THE MEAN FIELD EQUATION WITH CRITICAL PARAMETER
IN A PLANE DOMAIN

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Abstract. Consider the mean field equation with parameter \( \lambda = 8\pi \) in a bounded smooth domain \( \Omega \). Denote by \( E_{8\pi}(\Omega) \) the infimum of the associated functional \( I_{8\pi}(\Omega) \). We prove that if \( |\Omega| = \pi \), then \( E_{8\pi}(\Omega) \geq E_{8\pi}(B_1) \) and equality holds if and only if \( \Omega \) is a ball. We also give a sufficient condition for the existence of a minimizer for \( I_{8\pi}(\Omega) \).

1. Introduction

Let \( \Omega \subset \mathbb{R}^2 \) be a bounded domain with smooth boundary. Consider the following Mean Field equation

\[
\begin{cases}
-\Delta u = \frac{\lambda u}{\int_\Omega e^u}, & \text{in } \Omega, \\
u|_{\partial \Omega} = 0,
\end{cases}
\]

where \( \Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \) and \( \lambda \) is a real parameter. Equation (1.1) appears naturally in many physical problems. For example in [2] and [3], it has been derived from the mean field limit of the Gibbs measure associated to a system of \( N \) vortices. It also arises in the study of the Chern-Simons-Higgs model of superconductivity (see for example [8]). To study the existence of solutions to equation (1.1), we may use the variational approach. We consider the associated nonlinear functional \( I_\lambda \):

\[
I_\lambda(u, \Omega) = \frac{1}{2\lambda} \int_\Omega |\nabla u|^2 - \ln \left( \frac{1}{|\Omega|} \int_\Omega e^u \right),
\]

for \( u \in H^1_0(\Omega) \), and denote

\[
E_\lambda(\Omega) = \inf_{u \in H^1_0(\Omega)} I_\lambda(u, \Omega).
\]

A well known fact is that \( I_\lambda(u) \) is bounded below if and only if \( \lambda \leq 8\pi \). In particular, when \( \lambda < 8\pi \) Moser-Trudinger inequality [12] implies that the infimum of \( I_\lambda(u, \Omega) \) is always attained. However, in the critical case \( \lambda = 8\pi \), the existence of a minimizer of \( I_{8\pi} \) is a very difficult problem and depends on the geometry of \( \Omega \). When \( \Omega \) is a ball, the infimum of \( I_{8\pi} \) is never attained (see for example [2], [4]). Yet, when \( \Omega \) is thin, the infimum of \( I_{8\pi}(\Omega) \) can be achieved (see for example Proposition 1). For general domains, there are only a few results about the existence of a minimizer of \( I_{8\pi}(\Omega) \). For example, Chang, Chen and Lin proved that the set of domains \( \Omega \) on

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which the infimum of $I_{8\pi}$ is attained is open in the $C^1$ topology ([5]). In this paper we study the functional $I_\lambda$ in the critical case $\lambda = 8\pi$ and obtain the following

**Theorem 1.** Suppose that $|\Omega| = |B_1| = \pi$, where $|\Omega|$ is the area of $\Omega$ and $B_1$ is the unit ball in $\mathbb{R}^2$. Then

$$E_{8\pi}(\Omega) \geq E_{8\pi}(B_1),$$

and the equality holds if and only if $\Omega = B_1$.

If we view the infimum $E_{8\pi}(\Omega)$ as a Liouville type energy of the domain, then Theorem 1 says that we can use this energy to distinguish the unit ball from other domains with the same area, since the unit ball has the lowest energy among these domains. It is interesting to compare these with the Yamabe problem. Let $(M, g_0)$ be a Riemannian manifold of dimension $n > 2$. The Yamabe problem is to find a metric $g$ conformal to $g_0$ such that $(M, g)$ has constant scalar curvature $R$. If we write $g = u^{n-2}g_0$ with $q = 2n/(n-2)$, then $u$ satisfies the Yamabe equation:

$$-L_{g_0} u = R_u u^\frac{n}{n-2},$$

where $L_{g_0}$ is the conformal Laplacian. The associated variational problem is

$$\mu(M, g_0) = \inf \left\{ \int_M (a_n |du|^2 + Ru^2) dV_{g_0} : \int_M |u|^q dV_{g_0} = 1 \right\},$$

where $a_n = 4(n-1)/(n-2)$. Denote the scalar curvature of $g$ by $R_g$, we have

$$\mu(M, g_0) = \inf_{g \in [g_0]} \frac{\int_M R_g dV_g}{\int_M dV_g},$$

where $[g_0]$ is the conformal class of $g_0$. The solution to the famous Yamabe problem can be summarized as the following theorem:

**Theorem 2.** (Yamabe, Trudinger and Aubin) Let $(M, g_0)$ be a compact Riemannian manifold of dimension $n > 2$. Then $\mu(M) \leq \mu(S^n)$, where $S^n$ is the sphere with the standard metric. If $\mu(M) < \mu(S^n)$, the infimum of (1.2) is attained by a positive $C^\infty$ solution to the Yamabe equation.

Based on the work of Yamabe and Trudinger, Aubin and Schoen ([13]) completed the solution of the Yamabe problem by proving that $\mu(M, g_0) < \mu(S^n)$ unless $M$ is conformally equivalent to the standard sphere. This indicates that we can use $\mu(M, g_0)$ to distinguish the standard sphere from other compact manifolds in conformal sense. We are certainly wondering whether $E(\Omega)$ plays a similar role in the mean field equation as $\mu(M, g_0)$ in the Yamabe problem. Note that if $\mu(M) < \mu(S^n)$, then the infimum of (1.2) is attained by a positive $C^\infty$ solution to the Yamabe equation. It is interesting to consider, for a bounded smooth domain $\Omega \subset \mathbb{R}^2$ satisfying $|\Omega| = \pi$ and $E_{8\pi}(\Omega) > E_{8\pi}(B_1)$, when the infimum of $I(\cdot, \Omega)$ is attained by a smooth solution to the mean field equation. Chang, Chen and Lin ([5]) gave an example of a dumbell $\Omega_h$ which consists of two disjoint balls $B(r_1)$ and $B(r_2)$ connected with a tube of small width $h > 0$. They proved that when $r_1 < r_2$ and $h$ is sufficiently small, the infimum of $I(\cdot, \Omega_h)$ is not attained (see Proposition 7.3 in [5]). Therefore in order that the infimum of $I(\cdot, \Omega)$ is attained, we must add more conditions on the domain $\Omega$. For example, we could require that $\Omega$ is thin.

**Proposition 1.** Suppose $|\Omega| = \pi$ and $\Omega$ can be covered by a strip with width $d \leq \frac{\pi}{2\sqrt{\pi}} = 0.9527\ldots$, then the infimum of $I_{8\pi}(u, \Omega)$ can be achieved by a function in $H_0^1(\Omega)$. 
The paper is organized as follows. In section 2, we study the regular part $\gamma(\Omega)$ of the Green’s function of $\Omega$. The property of $\gamma(\Omega)$ will be used in the proof of Theorem \[\text{1}\] In section 3 we derive some standard estimates for $E_{8\pi}(\Omega)$. Theorem \[\text{1}\] then follows from these estimates. The existence result Proposition \[\text{1}\] is proved in Section 4.

We will suppress the subscript “$8\pi$” if no confusion would result.

2. Regular Part of the Green’s Function

Denote by $G(x,y)$ the Green’s function of $\Omega$:

$$
\begin{align*}
-\Delta_x G(x,y) &= \delta_y(x), \quad \text{in } \Omega, \\
G(\cdot,y)|_{\partial \Omega} &= 0.
\end{align*}
$$

Let

$$
\gamma(x,y) = G(x,y) - \frac{1}{2\pi} \ln \frac{1}{|x-y|},
$$

be the regular part of $G$, and set $\gamma(x) = \gamma(x,x)$, $\gamma(\Omega) = \sup_{x \in \Omega}(\gamma(x))$. Then we have the following Lemma.

**Lemma 1.** Suppose $\Omega \subset \mathbb{R}^2$ is an bounded domain with $|\Omega| = \pi$. Then $\gamma(\Omega) \leq 0$, and equality holds if and only if $\Omega = B_1$, the unit ball in $\mathbb{R}^2$.

**Proof.** For any $x_0 \in \Omega$,

$$
G(x_0,y) = \frac{1}{2\pi} \log \frac{1}{|x_0-y|} + \gamma(x_0) + O(|x_0-y|), \quad \text{as } y \to x_0.
$$

Therefore for any $\epsilon > 0$, there exists $\rho > 0$, s.t.

$$
\left| G(x_0,y) - \frac{1}{2\pi} \log \frac{1}{|x_0-y|} - \gamma(x_0) \right| \leq \epsilon \quad \text{whenever } |y-x_0| \leq \rho.
$$

It follows that when $\tau$ is sufficiently large,

$$
\text{(2.1)} \quad B_{x_0}(e^{-2\pi(\tau-\gamma(x_0)+\epsilon)}) \subset \Omega_\tau \subset B_{x_0}(e^{-2\pi(\tau-\gamma(x_0)-\epsilon)}),
$$

where $\Omega_\tau = \{G(x_0,y) > \tau\}$ and $B_{x_0}(r)$ is the ball in $\mathbb{R}^2$ with radius $r$ and centered at $x_0$.

Let $G^*(y) : B(1) \to \mathbb{R}$ be the rearrangement of $G(x_0,y)$ and let

$$
\mu(\tau) = |\Omega_\tau|, \quad \rho(\tau) = \sqrt{\mu(\tau)/\pi}.
$$

Then \[\text{2.1}\] implies

$$
\text{(2.2)} \quad e^{-2\pi(\tau-\gamma(x_0)+\epsilon)} \leq \rho(\tau) \leq e^{-2\pi(\tau-\gamma(x_0)-\epsilon)}.
$$

Define a function $\phi(t) : [0, \infty) \to \mathbb{R}$ by

$$
\phi(t) = \int_{\Omega \setminus \Omega_t} |\nabla_y G(x_0,y)|^2 dV_y - \int_{B(1) \setminus B(\rho(t))} |\nabla G^*(y)|^2 dV_y.
$$

From the properties of rearrangement, it is easy to see that $\phi(t)$ is increasing, $\phi(0) \geq 0$, $G^*(y)|_{\partial B(\rho(t))} = t$ and $G^*(y)|_{\partial B(1)} = 0$. It follows that

$$
\int_{B(1) \setminus B(\rho(t))} |\nabla G^*(y)|^2 dV_y \geq \int_{B(1) \setminus B(\rho(t))} |\nabla G_0(y)|^2 dV_y,
$$
where \( G_0(y) = t \log(|y|)/\log(t) \). Therefore using (2.2), for sufficiently large \( t \),

\[
\phi(t) = \int_{\Omega \setminus \Omega_t} |\nabla_y G(x_0, y)|^2 dV_y - \int_{B_1(1) \setminus B_{\rho(t)}} |\nabla G^*(y)|^2 dV_y \\
\leq \int_{\Omega \setminus \Omega_t} \nabla_y G(x_0, y) \cdot \nabla_y G(x_0, y) dV_y - \int_{B_1(1) \setminus B_{\rho(t)}} |\nabla G_0(y)|^2 dV_y \\
= -t \int_{\partial \Omega_t} \frac{\partial G(x_0, y)}{\partial n} dS + \frac{2\pi t^2}{\log \rho(t)} \\
\leq t - \frac{t^2}{t - \gamma(x_0) + \epsilon} = \frac{t}{t - \gamma(x_0) + \epsilon}.
\]

Since \( \phi(t) \geq 0 \) and \( \epsilon \) is arbitrary, we obtain that \( \gamma(x_0) \leq 0 \). Hence \( \gamma(\Omega) \leq 0 \). Furthermore, if \( \gamma(\Omega) = 0 \), it follows from the continuity of \( \gamma(x) \) that there exists \( x_0 \), s.t. \( \gamma(x_0) = 0 \). Then (2.3) implies \( \phi(t) \leq \epsilon \). Since \( \phi(t) \) is increasing, \( \phi(t) \geq 0 \) and \( \epsilon \) is arbitrary, we have \( \phi(t) = 0 \), for all \( t > 0 \). From the properties of rearrangement, we have \( \Omega = B(1) \).

**Remark 1.** The proof of Lemma 4 can be easily extended to higher dimensions. Let \( \Omega \subset \mathbb{R}^n \), \( n \geq 2 \) and let \( G_n(x, y) \) be the solution of the following differential equation:

\[
\begin{cases}
-\Delta_n u(x) = \delta_y(x), & \text{in } \Omega, \\
u|_{\partial \Omega} = 0,
\end{cases}
\]

where \( \Delta_n u(x) = \nabla(|\nabla u(x)|^{n-2} \nabla u(x)) \) is the \( n \)-Laplacian. We may consider as in the case \( n = 2 \) the regular part:

\[
\gamma_n(x, y) = G_n(x, y) - \frac{1}{n-1} \ln \frac{1}{|x - y|},
\]

where \( \omega_{n-1} \) is the volume of the \( (n-1) \)-dimensional sphere. Set \( \gamma_n(x) = \gamma_n(x, x) \), and \( \gamma(\Omega) = \sup_{x \in \Omega} (\gamma_n(x)) \). Then we have the following Lemma.

**Lemma 2.** Suppose \( \Omega \subset \mathbb{R}^n \) is a bounded domain with \( |\Omega| = |B_1^n| \), where \( B_1^n \) the unit ball in \( \mathbb{R}^n \). Then \( \gamma_n(\Omega) \leq 0 \), and equality holds if and only if \( \Omega = B_1^n \).

### 3. Estimates of \( E(\Omega) \)

In this section, we always assume that \( \Omega \subset \mathbb{R}^2 \) is a bounded domain with smooth boundary and \( |\Omega| = \pi \). We have the following estimates for \( E_{8\pi}(\Omega) \):

**Proposition 2.**

\[
E_{8\pi}(\Omega) \leq -1 - 4\pi \gamma(\Omega).
\]

**Proposition 3.** If the infimum of \( I_{8\pi}(\cdot, \Omega) \) is not attained, then

\[
E_{8\pi}(\Omega) \geq -1 - 4\pi \gamma(\Omega).
\]

These two estimates are very standard. For example, Proposition 3 has been proved in [2]. For the sake of completeness, we shall reprove these two estimates using a different method.
Suppose $G(x, y)$ is the Green’s function of $\Omega$, and $\gamma(x)$ is its regular part. We may assume that $\gamma(x)$ attains its maximum value at $x_0 \in \Omega$. Without loss of generality, suppose $x_0 = 0$. Let

$$G(x) = 8\pi G(x, 0) = 4 \ln \frac{1}{|x|} + A + \alpha(x),$$

where $A = 8\pi \gamma(0)$. For any $\Lambda, \epsilon > 0$, such that $\rho = \Lambda \epsilon < \text{dist}(0, \partial \Omega)$, we choose a test function

$$\phi(x) = \begin{cases} 
2 \ln \frac{\rho}{r^2 + \rho^2} - C_\epsilon, & \rho \leq |x| \\
G(x) - \eta(x)\alpha(x), & |x| < 2\rho \\
G(x), & |x| \geq 2\rho
\end{cases}$$

where $\eta(x)$ is a $C^\infty$ bump function,

$$\eta(x) = \begin{cases} 
1, & |x| \leq \rho \\
0, & |x| \geq 2\rho
\end{cases}$$

satisfying $|\nabla \eta(x)| \leq 2/\rho$, and

$$C_\epsilon = 2 \ln \frac{\Lambda^2}{1 + \Lambda^2} - A$$

so that the function $\phi(x) \in H^1_0(\Omega)$. It suffices to prove that

$$I(\phi, \Omega) \leq -1 - \frac{A}{2 \pi}.$$

First

$$\int_{B(\rho)} |\nabla \phi|^2 = 4 \int_{B(\rho)} \left(\frac{2r}{\epsilon^2 + r^2}\right)^2 \, dx = 32\pi \int_0^\rho \frac{r^2}{(\epsilon^2 + r^2)^2} \, dr = 16\pi \int_0^1 \frac{(1 - t)^2}{t} \, dt,$$

where

$$t = \frac{\epsilon^2}{\epsilon^2 + r^2} \quad \text{and} \quad \sigma = t(\rho) = \frac{1}{1 + \Lambda^2}.$$

Therefore

$$\frac{1}{16\pi} \int_{B(\rho)} |\nabla \phi|^2 = -\ln \sigma - 1 + \sigma.$$  

Second, since $G(x) = -4 \ln |x| + A + \alpha(x)$, $\alpha(x)$ is a smooth function and $\alpha(0) = 0$, we have when $\rho$ is sufficiently small,

$$\int_{\Omega \setminus B(\rho)} |\nabla G|^2 = -\int_{\partial B(\rho)} G \frac{\partial G}{\partial \mathbf{n}} = -\left(4 \ln \frac{1}{\rho} + A + O(\rho)\right) \int_{\partial B(\rho)} \frac{\partial G}{\partial \mathbf{n}} = 32\pi \ln \frac{1}{\rho} + 8\pi A + O(\rho).$$

For $|x| > \rho$, we have $\phi(x) = G(x) - \eta(x)\alpha(x)$. Therefore

$$|\nabla \phi|^2 = |\nabla G - \nabla(\eta \alpha)|^2 = |\nabla G|^2 + 2 \nabla G \cdot \nabla(\eta \alpha) + |\nabla(\eta \alpha)|^2.$$

Since $\alpha(x)$ is smooth and $\alpha(0) = 0$, $|\nabla G| = \frac{2}{\rho} (1 + O(\rho))$ for $2\rho > |x| > \rho$, and $|\nabla(\eta \alpha)| \leq C$ for any $x$. Hence,

$$\int_{\Omega \setminus B(\rho)} 2 \nabla G \cdot \nabla(\eta \alpha) + |\nabla(\eta \alpha)|^2 = \int_{B(2\rho) \setminus B(\rho)} 2 \nabla G \cdot \nabla(\eta \alpha) + |\nabla(\eta \alpha)|^2 = O(\rho),$$
which implies

\[ (3.3) \quad \int_{\Omega \setminus B(\rho)} |\nabla \phi|^2 = 32\pi \ln \frac{1}{\rho} + 8\pi A + O(\rho). \]

Combining (3.2) and (3.3), we obtain

\[ (3.4) \quad \frac{1}{16\pi} \int_{\Omega} |\nabla \phi|^2 = -\ln \sigma - 1 + \sigma - 2 \ln \rho + \frac{A}{2} + O(\rho). \]

Now we turn to the estimate of \( \int_{\Omega} e^\phi \).

\[ \int_{B(\rho)} e^\phi = \exp(-C) \int_{B(\rho)} \left( \frac{1}{\epsilon^2 + r^2} \right)^2 \ dx = 2\pi e^{-C} \int_0^\rho \frac{r}{\epsilon^2 + r^2} \ dr = \frac{\pi e^4}{\epsilon^2} \frac{1 + \Lambda^2}{\Lambda^2} \]

Since \( \phi(x) > 0 \) when \( \rho \) is small, we have

\[ (3.5) \quad -\ln \frac{1}{\pi} \int_{\Omega} e^\phi \leq -\ln \frac{1}{\pi} \int_{B(\rho)} e^\phi = -A + 2 \ln \epsilon - \ln \frac{1 + \Lambda^2}{\Lambda^2}. \]

Sending \( \Lambda \rightarrow \infty \) and using (3.4) and (3.5) we finally get

\[
I(\phi, \Omega) = \frac{1}{16\pi} \int_{\Omega} |\nabla \phi|^2 - \ln \frac{1}{\pi} \int_{\Omega} e^\phi \leq -1 - \frac{A}{2} = -1 - 4\pi \gamma(\Omega).
\]

This completes the proof of Proposition 2.

Next, we prove the opposite inequality, Proposition 3. We will use a similar argument given in [7]. Suppose the infimum of \( I(\cdot, \Omega) \) is not attained. For \( \epsilon > 0 \) we define

\[
I^\epsilon(u) = \frac{1}{16\pi} \int_{\Omega} |\nabla u|^2 - (1 - \epsilon) \ln \frac{1}{\pi} \int_{\Omega} e^u,
\]

and

\[
E^\epsilon = \inf_{u \in H^1_0(\Omega)} I^\epsilon(u).
\]

**Theorem 3.** ([11]) Let \( \Omega \subset \mathbb{R}^2 \) be a bounded smooth domain, \( \Omega^* \) be the ball in \( \mathbb{R}^2 \) which has the same area as \( \Omega \), and denote

\[
D_{a,b}(\Omega) = \{ f(x) - b \in H^1_0(\Omega) : \int_{\Omega} e^{2f} \ dx = a \}.
\]

We have the following sharp inequality:

\[
\inf_{w \in D_{a,b}(\Omega)} \int_{\Omega} |\nabla w|^2 \ dx \geq 4\pi \left( \ln \frac{ae^{-2b}}{\pi r^2} + \frac{\pi r^2}{ae^{-2b}} - 1 \right),
\]

where \( r \) is the radius of \( \Omega^* \).

It follows from the above sharp inequality that for any \( u \in H^1_0(\Omega) \)

\[
(3.6) \quad \int_{\Omega} e^u \ dx \leq \pi e \exp \left( \frac{1}{16\pi} \int_{\Omega} |\nabla u|^2 \ dx \right),
\]

which implies \( I^\epsilon(u) \) is bounded below by \(-1\).
Lemma 3. $E_\epsilon$ is achieved by a function $u_\epsilon \in H^1_0(\Omega)$, which is the solution of the following equation

\begin{equation}
\begin{aligned}
-\Delta u_\epsilon &= \frac{(1-\epsilon)8\pi e^{u_\epsilon}}{\int_\Omega e^{u_\epsilon}}, \text{ in } \Omega, \\
|u_\epsilon|_{\partial \Omega} &= 0,
\end{aligned}
\end{equation}

Proof. For fixed $\epsilon > 0$, let $\{u_n\}$ be a minimizing sequence of $\inf I^\epsilon$. It follows from (3.6) that

\begin{align*}
I^\epsilon(u_n) &= \frac{1}{16\pi} \int_\Omega |\nabla u_n|^2 - (1-\epsilon) \ln \frac{1}{\pi} \int_\Omega e^{u_n} \\
&= (1-\epsilon) I(u_n) + \epsilon \int_\Omega |\nabla u_n|^2 \\
&\geq - (1-\epsilon) + \epsilon \int_\Omega |\nabla u_n|^2.
\end{align*}

Therefore $||\nabla u_n||_{L^2} < C$. Since $u_n|_{\partial \Omega} = 0$, using Poincaré’s inequality we obtain that $||u_n||_{H^1} < C$. Hence $u_n \rightharpoonup u_\epsilon$ in $H^1_0(\Omega)$ for some $u_\epsilon \in H^1_0(\Omega)$. It follows from Trudinger’s inequality \cite{14} that $e^{ku_n} \to e^{k u_\epsilon}$ in $L^1(\Omega)$ for any $k > 0$. Therefore $u_\epsilon$ is a minimizer and satisfies the Euler-Lagrange equation (3.7). □

It follows from standard elliptic estimates that $u_\epsilon \in C^\infty(\Omega)$. Suppose $u_\epsilon$ attains its maximum at $x_\epsilon \in \Omega$ and set $\lambda_\epsilon = u_\epsilon(x_\epsilon) = \max_{x \in \bar{\Omega}} u_\epsilon(x)$. The following Lemma is immediate.

Lemma 4. There exists a subsequence $\epsilon_i \to 0$, such that

\[ \lim_{i \to \infty} \lambda_{\epsilon_i} = +\infty. \]

Proof. Suppose $\lambda_\epsilon$ is bounded above, $\lambda_\epsilon \leq C < +\infty$, then

\[ \int_\Omega e^{u_\epsilon} \leq C. \]

Since $E^\epsilon \leq I^\epsilon(0) = 0$, we have $\int_\Omega |\nabla u_n|^2 \leq C$. Hence there exists a subsequence of $u_\epsilon$ which converge weakly to $u_0$ in $H^1_0(\Omega)$. We can easily check that $u_0$ is a minimizer for $I(\cdot, \Omega)$, which contradict with the assumption that the infimum of $I(\cdot, \Omega)$ is not attained. □

In the following, for simplicity, we shall not distinguish a subsequence $\{\epsilon_i\}$ from the original $\{\epsilon\}$.

Next, we claim that $x_\epsilon$ will stay away from $\partial \Omega$, which implies

\[ x_\epsilon \to \bar{x} \in \Omega \text{ as } \lambda \to 0. \]

The claim can be proved by the moving plane method (see \cite{3}) and an interior integral estimate (cf. page 163-164 in \cite{10}). We shall omit the details here. Set $\tau_\epsilon = e^{\lambda_\epsilon/2}$ and

\[ \alpha_\epsilon = \left( \frac{(1-\epsilon)\pi}{\int_\Omega e^{u_\epsilon}} \right)^{1/2} \tau_\epsilon. \]

If $\alpha_\epsilon$ stays bounded as $\epsilon \to 0$, standard elliptic estimate of (3.7) implies that $u_\epsilon$ is uniformly bounded as $\epsilon \to 0$, which contradicts with the fact that $u_\epsilon$ blows up (Lemma 4). Therefore we have:
Lemma 5. \[
\lim_{\epsilon \to 0} \alpha_\epsilon = +\infty.
\]
Define
\[\phi_\epsilon(x) = u_\epsilon(\alpha_\epsilon^{-1} x + x_\epsilon) - 2 \ln \tau_\epsilon.\]
We can easily see that \(\phi_\epsilon\) satisfies
\[\begin{cases}
-\Delta \phi_\epsilon = 8 e^{\phi_\epsilon}, & \text{in } \Omega_\epsilon \\
\phi_\epsilon|_{\partial \Omega_\epsilon} = -2 \ln \tau_\epsilon,
\end{cases}\]
where \(\Omega_\epsilon = \alpha_\epsilon \cdot (\Omega - x_\epsilon)\). We claim that for any \(R > 0\), \(\phi_\epsilon\) is bounded in \(B(R)\) uniformly in \(\epsilon\). In fact, let \(\phi_\epsilon^{(1)}\) be the unique solution to
\[\begin{cases}
-\Delta \phi_\epsilon^{(1)} = 8 e^{\phi_\epsilon}, & \text{in } B(2R) \\
\phi_\epsilon^{(1)}|_{\partial B(2R)} = 0.
\end{cases}\]
Since \(x_\epsilon\) is a maximum point of \(u_\epsilon(x)\), we have \(\phi_\epsilon \leq \phi_\epsilon(0) = 0\) and \(e^{\phi_\epsilon} \leq 1\). It follows that \(\|\phi_\epsilon^{(1)}\|_{L^\infty} \leq C < +\infty\). Let \(\phi_\epsilon^{(2)} = \phi_\epsilon - \phi_\epsilon^{(1)}\). Then \(\phi_\epsilon^{(2)} \leq -\phi_\epsilon^{(1)} \leq C\). Since \(2C - \phi_\epsilon^{(2)}(0) = 2C - \phi_\epsilon(0) + \phi_\epsilon^{(1)}(0) \leq 3C\), Harnack’s inequality implies that \(\|2C - \phi_\epsilon^{(2)}\|_{L^\infty} \leq \tilde{C}\) in \(B(R)\). Hence \(\|\phi_\epsilon\|_{L^\infty(B(R))} \leq C + \tilde{C}\). Elliptic estimates yield that, up to a subsequence, \(\phi_\epsilon \to \phi_0\) in \(C^{1,\alpha}(B(R/2))\) for some \(\alpha \in (0, 1)\) and \(\phi_0\) satisfies
\[\begin{cases}
-\Delta \phi_0 = 8 e^{\phi_0}, & \text{in } \mathbb{R}^2 \\
\phi_0(0) = 0.
\end{cases}\]
Since \(\int_{\Omega_\epsilon} e^{\phi_\epsilon} = 8\pi(1 - \epsilon)\), we get
\[\int_{\mathbb{R}^2} e^{\phi_0} \leq \lim_{R \to \infty} \lim_{n \to \infty} \int_{B(R)} e^{\phi_\epsilon^{(1)}} \leq 8\pi(1 - \epsilon).
\]
The uniqueness theorem in [6] implies that
\[\phi_0(x) = 2 \ln \frac{1}{1 + |x|^2}.
\]
The following Lemma is due to Brezis and Merle [1]:

Lemma 6. Let \(D \subset \mathbb{R}^2\) be a bounded domain and \(u\) be a solution to the following equation,
\[\begin{cases}
-\Delta u = f(x), & \text{in } D, \\
u|_{\partial D} = 0.
\end{cases}\]
If \(f \in L^1(D)\), then for any \(\delta \in (0, 4\pi)\), there is a constant \(C(\delta)\) such that
\[\int_D \exp \left(\frac{(4\pi - \delta)|u(x)|}{\|f\|_{L^1(D)}}\right) \leq C(\delta).
\]
Using this lemma we have:

Lemma 7. For any \(K \subset \subset \Omega \setminus \{\bar{x}\}\), there exists a constant \(C(K)\), such that \(u_\epsilon(x) \leq C(K)\), for all \(x \in K\).
Proof. From (3.7) and Lemma 6 we know that \( e^{u_\epsilon} \in L^p(\Omega) \) for \( p \in (0, \frac{1}{2}) \). For any given \( K \subset \subset \Omega \setminus \{\tilde{x}\} \), we choose \( K' \) such that \( K \subset \subset K' \subset \subset \Omega \setminus \{\tilde{x}\} \). Since \( \phi_\epsilon \to \phi_0 \) in \( C^{1,\alpha} \) and \( \int_{\mathbb{R}^2} e^{\phi_0} = \pi \), we obtain that

\[
\lim_{\epsilon \to 0} \frac{\int_{\Omega} e^{u_\epsilon}}{\int_{\Omega} e^{u_\epsilon}} = 0.
\]

Let \( u^{(1)}_\epsilon \) be the unique solution to

\[
\begin{aligned}
&-\Delta u^{(1)}_\epsilon = (1-\epsilon)8\pi e^{u_\epsilon}, \text{ in } K', \\
&u^{(1)}_\epsilon|_{\partial \Omega} = 0,
\end{aligned}
\]

It follows from (3.11) and Lemma 6 that \( u^{(1)}_\epsilon \in L^p(K') \) for some \( p > 1 \). Since \( u^{(2)}_\epsilon := u_\epsilon - u^{(1)}_\epsilon \) is harmonic in \( K' \), Harnack’s inequality implies that

\[
\|u^{(2)}_\epsilon\|_{L^p(K')} \leq C\|u^{(2)}_\epsilon\|_{L^p(K')}
\]

\[
\leq C\|u_\epsilon\|_{L^p(K')} + \|u^{(1)}_\epsilon\|_{L^p(K')}
\]

\[
\leq C.
\]

Therefore, for some \( p > 1 \) we have

\[
\int_{K'} e^{pu_\epsilon} = \int_{K'} e^{pu^{(1)}_\epsilon} \cdot e^{pu^{(2)}_\epsilon} \leq C \int_{K'} e^{pu^{(1)}_\epsilon} \leq C
\]

It follows from the standard elliptic estimates of (3.7) that \( \|u^{(1)}_\epsilon\|_{L^\infty(K)} \leq C \), thus

\[
\|u_\epsilon\|_{L^\infty(K)} \leq C.
\]

We may assume without loss of generality that \( \tilde{x} = 0 \). Since \( u_\epsilon \) satisfies (3.7), it follows from Lemma 7 that \( u_\epsilon \to G(x) \in C^{1,\alpha}_e(\Omega \setminus \{\tilde{x}\}) \), where \( G(x) \) was defined in (3.1).

**Lemma 8.** For fixed \( R \), let \( r_\epsilon = R/\alpha_\epsilon \). Then for any \( x \in \Omega \setminus B(r_\epsilon) \),

\[
u_\epsilon(x) \geq G(x) - \lambda_\epsilon + 2\ln\left(\frac{1}{\pi} \int_\Omega e^{u_\epsilon}\right) + 2\ln\frac{R^2}{1+R^2} - A + o_\epsilon(1),
\]

where \( o_\epsilon(1) \) stands for some function that goes to 0 as \( \epsilon \to 0 \).

**Proof.** On \( \partial B(r_\epsilon) \), \( G(x) \) and \( u_\epsilon \) have the following asymptotic behavior:

\[
G(x) = -4\ln r_\epsilon + A + o_\epsilon(1),
\]

\[
u_\epsilon(x) = \lambda_\epsilon + 2\ln\frac{1}{1+R^2} + o_\epsilon(1).
\]

Therefore on \( \partial B(r_\epsilon) \),

\[
u_\epsilon - G = \lambda_\epsilon + 2\ln\frac{1}{1+R^2} + 4\ln\frac{R}{\alpha_\epsilon} - A + o_\epsilon(1)
\]

\[
= \lambda_\epsilon + 2\ln\frac{R^2}{1+R^2} - A - \ln\frac{1-\epsilon}{\pi} \int_\Omega e^{u_\epsilon} + o_\epsilon(1)
\]

\[
= -\lambda_\epsilon + 2\ln\left(\frac{1}{\pi} \int_\Omega e^{u_\epsilon}\right) + 2\ln\frac{R^2}{1+R^2} - A + o_\epsilon(1).
\]

Let

\[
D_\epsilon = -\lambda_\epsilon + 2\ln\left(\frac{1}{\pi} \int_\Omega e^{u_\epsilon}\right) + 2\ln\frac{R^2}{1+R^2} - A,
\]

\[
\text{\quad } = -\lambda_\epsilon + 2\ln\left(\frac{1}{\pi} \int_\Omega e^{u_\epsilon}\right) + 2\ln\frac{R^2}{1+R^2} - A.
\]
and consider the function \( u_\varepsilon - G - D_\varepsilon \) on \( \Omega \setminus B(r_\varepsilon) \). It satisfies
\[
\begin{cases}
\Delta (u_\varepsilon - G - D_\varepsilon) \leq 0, & \text{in } \Omega \setminus B(r_\varepsilon) \\
u_\varepsilon - G - D_\varepsilon \geq \alpha_\varepsilon(1), & \text{on } \partial(\Omega \setminus B(r_\varepsilon)).
\end{cases}
\]
Then the lemma follows immediately form the maximum principle. \( \square \)

Now we estimate \( I(u_\varepsilon) \). For fixed \( R \), let \( r_\varepsilon = R/\alpha_\varepsilon \) and choose \( \delta > r_\varepsilon \), such that \( B(\delta) \subset \Omega \). Then
\[
\int_{\Omega} |\nabla u_\varepsilon|^2 = \left( \int_{\Omega \setminus B(\delta)} + \int_{B(\delta) \setminus B(r_\varepsilon)} + \int_{B(r_\varepsilon)} \right) |\nabla u_\varepsilon|^2 := I_1 + I_2 + I_3.
\]
Since \( \phi_\varepsilon \to \phi_0 \) in \( C^{1,\alpha}(B(R/2)) \) (3.10), we obtain that
\[
I_3 = \int_{B(r_\varepsilon)} |\nabla u_\varepsilon|^2 = \int_{B(R)} \left| \nabla \left( 2 \ln \frac{1}{1+|x|^2} \right) \right|^2 + o_\varepsilon(1)
= 32\pi \int_0^R \frac{r^2}{(1+r^2)^2}dr + o_\varepsilon(1) = 16\pi \left( \ln(1+R^2) - \frac{R^2}{1+R^2} \right) + o_\varepsilon(1).
\]
From \( u_\varepsilon \to G(x) \) in \( C^{2}_{\text{loc}}(\Omega \setminus \{0\}) \), we know that
\[
I_1 = \int_{\Omega \setminus B(\delta)} |\nabla u_\varepsilon|^2 = \int_{\Omega \setminus B(\delta)} |\nabla G|^2 + o_\varepsilon(1) = -\int_{\partial B(\delta)} G \frac{\partial G}{\partial n} + o_\varepsilon(1).
\]
To estimate \( I_2 \), we apply Lemma 8
\[
I_2 = -\int_{B(\delta) \setminus B(r_\varepsilon)} \Delta u_\varepsilon \cdot u_\varepsilon + \int_{\partial(\delta \setminus B(r_\varepsilon))} u_\varepsilon \frac{\partial u_\varepsilon}{\partial n}
\geq \int_{B(\delta) \setminus B(r_\varepsilon)} (-\Delta u_\varepsilon) \cdot (G + D_\varepsilon + o_\varepsilon(1)) + \int_{\partial(\delta \setminus B(r_\varepsilon))} u_\varepsilon \frac{\partial u_\varepsilon}{\partial n}
= \int_{\partial(\delta \setminus B(r_\varepsilon))} (-\frac{\partial u_\varepsilon}{\partial n} G + \frac{\partial G}{\partial n} u_\varepsilon) + \int_{\partial(\delta \setminus B(r_\varepsilon))} (u_\varepsilon - D_\varepsilon) \frac{\partial u_\varepsilon}{\partial n} + o_\varepsilon(1).
\]
It follows that
\[
\int_{\Omega \setminus B(r_\varepsilon)} |\nabla u_\varepsilon|^2 \geq \int_{\partial B(\delta)} \left( u_\varepsilon \frac{\partial u_\varepsilon}{\partial n} - G \frac{\partial G}{\partial n} - D_\varepsilon \frac{\partial u_\varepsilon}{\partial n} + u_\varepsilon \frac{\partial G}{\partial n} - G \frac{\partial u_\varepsilon}{\partial n} \right)
- \int_{\partial B(r_\varepsilon)} \left( \frac{\partial G}{\partial n} u_\varepsilon + \frac{\partial u_\varepsilon}{\partial n} (u_\varepsilon - G - D_\varepsilon) \right) + o_\varepsilon(1).
\]
Since \( u_\varepsilon \to G(x) \) in \( C^{1,\alpha}_{\text{loc}}(\Omega \setminus \{0\}) \), we can easily see that
\[
\int_{\partial B(\delta)} u_\varepsilon \frac{\partial u_\varepsilon}{\partial n} - G \frac{\partial G}{\partial n} = o_\varepsilon(1), \quad \int_{\partial B(\delta)} u_\varepsilon \frac{\partial G}{\partial n} - G \frac{\partial u_\varepsilon}{\partial n} = o_\varepsilon(1),
\]
and on $\partial B(r_\epsilon)$,
\[
G(x) = 4 \ln \frac{1}{|x|} + A + o_\epsilon(1),
\]
\[
\frac{\partial G}{\partial n} = -4 \frac{1}{r_\epsilon} + o_\epsilon(1),
\]
\[
u_\epsilon(x) = \lambda_\epsilon + 2 \ln \frac{1}{1 + R^2} + o_\epsilon(1),
\]
\[
\frac{\partial \nu_\epsilon}{\partial n} = -\left( \frac{4R}{1 + R^2} + o_\epsilon(1) \right) \alpha_\epsilon.
\]
It follows that
\[
\int_{B(r_\epsilon)} \frac{\partial G}{\partial n} \nu_\epsilon = (\lambda_\epsilon + 2 \ln \frac{1}{1 + R^2} + o_\epsilon(1)) \int_{B(r_\epsilon)} \frac{\partial G}{\partial n} = -8\pi (\lambda_\epsilon + 2 \ln \frac{1}{1 + R^2} + o_\epsilon(1)),
\]
and
\[
\int_{B(r_\epsilon)} \frac{\partial \nu_\epsilon}{\partial n} (\nu_\epsilon - G - D_\epsilon) = o_\epsilon(1).
\]
Since $\nu_\epsilon$ satisfies (3.7), we have
\[
\int_{\partial (B_\delta)} \frac{\partial \nu_\epsilon}{\partial n} = \int_{B_\delta} \Delta \nu_\epsilon = \int_{B_\delta} \Delta \nu_\epsilon = \int_{\Omega} e^{\nu_\epsilon} \geq -(1 - \epsilon)8\pi.
\]
Combining the estimates for $I_1$, $I_2$ and $I_3$, we finally have
\[
\frac{1}{16\pi} \int_\Omega |\nabla \nu_\epsilon|^2 \geq \ln(1 + R^2) - \frac{R^2}{1 + R^2} + \frac{D_\epsilon}{2} (1 - \epsilon) + \frac{\lambda_\epsilon}{2} + \ln \frac{1}{1 + R^2} + o_\epsilon(1)
\]
\[
\geq (1 - \epsilon) \left( \ln \left( \frac{1}{\pi} \int_\Omega e^{\nu_\epsilon} \right) + \ln \frac{R^2}{1 + R^2} - \frac{A_\epsilon}{2} \right) - \frac{R^2}{1 + R^2} + o_\epsilon(1).
\]
Hence
\[
E^* = I^*(\nu_\epsilon) = \frac{1}{16\pi} \int_\Omega |\nabla \nu_\epsilon|^2 - (1 - \epsilon) \ln \left( \frac{1}{\pi} \int_\Omega e^{\nu_\epsilon} \right)
\]
\[
\geq -\frac{(1 - \epsilon)}{2} A + (1 - \epsilon) \ln \frac{R^2}{1 + R^2} - \frac{R^2}{1 + R^2}.
\]
We complete the proof of Proposition 3 by sending $\epsilon \to 0$ and $R \to \infty$.

Now we are ready to prove Theorem 1. Suppose the infimum of $I(\cdot, \Omega)$ is not attained, then Proposition 3 says $E(\Omega) \geq -1 - 4\pi\gamma(\Omega)$. Together with Proposition 2, we have
\[
E(\Omega) = -1 - 4\pi\gamma(\Omega).
\]
Since the infimum of $I(\cdot, B_1)$ is not attained, $E(B_1) = -1 - 4\pi\gamma(B_1) = -1$. Therefore Lemma 1 implies $E(\Omega) \geq E(B_1)$ and equality holds if and only if $\Omega = B_1$.

If on the other hand, the infimum of $I(\cdot, \Omega)$ is attained by some $u \in H^1_{\text{loc}}(\Omega)$. Then we have $\Omega \neq B_1$ and
\[
E(\Omega) = I(u, \Omega) \geq I(u^*, B_1) > E(B_1),
\]
where $u^* : B_1 \to \mathbb{R}$ is the rearrangement of $u$. This completes the proof of Theorem 1.
4. An existence result

In this section we provide an existence result. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with $|\Omega| = \pi$. Suppose $\Omega$ can be covered by a strip $D_d$ with width $d \leq \frac{\pi}{2\sqrt{e}}$, then we will show that the infimum of $I(\cdot, \Omega)$ can be achieved.

Without loss of generality, we may assume that $D_d = \{z \mid 0 < \text{Im} z < d \}$. Here we have identified $\mathbb{C}$ with $\mathbb{R}^2$. It is easy to see that $w = \phi(z) = e^{\pi z/d}$ maps $D_d$ to the upper half plane $\text{Im} z > 0$. Therefore for $\alpha \in (0, d)$, the Green’s function of $D_d$ with pole at $\alpha i$ is

$$G(z, \alpha i) = \frac{1}{2\pi} \log \frac{e^{\pi z/d} - e^{\alpha z/d}}{e^{\pi z/d} - e^{\alpha z/d}},$$

and the regular part of the Green’s function is given by

$$\gamma(z, \alpha i) = \frac{1}{2\pi} \log \frac{2 \sin(\alpha \pi/d)}{\pi/d} \leq \frac{1}{2\pi} \log \frac{2d}{\pi}.$$ 

Letting $z \to \alpha i$, we can easily see that

$$\gamma(\alpha i) = \lim_{z \to \alpha i} \gamma(z, \alpha i) = \frac{1}{2\pi} \log \frac{2 \sin(\alpha \pi/d)}{\pi/d} \leq \frac{1}{2\pi} \log \frac{2d}{\pi}.$$ 

Lemma 9. Suppose $z_0 \in \Omega_1 \subset \Omega_2$, then $\gamma_{\Omega_1}(z_0) \leq \gamma_{\Omega_2}(z_0)$.

Proof. Let $G_{\Omega_1}(z, z_0)$ and $G_{\Omega_2}(z, z_0)$ be the Green’s functions on $\Omega_1$ and $\Omega_2$ respectively. Then the lemma follows from applying maximum principle to $G_{\Omega_2} - G_{\Omega_1}$ on $\Omega_1$. □

Since $\Omega \subset D_d$, we have

$$(4.1) \quad \gamma(\Omega) = \sup_z \gamma(z, \Omega) \leq \gamma(D_d) = \frac{1}{2\pi} \log \frac{2d}{\pi}.$$ 

Suppose the infimum of $I(\cdot, \Omega)$ can not be achieved. From Proposition 2 and Proposition 3 we have

$$E(\Omega) = -1 - 4\pi \gamma(\Omega).$$

On the other hand $E(\Omega) \leq I(0, \Omega) = 0$. Hence $\gamma(\Omega) \geq -\frac{1}{4\pi}$. It follows from (4.1) that

$$\frac{1}{2\pi} \log(2d/\pi) \geq -\frac{1}{4\pi},$$

or $d \geq \pi/(2\sqrt{e})$, which contradicts with the assumption that $d < \pi/(2\sqrt{e})$. Hence the infimum of $I(\cdot, \Omega)$ is attained.

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References

[1] H. Brezis, F. Merle, Uniform estimates and blow up behavior for solutions of $-\Delta u = V(x)e^u$ in two dimensions, Comm. Partial Diff. Equation 16(1991), 1223-1253.
[2] E. Caglioti, P.L. Lions, C. Marchioro, and M. Pulvirenti, A Special Class of Stationary Flows for Two-Dimensional Euler Equations: A Statistical Mechanics Description, Commun. Math. Phys. 143(1992), 501-525
[3] E. Caglioti, P.L. Lions, C. Marchioro, and M. Pulvirenti, A Special Class of Stationary Flows for Two-Dimensional Euler Equations: A Statistical Mechanics Description, part II, Commun. Math. Phys. 174(1995), 229-260
[4] L. Carleson and S. Y. A. Chang, On the existence of an extremal funciton for an inequality of J. Moser, Bull. Sci. Math. 110(1986), no.2, 113-127
[5] S. Y. A. Chang, C. C. Chen and C. S. Lin, Extremal functions for a mean field equation in two dimension, Lecture on Partial Differential equations in honor of Louis Nirenberg's 75th birthday, Editors S. Y. A. Chang, C. S. Lin and H. T. Yau, Chapter 4, International Press 2003, pp 61-94.
[6] W. Chen and C. Li, Classification of solutions of some nonlinear elliptic equations, Duke Math. J. 63(1991). 615-622
[7] W. Ding, J. Jost, J. Li and G. Wang, The differential equation $\Delta u = 8\pi - 8\pi e^{u}$ on a compact Riemann surface, Asian J. Math. 1(1997), 230-248
[8] W. Ding, J. Jost, J. Li and G. Wang, An analysis of the two-vortex case in the Chern-Simons Higgs model, Calc. Var. and PDE 7(1998), no. 1, 87-97.
[9] B. Gidas, W. Ni, L. Nirenberg, Symmetry and related properties via the maximum principle, Comm. Math. Phys. 68(1979), no. 3, 209-243
[10] Z. C. Han, Asymptotic approach to singular solutions for nonlinear elliptic equations involving critical Sobolev exponent, Ann. Inst. H. Poincare Anal. Non Linéaire 8(1991), 159-174
[11] J. Li and M. Zhu, Sharp local embedding inequalities, to appear in CPAM.
[12] J. Moser, A sharp form of an inequality by N. Trudinger, Indiana Univ. Math. J. 11(1971), 1077-1092
[13] R. Schoen, Conformal deformation of a Riemannian metric to constant scalar curvature, J. Differential Geometry 20(1984) 479-495
[14] N. Trudinger, On imbeddings into Orlicz spaces and some applications, J. Math. Mech. 17(1967), 473-483