Prokhorov-Skorokhod continuity of random fields.

A natural approach.

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Abstract

We derive in this article sufficient conditions in the natural terms for belonging of almost all the trajectories of the certain separable continuous in probability random field to the multivariate Prokhorov-Skorokhod space.

We consider also as a consequence the Central Limit Theorem in this spaces.

Key words and phrases: Multivariate Prokhorov-Skorokhod space, separable random process (field), probability, quasy-distance, Rosenthal’s constant, function and transformation; net, metric entropy, rearrangement invariant space, weak convergence, method Monte-Carlo, key estimate, exponential tail estimate and exponential Orlicz space, increments, constructiveness, mixed moment, factorization, convergence almost everywhere, generalized module of continuity, natural way and choice, exponential estimate, Central Limit Theorem, Lebesgue-Riesz and Grand Lebesgue spaces.

1 Definitions. Notations. Statement of problem.

Let $X = [0, 1]^d$, $d = 1, 2, 3, \ldots$; $x = \bar{x} = (x_1, x_2, x_3, \ldots, x_d) \in X$, and let $\xi = \xi(x)$, $x \in X$ be separable stochastic continuous numerical valued random process (r.p.) or equally random field (r.f.,) in the general case. The correspondent probability space will be denoted by $(\Omega, B, P)$ with expectation $E$ and variance $\text{Var}$. Denote also by $D(X) = D[0, 1]^d$ the multivariate famous Prokhorov-Skorokhod space; we recall its definition further.

Our goal in this article is deriving some simple sufficient conditions for belonging of almost all trajectories of this random field to the Prokhorov-Skorokhod space:

$$P(\xi(\cdot) \in D(X)) = 1. \quad (1.0)$$
We will formulate our condition only in the simple and so-called natural, or equally constructive terms, which may be generated through the trajectory of the considered random field (r.f.), in contradiction to the many previous works; and we find also as a consequence the sufficient conditions for the weak compactness for the sequence of r.f. in these spaces.

We investigate also as a capacity of an application of obtained results the classical Central Limit Theorem (CLT) in these spaces.

As for the previous works: see for example [2],[3],[4],[5],[6],[7],[10],[19],[20],[25],[27],[30],[36],[37]-[38],[42],[43] etc.

The natural approach for the investigation of continuous random fields may be found in the articles [39],[12],[14],[26],[28]-[35] and so one.

The well-known application of the CLT in the Prokhorov-Skorokhod spaces in the non-parametric statistics were obtained in the classical works of Yu.V.Prokhorov [40], A.V.Skorokhod [43], see also [25]. The multivariate generalization is considered in the articles [3],[5],[7],[27],[37]-[38],[42] and so one. The important application in the multi-parametric Monte-Carlo method may be found in [18],[20]. A very interest application of these limit theorems in physics was investigated in [11],[37],[38].

The immediate predecessor of offered report is the preprint [30], in which was obtained the exact bilateral exponential bound for the used in practice the tail probability

\[ P_{\xi,X}(u) \overset{\text{def}}{=} P \left( \max_{x \in X} |\xi(x)| > u \right), \ u > u_0 = \text{const} \geq 1, \]

for discontinuous random field \( \xi = \xi(x), \ x \in X \). The one-dimensional case \( d = 1 \) was investigated in [36].

The paper is organized as follows. The section having even numbers: second, fourth and sixth contains used further auxiliary apparatus. The third section is devoted to the classical Lebesgue-Riesz approach for considered in this preprint problem. We offer in the fifth section a more general method based on the theory of the so-called Grand Lebesgue Spaces.

The investigation of the Central Limit Theorem in the Prokhorov-Skorokhod spaces in the natural terms is the subject of the seventh paragraph. We represent in the last section some concluding remarks.

We must introduce some notations and definitions. Let \( x = \vec{x} = \{ x_j \} \) and \( y = \vec{y} \) be two vectors from the source space \( X \). Let also \( q = q(x,y) \) be certain non zero non-negative numerical values continuous symmetric function such that \( q(x,x) = 0, \ x \in X, \) and let \( x^+ = \vec{x}^+ \geq x, \ x^- = \vec{x}^- \leq x, \) where the inequalities are understood coordinate-wise:

\[ x, y \in X, \ x = \{ x(i) \}, \ y = \{ y(i) \}, \ x < y \iff \forall i = 1, 2, \ldots, d \ x(i) < y(i). \]
Analogously may be defined the inequality \( x \leq y \) and so one.

We do not suppose that the function \( d(\cdot, \cdot) \) satisfies the triangle inequality. For instance, the function \( q(x, y) \) may has a form

\[
q(x, y) = C|x - y|^\alpha \overset{\text{def}}{=} C|x - y|^{\alpha}, \quad \alpha = \text{const} > 0,
\]

where \(|x|\) denotes the ordinary Euclidean norm of the vector \( x \). Other example:

\[
C^{-1} q(x, y) = \sum_{j=1}^{d} |x_j - y_j|^\alpha(j), \quad \alpha(j) = \text{const} > 0.
\]

We will name such a function \( q = q(x, y) \) as a quasy-distance.

We can and will suppose in the sequel without loss of generality \( \max_{x,y \in X} q(x, y) = 1 \).

These types spaces are introduced, applied and investigated in a recent report of Daniel J. Greenhoe [21], where was named as "distance spaces".

The "metric" covering numbers \( N(X, q, \epsilon) \), \( \epsilon \in (0, 1) \) of the set \( X \) relative this quasy-distance \( q = q(x, y) \) is defined quite analogously to classical metric-distance case, namely, as a minimal numbers of closed \( q-\text{"balls"} \)

\[
B_q(x, \epsilon) := \{ y, \ y \in X, \ q(x, y) \leq \epsilon \}
\]

which cover all the set \( X \):

\[
N(X, q, \epsilon) = \inf \{ N : \exists x_k \in X, k = 1, 2, \ldots, N : \cup_{k=1}^{N} B_q(x_k, \epsilon) = X \}.
\]

The natural logarithm of the quantity \( H(X, q, \epsilon) := \ln N(X, q, \epsilon) \) is named as metric entropy of the set \( X \) relative the quasy-distance \( q \) at the point \( \epsilon \), \( \epsilon \in (0, 1) \).

If for instance,

\[
q(x, y) = q_\alpha(x, y) \asymp |x - y|^\alpha, \quad \alpha = \text{const} \in (0, \infty),
\]

then

\[
N \left([0, 1]^d, q_\alpha, \epsilon \right) \asymp \epsilon^{-d/\alpha}, \quad \epsilon \in (0, 1].
\]

This notion play a very important role in the investigation of continuous random fields, see [12],[13]-[15],[28],[44]-[45].

Further, let \( x^{(1)} = \bar{x}^{(1)}, \ x^{(2)} = \bar{x}^{(2)}, \ x^{(3)} = \bar{x}^{(3)} \) be three (deterministic) vectors from the set \( X \) such that \( x^{(1)} \leq x^{(2)} \leq x^{(3)} \). Denote by \( T \) the set of all the indexes \( T = \{1, 2, \ldots, d\} \) and by \( M \) arbitrary subset of \( T \):

\[
M = \{i(1), i(2), \ldots, i(m)\}, \ 1 \leq i(1) < i(2) < i(m) \leq d.
\]  \hspace{1cm} (1.1)

Of course, \( m = 0 \) \( \Rightarrow M = \emptyset \) and \( m = d \) \( \Rightarrow M = T \). Define the following vector \( z_M = \bar{z}_M = \bar{z}_M \left(x^{(1)}, \ x^{(2)}, \ x^{(3)}\right) \) generated by means of the random vectors \( x^{(1)}, \ x^{(2)}, \ x^{(3)} \) and by the subset set \( M \):
\[ z_M(j) = x^{(1)}_j, \ j \notin M; \ z_M(j) = x^{(3)}_j, \ j \in M. \] (1.2)

Evidently, \( z_X = x^{(1)} \) and \( z_\emptyset = x^{(3)} \). Notice that the vector \( z_M \) dependent on the whole triple \( (x^{(1)}, x^{(2)}, x^{(3)}) \).

Define also for the source random field \( \xi = \xi(x) \) and for the certain ordered triple \( (x^{(1)}, x^{(2)}, x^{(3)}) \) the system of partial increments
\[
\Delta[\xi](M) = \Delta[\xi](M)(x^{(2)}) := \left\{ (\xi(x^{(2)}) - \xi(z_M)) \right\}, \ M \subset (T),
\]
\[
\tau[\xi] = \tau(x^{(1)}, x^{(2)}, x^{(3)})[\xi] = \min_{M \subset T} |\Delta[\xi](M)| = \min_{M \subset T} \left| \xi(x^{(2)}) - \xi(z_M) \right|. \tag{1.3}
\]

The generalized Prokhorov-Skorokhod \( \kappa[\xi](h) = \kappa[\xi]_q(h), \ h \in [0, 1] \) module of continuity for the random field \( \xi(\cdot) \) may be defined as follows. \( \kappa[\xi](h) = \kappa[\xi]_q(h) \)
\[
def \sup_{x^{(2)} \in X} \sup_{q(x^{(3)}, x^{(1)}) \leq h} \tau(x^{(1)}, x^{(2)}, x^{(3)})[\xi]. \tag{1.4}
\]

By definition, the random field \( \xi = \xi(x) \) belongs to the (multidimensional, in general case) Prokhorov-Skorokhod space \( D[0,1]^d \) almost everywhere, iff for some (equally, each) non zero quasy-distance function \( q = q(x,y), \ x,y \in [0,1]^d \)
\[
P \left( \lim_{h \to 0^+} \kappa[\xi]_q(h) = 0 \right) = 1. \tag{1.5}
\]

The complete investigation of these spaces in the multivariate case \( d \geq 2 \), in particular, the criterion for the tightness for the Borelian measures in these spaces, may be found in [3],[5],[10],[27],[36],[37],[42] etc.

## 2 Auxiliary estimates I. General approach.

**A.** We will denote as customary for arbitrary r.v. \( \eta \) its classical Lebesgue - Riesz \( L_p = L_p(\Omega) \) norm
\[
|\eta|_p = |\eta|_{L_p} := [E|\eta|^p]^{1/p}, \ p = \text{const} \geq 1.
\]

A multivariate generalization of this notion may be introduced as follows. Let \( \xi = \xi = \{\xi_1, \xi_2, \ldots, \xi_k\} \) be a random vector and let \( p = \vec{p} = \{p_1, p_2, \ldots, p_k\} \) be numerical deterministic \( k \) - dimensional vector such that \( \forall i \Rightarrow p_i \geq 1 \). We introduce the following mixed moment
\[
\mu = \mu(\{\xi_i\}, \{p_i\}) = \mu(\vec{\xi}, \vec{p}) =
\]
\[ \mu[\xi](p_1, p_2, \ldots, p_k) \overset{\text{def}}{=} E \left[ \prod_{i=1}^{k} |\xi_i|^{p_i} \right]. \]

Let \( \{ a(1), a(2), \ldots, a(k) \} \) be again \( k \)-tuple of real numbers greatest that one: \( a(j) > 1 \), and such that

\[ \sum_{i=1}^{k} \frac{1}{a(i)} = 1. \]

The set all of such the \( k \) tuples we will denote by \( A = A(k) \). We apply the famous Hölder’s inequality

\[ \mu \leq \prod_{j=1}^{k} \left[ E|\xi|^{a(j)p_i} \right]^{1/a(i)} = \prod_{i=1}^{k} \left\{ |\xi|_{a(i)p_i} \right\}^{p_i}. \tag{2.A} \]

The last estimate may can be strengthened as follows.

\[ \mu \leq \inf_{\vec{a} \in A(k)} \left[ \prod_{j=1}^{k} \left[ E|\xi|^{a(j)p_i} \right]^{1/a(i)} \right] = \inf_{\vec{a} \in A(k)} \left[ \prod_{i=1}^{k} |\xi|_{a(i)p_i} \right]^{p_i}. \tag{2.B} \]

**B.** Denote \( \zeta = \)

\[ \zeta_q[\xi] \left( x^{(1)}, x^{(3)}; u \right) = \zeta[\xi] \left( x^{(1)}, x^{(3)}; u \right) := \sup_{x^{(2)} \in X} \mathbb{P} \left( \min_{M \subset T} |\Delta[\xi](M)| > u \right) = \]

\[ \sup_{x^{(2)} \in X} \mathbb{P} \left( \min_{M \subset T} |\xi(x^{(2)}) - \xi(x_M)| > u \right), \]

\( x^{(1)}, x^{(2)}, x^{(3)} \in X, \ u > 0. \)

We will start from the following condition imposed by the natural way on the paths of the random field \( \xi(\cdot) \).

**Condition 2.0.** Assume that for certain continuous quasy-distance \( q = q(x, y) \), \( x, y \in X \)

\[ \zeta_q[\xi] \left( x^{(1)}, x^{(3)}; u \right) \leq \frac{q(x^{(1)}, x^{(3)})}{\lambda(u)}, \ u \geq 1, \tag{2.0} \]

is the so-called condition of key estimate, where \( \lambda = \lambda(u) \) is certain monotonically increasing function, continuous or not, satisfying the condition \( \lim_{u \to \infty} \lambda(u) = \infty. \)

**Remark 2.1.** Both the functions \( q(\cdot, \cdot) \) and \( \lambda(\cdot) \) in (2.0) may be introduced by the natural way as follows.

\[ q(x^{(1)}, x^{(3)}) = q_{\xi}(x^{(1)}, x^{(3)}) \overset{\text{def}}{=} \sup_{x^{(2)} \in X} \mathbb{P} \left[ \tau \left( x^{(1)}, x^{(2)}, x^{(3)} \right) \geq 1 \right] \tag{2.1} \]
and
\[
\frac{1}{\lambda(u)} = \frac{1}{\lambda_\xi(u)} \overset{\text{def}}{=} \sup_{0 \leq x^{(1)} < x^{(3)}} \left[ P \left( \tau \left( x^{(1)}, x^{(2)}, x^{(3)} \right) \geq u \right) \right].
\] (2.2)

Herewith the estimate (2.0) there holds, if obviously both these functions \( q(\cdot, \cdot), \lambda(\cdot) \) there exist.

C. An alternative but again constructive method for quasy-distance estimation (2.0) based of the classical Lebesgue-Riesz spaces \( L_p = L_p(\Omega) \) is follows. Introduce the following vectors and functions:

\[
s = \bar{s} = \{ s(M) \}, \quad M \subset T, \quad s(M) > 0.
\]

The set of such a vectors \( \{ s = s(M) \} \) will be denoted by \( L; \quad L = \{ \bar{s} \} \).

Further, introduce the following function

\[
\beta \left( x^{(1)}, x^{(3)} \right) = \beta \left( x^{(1)}, x^{(3)}; \bar{s} \right) = \beta[\xi] \left( x^{(1)}, x^{(3)}; \bar{s} \right) \overset{\text{def}}{=} \sup_{x^{(2)} \in X} \mathbf{E} \prod_{M \in T} \left[ \left| \Delta[\xi](M) \right|^{s(M)} \right].
\] (2.3)

The function \( \beta(x, y), \quad x, y \in X \) by fixed value of the vector \( \bar{s} \) is quasy-distance function on the set \( X \otimes X \), if of course there exists for some positive vector \( \bar{s} \).

We derive using the famous Tchebychev’s inequality the restriction of type (2.0)

\[
P_{\xi[\bar{s}]}(x^{(1)}, x^{(3)}, u) \leq \inf_{x^{(2)} \in X} \frac{\beta \left( x^{(2)}, \bar{s} \right)}{u \sum_M s(M)},
\] (2.4)

if evidently the right-hand side of the last inequality is finite. So, in this case \( \lambda(u) = u \sum_M s(M), \quad u > 0 \).

The right-hand side of the inequality (2.4) may be estimated in particular as follows.

\[
P_{\xi[\bar{s}]}(x^{(1)}, x^{(3)}, u) \leq \sup_{x^{(2)} \in X} \left[ \frac{\beta \left( x^{(2)}, t \cdot \bar{I} \right)}{u^{2dt}} \right],
\] (2.4a)

\( \bar{I} := \{ 1, 1, \ldots, 1 \}; \quad t = \text{const} > 0. \)

Obviously,

\[
P_{\xi[\bar{s}]}(x^{(1)}, x^{(3)}, u) \leq \inf_{t > 0} \sup_{x^{(2)} \in X} \left[ \frac{\beta \left( x^{(2)}, t \cdot \bar{I} \right)}{u^{2dt}} \right].
\] (2.4b)

Se for the detail explanation [36].

Somewhere or other, we grounded the key estimate (2.0).

Alike estimate with replacing the classical Lebesgue-Riesz space \( L_p \) by so-called Grand Lebesgue Space will be considered further.
Denote by $\varepsilon$ the set of all positive sequences $\varepsilon = \{\varepsilon(k)\}$ such that $\varepsilon(1) = 1$, $\varepsilon(k) \downarrow 0$, and by $\Theta$ the set of all positive sequences $\Theta = \{\theta(k)\}$ for which $\theta(k) \downarrow 0$ and

$$\sum_{k=0}^{\infty} \theta(k) = 1.$$  

Introduce the following variables:

$$Q = Q(X, q, \xi; u) \overset{\text{def}}{=} \inf_{\{\varepsilon\} \in \varepsilon} \inf_{\{\theta\} \in \Theta} \sum_{k=0}^{\infty} N(X, q, \varepsilon(k+1)) \cdot \frac{\epsilon(k)}{\lambda(u \cdot \theta(k))}.$$  

$$\sigma[q](h) \overset{\text{def}}{=} h^{-d} \sup_{|x^{(3)} - x^{(1)}| \leq 2h} q\left(x^{(3)}, x^{(1)}\right).$$

We need to use now a very important fact, which is an evident and slight generalization of the main result of the articles [30],[36]; see also the predecessor [42].

**Theorem 2.1.**

**A.** Suppose in addition to the assumption (2.0) that for the separable stochastic continuous random field $\xi = \xi(x)$, $x \in X = [0, 1]^d$ the following conditions holds true:

$$\lim_{u \to \infty} Q(X, q, \xi; u) = 0;$$

then

$$\mathbf{P}(\tau[\xi] > u) \leq Q(X, q, \xi; u).$$

(2.5)

**B.** As long as

$$\lim_{h \to 0+} \sigma[q](h) = 0,$$

we deduce

$$\mathbf{P}(\kappa[\xi](h) > u) \leq Q(X, q, \xi; u) \cdot \sigma[q](2h),$$

and as a consequence $\mathbf{P} \left( \xi(\cdot) \in D[0,1]^d = 1 \right)$.

**Example 2.1.** Suppose

$$\lambda(u) = u^{2\rho}, \ u \geq 1, \ \exists \rho = \text{const} > 0,$$

and that the quasy-distance $q(\cdot, \cdot)$ is such that
\[ N(X, q, \epsilon) \leq C_N \epsilon^{-\gamma}, \quad \gamma = \text{const} \in (0, 1). \]

One can apply the statement of theorem 2.1 choosing correspondingly

\[ \epsilon(k) := s^{k-1}, \quad \theta(k) = (1 - s)s^k, \quad s = \text{const} \in (0, 1) : \]

\[ \frac{Q(X, q, \xi; u)}{C_N u^{-2\rho}} \leq \frac{s^{-\gamma} (1 - s)^{-2\rho}}{1 - s^{1-\gamma-2\rho}}, \quad \gamma + 2\rho < 1, \]

therefore

\[ \frac{Q(X, q, \xi; u)}{C_N u^{-2\rho}} \leq \inf_{s \in (0, 1)} \left[ \frac{s^{-\gamma} (1 - s)^{-2\rho}}{1 - s^{1-\gamma-2\rho}} \right], \]

so that

\[ Q(X, q, \xi; u) \leq C(C_N, \gamma, \rho) u^{-2\rho}, \quad u \geq 1, \quad (2.7) \]

if of course \( \gamma + 2\rho < 1. \)

By virtue of (uniform) continuity of the quasy-distance function \( q(\cdot, \cdot) \)

\[ \lim_{h \to 0^+} \sigma[q](h) = 0, \]

following

\[ P(\kappa[\xi](h) > u) \leq C(C_N, \gamma, \rho) u^{-2\rho} \cdot \sigma[q](2h), \quad u \geq 1, \quad (2.8) \]

and as a consequence \( P \left( \xi(\cdot) \in D[0, 1]^d \right) = 1. \)

Assume in continuation that the family, or for simplicity the sequence of somehow dependent random fields \( \xi_n(X), \ n = 1, 2, \ldots \) be a given such that all the introduced conditions are satisfied uniformly on the parameter \( n. \)

In detail, suppose

\[ \sup_n P_{\xi[\xi_n]}(u) = P_{\kappa[\xi_n]}(x^{(1)}, x^{(3)}; u) \overset{def}{=} \]

\[ \sup_n \sup_{x^{(2)} \in X} \left[ P_{\tau[\xi_n]}(x^{(1)}, x^{(2)}, x^{(3)} \geq u) \right] \leq \]

\[ \frac{q(x^{(1)}, x^{(3)})}{\lambda(u)}, \quad u \geq 1, \quad (2.9) \]

which may be called as the uniform key estimate, where \( \lambda = \lambda(u) \) is a monotonically decreasing function, continuous or not, with condition \( \lim_{u \to \infty} \lambda(u) = \infty. \)

**Theorem 2.2.** Let the finite-dimensional distribution of the r.f. \( \xi_n(\cdot) \) converges to ones for some r.f. \( \xi_\infty(\cdot) \) belonging to at the same space \( D[0, 1]^d \) with probability one.
Suppose that for the random fields $\xi_n = \xi_n(x), \ x \in X = [0, 1]^d$ the following uniform conditions holds true:

$$\lim_{u \to \infty} \sup_n Q(X, q, \xi_n; u) = 0; \quad (2.10)$$

then

$$\sup_n \mathbf{P}(\Delta[\xi_n] > u) \leq \sup_n Q(X, q, \xi_n; u) =: \overline{Q}(X, q, \{\xi_n(\cdot); \} u). \quad (2.11)$$

Since

$$\lim_{h \to 0^+} \sigma[q](h) = 0, \quad (2.12)$$

we obtain

$$\sup_n \mathbf{P}(\kappa[\xi_n](h) > u) \leq \sup_n Q(X, q, \xi; u) \cdot \sigma[q](2h), \quad (2.13)$$

and as a consequence $\mathbf{P} \left( \xi_n(\cdot) \in D[0, 1]^d = 1 \right)$ and furthermore the sequence of the r.f. $\xi_n(\cdot)$ converges in distribution in the space $D[0, 1]^d$ to the distribution of the r.f. $\xi_\infty$.

**The Proof** follows immediately from the estimate

$$\sup_n \mathbf{P}(\kappa[\xi_n](h) > u) \leq \sup_n \overline{Q}(X, q, \{\xi_n\}; u) \cdot \sigma[q](2h),$$

therefore the sequence of distributions of the sequence $\xi_n(\cdot)$ is weakly compact in the considered space. See in detail [3], [9], [10], [27], [42].

Evidently, instead the integer index $n$ may be used a point $\alpha$ belonging on arbitrary set, in particular, some topological space.

### 3 Classical Lebesgue-Riesz approach.

We rely here on the introduced before the generalized Lebesgue-Riesz distance $\beta(x, y) = \beta(x, y, \vec{s}), \ x, y \in X = [0, 1]^d$ and the statement of theorem 2.1.

We adopt in this section the vector $s = \vec{s}$ to be arbitrary positive fixes constant vector.

Assumptions and notations:

$$p := \sum_M s(M) > 0, \ \gamma = \text{const} \in (0, 1), \ C(N) = \text{const} \in (0, \infty); \ \forall \epsilon \in (0, 1) \Rightarrow$$

$$N(X, \beta(\cdot, \cdot), \epsilon) \leq C(N) \epsilon^{-\gamma}, \quad (3.0)$$

following,

$$\lambda(u) = C(\lambda) u^{\sum_M s(M)} = C(\lambda) u^p, \ u > 0; \ C(\lambda) = \text{const} \in (0, \infty).$$
One can choose in the statement of theorem 2.1
\[ \epsilon(k) = 2^{-k}, \quad \theta(k) = (1 - \theta_0) \theta_0^k, \]
where \( \theta_0 = 2^{(\gamma-1)/2p} \in (0, 1) \). Denote
\[ W(\gamma) := \left(1 - 2^{-(1-\gamma)/2}\right)^{-1}. \]

**Theorem 3.1.** We get under our notations and conditions after some computations:
\[ Q(X, \beta, \xi; u) \leq \frac{2C(N)}{C(\lambda)} W^{p+1}(\gamma) \ u^{-p}, \ u \geq 1. \] (3.1)

If in addition
\[ \lim_{h \to 0^+} \sigma[\beta](h) = 0, \] (3.2)
then
\[ P(\kappa[\xi](h) > u) \leq \frac{2C(N)}{C(\lambda)} W^{p+1}(\gamma) \ u^{-p} \cdot \sigma[\beta](2h), \ u \geq 1, \] (3.2)
and as a consequence \( P \left( \xi(\cdot) \in D[0,1]^d \right) = 1. \)

**Example 3.1.** The conditions of theorem (3.1) are trivially satisfied for the r.f. \( \xi = \xi(x), x \in X \) of the form
\[ \xi(x) = I(\vec{\eta} < \vec{x}), \] (3.3)
where \( \vec{\eta} \) is a random vector with values in the set \( X \) having at last continuous function of distribution
\[ F(x) = F(\vec{x}) = P(\vec{\eta} < \vec{x}); \] (3.4)
the inequality between two vectors is understood as ordinary coordinate-wise, and the \( I(\cdot) \) is ordinary indicator function.

Analogous statement is true under at the same assumptions for the r.f.
\[ \xi(x) = I(\vec{\eta} < \vec{x}) - F(\vec{x}). \] (3.5)

The grounding follows immediately from the relation
\[ \beta(x^{(1)}, x^{(2)}, x^{(3)}, \vec{s}) = 0, \]
if \( \min_j x_j^{(1)} > 0 \) or \( \max_j x_j^{(3)} < 1. \)

**Remark 3.1.** Note that the variable \( \beta(\cdot, \cdot; \cdot) \) allows by virtue of Hölder’s inequality the following simple estimate
\[
\beta \left( x^{(1)}, x^{(3)}; s \right) \leq \\
\sup_{x^{(2)} \in X} \prod_{M \subset T} \left[ E|\Delta[\xi](M)|^{\alpha(M)} s(M) \right]^{1/s(M)} = \\
\sup_{x^{(2)} \in X} \prod_{M \subset T} |\Delta[\xi](M)|^{s(M)} = 1/s(M),
\]

where

\[
\alpha(M) > 1, \sum_{M} 1/\alpha(M) = 1.
\]

4 Auxiliary facts II: Grand Lebesgue Spaces (GLS).

We recall here some facts about the so-called Grand Lebesgue Spaces (GLS) spaces and deduce also mixed norm inequalities for the random vector belonging to these spaces.

Let \(\psi = \psi(p), p \in [1, b), b = \text{const} \in (1, \infty] \) (or \(p \in [1, b] \)) be certain bounded from below: \(\inf \psi(p) > 0\) continuous inside the semi-open interval \(p \in [1, b)\) numerical function. We can and will suppose \(b = \sup p, \psi(p) < \infty\), so that \(\text{supp} \psi = [1, b)\) or \(\text{supp} \psi = [1, b]\). The set of all such a functions will be denoted by \(\Psi(b) = \{\psi(\cdot)\}; \Psi := \Psi(\infty)\).

By definition, the (Banach) Grand Lebesgue Space (GLS) space \(G\psi = G\psi(b)\) consists on all the numerical valued random variables \(\zeta\) defined on our probability space and having a finite norm

\[
||\zeta|| = ||\zeta||_{G\psi} \overset{def}{=} \sup_{p \in [1,b]} \left\{ \frac{||\zeta||_p}{\psi(p)} \right\}.
\]

These spaces are Banach functional space, are complete, and rearrangement invariant in the classical sense, see [1], chapters 1,2; and were investigated in particular in many works, see e.g.[16]-[17],[23],[26],[28],[35],[36]. We refer here some used in the sequel facts about these spaces and supplement more.

Remark 4.1. Let us consider the so-called degenerate \(\Psi - \) function \(\psi_{(r)}(p)\), where \(r = \text{const} \in [1, \infty)\):

\[
\psi_{(r)}(p) \overset{def}{=} 1, \ p \in [1, r];
\]

so that the correspondent value \(b = b(r)\) is equal to \(r\). One can extrapolate formally this function onto the whole semi-axis \(R^1_+\):

\[
\psi_{(r)}(p) := \infty, \ p > r.
\]
The classical Lebesgue-Riesz $L_r$ norm for the r.v. $\eta$ is quite equal to the GLS norm $||\eta||_{G\psi(r)}$:

$$|\eta|_r = ||\eta||_{G\psi(r)}.$$ \hfill (4.1)

Thus, the ordinary Lebesgue-Riesz spaces are particular, more precisely, extremal cases of the Grand-Lebesgue ones.

Suppose now $0 < ||\zeta|| < \infty$. Define the function $v(p) = v_\psi(p) := p \ln \psi(p)$, $1 \leq p < b$ and put formally $(p) := \infty$, $p < 1$ or $p > b$.

Recall that the Young-Fenchel, or Legendre transform $f^*(y)$ for arbitrary function $f : \mathbb{R} \to \mathbb{R}$ is defined (in the one-dimensional case) as follows

$$f^*(y) \overset{def}{=} \sup_{x \in \text{Dom}(f)} (xy - f(x)).$$

It is known that

$$\mathbb{P}(|\zeta| > y) \leq \exp \left(-v_\psi^*(\ln(y/||\zeta||))\right), \quad y \geq e \cdot ||\zeta||.$$ \hfill (4.2)

Conversely, the last inequality may be reversed in the following version: if

$$\mathbb{P}(|\zeta| > y) \leq \exp \left(-v_\psi^*(\ln(y/K))\right), \quad y \geq e \cdot K, \quad K = \text{const} \in (0, \infty),$$

and if the function $v_\psi(p)$, $1 \leq p < \infty$ is positive, continuous, convex and such that

$$\lim_{p \to \infty} \ln \psi(p) = \infty,$$

then $\zeta \in G\psi$ and besides $||\zeta|| \leq C(\psi) \cdot K$.

Moreover, let us introduce the so-called exponential Orlicz space $L(M)$ over the source probability space with the generating Young-Orlicz function

$$M(u) := \exp(v_\psi(\ln |u|)), \quad |u| > e$$

and as ordinary $M(u) = \exp(Cu^2) - 1$, $|u| \leq e$. It is known, [26],[35],[36] that the $G\psi$ norm of arbitrary r.v. $\zeta$ is equivalent to the its norm in the (exponential) Orlicz space $L(M)$:

$$||\zeta||_{G\psi} \leq C_1||\zeta||_{L(M)} \leq C_2||\zeta||_{G\psi}, \quad 0 < C_1 < C_2 < \infty.$$ \hfill (4.3)

Furthermore, let now $\eta = \eta(z)$, $z \in Z$ be arbitrary family of random variables defined on any set $Z$ such that

$$\exists b \in (1, \infty) \forall p \in [1, b) \Rightarrow \psi_Z(p) := \sup_{z \in Z} |\eta(z)|_p < \infty.$$ \hfill (4.4)

The function $p \to \psi_Z(p)$ is named as natural function for the family of random variables $Z$. Obviously,
Let again $\xi = \vec{\xi} = \{\xi_1, \xi_2, \ldots, \xi_k\}$ be a random vector and let $p = \vec{p} = \{p_1, p_2, \ldots, p_k\}$ be numerical deterministic $k$-dimensional vector such that $\forall i \Rightarrow p_i \geq 1$. We introduced in the second section the following mixed moment

$$
\mu = \mu(\{\xi_i\}, \{p_i\}) = \mathbb{E} \left[ \prod_{i=1}^{k} |\xi_i|^{p_i} \right]
$$

and obtained the following estimate

$$
\mu \leq \inf_{\vec{a} \in A(k)} \left[ \prod_{i=1}^{k} \left[ |\xi_i| a(i)^{p_i} \right] \right].
$$

Suppose now that each r.v. $\xi_i$ belongs to certain $G_{\psi_i} = G_{\psi_i}(b(i))$, $1 < b(i) \leq \infty$ space:

$$
|\xi_i|_p \leq ||\xi_i|| G_{\psi_i} \cdot \psi_i(p), \ 1 \leq p \leq \infty;
$$

then

$$
\mu \leq Y = Y(\{||\xi_i|| G_{\psi_i}\}, \{p_i\}), \quad (4.5)
$$

where

$$
Y = Y(\{||\xi_i|| G_{\psi_i}\}, \{p_i\}) \overset{def}{=} \inf_{\vec{a} \in A(k)} \prod_{i=1}^{k} \left[ ||\xi_i|| G_{\psi_i} \cdot \psi_i(a(i)^{p_i}) \right]^{p_i}. \quad (4.6)
$$

If for instance

$$
\forall i \ ||\xi_i|| G_{\psi_i} = 1,
$$

for example when all the functions $\psi_i$ are the natural function for the r.v. $\xi_i$, then the estimate (4.6) may be evidently simplified as follows

$$
\mu \leq \inf_{\vec{a} \in A(k)} \prod_{i=1}^{k} \left[ (a(i)^{p_i}) \right]^{p_i}. \quad (4.6a)
$$

We deduce the following estimate for the tail of minimum of $\xi_i$:

$$
P \left( \min_i |\eta_i| > u \right) \leq \frac{Y(\{||\xi_i|| G_{\psi_i}\}, \{p_i\})}{u \sum_i p_i}. \quad (4.7)
$$

This relation play a very important role for the investigation of discontinuous random fields and will be used further.
5 Grand Lebesgue Spaces approach.

We rely here on the introduced before the generalized Lebesgue-Riesz distance $\beta(x, y) = \beta(x, y, \vec{s})$, $x, y \in X = [0, 1]^d$ and the statement of theorem 2.1 to extend the obtained therein results into the so-called Grand Lebesgue Spaces (GLS).

We adopt in the sequel the vector $s = \vec{s}$ to be arbitrary, i.e. variable positive vector.

Assumptions:

$\exists \gamma(s) \in [0, \infty)$, $C_s(N) = \text{const} \in (0, \infty)$, $\forall \epsilon \in (0, 1)$ $\Rightarrow$ $N(X, \beta(\cdot, \cdot; s), \epsilon) \leq C_s(N) \epsilon^{-\gamma(s)}$; (5.1)

following,

$\lambda(u) = \lambda_s(u) = C_s(\lambda) \ u^{\sum_s s(M)}$, $u > 0$; $C_s(\lambda) = \text{const} \in (0, \infty)$. (5.2)

The subset of the whole set $L$ of such vectors $s = \{s(M)\}$ for which $\gamma(s) \in [0, 1)$ will be denoted by $L^o$; $L^o = \{s = \vec{s} : \gamma(s) \in [0, 1); L^o \subset L\}$. We suppose in the sequel $L^o \neq \emptyset$.

Let us return now to the source problem concerning the Prokhorov-Skorokhod continuity of the r.f. $\xi = \xi(x)$.

We get under our notations and conditions by virtue of theorem 3.1 for all the values $s \in L^o$

$Q(X, \beta(\cdot, \cdot; s), \xi; u) \leq \frac{2C_s(N)}{C_s(\lambda)} W^{p(s)+1}(\gamma(s)) u^{-p(s)}$, $u \geq 1$. (5.3)

If in addition

$\exists s \in L^o \Rightarrow \lim_{h \to 0^+} \sigma[\beta(\cdot, \cdot; s)](h) = 0$, (5.3a)

i.e. if the function $(x, y) \to \beta(x, y; s)$ is (uniform) continuous, then $\exists s \in L^o \Rightarrow$

$P(\kappa[\xi](h) > u) \leq \frac{2C_s(N)}{C_s(\lambda)} W^{p(s)+1}(\gamma(s)) u^{-p(s)} \cdot \sigma[\beta(\cdot, \cdot; s)](2h)$, $u \geq 1$. (5.4)

Denote by $L^+$ the set of all the vectors $s = \vec{s}$ from the set $L^o$ for which the relation (5.3a) there holds. It is natural to suppose also $L^+ \neq \emptyset$.

The estimate (5.4) can be obviously strengthened as follows $P(\kappa[\xi](h) > u) \leq$

$\inf_{s \in L^+} \left\{ \frac{2C_s(N)}{C_s(\lambda)} W^{p(s)+1}(\gamma(s)) u^{-p(s)} \cdot \sigma[\beta(\cdot, \cdot; s)](2h) \right\}$, $u \geq 1$. (5.4a)
To summarize:

**Theorem 5.1.** We conclude under conditions of this section \( P(\kappa[\xi](h) > u) \leq \inf_{s \in L^+} \left[ \frac{2C_s(N)}{C_s(\lambda)} W^{p(s)+1}(\gamma(s)) \ u^{-p(s)} \cdot \sigma[\beta(\cdot, \cdot; s)](2h) \right] \), \( u \geq 1 \), \( (5.5) \)

and as a consequence \( P(\xi(\cdot) \in D[0,1]^d) = 1 \).

**Example 5.1.** Denote

\[
K(q) := \sup_{s \in L^p \cdot p(s) \leq q} \left[ \frac{2C_s(N)}{C_s(\lambda)} W^{p(s)+1}(\gamma(s)) \right], \quad 1 < q < \infty,
\]

then

\[
P(\zeta[\xi] > u) \leq K(q) \ u^{-q}, \quad (5.6)
\]

hence

\[
P(\zeta[\xi] > u) \leq \inf_{q > 0} \left[ K(q) \ u^{-q} \right]. \quad (5.7)
\]

If in addition

\[
\exists m = \text{const} > 0 \ \forall q > 1 \ K(q) \leq C_1 q^{q/m},
\]

then

\[
P \left( \zeta[\xi](x^{(1)}, x^{(3)}) > u \right) \leq C_1 \ \exp \left\{ -u^m/(me) \right\}, \ u \geq e.
\]

Thus, we obtained an exponential bound for tail of distribution of the r.v. \( \zeta[\xi] \), in accordance with the general theory of Grand Lebesgue Spaces, see fourth section.

More simple result may be obtained from the inequality (2.4a):

\[
P_{\zeta[\xi]}(x^{(1)}, x^{(3)}, u) \leq \inf_{t > 0} \left[ \sup_{x^{(2)} \in X} \beta \left( x^{(2)}, t \cdot \tilde{1} \right) \right]. \quad (5.8)
\]

**Remark 5.1.** Appearing in (1.3), (1.4) the r.v.

\[
\tau[\xi] = \tau \left( x^{(1)}, x^{(2)}, x^{(3)} \right) [\xi] = \min_{M \subset T} |\Delta[\xi](M)|
\]

may be estimated by means of the general theory of GLS as follows.

Suppose that each of the r.v. \( \Delta[\xi](M), \ M \subset T \) belongs uniformly over the space \( X \) to some \( G\psi_M \) - space:

\[
\forall M \subset T \ \exists b(M) \in (1, \infty) \ \forall p \in [1, b(M)] \ \Rightarrow \psi_M(p) := \sup_{x^{(1)}, x^{(2)}, x^{(3)} \in X} |\Delta_M|_p < \infty.
\]
We define formally as before the values $\psi_M(p)$ in the case when $b(M) < \infty$ for the values $p > b(M)$ as $\infty$, so that for all the values $p \geq 1$

$$\psi_M(p) = \sup_{x^{(1)}, x^{(2)}, x^{(3)} \in X} |\Delta_M|_p.$$  (5.9)

We deduce by means of inequality (4.7)

$$\mathbf{P}(\tau[\xi] > u) = \mathbf{P}\left(\min_{M \subset T} |\Delta[\xi](M)| > u\right) \leq \inf_{\{p_M\}} \frac{Y(\{||s_M||\psi_M\}, \{p_M\})}{u\sum_{M \subset T} p(M)}.$$  (5.10)

**Another approach.**

Let us define this time the correspondent natural $\psi -$ function $\upsilon = \upsilon(p)$ as follows

$$\upsilon(p) := \sup_{x^{(1)}, x^{(2)}, x^{(3)} \in X} \left|\tau\left(x^{(1)}, x^{(2)}, x^{(3)}\right)\right|_p,$$  (5.11)

if naturally $\upsilon(p)$ is finite in some non-trivial closed or semi-closed interval $p \in [1, b)$ or $p \in [1, b]$, $b = \text{const} \in (1, \infty]$.

Another equivalent version:

$$\upsilon(p) := \sup_{x^{(1)}, x^{(2)}, x^{(3)} \in X} \left|\Delta[\xi]\left(M\left(x^{(1)}, x^{(2)}, x^{(3)}\right)\right)\right|_p.$$  (5.11a)

The correspondent natural distance function $r(x, y)$ takes here a form

$$r = r(\cdot, \cdot) = r_{\upsilon}\left((x^{(1)}, x^{(3)})\right) = r\left((x^{(1)}, x^{(3)}\right) \overset{\text{def}}{=} \sup_{x^{(2)} \in X} \left||\tau\left(x^{(1)}, x^{(2)}, x^{(3)}\right)\right||_{G\upsilon},$$  (5.12)

and analogously may be defined as before the variable $\sigma_r(h)$.

We impose at the same condition

$$\forall \epsilon \in (0, 1) \ N(X, r, \epsilon) \leq C(N, r)\epsilon^{-\gamma}, \ \exists \gamma = \text{const} \in [0, 1).$$  (5.13)

It follows immediately from the direct definition of the norm for GLS

$$\beta(x, y; p) \leq \upsilon(p) \cdot r(x, y).$$

As long as

$$N(X, \beta_p, \epsilon) \leq N(X, \upsilon(p)r, \epsilon) = N(X, r, \epsilon/\upsilon(p)) \leq C(N, r) \epsilon^{-\gamma} \upsilon^\gamma(p),$$

we deduce on the basis of theorem 3.1 for all the admissible values $p$.  


\[ Q(X,r,\xi;u) \leq \frac{2C(N,r)}{C(\lambda)} v^\gamma(p) W^{p+1}(\gamma) u^{-p}, \; u \geq 1. \]

**Theorem 5.2.** We assert under the notations and conditions of this section

\[ Q(X,r,\xi;u) \leq \frac{2 C(N,r)}{C(\lambda)} \inf_{p\in[1,b]} \left\{ v^\gamma(p) W^{p+1}(\gamma) u^{-p} \right\}, \; u \geq 1. \quad (5.14) \]

If in addition

\[
\lim_{h \to 0^+} \sigma[r](h) = 0,
\]

then

\[
P(\kappa[\xi](h) > u) \leq \frac{2C(N,r)}{C(\lambda)} \times \]

\[
\inf_{p\in[1,b]} \left[ v^\gamma(p) W^{p+1}(\gamma) u^{-p} \cdot \sigma[r](2h) \right], \; u \geq 1, \quad (5.15)
\]

and as a consequence: \( P \left( \xi(\cdot) \in D[0,1]^d \right) = 1. \)

**Example 5.2.** Suppose the function \( v(p), \; p \in [1,\infty) \) is such that

\[
v^\gamma(p) \leq \exp \left( C(1) p^{1+v} \right)
\]

for certain positive finite constant \( v. \) It is easy to calculate

\[
Q(X,r,\xi;u) \leq \exp \left( -C_2(v,\gamma,p) (\ln u)^{1+1/v} \right), \; u \geq e. \quad (5.16)
\]

**Example 5.3.** Suppose the distance function \( r = r(x,y), \; x,y \in X \) is such that

\[
r = r(x,y) \asymp \sum_{j=1}^{d} |x(j) - y(j)|^{\alpha(j)}, \; \alpha(j) = \text{const} \in (0,\infty).
\]

The constant \( \gamma \) may be calculated by the formula

\[
\frac{1}{\gamma} = \frac{1}{\alpha(1)} + \frac{1}{\alpha(2)} + \ldots + \frac{1}{\alpha(d)}.
\quad (5.17)
\]

If for instance \( \forall j \; \alpha(j) = \alpha \in (0,\infty), \) then \( \gamma = d/\alpha; \) and the used before conditions

\[
\lim_{h \to 0^+} h^{-d} \sigma[r](h) = 0 \iff \gamma < 1
\]

are equivalent to the following restriction: \( \alpha > d. \)
6 Auxiliary estimates III. Mixed moment estimates for normalized sums of independent random variables.

We intend to extend in this section the results of penultimate one to the normalized sums of independent random variables.

We agree to take in the sequel during this and next sections that $p(j) \geq 2$; therefore $b(j) \in (2, \infty]$. Recall here the famous Rosenthal’s inequality, see [41], [22], [24], [33], [28] etc.

Let $\iota_i, i = 1, 2, \ldots; \iota = \iota_1$ be a sequence of the centered i., i.d. random variables with finite $p$th moment. Rosenthal’s inequality tell us:

$$
\sup_n \left| n^{-1/2} \sum_{i=1}^{n} \iota_i \right|_p \leq C(R) \frac{p}{\ln p} |\iota|_p,
$$

(6.0)

where $C(R)$ is an absolute constant. Denote for brevity $K(p) = K_R(p) := C(R) p/\ln p$– the so-called Rosenthal constant, more exactly, Rosenthal’s function on $p$.

We introduce in this connection the following Rosenthal transformation $\psi_R(p)$ for each $\Psi$– function $\psi(\cdot)$:

$$
\psi_R(p) \overset{def}{=} K_R(p) \cdot \psi(p) = K_R[\psi](p) := [C(R) \frac{p}{\ln p}] \cdot \psi(p).
$$

(6.1)

It is clear that if $\text{supp}\psi(\cdot) < \infty$, then $\psi_R(p) \asymp \psi(p)$.

The Rosenthal’s inequality may be rewritten by means of our notations as follows

$$
\sup_n \left| \left| n^{-1/2} \sum_{i=1}^{n} \iota_i \right| \right| G\psi_R \leq \left| \left| \iota \right| \right| G\psi.
$$

(6.2)

Note that the last estimate in essentially non-improvable.

We need more some generalization of Rosenthal’s inequality on the multivariate mixed moments, alike in the fourth section. Indeed, let $\{\xi^{(j)}_i\}, i = 1, 2, \ldots; j = 1, 2, \ldots, k$ be $k$ tuple of infinite sequences of centered random variables. We impose the following conditions of its independence:

$$
\text{for all the (fixed) upper index } j \text{ the centered random variables } \{\xi^{(j)}_i\}, i = 1, 2, \ldots; \xi^{(j)} := \xi^{(j)}_1 \text{ are independent and identical distributed.}
$$

Denote for every such the index $j$

$$
S_n(j) = n^{-1/2} \sum_{i=1}^{n} \xi^{(j)}_i,
$$

(6.3)

so that by virtue of Rosenthal’s inequality
\[
\sup_n \left| S_n(j) \right|_p \leq K_R(p) \left| \xi^{(j)} \right|_p = K(p) \left| \xi_1^{(j)} \right|_p, \quad (6.4)
\]

or equally
\[
\sup_n \left| S_n(j) \right| ||G_{\psi,j,R} \leq || \xi^{(j)} ||G_{\psi,j}; \quad (6.4a)
\]

and introduce the following mixed moment
\[
\nu = \nu \left( \{ \xi^{(j)} \}, \{ p_j \} \right) = \nu \left( \xi^\vec{\iota}, \vec{p} \right) =
\]
\[
\nu(p_1, p_2, \ldots, p_k) \overset{def}{=} \sup_n \mathbb{E} \left[ \prod_{j=1}^k \left| S_n(j) \right|^{p_j} \right], \quad p_j = \text{const} \geq 2. \quad (6.5)
\]

Let \( \{ a(1), a(2), \ldots, a(k) \} \) be again the \( k \)– tuple of real numbers greatest that one: \( a(j) > 1 \), and such that as before
\[
\sum_{j=1}^k \frac{1}{a(j)} = 1.
\]

The set all of such the \( k \) tuples we denoted by \( A = A(k) \). We apply again Hölder’s inequality
\[
\nu \leq \prod_{j=1}^k \left\{ K_R[\psi] \left( a(j) \left| p(j) \right| \left\| \xi^{(j)} \right\|_{a(j)p(j)} \right) \right\}^{p(j)}
\]
\[
\prod_{j=1}^k \left[ \psi_{j,R} \left( a(j) \left| p(j) \right| \right) \right]^{p(j)} := Z(\vec{a}, \vec{p}),
\]
thus
\[
\mu \left( \xi^\vec{\iota}, \vec{p} \right) \leq \inf_{\vec{a} \in A(k)} Z(\vec{a}, \vec{p}) =: \overline{Z}(\vec{p}). \quad (6.6)
\]

It follows immediately from the last inequality an uniform tail estimate
\[
\sup_n \mathbb{P} \left( \min_j \left| S_n(j) \right| > u \right) \leq \frac{\overline{Z}(\vec{p})}{u^{\sum_{j} p(j)}}, \quad u > 0, \quad (6.7)
\]
if of course the right-hand side of (6.7) is not only is finite but also tends to zero as \( u \to \infty \).

As a slight consequence:
\[
\sup_n \mathbb{P} \left( \min_j \left| S_n(j) \right| > u \right) \leq \inf_{\vec{p} \geq 1} \left[ \frac{\overline{Z}(\vec{p})}{u^{\sum_{j} p(j)}} \right], \quad u > 0. \quad (6.8)
\]
7 Central Limit Theorem in the multivariate
Prokhorov-Skorokhod space.

Let now \( \xi_i = \xi_i(x) \), \( \xi(x) = \xi_1(x) \), \( x \in X = [0, 1]^d \) be a sequence of separable stochastic continuous independent identically distributed (i.i.d) centered (mean zero): \( \mathbb{E}\xi_i(x) = 0 \), \( x \in X \) random fields with finite variance at each the point: \( \forall x \in X \) \( \text{Var}\xi_i(x) < \infty \), \( \xi(x) := \xi_1(x) \). Denote

\[
S_n(x) = n^{-1/2} \sum_{i=1}^{n} \xi_i(x). \tag{7.0}
\]

Denote also by \( S(x) = S_\infty(x) \) the centered separable stochastic continuous Gaussian random field with at the same covariation function as the source r.f. \( \xi(x) : \)

\[
R(x, y) := \mathbb{E}S(x)S(y) = \mathbb{E}\xi(x)\xi(y), \; x, y \in X.
\]

However, the stochastic continuity of the limiting Gaussian field \( S = S(x) \) is quite equivalent to the continuity of its covariation function \( R(x, y) \), which implies in turn the stochastic continuity of each r.f. \( \xi_i(x) \).

Note in addition

\[
\mathbb{E}S_n(x)S_n(y) = R(x, y); \quad R(x, x) = \text{Var}(\xi_i(x)) = \text{Var}(S_n(x)), \; x \in X.
\]

Evidently, the finite-dimensional distributions of the sequence of r.f. \( \{S_n(\cdot)\} \) converge to ones for \( S(\cdot) \). Therefore, we need to ground only the weak compactness of the sequence of distributions \( \text{Law}\{S_n(\cdot)\} \) in the Prokhorov-Skorokhod space, formulated as before in the natural terms for the source r.f. \( \xi(x) \). More precisely:

**Definition 7.1.**

We will say as ordinary that the sequence \( S_n(\cdot) \), or more simple the alone r.f. \( \xi(x) \) satisfies CLT in the Prokhorov-Skorokhod space \( D[0, 1]^d = D(X) \), if \( \xi(\cdot) \in D(X) \) with probability one and the sequence \( S_n(\cdot) \) converges weakly (in distribution) to the r.f. \( S(x) \). Briefly: \( \xi = \xi(x) \in CLT(PS) \) or more detail

\[
n \rightarrow \infty \Rightarrow S_n(\cdot) \overset{d}{\rightarrow} S_\infty(\cdot) = S(\cdot). \tag{7.1}
\]

The first examples of \( CLT(PS) \) with statistical applications belong to Yu.V.Prokhorov [40] and A.V.Skorokhod [43]. Another applications in the Monte-Carlo method and in the reliability theory may be found in [20]. The multivariate case \( d \geq 2 \) is investigated in the famous article of P.J.Bickel and M.J.Wichura [3].

Denote

\[
\Delta_n[\xi](M) = \Delta_n[\xi](M) (x^{(2)}) := (S_n(x^{(2)}) - S_n(z_M)), \; M \subset (T),
\]
\[ \Delta^{(i)}[\xi](M) = \Delta_n^{(i)}[\xi](M) \left( x^{(2)} \right) := \left( \xi_i(x^{(2)}) - \xi_i(z_M) \right) ; \]

\[ U[M](p) = U[M; x^{(1)}, x^{(3)}](p) := \sup_{x^{(2)} \in X} |\Delta[\xi](M)|_p, \quad p \geq 2, \]

\[ V[M](p) = V[M; x^{(1)}, x^{(3)}](p) := \sup_{x^{(2)} \in X} \sup_{n} |\Delta_n[\xi](M)|_p, \quad p \geq 2, \]

We have

\[ \Delta_n[\xi](M) := n^{-1/2} \sum_{i=1}^{n} \Delta^{(i)}[\xi](M), \]

and we derive following by virtue of Rosenthal’s inequality

\[ V[M](p) \leq K_R(p) \cdot U[M](p) =: KU[M](p). \] \hspace{1cm} (7.2)

Let us introduce the following quasy-distance function, more exactly, the family of quasy-distance functions

\[ \delta \left( x^{(1)}, x^{(3)} \right) = \delta \left( x^{(1)}, x^{(3)}; \tilde{s} \right) = \delta[\xi] \left( x^{(1)}, x^{(3)}; \tilde{s} \right) \overset{\text{def}}{=} \]

\[ \sup_{x^{(2)} \in X} \mathbb{E} \sup_{n} \prod_{M \subset T} \left[ |\Delta_n[\xi](M)|^{s(M)} \right]. \] \hspace{1cm} (7.3)

This function allows a very simple estimate by means of Hölder’s inequality. Namely, if we denote

\[ \tilde{\delta} \left( x^{(1)}, x^{(3)}; \tilde{s} \right) = \tilde{\delta}[\xi] \left( x^{(1)}, x^{(3)}; \tilde{s} \right) \overset{\text{def}}{=} \]

\[ \sup_{x^{(2)} \in X} \prod_{M \subset T} [KU[M](\alpha(M), s(M))]^{1/\alpha(M)}, \] \hspace{1cm} (7.4)

then

\[ \delta[\xi] \left( x^{(1)}, x^{(3)}; \tilde{s} \right) \leq \tilde{\delta} \left( x^{(1)}, x^{(3)}; \tilde{s} \right). \] \hspace{1cm} (7.5)

Further,

\[ \delta \left( x^{(1)}, x^{(3)}; \tilde{s} \right) \leq \tilde{\delta} \left( x^{(1)}, x^{(3)}; \tilde{s} \right) \leq \delta^+ \left( x^{(1)}, x^{(3)}; \tilde{s} \right), \]

where

\[ \delta^+ \left( x^{(1)}, x^{(3)}; \tilde{s} \right) = \delta^+[\xi] \left( x^{(1)}, x^{(3)}; \tilde{s} \right) \overset{\text{def}}{=} \]

\[ \inf_{\tilde{\alpha} \in A(T)} \sup_{x^{(2)} \in X} \prod_{M \subset T} [KU[M](\alpha(M), s(M))]^{1/\alpha(M)}, \] \hspace{1cm} (7.6)
where as was described $M \subset T,$
\[ \vec{\alpha} \in A(T) \Leftrightarrow \alpha(M) > 1, \sum_M 1/\alpha(M) = 1. \]

Assumptions and notations: $p := \sum_M s(M) > 0,$
\[ N(X, \delta[\xi](\cdot; \cdot; \vec{s}), \epsilon) \leq C_\delta(N) \epsilon^{-\gamma}, \epsilon \in (0,1), \]
\[ \gamma = \text{const} \in (0,1), C_\delta(N) = \text{const} \in (0,\infty), \]
following,
\[ \lambda(u) = C_\delta(\lambda) u^{\sum_M s(M)} = C_\delta(\lambda) u^p, u > 0; C(\lambda) = \text{const} \in (0,\infty). \]

**Theorem 7.1.** We get under our notations and conditions (7.7) etc. by virtue of theorem 2.2
\[ \sup_n Q(X, \delta, S_n; u) \leq \frac{2C_\delta(N)}{C_\delta(\lambda)} W^{p+1}(\gamma) u^{-p}, \ u \geq 1. \]  
(7.9)

If in addition
\[ \lim_{h \to 0^+} \sigma[\delta](h) = 0, \]  
(7.10)
then
\[ \sup_n P(\kappa[S_n](h) > u) \leq \frac{2C(N)}{C(\lambda)} W^{p+1}(\gamma) u^{-p} \cdot \sigma[\beta](2h), \ u \geq 1, \]  
(7.11)
and as a consequence the r.f. $\xi(\cdot)$ satisfies the CLT in the Prokhorov-Skorokhod space $X = D[0,1]^d.$

**Remark 7.1.** Obviously, instead the condition (7.7) may be used the following more easily verified one
\[ N(X, \delta[\xi](\cdot; \cdot; \vec{s}), \epsilon) \leq C_\delta(N) \epsilon^{-\gamma}, \]
\[ \epsilon \in (0,1), \gamma = \text{const} \in (0,1), C_\delta(N) = \text{const} \in (0,\infty), \]  
(7.12)
or in turn
\[ N(X, \delta^+[\xi](\cdot; \cdot; \vec{s}), \epsilon) \leq C_{\delta^+}(N) \epsilon^{-\gamma}, \]
\[ \epsilon \in (0,1), \gamma = \text{const} \in (0,1), C_{\delta^+}(N) = \text{const} \in (0,\infty). \]  
(7.13)
Remark 7.2. The conditions of theorem 7.1 are satisfied for instance for the so-called empirical random fields, see [40], [43]; see also [42]; as well as for multiple parametric integral computation by means of Monte-Carlo method [20].

8 Concluding remarks.

A. Necessary and sufficient condition for the Prokhorov-Skorokhod continuity of random fields.

Let $\xi = \xi(x)$, $x \in X$ be stochastic continuous separable r.f. The classical module of (uniform) continuity $\omega[\xi](h)$, $h \in [0,1]$ for the r.f. $\xi(\cdot)$ is defined as follows.

$$\omega[\xi](h) \overset{\text{def}}{=} \sup_{x,y: |x-y| \leq h} |\xi(x) - \xi(y)|. \quad (8.0)$$

The r.f. $\xi(\cdot)$ is continuous with probability one, or equally has a continuous almost surely sample path, iff

$$P \left( \lim_{h \to 0^+} \omega[\xi](h) = 0 \right) = 1. \quad (8.1)$$

The last relation is completely equivalent to the following natural condition

$$\lim_{h \to 0^+} E \arctan \omega[\xi](h) = 0, \quad (8.2)$$

as long as $\omega[\xi](h)$, $h \in [0,1]$ is monotonically increasing function.

Of course, our set $[0,1]^d$ may be replaced by arbitrary complete compact metric space.

Analogous statement, with at the same proof, holds true for the Prokhorov-Skorokhod space.

Proposition 8.1. Let $x_i(x)$, $x \in X$ be arbitrary separable continuous in probability r.f. In order to it belongs to the Prokhorov-Skorokhod space $D[0,1]^d$ almost everywhere, is necessary and sufficient

$$\lim_{h \to 0^+} E \arctan \kappa[\xi](h) = 0. \quad (8.3)$$

Analogously may be formulated the criterion for weak compactness the family of distributions of stochastic continuous r.f. $\{\xi_n(x)\}$, $x \in X$ in the Prokhorov-Skorokhod space $D[0,1]^d$ with converges all the finite-dimensional distributions:

$$\lim_{h \to 0^+} \sup_n E \arctan \kappa[\xi_n](h) = 0. \quad (8.3)$$

B. Factorability of module of continuity.
It is known, [28], chapter 4, that for arbitrary continuous with probability one random field $\xi(x), x \in X$ its classical (uniform) module of continuity $\omega[\xi](h), h \in [0,1]$ allows a factorization: there exists a random variable $\theta$ and a non-random module of continuity $\omega_o = \omega_o(h), 0 \leq h \leq 1$, such that

$$\omega[\xi](h) \leq \theta \cdot \omega_o(h). \quad (8.4)$$

Of course, the converse conclusion is also true.

Analogous factorization, with at the same proof, remains true for the Prokhorov-Skorokhod space. In detail, the r.f. $\xi(x)$ belongs to the space $D[0,1]^d$ a.e., or equally

$$\lim_{h \to 0^+} \kappa[\xi](h) = 0 \pmod{P},$$

if and only if

$$\kappa[\xi](h) \leq \Theta \cdot \omega_0(h)$$

for some finite r.v. $\Theta$ and for certain non-random module of continuity $\omega_0(h)$.

C. Minimum estimates by means of $B(\phi)$ spaces.

The exponential tail estimates for distribution for minimum of the finite set of random variables based on the theory of so-called $B(\phi)$ spaces with an application to the theory of discontinuous random fields may be found in [28], chapter 3, section 3.16; [29],[36].

D. Compact embedded support.

Let $B$ be separable Banach space and let $\mu$ be sigma-finite Borelian measure on the space $B$. It is known, see [28], chapter 4; [31] that there exists a separable compact embedded Banach subspace $Q$ of the space $B$ which is complete support of the measure $\mu$:

$$\mu(B \setminus Q) = 0.$$  

Recall that the Banach subspace $Q$ is said to be compact embedded into Banach space $B$, if any bounded set of the space $Q$ is pre-compact set of the space $B$.

As for the Linear Topological Spaces: note that this proposition is not true still for the classical Schwartz space $S$ of infinite differentiable functions having a finite support.

It is more interesting to note that this statement holds true yet for the considered in this report Prokhorov-Skorokhod spaces $D[0,1]^d$, despite the ones are not Linear Topological Spaces!

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