THE FACTORIZATION OF THE GIRY MONAD

KIRK STURTZ

Abstract. Using the density of the full subcategory of the category of convex spaces, consisting of the unit interval and the discrete convex space $\mathbb{2}$, we employ Isbell conjugate duality to construct the counit of the factorization of the Giry monad. We show that, provided that no measurable cardinals exist, the category of convex spaces is equivalent to the category of Giry-algebras.

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1. Introduction

In 1962, prior to the development of the theory of monads and their relationship with adjunctions, Lawvere\cite{7} constructed what he called the category of probabilistic mappings, which is, up to an equivalence, the Kleisi category of the Giry monad, Meas$^G$, where Meas is the category of measurable spaces. He also provided the adjunction between Meas and Meas$^G$, whose composite yields the Giry monad. In 1982 Giry\cite{4}, using the theory of monads, formally defined what we now refer to as the Giry monad, $(G, \eta, \mu)$. Giry also defined a similar monad on a subcategory of Meas, consisting of those measurable spaces arising from Polish topological spaces, Pol, with the arrows in the category being continuous maps.

The Kleisi category Meas$^G$ is well known and used by the science and engineering community, in stark contrast to the Eilenberg-Moore category of the Giry monad, Meas$^G$. This situation, in all likelihood, will continue unabated until the category Meas$^G$ is recognized as an equivalent category in which scientist and engineers can naturally model problems, permitting intuitive reasoning. (For the same reasons, the category Meas$^G$ has been embraced by the same community.) Progress towards this goal has been made by several researchers. Doberkat\cite{3} has characterized the algebras of Pol$^G$ using convex partitions on the space of all probability measures on $X$. Keimel\cite{6} has characterized the case of compact ordered spaces, generalizing an earlier result due to Swirszcz\cite{9}, stating that the algebras over the category of compact Hausdorff spaces are the compact convex...
sets $A$ embeddable in locally convex topological vector spaces with the barycenter maps as structure maps. In this paper we show, provided that no measurable cardinals exists, that $\text{Meas}^G$ is equivalent to the category of convex spaces, $\text{Cvx}$.

Recall the Giry monad $\mathcal{G}$ is defined on objects by mapping a measurable space $X$ to the measurable space $\mathcal{G}(X)$ of all probability measures on $X$, has a natural convex structure defined on it by taking convex sums of probability measures. On arrows, $X \xrightarrow{f} Y$, $\mathcal{G}(f)$ is the pushforward map, sending a probability measure $P \mapsto Pf^{-1}$, which is an affine map of convex spaces. These observations yield a functor $\text{Meas} \xrightarrow{\mathcal{P}} \text{Cvx}$, where the functor $\mathcal{P}$ is the same as $\mathcal{G}$, but viewed as a functor into the category of convex spaces, $\text{Cvx}$. To show that $\text{Meas}^\mathcal{G} \cong \text{Cvx}$ requires finding a factorization of the Giry monad, $\mathcal{G} = \Sigma \circ \mathcal{P}$, where $\text{Cvx} \xrightarrow{\Sigma} \text{Meas}$ is a left adjoint to $\mathcal{P}$, and then verifying this factorization yields $\text{Cvx} \cong \text{Meas}^G$.

The problem of factorizing the Giry monad through $\text{Cvx}$ is, for all intents and purposes, the problem of constructing the natural transformation defining the counit for a pair of functors, $\mathcal{P}$ and $\Sigma$, because the other aspects of the problem are relatively easy. There are only a couple of logical choices for $\Sigma$.

Our methodology to construct the counit, described in detail subsequently and outlined in this introduction, is motivated by the observation that the full subcategory $\mathcal{C}$ of $\text{Cvx}$ consisting of two objects, the unit interval with its natural convex structure, denoted by $I = [0, 1]$, and the discrete space $2$, is dense in $\text{Cvx}^\text{op}$. The density of $\mathcal{C}$ in $\text{Cvx}$ is equivalent to the statement that the truncated Yoneda mapping $\text{Cvx} \xrightarrow{\mathcal{Y}} \text{Set}^{\text{op}}$, specified by $\mathcal{Y}(A) = \text{Cvx}(\bot, A)$, is still full and faithful. Moreover, $\mathcal{Y}$ is also injective on objects, and hence $\mathcal{Y}$ is an embedding. This implies that $\text{Cvx} \xrightarrow{\mathcal{Y}} \text{Cvx}^{\text{op}}$ is also an embedding.\footnote{Isbell used the term left-adequate rather than the term dense. The fact that the unit interval is dense in $\text{Cvx}$ follows directly from \cite[2.2]{Isbell}. This fact is also straightforward to prove directly. We do not include the proof in the paper since Isbell conjugate duality is used only as a means for understanding the construction of the counit of the desired adjunction. Once we recognize the process, the whole scheme can be applied directly to $\text{Meas}$ and $\text{Cvx}$. A convex space $A$ is called discrete whenever $a_1, a_2 \in A$, the quantity $(1 - r)a_1 + ra_2$ is constant for all $r \in (0, 1)$.}

The two objects of $\mathcal{C}$ also have a natural measurable structure, given by the discrete $\sigma$-algebra for the discrete convex space $2$, and the Borel-$\sigma$-algebra on $I$, and hence are objects lying in $\text{Meas}$. There is also an evident functor $\text{Meas} \xrightarrow{\mathcal{Y}} (\text{Cvx}^{\text{op}})^{\text{op}}$ given by $\mathcal{Y}(X) = \mathcal{U}(\text{Meas}(X, \cdot))$ defined at the two component of $\mathcal{C}$, given by forgetting the measurable structure of the function spaces, and viewing the resulting sets, $2^X = \text{Meas}(X, 2)$ and $I^X = \text{Meas}(X, I)$, as convex spaces with the convex structure defined pointwise using the convex structure of $2$ and $I$.\footnote{Isbell conjugate duality, using the functor category $\mathcal{V}^{\text{op}}$, can be applied using any symmetric monoidal closed category $\mathcal{V}$, which is complete and cocomplete, as the base, provided $\mathcal{C}$ is enriched over $\mathcal{V}$.} The functor $\mathcal{Y}$ is, in general, not full because for any measurable space $X$, a probability measure $\hat{P} \in \mathcal{G}(X)$ determines an affine map $I^X \xrightarrow{\hat{P}} I$ defined by the integral, $f \mapsto \int_X f d\hat{P}$. Since $\mathcal{Y}(I \xrightarrow{x} X)(I) = I^X \xrightarrow{\text{ev}_x} I$, corresponding to the dirac measure $\delta_x \in \mathcal{G}(X)$, it follows that $\mathcal{Y}$ is not full.\footnote{Formally, we should write $\mathcal{Y}(X) = \mathcal{U}(\text{Meas}(X, \Sigma))$, where the functor $\Sigma$ is some functor $\text{Cvx} \rightarrow \text{Meas}$ assigning these two objects their natural measurable structure. However, we adopt the convention that, unless otherwise stated, function spaces will always be viewed as a convex structure unless specifically noted otherwise.}
Despite the fact $\hat{Y}$ is not an embedding, for purposes of constructing the counit of the desired factorization $\mathcal{P} \dashv \Sigma$, Isbell conjugate duality leads us to consider the $\text{Cat}$-diagram

$$
\begin{array}{ccc}
\text{Cvx}^{op} & \xrightarrow{\mathcal{O}} & (\text{Cvx}^{C})^{op} \\
\gamma & \downarrow & \mathcal{P} \\
\Sigma \text{Meas} & \xrightarrow{\Sigma} & \mathcal{C}
\end{array}
$$

where the desired adjoint pair $\mathcal{P} \dashv \Sigma$ should be related to the adjunction $\mathcal{O} \dashv \text{Spec}$, as characterized by the equivalence $\mathcal{Y} \circ \mathcal{P} \cong \text{Spec} \circ \hat{Y}$. To satisfy this equivalence requires defining $\text{Cvx} \xrightarrow{\Sigma} \text{Meas}$ to be the functor assigning a convex space $A$ the initial $\sigma$-algebra on the underlying set $A$ generated by the set of all affine maps $\{ A \xrightarrow{m} I \mid m \in \text{Cvx}(A, I) \}$.

Naturality leads to the view of a probability measure $\hat{P} \in \mathcal{G}(X)$ as the $I$-component of a natural transformation $\alpha \in \text{Spec}(\hat{Y}(X))[2]$.

$$
\begin{array}{ccc}
I & \xrightarrow{\alpha_I} & P \\
f & \mapsto & \int_X f \, d\hat{P}
\end{array}
$$

where we have used the fact that $I^2 = \text{Cvx}(2, I) \cong \text{Cvx}(1, I) \cong I$, and $\text{Spec}(\hat{Y}(X))[2] = \text{Nat}(\hat{Y}(X), \text{Cvx}(2, \_)) \cong \text{Nat}(\hat{Y}(X), \text{Cvx}(1, \_))$\footnote{Given any discrete convex space $B$, and any geometric space $A$, $B^A = \text{Cvx}(B, A) \cong \text{Cvx}(1, A)$. That is, the only affine maps from a discrete space to a geometric space are the constant affine maps. The convex space of natural transformations, $\text{Nat}(\hat{Y}(X), \text{Cvx}(1, \_))$ is a geometric space, and hence $\text{Nat}(\hat{Y}(X), \text{Cvx}(2, \_)) \cong \text{Nat}(\hat{Y}(X), \text{Cvx}(1, \_))^2 \cong \text{Nat}(\hat{Y}(X), \text{Cvx}(1, \_))$, where the first $\text{Cvx}$-isomorphism follows from the symmetric monoidal closed category structure of $\text{Cvx}$.}.

The right action of $\mathcal{C}$ on $\text{Cvx}$ forces the affine map $\alpha_I$ to satisfy the property of being a weakly-averaging affine map, $\alpha_I(\pi) = u$ for all constant functions $X \xrightarrow{\pi} I$ with value $u$. The fact that $I^X$ is a geometric spaces means it can be embedded into the real vector space $\mathbb{R}^X$, and similarly $I$ embeds into $\mathbb{R}$, and $\alpha_I$ extends uniquely to a linear map of Banach spaces, where $\mathbb{R}^X$ has the norm $||f||_\infty = \sup_{x \in X} |f(x)|$ and $\mathbb{R}$ has the norm given by the absolute value, $| \cdot |$. It thus follows that the extension of $\alpha_I$ is a continuous linear map with norm one. Consequently $\alpha_I$ preserves limits of pointwise convergent sequences, which corresponds to the continuity property of a probability measure, i.e., if $\{ U_i \}_{i=1}^\infty$ is a sequence of disjoint measurable sets, then $\hat{P}(\bigcup_{i \in \mathbb{N}} U_i) = \sum_{i \in \mathbb{N}} \hat{P}(U_i)$. For these reasons, we obtain a $\text{Cvx}$-isomorphism between $\mathcal{G}(X)$ and $\text{Spec}(\hat{Y}(X))[2]$.

Taking $X = \Sigma A$ for some convex space $A$, the naturality of any such $\alpha \in \text{Spec}(\hat{Y}(\Sigma A))[2]$ requires the commutativity of the $\text{Cvx}$-diagram

$$
\begin{array}{ccc}
1 & \xrightarrow{\epsilon^1} & \Sigma A \\
\alpha_1 & \downarrow & \downarrow & \alpha_2 \\
I & \xrightarrow{\epsilon^2} & 2
\end{array}
$$

where $I \xrightarrow{\epsilon^1} 2$ is the unique non-constant affine map in $\text{Cvx}(I, 2)$. We prove, under the condition that no measurable cardinals exist, this implies that $\alpha_2 = ev_a$ for some
unique point \( a \in A \). This process specifies the components of a natural transformation, \( \Sigma \circ P \Rightarrow id_{\text{Cvx}} \), given by \( \epsilon_A(P) = a \), which defines the counit of the adjunction \( P \dashv \Sigma \).

**Background** The category \( \text{Cvx} \) is the affine part of the theory of \( \mathcal{K} \)-modules, where \( \mathcal{K} \) is the rig \([0, \infty)\). This characterization, which is given by Meng\(^8\), immediately proves \( \text{Cvx} \) is complete, and a simple verification shows it is also cocomplete. The category \( \text{Meas} \) is also complete and cocomplete, as is well-know and simple to verify. We assume the reader is familiar with both categories being complete and cocomplete.

**Notation** The symbol \( X \) always refers to a measurable spaces while \( A \) and \( B \) always denote convex spaces. The notation "\( X \in ob \mathcal{C} \)" denotes an object in the category \( \mathcal{C} \) and "\( f \in ar \mathcal{C} \)" denotes an arrow in the category \( \mathcal{C} \). For an object \( X \) in any category the identity arrow on \( X \) is denoted \( id_X \). The notation \( u \) is used to denote a constant function with value \( u \) lying in the codomain of the function \( u \). In discussing singleton subsets, such as \( \{1\} \), we write \( m^{-1}(1) \) rather than \( m^{-1}(\{1\}) \), where \( m \) is any function into a space containing that element.

The symbol \( \Sigma \) is used as both a functor, and to denote the \( \sigma \)-algebra of a space. In the form "\( \Sigma A \)" , it refers to the functor applied to the convex space \( A \), whereas in the form \( \Sigma_A \) refers to the \( \sigma \)-algebra which the functor endows the (underlying set of the) convex space \( A \) with. Hence \( \Sigma A = (A, \Sigma_A) \).

The symbol \( \iota \) , with or without subscripts/superscripts, is used for inclusion maps, both inclusion as a subobject within a category, or as a subobject functor.

In the category \( \text{Cvx} \), the arrows \( A \xrightarrow{m} B \) are referred to as affine maps, and preserve "convex sums", 

\[
m((1 - \alpha)a_1 + \alpha a_2) = (1 - \alpha)m(a_1) + \alpha m(a_2) \quad \alpha \in \text{I}.
\]

For brevity, a convex sum is often denoted by 

\[
a_1 +_\alpha a_2 \overset{def}{=} (1 - \alpha)a_1 + \alpha a_2 \quad \alpha \in \text{I}.
\]

We make use of the result that every affine map \( \psi \in \text{Cvx}(\text{I}, A) \) can be characterized as a path map, \( \text{I} \xrightarrow{\gamma_{a_1,a_2}} A \), defined by \( \gamma_{a_1,a_2}(r) = a_1 +_r a_2 \). Taking \( A = \text{I} \) gives the characterization of the affine maps \( \text{Cvx}(\text{I}, \text{I}) \).

2. **The Property of Separability in Meas**

Given a measurable space \( X \), define an equivalence relation on \( X \) by

\[
x \sim y \quad \text{iff} \quad x \in U \Leftrightarrow y \in U \quad \forall U \in \Sigma_X.
\]

We say \( X \) is a separated measurable space if for any two points \( x, y \in X \), there is some \( U \in \Sigma_X \) with \( x \in U \) and \( y \notin U \). This definition is based upon the corresponding terminology used in point set topology. The equivalent categorical definition follows from

**Lemma 2.1.** A measurable space \( X \) is a separated measurable space iff \( 2 \) is a coseparator for the space \( X \).

**Proof.** Using the fact \( 1 \) is a separator for \( \text{Meas} \), the points \( x_1, x_2 \in X \) are separated if and only if there exist a characteristic map \( X \xrightarrow{\chi_U} 2 \) distinguishing the two points, for some measurable set \( U \) in \( X \). \( \square \)

Given any any space \( X \) let \( X_s \) denote the equivalence classes \( X \) under the relation \( \sim \), and

\[
X \xrightarrow{q_X} X_s \quad x \mapsto [x]
\]
the set map sending each point to its equivalence class, and endow the set $X_s$ with the largest $\sigma$-algebra such that $q_X$ is a measurable function. This $\sigma$-algebra is also referred to as the final $\sigma$-algebra generated by the projection map.

**Lemma 2.2.** The space $X_s$ is a separated measurable space.

*Proof.* Suppose $[x], [y] \in X_s$ are two distinct points in the quotient space. Thus there exist a measurable set $U \in \Sigma_X$ such that $x \in U$ and $y \in U^c$. These two measurable sets, $U$ and $U^c$ partition $X$ such that every element in $U$ is separated from every point of $U^c$. The image of these two measurable sets partitions $X_s$, and is measurable. Thus $x \in q_X(U)$ while $y \notin q_X(U^c) = q_X(U)^c$. □

**Lemma 2.3.** For $X \xrightarrow{f} Y$ a measurable map, the induced function $X_s \xrightarrow{f_s} Y_s$ mapping $[x] \mapsto [f(x)]$ is well defined and measurable. Hence the $\text{Meas}$-diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
q_X \downarrow & & \downarrow q_Y \\
X_s & \xrightarrow{f_s} & Y_s
\end{array}
\]

commutes.

*Proof.* Well-defined: Suppose $x_1 \sim x_2$ then $f(x_1) \sim f(x_2)$ in $Y$ are nonseparated because the existence of a $V \in \Sigma_Y$ such that $f(x_1) \in V$ and $f(x_2) \notin V$ would then yield the contradiction that $x_1 \sim x_2$ since $f^{-1}(V) \in \Sigma_X$ and $x_1 \in f^{-1}(V)$ while $x_2 \notin f^{-1}(V)$.

Thus we have $f(x_1) \sim f(x_2)$ in $Y$ and $[f(x_1)] = [f(x_2)]$ in $Y_s$. Hence $f_s$ is well-defined, and the diagram given in the Lemma commutes at the set theoretic level.

Measurability: If $W \subset \Sigma_{Y_s}$ then $q_{Y_s}^{-1}(W) \in \Sigma_Y$, and hence $f^{-1}(q_{Y_s}^{-1}(W)) \in \Sigma_X$. Since the $\text{Set}$-diagram

\[
\begin{array}{ccc}
X & \xrightarrow{q_X} & \Sigma_Y \\
\downarrow q_X & & \downarrow q_Y \circ f \\
X_s & \xrightarrow{f_s} & Y_s
\end{array}
\]

commutes and the two maps $q_X$ and $q_Y \circ f$ are measurable, it follows that $f_s$ is measurable since $\Sigma_{X_s}$ has the largest $\sigma$-algebra such that $q_X$ is measurable, i.e., $q_X^{-1}(f_s^{-1}(W)) = f^{-1}(q_Y^{-1}(W)) \in \Sigma_X$ implies $f_s^{-1}(W)$ is measurable. □

This result implies that this quotient space construction is functorial, yielding a functor to the full subcategory consisting of all the separated measurable spaces, $\text{Meas}_s$. The inclusion of this subcategory into $\text{Meas}$ then yields

**Lemma 2.4.** The inclusion functor is right adjoint to $\mathcal{S}$,

\[
\begin{array}{ccc}
\text{Meas} & \xrightarrow{\mathcal{S}} & \text{Meas}_s \\
\downarrow \iota_{\text{Meas}} & & \downarrow \iota_{\text{Meas}} \\
\mathcal{S} & \xleftarrow{\iota_{\text{Meas}}} & \text{Meas}_s
\end{array}
\]

The proof of this follows directly from the definition of the induced function on the quotient space. The monad determined by this adjunction is $(\iota_{\text{Meas}} \circ \mathcal{S}, q, \text{id})$, where $q$ is the natural transformation determined by the quotient projection maps at each component, and the multiplication is the identity because the monad is idempotent.

The result
Lemma 2.5. The object $2$ is a coseparator in $\text{Meas}_s$.

follows from the definition of the (sub)category $\text{Meas}_s$.

In $\text{Meas}$ function spaces are defined by

$$Y^X = (\text{Meas}(X, Y), \Sigma_{ev}^{Y^X})$$

where $\Sigma_{ev}^{Y^X}$ is the $\sigma$-algebra generated by all the point evaluation maps $Y^X \xrightarrow{ev} Y$, sending $f \mapsto f(x)$. We refer to the $\sigma$-algebras generated by the evaluation maps as *evaluation $\sigma$-algebras*, and context permitting, denote the evaluation $\sigma$-algebra as just $\Sigma_{ev}$.

Lemma 2.6. If $X, Y \in \text{ob } \text{Meas}_s$ then the function space $Y^X$ is a separated measurable space.

Proof. If $f, g \in Y^X$ are two distinct points, then there exist a point $x \in X$ such that $f(x) \neq g(x)$. Since $Y$ is separated there exist a $V \in \Sigma_Y$ such that $f(x) \in V$ while $g(x) \notin V$. Consequently the evaluation map $ev_x$ serves the purpose of separating the maps $f$ and $g$ since $f \in ev_x^{-1}(f^{-1}(V))$ while $g \notin ev_x^{-1}(f^{-1}(V))$. $\square$

3. Factorizing the Giry monad using separability

The unit of the Giry monad, which sends each point to a dirac measure, $x \mapsto \delta_x$, is in general, not an injective mapping because if $x_1, x_2 \in X$ are nonseparable, then $\delta_{x_1} = \delta_{x_2}$. However the unit of the Giry monad, restricted to the subcategory $\text{Meas}_s$, is injective.

The functor $\text{Meas} \xrightarrow{P} \text{Cvx}$, which we defined previously as $P(X) = G(X)$ and $P(f) = G(f)$, with both objects and arrows viewed as lying in $\text{Cvx}$ rather than $\text{Meas}$, factors through the subcategory $\text{Meas}_s$,

$$\xymatrix{ \text{Meas} \ar[rr]^-P & & \text{Cvx} \\
\text{Meas}_s \ar[urr]_-S \ar[rr]^-{\text{P}|} & & X \ar[l] \ar[u] \ar[urr]_-{\text{P}|(X_s) \ar[u]} \ar[rr]_-{X_s} & & \text{Meas}_s \ar[l] \ar[ll]_-S \ar[ll]_-{\text{P}|} \ar[ll]_-{\Sigma} }$$

As a result of this property, $P(X) = \text{P}|(X_s)$, we write $P(X)$ rather than $\text{P}|(X_s)$.

The consequence of this is that in factorizing the Giry monad it suffices to consider factorizations through the subcategory $\text{Meas}_s$, since, by Lemma 2.4, the inclusion functor $\text{Meas}_s \hookrightarrow \text{Meas}$ is right adjoint to $S$, so that any adjunct pair $\text{P}| \dashv \Sigma$ between $\text{Meas}_s$ and $\text{Cvx}$ can be composed with the adjunction $S \dashv t_{\text{Meas}}$ to obtain the composite adjunction.

$$\xymatrix{ \text{Meas} \ar[r]^-S & \text{Meas}_s \ar[r]^-{\text{P}|} & \text{Cvx} \\
\Sigma \ar[r]_-{(\text{P}| \circ S) \dashv (t \circ \Sigma)} & \text{Cvx} }$$

Factoring out the nonseparability associated with a space $X$ by the functor $S$ allows us to work in $\text{Meas}_s$, and the two categories, $\text{Meas}_s$ and $\text{Cvx}$, have virtually identical categorical properties. They are both complete, cocomplete, have a separator and coseparator, and are SMCC.

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5To prove $\text{Meas}^G \cong \text{Cvx}$, we invoke another argument based upon the existence of a right adjoint to $\Sigma$. The noted factorization of adjuncts is used to simplify the argument concerning the unit and counit of the composite adjunction.
4. The SMCC structure of \textbf{Meas} and \textbf{Cvx}

The SMCC structure of both \textbf{Meas} and \textbf{Cvx} arise from the usual \textit{hom tensor product} construction, giving adjunctions, for every object in the category, \( \underline{\otimes} A \vdash \underline{A} \), which have the evaluation maps \( B^A \otimes B \overset{ev}{\longrightarrow} B \) at each component specifying the counit of the adjunction \( \underline{\otimes} A \vdash \underline{A} \).

In \textbf{Meas}, for \( X, Y \in ob \textbf{Meas} \) the function spaces, denoted \( Y^X \), consist of the set \( \textbf{Meas}(X,Y) \) endowed with the smallest-\( \sigma \)-algebra such that all of the evaluation maps, \( \textbf{Meas}(X,Y) \overset{ev}{\longrightarrow} Y \) are measurable. We denote the \( \sigma \)-algebra generated by the evaluation maps by \( \Sigma_{ev} \), or just \( \Sigma_{ev} \) when the context make it clear, and refer to this \( \sigma \)-algebra as the evaluation \( \sigma \)-algebra. Hence, the function space \( Y^X = (\textbf{Meas}(X,Y), \Sigma_{ev}) \).

In constructing the monoidal structure so as to obtain a symmetric monoidal closed category, the tensor product \( Y^X \otimes_{\textbf{Meas}} X \) is the cartesian product \( Y^X \times X \) endowed with the final \( \sigma \)-algebra (largest \( \sigma \)-algebra) such that the constant graph maps \( \Gamma_{a} \) for \( a \in A \) and \( \Gamma_{m} \) for \( m \in B^A \) become affine maps. This tensor product makes \textbf{Cvx} a SMCC using the same argument as that used in \textbf{Meas}.

5. The measurable structure of a convex space

Given any convex space \( A \), the initial \( \sigma \)-algebra on the underlying set \( A \) generated by the affine maps into the unit interval, with the Borel \( \sigma \)-algebra, is the smallest \( \sigma \)-algebra \( \Sigma_A \) on the set \( A \) such that all the affine maps into \( I \) become measurable. Thus \( (A, \Sigma_A) \) is a measurable space and given any affine map \( \Gamma_{a} \) for \( a \in A \) and \( \Gamma_{m} \) for \( m \in B^A \) become affine maps. This tensor product makes \textbf{Cvx} a SMCC using the same argument as that used in \textbf{Meas}.

Finally we note that any measurable space \( X \) determines specifies a convex space \( I^X = \textbf{Meas}(X, I) \) using the pointwise construction, \((f +_a g)(x) = f(x) +_a g(x)\). Similarly one obtains the convex space \( 2^X = \textbf{Meas}(X, 2) \).

5. The measurable structure of a convex space

Given any convex space \( A \), the initial \( \sigma \)-algebra on the underlying set \( A \) generated by the affine maps into the unit interval, with the Borel \( \sigma \)-algebra, is the smallest \( \sigma \)-algebra \( \Sigma_A \) on the set \( A \) such that all the affine maps into \( I \) become measurable. Thus \( (A, \Sigma_A) \) is a measurable space and given any affine map \( A \overset{m}{\longrightarrow} B \) between convex spaces the set map \( \Sigma_A \overset{m}{\longrightarrow} (B, \Sigma_B) \) is measurable. This construction is clearly functorial and defines a functor \( \textbf{Cvx} \overset{\Sigma}{\longrightarrow} \textbf{Meas} \). In referring to the \( \sigma \)-algebra of the measurable space \( \Sigma A = (A, \Sigma_A) \) we will refer to it simply as \( \Sigma A \) rather than \( \Sigma_A \).
Let \( A \stackrel{m}{\rightarrow} I \) be an affine map and consider its composite with the unique affine map from the unit interval to the convex space \( \mathbf{2} \), where \( \mathbf{2} = \{0,1\} \) is the convex space defined by
\[
(1-\alpha)0 + \alpha 1 = \begin{cases} 
0 & \text{for all } \alpha \in [0,1) \\
1 & \text{otherwise}
\end{cases},
\]
and \( \chi_{m^{-1}(1)} \) in \( \text{Cvx} \).

**Diagram 2.** The affine maps \( A \stackrel{m}{\rightarrow} I \) determine affine characteristic functions.

The composite map, \( \chi_{m^{-1}(1)} = \epsilon_2 \circ m \) is an affine characteristic function. The set of all these affine characteristic functions (affine maps into \( \mathbf{2} \)) specify the initial \( \sigma \)-algebra on \( A \) also, which we denote by \( \Sigma_2A \). The existence of the affine map \( \epsilon_2 \) makes the \( \sigma \)-algebra \( \Sigma_2A \) sub \( \sigma \)-algebra of the initial \( \sigma \)-algebra on \( A \) generated by the affine maps into the unit interval, \( \Sigma_2A \subseteq \Sigma A \), and \( \Sigma_2 \) a subfunctor
\[
\Sigma_2 \hookrightarrow \Sigma.
\]

The affine characteristic functions can be characterized alternatively as specifying a pair of subobjects of a convex space, say \( A_0 \hookrightarrow A \) and its set-theoretic complement \( A_0^c \hookrightarrow A \), such that the map
\[
A \xrightarrow{\chi_{A_0}} \mathbf{2}
\]
is affine. Such a pair is called a *Boolean subobject pair* of \( A \), and we call the initial \( \sigma \)-algebra on \( A \) determined by \( \Sigma_2 \) the *Boolean \( \sigma \)-algebra* on \( A \).

**Lemma 5.1.** If \( \{A_0, A_0^c\} \) is a Boolean subobject pair then
\[
\Sigma_2(A) \cong \Sigma_2(A_0) \oplus_{\text{Meas}} \Sigma_2(A_0^c),
\]
where \( \oplus_{\text{Meas}} \) denotes the coproduct in \( \text{Meas} \).

**Proof.** The coproduct (sum) in \( \text{Meas} \) is constructed as the disjoint union of two measurable spaces. Since \( A \stackrel{\chi_{A_0}}{\rightarrow} \mathbf{2} \) partitions \( A \) into two disjoint parts, the result follows. \( \square \)

Let \( \mathbb{R} = (-\infty, \infty) \) with the natural convex structure. This convex structure extends to \( \mathbb{R}_\infty = (-\infty, \infty] \), by defining for all \( r \in \mathbb{R} \),
\[
\alpha \infty + (1-\alpha)r \overset{\text{def}}{=} \begin{cases} \infty & \text{for all } \alpha \in (0,1] \\
r & \text{for } \alpha = 0
\end{cases}.
\]

Borger and Kemp [1] have show this object \( \mathbb{R}_\infty \) is a coseparator in \( \text{Cvx} \).

**Lemma 5.2.** For every convex space \( A \), the measurable space \( \Sigma_2A \) is a separated measurable space.

**Proof.** Note that applying the functor \( \Sigma_2 \) to the convex space \( \mathbb{R} \) gives the standard Borel \( \sigma \)-algebra since the intervals \( (-\infty,u) \) generate the standard Borel \( \sigma \)-algebra, which then extends to \( \mathbb{R}_\infty \).

Let \( a_1, a_2 \in A \) be a pair of distinct points. Since \( \mathbb{R}_\infty \) is a coseparator in \( \text{Cvx} \), there exist an affine map \( m \) separating the pair, say \( m(a_1) < m(a_2) \). Consider the diagram
Since $A$ and $\mathbb{R}_\infty$ are convex spaces, with the indicated subobject Boolean pairs, the map $m$ becomes measurable under the functor $\Sigma_2$. Now, viewing the whole diagram in $\text{Meas}$, we can use Lemma 5.1 so $\Sigma A$ and $\Sigma \mathbb{R}_\infty$ are, up to an isomorphism, just the coproduct of the respective Boolean subobject pairs. Composing the measurable map $m$ with the (non affine) characteristic function $\chi_{\lbrack m(a_2), \infty \rbrack}$ we obtain the measurable map $\chi_{m^{-1}(\lbrack m(a_2), \infty \rbrack)}$ which coseparates the pair of elements, $\{a_1, a_2\}$. Thus $\Sigma_2 A$ is a separable measurable space. □

Using the fact $\Sigma_2$ is a subfunctor of $\Sigma$ it follows that

**Corollary 5.3.** For every convex space $A$ the measurable space $\Sigma A$ is a separated measurable space.

Abusing notation, we subsequently refer to $\Sigma$ as the functor $\text{Cvx} \xrightarrow{\Sigma} \text{Meas}_s$.

Since the Borel $\sigma$-algebra on the unit interval is generated by the subintervals (subobjects) $\{\lbrack 0, u \rbrack\}_{u \in I}$ we note

**Lemma 5.4.** Viewing $I$ as a convex space, the $\sigma$-algebra structure of the measurable space $\Sigma_2 I$ coincides with the Borel $\sigma$-algebra.

For $A$ a convex space, let us denote $I^{\Sigma A} \overset{\text{def}}{=} \text{Meas}(\Sigma A, I)$, and $I^A \overset{\text{def}}{=} \text{Cvx}(A, I)$. Both $I^{\Sigma A}$ and $I^A$ are convex spaces under the pointwise construction, and both can be given the evaluation $\sigma$-algebra, $\Sigma_{ev}$, to obtain the measurable inclusion map $I^A \subseteq I^{\Sigma A}$. Since the evaluation maps $I^{\Sigma A} \xrightarrow{ev_a} I$ and $I^A \xrightarrow{ev_a} I$ are affine we note that

**Lemma 5.5.** The evaluation $\sigma$-algebra on the function spaces $I^{\Sigma A}$ is a sub $\sigma$-algebra of the initial $\sigma$-algebra generated by the affine maps into $I$.

Since the evaluation maps can be viewed in either $\text{Meas}$ or $\text{Cvx}$, we let the context decide whether these evaluation maps refer to $\text{Cvx}$ or $\text{Meas}$ arrows.

6. **The integral as an affine map using the convex tensor product**

For any convex space $A$ and probability measure $\hat{P} \in G(\Sigma A)$ one obtains the weakly averaging affine functional given by

\[
I^{\Sigma A} \xrightarrow{P} I
\]

\[
f \longmapsto \int f \, d\hat{P}
\]

where $\int_A f \, d\hat{P}$ is the (Lebesgue) integral of the measurable function $\Sigma A \xrightarrow{f} I$. We refer to such functionals, arising from a probability measure, as probability functionals. Since
these probability functionals are affine they trivially becomes measurable when we apply the functor $\Sigma$.

Let $I^{(\Sigma A)}_{wa}$ denote the restriction of the function space $I^{(\Sigma A)}$, viewed in $Cvx$, consisting of the set of all probability functionals on the convex space $A$. For every measurable function $\Sigma A \xrightarrow{f} I$ the evaluation function

$$ I^{(\Sigma A)}_{wa} \xrightarrow{ev_f} I $$

$$ P \xrightarrow{f} P(f) = \int_A f \, d\hat{P} $$

is an affine function by the pointwise convex structure on the function space $I^{(\Sigma A)}$. Using the SMCC structure of $Cvx$ we obtain the commutative $Cvx$-diagram

$$ I^{(\Sigma A)}_{wa} \xrightarrow{\Gamma_f} I^{(\Sigma A)}_{wa} \otimes_{Cvx} I^{\Sigma A} \xrightarrow{\Gamma P} I^{\Sigma A} $$

where, for all convex sums $\{\alpha_i\}_{i=1}^n$,

$$ \int_A \left( \sum_{i=1}^n \alpha_i (P_i \otimes f_i) \right) \overset{def}{=} \sum_{i=1}^n \alpha_i \int_A f_i \, d\hat{P}_i, $$

This “integral map”, $\int_A$, is the restriction of the evaluation map to the subspace $I^{(\Sigma A)}_{wa} \otimes_{Cvx} I^{\Sigma A} \hookrightarrow I^{(\Sigma A)}$.

Upon application of the functor $\Sigma$ we obtain the above diagram viewed in $Meas$, with all the $\sigma$-algebras being given by the induced $\sigma$-algebra of affine maps into $I$.

### 7. Properties of the $Spec$ functor

Let $\mathcal{X}$ denote the image of the functor $\mathcal{Y}$ applied to a measurable space $X$, so that at each component of $\mathcal{C}$, $2^X$ and $I^X$, these spaces are convex spaces. Similiarly, let $\mathcal{I}$ denote the functor $Cvx(I, \mathcal{X}) \in_{ob} (Cvx^C)^{op}$. Then, by construction, the components of the natural transformation

$$ \alpha \in Spec(\mathcal{X})[2] = Nat(\mathcal{X}, 2) \cong Nat(\mathcal{X}, 1)^2 \cong Nat(\mathcal{X}, \mathcal{I}) $$

at component $I$ are affine maps, $I^X \xrightarrow{\alpha_I} I$.

**Lemma 7.1.** If $\alpha \in Spec(\mathcal{X})[2]$ then it satisfies the two properties

1. For every constant function $\overline{u} \in I^X$ with value $u \in I$,

$$ \alpha_I(\overline{u}) = u $$

2. For every $v \in I$ and every $f \in I^X$,

$$ \alpha_I(v \cdot f) = v \alpha_I(f). $$

There exist probability measures on convex spaces such that this functional need not be measurable with respect to the sub $\sigma$-algebra $\Sigma_2(I^{\Sigma A})$. I am indebted to Tomáš Crhák [2] for refuting the erroneous claim, given in an earlier draft of this paper, that the Boolean $\sigma$-algebra sufficed.
Proof. The naturality condition of $\alpha$ requires that for all $\gamma_{u,v} \in I$, and all $f \in I^X$, that the $\text{Cvx}$-diagram

\[
\begin{array}{ccc}
I^X & \xrightarrow{\alpha_I} & I \\
\downarrow \gamma_{u,v}^X & & \downarrow \gamma_{u,v}^I \\
I^X & \xrightarrow{\alpha_I} & I \\
\end{array}
\]

commutes.

To prove (1), consider the constant path map $\gamma_{u,u} \in \text{Cvx}(I, I)$. The naturality condition yields

\[
\alpha_I(\gamma_{u,u} \circ f) = \gamma_{u,u} \circ \alpha_I(f).
\]

To prove (2), consider the “scaling” path map $\gamma_{0,v}$. The naturality condition yields

\[
\alpha_I(v \cdot f) = \gamma_{0,v} \circ \alpha_I(f) = v \cdot \alpha_I(f).
\]

$\Box$

The (geometric) convex space $I^X$ embedds into the real vector space $\mathbb{R}^X$, with the norm $||f||_{\infty} = \sup_{x \in X} |f(x)|$, and $I$ embeds into $\mathbb{R}$ with the norm $|\cdot|$. Consequently every weakly-averaging affine map $I^X \xrightarrow{\alpha_I} I$ extends uniquely to a linear map $\mathbb{R}^X \xrightarrow{\hat{\alpha}_I} \mathbb{R}$ since every measurable function $f \in \mathbb{R}^X$ is uniquely determined by the values $\{\alpha_I(\chi_U) | U \in \Sigma_X\}$, i.e., every measurable map $X \xrightarrow{f} \mathbb{R}$ is the limit of a sequence of simple functions, $f_j = \sum_{i=1}^{N_j} \beta_{i,j} \chi_{U_{i,j}}$, and hence $\alpha_I$ uniquely determines the linear map by defining

\[
\hat{\alpha}_I(f_j) = \sum_{i=1}^{N_j} \beta_{i,j} \alpha_I(\chi_{U_{i,j}})
\]

It is a linear map since the constant zero map gets mapped to zero, $\hat{\alpha}_I(\chi_{\emptyset}) = 0$.

It follows the operator $\mathbb{R}^X \xrightarrow{\hat{\alpha}_I} \mathbb{R}$ is a bounded linear operator of norm 1, and since

\[
|\hat{\alpha}_I(f)| \leq ||\hat{\alpha}_I|| ||f||_{\infty} = ||f||_{\infty}.
\]

it follows that $\hat{\alpha}_I$ is a continuous linear map, and therefore its restriction to $I^X$, which coincides with $\alpha_I$, is also a continuous map. Consequently, we obtain

Lemma 7.2. Let $f, f_i \in I^X$ for all $i \in \mathbb{N}$. If $\{f_i\}_{i=1}^{\infty} \to f$ pointwise then every $\alpha \in \text{Spec}(\chi_X)[2]$ satisfies the property that

\[
\lim_{N \to \infty} \{\alpha_I(f_i)\} = \alpha_I(f)
\]

The preservation of a limit of a sequence of functions in $2^X$ by a natural transformation $\alpha \in \text{Spec}(\chi_X)[2]$, at component 2, of a sequence in $2^X$ need not preserve the limit of the sequence as can be seen by taking a sequence of functions in $2^X$ converging to the constant function $\chi_X$ of value one. (The space $2^X$ is not a geometric convex space.) However we do have

Lemma 7.3. Let $\alpha \in \text{Spec}(\chi_X)[2]$. If $\{U_i\}_{i=1}^{\infty}$ is a sequence of measurable sets in $X$ converging to $\emptyset$ then $\lim_{i \to \infty} \alpha_2(\chi_{U_i}) = 0$.

Proof. Viewing the sequence $\{\chi_{U_i}\}_{i=1}^{\infty}$ in $I^X$, by Lemma 7.2, it follows that $\lim_{i \to \infty} \alpha_1(\chi_{U_i}) = 0$. Composition with $\epsilon_2$ which preserves the ordering then yields, using naturality, that $\lim_{i \to \infty} \alpha_2(\chi_{U_i}) = 0$. $\Box$
Lemma 7.4. Let $X$ be a separated measurable space and $2^X \xrightarrow{\chi} 2$ a weakly averaging affine measurable map. Then the Boolean subobject pair $\{V, V^c\}$ of $2^X$ satisfies the property that if $U \in \Sigma_X$ and $\chi_U \in V$ then $\chi_{U^c} \in V^c$. In other words, $\chi_V(\chi_U) = 1$ if and only if $\chi_V(\chi_{U^c}) = 0$.

Proof. Suppose the contrary, that there exist a measurable set $U$ in $X$ such that $\chi_U, \chi_{U^c} \in V$. Then since $V$ is a Boolean subobject, for all $a \in (0, 1)$ it follows that the convex sum $\chi_{U} + a \chi_{U^c} = 0 \in V$ because evaluation at any point in $X$ gives either $1 + a = 0$ or $0 + a = 0$. But by hypothesis this convex sum is in $V = \chi_V(1)$, so $\chi_V(0) = 1$ which contradicts the hypothesis that $\chi_V$ is weakly averaging. \hfill \Box

Lemma 7.5. Let $A$ be a convex space, and assume no measurable cardinals exist. If $\alpha \in Spec(\Sigma A)[2]$ then, at component $2$, $2^{\Sigma A} \xrightarrow{\alpha \circ} 2$ is an evaluation map, $\alpha_2 = ev_a$ for a unique point $a \in A$.

Proof. Consider the set $\alpha_2^{-1}(1) = \{\chi_U | \alpha_2(\chi_U) = 1\}$. Suppose, to obtain a contradiction, that $\cap \alpha_2^{-1}(1) = \emptyset$. Using the hypothesis that no measurable cardinals exist, we can find a sequence of measurable sets $\{U_i\}_{i=1}^\infty$ in $X$, with each $\chi_{U_i} \in \alpha_2^{-1}(1)$, such that $\lim_{N \to \infty} \{\cap_{i=1}^N U_i\} = \emptyset$. By Lemma 7.3 it follows that $\lim_{N \to \infty} \{\alpha_2(\chi_{\cap_{i=1}^N U_i})\} = 0$. But for every $N \in \mathbb{N}$, since each $\chi_{U_i} \in \alpha_2^{-1}(1)$, it follows that $\alpha_2(\chi_{\cap_{i=1}^N U_i}) = 1$, and hence $\lim_{N \to \infty} \{\alpha_2(\chi_{\cap_{i=1}^N U_i})\} = 1$, yielding a contradiction. Thus we conclude that $\alpha_2^{-1}(1) \neq \emptyset$.

The uniqueness follows from the property, given in Lemma 2.6 that $\Sigma A$ is a separated measurable space. For let $U_a \overset{df}{=} \bigcap_{\chi_{U_i} \in \alpha_2^{-1}(1)} U_i$, and suppose that $a_1, a_2 \in U_a$ with $a_1 \neq a_2$, and with the measurable set $V \in \Sigma A$ separating the pair, with $a_1 \in V$ while $a_2 \in V^c$. By Lemma 7.4 either $\chi_V \in \alpha_2^{-1}(1)$ or $\chi_{V^c} \in \alpha_2^{-1}(1)$. Either choice contradicts the condition that every element $a \in U_a$ satisfies $\chi_{U_k}(a) = 1$ for all $\chi_{U_k} \in \alpha_2^{-1}(1)$.

\hfill \Box

Theorem 7.6. For any measurable space, we have a $Cvx$-isomorphism

$$Spec(\Sigma X)[2] \cong \mathcal{P}(X),$$

and this isomorphism is natural in $X$.

Proof. By lemmas 7.1 and 7.2 each $\alpha \in Spec(\Sigma X)[2]$, evaluated at $I$, satisfies the three basic properties of a probability functional $I^X \xrightarrow{\alpha_I} I$, given by

1. $\alpha_I(\mathcal{C}) = c$,
2. For all $s \in I$ we have $\alpha_I(sf) = s\alpha_I(f)$, and
3. If $\{f_i\}_{i=1}^\infty \to f$ pointwise (with each $f_i \in I^X$), then $\lim_{N \to \infty} \alpha_I(f_i) = \alpha_I(f)$.

---

Measureable cardinals are sets $X$ with a (large) cardinality $\kappa$ such that there exist countably additive $0 - 1$ measures $2^X \xrightarrow{\mu} 2$ with the property that $\cap F^{-1}(1) = \emptyset$ but for every subset $S \subset F^{-1}(1)$ of cardinality $\gamma < \kappa$, $\cap S \neq \emptyset$. Their existence using ZFC axioms can not be proven, and hence their existence would require an extension of the ZFC axioms.
Conversely, every probability functional determines an element \( \alpha \in \Spec(\mathcal{X})[2] \) at the component \( \mathbf{I} \), since it satisfies these three characteristic properties.

The naturality follows directly from the definitions of \( \Spec(\mathcal{X})[2] = \Nat(\mathcal{X}, \_\_) \), and \( \mathcal{P}(X) \). If \( X \xrightarrow{f} Y \) is a measurable function then \( \mathcal{P}(X \xrightarrow{f} Y) \) is the pushforward map of probability measures, mapping \( P \in \mathcal{P}(X) \mapsto Pf^{-1} \in \mathcal{P}(Y) \). On the other hand,

\[
\Spec(f) : \Spec(\mathcal{X}) \to \Spec(\mathcal{Y})
\]

\[
: \alpha \mapsto \alpha \circ f
\]

where \( (\alpha \circ f)_c = \alpha_c \circ c' \) is the evaluation at any component \( c \in \mathcal{C} \). That is, \( \Spec(f) \) is the “pushforward map” constructed using the natural transformation \( f \), as illustrated, for component \( \mathbf{I} \), in the \( \text{Cvx} \)-diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\mathbf{I}^X & \xrightarrow{\alpha} & \mathbf{I}^Y \\
\end{array}
\]

\[
\begin{array}{ccc}
\mathbf{I}^f & \xrightarrow{\alpha \circ f} & \mathbf{I}^f \\
\end{array}
\]

\[
\text{Lemma 7.7. For } X \text{ any measurable space,}
\]

\[
\text{Cvx}(\mathcal{I}^X, \mathcal{I}^1) \cong \text{Cvx}(\mathcal{I}^X, \mathcal{I})^1,
\]

and restriction of the set \( \text{Cvx}(\mathcal{I}^X, \mathcal{I}) \) to the set of weakly-averaging maps gives

\[
\text{Cvx}(\mathcal{I}^X, \mathcal{I})|_{wa} \cong \left( \text{Cvx}(\mathcal{I}^X, \mathcal{I})|_{wa} \right)^1.
\]

\[
\text{Proof. This fact follows from the SMCC structure of } \text{Cvx}, \text{ since we have the following sequence of bijections shown on the left-hand side, and their restriction to the weakly-averaging maps gives the sequence of bijections shown on the right-hand side,}
\]

\[
\begin{array}{ccc}
\mathcal{I}^X \xrightarrow{P} \mathcal{I}^1 \\
\mathcal{I} \xrightarrow{\bar{P}} \mathcal{I}^{(1^X)} \\
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{I}^X \xrightarrow{P} \mathcal{I}^1 \text{ weakly-averaging} \\
\mathcal{I} \xrightarrow{\bar{P}} \mathcal{I}^{(1^X)}|_{wa} \\
\end{array}
\]

\[
\text{By Theorem 7.6 and Lemma 7.7 it follows that}
\]

\[
\text{Corollary 7.8. For } X \text{ any measurable space,}
\]

\[
(\Spec(\mathcal{X})[2])^1 \cong \mathcal{P}(X)^1.
\]

\[
\text{Corollary 7.9. } \Spec \circ \hat{\mathcal{Y}} \cong \mathcal{Y} \circ \mathcal{P}.
\]

\[
\text{Proof. On objects we have the two composites, } \Spec \circ \hat{\mathcal{Y}} \text{ and } \mathcal{Y} \circ \mathcal{P}, \text{ giving the two paths}
\]

\[
\begin{array}{ccc}
\text{Cvx}(\mathcal{P}(X)) \cong \Spec(\mathcal{X}) \xrightarrow{\_\_ \_X} \mathcal{X} \\
\mathcal{P}(X) \xrightarrow{\_\_ \_X} \mathcal{X} \\
\end{array}
\]

\[
\text{Evaluation at the component } 2 \text{ gives, using Theorem 7.6}
\]

\[
\text{Cvx}(2, \mathcal{P}(X)) \cong \text{Cvx}(1, \mathcal{P}(X)) \cong \mathcal{P}(X) \cong \Spec(\mathcal{X})(2).
\]
Evaluation at the component \( I \) gives, using Corollary 7.8,
\[
\text{Cvx}(I, \mathcal{P}(X)) = \mathcal{P}(X)^I \
\approx (\text{Cvx}(I^X, I)|_{\text{wa}})^I \\
= \text{Cvx}(I^X, I^I)|_{\text{wa}} \\
= \text{Nat}(X, I) \\
= \text{Spec}(X[I])
\]

On arrows the equivalence is clear, since \( \mathcal{P}(X \xrightarrow{f} Y) = \mathcal{P}(X) \xrightarrow{f^{-1}} \mathcal{P}(Y) \) is the pushforward map of probability measures, and \( \text{Spec}(f) \) is also the pushforward map, as noted in the proof of the naturality condition for Theorem 7.6.

\[ \square \]

8. The adjunction between \( \mathcal{P} \) and \( \Sigma \)

The factorization of the Giry monad, through the category of convex spaces, reduces to showing that a "barycenter map" exist for each convex space \( A \). This barycenter map is the counit at the component \( A \).

**Lemma 8.1.** Provided that no measurable cardinals exist, for every convex space \( A \), it follows that
\[
\text{Spec}(\Sigma A)[2] \xrightarrow{\hat{\epsilon}_A} A
\]
where \( a \) is the unique element in \( A \) such that \( \alpha_2 = ev_a \), defines the components of a natural transformation
\[
ev_2 \circ \text{Spec} \circ \hat{\gamma} \circ \Sigma \xrightarrow{\hat{\epsilon}} \text{id}_{\text{Cvx}}
\]

where \( \hat{\epsilon} \) is the evaluation of the functor at component \( 2 \in \mathcal{C} \).

**Proof.** Every natural transformation \( \alpha \in \text{Spec}(\Sigma A)[2] \) evaluated at component \( 2 \) is, by Lemma 7.5, an affine map, \( 2^{\Sigma A} \xrightarrow{\alpha_2} 2 \), which is an evaluation map, \( ev_a \) for a unique point \( a \in A \). This defines the map \( \hat{\epsilon}_A \), and, for \( A \xrightarrow{m} B \) an affine map, we have the commutativity of the \textbf{Cvx}-diagram
\[
\begin{array}{ccc}
\text{Spec}(\Sigma A)[2] & \xrightarrow{\hat{\epsilon}_A} & A \\
\downarrow & & \downarrow \\
\text{Spec}(\Sigma B)[2] & \xrightarrow{\hat{\epsilon}_B} & B \\
\end{array}
\]

where \( \alpha \in \text{Spec}(\Sigma A)[2] = \text{Nat}(\Sigma A, \_ \_ \_) \) and its "pushforward" along \( m \) is the natural transformation \( \alpha \circ m \in \text{Spec}(\Sigma B)[2] = \text{Nat}(\Sigma B, \_ \_ \_) \). The evaluation of the latter map at component \( 2 \) is therefore
\[
\begin{array}{c}
2^{\Sigma A} \\
\xrightarrow{\alpha_2 = ev_{\hat{\epsilon}_A}(\alpha)} \\
\xrightarrow{(ev_{\hat{\epsilon}_A(\alpha)} \circ 2^{\Sigma m}) = ev_{\hat{\epsilon}_B(\alpha \circ m)}} \\
2^{\Sigma B}
\end{array}
\]

where the equality on the diagonal map follows because the affine map \( \alpha_2 \circ 2^{\Sigma m} = ev_{\hat{\epsilon}_A(\alpha)} \) is an evaluation map at the unique element \( \hat{\epsilon}_B(\alpha \circ \Sigma m) \). Evaluation of that equation at any characteristic function \( \chi_B \in 2^{\Sigma B} \) shows that \( m(\hat{\epsilon}_A(\alpha)) = \hat{\epsilon}_B(\alpha \circ m) \), thereby proving naturality.

\[ \square \]
Every probability measure $\hat{P} \in G(\Sigma A)$ specifies an affine functional $I^{\Sigma A} \xrightarrow{\epsilon} I$ using the integral, and by Theorem 7.6 every such functional is the $I$-component of a natural transformation $\alpha \in Spec(\Sigma A)[2]$. Thus the natural transformation $\hat{\epsilon}$, as defined above, can be characterized alternatively as in Lemma 8.2.

**Lemma 8.2.** Provided no measurable cardinals exist, there is a natural transformation $P \circ \Sigma \Rightarrow id_{Cvx}$ defined by components

\[
\begin{array}{ccc}
P(\Sigma A) & \xrightarrow{\epsilon} & A \\
\hat{P} & \xrightarrow{\hat{\epsilon} A(P)} & \hat{\epsilon}_A(P)
\end{array}
\]

where $\alpha_1 = P$, the functional determined by the probability measure $\hat{P}$ using the integral.

The natural transformation $\epsilon$ defined in this lemma yields the counit for an adjunction $P \dashv \Sigma$, such that the composite $\Sigma \circ P$ is the Giry monad.

**Theorem 8.3.** Assuming that no measurable cardinals exist, the functor $P$ is left adjoint to $\Sigma$, $P \dashv \Sigma$, and yields the Giry monad,

$$(G, \eta, \mu) = (\Sigma \circ P, \eta, \Sigma(\epsilon_P)).$$

**Proof.** The functor $\Sigma$ endows each convex space of probability measures $P(X)$ with the same $\sigma$-algebra as that associated with the Giry monad since the evaluation maps, which are used to define the $\sigma$-algebra for the Giry monad, pull back subobjects (=intervals) of $I$ to subobjects of $P(X)$

\[
\begin{array}{ccc}
ev_U^{-1}((a, b)) & \xrightarrow{\epsilon} & (a, b) \\
\mathcal{P}(X) & \xrightarrow{ev_U} & I
\end{array}
\]

where $U \in \Sigma X$.

The two natural transformations, $(P \circ \Sigma) \xrightarrow{\epsilon} id_{Cvx}$ and $id_{Meas} \xrightarrow{\eta} \Sigma \circ P$, together yield the required bijective correspondence. Given a measurable function $f$

\[
\begin{array}{ccc}
X & \xrightarrow{\eta_X} & \Sigma(P(X)) \\
\downarrow f & & \downarrow \Sigma(\hat{f}) \\
\Sigma A & \xrightarrow{\epsilon_A} & \mathcal{P}(\Sigma A)
\end{array}
\]

in $Meas$ and

\[
\begin{array}{ccc}
\mathcal{P}(X) & \xrightarrow{\epsilon_P(f)} & \mathcal{P}(\Sigma A) \\
\downarrow \hat{f} & & \downarrow \hat{\epsilon}_A \\
A & \xrightarrow{\hat{\epsilon}_A} & \hat{\epsilon}_A(P)
\end{array}
\]

in $Cvx$ define $\hat{f} = \epsilon_A \circ P(f)$, which yields

$$\Sigma(\epsilon_A \circ P(f)) \circ \eta_X(x) = \Sigma(\epsilon_A \circ \delta_{f(x)}) = f(x)$$

proving the existence of an adjunct arrow to $f$. The uniqueness follows from the fact that if $g \in \mathcal{Cvx}(P(X), A)$ also satisfies the required commutativity condition of the diagram on the left, $\Sigma g \circ \eta_X = f$, which says that for every $x \in X$ that

$$g(\delta_x) = f(x) = \epsilon_A(\delta_{f(x)}) = (\epsilon_A \circ P(f))(\delta_x) = \hat{f}(\delta_x).$$

We can now use the fact that $\epsilon_{P(X)} \circ P(\eta_X) = id_{P(X)}$ to conclude that for an arbitrary probability measure $P \in P(X)$ that $g(P) = \hat{f}(P)$ follows using $g(P) = g(\epsilon_{P(X)}(\delta_P))$ and naturality,

\[
\text{Conversely, the image of every convex subspace of } P(X) \text{ is a convex space under every affine map.}
\]
\[ \mathcal{P}(X) \xrightarrow{\mathcal{P}(\eta_X)} \mathcal{P}(\Sigma(\mathcal{P}(X))) \xrightarrow{\epsilon_{\mathcal{P}(X)}} \mathcal{P}(X) \]

\[ \mathcal{P}(f) \xrightarrow{\mathcal{P}(f)} \mathcal{P}(\Sigma g) \xrightarrow{\epsilon_A} A \]

\[ \mathcal{P}(\Sigma A) \xrightarrow{\epsilon_A} A \]

\[ \mathcal{P}(\chi_U) \xrightarrow{\mu_X} \mathcal{P}(\Sigma) \]

where the bottom path, \( \hat{f} = \epsilon_A \circ \mathcal{P}(f) \) yields \( \hat{f}(P) \), while the east-south path gives \( g(P) = g(\epsilon_{\mathcal{P}(X)} \circ \mathcal{P}(\eta_X)) \).

The unit of \( \mathcal{P} \dashv \Sigma \) is \( \eta \) (the same as the Giry monad), and the multiplication determined by the adjunction \( \mathcal{P} \dashv \Sigma \) is given by

\[ \hat{\mu}_X = \Sigma(\epsilon_{\mathcal{P}(X)}) \]

where the functor \( \Sigma \) just makes the affine map \( \epsilon_{\mathcal{P}(X)} \) a measurable function. We must show that this \( \hat{\mu} \) coincides with the multiplication \( \mu \) of the Giry monad which is defined componentwise by

\[ \mu_X(P[U]) = \int_{q \in G} ev_U(q) \, dP(q). \]

This follows by the naturality of \( \epsilon \) and the fact \( \mathcal{P}(\chi_U) = ev_U \) for all \( U \in \Sigma_X \),

\[ \Sigma(\mathcal{P}(\epsilon_{\mathcal{P}(X)})) \xrightarrow{\Sigma(\epsilon_{\mathcal{P}(X)})} \Sigma(\mathcal{P}(X)) \]

\[ \Sigma(\mathcal{P}(\epsilon_{\mathcal{P}(X)})) \xrightarrow{\epsilon_A} A \]

\[ \mathcal{P}(\Sigma A) \xrightarrow{\epsilon_A} A \]

The east-south path gives \( \hat{\mu}_{\mathcal{P}(X)} \Sigma(\epsilon_{\mathcal{P}(X)}(P))[U] = Q(U) \), while the south-east path gives the multiplication \( \mu_X \) of the Giry monad,

\[ \Sigma(\epsilon_{\mathcal{P}(2)})(Pev_U^{-1}) = \int_{q \in G} q(U) \, dP = \mu_X(P)[U]. \]

where we have used the fact \( \mu_2 = \Sigma(\epsilon_{\mathcal{P}(2)}) \).

\[ \square \]

9. The Equivalence of \textbf{Meas}^G and \textbf{Cvx}

To prove the equivalence \( \text{Cvx} \cong \text{Meas}^G \) requires that the comparison functor \( \Phi \),

\[ \text{Meas}^G \xrightarrow{\Phi} \text{Cvx} \]

yields an equivalence of categories. In this diagram the functors \( F^G \) and \( U^G \) are the well-known “free” and “forgetful” functors associated with algebras defined on a monad.
To show the equivalence $\mathbf{Cvx} \cong \mathbf{Meas}^G$ we show the unit and counit of the adjunction $\hat{\Phi} \dashv \Phi$ are naturally isomorphic to the identity functors on $\mathbf{Cvx}$ and $\mathbf{Meas}^G$, respectively. The functor $\hat{\Phi}$, applied to a Giry algebra $\mathcal{G}(X) \xrightarrow{h} X$, is the coequalizer (object) of the parallel pair $\epsilon_{\mathcal{P}(X)} \circ \mathcal{P}(h) \xrightarrow{q} \mathbf{CoEq}$.

The convex space $\mathbf{CoEq}$ is the $\mathbf{Cvx}$ object corresponding to the Giry algebra $h$. For the equivalence to hold, we must show that the map of $\mathcal{G}$-algebras

$$
\begin{array}{ccc}
\mathcal{G}(X) & \xrightarrow{\mathcal{G}(\theta)} & \Sigma(\mathcal{P}(\Sigma\mathbf{CoEq})) \\
\downarrow h & & \downarrow \Sigma(\epsilon_{\mathbf{CoEq}}) \\
X & \xrightarrow{\theta} & \Sigma\mathbf{CoEq}
\end{array}
$$

in $\mathbf{Meas}$

is an isomorphism. Towards this end, let $(\operatorname{Ker}(q), m_1, m_2)$ denote the kernel pair of the coequalizer $q$. Since $q \circ \epsilon_{\mathcal{P}(X)} = q \circ \mathcal{P}(h)$ there exist a unique map $\psi$ such that the $\mathbf{Cvx}$-diagram

commutes. Now apply the functor $\Sigma$ to this diagram to obtain the commutative $\mathbf{Meas}$-diagram on the right hand side of the above diagram. Define

$$
\theta = \Sigma q \circ \eta_X
$$

to obtain the (equivalent but redrawn) commutative diagram

where the outer paths commute since the $\mathcal{G}$-algebra $h$ satisfies the condition that $h \circ \mathcal{G}(h) = h \circ \mu_X$.

\footnote{Recall $\mu_X = \Sigma(\epsilon_{\mathcal{P}(X)})$.}
Since \((\Sigma(Ker(q)), \Sigma m_1, \Sigma m_2)\) is a pullback and \(h \circ \Sigma m_1 = h \circ \Sigma m_2\), there exist a unique map \(\Sigma CoEq \longrightarrow X\) which (necessarily) is the inverse of \(\theta\), and hence we obtain the isomorphism of measurable spaces \(X \cong \Sigma CoEq\). This proves that the unit of the adjunction \(\hat{\Phi} \dashv \Phi\) is naturally isomorphic to the identity functor \(id_{Meas}\).

Conversely, given a convex space \(C\), applying the functor \(\Sigma\) to the counit of the adjunction at \(C\) gives the \(\mathcal{G}\)-algebra

\[
\begin{align*}
\Sigma(\mathcal{P}(\Sigma C)) & \xrightarrow{\Sigma \epsilon_C} \Sigma C \\
\end{align*}
\]

This is precisely the process of applying the comparison functor \(\Phi\) to the convex space \(C\). Now if we apply the preceding process (apply the functor \(\hat{\Phi}\)) to this \(\mathcal{G}\)-algebra, we construct the coequalizer of the parallel pair

\[
\begin{align*}
\mathcal{P}(\Sigma(\mathcal{P}(\Sigma C))) & \xrightarrow{\epsilon_{\mathcal{P}(\Sigma C)}} \mathcal{P}(\Sigma C) \\
\end{align*}
\]

which is, up to isomorphism, just the convex space \(C\) and counit of the adjunction \(\mathcal{P} \dashv \Sigma\).

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E-mail address: kirksturtz@universalmath.com

\[10\] The proof this is a \(\mathcal{G}\)-algebra is a straightforward verification.