Decay of Massive Scalar Hair in the Background of Dilaton Gravity Black Hole

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We investigate analytically both the intermediate and late-time behaviour of the massive scalar field in the background of static spherically symmetric black hole solution in dilaton gravity with arbitrary coupling constant. The intermediate asymptotic behaviour of scalar field depends on the field’s parameter mass as well as the multiple number \( \ell \). On its turn, the late-time behaviour has the power law decay rate independent on coupling constant in the theory under consideration.

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I. INTRODUCTION

Late-time behaviour of various fields in the spacetime of a collapsing body plays an important role in black hole’s physics. It happens that regardless of details of the collapse or the structure and properties of the collapsing body the resultant black hole can be described only by few parameters such as mass, charge and angular momentum, black holes have no hair. On its own, it is interesting to investigate how these hair loss proceed dynamically.

For the first time the neutral external perturbations were studied by Price in [1]. It was shown that the late-time behavior is dominated by the factor \( t^{-(2l+3)} \), for each multipole moment \( \ell \). In Ref.[2] it was revealed that the decay-rate along null infinity and along the future event horizon was governed by the power laws \( u^{-(l+2)} \) and \( v^{-(l+3)} \), where \( u \) and \( v \) were the outgoing Eddington-Finkelstein (ED) and ingoing ED coordinates. On the other hand, Ref. [3] was devoted to the scalar perturbations on Reissner-Nordström (RN) background for the case when \(|Q| < M\) has the following dependence on time \( t^{-(2l+2)} \), while for \( |Q| = M \) the late-time behavior at fixed \( r \) is governed by \( t^{-(l+2)} \).

It turns out that a charged hair decayed slower than a neutral one [4]-[6], while the late-time tails in gravitational collapse of a self-interacting (SI) fields in the background of Schwarzschild solution was reported by Burko [7] and in RN solution at intermediate late-time was considered in Ref.[8]. At intermediate late-time for small mass \( m \) the decay was dominated by the oscillatory inverse power tails \( t^{-(l+3/2)} \sin(mt) \). This analytic prediction was verified at intermediate times, where \( mM \ll mt \ll 1/(mM)^2 \). It was found analytically [9] that for a nearly extreme RN spacetime the inverse power law behavior of the dominant asymptotic tail is of the form \( t^{-5/6} \sin(mt) \). Just it is independent of \( \ell \). The asymptotic tail behaviour of SI scalar field was also studied in Schwarzschild spacetime [10]. The oscillatory tail of scalar field has the decay rate of \( t^{-5/6} \) at asymptotically late time. The power-law tails in the evolution of a charged massless scalar field around a fixed background of dilaton black hole was studied in Ref.[11], where the inverse power-law relaxation of the fields at future timelike infinity, future null infinity and along the outer horizon of the considered black hole was found. The case of a massive scalar field was elaborated in [12] and it was envisaged numerically that at very late times the oscillatory tail decay is of the form \( t^{-5/6} \).

In addition, the late-time behaviour of massive Dirac fields were studied in the spacetime of Schwarzschild black hole [13]. It was found that the asymptotic behaviour of these fields is dominated by a decaying tail without any oscillation. The dumping exponent was independent on the multiple number of the wave mode and on the mass of the Dirac field. On the other hand, the decay of the massive Dirac fields was slower comparing the decay of massive scalar field. The above analysis was supplemented by the studies of charged massive Dirac fields in the spacetime of RN black hole [14]. The case of the decay of the fields in the stationary axisymmetric black hole background was studied numerically in Ref.[15] and it was found that in the case of Kerr black hole that the oscillatory inverse-power law of the dominant asymptotic tail behaviour is approximately depicted by the relation \( t^{-5/6} \sin(mt) \). In Ref.[16] both the intermediate late-time tail and the asymptotic behaviour of the charged massive Dirac fields in the background of Kerr-Newmann black hole was investigated. It was demonstrated that the intermediate late-time behaviour of the fields under consideration is dominated by an inverse power-law decaying tail without any oscillation.

The late-time behaviour of massive vector field obeying the Proca equation of motion in the background of Schwarzschild black hole was studied in [17]. It was revealed that at intermediate late times, three functions characterizing the field have different decay law depending on the multiple number \( \ell \). On the contrary, the late-time behaviour is independent on \( \ell \), i.e., the late-time decay law is proportional to \( t^{-5/6} \sin(mt) \).

As far as the \( n \)-dimensional static black holes is concerned, the no-hair theorem for them is quite well established [18]. The mechanism of decaying black hole hair in higher dimensional static black hole case concerning the evolution...
of massless scalar field in the \( n \)-dimensional Schwarzschild spacetime was determined in Ref.\[19\]. It was found that for odd dimensional spacetime the field decay had a power falloff like \( t^{-(2l+1)/2} \), where \( n \) is the dimension of the spacetime. This tail was independent of the presence of the black hole. For even dimensions the late-time behaviour is also in the power law form but in this case it is due to the presence of black hole \( t^{-(2l+2)/2} \). The late-time tails of massive scalar fields in the spacetime of \( n \)-dimensional static charged black hole was elaborated in Ref.\[20\] and it was found that the intermediate asymptotic behaviour of massive scalar field had the form \( t^{-((n+2)/2-1/2)} \). This pattern of decay was checked numerically for \( n = 5, 6 \). The massive scalar field quasi-normal modes in higher dimensional black hole were considered in \[21\]. Among all it was envisaged a qualitatively different dependence of the fundamental modes on the field mass for \( n \geq 6 \).

In our work we shall discuss analytically the intermediate and late-time behaviour of massive scalar field in the spacetime of black hole in dilaton gravity with arbitrary coupling constant. We confirmed the numerical result presented in Ref.\[12\]. Namely, we revealed that the intermediate asymptotic behaviour is not the final pattern of the decay of the massive scalar hair. At late times the inverse power law of the decay is relevant, which is independent on field’s parameters and the coupling constant of the theory. In Sec.II we gave the analytic arguments concerning the decay of scalar massive hair in the background of the considered black hole. We conclude our investigations and give some remarks concerning the extremal dilaton black holes in Sec.III.

II. GREEN’S FUNCTION ANALYSIS

The action for the dilaton gravity with arbitrary coupling constant is subject to the form

\[
S = \int d^4x \sqrt{-g} \left[ R - 2\nabla^{\mu}\phi \nabla_{\mu}\phi - e^{-2\alpha\phi} F_{\mu\nu} F^{\mu\nu} \right],
\]

where \( \phi \) is the dilaton field, \( \alpha \) coupling constant while \( F_{\mu\nu} = 2\nabla_{[\mu} A_{\nu]} \) is the strength of \( U(1) \) gauge field.

The static spherically symmetric solution of the Eqs. of motion for the underlying theory may be written as \[22\]

\[
ds^2 = - \left( 1 - \frac{r_-}{r} \right) \left( 1 - \frac{r}{r_-} \right)^{\frac{2\alpha^2}{1+\alpha^2}} dt^2 + \frac{dr^2}{\left( 1 - \frac{r}{r_-} \right) \left( 1 - \frac{r_-}{r} \right)^{\frac{2\alpha^2}{1+\alpha^2}}} + R^2(r)d\Omega^2,
\]

where \( R^2(r) = r^2 \left( 1 - \frac{r_-}{r} \right)^{\frac{2\alpha^2}{1+\alpha^2}} \), while \( r_- \) and \( r_+ \) are related to the mass \( M \) and the electric charge \( Q \) of the black hole

\[
e^{2\alpha\phi} = \left( 1 - \frac{r_-}{r_+} \right)^{\frac{2\alpha^2}{1+\alpha^2}}, \quad 2M = r_+ + \frac{1 - \alpha^2}{1 + \alpha^2} r_- \quad \text{and} \quad Q^2 = r_- r_+ \frac{1 - \alpha^2}{1 + \alpha^2}.
\]

The metric is asymptotically flat in the sense that the spacetime contains a data set \( (\Sigma_{end}, g_{ij}, K_{ij}) \) with gauge fields such that \( \Sigma_{end} \) is diffeomorphic to \( \mathbb{R}^3 \) minus a ball and the following asymptotic conditions are fulfilled:

\[
|g_{ij} - \delta_{ij}| + r|\partial_a g_{ij}| + ... + r^k |\partial_{a_1...a_k} g_{ij}| + r|K_{ij}| + ... + r^k |\partial_{a_1...a_k} K_{ij}| \leq O \left( \frac{1}{r} \right), \quad \phi = \phi_0 + O \left( \frac{1}{r} \right),
\]

\[
|F_{a\beta} + r|\partial_a F_{a\beta}| + ... + r^k |\partial_{a_1...a_k} F_{a\beta}| \leq O \left( \frac{1}{r^2} \right),\quad \phi = \phi_0 + O \left( \frac{1}{r} \right),
\]

where \( K_{ij} \) is the exterior curvature, \( \phi_0 \) is a constant value of the scalar field.

The massive scalar field \( \tilde{\psi} \) satisfies the following Eq. of motion:

\[
\nabla^i \nabla_i \tilde{\psi} - m^2 \tilde{\psi} = 0.
\]

Next, we define the tortoise coordinates \( y \) as

\[
dy = \frac{dr}{\left( 1 - \frac{r_-}{r} \right) \left( 1 - \frac{r}{r_-} \right)^{\frac{1}{1+\alpha^2}}}.
\]
Thus, the metric (2) can be rewritten in the form as

\[ ds^2 = \left(1 - \frac{r_+}{r}\right) \left(1 - \frac{r_-}{r}\right)^{\frac{1+\alpha^2}{4}} \left(-dt^2 + dy^2\right) + R^2(r)d\Omega^2. \]  

We resolve the field into spherical harmonics, which leads to the relation

\[ \tilde{\psi} = \sum_{l,m} \frac{1}{R(r)} \psi_{lm}(t,r) Y_{lm}(\theta, \phi). \]  

where \( Y_{lm} \) is a scalar spherical harmonics on the unit two-sphere. As a consequence of the above one gets the following equations of motion for each multipole moment

\[ \psi_{tt} - \psi_{yy} + V \psi = 0. \]  

The effective potential \( V \) implies the following relation:

\[ V = \left(1 - \frac{r_+}{r}\right) \left(1 - \frac{r_-}{r}\right)^{\frac{1+\alpha^2}{4}} \left[\frac{1}{R(r)} \frac{d}{dr} \left(\frac{1+\alpha^2}{2} \frac{dR(r)}{dr}\right) + \frac{l(l+1)}{R^2(r)} + m^2\right]. \]  

In order to analyze the time evolution of a massive field in the background of dilaton black hole we shall use the spectral decomposition method. In Refs. [3, 23] it was shown that the asymptotic tail is connected with the existence of a branch cut situated along the interval \(-m \leq \omega \leq m\). An oscillatory inverse power-law behaviour of the self-interacting scalar field arises from the integral of Green function \( \tilde{G}(y, y'; \omega) \) around branch cut. In what follows we shall denote it by \( G_\omega \) and our main aim will be to find the analytic form of this integral.

Namely, the time evolution of massive scalar field may be written in the following form:

\[ \psi(y, t) = \int dy' \left[ G(y, y'; t) \psi_l(y', 0) + G_l(y, y'; t) \psi(y', 0) \right], \]  

for \( t > 0 \), where the Green’s function \( G(y, y'; t) \) is given by the relation

\[ \left[ \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial y^2} + V \right] G(y, y'; t) = \delta(t) \delta(y - y'). \]  

In what follows, our main task will be to find the black hole Green function. In the first step we reduce equation (14) to an ordinary differential equation. To do it one can use the Fourier transform [23] \( \tilde{G}(y, y'; \omega) = \int_0^\infty dt \tilde{G}(y, y'; t)e^{i\omega t} \). This Fourier’s transform is well defined for \( Im \omega \geq 0 \), while the corresponding inverse transform yields

\[ G(y, y'; t) = \frac{1}{2\pi} \int_{-\infty+i\epsilon}^{\infty+i\epsilon} d\omega \tilde{G}(y, y'; \omega)e^{-i\omega t}, \]  

for some positive number \( \epsilon \). The Fourier’s component of the Green’s function \( \tilde{G}(y, y'; \omega) \) can be written in terms of two linearly independent solutions for homogeneous equation as

\[ \left( \frac{d^2}{dy^2} + \omega^2 - V \right) \psi_i = 0, \quad i = 1, 2. \]  

The boundary conditions for \( \psi_i \) are described by purely ingoing waves crossing the outer horizon \( H_+ \) of the charged dilaton black hole \( \psi_1 \approx e^{-i\omega y} \) as \( y \rightarrow -\infty \) while \( \psi_2 \) should be damped exponentially at \( i_+ \), namely \( \psi_2 \approx e^{-\sqrt{\omega^2 - \omega^2}y} \) at \( y \rightarrow \infty \).

To proceed further, we change the variables

\[ \psi_i = \frac{\xi}{\left(1 - \frac{r_+}{r}\right)^{1/2} \left(1 - \frac{r_-}{r}\right)^{\frac{1+\alpha^2}{2(1+\alpha^2)}}}, \]  

where \( \xi = i \int_0^\infty dt G_{\omega}(y, y'; t)e^{-i\omega t} \).
where \( i = 1, 2 \). Let us assume that the observer and the initial data are situated far away from the considered black hole. We expand Eq. 16 as a power series of \( r_\pm /r \) neglecting terms of order \( \mathcal{O}((r_\pm/r)^2) \) and higher. It leads us to the following expression:

\[
\frac{d^2}{dr^2}\xi + \left[ \omega^2 - m^2 + \frac{(2\omega^2 - m^2)(r_+ + \alpha_1 r_-)}{r} - \frac{l(l+1)}{r^2} \right]\xi = 0,
\]

(18)

where \( \alpha_1 = \frac{1}{r_+ r_-} \).

It turned out that the above equation can be solved in terms of Whittaker’s functions \( \text{[24]} \), namely the two basic solutions are needed to construct the Green function, with the condition that \( |\omega| \geq m \). Consequently, it implies the result as follows:

\[
\psi_1 = M_{\kappa, \tilde{\mu}}(2\tilde{\omega}r), \quad \psi_2 = W_{\kappa, \tilde{\mu}}(2\tilde{\omega}r),
\]

(19)

where we have denoted

\[
\tilde{\mu} = \sqrt{1/4 + l(l+1)}, \quad \kappa = (r_+ + \alpha_1 r_-) \left( \frac{m^2}{2\tilde{\omega}^2} - \tilde{\omega}^2 \right), \quad \tilde{\omega}^2 = m^2 - \omega^2.
\]

(20)

By virtue of the above relations the spectral Green function yields

\[
G_c(x, y; t) = \frac{1}{2\pi} \int_{-m}^{m} dw \left[ \frac{\psi_1(x, \omega e^{i\pi}) \psi_2(y, \omega e^{i\pi})}{W(\omega e^{i\pi})} - \frac{\psi_1(x, \tilde{\omega}) \psi_2(y, \tilde{\omega})}{W(\tilde{\omega})} \right] e^{-iwt}
\]

(21)

\[
= \frac{1}{2\pi} \int_{-m}^{m} dw f(\tilde{\omega}) e^{-iwt},
\]

where by we denoted Wronskian \( W(\tilde{\omega}) \).

First we discuss the intermediate asymptotic behaviour of the massive scalar field, i.e., when the range of parameters are \( M \ll r \ll t \ll M/(mM)^2 \). The intermediate asymptotic contribution to the Green function integral gives the frequency equal to \( \tilde{\omega} = \mathcal{O}(\sqrt{m/t}) \). It implies that \( \kappa \ll 1 \). One should have in mind that \( \kappa \) stems from the \( 1/r \) term in the massive scalar field equation of motion. Thus it depicts the effect of backscattering off the spacetime curvature and in the case under consideration the backscattering is negligible. In the case of intermediate asymptotic behaviour one finally gets

\[
f(\tilde{\omega}) = \frac{2^{\tilde{\mu}} - 1}{\tilde{\mu} \Gamma(2\tilde{\mu})} \Gamma(\frac{1}{2} + \tilde{\mu}) \Gamma(\frac{1}{2} - \tilde{\mu}) \left[ 1 + e^{(2\tilde{\mu}+1)i\pi} \right] (rr')^{\frac{1}{2} + \tilde{\mu}} \tilde{\omega}^{2\tilde{\mu}},
\]

(22)

where we have used the fact that \( \tilde{\omega} r \ll 1 \) and the form of \( f(\tilde{\omega}) \) can be approximated by means of the fact that \( M(a, b, z) = 1 \) as \( z \) tends to zero. The resulting Green function can be written as

\[
G_c(r, r'; t) = \frac{2^{\tilde{\mu}}}{\mu \sqrt{\pi}} \frac{\Gamma(-2\tilde{\mu}) \Gamma(\frac{1}{2} + \tilde{\mu}) \Gamma(\frac{1}{2} - \tilde{\mu})}{\mu \Gamma(2\mu) \Gamma(\frac{1}{2} - \mu)} \left[ 1 + e^{(2\tilde{\mu}+1)i\pi} \right] (rr')^{\frac{1}{2} + \tilde{\mu}} m^\mu t^{-1-\tilde{\mu}} \cos(mt - \frac{\pi}{2}(\tilde{\mu} + 1)).
\]

(23)

In the limit when \( t \gg 1/m \) one arrives at the conclusion that the spectral Green function provides

\[
G_c(r', r; t) = \frac{2^{\tilde{\mu} - 1}}{\mu \sqrt{\pi}} \frac{\Gamma(-2\tilde{\mu}) \Gamma(\frac{1}{2} + \tilde{\mu}) \Gamma(\frac{1}{2} - \mu + 1)}{\mu \Gamma(2\mu) \Gamma(\frac{1}{2} - \mu)} \left[ 1 + e^{(2\tilde{\mu}+1)i\pi} \right] (rr')^{\frac{1}{2} + \tilde{\mu}} m^\mu t^{-1-\tilde{\mu}} \cos(mt - \frac{\pi}{2}(\tilde{\mu} + 1)).
\]

(24)

Eq. (24) depicts the oscillatory inverse power-law behaviour. In our case the intermediate times of the power-law tail depends only on \( \tilde{\mu} \) which in turn is a function of the angular momentum parameter \( l \).

However, the different pattern of decay is expected when \( \kappa \gg 1 \), for the late-time behaviour, when the backscattering off the curvature is important. In this case we take into account the limit \( \text{[24]} \)

\[
M_{\kappa, \tilde{\mu}}(2\tilde{\omega}r) \approx \Gamma(1 + 2\tilde{\mu}) (2\tilde{\omega}r)^{2\tilde{\mu} - \tilde{\kappa}} J_{2\tilde{\mu}}(\sqrt{8\kappa\tilde{\omega}r}).
\]

(25)

Consequently, \( f(\tilde{\omega}) \) yields

\[
f(\tilde{\omega}) = \frac{\Gamma(1 + 2\tilde{\mu})}{2\tilde{\mu}} \left[ J_{2\tilde{\mu}}(\sqrt{8\kappa\tilde{\omega}r}) - I_{2\tilde{\mu}}(\sqrt{8\kappa\tilde{\omega}r}) \right] (rr')^{\frac{1}{2} + \tilde{\mu}} \tilde{\omega}^{2\tilde{\mu}}
\]

(26)

\[
+ \frac{\Gamma(1 + 2\tilde{\mu})^2}{2\tilde{\mu} \Gamma(2\mu) \Gamma(\frac{1}{2} - \tilde{\mu} - \kappa)} (rr')^{\frac{1}{2} - \kappa - 2\tilde{\mu}} J_{2\tilde{\mu}}(\sqrt{8\kappa\tilde{\omega}r}) J_{2\tilde{\mu}}(\sqrt{8\kappa\tilde{\omega}r})
\]

(27)

\[
+ e^{(2\tilde{\mu}+1)} I_{2\tilde{\mu}}(\sqrt{8\kappa\tilde{\omega}r}) I_{2\tilde{\mu}}(\sqrt{8\kappa\tilde{\omega}r}).
\]
It can be noticed that the first part of the above Eq. (26) the late time tail is proportional to $t^{-1}$. Just we calculate the second term of the right-hand side of Eq. (26). For the case when $\kappa \gg 1$ it can be brought to the form written as

$$G_e(2)(r, r'; t) = \frac{M}{2\pi} \int_{-\infty}^{\infty} dw \, e^{i(2\pi \kappa - wt)} e^{i\phi},$$

(27)

where we have defined

$$e^{i\phi} = \frac{1 + (-1)^{2\tilde{\mu}} e^{-2\pi i\kappa}}{1 + (-1)^{2\tilde{\mu}} e^{2\pi i\kappa}},$$

(28)

while $M$ provides the relation as follows:

$$M = \frac{(\Gamma(1 + 2\tilde{\mu}))^2 \Gamma(-2\tilde{\mu})}{2\tilde{\mu} \Gamma(2\tilde{\mu})} (r')^{\frac{5}{2}} \left[ J_{2\tilde{\mu}}(\sqrt{8\kappa \omega r}) J_{2\tilde{\mu}}(\sqrt{8\kappa \omega r'}) + I_{2\tilde{\mu}}(\sqrt{8\kappa \omega r}) I_{2\tilde{\mu}}(\sqrt{8\kappa \omega r'}) \right].$$

(29)

At very late time both terms $e^{iw t}$ and $e^{2\pi \kappa}$ are rapidly oscillating. It means that the scalar waves are mixed states consisting of the states with multipole phases backscattered by spacetime curvature. Most of them cancel with each others which have the inverse phase. In such a case, one can find the value of $G_e(2)$ by means of the saddle point method. It could be found that the value $2\pi \kappa - wt$ is stationary at the value of $w$ equal to the following:

$$a_0 = \left[ \frac{\pi m(r_+ + \alpha_1 r_-)}{2\sqrt{2}} \right]^{\frac{1}{\kappa}}.$$

(30)

Then approximating integration by the contribution from the very close nearby of $a_0$ is given by

$$F = \frac{im^{4/3}}{\sqrt{2}} (\pi)^{\frac{1}{6}} (r_+ + \alpha_1 r_-)^{\frac{2}{6}} (mt)^{-\frac{2}{6}} e^{i\phi(a_0)}.$$

(31)

In comparison to the late-time behaviour of the second term in Eq. (26), the first term can be neglected. The dominant role plays the behaviour of the second term, i.e., the late-time behaviour is proportional to $t^{-\frac{2}{6}}$. Thus, the asymptotic late-time behaviour of the Green’s function can be written in the form

$$G_e(r, r'; t) = \frac{m^{4/3}}{\sqrt{2}} (\pi)^{\frac{1}{6}} (r_+ + \alpha_1 r_-)^{\frac{2}{6}} (mt)^{-\frac{2}{6}} \sin(mt) \tilde{\psi}(r, m) \tilde{\psi}(r', m),$$

(32)

III. CONCLUSIONS

We have studied analytically the intermediate and the late-time behaviour of massive scalar field in the background of a dilaton black hole arising in the low energy string theory, the so-called dilaton gravity. We took into account the coupling constant $\alpha$ appearing in the action in dilaton gravity. In the case of the intermediate asymptotic behaviour we confirmed analytically the results found in [12], i.e., the oscillatory power law dependence only on the angular momentum parameter $l$ and the field parameter, the mass $m$ of it. However, as was claimed in Ref.[12] the intermediate asymptotic behaviour is not the final pattern of decay rate. The decay rate of the form $t^{-\frac{2}{6}}$ occurs when $\kappa \gg 1$. It stems from the resonance backscattering off the spacetime curvature. This tail rate behaviour is not dependent on mass of the black hole and scalar field, $\alpha$, coupling, and multiple number $l$. Our analytical results confirmed the numerical predictions [12] that oscillatory tail decay rate of the form $t^{-\frac{2}{6}}$. Our conclusions are also applicable for the extremal dilaton black holes, i.e., for which one has $r_+ = r_-$. Having in mind relation (33) one gets the following late-time behaviour for the extremal dilaton black hole. It provides the following:

$$G_e^{ext}(r, r'; t) = \frac{m^{4/3}}{\sqrt{2}} (\pi)^{\frac{1}{6}} \left( \frac{2Q}{\sqrt{1 + \alpha^2}} \right)^{\frac{1}{6}} (mt)^{-\frac{2}{6}} \sin(mt) \tilde{\psi}(r, m) \tilde{\psi}(r', m),$$

(33)

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