Perturbative Renormalizability of Chiral Two Pion Exchange
in Nucleon-Nucleon Scattering

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We study the perturbative renormalizability of chiral two pion exchange for the singlet and triplet channels within effective field theory, provided that the one pion exchange piece of the interaction has been fully iterated. We determine the number of counterterms/subtractions needed in order to obtain finite results when the cut-off is removed, resulting in three counterterms for the singlet channel and six for the triplet. The results show that perturbative chiral two pion exchange reproduce the data up to a center-of-mass momentum of \( k \sim 200 - 300 \text{MeV} \) in the singlet channel and \( k \sim 300 - 400 \text{MeV} \) in the triplet.

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I. INTRODUCTION

The effective field theory formulation of nuclear forces \(^1^4\) tries to exploit in a systematic manner the separation of scales between pion physics, which is known to dominate at large distances, and short range physics in the two-nucleon system. In Weinberg’s original proposal \(^5,^6\) the chiral nucleon-nucleon potential is organized as a power expansion (or counting) in terms of \( Q \)

\[ V_{\text{NN}}(r) = V^{(0)}(r) + V^{(2)}(r) + V^{(3)}(r) + O(Q^4), \]

where \( Q \) represents the low-energy scales of the system, usually the momentum \( p \) of the nucleons and the pion mass \( m_\pi \). The potential is then inserted into the Schrödinger or Lippmann-Schwinger equation in order to obtain theoretical predictions \(^7\)\(^{32}\). This prescription is usually referred to as Weinberg’s counting.

The resulting chiral potentials turn out to be singular, behaving at order \( Q^\nu \) as \( 1/r^{3+\nu} \) in coordinate space for short enough distances (\( m_\pi r \ll 1 \)). Therefore they need to be regularized in order to obtain well-defined results, usually by introducing a cut-off in the computations plus the necessary number of counterterms which ensure the renormalizability of the scattering amplitude \(^3\). This has been found to be in contradiction with Weinberg’s power counting, where the corresponding counterterms, determined by naive dimensional analysis, are not able to render the theory renormalizable. \(^1\)\(^9\)\(^{21}\)\(^{26}\)\(^{39}\)\(^{41}\) (or generate chiral inconsistencies \(^9\), prompting the KSW counting \(^8\)\(^{34}\)). Consequently one is forced to make a decision: either to follow an \textit{a priori} power counting or require renormalizability.

The direct and practical choice is to follow Weinberg’s original counting unaltered, leading to a framework amicable with large numerical computations, which demystifies nuclear forces and enjoys an undisputed phenomenological success \(^15\)\(^{16}\). The price to pay is that the cut-off must be fine tuned, lying inside a narrow window, a situation which we regard as unsatisfactory from a theoretical viewpoint. Recently, based on the renormalization philosophy of Lepage \(^35\)\(^{36}\), there has been interesting attempts to justify this particular approach \(^37\)\(^{38}\).

On the contrary, if one strives for a more robust theoretical foundation, one should be able to achieve cut-off independence. The results from non-perturbative renormalization in the case of singular interactions \(^19\)\(^{21}\)\(^{26}\)\(^{39}\)\(^{41}\) can be summarized as follows: one counterterm is needed to renormalize a channel where the potential is attractive and singular, while channels where the potential is singular and repulsive become insensitive to counterterms. The first condition can lead to an alarming loss of predictive power, as already at leading order (LO) there is an infinite number of attractive singular channels. The solution proposed in Ref. \(^19\) is to treat all partial waves with sufficiently high angular momentum perturbatively at LO, a procedure which is supported by the analysis of Ref. \(^42\)\(^2\). The second condition is particularly problematic: in the triplet channel the potential is attractive at LO but becomes repulsive at next-to-leading order (NLO), resulting in an unbound deuteron at this order when the cut-off is removed \(^20\).

As there is no way to predict what the sign of the interaction will be at higher orders, this represents a continuous threat to the non-perturbative renormalizability of the chiral potentials. In addition, there exists the risk that non-perturbative renormalization of the subleading...
pieces of the potential may lead to incompatibilities with the chiral expansion \[38\]. The previous issues can be avoided with the perturbative treatment of the higher order pieces of the potential, which respects power counting and renormalizability independently on whether the subleading contributions are repulsive or attractive. The problem is how to construct such a perturbation theory.

The purpose of this paper is to investigate the conditions under which perturbative chiral two pion exchange (TPE) can be renormalized in order to extend the power counting proposal of Nogga, Timmermans and van Kolck \[19\] to subleading orders. In the spirit of Refs. \[12, 43\], we use renormalizability as a guide to identify the required short distance operators. The technical meaning of renormalizability depends on whether we are in a perturbative or non-perturbative context. By perturbative renormalizability we refer to the elimination of all negative (positive) powers of the coordinate (momentum) space cut-off in the observables. In contrast, non-perturbative renormalizability deals with ambiguities instead of divergences: the scattering amplitude of an attractive singular interaction is finite but non-unique.

The cut-off lies within a sensible range. The perturbative renormalizability we refer to the elimination of all negative (positive) powers of the coordinate (momentum) space cut-off in the observables. In contrast, non-perturbative renormalizability deals with ambiguities instead of divergences: the scattering amplitude of an attractive singular interaction is finite but non-unique.

In the previous formulae \[u_k^{(0)}\] is a normalization factor, which is taken to be unity at \(k = 0\), and \(u_k^{(0)}\) is the LO reduced wave function in an energy independent normalization at the origin (or at the cut-off radius \(r_c\) if we are using a finite cut-off). The asymptotic normalization of \(u_k^{(0)}\) is determined by \(A^{(0)}(k) u^{(0)}(k) \rightarrow \sin (kr + \delta(0)) / \sin \delta(0)\) for \(r \rightarrow \infty\).

As can be easily checked, the perturbative integral diverges as \(1/r^{2+\delta}\) as a consequence of the short distance behaviour of the reduced wave function \(u_k^{(0)}(r) \sim 1\) and the potential \(V^{(s)}(r) \sim 1/r^{2+\delta}\). The divergences can be avoided with the perturbative treatment of the higher order pieces of the potential, which respects power counting and renormalizability independently on whether the subleading contributions are repulsive or attractive. The problem is how to construct such a perturbation theory.

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II. SINGLET CHANNEL

The present perturbative treatment of chiral TPE is based on distorted wave Born approximation. For simplicity, we will only consider in detail the singlet case. We can express the phase shifts as the following series

\[
\delta(k; r_c) = \delta(0)(k; r_c) + \delta(2)(k; r_c) + \delta(3)(k; r_c) + \mathcal{O}(Q^4),
\]

which is ordered according to the counting of the finite-range piece of the potential \[3\]. That is, power counting is now manifest in the amplitudes. The LO phase shift \(\delta(0)\) is computed non-perturbatively (and includes one counterterm \[4\]), while \(\delta(2)\) and \(\delta(3)\) are computed in first order perturbation theory \[4\]. The corresponding expression for the perturbative phase shifts is (see Appendix \[A\])

\[
\frac{\delta^{(\nu)}(k; r_c)}{\sin^2 \delta(0)} = -\frac{2\mu}{k} A^{(0)}(k; r_c)^2 I^{(\nu)}_{S_0}(k; r_c),
\]

where \(\nu = 2, 3\) and the perturbative integral \(I^{(\nu)}_{S_0}\) is defined as

\[
I^{(\nu)}_{S_0}(k; r_c) = \int_{r_c}^{\infty} dr V^{(s)}(r) u_k^{(0)}(r)^2.
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In the previous formulae \(\mu\) is the reduced mass, \(k\) the center of mass momentum, \(A^{(0)}\) is a normalization factor, which is taken to be unity at \(k = 0\), and \(u_k^{(0)}\) is the LO reduced wave function in an energy independent normalization at the origin (or at the cut-off radius \(r_c\) if we are using a finite cut-off).

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cured by making the adequate subtractions. Due to the energy-independent normalization of $u^{(0)}_k$ at the origin, the terms in the $k^2$ expansion of $u^{(0)}_k = \sum_n u^{(0)}_{2n} k^{2n}$ are progressively less singular, with $u^{(0)}_{2n} \sim r^{2n}$ for $r \to 0$. Expanding the previous integrals in terms of $k^2$ for $\nu = 2, 3$ we have

$$I^{(\nu)}_{1S_0}(k; r_c) = I^{(\nu)}_0(r_c) + k^2 I^{(\nu)}_2(r_c) + k^4 I^{(\nu)}_4(r_c) + I^{(\nu)}_{1S_0,R}(k; r_c), \quad (5)$$

where $I^{(\nu)}_{2,4}$ are the divergent pieces of the integral and $I^{(\nu)}_{1S_0,R}$ is the regular piece, as can be trivially checked. Therefore three subtractions or counterterms are needed in order to renormalize the perturbative results in the singlet. The specific method employed is not important. Here we modify the perturbative integral by adding three free parameters which are to be fitted to the data

$$\hat{I}^{(\nu)}_{1S_0}(k; r_c) = \lambda_0^{(\nu)} + \lambda_2^{(\nu)} k^2 + \lambda_4^{(\nu)} k^4 + I^{(\nu)}_{1S_0}(k; r_c). \quad (6)$$

By assuming the short range physics to be parametrized by an energy dependent delta-shell potential of the type

$$V^{(\nu)}_C(r; r_c) = \frac{\mu}{2\pi r_c^3} \sum_n C^{(\nu)}_{2n}(r_c) k^{2n} \delta(r - r_c), \quad (7)$$

we can easily relate the $\lambda^{(\nu)}_{2n}$ parameters to the $C^{(\nu)}_{2n}$ counterterms by

$$\lambda^{(\nu)}_{2n} = \frac{\mu}{2\pi r_c^3} C^{(\nu)}_{2n}(r_c) u^{(0)}_{2n}(r_c). \quad (8)$$

Equivalently, if one chooses to work in the momentum space formulation of Ref. [42], one could include the contact potential $\langle p | V^{(\nu)}_C | p' \rangle = C^{(\nu)}_0 (p^2 + p'^2) + C^{(\nu)}_2 (p^4 + p'^4)$. In either case, the first free parameter, $\lambda^{(\nu)}_0 (C^{(\nu)}_0)$, is only used to absorb the $k = 0$ divergence of the perturbative integral while its finite piece is redundant, as it only affects the zero energy behaviour of the phase shifts which has already been fixed at LO, meaning that we need to fix two additional observables, for example the effective range $a_0$ and the shape parameter $\eta$, in order to determine the NLO/N^4LO results. The number of counterterms agrees with the corresponding one predicted in the RGA of Ref. [42], where the power counting resulting from treating one pion exchange (OPE) non-perturbatively was analyzed in detail, and with the related deconstruction of Ref. [46], in which the short range physics for the singlet channel is determined by removing the non-perturbative OPE and perturbative TPE effects from the phenomenological phase shifts. Note that in Ref. [40] an incorrect number of counterterms was determined due to an improper normalization.

The results for the singlet $1S_0$ channel are shown in Fig. (1). Following [16, 17], we take $f_\pi = 92.4$ MeV, $m_\pi = 138.03$ MeV, $g_A = 1.26$, and $d_{18} = -0.97$ GeV^{-2}. For the chiral couplings we employ the customary values $c_1 = -0.81$ GeV^{-1}, $c_3 = -3.40$ GeV^{-1} and $c_4 = 3.40$ GeV^{-1}, which are compatible with the determination of Ref. [52]. The potential is taken from Ref. [53]. As can be seen, the results reproduce the $1S_0$ phase shifts up to $k \sim 200 - 300$ MeV, depending on the value of the cut-off. If the cut-off is small ($r_c = 0.1$ fm), the perturbative treatment of the subleading pieces of the interaction starts to fail already at $k \sim 200$ MeV, as a consequence of the relative weakness of OPE with respect to the enormous strength of TPE at short distances. The previous problems can be circumvented by using cut-offs of the order of $r_c \sim 1/2 m_\pi$ (0.7 fm), which are small enough as to guarantee the correct inclusion of the TPE tail. In particular we employ $r_c = 0.6 - 0.9$ fm as a range for which perturbative TPE calculations compete well with non-perturbative ones in the Weinberg counting at the same order [13, 17], though perturbative TPE is slightly less predictive due to the additional counterterm. On a different ground it should be noticed that OPE is perturbative in the singlet [33, 34, 54], even if iterated [55], suggesting that the previous results could be reinterpreted as an N^3LO/N^4LO computation in the KSW counting [33, 34].

The failure of perturbative subleading TPE at $r_c = 0.1$ fm raises interesting questions regarding the adequacy of the present power counting scheme and the role of chi-

FIG. 1. (Color online) Phase shifts for the $1S_0$ channel with non-perturbative OPE and perturbative TPE. The non-perturbative OPE computation contains one counterterm which is determined by fixing the $1S_0$ scattering length, $a_{0,\nu} = -23.14$ fm, while the perturbative TPE computation contains a correction to the LO counterterm plus two additional counterterms which are used to fit the Nijmegen II phase shifts [51] (equivalent to the Nijmegen PWA [51]) in the range $k = 40 - 160$ MeV. The error bands are generated varying the cut-off within the 0.6 - 0.9 fm range. The light blue band represents the N^2LO results from the standard Weinberg approach of Ref. [17]. The dashed dark blue line represents the N^3LO results for $r_c = 0.1$ fm. 

6 Taking into account the relationship $\Lambda = \pi/2r_c$ [23], the previous configuration space cut-off range is approximately equivalent to a momentum space (sharp) cut-off of $\Lambda \approx 350 - 500$ MeV.
rual TPE. Of course, the technical reasons why perturbation theory fails already at \( k \sim 200 \text{ fm} \) for small cut-offs are clear: OPE does not provide enough long range distortion as to avoid higher momentum waves to probe the van der Waals component of TPE, as has been discussed for example in Ref. [46]. This component originates from the behaviour of subleading TPE, which in the singlet channel can be schematically written as [53]

\[
2\mu V^{(\nu=3)}(r) = -\frac{R_6^4}{r^n} e^{-2m_\pi r} \sum_{\nu=0}^{5} a_\nu (2m_\pi r)^\nu, \quad (9)
\]

where the \( a_\nu \)'s are dimensionless parameters with \( a_0 = 1 \) and \( R_6 \) is a length scale related with the strength of TPE at short distances, which varies between \( R_6 = 1.6-1.8 \text{ fm} \) for typical values of the chiral couplings. The previous form implies that the chiral van der Waals component of subleading TPE should start to become apparent at distances below \( r \leq 1/2m_\pi \simeq 0.7 \text{ fm} \). This figure is supported by several renormalized non-perturbative TPE computations in the singlet [20, 21, 22], which usually reach cut-off independence at distances around or below \( 0.5 \text{ fm} \), signalling the onset of chiral van der Waals forces. For such cut-off radii the perturbative treatment of TPE generates terms like \( k R_6 \) and \( m_\pi R_6 \), which, taking into account the size of \( R_6 \), might cause the perturbative series to eventually diverge. The most consistent and straightforward solution to this problem is the use large enough cut-offs \( (r_c > 0.5 \text{ fm}) \) in order to avoid the conjectured breakdown of the perturbative series. The alternative solution, which will not be considered in the present work, is the iteration of chiral TPE or at least some parts of it [46]. Although interesting, this proposal seems difficult to harmonize within the EFT framework as it requires (i) to justify the promotion of an order \( Q^3 \) interaction to order \( Q^{-1} \) and (ii) the existence of a cut-off window for which subleading TPE dominates but the higher order corrections are still small compared to this contribution.

The employed cut-off window, \( r_c = 0.6-0.9 \text{ fm} \), represents a compromise between the requirements of the singlet and triplet channels. The optimum value of the cut-off in the singlet lies in the vicinity of \( r_c = 0.9-1.0 \text{ fm} \), a range for which the description of the triplet phases starts to worsen. This cut-off window may look soft, but it is not: the first deeply bound state (i.e. the first zero of the \( S \)-wave function) for the N\(^3\)LO potential happens at \( r_c = 0.70 \text{ fm} \), meaning that the lower range of the present cut-off window is already beyond what can be reached in the Weinberg scheme. It is interesting to notice that the previous cut-off range is similar to the radii at which most potential models of the NN interaction [50, 56, 57] have their minima, usually at \( r \sim 0.8 - 0.9 \text{ fm} \). The minima mark the distance at which the short range repulsion starts to overcome the long range attraction, and consequently can be understood as the separation point between short \( (r \lesssim 0.5 \text{ fm}) \) and long range \( (r \gtrsim 1.0 \text{ fm}) \) physics. In this sense, the cut-off is to be interpreted as a separation scale, as has been proposed within the context of RGA [42, 58, 59], rather than as a hard scale [37, 38].

In the calculations of Fig. (1) we also interpret the cut-off variation of the results as the error band of the theory. The previous is a sensible prospect in the sense that we expect the cut-off dependence of the scattering amplitudes to be a higher order effect. However, if the cut-off variation is to be understood as an error band, the size of the band should decrease at each new order to reflect the convergence properties of the theory. Paradoxically the N\(^2\)LO band is bigger than the NLO one, a worrisome situation which does not necessarily mean that we should abandon the previous interpretation. In fact the same happens in the Weinberg counting, as illustrated by the singlet channel results of Ref. [17]. The explanation is to be found in the surprisingly large size of the \( c_3 \) and \( c_4 \) chiral couplings, which causes the subleading TPE contribution to the chiral potential to be substantially bigger than the corresponding one from leading TPE. This is due to the large contributions from the \( \Delta \) resonance to the chiral couplings [60], \( c_3,\Delta = -2c_4,\Delta = -4h_\Delta^2/9\Delta \), with \( \Delta \) the nucleon-delta mass splitting and \( h_\Delta \) the \( \pi N \Delta \) axial coupling, ranging from \(-1.7 \) to \(-2.7 \text{ GeV}^{-1} \) depending on the value of \( h_\Delta \). In this sense, the increased size of the N\(^2\)LO error bands is just a reflection of the unexpected contribution from this low energy scale. The explicit inclusion of the \( \Delta \) resonance in the NN chiral potential, a theme which has been recurrently considered in the literature [8, 11, 22, 24–26], is presumed to solve the current issue with the error bands (see also the related discussion of Ref. [46]). This prospect does not appear to be unreasonable in view of the perturbative peripheral wave N\(^2\)LO-\(\Delta \) results of Ref. [22] and the related non-perturbative central and peripheral wave calculations of Refs. [24, 25], all of which indicate an enhancement in the convergence rate of the phase shifts as compared to the \( \Delta \)-less theory.

### III. Triplet Channel

In the case of the \( ^3S_1 - ^3D_1 \) channel the perturbative analysis is analogous to the previous one for the \( ^1S_0 \) channel, but more cumbersome due to the presence of coupled channels and the singular behaviour of the tensor piece of the LO potential in the triplet channels. The details of such analysis are shown in Appendix [B] but the essential point is that the inverse power law behaviour of the OPE tensor force \( (1/r^7) \) softens the perturbative integrals and reduce the necessary number of counterterms per phase. In fact we have that the s- and d-wave functions behave as \( u_k^{(s)} \sim r^{3/4} \) near the origin [61],

\[ \text{taking } h_\Delta \text{ between 1.08 and the SU}(4) \text{ value 1.34}, \text{ see [22]. The values for the chiral couplings once the } \Delta \text{ has been included can also be consulted in Ref. [22].} \]
and that each subtraction adds an $r^{5/2}$ suppression to the
short distance behaviour. This translates into two subtractions
for each of the three phases in the $^3S_1 - ^3D_1$
channel ($\delta S_1, \epsilon_1, \delta D_1$), meaning that we end up with six
counterterms at NLO/N^2LO in agreement with Ref. [20].
That is, the scattering amplitude can be completely
determined using six pieces of data, for example the value
of the three phase shifts at two different momenta.

The results are shown in Fig. (2). Perturbative TPE
provides a good description of the $^3S_1$ and $^3D_1$ phases
and the $\epsilon_1$ mixing angle up to moderately high momenta,
around $k \sim 300 - 400$ MeV, although it should be noted
that the results are quite sensitive to the choice of chiral
couplings, due to the linear dependence generated
by treating chiral TPE perturbatively. Contrary to the
singlet case, small cut-offs do not affect the momentum
range in which first order perturbation theory works, al-
though due to numerical limitations, the cut-off cannot
be reliably reduced below $r_c = 0.3$ fm. However, there
are reasons for keeping the cut-off in the proposed win-
dow, such as avoiding unphysical deeply bound states
in the leading order amplitudes (the first one appears
at $r_c = 0.45$ fm), or an excessive D-state probability in
the deuteron, yielding poor convergence in nuclear mat-
ter calculations. Larger cut-offs, of the order of 1 fm
and above, are also disfavoured as they lead to a worse
description of the $\epsilon_1$ mixing angle for momenta above
300 MeV, similar to the one obtained in the N^2LO Wein-
berg calculation of Ref. [17]. The proposed cut-off range
avoids the previous problems and, due to the stronger
long range distortion provided by the tensor component
of OPE, generate error bands which decrease in size order
by order.

In this regard, it is interesting to notice the opposite
cut-off preferences of the singlet and triplet channels.
The mismatch in the preferred cut-off windows is a re-
flection of the different physics at play in these waves.
In the singlet, all pion exchanges are perturbative and
the iteration of OPE is merely a short-cut to avoid the
computation of higher order perturbations, while in the
triplet tensor OPE really needs to be iterated. Different
power countings require different cut-off windows. In this
sense, large cut-off values worsen the convergence of the
triplet: the OPE tensor force starts to behave perturba-
tively, even if fully iterated in the Schrödinger equation.
This entails a change in the counting of the triplet chan-
el from the modified Weinberg scheme of Nogga, Tim-
mernans and van Kolck [19] eventually to KSW [33, 34],
thus reducing the convergence of the theory as the cut-off
is increased. The exact point at which the change takes
place is difficult to determine, but probably lies above
$r_c \gtrsim 1/m_\pi = 1.4$ fm. Of course, the fact that the $\epsilon_1$ mix-
ing angle is the phase which starts to feel the problem
earlier is not a surprise, as it depends on delicate cancel-
lations between short and long range effects. The singlet
channel, on the contrary, does not have any problem with
larger values of the cut-off as the power counting is not
changed: the use of larger cut-off values only entails a
rearrangement of the short range physics to account for those parts of the pion tail which have been ignored, but the assumption that all long range interactions are perturbative remains unchanged.

IV. THE ROLE OF THE CUT-OFF

In the previous calculations we have taken a very pragmatic point of view with regard to the cut-off: we have chosen the cut-off range $r_c = 0.6 - 0.9 \text{ fm}$ in order to improve the convergence of the theory and the description of the phase shifts up to $N^2 \text{LO}$. Of course, the choice of this range depends on a compromise between the specific requirements of the singlet and triplet channels, as explained in the previous sections. The important point is however that the proposed cut-off window generates leading order phase shifts which do not differ too much from the Nijmegen ones, an arrangement which minimizes the size of the subleading order corrections and, as a consequence, enhances the convergence of the theory.

This criterion basically coincides with the interesting cut-off philosophy of Beane, Kaplan and Vuorinen [63], in which the cut-off is merely a parameter controlling the convergence rate of the theory. The underlying idea behind this interpretation is an analogy with the role of renormalization scale dependence in QCD (see for example [64, 65]). A similar rationale can be provided by the observation that the full scattering amplitude, computed at all orders, is cut-off independent as a consequence of having an infinite number of counterterms. In this regard, cut-off dependence is just an artifact of finite order approximations which can be avoided by the careful selection of a cut-off window for which the particular power counting under consideration is realized.

However, for this interpretation to be complete within an EFT context it is necessary to determine first some formal aspects of the theory, such as the expansion parameter and the cut-off and momentum ranges for which the perturbative expansion converges. The knowledge of the expansion parameter is fundamental in order to be able to make rigorous error estimations of the results and to check the suitability of the selected cut-off window. On the other hand, the determination of the range of applicability of the EFT is necessary for avoiding power counting abuse, that is, claiming as legitimate the accidental description of data beyond the possibilities of the EFT under consideration.

Due to the mostly numerical nature of the present investigation, it is not clear how to extract the expansion parameter. However, the deconstruction of the $^1S_0$ singlet channel of Ref. [46] might provide some valuable clues regarding this important aspect of the theory. The energy dependence of the short range physics in this channel suggests a breakdown scale of $\Lambda_{0.5} \simeq 270 \text{ MeV}$. This estimation translates into an expansion parameter of $m_c/\Lambda_{0.5} \simeq 0.5$, a value which is compatible with the conjectured equivalence of the present approach with the KSW counting in the singlet channel. For the $^3S_1 - ^3D_1$ triplet channel there is no deconstruction yet which might provide a preliminary estimation of the breakdown scale, but if we assume the deconstruction of the p-wave uncoupled triplets [45] to hold in the $^3S_1 - ^3D_1$ coupled triplet, we obtain $\Lambda_{0.5} \simeq 340 \text{ MeV}$ [3]. The related expansion parameter would be $m_c/\Lambda_{0.5} \simeq 0.4$, a value which is compatible with the observation that the convergence in the triplet is better than in the singlet.

V. RELATION TO OTHER APPROACHES

In the present work we determine the power counting of the counterterms by requiring the renormalizability of the perturbative corrections to the scattering amplitude, where by renormalizability it is understood the elimination of all negative powers of the coordinate space cut-off $r_c$. There is still a residual cut-off dependence which is nominally of higher order, meaning that perturbative renormalizability implies the cut-off independence of the scattering amplitude at the order considered.

This is very similar to the RG approach of Birse [58], where the relative scaling (i.e. the power counting) of the counterterms is determined by requiring the cut-off independence of the scattering amplitude. Of course, exact cut-off independence is only achieved at infinite order. Finite order truncations will lead to a residual cut-off dependence involving positive powers of the cut-off $r_c$, but the renormalizability of the amplitudes is guaranteed. Therefore, it is not surprising a great degree of agreement between these two approaches.

This expectation is realized in the singlet channel, where renormalization group analysis [42] and deconstruction [46] are equivalent to perturbative renormalizability. For the triplet channel the situation is mixed: in the case of the $\epsilon_1$ mixing angle and the $^3D_1$ phase, the observation that two counterterms are needed to renormalize each of these phases is compatible with the deconstruction of the p- and d-wave uncoupled triplets of Ref. [45]. However, the RGA of Ref. [42] predicts one additional counterterm for the $^3S_1$ phase which should appear at order $Q^{5/2}$. This counterterm is not needed by perturbative renormalizability.

The previous discrepancy is surprising: we are making the same assumptions as Ref. [42] regarding which pieces of the interaction to iterate, yet the resulting power countings are slightly different. However, this is not new:

9 This corresponds to a laboratory energy of $250 \text{ MeV}$, above which the short range interaction in the $^3P_0$, $^3P_1$ and $^3D_2$ waves cannot be reliably described by two counterterms [45]. On the other hand, the assumption that the p-waves yield a good approximation for the breakdown scale of the s-waves is not unreasonable if we take into account that the deconstruction of the $^1P_1$ wave [17] basically suggests the same estimation as the $^1S_0$ wave [46].
the non-perturbative renormalizability of the OPE potential dictates that each attractive triplet requires one counterterm, while repulsive triplets do not. On the contrary, the RGA of Ref. [42] makes no distinction for the power counting of attractive and repulsive triplets. The paradigmatic example is given by the $3P_0$ (attractive) and $3P_1$ (repulsive) waves. As happened in the peripheral waves, the inconsistency can be circumvented in terms of the perturbative analysis of tensor OPE of Ref. [42]: for the $3P_0$ wave the perturbative treatment of OPE is expected to fail already at $k \sim 200$ MeV, while for the $3P_1$ wave this limit is extended up to $k \sim 400$ MeV. Therefore, in the range of momenta of interest for nuclear EFT the $3P_1$ wave can in principle be described in terms of the original Weinberg counting.

For the $3S_1$ phase the causes of the disagreement are to be found in the naive extrapolation of the idea of trivial and non-trivial fixed points to a problem where these concepts may not be applicable. The relevant observation in this context is that attractive singular potentials do not have a unique solution: the value of the scattering length oscillates indefinitely as the cut-off varies, a situation which is solved by the inclusion of a counterterm, stabilizing the solution. In this regard, for an attractive singular interaction all values of the scattering length are equally fine-tuned, implying that the distinction between trivial and non-trivial fixed points is artificial in this case. As analyzed in Ref. [40], the RG evolution of attractive singular potentials is driven in the infrared limit towards an oscillatory attractor-type solution resembling a limit cycle. However, the attractor-type solution does not have the discrete scaling properties of limit cycles (see [40] for details).

Alternatively, the previous observations can also be understood in terms of the behaviour of the squared reduced wave functions at short distances. For regular potentials there are two possible behaviours, the regular one, $|u(r_c)|^2 \sim r_c^2$, which can be identified with “natural” systems, and the irregular one, $|u(r_c)|^2 \sim 1$, which describes systems with unnaturally large scattering lengths. On the contrary, for attractive singular potential, the wave function always behaves as $|u(r_c)|^2 \sim r_c^{3/2}$ (times an oscillatory factor), independently of the value of the scattering length. That is, there is no additional short range enhancement due to large scattering lengths. In this regard, we should not expect the existence of two different kinds of fixed points in the RG flow of attractive singular interactions. The previous observations indicate that for attractive singular potentials (i) the $C_0$ counterterm must be of order $Q^{-1}$, as required by non-perturbative renormalizability and (ii) the first perturbation to the $C_0$ counterterms is of order $Q^{-3/2}$ as expected from the behaviour of the squared wave function, meaning that the attractor is a stable solution of the RG flow. Consequently, the correct RG analysis for channels with an attractive tensor force is the one termed “trivial” in Ref. [42], conveniently modified to incorporate the previous observation about the $C_0$ operator.

A recent work which is also relevant for the present approach is the toy model proposed by Epelbaum and Gegelia to address the role of regularization and renormalization in EFT [35]. In this work, the authors consider a two-body potential problem which shares many of the features of nuclear EFT, like the existence of a separation of scales or the possibility of expanding the long range interaction in terms of a power counting. The conclusions of the analysis of Epelbaum and Gegelia support most of the assumptions usually invoked in the Weinberg scheme, namely that naive dimensional analysis provides a good enough power counting and the ideal value of the cut-off should be chosen of the order of the hard scale of the problem. In addition, if the cut-off is taken much beyond the hard scale, the non-perturbative renormalization procedure may break the assumptions made in the first place by the power counting, a phenomenon which Epelbaum and Gegelia call “peratization”.

The lessons derived from a specific toy model may be however of limited significance. In particular, there is an essential feature of the chiral expansion which is not reproduced in the previous model, namely the appearance of singular interactions at leading and subleading orders. Contrary to the expectations of Epelbaum and Gegelia, the presence of singular potentials implies that (i) non-perturbative power counting will break down at cut-offs much softer than expected and that (ii) deviations from naive dimensional analysis may eventually happen. These aspects have been probably overlooked in the previous analysis due to the very good properties of the toy model: subleading contributions to the toy potential are only mildly divergent and, in addition, they are always suppressed by the expected ratio of low energy versus high energy scales. On the contrary, the subleading pieces of the chiral NN potential can receive unexpectedly large contributions from light degrees of freedom which have not been explicitly taken into account, like the $\Delta$ resonance. It is not surprising therefore that a toy model incorporating many of our naive expectations about EFT turns out to confirm them.

However, as far as we limit ourselves to soft enough cut-offs, the conclusions of Epelbaum and Gegelia re-

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10 One could think of extending this argument to the $3D_1$ phase, which is usually well reproduced in perturbation theory [10]. However, taking into account the coupled channel nature of the $3D_1$ phase, it is probably inconsistent to treat tensor OPE perturbatively in the $d$-wave channel but not in the $s$-wave channel.

11 Notice however that Ref. [40] uses a different language than Birse's RGA [16], what is called ultraviolet (long range) limit in [40] corresponds to the infrared (short range) limit of [16]. If we call the light and heavy scales $m_l$ and $m_h$, Ref. [40] is taking $m_{l/c} \to 0$, while Ref. [16] assumes $m_h \gg 1/r_c \gg m_l$, or equivalently $m_{l/c} \to \infty$, $m_{l/c} \to 0$ and $m_l/m_h \to 0$. Contrary to Ref. [16], Ref. [40] does not analyze the power counting of the short range operators but rather concentrates on issues such as the cut-off dependence of observables and the fixed points, limit cycles and attractors which result from the RG flow of regular and singular potentials.
Regarding naive dimensional analysis (i.e. Weinberg counting) are likely to hold. This observation is realized in the work of Shukla et al. [48], which, much in the spirit of deconstruction, analyzes the short range physics of the $^1S_0$ singlet channel with the chiral NN potential up to $N^2$LO. The authors observe that in the cut-off region $r_c = 1.0 - 1.8 \text{ fm}$ two counterterms are enough to parametrize the short range physics, a finding consistent with the idea that the Weinberg counting is better realized for soft values of the cut-off. A particularly interesting aspect of the previous work is the reanalysis of the short range physics for perturbative chiral TPE. For the cut-off range $r_c = 1.4 - 1.8 \text{ fm}$ the extracted short range physics can be accurately approximated by first order perturbative TPE, while for the region $r_c = 1.0 - 1.4 \text{ fm}$ one needs to go to second and third order in the perturbative series in order to reproduce the non-perturbative results, although there is still convergence. In the softer cut-off range the Weinberg scheme is perfectly realized as a perturbative power counting. For the harder cut-off range, Weinberg still is a consistent (non-perturbative) power counting scheme, as subleading order corrections are smaller than leading order ones. The efforts of Ref. [48] probably represent the best way to analyze the merits of the Weinberg counting in realistic cases. The extension to other partial waves, in particular the triplet, would be very welcomed.

If the cut-off is decreased below $R_0 = 1.0 \text{ fm}$, the authors of Ref. [48] observe that the contributions from subleading TPE start to grow uncontrollably, signalling the breakdown of the Weinberg counting. Below this cut-off, power counting is likely to be lost in non-perturbative calculations, as loop contributions from the subleading pieces will eventually dominate the amplitudes. The previous breakdown scale is however uncomfortably soft: using the equivalence $\Lambda = \pi/2r_c$ [20], $R_0$ naively corresponding to a (sharp) momentum cut-off of $\Lambda_0 \approx 310 \text{ MeV}$. Most Weinberg calculations use momentum space cut-offs of the order of $\Lambda \sim 0.5 \text{ GeV}$, which may be hard enough as to peratize the amplitudes. As suggested in Ref. [49], this may be already happening in the $^1S_0$ singlet channel for $\Lambda = 400 \text{ MeV}$ at $N^2$LO. These observations do not imply however that Weinberg counting is not useful, only that it should be employed within its specific range of applicability. In this respect, the most interesting feature of perturbative treatments is that they are guaranteed to respect the power counting independently of the value of the cut-off, precluding from the start the possibility of any power counting inconsistency.

A recent work which is also relevant for the discussion is the new KSW expansion of Beane, Kaplan and Vuorinen [63], which challenges one of the key premises of the present approach, namely that OPE should be fully iterated in the triplet, by constructing a viable nuclear EFT in which all pion exchanges are treated as perturbations. In this work the convergence problems of the original KSW counting [42, 55] are alleviated by the exchange of a fictitious meson of mass $\lambda$ which regulates the $1/r^3$ singularity of the tensor force at short distances. For the optimum value of the regulator ($\lambda = 750 \text{ MeV}$), the expansion apparently converges up to order $Q$, albeit slowly. At this order, the results of Ref. [63] for the $^3S_1$ and $^3D_1$ phases compare well with the LO results of the present approach. However, the order $Q$ results for the $\epsilon_1$ mixing angle is clearly worse than our LO computation and it does not seem to converge for $k > m_x$. This may be a good indicator that the tensor force really needs to be iterated, as the $\epsilon_1$ mixing angle is very sensitive to large cancellations between long and short range physics. In any case, a serious comparison of the present approach with the proposal of Beane, Kaplan and Vuorinen [63] requires (i) the extension of the previous results beyond order $Q$ and (ii) the consideration of the $^3P_0$ phase which according to Nogga, Timmermans and van Kolck [10] also demands the non-perturbative inclusion of tensor OPE.

The observation that OPE is perturbative in the singlet and non-perturbative in the triplet is closely related with the proposal of Beane, Bedaque, Savage and van Kolck (BBSvK) [54], which suggested the iteration of those pieces of the (leading order) chiral NN potential which survive in the chiral limit (that is, tensor OPE). This prescription is theorized to generate a convergent expansion of the scattering amplitudes around the chiral limit, therefore providing a consistent EFT expansion for two-nucleon systems. The existence of a deeper relationship with the present approach remains to be seen. However, the consideration of the subleading orders of the potential can break the correspondence, as there are pieces of these contributions to the potential which survive in the chiral limit and which are strong enough as to be iterated, particularly in the singlet channel. In this regard, the BBSvK scheme might provide a justification for the iteration of chiral van der Waals forces.

VI. CONCLUSIONS

The present approach determines the momentum and cut-off range for which chiral TPE behaves perturbatively when renormalizability is imposed. The use of small cut-offs is straightforward, but reduces the range of applicability of the theory in the singlet channel. The calculations turn out to confirm the viability of the counting proposal of Nogga, Timmermans and van Kolck [10], and corroborate to a large extent the related RGA by Birse [42], which predicted the power counting of the short range operators. There are some minor discrepancies however between perturbative renormalizability and RGA in the triplet channel, specifically for the $^3S_1$ phase, which are understood, suggesting minor modifications and possible improvements to the RGA of [42]. However there are some formal aspects of the present EFT formulation which need to be elucidated, like the role of the cut-off, the determination of the expansion parameter or the range of applicability of perturbative TPE. In
this regard, the deconstruction approach of Refs. 45, 47 is able to provide some interesting clues and preliminary answers. Of course, a complete evaluation of the renormalized perturbative treatment of chiral TPE should also include the calculation of the p- and d-wave phase shifts and the deuteron properties. The present analysis paves the way for such computations, which we leave for future works.

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Appendix A: Derivation of the DWBA for the Singlet Channel

In this appendix, we derive the DBWA formulae used along the present paper. We start by considering a potential which can be decomposed as a zeroth order approximation and a perturbation

\[ V(r) = V^{(0)}(r) + V^{(1)}(r) , \]  
(A1)

and the related reduced Schrödinger equations for the zeroth order and full reduced wave functions, \( u_k^{(0)} \) and \( u_k \)

\[- u_k^{(0)''} + 2\mu V^{(0)} u_k^{(0)} = k^2 u_k^{(0)} , \]  
(A2)

\[- u_k'' + 2\mu [V^{(0)} + V^{(1)}] u_k = k^2 u_k , \]  
(A3)

where \( \mu \) is the reduced mass of the system. The full reduced wave function can be perturbatively expanded as

\[ u_k^{(0+1)}(r) = u_k^{(0)}(r) + u_k^{(1)}(r) + \mathcal{O}((V^{(1)})^2) , \]  
(A4)

where, for the purposes of this work, it is enough to consider first order perturbation theory only.

In order to obtain the DWBA expressions we begin by (i) multiplying the zeroth order Schrödinger equation by the full solution \( u_k \), and (ii) the full Schrödinger equation by the zeroth order solution \( u_k^{(0)} \). Then we compute the difference between (i) and (ii), yielding

\[ (u_k^{(0)} u_k' - u_k^{(0)'} u_k) = 2\mu V^{(1)}(r) u_k^{(0)}(r) u_k(r) . \]  
(A5)

The expression above can be integrated to obtain the Wronskian identity

\[ W(u_k^{(0)}, u_k) \bigg|_{r_c} = 2\mu \int_{r_c}^\infty dr V^{(1)}(r) u_k^{(0)}(r) u_k(r) , \]  
(A6)

where \( W(f, g) = f(r)g'(r) - f'(r)g(r) \) is the Wronskian, and \( r_c \) and \( R \) are respectively the ultraviolet and infrared cutoffs. The infrared cutoff \( R \) can be eliminated by taking into account the long distance behaviour of the \( u_k^{(0)} \) and \( u_k \) reduced wave functions, which is given by

\[ u_k^{(0)}(r) \xrightarrow{r \to \infty} \frac{1}{\mathcal{A}^{(0)}(k)} \sin (kr + \delta^{(0)}) , \]  
(A7)

\[ u_k(r) \xrightarrow{r \to \infty} \frac{1}{\mathcal{A}(k)} \sin (kr + \delta) , \]  
(A8)

where \( \mathcal{A}^{(0)}(k) \) and \( \mathcal{A}(k) \) are normalization factors which ensure an energy independent normalization of the reduced wave functions at the cut-off radius. With the previous wave functions the Wronskian can be evaluated at \( R \to \infty \), resulting in

\[ W(u_k^{(0)}, u_k^{(0+1)}) = \frac{k}{\mathcal{A}^{(0)} A} \sin (\delta - \delta^{(0)}) . \]  
(A9)

Therefore, we arrive at the following expression

\[ \frac{k}{\mathcal{A}^{(0)} A} \sin (\delta - \delta^{(0)}) + f(r_c) = -2\mu \int_{r_c}^\infty dr V^{(1)}(r) u_k^{(0)}(r) u_k(r) , \]  
(A10)

where \( f(r_c) \) is just the Wronskian evaluated at \( r = r_c \), i.e. \( f(r_c) = W(u_k^{(0)}, u_k) \big|_{r_c} \), which does not depend on the momentum \( k \) as a consequence of the energy independent normalization at \( r = r_c \). The perturbative expansion of the previous formula can be obtained from the corresponding one of its components

\[ \delta(k) = \delta^{(0)}(k) + \delta^{(1)}(k) + \mathcal{O}((V^{(1)})^2) , \]  
(A11)

\[ u_k(r) = u_k^{(0)}(r) + u_k^{(1)}(r) + \mathcal{O}((V^{(1)})^2) , \]  
(A12)

\[ \mathcal{A}(k) = \mathcal{A}^{(0)}(k) + \mathcal{A}^{(1)}(k) + \mathcal{O}((V^{(1)})^2) , \]  
(A13)

\[ f(r_c) = f^{(1)}(r_c) + \mathcal{O}((V^{(1)})^2) , \]  
(A14)

yielding the following DWBA formula for the phase shift

\[ \frac{k}{\mathcal{A}^{(0)} A} \left( \frac{\delta^{(1)}(k; r_c)}{\sin \delta^{(0)}} \right)^2 + f^{(1)}(r_c) = -2\mu \int_{r_c}^\infty dr V^{(1)}(r) u_k^{(0)}(r)^2 , \]  
(A15)

where the Wronskian term \( f^{(1)} \) can be safely ignored in renormalized computations, as it vanishes once the first subtraction is done.
Appendix B: DWBA for the Triplet Channel

In this appendix we present the perturbative distorted wave formulas for the phase shifts in the $^3S_1 - ^3D_1$ triplet channel. For that, we express the phase shifts as the expansion
\[ \delta_\alpha(k; r_c) = \delta^{(0)}_\alpha + \delta^{(2)}_\alpha + O(Q^2), \]
\[ \delta_\beta(k; r_c) = \delta^{(0)}_\beta + \delta^{(2)}_\beta + O(Q^2), \]
\[ \epsilon(k; r_c) = \epsilon^{(0)} + \epsilon^{(2)} + O(Q^2), \]
where we have chosen the eigen parametrization of the phase shifts in this parametrization the DWBA formulas take their simplest form. The expansion of the nuclear bar phase shifts can be obtained from the previous one by reexpanding the relationships
\[ \bar{\delta}_1 + \bar{\delta}_2 = \delta_\alpha + \delta_\beta, \]
\[ \sin(\bar{\delta}_1 - \bar{\delta}_2) = \frac{\tan 2\epsilon}{\tan 2\epsilon}, \]
\[ \sin(\delta_\alpha - \delta_\beta) = \frac{\sin 2\epsilon}{\sin 2\epsilon}, \]
according to the counting. The LO phase shifts, $\delta^{(0)}_\alpha$, $\delta^{(0)}_\beta$ and $\epsilon^{(0)}$, are obtained by solving non-perturbatively the OPE potential with one counterterm, which is used for fixing the triplet scattering length $\alpha_t = 5.419$ fm. The exact procedure is explained in Ref. [61]. The expressions for the perturbative corrections to the LO phase shifts are the following
\[ \delta^{(\nu)}_\alpha(k; r_c) = -\frac{2\mu}{k} A^{(0)}_\alpha(k) I^{(\nu)}_\alpha(k; r_c), \]
\[ \delta^{(\nu)}_\beta(k; r_c) = -\frac{2\mu}{k^3} A^{(0)}_\beta(k) I^{(\nu)}_\beta(k; r_c), \]
\[ \epsilon^{(\nu)}(k; r_c) = \frac{2\mu}{k^3} A^{(0)}_\alpha(k) A^{(0)}_\beta(k) I^{(\nu)}_\alpha(k; r_c), \]
where the perturbative integrals $I^{(\nu)}_{\alpha\alpha}$, $I^{(\nu)}_{\alpha\beta}$ and $I^{(\nu)}_{\beta\beta}$ are defined as
\[ I^{(\nu)}_{\rho\sigma}(k; r_c) = \int_{r_c}^{\infty} dr \left[ V^{(\nu)}_{ss}(r) u^{(0)}_{k,\rho}(r) u^{(0)}_{k,\sigma}(r) + V^{(\nu)}_{sd}(r) \left( u^{(0)}_{k,\rho}(r) w^{(0)}_{k,\sigma}(r) + w^{(0)}_{k,\rho}(r) u^{(0)}_{k,\sigma}(r) \right) + V^{(\nu)}_{dd}(r) w^{(0)}_{k,\rho}(r) w^{(0)}_{k,\sigma}(r) \right], \]
with $\rho, \sigma = \alpha, \beta$. As in the singlet case, $\mu$ represents the reduced mass of the system, $u^{(0)}_{k,\alpha\beta}$ and $w^{(0)}_{k,\alpha\beta}$ are the leading order s- and d-wave reduced wave functions for the $\alpha(\beta)$ scattering states in an energy independent normalization at the origin / cut-off radius, and $A^{(0)}_\alpha$ and $A^{(0)}_\beta$ are the normalization factors which ensure the previous condition. The asymptotic normalization of the $\alpha$ and $\beta$ scattering states is taken to be
\[ A^{(0)}_{\alpha}(u^{(0)}_{k,\alpha}(r) \to \cos \epsilon^{(0)}(\cot \delta^{(0)}_\alpha \gamma_0(kr) - \gamma_0(kr)), \]
\[ A^{(0)}_{\beta}(u^{(0)}_{k,\beta}(r) \to \sin \epsilon^{(0)}(\cot \delta^{(0)}_\beta \gamma_2(kr) - \gamma_2(kr)), \]
\[ k^2 A^{(0)}_{\beta}(u^{(0)}_{k,\beta}(r) \to -\sin \epsilon^{(0)}(\cot \delta^{(0)}_\beta \gamma_0(kr) - \gamma_0(kr)), \]
\[ k^2 A^{(0)}_{\beta}(u^{(0)}_{k,\beta}(r) \to -\sin \epsilon^{(0)}(\cot \delta^{(0)}_\beta \gamma_2(kr) - \gamma_2(kr)), \]
where $\gamma_j(x) = x j_j(x)$, $\gamma_0(x)$, $\gamma_2(x)$, with $j_j(x)$ and $\gamma_j(x)$ the spherical Bessel functions. Due to the energy independent normalization of the wave functions at the cut-off radius, they can be expanded at short distances as
\[ u^{(0)}_{k,\alpha\beta}(r) \sim r^{5/4+5n/2} f(\sqrt{a/r}), \]
where the behaviour is given by
\[ u^{(0)}_{2n,\alpha\beta}(r) \sim r^{3/4+5n/2} f(\sqrt{a/r}) \]
with $f(x)$ some combination of $\sin x$, $\cos x$ and $e^{-\sqrt{2x}}$. The length scale $a$ is related to the strength of the tensor force. In principle the general solution of the Schrödinger equation for the tensor OPE potential also admits an $e^{-\sqrt{2x}}$ component which would destroy the renormalizability of the theory, as it generates divergences which cannot be absorbed by any finite number of counterterms. The previous component does not appear however if the LO wave functions have been properly renormalized. Therefore, what is essential is the power law behaviour of the wave functions, which dictates the divergence structure of the perturbative integrals $I^{(\nu)}_{\rho\sigma}$
\[ I^{(\nu)}_{\rho\sigma}(k; r_c) = I^{(\nu)}_{0,\rho\sigma}(r_c) + k^2 I^{(\nu)}_{2,\rho\sigma}(r_c) + I^{(\nu)}_{R,\rho\sigma}(k; r_c), \]
with $I^{(\nu)}_{0,\rho\sigma}$ and $I^{(\nu)}_{2,\rho\sigma}$ the divergent pieces of the integral and $I^{(\nu)}_{R,\rho\sigma}$ the regular piece. We can regularize the integral $I^{(\nu)}_{\rho\sigma}$ by including two free parameters
\[ I^{(\nu)}_{\rho\sigma}(k; r_c) = \lambda^{(\nu)}_{0,\rho\sigma} + \lambda^{(\nu)}_{2,\rho\sigma} k^2 + I^{(\nu)}_{\rho\sigma}(k; r_c), \]
which are to be fitted to the scattering data of the corresponding phase. The previous procedure yields a total of six counterterms for regularizing the NLO and N²LO phase shifts. As in the singlet case, the finite piece of one of these parameters ($\lambda^{(\nu)}_{0,\alpha\alpha}$) is redundant as it only affects the triplet scattering length $a_t$, which was already fixed at leading order. In other words, six pieces of data are enough to determine the NLO/N²LO phase shifts in the triplet.
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