Super controlled gates and controlled gates in two-qubit gate simulations

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Abstract

In two-qubit gate simulations an entangling gate is used several times together with single qubit gates to simulate another two-qubit gate. We show how a two-qubit gate’s simulation power is related to the simulation power of its mirror gate. And we show that an arbitrary two-qubit gate can be simulated by three applications of a super controlled gate together with single qubit gates. We also give the gates set that can be simulated by n applications of a controlled gate in a constructive way. In addition we give some gates which can be used four times to simulate an arbitrary two-qubit gate.

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We want to make a quantum computer because it can solve some difficult problems using smaller resource than that needed for classical computers. An n-qubit quantum computation can be accomplished by applying an n-qubit gate on a standard initial state followed by a measurement [1]. Any n-qubit gate can be decomposed as a sequence of the gates from a universal gate set, which contains all single qubit gates and an arbitrary two-qubit entangling gate [2, 3, 4, 5, 6, 7, 8, 9]. This problem is important because we do not know much even for the decomposition of two-qubit gates when the given gate is a general entangling gate [10].

In this paper we only investigate the decomposition or simulation of two-qubit gates. According to the canonical decomposition of two-qubit gate [10], we can always write a two-qubit gate in the form: $U_{AB} = (U^A \otimes U^B)U_d(V^A \otimes V^B)$, where $U^A$, $U^B$, $V^A$ and $V^B$ are single qubit gates and $U_d$ has a special form

$$U_d(\alpha_1, \alpha_2, \alpha_3) = \exp \left(i \sum_{j=1}^{3} \alpha_j \sigma_j^A \otimes \sigma_j^B \right),$$

where $\sigma_{1,2,3}$ are Pauli matrix. We can let the parameters satisfy $\pi/4 \geq \alpha_1 \geq \alpha_2 \geq |\alpha_3| \geq 0$. The special form $U_d$, which is locally equivalent to $U_{AB}$, is called the canonical form of the two-qubit gate $U_{AB}$. The canonical form of the CNOT, DCNOT and SWAP gates are $U_d(\pi/4, 0, 0)$, $U_d(\pi/4, \pi/4, 0)$, and $U_d(\pi/4, \pi/4, \pi/4)$ respectively [11]. A very important character of $U_d$ is that the magic basis states are its eigenstates [10, 12], i.e.,

$$U_d(\alpha_1, \alpha_2, \alpha_3)|\Phi_j\rangle = e^{i\lambda_j}|\Phi_j\rangle,$$

where $|\Phi_1\rangle = 1/\sqrt{2}(|00\rangle + |11\rangle)$, $|\Phi_2\rangle = i/\sqrt{2}(|01\rangle + |10\rangle)$, $|\Phi_3\rangle = 1/\sqrt{2}(|01\rangle - |10\rangle)$, $|\Phi_4\rangle = i/\sqrt{2}(|00\rangle - |11\rangle)$,

$$\lambda_1 = +\alpha_1 - \alpha_2 + \alpha_3, \lambda_2 = +\alpha_1 + \alpha_2 - \alpha_3,$$

$$\lambda_3 = -\alpha_1 - \alpha_2 - \alpha_3, \lambda_4 = -\alpha_1 + \alpha_2 + \alpha_3.$$

The controlled gates and super controlled gates in this paper denote the gates that have canonical forms $U_d(\alpha_1, 0, 0)$ and $U_d(\pi/4, \alpha_2, 0)$ respectively. In the geometric representation of two-qubit gates [12] controlled gates are represented by the line between the identity gate and the CNOT gate, and super controlled gates are represented by the line between the CNOT gate and the DCNOT gate. Assume that the two-qubit gates $U_1$ and $U_2$ have canonical forms $U_d(\alpha_1, \alpha_2, \alpha_3)$ and $U_d(\beta_1, \beta_2, \beta_3)$ respectively. We say that the two-qubit gate $U_2$ is a mirror gate of $U_1$, if $U_d(\alpha_1 + \pi/4, \alpha_2 + \pi/4, \alpha_3 + \pi/4)$ is locally equivalent to $U_d(\beta_1, \beta_2, \beta_3)$. It is not hard to see that the DCNOT gate is a mirror gate of the CNOT gate.

Now we present a general result about two-qubit simulation.

**Theorem 1** (mirror gate theorem) Assume that the gate $U_d(\gamma_1, \gamma_2, \gamma_3)$ is used in a quantum simulation circuit and the circuit simulates the two-qubit gate $U_d(\beta_1, \beta_2, \beta_3)$. Then we can replace the gate $U_d(\gamma_1, \gamma_2, \gamma_3)$ in the circuit by one of its mirror gate to simulate a mirror gate of $U_d(\beta_1, \beta_2, \beta_3)$.

**Proof** From the fact that the SWAP gate is locally equivalent to $U_d(\pi/4, \pi/4, \pi/4)$, we can write $U_{SWAP} = s_1^a s_2^b U_d(\pi/4, \pi/4, \pi/4) s_3^a s_4^b$ using some single qubit gates $s_1, s_2, s_3,$ and $s_4$. Without loss of generality, we denote all the gates applied after and before $U_d(\gamma_1, \gamma_2, \gamma_3)$ in the circuit by $k_1^{d_B}k_2^{d_A}U_d(a_1, a_2, a_3)k_3^{d_B}k_4^{d_A}$ and $U_{before}$ respectively, where $\{k_i\}_{i=1}^4$ is single qubit gate. Then we have

$$k_1^{d_B}k_2^{d_A}U_d(a_1, a_2, a_3)k_3^{d_B}k_4^{d_A}U_d(\gamma_1, \gamma_2, \gamma_3)U_{before} = U_d(\beta_1, \beta_2, \beta_3).$$

From the Equation (1) we can get

$$\sum_k^A S U_d(a_1, a_2, a_3)k_3^{d_B}k_4^{d_A}S^{-1}U_d(\gamma_1, \gamma_2, \gamma_3)U_{before} = SU_d(\beta_1, \beta_2, \beta_3),$$

where $S = U_d(\pi/2, 0, 0)$. If $\beta_1, \beta_2, \beta_3$ are the eigenstate of $U_d(a_1, a_2, a_3)$, then $SU_d(\beta_1, \beta_2, \beta_3)$ is also the eigenstate of $U_d(a_1, a_2, a_3)$ and $SU_d(\beta_1, \beta_2, \beta_3) = e^{i\lambda_j}SU_d(\beta_1, \beta_2, \beta_3)$, where $\lambda_j$ is the eigenvalue of $U_d(a_1, a_2, a_3)$.
where $S = U \text{SWAP} (s^A_3 s^B_3)^{-1}$. We have two facts about the two-qubit gate $S$:

$$S k^A_1 k^B_4 U_d (a_1, a_2, a_3) k^A_4 k^B_1 S^{-1}$$

$$(s^A_3 s^B_3)^{-1} k^A_1 k^B_4 U_d (a_1, a_2, a_3) k^A_4 k^B_1 (s^A_3 s^B_3),$$

and

$$SU_d (\gamma_1, \gamma_2, \gamma_3)$$

$$= s^A_4 s^B_2 U_d (\pi/4 + \gamma_1, \pi/4 + \gamma_2, \pi/4 + \gamma_3).$$

Now we rewrite the Equation (2) by using the two facts about $S$ and we have

$$k^A_1 k^B_2 U_d (a_1, a_2, a_3) k^A_3 k^B_4 MU_{before} = N,$$

where

$$M = (k^A_3 k^B_4)^{-1} k^A_1 k^B_3 (s^A_3 s^B_3) s^A_1 s^B_2$$

and

$$N = k^A_1 k^B_2 (k^A_2 k^B_1)^{-1} s^A_3 s^B_2 s^A_1 s^B_2$$

$$U_d (\pi/4 + \beta_1, \pi/4 + \beta_2, \pi/4 + \beta_3).$$

From the Equation (5) we can see that if $U_d (\gamma_1, \gamma_2, \gamma_3)$ is replaced by its mirror gate $M$ in the circuit, the simulated gate $U_d (\beta_1, \beta_2, \beta_3)$ is replaced by its mirror gate $N$.

Some important results can be derived from the mirror gate theorem.

**Corollary 1** If a two-qubit gate can be simulated by two applications of $U_{AB}$ together with single qubit gates, then the two qubit gate can also be simulated by two applications of the mirror gate of $U_{AB}$.

This result is from the fact that the mirror gate’s mirror gate is locally equivalent to the original gate. Since the DCNOT gate is a mirror gate of the CNOT gate, the gates set that can be simulated by two applications of the DCNOT gate is the same as that of the two applications of the CNOT gate. This gates set has been pointed out by Vidal and Dawson [5].

**Corollary 2** If an arbitrary two-qubit gate can be simulated by $n$ applications of $U_{AB}$ together with single qubit gates, then an arbitrary two-qubit gate can also be simulated by $n$ applications of the mirror gate of $U_{AB}$ [13].

It has been showed that an arbitrary two-qubit gate can be simulated by three applications of the CNOT gate [5, 4, 8, 9], so immediately we can conclude that an arbitrary two-qubit gate can also be simulated by three applications of the DCNOT gate. This result has been pointed out in [6].

The $B$ gate can be used two times together with single qubit gates to simulate an arbitrary two-qubit gate [7]. Then it comes to the question: what gate can be used three times to simulate an arbitrary two-qubit gate?

**Theorem 2** Three applications of the super controlled gate $U_d (\pi/4, \alpha_2, 0)$ together with single qubit gates can simulate any two-qubit gate $U_d (h_1, h_2, h_3)$. Two applications of the super controlled gate $U_d (\pi/4, \alpha_2, 0)$ together with single qubit gates can simulate any two-qubit gate $U_d (h_1, h_2, 0)$.

**Proof** We first define three two-qubit gates $U_A$, $U_B$, and $U_C$ by their function:

$$U_A (\Phi_1) = |00\rangle,$$

$$U_B (\Phi_2) = |01\rangle, U_A (\Phi_3) = |10\rangle, U_A (\Phi_4) = e^{-i\alpha_2} |11\rangle,$$

$$U_B (00) = (\cos 2\alpha_2 |00\rangle + \sin 2\alpha_2 |10\rangle)|00\rangle,$$

$$U_B (01) = |01\rangle, U_B (10) = e^{-i\alpha_2} |11\rangle,$$

$$U_C (\cos 2\alpha_2 |00\rangle + \sin 2\alpha_2 |10\rangle) = |\Phi_1\rangle,$$

$$U_C (01) = |\Phi_2\rangle, U_C (11) = e^{-i\alpha_2} |\Phi_3\rangle,$$

and

$$U_C (\sin 2\alpha_2 |00\rangle - \cos 2\alpha_2 |10\rangle) = |\Phi_4\rangle.$$ We can verified that the gate $e^{-2i\alpha_2} U_C e^{i\alpha_3} a^B_3 e^{-i\alpha_2} a^B_2 U_A$ is just the same gate as $U_d (h_1, h_2, h_3)$ by applying them on the magic basis states. It can also be verified that $U_A^{-1} e^{i\alpha_3} a^B_3 e^{-i\alpha_2} a^B_2 U_A$ is the same gate as $U_d (h_1, h_2, 0)$.

Using the methods given in [10, 12], we can find that $U_A$, $U_B$, $U_C$, and $U_A^{-1}$ are locally equivalent to the super controlled gate $U_d (\pi/4, \alpha_2, 0)$. So we can end our proof.

Now we turn to investigate the gates set that can be simulated by $n$ applications of a controlled gate.

**Theorem 3** Given two controlled gates $e^{i\gamma_1 a^A_3 \cdot a^B_3}$ and $e^{i\gamma_2 a^A_3 \cdot a^B_3}$ with $0 < \gamma_2 \leq \gamma_1 \leq \pi/2$. If a two-qubit gate can be simulated by these two controlled gates together with single qubit gates, it must be locally equivalent to a gate $U_d (h_1/2, h_2/2, 0)$ with $0 \leq h_2 \leq h_1 \leq \pi/2, \gamma_1 - \gamma_2 \leq h_1 - h_2$, and $h_1 + h_2 \leq \gamma_1 + \gamma_2$.

This problem has been investigated by Zhang et al. [6], but their result is incomplete. We will give the proof later. From theorem 3 we can derive the following result.

**Corollary 3** The two-qubit gate $U_d (h_1/2, h_2/2, 0)$ with $|h_1| + |h_2| \leq n \gamma$, can be simulated by $n (\geq 2)$ applications of the controlled gate $e^{i\gamma a^A_3 \cdot a^B_3}$ together with single qubit gates, where $0 < \gamma \leq \pi/2$.

**Proof** We only need to prove the result when $0 \leq h_2 \leq h_1 \leq \pi/2$, and $h_1 + h_2 \leq n \gamma$, because $U_d (h_1/2, h_2/2, 0)$ is locally equivalent to a gate $U_d (h_1/2, h_2/2, 0)$ with $0 \leq h_2 \leq h_1 \leq \pi/2$, and $h_1 + h_2 \leq n \gamma$. When $n = 2$ the corollary is obvious from theorem 3. Assume that the corollary is true for $n = m \geq 2$, we only need to prove that the corollary is also true for $n = m + 1$. Because two applications of the controlled gate $e^{i\gamma a^A_3 \cdot a^B_3}$ can simulate itself, the gate which can be simulated by $n$ applications of the controlled gate can also be simulated by $n + 1$ applications of the controlled gate $e^{i\gamma a^A_3 \cdot a^B_3}$ together with single qubit gates. It is not hard to find that the gate $U_d (h_1/2, h_2/2, 0)$ can be simulated by $U_d ((h_1 - \gamma)/2, h_2/2, 0)$ and $e^{i\gamma a^A_3 \cdot a^B_3}$ together with single qubit gates. So we only need to prove that the gate $U_d ((h_1 - \gamma)/2, h_2/2, 0)$ can be simulated by $m$ applications of the controlled gate $e^{i\gamma a^A_3 \cdot a^B_3}$. This is true.
since both $h_1 - \gamma$ and $h_2$ are in the interval $[0, \pi/2]$ and $h_1 - \gamma + h_2 \leq m\gamma$. So we can end our proof.

Zhang et. al. \cite{6} have given the gates set that can be simulated by $n$ ($n \geq 3$) applications of a controlled-unitary gate, but they have not given the explicit simulation method. Here we give an explicit simulation method for $n \geq 4$.

**Theorem 4** The gate $U_d(h_1/2, h_2/2, h_3/2)$ with $0 \leq |h_3| \leq h_2 \leq h_1 \leq \pi/2$, can be simulated by $n$ ($n \geq 3$) applications of the gate $e^{i\gamma\sigma_z^1 \otimes \sigma_3^0}$ together with single qubit gates if $h_1 + h_2 + |h_3| \leq m\gamma$, where $0 < \gamma \leq \pi/2$.

**Proof** This result is proved in \cite{6}, but their proof is not constructive. Here we give a constructive procedure to simulate the gates by $n$ ($n \geq 4$) applications of the controlled gate $e^{i\gamma\sigma_z^1 \otimes \sigma_3^0}$. First we have $|h_3| \leq n\gamma/3$. We denote $m = \lfloor n/3 \rfloor$, the function $|x|$ is defined as the smallest integer which is not smaller that $x$. When $n \geq 4$, we have $n - m \geq m \geq 2$. Based on corollary 3 we can find that the gate $e^{i\gamma\sigma_z^1 \otimes \sigma_3^0}$ can be simulated by $m$ applications of the controlled gate $e^{i\gamma\sigma_z^1 \otimes \sigma_3^0}$, because $|m\gamma - |h_3|| + |h_3| = m\gamma$. Similarly we can find that the gate $e^{i\gamma\sigma_z^1 \otimes \sigma_3^0}$ can be simulated by $(n - m)$ applications of the controlled gate $e^{i\gamma\sigma_z^1 \otimes \sigma_3^0}$, because $|h_1 - m\gamma + |h_3|| + h_2 \leq (n - m)\gamma$.

Notice that the gate $\exp \left( i \sum_{j=1}^{h_2} \sigma_j^1 \otimes \sigma_j^0 \right)$ is just the product of $e^{i\gamma\sigma_z^1 \otimes \sigma_3^0}$ and $e^{i\gamma\sigma_z^1 \otimes \sigma_3^0}$, and we can end our proof.

The condition $h_1 + h_2 + |h_3| \leq n\gamma$ is also a necessary condition for the gate that can be simulated by $n$ ($n \geq 3$) applications of the gate $e^{i\gamma\sigma_z^1 \otimes \sigma_3^0}$ together with single qubit gates \cite{6, 13}. Theorem 3 and theorem 4 tell us the gates set that can be simulated by $n$ ($\geq 2$) applications of the controlled gate. It is not hard to find out that the $\lceil \frac{3n}{2} \rceil$ applications of the controlled gate $e^{i\gamma\sigma_z^1 \otimes \sigma_3^0}$ can simulate an arbitrary two-qubit gate. According to the mirror gate theorem, we can easily find out the gates set that can be simulated by $n$ ($\geq 2$) applications of the gate $U_d(\pi/4, \pi/4, \pi/4 + \gamma/2)$. And we can conclude that the $\lceil \frac{3n}{2} \rceil$ applications of the gate $U_d(\pi/4, \pi/4, \pi/4 + \gamma/2)$ together with single qubit gates can simulate an arbitrary two-qubit gate.

We can only give part results for $n = 3$ in theorem 4. We first prove the following result.

**Theorem 5** The two-qubit gate

$$U_{AB} = U_d(a_1/2, a_2/2, a_3/2) \left( e^{i\pi_1 a_1^1} \otimes e^{i\pi_2 a_2^0} \right) U_d(b_1/2, b_2/2, b_3/2)$$

is locally equivalent to the gate

$$U_d(x/2, y/2, (a_2 + b_2)/2),$$

where

$$\cos(x + y) = \cos(a_1 + a_3) \cos(b_1 + b_3) - \cos(s_1 - s_2) \sin(a_1 + a_3) \sin(b_1 + b_3),$$

and

$$\cos(x - y) = \cos(a_1 - a_3) \cos(b_1 - b_3) - \cos(s_1 + s_2) \sin(a_1 - a_3) \sin(b_1 - b_3).$$

**Proof** Following the procedure in \cite{10, 12}, we first write the gate $U_{AB}$ in the magic basis. The gate $U_{AB}^T$ represents the transpose of $U_{AB}$ in the magic basis. Notice that \( \left( e^{i\pi_1 a_1^1} \otimes e^{i\pi_2 a_2^0} \right) \) can be regarded as a block diagnosed matrix in the magic basis. It is not hard to find that both the gate $U_{AB}$ and $M = U_{AB}^T U_{AB}$ can be regarded as block diagnosed matrix in the magic basis. As usual, we just compute the eigenvalues of the matrix $M$ to find the canonical form of the gate $U_{AB}$. Because the gate $M$ is represented by a block diagnosed matrix in the magic basis, we can easily find that the four eigenvalues of $M$ have the following relations:

$$x_1 x_2 = e^{-i(a_2 + b_2)} x_3 x_4 = e^{i(a_2 + b_2)},$$

$$x_1 + x_2 = 2e^{-i(a_2 + b_2)} \cos(x + y),$$

$$x_3 + x_4 = 2e^{i(a_2 + b_2)} \cos(x - y),$$

where $\cos(x + y)$ and $\cos(x - y)$ are given in the theorem. So we can write $x_1 = e^{-i(a_2 + b_2) + e^{-i(x+y)}},$ $x_2 = e^{i(a_2 + b_2) + e^{-i(x+y)}},$ $x_3 = e^{i(a_2 + b_2) + e^{-i(x-y)}},$ and $x_4 = e^{i(a_2 + b_2) + e^{-i(x-y)}}$. Compare with the eigenvalues of $M$ computed from a two-qubit gate in the canonical form, without loss of generality we think $U_{AB}$ is locally equivalent to the gate $\tilde{U}_d(x/2, y/2, (a_2 + b_2)/2)$.

**Theorem 6** The gate $U_d(h_1/2, h_2/2, h_3/2)$ can be simulated by 3 applications of the gate $e^{i\gamma\sigma_z^1 \otimes \sigma_3^0}$ together with single qubit gates if $0 \leq h_1 - h_2 \leq \min(3\gamma - |h_3|, \pi), 0 \leq h_1 + h_2 \leq \min(3\gamma + |h_3|, \pi),$ and $|h_2| \leq \gamma$, where $0 < \gamma \leq \pi/2$.

**Proof** We first simulate the gate $U_d(c_1/2, h_3/2, 0)$ with $0 \leq c_1 \leq \gamma \leq \gamma = \min(2\gamma - |h_3|, \pi - \gamma)$ by two applications of the controlled gate $e^{i\gamma\sigma_z^1 \otimes \sigma_3^0}$ based on corollary 3. Notice that $U_d(c_1/2, h_3/2, 0) \left( e^{i\pi_1 a_1^1} \otimes e^{i\pi_2 a_2^0} \right) U_d(0, 0, \gamma/2)$ is locally equivalent to the gate $U_d(h_1/2, h_2/2, h_3/2)$ with

$$\cos(h_1 + h_2) = \cos(c_1) \cos(\gamma) - \cos(s_1 - s_2) \sin(c_1) \sin(\gamma),$$

and

$$\cos(h_1 - h_2) = \cos(c_1) \cos(\gamma) + \cos(s_1 + s_2) \sin(c_1) \sin(\gamma).$$

From the above two equations, we can find that both $\cos(h_1 + h_2)$ and $\cos(h_1 - h_2)$ can be any value in the interval $[\cos(c_1 + \gamma), \cos(c_1 - \gamma)]$. So we have $c_1 \leq$
$h_1 - h_2 \leq c_1 + \gamma$ and $c_1 - \gamma \leq h_1 + h_2 \leq c_1 + \gamma$. When we vary $c_1$ from $\gamma$ to $\min (2\gamma - |h_3|, \pi - \gamma)$, we can find that $h_1$ and $h_2$ can be any value satisfying conditions: $0 \leq h_1 - h_2 \leq \min (3\gamma - |h_3|, \pi), 0 \leq h_1 + h_2 \leq \min (3\gamma - |h_3|, \pi)$, and $|h_2| \leq \gamma$.

Based on theorem 5 we can give a simple proof for theorem 3. Assume $a_1 = a_2 = b_1 = b_2 = 0$, and $0 < b_3 \leq a_3 \leq \pi/2$ in theorem 5. Then the gate $e^{i\tau_3\sigma^3_1 \otimes \sigma^3_2} e^{i\tau_1\sigma^1_1 \otimes \sigma^1_2} e^{i\tau_2\sigma^2_1 \otimes \sigma^2_2}$ is locally equivalent to the gate $U_d(x/2, y/2, 0)$ with $0 \leq y \leq x \leq \pi/2$, where

$$
cos (x + y) = \cos (a_3) \cos (b_3) - \cos (s_1 - s_2) \sin (a_3) \sin (b_3),
$$

and

$$
cos (x - y) = \cos (a_3) \cos (b_3) - \cos (s_1 + s_2) \sin (a_3) \sin (b_3).
$$

From the above two equations, we have $a_3 - b_3 \leq x - y$ and $x + y \leq a_3 + b_3$. According to ZYZ decomposition of single qubit gate [11], $e^{i\tau_3\sigma^3_1 \otimes \sigma^3_2} e^{i\tau_1\sigma^1_1 \otimes \sigma^1_2} e^{i\tau_2\sigma^2_1 \otimes \sigma^2_2}$ can be regarded as a representative of the gates that can be simulated by $e^{i\tau_3\sigma^3_1 \otimes \sigma^3_2}$ and $e^{i\tau_1\sigma^1_1 \otimes \sigma^1_2} e^{i\tau_2\sigma^2_1 \otimes \sigma^2_2}$. So we can end the proof of theorem 3.

In theorem 2 we have shown that three applications of a super controlled gate can simulate an arbitrary two-qubit gate, but it is still an open question to find out all the two-qubit gates that have the same simulation power as super controlled gates. Now we go on to find out some gates, which can be used four times to simulate any two-qubit gates. Since we know that two applications the gate $B = U_d(\pi/4, \pi/8, 0)$ can simulate any two-qubit gate [21], then four applications of the gate $U_{AB}$ can also simulate any two-qubit gate if two applications of $U_{AB}$ can simulate the gate $B$.

**Theorem 7** The gate $U_1 = U_d(a_{1/2}, 0, a_{3/2})$ can be used four times to simulate any two-qubit gate if $\cos (2a_1 + 2a_3) \leq -1/\sqrt{2}$ and $\cos (2a_1 - 2a_3) \leq 1/\sqrt{2}$.

**Proof** We only prove two applications of the gate $U_1$ can simulate the gate $B$. $U_2 = U_d(a_{3/2}, 0, a_{1/2})$ is locally equivalent to $U_1$. We assume that $U_1 \left( e^{i\tau_3\sigma^3_1 \otimes \sigma^3_2} e^{i\tau_1\sigma^1_1 \otimes \sigma^1_2} \right) U_2$ is locally equivalent to the gate $B$. Then according to theorem 5 we have

$$
cos (\pi/2 + \pi/4) = \cos^2 (a_1 + a_3) - \cos (s_1 - s_2) \sin^2 (a_1 + a_3),
$$

and

$$
cos (\pi/2 - \pi/4) = \cos^2 (a_1 - a_3) + \cos (s_1 + s_2) \sin^2 (a_1 - a_3).
$$

To ensure we can find suitable parameters $s_1$ and $s_2$ satisfying the above two equations, we only need

$$
\cos (2a_1 + 2a_3) \leq \cos (\pi/2 + \pi/4) = -1/\sqrt{2},
$$

$$
\cos (2a_1 - 2a_3) \leq \cos (\pi/2 - \pi/4) = 1/\sqrt{2}.
$$

So every gate, which is locally equivalent to $U_d(a_{1/2}, 0, a_{3/2})$, can be used four times to simulate any two-qubit gate if $\cos (2a_1 + 2a_3) \leq -1/\sqrt{2}$ and $\cos (2a_1 - 2a_3) \leq 1/\sqrt{2}$.

In conclusion, we have given a general result about two-qubit gate simulations and we have shown that some gates can be used three times or four times to simulate an arbitrary two-qubit gate. We also give the gates set that can be simulated by $n$ applications of a controlled gate through a constructive procedure. These results are important for quantum computer designers to exhibit the power of quantum computers since their design should base on the entangling gate that can generate directly from the experiment. The mirror gate theorem we present gives us another way to find the simulation power of entangling gates. This result may give some new insight into gate simulations.

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