Opinion Formation under Antagonistic Influences

Deepak Bhat\textsuperscript{1,*} and S. Redner\textsuperscript{1,\dagger}

\textsuperscript{1}Santa Fe Institute, 1399 Hyde Park Rd., Santa Fe, New Mexico 87501, USA

We study the opinion dynamics in a generalized voter model in which voters are additionally influenced by two antagonistic news sources, whose effect is to promote political polarization. We show that, as the influence of the news sources is increased, the mean time to reach a politically polarized state is quite short, and the steady-state opinion distribution exhibits a transition from a near consensus state to a politically polarized state.

A disheartening feature of current social discourse is its high degree of political polarization, particularly in the US and Europe (see, e.g., [17]). In recent decades, this polarization has increased to the point where, in the US, parents affiliated with a given political party are loathe to have their children wed someone affiliated with the other major party (see, e.g., [8]). We have no illusions of being able to explain the complex societal forces that have led to this situation. What we believe we can understand, however, are the consequences of this increased polarization on the dynamics of opinion formation.

Our modeling is based on the framework of the voter model [17,19] that is augmented by the influence of competing news sources. Many news sources promulgate a fixed political viewpoint [18,19] and news consumers predominantly consult sources that align with their own political persuasion. We therefore describe a society as being influenced by two news media sources of opposite political leanings (Fig. 1). These sources are effectively “zealots” in the framework of the voter model [17,19], in that they perpetually maintain their political opinion. While many variants of the voter model—inspired by real decision making—have been investigated (see, e.g., [20–29]), the role of news media has apparently not been considered (but see [30] for a study related to ours). Our goal is to determine the role of two news sources with opposing perspectives on the dynamics of public opinion.

Each individual voter has two possible opinion states, denoted as + and −. Individual opinions are updated according to voter model dynamics: a randomly selected voter adopts the opinion of a randomly selected neighbor. We account for the different propensities of news media and neighboring voters to influence a given voter as follows: for a voter linked to one news source and k other voters, the news source is picked with probability $p/R$ and a neighboring voter is picked with probability $(1-p)k/R$, where $R = p + k(1-p)$ is the total rate of picking any neighbor. The parameter $p$ thus quantifies the relative influence of a news source and a neighboring voter. (If a voter is connected to both news sources, then $R = 2p + k(1-p)$.) Once an interaction partner is selected, the voter adopts the opinion of this partner. This update step is repeated ad infinitum.

We treat two types of social networks (Fig. 1): (a) A complete graph of N voters, with $L_+$ ($L_-$) connections between voters and the + (−) news source. The news sources connect either to random voters or to disjoint voters. (b) More realistically, a two-clique graph with N voters in each clique, with $L_+$ connections between the + news source and random voters on clique $C_1$ (and correspondingly for $C_2$), and $L_0 = N^\beta$ links between nodes in different cliques. In both cases, $0 < L_\pm \leq N$, with corresponding link densities $\ell_\pm = L_\pm /N$.

We focus on four characteristics of the collective opinion state: (i) the consensus time $T_{\text{con}}$, defined as the average time to reach either + or − unanimity; (ii) the polarization time $T_{\text{pol}}$, defined as the average time to go from a state with non-zero magnetization (the difference in the fraction of + and − voters) to a politically polarized state of zero magnetization; (iii) the exit probability, defined as the probability to eventually reach + consensus when the initial density of + voters equals $x$, and (iv) the steady-state opinion distribution.

Our main results are: (i) $T_{\text{con}}$ typically grows algebraically with $N$, with a non-universal exponent that can be arbitrarily large. Based on an annealed-link approximation to be discussed below, we find, for voters on the complete graph:

\[
T_{\text{con}} \sim \begin{cases} 
N & 0 < \alpha < 1, \\
N \ln N & \alpha = 1, \\
N^\alpha & \alpha > 1,
\end{cases} \tag{1a}
\]

FIG. 1. Two antagonistic news sources (large circles) that influence voters of different opinions (small up and down triangles) that are situated on either: (a) a complete graph, or (b) a two-clique graph. The news sources have $L_\pm$ links to individuals. For the two-clique graph, there are $L_0$ links between voters in different cliques.
where \( \alpha = \min(\alpha_+, \alpha_-) \) and \( \alpha_\pm = p\ell_{\pm}/(1-p) \). For voters on the two-clique graph, in which the news sources have equal link densities \( \ell_+ = \ell_\pm = \ell \) and the cliques are sparsely interconnected \((\beta < 1)\)

\[
T_{\text{con}} \sim \begin{cases} 
N^{2-\beta} \alpha & 0 \leq \alpha < 1, \\
N^{2-\beta} \ln N \alpha & \alpha = 1, \\
N^{\alpha+1-\beta} \alpha & \alpha > 1. 
\end{cases}
\]

(1b)

For \( p \to 1 \), i.e., influential news sources, the exponent of the consensus time becomes arbitrarily large. That is, competing and well-connected news sources promote political polarization. Our results for \( T_{\text{con}} \) for \( p \to 0 \) for the two-clique graph are consistent with a previous study of the voter model on this graph [31].

(ii) When the two news sources are equally connected to the population, the polarization time \( T_{\text{pol}} \) scales as

\[
T_{\text{pol}} \sim N \frac{1-p}{p\ell}. \tag{2}
\]

Hence, political polarization occurs quickly when voters are better connected to competing news sources. (iii) The exit probability has an anti-sigmoidal shape (Fig. 2) because the competing news sources drive the population to a politically polarized state. (iv) For the complete and the two-clique graph, the opinion distribution undergoes a transition from a homogeneous to a polarized state as the influence of news sources on voters become stronger.

We now outline the calculations that underlie our results. Suppose that we know \( r_\pm(x) \), the rates for \( x \), the fraction of voters with + opinion, to change by \( \pm \frac{1}{N} \equiv \pm \delta x \). Let \( P(x,t) \) be the probability that the fraction of + voters lies between \( x \) and \( x + \delta x \). The Fokker-Planck equation for \( P \) is

\[
\frac{\partial P}{\partial t} = LP, \quad L = -\frac{\partial}{\partial x} V(x) + \frac{\partial^2}{\partial x^2} D(x), \tag{3}
\]

with drift velocity \( V(x) = [r_+(x) - r_-(x)]\delta x \) and diffusion coefficient \( D(x) = [r_+(x) + r_-(x)]\delta x^2/2 \). We can view the instantaneous opinion \( x \) as undergoing biased diffusion in the interval \([0, 1]\) in the presence of the effective potential

\[
\phi(x) = -\int_x^1 \frac{V(x')}{D(x')} dx'. \tag{4}
\]

A basic opinion characteristic is the exit probability \( E_+(x) \). This quantity satisfies the backward equation \( \mathcal{L}^\dagger E_+(x) = 0 \) [32,34], where the adjoint operator is

\[
\mathcal{L}^\dagger \equiv V(x) \frac{\partial}{\partial x} + D(x) \frac{\partial^2}{\partial x^2}, \tag{5}
\]

subject to the boundary conditions \( E_+(0) = 0, E_+(1) = 1 \). The formal solution for \( E_+(x) \) is

\[
E_+(x) = \frac{\int_0^x \exp[\phi(x')] dx}{\int_0^1 \exp[\phi(x')] dx'} \tag{6}
\]

By normalization, the fraction of trajectories that reach \( x = 0 \) without reaching \( x = 1 \) is \( E_-(x) = 1 - E_+(x) \).

Similarly, the consensus and polarization times satisfy the backward equation \( \mathcal{L}^\dagger T(x) = -1 \) [32,34]. The boundary conditions for \( T_{\text{con}} \) are \( T(0) = T(1) = 0 \), while the boundary conditions for \( T_{\text{pol}} \) are \( \frac{\partial T}{\partial x} |_{x=0} = 0 \) and \( T(1/2) = 0 \). The formal solutions are [35]

\[
T_{\text{con}}(x) = E_+(x) I(x, 1) - E_-(x) I(0, x), \tag{7}
\]

\[
T_{\text{pol}}(x) = I(x, 1/2),
\]

where \( I(a, b) = \int_a^b dx' \int_0^{x'} dx'' \exp[\phi(x') - \phi(x'')]/D(x'') \).

We now apply an annealed-link approximation to this formalism to determine \( E_+(x) \), \( T_{\text{con}} \), and \( T_{\text{pol}} \) for voters on the complete and the two-clique graphs (Figs. 1(a) and (b)). In this approximation, we replace the true transition rates for each voter on a given fixed-link network realization by the average transition rate, in which a link is present with probability proportional to its density.

**Complete Graph:** By straightforward enumeration of all relevant events, the transition rates \( r_\pm(x) \) for voters on the complete graph are:

\[
r_+(x) = \frac{1}{2} N A x (1-x) + B_+ (1-x), \tag{8}
\]

\[
r_-(x) = \frac{1}{2} N A x (1-x) + B_- x.
\]

The first term in \( r_\pm \) accounts for a voter that adopts the opinion of a neighboring voter and the second term accounts for adopting the opinion of the news source. The coefficients \( A \) and \( B_\pm \) are

\[
A = \frac{(1-\ell_+)(1-\ell_-)}{1-(1/N)} + \frac{(1-p)(\ell_+ + \ell_- - 2\ell_+\ell_-)}{(1-p) + (2p-1)/N} + \frac{(1-p)\ell_-}{(1-p) + (3p-1)/N}, \tag{9a}
\]

\[
B_\pm = \frac{p\ell_\pm}{2} \left[ \frac{1-\ell_\pm}{(1-p) + (2p-1)/N} + \frac{\ell_\pm}{(1-p) + (3p-1)/N} \right]. \tag{9b}
\]

Using (8) and (9) in the definitions of \( V(x) \) and \( D(x) \), their ratio is

\[
\frac{V(x)}{D(x)} = \frac{2[B_+(1-x) - B_- x]}{A x (1-x) + (1/N)[B_+(1-x) + B_- x]} \tag{10}
\]

Importantly, \( V/D \) is of order 1, except when \( x \) is of order 1/N away from the boundaries at 0 and 1. Within these boundary layers, the second term in the denominator of \( V/D \) ensures that \( V/D \) remains finite even when \( x = 0, 1 \). Considerable simplification arises by excluding these thin boundary layers and consequently dropping this second term. This approximation has a vanishingly small effect on the consensus time for large \( N \). We find the positions of the resulting slightly smaller interval
effective potential (12) tends to drive the population to the ant-sigmoidal shape of the probability for $\alpha = 1$ and $\alpha = 2$ arises by choosing $\ell = 1$ and $p = 2/3$.

$[a_-, 1 - a_+]$ by equating the two terms of the denominator of $V/D$. This gives $a_\pm = B_\pm /AN$. In this truncated interval, we have

$$V(x) = \frac{[B+(1-x) - B-x]}{N} D(x) \approx \frac{A(1-x)}{2N}.$$  

(11)

Using this approximation for $V(x)$ and $D(x)$, the effective potential in (4) becomes

$$\phi(x) = -\ln[x^{a_+}(1-x)^{a_-}],$$  

(12)

where $a_\pm = 2B_\pm /A$. We can also explicitly evaluate the integrals in Eqs. (6)–(7) for specific values of $a_\pm$. For simplicity, we specialize to the symmetric case of equally connected news sources, so that $a_+ = a_- = a$, and correspondingly $a_+ = a_0 \equiv a = \alpha/(2N)$. Performing the integral in Eq. (4) with the potential in (11), the exit probabilities for $\alpha = 1$ and $\alpha = 2$ are (Fig. 2)

$$E_+(x) = \frac{1}{2} \left[1 - \frac{H_1(x)}{H_1(a)}\right].$$  

(13)

where

$$H_1(x) = \ln \left(x^{-1} - 1\right)$$

and

$$H_2(x) = x^{-1} - (1-x)^{-1} + \ln \left(x^{-1} - 1\right)^2.$$  

The anti-sigmoidal shape of $E_+(x)$ arises because the effective potential (12) tends to drive the population to the politically polarized state of $x = \frac{1}{2}$.

To obtain the consensus time, we substitute Eq. (11) into the first of Eqs. (7) to give

$$T_{con}(x) = N \left[G_\alpha(a) - G_\alpha(x)\right]$$  

(14)

where, for simple rational values of $\alpha$, $G_\alpha$ is

$$G_{\frac{1}{2}}(x) = -4 \arcsin \sqrt{x} \arcsin \sqrt{1-x},$$

$$G_{1}(x) = -\ln \left[x(1-x)\right],$$

$$G_{\frac{3}{2}}(x) = (x^{-\frac{1}{2}}) \left[\arcsin \sqrt{x} - \arcsin \sqrt{1-x} \right] / \sqrt{x(1-x)},$$

$$G_{2}(x) = \frac{1}{6} \left[x^{-1}(1-x)^{-1} - 2 \ln \left[x(1-x)\right]\right].$$

These give $T_{con} \sim N$ for $\alpha = \frac{1}{2}$, $T_{con} \sim N \ln N$ for $\alpha = 1$, $T_{con} \sim N^{3/2}$ for $\alpha = \frac{3}{2}$ and $T_{con} \sim N^2$ for $\alpha = 2$, as in Eq. (1a).

We can understand the $N$ dependence of $T_{con}$ for arbitrary $\alpha$ in terms of the effective potential (12). According Kramers’ theory [30], the time to reach the boundaries at $a$ and at $1 - a$ are proportional to $\exp[\phi(a)]$ and to $\exp[\phi(1-a)]$, respectively. Because the potential scales logarithmically in $N$ as $x \to a$ or $x \to 1-a$, there is an algebraic, rather than an exponential, dependence of $T_{con}$ on $N$. This behavior contrasts with voter models with non-conserved dynamics [37, 38], where the effective potential leads to a consensus time that grows exponentially in $N$. For $\alpha < 1$, the effect of the logarithmic potential is subdominant with respect to fluctuations [39] and the latter drive the system to consensus, leading to $T_{con} \sim N$. These predictions agree with the simulation results in Fig. (3). When $\ell_+ \neq \ell_-$, the lowest barrier height in the potential determines the exponent; therefore $\alpha = \min(\alpha_+, \alpha_-)$ as in Eq. (1a).

Finally, we numerically verified that there is negligible difference in the consensus time when connections between the two news sources and the population are random or disjoint, with the same density of links.

To determine $T_{pol}$ in a simple way, consider the extreme case where each news source has a single link to the complete graph. This weak connectivity leads to the longest possible polarization time. Suppose that the system starts in the $-$ consensus state. At some point, the “informed” voter, the one that is linked to the $+$ news source, changes its opinion from $-$ to $+$. When this happens, this informed voter now disagrees with all its neighbors. From this excited state, subsequent opinion changes primarily occur among voters within the complete graph. Since there is only a single link to the news sources, they play a negligible role in subsequent opinion changes.

The state space of this reduced system is schematically represented in Fig. 4. Here $(0)$ denotes the consensus
state, $|1\rangle$ denotes the excited state where the informed voter has changed opinion, and $|p\rangle$ denotes the polarized state in which the fraction of $+$ and $-$ voters are equal, and $E$ is the exit probability to reach $|p\rangle$, which equals $\frac{2}{\beta}$ [35]. We can now write the following backward equations for the polarization time

$$T_{\text{pol}} = dt_0 + T'_{\text{pol}}, \quad T'_{\text{pol}} = (1-E)(dt_1 + T_{\text{pol}}) + E\tau.$$

(15)

Here $T_{\text{pol}}$ and $T'_{\text{pol}}$ are the times to reach the polarized state starting from the states $|0\rangle$ and $|1\rangle$, respectively, $dt_0 = 1/[r_+(0)+r_-(0)]$ is the time to leave the state $|0\rangle$, $dt_1 \approx 1$ is the time to leave the state $|1\rangle$, and $\tau = 2N(1-\ln2)$ is the conditional time to reach the state $|p\rangle$ from $|1\rangle$ by voter model dynamics [35]. Solving these equations gives Eq. (2). We emphasize that when the news sources are well connected to the population, the polarization time $T_{\text{pol}}$ is less than the consensus time because for $T_{\text{pol}}$ the state of the system is driven towards the minimum of the effective potential, while for $T_{\text{con}}$ the system has to surmount a potential barrier.

Finally, we obtain the steady-state opinion distribution, $P_{\text{ss}}(x) \equiv P(x,t \to \infty)$, by setting $\frac{\partial P}{\partial t} = 0$ in Eq. (3). We also need to apply reflecting boundary conditions because for all $\alpha > 0$, the endpoints are not fixed points of the stochastic dynamics. Imposing normalization, we find

$$P_{\text{ss}}(x) = \frac{x^{\alpha_+ - 1}(1-x)^{\alpha_- - 1}}{B[1-\alpha_+;\alpha_+,\alpha_-] - B[\alpha_-;\alpha_+,\alpha_-]},$$

(16)

where $B(x;y,z)$ is the incomplete beta function. For $\ell_+ = \ell_- = \ell$, $P_{\text{ss}}(x) \propto \left|x(1-x)\right|^{\alpha - 1}$. This distribution undergoes a bimodal to unimodal transition as $\alpha$ passes through 1.

Two-clique graph: We can adapt the above argument for the polarization time on the complete graph to obtain both $T_{\text{con}}$ and $T_{\text{pol}}$ on sparsely interconnected two-clique graphs, where $\beta \to 0$. Because the fraction of interclique links is negligible compared to intraclique links, the opinion dynamics when opinions in a single clique are not unanimous reduces to that of isolated cliques that are additionally influenced by news sources. Let $x_i$ be the fraction of $+$ voters on clique $C_i$ (Fig. 1) and denote the state of the system by $(x_1,x_2)$. It is convenient to take the initial condition as the maximally polarized (MP) state $(1,0)$. The population tends to remain close to the MP state because: (a) news sources tend to drive opinions to this state, and (b) the time $dt_0$, the inverse of the probability for a $\pm$ interaction between voters, which scales as $N^{1-\beta}$, is large for $\beta \to 0$.

For an isolated clique connected to a single news source, we obtain the probability to reach the state $x_1 = 0$ by setting $\ell_- = 0$, $\ell_+ = \ell$ for $V/D$ in Eq. (10) and using the resulting form in Eqs. (4) and (6) to give

$$E(x_1) = \begin{cases} 
1 - \frac{(\alpha+2N\alpha_1)^{1-\alpha_1-\alpha}}{(\alpha+2N)^{1-\alpha_1-\alpha}} & \alpha \neq 1 \\
1 - \frac{\ln(2N\alpha_1+1)}{\ln(2N+1)} & \alpha = 1.
\end{cases}$$

(17)

Using the same argument as in Eq. (15), where the MP state, the MP state with one opinion change, and consensus correspond $|0\rangle$, $|1\rangle$, and $|p\rangle$ respectively, we can compute $T_{\text{con}}$ and obtain Eq. (11). A closely related argument gives $T_{\text{pol}}$ in Eq. (2).

![FIG. 4. State space of the reduced system.](image)

![FIG. 5. Distribution of fraction $x_1$ of + opinion voters on clique $C_1$ of 128 voters on the two-clique graph, with $\ell = 1$.](image)

Finally, the steady-state distribution of $x_1$ normalized for each clique (Fig. 5) shows that the opinions in the two cliques indeed becomes more polarized as the number of interclique links is reduced or the interactions with news sources become stronger.

To summarize, the presence of two well-connected antagonistic news sources promotes political polarization in the voter model. The news sources give rise to an effective potential that leads to an anomalously long consensus time and a short time to reach a politically polarized state.

We thank Mirta Galesic for inspiring discussions and gratefully acknowledge financial support from NSF grant DMR-1608211.

---

*deepak.bhat@santafe.edu
redner@santafe.edu

[1] L. Adamic and N. Glance, LinkKDD 05, Proceedings of the 3rd International Workshop on Link Discovery, 36 (2005).
[2] D. Baldassarri and A. Gelman, Am. J. Sociol. 114, 408 (2008).
[3] M. P. Fiorina and S. J. Abrams, Annu. Rev. Pol. Sci. 16, 101 (2013).
