DETERMINANTS OF LAPLACIANS ON HILBERT MODULAR SURFACES

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Abstract. We study regularized determinants of Laplacians acting on the space of Hilbert-Maass forms for the Hilbert modular group of a real quadratic field. We show that these determinants are described by Selberg type zeta functions introduced in [4, 5].

1. Introduction

Determinants of the Laplacian $\Delta$ acting on the space of Maass forms on a hyperbolic Riemann surface $X$ are studied by many authors. (See, for example [13, 2, 8, 9].) It is known that the determinants of $\Delta$ are described by the Selberg zeta function (cf. [14]) for $X$.

On the other hand, two Laplacians $\Delta^{(1)}$, $\Delta^{(2)}$ act on the space of Hilbert-Maass forms on the Hilbert modular surface $X_K$ of a real quadratic field $K$. For this reason, it seems that there are no explicit formulas for "Determinants of Laplacians" on $X_K$ until now. In this article we consider regularized determinants of the first Laplacian $\Delta^{(1)}$ acting on its certain subspaces $V^{(2)}_m$, indexed by $m \in 2\mathbb{N}$. We show that these determinants are described by Selberg type zeta functions for $X_K$ introduced in [4, 5].

Let $K/\mathbb{Q}$ be a real quadratic field with class number one and $O_K$ be the ring of integers of $K$. Put $D$ be the discriminant of $K$ and $\epsilon_1$ be the fundamental unit of $K$. We denote the generator of $\text{Gal}(K/\mathbb{Q})$ by $\sigma$ and put $a':=\sigma(a)$ for $a \in K$. We also put $\gamma'=\left(\begin{array}{cc}a' & b' \\ c' & d' \end{array}\right)$ for $\gamma=\left(\begin{array}{cc}a & b \\ c & d \end{array}\right) \in \text{PSL}(2,O_K)$. Let $\Gamma_K = \{(\gamma,\gamma') \mid \gamma \in \text{PSL}(2,O_K)\}$ be the Hilbert modular group of $K$. It is known that $\Gamma_K$ is a co-finite (non-cocompact) irreducible discrete subgroup of $\text{PSL}(2,\mathbb{R}) \times \text{PSL}(2,\mathbb{R})$ and $\Gamma_K$ acts on the product $\mathbb{H}^2$ of two copies of the upper half plane $\mathbb{H}$ by component-wise linear fractional transformation. $\Gamma_K$ have only one cusp ($\infty$, $\infty$), i.e. $\Gamma_K$-inequivalent parabolic fixed point. $X_K := \Gamma_K \backslash \mathbb{H}^2$ is called the Hilbert modular surface.

Let $(\gamma,\gamma') \in \Gamma_K$ be hyperbolic-elliptic, i.e. $|\text{tr}(\gamma)| > 2$ and $|\text{tr}(\gamma')| < 2$. Then the centralizer of hyperbolic-elliptic $(\gamma,\gamma')$ in $\Gamma_K$ is infinite cyclic.

Definition 1.1 (Selberg type zeta function for $\Gamma_K$ with the weight $(0, m)$). For an even integer $m \geq 2$, we define

\begin{equation}
Z_m(s) := \prod_{(p,p') \in \text{Primes}} \prod_{n=0}^{\infty} \left(1 - \epsilon \left(m-2\omega\right) N(p)^{-(n+s)} \right)^{-1} \text{ for } \text{Re}(s) > 1.
\end{equation}

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Here, \((p, p')\) run through the set of primitive hyperbolic-elliptic \(\Gamma_K\)-conjugacy classes of \(\Gamma_K\), and \((p, p')\) is conjugate in \(\text{PSL}(2, \mathbb{R})\) to

\[
(p, p') \sim \begin{pmatrix} N(p)^{1/2} & 0 \\ 0 & N(p)^{-1/2} \end{pmatrix}, \begin{pmatrix} \cos \omega & -\sin \omega \\ \sin \omega & \cos \omega \end{pmatrix}
\]

Here, \(N(p) > 1\), \(\omega \in (0, \pi)\) and \(\omega \notin \pi\mathbb{Q}\). The product is absolutely convergent for \(\Re(s) > 1\).

Analytic properties of \(Z_m(s)\) are known.

**Theorem 1.2** ([3, Theorems 5.3 and 6.5]). For an even integer \(m \geq 2\), \(Z_m(s)\) a priori defined for \(\Re(s) > 1\) has a meromorphic extension over the whole complex plane.

In this article, we also consider “the square root of \(Z_2(s)\)."

**Definition 1.3** \((\sqrt{Z_2(s)})\).

\[
\sqrt{Z_2(s)} := \prod_{(p, p') \in \Gamma_K} \prod_{n=0}^{\infty} \left( 1 - N(p)^{-n+s} \right)^{-1/2}
\]

\[(1.2) \quad \sqrt{Z_2(s)} = \exp \left( \frac{1}{2} \sum_{(p, p') \in \Gamma_K} \sum_{k=1}^{\infty} \frac{1}{k} \frac{N(p)^{-ks}}{1 - N(p)^{-k}} \right) \text{ for } \Re(s) > 1.
\]

By [5, Theorem 6.5] and the fact that the Euler characteristic of \(X_K\) is even (See Lemma 2.2), we see that \(\frac{d}{ds} \log Z_2(s)\) has even integral residues at any poles. Therefore, we find that \(\sqrt{Z_2(s)}\) has a meromorphic continuation to the whole complex plane.

Let us introduce the completed Selberg type zeta functions \(\hat{Z}_2^{1/2}(s)\) and \(\hat{Z}_m(s)\) \((m \geq 4)\), which are invariant under \(s \to 1 - s\). (See [5, Theorems 5.4 and 6.6].)

**Definition 1.4** (Completed Selberg zeta functions).

\[(1.3) \quad \hat{Z}_2^{1/2}(s) := \sqrt{Z_2(s)} \ Z_{id}^{1/2}(s) \ Z_{\text{ell}}^{1/2}(s; 2) \ Z_{\text{par/sct}}^{1/2}(s; 2) \ Z_{\text{hyp2/sct}}^{1/2}(s; 2)
\]

with

\[
Z_{id}^{1/2}(s) := \left( \Gamma_2(s) \Gamma_2(s + 1) \right)^{2\zeta(-1)}, \quad Z_{\text{ell}}^{1/2}(s; 2) := \prod_{j=1}^{N} \prod_{l=0}^{\nu_j - 1} \Gamma \left( \frac{s+1}{\nu_j} + \frac{\nu_j - 1 - 2l}{2\nu_j} \right),
\]

\[
Z_{\text{par/sct}}^{1/2}(s; 2) := \varepsilon^{-s}, \quad Z_{\text{hyp2/sct}}^{1/2}(s; 2) := \zeta(s).
\]

\[(1.4) \quad \hat{Z}_m(s) := Z_m(s) \ Z_{id}(s) \ Z_{\text{ell}}(s; m) \ Z_{\text{hyp2/sct}}(s; m) \quad (m \geq 4)
\]

with

\[
Z_{id}(s) := \left( \Gamma_2(s) \Gamma_2(s + 1) \right)^{2\zeta(-1)}, \quad Z_{\text{ell}}(s; m) := \prod_{j=1}^{N} \prod_{l=0}^{\nu_j - 1} \Gamma \left( \frac{s+1}{\nu_j} + \frac{\nu_j - 1 - \alpha_l(m, j) - 2\ell}{2\nu_j} \right),
\]

\[
Z_{\text{hyp2/sct}}(s; m) := \zeta \left( s + \frac{m}{2} - 1 \right) \zeta \left( s + \frac{m}{2} - 2 \right)^{-1}.
\]

Here, \(\Gamma_2(s)\) is the double Gamma function (for definition, we refer to [10] or [3] Definition 4.10, p. 751), the natural numbers \(\nu_1, \nu_2, \ldots, \nu_N\) are the orders of the elliptic fixed points in \(X_K\) and the integers \(\alpha_l(m, j), \overline{\alpha_l}(m, j) \in \{0, 1, \ldots, \nu_j - 1\}\) are defined in (2.1). \(\zeta_K(s)\) is the Dedekind zeta function of \(K\); \(\zeta(s) := (1 - \varepsilon^{-2s})^{-1}\) and \(\varepsilon\) is the fundamental unit of \(K\).
Let \( m \in 2\mathbb{N} \). We recall that two Laplacians

\[
\Delta_0^{(1)} := -y_1^2 \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y_1^2} \right), \quad \Delta_m^{(2)} := -y_2^2 \left( \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial y_2^2} \right) + im y_2 \frac{\partial}{\partial x_2}
\]

are acting on \( L^2_{\text{dis}}(\Gamma \backslash \mathbb{H}^2; (0, m)) \), the space of Hilbert-Maass forms for \( \Gamma \) with weight \((0, m)\). (See Definition 2.3) We consider a certain subspace of \( L^2_{\text{dis}}(\Gamma \backslash \mathbb{H}^2; (0, m)) \) given by

\[
V_m^{(2)} = \left\{ f(z_1, z_2) \in L^2_{\text{dis}}(\Gamma \backslash \mathbb{H}^2; (0, m)) \left| \Delta_m^{(2)} f = \frac{m}{2} (1 - \frac{m}{2}) f \right. \right\}.
\]

The set of eigenvalues of \( \Delta_0^{(1)} \big|_{V_m^{(2)}} \) are enumerated as

\[
0 < \lambda_0(m) \leq \lambda_1(m) \leq \cdots \leq \lambda_n(m) \leq \cdots
\]

Let \( s \) be a fixed sufficiently large real number. We consider the spectral zeta function by using these eigenvalues.

\[
\zeta_m(w, s) = \sum_{n=0}^{\infty} \frac{1}{(\lambda_n(m) + s(s-1))^w} \quad (\text{Re}(w) > 0).
\]

We can show that \( \zeta_m(w, s) \) is holomorphic at \( w = 0 \). (See Proposition 1.3)

Let us define the regularized determinants of the Laplacian \( \Delta_0^{(1)} \big|_{V_m^{(2)}} \).

**Definition 1.5** (Determinants of restrictions of \( \Delta_0^{(1)} \)). Let \( m \in 2\mathbb{N} \). For \( s > 0 \), define

\[
\text{Det}\left( \Delta_0^{(1)} \big|_{V_m^{(2)}} + s(s-1) \right) := \exp \left( - \frac{\partial}{\partial w} \big|_{w=0} \sum_{n=0}^{\infty} \frac{1}{(\lambda_n(m) + s(s-1))^w} \right).
\]

We see later that \( \text{Det}\left( \Delta_0^{(1)} \big|_{V_m^{(2)}} + s(s-1) \right) \) can be extended to an entire function of \( s \). (See Corollary 1.7)

Our main theorem is as follows.

**Theorem 1.6** (Main Theorem). Let \( \Box_m := \Delta_0^{(1)} \big|_{V_m^{(2)}} \) for \( m \in 2\mathbb{N} \). We have the following determinant expressions of the completed Selberg type zeta functions.

1. \( \hat{Z}_2^\perp(s) = e^{(s - \frac{1}{2})^2 \zeta_K(-1) + \frac{1}{2}} \text{Det}(\Box_2 + s(s-1)) \).
2. \( \hat{Z}_4(s) = e^{2(s - \frac{1}{2})^2 \zeta_K(-1) + \frac{1}{2}} \frac{s(s-1) \cdot \text{Det}(\Box_4 + s(s-1))}{\text{Det}(\Box_2 + s(s-1))} \).
3. For \( m \geq 6 \), \( \hat{Z}_m(s) = e^{2(s - \frac{1}{2})^2 \zeta_K(-1) + \frac{1}{2}} \frac{\text{Det}(\Box_m + s(s-1))}{\text{Det}(\Box_{m-2} + s(s-1))} \).

Here, the constants \( C_m \) are given by

\[
C_2 = -\frac{1}{2} \log \varepsilon + \sum_{j=1}^{N} \frac{\nu_j^2 - 1}{12 \nu_j} \log \nu_j,
\]

\[
C_m = \sum_{j=1}^{N} \frac{\nu_j^2 - 1 - 12 \alpha_0(m, j) \nu_j - \alpha_0(m, j)}{6 \nu_j} \log \nu_j \quad (m \geq 4),
\]

the natural numbers \( \nu_1, \nu_2, \ldots, \nu_N \) are the orders of the elliptic fixed points in \( X_K \) and the integers \( \alpha_0(m, j) \in \{0, 1, \ldots, \nu_j - 1\} \) are defined in \( [2, 4] \).
We know the following Weyl’s law:
\[ N_m^+(T) := \# \{ j \mid \lambda_j(m) \leq T \} \sim \frac{(m-1)}{2} \cdot \zeta_K(-1) \cdot T \quad (T \to \infty). \]
(See [5 Theorem 6.11].) Therefore, we may say that \( Z_m(s) (m \geq 4) \) have “more” zeros than poles.

We have several corollaries from Theorem [1.6] by direct calculation.

**Corollary 1.7.** Let \( \Box_m = \Delta_0^{(1)} |_{V_m^{(2)}} \) for \( m \in 2\mathbb{N} \). For \( m \in 2\mathbb{N} \), we have

1. \( \det(\Box_2 + s(s-1)) = s(s-1) e^{-(s-\frac{1}{2})^2} \zeta_K(-1) - C_2 \tilde{Z}_2^1(s). \)
2. \( \det(\Box_m + s(s-1)) = e^{-(m-1)(s-\frac{1}{2})^2} \zeta_K(-1) - (C_2 + C_4 + \cdots + C_m) \tilde{Z}_m^1(s) \tilde{Z}_4(s) \cdots \tilde{Z}_m(s) \) for \( m \geq 4 \).

It follows from the above corollary that \( \det(\Box_m + s(s-1)) (m \in 2\mathbb{N}) \) can be extended to entire functions of \( s \).

By putting \( s = 1 \) in the above, we have

**Corollary 1.8.** For \( m \in 2\mathbb{N} \), we have

1. \( \det(\Box_2) = e^{-\frac{1}{4} \zeta_K(-1) - C_2} \text{Res}_{s=1} \tilde{Z}_2^1(s). \)
2. \( \det(\Box_4) = e^{-\frac{1}{4} \zeta_K(-1) - (C_2 + C_4)} \text{Res}_{s=1} \tilde{Z}_2^1(s) \cdot \tilde{Z}_4^1(1). \)
3. \( \det(\Box_m) = e^{-\frac{m-1}{4} \zeta_K(-1) - (C_2 + C_4 + \cdots + C_m)} \text{Res}_{s=1} \tilde{Z}_2^1(s) \cdot \tilde{Z}_4^1(1) \cdot \tilde{Z}_6^1(1) \cdots \tilde{Z}_m(1) \)

for \( m \geq 6 \).

Here, \( \Box_m = \Delta_0^{(1)} |_{V_m^{(2)}} \) for \( m \in 2\mathbb{N} \).

2. Preliminaries

We fix the notation for the Hilbert modular group of a real quadratic field in this section. We also recall the definition of Hilbert-Maass forms for the Hilbert modular group and review “Differences of the Selberg trace formula”, introduced in [3], which play a crucial role in this article.

2.1. Hilbert modular group of a real quadratic field. Let \( K/\mathbb{Q} \) be a real quadratic field with class number one and \( \mathcal{O}_K \) be the ring of integers of \( K \). Put \( D \) be the discriminant of \( K \) and \( \varepsilon > 1 \) be the fundamental unit of \( K \). We denote the generator of \( \text{Gal}(K/\mathbb{Q}) \) by \( \sigma \) and put \( a' := \sigma(a) \) for \( a \in K \). We also put \( \gamma' = \left( \begin{array}{cc} a' & b' \\ c' & d' \end{array} \right) \) for \( \gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{PSL}(2, \mathbb{O}_K) \).

Let \( G = \text{PSL}(2, \mathbb{R})^2 = \left( \frac{\text{SL}(2, \mathbb{R})}{\{ \pm I \}} \right)^2 \) and \( \mathbb{H}^2 \) be the direct product of two copies of the upper half plane \( \mathbb{H} := \{ z \in \mathbb{C} \mid \text{Im}(z) > 0 \} \). The group \( G \) acts on \( \mathbb{H}^2 \) by
\[
g.z = (g_1, g_2). (z_1, z_2) = (a_1z_1 + b_1, a_2z_2 + b_2) \quad (c_1z_1 + d_1, c_2z_2 + d_2) \in \mathbb{H}^2
\]
for \( g = (g_1, g_2) = \left( \begin{array}{cc} a_1 & b_1 \\ c_1 & d_1 \end{array} \right), \left( \begin{array}{cc} a_2 & b_2 \\ c_2 & d_2 \end{array} \right) \) and \( z = (z_1, z_2) \in \mathbb{H}^2 \).

A discrete subgroup \( \Gamma \subset G \) is called irreducible if it is not commensurable with any direct product \( \Gamma_1 \times \Gamma_2 \) of two discrete subgroups of \( \text{PSL}(2, \mathbb{R}) \). We have classification of the elements of irreducible \( \Gamma \).
Proposition 2.1 (Classification of the elements). Let \( \Gamma \) be an irreducible discrete subgroup of \( G \). Then any element of \( \Gamma \) is one of the followings.

1. \( \gamma = (I, I) \) is the identity
2. \( \gamma = (\gamma_1, \gamma_2) \) is hyperbolic \( \iff |\text{tr}(\gamma_1)| > 2 \) and \( |\text{tr}(\gamma_2)| > 2 \)
3. \( \gamma = (\gamma_1, \gamma_2) \) is elliptic \( \iff |\text{tr}(\gamma_1)| < 2 \) and \( |\text{tr}(\gamma_2)| < 2 \)
4. \( \gamma = (\gamma_1, \gamma_2) \) is hyperbolic-elliptic \( \iff |\text{tr}(\gamma_1)| > 2 \) and \( |\text{tr}(\gamma_2)| < 2 \)
5. \( \gamma = (\gamma_1, \gamma_2) \) is elliptic-hyperbolic \( \iff |\text{tr}(\gamma_1)| < 2 \) and \( |\text{tr}(\gamma_2)| > 2 \)
6. \( \gamma = (\gamma_1, \gamma_2) \) is parabolic \( \iff |\text{tr}(\gamma_1)| = |\text{tr}(\gamma_2)| = 2 \)

Note that there are no other types in \( \Gamma \). (parabolic-elliptic etc.)

Let us consider the Hilbert modular group of the real quadratic field \( K \) with class number one,

\[ \Gamma_K := \left\{ (\gamma, \gamma') = \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \right) \left| \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \in \text{PSL}(2, \mathbb{O}_K) \right\}. \]

It is known that \( \Gamma_K \) is an irreducible discrete subgroup of \( G = \text{PSL}(2, \mathbb{R})^2 \) with the only one cusp \( \infty := (\infty, \infty) \), i.e. \( \Gamma_K \)-inequivalent parabolic fixed point. \( X_K = \Gamma_K \backslash \mathbb{H}^2 \) is called the Hilbert modular surface.

We have a lemma about the Euler characteristic of the Hilbert modular surface \( X_K \).

Lemma 2.2. Let \( E(X_K) \) be the Euler characteristic of the Hilbert modular surface \( X_K = \Gamma_K \backslash \mathbb{H}^2 \). Then we have \( E(X_K) \in 2\mathbb{N} \).

Proof. By noting the formula \( E(X_K) = 2\zeta_K(-1) + \sum_{j=1}^{N} \frac{\nu_j - 1}{\nu_j} \) (see (2), (4) on [7, pp.46-47]), \( E(X_K) \) is a positive integer. Let \( Y_K \) and \( Y_K^- \) be the non-singular algebraic surfaces resolved singularities, in the canonical minimal way, of compactifications of \( \Gamma_K \backslash \mathbb{H}^2 \) and \( \Gamma_K \backslash (\mathbb{H} \times \mathbb{H}^-) \) respectively. Here \( \mathbb{H}^- \) is the lower half plane. Let \( \chi(Y_K) \) and \( \chi(Y_K^-) \) be the arithmetic genera of \( Y_K \) and \( Y_K^- \) respectively. By the formulas (12) and (14) on [7, p.48], we have

\[ E(X_K) = 2(\chi(Y_K) + \chi(Y_K^-)) \]

We complete the proof. \( \square \)

We fix the notation for elliptic conjugacy classes in \( \Gamma_K \). Let \( R_1, R_2, \ldots, R_N \) be a complete system of representatives of the \( \Gamma_K \)-conjugacy classes of primitive elliptic elements of \( \Gamma_K \).

\( \nu_1, \nu_2, \ldots, \nu_N \) \((\nu_j \in \mathbb{N}, \nu_j \geq 2)\) denote the orders of \( R_1, R_2, \ldots, R_N \). We may assume that \( R_j \) is conjugate in \( \text{PSL}(2, \mathbb{R})^2 \) to

\[ R_j \sim \left( \begin{pmatrix} \cos \frac{\pi}{\nu_j} & -\sin \frac{\pi}{\nu_j} \\ \sin \frac{\pi}{\nu_j} & \cos \frac{\pi}{\nu_j} \end{pmatrix}, \begin{pmatrix} \cos \frac{l \pi}{\nu_j} & -\sin \frac{l \pi}{\nu_j} \\ \sin \frac{l \pi}{\nu_j} & \cos \frac{l \pi}{\nu_j} \end{pmatrix} \right), \quad (t_j, \nu_j) = 1. \]

For even natural number \( m \geq 2 \) and \( l \in \{0, 1, \ldots, \nu_j - 1\} \), we define \( \alpha_l(m, j), \bar{\alpha}_l(m, j) \in \{0, 1, \ldots, \nu_j - 1\} \) by

\[ l + \frac{t_j(m - 2)}{2} \equiv \alpha_l(m, j) \pmod{\nu_j}, \]
\[ l - \frac{t_j(m - 2)}{2} \equiv \bar{\alpha}_l(m, j) \pmod{\nu_j}. \]

We divide hyperbolic conjugacy classes of \( \Gamma_K \) into two subclasses according to their types.

Definition 2.3 (Types of hyperbolic elements). For a hyperbolic element \( \gamma \), we define that
(1) $\gamma$ is type 1 hyperbolic $\iff$ whose all fixed points are not fixed by parabolic elements.
(2) $\gamma$ is type 2 hyperbolic $\iff$ not type 1 hyperbolic.

We denote by $\Gamma_{\mathrm{H}1}$, $\Gamma_{\mathrm{E}}$, $\Gamma_{\mathrm{HE}}$ and $\Gamma_{\mathrm{H}2}$, type 1 hyperbolic $\Gamma_K$-conjugacy classes, elliptic $\Gamma_K$-conjugacy classes, hyperbolic-elliptic $\Gamma_K$-conjugacy classes, elliptic-hyperbolic $\Gamma_K$-conjugacy classes and type 2 hyperbolic $\Gamma_K$-conjugacy classes of $\Gamma_K$ respectively.

2.2. **The space of Hilbert-Maass forms.** Fix the weight $(m_1, m_2) \in (\mathbb{Z})^2$. Set the automorphic factor $j_\gamma(z_j) = \frac{c_{m_j} + d}{|cz_j + a|}$ for $\gamma \in \text{PSL}(2, \mathbb{R})$ $(j = 1, 2)$.

Let $\Delta_{m_j} := -y_j^2 \left( \frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2} \right) + im_j y_j \frac{\partial}{\partial x_j}$ $(j = 1, 2)$ be the Laplacians of weight $m_j$ for the variable $z_j$.

Let us define the $L^2$-space of automorphic forms of weight $(m_1, m_2)$ with respect to the Hilbert modular group $\Gamma_K$.

**Definition 2.4** ($L^2$-space of automorphic forms of weight $(m_1, m_2)$).

$$L^2(\Gamma_K \backslash \mathbb{H}^2; (m_1, m_2)) := \left\{ f : \mathbb{H}^2 \to \mathbb{C}, C^\infty \ | \ (i) \ f((\gamma, \gamma')(z_1, z_2)) = j_\gamma(z_{1'})j_{\gamma'}(z_{2'})f(z_1, z_2) \ \forall (\gamma, \gamma') \in \Gamma_K \\
(ii) \exists (\lambda^{(1)}, \lambda^{(2)}) \in \mathbb{R}^2 \Delta_{m_1} f(z_1, z_2) = \lambda^{(1)} f(z_1, z_2), \ \Delta_{m_2} f(z_1, z_2) = \lambda^{(2)} f(z_1, z_2) \\
(iii) \|f\|^2 = \int_{\Gamma_K \backslash \mathbb{H}^2} f(z)\overline{f}(z) \ d\mu(z) < \infty \right\}.$$  

Here, $d\mu(z) = \frac{dx_1 dy_1 dx_2 dy_2}{y_2^2}$ for $z = (z_1, z_2) \in \mathbb{H}^2$.

Then, it is known that

**Proposition 2.5.** Let $L^2_{\text{dis}}(\Gamma_K \backslash \mathbb{H}^2; (m_1, m_2))$ be the subspace of the discrete spectrum of the Laplacians and $L^2_{\text{con}}(\Gamma_K \backslash \mathbb{H}^2; (m_1, m_2))$ be the subspace of the continuous spectrum. Then, we have a direct sum decomposition:

$$L^2(\Gamma_K \backslash \mathbb{H}^2; (m_1, m_2)) = L^2_{\text{dis}}(\Gamma_K \backslash \mathbb{H}^2; (m_1, m_2)) \oplus L^2_{\text{con}}(\Gamma_K \backslash \mathbb{H}^2; (m_1, m_2))$$

and there is an orthonormal basis $\{\phi_j\}_{j=0}^\infty$ of $L^2_{\text{dis}}(\Gamma_K \backslash \mathbb{H}^2; (m_1, m_2))$.

**Definition 2.6** (Hilbert Maass forms of weight $(m_1, m_2)$). Let $(m_1, m_2) \in (2\mathbb{Z})^2$. We call

$$L^2_{\text{dis}}(\Gamma_K \backslash \mathbb{H}^2; (m_1, m_2))$$

the space of Hilbert Maass forms for $\Gamma_K$ of weight $(m_1, m_2)$.

Let $\{\phi_j\}_{j=0}^\infty$ be an orthonormal basis of $L^2_{\text{dis}}(\Gamma_K \backslash \mathbb{H}^2; (m_1, m_2))$ and $(\lambda_j^{(1)}, \lambda_j^{(2)}) \in \mathbb{R}^2$ such that

$$\Delta_{m_1} \phi_j = \lambda_j^{(1)} \phi_j \quad \text{and} \quad \Delta_{m_2} \phi_j = \lambda_j^{(2)} \phi_j.$$  

We write $\lambda_j^{(l)} = \frac{1}{4} + (r_j^{(l)})^2$ and $r_j^{(l)}$ are defined by

$$r_j^{(l)} := \begin{cases} \sqrt{\lambda_j^{(l)} - \frac{1}{4}} & \text{if } \lambda_j^{(l)} \geq \frac{1}{4}, \\
 i\sqrt{\frac{1}{4} - \lambda_j^{(l)}} & \text{if } \lambda_j^{(l)} < \frac{1}{4}, \end{cases}$$

for $l = 1, 2$. 

2.3. Double differences of the Selberg trace formula. Let \( m \) be an even integer. We studied and derived the full Selberg trace formula for \( L^2(\Gamma \setminus \mathbb{H}^2; (0, m)) \) in \([5]\). (See \([5\), Theorem 2.22].) Let \( h(r_1, r_2) \) be an even “test function” which satisfy certain analytic conditions. Roughly speaking, \([5\), Theorem 2.22] is as follows.

\[
\sum_{j=0}^{\infty} h(r^{(1)}_j, r^{(2)}_j) = I(h) + II_a(h) + II_b(h) + III(h).
\]

Here, the right hand side is a sum of distributions of \( h \) contributed from several conjugacy classes of \( \Gamma_K \) and Eisenstein series for \( \Gamma_K \). Assuming that the test function \( h(r_1, r_2) \) is a product of \( h_1(r_1) \) and \( h_2(r_2) \), we derived “differences of STF” (\([5\), Theorem 4.1]) and “double differences of STF” (\([5\), Theorem 4.4]). We explain for this.

Let us consider the subspace of \( L^2_{\text{dis}}(\Gamma \setminus \mathbb{H}^2; (0, m)) \) given by

\[
V^{(2)}_m = \left\{ f \in L^2_{\text{dis}}(\Gamma \setminus \mathbb{H}^2; (0, m)) \mid \Delta^{(2)}_m f = \frac{m}{2} \frac{1 - m^2}{2} f \right\}.
\]

Let \( h_1(r) \) be an even function, analytic in \( \text{Im}(r) < \delta \) for some \( \delta > 0 \),

\[
h_1(r) = O((1 + |r|^2)^{-2-\delta})
\]

for some \( \delta > 0 \) in this domain. Let \( g_1(u) := \frac{1}{2\pi} \int_{-\infty}^{\infty} h_1(r)e^{-iru} dr \). Then we have

**Proposition 2.7** (Double differences of STF for \( L^2(\Gamma \setminus \mathbb{H}^2; (0, 2)) \)). Let \( m = 2 \). We have

\[
\sum_{j=0}^{\infty} h_1\left(\rho_j^{(2)}\right) - h_1\left(\frac{i}{2}\right) = \frac{\text{vol}(\Gamma \setminus \mathbb{H}^2)}{16\pi^2} \int_{-\infty}^{\infty} rh_1(r) \tanh(\pi r) dr
\]

\[
- \sum_{R(\theta_1, \theta_2) \in \Gamma_E} \frac{i e^{-i\theta_1}}{8\nu R \sin \theta_1} \int_{-\infty}^{\infty} g_1(u) e^{-u/2} \left[ \frac{e^u - e^{2i\theta_1}}{\cosh u - \cos 2\theta_1} \right] du
\]

\[
- \frac{1}{2} \sum_{(\gamma, \omega) \in \Gamma_{\text{HE}}} \log \frac{N(\gamma_0)}{N(\gamma)} \frac{g_1(\log N(\gamma))}{N(\gamma)^{1/2} - N(\gamma)^{-1/2}} - \log \varepsilon g_1(0) - 2 \log \varepsilon \sum_{k=1}^{\infty} g_1(2k \log \varepsilon) \varepsilon^{-k}.
\]

Here, \( \{\lambda_j^{(2)} = 1/4 + \rho_j^{(2)^2}\}_{j=0}^{\infty} \) is the set of eigenvalues of the Laplacian \( \Delta_0^{(1)} \) acting on \( V^{(2)}_2 \).

**Proof.** See \([5\), Corollary 6.3].
Proposition 2.8. (Double differences of STF for $L^2(\Gamma_K/\mathbb{H}^2; (0, m))$). Let $m \in 2\mathbb{N}$ and $m \geq 4$. We have

$$\sum_{j=0}^{\infty} h_1(\rho_j(m)) - \sum_{j=0}^{\infty} h_1(\rho_j(m - 2)) + \delta_{m,4} h_1\left(\frac{i}{2}\right)$$

$$= \frac{\text{vol}(\Gamma_K/\mathbb{H}^2)}{8\pi^2} \int_{-\infty}^{\infty} rh_1(r) \tanh(\pi r) \, dr$$

$$- \sum_{R(\theta_1, \theta_2) \in \Gamma_E} \frac{ie^{-i\theta_1} e^{i(m-2)\theta_2}}{4\nu R \sin \theta_1} \int_{-\infty}^{\infty} g_1(u) e^{-u/2} \left[ \frac{e^u - e^{2i\theta_1}}{\cosh u - \cos 2\theta_1} \right] \, du$$

$$- \sum_{(\gamma, \omega) \in \Gamma_{HE}} \log N(\gamma_0) \frac{N(\gamma)^{1/2} - N(\gamma)^{-1/2}}{2\pi} g_1(\log N(\gamma)) e^{i(m-2)\omega}$$

$$- 2 \log \varepsilon \sum_{k=1}^{\infty} g_1(2k \log \varepsilon) \left( e^{-k(m-1)} - e^{-k(m-3)} \right).$$

Here, $\{\lambda_j(q) = 1/4 + \rho_j(q)^2\}_{j=0}^{\infty}$ is the set of eigenvalues of the Laplacian $\Delta_0^{(1)}$ acting on $V_q^{(2)} (q = m, m - 2)$.

Proof. See [5, Theorem 4.4] and [5, (5.3)]. \qed

3. Asymptotic behavior of the completed Selberg zeta functions

We have to know the asymptotic behavior of the completed Selberg zeta functions $\hat{Z}_2^{1/2}(s)$ and $\hat{Z}_m(s)$ $(m \geq 4)$ when $s \to \infty$, to prove Main Theorem (Theorem 1.6). We calculate their asymptotic behavior in this section.

Lemma 3.1. (Stirling’s formula for $\Gamma_2(z)$). Let $\Gamma_2(z) := \exp\left(\frac{\partial}{\partial s}\bigg|_{s=0} \sum_{m,n=0}^{\infty} (m + n + z)^{-s}\right)$ be the double Gamma function. Then we have

$$\log \Gamma_2(z + 1) = \frac{3}{4} z^2 - \left(\frac{z^2}{2} - \frac{1}{12}\right) \log z + o(1) \quad (z \to \infty).$$

Proof. Let $G(z)$ be the Barnes $G$-function defined by (See [11, p.268].)

$$G(z) = (2\pi)^{\frac{z}{2}} e^{-\frac{z^2}{2}(1+\gamma)} \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right) e^{-z + \frac{z^2}{k}}.$$ 

Here, $\gamma = -\Gamma'(1)$ is the Euler constant. By using the relation (See [12, Proposition 4.1].)

$$\Gamma_2(z) = e^{\zeta'(-1)} (2\pi)^{\frac{z-1}{2}} G(z)^{-1},$$

and the asymptotic formula (See [11, p.269].)

$$\log G(z + 1) = \frac{z}{2} \log(2\pi) + \zeta'(-1) - \frac{3}{4} z^2 + \left(\frac{z^2}{2} - \frac{1}{12}\right) \log z + o(1) \quad (z \to \infty),$$

we have the desired formula. \qed
**Lemma 3.2** (Asymptotics of the identity factors). We have

(3.2) \( \log Z_{\text{id}}^2(s) = \zeta_K(-1) \left\{ \frac{3}{2}s^2 - s - \left( s^2 - s + \frac{1}{3} \right) \log s \right\} + o(1) \quad (s \to \infty) \),

(3.3) \( \log Z_{\text{id}}(s) = 2\zeta_K(-1) \left\{ \frac{3}{2}s^2 - s - \left( s^2 - s + \frac{1}{3} \right) \log s \right\} + o(1) \quad (s \to \infty) \).

**Proof.** By Definition 1.4

\[
\log Z_{\text{id}}^2(s) = \zeta_K(-1) \left( \log \Gamma_2(s) + \log \Gamma_2(s+1) \right)
\]

and Lemma 3.1, we have the desired (3.2). We see that the relation \( \log Z_{\text{id}}(s) = 2 \log Z_{\text{id}}^2(s) \) implies (3.3). It completes the proof. \( \square \)

**Lemma 3.3** (Asymptotics of the elliptic factors). We have

(3.4) \( \log Z_{\text{ell}}^1(s; 2) = \sum_{j=1}^{N} \frac{\nu_j^2 - 1}{12\nu_j} \log \frac{s}{\nu_j} + o(1) \quad (s \to \infty) \),

(3.5) \( \log Z_{\text{ell}}(s; m) = -\sum_{j=1}^{N} \frac{\nu_j^2 - 1}{6\nu_j} \frac{\nu_j - \alpha_0(m, j)}{\nu_j} \log \frac{s}{\nu_j} + o(1) \quad (s \to \infty) \)

for \( m \in 2\mathbb{N} \) and \( m \geq 4 \). Here \( \alpha_0(m, j) \in \{0, 1, \ldots, \nu_j - 1\} \) are defined in (2.1).

**Proof.** We use Stirling’s formula of \( \Gamma(z) \). (See [11] p.12.)

\[
\log \Gamma(z) = \left( z - \frac{1}{2} \right) \log z - z + \frac{1}{2} \log(2\pi) + o(1) \quad (z \to \infty).
\]

By Definition 1.4

\[
\log Z_{\text{ell}}(s; m) = \sum_{j=1}^{N} \sum_{l=0}^{\nu_j-1} \frac{\nu_j - 1 - \alpha_l(m, j) - \overline{\alpha_l}(m, j)}{\nu_j} \log \left( \frac{s + l}{\nu_j} \right).
\]
We see that \( \{\alpha_l(m, j) \mid 0 \leq l \leq \nu_j - 1\} = \{\overline{\alpha}_l(m, j) \mid 0 \leq l \leq \nu_j - 1\} = \{0, 1, 2, \ldots, \nu_j - 1\} \) for each \( j \). Thus we have \( \sum_{l=0}^{\nu_j-1} (\nu_j - 1 - \alpha_l(m, j) - \overline{\alpha}_l(m, j)) = 0 \), and find that

\[
\sum_{l=0}^{\nu_j-1} \frac{\nu_j - 1 - \alpha_l(m, j) - \overline{\alpha}_l(m, j)}{\nu_j} \log \Gamma\left(\frac{s+l}{\nu_j}\right)
\]

\[
= \sum_{l=0}^{\nu_j-1} \frac{\nu_j - 1 - \alpha_l(m, j) - \overline{\alpha}_l(m, j)}{\nu_j} \left( \left( \frac{s+l}{\nu_j} - \frac{1}{2} \right) \log \left( \frac{s+l}{\nu_j} \right) - \frac{s+l}{\nu_j} + \frac{1}{2} \log(2\pi) \right) + o(1)
\]

\[
= \sum_{l=0}^{\nu_j-1} \frac{\nu_j - 1 - \alpha_l(m, j) - \overline{\alpha}_l(m, j)}{\nu_j} \left( \left( \frac{s}{\nu_j} - \frac{1}{2} \right) \log(s + l) - \frac{l}{\nu_j} \log \nu_j - \frac{l}{\nu_j} \right) + o(1)
\]

\[
= \sum_{l=0}^{\nu_j-1} \frac{\nu_j - 1 - \alpha_l(m, j) - \overline{\alpha}_l(m, j)}{\nu_j} \left( \left( \frac{s}{\nu_j} - \frac{1}{2} \right) \log s + \frac{l}{\nu_j} \log \frac{s}{\nu_j} \right) + o(1)
\]

\[
= \sum_{l=0}^{\nu_j-1} \frac{\nu_j - 1 - \alpha_l(m, j) - \overline{\alpha}_l(m, j)}{\nu_j} \cdot \frac{l}{\nu_j} \log \frac{s}{\nu_j} + o(1)
\]

\[
= \frac{(\nu_j - 1)^2}{2\nu_j} \log \frac{s}{\nu_j} - \sum_{l=0}^{\nu_j-1} \frac{\alpha_l(m, j) + \overline{\alpha}_l(m, j)}{\nu_j} \cdot \frac{l}{\nu_j} \log \frac{s}{\nu_j} + o(1) \quad (s \to \infty).
\]

By (2.1), we can check that

\[
\alpha_l(m, j) = \begin{cases} 
\alpha_0(m, j) + l & (0 \leq l \leq \nu_j - \alpha_0(m, j) - 1) \\
\alpha_0(m, j) - \nu_j + l & (\nu_j - \alpha_0(m, j) \leq l \leq \nu_j - 1) 
\end{cases}
\]

hence we calculate further,

\[
\sum_{l=0}^{\nu_j-1} \frac{l \alpha_l(m, j)}{\nu_j^2} = \sum_{l=0}^{\nu_j-1} \frac{l \alpha_0(m, j) + l}{\nu_j^2} + \sum_{l=\nu_j - \alpha_0(m, j)}^{\nu_j-1} \frac{l \alpha_0(m, j) - \nu_j + l}{\nu_j^2}
\]

\[
= \frac{(\nu_j - 1)(2\nu_j - 1)}{6\nu_j} + \frac{\alpha_0(m, j)(\alpha_0(m, j) - \nu_j)}{\nu_j}.
\]

By noting \(\alpha_0(m, j)(\alpha_0(m, j) - \nu_j) = \overline{\alpha}_0(m, j)(\overline{\alpha}_0(m, j) - \nu_j)\), we have

\[
\sum_{l=0}^{\nu_j-1} \frac{\nu_j - 1 - \alpha_l(m, j) - \overline{\alpha}_l(m, j)}{\nu_j} \log \Gamma\left(\frac{s+l}{\nu_j}\right)
\]

\[
= \frac{(\nu_j - 1)^2}{2\nu_j} \log \frac{s}{\nu_j} - \sum_{l=0}^{\nu_j-1} \frac{\alpha_l(m, j) + \overline{\alpha}_l(m, j)}{\nu_j} \cdot \frac{l}{\nu_j} \log \frac{s}{\nu_j} + o(1)
\]

\[
= \frac{(\nu_j - 1)^2}{2\nu_j} \log \frac{s}{\nu_j} - \frac{(\nu_j - 1)(2\nu_j - 1)}{6\nu_j} + \frac{\alpha_0(m, j)(\alpha_0(m, j) - \nu_j)}{\nu_j} \log \frac{s}{\nu_j} + o(1)
\]

\[
= \frac{-\nu_j^2 - 1 - 12\alpha_0(m, j)\{\nu_j - \alpha_0(m, j)\}}{6\nu_j} \log \frac{s}{\nu_j} + o(1) \quad (s \to \infty).
\]
Thus we have (3.5). In addition, we note that
\[ \log Z_{\text{ell}}^2(s; 2) = \frac{1}{2} \log Z_{\text{ell}}(s; m) \bigg|_{m=2}. \]
Since \( \alpha_l(2, j) = l \), we see that \( \alpha_0(2, j) = 0 \) for any \( j \). Therefore we have (3.4). It completes the proof. \( \square \)

**Proposition 3.4** (Asymptotics of the completed Selberg zeta functions). We have
\[ \log \hat{Z}^\frac{3}{2}(s) = \zeta_K(-1) \left( \frac{3}{2} s^2 - s - \left( s^2 - s + \frac{1}{3} \right) \log s \right) \]
\[ - \sum_{j=1}^{N} \frac{\nu_j^2 - 1}{12\nu_j} \log \frac{s}{\nu_j} - s \log \varepsilon + o(1) \quad (s \to \infty), \]
\[ \log \hat{Z}_m(s) = 2\zeta_K(-1) \left( \frac{3}{2} s^2 - s - \left( s^2 - s + \frac{1}{3} \right) \log s \right) \]
\[ - \sum_{j=1}^{N} \frac{\nu_j^2 - 1 - 12\alpha_0(m, j)}{6\nu_j} \log \frac{s}{\nu_j} + o(1) \quad (s \to \infty), \]
for \( m \in 2\mathbb{N} \) and \( m \geq 4 \). Here \( \alpha_0(m, j) \in \{0, 1, \ldots, \nu_j - 1\} \) are defined in (2.1).

**Proof.** We note that \( \log \sqrt{Z_2(s)}, \log Z_m(s) = o(1) \) \( (s \to \infty) \). By Definition 1.4 and Lemmas 3.2 and 3.3 we complete the proof. \( \square \)

4. **Asymptotic behavior of the regularized determinants**

To investigate the analytic nature of the spectral zeta function \( \zeta_m(w, s) \) at \( w = 0 \), we introduce the theta function \( \theta_m(t) \) in this section. Since the regularized determinants of the Laplacians \( \text{Det}(\Box_m + s(s - 1)) \) are defined by the derivative of \( -\zeta_m(w, s) \) at \( w = 0 \), we need to know the asymptotics of \( -\frac{\partial}{\partial w} \zeta_m(w, s) \bigg|_{w=0} \) when \( s \to \infty \). We calculate their asymptotics in this section.

**Definition 4.1.** For \( m \in 2\mathbb{N} \) and \( t > 0 \), define
\[ \theta_m(t) := \sum_{j=0}^{\infty} e^{-t \lambda_j(m)}. \]

We investigate the asymptotic behavior of \( \theta_m(t) \) as \( t \to +0 \) by using Propositions 2.7 and 2.8 which are called “Double differences of the Selberg trace formula for Hilbert modular surfaces” introduced and proved in [5].

**Proposition 4.2.** We have the following asymptotic formulas.
\[ \theta_2(t) = \frac{1}{2} \zeta_K(-1) \frac{1}{t} - \frac{\log \varepsilon}{2\sqrt{\pi}} \frac{1}{\sqrt{t}} + \left( -\frac{1}{6} \zeta_K(-1) + b_0(2) + 1 \right) + o(1) \quad (t \to +0), \]
\[ \theta_m(t) = \frac{m-1}{2} \zeta_K(-1) \frac{1}{t} - \frac{\log \varepsilon}{2\sqrt{\pi}} \frac{1}{\sqrt{t}} + \left( -\frac{m-1}{6} \zeta_K(-1) + b_0(2) + b_0(4) + \cdots + b_0(m) \right) \]
\[ + o(1) \quad (m \in 2\mathbb{N}, m \geq 4). \]
Here, \( b_0(2) = - \sum_{j=1}^{N} \frac{\nu_j^2 - 1}{24\nu_j}, \) \( b_0(m) = - \sum_{j=1}^{N} \frac{\nu_j^2 - 1 - 12\alpha_0(m,j)\{\nu_j - \alpha_0(m,j)\}}{12\nu_j} \) \( (m \geq 4). \)

**Proof.** For \( t > 0, \) let us take the pair of test functions \( h_1(r) = e^{-t(r^2 + 1/4)} \) and \( g_1(u) = \frac{1}{\sqrt{4\pi t}} \exp(-\frac{u}{4t} - \frac{\nu^2}{4}) \) in Proposition 2.10, then we have

\[
\theta_2(t) - 1 = I_2(t) + E_2(t) + HE_2(t) + PS_2(t) + HS_2(t).
\]

Here,

- \( I_2(t) = \frac{\text{vol}(GK)\|H^2\|}{16\pi^2} \int_{-\infty}^{\infty} \exp\left(-\frac{r^2}{t} + 1/4\right) r \tanh(\pi r) \, dr, \)
- \( E_2(t) = - \sum_{R(\theta_1, \theta_2) \in \Gamma_E} \frac{i6\nu R^5 \sin \theta_1}{8\nu^3 \sin \theta_1} \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{r^2}{4} - \frac{\nu^2}{4t}\right) e^{-u/2} \left[ \frac{e^{-2i\theta_1}}{\cosh u - \cos 2\theta_1} \right] \, du, \)
- \( HE_2(t) = - \frac{1}{2} \sum_{(\gamma, \omega) \in \Gamma_{HE}} \frac{\log N(\gamma)}{N(\gamma)^{1/2} - N(\gamma)^{-1/2}} \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{t^2}{4} - \frac{\log N(\gamma)^2}{4t}\right), \)
- \( PS_2(t) = - \log \epsilon \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{t^2}{4}\right), \)
- \( HS_2(t) = - 2 \log \epsilon \sum_{k=1}^{\infty} \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{t^2}{4} - \frac{(2k \log \epsilon)^2}{4t}\right) e^{-k}. \)

Firstly, we see that \( HE_2(t) \) and \( HS_2(t) \) are exponentially decreasing as \( t \to +0. \) Secondly, by changing the variable \( u \) to \( \sqrt{t}u \) in \( E_2(t), \) we see that there is a constant \( b_0(2) \) such that \( E_2(t) = b_0(2) + o(1) \) \( (t \to +0). \) Thirdly, \( PS_2(t) = - \log \epsilon \frac{1}{\sqrt{4\pi t}} \left(1 - t/4 + o(t)\right) \) \( (t \to +0). \)

Lastly, noting \( \frac{\text{vol}(GK\|H^2)}{8\pi^2} = \zeta_K(-1) \) and integration by parts, we have

\[
I_2(t) = \frac{1}{2} \zeta_K(-1) \frac{1}{2t} \int_{-\infty}^{\infty} \exp\left(-t\left(r^2 + \frac{1}{4}\right)\right) \frac{\pi}{\cosh^2(\pi r)} \, dr
= \frac{\pi}{4} \zeta_K(-1) \sum_{n=0}^{\infty} \frac{(-1)^n n!}{n!} \int_{-\infty}^{\infty} \frac{r^2}{\cosh^2(\pi r)} \, dr
= \frac{a_{-1}(2)}{t} + a_0(2) + o(1) \quad (t \to +0).
\]

We calculate the coefficients \( a_n(2) \) \( (n = -1, 0). \)

\[
a_{-1}(2) = \frac{\pi}{4} \zeta_K(-1) \int_{-\infty}^{\infty} \frac{dx}{\cosh^2(\pi r)} = \frac{\pi}{4} \zeta_K(-1) \cdot \frac{4}{\pi} \int_{0}^{\infty} \frac{x}{(x^2 + 1)^2} \, dx = \frac{1}{2} \zeta_K(-1),
\]
\[
a_0(2) = - \frac{\pi}{4} \zeta_K(-1) \left\{ \int_{-\infty}^{\infty} \frac{r^2}{\cosh^2(\pi r)} \, dr + \frac{1}{4} \int_{-\infty}^{\infty} \frac{dx}{\cosh^2(\pi r)} \right\} = - \frac{\pi}{4} \zeta_K(-1) \left( \frac{1}{6\pi} + \frac{1}{4} \cdot \frac{2}{\pi} \right)
= - \frac{1}{6} \zeta_K(-1).
\]

Here, we used the formula: \( \int_{0}^{\infty} \frac{r^2}{\cosh^2(\pi r)} \, dr = \frac{(2^2 - 2)\pi^2}{(2\pi)^2 \pi} \cdot \frac{1}{6} = \frac{1}{12\pi} \) on [3.527 no.5]. Besides, we calculate the coefficient \( b_0(2) \) appearing in \( E_2(t). \)

\[
b_0(2) = - \sum_{R(\theta_1, \theta_2) \in \Gamma_E} \frac{ie^{-i\theta_1}}{8\nu R \sin \theta_1} \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{u^2}{4}\right) \left[ \frac{1 - e^{-2i\theta_1}}{1 - \cos 2\theta_1} \right] \, du
= - \sum_{j=1}^{N} \sum_{k=1}^{N} \frac{1}{4\nu_j} \cdot \frac{1}{1 - \cos(\frac{2\nu_k}{\nu_j})} = - \sum_{j=1}^{N} \frac{\nu_j^2 - 1}{24\nu_j}.
\]
Summing up each terms appearing in the right hand side of (4.3), we have the desired formula (4.2).

Let us prove (4.3) with $m = 4$. For $t > 0$, we also take the pair of test functions $h_1(r) = e^{-t(r^2 + 1/4)}$ and $g_1(u) = \frac{1}{4\pi t} \exp(-\frac{t}{4} - \frac{u^2}{4t})$ in Proposition 2.5 with $m = 4$, then we have

$$\theta_4(t) - \theta_2(t) + 1 = I_4(t) + E_4(t) + HE_4(t) + HS_4(t).$$

Here,

- $I_4(t) = \frac{\text{vol} \left( \Gamma / \mathbb{H}^2 \right)}{8\pi^2} \int_{-\infty}^{\infty} \exp \left(-t(r^2 + 1/4) \right) r \tanh(\pi r) \, dr$,
- $E_4(t) = -\sum_{\gamma \in \Gamma} \frac{\log N(\gamma)}{4\pi t \sin^2 \frac{\pi}{4t}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} \exp \left(-\frac{t}{4} - \frac{u^2}{4t} \right) e^{-u/2} \left[ \frac{\exp \left( \frac{u}{4} \right) - \exp \left( -\frac{u}{4} \right)}{\cos \frac{u}{2\pi} - \cos 2\pi t} \right] \, du$,
- $HE_4(t) = -\sum_{\gamma \in \Gamma} \frac{\log N(\gamma)}{4\pi t} \frac{1}{\sqrt{4\pi t}} \exp \left(-\frac{t}{4} - \frac{\log N(\gamma)^2}{4t} \right) e^{2i\omega}$,
- $HS_4(t) = -2 \log \varepsilon \sum_{s=1}^{\infty} \frac{1}{\sqrt{4\pi t}} \exp \left(-\frac{t}{4} - \frac{(2k^2\varepsilon)^2}{4t} \right) \varepsilon^{-3k} - \varepsilon^{-k}$.

Similarly, we see that $HE_4(t)$ and $HS_4(t)$ are exponentially decreasing as $t \to +0$, and there is a constant $b_0(4)$ such that $E_4(t) = b_0(4) + o(1)$ ($t \to +0$), and $I_4(t) = \zeta_K(-1)(1/t - 1/3) + o(1)$ ($t \to +0$). Summing up each terms appearing in the right hand side of (4.3) and using (4.2) in the left side, we have the desired formula (4.3) with $m = 4$. One can prove (4.3) for $m \geq 6$ similarly. We complete the proof.

**Proposition 4.3.** Let $s$ be a fixed sufficiently large real number. For $m \in 2\mathbb{N}$, let

$$\zeta_m(w, s) := \sum_{n=0}^{\infty} \frac{1}{(\lambda_n(m) + s(s - 1))^{w}} \quad (\text{Re}(w) > 0).$$

be the spectral zeta function for $\Box_m$. Then $\zeta_m(w, s)$ is holomorphic at $w = 0$.

**Proof.** We follow [p.448]. For $w \in \mathbb{C}$ with $\text{Re}(w) > 0$, we have

$$\zeta_m(w, s) = \frac{1}{\Gamma(w)} \int_{0}^{\infty} \theta_m(t) e^{-s(s-1)t} t^w \, dt.$$  \hspace{1cm} (4.6)

We consider the first three terms of $\theta_m(t)$ in Proposition 1.2. Let

$$ \eta_p(w, s) := \frac{1}{\Gamma(w)} \int_{0}^{\infty} t^{-p} e^{-s(s-1)t} t^{w-1} \, dt = \frac{1}{\Gamma(w)} \left( s(s-1) \right)^{p-w} \Gamma(w-p) \quad (p = 0, \frac{1}{2}, 1).$$ \hspace{1cm} (4.7)

with $p = 0, \frac{1}{2}, 1$. Then we see that $\eta_p(w, s)$ ($p = 0, \frac{1}{2}, 1$) are holomorphic at $w = 0$. The reminder term is

$$\eta_f(w, s) := \frac{1}{\Gamma(w)} \int_{0}^{\infty} f(t) e^{-s(s-1)t} t^w \, dt \quad \text{with} \quad f(t) = o(1) \quad (t \to +0) \quad \text{and} \quad O(1) \quad (t \to \infty).$$  \hspace{1cm} (4.8)

Since $\frac{1}{\Gamma(w)}$ vanishes at $w = 0$, it completes the proof.

**Proposition 4.4.** Let $m$ be an even natural number. We have

$$\frac{\partial}{\partial w} \zeta_2(w,s) \bigg|_{w=0} = -\zeta_K(-1) \left( s^2 - s + \frac{1}{3} \right) \log s + \frac{1}{2} \zeta_K(-1) \cdot s^2 - s \log \varepsilon$$  \hspace{1cm} (4.9)

$$+ \left( 2b_0(2) + 2 \right) \log s - \frac{1}{4} \zeta_K(-1) + \frac{1}{2} \log \varepsilon + o(1) \quad (s \to \infty),$$

where $b_0(2)$ is the constant such that $E_4(t) = b_0(2) + o(1)$ ($t \to +0$).
and for $m \geq 4$,
\begin{equation}
-\frac{\partial}{\partial w} \zeta_m(w, s) \bigg|_{w=0} = -(m-1)\zeta_K(-1) \left(s^2 - s + \frac{1}{3}\right) \log s + \frac{m-1}{2} \zeta_K(-1) \cdot s^2 - s \log \varepsilon + \left(2b_0(2) + \cdots + 2b_0(m)\right) \log s
\end{equation}
\begin{equation}
- \frac{m}{4} \zeta_K(-1) + \frac{1}{2} \log \varepsilon + o(1) \quad (s \to \infty).
\end{equation}

Besides, we have for $m \geq 4$,
\begin{equation}
-\frac{\partial}{\partial w} \zeta_m(w, s) \bigg|_{w=0} + \frac{\partial}{\partial w} \zeta_{m-2}(w, s) \bigg|_{w=0}
\end{equation}
\begin{equation}
= -2\zeta_K(-1) \left(s^2 - s + \frac{1}{3}\right) \log s + \zeta_K(-1) \cdot s^2 + \left(2b_0(m) - 2\delta_{4,m}\right) \log s
\end{equation}
\begin{equation}
- \frac{1}{2} \delta_{4,m} \log s + o(1) \quad (s \to \infty).
\end{equation}

Proof. By the formulas (4.7) and (4.8), we find that
\begin{equation}
\frac{\partial}{\partial w} \eta_0(w, s) \bigg|_{w=0} = -\log (s(s-1)) = -2\log s + o(1) \quad (s \to \infty),
\end{equation}
\begin{equation}
\frac{\partial}{\partial w} \eta_1(w, s) \bigg|_{w=0} = -2\sqrt{\pi} (s(s-1))^{\frac{1}{2}} = -2\sqrt{\pi} \left(s - \frac{1}{2}\right) + o(1) \quad (s \to \infty),
\end{equation}
\begin{equation}
\frac{\partial}{\partial w} \eta_2(w, s) \bigg|_{w=0} = s(s-1) \left(\log (s(s-1)) - 1\right)
\end{equation}
\begin{equation}
= 2s(s-1) \log s + \frac{1}{2} - s^2 + o(1) \quad (s \to \infty),
\end{equation}
\begin{equation}
\frac{\partial}{\partial w} \eta_f(w, s) \bigg|_{w=0} = o(1) \quad (s \to \infty).
\end{equation}

Therefore, by using (4.12), we have
\begin{equation}
-\frac{\partial}{\partial w} \zeta_2(w, s) \bigg|_{w=0} = -\frac{1}{2} \zeta_K(-1) \left(2s(s-1) \log s + \frac{1}{2} - s^2\right) - \left(s - \frac{1}{2}\right) \log \varepsilon
\end{equation}
\begin{equation}
+ \left(-\frac{1}{6} \zeta_K(-1) + b_0(2) + 1\right) \cdot 2 \log s + o(1) \quad (s \to \infty)
\end{equation}
\begin{equation}
= -\zeta_K(-1) \left(s^2 - s + \frac{1}{3}\right) \log s + \frac{1}{2} \zeta_K(-1) \cdot s^2 - s \log \varepsilon
\end{equation}
\begin{equation}
+ \left(2b_0(2) + 2\right) \log s - \frac{1}{4} \zeta_K(-1) + \frac{1}{2} \log \varepsilon + o(1) \quad (s \to \infty).
\end{equation}

For $m \geq 4$, by using (4.13), we have
\begin{equation}
-\frac{\partial}{\partial w} \zeta_m(w, s) \bigg|_{w=0} = -(m-1)\zeta_K(-1) \left(s^2 - s + \frac{1}{3}\right) \log s + \frac{m-1}{2} \zeta_K(-1) \cdot s^2 - s \log \varepsilon
\end{equation}
\begin{equation}
+ \left(2b_0(2) + \cdots + 2b_0(m)\right) \log s - \frac{m-1}{4} \zeta_K(-1) + \frac{1}{2} \log \varepsilon
\end{equation}
\begin{equation}
+ o(1) \quad (s \to \infty).
\end{equation}

We complete the proof.
5. Proof of Main Theorem

In this section we prove Theorem 1.6. We prove the following two propositions. The first proposition connect the completed Selberg zeta functions:

\[ \widehat{Z}_2^*(s), \widehat{Z}_4(s), \ldots, \widehat{Z}_m(s) \]

with the regularized determinants of Laplacians:

\[ \Det(\Box_2 + s(s-1)), \Det(\Box_4 + s(s-1)), \ldots, \Det(\Box_m + s(s-1)). \]

The second proposition determines the explicit relations among them. Theorem 1.6 is deduced from these two propositions.

**Proposition 5.1.** Let \( \Box_m := \Delta_0^{(1)} |_{w_m} \) for \( m \in 2\mathbb{N} \). There exist polynomials \( P_2(s), \ldots, P_m(s) \) such that

\[ \widehat{Z}_2^*(s) = e^{P_2(s)} \frac{\Det(\Box_2 + s(s-1))}{s(s-1)}, \quad \widehat{Z}_4(s) = e^{P_4(s)} \frac{s(s-1) \cdot \Det(\Box_4 + s(s-1))}{\Det(\Box_2 + s(s-1))}, \]

\[ \widehat{Z}_m(s) = e^{P_m(s)} \frac{\Det(\Box_m + s(s-1))}{\Det(\Box_{m-2} + s(s-1))} \quad (m \geq 6). \]

**Proof.** Let \( k \) be a sufficiently large natural number. We note that

\[ \left( -\frac{1}{2s-1} \frac{d}{ds} \right)^{k+1} \zeta_m(w, s) = w(w+1) \cdots (w+k) \zeta_m(w+k+1, s). \]

Taking \( \frac{\partial}{\partial w} \bigg|_{w=0} \) of both sides, we have

\[ \left( -\frac{1}{2s-1} \frac{d}{ds} \right)^{k+1} \log \Det(\Box_m + s(s-1)) = -\sum_{j=0}^{\infty} \frac{k!}{(\lambda_j(m) + s(s-1))^{k+1}}. \]

Let \( m = 2 \), we use the following double differences of STF with the certain test function: (See 3. Theorem 6.4.)

\[ \sum_{j=0}^{\infty} \left[ \frac{\rho_j(2)^2}{s+k} + \sum_{h=1}^{2} \frac{c_h(s)}{\beta_h + \frac{1}{2} + k} \right] - \left[ \frac{1}{s(s-1)} + \sum_{h=1}^{2} \frac{c_h(s)}{\beta_h^2 - \frac{1}{4}} \right] \]

\[ = \zeta_K(-1) \sum_{k=0}^{\infty} \left[ \frac{1}{s+k} + \frac{2}{\beta_h + \frac{1}{2} + k} \right] + \frac{1}{2s-1} \frac{d}{ds} \sqrt{Z_2(s)} + 2 \beta_h \frac{d}{ds} \sqrt{Z_2(s)} \]

\[ + \frac{1}{2s-1} \sum_{j=1}^{N} \sum_{l=0}^{\nu_j-1} \frac{\nu_j - 1 - 2l}{2
u_j^2} \psi(s + l) + \frac{2 \beta_h \sum_{j=1}^{N} \sum_{l=0}^{\nu_j-1} \psi(s + l)}{2
u_j^2} \psi \left( \frac{1}{2} + \beta_h + l \right) \]

\[ + \frac{1}{2s-1} \frac{d}{ds} \log(\varepsilon^{-s}) + 2 \beta_h \frac{d}{ds} \log(\varepsilon^{-(\beta_h + 1/2)}) \]

\[ + \frac{1}{2s-1} \frac{d}{ds} \left\{ \frac{1}{(1-\varepsilon^{-2s})} \right\} + \sum_{h=1}^{2} \frac{c_h(s)}{2\beta_h} \frac{d}{d\beta_h} \log \left\{ \frac{1}{(1-\varepsilon^{-(2\beta_h + 1)})} \right\}. \]
Here, \( \psi(z) \) is the digamma function, \( \beta_1 \neq \beta_2 \) are constants and \( c_1(s), c_2(s) \) are quadratic polynomials invariant under \( s \to 1 - s \). Operating \( \left( -\frac{1}{2s - 1} \frac{d}{ds} \right)^k \) on both sides, we have

\[
\sum_{j=0}^{\infty} \frac{k!}{(\lambda_j(2) + s(s-1))^{k+1}} = \left( -\frac{1}{2s - 1} \frac{d}{ds} \right)^k \frac{1}{2s - 1} \frac{d}{ds} \log \left( \hat{Z}_2^\beta(s) \cdot s(s-1) \right).
\]

By (5.1) and (5.2), we have

\[
\left( -\frac{1}{2s - 1} \frac{d}{ds} \right)^{k+1} \log \det(\Box_2 + s(s-1)) = \left( -\frac{1}{2s - 1} \frac{d}{ds} \right)^{k+1} \log \left( \hat{Z}_2^\beta(s) \cdot s(s-1) \right).
\]

Therefore, we find that there exists a polynomial \( P_2(s) \) such that

\[
\log \det(\Box_2 + s(s-1)) + P_2(s) = \log \left( \hat{Z}_2^\beta(s) \cdot s(s-1) \right).
\]

Thus we have

\[
\hat{Z}_2^\beta(s) = e^{P_2(s)} \frac{\det(\Box_2 + s(s-1))}{s(s-1)}.
\]

Let \( m \geq 4 \) be an even integer. We use the following double differences of STF with the certain test function: (See [5, Theorem 5.2].)

\[
\sum_{j=0}^{\infty} \left[ \frac{1}{\rho_j(m)^2 + (s - \frac{1}{2})^2} + \sum_{h=1}^{2} \frac{c_h(s)}{\rho_j(m)^2 + \beta_h^2} \right]
- \sum_{j=0}^{\infty} \left[ \frac{1}{\rho_j(m-2)^2 + (s - \frac{1}{2})^2} + \sum_{h=1}^{2} \frac{c_h(s)}{\rho_j(m-2)^2 + \beta_h^2} \right] + \delta_{m,4} \left[ \frac{1}{s(s-1)} + \sum_{h=1}^{2} \frac{c_h(s)}{\beta_h^2 - \frac{1}{4}} \right]
= 2\zeta_K(-1) \sum_{k=1}^{N} \left[ \frac{1}{s + k} + \frac{c_h(s)}{\beta_h + \frac{1}{2} + k} \right] + \frac{1}{2s - 1} \frac{Z_m'(s)}{Z_m(s)} + \frac{1}{2s - 1} Z_m(s) \frac{2 c_h(s) Z_m(\frac{1}{2} + \beta_h)}{2 \beta_h Z_m(s)}
+ \frac{1}{2s - 1} \sum_{j=1}^{N} \sum_{l=0}^{\nu_j - 1} \frac{\nu_j - 1 - \alpha_l(m, j) - \alpha_l(m, j)}{\nu_j^2} \psi \left( \frac{s + l}{\nu_j} \right)
+ \frac{2 c_h(s)}{2 \beta_h} \sum_{j=1}^{N} \sum_{l=0}^{\nu_j - 1} \frac{\nu_j - 1 - \alpha_l(m, j) - \alpha_l(m, j)}{\nu_j^2} \psi \left( \frac{1}{\nu_j} + \beta_h + l \right)
+ \frac{1}{2s - 1} \frac{d}{ds} \log \left( \frac{1 - \varepsilon^{-(2s+m-4)}}{1 - \varepsilon^{-(2s+m-2)}} \right) + \frac{2 c_h(s)}{2 \beta_h} \frac{d}{ds} \log \left( \frac{1 - \varepsilon^{-(2\beta_h+m-3)}}{1 - \varepsilon^{-(2\beta_h+m-1)}} \right).
\]

Here, \( \beta_1 \neq \beta_2 \) are constants and \( c_1(s), c_2(s) \) are quadratic polynomials invariant under \( s \to 1 - s \).

Operating \( \left( -\frac{1}{2s - 1} \frac{d}{ds} \right)^k \) on both sides, we have

\[
\sum_{j=0}^{\infty} \frac{k!}{(\lambda_j(m) + s(s-1))^{k+1}} = \sum_{j=0}^{\infty} \frac{k!}{(\lambda_j(m-2) + s(s-1))^{k+1}}
\]

\[
= \left( -\frac{1}{2s - 1} \frac{d}{ds} \right)^k \frac{1}{2s - 1} \frac{d}{ds} \left( \log \hat{Z}_m(s) - \delta_{m,4} \log(s(s-1)) \right).
\]
By (5.1) and (5.4), there exists a polynomial $P_n(s)$ such that

$$\log \text{Det}(\square_m + s(s - 1)) - \log \text{Det}(\square_{m-2} + s(s - 1)) + P_m(s)$$

(5.5)

We complete the proof.

**Proposition 5.2.** We have

$$P_2(s) = \left( s - \frac{1}{2} \right)^2 \zeta_K(-1) - \frac{1}{2} \log s + \sum_{j=1}^N \frac{\nu_j^2 - 1}{12\nu_j} \log \nu_j,$$

$$P_m(s) = 2 \left( s - \frac{1}{2} \right)^2 \zeta_K(-1) + \sum_{j=1}^N \frac{\nu_j^2 - 1 - 12\alpha_0(m, j)(\nu_j - \alpha_0(m, j))}{6
\nu_j} \log \nu_j \quad (m \geq 4).$$

**Proof.** Substituting (3.6) and (4.9) in (5.3), we have

$$P_2(s) = \log (\hat{Z}_2^s(s) \cdot s(s - 1)) - \log \text{Det}(\square_2 + s(s - 1))$$

$$= \zeta_K(-1) \left\{ \frac{3}{2} s^2 - s - \left( s^2 - s + \frac{1}{3} \right) \log s \right\} - \sum_{j=1}^N \frac{\nu_j^2 - 1}{12
\nu_j} \log \frac{s}{\nu_j} - s \log \varepsilon$$

$$+ 2 \log s + \zeta_K(-1) \left( s^2 - s + \frac{1}{3} \right) \log s - \frac{1}{2} \zeta_K(-1) \cdot s^2 + s \log \varepsilon$$

$$- \left( 2b_0(2) + 2 \right) \log s + \frac{1}{4} \zeta_K(-1) - \frac{1}{2} \log \varepsilon + o(1) \quad (s \to \infty)$$

$$= \left( s - \frac{1}{2} \right)^2 \zeta_K(-1) - \frac{1}{2} \log \varepsilon + \sum_{j=1}^N \frac{\nu_j^2 - 1}{12 \nu_j} \log \nu_j + o(1) \quad (s \to \infty).$$

Since $P_2(s)$ is a polynomial, we have the desired formula for $P_2(s)$.

Let $m \geq 4$. Substituting (3.7) and (4.11) in (5.5), we have

$$P_m(s) = \log (\hat{Z}_m(s) - \delta_{m, s} \log (s(s - 1))$$

$$- \log \text{Det}(\square_m + s(s - 1)) + \log \text{Det}(\square_{m-2} + s(s - 1))$$

$$= 2 \left( s - \frac{1}{2} \right)^2 \zeta_K(-1) + \sum_{j=1}^N \frac{\nu_j^2 - 1 - 12\alpha_0(m, j)(\nu_j - \alpha_0(m, j))}{6 \nu_j} \log \nu_j + o(1) \quad (s \to \infty).$$

Since $P_m(s)$ is a polynomial, we have the desired formula for $P_m(s)$. It completes the proof. 

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