THE LOCALIZED INDUCTION EQUATION, 
THE HEISENBERG CHAIN, 
AND THE 
NON-LINEAR SCHRÖDINGER EQUATION

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ABSTRACT. The three equations named in the title are examples of infinite-dimensional completely integrable Hamiltonian systems, and are related to each other via simple geometric constructions. In this paper, these interrelationships are further explained in terms of the recursion operator for the Localized Induction Equation, and the recursion operator is seen to play a variety of roles in key geometric variational formulas.

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0. INTRODUCTION

Among the natural variational integrals in geometry are the inevitable integrals on space curves $\gamma(s)$; these include length $L(\gamma) = \int ds$, total torsion $T(\gamma) = \int \tau ds$, total squared curvature $K(\gamma) = \frac{1}{2} \int \kappa^2 ds$, the integral $F(\gamma) = \frac{1}{2} \int \kappa^2 \tau ds = \frac{1}{2} \int \gamma_s \times \gamma_{ss} \cdot \gamma_{sss} ds$, and an infinite sequence of integrals following these four. A closely related sequence, for curves $T(\sigma)$ on the sphere, begins with total geodesic curvature $A(T) = \int \kappa_T ds_T$, and energy
$\mathcal{E}(T) = \int \frac{1}{2} |T_\sigma|^2 d\sigma$. (By the Gauss-Bonnet theorem we can view $A(T)$ as the spherical area bounded by $T$, when $T$ is closed). Though the whimsical term inevitable will be explained here by some simple geometric considerations, these sequences first arose via some physical models (which the geometric point of view has helped illuminate).

In fact, the length integral $L$ may be regarded as the Hamiltonian for a model of thin vortex tubes in three dimensional hydrodynamics, where the evolution of the tube’s centerline $\gamma(s,t)$ is governed by the Localized Induction Equation (LIE): $\gamma_t = \gamma_s \times \gamma_{ss} = \kappa B$ (see [M-W]). Then the remaining integrals in the first list are constants of motion for the evolving curves. Similarly, the energy $E$ is the Hamiltonian for the (continuous) Heisenberg Chain (HC) (see [F-T]), $T_t = T \times T_\sigma$, while $A$ and the integrals following $E$ are constants of motion for the curves $T(\sigma,t)$ evolving on the sphere.

The simple geometric relationship between these two soliton equations is this: if $\gamma$ satisfies (LIE), then the tangent indicatrix of $\gamma$, $T = \gamma_s$, satisfies (HC). By the way, it pays not to check this fact too hastily. After differentiating the first equation with respect to $s$, one wants to replace $\gamma_{ts}$ by $\gamma_{st} = T_t$ to get the left hand side of (HC). To justify this one has to observe that $W = \kappa B$ happens to belong to the special class of locally arclength preserving (LAP) variation vectorfields. $W$ is LAP if the speed along $\gamma$ is preserved under the evolution $\gamma_t = W$ – in particular, an arclength parameter $s$ remains an arclength parameter – which is equivalent to the condition that $W_s$ is orthogonal to $T$ (obviously satisfied by $W = \kappa B = T \times T_s$).
The two lists of constants are also related by the tangent indicatrix construction; if $T = \gamma_s$, then $T(\gamma) = \mathcal{A}(T), \mathcal{K}(\gamma) = \mathcal{E}(T)$, etc. (Note that the second list is ‘missing’ one integral, namely $\mathcal{L}$.) The two lists are essentially ‘equivalent’, but the first is of perhaps greater interest in the study of geometric variational problems. For example, one of the oldest problems in the calculus of variations is that of the Bernoulli elastic rod, to ‘minimize’ $\mathcal{K}$ with a constraint on $\mathcal{L}$. More generally, the extrema for sums $\alpha_1 \mathcal{L} + \alpha_2 T + \alpha_3 \mathcal{K}$, $\alpha_i$ constants, are precisely the (centerlines of) Kirchhoff elastic rods in equilibrium. (Actually, this simple characterization seems to be relatively unknown; the twist energy, which the Kirchhoff model adds to the Bernoulli model is given by a quadratic integral on framed curves [L-S]). To our knowledge, no one has classified the extrema for sums $\alpha_1 \mathcal{L} + \alpha_2 T + \alpha_3 \mathcal{K} + \alpha_4 \mathcal{F}$.

The first of these variational problems was introduced into the study of (LIE) in 1971 by Hasimoto, who observed that if $\gamma(s,0)$ represents a (Bernoulli) elastic rod in equilibrium, then $\gamma$ moves through space under (LIE) without changing shape [Has 1]. More generally (not part of Hasimoto’s original observation) such one-soliton solutions to (LIE) are precisely the equilibria for the Kirchhoff model.

With this beginning, Hasimoto went on to relate the filament equation itself to yet another physical model – already known to exhibit soliton behavior – the Non-linear Schrödinger Equation (NLS) : $-i\psi_t = (\psi_{ss} + \frac{1}{2}|\psi|^2\psi)$. Hasimoto showed that if a space curve $\gamma$ evolves according to (LIE), then the curvature and torsion of $\gamma$, $\kappa(s,t)$ and $\tau(s,t)$, can be combined into a complex curvature function $\psi(s,t) = \kappa e^{i \int \tau ds}$, which satisfies
(NLS) [Has 2]. The proof of this fact amounts to computing variational formulas for \(\kappa\) and \(\tau\) or equivalently, for the complex curvature \(\psi\). It turns out that these formulas have a special structure, which is related to the infinite list of constants of motion for (LIE), and which also clarifies the precise nature of the equivalence of the three models. The study of these formulas and their relatives, their geometric interpretation, and their geometric applications form the central theme of this paper.

To describe this special structure, let \(\mathcal{H}(\gamma) = \psi\) denote the Hasimoto transformation taking space curves to complex curvature functions. Then the desired variational formula for \(\psi\) is contained in the following general formula for the differential of \(\mathcal{H}\) at \(\gamma\) in the direction of \(W\), a LAP vectorfield:

\[
(1) \quad d\mathcal{H}(W) \equiv -\mathcal{Z}\mathcal{R}^2(W).
\]

The most important feature of this formula (whose proof is included below) is the appearance of the (squared) recursion operator \(\mathcal{R}\), a linear integro-differential operator on vectorfields. The operator \(\mathcal{Z}\) can be thought of as a simple isomorphism from LAP vectorfields to complex functions of \(s\); formulas for \(\mathcal{R}\) and \(\mathcal{Z}\) will be given below. We write \(\equiv\) to denote equivalence modulo addition of terms of the form \(ic\psi = ic\mathcal{H}(\gamma), c\) a real constant. Note we have defined \(\mathcal{H}\) itself only up to multiplicative factors of the form \(e^{i\theta}\), \(\theta\) a ‘constant of integration’ – this ambiguity is not reflected in \(\kappa\) or \(\tau\). For a varying curve, differentiation of \(\theta = \theta(t)\) produces an extra term \(ic\psi\) in the variation of \(\psi\). Setting \(W = \kappa B\) in (1) easily yields \(\psi_t = d\mathcal{H}(W) = i(\psi_{ss} + \frac{1}{2}(|\psi|^2 + c(t)))\psi\), which is the precise statement of Hasimoto’s result.
1. THE RECURSION OPERATOR AND NATURAL FRAMES

To discuss formulas for $R$ and $Z$ we use natural frames along $\gamma$, in place of the standard Frenet frame $\mathcal{F} = \{T, N, B\}$. Recall the Frenet equations may be written in the form $T_s = \Omega \times T, N_s = \Omega \times N, B_s = \Omega \times B$, where $\Omega$ is the Darboux vector (s-angular velocity of $\mathcal{F}$) given by $\omega = \tau T + \kappa B$. Similarly, the natural frames are those orthonormal frames $\mathcal{N} = \{T, U, V\}$ along $\gamma$ having s-angular velocity $\omega^s = -vU + uV$, for some ‘curvature functions’ $u = u(s)$ and $v = v(s)$; so $\mathcal{N}$ satisfies the ‘Frenet Equations’ $T_s = \omega^s \times T = uU + vV, U_s = -uT, V_s = -vT$. Note that $\omega^s = T \times T_s = \kappa B$, the vectorfield defining (LIE)! In other words, among all (adapted orthonormal) frames $\mathcal{N} = \{T, U, V\}$, the natural frames are characterized by: $\mathcal{N}$ has zero tangential component of angular velocity. One might prefer to call $\mathcal{N}$ natural if it has constant $T$-component of angular velocity – think of a bead sliding and turning without friction along a wire $\gamma$ – but we will not use this more general class here.

An alternate way to describe natural frames is to say that the vectors $\{U, V\}$ normal to $\gamma$ are obtained by parallel translation in the sphere along the tangent indicatrix $T$. Note that while $\mathcal{F}$ is uniquely determined, $\mathcal{N}$ is determined only after an ‘initial frame’ has been chosen at some point along $T$. An advantage of $\mathcal{N}$ (which plays no role presently) is that $\mathcal{N}$ can be defined even where $\gamma$ has vanishing curvature.

We give formulas for $Z(W)$ and $R(W)$, where $W = aT + bU + cV$, and we also provide a simpler expression for the complex curvature function $\psi$: 

(LIE), (HC), AND (NLS)
\[
\psi = \mathcal{H}(\gamma) = u + iv,
\]
\[
Z(W) = b + ic,
\]
\[
R(W) = -\mathcal{P}(T \times W_s),
\]
\[
\mathcal{P}(W) = (\int bu + cv \, ds) T + bU + cV
\]

Here we have also introduced a *parameterization operator* \( \mathcal{P} \), which leaves the normal part of an arbitrary vectorfield \( W \) alone but turns \( W \) into a LAP vectorfield (since \( \frac{d}{ds} \mathcal{P}(W) \) has no \( T \) component). It follows that the recursion operator \( R \) preserves the class of LAP vectorfields. Note that we are again being casual about constants of integration; \( \mathcal{P}(W) \) has only been defined up to addition of terms of the form constant \( \cdot T \), and by the same token, \( Z \) is not quite an isomorphism of LAP vectorfields.

Until now, we haven’t said much about the meaning of the recursion operator \( R \) – we have only indicated that it allows for a very compact expression for the variation of the complex curvature. To get some experience with \( R \), let’s apply it to the humblest of all LAP vectorfields: \( R(T) = -\mathcal{P}(T \times T_s) = -\mathcal{P}(\kappa B) = -\kappa B \), an old friend! In this computation we simply ignored \( \mathcal{P} \) since \( \kappa B \) is already LAP.

Since this worked out so nicely, let’s apply \( R \) again (keeping in mind that tangential terms inside of \( \mathcal{P}() \) can be discarded): \( R^2(T) = R(-\kappa B) = \mathcal{P}(T \times (T \times T_s)_s) = \mathcal{P}(-T_{ss}) = -\mathcal{P}(u_s U + v_s V) = -\int (u_s u + v_s v) ds T - u_s U - v_s V = -\frac{1}{2} (u^2 + v^2) T - u_s U - v_s V \). Note our good fortune in being able to compute the antiderivative explicitly! As it turns out, this will continue to happen as we successively apply \( R \), and the LAP
vectorfields $X_n = \mathcal{R}^n(-T), n = 1, 2, 3, \ldots$, are none other than the Hamiltonian vector-fields of the constants of motion for the Localized Induction Equation – hence the names “recursion operator” and “inevitable integrals”.

2. THE VARIATIONAL FORMULAS

We have explained the name, but only part of the significance of $\mathcal{R}$. Recall that the formula (1) relates variations in the ‘first’ model (LIE) to variations in the ‘third’ model (NLS). Since the square of $\mathcal{R}$ occurs in this formula, it seems reasonable that a first power of $\mathcal{R}$ ought to relate the ‘second’ model (HC) – this is clearly the intermediate model – to (LIE) and (NLS). To see how this is true, it is useful to modify the (HC) model slightly, replacing the curve $T$ by a natural frame $\mathcal{N} = \{T, U, V\}$. Note that in doing so, we have only enlarged our ‘state space’ by one dimension, corresponding to our usual ambiguity (say, the choice of ‘initial frame’ $\{U_0, V_0\}$, or choice of ‘phase factor’ $e^{i\theta}$ for $\psi$).

Now suppose $\mathcal{N}(s, t)$ evolves in time according to $T_t = \omega_t \times T, U_t = \omega_t \times U, V_t = \omega_t \times V$, for some $t$-angular velocity $\omega_t = \omega_t(s, t)$. For $\mathcal{N}$ to represent an evolving natural frame, it must satisfy $0 = U_s \cdot V$, for all $s, t$. It follows that for $\omega_t$ to be naturality-preserving (NP) it must satisfy: $0 = (U_s \cdot V)_t = U_{ts} \cdot V + U_s \cdot V_t = (\omega_t \times U)_s \cdot V + U_s \cdot (\omega_t \times V) = (\omega_t)_s \times U \cdot V = (\omega_t)_s \cdot T$, a familiar condition! In other words, allowing the same vectorfield $(aT + bU + cV)$ to play two different roles: $W = aT + bU + cV$ is LAP if and only if $\omega_t = aT + bU + cV$ is NP. In particular, this means that if $\omega_t$ is NP, so is $\mathcal{R}^n(\omega_t), n = 1, 2, 3, \ldots$. 
Without further ado, we state the promised relationship between the variations of curves and variations of the corresponding natural frames: \textit{If } $W$ \textit{is LAP, and } $\gamma(s,t)$ \textit{evolves according to } $\gamma_t = W$, \textit{then the } $t$\textit{-angular velocity of a natural frame } $N = \{T,U,V\}$ \textit{along } $\gamma$ \textit{is given by}

\begin{equation}
\omega^t = -R W
\end{equation}

\textbf{Proof:} Since $W = aT + bU + cV$ is assumed to be LAP, we have $\gamma_{st} = \gamma_{ts}$, so $\omega^t \times T = T_t = \gamma_{ts} = (aT + bU + cV)_s = (au + bs)U + (av + cs)V$. It follows that $\omega^t = \eta T - (av + cs)U + (au + bs)V$, for some function $\eta = \eta(s,t)$. To determine $\eta$, we compare expressions for $U_{st}$ and $U_{ts}$: $U_{st} = (-uT)_t = -u_tT - u(au + bs)U - u(av + cs)V$, while $U_{ts} = (\omega^t \times U)_s = ((\eta T + (au + bs)V) \times U)_s = -(au + bs)T + \eta V)_s = -(bs_s + asu + au_s + \eta v) T - u(au + bs)U + (\eta_s - auv - vbs)V$, so equating $V$-coefficients gives $\eta_s - bs v = -uc_s$. Therefore $\omega^t = (\int vb_s - uc_s ds)T - (av + cs)U + (au + bs)V$. Finally,

$-R(W) = \mathcal{P}(T \times (aT + bU + cV)_s) = \mathcal{P}(T \times ((au + bs)U + (av + cs)V) = \mathcal{P}(-(av + cs)U + (au + bs)V) = (\int -(av + cs)u + (au + bs)v ds)T - (av + cs)U + (au + bs)V = \omega^t$.

It remains to see how $\mathcal{R}$ takes us from the second model to the third model. Given a complex function $\psi = \psi(s,t)$, we can always write the $t$-derivative $\psi_t$ in the form $\psi_t = -Z(\phi^t)$, for some vector field $\phi^t = aT + bU + cV$. Of course, $\phi^t$ is not unique since $Z$ ignores tangential terms. But if $\phi^t$ is also required to be LAP, then it will be unique up to the usual terms, \textit{constant } $\cdot$ $T$. Now the question is, if a natural frame $N = \{T,U,V\}$ evolves with $t$-angular velocity $\omega^t$, and if $\psi = u + iv = Z(uU + vV) = Z(T_s)$ evolves
accordingly, what is the vectorfield $\phi^t$? The required computation is simplified by the abuse of notation $Z = U + iV, Z(W) = Z \cdot W$ (complex inner product): $\psi_t = (Z \cdot T_s)_t = Z_t \cdot T_s + Z \cdot T_{ts} = \omega^t \times Z \cdot T_s + Z \cdot (\omega^t \times T)_s = Z \cdot (-T \times (\omega^t)_s) = Z(\mathcal{R}(\omega^t))$. In other words, we have the following formula relating the variations in natural frames and complex wave functions:

$$\phi^t = -\mathcal{R}(\omega^t).$$

Finally, note that in case $\mathcal{N}(s,t)$ comes from a curve $\gamma(s, t)$, and $\gamma_t = W$ is LAP, we have $\psi_t = Z(\mathcal{R}(\omega^t)) = Z(\mathcal{R}(\mathcal{R}(W))) = -Z\mathcal{R}^2(W)$, which proves (1).

In order to summarize the various roles of the recursion operator $\mathcal{R}$ we have included a diagram below. Note that $\mathcal{R}$ connects levels corresponding to the three models (LIE), (HC), and (NLS) (for the actual equations (LIE), (HC), and (NLS), set $n = 1$) and $\mathcal{R}$ also connects $t$ and $s$ directions of motion. Some of the arrows have not been explained, but are easily interpreted and checked.

$$\begin{array}{c}
\gamma_s \xrightarrow{-\mathcal{R}^n} \gamma_t \\
\downarrow \quad \downarrow \quad \downarrow \\
\omega^s \xrightarrow{-\mathcal{R}^n} \omega^t \\
\downarrow \quad \downarrow \quad \downarrow \\
\phi^s \xrightarrow{-\mathcal{R}^n} \phi^t
\end{array}$$

We have not discussed here the Hamiltonian nature of the equations (LIE), (HC), or (NLS), or the actual definitions of the spaces on which these Hamiltonian flows are defined. For background on such points see [L-P 2], where (1) is first proved and used to
establish the precise nature of the Hasimoto transformation as a Poisson map. Arguments similar to those used in [L-P 2] can be combined with formulas (3) and (4) to give similar characterizations of the intermediate maps discussed here (thus interpreting $\mathcal{H}$ as a composite of Poisson maps). However, our intent here has been to emphasize the ubiquity of the recursion operator $\mathcal{R}$ and its relation to geometric variational problems and flows of geometric origin.

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