ROTATING WAVES IN NONLINEAR MEDIA AND CRITICAL DEGENERATE SOBOLEV INEQUALITIES

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Abstract. We investigate the presence of rotating wave solutions of the nonlinear wave equation
\[ \partial_t^2 v - \Delta v + mv = |v|^{p-2} v \] in \( \mathbb{R} \times B \), where \( B \subset \mathbb{R}^N \) is the unit ball, complemented
with Dirichlet boundary conditions on \( \mathbb{R} \times \partial B \). Depending on the prescribed angular velocity \( \alpha \) of the rotation, this leads to a Dirichlet problem for a semilinear elliptic or degenerate
elliptic equation. We show that this problem is governed by an associated critical degenerate Sobolev inequality in the half space. After proving this inequality and the existence
of associated extremal functions, we then deduce necessary and sufficient conditions for the
existence of ground state solutions. Moreover, we analyze under which conditions on \( \alpha \), \( m \) and \( p \) these ground states are nonradial and therefore give rise to truly rotating waves. Our
approach carries over to the corresponding Dirichlet problems in an annulus and in more general Riemannian models with boundary, including the hemisphere. We briefly discuss these
problems and show that they are related to a larger family of associated critical degenerate Sobolev inequalities.

1. Introduction

Within a standard model, the analysis of wave propagation in an ambient medium with
nonlinear response leads to the study of a nonlinear wave equation of the type
\[ \partial_t^2 v - \Delta v + mv = f(v) \quad \text{in } \mathbb{R} \times \Omega, \] (1.1)
in an ambient domain \( \Omega \subset \mathbb{R}^N \) with mass parameter \( m \geq 0 \) and nonlinear response function \( f \).
In the case \( m = 0 \), (1.1) is the classical nonlinear wave equation, while the case \( m > 0 \) is also
known as a nonlinear Klein-Gordon equation. For nonlinearities of the form \( f(v) = g(|v|^2)v \) with a real-valued function \( g \), standing wave solutions can be found by the ansatz
\[ v(t, x) = e^{-ikt} u(x), \quad k > 0 \] (1.2)
with a real-valued function \( u \). Depending on the frequency parameter \( k \), this reduces (1.1)
either to a stationary nonlinear Schrödinger or a nonlinear Helmholtz equation (see e.g. [14]
for more details). The resulting stationary nonlinear Schrödinger equation has been studied
extensively in the past four decades by variational methods, see e.g. the monograph [3]
and the references therein. Due to a lack of a direct variational framework, the nonlinear
Helmholtz equation requires a different approach and has been studied more recently e.g. in
[21, 14, 8, 30, 31] by dual variational methods and bifurcation theory.

Clearly, the amplitude \( |v| \) of a solution \( v \) of (1.1) given by the ansatz (1.2) remains time-
dependent. As a consequence, the analysis of standing wave solutions does not lead to
a full understanding of (1.1) from a dynamical point of view and should be complemented,
in particular, by the study of non-stationary real-valued time-periodic solutions, travelling wave solutions and scattering solutions. We stress that the ansatz (1.2) does not give rise to non-stationary real-valued time-periodic solutions since the nonlinearity of the problem does not allow to pass to real and imaginary parts.

In the case where \(\Omega = \mathbb{R}^N\) and \(f(v)\) in (1.1) is replaced by \(q(x)f(v)\) with a compactly supported weight function \(q\), spatially localized real-valued time-periodic solutions, also called breathers, have attracted increasing attention recently, see e.g. [24, 32] and the references therein. In the case where \(\Omega\) is a radial domain, a further interesting type of real-valued time-periodic solution is given by rotating wave solutions. In particular, if \(\Omega\) is a bounded radial domain and (1.1) is complemented with the Dirichlet boundary condition \(v = 0\) on \(\mathbb{R} \times \partial\Omega\), the existence of rotating waves and their variational characterization arises as a natural question which, up to our knowledge, has not been addressed systematically so far.

The main purpose of the present paper is to provide such a systematic study. While we mainly focus on the case where \(\Omega = B\) is the unit ball in \(\mathbb{R}^N\), we will also address the case where \(\Omega\) is an annulus or a general Riemannian model with boundary, see Sections 5 and 6 below. Specifically, we study the case of a focusing nonlinearity of the form \(f(v) = |v|^{p-2}v\), which leads to the superlinear problem

\[
\begin{align*}
\begin{cases}
\partial_t^2 v - \Delta v + mv = |v|^{p-2}v & \text{in } \mathbb{R} \times B \\
v = 0 & \text{on } \mathbb{R} \times \partial B
\end{cases}
\end{align*}
\tag{1.3}
\]

for \(N \geq 2\), where \(2 < p < 2^*\) and \(m > -\lambda_1(B)\). Here, \(\lambda_1(B)\) denotes the first Dirichlet eigenvalue of \(-\Delta\) on \(B\) and \(2^*\) denotes the critical Sobolev exponent given by \(2^* = \frac{2N}{N-2}\) for \(N \geq 3\) and \(2^* = \infty\) for \(N = 2\). The ansatz for time-periodic rotating solutions of (1.3) is given by

\[
v(t, x) = u(R_{\alpha t}(x))
\tag{1.4}
\]

where, for \(\theta \in \mathbb{R}\), we let \(R_\theta \in O(N)\) denote a planar rotation in \(\mathbb{R}^N\) with angle \(\theta\), so the constant \(\alpha > 0\) in (1.4) is the angular velocity of the rotation. Without loss of generality, we may assume that

\[
R_\theta(x) = (x_1 \cos \theta + x_2 \sin \theta, -x_1 \sin \theta + x_2 \cos \theta, x_3, \ldots, x_N)
\quad \text{for } x \in \mathbb{R}^N,
\]

so \(R_\theta\) is the rotation in the \(x_1, x_2\)-plane with fixed point set \(\{0_{\mathbb{R}^2}\} \times \mathbb{R}^{N-2}\). In the following, we call a function \(u\) on the unit ball \(x_1, x_2\)-nonradial if it is not \(R_\theta\)-invariant for at least one angle \(\theta \in \mathbb{R}\). If the profile function \(u\) in (1.4) is \(x_1, x_2\)-nonradial, then the corresponding solution \(v\) can be interpreted as a rotating wave in a medium with nonlinear response given by the right hand side of (1.3). The ansatz (1.4) reduces (1.3) to

\[
\begin{align*}
\begin{cases}
-\Delta u + \alpha^2 \partial_\theta^2 u + mu = |u|^{p-2}u & \text{in } B \\
u = 0 & \text{on } \partial B
\end{cases}
\end{align*}
\tag{1.5}
\]

where \(\partial_\theta = x_1 \partial_{x_2} - x_2 \partial_{x_1}\) denotes the associated angular derivative operator. We point out that a seemingly closely related equation, with the term \(\alpha^2 \partial_\theta^2 u\) replaced by \(-\alpha^2 \partial_\theta^2 u\), arises in an ansatz for solutions of nonlinear Schrödinger equations in \(\mathbb{R}^3\) with invariance with respect to screw motion, see [2] and also [13] for a related work on Allen-Cahn equations. Note, however, that the positive sign of the term \(\alpha^2 \partial_\theta^2 u\) results in a drastic change of the nature of the problem, as the operator \(-\Delta + \alpha^2 \partial_\theta^2\) loses uniform ellipticity in \(B\) if \(\alpha \geq 1\). This also distinguishes the study of (1.5) from the related study of rotating solutions to nonlinear Schrödinger equations, where the angular velocity \(\alpha\) appears within a first order term which
does not affect the ellipticity of the associated Schrödinger operator, see e.g. [39, 28] and the references therein.

If a solution $u$ of (1.5) satisfies $\partial_\theta u \equiv 0$ in $B$, then $u$ solves the classical stationary nonlinear Schrödinger equation $-\Delta u + mu = |u|^{p-2}u$ in $B$ with Dirichlet boundary conditions on $\partial B$, so it satisfies (1.5) with $\alpha = 0$. If, in addition, $u$ is positive, then $u$ has to be a radial function as a consequence of the symmetry result of Gidas, Ni and Nirenberg [19]. Thus, the ansatz (1.4) then merely gives rise to a radial stationary solution of (1.3). We mention here that radially symmetric non-stationary solutions of (1.1) in $\Omega = B$ were first studied by Ben-Naoum and Mahwin [5] for sublinear nonlinearities and more recently by Chen and Zhang [9, 10, 11]. In this problem, the spectral properties of the radial wave operator lead to delicate assumptions on the dimension as well as the ratio between the radius of the ball and the period length. The main purpose of the present paper is to analyze for which range of parameters $\alpha$, $m$ and $p$ ground state solutions of (1.5) exist and to distinguish under which assumptions on $\alpha$, $m$ and $p$ they are radial or $x_1$-$x_2$-nonradial and therefore correspond to rotating waves via the ansatz (1.4).

By a ground state solution of (1.5), we mean a solution characterized as a minimizer of the minimization problem for

$$\mathcal{E}_{\alpha,m,p}(B) := \inf_{u \in H^1_0(B) \setminus \{0\}} R_{\alpha,m,p}(u),$$

where, for $m \in \mathbb{R}$, $\alpha \geq 0$ and $p \in [2, 2^*)$, we consider the associated Rayleigh quotient $R_{\alpha,m,p}$ given by

$$R_{\alpha,m,p}(u) = \frac{\int_B (|\nabla u|^2 - \alpha^2 |\partial_\theta u|^2 + mu^2) \, dx}{(\int_B |u|^p \, dx)^{2/p}}, \quad u \in H^1_0(B) \setminus \{0\}.$$  

As we shall see in Remark 4.3 below, this minimization problem is only meaningful for $0 \leq \alpha \leq 1$, since for every $p \in [2, 2^*)$ and $m \in \mathbb{R}$ we have

$$\mathcal{E}_{\alpha,m,p}(B) = -\infty \quad \text{for } \alpha > 1.$$  

Moreover, for every $p \in [2, 2^*)$ and $m \in \mathbb{R}$,

the function $\alpha \mapsto \mathcal{E}_{\alpha,m,p}(B)$ is continuous and nonincreasing on $[0, 1]$.  

In the case $0 < \alpha < 1$, the operator $-\Delta + \alpha^2 \partial_\theta^2$ is uniformly elliptic, as can be seen by writing the operator in polar coordinates as

$$-\Delta + \alpha^2 \partial_\theta^2 = -\Delta_r u - \frac{1}{r^2} \Delta_{S^{N-1}} u + \alpha^2 \partial_\theta^2 u,$$

where $\Delta_{S^{N-1}}$ denotes the Laplace-Beltrami operator on the unit sphere $S^{N-1}$. In this case the existence of minimizers of $R_{\alpha,m,p}$ on $H^1_0(B) \setminus \{0\}$ follows by a standard compactness and weak lower semicontinuity argument. However, even in this case it is difficult to decide in general whether minimizers are radial or nonradial functions. This is due to competing effects. Firstly, the additional term $-\alpha^2 |\partial_\theta u|^2_{L^2(B)}$ favours $x_1$-$x_2$-nonradial functions as energy minimizers. On the other hand, the Pólya-Szegő inequality yields $\int_B |\nabla u^*|^2 \, dx \leq \int_B |\nabla u|^2 \, dx$, where $u^*$ denotes the (radial) Schwarz symmetrization of a function $u \in H^1_0(B)$.

Since $R_{\alpha,m,p}(u) = R_{0,m,p}(u)$ for every radial function $u \in H^1_0(B) \setminus \{0\}$ and every $\alpha \in [0, 1]$, a sufficient condition for the $x_1$-$x_2$-nonradiality of all ground state solutions is the inequality

$$\mathcal{E}_{\alpha,m,p}(B) < \mathcal{E}_{0,m,p}(B).$$
In particular, we will be interested in proving this inequality for \( \alpha \) close to 1. We point out that the borderline case \( \alpha = 1 \) differs significantly from the case \( 0 \leq \alpha < 1 \), as the differential operator \(-\Delta + \partial_\theta^2\) is no longer uniformly elliptic on \( B \). In fact, it follows from the representation (1.9) in the case \( \alpha = 1 \) that the operator \(-\Delta + \partial_\theta^2\) fails to be uniformly elliptic in a neighborhood of the great circle \( \{ x \in \partial B : x_3 = \cdots = x_N = 0 \} \) (which equals \( \partial B \) in the case \( N = 2 \)). We shall see in this paper that the minimization problem in the case \( \alpha = 1 \) is essentially governed by a degenerate anisotropic critical Sobolev inequality in the half space. The corresponding critical exponent in this Sobolev inequality is given by

\[
2^*_1 := \frac{4N + 2}{2N - 3}.
\]

The relevance of this exponent is indicated by our first main result which yields the following characterization.

**Theorem 1.1.** Let \( m > -\lambda_1(B) \) and \( p \in (2, 2^*). \)

(i) If \( \alpha \in (0, 1) \), then there exists a ground state solution of (1.5).

(ii) We have

\[
\mathcal{C}_{1,m,p}(B) = 0 \quad \text{for } p > 2^*_1, \quad \text{and} \quad \mathcal{C}_{1,m,p}(B) > 0 \quad \text{for } p \leq 2^*_1. \tag{1.11}
\]

Moreover, for any \( p \in (2^*_1, 2^*) \), there exists \( \alpha_p \in (0, 1) \) with the property that

\[
\mathcal{C}_{\alpha,m,p}(B) < \mathcal{C}_{0,m,p}(B) \quad \text{for } \alpha \in (\alpha_p, 1]
\]

and therefore every ground state solution of (1.5) is \( x_1 \cdot x_2 \)-nonradial for \( \alpha \in (\alpha_p, 1) \).

The following new degenerate Sobolev inequality is an immediate consequence of the special case \( m = 0, \alpha = 1 \) in Theorem 1.1.

**Corollary 1.2.**

\[
\left( \int_B |u|^{2^*_1} \, dx \right)^{\frac{2}{2^*_1}} \leq \frac{1}{\mathcal{C}_{1,0,p}(B)} \int_B \left( |\nabla u|^2 - |\partial_\theta u|^2 \right) \, dx \quad \text{for } u \in H^1_0(B).
\]

Moreover, the exponent \( 2^*_1 \) is optimal in the sense that no such inequality holds for \( p > 2^*_1 \).

Theorem 1.1 yields symmetry breaking of ground states for suitable parameter values of \( p, \alpha \) and \( m \), but the precise parameter range giving rise to this symmetry breaking remains largely open. To shed further light on this question, we state the following result which establishes uniqueness and radial symmetry of ground state solutions for \( \alpha \) close to zero and every \( m \geq 0, 2 < p < 2^* \).

**Theorem 1.3.** Let \( m \geq 0 \) and \( 2 < p < 2^* \). Then there exists \( \alpha_0 > 0 \) such that

\[
\mathcal{C}_{\alpha,m,p}(B) = \mathcal{C}_{0,m,p}(B) \quad \text{for } \alpha \in [0, \alpha_0).
\]

Moreover, for \( \alpha \in [0, \alpha_0) \), there is, up to sign, a unique ground state solution of (1.5) which is a radial function.

Combining Theorems 1.1 and 1.3, we find that, for fixed \( p > 2^*_1 \), symmetry breaking of ground state solutions occurs when passing a critical parameter \( \alpha = \alpha(p) \) which lies in the interval \( [\alpha_0, \alpha^*_p] \). However, so far it remains unclear whether symmetry breaking also occurs in the case \( p \leq 2^*_1 \). Before stating a partial answer to this question for \( 2 < p < 2^*_1 \), we first
note that symmetry breaking does not occur in the linear case \( p = 2 \). More precisely, we shall observe in Section 4 below that

\[
\mathcal{E}_{\alpha,m,2}(B) = \mathcal{E}_{0,m,2}(B) = \lambda_1(B) + m \quad \text{for all } \alpha \in [0,1], \ m \in \mathbb{R}.
\]

Moreover, every Dirichlet eigenfunction of \((1.6)\) is radial in this linear case. On the other hand, for every \( p \) strictly greater than 2, symmetry breaking occurs for sufficiently large values of the parameter \( m \), as the following result shows.

**Theorem 1.4.** Let \( \alpha \in (0,1) \) and \( 2 < p < 2^* \). Then there exists \( m_0 > 0 \) with the property that \((1.10)\) holds for \( m \geq m_0 \) and therefore every ground state solution of \((1.5)\) is \( x_1\)-\( x_2\)-nonradial for \( m \geq m_0 \).

Next, we discuss the limit case \( \alpha = 1 \) in the minimization problem \((1.6)\). We may study this limit case based on Corollary 1.2, but we need to look for minimizers in a space larger than \( H^1_0(B) \). More precisely, we let \( \mathcal{H} \) be given as the closure of \( C^{1,\alpha}_B(B) \) in

\[
\left\{ u \in L^2(B) : \|u\|^2_{\mathcal{H}} := \int_B \left( |\nabla u|^2 - |\partial_x u|^2 \right) \, dx < \infty \right\}
\]

with respect to the norm \( \| \cdot \|_{\mathcal{H}} \). We then have the following result, which complements Theorems 1.1 and 1.4 in the case \( \alpha = 1 \).

**Theorem 1.5.** Let \( 2 < p < 2^* \) and \( \alpha = 1 \).

(i) For every \( m > -\lambda_1(B) \), there exists a ground state solution of \((1.5)\).

(ii) There exists \( m_0 > 0 \) with the property that \((1.10)\) holds for \( m \geq m_0 \) and therefore every ground state solution \( u \in \mathcal{H} \) of \((1.5)\) is \( x_1\)-\( x_2\)-nonradial for \( m \geq m_0 \).

The critical case \( \alpha = 1, \ p = 2^* \) remains largely open, but we have a partial result on the existence of ground state solutions which relates problem \((1.5)\) to a degenerate Sobolev inequality of the form

\[
\|u\|_{L^{2^*}_p(\mathbb{R}^N_+)} \leq C \left( \int_{\mathbb{R}^N_+} \sum_{i=1}^{N-1} |\partial_i u|^2 + x_1^s |\partial_N u|^2 \, dx \right)^{1/2}
\]

in the half space

\[
\mathbb{R}^N_+ := \{ x \in \mathbb{R}^N : x_1 > 0 \}.
\]

This inequality seems new and of independent interest, and it is the key ingredient in the proof of Theorem 1.1. Our main result related to this half space inequality is the following.

**Theorem 1.6.** Let \( s > 0 \) and set \( 2^*_s := \frac{4N+2s}{2N-4s} \). Then we have

\[
\mathcal{S}_s(\mathbb{R}^N_+) := \inf_{u \in C^{1,\alpha}_B(\mathbb{R}^N_+)} \int_{\mathbb{R}^N_+} \sum_{i=1}^{N-1} |\partial_i u|^2 + x_1^s |\partial_N u|^2 \, dx > 0.
\]

Moreover, the value \( \mathcal{S}_s(\mathbb{R}^N_+) \) is attained in \( H_s \setminus \{0\} \), where \( H_s \) denotes the closure of \( C^{1}_s(\mathbb{R}^N_+) \) in the space

\[
\left\{ u \in L^{2^*_s}(\mathbb{R}^N_+) : \|u\|^2_{H_s} := \int_{\mathbb{R}^N_+} \sum_{i=1}^{N-1} |\partial_i u|^2 + x_1^s |\partial_N u|^2 \, dx < \infty \right\}
\]

with respect to the norm \( \| \cdot \|_{H_s} \).
Here, distributional derivatives are considered in \((1.14)\). Several remarks regarding Theorem 1.6 are in order. First, we point out that the criticality of the exponent \(2^*_s := \frac{4N+2s}{2N-4+s}\) in 1.6 corresponds to the fact that the quotient in \((1.13)\) is invariant under an anisotropic rescaling given by \(u \mapsto u_\lambda\) for \(\lambda > 0\) with \(u_\lambda(x) := u(\lambda x_1, \lambda x_2, \ldots, \lambda x_{N-1}, \lambda^\frac{2}{4} x_N)\). This invariance leads to a lack of compactness, and we have to apply concentration-compactness methods to deduce the existence of minimizers. We further note that the existence of minimizers in the half space problem is in striking contrast to the case \(s = 0\) which is excluded in Theorem 1.6. Indeed, the case \(s = 0\) corresponds to the classical Sobolev inequality which only admits extremal functions in the entire space \(\mathbb{R}^N\).

We have already noted that the case \(s = 1\) in Theorem 1.6 is of key importance in the proof of Theorem 1.1. The more general case \(s \in (0,2]\) arises in a similar way when \((1.5)\) is studied in Riemannian models with boundary in place of \(B\), and we will discuss this case in Section 6 below. We point out that the setting of Riemannian models includes hypersurfaces of revolution with boundary in \(\mathbb{R}^{N+1}\), and that the particular case of a hemisphere corresponds to the case \(s = 2\). The latter is no surprise in view of the recent work of Taylor \([42]\) and Mukherjee \([36, 37]\), who studied the problem of rotating solutions on the unit sphere. In particular, their work relies on degenerate Sobolev embeddings on the unit sphere where also the value \(2^*_s = \frac{2(N+1)}{N-1}\) appears as a critical exponent. In fact, our approach allows to use the case \(s = 2\) in Theorem 1.6 and the corresponding inequality in \(\mathbb{R}^N\) (see Theorem 2.1 below) to give new proofs of these degenerate Sobolev embeddings which does not rely on Fourier analytic and pseudodifferential arguments as in \([42]\).

Next we remark that degenerate Sobolev type inequalities have been studied extensively in the context of Grushin operators which take the form

\[
\mathcal{L} = \Delta_x + c|x|^{2s} \Delta_y
\]

on \(\mathbb{R}^N = \mathbb{R}^m \times \mathbb{R}^k\), where \(x \in \mathbb{R}^m\), \(y \in \mathbb{R}^k\) and \(s > 0\). For a comprehensive survey of the properties of these operators, see e.g. \([22]\). In particular, an associated Sobolev type inequality of the type

\[
\|u\|_{L^\frac{2m+2k(s+1)}{m+k(s+1)-2}(\mathbb{R}^N)} \leq C \left( \int_{\mathbb{R}^N} |\nabla_x u|^2 + c|x|^{2s} |\nabla_y u|^2 \, d(x, y) \right)^{1/2}, \quad u \in C^1_c(\mathbb{R}^N) \quad (1.15)
\]

has been established. Here, the associated critical exponent is related to the homogeneous dimension in the context of more general weighted Sobolev inequalities. We also mention symmetry results for positive entire solutions to semilinear problems involving \(\mathcal{L}\) in \([35]\), as well as the existence of extremal functions on \(\mathbb{R}^N\) shown in \([4]\) and \([34]\).

We point out that the restriction of inequality \((1.15)\) to the half space coincides with the inequality \((1.12)\) in the case \(N = 2\). On the other hand, for \(N \geq 3\), the inequality \((1.12)\) is not associated to a Grushin operator in the sense above. Nonetheless, it is worth noting that for \(m = N - 1\), \(k = 1\) and \(s = \frac{1}{2}\), the critical exponents coincide.

More closely related to Theorem 1.6 in the case \(N \geq 3\) is \([16, \text{Theorem 1.7}]\) where a more general family of Grushin type operators and their associated inequalities has been considered. However, the inequality \((1.12)\) associated to \((1.10)\) is a limit case which is not part of the family of inequalities considered in \([16, \text{Theorem 1.7}]\).

Coming back to the existence of ground state solutions of \((1.5)\) in the critical case \(\alpha = 1\), \(p = 2^*_1\), we state the following result.
Theorem 1.7. If
\[ \mathcal{C}_{1,m,2}^1(\mathcal{B}) < 2^{\frac{1}{2} - \frac{1}{s'}}S_1(\mathbb{R}^N) \] (1.16)
for some \( m > -\lambda_1(\mathcal{B}) \), then the value \( \mathcal{C}_{1,m,2}^1(\mathcal{B}) \) is attained in \( \mathcal{H} \setminus \{0\} \) by a ground state solution of (1.5). Moreover, there exists \( \varepsilon > 0 \) with the property that (1.16) holds for every \( m \in (-\lambda_1(\mathcal{B}), -\lambda_1(\mathcal{B}) + \varepsilon) \).

Here, the factor \( 2^{\frac{1}{2} - \frac{1}{s'}} \) is due to the scaling properties of a more general quotient related to (1.13), see Remark 2.3(ii) below.

The paper is organized as follows. We first study the degenerate Sobolev inequality (1.12) and hence prove Theorem 1.6 in Section 2. This is subsequently used in Section 3 to prove the second part of Theorem 1.1. In Section 4 we then discuss the properties of ground state solutions of (1.5) in detail and give the proofs of Theorems 1.3 and 1.4. This also includes the degenerate case \( \alpha = 1 \) and the proof of Theorem 1.5. Section 5 is then devoted to the properties of rotating waves when \( \mathcal{B} \) is replaced by an annulus. In this case, our methods give rise to an analogue of Theorem 1.1 with more explicit conditions for \( x_1-x_2 \)-nonradiality of ground states. In Section 6 we discuss how the general degenerate Sobolev inequality (1.12) can be used to give an analogue of Theorem 1.1 for Riemannian models. Finally, in the appendix, we prove uniform \( L^\infty \)-bounds for weak solutions of (1.5) in the case \( \alpha = 1 \).

2. A family of degenerate Sobolev inequalities

In this section, we give the proof of Theorem 1.6. More precisely, in the first part of the section, we prove the corresponding degenerate Sobolev inequality
\[ (\int_{\mathbb{R}^N} |u|^{p*} \, dx)^{\frac{2}{p*}} \leq C \int_{\mathbb{R}^N} \left( \sum_{i=1}^{N-1} |\partial_i u|^2 + |x_1|^\alpha |\partial_N u|^2 \right) \, dx \quad \text{for } u \in C^1_c(\mathbb{R}^N) \] (2.1)
in the entire space with a constant \( C > 0 \), from which the positivity of \( S_\alpha(\mathbb{R}^N) \) in (1.13) follows.

In the second part of the section, we then prove the existence of minimizers of the quotient in (1.13) in the larger space \( H_\alpha \) defined in Theorem 1.6.

2.1. Degenerate Sobolev inequality on \( \mathbb{R}^N \). The first step in the proof of (2.1) is the following key inequality.

Lemma 2.1. Let \( \alpha > 0 \) and \( p > 2 \) be given. Then we have
\[ \int_{\mathbb{R}^N} |u|^p \, dx \leq \kappa \left( \int_{\mathbb{R}^N} |x_1|^\alpha |u|^q \, dx \right)^{\frac{1}{q'}} \left( \int_{\mathbb{R}^N} |\partial_1 u|^2 \, dx \right)^{\frac{1}{2}} \quad \text{for } u \in C^1_c(\mathbb{R}^N) \] (2.2)
with
\[ q = \frac{p(2 + \alpha) - 2\alpha}{2} \quad \text{and} \quad \kappa > 0. \] (2.3)

Proof. Let \( u \in C^1_c(\mathbb{R}^N) \). By Hölder’s inequality, we have
\[ \int_{\mathbb{R}^N} |u|^p \, dx \leq \left( \int_{\mathbb{R}^N} |x_1|^\alpha |u|^q \, dx \right)^{\frac{1}{q'}} \left( \int_{\mathbb{R}^N} |x_1|^{-\sigma} |u|^{(p-r)\sigma} \, dx \right)^{\frac{1}{r}} \] (2.4)
for $s > 0$, $\sigma \in (1, \infty)$ and $r \in (0, p)$. It is convenient to write $s = \frac{t}{2}$ and $m = (p - r)\sigma$, then (2.4) becomes
\[
\int_{\mathbb{R}^N} |u|^p \, dx \leq \left( \int_{\mathbb{R}^N} |x|^\frac{t}{p-1} |u|^{p\sigma'} - \frac{m}{\sigma - 1} \, dx \right)^{\frac{1}{p}} \left( \int_{\mathbb{R}^N} |x|^{-t} |u|^m \, dx \right)^{\frac{1}{m}}
\] (2.5)
for $t > 0$, $\sigma \in (1, \infty)$ and $m \in (0, \sigma)$. If, more specifically,
\[
t \in (0, 1), \quad \sigma \in (1, \infty), \quad m \in (1, p\sigma), \quad \tau > 1 \quad \text{and} \quad \theta \in (0, 1),
\] (2.6)
we may integrate by parts and use Hölder’s inequality to get
\[
\int_{\mathbb{R}^N} |x|^{-t} |u|^m \, dx = -\frac{m}{1-t} \int_{\mathbb{R}^N} x |x|^{-t} |u|^{m-1} \partial_1 u \, dx
\]
and choose, specifically,
\[
\tau = \frac{t}{2(1-t)(\sigma - 1)}
\] (2.9)
which satisfies $\tau > 1$ by (2.8) and $2(1-t)\tau = \frac{t}{\sigma - 1}$. Therefore (2.7) reduces to
\[
\int_{\mathbb{R}^N} |x|^{-t} |u|^m \, dx
\] (2.10)
\[
\leq c \left( \int_{\mathbb{R}^N} |x|^{\frac{t}{\sigma - 1}} |u|^{2\theta(m-1)\tau} \, dx \right)^{\frac{1}{2\theta}} \left( \int_{\mathbb{R}^N} |u|^{2(1-\theta)(m-1)\tau'} \, dx \right)^{\frac{1}{2\tau'}} \left( \int_{\mathbb{R}^N} |\partial_1 u|^2 \, dx \right)^{\frac{1}{2}}.
\] (2.7)

We now restrict our attention to values
\[
1 > t > \frac{2\sigma - 2}{2\sigma - 1}
\] (2.8)
and choose, specifically,
\[
\tau = \frac{t}{2(1-t)(\sigma - 1)}
\] (2.9)
which satisfies $\tau > 1$ by (2.8) and $2(1-t)\tau = \frac{t}{\sigma - 1}$. Therefore (2.7) reduces to
\[
\int_{\mathbb{R}^N} |x|^{-t} |u|^m \, dx
\] (2.10)
\[
\leq c \left( \int_{\mathbb{R}^N} |x|^{\frac{t}{\sigma - 1}} |u|^{2\theta(m-1)\tau} \, dx \right)^{\frac{1}{2\theta}} \left( \int_{\mathbb{R}^N} |u|^{2(1-\theta)(m-1)\tau'} \, dx \right)^{\frac{1}{2\tau'}} \left( \int_{\mathbb{R}^N} |\partial_1 u|^2 \, dx \right)^{\frac{1}{2}}.
\] (2.7)

Next we define
\[
m := \frac{p(\sigma - 1)(\tau - 1) + \sigma p - 1}{2\tau(\sigma - 1) + 1} + 1 = \frac{p(\sigma - 1)(\tau - 1) + \sigma p + 2\tau(\sigma - 1)}{2\tau(\sigma - 1) + 1}
\] (2.11)
and
\[
\theta = \frac{(m-1) - \frac{p}{2\tau}}{m - 1}.
\] (2.12)

A short computation shows that these values are chosen such that the conditions
\[
2\theta(m-1)\tau = p\sigma' - \frac{m}{\sigma - 1} \quad \text{and} \quad 2(1-\theta)(m-1)\tau' = p
\] (2.13)
hold for the exponents in (2.10). In order to use the inequalities with these values of $\theta$ and $m$, we have to ensure that these values are admissible in the sense of (2.6). By definition, we have $m > 1$. Moreover, we note that $m < \sigma p$ since
\[
\sigma \geq 1 \geq \frac{1}{2\tau'} + \frac{1}{p}, \quad \text{i.e.,} \quad p(\tau - 1) + 2\tau \leq 2\sigma p\tau,
\]
and hence
\[
p(\sigma - 1)(\tau - 1) + \sigma p + 2\tau(\sigma - 1) \leq \sigma p(2\tau(\sigma - 1) + 1).
\]
Hence $m \in (1, \sigma p)$, as required. Moreover, we have $\theta < 1$ by definition. To see that $\theta > 0$, we note that, since $p > 2$, we have $\tau' > 1 \geq \frac{p}{2(p-1)} \geq \frac{2}{2(\sigma - 1)}$ and therefore

$$2\left(p(\sigma - 1)\tau + \tau'(\sigma p - 1)\right) > p \left(2\tau(\sigma - 1) + 1\right),$$

which shows that

$$2(m - 1)\tau' = \frac{2p(\sigma - 1)\tau + \tau'(\sigma p - 1)}{2\tau(\sigma - 1) + 1} > p.$$

Consequently, $\theta > 0$, and thus $\theta \in (0, 1)$, as required in (2.6). So we may consider these values of $\tau$, $m$ and $\theta$ in (2.5) and (2.10). With (2.13), this yields the inequalities

$$\int_{\mathbb{R}^N} |u|^p \, dx \leq \left( \int_{\mathbb{R}^N} |x_1|^{\frac{\sigma}{\sigma - 1}} |u|^q \, dx \right)^{\frac{1}{q}} \left( \int_{\mathbb{R}^N} |x_1|^{-\tau} |u|^m \, dx \right)^{\frac{1}{m}}$$

and

$$\int_{\mathbb{R}^N} |x_1|^{-\tau} |u|^m \, dx \leq c \left( \int_{\mathbb{R}^N} |x_1|^{\frac{\sigma}{\sigma - 1}} |u|^q \, dx \right)^{\frac{1}{q}} \left( \int_{\mathbb{R}^N} |u|^p \, dx \right)^{\frac{1}{p}} \left( \int_{\mathbb{R}^N} |\partial_1 u|^2 \, dx \right)^{\frac{1}{2}},$$

with

$$q := 2\theta(m - 1)\tau = p\sigma' - \frac{m}{\sigma - 1}. \tag{2.14}$$

Combining these inequalities yields

$$\int_{\mathbb{R}^N} |u|^p \, dx \leq c \left( \int_{\mathbb{R}^N} |x_1|^{\frac{\sigma}{\sigma - 1}} |u|^q \, dx \right)^{\frac{1}{q}} \left( \int_{\mathbb{R}^N} |u|^p \, dx \right)^{\frac{1}{p}} \left( \int_{\mathbb{R}^N} |\partial_1 u|^2 \, dx \right)^{\frac{1}{2}},$$

and therefore

$$\int_{\mathbb{R}^N} |u|^p \, dx \leq \left( \int_{\mathbb{R}^N} |x_1|^{\frac{\sigma}{\sigma - 1}} |u|^q \, dx \right)^{\frac{1}{q}} \left( \int_{\mathbb{R}^N} |\partial_1 u|^2 \, dx \right)^{\frac{1}{2}}. \tag{2.15}$$

To obtain (2.2), it is convenient to set $\alpha := \frac{\tau}{\sigma - 1} > 0$, noting that the admissibility condition (2.8) translates to

$$\frac{1}{\sigma - 1} > \alpha > \frac{2}{2\sigma - 1}. \tag{2.16}$$

Note that, if $\alpha > 0$ is given, we always find $\sigma \in (1, \infty)$ with the property that (2.16) holds. Moreover, the exponents in (2.15) then satisfy

$$\frac{\tau}{2\sigma \tau - \tau + 1} = \frac{\alpha}{2 + \alpha}, \quad \frac{2\sigma \tau - 2\tau + 1}{2\sigma \tau - \tau + 1} = \frac{2}{2 + \alpha},$$

so (2.15) becomes

$$\int_{\mathbb{R}^N} |u|^p \, dx \leq c \left( \int_{\mathbb{R}^N} |x_1|^\alpha |u|^q \, dx \right)^{\frac{2}{2 + \alpha}} \left( \int_{\mathbb{R}^N} |\partial_1 u|^2 \, dx \right)^{\frac{2\alpha}{2 + \alpha}}. \tag{2.8}$$

This is already the inequality in (2.2). So it only remains to show that the two definitions of $q$ given in (2.14) and (2.14) are consistent, i.e., we have the identity

$$2\theta(m - 1)\tau = \frac{p(2 + \alpha) - 2\alpha}{2}.$$

The latter follows by a somewhat tedious but straightforward computation, so the proof of the lemma is complete.

We may now complete the proof of the main result of this section, given as follows.
Theorem 2.2. Let $s > 0$ and $2^*_s = \frac{4N + 2s}{2N - 4 + s}$ as in Theorem 1.6. Then inequality (2.1) holds with some constant $C > 0$.

We remark that this may be proven by combining the previous results with a suitable adaption of the inequality on the halfspace given in [16, Theorem 1.7] to the setting of the entire space $\mathbb{R}^N$. For the convenience of the reader, we give a self-contained proof.

Proof. In the following, the letter $c > 0$ stands for a constant which may change from line to line. Let $\alpha = \frac{s}{2(N-1)}$. Then Lemma 2.1 yields

$$\int_{\mathbb{R}^N} |u|^{2q_s} \, dx \leq \kappa \left( \int_{\mathbb{R}^N} |x_1|^{\alpha |u|^{q_s} \, dx} \right)^{\frac{2}{2 + \alpha}} \left( \int_{\mathbb{R}^N} \left| \partial_1 u \right|^2 \, dx \right)^{\frac{\alpha}{2 + \alpha}}$$

with $q_s := \frac{N(2^*_s + 2)}{2(N-1)}$. To estimate the term $\int_{\mathbb{R}^N} |x_1|^{\alpha |u|^{q_s} \, dx}$, we define, for $i = 1, \ldots, N$, the functions $a_i \in C_c(\mathbb{R}^{N-1})$ by

$$a_i(x_i) := \int_{\mathbb{R}} |u|^{\frac{q(N-1)}{N} - 1} \left| \partial_i u \right| \, dx_i$$

where

$$\hat{x}_i := (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N) \in \mathbb{R}^{N-1} \quad \text{for} \quad x \in \mathbb{R}^N \quad \text{and} \quad i = 1, \ldots, N.$$ Integrating the derivative $\partial_i |u|^{q(N-1)/N}$ in the $x_i$-direction, we find that $|u(x)|^{q(N-1)/N} \leq ca_i(\hat{x}_i)$ for all $x \in \mathbb{R}^N$, $i = 1, \ldots, N$ and therefore

$$|u(x)|^{q(N-1)/N} \leq c \prod_{i=1}^N a_i(\hat{x}_i) \quad \text{for} \quad x \in \mathbb{R}^N.$$

Applying Gagliardo’s Lemma [18, Lemma 4.1] to the functions $\frac{1}{a_1 \cdots a_N}$ and $x \mapsto |x_1|^\alpha a_N^{1/(N-1)}(x)$, we thus find that

$$\int_{\mathbb{R}^N} |x_1|^\alpha |u|^{q_s} \, dx \leq c \left( \int_{\mathbb{R}^{N-1}} |x_1|^{(N-1)\alpha} a_N(\hat{x}_N) \prod_{i=1}^{N-1} \int_{\mathbb{R}^{N-1}} a_i(\hat{x}_i) \, d\hat{x}_i \right)^{\frac{1}{N-1}}$$

$$= c \left( \int_{\mathbb{R}^N} |x_1|^{\frac{q(N-1)}{N} - 1} |\partial_N u| \, dx \prod_{i=1}^{N-1} \int_{\mathbb{R}^N} |u|^{q(N-1)/N} \left| \partial_i u \right| \, dx \right)^{\frac{1}{N-1}}$$

$$\leq c \left( \int_{\mathbb{R}^N} |u|^{2q(N-1)/N - 2} \, dx \right)^{\frac{2}{2(N-1)}} \left( \int_{\mathbb{R}^N} |x_1|^\alpha |\partial_N u|^2 \, dx \prod_{i=1}^{N-1} \int_{\mathbb{R}^N} |\partial_i u|^2 \, dx \right)^{\frac{N-1}{2(N-1)}}.$$

Since $\frac{2(N-1)q_s}{N} - 2 = 2^*_s$, we conclude that

$$\int_{\mathbb{R}^N} |u|^{2q_s} \, dx \leq c \left( \int_{\mathbb{R}^N} |x_1|^\alpha |u|^{q_s} \, dx \right)^{\frac{2}{2 + \alpha}} \left( \int_{\mathbb{R}^N} \left| \partial_1 u \right|^2 \, dx \right)^{\frac{\alpha}{2 + \alpha}}$$

$$\leq c \left( \int_{\mathbb{R}^N} |u|^{2q_s} \, dx \right)^{\frac{2}{2(N-1)}} \left( \int_{\mathbb{R}^N} |x_1|^\alpha |\partial_N u|^2 \, dx \prod_{i=1}^{N-1} \int_{\mathbb{R}^N} |\partial_i u|^2 \, dx \right)^{\frac{1}{N-1}}$$

$$\times \left( \int_{\mathbb{R}^N} \left| \partial_1 u \right|^2 \, dx \right)^{\frac{\alpha}{2 + \alpha}}.$$

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\[ = c \left( \int_{\mathbb{R}^N} |u|^{2^*} \, dx \right) \frac{1}{2(N-1)+2} \left( \int_{\mathbb{R}^N} |x_1| |\partial_N u|^2 \, dx \prod_{i=2}^{N-1} \int_{\mathbb{R}^N} |\partial_i u|^2 \, dx \right)^{\frac{1}{2(N-1)+2}} \]

and therefore
\[ \left( \int_{\mathbb{R}^N} |u|^{2^*} \, dx \right)^{\frac{N-2+\frac{2}{p}}{2(N-1)+2}} \leq c \left( \int_{\mathbb{R}^N} |x_1| |\partial_N u|^2 \, dx \prod_{i=2}^{N-1} \int_{\mathbb{R}^N} |\partial_i u|^2 \, dx \right)^{\frac{N}{2(N-1)+2}} \left( \int_{\mathbb{R}^N} |\partial_1 u|^2 \, dx \right)^{\frac{1+\frac{2}{p}}{2(N-1)+2}}. \]

Finally, Young’s inequality gives
\[ \left( \int_{\mathbb{R}^N} |u|^{2^*} \, dx \right)^{\frac{2}{2^*}} \leq c \left( \int_{\mathbb{R}^N} |x_1| |\partial_N u|^2 \, dx \prod_{i=2}^{N-1} \int_{\mathbb{R}^N} |\partial_i u|^2 \, dx \right)^{\frac{N}{2(N-1)+2}} \left( \int_{\mathbb{R}^N} |\partial_1 u|^2 \, dx \right)^{\frac{2}{2^*}}, \]

\[ \leq c \left( \int_{\mathbb{R}^N} |x_1| |\partial_N u|^2 \, dx + \sum_{i=1}^{N-1} \int_{\mathbb{R}^N} |\partial_i u|^2 \, dx \right). \]

\[ \square \]

In particular, this implies
\[ S_\lambda(\mathbb{R}^N_+) = \inf_{u \in C^1_c(\mathbb{R}^N_+)} \frac{\int_{\mathbb{R}^N_+} \sum_{i=1}^{N-1} |\partial_i u|^2 + x_1^* |\partial_N u|^2 \, dx}{\left( \int_{\mathbb{R}^N_+} |u|^{2^*} \, dx \right)^{\frac{2}{2^*}}} > 0 \]

and thus the first part of Theorem 1.6.

**Remark 2.3. (Optimality and Variants)**

(i) The exponent 2^*_s in (1.13) is optimal in the sense that
\[ \inf_{u \in C^1_c(\mathbb{R}^N)} \frac{\int_{\mathbb{R}^N} \left( \sum_{i=1}^{N-1} |\partial_i u|^2 + |x_1|^s |\partial_N u|^2 \right) \, dx}{\|u\|^{2^*_s}_{L^p(\mathbb{R}^N)}} = 0 \quad \text{for } p \neq 2^*_s. \quad (2.17) \]

This follows by considering the rescaling \( u \mapsto u_\lambda, \lambda > 0 \) with
\[ u_\lambda(x) := u(\lambda x_1, \lambda x_2, \ldots, \lambda x_{N-1}, \lambda^{1+\frac{2}{p}} x_N). \]

Indeed, for \( u \in C^1_c(\mathbb{R}^N) \) we have
\[ \int_{\mathbb{R}^N_+} \left( \sum_{i=1}^{N-1} |\partial_i u_\lambda|^2 + x_1^* |\partial_N u_\lambda|^2 \right) \, dx = \lambda^{\frac{2N+4-4}{2}} \int_{\mathbb{R}^N_+} \left( \sum_{i=1}^{N-1} |\partial_i v|^2 + x_1^* |\partial_N u|^2 \right) \, dx \]
and, for \( 1 < p < \infty, \)
\[ \left( \int_{\mathbb{R}^N_+} |u_\lambda|^p \, dx \right)^{\frac{2}{p}} = \lambda^{-\frac{2}{p}(N+\frac{2}{p})} \left( \int_{\mathbb{R}^N_+} |u|^p \, dx \right)^{\frac{2}{p}}. \]

Since \( \frac{2N+4-4}{2} = \frac{2}{p}(N+\frac{2}{p}) \) if and only if \( p = 2^*_s, \) (2.17) follows.
(ii) For \( \kappa > 0, u \in C^1_c(\mathbb{R}^N) \), we may consider a rescaled function of the form
\[
v(x) = u \left( x_1, \ldots, x_{N-1}, \frac{x_n}{\sqrt{\kappa}} \right).
\]
Comparing the associated quotients then yields
\[
\inf_{u \in C^1_c(\mathbb{R}^N)} \frac{\int_{\mathbb{R}^N} \left( \sum_{i=1}^{N-1} |\partial_i u|^2 + \kappa |x_1|^4 |\partial_N u|^2 \right) dx}{\|u\|_{L^{2s}^\ast(\mathbb{R}^N)}^2} = \kappa^{\frac{1}{2^\ast} - \frac{1}{2s}} S_s(\mathbb{R}^N_+).
\]  
(2.18)

In the special case \( \kappa = 2 \), this quotient will appear later when we connect \( \mathcal{C}_{1,m,2^\ast}(\mathcal{B}) \) and \( S_1(\mathbb{R}^N_+) \), in particular in the proof of Theorem 1.7.

Recalling the space \( H_s \) defined in Theorem 1.6, we see that Theorem 2.2 immediately implies that \( H_s \) is continuously embedded into \( L^{2s}_\ast(\mathbb{R}^N_+) \).

2.2. Existence of minimizers. In the following, we fix \( s > 0 \) and study minimizing sequences for
\[
S := S_s(\mathbb{R}^N_+) = \inf_{u \in H_s \setminus \{0\}} \frac{\int_{\mathbb{R}^N} \left( \sum_{i=1}^{N-1} |\partial_i u|^2 + \kappa |x_1|^4 |\partial_N u|^2 \right) dx}{(\int_{\mathbb{R}^N} |u|^{2s} dx)^{\frac{1}{2s}}} > 0.
\]

First, consider the following classical lemma due to Lions [29], which we give in the form presented in [40]:

**Lemma 2.4. (Concentration-Compactness Lemma)**

Suppose \( (\mu_n)_n \) is a sequence of probability measures on \( \mathbb{R}^N \). Then, after passing to a subsequence, one of the following three conditions holds:

(i) **(Compactly)** There exists a sequence \( (x_n)_n \subset \mathbb{R}^N \) such that for any \( \varepsilon > 0 \) there exists \( R > 0 \) such that
\[
\int_{B_R(x_n)} d\mu_n \geq 1 - \varepsilon.
\]

(ii) **(Vanishing)** For all \( R > 0 \) it holds that
\[
\lim_{n \to \infty} \left( \sup_{x \in \mathbb{R}^N} \int_{B_R(x)} d\mu_n \right) = 0.
\]

(iii) **(Dichotomy)** There exists \( \lambda \in (0, 1) \) such that for any \( \varepsilon > 0 \) there exists \( R > 0 \) and \( (x_n)_n \subset \mathbb{R}^N \) with the following property: Given \( R' > R \) there are nonnegative measures \( \mu_n^1, \mu_n^2 \) such that
\[
0 \leq \mu_n^1 + \mu_n^2 \leq \mu_n
\]
\[
\text{supp} \mu_n^1 \subset B_R(x_n), \quad \text{supp} \mu_n^2 \subset \mathbb{R}^N \setminus B_{R'}(x_n)
\]
\[
\limsup_{n \to \infty} \left( |\lambda - \int_{\mathbb{R}^N} d\mu_n^1| + |(1 - \lambda) - \int_{\mathbb{R}^N} d\mu_n^2| \right) \leq \varepsilon.
\]

A characterization of minimizing sequences in the sense of measures is given in the following lemma, which is a straightforward adaption of [40, Lemma 4.8]:
Lemma 2.5. (Concentration-Compactness Lemma II)

Let \( s > 0 \) and suppose \( u_n \to u \) in \( H_s \) and \( \mu_n := \left( \sum_{i=1}^{N-1} |\partial_i u_n|^2 + x_1^4 |\partial_N u_n|^2 \right) dx \to \mu \), \( \nu_n := |u_n|^{2s} dx \to \nu \) weakly in the sense of measures where \( \mu \) and \( \nu \) are finite measures on \( \mathbb{R}^N \). Then:

(i) There exists an at most countable set \( J \), a set \( \{x^j : j \in J\} \subset \mathbb{R}^N_+ \) and \( \{\nu^j : j \in J\} \subset (0, \infty) \) such that

\[
\nu = |u|^{2s} dx + \sum_{j \in J} \nu^j \delta_{x^j}.
\]

(ii) There exists a set \( \{\mu^j : j \in J\} \subset (0, \infty) \) such that

\[
\mu \geq \left( \sum_{i=1}^{N-1} |\partial_i u|^2 + x_1^4 |\partial_N u|^2 \right) dx + \sum_{j \in J} \mu^j \delta_{x^j}
\]

where

\[
S(\nu^j)^{\frac{2}{s}} \leq \mu^j
\]

for \( j \in J \). In particular, \( \sum_{j \in J} (\nu^j)^{\frac{2}{s}} < \infty \).

Our main result then states that \( S \) is attained in \( H_s \) and completes the proof of Theorem 1.6.

Theorem 2.6. Let \( s > 0 \) and suppose \( (u_n) \) is a minimizing sequence for

\[
S = \inf_{u \in H_s \setminus \{0\}} \frac{\int_{\mathbb{R}^N_+} \left( \sum_{i=1}^{N-1} |\partial_i u|^2 + x_1^4 |\partial_N u|^2 \right) dx}{\left( \int_{\mathbb{R}^N_+} |u|^{2s} dx \right)^{\frac{2}{s}}}
\]

with \( \|u_n\|_{L^{2s}} = 1 \). Then, up to translations orthogonal to \( x_1 \) and anisotropic scaling, \( (u_n) \) is relatively compact in \( H_s \).

Proof. For \( r > 0 \) we define the family of rectangles

\[ Q_r := \left\{ (0, r^2) \times \left( y + (-r^2, r^2)^{N-2} \times (-r^{2+s}, r^{2+s}) \right) : y \in \mathbb{R}^{N-1} \right\} \]

It is important to note that for \( R > 0 \), with respect to the transformation

\[ \tau_R(x) = (R^2 x_1, R^2 x_2, \ldots, R^2 x_{N-1}, R^{2+s} x_N) \]

these sets satisfy

\[ \tau_R(Q_r) = Q_{rR} \]

Moreover, the functions

\[ Q_n(r) := \sup_{E \in Q_r} \int_E |u_n|^{2s} dx \]

are continuous on \( [0, \infty) \) and satisfy

\[ \lim_{r \to 0} Q_n(r) = 0, \quad \lim_{r \to \infty} Q_n(r) = 1. \]

Hence we may choose \( A_n > 0, y_n \in \mathbb{R}^{N-1} \) such that the rescaled sequence \( v_n \in H_s \) given by

\[ v_n(x) := A_n^{\frac{N-4+s}{2}} u_n(A_n^2 x_1, A_n^2 (x_2 + (y_n)_1), \ldots, A_n^{2+s}(x_N + (y_n)_{N-1}) \]

is a minimizing sequence for

\[
S_n(r) := \inf_{u \in H_s \setminus \{0\}} \frac{\int_{\mathbb{R}^N_+} \left( \sum_{i=1}^{N-1} |\partial_i u|^2 + x_1^4 |\partial_N u|^2 \right) dx}{\left( \int_{\mathbb{R}^N_+} |u|^{2s} dx \right)^{\frac{2}{s}}}
\]

with \( \|u_n\|_{L^{2s}} = 1 \). Then, up to translations orthogonal to \( x_1 \) and anisotropic scaling, \( (v_n) \) is relatively compact in \( H_s \).
satisfies
\[ Q_n(1) = \sup_{E \subseteq \Omega} \int_E |v_n|^2 dx = \int_{(0,1) \times (-1,1)^{N-1}} |v_n|^2 dx = \frac{1}{2}. \]

After passing to a subsequence, we may assume \( v_n \to v \) in \( H_\sigma \) and in \( L^{2^*}(\mathbb{R}^N) \). We now consider the measures
\[ \mu_n := \left( \sum_{i=1}^{N-1} |\partial_i v_n|^2 + x_i^1 |\partial_N v_n|^2 \right) dx, \quad \nu_n := |v_n|^2 dx \]
and apply Lemma 2.4 to \((\nu_n)_n\), where we note that \( \mu_n \) and \( \nu_n \) are initially measures on \( \mathbb{R}^N_+ \) but can trivially be extended to \( \mathbb{R}^N \). By our normalization, vanishing cannot occur. We assume that we have dichotomy and thus let \( \lambda \in (0,1) \) be as in Lemma 2.4(iii). Then, considering a sequence \( \varepsilon_n \downarrow 0 \), for any \( n \in \mathbb{N} \) there exist \( R_n > 0 \), \( x_n \in \mathbb{R}^N_+ \) as well as nonnegative measures \( \nu_n^1, \nu_n^2 \) on \( \mathbb{R}^N_+ \) such that
\[ 0 \leq \nu_n^1 + \nu_n^2 \leq \nu_n \]
\[ \text{supp} \nu_n^1 \subset \mathbb{R}^N_+ \cap B_{R_n}(x_n), \quad \text{supp} \nu_n^2 \subset \mathbb{R}^N_+ \setminus B_{\frac{2R_n}{2R_n+1}}(x_n) \]
\[ \lambda - \int_{\mathbb{R}^N_+} d\nu_n^1 + (1 - \lambda) - \int_{\mathbb{R}^N_+} d\nu_n^2 \leq 2\varepsilon_n \]
and thus
\[ \limsup_{n \to \infty} \left( \lambda - \int_{\mathbb{R}^N_+} d\nu_n^1 + (1 - \lambda) - \int_{\mathbb{R}^N_+} d\nu_n^2 \right) = 0. \]

From the proof of the Lemma 2.4 (see [40]) we can assume \( R_n \to \infty \) and, in particular, \( R_n \geq 1 \).

For \( r > 0 \), let the anisotropic scaling \( \tau_r \) be defined as in (2.19). We crucially note that
\[ B_{R_1}(0) \subset \tau_{\sqrt{\mu_n}}(B_1(0)) \]
and
\[ \mathbb{R}^N_+ \setminus B_{\frac{2R_n}{2R_n+1}}(0) \subset \mathbb{R}^N_+ \setminus \tau_{\sqrt{\mu_n}}(B_2(0)). \]
We take \( \varphi \in C_c^\infty(B_2(0)) \) with \( 0 \leq \varphi \leq 1 \) and \( \varphi \equiv 1 \) in \( B_1(0) \). For \( n \in \mathbb{N} \), let \( \varphi_n(x) := \varphi(\tau_{\sqrt{\mu_n}}(x - x_n)) \), so that
\[ \varphi_n \equiv 1 \quad \text{on} \quad x_n + \tau_{\sqrt{\mu_n}}(B_1(0)), \quad \varphi_n \equiv 0 \quad \text{on} \quad \mathbb{R}^N \setminus (x_n + \tau_{\sqrt{\mu_n}}(B_2(0))), \]
and thus, in particular,
\[ \varphi_n \equiv 1 \quad \text{on} \quad \text{supp } \nu_n^1, \quad \varphi_n \equiv 0 \quad \text{on} \quad \text{supp } \nu_n^2. \]

Note that
\[ |\partial_1 v_n|^2 + x_i^1 |\partial_2 v_n|^2 \geq \left( |\partial_1 v_n|^2 + x_i^1 |\partial_2 v_n|^2 \right) \left( \varphi_n^2 + (1 - \varphi_n)^2 \right). \]
We have
\[ \left( \int_{\mathbb{R}^N_+} \left( \sum_{i=1}^{N-1} |\partial_i (\varphi_i v_n)|^2 + x_i^1 |\partial_N (\varphi_i v_n)|^2 \right) dx \right)^{\frac{1}{2}} \]
\[ \leq \left( \int_{\mathbb{R}^N_+} \varphi_n^2 \left( \sum_{i=1}^{N-1} |\partial_i v_n|^2 + x_i^1 |\partial_N v_n|^2 \right) dx \right)^{\frac{1}{2}} + \left( \int_{\mathbb{R}^N_+} \nu_n^2 \left( \sum_{i=1}^{N-1} |\partial_i \varphi_n|^2 + x_i^1 |\partial_N \varphi_n|^2 \right) dx \right)^{\frac{1}{2}} \]
and analogously for \((1 - \varphi_n)\) instead of \(\varphi_n\). Squaring and adding these estimates gives

\[
\|\varphi_nv_n\|_{H^1}^2 + \|(1 - \varphi_n)v_n\|_{H^1}^2 \\
\leq \int_{\mathbb{R}^N} \left( \sum_{i=1}^{N-1} |\partial_iv_n|^2 + x_i^s|\partial_N\varphi_n|^2 \right) dx + 2\int_{\mathbb{R}^N} v_n^2 \left( \sum_{i=1}^{N-1} |\partial_i\varphi_n|^2 + x_i^s|\partial_N\varphi_n|^2 \right) dx \\
+ 4 \left( \int_{\mathbb{R}^N} \left( \sum_{i=1}^{N-1} |\partial_iv_n|^2 + x_i^s|\partial_N\varphi_n|^2 \right) dx \right)^\frac{1}{2} \left( \int_{\mathbb{R}^N} v_n^2 \left( \sum_{i=1}^{N-1} |\partial_i\varphi_n|^2 + x_i^s|\partial_N\varphi_n|^2 \right) dx \right)^\frac{1}{2}.
\]

Setting

\[
\beta_n := 2\int_{\mathbb{R}^N} v_n^2 \left( \sum_{i=1}^{N-1} |\partial_i\varphi_n|^2 + x_i^s|\partial_N\varphi_n|^2 \right) dx \\
+ 4 \left( \int_{\mathbb{R}^N} \left( \sum_{i=1}^{N-1} |\partial_iv_n|^2 + x_i^s|\partial_N\varphi_n|^2 \right) dx \right)^\frac{1}{2} \left( \int_{\mathbb{R}^N} v_n^2 \left( \sum_{i=1}^{N-1} |\partial_i\varphi_n|^2 + x_i^s|\partial_N\varphi_n|^2 \right) dx \right)^\frac{1}{2}
\]

we thus have

\[
\int_{\mathbb{R}^N} \left( \sum_{i=1}^{N-1} |\partial_i\varphi_n|^2 + x_i^s|\partial_N\varphi_n|^2 \right) dx \geq \|\varphi_nv_n\|_{H^1}^2 + \|(1 - \varphi_n)v_n\|_{H^1}^2 - \beta_n.
\]

Next, we define the anisotropic annulus

\[
A_n := x + \tau\sqrt{\text{Re}}(B_2(0)) \setminus \tau\sqrt{\text{Re}}(B_1(0))
\]

and consider \(\delta > 0\). Using Young’s inequality and the fact that any derivative of \(\varphi_n\) vanishes outside of \(A_n\), we can estimate

\[
\beta_n \leq \delta \int_{A_n} \left( \sum_{i=1}^{N-1} |\partial_i\varphi_n|^2 + x_i^s|\partial_N\varphi_n|^2 \right) dx + C(\delta) \int_{A_n} v_n^2 \left( \sum_{i=1}^{N-1} |\partial_i\varphi_n|^2 + x_i^s|\partial_N\varphi_n|^2 \right) dx.
\]

Note that

\[
\sum_{i=1}^{N-1} |\partial_i\varphi_n|^2 + x_i^s|\partial_N\varphi_n|^2 = R_n^{-2} \sum_{i=1}^{N-1} ||\partial_i\varphi|||\tau_n(x)||^2 + x_i^sR_n^{-2-s}||\partial_N\varphi|||\tau_n(x)||^2
\]

\[
= R_n^{-2} \left( \sum_{i=1}^{N-1} ||\partial_i\varphi||^2 + (\cdot)^s||\partial_N\varphi||^2 \right) \circ \tau^{-1}\sqrt{\text{Re}},
\]

and thus

\[
\sum_{i=1}^{N-1} |\partial_i\varphi_n|^2 + x_i^s|\partial_N\varphi_n|^2 \leq CR_n^{-2}
\]

for some \(C > 0\) independent of \(n\). This gives

\[
\int_{A_n} v_n^2 \left( \sum_{i=1}^{N-1} |\partial_i\varphi_n|^2 + x_i^s|\partial_N\varphi_n|^2 \right) dx \leq CR_n^{-2}\|v_n\|_{L^2(A_n)}^2.
\]

Using Hölder’s inequality then further yields

\[
R_n^{-1}\|v_n\|_{L^2(A_n)} \leq R_n^{-1}|A_n|^{\frac{1}{2N+1}}\|v_n\|_{L^2(A_n)} \leq C\|v_n\|_{L^2(A_n)}.
\]
\[ \leq C \left( \int_{\mathbb{R}^N} d\nu_n - \left( \int_{\mathbb{R}^N} d\nu_n^1 + \int_{\mathbb{R}^N} d\nu_n^2 \right) \right)^{\frac{1}{2}} \rightarrow 0 \]
as \(n \to \infty\). Here we used
\[ |A_n| = |\tau_{\sqrt{R_n^+}}(B_2(x_n))| - |\tau_{\sqrt{R_n^+}}(B_1(x_n))| = R_n^{\frac{2N+2}{2}} (|B_2(0)| - |B_1(0)|). \]
Overall, we find that, for any \(\delta > 0\),
\[ \limsup_{n \to \infty} \beta_n \leq \delta \sup_n \|v_n\|_{H^s}^2, \]
and since \((v_n)_n\) remains bounded in \(H_s\), we conclude
\[ \int_{\mathbb{R}^N} \left( \sum_{i=1}^{N-1} |\partial_i v_n|^2 + x_1^\alpha |\partial_N v_n|^2 \right) dx \geq \|\varphi v_n\|_{H_s}^2 + \|(1 - \varphi)v_n\|_{H_s}^2 - \beta_n \]
\[ \geq S \left( \|\varphi v_n\|_{L^2_s(\mathbb{R}^N)}^2 + \|(1 - \varphi)v_n\|_{L^2_s(\mathbb{R}^N)}^2 \right) + o(1) \]
\[ \geq S \left( \int_{B_{R_n}(x_n)} d\nu_n \right)^{\frac{2}{s'}} + \left( \int_{\mathbb{R}^N \setminus B_{R_n}(x_n)} d\nu_n \right)^{\frac{2}{s'}} + o(1) \geq S \left( \lambda_s^{\frac{2}{s'}} + (1 - \lambda)^{\frac{2}{s'}} \right) + o(1). \]
But since \(\lambda \in (0, 1)\), we have \(\lambda_s^{\frac{2}{s'}} + (1 - \lambda)^{\frac{2}{s'}} > 1\) and thus
\[ S = \lim_{n \to \infty} \int_{\mathbb{R}^N} \left( \sum_{i=1}^{N-1} |\partial_i v_n|^2 + x_1^\alpha |\partial_N v_n|^2 \right) dx \]
\[ \geq \liminf_{n \to \infty} \left( S \left( \lambda_s^{\frac{2}{s'}} + (1 - \lambda)^{\frac{2}{s'}} \right) + o(1) \right) > S, \]
a contradiction. Hence we cannot have dichotomy.
Since we are therefore in case (i) of the Lemma 2.4, there exists a sequence \((x_n)_n\) such that for any \(\varepsilon > 0\) there exists \(R = R(\varepsilon) > 0\) with
\[ \int_{B_R(x_n)} d\nu_n \geq 1 - \varepsilon. \]
Since we normalized so that
\[ \int_{(0,1) \times (-1,1)^{N-1}} |v_n|^{2\ast} dx = \frac{1}{2}, \]
we must have \((0, 1) \times (-1, 1)^{N-1} \cap B_R(x_n) \neq \emptyset\) if \(\varepsilon < \frac{1}{R}\). By making \(R\) larger if necessary, we can thus assume
\[ \int_{B_R(0)} d\nu_n \geq 1 - \varepsilon. \]
In particular, we may therefore pass to a subsequence such that \(\nu_n \rightharpoonup \nu\) weakly in the sense of measure, where \(\nu\) is a finite measure on \(\mathbb{R}^N\). By weak lower (and upper) semicontinuity (of measures), we then have
\[ \int_{\mathbb{R}^N} d\nu = 1. \]
By Lemma 2.5 we may now assume
\[ \mu_n \to \mu \geq \sum_{i=1}^{N-1} \left( |\partial v|^2 + x^i N v^2 \right) dx + \sum_{j \in J} \mu^j \delta_{x^j} \quad \text{and} \quad \nu_n \to |v|^2 dx + \sum_{j \in J} \nu^j \delta_{x^j} \]
for points \( x^j \in \mathbb{R}^N_+ \) and positive \( \mu^j, \nu^j \) satisfying \( S(\nu^j) \frac{1}{2^*} \leq \mu^j \). We have
\[
S + o(1) = \|\nu_n\|_{H^s}^2 = \int_{\mathbb{R}^N_+} d\nu_n \geq \int_{\mathbb{R}^N_+} d\mu + o(1) \geq \|v\|_{H^s}^2 + \sum_{j \in J} \mu^j + o(1)
\]
\[ \geq S \left( \|v\|_{L^{2^*}(\mathbb{R}^N_+)}^2 + \sum_{j} \nu^j \right) + o(1) \]
\[ \geq S \left( \|v\|_{L^{2^*}(\mathbb{R}^N_+)}^2 + \sum_{j} \nu^j \right) + o(1) \]
\[ = S \left( \int_{\mathbb{R}^N} d\nu \right)^{\frac{2}{2^*}} + o(1) = S + o(1) \quad (2.20) \]
as \( n \to \infty \). In the second inequality, we used the fact that the map \( t \mapsto t^{\frac{2}{2^*}} \) is strictly concave and hence subadditive. Moreover, the strict concavity implies that equality can only hold, if at most one of the terms \( \|v\|_{L^{2^*}(\mathbb{R}^N_+)}^2 \) and \( \nu^j \), \( j \in J \) is nonzero.

**Claim:** \( \nu^j = 0 \) for all \( j \).

Assuming that this is false, we have \( \nu_n \to \delta_{x^j} \) for some \( x^1 \in \overline{\mathbb{R}^N_+} \). By our normalization and weak lower semicontinuity (of measures), \( x^1 \not\in Q := (0,1) \times (-1,1)^{N-1} \) since
\[ \delta_{x^1}(Q) \leq \liminf_{n \to \infty} \nu_n(Q) = \frac{1}{2}. \]
Moreover, if \( \text{dist}(x^1, Q) > 0 \), there exists \( \varepsilon > 0 \) such that \( B_\varepsilon(x^1) \cap Q \neq \emptyset \) and thus
\[ 1 = \delta_{x^1}(B_\varepsilon(x^1)) \leq \liminf_{n \to \infty} \nu_n(B_\varepsilon(x^1)) \leq \frac{1}{2}, \]
which is a contradiction. Hence it only remains to consider the case \( x^1 \in \partial Q \). Due to the normalization
\[ \sup_{E \in Q_1} \int_E |v_n|^2 dx = \int_{(0,1) \times (-1,1)^{N-1}} |v_n|^2 dx = \frac{1}{2}, \]
we have \( x^1 \not\in \{(0,y) + Q \} \) for all \( y \in \mathbb{R}^{N-1} \), so \( x^1 \) must be of the form \( x^1 = (1,y) \) or \( (0,y) \) for some \( y \in (-1,1)^{N-1} \). The latter case can be excluded, since, for \( \varepsilon \in (0,\frac{1}{2}) \),
\[ \delta_{x^1}(B_\varepsilon(0,y)) \leq \liminf_{n \to \infty} \nu_n(B_\varepsilon(0,y)) \leq \liminf_{n \to \infty} \nu_n((0,y) + Q) \leq \frac{1}{2}. \]
After a translation orthogonal to the \( x_1 \)-direction, we may therefore assume \( x^1 = (1,0,\ldots,0) \) and first note that \( v \equiv 0 \) and hence \( \mu \geq S \delta_{x^1} \) by (2.20). On the other hand,
\[ \int_{\mathbb{R}^N} d\mu \leq \liminf_{n \to \infty} \int_{\mathbb{R}^N} d\mu_n = S, \]
whence we conclude \( \mu = S \delta_{x^1} \).
For any $0 < \delta < \frac{1}{2}$, $B_\delta := B_\delta(x_1)$ is a continuity set of $\nu = \delta_{x_1}$, hence

$$\nu_n(B_\delta) \to 1$$

and similarly

$$\mu_n(B_\delta) \to S$$

as $n \to \infty$. In particular, for fixed $\varepsilon > 0$ we find $n_0 = n_0(\varepsilon, \delta)$ such that

$$\int_{B_\delta} |v_n|^{2^*} \, dx \geq 1 - \varepsilon, \quad S - \varepsilon \leq \int_{B_\delta} \left( \sum_{i=1}^{N-1} |\partial_i v_n|^2 + x_1^s |\partial_N v_n|^2 \right) \, dx \leq S + \varepsilon$$

for $n \geq n_0$. Furthermore,

$$\frac{1}{1+\delta} \int_{B_\delta} \left( \sum_{i=1}^{N-1} |\partial_i v_n|^2 + x_1^s |\partial_N v_n|^2 \right) \, dx \leq \int_{B_\delta} \sum_{i=1}^{N} |\partial_i v_n|^2 \, dx$$

and

$$\int_{B_\delta} \sum_{i=1}^{N} |\partial_i v_n|^2 \, dx \leq \frac{1}{1-\delta} \int_{B_\delta} \left( \sum_{i=1}^{N-1} |\partial_i v_n|^2 + x_1^s |\partial_N v_n|^2 \right) \, dx$$

imply

$$\frac{S - \varepsilon}{1+\delta} \leq \int_{B_\delta} \sum_{i=1}^{N} |\partial_i v_n|^2 \, dx \leq \frac{S + \varepsilon}{1-\delta}$$

for $n \geq n_0$. It is important to note that the weak convergence $\nu_n \rightharpoonup \delta_{x_1}$ implies that, for any $t \in (0, \delta)$ and $q \in (2^*_s, 2^*_s)$, we have

$$1 = \liminf_{n \to \infty} \int_{B_t} |v_n|^{2^*_s} \, dx \leq |B_t|^{2^*_s} \liminf_{n \to \infty} \left( \int_{B_t} |v_n|^q \, dx \right)^{\frac{2^*_s}{q}}$$

$$\leq |B_t|^{1-\frac{2^*_s}{q}} \liminf_{n \to \infty} \left( \int_{B_\delta} |v_n|^q \, dx \right)^{\frac{2^*_s}{q}}.$$ 

In particular, this implies

$$\liminf_{n \to \infty} \left( \int_{B_\delta} |v_n|^q \, dx \right)^{\frac{2^*_s}{q}} \geq |B_t|^{\frac{2^*_s}{q}-1}, \quad (2.21)$$

and since $t \in (0, \delta)$ was arbitrary, we conclude that $\|v_n\|_{L^q(B_\delta)} \to \infty$ as $n \to \infty$ for any $q \in (2^*_s, 2^*_s)$.

Now let $\varphi \in C_c^\infty(\mathbb{R}^N)$ such that $\varphi \equiv 1$ on $B_1(0)$ and $\varphi \equiv 0$ on $\mathbb{R}^N \setminus B_2(0)$, and set

$$\varphi_\delta(x) := \varphi \left( \frac{x-x_1}{\delta} \right)$$

so that $\varphi_\delta \equiv 1$ on $B_\delta(x_1)$, $\varphi \equiv 0$ on $\mathbb{R}^N \setminus B_{2\delta}(x_1)$. Then, by Sobolev’s inequality

$$\left( \int_{\mathbb{R}^N} |\varphi_\delta v_n|^q \, dx \right)^{\frac{2}{q}} \leq C_q \left( \int_{\mathbb{R}^N} \sum_{i=1}^{N} |\partial_i (\varphi_\delta v_n)|^2 \, dx + \int_{\mathbb{R}^N} |\varphi_\delta v_n|^2 \, dx \right). \quad (2.22)$$

Note that (2.21) implies that the left hand side goes to infinity as $n \to \infty$ since

$$\int_{B_\delta} |v_n|^q \, dx \leq \int_{\mathbb{R}^N} |\varphi_\delta v_n|^q \, dx.$$
On the other hand,
\[
\int_{\mathbb{R}^N_+} |\varphi_\delta v_n|^2 \, dx \leq |B_{2\delta}|^{1 - \frac{2}{s}} \left( \int_{B_{2\delta}} |v_n|^{2s} \, dx \right)^{\frac{2}{s}} \leq |B_2|^{1 - \frac{2}{s}},
\]
and, noting that \( \nabla \varphi_\delta(x) = \delta^{-1} |\nabla \varphi| (\frac{s - 2}{\delta}) \),
\[
\left( \int_{\mathbb{R}^N_+} \sum_{i=1}^{N} |\partial_i (\varphi_\delta v_n)|^2 \, dx \right)^{\frac{1}{2}} \leq \left( \int_{\mathbb{R}^N_+} \varphi_\delta^2 \sum_{i=1}^{N} |\partial_i v_n|^2 \, dx \right)^{\frac{1}{2}} + \left( \int_{\mathbb{R}^N_+} v_n^2 \sum_{i=1}^{N} |\partial_i \varphi_\delta|^2 \, dx \right)^{\frac{1}{2}}
\leq \left( \int_{B_{2\delta}} \sum_{i=1}^{N} |\partial_i v_n|^2 \, dx \right)^{\frac{1}{2}} + \sqrt{N} \delta^{-1} \| \nabla \varphi \| \infty \left( \int_{B_{2\delta} \setminus B_\delta} |v_n|^2 \right)^{\frac{1}{2}}
\leq \sqrt{\frac{S + \varepsilon}{1 - 2\delta}} + \sqrt{N} \delta^{-1} \| \nabla \varphi \| \infty \left( \int_{B_{2\delta} \setminus B_\delta} |v_n|^{2s} \right)^{\frac{1}{s}} \leq \sqrt{\frac{S + \varepsilon}{1 - 2\delta}} + \sqrt{N} \delta^{-1} \| \nabla \varphi \| \infty \left( \int_{B_{2\delta} \setminus B_\delta} |v_n|^{2s} \right)^{\frac{1}{s}}.
\]
This implies that the right hand side of (2.22) remains bounded as \( n \to \infty \), a contradiction.

We conclude \( \nu^j = 0 \) for all \( j \) and hence \( \| v \|_{L^{2s}(\mathbb{R}^N_+)} = 1 \). Since \( L^{2s}(\mathbb{R}^N_+) \) is uniformly convex, this implies \( v_n \to v \) in \( L^{2s}(\mathbb{R}^N_+) \). Moreover, since \( \| v \|^{2}_{H_s} \geq S \), weak lower semicontinuity gives \( \| v_n \|^2_{H_s} \to S = \| v \|^2_{H_s} \) and hence \( v_n \to v \) in \( H_s \) again by uniform convexity of the Hilbert space \( H_s \). This completes the proof. \( \square \)

**Remark 2.7. (Existence of minimizers on \( \mathbb{R}^N \))**

We note that Theorem 2.2 implies
\[
S_s(\mathbb{R}^N) := \inf_{u \in C^1_c(\mathbb{R}^N)} \frac{\int_{\mathbb{R}^N} \sum_{i=1}^{N-1} |\partial_i u|^2 + |x_1|^s |\partial N u|^2 \, dx}{(\int_{\mathbb{R}^N} |u|^{2s} \, dx)^{\frac{2}{s}}} > 0.
\]
Consequently, we can look for minimizers in the closure of \( C^1_c(\mathbb{R}^N) \) in
\[
\left\{ u \in L^{2s}(\mathbb{R}^N) : \int_{\mathbb{R}^N} \sum_{i=1}^{N-1} |\partial_i u|^2 + |x_1|^s |\partial N u|^2 \, dx < \infty \right\}.
\]
The previous arguments can then easily be adapted to prove the existence of minimizers of \( S_s(\mathbb{R}^N) \) similar to Theorem 2.6.

3. A Degenerate Sobolev Inequality on B

In this section we shall prove the second part of Theorem 1.1, namely the properties of \( \mathcal{C}_{1,m,p}(B) \) given in (1.11).

We first use the scaling properties discussed in Remark 2.3(i) to prove the following.

**Proposition 3.1.** Let \( p > 2^*_s \) and \( m > -\lambda_1(B) \). Then \( \mathcal{C}_{1,m,p}(B) = 0 \), i.e.
\[
\inf_{u \in C^2(B) \setminus \{0\}} \frac{\| \nabla u \|_2^2 - \| \partial_\theta u \|_2^2 + m \| u \|_p^2}{\| u \|_p^2} = 0.
\]
Proof. Let $\varepsilon > 0$. By (2.7), there exists $v \in C_0^1(\mathbb{R}^N_+)$ with the property that
\[
\int_{\mathbb{R}^N_+} \left( \sum_{i=1}^{N-1} |\partial_i v|^2 + 2x_1 |\partial_N v|^2 \right) dx < \varepsilon \left( \int_{\mathbb{R}^N_+} |v|^p dx \right)^{\frac{2}{p}}.
\]
For $\lambda \in (0, 1)$, let
\[
\tau_\lambda : \mathcal{B} \to \mathbb{R}^N_+, \quad \tau_\lambda(x) = (\lambda^{-2} (x_1 + 1), \lambda^{-2} x_3, \ldots, \lambda^{-2} x_{N-1}, \lambda^{-3} x_2)
\]
and set $u_\lambda := v \circ \tau_\lambda$. If $\lambda$ is chosen sufficiently small, we have $u \in C_0^1(\mathcal{B})$ and
\[
\|\nabla u\|_{L^2(\mathcal{B})}^2 - \|\partial_N u\|_{L^2(\mathcal{B})}^2 = \int_{\mathcal{B}} \left( \sum_{i=1}^{N-1} |\lambda^{-2} [\partial_i v] \circ \tau_\lambda|^2 + |\lambda^{-3} [\partial_N v] \circ \tau_\lambda|^2 - |x_1 \lambda^{-3} [\partial_N v] \circ \tau_\lambda - x_2 \lambda^{-2} [\partial_1 v] \circ \tau_\lambda|^2 \right) dx
\]
\[
= \lambda^{2N+1} \int_{\mathbb{R}^N_+} \left( \sum_{i=1}^{N-1} [\lambda^{-4} [\partial_i v]^2 + \lambda^{-6} [\partial_N v]^2] - (\lambda^2 x_1 - 1) \lambda^{-3} [\partial_N v] - \lambda^3 x_2 \lambda^{-2} [\partial_1 v]^2 \right) dx
\]
\[
= \lambda^{2N-3} \int_{\mathbb{R}^N_+} \left( \lambda^{-2} x_1 [\partial_N v]^2 - 2x_2 \lambda^2 (\lambda^2 x_1 - 1) [\partial_1 v] [\partial_N v] + \lambda^6 x_2^2 [\partial_1 v]^2 \right) dx,
\]
while
\[
\|u\|_{L^2(\mathcal{B})}^2 \leq \lambda^{2N+1} \|v\|_{L^2(\mathbb{R}^N_+)}^2 \quad \text{and} \quad \|u\|^2_{L^p(\mathcal{B})} = \lambda^{\frac{4N+2}{p}} \|v\|^2_{L^p(\mathbb{R}^N_+)}. \tag{3.1}
\]
We conclude that
\[
\mathcal{C}_{1,m,p}(\mathcal{B}) \leq \frac{\|\nabla u\|_{L^2(\mathcal{B})}^2 - \|\partial_N u\|_{L^2(\mathcal{B})}^2 + m\|u\|_{L^2(\mathcal{B})}^2}{\|u\|^2_{L^p(\mathcal{B})}} \leq \lambda^{\frac{4N-3}{p}} \int_{\mathbb{R}^N_+} \left( \sum_{i=1}^{N-1} [\partial_i v]^2 + 2x_1 [\partial_N v]^2 \right) dx \leq o \left( \lambda^{\frac{4N-3}{p}} \right) < \varepsilon
\]
for $\lambda > 0$ small enough, since $p > 2^*_+ = \frac{4N+2}{2N-3}$. Recalling that $\varepsilon > 0$ was arbitrary, this yields the claim. \hfill \square

To prove the second assertion on $\mathcal{C}_{1,m,p}(\mathcal{B})$ in (1.11), we now transfer the information given by Theorem 1.6 in the case $s = 1$ to the ball $\mathcal{B}$. To this end, we consider the great circle
\[
\gamma := \{ x \in \partial \mathcal{B} : x_3 = \cdots = x_N = 0 \}. \tag{3.2}
\]
We have the following key lemma.

**Lemma 3.2.** Let $\varepsilon > 0$. Then there exists $\delta > 0$ with the property that, for any $x_0 \in \gamma$,
\[
\frac{\int_{\Omega_{x_0,\delta}} (|\nabla u|^2 - |\partial_N u|^2) \ dx}{\|u\|^2_{L^2(\Omega_{x_0,\delta})}} \geq (1 - \varepsilon) 2^{\frac{4N-1}{4N}} \mathcal{S}_1(\mathbb{R}^N_+) \quad \text{for} \ u \in C_0^1(\Omega_{x_0,\delta}) \setminus \{0\},
\]
where $\mathcal{S}_1(\mathbb{R}^N)$ is given in Theorem 1.6 and

$$\Omega_{x_0,\delta} := B \cap B_\delta(x_0) = \{ x \in B : |x - x_0| < \delta \}. \tag{3.3}$$

**Proof.** We may assume $x_0 = e_2 = (0, 1, 0, \ldots, 0)$ is the second coordinate vector. We fix $\delta > 0$ and consider a function $u \in C^1_0(\Omega_{e_2,\delta})$ which we extend trivially to a function $u \in C^1_0(\mathbb{R}^N)$. Moreover, we write $u$ in $N$-dimensional polar coordinates, so we consider $U := [0, 1] \times (-\pi, \pi) \times (0, \pi)^{N-2}$ and the function

$$v = u \circ P : U \to \mathbb{R}$$

with $P : U \to \mathbb{R}^N$ given by

$$P(r, \theta, \vartheta_1, \ldots, \vartheta_{N-2}) = (r \cos \theta \sin \vartheta_1 \cdots \sin \vartheta_{N-2}, r \sin \theta \sin \vartheta_1 \cdots \sin \vartheta_{N-2},$$

$$r \cos \vartheta_1, r \sin \vartheta_1 \cos \vartheta_2, \ldots, r \sin \vartheta_{N-1} \cos \vartheta_{N-2}, r \sin \vartheta_1 \cdots \sin \vartheta_{N-3} \cos \vartheta_{N-2}) \tag{3.4}$$

We then have

$$\int_B \left( |\nabla u|^2 - |\partial_\theta u|^2 \right) \, dx \tag{3.5}$$

$$= \int_0^1 \int_{-\pi}^\pi \int_0^{\pi-N-2} \left( |\partial_r u|^2 + \frac{1}{r^2} \sum_{i=1}^{N-2} g_i |\partial_{\vartheta_i} u|^2 + \left( \frac{gN-1}{r^2} - 1 \right) |\partial_\theta u|^2 \right) g \, d\vartheta_1 \cdots d\vartheta_{N-2} \, d\theta \, dr \tag{3.6}$$

with the functions $g, g_i : U \to \mathbb{R}$, $i = 1, \ldots, N - 1$ given by

$$g(r, \theta, \vartheta_1, \ldots, \vartheta_{N-2}) = r^{N-1} \prod_{k=1}^{N-2} \sin^{N-1-k} \vartheta_k, \quad g_i(r, \theta, \vartheta_1, \ldots, \vartheta_{N-2}) = \prod_{k=1}^{i-1} \frac{1}{\sin^{2} \vartheta_k} \tag{3.7}$$

In particular, we have $g \leq 1$ and $g_i \geq 1$ in $U$ for $i = 1, \ldots, N - 1$. Moreover, since $P^{-1}(e_2) = (1, \frac{\pi}{2}, \ldots, \frac{\pi}{2})$ and $g(1, \frac{\pi}{2}, \ldots, \frac{\pi}{2}) = 1$, we can choose $\delta > 0$ sufficiently small so that

$$P^{-1}(\Omega_{e_2,\delta}) \subset (0, 1) \times (0, \pi)^{N-1} \quad \text{and} \quad g \geq (1 - \varepsilon) \text{ in } P^{-1}(\Omega_{e_2,\delta}). \tag{3.8}$$

Therefore

$$\int_B \left( |\nabla u|^2 - |\partial_\theta u|^2 \right) \, dx \geq (1 - \varepsilon) \int_0^1 \int_{-\pi}^\pi \int_0^{\pi-N-2} \left( |\partial_r u|^2 + \sum_{i=1}^{N-2} |\partial_{\vartheta_i} u|^2 + \frac{(1-r)(1+r)}{r^2} |\partial_\theta u|^2 \right) \, d\vartheta_1 \cdots d\vartheta_{N-2} \, d\theta \, dr. \tag{3.9}$$

Noting that

$$\frac{(1-r)(1+r)}{r^2} \geq \frac{2-\delta}{(1-\delta)^2} \geq 2(1-r)$$

and substituting $t = 1-r$ we thus find that

$$\int_B \left( |\nabla u|^2 - |\partial_\theta u|^2 \right) \, dx \geq (1 - \varepsilon) \int_0^1 \int_{-\pi}^\pi \int_0^{\pi-N-2} \left( |\partial_r \tilde{v}|^2 + \sum_{i=1}^{N-2} |\partial_{\vartheta_i} \tilde{v}|^2 + 2t |\partial_\theta \tilde{v}|^2 \right) \, d\vartheta_1 \cdots d\vartheta_{N-2} \, d\theta \, dt. \tag{3.10}$$

with

$$\tilde{v} : U \to \mathbb{R}, \quad \tilde{v}(t, \vartheta_1, \ldots, \vartheta_{N-2}, \theta) = u(P(1-t, \vartheta_1, \ldots, \vartheta_{N-2}, \theta))$$
Note that \( u \in C^1_c(\Omega_{\varepsilon_2, \delta}) \) implies, by (3.8), that \( \tilde{v} \) is compactly supported in \((0, 1) \times (0, \pi)^{N-1} \subset \mathbb{R}^N_+\), so we may regard \( \tilde{v} \) as a function in \( C^1_c(\mathbb{R}^N_+) \) and deduce that

\[
\int_{B} \left( |\nabla u|^2 - |\partial_\theta u|^2 \right) \, dx \geq (1 - \varepsilon) \int_{\mathbb{R}^N_+} \left( \sum_{i=1}^{N-1} |\partial_i \tilde{v}|^2 + 2x_1 |\partial_N \tilde{v}|^2 \right) \, dx.
\]

Rather directly, we also find that, by a change of variables,

\[
\int_{\Omega} |u|^2 \, dx = \int_{U} |v|^2 \, g(\rho, \theta, \vartheta_1, \ldots, \vartheta_{N-2}) \, dx \leq \int_{U} |v|^2 \, d(\rho, \theta, \vartheta_1, \ldots, \vartheta_{N-2}) = \int_{\mathbb{R}^N_+} |\tilde{v}|^2 \, dx.
\]

Using (2.18) with \( \kappa = 2 \), we conclude that

\[
\frac{\int_{\Omega} \left( |\nabla u|^2 - |\partial_\theta u|^2 \right) \, dx}{\|u\|^2_{L^2(\Omega)}} \geq (1 - \varepsilon) \frac{\int_{\mathbb{R}^N_+} \left( \sum_{i=1}^{N-1} |\partial_i \tilde{v}|^2 + 2x_1 |\partial_N \tilde{v}|^2 \right) \, dx}{\left( \int_{\mathbb{R}^N_+} |\tilde{v}|^2 \, dx \right)^{\frac{2}{2+\frac{2}{p}}}} \geq (1 - \varepsilon)^{2+\frac{2}{p}} S_1(\mathbb{R}^N_+),
\]

as claimed. \( \square \)

We can now prove the main result of this section.

**Theorem 3.3.** For any \( 1 \leq p \leq 2^*_N \) there exists \( C > 0 \), such that any \( u \in C^1_c(\mathcal{B}) \) satisfies

\[
\|u\|_{L^p(\mathcal{B})} \leq C \int_{\mathcal{B}} \left( |\nabla u|^2 - |\partial_\theta u|^2 \right) \, dx.
\]

**Proof.** Since \( \mathcal{B} \) is bounded, it suffices to consider the case \( p = 2^*_N \). In the following, \( C > 0 \) denotes a constant independent of \( u \), which may change from line to line. Fix \( \varepsilon \in (0, \frac{1}{2}) \) and let \( \delta > 0 \) be given as in Lemma 3.2. Take points \( x_1, \ldots, x_m \in \gamma \) such that the sets \( U_k := B_\delta(x_k) \) satisfy

\[
\gamma \subset \bigcup_{k=1}^{m} U_k
\]

and let \( \delta_0 := \text{dist}(\gamma, \mathcal{B} \setminus \bigcup_{k=1}^{m} U_k) \). We then let \( U_0 := \left\{ x \in \mathcal{B} : \text{dist}(x, \gamma) > \frac{\delta_0}{2} \right\} \) and thus have \( \mathcal{B} \subset \bigcup_{k=0}^{m} U_k \). We may then choose a partition of unity \( \eta_0, \ldots, \eta_m \) subordinate to this covering. Then

\[
\|u\|_{L^{2^*_N}(\mathcal{B})} \leq \sum_{k=0}^{m} \|\eta_k u\|_{L^{2^*_N}(U_k)} \leq C \sum_{k=0}^{m} \left( \int_{U_k} \left( |\nabla (\eta_k u)|^2 - |\partial_\theta (\eta_k u)|^2 \right) \, dx \right)^{\frac{1}{2^*_N}},
\]

where we used Lemma 3.2 and the fact that \( v \mapsto \int_{U_0} \left( |\nabla v|^2 - |\partial_\theta v|^2 \right) \, dx \) induces an equivalent norm on \( H^1_0(U_0) \). Note that, for \( k = 0, \ldots, m \), we have

\[
\int_{U_k} \left( |\nabla (\eta_k u)|^2 - |\partial_\theta (\eta_k u)|^2 \right) \, dx \leq 2 \left( \int_{U_k} \eta_k^2 \left( |\nabla u|^2 - |\partial_\theta u|^2 \right) \, dx \right) + \int_{U_k} u^2 \left( |\nabla \eta_k|^2 + |\partial_\theta \eta_k|^2 \right) \, dx
\]

\[
\leq C \int_{U_k} \left( |\nabla u|^2 - |\partial_\theta u|^2 + u^2 \right) \, dx,
\]

where we used Lemma 3.2 and the fact that \( v \mapsto \int_{U_0} \left( |\nabla v|^2 - |\partial_\theta v|^2 \right) \, dx \) induces an equivalent norm on \( H^1_0(U_0) \). Note that, for \( k = 0, \ldots, m \), we have

\[
\int_{U_k} \left( |\nabla (\eta_k u)|^2 - |\partial_\theta (\eta_k u)|^2 \right) \, dx \leq 2 \left( \int_{U_k} \eta_k^2 \left( |\nabla u|^2 - |\partial_\theta u|^2 \right) \, dx \right) + \int_{U_k} u^2 \left( |\nabla \eta_k|^2 + |\partial_\theta \eta_k|^2 \right) \, dx
\]

\[
\leq C \int_{U_k} \left( |\nabla u|^2 - |\partial_\theta u|^2 + u^2 \right) \, dx,
\]
with some fixed $C > 0$. We conclude that

$$
\|u\|_{L^2(B)} \leq C \sum_{k=0}^{m} \left( \int_{U_k} \left( |\nabla u|^2 - |\partial_\theta u|^2 + u^2 \right) \, dx \right)^{1/2},
$$

and thus

$$
\|u\|_{L^2(B)}^2 \leq C \sum_{k=0}^{m} \int_{U_k} \left( |\nabla u|^2 - |\partial_\theta u|^2 + u^2 \right) \, dx = C \int_B \left( |\nabla u|^2 - |\partial_\theta u|^2 + u^2 \right) \, dx. \tag{3.9}
$$

In order to complete the proof, we note that Proposition 4.1 implies

$$
\inf_{u \in C^2_0(B) \setminus \{0\}} \frac{\int_B \left( |\nabla u|^2 - |\partial_\theta u|^2 \right) \, dx}{\int_B u^2 \, dx} = \lambda_1(B) > 0
$$

and hence

$$
\int_B u^2 \, dx \leq \frac{1}{\lambda_1(B)} \int_B \left( |\nabla u|^2 - |\partial_\theta u|^2 \right) \, dx.
$$

In view of (3.9) this finishes the proof. \hfill \Box

4. The variational setting for and preliminary results on ground state solutions

4.1. The variational setting. In this section, we set up the variational framework for (1.5) and prove some preliminary estimates for the quantities $\mathcal{C}_{\alpha,0,2}(B)$ and $R_{\alpha,m,p}$ defined in (1.6) and (1.7). We first show a Poincaré type estimate. Recall here that $\lambda_1(B)$ is the first Dirichlet eigenvalue of $-\Delta$ on the unit ball $B$.

**Proposition 4.1.** For $0 \leq \alpha \leq 1$, we have

$$
\mathcal{C}_{\alpha,0,2}(B) = \inf_{u \in C^2_0(B) \setminus \{0\}} \frac{\int_B \left( |\nabla u|^2 - \alpha^2 |\partial_\theta u|^2 \right) \, dx}{\int_B u^2 \, dx} = \lambda_1(B). \tag{4.1}
$$

Moreover, minimizers are precisely the Dirichlet eigenfunctions of $-\Delta$ on $B$ corresponding to the eigenvalue $\lambda_1(B)$ and are therefore radial.

**Proof.** By (1.8) and since $\mathcal{C}_{0,0,2}(B) = \lambda_1(B)$ by the variational characterization of $\lambda_1(B)$, it suffices to prove (4.1) in the case $\alpha = 1$. In the following, we let $\{Y_{\ell,k} : \ell \in \mathbb{N} \cup \{0\}, \ k = 1, \ldots, d_\ell\}$ denote an $L^2$-orthonormal basis of $L^2(S^{N-1})$ of spherical harmonics of degree $\ell$. More precisely, we can choose $Y_{\ell,k}$ in such a way that, for every $\ell \in \mathbb{N} \cup \{0\}$, the functions $Y_{\ell,k}, k = 1, \ldots, d_\ell$ form a basis of the eigenspace of the Laplace Beltrami operator $-\Delta_{S^{N-1}}$ corresponding to the eigenvalue $\ell(\ell + N - 2)$ and such that

$$
-\partial_\omega^2 Y_{\ell,k} = \ell^2 Y_{\ell,k} \quad \text{for } k = 1, \ldots, d_\ell
$$

where $|\ell| \leq \ell$, see e.g. [23]. Let $\varphi \in C^1_c(B)$, and let $\varphi_{\ell,k} \in C^1([0,1])$ be the angular Fourier coefficient functions defined by

$$
\varphi_{\ell,k}(r) = \int_{S^{N-1}} \varphi(r\omega) Y_{\ell,k}(\omega) \, d\omega, \quad 0 \leq r \leq 1.
$$

For fixed $r \in [0,1]$, we then have the Parseval identities

$$
\|\varphi(r \cdot)\|_{L^2(S^{N-1})}^2 = \sum_{\ell,k} |\varphi_{\ell,k}(r)|^2 \|Y_{\ell,k}\|_{L^2(S^{N-1})}^2,
$$

$$
\|\partial_\varphi \varphi(r \cdot)\|_{L^2(S^{N-1})}^2 = \sum_{\ell,k} |\partial_r \varphi_{\ell,k}(r)|^2 \|Y_{\ell,k}\|_{L^2(S^{N-1})}^2.
$$


(4.1) \( \|\nabla_{S^{N-1}} \varphi(r \cdot)\|_{L^2(S^{N-1})}^2 = \sum_{\ell,k} (\ell + N - 2)|\varphi_{\ell,k}(r)|^2 \|Y_{\ell,k}\|_{L^2(S^{N-1})}^2 \) and
\( \|\partial_\theta \varphi(r \cdot)\|_{L^2(S^{N-1})}^2 = \sum_{\ell,k} \ell^2 |\varphi_{\ell,k}(r)|^2 \|Y_{\ell,k}\|_{L^2(S^{N-1})}^2 \)

in \( L^2(S^{N-1}) \). Here and in the following, we simply write \( \sum_{\ell,k} \) in place of \( \sum_{\ell=0}^{\infty} \sum_{k=1}^{\infty} \). Since \( \frac{\ell(\ell+N-2)}{r^2} \geq \ell^2_k \) for \( r \in [0, 1] \) and every \( \ell, k \), we estimate that
\[
\int_B \left( \|\nabla \varphi\|^2 - |\partial_\theta \varphi|^2 \right) \, dx
= \int_0^1 r^{N-1} \int_{S^{N-1}} \left( |\partial_r \varphi(r\omega)|^2 + \frac{1}{r^2} |\nabla_{S^{N-1}} \varphi(r\omega)|^2 - |\partial_\theta \varphi(r\omega)|^2 \right) \, d\omega \, dr
= \sum_{\ell,k} \|Y_{\ell,k}\|_{L^2(S^{N-1})}^2 \int_0^1 r^{N-1} |\partial_r \varphi_{\ell,k}(r)|^2 + \left( \frac{\ell(\ell+N-2)}{r^2} - \ell^2_k \right) |\varphi_{\ell,k}(r)|^2 \, dr
\geq \sum_{\ell,k} \|Y_{\ell,k}\|_{L^2(S^{N-1})}^2 \int_0^1 r^{N-1} |\partial_r \varphi_{\ell,k}(r)|^2 \, dr
\geq \lambda_1(B) \sum_{\ell,k} \|Y_{\ell,k}\|_{L^2(S^{N-1})}^2 \int_0^1 r^{N-1} |\varphi_{\ell,k}(r)|^2 \, dr
= \lambda_1(\Delta, B) \int_0^1 r^{N-1} \|\varphi(r \cdot)\|_{L^2(S^{N-1})}^2 \, dr = \lambda_1(B) \int_B |\varphi|^2 \, dx.
\]

Clearly, equality holds if and only if \( \varphi_{\ell,k} \equiv 0 \) for \( \ell \geq 1 \) and \( \varphi_0 \) corresponds to a first eigenfunction of the Dirichlet Laplacian on \( B \). \( \square \)

**Corollary 4.2.**

(i) We have \( \mathcal{C}_{\alpha,2}(B) = \mathcal{C}_{0,2}(B) = \lambda_1(B) + m \) for \( \alpha \in [0, 1] \), \( m \in \mathbb{R} \).

(ii) For \( \alpha \in [0, 1] \), \( m > -\lambda_1(B) \), \( 2 \leq p < 2^* \) and \( u \in H^1_0(B) \setminus \{0\} \) we have \( R_{\alpha,m,p}(u) > 0 \).

**Proof.** (i) This follows immediately from Proposition 4.1.

(ii) For \( \alpha \in [0, 1] \), \( m \geq -\lambda_1(B) \), \( 2 \leq p < 2^* \) and \( u \in H^1_0(B) \setminus \{0\} \) we have \( R_{\alpha,m,p}(u) > R_{1-\lambda_1(B),p}(u) \) and
\[
R_{1-\lambda_1(B),p}(u) \left( \int_B |u|^p \, dx \right)^\frac{2}{p} = \left( \int_B |u|^p \, dx \right)^\frac{2}{p} \int_B \left( |\nabla u|^2 - |\partial_\theta u|^2 - \lambda_1(B) u^2 \right) \, dx \geq 0
\]
by Proposition 4.1. \( \square \)

**Remark 4.3. (The case \( \alpha > 1 \))**

It is natural to ask what happens for \( \alpha > 1 \). In fact, in this case, the infimum \( \mathcal{C}_{\alpha,m,p}(B) \) in (1.6) satisfies
\[
\mathcal{C}_{\alpha,m,p}(B) = -\infty \quad \text{for every } m \in \mathbb{R}, \ p \in [2, \infty).
\]

To see this, we fix \( \varepsilon \in (0, 1) \) and nonzero functions \( \varphi \in C^1_c(1 - \varepsilon, 1) \), \( \psi \in C^1_c(\frac{\pi}{2} - \varepsilon, \frac{\pi}{2} + \varepsilon) \). Moreover, we consider the sequence of functions \( u_k \in C^1_c(B) \) which, in the polar coordinates from (3.4), are given by
\[
(r, \theta_1, \ldots, \theta_{N-2}) \mapsto \varphi(r) \psi(\theta_1) \cdots \psi(\theta_{N-2}) X_k(\theta), \quad \text{where } X_k(\theta) = \sin(k\theta).
\]
Similarly as in (3.5), we then find, with \( U_{\varepsilon} := (1 - \varepsilon, 1) \times (-\pi, \pi) \times (\frac{\pi}{2} - \varepsilon, \frac{\pi}{2} + \varepsilon)^{N-2} \), that
\[
\int_{B} \left( |\nabla u_k|^2 - \alpha^2 |\partial_{\theta} u_k|^2 \right) \, dx
= \int_{U_{\varepsilon}} \left( |\psi'(r)|^2 |X_k(\theta)|^2 \prod_{i=1}^{N-2} |\psi(\vartheta_i)|^2 + \frac{1}{r^2} \sum_{i=1}^{N-2} g_i |\psi'(\vartheta_i)|^2 |\varphi(r)|^2 |X_k(\theta)|^2 \prod_{j=1}^{N-2} |\psi(\vartheta_j)|^2 
+ \left( \frac{g_{N-1}}{r^2} - \alpha^2 \right) |X_k'(\theta)|^2 |\varphi(r)|^2 \prod_{i=1}^{N-2} |\psi(\vartheta_i)|^2 \right) g \, d(r, \theta, \vartheta_1, \ldots, \vartheta_{N-2})
\]
with the functions \( g, g_i : U \rightarrow \mathbb{R}, \ i = 1, \ldots, N-1 \) given in (3.7). We may now choose \( \varepsilon = \varepsilon(\alpha) > 0 \) so small that
\[
\frac{1}{2} \leq g \leq 1 \quad \text{and} \quad \alpha^2 - \frac{g_{N-1}}{r^2} \geq \varepsilon \quad \text{on} \ U_{\varepsilon}.
\]
Since also \(|X_k| \leq 1\) by definition, we estimate
\[
\int_{B} \left( |\nabla u_k|^2 - \alpha^2 |\partial_{\theta} u_k|^2 \right) \, dx \leq c - d(k),
\]
where
\[
c := \int_{U_{\varepsilon}} \left( |\psi'(r)|^2 \prod_{i=1}^{N-2} |\psi(\vartheta_i)|^2 + \frac{1}{r^2} \sum_{i=1}^{N-2} g_i |\psi'(\vartheta_i)|^2 |\varphi(r)|^2 \prod_{j=1}^{N-2} |\psi(\vartheta_j)|^2 \right) d(r, \theta, \vartheta_1, \ldots, \vartheta_{N-2})
\]
and
\[
d(k) := \int_{U_{\varepsilon}} \left( \alpha^2 - \frac{g_{N-1}}{r^2} \right) |X_k'(\theta)|^2 |\varphi(r)|^2 \prod_{i=1}^{N-2} |\psi(\vartheta_i)|^2 g \, d(r, \theta, \vartheta_1, \ldots, \vartheta_{N-2}) \geq \frac{k^2 \varepsilon}{2} \int_{1-\varepsilon}^{1} |\varphi(r)|^2 \, dr \int_{-\pi}^{\pi} \cos^2(k\theta) \, d\theta \left( \int_{\frac{\pi}{2}-\varepsilon}^{\frac{\pi}{2}+\varepsilon} |\psi(\vartheta)|^2 \, d\vartheta \right)^{N-2} = \frac{\pi}{2} d_2 k^2 \]
with \( d_2 := \left( \int_{1-\varepsilon}^{1} |\varphi(r)|^2 \, dr \left( \int_{\frac{\pi}{2}-\varepsilon}^{\frac{\pi}{2}+\varepsilon} |\psi(\vartheta)|^2 \, d\vartheta \right) \right)^{N-2} \). Hence \( d(k) \to \infty \) as \( k \to \infty \). Moreover, for every \( p \in [2, \infty) \) we have
\[
\int_{B} |u_k|^p \, dx = \int_{U_{\varepsilon}} |\varphi(r)|^p |X_k(\theta)|^p \prod_{i=1}^{N-2} |\psi(\vartheta_i)|^p g \, d(r, \theta, \vartheta_1, \ldots, \vartheta_{N-2}) \leq d_p
\]
with
\[
d_p := 2\pi \int_{1-\varepsilon}^{1} |\varphi(r)|^p \, dr \left( \int_{\frac{\pi}{2}-\varepsilon}^{\frac{\pi}{2}+\varepsilon} |\psi(\vartheta)|^p \, d\vartheta \right)^{N-2} < \infty.
\]
It thus follows that
\[
\frac{\int_{B} \left( |\nabla u_k|^2 - \alpha^2 |\partial_{\theta} u_k|^2 + m |u_k|^2 \right) \, dx}{\left( \int_{B} |u_k|^p \, dx \right)^{\frac{2}{p}}} \leq \frac{c - d(k) - m d_2}{(d_p)^{\frac{2}{p}}} \to -\infty \quad \text{as} \quad k \to \infty.
\]
for every \( p \in [2, \infty), \ m \in \mathbb{R} \). This shows (4.2).

Consequently, the study of ground state solutions of (1.5) requires a completely different approach in the case \( \alpha > 1 \). This is further treated in the forthcoming paper [25].
In the following, we show that, for $\alpha \in [0,1)$ and $2 < p < 2^*$, the value $\mathcal{C}_{\alpha,m,p}(B) > 0$ is attained in $H_0^1(B) \setminus \{0\}$ and that any minimizer gives rise to a weak solution of (1.5).

**Lemma 4.4.** Let $0 \leq \alpha < 1$, $2 < p < 2^*$ and $m > -\lambda_1(B)$. Then the value $\mathcal{C}_{\alpha,m,p}(B)$ is positive and attained at a function $u_0 \in H_0^1(B) \setminus \{0\}$. Moreover, after multiplication by a positive constant, $u_0$ is a weak solution of (1.5), and $u_0 \in C^{2,\sigma}(\overline{B})$ for some $\sigma > 0$.

**Proof.** We first note that

$$\int_B \left( |\nabla u|^2 - \alpha^2 |\partial_\theta u|^2 \right) \, dx \geq (1 - \alpha^2) \int_B |\nabla u|^2 \, dx \quad \text{for } u \in H_0^1(B).$$

Since $\alpha \in [0,1)$, it therefore follows from Sobolev embeddings that

$$R_{\alpha,m,p}(u) \geq C_{m,\alpha} \frac{\int_B |\nabla u|^2 \, dx}{(\int_B |u|^p \, dx)^{\frac{2}{p}}} \quad u \in H_0^1(B) \setminus \{0\} \tag{4.3}$$

with a constant $C_{m,\alpha} > 0$. We take a minimizing sequence $(u_n)_n$ for the Rayleigh quotient $R_{\alpha,m,p}$, normalized such that $\int_B |u|^p \, dx = 1$ for all $n$. By (4.3), $(u_n)_n$ remains bounded in $H_0^1(B)$ and we may pass to subsequence that weakly converges to $u_0 \in H_0^1(B)$. The compactness of the embedding $H_0^1(B) \hookrightarrow L^p(B)$ and the weak lower semicontinuity of the quadratic form $u \mapsto \int_B \left( |\nabla u|^2 - \alpha^2 |\partial_\theta u|^2 \right) \, dx$ then imply that $\int_B |u_0|^p \, dx = 1$ and $R_{\alpha,m,p}(u_0) = \mathcal{C}_{\alpha,m,p}(B)$. Hence $\mathcal{C}_{\alpha,m,p}(B)$ is attained, and $\mathcal{C}_{\alpha,m,p}(B) > 0$ by Corollary 4.2.

Next, standard variational arguments show that every $L^p$-normalized minimizer $u_0$ must be a weak solution of

$$\begin{cases} -\Delta u + \alpha^2 \partial^2_\theta u + mu = \mathcal{C}_{\alpha,m,p}(B)|u|^{p-2}u & \text{in } B \\ u = 0 & \text{on } \partial B. \end{cases}$$

We then conclude that $[\mathcal{C}_{\alpha,m,p}(B)]^{\frac{1}{p}} u_0$ solves (1.5). Finally, classical elliptic regularity theory yields $C^{2,\sigma}(\overline{B})$ since the operator $-\Delta - \alpha^2 \partial_\theta$ is uniformly elliptic in $B$ in the case $0 \leq \alpha < 1$. \hfill \Box

**Definition 4.5.** Let $0 \leq \alpha < 1$, $2 < p < 2^*$ and $m > -\lambda_1(B)$. A weak solution $u \in H_0^1(B) \setminus \{0\}$ of (1.5) such that $R_{\alpha,m,p}(u) = \mathcal{C}_{\alpha,m,p}(B)$ will be called a **ground state solution**.

**4.2. The degenerate elliptic case** $\alpha = 1$. In the limiting case $\alpha = 1$, problem (1.5) becomes degenerate and requires to work in a function space different from $H_0^1(B)$. From Proposition 4.1, we deduce that

$$(u,v) \mapsto \langle u,v \rangle_{\mathcal{H}} := \int_B (\nabla u \cdot \nabla v - \partial_\theta u \partial_\theta v) \, dx$$

defines a scalar product on $C_0^1(B)$. The induced norm will be denoted by $\| \cdot \|_{\mathcal{H}}$.

**Lemma 4.6.** Let $\mathcal{H}$ denote the completion of $C_0^1(B)$ with respect to $\| \cdot \|_{\mathcal{H}}$. Then $\mathcal{H}$ is a Hilbert space which is embedded in $L^p(B)$ for $p \in [2,2^*_1]$, where $2^*_1 := \frac{4N+2}{N-2}$ as before. Moreover, we have:

(i) If $1 \leq p < 2^*_1$, then the embedding $\mathcal{H} \hookrightarrow L^p(B)$ is compact.

(ii) If $m > -\lambda_1(B)$ and $p \in [2,2^*_1]$, then the Rayleigh quotient $R_{1,m,p}(u)$ is well defined by (1.7) and positive for functions $u \in \mathcal{H} \setminus \{0\}$,
Proof. The embedding \( \mathcal{H} \hookrightarrow L^p(B) \) for \( p \in [2, 2^*_1] \) is an immediate consequence of Theorem 3.3.

To prove (i), we fix \( p \in [1, 2^*_1) \), and we let \( (u_n)_n \subset \mathcal{H} \) be a bounded sequence. Moreover, we put \( B_m := B_{1-m}(0) \subset B \) for \( m \geq 2 \). Then \( u_m := 1_{B_m}u_n \) defines a bounded sequence in \( H^1(B_m) \) for every \( m \geq 2 \). After passing to a subsequence, \( (u_m)_n \) converges in \( L^p(B_m) \) by Rellich-Kondrachov. After passing to a diagonal sequence we may therefore assume that there exists \( u \in L^p(B) \) with the property that \( u_n \to u \) for \( m \in \mathbb{N} \). Moreover,

\[
\|u - u_n\|_{L^p(B)} \leq \|u - u_n\|_{L^p(B_m)} + \|u - u_n\|_{L^q(B \setminus B_m)}|B \setminus B_m|^\frac{1}{q}.
\]

Since \( \|u - u_n\|_{L^q(B \setminus B_m)} \leq \|u - u\|_{L^q(B)} \) remains bounded independently of \( m \) and \( n \), this gives

\[
\limsup_{n \to \infty} \|u - u_n\|_{L^p(B)} \leq C |B \setminus B_m|^\frac{1}{q - \frac{1}{q}}
\]

for some \( C > 0 \) independent of \( m \), where the right hand side tends to zero as \( m \to \infty \). This proves that \( u_n \to u \) in \( L^p(B) \).

Finally, we note that (ii) is an immediate consequence of Proposition 4.1 and the embedding \( \mathcal{H} \hookrightarrow L^p(B) \) for \( p \in [2, 2^*_1] \). \( \square \)

Lemma 4.6 allows the following definition of a weak solution of (1.5) with \( \alpha = 1 \) in the case where \( p \in [2, 2^*_1] \).

**Definition 4.7.** Let \( m > -\lambda_1(B) \), \( p \in [2, 2^*_1] \).

(i) We call \( u \in \mathcal{H} \) a weak solution of (1.5) with \( \alpha = 1 \) if

\[
\langle u, v \rangle_{\mathcal{H}} = \int_B (\|p^{\alpha/2}uv - mvu\|_{L^q}^q) \quad \text{for every } v \in \mathcal{H}.
\]

(ii) A weak solution \( u \in \mathcal{H} \) of (1.5) with \( \alpha = 1 \) will be called a ground state solution if \( u \) is a minimizer for \( R_{1,m,p} \), i.e., we have \( R_{1,m,p}(u) = \mathcal{C}_{1,m,p}(B) \).

We then have the following existence result which replaces Proposition 4.4 in the degenerate elliptic case \( \alpha = 1 \).

**Proposition 4.8.** Let \( 1 < p < 2^*_1 \) and \( m > -\lambda_1(B) \). Then we have

\[
\mathcal{C}_{1,m,p}(B) > 0,
\]

and there exists \( u_0 \in \mathcal{H} \setminus \{0\} \) with \( R_{1,m,p}(u_0) = \mathcal{C}_{1,m,p}(B) \), i.e., \( u_0 \) minimizes \( R_{1,m,p} \) in \( \mathcal{H} \setminus \{0\} \). Furthermore, after multiplication by a positive constant, \( u_0 \) is ground state solution of (1.5) with \( \alpha = 1 \) and \( u_0 \in C^{2,\sigma}_{\text{loc}}(B) \) for some \( \sigma > 0 \).

**Proof.** Proving the existence of \( u_0 \) is completely analogous to the proof of Lemma 4.4, making use of the Rellich-Kondrachov type result stated in Lemma 4.6(i).

In order to prove the regularity result, we first note that a Moser iteration scheme can be used to show that \( u_0 \in L^\infty(B) \), see Lemma A.1 in the appendix for a detailed proof. For any fixed \( s \in (0, 1) \) we may then use the fact that the operator \( -\Delta + \partial_0^s \) is uniformly elliptic in the ball \( B_s = \{ x \in \mathbb{R}^N : |x| < s \} \) and classical elliptic regularity theory, to show \( u_0 \in C^{2,\sigma}_{\text{loc}}(B_s) \). \( \square \)

Next, we treat the critical case \( p = 2^*_1 \), and first show that \( \mathcal{C}_{1,m,2^*_1}(B) \) is attained, provided it is small enough.

**Theorem 4.9.** Let \( m > -\lambda_1(B) \) such that \( \mathcal{C}_{1,m,2^*_1}(B) < 2^{\frac{1}{2^*_1 - 1}} S_1(\mathbb{R}^N_+) \). Then the value \( \mathcal{C}_{1,m,2^*_1}(B) \) is attained in \( \mathcal{H} \setminus \{0\} \).
In particular, this proves the first part of Theorem 1.7. The strategy of the proof is inspired by [17] and first requires the following characterization of sequences in H:

**Lemma 4.10.** Let

\[ Z(v) := \int_B \left( |\nabla v|^2 - |\partial_\theta v|^2 + mv^2 \right) \, dx \quad \text{and} \quad N(v) := \int_B |v|^{2^*_m} \, dx \quad \text{for} \ v \in \mathcal{H}. \]

Then we have

\[ 2^{\frac{1}{2}} - \frac{1}{2} S_1(\mathbb{R}^N) \leq \inf \left\{ \liminf_{n \to \infty} Z(w_n) : (w_n)_n \subset \mathcal{H}, \ N(w_n) = 1, \ w_n \to 0 \ \text{in} \ \mathcal{H} \right\}. \]

**Proof.** Let \((w_n)_n \subset \mathcal{H}\) such that \(N(w_n) = 1, \ w_n \to 0 \ \text{in} \ \mathcal{H}\). Let \(\varepsilon > 0\) and choose \(U_0, \ldots, U_m \subset B\) as in the proof of Theorem 3.3, so that

\[ B \subset \bigcup_{k=0}^m U_k. \]

We may then choose functions \(\eta_0, \ldots, \eta_m \in C^2_c(B)\) such that \(\text{supp} \ \eta_k \subset U_k\) and \(\sum_{k=0}^m \eta_k^2 = 1\) on \(B\). Then

\[ \int_B \left( |\nabla (\eta_k w_n)|^2 - |\partial_\theta \eta_k w_n|^2 \right) \, dx = \int_B \left( \eta_k^2 |\nabla w_n|^2 + 2 \eta_k \nabla w_n \cdot \nabla \eta_k + w_n^2 |\nabla \eta_k|^2 \right) \, dx \]

\[ - \int_B \left( \eta_k^2 |\partial_\theta w_n|^2 + 2 \eta_k \partial_\theta w_n \cdot \partial_\theta \eta_k + w_n^2 |\partial_\theta \eta_k|^2 \right) \, dx \]

and thus

\[ \int_B \left( |\nabla w_n|^2 - |\partial_\theta w_n|^2 + mw_n^2 \right) \, dx \geq \sum_{k=0}^m \int_B \left( |\nabla (\eta_k w_n)|^2 - |\partial_\theta \eta_k w_n|^2 \right) \, dx - C \int_B w_n^2 \, dx \]

with a constant \(C > 0\) independent of \(n\). Here we used the fact that the mixed terms can be estimated as follows:

\[ \int_B w_n^2 \left( |\nabla \eta_k|^2 - |\partial_\theta \eta_k|^2 \right) \, dx \leq 2 \sup_{k \in \{0, \ldots, m\}} \| \nabla \eta_k \|_\infty^2 \int_B w_n^2 \, dx \]

\[ \int_B \eta_k w_n \left( \nabla w_n \cdot \nabla \eta_k - \partial_\theta w_n \partial_\theta \eta_k \right) \, dx \leq \int_B \eta_k w_n^2 \left| -\Delta \eta_k + \partial_\theta^2 \eta_k \right| \, dx \]

\[ \leq \sup_{k \in \{0, \ldots, m\}} \left\| -\Delta \eta_k + \partial_\theta^2 \eta_k \right\|_\infty \int_B |w_n|^2 \, dx. \]

We first note that \(w_n \to 0\) in \(L^2(B)\), since the embedding \(\mathcal{H} \hookrightarrow L^2(B)\) is compact by Lemma 4.6(i). Moreover, it is easy to see that \(\| \cdot \|_H\) induces an equivalent norm on \(H^1_{0}(U_0)\), which implies that \(\eta_0 w_n \to 0\) in \(H^1_{0}(U_0)\). In particular, noting that by \(2^*_1 < 2^*\) the classical Rellich-Kondrachov theorem implies \(\eta_0 w_n \to 0\) in \(L^{2^*}(B)\), we conclude

\[ \liminf_{n \to \infty} \int_B \left( |\nabla (\eta_0 w_n)|^2 - |\partial_\theta (\eta_0 w_n)|^2 + m(\eta_0 w_n)^2 \right) \, dx \geq \liminf_{n \to \infty} \left( \int_B |\eta_0 w_n|^{2^*_m} \, dx \right)^{\frac{2}{2^*_m}}. \]

On the other hand, Lemma 3.2 gives

\[ \int_B \left( |\nabla (\eta_k w_n)|^2 - |\partial_\theta \eta_k w_n|^2 \right) \, dx \geq (1 - \varepsilon) 2^{\frac{1}{2} - \frac{1}{2^*_m}} S_1(\mathbb{R}^N) \left( \int_B |\eta_k w_n|^{2^*_m} \, dx \right)^{\frac{2}{2^*_m}}. \]
for $k = 1, \ldots, m$ and hence
\[
\liminf_{n \to \infty} \int_B \left( |\nabla w_n|^2 - |\partial_y u_n|^2 + mw_n^2 \right) \, dx \geq \liminf_{n \to \infty} \sum_{k=0}^m \int_B \left( |\nabla (\eta_k w_n)|^2 - |\partial_y \eta_k w_n|^2 \right) \, dx
\]
\[
\geq (1 - \varepsilon) 2^{1 - \frac{N}{N_1}} \mathcal{S}_1(\mathbb{R}^N_+) \liminf_{n \to \infty} \sum_{k=0}^m \left( \int_B |\eta_k w_n|^2 \, dx \right)^{\frac{2}{N_1}}
\]
\[
= (1 - \varepsilon) 2^{1 - \frac{N}{N_1}} \mathcal{S}_1(\mathbb{R}^N_+) \liminf_{n \to \infty} \sum_{k=0}^m \left\| \eta_k w_n \right\|_{L^2}^{2^{\frac{N}{N_1}}}
\]
\[
= (1 - \varepsilon) 2^{1 - \frac{N}{N_1}} \mathcal{S}_1(\mathbb{R}^N_+) \liminf_{n \to \infty} \| w_n \|_{L^2}^{2^{\frac{N}{N_1}}}
\]
\[
= (1 - \varepsilon) 2^{1 - \frac{N}{N_1}} \mathcal{S}_1(\mathbb{R}^N_+).
\]
Since $\varepsilon > 0$ was arbitrary, we conclude that
\[
\liminf_{n \to \infty} \int_B \left( |\nabla w_n|^2 - |\partial_y u_n|^2 + mw_n^2 \right) \, dx \geq 2^{1 - \frac{N}{N_1}} \mathcal{S}_1(\mathbb{R}^N_+)
\]
as claimed. \hfill \Box

We may now complete the proof of our main result.

**Proof of Theorem 4.9.** Consider a minimizing sequence $(u_n)_n \subset \mathcal{H}$ for $\mathcal{C}_{1,m,2^*_1}(B)$ with $\|u_n\|_{2^*_1} = 1$. Then $(u_n)_n$ is bounded in $\mathcal{H}$, hence, after passing to a subsequence, we may assume $u_n \rightharpoonup u_0$ in $\mathcal{H}$. We set $v_n := u_n - u_0$ and note that, by Sobolev embeddings,
\[
v_n \to 0 \quad \text{in} \quad L^q(B_s)
\]
for $1 \leq q < 2^*_1$ and $0 < s < 1$, where $B_s := \{ x \in \mathbb{R}^N : |x| < s \}$. Weak convergence implies
\[
\mathcal{C}_{1,m,2^*_1}(B) = \lim_{n \to \infty} Z(u_n) = Z(u_0) + \lim_{n \to \infty} Z(v_n),
\]
whereas the Brezis-Lieb Lemma yields
\[
1 = N(u_n) = N(u_0) + N(v_n) + o(1).
\]
In particular, the limits
\[
T := \lim_{n \to \infty} N(v_n), \quad M := \lim_{n \to \infty} Z(v_n)
\]
exist. If $T = 0$, it follows that $N(u_0) = 1$ and we are finished. For $T > 0$, Lemma 4.10 implies
\[
M \geq \mathcal{S}_1(\mathbb{R}^N_+) T^{\frac{2}{N_1}}
\]
and hence
\[
\mathcal{C}_{1,m,2^*_1}(B) = Z(u_0) + M \geq Z(u_0) + 2^{1 - \frac{N}{N_1}} \mathcal{S}_1(\mathbb{R}^N_+) T^{\frac{2}{N_1}}
\]
\[
\geq Z(u_0) + \left( 2^{1 - \frac{N}{N_1}} \mathcal{S}_1(\mathbb{R}^N_+) - \mathcal{C}_{1,m,2^*_1}(B) \right) T^{\frac{2}{N_1}} + \mathcal{C}_{1,m,2^*_1}(B) (1 - N(u_0))^\frac{2}{N_1}
\]
\[
\geq Z(u_0) + (2^{1 - \frac{N}{N_1}} \mathcal{S}_1(\mathbb{R}^N_+) - \mathcal{C}_{1,m,2^*_1}(B)) T^{\frac{2}{N_1}} + \mathcal{C}_{1,m,2^*_1}(B) - \mathcal{C}_{1,m,2^*_1}(B) N(u_0)^\frac{2}{N_1},
\]
where we used the inequality $(a - b)^\tau \geq a^\tau - b^\tau$ for $a \geq b \geq 0$ and $0 \leq \tau \leq 1$. It follows that
\[
Z(u_0) + (2^{\frac{1}{2} - \frac{1}{2\tau}} S_1(\mathbb{R}^N_+) - \mathcal{C}_{1,m,2^*_1}(B)) T^{\frac{2}{2\tau}} - \mathcal{C}_{1,m,2^*_1}(B) N(u_0) T^{\frac{2}{2\tau}} \leq 0,
\]
and therefore
\[
\int_B \left( |\nabla u_0|^2 - |\partial_\theta u_0|^2 + m u_0^2 \right) dx - \mathcal{C}_{1,m,2^*_1}(B) \left( \int_B |u_0|^{2^*_1} dx \right)^{\frac{2}{2^*_1}} + (2^{\frac{1}{2} - \frac{1}{2\tau}} S_1(\mathbb{R}^N_+) - \mathcal{C}_{1,m,2^*_1}(B)) T^{\frac{2}{2\tau}} \leq 0.
\]
By definition, we have $\int_B \left( |\nabla u_0|^2 - |\partial_\theta u_0|^2 + m u_0^2 \right) dx - \mathcal{C}_{1,m,2^*_1}(B) \left( \int_B |u_0|^{2^*_1} dx \right)^{\frac{2}{2^*_1}} \geq 0$ and since $2^{\frac{1}{2} - \frac{1}{2\tau}} S_1(\mathbb{R}^N_+) - \mathcal{C}_{1,m,2^*_1}(B) > 0$ by assumption, we must have $T = 0$, i.e. $v_n \to 0$ in $L^p(\Omega)$. It follows that $u_0 \neq 0$ and $\int_B |u_0|^{2^*_1} dx = 1$, and (4.5) gives
\[
\int_B \left( |\nabla u_0|^2 - |\partial_\theta u_0|^2 + m u_0^2 \right) dx \leq \mathcal{C}_{1,m,2^*_1}(B) \left( \int_B |u_0|^{2^*_1} dx \right)^{\frac{2}{2^*_1}},
\]
which implies that $u_0$ is a minimizer. \qed

We note the following consequence of Theorem 4.9, which extends (4.4) to the critical case.

**Corollary 4.11.** We have $\mathcal{C}_{1,m,2^*_1}(B) > 0$.

**Proof.** If the value $\mathcal{C}_{1,m,2^*_1}(B)$ is attained in $H \setminus \{0\}$, then we have $\mathcal{C}_{1,m,2^*_1}(B) > 0$ by Lemma 4.6(ii). If not, we have $\mathcal{C}_{1,m,2^*_1}(B) \geq S_1(\mathbb{R}^N_+) > 0$ by Theorem 4.9 and Theorem 2.2. \qed

In general, the existence of ground state solutions in the case $\alpha = 1$, $p = 2^*_1$ remains an open problem and might depend on the parameter $m > -\lambda_1(B)$. We have the following partial existence result in the critical case.

**Theorem 4.12.** There exists $\varepsilon > 0$, such that for $m \in (-\lambda_1(B), -\lambda_1(B) + \varepsilon)$ there exists $u_0 \in H \setminus \{0\}$ such that
\[
R_{1,m,2^*_1}(u_0) = \inf_{u \in H \setminus \{0\}} R_{1,m,2^*_1}(u),
\]
i.e. $u_0$ minimizes $R_{1,m,2^*_1}$. Furthermore, after multiplication by a positive constant, $u_0$ is a weak solution of
\[
\begin{cases}
-\Delta u + \partial_\theta^2 u + mu = |u|^{2^*_1-2} u & \text{in } B, \\
u = 0 & \text{on } \partial B,
\end{cases}
\]
i.e., $u_0$ satisfies
\[
\int_B \nabla u \cdot \nabla \varphi - \partial_\theta u \partial_\theta \varphi + m u \varphi dx = \int_B |u|^{2^*_1-2} u \varphi dx
\]
for all $\varphi \in H$.

**Proof.** For a (necessarily radial) eigenfunction $\varphi_1$ of $-\Delta$ on $B$ corresponding to $\lambda_1(B)$, we have
\[
\mathcal{C}_{1,m,2^*_1}(B) \leq R_{1,m,2^*_1}(\varphi_1) = \frac{(\lambda_1(B) + m) \int_B \varphi_1^2 dx}{(\int_B |\varphi_1|^p dx)^{\frac{2}{p}}},
\]
which implies \( \mathcal{G}_{1,m,2}^+(B) \to 0 \) as \( m \to -\lambda_1(B)^+ \). In particular, it follows that there exists \( \varepsilon > 0 \) such that
\[
\mathcal{G}_{1,m,2}^+(B) < 2^{\frac{1}{p-2}} \mathcal{S}_1(\mathbb{R}^N)
\]
holds for \( m \in (-\lambda_1(B), -\lambda_1(B) + \varepsilon) \). By Theorem 4.9, this finishes the proof.

Note that this completes the proof of Theorem 1.7.

4.3. Radiality versus \( x_1-x_2 \)-nonradiality of ground state solutions. By classical results due to McLeod and Serrin [33], Kwong [26], Kwong and Li [27] (see also references in [12]), the problem
\[
\begin{cases}
-\Delta u + mu = |u|^{p-2}u & \text{in } B \\
u = 0 & \text{on } \partial B,
\end{cases}
\]
has a unique radial positive solution \( u_{rad} \in H_0^1(B) \) which is a minimizer for \( \mathcal{G}_{0,m,p}(B) \). Clearly, \( u_{rad} \) is also a weak solution of \((1.5)\) for every \( \alpha > 0 \), but it might not be a ground state solution. In fact, we have the following.

**Lemma 4.13.** Let \( 2 < p < 2^* \) and \( m > -\lambda_1(B) \) be fixed.

(i) The map
\[
[0,1] \to \mathbb{R}, \quad \alpha \mapsto \mathcal{G}_{\alpha,m,p}(B)
\]
is continuous and nonincreasing.

(ii) Let \( \alpha \in (0,1] \), and suppose that \( p \leq 2^*_1 \) in the case \( \alpha = 1 \). Then the following properties are equivalent:

(i) \( \mathcal{G}_{\alpha,m,p}(B) < \mathcal{G}_{0,m,p}(B) \).

(ii) Every ground state solution of \((1.5)\) is \( x_1-x_2 \)-nonradial.

**Proof.** (i) The monotonicity of \( \mathcal{G}_{\alpha,m,p}(B) \) in \( \alpha \) follows immediately from the definition. In order to prove continuity, we first consider \( \alpha_0 \in (0,1] \) and let \( \varepsilon > 0 \). Moreover, we let \( u_0 \in H_0^1(B) \setminus \{0\} \) be a function with \( R_{\alpha_0,m,p}(u_0) < \mathcal{G}_{\alpha_0,m,p}(B) + \varepsilon \). For \( \alpha \leq \alpha_0 \), we then have
\[
\mathcal{G}_{\alpha_0,m,p}(B) \leq \mathcal{G}_{\alpha,m,p}(B) \leq R_{\alpha,m,p}(u_0)
\]
\[
\leq R_{\alpha_0,m,p}(u_0) + (\alpha_0^2 - \alpha^2) \int_B |\partial_\theta u_0|^2 dx
\]
\[
\leq \mathcal{G}_{\alpha_0,m,p}(B) + (\alpha_0^2 - \alpha^2) \int_B |\partial_\theta u_0|^2 dx
\]
which implies that \( \limsup_{\alpha \to \alpha_0} |\mathcal{G}_{\alpha,m,p}(B) - \mathcal{G}_{\alpha_0,m,p}(B)| \leq \varepsilon \). This shows the continuity from the left in \( \alpha_0 \).

Next we let \( \alpha_0 \in [0,1) \) and show continuity from the right in \( \alpha_0 \). For this we fix \( \delta > 0 \) such that \( (\alpha_0, \alpha_0 + \delta) \subset (0,1) \). For \( \alpha \in (\alpha_0, \alpha_0 + \delta) \), Lemma 4.4 implies that the value \( \mathcal{G}_{\alpha,m,p}(B) \) is attained at a function \( u_\alpha \in H_0^1(B) \setminus \{0\} \) with \( \int_B |u_\alpha|^p dx = 1 \). Therefore
\[
\mathcal{G}_{\alpha_0,m,p}(B) \geq \mathcal{G}_{\alpha,m,p}(B) = R_{\alpha,m,p}(u_\alpha) = R_{\alpha_0,m,p}(u_\alpha) + (\alpha_0^2 - \alpha^2) \int_B |\partial_\theta u_\alpha|^2 dx
\]
\[
\geq \mathcal{G}_{\alpha_0,m,p}(B) - |\alpha_0^2 - \alpha^2| \int_B |\nabla u_\alpha|^2 dx,
\]
whence, using the fact that
\[(1 - \alpha^2) \int_B |\nabla u_\alpha|^2 \, dx \leq \int_B \left( |\nabla u_\alpha|^2 - \alpha^2 |\partial_\theta u_\alpha|^2 \right) \, dx = \mathcal{E}_{\alpha,m,p}(B) \leq \mathcal{E}_{0,m,p}(B),\]
we conclude
\[\mathcal{E}_{\alpha_0,m,p}(B) \geq \mathcal{E}_{\alpha,m,p}(B) \geq \mathcal{E}_{\alpha_0,m,p}(B) - \frac{\alpha_0^2 - \alpha^2}{1 - \alpha^2} \mathcal{E}_{0,m,p}(B)\]
\[\geq \mathcal{E}_{\alpha_0,m,p}(B) - \frac{\alpha_0^2 - \alpha^2}{1 - (\alpha_0 + \delta)^2} \mathcal{E}_{0,m,p}(B).\]
This shows the continuity from the right in \(a_0\).

(ii) As noted above, \(\mathcal{E}_{0,m,p}(B)\) is attained by a radial positive solution \(u_{rad}\) of (4.6) and we have \(R_{0,m,p}(u_{rad}) = R_{\alpha,m,p}(u_{rad})\). Hence, if \(\mathcal{E}_{0,m,p}(B) = \mathcal{E}_{\alpha,m,p}(B)\), then \(u_{rad}\) is also a radial ground state solution of (1.5). Hence (ii)2 and (i) imply that \(\mathcal{E}_{\alpha,m,p}(B) < \mathcal{E}_{0,m,p}(B)\).
If, conversely, there exists a radial ground state solution \(u\) of (1.5), then we have
\[\mathcal{E}_{0,m,p}(B) \leq R_{0,m,p}(u) = R_{\alpha,m,p}(u) = \mathcal{E}_{\alpha,m,p}(B)\]
and therefore equality holds by (i). Consequently, the \(\mathcal{E}_{\alpha,m,p}(B) < \mathcal{E}_{0,m,p}(B)\) implies that every ground state solution of (1.5) is \(x_1-x_2\)-nonradial. \(\square\)

The second part of this section is devoted to the proof of Theorem 1.3, which yields radiality of ground state solutions for \(\alpha\) close to zero. For this, we fix \(m \geq 0\) and \(2 < p < 2^*\). Moreover, we consider a sequence of numbers \(\alpha_n \in (0,1)\), \(\alpha_n \to 0\) and, for every \(n \in \mathbb{N}\), a positive ground state solution \(u_n \in H^1_0(B)\) of (1.5) with \(\alpha = \alpha_n\). Recall that the existence of \(u_n\) is proved in Lemma 4.4. To prove Theorem 1.3, it then suffices to show that
\[u_n = u_{rad}\]
for \(n\) sufficiently large, (4.7)
where \(u_{rad}\) is the unique positive solution of (4.6). We first claim the following.

Lemma 4.14. \(u_n \to u_{rad}\) in \(H^1_0(B)\) as \(n \to \infty\).

Proof. We put \(v_n := \frac{u_n}{\|u_n\|_{L^p(B)}}\), so \(v_n\) is an \(L^p\)-normalized minimizer for \(\mathcal{E}_{\alpha_n,m,p}(B)\). Then \((v_n)\) is bounded in \(H^1_0(B)\) by definition of \(\mathcal{E}_{\alpha_n,m,p}(B)\). Consequently, we have \(v_n \to v_0\) in \(H^1_0(B)\) after passing to a subsequence, which implies that \(v_n \to v_0\) in \(L^p(B)\) and therefore \(\int_B |v_0|^p \, dx = 1\). We show that \(v_0\) is minimizer for \(\mathcal{E}_{0,m,p}(B)\). Indeed, by weak lower semicontinuity, we have
\[\mathcal{E}_{0,m,p}(B) \leq R_{0,m,p}(v_0) \leq \liminf_{n \to \infty} R_{0,m,p}(v_n) \leq \lim_{n \to \infty} \left( R_{\alpha_n,m,p}(v_n) + \alpha_n^2 \|\partial_\theta v_n\|_{L^2(B)}^2 \right)\]
\[\leq \lim_{n \to \infty} \mathcal{E}_{\alpha_n,m,p}(B) + \alpha_n \|u_n\|_{H^1_0(B)}^2 = \mathcal{E}_{0,m,p}(B),\]
where we used Lemma 4.13 in the last step. Hence \(v_0\) is a minimizer of \(\mathcal{E}_{0,m,p}(B)\), and a posteriori we find that
\[\|\nabla v_n\|_{L^2(B)}^2 + m\|v_n\|_{L^2(B)}^2 = R_{\alpha_n,m,p}(v_n) + \alpha_n^2 \|\partial_\theta v_n\|_{L^2(B)}^2\]
\[\to R_{0,m,p}(v_0) = \|\nabla v_0\|_{L^2(B)}^2 + m\|v_0\|_{L^2(B)}^2\]
as \(n \to \infty\).

By uniform convexity of \(H^1_0(B)\), we thus conclude that \(v_n \to v_0\) in \(H^1_0(B)\). Next we recall that, as noted in the proof of Lemma 4.4, we have
\[u_n := \left[ \mathcal{E}_{\alpha_n,m,p}(B) \right]^{\frac{1}{p-2}} v_n\]and, by uniqueness,
\[u_{rad} := \left[ \mathcal{E}_{\alpha,m,p}(B) \right]^{\frac{1}{p-2}} v_0.\]
Hence Lemma 4.13 implies that \( u_n \to u_{\text{rad}} \) in \( H^1_0(B) \). Although we have proved this only after passing to a subsequence, the convergence of the full sequence now follows from the uniqueness of \( u_{\text{rad}} \). The proof is thus finished. \( \square \)

Next, we improve Lemma 4.14 by noting that

\[ u_n \to u_{\text{rad}} \quad \text{in} \quad H^2(B). \]  \( (4.8) \)

This follows in a standard way from Lemma 4.14 and standard elliptic regularity theory (see e.g. [20, Theorem 8.12]), since \( u_n = u_{\text{rad}} - u_n \in H^1_0(B) \) is a weak solution of

\[
\begin{align*}
-\Delta w_n + \alpha^2 \partial_\theta w_n + mw_n &= |v_{\text{rad}}|^{p-2} v_{\text{rad}} - |v_n|^{p-2} v_n \quad \text{in} \ B \\
w_n &= 0 \quad \text{on} \ \partial B,
\end{align*}
\]

and the coefficients of the differential operator \(-\Delta + \alpha^2 \partial_\theta\) are uniformly bounded and elliptic in \( n \in \mathbb{N} \).

We may now complete the proof of our main result as follows.

**Proof of Theorem 1.3.** To complete the proof of \((4.7)\), we consider the map

\[ F : (-1, 1) \times H^2(B) \cap H^1_0(B) \to L^2(B), \quad F(\alpha, u) := -\Delta u + \alpha^2 \partial_\theta u + mu - |u|^{p-2} u, \]

and we note that weak solutions of \((1.5)\) correspond to zeroes of \( F \). We also note that \( F(\alpha, u_{\text{rad}}) = 0 \) for all \( \alpha \). We wish to apply the implicit function theorem at \((0, u_{\text{rad}})\), so we need to check that

\[ [\partial_u F](0, u_{\text{rad}}) = -\Delta + m - (p-1)|u_{\text{rad}}|^{p-2} \]

is invertible as a map \( H^2(B) \cap H^1_0(B) \to L^2(B) \). This is equivalent to the nondegeneracy of \( u_{\text{rad}} \) as a solution of \((4.6)\) which is noted e.g. in [12, Theorem 4.2] for \( m = 0 \) and in [1, Theorem 1.1] in the case \( m > 0 \). Now the implicit function theorem yields \( \varepsilon > 0 \) with the following property: If \( u \in H^2(B) \cap H^1_0(B) \) satisfies \( \|u - u_{\text{rad}}\|_{H^2(B)} < \varepsilon \) and \( F(\alpha, u) = 0 \) for some \( \alpha \in (-\varepsilon, \varepsilon) \), then \( u = u_{\text{rad}} \).

Hence Lemma 4.8 implies that \( u_n = u_{\text{rad}} \) for \( n \) sufficiently large, which shows \((4.7)\), as claimed. \( \square \)

In the remainder of this section, we show \( x_1-x_2\)-nonradial ground states for large \( m \), as claimed in Theorem 1.4. We restate this theorem here in an equivalent form.

**Theorem 4.15.** Let \( \alpha \in (0, 1) \) and \( 2 < p < 2^* \). Then there exists \( \varepsilon_0 > 0 \), such that the ground states of

\[
\begin{align*}
-\Delta u + \alpha^2 \partial_\theta^2 u + \frac{1}{\varepsilon^2} u &= |u|^{p-2} u \quad \text{in} \ B, \\
u &= 0 \quad \text{on} \ \partial B,
\end{align*}
\]

are \( x_1-x_2\)-nonradial for \( \varepsilon \in (0, \varepsilon_0) \). Moreover, if \( p < 2^*_1 \), the same result holds for \( \alpha = 1 \).

**Proof.** We first treat the case \( \alpha \in (0, 1) \). In the following, for \( u \in H^1_0(B) \) and \( \varepsilon > 0 \), we consider \( B_{1/\varepsilon} := B_{1/\varepsilon}(0) \) and the rescaled function \( u_\varepsilon \in H^1_0(B_{1/\varepsilon}) \), \( u_\varepsilon(x) = u(\varepsilon x) \). A direct computation then shows that

\[
\frac{\int_{B_{1/\varepsilon}} (|\nabla u_\varepsilon|^2 - \alpha^2 \varepsilon^2 |\partial_\theta u_\varepsilon|^2 + u_\varepsilon^2) \, dx}{\left(\int_{B_{1/\varepsilon}} |u_\varepsilon|^p \, dx\right)^{\frac{2}{p}}} = \varepsilon^{2-N+2\frac{N}{p}} \frac{1}{R_{\alpha, p}}(u).
\]  \( (4.10) \)
As a consequence, we have
\[
\mathcal{C}_{\alpha,1,p}(B_{1/\varepsilon}) := \inf_{v \in H^1_0(B_{1/\varepsilon}) \setminus \{0\}} \frac{\int_{B_{1/\varepsilon}} (|\nabla v|^2 - \alpha^2 \varepsilon^2 |\partial_\eta v|^2 + v^2) \, dx}{\left( \int_{B_{1/\varepsilon}} |v|^p \, dx \right)^{2/p}} = \varepsilon^{2-N+2} \mathcal{C}_{\alpha,1,2^N/p}(B).
\]

It therefore suffices to show that there exists \( \varepsilon_0 > 0 \) such that all minimizers for \( \mathcal{C}_{\alpha,1,p}(B_{1/\varepsilon}) \) in \( H^1_0(B_{1/\varepsilon}) \setminus \{0\} \) are \( x_1-x_2 \)-nonradial if \( \varepsilon \in (0, \varepsilon_0) \). We argue by contradiction and suppose that there exists a sequence \( \varepsilon_n \to 0 \) and, for every \( n \in \mathbb{N} \), a minimizer \( v_{\varepsilon_n} \in H^1_0(B_{1/\varepsilon_n}) \setminus \{0\} \) for \( \mathcal{C}_{\alpha,\varepsilon_n,1,p}(B_{1/\varepsilon_n}) \) which satisfies
\[
\partial_\varepsilon v_{\varepsilon_n} \equiv 0 \quad \text{in } B_{1/\varepsilon_n}.
\] (4.11)

To simplify the notation, we continue writing \( \varepsilon \) in place of \( \varepsilon_n \) in the following. From (4.11) and the inclusion \( H^1_0(B_{1/\varepsilon}) \subset H^1(\mathbb{R}^N) \), we then deduce that
\[
\mathcal{C}_{\alpha,1,p}(B_{1/\varepsilon}) = \frac{\int_{B_{1/\varepsilon}} (|\nabla v_{\varepsilon}|^2 + v^2) \, dx}{\left( \int_{B_{1/\varepsilon}} |v|^p \, dx \right)^{2/p}} \geq \inf_{v \in H^1(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} (|\nabla v|^2 + v^2) \, dx}{\left( \int_{\mathbb{R}^N} |v|^p \, dx \right)^{2/p}} =: \mathcal{C}_{0,1,p}(\mathbb{R}^N). \] (4.12)

We will now derive a contradiction to this inequality by constructing suitable functions in \( H^1_0(B_{1/\varepsilon}) \) to estimate \( \mathcal{C}_{\alpha,1,1,p}(B_{1/\varepsilon}) \). To this end, we first note that the value \( \mathcal{C}_{0,1,p}(\mathbb{R}^N) \) is attained by any translation of the unique positive radial solution \( \tilde{u}_0 \in H^1(\mathbb{R}^N) \) of the nonlinear Schrödinger equation
\[
-\Delta u + u = |u|^{p-2}u \quad \text{in } \mathbb{R}^N.
\]

Now take a radial function \( \eta \in C^1_0(B) \) such that \( 0 \leq \eta \leq 1 \) and \( \eta \equiv 1 \) in \( B_{1/2} \), and let \( u_0(x) := \tilde{u}_0(x-e_1) \) where \( e_1 = (1,0,\ldots,0) \). We then define
\[
\eta_\varepsilon, \ w_\varepsilon \in C^1_0(B_{1/\varepsilon}) \quad \text{by} \quad \eta_\varepsilon(x) = \eta(\varepsilon x), \ w_\varepsilon(x) = \eta_\varepsilon(x)u_0(x).
\]

Then we have \( w_\varepsilon \equiv u_0 \) in \( B_{1/(2\varepsilon)} \), and
\[
\mathcal{C}_{\alpha,1,p}(B_{1/\varepsilon}) \leq \frac{\int_{B_{1/\varepsilon}} (|\nabla w_\varepsilon|^2 - \alpha^2 \varepsilon^2 |\partial_\eta w_\varepsilon|^2 + w_\varepsilon^2) \, dx}{\left( \int_{B_{1/\varepsilon}} |w_\varepsilon|^p \, dx \right)^{2/p}} \leq \frac{\int_{B_{1/\varepsilon}} (|\nabla u_0|^2 + u_0^2) \, dx}{\left( \int_{B_{1/\varepsilon}} |u_0|^p \, dx \right)^{2/p}} \leq \frac{\int_{B_{1/\varepsilon}} (u_0^2 |\nabla \eta_\varepsilon|^2 + 2\eta_\varepsilon |\nabla u_0| |\nabla \eta_\varepsilon| \cdot |\nabla u_0| - \alpha^2 \varepsilon^2 |\partial_\eta u_0|^2) \, dx}{\left( \int_{B_{1/\varepsilon}} |u_0|^p \, dx \right)^{2/p}}.
\] (4.13)

We first estimate the second term and note that classical results (see [7]) imply that there exist \( C_0, \delta_0 > 0 \), such that
\[
|u_0(x)|, |\nabla u_0(x)| \leq C_0 e^{-\delta_0 |x|} \quad \text{for } x \in \mathbb{R}^N.
\] (4.14)

Noting that \( \nabla \eta_\varepsilon \equiv 0 \) on \( B_{1/(2\varepsilon)} \), this readily implies
\[
\int_{B_{1/\varepsilon}} \left( u_0^2 |\nabla \eta_\varepsilon|^2 + 2\eta_\varepsilon u_0 |\nabla \eta_\varepsilon| \cdot |\nabla u_0| \right) \, dx \leq C_1 e^{-\frac{4\delta_0}{\varepsilon}}
\]
for some constants $C_1, \delta_1 > 0$. Moreover, for $\varepsilon \in (0, 1)$ we have
\[
\alpha^2 \varepsilon^2 \int_{B_{1/\varepsilon}} \eta^2 \partial_\theta u_0^2 \, dx \geq C_2 \varepsilon^2 \quad \text{with} \quad C_2 := \alpha^2 \int_B |\partial_\theta u_0|^2 \, dx > 0,
\]
since $u_0$ is an $x_1$-$x_2$-nonradial function. After possibly modifying $C_1, C_2 > 0$, this gives
\[
\int_{B_{1/\varepsilon}} (u_0^2 |\nabla \eta_e|^2 + 2 \eta_e u_0 \nabla \eta_e \cdot \nabla u_0 - \alpha^2 \varepsilon^2 |\partial_\theta u_0|^2) \, dx \leq C_1 e^{-\frac{\delta_1}{\varepsilon^2}} - C_2 \varepsilon^2.
\]
Next we consider the first term in (4.13) and note that
\[
\int_{B_{1/\varepsilon}} \frac{\eta^2_e (|\nabla u_0|^2 + u_0^2)}{\left(\int_{B_{1/\varepsilon}} \eta^p_e |u_0|^p \, dx\right)^{\frac{2}{p}}} \leq \int_{\mathbb{R}^N} (|\nabla u_0|^2 + u_0^2) \, dx,
\]
while (4.14) implies
\[
\int_{\mathbb{R}^N \setminus B_{1/2\varepsilon}} |u_0|^p \, dx \leq C_3 e^{-\frac{\delta_2}{\varepsilon^2}}
\]
for some $C_3, \delta_2 > 0$. It thus follows that
\[
\int_{B_{1/\varepsilon}} \frac{\eta^2_e (|\nabla u_0|^2 + u_0^2)}{\left(\int_{B_{1/\varepsilon}} \eta^p_e |u_0|^p \, dx\right)^{\frac{2}{p}}} \leq \int_{\mathbb{R}^N} (|\nabla u_0|^2 + u_0^2) \, dx \leq \int_{\mathbb{R}^N} (|\nabla u_0|^2 + u_0^2) \, dx - \frac{e^{-\frac{\delta_2}{\varepsilon^2}}}{C_4} + C_4 e^{-\frac{2\delta_2}{\varepsilon^2}} = \mathcal{C}_{0,1,p}(\mathbb{R}^N) + C_4 e^{-\frac{2\delta_2}{\varepsilon^2}}
\]
for $\varepsilon > 0$ sufficiently small with some constant $C_4 > 0$, since $u_0$ attains $\mathcal{C}_{0,1,p}(\mathbb{R}^N)$. In view of (4.12) and (4.13), this yields that
\[
\mathcal{C}_{0,1,p}(\mathbb{R}^N) \leq \mathcal{C}_{0,1,p}(B_{1/\varepsilon}) \leq \mathcal{C}_{0,1,p}(\mathbb{R}^N) - C_2 \varepsilon^2 + C_1 e^{-\frac{\delta_1}{\varepsilon^2}} + C_4 e^{-\frac{2\delta_2}{\varepsilon^2}},
\]
and the right hand side of this inequality is smaller than $\mathcal{C}_{0,1,p}(\mathbb{R}^N)$ if $\varepsilon > 0$ is sufficiently small. This is a contradiction, and thus the claim follows in this case.

In the case $\alpha = 1$, the argument is the same up to replacing $H^1_0(B)$ by $H$ and by considering the corresponding rescaled function space $H_{\varepsilon}$ on $B_{1/\varepsilon}$. Then the contradiction argument can be carried out in the same way, since radial functions in $H_{\varepsilon}$ belong to $H^1_0(B_{1/\varepsilon}) \subset H^1(\mathbb{R}^N)$. \hfill \Box

5. The case of an annulus

In this section, we consider rotating solutions of (1.3) in the case where $B$ is replaced by an annulus
\[
A_r := \{x \in \mathbb{R}^N : r < |x| < 1\}
\]
for some $r \in (0, 1)$. The ansatz (1.4) then leads to the reduced problem
\[
\begin{cases}
-\Delta u + \alpha^2 \partial_\theta^2 u + mu = |u|^{p-2} u & \text{in } A_r, \\
u = 0 & \text{on } \partial A_r
\end{cases}
\]
where \( m > -\lambda_1(A_r) \), \( p \in (2, \frac{2N}{N-2}) \) and \( \partial \theta = x_{N-1} \partial_{x_N} - x_N \partial_{x_{N-1}} \) as before. Here, \( \lambda_1(A_r) \) denotes the first Dirichlet eigenvalue of \( -\Delta \) on \( A_r \). As in (1.6), we may then define
\[
G_{\alpha,m,p}(A_r) := \inf_{u \in H^1_0(A_r) \setminus \{0\}} R_{\alpha,m,p}(u) \tag{5.2}
\]
with the Rayleigh quotient \( R_{\alpha,m,p}(u) \) given by (1.7) for functions \( u \in H^1_0(A_r) \). In the following, a weak solution of (5.1) will be called a ground state solution if it is a minimizer for (5.2).

We then have the following analogue of Theorem 5.1.

**Theorem 5.1.** Let \( r \in (0,1) \), \( m > -\lambda_1(A_r) \) and \( p \in (2,2^*) \).

(i) If \( \alpha \in (0,1) \), then there exists a ground state solution of (5.1).

(ii) We have
\[
G_{1,m,p}(A_r) = 0 \quad \text{for} \quad p > 2^*_1, \quad \text{and} \quad G_{1,m,p}(A_r) > 0 \quad \text{for} \quad p \leq 2^*_1.
\]

Moreover, for any \( p \in (2^*_1,2^* \] \), there exists \( \alpha_p \in (0,1) \) with the property that
\[
G_{\alpha,m,p}(A_r) < G_{0,m,p}(A_r) \quad \text{for} \quad \alpha \in (\alpha_p,1]
\]
and therefore every ground state solution of (5.1) is \( x_1-x_2 \)-nonradial for \( \alpha \in (\alpha_p,1] \).

This theorem does not come as a surprise and is proved by precisely the same arguments as Theorem 1.1, so we omit the proof. Instead, we now discuss an interesting additional feature of the annulus \( A_r \). Unlike in the case of the ball, we can formulate explicit sufficient conditions for the parameters \( p, \alpha, m \) and \( r \) which guarantee that every ground state solution of (5.1) is \( x_1-x_2 \)-nonradial. This is the content of the following theorem.

**Theorem 5.2.** Let \( N \geq 2 \), \( m \geq 0 \), \( r, \alpha \in (0,1) \) and assume \( p > \frac{N-1-r^2}{\kappa(r,m)} + 2 \) with
\[
\kappa(r,m) = \begin{cases} \frac{mr^2 + \max \left\{ \left( \frac{N-2}{2} \right)^2, \left( \frac{\pi}{1-r} \right)^2 r^{N-1} \right\} }{N \geq 3;} \\ \frac{mr^2 + \left( \frac{\pi}{1-r} \right)^2 r^N }{N = 2.} \end{cases}
\tag{5.3}
\]

Then every ground state solution of (5.1) is \( x_1-x_2 \)-nonradial.

We point out that \( \kappa(m,r) \to \infty \) if \( m \to \infty \) or \( r \to 1^- \). Consequently, for given \( p > 2 \), ground states of (5.1) are nonradial if either \( m \) is large or the annulus is thin, i.e. \( r \) is close to 1. The proof is based on the following lemma.

**Lemma 5.3.** Suppose that \( m \geq 0 \), and that there exists a function \( v \in H^1_0(A_r) \) satisfying
\[
\int_{S^{N-1}} v(s,\cdot) \, d\sigma = 0 \quad \text{for every} \quad s \in (r,1) \tag{5.4}
\]
and
\[
\int_{A_r} \left( |\nabla v|^2 - \alpha^2 |\partial_{\theta} v|^2 + m v^2 \right) \, dx - (p-1) \int_{A_r} |u_0|^{p-2} v^2 \, dx < 0. \tag{5.5}
\]
Then we have
\[
G_{\alpha,m,p}(A_r) < R_{\alpha,m,p}(u_0), \tag{5.6}
\]
where \( u_0 \in H^1_0(A_r) \) is the unique positive radial solution of (5.1).

Here we note that in the case \( m = 0 \), the uniqueness of the positive radial solution \( u_0 \) of (5.1) has been first proved by Ni and Nussbaum [38]. In the case \( m > 0 \), the uniqueness is due to Tang [41] and Felmer, Martínez and Tanaka [15] for \( N \geq 3 \) and \( N = 2 \), respectively.
Proof. We argue by contradiction and assume that equality holds in (5.6). Then \( u_0 \) is a minimizer for \( \mathcal{C}_{\alpha,m,p}(A_r) \), which implies, in particular, that
\[
R'_{\alpha,m,p}(u_0)v = 0 \quad \text{and} \quad R''_{\alpha,m,p}(u_0)(v,v) \geq 0. \tag{5.7}
\]
In the following, we write \( R_{\alpha,m,p} = \frac{Z(u_0)}{N(u_0)} \) for \( u \in H^1(A_r) \setminus \{0\} \) with
\[
Z(u) := \int_{A_r} \left( |\nabla u|^2 - \alpha^2 |\partial_\theta u|^2 + mu^2 \right) \, dx \quad \text{and} \quad N(u) := \left( \int_{A_r} |u|^p \, dx \right)^{\frac{2}{p}}.
\]
The first property in (5.7) then implies \( N(u_0)Z'(u_0)v = Z(u_0)N'(u_0)v \) and consequently
\[
N(u_0)^3[R_{\alpha,m,p}]''(u_0)(v,v) = N(u_0)^2Z''(u_0)(v,v) - Z(u_0)N(u_0)N''(u_0)(v,v)
\]
for \( v \in H^1_0(A_r) \). Therefore, the second property in (5.7) yields
\[
Z''(u_0)(v,v) - \frac{Z(u_0)}{N(u_0)^2}N''(u_0)(v,v) \geq 0.
\]
Moreover, noting that \( u_0 \) is a weak solution of (5.1) and therefore \( Z(u_0) = N(u_0)^{\frac{p}{2}} \), we conclude that
\[
0 \leq \frac{1}{2} \left( Z''(u_0)(v,v) - \frac{Z(u_0)}{N(u_0)^2}N''(u_0)(v,v) \right)
= \int_{A_r} \left( |\nabla v|^2 - \alpha^2 |\partial_\theta v|^2 + mv^2 \right) \, dx - (p-1) \int_{A_r} |u_0|^{p-2}v^2 \, dx
+ (p-2)N(u_0)^{-\frac{p}{2}} \left( \int_{A_r} |u_0|^{p-2}v \, dx \right)^2.
\]
This, however, contradicts (5.5) since \( \int_{A_r} |u_0|^{p-2}u \, dx = 0 \) by (5.4). The proof is thus finished.

Proof of Theorem 5.2. Our goal is to construct a function that satisfies the conditions of Lemma 5.3. To this end, let \( \mu_1 \) be the first eigenvalue of the weighted eigenvalue problem
\[
\begin{cases}
-\Delta w - \frac{N - 1}{r} + mw - (p-1)|u_0(r)|^{p-2}w = \frac{\mu}{r^2}w & \text{in } (r,1), \\
w(r) = w(1) = 0,
\end{cases}
\]
and let \( w \) be the up to normalization unique positive eigenfunction. Moreover, let \( Y \in C^\infty(S^{N-1}) \) be a spherical harmonic of degree 1 such that \( \partial_\theta^2 Y = -Y \) and set \( v(r,\omega) := w(r)Y(\omega) \). Then condition (5.4) of Lemma 5.3 is satisfied. By construction, \( v \) also satisfies
\[
-\Delta v + \alpha^2 \partial_\theta^2 v + (m - \alpha^2)v - (p-1)|u_0|^{p-2}v = \frac{\mu_1 + N - 1}{|x|^2}v - \alpha^2 v
\]
and testing this equation with \( v \) itself then yields
\[
\int_{A_r} \left( |\nabla v|^2 - \alpha^2 |\partial_\theta v|^2 + mv^2 - (p-1)|u_0|^{p-2}v^2 \right) \, dx
= (\mu_1 + (N - 1)) \int_{A_r} \frac{v^2}{|x|^2} \, dx - \alpha^2 \int_{A_r} v^2 \, dx \leq (\mu_1 + (N - 1) - r^2\alpha^2) \int_{A_r} \frac{v^2}{|x|^2} \, dx.
\]
We recall that $\mu_1$ can be characterized by
\[
\mu_1 = \min_{\varphi \in H^1_{0,rad}(A_r) \setminus \{0\}} \frac{\int_{A_r} (|\nabla \varphi|^2 + m\varphi^2) \, dx - (p-1) \int_{A_r} |u_0|^{p-2}\varphi^2 \, dx}{\int_{A_r} \frac{\varphi^2}{|x|^2} \, dx}.
\]
Taking $\varphi = u_0$ in this quotient, we obtain the estimate
\[
\mu_1 \leq \frac{\int_{A_r} (|\nabla u_0|^2 + mu_0^2) \, dx - (p-1) \int_{A_r} |u_0|^p \, dx}{\int_{A_r} \frac{u_0^2}{|x|^2} \, dx}
= -(p-2)\frac{\int_{A_r} (|\nabla u_0|^2 + mu_0^2) \, dx}{\int_{A_r} \frac{u_0^2}{|x|^2} \, dx} \leq -(p-2)\left(\frac{\int_{A_r} |\nabla u_0|^2 \, dx}{\int_{A_r} \frac{u_0^2}{|x|^2} \, dx} + mu_0^2\right).
\] (5.9)

We now distinguish the cases $N \geq 3$ and $N = 2$. If $N \geq 3$, Hardy’s inequality gives
\[
\int_{A_r} |\nabla u_0|^2 \, dx \geq \left(\frac{N-2}{2}\right)^2 \int_{A_r} \frac{u_0^2}{|x|^2} \, dx.
\] (5.10)

Alternatively, we may also estimate, since $u_0$ is radial,
\[
\int_{A_r} |\nabla u_0|^2 \, dx = \int_r^1 \rho^{N-1} |\partial_\rho u_0(\rho)|^2 \, d\rho \geq r^{N-1} \int_r^1 |\partial_\rho u_0(\rho)|^2 \, d\rho \\
\geq \left(\frac{\pi}{1-r}\right)^2 r^{N-1} \int_r^1 u_0^2(\rho) \, d\rho \geq \left(\frac{\pi}{1-r}\right)^2 r^{N-1} \int_r^1 \rho^{N-3} u_0^2(\rho) \, d\rho \\
= \left(\frac{\pi}{1-r}\right)^2 r^{N-1} \int_{A_r} \frac{u_0^2}{|x|^2} \, dx.
\] (5.11)

Thus (5.9) gives $\mu_1 < -(p-2)\kappa(r,m)$ with $\kappa(r,m)$ given in (5.3) for $N \geq 3$. Inserting this into (5.8) yields
\[
\int_{A_r} \left(|\nabla v|^2 - \alpha^2 |\partial_\theta v|^2 + mu^2 - (p-1)|u_0|^{p-2}v^2\right) \, dx \\
\leq -(p-2)\kappa + N - 1 - r^2\alpha^2,
\]
i.e., condition (5.5) of Lemma 5.3 is satisfied if $p > \frac{N-1-r^2\alpha^2}{\kappa} + 2$, which holds by assumption.

Hence $v$ satisfies the assumptions of Lemma 5.3, which implies that (5.6) holds and therefore every minimizer for (5.2) is nonradial. Let $u$ denote such a nonradial ground state solution, and suppose by contradiction that $\partial_\theta u_0 \equiv 0$. The nonradiality of $u$ implies that there exists an isometry $A \in O(N)$ such that $\hat{u} := u \circ A \in H^1_0(A_r)$ satisfies $\partial_\theta \hat{u} \neq 0$. Since $A$ is an isometry, this implies
\[
R_{\alpha,m,p}(\hat{u}) = R_{\alpha,m,p}(u) - \alpha^2 \int_{A_r} |\partial_\theta \hat{u}|^2 \, dx \left(\int_{A_r} |u|^p \, dx\right)^{-\frac{p-2}{p}} < R_{\alpha,m,p}(u) = \mathcal{C}_{1,m,p}(A_r),
\]
which contradicts (5.2). Consequently, we have $\partial_\theta u_0 \neq 0$, which yields that $u_0$ is $x_1$-$x_2$-nonradial. This finishes the proof in the case $N \geq 3$.

It remains to consider the case $N = 2$. In this case, we replace the estimates (5.10) and (5.11) by
\[
\int_{A_r} |\nabla u_0|^2 \, dx \geq \left(\frac{\pi}{1-r}\right)^2 r^{N-1} \int_r^1 u_0^2(\rho) \, d\rho \geq \left(\frac{\pi}{1-r}\right)^2 r^N \int_{A_r} \frac{u_0^2}{|x|^2} \, dx.
\]
Combining this with (5.9) we again get \( \mu_1 < -(p - 2)\kappa(r, m) \) with \( \kappa(r, m) \) given in (5.3) for \( N = 2 \). We may thus complete the proof as above.

6. RIEMANNIAN MODELS

So far we only used the inequality stated in Theorem 2.2 in the case \( s = 1 \). We shall now consider an application for general \( s \in (0, 2] \) by considering (1.3) on a special class of Riemannian manifolds with boundary.

Indeed, consider a Riemannian model, i.e., a Riemannian manifold \( (M, g) \) of dimension \( N \geq 2 \) admitting a pole \( o \in M \) and whose metric is (locally) given by

\[
 ds^2 = dr^2 + (\psi(r))^2 d\Theta^2
\]

for \( r > 0 \), \( \Theta \in S^{N-1} \), where \( d\Theta^2 \) denotes the canonical metric on \( S^{N-1} \) and \( \psi \) is a smooth function that is positive on \((0, \infty)\). Moreover, we assume

\[
 \psi'(0) = 1 \quad \text{and} \quad \psi^{(2k)}(0) = 0 \quad \text{for} \ k \in \mathbb{N}_0.
\]

For such a Riemannian model, the associated Laplace-Beltrami operator becomes

\[
 \Delta_g f = \frac{1}{\psi^{N-1}} \partial_r \left( \psi^{N-1} \partial_r f \right) + \frac{1}{\psi^2} \Delta_{S^{N-1}} f
\]

where \( \Delta_{S^{N-1}} \) denotes the Laplace-Beltrami operator on \( S^{N-1} \). Riemannian models are of independent geometric interest, we refer to [6] and the references therein for a broader overview.

In the following, we focus on the case \( M = B \), \( o = 0 \in \mathbb{R}^N \) and again study the problem

\[
 \begin{cases}
 \partial_t^2 v - \Delta_g v + mv = |v|^{p-2}v & \text{in } M \\
 v = 0 & \text{on } \partial M
 \end{cases}
\]

where \( 0 < p < \frac{2N}{N-2} \) and \( m > -\lambda_1(M) \) with \( \lambda_1(M) \) denoting the first Dirichlet eigenvalue of \( -\Delta_g \) on \( M \). We stress that the case \( \psi(r) = r \) corresponds to the classical flat metric on \( B \) considered in detail in the previous sections. As a further example, the hemisphere of radius \( \frac{2}{\pi} \) given by \( S^{N-1}_{2/\pi, +} := \{ x \in S^{N-1} : x_N > 0 \} \) can be interpreted as a Riemannian model. Indeed, using polar coordinates \((r, \omega) \in (0, 1) \times S^{N-1} \), a parametrization \( B \to S^{N-1}_{2/\pi, +} \) is given by \((r, \omega) \mapsto \frac{2}{\pi} (\sin(\frac{2}{\pi} r) \omega, \cos(\frac{2}{\pi} r)) \). This yields a Riemannian model with \( \psi(r) = \sin(\frac{2}{\pi} r) \). Similarly, Riemannian models can also be used for spherical caps.

As in the flat case, we restrict our attention to solutions of (6.3) of the form \( v(t, x) = u(R_\theta(x)) \), where \( R_\theta \in O(N+1) \) denote a planar rotation in \( \mathbb{R}^N \) with angle \( \theta \). We may again assume, without loss of generality, that

\[
 R_\theta(x) = (x_1 \cos \theta + x_2 \sin \theta, -x_1 \sin \theta + x_2 \cos \theta, x_3, \ldots, x_N)
\]

for \( x \in \mathbb{R}^N \), so \( R_\theta \) is the rotation in the \( x_1 - x_2 \)-plane. This leads to the reduced equation

\[
 \begin{cases}
 -\Delta_g u + \alpha^2 \partial_\theta^2 u + mu = |u|^{p-2} u & \text{in } M \\
 u = 0 & \text{on } \partial M
 \end{cases}
\]

with the differential operator \( \partial_\theta = x_1 \partial_{x_2} - x_2 \partial_{x_1} \) associated to the Killing vector field \( x \mapsto (-x_2, x_1, 0, \ldots, 0) \) on \( M \). We may then again study the quotient

\[
 R^M_{\alpha,m,p} : H^1_0(M) \setminus \{0\} \to \mathbb{R}, \quad R^M_{\alpha,m,p}(u) := \frac{\int_M (|\nabla_g u|^2 - \alpha^2 |\partial_\theta u|^2 + mu^2) \, dg}{\|u\|^2_{L^p(M)}},
\]
and its minimizers, i.e.
\[
\mathcal{C}_{\alpha,m,p}(M) := \inf_{u \in C^1_\mathcal{E}(B) \setminus \{0\}} P^M_{\alpha,m,p}(u).
\]
Analogously to Theorem 1.1, we can use the general inequality stated in Theorem 2.2 to give the following result.

**Theorem 6.1.** Let \( s \in (0,2] \), and let \((M,g)\) be a Riemannian model with \( M = B \) and associated function \( \psi \in C^\infty[0,1) \) satisfying (6.2) and
\[
c_1(1-r)^s \leq 1 - \psi(r) \leq c_2(1-r)^s \quad \text{for } r \in (0,1) \text{ with constants } c_1,c_2 > 0.
\]
Moreover, let \( m > -\lambda_1(M) \), and let \( 2 < p < 2^* \).

(i) If \( \alpha \in (0,1) \), then there exists a ground state solution of (1.5).

(ii) We have
\[
\mathcal{C}_{1,m,p}(M) = 0 \quad \text{for } p > 2^*_\alpha, \quad \text{and} \quad \mathcal{C}_{1,m,p}(M) > 0 \quad \text{for } p \leq 2^*_\alpha.
\]
Moreover, for any \( p \in (2^*_\alpha,2^*) \), there exists \( \alpha_p \in (0,1) \) with the property that
\[
\mathcal{C}_{\alpha,m,p}(M) < \mathcal{C}_{0,m,p}(M) \quad \text{for } \alpha \in (\alpha_p,1).
\]
and therefore every ground state solution of (6.4) is \( x_1,x_2 \)-nonradial for \( \alpha \in (\alpha_p,1) \).

**Proof.** Since the proof is completely parallel to the proof of Theorem 1.1, we omit some details and focus our attention on showing where condition (6.5) enters. It is again useful to introduce polar coordinates \((r,\theta,\vartheta_1,\ldots,\vartheta_{N-2}) \in U := (0,1) \times (-\pi,\pi) \times (0,\pi)^{N-2}\) given by
\[
(x_1,\ldots,x_N) = (r \cos \theta \sin \vartheta_1 \cdots \sin \vartheta_{N-2}, r \sin \theta \sin \vartheta_1 \cdots \sin \vartheta_{N-2},
\]
\[
r \cos \vartheta_1, r \sin \vartheta_1 \cos \vartheta_2, \ldots, r \sin \vartheta_1 \ldots \sin \vartheta_{N-3} \cos \vartheta_{N-2}, r \sin \vartheta_1 \ldots \sin \vartheta_{N-3} \sin \vartheta_{N-2}.
\]
In the following, we will abbreviate the coordinates \((\theta,\vartheta_1,\ldots,\vartheta_{N-2}) \) to \( \Theta \) for simplicity. In these coordinates, the metric (6.1) becomes
\[
g = dr^2 + (\psi(r))^2 \left( \sum_{i=1}^{N-2} \left( \prod_{k=1}^{i-1} \sin^2 \vartheta_k \right) d\vartheta_i^2 + \left( \prod_{k=1}^{N-1} \sin^2 \vartheta_k \right) d\vartheta_k^2 \right),
\]
and therefore the quadratic form associated to the operator \(-\Delta_g + \alpha^2 \partial_\vartheta^2\) is given by
\[
\int_M \left( |\nabla_g u|^2 - |\partial_\vartheta u|^2 \right) dg
\]
\[
= \int_U \left[ |\partial_\vartheta u|^2 + \frac{1}{\psi^2} \sum_{i=1}^{N-2} h_i |\partial_\vartheta \partial_\vartheta_i u|^2 + \left( \frac{h_{N-1}}{\psi^2} - 1 \right) |\partial_\vartheta u|^2 \right] |g| d(r,\Theta)
\]
for \( u \in C^1_\mathcal{E}(M) \) with
\[
|g|(r,\Theta) = (\psi(r))^{N-1} \prod_{k=1}^{N-2} \sin^{N-1-k} \vartheta_k, \quad h_i(r,\Theta) = \prod_{k=1}^{i-1} \frac{1}{\sin^2 \vartheta_k},
\]
Moreover,
\[
\int_M |u|^p dg = \int_U |u|^p d(r,\Theta) \quad \text{for } u \in C^1_\mathcal{E}(M) \text{ and } p > 1.
\]
Next we note that, as a consequence of (6.5), we have
\[
|g|(\Theta_0) = 1 \quad \text{and} \quad h^i(\Theta_0) = 1 \quad \text{for } i = 1,\ldots,N \text{ with } \Theta_0 := \left( 1, 0, \frac{\pi}{2}, \ldots, \frac{\pi}{2} \right). \quad (6.7)
\]
Moreover, by assumption (6.5), the function \( \frac{h_{N-1}}{\psi^2} - 1 \) satisfies
\[
\hat{c}_1 \left( (1 - r)^s + \sum_{k=1}^{N-2} (\vartheta_k - \frac{\pi}{2})^2 \right) \leq \frac{h_{N-1}}{\psi^2} (r, \Theta) - 1 \leq \hat{c}_2 \left( (1 - r)^s + \sum_{k=1}^{N-2} (\vartheta_k - \frac{\pi}{2})^2 \right)
\]
(6.8)
for \((r, \theta, \vartheta_1, \ldots, \vartheta_{N-2}) \in U_0\) with suitable constants \(\hat{c}_1, \hat{c}_2 > 0\), where
\[
U_0 := \left( \frac{1}{2}, 1 \right) \times (-\pi, \pi) \times \left( \frac{\pi}{4}, \frac{3\pi}{4} \right)^{N-2} \subset U.
\]
We now consider a fixed function \(u \in C^1_c(U_0) \setminus \{0\} \subset C^1_c(U) \setminus \{0\}\), which, regarded as a function of polar coordinates, gives rise to a function in \(C^1_c(M)\). For \(\lambda \in (0, 1)\) we consider the map
\[
\Lambda_\lambda : U_0 \to U_0, \quad (r, \Theta) \mapsto \left( 1 + \lambda (1 - r), \lambda^{\frac{N-2}{2}} \vartheta_1, \vartheta_2, \ldots, \vartheta_{N-2} + \lambda \left( \frac{\vartheta_1 - \frac{\pi}{2}}{2} \right), \ldots, \vartheta_{N-2} + \lambda \left( \frac{\vartheta_2 - \frac{\pi}{2}}{2} \right) \right),
\]
and we define \(u_\lambda := u \circ \Lambda_\lambda^{-1} \in C^1_c(U_0) \setminus \{0\}\) for \(\lambda \in (0, 1)\).

Using (6.7) and (6.8), we find that
\[
\lambda^{\frac{2N-s}{p}} \left( \int_U g |u_\lambda|^p d(r, \Theta) \right)^{\frac{1}{p}} = \left( \int_U g \circ \Lambda_\lambda |u|^p d(r, \Theta) \right)^{\frac{1}{p}} \to \left( \int_U |u|^p d(r, \Theta) \right)^{\frac{1}{p}} =: c_u(p)
\]
as \(\lambda \to 0^+\) and
\[
\limsup_{\lambda \to 0^+} \lambda^{2-s-N} \int_U \left( |\partial_r u_\lambda|^2 + \frac{1}{\psi^2} \sum_{i=1}^{N-2} h_i |\partial_{\vartheta_i} u_\lambda|^2 + \left( \frac{h_{N-1}}{\psi^2} - 1 \right) |\partial_\vartheta u_\lambda|^2 \right) |g| d(r, \Theta)
\]
\[= \limsup_{\lambda \to 0^+} \int_U \left( |\partial_r u|^2 + \frac{1}{\psi^2} \circ \Lambda_\lambda \sum_{i=1}^{N-2} h_i \circ \Lambda_\lambda |\partial_{\vartheta_i} u|^2 + \lambda^{-s} \left( \frac{h_{N-1}}{\psi^2} \circ \Lambda_\lambda - 1 \right) |\partial_\vartheta u|^2 \right) |g| \circ \Lambda d(r, \Theta)
\]
\[\leq d_u^1 + d_u^2
\]
with
\[
d_u^1 := \int_U \left( |\partial_r u|^2 + \sum_{i=1}^{N-2} |\partial_{\vartheta_i} u|^2 \right) d(r, \Theta)
\]
and
\[
d_u^2 = \hat{c}_2 \limsup_{\lambda \to 0^+} \int_U \left( (1 - r)^s + \lambda^{2-s} \sum_{k=1}^{N-2} (\vartheta_k - \frac{\pi}{2})^2 \right) |\partial_\vartheta u|^2 d(r, \Theta)
\]
\[= \begin{cases} \hat{c}_2 \int_U (1 - r)^s |\partial_\vartheta u|^2 d(r, \Theta), & s \in (0, 2), \\ \hat{c}_2 \int_U (1 - r)^2 + \sum_{i=1}^{N-2} (\vartheta_k - \frac{\pi}{2})^2 |\partial_\vartheta u|^2 d(r, \Theta), & s = 2. \end{cases}
\]
It thus follows that
\[
\mathcal{G}_{1,m,p}(M) \leq \limsup_{\lambda \to 0^+} R_{1,m,p}(u_\lambda) = \limsup_{\lambda \to 0^+} \frac{\lambda^{N+s-2} (d_u^1 + d_u^2) + \lambda^{\frac{2N+2s}{p}} c_u(2)}{\lambda^{\frac{2N+s}{p}} c_u(p)} = 0 \quad \text{if } p > 2s.
\]
This shows the first identity in (6.6). To see the second identity in (6.6), we argue as in Section 3. More precisely, we first note that it is sufficient to consider the case \( p = 2^* \), and then we show the inequality
\[
\left( \int_U |u|^{2^*} d(r, \Theta) \right)^{\frac{2}{2^*}} \leq C \int_U \left( |\partial_r u|^2 + \frac{1}{\psi^2} \sum_{i=1}^{N-2} h_i |\partial_{\theta_i} u|^2 + \left( \frac{h_{N-1}}{\psi^2} - 1 \right) |\partial_{\theta_1} u|^2 \right) |g| d(r, \Theta)
\]
for functions \( u \in C^1_c(U_0) \) with a suitable constant \( C > 0 \). For this, we use Theorem 1.6 and the first inequality in (6.8). The argument is then completed by using the rotation invariance of the problem and a partition of unity argument to localize the problem. \( \square \)

**Remark 6.2.**

(i) In the case of a hemisphere mentioned earlier, i.e. \( \psi(r) = \frac{2}{3} \sin(\frac{\pi}{4}r) \), Theorem 6.1 yields nonradial ground state solutions for \( p > 2^* = \frac{2(N+1)}{N-1} \). Notably, this corresponds to the critical exponent for generalized travelling waves on the sphere \( S^N \) found in [42, 36, 37]. In fact, our approach based on Theorem 1.6 can be used to give an alternative proof for the existence of nontrivial solutions and the embeddings stated in [42, Proposition 3.2] and [37, Proposition 1.2 + Lemma 1.3].

(ii) Theorem 6.1 leaves open the case \( s > 2 \). Note that the two-sided estimate (6.8) needs to be analyzed more carefully if \( s > 2 \) and \( N \geq 3 \), as the leading order term is then 2 in place of \( s \). In this case, if (6.5) holds for some \( s > 2 \), the conclusion will instead be
\[
\mathcal{E}_{1,m,p}(M) = 0 \quad \text{for } p > 2^*, \quad \text{and} \quad \mathcal{E}_{1,m,p}(M) > 0 \quad \text{for } p \leq 2^*.
\]
For \( N = 2 \), on the other hand, (6.8) suggests that Theorem 6.1 also holds for \( s > 2 \).

**Appendix A. Boundedness of solutions**

In the proof of the regularity properties of ground states in the case \( \alpha = 1 \) stated in Proposition 4.8, we used the following:

**Lemma A.1.** Let \( 2 < p < 2^*_1 \), \( m > -\lambda_1 \) and let \( u \in \mathcal{H} \) be a weak solution of
\[
- \Delta u + \partial_\theta^2 u + mu = |u|^{p-2}u \quad \text{in } B.
\]
Then \( u \in L^\infty(B) \). Furthermore, there exist constants \( C = C(N,m), \sigma > 0 \) such that
\[
|u|_\infty \leq C ||u||_H^\sigma.
\]
For \( m \geq 0 \), the constant \( C = C(N) > 0 \) can be chosen independent of \( m \).

**Proof.** The proof is based on Moser iteration, cf. [40, Appendix B] and the references therein.

We fix \( L, s \geq 2 \) and consider auxiliary functions \( h, g \in C^1([0, \infty)) \) defined by
\[
h(t) := s \int_0^t \min\{\tau^{s-1}, L^{s-1}\} d\tau \quad \text{and} \quad g(t) := \int_0^t |h'(\tau)|^2 d\tau.
\]
We note that
\[
h(t) = t^s \quad \text{for } t \leq L \quad \text{and} \quad g(t) \leq t g'(t) = t h'(t)^2 \quad \text{for } t \geq 0,
\]
since the function \( t \mapsto h'(t) = s \min\{t^{s-1}, L^{s-1}\} \) is nondecreasing. We shall now show that \( w := u^+ \in L^\infty(B) \), and that \( ||w||_\infty \) is bounded by the right hand side of (A.2). Since we may replace \( u \) with \(-u\), the claim will then follow.

We note that \( w \in \mathcal{H} \) and \( \varphi := g(w) \in \mathcal{H} \) with
\[
\nabla w = 1_{\{u>0\}} \nabla u, \quad \nabla \varphi = g'(w) \nabla w, \quad \partial_\theta w = 1_{\{u>0\}} \partial_\theta u, \quad \partial_\theta \varphi = g'(w) \partial_\theta w.
\]
This follows from the boundedness of $g'$ and the estimate $g(t) \leq s^2 t^{2s-1}$ for $t \geq 0$. Testing (A.1) with $\varphi$ gives
\[
\int_B \left( \nabla u \cdot \nabla \varphi - (\partial_t u \partial_t \varphi) + mu \varphi \right) dx = \int_B |u|^{p-2} u \varphi dx,
\]
from where we estimate
\[
\int_B \left( |\nabla h(w)|^2 - (\partial_t h(w))^2 \right) + mwg(w) dx = \int_B \left( g'(w) \left( |\nabla w|^2 - (\partial_t w)^2 \right) + mwg(w) \right) dx
\]
\[
= \int_B |u|^{p-2} u g(w) dx
\]
\[
\leq \int_B w^p (h'(w))^2 dx.
\]  
(A.4)

Here we used (A.3) in the last step. We now fix $r > 1$ with $\frac{(p-2)r}{2} \geq 2$ and $q > 4r$. Combining (A.4) with Proposition 4.1 and Theorem 3.3, we obtain the inequality
\[
|h(w)|^2_{pr} \leq c_0 \int_B w^p (h'(w))^2 dx
\]  
(A.5)
with a constant $c_0 = c_0(N, m) > 0$. Note that for $m \geq 0$, $c_0$ only depends on $N$. Since
\[
h(t) = t^s, \quad h'(t) = st^{s-1} \quad \text{and} \quad g(t) = s^2 t^{2s-2} \quad \text{for} \quad t \leq L,
\]
we may let $L \to \infty$ in (A.5) and apply Lebesgue’s theorem to obtain
\[
|w^s|_{p^*}^2 \leq c_0 s^2 \int_B w^{p+2s-2} dx \leq c_0 s^2 |w|_{p^*}^{p-2} |w|^{2s}_{2sq},
\]
where $q = \frac{p^*}{p^* - p + 2}$ is the the conjugated exponent to $\frac{p^*}{p-2}$. This yields
\[
|w|_{sp^*} \leq (c_1 s)^\frac{1}{2} |w|^{2s}_{2sq} \quad \text{with} \quad c_1 := \left( c_0 |w|_{p^*}^{p-2} \right)^\frac{2}{2},
\]  
(A.6)
whenever $w \in L^{2sq}(B)$. We now consider $s = s_n = \rho^n$ for $n \in \mathbb{N}$ with $\rho := \frac{p^*}{2q} = \frac{2 + p - p^*}{2} > 1$, so that
\[
2s1q = p^* \quad \text{and} \quad 2s_{n+1}q = s_n p^* \quad \text{for} \quad n \in \mathbb{N}.
\]
Iteration of (A.6) then gives
\[
|w|_{p^*} = |w|_{s_n p^*} \leq |w|_{p^*} \prod_{j=1}^{n} (c_1 \rho^j)^{p^* - j} \leq c_1^{-1} c_2 |w|_{p^*}
\]
for all $n$ with
\[
c_2 := \rho^{\sum_{j=1}^{n} j p^* - j} < \infty.
\]
It follows that
\[
|w|_{\infty} = \lim_{n \to \infty} |w|_{p^*} \leq c_1^{\frac{p}{p^*}} c_2 |w|_{p^*}.
\]  
(A.7)
Moreover, by Theorem 3.3, we have
\[
c_1 \leq c_1' \|w\|_{H^2}^{\frac{p}{2}} \leq c_1' \|u\|_{H^2}^{\frac{p}{2}} \quad \text{and} \quad |w|_{q} \leq \hat{c} \|w\|_{H} \leq \hat{c} \|u\|_{H}
\]
with constants $c_1', \hat{c} > 0$ depending only on $N$. It thus follows from (A.7) that
\[
|w|_{\infty} \leq C \|u\|_{H}^{\frac{(p-2)p}{2(p-1)}} \quad \text{with} \quad C := c_2 (c_1')^{\frac{p}{p-1} \hat{c}}.
\]
The proof is thus finished.

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