Morel Homotopy Modules and Milnor-Witt Cycle Modules

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Abstract

This article is a sequel of [Fel18]. We study the cohomology theory and the canonical Milnor-Witt cycle module associated to a motivic spectrum. We prove that the heart of Morel-Voevodsky stable homotopy category over a perfect field (equipped with its homotopy t-structure) is equivalent to the category of Milnor-Witt cycle modules, thus generalising Déglise’s thesis [Dé11]. As a corollary, we recover a theorem of Ananyevskiy and Neshitov [AN18] and we prove that Chow-Witt groups in degree zero are birational invariants.

Keywords — Cycle modules, Milnor-Witt K-theory, Chow-Witt groups, A1-homotopy

MSC — 14C17, 14C35, 11E81

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1 Introduction

In the fundamental paper [Ros96], Rost introduced the notion of a cycle module. The idea was to find a good axiomatization of the main properties encountered in the study of Milnor K-theory, Quillen K-theory or Galois cohomology. According to [Ros96], a cycle module $M$ over a perfect field $k$ is the data of a $\mathbb{Z}$-graded abelian group $M(E)$ for every finitely generated field extension $E/k$, equipped with restriction maps, (finite) corestriction maps, a Milnor K-theory module action and residue maps $\partial$. Moreover, these data are subject to certain compatibility relations $(r1a), \ldots, (r3e), (fd)$ and $(c)$. The theory results in the construction of Gersten type complexes whose cohomology groups are called Chow groups with coefficients and can be used, for instance, to extend to the left the localization sequence of Chow groups associated with a closed embedding.

In order to construct the derived category of motives $\text{DM}(k, \mathbb{Z})$, Voevodsky introduced so-called homotopy sheaves which are homotopy invariant Nisnevich sheaves with transfers. One important example is given by $\mathcal{G}_m$, the sheaf of global units. Voevodsky proved that any homotopy sheaf $F$ has a Gersten resolution, implying that $F$ is determined in some sense by the data of its fibers in every function fields. This statement was made more precise in Déglise’s thesis: the heart of $\text{DM}(k, \mathbb{Z})$ with respect to its homotopy $t$-structure has a presentation given by the category of Rost cycle modules over $k$.

Morel’s point of view on the heart of $\text{DM}(k, \mathbb{Z})$ is given by the category of oriented homotopy modules. We recall that a homotopy module is a strictly $\mathbb{A}^1$-invariant Nisnevich sheaf with an additional structure defined over the category of smooth schemes (see Definition 4.3). It is called oriented when the Hopf map $\eta$ acts on it by 0. Déglise’s theorem proves that oriented homotopy modules form a subcategory of the category of homotopy modules which is equivalent to the category of Rost cycle modules. Morel’s natural conjecture [Mor12, Remark 2.49] was that there is a presentation of the heart of the stable homotopy category $\text{SH}(k)$ (or equivalently, the category of homotopy modules) in terms of some non-oriented version of cycle modules.

Current and future work

In [Fel18], the author introduced the theory of Milnor-Witt cycle modules, generalising the work of Rost [Ros96] on Milnor cycle modules and Schmid’s thesis [Sch98]. Indeed, we have studied general cycle complexes $C^*(X, M, \mathcal{V}_X)$ and their (co)homology groups $A^*(X, M, \mathcal{V}_X)$ (called Chow-Witt groups with coefficients) in a quadratic setting over a perfect base field of any characteristic. The general coefficient systems $M$ for these complexes are called Milnor-Witt cycle modules. The main example of such a cycle module is given by the Milnor-Witt K-theory (see [Fel18 Theorem 4.13]); other examples will be deduced from Theorem 4.1 or Theorem 5.4 (e.g. the representability of hermitian K-theory in $\text{SH}(k)$ will lead to a MW-cycle module, associated with hermitian K-theory). A major difference with Rost’s theory is that the grading to be considered is not $\mathbb{Z}$ but the category of virtual bundles (or, equivalently, the category of virtual vector spaces), where a virtual bundle $\mathcal{V}$ is, roughly speaking, the data of an integer $n$ and a line bundle $\mathcal{L}$ (see [Fel18 Appendix A]). Intuitively, Milnor-Witt cycle modules are given by (twisted) graded abelian groups equipped with extra data (restriction, corestriction, $K^{MW}$-action and residue maps).

For any scheme $X$, any virtual bundle $\mathcal{V}_X$ and any Milnor-Witt cycle module $M$, we have proved that there exists a complex $C^*(X, M, \mathcal{V}_X)$ equipped with pushforwards, pullbacks, a Milnor-Witt K-theory action and residue maps satisfying standard functoriality properties. A fundamental theorem is that the associated cohomology groups $A^*(X, M, \mathcal{V}_X)$ satisfy the homotopy invariance property (see [Fel18 Theorem 9.4]).

In this paper, we prove that Milnor-Witt cycle modules are closely related to Morel’s $\mathbb{A}^1$-homotopy theory: they can be realized geometrically as elements of the stable homotopy category. Precisely, we prove the following theorem.

**Theorem 1.** (Theorem 4.7) Let $k$ be a perfect field. The category of Milnor-Witt cycle modules is equivalent to the heart of Morel-Voevodsky stable homotopy category (equipped with the homotopy $t$-structure):
In order to prove this theorem, we study the cohomology theory associated with a motivic spectrum. This notion is naturally dual to the bivariant theory developed in [DJK18] and recalled in Section 2 (see Theorem 2.18). A motivic spectrum $E$ leads to a functor $\hat{E}$ from the category of finitely generated fields over $k$ to the category of graded abelian groups (be careful that the grading is not $\mathbb{Z}$ but is given by the category of virtual vector spaces). We prove that the functor $\hat{E}$ is a Milnor-Witt cycle premodule [Fel18, Definition 3.1]. Indeed, most axioms are immediate consequence of the general theory [DJK18]. Moreover, in Theorem 2.16 we prove a ramification theorem of independent interest that can be applied to prove rule [R3a]. Furthermore, we check axioms [FD] and [C] so that $\hat{E}$ is a Milnor-Witt cycle module. These two axioms follows from the study of a spectral sequence defined in Section 3 (see Theorem 3.4); another – more elementary – proof may result from an adaptation of [Ros96, Theorem 2.3] to the context of Milnor-Witt cycle modules but this method would rely heavily on the fact that the base field is perfect.

In Section 4, we construct a homotopy module for any Milnor-Witt cycle module and proceed to prove that the heart of the stable homotopy category (which is known to be equivalent to the category of homotopy module according to [Mor13]) is equivalent to the category of Milnor-Witt cycle modules. This result generalizes Déglise’s thesis (see Theorem 5.3) and answers affirmatively an old conjecture of Morel (see [Mor12, Remark 2.49]). An important corollary is the following result (which was proved independently by Ananyevskiy and Neshitov in [AN18, Theorem 8.12]):

**Theorem 2.** (Theorem 5.2) The heart of Morel-Voevosky stable homotopy category is equivalent to the heart of the category of MW-motives [DF18] (both equipped with their respective homotopy t-structures):

$$\text{SH}(k) \cong \text{DM}(k).$$

In future work, we will study the theory of Milnor-Witt cycle modules over more general base scheme $S$ and prove Theorem 4.1 in this more general setting. Assuming such task accomplished, we may already claim the corollaries stated in Section 5 to be true (including Theorem 5.2 whose original proof [AN18] needs the base field to be perfect).

**Outline of the paper**

In Section 2, we follow [DJK18] and define the bivariant theory associated to a motivic spectrum. We extend the main results for the associated cohomology theory. We study the basic properties of fundamental classes and prove a ramification formula.

In Section 3, we recall the theory of Milnor-Witt cycle modules developed in [Fel18]. For any motivic spectrum, we then construct a Milnor-Witt cycle modules in a functorial way.

The heart of the paper is Section 4 where we define a homotopy module for any Milnor-Witt cycle module and prove our main theorem: the heart of the stable homotopy category (which is known to be equivalent to the category of homotopy module according to [Mor13]) is equivalent to the category of Milnor-Witt cycle modules.

Finally in Section 5, we give some corollaries of the main theorem. In particular, we show that the heart of stable homotopy category is equivalent to the heart of the category of MW-motives [DF18]. We also prove that Chow-Witt groups and Morel homotopy modules are birational invariants.

**Conventions and Notations**

Throughout the paper, we fix a (commutative) field $k$ and we assume moreover that $k$ is perfect (of arbitrary characteristic). We consider only schemes that are noetherian and essentially of finite type over $k$. All schemes and morphisms of schemes are defined over $k$.

We denote by $S = \text{Spec} \, k$ the spectrum of $k$.

---

*That is, isomorphic to a limit of finite type schemes with affine étale transition maps.*
We use the term \( s\)-morphism as an abbreviation for \( s\)-separated morphism of finite type.

By a field \( E \) over \( k \), we mean a \( k \)-finitely generated field \( E \). Since \( k \) is perfect, notice that \( \text{Spec} \, E \) is essentially smooth over \( S \).

Let \( f : X \to S \) be a morphism of schemes and \( V_S \) be a virtual bundle over \( S \). We denote by \( V_X \) or by \( f^*V_S \) or by \( V_S \times_S X \) the pullback of \( V_S \) by \( f \).

Let \( f : X \to Y \) be a morphism of schemes. Denote by \( L_f \) or by \( L_{X/Y} \) the virtual vector bundle over \( Y \) representing the cotangent complex of \( f \). If \( p : X \to Y \) is a smooth morphism, then \( L_p \) is (isomorphic to) \( T_p = \Omega_{X/Y} \) the space of \( (\text{Kähler}) \) differentials. If \( i : Z \to X \) is a regular closed immersion, then \( L_i \) is the normal cone \( -N_Z X \). If \( f \) is the composite \( Y \xrightarrow{i} \mathbb{P}^{n}_X \xrightarrow{p} X \) with \( p \) and \( i \) as previously (in other words, if \( f \) is lci projective), then \( L_f \) is isomorphic to the virtual tangent bundle \( \pi^*\mathcal{T}_{\mathbb{P}^{n}_X/X} - N_Y(\mathbb{P}^{n}_X) \) (see also [Fel18, Section 9]).

Let \( E \) be a field (over \( k \)) and \( v \) a (discrete) valuation on \( E \). We denote by \( \mathcal{O}_v \), its valuation ring, by \( \mathfrak{m}_v \) its maximal ideal and by \( \kappa(v) \) its residue class field. We consider only valuations of geometric type, that is we assume: \( k \subset \mathcal{O}_v \), the residue field \( \kappa(v) \) is finitely generated over \( k \) and satisfies \( \text{tr.deg}_k(\kappa(v)) + 1 = \text{tr.deg}_k(E) \).

For \( E \) a field (resp. \( X \) a scheme), we denote by \( \langle n \rangle \) the virtual space \( A^d_E \) (resp. \( A^d_X \)).

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## 2 Bivariant Theory

### 2.1 Recollection and notations

In this subsection, we recall some results from [DJK18, §2]. Let \( S \) be a base scheme. Denote by \( \mathcal{V}(S) \) the Picard groupoid of virtual vector bundles on \( S \) (see [Del87] §4 or [Fel18, Appendix A]). If \( \mathcal{V}_S \) is a virtual vector bundle over \( S \), we denote by \( \text{Th}_S(\mathcal{V}_S) \) its associated Thom space (this is an \( \otimes \)-invertible motivic spectra over \( S \), see [CD12, Remark 2.4.15]).

**Definition 2.1.** Let \( E \in \text{SH}(S) \) be a motivic spectrum. Given an \( s\)-morphism \( p : X \to S \), a integer \( n \in \mathbb{Z} \) and a virtual bundle \( \mathcal{V}_X \in \mathcal{V}(X) \), we define the bivariant theory of \( X/S \) in bidegree \( (n, \mathcal{V}_X) \), with coefficients in \( E \), as the abelian group:

\[
E_n(X/S, \mathcal{V}_X) = \left[ \text{Th}_X(\mathcal{V}_X)[n], p'([E]) \right] = \left[ p_* \text{Th}_X(\mathcal{V}_X)[n], E \right].
\]

The cohomology theory represented by \( E \) is defined by the formula:

\[
E^n(X, \mathcal{V}_X) = E_{-n}(X/X, -\mathcal{V}_X) = \left[ \mathbb{1}_X, \mathcal{E}_X \otimes \text{Th}_X(\mathcal{V}_X)[n] \right]
\]

for any scheme \( X \) over \( S \) and any pair \( (n, \mathcal{V}_X) \in \mathbb{Z} \times \mathcal{V}(X) \).

In the special case where \( E = \mathbb{1} \) is the sphere spectrum, we will use the notation

\[
H_n(X/S, \mathcal{V}_X) = \mathbb{1}_n(X/S, \mathcal{V}_X) = \left[ \text{Th}_X(\mathcal{V}_X)[n], p'([\mathbb{1}_S]) \right]
\]

and we will refer to this simply as the bivariant \( K^1 \)-theory.

Similarly, we set \( H^n(X, \mathcal{V}_X) = \mathbb{1}^n(X, \mathcal{V}_X) \) and refer to this as the \( K^1 \)-cohomology.

### 2.2. Base change

For any cartesian square

\[
\begin{array}{ccc}
Y & \xrightarrow{g} & X \\
\downarrow & \Delta & \downarrow \\
T & \xrightarrow{f} & S
\end{array}
\]
one gets a map

\[ \Delta^* : \mathbb{E}_n(X/S, \mathcal{V}_X) \to \mathbb{E}_n(Y/T, \mathcal{V}_T) \]

by applying the functor \( g^* : \mathbf{SH}(X) \to \mathbf{SH}(Y) \) and using the exchange transformation \( \text{Ex}^*: g^*p^! \to q^*f^* \) associated with the square \( \Delta \).

2.3. Covariance for proper morphisms Let \( f : Y \to X \) be a proper morphism. We have a map

\[ f_* : \mathbb{E}_n(Y, \mathcal{V}_Y) \to \mathbb{E}_n(X, \mathcal{V}_X) \]

coming from the unit map \( f! f^! \to \text{Id} \) and the fact that \( f^! = f_* \) since \( f \) is proper.

2.4. Contravariance for étale morphisms Let \( f : Y \to X \) be an étale morphism, we have a map

\[ f^* : \mathbb{E}_n(X/S, \mathcal{V}_X) \to \mathbb{E}_n(Y/S, \mathcal{V}_Y) \]

obtained by applying the functor \( f^! : \mathbf{SH}(X) \to \mathbf{SH}(S) \) and using the purity isomorphism \( f^! = f^* \) as \( f \) is étale.

2.5. Products Consider a multiplication map \( \mu : \mathbb{E} \otimes \mathbb{E}' \to \mathbb{E}'' \) between motivic spectra. For any \( s \)-schemes \( Y \to X \to S \), any integers \( n, m \) and any virtual vector bundles \( \mathcal{W}_Y/Y \) and \( \mathcal{V}_X/X \), there is a multiplication map

\[ \mathbb{E}_m(Y/X, \mathcal{W}_Y) \otimes \mathbb{E}_n(X/S, \mathcal{V}_X) \to \mathbb{E}_{m+n}(Y/S, \mathcal{W}_Y + \mathcal{V}_Y). \]

Definition 2.6. Let \( X \to S \) be an \( s \)-morphism.

- A fundamental class\(^2\) of \( f \) is an element

\[ \eta_f \in H(X/S, \mathcal{V}_f) \]

for a given virtual vector bundle \( \mathcal{V}_f \) over \( X \).

- Let \( \mathcal{C} \) be a subcategory of the category of (quasi-compact and quasi-separated) schemes. A system of fundamental classes for \( \mathcal{C} \) is the data, for each morphism \( f : X \to Y \) in \( \mathcal{C} \), of a virtual bundle \( \mathcal{V}_f \in \mathsf{V}(X) \) and an orientation \( \eta_f \in H_0(f, \mathcal{V}_f) \) such that the following relations hold:

  1. Normalisation If \( f = \text{Id}_S \), then \( \mathcal{V}_f = 0 \) and the orientation \( \eta_f \in H_0(\text{Id}_S, 0) \) is given by the identity \( \text{Id} : \mathbb{1}_S \to \mathbb{1}_S \).

  2. Associativity formula For any composable morphisms \( f \) and \( g \) in \( \mathcal{C} \), one has an isomorphism:

\[ \mathcal{V}_{f \circ g} \simeq \mathcal{V}_g + g^* \mathcal{V}_f \]

and, modulo this identification, the following relation holds:

\[ \eta_{f \circ g}^\mathcal{C} \cdot \eta_f^\mathcal{C} = \eta_g^\mathcal{C}. \]

- Suppose the category \( \mathcal{C} \) admits fibred products. We say that a system of fundamental classes \( (\eta_f^\mathcal{C})_f \) is stable under transverse base change if it satisfies the following condition: for any cartesian square

\[
\begin{array}{ccc}
Y & \xrightarrow{g} & T \\
\downarrow{\eta} & \Delta & \downarrow{p} \\
X & \xrightarrow{f} & S
\end{array}
\]

such that \( f \) and \( g \) are in \( \mathcal{C} \) and \( p \) is transverse to \( f \), then one has \( \mathcal{V}_g = q^* \mathcal{V}_f \) and the following formula holds in \( H_0(g, \mathcal{V}_g) \): \( \Delta^*(\eta_f^\mathcal{C}) = \eta_g^\mathcal{C}. \)

\(^2\)Also called orientation in [DJK18, Definition 2.2.1].
In [DJK18, Theorem 3.3.2], the authors prove the following theorem.

**Theorem 2.7.** There exists a unique system of fundamental classes \( \eta_f \in H_0(X/S, L_f) \) associated with the class of quasi-projective lci morphisms \( f \) such that:

1. For any smooth s-morphism \( p \), the class \( \eta_p \) agrees with the fundamental class defined in [DJK18, Example 2.2.3] thanks to the purity isomorphism.
2. For any regular closed immersion \( i : Z \rightarrow X \), the class \( \eta_i \) agrees with the fundamental class defined in [DJK18, Example 3.2.3] by deformation to the normal cone.

Thanks to this system of fundamental classes, we can define Gysin morphisms as follows.

**2.8. Contravariance for lci morphisms**

Let \( f : Y \rightarrow X \) be an lci quasi-projective morphism \( f : Y/S \rightarrow X/S \) of s-morphisms with virtual cotangent bundle \( L_f \). For any motivic spectrum \( E \in SH(S) \), for any integer \( n \) and any virtual bundle \( V_X \) over \( X \), there exists a Gysin morphism:

\[
\begin{align*}
& f^* : E_n(X/S, V_X) \\
& \quad \rightarrow E_n(Y/S, L_f + V_Y)
\end{align*}
\]

\( x \mapsto \eta_f \cdot x \)

using the product defined in 2.5. These Gysin morphisms satisfy the following formulas:

1. **Functoriality:** For any suitable morphisms \( f \) and \( g \), one has \((fg)^* = g^* f^*\).
2. **Base change:** For any cartesian square

\[
\begin{array}{ccc}
Y & \xrightarrow{q} & T \\
\downarrow{\eta} \quad \Delta & & \downarrow{p} \\
X & \xrightarrow{f} & S
\end{array}
\]

such that \( f \) is quasi-projective lci and transverse to \( p \), one has \( f^* p_* = q_* g^* \).

**2.9. Localization long exact sequence**

Another essential property is the following long exact sequence following from the usual localization triangle in the six functors formalism in \( SH \).

Indeed, let \( E \in SH(S) \) be a motivic spectrum. For any closed immersion \( i : Z \rightarrow X \) of separated schemes over \( S \), with (quasi-compact) complementary open immersion \( j : U \rightarrow X \), there exists a canonical localization long exact sequence of the form:

\[
\cdots \rightarrow E_n(Z/S, V_Z) \xrightarrow{i^*} E_n(X/S, V_X) \xrightarrow{j^*} E_n(U/S, V_U) \xrightarrow{\partial} E_{n-1}(Z/S, V_Z) \rightarrow \cdots
\]

We will need the following properties of localizations long exact sequences.

**Proposition 2.10.** Let \( E \in SH(S) \) be a motivic spectrum and consider the following commutative square

\[
\begin{array}{ccc}
T & \xrightarrow{k} & Y \\
\downarrow{q} \quad \Delta & & \downarrow{p} \\
Z & \xrightarrow{i} & X
\end{array}
\]

of closed immersions of separated schemes over \( S \). For \( V_X \) a virtual vector bundle over \( X \), we have the following diagram
Proposition 2.11 (Base change for lci morphisms). Consider a cartesian square of schemes

\[
\begin{array}{ccc}
X' & \xrightarrow{f'} & Y' \\
| & \searrow & \nwarrow \\
X & \xrightarrow{f} & Y
\end{array}
\]

with \( f \) proper and essentially smooth, and \( g \) lci. Suppose moreover that the square is tor-independent, that is for any \( x \in X \), \( y' \in Y' \) with \( y = f(x) = g(y') \) and for any \( i > 0 \) we have

\[
\text{Tor}_i^{\mathcal{O}_{Y',y'}}(\mathcal{O}_{X,x}, \mathcal{O}_{Y',y'}) = 0.
\]

Up to the canonical isomorphism \( f^*\mathcal{L}_y \simeq \mathcal{L}_{y'} \), we have:

\[
f_i^* \circ g^* = g_i^* \circ f_*.
\]

The bivariant theory of a spectrum satisfies some \( A^1 \)-homotopy invariance property.

Proposition 2.12. Let \( E \in SH(S) \) be a motivic spectrum. Let \( X \) be an s-scheme over \( S \) and let \( p : V \to X \) be a vector bundle with virtual tangent bundle \( \mathcal{L}_p = p^* \mathcal{V} \). Then the Gysin morphism

\[
p^* : E_n(X/S, \mathcal{V}_X) \to E_n(V/S, \mathcal{L}_p + \mathcal{V})
\]

is an isomorphism for any integer \( n \) and any virtual bundle \( \mathcal{V}_X \) over \( X \).

Proof. See [DJK18] Lemma 2.2.6] when \( E \) is the sphere spectrum \( \mathbb{1} \). The general case follows from the definitions.

Definition 2.13. Keeping the notations of 2.12, we define the Thom isomorphism

\[
\Phi_{V/X} : E_n(V/S, \mathcal{V}_V) \to E_n(X/S, \mathcal{V} - \langle V \rangle),
\]

associated with \( V/X \), as the inverse of the Gysin morphism \( p^* : E_n(X/S, \mathcal{V}_X - \langle E \rangle) \to E_n(V/S, \mathcal{V}_V) \).
2.2 Ramification formula

2.15. In the following, we study ramifications.

Consider a topologically cartesian square

\[
\begin{array}{ccc}
T & \xrightarrow{k} & Y \\
\downarrow q & & \downarrow p \\
Z & \xrightarrow{i} & X
\end{array}
\]

where \( k \) and \( i \) are regular closed immersions of codimension 1 and \( T \) is a reduced connected scheme. We have \( T = (Z \times_X Y)_{\text{red}} \). Denote by \( \mathfrak{J}' \) the ideal of the immersion \( Z \times_X Y \to Y \), by \( \mathfrak{J} \) the ideal of \( k \) and by \( \mathfrak{I} \) the ideal of \( i \). Assume moreover that the square is ramified with ramification index \( e \) in the sense that there exists a nonzero natural number \( e \) such that \( \mathfrak{J}' = \mathfrak{J}^e \).

We consider the morphism of deformation spaces \( \nu^{(e)} : D_T Y \to D_Z X \) defined as the spectrum of the composite

\[
\bigoplus_{n \in \mathbb{Z}} \mathfrak{J}^n \cdot t^{-n} \to \bigoplus_{n \in \mathbb{Z}} (\mathfrak{J}'^e)^n \cdot t^{-n} \to \bigoplus_{m \in \mathbb{Z}} \mathfrak{J}^m \cdot t^{-m}
\]

where the first map is induced by the morphism \( \tilde{f} : \mathfrak{J} \to \mathfrak{J}' \) defined via \( f \) and where the second map takes the parameter \( t \) to its power \( t^e \).

The map \( \nu^{(e)} \) factors naturally making the following diagram commutative:

\[
\begin{array}{ccc}
N_T Y & \xrightarrow{\Phi_{D_T Y}} & D_T Y \\
\downarrow \Delta & & \downarrow \gamma_m Y \\
q^* N_Z X & \xrightarrow{\Phi_{D_Z X}} & q^* D_Z X \\
\downarrow \Delta^* & & \downarrow \gamma_m X
\end{array}
\]

Hence, we have the following commutative diagram:

\[
\begin{array}{ccc}
E_{n+1}(G_m Y/Y, *) & \xrightarrow{\Delta^*} & E_n(N_T Y/Y, *) \\
\downarrow \partial_T/Y & & \downarrow \Phi_{N_T Y/T} \\
E_{n+1}(G_m Y/Y, *) & \xrightarrow{\Delta^*} & E_n(N_Z X/X, *) \\
\downarrow \partial_{Z/X} & & \downarrow \Phi_{N_Z X/Z} \\
E_{n+1}(G_m X/X, *) & \xrightarrow{\Delta^*} & E_n(N_Z X/X, *) \\
\end{array}
\]

where the arrows \( \Delta^* \) denote the obvious maps induced by the corresponding squares. Square (1) (resp. square (2)) is commutative because of the naturality of localization long exact sequences with respect to the proper covariance (resp. base change). The map \( \nu^{(e)} \) is defined so that the square (3) commutes. Square (4) commutes by compatibility of Thom isomorphisms with respect to base change [2.14].

From this, we deduce that

\[
\Delta^*(\eta_k) = \nu^{(e)}_*(\eta_k).
\]

The map \( \nu^{(e)} \) induces a morphism of Thom spaces.
Th(N_T Y) → Th(q^* N_Z X)

which corresponds after delooping to the quadratic form e_ε in \( K^{MW}(\kappa(v)) \). We have proved the following theorem:

**Theorem 2.16.** Keeping the previous notations, the following holds in \( E_n(T/Y, N_T Y + V) \)

\[ \Delta^*(\eta_k) = e_\varepsilon \cdot \eta_k. \]

**Corollary 2.17.** Consider the topologically cartesian square

\[
\begin{array}{ccc}
T & \xrightarrow{k} & Y \\
\downarrow{q} & \Delta & \downarrow{p} \\
Z & \xrightarrow{i} & X
\end{array}
\]

where \( k \) and \( i \) are regular closed immersions of codimension 1 and \( T \) is a reduced connected scheme. Assume moreover that the square is ramified with ramification index \( e \) as before. Then we have the following ramification formula:

\[ q^* i_* = e_\varepsilon k_* p^*. \]

### 2.3 Application to cohomology

For \( E \in SH(S) \), \( n \in \mathbb{N} \), \( p : X \to S \) a scheme over \( S \) and \( V_X \) a virtual vector bundle over \( X \), we define the cohomology group

\[ E^n(X, V_X) = \text{Hom}_{SH(X)}(\underline{1}_X, E_X \otimes \text{Th}_X(V_X)[n]) \]

where \( E_X = p^* E_S \).

It is dual to the bivariant theory \( \mathbb{E}_*(-, \cdot) \) defined previously. Indeed, we have the following theorem:

**Theorem 2.18.** Let \( f : X \to S \) be an essentially\footnote{In [DJK18], the authors worked only with separated \( S \)-schemes of finite type but we can extend in a canonical way most of the results for separated \( S \)-schemes essentially of finite type (see also [ADNT18 §2.1.1]).} smooth scheme and \( E \in SH(S) \) a motivic spectrum. Then for any integer \( n \) and any virtual bundle \( V_X \) over \( X \), there is a canonical isomorphism

\[ E^n(X, V_X) \simeq \mathbb{E}_{-n}(X/S, L_f - V_X) \]

which is contravariantly natural in \( X \) with respect to étale morphisms.

**Example 2.19.** A crucial example follows from the work of Morel: if \( E \) is the unit sphere \( \underline{1} \), then the group \( H^n(X, \langle n \rangle) \) is isomorphic to the Milnor-Witt theory \( K^{MW}(X) \).

In the following, we give the usual properties of the bivariant cohomology theory. The proofs follow the same ideas than in the homological case (see [DJK18 §2]). In practice (since our base field \( k \) is perfect) we work mainly with essentially smooth schemes, hence we could also apply Theorem 2.18.

**2.20. CONTRAVARIANCE** Let \( f : Y \to X \) be a morphism of schemes and \( V_Y \) be a virtual bundle over \( X \). There exists a map

\[ f^* : E^n(X, V_X) \to E^n(Y, V_Y). \]

**2.21. COVARIANCE** Let \( f : Y \to X \) be an lci projective map. There exists a Gysin morphism

\[ f_* : E^n(Y, L_f + V_Y) \to E^n(X, V_X). \]
As always, the definition follows from general considerations using the six functors formalism. For instance, assume $f$ smooth. By adjunction, the set $\text{Hom}_{\mathcal{SH}(Y)}(\mathbb{1}_Y, E_Y \otimes \text{Th}(L_f))$ is in bijection with $\text{Hom}_{\mathcal{SH}(X)}(\mathbb{1}_X, f_*(E_Y \otimes \text{Th}(L_f)))$. The purity isomorphism and the fact that $f_* = f_!$ (since $f$ is proper) lead a bijection with the set $\text{Hom}_{\mathcal{SH}(X)}(\mathbb{1}_X, f_! E_X)$. Any element of this set can be composed with the counit map $f_! f^* \to \mathbb{1}$ so that we obtain an element in $\text{Hom}_{\mathcal{SH}(X)}(\mathbb{1}_X, E_X)$.

More generally, the group $E'(Y, L_f + \mathcal{V}_Y)$ is isomorphic to $[f^*(\mathbb{1}_X), f^*(E_X \otimes \text{Th}(\mathcal{V}_X)[r]) \otimes \text{Th}(L_f)]_Y$ which is, by adjunction, isomorphic to $[\mathbb{1}_X, f_*(f^*(E_X \otimes \text{Th}(\mathcal{V}_X)[r]) \otimes \text{Th}(L_f))]_X$. We can then compose with the trace map $\text{Tr}_f : f_* \Sigma^\mathcal{L}_f f^* \to \mathbb{1}$ defined in [DJK18, §2.3.3] in order to obtain a map in the group $E'(X, \mathcal{V}_X)$.

2.22. Milnor-Witt action Any motivic spectrum $E$ equipped with a unit isomorphism $\mathbb{1} \otimes E \to E$ defines an action by composition on the left

$$\gamma : H^m(X, \mathcal{W}_X) \otimes E^n(X, \mathcal{V}_X) \to E^{m+n}(X, \mathcal{W}_X + \mathcal{V}_X).$$

2.23. Localization long exact sequence Another essential property is the following long exact sequence following from the usual localization triangle in the six functors formalism in $\mathcal{SH}$.

Indeed, let $E \in \mathcal{SH}(S)$ be a motivic spectrum. For any closed immersion $i : Z \to X$ of separated schemes over $S$, with (quasi-compact) complementary open immersion $j : U \to X$, there exists a canonical localization long exact sequence of the form:

$$\cdots \to E^n(Z, \mathcal{L}_Z + \mathcal{V}_Z) \xrightarrow{i_*} E^n(X, \mathcal{L}_X + \mathcal{V}_X) \xrightarrow{j^*} E^n(U, \mathcal{L}_U + \mathcal{V}_U) \xrightarrow{\partial} E^{n+1}(Z, \mathcal{L}_Z + \mathcal{V}_Z) \to \cdots$$

where the residue map $\partial$ can be defined as the following composition

$$E^n(X - Z, \mathcal{V}_{X-Z}) \xrightarrow{\partial_i} E_{-n}(X - Z/X, -\mathcal{V}_{X-Z}) \xrightarrow{\partial} E_{-n}(X - Z/X, -\mathcal{V}_{X-Z}).$$

**Proposition 2.24.** Let $E \in \mathcal{SH}(S)$ be a motivic spectrum and consider the following commutative square

$$\begin{array}{ccc}
T & \xrightarrow{k} & Y \\
\downarrow q & & \downarrow q \\
Z & \xrightarrow{i} & X
\end{array}$$

of closed immersions of separated schemes over $S$. For $\mathcal{V}_X$ a virtual vector bundle over $X$, we have the following diagram

$$\begin{array}{ccccccc}
E^n(T, \mathcal{V}_T^Z) & \xrightarrow{k_*} & E^n(Y, \mathcal{V}_Y^Z) & \xrightarrow{k^*} & E^n(Y - T, \mathcal{V}_Y^Z) & \xrightarrow{\partial_k} & E^{n+1}(T, \mathcal{V}_T^Z) \\
\downarrow q_* & & \downarrow p_* & & \downarrow p & & \downarrow q_* \\
E^n(Z, \mathcal{V}_Z^Z) & \xrightarrow{i_*} & E^n(X, \mathcal{V}_X^Z) & \xrightarrow{i^*} & E^n(X - Z, \mathcal{V}_X^Z) & \xrightarrow{\partial_i} & E^{n+1}(Z, \mathcal{V}_Z^Z) \\
\downarrow q_* & & \downarrow p_* & & \downarrow p & & \downarrow q_* \\
E^n(Z - T, \mathcal{V}_Z^Z) & \xrightarrow{\iota} & E^n(X - Y, \mathcal{V}_X^Z - \mathcal{Y}) & \xrightarrow{\jmath^*} & E^n(X - (Z \cup Y), \mathcal{V}_X^Z - (Z \cup Y)) & \xrightarrow{\partial} & E^{n+1}(Z - T, \mathcal{V}_Z^Z) \\
\downarrow \partial_k & & \downarrow \partial_p & & \downarrow \partial & & \downarrow \partial \\
E^{n+1}(T, \mathcal{V}_T^Z) & \xrightarrow{k_*} & E^{n+1}(Y, \mathcal{V}_Y^Z) & \xrightarrow{k^*} & E^{n+1}(Y - T, \mathcal{V}_Y^Z) & \xrightarrow{\partial_k} & E^{n+2}(T, \mathcal{V}_T^Z)
\end{array}$$
with obvious maps. Each squares of this diagram is commutative except for $(*)$ which is anti-commutative.

**Proof.** This follows from Proposition 2.10.

**Proposition 2.25.** Let $E \in S\text{H}(S)$ be a motivic spectrum. Let $i : Z \to X$ a closed immersion with complementary open immersion $j : U \to X$. Let $x \in H^n(Y, V_Y)$. Then the following diagram is commutative:

\[
\begin{array}{c}
\mathbb{E}^n(Z, \mathcal{L}_i + W_Z) \xrightarrow{\nu_*} \mathbb{E}^n(X, W_X) \xrightarrow{j^*} \mathbb{E}^n(U, W_U) \\
\downarrow \text{co} \gamma_{i(*)} \downarrow \downarrow \gamma_{i(*)} \\
\mathbb{E}^{n+m}(Z, \mathcal{L}_i + V_Z + W_Z) \xrightarrow{\nu_*} \mathbb{E}^{n+m}(X, V_X + W_X) \xrightarrow{j^*} \mathbb{E}^{n+m}(U, V_U + W_U) \\
\downarrow \text{co} \gamma_{i(*)} \downarrow \downarrow \gamma_{i(*)}
\end{array}
\]

where $\gamma_i$ is the multiplication map defined in \[2.29\] (see also \[2.28\]) and where $\varepsilon$ is the isomorphism induced by the switch isomorphism $\mathcal{L}_i + V_Z \simeq V_Z + \mathcal{L}_i$.

**Proof.** This follows from \[DJK18\ Proposition 2.1.12\].

**Proposition 2.26** (Base change for lci morphisms). Consider a cartesian square of schemes

\[
\begin{array}{c}
X' \xrightarrow{f'} Y' \\
\downarrow g' \downarrow g \\
X \xrightarrow{f} Y
\end{array}
\]

with $f$ proper and essentially smooth, and $g$ lci. Suppose moreover that the square is tor-independent, that is for any $x \in X$, $y' \in Y'$ with $y = f(x) = g(y')$ and for any $i > 0$ we have

$\text{Tor}_i^{O_{X,Y}}(O_{X,x}, O_{Y',y'}) = 0$.

Up to the canonical isomorphism $f'^* \mathcal{L}_g \simeq \mathcal{L}_{g'}$, we have:

$f'_* \circ g'^* = g'^* \circ f_*$.

**Proof.** See Proposition 2.2.

We will need the following proposition:

**Proposition 2.27.** Let $\nu : Z \to X$ a closed immersion of smooth schemes. Consider the canonical decomposition $Z = \bigsqcup_{i \in I} Z_i$ and $X = \bigsqcup_{j \in J} X_j$ into connected components. Denote by $\tilde{Z}_j = Z \times_X X_j$. For any $i \in I$, let $j \in J$ be the unique element such that $Z_i \subset X_j$ and denote by $\nu'_i : Z_i \to \tilde{Z}_j$ the induced immersion. Consider the complement $Z_i'$ such that $\tilde{Z}_j = Z_i \sqcup Z_i'$. The following diagram is commutative:

\[
\begin{array}{c}
\mathbb{E}^n(X - Z, V_X^C - Z) \xrightarrow{\partial_{X,Z}} \mathbb{E}^n(Z, V_Z^C) \xrightarrow{\nu_*} \mathbb{E}^n(X, V_X^C) \\
\downarrow \Downarrow \downarrow \Downarrow \\
\bigoplus_{j \in J} \mathbb{E}^n(X_j - \tilde{Z}_j, V_{X_j}^C - \tilde{Z}_j) \xrightarrow{\bigoplus_{j \in J} \partial_{X_j,Z_j}} \bigoplus_{j \in J} \mathbb{E}^n(Z_j, V_{Z_j}^C) \xrightarrow{\bigoplus_{j \in J} \nu_{j(*)}} \bigoplus_{j \in J} \mathbb{E}^n(X_j, V_{X_j}^C)
\end{array}
\]

where the vertical maps are the canonical isomorphisms and where, for any $(i,j) \in I \times J$, if $Z_i \subset X_j$, then $\nu_{ij} = (\nu'_i)_*$ and $\partial_{ij} = \partial_{X_j,Z_j} = 0$; otherwise $\nu_{ij} = 0$ and $\partial_{ij} = 0$.

**Proof.** Straightforward.
3 From Homotopy Modules to Milnor-Witt Cycle Modules

3.1 Recollection on Milnor-Witt cycle modules

We denote by $\mathcal{G}_k$ the category whose objects are the couple $(E, \mathcal{V}_E)$ where $E$ is a field over $k$ and $\mathcal{V}_E \in \mathfrak{W}(E)$ is a virtual vector space (of finite dimension over $F$). A morphism $(E, \mathcal{V}_E) \to (F, \mathcal{V}_F)$ is the data of a morphism $E \to F$ of fields over $k$ and an isomorphism $\mathcal{V}_E \otimes_E F \cong \mathcal{V}_F$ of virtual $F$-vector spaces.

A morphism $(E, \mathcal{V}_E) \to (F, \mathcal{V}_F)$ in $\mathcal{G}_k$ is said to be finite (resp. separable) if the field extension $F/E$ is finite (resp. separable).

We recall that a Milnor-Witt cycle module $M$ over $k$ is a functor from $\mathcal{G}_k$ to the category $\text{Ab}$ of abelian groups equipped with data

1. (restriction maps) Let $\varphi : (E, \mathcal{V}_E) \to (F, \mathcal{V}_F)$ be a morphism in $\mathcal{G}_k$. The functor $M$ gives a morphism $\varphi^* : M(E, \mathcal{V}_E) \to M(F, \mathcal{V}_F)$,
2. (corestriction maps) Let $\varphi : (E, \mathcal{V}_E) \to (F, \mathcal{V}_F)$ be a morphism in $\mathcal{G}_k$ where the morphism $E \to F$ is finite. There is a morphism $\varphi^* : M(F, \Omega_{E/k} + \mathcal{V}_F) \to M(E, \Omega_{E/k} + \mathcal{V}_E)$,
3. (Milnor-Witt K-theory action) Let $(E, \mathcal{V}_E)$ and $(E, \mathcal{W}_E)$ be two objects of $\mathcal{G}_k$. For any element $x$ of $K^{MW}(E, \mathcal{W}_E)$, there is a morphism

$$\gamma_x : M(E, \mathcal{V}_E) \to M(E, \mathcal{W}_E + \mathcal{V}_E)$$

so that the functor $M(E, -) : \mathfrak{W}(E) \to \text{Ab}$ is a left module over the lax monoidal functor $K^{MW}(E, -) : \mathfrak{W}(E) \to \text{Ab}$ (see [Yet03, Definition 39]; see also [FEL18, Definition 3.1]),
4. (residue maps) Let $E$ be a field over $k$, let $v$ be a valuation on $E$ and let $\mathcal{V}$ be a virtual projective $O_v$-module of finite type. Denote by $\mathcal{V}_E = \mathcal{V} \otimes_{O_v} E$ and $\mathcal{V}_{k(v)} = \mathcal{V} \otimes_{O_v} k(v)$. There is a morphism

$$\partial_v : M(E, \mathcal{V}_E) \to M(k(v), -N_v + \mathcal{V}_{k(v)})$$

and satisfying rules

1. (functoriality of restriction maps),
2. (functoriality of corestriction maps),
3. (base change property),
4. (projection formulae),
5. (ramification formula),
6. (compatibility between residue maps and the first three data),
7. (compatibility with orientations).

Moreover, a Milnor-Witt cycle module $M$ satisfies axioms [FD] (finite support of divisors) and [C] (closedness) that enable us to define a complex $(C_p(X, M, \mathcal{V}_X), d_p)_{p \in \mathbb{Z}}$ for any scheme $X$ and virtual bundle $\mathcal{V}_X$ over $X$ where

$$C_p(X, M, \mathcal{V}_X) = \bigoplus_{x \in X(p)} M(k(x), \Omega_{k(x)/k} + \mathcal{V}_x)$$

and the differential $d_p = (\partial_x^p)_{(x, y) \in X(p) \times X(p-1)}$ is defined as follows (see [FEL18, Section 4]).
$M(x, \mathcal{V}_X) = M(\kappa(x), \Omega_{\kappa(x)/k} + \mathcal{V}_x)$.

If $X$ is normal, then for any $x \in X^{(i)}$ the local ring of $X$ at $x$ is a valuation ring so that \([D4]\) gives us a map $\partial_x : M(\xi, \mathcal{V}_X) \to M(x, \mathcal{V}_X)$ where $\xi$ is the generic point of $X$.

If $X$ is any scheme, let $x, y$ be any points in $X$. We define a map

$$\partial_y^x : M(x, \mathcal{V}_X) \to M(y, \mathcal{V}_X)$$

as follows. Let $Z = \{x\}$. If $y \not\in Z$, then put $\partial_y^x = 0$. If $y \in Z$, let $\tilde{Z} \to Z$ be the normalization and put

$$\partial_y^x = \sum_{z|y} \text{cores}_{\kappa(z)/\kappa(y)} \circ \partial_z$$

with $z$ running through the finitely many points of $\tilde{Z}$ lying over $y$.

The complex $(C_p(X, M, \mathcal{V}_X), d)_{p \geq 0}$ is called the Milnor-Witt complex of cycles on $X$ with coefficients in $M$ and we denote by $A_p(X, M, \mathcal{V}_X)$ the associated homology groups (called Chow-Witt groups with coefficients in $M$). We can define five basic maps on the complex level (see [Fel18, Section 4]):

**Pushforward** Let $f : X \to Y$ be a $k$-morphism of schemes, let $\mathcal{V}_Y$ be a virtual bundle over the scheme $Y$. The data \([D2]\) induces a map

$$f_* : C_p(X, M, \mathcal{V}_X) \to C_p(Y, M, \mathcal{V}_Y).$$

**Pullback** Let $g : X \to Y$ be an essentially smooth morphism of schemes. Let $\mathcal{V}_Y$ a virtual bundle over $Y$. Suppose $X$ connected (if $X$ is not connected, take the sum over each connected component) and denote by $s$ the relative dimension of $g$. The data \([D1]\) induces a map

$$g^* : C_p(Y, M, \mathcal{V}_Y) \to C_{p+s}(X, M, -L_{X/Y} + \mathcal{V}_X).$$

**Multiplication with units** Let $X$ be a scheme of finite type over $k$ with a virtual bundle $\mathcal{V}_X$. Let $a_1, \ldots, a_n$ be global units in $\mathcal{O}_X$. The data \([D3]\) induces a map

$$[a_1, \ldots, a_n] : C_p(X, M, \mathcal{V}_X) \to C_p(X, M, \langle n \rangle + \mathcal{V}_X).$$

**Multiplication with $\eta$** Let $X$ be a scheme of finite type over $k$ with a virtual bundle $\mathcal{V}_X$. The Hopf map $\eta$ and the data \([D3]\) induces a map

$$\eta : C_p(X, M, \mathcal{V}_X) \to C_p(X, M, -\mathbb{A}^1_X + \mathcal{V}_X).$$

**Boundary map** Let $X$ be a scheme of finite type over $k$ with a virtual bundle $\mathcal{V}_X$, let $i : Z \to X$ be a closed immersion and let $j : U = X \setminus Z \to X$ be the inclusion of the open complement. The data \([D4]\) induces (as in \([3.1]\)) a map

$$\partial = \partial^Z_U : C_p(U, M, \mathcal{V}_U) \to C_{p-1}(Z, M, \mathcal{V}_Z).$$

These maps satisfy the usual compatibility properties ([Fel18, Section 5]). In particular, they induce maps $f_*, g^*, [a], \eta, \partial^Z_U$ on the homology groups $A_*(X, M, *)$.

We end this subsection with a lemma illustrating the importance of the rule \([R4a]\). This will be useful in Section 4.

**Lemma 3.2.** Let $M$ be a Milnor-Witt cycle module over $k$. For any field $E/k$ and any virtual vector bundle $\mathcal{V}_E$ over $E$, we have a canonical isomorphism

$$M(E, \mathcal{V}_E) \simeq M(E, \langle n \rangle) \otimes_{\mathbb{Z}[E^\times]} \mathbb{Z}[\det(\mathcal{V}_E)^\times]$$

where $n$ is the rank of $\mathcal{V}_E$.

**Proof.** Any element $u \in \det(\mathcal{V}_E)^\times$ defines an isomorphism
Let \( \Theta_u : M(E, V_E) \simeq M(E, \langle n \rangle) \otimes_{\mathbb{Z}[E^{\times}]} \mathbb{Z}[\det(V_E)^{\times}] \)

in a obvious way thanks to \([\text{D1}]\). One can check that this map does not depend on the choice of \( u \) according to rule \([\text{R1a}]\).

Note that this lemma is true for Milnor-Witt K-theory \( K^{MW} \) by definition.

### 3.2 Cycle premodule structure

Let \( \mathcal{E} \in \text{SH}(S) \) be a motivic ring spectrum. For any field \( E \) and any virtual vector space \( V_E \) of rank \( r \) over \( E \), we put

\[
\tilde{E}(E, V_E) = E^{-r}(X, V_X) = \text{Hom}_{\text{SH}(X)}(\mathbb{1}_X, \mathbb{E}_X \otimes \text{Th}_X(V_X)[-r]),
\]

where \( X = \text{Spec} \ E \). We prove that this defines a functor \( \tilde{E} : \mathfrak{G}_k \to \text{Ab} \) which is a Milnor-Witt cycle module. Indeed we have the following data:

\begin{enumerate}[\textbf{(D1)}]
\item Let \( \varphi : (E, V_E) \to (F, V_F) \) be a morphism in \( \mathfrak{G}_k \). The cohomology theory \( E^*(-, \ast) \) being contravariant (see \([2.20]\)), we obtain a map \( \varphi_* : \tilde{E}(E, V_E) \to \tilde{E}(F, V_F) \).
\item Let \( \varphi : (E, V_E) \to (F, V_F) \) be a morphism in \( \mathfrak{G}_k \) where the morphism \( E \to F \) is finite. The (twisted) covariance described in \([2.21]\) leads to a morphism \( \varphi^* : \tilde{E}(F, \Omega_{E/k} + V_F) \to \tilde{E}(E, \Omega_{E/k} + V_E) \).
\item Let \( (E, V_E) \) and \( (E, W_E) \) be two objects of \( \mathfrak{G}_k \). For any element \( x \) of \( K^{MW}(E, W_E) \), there is a morphism

\[
\gamma_x : \tilde{E}(E, V_E) \to \tilde{E}(E, W_E + V_E)
\]

given by composition on the left by \( x \) (as in \([2.22]\)) as we can identify \( K^{MW}(E, W_E) \) with \( \mathbb{1}(E, W_E) \) (see Example \([2.19]\)). We can check that the functor \( \tilde{E}(E, -) : \mathfrak{B}(E) \to \text{Ab} \) is then a left module over the lax monoidal functor \( K^{MW}(E, -) : \mathfrak{B}(E) \to \text{Ab} \) (see \([\text{Yel03}]\) Definition 39, see also \([\text{Fel18}]\) Remark 2.5).
\item Let \( E \) be a field over \( k \), let \( v \) be a valuation on \( E \) and let \( V \) be a virtual projective \( O_v \)-module of finite type. As usual, denote by \( V_E = V \otimes_{O_v} E \) and \( V_{\kappa(v)} = V \otimes_{O_v} \kappa(v) \). There is a morphism

\[
\partial_v : \tilde{E}(E, V_E) \to \tilde{E}(\kappa(v), -\mathcal{N}_v + V_{\kappa(v)})
\]

given by the long exact sequence \([2.23]\) where the closed immersion is

\[
\text{Spec} \ \kappa(v) \hookrightarrow \text{Spec} \ O_v.
\]

It is clear that the data \([\text{D1}]\) and \([\text{D2}]\) are functorial so that the two following rules hold:

\begin{enumerate}[\textbf{(R1a)}]
\item Let \( \varphi \) and \( \psi \) be two composiable morphisms in \( \mathfrak{G}_k \). One has

\[
(\psi \circ \varphi)_* = \psi_* \circ \varphi_*.
\]
\item Let \( \varphi \) and \( \psi \) be two composable finite morphisms in \( \mathfrak{G}_k \). One has

\[
(\psi \circ \varphi)^* = \varphi^* \circ \psi^*.
\]

The base change theorem \([2.26]\) leads to the following rule \([\text{R1c}]\)

\begin{enumerate}[\textbf{(R1c)}]
\item Consider \( \varphi : (E, V_E) \to (F, V_F) \) and \( \psi : (E, V_E) \to (L, V_L) \) with \( \varphi \) finite and \( \psi \) separable. Let \( R \) be the ring \( F \otimes_E L \). For each \( p \in \text{Spec} \ R \), let \( \varphi_p : (L, V_L) \to (R/p, V_{R/p}) \) and \( \psi_p : (F, V_F) \to (R/p, V_{R/p}) \) be the morphisms induced by \( \varphi \) and \( \psi \). One has
\end{enumerate}
ψ∗ ◦ φ∗ = \sum_{p \in \text{Spec } R} (φ_p)^∗ \circ (ψ_p)_∗.

The general formalism of Fulton-McPherson gives the usual projection formulas (see also [DJK18 2.2.7], or [Dé17 1.2.8]):

(R2) Let φ : (E, V_E) → (F, V_F) be a morphism in \( \mathcal{F}_k \), let \( x \) be in \( K_{MW}(E, W_E) \) and \( y \) be in \( K_{MW}(F, \Omega_{F/k} + W'_F) \) where \( (E, W_E) \) and \( (F, W'_F) \) are two objects of \( \mathcal{F}_k \).

(R2a) We have \( φ^∗ \circ γ_x = γ_{φ^∗(x)} \circ φ^∗ \).

(R2b) Suppose \( φ \) finite. We have \( φ^∗ \circ γ_x \circ φ^∗ = γ_{φ^∗(x)} \circ φ^∗ \).

(R2c) Suppose \( φ \) finite. We have \( φ^∗ \circ γ_y \circ φ^∗ = γ_{φ^∗(y)} \).

We now prove the remaining rules.

(R3a) Let \( E \to F \) be a field extension and \( w \) be a valuation on \( F \) which restricts to a non trivial valuation \( v \) on \( E \) with ramification \( e \). Let \( V \) be a virtual \( \mathcal{O}_v \)-module so that we have a morphism \( φ : (E, V_E) \to (F, V_F) \) which induces a morphism \( φ : (κ(v), -N_v + Vκ(v)) \to (κ(w), -N_w + Vκ(w)) \). We have

\[ \partial_w \circ φ^∗ = γ_{e_w} \circ \mathcal{T}_φ \circ \partial_v. \]

Proof. We apply Theorem 2.16 to the commutative diagram

(R3b) Let \( φ : E \to F \) be a finite morphism of fields, let \( v \) be a valuation over \( E \) and let \( V \) be a virtual vector bundle over \( \mathcal{O}_v \). For each extension \( w \) of \( v \), we denote by \( φ_w : κ(v) \to κ(w) \) the map induced by \( φ \). We have

\[ \partial_v \circ φ^∗ = \sum_w φ_w^∗ \circ \partial_w. \]

Proof. There exists a semilocal ring \( A \) over \( \mathcal{O}_v \) such that the set of maximal ideals consists of the ideals \( m_w \) where \( w \) is an extension of \( v \).

Denote by \( T = \oplus_{w|v} \text{Spec } κ(w), Y = \text{Spec } A, Z = \text{Spec } κ(v) \) and \( X = \text{Spec } \mathcal{O}_v \) so that we have the following commutative diagram of cartesian squares:

\[
\begin{array}{ccc}
T & \xrightarrow{k} & Y' & \xrightarrow{i'} & Y - T \\
g & | & f & | & h \\
Z & \xrightarrow{i} & X & \xleftarrow{i'} & X - Z
\end{array}
\]

where \( k, i \) are the canonical closed immersions with complementary open immersions \( k', i' \) respectively and where \( f, g, h \) are the canonical maps.

According to Proposition 2.24 this leads to the following commutative diagram:
\[ E^r(T, \mathcal{L}_T + \mathcal{V}_T) \xrightarrow{\partial} E^r(Y, \mathcal{L}_Y + \mathcal{V}_Y) \xrightarrow{\partial} E^r(Y - T, \mathcal{L}_{Y - T} + \mathcal{V}_{Y - T}) \]

where \( r \) is the rank of \( \mathcal{V}_X \). The rule (R3b) follows from the commutativity of the square (\(*\)).

(R3c) Let \( \varphi : (E, \mathcal{V}_E) \to (F, \mathcal{V}_F) \) be a morphism in \( \mathfrak{S}_k \) and let \( w \) a valuation on \( F \) which restricts to the trivial valuation on \( E \). Then

\[ \partial_w \circ \varphi_* = 0. \]

Proof. Consider the closed inclusion \( i : Z \to X \) and its open complementary map \( j : U \to X \) where \( X = \text{Spec} \mathcal{O}_v \), \( Z = \text{Spec} \kappa(w) \) and \( U = \text{Spec} F \). According to the long exact sequence 2.23, the composite

\[ E^r(X, \mathcal{V}_X) \xrightarrow{f^*} E^r(U, \mathcal{V}_U) \xrightarrow{\partial_{Z,X}} E^{r+1}(Z, \mathcal{L}_{Z/X} + \mathcal{V}_Z) \]

is zero. The result follows from the fact that the map \( \text{Spec} E \to \text{Spec} F \) factors through \( j \) since \( w \) restricts to the trivial valuation on \( E \).

(R3d) Let \( \varphi \) and \( w \) be as in (R3c), and let \( \overline{\varphi} : (E, \mathcal{V}_E) \to (\kappa(w), \mathcal{V}_{\kappa(w)}) \) be the induced morphism. For any prime \( \pi \) of \( v \), we have

\[ \partial_w \circ \gamma_{[-\pi]} \circ \varphi_* = \overline{\varphi}_*. \]

Proof. Denote by \( Z = \text{Spec} \kappa(w) \), \( U = \text{Spec} F \), \( X = \text{Spec} \mathcal{O}_w \) and \( Y = \text{Spec} E \) and consider the induced maps as defined in the following commutative diagram:

\[ \begin{array}{ccc}
Z & \xrightarrow{i} & X \\
\downarrow f & & \downarrow j \\
Y & \xrightarrow{j} & U
\end{array} \]

We want to prove that the following diagram is commutative:

\[ \begin{array}{ccc}
E^{-r}(Y, \mathcal{V}_Y) & \xrightarrow{f^*} & E^{-r-1}(U, \mathcal{V}_U) \\
\downarrow f & & \downarrow \gamma_{[-\pi]} \\
E^{-r}(Z, \mathcal{V}_Z) & \xleftarrow{\partial} & E^{-r}(U, \mathcal{A}_1^1 + \mathcal{V}_U)
\end{array} \]

where we use the isomorphism \( \mathcal{N}_Z X \simeq \mathcal{A}_1^1 \) defined by the choice of prime \( \pi \). We can split this diagram into the following one:
of this subsection is to prove that the Milnor-Witt cycle pre module

Proof. This follows from Proposition 2.25 since $d$ commutative by functoriality of the pullback maps.

(R3e) Let $E$ be a field over $k$, $v$ be a valuation on $E$ and $u$ be a unit of $v$. Then

$$\partial_v \circ \gamma_u = \gamma_{\frac{u}{v}} \circ \partial_v$$

and

$$\partial_v \circ \gamma_{\eta} = \gamma_{\eta} \circ \partial_v.$$

Proof. This follows from Proposition 2.25 since $\varepsilon \eta = \eta$ (where $\varepsilon = -(-1))$.

(R4a) Let $(E, V_{E}) \in \mathfrak{S}_k$ and let $\Theta$ be an automorphism of $V$. Denote by $\Delta$ the canonical map from the group of automorphism of $V_{E}$ to the group $K^{MW}(E, 0)$. Then

$$\Theta_{*} = \gamma_{\Delta(\Theta)} : \hat{E}(E, V_{E}) \to \hat{E}(E, V_{E}).$$

Proof. One reduce to the case where $E = \mathbb{1}$. In this case, $\hat{E}$ is (isomorphic to) the Milnor-Witt K-theory $K^{MW}$ (see Example 2.19), hence the result.

We have proved that $\hat{E}$ is a Milnor-Witt cycle premodule. In the following subsection, we prove that it satisfies axioms (FD) and (C).

### 3.3 Cycle module structure

Put $S = \text{Spec}(k)$. Let $f : X \to S$ be a scheme and $V_{X}$ be a virtual vector bundle over $X$. Let $E_{S} \in \text{SH}(S)$ be a motivic spectrum. Recall that we denote by $E_{X}$ the spectrum $f^{*}(E_{S})$. The purpose of this subsection is to prove that the Milnor-Witt cycle premodule $\hat{E}$ is in fact a module. Roughly speaking, this means that the graded group $C^{*}(X, \hat{E}, *)$ forms a complex.

Consider a flag $\mathfrak{F} = (Z_{p})_{p \in \mathbb{Z}}$ a flag over $X$, that is a sequence a closed subschemes of $X$

$$\emptyset \subset Z_{1} \subset Z_{2} \subset \cdots \subset Z_{n} \subset X$$

where $\dim Z_{p} \leq p$.

For $p \in \mathbb{Z}$ denote by $U_{p} = X - Z_{p}$ and $T_{p} = Z_{p} - Z_{p-1}$. Consider the canonical maps $j_{p} : U_{p} \subset U_{p-1}$ and $i_{p} : T_{p} \subset U_{p-1}$.

For $p, q \in \mathbb{Z}$, denote by

$$E^{p, q}_{p, q} = E^{p, q}(T_{p}, V_{T_{p}}) = [1_{T_{p}}, E_{T_{p}} \otimes \theta_{T_{p}}(V_{T_{p}})[q - p]]_{T_{p}}$$

and
\[ D_{p,q}^3 = E^{r-p-1}(U_p, \mathcal{V}_X) = [1_{U_p}, E_{U_p} \otimes \text{Th}_{U_p}(V^C_{U_p})[q-p-1]]_{U_p}. \]

According to [223] we have a long exact sequence

\[ \ldots \rightarrow D_{p+1,q}^3 \rightarrow D_{p,q}^3 \rightarrow D_{p-1,q}^3 \rightarrow \ldots \]

so that \((D_{p,q}^3, E_{p,q}^3)_{p,q \in \mathbb{Z}}\) is an exact couple. By the general theory (see [McC01], Chapter 3), this defines a spectral sequence. In particular, we have canonical differentials maps \(d\) that well-defined and satisfying \(d \circ d = 0\). Moreover, we can prove that this spectral sequence converges to \(E^{p+q}(X, \mathcal{E}_X/S + \mathcal{V}_X)\) (because the \(E_1^{p,q}\)-term is bounded) but we do not need this fact.

For \(p, q \in \mathbb{Z}\), denote by

\[ D_{p,q}^{1,X} = \text{colim}_{x \in \text{Flag}(X)_{op}} D_{p,q}^3, \]
\[ E_{p,q}^{1,X} = \text{colim}_{x \in \text{Flag}(X)_{op}} E_{p,q}^3, \]

where the colimit is taken over the flags \(\mathcal{F}\) of \(X\).

This gives rise to the following theorem.

We need to compute this spectral sequence. This is done in the following.

**Theorem 3.3.** For \(p, q \in \mathbb{Z}\), we have a canonical isomorphism

\[ E_{p,q}^{1,X} \simeq \bigoplus_{x \in X_{(p)}} [1_{\kappa(x)}, E_{\kappa(x)} \otimes \text{Th}_{\kappa(x)}(V^C_{\kappa(x)})[q-p]]_{\kappa(x)}. \]

In particular, if \(r\) is the rank of \(\mathcal{V}_X\), then

\[ E_{p,-r}^{1,X} \simeq C_p(X, \hat{E}, \mathcal{V}_X). \]

**Proof.** The proof is the same as [Fel18 Theorem 8.2].

**Theorem 3.4.** Assume \(X\) is a smooth scheme. Keeping the previous notations, the following diagram is commutative:

\[ \begin{array}{ccc}
E_{p,q}^{1,X} & \xrightarrow{d_{p,q}} & E_{p-1,q}^{1,X} \\
\downarrow & & \downarrow \\
E^{q-p}(\text{Spec}(\kappa(y), V^C_y)) & \xrightarrow{\partial_y} & E^{q-p+1}(\text{Spec}(\kappa(x), V^C_x))
\end{array} \]

where \(d_{p,q}\) is the differential canonically associated to the spectral sequence and where the vertical maps are the canonical projections associated to isomorphism of Theorem 3.3.

**Proof.** (see also [Dc12 Proposition 1.15])

By definition, \(d_{p,q}\) is the colimit of arrows

\[ E^{q-p}(Z - Y, V^C_{Z-Y}) \xrightarrow{i_{Z,Y}} E^{q-p}(X - Y, V^C_{X-Y}) \xrightarrow{\partial_{Z,Y}} E^{q-p+1}(W - Y, V^C_{W-Y}) \]

where \(W \subset Y \subset Z\) are large enough closed subschemes with \(\dim_X(Z) = p, \ dim_X(Y) = p - 1\) and \(\dim_X(W) \leq p - 2\). In the following, we consider \(W, Y, Z\) as above. For simplification, we may replace \(X\) by \(X - W\) so that we can remove any subset of \(X\) if its dimension is \(\leq p - 2\).

Enlarging \(Y\), we may assume that \(Y\) contains \(Z_{\text{sing}}\) the singular locus of \(Z\). Since the singular locus of \(Y\) has dimension strictly less than \(p - 1\), we may assume that \(Y\) is smooth. In short, we study the composite:

\[ E^{q-p}(Z - Y, V^C_{Z-Y}) \xrightarrow{i_{Y,Z}} E^{q-p}(X - Y, V^C_{X-Y}) \xrightarrow{\partial_Y} E^{q-p+1}(Y, V^C_X) \]
where \( i_Y : Z - Y \to X - Y \) is the restriction of the canonical closed immersion \( Y \to Z \).

We denote by \( Y_y \) (resp. \( Z_x \)) the irreducible component of \( Y \) (resp. \( Z \)) containing \( y \) (resp. \( x \)). We may decompose \( Y \) as \( Y = Y_y \sqcup Y_y' \) with \( Y_y' = Y \setminus Y_y \) since \( Y \) is smooth. Denote by \( Y_x = Y \times_Z Z_x \) so that \( Z_x - \hat{Y}_x \) is a connected component of the (smooth) scheme \( Z - Y \). Denote by \( i_x : Z_x - \hat{Y}_x \to X - Y \) the canonical inclusion. According to Proposition 2.27, we get the following commutative diagram

\[
\begin{array}{c}
\mathbb{E}^{q-p}(Z - Y, \mathcal{V}_{Z - Y}^\xi) \\
\mathbb{E}^{q-p}(Z_x - Y_x, \mathcal{V}_{Z_x - Y_x}^\xi) \\
\end{array}
\xrightarrow{\partial_{x,y}}
\begin{array}{c}
\mathbb{E}^{q-p}(X - Y, \mathcal{V}_{X - Y}^\xi) \\
\mathbb{E}^{q-p+1}(Y, \mathcal{V}_Y^\xi) \\
\end{array}
\xrightarrow{\partial_{x,y}}
\begin{array}{c}
\mathbb{E}^{q-p+1}(Y_y, \mathcal{V}_{Y_y}^\xi) \\
\end{array}
\]

where the vertical maps are the canonical projections. The theorem is equivalent to proving that the differential \( \partial_y^x \) defined in 2.24 is the colimit of the maps \( \partial_{Y_y,x}^x \) defined in the above diagram.

Assume that \( y \) is not a specialization of \( x \), that is \( y \notin \{x\} \). Then \( \dim_X(Y_x \cap Z_x) \leq p - 2 \) hence (reducing \( X \) to \( X = (Y_x \cap Z_x) \)) we may assume that \( Y_x \cap Z_x = \emptyset \). Thus \( Y_x \cap (Z_x - \hat{Y}_x) = \emptyset \) and we get the following cartesian square of closed immersions

\[
\begin{array}{c}
\emptyset \\
Z_x - \hat{Y}_x \\
\end{array}
\xrightarrow{\sigma} 
\begin{array}{c}
X - Y_y' \\
\end{array}
\xrightarrow{i_x} 
\begin{array}{c}
Y_y \\
\end{array}
\]

which spawns the equality \( \partial_{X - Y_y', Y_y} \circ i_{x,x} \) by naturality of the residue maps (see Proposition 2.24). This proves the proposition in this case.

Assume that \( y \) is a specialization of \( x \) so that \( Y_y \subset Z_x \) and \( Y_y \subset \hat{Y}_x \). For simplification, we assume that \( Z = Z_x \), that is \( Z \) is irreducible with generic point \( x \). Consider the normalization \( f : \tilde{Z} \to Z \) of \( Z \). The singular locus \( Z_{\text{sing}} \) is of codimension greater than 1 in \( \tilde{Z} \) hence \( f(Z_{\text{sing}}) \) is of dimension strictly less than \( p - 1 \) in \( \tilde{Z} \) and (reducing \( X \)) we may assume that \( \tilde{Z} \) is smooth.

Denote by \( \hat{Y} \) (resp. \( \hat{Y}_y, \hat{Y}_y' \)) the reduced inverse image of \( Y \) (resp. \( Y_y, Y_y' \)) along \( f \). Reducing \( X \) again, we may assume that \( \hat{Y}_y \) is smooth and \( \hat{Y}_y \cap \hat{Y}_y' = \emptyset \). We can also assume that every connected component of \( \hat{Y}_y \) dominates \( Y_y \) (by reducing \( X \), we can remove the non-dominant connected components). From this, we see that the map \( g_y : \hat{Y}_y \to Y_y \) induced by \( f \) is finite and equidimensional. Consider the following topologically cartesian square:

\[
\begin{array}{c}
\hat{Y}_y \\
\end{array}
\xrightarrow{\tilde{\sigma}} 
\begin{array}{c}
\tilde{Z} - \hat{Y}_y' \\
\end{array}
\xrightarrow{g_y} 
\begin{array}{c}
Y_y \\
\end{array}
\xrightarrow{\sigma} 
\begin{array}{c}
X - Y_y' \\
\end{array}
\]

where \( \sigma \) and \( \tilde{\sigma} \) are the canonical closed immersions and the right vertical map is induced by the composite

\[
\tilde{Z} \xrightarrow{f} Z \xrightarrow{i} X.
\]

By taking complements of \( \tilde{\sigma} \) and \( \sigma \), it induces the map

\[
\tilde{Z} - \hat{Y}_y \xrightarrow{h} Z - Y \xrightarrow{i} X - Y.
\]

By naturality of residues with respect to Gysin morphisms and by functoriality of the Gysin morphisms, we get the commutative diagram
we have the following commutative diagram
\[ \begin{array}{c}
\mathbb{E}^{q-p}(Z - Y, \mathcal{V}_{Z - Y}) \xrightarrow{\partial_{x, \mathcal{V}_{Z - Y}}} \mathbb{E}^{q-p}(X - Z, \mathcal{V}_{X - Z}) \xrightarrow{\partial_{x, \mathcal{V}_{X - Z}} y_{x}} \mathbb{E}^{q-p+1}(Y, \mathcal{V}_{Y}) \\
\mathbb{E}^{q-p}(\hat{Z} - \hat{Y}, \mathcal{V}_{\hat{Z} - \hat{Y}}) \xrightarrow{\partial_{x, \mathcal{V}_{\hat{Z} - \hat{Y}}}} \mathbb{E}^{q-p}(\hat{Y}, \mathcal{V}_{\hat{Y}}) \xrightarrow{g_{x}} \mathbb{E}^{q-p+1}(Y, \mathcal{V}_{Y}).
\end{array} \]

For any \( t \in f^{-1}(y) \), there exists a unique connected component \( \hat{Y}_{t} \) in the (smooth) scheme \( \hat{Y}_{y} \) so that \( \hat{Y}_{y} = \bigcup_{t \in f^{-1}(y)} \hat{Y}_{t} \). Note that \( Y_{t} \) is also a connected component of \( \hat{Y} \). Denote by \( \bar{Z}_{t} = \bar{Z} - (\bar{Y} - Y_{t}) \); this is an open subscheme of \( \bar{Z} \) containing \( \bar{Y}_{t} \) and \( \bar{Z}_{t} - \bar{Y}_{t} = \bar{Z} - \bar{Y} \). According to Proposition 2.27 we have the following commutative diagram
\[ \begin{array}{c}
\mathbb{E}^{q-p}(\bar{Z} - \bar{Y}, \mathcal{V}_{\bar{Z} - \bar{Y}}) \xrightarrow{\partial_{x, \mathcal{V}_{\bar{Z} - \bar{Y}}} \bar{Y}_{y}} \mathbb{E}^{q-p+1}(\bar{Y}, \mathcal{V}_{\bar{Y}}) \xrightarrow{g_{x}} \mathbb{E}^{q-p+1}(Y, \mathcal{V}_{Y}) \\
\mathbb{E}^{q-p}(\bar{Z} - \bar{Y}, \mathcal{V}_{\bar{Z} - \bar{Y}}) \xrightarrow{\partial_{x, \mathcal{V}_{\bar{Z} - \bar{Y}}} \bar{Y}_{y}} \mathbb{E}^{q-p+1}(\bar{Y}, \mathcal{V}_{\bar{Y}}) \xrightarrow{\partial_{x, \mathcal{V}_{\bar{Y}}} \bar{Y}_{y}} \mathbb{E}^{q-p+1}(Y, \mathcal{V}_{Y}) \\
\mathbb{E}^{q-p}(\bar{Z} - \bar{Y}, \mathcal{V}_{\bar{Z} - \bar{Y}}) \xrightarrow{\partial_{x, \mathcal{V}_{\bar{Z} - \bar{Y}}} \bar{Y}_{y}} \mathbb{E}^{q-p+1}(\bar{Y}, \mathcal{V}_{\bar{Y}}) \xrightarrow{\partial_{x, \mathcal{V}_{\bar{Y}}} \bar{Y}_{y}} \mathbb{E}^{q-p+1}(Y, \mathcal{V}_{Y}) \\
\mathbb{E}^{q-p}(\bar{Z} - \bar{Y}, \mathcal{V}_{\bar{Z} - \bar{Y}}) \xrightarrow{\partial_{x, \mathcal{V}_{\bar{Z} - \bar{Y}}} \bar{Y}_{y}} \mathbb{E}^{q-p+1}(\bar{Y}, \mathcal{V}_{\bar{Y}}) \xrightarrow{\partial_{x, \mathcal{V}_{\bar{Y}}} \bar{Y}_{y}} \mathbb{E}^{q-p+1}(Y, \mathcal{V}_{Y})
\end{array} \]
where the middle vertical map is the canonical isomorphism.

We can now identify \( \partial_{y} \) with the formal colimit of \( \partial_{y, x} \) for \( y, \tilde{y} \). In view of 3.1 this is justified because:

- \( h \) is birational and \( \bar{Z} - \bar{Y} \) is smooth with function field \( \kappa(x) \).
- The closed pair \( (\bar{Z}_{t}, \bar{Y}_{t}) \) is smooth of codimension 1 and the local ring of \( \mathcal{O}_{\bar{Z}_{t}, \bar{Y}_{t}} \) is isomorphic (through \( h \)) to the valuation ring \( \mathcal{O}_{v_{t}} \) corresponding to the valuation \( v_{t} \) on \( \kappa(x) \) considered in 3.1.

From Theorem 3.3 and Theorem 3.4 we deduce that the differentials coincide so that \( C_{*}(X, \mathbb{E}, \mathcal{V}_{X}) \) is a complex when the scheme \( X \) is smooth. We use this to prove that the premodule \( \mathbb{E} \) is a Milnor-Witt cycle module:

**FD** Finite Support of Divisors. Let \( X \) be a normal scheme, \( \mathcal{V}_{X} \) be a virtual vector bundle over \( X \) and \( \rho \) be an element of \( M(\xi_{X}, \mathcal{V}_{X}) \). Then \( \partial_{x}(\rho) = 0 \) for all but finitely many \( x \in X^{(1)} \).

*Proof.* We can assume without loss of generality that \( X \) is affine of finite type. Then there exist a virtual vector bundle \( \mathcal{V}_{\tilde{X}} \) over \( \tilde{X}^{r} \) and a closed immersion \( i : X \to \tilde{X}^{r} \) for some \( r \geq 0 \) which induces an inclusion
\[ C^{*}(X, \mathbb{E}, \mathcal{V}_{X}) \subset C^{*}(\tilde{X}^{r}, \mathbb{E}, \mathcal{V}_{\tilde{X}}) \]
compatible with the differentials thanks to the previous theorem. Hence [FD] holds.

**C** Closedness. Let \( X \) be integral and of dimension 2 and \( \mathcal{V}_{X} \) be a virtual bundle over \( X \). Then
\[ 0 = \sum_{x \in X^{(1)}} \partial_{x}^{0} \circ \partial_{x}^{0} : M(\xi_{X}, \mathcal{V}_{X}) \to M(x_{0}, \mathcal{V}_{X}) \]
where \( \xi \) is the generic point and \( x_{0} \) the closed point of \( X \).

*Proof.* According to (FD), the differentials \( d \) of \( C^{*}(X, \mathbb{E}, \mathcal{V}_{X}) \) are well-defined. We want to prove that \( d \circ d = 0 \). Again, we can assume \( X \) to be affine of finite type over \( k \). Then there exist a virtual vector bundle \( \mathcal{V}_{\tilde{X}} \) over \( \tilde{X}^{r} \) and a closed immersion \( i : X \to \tilde{X}^{r} \) for some \( r \geq 0 \) which induces an inclusion

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Definition 4.3. A morphism of homotopy modules is a homogeneous natural transformation of $	ext{Sm}_k$ on $	ext{homotopy invariant}$ if the Nisnevich cohomology sheaf $H_n^F$ the morphism $\text{the heart of Morel-Voevodsky stable homotopy category (equipped with the homotopy t-structure)}$:

We recall some facts about the heart of the stable homotopy category (see [Mor03, §5.2] or [Dé11, §1]).

4.1 Associated homotopy module

Theorem 4.1. Let $k$ be a perfect field. The category of Milnor-Witt cycle modules is equivalent to the heart of Morel-Voevodsky stable homotopy category (equipped with the homotopy t-structure):

$\mathfrak{M}_k^{MW} \simeq \text{SH}(k)\bigcirc$.

4.1 Associated homotopy module

We recall some facts about the heart of the stable homotopy category (see [Mor03, §5.2] or [Dé11, §1]).

Definition 4.2. Let $F$ be an abelian Nisnevich sheaf on $\text{Sm}_k$. We denote by $F_{-1}(X)$ the kernel of the morphism $F(X \times \mathbb{G}_m) \rightarrow F(X)$ induced by the unit section of $\mathbb{G}_m$. We say that $F$ is strictly homotopy invariant if the Nisnevich cohomology sheaf $H_n^\text{Sis}(-, F)$ is homotopy invariant.

Definition 4.3. A homotopy module is a pair $(F_*, \omega_*)$ where $F_i$ is a $\mathbb{Z}$-graded abelian Nisnevich sheaf on $\text{Sm}_k$ which is strictly homotopy invariant and $\omega_n : F_{n-1} \rightarrow (F_n)_{-1}$ is an isomorphism (called desuspension map). A morphism of homotopy modules is a homogeneous natural transformation of $\mathbb{Z}$-graded sheaves compatible with the given isomorphisms. We denote by $\text{HM}(k)$ the category of homotopy modules over $k$.

4.4. For any spectrum $E$, the spectrum $\underline{E}(E)$ has a canonical structure of a homotopy module. Moreover, the functor $\underline{E} : E \mapsto \underline{E}(E)$ induces an equivalence of categories between the heart of $\text{SH}(k)$ for the homotopy t-structure and the category $\text{HM}(k)$. We denote its inverse by $H : \text{HM}(k) \rightarrow \text{SH}(k)\bigcirc$.

We continue with two lemmas of independent interest.

Lemma 4.5. Let $g : Y \rightarrow X$ be a smooth morphism of schemes of finite type over $k$ of constant fiber dimension 1, let $\sigma$ be a section of $g$, let $\mathcal{V}_X$ be a virtual vector bundle over $X$ and let $t \in \mathcal{O}_Y$ be a global parameter defining the subscheme $\sigma(X)$.

Then $\sigma_* : C_*(X, M, \mathcal{V}_X) \rightarrow C_*(Y, M, \mathcal{V}_Y)$ is null homotopic.

Proof. Consider the open subscheme $j : U = Y \setminus \sigma(X) \rightarrow Y$ and let $\tilde{g} = g \circ j$ the restriction of $g$. Let $\partial$ be the boundary map associated to $\sigma$. According to [Feld18, Lemma 5.5] we have $\sigma_* = \sigma_* \circ \partial \circ [t] \circ \tilde{g}^* = d \circ j_* \circ [t] \circ \tilde{g}^*$.

With the same proof, we have a slightly more general result:

Lemma 4.6. Let $g : Y \rightarrow X$, $\sigma : X \rightarrow Y$ and $\mathcal{V}_X$ as previously. Let $i : Z \rightarrow X$ be a closed immersion and consider $\tilde{Z} = g^{-1}(Z)$ the pullback along $g$. The induced map $\bar{\sigma} : Z \rightarrow \tilde{Z}$ is such that the pushforward $\bar{\sigma}_* : C_*(\tilde{Z}, M, \mathcal{V}_Z) \rightarrow C_*(Z, M, \mathcal{V}_Z)$ is null homotopic.
Remark 4.7. This result may be compared to [FS08 Corollary 3.5].

Fix $M$ a Milnor-Witt cycle module over $k$. We associate to $M$ a homotopy module $F^M$, that is a homotopy invariant Nisnevich sheaf of $\mathbb{Z}$-graded abelian groups. Indeed, let $X$ be a smooth scheme over $S$. For any integer $n$, we put

$$F^M_n(X) = A^0(X, M, -\Omega_{X/k} + \langle n \rangle).$$

This defines a presheaf $F^M = (F^M_n)_{n \in \mathbb{Z}}$ of graded abelian groups satisfying the homotopy invariance property (see [Fel18 Theorem 8.3]).

Denote by $s_1 : X = \{1\} \times X \to G_mX$ the induced closed immersion. We have $(F^M_n)_{-1}(X) = \ker s_1^*$. By homotopy invariance, we have also $F^M_n(\mathbb{A}_k^1) \cong F^M_n(X)$ hence $(F^M_n)_{-1}(X) = \coker j^*$ where $j$ is the open immersion $G_mX \to \mathbb{A}_k^1$.

As usual, we get the following long exact sequence:

$$0 \longrightarrow F^M_n(X) \xrightarrow{j^*} F^M_n(G_mX) \xrightarrow{\hat{\partial}} F^M_{n-1}(X) \xrightarrow{i_*} A^1(\mathbb{A}_k^1, M, -\Omega_{\mathbb{A}_k^1/k} + \langle n \rangle).$$

Thus we see that $\hat{\partial}$ induces a map $(F^M_n)_{-1}(X) \to F^M_{n-1}(X)$ which is an isomorphism because $i_*$ is zero (according to the Lemma 4.6).

We prove that $F^M$ is a Nisnevich sheaf. We start with the complex $C_*(\mathbb{Z}, M, *)$ and consider a Nisnevich square

$$
\begin{array}{ccc}
U_V & \xrightarrow{j} & V \\
\downarrow{i} & & \downarrow{p} \\
U & \xrightarrow{i} & X
\end{array}
$$

where $i$ is open and $p$ etale. Denote by $Z = (X - U)_{\text{red}}$ so that we have the decomposition

$$C_*(X, M, *) = C_*(U, M, *) \oplus C_*(Z, M, *)$$

and

$$C_*(V, M, *) = C_*(U_V, M, *) \oplus C_*(Z_V, M, *).$$

By assumption the induced map $p : Z_V \to Z$ is an isomorphism, hence the canonical map $p_* : C_*(Z_V, M, *) \to C_*(Z, M, *)$ is an isomorphism. Hence we can see that the image of the Nisnevich square by $C_*(-, M, *)$ is cocartesian. This proves that $C_*(-, M, *$) is a Nisnevich sheaf and so is $F^M$.

We have proved the following theorem.

**Theorem 4.8.** Let $M$ be a Milnor-Witt cycle module over $k$. The graded presheaf $F^M$ of abelian groups, defined by

$$F^M_n(X) = A^0(X, M, -\Omega_{X/k} + \langle n \rangle)$$

for any smooth scheme $X/S$ and any integer $n$, is a homotopy module.

### 4.2 First isomorphism

Let $M$ be a Milnor-Witt cycle module. Since the category of homotopy modules is equivalent to the heart of the stable homotopy category $\text{SH}(S)$ (see [1.4], Theorem 1.8 implies that there is an object $\mathbb{M}$ of $\text{SH}(S)^{\vee}$ equipped with isomorphisms

$$\alpha_X : \mathbb{M}^{-n}(X, \langle n \rangle) \to F^M_n(X)$$
for any irreducible smooth scheme $X$ of dimension $d$ and any integer $n$ (we recall that $F^0_n(X) = A^n(X, M, -\Omega_{X/k} + \langle n \rangle)$). The maps $\alpha$ are compatible with the right-way maps (contravariance) and the desuspension functor $(-)_1$ in the sense that the following diagrams commute

$$
\begin{array}{ccc}
M^{-n}(X, \langle n \rangle) & \xrightarrow{f^*} & M^{-n}(Y, \langle n \rangle) \\
\downarrow \alpha_X & & \downarrow \alpha_Y \\
A^0(X, M, -\Omega_{X/k} + \langle n \rangle) & \xrightarrow{f^*} & A^0(Y, M, -\Omega_{Y/k} + \langle n \rangle)
\end{array}
$$

for any morphism $f : Y \to X$ of smooth schemes and

$$
\begin{array}{ccc}
M^{-n}(X, \langle n - 1 \rangle) & \xrightarrow{\omega_n} & (M^{-n}(X, \langle n \rangle))_{-1} \\
\downarrow \alpha_X & & \downarrow (\alpha_Y)_{-1} \\
A^0(X, M, -\Omega_{X/k} + \langle n - 1 \rangle) & \xrightarrow{\omega'_n} & (A^0(X, M, -\Omega_{X/k} + \langle n \rangle))_{-1}
\end{array}
$$

where $\omega_n$ and $\omega'_n$ are the structural desuspension maps associated the two homotopy modules for any integer $n$.

Fix $E/k$ a field and $n$ an integer. Using the previous isomorphism $\alpha_X$ with $X = \text{Spec } A$ a smooth model of $E$ and taking the limit over all such $X$, we obtain an isomorphism of abelian groups

$$
\alpha_E : \tilde{M}(E, \langle n \rangle) \to M(E, \langle n \rangle).
$$

According to 3.2, this also defines in a canonical way an isomorphism

$$
\alpha_E : \tilde{M}(E, \mathcal{V}_E) \to M(E, \mathcal{V}_E)
$$

for any virtual vector bundles $\mathcal{V}_E$ over $E$.

It suffices to prove that $\alpha_E$ is natural in the data $\textbf{[D1] [D2] [D3] and [D4]}$ (see [FELLS, Definition 3.5]).

**[D1]** For any morphism $f : Y \to X$ of smooth schemes, the maps $\alpha$ are compatible with right-way (pullbacks) morphisms thus the following diagram is commutative

$$
\begin{array}{ccc}
\tilde{M}(E, \mathcal{V}_E) & \xrightarrow{\text{resp} / E} & \tilde{M}(F, \mathcal{V}_F) \\
\downarrow \alpha_E & & \downarrow \alpha_F \\
M(E, \mathcal{V}_E) & \xrightarrow{\text{resp} / E} & M(F, \mathcal{V}_F)
\end{array}
$$

where $F/E$ is a field extension and $\mathcal{V}_E$ is a virtual vector bundle over $E$.

**[D4]** Let $Z$ be a smooth scheme over $S$. Since the maps $\alpha$ commute with the functor $(-)_{-1}$, we have the following commutative diagram

$$
\begin{array}{ccc}
M^{-n}(A^1_Z, \langle n \rangle) & \xrightarrow{j^*} & M^{-n}(G_m Z, \langle n \rangle) \\
\downarrow \alpha & & \downarrow \alpha \\
A^0(A^1_Z, -\Omega_{A^1_Z/S} + \langle n \rangle) & \xrightarrow{j^*} & A^0(G_m Z, -\Omega_{G_m Z/S} + \langle n \rangle) \\
\downarrow \alpha & & \downarrow \alpha \\
A^0(Z, -\Omega_{Z/S} + \langle n \rangle)
\end{array}
$$

where $j : G_m Z \to A^1_Z$ is the open immersion complementary to the zero section $i : Z \to A^1_Z$.

By deformation to the normal cone, we have the same commutative diagram when $j : X - Z \to X$ is the open immersion associated with a regular immersion $i : Z \to X$ of codimension 1. In particular, when $X = \text{Spec } \mathcal{O}_v$ is the spectrum of a valuation ring and $Z = \text{Spec } k(v)$, we find that the maps $\alpha$ are compatible with the residue maps.
In order to prove that the maps $\alpha$ commute with the $K_{MW}$-action on the left, it suffices to do it for any generator $[u]$ (where $u$ is a unit) and the Hopf map $\eta$.

Let $E$ be a field over $k$ and $u$ be a unit of $E$. Denote by $X$ the essentially smooth scheme Spec $E$. The unit defines a map $u : X \to \mathbb{G}_m X$ which induces a map

$$u^* : \mathbb{M}^{-n}(\mathbb{G}_m X, \langle n \rangle) \to \mathbb{M}^{-n}(X, \langle n \rangle)$$

for any integer $n$. Moreover, we consider the canonical maps

$$\omega_n : \mathbb{M}^{-(n-1)}(X, (-1 + n)) \to (\mathbb{M}^{-n}(X, \langle n \rangle))_{-1}$$

and

$$\nu_n : (\mathbb{M}^{-n}(X, \langle n \rangle))_{-1} \subset \mathbb{M}^{-n}(\mathbb{G}_m X, \langle n \rangle).$$

Now consider the canonical morphism

$$\eta^* : \mathbb{M}^{-n}(X, \langle n \rangle) \to \mathbb{M}^{-n}(\mathbb{G}_m X, \langle n \rangle)$$

induced by the Hopf map and the canonical projection

$$\pi_n : \mathbb{M}^{-n}(\mathbb{G}_m X, \langle n \rangle) \to (\mathbb{M}^{-n}(X, \langle n \rangle))_{-1}.$$

One can check that the data D3 satisfies

$$\gamma_u = u^* \nu_n \omega_n : \mathbb{M}^{-(n-1)}(X, (-1 + n)) \to \mathbb{M}^n(X, \langle n \rangle).$$

and

$$\gamma_0 = \omega_n \pi_n \eta^* : \mathbb{M}^n(X, \langle n \rangle) \to \mathbb{M}^{-(n-1)}(X, (-1 + n)).$$

We have the same description for the Milnor-Witt cycle module $M$. Since the maps $\alpha$ commute with pullbacks and transition maps $\tau_n$, we see that they also commute with the $K_{MW}$-action.

\[ \begin{array}{c}
\hat{\mathcal{M}}(E, \mathcal{V}_E) \xrightarrow{\partial_x} \hat{\mathcal{M}}(\kappa(v), -\mathcal{N}_v + \mathcal{V}_{\kappa(v)}) \\
\downarrow \alpha \quad \downarrow \alpha \\
\hat{\mathcal{M}}(E, \mathcal{V}_E) \xrightarrow{\partial_x} \hat{\mathcal{M}}(\kappa(v), -\mathcal{N}_v + \mathcal{V}_{\kappa(v)})
\end{array} \]

is a commutative square.

\[(\mathbf{D2})\] Let $E$ be a field. The homotopy invariance property $[\mathbf{H}]$ states that the following sequence is split exact (see [FGILS] Proposition 3.9):

\[ \begin{array}{c}
0 \xrightarrow{\cdot} \hat{\mathcal{M}}(E, \mathcal{A}_E^1 + \Omega_{E/k} + \mathcal{V}_E) \xrightarrow{\text{res}_{E/k}} \hat{\mathcal{M}}(E(t), \Omega_{E(u)/k} + \mathcal{V}_{E(u)}) \xrightarrow{d} \\
\end{array} \]

where $d = \sum_{x \in (\mathbb{A}_E^1)^1} \partial_x$ and where $\mathcal{V}_E$ is a virtual vector bundle over $E$ (this is true for any Milnor-Witt cycle module hence in particular for $\hat{\mathcal{M}}$).

We can use this property $[\mathbf{H}]$ and the data $[\mathbf{D1}]$ to characterize the data $[\mathbf{D2}]$. Indeed, let $F/E$ be a finite field extension and assume $F = E(x)$ to be monogene where $x$ corresponds to a point in $(\mathbb{A}_E^1)^1$. For any $\beta \in M(F, \Omega_{F/k} + \mathcal{V}_F)$ there exists $\gamma \in M(E(t), \Omega_{E(u)/k} + \mathcal{V}_{E(u)})$ with the property that $d(\gamma) = \beta$. Now the valuation at $\infty$ yields a morphism $\partial_{\infty}^{-1/2} : M(E(t), \Omega_{E(t)/k} + \mathcal{V}_{E(t)}) \to M(E, \Omega_{E/k} + \mathcal{V})$ which vanishes on the image of $\text{res}_{E(t)/E}$. The element $-\partial_{\infty}^{-1/2}(\gamma)$ does not depend on the choice of $\gamma$ and is in fact equal to $\text{cores}_{F/E}(\beta)$. Using this characterization, we see that $[\mathbf{D2}]$ commutes with the maps $\alpha$ since they commute with $[\mathbf{D4}]$. 

\[(\mathbf{D3})\] In order to prove that the maps $\alpha$ commute with the $K_{MW}$-action on the left, it suffices to do it for any generator $[u]$ (where $u$ is a unit) and the Hopf map $\eta$. 

We have the same description for the Milnor-Witt cycle module $M$. Since the maps $\alpha$ commute with pullbacks and transition maps $\tau_n$, we see that they also commute with the $K_{MW}$-action.
4.3 Second isomorphism

Let $M \in \text{SH}(k)^\vee$. Let $X$ be a smooth scheme over $k$ and let $x$ be a generic point of $X$. For any integer $n$, we have a canonical map $M^{-n}(X, \langle n \rangle) \to M^{-n}(\kappa(x), \langle n \rangle) = M(\kappa(x), \langle n \rangle)$. Thus we have a map $b_X : M^{-n}(X, \langle n \rangle) \to C^0(X, \hat{M}, - \Omega_{X/k} + \langle n \rangle)$ which factors through $A^0(X, \hat{M}, - \Omega_{X/k} + \langle n \rangle)$. We want to prove that the arrow

$$b : M \to A^0(-, \hat{M}, *)$$

is an isomorphism of homotopy modules.

We prove that $b_X$ are natural in $X$ (with respect to Gysin morphisms). If $p : Y \to X$ is a smooth projective map of smooth schemes, it is clear by the definition of pullbacks for Chow-Witt groups with coefficients in $M$ (see [Fel18, §4.5]) that $p^*$ commutes with $b$.

Now consider a regular closed immersion $i : Z \to X$ of smooth schemes. Recall that (by definition, see [Fel18, Definition 9.1]) the Gysin morphism $i^*$ (for Chow-Witt groups with coefficients in $M$) makes the following diagram commutative

$$
\begin{array}{c}
A^0(X, M, \langle n \rangle) \\
| \\
| q^* \\
\downarrow \\
A^0(Z, M, -L_i + \langle n \rangle) \\
\end{array}
\xrightarrow{t}\ A^0(\mathbb{G}_m X, M, \langle n \rangle)
$$

where $q : \mathbb{G}_m X \to X$ is the canonical projection and $t$ is a parameter such that $A^1_k = \text{Spec} k[t]$ and where $L_i = - N_Z X$.

Similarly, the Gysin morphism $i^*$ (for the cohomology theory $M$) makes the following diagram commutative

$$
\begin{array}{c}
M^{-n}(X, \langle n \rangle) \\
| \\
| q^* \\
\downarrow \\
M^{-n}(Z, \langle n \rangle) \\
\end{array}
\xrightarrow{t}\ M^{-n}(\mathbb{G}_m X, \langle n \rangle)
$$

where $q, \pi$ and $t$ are defined as previously. Putting things together, we see that the maps $\beta$ commute with $i^*$ hence with any pullbacks (of lci morphisms).

Moreover, we prove that $b$ is compatible with the desuspension maps $\omega_n : M^{-n-1}_\text{lc} \simeq (M^{-n}_\text{lc})^{-1}$ defining the homotopy modules $M$ and $A^0(-, \hat{M}, *)$. Let $X$ be an irreducible smooth scheme, we have the following diagram:

$$
\begin{array}{c}
0 \xrightarrow{0} M^{-n}(A^1_k, \langle n \rangle) \xrightarrow{j^*} M^{-n}(\mathbb{G}_m X, \langle n \rangle) \xrightarrow{\partial_*} M^{-n-1}(X, \langle n - 1 \rangle) \\
\downarrow b \downarrow (1) \downarrow b \downarrow (2) \\
0 \xrightarrow{0} A^0(A^1_k, \hat{M}, *) \xrightarrow{j^*} A^0(\mathbb{G}_m X, \hat{M}, *) \xrightarrow{\partial} A^0(X, \hat{M}, * - 1) \\
\end{array}
$$

We have already seen that the square (1) commutes. The map $\partial$ is defined in [Fel18] using the data [D4] of the cycle module $\hat{M}$ which corresponds to the map $\partial$. Hence the square (2) commutes. We can see that the desuspension map $\omega_n : M^{-n-1}_\text{lc} \simeq (M^{-n}_\text{lc})^{-1}$ is induced by $\partial \hat{M}$. Thus $b$ is a morphism of homotopy modules.

Finally, when $X$ is the spectrum of a field, the map $b_X$ is an isomorphism and so $b$ is an isomorphism of homotopy modules.

We thus have proved Theorem [4.4].
5 Applications

Hermitian K-theory and Witt groups. We assume that the characteristic of $k$ is different from 2. In [Hor05], Hornbostel proved that both hermitian K-theory and Witt groups are representable in the stable homotopy category of Morel and Voevodsky. According to Theorem 4.1, there are Milnor-Witt cycle modules $\text{KO}$ and $\text{KW}$ respectively associated to these theories in a canonical way.

Monoidal structure

Theorem 5.1. The category $\mathfrak{M}^{\text{MW}}_k$ of Milnor-Witt cycle modules an (abelian) Grothendieck category with products. Moreover, there is a canonical symmetric closed monoidal structure on $\mathfrak{M}^{\text{MW}}_k$ such that the unit element is the cycle module $\text{K}^{\text{MW}}_k$. Moreover, the monoidal tensor product commutes with the shifting functor defined in [Fel18, Example 4.7].

Equivalences of categories

Theorem 5.2. Let $k$ be a perfect field. The category of Milnor-Witt cycle modules is equivalent to the heart of the category of MW-motives $\widetilde{\text{DM}}(k)$ (equipped with the homotopy t-structure):

$$\mathfrak{M}^{\text{MW}}_k \simeq \text{DM}(k)^\heartsuit.$$  

In particular, the heart of Morel-Voevodsky stable homotopy category is equivalent to the heart of the category of MW-motives $\widetilde{\text{DM}}(k)$ (both equipped with their respective homotopy t-structures):

$$\text{SH}(k)^\heartsuit \simeq \widetilde{\text{DM}}(k)^\heartsuit.$$  

Proof. Let $M$ be a Milnor-Witt cycle module. It corresponds to a homotopy module $F^M$ according to the previous section. Since we have an action of the Milnor-Witt K-theory on $F^M$, we can prove (as in [CF17] Lemma 5.3) that $F^M$ has in fact MW-transfers. We can then proceed as for Theorem 4.1 to prove the first equivalence of categories. The second equivalence follows from Theorem 4.1.

Theorem 5.3 (Déglise). Let $k$ be a perfect field. The category of Rost cycle modules over $k$ is equivalent to the heart of the category of Voevodsky’s motives $\text{DM}(k,\mathbb{Z})$ with respect to the homotopy t-structure:

$$\mathfrak{M}^\text{M}_k \simeq \text{DM}(k,\mathbb{Z})^\heartsuit.$$  

Proof. One can see that the category of Rost cycle modules is equivalent to the full subcategory of $\mathfrak{M}^\text{MW}_k$ of Milnor-Witt cycle modules with trivial action by the generator $\eta$. Thanks to Theorem 5.2 this subcategory is equivalent to the full subcategory of $\text{DM}(k)^\heartsuit$ of homotopy modules with transfers and with trivial action of the Hopf map $\eta$. This last category is equivalent to $\text{DM}(k,\mathbb{Z})^\heartsuit$.

Adjunction between MW-cycle modules and Rost cycle modules

Theorem 5.4 (Adjunction Theorem). There is an adjunction between the category of Milnor-Witt cycle modules and the category of classical cycle modules:

$$\mathfrak{M}^\text{MW}_k \rightleftarrows \mathfrak{M}^\text{M}_k.$$  

Proof. We gave an elementary proof of this result in [Fel18, Section 12] with an explicit description of the adjoint functors. For a second proof, simply combine Theorem 4.1 and Theorem 5.3.

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Birational invariance

Theorem 5.5. Let $X$ be a proper smooth integral scheme over $k$, let $V_k$ a virtual vector bundle over $k$ and let $M$ be a Milnor-Witt cycle module. Then the group $A^0(X, M, -\Omega_{X/k} + V_X)$ is a birational invariant of $X$.

Proof. (see also [Voi19, Lemma 1.3]). Denote by $I(X) = A^0(X, M, -\Omega_{X/k} + V_X)$. This defines a contravariant functor that satisfies:

1. If $U \subset X$ is a Zariski open set, then the map $I(X) \rightarrow I(U)$ is injective,
2. If $U \subset X$ is a Zariski open set such that $\text{codim} X(X \setminus U) \geq 2$, then the map $I(X) \rightarrow I(U)$ is an isomorphism.

Indeed, this follows from the usual localization long exact sequence (see [Fel18, §6.4]).

Now let $\Phi : X \dashrightarrow Y$ be a birational map between smooth and proper integral schemes over $k$. Then there is an open set $U \subset X$ such that $\text{codim} X(X \setminus U) \geq 2$ and $\Phi U$ is an morphism. Then we have $I(X) \simeq I(U)$ and, by functoriality, a morphism $\Phi^*_U : I(Y) \rightarrow I(U)$, hence a morphism $\Phi_U : I(Y) \rightarrow I(X)$. Replacing $\Phi$ by $\Phi^{-1}$, we get $\Phi^{-1}_U : I(X) \rightarrow I(V)$ for some Zariski open set $V$ of $Y$ such that $I(Y) \simeq I(V)$. Let $U' \subset U$ be defined as $\Phi^{-1}_U(V)$. Then $\Phi^{-1} \circ \Phi$ is the identity on $U'$, hence $(\Phi^{-1})_* \circ \Phi_* : I(X) \rightarrow I(X)$ is the identity. Since $I(X) \rightarrow I(U')$ is injective, we can conclude that $\Phi_*$ is an isomorphism.

As a corollary, we have the following theorem.

Theorem 5.6. Let $X$ be a proper smooth integral scheme over $X$. Then the Chow-Witt groups $\tilde{\text{CH}}^0(X)$ are birational invariants of $X$. Moreover, let $F_* \in \text{HM}(k)$ be a homotopy module, then $F_*(X)$ is a birational invariant.

Proof. See [Fel18, §6.5] for the definition of $\tilde{\text{CH}}^0(X)$. The first statement follows from Theorem 5.5 applied with $M = K^{MW}$ (see [Fel18, Proposition 6.6]). The second statement follows from Theorem 5.5 and Theorem 4.1.

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