Probability distribution functions of gas in M31 and M51

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ABSTRACT
We present probability distribution functions (PDFs) of the surface densities of ionized and neutral gas in the nearby spiral galaxies M31 and M51, as well as of dust emission and extinction $A_V$ in M31. The PDFs are close to lognormal and those for H and $A_V$ in M31 are nearly identical. However, the PDFs for H$_2$ are wider than the H$_i$ PDFs and the M51 PDFs have larger dispersions than those for M31. We use a simple model to determine how the PDFs are changed by variations in the line-of-sight (LOS) path length $L$ through the gas, telescope resolution and the volume filling factor of the gas, $f_v$. In each of these cases the dispersion $\sigma$ of the lognormal PDF depends on the variable with a negative power law. We also derive PDFs of mean LOS volume densities of gas components in M31 and M51. Combining these with the volume density PDFs for different components of the ISM in the Milky Way (MW), we find that $\sigma$ decreases with increasing length $L$ with an exponent of $-0.76 \pm 0.06$, which is steeper than expected. We show that the difference is due to variations in $f_v$. As $f_v$ is similar in M31, M51 and the MW, the density structure in the gas in these galaxies must be similar. Finally, we demonstrate that an increase in $f_v$ with increasing distance to the Galactic plane explains the decrease in $\sigma$ with latitude of the PDFs of emission measure and far-ultraviolet emission observed for the MW.

Key words: turbulence – ISM: structure – galaxies: individual: M31 – galaxies: individual: M51 – galaxies: individual: MW.

1 INTRODUCTION
Many processes can leave an imprint on the complicated structure of the interstellar medium (ISM) observed in the Milky Way (MW) and nearby galaxies. For example, variations in the gas density can be due to: an evolving population of gravitationally bound clouds; expanding supernova remnant shells compressing the gas; cooling and heating of the gas leading to phase transitions; the turbulent and compressible nature of the flow.

During the last two decades compressible turbulence has been recognized as an important cause of the density structure of the ISM in the MW. Both isothermal (Elmegreen & Scalo 2004, and references therein) and multiphase (de Avillez & Breitschwerdt 2005; Wada & Norman 2007) simulations of the ISM produced lognormal volume density distributions. Several authors noted that for a large enough volume and for a long enough simulation run, the physical processes causing the density fluctuations in the ISM in a galactic disc can be regarded as random, independent, multiplicative events (e.g. Vázquez-Semadeni 1994; Passot & Vázquez-Semadeni 1998; Wada & Norman 2007). Therefore, the probability distribution function (PDF) of log(density) becomes Gaussian and the density PDF lognormal, as long as the local Mach number and the density are not correlated (Kritsuk et al. 2007; Federrath et al. 2010).

However, Tassis et al. (2010) show that lognormal column density distributions in molecular clouds are a generic statistical representation of the inhomogeneous distribution of gas density, whether the inhomogeneity is a result of turbulence, gravity or ambipolar diffusion. Recent reviews which discuss density PDFs of molecular clouds and their connection to turbulence and star formation include McKee & Ostriker (2007), Hennebelle & Falgarone (2012) and Padoan et al. (2014) and recent theoretical modelling can be found in Federrath, Klessen & Schmidt (2008), Federrath (2013) and Federrath & Klessen (2013).

Observational evidence to test the simulation results is slowly growing. Padoan, Jones & Nordlund (1997), Goodman, Pineda & Schnee (2009), Kainulainen et al. (2009), Schmalzl et al. (2010), Schneider et al. (2012) and Kainulainen, Federrath & Henning (2014) obtained lognormal column density PDFs for the extinction through dust in molecular clouds in the MW. The latter authors discuss the relationship between the density PDFs and star formation. Schneider et al. (2013) presented lognormal PDFs of 12CO(1–0) integrated intensities for four dust clouds in the MW, and Hughes et al. (2013) found lognormal PDFs of the 12CO(1–0) brightness temperature and integrated intensities in M51. Also the PDF of the H$_2$ column densities in M33 is lognormal, but with a high-density
We call our PDFs of the gas densities probability distribution functions, of M31. Each PDF was calculated from independent data points (at least $1.67 \times$ beamwidth apart) above $2 \times$ rms noise.

M51 is a bright Sbc galaxy at a distance of 7.6 Mpc (Ciardullo et al. 2002). It has an angular extent of about 10 arcmin, a position angle of PA $= -10^\circ$ and an inclination of $i = 20^\circ$ (Tully 1974; Pety et al. 2013). For the PDFs we used the H I data from the THINGS survey (Walter et al. 2008) at a resolution of $5.82 \times 5.56$ arcsec$^2$, the BIMA 12CO(1–0) survey of Helfer et al. (2003) at a resolution of $5.8 \times 5.1$ arcsec$^2$, which was corrected for missing spacings using single-dish observations and the H I map of Greenanalt et al. (1998) at a resolution of a few arcsec. All maps were reduced to a common field of about $8.5 \times 10.5$ arcmin$^2$ covering most of M51 apart from the extended southern outer arm in H I. They were then smoothed to a resolution of 8 arcsec corresponding to $295 \times 315$ pc$^2$ along major $\times$ minor axis in the plane of M51. We converted the CO data to column densities of H$_2$ using the conversion factor $4.75 \times 10^{19}$ cm$^{-2}$ (K km s$^{-1}$)$^{-1}$, one quarter of that for the MW (Guélin et al. 1995). Of each map, only independent data points above $2 \times$ rms noise were used to produce the PDF.

3 PROBABILITY DISTRIBUTION FUNCTIONS

In this section we present the observed surface density PDFs$^1$ for M31 and M51 and characterize them by fitting lognormal distributions to the histograms. Since there are various ways to parametrize a lognormal we shall briefly describe the exact equations we use. To allow for easy comparison with Berkhuijsen & Fletcher (2008) and other work on observed PDFs we use common, or base 10, logarithms. The lognormal PDF for the random variable $x$ is given by

$$\Lambda(\mu, \sigma) = \frac{1}{\sigma x \sqrt{2\pi}} \exp\left[-\frac{(\log_{10} x - \mu)^2}{2\sigma^2}\right] \, dx,$$

which is equivalent to a Gaussian distribution for $\log_{10}(x)$,

$$N[\log_{10}(x)]|\mu, \sigma| = \frac{1}{\sigma \sqrt{2\pi}} \exp\left[-\frac{(\log_{10} x - \mu)^2}{2\sigma^2}\right] \, d\log_{10}x.$$ (2)

We use equation (2) to derive $\mu$ and $\sigma$ from fits to histograms of $\log_{10}(x)$, taking into account the logarithmic bin-width to ensure that the integral of the PDF is unity. It is important to note that $\mu$ and $\sigma$ are the mean and standard deviation of the Gaussian distribution of $\log_{10}(x)$; the mean and standard deviation of the distribution of $x$ are given by

$$E[x] = \exp(\mu + \sigma^2/2),$$

and

$$S[x] = \sqrt{(\exp^2\sigma^2 - 1)\exp(2\mu + \sigma^2)},$$

respectively. Thus the variance of $x$ depends on both the mean and standard deviation of $\log_{10}(x)$. We shall try to avoid confusion by calling $\sigma$ the dispersion of the PDF and using $X_0 = 10^\sigma$ to describe the peak of the PDF (note that $10^\sigma$ is the median of the distributions of both $x$ and $\log_{10}(x)$). For the Gaussian PDFs of $\log_{10}(x)$, which is the form we show in our figures, small (large) values of $\sigma$ result in a narrow (broad) Gaussian PDF.

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$^1$ We call our PDFs of the gas densities probability distribution functions, rather than probability density functions, to avoid referring to the ‘probability density of the density’.

2 THE DATA

M31, the spiral galaxy nearest to us, is at a distance of 780 $\pm$ 40 kpc (Stanek & Garnavich 1998). The Sb galaxy of large angular extent (more than $3^\circ \times 1^\circ$) and low surface brightness has an inclination of $i = 77.5$ (Braun 1991; Chemin, Carignan & Foster 2009) and so is nearly edge-on. The position angle is PA $= 37.7^\circ$. Our gas density PDFs are based on the Westerbork H I column density maps of Greenawalt et al. (2008) and the H I map of Greenanalt et al. (1998) covering most of M51

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containing the bright emission ring (35 arcmin < R < 60 arcmin or 7.9 < R < 13.6 kpc) and for the weaker emission interior to this ‘ring’ (0 arcmin < R < 35 arcmin). For Hα and H2 the fits of the PDFs for the subregions are nearly the same, but for H1 the position of the peak of the PDF, X0, for the bright ‘ring’ is at a significantly higher density than for the inner region and the dispersion is smaller. This difference in X0 causes the low-density excess – and hence the asymmetry – in the total PDF and its larger dispersion compared to that of the subregions.

Fig. 2 shows the M51 PDFs of the emission measures of the ionized gas and of the column densities of H1 and H2 at radii R < 3 arcmin. The parameters of the lognormal fits are given in Table 2. Again the H1 PDF has an excess at low densities. The peak at high densities is due to gas in the outer spiral arms. We have tried to construct separate PDFs for spiral arms and the interarm regions by using a spiral-arm mask; this was created using the coefficients of a wavelet transform of the CO map with a scale of about 1 kpc, the positive coefficients tracing a clear two-armed spiral. Although the dispersions of these PDFs could not be reliably determined, the values of X0 of the arm PDFs are significantly higher than of the PDFs for the interarm regions. Recently, Hughes et al. (2013) found that PDFs of the integrated 12CO(1–0) intensities in M51 for arm regions are clearly wider than for interarm regions, and have an about 25 per cent larger X0. They also presented PDFs for the inner 4.5 arcmin × 2.8 arcmin of M51. At a spatial resolution of 53 pc, using all pixels with an intensity above three times the noise, they obtained a dispersion of σ = 0.62, the same as our value for a larger area of M51 observed at a lower resolution.

The PDFs of Hα, H1 and H2 for M31 and M51 presented in Figs 1 and 2 are close to lognormal. That the PDFs of H2 are lognormal may seem surprising in view of the recent work of Shetty et al. (2011). In their simulations of molecular clouds, the authors found that the PDFs of 12CO(1–0) intensities and of CO column densities are generally not lognormal and have different shapes when CO is optically thick. But at low densities and for the model of a MW cloud of medium density the PDFs are lognormal. The box size in their simulations is 20 pc, much smaller than the typical resolution of 300 pc used for the PDFs of the gas surface densities in M31 and M51. Hence, the fine intensity seen by the radio beam is the average intensity of a large sample of individual clouds. Saturated lines will hardly be observed because the filling factor of the densest clouds is very small. Therefore the observed CO PDFs become lognormal and will give a fair representation of the PDF of the H2 column density.

3.1 PDFs of surface densities

In Fig. 1 we present the M31 PDFs of the emission measures of the ionized gas and of the column densities of H1 and H2 for the radial range 0 arcmin < R < 60 arcmin in the plane of the galaxy, corresponding to 0 < R < 13.6 kpc. In Table 1 we list the parameters of the lognormal fits together with those for the area of M51 observed at a lower resolution.

![Figure 1. PDFs of the surface densities of ionized and neutral gas in the radial interval 0 arcmin < R < 60 arcmin in M31. Top: Hα (in cm⁻² pc); middle: N(H1) (in 10¹⁹ atom cm⁻²); bottom: N(H2) (in 10¹⁹ mol cm⁻²). Dashed lines show the lognormal fits given in Table 1.](https://academic.oup.com/mnras/article-abstract/448/3/2469/1097955/figure1)

Table 1. Lognormal fits to the PDFs of gas surface densities in M31, half-power beam width (HPBW) 45 arcsec.

|      | Units       | R (arcmin) | N   | Position of maximum | Dispersion | X²  |
|------|-------------|------------|-----|---------------------|------------|-----|
|      |             |            |     | μ                  | X₀         | σ   |     |
| N(H₁) 10¹⁹ atom cm⁻² | 0–60       | 1249       | 2.16 ± 0.03 | 145 ± 10 | 0.27 ± 0.02 | 12.7 |
|      |            | 0–35       | 368 | 1.86 ± 0.01 | 72 ± 2 | 0.23 ± 0.01 | 0.8 |
|      |            | 35–60      | 881 | 2.32 ± 0.02 | 209 ± 10 | 0.19 ± 0.01 | 5.5 |
| N(H₂) 10¹⁹ mol cm⁻² | 0–60       | 971        | 1.19 ± 0.04 | 15.5 ± 1.5 | 0.41 ± 0.02 | 2.9 |
|      |            | 0–35       | 250 | 1.11 ± 0.05 | 12.9 ± 1.6 | 0.39 ± 0.04 | 1.5 |
|      |            | 35–60      | 721 | 1.22 ± 0.04 | 16.6 ± 1.6 | 0.40 ± 0.03 | 3.0 |
| Hα cm⁻⁶ pc   | 0–60       | 1097       | 0.88 ± 0.03 | 7.6 ± 0.5 | 0.34 ± 0.02 | 2.9 |
|      |            | 0–35       | 397 | 0.93 ± 0.03 | 8.5 ± 0.6 | 0.34 ± 0.02 | 1.2 |
|      |            | 35–60      | 700 | 0.85 ± 0.03 | 7.1 ± 0.5 | 0.34 ± 0.02 | 2.3 |

Notes. N is the total number of data points; the error in each bin δᵢ is estimated as δᵢ = √Nᵢ for (number of bins – 2) degrees of freedom; X² is the reduced chi-squared goodness of fit parameter.
Comparing Figs 1 and 2 we note the following:

(a) Both PDFs of Hα show a low-intensity cut-off at the sensitivity limit. A similar cut-off occurs in the PDF of H2 for M31.

(b) The Hα PDF for M51 extends to higher emission measures than that for M31 and the emission measure of the maximum is about 35 per cent higher than that of M31. Since the M51 data were not corrected for extinction and the inclination of M51 is smaller than that of M31, this means that the face-on surface brightness in Hα of M51 is much larger than that of M31.

(c) The steep decrease seen in the H1 PDFs at high column densities may be attributed to opacity in the H1 lines. In scatter plots of star formation rate against H1 mass surface density a cut-off near \(10 \, \text{M}_\odot \, \text{pc}^{-2} \) is visible in M31 (Tabatabaei & Berkhuijsen 2010), M51 and many other galaxies (Bigiel et al. 2008). Braun et al. (2009) have shown that after correcting the H1 column densities of M31 for opacity the apparent cut-off vanishes.

(d) The H1 column densities of M31 extend to higher values than those of M51 and the maximum of the PDF of M31 occurs at a column density that is five times that of M51. On the sky, M31 is much brighter in H1 than M51, but the face-on value of \(X_0\), defined as \(X_0 \cos i\), of M31 is only about 1.2 times that of M51. The H1 PDF for M51 has a pronounced low-density tail, while that for M31 only shows a mild low-density excess. The low-density tail of the M51 PDF may be a relic of the last encounter with the companion galaxy NGC5195 (Howard & Byrd 1990) that gave rise to the extended H1 arm in the south-east (Rots et al. 1990).

(e) The H2 column densities of M51 are much higher than those of M31: the face-on value of \(X_0\) of M51 is 11 times that of M31. The star formation rate of M51 is five times higher than that of M31 (Tabatabaei et al. 2013), which may be related to the higher column densities reached in M51 than in M31 (Kainulainen et al. 2014). H2 is the dominant gas phase in the inner part of M51 (García-Burillo, Guélin & Cernicharo 1993; Schuster et al. 2007), while in M31 the atomic gas is the dominant component (Dame et al. 1993; Nieten et al. 2006).

### 3.2 PDFs of volume densities in M31

We calculated the mean H1 volume densities, \((n_{\text{H1}})\), from the column densities using the formula for the H1 scaleheight increasing with radius, \(h(R)\), given by Braun (1991), scaled to the distance \(D = 780 \, \text{kpc}\). This gives \(h(10 \, \text{arcmin}) = 240 \, \text{pc}\) and \(h(60 \, \text{arcmin}) = 425 \, \text{pc}\). The volume density then is \(n_{\text{H1}} = N_{\text{H1}} / L\) where the LOS \(L = 2h(R) / \cos i\). Assuming a scaleheight of the molecular gas equal to \(h(R)/2\) yielded the mean volume densities of the molecular gas, \((n_{\text{H2}})\). The volume density of all neutral gas then is \((n) = (n_{\text{H1}}) + (n_{\text{H2}})\). As an example, we present the PDFs of these volume densities for the ‘ring’ in Fig. 3 and the lognormal fits for all regions in Table 3. The H1 PDF is narrower and more symmetric than that for the full area shown in Fig. 1 because the low-density extension is not included. It is shifted by a factor of 2.4 w.r.t. the PDF for the inner region. Comparison with Table 1 shows that the dispersions of the PDFs of \((n_{\text{H1}})\) and \((n_{\text{H2}})\) are the same as those of the column density PDFs within the errors (see Section 4 for a simple explanation for this equivalence.). These estimates for the mean volume density PDFs are useful in that they allow us to compare the properties of M31 with those of the MW in Section 5.2.

### Table 2. Lognormal fits to the PDFs of gas surface densities in M51 at \(R < 3 \, \text{arcmin}\)

| \(X\) | Units | \(N\) | \(\mu\) | \(X_0\) | \(\sigma\) | \(\chi^2\) |
|-----|-------|-----|-----|-----|-----|-----|
| \(N(\text{H1})\) | \(10^9\) \text{atom cm}^{-2} | 2605 | 1.46 ± 0.05 | 28.7 ± 3.3 | 0.51 ± 0.05 | 6.4 |
| \(N(\text{H2})\) | \(10^9\) \text{atom cm}^{-2} | 282 | 1.60 ± 0.05 | 39.4 ± 4.5 | 0.64 ± 0.05 | 1.1 |
| Hα | \(\text{cm}^{-6} \text{pc}\) | 728 | 1.01 ± 0.04 | 10.2 ± 1.0 | 0.59 ± 0.03 | 1.4 |

Notes. \(N\) is the total number of data points; the error in each bin \(\delta_i\) is estimated as \(\delta_i = \sqrt{\frac{N_i}{(\text{number of bins} - 2)}\text{ degrees of freedom}}\); \(\chi^2\) is the reduced chi-squared goodness of fit parameter.
are merely compatible with this assumption (Tassis et al. 2010; Hennebelle & Falgarone 2012). One could consider two contributions to the column density PDFs: one from the density structure arising from compressible turbulence in the ISM and the other from the densities of specific, discrete objects that are not caused by turbulence, such as supernova remnants and H\textsc{ii} regions (both of which may, however, be drivers of turbulence). How to separate these contributions to the PDF is not clear. Both types of structure have sizes less than the beam. The dominant energy input driving turbulence in the ISM is probably due to stellar cluster winds and superbubbles produced by clustered supernovae, with a typical scale in the range 50–500 pc (Elmegreen & Scalo 2004, Section 3). Many other drivers of turbulence, such as spiral shocks, the magnetorotational instability and protostellar jets have also been proposed (see Federrath & Klessen 2013, for a discussion of these in the context of PDFs). The H\textsubscript{2} at the largest scale is collected in giant molecular clouds (GMCs), with a typical size of 50 pc (Blitz et al. 2007) to 100 pc in M51 (Colombo et al. 2014), and the H\textalpha\ has both a diffuse component as well as higher density H\textsc{ii} regions of typical size 30 pc (Habing & Israel 1979; Scoville et al. 2001; Azimlu, Marcink & Barmby 2011). The thickness of the diffuse gas layer is of the order of 500 pc and that of the H\textalpha\ 150 pc, so each telescope-beam cylinder will contain one or many turbulent cells or GMCs or H\textsc{ii} regions. Furthermore, the ISM consists of several phases, so the emission that we use to construct the average volume density PDFs is not uniformly distributed along the total LOS but only arises from a fraction of this distance. In order to understand how averaging over multiple cells or clouds, both along and across the beam, and how the filling factor of the ISM occupied by the phase we observe affect the density PDFs, we constructed a simple model.

4.1 A simple model for the random gas density distribution in the ISM

Independent random numbers drawn from a lognormal distribution with parameters $\mu = 0$, $\sigma = 1$—see equations (1) and (2)—were distributed over a Cartesian $(x, y, z)$ grid of $(1000 \times 1000 \times L)$ mesh points. The value of each mesh point represents the gas density of a single gas cloud and the unit of distance in the model is thus the cloud size. The parameter $L$ represents the LOS and for $L > 1$ the random numbers were summed along the $z$-axis to give a map of the column density of the model, with $1000 \times 1000$ LOS. An average volume density along each LOS was also computed by dividing the column density by $L$. Convolutions of the column density

### Table 3. Lognormal fits to the PDFs of gas volume densities in M31, HPBW 45 arcsec.

| $X$          | Units       | R (arcmin) | N  | Position of maximum $\mu$ | $\chi^2$ | $\sigma$ |
|--------------|-------------|------------|----|---------------------------|----------|----------|
| $\langle n_{\text{H}_1} \rangle$ | atom cm$^{-3}$ | 0–60 | 1457 | $-0.75 \pm 0.03$ | 0.78 $\pm 0.012$ | 0.25 $\pm 0.02$ | 12.5 |
| $\langle n_{\text{H}_1} \rangle$ | atom cm$^{-3}$ | 0–35 | 459  | $-1.00 \pm 0.02$ | 1.00 $\pm 0.005$ | 0.23 $\pm 0.01$ | 1.6 |
| $\langle n_{\text{H}_1} \rangle$ | atom cm$^{-3}$ | 35–60 | 998  | $-0.62 \pm 0.02$ | 0.240 $\pm 0.011$ | 0.19 $\pm 0.01$ | 5.7 |
| $\langle n_{2\text{H}_2} \rangle$ | atom cm$^{-3}$ | 0–60 | 920  | $-1.05 \pm 0.03$ | 0.089 $\pm 0.006$ | 0.39 $\pm 0.02$ | 2.6 |
| $\langle n_{2\text{H}_2} \rangle$ | atom cm$^{-3}$ | 0–35 | 248  | $-1.08 \pm 0.05$ | 0.083 $\pm 0.010$ | 0.41 $\pm 0.03$ | 1.3 |
| $\langle n_{2\text{H}_2} \rangle$ | atom cm$^{-3}$ | 35–60 | 672  | $-1.04 \pm 0.03$ | 0.091 $\pm 0.007$ | 0.37 $\pm 0.02$ | 2.0 |
| $\langle n \rangle$ | atom cm$^{-3}$ | 0–60 | 1451 | $-0.61 \pm 0.02$ | 0.245 $\pm 0.012$ | 0.30 $\pm 0.01$ | 5.1 |
| $\langle n \rangle$ | atom cm$^{-3}$ | 0–35 | 457  | $-0.82 \pm 0.02$ | 0.151 $\pm 0.007$ | 0.32 $\pm 0.01$ | 1.1 |
| $\langle n \rangle$ | atom cm$^{-3}$ | 35–60 | 994  | $-0.51 \pm 0.01$ | 0.309 $\pm 0.007$ | 0.25 $\pm 0.01$ | 2.6 |

*Notes.* $\chi^2$ is the reduced chi-squared goodness of fit parameter; the error in each bin $\delta_i$ is estimated as $\delta_i = \sqrt{\chi^2}$ for (number of bins–2) degrees of freedom.
with a 2D Gaussian of a given full width at half-maximum (FWHM) was used to model the consequences of smoothing by a beam of size $D = \text{FWHM}$.

Making $(1 - f_i)$ of the mesh points empty allowed us to control the fractional volume occupied by the clouds, $f_i = V/(1000 \times 1000 \times L)$, where $V$ is the number of mesh points for which the gas density is non-zero. The empty mesh-points were chosen at random. We shall refer to $f_i$ as the volume filling factor but stress that there are at least two other quantities that are also called filling factors. Both measure the ‘clumpiness’ of a given gas phase, either averaged over the entire volume or only the volume occupied by that phase, and calculated using the ratio $(n)^2/(\sigma^2)$, where $n$ is the gas density and $(\ldots)$ represents the appropriate average: note that the relation between these filling factors and $f_i$ is not straightforward (see Gent et al. 2013, section 5). Under the assumption that the gas is confined to clouds which all have the same density, which is commonly made when deriving filling factors from observational data (see e.g. Berkhuijsen & Fletcher 2008), the fractional volume occupied by the gas and the volume filling factor defined as $(n)^2/(\sigma^2)$ are the same. We adopt this assumption in what follows; in other words we assume that the fractional volume and the volume filling factor are one and the same quantity.

Normalized histograms of the column or average volume density gave the model PDFs, and the best-fitting lognormal of density to the PDF, using a least-squares method, was used to compute $\mu$ and $\sigma$ for each model. 10 independent realizations were calculated for each set of model parameters and the resulting $\mu$ and $\sigma$ averaged. For $L = 1$, $f_i = 1$ and perfect resolution (no smoothing), the model PDFs have the same parameters, $\mu = 0$ and $\sigma = 1$, as the lognormal distribution used to populate the model (within a numerical error due to the finite size of the mesh and the constraints of the fitting algorithm). We used the model to explore how $\sigma$, as this is by far the most useful of the two parameters in interpreting observations, varies as $L$, $f_i$, and $D$ change.

### 4.2 PDF dispersion and the path length

Fig. 4(a) shows how the dispersion of the lognormal PDF, $\sigma$, changes as the path length along the LOS, $L$, is increased. Here $f_i = 1$ and no smoothing is applied. For $L = 1$ the model recovers $\sigma = 1$, as required by the parameters of the original distribution. As $L$ increases $\sigma$ decreases as a power law $\sigma \propto L^{-\alpha}$. The fitted power law in Fig. 4(a) has $\alpha = -0.4$ for the range $1 < L < 64$, but as the lower end of the range in $L$ increases (i.e. for $L \gg 1$) we find that $\alpha \to -0.5$. This scaling is in agreement with Fischera & Dopita (2004) who studied the relationship between $\sigma$ of the column density PDF and the ratio of $L$ to the size of a turbulent cell, $L/s_{\text{max}}$: for thick discs, i.e. $L/s_{\text{max}} = 1$, they derived $\sigma \propto (L/s_{\text{max}})^{-0.5}$. In our simple models $s_{\text{max}} = 1$ and so $L/s_{\text{max}} \geq 1$.

### 4.3 PDFs of column and average volume densities

The scaling of $\sigma$ with $L$ for the lognormal PDFs of the average volume density, defined as $\langle n \rangle = \Sigma/L$, follows the same power-law behaviour as the column density PDFs: for $n$ we also find that $\sigma \propto L^{-0.5}$ for $L \gg 1$. This is not surprising, as dividing a lognormal by a constant changes $\mu$ but not $\sigma$ (e.g. Aitchison & Brown 1957, Theorem 2.1). In these models we set $f_i = 1$ and applied no smoothing.

### 4.4 PDF dispersion and beam smoothing

Fig. 4(b) shows how $\sigma$ changes under smoothing of the model column (and volume) densities, where we have fixed $f_i = 1$. Smoothing reduces $\sigma$ and for beamwidths $D \geq 1$ the scaling follows a power law, shown in the inset to the figure, with $\sigma \propto D^{-1}$ irrespective of $L$. This power-law behaviour means that when $D \geq 4$–5 clouds for shallow $L$ and $D \geq 2$–3 clouds for longer $L$, the effect on $\sigma$ of further moderate smoothing is marginal: the resolution of observations of nearby galaxies is typically in the range $1 < D < 10$. In principle, if observations are available with a resolution of $D < s_{\text{max}}$ and repeated smoothing results in $\sigma \propto D^{-1}$ beyond a certain beamwidth, one could recover $s_{\text{max}}$ from the column density PDFs.

### 4.5 PDF dispersion and the filling factor

Fig. 5(a) shows that as the filling factor increases, i.e. as the gas occupies more of the total volume, the dispersion of the fitted lognormal PDFs for the average volume density decreases linearly. However, the effect on $\sigma$ depends on both $L$ and $f_i$. If $f_i$ is a constant for all LOS, then the power-law scaling $\sigma \propto L^\alpha$ with $\alpha = -0.5$
still holds. However, if \( f_v \) depends on L this scaling can change. This motivates us to define the effective path length

\[
L_{\text{eff}} = f_v L, \tag{3}
\]

which is the fraction of the total LOS that is occupied by clouds or turbulent cells. Fig. 5 shows that \( \sigma \) scales as a power law \( \sigma \propto L_{\text{eff}}^\alpha \) for \( L_{\text{eff}} > 1 \). The best-fitting power law has an index \( \alpha = -0.5 \), as in the models where \( f_v = 1 \) shown in Fig. 4(a).

4.6 Summary of model results

For convenience we summarize the results obtained using this simple model for a lognormal distribution of gas density in the ISM. We shall use these results to interpret the observations.

(i) The dispersions of the lognormal column density and average volume density PDFs both scale as a power law with the total LOS, \( \sigma \propto L^\alpha \). When the filling factor \( f_v = 1 \) both have \( \alpha = -0.5 \).

(ii) Smoothing by the telescope beam, when the beam is bigger than the size of a cloud, results in a power-law scaling \( \sigma \propto D^{-1} \). For \( D > 5s_{\text{max}} \) in a thin layer (\( L/s_{\text{max}} \sim 1 \)) and \( D > 2s_{\text{max}} \) in a thicker layer (\( L/s_{\text{max}} \sim 10 \)), further smoothing only produces a weak effect on \( \sigma \).

(iii) \( \sigma \) is linearly related to the filling factor of the gas but the magnitude of the effect also depends on \( L \). Defining the effective path length \( L_{\text{eff}} = f_v L \) results in the scaling \( \sigma \propto L_{\text{eff}}^{-0.5} \).

5 DISCUSSION

Two differences between the PDFs shown in Figs 1–3 and Tables 1–3 are immediately obvious: the M51 PDFs are significantly wider than the M31 PDFs and for both galaxies the H\( _2 \) PDF is wider than the H\( _1 \) PDF. Which factors could cause these differences?

Simulations of the ISM have shown that the dispersion of a PDF depends on variables like disc mass, temperature, magnetic field strength, length of the LOS and density scale. Furthermore, we showed in Section 4 that also the filling factor plays a role. Below we consider each of these factors separately.

The width of the PDF increases with the gas density in the disc (Tassis 2007; Wada & Norman 2007). In the HD simulations of Wada & Norman (2007) an increase of a factor of 10 in disc mass leads to an increase in the dispersion of 30 per cent. As the mean face-on column density of the total gas in M51 is about five times higher than that in M31 (calculated from the H\( _1 \) and H\( _2 \) PDFs with \( \langle N \rangle = X_0 \exp(\sigma_{\text{HD}}^2/2) \) (Ostriker, Stone & Gammie 2001, and added), this could account for a difference of about 15 per cent in the dispersions of the gas PDFs for the two galaxies.

In the magnetohydrodynamic (MHD) simulations of de Avillez & Breitschwerdt (2005), the dispersion of the PDF of cold gas is wider than that of warm or hot gas. This was confirmed by the HD simulations of Robertson & Kravtsov (2008) and Gent et al. (2013). Since the molecular gas is generally cooler than the atomic gas, the temperature difference could cause the H\( _1 \) PDF to be wider than the H\( _1 \) PDF. Unfortunately, we cannot estimate the expected difference in dispersion because the temperature ranges in the figures of these simulation papers are too large.

Molina et al. (2012) simulated the development of molecular clouds with time using an MHD code. They derived relations between the dispersion of the PDF, the Mach number \( M \) and plasma \( \beta \), the ratio between thermal and magnetic pressure. They found that increasing \( M \), i.e. more cooler and denser gas, widens the PDFs of the clouds (see also Federrath et al. 2008; Price, Federrath & Brunt 2011; Federrath 2013), consistent with our observations, whereas decreasing \( \beta \) makes the PDFs somewhat narrower. We can estimate the ratio of mean \( \beta \) in M51 to that in M31 by comparing the thermal and magnetic pressures in the two galaxies. As the total thermal pressure is dominated by the ionized gas, we use \( X_0 \) of the H\( \alpha \) PDFs (Tables 1 and 2) and the LOS \( L \) given in Tables 4 and 5 to calculate the rms electron density \( \sqrt{n_e^2} \) for each galaxy. This is 2.9 times higher in M51 than in M31 suggesting that in M51 the thermal pressure is about three times higher than in M31. As the magnetic field in M51 (Fletcher et al. 2011) is about three times stronger than in M31 (Fletcher et al. 2004), the magnetic pressure in M51 is about nine times higher than in M31; hence, \( \beta \) in M51 is about one-third of that in M31. Molina et al. (2012) set \( \beta \propto M^{-2} \) in their simulations of molecular clouds. If this relationship also holds in less dense gas phases, a three times smaller \( \beta \) would imply an \( \sqrt{3} = 1.7 \) times higher mean \( M \) in M51 than in M31. Since \( \beta \) affects the PDF dispersion more strongly than \( \beta \) (see Molina et al. 2012), the PDFs for M51 will be wider than those for M31, in agreement with our observations. Separate measurements of \( M \) and \( \beta \) will be necessary to see how large this effect is compared to other influences on the width of the PDF.

The scale of the density fluctuations influences the dispersion of the PDF because an increase in the number of clouds seen by the
Table 4. Effective LOS $L_{\text{eff}}$ and filling factors.

| Gal. | Gas  | Zone        | $N$  | $L$ (pc) | $L_{\text{eff}}$ (pc) | $f_e$ | Dispersion | Ref. |
|------|------|-------------|------|---------|------------------------|------|------------|------|
| M31  | 2H$_2$ | $R < 35$ arcmin | 248  | 1370    | 60 $\pm$ 15           | 0.04 $\pm$ 0.02 | 0.41 $\pm$ 0.03 | 1    |
|      |      | $R > 35$ arcmin | 672  | 1770    | 70 $\pm$ 15           | 0.04 $\pm$ 0.01 | 0.37 $\pm$ 0.02 | 1    |
|      | H$_1$  | $R < 35$ arcmin | 459  | 2730    | 180 $\pm$ 40          | 0.07 $\pm$ 0.02 | 0.23 $\pm$ 0.01 | 1    |
|      |      | $R > 35$ arcmin | 998  | 3560    | 270 $\pm$ 60          | 0.08 $\pm$ 0.02 | 0.19 $\pm$ 0.01 | 1    |
| MW   | H$_{\text{cold}}$ | $|b| > 5^\circ$ | 64   | 2050    | 130 $\pm$ 40          | 0.06 $\pm$ 0.03 | 0.27 $\pm$ 0.03 | 3    |
|      |      | $|b| < 5^\circ$ | 98   | 2300    | 90 $\pm$ 25           | 0.04 $\pm$ 0.01 | 0.33 $\pm$ 0.03 | 2    |
|      | N$_e$  | $|b| > 5^\circ$ | 42   | 4630    | 300 $\pm$ 65          | 0.06 $\pm$ 0.02 | 0.18 $\pm$ 0.01 | 2    |
|      |      | $|b| < 5^\circ$ | 34   | 2480    | 200 $\pm$ 50$^*$      | 0.08 $\pm$ 0.02 | 0.22 $\pm$ 0.02$^*$ | 2   |
|      | O$_{\text{vi}}$ | $|z| > 200$ pc | 190  | 2040    | 100 $\pm$ 25          | 0.05 $\pm$ 0.02 | 0.31 $\pm$ 0.02 | 4    |
|      | N$_e^2$ | $|b| > 5^\circ$ | 157  | 1700    | 85 $\pm$ 20           | 0.05 $\pm$ 0.02 | 0.29 $\pm$ 0.02 | 2    |
|      |      | $|b| < 5^\circ$ | 728  | 750     | 28 $\pm$ 6            | 0.04 $\pm$ 0.01 | 0.59 $\pm$ 0.03 | 5    |

Notes. $N$ = number of LOS; assumed errors in $L$: 25 per cent; $^*$ = 0.22 $\pm$ 0.02 and $L_{\text{eff}} = 200$ $\pm$ 50 pc used to calibrate $L_{\text{eff}}$ (see text). References – 1: this work; 2: Berkhuijsen & Fletcher (2008); 3: Berkhuijsen & Fletcher (2012); 4: obtained from data in Bowen et al. (2008); 5: Seon (2009).

Table 5. Estimates of LOS dispersion of column density PDF.

| Gal. | Comp.   | $L$ (pc) | $L_{\text{eff}}$ (pc) | $f_e$ | Dispersion |
|------|---------|----------|------------------------|------|------------|
| M31  | H$_2$   | 1750 $\pm$ 1500 | 85 $\pm$ 20        | 0.05 $\pm$ 0.04 | 0.34 $\pm$ 0.02 |
| M51  | H$_\alpha$ | 1030 $\pm$ 900 | 37 $\pm$ 10        | 0.04 $\pm$ 0.03 | 0.51 $\pm$ 0.05 |

5.1 Does the observed dispersion depend on beamwidth?

In Section 4 we showed that the dispersion $\sigma$ of the column density PDFs varies strongly under smoothing when the smoothing beamwidth is smaller than about five clouds, but that once the beamwidth is larger than this the effect of further smoothing becomes negligible. Thus the behaviour of the best fit $\sigma$ under repeated smoothing can, in principle, be used to set a rough upper limit on the correlation size of the clouds.

We investigated this point for the column densities of H$_1$ and H$_2$ of M31. To this end we smoothed the H$_1$ and H$_2$ maps with Gaussian functions to angular resolutions of 24 arcsec $\times$ 36 arcsec (indicated by 30 arcsec), 90 arcsec and 180 arcsec, and calculated the PDFs. Including the PDFs at 45 arcsec and the original resolution of 23 arcsec of the H$_2$ map, the range spans a factor of 7.8 in scale. We found that for the full region $R < 60$ arcmin as well as for the subregions the dispersions of the PDFs are the same within the errors for both H$_1$ and H$_2$ (see Fig. 6). Hence, the dispersion of the gas PDFs for M31 is independent of smoothing scale between about 90 pc $\times$ 400 pc and 700 pc $\times$ 3150 pc in the plane of M31.

Since the LOS through the disc of M31 is large (on average 3000 pc for the H$_1$ disc), Fig. 4(b) suggests that smoothing will not affect $\sigma$ when the FWHM is wider than about two clouds. We can use this to make an estimate of the upper limit to the size of the clouds in M31. The smallest of the smoothing scales we consider, 90 pc $\times$ 400 pc, must contain at least $2^2 = 4$ clouds, giving an estimate for the upper limit of the size of the clouds of $s_{\text{max}} < \sqrt{(400 \times 90)/4} = 95$ pc.

Note that the linear resolution of the M51 observations is about 300 pc, similar to that of the best resolved H$_1$ observations of M31. Therefore, the difference in dispersion of the PDFs of the gas surface...
densities for M31 and M51 cannot be caused by a difference in angular resolution, provided the scales of the clouds are the same.

5.2 LOS effects

In the case of column density PDFs, Fig. 4(a) shows that at a fixed resolution the fitted σ will be smaller by about a factor of 2 for a four times longer LOS. In particular, we found that σ decreases with $L^{-0.5}$. We observe this effect in the PDFs of M31 and M51 in several ways.

First, the PDFs of M31 (Fig. 1) have a smaller dispersion than those of M51 (Fig. 2). The LOS through the strongly inclined galaxy M31 is typically about four times longer than the LOS through M51 that is nearly seen face-on, provided that the scaleheights are similar. Assuming that the cloud scales in the two galaxies are comparable, we then expect more clouds along the LOS through M31 than along the LOS through M51, and hence smaller dispersions of the PDFs for M31 than for M51.

Secondly, for both M31 and M51 the PDFs of H$_2$ are significantly wider than those of H I. Since the molecular gas is more concentrated in the disc mid-plane than H I (due to the larger gravitational forces) and molecular clouds are generally denser and smaller than H I clouds, the LOS through molecular gas are shorter than through H I gas leading to a larger dispersion of the H$_2$ PDF than of the H I PDF. Also the Mach number of the H$_2$ gas is higher than that of the H I gas (because the temperature is much lower whereas the turbulent velocity is similar), causing a higher compression of the gas in the molecular clouds by turbulent shocks and larger dispersions of the cloud PDFs (Molina et al. 2012 and references therein). However, this effect may be reduced if the LOS intersects many clouds, or if the beamwidth is larger than the typical cloud size: in this case the peak of the PDF may move to a higher density but the dispersion can be unchanged. Which of these two effects is responsible for the observed difference in the PDF dispersions requires further study.

Thirdly, the H I PDF for the inner region of M31 is slightly wider ($\sigma = 0.23 \pm 0.01$) than that for the bright ‘ring’ ($\sigma = 0.19 \pm 0.01$) where the LOS is longer and goes through more clouds than in the inner region. This interpretation is supported by the higher value of $\chi^2$ in the ‘ring’. The same trend is visible in the H$_2$ PDFs (see Table 3).

Furthermore, the dispersions of gas PDFs derived in numerical simulations of the ISM (e.g. Vázquez-Semadeni & Garcia 2001; de Avillez & Breitschwerdt 2005; Wada & Norman 2007) are usually larger than observed PDFs. Berkhuijsen & Fletcher (2008) attributed the difference to the higher densities used in the simulations compared to observed densities. However, the much smaller LOS available in simulations than in observations may be the main cause of the larger dispersions of PDFs derived from simulations (see also Fischera & Dopita 2004).

It is possible to quantify the relationship between the dispersion of column density PDFs and LOS. Because the scaleheight of the gas in M31 increases outwards, we can test this relationship using the PDFs of the average volume densities of the subregions at different radii given in Table 3.

In Fig. 7 we show the dispersion of the PDFs of average volume density of H I and H$_2$ as function of the mean LOS $L$ of the radial ranges 0 arcmin < $R$ < 35 arcmin and 35 arcmin < $R$ < 60 arcmin, where $L = 2h(R)/\cos i$ and $h(R)$ is the scaleheight as derived in Section 3.2.

We have added three points for the mean volume density of diffuse H I (cool and warm H I) towards stars in the solar neighbourhood and for the mean volume density of free electrons in the DIG at Galactic latitudes |$b|$ > 5° within a few kpc from the sun as well as two points of mean electron density squared obtained from emission measures towards pulsars, all taken from Berkhuijsen & Fletcher (2008, 2012). We also show the dispersion of the OVI PDF calculated along the LOS to 190 stars using fig. 9 (left-hand panels) in Bowen et al. (2008).

For M51 scaleheights of molecular and ionized gas are available. Pety et al. (2013) combined CO surveys observed with Plateau de Bure and the 30 m IRAM telescope. An extensive analysis showed that the molecular gas not only has a thin disc with a Gaussian scaleheight of $h = 40$ pc, but also a thick disc with a scaleheight of about 250 pc. Taking the about eight times higher volume density of the thin disc in the mid-plane into account, their sum can be described by a single exponential with a scaleheight of about 75 pc. Seon (2009) modelled the Hα emission from M51 observed by Thrilker, Braun & Walterbos (2000) and derived a thin disc of HI regions with exponential scaleheight of 300 pc and a thick disc of DIG with 1 kpc scaleheight. In the mid-plane the emission from the thin disc is about seven times stronger than that from the DIG. The sum of the two distributions can be approximated by a single exponential disc with a scaleheight of 350 pc. We have added the dispersions of N(H$_2$) and Hα given in Table 2 using the single scaleheights obtained above for the LOS $L$. Details on the data points in Fig. 7 are given in Table 4.

Apart from the N(H$_2$) point of M51 (that we discuss below), the dispersions in Fig. 7 systematically decrease with increasing $L$. A weighted fit through the 12 available points yields the power law

$$\log[\sigma(L)] = (2.0 \pm 1.7) - (0.76 \pm 0.06) \log[L],$$

with $\chi^2 = 0.3$ and $L$ in pc.

We conclude that the length of the LOS through the gas in M31, the MW and M51 is the main influence on the width of the PDFs of average volume density. This implies that the number of clouds along the LOS determines the dispersion of the PDF.

\(^2\) We use the upper value of the range 190–250 pc given by Pety et al. (2013) because it applies to the larger radii covered by our data.
Since the data points for the MW and Hα for M51 in Fig. 7 are consistent with the M31 points, the variation in cloud sizes must be similar in these galaxies. This conclusion is supported by the work of Azimou et al. (2011), who found that the size distributions of H II regions in different galaxies are very similar, but the total number of H II regions differs depending on the star formation rate.

We note that most of the data points in Fig. 7 forming the σ – L relation refer to diffuse gas. Much of the H i in M31 is known to be diffuse (Braun & Walterbos 1992) and the small amount of molecular gas in M31 may ‘look’ diffuse in our large radio beam because it does not contain many small, dense H2 clouds.

5.3 Importance of volume filling factor

The power-law dependence of σ on L presented in Fig. 7 has an exponent of −0.76 ± 0.06, which is −0.26 smaller than expected from the models in Section 4. However, the density structure in the ISM contains holes and voids in and between gas clouds (e.g. Elmegreen 1997, 1998; de Avillez & Breitschwerdt 2005) and so we should also take into account the volume filling factor, f_v, of the gas using equation (3).

Estimates of f_v from observations (Dickey 1993; Berkhuijsen 1999) and simulations (de Avillez & Breitschwerdt 2005; Gent et al. 2013) have shown that H i gas has a larger filling factor than the denser molecular gas. So in Fig. 7 f_v will increase with increasing L and L_{eff} will increase faster than L.

For the (n_e) point for the MW at L = 2480 pc in Fig. 7, Berkhuijsen & Müller (2008) derived a volume filling factor of f_v = 0.08 ± 0.02. Hence the effective LOS producing the dispersion of 0.22 ± 0.02 is L_{eff} = (0.08 ± 0.02) × 2480 ∼ 200 ± 50 pc. Assuming that the relation σ ∝ L^{−0.5} found in Section 4 holds in the ISM, we can estimate L_{eff} for the other points in Fig. 7 using the observed σ in σ/(0.22 ± 0.02) = (L_{eff}/(200 ± 50))^{−0.5}. The volume filling factor then is f_v = L_{eff}/L_v.

We list the observed and derived properties of the 13 points in Fig. 7 in Table 4. The errors given are random errors only. The error of 25 per cent in L_{eff} of the calibration point is systematic and affects all L_{eff} and f_v in the same way. As expected, the values of L_{eff} increase with increasing L varying between about 25 pc in M51 and more than 200 pc in the nearly edge-on galaxy M31. The filling factors, though, vary much less: in M31 between 0.04 ± 0.01 for H2 and 0.08 ± 0.02 for the H i point at L = 3560 pc. Apart from the deviating H2 point for M51, also the other filling factors are in this range. These values seem amazingly small, but they compare well with other information. For the cold (H2 and H i) gas at temperatures T < 200 K, Breitschwerdt & de Avillez (2005) found from their MHD simulations f_v = 0.06. Gent et al. (2013) derived from their simulations f_v = 0.03 for cold gas at temperatures T < 500 K. In their OVI-line study, Bowen et al. (2008) estimate that the size of the gas regions causing these lines is about 100–200 pc. Since the mean distance is L = 2040 pc, f_v = 0.049–0.098 for these regions, consistent with our value of f_v = 0.05 ± 0.02. We note that at L = 2480 pc σ of (n_e^2) is larger than σ of (n_e), as one would expect. The larger σ of (n_e^2) leads to values of L_{eff} and f_v that are nearly half of those of (n_e), in agreement with the smaller scaleheight and larger clumpiness of H ii regions compared to the DIG.

In Fig. 7 the H2 point for M51 is located at a much smaller LOS than is expected from the relation formed by the other 12 data points. This may be due to the large filling factor of f_v = 0.15 ± 0.05, which is three to four times larger than f_v of the H2 points for M31 (see Table 4). In the inner parts of M51 (R < 3 arcmin) the dominant gas component is H2 (e.g. Pety et al. 2013), while in M31 H i is the dominant component (e.g. Nieten et al. 2006). At the end of Section 3.1 we showed that the mean face-on column density of H2 in M51 is 11 times higher than that of M31, which explains the higher value of f_v in M51. When the filling factor is high, already a small effective LOS contains so many clouds that the PDF becomes lognormal. As L_{eff} in M51 is only about 25 pc, the maximum size of the clouds along the LOS must be much less than 25 pc.

Since the dispersion of the column density PDF is the same as that of the PDF of the average volume density (see Section 4), we can estimate L of Hα in M31 and of H i in M51 from equation (4) using the observed dispersions. This yields L = 1750 ± 1500 pc for Hα in M31, similar to that of H i at R > 35 arcmin. Using the same scaling for L_{eff} as above, we find L_{eff} = 85 ± 20 pc and f_v = 0.05 ± 0.02 (see Table 5). The value of σ implies a scaleheight of 190 ± 160 pc, the same as for the H2 in the emission ring. This scaleheight of Hα is consistent with the observation of Walterbos & Braun (1994) that the z-extend of the DIG in M31 is less than about 500 pc. If equation (4) and the scaling for L_{eff} also hold for the H i gas in M51, then the observed dispersion in Table 2 indicates an LOS of about 1 kpc (see Table 5). The corresponding L_{eff} and filling factor of f_v = 37 ± 10 pc and f_v = 0.04 ± 0.01, respectively, are smaller than in M31, which may be due to the small H i content in M51.

Interestingly, Seon (2009) presented separate Hα PDFs for the DIG and the H ii regions in M51 based on the data of Thilker et al. (2000). Although the scaleheights differ by a factor of 3 (300 and 1000 pc for H ii regions and the DIG, respectively), both PDFs have a dispersion of 0.36 ± 0.03, but the position of their maxima differs by about a factor of 7. The LOS through the DIG would be 2130 pc and the dispersion of 0.36 ± 0.03 is not far from the value of 0.29 ± 0.09 expected from equation (4), but the point for the H ii regions’ PDF with an LOS of 640 pc disagrees with the value of σ = 0.73 ± 0.14 expected from equation (4). The same dispersion for the two layers indicates that L_{eff} is the same and the difference in LOS then means that the volume filling factors and/or the maximum cloud scales differ. Using the same scaling as before, we find that σ = 0.36 ± 0.03 corresponds to L_{eff} = 75 ± 20 pc giving f_v = 0.04 ± 0.01 for the DIG and f_v = 0.12 ± 0.04 for the disc of H ii regions. The filling factor of the H ii regions is similar to that of the H2 clouds in M51 suggesting that their density structure is similar.

It is worth noting that as the PDFs for the disc and the DIG in M51 are strongly shifted in log (X_φ), their combination yields a much wider PDF than that of the separate components. Thus our Hα PDF for M51 is not representative for small regions or specific components. This poses a general problem for the interpretation of PDFs for large regions in galaxies containing diffuse, low-density regions as well as dense spiral arms.

5.4 Estimating Mach numbers using PDFs: a note of caution

A connection between the dispersion of the density PDF σ and the Mach number M of interstellar turbulence has been identified theoretically (e.g. Padoan et al. 1997; Passot & Vázquez-Semadeni 1998; Ostriker et al. 2001; Federrath et al. 2008, 2010; Price et al. 2011; Molina et al. 2012) and used to make an estimate of M from observations (e.g. Padoan et al. 1997; Berkhuijsen & Fletcher 2008; Hill et al. 2008). However, since the magnitude of σ depends on the length of the LOS L, the telescope resolution Δ and the volume filling factor of the gas f_v, caution should be exercised in applying the theoretical relation to real data. To make a reliable estimate
of $M$ one needs to know something about the distribution of the observed gas and so correct the observed $\sigma$ for the effects of $L$, $D$ and $f_c$.

5.5 M31: PDFs of dust emission and extinction

In view of the generally lognormal shapes of the gas PDFs, one may ask whether other constituents of the ISM also have lognormal PDFs. The dust emission is a good candidate since dust optical depth and gas column density are well correlated in the SN (Bohlin, Savage & Drake 1978) and in M31 (Tabatabaei & Berkhuijsen 2010), and lognormal PDFs of molecular gas column density and of extinction through dust clouds in the MW have been reported (e.g. Padoan et al. 1997; Goodman et al. 2009; Kainulainen et al. 2009; Schneider et al. 2013).

We calculated the PDFs of the dust emission from M31 at 24, 70 and 160 $\mu$m using the Spitzer MIPS maps of Gordon et al. (2006) smoothed to a resolution of 45 arcsec by Tabatabaei & Berkhuijsen (2010). The lognormal fits are given in Table 6. At 24 and 70 $\mu$m the fits for the three regions closely agree, as is also the case for the PDFs of $N(H_2)$ and H$\alpha$ (see Table 1). The dispersion of 0.34 $\pm$ 0.01 of the 24 $\mu$m PDFs is the same as that of H$\alpha$ and the dispersion of 70 $\mu$m PDFs agrees with that of H$\alpha$. As both the ionized gas and the warm dust are mainly heated by young OB stars, the agreement indispersion suggests that the density structure of the ionized gas is similar to that of the warm dust particles. This is consistent with the good pixel-to-pixel correlation between these dust emissions and that of ionized gas reported by Tabatabaei & Berkhuijsen (2010).

Although at 160 $\mu$m the dispersion of the PDFs is also the same for each region, the value of $X_0$ for the “ring” is about 25 per cent higher than for the inner region. The dispersion of 0.27 $\pm$ 0.01 is close to that of the H$\alpha$ PDFs (see Table 1). Since the dust emitting at 160 $\mu$m is mainly heated by the smooth ISRF (Xu & Helou 1996), this suggests that the density structures of the cool dust and the H$\alpha$ gas are similar. Indeed, Tabatabaei & Berkhuijsen (2010) found that the wavelet correlation on scales below about 1 kpc between 160 $\mu$m and H$\alpha$ is much better than between the other wavelengths and H$\alpha$.

We also calculated the PDFs of the dust extinction $A_\alpha$ through M31 using the map of optical depth at the wavelength H$\alpha$, $\tau_{H\alpha}$, derived by Tabatabaei & Berkhuijsen (2010), and $A_\alpha = 1.234 \tau_{H\alpha}$. The parameters of the lognormal fits are given at the end of Table 6. The dispersions in the $A_\alpha$ PDFs for the subregions closely agree with those of the column densities of H$\alpha$ (see Table 1), the dominant gas phase in M31. This indicates that in spite of the radial increase in the atomic gas-to-dust ratio the density structure in the atomic gas and the cool dust are very similar, in agreement with the nearly linear correlation between extinction and $N(H_\alpha)$ obtained by Tabatabaei & Berkhuijsen (2010).

5.6 Comparison with the MW

The H$\alpha$ PDFs for M31 and M51 may be compared to those obtained for the MW by Hill et al. (2008) from the Wisconsin H$\alpha$-Mapper survey (Haffner et al. 2003). For the PDF of the emission measure perpendicular to the Galactic plane, EM $\sin|b|$, of the DIG at $|b| > 10^\circ$ they found a dispersion of 0.190 $\pm$ 0.001, which is nearly half the value for the DIG in M51 (Seon 2009) and for M31 (see Table 1). However, in order to derive the PDF of DIG only, Hill et al. (2008) removed all sight-lines towards classical H$\alpha$ regions from their sample, whereas in our PDFs for M31 and M51 these regions are included. The MW PDF including classical H$\alpha$ regions has a much larger dispersion than 0.19 and shows a high-density excess like the H$\alpha$ PDFs for M31 and M51. High-density tails are expected (Ostriker et al. 2001) if the temperature in the medium decreases with increasing density, as is the case in the ionized medium in the MW (Madsen, Reynolds & Haffner 2006).

Hill et al. (2008) showed that the dispersion of the PDF of EM $\sin|b|$ decreases with increasing latitude. Seon (2011) found the same effect for the diffuse background emission from the MW in FUV, which mainly consists of starlight scattered by interstellar dust. Because the LOS decreases with increasing $|b|$, we would expect the opposite behaviour (see Section 5.2). However, the decrease in $\sigma$ is consistent with the observation of Berkhuijsen & Müller (2008) that the spread in $(n_e)$ decreases with increasing distance to the plane $|z|$ (also visible in Fig. 5 of Gaensler et al. 2008), which is accompanied by an increase in the size of ionized clouds. Savage & Wakker (2009) arrived at a similar conclusion based on O VI-line observations. We show below that the decrease in dispersion with increasing $|b|$ is due to an increase in the volume filling

### Table 6. Lognormal fits to the PDFs of dust emission from M31, HPBW 45 arcsec. The fitted function is $Y = (\sqrt{2\pi\sigma})^{-1} \exp[-(\log_{10} X - \mu)^2/2\sigma^2]$. $Y$ is the fraction of sightlines in each bin divided by the logarithmic bin-width $\Delta(\log_{10} X)$. $\chi^2$ is the reduced chi-squared goodness of fit parameter, with the error in each bin $\delta_i$ estimated as $\delta_i = \sqrt{N_i}$ and for (number of bins $−2$) degrees of freedom.

| $X$ | Units | $R$ (arcmin) | $N$ | $\mu$ | $X_0$ | $\sigma$ | $\chi^2$ |
|-----|--------|-------------|-----|-------|------|--------|--------|
| 24 $\mu$m | MJy sr$^{-1}$ | 0–60 | 1508 | -0.47 $\pm$ 0.01 | 0.34 $\pm$ 0.01 | 0.34 $\pm$ 0.01 | 1.2 |
| 35–60 | 893 | -0.46 $\pm$ 0.01 | 0.35 $\pm$ 0.01 | 0.35 $\pm$ 0.01 | 0.8 |
| 70 $\mu$m | MJy sr$^{-1}$ | 0–60 | 1426 | 0.43 $\pm$ 0.02 | 2.69 $\pm$ 0.13 | 0.39 $\pm$ 0.01 | 1.6 |
| 35–60 | 924 | 0.43 $\pm$ 0.02 | 2.69 $\pm$ 0.13 | 0.39 $\pm$ 0.01 | 1.3 |
| 160 $\mu$m | MJy sr$^{-1}$ | 0–60 | 1516 | 1.36 $\pm$ 0.01 | 23.4 $\pm$ 0.6 | 0.26 $\pm$ 0.01 | 1.8 |
| 35–60 | 981 | 1.40 $\pm$ 0.01 | 25.1 $\pm$ 0.6 | 0.26 $\pm$ 0.01 | 1.4 |
| $A_\alpha$ | mag | 0–60 | 1406 | -0.04 $\pm$ 0.02 | 0.91 $\pm$ 0.04 | 0.20 $\pm$ 0.01 | 6.6 |
| 35–60 | 910 | 0.01 $\pm$ 0.01 | 1.02 $\pm$ 0.03 | 0.18 $\pm$ 0.01 | 4.4 |
factors of the ionized gas and the dust grains, leading to an increase in the effective LOS.

Hill et al. (2008) and Seon (2011) derived the dispersion for the latitude intervals $10^\circ < |b| < 30^\circ$, $30^\circ < |b| < 60^\circ$ and $60^\circ < |b| < 90^\circ$. From the dispersion and the scaling explained in Section 5.3, we calculated $L_{\text{eff}}$ for each case, which increases when $\sigma$ decreases. Assuming a maximum LOS equal to the scaleheight of the DIG of $L = 1$ kpc (Berkhuijsen & Müller 2008; Schnitzeler 2012), we obtain the volume filling factor $f_v = L_{\text{eff}}/L$. Table 7 shows that $f_v$ of both the DIG and the scattering dust increases with increasing $|b|$. The largest part of the DIG and dust layers is sampled by LOS in the interval $10^\circ < |b| < 30^\circ$, whereas at the highest latitudes mainly DIG and dust at high $|z|$ near the Sun are observed. Therefore, the value of $f_v$ = 0.26 ± 0.05 for the DIG at $10^\circ < |b| < 30^\circ$ best represents the mean filling factor of the DIG seen towards $L = 1$ kpc perpendicular to the Galactic plane. This value agrees well with the value of $f_v = 0.22 ± 0.05$ obtained by Berkhuijsen, Mitra & Müller (2006) from emission measures and dispersion measures towards 157 pulsars at $|b| > 10^\circ$. These authors found an increase of the local filling factor $f_v(z)$ with increasing $|z|$, which was confirmed by Gaensler et al. (2008) using DM and EM towards 51 pulsars with known distance. An increase of $f_v(z)$ with $|z|$ is also indicated by the increase of the mean $f_v$ towards higher latitudes in Table 7.

The dust filling factors for the two lower intervals of $|b|$ are about half of those of the DIG, but at high $|b|$ the values of $f_v$ are the same. Near the plane the clumpiness of the diffuse dust seems larger than that of the DIG. On the other hand, the values of $f_v$ will be too low if the scaleheight of the diffuse dust is smaller than 1 kpc.

### 6 SUMMARY

We have presented PDFs of the surface densities of neutral and ionized gas at radii $R \lesssim 3$ arcmin in M51 and in the radial ranges $0 \text{arcmin} < R < 15$ arcmin, $15 \text{arcmin} < R < 60$ arcmin and $0 \text{arcmin} < R < 60$ arcmin in M31 (Section 3.1, Figs 1 and 2). For M31 we also presented the PDFs of the average volume densities along the LOS of $H_1$, $H_2$ and total gas (Fig. 3) and derived the PDFs of the dust emission and extinction (Table 6). The main conclusions from Section 3, Section 4 and the discussion in Section 5 may be summarized as follows.

1. All PDFs are close to lognormal, but their dispersions differ (Tables 1–6). We have investigated which factors determine the dispersion.

2. The dispersion of the lognormal PDFs of column and average volume densities depend on the path length $L$, with $\sigma \propto L^{-0.5}$, and telescope resolution $D$, with $\sigma \propto D^{-1}$. The PDFs of column and average volume density are also affected by the filling factor of the gas; if an effective LOS is defined as $L_{\text{eff}} = f_v L$ then $\sigma \propto L_{\text{eff}}^{-0.5}$.

3. The length of the LOS through the medium, $L$, is the dominant factor shaping the PDF. Short LOS through the nearly face-on galaxy M51 cause wider PDFs than long LOS through the nearly edge-on galaxy M31, and for both galaxies the $H_2$ PDFs are wider than the $H_1$ PDFs because the scaleheight of molecular gas is only half that of atomic gas. The contribution of the higher Mach number in the $H_2$ to the widening of the observed PDFs needs further study.

4. The dispersions of the PDFs of $H_1$ and $H_2$ column densities in M31 are independent of the beamwidth for angular resolutions of 23–180 arsec (Fig. 6) corresponding to linear resolutions in the plane of M31 of 90 pc × 400 pc to 700 pc × 3150 pc along major × minor axis. This suggests that the maximum cloud size $x_{\text{max}} \lesssim 95$ pc.

5. The dispersions of the volume density PDFs of $H_1$ and $H_2$ for M31 (Fig. 3), of diffuse $H_1$ for the SN (Berkhuijsen & Fletcher 2008, 2012), and of DIG for M51 vary with $L$ as $\sigma \propto L^{-0.76 ± 0.06}$ (Fig. 7), indicating that the density structures are very similar in M31, M51 and the MW.

6. The exponent of the power-law $\sigma - L$ relation is steeper than the $-0.5$ expected from analytic calculations (Fischera & Dopita 2004) and the models in Section 4. We show that the difference can be explained by an increase in the volume filling factor between dense and less dense regions causing an increase in the effective LOS, $L_{\text{eff}} = f_v L$. An increase in $f_v$ also explains the decrease in $\sigma$ with increasing latitude of the PDFs observed for the DIG and the scattering dust in the MW (Hill et al. 2008; Seon 2011).

7. The dispersions of the $A_1$ PDFs for M31 closely agree with those of the column densities of $H_1$, the dominant gas phase in M31, suggesting that the density structures in the cool dust and in $H_1$ are similar.

8. The lognormal PDFs we obtained for M31 and M51 do not, taken in isolation, show that compressible turbulence is shaping the gas density distribution. However, they are compatible with this interpretation: both in terms of their shape and in how their properties vary with beamwidth and LOS. By combining the PDF parameters with independent measures of, for example, the Mach number, filling factor and plasma beta, a deeper understanding of turbulence in a multiphase ISM should become possible.

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