Some interesting class of integrable partial differential equation systems

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Abstract

We determine a considerable class of nonlinear partial differential equation systems, which have global regular solutions. Uniqueness is not a direct general consequence of this method. The scheme can be applied to the incompressible Navier Stokes equation.

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1 Definition of an integrable class of nonlinear partial differential equation systems

For a positive viscosity constant \( \nu > 0 \) consider the Cauchy problem

\[
v_{i,t} - \nu \Delta v_{i} + f_{i}(v, \nabla v) = 0, \quad 1 \leq i \leq D, \tag{1}
\]

where \( v = (v_{1}, \cdots, v_{D})^{T} \) and \( \nabla v = (\nabla v_{1}, \cdots, \nabla v_{D})^{T} \) on the domain \([0, \infty) \times \mathbb{R}^{D}\) with data

\[
v_{i}(0, .) = h_{i}, \quad 1 \leq i \leq D. \tag{2}
\]

Here, \( D \geq 3 \) denotes the dimension of the problem and \( \mathbb{R} \) denotes the field of real numbers.

Remark 1.1. The operators \( f_{i} \) in (1) can be global operators. In case of global operators certain conditions which determine the class are formulated with respect to global norms.

Assume that the following conditions are satisfied.

a) For \( m \geq 2 \) and \( 1 \leq i \leq D \) the data satisfy

\[
h_{i} \in H^{m} \cap C^{m}. \tag{3}
\]

Here, \( C^{m} \) is the function space of continuous functions with continuous multivariate derivatives up to order \( m \), and \( H^{m} \) is the standard Sobolev space of order \( m \). For \( m = 0 \) \( C \) is the set of continuous functions. We shall use the norm we have

\[
|h_{i}|_{H^{m} \cap C^{m}} = \sum_{0 \leq |\alpha| \leq m} |D_{\alpha}^{2} h_{i}|_{L^{2} \cap C}, \tag{4}
\]
where for a continuous and bounded function \( g \) we may define the norm

\[
|g|_{L^2(\mathbb{R}^n)} := \max \left\{ |g|_{L^2}, \sup_{x \in \mathbb{R}^n} |g(x)| \right\}.
\]  

(5)

If \( \Omega \subset \mathbb{R}^n \) is compact, then \( |g|_{L^2(\Omega) \cap C(\Omega)} \) denotes a local form of this norm.

In general we need local conditions, and do not even need the closure of the space related to the norm \( |g|_{L^2(\mathbb{R}^n)} \).

b) The nonlinear terms \( f_i \) satisfy a sub-homogeneity condition in the sense that for spatial scaling \( y = rx \), \( r > 0 \), and for functions \( v_i \), \( 1 \leq i \leq D \) with \( v_i(t,.) \in H^m \cap C^m \), \( m \geq 2 \), we have for all \( 1 \leq i \leq D \), all \( t \geq 0 \), and all \( x = ry \in \mathbb{R}^D \)

\[
|f_i(v(t, x), \nabla v(t, x))| \leq r |f_i(v^r(t, x), \frac{1}{r} \nabla v(t, x))|.
\]  

(6)

Here, \( r \) is a positive real number. Note that for the transformation \( v_i^r(\tau, y) = v_i(t, x) \) we have

\[
r f_i \left( v(t, x), \frac{1}{r} \nabla v(t, x) \right) = r f_i(v^r(\tau, y), \nabla v^r(t, y)).
\]  

(7)

c) In the following for functions \( F_j \) we use the notation

\[
F_{j, i}(v, \nabla v) := \frac{\partial}{\partial x_j} F_j^i(v(t, x), \nabla v(t, x)),
\]

i.e., the notation refers to derivatives with respect to the argument \( x_j \) of the composition; some readers may prefer the notation

\[
\frac{\partial}{\partial x_j} F_j^i(v(t, x), \nabla v(t, x)) = \left( F_j^i(v(t, x), \nabla v(t, x)) \right)_{, j},
\]

which we avoid for simplicity of notation. If \( F_j \) has a local interpretation, then derivatives with respect to the arguments of \( F_j \) may be denoted by \( D_{\alpha, \nabla} F_j^i \), where in this case \( \gamma \) is a multiindex of length \( D + D^2 \). Note that these are usual multivariate derivatives. We assume that there exists a matrix of regular functions \( (F_j^i)_{1 \leq i,j \leq D} \) such that for all \( 1 \leq i \leq D \)

\[
\sum_{j=1}^{D} F_{j, i}(v, \nabla v) = f_i(v, \nabla v).
\]  

(8)

Furthermore, if \( F_j \) has a local interpretation such that for all multiindices \( 0 \leq |\beta| \leq m \) the functions \( D_{\alpha, \nabla} F_j^i : \mathbb{R}^{D+D^2} \to \mathbb{R} \) are locally Lipschitz continuous (on any compact domain in \( D \subset \mathbb{R}^D \) and with respect to all arguments). If an operator \( F_j^i \) is global, then we assume Lipschitz continuity with respect to the norm \( |.|_{L^2(\mathbb{R}^n)} \), i.e., in this case we assume that for functions \( u, u' \in H^m \cap C^m \) and for \( 0 \leq |\beta| \leq m \) we have

\[
|D_{\beta}^j F_j^i(u, \nabla u)(.) - D_{\beta}^j F_j^i(u', \nabla u')(.)|_{L^2(\mathbb{R}^n)} \leq L |D_{\beta}^j u - D_{\beta}^j u'|_{L^2(\mathbb{R}^n)}.  
\]  

(9)
for some finite constant $L > 0$. Note that this implies a similar (one order lower) condition for $f_i$ for all $0 \leq |\gamma| \leq m - 1$.

In general, in $\mathbb{R}^3$ a local form with respect to norms $\| \cdot \|_{L^2(\Omega) \cap C(\Omega)}$ for all compact $\Omega \subseteq \mathbb{R}^n$ is sufficient for our purposes. For technical reasons we also require that for any $g = (g_1, \cdots, g_D)$ with $|g_i|_{H^m \cap C^m} \leq C < \infty$ we have Lipschitz continuity of $g \to D^2_j F^i_j (g(y), \nabla g(y))$ with finite Lipschitz constants $L^\gamma_j$ (dependent on $C$) in the sense that for all $1 \leq i, j \leq D$ and all $0 \leq |\gamma| \leq m - 1$ and $y, y' \in \mathbb{R}^D$

$$\left| F^i_j (g(y), \nabla g(y)) - F^i_j (g(y'), \nabla g(y')) \right| \leq L^0_j |y - y'|,$$  \hspace{1cm} (10)

and

$$\left| D^2_j F^i_{j,j} (g(y), \nabla g(y)) - D^2_j F^i_{j,j} (g(y'), \nabla g(y')) \right| \leq L^\gamma_j |y - y'|.$$ \hspace{1cm} (11)

Here, $D^2_j$ denotes the multivariate spatial derivative with respect to the multiindex $\gamma = (\gamma_1, \cdots, \gamma_D)$ and with respect to the argument of $g$.

d) The verification of the technical condition in c) can be simplified for specific models if we add a stronger assumption. This additional assumption is also useful, if generalisations of the diffusion term or viscosity limits are considered, e.g., if we replace the Laplacian term by a H"ormander vector-field condition for highly degenerate operators of second order. Let

$$h_i \in H^m \cap C^m \cap C_\text{pol,m}^{m(D+1)},$$ \hspace{1cm} (12)

along with

$$C_\text{pol,m}^{l} = \left\{ f : \mathbb{R}^D \to \mathbb{R} : \exists c > 0 \forall \|x\| \geq 1 \forall 0 \leq |\gamma| \leq m \left| D^\gamma f(x) \right| \leq \frac{c}{1 + |x|^l} \right\}$$ \hspace{1cm} (13)

for $l \geq 1$. Note that the latter function space has a multiplicative property, i.e., $g, h \in C_\text{pol,m}^{l}$ implies that $gh \in C_\text{pol,m}^{l}$. In addition to $C_\text{pol,m}^{l}$, we then require a submultiplicative property of order $k \in \{0, \cdots, m(D + 1) - 1\}$, i.e., for $g = (g_1, \cdots, g_D)$ along with $g_i \in C_\text{pol,m}^{m(D+1)}$ we require that for some $k \in \{0, \cdots, m(D + 1) - 1\}$ we have

$$g_i \in C_\text{pol,m}^{m(D+1)}, (\nabla g_i) \in C_\text{pol,m}^{m(D+1)} \implies D^\beta F^i_j (g, \nabla g) \in C_\text{pol,m}^{2m(D+1) - k}.$$ \hspace{1cm} (14)

seems sufficient. This 'multiplicative property' of the nonlinear term holds for the incompressible Navier-Stokes equation operator. The number $m(D + 1)$ in the upperscript of $C_\text{pol,m}^{m(D+1)}$ is not a sharp choice, but a choice which may be also sufficient for the consideration of viscosity limits.

We have

**Theorem 1.2.** If the set of conditions a), b), c) or the stronger set of conditions a), b), c) and d) are satisfied, then the Cauchy problem has a global classical solution $v_i \in C^1 ([0, \infty), H^m \cap C^m), 1 \leq i \leq D$.

Some remarks are in order.
Remark 1.3. We have described a quadratic system where a vector \( v \) with \( D \) components solves \( D \) equations. This is not an essential restriction, and it is convenient.

Remark 1.4. We do not claim uniqueness in [12] although in many special situations standard arguments may lead to uniqueness. Note that in some situations the method may be applied in order to get global solution branches of viscosity limits of the equations in [1], and then there are examples, where determinism is lost and global regular solution branches exist next to singular solutions.

Remark 1.5. We use Gaussian upper bound estimate. We cannot prove a strong contraction semi-group property for a class of nonlinear operators which includes the Navier Stokes equation operator. The deviation from a strong semigroup property is indicated directly, if we estimate for \( u \pm d \) or the weaker assumption that for 

Remark 1.6. We may reduce to a local Lipschitz continuity assumption in c) if we add d) or the weaker assumption that for \( u = (u_1, \cdots, u_d) \) with \( u_i \in H^m \cap C^m \) for \( m \geq 2 \) we have that \( F^i_j(u, \nabla u) \in L^2 \).

Example 1.7. In case of the incompressible Navier Stokes equation (cf. [1] for the modeling) the incompressibility condition implies

\[
\sum_{j=1}^{D} \frac{\partial (v_i v_j)}{\partial x_j} = \sum_{j=1}^{D} v_j \frac{\partial v_i}{\partial x_j} + v_i \sum_{j=1}^{D} \frac{\partial v_j}{\partial x_j} = \sum_{j=1}^{D} v_j \frac{\partial v_i}{\partial x_j} 
\]

(15)

Therefore we may define

\[
F^i_j = v_j v_i - \delta_{ij} K_D *_{sp} \sum_{l,m=1}^{D} v_{l,m} v_{m,l}, 
\]

(16)

where \( K_D \) denotes the Laplacian kernel of dimension \( D \) and \( \delta_{ij} \) is the Kronecker-\( \delta \). Here \( *_{sp} \) denotes the convolution with respect to the spatial variables. Note that for \( y = rx, v^i_j(t, y) = v_j(t, x) \), and \( z = rw, \) and the first spatial derivative of the Laplacian kernel \( x \rightarrow K_D(x) = \frac{x}{|x|^D} \) the Leray projection term (written as an operator on the Jacobian \( J(v) = (v_{l,m})_{1 \leq l,m \leq D} \) term satisfies

\[
L(J(v)) = \int_{\mathbb{R}^D} \frac{z - w}{|z - w|} \sum_{l,m=1}^{D} v_{l,m}(t, w) v_{m,l}(t, w) dw \\
= \int_{\mathbb{R}^D} \frac{z}{|z|} \sum_{l,m=1}^{D} r v^r_{l,m}(t, z) r v^r_{m,l}(t, z) \frac{1}{r^D} dz
\]

(17)

This linear spatial scaling of the Leray projection term can be observed also from the linear scaling of the pressure gradient. The Poisson equation for the scaled equation becomes \( r^2 \sum_{l,m=1}^{D} v^r_{l,m} v^r_{m,l} = r^2 \Delta p \) and is identical to the pressure elimination equation in original coordinates as the factor \( r^2 \) cancels. If the additional assumption d) holds, then the technical Lipschitz condition in c) is
rather obvious, but it can also be verified under the weaker set of assumptions a), b) and c). Furthermore, a local Lipschitz condition with respect to the \( |.|_{L^2(\Omega)} \cap C(\Omega) \)-norm is satisfied for compact \( \Omega \subset \mathbb{R}^n \) (needed in the context of local time contraction results). Application of Theorem 1.2 implies the existence of a global regular solution branch. In this special case uniqueness is implied.

2 Proof Theorem 1.2

i) We rewrite the equation using the integrability of the nonlinear terms. From (1) and assumption c) we have

\[
v_{i,t} - \nu \Delta v_i + \sum_{j=1}^{D} F_{j,j}^{i}(v, \nabla v) = 0, \quad 1 \leq i \leq D. \tag{18}
\]

Recall the notation \( F_{j,j}^{i}(v, \nabla v) = \left( F_{j}^{i}(v, \nabla v) \right)_j \), where the derivative is with respect to the \( j \)th spatial argument \( x_j \) of the composition of \( F_{j}^{i} \) with \((v, \nabla v)\).

ii) Assume that data \( v_i(t_0, .), \quad 1 \leq i \leq D \) are given. We have local time contraction in spatial \( H^m \cap C^m \) space. More precisely, define a time local iteration scheme \( v_{i}^{k}, \quad 1 \leq i \leq D, \quad k \geq 0 \) in a time interval \([t_0, t_0 + \Delta]\), where \( t_0 \geq 0 \) and

\[
v_i^0(t_0, .) = v_i(t_0, .), \quad (\text{n.b.} \quad v_i^0(0, .) = h_i(\cdot)), \tag{19}
\]

and \( v_i^{k} \) is a solution of the linearized equation

\[
v_{i,t}^{k} - \nu \Delta v_i^{k} + \sum_{j=1}^{D} F_{j,j}^{i}(v^{k-1}, \nabla v^{k-1}) = 0, \quad 1 \leq i \leq D. \tag{20}
\]

Lemma 2.1. Let \( t_0 \geq 0 \) and assume that for some \( m \geq 2 \) and a finite constant \( C_0 > 0 \) we have

\[
|v_i(t_0, .)|_{H^m \cap C^m} \leq C_0. \tag{21}
\]

Then there exists a \( \Delta > 0 \) dependent only on dimension and on the constant \( |v_i(t_0, .)|_{H^m \cap C^m} \) such that on the time interval \([t_0, t_0 + \Delta]\) the functional increments \( \delta v_j^{k+1} = v_j^{k+1} - v_j^{k}, \quad 1 \leq j \leq D \) satisfy

\[
\sup_{t \in [t_0, t_0 + \Delta]} |\delta v_j^{k+1}(t, .)|_{H^m \cap C^m} \leq \frac{1}{2} \sup_{t \in [t_0, t_0 + \Delta]} |\delta v_j^{k}(t, .)|_{H^m \cap C^m} \tag{22}
\]

and

\[
\sup_{t \in [t_0, t_0 + \Delta]} |\delta v_1^{k}(t, .)|_{H^m \cap C^m} \leq \frac{1}{2}. \tag{23}
\]

The proof uses classical representations and the Lipschitz continuity of (spatial derivatives of) \( F_{j}^{i} \) and is given in the appendix.

iii) The local time contraction result of item ii) implies that there exist regular local time solutions \( v_i \in C^1([t_0, t_0 + \Delta], H^m \cap C^m), \quad 1 \leq i \leq D. \)
Remark 2.2. If the \( f_i \) satisfy the condition in d), then we have

\[
v_i^k \in C^4 \left( [t_0, t_0 + \Delta], H^m \cap C^m \cap C_{pol,m}^{m(D+1)} \right), 1 \leq i \leq D,
\]

and this holds also in the limit for \( v_i, 1 \leq i \leq D \). Here note that convolutions with a first order spatial derivative of a Gaussian has an upper bound which decreases the order of polynomial decay by one order. For more complicated second order diffusions we have a convolution with a density which can cause a stronger decrease of polynomial order decrease at spatial infinity, and we need a stronger submultiplicative property (smaller \( m \) in assumption d).

We consider local classical representations of solutions and use the convolution rule in order to get for all \( t \in [t_0, t_0 + \Delta] \) and \( x \in \mathbb{R}^D \)

\[
v_i(t, x) = v_i(t_0, .) \ast_{sp} G_\nu + \sum_{j=1}^D F_{j,j}^i (v, \nabla v) \ast G_\nu
\]

\[
= v_i(t_0, .) \ast_{sp} G_\nu + \sum_{j=1}^D F_{j,j}^i (v, \nabla v) \ast G_{\nu,j}
\]

Again remember that \( ., j \) refers to derivative with respect to \( x_j \), where we use the notation \( F_{j,j}^i (v, \nabla v) = (F_{j,j}^i (v, \nabla v)) \). Furthermore, the symbol \( \ast \) denotes convolution with respect to space and time. In the last line the nonlinear terms are convoluted with first order spatial derivatives of the Gaussian, while the first term on the right side of (24) is a convolution with the Gaussian, which behaves completely different. We shall observe that small damping of the latter term offsets possible growth caused by the former. Note that for spatial multivariate derivatives of order \(|\beta|\) and for \( 0 \leq |\gamma| + 1 = |\beta| \leq m \), \( \beta_k = \gamma_k + 1 \), and \( \beta_l = \gamma_l \) for \( l \neq k \) we have representations of the form

\[
D_x^\beta v_i(t, x) = D_x^\beta v_i(t_0, .) \ast_{sp} G_\nu + \sum_{j=1}^D D_x^\gamma F_{j,j}^i (v, \nabla v) \ast G_{\nu,k}.
\]

iv) Using Lipschitz estimates we get an upper bound of the nonlinear terms. First note that first order spatial derivatives of the Gaussian \( G_{\nu,j} \) have a symmetry which can be combined with Lipschitz continuity of the nonlinear function terms. We have

\[
G_{\nu,j}(t, y) = \frac{-y_j}{2\pi \nu t} \frac{1}{\sqrt{4\pi t}} \exp \left( -\frac{|y|^2}{4\nu t} \right).
\]

Define \( y^{j-} = (y_1^{j-}, \ldots, y_{j-}^{j-}) \), where \( y_k^{j-} = y_k \) for \( k \neq j \) and \( y_j^{j-} = -y_j \). Assuming \( |v_i(t_0, .)|_{H^{\gamma} \cap C^m} = C_{t_0} < \infty \) the local contraction results tells us that for some \( \Delta > 0 \) and all time \( t \in [t_0, t_0 + \Delta] \) we have

\[
|v_i(t, .)|_{C^m \cap H^m} \leq C_{t_0} + 1.
\]

Recall that functions \( F_{j,j}^i : \mathbb{R}^{D+D^2} \rightarrow \mathbb{R} \) are Lipschitz on any finite ball, especially on a ball of \( \mathcal{B}_{C_{t_0} + 1}(0) \) of radius \( C_{t_0} + 1 \) around zero. It follows
that for all $1 \leq i, j \leq D$
\[
\left| \int_{B^D} F_{ij}^i(v, \nabla v)(t, x-y) \frac{e^{-|y|^2}}{2\pi v t} \exp\left(-\frac{|y|^2}{4vt}\right) dy \right|
\]
\[
= \left| \left( \int_{B^D} F_{ij}^i(v, \nabla v)(t, x-y) - F_{ij}^j(v, \nabla v)(t, x-y^{i^-}) \right) \times\right.
\]
\[
\times \frac{e^{-|y|^2}}{4vt} \exp\left(-\frac{|y|^2}{4vt}\right) dy \right|
\]
\[
\leq 2L_j \left| \int_{B^D} \frac{y_j^2}{4vt} \exp\left(-\frac{|y|^2}{4vt}\right) dy \right|,
\]
where we use assumption c). Analogous estimates hold for spatial derivatives with the related Lipschitz constants $L_j$. 

v) The nonlinear upper bound of item iv) has a scaling which is different from a normal Gaussian. Moreover the linear and the nonlinear part of the equation have scaling constraints due to the sub homogeneity condition. For $y = \tau \rho$ and $x = y$, and $v_{\rho,r}^{\rho,r}(\tau, y) = v_i(t, x)$ we have $v_{i,t} = v_{\rho,r}^{\rho,r} \frac{d\tau}{dt} = v_{\rho,r}^{\rho,r} \frac{\rho}{\rho, r}, \quad v_{i,j} = v_{\rho,r}^{\rho,r} r, \quad$ and $v_{i,j,j} = v_{\rho,r}^{\rho,r} r^2$.
\[
v_{\rho,r}^{\rho,r} = \rho^2 \nu \Delta v_{\rho,r}^{\rho,r} + \rho f_i(v_{\rho,r}^{\rho,r}, \nabla v_{\rho,r}^{\rho,r}) = 0, \quad 1 \leq i \leq D.
\] (29)

Now,
\[
f_i(v_{\rho,r}^{\rho,r}(t, y), \nabla v_{\rho,r}^{\rho,r}(t, y)) = f_i(v(t, x), \nabla v(t, x))
\]
\[
\leq r f_i(v(t, x), \nabla v(t, x)) = r f_i(v^v(t, y), \nabla v^v(t, y)).
\] (30)

Hence,
\[
v_{\rho,r}^{\rho,r} = \rho^2 \nu \Delta v_{\rho,r}^{\rho,r} + \rho \sum_{j=1}^{D} F_{j,i}^i(v_{\rho,r}^{\rho,r}, \nabla v_{\rho,r}^{\rho,r}) = 0, \quad 1 \leq i \leq D,
\] (31)

where
\[
|\rho \sum_{j=1}^{D} F_{j,i}^i(v_{\rho,r}^{\rho,r}, \nabla v_{\rho,r}^{\rho,r})| = |\rho f_i(v_{\rho,r}^{\rho,r}, \nabla v_{\rho,r}^{\rho,r})|
\]
\[
\leq \rho r |f_i(v_{\rho,r}^{\rho,r}, \nabla v_{\rho,r}^{\rho,r})| = \rho r |\sum_{j=1}^{D} F_{j,i}^i(v_{\rho,r}^{\rho,r}, \nabla v_{\rho,r}^{\rho,r})|.
\] (32)

The local contraction result transfers to the scaled situation. It follows that on a time interval $[t_0, t_0 + \Delta_0]$ with $\Delta_0 = \rho \Delta$ we have
\[
\sup_{t \in [t_0, t_0+\Delta_0]} \left| v_i(t, .) \right|_{H_{m+1} \cap C_m} \left| v_i(t_0, .) \right|_{H_{m+1} \cap C_m} + 1 := C_{t_0 m + 1}
\] (33)

Note that this holds for $t_0 = 0$ especially.

For the scaled function we have the local representation
\[
v_{\rho,r}^{\rho,r}(\tau, x) = v_{\rho,r}^{\rho,r}(t_0, .) *_{sp} G_{\rho,r}^{\rho,r} + \rho r \sum_{j=1}^{D} F_{j,i}^i(v_{\rho,r}^{\rho,r}, \nabla v_{\rho,r}^{\rho,r}) * G_{\rho,r}^{\rho,r}
\]
\[
= v_{\rho,r}^{\rho,r}(t_0, .) *_{sp} G_{\rho,r}^{\rho,r} + \rho r \sum_{j=1}^{D} F_{j,i}^i(v_{\rho,r}^{\rho,r}, \nabla v) * G_{\rho,r}^{\rho,r}
\] (34)
For the scaled Gaussian $G^{\rho,r}_{\nu,i}(\tau, x) := G_{\nu}(t, y)$ we have

$$
\left| G^{\rho,r}_{\nu,i}(\tau, y) \right| \leq \frac{2}{(4\pi \rho r^2 \nu)^{\delta} |y|} \left( \frac{|y|^2}{4\pi \rho r^2 \nu} \right)^{D/2 + 1 - \delta} \exp \left( - \frac{|y|^2}{4\pi \rho r^2 \nu} \right) \tag{35}
$$

Hence we have for $\delta \in (0, 1)$ and all $\rho, r > 0$

$$
\left| G^{\rho,r}_{\nu,i}(\tau, y) \right| \leq \frac{C}{(4\pi \rho r^2 \nu)^{\delta} |y|^{D+1-2\delta}}, \tag{36}
$$

where the upper bound constant

$$
C = \sup_{|z| > 0} (z)^{D/2 + 1 - \delta} \exp \left( - z^2 \right) > 0
$$

is sufficient and independent of $\nu > 0$. We are interested in the scaling of convolutions of these Gaussians with Lipschitz continuous functions $y \to l(x - y)$ with upper bound $l_0|y|$ (with a Lipschitz constant $l_0$ independent of $x$). Now let the spatial parameter $r$ be large enough such that

$$
4\pi \rho r^2 \nu \geq 1. \tag{37}
$$

Using the abbreviations $\sigma_\tau = \tau - \sigma$, and $B = \{ y ||y| \leq 4\pi \rho r^2 \nu \}$, we find

$$
\int_0^\tau \int_B |y| |G^{\rho,r}_{\nu,i}(\sigma_\tau, y)| dyd\sigma \leq \int_0^\tau \int_B \left( \frac{|y|^2}{4\pi \rho r^2 \nu} \right)^{\delta - D/2} \left( \frac{|y|^2}{4\pi \rho r^2 \nu} \right)^{D/2 + 1 - \delta} \exp \left( - \frac{|y|^2}{4\pi \rho r^2 \nu} \right) dyd\sigma
$$

$$
\leq \int_0^\tau \int_B \left( \frac{|y|^2}{4\pi \rho r^2 \nu} \right)^{\delta - D/2} C dyd\sigma \leq l_0 C^* (\tau - t_0)^{1 - \delta} (4\pi \rho r^2)^{\delta}, \tag{38}
$$

for any $\delta \in (0, 1)$ and for some finite constant $C^*$ which is independent of the parameters $\rho, r, \nu$ (radial coordinates with measure $r^{D-1}dr$ may be used for the latter observation). Note that for small time and/or small parameters the corresponding integral on the complementary domain $\int_0^\tau \int_{B^c} \cdots$ becomes small such that \textit{28} is indeed the essential estimate in the sense that it gives the whole upper bound up to an $\epsilon'$. Now we can estimate the nonlinear term rephrasing \textit{28} above and including time. Using the sub- homogeneity property we have the upper
vi) We use small damping of the heat convolution on infinite space, if a threshold is exceeded, i.e., we may assume that for the data we have for all

\[ C^* (\tau - t_0)^{1 - \delta} + \epsilon = 2D \int_{t_0}^{\tau+\Delta} \int_{B_1(0)} \frac{\rho \nu C}{\sigma^3} dy d\sigma \]

(40)

where we have a finite constant \( C^* \) with

\[ C^* (\tau - t_0)^{1 - \delta} + \epsilon = 2D \int_{t_0}^{\tau+\Delta} \int_{B_1(0)} \frac{\rho \nu C}{\sigma^3} dy d\sigma \]

(40)

Here \( \epsilon \) is defined by the second term on the right side of (39). The first summand on the right side of (40) is independent of \( r, \rho, \nu \). The last summand on the right side of (40) is relatively small (goes to zero with exponential decay as \( \rho r^2 \nu \) or \( \Delta_0 \) go to zero). Hence \( \epsilon \) is small compared to any of the polynomials of lower order which determine the main part of \( C^* \). Summing up we have

\[ \left| \sum_{j=1}^{D} \rho r \int_{t_0}^{\tau} \int_{R^D} F_j^i (v^r, \nabla v^r) (s, y - z) \times \right. \]

\[ \times \frac{-2z_i}{4\pi \nu (\tau - s)} \frac{1}{4\pi \nu (\tau - s)^{\frac{3}{2}}} \exp \left( -\frac{|z|^2}{4\pi \nu \tau (\tau - s)} \right) \]

(41)

where \( L_0 := \left( \sum_{j=1}^{D} 2L_{j0}^0 \right) \) and with an \( \epsilon > 0 \) of exponential decay with respect to \( \nu \rho r^2 \Delta_0 \) as \( \nu \rho r^2 \Delta_0 \) becomes small. Analogous considerations lead to an upper bound

\[ \left| \sum_{j=1}^{D} \rho r \int_{t_0}^{\tau} \int_{R^D} D^2 F_j^i (v^r, \nabla v^r) (\tau - s, y - z) \times \right. \]

\[ \times \frac{-2z_i}{4\pi \nu (\tau - s)} \frac{1}{4\pi \nu (\tau - s)^{\frac{3}{2}}} \exp \left( -\frac{|z|^2}{4\pi \nu \tau (\tau - s)} \right) \]

(42)

for some constant \( L_m \) which is independent of \( \rho, r \) and \( \nu \) and which we choose such that it serves for all spatial derivatives up to order \( m \). We may assume that \( L_m \geq 0 \) such that we may use \( L_m \) for all these upper bounds as a constant which is independent of specific multiindices less or equal to \( m \).
\[ 0 \leq |\beta| \leq m \]
\[ \max_{1 \leq i \leq n} |D_x^\beta u_i(t_0, \cdot)|_{L^2 \cap C} \geq 1. \]  

(43)

If the latter condition is not satisfied for some \( \beta \) then there is a \( \Delta > 0 \) such that the respected norm is less equal 1 for some time \( t \in [t_0, t_0 + \Delta] \), and we need no damping estimate for this part of the \( H^m \cap C^m \)-norm in the interval \( [t_0, t_0 + \Delta] \). This way we construct an upper bound close to a constant \( C_m \), where \( C_m \) is the number of terms in the standard definition of the \( H^m \cap C^m \)-norm. We consider \( L^2 \)-estimates. We apply a Fourier transform with respect to the spatial variables, i.e., the operation

\[ \mathcal{F}(u)(\tau, \xi) = \int_{\mathbb{R}^D} \exp(-2\pi i x \xi) u(\tau, x) dx, \]  

(44)

in order to analyze the viscosity damping encoded in the first term on the right side of (34) on a time interval \( [t_0, t_0 + \Delta_0] \), where \( \Delta_0 = \rho \Delta \). For \( \tau \in [t_0, t_0 + \Delta_0] \) and parameters \( r, \rho > 0 \) we have

\[ \mathcal{F}(v_i^{\rho,r,\nu}(t_0, \cdot) *_{sp} G_{\nu}^{\rho,r}(\tau - t_0)) = \mathcal{F}(v_i^{\rho,r,\nu}(t_0, \cdot)) \mathcal{F}(G_{\nu}^{\rho,r}(\tau - t_0, \cdot)) \]
\[ = \mathcal{F}(v_i^{\rho,r,\nu}(t_0, \cdot)) \exp \left( -4\pi^2 \rho r^2 \nu (\tau - t_0)(\cdot)^2 \right), \]  

(45)

where we use (let \( t_0 = 0 \) for simplicity)

\[ \mathcal{F}(G_{\nu}^{\rho,r}(\tau, \cdot))(\tau, \xi) = \mathcal{F} \left( \frac{1}{\sqrt{4\pi \rho r^2 \nu}} \exp \left( -\frac{\xi^2}{4\rho r^2 \nu} \right) \right)(\tau, \xi) \]
\[ = \exp \left( -4\pi^2 \rho r^2 \nu |\xi|^2 \right). \]  

(46)

In the following we let \( t_0 = 0 \) and remark that the following estimates hold for \( t_0 > 0 \) if \( \tau \) is replaced by \( \tau - t_0 \). For \( \Delta > 0 \) small enough (such
that, say, $8\pi^2 pr^2 \nu \tau \Delta_0^2 \leq 1$, and for $\tau \in [0, \Delta_0]$ we get
\[
|v_i^{\rho,\tau,\nu}(t_0,.) \ast_{sp} G(\rho,\pi,\tau,.)|^2_{L^2} = \int_{\mathbb{R}^D} (\mathcal{F}(v_i^{\rho,\tau,\nu}(t_0,.))(\xi) \exp(-4\pi^2 pr^2 \nu \tau |\xi|^2))^2 \, d\xi
\]
\[
= \int_{\mathbb{R}^D} (\mathcal{F}(v_i^{\rho,\tau,\nu}(t_0,.)))(\xi) \exp(-8\pi^2 pr^2 \nu \tau |\xi|^2) \, d\xi
\]
\[
= \int_{\mathbb{R}^D \times [0, \Delta_0] \times [1, D]} (\mathcal{F}(v_i^{\rho,\tau,\nu}(t_0,.)))(\xi) \exp(-8\pi^2 pr^2 \nu \tau |\xi|^2) \, d\xi
\]
\[
\lesssim \left| \int_{\mathcal{D}_\Sigma} \mathcal{F}(v_i^{\rho,\tau,\nu}(t_0,.)) \right|_{L^2} \exp\left(-8\pi^2 pr^2 \nu \tau \Delta_0^2\right) + c_n^\Delta \left(8D\pi^2 pr^2 \nu \tau \Delta_0^{1+D}\right). \tag{47}
\]
Here, we use the assumption that $\Delta_0 > 0$ is small enough (especially $8\pi^2 pr^2 \nu \tau \Delta_0 \leq 1$) and use the abbreviation
\[
c_n^\Delta := \sup_{|\xi| \leq \Delta} \left| \mathcal{F}(D^2_x v_i^{\rho,\tau,\nu}(t_0,.)) \right|^2(\xi). \tag{48}
\]
which is a finite constant (since $|v_i^{\rho}(t_0,.)|_{H^2(\mathbb{R}^2)}$ is finite.

If we take the square root we may use the asymptotics $\sqrt{1 + a} = 1 + \frac{1}{2} a + O(a^2)$. For $\tau \in [0, \Delta_0]$ and
\[
0 < \Delta_0 \leq \max\left\{\frac{1}{8\pi^2 pr^2 \nu \max\{c_n^\Delta, 1\}}, \frac{1}{2}\right\} \tag{49}
\]
we get (the generous) estimate
\[
|v_i^{\rho,\tau,\nu}(t_0,.) \ast_{sp} G(\rho,\pi,\tau,.)|_{L^2} \leq |\mathcal{F}(v_i^{\rho,\nu})(t_0,.)|_{L^2} \exp\left(-4\pi^2 pr^2 \nu \tau \Delta_0^2\right)
\]
\[
+ c_n^\Delta \left(8D\pi^2 pr^2 \nu \tau \Delta_0^{1+D}\right)
\]
\[
\leq |v_i^{\rho,\nu}(t_0,.)|_{L^2} \exp\left(-4\pi^2 pr^2 \nu \tau \Delta^2\right) + c_n^\Delta \left(8D\pi^2 pr^2 \nu \tau \Delta_0^{1+D}\right). \tag{50}
\]
If $|v_i^{\rho,\nu}(t_0,.)|_{L^2}$ becomes large or $\Delta > 0$ is small enough compared to $|v_i^{\rho,\nu}(t_0,.)|_{L^2}$, then the second summand on right side of (50) is small compared to the first summand. Note that we may replace $v_i^{\rho,\nu}(t_0,.)$ by multivariate spatial derivatives $D^\beta_x v_i^{\rho,\nu}(t_0,.)$ for $0 \leq |\beta| \leq m$ such that an analogous estimate holds for spatial derivatives $|D^\beta_x v_i^{\rho,\nu}(t,.)|_{L^2}$.
for $0 \leq |\beta| \leq m$. We shall observe below that this small damping after
Discrete time (under the assumption that some $|D^2v_{i}^{\rho,r,\nu}(t_0,.))_{L^2}$ exceeds
a certain level (say $1$) is strong enough in order to offset possible growth
cased by the nonlinear terms.

vii) We compare the damping with the upper bound of the nonlinear term.
Summing up the preceding argument recall that we have to replace $\tau$ by
$\tau - t_0$ on order to have for given $t_0 \geq 0$ and $\tau \in [t_0, t_0 + \Delta_0]$
$$
|D^2v_i^{\rho,r,\nu}(\tau, x)| \leq |D^2v_i^{\rho,r,\nu}(t_0, .)|_{L^2} \exp (-4\pi^2 \nu \rho \tau^2 (\tau - t_0)\Delta_0^2) \\
+ c^A_m \left(8D^2 \nu \rho \tau \Delta_0^{1+D}\right) + prL_m(4\pi \nu \rho ^2 \nu \Delta_0^{1+D}) .
$$
(51)

Recall from (50) and analogous estimates for spatial derivatives that we
have a damping estimate
$$
|D^2v_i^{\rho,r,\nu}(t_0, .)|_{L^2} + \exp (-4\pi^2 \nu \rho \tau^2 (\tau - t_0)\Delta_0^2) + c^A_m \left(8D^2 \nu \rho \tau \Delta_0^{1+D}\right) .
$$
(52)

These estimates become effective for small $\Delta_0 > 0$ if the norm of (some spatial derivative) of the initial data exceeds a certain threshold, i.e., if
$|D^2v_i^{\rho,r,\nu}(t_0, .)|_{L^2} \geq 1$. If this threshold is realised for some $0 \leq |\beta| \leq m$, then the last upper bound term in (51) or in (52) is relatively small for small $\Delta_0 \geq \Delta_0 (\tau - t_0)$ .
Note that this upper bound term contains no spatial variables, i.e., it behaves like a constant with respect to spatial norms.

Define
$$
c^A_D := c^A_m \left(8D^2 \nu \rho \tau \Delta_0^{1+D}\right) .
$$
(53)

For $\tau \in [t_0, t_0 + \Delta_0]$ we have
$$
|D^2v_i^{\rho,r,\nu}(t_0 + \Delta_0, .)|_{L^2} \leq |v_i^{\rho,r,\nu}(t_0, .)|_{L^2} \exp (-4\pi^2 \nu \rho \tau^2 \Delta_0^2) + c^A_D
+ prL_m(4\pi \nu \rho ^2 \nu \Delta_0^{1+D}) .
$$
(54)

It is sufficient to consider the case $t_0 = 0$ as the same following estimates hold for $t_0 \geq 0$ if $\tau$ is replaced by $\tau - t_0$. In this case the relation on (54) shows us that
$$
|D^2v_i^{\rho,r,\nu}(\Delta_0, .)|_{L^2} \leq |D^2v_i^{\rho,r,\nu}(0, .)|_{L^2} \\
+ prL_m(4\pi \nu \rho ^2 \nu \Delta_0^{1+D}) .
$$
(55)

if
$$
|D^2v_i^{\rho,r,\nu}(0, .)|_{L^2} \left(\exp (-4\pi^2 \nu \rho \tau^2 \Delta_0^2) - 1\right) + c^A_D
+ prL_m(4\pi \nu \rho ^2 \nu \Delta_0^{1+D}) .
$$
(56)

Note that the positive real number $\epsilon$ in (50) is of exponential decay on a
small time interval. More precisely,
$$
\epsilon \sim \sum^D_{i=1} \int_{t_0}^{t_0 + \Delta} \int_{\mathbb{R}^D \setminus \delta B} ||G_{\nu r}^{\rho}(\sigma, y)||d\sigma dy \downarrow 0 \text{ as } \Delta \downarrow 0,
$$
(57)
hence $\epsilon$ is comparatively small as $4\pi \nu \rho^2 \nu (\tau - t_0) \leq 4\pi \nu \rho^2 \nu \Delta_0$ becomes small. Furthermore, as $c^A_D \downarrow 0$ as $\Delta_0 \downarrow 0$ with $\Delta^{D+1}$, and the damping
factor \(|v^{\rho,r,\nu}_i(0,.)|_{L^2} \leq \left(1 - \exp\left(-4\pi^2 \nu \rho^2 \tau \Delta_0^2\right)\right)\) is dominant for \(\tau = \Delta_0\) if \(|v^{\rho,r,\nu}_i(0,.)|_{L^2} \geq 1\) such that \(c_D^2\) is relatively small compared to the modulus of the main part of this damping term

\[
|v^{\rho,r,\nu}_i(0,.)|_{L^2} \left(-4\pi^2 \nu \rho^2 \Delta_0 \Delta_0^2\right).
\] (58)

Next we consider the conditions such that the modulus of the main damping part is larger than the last term (54) (the last term for \(\epsilon\) which is comparatively small and can be neglected). Here we observe the exponents of the parameters \(\rho, r, \nu\) and \(\Delta_0\) in (58) compared to the exponents of the parameters \(\rho, r\) and \(\Delta_0\) of the last term in (54). For \(\tau = \Delta_0\) for the damping term in (58) we have the dependence

\[
\sim \nu \rho r^2 \Delta_0^3,
\] (59)

which has to be compared with the last term in (54), where we have the dependence

\[
(\rho)^{1+\delta} (r)^{1+2\delta} (\nu)^{1-\delta}.
\] (60)

The estimate of convolutions of a Lipschitz function with first order spatial derivatives of the Gaussian forced us to assume \(\nu \rho r^2\) which implies large \(r > 1\) in general. Hence we have to impose

\[
\delta < \frac{1}{2} \text{ such that } r^{1+2\delta} < r^2.
\] (61)

Choosing a small step size parameter \(\rho\), say \(r = \rho = \Delta_0^\mu\), we have

\[
(\rho)^{\mu(1+\delta)} (r)^{1+2\delta} \Delta_0^{1-\delta} = \Delta_0^{\mu(1+\delta)+1-\delta}
\] (62)

which has to be compared with the damping term, where for \(\rho = \Delta_0^\mu\) the latter has the dependence

\[
\rho \rho^2 \Delta_0^3 = \Delta_0^{\mu+3} r^2.
\] (63)

For small \(\Delta_0\) we get the time step size condition

\[
\mu(1+\delta) + 1 - \delta > \mu + 3 \text{ iff } \mu > \frac{2+\delta}{\delta}
\] (64)

which implies \(\mu > 5\) for \(\delta < 0.5\). Hence any choice with \(r\) as above, \(\delta \in (0,0.5)\), and \(\rho\) as above with \(\mu > \frac{2+\delta}{\delta}\) implies a regular upper bound for \(v_i^t\), \(1 \leq i \leq D\). Note that \(r > 1\) is related to a deviation from a strong semigroup contraction principle of the operator. Note that in any case the upper bound constructed is at discrete times \(l \Delta_0, l \in \mathbb{N}\), where \(\mathbb{N}\) denotes the set of natural numbers. However, using the local contraction result and the semigroup property we get a regular upper bound for all time.

3 Appendix: Proof of local time contraction

Again remember the notation discussed in assumption c) above. It is sufficient to prove the theorem for the functions \(v_i^t(\tau,x) = v_i(t,x)\), where \(v_i(t,x) = \)
where $G^\rho_v = G^\rho_v$ in the notation above, i.e., $r = 1$. For $0 \leq |\gamma| \leq m - 1$ and related $1 \leq |\beta| \leq m$, where $\beta_j = \gamma_j + 1$ and $\beta_i = \gamma_i$ for $i \neq j$ we have
\[
D_x^\beta v^{i,k} = \rho \sum_{j=1}^D \frac{D_x^\gamma v^{i,j}}{F_{j,j}} (v^{\rho,k-1}, \nabla v^{\rho,k-1}) * G^\rho_{v,j} = \rho D_x^\beta f^i (v^{\rho,k-1}, \nabla v^{\rho,k-1}) * G^\rho_{v,i}.
\]
According to our notation discussed in item c) above, we have $\sum_i F^i_{j,j} = f_i$ for all $1 \leq i \leq D$, so for fixed time $\tau \in [t_0, t_0 + \Delta_0]$ from (65) we get
\[
|v^{i,k}_\tau (\tau, \cdot) - v^{i,k-1}_\tau (\tau, \cdot)|_{H^m \cap C^m}
= \sum_{0 \leq |\beta| \leq m} |D_x^\beta v^{i,k}_\tau (\tau, \cdot) - D_x^\beta v^{i,k-1}_\tau (\tau, \cdot)|_{L^2 \cap C}.
\]
\[
\leq \rho \left| \left( f_i (v^{\rho,k-1}, \nabla v^{\rho,k-1}) - f_i (v^{\rho,k-1}, \nabla v^{\rho,k-2}) \right) * G^\rho_v \right|_{L^2 \cap C}
+ \rho \left| \left( f_i (v^{\rho,k-1}, \nabla v^{\rho,k-2}) - f_i (v^{\rho,k-2}, \nabla v^{\rho,k-2}) \right) * G^\rho_v \right|_{L^2 \cap C}
+ \sum_{0 \leq |\gamma| \leq m-1} \rho \left| \left( D_x^\gamma f_i (v^{\rho,k-1}, \nabla v^{\rho,k-1}) - D_x^\gamma f_i (v^{\rho,k-1}, \nabla v^{\rho,k-2}) \right) * G^\rho_{v,j} \right|_{L^2 \cap C}.
\]
\[
= \rho \left| \left( f_i (v^{\rho,k-1}, \nabla v^{\rho,k-1}) - f_i (v^{\rho,k-1}, \nabla v^{\rho,k-2}) \right) * G^\rho_v \right|_{L^2 \cap C}
+ \rho \left| \left( f_i (v^{\rho,k-1}, \nabla v^{\rho,k-2}) - f_i (v^{\rho,k-2}, \nabla v^{\rho,k-2}) \right) * G^\rho_v \right|_{L^2 \cap C}
+ \sum_{0 \leq |\gamma| \leq m-1} \rho \left| \left( D_x^\gamma f_i (v^{\rho,k-1}, \nabla v^{\rho,k-1}) - D_x^\gamma f_i (v^{\rho,k-1}, \nabla v^{\rho,k-2}) \right) * G^\rho_{v,j} \right|_{L^2 \cap C}.
\]
(66)

If $F^i_j$ has a local interpretation, then for all $0 \leq |\beta| \leq m$ $D_x^\beta F^i_j$ is assumed to be Lipschitz continuous with respect to all arguments on compact domains, hence $D_x^\beta f$ is also Lipschitz as a sum of Lipschitz functions. We may use an uniform Lipschitz constant $L$ which serves for all arguments and get
\[
|v^{i,k}_\tau (\tau, \cdot) - v^{i,k-1}_\tau (\tau, \cdot)|_{H^m \cap C^m}
\leq \rho \left| \left( f_i (v^{\rho,k-1}, \nabla v^{\rho,k-1}) - f_i (v^{\rho,k-1}, \nabla v^{\rho,k-2}) \right) * G^\rho_v \right|_{L^2 \cap C}
+ \rho \left| \left( f_i (v^{\rho,k-1}, \nabla v^{\rho,k-2}) - f_i (v^{\rho,k-2}, \nabla v^{\rho,k-2}) \right) * G^\rho_v \right|_{L^2 \cap C}
+ \sum_{0 \leq |\gamma| \leq m-1} \rho \left| \left( D_x^\gamma f_i (v^{\rho,k-1}, \nabla v^{\rho,k-1}) - D_x^\gamma f_i (v^{\rho,k-1}, \nabla v^{\rho,k-2}) \right) * G^\rho_{v,j} \right|_{L^2 \cap C}.
\]
(68)

For a global operator $F^i_j$ with Lipschitz continuity with respect to $|.|_{L^2 \cap C}$ norm a similar relation as in [18] holds. We have considered this elsewhere in the case of the Navier Stokes operator, and may reconsider this in a journal version.
of this paper for completeness. As an essential case we estimate a summand of the third term on the right side of (68). We write

\[ G^{\rho}_{\nu,j} = 1_{B^1_0} G^{\rho}_{\nu,j} + 1_{(R^D \setminus B^1_0)} G^{\rho}_{\nu,j}, \]  

(69)

where \( 1_{B^1_0} \) is the characteristic function of the ball of radius 1 around 0.

It has the upper bound

\[ \rho \left| L \left| D^2 v^{\rho,k-1}_{i,j} (\tau, \cdot) - D^2 v^{\rho,k-2}_{i,j} (\tau, \cdot) \right| * 1_{B^1_0} G^{\rho}_{\nu,j} \right|_{L^2 \cap C} \]

\[ + \rho \left| L \left| D^2 v^{\rho,k-1}_{i,j} (\tau, \cdot) - D^2 v^{\rho,k-2}_{i,j} (\tau, \cdot) \right| * 1_{(R^D \setminus B^1_0)} G^{\rho}_{\nu,j} \right|_{L^2 \cap C}. \]  

(70)

where

\[ C_{1,\gamma} := \int_{R^D} \left| 1_{B^1_0} G^{\rho}_{\nu,j} (\tau, x) \right| dx dt + \left| \mathcal{F} \left( 1_{(R^D \setminus B^1_0)} G^{\rho}_{\nu,j} \right) \right|_{L^2 \cap C}. \]  

(71)

The convolutions with \( G^{\rho}_{\nu} \) can be treated similarly. We may define

\[ C_{0,\gamma} := \int_{R^D} \left| 1_{B^1_0} G^{\rho}_{\nu,j} (\tau, x) \right| dx dt + \left| \mathcal{F} \left( 1_{(R^D \setminus B^1_0)} G^{\rho}_{\nu,j} \right) \right|_{L^2 \cap C}. \]  

(72)

Hence

\[ \left| v^\rho (\tau, \cdot) - v^\rho (\tau, \cdot) \right|_{H^m \cap C^m} \]

\[ + \sum_{0 \leq |\beta| \leq m} \rho L \left( C_{0,\gamma} + C_{1,\gamma} \right) \left| D^2 v^{\rho,k-1}_{i,j} (\tau, \cdot) - D^2 v^{\rho,k-2}_{i,j} (\tau, \cdot) \right|_{L^2 \cap C}. \]  

(73)

We may then choose \( \rho = \frac{1}{4L(\sum_{0 \leq |\beta| \leq m} (C_{0,\gamma} + C_{1,\gamma}))} \) and get the desired contraction.

References

[1] Landau, L., Lifschitz, E. Lehrbuch der Theoretischen Physik VI, Hydrodynamik, Akademie Verlag, Berlin. J., (1978).