = 1 \) or higher):

![Graph](image)
For every valid starlike set $\Omega$ and every $n \in \mathbb{N}$, the set $\Omega_n$ separates the endpoints of at least two edges of $\mathcal{G}$ so that $P_nA_P + Q_n$ is non-invertible.

For any dimension $N \in \mathbb{N}$, any valid set $\Omega \in \mathbb{R}^N$ and any given sequence $n_1 < n_2 < \cdots$ of naturals, one can construct a graph $\mathcal{G}$ in the style of Example 2.5 b) such that $(P_nA_P + Q_n)_{n \in \mathcal{I}}$ is stable iff $\mathcal{I}$ is a subset of $\{n_1, n_2, \ldots\}$.

2.4 Some Words on the Case $N = 1$

Not surprisingly, the results are most complete in dimension $N = 1$, where one can sharpen and extend much of what was said previously (also see [20, 21, 24, 25, 12]). This is clearly due to the simple geometry of this setting: Firstly, to infinity there are only two ways to go: right or left, and secondly, all valid starlike sets are intervals from $a$ to $b$ with reals $a < 0 < b$ so that there are only two possibilities for $\Omega_h$ in (10): $\{0, 1, \ldots\}$ and $\{\ldots, -1, 0\}$.

The main result on $N = 1$ is by Rabinovich, Roch and Silbermann. It highlights an important benefit from extending the stability theorem from the full finite section sequence to subsequences. A proof can be found in [20] or, slightly generalized, in [12, §6].

**Proposition 2.6** [20] The uniform boundedness condition in Theorem 2.1 (iii) is redundant if $N = 1$.

As a next result in dimension $N = 1$, we mention that if the full FSM (5) is stable for one valid $\Omega$ then (5) has a stable subsequence for all valid $\Omega$. So conversely, if there exists a valid $\Omega$ for which (5) has no stable subsequence then there is no valid $\Omega$ for which the whole sequence (5) is stable. A proof can be found in [12, §5].

Example 2.5 b) has shown that, for some operators, the finite section method cannot be “adjusted”, via choosing $\Omega$ and $\mathcal{I}$, to become stable. We now give a necessary criterion for the existence of an index set $\mathcal{I} \subseteq \mathbb{N}$ and a valid $\Omega$ such that (7) is stable.

**Proposition 2.7** Let $E = \ell^p(\mathbb{Z}, \mathbb{C})$ with $p \in [1, \infty]$ and $A \in \text{BDO}(E)$. For the existence of a valid starlike set $\Omega \subset \mathbb{R}$ and an infinite index set $\mathcal{I} \subseteq \mathbb{N}$ such that the sequence $(P_nA_P + Q_n)_{n \in \mathcal{I}}$ is stable it is necessary, but not sufficient, that $A$ is invertible and the Fredholm index $\text{ind}_+(A) := \text{ind}(P_{\mathcal{N}}A|_{\text{im} P_{\mathcal{N}}})$ is zero.

A proof, based on work on the Fredholm index of band-dominated operators in [14], can be found in [12, §5]. In [13] (see [22, 11] for $p \neq 2$) we have shown that, under the additional condition that all diagonals of $[A]$ are slowly oscillating, invertibility of $A$ and $\text{ind}_+(A) = 0$ are even sufficient for the stability of the full finite section sequence (5) for all valid $\Omega$. Here we call a sequence $(b_k)_{k \in \mathbb{Z}}$ slowly oscillating if $b_{k+1} - b_k \to 0$ as $k \to \pm \infty$.

1The idea is to take the graph from Example 2.5 b) and to place “gaps” between $a_i := [a_{ni}]$ and $a_i - 1$ and between $b_i := [b_{ni}]$ and $b_i + 1$ for $i = 1, 2, \ldots$, where $a < 0$ and $b > 0$ are the unique intersection points of $\Gamma = \partial \Omega$ with the $x$-axis and $\lceil \cdot \rceil$ and $\lfloor \cdot \rfloor$ stand for rounding up and down to the next integer, respectively.
Remark 2.8 By Proposition 2.7, for an invertible operator $A$ with $\kappa := \text{ind}_+(A) \neq 0$, there is no valid $\Omega$ and no index set $I \subseteq \mathbb{N}$ for which (7) is stable. This problem of a nonzero plus-index $\kappa$ can be overcome as follows: Instead of solving $Au = b$, one looks at $V_\kappa Au = V \kappa b$ with $V_\kappa$ as in Example 1.2. Since $V_\kappa$ is invertible, these two equations are equivalent. Moreover, also $A' := V_\kappa A$ is invertible and

$$\text{ind}_+(A') = \text{ind}_+(V_\kappa A) = \text{ind}_+(V_\kappa) + \text{ind}_+(A) = -\kappa + \kappa = 0.$$ 

This preconditioning-type procedure of shifting the whole system (all matrix entries and the right hand side $b$) down by $\kappa$ rows is reminiscent of Gohberg’s statement that, in a two-sided infinite matrix, “it is every diagonal’s right to claim to be the main one” (see page 51f in [4]). Our results show that, however, from the perspective of the FSM, there is one diagonal that deserves being the main diagonal a bit more than the others. □

3 Strategy Two: Rectangular Subsystems

3.1 Motivation

Let us go back to Example 1.2 and try to fix one of the basic problems of the FSM. For simplicity, think of dimension $N = 1$ and $\Omega = [-1,1]$ so that $\Omega_n = \{-n,\ldots,n\}$ for all $n \in \mathbb{N}$. Look at the shift operator $A = V_k$ with $k = 1$, say. The FSM for the solution of $Au = b$, that is

$$P_n AP_n u_n = P_n b, \quad n = 1, 2, \ldots,$$

(13)

thinks of an approximate solution $u_n$ with support in $\{-n,\ldots,n\}$, then applies the operator — in our case the forward shift by 1 component — and afterwards cuts off at $\{-n,\ldots,n\}$ again, hereby trying to match the restriction of the right-hand side $b$ to $\{-n,\ldots,n\}$. It is clear that this truncated equation (13) is in general not solvable since the left-hand side of (13) always has a 0 at component $-n$ whereas the right-hand side has the same component $-n$ as $b$ has. Even if $b(-n) = 0$ and (13) is solvable then the solution $u_n$ is not unique\(^2\) since its $n$th component got shifted and then cut off whence it is irrelevant for (13).

The observation generalizes to band and band-dominated operators of course. If we truncate $u_n$ at $\{-n,\ldots,n\}$ and apply a band operator $A$ with band-width $w$ then $AP_n u_n$ is supported in $\{-n-w,\ldots,n+w\}$ whence, for the same reasons as illustrated for the shift $A = V_1$, it is better to cut off at $\{-m,\ldots,m\}$ with $m = n + w$ and not at $\{-n,\ldots,n\}$. The resulting system

$$P_m AP_n u_n = P_m b, \quad n = 1, 2, \ldots,$$

(14)

with $m = n + w$ is over-determined — it has rectangular matrices that have $2w$ more rows than they have columns. But one can still try to solve it approximately (by least squares, say).

\(^2\)Of course, for a finite quadratic system, solvability for all right-hand sides (i.e. surjectivity of the finite matrix operator) is equivalent to uniqueness of the solution (i.e. injectivity). The approach here is to say that lack of surjectivity can be overcome by looking for approximate rather than exact solutions, whereas lack of injectivity is a more serious problem that will be dealt with by adding more equations (i.e. more matrix rows) to the finite system.
From the matrix point of view, \( [AP_n] \) is the same as \([A]\), only with all but columns number \(-n,\ldots, n\) put to zero. If the horizontal cut-off \([P_m A P_n]\) (that one also has to do to get a finite system for the computer) is done at \(m = n\), like in (13), then some ‘large’ entries of \([AP_n]\) will get cut off (recall \(A = V_1\)) which might cause problems as mentioned earlier; so it could be good to choose \(m\) a bit larger. In fact, if \(A\) has the property

\[
\|P_m A P_n - A P_n\| \to 0, \quad \text{i.e.} \quad \|Q_m A P_n\| \to 0 \quad \text{as} \quad m \to \infty \tag{15}
\]

for all \(n \in \mathbb{N}\) then it seems possible to work with this rectangular cut-off idea, where \(m\) in (14), depending on \(n\), is chosen large enough to make \(\|P_m A P_n - A P_n\| = \|Q_m A P_n\|\) small enough. The class of operators with property (15) clearly contains all of BDO(\(E\)).

The above idea is so natural that it can hardly be new. Indeed, it is already used by some of the numerical community and it goes back at least to the 1960’s when Cleve Moler suggested, roughly speaking: If square submatrices give you problems, make them higher and use least squares. In [5] we have not only reinvented this method, we have (and that seems to be new) given a proof that, in the setting of a rather general Banach space \(E\), the method is applicable as soon as \(A\) is invertible and subject to (15). We will now recall the main steps of this proof.

### 3.2 The Rectangular Finite Section Method (rFSM)

We will work with the same spaces \(E = \ell^p(\mathbb{Z}^N, X)\) and the same projection operators \(P_n\) and \(Q_n\) here as defined above, but we will now exclude the case \(p = \infty\) because we require strong convergence \(P_n \to I\), i.e. \(P_n u \to u\) for all \(u \in E\), as \(n \to \infty\). The subspace \(c_0(\mathbb{Z}^N, X) = \{u \in \ell^\infty(\mathbb{Z}^N, X) : P_n u \to u\}\) of \(\ell^\infty(\mathbb{Z}^N, X)\) is however a valid choice for \(E\).

Now suppose \(A : E \to E\) is a bounded and invertible linear operator with (15), that means \(\|a_{ij}\| \to 0\) as \(|i| \to \infty\) for every fixed \(j \in \mathbb{Z}^N\), where \([A] = (a_{ij})\) is the matrix representation of \(A\). Then the equation \(A u = b\) has a unique solution \(u =: u_0\) for every right-hand side \(b \in E\). For the approximate computation of \(u_0\) we propose the following method: For given precision \(\delta > 0\) and cut-off parameters \(m\) and \(n \in \mathbb{N}\), calculate a solution \(u \in \text{im} P_n\) of the inequality

\[
\|P_m A P_n u - P_m b\| < \delta. \quad \text{(rFSM)}
\]

We start with a result about the existence of solutions of (rFSM).

**Definition 3.1** We say that \(n_0 \in \mathbb{N}\) is an admissible \(n\)-bound for \(A\), \(b\) and a given precision \(\delta > 0\) if (rFSM) is solvable in \(E\) for all \(m \in \mathbb{N}\) and all \(n \geq n_0\).

**Proposition 3.2** For every \(\delta > 0\), there is an admissible \(n\)-bound \(n_0 \in \mathbb{N}\).

**Proof.** We demonstrate how to choose \(n_0\) so that \(u := P_n u_0 = P_n A^{-1} b\) solves (rFSM) for every \(n \geq n_0\). For all \(m \in \mathbb{N}\) and \(n \in \mathbb{N}\), we have

\[
\|P_m A P_n u - P_m b\| = \|P_m A P_n^2 A^{-1} b - P_m b\| \\
\leq \|P_m A A^{-1} b - P_m b\| + \|P_m A Q_n A^{-1} b\| \\
\leq 0 + \|A\| \cdot \|Q_n A^{-1} b\|.
\]
But, by our assumption $Q_n \to 0$, there is a $n_0 \in \mathbb{N}$ such that
\[
\|Q_n A^{-1} b\| \leq \frac{\delta}{\|A\|}
\] (16)
for all $n \geq n_0$, so that $\|P_m A P_n u - P_m b\| < \delta$ holds, and hence $u$ solves (rFSM) for all $n \geq n_0$ and $m \in \mathbb{N}$. ■

Lemma 3.3 Let $n_0 \in \mathbb{N}$ be an admissible $n$-bound for $A$, $b$ and a given precision $\delta > 0$. If $n \geq n_0$ and $m \in \mathbb{N}$ are such that $\|Q_m A P_n\| < 1/\|A^{-1}\|$ then the set of all solutions of (rFSM) is a bounded subset of $E$. Precisely, every solution $u \in \text{im} P_n$ of (rFSM) is subject to $\|u\| \leq M$ with $M$ given by (17).

Proof. Suppose $u \in \text{im} P_n$ solves (rFSM) for given parameters $\delta, m, n$. Then
\[
\|Au\| - \|P_m b\| \leq \|Au - P_m b\| = \|AP_n u - P_m b\|
\]
\[
\leq \|AP_n u - P_m A P_n u\| + \|P_m A P_n u - P_m b\|
\]
\[
\leq \|Q_m A P_n\| \cdot \|u\| + \delta
\]
together with $\|u\| \leq \|A^{-1}\| \cdot \|Au\|$ implies that
\[
\frac{\|u\|}{\|A^{-1}\|} \leq \|Au\| \leq \|P_m b\| + \|Q_m A P_n\| \cdot \|u\| + \delta
\]
\[
\leq \|b\| + \|Q_m A P_n\| \cdot \|u\| + \delta
\]
and hence
\[
\|u\| \leq M := \frac{\|b\| + \delta}{1/\|A^{-1}\| - \|Q_m A P_n\|}.
\] (17) ■

Now we are ready for the key result showing that every solution of (rFSM) is indeed close to the solution $u_0$ of $Au = b$.

Theorem 3.4 For every $\varepsilon > 0$, there are parameters $\delta, m, n$ such that every solution $u$ of the system (rFSM) is an approximation
\[
\|u - u_0\|_E < \varepsilon
\] (18)
of the exact solution $u_0$ of $Au = b$. Precisely, there are three functions $\delta_0 : \mathbb{R}_+ \to \mathbb{R}_+$, $n_0 : \mathbb{R}_+ \to \mathbb{N}$ and $m_0 : \mathbb{R}_+^2 \times \mathbb{N} \to \mathbb{N}$ such that if $\delta < \delta_0(\varepsilon)$, $n \geq n_0(\delta)$ and $m \geq m_0(\varepsilon, \delta, n)$, then every solution $u \in \text{im} P_n$ of (rFSM) is subject to (18).

Proof. Let $\varepsilon > 0$ be given. We start the proof with three preliminary steps.

(a) Choose $\delta < \delta_0 := \frac{\varepsilon}{3\|A^{-1}\|}$.

(b) Choose $n_0 \in \mathbb{N}$ such that $(\|Q_n u_0\| = \|Q_n A^{-1} b\| < \frac{\delta}{\|A\|}$ for all $n \geq n_0$, so that $n_0$ is an admissible $n$-bound for $\delta$ (see inequality (16)). Now let $n \geq n_0$.
(c) Choose \( m_0 \in \mathbb{N} \) such that both \( \|Q_mb\| < \frac{\varepsilon}{3\|A^{-1}\|} \) and

\[
\|Q_mAP_n\| < \frac{1}{\|A^{-1}\|} \left(1 - \frac{1}{1 + \frac{\varepsilon}{3\|b\| + \delta \cdot \|A^{-1}\|}}\right)
\]  

(19)

hold for all \( m \geq m_0 \), and fix some \( m \geq m_0 \).

Now let \( u \in \text{im } P_n \) be a solution of (rFSM) with parameters \( \delta, n \) and \( m \) as chosen above. From (19) we get \( \|Q_mAP_n\| < 1/\|A^{-1}\| \), and hence, by Lemma 3.3,

\[
\|u\| \leq M
\]  

(20)

with \( M \) as defined in (17). Moreover, inequality (19) is equivalent to

\[
\|Q_mAP_n\| < \frac{1}{\|A^{-1}\|} \cdot \frac{\varepsilon}{3\|b\| + \delta \cdot \|A^{-1}\|}
\]  

and hence to

\[
\left(1 + \frac{\varepsilon}{3\|b\| + \delta \cdot \|A^{-1}\|}\right) \cdot \|Q_mAP_n\| < \frac{1}{\|A^{-1}\|} \cdot \frac{\varepsilon}{3\|b\| + \delta \cdot \|A^{-1}\|}
\]  

This, moreover, is equivalent to

\[
\|Q_mAP_n\| < \frac{1}{\|A^{-1}\|} \cdot \frac{\varepsilon}{3\|b\| + \delta \cdot \|A^{-1}\|} - \frac{\varepsilon}{3\|b\| + \delta \cdot \|A^{-1}\|} \cdot \|Q_mAP_n\| = \frac{\varepsilon}{3\|b\| + \delta \cdot \|A^{-1}\|}
\]  

(21)

with \( M \) as defined in (17). Then we have

\[
\|u - u_0\| = \|P_nu - u_0\| = \|A^{-1}AP_nu - A^{-1}b\| \leq \|A^{-1}\| \cdot \|AP_nu - b\| \\
\leq \|A^{-1}\| \cdot \left(\|AP_nu - P_mAP_nu\| + \|P_mAP_nu - P_mb\| + \|P_mb - b\|\right) \\
< \|A^{-1}\| \cdot \left(\|Q_mAP_n\| \cdot \|u\| + \delta + \|Q_mb\|\right) \\
< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,
\]

using inequalities (21) and (20) and the bounds on \( \delta \) and \( \|Q_mb\| \) in the last step. ■

Remark 3.5 One way to effectively solve the system (rFSM) for given parameters \( m, n \) and \( \delta \) is to compute a \( u \in \text{im } P_n \) that minimizes the discrepancy in (6), for example using a gradient method or, if possible, by directly applying the Moore-Penrose pseudo-inverse \( B^+ \) of \( B := P_mAP_n \) to the right-hand side \( P_mb \).

If \( E \) is a Hilbert space (i.e. if \( p = 2 \) and \( X \) is a Hilbert space) then it is well-known that \( u \in \text{im } P_n \) minimizes the residual \( \|Bu - P_mb\| \) if and only if \( B^*(Bu - P_mb) = 0 \). If, in addition, \( P_m \) is self-adjoint for all \( m \in \mathbb{N} \), then, after re-substituting \( B \), the latter is equivalent to

\[
P_nA^*P_mAP_nu = P_nA^*P_mb.
\]  

(22)
However, if \( m \) is sufficiently large, then, by (15) and \( P_n \to I \), the equation (22) is just a small perturbation of

\[
P_n A^* A P_n u = P_n A^* b,
\]

which is nothing but the finite section method for the equation

\[
A^* A u = A^* b.
\]

Note that the finite section method (23) is applicable since \( A^* A \) is positive definite (see, e.g. Theorem 1.10 b in [4]). Clearly, if \( A \) is invertible, as we require, then also its adjoint \( A^* \) is invertible, and (24) is equivalent to our original equation \( A u = b \).

Summarizing, if \( E \) is a Hilbert space and all \( P_m \) are self-adjoint, then minimizing \( \| P_m A P_n u - P_m b \| \) is equivalent to solving a slight perturbation (22) of the finite section method (23) for (24). \( \Box \)

### 4 Summary

Compared to the FSM and its subsequence version fully characterized in Theorem 2.1, the rFSM imposes no further conditions on the operator \( A \) (such as richness, conditions on its limit operators, etc.) other than its invertibility and the rather mild decay property (15). Of course, on the down side, we are restricted to \( p < \infty \), and, more seriously, for general operators \( A \) we do not really know yet how to choose \( m \) in dependence on \( n \). However, the choice \( m = n + w \) is clear for operators with band-width \( w \), and something similar is possible for a band-dominated operator \( A \) (where \( w \) must be fitted to the function \( f_A \) from [10, p. 32ff]). In contrast, in [5] we have chosen \( m = \frac{6}{5} n \).

We close with the following example.

**Example 4.1** Let \( E = \ell^2(\mathbb{Z}, \mathbb{C}) \), \( b = (b(i))_{i \in \mathbb{Z}} \in E \) with \( b(i) = 2^{-|i|} \), and let \( A : E \to E \) be given in block matrix notation by

\[
A = \begin{pmatrix}
\ddots & C \\
B & C \\
B & \ddots \\
\end{pmatrix},
\]

where

\[
B = \begin{pmatrix}
1 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\quad \text{and} \quad
C = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 1 & 1
\end{pmatrix},
\]

and where one of the \( B \) blocks is located at position \( \{(i, j) : i, j \in \{-1, 0, 1\}\} \) in \( A \). Then \( A \) is a band operator with band-width \( w = 3 \), it is invertible but has a nonzero plus-index \( \text{ind}_+(A) = \text{ind}(P_N A P_N|_{\text{im} P_N}) = 1 \). From Proposition 2.7 we know that there is no choice of \( \Omega \) and \( I \subset \mathbb{N} \) that makes the FSM (7) for \( A u = b \) stable. But Remark 2.8 shows that
passing to the equivalent system $V_1 Au = V_1 b$ might solve this problem since $A' := V_1 A$
has plus-index zero. This approach leads us to looking at finite sections of

$$A' = V_1 A = \begin{pmatrix} \ddots & D \\ D & \ddots & D \\ & D & \ddots \end{pmatrix} \quad \text{with} \quad D = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and one of the $D$ blocks located at position $\{ (i,j) : i, j \in \{-1, 0, 1\} \}$ in $A'$. For $\Omega = [-1, 1]$,
we get that $\sigma_{\Omega, N}^{\text{stab}}(A') = \{ A', F, G, H, J, K, L \}$, where

$$F = \text{diag} \left( \cdots, 1, \frac{1}{N}, D, D, \cdots \right), \quad J = \text{diag} \left( \cdots, D, D, \left( \begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix} \right), \overline{1}, \overline{1}, \cdots \right),$$

$$G = \text{diag} \left( \cdots, 1, \frac{1}{N}, D, D, \cdots \right), \quad K = \text{diag} \left( \cdots, D, D, D, \overline{1}, \overline{1}, \cdots \right),$$

$$H = \text{diag} \left( \cdots, 1, \frac{1}{N}, D, D, \cdots \right), \quad L = \text{diag} \left( \cdots, D, D, 1, \overline{1}, 1, \cdots \right)$$

with the underlined 1’s at position $(-1, -1)$ and the overlined 1’s at position $(1, 1)$ in the
respective matrix. Out of the seven elements of $\sigma_{\Omega, N}^{\text{stab}}(A')$ only $A', G, K$ and $L$
are invertible (noting that $D$ is invertible). Looking at certain index subsets $\mathcal{I}$ of $\mathbb{N}$, we see that

$$\sigma_{\Omega, 3N}^{\text{stab}}(A') = \{ A', F, J \}, \quad \sigma_{\Omega, 3N+1}^{\text{stab}}(A') = \{ A', G, K \}, \quad \sigma_{\Omega, 3N+2}^{\text{stab}}(A') = \{ A', H, L \},$$

so that the set $\mathcal{I} = 3N + 1$ yields a stable subsequence (7), whereas $3N$ and $3N + 2$ don’t.
After these considerations (shift the system by 1 row, single out a stable subsequence) it
is now straightforward to approximately solve the equation via the FSM.

In contrast, the rFSM immediately applies to $Au = b$ if we choose $m = n + 3$ (since
$A$ has band-width $w = 3$) and solve the systems $P_{n+3} A P_n u \approx P_{n+3} b$ approximately for
$n = 1, 2, \ldots$ using the Moore-Penrose pseudo-inverse of $P_{n+3} A P_n$. The latter means that
we compute

$$u_n := (P_{n+3} A P_n)^+ P_{n+3} b, \quad n = 1, 2, \ldots$$

and get that $u_n$ (if extended by zeros to an infinite vector) converges to the exact solution
$u_0 = A^{-1} b$ with

$$\| u_n - u_0 \| \leq \| A^{-1} \| \left( \| P_{n+3} A P_n u_n - P_{n+3} b \| + \| Q_{n+3} b \| \right) \leq 2 \left( \frac{3 \cdot 2}{2^n} + \frac{1}{2^{n+3}} \right) = \frac{49}{2^{n+2}},$$

which follows from the computations in the proofs of Proposition 3.2 and Theorem 3.4
together with $\| A \| \leq 3$ and $\| A^{-1} \| \leq 2$. □

Acknowledgements. I would like to thank the organizers, especially Robert Edward
and Golden Thambithurai, for their invitation to and great hospitality at the Interna-
tional Conference on Functional Analysis at Nagercoil, India. The research presented in
this paper was financially supported by Marie-Curie Grants MEIF-CT-2005-009758 and
PERG02-GA-2007-224761 of the EU and was largely carried out on the very pleasant trip
to Nagercoil.
References

[1] G. Baxter: A norm inequality for a ‘finite-section’ Wiener-Hopf equation, *Illinois J. Math.*, 1962, 97–103.

[2] S. N. Chandler-Wilde and M. Lindner: Limit Operators, Collective Compactness, and the Spectral Theory of Infinite Matrices, in publication (also see TU Chemnitz Preprint 7, 2008).

[3] I. Gohberg and I. A. Feldman, *Convolution equations and projection methods for their solution*, Transl. of Math. Monographs, 41, Amer. Math. Soc., Providence, R.I., 1974 [Russian original: Nauka, Moscow, 1971].

[4] R. Hagen, S. Roch and B. Silbermann: $C^*$—Algebras and Numerical Analysis, Marcel Dekker, Inc., New York, Basel, 2001.

[5] E. Heinemeyer, M. Lindner and R. Potthast: Convergence and numerics of a multi-section method for scattering by three-dimensional rough surfaces, *SIAM Journal on Numerical Analysis*, 46 (2008), 1780–1798.

[6] G. Heinig and F. Hellinger: The finite section method for Moore-Penrose inversion of Toeplitz operators, *Integral Equations Operator Theory* 19 (1994), 419–446.

[7] A. V. Kozak: A local principle in the theory of projection methods, *Dokl. Akad. Nauk SSSR* 212 (1973), 12871289; English transl. *Soviet Math. Dokl.* 14 (1973).

[8] A. V. Kozak and I. V. Simonenko: Projectional methods for solving multidimensional discrete equations in convolutions, *Sib. Mat. Zh.* 21 (1980), 119–127.

[9] M. Lindner: The finite section method in the space of essentially bounded functions: An approach using limit operators, *Numer. Func. Anal. & Optim.* 24 (2003) no. 7&8, 863–893.

[10] M. Lindner: *Infinite Matrices and their Finite Sections: An Introduction to the Limit Operator Method*, Frontiers in Mathematics, Birkhäuser 2006.

[11] M. Lindner: Fredholmness and index of operators in the Wiener algebra are independent of the underlying space, *Operators and Matrices* 2 (2008), 297–306.

[12] M. Lindner: Stable subsequences of the finite section method, *Preprint 2008-15, TU Chemnitz*.

[13] M. Lindner, V. S. Rabinovich and S. Roch: Finite sections of band operators with slowly oscillating coefficients, *Linear Algebra and Applications* 390 (2004), 19–26.

[14] V. S. Rabinovich, S. Roch and J. Roe: Fredholm indices of band-dominated operators, *Integral Equations Operator Theory* 49 (2004), no. 2, 221–238.

[15] V. S. Rabinovich, S. Roch and B. Silbermann: Fredholm Theory and Finite Section Method for Band-dominated operators, *Integral Equations Operator Theory* 30 (1998), no. 4, 452–495.
[16] V. S. Rabinovich, S. Roch and B. Silbermann: Band-dominated operators with operator-valued coefficients, their Fredholm properties and finite sections, *Integral Equations Operator Theory* **40**(2001), no. 3, 342–381.

[17] V. S. Rabinovich, S. Roch and B. Silbermann: Algebras of approximation sequences: Finite sections of band-dominated operators, *Acta Appl. Math.* **65**(2001), 315–332.

[18] V. S. Rabinovich, S. Roch and B. Silbermann: *Limit Operators and Their Applications in Operator Theory*, Birkhäuser 2004.

[19] V. S. Rabinovich, S. Roch and B. Silbermann: Finite sections of band-dominated operators with almost periodic coefficients, *Operator Theory: Advances and Applications* **170**(2007), 205–228.

[20] V. S. Rabinovich, S. Roch and B. Silbermann: On finite sections of band-dominated operators, *Operator Theory: Advances and Applications* **181**(2008), 385–391.

[21] V. S. Rabinovich, S. Roch and B. Silbermann: The finite sections approach to the index formula for band-dominated operators, *Operator Theory: Advances and Applications* **187**(2008), 185–193.

[22] S. Roch: Band-dominated operators on $\ell^p$–spaces: Fredholm indices and finite sections, *Acta Sci. Math.* **70**(2004), no. 3–4, 783–797.

[23] S. Roch: *Finite sections of band-dominated operators*, Memoirs AMS, Vol. 191, Nr. 895, 2008.

[24] M. Seidel and B. Silbermann: Finite Sections of Band-Dominated Operators: $\ell^p$-Theory, *Complex Analysis and Operator Theory*, **2**(2008), 683–699.

[25] M. Seidel and B. Silbermann: Banach Algebras of Structured Matrix Sequences, *Linear Algebra and Applications* **430**(2009), 1243–1281.

[26] B. Silbermann: Modified finite sections for Toeplitz operators and their singular values, *SIAM J. Math. Anal.* **24**(2003), 678–692.

**Author’s address:**

Marko Lindner
Fakultät Mathematik
TU Chemnitz
D-09107 Chemnitz
GERMANY

marko.lindner@mathematik.tu-chemnitz.de
This figure "graph1.jpg" is available in "jpg" format from:

http://arxiv.org/ps/1002.4645v1
This figure "graph2.jpg" is available in "jpg" format from:

http://arxiv.org/ps/1002.4645v1