The quaternionic second weighted zeta function of a graph and the Study determinant

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Abstract
We establish a generalization of the second weighted zeta function of a graph to the case of quaternions. For an arc-weighted graph whose weights are quaternions, we define the second weighted zeta function by using the Study determinant that is a quaternionic determinant for quaternionic matrices defined by Study. This definition is regarded as a quaternionic analogue of the determinant expression of Hashimoto type for the Ihara zeta function of a graph. We derive the Study determinant expression of Bass type and the Euler product for the quaternionic second weighted zeta function.

Keywords: Quaternionic matrix; Study determinant; Ihara zeta function

1 Introduction
The Ihara zeta function of a graph has achieved success in spectral theory of graphs. Zeta functions of graphs started from Ihara zeta functions of regular graphs by Ihara. Ihara defined the $p$-adic Selberg zeta function $Z_p(u)$ of a torsion-free discrete cocompact subgroup $\Gamma$ of $PGL_2$ over a locally compact field under a discrete valuation, and showed that its reciprocal...
is a explicit polynomial. Ihara’s motivation to define $Z_T(u)$ was to count the number of primitive conjugacy classes of torsion-free discrete cocompact subgroups. Regarding Ihara’s work, Serre [18] pointed out that the Ihara zeta function is the zeta function of the quotient $\Gamma \backslash T$ (a finite regular graph) of the one-dimensional Bruhat-Tits building $T$ (an infinite regular tree) associated with $\text{SL}_2(\mathbb{Q}_p)$. This observation led to rapid developments of zeta functions of graphs. Sunada [22], [23] developed zeta functions of regular graphs equipped with unitary representations of fundamental groups of the graphs. Hashimoto [5] explored multivariable zeta functions of bipartite graphs and also gave a determinant expression for the Ihara zeta function of a general graph by using its edge matrix. Bass [2] generalized Ihara’s result on zeta functions of regular graphs to irregular graphs, and showed that their reciprocals are again polynomials. Subsequently, various proofs of Bass’ Theorem were given by Stark and Terras [19], Foata and Zeilberger [4], Kotani and Sunada [12]. Zeta functions of edge-weighted graphs were proposed by Hashimoto [6] and those of arc-weighted graphs by Stark and Terras [19] which are called edge zeta functions. Stark and Terras [19] gave their determinant expressions by using their edge matrices. Mizuno and Sato [14] focused on a special version of edge zeta functions, and defined the weighted zeta function by incorporating a variable $t$ that measures the length of cycles into the edge zeta function. Subsequently, Sato [17] also defined a new class of zeta functions of graphs by modifying the determinant expression of the weighted zeta function defined in [14]. This new zeta function, which was named the second weighted zeta function by Sato, played essential roles in the concise proof of the spectral mapping theorem for the Grover walk on a graph in [11] and of the Smilansky’s formula [21] for the characteristic polynomial of the bond scattering matrix of a graph in [15]. Thereby we expect that the second weighted zeta function brings about rich outcomes in quantum dynamics on graphs such as quantum walks on graphs or quantum graphs. In this paper, we aim to establish a quaternionic analogue of the second weighted zeta function of a graph and to derive its basic properties. Our results will play crucial roles in our future work on quaternionic quantum walks which was established by Konno [9] recently.

The quaternion was discovered by Hamilton in 1843. It can be considered as an extension of the complex number. However, quaternions do not commute mutually in general and the definition of determinant is invalid for quaternionic matrices. For many years, a number of researchers, for example Cayley, Study, Moore, Dieudonné, Dyson, Mehta, Xie, Chen, have given different definitions of determinants of quaternionic matrices. Detailed accounts on the determinants of quaternionic matrices can be found in, for example, [1, 24]. In this paper, we extend the second weighted zeta function of a graph to the case of quaternions by using the quaternionic determinant defined by Study [20]. An advantage of this approach is that one can reduce a calculation of the Study determinant to that of the ordinary determinant, so is easier to handle and apply than other general and abstract approaches. The Study determinant enables us to derive the explicit determinant expression and the Euler product for the quaternionic second weighted zeta function analogous to the ordinary determinant.

The rest of the paper is organized as follows. In Section 2, we provide a summary of the Ihara zeta function and its variants. We give various zeta functions of graphs, including the second weighted zeta function, and present their determinant expressions. In the end of this section, we explain briefly that the second weighted zeta function can be viewed as a natural generalization of the Ihara zeta function of a tree lattice and is related to quantum systems on graphs. In Section 3, we explain the Study determinant of a quaternionic matrix
and give some properties of it which are needed in later sections. In Section 4, we define
the quaternionic second weighted zeta function of a graph by using the Study determinant
which is considered as a quaternionic analogue of the determinant expression of Hashimoto
type, and determine its determinant expression of Bass type (Theorem 4.1). In Section 5,
we derive the Euler product for the quaternionic second weighted zeta function of a graph.

2 The Ihara zeta function of a graph

In this section, we provide a summary of the Ihara zeta function of a graph and its develop-
ment which led the Ihara zeta function to the second weighted zeta function. Let $G = (V(G),
E(G))$ be a finite connected graph with the set $V(G)$ of vertices and the set $E(G)$ of undi-
rected edges $uv$ joining two vertices $u$ and $v$. We assume that $G$ has neither loops nor
multiple edges throughout. For $uv \in E(G)$, an arc $(u, v)$ is the directed edge from $u$ to $v$.
Let $D(G) = \{ (u, v), (v, u) \mid uv \in E(G) \}$ and $|V(G)| = n$, $|E(G)| = m$, $|D(G)| = 2m$. For
$e = (u, v) \in D(G)$, $o(e) = u$ denotes the origin and $t(e) = v$ the terminal of $e$ respectively.
Furthermore, let $e^{-1} = (v, u)$ be the inverse of $e = (u, v)$. The degree $\deg v = \deg_G v$ of
a vertex $v$ of $G$ is the number of edges incident to $v$. We denote by $D_G$ the symmetric
digraph whose vertex set is $V(G)$ and directed edge set is $D(G)$. A path $P$ of length $\ell$ in
$G$ is a sequence $P = (e_1, \cdots , e_\ell)$ of $\ell$ arcs such that $e_r \in D(G)$ and $t(e_r) = o(e_{r+1})$
for $r \in \{ 1, \cdots , \ell - 1 \}$. We set $o(P) = o(e_1)$ and $t(P) = t(e_\ell)$. $|P|$ denotes the length of $P$.
We say that a path $P = (e_1, \cdots , e_\ell)$ has a backtracking if $e_{r+1} = e_r^{-1}$ for some $r$
($1 \leq r \leq \ell - 1$), and that $P = (e_1, \cdots , e_\ell)$ has a tail if $e_\ell = e_1^{-1}$. A path $P$ is said to be a cycle if $t(P) = o(P)$.
The inverse of a path $P = (e_1, \cdots , e_\ell)$ is the path $(e_\ell^{-1}, \cdots , e_1^{-1})$ and is denoted by $P^{-1}$.

Two cycles $C_1 = (e_1, \cdots , e_s)$ and $C_2 = (f_1, \cdots , f_\ell)$ are said to be equivalent if there exists
$s$ such that $f_r = e_{r+s}$ for all $r$ where indices are treated modulo $\ell$. Let $[C]$ be the equivalence
class which contains the cycle $C$. Let $B'$ be the cycle obtained by going $r$ times around a
cycle $B$. Such a cycle is called a power of $B$. A cycle $C$ is said to be reduced if both $C$ and
$C^2$ have no backtracking. Furthermore, a cycle $C$ is said to be prime if it is not a power of a
strictly smaller cycle.

The Ihara zeta function of a graph $G$ is a function of $t \in \mathbb{C}$ with $|t|$ sufficiently small,
defined by

$$Z(G, t) = Z_G(t) = \prod_{|C|} (1 - t^{|C|})^{-1},$$

where $[C]$ runs over all equivalence classes of prime, reduced cycles of $G$. Ihara’s original
definition was group theoretic and defined for a torsion-free discrete cocompact subgroup $\Gamma$
of $PGL_2$ over a locally compact field under a discrete valuation. In the case of regular graphs,
equivalence classes of prime, reduced cycles of $G$ correspond to primitive conjugacy classes of
$\Gamma$ and $|C|$ the degree of the corresponding primitive conjugacy class. For the details, see
[8]. Determinant expressions of Ihara zeta functions for finite graphs are obtained in the
following way.

Let $B = (B_{ef})_{e,f \in D(G)}$ and $J_0 = (J_{ef})_{e,f \in D(G)}$ be $2m \times 2m$ matrices defined as follows:

$$B_{ef} = \begin{cases} 1 & \text{if } t(e) = o(f), \\ 0 & \text{otherwise}, \end{cases}$$

$$J_{ef} = \begin{cases} 1 & \text{if } f = e^{-1}, \\ 0 & \text{otherwise}. \end{cases}$$
Then the matrix $B - J_0$ is called the edge matrix of $G$.

**Theorem 2.1** (Hashimoto [5]; Bass [2]). Let $G$ be a connected graph. Then the reciprocal of the Ihara zeta function of $G$ is given by

$$ Z(G, t)^{-1} = \det(I_{2m} - t(B - J_0)) = (1 - t^2)^{r-1} \det(I_n - tA + t^2(D - I_n)), \quad (2.1) $$

where $r$ and $A$ are the Betti number and the adjacency matrix of $G$ respectively, and $D = (D_{uv})_{u,v \in V(G)}$ is the diagonal matrix with $D_{uu} = \deg u$ for all $u \in V(G)$.

We call the middle formula the determinant expression of Hashimoto type and the right hand side the determinant expression of Bass type in (2.1).

Now we shall give the definition of the second weighted zeta function. Consider an $n \times n$ complex matrix $W = (W_{uv})_{u,v \in V(G)}$ with $(u, v)$-entry equals 0 if $(u, v) \notin D(G)$. We call $W$ a weighted matrix of $G$. Furthermore, let $w(u, v) = W_{uv}$ for $u, v \in V(G)$ and $w(e) = w(u, v)$ if $e = (u, v) \in D(G)$. For a path $P = (e_1, \cdots, e_\ell)$ of $G$, the norm $w(P)$ of $P$ is defined by $w(P) = w(e_1)w(e_2) \cdots w(e_\ell)$. For a weighted matrix $W$ of $G$, let $B_w = (B_{ef}^{(w)})_{e,f \in D(G)}$ be the $2m \times 2m$ complex matrix as follows:

$$ B_{ef}^{(w)} = \begin{cases} w(f) & \text{if } t(e) = o(f), \\ 0 & \text{otherwise.} \end{cases} $$

Then the second weighted zeta function of $G$ is defined by

$$ Z_1(G, w, t) = \det(I_{2m} - t(B_w - J_0))^{-1}. $$

We call $B_w - J_0$ the $B$-weighted edge matrix of $G$. If $w(e) = 1$ for any $e \in D(G)$, then the second weighted zeta function of $G$ coincides with the Ihara zeta function of $G$.

**Theorem 2.2** (Sato [17]). Let $G$ be a connected graph, and let $W$ be a weighted matrix of $G$. Then the reciprocal of the second weighted zeta function of $G$ is given by

$$ Z_1(G, w, t)^{-1} = (1 - t^2)^{m-n} \det(I_n - tW + t^2(D_w - I_n)), $$

where $n = |V(G)|$, $m = |E(G)|$ and $D_w = (D_{uv}^{(w)})_{u,v \in V(G)}$ is the diagonal matrix with $D_{uu}^{(w)} = \sum_{e,o(e) = u} w(e)$ for all $u \in V(G)$.

In [15], Mizuno and Sato obtained the Euler product for $Z_1(G, w, t)$. Let $\tilde{w}(e, f)$ be the $(e, f)$-entry of the matrix $B_w - J_0$. $\tilde{w}(e, f)$ is given by the following formula:

$$ \tilde{w}(e, f) = \begin{cases} w(f) & \text{if } t(e) = o(f) \text{ and } f \neq e^{-1}, \\ w(f) - 1 & \text{if } f = e^{-1}, \\ 0 & \text{otherwise.} \end{cases} \quad (2.2) $$

We set $\hat{w}(P) = \tilde{w}(e_1, e_2)\tilde{w}(e_2, e_3) \cdots \tilde{w}(e_{\ell-1}, e_\ell)$ for a path $P = (e_1, \cdots, e_\ell)$. 

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Theorem 2.3 (Mizuno and Sato [15]). Let \( G \) be a connected graph. Then
\[
Z_1(G, w, t) = \prod_{[C]} (1 - \tilde{w}(C)t^{|C|})^{-1},
\]
where \([C]\) runs over all equivalence classes of prime cycles of \( G \).

We shall give some notable comments on the second weighted zeta function of a graph.

Let \( T \) be a locally finite (possibly infinite) tree and \( \Gamma \) a group with an action on the symmetric digraph \( X = D_T = (V(X), \overset{\rightarrow}{E}(X)) \) where \( V(X) = V(T) \) and \( \overset{\rightarrow}{E}(X) = D(T) \). According to [2], we assume that \( \Gamma \) satisfies the following conditions:

(I) \( \Gamma \) acts without inversions, that is, \( \Gamma e \neq \Gamma e^{-1} \) for all \( e \in \overset{\rightarrow}{E}(X) \).

(D) \( \Gamma \) is discrete, that is, the stabilizer \( \Gamma_u \) of \( u \) is finite for every \( u \in V(X) \).

(F) \( \Gamma \) is uniform (= cocompact), that is, \( \Gamma \setminus X \) is finite.

\( \Gamma \) is called a uniform tree lattice. Let \( Y = (V(Y), \overset{\rightarrow}{E}(Y)) = \Gamma \setminus X \) be the quotient digraph and \( p : X \rightarrow Y \) the projection. For \( x \in X \) and \( y = p(x) \in Y \), the index \( i(e) \) of \( e \in \overset{\rightarrow}{E}(Y) \) which satisfies \( o(e) = y \) is defined by
\[
i(e) = |\{ e' \in \overset{\rightarrow}{E}(X) | o(e') = x, p(e') = e \}|.
\]
The pair \((Y, i)\) is called an edge-indexed graph. Now we assume \( Y \) has no loop and set \( D_G = Y \) and \( w(e) = i(e) \) for all \( e \in \overset{\rightarrow}{E}(Y) \). Then \( Z_1(G, w, t) \) coincides with the zeta function of the edge-indexed graph \((Y, i)\) which was defined by Bass [2]. Therefore we can consider \( Z_1(G, w, t) \) as a natural generalization of the zeta function of the edge-indexed graph.

The second weighted zeta function has remarkable connections with quantum dynamics on graphs. Replacing \( t \) with \( 1/\lambda \) in Theorem 2.2 we have
\[
det(\lambda I_{2m} - (B_w - J_0)) = (\lambda^2 - 1)^{m-n} \det(\lambda^2 I_n - \lambda W + (D_w - I_n)). \tag{2.3}
\]
Using this equation, several quantum systems on graphs have been investigated. Consequently, new proofs of spectral properties of them were given in [15, 11] and spectra were newly determined by easy derivable parameters from \( W \) in [7, 10]. Interestingly, time evolution operators of important quantum systems on graphs, for example, the Grover walk, the Szegedy walk, the bond scattering matrix of Smilansky, can be expressed by \( B \)-weighted edge matrices. It enables us to apply (2.3) to derive spectra of time evolution operators. In this way, the generalization of Ihara zeta functions of tree lattices has significant applications in quantum mechanics. We expect that the second weighted zeta function brings about further outcomes in quantum dynamics on graphs such as quantum walks or quantum graphs.

### 3 The Study determinant of a quaternionic matrix

We shall establish the quaternionic second weighted zeta function of a graph from now on. In this section, we explain quaternions and quaternionic matrices that are needed in later
sections. Let $\mathbb{H}$ be the set of quaternions. $\mathbb{H}$ is a noncommutative associative algebra over $\mathbb{R}$, whose underlying real vector space has dimension 4 with a basis $1, i, j, k$ which satisfy the following relations:

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$ 

For $x = x_0 + x_1 i + x_2 j + x_3 k \in \mathbb{H}$, $x^*$ denotes the conjugate of $x$ in $\mathbb{H}$ which is defined by $x^* = x_0 - x_1 i - x_2 j - x_3 k$, and $\text{Re} \, x = x_0$ the real part of $x$. One can easily check $xx^* = x^*x$, $(x^n)^* = (x^*)^*$, and $x^{-1} = x^* / |x|^2$ for $x \neq 0$. Hence, $\mathbb{H}$ constitutes a skew field. We call $|x| = \sqrt{xx^*} = \sqrt{x^*x} = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}$ the norm of $x$. Indeed, $| \cdot |$ satisfies

1. $|x| \geq 0$, and moreover $|x| = 0 \iff x = 0$,
2. $|xy| = |x||y|$,
3. $|x + y| \leq |x| + |y|$.

We notice that $\mathbb{H}$ is a complete space with respect to the metric derived from the norm of quaternions.

Any quaternion $x$ can be presented by two complex numbers $x = a + jb$ uniquely. Explicitly, if $x = x_0 + x_1 i + x_2 j + x_3 k$ then $a = x_0 + x_1 i$ and $b = x_2 - x_3 i$. Such a presentation is called the symplectic decomposition. Two complex numbers $a$ and $b$ are called the simplex part and the perplex part of $x$ respectively. We mean by a quaternionic matrix a matrix whose entries are quaternions. The symplectic decomposition is also valid for a quaternionic matrix. $\text{Mat}(m \times n, \mathbb{H})$ denotes the set of $m \times n$ quaternionic matrices and $\text{Mat}(m, \mathbb{H})$ the set of $m \times m$ quaternionic matrices. For $M \in \text{Mat}(m \times n, \mathbb{H})$, we can write $M = M^S + j M^P$ uniquely where $M^S, M^P \in \text{Mat}(m, n, \mathbb{C})$. $M^S$ and $M^P$ are called the simplex part and the perplex part of $M$ respectively. We define $\psi$ to be the map from $\text{Mat}(m \times n, \mathbb{H})$ to $\text{Mat}(2m \times 2n, \mathbb{C})$ as follows:

$$\psi : \text{Mat}(m \times n, \mathbb{H}) \rightarrow \text{Mat}(2m \times 2n, \mathbb{C}) \quad M \mapsto \begin{bmatrix} M^S & -M^P \\ M^P & M^S \end{bmatrix},$$

where $\overline{A}$ is the complex conjugate of a complex matrix $A$. Then $\psi$ is an $\mathbb{R}$-linear map. We also have

**Lemma 3.1.** Let $M \in \text{Mat}(m \times n, \mathbb{H})$ and $N \in \text{Mat}(n \times m, \mathbb{H})$. Then

$$\psi(MN) = \psi(M)\psi(N).$$

**Proof.** Let $M = A + jB$ and $N = C + jD$ be symplectic decompositions of $M$ and $N$. Then

$$MN = (A + jB)(C + jD) = AC + A jD + jBC + jBjD.$$ 

Since $Xj = j\overline{X}$ for every complex matrix $X$, we obtain

$$MN = AC - BD + j(\overline{A}D + BC),$$ 

and therefore

$$\psi(MN) = \begin{bmatrix} AC - BD & -A\overline{D} - BC \\ A\overline{D} + BC & AC - BD \end{bmatrix}.$$ 

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On the other hand,

\[ \psi(M)\psi(N) = \begin{bmatrix} A & -B \\ B & A \end{bmatrix} \begin{bmatrix} C & -D \\ D & C \end{bmatrix} = \begin{bmatrix} AC - BD & -AD - BC \\ BC + AD & -BD + AC \end{bmatrix}. \]

Thus \( \psi(MN) = \psi(M)\psi(N) \) holds. \( \square \)

From Lemma 3.1 it follows immediately that

**Proposition 3.2.** If \( m = n \), then \( \psi \) is an injective \( \mathbb{R} \)-algebra homomorphism.

In [20], Study defined a determinant of an \( n \times n \) quaternionic matrix which we denote by \( \text{Sdet}(M) = \det(\psi(M)) \), where \( \det \) is the ordinary determinant. We call \( \text{Sdet} \) the **Study determinant**. The Study determinant is the unique, up to a real power factor, functional \( d_H \) which satisfies the following three axioms [1]:

(A1) \( d_H(A) = 0 \iff A \) is singular.

(A2) \( d_H(AB) = d_H(A)d_H(B) \) for all \( A, B \in \text{Mat}(n, \mathbb{H}) \).

(A3) If \( A' \) is obtained from \( A \) by adding a left-multiple of a row to another row or a right-multiple of a column to another column, then \( d_H(A') = d_H(A) \).

Therefore it is reasonable to adopt the Study determinant to investigate the quaternionization of determinant expressions for zeta functions of graphs. Before stating properties of \( \text{Sdet} \), we mention a useful formula whose proof can be found in for example [25]:

**Lemma 3.3.** If \( A, B, C, D \) are complex square matrices with same size and \( AC = CA \), then

\[ \det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(AD - CB). \]

\( \text{Sdet} \) has several basic properties as follows:

**Proposition 3.4.**

(i) \( \text{Sdet}(M) \in \mathbb{R}_{\geq 0} = \{a \in \mathbb{R} \mid a \geq 0\} \) for \( M \in \text{Mat}(n, \mathbb{H}) \).

(ii) \( \text{Sdet}(M) = 0 \iff M \) has no inverse.

(iii) \( \text{Sdet}(MN) = \text{Sdet}(M)\text{Sdet}(N) \) for \( M, N \in \text{Mat}(n, \mathbb{H}) \).

(iv) If \( N \) is obtained from \( M \) by adding a left-multiple of a row to another row or a right-multiple of a column to another column, then \( \text{Sdet}(N) = \text{Sdet}(M) \).

(v) \( \text{Sdet}(\alpha M) = \text{Sdet}(M \alpha) = |\alpha|^{2n} \text{Sdet}(M) \) for \( M \in \text{Mat}(n, \mathbb{H}), \alpha \in \mathbb{H} \).

(vi) If \( M \in \text{Mat}(n, \mathbb{H}) \) is of the form:

\[ M = \begin{bmatrix} \lambda_1 & * & \cdots & * \\ 0 & \lambda_2 & * & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ * & \lambda_2 & 0 & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ * & * & \cdots & \lambda_n \end{bmatrix}, \]

then \( \text{Sdet}(M) = \prod_{r=1}^{n} |\lambda_r|^2 \).
(vii) Let $A$ be an $m \times n$ matrix and $B$ an $n \times m$ matrix. Then
\[
Sdet(I_m - AB) = Sdet(I_n - BA).
\]

Proof. The proofs of (i), (ii), (iii), (iv) can be found in [1]. We prove (v), (vi) and (vii).

(v) Let $M \in \text{Mat}(n, \mathbb{H})$ and $\alpha = \alpha_s + j\alpha_p \in \mathbb{H} (\alpha_s, \alpha_p \in \mathbb{C})$. Then using Lemma 3.1 and Lemma 3.3 we have
\[
Sdet(\alpha M) = \det(\psi(\alpha M)) = \det(\psi(\alpha I_n)\psi(M))
= \det \begin{bmatrix}
\alpha_s I_n & -\alpha_p I_n \\
\alpha_p I_n & \alpha_s I_n
\end{bmatrix} \det(\psi(M))
= \det((\alpha_s I_n)(\alpha_s I_n) + (\alpha_p I_n)(\alpha_p I_n)) Sdet(M)
= \det((|\alpha_s|^2 + |\alpha_p|^2)I_n) Sdet(M)
= |\alpha|^{2n} Sdet(M).
\]

In the same way, we can deduce $Sdet(M\alpha) = |\alpha|^{2n} Sdet(M)$.

(vi) For a $2n \times 2n$ matrix $N$ and any two subsets $I = \{i_1, i_2, \cdots, i_r\}$, $J = \{j_1, j_2, \cdots, j_s\}$ of $[2n]$, $N^{IJ}$ denotes the submatrix obtained from $N$ by deleting $i_1, i_2, \cdots, i_r$ th rows and $j_1, j_2, \cdots, j_s$ th columns. Then by definitions of Sdet and $\psi$, we get the following:
\[
Sdet(M) = \det(\psi(M)) = \det \begin{bmatrix}
M^S & -M^P \\
M^P & M^S
\end{bmatrix}
= \det \begin{bmatrix}
\lambda_1^S & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \lambda_2^S & \cdots & 0 & -\lambda_2^P & \cdots & \cdots & \cdots \\
\vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
0 & 0 & \cdots & \lambda_n^S & 0 & 0 & \cdots & -\lambda_n^P \\
\lambda_1^P & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \lambda_2^P & \cdots & 0 & \lambda_2^S & \cdots & \cdots & \cdots \\
\vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
0 & 0 & \cdots & \lambda_n^P & 0 & 0 & \cdots & \lambda_n^S
\end{bmatrix}
= \lambda_1^S \det(\psi(M)^{\{1\}\{1\}}) + (-1)^{n+2} \lambda_1^P \det(\psi(M)^{\{n+1\}\{1\}})
= \lambda_1^S \lambda_1^P \det(\psi(M)^{\{1,n+1\}\{1,n+1\}}) + \lambda_2^P \lambda_2^S \det(\psi(M)^{\{1,n+1\}\{1,n+1\}})
= |\lambda_1|^2 \det(\psi(M)^{\{1,n+1\}\{1,n+1\}})
= |\lambda_1|^2 |\lambda_2|^2 \det(\psi(M)^{\{1,2,n+1,n+2\}\{1,2,n+1,n+2\}}) = \cdots
= \prod_{r=1}^{n} |\lambda_r|^2.
\]

(vii) Let $A$ be an $m \times n$ matrix and $B$ an $n \times m$ matrix. Then by the definition of the Study determinant, we have
\[
Sdet(I_m - AB) = Sdet(\psi(I_m - AB)) = det(I_{2m} - \psi(A)\psi(B))
= det(I_{2m} - \psi(B)\psi(A)) = det(\psi(I_n - BA))
= Sdet(I_n - BA).
\]
Here the equation \( \det(I_{2n} - \psi(A)\psi(B)) = \det(I_{2n} - \psi(B)\psi(A)) \) is based on the property of determinant that can be found in, for example, Problem 5 on page 47 in [25].

\[ \square \]

Remark 3.5. \( \text{Sdet} \) is not multilinear as \( \det \) is. Furthermore, \( \text{Sdet}(^TM) = \text{Sdet}(M) \) does not hold in general where \( ^TM \) is the transpose of \( M \).

4 The quaternionic second weighted zeta function of a graph

We follow symbols and notations in the previous section. We shall give the definition of a quaternionic analogue of the second weighted zeta function and derive its determinant expression of Bass type by the Study determinant. In the same way as the complex case, consider a quaternionic matrix \( W = (W_{uv})_{u,v \in V(G)} \in \text{Mat}(n, \mathbb{H}) \) with \( (u, v) \)-entry equals 0 if \( (u, v) \notin D(G) \). We call \( W \) a quaternionic weighted matrix of \( G \). Furthermore, let \( w(u, v) = W_{uv} \) for \( u, v \in V(G) \) and \( w(e) = w(u, v) \) if \( e = (u, v) \in D(G) \). For each path \( P = (e_1, \ldots, e_t) \) of \( G \), the norm \( w(P) \) of \( P \) is defined by \( w(P) = w(e_1)w(e_2)\cdots w(e_t) \).

For a quaternionic weighted matrix \( W \) of \( G \), the \( 2m \times 2m \) quaternionic matrix \( B_w = (B_{ef}^{(w)})_{e,f \in D(G)} \in \text{Mat}(2m, \mathbb{H}) \) is defined as follows:

\[
B_{ef}^{(w)} = \begin{cases} 
  w(f) & \text{if } t(e) = o(f), \\
  0 & \text{otherwise.} 
\end{cases} \tag{4.1}
\]

We define the quaternionic second weighted zeta function of \( G \) to be as follows:

\[
Z^B_1(G, w, t) = \text{Sdet}(I_{2m} - t(B_w - J_0))^{-1},
\]

where \( t \) is a quaternionic variable.

The Study determinant expression of Bass type for the quaternionic second weighted zeta function of a graph is given as follows. The block diagonal sum \( M_1 \oplus \cdots \oplus M_s \) of square matrices \( M_1, \ldots, M_s \) is defined as the square matrix:

\[
\begin{bmatrix} 
  M_1 & 0 \\
  \vdots & \ddots \\
  0 & M_s 
\end{bmatrix}.
\]

If \( M_1 = M_2 = \cdots = M_s = N \), then we write \( s \circ N = M_1 \oplus \cdots \oplus M_s \). The Kronecker product \( A \otimes B \) of matrices \( A \) and \( B \) is considered as the matrix \( A \) having the element \( a_{rs} \) replaced by the matrix \( a_{rs}B \).

**Theorem 4.1.** Let \( G \) be a connected graph, and let \( W \) be a quaternionic weighted matrix of \( G \). Then the reciprocal of the quaternionic second weighted zeta function of \( G \) is given by

\[
Z^B_1(G, w, t)^{-1} = |1 - t^2|^{2m-2n}\text{Sdet}(I_n - Wt + (D_w - I_n)t^2),
\]

where \( n = |V(G)| \), \( m = |E(G)| \) and \( D_w = (D_{uv}^{(w)})_{u,v \in V(G)} \) is the diagonal matrix with \( D_{uu}^{(w)} = \sum_{e: o(e)=u} w(e) \).
Proof. Let $D(G) = \{ f_1, \ldots, f_m, f_1^{-1}, \ldots, f_m^{-1} \}$. Arrange arcs of $G$ as follows:

\[ f_1, f_1^{-1}, \ldots, f_m, f_m^{-1}. \]

By the definition of the second weighted zeta function of $G$ and Proposition \ref{P:3.3}, we have

\[
Z_{11}(G, w, t)^{-1} = \text{Sdet}(I_{2m} - t(B_w - J_0)) \\
= \text{Sdet}(I_{2m} + tJ_0 - tB_w) \\
= \text{Sdet}(I_{2m} - tB_w(I_{2m} + tJ_0)^{-1})\text{Sdet}(I_{2m} + tJ_0). \tag{4.2}
\]

Let $t = ts + jt_p \in \mathbb{H}$ be the symplectic decomposition. Then we have

\[
\text{Sdet}(I_{2m} + tJ_0) = \det \begin{bmatrix} I_{2m} + tsJ_0 & -t_pJ_0 \\ t_pJ_0 & I_{2m} + tsJ_0 \end{bmatrix}, \tag{4.3}
\]

where $I_{2m} + tsJ_0$ and $t_pJ_0$ are given by

\[
I_{2m} + tsJ_0 = m \circ \begin{bmatrix} 1 & ts \\ ts & 1 \end{bmatrix}, \quad t_pJ_0 = m \circ \begin{bmatrix} 0 & t_p \\ t_p & 0 \end{bmatrix}.
\]

For any two complex numbers $\alpha$ and $\beta$,

\[
\begin{bmatrix} 1 & \alpha \\ \alpha & 1 \end{bmatrix} \begin{bmatrix} 0 & \beta \\ \beta & 0 \end{bmatrix} = \begin{bmatrix} 0 & \beta \\ \beta & 0 \end{bmatrix} \begin{bmatrix} 1 & \alpha \\ \alpha & 1 \end{bmatrix}
\]

holds. This implies any two blocks in the right-hand side of (4.3) commute. Thus by Lemma 3.3 we have

\[
\text{Sdet}(I_{2m} + tJ_0) = \det((I_{2m} + tsJ_0)(I_{2m} + tsJ_0) + t_p\overline{t_p}J_0^2) \\
= \det(I_{2m} + (ts + t_p)J_0 + (|ts|^2 + |t_p|^2)I_{2m}) \\
= \det I_m \bigotimes \begin{bmatrix} 1 + |t|^2 & 2\text{Re }t \\ 2\text{Re }t & 1 + |t|^2 \end{bmatrix} \\
= \{(1 + |t|^2)^2 - 4(\text{Re }t)^2\}^m \tag{4.4} \\
= \{(1 + t^*)^2 - (t + t^*)^2\}^m \\
= \{(1 - t^2)(1 - (t^*)^2)\}^m \\
= \{(1 - t^2)(1 - t^*)^2\}^m \\
= |1 - t^2|^{2m}
\]

On the other hand, since the following holds:

\[
t \frac{1}{1 - t^2} = t \frac{(1 - (t^2))^s}{|1 - t^2|^2} = t \frac{1 - t^*t^*}{|1 - t^*|^2} = t \frac{1 - t^*t}{|1 - t|^2} = t \frac{1}{1 - t^2}, \tag{4.5}
\]

and hence

\[
(I_{2m} + tJ_0)(I_{2m} - tJ_0) \frac{1}{1 - t^2} = (I_{2m} - tJ_0) \frac{1}{1 - t^2} (I_{2m} + tJ_0) = I_{2m}. \tag{4.6}
\]
Thus, we have

\[(I_{2m} + tJ_0)^{-1} = (I_{2m} - tJ_0)\frac{1}{1 - t^2}.\]

Now, let \(K = (K_{ev})_{e \in D(G), v \in V(G)}\) and \(L = (L_{ev})_{e \in D(G), v \in V(G)}\) be \(2m \times n\) matrices defined as follows:

\[K_{ev} = \begin{cases} w(e) & \text{if } o(e) = v, \\ 0 & \text{otherwise,} \end{cases} \quad L_{ev} = \begin{cases} 1 & \text{if } t(e) = v, \\ 0 & \text{otherwise.} \end{cases}\]

Then

\[L^TK = B_w\]

holds, where \(^TK\) is the transpose of \(K\). Thus, by Lemma 3.4, (4.5) and (4.6), we can show that

\[Sdet(I_{2m} - tB_w(I_{2m} + tJ_0)^{-1}) = Sdet(I_n - tWt + (t^2 - 1)I_n)\frac{1}{1 - t^2}1_{2n}.\]

For an arc \((u,v) \in D(G)\),

\[(^TK(I_{2m} - tJ_0)L)_{uv} = w(u,v).\]

In the case of \(u = v\),

\[(^TK(I_{2m} - tJ_0)L)_{uu} = -\sum_{o(e)=u} w(e)t.\]

Thus by Proposition 3.4 we have

\[Sdet(I_{2m} - tB_w(I_{2m} + tJ_0)^{-1}) = Sdet(I_n - (W - D_w)t + (t^2 - 1)I_n)\frac{1}{1 - t^2}1_{2n}.\]

Finally, we conclude from (4.2) and (4.4) that

\[Z_H^1(G, w, t)^{-1} = Sdet(I_{2m} - tB_w(I_{2m} + tJ_0)^{-1})Sdet(I_{2m} + tJ_0)\]

\[= Sdet(I_n - Wt + (D_w - I_n)t^2)\frac{1}{1 - t^2}1_{2n}2^{2m}\]

\[= |1 - t^2|^{2m-2n}Sdet(I_n - Wt + (D_w - I_n)t^2).\]

\[\square\]

5 The Euler product for the quaternionic second weighted zeta function

In this section, we derive the Euler product of the quaternionic second weighted zeta function. In order to obtain the Euler product, we make use of the notion of noncommutative formal
power series. Our argument on this subject is based on the proofs of Amitsur’s identity in [16] or [4] until (5.6). For the sake of argument, we will give a brief account of noncommutative formal power series at first. A detailed exposition of formal series can be found in [3]. Let 
\[ X = \{x_1, \ldots, x_N\} \]
be a finite nonempty totally ordered set in which elements are arranged ascendingly. \( X^* \) denotes the free monoid generated by \( X \). Let \( \prec \) be the lexicographic order on \( X^* \) derived from the total order on \( X \). For a word \( w = x_{i_1}x_{i_2}\cdots x_{i_d} \in X^* \), \( d \) is called the length of \( w \) which is denoted by \(|w|\). The length of the empty word is defined to be 0. A nonempty word \( w \) in \( X^* \) is called a Lyndon word if \( w \) is prime, namely, not a power \( w^r \) of any other word \( w' \) for any \( r \geq 2 \), and is minimal in the cyclic rearrangements of \( w \). We denote by \( L_X \) the set of Lyndon words in \( X^* \). It is well known that any nonempty word \( w \) can be formed uniquely as a nonincreasing sequence of Lyndon words.

**Theorem 5.1.** For any nonempty word \( w \in X^* \), there exists a unique nonincreasing sequence of Lyndon words \( l_1, l_2, \ldots, l_d \) such that \( w = l_1l_2\cdots l_d \).

**Proof.** For the proof, see for example [13].

Let us consider the ring of noncommutative formal power series \( \mathbb{R}\langle\langle X^*\rangle\rangle \). Each element \( f \) of \( \mathbb{R}\langle\langle X^*\rangle\rangle \) is displayed as
\[ f = \sum_{w \in X^*} f_ww \quad (f_w \in \mathbb{R}). \]
\( \mathbb{R}\langle\langle X^*\rangle\rangle \) can be equipped with the topology defined by the following manner. Let \( \omega \) be the function defined as follows:
\[ \omega(\alpha, \beta) = \begin{cases} \infty & \text{if } \{w \in X^* \mid \alpha_w \neq \beta_w\} = \emptyset, \\ \inf\{n \in \mathbb{N} \mid \exists w \in X^*, |w| = n, \alpha_w \neq \beta_w\} & \text{otherwise}. \end{cases} \]
Then an ultrametric distance \( d_\omega \) on \( \mathbb{R}\langle\langle X^*\rangle\rangle \) is given by \( d_\omega(\alpha, \beta) = 2^{-\omega(\alpha, \beta)} \) and a topology on \( \mathbb{R}\langle\langle X^*\rangle\rangle \) is derived from \( d_\omega \). We notice that \( \mathbb{R}\langle\langle X^*\rangle\rangle \) is complete for this topology. Since \( (1 - l)^{-1} = 1 + l + l^2 + \cdots \) for every \( l \in X^* \) in \( \mathbb{R}\langle\langle X^*\rangle\rangle \), Theorem 5.1 implies
\[ \prod_{l \in L_X} (1 - l)^{-1} = \sum_{w \in X^*} w, \quad (5.1) \]
in \( \mathbb{R}\langle\langle X^*\rangle\rangle \), where \( \prod_{l \in L_X} \) means that the factors are multiplied in decreasing order. On the other hand, it follows that
\[ \sum_{w \in X^*} w = (1 - (x_1 + \cdots + x_N))^{-1}. \quad (5.2) \]
(5.1) and (5.2) imply the following equation:
\[ (1 - (x_1 + \cdots + x_N))^{-1} = \prod_{l \in L_X} (1 - l)^{-1}. \quad (5.3) \]
From (5.3), we obtain
Proposition 5.2.\[1 - (x_1 + \cdots + x_N) = \prod_{l \in \mathcal{L}_X} (1 - l), \tag{5.4}\]
where \( \prod_{l \in \mathcal{L}_X} \) means that the factors are multiplied in increasing order.

Proof. In order to show that
\[
\left\{ \prod_{l \in \mathcal{L}_X} (1 - l)^{-1} \right\} \left\{ \prod_{l \in \mathcal{L}_X} (1 - l) \right\} = 1, \tag{5.5}\]
we check that for an arbitrary nonnegative integer \( r \geq 0 \) the sum of words of length \( r \) equals
1 if \( r = 0 \) and 0 if \( r > 0 \). Since \( \prod_{l \in \mathcal{L}_X} (1 - l)^{-1} = \prod_{l \in \mathcal{L}_X} (1 + l + l^2 + \cdots) \), the sum of words of length at most \( d \) in the left hand side of [5.5] is the same as that of the product:
\[
\left\{ \prod_{l \in \mathcal{L}_X, |l| \leq d} (1 + l + l^2 + \cdots) \right\} \left\{ \prod_{l \in \mathcal{L}_X, |l| \leq d} (1 - l) \right\}.
\]
This is a finite product since \( |X| < \infty \) and therefore is equal to 1. Since \( d \) is arbitrary, [5.5] holds.

Let \([2m] = \{1, 2, \cdots, 2m\}\) and \([2m] \times [2m]\) the cartesian product with the lexicographic order derived from the natural order on \([2m]\). We say that a word \( w = (i_1, j_1)(i_2, j_2)\cdots(i_d, j_d) \in ([2m] \times [2m])^* \) is connected if \( j_r = i_{r+1} \) for \( r = 1, 2, \cdots, d - 1 \). For a connected word \( w = (i_1, i_2)(i_2, i_3)\cdots(i_d, j_d) \in ([2m] \times [2m])^* \), we set \( o(w) = i_1 \) and \( t(w) = j_d \). Consider the finite nonempty set \( X = \{x(r, s) \mid (r, s) \in [2m] \times [2m]\} \) equipped with the total order derived from \([2m] \times [2m]\). For each matrix \( A = (a_{rs}) \in \text{Mat}(2m, \mathbb{H}) \), we define \( \rho^A \) to be the \( \mathbb{R}\)-algebra homomorphism from the monoid ring \( \mathbb{R}[X^*] \) to \( \text{Mat}(2m, \mathbb{H}) \) defined by \( \rho^A(x(r, s)) = a_{rs} \mathbf{E}_{rs} \), where \( \mathbf{E}_{rs} \) denotes the \( (r, s) \)-matrix unit. Let \( A(r, s) = a_{rs} \mathbf{E}_{rs} \). For \( a_{rs} \) \( (1 \leq r, s \leq 2m) \) with \( |a_{rs}| \) sufficiently small, we can apply \( \rho^A \) to [5.4] so that
\[
\mathbf{I}_{2m} - \{ A(1, 1) + A(1, 2) + \cdots + A(2m, 2m) \} = \prod_{l \in \mathcal{L}_{[2m] \times [2m]}} (\mathbf{I}_{2m} - A_l), \tag{5.6}\]
where \( A_l = A(i_1, j_1)A(i_2, j_2)\cdots A(i_d, j_d) \) for each \( l = (i_1, j_1)(i_2, j_2)\cdots(i_d, j_d) \in \mathcal{L}_{[2m] \times [2m]} \). Indeed, the following holds:

Proposition 5.3. For \( a_{rs} \) \( (1 \leq r, s \leq 2m) \) with \( |a_{rs}| \) sufficiently small, all entries in the right hand side of [5.6] converge absolutely with respect to the norm of quaternions. Particularly all entries in the right hand side of [5.6] converge.

Proof. We put \( a_l = a_{i_1j_1}a_{i_2j_2}\cdots a_{i_dj_d} \) and \( \mathbf{E}_l = \mathbf{E}_{i_1j_1}\mathbf{E}_{i_2j_2}\cdots \mathbf{E}_{i_dj_d} \) for a Lyndon word \( l = (i_1, j_1)(i_2, j_2)\cdots(i_d, j_d) \in \mathcal{L}_{[2m] \times [2m]} \). Since \( A_l = A_{(i_1, j_1)\cdots(i_d, j_d)} = a_{i_1j_1}a_{i_2j_2}\cdots a_{i_dj_d} \mathbf{E}_l \), it
follows that
\[
\left( \prod_{l \in L[2m] \times [2m]} (I_{2m} - A_l) \right) = \prod_{l \in L[2m] \times [2m]} (I_{2m} - a_l E_l)
\]
\[
= I_{2m} + \sum_{h=1}^{\infty} \sum_{w=I_{n(w)}, l_1 < \cdots < l_{n(w)}} (-1)^{n(w)} a_w E_w,
\]
(5.7)

where \(a_w = a_{i_1} a_{i_2} \cdots a_{i_{n(w)}}, E_w = E_{i_1} E_{i_2} \cdots E_{i_{n(w)}}\) and \(n(w)\) is the number of Lyndon words that are multiplied in \(w\). If \(|w| = h\) then \(a_w\) can be expressed by \(a_w = a_{i_1 j_1} \cdots a_{i_k j_k}\). Then we notice that \(|a_w| = |a_{i_1 j_1}| \cdots |a_{i_k j_k}|\). We can easily see that \(E_w = E_{o(w)(w)}\) if \(w\) is connected, and \(E_w = O_{2m}\) otherwise. Therefore \((r,s)\)-entry of (5.7) is expressed as follows:
\[
\left( \prod_{l \in L[2m] \times [2m]} (I_{2m} - A_l) \right)_{rs} = \delta_{rs} + \sum_{h=1}^{\infty} \sum_{w=I_{n(w)}, l_1 < \cdots < l_{n(w)}} (-1)^{n(w)} a_w.
\]
(5.8)

In (5.8), the number of \(w\) of length \(h\) is at most the number of words in \((2m \times 2m)^*\) of length \(h\) which equals \((2m)^{2h}\). Hence if \(|a_{rs}| < 1/(8m^2)\) for all \(r, s = 1, \ldots, 2m\), then
\[
|\delta_{rs}| + \sum_{h=1}^{\infty} \sum_{w=I_{n(w)}, l_1 < \cdots < l_{n(w)}} (-1)^{n(w)} a_w
\]

\[
\leq |\delta_{rs}| + \sum_{h=1}^{\infty} \sum_{w=I_{n(w)}, l_1 < \cdots < l_{n(w)}} |a_w|
\]

\[
< 1 + \sum_{h=1}^{\infty} \frac{(2m)^{2h}}{(8m^2)^h} = 1 + \sum_{h=1}^{\infty} \frac{1}{2^h} = 2
\]

Thus the right hand side of (5.8) converges absolutely with respect to the norm of quaternions.

Since \(A(1, 1) + A(1, 2) + \cdots + A(2m, 2m) = A\), it follows from (5.6) that

**Proposition 5.4.** Let \(A = (a_{rs})\) be a \(2m \times 2m\) quaternionic matrix with \(|a_{rs}|\) sufficiently small. Then
\[
I_{2m} - A = \prod_{(i_1 j_1) \cdots (i_d j_d) \in L(2m) \times (2m)} (I_{2m} - a_{i_1 j_1} a_{i_2 j_2} \cdots a_{i_{d-1} j_{d-1}} a_{i_d j_d} E_{i_d j_d}),
\]
(5.9)
in \(\text{Mat}(2m, \mathbb{H})\).
Now we take Study determinants of both sides in (5.9).

\[ \text{Sdet}(I_{2m} - A) = \text{Sdet} \left( \prod_{(i_1,j_1) \cdots (i_d,j_d) \in L_{[2m]}^{[2m]}} (I_{2m} - a_{i_1j_1}a_{i_2j_2} \cdots a_{i_dj_d}e_{i_1j_1}) \right) \]

\[ = \prod_{(i_1,j_1) \cdots (i_d,j_d) \in L_{[2m]}^{[2m]}} \text{Sdet}(I_{2m} - a_{i_1j_1}a_{i_2j_2} \cdots a_{i_dj_d}e_{i_1j_1}). \]  

(5.10)

We notice that the last formula does not depend on the order in which factors are multiplied since Study determinants take values in \( \mathbb{R} \). It follows from Proposition 3.4(vi) that if \( j_d = i_1 \), then

\[ \text{Sdet}(I_{2m} - a_{i_1j_2}a_{i_2j_3} \cdots a_{i_dj_1}e_{i_1j_1}) = |1 - a_{i_1j_2}a_{i_2j_3} \cdots a_{i_dj_1}|^2, \]

and otherwise,

\[ \text{Sdet}(I_{2m} - a_{i_1j_2}a_{i_2j_3} \cdots a_{i_dj_1}e_{i_1j_1}) = 1. \]

Let \( W = (W_{uv})_{u,v} \in V(G) \) be an arbitrary quaternion weighted matrix of \( G \) and \( t \) a quaternion with \( |t| \) sufficiently small so that \( |t\tilde{w}(e,f)| < 1/(8m^2) \) for all \( e, f \in D(G) \), where \( \tilde{w}(e,f) \) is as in (2.2). Putting \( A = t(B_w - J_0) \) and indexing rows and columns with \( e_1, e_2, \cdots, e_{2m} \in D(G) \), then we have \( a_{rs} = a_{e_re_s} = t\tilde{w}(e_r, e_s) \). Therefore, (5.10) yields

\[ \text{Sdet}(I_{2m} - t(B_w - J_0)) = \prod_{(i_1,j_1) \cdots (i_d,j_d) \in L_{[2m]}^{[2m]}} \text{Sdet}(I_{2m} - t\tilde{w}(e_{i_1}, e_{i_2})t\tilde{w}(e_{i_2}, e_{i_3}) \cdots t\tilde{w}(e_{i_d}, e_{i_1})e_{i_1j_1}) \]

\[ = \prod_{(i_1,j_1) \cdots (i_d,j_d) \in L_{[2m]}^{[2m]}} |1 - t\tilde{w}(e_{i_1}, e_{i_2})t\tilde{w}(e_{i_2}, e_{i_3}) \cdots t\tilde{w}(e_{i_d}, e_{i_1})|^2 \]

Each Lyndon word \( (i_1, i_2) \cdots (i_d, i_1) \) in \( L_{[2m]}^{[2m]} \) corresponds to a Lyndon word \( i_1i_2 \cdots i_d \) in \( L_{[2m]} \) bijectively. Hence we obtain

**Theorem 5.5.** Let \( t \) be a quaternion with \( |t| \) sufficiently small. Then

\[ Z_{11}^{\mathbb{H}}(G, w, t) = \prod_{i_1i_2 \cdots i_d \in L_{[2m]}} |1 - t\tilde{w}(e_{i_1}, e_{i_2})t\tilde{w}(e_{i_2}, e_{i_3}) \cdots t\tilde{w}(e_{i_d}, e_{i_1})|^{-2}. \]

Since real numbers are central in \( \mathbb{H} \), it follows that

**Corollary 5.6.** Let \( t \) be a real number with \( |t| \) sufficiently small. Then

\[ Z_{11}^{\mathbb{H}}(G, w, t) = \prod_{i_1i_2 \cdots i_d \in L_{[2m]}} |1 - \tilde{w}(e_{i_1}, e_{i_2})\tilde{w}(e_{i_2}, e_{i_3}) \cdots \tilde{w}(e_{i_d}, e_{i_1})t^d|^{-2}. \]

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References

[1] Aslaksen, H.: Quaternionic Determinants. Math. intelligencer 18, no3, pp. 57–65 (1996)

[2] Bass, H.: The Ihara-Selberg zeta function of a tree lattice. Internat. J. Math. 3, pp. 717–797 (1992)

[3] Berstel, J., Reutenauer, C.: Noncommutative rational series with applications, Cambridge University press, Cambridge, 2011.

[4] Foata, D., Zeilberger, D.: A combinatorial proof of Bass’s evaluations of the Ihara-Selberg zeta function for graphs. Trans. Amer. Math. Soc. 351, pp. 2257–2274 (1999)

[5] Hashimoto, K.: Zeta Functions of Finite Graphs and Representations of p-Adic Groups. in "Adv. Stud. Pure Math". Vol. 15, pp. 211–280, Academic Press, New York, 1989.

[6] Hashimoto, K.: On Zeta and L-Functions of Finite Graphs. Internat. J. Math. 1, pp. 381–396 (1990)

[7] Higuchi, H., Konno, N., Sato, I., Segawa, E.: A note on the discrete-time evolutions of quantum walk on a graph. J. Math-for-Industry, 5 (2013B-3), pp.103–109, (2013)

[8] Ihara, Y.: On discrete subgroups of the two by two projective linear group over p-adic fields. J. Math. Soc. Japan 18, pp. 219–235 (1966)

[9] Konno, N.: Quaternionic quantum walks. Quantum Stud.: Math. Found. 2, 63–76 (2015)

[10] Konno, N., Mitsuhashi, H., Sato, I.: The discrete-time quaternionic quantum walk on a graph. Quantum Inf. Process. 15, 651–673 (2016)

[11] Konno, N., Sato, I.: On the relation between quantum walks and zeta functions. Quantum Information Processing 11 Issue 2, pp. 341–349 (2012)

[12] Kotani, M., Sunada, T.: Zeta functions of finite graphs. J. Math. Sci. U. Tokyo 7, pp. 7–25 (2000)

[13] Lothaire, M.: Combinatorics on words. Cambridge University Press, (1997)

[14] Mizuno, H., Sato, I.: Weighted zeta functions of graphs. J. Combin. Theory Ser. B 91, pp.169–183 (2004)

[15] Mizuno, H., Sato, I. : The scattering matrix of a graph. Electron. J. Combin. 15, R96 (2008)

[16] Reutenauer, C., Schützenberger, M-P.: A formula for the Determinant of a Sum of Matrices. Lett. Math. Phys. 13, pp. 299–302 (1987)

[17] Sato, I. : A new Bartholdi zeta function of a graph. Int. J. Algebra 1, pp. 269–281 (2007)
[18] Serre, J. -P.: Trees, Springer-Verlag, New York, 1980.

[19] Stark, H. M., Terras, A. A.: Zeta functions of finite graphs and coverings. Adv. Math. 121, pp. 124-165 (1996)

[20] Study, E.: Zur Theorie der lineare Gleichungen. Acta. Math. 42, pp. 1-61 (1920)

[21] Smilansky, U.: Quantum chaos on discrete graphs. J. Phys. A: Math. Theor. 40, F621–F630 (2007)

[22] Sunada, T.: L-Functions in Geometry and Some Applications. in ”Lecture Notes in Math.”, Vol. 1201, pp. 266–284, Springer-Verlag, New York, 1986.

[23] Sunada, T.: Fundamental Groups and Laplacians(in Japanese), Kinokuniya, Tokyo, 1988.

[24] Zhang, F.: Quaternions and matrices of quaternions. Linear Algebra Appl. 251, pp. 21–57 (1997)

[25] Zhang, F.: Matrix Theory 2nd ed., Springer, New York, 2011.