On the Non-flatness Nature of Noncommutative Minkowski Spacetime

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Abstract

In the framework of twisted-diffeomorphism approach to noncommutative gravity with canonical/Moyal-Weyl type noncommutative (NC) coordinate structure, we show that the NC Minkowski spacetime parametrized either with spherical polar coordinates or with parabolic coordinates has nontrivial NC corrections to Riemann curvature tensor, Ricci tensor and curvature scalar. Apparently, there are no such corrections if we choose rectilinear coordinates or even cylindrical coordinates, for which the metric in the commutative-counterpart is dependent on at most one coordinate only. We present both first order and second order calculations. The emergent curvature corrections might seem to raise the question of whether the statement of curvature is coordinate-dependent, but note, for example, that the transformation from spherical polar system to Cartesian system is not a diffeomorphism as such since its non-injective nature makes it a local diffeomorphism. In other words, if the flat-spacetime metric tensor in the commutative case depends on more than one curvilinear coordinates, the introduction of noncommutativity among these coordinates can possibly make the spacetime curved. It is worth remarking that such a curvature emerges in the context of NC Minkowski spacetime in the absence of any gauge or matter fields. It is purely an NC geometric effect.

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1 Introduction

Spacetime noncommutativity is expected to play a significant role in the physics of Planck length scale [1]. In addition to the possibility of a minimal length associated with such noncommutative (NC) spacetimes, the quantization of length also shows up in certain cases [2]. On the one hand, the idea of spacetime noncommutativity emerges from the attempts to unify the gravity and the quantum [3], the emergent gravity phenomena in noncommutative gauge field theories, on the other hand, take the quantum field theories a step closer to bring the gravitational effects into the quantum field theories [4]- [13], and in the same vein, the gravitational theories in NC spacetimes are expected to bring the quantum effects into the gravitational theories.

There are a number of different approaches to NC gravity. With the idea of the Seiberg-Witten map [14] being extended to enveloping algebra [15], gauge theories of gravity in NC spacetime have been studied by mapping the commutative gauge theories of gravitation to their NC counterparts. In one approach, the noncommutative SO(4,1) de sitter group is gauged and contracted to the Poincare group ISO(3,1) using the Seiberg-Witten map [16]. Using the concept of twisted Poincare algebra [17, 18] and extending the idea to general coordinate transformations, gravity theory in NC spacetime is also constructed [19]. When commutation relations between the spacetime coordinates have a Lie Algebra structure, there is a class of volume preserving general coordinate transformation which preserves the symmetry of the noncommutative algebra [16, 20–22]. The theory of gravity corresponding to volume preserving diffeomorphism is unimodular theory of gravity. In this case NC General theory of relativity is constructed by gauging the Lorenz gauge group SO(3,1) in the enveloping algebra framework. Two dimensional NC gravity theories have also been proposed based on NC gauge framework. Since the general theory of relativity is invariant under diffeomorphism, a theory of noncommutative gravity is formulated using the mathematical framework of twist-deformed diffeomorphism [19]. In this theory of gravitation, which is invariant under deformed diffeomorphism, Christoffel symbol, Riemann curvature tensor, Ricci tensor and curvature scalar etc., are mathematically constructed in NC spacetime and the effect of non-commutation of spacetime coordinates manifests as the NC corrections to these geometrical elements.

In our work, we consider the twist-deformed diffeomorphism framework of NC gravity in Moyal-Weyl-like canonical noncommutative spacetime. We briefly review the idea of twist-deformed diffeomorphism in Section 2. The key point in Section 3 is that when the metric tensor components are functions of only a subset of coordinates that commute among themselves, there are no NC corrections to curvature tensors and curvature scalar even if the spacetime has other noncommuting coordinate pairs. In this case the NC gravity theory is the same as the classical theory in the pure gravity/geometry sector. Only the interaction with other gauge or matter can make the noncommutativity observable. Noncommutative effects take place in the pure gravity/geometry sector only if the metric tensor depends on at least two coordinates that do not commute.
In Section 4, we deal with the NC curvature corrections to NC Minkowski spacetime parametrized by spherical polar coordinates. We have done both first order and second order noncommutative corrections to inverse metric tensor, Christoffel symbols, Riemann curvature tensor, Ricci tensor and curvature scalar in the twist-deformed diffeomorphism framework of NC gravity. We repeat these calculations in Section 5 for the NC Minkowski spacetime parametrized by parabolic coordinates. All of these calculations are based on the formulas derived in [19].

We have extensively used Maple 2021 to get our results. All of the results presented in Sections 4 and 5 are the ones displayed as outputs in Maple (except for the indicial values that we denote like \((r, \theta, \phi, t)\) for the ease of understanding). Some of these results have been cross-checked by manual calculations to verify the Maple code.

The main result of our calculations is that the NC corrections to Riemann curvature tensor, Ricci tensor and the curvature scalar do not vanish for the NC Minkowski spacetime parametrized either by spherical polar or by parabolic coordinates. This turns out to be the general feature of a space in which the metric tensor is a function of at least two noncommuting coordinates. The concluding remarks are given in Section 6.

2 Deformed Diffeomorphism on NC Spacetime

NC spacetime is defined by the commutation relations among spacetime coordinates:

\[ [\hat{x}^\mu, \hat{x}^\nu] = i\Theta^{\mu\nu}, \]

\[ \text{where } (\Theta^{\mu\nu}) \text{ is a constant and real antisymmetric matrix. The above operator algebra can be realized on a linear space of complex functions } f(x) \text{ of commuting variables if the algebra of functions is given a noncommutative multiplication structure called Moyal-Weyl star-product } (\ast\text{-product}) [24]: \]

\[ f(x) \ast g(x) = e^{i\Theta^{\mu\nu} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x'^\nu}} f(x)g(x') \bigg|_{x'=x}. \]

The concept of deformed diffeomorphism relies on the idea that the general coordinate transformations in general relativity are, in general, diffeomorphism transformations. If \(\delta_\xi\) denotes the infinitesimal change in a scalar field \(\phi(x)\) under the infinitesimal general coordinate transformation \(x^\mu \rightarrow x'^\mu = x^\mu + \xi^\mu(x)\), then the invariance relation \(\phi'(x') = \phi(x)\) implies the field transformation \(\phi(x) \rightarrow \phi'(x') = \phi(x) + \delta_\xi \phi = \phi(x) - \xi^\mu(x)\partial_\mu \phi\). Also, for the pointwise multiplication of two transformed fields, the following equality holds:

\[ \phi'(x') \cdot \chi'(x') = \phi(x) \cdot \chi(x) + \delta_\xi (\phi(x) \cdot \chi(x)) \]

because of the Leibniz rule

\[ (\delta_\xi \phi) \cdot \chi + \phi \cdot (\delta_\xi \chi) = \delta_\xi (\phi \cdot \chi). \]
If $\mu$ denotes the map that maps a tensor product to the pointwise multiplication of commutative algebra, i.e., $\mu(\phi \otimes \chi) = \phi \cdot \chi$, then using the coproduct $\Delta(\delta_\xi) = \delta_\xi \otimes 1 + 1 \otimes \delta_\xi$, Eq. (4a) from right to left can be written as

$$\delta_\xi(\phi \cdot \chi) = \delta_\xi \mu(\phi \otimes \chi) = \mu(\delta_\xi \otimes 1 + 1 \otimes \delta_\xi)(\phi \otimes \chi) = \mu(\Delta(\delta_\xi)(\phi \otimes \chi)). \quad (4b)$$

The key point here is the commutation relation $\delta_\xi \mu = \mu \Delta(\delta_\xi)$. For three successive infinitesimal transformations involving $-\xi, \eta$ and $\xi$, the commutator $[\delta_\xi, \delta_\eta] = \delta_{\xi \times \eta}$ implies the closure property of the algebra of vector fields $\xi^\mu \partial_\mu$ acting on a differential manifold. Essentially, the commutator denotes the algebra of diffeomorphism, and its coproduct-counterpart is worked out to be $[\Delta(\delta_\xi), \Delta(\delta_\eta)] = \Delta(\delta_{\xi \times \eta})$. The Leibniz rule (4) holds regardless of whether $\delta_\xi$ depends on $x$ and the pointwise product of the fields is replaced by *-product, then Eq.(4) does not hold, i.e., $(\delta_\xi \phi) \ast \chi + \phi \ast (\delta_\xi \chi) \neq \delta_\xi(\phi \ast \chi)$. This is exactly where the concept of twist-deformed diffeomorphism comes into the picture.

If $\mu_*$ denotes the mapping of tensor product to the *-product of noncommutative coordinate algebra, i.e.,

$$\mu_*\{\phi \otimes \chi\} = \mu_\mathcal{F}\{\phi \otimes \chi\} = \phi \ast \chi, \quad (5)$$

where $\mathcal{F} = e^{i \Theta^{\mu \nu}} \frac{\partial}{\partial x^\mu \otimes \partial x^\nu}$ is called the Drinfeld twist, then using the twisted coproduct defined by

$$\Delta_t(\delta_\xi) = e^{-i \Theta^{\mu \nu}} \frac{\partial}{\partial x^\mu \otimes \partial x^\nu} \left( \delta_\xi \otimes 1 + 1 \otimes \delta_\xi \right) e^{i \Theta^{\mu \nu}} \frac{\partial}{\partial x^\mu \otimes \partial x^\nu} = \mathcal{F}^{-1} (\delta_\xi \otimes 1 + 1 \otimes \delta_\xi) \mathcal{F}, \quad (6)$$

the analogue of (4b) in noncommutative algebra can be written as

$$\delta_\xi u_*\{\phi \otimes \chi\} = \delta_\xi \mu_*\{\mathcal{F}(\phi \otimes \chi)\} = \mu_*\{\mathcal{F}(\delta_\xi)(\phi \otimes \chi)\} = \mu_*\{\Delta_t(\delta_\xi)(\phi \otimes \chi)\}, \quad (7)$$

where an $\mathcal{F}\mathcal{F}^{-1}(=1)$ has been inserted into the last equality. In order to take the $\delta_\xi$ out of the tensor product on the rightmost side of Eq.(7), to the left of $\mu_*$, as in the leftmost side of Eq.(7), the coproduct needs to be twisted: $\mu_* \Delta_t(\delta_\xi) = \delta_\xi \mu_*$. The above rule Eq.(7) is easily generalized to the product of arbitrary tensor fields leading to a covariant construction of theories.

Our idea here is not to review the entire formulation of twist-deformed diffeomorphism algebra, but to stress how the twist-deformation of the tensor product is at the core of this formalism and how it is linked to the *-product.

In this twisted-diffeomorphism approach to NC gravity, the basic element is the classical vierbein $e^a_\mu(x)$, also called tetrad in 3+1 dimensions, and this is taken to be the full NC version of vierbein to all orders in the NC parameter $\Theta$ [19]. But the metric in NC spacetime is defined to have the following symmetric form with $\Theta$-dependency:

$$G_{\mu \nu} = \frac{1}{2} (e^a_\mu \ast e^b_\nu + e^a_\nu \ast e^b_\mu) \eta_{ab}, \quad (8)$$
with its zeroth order being the metric in the commutative case, i.e., $G^{(0)}_{\mu\nu} = g_{\mu\nu} = e_{\mu}^{a} \cdot e_{\nu}^{a} \eta_{ab}$, where $\eta_{ab}$ is the constant symmetric metric of flat Minkowski spacetime.

With the conditions that the metric is a covariant tensor of rank two, that its inverse $G^{\mu\nu}$ is also a tensor of rank two such that $G_{\mu\nu} \ast G^{\rho\sigma} = \delta^{\rho}_{\mu}$, and that the covariant derivative of $G_{\mu\nu}$ is zero, the other elements like Christoffel symbols, covariant derivative, Riemann curvature tensor, Ricci tensor and curvature scalar are constructed in terms of the metric tensor and the inverse metric tensor [19].

3 Remarks on the Tetrads and the Metrics

The metric tensors of the four-dimensional spacetime considered in this paper are diagonal and the tetrads associated with these metrics are also diagonal.

3.1 At most One Coordinate Dependency of Tetrads

When the tetrads are constants, it is clear from (8) that the metric tensor $G_{\mu\nu} = g_{\mu\nu}$ to all orders in $\Theta$. Other geometrical elements are also not different from the commutative counterparts. The Cartesian system of coordinates falls under this category.

If the tetrads depend on only one coordinate, then also $e_{\mu}^{a} \ast e_{\nu}^{b} = e_{\mu}^{a} \cdot e_{\nu}^{b}$ and all the geometrical elements are the same as in the commutative theory. Because any nontrivial $\Theta$-corrections require the basic element $e_{\mu}^{a}$ to be dependent on at least two distinct coordinates such that the corresponding $\Theta^{\alpha\beta} \neq 0$. An example for this type is the cylindrical system of coordinates.

In the above two cases, only the interactions with other fields can reveal the underlying NC structure of spacetime.

3.2 More than One Coordinate Dependency of Tetrads

If the vierbein depends only on $x^{1}$ and $x^{2}$ and if these two coordinates commute, then in the pure gravity sector there are no NC corrections even if the spacetime has other noncommuting coordinate pairs. Because all the basic geometric elements are constructed out of the *-product of tetrads as in Eq.(8) and

$$e_{\mu}^{a}(x^{1}, x^{2}) \ast e_{\nu}^{b}(x^{1}, x^{2}) = e_{\mu}^{a} e_{\nu}^{b} \text{ if } \Theta^{12} = 0. \quad (9)$$

So the metric tensor will have no $\Theta$-dependency. The inverse metric, which has an expansion in $\Theta$ that depends only on the derivatives of $e_{\mu}^{a}$ in its expressions for $\Theta$-corrections [19], will also have no $\Theta$-corrections. The reason is both derivatives with respect to $x^{\alpha}$ and $x^{\beta}$ appear as multiplicative factors in each of the terms involving $\Theta^{\alpha\beta}$, and it is either that for nontrivial derivative $\Theta^{\alpha\beta}$ is zero or that for nontrivial $\Theta^{\alpha\beta}$ the derivative is zero, making each term to vanish. So the metric and the inverse metric will be the same as their commutative-counterparts. By the same token, the Christoffel symbols and Riemann tensor will also be the
same as their commutative-counterparts. Only the interaction with other gauge or matter fields, which in general depend on all coordinates, can reveal such underlying noncommuting structure of coordinates if $\Theta^{\alpha\beta} \neq 0$ for other pairs of distinct values of $\alpha$ and $\beta$.

But if the vierbein $e^a_\mu$ is a function of, for example, $x^1$ and $x^2$, and if these two coordinates are promoted as noncommuting operators, then nontrivial NC curvature corrections do show up.

4 NC Minkowski Spacetime in Spherical Polar Coordinates

The convention we adopt is $x^\mu = (x^1, x^2, x^3, x^4) = (r, \theta, \phi, t)$, in which the spacetime indices take the values 1, 2, 3 and 4. The signature of $\eta_{ab}$ is taken to be $\ (+, +, +, -)$. In the commutative case, the tetrad, the metric tensor and the inverse of the metric tensor in this system of coordinates are given by

$$e^a_\mu = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & r & 0 & 0 \\ 0 & 0 & r \sin \theta & 0 \\ 0 & 0 & 0 & -c \end{bmatrix}; \quad g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & r^2 & 0 & 0 \\ 0 & 0 & r^2 \sin^2 \theta & 0 \\ 0 & 0 & 0 & -c^2 \end{bmatrix}; \quad g^{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 & 0 \\ 0 & 0 & \frac{1}{r^2 \sin^2 \theta} & 0 \\ 0 & 0 & 0 & \frac{1}{c^2} \end{bmatrix}. \quad (10)$$

There are two reasons why this coordinate system is chosen for the curvature analysis: 1) The tetrad depends on more than one coordinate. 2) The Jacobian determinant of the transformation $(r, \theta, \phi, t) \rightarrow (r \sin \theta \cos \phi, r \sin \theta \sin \phi, r \cos \theta, t)$, which is equal to $r^2 \sin \theta$, vanishes in a nontrivial subset of the domain of the transformation.

In the commutative case, the non-zero Christoffel symbols of the second kind are given by (see for example [25]):

$$\Gamma^{(0)}_{\theta\theta} = -r, \quad \Gamma^{(0)}_{\phi\phi} = -r \sin^2 \theta, \quad \Gamma^{(0)}_{\theta \phi} = \frac{1}{r} \quad (11a)$$

$$\Gamma^{(0)}_{\phi\theta} = -\sin \theta \cos \theta, \quad \Gamma^{(0)}_{r \phi} = \frac{1}{r}, \quad \Gamma^{(0)}_{\theta \phi} = \cot \theta \quad (11b)$$

with the proviso that $\Gamma^{(0)}_{\mu \nu} = \Gamma^{(0)}_{\nu \mu}$. The Riemann tensor $R^{(0)}_{\mu\nu\rho\sigma}$, the Ricci tensor $R^{(0)}_{\mu\nu}$ and the curvature scalar $R^{(0)}$ are all zero.

Since the tetrad depends only on $r$ and $\theta$, when the coordinates are promoted as noncommuting operators, the only noncommutation relation that will lead to NC curvature effects is

$$[\hat{r}, \hat{\theta}] = i \Theta^{r\theta}. \quad (12)$$

No other nontrivial $\Theta^{\mu\nu}$ will result in any change in the nature of curvature of the spacetime. If $\Theta^{r\theta} = 0$, then the NC Minkowski spacetime is flat regardless of the presence of any other nontrivial $\Theta^{\mu\nu}$ in the theory.
4.1 First Order NC Corrections

4.1.1 Inverse Metric Tensor and Christoffel Symbols

The metric tensor in the NC case, Eq.(8), is symmetric and real by its construction. Therefore, there is no first order correction to it.

The first order NC correction to the inverse of metric tensor is given by [19]

\[ G^{(1)}_{\mu\nu} = \frac{i}{2} \Theta^{\alpha\beta} \partial_\alpha (g^{\gamma\mu}) \partial_\beta (g^{\delta\gamma}) g^{\delta\nu}. \] (13)

Substituting for the zeroth order metric and inverse metric components from Eq.(10) into Eq.(13) gives \( G^{(1)}_{\mu\nu} = 0 \), i.e., there are no first order corrections to the inverse metric.

The Christoffel symbols of the second kind have the following expression for their first order correction [19]:

\[ \Gamma^{(1)}_{\mu\nu} = \frac{i}{2} \Theta^{\alpha\beta} \left( \partial_\alpha \Gamma^{(0)}_{\mu\nu} \right) g^{\sigma\tau} \left( \partial_\beta g^{\tau\rho} \right), \] (14)

and these are symmetric under the exchange of \( \mu \) and \( \nu \). For the spacetime under consideration, the independent non-zero symbols are calculated to be

\[ \Gamma^{(1)}_{\phi\phi} = i \Theta^{\theta\phi} \frac{(1 - 2 \cos^2 \theta)}{r}, \quad \Gamma^{(1)}_{r\phi} = i \Theta^{\theta\phi} \frac{\cos \theta}{r^2 \sin \theta}, \quad \Gamma^{(1)}_{\phi\theta} = -i \Theta^{\theta\phi} \frac{1}{r \sin^2 \theta}, \] (15)

i.e., only the zeroth order symbols corresponding to Eq.(11b) have first order \( \Theta \)-corrections.

4.1.2 Curvature Tensors and Curvature Scalar

Riemann tensor has the following first order expression [19]

\[ R^{(1)}_{\mu\nu\rho\sigma} = -\frac{i}{2} \Theta^{\alpha\lambda} \left( \partial_\alpha R^{(0)}_{\mu\nu\rho\tau} (\partial_\lambda g^{\tau\gamma}) g^{\gamma\sigma} \right) + \frac{i}{2} \Theta^{\lambda\kappa} \left( (\partial_\kappa R^{(0)}_{\mu\nu\rho\tau}) (\partial_\lambda g^{\tau\gamma}) g^{\gamma\sigma} \right. \]
\[ \left. \quad - \left( \partial_\rho \Gamma^{(0)}_{\mu\nu} \right) g^{\gamma\sigma} + \partial_\mu \left( (\partial_\lambda g^{\beta\gamma}) g^{\gamma\sigma} \right) + (\partial_\lambda \Gamma^{(0)}_{\mu\nu\sigma}) \right) - (\mu \leftrightarrow \nu). \] (16)

It is antisymmetric under the exchange of \( \mu \) and \( \nu \). There are as many as 14 nonzero components and 7 of them are independent for the spacetime under consideration. They are the following:

\[ R^{(1)}_{r\phi\phi} = -\frac{3 i \Theta^{r\phi}}{r^2 \sin^2 \theta}, \quad R^{(1)}_{r\phi\phi} = \frac{i \Theta^{r\phi} (\cos^2 \theta - 2)}{r^2 \sin^2 \theta}, \]
\[ R^{(1)}_{r\phi\phi} = \frac{i \Theta^{r\phi} (2 - 3 \cos^2 \theta)}{r^2}, \quad R^{(1)}_{r\phi\phi} = \frac{i \Theta^{r\phi} (1 + \cos^2 \theta)}{r^2 (1 - \cos^2 \theta)}, \]
\[ R^{(1)}_{\theta\phi\phi} = \frac{i \Theta^{\theta\phi} \cot \theta}{r}, \quad R^{(1)}_{\theta\phi\phi} = -i \Theta^{\theta\phi} \frac{\sin \theta}{r^2}, \] (17)
\[ R^{(1)}_{\theta\phi\phi} = \frac{i \Theta^{\theta\phi} \sin \theta \cos \theta}{r}. \]
The first order correction to the Ricci tensor is simply given by the sum over two indices in the first order Riemann tensor,
\[ R^{(1)}_{\mu\nu} = R^{(1)}_{\mu\sigma\nu\sigma} \], and it has 4 nonzero components:
\[ R^{(1)}_{r\theta} = \frac{i}{8} \Theta^{r\theta} \cos^2 \theta - \frac{3}{2} r^2 \sin^2 \theta, \]
\[ R^{(1)}_{\theta r} = \frac{i}{8} \Theta^{r\theta} (1 + \cos^2 \theta) r^2 \left(1 - \cos^2 \theta\right), \]
\[ R^{(1)}_{\theta\theta} = \frac{i}{8} \Theta^{r\theta} \cot \theta r, \]
\[ R^{(1)}_{\phi\phi} = -\frac{i}{4} \Theta^{r\theta} \sin \theta \cos \theta r. \] (18)

The first order correction to the curvature scalar is, however, zero. This can be easily seen since in our case
\[ R^{(1)} = G_{\mu\nu}^{\alpha\beta} R^{(0)}_{\nu\mu} + g_{\mu\nu} R^{(1)}_{\nu\mu} + \frac{i}{2} \Theta^{\alpha\beta} (\partial_\alpha g^{\mu\nu}) (\partial_\beta R^{(0)}_{\nu\mu}) = g_{\theta\theta} R^{(1)}_{\theta\theta} + g_{\phi\phi} R^{(1)}_{\phi\phi} = 0. \]

### 4.2 Second Order NC Corrections

#### 4.2.1 Metric, Inverse Metric and Christoffel Symbols

Because of the nature of the definition Eq.(8) of the metric tensor in the NC spacetime, it has the second order correction [19]:
\[ G^{(2)}_{\mu\nu} = -\frac{1}{8} \theta^{\alpha_1\beta_1} \theta^{\alpha_2\beta_2} (\partial_{\alpha_1} \partial_{\alpha_2} e^a_\mu)(\partial_{\beta_1} \partial_{\beta_2} e^b_\nu) \eta_{ab}. \] (19)

For the case under consideration, the only nonzero component is
\[ G^{(2)}_{\phi\phi} = \frac{1}{4} (\Theta^{r\theta})^2 \cos^2 \theta. \] (20)

The second order correction to the inverse metric tensor in the general case is given by [19]:
\[ G^{(2)\mu\nu\star} = \frac{1}{8} \theta^{\alpha_1\beta_1} \theta^{\alpha_2\beta_2} \left( (\partial_{\alpha_1} \partial_{\alpha_2} g^{\mu\gamma})(\partial_{\beta_1} \partial_{\beta_2} g_{\gamma\eta}) + g^{\mu\gamma}(\partial_{\alpha_1} \partial_{\alpha_2} e^a_\mu)(\partial_{\beta_1} \partial_{\beta_2} e^b_\nu) \eta_{ab} \right. 
\[ - 2 \partial_{\alpha_1} (\partial_{\alpha_2} g^{\mu\gamma})(\partial_{\beta_2} g_{\gamma\delta}) \delta^\delta_\eta (\partial_{\beta_1} g_{\eta\nu}) \left) g^{\eta\nu}, \] (21)

and for the NC Minkowski spacetime in spherical polar coordinates, the only component that has nontrivial correction is
\[ G^{(2)\phi\phi\star} = - (\Theta^{r\theta})^2 \frac{4 + \cos^2 \theta}{4 r^4 \sin^4 \theta}. \] (22)

The Christoffel symbols of the second kind have a rather lengthy expression for its second order correction [19]. We give here only the independent nonzero components:
\[ \Gamma^{(2)}_{\phi\phi} = -\frac{13 (\Theta^{r\theta})^2 \sin 2 \theta}{8 r^2}, \quad \Gamma^{(2)}_{\phi\theta} = \frac{(\Theta^{r\theta})^2 (2 - 5 \cos^2 \theta)}{4 r^3 \sin^2 \theta}, \quad \Gamma^{(2)}_{\phi\phi} = -\frac{3 (\Theta^{r\theta})^2 \cos \theta}{4 r^2 \sin^3 \theta}. \] (23)

So it turns out that only those Christoffel symbols that have first order corrections acquire nontrivial second order corrections as well.
4.2.2 Curvature Tensors and Curvature Scalar

The following general expression for the second order correction to Riemann tensor [19] involves only the Christoffel symbols and their derivatives,

\[
R^{(2)}_{\mu\nu\rho} = \partial_\rho \Gamma^{(2)}_{\mu\nu} + \Gamma^{(2)}_{\nu\rho} \Gamma^{(0)}_{\mu\gamma} + \Gamma^{(0)}_{\nu\rho} \Gamma^{(2)}_{\mu\gamma} + \frac{i}{2} \theta^{\alpha\beta} \left( \partial_\alpha \Gamma^{(1)}_{\nu\rho} \Gamma^{(0)}_{\mu\gamma} + \partial_\beta \Gamma^{(0)}_{\nu\rho} \Gamma^{(1)}_{\mu\gamma} \right) - \frac{1}{8} \theta^{\alpha_1\beta_1} \theta^{\alpha_2\beta_2} \left( \partial_{\alpha_1} \partial_{\alpha_2} \Gamma^{(0)}_{\nu\rho} \Gamma^{(0)}_{\mu\gamma} \right) - (\mu \leftrightarrow \nu). \tag{24}
\]

These are antisymmetric under the exchange of \( \mu \) and \( \nu \). On the whole, there are 18 nonzero components in the present case and 9 of them are independent. They are as follows:

\[
\begin{align*}
R^{(2)}_{\theta\phi\phi} &= \frac{-(\Theta^{r\theta})^2 \cos \theta}{r^3 \sin^3 \theta}, \\
R^{(2)}_{\phi\theta\theta} &= \frac{-(\Theta^{r\theta})^2 (5 \cos^2 \theta + 16) \cos \theta}{4 r^3 \sin^3 \theta}, \\
R^{(2)}_{\phi\phi\theta} &= \frac{5 (\Theta^{r\theta})^2 (21 \cos^2 \theta - 20) \cos \theta}{4 r^3 \sin \theta}, \\
R^{(2)}_{\phi\phi\phi} &= \frac{5 (\Theta^{r\theta})^2 (5 \cos^4 \theta - 21 \cos^2 \theta + 1)}{4 r^2 \sin^4 \theta}, \\
R^{(2)}_{\theta\theta\phi} &= \frac{2 \Theta^{r\theta} (55 \cos^2 \theta - 43 \cos^4 \theta - 15)}{4 r^2 \sin^2 \theta}.
\end{align*}
\]

The sum over the repeated index in \( R^{(2)}_{\mu\nu} \) leads to nontrivial second order corrections to five Ricci tensor components:

\[
\begin{align*}
R^{(2)}_{rr} &= \frac{9 (\Theta^{r\theta})^2 (2 - 5 \cos^2 \theta)}{4 r^4 \sin^2 \theta}, \\
R^{(2)}_{r\theta} &= \frac{-(\Theta^{r\theta})^2 (5 \cos^2 \theta + 16) \cos \theta}{4 r^3 \sin^3 \theta}, \\
R^{(2)}_{\theta r} &= \frac{-(\Theta^{r\theta})^2 (12 + 5 \cos^2 \theta)}{4 r^3 \sin^3 \theta}, \\
R^{(2)}_{\theta\theta} &= \frac{(\Theta^{r\theta})^2 (5 \cos^4 \theta - 21 \cos^2 \theta + 1)}{4 r^2 \sin^4 \theta}, \\
R^{(2)}_{\phi\phi} &= \frac{2 \Theta^{r\theta} (43 \cos^4 \theta - 55 \cos^2 \theta + 15)}{4 r^2 \sin^2 \theta} - \left( \frac{7 \cos^2 \theta + 2}{\sin^2 \theta} \right).
\end{align*}
\]

The curvature scalar in the second order is expressed as [19]

\[
R^{(2)} = G^{(2)\mu\nu} R^{(0)}_{\mu\nu} + g^{\mu\nu} R^{(2)}_{\nu\rho} + G^{(1)\mu\nu} R^{(1)}_{\nu\mu} \tag{27}
\]

\[
+ \frac{i}{2} \theta^{\alpha\beta} \left( \partial_\alpha g^{\mu\nu} \Gamma^{(1)}_{\nu\rho} \Gamma^{(0)}_{\mu\gamma} + \partial_\beta \Gamma^{(0)}_{\nu\rho} \Gamma^{(1)}_{\mu\gamma} \right) - \frac{1}{8} \theta^{\alpha_1\beta_1} \theta^{\alpha_2\beta_2} \left( \partial_{\alpha_1} \partial_{\alpha_2} g^{\mu\nu} \Gamma^{(0)}_{\nu\rho} \Gamma^{(0)}_{\mu\gamma} \right) - (\mu \leftrightarrow \nu).
\]

Unlike the first order, this second order picks up nontrivial curvature scalar correction and the result is

\[
R^{(2)} = \frac{4 (\Theta^{r\theta})^2 (7 \cos 4\theta - 11 \cos 2\theta - 2)}{r^4 (\cos 4\theta - 4 \cos 2\theta + 3)}. \tag{28}
\]
5 NC Minkowski Spacetime in Parabolic Coordinates

For the definition and discussion of this coordinate system in the 3-D commutative case, see for example [26]. If we denote the Minkowski spacetime coordinates as \( x^\mu = (x^1, x^2, x^3, x^4) \) = \((u, v, \phi, t)\), then the tetrad can be compactly specified by

\[
e_{\mu}^a = \text{diag} \left( \sqrt{u^2 + v^2}, \sqrt{u^2 + v^2}, uv, -c \right),
\]

and the corresponding metric tensor and the inverse metric tensor are given by

\[
g_{\mu\nu} = \text{diag} \left( u^2 + v^2, u^2 + v^2, u^2v^2, -c^2 \right), \quad g^{\mu\nu} = \text{diag} \left( \frac{1}{u^2 + v^2}, \frac{1}{u^2 + v^2}, \frac{1}{u^2v^2}, -\frac{1}{c^2} \right).
\]

We choose this coordinate system because the corresponding tetrad is dependent on two coordinates. Also, for the transformation \((u, v, \phi, t) \rightarrow (uv \cos \phi, uv \sin \phi, \frac{1}{2}(u^2 - v^2), t)\) from parabolic system to Cartesian system, the Jacobian determinant, which is equal to \((u^2 + v^2)uv\), vanishes in a nontrivial subset of the domain of the transformation.

In addition to the symmetric nature of the metric, we also have \(g_{uu} = g_{vv}\). This results in 6 distinct nonzero Christoffel symbols in the zeroth order case, and they are worked out to be

\[
\Gamma^{(0)}_{uu} = -\Gamma^{(0)}_{vv} = \Gamma^{(0)}_{uv} = \frac{u}{u^2 + v^2}, \quad \Gamma^{(0)}_{vu} = -\Gamma^{(0)}_{uv} = \frac{v}{u^2 + v^2},
\]

\[
\Gamma^{(0)}_{\phi\phi} = \frac{-uv^2}{u^2 + v^2}, \quad \Gamma^{(0)}_{\phi\phi} = \frac{-u^2v}{u^2 + v^2}, \quad \Gamma^{(0)}_{u\phi} = \frac{1}{u}, \quad \Gamma^{(0)}_{v\phi} = \frac{1}{v}.
\]

But in this commutative case, the Riemann tensor \(R^{(0)}_{\mu\nu\rho\sigma}\), the Ricci tensor \(R^{(0)}_{\mu\nu}\) and the curvature scalar \(R^{(0)}\) are all zero.

Let’s promote the two coordinates \(u\) and \(v\) as noncommuting operators with the commutation relation

\[
[\hat{u}, \hat{v}] = i\Theta^{uv}.
\]

Any other nontrivial commutation relation in the theory is not of our concern as they won’t be changing the curvature properties of the geometry.

5.1 First Order NC Corrections

5.1.1 Inverse Metric Tensor and Christoffel Symbols

Substitution of the zeroth order metric components Eq.(30) into Eq.(13) results in \(G^{(1)\mu\nu} = 0\), i.e., there are no first order corrections to the inverse metric tensor.
The first order nonzero Christoffel symbols Eq.(14) are worked out to be

\[
\Gamma^{(1)}_{uu} = -\frac{i\Theta^{uv} u}{(u^2 + v^2)^2}, \quad \Gamma^{(1)}_{uv} = -\frac{i\Theta^{vu} v}{(u^2 + v^2)^2}, \quad \Gamma^{(1)}_{vv} = -\frac{i\Theta^{vu} v}{(u^2 + v^2)^2} \quad \Gamma^{(1)}_{uu} = -\frac{i\Theta^{uv} u}{(u^2 + v^2)^2} \quad \Gamma^{(1)}_{uv} = -\frac{i\Theta^{vu} v}{(u^2 + v^2)^2}, \quad \Gamma^{(1)}_{vv} = -\frac{i\Theta^{vu} v}{(u^2 + v^2)^2}.
\]

5.1.2 Curvature Tensors and Curvature Scalar

There are 7 distinct nonzero components of Riemann tensor in the first order:

\[
R^{(1)}_{uu} = \frac{2i\Theta^{uv} u}{(u^2 + v^2)^2}, \quad R^{(1)}_{uv} = -\frac{3i\Theta^{uv} u}{u^2v^2}, \quad R^{(1)}_{uv} = \frac{i\Theta^{uv} u(u^2 - 2u^2)}{(u^2 + v^2)^2}, \quad R^{(1)}_{uv} = \frac{i\Theta^{uv} u(2u^2 + v^2)}{(u^2 + v^2)u^2v^2},
\]

There are only 3 nonzero first order components for Ricci tensor:

\[
R^{(1)}_{uv} = \frac{-i\Theta^{uv} (u^4 + u^2v^2 + 2v^4)}{(u^2 + v^2)^2u^2v^2}, \quad R^{(1)}_{uv} = \frac{i\Theta^{uv} (2u^4 + u^2v^2 + v^4)}{(u^2 + v^2)^2u^2v^2}, \quad R^{(1)}_{uv} = \frac{8i\Theta^{uv} (u^2 - v^2)uv}{(u^2 + v^2)^3}.
\]

Unlike the case of spherical polar coordinates, the parabolic system leads to nontrivial first order correction to the curvature scalar:

\[
R^{(1)} = \frac{8i\Theta^{uv} (u^2 - v^2)}{(u^2 + v^2)^3}.
\]

5.2 Second Order NC Corrections

5.2.1 Metric, Inverse Metric and Christoffel Symbols

The only component of the metric tensor that picks up the second order NC correction is

\[
G^{(2)}_{\phi\phi} = \frac{(\Theta^{uv})^2}{4}.
\]

But there are three nonzero second order components of the inverse metric tensor and all of them are diagonal:

\[
G^{(2)}_{uu} = \frac{(\Theta^{uv})^2}{(u^2 + v^2)^3}, \quad G^{(2)}_{vv} = \frac{(\Theta^{uv})^2}{(u^2 + v^2)^3}, \quad G^{(2)}_{\phi\phi} = \frac{-5(\Theta^{uv})^2}{u^4v^4}.
\]
The Christoffel symbols pick up the following 6 distinct nonzero second order corrections:

\[
\begin{align*}
\Gamma_{uu}^{(2)} &= \frac{(\Theta^{uv})^2 u}{(u^2 + v^2)^3} = \Gamma_{vv}^{(2)} = -\Gamma_{uv}^{(2)}, \\
\Gamma_{u\phi}^{(2)} &= -\frac{3(\Theta^{uv})^2}{4u^3v^2}, \\
\Gamma_{\phi\phi}^{(2)} &= \frac{(\Theta^{uv})^2 u (3u^2 - 11v^2)}{2(u^2 + v^2)^3}, \\
-\Gamma_{\phi\phi}^{(2)} &= \frac{(\Theta^{uv})^2 v (11u^2 - 3v^2)}{2(u^2 + v^2)^3}. \\
\end{align*}
\]

(39)

5.2.2 Curvature Tensors and Curvature Scalar

In NC Minkowski spacetime in parabolic coordinates, there are 11 Riemann tensor components that receive second corrections, apart from the antisymmetric ones under the exchange of first two indices:

\[
\begin{align*}
R_{uuu}^{(2)} &= -R_{uu}^{(2)} = \frac{12(\Theta^{uv})^2}{(u^2 + v^2)^3}, \\
R_{u\phi\phi}^{(2)} &= \frac{2(\Theta^{uv})^2(-u^2 + v^2)}{(u^2 + v^2)^3}, \\
R_{u\phi\phi}^{(2)} &= -\frac{3(\Theta^{uv})^2(13u^2 + 5v^2)}{4(u^2 + v^2)^3}, \\
R_{\phi\phi\phi}^{(2)} &= \frac{-3(\Theta^{uv})^2(5u^2 + 13v^2)}{4(u^2 + v^2)^2u^4v^2}, \\
R_{\phi\phi\phi}^{(2)} &= \frac{-3(\Theta^{uv})^2(11u^6 + 175u^4v^2 + 341u^2v^4 - 9v^6)}{4(u^2 + v^2)^4uv}, \\
R_{\phi\phi\phi}^{(2)} &= \frac{-3(\Theta^{uv})^2(3u^6 - 39u^4v^2 + 393u^2v^4 - 101v^6)}{4(u^2 + v^2)^4u^2v^2}. \\
\end{align*}
\]

(40)

The sum over the repeated index in $R_{\mu\nu\sigma\tau}^{(2)}$ results in the second order Ricci tensor, and there are 5 nontrivial components:

\[
\begin{align*}
R_{uu}^{(2)} &= \frac{3(\Theta^{uv})^2(13u^6 + 15u^4v^2 + 23u^2v^4 + 5v^6)}{4(u^2 + v^2)^3u^4v^2}, \\
R_{uv}^{(2)} &= \frac{-3(\Theta^{uv})^2(29u^4 + 42u^2v^2 + 21v^4)}{4(u^2 + v^2)^2u^3v^3}, \\
R_{vu}^{(2)} &= \frac{-3(\Theta^{uv})^2(21u^4 + 42u^2v^2 + 29v^4)}{4(u^2 + v^2)^2u^3v^3}, \\
R_{vv}^{(2)} &= \frac{-3(\Theta^{uv})^2(5u^6 + 23u^4v^2 + 15u^2v^4 + 13v^6)}{4(u^2 + v^2)^3u^2v^4}, \\
R_{\phi\phi}^{(2)} &= \frac{(\Theta^{uv})^2(3u^8 - 140u^6v^2 + 786u^4v^4 - 140u^2v^6 + 3v^8)}{4(u^2 + v^2)^4u^2v^2}. \\
\end{align*}
\]

(41)

The correction to the scalar curvature is worked out to be

\[
R^{(2)} = \frac{(\Theta^{uv})^2(u^8 - 2u^6v^2 - 54u^4v^4 - 2u^2v^6 + 3v^8)}{(u^2 + v^2)^4u^2v^2}. 
\]

(42)
6 Concluding Remarks

An integral representation of \(*\)-product has been given in [27] (see also [28]) which holds for any odd or even dimension \(n > 1\):

\[
(f * g)(x) = \frac{1}{(2\pi)^n} \int d^n y \, d^n z \, f(x - \frac{1}{2}\Theta y) \, g(x + z) \, e^{-iyz}.
\]  
(43)

This representation clearly implies that the value of \(f \ast g\) at a particular point \(x\) depends on the values of \(f\) and \(g\) at all points in the domain. Therefore, any coordinate transformation cannot restrict the domain in the new coordinate system, else the nature of \(f \ast g\) will change. In the commutative case, in which the value of \(f \cdot g\) at a point \(x\) depends only on the values of \(f\) and \(g\) at \(x\), the restricted domain in the new system doesn’t lead to any interesting consequences. But with the inclusion of the points \(r = 0\) and \(\theta = \pi\), the transformation from spherical polar to Cartesian coordinates is not a bijective local diffeomorphism. Only a bijective local diffeomorphism is a diffeomorphism [29]. Since the Jacobian determinant of the transformation is equal to \(r^2 \sin \theta\) which vanishes at the values of \(r = 0\), \(\theta = 0\) and \(\theta = \pi\), the transformation does not have unique inverse resulting in the non-injective nature of the mapping \((r, \theta, \phi, t) \mapsto (r \sin \theta \cos \phi, r \sin \theta \sin \phi, r \cos \theta, t)\). Hence, the NC Minkowski spacetime parametrized with spherical polar coordinates and the NC Minkowski spacetime parametrized with Cartesian coordinates are not \(\Theta\)-diffeomorphic to each other, and hence the introduction of noncommutativity among \(r\) and \(\theta\) leads to an entirely different NC spacetime with curvature effects. These curvature effects vanish completely in the limit \(\Theta \rightarrow 0\). Similar arguments hold for the case of parabolic coordinates for which the Jacobian determinant of the surjective-only transformation \((u, v, \phi, t) \mapsto (uv \cos \phi, uv \sin \phi, \frac{1}{2}(u^2 - v^2), t)\) is equal to \((u^2 + v^2)uv\). Also, in the non-relativistic limit, similar NC corrections would show up for the physical 3-D space parametrized with such curvilinear coordinates.

In general, if the metric tensor in the commutative case depends on more than one curvilinear coordinates and the Jacobian determinant of the transformation to another coordinate system is vanishing in a nontrivial subset of the domain of the transformation, thereby making the transformation non-injective, then the introduction of coordinate-noncommutativity in the two cases can lead to different curvature properties. A detailed study on this aspect of different spaces, including the NC versions of curved spaces of the commutative case, will be published elsewhere [30].

Acknowledgments

MR would like to thank the DST for the INSPIRE fellowship. The authors gratefully acknowledge the funding No.PU/PS2/PHYS/Minor-Equip/21-22/241 by Pondicherry University under the Minor Equipment Grant No.PU/PD2/Minor-Equip/2022/536. Thanks are also due to Dr.RSK for his efforts to develop the computational lab in the department.
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