Plane Gossip: Approximating rumor spread in planar graphs

Jennifer Iglesias†, Rajmohan Rajaraman‡, R. Ravi, and Ravi Sundaram

1Carnegie Mellon University, Pittsburgh PA, USA, \{jiglesia,ravi\}@andrew.cmu.edu
2Northeastern University, Boston MA, USA, \{rraj,koods\}@ccs.neu.edu

July 18, 2017

Abstract

We study the design of schedules for multi-commodity multicast. In this problem, we are given an undirected graph $G$ and a collection of source-destination pairs, and the goal is to schedule a minimum-length sequence of matchings that connects every source with its respective destination. Multi-commodity multicast models a classic information dissemination problem in networks where the primary communication constraint is the number of connections that a given node can make, not link bandwidth. Multi-commodity multicast and its special cases, (single-commodity) broadcast and multicast, are all NP-complete and the best approximation factors known are $2^{O(\sqrt{\log n})}$, $O(\log n / \log \log n)$, and $O(\log k / \log \log k)$, respectively, where $n$ is the number of nodes and $k$ is the number of terminals in the multicast instance.

Multi-commodity multicast is closely related to the problem of finding a subgraph of optimal poise, where the poise is defined as the sum of the maximum degree of the subgraph and the maximum distance between any source-destination pair in the subgraph. We first show that for any instance of the single-commodity multicast problem, the minimum poise subgraph can be approximated to within a factor of $O(\log k)$ with respect to the value of a natural LP relaxation in an instance with $k$ terminals. This is the first upper bound on the integrality gap of the natural LP; all previous algorithms, both combinatorial and LP-based, yielded approximations with respect to the integer optimum. Using this integrality gap upper bound and shortest-path separators in planar graphs, we obtain our main result: an $O(\log \sqrt[3]{k \log n / (\log \log n)})$-approximation for multi-commodity multicast for planar graphs, where $k$ is the number of source-destination pairs.

We also study the minimum-time radio gossip problem in planar graphs where a message from each node must be transmitted to all other nodes under a model where nodes can broadcast to all neighbors in a single step but only nodes with a single broadcasting neighbor get a non-interfered message. In earlier work (Iglesias et al., FSTTCS 2015), we showed a strong $\Omega(n^{1-\epsilon})$-hardness of approximation for computing a minimum gossip schedule in general graphs. Using our techniques for the telephone model, we give an $O(\log^2 n)$-approximation for radio gossip in planar graphs breaking this barrier. Moreover, this is the first bound for radio gossip given that does not rely on the maximum degree of the graph.

Finally, we show that our techniques for planar graphs extend to graphs with excluded minors. We establish polylogarithmic-approximation algorithms for both multi-commodity multicast and radio gossip problems in minor-free graphs.

*This material is based upon research supported in part by the U. S. Office of Naval Research under award number N00014-12-1-1001 and National Science Foundation under award number CCF-1527032.

†Supported by the National Science Foundation Graduate Research Fellowship Program under Grant No. 2013170941.
1 Introduction

Rumor spreading in networks has been an active research area with questions ranging from finding the minimum possible number of messages to spread gossip around the network [32, 3, 17] to finding graphs with minimum number of edges that are able to spread rumors in the minimum possible time in the network [16]. There is also considerable work in the distributed computing literature on protocols for rumor spreading and gossip based on simple push and pull paradigms (e.g., see [10, 19, 15, 11]).

The focus of this paper is the class of problems seeking to minimize the time to complete the rumor spread, the prototypical example being the minimum broadcast time problem where a message at a root node must be sent to all nodes via connections represented by an undirected graph in the minimum number of rounds. Under the popular “telephone” model, every node can participate in a telephone call with at most one other neighbor in each round to transmit the message, and the goal is to minimize the number of rounds. This problem has seen active work in designing approximation algorithms [21, 29, 4, 9]. One generalization of broadcast is the minimum multicast time problem: We are given an undirected graph $G(V, E)$ representing a telephone network on $V$, where two adjacent nodes can place a telephone call to each other. We are given a source vertex $r$ and a set of terminals $R \subseteq V$. The source vertex has a message and it wants to inform all the terminals of the message. To do this, the vertices of the graph can communicate in rounds using the telephone model. The goal is to deliver the message to all terminals in the minimum number of rounds.

Recently, a more general demand model called the multicommodity multicast was introduced in [27]. In the minimum multicommodity multicast time problem, a graph $G(V, E)$ is given along with a set of pairs of nodes $P = \{(s_i, t_i) | 1 \leq i \leq k\}$, known as demand pairs. Each vertex $s_i$ has a message $m_i$ which needs be delivered to $t_i$. The vertices communicate similar to the multicast problem. The goal is to deliver the message from each source to its corresponding sink in the minimum number of rounds. Note that there is no bound on the number of messages that can be exchanged in a telephone call. In this sense, the telephone model captures a classic information dissemination problem where the primary communication constraint is the number of connections that a given node can make in each round, not link bandwidth.

1.1 Poly-logarithmic approximation for planar multicommodity multicast.

While even sub-logarithmic ratio approximations have been known for the minimum time multicast problem [21, 29, 4, 9], the best known approximation guarantees for the multicommodity case [27] is $O(2^\sqrt{\log k})$ where $k$ is the number of different source-sink pairs.

Theorem 1.1. There is a polynomial time algorithm for minimum time multicommodity multicast with $k$ source-sink pairs in $n$-node undirected planar graphs that constructs a schedule of length $O(\text{OPT} \log^3 k \frac{\log n}{\log \log n})$ where $\text{OPT}$ is the length of the optimal schedule.

This result extends in a natural way to bounded genus graphs. Our results make critical use of the fact that planar graphs admit small-size balanced vertex separators that are a combination of three shortest paths starting from any given node [31]. We aggregate messages at the paths, move them along the path and then move them onto their destinations using a local multicast. To break the overall multi-commodity multicast problem into recursive subproblems, we solve an LP relaxation for the overall problem and for those pairs for which the LP uses the separator path nodes in sending messages by a "large" amount, we aggregate them to the separator paths and move them along the paths. However, to define this aggregation automatically we need to use a
linear program which requires us to relate another lower bound for the schedule length that we describe next.

1.2 Poise and a new LP rounding algorithm.

Suppose that the (single-commodity) multicast problem in a graph \( G \) with root \( r \) and terminals \( R \) admits a multicast schedule of length \( L \). Consider all the nodes \( I \subseteq V \) in the graph that are informed of the message from the root in the course of the schedule. For every node \( v \in I \) consider the edge through which \( v \) first heard the message and direct this edge into \( v \). It is easy to verify that this set of arcs forms an out-arborescence \( T \) rooted at \( r \) and spanning \( I \). In particular, every node in \( I \) except \( r \) has in-degree exactly one and there is a directed path from \( r \) to every vertex in \( I \).

**Definition 1.2.** Define the poise of an undirected tree \( T \) to be the sum of the diameter of the tree and the maximum degree of any node in it. Define the poise of a directed tree to be that of its undirected version (ignoring directions).

The discussion above of constructing a directed tree from a multicast schedule implies that the poise of the tree constructed from a multicast schedule of length \( L \) is at most \( 3L \) (see also \[29\]).

The following lemma gives the relation in the other direction.

**Lemma 1.3.** \[29\] Given a tree on \( n \) nodes of poise \( L \), there is a polynomial time algorithm to construct a broadcast scheme of length \( O(L \cdot \frac{\log n}{\log \log n}) \) from any root.

Note that a complete \( d \)-ary regular tree of depth \( d \) requires time \( d^2 \) to finish multicast from the root; If the size of the tree is \( n \), then \( d = O(\frac{\log n}{\log \log n}) \). For this tree \( L = O(\frac{\log n}{\log \log n}) \) while any broadcast scheme takes \( \Omega((\frac{\log n}{\log \log n})^2) \) steps showing that the multiplicative factor is necessary.

Even though approximation algorithms for minimum poise trees connecting a root to a set of terminals were known from earlier work \[29, 9, 4\], their guarantees are with respect to an optimal (integral) solution and not any specific LP relaxation. In particular, the LP-based algorithm of \[29\] rounds a solution to the poise LP in phases without preserving the relation of the residual LPs that arise in the phases to the LP for the poise of the whole graph. Similarly, the LP-based algorithm of \[4\] solves a series of LPs determining how to hierarchically pair terminals and form the desired broadcast tree with cost within a logarithmic factor of the integral optimum poise, but without relating the resulting tree to the LP value of the poise of the original graph. It is not straightforward to use these methods to derive an integrality gap for the minimum poise LP, and this has remained an open problem. Deriving an approximation algorithm for minimum poise subgraphs for the single-commodity multicast version with a small integrality gap is a critical ingredient in our approximation algorithm for multicommodity multicast problem in planar graphs (Theorem 1.1). We derive the first such result.

**Theorem 1.4.** Given a fractional feasible solution of value \( L \) to a natural linear programming relaxation of the minimum poise of a tree connecting a root \( r \) to terminals \( R \) (POISE-L LP, see Section 2), there is a polynomial time algorithm to construct a tree spanning \( r \cup R \) of poise \( O(L \log k) \) where \( k = |R| \) and \( n = |V| \).

Our LP rounding for minimum poise are based on exploiting a connection to the theory of multiflows \[24, 5, 12\]; this is an interesting technique in its own right that we hope will be useful in obtaining other LP rounding results for connectivity structures while preserving degrees and distances.
1.3 Radio Gossip in Planar Graphs.

Our techniques for addressing multicommodity multicast are also applicable to radio gossip in planar graphs. In the radio model of communication that also occurs in rounds, a transmitting node may broadcast to multiple nodes in around but a node may receive successfully in a given time step only if exactly one of its neighbors transmits. The gossip problem is a special case of the multicommodity multicast problem where the demand pairs include all possible pairs of nodes (alternately, every node’s message must be transmitted to every other node). The minimum gossip problem in the radio model has been widely studied [14] but all known upper bounds involve both the diameter and degree of the network. In particular, for general \( n \)-node graphs, there is an \( \Omega(n^{1-\epsilon}) \)-hardness of approximation result for computing a minimum gossip schedule [18]. Our next result breaks this barrier for planar graphs (the proof and algorithm are in Section 4).

**Theorem 1.5.** There is a polynomial time algorithm for minimum time radio gossip in an \( n \)-node undirected planar graph that constructs a schedule of length \( O(OPT \cdot \log^2 n) \) where \( OPT \) is the length of the optimal gossip schedule.

Since radio broadcast from any node can already be achieved with additive poly-logarithmic time overhead above the optimum [25], our algorithm for radio gossip focuses on gathering all the messages to a single node. For this, we use the path-separator decomposition in planar graphs to recursively decompose the graph and gather messages bottom up. However, the diameter of subgraphs formed by the decomposition are not guaranteed to be bounded so we use a carefully constructed degree-bounded matching subproblem to accomplish the recursive gathering: these techniques adapt and extend the methods used for constructing telephone multicast schedules [27] but apply them for the first time to the radio gathering case.

1.4 Minor-free Graphs

Both our results on planar graphs also naturally extend to minor-free graphs, as similar path separator results are also known for minor-free graphs [1]. They are detailed in Section 5 in the Appendix for completeness.

**Theorem 1.6.** There is a polynomial time algorithm for minimum time multicommodity multicast with \( k \) source-sink pairs in \( n \)-node undirected \( H \)-minor-free graph for a constant sized \( H \) that constructs a schedule of length \( O(OPT \log^3 k \cdot \frac{\log n}{\log \log n}) \) where \( OPT \) is the length of the optimal schedule.

**Theorem 1.7.** There is a polynomial time algorithm for minimum time radio gossip in an \( n \)-node undirected \( H \)-minor-free graph for a constant sized \( H \) that constructs a schedule of length \( O(OPT \log^2 n) \) where \( OPT \) is the length of the optimal gossip schedule.

1.5 Previous Work

**Minimum time multicast in the telephone model.** Finding optimal broadcast schedules for trees was one of the first theoretical problems in this setting and was solved using dynamic programming [28]. For general graphs, Kortsarz and Peleg [21] developed an additive approximation algorithm which uses at most \( c \cdot OPT + O(\sqrt{n}) \) rounds for some constant \( c \) in an \( n \)-node graph. They also present algorithms for graphs with small balanced vertex separators with approximation ratio \( O(\log n \cdot S(n)) \) where \( S(n) \) is the size of the minimum balanced separator on graphs of size \( n \) from the class. The first poly-logarithmic approximation for minimum broadcast time was achieved by Ravi [29] and the current best known approximation ratio is \( O(\frac{\log n}{\log \log n}) \) due to Elkin.
and Korsartz [9]. The best known lower bound on the approximation ratio for telephone broadcast is $3 - \epsilon$ [6].

In his study of the telephone broadcast problem, Ravi [29] introduced the idea of finding low poise spanning trees to accomplish broadcast. In the course of deriving a poly-logarithmic approximation, Ravi also showed how a tree of poise $P$ in an $n$-node graph can be used to complete broadcast starting from any node in $O(P \cdot \frac{\log n}{\log \log n})$ steps. His result provided an approximation guarantee with respect to the optimal poise of a tree but not its natural LP relaxation that we investigate.

Guha et al. [4] improved the approximation factor for multicasting in general graphs to $O(\log k)$ where $k$ is the number of terminals. The best known approximation factor for the multicast problem is $O(\frac{\log k}{\log \log k})$ [9]. Both of [9, 4] present a recursive algorithm which reduces the total number of uninformed terminals in each step of the recursion, while using $O(OPT)$ number of rounds in that step. In [4], they reduce the number of uninformed terminals by a constant factor in each step and so they obtain a $O(\log k)$-approximation, but in [9], the number of uninformed terminals is reduced by a factor of $OPT$ which gives a $O(\frac{\log k}{\log \log k})$-approximation due to the fact that $OPT = \Omega(\log k)$.

These papers also imply an approximation algorithm with factors $O(\log k)$ and $O(\frac{\log k}{\log \log k})$ for the Steiner minimum poise subgraph problem; however, these guarantees are again with respect to the optimum integral value for this problem and not any fractional relaxation.

For the multicommodity multicast problem, Nikzad and Ravi [27] adapt the methods of [8, 9] to present an algorithm with approximation ratio $\tilde{O}(2\sqrt{\log k})$ where $k$ is the number of different source-sink pairs. They also show that there is a poly-logarithmic approximation inter-reducibility between the problem of finding a minimum multicommodity multicast schedule and that of finding a subgraph of minimum generalized Steiner poise (i.e., a subgraph that connect all source-sink pairs, but is not necessarily connected overall, and has minimum sum of maximum degree and maximum distance in the subgraph between any source-sink pair).

**Radio Gossip.** The radio broadcast and gossip problems have been extensively studied (see the work reviewed in the survey [13]). The best-known scheme for radio broadcast is by Kowalski and Pelc [22] which completes in time $O(D + \log^2 n)$, where $n$ is the number of nodes, and $D$ is the diameter of the graph and is a lower bound to get the message across the graph from any root. The $O(\log^2 n)$ term is also unavoidable as demonstrated by Alon et al. [2] in an example with constant diameter that takes $\Omega(\log^2 n)$ rounds for an optimal broadcast scheme to complete. Elkin and Korsartz [7] also show that achieving a bound better than additive log-squared is not possible unless $NP \subseteq DTIME(n^{\log \log n})$. For planar graphs, the best upper bound for radio broadcast time is $D + O(\log n)$ given by [25]. The best bound for radio gossip known so far, however, is $O(D + \Delta \log n)$ steps in an $n$-node graph with diameter $D$ and maximum degree $\Delta$ [14], even though there is no relation in general between the optimum radio gossip time and the maximum degree. Indeed, for general graphs, there is a polynomial inapproximability lower-bound for the minimum time radio gossip problem [18].

**Planar path separators.** For our results on planar graphs, we rely on the structure of path-separators. Lipton and Tarjan first found small $O(\sqrt{n})$-sized separators for $n$-node undirected planar graphs [23]. More recently, planar separators based on any spanning tree of a planar graph were found [31] with the following key property: these balanced vertex separators can be formed by starting at any vertex and taking the union of three shortest paths from this vertex. Minor-free graphs also admit small path-separators as found by [1]; in this case, the number of paths used depends on the graphs which are excluded minors, but stays constant for constant-sized excluded minors.
2 LP Rounding for Multicast in General Graphs

In this section we present an approximation algorithm for finding a minimum poise Steiner subgraph, and establish an LP integrality gap upper bound, thus proving Theorem 1.4. We begin by presenting a linear program for a multicommodity generalization of minimum poise Steiner subgraph, which is useful for the multicommodity multicast problem. This linear program, when specialized to the case where we need to connect a root $r$ to a subset $R$ of terminals, is our LP for the minimum poise Steiner subgraph problem.

2.1 Linear Program for Poise

The generalized Steiner poise problem is to determine the existence of a subgraph containing paths for every demand pair in $K = \{(s_i, t_i)\}_{1 \leq i \leq k}$ of poise at most $L$, i.e. every demand pair is connected by a path of length at most $L$ and every node in the subgraph has degree at most $L$.

We use indicator variables $x(e)$ to denote the inclusion of edge $e$ in the subgraph. Since the poise is at most $L$, this is also an upper bound on the length of the path from any terminal to the root. For every terminal $(s_i, t_i) \in K$, define $P_i$ to be the set of all (simple) paths from $s_i$ to $t_i$. We use a variable $y_t(P)$ for each path $P \in P_i$ that indicates whether this is the path used by $s_i$ to reach $t_i$ in the subgraph. For a path $P$, let $\ell(P)$ denote the number of hops in $P$. The integer linear program for finding a subgraph of minimum poise is given below.

$$\begin{align*}
\text{minimize} & \quad L = L_1 + L_2 \\
\text{subject to} & \quad \sum_{e \in \ell(v)} x(e) \leq L_1 \quad \forall v \in V \\
& \quad \sum_{P \in \mathcal{P}(t,r)} y_t(P) = 1 \quad \forall t \in R \\
& \quad \sum_{P \in \mathcal{P}(t,r)} \ell(P)y_t(P) \leq L_2 \quad \forall t \in R \\
& \quad \sum_{P \in \mathcal{P}(t,r), e \in P} y_t(P) \leq x(e) \quad \forall e \in E, t \in R \\
& \quad x(e) \in \{0, 1\} \quad \forall e \in E \\
& \quad y_t(P) \in \{0, 1\} \quad \forall t \in R, P \in \mathcal{P}_L(t, r).
\end{align*}$$

The first set of constraints specifies that the maximum degree of any node using the edges in the subgraph is at most $L_1$. The second set insists that there is exactly one path chosen between every pair $(s_i, t_i) \in K$. The third set ensures that the length of the path thus selected is at most $L_2$. The fourth set requires that if the path $P \in P_i$ is chosen to connect $s_i$ to $t_i$, all the edges in the path must be included in the subgraph.

We will solve the LP obtained by relaxing the integrality constraints to nonnegativity constraints and get an optimal solution $x, y \geq 0$.

For the remainder of this section, we will focus on the rooted version of this problem. In particular, there will be a root $r$ and set of terminals $R$, then we will make $K = \{(r, t) | t \in R\}$. It still remains to round a solution to POISE-LP to prove Theorem 1.4. Before presenting the rounding algorithm in Section 2.3, we describe a result on multiflows that will be useful in decomposing our LP solution into a set of paths that match terminals with each other.

2.2 Preliminaries

Given an undirected multigraph $G$ with terminal set $T \subset V$ of nodes, a multiflow is an edge-disjoint collection of paths each of which start and end in two distinct terminals in $T$. The value of the multiflow is the number of paths in the collection. Such a path between two distinct terminals is

\footnote{Even though the number of path variables is exponential, it is not hard to convert this to a compact formulation on the edge variables that can be solved in polynomial time. See e.g., \cite{29}.}
called a $T$-path and a multiflow is called a $T$-path packing. For any terminal $t \in T$, let $\lambda(t, T \setminus t)$ denote the minimum cardinality of an edge cut separating $t$ from $T \setminus t$ in $G$. Note that in any multiflow, the maximum number of paths with $t$ as an endpoint is at most $\lambda(t, T \setminus t)$. Furthermore, since every path in a multiflow has to end in distinct vertices in $T$, the maximum value of any multiflow for $T$ is upper bounded by $\sum_{t \in T} \frac{\lambda(t, T \setminus t)}{2}$, by summing over the maximum number of possible paths from each terminal and dividing by two to compensate for counting each path from both sides. This upper bound can be achieved if a simple condition is met.

**Theorem 2.1.** [24, 5] If every vertex in $V \setminus T$ has even degree, then there exists a multiflow for $T$ of value $\sum_{t \in T} \frac{\lambda(t, T \setminus t)}{2}$.

The following simple construction will be useful in the rounding algorithm to identify good paths to merge clusters. It is based off of a lemma from [29].

**Lemma 2.2.** Let $G$ be a digraph where every node has at most one outgoing edge (and no self loops). In polynomial time, one can find an edge-induced subgraph $H$ of $G$ such that $H$ is a partition of the nodes of $G$ into a forest of directed trees each being an inward arborescence, and with $|E(H)| \geq |E(G)|/2$.

**Proof.** Consider any connected component of $G$, if there are $v$ vertices, then there are either $v$ or $v-1$ edges (as each vertex has outdegree at most 1). If there are $v$ edges, there is a cycle. When we remove an edge from the cycle, we now have a connected component with $v-1$ edges. If there are $v-1$ edges and all the vertices have outdegree at most 1, then it is already an inward arborescence.

An algorithm to find $H$ would simply check the graph for directed cycles, and if any cycle exists, it would remove an edge from that cycle. Any component which had a cycle has at least 2 edges, and we remove at most 1 edge from every component. So, the resulting graph $H$ has at least half the edges that the original graph $G$ had. \qed

### 2.3 The Rounding Algorithm

The main idea of Algorithm 1 is to work in $O(\log k)$ phases, reducing the number of terminal-containing components in the subgraph being built by a constant fraction at each stage [30]. We begin with an empty tree containing only the terminals $R$, each in a cluster by themselves. In each phase, we will merge a constant fraction of the clusters together carefully so that the diameter of any cluster increases by at most an additive $O(L)$ per phase: for this, we choose a terminal as a center of each cluster. When we merge clusters, we partition the clusters into stars where we have paths of length $O(L)$ from the centers of the star leaf clusters to the center of the star center-cluster. These steps closely follow those in [29]. The crux of the new analysis is to extract a set of stars that merge a constant fraction of the current cluster center clusters using a solution to POISE-L LP.

The key subroutine to determine paths to merge centers is presented in Algorithm 2. This uses the multiflow packing theorem of [24, 5].

### 2.4 Performance Ratio

In this section, we prove Theorem 1.4. The performance ratio of the rounding algorithm in the theorem is a consequence of the following claims, the first of which follows directly from the path pruning in Algorithm 2.

**Lemma 2.3.** The length of each path output by Merge-Centers($C^*$) is at most $4L$.

**Lemma 2.4.** The expected number of paths output by Merge-Centers($C^*$) is $\Omega(|C^*|)$. 
Algorithm 1 LP Rounding for Poise-L tree

1: Clusters $C \leftarrow R$; Centers $C^* \leftarrow R$; Solution graph $H \leftarrow \emptyset$; Iteration $i \leftarrow 1$.
2: while $|C| > 1$ do
3: Use Algorithm Merge-Centers($C^*$) to identify a subgraph $F_i$ whose addition reduces the number of clusters by a constant fraction;
4: $H \leftarrow H \cup F_i$; Update $C$ to be the set of clusters after adding the subgraph $F_i$, and update $C^*$ to be the centers of the updated clusters based on the star structure from Algorithm Merge-Centers($C^*$). Increment $i$.
5: end while
6: Add a path of length at most $L$ from $r$ to the center of the final cluster in $H$. Find a shortest path tree in $H$ rooted at $r$ reaching all the terminals in $R$ and output it.

Proof. The main observation here is that the total number of edges in the multigraph $G$ is at most $|C^*| \cdot L \cdot M$. To see this, note that each terminal $t$ retains its flow of value 1 in POISE-LP corresponding to the paths with nonzero value for $y_t$. Thus in the scaled version, it retains $M$ paths to the root, and the average number of hops in these paths is at most $L_2 \leq L$ hops, for each terminal. Summing over all terminals, the number of edges in $G$ is at most $|C^*| \cdot L \cdot M$.

The total number of paths discarded cannot exceed $\frac{|C^*| \cdot M}{4}$. Otherwise the paths each of length at least $4L$ each would need more edges in $G$ than we started with. After discarding, we expect to still collect at least $\frac{1}{M} \cdot (|C^*| \cdot M - \frac{|C^*| \cdot M}{4}) = \frac{3|C^*|}{4}$ paths fractionally. Hence the expected number of terminals in the subgraph $H$ is at least $\frac{3|C^*|}{4}$. The set of arcs finally retained in $H''$ is at least one third of the nodes of $H$, the worst case being a path of two arcs. This leads to an expectation of at least $\frac{|C^*|}{4}$ paths finally output to merge the initial clusters.

Lemma 2.5. The distance of any node in a cluster to its center increases by at most $4L$ in the newly formed cluster by merging paths corresponding to stars in $H''$. Thus, the diameter of any cluster in iteration $i$ is at most $8iL$.

Proof. The proof is by induction over $i$, and is immediate by observing that any node can reach the new merged cluster center say $c$ by first following the path to its old center, say $t$ and then following the path $P_t$ corresponding to the arc in $H''$ from $t$ to $c$. By Lemma 2.3 above, the length of $P_t$ is at most $4L$ and the claim follows.

Lemma 2.6. The maximum degree at any node of $G$ induced by the union of paths output by Merge-Centers($C^*$) is $O(L)$.

Proof. This is a simple consequence of the performance guarantee of rounding the LP solution obtained for the collection of paths. Since the paths we found pack into the LP solution $2x$ (from the property of the multflow packing), the expected congestion due to the chosen random paths on any edge $e$ is at most $2x(e)$. From the first constraint in the LP, the expected congestion at any node due to paths incident on it is at most $2L_1 \leq 2L$, by linearity of expectation.

We apply the classic rounding algorithm of [20]. Since the length of each path in the collection is at most $4L$ and the expected congestion is at most $2L$, we obtain that there is a rounding, which can be determined in polynomial time, such that the node congestion (degree) in the rounded solution of at most $4L$. $4L$.

By Lemma 2.4, the number of iterations of the main Algorithm 1 is $O(\log k)$ where $k$ is the number of terminals. Lemma 2.5 guarantees that the subgraph of the final cluster containing all the
Algorithm 2 Merge-Centers \((C^*)\) using LP solution \(x\)

1: Multiply the POISE-LP solution \(x\) by the least common multiple \(M\) of the denominators in the nonzero values of \(x\) to get a multigraph.

2: For every terminal \(t \in C^*\), retain the edges in the paths corresponding to the paths in its LP-solution with nonzero value (i.e., paths \(P\) with nonzero \(y_t(P)\)), for a total of \(M\) connectivity from \(t\) to \(r\). Note that the union of all the retained edges gives connectivity \(M\) from every \(t \in C^*\) to \(r\) and hence by transitivity, between each other.

3: Double each edge in the multigraph and use Theorem 2.1 to find a multiflow of value \(\sum_{t \in C^*} \frac{\lambda(t,C^* \setminus t)}{2} \geq \sum_{t \in C^*} \frac{2M}{2} = |C^*| \cdot M\). Note that each terminal in \(C^*\) has at least \(M\) paths in the multiflow.

4: Eliminate all the paths in the multiflow of length longer than \(4L\).

5: For every terminal \(t\), pick one of the \(M\) paths incident on it uniformly at random and set this path to be \(P_t\). If the chosen path is eliminated due to the length restriction, set \(P_t \leftarrow \emptyset\).

6: Let \(H\) be an auxiliary graph on vertex set \(C^*\) with at most one arc coming out of each \(t \in C^*\) pointing to the other endpoint of \(P_t\) (or add no edge if \(P_t = \emptyset\)).

7: Apply Lemma 2.2 to the subgraph of \(H\) made of nodes, to get a collection \(H'\) of in-trees. For each in-tree, partition the arcs into those in odd and even levels of the tree and pick the set with the larger number of arcs. Note that these sets form stars originating from a set of centers and going to a single center. Let \(H''\) denote the set of these stars.

8: For each arc of the stars in \(H''\), include the path \(P_t\) originating at the leaf of the star corresponding to the arc in \(H''\), and output the collection of paths.

terminals has distance \(O(\log k \cdot L)\) between any pair of terminals. Since the final output is a shortest path tree of this subgraph rooted at \(r\), its diameter is also of the same order. Lemma 2.6 ensures that the total degree of any node in the subgraph of the final cluster is \(O(\log k \cdot L)\), and this is also true for the tree finally output. This completes the proof of Theorem 1.4. We can derandomize the above randomized algorithm using the standard method of pessimistic estimators [26].

3 Approximating multicommodity multicast on planar graphs

In this section we prove Theorem 1.1. Let \(G = (V, E)\) be the given planar graph, with \(n = |V|\), and let \(K = \{(s_i, t_i) : 1 \leq i \leq k\}\) be the set of the \(k\) source-destination pairs that need to be connected. Let \(\gamma = 1/\log k\). We given a brief overview of our algorithm PlanarMCMulticast, which is fully described in Algorithm 3.

PlanarMCMulticast is a recursive algorithm, breaking the original problem into smaller problems each with at most a constant fraction of the demand pairs in \(K\) in each recursive call, thus having \(O(\log k)\) depth in the recursion. For a given graph, the algorithm proceeds as follows.

- Find a node separator composed of three shortest paths from an arbitrary vertex [31] to break the graph into pieces each with a constant fraction of the original nodes.
- Solve a generalized Steiner poise LP on the given pairs to identify demand pairs that cross the separator nodes to an extent at least \(\Omega(\gamma)\).
- Satisfy these demand pairs by routing their messages from the sources to the separator, moving the messages along the separator (since they are shortest paths, so this movement takes minimal time) and back to the destinations, by scaling the LP values by a factor of \(O(\frac{1}{\gamma})\) and using Theorem 2.4 to find a low poise tree to route to/from the separator.
• For the remaining demand pairs (which are mainly routed within the components after removing the separators), PlanarMCMulticast recurses on the pieces.

The key aspect of planarity that is used here is the structure theorem that planar graphs contain small-size balanced vertex separators that are a combination of three shortest paths starting from any given node.

**Algorithm 3** PlanarMCMulticast$(G, K)$

1. **Base case:** When $K = \{(s_1, t_1)\}$ has one demand pair, schedule the message on the shortest path between the source, $s_1$, and destination, $t_1$.

2. **Separate the graph:** Define the weight of a node as the number of source-destination pairs it is part of, and the weight of a subset of nodes as the sum of their weights. Find a 3-path separator $P$ of $G$, given by shortest paths $P_1$, $P_2$, and $P_3$, whose removal partitions the graph into connected components each of which has weight at most half that of the graph.

3. **Partition the terminal pairs:** Partition the set $K$ into two subsets, by solving the POISE-LP.
   - Let $K_1$ consist of pairs $(s_i, t_i)$ such that in POISE-LP, the fraction of the unit flow from $s_i$ to $t_i$ that intersects $P$ is at least $\gamma$.
   - Let $K_2 = K - K_1$ consist of the remaining pairs, i.e. pairs $(s_i, t_i)$ such that in the LP, the fraction of the unit flow from $s_i$ to $t_i$ that intersects $P$ is less than $\gamma$.

4. **Scale flow for pairs in $K_1$:** For each pair $(s_i, t_i)$ in $K_1$, scale the flow between $s_i$ and $t_i$ in the POISE-LP by $\frac{3}{\gamma}$ so there exists a path $P_j$ which intersects a unit of this scaled $s_i$-$t_i$ flow; remove other $s_i$-$t_i$ flows that do not intersect $P_j$ up to a unit. Assign $(s_i, t_i)$ to a set $S_j$.

5. **Create 3 minimum poise Steiner tree problems for $K_1$:** For each path $P_j$, create a minimum Steiner poise problem as follows: (i) attach, to the graph, an auxiliary binary tree $T_j$ with nodes of $P_j$ forming the leaves, and adding new dummy internal nodes (This step is similar to [27]); (ii) set the root of the binary tree to be the root for the Steiner poise problem, and the terminals to be all the $s_i$ and $t_i$ in $S_j$.

6. **Round the POISE-LP solution:** For each $P_j$, round the LP to obtain a Steiner tree $T_j$ of small poise connecting all the terminals in $S_j$ with the root using the algorithm from Theorem 1.4.

7. **Construct schedule for $K_1$:** Use Lemma 1.3 on the tree $T_j$ to perform a multicast between all terminals in it as follows: use the multicast schedule to move the messages, from the sources, till they hit the path $P_j$, then move messages along the path followed by the multicast schedule in reverse to move them towards the destinations. (Moving messages along a path can be achieved by a schedule that alternates between the even and odd matchings in the path for as many steps as the target length of the schedule)

8. **Scale flow for $K_2$:** For each pair $(s_i, t_i)$ in $K_2$, remove any flow that intersects $P$ and scale the remaining flow (by a factor of at most $\frac{1}{1-\gamma}$) so as to continue to have unit total flow between the pair.

9. **Recurse for $K_2$:** For each connected component $C_j$, let $K_2^j$ denote the subset of $K_2$ with both terminals in $C_j$. Run PlanarMCMulticast($C_j, K_2^j$) in parallel.

We now prove that PlanarMCMulticast constructs, in polynomial-time, a multicommodity multicast schedule a schedule of length $O((OPT \log^3 k \cdot \frac{\log n}{\log \log n})$ where $OPT$ is the length of the optimal schedule.

1. Observe that $3OPT$ is an upper bound for the value $L$ for the POISE-LP for this instance.
2. In Step 2 of PlanarMCMulticast, the separator is obtained using the algorithm in [31]. In Step 3, we use POISE-LP, as specified in Section 2 to find the fractional solution. In Step 4 of PlanarMCMulticast, unit $s_i - t_i$ flow is achieved by scaling up the LP cost by at most $\frac{3}{\gamma}$ since at least $\frac{\gamma}{3}$ flow intersects one of the three paths in $P$. Now, observe that this scaled LP solution immediately yields a valid solution to POISE-LP in Step 5. Applying Theorem 1.4 in Step 6, with the value of $L = O(OPT \log k)$ gives a tree of poise $O(OPT \log^2 k)$.

3. The algorithm performs $O(\log k)$ (recursive) phases; the poise of the tree at the $i$th level of recursion is itself based on an LP that has been scaled by a factor of at most $1 - \frac{\gamma^i}{1 - \gamma}$ (in Step 8) in the previous $i - 1 = O(\log k)$ iterations followed by a final scaling of $\frac{3}{\gamma}$ in the last iteration. In any iteration, the total factor by which the initial LP value of $OPT$ is scaled is at most $(\frac{1}{1 - \gamma})^{i-1} \cdot \frac{3}{\gamma} \leq (1 + \gamma)^{\frac{i}{\log k}} \cdot \frac{3}{\gamma} = O(\log k)$ since $\gamma = \frac{1}{\log k}$.

4. In Step 7, we incur a multiplicative factor of $O(\frac{\log n}{\log \log n})$ in going from a small poise tree to a schedule. Here, we crucially use the fact that the separator paths are shortest paths - for a demand pair $(s_i, t_i)$ let $f_i$ denote the first vertex on the separator path that the message arrives at after leaving source $s_i$ and let $l_i$ denote the last vertex (on the separator path) that the message departs from, on its way to the destination $t_i$; then $f_i$ and $l_i$ must be at most an additive $O(OPT)$ of the sum of the lengths of the paths from $s_i$ to $f_i$ and $l_i$ to $t_i$ along the separator path, since every demand pair has a path of length $O(OPT)$ between them in the LP solution in this subgraph. Thus in Step 7, we can wait to aggregate all messages from the sources at the separator path, then move all the messages one way along the path and then the opposite way, for as many time steps as the poise of the integral tree, without more than tripling the total schedule.

5. Since there are $O(\log k)$ recursive phases, the final schedule has length $O(OPT \log^3 k \frac{\log n}{\log \log n})$.

This proves Theorem 1.5.

4 A polylogarithmic approximation for radio gossip on planar graphs

In this section, we present an $O(\log^2 n)$-approximation algorithm for finding a radio gossip schedule on planar graphs, and prove Theorem 1.5.

Let $G = (V, E)$ be a given planar graph. Once the messages from all nodes have all been gathered together at a node we can easily broadcast them back out in $O(OPT + \log^2 n)$ rounds using [22]. So we focus on gathering the messages together at one node. To do this, we recursively find 3-path separators in the graph [31] to decompose it into connected components. Then, working backwards, we gather messages from the 3-path separators found in an iteration at the nodes of the 3-path separators found in previous iterations, using techniques from telephone multicast [27]. The key properties used in the recursive algorithm are that the number of paths in the separator is a constant 3 and the paths are all shortest paths in the component they separate from some vertex.

We assume the optimal schedule has length $L$. Note that $L \leq 2n$ since gossip can be achieved by simply choosing any spanning tree and broadcasting one node at a time in post-order (to gather all messages at the root) and then in reverse post-order (to spread all messages back to all nodes).
Algorithm 4 A gathering procedure for radio gossip in planar graphs.

1: Clusters $C_0 \leftarrow \{V\}$; Vertices $V_0 \leftarrow V$; Graph $G_0 \leftarrow G$; Iteration $i \leftarrow 1$.
2: while $V_{i-1} \neq \emptyset$ do
3:     for all connected component $C \in C_{i-1}$ do
4:         Choose some $v \in C$. Find shortest paths $p_1, p_2, p_3$ from $v$ that form a 3-path separator in $C$ using [31]; Add these to $P_i$, the paths found in the $i$th iteration.
5:     Add $v$ and every $(2L + 1)$st vertex along paths $p_1, p_2, p_3$ to $N_i$
6: end for
7: Remove the vertices in $P_i$ from $V_{i-1}$ to get $V_i$; Let $G_i$ be $G[V_i]$ and $C_i$ denote the connected components of $G_i$; Increment $i$.
8: end while
9: while $i > 0$ do
10:    Do $2L$ rounds of radio broadcasts in series on nodes that are $2L + 1$ apart from each other along the paths in $P_i$ to gather all the messages on $P_i$ at the nodes $N_i$.
11:    Form $G'_i$ a bipartite graph from $N_i$ to $U_i = \bigcup_{j=1}^{i-1} P_j$. Add an edge $uv \in E'_i$ if there is a path from $u \in N_i$ to $v$ in $G[C_{i-1} \cup \{v\}]$ of length at most $L$. Find a $3L$-matching in $G'$ where every vertex of $U_i$ has degree at most $3L$.
12:    Do up to $L$ rounds of radio broadcast to get the messages from $N_i$ to within one node of $U_i$, along the paths in the $3L$-matching found above. Note that the messages stay within the component in $C_{i-1}$ containing $u$ for this part.
13:    Move the messages from the last nodes in $C_{i-1}$ to their destination nodes in $U_i$ in the $3L$-matching using at most $9L$ rounds for each of the paths ($27L \log n$ total).
14:    Decrement $i$
15: end while

We also assume that $L > 2$ ( $L \leq 2$ only occurs if there are 1 or 2 nodes total). The details of the algorithm are given in Algorithm 4.

We will first prove a couple lemmas needed for the proof of Theorem 1.5.

**Lemma 4.1.** The graph $G'_i$ has a $3L$ matching which matches every vertex of $N_i$ to some vertex of $U_i$ and every vertex of $U_i$ has degree at most $3L$.

**Proof.** Consider the graph $G'_i = (N_i, U_i, E'_i)$. Let $p_v$ be a path that $v$’s message take from $v$ to $r$ in the optimal solution. Let $p'_v$ be the prefix of $p_v$ until the first vertex of $U_i$. All the paths $p_v$ have length at most $L$ (since this is the length of the optimal schedule). For each node $w \in V_i$, $w$ is in at most one of the $p'_v$ for $v \in N_i$. This is because if two $p'_v$’s from the same path in $P_i$ arrive at a node, there would be a path of length at most $2L$ between two nodes in $N_i$ from the same path $P_i$; But the paths in $P_i$ were chosen to be shortest paths in $C_{i-1}$, and $N_i$ were nodes that were pairwise distance $2L + 1$ from each other, a contradiction. Now consider $u \in U_i$ and the $p'_v$ for $N_i$ that match to $u$: there can be at most $L$ nodes from which messages go from $V_i$ to $u$ (since that is an upper bound on its message receiving degree in the optimal solution). Thus, in the optimal solution there are at most $3L$ paths from $N_i$ to any specific node in $U_i$ and these paths have length at most $L$. So, there must exist a $3L$-matching in $G'$ which matches every vertex of $N_i$ to some vertex of $U_i$ and no vertex of $U_i$ has degree more than $3L$.

**Lemma 4.2.** Each iteration of the step 13 in algorithm 4 takes time at most $O(L \log n)$ and moves the messages to $U_i$. 

---

11
Proof. In Algorithm 4, step 13 is just to achieve the last step of message movement in the \( q_v \) paths. Each node \( w \in V_i \) can be adjacent to at most 3 nodes in a given path of \( P_j \) for any \( j < i \), as these paths are shortest paths in \( G_{j-1} \), and in particular these nodes are within 2 of each other in the path. Also, for a given \( w \in V_i \), \( w \) can be adjacent to multiple paths in \( P_j \) but they must all be in the same component of \( C_{j-1} \), and there is at most three such paths. Let \( S_j^k(\ell) \) be every third vertex on the paths \( P_j^k \) starting with the \( \ell \)th vertex. In \( 3L \) steps (the maximum degree of the matching at these nodes) we can gather the messages that need to be received at \( S_j^k(\ell) \) as no node is adjacent to two nodes in this set. Doing this gathering for every shift \( \ell \) from one to three, and each of three choice of which path \( k \), a total of \( 27L \) steps gets the message from the \( q_v \) to \( P_j \). This process is repeated for each collection \( P_j \) with \( j < i \). Now all the messages that were along \( P_i \) have been moved to some node in \( U_i = \bigcup_{j=1}^{i-1} P_j \) in \( 27L \log n \) steps.

Having established the lemmas, we now give the proof of Theorem 1.5.

Proof of Theorem 1.5. First, we establish that the algorithm runs correctly. Let \( r \) be the root (chosen in the first iteration). First the algorithm gathers the messages on the \( P_i \) to \( N_i \). We will divide the paths into \( P_i^1, P_i^2, P_i^3 \), so that each component of \( C_{i-1} \) that has three paths puts one path in each of these sets. Now, we will handle each of the \( P_i^j \) one at a time. To deal with \( P_i^j \) in the \( k \)th step, the nodes which are \( 2L - k \) further from a node in \( N_i \) broadcast all their messages to the node one closer to \( N_i \). There is no interference amongst the nodes in \( V_{i-1} \) as only nodes at distance at least \( 2L \) in each component of \( C_{i-1} \) are broadcasting at a time, and nodes in different components are non-adjacent (since they are disconnected by the separators \( P_{i-1} \)). This will gather all the messages along \( P_i^j \). Doing this once for each of the \( P_i^j \) in \( 2L \) steps, all the messages currently on \( P_i \) will be gathered at the \( N_i \) in \( 6L \) steps. The messages are all currently at \( N_i \) or \( U_i \).

Lemma 4.1 tells us that \( G_i' \) has a \( 3L \)-matching as desired, and so we can find such a matching. Once a matching is chosen, let \( q_v \) be a shortest path within \( G_{i-1} \) from \( v \in N_i \) to the vertex it is matched with in \( U_i \). Within each component of \( C_i \), we can broadcast along the \( q_v \) for every \( N_i \) in one of the paths simultaneously; There will be no interference as the \( N_i \)'s and their matching paths are far apart within \( C_{i-1} \). Thus, it takes at most \( L \) rounds of radio broadcasting to move the messages from \( N_i \) along their \( q_v \) to the vertex before \( U_i \).

Lemma 4.2 gives us that in time \( O(L \log n) \) we move the messages onto \( U_i \).

In the last iteration of the process, we will have all the messages on the first path separator \( P_1 \). \( P_1 \) is a path separator of shortest paths on the whole graph and the diameter of \( G \) is a lower bound on \( L \). So, in \( 3L \) steps we can move the messages from \( P_1 \) to \( r \). We have now successfully gathered all the messages to \( r \).

The time it takes to deliver all the messages to \( r \) is at most \( O(L \log^2 n) \). The path separator ensures that each component has at most a constant fraction of the number of vertices of the original graph. Therefore, the final \( i \leq \log n \). Each iteration, the number of rounds of broadcast we do is \( 6L \) in the first part and \( 27L \log n \) in the last step. So, this schedule uses \( O(L \log^2 n) \) steps to gather all the messages at \( r \).

5 Minor-free graphs

In this section, we prove that both results on planar graphs can be extended to any family of minor-free graphs. For this section, we will have to use the more general definition for path-separators.
Definition 5.1. A vertex-weighted graph $G$ is $k$-path separable if there exists a subgraph $S$, called a $k$-path separator, such that:

1. $S = P_0 \cup P_1 \cup \ldots$ where each subgraph $P_i$ is the union of $k_i$ shortest paths in $G \setminus \cup_{j<i} P_j$
2. $\sum_i k_i \leq k$
3. either $G \setminus S$ is empty, or each connected component of $G \setminus S$ is $k$-path separable and has total vertex weight at most half of the original.

This definition of path separator while more complicated can be integrated into our PlanarM-CMulticast algorithm and algorithm 3 with only small adjustments. We will use the following to theorem which tells us when a $k$-path separator exists and can be found.

Theorem 5.2. Every $H$-minor-free weighted connected graph is $k(H)$-path separable, and a $k(H)$-path separator can be computed in polynomial time.

Note that $k(H) = O(|H|)$ in the above construction.

5.1 Multicommodity multicast in minor-free graphs

Consider that our graph is $H$-minor free and $k(H)$ is the number of paths needed for the path separator. We will need to repeat steps 3-8 of the original algorithm for each subgraph of shortest paths. Other than that, the algorithm remains the same.

The main changes to the algorithm are as follows.

- The path separator we find is different.
- We may need to iterate through steps 3-8 of the original algorithm multiple times.

Since these are the only changes to the algorithm, we first see that finding the path separator takes polynomial time, and the number of iterations increases by at most a factor of $k(H)$ (a constant) so the algorithm still runs in polynomial time.

The only other change to the analysis is that the number of recursions is $k(H) \log k$. This gives that we incur a factor of $(1 - \gamma)k(H)\gamma^2$ when scaling the LP. Since $\gamma = \frac{1}{\log k}$, the LP gets scaled by a factor of $O(e^{k(H)} \log k)$. Since $|H|$ and hence $k(H)$ is a constant, we get that the resulting schedule from this algorithm is at most $O((OPT \log k + \log n) \log^2 k \log n \log \log n)$. In terms of $k(H)$, our algorithm builds a schedule that takes $O((OPT \log k + \log n) \log^2 k \log n \log \log n k(H)e^{k(H)})$

This proves theorem 1.6.

5.2 Radio Gossip in Minor-free Graphs

The modification to go from radio gossip on planar graphs to radio gossip on minor-free graphs is even more simple. Let $k = k(H)$ where our graph $G$ is $H$-minor-free. We only change how we set up all of our initial sets; $P_i, C_i, V_i, N_i, U_i$. For each iteration, we will define up to $k$ of these sets one for each of the (potential) subgraphs which compose the path separator.

The only major change to this algorithm is the set-up. In the set-up, we have to process the subgraphs which compose the path separator one at a time (as opposed to there only being one subgraph which is the whole path separator). This only increases the number of iterations by a factor a $k(H)$.

The other change is that everywhere we had a 3 before it now becomes a $k = k(H)$. All our previous lemmas and theorems hold if we change the 3 arising from the planar case to $k$. Therefore,
we have shown that this algorithm produces a schedule for gathering which runs in time $O(L \log^2 n)$ (or $O(Lk^3 \log^2 n)$ if $k$ is not constant) to gather all the messages in one place.

We again use the result of Kowalski and Pelc to broadcast the messages once they have been gathered [22]. This broadcast takes time $L + O(\log^2 n)$, so the whole gossip schedule takes time $O(L \log^2 n)$ proving Theorem 1.7 as desired. We again use the result of Kowalski and Pelc to broadcast the messages once they have been gathered [22]. This broadcast takes time $L + O(\log^2 n)$, so the whole gossip schedule takes time $O(L \log^2 n)$ proving Theorem 1.7 as desired. If we don’t assume $k = k(H)$ is a constant, then the algorithm takes time $O(Lk^3 \log^2 n)$.

References

[1] I. Abraham and C. Gavoille. Object location using path separators. In PODC, pages 188–197, 2006.

[2] N. Alon, A. Bar-Noy, N. Linial, and D. Peleg. A lower bound for radio broadcast. J. Comput. Syst. Sci., 43:290–298, 1991.

[3] B. Baker and R. Shostak. Gossips and telephones. Discrete Mathematics, 2(3):191–193, 1972.

[4] A. Bar-Noy, S. Guha, J. Naor, and B. Schieber. Message multicasting in heterogeneous networks. SIAM J. Comput., 30(2):347–358, 2000.

[5] B. Cherkassky. Mnogopolyusnye dvukhproduktovye zadachi [russian: Multiterminal two commodity problems]. Issledovaniya po Diskretnoi Optimizatsii [Russian: Studies in discrete optimization], pages 261–289, 1976.

[6] M. Elkin and G. Kortsarz. A combinatorial logarithmic approximation algorithm for the directed telephone broadcast problem. SIAM J. Comput., 35(3):672–689, 2005.

[7] M. Elkin and G. Kortsarz. Polylogarithmic additive inapproximability of the radio broadcast problem. SIAM J. Discrete Math, 19(4):881–899, 2005.

[8] M. Elkin and G. Kortsarz. An approximation algorithm for the directed telephone multicast problem. Algorithmica, 45(4):569–583, 2006.

[9] M. Elkin and G. Kortsarz. Sublogarithmic approximation for telephone multicast. J. Comput. Syst. Sci., 72(4):648–659, 2006.

[10] U. Feige, D. Peleg, P. Raghavan, and E. Upfal. Randomized broadcast in networks. Random Structures & Algorithms, pages 447–460, 1990.

[11] N. Fountoulakis, K. Panagiotou, and T. Sauerwald. Ultra-fast rumor spreading in social networks. In Proceedings of the Twenty-third Annual ACM-SIAM Symposium on Discrete Algorithms, SODA ’12, pages 1642–1660. SIAM, 2012.

[12] A. Frank. Connections in Combinatorial Optimization. Oxford Lecture Series in Mathematics and Its Applications. OUP Oxford, 2011.

[13] L. Gasieniec. On efficient gossiping in radio networks. In S. Kutten and J. erovnik, editors, Structural Information and Communication Complexity, volume 5869 of Lecture Notes in Computer Science, pages 2–14. Springer Berlin Heidelberg, 2010.
L. Gasieniec, D. Peleg, and Q. Xin. Faster communication in known topology radio networks. *Distributed Computing*, 19(4):289–300, 2007.

G. Giakkoupis. Tight bounds for rumor spreading in graphs of a given conductance. In T. Schwentick and C. Dür, editors, *28th International Symposium on Theoretical Aspects of Computer Science (STACS 2011)*, volume 9 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 57–68, Dagstuhl, Germany, 2011. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik.

M. Grigni and D. Peleg. Tight bounds on minimum broadcast networks. *SIAM J. Discrete Math.*, 4(2):207–222, 1991.

A. Hajnal, E. C. Milner, and E. Szemerédi. A cure for the telephone disease. *Canad. Math. Bull*, 15(3):447–450, 1972.

J. Iglesias, R. Rajaraman, R. Ravi, and R. Sundaram. Rumors across radio, wireless, telephone. In *FSTTCS*, pages 517–528, 2015.

R. Karp, C. Schindelhauer, S. Shenker, and B. Vocking. Randomized rumor spreading. In *Proceedings of the 41st Annual Symposium on Foundations of Computer Science, FOCS ’00*, pages 565–, Washington, DC, USA, 2000. IEEE Computer Society.

R. M. Karp, F. T. Leighton, R. L. Rivest, C. D. Thompson, U. V. Vazirani, and V. V. Vazirani. Global wire routing in two-dimensional arrays. *Algorithmica*, 2(1):113–129, Nov 1987.

G. Kortsarz and D. Peleg. Approximation algorithms for minimum-time broadcast. *SIAM J. Discrete Math.*, 8(3):401–427, 1995.

D. R. Kowalski and A. Pelc. Optimal deterministic broadcasting in known topology radio networks. *Distributed Computing*, 19(3):185–195, 2007.

R. J. Lipton and R. E. Tarjan. A separator theorem for planar graphs. *SIAM J. Applied Mathematics*, 36(2):177–189, 1979.

L. Lovász. On some connectivity properties of eulerian graphs. *Acta Mathematica Academiae Scientiarum Hungaricae*, 28:129–138, 1976.

F. Manne, S. Wang, and Q. Xin. Faster radio broadcast in planar graphs. In *WONS*, pages 9–13, 2008.

R. Motwani and P. Raghavan. *Randomized Algorithms*. Cambridge International Series on Parallel Computation. Cambridge University Press, 1995.

A. Nikzad and R. Ravi. Sending secrets swiftly: Approximation algorithms for generalized multicast problems. In *Automata, Languages, and Programming*, pages 568–607. Springer, 2014.

A. Proskurowski. Minimum broadcast trees. *IEEE Trans. Computers*, 30(5):363–366, 1981.

R. Ravi. Rapid rumor ramification: Approximating the minimum broadcast time. In *FOCS*, pages 202–213. IEEE, 1994.

R. Ravi. Matching based augmentations for approximating connectivity problems. In *LATIN*, pages 13–24. Springer, 2006.
[31] M. Thorup. Compact oracles for reachability and approximate distances in planar digraphs. *J. ACM*, 51(6):993–1024, 2004.

[32] R. Tijdeman. On a telephone problem. *Nieuw Archief voor Wiskunde*, 3(19):188–192, 1971.