**GL(n, R)** wormholes and waves in diverse dimensions

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Received 10 February 2009
Published 3 April 2009
Online at stacks.iop.org/CQG/26/085020

Abstract

We construct the most general Ricci-flat metrics in \((D + n)\)-dimensions that preserve the \(\mathbb{R}^{1,n-1} \times SO(D)\) isometry. The equations of motion are governed by the system of a \(GL(n, \mathbb{R})/SO(1,n-1)\) scalar coset coupled to \(D\)-dimensional gravity. Among the solutions, we find a large class of smooth Lorentzian wormholes that connect two asymptotic flat spacetimes. In addition, we obtain new vacuum tachyonic wave solutions in \(D \geq 4\) dimensions, which fit the general definition of pp-waves in that there exists a covariantly constant null vector. The momenta of the tachyon waves are larger than their ADM masses. The world volume of the tachyon wave is \(\mathbb{R}^{1,2}\), instead of \(\mathbb{R}^{1,1}\) for the usual vacuum pp-wave. We show that the tachyon wave solutions admit no Killing spinors, except in \(D = 4\), in which case it preserves half of the supersymmetry. We also obtain a general class of \(p\)-brane wormhole and tachyon wave solutions where the \(\mathbb{R}^{1,n-1}\) part of the spacetime lies in the world volume of the \(p\)-branes. These include examples of M-branes and D3-brane. Furthermore, we obtain AdS tachyon waves in \(D \geq 4\) dimensions.

PACS numbers: 04.50.-h, 04.50.Gh

1. Introduction

Recently, a large class of smooth Lorentzian wormholes were constructed in \(D \geq 5\) dimensions [1]. These include the previously known example in \(D = 5\) [2]. Such a wormhole can be viewed as a gravitational string that carries a momentum propagating in one space direction. The metric has an isometry of \(\mathbb{R}^{1,1} \times SO(D)\) in \((D + 2)\) dimensions, where \(\mathbb{R}^{1,1}\) denotes the world volume, comprising the time and the momentum-carrying space. If one performs Kaluza–Klein reduction in the \(\mathbb{R}^{1,1}\) directions, the \(D\)-dimensional solution is then a spherical symmetric wormhole supported by a \(GL(2, \mathbb{R})/O(1, 1)\) scalar coset [3].
In this paper, we generalize the above construction to obtain the most general Ricci-flat metrics in \((D + n)\)-dimensions that preserve the \(\mathbb{R}^{1,n-1} \times SO(D)\) isometry. The metric ansatz has the form

\[
ds_{\text{D+n}}^2 = r^2 \Omega_{D-1}^2 + \frac{dr^2}{f} + \sum_{\mu, \nu = 0}^{n-1} g_{\mu\nu} \, dz^\mu \, dz^\nu,
\]

(1.1)

where \(f\) and \(g_{\mu\nu}\) depend only on the \(D\)-dimensional radial variable \(r\). Note that this ansatz encompasses all the spherical symmetric \(p\)-branes and pp-waves. The Kaluza–Klein reduction on \(z^\mu\)'s, which includes a time direction, was performed in [3]. The \(D\)-dimensional system consists of the Euclidean-signatured metric and a \(GL(n, \mathbb{R})/SO(1, n-1)\) scalar coset [3–5]. The task is then reduced to construct spherical symmetric solutions in \(D\)-dimensions that are supported by the \(GL(n, \mathbb{R})/SO(1, n-1)\) scalar coset.

The group \(GL(n, \mathbb{R})\) is a direct product of the \(\mathbb{R}\) and \(SL(n, \mathbb{R})\) groups, where the \(\mathbb{R}\) factor is the breathing mode that measures the overall scale size of the \(\mathbb{R}^{1,n-1}\) spacetimes. It is consistent to truncate out this mode, leaving the \(SL(n, \mathbb{R})/SO(1, n-1)\) coset. In sections 2–5, we obtain the most general solutions supported by this scalar coset. We first present the general formalism in section 2, then proceed with the simplest \(SL(2, \mathbb{R})\) case in section 3 and \(SL(3, \mathbb{R})\) in section 4. The most general solutions for arbitrary \(SL(n, \mathbb{R})\) are presented in section 5.

Among the solutions we obtain, there is a large class of new smooth Lorentzian wormholes. The role that wormholes play in string theory has been studied recently in the context of AdS/CFT correspondence. In Lorentzian signature, wormholes that connect two asymptotic AdS spacetimes appear unlikely, and disconnected boundaries can only be separated by horizons [6]. Thus, the recent studies of wormholes in string theory and in the context of the AdS/CFT correspondence have so far concentrated on Euclidean-signatured spaces [7–11]. By combining the standard brane ansatz and wormhole solutions obtained in [1], a large class of Lorentzian brane wormholes were obtained in [12]. These solutions include examples of wormholes that connect \(\text{AdS} \times \text{Sphere}\) in one asymptotic region to a Minkowski spacetime in the other. In section 7, we obtain analogous brane solutions for our new Ricci-flat wormholes.

We also obtain new vacuum gravitational wave solutions (4.15) and (4.19) for all \(D \geq 4\) dimensions. We verify, up to the cubic order, that the polynomial scalar invariants of the Riemann tensor vanish identically. We expect that, as in the case of the vacuum pp-wave solution, all these invariants vanish identically. The solutions fit the general definition of pp-waves in that there exists a covariantly constant null vector. However, the solutions have some distinct features. One odd property of our new wave solutions is that the linear momenta are greater than their AdM masses. Thus, we call these solutions vacuum tachyon waves. The world volume of the tachyon wave has three dimensions, instead of the two dimensions for the pp-wave. In appendix B, we show that these tachyon wave metrics admit no Killing spinors, except for \(D = 4\), in which case, the solution preserves half of the supersymmetry. We also construct \(p\)-brane solutions with such a tachyon wave propagating in the world volume of the brane in section 7. Furthermore, we obtain AdS tachyon waves in all \(D \geq 4\) dimensions.

In section 6, we construct the most general solution for the case where the world-volume spacetime \(\mathbb{R}^{1,n-1}\) is replaced by the Euclidean space \(\mathbb{R}^n\). In section 8, we include the breathing mode and hence the scalar coset is \(GL(n, \mathbb{R})/SO(1, n-1)\). We also obtain the most general solutions for this case. We conclude our paper in section 9. In appendix A, we present the discussion of the properties of a constant matrix that is crucial for solving the scalar equations of motion.
2. General formalism

One goal of this paper is to construct the most general Ricci-flat metrics with an \( R^{1,n-1} \times SO(D) \) isometry in \((D + n)\)-dimensions. The Kaluza–Klein reduction on \( R^{1,n-1} \) gives rise to a scalar coset of \( GL(n, \mathbb{R})/SO(1, n - 1) \) in \( D \)-dimensions \cite{3}. In this section, we set up the general formalism for the case where the breathing mode, associated with the \( R \) factor of \( GL(n, \mathbb{R}) = \mathbb{R} \times SL(n, \mathbb{R}) \), is (consistently) truncated out.

We begin by reviewing the \( SL(n, \mathbb{R})/SO(1, n - 1) \) scalar coset. It can be parameterized by the upper-triangular Borel gauge, which includes all the positive-root generators \( E_i \) with \( i < j \) Cartan generators \( \hat{H} \). They satisfy the Borel subalgebra

\[
[\hat{H}, E_i] = \delta_{i,i} E_i, \quad [E_i, E_j] = E_i^\ell E_j^\ell - E_j^\ell E_i^\ell, \tag{2.1}
\]

where \( \delta_{i,j} \) are the positive-root vectors. Following the general discussion in \cite{13, 14}, one can parameterize the \( SL(n, \mathbb{R})/SO(1, n - 1) \) coset representative \( \mathcal{V} = V_1 V_2 \), with

\[
V_1 = e^{\frac{1}{2} \hat{H}_i}, \quad V_2 = \prod_{i<j} U_{i,j} = \cdots U_{23} U_{23} \cdots U_{14} U_{13} U_{12}, \quad U_{i,j} \equiv e^{\chi_i E_i^j}. \tag{2.2}
\]

Here, the generators \( \hat{H} \) and \( E_i^j \) are represented by \( n \times n \) matrices. Defining a symmetric matrix \( \mathcal{M} = \mathcal{V}^T \eta \mathcal{V} \), \( \eta = \text{diag}(-1, 1, \ldots , 1) \),

\[
\mathcal{M} = \mathcal{V}^T \eta \mathcal{V}, \quad \eta = \text{diag}(-1, 1, \ldots , 1), \tag{2.3}
\]

we propose that the metric ansatz for the \((D + n)\)-dimensional spacetime is given by

\[
dx_{D+n}^2 = dz_D^2 + dz_M^2 \mathcal{M} dz, \tag{2.4}
\]

where \( dz = (dt, dz_1, \ldots , dz_{n-1}) \). If we had chosen \( \eta \) in (2.3) to be the identity matrix instead, we would have a Euclidean-signatured space for the coordinates \( t \) and \( z_i \)'s. For a vacuum solution with all the scalars \( \hat{\phi} \) and \( \chi^i_j \) vanishing, the metric in the \( t \) and \( z_i \) directions is given by

\[
dx_n^2 = -dt^2 + dz_1^2 + \cdots + dz_{n-1}^2. \]

We shall refer the \( \mathbb{R}^{1,n-1} \) spacetimes of \( t \) and \( z_i \) coordinates as the world volume of our solutions. It was shown in \cite{3} that the Kaluza–Klein reductions on the time and on space directions commute. Thus, any permutation of the ‘−1’ and ‘1’ entries in \( \eta \) given by (2.3) is equivalent.

The Kaluza–Klein reduction on \( R^n \) (or \( T^n \)) with this type of ansatz, corresponding to \( \eta \) being the identity matrix, was considered \cite{13, 14}, which effectively implements successively \cite{15–17} the \( S^1 \) reduction. The reduction on \( \mathbb{R}^{1,n-1} \) is analogous, and was performed in \cite{3}. The \( D \)-dimensional effective Lagrangian is given by

\[
\mathcal{L} = \sqrt{\mathcal{M}} (R + \frac{1}{2} \text{tr}(\partial_{\mu} \mathcal{M}^{-1} \partial^{\mu} \mathcal{M})). \tag{2.5}
\]

Owing to the consistency of the reduction, the Ricci-flatness of the \((D+n)\)-dimensional metric (2.4) becomes equivalent to solutions to the lower \( D \)-dimensional system (2.5).

The Lagrangian (2.5) is invariant under the global \( SL(n, \mathbb{R}) \) symmetry, with the transformation rule

\[
\mathcal{M} \rightarrow \Lambda^T \mathcal{M} \Lambda, \tag{2.6}
\]

where \( \Lambda \) is any \( n \times n \) constant matrix satisfying \( \det \Lambda = 1 \). This symmetry has an origin as rather simple general coordinate transformations in the \((D + n)\)-dimensions. Indeed, the metric (2.4) is invariant under (2.6) together with \( dz \rightarrow \Lambda^{-1} dz \).

Throughout the paper, we consider solutions that are spherical symmetric for the \( D \)-dimensional metric \( ds_D^2 \). Without loss of generality, we take the metric to have the form

\[
ds_D^2 = \frac{dr^2}{f} + r^2 d\Omega_{D-1}^2, \tag{2.7}
\]
where \( f \) and all the scalars depend on the radial variable \( r \) only. Einstein equations in the foliating sphere \( S^{D-1} \) directions imply that
\[
\frac{(D-2)(1-f)}{r^2} - \frac{f'}{2r} = 0, \tag{2.8}
\]
where the prime denotes a derivative with respect to \( r \). Thus, we have
\[
f' = 1 - \left( \frac{a}{r} \right)^{2(D-2)}. \tag{2.9}
\]
The Einstein equation associated with the \( R_{rr} \) term implies that
\[
-\frac{(D-1)f'}{2rf} + \frac{1}{4} \text{tr}(\mathcal{M}^{-1}\mathcal{M}') = 0. \tag{2.10}
\]
The scalar equations of motion are given by
\[
(\mathcal{M}^{-1}\mathcal{M}) = 0, \tag{2.11}
\]
where the dot denotes a derivative with respect to \( \rho \), defined by
\[
d\rho = \frac{dr}{r^{D-1}\sqrt{f}}. \tag{2.12}
\]
The second-order differential equation (2.11) can be easily integrated to give rise to a set of first-order equations, given by
\[
\mathcal{M}^{-1}\mathcal{M} = C, \tag{2.13}
\]
where \( C \) is a Lie-algebra-valued constant matrix. Substituting this and (2.9) into (2.10), we have
\[
\mathcal{I} = -\frac{1}{4} \text{tr}(C^2) = 2(D-1)(D-2)a^{2(D-2)}. \tag{2.14}
\]
For the solution to be absent from a naked curvature power-law singularity at \( r = 0 \), it is necessary to have \( a^{2(D-2)} \geq 0 \), which implies that \( \mathcal{I} \geq 0 \). For the case with \( \mathcal{I} < 0 \), and hence \( a^{2(D-2)} < 0 \), a naked curvature power-law singularity at \( r = 0 \) is unavoidable. Note that the quantity \( \mathcal{I} \) is invariant under the global symmetry transformation, under which \( C \) transforms as
\[
C \to \Lambda^{-1}C\Lambda. \tag{2.15}
\]

It is worth mentioning that the solution for \( f \) and equations for \( \mathcal{M} \) apply to any scalar coset \( \mathcal{M} \), and not just the \( SL(n, \mathbb{R})/SO(1, n-1) \) case which we consider. In the following three sections, we shall obtain the most general solutions for \( SL(2, \mathbb{R}) \), \( SL(3, \mathbb{R}) \) and \( SL(n, \mathbb{R}) \) respectively.

3. General \( SL(2, \mathbb{R}) \) solutions

In this section, we give a detailed discussion for the case with \( n = 2 \). The Borel subalgebra of \( SL(2, \mathbb{R}) \) is generated by \( H \) and \( E_+ \), given by
\[
H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{3.1}
\]
The coset can be parameterized by
\[
\mathcal{V} = e^{i\phi}H e^{\chi}E_+ = \begin{pmatrix} e^{i\phi} & \chi e^{i\phi} \\ 0 & e^{-i\phi} \end{pmatrix}. \tag{3.2}
\]
It follows that $\mathcal{M}$ is a symmetric $2 \times 2$ matrix, given by

$$\mathcal{M} = \begin{pmatrix} -e^\phi & -e^\phi \\ -e^{-\phi} & e^{-\phi} - \chi e^\phi \end{pmatrix}, \quad \eta = \text{diag}(-1, 1). \quad (3.3)$$

Substituting this into (2.5), we have

$$L = \sqrt{\gamma} \left( R - \frac{1}{2} (\partial \phi)^2 + \frac{1}{2} e^{2\phi} (\partial \chi)^2 \right). \quad (3.4)$$

The equations of motion (2.11) for the scalars can be solved by the following general ansatz:

$$\mathcal{M}^{-1} \mathcal{M} = \mathcal{C} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}, \quad (3.5)$$

where $c_{ij}$ are constants subject to the traceless condition $c_{22} = -c_{11}$. Equation (3.5) can be written explicitly by

$$\dot{\phi} = c_{11} + c_{21} \chi, \quad \dot{\chi} = -c_{21} e^{-2\phi}, \quad (3.6)$$

together with an algebraic constraint

$$c_{21} e^{-2\phi} - c_{21} \chi^2 - 2c_{11} \chi + c_{12} = 0. \quad (3.7)$$

It is easy to verify that this algebraic constraint is consistent with the first-order equations (3.6) and hence a partial solution to these equations. Substituting (3.6), (3.7) and (2.9) into (2.10), we have the following constraint:

$$I = -c_{11}^2 - c_{12} c_{21} = 2(D - 1)(D - 2)a^{2(D-2)}. \quad (3.8)$$

Thus the solution is parameterized by four parameters, namely $c_{11}, c_{12}, c_{21}$ and one extra from solving (3.6) and (3.7).

As we discussed earlier, the system has an $SL(2, \mathbb{R})$ global symmetry which can be used to fix three of the four parameters, leaving one parameter arbitrary\(^4\). To do this explicitly, we first note that we can use the Borel subgroup to diagonalize $\mathcal{V}$ and hence $\mathcal{M}$, at one point in spacetime. We choose this point to be at asymptotic infinity $r = \infty$, such that the solutions are asymptotic Minkowskian with the standard diagonal metric. This can be achieved by making use of the Borel transformation, $\phi \rightarrow \phi + c_1$ and $\chi \rightarrow \chi + c_2$, to set $\phi = 0$ and $\chi = 0$ at $r = \infty$, corresponding to $\mathcal{M}_\infty = \text{diag}(-1, 1)$. After imposing this boundary condition, it follows from (3.7) that we have

$$c_{21} = -c_{12}. \quad (3.9)$$

Thus at asymptotic infinity, we have

$$\mathcal{M}_{|r\rightarrow\infty} \equiv \tilde{\mathcal{C}} = -\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}. \quad (3.10)$$

We now apply the residual $O(1, 1)$ symmetry, whose group elements can be parameterized as

$$\Lambda = \begin{pmatrix} c & s \\ s & c \end{pmatrix}, \quad (3.11)$$

where $c = \cosh \delta$ (and $s = \sinh \delta$) is the boost parameter. This symmetry leaves $\mathcal{M}_\infty = \text{diag}(-1, 1)$ invariant and transforms $\tilde{C}$ and equivalently $\tilde{\mathcal{C}}$ as follows:

$$\mathcal{C} \rightarrow \Lambda^{-1} \mathcal{C} \Lambda, \quad \tilde{\mathcal{C}} \rightarrow \Lambda^T \tilde{\mathcal{C}} \Lambda. \quad (3.12)$$

\(^4\) If one takes into account that the metric is Ricci-flat, and hence it remains a solution with any constant scaling of the metric, then this parameter can be viewed as trivial as well.
Thus, we have
\[ c_{11} \rightarrow (e^2 + s^2)c_{11} + 2csc_{12}, \quad c_{12} \rightarrow (e^2 + s^2)c_{12} + 2csc_{11}. \] (3.13)
Depending on the values of \( c_{ij} \), three inequivalent classes arise.

**Class I.** \( c_{11} < c_{12} \)

In this class, \( C \) cannot be diagonalized to a real matrix. For simplification, we can use the \( O(1, 1) \) symmetry to set \( c_{11} = 0 \) instead. It follows that we have the following solution:
\[ \chi = \tan \left( \sqrt{\frac{2(D - 1)}{D - 2}} \arcsin \left( \frac{a}{r} \right)^{D-2} \right), \quad e^{2\phi} = \frac{1}{1 + \chi^2}. \] (3.14)

The corresponding \((D + 2)\)-dimensional metric is then given by
\[ ds^2_{D+2} = \frac{dr^2}{1 - (\frac{a}{r})^{2(D-2)}} + r^2 d\Omega_{D-1}^2 + \frac{1}{\sqrt{1 + \chi^2}} \left( -dr^2 + dz_1^2 - 2\chi dr dz_1 \right). \] (3.15)

For \( a^{2(D-2)} > 0 \), this is precisely the Ricci-flat Lorentzian wormhole solution obtained in [1], although now with a different radial coordinate. For \( a^{2(D-2)} < 0 \), the solution has a naked curvature power-law singularity at \( r = 0 \).

**Class II.** \( c_{11} > c_{12} \)

In this class, we can find a boost parameter \( \delta \) so set \( c_{12} = 0 \), so that the matrix \( C \) is diagonal. We send \( a^{2(D-2)} \rightarrow -a^{2(D-2)} \) since in this case we have \( \mathcal{I} \leq 0 \). The solution can be straightforwardly obtained, given by
\[ ds^2_{D+2} = r^2 d\Omega_{D-1}^2 + \frac{dr^2}{1 + (\frac{a}{r})^{2(D-2)}} - e^{2\phi} dr^2 + e^{-2\phi} dz_1^2, \]
\[ \rho = -\frac{1}{(D - 2)a^{D-2}} \arcsinh \left( \frac{a}{r} \right)^{D-2}, \quad \lambda = \sqrt{2(D - 1)(D - 2)a^{D-2}}. \] (3.16)
The solution has a naked curvature power-law singularity at \( r = 0 \).

**Class III.** \( c_{11} = c_{12} \)

We shall parameterize \( C \) to be
\[ C_2 = a \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}. \] (3.17)
This \( 2 \times 2 \) matrix is of rank 1 with all eigenvalues vanishing; it cannot be diagonalized. Condition (3.8) implies that \( a = 0 \), and hence \( f = 1 \). The scalars \( \chi \) and \( \phi \) can be easily solved, given by
\[ e^\phi = 1 - \frac{q}{r^{D-2}}, \quad \chi = 1 - e^{-\phi}, \quad q = \frac{a}{D - 2}. \] (3.18)
The corresponding \((D + 2)\)-dimensional metric describes the standard pp-wave, given by
\[ ds^2_{D+2} = -du dv + \frac{q}{r^{D-2}} du^2 + dv^2 + r^2 d\Omega_{D-1}^2. \] (3.19)
where \( u = t - z_1 \) and \( v = t + z_1 \) are the asymptotic light-cone coordinates.

Thus, we have demonstrated that there are total three classes of solutions of the \( SL(2, \mathbb{R}) \) system. By requiring the absence of naked curvature power-law singularity, the most general Ricci-flat metric in \((D + 2)\) dimensions with the \( \mathbb{R}^{1,1} \times SO(D) \) isometry with a fixed \( \mathbb{R}^{1,1} \) volume is either a wormhole or a pp-wave. As was observed in [1], the latter can be obtained as a singular infinite boost of the former.

It is worth pointing out that the boosted Schwarzschild black hole is not included in the solution space. This is because the volume form for world volume \( \mathbb{R}^{1,1} \) in this case is not a constant.
4. General $SL(3, \mathbb{R})$ solutions

The coset $SL(3, \mathbb{R})/SO(1, 2)$ representative $\mathcal{V}$ in a Borel gauge is given by [3]

$$
\mathcal{V} = e^\tilde{H} e^{\phi_1 E_{23}} e^{\phi_2 E_{31}} e^{\phi_3 E_{12}}
\begin{pmatrix}
4\psi_1, & 1\psi_1, & 1\psi_1, & 0 & 0 \\
-1\psi_1, & -1\psi_1, & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & -1 & 1 \\
\end{pmatrix},
$$

(4.1)

where $\tilde{H}$ represents the two Cartan generators and $E_{ij}$ denote the positive-root generators of $SL(3, \mathbb{R})$. It follows that we have

$$
\mathcal{M} = \mathcal{V}^T \eta \mathcal{V}, \quad \eta = \text{diag}(-1, 1, 1).
$$

(4.2)

The $D$-dimensional Lagrangian (2.5) is then given by

$$
(\sqrt{g})^{-1} \mathcal{L} = R - \frac{1}{2} (\partial \phi_1)^2 - \frac{1}{2} (\partial \phi_2)^2 + \frac{1}{2} e^{-2\phi_1} (\partial \chi_{13} - \chi_{23} \partial \chi_{12})^2
\begin{aligned}
&+ \frac{1}{2} e^{-\sqrt{3} \phi_1} (\partial \chi_{23})^2 + \frac{1}{2} e^{\sqrt{3} \phi_1} (\partial \chi_{12})^2.
\end{aligned}
$$

(4.3)

The scalar equations of motion (2.11) can be solved by the following constant matrix:

$$
\mathcal{M}^{-1} \mathcal{M} = \mathcal{C} \equiv \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & -c_{11} - c_{22} \end{pmatrix}.
$$

(4.4)

One can read off five first-order equations for the five scalars $\phi_1, \phi_2, \chi_{12}, \chi_{13}$ and $\chi_{23}$ respectively. In addition, there are three algebraic constraints which are partial solutions to the equations. Substituting (4.4) into (2.10), we obtain

$$
\mathcal{I} = -\frac{1}{4} \text{tr}(\mathcal{C}^2) = -\left(c_{11}^2 + c_{22}^2 + c_{12} c_{21} + c_{13} c_{31} + c_{23} c_{32}\right)
= 2(D - 1)(D - 2) a^{2(D - 2)}.
$$

(4.5)

Thus we see that the solution is parameterized by a total of ten constant parameters, eight $c_{ij}$’s and two extra by solving for the first-order equations and algebraic constraints. The system has an $SL(3, \mathbb{R})$ global symmetry, which can remove eight parameters, leaving the solution with two arbitrary parameters.

To apply the global $SL(3, \mathbb{R})$ symmetry, we first use the Borel transformations to set $\phi_i = 0$ and $\chi_{ij} = 0$ at the asymptotic infinity, $r = \infty$. The spacetimes at $r = \infty$ are then Minkowskian since we have $\mathcal{M}_\infty = \eta$. Applying this boundary condition to the three algebraic constraints, we find that

$$
c_{21} = -c_{12}, \quad c_{31} = -c_{13}, \quad c_{32} = c_{23}.
$$

(4.6)

Thus, we have

$$
\mathcal{M}_{|r \to \infty} \equiv \mathcal{C} = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{12} & -c_{22} & -c_{23} \\ c_{13} & -c_{23} & c_{11} + c_{22} \end{pmatrix}.
$$

(4.7)

The upshot of this discussion is that the constant traceless matrix $\mathcal{C}$ defined in (4.4), after applying the Borel subgroup of $SL(3, \mathbb{R})$ global transformations, can be expressed as $\mathcal{C} = \eta \mathcal{C}$, where $\mathcal{C}$ is a symmetric constant matrix.

As we shall discuss in detail in appendix A for the general $SL(n, \mathbb{R})$, three different classes of solutions emerge depending on the value of the matrix $\mathcal{C}$. In this section, we shall just present the results.
Class I.

The first class corresponds to the case where \( C \) has a pair of complex eigenvalues. It is isomorphic to

\[
C = \begin{pmatrix}
\alpha & -\beta & 0 \\
\beta & \alpha & 0 \\
0 & 0 & -2\alpha
\end{pmatrix}.
\] (4.8)

We find that the corresponding \((D + 3)\)-dimensional metric is given by

\[
ds^2 = r^2 \Omega_1^2 + \frac{dr^2}{1 - \left(\frac{a}{r}\right)^{2(D-2)}} + e^{\lambda_0 \rho} \left[-\lambda_0 \rho dz_0^2 + \cos(\rho) \left( -dr^2 + dz_1^2 \right) + 2 \sin(\rho) d \tau \ dz_1 \right] + e^{-2\lambda_0 \rho} dz_3^2,
\] (4.9)

with

\[
\beta^2 - 3\alpha^2 = \frac{2}{D-1} \left( D - 2 \right) a^2(D-2).
\] (4.10)

For \( a^2(D-2) > 0 \), the solution describes a smooth Lorentzian wormhole. In a special case with \( \alpha = 0 \), it describes a direct product of an \( SL(2, \mathbb{R}) \) wormhole, discussed in section 3, with a real line. For \( a^2(D-2) < 0 \), the solution has a naked curvature power-law singularity at \( r = 0 \). When \( \beta^2 = 3\alpha^2 \), corresponding to having \( a = 0 \), the solution has a naked curvature power-law singularity at \( r = 0 \), since in this case \( \rho \sim 1/r^{D-2} \).

Class II.

The second class corresponds to the fact that \( C \) has three real eigenvalues, with one time-like and two space-like eigenvectors. The matrix \( C \) can be diagonalized by a certain \( SO(1, 2) \) matrix to give \( C = \text{diag}(\lambda_0, \lambda_1, \lambda_2) \) with \( \lambda_2 = -\lambda_0 - \lambda_1 \). In this case, we have \( I = -\lambda_0 \lambda_1^2 - \frac{1}{2} \lambda_1^2 - \frac{1}{2} (\lambda_0 + \lambda_1)^2 \). It follows from (4.5) that the reality condition implies that \( f = 1 + a^{2(D-2)}/r^{2(D-2)} \). Thus, we find that the corresponding \((D + 3)\)-dimensional metric is given by

\[
ds^2 = r^2 \Omega_1^2 + \frac{dr^2}{1 + \left(\frac{a}{r}\right)^{2(D-2)}} - e^{\lambda_0 \rho} \ dz_0^2 + e^{\lambda_1 \rho} \ dz_1^2 + e^{-2(\lambda_0 + \lambda_1) \rho} dz_3^2,
\] (4.11)

where

\[
\lambda_0^2 + \lambda_1^2 + (\lambda_0 + \lambda_1)^2 = 4(D-1)(D-2) a^2(D-2).
\] (4.12)

This solution has a naked curvature power-law singularity at \( r = 0 \).

Class III.

In the third class, the matrix \( C \) has three real eigenvalues, but with degenerate eigenvectors. There are two inequivalent cases. The first case corresponds to the fact that \( C \) is of rank 2 and all of its eigenvalues vanish. Such \( C \) is isomorphic to

\[
C_3 = \alpha \begin{pmatrix}
\cos \beta & -\cos \beta & \frac{1}{2} \sin \beta \\
\cos \beta & \cos \beta & \frac{1}{2} \sin \beta \\
-\frac{1}{2} \sin \beta & -\frac{1}{2} \sin \beta & 0
\end{pmatrix}.
\] (4.13)
We find that the corresponding \((D + 3)\)-dimensional metric is given by
\[
\begin{align*}
\text{d}s^2_{D+3} &= r^2 \text{d}z_1^2 + \text{d}r^2 + \text{d}\Omega^2
\end{align*}
\]
\[
\begin{align*}
\text{d}z_2^2 &= -(1 + \rho \cos \beta - \frac{1}{8} \rho^2 \sin^2 \beta) \text{d}r^2 + \left(1 - \rho \cos \beta - \frac{1}{8} \rho^2 \sin^2 \beta\right) \text{d}z_1^2 \\
&\quad + 2\left(\rho \cos \beta - \frac{1}{8} \rho^2 \sin^2 \beta\right) \text{d}r \text{d}z_1 - \rho \sin \beta \text{d}r \text{d}z_2 \\
&\quad + \rho \sin \beta \text{d}z_1 \text{d}z_2 + \text{d}z_3^2,
\end{align*}
\] 
with \(\rho = -\frac{u}{(D-2)v}\). In the asymptotic light-cone coordinates \(u = t + z_1\) and \(v = t - z_1\), the metric becomes
\[
\begin{align*}
\text{d}s^2_{D+3} &= -\text{d}u \text{d}v + \text{d}u^2 - \left(\rho \cos \beta - \frac{1}{8} \rho^2 \sin^2 \beta\right) \text{d}v^2 - \rho \sin \beta \text{d}v \text{d}w + \text{d}r^2 + r^2 \text{d}\Omega^2.
\end{align*}
\] 
(4.15)

Here, we rename the \(z_2\) coordinate to be \(w\). This metric has non-vanishing Riemann tensor components, and hence it is not flat. We also verify that the Riemann tensor square and cubic scalar invariants all vanish identically. We expect that as in the case of the vacuum pp-wave solution, all the polynomial scalar invariants of the Riemann tensor for the metric (4.15) vanish identically.

The new wave solutions fit the general definition of pp-waves in that there exists a covariantly constant null vector \(k = \partial/\partial t\). However, there are several differences comparing this new wave solution to the usual pp-wave. The world volume for the usual pp-wave solution is two dimensional, whilst it has three dimensions spanned by \((t, z_1, z_2)\) (or \((u, v, w)\)) for the new solution. The mass and the momentum are given by
\[
M = \alpha \cos \beta, \quad P_1 = \alpha \cos \beta, \quad P_2 = -\frac{1}{2} \alpha \sin \beta.
\] 
(4.16)

Note that the mass and the momentum components are evaluated from the Komar integrals for the Killing vectors \(\partial/\partial t\) and \(\partial/\partial z_1\) respectively. We omitted a certain overall inessential constant factor in presenting the above quantities. Thus, the solution is of tachyonic nature, since \(M^2 - P_1^2 - P_2^2 = -\frac{1}{4} \alpha^2 \sin^2 \beta \leq 0\). We shall call this solution a tachyon wave. We can make an orthonormal transformation
\[
\begin{align*}
\text{z}_1 &= \frac{2 \cos \beta z_1 + \sin \beta \text{d}z_3}{\sqrt{4 \cos^2 \beta + \sin^2 \beta}}, \\
\text{z}_2 &= \frac{2 \cos \beta z_2 - \sin \beta \text{d}z_3}{\sqrt{4 \cos^2 \beta + \sin^2 \beta}},
\end{align*}
\] 
(4.17)
such that the momentum has only the \(z_1\) component. In these coordinates, we have
\[
M = \alpha \cos \beta, \quad P_1 = \alpha \cos \beta \sqrt{1 + \frac{1}{4} \tan^2 \beta}, \quad P_2 = 0.
\] 
(4.18)

The vacuum pp-wave (3.19) has half of Killing spinors. As we show in appendix B, there is no Killing spinor in this tachyon wave solution. This is consistent with the fact that the BPS condition for a tachyon is obviously not satisfied. When \(\beta = 0\), we have \(P_2 = 0\) and \(M = P_1\); the BPS condition is then satisfied and the solution becomes the pp-wave. When \(\beta = \frac{1}{2} \pi\), the solution becomes massless, but with non-vanishing linear momentum.

The tachyon wave metric (4.15) is also valid for \(D = 1\) and \(D = 2\), corresponding to total four and five spacetime dimensions. The corresponding \(\rho\) is given by \(\rho = \alpha \log(r)\) and \(\rho = \alpha r\) respectively. The metrics for these two cases are given by
\[
\begin{align*}
\text{d}s^2 &= -\text{d}u \text{d}v + \text{d}y^2 - \left(\alpha \cos \beta - \frac{1}{8} \alpha^2 \sin^2 \beta\right) \text{d}v^2 - \alpha \sin \beta \text{d}v \text{d}w + \text{d}x^2, \\
\text{d}s^2 &= -\text{d}u \text{d}v + \text{d}v^2 - \left(\log \alpha \cos \beta - \frac{1}{8} \left(\log r\right)^2 \alpha^2 \sin^2 \beta\right) \text{d}v^2 \\
&\quad - \log \alpha \sin \beta \text{d}v \text{d}w + \text{d}r^2 + r^2 \text{d}\phi^2.
\end{align*}
\] 
(4.19)

However, in these two cases, there is no well-defined asymptotic region. Note that the four-dimensional metric belongs to type N in the Petrov classification. When \(\beta = 0\), the four-dimensional metric is flat.
We can perform Kaluza–Klein reduction on the $w$ direction for the tachyon wave (4.15). We have

$$ds_{D+2}^2 = -du \, dv - \left( \rho \cos \beta + \frac{1}{8} \rho^2 \sin^2 \beta \right) dv^2 + dr^2 + r^2 d\Omega_{D-1}^2,$$

$$\mathcal{A} = -\frac{1}{2} \rho \sin \beta \, dv.$$  

(4.20)

Thus, the lower dimensional solution is a usual pp-wave supported by a Maxwell field.

The second degenerate case for $C$ is given by

$$C = \begin{pmatrix}
\alpha + \beta & \alpha & 0 \\
-\alpha & -\alpha + \beta & 0 \\
0 & 0 & -2\beta
\end{pmatrix}.$$  

(4.21)

The solution is given by

$$ds_{D+3}^2 = r^2 d\Omega_{D-1}^2 + \frac{dr}{1 + \left( \frac{a}{r} \right)^{2(D-2)}} + e^{\beta \rho} (-du \, dv + g \rho \, dv^2) + e^{-2\beta \rho} \, dz^2,$$

$$\rho = -\frac{1}{(D-2)a^2} \arcsinh \left( \frac{a}{r} \right)^{(D-2)}, \quad \beta = \sqrt{\frac{2}{3}(D-1)(D-2)a^{2(D-2)}}.$$  

(4.22)

When $\beta = 0$, the matrix (4.21) reduces to (3.17) and the above metric becomes the vacuum pp-wave. When $\alpha = 0$, corresponding to $q = 0$, the solution becomes that of class I. In general the solution has a naked curvature power-law singularity at $r = 0$.

5. General $SL(n, \mathbb{R})$ solutions

Having discussed the $SL(2, \mathbb{R})$ and $SL(3, \mathbb{R})$ examples, it is straightforward to generalize to obtain the most general spherical symmetric solutions for any $SL(n, \mathbb{R})$. We can use the Borel transformation to diagonalize $\mathcal{M}$ at the asymptotic infinity $r = \infty$, namely $\mathcal{M}_\infty = \eta$. This implies that the traceless matrix $C$ has the property that $\eta C$ is symmetric. To be specific, for $\mathcal{M}$ given by (2.3), $C$ is given by

$$C = \begin{pmatrix}
-\sum_{i=1}^{n-1} \Phi_i & -\check{x}_{01} & \cdots & -\check{x}_{0,n-1} \\
\check{x}_{01} & \Phi_1 & \cdots & \check{x}_{1,n-1} \\
\vdots & \vdots & \ddots & \vdots \\
\check{x}_{0,n-1} & \check{x}_{1,n-1} & \cdots & \Phi_{n-1}
\end{pmatrix} \bigg|_{r \to \infty}.  

(5.1)

As we demonstrate in appendix A, three classes of solutions emerge depending on the values of $C$.

Class I.

The first class corresponds to $C$ with a pair of complex eigenvalues. Using the $SO(1, 1-n)$ residual global symmetry, we can simplify $C$ as follows:

$$C = \begin{pmatrix}
\alpha & -\beta \\
\beta & \alpha
\end{pmatrix}$$

$$\begin{pmatrix}
\lambda_2 \\
\vdots \\
\lambda_{n-1}
\end{pmatrix},$$

(5.2)

with $\text{tr}(C) = 0$. We find that the corresponding $(D+n)$-dimensional Ricci-flat metric can be straightforwardly obtained, given by
\[
\frac{ds^2_{D+n}}{d\Omega^2_{D-1}} = r^2 + \frac{dr^2}{1 - \left(\frac{a}{r}\right)^{2(D-2)}} + e^{\alpha\rho}[\cos(\beta\rho)(-dr^2 + dz_i^2) + 2\sin(\beta\rho)drdz_i] + \sum_{i=2}^{n-1} \lambda_i^{\rho} dz_i^2,
\]

\[
\sum_{i=2}^{n-1} \lambda_i + 2\alpha = 0, \quad 2\beta^2 - 2\alpha^2 - \sum_{i=2}^{n-1} \lambda_i^2 = 4(D - 1)(D - 2)a^{2(D-2)},
\]

\[
\rho = -\frac{1}{(D - 2)a^{D-2}} \arcsinh \left(\frac{a}{r}\right)^{D-2}.
\]

For \(a^{2(D-2)} > 0\), the solution describes a smooth Lorentzian wormhole. For \(a^{2(D-2)} \leq 0\), the solution has a naked curvature power-law singularity at \(r = 0\). The global structure and the traversability of the \(SL(2, \mathbb{R})\) Lorentzian wormhole in \(D = 5\) were discussed in detail in \([1, 18, 19]\). Note that there are no further off-diagonal terms in \(SL(n, \mathbb{R})\) wormholes than that with \(SL(2, \mathbb{R})\). We expect that the property of the general wormholes is analogous to the \(SL(2, \mathbb{R})\) one. We shall discuss their properties in detail in a future publication.

Class II.

The second class corresponds to \(C\) with all real eigenvalues and one of the eigenvectors being time-like. In this case, it can be diagonalized by an \(SO(1, n - 1)\) transformation. We thus have

\[
C = \text{diag}(\lambda_0, \lambda_1, \ldots, \lambda_{n-1}),
\]

with \(\text{tr}(C) = 0\). We find that the corresponding \((D + n)\)-dimensional Ricci-flat metric solution is

\[
\frac{ds^2_{D+n}}{d\Omega^2_{D-1}} = r^2 + \frac{dr^2}{1 - \left(\frac{a}{r}\right)^{2(D-2)}} - e^{\alpha\rho} dr^2 + \sum_{i=1}^{n-1} e^{\lambda_i\rho} dz_i^2,
\]

\[
\sum_{i=0}^{n-1} \lambda_i = 0, \quad \sum_{i=0}^{n-1} \lambda_i^2 = 4(D - 1)(D - 2)a^{2(D-2)},
\]

\[
\rho = -\frac{1}{(D - 2)a^{D-2}} \arcsinh \left(\frac{a}{r}\right)^{D-2}.
\]

The solution is asymptotic Minkowskian; it has a naked curvature power-law singularity at \(r = 0\).

Class III.

In this class, all the eigenvalues of the matrix \(C\) are real, but there is no time-like eigenvector. As we demonstrate in appendix A, there are two cases of \(C\). They can be constructed from rank-1 \(C_2\) and rank-2 \(C_3\) given in \((3.17)\) and \((4.13)\), respectively.

The first case is given by

\[
C = \begin{pmatrix}
C_2 + \lambda_0 \mathbb{I} \\
\lambda_2 & \lambda_3 & \cdots & \lambda_{n-1}
\end{pmatrix},
\]

\[(5.6)\]
with $\text{tr}(C) = 0$. We find that the corresponding $(D + n)$-dimensional Ricci-flat metric is given by

$$
\text{ds}_{D+n}^2 = r^2 \Omega_{D-1}^2 + \frac{dr^2}{1 + \left(\frac{r}{a}\right)^{D-2}} + e^{2\phi}(-du + q\varphi \, dv + \sum_{i=2}^{n-1} e^{\lambda_i \varphi} \, dz_i^2),
$$

$$
\rho = \frac{1}{(D-2)\alpha^{D-2}} \arcsinh \left( \frac{a}{r} \right)^{(D-2)},
$$

(5.7)

$$
3\lambda_0 + \sum_{i=2}^{n-1} \lambda_i = 0, \quad 2\lambda_0^2 + \sum_{i=2}^{n-1} \lambda_i^2 = 4(D-1)(D-2)\alpha^{2(D-2)}.
$$

The solution reduces to the standard pp-wave when all $\lambda_i$’s vanish. In general, the solution has a naked curvature power-law singularity at $r = 0$.

In the second case, we have

$$
\mathcal{C} = \begin{pmatrix}
C_3 + \lambda_0 \mathbb{1} & \lambda_3 \\
\lambda_3 & \lambda_4 \\
\vdots & \ddots \\
\lambda_{n-1}
\end{pmatrix},
$$

(5.8)

again with $\text{tr}(C) = 0$. We find that the corresponding $(D + n)$-dimensional Ricci-flat metric is given by

$$
\text{ds}_{D+n}^2 = r^2 \Omega_{D-1}^2 + \frac{dr^2}{1 + \left(\frac{r}{a}\right)^{D-2}} + e^{2\phi}(-du + q\varphi \, dv + \sum_{i=1}^{n-1} e^{\lambda_i \varphi} \, dz_i^2),
$$

$$
\rho = \frac{1}{(D-2)\alpha^{D-2}} \arcsinh \left( \frac{a}{r} \right)^{(D-2)},
$$

(5.9)

$$
3\lambda_0 + \sum_{i=3}^{n-1} \lambda_i = 0, \quad 3\lambda_0^2 + \sum_{i=3}^{n-1} \lambda_i^2 = 4(D-1)(D-2)\alpha^{2(D-2)}.
$$

In general, the solution also has a naked curvature power-law singularity at $r = 0$. The tachyon wave arises when all $\lambda_i$’s vanish.

To summarize, in this section, we obtain a complete set of spherical symmetric solutions for the $\text{SL}(n, \mathbb{R})/\text{SO}(1, n-1)$ coset scalar, and we obtain the corresponding $(D + n)$-dimensional Ricci-flat metrics.

6. Euclidean signature solutions

In the previous sections, we consider solutions that are asymptotic Minkowskian. The solution space becomes much simpler if we choose the Euclidean signature. In this case, the matrix $C$ is symmetric and can always be diagonalized by an $\text{SO}(n)$ transformation. The most general solution is then given by

$$
\text{ds}_{D+3}^2 = r^2 \Omega_{D-1}^2 + \frac{dr^2}{1 + \left(\frac{r}{\alpha}\right)^{2(D-2)}} + \sum_{i=1}^{n} e^{\alpha_i \varphi} \, dz_i^2
$$

(6.1)

$$
12
with
\[ \sum_{i=1}^{n} \alpha_i = 0, \quad \sum_{i=1}^{n} \alpha_i^2 = 4(D - 1)(D - 2)a^{2(D - 2)}, \]
\[ \rho = -\frac{1}{(D - 2)a^{D - 2}} \arcsinh \left( \frac{a}{r} \right)^{(D - 2)}. \]  

(6.2)

7. General p-brane wormholes and waves

In sections 2–5, we constructed the most general Ricci-flat metrics in \((D+n)\)-dimensions with the \(\mathbb{R}^{1,n-1} \times SO(D)\) isometry. There are three different classes of such solutions. Here we construct charged solutions to \(\hat{D}\)-dimensional Einstein gravity coupled to a \((p+2)\)-form field strength, together with a dilaton. The Lagrangian has the following general form:
\[ \mathcal{L}_{\hat{D}} = \sqrt{-g} \left( R - \frac{1}{2} (\partial \phi)^2 - \frac{1}{2(p + 2)!} e^{\alpha \phi} F_{p+2}^2 \right). \]  

(7.1)

where \(F_{p+2} = dA_{p+1}\). The constant \(\alpha\) can be parameterized as [15]
\[ \alpha^2 = \Delta - \frac{2(p + 1)(\hat{D} - p - 3)}{\hat{D} - 2}. \]  

(7.2)

The Lagrangian (2.5) is of the form that typically arises as a truncation of the full Lagrangian in many supergravities, with \(\Delta\) being given by
\[ \Delta = \frac{4}{N}, \]  

(7.3)

for integer \(N\) that can arise depend on the spacetime dimensions; they are classified in [16].

We start with the Ricci-flat solutions in \((D+n)\)-dimensions constructed in the paper, and augment the spacetime with flat directions to become \(\hat{D} = D + n + m\) dimensions. The Ricci-flat \(\hat{D}\)-dimensional metric has the form
\[ ds^2_{\hat{D}} = ds^2_D + ds^2_n + \sum_{i=1}^{m} (dx^i)^2, \]  

(7.4)

where the first two terms denote the general SL\((n, \mathbb{R})\) solutions we obtained in the previous sections. We now follow [12] and consider an electric \(p\)-brane solution where \(p = m + n - 1\). We find that the solution is given by
\[ ds^2_{\hat{D}} = H^{\frac{1}{N}} ds^2_D + H^{-\frac{(D-2)N}{D-2}} \left( ds^2_n + \sum_{i=1}^{m} (dx^i)^2 \right) \]
\[ e^{\alpha \phi} = H^{\frac{\alpha p}{(D-2)N}}, \]
\[ F_{p+2} = \sqrt{N} dz_1 \wedge \cdots \wedge dz_n \wedge dx_1 \wedge \cdots \wedge dx_m \wedge dH^{-1}, \]  

(7.5)

where \(H\) is a harmonic function on the metric \(ds^2_D\). Thus for spherical symmetric branes, we have
\[ f = 1 : H = 1 + \frac{Q}{r^{D-2}}, \]
\[ f = 1 - \left( \frac{a}{r} \right)^{2(D-2)} : H = 1 + Q \arcsin \left( \frac{a}{r} \right), \]
\[ f = 1 + \left( \frac{a}{r} \right)^{2(D-2)} : H = 1 + Q \arcsinh \left( \frac{a}{r} \right). \]  

(6.7)
Note that we can also consider ‘magnetic $p$-brane wormholes’, which are equivalent to the previously discussed electric cases, but constructed using the $(D - p - 2)$-form dual of the $(p + 2)$-form field strength $F_{p+2}$. In other words, we can introduce the dual field strength

$$
\tilde{F}_{p+2} = e^{a\phi} F_{p+2},
$$

(7.7)

where $\tilde{p} = D - p - 4$, in terms of which the Lagrangian (7.1) can be rewritten as

$$
\mathcal{L}_D = \sqrt{-\tilde{g}} \left( R - \frac{1}{2} (\bar{\partial} \phi)^2 - \frac{1}{2(\tilde{p} + 2)!} e^{-a\phi} \tilde{F}_{p+2}^2 \right),
$$

(7.8)

where $\tilde{F}_{p+2} = d\tilde{A}_{p+1}$. The electric solution (7.5) to (7.1) can then be reinterpreted as a magnetic solution to (7.8), with $\tilde{F}_{p+2}$ being given by

$$
\tilde{F}_{p+2} = \sqrt{N}(D - 2)Q\Omega_{D-1}.
$$

(7.9)

A particularly interesting class of brane solutions are those where the dilaton decouples. These include the M-branes and D3-brane. The global structure of such a brane with an $SL(n, \mathbb{R})$ wormhole was discussed in [12]. These solutions connect $AdS \times$ Sphere in one asymptotic region to a flat Minkowski spacetime in the other. We expect that this property maintains when the general allowed $SL(n, \mathbb{R})$ wormholes are added to the branes.

Here we present explicitly the M-branes and D3-brane on the background of the tachyon wave (4.15) we constructed in section 4. We find that the M2-brane tachyon wave is given by

$$
dx^2_{11} = H^{-\frac{3}{2}} \left( -du \, dv + dw^2 - \left( \frac{\alpha \cos \beta}{r^6} - \frac{\alpha^2 \sin^2 \beta}{8r^6} \right) \, dv^2 - \frac{\alpha \sin \beta}{r^6} \, dv \, dw \right)
+ H^{\frac{1}{2}} (dr^2 + r^2 \, d\Omega_2^2),
$$

(7.10)

$$
F_{ai} = du \wedge dv \wedge dw \wedge dH^{-1}, \quad H = 1 + \frac{Q}{r^6}.
$$

The M5-brane tachyon wave is

$$
dx^2_{11} = H^{-\frac{3}{2}} \left( -du \, dv + dw^2 - \left( \frac{\alpha \cos \beta}{r^3} - \frac{\alpha^2 \sin^2 \beta}{8r^6} \right) \, dv^2 - \frac{\alpha \sin \beta}{r^3} \, dv \, dw + dx^i \, dx^i \right)
+ H^{\frac{1}{2}} (dr^2 + r^2 \, d\Omega_5^2),
$$

(7.11)

$$
F_{ai} = 3Q \Omega_{ai}, \quad H = 1 + \frac{Q}{r^3}.
$$

Finally, the D3-brane tachyon wave of the type IIB supergravity is

$$
dx^2_{10} = H^{-\frac{3}{2}} \left( -du \, dv + dw^2 - \left( \frac{\alpha \cos \beta}{r^4} - \frac{\alpha^2 \sin^2 \beta}{8r^6} \right) \, dv^2 - \frac{\alpha \sin \beta}{r^4} \, dv \, dw + dx^i \, dx^i \right)
+ H^{\frac{1}{2}} (dr^2 + r^2 \, d\Omega_5^2),
$$

(7.12)

$$
F_{ai} = 4Q (\Omega_{ai} + * \Omega_{ai}), \quad H = 1 + \frac{Q}{r^4}.
$$

In the decoupling limit where the constant 1 in function $H$ can be dropped, the metric becomes a direct product of the $AdS$ tachyon wave and a sphere. The $AdS$ tachyon wave in arbitrary dimensions is given by

$$
dx^2_D = \frac{dy^2}{y^2} + y^2 \left( -du \, dv + dw^2 - \left( \frac{\alpha \cos \beta}{y^{D-1}} - \frac{\alpha^2 \sin^2 \beta}{8y^{2(D-1)}} \right) \, dv^2 - \frac{\alpha \sin \beta}{y^{D-1}} \, dv \, dw + dx^i \, dx^i \right),
$$

(7.13)

where $i = 1, \ldots, (D - 4)$, and the cosmological constant is scaled to be a unit. In other words, the metrics satisfy $R_{\mu\nu} = -(D - 1) g_{\mu\nu}$. At asymptotic infinity $r = \infty$, the metric
is AdS\(_D\) in holospherical coordinates. The solution describes a tachyon wave propagating in the AdS\(_D\) spacetime. It is easy to verify that the polynomial scalar invariants of the Riemann tensors are all independent of the parameters \(\alpha\) and \(\beta\). For \(\beta = 0\), it becomes a pp-wave in AdS. For \(\beta = \frac{1}{2}\pi\), the tachyon wave is massless.

8. Generalize to \(GL(n, \mathbb{R})\)

In the \(\mathbb{R}^{1,n-1}\) reduction we considered in section 2, we can also turn on the breathing mode that scales the volume of the \(\mathbb{R}^{1,n-1}\) spacetime. The resulting scalars in \(D\)-dimensions parameterize a coset of \(GL(n, \mathbb{R})/O(1, n-1)\). The reduction ansatz is given by

\[
d s^2_{D,n} = e^{-\sqrt{\frac{2}{D-2}} \rho} d s^2_D + e^{\sqrt{\frac{2}{D-2}} \rho} d z^T M d z. \tag{8.1}
\]

The \(D\)-dimensional Lagrangian is given by

\[
\mathcal{L} = \sqrt{g} \left( R - \frac{1}{2} (\partial \phi)^2 + \frac{1}{4} \text{tr}(\partial M^{-1} \partial M) \right). \tag{8.2}
\]

We now define

\[
\hat{M} = e^{\frac{1}{2} \rho} M. \tag{8.3}
\]

The Lagrangian becomes

\[
\mathcal{L} = \sqrt{g} \left( R + \frac{1}{4} \text{tr}(\partial \hat{M}^{-1} \partial \hat{M}) \right). \tag{8.4}
\]

It follows that the constant matrix \(\hat{C}\), defined by

\[
\hat{M}^{-1} \hat{C} \equiv \hat{C}, \tag{8.5}
\]

has the same property as \(C\) except that it is no longer traceless. The classification of the \(C\) matrix also holds for \(\hat{C}\). This leads to the following three classes of solutions.

Class I.

This generalizes the first class of the \(SL(n, \mathbb{R})\) solutions. It is given by

\[
d s^2_{D,n} = e^{-\rho} \left( r^2 d \Omega^2_{D-1} + \frac{d r^2}{1 - \left( \frac{a}{r} \right)^{2(D-2)}} \right)
+ e^{\rho} \left[ \cos(\beta \rho) \left( -d \varphi^2 + d z_i^2 \right) + 2 \sin(\beta \rho) d t d z_i \right] + \sum_{i=2}^{n-1} e^{\lambda_i} d z_i^2, \tag{8.6}
\]

where

\[
\rho = -\frac{1}{(D-2)a^{(D-2)}} \arcsin \left( \frac{a}{r} \right)^{D-2}, \quad 2\alpha + \sum_{i=2}^{n-1} \lambda_i = \lambda, \tag{8.7}
\]

\[
2(\beta^2 - \alpha^2) - \sum_{i=2}^{n-1} \lambda_i^2 - \frac{\lambda^2}{D-2} = 4(D-1)(D-2)a^{2(D-2)}.
\]

For \(a^{2(D-2)} > 0\), the solution describes a smooth Lorentzian wormhole that connects two asymptotic flat spacetimes. For \(a^{2(D-2)} \leq 0\), the solution has a naked curvature power-law singularity at \(r = 0\).
Class II.

This generalizes the second class of the $\text{SL}(n, \mathbb{R})$ solutions. $\hat{C}$ can be diagonalized, and there is no traceless condition on the eigenvalues $\lambda_i$. We find that the $(D + n)$-dimensional Ricci-flat metric is given by

$$
\begin{align*}
\text{ds}^2_{D+n} &= e^{-2\rho} \left( r^2 \text{d}O^2_{D-1} + \frac{\text{dr}^2}{1 + \left( \frac{r}{\Lambda} \right)^{2(D-2)}} \right) - e^{4\rho} \text{ds}^2 + \sum_{i=1}^{n-1} e^{2\rho} \text{d}z_i^2, \\
\sum_{i=0}^{n-1} \lambda_i &= \lambda, \\
p\lambda_0 + \sum_{i=p}^{n-1} \lambda_i &= \lambda, \\
p\lambda_0 + \sum_{i=p}^{n-1} \lambda_i + \frac{\lambda^2}{D-2} &= 4(D-1)(D-2)a^{2(D-2)}, \quad (8.8)
\end{align*}
$$

In general, the solution has a naked singularity at $r = 0$. However, when $\lambda_i = 0$ for all $i$ except $\lambda_0$, we have $\lambda_0 = \lambda = 2(D-2)a^{D-2}$. The prefactor of $\text{d}O^2_{D-1}$ becomes a finite and non-vanishing constant at $r = 0$. The solution is the Schwarzschild black hole with the horizon located at $r = 0$.

Class III.

This generalizes the third class of the $\text{SL}(n, \mathbb{R})$ solutions. The $(D + n)$-dimensional Ricci-flat metrics are given by

$$
\begin{align*}
\text{ds}^2_{D+n} &= e^{-2\rho} \left( r^2 \text{d}O^2_{D-1} + \frac{\text{dr}^2}{1 + \left( \frac{r}{\Lambda} \right)^{2(D-2)}} \right) - e^{4\rho} \text{ds}^2 + \sum_{i=0}^{n-1} e^{2\rho} \text{d}z_i^2, \\
p\lambda_0 + \sum_{i=p}^{n-1} \lambda_i &= \lambda, \\
p\lambda_0 + \sum_{i=p}^{n-1} \lambda_i + \frac{\lambda^2}{D-2} &= 4(D-1)(D-2)a^{2(D-2)}, \quad (8.9)
\end{align*}
$$

where $p = 2, 3$ and $\text{ds}^2_p$'s are given by

$$
\begin{align*}
\text{ds}_2^2 &= -\text{d}u \text{d}v + \alpha \rho \text{d}v^2, \\
\text{ds}_3^2 &= -\text{d}u \text{d}v + \text{dw}^2 - \left( \alpha \cos \beta \rho - \frac{1}{8} a^2 \sin^2 \beta \rho \right) \text{d}v^2 - \alpha \sin \beta \rho \text{d}v \text{d}w. \quad (8.10)
\end{align*}
$$

In general the solution has a naked curvature power-law singularity at $r = 0$. When all $\lambda_i$ vanish, the solution reduces to either the pp-wave for $p = 2$ or the tachyon wave for $p = 3$.

9. Conclusions

In this paper, we construct the most general Ricci-flat metrics in $(D + n)$-dimensions with the $\mathbb{R}^{1,n-1} \times \text{SO}(D)$ isometry. We find that there are three classes of solutions. The first class describes the smooth Lorentzian wormholes that connect two asymptotic-flat spacetimes, as well as its analytic extension which has a naked curvature power-law singularity in the middle. All the wormhole metrics have one off-diagonal component in the $\mathbb{R}^{1,n-1}$ directions so that the solutions have both mass and a linear momentum, which propagates in one space direction. In the second class, the metric in the $\mathbb{R}^{1,n-1}$ direction is diagonal, and hence the solution has only the mass, with no linear momentum. The solution is asymptotic Minkowskian, but with a naked curvature power-law singularity in the middle. The third class describes a tachyon wave whose linear momentum is larger than its mass. The solution fits the general definition
of a pp-wave in that there exists a covariantly constant null vector. However, the tachyon wave has its own distinct features. The world volume for the tachyon wave is $\mathbb{R}^{1,2}$ instead of $\mathbb{R}^{1,1}$ for the usual pp-waves. We verify, up to the cubic order, that the polynomial scalar invariants of the Riemann tensor vanish identically. We expect that they all vanish identically, as in the case of the pp-waves. We also show in appendix B that the tachyon waves admit no Killing spinor except in $D = 4$, in which case, it preserves half of the supersymmetry.

We also obtain $p$-brane solutions where the $\mathbb{R}^{1,a-1}$ part of the spacetime lies in the world volume of the $p$-branes. Particularly interesting examples include M-branes and D3-brane, for which AdS can arise in certain decoupling limits. For the first class, solutions become AdS wormholes that connect AdS $\times$ Sphere in one asymptotic region to a flat spacetime in the other. The global structure and traversability of the $SL(2,\mathbb{R})$ wormholes were discussed in detail in [1, 12]. (See also [18, 19].) We shall discuss these properties of the general $GL(n,\mathbb{R})$ wormholes in a future publication. For the third class, we obtain $p$-brane solutions with a tachyon wave propagating in the world volume. In the decoupling limit of the corresponding M-branes and D3-brane, the metric becomes a direct product of a sphere and an AdS tachyon wave. We present the AdS tachyon wave for all dimensions $D \geq 4$. These solutions and the AdS wormholes provide interesting backgrounds for the AdS/CFT correspondence.

Acknowledgments

The research of HL and JM is supported in part by DOE grant DE-FG03-95ER40917. ZLW acknowledges support by grants from the Chinese Academy of Sciences, a grant from 973 Program with grant no 2007CB815401 and grants from the NSF of China with grant nos 10588503 and 10535060. We are grateful to Malcolm Perry for useful discussions.

Appendix A. Properties of the matrix $C$

In this appendix, we study the properties of the $SL(n,\mathbb{R})$ Lie-algebra-valued integration constant matrix $C$, defined by

$$\mathcal{M}^{-1}\mathcal{M} \equiv C.$$  \hspace{1cm} (A.1)

(See section 2 for detail.) We require all the elements of $C$ to be real. Under the $SL(n,\mathbb{R})$ transformation

$$d\vec{z} \rightarrow \Lambda^{-1} d\vec{z}, \quad \mathcal{M} \rightarrow \Lambda^T \mathcal{M} \Lambda,$$  \hspace{1cm} (A.2)

$C$ transforms as

$$C \rightarrow \Lambda^{-1} C \Lambda.$$  \hspace{1cm} (A.3)

We use the Borel transformation to fix the boundary condition for $\mathcal{M}$ so that

$$\mathcal{M}_\infty = \eta \equiv \text{diag}(-1,1,\ldots,1).$$  \hspace{1cm} (A.4)

It follows that with this boundary condition, $\eta C$ must be symmetric. The general $C$ is given by (5.1).

There is a residual $SO(1,n-1)$ symmetry of $SL(n,\mathbb{R})$ that preserves asymptotic $\mathcal{M}_\infty$. We can use the $SO(n-1)$ subgroup to simplify $C$ so that we have

$$C = \begin{pmatrix}
-a_0 & -\beta_1 & -\beta_2 & \cdots & -\beta_{n-1} \\
\beta_1 & a_1 & 0 & \cdots & 0 \\
\beta_2 & 0 & a_2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\beta_{n-1} & 0 & 0 & \cdots & a_{n-1}
\end{pmatrix},$$  \hspace{1cm} (A.5)
Here, $\alpha_0 = \sum_{i=1}^{n-1} \alpha_i$ so that the matrix is traceless. Note that $C$ can have real or complex eigenvalues. Since $C$ is real, complex eigenvalues should arise in pairs. The eigenvectors, satisfying

$$C_v u^{(a)v} = \lambda_v^{(a)} v^{(a)u}, \quad C_u v^{(a)u} = \lambda_v^{(a)} v^{(a)u},$$  \hspace{1cm} (A.6)

form a linear space. The inner product of the eigenvectors is defined by contracting indices with the Minkowski metric given in equation (A.4). It follows that any real eigenvector $\tilde{v}$, associated with a real eigenvalue, can be time-like, space-like or null, depending on whether $v^2 \equiv v_u v^u = -1$, $1$ or $0$.

We now present a set of properties of $C$, which will be useful for its classification.

(i) Using $\eta$ to raise or lower indices, we have $C_u = C_v$. It follows that for any eigenvector defined in equation (A.6), we have

$$v^{(a)} u^{(a)} C_v v^{(a)} = \lambda_v^{(a)} v^{(a)}.$$

(ii) Eigenvectors of different eigenvalues are orthogonal to each other. Proof. By using (i), we have

$$v^{(b)} u^{(b)} C_v v^{(b)} = \lambda_v^{(b)} v^{(b)}.$$

$$\Rightarrow v^{(b)} u^{(b)} = 0 \quad \text{if} \quad \lambda_v^{(a)} \neq \lambda_v^{(b)}.$$  \hspace{1cm} (A.8)

As a result, the eigenvector of any real eigenvalue is orthogonal to both the real and imaginary parts of the eigenvector of any complex eigenvalue.

(iii) All the time-like and linearly independent null vectors have a non-vanishing inner product with each other. Proof.

- For any two time-like vectors, one can always choose a normal orthogonal basis such that they are in the form

$$\tilde{v}^{(1)} = a[1, 0, \ldots, 0], \quad \tilde{v}^{(2)} = b[\cosh \delta, \sinh \delta, 0, \ldots, 0].$$  \hspace{1cm} (A.9)

- For any two linearly independent null vectors, we can always choose a normal orthogonal basis so that they are in the form

$$\tilde{v}^{(1)} = a[1, 1, 0, \ldots, 0], \quad \tilde{v}^{(2)} = b[1, \cos \delta, \sin \delta, 0, \ldots, 0].$$  \hspace{1cm} (A.10)

- For a time-like vector and a null vector, we can always choose a normal orthogonal basis such that they are in the form

$$\tilde{v}^{(1)} = a[1, 0, 0, \ldots], \quad \tilde{v}^{(2)} = b[1, 1, 0, \ldots].$$  \hspace{1cm} (A.11)

It is clear that $\tilde{v}^{(1)}$ and $\tilde{v}^{(2)}$ have a non-vanishing inner product in all the above three cases.

(iv) The matrix $C$ can have at most one pair of complex eigenvalues. Proof. Suppose there exist two pairs of different complex eigenvalues $\lambda^{(1)}, \lambda^{(2)*}$ and $\lambda^{(2)}, \lambda^{(2)*}$, and the corresponding eigenvectors $\tilde{x} \pm i \tilde{y}$ and $\tilde{u} \pm i \tilde{v}$. Because of (ii), one has

$$\tilde{x}^2 + \tilde{y}^2 = \tilde{u}^2 + \tilde{v}^2,$$

$$\tilde{x} \cdot \tilde{u} = \tilde{x} \cdot \tilde{v} = \tilde{y} \cdot \tilde{u} = \tilde{y} \cdot \tilde{v} = 0.$$  \hspace{1cm} (A.12)

Equation (A.12) implies that one in $\tilde{x}$, $\tilde{y}$ and one in $\tilde{u}$, $\tilde{v}$ must be either time-like or null. However, this is in contradiction with (iii) and equation (A.13). So there can be at most one pair of complex eigenvalues.
Any null subspace can only have one null direction.

**Proof.** Suppose that there are two linearly independent null vectors \( \vec{v}^{(1)} \) and \( \vec{v}^{(2)} \) respectively. Without loss of generality, we can choose

\[
\vec{v}^{(1)} = a(1, 1, 0, 0 \ldots), \quad \vec{v}^{(2)} = b(1, \cos \theta, \sin \theta, 0 \ldots);
\]

then \( \frac{\vec{v}^{(1)}}{a} + \frac{\vec{v}^{(2)}}{b} \) is time-like and thus the space is time-like. Therefore, the null subspace can have only one null direction.

We can now use these properties to classify the matrix \( C \).

**Class I.**

In this class, \( C \) has one and only pair of complex eigenvalues, as is allowed by (iv). In this case, we should not diagonalize \( C \) since we require \( C \) to be a real matrix.

From the proof of (iv), we see that the existence of a pair of eigenvalues means the existence of a time-like or null vector. Then from (ii) and (iii), all the remaining real eigenvalues of \( C \) must have space-like eigenvectors. These space-like eigenvectors can be used to partially diagonalize \( C \). So if the pair of complex eigenvalues is \( p \)-fold degenerate, we have

\[
C = \begin{pmatrix}
\lambda_2 & \cdots & \lambda_{2p-1} \\
\vdots & \ddots & \vdots \\
\alpha_1 - \beta_1 & \alpha_2 & \cdots & 0 \\
\beta_2 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\beta_{2p-1} & 0 & \cdots & \alpha_{2p-1}
\end{pmatrix},
\]

(A.15)

with

\[
N_{2p \times 2p} = \begin{pmatrix}
-\alpha_0 & -\beta_1 & -\beta_2 & \cdots & -\beta_{2p-1} \\
\beta_1 & \alpha_1 & 0 & \cdots & 0 \\
\beta_2 & 0 & \alpha_2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\beta_{2p-1} & 0 & 0 & \cdots & \alpha_{2p-1}
\end{pmatrix}.
\]

(A.16)

As we shall see presently, all \( \beta_i \) are non-vanishing, and

\[
\alpha_i \neq \alpha_j, \quad \text{if} \quad i \neq j \quad \text{for} \quad i, j = 1, \ldots, 2p - 1.
\]

(A.17)

Since \( N_{2p \times 2p} \) has a pair of \( p \)-fold degenerate complex eigenvalues,

\[
\text{Det}(N_{2p \times 2p} - \lambda I_{2p \times 2p}) = (a + ib - \lambda)^p(a - ib - \lambda)^p,
\]

\[
\Rightarrow \beta_i = \sqrt{(\alpha_i - \alpha_j)^2 + b^2} \prod_{j \neq i}(\alpha_i - \alpha_j), \quad i, j = 1, \ldots, 2p - 1,
\]

(A.18)

It is easy to see that the constant \( \beta_i \) cannot be all real for \( p \geq 2 \). However for \( p = 1 \), we have

\[
a = \frac{\alpha_1 - \alpha_0}{2}, \quad \beta_1 = \sqrt{(\alpha_1 - a)^2 + b^2}.
\]

(A.19)

So \( N_{2p \times 2p} \) exists only for \( p = 1 \). In this case, we can use \( O(1, 1) \) symmetry to set \( \alpha_0 = \alpha_1 \). Thus, \( C \) in this case can all be simplified by the \( SO(1, n - 1) \) symmetry to be

\[
C = \begin{pmatrix}
\alpha & -\beta \\
\beta & \alpha \\
\vdots & \ddots \\
\lambda_2 & \cdots & \lambda_{n-1}
\end{pmatrix}.
\]

(A.20)
Class II.

All the eigenvalues are real and $C$ has one time-like eigenvector $\vec{x}(0)$. In fact, as we can see from (iii), there can be no more than one time-like eigenvector in constructing an orthogonal basis. We build a normal orthogonal basis $\vec{x}(\mu)$ based on $\vec{x}(0)$:

$$x^\mu(\nu) = \eta(\mu)(\nu).$$  \hspace{1cm} (A.21)

Making an $SO(1, n - 1)$ transformation as

$$N^\mu_\mu = x^\mu(\mu),$$  \hspace{1cm} (A.22)

$C$ can be reduced to

$$C \rightarrow \begin{pmatrix} \lambda & 0 \\ 0 & C_{(n-1)\times(n-1)} \end{pmatrix}.$$  \hspace{1cm} (A.23)

The Euclidean subspace associated with $C_{(n-1)\times(n-1)}$ can be diagonalized by a further $SO(n - 1)$ transformation. This is consistent with the fact that there could be no more time-like eigenvectors orthogonal to $\vec{x}(0)$. Thus in this class, $C$ can be diagonalized by an $SO(1, n - 1)$ transformation, given by

$$C = \text{diag}(\lambda_0, \lambda_1, \ldots, \lambda_{n-1}).$$  \hspace{1cm} (A.24)

Class III.

In this class, all the eigenvalues are real, but there is no time-like eigenvector. It follows that all the eigenvectors must be space-like or null. We see from (i) and (ii) that all null eigenvectors must share the same eigenvalue and thus they belong to the same eigenspace. In addition, with (v), we conclude further that there is actually only one null eigenvector.

If there is a space-like eigenvector $\vec{x}(n-1)$, we can build a normal orthogonal basis $\vec{x}(\mu)$ based on $\vec{x}(n-1)$ and construct the corresponding $SO(1, n - 1)$ transformation $\Lambda$ as in class II, which transforms $C$ to

$$C \rightarrow \begin{pmatrix} C_{(n-1)\times(n-1)} & 0 \\ 0 & \lambda \end{pmatrix}.$$  \hspace{1cm} (A.25)

This procedure can be repeated until there is no more space-like eigenvector in the remaining off-diagonal subspace. This leads to

$$C = \begin{pmatrix} N_{p\times p} & \\ & \lambda_p \\ & & \cdots \\ & & & \lambda_{n-1} \end{pmatrix},$$  \hspace{1cm} (A.26)

where $N_{p\times p}$ is a degenerate matrix with only one $p$-fold eigenvalue $\lambda$ but only one eigenvector, which is null. Obviously, we must have $p > 1$; otherwise, the system would be reduced to class II.

By $SO(p - 1)$ transformations, we can take

$$N_{p\times p} = \begin{pmatrix} -\alpha_0 & -\beta_1 & -\beta_2 & \cdots & -\beta_{p-1} \\ \beta_1 & \alpha_1 & 0 & \cdots & 0 \\ \beta_2 & 0 & \alpha_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \beta_{p-1} & 0 & 0 & \cdots & \alpha_{p-1} \end{pmatrix}.$$  \hspace{1cm} (A.27)

As we see presently, all the constants are non-vanishing, and

$$\alpha_i \neq \alpha_j \quad \text{if} \quad i \neq j \quad \text{for} \quad i, j = 1, \ldots, p - 1.$$  \hspace{1cm} (A.28)

\[\text{...}\]
Furthermore, since $N_{p \times p}$ has a $p$-fold degenerate eigenvalue, it follows that

$$\det(N_{p \times p} - \lambda I_{p \times p}) = (\lambda_0 - \lambda)^p$$

$$\implies \beta_i = \frac{(\alpha_i - \lambda_0)^p}{\prod_{j \neq i}(\alpha_i - \alpha_j)}, \quad i, j = 1, \ldots, p - 1,$$

$$\lambda_0 = \frac{1}{p} \text{Tr}(N_{p \times p}).$$

(A.29)

It can be easily seen that $\beta_i$ cannot be all real for $p \geq 4$. So $N_{p \times p}$ exists only for $p = 2$ and $p = 3$. Indeed, both cases have one null eigenvector.

Let us first look at the case with $p = 2$. We have

$$\lambda_0 = \frac{\alpha_1 - \alpha_0}{2}, \quad \beta_1 = |\alpha_1 - \lambda_0| = \frac{|\alpha_0 + \alpha_1|}{2}.$$  

For $n = 2$, the traceless condition implies that

$$C_2 = \alpha \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}.$$  

(A.31)

For $n \geq 3$, we have

$$C = \begin{pmatrix} C_2 + \lambda_0 \mathbb{I} \\ \lambda_2 \\ \vdots \\ \lambda_{n-1} \end{pmatrix}.$$  

(A.32)

Now let us consider $p = 3$. We have

$$\lambda_0 = \frac{1}{3} (\alpha_1 + \alpha_2 - \alpha_0), \quad \beta_1 = \sqrt{\frac{(\alpha_1 - \lambda_0)^3}{\alpha_1 - \alpha_2}}, \quad \beta_2 = \sqrt{\frac{(\alpha_2 - \lambda_0)^3}{\alpha_2 - \alpha_1}}.$$  

(A.33)

For $n = 3$, the traceless condition implies that $\lambda_0 = 0$. We can parameterize $\alpha_1 = \alpha \sin^2 \frac{1}{2} \beta$ and $\alpha_2 = -\alpha \cos^2 \frac{1}{2} \beta$. Thus, we have

$$C_3 = \alpha \begin{pmatrix} \cos \beta & -\sin^3 \frac{1}{2} \beta & \cos^3 \frac{1}{2} \beta \\ \sin^3 \frac{1}{2} \beta & \sin^2 \frac{1}{2} \beta & 0 \\ \cos^3 \beta & 0 & -\cos^2 \frac{1}{2} \beta \end{pmatrix}.$$  

(A.34)

$C_3$ given in sections 4 and 5 is related to this by an $SO(2)$ transformation. For $n \geq 4$, we thus have

$$C = \begin{pmatrix} C_3 + \lambda_0 \mathbb{I} \\ \lambda_3 \\ \vdots \\ \lambda_{n-1} \end{pmatrix}.$$  

(A.35)

Note that the matrices $C_p$ have rank $p - 1$ for $p = 2, 3$. In both cases, all eigenvalues are zero, with just one eigenvector, which is null. Also, the proportion factor $\alpha$ in $C_2$ and $C_3$ can be set to 1 by a further boost. We keep it since it is related to the momentum charge for the corresponding waves.

Finally it is worth pointing out that the above classification also applies for the $GL(n, \mathbb{R})$ system, in which case, the traceless condition for $C$ is relaxed.
Appendix B. Killing spinor analysis for the tachyon wave

The tachyon wave solution in $D \geq 4$ dimensions we obtained in section 4 can be rewritten as
\[ ds_D^2 = -du dv + dw^2 - \left( \alpha \rho \cos \beta - \frac{1}{8} \alpha^2 \rho^2 \sin^2 \beta \right) dv^2 - \alpha \rho \sin \beta dv dw + \sum_{i=1}^{D-3} dx^i dx^i, \]  
(B.1)

where $\rho$ is given by
\[ \rho = \begin{cases} r^{5-D}, & \text{for } D \geq 6, \\ \log r, & \text{for } D = 5, \\ r, & \text{for } D = 4, \end{cases} \]  
(B.2)

with $r \equiv \sqrt{x^ix^i}$ being the radial coordinate of the transverse space $dx^i dx^i$. We choose the following vielbein basis:
\[ e^+ = \frac{1}{2} dv, \quad e^- = du + \left( \alpha \rho \cos \beta + \frac{1}{8} \alpha^2 \rho^2 \sin \beta \right) dv, \]
\[ e^w = dw - \frac{1}{2} \alpha \rho \sin \beta dv, \quad e^i = dx^i; \]  
(B.3)

then we have
\[ ds^2 = -2e^+ e^- + (e^w)^2 + e^i e^i. \]  
(B.4)

Note that we use a hat to denote tangent indices. The non-vanishing spin connections are given by
\[ \omega^w_{\hat{+}} = -\frac{1}{2} (D-5) \alpha \sin \beta \frac{x_i}{r^{D-3}} dx^i, \]
\[ \omega^i_{\hat{+}} = -(D-5) \alpha x_i \frac{x_j}{r^{D-3}} \left( \cos \beta dv + \frac{1}{2} \sin \beta dw \right), \]  
(B.5)

\[ \omega^i_{\hat{0}} = -\frac{1}{4} (D-5) \alpha \sin \beta \frac{x_i}{r^{D-3}} dv. \]

Note that for $D = 5$, we will take $(D-5)\alpha \equiv \tilde{\alpha}$ to be non-vanishing. Substituting these into the Killing spinor equation
\[ \delta \psi_M = D_M \epsilon = (\partial_M + \frac{1}{4} \omega_A^{AB} \Gamma_{AB}) \epsilon = 0, \]  
(B.6)

we have
\[ \left( \partial_\hat{+} - \frac{1}{4} (D-5) \alpha \sin \beta \frac{x_i}{r^{D-3}} \Gamma^w \Gamma^i \right) \epsilon = 0, \]  
(B.7)
\[ \left( \partial_\hat{+} - \frac{1}{2} (D-5) \alpha x_i \frac{x_j}{r^{D-3}} \left( \cos \beta \Gamma^i \Gamma^j + \frac{1}{4} \sin \beta \Gamma^i \Gamma^j \right) \right) \epsilon = 0, \]  
(B.8)
\[ \left( \partial_\hat{0} - \frac{1}{4} (D-5) \alpha \sin \beta \frac{x_i}{r^{D-3}} \Gamma^i \Gamma^w \right) \epsilon = 0, \]  
(B.9)

\[ \partial_\hat{=} \epsilon = 0. \]  
(B.10)

In $D = 4$, for a generic $\beta$ value, the solution is $\epsilon = \exp \left( -\frac{1}{2} \alpha \rho \sin \beta \Gamma^i \Gamma^w \right) \epsilon_0$, where $\epsilon_0$ is a constant spinor with $\Gamma^i \epsilon_0 = 0$, and hence the solution preserves half of the supersymmetry. When $\beta = 0$, the metric is flat and hence it preserves full supersymmetry. Indeed, the Killing spinor is given by $\epsilon = (1 - \frac{1}{2} \alpha v \Gamma^i \Gamma^w) \epsilon_0$, where $\epsilon_0$ is an arbitrary constant spinor.
For $D \geq 5$, the $\beta = 0$ case gives rise to the vacuum pp-wave solution. The corresponding Killing spinors are constant spinors $\epsilon_0$ subject to $\Gamma^i \epsilon_0 = 0$. For generic $\beta \neq 0$, equations (B.7) and (B.10) can be easily solved, given

$$
\epsilon = \exp \left[ -\frac{1}{4} \alpha \sin \beta \frac{1}{r^{D-3}} \Gamma^\phi \Gamma^i \right] \tilde{\epsilon}(v, w) = \left( 1 - \frac{1}{4} \alpha \sin \beta \frac{1}{r^{D-3}} \Gamma^\phi \Gamma^i \right) \tilde{\epsilon}(v, w).
$$

(B.11)

Substituting this into (B.8), we have

$$
\left( \partial_v - \frac{1}{4} (D - 5) \alpha \sin \frac{x_i}{r^{D-3}} \Gamma^i \Gamma^\phi \right) \tilde{\epsilon}(v, w) = 0.
$$

(B.12)

Since this equation holds for arbitrary values of $x_i$, it follows that $\Gamma^i \tilde{\epsilon}(v, w) = 0$ and $\tilde{\epsilon}(v, w) = \tilde{\epsilon}(v)$. Equation (B.9) then becomes

$$
\left( \partial_v - \frac{1}{8} \alpha \sin \frac{x_i}{r^{D-3}} \Gamma^i \Gamma^\phi \right) \tilde{\epsilon}(v) = 0.
$$

(B.13)

It is clear that this equation has no solution for non-vanishing $\beta$.

It is also instructive to examine the integrability condition

$$
R_{vwab} \Gamma^{ab} \epsilon = 0.
$$

(B.14)

The non-zero components of the curvature 2-form are

$$
\Theta_{\hat{u} \hat{i}} = -\frac{(D - 5)^2 \alpha^2 \sin^2 \beta}{8 \alpha^2 (D-4)} dv \wedge dw,
$$

$$
\Theta_{\hat{t} \hat{i}} = (D - 5) \alpha \cos \beta \frac{x_i}{r^{D-3}} dv \wedge d \left( \frac{x_i}{r^{D-3}} \right) + \frac{1}{4} (D - 5) \alpha \sin \beta \frac{X_i X_j}{r^{2(D-3)}} dv \wedge dx_j,
$$

$$
\Theta_{\hat{t} \hat{w}} = \frac{1}{4} (D - 5) \alpha \sin \beta \wedge d \left( \frac{x_i}{r^{D-3}} \right).
$$

(B.15)

Thus for non-vanishing $\beta$, we have

$$
0 = R_{vwab} \Gamma^{ab} \epsilon = -\frac{(D - 5)^2 \alpha^2 \sin^2 \beta}{8 \alpha^2 (D-4)} \Gamma^\phi \Gamma^i \epsilon = \Gamma^\phi \epsilon = 0.
$$

(B.16)

The remaining conditions become

$$
R_{vwab} \Gamma^{ab} \epsilon = \frac{(D - 5) \alpha \sin \beta}{r^{D-1}} \left( \alpha^2 \Gamma^i \Gamma^\phi \Gamma^i - (D - 3) x_i x_j \Gamma^i \Gamma^\phi \right) \epsilon = 0.
$$

(B.17)

Contracting the equation with $x^i$, we have

$$
(D - 4) x_i \Gamma^i \Gamma^\phi \epsilon = 0.
$$

(B.18)

It has no non-trivial solution except in four dimensions.

Thus we have demonstrated that the general tachyon wave solutions (B.1), with $\beta \neq 0$, admit no Killing spinors except in $D = 4$, in which case, it preserves half of the supersymmetry.

References

[1] Lü H and Mei J 2008 Ricci-flat and charged wormholes in five dimensions Phys. Lett. B 666 511 (arXiv: 0806.3111 [hep-th])

[2] Chodos A and Detweiler S 1982 Spherically symmetric solutions in five-dimensional general relativity Gen. Rel. Grav. 14 879

[3] Cremmer E, Lavrenenko I V, Lü H, Pope C N, Stelle K S and Tran T A 1998 Euclidean-signature supergravities, dualities and instantons Nucl. Phys. B 534 40 (arXiv:hep-th/9803259)
[4] Stelle K S 1998 BPS branes in supergravity arXiv:hep-th/9803116
[5] Hull C M and Julia B 1998 Duality and moduli spaces for time-like reductions Nucl. Phys. B 534 250 (arXiv: hep-th/9803239)
[6] Galloway G J, Schleich K, Witt D and Woolgar E 2001 The AdS/CFT correspondence conjecture and topological censorship Phys. Lett. B 505 255 (arXiv:hep-th/9912119)
[7] Maldacena J M and Maoz L 2004 Wormholes in AdS J. High Energy Phys. JHEP02(2004)053 (arXiv: hep-th/0401024)
[8] Bergshoeff E, Collinucci A, Ploegh A, Vandoren S and Riet T Van 2006 Non-extremal D-instantons and the AdS/CFT correspondence J. High Energy Phys. JHEP01(2006)061 (arXiv:hep-th/0510048)
[9] Arkani-Hamed N, Orgsra J and Polchinski J 2007 Euclidean wormholes in string theory J. High Energy Phys. JHEP12/2007/018 (arXiv:0705.2768 [hep-th])
[10] Bergman A and Distler J 2007 Wormholes in maximal supergravity arXiv:0707.3168 [hep-th]
[11] Bergshoeff E, Chemissamy W, Ploegh A, Trigiante M and Riet T Van 2009 Generating geodesic flows and supergravity solutions Nucl. Phys. B 812 343–401 (arXiv:0806.2310 [hep-th])
[12] Bergman A, Lü H, Mei J and Pope C N 2009 AdS wormholes Nucl. Phys. B 810 300–15 (arXiv:0808.2481 [hep-th])
[13] Cremmer E, Julia B, Lü H and Pope C N 1998 Dualisation of dualities: I Nucl. Phys. B 523 73 (arXiv: hep-th/9710119)
[14] Cremmer E, Julia B, Lü H and Pope C N 1998 Dualisation of dualities: II. Twisted self-duality of doubled fields and superdualities Nucl. Phys. B 535 242 (arXiv:hep-th/9806106)
[15] Lü H, Pope C N, Sezgin E and Stelle K S 1995 Stainless super p-branes Nucl. Phys. B 456 669 (arXiv: hep-th/9508042)
[16] Lü H and Pope C N 1996 P-brane solitons in maximal supergravities Nucl. Phys. B 465 127 (arXiv: hep-th/9512012)
[17] Lü H, Pope C N and Stelle K S 1996 Vertical versus diagonal dimensional reduction for p-branes Nucl. Phys. B 481 313 (arXiv:hep-th/9605082)
[18] Azreg-Aïnou M and Clement G 1990 The geodesics of the Kaluza–Klein wormhole soliton Gen. Rel. Grav. 22 1119
[19] Azreg-Aïnou M, Clement G, Constantinidis C P and Fabris J C 2000 Electrostatic solutions in Kaluza–Klein theory: geometry and stability Grav. Cosmol. 6 207 (arXiv:gr-qc/9911107)