FINITE NON-METABELIAN SCHUR $\sigma$-GALOIS GROUPS
OF CLASS FIELD TOWERS

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Abstract. For each odd prime $p \geq 5$, there exist finite $p$-groups $G$ with derived quotient $G/G' \cong C_p \times C_p$ and nearly constant transfer kernel type $\tau(G) = (\kappa_1, \ldots, \kappa_{p+1})$ having two fixed points $\kappa_1 = 1$, $\kappa_2 = 2$, and $\kappa_i = 2$ for $3 \leq i \leq p+1$. It is proved that, for $p = 7$, this type $\tau(G)$ with the simplest possible case of logarithmic abelian quotient invariants $\tau(G) = (H_i/H_i')_{1 \leq i \leq 5} = (1^5, 1^3, (21)^0)$ of the eight maximal subgroups $H_i$, is realized by exactly $98$ non-metabelian Schur $\sigma$-groups $S$ of order $7^{11}$ with fixed derived length $\text{dl}(S) = 3$ and metabelianizations $S/S''$ of order $7^7$. For $p = 5$, the type $\tau(G)$ with $\tau(G) = (H_i/H_i')_{1 \leq i \leq 5} = (21^3, 1^3, (21)^4)$ leads to infinitely many non-metabelian Schur $\sigma$-groups $S$ of order at least $5^{14}$ with unbounded derived length $\text{dl}(S) \geq 3$ and metabelianizations $S/S''$ of fixed order $5^7$. These results admit the conclusion that $d = -159592$ is the first known discriminant of an imaginary quadratic field with $7$-class field tower of precise length $\ell_7(K) = 3$, and $d = -90868$ is a discriminant of an imaginary quadratic field with $5$-class field tower of length $\ell_5(K) \geq 3$, whose exact length remains unknown.

1. Introduction

Let $p$ be an odd prime number. Using the Shafarevich theorem [31] Thm. 5.1, p. 28 on the relation
\[ \text{rank} \quad \text{dim}_{\mathbb{Q}}(H^2(G, \mathbb{F}_p)) \]
with fundamental discriminant $d < 0$ the Galois group $G$ must be a Schur $\sigma$-group [1], and that this $p$-pro-group can only be a finite $p$-group when the $p$-class rank $\varrho_p = \text{dim}_{\mathbb{F}_p}(\text{Cl}_p(K)/\text{Cl}_p(K)^p)$ of $K$ is bounded by $\varrho_p \leq 2$.

The aim of the present work is to grant access to the most recent information on finite non-metabelian Schur $\sigma$-groups $S$ with odd prime power order $\#S = p^n$, and their actual realization $S \cong \text{Gal}(\mathbb{F}_p^\omega(K)/K)$ as Galois groups of finite $p$-class field towers with at least three stages over imaginary quadratic fields $K$. In view of the necessary condition $\varrho_p \leq 2$ we restrict our investigations to elementary bicyclic $p$-class groups $\text{Cl}_p(K) \cong C_p \times C_p$.

In order to reduce the probability of infinite families $S$ of Schur $\sigma$-groups $S$ with unbounded derived length $\text{dl}(S)$, we recall the following terminology $[28, 30, 32, 40, 41]$.

Definition 1. A finite $p$-group is called periodic if its isomorphism class is vertex of an infinite descendant tree with fixed coclass. Otherwise it is called sporadic.

Unbounded descendant trees whose edges are restricted to step size $s = 1$, and whose vertices consequently share a common coclass, are called coclass trees $[28]$. We shall focus our attention on Schur $\sigma$-groups $S$ with periodic metabelianizations $M = S/S''$ on coclass trees.
2. Theoretical foundations

For the convenience of the reader we begin with a collection of concepts which appear in the title of this work: non-metabelian groups, Schur-groups, and \(\sigma\)-groups.

The derived series \(G^{(0)} \geq G^{(1)} \geq G^{(2)} \geq \ldots \geq G^{(n)} \geq \ldots\) of a group \(G\) is defined recursively by \(G^{(0)} := G\) and \((\forall i \in \mathbb{N}) G^{(i)} := [G^{(i-1)}, G^{(i-1)}]\). If this series becomes stationary in the trivial group 1, that is, either \(G^{(0)} = 1\) or \((\exists n \in \mathbb{N}) G^{(n-1)} > G^{(n)} = 1\), then \(G\) is called solvable. The derived length of \(G\) is defined by \(dl(G) := \inf\{i \in \mathbb{N}_0 \mid G^{(i)} = 1\} \in \mathbb{N}_0 \cup \{\infty\}\). This invariant admits a coarse classification of groups:

- \(dl(G) = 0 \iff G = 1\) is the trivial group.
- \(dl(G) = 1 \iff G > G' > 1\), i.e. \(G\) is (non-trivial) abelian.
- \(dl(G) = 2 \iff G > G' > G'' = 1\), i.e. \(G'\) is abelian, and \(G\) is non-abelian but metabelian.
- \(dl(G) = \infty \iff (\forall i \in \mathbb{N}_0) G^{(i)} > 1 \iff G\) is non-solvable.

**Definition 2.** A finite group \(G\) is called non-metabelian if it possesses derived length \(dl(G) \geq 3\).

Let \(p\) be a prime number, and \(G\) be a pro-\(p\) group. Then the generator rank \(d_1(G) := \dim_{\mathbb{F}_p}(H^1(G, \mathbb{F}_p))\) and the relation rank \(d_2(G) := \dim_{\mathbb{F}_p}(H^2(G, \mathbb{F}_p))\) are two cohomological invariants of \(G\).

**Definition 3.** A pro-\(p\) group \(G\) is called a Schur-group if \(d_2(G) = d_1(G)\). In this case \(G\) is said to possess a balanced presentation, since the relation rank equals the generator rank [1] [21].

Let \(\sigma \in \text{Aut}(G)\) be an automorphism of a group \(G\). Since the derived subgroup \(G'\) of \(G\) is characteristic, \(\sigma\) induces an automorphism on the abelianization \(G/G'\) of \(G\). If the induced automorphism is denoted by \(\hat{\sigma}\), then \((\forall n \in \mathbb{N})(\forall x \in G) (\hat{\sigma}^n(xG')) = \sigma^n(x)G'\), which shows that the order of \(\hat{\sigma}\) divides the order of \(\sigma\), if the latter is finite.

**Definition 4.** A group \(G\) is called a \(\sigma\)-group if it possesses an automorphism \(\sigma \in \text{Aut}(G)\) which acts as inversion on the abelianization, that is, \((\forall x \in G) (\sigma(x)G') = x^{-1}G'\).

Observe that a group \(G\) is abelian if and only if the inversion is an automorphism of \(G\). Consequently, every abelian group is a \(\sigma\)-group, and a finite \(\sigma\)-group \(G\) has an automorphism group with even order \#Aut\((G)\), since the inversion on \(G/G'\), induced by \(\sigma\), is an involution of order 2.

3. 7-class field towers with precise length three

In this section we present our main result concerning the discovery of the first three-stage towers of 7-class fields, which do not occur in the previous literature. In order to ensure the exact length three, we abstain from transfer kernel types (TKT) of sporadic groups, which are permutations.

**Theorem 5.** (Sufficient criterion for a 7-class field tower with exactly three stages)

An imaginary quadratic number field \(K = \mathbb{Q} (\sqrt{d})\) with fundamental discriminant \(d < 0\), elementary bicyclic 7-class group \(\text{Cl}_7(K) \simeq C_7 \times C_7\), and Artin pattern \(\text{AP}(K) = (\varkappa(K), \tau(K))\) with capitulation type \(\varkappa(K) \sim (1, 2, 2, 2, 2, 2, 2, 2, 2)\) in the eight unramified cyclic septic extensions \((E_i)_{1 \leq i \leq 8}\) of \(K\) and logarithmic abelian type invariants \(\tau(K) \sim (1^5, 1^3, 21, 21, 21, 21, 21, 21, 21)\) of the 7-class groups \((\text{Cl}_7(E_i))_{1 \leq i \leq 8}\) possesses a finite 7-class field tower \(K < F_1^2(K) < F_2^2(K) < F_3^2(K) = F_3^2(K)\) with precise length \(f_7(K) = 3\).

Now we supplement the purely arithmetical information of Theorem 5 with group theoretical data. We use the notation of the SmallGroups database [9] [10] and the ANUPQ package [17].

**Corollary 6.** Under the conditions of Theorem 5, there are only two possibilities for the metabelian second 7-class group \(M = \text{Gal}(F_3^2(K)/K)\) of \(K\), namely \(M \simeq \text{SmallGroup}(823543, m)\) with either \(m = 1990\) or \(m = 1991\). Both share the invariants \(l_0 = 7\), \(c_l = 5\), \(c_c = 2\), \(d_2 = 3\). See Figure 4.

There are 98 candidates for the non-metabelian Schur \(\sigma\)-Galois group \(S = \text{Gal}(F_3^2(K)/K)\) of the 7-class field tower of \(K\). In dependence on the metabelianization \(M \simeq S/S''\), they are given by...
\[ S \simeq \text{SmallGroup}(117\,649, 708) - \#2; \ell - \#1, k - \#2; j \text{ with } 1 \leq j \leq 7 \text{ and } \]

\[
\ell \in \{4 + 17v \mid 0 \leq v \leq 6\}, \quad k = \begin{cases} 
1 & \text{for } \ell = 4, \\
2 & \text{otherwise},
\end{cases} \quad \text{if } m = 1990,
\]

\[
\ell \in \{11 + 17v \mid 0 \leq v \leq 6\}, \quad k = \begin{cases} 
1 & \text{for } \ell = 11, \\
4 & \text{otherwise},
\end{cases} \quad \text{if } m = 1991.
\]

All of them share the invariants \( \ell_0 = 11, \text{cl} = 7, \text{cc} = 4, \) and \( d_2 = 2 = d_1 = \dim_{\overline{\tau}}(H^1(S, F_7)) = g_7. \)

**Proof.** The next section is devoted to the rigorous justification of Theorem 5 and Corollary 6. \( \square \)

**Example 7.** The three absolutely smallest discriminants of imaginary quadratic number fields \( K = \mathbb{Q}(\sqrt{d}) \) which satisfy the conditions in Theorem 5 are \( d = -159\,592, \) \( d = -611\,076, \) and \( d = -839\,147. \) The 7-class field tower over these three fields has exactly three stages.

4. **7-groups connected with 7-class towers**

We apply the strategy of pattern recognition via Artin transfers [43] to the situation described in Theorem 6. The number theoretic Artin pattern \( \text{AP}(K) = ((\ker(T_{K,E})_{1 \leq i \leq g}), (C_{\ell}(E_i))_{1 \leq i \leq g}) \) of the class extension homomorphisms \( T_{K,E_i} : C_{\ell}(K) \to C_{\ell}(E_i), \) \( aP_K \mapsto (aO_{E_i})P_{E_i}, \) from \( K \) to its eight unramified cyclic septic extension fields \( E_i \) interpreted as group theoretic Artin pattern \( \text{AP}(G) = ((\ker(T_{G,H_i}))_{1 \leq i \leq g}, (H_i/H'_i)_{1 \leq i \leq g}) \) of the Artin transfer homomorphisms \( T_{G,H_i} : G/G' \to H_i/H'_i \) from any of the \( n \)-th 7-class groups \( G = \text{Gal}(F^7_7(K)/K), \) of \( K, \) with \( n \geq 2, \) to its eight maximal subgroups \( H_i \) of index 7 [2, 3, 30, 35, 36]. The following proof is visualized by Figure 1.

We begin our search for the desired transfer kernel type (TKT) \( \simeq (1, 2, 2, 2, 2, 2, 2) \) at the abelian root SmallGroup(49, 2) \( \simeq C_7 \rtimes C_7 \) of the tree of all finite 7-groups \( G \) with abelianization \( G/G' \) of type \((7, 7). \) Groups \( G \) with coclass \( cc(G) = 1 \) in the coclass tree \( T^1 (49, 2) \) are discouraged, since they possess at least 7 total transfer kernels, that is, \( \simeq (*, 0, 0, 0, 0, 0, 0) \) with a place holder \(*\), but imaginary quadratic fields with \( \overline{\tau} = 2 \) cannot have total capitulation [23, 33, 34].

Thus we follow the edges with step size \( s = 2 \) at the metabelian bifurcation of SmallGroup(343, 3). Many of these immediate descendants have permutations as TKT, but the search can be restricted to the tree of SmallGroup(16807, 4) with TKT \( \simeq (0, 2, 2, 2, 2, 2, 2), \) since this type is able to shrink to \((1, 2, 2, 2, 2, 2, 2)\) with two fixed points. Note that the trees of SmallGroup(16807, i) with \( 5 \leq i \leq 6 \) are irrelevant, since the TKT \( \simeq (0, 1, 1, 1, 1, 1, 1, 1) \) can only shrink to the type \((1, 1, 1, 1, 1, 1, 1, 1)\) with a single fixed point [2].

The possible metabelian second 7-class groups \( \text{Gal}(F^7_7(K)/K) \) of \( K \) are easy to find as leaves of the second branch of the coclass tree \( T^2 (16807, 4). \) With respect to the branch root, they are given by the identifiers SmallGroup(823543, 1990) = SmallGroup(117649, 708) - #1; 11 and SmallGroup(823543, 1991) = SmallGroup(117649, 708) - #1; 18. They are not unique with TKT \((1, 2, 2, 2, 2, 2, 2), \) but they are unique in combination with the required transfer target type (TTT) \( \tau \approx (1^5, 1^3, 21, 21, 21, 21, 21, 21). \) Since they have relation rank \( d_2 = 3 > 2 = d_1, \) their presentation is not balanced, and the 7-class field tower must have length \( \ell_7(K) \geq 3 [49]. \)

In order to find the non-metabelian Schur \( \sigma \)-Galois group \( \text{Gal}(F^\infty_7(K)/K) \) of the maximal unramified pro-7 extension of \( K, \) we follow the paths with step size \( s = 2 \) at the non-metabelian bifurcation of SmallGroup(117649, 708) with nuclear rank \( \nu = 2, \) similarly as demonstrated for \( p = 3 \) in [10]. In contrast to \( p = 3, \) none of the immediate descendants of order \( 7^8 \) is a Schur \( \sigma\)-group. The siblings \( G = \text{SmallGroup}(117649, 708) - #2; \ell \text{ with } \)

- \( \ell \in \{1 + 17v \mid 0 \leq v \leq 6\} \) have type \((0, 2, 2, 2, 2, 2, 2, 2), \)
- \( \ell \in \{4 + 17v \mid 0 \leq v \leq 6\} \) have type \((1, 2, 2, 2, 2, 2, 2, 2)\) and \( G/G'' \simeq (823\,543, 1990), \)
- \( \ell \in \{11 + 17v \mid 0 \leq v \leq 6\} \) have type \((1, 2, 2, 2, 2, 2, 2, 2)\) and \( G/G'' \simeq (823\,543, 1991). \)

The danger that descendants of the groups with \( \ell \in \{1 + 17v \mid 0 \leq v \leq 6\} \) give rise to further relevant Schur \( \sigma\)-groups is eliminated by the monotony principle [32], since the polarization of the TTT \( \tau \) increases from \( \tau_1 = 1^5 \) with rank 5 to \( \tau_1 = 1^6 \) with rank 6.
After a second non-metabelian bifurcation at the unique \( \sigma \)- descendant of a group with \( \ell \in \{4 + 17v \mid 0 \leq v \leq 6\} \) or \( \ell \in \{11 + 17v \mid 0 \leq v \leq 6\} \), we arrive at the desired Schur \( \sigma \)-groups \( S \) with \( \# S = 7^{11} \) and \( \text{dl}(S) = 3 \). These are 49 candidates for each of the two metabelianizations. Their identifiers are given by SmallGroup(117 649, 708) – \#2; \#1; \ell – \#1, \kappa – \#2, \jmath \) as in Formula (1).

In Table 1, invariants for the pre-period and period of the tree with root \( \langle 7^5, 4 \rangle \) and its descendants of fixed coclass \( cc = 2 \) on the mainline or with type \( \times \) in Theorem [4] are given. The identifier \( \text{id} \) of each vertex is either the absolute number in the SmallGroups database \([9, 10]\) or the relative counter with respect to the parent, in the notation of the ANUPQ package \([17]\). The order \( \text{ord} \) and nilpotency class \( \text{cl} \) are arranged in ascending order. The Artin pattern \( \text{AP} = (\kappa, \tau) \) consists of the stabilization \([32]\).

\[ \times \sim (*, 2, 2, 2, 2, 2, 2), \quad \tau \sim (*, 1^3, 21, 21, 21, 21, 21) \]

with a placeholder * for the polarization \((\kappa_1, \tau_1)\). The stabilization remains constant along the entire coclass tree, whereas the polarization varies from branch to branch with strictly increasing \( \tau_1 \). Therefore, it suffices to list the polarization. The graph theoretic structure of the tree is determined by the nuclear rank \( \nu \) and the descendant numbers \( N_1/C_1 \) with step size \( s = 1 \), respectively \( N_2/C_2 \) with step size \( s = 2 \), when \( \nu = 2 \). The latter phenomenon, which is called bifurcation to bigger coclass, occurs three times in the irregular pre-period of length 6. Starting with the first primitive period, also of length 6, the value of the nuclear rank \( \nu = 1 \) remains settled, and no further information is required in order to describe the entire infinite tree. The \( p \)-multiplicator rank \( \mu \) coincides with the relation rank \( d_2 \) in the Shafarevich theorem \([49]\).

Table 1. Structure and periodicity of the coclass tree \( \mathcal{T}^2(16 807, 4) \)

| id  | \text{ord} | \text{cl} \times | \nu \kappa | \tau \tau_1 | \text{N}_1/C_1 | \text{N}_2/C_2 |
|-----|------------|----------------|------------|------------|----------------|----------------|
| 4   | 16 807     | \#4           | 3          | 0          | 1^3            | 1 3            | 11/11          |
| 708 | 117 649    | \#6           | 4          | 0          | 1^4            | 2 4            | 38/3 119/119  |
| 2004| 823 543    | \#7           | 5          | 0          | 1^5            | 1 4            | 32/14          |
| 1990| 823 543    | \#7           | 5          | 1          | 1^5            | 0 3            |               |
| 1991| 823 543    | \#7           | 5          | 1          | 1^5            | 0 3            |               |
|     | 57 640 801 | \#8           | 6          | 0          | 1^6            | 2 5            | 52/9 805/217  |
|     | 40 353 607 | \#9           | 7          | 0          | 2^1            | 1 5            | 25/13          |
|     | 282 475 249| \#10          | 8          | 0          | 2^1^4          | 2 6            | 80/11 833/119 |
|     | 1977 326 743| \#11          | 9          | 0          | 2^1^4          | 1 6            | 28/6           |
|     | 13 841 287 201| \#12          | 10         | 0          | 2^1^2          | 1 6          | 51/11          |
|     | 96 889 010 407| \#13          | 11         | 0          | 2^1^2          | 1 6          | 40/6           |
|     | 678 223 072 849| \#14          | 12         | 0          | 2^6            | 1 6          | 55/11          |
|     | 47 475 547 493| \#15          | 13         | 0          | 3^2            | 1 6          | 26/6           |
|     | 33 232 930 569 601| \#16         | 14         | 0          | 3^2^4          | 1 6          | 79/11          |

In view of Table 1, it is obvious that SmallGroup(823 543, 1990) = SmallGroup(117 649, 708) – \#1; 11 and SmallGroup(823 543, 1991) = SmallGroup(117 649, 708) – \#1; 18 are the only two possibilities for the metabelianization \( M \simeq S/S'' \) of the Galois group \( S \) of the 7-class field tower of any algebraic number field \( K \) satisfying the conditions in Theorem [5] because firstly the coclass tree \( \mathcal{T}^2(7^5, 4) \) with metabelian main line is unique with stabilization in Formula (2), and secondly, although there occur vertices with \( \times \sim (1, 2, 2, 2, 2, 2, 2) \) on every branch of the tree, the mentioned groups on branch \( B(2) \) with root SmallGroup(117 649, 708) are the only two which possess the required polarization \( \tau_1 = 1^5 \). Since their relation rank is given by \( d_2 = \mu = 3 \), bigger than the required \( d_2 = 2 \), a first consequence is the lower bound \( \ell_7(K) \geq 3 \) for the length of the 7-class tower of \( K \).

In Table 2, we give an exemplary root path to one of the non-metabelian Schur \( \sigma \)-groups \( S \simeq \text{Gal}(\mathbb{F}_{7^2}(K)/K) \) and corresponding invariants. Figure 4 illustrates Table 1 and Table 2.
Table 2. Exemplary root path to $S \simeq \text{Gal}(F_{7}^\infty(K)/K)$ with lo = 11

| id       | ord      | cl | dl | κ | ν | µ | $N_1/C_1$ | $N_2/C_2$ |
|----------|----------|----|----|---|---|---|------------|------------|
| 708      | 117649 = $7^6$ | 4  | 2  | 0 | 4 | 14 | 38/3      | 119/119    |
| -#2; 62  | 5764801 = $7^8$ | 5  | 3  | 1 | 5 | 2  | 4/4       |            |
| -#1; 4   | 40353607 = $7^9$ | 6  | 3  | 1 | 5 | 2  | 14/0      | 7/0        |
| -#2; 1   | 1977326743 = $7^{11}$ | 7  | 3  | 1 | 5 | 0  | 2         |            |

Figure 1. 7-class field tower groups of order $7^{11}$ on the tree $T(16807, 4)$

5. 5-CLASS FIELD TOWERS WITH LENGTH AT LEAST THREE

The analogue of the results in § 3 for $p = 5$ is astonishing and disappointing for two reasons:

- Firstly, the smallest Schur $\sigma$-groups with desired Artin pattern set in at the annoying logarithmic order 14 instead of 11,
and secondly, even more annoying, there is not a finite number of such Schur $\sigma$-groups but rather an infinity of them with unbounded derived length, similarly as known from the situations with sporadic metabelianizations \[8\].

**Theorem 8. (Sufficient criterion for a 5-class field tower with at least three stages)**

An imaginary quadratic number field $K = \mathbb{Q}(\sqrt{d})$ with fundamental discriminant $d < 0$, elementary bicyclic 5-class group $\text{Cl}_5(K) \simeq C_5 \times C_5$, and Artin pattern $\text{AP}(K) = (\varphi(K), \tau(K))$ with capitulation type $\varphi(K) \sim (1, 2, 2, 2, 2)$ in the six unramified cyclic septic extensions $(E_i)_{1 \leq i \leq 6}$ of $K$ and logarithmic abelian type invariants $\tau(K) \sim (21^3, 1^3, 21, 21, 21, 21)$ of the 5-class groups $(\text{Cl}_5(E_i))_{1 \leq i \leq 6}$ possesses a finite 5-class field tower $K < F^2_5(K) < F^3_5(K) < F^4_5(K)$ with length $\ell_5(K) \geq 3$.

Now we supplement the purely arithmetical information of Theorem 8 with group theoretical data. We use the notation of the SmallGroups database \[9, 10\] and the ANUPQ package \[17\].

**Corollary 9.** Under the conditions of Theorem 8 there are only two possibilities for the metabelian second 5-class group $M = \text{Gal}(F^2_5(K)/K)$ of $K$, namely $M \simeq \text{SmallGroup}(78, 125)$ with either $m = 889$ or $m = 890$. Both share the invariants $l_0 = 7$, $c_1 = 5$, $c_2 = 2$, $d_2 = 3$. See Figure 3.

There are infinitely many candidates for the non-metabelian Schur $\sigma$-Galois group $S = \text{Gal}(F^2_5(K)/K)$ of the 5-class field tower of $K$. In dependence on the metabelianization $M \simeq S/S''$, the 200 candidates with smallest possible order $5^{21}$ and derived length 3 are given by

$$S \simeq \text{SmallGroup}(15625, 564) - \#2; \ell - \#1; k - \#2; j - \#1; i - \#2; h$$

$$\text{with } 1 \leq h \leq 5 \text{ and}$$

$$\ell(k) = (21, 1), (j, i) \in \{(1, 11), (2, 3), (3, 12), (4, 13), (5, 8)\} \text{ or}$$

$$\ell(k) = (22, 3), (j, i) \in \{(1, 6), (2, 13), (3, 2), (4, 13), (5, 13)\} \text{ or}$$

$$\ell(k) = (23, 3), (j, i) \in \{(1, 13), (2, 6), (3, 13), (4, 2), (5, 13)\} \text{ or}$$

$$\ell(k) = (25, 2), (j, i) \in \{(1, 3), (2, 11), (3, 13), (4, 7), (5, 13)\}, \text{ if } m = 889,$$

$$\ell(k) = (26, 1), (j, i) \in \{(1, 13), (2, 12), (3, 13), (4, 3), (5, 6)\} \text{ or}$$

$$\ell(k) = (27, 3), (j, i) \in \{(1, 7), (2, 13), (3, 13), (4, 11), (5, 3)\} \text{ or}$$

$$\ell(k) = (29, 3), (j, i) \in \{(1, 11), (2, 3), (3, 7), (4, 13), (5, 13)\} \text{ or}$$

$$\ell(k) = (30, 2), (j, i) \in \{(1, 12), (2, 13), (3, 3), (4, 6), (5, 13)\}, \text{ if } m = 890.$$

The latter share the invariants $l_0 = 14$, $c_1 = 9$, $c_2 = 5$, and $d_2 = 2 = d_1 = \text{dim}_{\mathbb{F}_5}(H^1(S, F_5)) = 85$.

However, there also exist 25000 candidates with order $5^{21}$ and derived length 4 among the descendants of the two groups $\text{SmallGroup}(15625, 564) - \#2; \ell$ with $\ell \in \{24, 28\}$. See Figure 3.

**Proof.** The next section is devoted to the rigorous justification of Theorem 8 and Corollary 9. \(\square\)

**Example 10.** The absolutely smallest discriminant of an imaginary quadratic number field $K = \mathbb{Q}(\sqrt{d})$ which satisfies the conditions in Theorem 8 is $d = -90868$. The 5-class field tower over this field has at least three stages, but the precise length $\ell_5(K) \geq 3$ remains unknown.

6. **5-groups connected with 5-class towers**

We employ the strategy of pattern recognition via Artin transfers \[43\] to the situation described in Theorem 8. The number theoretic Artin pattern $\text{AP}(K) = (\ker(T_{K,E}), (\text{Cl}_5(E_i))_{1 \leq i \leq 6})$ of the class extension homomorphisms $T_{K,E} : \text{Cl}_5(K) \to \text{Cl}_5(E_i)$, $\mathfrak{ap}_K \leftrightarrow (\mathfrak{ap}_{E_i})_{E_i}$, from $K$ to its six unramified cyclic quintic extension fields $E_i$ is interpreted as group theoretic Artin pattern $\text{AP}(G) = (\ker(T_{G,H_i}), (H_i/H'_i)_{1 \leq i \leq 6})$ of the Artin transfer homomorphisms $T_{G,H_i} : G/G' \to H_i/H'_i$ from any of the $n$-th 5-class groups $G = \text{Gal}(F^5_5(K)/K)$ of $K$, with $n \geq 2$, to its six maximal subgroups $H_i$ of index 5 \[2, 3, 30, 35, 36\]. The following proof is visualized by Fig. 2.

We begin our search for the desired transfer kernel type (TKT) $\simeq (1, 2, 2, 2, 2, 2)$ at the abelian root $\text{SmallGroup}(25, 2) \simeq C_5 \times C_5$ of the tree of all finite 5-groups $G$ with abelianization $G/G'$ of type $(5, 5)$. Groups $G$ with coclass $\text{cc}(G) = 1$ in the coclass tree $T^1(25, 2)$ are discouraged,
since they possess at least 5 total transfer kernels, that is, $\kappa \sim (\ast, 0, 0, 0, 0, 0)$ with a placeholder $\ast$, but imaginary quadratic fields with $\nu_5 = 2$ cannot have total capitulation $23$. Thus we follow the edges with step size $s = 2$ at the metabelian bifurcation of SmallGroup(125, 3).

Many of these immediate descendants have permutations as TKT, but the search can be restricted to the tree of SmallGroup(3125, 4) with TKT $\kappa \sim (0, 2, 2, 2, 2, 2)$, since this type is able to shrink to $(1, 2, 2, 2, 2, 2)$ with two fixed points. Note that the trees of SmallGroup(3125, i) with $5 \leq i \leq 6$ are irrelevant, since the TKT $\kappa \sim (0, 1, 1, 1, 1, 1)$ can only shrink to the type $(1, 1, 1, 1, 1, 1)$ with a single fixed point $41$.  The possible metabelian second 5-class groups $\text{Gal}(\mathbb{F}_5^\infty(K)/K)$ of $K$ are easy to find as leaves of the second branch of the coclass tree $T^2(3125, 4)$. With respect to the branch root, they are given by the identifiers $\langle 78, 125, 889 \rangle = \text{SmallGroup}(15, 625, 564) - \#1; 5$ and $\langle 78, 125, 890 \rangle = \text{SmallGroup}(15, 625, 564) - \#1; 6$. They are not unique with TKT $(1, 2, 2, 2, 2, 2)$, but they are unique in combination with the required transfer target type (TTT) $\tau \sim (21^3, 13, 21, 21, 21, 21)$. Since they have relation rank $d_2 = 3 > 2 = d_1$, their presentation is not balanced, and the 5-class field tower must have length $\ell_5(K) \geq 3$ $44$.

In order to find the non-metabelian Schur $\sigma$-Galois group $\text{Gal}(\mathbb{F}_5^\infty(K)/K)$ of the maximal unramified pro-5 extension of $K$, we follow the paths with step size $s = 2$ at the non-metabelian bifurcation of SmallGroup(15, 625, 564) with nuclear rank $\nu = 2$, similarly as demonstrated for $p = 3$ in $10$. In contrast to $p = 3$, none of the immediate descendants of order $5^8$ is a Schur $\sigma$-group. The siblings $G = \text{SmallGroup}(15, 625, 564) - \#2; \ell$ with

- $16 \leq \ell \leq 20$ have type $(0, 2, 2, 2, 2, 2)$,
- $21 \leq \ell \leq 25$ have type $(1, 2, 2, 2, 2, 2)$ and $G/G'' \simeq \langle 78, 125, 889 \rangle$,
- $26 \leq \ell \leq 30$ have type $(1, 2, 2, 2, 2, 2)$ and $G/G'' \simeq \langle 78, 125, 890 \rangle$.

The danger that descendants of the groups with $16 \leq \ell \leq 20$ give rise to further relevant Schur $\sigma$-groups is eliminated by the monotony principle $32$, since the polarization of the TTT $\tau$ increases from $\tau_1 = 21^3$ to $\tau_1 = 2^{2\cdot}1^2$, both with rank 4.

After a second non-metabelian bifurcation at the unique $\sigma$-descendant of a group with $\ell \in \{21, 22, 23, 25\}$, resp. $\ell \in \{26, 27, 29, 30\}$, and a third non-metabelian bifurcation at the unique $\sigma$-descendant of SmallGroup(15, 625, 564) − $\#2; \ell - \#1; k - \#2; j$ with $1 \leq j \leq 5$ we arrive at the desired Schur $\sigma$-groups $S$ with $\#S = 5^{14}$ and $\text{d}(S) = 3$. These are 100 candidates for each of the two metabelianizations. Their identifiers are given by SmallGroup(15, 625, 564) − $\#2; \ell - \#1; k - \#2; j - \#1; i - \#2; h$ as in Formula $49$.

Unfortunately, the two groups with $\ell \in \{24, 28\}$ are exceptional. They destroy the warranty for a 5-class field tower with precise length $\ell_5(K) = 3$. See Table $5$ and Figure $5$.

In Table $3$ invariants for the pre-period and period of the tree with root $(5^3, 4)$ and its descendants of fixed coclass $cc = 2$ on the mainline or with type $\kappa$ in Theorem $8$ are given. The identifier id of each vertex is either the absolute number in the SmallGroups database $9, 10$ or the relative counter with respect to the parent, in the notation of the ANUPQ package $17$. The order ord and nilpotency class cl are arranged in ascending order. The Artin pattern $\text{AP} = (\kappa, \tau)$ consists of the stabilization $32$.

$$\kappa \sim (\ast, 2, 2, 2, 2, 2), \quad \tau \sim (\ast, 1^3, 21, 21, 21, 21)$$

with a placeholder $\ast$ for the polarization $(\kappa_1, \tau_1)$. The stabilization remains constant along the entire coclass tree, whereas the polarization varies from branch to branch with strictly increasing $\tau_1$. Therefore, it suffices to list the polarization. The graph theoretic structure of the tree is determined by the nuclear rank $\nu$ and the descendant numbers $N_1/C_1$ with step size $s = 1$, respectively $N_2/C_2$ with step size $s = 2$, when $\nu = 2$. The latter phenomenon, which is called bifurcation to bigger coclass, occurs twice in the irregular pre-period of length 4. Starting with the first primitive period, also of length 4, the value of the nuclear rank $\nu = 1$ remains settled, and no further information is required in order to describe the entire infinite tree. The $p$-multiplicator rank $\mu$ coincides with the relation rank $d_2$ in the Shafarevich theorem $49$. 
In Table 4 resp. 5, we give an exemplary, resp. exceptional, root path to one of the non-metabelian Schur $\sigma$-groups $S \simeq \text{Gal}(F_5^\infty(K)/K)$ of order $5^{14}$, derived length 3, resp. order $5^{21}$, derived length 4, and corresponding invariants. In Table 5 there occurs a trifurcation at the vertex with order $5^{13}$, nuclear rank $\nu = 3$, and descendant numbers $N_3/C_3$.

**Table 3. Structure and periodicity of the coclass tree $T^2(3\,125, 4)$**

| id   | ord     | cl | $\mathfrak{z}_1$ | $\tau_1$ | $\nu$ | $\mu$ | $N_1/C_1$ | $N_2/C_2$ |
|------|---------|----|------------------|----------|-------|-------|-----------|-----------|
| 4    | 3\,125  = 5$^4$ | 3   | 0 | 1$^3$ | 1 | 3    | 7/7       |           |
| 564  | 15\,625  = 5$^6$ | 4   | 0 | 1$^4$ | 2 | 4    | 30/5 | 75/75     |
| 888  | 78\,125  = 5$^7$ | 5   | 0 | 21$^3$ | 1 | 4    | 12/8  |           |
| 889  | 78\,125  = 5$^7$ | 5   | 1 | 21$^3$ | 0 | 3    |       |           |
| 890  | 78\,125  = 5$^7$ | 5   | 1 | 21$^3$ | 0 | 3    |       |           |
| -#1:6 | 390\,625  = 5$^8$ | 6   | 0 | 2$^2$$^1$ | 2 | 5    | 38/7 | 75/15     |
| -#1:1 | 1\,953\,125  = 5$^9$ | 7   | 0 | 2$^1$$^1$ | 1 | 5    | 13/3  |           |
| -#1:6 | 9\,765\,625  = 5$^{10}$ | 8   | 0 | 2$^4$ | 1 | 5    | 29/7  |           |
| -#1:1 | 48\,828\,125  = 5$^{11}$ | 9   | 0 | 32$^3$ | 1 | 5    | 13/3  |           |
| -#1:6 | 244\,140\,625  = 5$^{12}$ | 10  | 0 | 3$^2$$^2$ | 1 | 5    | 37/7  |           |

**Table 4. Exemplary root path to $S \simeq \text{Gal}(F_5^\infty(K)/K)$ with lo = 14**

| id   | ord     | cl | dl | $\mathfrak{z}_1$ | $\tau_1$ | $\nu$ | $\mu$ | $N_1/C_1$ | $N_2/C_2$ |
|------|---------|----|----|------------------|----------|-------|-------|-----------|-----------|
| 564  | 15\,625  = 5$^6$ | 4   | 2  | 0 | 1$^4$ | 2 | 4    | 30/5 | 75/75     |
| -#2:27 | 390\,625  = 5$^8$ | 5   | 3  | 1 | 1$^5$ | 1 | 3    | 3/3 |           |
| -#1:3 | 1\,953\,125  = 5$^9$ | 6   | 3  | 1 | 1$^5$ | 2 | 4    | 10/5 | 5/5       |
| -#2:1 | 48\,828\,125  = 5$^{11}$ | 7   | 3  | 1 | 1$^5$ | 1 | 3    | 13/13 |           |
| -#1:6 | 244\,140\,625  = 5$^{12}$ | 8   | 3  | 1 | 1$^5$ | 2 | 4    | 6/0 | 5/0       |
| -#2:1 | 6\,103\,515\,625  = 5$^{14}$ | 9   | 3  | 1 | 1$^5$ | 0 | 2    |       |           |

**Table 5. Exceptional root path to $S \simeq \text{Gal}(F_5^\infty(K)/K)$ with lo = 21**

| id   | ord     | cl | dl | $\mathfrak{z}_1$ | $\tau_1$ | $\nu$ | $\mu$ | $N_1/C_1$ | $N_2/C_2$ | $N_3/C_3$ |
|------|---------|----|----|------------------|----------|-------|-------|-----------|-----------|-----------|
| 564  | 15\,625  = 5$^6$ | 4   | 2  | 0 | 1$^4$ | 2 | 4    | 30/5 | 75/75     |           |
| -#2:24 | 390\,625  = 5$^8$ | 5   | 3  | 1 | 1$^5$ | 1 | 3    | 3/3 |           |           |
| -#1:3 | 1\,953\,125  = 5$^9$ | 6   | 3  | 1 | 1$^5$ | 2 | 4    | 30/5 | 25/25     |           |
| -#2:1 | 48\,828\,125  = 5$^{11}$ | 7   | 3  | 1 | 1$^5$ | 2 | 4    | 78/78 | 313/313   |           |
| -#2:306 | 1\,220\,703\,125  = 5$^{13}$ | 8   | 3  | 1 | 1$^5$ | 3 | 5    | 39/5 | 95/50 | 25/25 |
| -#3:1 | 1\,525\,878\,906\,625  = 5$^{16}$ | 9   | 3  | 1 | 1$^5$ | 1 | 3    | 13/13 |           |           |
| -#1:11 | 762\,939\,453\,125  = 5$^{17}$ | 10  | 3  | 1 | 1$^5$ | 2 | 4    | 10/0 | 5/5       |           |
| -#2:1 | 1\,907\,348\,328\,125  = 5$^{19}$ | 11  | 3  | 1 | 1$^5$ | 1 | 3    | 3/3 |           |           |
| -#1:3 | 95\,367\,431\,640\,625  = 5$^{20}$ | 12  | 3  | 1 | 1$^5$ | 1 | 3    | 5/0 |           |           |
| -#1:1 | 476\,837\,158\,203\,125  = 5$^{21}$ | 13  | 4  | 1 | 1$^5$ | 0 | 2    |       |           |           |
Figure 2. 5-class field tower groups of order $5^{14}$ on the tree $T(3125, 4)$
FIGURE 3. Exceptional 5-class field tower groups of order $5^{21}$ on the tree $\mathcal{T}(3125,4)$

Order
390 625
1 953 125
9 765 625
48 828 125
244 140 625
1 220 703 125
6 103 515 625
30 517 578 125
152 587 890 625
762 939 453 125
3 814 697 265 625
1 907 486 328 125
95 367 431 640 625
476 837 158 203 125

Legend:
- non-metabelian with TKT L
- exceptional non-metabelian with TKT L
- non-metabelian Schur $\sigma$-group with TKT L
- non-metabelian with TKT M

$S = \text{Gal}(F_{5}^{\infty}(K)/K)$

5 Schur $\sigma$-groups with $dl = 4$. 

7. Conclusion

For the prime numbers \( p \in \{5, 7\} \), we have demonstrated impressively that descendant trees \( T \) of finite \( p \)-groups with edges of fixed step size \( s = 1 \) and vertices of fixed coclass \( cc = 2 \) and \( p \)-power order reveal some general structural principles of so-called coclass trees:

- The branches \( B(1), \ldots, B(p-1) \) form an irregular pre-period with \( \frac{p-1}{2} \) bifurcations to the next coclass \( cc = 3 \) at the roots of the branches \( B(2), B(4), \ldots, B(p-1) \) with even counter.

- With respect to their isomorphism as subgraphs of the tree, the branches show a periodicity \( \left( \binom{2}{p} \right) \) with length \( \ell = p - 1 \), that is,

\[
(\forall i \geq 0) \quad B(i + \ell) \simeq B(i).
\]

- In view of the realization of the vertices \( V \in T \) as second \( p \)-class groups \( V \simeq \text{Gal}(\mathbb{F}_p^2(K)/K) \) of imaginary quadratic number fields \( K = \mathbb{Q}(\sqrt{\Delta}) \) with \( d < 0 \), the terminal leaves of the branches \( B(2), B(4), \ldots \) with even counters are admissible \([23, 25, 26, 27]\).

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