Random Relation Algebras

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What does a “typical” finite relation algebra look like? In graph theory, one has the “random graph” $G_{n,p}$, which is actually a probability space of graphs [3]. (If one sets $p = \frac{1}{2}$, $G_{n,p}$ corresponds to the uniform distribution on the set of all labelled graphs on $n$ vertices.) Then a graph property $P$ (like being connected) is said to hold in “most” graphs if the probability that $P$ holds in $G_{n,p}$ goes to one as $n \to \infty$.

In this paper, we develop a random model for finite symmetric integral relation algebras, and prove some preliminary results.

**Definition 1.** Let $R(n, p)$ denote the probability space whose events are the finite symmetric integral not-necessarily-associative relation algebras with $n$ diversity atoms. For each diversity cycle $abc$, make it mandatory with probability $p$ (and forbidden otherwise), with these choices independent of one another.

**Example 2.** Let $n = 3$, and $p = \frac{1}{2}$. Given diversity atoms $a, b, c$, the possible diversity cycles are $aaa, bbb, ccc, abb, baa, acc, caa, bcc, cbb, abc$. The random selection of all cycles except $bbb$ and $cbb$ gives relation algebra $59_{65}$, while the selection of only $abb, acc$, and $bcc$ gives $1_{65}$. Clearly, some selections will fail to give a relation algebra.

**Theorem 3.** For any fixed $0 < p \leq 1$, the probability that $R(n, p)$ is a relation algebra goes to one as $n \to \infty$.

**Proof.** We must show that $R(n, p)$ is associative, for which it suffices to show the following: for all mandatory $abc$ and $xye$, there is a $z$ such that $axz$ and $byz$ are mandatory. There are $n + 2\binom{n}{2} + \binom{n}{3}$ diversity cycles, which is asymptotically $\frac{n^3}{6}$. There are thus $\left(\frac{n^3}{2}\right)$ possible pairs of cycles, which is
asymptotically \( \frac{n^6}{72} \). (This is over-counting, since some of those pairs won’t “match up” with a common diversity atom, but it won’t matter.) For any given pair \( abc, xyc \), the probability that, for a particular atom \( z \), \( axz \) and \( byz \) are not both mandatory is \( 1 - p^2 \). The probability that no such \( z \) works is then \( \prod_z (1 - p^2) \). Hence the overall probability of failure of associativity is bounded above by

\[
\sum_{abc} \prod_z (1 - p^2) = \sum_{abc} (1 - p^2)^n,
\]

which is asymptotically \( \frac{n^6}{72} (1 - p^2)^n \), which goes to zero for fixed \( p \).

Now we turn to the question of representability. We use the fact that having a flexible atom is sufficient for representability over a countable set.

**Theorem 4.** Let \( p \geq n \frac{1}{(n+1)^2} \). Then the expected number of flexible atoms is \( R(n, p) \) is at least one.

**Proof.** Given an atom \( z \), the probability that it is flexible is \( p \frac{1}{n+1} \), since all of the \( \binom{n+1}{2} \) cycles involving \( z \) must be mandatory. Then by linearity of expectation we have

\[
\mathbb{E}[\text{number of flexible atoms}] = \sum_z p \frac{1}{n+1} = np \frac{1}{n+1}.
\]

Set \( p \geq n \frac{1}{(n+1)^2} \). Then \( np \frac{1}{n+1} \geq n \left( n \frac{1}{(n+1)^2} \right) \frac{1}{n+1} = 1 \).

Theorem 3 has two rather glaring shortcomings. First, it doesn’t show that the probability of representability goes to one as \( n \to \infty \), as one usually wants. Second, using the presence of a flexible atom as a sufficient condition for representability is overkill. It seems like it ought to be possible to strengthen Theorem 4 to prove that almost all finite symmetric integral relation algebras are representable, and a more general definition of \( R(n, p) \) might allow a positive solution to problem 20 from [4]: If \( RA(n) \) (respectively, \( RRA(n) \)) is the number of isomorphism types of relation algebras (respectively, representable relation algebras) with no more than \( n \) elements, is it
the case that
\[
\lim_{n \to \infty} \frac{RRA(n)}{RA(n)} = 1?
\]

However, what is really desired (by this author, at least) is a notion
of a quasirandom relation algebra. There are many graph properties, all
asymptotically equivalent, that hold almost surely in \( G_{n,1/2} \) and therefore
can be taken as a definition of a quasirandom graph. One such example
is the property of having all but \( o(n) \) vertices of degree \( (1 + o(1)) \frac{n}{2} \). Such
properties serve as proxies for “randomness”.

In a similar fashion, quasirandom subsets of \( \mathbb{Z}/n\mathbb{Z} \) were defined in [1].
Again, a number of properties were proved to be asymptotically equivalent.
One such property is that of the characteristic function of the subset having
small (as in \( o(n) \)) nontrivial Fourier coefficients.

What would be a quasirandom relation algebra? Restricting attention
once again to symmetric integral relation algebras, here is one possibility. For
each atom \( a \), form a graph \( G_a \) with vertices labeled with the other diversity
atoms, with an edge between \( b \) and \( c \) if \( abc \) is mandatory (or a loop on \( b \) if
\( abb \) is mandatory). Then call the algebra quasirandom if all but \( o(n) \) of the
graphs \( G_a \) are quasirandom.

Is this a good definition? Probably not. (It completely ignores 1-cycles,
for example. Does that matter? The fraction of diversity cycles that are
1-cycles is asymptotically zero.) I offer it merely as an example of the sort of
thing one might propose. My purpose is to start a conversation that might
lead to a significant interaction between the field of relation algebra and
the subfield of combinatorics that is concerned with quasirandom structures.
This paper is a first step.

Here are a few problems to consider.

**Problem 1.** Is there a function \( p(n) \) such that \( R(n,p(n)) \) is asymptoti-
cally the uniform distribution on symmetric integral relation algebras of order
\( 2^{n+1} \)?

**Problem 2.** Improve the bound on \( p \) in Theorem 4.

**Problem 3.** Formulate several notions of quasirandomness for relation al-
gebras, and show that they are equivalent, as in [1, 2]. Maddux’s work on
algebras with no mandatory 3-cycles [3] suggests that the difficult part of
representability lies in the 3-cycles. Results on quasirandom 3-uniform hy-
pergraphs might be relevant.
Problem 4. First-order graph properties obey a 0-1 law in the standard uniform random graph model, i.e., every property holds with asymptotic probability 1 or asymptotic probability 0 in $G_{n,1/2}$. Does the same hold for $R(n,p)$?

References

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