On properties of Tribonacci-Lucas polynomials

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Abstract

In this paper, we investigated properties of Tribonacci-Lucas polynomials which generalized Tribonacci-Lucas numbers. From this generalization, we also obtain some new algebraic properties on these numbers and polynomials as Binet formula, summation, binomial sum and generating function.

Keywords: Tribonacci-Lucas numbers, Tribonacci-Lucas polynomials, Binomial sums, Generating functions.

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1 Introduction

Recently, Fibonacci and Lucas numbers have investigated very largely and authors tried to developed and give some directions to mathematical calculations using these type of special numbers. One of these directions goes through to the Tribonacci and the Tribonacci-Lucas numbers. In fact Tribonacci numbers have been firstly defined by M. Feinberg in 1963 and then some important properties over this numbers have been created by [3,6-10,13]. On the other hand, Tribonacci-Lucas numbers have been introduced and investigated by authors in [2,4]. In addition, there exists another mathematical term, namely to be incomplete, on Tribonacci and Tribonacci-Lucas numbers. As a brief background, the incomplete Tribonacci and Tribonacci-Lucas numbers were introduced by authors [11,12], and further the generating functions of these numbers were presented by authors.

For $n \geq 2$, it is known that while the Tribonacci sequence $\{T_n\}_{n \in \mathbb{N}}$ is defined by

$$T_{n+1} = T_n + T_{n-1} + T_{n-2} \quad (T_0 = 0, \ T_1 = T_2 = 1),$$

(1)

and the Tribonacci-Lucas sequence $\{K_n\}_{n \in \mathbb{N}}$ is defined by

1
\[ K_{n+1} = K_n + K_{n-1} + K_{n-2} \ (K_0 = 3, \ K_1 = 1, \ K_2 = 3). \] (2)

There is also well known that each of the Tribonacci and Tribonacci-Lucas numbers is actually a linear combination of \( \alpha^n, \beta^n \) and \( \gamma^n \). In other words,

\[
T_n = \frac{\alpha^{n+1}}{(\alpha - \beta)(\alpha - \gamma)} + \frac{\beta^{n+1}}{(\beta - \alpha)(\beta - \gamma)} + \frac{\gamma^{n+1}}{(\gamma - \alpha)(\gamma - \beta)} \\
K_n = \alpha^n + \beta^n + \gamma^n, \tag{3}
\]

where \( \alpha, \beta \) and \( \gamma \) are roots of the characteristic equations of (1) and (2) such that

\[
\alpha = 1 + \frac{1}{3} \sqrt[3]{19 + 3 \sqrt[3]{33}} + \frac{1}{3} \sqrt[3]{19 - 3 \sqrt[3]{33}}, \quad \beta = 1 + w \frac{1}{3} \sqrt[3]{19 + 3 \sqrt[3]{33}} + w^2 \frac{1}{3} \sqrt[3]{19 - 3 \sqrt[3]{33}}, \quad \gamma = 1 + w^2 \frac{1}{3} \sqrt[3]{19 + 3 \sqrt[3]{33}} + w \frac{1}{3} \sqrt[3]{19 - 3 \sqrt[3]{33}},
\]

where \( w = \frac{-1 + i \sqrt{3}}{2} \).

Moreover, we note that equations in (3) are called the Binet formulas for Tribonacci and Tribonacci-Lucas numbers, respectively.

Moreover, authors studied a large class of polynomials by Fibonacci and Tribonacci numbers [1,5]. The Fibonacci polynomials are defined by

\[ F_{n+2} (x) = xF_{n+1} (x) + F_n (x), \]

where initial conditions \( F_0 (x) = 0 \) and \( F_1 (x) = 1 \). And, in 1973, Hoggatt and Bicknell [5] introduced Tribonacci polynomials. The Tribonacci polynomials \( T_n (x) \) are defined by the recurrence relation

\[ T_{n+3} (x) = x^2 T_{n+2} (x) + xT_{n+1} (x) + T_n (x), \] (4)

where \( T_0 (x) = 0, \ T_1 (x) = 1, \ T_2 (x) = x^2 \). Note that \( T_n (1) = T_n, \ n \in \mathbb{N} \). In addition to, they gave the binomial sums of these polynomials as

\[ F_n (x) = \left. \sum_{j=0}^{[\frac{n-1}{2}]} \binom{n-j-1}{j} x^{n-2j-1} \right|, \]

where \([x]\) is the greatest integer contained in \( x \), and

\[ T_n (x) = \sum_{j=0}^{\binom{n-j-1}{j}} x^{2n-3j-2}, \]

where \( \binom{n-j-1}{j} \) is the trinomial coefficient in the \( n \text{th} \) row and \( j \text{th} \) column where, as is usual, the left-most column is the zero \( j \text{th} \) column and the top row is the zero \( j \text{th} \) row, and \( \binom{n-j-1}{j} = 0 \) if \( j > n \).
Also, in [11], authors defined Tribonacci-Lucas polynomials, incomplete Tribonacci-Lucas numbers and incomplete Tribonacci-Lucas polynomials. That is, Tribonacci-Lucas polynomials are defined by

\[ K_{n+3}(x) = x^2K_{n+2}(x) + xK_{n+1}(x) + K_n(x), \]

(5)

where \( K_0(x) = 3, \ K_1(x) = x^2, \ K_2(x) = x^4 + 2x. \)

Incomplete Tribonacci-Lucas numbers and polynomials are defined by

\[ K_n^{(s)}(x) = \sum_{i=0}^{s} \sum_{j=0}^{i} \frac{n}{n-i-j} \binom{i}{j} \binom{n-i-j}{i} x^{2n-3(i+j)}, \]

and

\[ K_n(s) = \sum_{i=0}^{s} \sum_{j=0}^{i} \frac{n}{n-i-j} \binom{i}{j} \binom{n-i-j}{i}, \]

respectively, where \( 0 \leq s \leq \left\lfloor \frac{n}{2} \right\rfloor \) for \( n \in \mathbb{Z}^+. \)

In the light of the above paragraph, the main goal of this paper is to improve the Tribonacci-Lucas polynomials and numbers with a different viewpoint. In order to do that we first obtain Binet formula, sum and binomial summation of Tribonacci-Lucas polynomials. Then, we give the generating function of this polynomial.

2 Main Results

In this section, we will mainly focus on the Tribonacci-Lucas polynomials to get some important results. In fact, we will present the related Binet formula, relationship of between Tribonacci polynomials, summation, binomial sum and generating function. Besides, we will get similar result for the Tribonacci-Lucas numbers by using this polynomials.

Hence, we firstly derive the Binet formula of the Tribonacci-Lucas polynomials.

**Theorem 1** For \( n \in \mathbb{N} \), we can write the Binet formulas for the Tribonacci-Lucas polynomials as the form

\[ K_n(x) = A\lambda_1^n + B\lambda_2^n + C\lambda_3^n, \]

where

\[ A = \frac{K_2(x) - (\lambda_2 + \lambda_3) K_1(x) + \lambda_2\lambda_3 K_0(x)}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}, \]

\[ B = \frac{K_2(x) - (\lambda_1 + \lambda_3) K_1(x) + \lambda_1\lambda_3 K_0(x)}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)}, \]

\[ C = \frac{K_2(x) - (\lambda_1 + \lambda_2) K_1(x) + \lambda_1\lambda_2 K_0(x)}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)}, \]

such that \( \lambda_1, \lambda_2, \lambda_3 \) are roots of characteristic equation of (5).
Proof. We will show the truthness of the Binet formula for Tribonacci-Lucas polynomials.

So let us consider (5). By the assumption, the roots of the characteristic equation of (5) are $\lambda_1$, $\lambda_2$ and $\lambda_3$. Hence the general solution of it is given by

$$K_n(x) = C_1\lambda_1^n + C_2\lambda_2^n + C_3\lambda_3^n.$$ 

Using initial conditions of equation (5) and also applying fundamental linear algebra operations, we clearly get the coefficient $C_1 = A$, $C_2 = B$ and $C_3 = C$, as desired. This implies the formula for $K_n(x)$. ■

**Theorem 2** The relation between the Tribonacci polynomials $T_n(x)$ and the Tribonacci-Lucas polynomials $K_n(x)$ is

$$K_n(x) = x^2T_n(x) + 2xT_{n-1}(x) + 3T_{n-2}(x),$$

where $n \geq 2$.

**Proof.** Let us show this by induction, for $n = 2$, we can write

$$K_2(x) = x^2T_2(x) + 2xT_1(x) + 3T_0(x) = x^4 + 2x.$$ 

Now, assume that, it is true for all positive integers $m$, i.e.

$$K_m(x) = x^2T_m(x) + 2xT_{m-1}(x) + 3T_{m-2}(x). \quad (6)$$

Then, we need to show that above equality holds for $n = m + 1$, that is,

$$K_{m+1}(x) = x^2T_{m+1}(x) + 2xT_m(x) + 3T_{m-1}(x). \quad (7)$$

By considering the left hand side of Equation (7), we can expand the recurrence relation as

$$K_{m+1}(x) = x^2K_m(x) + xK_{m-1}(x) + K_{m-2}(x).$$

Then, using Equation (6), we have

$$K_{m+1}(x) = x^2\left(x^2T_m(x) + 2xT_{m-1}(x) + 3T_{m-2}(x)\right) + x\left(x^2T_{m-1}(x) + 2xT_{m-2}(x) + 3T_{m-3}(x)\right) + \left(x^2T_{m-2}(x) + 2xT_{m-3}(x) + 3T_{m-4}(x)\right)$$

Finally, by considering (6), we obtain

$$K_{m+1}(x) = x^2T_{m+1}(x) + 2xT_m(x) + 3T_{m-1}(x)$$

which ends up the induction. ■

In [4], the author obtained the relationship for Tribonacci and Tribonacci-Lucas numbers. However, in here, we will obtain this relationship in terms of Tribonacci-Lucas polynomials as a consequence of Theorem 2. To do that we will take $x = 1$ in Theorem 2.
Corollary 3 The relation between of Tribonacci numbers $T_n$ and Tribonacci-Lucas numbers $K_n$ is

$$K_n = T_n + 2T_{n-1} + 3T_{n-2},$$

where $n \geq 2$.

Now, we will give binomial summation of Tribonacci-Lucas polynomials as follows:

Theorem 4 For $n \in \mathbb{N}$, we have the equality

$$K_{3n}(x) = \sum_{i=0}^{n} \sum_{j=0}^{i} \binom{n}{i} \binom{i}{j} x^{i+j} K_{i+j}(x). \quad (8)$$

Proof. Let $\lambda$ stand for a root of the characteristic equation of equation (5). Then, we have

$$\lambda^3 = x^2 \lambda^2 + x \lambda + 1$$

and we can write by considering binomial expansion

$$(\lambda^3)^n = (\lambda^3 - 1 + 1)^n$$

$$= \sum_{i=0}^{n} \binom{n}{i} (\lambda^3 - 1)^i$$

$$= \sum_{i=0}^{n} \binom{n}{i} (x^2 \lambda^2 + x \lambda)^i$$

$$= \sum_{i=0}^{n} \binom{n}{i} (x \lambda)^i \sum_{j=0}^{i} \binom{i}{j} (x \lambda)^j$$

$$= \sum_{i=0}^{n} \sum_{j=0}^{i} \binom{n}{i} \binom{i}{j} (x \lambda)^{i+j}$$

If we replace to $\lambda_1, \lambda_2, \lambda_3$ by $\lambda$ and rearrange, then we obtain

$$K_{3n}(x) = \sum_{i=0}^{n} \sum_{j=0}^{i} \binom{n}{i} \binom{i}{j} x^{i+j} K_{i+j}(x).$$

In [13], the authors presented the binomial sum for Tribonacci-Lucas numbers. However, in here, we will derive this sum by using Tribonacci-Lucas polynomials. To do that we will take $x = 1$ in Theorem 4. Hence we have the following corollary.

Corollary 5 For $n \in \mathbb{N}$, we have the following equality:

$$K_{3n} = \sum_{i=0}^{n} \sum_{j=0}^{i} \binom{n}{i} \binom{i}{j} K_{i+j}. \quad (9)$$
For Tribonacci-Lucas polynomials, we give the summations according to specified rules as we depicted at the beginning of this section.

**Theorem 6** For \( n > 0 \) and \( m > j \geq 0 \), there exist

\[
\sum_{i=0}^{n-1} K_{mi+j} (x) = \frac{K_{mn+j+m} (x) + K_{mn+j-m} (x) + (1 - X_m) K_{mn+j} (x)}{X_m - X_m} - \frac{K_{j+m} (x) + K_{j-m} (x) + (1 - X_m) K_{j} (x)}{X_m - X_m}
\]

where \( X_m = \lambda_1^m + \lambda_2^m + \lambda_3^m \).

**Proof.** The main point of the proof will be touched just the result Theorem 1, in other words the Binet formulas of related polynomials. Thus

\[
\sum_{i=0}^{n-1} K_{mi+j} (x) = \sum_{i=0}^{n-1} \left( A \lambda_1^{mi+j} + B \lambda_2^{mi+j} + C \lambda_3^{mi+j} \right),
\]

\[
= A \lambda_1^j \left( \frac{\lambda_1^{mn} - 1}{\lambda_1 - 1} \right) + B \lambda_2^j \left( \frac{\lambda_2^{mn} - 1}{\lambda_2 - 1} \right) + C \lambda_3^j \left( \frac{\lambda_3^{mn} - 1}{\lambda_3 - 1} \right).
\]

In here, simplifying the last equality in above will be implied (10) as required.

If we choose specific values for \( m \) and \( j \), then the following result will be clear for the summation of Tribonacci-Lucas polynomials as a consequence of Theorem 6.

**Corollary 7** For \( n > 0 \), \( m = 1 \) and \( j = 0 \), the general sum of Tribonacci-Lucas polynomials

\[
\sum_{i=0}^{n-1} K_i (x) = \frac{K_{n+1} (x) + K_{n-1} (x) + (1 - x^2) K_n (x) + 2x^2 + x - 3}{x^2 + x}
\]

and for \( m = 2 \) and \( j = 0 \), the sum with even indices of Tribonacci-Lucas polynomials is

\[
\sum_{i=0}^{n-1} K_{2i} (x) = \frac{K_{2n+2} (x) + K_{2n-2} (x) + (1 - X_2) K_{2n} (x) + 3X_2 - 2x^4 - 4x - 3}{X_2 - X_2},
\]

and for \( m = 2 \) and \( j = 1 \), the sum with odd indices of Tribonacci-Lucas polynomials is

\[
\sum_{i=0}^{n-1} K_{2i+1} (x) = \frac{K_{2n+3} (x) + K_{2n-1} (x) + (1 - X_2) K_{2n+1} (x) + 2x^2X_2 - x^6 - 3x^3 - 2x^2 - 3}{X_2 - X_2},
\]

where \( X_2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \).
As we noted in the beginning of this section, the other aim of this paper is to present generating function of this polynomial.

**Theorem 8** For Tribonacci-Lucas polynomials, we have the generating function

$$G(z) = \frac{3 - 2x^2 z - xz^2}{1 - x^2 z - xz^2 - z^3}.$$  \hfill (11)

**Proof.** Assume that $G(z)$ is the generating function for the polynomials $\{K_n(x)\}_{n \in \mathbb{N}}$. Then we have

$$G(z) = K_0(x) + K_1(x) z + K_2(x) z^2 + \ldots + K_n(x) z^n + \ldots \quad (12)$$

If we multiply the $G(z)$ given in (12) with $x^2 z$, $xz^2$ and $z^3$, respectively, then we get

\[
\begin{align*}
x^2 z G(z) &= x^2 K_0(x) z + x^2 K_1(x) z^2 + x^2 K_2(x) z^3 + \ldots + x^2 K_n(x) z^{n+1} + \\
x^2 z G(z) &= x K_0(x) z^2 + x K_1(x) z^3 + x K_2(x) z^4 + \ldots + x K_n(x) z^{n+2} + \\
z^3 G(z) &= K_0(x) z^3 + K_1(x) z^4 + K_2(x) z^5 + \ldots + K_n(x) z^{n+3} + \ldots \quad (13)
\end{align*}
\]

Consequently, by subtracting the sum of (13) from (12), it is obtained the equation

$$G(z) = \frac{K_0(x) + z (K_1(x) - x^2 K_0(x)) + z^2 (K_2(x) - x^2 K_1(x) - x K_0(x))}{1 - x^2 z - x z^2 - z^3}$$

which completes the proof of the Theorem. \hfill \blacksquare

In [4], the author obtained the generating function for the Tribonacci-Lucas numbers. However, in here, we will obtain this function in terms of the Tribonacci-Lucas polynomials as a consequence of the Theorem 8. To do that we will again take $x = 1$ Theorem 8. Hence we have the following corollary.

**Corollary 9** The generating function of the Tribonacci-Lucas numbers $K_n$ is given by

$$g(z) = \sum_{n=0}^{\infty} K_n z^n = \frac{3 - 2z - z^2}{1 - z - z^2 - z^3}.$$  

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