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Low discrepancy sequences failing Poissonian pair correlations

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Abstract. M. Levin defined a real number $x$ that satisfies that the sequence of the fractional parts of $(2^n x)_{n \geq 1}$ are such that the first $N$ terms have discrepancy $O((\log N)^2 / N)$, which is the smallest discrepancy known for this kind of parametric sequences. In this work we show that the fractional parts of the sequence $(2^n x)_{n \geq 1}$ fail to have Poissonian pair correlations. Moreover, we show that all the real numbers $x$ that are variants of Levin’s number using Pascal triangle matrices are such that the fractional parts of the sequence $(2^n x)_{n \geq 1}$ fail to have Poissonian pair correlations.

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1. Introduction and statement of results. A sequence $(x_n)_{n \geq 1}$ of real numbers in the unit interval is said to have Poissonian pair correlations if for all non-negative real numbers $s$, 

$$
\lim_{N \to \infty} F_N(s) = 2s
$$

where 

$$
F_N(s) = \frac{1}{N} \# \left\{ (i, j) : 1 \leq i \neq j \leq N \text{ and } \|x_i - x_j\| < \frac{s}{N} \right\}
$$

and $\|x\|$ is the distance between $x$ and its nearest integer. The function $F_N(s)$ counts the number of pairs $(x_n, x_m)$ for $1 \leq m, n \leq N$, $m \neq n$, of points which are within distance at most $s/N$ of each other, in the sense of distance on the torus. If $\lim_{N \to \infty} F(s) = 2s$ for all $s \geq 0$, then the asymptotic distribution of the pair correlations of the sequence is Poissonian, and this explains that the property is referred to as having Poissonian pair correlations. Almost surely a sequence of independent identically distributed random variables in the unit
interval has this property. Several particular sequences have been proved to have the property, for example, \((\sqrt{n} \mod 1)_{n \geq 1}\) \([4]\). It is known that for almost all real numbers \(x\) \((a_n x \mod 1)_{n \geq 1}\) has the property when \(a_n\) is integer valued and \((a_n)_{n \geq 1}\) is lacunary \([11]\); also when \(a_n = n^2\) (or a higher polynomial) \([10, 14]\). However, for specific values such as \(x = \sqrt{2}\) and \(a_n = n^2\) it is not known whether \((a_n x \mod 1)_{n \geq 1}\) has Poissonian pair correlations or not.

The property of Poissonian pair correlations implies uniform distribution modulo 1, this was only recently proved in \([1, \text{Theorem 1}]\) and also in \([5, \text{Corollary 1.2}]\). The converse does not always hold. Several uniformly distributed sequences of the form \((b^n x \mod 1)_{n \geq 1}\) where \(b\) is an integer greater than 1 and \(x\) is a constant were proved to fail the property of Poissonian pair correlations. Pirsic and Stockinger \([9]\) proved it for Champernowne’s constant (defined in base \(b\)). Larcher and Stockinger \([6]\) proved it for \(x\) a Stoneham number \([12]\) and for every real number \(x\) having an expansion which is an infinite de Bruijn word (see \([3, 13]\) for the presentation of these infinite words). Larcher and Stockinger also show in \([7]\) the failure of the property for other sequences of the form \((a_n x \mod 1)_{n \geq 1}\).

In this paper we show that the sequence \((2^n \lambda \mod 1)_{n \geq 1}\), where \(\lambda\) is the real number defined by Levin in \([8, \text{Theorem 2}]\), fails to have Poissonian pair correlations. Levin’s number \(\lambda\) is defined constructively using Pascal triangle matrices and satisfies that the discrepancy of the first \(N\) terms of the sequence \((2^n \lambda \mod 1)_{n \geq 1}\) is \(O((\log N)^2/N)\). This is the smallest discrepancy bound known for sequences of the form \((2^n x \mod 1)_{n \geq 1}\) for some real number \(x\).

We also show that each of the real numbers \(\rho\) considered by Becher and Carton in \([2]\) are such that the sequence \((2^n \rho \mod 1)_{n \geq 1}\) fails to have Poissonian pair correlations. These numbers \(\rho\) are variants of Levin’s number \(\lambda\) because they are defined using rotations of Pascal triangle matrices and the sequence \((2^n \rho \mod 1)_{n \geq 1}\) has the same low discrepancy as that obtained by Levin.

1.1. Levin’s number. We start by defining the number \(\lambda\) given by Levin in \([8, \text{Theorem 2}]\) and further examined in \([2]\). As usual, we write \(F_2\) to denote the field of two elements. In this work, we freely make the identification between binary words and vectors on \(F_2\). We define recursively a sequence of matrices on \(F_2\):

\[
M_0 = (1) \quad \text{and for every } d \geq 0, \quad M_{d+1} = \begin{pmatrix} M_d & M_d \\ 0 & M_d \end{pmatrix}.
\]

The first elements of this sequence, for example, are:

\[
M_0 = (1) \quad M_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad M_2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\]

Let \(d\) be a non-negative integer and let \(e = 2^d\). The matrix \(M_d \in F_2^{e \times e}\) is upper triangular with 1s on the diagonal, hence it is non-singular. Then, if 

\[
w_0, \ldots, w_{2^e-1}
\]
is the enumeration of all vectors of length \( e \) in lexicographical order, the sequence
\[
M_d w_0, \ldots, M_d w_{2^e - 1}
\]
ranges over all vectors of length \( e \). We obtain the \( d \)-th block of \( \lambda \) by concatenation of the terms of that sequence:
\[
\lambda_d = (M_d w_0)(M_d w_1) \ldots (M_d w_{2^e - 1}).
\]
Levin’s constant \( \lambda \) is defined as the infinite concatenation
\[
\lambda = \lambda_0 \lambda_1 \lambda_2 \ldots
\]
The expansion of \( \lambda \) in base 2 starts as follows (the spaces are just for convenience):
\[
\begin{array}{cccccccccccccccc}
\lambda_0 & \lambda_1 & \lambda_2 & \lambda_3 \\
0000 & 0111 & 1010 & 0101 & 1100 & 0011 & 0110 & 0011 & 0110 & 1001 & 1011 & 1110 & 0001 \\
\end{array}
\]
Now we introduce a family \( \mathcal{L} \) of constants which have similar properties to those of \( \lambda \). Let \( \sigma \) be the rotation that takes a word and moves its last letter at the beginning: that is, \( \sigma(a_1 \ldots a_n) = a_n a_1 \ldots a_{n-1} \). We are going to use \( \sigma \) to define a family of matrices obtained by selectively rotating some of the columns of \( M_d \).

As before, assume \( d \) is a non-negative integer and let \( e = 2^d \). We say that a tuple \( \nu = (n_1, \ldots, n_e) \) of non-negative integers is suitable if
\[
n_e = 0 \text{ and } n_i+1 \leq n_i \leq n_i+1 + 1 \text{ for each } 1 \leq i \leq e - 1.
\]
Let \( C_1, \ldots, C_e \) denote the columns of \( M_d \), and let \( \sigma^n \) denote the composition of the rotation \( \sigma \) with itself \( n \) times. Then, define
\[
M_d^{\nu} = (\sigma^{n_1} (C_1), \ldots, \sigma^{n_e} (C_e)).
\]
For example, by taking \( d = 2 \) we have 8 different possible matrices, one for every choice of \( \nu \):
\[
\begin{array}{cccccccc}
M_2^{(0,0,0,0)} & M_2^{(1,0,0,0)} & M_2^{(1,1,0,0)} & M_2^{(2,1,0,0)} \\
(1 \ 1 \ 1 \ 1) & (0 \ 1 \ 1 \ 1) & (0 \ 0 \ 1 \ 1) & (0 \ 0 \ 1 \ 1) \\
(0 \ 1 \ 0 \ 1) & (1 \ 1 \ 0 \ 1) & (1 \ 1 \ 0 \ 1) & (0 \ 1 \ 0 \ 1) \\
(0 \ 0 \ 1 \ 1) & (0 \ 0 \ 1 \ 1) & (0 \ 1 \ 1 \ 1) & (0 \ 0 \ 0 \ 1) \\
(0 \ 0 \ 0 \ 1) & (0 \ 0 \ 0 \ 1) & (0 \ 0 \ 0 \ 1) & (0 \ 0 \ 0 \ 1) \\
\end{array}
\]
As before, we let \( e = 2^d \) and \( w_0, \ldots, w_{2^e - 1} \) be the increasingly ordered sequence of all vectors in \( \mathbb{F}_2^e \). We say that a word is an \( e \)-affine necklace if it can be written as the concatenation \( (M w'_0)(M w'_1) \ldots (M w'_{2^e - 1}) \) for some \( z \in \mathbb{F}_2^e \) and a suitable tuple \( \nu \) with \( M = M_d^{\nu} \) and \( w'_i = w_i + z \) for \( 0 \leq i \leq 2^e - 1 \).
Finally, we define $L$ to be the set of all binary words that can be written as an infinite concatenation $\rho_0\rho_1\rho_2\ldots$ where every $\rho_d$ is a $2^d$-affine necklace. Note that $\lambda \in L$, by taking $\nu = (0, \ldots, 0)$ everywhere.

The rest of this note is devoted to proving the following result:

**Theorem 1.** For all $\rho \in L$, the sequence of fractional parts of $(2^n\rho)_{n \geq 1}$ does not have Poissonian pair correlations.

2. **Lemmas.** First we prove some necessary results. We present in an alternating manner results about $M_d$ and its corresponding generalizations to the family of matrices $M_d^\nu$.

**Lemma 1.** For all $d$, $M_d$ is triangular and all entries in its diagonal are ones. In particular, $M_d$ is non-singular.

*Proof.* This is easily proven with induction. $M_0 = (1)$ satisfies the lemma, and if $M_d$ satisfies it, then $M_{d+1} = \begin{pmatrix}M_d & M_d \\ 0 & M_d\end{pmatrix}$ satisfies it too. \(\Box\)

**Lemma 2.** For all non-negative $d$ and for every suitable tuple $\nu$, $M_d^\nu$ is non-singular.

*Proof.* This fact is proven in [2, Lemma 4]. \(\Box\)

**Lemma 3.** For all $d$ and for all even $n$, $M_dw_n$ and $M_dw_{n+1}$ are complementary vectors. That is, the $i$-th coordinate of $M_dw_n$ equals zero if and only if the $i$-th coordinate of $M_dw_{n+1}$ equals one.

*Proof.* The sequence $w_0, w_2, \ldots w_{2^e-1}$ is lexicographically ordered and hence the last entry of $w_n$ is zero whenever $n$ is even. Therefore, $w_{n+1}$ only differs from $w_n$ in the last entry

$$M_dw_{n+1} = M_dw_n + M_d\begin{pmatrix}0 \\ \vdots \\ 0 \\ 1\end{pmatrix} = (M_dw_n) + \begin{pmatrix}1 \\ \vdots \\ 1 \\ 1\end{pmatrix}.$$

\(\Box\)

To simplify notation, from now on we write $\overline{z}$ for the complementary vector of $z$. Note that Lemma 3 implies that $\lambda_d$ can be written as a concatenation of words of the form $w\overline{w}$.

**Lemma 4.** For all non-negative $d$, for all even $n$, and for every suitable tuple $\nu$, the vectors $M_d^\nu w_n$ and $M_d^\nu w_{n+1}$ are complementary.

*Proof.* The last coordinate of $\nu$ is zero by definition. Therefore, the last column of $M_d^\nu$ is the same as the last column of $M_d$; that is, it is the vector of ones. The same argument used to prove Lemma 3 applies. \(\Box\)

We say that a vector is *even* if its last entry is 0. Hence, when $n$ is even, $w_n$ is an even vector.
**Lemma 5.** Let \( d \) be a non-negative integer and \( e = 2^d \). The subspace of all even vectors of length \( e \),
\[
\mathcal{P} = \{ v \in \mathbb{F}_2^e \mid v_e = 0 \}
\]
is invariant under \( M_d \). Furthermore, \( M_d w_n \) is an even vector if and only if \( w_n \) is an even vector.

**Proof.** By Lemma 1, \( M_d \) is upper triangular and its diagonal is comprised by ones. This implies that all of its columns except the last are even vectors. Therefore, the only way to obtain an odd vector via the computation \( M_d w \) is that \( w \) itself is odd.

**Lemma 6.** Let \( d \) be a non-negative integer and \( e = 2^d \). Depending on \( \nu \), there are two distinct possibilities:

1. The subspace of even vectors \( \mathcal{P} \) is invariant under \( M_d^\nu \). In this case, \( w \) is an even vector if and only if \( M_d^\nu w \) is an even vector.
2. The subspace \( \mathcal{P} \) is in bijection with the subspace \( \{(v_1, \ldots, v_e) \in \mathbb{F}_2^e \mid v_1 = 0\} \) via \( M_d^\nu \). In this case, \( w \) is an even vector if and only if \( M_d^\nu w \) has a zero in its first coordinate.

**Proof.** Take \( \nu = (n_1, \ldots, n_e) \) such that \( n_e = 0 \) and \( n_{i+1} \leq n_i \leq n_{i+1} + 1 \) for all \( 1 \leq i \leq e - 1 \). Combining \( n_i \leq n_{i+1} + 1 \) for \( i = e - 1 \) and \( n_e = 0 \) gives that \( n_{e-1} \) is either zero or one. We consider both possibilities separately: the case \( n_{e-1} = 0 \) will yield (1) and the case \( n_{e-1} = 1 \) will yield (2).

First, suppose that \( n_{e-1} \) equals one. Then, all entries of \( \nu \) before it must be greater than one. That means that, when building \( M_d^\nu \) from \( M_d \), all of its columns except the last are rotated at least one position. Fix an index \( i \) such that \( 1 \leq i \leq e - 1 \), and consider \( c \) the \( i \)-th column of the matrix \( M_d \). We show that the first element of \( \sigma^{n_i}(c) \) is zero.

By Lemma 1, we know that \( M_d \) is triangular. That means that the elements \( c_{i+1}, \ldots, c_e \) are necessarily zeros. But the first element of \( \sigma^{n_i}(c) \) is \( c_{e-n_{i+1}} \); and from the inequality \( n_i \leq e - i \) it follows that \( e - n_i + 1 \) is greater or equal than \( i + 1 \). Therefore, the first element of \( \sigma^{n_i}(c) \) is zero.

Because \( i \) is any index between 1 and \( e - 1 \), it follows that the first \( e - 1 \) columns of \( M_d^\nu \) have a zero as their first coordinate. If \( w \) is an even vector, \( M_d w \) is a linear combination of vectors which start with zero, and therefore \( M_d w \) also starts with a zero. Conversely, if \( w \) begins with a one, then \( M_d w \) must be an odd vector, because it’s a linear combination of elements that start with a zero and the last column of \( M_d^\nu \), which is the vector of ones. We conclude that \( w \) is an even vector if and only if \( M_d^\nu w \) starts with a zero.

The case where \( n_{e-1} \) equals zero is analogous, and we give an outline of the proof. First, prove that the first \( e - 1 \) columns of \( M_d^\nu \) are even vectors. Then, \( M_d^\nu w \) is even if and only if \( w \) is even.

**3. Proof of the main theorem.** We first prove the theorem for \( \lambda \) and at the end we explain how to generalize the result to each number in the family \( \mathcal{L} \). For any given non-negative \( d \), we set \( e = 2^d \) and show that for an appropriate choice of increasing \( N \) which depends on \( d \) and \( e \), \( F_N(2) \) diverges. For this
we show that some selected patterns have too many occurrences in $\lambda_d$. More precisely, we count occurrences of binary words of length $d+e$,

$$a = a_1 a_2 \ldots a_{d+e}$$

such that

$$a_1 \ldots a_d = a_{e+1} \ldots a_{e+d}.$$ 

The reason for this choice comes from Lemma 3 and it will soon become clear. We need some terminology. Given a word $a$ as above and an occurrence of $a$ in $\lambda_d$,

(1) let $k$ be the number of zeros in $a_d \ldots a_e$;

(2) let $n$ be the index such that the $a$ occurs in $M_d w_n$;

(3) let $z$ be the position in $a$ that matches with the $e$-th (that is the last) symbol of $M_d w_n$ (Fig. 1).

We require $n$ to be an even number and $z$ to be in the range $d \leq z \leq e$. The latter is to prevent the word $a$ from spanning over more than two words, and the former is to ensure that a match for the first $d$ letters automatically yields a match for the last $d$ letters (a combination of Lemma 3 and the hypothesis over $a$). In addition, by Lemma 5 we know that $M_d w_n$ is an even word, and therefore $a_z$ must be zero.

We fix $k$ and count all possible occurrences in $\lambda_d$ of every possible word $a$. There are exactly

$$2^{d-1} \binom{e - d + 1}{k}$$

words $a$ with $k$ zeros in $a_d \ldots a_e$. For every one of them, we have a choice of $k$ different $z$, because we know that $a_z$ must be zero. We claim that each of those choices for $z$ correspond to an actual occurrence of $a$ in $\lambda_d$. Let us suppose that the binary word $a$, whose length is $d+e$, starts in $M_d w_n$ and continues in $M_d w_{n+1}$. Then, it must hold that, for some $z$ (Fig. 2),

$$a_1 \ldots a_z = (M_d w_n)_{e-z+1} \ldots (M_d w_n)_e$$

and

$$a_{z+1} \ldots a_e = (M_d w_{n+1})_1 \ldots (M_d w_{n+1})_{e-z}.$$ 

So, Lemma 3 allows us to conclude that
Figure 2. Given an $a$ and a choice of $z$, it is possible to find an unique position within $\lambda_d$ where the word $a$ occurs with alignment $z$

$M_dw_n = \overline{a_{z+1}} \ldots a_e a_1 \ldots a_z$.

By Lemma 1, we know that there exists some $w_n$ that satisfies this equation and by Lemma 5, $n$ must be an even number. Therefore, given a choice of $z$, there is an occurrence of $a$. We conclude that for every choice of $k$, and for every word $a \in \{0, 1\}^{e+d}$ with $k$ zeros in $a_d \ldots a_e$, we have exactly $k$ occurrences within $\lambda_d$.

We now prove that the sequence of fractional parts of $(2^n \lambda)_{n \geq 1}$ does not have Poissonian pair correlations. Take $s = 2$ and $N = 2^{d+e+1}$. We prove that $\lim_{d \to \infty} F_N(s) = \infty$. In order to do that, we note that two different occurrences of the same word $a \in \{0, 1\}^{e+d}$ correspond to two different suffixes of $\lambda$ that share its first $e + d$ digits.

We write $\{x\}$ to denote $x - \lfloor x \rfloor$, the fractional expansion of $x$. If $a$ has two different occurrences within $\lambda$ at positions $i$ and $j$, then

$$\| \{ 2^i \lambda \} - \{ 2^j \lambda \} \| = \| 0.a_1 \ldots a_e + d \lambda_{i+d+e+1} \ldots - 0.a_1 \ldots a_e + d \lambda_{j+d+e+1} \ldots \| \\
\leq | 0.a_1 \ldots a_e + d \lambda_{i+d+e+1} \ldots - 0.a_1 \ldots a_e + d \lambda_{j+d+e+1} \ldots | \\
< 2^{-(e+d)} \\
= \frac{s}{N}.$$ 

Therefore, if $i$ and $j$ are both no greater than $N$, the pairs $(i, j)$ and $(j, i)$ count for $F_N(s)$. For indices $i$ and $j$ of $\lambda$ which correspond to elements of $\lambda_d$ this is the case:

$$|\lambda_0 \ldots \lambda_d| = \sum_{i=0}^d |\lambda_i| = \sum_{i=0}^d 2^i 2^{2^i} = \sum_{i=0}^d 2^{i+2^i} < 2^{2^{d+d+1}} = N.$$ 

We are now able to give a lower bound for $F_N(s)$. To do this, we count all possible pairs of occurrences of every word $a$ satisfying the condition $\overline{a_{e+1}} \ldots a_{e+d} = a_{e+1} \ldots a_{e+d}$ in $\lambda_d$. Recall that a word $a$ with $k$ zeros in the middle has exactly $k$ occurrences in $\lambda_d$, so by summing over $k$ we get:

$$F_N(2) \geq \frac{1}{N} \sum_{k=0}^{e-d+1} 2^{e-d-1} \binom{e-d+1}{k} \binom{k}{2}.$$
\begin{align*}
&= \frac{1}{2^{e+1}} \sum_{k=0}^{e-d+1} \binom{e - d + 1}{k} \binom{k}{2} \\
&= \frac{1}{2^{e+1}} \binom{e - d + 1}{2} \cdot 2^{e-d+1-2} \\
&= \frac{1}{8e} (e - d + 1)(e - d).
\end{align*}

In the third step, we applied the identity
\[
\sum_{k=0}^{n} \binom{n}{k} \binom{k}{2} = 2^{n-2} \binom{n}{2}
\]
with \(n = (e - d + 1)\). Thus,
\[
F_N(2) \geq \frac{(e - d + 1)(e - d)}{8e}
\]
and the last expression diverges as \(d \to \infty\) because \(e\) is squared in the numerator but linear in the denominator and \(d\) is insignificant with respect to \(e\). This concludes the proof that the sequence of the fractional parts of \((2^n \lambda)_{n \geq 1}\) does not have Poissonian pair correlations.

We now explain how to extend the proof for any given constant in \(L\). Take \(\rho \in L\). Then \(\rho\) can be written as a concatenation
\[
\rho = \rho_0 \rho_1 \rho_2 \ldots
\]
where each \(\rho_d\) is a \(2^d\)-affine necklace. That means that for every \(d\), there exists a suitable tuple \(\nu\) such that
\[
\rho_d = (M^\nu_d w_0)(M^\nu_d w_1) \ldots (M^\nu_d w_{2^d-1}).
\]
We take \(s = 2\) and \(N_d = 2^{e+d+1}\) and we prove that the sequence \(F_{N_d}(s)\) diverges as \(d \to \infty\). As we did for \(\lambda\), it is possible to give a lower bound for \(F_{N_d}(s)\) by counting occurrences within \(\rho_d\) of words of length \(e + d\).

Fix a non-negative integer \(d\). By Lemma 6, there are two possibilities: either \(M^\nu_d\) maps the subspace of even vectors \(P\) to itself or it maps it to the set of vectors beginning with zero. In the first case, we can replicate essentially verbatim the procedure we followed for \(\lambda\) to get a lower bound for \(F_{N_d}(s)\). In the second case, we have to slightly alter the argument: \(z\) is redefined to be the index of \(a\) such that \(a_z\) matches the first letter of \((M^\nu_d w_{n+1})\), and \(k\) is redefined to be the number of ones in \(a_{d+1} \ldots a_{e+1}\). Despite these modifications, we reach the same lower bound for \(F_{N_d}(s)\). Since it diverges as \(d \to \infty\) we conclude that the sequence of the fractional parts of \((2^n \rho)_{n \geq 1}\) does not have Poissonian pair correlations.

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