Generalized essential maps and coincidence type theory for compact multifunctions

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Abstract

In this paper we discuss generalized essential maps. By establishing a very simple result we are able to present a variety of topological transversality theorems in a general setting.

Keywords: Essential maps, homotopy, admissible maps.

2020 MSC: 47H10, 54H25, 55M20.

1. Introduction

The topological transversality theorem [4] for continuous compact maps states that for continuous compact maps $F$ and $G$ with $F \approx G$ then $F$ is essential if and only if $G$ is essential. The essential map theory was extended to set valued maps and to $d$–essential maps [6–8]. In this paper we consider admissible maps (see below) and we establish a very general topological transversality theorem. To do this we first present a very simple result which we will then use to establish topological transversality theorems in a variety of settings.

Let $X$, $Y$ be metric spaces and $\Gamma$ paracompact. A continuous single valued map $p : \Gamma \rightarrow X$ is called a Vietoris map (written $p : \Gamma \Rightarrow X$) if the following two conditions are satisfied:

(i). for each $x \in X$, the set $p^{-1}(x)$ is acyclic (with respect to the Čech cohomology functor),
(ii). $p$ is a perfect map i.e., $p$ is closed and for every $x \in X$ the set $p^{-1}(x)$ is nonempty and compact.

Let $D(X, Y)$ be the set of all admissible pairs $X \overset{p}{\leftarrow} \Gamma \overset{q}{\rightarrow} Y$ where $p$ is a Vietoris map and $q$ is continuous. We will denote every such diagram by $(p, q)$. Given two diagrams $(p, q)$ and $(p', q')$, where $X \overset{p'}{\leftarrow} \Gamma' \overset{q'}{\rightarrow} Y$, we write $(p, q) \sim (p', q')$ if there a homeomorphism $f : \Gamma \rightarrow \Gamma'$ such that $p' \circ f = p$ and $q' \circ f = q$. The equivalence class of a diagram $(p, q) \in D(X, Y)$ with respect to $\sim$ is denoted by

$$\phi = (X \overset{p}{\leftarrow} \Gamma \overset{q}{\rightarrow} Y) : X \rightarrow Y$$

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doi: 10.22436/jmcs.021.02.02
Received: 2019-09-05 Revised: 2019-10-22 Accepted: 2020-03-03
or \( \phi = ([p, q]) \) and is called a morphism from \( X \) to \( Y \). We let \( M(X, Y) \) be the set of all such morphisms. Note if \( (p, q), (p_1, q_1) \in D(X, Y) \) (where \( X \not\ni \Gamma \ni \gamma \ni Y \) and \( X \not\ni \Gamma' \ni \gamma' \ni Y \) and \( (p, q) \sim (p_1, q_1) \) then it is easy to see that for \( x \in X \) we have \( q_1(p_1^{-1}(x)) = q(p^{-1}(x)) \). For any \( \phi \in M(X, Y) \) a set \( \phi(x) = q \circ p^{-1}(x) \) where \( \phi = ([p, q]) \) is called an image of \( x \) under a morphism \( \phi \). Let \( \phi \in M(X, Y) \) and \( (p, q) \) a representative of \( \phi \). We define \( \phi(X) \subseteq Y \) by \( \phi(X) = q(p^{-1}(X)) \). Note \( \phi(X) \) does not depend on the representative of \( \phi \). Now \( \phi \in M(X, Y) \) is called compact provided the set \( \phi(X) \) is relatively compact in \( Y \). We say a map \( \phi \) is admissible or determined by a morphism \( X \ni p \ni \Gamma \ni \gamma \ni Y \) provided \( \phi(x) = q \circ p^{-1}(x) \) for any \( x \in X \) and we write \( \phi \in \text{Adm}(X, Y) \) (note \( \phi \) is upper semicontinuous) i.e., \( \text{Adm}(X, Y) \) denotes the class of all admissible set-valued maps \( \phi : X \to 2^Y \) (note a set-valued map \( \phi : X \to 2^Y \) is admissible if it is represented by an admissible pair). Let \( U \) be open in \( X \) and let \( F, G \in \text{Adm}_{\partial U}(U, X) \) (i.e., \( F, G \in \text{Adm}(\overline{U}, X) \) with \( x \notin F(x), x \notin G(x) \) for \( x \in \partial U \)) be compact maps. We say \( F \equiv G \) (compactly) in \( \text{Adm}_{\partial U}(U, X) \) if there exists a (compact) admissible \( \psi : U \times [0, 1] \to 2^X \) with \( x \notin \psi(x, t) \) for any \( x \in \partial U \) and \( t \in (0, 1) \), \( \psi_0 = F \) and \( \psi_1 = G \) (here \( \psi_t(x) = \psi(x, t) \)). Note \( \psi_t(x) \) (compact) in \( \text{Adm}_{\partial U}(U, X) \) is an equivalence relation; see [3, Section 46], [5, Section 5]. Suppose \( F \in \text{Adm}_{\partial U}(U, X) \) is a compact map and \( f : \overline{U} \to X \) is a single valued continuous compact map with \( x \neq f(x) \) for \( x \in \partial U \). For a condition (clearly satisfied if \( f \) is the zero map) to guarantee that \( F \equiv f \) (compactly) in \( \text{Adm}_{\partial U}(U, X) \) see [3, (Section 46), Proposition 46.3].

2. Topological Transversality Theorem

We will consider classes \( A \) and \( B \) of maps. Let \( E \) be a completely regular space and \( U \) an open subset of \( E \).

**Definition 2.1.** We say \( F \in A(U, E) \) if \( F \in A(\overline{U}, E) \) and \( F : U \to K(E) \) is an upper semicontinuous (u.s.c) compact map; here \( \overline{U} \) denotes the closure of \( U \) in \( E \) and \( K(E) \) denotes the family of nonempty compact subsets of \( E \).

**Remark 2.2.** Examples of \( F \in A(\overline{U}, E) \) might be that \( F \) has convex values or \( F \) has acyclic values or \( F \) is admissible (as described in Section 1).

In this paper we fix a \( \Phi \in B(\overline{U}, E) \) (i.e., \( \Phi \in B(\overline{U}, E) \) and \( \Phi : \overline{U} \to K(E) \) is a u.s.c. map).

**Definition 2.3.** We say \( F \in A_{\partial U}(U, E) \) if \( F \in A(U, E) \) and \( \Phi(x) \cap F(x) = \emptyset \) for \( x \in \partial U \); here \( \partial U \) denotes the boundary of \( U \) in \( E \).

**Definition 2.4.** Let \( F, G \in A_{\partial U}(U, E) \). We say \( F \equiv G \) in \( A_{\partial U}(U, E) \) if there exists a u.s.c. compact map \( \Psi : U \times [0, 1] \to K(E) \) with \( \Psi \in A(\overline{U} \times [0, 1], E) \), \( \Phi(x) \cap \Psi_t(x) = \emptyset \) for any \( x \in \partial U \) and \( t \in (0, 1) \) (here \( \Psi_t(x) = \Psi(x, t) \)), \( \Psi_0 = F \) and \( \Psi_1 = G \). In addition here we always assume for any map \( \Theta \in A(U \times [0, 1], E) \) and any maps \( g \in C(U, U \times [0, 1]) \) and \( f \in C(\overline{U} \times [0, 1], U \times [0, 1]) \) then \( \Theta \circ g \in A(U, E) \) and \( \Theta \circ f \in A(U \times [0, 1], E) \); here \( C \) denotes the class of single valued continuous functions.

**Remark 2.5.**

(a). In our results below alternatively we could use the following definition for \( \equiv \) in \( A_{\partial U}(U, E) \): \( F \equiv G \) in \( A_{\partial U}(U, E) \) if there exists a u.s.c. compact map \( \Psi : U \times [0, 1] \to K(E) \) with \( \Psi(., \eta(., .)) \in A(U, E) \) for any continuous function \( \eta : U \to [0, 1] \) with \( \eta(\partial U) = 0 \), \( \Phi(x) \cap \Psi_t(x) = \emptyset \) for any \( x \in \partial U \) and \( t \in (0, 1) \) (here \( \Psi_t(x) = \Psi(x, t) \)), \( \Psi_0 = F \) and \( \Psi_1 = G \). [Note the additional assumption in Definition 2.4 is not needed here].

(b). Throughout the paper we assume \( \equiv \) in \( A_{\partial U}(U, E) \) is a reflexive, symmetric relation.

**Remark 2.6.** Let \( F \in A_{\partial U}(U, E) \). We say \( F \) is \( \Phi \)-essential in \( A_{\partial U}(U, E) \) if for every map \( J \in A_{\partial U}(U, E) \) with \( J|_{\partial U} = F|_{\partial U} \) and \( J \equiv F \) in \( A_{\partial U}(U, E) \) there exists a \( x \in U \) with \( \Phi(x) \cap J(x) \neq \emptyset \).

We now present a simple result which will more or less immediately yield a very general topological transversality theorem.
**Theorem 2.7.** Let $E$ be a completely regular topological space, $U$ an open subset of $E$, $F \in A_{\partial U}(\overline{U}, E)$ and $G \in A_{\partial U}(\overline{U}, E)$ is $\Phi$–essential in $A_{\partial U}(\overline{U}, E)$. Also suppose

\[
\left\{ \begin{array}{l}
\text{for any map } J \in A_{\partial U}(\overline{U}, E) \text{ with } J|_{\partial U} = f|_{\partial U} \text{ and } \\
J \cong F \text{ in } A_{\partial U}(\overline{U}, E) \text{ we have } G \cong J \text{ in } A_{\partial U}(\overline{U}, E).
\end{array} \right. 
\tag{2.1}
\]

Then $F$ is essential in $A_{\partial U}(\overline{U}, E)$.

**Proof.** Without loss of generality assume $\cong$ in $A_{\partial U}(\overline{U}, E)$ is as in Definition 2.4. Consider any map $J \in A_{\partial U}(\overline{U}, E)$ with $J|_{\partial U} = f|_{\partial U}$ and $J \cong F$ in $A_{\partial U}(\overline{U}, E)$. From (2.1) there exists a u.s.c. compact map $H^f : U \times [0, 1] \to \mathbb{K}(E)$ with $H^f \in A(U \times [0, 1], E)$, $\Phi(x) \cap H^f(x) = \emptyset$ for any $x \in \partial U$ and $t \in (0, 1)$ (here $H^f(x) = H^f(x, t)$), $H^0_0 = G$ and $H^1_0 = J$. Let

\[ K = \{ x \in \overline{U} : \Phi(x) \cap H^f(x, t) \neq \emptyset \text{ for some } t \in [0, 1] \} \]

and

\[ D = \{ (x, t) \in \overline{U} \times [0, 1] : \Phi(x) \cap H^f(x, t) = \emptyset \} . \]

Now $D \neq \emptyset$ (note $G$ is $\Phi$–essential in $A_{\partial U}(\overline{U}, E)$) and $D$ is closed (note $\Phi$ and $H^f$ are u.s.c.) and so $D$ is compact (note $H^f$ is a compact map). Let $\pi : \overline{U} \times [0, 1] \to \overline{U}$ be the projection. Now $K = \pi(D)$ is closed (see Kuratowski’s theorem [2, pp 126]) and so in fact compact (recall projections are continuous). Also note $K \cap \partial U = \emptyset$ (since $\Phi(x) \cap H^f_t(x) = \emptyset$ for any $x \in \partial U$ and $t \in [0, 1]$) so since $E$ is Tychonoff there exists a continuous map $\mu : \overline{U} \to [0, 1]$ with $\mu(\partial U) = 0$ and $\mu(K) = 1$. Define the map $R$ by $R(x) = H^f(x, \mu(x))$. Now $R \in A_{\partial U}(\overline{U}, E)$ (note $H^f(x, \mu(x)) = H^f \circ g(x)$ where $g : \overline{U} \to \overline{U} \times [0, 1]$ is given by $g(x) = (x, \mu(x))$) with $\partial_{\overline{U}} = G|_{\partial U}$ (note if $x \in \partial U$ then $R(x) = H^f(x, 0) = G$ and so $R(x) \cap \Phi(x) = \emptyset$). We now show $R \cong G$ in $A_{\partial U}(\overline{U}, E)$. To see this let $Q : \overline{U} \times [0, 1] \to \mathbb{K}(E)$ be given by $Q(x, t) = H^f_t(x, t \mu(x)) = H^f \circ f(x, t)$ where $f : \overline{U} \times [0, 1] \to \overline{U} \times [0, 1]$ is given by $f(x, t) = (x, t \mu(x))$. Note $Q \in A(\overline{U} \times [0, 1], E)$, $Q_0 = G$, $Q_1 = R$ and $\Phi(x) \cap Q_t(x) = \emptyset$ for any $x \in \partial U$ and $t \in (0, 1)$ (since if $t \in (0, 1)$ and $x \in \partial U$ then $\Phi(x) \cap H^f_t(x, t \mu(x)) = \emptyset$ so $x \in K$ and as a result $\mu(x) = 1$ i.e., $\Phi(x) \cap H^f_t(x, t \mu(x)) = \emptyset$). Thus $R \cong G$ in $A_{\partial U}(\overline{U}, E)$. Since $G$ is $\Phi$–essential in $A_{\partial U}(\overline{U}, E)$ there exists a $x \in U$ with $\Phi(x) \cap R(x) \neq \emptyset$ (i.e., $\Phi(x) \cap H^f\mu(x)(x) \neq \emptyset$). Thus $x \in K$, $\mu(x) = 1$ and so $\emptyset \neq \Phi(x) \cap H^f_1(x) = \Phi(x) \cap J(x)$.

□

**Remark 2.8.**

(i). In the proof of Theorem 2.7 it is simple to adjust the proof if we use $\cong$ in $A_{\partial U}(\overline{U}, E)$ from Remark 2.5 if we note $R(\cdot) = H^f(\cdot, \mu(\cdot))$ and $Q(\cdot, \nu(\cdot)) = H^f(\cdot, \nu(\cdot) \mu(\cdot)) = H^f(\cdot, w(\cdot))$ (with $w(\cdot) = \nu(\cdot) \mu(\cdot)$) for any continuous $\nu : \overline{U} \to [0, 1]$ with $\nu(\partial U) = 0$ (note $w : \overline{U} \to [0, 1]$ is continuous and $w(\partial U) = 0$).

(ii). One could replace u.s.c. in the definition of $A(U, E)$, $B(U, E)$, Definition 2.4 and Remark 2.5 with any condition that guarantees that $K$ in the proof of Theorem 2.7 is closed; this is all that is needed if $E$ is normal. If $E$ is Tychonoff and not normal the one can also replace the compactness of the map in the proof of Theorem 2.7.

**Example 2.9.** Theorem 2.7 immediately yields a general Leray–Schauder type alternative for coincidences. Let $E$ be a completely metrizable locally convex space, $U$ an open subset of $E$, $F \in A_{\partial U}(\overline{U}, E)$, $G \in A_{\partial U}(\overline{U}, E)$ is $\Phi$–essential in $A_{\partial U}(\overline{U}, E)$ and $\Phi(x) \cap \{ t F(x) + (1 - t) G(x) \} = \emptyset$ for any $x \in \partial U$ and $t \in (0, 1)$. For any map $J \in A_{\partial U}(\overline{U}, E)$ with $J|_{\partial U} = f|_{\partial U}$ suppose $H^f \in A(\overline{U} \times [0, 1], E)$ where $H^f(x, t) = t J(x) + (1 - t) G(x)$. [Also here we assume for any map $\Theta \in A(\overline{U} \times [0, 1], E)$ and any maps $g \in C(\overline{U} \times [0, 1], E)$ and $f \in C(\overline{U} \times [0, 1], E)$ then $\Theta \circ g \in A(\overline{U}, E)$ and $\Theta \circ f \in A(\overline{U} \times [0, 1], E)$]. Then $F$ is $\Phi$–essential in $A_{\partial U}(\overline{U}, E)$. 

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The proof follows from Theorem 2.7 since topological vector spaces are completely regular and note if \( J \in A_{\partial U}(\overline{U}, E) \) with \( J_{|\partial U} = F_{|\partial U} \) then with \( H^t(x, t) = t J(x) + (1 - t) G(x) \) note \( H^t_0 = G, H^t_1 = J, H^t : \overline{U} \times [0, 1] \to K(E) \) is a u.s.c. compact (see [1, Theorem 4.18]) map, \( H^t \in A(\overline{U} \times [0, 1], E) \) and \( \Phi(x) \cap H^t_1(x) = \emptyset \) for \( x \in \partial U \) and \( t \in (0, 1) \) then since \( J_{|\partial U} = F_{|\partial U} \) we note that \( \Phi(x) \cap H^t_1(x) = \emptyset \) so as a result \( G \equiv \Phi \) (Definition 2.4) in \( A_{\partial U}(\overline{U}, E) \) (i.e., (2.1) holds). [Note \( E \) being a completely metrizable locally convex space can be replaced by any (Hausdorff) topological vector space \( E \) which has the property that the closed convex hull of a compact set in \( E \) is compact. In fact it is easy to see, if we argue differently, that all we need to assume is that \( E \) is a topological vector space].

We now present the topological transversality theorem in a general setting. Assume
\[ \equiv \text{ in } A_{\partial U}(\overline{U}, E) \] is an equivalence relation. (2.2)

**Theorem 2.10.** Let \( E \) be a completely regular topological space, \( U \) an open subset of \( E \) and assume (2.2) holds. Suppose \( F \) and \( G \) are two maps in \( A_{\partial U}(\overline{U}, E) \) with \( F \equiv G \) in \( A_{\partial U}(\overline{U}, E) \). Then \( F \) is \( \Phi \)-essential in \( A_{\partial U}(\overline{U}, E) \) if and only if \( G \) is \( \Phi \)-essential in \( A_{\partial U}(\overline{U}, E) \).

**Proof.** Assume \( G \) is \( \Phi \)-essential in \( A_{\partial U}(\overline{U}, E) \). To show \( F \) is \( \Phi \)-essential in \( A_{\partial U}(\overline{U}, E) \) let \( J \in A_{\partial U}(\overline{U}, E) \) with \( J_{|\partial U} = F_{|\partial U} \) and \( J \equiv F \) in \( A_{\partial U}(\overline{U}, E) \). Now since \( F \equiv G \) in \( A_{\partial U}(\overline{U}, E) \) then (2.2) guarantees that \( G \equiv J \) in \( A_{\partial U}(\overline{U}, E) \) i.e., (2.1) holds. Then Theorem 2.7 guarantees that \( F \) is \( \Phi \)-essential in \( A_{\partial U}(\overline{U}, E) \). A similar argument shows that if \( F \) is \( \Phi \)-essential in \( A_{\partial U}(\overline{U}, E) \) then \( G \) is \( \Phi \)-essential in \( A_{\partial U}(\overline{U}, E) \).

Assume (2.2) holds. If \( F \) and \( G \) are maps in \( A_{\partial U}(\overline{U}, E) \) with \( F_{|\partial U} = G_{|\partial U} \) is \( F \equiv G \) in \( A_{\partial U}(\overline{U}, E) \)? We will discuss this now.

We assume the following conditions:

\[ E \text{ is a (Hausdorff) topological vector space and } U \text{ is convex} \]
(2.3)

there exists a retraction \( r : \overline{U} \to \partial U \)
(2.4)

and
\[
\left\{ \begin{array}{l}
\text{for any map } \Theta \in A(\overline{U}, E) \text{ and } f \in C(\overline{U} \times [0, 1], \overline{U}) \\
\quad \text{then } \Theta \circ f \in A(\overline{U} \times [0, 1], E).
\end{array} \right.
\]
(2.5)

**Remark 2.11.** Note topological vector spaces are completely regular. Also if \( E \) is an infinite dimensional Banach space and \( U \) is convex then (2.4) holds. Also note if \( A \) is closed under composition then (2.5) holds.

Let \( r \) be in (2.4) and let \( F \) and \( G \) be maps in \( A_{\partial U}(\overline{U}, E) \) with \( F_{|\partial U} = G_{|\partial U} \). Consider the map \( F^* \) given by \( F^*(x) = F(r(x)) \) for \( x \in \overline{U} \). Note \( F^*(x) = G(r(x)) \) for \( x \in \overline{U} \) since \( F_{|\partial U} = G_{|\partial U} \). Let
\[ H(x, \lambda) = G(2 \lambda r(x) + (1 - 2 \lambda) x) = G \circ j(x, \lambda) \text{ for } (x, \lambda) \in \overline{U} \times \left[ 0, \frac{1}{2} \right] \]
(2.6)

(here \( j : \overline{U} \times [0, \frac{1}{2}] \to \overline{U} \) (note \( \overline{U} \) is convex) is given by \( j(x, \lambda) = 2 \lambda r(x) + (1 - 2 \lambda) x \). Now \( H : \overline{U} \times [0, \frac{1}{2}] \to K(E) \) is a u.s.c. compact map. Also from (2.5) note \( H \in A(\overline{U} \times [0, \frac{1}{2}], E) \) with \( \Phi(x) \cap H_{\lambda}(x) = \emptyset \) for \( x \in \partial U \) and \( \lambda \in [0, \frac{1}{2}] \) (note if \( x \in \partial U \) and \( \lambda \in [0, \frac{1}{2}] \) then since \( r(x) = x \) we have \( \Phi(x) \cap H_{\lambda}(x) = \emptyset \) since \( x \lambda \)). Thus \( G \equiv F^* \) in \( A_{\partial U}(\overline{U}, E) \) (Definition 2.4). Similarly with
\[ Q(x, \lambda) = \Phi((2 - 2 \lambda) r(x) + (2 \lambda - 1) x) \text{ for } (x, \lambda) \in \overline{U} \times \left[ \frac{1}{2}, 1 \right] \]
(2.7)

we see that \( F^* \equiv F \) in \( A_{\partial U}(\overline{U}, E) \) (Definition 2.4). Combining gives \( F \equiv G \) in \( A_{\partial U}(\overline{U}, E) \) (Definition 2.4).

In this situation we could replace Definition 2.6 with:
Definition 2.12. Let $F \in A_{\partial U}(\overline{U}, E)$. We say $F$ is essential in $A_{\partial U}(\overline{U}, E)$ if for every map $J \in A_{\partial U}(\overline{U}, E)$ with $J_{|\partial U} = F_{|\partial U}$ there exists an $x \in U$ with $\Phi(x) \cap J(x) \neq \emptyset$.

Now from Theorem 2.7 (in fact here the argument would be shorter since the map $Q$ is not needed and the assumption $\Theta \circ f \in A(U \times [0,1], E)$ is not needed in Definition 2.4) and Theorem 2.10 we have:

Theorem 2.13. Let $E$ be a topological vector space, $U$ an open convex subset of $E$ and assume (2.2), (2.4) and (2.5) hold. Suppose $F$ and $G$ are two maps in $A_{\partial U}(\overline{U}, E)$ with $F \equiv G$ in $A_{\partial U}(\overline{U}, E)$ (as in Definition 2.4). Then $F$ is $\Phi$–essential (Definition 2.12) in $A_{\partial U}(\overline{U}, E)$ if and only if $G$ is $\Phi$–essential (Definition 2.12) in $A_{\partial U}(\overline{U}, E)$.

Remark 2.14.

(i). Suppose (2.4) and (2.5) hold and in addition assume

$$
\begin{aligned}
&\text{for any map } \Theta \in A(U, E) \text{ then } \Theta(. , \eta(.)) = \Theta \circ f(. , \eta(.)) \in A(U, E) \\
&\text{for any continuous function } \eta : \overline{U} \rightarrow [0,1] \text{ with } \eta(\partial U) = 0 \text{ where } f(x, t) = t \eta(x) + (1 - t) \lambda, \ t \in [0,1], x \in U.
\end{aligned}
$$

Let $F$ and $G$ be maps in $A_{\partial U}(\overline{U}, E)$ with $F_{|\partial U} = G_{|\partial U}$. It is simple to adjust the proof above (use (2.6) instead of (2.5)) to establish $F \equiv G$ in $A_{\partial U}(\overline{U}, E)$ (as in Remark 2.5). As a result we get immediately Theorem 2.13 (with (2.5) replaced by (2.6) and $\equiv$ in $A_{\partial U}(\overline{U}, E)$ (Definition 2.4) replaced by $\equiv$ in $A_{\partial U}(\overline{U}, E)$ (Remark 2.5)).

(ii). Let $F$ and $G$ be maps in $A_{\partial U}(\overline{U}, E)$ with $F_{|\partial U} = G_{|\partial U}$. Assume the following conditions:

$E$ is a completely metrizable locally convex space

(2.7)

$$
\Phi(x) \cap \{ tF(x) + (1 - t)G(x) \} = \emptyset \text{ for } x \in \partial U \text{ and } t \in (0,1)
$$

and

$$
\begin{aligned}
&\text{for any continuous function } \eta : \overline{U} \rightarrow [0,1] \text{ with } \eta(\partial U) = 0.
\end{aligned}
$$

(2.9)

Let $H(x, \lambda) = \lambda F(x) + (1 - \lambda) G(x)$ for $(x, \lambda) \in \overline{U} \times [0,1]$. Note $H : \overline{U} \times [0,1] \rightarrow K(E)$ is a u.s.c. compact (see [1, Theorem 4.18]) map and by (2.9) note $H(. , \eta(.)) \in A(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \rightarrow [0,1]$, and from (2.8) note $\Phi(x) \cap H(x) = \emptyset$ for $x \in \partial U$ and $t \in (0,1)$ so as a result $F \equiv G$ in $A_{\partial U}(\overline{U}, E)$ (Remark 2.5). (Note (2.7) can be replaced by any topological vector space $E$ which has the property that the closed convex hull of a compact set in $E$ is compact). As a result in this setting we get immediately Theorem 2.13 (with (2.3), (2.4), (2.5) replaced by (2.7), (2.8), (2.9) and $\equiv$ in $A_{\partial U}(\overline{U}, E)$ (Definition 2.4) replaced by $\equiv$ in $A_{\partial U}(\overline{U}, E)$ (Remark 2.5)).

Now we present an example of a $\Phi$–essential (Definition 2.12) map.

Example 2.15. Let $E$ be a (Hausdorff) topological space, $U$ an open subset of $E$, $\Phi \in B(E, E)$ (i.e., $\Phi \in B(E, E)$ and $\Phi : E \rightarrow K(E)$ is a u.s.c. map) and $F \in A_{\partial U}(\overline{U}, E)$. Assume the following conditions hold:

$$
\begin{aligned}
&\text{there exists an } x \in \overline{U} \text{ with } \Phi(x) \cap \{ 0 \} \neq \emptyset \\
&\text{there exists a retraction } r : E \rightarrow \overline{U}
\end{aligned}
$$

(2.10)

$$
\begin{aligned}
&\Phi(x) \cap \lambda F(x) = \emptyset \text{ for } x \in \partial U \text{ and } \lambda \in (0,1)
\end{aligned}
$$

(2.11)

$$
\begin{aligned}
&\text{for any continuous map } \mu : E \rightarrow [0,1] \text{ with } \mu(E \setminus U) = 0 \\
&\text{and any map } J \in A_{\partial U}(\overline{U}, E) \text{ with } J_{|\partial U} = F_{|\partial U} \\
&\text{there exists an } w \in E \text{ with } \Phi(w) \cap \mu(w) J(r(w)) \neq \emptyset
\end{aligned}
$$

(2.12)

and

$$
\begin{aligned}
&\text{there is no } z \in E \setminus U \text{ with } \Phi(z) \cap \{ 0 \} \neq \emptyset.
\end{aligned}
$$

(2.13)
Then $F$ is $\Phi$–essential (Definition 2.12) in $A_{\partial U}(\overline{U}, E)$.

To see this let $J \in A_{\partial U}(\overline{U}, E)$ with $J_{\partial U} = F|_{\partial U}$. Now let

$$K = \left\{ x \in \overline{U} : \Phi(x) \cap \lambda J(x) \neq \emptyset \text{ for some } \lambda \in [0,1] \right\}.$$ 

Now $K \neq \emptyset$ (see (2.10)) is compact and $K \subseteq \overline{U}$. In fact $K \subseteq U$ from (2.12) (note if $x \in \partial U$ and $x \in K$ then for some $\lambda \in [0,1]$ we have $\emptyset \neq \Phi(x) \cap \lambda J(x) = \Phi(x) \cap \lambda F(x)$, a contradiction). Then there exists a continuous map $\mu : E \to [0,1]$ with $\mu(E \setminus U) = 0$ and $\mu(K) = 1$. Let $r$ be as in (2.11) and (2.13) guarantees that there exists a $x \in E$ with $\Phi(x) \cap \mu(x) J(r(x)) \neq \emptyset$. If $x \in E \setminus U$ then $\mu(x) = 0$ so $\Phi(x) \cap \{0\} \neq \emptyset$, and this contradicts (2.14). Thus $x \in U$ so $\Phi(x) \cap \mu(x) J(x) \neq \emptyset$, so $x \in K$, $\mu(x) = 1$ and consequently $\Phi(x) \cap J(x) \neq \emptyset$.

**Remark 2.16.** It is very easy to extend the above ideas to the $(L, T)$ $\Phi$–essential maps in [6].

Now we consider a generalization of $\Phi$–essential maps, namely the $d$–$\Phi$–essential maps. Let $E$ be a completely regular topological space and $U$ an open subset of $E$. For any map $F \in A(\overline{U}, E)$ write $F^* = I \times F : \overline{U} \to K(\overline{U} \times E)$, with $I : U \to \overline{U}$ given by $I(x) = x$, and let

$$d : \left\{ (F^*)^{-1} (B) \right\} \cup \{ \emptyset \} \to \Omega$$

be any map with values in the nonempty set $\Omega$ where $B = \{ (x, \Phi(x)) : x \in \overline{U} \}$.

**Definition 2.17.** Let $F \in A_{\partial U}(\overline{U}, E)$ and write $F^* = I \times F$. We say $F^* : \overline{U} \to K(\overline{U} \times E)$ is $d$–$\Phi$–essential if for every map $J \in A_{\partial U}(\overline{U}, E)$ (write $J^* = I \times J$ with $J_{\partial U} = F|_{\partial U}$ and $J \cong F$ in $A_{\partial U}(\overline{U}, E)$ we have that

$$d \left( (F^*)^{-1} (B) \right) = d \left( (J^*)^{-1} (B) \right) = d \left( (G^*)^{-1} (B) \right)$$

for any map $J \in A_{\partial U}(\overline{U}, E)$ with $J_{\partial U} = F|_{\partial U}$ and $J \cong F$ in $A_{\partial U}(\overline{U}, E)$ and $G \cong J$ in $A_{\partial U}(\overline{U}, E)$.

**Theorem 2.19.** Let $E$ be a completely regular topological space, $U$ an open subset of $E$, $B = \{ (x, \Phi(x)) : x \in \overline{U} \}$, $d$ is defined in (2.15), $F \in A_{\partial U}(\overline{U}, E)$ and $G \in A_{\partial U}(\overline{U}, E)$ (write $F^* = I \times F$ and $G^* = I \times G$). Suppose $G^*$ is $d$–$\Phi$–essential and

$$\left\{ \begin{array}{ll}
\text{for any map } J \in A_{\partial U}(\overline{U}, E) \\
J \cong F \text{ in } A_{\partial U}(\overline{U}, E) \\
\text{we have } G \cong J \text{ in } A_{\partial U}(\overline{U}, E) \\
\text{and } d \left( (F^*)^{-1} (B) \right) = d \left( (G^*)^{-1} (B) \right).
\end{array} \right.$$ 

Then $F^*$ is $d$–$\Phi$–essential.

**Proof.** Without loss of generality assume $\cong$ in $A_{\partial U}(\overline{U}, E)$ as in Definition 2.4. Consider any map $J \in A_{\partial U}(\overline{U}, E)$ (write $J^* = I \times J$ with $J_{\partial U} = F|_{\partial U}$ and $J \cong F$ in $A_{\partial U}(\overline{U}, E)$). From (2.16) there exists a u.s.c. compact map $H^j : \overline{U} \times [0,1] \to K(E)$ with $H^j \in A(\overline{U} \times [0,1], E)$, $\Phi(x) \cap H^j(x,t) = \emptyset$ for any $x \in \partial U$ and $t \in (0,1)$ (here $H^0(x) = H^j(x,t)$, $H_0 = G$, $H_1 = J$ and $d \left( (F^*)^{-1} (B) \right) = d \left( (G^*)^{-1} (B) \right)$). Let $(H^j)^* : \overline{U} \times [0,1] \to K(\overline{U} \times E)$ be given by $(H^j)^*(x,t) = (x, H^j(x,t))$ and let

$$K = \left\{ x \in \overline{U} : (x, \Phi(x)) \cap (H^j)^*(x,t) \neq \emptyset \text{ for some } t \in [0,1] \right\}.$$ 

Now $K \neq \emptyset$ is closed, compact and $K \cap \partial U = \emptyset$ so since $E$ is Tychonoff there exists a continuous map $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(K) = 1$. Let $R(x) = H^j(x, \mu(x))$ and write $R^* = I \times R$. Now as in Theorem 2.7, $R \in A_{\partial U}(\overline{U}, E)$ with $R_{\partial U} = G|_{\partial U}$ and $R \cong G$ in $A_{\partial U}(\overline{U}, E)$. Since $G^*$ is $d$–$\Phi$–essential then

$$d \left( (G^*)^{-1} (B) \right) = d \left( (R^*)^{-1} (B) \right) \neq d(\emptyset).$$
Now since \( \mu(K) = 1 \) we have
\[
(R^*)^{-1}(B) = \{ x \in \overline{U} : (x, \Phi(x)) \cap (x, H^{I}(x, \mu(x))) \neq \emptyset \}
\]
\[
= \{ x \in \overline{U} : (x, \Phi(x)) \cap (x, H^{I}(x, 1)) \neq \emptyset \} = (J^*)^{-1}(B),
\]
so from (2.17) we have \( d\left((G^*)^{-1}(B)\right) = d\left((J^*)^{-1}(B)\right) \neq d(\emptyset) \). Now combine with the above and we have
\( d\left((F^*)^{-1}(B)\right) = d\left((J^*)^{-1}(B)\right) \neq d(\emptyset) \). \( \square \)

Note again it is simple to adjust the proof in Theorem 2.19 if we use \( \cong \) in \( A_{\partial U}(\overline{U}, E) \) from Remark 2.5.

**Theorem 2.20.** Let \( E \) be a completely regular topological space, \( U \) an open subset of \( E \), \( B = \{(x, \Phi(x)) : x \in \overline{U}\} \), \( d \) is defined in (2.15) and assume (2.2) holds. Suppose \( F \) and \( G \) are two maps in \( A_{\partial U}(\overline{U}, E) \) (write \( F^* = I \times F \) and \( G^* = I \times G \)) and \( F \cong G \) in \( A_{\partial U}(\overline{U}, E) \). Then \( F^* \) is \( d-\Phi \)-essential if and only if \( G^* \) is \( d-\Phi \)-essential.

**Proof.** Without loss of generality assume \( \cong \) in \( A_{\partial U}(\overline{U}, E) \) is as in Definition 2.4. Assume \( G^* \) is \( d-\Phi \)-essential. Let \( J \in A_{\partial U}(\overline{U}, E) \) (write \( J^* = I \times J \)) with \( \partial U \) and \( J \cong F \) in \( A_{\partial U}(\overline{U}, E) \). If we show (2.16) then \( F^* \) is \( d-\Phi \)-essential from Theorem 2.19. Now (2.2) implies that \( G \cong J \) in \( A_{\partial U}(\overline{U}, E) \). To complete (2.16) we need to show \( d\left((F^*)^{-1}(B)\right) = d\left((G^*)^{-1}(B)\right) \). We will follow the argument in Theorem 2.19. Note since \( G \cong F \) in \( A_{\partial U}(\overline{U}, E) \), let \( H : \overline{U} \times [0, 1] \to K(E) \) be a u.s.c. compact map with \( H \in A(\overline{U} \times [0, 1], E) \), \( \Phi(x) \cap H_t(x) = \emptyset \) for any \( x \in \partial U \) and \( t \in (0, 1) \) (here \( H_t(x) = H(x, t) \), \( H_0 = G \) and \( H_1 = F \). Let \( H^* : \overline{U} \times [0, 1] \to K(\overline{U} \times E) \) be given by \( H^*(x, t) = (x, H(x, t)) \) and let
\[
K = \{ x \in \overline{U} : (x, \Phi(x)) \cap H^*(x, t) \neq \emptyset \text{ for some } t \in [0, 1] \}.
\]
Now \( K \neq \emptyset \) and there exists a continuous map \( \mu : \overline{U} \to [0, 1] \) with \( \mu(\partial U) = 0 \) and \( \mu(K) = 1 \). Let \( R(x) = H(x, \mu(x)) \) and write \( R^* = I \times R \). Now \( R \in A_{\partial U}(\overline{U}, E) \) with \( R|_{\partial U} = G|_{\partial U} \) and \( R \cong G \) in \( A_{\partial U}(\overline{U}, E) \) so since \( G^* \) is \( d-\Phi \)-essential then \( d\left((G^*)^{-1}(B)\right) = d\left((R^*)^{-1}(B)\right) \neq d(\emptyset) \). Now since \( \mu(K) = 1 \) we have
\[
(R^*)^{-1}(B) = \{ x \in \overline{U} : (x, \Phi(x)) \cap (x, H(x, \mu(x))) \neq \emptyset \}
\]
\[
= \{ x \in \overline{U} : (x, \Phi(x)) \cap (x, H(x, 1)) \neq \emptyset \} = (F^*)^{-1}(B),
\]
so \( d\left((F^*)^{-1}(B)\right) = d\left((G^*)^{-1}(B)\right) \). \( \square \)

Note again it is simple to adjust the proof in Theorem 2.20 if we use \( \cong \) in \( A_{\partial U}(\overline{U}, E) \) from Remark 2.5.

**Remark 2.21.** It is very easy to extend the above ideas to the \( (L, T) \) \( d-\Phi \)-essential maps in [7].

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