THE LAPLACE TRANSFORM METHOD
FOR LINEAR DIFFERENTIAL EQUATIONS
OF THE FRACTIONAL ORDER

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Preface

Differential equations of the fractional order appear more and more frequently in different research areas and engineering applications. An effective and easy-to-use method for solving such equations is needed.

However, known methods have certain disadvantages. Methods, described in details in [1, 2, 3] for fractional differential equations of the rational order, do not work in the case of an arbitrary real order. On the other hand, there is an iteration method described in [5], which allows solution of fractional differential equations of an arbitrary real order, but it works effectively only for relatively simple equations, as well as the series method [1, 17]. Other authors (e.g. [3, 4]) used in their investigations the one-parameter Mittag-Leffler function

\[ E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)} \]

Still other authors [5, 6] prefer the Fox H-function [8], which seems to be too general to be frequently used in applications.

Instead of this variety of different methods, we introduce a method which is free of the mentioned disadvantages and suitable for a wide class of initial value problems for fractional differential equations. The method uses the Laplace transform technique and is based on the formula of the Laplace transform of the Mittag-Leffler function in two parameters \( E_{\alpha,\beta}(z) \). We hope that the described method could be useful for obtaining solutions of different applied problems, appearing in physics, chemistry, electrochemistry, engineering, financing and banking, etc. To outline the area of the method’s applicability, we have included in the bibliography also the works by different authors [30]–[54], in which fractional linear differential equations appears or which could serve as a basis for obtaining such equations.

This work deals with solution of the fractional linear differential equations with constant coefficients and consists of four short chapters.

In Chapter 1 we present some auxiliary tools which are necessary for using the method. The reader can find the definition there and some important properties of the Mittag-Leffler function in two parameters and the Wright function. The basic result, presented in Chapter 1, is the Laplace transform of the Mittag-Leffler function and its derivatives. Besides that, we introduce two tools necessary for testing candidate solutions by direct substitution in corresponding equations: fractional derivatives of the Mittag-Leffler function and a rule for the fractional differentiation of integrals depending on a parameter.

In Chapter 2 we give solutions to some initial-value problems for ”standard” fractional differential equations. Some of them were solved by other authors earlier by other methods, and the comparison in such cases just underline the
simplicity and the power of our approach.

In Chapter 3 we extend the proposed method for the case of so-called "sequential" fractional differential equations (we adopted the convenient terminology of Miller and Ross). For this purpose, we obtained the Laplace transform for the "sequential" fractional derivative. The "sequential" analogues of the problems, solved in Chapter 2, are considered. Naturally, we arrive at solutions which are different from those obtained in Chapter 2.

However, there is something common in solutions of the corresponding "standard" and "sequential" fractional differential equations: they both have the same fractional Green’s function. In Chapter 4 we give our definition of the fractional Green’s function and some of its properties, necessary for constructing solutions of initial-value problems for fractional linear differential equations with constant coefficients.

We give the solution of the initial-value problem for the ordinary fractional linear differential equation with constant coefficients using only its Green’s function. Due to this result, the solution of such initial value problems reduces to finding the fractional Green’s functions. We obtained the explicit expressions for the fractional Green’s function for some special cases (one-, two-, three- and four-term equations).

The explicit expression for an arbitrary fractional linear ordinary differential equation with constant coefficients ends this work.

In what follows below, \( aD_t^\alpha \) means the Riemann-Liouville fractional derivative of order \( \alpha \) (e.g., [1, 2]):

\[
aD_t^\alpha f(t) = \frac{1}{\Gamma(n - \alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t - \tau)^{n-\alpha+1}} d\tau, \quad (n - 1 < \alpha < n)
\]

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Chapter 1

Introduction

1.1 Mittag-Leffler function in two parameters

The function, which plays a very important role in this work, was in fact introduced by Agarwal [23]. A number of relationships for this function were obtained by Agarwal and Humbert [24] using the Laplace transform technique. This function could have been called the Agarwal function. However, Agarwal and Humbert generously left the same notation as for the one-parameter Mittag-Leffler function, and that was the reason that now the two-parameter function is called the Mittag-Leffler function. We will use the name and the notation used in the fundamental handbook on special functions [10]. In spite of using the same notation as Agarwal, the definition given there differs from Agarwal’s definition by a non-constant factor.

**Definition.** A two-parameter function of the Mittag-Leffler type is defined by the series expansion [10]

\[ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad (\alpha > 0, \beta > 0) \] (1.1)

**Relation to other functions.** There are some relationships given in [10, §18.1]:

\[ E_{1,1}(z) = e^z, \quad E_{1,2}(z) = \frac{e^z - 1}{z}, \] (1.2)

\[ E_{2,1}(z) = \cosh(\sqrt{z}), \quad E_{2,2}(z) = \frac{\sinh(\sqrt{z})}{\sqrt{z}}, \] (1.3)

\[ E_{1/2,1}(\sqrt{z}) = \frac{2}{\sqrt{\pi}} e^{-z} \text{erfc}(-\sqrt{z}). \] (1.4)

For \( \beta = 1 \) we obtain the Mittag-Leffler function in one parameter:

\[ E_{\alpha,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)} \equiv E_{\alpha}(z). \] (1.5)
The function $\mathcal{E}_t(\nu, a)$, introduced in [2] for solving differential equations of the rational order, is the particular case of the Mittag-Leffler function (1.1):

$$
\mathcal{E}_t(\nu, a) = t^\nu \sum_{k=0}^{\infty} \frac{(at)^k}{\Gamma(\nu + k + 1)} = t^\nu E_{1, \nu+1}(at) .
$$

(1.6)

Rabotnov’s [11] function $\mathcal{E}_\alpha(\beta, t)$ is the particular case of the Mittag-Leffler function (1.1) too:

$$
\mathcal{E}_\alpha(\beta, t) = t^\alpha \sum_{k=0}^{\infty} \frac{\beta^k t^{k(\alpha+1)}}{\Gamma((k+1)(1+\alpha))} = t^\alpha E_{\alpha+1, \alpha+1}(\beta t^{\alpha+1}).
$$

(1.7)

It follows from the relationships (1.6) and (1.7) that the properties of the Miller-Ross function and Rabotnov’s function follows from the properties of the Mittag-Leffler function in two parameters (1.1).

Plotnikov [25, cf. [26]] and Tseytlin [26] used in their investigations two functions $\mathcal{E}_\alpha(z)$ and $\mathcal{E}_\alpha(z)$, which they call the fractional sine and cosine. Those functions are also just the particular cases of the Mittag-Leffler function in two parameters:

$$
\mathcal{E}_\alpha(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{(2-\alpha)n+1}}{\Gamma((2-\alpha)n+2)} = z E_{2-\alpha, 2}(-z^{2-\alpha})
$$

(1.8)

$$
\mathcal{E}_\alpha(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{(2-\alpha)n}}{\Gamma((2-\alpha)n+1)} = z E_{2-\alpha, 2}(-z^{2-\alpha})
$$

(1.9)

Of course, properties of the fractional sine and cosine follow from the properties of the Mittag-Leffler function (1.1).

1.2 The Laplace transform of the Mittag-Leffler function in two parameters

As follows from relationship (1.2), the Mittag-Leffler function $E_{\alpha, \beta}(z)$ is a generalization of the exponential function $e^z$ and the exponential function is a particular case of the Mittag-Leffler function.

We will outline here the way to obtain the Laplace transform of the Mittag-Leffler function with the help of the analogy between this function and the function $e^z$. For this purpose, let us obtain the Laplace transform of the function $t^k e^{\alpha t}$ in an untraditional way.

First, let us prove that

$$
\int_0^\infty e^{-t} e^{\pm z t} dt = \frac{1}{1 \mp z}, \quad |z| < 1.
$$

(1.10)

Indeed, using the series expansion for $e^z$, we obtain

$$
\int_0^\infty e^{-t} e^{zt} dt = \frac{1}{1 - z} = \sum_{k=0}^{\infty} \frac{(\pm z)^k}{k!} \int_0^\infty e^{-t} t^k dt = \sum_{k=0}^{\infty} (\pm z)^k = \frac{1}{1 \mp z}.
$$

(1.11)
Second, we differentiate both sides of equation (1.10) with respect to $z$. The result is
\[ \int_0^\infty e^{-t} t^k e^{zt} dt = \frac{k!}{(1-z)^{k+1}}, \quad (|z| < 1), \quad (1.12) \]
and after obvious substitutions we obtain the well-known pair of the Laplace transform of the function $t^k e^{±at}$:
\[ \int_0^\infty e^{-pt} t^k e^{±at} dt = \frac{k!}{(p ± a)^{k+1}}, \quad (Re(p) > |a|). \quad (1.13) \]

Let us now consider the Mittag-Leffler function (1.1). Substitution of (1.1) in the integral below leads to
\[ \int_0^\infty e^{-t} t^{β-1} E_{α,β}(zt^α) dt = \frac{1}{1-z}, \quad (|z| < 1), \quad (1.14) \]
and we obtain from (1.14) a pair of the Laplace transforms of the function $t^{αk+β-1} E_{α,β}(±zt^α)$, $(E_{α,β}(y) = \frac{d}{dy} E_{α,β}(y))$:
\[ \int_0^\infty e^{-pt} t^{αk+β-1} E_{α,β}(±at^α) dt = \frac{k!}{(p^α ± a)^{k+1}}, \quad (Re(p) > |a|^{1/α}). \quad (1.15) \]
The particular case of (1.15) for $α = β = \frac{1}{2}$
\[ \int_0^\infty e^{-pt} t^{k-1} E_{1/2,1/2}(±a\sqrt{t}) dt = \frac{k!}{(\sqrt{p} ± a)^{k+1}}, \quad (Re(p) > a^2). \quad (1.16) \]
is useful for solving semidifferential equations considered in [1, 2].

### 1.3 The Wright function

This function, related to the Mittag-Leffler function in two parameters $E_{α,β}(z)$, was introduced by Wright [29, cf. [10, 24]]. A number of useful relationships was obtained by Humbert and Agarwal [24] with the help of the Laplace transform.

For convenience we adopt here Mainardi’s notation for the Wright function $W(z; α, β)$.

**Definition.** The Wright function is defined as [10, formula 18.1(27)]
\[ W(z; α, β) = \sum_{k=0}^\infty \frac{z^k}{k! \Gamma(αk + β)} \quad (1.17) \]

**Integral representation.** This function can be represented by the following integral [10, formula 18.1(29)]:
\[ W(z; α, β) = \frac{1}{2\pi i} \int_{Ha} u^{-β} e^{u±zu^{-α}} du \quad (1.18) \]
where $(Ha)$ denotes Hankel’s contour.
To prove (1.18), let us write the integrated function in the form of a power series in \( z \) and perform term-by-term integration using the well-known formula (1.16(2)):

\[
\frac{1}{\Gamma(z)} = \int_{H_a} e^{u} u^{-z} du.
\]

**Relation to other functions.** It follows from the definition (1.17) that

\[
W(z; 0, 1) = e^z
\]

(1.19)

\[
\left(\frac{z}{2}\right)^{\nu} W\left(\mp \frac{z^2}{4}; 1, \nu + 1\right) = \begin{cases} J_\nu(z) \\ I_\nu(z) \end{cases}
\]

(1.20)

Taking \( \beta = 1 - \alpha \), we obtain Mainardi’s function \( M(z; \alpha) \):

\[
W(-z; -\alpha, 1 - \alpha) = M(z; \alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{k! \Gamma(-\alpha(k+1) + 1)}
\]

(1.21)

The following particular case of the Wright function was given by Mainardi [15]:

\[
W(-z; -1/2, -1/2) = M(z; 1/2) = \frac{1}{\sqrt{\pi}} \exp\left(-\frac{z^2}{4}\right).
\]

(1.22)

We see that the Wright function is a generalization of the exponential function and the Bessel functions. For \( \alpha > 0 \) and \( \beta > 0 \) it is an entire function in \( z \).

Recently Mainardi [15] pointed out that \( W(z; \alpha, \beta) \) is an entire function in \( z \) also for \( -1 < \alpha < 0 \).

Let us prove this statement. Using the well-known relationship

\[
\Gamma(y)\Gamma(1-y) = \frac{\pi}{\sin(\pi y)},
\]

we can write the Wright function in the form

\[
W(z; \alpha, \beta) = \frac{1}{\pi} \sum_{k=0}^{\infty} z^k \frac{\Gamma(1-\alpha k - \beta) \sin \pi (\alpha k + \beta)}{k!}
\]

(1.24)

Let us introduce an auxiliary majorizing series

\[
S = \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{\Gamma(1-\alpha k - \beta)}{k!} |z|^k
\]

(1.25)

The convergence radius of series (1.25) for \( -1 < \alpha < 0 \) is infinite:

\[
R = \lim_{k \to \infty} \frac{\Gamma(1-\alpha k - \beta)}{k! \Gamma(1-\alpha k - \alpha - \beta)} = \lim_{k \to \infty} \frac{k + 1}{\alpha^k k!} = \infty.
\]

(1.26)

(We use here relationship (3, formula 1.18(4)).)

It follows the comparison of the series (1.17) and (1.23) that for \( \alpha > -1 \) and arbitrary \( \beta \) the convergence radius of the series representation of the Wright function \( W(z; \alpha, \beta) \) is infinite, and the Wright function is an entire function.

The Wright function plays an important role in the solution of linear partial fractional differential equations, e.g. fractional diffusion–wave equation.
1.4 Tools for testing candidate solutions

The following tools are necessary for checking by substitution if candidate solutions, obtained by any more or less formal method, satisfy corresponding equations and initial conditions.

**Fractional derivatives of the Mittag-Leffler function.**

By fractional-order differentiation \( 0^{D_{t}^\gamma} \) (\( \gamma \) is an arbitrary real number) of series representation (1.1) we obtain

\[
0^{D_{t}^\gamma} (t^{\alpha k + \beta - 1} E_{\alpha,\beta}^{(k)}(\lambda t^\alpha)) = t^{\alpha k + \beta - \gamma - 1} E_{\alpha,\beta - \gamma}^{(k)}(\lambda t^\alpha) \quad (1.27)
\]

The particular case of relationship (1.27) for \( k = 0 \) and integer \( \gamma \) is given in [10], equation 18.1(25).

**A rule for fractional differentiation of an integral depending on a parameter**, when the upper limit also depends on the parameter:

\[
0^{D_{t}^\alpha} \int_{0}^{t} K(t, \tau) d\tau = \int_{0}^{t} 0^{D_{\tau}^\alpha} K(t, \tau) d\tau + \lim_{\tau \to t-0} \tau^{\alpha - 1} K(t, \tau), \quad (0 < \alpha < 1). \quad (1.28)
\]

The proof is straightforward:

\[
0^{D_{x}^\alpha} \int_{0}^{x} K(x, \xi) d\xi = \frac{1}{\Gamma(1 - \alpha)} \frac{d}{dx} \int_{0}^{x} \frac{dt}{(x - t)^\alpha} \int_{0}^{t} K(t, \xi) dt \\
= \frac{1}{\Gamma(1 - \alpha)} \frac{d}{dx} \int_{0}^{x} d\xi \int_{\xi}^{x} K(t, \xi) dt \\
= \frac{d}{dx} \int_{0}^{x} \tilde{K}(x, \xi) d\xi = \int_{0}^{x} \frac{d}{d\xi} \tilde{K}(x, \xi) d\xi + \lim_{\xi \to x-0} \tilde{K}(x, \xi) \\
= \int_{0}^{x} \xi^{\alpha - 1} K(x, \xi) d\xi + \lim_{\xi \to x-0} \xi^{\alpha - 1} K(x, \xi),
\]

where

\[
\tilde{K}(x, \xi) = \frac{1}{\Gamma(1 - \alpha)} \int_{\xi}^{x} \frac{dt}{(x - t)^\alpha}.
\]

The following important particular case must be mentioned. If we have \( K(t-\tau)f(\xi) \) instead of \( K(t, \tau) \), then relationship (1.28) takes the form:

\[
0^{D_{t}^\alpha} \int_{0}^{t} K(t-\tau)f(\tau) d\tau = \int_{0}^{t} 0^{D_{\tau}^\alpha} K(\tau)f(t-\tau) d\tau + \lim_{\tau \to +0} f(t-\tau) 0^{D_{\tau}^\alpha - 1} K(\tau)). \quad (1.29)
\]
Chapter 2

Standard fractional differential equations

The following examples illustrate the use of (1.15) for solving fractional-order differential equations with constant coefficients. In this chapter we use the classic formula for the Laplace transform of the fractional derivative, as given, e.g., in [1], p.134 or [2], p.123:

\[
\int_0^\infty e^{-pt} \, 0D_t^\alpha f(t) \, dt = p^\alpha F(p) - \sum_{k=0}^{n-1} p^k \, 0D_t^{\alpha-k-1} f(t) \bigg|_{t=0}, \quad (n - 1 < \alpha \leq n).
\]  

(2.1)

2.1 Ordinary linear fractional differential equations

In this section we give some examples of solution of ordinary linear differential equations of the fractional order.

Example 1. A slight generalization of an equation solved in ([1], p.157):

\[0D_t^{1/2} f(t) + af(t) = 0, \quad (t > 0); \quad 0D_t^{-1/2} f(t) \bigg|_{t=0} = C \quad (2.2)\]

Applying the Laplace transform we obtain

\[F(p) = \frac{C}{p^{1/2} + a}, \quad C = 0D_t^{-1/2} f(t) \bigg|_{t=0}\]

and the inverse transform with a help of (1.10) gives the solution of (2.2):

\[f(t) = Ct^{-1/2} E_{1, 1/2}(-a \sqrt{t}).\]  

(2.3)

Using series expansion (1.1) of \(E_{\alpha, \beta}(t)\), it is easy to check that for \(a = 1\) solution (2.3) is identical to the solution \(f(t) = C(\frac{1}{\sqrt{\pi t}} - e^t \text{erfc}(\sqrt{t}))\), obtained in [1] in a more complex way.

Example 2. Let us consider the following equation:

\[0D_t^Q f(t) + 0D_t^q f(t) = h(t), \quad (2.4)\]
which "encounters very great difficulties except when the difference \( q - Q \) is integer or half-integer" (\([1]\), p.156).

Suppose that \( 0 < q < Q < 1 \). Laplace transform of (2.4) leads to
\[
(p^Q + p^q)F(p) = C + H(p), \quad \text{where} \quad C = \left( 0D_t^{q-1}f(t) + 0D_t^{Q-1} \right)_{t=0},
\]
and
\[
F(p) = \frac{C + H(p)}{p^Q + p^q} = \frac{C + H(p)}{p^q(p^{Q-q} + 1)} = \left( C + H(p) \right) \frac{p^{-q}}{p^{Q-q} + 1}.
\]

After inversion with a help of (1.15) for \( \alpha = Q - q \) and \( \beta = Q \), we obtain the solution:
\[
f(t) = C G(t) + \int_0^t G(t - \tau)h(\tau)d\tau,
\]
where
\[
C = \left( 0D_t^{Q-1}f(t) + 0D_t^{Q-1}f(t) \right)_{t=0}, \quad G(t) = t^{Q-1}E_{Q-Q}(t^{Q-1}).
\]

The case \( 0 < q < Q < n \) (for example, equation obtained in \([12]\)) can be solved similarly.

**Example 3.** Let us consider the following initial value problem for a non-homogeneous fractional differential equation under non-zero initial conditions:
\[
0D_t^\alpha y(t) - \lambda y(t) = h(t), \quad (t > 0); \quad 0D_t^{\alpha-k}y(t) \bigg|_{t=0} = b_k, \quad (k = 1, 2, \ldots, n)
\]
where \( n - 1 < \alpha < n \). Problem (2.8) was analytically solved in \([3]\) by the iteration method. With the help of the Laplace transform and formula (1.15) we obtain the same solution directly and more quickly.

Indeed, taking into account initial conditions (2.8), the Laplace transform of equation (2.8) yields
\[
p^{\alpha}Y(p) - \lambda Y(p) = H(p) + \sum_{k=1}^{n} b_k p^{k-1},
\]
and the inverse transform using (1.15) gives the solution:
\[
y(t) = \sum_{k=1}^{n} b_k t^{\alpha-k}E_{\alpha,\alpha-k+1}(\lambda t^\alpha) + \int_0^t (t - \tau)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t - \tau)^\alpha)h(\tau)d\tau.
\]
2.2 Partial linear fractional differential equations

The proposed approach can be successfully used for solving partial linear differential equations of the fractional order.

Example 4. Nigmatullin's fractional diffusion equation.

Let us consider the following initial boundary value problem for the fractional diffusion equation in one space dimension:

\[ 0D_t^\alpha u(x, t) = \lambda^2 \frac{\partial^2 u(x, t)}{\partial x^2}, \quad (t > 0, \ -\infty < x < \infty); \quad (2.11) \]

\[ \lim_{x \to \pm \infty} u(x, t) = 0; \quad 0D_t^{\alpha-1} u(x, t) \bigg|_{t=0} = \varphi(x). \quad (2.12) \]

We assume here \( 0 < \alpha < 1 \). Equation of the type (2.11) was deduced by Nigmatullin \[13\] and by Westerlund \[14\] and studied by Mainardi \[15\]. We will give a simple solution of problem (2.11) demonstrating once again the advantage of using the Mittag-Leffler function in two parameters (1.1).

Taking into account boundary conditions (2.12), the Fourier transform with respect to variable \( x \) gives:

\[ 0D_t^\alpha \overline{u}(\beta, t) + \lambda^2 \beta^2 \overline{u}(\beta, t) = 0 \quad (2.13) \]

\[ 0D_t^{\alpha-1} \overline{u}(x, t) \bigg|_{t=0} = \overline{\varphi}(\beta), \quad (2.14) \]

where \( \beta \) is the Fourier transform parameter. Applying Laplace transform to (2.13) and using initial condition (2.14) we obtain

\[ \overline{U}(\beta, p) = \frac{\varphi(\beta)}{p^{\alpha} + \lambda^2 \beta^2}. \quad (2.15) \]

Inverse Laplace transform of (2.15) using (1.15) gives

\[ \overline{u}(\beta, t) = \varphi(\beta) t^{\alpha-1} E_{\alpha, \alpha}(-\lambda^2 \beta^2 t^\alpha), \quad (2.16) \]

and inverse Fourier transform produces the solution of the problem (2.11)-(2.12):

\[ u(x, t) = \int_{-\infty}^{\infty} G(x - \xi, t) \varphi(\xi) d\xi, \quad (2.17) \]

\[ G(x, t) = \frac{1}{\pi} \int_0^{\infty} t^{\alpha-1} E_{\alpha, \alpha}(-\lambda^2 \beta^2 t^\alpha) \cos \beta x d\beta. \quad (2.18) \]

Let us evaluate integral (2.18). The Laplace transform of (2.18) and formula 1.2(11) from \[19\] produce

\[ g(x, p) = \frac{1}{\pi} \int_0^{\infty} \frac{\cos(\beta x) d\beta}{\lambda^2 \beta^2 + p^{\alpha}} = \frac{1}{2\lambda} p^{-\alpha/2} e^{-|x|\lambda^{-1} p^{\alpha/2}}, \quad (2.19) \]

and the Laplace inversion gives

\[ G(x, t) = \frac{1}{4\lambda \pi i} \int_{Br} e^{pt} p^{-\alpha/2} \exp(-|x|\lambda^{-1} p^{\alpha/2}) dp. \quad (2.20) \]
Performing substitutions \( \sigma = pt \) and \( z = |x|\lambda^{-1}t^{-\rho} \) \((\rho = \alpha/2)\) and transformation of the Bromwich contour \((Br)\) to the Hankel contour \((Ha)\), as was done in a similar case by Mainardi \([13]\), we obtain
\[
G(x,t) = \frac{t^{1-\rho}}{2\lambda} \int_{Ha} e^{\sigma - z\sigma^\rho \frac{d\sigma}{\sigma^\rho}} = \frac{1}{2\lambda} t^{\rho-1} W(-z, -\rho, \rho), \quad z = \frac{|x|}{\lambda^\rho} \tag{2.21}
\]
where \( W(z, \lambda, \mu) \) is the Wright function \((1.17)\). We would like to note that, in fact, we have just evaluated the Fourier cosine-transform of the function \( u_1(\beta) = t^{\alpha-1} E_{\alpha,\alpha}(-\lambda^2 \beta^2 t^\alpha) \).

It is easy to check that for \( \alpha = 1 \) (traditional diffusion equation) fractional Green function \((2.21)\) reduces to the classic expression
\[
G(x,t) = \frac{1}{2\sqrt{\pi t}} \exp\left(-\frac{x^2}{4\lambda^2 t}\right). \tag{2.22}
\]

**Example 5. Wyss’s fractional diffusion equation.**

The following example shows the effectiveness of the proposed method also for fractional integral equations. Let us consider the Wyss’s type formulation of the diffusion equation \([3]\) (for simplicity and comparison with the previous example — in one space dimension):
\[
u(x,t) = \varphi(x) + \lambda^2 \frac{D_t^{-\alpha}}{0} \frac{\partial^2 u(x,t)}{\partial x^2}, \quad (-\infty < x < \infty, \quad t > 0); \tag{2.23}
\]
\[
\lim_{x \to \pm \infty} u(x,t) = 0, \quad u(x,0) = \varphi(x), \tag{2.24}
\]

Applying the Fourier transform with respect to space variable \( x \) and the Laplace transform with respect to time \( t \), we obtain:
\[
\overline{U}(\beta, p) = \frac{\varphi(\beta)p^{\alpha-1}}{p^\alpha + \lambda^2 \beta^2}, \tag{2.25}
\]
where \( \overline{U}(\beta, p) \) is the Fourier-Laplace transform of \( u(x,t) \), \( \beta \) is the Fourier transform parameter and \( p \) is the Laplace transform parameter.

Inverting Laplace and Fourier transforms as it was done in Example 4, we obtain the solution of problem \((2.23)\):
\[
u(x,t) = \int_{-\infty}^{\infty} G(x - \xi, t) \varphi(\xi)d\xi, \tag{2.26}
\]
\[
G(x,t) = \frac{1}{\pi} \int_{0}^{\infty} E_{\alpha,1}(-\lambda^2 \beta^2 t^\alpha) \cos \beta xd\beta. \tag{2.27}
\]

Let us evaluate integral \((2.27)\). The Laplace transform of \((2.27)\) and formula 1.2(11) from \([13]\) produce
\[
g(x,p) = \frac{p^{\alpha-1}}{\pi} \int_{0}^{\infty} \cos(\beta x)d\beta = \frac{1}{2\lambda^{\rho-1}} \exp(-|x|\lambda^{-1}p^{\alpha/2}), \tag{2.28}
\]

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and the inverse Laplace transform gives:

\[ G(x,t) = \frac{1}{4\lambda \pi i} \int_{Br} e^{pt} p^{\frac{\alpha}{2} - 1} \exp(-|x|\lambda^{-1} p^{\alpha/2}) dp. \quad (2.29) \]

Performing substitutions \( \sigma = pt \) and \( z = |x|\lambda^{-1} t^{-\rho} \) \((\rho = \alpha/2)\) and transformation of the Bromwich contour \((Br)\) to the Hankel contour \((Ha)\), as it was done in a similar case by Mainardi \([15]\), we obtain

\[ G(x,t) = \frac{t^{-\rho}}{2\lambda} \frac{1}{2\pi i} \int_{Ha} e^{\sigma - z\sigma^\rho} \frac{d\sigma}{\sigma^{1-\rho}} = \frac{1}{2\lambda} t^{-\rho} M(z,\rho), \quad z = \frac{|x|}{\lambda t^\rho} \quad (2.30) \]

where \( M(z,\rho) = W(-z,-\rho,1-\rho) \) is the Mainardi function \((1.21)\).

The last expression is identical to the expression, obtained by Mainardi \([15]\) in another way.

We would like to note at this point, as in the previous example, that we have just evaluated the Fourier cosine-transform of the function \( u_2(\beta) = E_{\alpha,1}(-\lambda^2 \beta^2 t^\alpha). \)

For \( \alpha = 1 \) the fractional Green’s function \((2.30)\) also reduces to the classic expression \((2.22)\). The case of arbitrary number of space dimensions can be solved similarly.

For \( \alpha = 1 \) both generalizations (Nigmatullin’s as well as Wyss’s) of the diffusion problem give the standard diffusion problem, and the solutions reduce to the classic solution. However, it is obvious that the asymptotic behavior of \((2.17)\) and \((2.26)\) for \( t \to 0 \), and \( t \to \infty \) is different (see also discussion in \([7]\) on two different generalizations of the standard relaxation equation and discussion in \([16]\) on two fractional models — one based on fractional derivatives and the other based on fractional integrals — for mechanical stress relaxation).

This difference was caused by initial conditions of different types. The class of solutions is determined by the number and the type of initial conditions.
Chapter 3

Sequential fractional differential equations

3.1 The Laplace transform of a sequential fractional differential operator

Let us consider the following initial value problem:

\[ 0 \mathcal{L}_t y(t) = f(t); \quad a D_{t}^{\sigma_k} y(t) \big|_{t=0} = b_k, \quad k = 1, \ldots, n \]  

(3.1)

These are sequential fractional differential equations, according to the terminology used by Miller and Ross [2]. To extend the Laplace transform method using the advantages of (1.15) for such equations with constant coefficients, we obtained the following formula:

\[ \int_0^\infty e^{-pt} 0 \mathcal{L}_t^{\sigma_m} f(t) dt = p^{\sigma_m} F(p) - \sum_{k=0}^{m-1} p^{\sigma_m-\sigma_m-k} a D_{t}^{\sigma_m-k-1} f(t) \big|_{t=0}, \]  

(3.3)

The particular case of (3.3) for \( f(t) \) m-times differentiable, \( \alpha_m = \mu, \alpha_k = 1, \) \((k = 1, 2, \ldots, m - 1)\) was obtained by Caputo [18, p.41] much earlier. Taking \( \alpha_1 = \mu, \alpha_k = 1, \) \((k = 2, 3, \ldots, m)\) leads under obvious assumptions to the classic formula (2.1).
3.2 Ordinary linear fractional differential equations

In this section we give solutions of the "sequential" analogues of the "standard" linear ordinary fractional differential equations with constant coefficients. Of course, we must take appropriate initial conditions, also in terms of sequential fractional derivatives.

Example 6. Let us consider the sequential analogue of Example 1:

\[
0 \, D_t^\alpha \left( 0 \, D_t^\beta y(t) \right) + ay(t) = 0 \tag{3.4}
\]

\[
0 \, D_t^{\alpha-1} \left( 0 \, D_t^\beta y(t) \right) \bigg|_{t=0} = b_1, \quad 0 \, D_t^{\beta-1} y(t) \bigg|_{t=0} = b_2, \tag{3.5}
\]

where \(0 < \alpha < 1, 0 < \beta < 1, \alpha + \beta = 1/2\).

Our formula (3.3) of the Laplace transform of the sequential fractional derivative allows us to utilize the initial conditions (3.5). To use (3.3), we take \(\sigma_1 = \alpha, \sigma_2 = \beta\) and \(m = 2\). Therefore, \(\sigma_1 = \alpha, \sigma_2 = \alpha + \beta\). Then the Laplace transform (3.3) of equation (3.4) gives:

\[
(p^{\alpha+\beta} + a)Y(p) = p^\beta b_2 + b_1, \tag{3.6}
\]

\[
Y(p) = b_2 \frac{p^\beta}{p^{\alpha+\beta} + a} + b_1 \frac{1}{p^{\alpha+\beta} + a}, \tag{3.7}
\]

and after the Laplace inversion with the help of (1.15) we find the solution to the problem (3.4)-(3.5):

\[
y(t) = b_2 t^{\alpha-1} E_{\alpha+\beta,\alpha}(-at^{\alpha+\beta}) + b_1 t^{\alpha+\beta-1} E_{\alpha+\beta,\alpha+\beta}(-at^{\alpha+\beta}). \tag{3.8}
\]

For \(\beta = 0\) and \(\alpha = 1/2\) (and assuming, of course, \(b_2 = 0\)), we can obtain from (3.8) the solution of Example 1.

Example 7. Let us now consider the following sequential analogue for the equation, considered in Example 2:

\[
0 \, D_t^\alpha \left( 0 \, D_t^\beta y(t) \right) + 0 \, D_t^q y(t) = h(t), \tag{3.9}
\]

where \(0 < \alpha < 1, 0 < \beta < 1, \alpha + \beta = Q > q\).

The Laplace transform (3.3) of equation (3.9) gives:

\[
(p^{\alpha+\beta} + p^q)Y(p) = H(p) + p^\beta b_2 + b_1, \tag{3.10}
\]

\[
b_1 = 0 \, D_t^{\alpha-1} \left( 0 \, D_t^\beta y(t) \right) \bigg|_{t=0} + 0 \, D_t^{\beta-1} y(t) \bigg|_{t=0},
\]

\[
b_2 = 0 \, D_t^{\beta-1} y(t) \bigg|_{t=0}.
\]

Writing \(Y(p)\) in the form

\[
Y(p) = \frac{p^{-q}H(p)}{p^{\alpha+\beta-q} + 1} + b_2 \frac{p^{\beta-q}}{p^{\alpha+\beta-q} + 1} + b_1 \frac{p^{-q}}{p^{\alpha+\beta-q} + 1}, \tag{3.11}
\]

and after the Laplace inversion with the help of (1.15) we find the solution to the problem (3.9):

\[
y(t) = b_2 t^{\alpha-1} E_{\alpha+\beta,\alpha}(-at^{\alpha+\beta}) + b_1 t^{\alpha+\beta-1} E_{\alpha+\beta,\alpha+\beta}(-at^{\alpha+\beta}). \tag{3.12}
\]
and finding the inverse Laplace transform with the help of (1.15), we obtain the solution:

\[
y(t) = b_2 t^{\alpha-1} E_{\alpha+\beta,\alpha}(t) + b_1 t^{\alpha+\beta} E_{\alpha+\beta,\alpha}(t) + \int_0^t (t - \tau)^{\alpha+\beta-1} E_{\alpha+\beta,\alpha}(t) h(\tau) d\tau.
\]  

(3.12)

It is easy to see that this solution contains solution of Example 2 as a special case.

**Example 8.** Let us consider the following initial value problem for the sequential fractional differential equation:

\[
\begin{align*}
0D_t^{\alpha_2} (0D_t^{\alpha_1} y(t)) - \lambda y(t) = h(t); & \quad (0 < \alpha_1, \alpha_2 < 1) \quad (3.13) \\
0D_t^{\alpha_2-1} 0D_t^{\alpha_1} y(t)|_{t=0} = b_1, \quad 0D_t^{\alpha_1-1} y(t)|_{t=0} = b_2; & \quad (3.14)
\end{align*}
\]

The Laplace transform (3.3) of equation (3.13) gives

\[
(p^{\alpha_1+\alpha_2} - \lambda) Y(p) = p^{\alpha_2} b_2 + b_1,
\]

and after inversion using (1.15) we obtain the solution:

\[
y(t) = b_2 t^{\alpha_1-1} E_{\alpha,\alpha_1}(\lambda t^{\alpha}) + b_1 t^{\alpha-1} E_{\alpha,\alpha}(\lambda t^{\alpha}) + \\
\int_0^t (t - \tau)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t - \tau)^\alpha) h(\tau) d\tau, \quad (\alpha = \alpha_1 + \alpha_2) \quad (3.15)
\]

Let us take \(\alpha\) the same as in Example 3. Using (1.1), (1.27) and (1.29), it is easy to verify that (3.15) is the solution of (3.13). It is also worthwhile to note that if \(b_1 \neq 0, b_2 \neq 0\), then (3.15) is not a solution of equation \(0D_t^{\alpha} y(t) - \lambda y(t) = h(t)\) from Example 3; also (2.10) is not a solution of equation (3.13). On the other hand, equations (2.8) and (3.13) are very close one to another: the fractional Green’s function in both cases is \(G(t) = t^{\alpha-1} E_{\alpha,\alpha}(\lambda t^{\alpha})\).

### 3.3 Partial linear fractional differential equations

**Example 9.** Let us consider Mainardi’s [15] initial value problem for the fractional diffusion–wave equation:

\[
0D_t^{\alpha} u(x, t) = \lambda^2 \frac{\partial^2 u(x, t)}{\partial x^2}, \quad (|x| < \infty, t > 0) \quad (3.16)
\]

\[
u(x, 0) = f(x), \quad (|x| < \infty) \quad (3.17)
\]

\[
\lim_{x \to \pm \infty} u(x, t) = 0, \quad (t > 0) \quad (3.18)
\]

where \(0 < \alpha < 1\).
The Laplace transform of problem (3.16)–(3.18) using (3.3) for \( k = 2, \alpha_1 = \alpha - 1, \alpha_2 = 1 \) (this gives Caputo’s formula [18]), i.e.,

\[
L\{\, 0^\alpha D_t^\alpha y(t) \,\} = p^\alpha Y(p) - p^{\alpha - 1} y(0)
\]

produces:

\[
p^\alpha \overline{u}(x,p) - p^{\alpha - 1} f(x) = \lambda^2 \frac{\partial^2 \overline{u}(x,p)}{\partial x^2}, \quad (|x| < \infty)
\]

\[
\lim_{x \pm \infty} \overline{u}(x,p) = 0, \quad (t > 0)
\]

Applying now the Fourier exponential transform to equation (3.20) and utilizing boundary conditions (3.21), we obtain:

\[
U(\beta, p) = \frac{p^{\alpha - 1}}{p^2 + \lambda^2 \beta^2} F(\beta),
\]

where \( U(\beta, p) \) and \( F(\beta) \) are the Fourier transforms of \( \overline{u}(x,p) \) and \( f(x) \).

The inverse Laplace transform of the fraction \( \frac{p^{\alpha - 1}}{p^2 + \lambda^2 \beta^2} \) is \( E_{\alpha,1}(-\lambda^2 \beta^2 t^\alpha) \) \( (E_{\lambda,\mu}(z) \) is the Mittag-Leffler function in two parameters). Therefore, the inversion of the Fourier and the Laplace transform gives the solution in the following form:

\[
u(x,t) = \int_{-\infty}^{\infty} G(x - \xi, t)f(\xi)d\xi,
\]

\[
G(x, t) = \frac{1}{\pi} \int_0^\infty E_{\alpha,1}(-\lambda^2 \beta^2 t^\alpha) \cos(\beta x) d\beta = \frac{1}{2\lambda} t^{-\rho} W(-z, -\rho, 1 - \rho),
\]

where \( W(z, \lambda, \mu) \) is the Wright function [1.17]. This solution is identical to the solution of the Wyss fractional (integral) diffusion equation (2.26).
Chapter 4

Fractional Green’s function

4.1 Definition and some properties

In this section we consider equation (3.1) under homogeneous initial conditions $b_k = 0$, $(k = 1, \ldots, n)$, i.e.

$$0 \mathcal{L}_t y(t) = f(t); \quad 0 \mathcal{D}_t^\alpha y(t) \bigg|_{t=0} = 0, \quad k = 1, \ldots, n \quad (4.1)$$

$$a \mathcal{L}_t y(t) = a \mathcal{D}_t^\alpha y(t) + \sum_{k=1}^{n-1} p_k(t) a \mathcal{D}_t^\alpha y(t) + p_n(t)y(t),$$

$$a \mathcal{D}_t^\alpha \equiv a \mathcal{D}_t^{\alpha_k} \mathcal{D}_t^{\alpha_{k-1}} \ldots \mathcal{D}_t^{\alpha_1}; \quad a \mathcal{D}_t^\alpha \equiv a \mathcal{D}_t^{\alpha_k} \mathcal{D}_t^{\alpha_{k-1}} \ldots \mathcal{D}_t^{\alpha_1};$$

$$\sigma_k = \sum_{j=1}^k \alpha_j, \quad (k = 1, 2, \ldots, n); \quad 0 \leq \alpha_j \leq 1, \quad (j = 1, 2, \ldots, n).$$

The following definition is a "fractionalization" of the definition given in [27].

**Definition.** Function $G(t, \tau)$ satisfying the following conditions

a) $\tau \mathcal{L}_t G(t, \tau) = 0$ for every $\tau \in (0, t)$;

b) $\lim_{\tau \to t^-} (\tau \mathcal{D}_t^\sigma G(t, \tau)) = \delta_{k,n}, \quad k = 0, 1, \ldots, n,$

$(\delta_{k,n}$ is Kronecker’s delta);  

c) $\lim_{\tau, t \to +0} (\tau \mathcal{D}_t^\sigma G(t, \tau)) = 0, \quad k = 0, 1, \ldots, n - 1$

is called Green’s function of equation (4.1).

**Properties.**

1. Using (1.28), it can be shown that $y(t) = \int_0^t G(t, \tau)f(\tau)d\tau$ is the solution of problem (4.1).

Let us outline the proof of this statement. Evaluating $0 \mathcal{D}_t^\alpha y(t)$, $0 \mathcal{D}_t^\alpha y(t)$, $\ldots, 0 \mathcal{D}_t^\alpha y(t)$ using the rule (1.28) and condition (b) of the definition of Green’s function, we obtain:

$$0 \mathcal{D}_t^\alpha y(t) = 0 \mathcal{D}_t^\alpha \int_0^t G(t, \tau)f(\tau)d\tau$$
\[\int_0^t \tau D_t^{\alpha_1} G(t, \tau) f(\tau) d\tau + \lim_{\tau \to t^-} \tau D_t^{\alpha_1 - 1} G(t, \tau) f(\tau) = \int_0^t \tau D_t^{\alpha_1} G(t, \tau) f(\tau) d\tau \quad (4.2)\]

\[0D_t^{\alpha_2} y(t) = 0D_t^{\alpha_2} (0D_t^{\alpha_1}) = 0D_t^{\alpha_2} \int_0^t \tau D_t^{\alpha_1} G(t, \tau) f(\tau) d\tau = \int_0^t \tau D_t^{\alpha_2} (\tau D_t^{\alpha_1} G(t, \tau)) f(\tau) d\tau + \lim_{\tau \to t^-} \tau D_t^{\alpha_2 - 1} (\tau D_t^{\alpha_1} G(t, \tau)) f(\tau) = \int_0^t \tau D_t^{\alpha_2} G(t, \tau) f(\tau) d\tau \quad (4.3)\]

\[\cdots \cdots \cdots \]

\[0D_t^{\sigma_{n-1}} y(t) = 0D_t^{\sigma_{n-1}} (0D_t^{\sigma_{n-2}}) = 0D_t^{\sigma_{n-1}} \int_0^t \tau D_t^{\sigma_{n-2}} G(t, \tau) f(\tau) d\tau = \int_0^t \tau D_t^{\sigma_{n-1}} (\tau D_t^{\sigma_{n-2}} G(t, \tau)) f(\tau) d\tau + \lim_{\tau \to t^-} \tau D_t^{\sigma_{n-1} - 1} (\tau D_t^{\sigma_{n-2}} G(t, \tau)) f(\tau) = \int_0^t \tau D_t^{\sigma_{n-1}} G(t, \tau) f(\tau) d\tau \quad (4.4)\]

\[0D_t^{\sigma_n} y(t) = 0D_t^{\sigma_n} (0D_t^{\sigma_{n-1}}) = 0D_t^{\sigma_n} \int_0^t \tau D_t^{\sigma_{n-1}} G(t, \tau) f(\tau) d\tau = \int_0^t \tau D_t^{\sigma_n} (\tau D_t^{\sigma_{n-1}} G(t, \tau)) f(\tau) d\tau + \lim_{\tau \to t^-} \tau D_t^{\sigma_n - 1} (\tau D_t^{\sigma_{n-1}} G(t, \tau)) f(\tau) = \int_0^t \tau D_t^{\sigma_n} G(t, \tau) f(\tau) d\tau + f(t) \quad (4.5)\]

Multiplying these equations by the corresponding coefficients and summarizing, we obtain

\[0L_t y(t) = \int_0^t \tau L_t G(t, \tau) f(\tau) d\tau + f(t) = f(t), \quad (4.6)\]

which completes this proof.

2. For fractional differential equations with constant coefficients \(G(t, \tau) \equiv G(t - \tau)\). This is obvious because in such a case Green’s function can be obtained by Laplace transform method.
The type (standard or sequential) of the equation is not important for determining Green’s function, because due to condition (b) in the Green’s function definition all non-integral addends vanish.

3. Appropriate derivatives of Green’s function \( G(x, \tau) \) form a set of linearly independent solutions of a homogeneous \((f(t) \equiv 0)\) equation (3.1) (for a simple illustration, see Examples 3 and 8).

Let us demonstrate this for the case of the linear fractional differential equations with constant coefficients, which are the main subject for study in this work and for which we have \( G(t, \tau) \equiv G(t - \tau) \).

Let us take \( 0 < \lambda < \sigma_n \). First, the function
\[
y_{\lambda}(t) = D_{0+}^\lambda G(t)
\]
is a solution of the corresponding homogeneous equation. Indeed,
\[
0L_t y_{\lambda}(t) = 0L_t \left( D_{0+}^\lambda G(t) \right) = D_{0+}^\lambda \left( 0L_t G(t) \right) = 0.
\]
We used here that \( 0L_t D_{0+}^\lambda = D_{0+}^\lambda 0L_t \), which follows from condition (c) in the definition of the fractional Green’s function.

Second,
\[
D_{0+}^{\sigma_n - \lambda - 1} y_{\lambda}(t) \bigg|_{t=0} = 0.
\]
In fact,
\[
D_{0+}^{\sigma_n - \lambda - 1} y_{\lambda}(t) \bigg|_{t=0} = D_{0+}^{\sigma_n - \lambda - 1} \left( D_{0+}^\lambda G(t) \right) \bigg|_{t=0} = D_{0+}^{\sigma_n - 1} G(t) \bigg|_{t=0} = 0.
\]
We first used here the relationship
\[
D_{0+}^{\sigma_n - \lambda - 1} 0L_t G(t) \equiv D_{0+}^{\left( \sigma_n - \lambda - 1 \right) + \lambda} G(t),
\]
which follows from condition (c) of the definition of Green’s function, and then condition (b).

We see that having the fractional Green’s function of equation (4.1), we can determine particular solutions of the corresponding homogeneous equation, which are necessary for satisfying inhomogeneous initial conditions.

Therefore, solution of linear fractional differential equations with constant coefficients reduces to finding the fractional Green’s function. After that, we can immediately write the solution of the inhomogeneous equation satisfying given inhomogeneous initial conditions.

This solution has the form
\[
y(t) = \sum_{k=1}^{n} b_k \psi_k(t) + \int_0^t G(t - \tau) f(\tau) d\tau,
\]
\[
b_k = \left. D_{0+}^{\sigma_n - 1} y(t) \right|_{t=0}
\]
\[
\psi_k(t) = D_{0+}^{\sigma_n - \sigma_k} G(t), \quad D_{0+}^{\sigma_n - \sigma_k} \equiv D_{0+}^{\alpha_n} D_{0+}^{\alpha_n - 1} \cdots D_{0+}^{\alpha_k + 1}
\]
Because of this, in the following sections we find some explicit expressions for fractional Green’s functions, including a general linear fractional differential equation.
4.2 Fractional Green’s function for the one-term fractional differential equation

The fractional Green’s function $G_1(t)$ for the one-term fractional-order differential equation with constant coefficients

$$a_0D_t^\alpha y(t) = f(t), \quad (4.15)$$

where the derivative can be either ”classic” (i.e., considered in the book by Oldham and Spanier) or ”sequential” (Miller and Ross), is found by the inverse Laplace transform of the following expression:

$$g_1(p) = \frac{1}{ap^\alpha}. \quad (4.16)$$

The inverse Laplace transform then gives

$$G_1(t) = \frac{1}{a} \frac{\tau^{\alpha-1}}{\Gamma(\alpha)}. \quad (4.17)$$

The solution of equation (4.15) under homogeneous initial conditions is

$$y(t) = \frac{1}{a\Gamma(\alpha)} \int_0^t \frac{f(\tau)d\tau}{(t-\tau)^{1-\alpha}} = \frac{1}{a} \, 0D_t^{-\alpha}f(t). \quad (4.18)$$

Using [28, lemma 3.3], we can easily verify that expression (4.18) gives the solution of equation (4.15), if $f(x)$ is continuous in $[0, \infty)$.

4.3 Fractional Green’s function for the two-term fractional differential equation

The fractional Green’s function $G_2(t)$ for the two-term fractional-order differential equation with constant coefficients

$$a_0D_t^\alpha y(t) + by(t) = f(t), \quad (4.19)$$

where the derivative can be either ”classic” (i.e., considered in the book by Oldham and Spanier) or ”sequential” (Miller and Ross), is found by the inverse Laplace transform of the following expression:

$$g_2(p) = \frac{1}{ap^\alpha + b} = \frac{1}{a} \, \frac{1}{p^\alpha + \frac{b}{a}}, \quad (4.20)$$

which leads to

$$G_2(t) = \frac{1}{a} \, t^{\alpha-1} E_{\alpha,\alpha} (-\frac{b}{a} \, t^\alpha). \quad (4.21)$$

For instance, function $G_2(t)$ plays the key role in solution of Example 1 and 3.

Taking in (4.21) $b = 0$ and using the definition of the Mittag-Leffler function (1.1), we obtain Green’s function $G_1(t)$ for the one-term equation.
4.4 Fractional Green’s function for the three-term fractional differential equation

The fractional Green’s function \( G_3(t) \) for the three-term fractional-order differential equation with constant coefficients

\[
a_0 D_t^\beta y(t) + b_0 D_t^\alpha y(t) + c y(t) = f(t), \tag{4.22}
\]

where the derivatives can be either "classic" (i.e., considered in the book by Oldham and Spanier) or "sequential" (Miller and Ross), is found by the inverse Laplace transform of the following expression:

\[
g_3(p) = \frac{1}{ap^\beta + bp^\alpha + c} \tag{4.23}
\]

Assuming \( \beta > \alpha \), we can write \( g_3(p) \) in the form

\[
g_3(p) = \frac{1}{c} \frac{cp^{-\alpha}}{ap^{\beta-\alpha} + b} + \frac{1}{c} \frac{cp^{-\alpha}}{ap^{\beta-\alpha} + b} = \frac{1}{c} \sum_{k=0}^{\infty} (-1)^k \left( \frac{c}{a} \right)^{k+1} \frac{p^{-\alpha k - \alpha}}{(p^{\beta-\alpha} + b)^{k+1}} \tag{4.24}
\]

The term-by-term inversion, based on the general expansion theorem for the Laplace transform given in \[20, \S 22\], using (1.15) produces

\[
G_3(t) = \frac{1}{c} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left( \frac{c}{a} \right)^k \frac{t^k}{(p^{\beta-\alpha} + b)^{k+1}} \tag{4.25}
\]

where \( E_{\lambda,\mu}(z) \) is the Mittag-Leffler function in two parameters,

\[
E_{\lambda,\mu}(y) \equiv \frac{d^k}{dy^k} E_{\lambda,\mu}(y) = \sum_{j=0}^{\infty} \frac{(j + k)! y^j}{j! \Gamma(\lambda j + \lambda k + \mu)}. \quad (k = 0, 1, 2, \ldots) \tag{4.26}
\]

We assume in this solution that \( a \neq 0 \), because otherwise we have the two-term equation (4.19). We can also assume \( c \neq 0 \), because for \( c = 0 \)

\[
g_3(p) = \frac{1}{ap^\beta + bp^\alpha} = \frac{p^{-\alpha}}{ap^{\beta-\alpha} + b},
\]

and the Laplace inversion can be done in the same way as in the case of the two-term equation.

Two special cases of equation (4.22) were considered by Bagley and Torvik [21] (for \( \beta = 2 \) and \( \alpha = 3/2 \)) and by Caputo [18] (for \( \beta = 2 \) and \( 0 < \alpha < 1 \)). It is easy to show that our solution (4.25) contains Caputo’s solution as a particular case.

Indeed, substituting (4.26) into (4.25) and changing the order of summation, we obtain:

\[
G_3(t) = \frac{1}{c} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left( \frac{c}{a} \right)^k \sum_{j=0}^{\infty} (-1)^j \left( \frac{b}{a} \right)^j \frac{(j + k)!}{k! j! \Gamma(\beta(j + k + 1) - \alpha j)}
\]

\[
= \frac{1}{c} \sum_{j=0}^{\infty} \frac{(-b)^j}{a} \sum_{k=0}^{\infty} (-1)^k \left( \frac{c}{a} \right)^{k+1} \frac{(j + k)!}{k! j! \Gamma(\beta(j + k + 1) - \alpha j)} \tag{4.27}
\]

For \( \beta = 2 \) this expression is identical with the expression obtained by Caputo [18, formula (2.27)].
4.5 Fractional Green’s function for the four-term fractional differential equation

The fractional Green’s function $G_4(t)$ for the four-term fractional-order differential equation with constant coefficients

$$\frac{a}{0} D_t^{\gamma} y(t) + \frac{b}{0} D_t^{\beta} y(t) + \frac{c}{0} D_t^{\alpha} y(t) + d y(t) = f(t), \quad (4.28)$$

where the derivatives can be, as in the previous section, either "classic" or "sequential", is found by the inverse Laplace transform of the following expression:

$$g_4(p) = \frac{1}{ap^\gamma + bp^\beta + cp^\alpha + d} (4.29)$$

Assuming $\gamma > \beta > \alpha$, we can write $g(p)$ in the form

$$g_4(p) = \frac{1}{ap^\gamma + bp^\beta + cp^\alpha + d} \left( \frac{c}{a} p^\alpha + d \right)$$

$$= \frac{a^{-1} p^{-\beta}}{p^{\gamma-\beta} + a^{-1} b} \frac{1}{1 + \frac{a^{-1} c p^{\alpha-\beta} + a^{-1} d p^{-\beta}}{p^{\gamma-\beta} + a^{-1} b}}$$

$$= \sum_{m=0}^{\infty} (-1)^m \frac{a^{-1} p^{-\beta}}{(p^{\gamma-\beta} + a^{-1} b)^{m+1}} \left( \frac{c}{a} p^\alpha + d p^{-\beta} \right)^m$$

$$= \frac{1}{a} \sum_{m=0}^{\infty} (-1)^m \left( \frac{d}{a} \right)^m \frac{m}{k} \left( \frac{c}{d} \right)^k p^{ak-\beta m} \right)$$

The term-by-term inversion, based on the general expansion theorem for the Laplace transform given in [20, §22], using (1.15) gives the final expression for the fractional Green’s function for equation (4.28):

$$G_4(t) = \frac{1}{a} \sum_{m=0}^{\infty} \frac{1}{m!} \left( \frac{d}{a} \right)^m \sum_{k=0}^{m} \binom{m}{k} \left( \frac{c}{d} \right)^k t^{\gamma(m+1) - \alpha k - 1} E_{\gamma - \beta, \gamma + \beta m - \alpha k} \left( -\frac{b}{a} t^{\gamma - \beta} \right)$$

We assumed in this solution $a \neq 0$, because in the opposite case we have the three-term equation (1.22). We can also assume $d \neq 0$, because in the case of $d = 0$, after writing

$$g_4(p) = \frac{p^{-\alpha}}{ap^{-\alpha} + bp^{-\alpha} + c} \quad (4.32)$$

the Laplace inversion can be done in the same way as in the case of the three-term equation.
4.6 Fractional Green’s function for the general linear fractional differential equation

The above results can be essentially generalized.

The fractional Green’s function $G_n(t)$ for the $n$-term fractional-order differential equation with constant coefficients

$$a_n D^{\beta_n} y(t) + a_{n-1} D^{\beta_{n-1}} y(t) + \ldots + a_1 D^{\beta_1} y(t) + a_0 D^{\beta_0} y(t) = f(t),$$  \hspace{1cm} (4.33)

where derivatives $D^\alpha \equiv 0 D^\alpha$ can be, as in the previous sections, either "classic" or "sequential", is found by the inverse Laplace transform of the following expression:

$$g_n(p) = \frac{1}{a_n p^{\beta_n} + a_{n-1} p^{\beta_{n-1}} + \ldots + a_1 p^{\beta_1} + a_0 p^{\beta_0}}$$  \hspace{1cm} (4.34)

Let us assume $\beta_n > \beta_{n-1} > \ldots > \beta_1 > \beta_0$ and write $g_n(p)$ in the form:

$$g_n(p) = \frac{1}{a_n p^{\beta_n} + a_{n-1} p^{\beta_{n-1}}} \left( 1 + \frac{n-2}{a_k p^{\beta_k}} \right)$$

$$= \frac{a_n^{-1} p^{-\beta_n-1}}{p^{\beta_n-\beta_n-1} + \frac{a_{n-1}}{a_n}} \left( 1 + \frac{n-2}{a_k p^{\beta_k}} \right)$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m a_n^{-1} p^{-\beta_n-1}}{(p^{\beta_n-\beta_n-1} + \frac{a_{n-1}}{a_n})^{m+1}} \left( \sum_{k=0}^{n-2} \frac{a_k}{a_n} p^{\beta_n-\beta_k} \right)^m$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m a_n^{-1} p^{-\beta_n-1}}{(p^{\beta_n-\beta_n-1} + \frac{a_{n-1}}{a_n})^{m+1}} \left( \sum_{k_0+k_1+\ldots+k_{n-2}=m} (m; k_0, k_1, \ldots, k_{n-2}) \prod_{i=0}^{n-2} \frac{a_i}{a_n} p^{(\beta_i-\beta_n-1)k_i} \right)$$

$$= \frac{1}{a_n} \sum_{m=0}^{\infty} (-1)^m \sum_{k_0+k_1+\ldots+k_{n-2}=m} (m; k_0, k_1, \ldots, k_{n-2}) \prod_{i=0}^{n-2} \frac{a_i}{a_n} p^{(\beta_i-\beta_n-1)k_i}$$

$$= \prod_{i=0}^{n-2} \frac{a_i}{a_n} p^{-\beta_n-1} \sum_{i=0}^{n-2} (\beta_i-\beta_n-1)k_i$$

$$= \frac{1}{a_n} \prod_{i=0}^{n-2} \frac{a_i}{a_n} p^{-\beta_n-1} + \frac{a_{n-1}}{a_n} \prod_{i=0}^{n-2} \frac{a_i}{a_n} p^{-\beta_n-1}$$  \hspace{1cm} (4.35)

where $(m; k_0, k_1, \ldots, k_{n-2})$ are the multinomial coefficients [22, chapter 24]

The term-by-term inversion, based on the general expansion theorem for the Laplace transform given in [20, §22], using (1.13) gives the final expression.
for the fractional Green’s function for equation (4.33):

\[ G_n(t) = \frac{1}{a_n} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \sum_{k_0 + k_1 + \ldots + k_{n-2} = m \atop k_0 \geq 0; \ldots; k_{n-2} \geq 0} (m; k_0, k_1, \ldots, k_{n-2}) \] (continued)

\[ \prod_{i=0}^{n-2} \left( \frac{a_i}{a_n} \right)^{k_i} t^{(\beta_n - \beta_{n-1})m + \beta_n + \sum_{j=0}^{n-2} (\beta_{n-1} - \beta_j)k_j} \left( -\frac{a_{n-1}}{a_n} t^{\beta_{n-1}} \right) \] (4.36)
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