A MULTIFRACTAL ANALYSIS FOR CUSPIDAL WINDINGS ON HYPERBOLIC SURFACES

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ABSTRACT. In this paper we investigate the multifractal decomposition of the limit set of a finitely generated, free Fuchsian group with respect to the mean cusp winding number. We will completely determine its multifractal spectrum by means of a certain free energy function and show that the Hausdorff dimension of sets consisting of limit points with the same scaling exponent coincides with the Legendre transform of this free energy function. As a by-product we generalise previously obtained results on the multifractal formalism for infinite iterated function systems to the setting of infinite graph directed Markov systems.

1. INTRODUCTION AND STATEMENT OF RESULTS

In this paper we carry out a multifractal analysis of cusp windings of the geodesic flow on $\mathbb{H}/G$, for a finitely generated, free, non-elementary Fuchsian group $G$ with parabolic elements acting on the upper half-space model $(\mathbb{H}, d)$ of 2-dimensional hyperbolic space. Recall that to each $x$ in the radial limit set $L_r(G)$ of $G$ one can associate an infinite word whose letters come from the symmetric set $G_0$ of generators of $G$. Note that $G$ can be written as a free product $G = H \ast \Gamma$, where $H = \langle h_1 \rangle \ast \ldots \ast \langle h_u \rangle$ denotes the free product of finitely many elementary hyperbolic groups, and $\Gamma = \langle \gamma_1 \rangle \ast \ldots \ast \langle \gamma_v \rangle$ denotes the free product of finitely many parabolic subgroups of $G$ such that $\langle \gamma_i \rangle$ is the parabolic subgroup of $G$ associated with the parabolic fixed point $p_i$ (see also [KS04]). We will always assume that $v \geq 1$; furthermore, the fact that $G$ is non-elementary implies that $u + v > 1$. Clearly, $\langle \gamma_i \rangle \cong \mathbb{Z}$ and $\gamma_i^{-1}(p_i) = p_i$, for all $i = 1, \ldots, v$. It is well known [B97] that the Poincaré exponent $\delta = \delta_G$ of $G$ coincides with the Hausdorff dimension $\dim_H(L_r(G))$ of the radial limit set of $G$.

There is a natural coding of the limit set by infinite sequences over the set of generators. That is, with $F$ referring to the Dirichlet fundamental domain of $G$ containing $i \in \mathbb{H}$, the images of $F$ under $G$ tessellate $\mathbb{H}$ and each side of each of the tiles is uniquely labelled by an element of $G_0$. The hyperbolic ray $s_i$ from $i$ towards $x \in L_r(G)$ must traverse infinitely many of these tiles, and the infinite word expansion associated with $x$ is then obtained by progressively recording, starting at $i$, the generators of the sides at which $s_i$ exits the tiles. In this way we derive an infinite word $\omega = (\omega_1, \omega_2, \ldots) \in G_0^\mathbb{N}$, which is necessarily reduced, where reduced means that $\omega_n \omega_{n+1} \neq \text{id}$, for all $n \in \mathbb{N}$. We then form a sequence of blocks $(B_n)_{n \in \mathbb{N}}$ in this word in the following way. Each hyperbolic generator that appears in the word is called a block of length 1. Further, if the same parabolic generator appears consecutively exactly $n$ times, then this is called a block of length $n$. By construction, such a block of length $n$ corresponds to the event that the projection of $s_i$ onto $\mathbb{H}/G$ spirals
We say that the mean cusp-winding number whenever the limit exists. Here, \( \log \)
where \( B_k \) denotes the \( k \)-th block in the infinite word associated to \( x \) and \( |B_k| \) denotes its length. Our main aim is to investigate the fluctuation of a certain asymptotic exponential scaling associated to this process, thereby extending results in \([JK10, JK11]\) for \( \text{PSL}_2(\mathbb{Z}) \).

We say that the mean cusp-winding number of \( x \in L_r(G) \) is given by
\[
\lim_{n \to \infty} \frac{2 \sum_{k=1}^{n} \log^+(a_k)}{d(B_1 \cdots B_n(i), i)},
\]
whenever the limit exists. Here, \( \log^+(a) \) is defined as \( \begin{cases} \log(a), & \text{if } a > 0; \\ 0, & \text{if } a = 0. \end{cases} \) The fluctuation of this quantity is captured in the following level sets with prescribed scaling constant \( \alpha \in \mathbb{R} \) given by
\[
\mathcal{F}_\alpha := \left\{ x \in L_r(G) \mid \lim_{n \to \infty} \frac{2 \sum_{k=1}^{n} \log^+(a_k)}{d(B_1 \cdots B_n(i), i)} = \alpha \right\},
\]
and
\[
\mathcal{F}_\alpha^+ := \left\{ x \in L_r(G) \mid \limsup_{n \to \infty} \frac{2 \sum_{k=1}^{n} \log^+(a_k)}{d(B_1 \cdots B_n(i), i)} \geq \alpha \right\}, \quad \alpha \geq \alpha_0,
\]
\[
\mathcal{F}_\alpha^- := \left\{ x \in L_r(G) \mid \liminf_{n \to \infty} \frac{2 \sum_{k=1}^{n} \log^+(a_k)}{d(B_1 \cdots B_n(i), i)} \leq \alpha \right\}, \quad \alpha < \alpha_0.
\]
The following facts will be proved in Section 3.

**Fact 1.1.** The subset of the limit set
\[
L_c = \{ x \in L_r(G) \mid a_k(x) \leq 1 \text{ for all } k \in \mathbb{N} \},
\]
encoding those geodesic rays whose maximal winding around any given cusp is at most one, is contained in \( \mathcal{F}_0 \). In particular, where \( H_0 := \{ h_i^i h_i^{-1} \mid 1 \leq i \leq u \} \) denotes the symmetric set of hyperbolic generators, the set of limit points \( L(\langle H_0 \rangle) \) that can be coded exclusively via hyperbolic elements is contained in \( \mathcal{F}_0 \). That is,
\[
L(\langle H_0 \rangle) \subset L_c \subset \mathcal{F}_0.
\]

**Fact 1.2.** We have that the set \( \{ x \in L_r(G) \mid \lim_{n \to \infty} n^{-1} \sum_{k=1}^{n} \log a_k(x) = \infty \} \) is contained in \( \mathcal{F}_1 \). In particular, for the Jarník set
\[
\mathcal{J} := \left\{ x \in L_r(G) \mid \lim_{n \to \infty} n^{-1} \sum_{k=1}^{n} \log a_k(x) = \infty \text{ and } \lim_{n \to \infty} \frac{\log a_n(x)}{d(B_1 \cdots B_n(i), i)} = 0 \right\}
\]
considered in \([Mun12]\) we have \( \mathcal{J} \subset \mathcal{F}_1 \) and \( \dim_H(\mathcal{J}) = 1/2 \).

**Fact 1.3.** For every \( x \in \mathcal{F}_1 \) we have that the sequence \( \{ a_n(x) \}_{n \in \mathbb{N}} \) is unbounded.

**Fact 1.4.** We have \( \mathcal{F}_\alpha \neq \emptyset \) if and only if \( \alpha \in [0, 1] \).

Since these level sets are generally Lebesgue null sets, the Hausdorff dimension \( \dim_H(\mathcal{F}_\alpha) \) is an appropriate quantity to measure the size of the sets \( \mathcal{F}_\alpha \). In this paper we will give a complete analysis of the cusp-winding spectrum
\[
f(\alpha) := \dim_H(\mathcal{F}_\alpha), \quad \alpha \in \mathbb{R}.
\]
Using the Thermodynamic Formalism we will be able to express the function \( f \) on \([0, 1]\) implicitly in terms of the cusp-winding pressure function
\[
P(t, \beta) := \lim_{n \to \infty} \frac{1}{n} \log \sum_{B_1 \cdots B_n \in \mathcal{G}_n} e^{-t d(B_1 \cdots B_n(i), i) - 2 \beta \sum_{k=1}^{n} \log^+(a_k)}, \quad t, \beta \in \mathbb{R},
\]
Further, the set of points for which the mean cusp winding number does not exist has full Hausdorff dimension.

Theorem 1.5

The Hausdorff dimension for the cusp winding spectrum is given by

\[
\delta, \quad \frac{1}{2}, \quad \frac{1}{2}
\]

The function \( f \) on \((0, 1)\) is strictly concave, continuous, real-analytic on \((0, 1)\) and its maximal value is equal to \( \delta(G) \) (cf. Fig 1.1). For the boundary points we have

\[
f(0) = \delta, \quad f(1) = \frac{1}{2}, \text{ and } \lim_{\alpha \to 0} f'(\alpha) = -1, \lim_{\alpha \to 1} f'(\alpha) = +\infty.
\]

Further, the set of points for which the mean cusp winding number does not exist has full Hausdorff dimension \( \delta(G) \).

If we could apply Theorem 1.5 to the modular surface we would relate the cuspidal winding spectrum to the arithmetic-geometric spectrum investigated in [JK10]. In fact, if instead of the modular group \( \text{PSL}_2(\mathbb{Z}) \) we consider a torsion-free normal modular subgroup \( \Gamma(2) \)
of index 6, the condition of freeness would be achieved for $\Gamma(2)$. Then the arithmetic-geometric scaling level sets in [JK10] correspond to the level sets of the mean cuspidal winding for $\Gamma(2)$.

Another main result of this paper is the multifractal formalism for level sets of quotients of Birkhoff sums in the context of conformal graph directed Markov systems (see Theorem 4.1). This result is obtained by extending our previous results in the context of conformal iterated function systems ([JK11]). By using the methods of [JK11], we are able to develop the multifractal formalism without any technical summability assumptions on the thermodynamic potentials, which have been imposed, for example, in [MU03, Section 4.9].

2. Ergodic Theory for Fuchsian Groups

In this section we study the ergodic theory for Fuchsian groups with parabolic elements. In Section 2.1 we describe the action of $G$ on $\partial \mathbb{H}$ by a canonical Markov map which is referred to as the Bowen–Series map. In Section 2.2 we give the crucial geometric estimates which underlie the whole multifractal analysis. In Section 2.3 we set up the induced dynamics of the Bowen–Series map on the complement of some neighbourhood of the parabolic fixed points of $G$. For the induced system we can later use Gibbs theory to prove the multifractal formalism.

2.1. The canonical Markov map. As already mentioned in the introduction, throughout we exclusively consider a finitely generated, free Fuchsian group $G$. Recall that $G$ can be written as a free product $G = H \ast \Gamma$, where $H = \langle h_1 \rangle \ast \ldots \ast \langle h_v \rangle$ denotes the free product of finitely many elementary, hyperbolic groups, and $\Gamma = \langle \gamma_1 \rangle \ast \ldots \ast \langle \gamma_i \rangle$ denotes the free product of finitely many parabolic subgroups $\langle \gamma_i \rangle$ of $G$ with the parabolic fixed point $p_i = \gamma_i^*(p_i)$, $i = 1, \ldots, v$. Since $G$ is finitely generated, $G$ admits the choice of a Poincaré polyhedron $F$ with a finite set $\mathcal{F}$ of faces. Let us now first recall from [SS05] the construction of the relevant coding map $T$ associated with $G$, which maps the radial limit set $L_r(G)$ into itself. This construction parallels the construction of the well-known Bowen–Series map (cf. [BS79, Stat04]). For $\xi, \eta \in L_r(G)$, let $\gamma_{\xi, \eta} : \mathbb{R} \to \mathbb{H}$ denote the directed geodesic from $\eta$ to $\xi$ such that $\gamma_{\xi, \eta}$ intersects the closure $\overline{F}$ of $F$ in $\mathbb{H}$, and normalised such that $\gamma_{\xi, \eta}(0)$ is the point on the geodesic from $\eta$ to $\xi$ which is closest to $i$. The exit time $e_{\xi, \eta}$ is defined by

$$e_{\xi, \eta} := \sup \{ s \mid \gamma_{\xi, \eta}(s) \in \overline{F} \}.$$  

Since $\xi, \eta \in L_r(G)$, we clearly have that $|e_{\xi, \eta}| < \infty$. By Poincaré’s Polyhedron Theorem (cf. [EP94]), we have that the set $\mathcal{F}$ carries an involution $\mathcal{F} \to \mathcal{F}$, given by $s \mapsto s'$ and $s'' = s$. In particular, for each $s \in \mathcal{F}$ there is a unique face-pairing transformation $g_s \in G$ such that $g_s(\overline{F}) \cap \overline{F} = s'$. We then let

$$\mathcal{L}_r(G) := \{ (\xi, \eta) \mid \xi, \eta \in L_r(G), \xi \neq \eta \text{ and } \exists t \in \mathbb{R} : \gamma_{\xi, \eta}(t) \in \overline{F} \},$$

and define the map $S : \mathcal{L}_r(G) \to \mathcal{L}_r(G)$, for all $(\xi, \eta) \in \mathcal{L}_r(G)$ such that $\gamma_{\xi, \eta}(e_{\xi, \eta})$ is in $s$, for some $s \in \mathcal{F}$, by

$$S(\xi, \eta) := (g_s(\xi), g_s(\eta)).$$

In order to show that the map $S$ admits a Markov partition, we introduce the following collection of subsets of the boundary $\partial \mathbb{H}$ of $\mathbb{H}$. For $s \in \mathcal{F}$, let $A_s$ refer to the open hyperbolic half-space for which $F \subset \mathbb{H} \setminus A_s$ and $s \subset \partial A_s$. We then define the projection $b_s$ of the side $s$ to $\partial \mathbb{H}$ by

$$b_s := \text{Int} \left( \overline{\text{Cl}_\mathbb{H}(A_s)} \cap \partial \mathbb{H} \right).$$

Clearly, $b_s \cap b_t = \emptyset$, for all $s, t \in \mathcal{F}, s \neq t$. Hence, by the convexity of $F$, we have that $\gamma_{\xi, \eta}(e_{\xi, \eta}) \in s$ if and only if $\xi \in b_s$. In other words, $S(\xi, \eta) = (g_s(\xi), g_s(\eta))$ for all $\xi \in b_s$. 


This immediately gives that the projection map $p_1: (\xi,\eta) \mapsto \xi$ onto the first coordinate of $\mathcal{L}_r(G)$ leads to a canonical factor $T$ of $S$, that is, we obtain the map

$$T : L_r(G) \to L_r(G), \text{ given by } T_{|b_i \cap L_r(G)} := g_i.$$  

Clearly, $T$ satisfies $p_1 \circ S = T \circ p_1$. Since $T(b_s) = g_s(b_s) = \text{Int}(\partial H \setminus b_s)$, it follows that $T$ is a non-invertible Markov map with respect to the partition $\{b_i \cap L_r(G) \mid s \in \mathcal{F}\}$. For this so-obtained expansive map $T$ we then have the following result.

**Proposition 2.1** ([SS05 Proposition 2, Proposition 3]). *The map $T$ is a topologically mixing Markov map with respect to the partition generated by $\{b_i \cap L_r(G) \mid s \in \mathcal{F}\}$. Moreover, the map $S$ is the natural extension of $T$.***

### 2.2. Horocircles and basic estimates.

Recall that for each $x \in L_r(G)$ we let $B_k(x)$ denote the $k$-th block of the infinite word expansion of the geodesic ray from $i$ towards $x$ and $a_k(x) := |B_k(x)| - 1$. In order to obtain estimates for $d(B_1 \cdots B_n(i),i)$, we introduce the following horocircles. For each parabolic generator $\gamma \in \Gamma_0$ with fixed point $p_1$, we define the horocircle $H_\gamma$ of Euclidean height $1/2$ which is given as follows: there exists a unique $\Delta \in \text{PSL}_2(\mathbb{R})$ such that $\Delta(p_1) = \infty$ and $\Delta^{-1} \gamma \Delta(z) = z + 1$. Then we set $H_\gamma := \Delta^{-1}(\{\text{Im}(z) = 1/2\})$. Another way to define $H_\gamma$ is to require that the collar geodesic, i.e., the projection of $H_\gamma$ to $\mathbb{H}/G$, has hyperbolic length $2$. These horocircles are pairwise disjoint. To see this, first note that without loss of generality we can assume that one of the parabolic generators of $G$ is of the form $z \mapsto z + 1$, with fixed point at infinity. The fact that $G$ is a free group ensures that the edges of the Dirichlet fundamental domain for $G$ do not intersect. An example is shown in the left-hand side of Figure 2.1. In the configuration on the right of Figure 2.1, the horocircle has maximal Euclidean height. It is therefore sufficient to verify that the horocircles are disjoint in that case. Note that this depicts a group which is not free, to have a free group we would need to shrink the edges labelled $u$ and $u'$ so that they don’t touch the vertical lines. This necessarily shrinks the Euclidean height of the horocircle, as otherwise the length of the projected collar geodesic would be greater than 2. Let $\Delta \in \text{PSL}_2(\mathbb{R})$ be given by $\Delta(z) := -1/(4z)$. Then the geodesic $u$ in Figure 2.1 (right) is mapped to the vertical line through $\Delta(-1/2) = 1/2$. Similarly, the geodesic $u'$ is mapped to the vertical line through $\Delta(1/2) = -1/2$. Hence, the parabolic generator $\gamma$ corresponding to the side-pairing of $u$ and $u'$ satisfies $\Delta^{-1} \gamma \Delta(z) = z + 1$ and the dashed horocircle through

**Figure 2.1.** On the left we show a typical fundamental domain with two parabolic and one hyperbolic generator. On the right we show the largest possible horocircle with Euclidean height $1/2$ corresponding to the cusp at 0 just touching another horocircle with Euclidean height $1/2$ corresponding to the cusp at infinity.
0 is represented by the horizontal line through $\Delta(i/2) = i/2$. We have thus shown that the dashed horocircle through zero has Euclidean height $1/2$.

To estimate $d(B_1 \cdots B_n(i), i)$, we will partition the directed geodesic segment $\xi$ which goes from $i$ to $B_1 \cdots B_n(i)$ and corresponds to $x \in \mathcal{L}_r(G)$ into $(n + 1)$ arcs $\xi_1, \ldots, \xi_{n+1}$ as follows. The first arc $\xi_1$ starts at $i$ and the last arc $\xi_{n+1}$ terminates at $B_1 \cdots B_n(i)$. For $k = 1, \ldots, n$, the arc $\xi_k$ terminates at the first intersection of $\xi$ and an image of a side of the Dirichlet fundamental domain corresponding to $G$ centred in $i$, whenever $|B_k| = 1$, or the intersection of $\xi$ with the first horocircle $H_j$, $j = 1, \ldots, \nu$, when leaving this horocircle. The terminating point of $\xi_k$ coincides with the initial point of the following arc $\xi_{k+1}$.

If we cut off the cusps of $\mathbb{H}/G$ along the horocircles $H_1, \ldots, H_{\nu}$, then we obtain a compact manifold which in particular has a finite diameter $C_0 > 0$. Since $\xi_{n+1}$ is contained in this compact set we have for all $n \in \mathbb{N}$ that,

\[\sum_{k=1}^{n} l(\xi_k) \leq d(B_1 \cdots B_n(i), i) = \sum_{k=1}^{n+1} l(\xi_k) \leq \sum_{k=1}^{n} l(\xi_k) + C_0,\]

where $l(\xi_k)$ refers to the hyperbolic length of the arc $\xi_k$ (see Figure 2.2).

Before proving the facts stated in the introduction we will need the following geometric observation.

\[\text{Figure 2.2. The first three geodesic arcs } \xi_1, \xi_2, \xi_3 \text{ connecting } i \text{ and } g\gamma h(i), \gamma \in \Gamma_0, g, h \in H_0.\]
To complete the proof, observe that, for all $a_i$ as defined in Section 2.2.

**Proposition 2.2.** For $n \geq 3$ we have that the length of the geodesic arc $\xi$ lying above \{Im$(z) = 1/2$\} and connecting $\pm(n - 1)/2$ (cf. Figure 2.3) lies between the two constants $\log \left(3 \cdot (n - 1)^2\right)$ and $\log \left(4 \cdot (n - 1)^2\right)$.

*Proof.* Let $0 < h < r$ and put $a := \sqrt{r^2 - h^2}$. We parametrise the circular arc of the circle with centre 0 and radius $r$ which lies above the line \{Im$(z) = h$\}. The parametrisation is given by $\gamma : [-a, a] \to \mathbb{H}, \gamma(t) := \left(t, \sqrt{r^2 - t^2}\right)$. We have

$$\gamma'(t) = \left(1, -t(r^2 - t^2)^{-1/2}\right) \quad \text{and} \quad \|\gamma'(t)\|_2 = \int_{-a}^{a} \frac{1}{\sqrt{r^2 - t^2}} dt = \int_{-a}^{a} \frac{1}{1 - (t/r)^2} dt = \int_{-a/r}^{a/r} \frac{1}{1 - \gamma^2} dy = \log \left(\frac{1 + a/r}{1 - a/r}\right).$$

Hence, the hyperbolic length of $\gamma$ is given by

$$l(\gamma) = \int_{-a}^{a} \|\gamma'(t)\|_2 dt = \frac{1}{r} \int_{-a}^{a} \frac{1}{1 - (t/r)^2} dt = \int_{-a/r}^{a/r} \frac{1}{1 - \gamma^2} dy = \log \left(\frac{1 + a/r}{1 - a/r}\right).$$

It will be convenient to write

$$l(\gamma) = \log \left(\frac{1 + a/r}{1 - a/r}\right) = \log \left(\frac{(1 + a/r)^2}{1 - (a/r)^2}\right).$$

To verify the statement in the proposition, we must consider the case that $h := 1/2$ and $r := (n - 1)/2$. Then $a = \sqrt{n^2 - 2n}/2$ and by our previous calculation we obtain

$$l(\xi) = \log \left(\frac{1 + \frac{\sqrt{n^2 - 2n}}{n - 1}}{1 - \frac{\sqrt{n^2 - 2n}}{n - 1}}\right)^2 = \log \left((n - 1)^2 \left(1 + \frac{\sqrt{n^2 - 2n}}{n - 1}\right)^2\right).$$

To complete the proof, observe that, for all $n \geq 3$, we have $4 \geq \left(1 + \frac{\sqrt{n^2 - 2n}}{n - 1}\right)^2 \geq 3$. □

**Corollary 2.3.** For any geodesic arc $\xi_i$ corresponding to a given block $B_i$ with $a_i \geq 2$, we have

$$2 \log \left(a_i\right) + \log 3 \leq l(\xi_i).$$

If $a_i \geq 0$ then we still have

$$2 \log^+ \left(a_i\right) + C_1 \leq l(\xi_i) \leq 2 \log (a_i + 1) + C_2,$$

where $C_1 := \min \{\log 3, d(s, t) \mid s, t \in \mathcal{F}, g_i \neq id\}$ and $C_2 := \max \{\log 4, C_0\}$, where $C_0$ as defined in Section 2.2.
2.3. Inducing and the topological Markov chain with infinite state space. Let us set
\[ \mathcal{D} := \{ y \in L_r(G) \mid a_1(y) = 0 \}. \]

The induced transformation \( T_\rho \) on \( \mathcal{D} := \{ a_1 = 0 \} \) is defined by \( T_\rho (\xi) := T^{\rho(\xi)}(\xi) \), where \( \rho \) denotes the return time function, given by \( \rho(\xi) := \min\{ n \in \mathbb{N} \mid T^n(\xi) \in \mathcal{D} \} \). Denote by \( \Gamma_0 := \{ \gamma, \gamma_i^{-1} \mid 1 \leq i \leq \nu \} \) the symmetric set of parabolic generators. Recall that \( G_0 = \Gamma_0 \cup H_0 \). Define the induced partition
\[ \alpha_{\rho} := \bigcup_{n \in \mathbb{N}} \mathcal{D} \cap \{ \rho = n \} \cap \bigcap_{k=0}^{n+1} T^{-k}(\{ b_s \cap L_r(G) \mid s \in \mathcal{D} \}) \]
and the infinite alphabet
\[ I := \{ g_1 \gamma^g_2 \mid \gamma \in \Gamma_0, g_1, g_2 \in G_0 \backslash \{ \gamma \} \in \mathbb{N} \} \cup
\{ g_1 h g_2 \mid g_1 \in G_0, h \in H_0 \backslash \{ g_1 \} \} \in \mathbb{N} \}. \]

The incidence matrix \( A \in \{0,1\}^{I \times I} \) is for \( \eta = \eta_1 \ldots \eta_n \in I \) and \( \tau = \tau_1 \ldots \tau_m \in I \), given by \( A(\eta, \tau) = 1 \) if and only if \( \eta_{n-1} = \tau_1 \) and \( \eta_n = \tau_2 \). We consider the Markov shift \( (\Sigma_A, \sigma) \) where
\[ \Sigma_A := \{ \omega = (\omega_1, \omega_2, \ldots) \in I^\mathbb{N} \mid \forall k \in \mathbb{N} \ A(\omega_k, \omega_{k+1}) = 1 \} \]
and \( \sigma : \Sigma_A \to \Sigma_A \), \( \sigma(\omega) \) is the left shift map.

Recall that \( \mathcal{D} := \{ a_1 = 0 \} \). The induced Bowen–Series map \( (\mathcal{D}, T_\rho) \) is conjugated to the left shift \( (\Sigma_A, \sigma) \) via the coding map \( \pi : \mathcal{D} \to \Sigma_A \) which is defined in terms of the infinite word expansion, as given in the introduction, of limit points. Note that \( \pi \) is surjective, because any limit point with expansion having first block of length 1 necessarily corresponds to a word starting with a letter in \( I \). That is, we have the following commutative diagram
\[ \begin{array}{ccc}
\mathcal{D} & \xrightarrow{T_\rho} & \mathcal{D} \\
\pi \downarrow & & \pi \downarrow \\
\Sigma_A & \xrightarrow{\sigma} & \Sigma_A
\end{array} \]

For \( \omega \in \Sigma_A \) and \( n \in \mathbb{N} \) let \( \omega_{[n]} := (\omega_1, \ldots, \omega_n) \) and let \( \omega|_n := \{ \tau \in \Sigma_A \mid \tau_1 = \omega_1, \ldots, \tau_n = \omega_n \} \) denote the \( n \)-cylinder of \( \omega \). We have \( \pi(\mathcal{D}) = \{ \omega \mid \omega \in I \} \).

We now aim to express the cusp-winding scaling limit in dynamical terms. For this we introduce the two potential functions
\[ \begin{align*}
\psi : \Sigma_A & \to \mathbb{R}_+^\times, \quad (\omega_1, \omega_2, \ldots) \mapsto -2 \log^+ |(\omega_1| - 3), \\
\phi : \Sigma_A & \to \mathbb{R}^\times, \quad (\omega_1, \omega_2, \ldots) \mapsto -\log \left( \left(g_1 \cdots g_{|\omega_1| - 2}\right) \left(\pi^{-1}(\omega)\right) \right),
\end{align*} \]
with \( \omega_1 = g_1 \cdots \in \mathbb{N} \in I \),

where \( \psi \) describes the cusp-winding number, while \( \phi \) describes the geometric properties of the geodesic flow. We will equip \( \Sigma_A \) with the metric \( d \) which is given for each \( \omega, \tau \in I^\mathbb{N} \) by
\[ d(\omega, \tau) := \exp \left(-|\omega \wedge \tau|\right) \],
where \( |\omega \wedge \tau| \) denotes the length of the longest common initial block \( \omega \wedge \tau \) of \( \omega \) and \( \tau \). Since \( \psi \) depends only on the first symbol, we immediately see that \( \psi \) is Hölder continuous with respect to this metric. For the proof that \( \phi \) is also Hölder continuous we refer to [KS04].
2.4. Topological pressure. First, let us fix some notation. For two sequences \((a_n), (b_n)\) we will write \(a_n \ll b_n\), if \(a_n \leq Kb_n\) for some \(K > 0\) and all \(n \in \mathbb{N}\), and if \(a_n \ll b_n\) and \(b_n \ll a_n\) then we write \(a_n \asymp b_n\).

For \(n \in \mathbb{N}\) put

\[
\Sigma_n^0 = \{ \omega = (\omega_1, \ldots, \omega_n) \in \mathcal{L}^n \mid A(\omega_i, \omega_{i+1}) = 1 \text{ for all } 1 \leq i \leq n - 1 \}.
\]

The topological pressure \(\mathcal{P}(\varphi + \psi)\) of the potential \(\varphi + \psi\) for \(t, \beta \in \mathbb{R}\) is defined to be

\[
(2.5) \quad \mathcal{P}(\varphi + \psi) := \lim_{n \to \infty} \frac{1}{n} \log \sum_{\omega \in \Sigma_n^0} \exp \sup_{\tau \in [0]} (S_\beta(\varphi + \psi))(\tau),
\]

where we set \(S_\beta f := \sup_{\omega \in [\omega]} S_n f(x)\) with \(S_n f(x) := \sum_{i=0}^{n-1} f \circ \sigma^i(x)\) and \(S_0 f = 0\). By a standard argument involving sub-additivity the above limit always exists (although it is possibly equal to infinity).

The next lemma shows that the set \(\mathcal{P}_\alpha\) can be characterised by the potentials \(\varphi\) and \(\psi\) and that the cusp-winding pressure \(P(t, \beta)\) agrees with \(\mathcal{P}(\varphi + \psi)\) for all \(t, \beta \in \mathbb{R}\). In the proof of the following lemma we make use of the fact that the topological Markov chain \((\Sigma, \sigma)\) is finitely primitive, that is, there exists \(l \in \mathbb{N}\) and a finite set \(F \subset \Sigma_l\) such that for all \(a, b \in I\) there exists \(u \in \Sigma_l^0\) such that the word \(aub\) belongs to \(\Sigma_{l+2}\). For more details, see [MU03]. Here the finite set \(F\) can be constructed due to the fact that there are only finitely many generators of \(G\).

Lemma 2.4. For \(\alpha \in \mathbb{R}, x \in \mathcal{P}\) and \(\omega := \pi(x) \in \Sigma_A\) we have

\[
\lim_{n \to \infty} \frac{S_n \varphi(\omega)}{S_n \psi(\omega)} = \alpha \iff \lim_{n \to \infty} \frac{2 \sum_{i=1}^{n} \log(a_i(x))}{d(B_1 \ldots B_n, i, i)} = \alpha,
\]

and for all \(t, \beta \in \mathbb{R}\) we have

\[
P(t, \beta) = \mathcal{P}(\varphi + \psi).
\]

Moreover, \(\mathcal{P}(\varphi + \psi) < \infty\) if and only if \(t + \beta > 1/2\). Further, for each \(\beta \in \mathbb{R}\) we have \(\lim_{t \to 1/2^-} P(t, \beta) = \infty\) and \(\lim_{t \to \infty} P(t, \beta) = -\infty\).

Proof. Following [KSS04], we find a constant \(D > 0\) such that for all \(x \in L_r(G), \omega := \pi(x) \in \Sigma_A\) and \(n \in \mathbb{N}\),

\[
(2.6) \quad |d(B_1 \ldots B_n, i, i) - S_n \varphi(\omega)| \leq D.
\]

This proves the first two assertions.

Since \((\Sigma, \sigma)\) is finitely primitive, it follows from [MU03, Proposition 2.1.9] that

\[
\mathcal{P}(\varphi + \psi) < \infty \quad \text{if and only if} \quad \sum_{x \in \mathcal{P}_\alpha} e^{\sup_{\tau \in [0]} (\varphi(\psi(x)))) < \infty.
\]

Now, we deduce with \((2.6), (2.2)\) and \((2.4)\) that for all \(t, \beta \in \mathbb{R}\)

\[
(2.7) \quad \sum_{x \in \mathcal{P}_\alpha} e^{\sup_{\tau \in [0]} (\varphi(x))} \leq \sum_{n \geq 1} e^{-2\beta \log(n) - 2\beta \log(n)} \lesssim \sum_{n \geq 1} n^{-2(\beta + 1)}.
\]

which converges if and only if \(t + \beta > 1/2\). This also shows \(\lim_{t \to 1/2^-} P(t, \beta) = \infty\).

Finally, by \((1.1)\) and Corollary \((2.3)\) (with the constant \(C_1\) as defined in Corollary \((2.3)\), we have for fixed \(n > C_1/(2C_0)\) and \(t > 0\),
\[ P(t, \beta) \leq \frac{1}{n} \log \sum_{B_1 B_2 \in \mathcal{A}_n} e^{-t d(B_1, B_2)} - 2\beta \sum_{i=1}^n \log^+ \left( |B_i| - 1 \right) + \frac{2C_0}{n} \]
\[ \leq \frac{1}{n} \log \sum_{B_1 B_2 \in \mathcal{A}_n} e^{-t \sum_{i=1}^n (2\log^+(a_i) + C_1)} - 2\beta \sum_{i=1}^n \log^+ \left( a_i \right) + \frac{2C_0}{n} \]
\[ \leq t \left( \frac{2C_0}{n} - C_1 \right) + \log \left( 2^V \sum_{k=1}^\infty k^{-2(t+\beta)} + 2u \right) \to -\infty \text{ for } t \to \infty. \]

2.5. The cusp-winding free energy. The following proposition shows the existence of the cusp-winding free energy function and some of its basic properties.

**Proposition 2.5.** For each \( \beta \in \mathbb{R} \) there exists a unique number \( t(\beta) \) such that

(2.8) \[ P(t(\beta), \beta) = 0. \]

The cusp-winding free energy function \( t : \mathbb{R} \to \mathbb{R} \) defined in this way is real-analytic and strictly convex.

**Proof.** We have to verify that the function \( (t, \beta) \mapsto P(t, \beta) \) is real-analytic on the set \( \{(t, \beta) \in \mathbb{R}^2 | 2(t + \beta) > 1\} \). Recall that \( \Sigma_A \) is a finitely primitive Markov shift. Let \( (t, \beta) \in \mathbb{R}^2 \) with \( 2(t + \beta) > 1 \). By [MU03, Proposition 2.6.13] it suffices to show that \( \phi, \psi \in L^1(\Sigma_A, \mu_{\phi+\beta\psi}) \), where \( L^1(\Sigma_A, \mu_{\phi+\beta\psi}) \) denotes the space of \( \mu_{\phi+\beta\psi} \)-integrable functions on \( \Sigma_A \). Here, \( \mu_{\phi+\beta\psi} \) denotes the unique equilibrium state of \( t\phi + \beta\psi \) for the dynamical system \( (\Sigma_A, \sigma) \), that is, \( \mu_{\phi+\beta\psi} \) is the unique \( \sigma \)-invariant Borel probability measure on \( \Sigma_A \) such that \( h(\mu_{\phi+\beta\psi} + f\phi + \beta\psi d\mu_{\phi+\beta\psi}) = 0 \), where \( h(\mu_{\phi+\beta\psi}) \) refers to the metric entropy of the measure-theoretic dynamical system \( (\Sigma_A, \sigma, \mu_{\phi+\beta\psi}) \). It is well known that \( \mu_{\phi+\beta\psi} \) is a Gibbs measure with respect to \( t\phi + \beta\psi \). In particular, using the estimate in (2.7), we have that \( \mu_{t\phi + \beta\psi}(\{a_i\}) \ll \exp \left( -2t \log |a_i| - 2\beta \log |a_i| \right) \) for all \( a_i \in I \). Since \( 2(t + \beta) > 1 \), it follows that \( \phi, \psi \in L^1(\Sigma_A, \mu_{\phi+\beta\psi}) \). By [MU03, Theorem 2.6.12] we know that the pressure \( P \) is real-analytic on \( \{(t, \beta) \in \mathbb{R}^2 | P(t, \beta) < \infty \} \), which by Lemma 2.4 is equal to \( \{(t, \beta) \in \mathbb{R}^2 | 2(t + \beta) > 1\} \).

Now let \( \beta \in \mathbb{R} \). By Lemma 2.4 we have that \( P(t, \beta) < \infty \), if and only if \( t > 1/2 - \beta \). Also by Lemma 2.4 we have \( \lim_{t \to 1/2 - \beta} P(t, \beta) = \infty \) and \( \lim_{t \to \infty} P(t, \beta) = \infty \), which conclude that there exists a unique \( t = t(\beta) \) with \( P(t(\beta), \beta) = 0 \).

The strict convexity of \( t \) follows from convexity and real-analyticity, unless \( t' \) is constant. That \( t' \) is not constant follows from the asymptotic behaviour of \( t \) for \( \beta \to \pm \infty \) as described in the following Lemmas 2.6 and 2.7.

**Lemma 2.6.** For the right asymptotic of \( t \) we have that \( \lim_{\beta \to \infty} t(\beta) = \delta_c \).

**Proof.** For \( s \in \mathbb{R} \) and \( n \in \mathbb{N} \), set
\[ p_n(s) := \frac{1}{n} \log \sum_{B_1 B_2 \in \mathcal{A}_n} e^{-s d(B_1, B_2)} \text{ and } p(s) := \lim_{n \to \infty} p_n(s). \]

Then for all \( \beta > 0 \), we have
\[ 0 = \lim_{n \to \infty} \frac{1}{n} \log \sum_{B_1 B_2 \in \mathcal{A}_n} e^{-\beta d(B_1, B_2)} \sum_{a_i} e^{-2\beta \log^+(a_i)} \geq \lim_{n \to \infty} p_n(t(\beta)) - p(t(\beta)). \]

By the formula of Bishop and Jones ([BJ97]) it follows that \( t(\beta) \geq \delta_c \).
Now, let $C := 2C_0$ with $C_0$ as in Section 2.2 and recall that the constant $v$ denotes the number of cusps. Let $\zeta$ denote the Riemann zeta-function and choose $\varepsilon > 0$. Then by using the triangle inequality $3(n-k)$ times corresponding to blocks of length greater than 4, we obtain

\[
\frac{1}{n} \log \sum_{B_1 \ldots B_n \in \mathcal{G}_n} e^{-t_0(B_1 \ldots B_n)} - 2\varepsilon \sum_{i=1}^{n} \log (\lambda_i) \leq \frac{1}{n} \log \sum_{k=0}^{n} \binom{n}{k} e^{k\varepsilon} \sum_{(a_1 \ldots a_n) \in \mathcal{G}_n} e^{\sum_{i=1}^{n-k} \log (\lambda_i - 2\varepsilon \log (a_i))} \leq \frac{1}{n} \log \sum_{k=0}^{n} \binom{n}{k} e^{k\varepsilon} \sum_{(a_1 \ldots a_n) \in \mathcal{G}_n} \prod_{i=1}^{n-k} a_i^{-2\varepsilon} \leq \frac{1}{n} \log \sum_{k=0}^{n} \binom{n}{k} e^{k\varepsilon} \sum_{(a_1 \ldots a_n) \in \mathcal{G}_n} \prod_{i=1}^{n-k} a_i^{-2\varepsilon}
\]

for $\beta$ large enough. Here we used the general observation that for $\lambda_k \to \lambda > 0$ and $b > 0$ we have

\[
\lim_{n \to \infty} \left( \sum_{k=0}^{n} \binom{n}{k} \lambda_k^b n^{-k} \right) = \lambda + b.
\]

This shows that for all sufficiently large $\beta$ we have $\delta_\varepsilon \leq t(\beta) \leq \delta_\varepsilon + \varepsilon$. Since $\varepsilon > 0$ was arbitrary, it follows that $\lim_{\beta \to \infty} t(\beta) = \delta_\varepsilon$.

\[\square\]

**Lemma 2.7.** For all $\varepsilon \in (0, 1/2)$ we find that for all $-\beta$ large enough we have

\[
\frac{1}{2} - \beta < t(\beta) < \frac{1}{2} + \beta + \frac{\varepsilon}{2}.
\]

**Proof.** Combining Lemma 2.4 and Proposition 2.5 we deduce that $t(\beta) \geq 1/2 - \beta$.

For the proof of the upper bound it suffices to show that for a given $\varepsilon \in (0, 1/2)$ and $t_\beta := 1/2 - \beta + \varepsilon/2$ we have $P(t_\beta, \beta) < 0$ for all $-\beta$ large. Recall once more that $v$ denotes the number of cusps and $u = \text{card}(H_0)$. Fix a natural number $n > 2C_0/C_1$ with $C_0 > 0$ taken from (1.1) and $C_1$ taken from (2.4). By (1.1) and (2.4) combined with (2.2) we have for all $\beta$ negative

\[
P(t_\beta, \beta) \leq \frac{1}{n} \log \sum_{B_1 \ldots B_n \in \mathcal{G}_n} e^{-t_\beta d(B_1 \ldots B_n)} - 2\beta \sum_{i=1}^{n} \log (\lambda_i - 1) + 2\beta C_0 \frac{n}{n} + \left( \frac{2C_0}{n} - C_1 \right) + \frac{1}{n} \log \left( \left( \frac{2v}{\sum_{k=1}^{n} k^{-1 + \varepsilon}} + 2u \right) \right) \to -\infty \text{ for } \beta \to -\infty.
\]

\[\square\]
3. PROOF OF FACTS

Proof of Fact 1.1. The inclusion $L_c \subset \mathcal{F}_0$ follows immediately from the definition of $\mathcal{F}_0$.

Proof of Fact 1.2. Using (2.2) and (2.4) along with the property that $\log(k+1) \leq \log^+(k) + 1$, for $k \in \mathbb{N}$, we have
\[
\liminf_{n \to \infty} \frac{2 \sum_{i=1}^n \log^+(a(x))}{d(B_1 \cdots B_n(i), i)} = \liminf_{n \to \infty} \frac{\sum_{i=1}^n \log^+(a(x))}{\sum_{i=1}^n I(\xi)} < 1.
\]
By assumption, the Cesàro average of $\log(a(x))$ tends to infinity, which implies that
\[
\liminf_{n \to \infty} \left(1 + \frac{n(C_2 + 2) + C_0/2}{\sum_{i=1}^n \log^+(a(x))}\right)^{-1} = 1.
\]
Combining this with (3.1) below finishes the proof. For the proof of the equality
\[
\dim_H(\mathcal{F}) = 1/2
\]
we refer to [Mun12].

Proof of Fact 1.3. We prove that if $a_i(x) \leq K$ for all $i \in \mathbb{N}$, then $x \notin \mathcal{F}_1$. By (2.4) we have for $a_i \geq 0$
\[
2 \log^+(a_i) + C_1 \leq I(\xi).
\]
Then we have by (2.2)
\[
\limsup_{n \to \infty} \frac{2 \sum_{i=1}^n \log^+(a(x))}{d(B_1 \cdots B_n(i), i)} \leq \limsup_{n \to \infty} \frac{2 \sum_{i=1}^n \log^+(a(x))}{\sum_{i=1}^n I(\xi)} \leq \left(1 + \frac{C_1}{2 \log(K)}\right)^{-1} < 1.
\]

Proof of Fact 1.4. It is well known that $\{\alpha \in \mathbb{R} | \mathcal{F}_\alpha \neq \emptyset\}$ is an interval (cf. [Sch99]). By the Facts 1.2 and 1.1 we have $\mathcal{F}_0 \neq \emptyset$ and $\mathcal{F}_1 \neq \emptyset$. Further, we clearly have on the one hand $0 \leq \liminf_{n \to \infty} \frac{2 \sum_{i=1}^n \log^+(a(x))}{d(B_1 \cdots B_n(i), i)}$ and on the other hand, using (2.2) and (2.4), we estimate for all $x \in L_r(G)$,
\[
\limsup_{n \to \infty} \frac{2 \sum_{i=1}^n \log^+(a(x))}{d(B_1 \cdots B_n(i), i)} \leq \limsup_{n \to \infty} \frac{2 \sum_{i=1}^n \log^+(a(x))}{\sum_{i=1}^n I(\xi)} \leq \lim_{n \to \infty} \frac{\sum_{i=1}^n I(\xi)}{\sum_{i=1}^n I(\xi)} = 1.
\]
This proves the fact.

4. MULTIFRACTAL FORMALISM FOR CONFORMAL GRAPH DIRECTED MARKOV SYSTEMS

In this section we develop the multifractal formalism for quotients of Birkhoff sums in the framework of conformal graph directed Markov systems (cf. [MU03] Section 4.9). Let us first briefly recall the definition a conformal graph directed Markov system (MU03). A conformal graph directed Markov system $\Phi$ is given by a finite set of vertices $V$, a family $(X_v)_{v \in V}$ of compact connected subsets of $\mathbb{R}^d$ with $X_v = \text{Int}(X_v)$ and an edge set $E \subset V \times V$ together with contractions $(\Phi_e)_{e \in E}$, where $\Phi_e : X_{i(e)} \to X_{j(e)}$. Here, $t(e) \in V$ denotes the
terminal vertex and \( i(e) \in V \) denotes the initial vertex of the edge \( e \). Moreover, \( \Phi \) is endowed with an (edge) incidence matrix \( A \in \{0,1\}^{E \times E} \) satisfying \( A_{i,j} = 1 \) only if \( X_i(f) = X_j(e) \). We denote the associated Markov shift with alphabet \( E \) and incidence matrix \( A \) by \( \Sigma := \Sigma_{\Phi} \).

We always assume that each \( \Phi \) has a \( C^1 \)-conformal extension satisfying a Hölder condition as stated in (4c, 4e) of [MU03, Section 4.2]. Moreover, we assume that \( \Phi \) satisfies the open set condition and the cone condition as stated in (4b, 4d) [MU03, Section 4.2]. Furthermore, we assume that the Markov shift \( \Sigma_{\Phi} \) is finitely irreducible (cf. [MU03, page 5]).

There is a natural coding map \( p : \Sigma \to \bigcup_{V \in \mathcal{V}} X_v \). The associated geometric potential \( \zeta := \zeta_{\Phi} : \Sigma \to \mathbb{R} \) is Hölder continuous with respect to the shift metric. Let \( \psi : \Sigma \to \mathbb{R} \) denote another Hölder continuous map. As in (JK11), we define the associated free energy function \( t : \mathbb{R} \to \mathbb{R} \cup \{+\infty\} \) which is for \( \beta \in \mathbb{R} \) given by

\[
  t(\beta) := \inf \{ t \in \mathbb{R} \mid \mathcal{P}(t\zeta + \beta \psi) \leq 0 \}.
\]

The function \( t \) is a closed convex function with domain \( \text{dom}(t) \), and we denote by \( t^- \) (resp. \( t^+ \)) its left (resp. right) derivative. We denote by \( \hat{t} \) the Legendre transform of \( t \). Define also

\[
  \alpha_- := \inf \{-t^-(x) \mid x \in \text{dom}(t)\} \quad \text{and} \quad \alpha_+ := \sup \{-t^+(x) \mid x \in \text{dom}(t)\}.
\]

For \( \alpha \in \mathbb{R} \) define the sets

\[
  \mathcal{F}_\alpha(\Phi, \psi) := \{ \omega \in \Sigma_{\Phi} \mid \lim_{k \to +\infty} \frac{S_k \psi(\omega)}{S_k \zeta_{\Phi}(\omega)} = \alpha \}.
\]

Set \( \alpha_0 := -t^+(0) \) and let

\[
  \mathcal{F}_\alpha^+(\Phi, \psi) := \begin{cases} \{ \omega \in \Sigma_{\Phi} \mid \limsup_{k \to +\infty} \frac{S_k \psi(\omega)}{S_k \zeta_{\Phi}(\omega)} \geq \alpha \} & \alpha \geq \alpha_0, \\ \{ \omega \in \Sigma_{\Phi} \mid \liminf_{k \to +\infty} \frac{S_k \psi(\omega)}{S_k \zeta_{\Phi}(\omega)} \leq \alpha \} & \alpha < \alpha_0. \end{cases}
\]

Our main result is an extension of our multifractal formalism for conformal iterated function system (JK11).

**Theorem 4.1.** For every conformal graph directed Markov system \( \Phi \) satisfying the above assumptions, for every Hölder continuous potential \( \psi \) on the associated Markov shift and for every \( \alpha \in (\alpha_-, \alpha_+) \) we have

\[
  \dim_H(\mathcal{F}_\alpha(\Phi, \psi)) = \dim_H(\mathcal{F}_\alpha^+(\Phi, \psi)) = -\hat{t}(-\alpha).
\]

For every \( \alpha \in \mathbb{R} \) we have \( \dim_H(\mathcal{F}_\alpha(\Phi, \psi)) \leq \max\{ -\hat{t}(\alpha), 0 \} \) and if \( -\hat{t}(\alpha) < 0 \) then \( \mathcal{F}_\alpha^+(\Phi, \psi) = \emptyset \).

Theorem 4.1 can be proved by the same methods as in (JK11). The proof of the upper bound of the Hausdorff dimension follows from standard covering arguments (see for instance MU03, Proof of Theorem 4.2.13). To prove the lower bound of the Hausdorff dimension of the multifractal level sets, the key is to approximate the infinitely-generated conformal graph directed Markov system \( \Phi \) by finitely-generated subsystems. Previously, this method has been used in (MU03, Theorem 4.2.13) to obtain the lower bound of the Hausdorff dimension of the limit set of a conformal graph directed Markov system. For the special case of conformal iterated function systems, the same method proved successful also for the level sets of multifractal decompositions of limit sets (JK11). It is straightforward to extend the proof in (JK11) to conformal graph directed Markov systems. The key technical detail is the approximation property for the topological pressure for Hölder continuous potentials on finitely-irreducible topological Markov shifts [MU03, Theorem 2.1.5].
5. Proof of Theorem 1.5

In this section we give the proof of Theorem 1.5. For the interior points of the spectrum, the theorem is an application of our general multifractal formalism for conformal graph directed Markov systems. For the boundary points, additional arguments are required.

Multifractal formalism for the interior of the spectrum. It is known that the radial limit set \( L_t(G) \) of a free Fuchsian group \( G \) has a representation as an infinite conformal graph directed Markov system \( \Phi \) (see, for example, [KS07]). The vertex set \( V \) is given by the symmetric set of generators \( G_0 \). For each \( g \in G_0 \) the compact connected set \( \Sigma_g \subset \mathbb{R} \) is given by the closure of \( b_g \), where \( b_g \) is the projection of the face \( s \in \mathcal{F} \) such that \( g=s \), the face-pairing transformation with respect to Dirichlet fundamental domain of \( G \) (see (2.1) in Section 2.1 for the details). Note that by conjugating the group \( G \), we may assume that the point at infinity does not belong to the limit set of \( G \). The edge set \( E \) is given by

\[
E := I := \{ g_1^{\gamma} g_2 \mid \gamma \in \Gamma_0, g_1, g_2 \in G_0 \setminus \{ \gamma^{-1} \}, n \in \mathbb{N} \} \cup \\
\cup \{ g_1 h g_2 \mid g_1 \in G_0, h \in H_0 \setminus \{ g_1^{-1} \}, g_2 \in G_0 \setminus \{ h^{-1} \} \}.
\]

For \( \eta_1 \ldots \eta_n \in E \) we define \( \iota(\eta_1 \ldots \eta_n) := \eta_1 \) and \( \iota(\eta_1 \ldots \eta_n) := \eta_{n-2} \). We consider the incidence matrix \( A \in \{0,1\}^{E \times E} \) which satisfies for \( e = \eta_1 \ldots \eta_n \) and \( f = \tau_1 \ldots \tau_m \in E \) that \( A(e,f) = 1 \) if and only if \( \eta_{m-1} = \tau_1 \) and \( \eta_m = \tau_2 \). For such edges we define

\[
\Phi_{e,f} : X_{\iota(e)} \to X_{\iota(f)}, \quad \Phi_{e,f} := \eta_1^{-1} \ldots \eta_{n-2}^{-1}.
\]

Note that the Markov shift \( \Sigma_\Phi \) associated with the graph directed Markov system \( \Phi \) coincides with the Markov shift \( \Sigma_\Lambda \) in Section 2.3. Also note that the coding map \( \rho \) of \( \Phi \) is the inverse of the coding map \( \pi \) defined in Section 2.3. Observe that \( \zeta_\Phi \) is equal to the potential \( \varphi \) defined in Section 2.3. Define \( \psi : \Sigma_\Phi \to \mathbb{R} \) as in Section 2.3 and recall from Lemma 2.4 and Proposition 2.5 that \( \iota \) associated with \( \Phi \) and \( \psi \) coincides with the real-analytic function \( t \) defined in Section 2.3. Hence, by Theorem 4.1, we have for \( \alpha \in (\alpha_-, \alpha_+) \),

\[
\dim_H (\mathcal{F}_\alpha(\Phi, \psi)) = \dim_H (\mathcal{F}_\alpha^*(\Phi, \psi)) = -\tilde{t}(-\alpha).
\]

By the definition of \( \Phi \) and the potential \( \psi \), and by Lemma 2.4 we have

\[
\mathcal{F}_\alpha(\Phi, \psi) = \mathcal{F}_\alpha \cap \mathcal{D} \quad \text{and} \quad \mathcal{F}_\alpha^*(\Phi, \psi) = \mathcal{F}_\alpha^* \cap \mathcal{D}.
\]

Combining this with Fact 1.4 we see that \( \alpha_- = 0 \) and \( \alpha_+ = 1 \). Finally, since \( \mathcal{F}_\alpha \) (resp. \( \mathcal{F}_\alpha^* \)) is a countable union of bi-Lipschitz images of \( \mathcal{F}_\alpha \cap \mathcal{D} \) (resp. \( \mathcal{F}_\alpha^* \cap \mathcal{D} \)), we conclude that \( \dim_H (\mathcal{F}_\alpha) = \lim_{\alpha \to 0} \dim_H (\mathcal{F}_\alpha^*) = -\tilde{t}(-\alpha) \).

Asymptotic behaviour for the left boundary point. By Fact 1.1 we have

\[
\dim_H (\mathcal{F}_0) \geq \dim_H (L_c) = \delta_c.
\]

Combining (5.1) and Lemma 2.6 the lower bound follows from

\[
\dim_H (\mathcal{F}_0) \leq \lim_{\alpha \to 0} \dim_H (\mathcal{F}_\alpha^*) = \delta_c.
\]

Asymptotic behaviour for the right boundary point. Combining Fact 1.2 and 1.3 we obtain \( \dim_H (\mathcal{F}_1) \geq \dim_H (J) = 1/2 \).

Now this observation and the fact that \( \mathcal{F}_1 \subset \mathcal{F}_\alpha \) for every \( \alpha_0 < \alpha < 1 \) together with Lemma 2.7 finally gives

\[
\dim_H (\mathcal{F}_1) \leq \lim_{\alpha \to 1} \dim_H (\mathcal{F}_\alpha) \leq 1/2.
\]

This proves the claim for the properties of the right boundary point of the spectrum.
The derivatives in the boundary points. For the derivative of $f$ at the boundary points we use the general identity for Legendre transforms

$$f'(\alpha) = (\hat{t}')'(\alpha) = - (t')^{-1}(-\alpha), \alpha \in (0,1).$$

The real-analyticity then gives

$$\lim_{\alpha \searrow 0} f'(\alpha) = - \lim_{\alpha \nearrow 1} f'(\alpha) = +\infty.$$

Irregular set. The fact that the set of points for which the mean cusp winding number does not exist has full Hausdorff dimension $\delta(G)$ follows from [BS00] by exhaustion of $\Sigma_A$ with finite alphabet subsystems.

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