Decomposition matrices for groups of Lie type in non-defining characteristic

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Abstract

We determine approximations to the decomposition matrices for unipotent \( \ell \)-blocks of several series of finite reductive groups of classical and exceptional type over \( \mathbb{F}_q \) of low rank in non-defining good characteristic \( \ell \).
1. Introduction

It is a fundamental problem in representation theory of finite groups to construct all irreducible representations of the finite nearly simple groups, or at least determine their dimensions. This question is open even for the family of symmetric groups. While it was solved by Frobenius more than a hundred years ago for representations over the complex numbers, it remains a challenging problem when the characteristic of the base field is positive. In this manuscript we consider certain families of finite quasi-simple groups of Lie type. For these, the ordinary characters were classified by G. Lusztig in the 1980s (see [40]). For positive characteristic representations it is open in general. It comes in two fundamentally different flavours. In the defining characteristic case, the representations of the finite group are closely related to the representations of the ambient algebraic group, to which a highest weight theory applies, but still deep problems remain. Here, we are concerned with the other, the cross characteristic case. The main problem here can be phrased as finding the $\ell$-modular decomposition matrices for the so-called unipotent blocks.

We obtain very close approximations to these decomposition matrices for all finite groups of Lie type and Lie rank at most 6, and sometimes even beyond, that is to say, we either determine the matrices completely or up to at most a very small number of undetermined entries.

Our motivation for doing this is at least threefold. First, it is of interest for many applications to know the dimensions of all irreducible modules of the “small” finite simple groups, or at least the few smallest such dimensions. Secondly, we expect that the data obtained in this work may be useful to derive and test conjectures for example on

- the number and labels of cuspidal unipotent Brauer characters,
- the parameters of modular Hecke algebras,
- the subdivision of unipotent characters into $\ell$-modular Harish-Chandra series,
- and last not least on the shape of decomposition matrices and on individual decomposition numbers, for example Craven’s conjecture [10].

For example, the parametrisation of cuspidal Brauer characters and of the distribution into Harish-Chandra series for the class of unitary groups was predicted and proved based on similar data obtained in [27]. Thirdly, we expect further applications and connections based on the data obtained here. In fact our previous results on unitary groups [17] and on exceptional groups [18] have already been used by E. Norton [44] to predict and then prove the distribution of characters into modular Harish-Chandra series.

Our results are mostly phrased under the assumption (T_\ell) that unipotent $\ell$-blocks of finite reductive groups $G$ have uni-triangular decomposition matrix. This is known to hold whenever $\ell$ and the defining characteristic of $G$ are not too small, see [5, Thm. A].

The work is structured as follows. The first part contains definitions and some general results. We recall some notions and results from the ordinary and modular representation theory of finite reductive groups in Chapter [1]. In Chapter [2] we derive criteria for when the parameters of modular Hecke algebras can be obtained from a corresponding characteristic zero situation. The second part is devoted to the determination of decomposition matrices in the various groups under consideration. It is divided up according to the order $d = d_{\ell}(q)$ of the size $q$ of the finite field over which our group is defined, modulo $\ell$, the characteristic
of the representations we consider. We start off with the most difficult case of $d = 2$, for which the ranks of Sylow $\ell$-subgroups are the largest. Here, in Section 2.2 we prove a general result on multiplicities in the $\ell$-modular reduction of the Steinberg character, before we go on to treat the different families of finite quasi-simple reductive groups.

In the subsequent Chapters 5 and 6 we consider decomposition matrices for the cases $d = 3$ and $d = 6$ respectively. At the end of each section we give an overview of the distribution into Harish-Chandra series observed in the blocks under consideration. Finally, in Chapter 7 we discuss the $\ell$-modular Brauer trees when $d \geq 7$. In the last chapter we check that our results are in agreement with a conjecture of Craven, and indicate in Proposition 8.2 which further decomposition numbers would follow if this conjecture holds.

**Acknowledgement:** The computation of Harish-Chandra and Deligne–Lusztig induction (and thus restriction) of unipotent characters was done in Jean Michel’s development version of the Chevie system [43] using in particular the command LUSZTIGINDUCTIONTABLE. We thank him for having provided this and various further very convenient functionalities. Chapter 8 was written after an observation by Emily Norton. We thank her for her valuable suggestion as well as for her help in the proof of Proposition 6.15.
Part 1

Setting and general methods
CHAPTER 1

Finite groups of Lie type

1. Basic sets and uni-triangularity

Let \( \ell \) be a prime number and \((K, O, k)\) be an \( \ell \)-modular system which we assume to be large enough for all the finite groups encountered. More specifically, since we will be working with \( \ell \)-adic cohomology we assume throughout this paper that \( K \) is a finite extension of the field \( \mathbb{Q}_\ell \) of \( \ell \)-adic numbers.

Let \( H \) be a finite group. A representation of \( H \) will always be assumed to be finite-dimensional. The set of irreducible characters (resp. irreducible Brauer characters) will be denoted by \( \text{Irr}_H \) (resp. \( \text{IBr}_H \)). Given a simple \( kH \)-module \( N \), we shall denote by \( \varphi_N \) (resp. \( P_N \), resp. \( \Psi_N \)) its Brauer character (resp. its projective cover, resp. the ordinary character of its projective cover). The restriction of an ordinary character \( \chi \) of \( H \) to the set of \( \ell' \)-elements will be denoted by \( \chi^{\circ} \), and referred to as the \( \ell \)-modular reduction of \( \chi \) (or sometimes just the \( \ell \)-reduction of \( \chi \)). It decomposes uniquely on the family of irreducible Brauer characters of \( H \) as

\[
\chi^{\circ} = \sum_{\varphi \in \text{IBr}_H} d_{\chi, \varphi} \varphi.
\]

The non-negative integral coefficients \( (d_{\chi, \varphi})_{\chi \in \text{Irr}_H, \varphi \in \text{IBr}_H} \) form the \( \ell \)-modular decomposition matrix of \( H \). Equivalently, if \( \varphi = \varphi_N \) is the Brauer character of a simple \( kH \)-module \( N \), then the character of its projective cover satisfies

\[
\langle \varphi_N; \varphi \rangle = \dim \text{Hom}_{kH}(P, N)
\]

by Brauer reciprocity.

We denote by \( \langle -; - \rangle_H \) the usual inner product on the space of \( K \)-valued class functions on \( H \). If \( N \) is a simple \( kH \)-module with Brauer character \( \varphi \) and \( P \) a projective \( kH \)-module with ordinary character \( \Psi \) then \( \langle \Psi; \varphi \rangle = \dim \text{Hom}_{kH}(P, N) \) gives the multiplicity of \( P_N \) as a direct summand of \( P \).

Recall that a basic set of characters is a set \( B \) of complex irreducible characters of \( H \) such that \( B^o = \{ \chi^{\circ} \mid \chi \in B \} \) is a \( \mathbb{Z} \)-basis of \( \mathbb{Z}\text{IBr}_H \). This means that the restriction of the decomposition matrix to \( B \) is invertible over \( \mathbb{Z} \).

According to Brauer, the ordinary and \( \ell \)-modular irreducible characters of \( H \) are subdivided into \( \ell \)-blocks such that the decomposition matrix is block diagonal with respect to this partition. It then also makes sense to speak of basic sets of individual \( \ell \)-blocks. Now assume that we have a basic set \( B \) for an \( \ell \)-block \( b \) of \( H \) ordered in such a way that the decomposition matrix of \( b \) with respect to \( B \) is known \textit{a priori} to be lower uni-triangular. Then this facilitates very much the explicit determination of this decomposition matrix as explained in the following example.

**Example 1.1.** Let \( B = \{ \rho_1, \ldots, \rho_r \} \subseteq \text{Irr}_b \) be a basic set of a block \( b \) of \( H \) such that the decomposition matrix is known to be uni-triangular with respect to this ordering,
and let \( \{ \Psi_1, \ldots, \Psi_r \} \) be the corresponding projective indecomposable modules (PIMs for short) in \( b \). Then:

(a) If \( \Psi \) is a projective character in \( b \) that involves \( \rho_i \) but no \( \rho_j \) with \( j < i \) then \( \Psi_i \) is a direct summand of \( \Psi \).

(b) More generally, let \( \Psi, \Psi' \) be two projective characters such that \( \Psi - \Psi' \) involves \( \rho_i \) with positive multiplicity, but no \( \rho_j \) with \( j < i \). Then \( \Psi_i \) is a direct summand of \( \Psi \).

Thus, under certain conditions known PIMs can be subtracted from given projectives, and in this way a set of projective characters can be partially echelonised.

2. Unipotent characters and unipotent blocks

Let \( G \) be a connected reductive linear algebraic group over \( \overline{\mathbb{F}}_p \), and \( F : G \to G \) be a Frobenius endomorphism with respect to an \( \mathbb{F}_q \)-structure. (In particular, we do not consider Ree and Suzuki groups here.) We let \((G^*, F)\) be a Langlands dual group to \((G, F)\). We let \( G := G^F \) and \( G^* := G^{*F} \) denote the associated finite reductive groups. Throughout, whenever \( H \leq G \) is an \( F \)-stable subgroup we write \( H = H^F \).

Recall from \([12, \S13]\) that the irreducible characters of \( G \) over \( K \) fall into rational Lusztig series attached to conjugacy classes of semisimple elements in \( G^* \). Given a semisimple element of \( G^* \) we denote by \( \mathcal{E}(G, s) \) the corresponding Lusztig series. The unipotent characters of \( G \) are by definition the elements of \( \cup \mathcal{E}(G, t) \), for \( t \) running over the \( \ell \)-elements of \( G^* \). Unipotent characters were classified by Lusztig \([40]\). When working with explicit characters later, we will use the notation in \([8]\) for exceptional groups. For classical groups, we will denote by \([\lambda]\) a unipotent character parametrised by a partition or a bipartition \( \lambda \).

Assumption 1.2. Throughout this work we shall make the following assumptions on \( \ell \):

- \( \ell \neq p \) (non-defining characteristic),
- \( \ell \) is very good for \( G \). (i.e., \( \ell \) is good for \( G \) and \( \ell \) divides neither \( |Z(G)/Z^e(G)| \) nor \( |Z(G^*)/Z^e(G^*)| \)).

Note that both conditions are inherited by all Levi subgroups of \( G \), so that inductive arguments can be applied.

In this situation, the unipotent characters lying in a given unipotent \( \ell \)-block \( b \) of \( G \) form a basic set of this block (see \([24, 22]\)). Consequently, the restriction of the decomposition matrix of \( b \) to the unipotent characters is invertible and the whole decomposition matrix of the unipotent blocks can be recovered from these unipotent parts together with the ordinary character table. This square matrix will be referred to as the unipotent \( \ell \)-decomposition matrix of \( G \). In particular every (virtual) unipotent character is a virtual projective character, up to adding and removing some non-unipotent characters. In addition, under our Assumption 1.2 on \( \ell \), not only the parametrisation of unipotent characters but also the decomposition matrix of the unipotent \( \ell \)-blocks is independent of the isogeny type of \( G \) (see \([7, \text{Thm. 17.7}]\)). Therefore, we will not specify it in our statements and proofs.
Furthermore, under these assumptions on \( \ell \), the basic set is widely expected to be uni-triangular. This means that one can order the set of unipotent characters and unipotent Brauer characters in such a way that the unipotent decomposition matrix has uni-triangular shape (see [25, Conj. 3.4]). More precisely, in many of our proofs we will assume the following property of a finite group of Lie type \( G \) at a prime \( \ell \):

\[(T_\ell) \text{ The decomposition matrix of the unipotent } \ell\text{-blocks of } G \text{ has a uni-triangular shape with respect to any total ordering of the unipotent characters compatible with the order on families.}\]

This has recently been shown when \( p \) is a good prime for \( G \) (see [5, Thm. A]).

The unipotent \( \ell \)-blocks of the finite reductive groups are known by the work of Fong–Srinivasan [19], Broué–Malle–Michel [4] and Cabanes–Enguehard [6]. We will use these results throughout without further reference.

3. Deligne–Lusztig theory

We fix an \( F \)-stable maximal torus \( T \) of \( G \) contained in an \( F \)-stable Borel subgroup \( B \) of \( G \). Such a torus is said to be maximally split (or sometimes quasi-split as in [12]).

Given an \( F \)-stable Levi complement \( L \) of a parabolic subgroup \( P \) of \( G \), we denote by \( R^G_L \) and \( {}^*R^G_L \) the Deligne–Lusztig induction and restriction maps

\[ R^G_L : \mathbb{Z} \text{Irr}_A L \rightarrow \mathbb{Z} \text{Irr}_A G \quad \text{and} \quad {}^*R^G_L : \mathbb{Z} \text{Irr}_A G \rightarrow \mathbb{Z} \text{Irr}_A L \]

where \( A \) is the field \( K \) or \( k \). We will only use them when in a situation where they do not depend on \( P \), which justifies our notation. We refer to [1] for conditions ensuring the independence from the parabolic subgroup. When \( P \) can be chosen to be \( F \)-stable — in which case we will say that \( L \) is a 1-split Levi subgroup of \( G \) — these coincide with the Harish-Chandra induction and restriction map induced by the Harish-Chandra induction and restriction functors. These will be also denoted by \( R^G_L \) and \( {}^*R^G_L \).

Let \( t \) be a semisimple element of \( G^* \). Assume that \( L^* := C_{G^*}(t) \) is a Levi subgroup of \( G^* \), in duality with an \( F \)-stable Levi subgroup \( L \) of \( G \). Since \( t \in Z(L^*)^F \), the element \( t \) corresponds to a linear character of \( L = L^F \), which we will denote by \( \hat{t} \). By [12, Thm. 13.25] there is a sign \( \varepsilon \in \{ \pm 1 \} \) which depends only on \( L, G \) and \( F \) such that the maps

\[ E(L, 1) \xrightarrow{-\otimes t} E(L, t) \xrightarrow{\varepsilon R^G_L} E(G, t) \]

are bijections. When \( t \) is an \( \ell \)-element and \( \rho \) a unipotent character of \( L \), the irreducible character \( \varepsilon R^G_L(\rho \otimes \hat{t}) \) lies in a unipotent block of \( G \). Consequently, its \( \ell \)-reduction can be written uniquely as a linear combination of the \( \ell \)-reductions of unipotent characters, which form a basic set. On the other hand, the \( \ell \)-reduction of \( t \) is trivial, therefore by [12, Prop. 12.2] the character \( \varepsilon R^G_L(\rho \otimes \hat{t}) \) has the same \( \ell \)-reduction as the virtual unipotent character \( \varepsilon R^G_L(\rho) \). This will prove to be a powerful tool to derive relations on the decomposition numbers of \( G \), as shown in Example [1.3]

**Lemma 1.3.** Let \( t \) be a semisimple \( \ell \)-element of \( G^* \) such that \( C_{G^*}(t) = L^* \) is a Levi subgroup of \( G^* \) with dual \( L \leq G \). Then for every unipotent character \( \rho \) of \( L \), \( \varepsilon R^G_L(\rho)^\circ \) is a non-negative linear combination of irreducible Brauer characters.
Example 1.4. Let $G = \text{PGL}_2(q)$ and hence $G^* = \text{SL}_2(q)$. The unipotent characters of $G$ are $\text{St}_G$ and $1_G$. There are semisimple element $t \in G^*$ with eigenvalues $\{\lambda, \lambda^{-1}\}$ for every $\lambda \in \mathbb{F}_q^*$ satisfying $\lambda^q = \lambda^{-1}$. When $\lambda \neq \pm 1$, such an element is regular. If $\ell \mid (q+1)$ is an odd prime number, then one can take $\lambda$ to be a non-trivial $\ell$-element. The corresponding semisimple element $t$ is a regular $\ell$-element of $G^*$, therefore $L^* = C_{G^*}(t)$ is a maximal torus of $G^*$. By Lemma [1.3] the $\ell$-modular reduction of the virtual character $-R^G_L(1_L) = \text{St}_G - 1_G$ is a non-negative combination of irreducible Brauer characters. This shows that when $\ell$ is an odd prime number dividing $q+1$, then the $\ell$-reduction of $\text{St}_G$ contains at least one copy of the trivial Brauer character and the unipotent $\ell$-decomposition matrix has the following shape

$$
\begin{array}{ccc}
1_G & 1 \\
\text{St}_G & \alpha & 1
\end{array}
$$

with $\alpha \geq 1$ (the coefficient 1 in the bottom right corner comes from the fact that this square matrix is invertible since the unipotent characters form a basic set for the unipotent blocks). We have $-R^G_L(1_L) = (\alpha - 1)(\varphi_k + \varphi_{\text{St}})$.

Let $W := N_G(T)/T$ denote the Weyl group of $G$. It is a finite Coxeter group with distinguished set of generators $S$ and associated length function $l$. Recall from [8 Prop. 3.3.3] that the $G$-conjugacy classes of $F$-stable maximal tori of $G$ are parametrised by the $F$-conjugacy classes of $W$. Given $w \in W$, we will denote by $T_w$ a torus of type $w$ with respect to $T$ and we will write $R_w := R^G_L(1_{T_w})$. This is a virtual character of $G$ all of whose constituents are unipotent characters.

As pointed out above, under our Assumption[12] the unipotent characters form a basic set for the unipotent blocks. Therefore $R_w$ can be seen as the unipotent part of a virtual projective module $\tilde{R}_w$. The decomposition of $\tilde{R}_w$ on the basis of characters of PIMs can be read off from the decomposition of $R_w$ on the family of unipotent parts of characters of PIMs. Given a simple $kG$-module $N$, the coefficient of $\Psi_N$ on $R_w$ will refer to the coefficient of $\Psi_N$ on $\tilde{R}_w$, which is given by $\langle \tilde{R}_w, \varphi_N \rangle$. The following proposition gives some control on the sign of this coefficient (see [15 Prop. 1.5]).

Proposition 1.5. Let $w \in W$. If $\Psi_N$ does not occur in $R_v$ for any $v < w$ in the Bruhat order on $W$ then the coefficient of $\Psi_N$ in $(-1)^{l(w)}R_w$ is non-negative.

Example 1.6. As in Example[14] we work with $G = \text{PGL}_2(q)$ and an odd prime number $\ell$ dividing $q+1$. There are two elements in the Weyl group of $G$, namely the trivial element $e$ and the simple reflection $s$. The Deligne–Lusztig characters are $R_e = 1_G + \text{St}_G$ and $R_s = 1_G - \text{St}_G$. With the decomposition matrix given in Example[14] they decompose on the basis of PIMs as

$$
\tilde{R}_e = \Psi_k + (1 - \alpha)\Psi_{\text{St}} \quad \text{and} \quad \tilde{R}_s = \Psi_k - (1 + \alpha)\Psi_{\text{St}}.
$$

By Proposition[1.5] we have $1 - \alpha \geq 0$, which forces $\alpha = 1$. Therefore $\Psi_{\text{St}}$ does not occur in $\tilde{R}_e$; it has a cuspidal head and $\Psi_{\text{St}}$ occurs with multiplicity $1 + \alpha = 2$ in $-R_s$, a non-negative number, as predicted by Proposition[1.5]. The unipotent $\ell$-decomposition matrix is hence

$$
\begin{array}{ccc}
1_G & 1 \\
\text{St}_G & 1 & 1
\end{array}
$$

with $\alpha = 1$. FINITE GROUPS OF LIE TYPE
The projective character corresponding to the first column lies in the principal series (indicated by the label “ps”), while the second one is cuspidal (and therefore labelled “c”).

If \( w \in W \) is such that \( wF \) is a \( d \)-regular element (in the sense of Springer [47]) for some \( d \geq 1 \) then \( T_w \) contains a (Sylow) \( \Phi_d \)-torus \( S \) of \( G \) such that \( C_G(S) = T_w \). For \( d \) the order of \( q \) modulo \( \ell \), under suitable conditions on the \( \ell \)-part \( \Phi_d(q) \) of the \( d \)th cyclotomic polynomial \( \Phi_d \) evaluated at \( q \), the dual torus \( (S^*)^F \) contains an \( \ell \)-element \( t \) such that \( C_{G^*}(t) = T_{w^*} \) (see Proposition [14] for the case \( d = 2 \) and the examples below). In that case by Lemma [1.3], \((-1)^{\ell(w)}(R_w)^o\) is a non-negative linear combination of irreducible Brauer characters.

**Example 1.7.** (a) Let \( G \) be a group of type \( C_n(q) \). Then a regular semisimple element of \( G \) is an element of \( G \) with \( 2n \) distinct eigenvalues of the form \( \{\lambda_1^{\pm 1}, \ldots, \lambda_n^{\pm 1}\} \) in the natural \( 2n \)-dimensional matrix representation that are permuted under the Frobenius map \( \lambda \mapsto \lambda^q \). If the eigenvalues are \( \ell \)-elements and \( d \) is the order of \( q \) modulo \( \ell \), then each eigenvalue has an orbit under \( F \) of size \( d \), therefore \( d \) must divide \( 2n \). In that case regular \( \ell \)-elements of \( G \) exist whenever \( \Phi_d(q) = 2 \).

(b) Let \( G \) be a group of type \( D_n(q) \). There are two types of regular semisimple elements of odd order: elements of \( G \) with \( 2n \) distinct eigenvalues of the form \( \{\lambda_1^{\pm 1}, \ldots, \lambda_n^{\pm 1}\} \), or elements with \( 2n - 2 \) distinct eigenvalues \( \{\lambda_1^{\pm 1}, \ldots, \lambda_{n-1}^{\pm 1}\} \) and two eigenvalues both equal to 1. If such an element is \( F \)-stable then \( d \), the order of \( q \) modulo \( \ell \), divides \( 2n \) or \( 2n - 2 \) respectively. It can be chosen to be an \( \ell \)-element provided that \( \Phi_d(q) > 2n \) or \( \Phi_d(q)^{\ell} > 2n - 2 \) respectively.

(c) Let \( G \) be a group of type \( 2D_n(q) \) with \( n \) odd. Then \( w_0F \) acts as \(-q\) on the set of characters and cocharacters of \( T \). Let \( e \) be the order of \(-q \) modulo \( \ell \). Using Ennola duality we deduce from (b) that when \( e \) divides \( 2n \) and \( \Phi_e(q)^{\ell} > 2n \) (resp. when \( e \) divides \( 2n - 2 \) and \( \Phi_e(q)^{\ell} > 2n - 2 \)) then there exists a regular \( \ell \)-element of \( G \).

(d) Let \( S \) be an \( F \)-stable torus of \( G^* \) with \( S = S^F \) of order \( \Phi_d(q) \) (a \( \Phi_d \)-torus of rank 1). If \( \ell \) is good and does not divide the order of \((Z(G^*)/Z^o(G^*))^F\) then by [7, Lemma 13.17], the \( d \)-split Levi subgroup \( L^* = C_{G^*}(S) \) of \( G^* \) is also equal to \( C_{G^*}(S_{\ell}) \), the centraliser of a Sylow \( \ell \)-subgroup \( S_{\ell} \) of \( S \). Consequently, if \( t \in S_{\ell} \) is any generator of the cyclic group \( S_{\ell} \) then \( L^* = C_{G^*}(t) \).
CHAPTER 2

Hecke algebras

Let $\Lambda$ be one of $K$, $O$ or $k$. Throughout this chapter, let $L$ be a 1-split Levi subgroup of $G$, that is, an $F$-stable Levi complement of an $F$-stable parabolic subgroup of $G$. In this case the maps $R^G_L$ and $^*R^G_L$ introduced in §1.3 are induced by the Harish-Chandra induction and restriction functors, which we still denote by $R^G_L$ and $^*R^G_L$.

Throughout this section we shall assume that $Z(G)$ is connected so that the results in [40, §8] apply.

1. Reduction stability

A $\Lambda L$-module $X$ is cuspidal if it is killed under every proper Harish-Chandra restriction. In that case the algebra $H^G_G(L,X) := \text{End}_{\Lambda G}(R^G_L(X))$ has a very specific structure. When $\Lambda = K$ and $X$ is simple, Lusztig showed in [40, §8.6] that the group $W^G_G(L,X) := N^G_G(L,X)/L$ admits a structure of a Coxeter system and $H^G_G(L,X)$ is an Iwahori–Hecke algebra associated to $W^G_G(L,X)$. More precisely, if $S^G_G(L,X)$ denotes the set of simple reflections associated to the Coxeter structure the $H^G_G(L,X)$ has a $K$-basis $\{T_w\}_{w \in W^G_G(L,X)}$ satisfying, for all $s \in S^G_G(L,X)$ and $w \in W^G_G(L,X)$

$$T_s T_w = \begin{cases} T_{sw} & \text{if } sw > w, \\ (q_s - 1)T_{sw} + q_s T_w & \text{otherwise.} \end{cases}$$

In addition the parameters $\{q_s\}_{s \in S^G_G(L,X)}$ can be computed explicitly. They are actually already determined by the Hecke algebras of the cuspidal module $X$ inside the minimal Levi overgroups of $L$ in $G$. When $\Lambda = k$, the endomorphism algebra was studied for example in [27]; it can still be shown to be closely related to an Iwahori–Hecke algebra, but much less is known about the parameters. The best situation occurs when the normaliser of the cuspidal lattice is invariant under change of scalars, as studied in [21, §2.6]. More precisely, given an $OL$-lattice X we will say that $R^G_L(X)$ is reduction stable if

\begin{enumerate}
\item $KX$ is irreducible, and
\item $N^G_G(L,X) = N^G_G(L,KX) = N^G_G(L,kX)$.
\end{enumerate}

In that case $H^G_G(L,kX)$ is an Iwahori–Hecke algebra whose parameters are the $\ell$-reduction of the parameters of the Iwahori–Hecke algebra $H^G_G(L,KX)$.

**Proposition 2.1** (Geck). Let $X$ be a cuspidal $OL$-module. If $R^G_L(X)$ is reduction stable then there is an $O$-basis $\{T_w\}_{w \in W^G_G(L,X)}$ of $H^G_G(L,X)$ endowing it with a structure of an Iwahori–Hecke algebra.

Furthermore, if $\Lambda$ is one of $K$ or $k$ then $\{1 \otimes_{O} T_w\}_{w \in W^G_G(L,X)}$ is a $\Lambda$-basis of $H^G_G(L,\Lambda X)$. 

9
The standard setup for reduction stability is when both $KX$ and $kX$ are simple modules. In that case it is enough to check that $N_G(L, KX) = N_G(L, kX)$. We will often need to work with a slightly more general setup.

**Proposition 2.2.** Let $X$ be a non-zero cuspidal $OL$-lattice. We assume that

1. the head $Y$ of $kX$ is a simple module;
2. $N_G(L, KX) = N_G(L, Y)$; and
3. $\text{Hom}_{kL}(w(kX), \text{rad}(kX)) = 0$ for all $w \in N_G(L)$.

Then $R^G_L(X)$ is reduction stable. Furthermore, $\mathcal{H}_G(L, kX) \simeq \mathcal{H}_G(L, Y)$.

**Proof.** First note that $N_G(L, X) \subset N_G(L, kX) \subset N_G(L, Y) = N_G(L, KX)$ where the second inclusion comes from the fact that the head of $kX$ is the simple module $Y$. Therefore we only need to show that $N_G(L, KX) \subset N_G(L, X)$ to prove the reduction stability.

Let $w \in N_G(L)$. Consider the short exact sequence of $kL$-modules

$$0 \longrightarrow \text{rad}(kX) \longrightarrow kX \longrightarrow Y \longrightarrow 0.$$ 

Since $\text{Hom}_{kL}(w(kX), \text{rad}(kX)) = 0$ by (4), the covariant functor $\text{Hom}_{kL}(w(kX), -)$ gives an injective map

$$\psi_w: \text{Hom}_{kL}(w(kX), kX) \hookrightarrow \text{Hom}_{kL}(w(kX), Y).$$

On the other hand, with the head of $w(kX)$ being $wY$ we have a natural isomorphism

$$\phi_w: \text{Hom}_{kL}(wY, Y) \simeq \text{Hom}_{kL}(w(kX), Y)$$

induced by the map $w(kX) \to wY$. This shows that $\text{Hom}_{kL}(w(kX), kX)$ has dimension at most 1. Now let $w \in N_G(L, KX)$. Since $K\text{Hom}_{OL}(wX, X) \simeq \text{Hom}_{kL}(w(KX), KX)$ we must have that $\text{Hom}_{OL}(wX, X)$ is non-zero. On the other hand, the natural map

$$k\text{Hom}_{OL}(wX, X) \hookrightarrow \text{Hom}_{kL}(w(kX), kX)$$

is an embedding. By the previous observation on the dimension of $\text{Hom}_{kL}(w(kX), kX)$, we deduce that it must be an isomorphism and that $\text{Hom}_{kL}(w(kX), kX) \simeq k$. Consequently $\psi_w$ is also an isomorphism. In particular, any non zero morphism from $w(kX)$ to $kX$ will send the head of $wKX$ to the head of $kX$ and therefore must be an isomorphism. This shows that $w(kX) \simeq kX$. In addition, since $k\text{Hom}_{OL}(wX, X) \simeq \text{Hom}_{kL}(w(kX), kX)$ then we also have $wX \simeq X$ by Nakayama’s lemma. This proves that $R^G_L(X)$ is reduction stable. Note that the fact that $\text{End}_{kL}(kX)$ has dimension 1 forces $KX$ to be a simple $KL$-module.

Note that if $w \notin N_G(L, Y)$ then $\psi_w$ is the zero map, therefore it is also an isomorphism in that case. Let $\pi: kX \to Y$. By adjunction and the Mackey formula, the natural map $\text{End}_{kG}(R^G_L(kX)) \longrightarrow \text{Hom}_{kG}(R^G_L(kX), R^G_L(Y))$ induced by $R^G_L(\pi)$ is an isomorphism since for all $w \in N_G(L)$ each map $\psi_w: \text{Hom}_{kL}(w(kX), kX) \longrightarrow \text{Hom}_{kG}(w(kX), Y)$ is an isomorphism. The same holds for the natural map $\text{End}_{kG}(R^G_L(Y)) \longrightarrow \text{Hom}_{kG}(R^G_L(kX), R^G_L(Y))$ since $\phi_w$ is an isomorphism for all $w \in N_G(L)$. The combination of the two gives the asserted isomorphism $\text{End}_{kG}(R^G_L(kX)) \simeq \text{End}_{kG}(R^G_L(Y))$. \qed
2. Embedding of decomposition matrices

Let \( X \) be a cuspidal simple \( kL \)-module. Then the simple representations of the algebra \( \mathcal{H}_G(L, X) \) encode the simple \( kG \)-modules occurring in the head of \( R_L^G(X) \), that is, the simple modules lying in the Harish-Chandra series above the cuspidal pair \((L, X)\). In addition, when \( X \) comes from a reduction stable \( O\Lambda L \)-lattice \( \tilde{X} \), one can compute the decomposition of \( R_L^G(P_X) \) into PIMs using the decomposition matrix of \( \mathcal{H}_G(L, \tilde{X}) \), as explained in [14, §3]. This gives a powerful method to compute the decomposition numbers of \( G \) for PIMs whose heads lie in the series above \((L, X)\). As in the previous section, we explain how to generalise this method when \( X \) is no longer simple.

**Proposition 2.3.** Let \( X \) be a cuspidal \( O\Lambda L \)-lattice. We assume that

1. the head \( Y \) of \( kX \) is a simple module;
2. \( N_G(L, kX) = N_G(L, Y) \); and
3. \( \text{Hom}_{k\Lambda L}(wP_Y, \text{rad}(kX)) = 0 \) for all \( w \in N_G(L) \).

Then \( R_L^G(X) \) is reduction stable, \( \mathcal{H}_G(L, kX) \cong \mathcal{H}_G(L, Y) \) and the decomposition matrix of \( \mathcal{H}_G(L, X) \) embeds into that of \( G \).

**Proof.** Let \( w \in N_G(L) \). Condition (3) implies that \( wY \) is not a composition factor of \( \text{rad}(kX) \) which shows in particular that \( \text{Hom}_{k\Lambda L}(w(kX), \text{rad}(kX)) = 0 \). Therefore the conditions of Proposition 2.2 are satisfied and we deduce that \( R_L^G(X) \) is reduction stable and \( \mathcal{H}_G(L, kX) \cong \mathcal{H}_G(L, Y) \).

To show the statement on the decomposition matrices we only need to check the condition (D) given in [29, 4.1.13, 4.1.14]. Since the head of \( k\tilde{X} \) is simple, equal to \( Y \), we have a surjective map \( P_Y \rightarrow kX \), which induces a surjective map \( R_L^G(P_Y) \rightarrow R_L^G(kX) \). Let \( \tilde{P}_Y \) be the projective \( O\Lambda L \)-module which is the (unique) lift of \( P_Y \), and let \( P := R_L^G(\tilde{P}_Y) \). Since \( P \) is projective we have a map \( P \rightarrow R_L^G(X) \) which lifts the surjective map \( R_L^G(P_Y) \rightarrow R_L^G(kX) \). By Nakayama’s lemma it should also be surjective. Now condition (D) is equivalent to

\[
\langle KP; R_L^G(KX) \rangle_G = \langle R_L^G(KX); R_L^G(KX) \rangle_G.
\]

To verify it we use the condition (3), the Mackey formula and cuspidality. We have

\[
\langle KP; R_L^G(KX) \rangle_G = \langle R_L^G(KP_Y); R_L^G(KX) \rangle_G = \sum_{w \in N_G(L)/L} \langle KP_Y; wKX \rangle_G.
\]

Now \( \langle KP_Y; wKX \rangle_G \) equals the multiplicity of \( Y^w \) as a composition factor of \( kX \), which by (3) is also the multiplicity of \( Y^w \) in \( Y = kX/\text{rad}(kX) \). It is therefore 1 if \( w \in N_G(L, Y) \) or 0 otherwise. By (2) we have \( N_G(L, Y) = N_G(L, KX) \) hence \( \langle KP_Y; wKX \rangle_G = \langle KX; wKX \rangle_G \) and we are done. \( \square \)

**Corollary 2.4.** Let \( X \) be a cuspidal \( O\Lambda L \)-lattice in a block \( b \) of \( O\Lambda L \) such that

1. the head \( Y \) of \( kX \) is a simple module;
2. \( N_G(L, Y) = N_G(L, KX) = N_G(L, b) \); and
3. \( Y \) occurs only once as a composition factor of \( kX \).
Then $R_G^G(X)$ is reduction stable, $\mathcal{H}_G(L, kX) \simeq \mathcal{H}_G(L, Y)$ and the decomposition matrix of $\mathcal{H}_G(L, X)$ embeds into that of $G$.

**Example 2.5.** Let $\chi$ be a cuspidal ordinary irreducible character of a 1-split Levi subgroup $L$ of $G$.

(a) Assume that $\chi$ lies in a block $b$ with cyclic defect groups. Let $Y$ be any $kL$-composition factor of the $\ell$-reduction of $\chi$. Then by Green [32] there exists an $OL$-lattice $X$ with character $\chi$ such that $kX$ is uniserial, has mutually distinct composition factors, and has $X$ as its head (the various composition factors are labelling the edges incident to the vertex labelled by $\chi$ in the Brauer tree of $b$). Then the assumptions of Corollary 2.4 are for example satisfied whenever $N_G(L, b)$ acts trivially on the Brauer tree and on the character $\chi$.

(b) Assume that the $\ell$-modular reduction of $\chi$ has only two composition factors, say $Y$ and $Z$, and that $Y$ is self-dual. Then there exists an $OL$-lattice $X$ with character $\chi$ or $\chi^*$ such that $kX$ is uniserial with head $Y$. In that case it is enough to check assumption (2) of Corollary 2.4. Note that it can be checked equivalently on $Y$ or $Z$.

(c) More generally, assume that $Y$ is a simple $kL$-module occurring in the $\ell$-modular reduction of $\chi$ with multiplicity one. In other words we assume that $\langle \Psi_Y; \chi \rangle_L = 1$. Denote by $\tilde{P}_Y$ a projective $OL$-module lifting $P_Y$. Let

$$e := \sum_{\rho \in \text{Irr}_K(b \setminus \{\chi\})} e_\rho$$

where $e_\rho$ is the central idempotent associated to the irreducible character $\rho$. Then $N := e\tilde{P}_Y \cap \tilde{P}_Y$ is a pure $OL$-submodule of $\tilde{P}_Y$ and $X := \tilde{P}_Y / N$ is an $OL$-lattice with character $\chi$ such that $kX$ has simple head $Y$. In particular, $X$ satisfies conditions (1) and (3) of Corollary 2.4.

### 3. Verifying reduction stability

Let us comment on how to guarantee the assumptions of Corollary 2.4 in certain situations. In this work, we will solely be concerned with unipotent blocks of finite reductive groups $G = GF$. By [7 Thm. 17.7] unipotent characters and unipotent Brauer characters are insensitive to the centre of the group whenever $\ell$ is very good, therefore we may assume that $G$ has connected centre. Now let $L$ be an $F$-stable Levi subgroup with a cuspidal $OL$-lattice $X$ such that the head of $kX$ is simple. If $L$ lies in a unipotent block then it is trivial on $Z(L)$. Now $N_G(L) = N_G(L)^F$ induces algebraic automorphisms of $L$, hence inner, diagonal and graph automorphisms. If $G$ has connected centre, then so has $L$, and all diagonal automorphisms of $L$ are inner; hence $N_G(L)^F$ induces graph automorphisms on $L$. Then there exists an $N_G(L)$-stable $OL$-lattice $X'$ with $kX' = kX$ if one of the following holds:

- $[L, L]$ does not have non-trivial graph automorphisms;
- $[L, L]$ is a product of simple factors regularly permuted by the graph automorphisms induced by $N_G(L)$ (since then we may choose a lift in one of the factors and then take the product over the $N_G(L)$-orbit).

This will deal with most of the cases we encounter. It thus remains to discuss situations in which $N_G(L)^F$ induces non-trivial graph automorphisms on a simple factor of $L^F$. 

Part 2

Decomposition matrices
CHAPTER 3

Description of the strategy

We keep the notation and setup from the previous chapter. In particular, \( G \) is a connected reductive linear algebraic group, \( F : G \to G \) is a Frobenius endomorphism with respect to an \( \mathbb{F}_q \)-rational structure and \( G = G^F \) is the corresponding finite group of Lie type.

In our proofs we shall use the following tools, most of which already served well in our papers [17, 18]:

\begin{itemize}
  \item[(HCi)] Harish-Chandra inducing PIMs from proper Levi subgroups and cutting by a block of \( G \) gives projective characters, which are hence non-negative integral linear combinations of PIMs of \( G \). Similarly, projective characters can also be obtained by Harish-Chandra restricting projectives from a larger group containing \( G \) as a Levi subgroup, or by a succession of such steps.
  \item[(HCr)] If \( \Psi \) is a projective character of \( G \) such that no non-zero proper subcharacter has the property that its Harish-Chandra restriction to any Levi subgroup \( L \) of \( G \) decomposes non-negatively on the PIMs of \( L \), then \( \Psi \) is the character of a PIM.
  \item[(H)] The number of Brauer characters in a Harish-Chandra series equals the number of simple modules of the Hecke algebra \( \mathcal{H} \) of the corresponding cuspidal Brauer character. More precisely, the decomposition of induced PIMs in that series can be read off from the corresponding decomposition for \( \mathcal{H} \) (see Proposition 2.3).
  \item[(Csp)] There exist cuspidal unipotent Brauer characters for \( G \) if and only if the centraliser of a Sylow \( \ell \)-subgroup of \( G \) is not contained in any proper 1-split Levi subgroup (see [27, Cor. 2.7]).
  \item[(St)] The \( \ell \)-modular Steinberg character of \( G \), i.e., the unique unipotent Brauer constituent in the \( \ell \)-modular reduction of an ordinary Gelfand–Graev character of \( G \), is cuspidal if and only if a Sylow \( \ell \)-subgroup of \( G \) is not contained in any proper 1-split Levi subgroup of \( G \) (see [27, Thm. 4.2]).
  \item[(DL)] Let \( w \in W \) and \( \Psi \) be the character of a PIM of \( G \). If \( \Psi \) does not occur in the Deligne–Lusztig character \( R_v \) for any \( v < w \) then the coefficient of \( \Psi \) in \((-1)^{(w)}R_w\) is non-negative (see Proposition 1.5).
  \item[(Red)] Let \( L \) be an \( F \)-stable Levi subgroup of \( G \) such that \( L^* \) is the centraliser of a semisimple \( \ell \)-element of \( G^* \). Then there is a sign \( \varepsilon \in \{ \pm 1 \} \) such that for any unipotent character \( \rho \) of \( L \), \( \varepsilon R_L^G(\rho) \) is a non-negative linear combination of irreducible Brauer characters (see Lemma 1.3).
  \item[(Tri)] Assume that the (unipotent) decomposition matrix of \( G \) is uni-triangular with respect to some total ordering of unipotent characters compatible with increasing \( a \)-values. Then, we can partially echelonise any set of projective characters of \( G \) (as explained in Example 1.1).
\end{itemize}
Remark 3.1. When $\ell$ is very good for $G$ there is an $F$-equivariant bijection between the conjugacy classes of $\ell$-elements of $G$ and of $G^*$ preserving the centralisers (see [24, Prop. 4.2]). Therefore in that case the assumptions of (Red) hold for $G$ whenever they hold for the dual group $G^*$.

Throughout, for a prime $\ell$ and a finite reductive group $G$ defined over $F_q$ we will denote by $d = d_{\ell}(q)$ the order of $q$ modulo $\ell$. Our results turn out to be uniform in $d$, not depending on the prime $\ell$ (once $\ell$ is large enough with respect to the root system of $G$).

We will not consider the case $d = 1$, since there by a result of Puig for all large enough primes the unipotent decomposition matrix is always the identity matrix. Indeed, in this case the $\ell$-blocks are unions of Harish-Chandra series and all Hecke algebras are semisimple after reduction modulo $\ell$. Furthermore, we will not deal with the case when $d = 4$ since this was already considered in our predecessor paper [18]; we shall only indicate how some of the remaining unknowns can be computed using $(T_\ell)$, see Remark 8.4. Finally, we will not consider the general linear groups, as their unipotent decomposition matrices were determined up to rank 10 by James [37], nor the general unitary groups whose unipotent decomposition matrices up to rank 10 were computed in [27] for linear primes and in [17] for unitary primes.
CHAPTER 4

Decomposition matrices at \( d_\ell(q) = 2 \)

In this section we determine decomposition matrices for unipotent \( \ell \)-blocks of various classical and exceptional groups over \( F_q \) for primes \( \ell \) dividing \( q + 1 \). This is by some measure the most complicated case since the ranks of Sylow \( \ell \)-subgroups for such primes are generally larger than for any prime \( \ell \) with \( d_\ell(q) \geq 3 \). Nevertheless, we obtain almost complete results in the cases considered.

1. Centralizers of \( \ell \)-elements

Recall that \( G \) is connected reductive with Frobenius map \( F \), \( T \) is a maximally split torus of \( G \) and \( B \) is an \( F \)-stable Borel subgroup containing it. We denote by \( \Phi \) the set of roots of \( G \) with respect to \( T \), by \( \Phi^+ \) the set of positive roots corresponding to \( B \), and by \( \Delta \) the set of simple roots. Given \( \alpha \in \Phi \) we write \( \text{ht}(\alpha) \) for the height of \( \alpha \), that is the sum of the coefficients of \( \alpha \) expressed in the basis \( \Delta \). The Weyl group of \( \Phi \) can be identified with the Weyl group \( W \) of \( G \).

To any subset \( I \) of \( S \) there is a corresponding parabolic subgroup \( W_I \) of \( W \) and a standard Levi subgroup \( L_I = \langle T, W_I \rangle \) of \( G \). When \( I \) is \( F \)-stable then \( L_I \) is \( F \)-stable and is a 1-split Levi subgroup of \( G \). In this section we shall rather be interested in the 2-split Levi subgroups as defined, e.g., in \([4], p. 17\). They are obtained as centralisers of \( \Phi_2 \)-tori, which are \( F \)-stable tori of \( G \) of order \((q + 1)^r \) for some \( r \geq 0 \). Let \( w_0 \) be the longest element of \( W \) and by \( S \) the set of simple reflections in \( W \) corresponding to \( \Delta \). In the case where \( w_0 F \) acts trivially on \( S \setminus I \) then the pair \((L_I, w_0 F)\) is conjugate to a pair \((L, F)\) where \( L \) is a 2-split Levi subgroup. We give here some further conditions on \( \ell \) to ensure the existence of an \( \ell \)-element whose centralizer is \( L \). This will be needed in order to use the method (Red) from Chapter 3.

**Proposition 4.1.** Let \( I \subseteq S \) be a subset of the set of simple reflections of \( W \) and \( L_I \) be the corresponding standard Levi subgroup of \( G \). We assume that

1. \( w_0 F \) acts trivially on \( S \setminus I \);
2. \( \ell \) is very good for \( G \); and
3. \( (q + 1)_\ell > \text{ht}(\pi_I(\alpha)) \) for all \( \alpha \in \Phi^+ \), where \( \pi_I \) is the projection of \( Z\Phi^+ \) to \( Z\Phi^+_{S \setminus I} \).

Then there exists \( t \in Z^\circ(L_I)^{w_0 F} \) such that \( C_G(t) = L_I \).

**Proof.** Let \( \pi_{\text{ad}}: G \rightarrow G_{\text{ad}} := G/Z(G) \). From \([12]\) Prop. 2.3 it follows that \( \pi_{\text{ad}}(C_{G}(t)) = C_{G_{\text{ad}}}^{\circ}(\pi_{\text{ad}}(t)) \) for any semisimple element \( t \in G \). Since \( \ell \) is very good, both \( Z(G)/Z^\circ(G) \) and \( Z(G^*)/Z^\circ(G^*) \) are \( \ell \)-groups. The first one ensures that any \( \ell \)-element of \( G_{\text{ad}} \) lifts to an \( \ell \)-element of \( G \), whereas the second implies that the centralisers of \( \ell \)-elements are connected by \([12] 13.14(\text{iii}) \) and 13.15(i)]). Finally, by \([7]\) Prop. 13.12 we also have that \( Z(L)/Z^\circ(L) \) is an \( \ell \)-group for any Levi subgroup of \( G \). This shows that we can assume that \( G \) is semisimple of adjoint type without loss of generality.
Write $\Delta = \{\alpha_1, \ldots, \alpha_n\} \subset X(T)$ and denote by $\{\varpi_1, \ldots, \varpi_n\} \subset Y(T)$ the dual basis for the pairing between characters and cocharacters. Here $X(T)$ is the lattice of characters of $T$ whereas $Y(T)$ is the lattice of cocharacters. We reorder the simple roots so that $\{\alpha_1, \ldots, \alpha_m\}$ are the simple roots corresponding to the simple reflections in $S \setminus I$.

Given $\lambda \in \mathbb{F}_p^\times$, we consider the semisimple element

$$t(\lambda) := \varpi_1(\lambda)\varpi_2(\lambda) \cdots \varpi_m(\lambda)$$

of $G$. If $\alpha = \sum_{i=1}^n x_i \alpha_i \in X(T)$ then

$$\alpha(t(\lambda)) = \lambda \sum_{i=1}^m x_i = \lambda^{|\pi_I(\alpha)|},$$

where $\pi_I$ is the projection of $\mathbb{Z}^\Phi^+$ to $\mathbb{Z}^{\Phi^+_I}$. In particular $t(\lambda)$ lies in the kernel of every root $\alpha \in \Phi_I$, therefore it lies in $Z(L_I)$. In addition, $w_0 F(\varpi_i) = -q \varpi_i$ for every $i \leq m$. Indeed, by assumption (1) $w_0 F$ permutes the elements in $I$ but fixes the elements in $S \setminus I$. Therefore we have

$$\langle w_0 F(\varpi_i) + q \varpi_i, \alpha_j \rangle = \langle w_0 F(\varpi_i), \alpha_j \rangle + q \langle \varpi_i, \alpha_j \rangle$$

$$= \langle \varpi_i, w_0 F(\alpha_j) \rangle + q \langle \varpi_i, \alpha_j \rangle$$

$$= \langle \varpi_i, w_0 F(\alpha_j) \rangle + q \delta_{i,j}.$$ 

If $j > m$ then $w_0 F(\alpha_j) \in \mathbb{Z}^\Phi_I$ therefore $\langle \varpi_i, w_0 F(\alpha_j) \rangle = 0$ whereas if $j \leq m$ we have $w_0 F(\alpha_j) = -q\alpha_j$ and hence $\langle \varpi_i, w_0 F(\alpha_j) \rangle = -q \delta_{i,j}$. In each case $\langle w_0 F(\varpi_i) + q \varpi_i, \alpha_j \rangle = 0$ which proves that $w_0 F(\varpi_i) = -q \varpi_i$ for all $i \leq m$. We deduce that $t(\lambda)$ is $w_0 F$-stable whenever $\lambda$ is an $\ell$-element satisfying $\lambda^\ell = \lambda^{-1}$.

By (1) and assumption (3), there exists an $\ell$-element $\lambda \in \mathbb{F}_p^\times$ such that $\lambda^{\ell+1} = 1$ and $\alpha(t(\lambda)) \neq 1$. By [12] Prop. 2.3 this implies that $C_G^\circ(t(\lambda)) = L_I$. But since $\ell$ is very good the centraliser of every $\ell$-element is connected by [12] 13.14(iii)].

2. Multiplicities in the Steinberg character

Before considering individual series of groups, we first prove a general result. It demonstrates how Deligne–Lusztig characters can be used to obtain relations between the entries of decomposition matrices of unipotent $\Phi_2$-block, yielding new decomposition numbers in the “bottom right corner” of the matrix. This is the part of the decomposition matrix about which Harish-Chandra methods usually yield the least information.

We call

$$h := 1 + \max\{\text{ht}(\alpha) \mid \alpha \in \Phi^+\}$$

the Coxeter number of $G$. When $\Phi$ is irreducible, it equals the order of any Coxeter element of $W$ (see [3], Prop. VI.1.31]).

**Theorem 4.2.** Let $G$ be connected reductive. We assume that

1. $w_0 F$ acts trivially on $W$;
2. $\ell$ is very good for $G$; and
3. $(q+1)\ell \geq h$, where $h$ is the Coxeter number of $G$.

Then there exists a linear character $\theta$ of $T_{w_0}$ in general position such that $\theta^\circ = 1$ and

$$(-1)^{i(w_0)} R_{T_{w_0}}^G(\theta)^\circ = \varphi_{\text{St}}.$$
2. MULTICIPALITIES IN THE STEINBERG CHARACTER

PROOF. Let \((G^*, T^*, F)\) be in duality with \((G, T, F)\). By Proposition 13 applied to \(G^*\) and \(I = \emptyset\), there exists an \(\ell\)-element \(t \in T_{w_0}^*\) such that \(C_{G^*}(t) = T^*\). Therefore \(t\) is a regular \(\ell\)-element. Under the duality, this shows that there exists an irreducible character \(\theta\) of \(T_{w_0}\) in general position such that \(\theta^\circ = 1\).

By [39] Cor. 2.10 the property that \(\theta\) is regular implies that \(\chi_{\theta} = (-1)^{l(w_0)} R_{T_{w_0}}^G(\theta)\) is an irreducible character. Furthermore, since \(w_0 F\) is central \(w_0\) lies in a cuspidal \(F\)-conjugacy class of \(W\) which implies that \(\chi_{\theta}\) is a cuspidal character by [39] Cor. 2.19. In particular, any irreducible Brauer character \(\varphi\) occurring in \(\chi_{\theta}\) is also cuspidal.

Since \(\theta\) is an \(\ell\)-character then \(\chi_{\theta}^\circ = (-1)^{l(w_0)} R_{T_{w_0}}^G(1)^\circ\), which gives the expression of \(\chi_{\theta}^\circ\) on the basic set of unipotent characters. We denote by \(P_{St}\) the unique projective indecomposable summand of a Gelfand–Graev character which contains the Steinberg character \(St\) of \(G\), and by \(\Psi_{St}\) its character. The Steinberg character occurs with multiplicity 1 in \(\Psi_{St}\) and any other constituent is non-unipotent. Since \(\langle R_{T_{w_0}}^G(1); St \rangle = \langle R_{T_{w_0}}^G(1); \Psi_{St} \rangle = (-1)^{l(w_0)}\) (see for example [12] Cor. 12.18(ii)) we deduce that \(\varphi_{St}\) occurs with multiplicity one in \(\chi_{\theta}^\circ\).

We need to show that no other Brauer character can occur. Recall from \[\text{3}\] that for \(w \in W\), we denote by \(\tilde{R}_w\) a virtual projective character obtained by adding and removing suitable non-unipotent characters to \(R_w = R_{T_w}^G(1)\). The orthogonality relations for Deligne–Lusztig characters, together with the fact that \(R_{w_0}\) contains only unipotent constituents, yield the following relation for \(w \neq w_0\):

\begin{equation}
0 = \langle R_{T_{w_0}}^G(1); (-1)^{l(w_0)} R_{T_{w_0}}^G(1) \rangle = \langle \tilde{R}_w; \varphi \rangle = \sum_{\varphi \in \hat{Br} G} \langle \tilde{R}_w; \varphi \rangle \langle \Psi_{\varphi}; \chi_{\theta}^\circ \rangle.
\end{equation}

Note that since \(\chi_{\theta}\) is cuspidal, the Brauer characters contributing to this sum are all cuspidal. Let \(w \neq w_0\) be of minimal length in its conjugacy class. We prove by induction on its length \(l(w)\) that for every cuspidal Brauer character \(\varphi\), if \(\langle \tilde{R}_w; \varphi \rangle \neq 0\) then \(\langle \Psi_{\varphi}; \chi_{\theta}^\circ \rangle = 0\). This already holds for any element \(w\) lying in a proper \(F\)-stable parabolic subgroup since in that case there are no cuspidal Brauer characters \(\varphi\) such that \(\langle \tilde{R}_w; \varphi \rangle \neq 0\). Assume now that the property holds for any \(v \in W\) such that \(l(v) < l(w)\). If \(\varphi\) is an irreducible Brauer character such that \(\langle \tilde{R}_w; \varphi \rangle \langle \Psi_{\varphi}; \chi_{\theta}^\circ \rangle \neq 0\), then by induction, \(\Psi_{\varphi}\) cannot occur in any \(\tilde{R}_v\) for \(l(v) < l(w)\). It follows from Proposition 1.5 that \((-1)^{l(w)}\langle \tilde{R}_w; \varphi \rangle > 0\), so that \((-1)^{l(w)}\langle \tilde{R}_w; \varphi \rangle \langle \Psi_{\varphi}; \chi_{\theta}^\circ \rangle > 0\) which contradicts \[2\].

In other words, if \(\Psi_{\varphi}\) occurs in some \(\tilde{R}_w\) for \(w \neq w_0\) then \(\varphi\) is not a constituent of \(\chi_{\theta}^\circ\). It remains to see that all the PIMs but one will actually occur. Note that since \(\langle \chi_{\theta}; \varphi_{St} \rangle = 1\) we already know that \(\Psi_{St}\) occurs only in \(\tilde{R}_{w_0}\). Let \(\varphi \neq \varphi_{St}\) be an irreducible Brauer character. Let us consider the virtual projective character \(\tilde{Q} = \sum_{w \in W} (-1)^{l(w)} \tilde{R}_w\).

Its unipotent part equals \(St\), therefore we must have \(\tilde{Q} = \Psi_{St}\). In particular, \(\Psi_{\varphi}\) does not occur in \(\tilde{Q}\). We deduce that if \(\Psi_{\varphi}\) does not occur in \(\tilde{R}_w\) for all \(w \neq w_0\), then it does not occur in any \(\tilde{R}_w\), which contradicts \[2\] Thm. A].

\[\square\]
We can use the previous theorem to compute non-trivial decomposition numbers of the Steinberg character.

**Corollary 4.3.** Let \((G, F)\) be simple of classical type \(2A_{n-1}(q), D_{2n}(q), 2D_{2n+1}(q)\) with \(n \geq 2\), or of exceptional type \(2E_6(q), E_7(q),\) or \(E_8(q)\). Assume that \(p\) is good and \(\ell\) is very good for \(G\). Then:

(a) There is a unique unipotent character \(\rho\) of \(G^F\) with \(a\)-value 1.

(b) Let \(\rho^*\) be the Alvis–Curtis dual of \(\rho\). If \((q+1)\ell \geq h\), there exists a PIM of \(G^F\) whose unipotent part is given by \(\rho^* + (\text{rank } G)\) St.

**Proof.** When \(\ell\) is very good unipotent characters and Brauer characters are insensitive to the centre by [47, Thm. 17.7]. Therefore we may and will assume that \(G\) has trivial centre. Let \(O\) be the subregular unipotent class of \(G\), that is the maximal unipotent class outside the regular unipotent class. It is the unique class of codimension 2 in the variety of unipotent elements in \(G\) and thus \(F\)-stable. From the classification of unipotent classes in good characteristic (see e.g. [46]) one can check that it is special. We list below for each type considered the class (with Jordan form in the natural matrix representation for classical types), the special unipotent character \(\rho\) of \(G^F\) with unipotent support \(O\) and its Alvis–Curtis dual \(\rho^*\). Recall from [2] that a unipotent character corresponding to a bipartition \(\lambda\) is denoted by \([\lambda]\).

| Type | \(O\) | \(\rho\) | \(\rho^*\) |
|------|------|------|------|
| \(2A_{n-1}\) | \((n-1, 1)\) | \((n-1, 1)\) | \(21^{n-2}\) |
| \(D_n\) | \((2n-3, 3)\) | \((n-1, 1)\) | \(11^{n-1}\) |
| \(2D_n\) | \((2n-3, 3)\) | \((n-2, 1)\) | \(21^{n-3}\) |
| \(E_6\) | \(E_6(a_1)\) | \(\phi_2, 4\) | \(\phi_2, 16\) |
| \(E_7\) | \(E_7(a_1)\) | \(\phi_7, 1\) | \(\phi_7, 46\) |
| \(E_8\) | \(E_8(a_1)\) | \(\phi_8, 1\) | \(\phi_{8, 91}\) |

In each of these cases, the \(a\)-value of \(\rho\) equals 1, and \(\rho\) (as well as \(\rho^*\)) is alone in its family. By maximality of \(O\), every other non-trivial unipotent character has \(a\)-value at least 2, which proves (a).

Since the Springer correspondence sends the trivial local system on \(O\) to the reflection representation \(\phi_{\text{ref}}\) of \(W\), then \(\rho\) is equal to the almost character associated with \(\phi_{\text{ref}}\) (see for example [8, §13.3]). More precisely, there exists an extension \(\tilde{\phi}_{\text{ref}}\) of \(\phi_{\text{ref}}\) to \(W \rtimes \langle F \rangle\) such that

\[
\rho = R_{\tilde{\phi}_{\text{ref}}} := \frac{1}{|W|} \sum_{w \in W} \tilde{\phi}_{\text{ref}}(wF)R_w.
\]

When \((G, F)\) is split, \(\rho\) is just the principal series character corresponding to \(\phi_{\text{ref}}\).

By [5, Thm. A] there exists a generalised Gelfand–Graev module \(\Gamma\) of \(OG\) whose character has unipotent part \(\rho^* + x\text{St}\) for some non-negative integer \(x\) (depending on \(q\)). Let \(P\) be the unique direct summand of \(\Gamma\) whose character \(\Psi\) has \(\rho^*\) as a constituent. Then the unipotent part of \(\Psi\) equals \(\rho^* + y\text{St}\) for some non-negative integer \(y\). Let \(\varphi\) be the irreducible Brauer character such that \(\Psi = \Psi_{\varphi}\). By Theorem 4.2 the multiplicity of \(\varphi\) in \((R_{\varphi_0})^\circ\) is zero, yielding the equation

\[
0 = \langle \Psi; R_{\varphi_0} \rangle = \langle \rho^*; R_{\varphi_0} \rangle + (-1)^{(\omega_0)}y.
\]
Now, the Alvis–Curtis dual of \( R_w \) is \((-1)^{l(w_0)} R_{w_0}\) (see for example [12, Thm. 12.8]). Using equation (3) we get

\[
\langle \rho^*; R_{w_0} \rangle = (-1)^{l(w_0)} \langle \rho; R_{w_0} \rangle = (-1)^{l(w_0)} \tilde{\phi}_{\text{ref}}(w_0 F) = -(-1)^{l(w_0)} \text{rank } G.
\]

Then (b) follows from (4).

For the groups not listed in Corollary 1.3, but for which \( w_0 F \) acts trivially on \( W \), there are several unipotent characters with minimal non-zero \( a \)-value and they form a non-trivial family. We can still give an approximation of the previous decomposition number as we illustrate for groups of type \( B \) and \( C \).

**Corollary 4.4.** Let \((G, F)\) be simple of type \( B_n(q) \) or \( C_n(q) \), \( n \geq 2 \). Assume that \( p \) and \( \ell \) are odd. If \((q + 1) \ell > 2n\), then there exist two PIMs whose unipotent parts are given by

\[
[1^n.] + (n - \delta) \text{St} \quad \text{and} \quad [B_2; 1^{n-2}] + (n - 1 + \delta) \text{St},
\]

where \( \delta = 1 \) if \( n \) is even, and 0 otherwise.

**Proof.** As above, we may and we will assume that the centre of \( G \) is trivial. The unique family of unipotent characters of \( G \) with \( a \)-value 1, which corresponds to the subregular unipotent class, is \( \mathcal{F} = \{[n - 1, 1], [n], [(n - 1, 1)], [B_2; n - 2]\} \). In terms of symbols, these unipotent characters are given in the same order by

\[
\mathcal{F} = \left\{ \begin{pmatrix} 0 & n \\ 1 & \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ n & \end{pmatrix}, \begin{pmatrix} 1 & n \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & \end{pmatrix} \right\}.
\]

Let us focus on the characters \([n]\) and \([B_2; n - 2]\). Their uniform parts can be expressed in terms of almost characters (see [40, Chap. 4]), from which we can compute the scalar product with any Deligne–Lusztig character \( R_w, w \in W \). This yields

\[
\langle [n]; R_w \rangle = \frac{1}{2} \left( \phi^{n-1,1}(w) - \phi^{n-1,1}(w) + \phi^n(w) \right),
\]

\[
\langle [B_2; n - 2]; R_w \rangle = \frac{1}{2} \left( \phi^{n-1,1}(w) - \phi^{n-1,1}(w) - \phi^n(w) \right),
\]

where \( \phi^\mu \) denotes the irreducible character of \( W \) corresponding to the bipartition \( \mu \) of \( n \). One can compute easily the values of these characters at the central element \( w_0 \): the character \( \phi^{n-1,1} \) is the reflection character, therefore \( \phi^{n-1,1}(w_0) = -n \). The character \( \phi^n \) is linear, with value \(-1\) on the first simple reflection and \( 1 \) on the others, and thus \( \phi^n(w_0) = (-1)^n \). Finally, for the value of \( \phi^{n-1,1} \), we use that the induction of \( \phi^{n-1} \) from \( B_{n-1} \) to \( B_n \) decomposes as \( \phi^{n-1,1} + \phi^n + \phi^{n-1,1} \) which gives \( \phi^{n-1,1}(w_0) = n - 1 \). This yields

\[
\langle [n]; R_{w_0} \rangle = -n + \delta_n \text{ even},
\]

\[
\langle [B_2; n - 2]; R_{w_0} \rangle = -n + \delta_n \text{ odd}.
\]

(5)

The Alvis–Curtis duals of \([n]\) and \([B_2; n - 2]\) are \([1^n]\) and \([B_2; 1^{n-2}]\) respectively. By [5, Thm. B], there exist Kawanaka modules \( K' \) and \( K'' \) which are projective modules whose characters have unipotent parts \([1^n] + x\text{St} \) and \([B_2; 1^{n-2}] + y\text{St} \), for some non-negative integers \( x \) and \( y \). Therefore there is an indecomposable summand \( P' \) (resp. \( P'' \)) of \( K' \) (resp. \( K'' \)), with \([1^n]\) (resp. \([B_2; 1^{n-2}]\)) occurring in the character of \( P' \) (resp. \( P'' \)).
Now using Theorem 4.2 and (5) we can compute the multiplicity of St in both of these projective characters as given in the statement. □

3. Unipotent decomposition matrix of \( D_4(q) \)

We are now ready to compute decomposition matrices for specific series of finite reductive groups. We start with the orthogonal groups \( D_4(q) \). As customary, we will label the unipotent characters in the principal series by characters of the Weyl group of type \( D_4 \), that is, by unordered pairs of partitions of 4, while the (unique) cuspidal unipotent character will be denoted “\( D_4 \)”. This is also the labelling used in the Chevie system [43].

In our tables of decomposition matrices the second column lists the degrees of the unipotent characters as a product of cyclotomic polynomials \( \Phi_e \) evaluated at \( q \). In the last line, we give the \( \ell \)-modular Harish-Chandra series of the Brauer characters (and, by abuse of notation, also of the corresponding PIMs), by either writing “ps” for characters in the principal series, or a type of Levi subgroup if that Levi subgroup has a unique cuspidal unipotent Brauer character, or by the label of a cuspidal unipotent Brauer character of that Levi subgroup. The root system of type \( D_4 \) has three conjugacy classes of subsystems of type \( A_2 \), cyclically permuted by the graph automorphism of order 3; we denote them by \( D_2 \), \( A_2^+ \) and \( A_2^- \). The symbol “c” denotes cuspidal PIMs. Also, in all of our tables for better readability we print “.” in place of “0”.

The groups of type \( D_4 \) have 14 unipotent characters. For primes \( \ell > 2 \) with \( \ell | (q + 1) \), thirteen of them lie in the principal \( \ell \)-block, while the one with label 1.21 is of \( \ell \)-defect zero.

**Theorem 4.5.** The decomposition matrix for the unipotent \( \ell \)-blocks of \( D_4(q) \), \( q \) odd, \( q \equiv -1 \pmod{\ell} \), \( \ell \geq 11 \), is as given in Table 1.

**Proof.** By [24, Thm. 5.1] the unipotent characters form a basic set for the unipotent \( \ell \)-blocks of \( G = D_4(q) \) when \( \ell \neq 2 \). It was shown in [31], by constructing projective characters by Harish-Chandra induction and from generalised Gelfand–Graev characters, that when \( q \) is odd the decomposition matrix of the unipotent blocks of \( G \) is uni-triangular. This provides a unique labelling for the irreducible unipotent Brauer characters; we denote them by \( \varphi_x \) if \( x \) is the label of the corresponding ordinary unipotent character.

Let us denote by \( \Psi_i \), \( 1 \leq i \leq 14 \), the linear combinations of unipotent characters given by the columns in Table 1. Harish-Chandra induction from proper Levi subgroups now yields these projectives except for \( \Psi_9 \), \( \Psi_{13} \) and \( \Psi_{14} \). The Steinberg-PIM is cuspidal by (St) since no proper Levi subgroup contains a Sylow \( \ell \)-subgroup of \( G \).

The first two columns correspond to PIMs as can be seen from the decomposition matrix of the Hecke algebra for the principal series. By Harish-Chandra theory, if any of the listed induced projective characters is decomposable, it can only contain constituents from lower Harish-Chandra series. It is then easily seen that all other projective characters given in Table 1 must also be indecomposable.

It remains to show that \( \Psi_9 \) and \( \Psi_{13} \) are the unipotent parts of PIMs. The graph automorphism of \( G \) of order 3 fixes the unique cuspidal unipotent character but permutes the characters with labels \( 1^2+, 1^2-, .21^2 \) cyclically. It follows that the first three entries below the diagonal in the 9th column of the decomposition matrix must be equal. We
Table 1. $D_4(q)$, $11 \leq \ell | (q + 1)$

| .4 | 1 | 1 |
| 1.3 | $q^2\Phi_4^2$ | 2 1 |
| 2+ | $q^2\Phi_3\Phi_6$ | 1 1 1 |
| 2− | $q^2\Phi_3\Phi_6$ | 1 1 . 1 |
| .31 | $q^2\Phi_3\Phi_6$ | 1 1 . . 1 |
| .2$^2$ | $\frac{1}{2}q^3\Phi_1^2\Phi_6$ | . 1 1 1 1 1 |
| 1$^2$.2 | $\frac{1}{2}q^3\Phi_3\Phi_4^2$ | 2 2 1 1 1 . 1 |
| 1.21 | $\frac{1}{2}q^3\Phi_2^2\Phi_6$ | . . . . . . . . 1 |
| $D_4$ | $\frac{1}{2}q^3\Phi_1^4\Phi_3$ | . . . . . . . . 1 |
| 1$^2$.+ | $q^6\Phi_3\Phi_6$ | 1 1 1 1 1 1 1 . 2 1 |
| 1$^2$.− | $q^6\Phi_3\Phi_6$ | 1 1 1 1 1 1 1 . 2 . 1 |
| .2$^1$ | $q^6\Phi_3\Phi_6$ | 1 1 1 1 1 1 1 . 2 . . 1 |
| 1.3 | $q^7\Phi_3^4$ | 2 1 1 1 1 1 1 . 2 . 4 1 1 1 1 |
| .1$^4$ | $q^{12}$ | 1 . . . . . . 1 1 . 6 1 1 1 4 1 |

$\Psi_{\Phi} = [D_4] + a_4([1^2+] + [1^2−] + [21^2]) + a_2[1.1^3] + a_3[1^4]$, $\Psi_{\Phi} = [1.1^3] + a_4[1^4]$.

By Theorem 4.12 we know that $R_{w_0}$ has $\varphi_{1^4}$ as only Brauer constituent. The relation $\langle \Psi_{\Phi}; R_{w_0} \rangle = 0$ shows that $a_4 = 4$, and the relation $\langle \Psi_{D_4}; R_{w_0} \rangle = 0$ yields $a_3 = 8 - 9a_1 + 4a_2$.

Let $w \in W$ be a Coxeter element. The coefficient of $\Psi_{\Phi}$ on $R_w$ equals $-1 + 3a_1 - a_2$. By (DL) this forces $-1 + 3a_1 - a_2 \geq 0$. On the other hand by Proposition 4.11 for $\ell > 4$ a 2-split Levi subgroup $\mathfrak{z}A_2(q).(q + 1)^2$ of $G$ has a linear $\ell$-character in general position. Then (Red) with $\rho$ the unique cuspidal unipotent character of $L$ gives $3a_1 - a_2 \leq 3$. Therefore there exists $c \in \{-1, 0, 1\}$ such that $a_2 = 3a_1 - 2 + c$. With this notation we have $a_3 = 4c + 3a_1$. We will see in the proof of Theorem 4.8 that $a_1 = 2$ and in the proof of Theorem 4.9 that $c = 0$.

Remark 4.6. Assuming $(T_\ell)$, the result in Table II remains true for even $q$. In fact, it was shown by Paolini [45] that $(T_\ell)$ holds for $D_4(q)$ in characteristic 2 and thus the unipotent decomposition matrix of $D_4(2^\ell)$ agrees with the one given in Table II up to the knowledge of $a_1$ and $c$ which will require $(T_\ell)$ for the types $D_5$ and $D_6$.

4. Unipotent decomposition matrix of $D_5(q)$

We now turn to the orthogonal groups $D_5(q)$. For the determination of the decomposition matrices for unipotent blocks of non-cyclic defect we will need to know the structure and parameters of the Hecke algebras attached to cuspidal $\ell$-modular Brauer characters of
certain Levi subgroups. Here and later, by convention, $\mathcal{H}(X; q_1)$ for $X \in \{A_n, D_n\}$ denotes
the Iwahori–Hecke algebra over $k$ of type $X$ with parameters $(q_1, -1)$, and $\mathcal{H}(B_n; q_1; q_2)$
denotes the Iwahori–Hecke algebra over $k$ of type $B_n$ with parameters $(q_1, -1)$ at the type
$B_1$-node, and parameters $(q_2, -1)$ at the remaining nodes on the $A_{n-1}$-branch. In Table 2
we have collected the number $|\text{Irr } \mathcal{H}|$ for some small rank modular Iwahori–Hecke $\mathcal{H}$
 algebras occurring later. They can easily be computed using for example the programme of Jacon [36].

Table 2. $|\text{Irr } \mathcal{H}|$ for some modular Hecke algebras

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 |
|-----|---|---|---|---|---|---|
| $\mathcal{H}(B_n; 1; 1)$ | 2 | 5 | 10 | 20 | 36 | 65 |
| $\mathcal{H}(B_n; -1; 1)$ | 1 | 2 | 3 | 5 | 7 | ? |
| $\mathcal{H}(B_n; 1; -1)$ | 2 | 2 | 4 | 6 | 8 | 12 |
| $\mathcal{H}(B_n; -1; -1)$ | 1 | 2 | 3 | 4 | 6 | 9 |
| $\mathcal{H}(D_n; 1)$ | 1 | 2 | 3 | 13 | 18 | 37 |
| $\mathcal{H}(D_n; -1)$ | 1 | 1 | 2 | 3 | 4 | 6 |

In this as well as in all later tables, $W(D_1)$ has to be interpreted as the trivial group.

**Lemma 4.7.** Let $q$ be a prime power and $2 < \ell ((q + 1)$. The Hecke algebras of various
$\ell$-modular cuspidal pairs $(L, \lambda)$ of Levi subgroups $L$ in $D_n(q)$ and their respective numbers
of simple modules are as given in Table 3.

Table 3. Hecke algebras and $|\text{Irr } \mathcal{H}|$ in $D_n(q)$ for $d_\ell(q) = 2$

| $(L, \lambda)$ | $\mathcal{H}$ | $n = 4$ | 5 | 6 |
|----------------|-------------|-------|---|---|
| $(A_1, \varphi_{12})$ | $\mathcal{H}(A_1; q) \otimes \mathcal{H}(D_{n-2}; q)$ | 1 | 2 | 2 | 1 |
| $(A_2, \varphi_{12}^{22})$ | $\mathcal{H}(B_2; q^2; q) \otimes \mathcal{H}(D_{n-4}; q)$ | 2 | 2 | 2 |
| $(A_2, \varphi_{12}^{32})$ | $\mathcal{H}(B_3; q^3; q^2) \otimes \mathcal{H}(D_{n-6}; q)$ | 2 | 4 | 4 | 2 |
| $(D_2, \varphi_{12})$ | $\mathcal{H}(B_{n-2}; q^2; q)$ | 1 | 2 | 2 |
| $(D_2 A_1, \varphi_{12} \boxtimes \varphi_{12})$ | $\mathcal{H}(A_1; q) \otimes \mathcal{H}(B_{n-4}; q^2; q)$ | 1 | 2 | 2 |
| $(D_2 A_1, \varphi_{12} \boxtimes \varphi_{12}^{22})$ | $\mathcal{H}(B_2; q; q^2; q) \otimes \mathcal{H}(B_{n-6}; q^2; q)$ | 1 | 2 | 2 |
| $(D_2 A_1, \varphi_{12}^{22})$ | $\mathcal{H}(B_{n-4}; q^4; q)$ | 1 | 2 | 2 |
| $(D_2 A_1, \varphi_{12}^{32})$ | $\mathcal{H}(B_{n-4}; q^2; q)$ | 1 | 2 | 2 |
| $(D_2 A_1, D_4 \boxtimes \varphi_{12})$ | $\mathcal{H}(A_1; q^9) \otimes \mathcal{H}(B_{n-6}; q^4; q^2)$ | 1 | 2 | 2 |

**Proof.** The relative Weyl group of a Levi subgroup $L$ of $D_n(q)$ of type $A_1$ has type
$D_{n-2}A_1$ (see either [35], p. 72) or use Chevie [26]). As the modular Steinberg character
$\varphi_{12}$ of $A_1(q)$ is liftable to a cuspidal character in characteristic 0, the parameters of the
Hecke algebra are the same as in characteristic 0 by Proposition 2.2. They can hence be
determined locally inside the minimal Levi overgroups of $L$: in type $A_1^2$ the parameter
is $q$, and similarly in a Levi subgroup of type $D_2 A_1$

The relative Weyl group of a Levi subgroup of type $A_1^2$ has type $B_2 D_{n-4}$. The modular
Steinberg character of $A_1(q)^2$ is liftable, so we can determine the parameters locally inside
4. UNIPOTENT DECOMPOSITION MATRIX OF $D_5(q)$

$A_3(q)$ (here the parameter is $q^2$), in $D_2(q)A_1(q)$ and in $A_1(q)^3$ (where the parameters clearly are $q$). The relative Weyl group of a Levi subgroup of type $A_1^1$ has type $B_3D_{n-6}$, and it is contained in minimal Levi overgroups of types $A_3A_1D_2A_2^1$ and $A_1^1$.

The relative Weyl group of a Levi subgroup of type $D_2$ (the two end nodes interchanged by the graph automorphism) has type $B_{n-2}$ by [35, p. 71]. Again the modular Steinberg character $\varphi_{12}$ of $D_2(q)$ is liftable to a cuspidal character in characteristic 0, and the parameters of the Hecke algebra can hence be determined locally: inside a Levi subgroup of type $D_3$ the parameter is $q^2$, while inside a Levi subgroup of type $D_2A_1$, it clearly is $q$.

The relative Weyl group of a Levi subgroup $D_2A_1$ is of type $A_1B_{n-4}$; the parameters of the Hecke algebra for its cuspidal $\ell$-modular (liftable) Steinberg character $\varphi_{12} \boxtimes \varphi_{12}$ are again determined locally inside proper Levi subgroups $D_1$, $D_3A_1$ and $D_2A_1^2$ of $G$.

In all cases discussed so far, the assumptions of reduction stability in Corollary 2.3 are satisfied as all graph automorphisms of $L$ just permute simple components. The same considerations apply to the cuspidal Brauer character of a Levi subgroup of type $D_2A_1^2$.

The ordinary cuspidal unipotent character of $G = D_4(q)$ remains irreducible upon reduction modulo $\ell$, by Table 1, hence the corresponding cuspidal Brauer character is reduction stable. The parameters of the Hecke algebra in characteristic 0 are known, see [8, p. 464]. The parameters in the remaining cases are determined analogously.

The cuspidal modular Steinberg character is the $\ell$-modular reduction of a Deligne–Lusztig character $R^\ell_2(\theta)$ for $\theta \in \text{Irr} T$ in general position with $T$ a Sylow $\Phi_2$-torus. The normaliser of a Levi subgroup of type $D_4$ in larger groups of type $D_n$ induces the graph automorphism of order 2 on $G$. (This can be seen from the inclusion $(GO_2GO_{2n-8}) \cap SO_{2n} \leq SO_{2n}$.) It is clear that there exist $\ell$-characters $\theta \in \text{Irr} T$ in general position invariant under this graph automorphism. Finally, the cuspidal Brauer character $\varphi_{12}$ occurs as one of two composition factors in the $\ell$-modular reduction of a character $R^\ell_2(\theta)$, with $L^F = 2A_2(q).((q + 1)^2$ and $\theta = \theta_1 \boxtimes \theta_2$ with $\theta_1$ the cuspidal unipotent character of $2A_2(q)$, and $\theta_2 \in \text{Irr} Z$ an $\ell$-character in general position of the $\Phi_2$-torus $Z = Z(L)^F$. Here, $\theta_1$ is certainly stable under all automorphisms, and for the two dimensional torus $Z$ it is immediate that there is a stable $\ell$-character in general position. Thus, Corollary 2.4 applies.

The groups of type $D_5$ have 20 unipotent characters. All of them lie in the principal $\ell$-block for primes $\ell$ dividing $q + 1$.

Theorem 4.8. Assume $(T_\ell)$. The decomposition matrix for the principal $\ell$-block of $D_5(q)$, $11 \leq \ell|(q + 1)$, is as given in Table 2, where $d \in \{0, 1\}$.

For reasons of space, in the 2nd column of Table 2 we just give the leading coefficient and $q$-power of the degree polynomials of the corresponding unipotent characters. The Harish-Chandra series of the cuspidal PIMs of $D_4(q)$ are indicated by the label of the unipotent character corresponding to it by triangularity of the decomposition matrix given in Table 1.

Proof. Let $G = D_5(q)$. Again, the unipotent characters form a basic set for the principal $\ell$-block. Note that the $\ell$-modular decomposition matrices for unipotent blocks of all proper Levi subgroups are known, except for the one PIM in the decomposition
matrix of the principal $\ell$-block of $D_4(q)$ whose unipotent part is given by

$$[D_4] + a_1([1^2]+) + [1^2]- + ([21^2]) + a_2[1.1^3] + a_3[.1^4]$$

with $a_2 = c + 3a_1 - 2$, $a_3 = 4c + 3a_1$ and $c \in \{-1,0,1\}$ (see the proof of Theorem 4.5).

Let us again denote by $\Psi_i$, $1 \leq i \leq 20$, the linear combinations of unipotent characters given by the columns in Table 4. We shall show that these are the unipotent parts of projective indecomposable characters of $G$.

The columns $\Psi_1$, $\Psi_2$, $\Psi_4$ and $\Psi_7$ are obtained from the decomposition matrix of the principal series Hecke algebra $\mathcal{H}(D_5; q)$, which in turn is easily deduced from the one for $\mathcal{H}(B_5; 1; q)$; see the programme of Jacon [36]. Harish-Chandra induction of PIMs from proper Levi subgroups gives $\Psi_9$, $\Psi_{12}$, $\Psi_{17}$ and the projectives with unipotent parts

$$\tilde{\Psi}_3 = \Psi_3 + 2\Psi_6,$$

$$\tilde{\Psi}_5 = \Psi_5 + 2\Psi_{12},$$

$$\tilde{\Psi}_8 = \Psi_8 + a_1\Psi_{12} + (a_2 - a_1)\Psi_{15} + \Psi'_{16},$$

$$\tilde{\Psi}_{10} = \Psi_{10} + \Psi_{14},$$

$$\tilde{\Psi}_{11} = \Psi_{11} + 2\Psi_{15},$$

$$\tilde{\Psi}_{13} = \Psi_{13} + \Psi_{15} + \Psi_{19},$$

$$\tilde{\Psi}_{18} = \Psi_{18} + \Psi_{20}.$$

### Table 4. $D_5(q)$, $11 \leq \ell | (q + 1)$, principal block

| .5 | 1 | 1 |
| 1.4 | q | 1 | 1 |
| .41 | $q^2$ | . | 1 | 1 |
| 2.3 | $q^2$ | 1 | 1 | . | 1 |
| 1.23 | $q^3$ | 1 | 1 | 1 | 1 |
| .32 | $q^3$ | . | 1 | 1 | 1 |
| 1.31 | $q^3$ | 1 | 1 | 1 | . | 1 |
| 1.2 | $q^2$ | 1 | . | 1 | 1 | . | 2 | . | 1 |
| 1.21 | $q^6$ | 1 | 1 | 1 | 1 | 1 | 2-$\ell$ | 1 | 1 | . | 1 |
| 1.32 | $q^7$ | 1 | 1 | 1 | 1 | 1 | 2-$\ell$ | . | 1 | 1 |
| 1.21 | $q^7$ | . | 1 | 1 | 1 | 1 | 2-$\ell$ | 1 | 1 | 1 | . | 1 |
| 1.2 | $q^7$ | . | 1 | 1 | 2-$\ell$ | 1 | 1 | . | 1 | . | . | 1 |
| 1.2 | $q^7$ | . | 1 | 1 | 1 | 1 | 1 | 1 | . | 1 | . | 1 | . | 1 |
| 1.2 | $q^7$ | . | 1 | 1 | 1 | 1 | 1 | 1 | . | 1 | . | 1 | . | 1 |
| 1.2 | $q^7$ | . | 1 | 1 | 1 | 1 | 1 | 1 | . | 1 | . | 1 | . | 1 |
| 1.2 | $q^7$ | . | 1 | 1 | 1 | 1 | 1 | 1 | . | 1 | . | 1 | . | 1 |
| 1.2 | $q^7$ | . | 1 | 1 | 1 | 1 | 1 | 1 | . | 1 | . | 1 | . | 1 |
| 1.2 | $q^7$ | . | 1 | 1 | 1 | 1 | 1 | 1 | . | 1 | . | 1 | . | 1 |
4. UNIPOTENT DECOMPOSITION MATRIX OF $D_2(q)$

The Hecke algebra for the cuspidal $\ell$-modular Steinberg character of a Levi subgroup of $G$ of type $D_2$ (the two end nodes interchanged by the graph automorphism) was determined in Lemma 4.7 to be $H = H(B_3; q^2; q)$. It has four PIMs, thus there will be four PIMs of $G$ in the Harish-Chandra series $D_2$. Now the projectives $\tilde{\Psi}_3$ and $\tilde{\Psi}_{11}$ contain summands in that series. The only proper subsums of $\tilde{\Psi}_3$ satisfying (HCr) are $\Psi_3$, $\Psi_6$, $\Psi_3 + \Psi_6$ and $2\Psi_6$. On the other hand, from the decomposition matrix of $H$ we see that this projective has to have two distinct summands in the $D_2$-Harish-Chandra series. It follows that $\Psi_3$ and $\Psi_6$ are PIMs for $G$. Exactly the same reasoning applies to the projective $\Psi_{11}$: it must decompose as $\Psi_{11} + 2\Psi_{15}$, and both $\Psi_{11}$ and $\Psi_{15}$ are PIMs by (HCr).

The Hecke algebra $H(A_1; q) \otimes H(A_1; q^2)$ for the cuspidal modular Steinberg character of a Levi subgroup $D_2A_1$ has two irreducible characters (see Table 4). It follows that the corresponding Harish-Chandra series contains two PIMs. The projective $\tilde{\Psi}_{10}$ contains summands in that series, and the only splitting satisfying (HCr) is $\Psi_{10} + \Psi_{14}$; this yields that $\Psi_{10}$ and $\Psi_{14}$ are PIMs.

For the Harish-Chandra series above the Steinberg PIM of a Levi subgroup of type $A_1$, the Hecke algebra $H = H(A_1; q) \otimes H(D_3; q)$ determined in Lemma 4.7 has two PIMs; by (HCr) and the decomposition matrix of $H$ the only admissible splitting of the projective $\tilde{\Psi}_5$ in this series is as $\Psi_5 + 2\Psi_{12}$, thus we find the PIM $\Psi_5$. The Harish-Chandra induction $\Psi_{18}$ of the Steinberg PIM from $D_4$ contains the Steinberg PIM of $G$ by (St); this gives $\Psi_{18}$ and $\Psi_{20}$, both of which are indecomposable by (HCr).

The Hecke algebra for the ordinary unipotent cuspidal character of $D_4$ is $H(A_1; q^4)$, so $\Psi_8$ has at least two projective summands $\tilde{\Psi}_8, \tilde{\Psi}_{16}$, the first containing $[D_4; 2]$, the second $[D_4; 1^2]$. On the other hand by (Tri) there is a PIM involving $[D_4; 1^2], [2^21]$, and unipotent characters of larger $a$-value. As $\tilde{\Psi}_{16}$ must be a summand of this, it only contains unipotent characters with $a$-value at least 7. The only possible summand of $\Psi_8$ satisfying (HCr) and containing none of $[1^32]$ and $[1.21^2]$ is $\tilde{\Psi}_8'$ as listed below, with a parameter $b \geq 0$ satisfying $a_2 \leq b \leq a_3$, and so $\Psi_8' = \Psi_8 + \alpha \Psi_{12} + (a_2 - a_1)\Psi_{15}$ is projective:

| $D_4;2$ | $1$ |
| $31^2$ | $a_1$ |
| $1^2.21$ | $a_1$ |
| $1^3.2$ | $a_2-a_1$ |
| $2.2^1$ | $a_1$ |
| $1.2^1$ | $a_1$ |
| $D_4;1^2$ | . | 1 |
| $1^2.1^3_1$ | $a_2$ | $a_1$ |
| $2.1^3$ | $b-a_2$ | $a_1+a_3-b$ |
| $1.1^4$ | $b+a_1-a_2$ | $a_2+a_3-a_1-b$ |
| $1^5$ | $a_2+a_3-b$ | $b-a_2$ |

$\Psi_8'$ $\tilde{\Psi}_{16}$

As the centraliser of a Sylow $\ell$-subgroup of $G$ is contained inside a proper parabolic subgroup of type $D_4$, $G$ cannot have cuspidal Brauer characters by (Csp). Since we already accounted for all Harish-Chandra series except for the one above $\varphi_{1,13}$ of $D_4$, $\Psi_{19}$ must lie in that series, and must hence be a summand of the Harish-Chandra induction $\tilde{\Psi}_{13}$ of the cuspidal PIM $\Psi_{1,13}$ of $D_4(q)$. The only such sumsub of $\tilde{\Psi}_{13}$ satisfying (HCr)
is $\Psi_{19}$, so this gives the PIM $\Psi_{19}$. Harish-Chandra inducing $\Psi_{19}$ to $D_6(q)$, restricting back again and decomposing shows that $\Psi_{13} - \Psi_{19} = \Psi_{13} + \Psi_{15}$ must be decomposable, and (HCr) then yields $\Psi_{13}$.

Modulo the knowledge of $a_1, b$ and $c$ we have now obtained all columns in Table 4 except for $\Psi_8$. When Harish-Chandra inducing $\Psi_{16}$ to $D_6(q)$ and restricting back again, the decomposition in terms of the projectives obtained so far has coefficient 1 on $\Psi_8$ and negative coefficients $-a_1$ on $\Psi_{12}$ and $a_1 - a_2$ on $\Psi_{15}$. So $\Psi_8$ is not indecomposable, but contains $\Psi_{15}$ at least $a_2 - a_1$ times. This shows that $\Psi_8$ is projective. In addition, (HCr) shows the inequalities $2(a_2 - a_1) \leq b \leq a_3$. Note however that at this stage we cannot show that $\Psi'_8$ is indecomposable since it could still contain some copies of $\Psi_8$.

Let us consider the Deligne–Lusztig character $R_w$ associated to the Coxeter element $w \in W$. The PIMs $\Psi_{18}$ and $\Psi_{20}$ do not occur in any $R_v$ for $v < w$. The multiplicity of $R_w$ on $\Psi_{20}$ is equal to $4 - 8a_1 + 2b - 8c$ whereas the multiplicity on $\Psi_{18}$ is the opposite. We deduce from (DL) that both quantities are zero and therefore $b = -2 + 4a_1 + 4c$. Now since $b \leq a_3$ and $a_3 = 4c + 3a_1$ we obtain $a_1 \leq 2$. On the other hand, if $\ell > 6$ then by Proposition 4.1 there exists a linear $\ell$-character of a Levi subgroup $L$ of $G$ of type $D_2(q), (q-1)(q+1)^2$. Using (Red) with the trivial character of $L$ we get $a_1 \geq 2$, therefore $a_1 = 2$ and $b = a_3 = 4c + 6$ and thus $a_2 = c + 4$. In addition, the relation $a_2 \leq b$ forces $c \geq 0$ hence $c \in \{0, 1\}$.

Finally we use (Red) with the following pairs $(L, \rho)$ to show that $\Psi'_8$ is almost indecomposable:

- $(A_3(q), (q+1)^2, [4])$ shows that $\Psi_{11}$ cannot be a direct summand of $\Psi'_8$;
- $(A_1(q), A_2(q), (q + 1), [2] \boxtimes [31])$ shows that $\Psi_{12}$ can be a direct summand of $\Psi'_8$ with multiplicity at most 1.

Therefore $\Psi_8 := \Psi_8' - d\Psi_{12}$ is indecomposable for some $d \in \{0, 1\}$. \hfill $\square$

5. Unipotent decomposition matrix of $D_6(q)$

The groups of type $D_6$ have 42 unipotent characters. For primes $\ell > 2$ with $\ell |(q+1)$, 37 of them lie in the principal $\ell$-block, the other five lie in a block of defect $(q + 1)^2$. 

**Theorem 4.9.** Assume $(T_\ell)$. The decomposition matrices for the unipotent $\ell$-blocks of $D_6(q)$, for $11 \leq \ell |(q+1)$, are as given in Tables 3, 7, where $d \in \{0, 1\}$ is as in Theorem 4.8 and the unknown entries satisfy moreover

$$c_{20} = 24 - 15c_{17} - 5c_{18} + 6c_{19},$$

$$5 \leq 4c_{17} + c_{18} - c_{19} \leq 7, \ c_4, c_{12} + c_4 \geq 1, \ c_6 \geq 2, \ c_8 \geq 4.$$

**Proof.** For the non-principal unipotent block all columns as given in Table 7 are obtained directly from (HCl) and are indecomposable by (HCr). So we are left to consider the principal block.

Denote by $\Psi_i$, $1 \leq i \leq 37$, the linear combinations corresponding to the columns of Tables 5 and 6. All $\Psi_i$ admit from the ones with

$$i \in \{3, 9, 10, 25, 27, 31, 34, 36, 37\}$$
are obtained by (HCi) from Levi subgroups of types $D_5$, $A_5$, $D_4A_1$ and $D_2A_3$, up to the knowledge of $c \in \{0,1\}$ which we will determine later in the proof. Moreover we obtain
4. DECOMPOSITION MATRICES AT $d_6(q) = 2$

**Table 6.** $D_6(q)$, $11 \leq \ell |(q + 1)$, cntd.

| $q^8$ | 1 |
|--------|---|
| $q^6$ | . 1 |
| $q^6$ | 1 1 1 |
| $q^{10}$ | . 1 1 |
| $q^{10}$ | 1 1 1 | . 1 |
| $q^{12}$ | . 1 1 1 | . 1 |
| $q^{13}$ | 1 4 1 1 | 2 1 1 |
| $q^{13}$ | . 1 . 1 . 1 . 1 |
| $q^{12}$ | 1 1 1 |
| $q^{15}$ | 1 1 1 c$_4$ 1 c$_{12}$ . . . 1 |
| $q^{15}$ | 1 1 1 c$_4$ 1 c$_{12}$ . . . 1 |
| $q^{16}$ | 2 5 1 2c$_4$ 2 2c$_{12}$ 1 1 1 c$_{17}$ 1 1 1 |
| $q^{20}$ | 1 4 1 3 1 2 1 1 3 c$_{18}$ . . . 1 |
| $q^{21}$ | 1 5 1 2c$_4$ 2 2c$_{12}$ 1 1 4 c$_{19}$ 1 1 4 1 1 |
| $q^{30}$ | 1 3 2c$_4$ 4+2c$_{12}$ . 1 3 c$_{20}$ 1 1 1 6 1 |

| A$_1^2$ A$_1$ 1.1$^3$ D$_2$A$_1$ c | A$_4^2$ c | .1$^4$ A$_1$ 1.1$^4$ 1.1$^3$ c | A$_4^2$ A$_1^3$ c | .1$^4$ c c |

**Table 7.** $D_6(q)$, $11 \leq \ell |(q + 1)$, block of defect $\Phi_2^2$.

| 1.41 | $q^3\Phi_2^4\Phi_3\Phi_6^2\Phi_{10}$ | 1 |
| 21.3 | $q^4\Phi_2^2\Phi_3\Phi_6^3\Phi_{10}$ | 1 1 |
| .321 | $q^7\Phi_2^3\Phi_3\Phi_6\Phi_{10}$ | 1 1 |
| 1.21$^3$ | $q^{10}\Phi_2^4\Phi_3\Phi_6^2\Phi_{10}$ | 1 1 1 1 |
| 1.21$^3$ | $q^{13}\Phi_2^3\Phi_3\Phi_6^3\Phi_{10}$ | 1 . . 1 1 |

$\Psi_{10} + \Psi_{11}$ and $\Psi_{10} + \Psi_{22}$ from which (Tri) yields $\Psi_{10}$. We can also determine the Harish-Chandra series of all the $\Psi_i$ obtained so far, and comparing with Table 3 it ensures that all non-cuspidal series have been accounted for except for one PIM in series $D_2$, which must be a summand of $\Psi_3 + \Psi_{11}$, and one PIM in series $D_4$ which must be a summand of $\Psi_9$.

Thus the principal block must contain six cuspidal Brauer characters corresponding to the columns with indices 25, 27, 31, 34, 36, 37. By our assumption $(T_3)$ there will be projectives for these columns with non-zero entries only on and below the diagonal, for characters lying in smaller families. We denote these unknown entries by $c_1, \ldots, c_{24}$ according to the following table, where we have used that the non-trivial graph automorphism of $D_6(q)$ of order 2 interchanges the unipotent characters $[1^3+]$ and $[1^3-]$ but fixes the six cuspidal
5. UNIPOTENT DECOMPOSITION MATRIX OF $D_6(q)$

PIMs, which means that their multiplicities in these PIMs must agree.

| $2^3$ | 1 |
|-------|---|
| $1^2.2^1^2$ | . |
| $D_4'.2$ | . 1 |
| $.3^1^3$ | . |
| $.2^2^1^2$ | $c_1$ $c_9$ |
| $1^4.2$ | $c_2$ $c_1^0$ |
| $D_4'.1^2$ | $c_3$ $c_1^1$ 1 |
| $1^3^+$. $1^3^-$ | $c_4$ $c_1^2$ . |
| $1^2$. $1^4$ | $c_5$ $c_1^3$ $c_1^7$ 1 |
| $.2^1^4$ | $c_6$ $c_1^4$ $c_1^8$ $c_2^1$ |
| $1.1^5$ | $c_7$ $c_1^5$ $c_1^9$ $c_2^2$ 1 |
| $1^6$ | $c_8$ $c_1^6$ $c_2^0$ $c_2^3$ $c_2^4$ |
| $\Psi_2^5$ $\Psi_2^7$ $\Psi_3^1$ $\Psi_3^4$ $\Psi_3^6$ |

Note that the missing cuspidal columns can not occur as summands of any of the projectives obtained so far. The value $c_{24} = 6$ follows from Corollary 4.3(b). To derive conditions on the other unknowns we start by looking at the coefficients of the various PIMs on the Deligne–Lusztig character $R_w$ attached to a Coxeter element $w \in W$. The PIMs corresponding to the cuspidal simple modules, as well as the PIMs $\Psi_3^5$ and $\Psi_2^9$ do not occur in $R_v$ for $v < w$, therefore their coefficients must be non-negative by (DL).

The coefficients on $\Psi_2^9$ and $\Psi_3^1$ are $3-c_1-c_9$ and $3-c_3-c_1^1$ respectively. On the other hand, if $\ell > 4$ we can invoke (Red) with Proposition 4.1 for $\rho$ the unipotent character labelled by $2^2$ (resp. the cuspidal unipotent character) of $D_4(q).(q+1)^2$ to get $c_9 \geq 2$ and $c_1 \geq 1$ (resp. $c_3 \geq 2$ and $c_1^1 \geq 1$). This shows that $c_1 = c_1^1 = 1$ and $c_3 = c_9 = 2$.

At this point, Harish-Chandra inducing the cuspidal PIM $\Psi_2^3$ to $D_7(q)$ and restricting the result to $D_5(q).A_1(q)$ only decomposes non-negatively when the parameter $c$ left open in the decomposition matrix for $D_4(q)$ satisfies $c = 0$. This furnishes the final step in the proofs of Theorems 4.5 and 4.8.

The coefficient on $\Psi_3^4$ is $c_{10} + 2c_{12} - c_{13} + c_2 + 2c_4 - c_5$. If $\ell > 4$, (Red) applied to the cuspidal unipotent character of a 2-split Levi subgroup $2A_2(q)^2.(q+1)^2$ gives

$$c_{10} + 2c_{12} - c_{13} \leq 0 \quad \text{and} \quad c_2 + 2c_4 - c_5 \leq 0.$$  

Therefore they must both be zero.

The coefficient on $\Psi_3^5$ is $5 + 3c_{10} - c_{14} + 3c_2 - c_6$. If $\ell > 8$ then by (Red) for the trivial character of $A_1(q)^2.(q+1)^4$ we get

$$2 + 3c_{10} - c_{14} \leq 0 \quad \text{and} \quad 3 + 3c_2 - c_6 \leq 0.$$  

Hence $c_{14} = 2 + 3c_{10}$ and $c_6 = 3 + 3c_2$.

Finally, the coefficient on $\Psi_3^6$ is $4c_{10} + 2c_{12} - c_{15} + 4c_2 + 2c_4 - c_7$, and hence is non-negative. On the other hand, the trivial character of $A_1(q). (q+1)^5$ when $\ell > 10$ gives, by (Red), the relations

$$4c_{10} + 2c_{12} - c_{15} \leq 0 \quad \text{and} \quad 4c_2 + 2c_4 - c_7 \leq 0,$$
and therefore these expressions must both vanish. Using this first set of relations we obtain by Theorem 4.2 the relations

\[ c_8 = 3 + 3c_2 + 2c_4 \quad \text{and} \quad c_{16} = 4 + 3c_{10} + 2c_{12} \]

for the multiplicities in the Steinberg character.

We continue by analysing the coefficients of the Deligne–Lusztig character \( R_w \) for \( w = s_1s_3s_1s_2s_3s_4s_5s_6 \). The coefficients on \( \Psi_{15} \) and \( \Psi_{36} \) are \(-4c_{21}\) and \(16 + 4c_{21} - 4c_{22}\) respectively. Hence \( c_{21} = 0 \) by (DL). On the other hand, when \( \ell > 10 \), (Red) with the trivial character of \( A_1(q). (q + 1)^3 \) gives \( 4 - c_{22} \leq 0 \), and we deduce that \( c_{22} = 4 \).

Theorem 4.2 finally yields \( c_{23} = 9 \).

For the last relation we consider the coefficient of \( \Psi_{36} \) in the Deligne–Lusztig character \( R_w \) for \( w = s_1s_2s_3s_1s_2s_3s_4s_5s_6 \). It gives

\[ X := -120 + 96c_{18} + 24c_{19} - 24c_{20} \geq 0. \]

On the other hand (Red) applied to the cuspidal unipotent character of \( 2A_2(q). (q + 1)^4 \) gives \( X/24 \leq 2 \). Theorem 4.2 finally yields \( c_{20} = 24 - 15c_{17} - 5c_{18} + 6c_{19} \).

The Harish-Chandra series above \([2^3]\) in \( D_7(q) \) has two summands; decomposing the Harish-Chandra induction shows that we must have \( c_2 = 0 \). Similarly, decomposing the Harish-Chandra series above \([D_3.2]\) in \( D_7(q) \) shows that we must have \( c_{10} = 0 \).

Now (HCr) proves that all \( \Psi_i \) are indecomposable, apart possibly from \( \Psi_9 \) and \( \Psi_{12} \): The projective \( \Psi_9 \) can only contain \( \Psi_{14}, \Psi_{16}, \Psi_{18} \). When \( \ell > 4 \) one can invoke (Red) for the trivial character of a Levi subgroup \( D_4(q). (q + 1)^2 \) to exclude the possibility that \( \Psi_{14} \) occurs. Moreover, using (Red) with the unipotent character \([2^2] \boxtimes [2^-]\) of \( A_1(q)D_4(q). (q + 1) \) and the character \([2.2]\) of \( 2D_5(q). (q + 1) \) we can check that \( \Psi_{16} \) and \( \Psi_{18} \) each can occur at most once.

The projective module of character \( \Psi_{12} \) can only contain \( \Psi_{20}, \Psi_{21}, \Psi_{22}, \Psi_{26}, \Psi_{27}, \Psi_{32}, \Psi_{33} \). When \( \ell > 6 \) one can use (Red) with the trivial character of a Levi subgroup \( A_1(q)^3. (q + 1)^3 \) to check that in fact only \( \Psi_{22} \) and \( \Psi_{26} \) might occur. In addition, they can occur only once, which follows from (Red) with the unipotent characters \([1^2.2]\) of \( 2A_4(q). (q + 1)^2 \) and \([1^2.2] \boxtimes [2.2]^2\) of \( 2A_5(q)A_1(q)^2. (q + 1) \).

The lower bounds on the \( c_i \)’s are obtained using (Red) in the following cases:

- \( L^* = 2A_3(q). A_1(q). (q + 1)^2 \) with \( \rho = [2^2] \boxtimes [2] \) gives \( c_4 \geq 2 \);
- \( L^* = A_1(q)^3. (q + 1)^3 \) with the trivial representation gives \( c_{12} \geq 2 \); and
- \( L^* = A_1(q)^2. (q + 1)^4 \) (two non-conjugate) with the trivial representation gives \( c_{17} \geq 2 \) and \( c_{18} \geq 4 \).

Note that the previous relations imply \( c_{19} \geq 5 \). \( \square \)

Remark 4.10. In [16] we introduced, for any \( w \in W \) a virtual character \( Q_w \) representing the Alvis–Curtis dual of the intersection cohomology of the Deligne–Lusztig variety corresponding to \( w \). Let \( w = s_1s_3s_1s_2s_3s_4s_5s_6 \) and \( w' = (s_1s_2s_3)^2s_4s_3s_1s_2s_3s_4s_5s_6 \). Then

\[
\langle Q_w; \varphi_{1^3+} \rangle = 13 - 5c_{12} - c_4,
\]

\[
\langle Q_{w'}; \varphi_{12,11} \rangle = 48 - 24c_{17},
\]

\[
\langle Q_{w'}; \varphi_{214} \rangle = 96 - 24c_{18}.
\]
If the conjecture in [16] Conj. 1.2 holds then these multiplicities must be non-negative. With the lower bounds on the $c_i$’s given in Theorem 4.9 this would imply $c_{12} = c_{17} = 2$, $c_4 \in \{2,3\}$, $c_{18} = 4$ and $c_{19} \in \{5,\ldots,7\}$.

6. Unipotent decomposition matrix of $E_6(q)$

For the groups $E_6(q)$ we again first determine some Hecke algebras:

**Lemma 4.11.** Let $q$ be a prime power and $\ell$ odd such that $(q + 1)\ell > 5$. The Hecke algebras of various $\ell$-modular cuspidal pairs $(L, \lambda)$ of Levi subgroups $L$ in $E_6(q)$ and their respective numbers of simple modules are as given in Table 8.

**Table 8.** Hecke algebras in $E_6(q)$ for $d_\ell(q) = 2$

| $(L, \lambda)$ | $\mathcal{H}$ | $|\text{Irr} \mathcal{H}|$ |
|-----------------|----------------|---------------------------|
| $(A_1, \varphi_{12})$ | $\mathcal{H}(A_5; q)$ | 3 + 1 |
| $(A_2, \varphi_{12})$ | $\mathcal{H}(B_3; q^2; q)$ | 4 |
| $(A_3, \varphi_{12})$ | $\mathcal{H}(A_1; q) \otimes \mathcal{H}(A_2; q^2)$ | 3 |
| $(D_4, D_4)$ | $\mathcal{H}(A_2; q^4)$ | 3 |
| $(D_4, \varphi_{11})$ | $\mathcal{H}(A_2; q^2)$ | 3 |
| $(D_4, \varphi_{11})$ | $\mathcal{H}(A_2; q^2)$ | 3 |

**Proof.** The arguments are very similar to the ones used in the proof of Lemma 4.7. In fact, most of the parameters of the relevant Hecke algebras were already determined there. As an example let us consider the $\ell$-modular Steinberg character of a Levi subgroup of type $A_1$. By [35] p. 75 its relative Weyl group in $E_6(q)$ is of type $A_3$, and as the character is liftable, its parameters are determined locally, inside a Levi subgroup of type $A_2$, to be equal to $q$. For the (liftable) modular Steinberg character $\varphi^4$ of a Levi subgroup of type $D_4$ the relative Weyl group has type $A_2$, and the parameter was again already in Lemma 4.7 shown to be equal to $q^2$.

Reduction stability for the first four cases has already been argued in the proof of that lemma. The normaliser of a Levi subgroup of type $D_4$ inside $E_6$ induces the full group $G_3$ of graph automorphisms. Thus, by our description of $\varphi_{11}$ in Lemma 4.7 we need to see that there exist $\ell$-characters in general position for a Sylow $\Phi_2$-torus which are stable under the graph automorphisms. For this we use that the extension of $D_4$ by $G_3$ is realised inside the groups of type $F_4$ (see e.g. [42] Exmp. 13.9): the long root subgroups generate a subgroup of type $D_4$, and it is normalised by the Weyl group of type $A_2$ generated by two reflections at short roots. By [38] Tab. T.A.133] there exist semisimple $\ell$-elements with centralisers a short root $A_2$ in $F_4(q)$ whenever $q > 5$ and thus there is an $\ell$-character $\theta$ of a Sylow $\Phi_2$-torus $T$ of $G = D_4(q)$ such that $R^G_T(\theta)$ lifts the modular Steinberg character $\varphi_{11}$. The argument for $\varphi_{11}$ is similar.

The groups of type $E_6$ have 30 unipotent characters. For primes $\ell > 3$ with $\ell | (q + 1)$, 25 of them lie in the principal $\ell$-block and three are of defect zero. The remaining two lie in a unipotent block of cyclic defect, with Brauer tree

\[
\begin{array}{c}
\phi_{64,4} & \phi_{64,13} & \circ \\
\text{ps} & A_1
\end{array}
\]
4. Decomposition Matrices at \(d_c(q) = 2\)

**Theorem 4.12.** Assume \((T_\ell)\). The decomposition matrix for the principal \(\ell\)-block of \(E_6(q), \ 11 \leq \ell \mid (q + 1)\), is as given in Table 4 where \(d \in \{0, 1\}\) is as in Theorem 4.8. Here, the projectives in columns 7 and 11 might be decomposable.

**Table 9.** \(E_6(q), \ 11 \leq \ell \mid (q + 1)\), principal block

| \(\phi_{1,0}\) | 1 | 1 |
| \(\phi_{6,1}\) | \(q\) | . | 1 |
| \(\phi_{20,2}\) | \(q^2\) | . | 1 |
| \(\phi_{30,3}\) | \(q^3\) | . | 1 |
| \(\phi_{15,5}\) | \(q^5\) | . | 1 |
| \(\phi_{15,4}\) | \(q^6\) | . | 1 |
| \(D_4: 3\) | \(q^7\) | . | 1 |
| \(\phi_{60,5}\) | \(q^8\) | . | 1 |
| \(\phi_{24,6}\) | \(q^9\) | . | 1 |
| \(\phi_{81,6}\) | \(q^{10}\) | 1 | 1 |
| \(\phi_{20,10}\) | \(q^{11}\) | . | 1 |
| \(\phi_{90,8}\) | \(q^{12}\) | . | 1 |
| \(\phi_{10,9}\) | \(q^{13}\) | . | 1 |
| \(\phi_{60,8}\) | \(q^{14}\) | . | 1 |
| \(D_4: 21\) | \(q^{15}\) | . | 1 |
| \(\phi_{81,10}\) | \(q^{16}\) | 1 | 1 |
| \(\phi_{60,11}\) | \(q^{17}\) | . | 1 |
| \(\phi_{24,12}\) | \(q^{18}\) | . | 1 |
| \(\phi_{30,15}\) | \(q^{19}\) | . | 1 |
| \(\phi_{15,17}\) | \(q^{20}\) | 1 | 1 |
| \(\phi_{15,16}\) | \(q^{21}\) | . | 1 |
| \(D_4: 1^3\) | \(q^{22}\) | . | 1 |
| \(\phi_{20,20}\) | \(q^{23}\) | . | 1 |
| \(\phi_{6,25}\) | \(q^{24}\) | . | 1 |
| \(\phi_{1,36}\) | \(q^{25}\) | . | 1 |

\[
\begin{array}{ccccccc}
& ps & ps & ps & ps & A_1 & ps & \ast \\
D_4 & A_1 & A_1 & A_1 & A_1 & A_1 & A_1 & \ast \\
& \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\end{array}
\]

**Proof.** Let \(G = E_6(q)\). We denote by \(\Psi_i, \ 1 \leq i \leq 25\), the linear combinations of unipotent characters given by the columns in Table 9. We shall show that these are the unipotent parts of projective indecomposable characters of \(G\).

The columns \(\Psi_i\) with \(i \in \{1, 2, 3, 4, 6, 8\}\) are obtained from the decomposition matrix of the Hecke algebra of type \(E_6\) which has been determined by Geck [23, Table D].

By Table 9 the \(A_1\)-Harish-Chandra series contains three Brauer characters in the principal block. Harish-Chandra induction of the two PIMs of \(D_3(q)\) in that series yields \(\Psi_5 = \Psi_5 + \Psi_8 + 2\Psi_{10} + \Psi_{12}\) and \(\Psi_{10} = \Psi_{10} + 2\Psi_{12}\). From the decomposition matrix of the Hecke algebra \(H(A_5; q)\) it follows that \(\Psi_{10}\) must contain one PIM in that series plus two copies of another one. But the only splitting of \(\Psi_{10}\) compatible with (HCr) is into
7. Unipotent decomposition matrix of $^2D_4(q)$

We now turn to the groups of twisted type, where again we start with the orthogonal groups. Note that $^2D_3(q) \cong A_4(q)$ is a unitary group, whose unipotent decomposition matrix we already determined in [17].

The groups of type $^2D_4$ have 10 unipotent characters, all of which lie in the principal $\ell$-block for primes $\ell$ dividing $q + 1$. For the following result, we need no restriction on $q$:

**Theorem 4.13.** The decomposition matrix for the principal $\ell$-block of $^2D_4(q)$, with $2 < \ell | (q + 1)$, is as given in Table 10.

Here $\alpha + 1$ is the multiplicity of the cuspidal Brauer character $\varphi_{2\alpha}$ in the $\ell$-modular reduction of the Steinberg character $[1^2]$ of $^2D_3(q)$; in particular $\alpha = 2$ if $(q + 1)_\ell > 3$ by [17] Tab. 1, and $\alpha \leq 2$ always.

**Proof.** We employ similar arguments as in the untwisted case. All unipotent characters of $G = ^2D_4(q)$ lie in the principal $\ell$-block, and they form a basic set. Let $\Psi_1, \ldots, \Psi_{10}$ denote the linear combinations of unipotent characters corresponding to the ten columns of Table 10. We construct projective characters as follows: the decomposition matrix of

$\Psi_{10}$ plus two times $\Psi_{12}$. So these two lie in the $A_1$-series. Then, $\Psi_5$ has to contain two copies of $\Psi_{10}$ and one copy of $\Psi_{12}$, so that $\Psi'_5 = \Psi_5 + \Psi_8$ is projective, and it contains one indecomposable summand from the $A_1$-series. Inducing the first $A_1$-PIM from $A_5(q)$ we see that the same holds for $\Psi''_5 = \Psi_4 + \Psi_5$. Hence neither of $\Psi'_5, \Psi''_5$ is indecomposable, and all of their summands apart from the one in the $A_1$-series lie in the principal series. This yields $\Psi_5$.

The $A_2^2$-series contains four Brauer characters by Table 8. Harish-Chandra induction of the two PIMs of $A_5(q)$ in that series yields $\Psi_9 = \Psi_9 + 2\Psi_{14}$ and $\Psi_{16} = \Psi_{16} + 2\Psi_{19}$. Thus, both $\Psi_9$ and $\Psi_{16}$ contain two summands from that series, one with multiplicity 2. The only splitting consistent with (HCr) is just given by $\Psi_9 + 2\Psi_{14}$, respectively $\Psi_{16} + 2\Psi_{19}$.

The $A_2^2$-series contains three Brauer characters. From (HCl) and using (Tri) we obtain $\Psi_{13} + \Psi_{14}, \Psi_{13} + \Psi_{14} + \Psi_{17}$ (which together yield $\Psi_{17}$), $\Psi_{14} + \Psi_{17} + \Psi_{19}$ and $\Psi_{21}$. By (Tri) we have that $\Psi_{13} + \Psi_{14}$ cannot be indecomposable, and the only admissible splitting with (HCr) is into $\Psi_{13}$ plus $\Psi_{14}$.

The Harish-Chandra series above the three cuspidal unipotent Brauer characters of $D_4(q)$ all contain three Brauer characters by Table 8. For $1.1^3$, (HCl) gives $\Psi_{11} + \Psi_{12} + \Psi_{20}$ and $\Psi_{20} + \Psi_{24}$, which both must contain two PIMs from this series. Now, by (Tri), $\Psi_{11} + \Psi_{12}$ cannot be indecomposable, and the only possible splitting with (HCr) leads to $\Psi_{11}, \Psi_{20}$ and $\Psi_{24}$. The situation for $1.1^4$ is entirely similar, giving $\Psi_{18}, \Psi_{23}$ and $\Psi_{25}$, all three of which are indecomposable by (HCl). We also find $\Psi_{7} + \Psi_{15} + 2\Psi_{19}$ and $\Psi_{15} + 2\Psi_{19} + \Psi_{22}$ which split up at least into $\Psi_{7}, \Psi_{15} + 2\Psi_{19}$ and $\Psi_{22}$.

Harish-Chandra inducing $\Psi_{22}$ to $E_7(q)$, cutting by the principal block, and restricting back shows that $\Psi_{19}$ is twice contained in $\Psi_{15} + 2\Psi_{19}$.

We have thus accounted for all columns in the table. An application of (HCr) shows that all of them are indecomposable, except possibly for $\Psi_7$ and $\Psi_{11}$: $\Psi_{11}$ might contain $\Psi_{19}$ once, and $\Psi_7$ might contains copies of $\Psi_{10}$ and of $\Psi_{12}$.

□
the Hecke algebra $H(B_3; q^2; q)$ for the principal series gives the four PIMs $\Psi_4, \Psi_2, \Psi_3, \Psi_7$ labelled by “ps” in Table 10. Next Harish-Chandra induction of the $A_1$-series PIM of a Levi subgroup of type $A_2$ gives a projective character with unipotent part $\Psi_4 = \Psi_4 + \Psi_6$, but the induction of neither of the $A_1$-series PIMs from a Levi subgroup of type $2D_2A_1$ contains this character, so $\Psi_4$ must be decomposable. The only subsums compatible with $(HCr)$ are $\Psi_4$ and $\Psi_6$, so these are the two PIMs of $G$ in the $A_1$-series.

Note that the centraliser of a Sylow $\Phi_2$-torus of $G$ lies inside a Levi subgroup of type $2D_3$, so by (Csp), all Brauer characters of $G$ have a Harish-Chandra vertex contained in that Levi subgroup. Furthermore, by [27, Thm. 4.2], the Harish-Chandra vertex of the modular Steinberg character of $G$ is the (cuspidal) modular Steinberg character $\varphi_{12}$ of $2D_3(q)$. But the Harish-Chandra induction of this PIM has unipotent part $\Psi_8 = \Psi_8 + \Psi_{10}$, so $\Psi_8$ is decomposable and yields the two PIMs $\Psi_8$ and $\Psi_{10}$. We have now accounted for all Harish-Chandra series except for that of the cuspidal character $\varphi_2$ of $2D_3(q)$. Hence the two remaining PIMs must lie in that series.

The column $\Psi_9$ (with a yet undetermined entry $a \geq 0$ in the last row) comes from the tensor product of the unipotent character [21.] with an irreducible Deligne–Lusztig character for a Coxeter torus (which is projective). This can be computed using the table of unipotent characters for $2D_4(q)$ available in Chevie [26]. Harish-Chandra induction of $\Psi_2$ from $2D_3(q)$ gives a projective character with unipotent part $\Psi_5 = \Psi_5 + \Psi_7 + \Psi_9$. As the remaining two PIMs must lie in the Harish-Chandra series above $\Psi_2$, $\Psi_5$ has at least two summands, and it has three if $\Psi_7$ is a summand of $\Psi_5$. We thus obtain the following lower right-hand corner of the decomposition matrix, for some $b \in \{0,1\}$ (with $b = 0$ if

\begin{table}
\centering
\begin{tabular}{c|cccccccc}
3. & 1 & 1 &  &  &  &  &  & \\
21. & $q\Phi_8$ & . & 1 &  &  &  &  & \\
2.1 & $q^2\Phi_3\Phi_6$ & . & 1 & 1 &  &  &  & \\
13. & $\frac{1}{2}q^3\Phi_6\Phi_8$ & 1 & . & 1 &  &  &  & \\
 .3 & $\frac{1}{2}q^3\Phi_6\Phi_8$ & . & 1 & . & 1 &  &  & \\
12.1 & $\frac{1}{2}q^3\Phi_3\Phi_8$ & . & 1 & 1 & . & . & 1 & \\
1.2 & $\frac{1}{2}q^3\Phi_3\Phi_8$ & 1 & . & . & . & . & 1 & \\
1.1 & $q^6\Phi_3\Phi_6$ & 1 & . & . & 1 & $\alpha$ & 1 & 1 & \\
 .21 & $q^7\Phi_8$ & . & . & . & $\alpha$ & 1 & 1 & 1 & \\
 .13 & $q^{12}$ & . & . & 1 & 1 & 1 & . & $\alpha$ & 1 & \\
 & $ps$ & $ps$ & $ps$ & $A_1$ & $A_1$ & $ps$ & $A^2$ & $2$ & $1^2$
\end{tabular}
\end{table}
$\Psi_5$ has two summands, and $b = 1$ if it has three).

\[
\begin{array}{c|cccc}
\hat{\nu} & 1 \\
1^2.1 & 0 \\
1.2 & 1 - b & 1 \\
1.1^2 & u - b & 1 & 1 \\
.21 & 1 & 1 & 1 \\
.1^3 & u - v & 1 & \ldots & v & 1 \\
\end{array}
\]

Note that for $(q + 1)\ell = 3$, the result in [17, Tab. 1] only gives an upper bound 3 for the multiplicity of the Brauer character $\phi_2$ in the 3-modular reduction of the Steinberg character. We write $u$ for this multiplicity (which equals 3 unless $(q + 1)\ell = 3$), and we have $0 \leq v \leq u$.

The Harish-Chandra restriction to $^2D_3(q)$ of the two projectives in the $\phi_2$-series decomposes as $(1 - 2b + v)\Psi_{1,1} + \Psi_{1,2} + (u - v - 1)\Psi_{1,2}$, respectively $\Psi_{1,2} + (1 + v - u)\Psi_{1,2}$ (see [17, Table 1]). This shows that $v = u - 1$. Finally, Harish-Chandra induction of $\Psi_5$ to $^2D_5(q)$ and restriction back to $^2D_4(q)$ should decompose non-negatively into PIMs. This forces $b = 1$. Our claim follows by setting $\alpha := v$. \hfill \Box

**Remark 4.14.** From the known 3-modular decomposition matrices it can be seen that $\alpha = 1$ for $\text{SO}_6^\circ(2) \cong \text{U}_4(2)$ and $\text{SO}_8^\circ(2)$, so the case $(q + 1)\ell = 3$ does indeed behave differently from the generic one where $\alpha = 2$.

### 8. Unipotent decomposition matrix of $^2D_5(q)$

We now turn to the groups $^2D_5(q)$, where we first need to determine the structure of certain Hecke algebras.

**Lemma 4.15.** Let $q$ be a prime power and $\ell \neq 2$ with $(q + 1)\ell \geq 7$. The Hecke algebras of various $\ell$-modular cuspidal pairs $(L, \lambda)$ of Levi subgroups $L$ in $^2D_n(q)$ and their respective numbers of irreducible characters are as given in Table 11.

| $(L, \lambda)$          | $\mathcal{H}$ | $n = 4$ | $5$ | $6$ | $7$ |
|------------------------|---------------|--------|----|----|----|
| $(A_1, \varphi_{12})$  | $\mathcal{H}(A_1; q) \otimes \mathcal{H}(B_{n-3}; q^2; q)$ | 2     | 2  | 4  | 6  |
| $(A_2, \varphi_{12}^{\circ2})$ | $\mathcal{H}(B_2; q^2; q) \otimes \mathcal{H}(B_{n-5}; q^2; q)$ | -    | 2  | 4  | 6  |
| $(^2D_3, \varphi_2)$   | $\mathcal{H}(B_{n-3}; q^2; q)$ | 2     | 2  | 4  | 6  |
| $(^2D_3, \varphi_{12})$ | $\mathcal{H}(B_{n-3}; q^2; q)$ | 2     | 2  | 4  | 6  |
| $(^2D_3A_1, \varphi_2 \boxtimes \varphi_{12})$ | $\mathcal{H}(A_1; q^3) \otimes \mathcal{H}(B_{n-5}; q^2; q)$ | -    | 1  | 2  | 2  |
| $(^2D_3A_1, \varphi_{12} \boxtimes \varphi_{12})$ | $\mathcal{H}(A_1; q) \otimes \mathcal{H}(B_{n-5}; q^2; q)$ | -    | 1  | 2  | 2  |

**Proof.** Recall that $^2D_n(q)$ has Weyl group of type $B_{n-1}$. The relative Weyl group of a Levi subgroup of type $A_1$ inside $^2D_n(q)$ is of type $A_1B_{n-3}$, see [35, p. 70]. Since the modular Steinberg character $\varphi_2$ of $A_1(q)$ is liftable, we may determine the parameters locally, inside minimal Levi overgroups of types $^2D_3$, $^2D_2A_1$ and $A_1^2$. The relative Weyl
group for a Levi of type $A_1^2$ is of type $2D_3 B_{n-5}$, and the minimal Levi overgroups have types $A_3$, $2D_3 A_1$ $2D_2 A_1$ and $A_1^3$. For the modular Steinberg character of a Levi subgroup $2D_3(q)$ the relative Weyl group has type $B_{n-3}$ and the minimal Levi overgroups are of types $2D_4$ and $2D_3 A_1$, with corresponding parameters $q^2$ and $q$. Finally, the minimal Levi overgroups for a Levi subgroup of type $2D_3 A_1$ have types $2D_5$, $2D_4 A_1$ and $2D_4 A_1^2$, with parameters $q^2$ and $q$ in the latter two. The parameter for the containment in $2D_5(q)$ will be determined in the proof of Theorem 4.16.

The cuspidal Brauer character $\varphi_2$ of $G = 2D_3(q)$ lift to an ordinary character $R^G_L(\theta)$ for $\theta \in \Irr L$ by [17 Prop. 5.4], where $L = 2A_2(q) (q+1)$ and $\theta$ is the product of the cuspidal unipotent character $\theta_1$ of $2A_2(q)$ with an $\ell$-character $\theta_2$ of $Z(L)$ in general position. Now $R^G_L(\theta_1 \boxtimes \theta_2) = R^G_L(\theta_1 \boxtimes \theta_2^1)$, so $R^G_L(\theta)$ is invariant under the graph automorphism of $2D_3(q)$ and hence reduction stable. Similarly, the cuspidal modular Steinberg character $\varphi_{1^2}$ lifts to $R^G_L(\theta)$ for $\theta$ an $\ell$-character in general position of a Sylow $\Phi_2$-torus $T$ of $G$. Here the normaliser of $G$ in a larger type $D$-group induces the Weyl group $W$ of type $B_3$ on $T$, and direct calculation then shows that there is an $\ell$-character of $T$ in general position stabilised by a short root reflection of $W$ when $(q+1)_\ell \geq 7$.

The groups of type $2D_5$ have 20 unipotent characters. For primes $\ell > 2$ with $\ell|(q+1)$, 18 of them lie in the principal $\ell$-block, and there is a further unipotent $\ell$-block of cyclic defect, with Brauer tree

$$
\begin{array}{c}
21.1 & \bigcirc & 1.21 \\
\text{ps} & \text{ps}
\end{array}
$$

**Theorem 4.16.** Assume $(T_\ell)$. The decomposition matrix for the principal $\ell$-block of $2D_5(q)$, $11 \leq \ell|(q+1)$, is as given in Table 12, where $b \geq 2$ and $d \in \{0, 1\}$.

**Proof.** The group $G = 2D_3(q)$ has 18 unipotent characters in its principal $\ell$-block. Since $\ell > 2$ these unipotent characters form a basic set. As in the previous proofs let $\Psi_1, \ldots, \Psi_{18}$ denote projective characters with unipotent part as in the columns of the matrix in Table 12. All projectives $\Psi_i$, except for $i \in \{1, 15, 17, 18\}$ are found by (HC4).

Since the Sylow $\ell$-subgroups of $G$ are not contained in any proper Levi subgroup of $G$, the PIM $\Psi_{18}$ is cuspidal by (St). Application of (HC5) now shows that all projectives obtained so far are in fact indecomposable, except possibly for $\Psi_5$. (For $\Psi_4$ and $\Psi_8$, there are two possible splittings consistent with (HC4), but in both cases, neither summand occurs among the other PIMs, so that a splitting would lead to a non-independent set of PIMs.) This implies in particular that the series above the two cuspidal unipotent Brauer characters of $2D_3(q) A_1(q)$ both just contain one Brauer character and hence the parameter of the Hecke algebra has to be $-1 \in k$, as claimed in Lemma 4.15.

As $\ell > 10$, by Theorem 4.12 there exists a non-unipotent cuspidal character $\rho$ with $\rho^\circ = R^G_{w_0}(\rho) = \varphi_{14}$. Assume $\varphi \in \{\varphi_{14}, \varphi_{22}, \varphi_{212}\}$ is not cuspidal. Then the corresponding projective cover can be obtained from the other columns of the decomposition matrix. The condition $(\langle \rho \varphi \rangle : \rho) = 0$ would then force $\varphi = \varphi_{22}$ and the unipotent part of $\Psi_{15}$ to be either $\rho_{22} + \rho_{113} + 3\rho_{212} + \rho_{14}$ or $\rho_{22} + 2\rho_{113} + 4\rho_{212} + 2\rho_{14}$. This contradicts the fact that the entry at $[1,1^3]$ in $\Psi_{15}$ vanishes by (Tri).

Now, denote by $b_1, \ldots, b_7$ the yet unknown decomposition numbers below the diagonal in the columns corresponding to $\varphi_{14}$, $\varphi_{22}$ and $\varphi_{212}$ (recall that the first entry below the
Table 12. $^2D_5(q)$, $11 \leq \ell|(q + 1)$, principal block

|   |     |     |     |
|---|-----|-----|-----|
| 4 | $q\Phi_3\Phi_{10}$ | 1   | 1   |
| 31| $q^2\Phi_4\Phi_8$  | 1   | 1   |
| 2 | $q^2\Phi_5\Phi_{10}$| 1   | 1   |
| 4 | $\frac{1}{2}q^3\Phi_6\Phi_8\Phi_{10}$| .  | 1   |
| 21 | .   | .   | 1   |
| 21 | $\frac{1}{2}q^3\Phi_6\Phi_8\Phi_{10}$| 1   | 1   |
| 2 | $\frac{1}{2}q^3\Phi_3\Phi_2\Phi_{10}$ | 1   | 1   |
| 1.3 | $\Phi_4\Phi_8\Phi_{10}$ | .   | 1   |
| 12 | $\frac{1}{2}q^3\Phi_3\Phi_8\Phi_{10}$| 1   | 1   |
| .31 | $\Phi_3\Phi_8\Phi_{10}$ | .   | 1   |
| 2 | $\frac{1}{2}q^7\Phi_8\Phi_{10}$ | 1   | 1   |
| 1.4 | $\frac{1}{2}q^7\Phi_3\Phi_8\Phi_{10}$| .  | 1   |
| .31 | $\frac{1}{2}q^7\Phi_3\Phi_8\Phi_{10}$| .  | 1   |
| 12.2 | $\frac{1}{2}q^7\Phi_2\Phi_8\Phi_{10}$| 1   | 1   |
| .2 | $q^{10}\Phi_8\Phi_{10}$ | .   | 1   |
| 1.1 | $q^{12}\Phi_4\Phi_8$ | .   | 1   |
| .21 | $q^{13}\Phi_3\Phi_{10}$ | .   | 1   |
| .14 | $q^{20}$ | .   | 1   |

\[ \Psi_{12} = [1^4] + b_1[.2^2] + b_2[1.1^3] + b_3[.21^2] + b_4[.1^4], \]
\[ \Psi_{15} = [.2^4] + b_5[.21^2] + b_6[.1^4], \]
\[ \Psi_{17} = [.21^2] + b_7[.1^4]. \]

From Theorem 1.2 we obtain the relations
\[-15 + 10b_1 + 4b_2 - 5b_3 + b_4 = 0, \quad 10 - 5b_5 + b_6 = 0, \quad b_7 = 5.\]

To obtain further relations we decompose suitable Deligne–Lusztig characters $R_w$ in terms of projective characters and then apply (DL). We start with a Coxeter element $w = s_2s_3s_4s_5$, whose coefficient on $\Psi_{17}$ is $3 - b_5$, forcing $b_5 \geq 3$. On the other hand, if $\ell > 8$ then by Proposition 1.1 there exists a linear $\ell$-character in general position of a 2-split Levi subgroup $L$ of type $A_1(q).A_1(q + 1)^4$. Then (Red) with $\rho$ being the Steinberg character of $L$ shows that $b_5 \geq 3$, which yields $b_5 = 3$ and $b_6 = 5$ using the previous equations. Note that as a consequence neither $\Psi_{12}$, $\Psi_{17}$ nor $\Psi_{18}$ occurs in $R_w$.

With $w = s_1s_2s_3s_4s_5s_6$, the coefficient of $\Psi_{17}$ on $R_w$ equals $-3 + 3b_1 + b_2 - b_3$ and must be non-negative by (DL). On the other hand, if $\ell > 6$ one can use (Red) with a Levi subgroup of type $2A_2(q).A_1(q + 1)^3$ and its cuspidal unipotent character to find that $5 - 3b_1 - b_2 + b_3 \geq 0$. This shows that there exists $d \in \{0, 1, 2\}$ such that $b_3 = -3 + 3b_1 + b_2 - d$. 

Diagonal in $\Psi_{12}$ and in $\Psi_{15}$ vanishes). Thus, we have

\[ \Psi_{12} = [1^4] + b_1[.2^2] + b_2[1.1^3] + b_3[.21^2] + b_4[.1^4], \]
\[ \Psi_{15} = [.2^4] + b_5[.21^2] + b_6[.1^4], \]
\[ \Psi_{17} = [.21^2] + b_7[.1^4]. \]
At this stage we can only find lower bounds on the remaining unknowns $b_1, b_2, d$. We have already seen that $d \in \{0, 1, 2\}$. Using (Red) for the two non-conjugate pairs of the form $(L, \rho) = (A_1(q)^2, (q + 1)^3, [2]^{132})$ gives $b_1 \geq 2$ and $b_2 \geq 3$ whenever $\ell > 6$. The other relations will be obtained in the proof of Theorem 4.18. In the table we have set $b := b_1$.

Remark 4.17. As in Remark 4.10 consider the virtual character $Q_w$ introduced in [16], for $w = (s_1 s_2 s_3)^2 s_4 s_5 s_4 s_3 s_1 s_2 s_3$. Then in the principal block $B_0$ of $^{2D_5}(q)$ we have

$$B_0 Q_w = 48([1^4] + [3.1] + 3[.2^2] + 5[1.1^3] + 9[.21^2] + 15[.1^4]).$$

If the conjecture in [16], Conj. 1.2 holds, this should have $\Psi_{14}$ as a direct summand, which would prove that $\Psi_{14}$ is a direct summand of $[1^3] + 2[.2^2] + 3[1.1^3] + 6[.21^2] + 13[.1^4]$. This would force in particular $b \leq 2$ hence $b = 2$. We deduce that assuming the conjecture, there are the following two possibilities left for the character of $\Psi_{14}$:

| $^4$  | 1  | 1   |
|------|----|-----|
| $^2$  | 2  | 2   |
| $1.1^3$ | 3 | 3   |
| $21^2$ | 6 | 5   |
| $.1^4$ | 13| 8   |

9. Unipotent decomposition matrix of $^{2D_6}(q)$

The groups of type $^{2D_6}$ have 36 unipotent characters, all of which lie in the principal $\ell$-block for primes $\ell$ dividing $q + 1$.

Theorem 4.18. Assume (T). The decomposition matrix for the principal $\ell$-block of $^{2D_6}(q)$, $11 \leq \ell | (q + 1)$, is as given in Table 13 where $b \geq 2$ and $d \in \{0, 1\}$ are as in Theorem 4.16. All columns but possibly $\Psi_5$ are indecomposable.

Proof. Since $\ell > 2$ the unipotent characters form a basic set for the unipotent blocks. As before, we denote by $\Psi_i$, $1 \leq i \leq 36$, (virtual) projective characters of $G$ whose unipotent parts decompose as given in the respective columns of Table 13 and we propose to show that these are the unipotent PIMs of $G$.

Note that the decomposition matrices of the unipotent blocks of all proper Levi subgroups are known, up to the undetermined entries in the PIM $\Psi_{14}$ of $^{2D_5}(q)$. The $\Psi_i$, $i \in \{1, 2, 3, 4, 6, 7, 10, 18\}$ are the PIMs in the principal series, so are obtained from the decomposition matrix of the Hecke algebra $H(B_6; q^2; q)$ (which can be computed with [36]). Now consider the Harish-Chandra series above the cuspidal $\ell$-modular Steinberg character $\varphi_{12}$ of a Levi subgroup of type $A_1$. Harish-Chandra inducing the two PIMs in that series from a Levi subgroup of type $^{2D_5}$ gives $\Psi_8 + 2\Psi_{16}$ and $\Psi_{13} + 2\Psi_{22}$. The decomposition matrix of the corresponding Hecke algebra $H(A_1; q) \otimes H(B_6; q^2; q)$ (see Lemma 4.15) shows that each of these must have two summands in the $A_1$-series, one with multiplicity 2. The only possible splitting compatible with (HCr) gives the four PIMs $\Psi_8, \Psi_{13}, \Psi_{16}$ and $\Psi_{22}$.

Next consider the Harish-Chandra series of the cuspidal $\ell$-modular Steinberg character $\varphi_{212}$ of a Levi subgroup of type $A_1^2$. HC-inducing the two PIMs in that series from a Levi
subgroup of type $A_4$ gives $\Psi_{11} + \Psi_{12}$ and $\Psi_{26} + \Psi_{27}$. Comparison with the corresponding Hecke algebra $\mathcal{H}(B_2; q^2; q) \otimes \mathcal{H}(B_1; q^2)$ shows that both have to have two PIMs as summands in this series. Again splitting via (HCr) yields $\Psi_{11}, \Psi_{12}, \Psi_{26}$ and $\Psi_{27}$.
Now consider the Harish-Chandra series of the cuspidal \( \ell \)-modular Steinberg character \( \varphi_{12} \) of a Levi subgroup of type \( 2D_3 \). Here, Harish-Chandra inducing the two PIMs in that series from a Levi subgroup of type \( 2D_5 \) gives \( \Psi_{14} + 2\Psi_{19} \) and \( \Psi_{28} + 2\Psi_{30} \). Comparison with the decomposition matrix of the Hecke algebra \( \mathcal{H}(B_3; q^2; q) \) and application of (HCr) then shows that \( \Psi_{14}, \Psi_{19}, \Psi_{28} \) and \( \Psi_{30} \) are projective.

The situation for the Harish-Chandra series of the cuspidal unipotent Brauer character \( \varphi_2 \) of a Levi subgroup of type \( 2D_3 \) is precisely as in the previous case, yielding the columns \( \Psi_5, \Psi_9, \Psi_{17} \) and \( \Psi_{24} \). Now consider the Harish-Chandra series above the cuspidal unipotent character \( \varphi_{12} \oplus \varphi_{12} \) of a Levi subgroup of type \( 2D_3A_1 \). By Table 12 this series will contain two PIMs, whence the Harish-Chandra induction of the unique PIM of a Levi subgroup of type \( 2D_3A_2 \) in that series, viz. \( \Psi_{15} + \Psi_{21} + \Psi_{22} \), must have at least two summands in that series. The only minimal summands compatible with (HCr) are \( \Psi_{15}, \Psi_{21}, \Psi_{22} \), of which the later lies in a lower Harish-Chandra series. But \( \Psi_{22} \) cannot be a summand in the PIMs induced from \( 2D_4A_1 \), so this yields \( \Psi_{15} \) and \( \Psi_{21} \). Next consider the Harish-Chandra series of the modular Steinberg character \( \varphi_{12} \otimes \varphi_{12} \) of a Levi subgroup of type \( 2D_3A_1 \). The Harish-Chandra induction of the corresponding PIM from \( 2D_3A_2 \) gives \( \Psi_{23} = \Psi_{23} + \Psi_{31} \). As a Levi subgroup of type \( 2D_4A_1 \) has two PIMs in this series, \( \Psi_{23} \) must have at least two summands in this series, and the only possibility according to (HCr) is \( \Psi_{23} \) plus \( \Psi_{31} \).

Since a Sylow \( \ell \)-subgroup of \( G \) is contained in a Levi subgroup of \( G \) of type \( 2D_5 \), \( G \) does not possess cuspidal unipotent Brauer characters by (Csp) and moreover, by (St) the Harish-Chandra induction of the Steinberg PIM from \( 2D_5(q) \) must contain the Steinberg PIM of \( G \), which gives \( \Psi_{34} \) and \( \Psi_{36} \).
We have now accounted for all Harish-Chandra series except for those above the three cuspidal Brauer characters $\varphi_{1^4}$, $\varphi_{22}$ and $\varphi_{21^2}$ of $^2D_5(q)$. As we are still missing six projectives, each of these three series will contribute two PIMs, and they must occur as summands of the Harish-Chandra induction $\Psi_{20}$, $\Psi_{25}$ and $\Psi_{29}$ of the respective cuspidal PIMs from $^2D_5(q)$. The only way that $\Psi_{25}$ can split according to (HCr) is as $\Psi_{25} = \Psi_{25} + \Psi_{33}$. By (Tri) there is a PIM of $G$ containing $[.21^3]$ once plus some copies of the Steinberg character. This cannot be a summand of $\Psi_{20}$, so it must occur in $\Psi_{29}$. Then (HCr) shows that $\Psi_{35}$ is a summand of $\Psi_{29}$. Harish-Chandra restriction of a PIM from $^2D_5(q)$ which contains the unipotent character $[.31^3]$ once and other unipotent characters of at least the same $\alpha$-value decomposes as $\Psi_{29} = \Psi_{30} + \Psi_{31}$ with higher $\alpha$-value, so $\Psi_{30}$ is a summand of $\Psi_{29}$ as well. This yields $\Psi_{29}$.

Finally, the PIM involving $[1^2.1^3]$ and constituents with higher $\alpha$-value must be a summand of $\Psi_{20}$. Again, (HCr) then yields two projective modules $\Psi_{20}$ and $\Psi_{32}$ as given below:

| $\Psi_{20}$ | $\Psi_{32}$ |
|-------------|-------------|
| 31^3        | b_1        |
| 1.2^2       | b_1        |
| 1^3.1^2     | .           |
| 1.1^4       | 1           |
| 2.1^3       | b_2        |
| 1.2^1       | 3b_1+b_2-d-3 |
| 1.2^1.1^3   | 3b_1+2b_2-d-3 |
| 1.5.1^5     | b_2        |
| 2.1^2       | 3b_1+b_2-d-3 |
| 1.1^4       | 5b_1+b_2-5d-c |
| 2.1^3       | 6b_1+b_2-5d-c |
| .1^5        | 2b_1+2b_2-d+c-3 |
|             | 3+3b_1-b_2-4d-c |

where $c$ is some non-negative integer.

Let $w = s_2s_1s_3s_2s_3s_4s_5s_6$. The coefficients of $\Psi_{34}$ and $\Psi_{36}$ on $R_w$ since these PIMs cannot occur in $\Psi_{20}$ or $\Psi_{32}$. We find that the coefficients are opposed therefore they must be zero by (DL). This gives $-24 + 8b_2 + 8c = 0$, hence $c = 3 - b_2$. But we showed that $b_2 \geq 3$ at the end of the proof of Theorem 4.16, therefore we must have $b_2 = 3$ and $c = 0$. In addition, if $l > 4$ then (Red) with $(^2A_2(q)A_1(q^2), (q + 1)^2, ^2A_2 \otimes [2])$ gives $d \leq 1$.

All columns are indecomposable, except possibly for $\Psi_3$ which might contain once $\Psi_{18}$, and for $\Psi_{20}$ which could contain the PIMs $\Psi_{24}$, $\Psi_{28}$ and $\Psi_{30}$. Now Harish-Chandra inducing $\Psi_{32}$ to $^2D_5(q)$, restricting it back and decomposing it in the $\Psi_1$ shows that $\Psi_{20}'$ must indeed contain $\Psi_{24}$ at least $b_1$ times, and $\Psi_{30}$ at least $3-d$ times. The new character could only contain $\Psi_{28}$. However, when $l > 6$, one can use (Red) with the trivial character of a Levi subgroup of type $A_3(q). (q + 1)^3$ to see that $\Psi_{28}$ cannot be a summand of $\Psi_{20}$. \qed
10. Unipotent decomposition matrix of $^2E_6(q)$

**Lemma 4.19.** Let $q$ be a prime power and $2 < \ell | (q + 1)$. The Hecke algebras of various $\ell$-modular cuspidal pairs $(L, \lambda)$ of Levi subgroups $L$ in $^2E_6(q)$ and their respective numbers of irreducible characters are as given in Table [17].

| $(L, \lambda)$       | $\mathcal{H}$                  | $|\text{Irr} \mathcal{H}|$ |
|----------------------|--------------------------------|-----------------------------|
| $(A_1, \varphi_{12})$| $\mathcal{H}(B_3; q^2)$        | 3                           |
| $(^2D_3, \varphi_{2})$ | $\mathcal{H}(B_2; q^2; q)$     | 2                           |
| $(^2D_3, \varphi_{12})$ | $\mathcal{H}(B_2; q^2; q)$     | 2                           |
| $(^2A_5, \varphi_{321})$ | $\mathcal{H}(A_1; q^3)$       | 1                           |
| $(^2A_5, \varphi_{2212})$ | $\mathcal{H}(A_1; q)$         | 1                           |
| $(^2A_5, \varphi_{214})$ | $\mathcal{H}(A_1; q)$         | 1                           |
| $(^2A_5, \varphi_{16})$   | $\mathcal{H}(A_1; -1)$       | 1                           |

**Proof.** For type $A_1$ the minimal Levi overgroups in $^2E_6(q)$ are of types $^2D_3$ and $A_1^2D_2$; for $^2D_3$ the minimal overgroups are of types $^2D_4$ and $^2A_5$. For the ordinary cuspidal unipotent character $[321]$ of $^2A_5(q)$ the Hecke algebra inside $^2E_6(q)$ is known to have parameter $q^4$ (see [8]). The induction of the Steinberg PIM of $^2A_5(q)$ to $^2E_6(q)$ is indecomposable by (HCr), so the parameter here is $-1 \in k$. By [17] Prop. 5.4, the cuspidal Brauer character $\varphi_{214}$ of $^2A_5(q)$ lifts to an ordinary cuspidal character in the Lusztig series of an $\ell$-element with centraliser $^2A_2(q)(q + 1)^4$, whose centraliser in $^2E_6(q)$ is of type $^2A_2(q)A_1(q)(q + 1)^3$. Similarly, $\varphi_{2212}$ lifts to an ordinary cuspidal character in the Lusztig series of an $\ell$-element with centraliser $^2A_2(q)^2(q + 1)^2$ (resp. $^2A_2(q)A_1(q)(q + 1)$) in $^2A_5(q)$ (resp. $^2E_6(q)$). Thus the parameter of the Hecke algebra is $q$ in either case.

The reduction stability of the cuspidal unipotent Brauer characters of $^2D_3(q)$ was already discussed in Lemma 4.15. From the extended Dynkin diagram it can be seen that the relative Weyl group of an $A_5$-Levi subgroup inside $E_6(q)$ acts trivially on the $A_5$-factor, so all cuspidal Brauer characters of $^2A_5(q)$ are reduction stable.

The groups $^2E_6(q)$ have 30 unipotent characters. For primes $\ell > 3$ dividing $q + 1$, 25 of these lie in the principal $\ell$-block and three more, namely those labelled $\phi_{8,3}, \phi_{8,9}, \phi_{18,5}$, lie in a block of defect $(q + 1)^2$. The unipotent part of the decomposition matrix of this latter $\ell$-block is easily shown to be the identity matrix. The last two unipotent characters, both of which are cuspidal, are of defect 0. We obtain the following approximation to the decomposition matrix of the principal block:

**Theorem 4.20.** The decomposition matrix for the principal $\ell$-block of $^2E_6(q)$, $\ell | (q + 1)$, $\ell > 3$ and $(q + 1)\ell > 11$, is approximated as given in Table [18]. The unknown entries satisfy...
All projectives listed are indecomposable except possibly for \( \Psi_6 \) and \( \Psi_7 \).

Table 16. \( ^2E_6(q) \), \( \ell|q+1 \), \( \ell > 3 \) and \((q+1)\ell > 11\)

| \( f \in \{0, 1, 2\} \) and | \( x_5 = 6x_1 - x_2 + x_3 + 2x_4 - 7 \) |
|-----------------------------|-----------------------------------|
| \( x_7 = 3x_1 - 3x_2 + x_3 + 2x_4 - 5 \) |
| \( x_8 = 6x_1 - 3x_2 + x_3 + 2x_4 + 2x_6 - 12 \) |
| \( x_9 = 6x_1 - 5x_2 + 4x_4 + 5x_6 - 15 \) |
| \( x_{10} = 15x_1 - 15x_2 + x_3 + 10x_4 + 5x_6 - 25 \) |
| \( v_2 = 5v_1 + f - 9 \) |
| \( v_3 = 10v_1 + 6f - 24 \) |

Here the symbols \( "2" \), \( "1^2" \), \( "321" \), \( "2^21^2" \), \( "2^41^4" \) and \( "1^6" \) stand for Harish-Chandra series of cuspidal Brauer characters of \( ^2D_5(q) \), \( ^2A_5(q) \) respectively, see [17 Table 1].
PROOF. As usual, we write $\Psi_i$ for linear combinations of unipotent characters corresponding to the column $i$ of Table 16. For $i \in \{6, 7, 10, 12, 15, 16, 18, 20\}$ they are obtained by (HCi). We also find $\Psi_2 + \Psi_4$ and $\Psi_2 + \Psi_5$ which by (Tri) and (HCr) yield $\Psi_2, \Psi_4$ and $\Psi_5$. Then $\Psi_1 + 2\Psi_2$ and $\Psi_1 + \Psi_3$ give $\Psi_1$ and $\Psi_3$. From this, $\Psi_3 + \Psi_9$ allows to isolate $\Psi_9$. Also $\Psi_4 + \Psi_8$ again by (Tri) yields $\Psi_8$. Then $\Psi_8 + \Psi_{14}$ leads to $\Psi_{14}$ and $\Psi_{12} + \Psi_{13}$ gives $\Psi_{13}$.

Counting the columns obtained so far and comparing with the number of characters in non-cuspidal Harish-Chandra series with Lemma 4.19 shows that the eight columns with indices $i \in \{11, 17, 19, 21, 22, 23, 24, 25\}$ must be cuspidal.

By Corollary 4.3 the second to last entry in the last row equals $6 = \text{rank } G$. We then use (DL) to find some of the entries in columns corresponding to the other cuspidal Brauer characters. Note that by Proposition 4.1 we can use (Reg) for any $d$-split Levi subgroup whenever $(q + 1)_\ell > 12$. We start with $\Psi_{17}$: the decomposition numbers in that column will be denoted by $y_1, \ldots, y_8$ so that the unipotent part of $\Psi_{17}$ equals

$$\Psi_{17} = \phi_4^n y + y_1 \phi_8^s + \cdots + y_7 \phi_9^s + y_8 \phi_{1,24}.$$

Let us consider the Deligne–Lusztig character $R_w$ associated to a Coxeter element $w$. One first checks that $\Psi_{18}$ does not occur in any $R_w$ for $v < w$. Its coefficient at $\Psi_{18}$ is $-y_1$, therefore by (DL) the entry $y_1$ must be zero. The coefficient of $\Psi_{19}$ equals $2 - y_2$ which forces $y_2 \leq 2$. One can use (Red) with the character $^2A_2 \boxtimes [2]$ of a Levi subgroup $^2A_2(q^2).(q + 1)^2$ to see that we also have $y_2 \geq 2$, which proves that $y_2 = 2$. The coefficients of $\Psi_{20}, \Psi_{21}$ and $\Psi_{22}$ are $3 - y_3, -y_4$ and $1 - y_5$ respectively. Using (Red) for a Levi subgroup $D_4(q).(q + 1)^2$ and the character $[2^2]$ (resp. the cuspidal unipotent character) we have $y_3 \geq 3$ (resp. $y_5 \geq 1$). We deduce that

$$y_3 = 3, \quad y_4 = 0 \quad \text{and} \quad y_5 = 1.$$

The coefficient of $\Psi_{23}$ is $1 - y_6$, and (Red) for the trivial character of a Levi subgroup $A_1(q^2).(q + 1)^2$ gives $y_6 \geq 1$, whence $y_6 = 1$. Finally, the coefficient of $\Psi_{24}$ is $3 - y_7$; one can invoke (Red) with the trivial character of a Levi subgroup $A_1(q).(q + 1)^5$ to ensure that $y_7 \geq 3$, hence $y_7 = 3$. We conclude that $y_8 = 5$ using Theorem 4.2.

We now turn to $\Psi_{21}$ using the Deligne–Lusztig character $R_w$ with $w = s_1 s_2 s_3 s_4 s_3$. We denote by $u_i$ the unknown decomposition numbers in the 21st column: under the assumption (Tri) there are $u_i \geq 0$ such that

$$\Psi_{21} = \phi_{1,12}^n + u_1 \phi_{1,13} + u_2 \phi_{2,16} + u_3 \phi_{1,24}.$$

The coefficient of $\Psi_{23}$ on $R_w$ is $2 - u_1$; on the other hand, (Red) with the trivial character of a Levi subgroup $A_1(q^2).(q + 1)^4$ gives $u_2 \geq 2$. The coefficient of $\Psi_{24}$ is $5 - u_2$ and (Red) with the trivial character of $A_1(q).(q + 1)^5$ gives $u_5 \geq 5$. This shows that $u_1 = 2$ and $u_2 = 5$. Theorem 4.2 gives $u_3 = 5$.

We continue with the PIM $\Psi_{19}$ and the Deligne–Lusztig character $R_w$ with $w = s_4 s_5 s_4 s_2 s_3 s_1 s_4 s_5$. The unknown entries will be denoted by $z_1, \ldots, z_6$. The coefficients of the PIMs $\Psi_{20}$ and $\Psi_{22}$ on $R_w$ are $4 - 2z_1$ and $4 - 2z_3$ respectively. The unipotent characters $[2^2]$ and $[D_4]$ of $D_4(q).(q + 1)^2$ give, by (Red), the relations $z_1, z_3 \geq 2$. We deduce that $z_1 = z_3 = 2$. The PIM $\Psi_{23}$ has coefficient $4 + 4z_2 - 2z_4$; using (Red) with the cuspidal unipotent character of $^2A_2(q^2).(q + 1)^2$ we get $-2 - 2z_2 + z_4 \geq 0$, which proves that $z_4 = 2 + 2z_2$. Next, $\Psi_{24}$ has coefficient $8 + 10z_2 - 2z_5$ and by (Red) with the trivial
character of $A_1(q).(q + 1)^5$ we must also have $-4 - 5z_2 + z_5 \geq 0$, whence $z_5 = 4 + 5z_2$.

Finally, Theorem 4.2 shows that $z_6 = 10 + 5z_2$. We set $z := z_2$.

Let us now consider the PIM $\Psi_{11}$ for which many entries are unknown. Using (Tri), only 10 entries below the diagonal can be non-zero, starting with the row corresponding to $\phi''_9$. We denote these by $x_i$. Let $w = s_1s_2s_4s_3s_1s_5s_4s_3s_6s_5s_4s_3$. The coefficients of $\Psi_{20}$ and $\Psi_{22}$ in $R_w$ are

$$X := 6x_1 - x_2 + 3x_3 + 2x_4 - x_5 - 7 \quad \text{and} \quad Y := 3x_1 - 3x_2 + x_3 + 2x_4 - x_7 - 5$$

respectively. By (Red) with the trivial character of $^2A_2(q).(q + 1)^4$, the sum $X + Y$ must be non-positive. Therefore (DL) forces $X = Y = 0$ which gives the values of $x_5$ and $x_7$ in terms of $x_1, \ldots, x_4$. The coefficient of $\Psi_{23}$ is $-12 + 6x_1 - 3x_2 + x_3 + 2x_4 + 2x_6 - x_8$. By (Red) applied to the cuspidal unipotent character of $^2A_2(q)^2.(q + 1)^2$ it is also non-positive, therefore it must be zero, so that

$$x_8 = 6x_1 - 3x_2 + x_3 + 2x_4 + 2x_6 - 12.$$  

Finally, the coefficient of $\Psi_{24}$ is $-15 + 6x_1 - 5x_2 + 4x_4 + 5x_6 - x_9$; it is non-negative by (DL) and non-positive by (Red) applied to the trivial character of $A_1(q).(q + 1)^5$, forcing $x_9 = 6x_1 - 5x_2 + 4x_4 + 5x_6 - 15$.

Theorem 4.2 then gives $x_{10} = 15x_1 - 15x_2 + x_3 + 10x_4 + 5x_6 - 25$.

Let us denote by $w_1$ and $w_2$ the two unknown entries in the 23rd column. Let $w = s_2s_4s_2s_3s_4s_6s_5s_4s_3s_6$. The coefficient of $\Psi_{23}$ in $R_w$ is $60 - 12w_1$ therefore $w_1 \leq 5$ by (DL). On the other hand, (Red) applied to the trivial character of $A_1(q).(q + 1)^5$ forces $w_1 \geq 5$, hence $w_1 = 5$. We get $w_2 = 10$ using Theorem 4.2.

The last Deligne–Lusztig character $R_w$ we look at is the one associated to the element $w = s_1s_2s_3s_1s_4s_5s_4s_3s_6s_5s_4s_3s_1$. Let $v_1, v_2, v_3$ be the three unknown entries in the 22nd column. The coefficient of $\Psi_{24}$ on $R_w$ equals $-252 + 180v_1 - 36v_2 = 36(-7 + 5v_1 - v_2)$. By (DL) this implies that $v_2 \leq 5v_1 - 7$. On the other hand, (Red) for the cuspidal unipotent character of $^2A_2(q).(q + 1)^4$ yields the relation $9 - 5v_1 + v_2 \geq 0$. We deduce that $v_2 = 5v_1 - 9 + f$ with some $f \in \{0, 1, 2\}$. Theorem 4.2 then gives $v_3 = 30 - 20v_1 + 6v_2 = 10v_1 + 6f - 24$.

Now all projectives in the table are indecomposable except possibly for $\Psi_6$ which might contain $\Psi_{12}$ once, and for $\Psi_7$ which might contain $\Psi_9$ (twice), $\Psi_{12}$ (four times), $\Psi_{14}$ and $\Psi_{15}$ (once each). \hfill \Box

### 11. Unipotent decomposition matrices of $B_4(q)$ and $C_4(q)$

Here, we find the decomposition matrices for the unipotent blocks of odd-dimensional orthogonal groups $B_4(q)$ and symplectic groups $C_4(q)$, assuming $(T_l)$.

The decomposition matrix for the principal $\ell$-block of $SO_5(q)$, $2 < \ell | (q + 1)$, was determined by Himstedt–Noeske [34]. Again we first record the parameters of certain Hecke algebras.

**Lemma 4.21.** Let $q$ be a prime power and $2 < \ell | (q + 1)$. The Hecke algebras of various $\ell$-modular cuspidal pairs $(L, \lambda)$ of Levi subgroups $L$ in $B_4(q)$ and their respective numbers of irreducible characters are as given in Table 17.
Table 17. Hecke algebras in $B_4(q)$ for $d_e(q) = 2$

| $(L, \lambda)$ | $\mathcal{H}$ | $|\text{Irr } \mathcal{H}|$ |
|-----------------|---------------|-----------------|
| $(A_1, \varphi_{1^2})$ | $\mathcal{H}(A_1; q) \otimes \mathcal{H}(B_2; q; q)$ | 2 |
| $(B_1, \varphi_{1^3})$ | $\mathcal{H}(B_3; q; q)$ | 3 |
| $(A_1^2, \varphi_{1^2_{1^2}})$ | $\mathcal{H}(B_2^2; q^2; q)$ | 2 |

Proof. The relative Weyl groups of the relevant Levi subgroups can be found in [35] p. 70. All three cuspidal characters are the $\ell$-modular Steinberg characters of the respective groups and thus liftable. The minimal Levi overgroups for type $A_1$ are of types $B_2$, $B_1 A_1$ and $A_1^2$ and thus lead to parameters $q$ in all three cases. For a Levi subgroup of type $B_1$ the minimal overgroups have types $B_2$ and $B_1 A_1$, with parameters $q$ as just seen. Finally for a Levi subgroup of type $A_1^2$ the minimal overgroups are of types $A_3$ and $B_2 A_1$, with parameters $q^2$, $q$ respectively.

As for reduction stability, by the considerations in Section 2.3 we only need to worry about the last case. Here, the embedding $SO_5SO_4 \leq SO_9$ shows that the relative Weyl group $B_2$ centralises the $A_1^2$ Levi subgroup and so any lift of the cuspidal Brauer character will be reduction stable.

The groups of type $B_3$ have 25 unipotent characters. For primes $\ell > 2$ dividing $q + 1$, 20 of these lie in the principal $\ell$-block, the other five lie in a block of defect $(\Phi_2^2)_{1^\ell}$.

Theorem 4.22. Assume $(T_i)$. The $\ell$-modular decomposition matrices of the unipotent $\ell$-blocks of $G = B_4(q)$, $7 < \ell | (q + 1)$, are as given in Table 18 and 19.

Here $B_4^\lambda$ denotes the Harish-Chandra series of the cuspidal unipotent Brauer character $B_4^1$ of $B_4(q)$, $B_4^0$, $B_4^2$ the ones of the cuspidal unipotent Brauer characters $B_2 \boxtimes \varphi_{1^2}$ and $\varphi_{1^2} \boxtimes \varphi_{1^2}$ of $B_2(q)A_1(q)$ and $A_2^1$ the one of the $\ell$-modular Steinberg character $\varphi_{1^2} \boxtimes \varphi_{1^2}$ of $B_1(q)A_1(q)$.

Proof. The five projectives in the non-principal unipotent $\ell$-block are obtained by (HCl) and are easily seen to be indecomposable. This proves Table 19. So now consider the principal block. The three columns labelled “ps” come from the decomposition matrix of the Hecke algebra $\mathcal{H}(B_4; q; q)$. Harish-Chandra inducing the unipotent PIMs from proper Levi subgroups $L$ of $G$ and cutting by the principal block we obtain projectives which are non-negative integral linear combinations of the sixteen columns $\Psi_1, \ldots, \Psi_{14}, \Psi_{16}, \Psi_{19}$ in our table. (For Levi subgroups of type $B_3$ the decomposition matrix, depending on $(q + 1)_\ell$, was obtained in [34 Table 5].) Furthermore, among these induced projectives we actually find all of the $\Psi_i$ not labelled “ps”, except that instead of $\Psi_6$ we obtain $\Psi_6 + \Psi_7$ and $\Psi_6 + \Psi_{19}$. Since the space spanned by these projectives together with $\Psi_7$ and $\Psi_{19}$ is only 3-dimensional, we conclude that $\Psi_6$ is also a projective character.

We next claim that all of the $\Psi_i$ are indecomposable. For $i \notin \{3, 6, 8\}$ this follows by application of (HCr). For the remaining three columns there is only one possible non-trivial decomposition each, into

$$
\Psi_3 = \Psi'_3 + \Psi'_7, \quad \Psi_6 = \Psi'_6 + \Psi'_8, \quad \Psi_8 = \Psi'_8 + \Psi'_{10}
$$

(with $\Psi'_7 + \Psi'_8 = \Psi_7$). The three projective characters are induced from Steinberg PIMs in the series $B_1$, $A_1^2$, $A_1$. The corresponding Hecke algebras were determined in Lemma 4.21.
11. Unipotent Decomposition Matrices of $B_4(q)$ and $C_4(q)$

Table 18. $B_4(q)$, $7 < \ell | (q + 1)$, principal block

|        | 1   | 1   |
|--------|-----|-----|
| $B_2$:2| $\frac{1}{2}q$ | 1   |
| 4      | $\frac{1}{2}q$ | 1   |
| 31     | $\frac{1}{2}q$ | 1   | 1   |
| $B_2$:12| $\frac{1}{2}q^2$ | 1 | . . | 1 |
| $2^2$  | $\frac{1}{2}q^2$ | . . | 1 | . 1 |
| 2.2    | $\frac{1}{2}q^2$ | 1 | . 1 | . . | 1 |
| 21.1   | $q^3$ | 1   | . 1 | 1 | 1 |
| 31     | $\frac{1}{2}q^4$ | 1 | . 1 | 1 | 1 |
| 212    | $\frac{1}{2}q^4$ | 1 | . 1 | 1 | 1 | 1 |
| 2.12   | $\frac{1}{2}q^4$ | 1 | 2 | 1 | . . | 1 | . 1 |
| 1.22   | $q^4$ | 1 | . 1 | . . | 1 | 1 | . 1 |
| 1.21   | $q^5$ | 1 | 2 | 2 | 1 | 2 | 1 | 1 | 1 |
| $B_2$:2| $\frac{1}{2}q^6$ | . 1 | . . | . . | . . | . 1 |
| $2^2$  | $\frac{1}{2}q^6$ | . . | 1 | 2 | 1 | . . | 1 | . 1 |
| 1.2.2  | $\frac{1}{2}q^8$ | 1 | 2 | 1 | . 2 | 1 | 1 | 2 | . . | 1 |
| $B_2$:12| $\frac{1}{2}q^9$ | . 1 | . | 1 | . . | . . | . . | 1 | 2 | 1 |
| 1.4    | $\frac{1}{2}q^9$ | 1 | . . | . . | 1 | 1 | . . | . . | 1 | 1 |
| $2^2$  | $\frac{1}{2}q^9$ | 1 | 2 | 1 | 1 | 2 | 1 | 1 | 3 | 1 | 1 | 2 | 1 | . . | 1 |
| $1^4$  | $q^{16}$ | 1 | 2 | 1 | 1 | 2 | . . | 1 | 3 | 1 | 1 | 2 | 3 | 1 | 4 | 3 | 1 | 1 |

Table 19. $B_4(q)$, $5 \leq \ell | (q + 1)$, block of defect $\Phi_2^2$

|        | $\frac{1}{2}q\Phi_2^2\Phi_4\Phi_6$ | 1   |
|--------|----------------------------------|-----|
| 3.1    | $\frac{1}{2}q\Phi_2^2\Phi_4\Phi_6$ | 1   |
| 1.3    | $\frac{1}{2}q\Phi_2^2\Phi_4\Phi_6\Phi_8$ | 1 | 1 |
| $B_2$:11| $\frac{1}{2}q^4\Phi_2^2\Phi_4^2\Phi_6\Phi_8$ | . . | 1 |
| 1.3.1  | $\frac{1}{2}q^6\Phi_2^2\Phi_4\Phi_6\Phi_8$ | 1 | . | 1 |
| 1.1.3  | $\frac{1}{2}q^8\Phi_2^3\Phi_4\Phi_6$ | 1 | 1 | 2 | 1 | 1 |

by Table 11, they have 3, 2, 2 modular irreducibles, respectively. We already saw that the other unipotent block of $G$ has one PIM in each of the series $B_1$ and $A_1$. If $\Psi_3$ decomposes, its summands must lie in the $B_1$-series, which would produce four PIMs in that series, but we just argued that there should be exactly three. The same argument applies to the other two series. Thus all columns not labelled “c” in Table 18 correspond to PIMs. Since $(q + 1)\ell > 8$, Corollary 4.4 gives the columns $\Psi_{17}$ and $\Psi_{18}$.

Now by uni-triangularity there are $x_i \geq 0$ such that

$$\Psi_{15} = [2^2] + x_1[B_2:.1^2] + x_2[1^4.] + x_3[.21^2] + x_4[.1^4].$$
For determining the $x_i$’s we proceed as usual. We first use Theorem 4.22 to get $x_4 = 4x_1 + 3x_2 + x_3 - 6$. Then we decompose the Deligne–Lusztig character $R_w$ for $w$ a Coxeter element on the basis of PIMs. We have, up to adding and removing non-unipotent characters

$$R_w = \Psi_1 + \Psi_2 - \Psi_3 - \Psi_4 - \Psi_5 + \Psi_6 - \Psi_8 + \Psi_9 + \Psi_{12} - \Psi_{14} + \Psi_{15} - \Psi_{16} + (2 - x_1)\Psi_{17} - x_2\Psi_{18} + (1 - x_3)\Psi_{19}.$$  

By decomposing $R_v$ for $v < w$, one checks that none of $\Psi_{17}$, $\Psi_{18}$, $\Psi_{19}$ occurs (for $\Psi_{17}$ and $\Psi_{18}$ one could also invoke the fact that they correspond to cuspidal modules). Then (DL) yields $x_1 \leq 2$, $x_2 = 0$ and $x_3 \leq 1$. On the other hand, for a 2-split Levi subgroup $B_2(q).(q + 1)^2$ and $\ell > 4$ one can use (Red) with the unipotent characters $[B_2]$ and $[2]$ to get respectively $x_1 \geq 2$ and $x_3 \geq 1$, showing $x_1 = 2$ and $x_3 = 1$. □

As Lusztig has shown, the unipotent characters of groups of types $B_n$ and $C_n$ are parametrised by the same combinatorial objects. With this we may state:

**Theorem 4.23.** Assume $(T_2)$. Then the $\ell$-modular decomposition matrices for the unipotent blocks of $G = C_4(q)$, for $8 < \ell | (q + 1)$, are as given in Tables 18 and 19 for $B_4(q)$ above.

**Proof.** All of the arguments in the proof of Theorem 4.22 go through for $C_4(q)$ as well. □

**12. Unipotent decomposition matrix of $F_4(q)$**

The groups of type $F_4$ have 37 unipotent characters. For $d_l(q) = 2$ and $\ell > 3$, 25 of these lie in the principal $\ell$-block, five more lie in a block of defect $(q + 1)^2$, and the last seven are of defect 0. For $p$ good, the decomposition matrices of the unipotent $\ell$-blocks of $F_4(q)$ were partially computed by Köhler in [38]. He completely determined the matrix for the non-principal block of positive defect. We obtain here most of the entries that were left undetermined for the principal block.

**Theorem 4.24.** The decomposition matrix for the principal $\ell$-block of $F_4(q)$, $(q, 6) = 1$, with $3 < \ell | (q + 1)$ and $(q + 1)_\ell > 11$ is approximated as given in Table 20.

Here, the unknown parameters satisfy $x_6 = 4 - 2x_1 - 2x_2 + 2x_3$ and $x_7 = 4 - x_3 + 2x_4 + 2x_5$.

Furthermore, $x_1, x_2, x_3, z \geq 2$.

Here $B_3^q$ denotes the Harish-Chandra series of the cuspidal unipotent Brauer character $B_2: \varphi_{12}$ of $B_3(q)$, and $A_1^* \Xi \varphi_{12}$ of $A_1(q)A_1(q)$.

**Proof.** The values $a, b, c, d$ left undetermined in [38] T.A.157 are obtained by Harish-Chandra induction from the decomposition matrices of the Levi factors $B_3(q)$ and $C_3(q)$: using the values given in [34] Table 5 and Thm. 4.3, we get $b, d \leq \gamma = 2$ and $a, c \leq \beta - 1 = 2$ (recall that $(q + 1)_\ell > 5$). This forces $a = b = c = d = 2$. Furthermore
Table 20. $F_4(q)$, $\ell > 3$ and $(q+1)_\ell > 11$

| \phi_{1,0} | 1 | 1 |
| \phi_{2,4} | q^\frac{1}{2} | 1 |
| \phi_{2,4} | q^\frac{1}{2} | 1 |
| B_{2,-2} | q^\frac{1}{2} | 1 |
| \phi_{9,2} | q^2 | 1 | 1 | 1 |
| \phi_{8,3} | q^3 | 1 | 1 | 1 |
| \phi_{8,3} | q^3 | 1 | 1 | 1 |
| \phi_{6,6} | q^{12} | 1 | 1 | 1 |
| \phi_{9,6} | q^{12} | 1 | 1 | 1 |
| \phi_{1,12} | q^{12} | 1 | 1 | 1 |
| \phi_{4}[1] | q^{12} | 1 | 1 | 1 |
| \phi_{6,6} | q^{12} | 1 | 1 | 1 |
| \phi_{6,6} | q^{12} | 1 | 1 | 1 |
| \phi_{9,10} | q^{10} | 1 | 1 | 1 |
| \phi_{9,10} | q^{10} | 1 | 1 | 1 |
| \phi_{9,10} | q^{10} | 1 | 1 | 1 |
| \phi_{9,10} | q^{10} | 1 | 1 | 1 |
| \phi_{9,10} | q^{10} | 1 | 1 | 1 |
| B_{2,-1}^{12} | q^{24} | 1 | 1 | 1 |
| B_{2,-1}^{12} | q^{24} | 1 | 1 | 1 |

\[ \text{T.A.157} \] gives the following approximation to the lower right hand corner of the decomposition matrix:

\[
\begin{align*}
\phi_{9,10} & \quad 1 \\
\phi_{9,10} & \quad 1 \\
\phi_{9,10} & \quad 1 \\
B_{2,-1}^{12} & \quad 1 \\
\phi_{1,24} & \quad 1 \\
\end{align*}
\]

Now, Theorem 12 yields the values of $b_3, b_4$ and $b_5$, together with the relation $b_2 = 4b_1 - 5$.

Let $w$ be a Coxeter element of the Weyl group of $G = F_4(q)$. The corresponding Deligne–Lusztig character $R_w$ has the following decomposition in terms of projective characters in the principal block $B_0$:

\[ B_0 R_w = \Psi_1 + \Psi_4 - \Psi_6 - \Psi_7 + \Psi_8 - \Psi_{13} + \Psi_{14} + \Psi_{17} - \Psi_{18} + \Psi_{21} + (2 - b_1) \Psi_{24}. \]

By [15] Prop. 1.5, we deduce that $b_1 \leq 2$, and therefore $b_1 = 2$ and $b_2 = 3$. 

We now focus on the columns corresponding to the ordinary cuspidal unipotent characters, whose entries will be denoted by $x_1, \ldots, x_{21}$ as follows:

|   | $F_4^I[1]$ | $F_4^I[-1]$ | $F_4[-1]$ |
|---|---|---|---|
| $\phi_{8,9}$ | $x_1$ | $x_9$ | $x_{15}$ |
| $\phi_{8,9}$ | $x_2$ | $x_9$ | $x_{16}$ |
| $\phi_{9,10}$ | $x_3$ | $x_{10}$ | $x_{17}$ |
| $\phi_{2,16}$ | $x_4$ | $x_{11}$ | $x_{18}$ |
| $\phi_{2,16}$ | $x_5$ | $x_{12}$ | $x_{19}$ |
| $\phi_{s,15}$ | $x_6$ | $x_{13}$ | $x_{20}$ |
| $\phi_{1,24}$ | $x_7$ | $x_{14}$ | $x_{21}$ |

Let $w = s_1s_2s_3s_4s_2s_3$. The coefficients of the PIMs $\Psi_{19}$, $\Psi_{22}$ and $\Psi_{24}$ are respectively $2 - x_{15}$, $2 - x_{15} + x_{17} - x_{18}$ and $4 - 2x_{15} - 2x_{16} + 2x_{17} - x_{20}$. They are all non-negative by (DL) and they add up to $8 - 4x_{15} - 2x_{16} + 3x_{17} - x_{18} - x_{20}$. But when $(q + 1)_\ell > 7$, this sum should also be non-positive by (Red) applied to the trivial character of $\tilde{A}_1(q). (q + 1)^3$ (see Proposition 4.1). Therefore we must have

\[
x_{15} = 2, \quad x_{18} = x_{17} \quad \text{and} \quad x_{20} = 2x_{17} - 2x_{16}.
\]

Similarly, the coefficients of $\Psi_{20}$, $\Psi_{23}$ and $\Psi_{24}$ are respectively $2 - x_{16}$, $2 - x_{16} + x_{17} - x_{19}$ and $4 - 2x_{15} - 2x_{16} + 2x_{17} - x_{20}$. They add up to a number which is both non-negative by (DL) and non-positive by (Red) applied to the trivial character of $A_1(q). (q + 1)^3$. Therefore

\[
x_{16} = 2, \quad x_{19} = x_{17} \quad \text{and} \quad x_{20} = 2x_{17} - 4.
\]

Then by Theorem 1.2 we get $x_{21} = 3x_{17}$. We set $z := x_{17}$.

Let $w = (s_1s_2s_3s_4)^2$ and $R_w$ be the corresponding Deligne–Lusztig character. We proceed as in the previous paragraph. The sum of the coefficients of $\Psi_{19}$, $\Psi_{22}$ and $\Psi_{24}$ is

\[
(-x_8) + (2 - x_8 + x_{10} - x_{11}) + (-2x_8 - 2x_9 + 2x_{10} - x_{13})
\]

and the sum of the coefficients of $\Psi_{20}$, $\Psi_{24}$ and $\Psi_{24}$ is

\[
(-x_9) + (2 - x_9 + x_{10} - x_{12}) + (-2x_8 - 2x_9 + 2x_{10} - x_{13}).
\]

Both are sums of non-negative integers by (DL) and are non-positive by (Red) (for the same unipotent characters as before). We deduce that

\[
x_8 = x_9 = 0, \quad x_{11} = x_{12} = x_{10} + 2, \quad \text{and} \quad x_{13} = 2x_{10}.
\]

With Theorem 1.2 we get $x_{14} = 3x_{10} + 4$. We set $y := x_{10}$.

We now consider the Deligne–Lusztig character associated with $w = (s_1s_2s_3s_4)^3$. Only the PIMs $\Psi_{24}$ and $\Psi_{25}$ do not occur in $R_v$ for $v < w$. The coefficient of $\Psi_{24}$ on $R_w$ equals $4 - 2x_1 - 2x_2 + 2x_3 - x_6$ and therefore it must be non-negative by (DL). On the other hand, if $(q + 1)_\ell > 4$ on can invoke (Red) for the cuspidal unipotent character of the Levi subgroup $B_2(q). (q + 1)^2$ to ensure that it is also non-positive. We deduce that $x_6 = 4 - 2x_1 - 2x_2 + 2x_3$. The relation $x_7 = 4 - x_3 + 2x_4 + 2x_5$ now follows from Theorem 1.2.
Finally, we use (Red) to obtain lower bounds on some of the missing entries. With the trivial character of the 2-split Levi subgroups of type $^2A_2(q). (q + 1)^2$ we find $x_1 \geq 2$ and $x_2 \geq 2$. Using the character $[21] \boxtimes [2]$ of a 2-split Levi subgroup of type $^2A_2(q). A_1(q). (q + 1)$ we find $x_3 \geq x_1 + x_2 - 2$ hence $x_3 \geq 2$.

**Remark 4.25.** As before we can obtain upper bounds on the missing decomposition numbers if [16] Conj. 1.2 holds. Let $w_1 = s_2s_3s_2s_1s_3s_2s_3s_4s_3s_2s_1s_3s_4$. We have

$$\langle Q_{w_1}; \varphi_{19} \rangle = 4 - 2x_1 \quad \text{and} \quad \langle Q_{w_1}; \varphi_{20} \rangle = 4 - 2x_2,$$

which would show that $x_1 = x_2 = 2$. Now with $w_2 = s_1s_2s_1s_3s_2s_1s_4s_3s_2s_1s_3s_2s_4s_3s_2s_1$ and $w_3 = s_3s_4s_3s_2s_1s_3s_2s_3s_4s_3s_2s_1s_3s_2s_3s_4$ we have

$$\langle Q_{w_2}; \varphi_{22} \rangle = 72 + 12x_3 - 12x_4,$$
$$\langle Q_{w_3}; \varphi_{21} \rangle = 156 - 12z - 12x_3,$$
$$\langle Q_{w_3}; \varphi_{23} \rangle = 72 + 12x_3 - 12x_5.$$  

This gives $x_4, x_5 \leq x_3 + 6$ and $x_3 + z \leq 13$. Finally with $w_4 = s_3s_2s_3s_4s_3s_2s_1s_3s_4$ and $w_5 = s_2s_1s_4s_3s_2s_1s_3s_2s_3$ we find

$$\langle Q_{w_4}; \varphi_{21} \rangle = 18 - 6z \quad \text{and} \quad \langle Q_{w_5}; \varphi_{21} \rangle = 14 - 2y.$$  

We conclude that $y \leq 7$ and $z \in \{2, 3\}$. Using the previous inequalities we obtain $x_3 \leq 11$ and $x_4, x_5 \leq 17$. 


Decomposition matrices at $d_\ell(q) = 3$

Here, we consider decomposition matrices of unipotent blocks of groups of Lie type $G = G(q)$ for primes $\ell$ with $d_\ell(q) = 3$, so $\ell|(q^2 + q + 1)$ and in particular $\ell \geq 7$. If $G$ is of classical type, then such primes $\ell$ with are linear for $G$, and so the decomposition numbers are known by work of Gruber and Hiss [33] to be given by suitable $q$-Schur algebras. (This implies, for example, that the block distribution of unipotent characters refines the subdivision into ordinary Harish-Chandra series, and that $(T_\ell)$ is always satisfied.) Nevertheless, to our knowledge they have never been written out explicitly, so we derive them here, also as an induction base for blocks of groups of exceptional type for which the theory of linear primes from [33] does not apply.

1. Even-dimensional split orthogonal groups

We begin with groups of type $D_n$ for $n \leq 7$. The Brauer trees of unipotent blocks with cyclic defect were first determined by Fong and Srinivasan [20] and in our situation can also easily be obtained by Harish-Chandra induction:

**Proposition 5.1.** Let $q$ be a prime power and $\ell$ a prime with $d_\ell(q) = 3$. The Brauer trees of the unipotent $\ell$-blocks of $D_n(q)$, $3 \leq n \leq 7$, with cyclic defect are as given in Table 7.

Here and later on, we label the ordinary unipotent characters by their Harish-Chandra series; for the principal series this means by the irreducible characters of a Weyl group of type $D_n$, hence by unordered pairs of partitions of $n$, and for characters in the Harish-Chandra series of the cuspidal unipotent character of a Levi subgroup of type $D_4$ by the symbol “$D_4$” and a character of the relative Weyl group, which is of type $B_{n-4}$, hence by a bipartition of $n-4$.

Under the edges of the Brauer trees, which represent the irreducible Brauer characters (or equivalently the PIMs) of the block, we have indicated their corresponding $\ell$-modular Harish-Chandra series; here "ps" stands for the principal series, while "$A_2$" stands for the series of the cuspidal $\ell$-modular Steinberg character of a Levi subgroup of type $A_2$.

Again, we first need to determine the parameters of the Hecke algebras attached to cuspidal $\ell$-modular Brauer characters of certain Levi subgroups:

**Lemma 5.2.** Let $q$ be a prime power and $\ell|(q^2 + q + 1)$.

(a) The Hecke algebra for the cuspidal $\ell$-modular Steinberg character $\varphi_{13}$ of a Levi subgroup of type $A_2$ inside $D_n(q)$, $n \geq 4$, is $\mathcal{H}(B_{n-3};1;q)$.

(b) The Hecke algebra for the cuspidal $\ell$-modular Steinberg character $\varphi_{23}$ of a Levi subgroup of type $A_2$ inside $D_n(q)$ is $\mathcal{H}(A_1;q^3) \otimes \mathcal{H}(A_1;q^3)$ if $n = 6$ and $\mathcal{H}(B_2;q^3;1) \otimes \mathcal{H}(D_{n-6};q)$ when $n \geq 7$ (where $D_1$ has to be interpreted as the trivial group).
Table 1. Brauer trees for $D_n(q)$ ($3 \leq n \leq 7$), $7 \leq \ell | (q^2 + q + 1)$

| $D_3(q)$ | $D_5(q)$ | $D_7(q)$ | $1^2.41$ | $1^2.32$ | $1^2.1^5$ | $A_2$ |
|----------|----------|----------|----------|----------|----------|-------|
| .3       | 1.4      | 2.5      | .3       | .21      | .1^3     |       |
| .21      | 2.2^2    | 2.21     | .21      | .21^3    | .2.1^3   |       |
| .1^3     | .2.1^3   | .2.12^3  | .2.12^3  | .2.12^3  | .2.12^3  |       |
| .1        | .21      | .2.12^3  | .2.12^3  | .2.12^3  | .2.12^3  |       |

Proof. First, by [35, p. 72] the relative Weyl group of $A_2$ inside $D_n$ is of type $B_n-3$. The parameters of the corresponding Hecke algebra are determined by the parameters inside the minimal Levi subgroups above $A_2$, viz. those of types $D_4$ and $A_2A_1$. Clearly the parameter inside the product $A_2(q)A_1(q)$ is equal to $q$, by [28, Lemma 3.19]. Now the $\ell$-modular cuspidal Steinberg character of $L = A_2(q)$ is liftable to an ordinary cuspidal character $\lambda$ by [27, Thm. 7.8], lying in the Lusztig series of a regular semisimple $\ell$-element of $L^\ast$. Then the parameters of the relative Hecke algebra inside $M = D_4(q)$ can be determined as the $\ell$-modular reduction of the quotient of the degrees of the two ordinary constituents of $R_M^L(\lambda)$, see [28, Lemma 3.17]. This equals 1 as claimed in (a). Reduction stability holds by Example [2.5(a)].

In (b), the cuspidal modular Steinberg character of a Levi subgroup of type $A_2^3$ is the exterior tensor product of those of the two factors, so is again liftable to characteristic zero by [27, Thm. 7.8]. Again by [35, p. 72] the relative Weyl group has type as stated. The minimal Levi overgroups here are of types $A_5$ (twice) inside $D_6$, and of type $D_4A_2$ and $A_3^2A_1$ in $D_7$, $D_8$ respectively, and the parameters in either case can again be determined...
using [28] Lemma 3.17. Here, reduction stability holds if we choose the same lift to characteristic 0 in both factors.

The only unipotent $\ell$-blocks of non-cyclic defect of $D_6(q)$ and $D_7(q)$ are their principal blocks.

**Proposition 5.3.** The decomposition matrix for the principal $\ell$-block of $D_6(q)$ for primes $7 \leq \ell | (q^2 + q + 1)$ is as given in Table 2.

Here, in the second column we print the degrees of the corresponding unipotent characters as products of cyclotomic polynomials evaluated at $q$.

**Table 2. $D_6(q)$, $7 \leq \ell | (q^2 + q + 1)$**

| $q^6$ | $q^2\Phi_5\Phi_{10}$ | 1 | 1 |
|-------|---------------------|---|---|
| $.51$ | $q^3\Phi_5\Phi_8\Phi_{10}$ | . | 1 |
| $3+$  | $q^3\Phi_5\Phi_8\Phi_{10}$ | . | 1 |
| $3-$  | $q^3\Phi_5\Phi_8\Phi_{10}$ | . | 1 |
| $.3^2$| $q^4\Phi_5\Phi_6\Phi_8\Phi_{10}$ | . | 1 |
| $21.3$| $q^4\Phi_5\Phi_6^2\Phi_{10}$ | . | 1 |
| $.41^2$| $q^6\Phi_5\Phi_8\Phi_{10}$ | . | 1 |
| $1^3.3$| $q^8\Phi_5\Phi_8\Phi_{10}$ | . | 1 |
| $.321$| $q^8\Phi_5\Phi_8\Phi_{10}$ | . | 1 |
| $21^+$| $q^{10}\Phi_5\Phi_8\Phi_{10}$ | . | 1 |
| $21^-$| $q^{10}\Phi_5\Phi_8\Phi_{10}$ | . | 1 |
| $.2^3$| $q^{10}\Phi_5\Phi_8\Phi_{10}$ | . | 1 |
| $1^3.21$| $q^{12}\Phi_5\Phi_8\Phi_{10}$ | . | 1 |
| $.31^3$| $q^{15}\Phi_5\Phi_8\Phi_{10}$ | . | 1 |
| $1^3^+$| $q^{15}\Phi_5\Phi_8\Phi_{10}$ | . | 1 |
| $1^3^-$| $q^{15}\Phi_5\Phi_8\Phi_{10}$ | . | 1 |
| $.21^4$| $q^{20}\Phi_5\Phi_{10}$ | . | 1 |
| $.1^6$| $q^{30}$ | . | 1 |

Proof. By Harish-Chandra induction we obtain the projectives in the principal series as well as those in the $A_2$-series, which are easily seen to be indecomposable by (HCr). Since the centraliser of a Sylow $\ell$-subgroup of $D_6(q)$ is contained in a Levi subgroup of type $A_5$, there are no cuspidal Brauer characters by (Csp) so the remaining four PIMs must belong to Brauer characters lying in the Harish-Chandra series above the cuspidal Steinberg PIM of a Levi subgroup of type $A_2$. The corresponding Hecke algebra $\mathcal{H}(A_1; q^3)^{32}$, determined in Lemma 5.2(b), remains semisimple modulo $\ell$. Now Harish-Chandra induction also yields the projectives $\Psi_{10} + \Psi_{12} + \Psi_{15}$, $\Psi_{16} + \Psi_{18}$, $\Psi_{11} + \Psi_{12} + \Psi_{16}$, $\Psi_{15} + \Psi_{18}$, and $\Psi_{12} + \Psi_{15} + \Psi_{16} + \Psi_{18}$, where $\Psi_i$ denotes the linear combination of unipotent characters with coefficients as given in the $i$th column of our table. The only way that these can split into four characters all satisfying (HCr) is the one given. \qed
5. Decomposition Matrices at $d_t(q) = 3$

**Proposition 5.4.** The decomposition matrix for the principal $\ell$-block of $D_7(q)$ for primes $7 \leq \ell | (q^2 + q + 1)$, is as given in Table 3.

**Table 3.** $D_7(q), 7 \leq \ell | (q^2 + q + 1)$

| .7 | 1 | 1 |
| 1.6 | $q$ | 1 |
| 1.51 | $q^3$ | 1 1 |
| .52 | $q^3$ | 1 1 |
| 3.4 | $q^4$ | 1 1 |
| .43 | $q^4$ | 1 1 |
| 21.4 | $q^4$ | 1 1 |
| 2.2.3 | $q^6$ | 1 1 |
| 2.3 | $q^6$ | 1 1 |
| .421 | $q^7$ | 1 1 1 1 |
| 1.3.4 | $q^7$ | 1 1 |
| 1.41 | $q^7$ | 1 1 |
| 21.2.2 | $q^7$ | 1 1 1 1 |
| 1.3.2 | $q^9$ | 1 1 1 1 |
| 1.2.3 | $q^{12}$ | 1 1 |
| 1.3.2 | $q^{12}$ | 1 1 |
| .41 | $q^{12}$ | 1 1 |
| 1.31 | $q^{13}$ | 1 1 |
| .321 | $q^{13}$ | 1 1 |
| 1.4.3 | $q^{13}$ | 1 1 |
| .2 | $q^{16}$ | 1 1 |
| 1.2.1 | $q^{16}$ | 1 1 |
| 1.21 | $q^{21}$ | 1 1 |
| 1.21 | $q^{21}$ | 1 1 |
| .2 | $q^{21}$ | 1 1 |
| 1.3.14 | $q^{21}$ | 1 1 |
| 1.1 | $q^{31}$ | 1 1 |
| .17 | $q^{42}$ | 1 1 |

Proof. Again, since the centraliser of a Sylow $\ell$-subgroup of $D_7(q)$ is contained in a Levi subgroup of type $A_5$, there are no cuspidal Brauer characters by (Csp) so all Brauer characters must lie in the Harish-Chandra series above cuspidal characters of proper Levi subgroups. We investigate these in turn.

The Hecke algebra $H(D_7; q)$ for the principal series can be considered as a subalgebra of index 2 of an Iwahori–Hecke algebra $H(B_7; q)$. The latter is Morita equivalent to a sum of tensor products of Hecke algebras of type $A_n$, $n \leq 7$, by [13, 4.7], and the decomposition matrices of the latter are known by the work of James [37, p. 259]. This gives the 14 columns corresponding to the principal series, for all $\ell \geq 7$. Next, the columns $\Psi_i$, for $i \in \{18, 20, 22, 23, 25\}$ are obtained by Harish-Chandra induction. Further, the
projectives $\Psi_3 + \Psi_7 + \Psi_{17}$, $\Psi_{11} + \Psi_{20}$ and $\Psi_{17} + \Psi_{18}$ yield $\Psi_{11}$ and $\Psi_{17}$; while $\Psi_9 + \Psi_{16} + \Psi_{19} + \Psi_{24}$, $\Psi_{14} + \Psi_{16} + \Psi_{20} + 2\Psi_{22} + \Psi_{23}$ and $\Psi_{23} + \Psi_{24}$ yield $\Psi_{16}$ and $\Psi_{24}$. Harish-Chandra induction also gives the projectives $\Psi_{15} + \Psi_{21}$, $\Psi_{15} + \Psi_{26}$, $\Psi_{21} + \Psi_{27}$ and $\Psi_{26} + \Psi_{27}$ (modulo addition of known projectives). The Hecke algebra $H(B_2; q^2; 1)$ for the cuspidal $\ell$-modular Steinberg character $\varphi_{13}^{22}$ of a Levi subgroup of type $A_2^3$ remains semisimple modulo $\ell$ by Lemma 5.2(b). This yields the remaining four columns of the asserted decomposition matrix. Then (HCr) shows that all columns are indecomposable. □

2. Unipotent decomposition matrix of $E_6(q)$

The triangular shape of the decomposition matrix for the unipotent blocks of $E_6(q)$, when $q$ is a power of a good prime for $E_6$, has been shown by Geck and Hiss [25, 7.5] in the case when $d_\ell(q) = 3, 6$, by using generalised Gelfand–Graev characters. This shows that $(T_i)$ is satisfied under these assumptions on $q$. In fact, for $d_\ell(q) = 3$, Geck and Hiss [25, Table 3] give an approximation to the decomposition matrix involving four unknown entries, plus the unknown decomposition of the two ordinary cuspidal characters.

**Lemma 5.5.** Let $q$ be a prime power and $\ell | (q^2 + q + 1)$. The Hecke algebra for the cuspidal $\ell$-modular Steinberg character $\varphi_{13}^{22}$ of a Levi subgroup of type $A_2^3$ inside $E_6(q)$ is $\mathcal{H}(G_2; q^3; q)$, with four irreducible characters.

This is easily seen as in the previous cases; reduction stability holds since the normaliser just interchanges the two $A_2$-factors and we can choose the same Steinberg module in both factors.

**Theorem 5.6.** Let $(q, 6) = 1$. Then the decomposition matrix for the principal $\ell$-block of $E_6(q)$ for primes $\ell > 3$ with $(q^2 + q + 1)_\ell > 7$ is as given in Table 4.

Here, the unknown parameters satisfy $a_8 \leq -1 - a_3 - a_4 + a_6 + a_7$ and the relations

$$a_5 = 1 - a_1 - a_2 + a_3, \quad \text{and} \quad a_9 = 3 + 2a_3 + 3a_4 - 3a_6 - 2a_7 + 3a_8.$$  

There is a further unipotent $\ell$-block of cyclic defect with Brauer tree

$$D_4 : 3 \overset{c}{\longrightarrow} D_4 : 21 \overset{1^3}{\longrightarrow} D_4.$$

**Proof.** We explain how to find projective characters $\Psi_i$, $1 \leq i \leq 24$, with unipotent parts as given in the columns of Table 4. First, (HCl) gives all columns except for those with index $i = 9, 12, 14, 15, 19, 20, 21, 23, 24$. (Alternatively, the ten PIMs in the principal series can also be read off from the decomposition matrix of the Hecke algebra $\mathcal{H}(E_6(q)$, given in [29, Tab. 7.13].) Furthermore (HCl) yields $\Psi_9 + \Psi_{11} + \Psi_{12}$ and $\Psi_9 + \Psi_{11} + \Psi_{21}$. From these and (Tri) we get $\Psi_9 + \Psi_{11}$, $\Psi_{12}$ and $\Psi_{21}$. Harish-Chandra inducing the 16th PIM from a Levi subgroup of type $D_6$ to $E_7(q)$ and restricting it back to $E_6(q)$ yields $\Psi_9 + \Psi_{12} + \Psi_{16}$, which shows that $\Psi_9$ is a projective character. Finally, (HCl) from a Levi subgroup of type $A_5$ also yields a projective character $\Psi_{19} = \Psi_{19} + \Psi_{21}$. Now the decomposition matrix of the Hecke algebra for the $A_2^3$-series, determined in Lemma 5.5 shows that $\Psi_{19}$ must have two summands in that series, and then (HCl) leads to the only admissible splitting $\Psi_{19} \oplus \Psi_{21}$.

At this point we have accounted for projectives in all non-cuspidal Harish-Chandra series, so the remaining five PIMs must belong to cuspidal Brauer characters. (A priori,
Table 4. $E_6(q), 7 < (q^2 + q + 1)\ell, (q, 6) = 1$

| $\phi_{i,j}$ | $E_6[\zeta_3]$ | $E_6[\zeta_2^3]$ |
|--------------|----------------|------------------|
| $\phi_{1,0}$ | 1              | 1                |
| $\phi_{6,1}$ | $q$            | 1                |
| $\phi_{20,2}$ | $q^2$          | 1                |
| $\phi_{30,3}$ | $\frac{1}{3}q^3$ | 1                |
| $\phi_{15,5}$ | $\frac{1}{3}q^3$ | 1                |
| $\phi_{15,4}$ | $\frac{1}{3}q^3$ | 1                |
| $\phi_{64,4}$ | $q^4$          | 1                |
| $\phi_{60,5}$ | $q^5$          | 1                |
| $\phi_{24,6}$ | $q^6$          | 1                |
| $\phi_{20,10}$ | $\frac{1}{3}q^7$ | 1                |
| $\phi_{80,7}$ | $\frac{1}{3}q^7$ | 1                |
| $\phi_{10,9}$ | $\frac{1}{3}q^7$ | 1                |
| $E_6[\zeta_3]$ | $\frac{1}{3}q^7$ | 1                |
| $E_6[\zeta_3^2]$ | $\frac{1}{3}q^7$ | 1                |
| $\phi_{60,11}$ | $q^{11}$       | 1                |
| $\phi_{24,12}$ | $q^{12}$       | 1                |
| $\phi_{64,13}$ | $q^{13}$       | 1                |
| $\phi_{15,16}$ | $\frac{1}{3}q^{15}$ | 1                |
| $\phi_{15,17}$ | $\frac{1}{3}q^{15}$ | 1                |
| $\phi_{30,15}$ | $\frac{1}{3}q^{15}$ | 1                |
| $\phi_{20,20}$ | $q^{20}$       | 1                |
| $\phi_{6,25}$ | $q^{25}$       | 1                |
| $\phi_{1,36}$ | $q^{36}$       | 1                |

by (St), the modular Steinberg character is known to be cuspidal.) Let us denote the unknown entries below the diagonal of those PIMs as follows:

| $E_6[\zeta_3]$ | 1 |
| $E_6[\zeta_2^3]$ | 1 |
| $\phi_{60,11}$ | $a_1$ $a_1$ |
| $\phi_{24,12}$ | $a_2$ $a_2$ |
| $\phi_{64,13}$ | $a_3$ $a_3$ |
| $\phi_{15,16}$ | $a_4$ $a_4$ |
| $\phi_{15,17}$ | $a_5$ $a_5$ |
| $\phi_{30,15}$ | $a_6$ $a_6$ |
| $\phi_{20,20}$ | $a_7$ $a_7$ $b_1$ |
| $\phi_{6,25}$ | $a_8$ $a_8$ $b_2$ |
| $\phi_{1,36}$ | $a_9$ $a_9$ $b_3$ $b_4$ |

$p, s, p, s, p, s, p, s, A_2^2, A_2 A_2 C, c, A_2, A_2, A_2 A_2 C, c, A_2^2, A_2 C, c$
Note here that since the ordinary cuspidal unipotent characters $E_6[\zeta_3]$ and $E_6[\zeta_3^2]$ are Galois conjugate, while all other unipotent characters are rational valued, the unknown decomposition numbers in the two corresponding columns must agree. Now, by (T), the PIM of $E_7(q)$ starting at $\phi_{21,33}$ has coefficient 0 on $\phi_{56,30}$ and $\phi_{21,36}$. Harish-Chandra restricting this to $E_6(q)$ gives a projective starting at $\phi_{15,17}$, but not involving $\phi_{20,30}$. Thus again by (Tri) we conclude that $b_1 = 0$.

We now use the combination of (DL) and (Red) to determine some of the unknown entries. We start with the Deligne–Lusztig character $R_w$ associated to a Coxeter element $w$. The coefficient on $\Psi_{21}$ is $2 - 2a_1 - 2a_2 + 2a_3 - 2a_5$ and hence is non-negative by (DL). On the other hand, if $\ell > 3$ then (Red) for the cuspidal unipotent character of the 3-split Levi subgroup $^3D_4(q).(q^2 + q + 1)$ gives the relation $-1 + a_1 + a_2 - a_3 + a_5 \geq 0$ (see Example [17,d]). We deduce that $a_5 = 1 - a_1 - a_2 + a_3$. The coefficients of $\Psi_{23}$ and $\Psi_{24}$ are
\[
X = -1 - 2a_3 - 2a_4 + 2a_6 + 2a_7 - 2a_8 \geq 0
\]
and
\[
Y = 3 - 2a_5 + 2a_7 - 2a_9 - b_4 X \geq 0.
\]
Note that $X$ cannot be equal to zero by parity. Using explicit computations in Chevie [26], one can see that there exists a regular $\ell$-element in $G^*$ whenever $(q^2 + q + 1)\ell > 7$. Then (Red) applied to the case of a maximal torus gives $b_4 \geq 3$ and $-3 - 2a_3 - 3a_4 + 3a_6 + 2a_7 - 3a_8 + a_9 \geq 0$. Then
\[
0 \leq Y = 3 - 2a_3 + 2a_7 - 2a_9 - b_4 X \\
\leq 3 - 2a_3 + 2a_7 - 2a_9 - 3X \\
= 2(3 + 2a_3 + 3a_4 - 3a_6 - 2a_7 + 3a_8 - a_9) \leq 0
\]
forces $b_4 = 3$ and $3 + 2a_3 + 3a_4 - 3a_6 - 2a_7 + 3a_8 - a_9 = 0$.

Finally, the coefficient of $\Psi_{24}$ on the Deligne–Lusztig character $R_w$ associated with $w = s_1s_2s_3s_1s_5s_4s_6s_5s_4s_2s_3s_4$ is equal to $-18 + 9b_2 - 3b_3$. With (Red) applied to a maximal torus again we have also $6 - 3b_2 + b_3 \leq 0$ hence (DL) forces $b_3 = -6 + 3b_2$. In the table, we write $b$ for $b_2$.

It now follows with (HCr) that all projectives in the table are indecomposable. □

**Remark 5.7.** As before, we can use the virtual characters $Q_w$ for $w \in W$ to get conjectural upper bounds for the unknown entries. With $w$ being the Coxeter element, we find the following inequalities:
\[
\begin{align*}
 a_1 & \leq 13, & a_2 & \leq 1, & a_3 & \leq 4 + a_1 + a_2, \\
 a_4 & \leq 9 + a_1, & a_6 & \leq 2 - a_2 + a_3, & a_7 & \leq 21 - a_1 + a_3 + a_4, \\
 a_8 & \leq -1 - a_3 - a_4 + a_6 + a_7.
\end{align*}
\]
On the other hand, the trivial character of a 2-split Levi subgroup of type $A_2(q).A_2(q).(q^2 + q + 1)$ (resp. $A_2(q).(q^2 + q + 1)^2$) gives $a_2 \geq 1$ (resp. $a_6 \geq 2 - a_2 + a_3$) by (Red). Therefore $a_2 = 1$ and $a_6 = 1 + a_3$. From this we deduce that we should have $a_1 \leq 13$, $a_3 \leq 18$, $a_4 \leq 22$, $a_5 \leq 5$, $a_6 \leq 14$, $a_7 \leq 48$ and $a_8 \leq 26$.

To obtain a bound on $b$ we use $Q_w'$ with $w' = s_1s_2s_3s_1s_4s_3s_1s_5s_4s_3s_1s_6s_5s_4s_3s_1$ from which we get $b \leq 8$ if [16, Conj. 1.2] holds. Most of these upper bounds are probably not sharp.
3. Even-dimensional non-split orthogonal groups

Next, we consider the unipotent blocks of twisted orthogonal groups $2D_n$, $4 \leq n \leq 7$. Again, the Brauer trees were described in [20] (and can readily be determined):

**Proposition 5.8.** Let $q$ be a prime power and $\ell$ a prime with $d_\ell(q) = 3$. The Brauer trees of the unipotent $\ell$-blocks of $2D_n(q)$, $4 \leq n \leq 7$, with cyclic defect are as given in Table 5.

**Table 5.** Brauer trees for $2D_n(q)$ ($4 \leq n \leq 7$), $7 \leq \ell|(q^2 + q + 1)$

\[
\begin{array}{cccccccc}
2D_4(q): & 3 & 21 & 13 & . & . & 3 & . & . \\
2D_5(q): & 4 & 2^2 & 14 & . & . & 1.1 & 1.21 & 1.3 \\
 & & .4 & .2^2 & .1 & . & 1.3,1 & 21.1 & 21.3 & 3.1 \\
2D_6(q): & 5 & 2^21 & 21^3 & . & . & 2.1 & 2.21 & 2.3 \\
 & & .5 & .2^21 & .21^3 & . & 1.3^2 & 21.2 & 3.2 & . \\
 & & 41 & .41 & .32 & .15 & . & 21^2 & 21^3 & 3.3 \\
 & & 41 & .41 & .32 & .15 & . & 21^2 & 21^3 & 3.3 \\
 & & 41 & .41 & .32 & .15 & . & 21^2 & 21^3 & 3.3 \\
2D_7(q): & 51 & 2^21.1 & 21^3.1 & . & . & 2.1 & 2.2^2 & . & 2.4 \\
 & & 1.5 & 1.2^21 & 1.21^3 & . & 21^2 & 21^3 & 3.2 & 4.2 \\
 & & 41.1 & 32.1 & 15.1 & . & 1.2^2 & 1.2^2 & 1.2^2 & 4.2 \\
 & & 1.41 & 1.32 & 1.15 & . & 1.2^2 & 2^2 & 2^2 & 4.1^2 \\
\end{array}
\]

Here and later, the unipotent characters of $2D_n(q)$ in the principal series are denoted by the corresponding character of the Weyl group, which in this case is of type $B_{n-1}$, hence by bipartitions of $n - 1$. Note that the order of $2D_n(q)$ with $n \leq 3$ is not divisible by primes $\ell$ with $d_\ell(q) = 3$.

Again, we first collect some information on Hecke algebras of cuspidal characters.

**Lemma 5.9.** Let $q$ be a prime power and $\ell|(q^2 + q + 1)$.

(a) The Hecke algebra for the cuspidal $\ell$-modular Steinberg character $\varphi_{13}$ of a Levi subgroup of type $A_2$ inside $2D_n(q)$, $n \geq 4$, is $\mathcal{H}(A_2; 1) \otimes \mathcal{H}(B_{n-4}; q^2; q)$.

(b) The Hecke algebra for the cuspidal $\ell$-modular Steinberg character $\varphi_{15}^{\otimes 2}$ of a Levi subgroup of type $A_2^2$ inside $2D_n(q)$, $n \geq 7$, is $\mathcal{H}(B_2; q^3; 1) \otimes \mathcal{H}(B_{n-7}; q^2; q)$. 
4. SYMPLECTIC AND ODD-DIMENSIONAL ORTHOGONAL GROUPS

**Proof.** Note that $^2D_n(q)$ has Weyl group of type $B_{n-1}$. By [35, p. 70] the relative Weyl group of $A_2$ inside $B_{n-1}$ is of type $A_1B_{n-4}$, and that of $A_2^+$ is of type $B_2B_{n-7}$. The cuspidal characters in question are the same as those in Lemma 5.2 and thus lift to characteristic zero. The relevant minimal Levi overgroups in (a) are of types $^2D_4$ when $n = 4$, $^2D_2A_2$ when $n = 5$ and $A_2A_1$ when $n = 6$, leading to the parameters $1, q^2, q$ respectively. In (b) they are of types $A_3, ^2D_4A_2$ when $n = 7$, $^2D_2A_2^+$ when $n = 8$ and $A_2^2A_1$ when $n = 9$, leading to the parameters $q^3, 1, q^2$ and $q$. The argument is now completely analogous to the one employed in the proof of Lemma 5.2.

The Steinberg character of a Levi subgroup of type $A_2$ is reduction stable by Example 2.5(a), and for a Levi subgroup $A_2^+$ we take the same Steinberg module in both factors. □

The Sylow $\ell$-subgroups of $^2D_6(q)$ are cyclic when $d_\ell(q) = 3$, so the smallest rank not covered by Proposition 5.8 is $n = 7$.

**Proposition 5.10.** The decomposition matrix for the principal $\ell$-block of $^2D_7(q)$ for primes $7 \leq \ell|(q^2 + q + 1)$ is as given in Table 6.

**Proof.** First, observe that the centraliser of a Sylow $\ell$-subgroup of $^2D_7(q)$ is contained in a Levi subgroup of type $A_3$, so there do not exist cuspidal unipotent $\ell$-modular Brauer characters by (Csp). The Hecke algebra for the principal series is $H(B_6; q^2; q)$ (see e.g. [8, p. 464]). By [13, 4.7] it is Morita equivalent to a sum of products of Hecke algebras of type $A_n$, $n \leq 6$, whose decomposition matrices are known by [37, p. 259] for all $\ell \geq 7$. This yields the columns of the principal series PIMs. All columns labelled $A_2$ are obtained by (HCi). Then (HCr) shows that these are indeed indecomposable. Finally, by Lemma 5.9(b) the Hecke algebra for the cuspidal $\ell$-modular Steinberg character of a Levi subgroup of type $A_2^+$ is $H(B_2; q^3; 1)$ and hence semisimple modulo $\ell$. Then splitting up the Harish-Chandra induction of the corresponding PIMs using (HCr) yields the last five missing columns. □

4. Symplectic and odd-dimensional orthogonal groups

Next, we consider the symplectic groups $C_n(q)$ and the odd-dimensional orthogonal groups $B_n(q)$ with $n \leq 6$. Note that according to Lusztig’s classification, the unipotent characters of both series of groups are parametrised in the same way, see e.g. [8, §13.8].

**Proposition 5.11.** Let $q$ be a prime power and $\ell$ a prime with $d_\ell(q) = 3$. The Brauer trees of the unipotent $\ell$-blocks of $B_n(q)$ and $C_n(q)$, $3 \leq n \leq 6$, with cyclic defect are the same as for the unipotent $\ell$-blocks of $^2D_{n+1}(q)$ in Table 6 plus the three additional trees given in Table 7.

Here, $B_2A_2$ denotes the Harish-Chandra series of the cuspidal unipotent $\ell$-modular Brauer character $B_2 \boxtimes \varphi_1$, of a Levi subgroup of type $B_2A_2$.

**Proposition 5.12.** The decomposition matrix for the principal $\ell$-block of $B_6(q)$ and of $C_6(q)$ for primes $7 \leq \ell|(q^2 + q + 1)$ is as given in Table 8.
Table 6. $2D_7(q)$, $7 \leq \ell |(q^2 + q + 1)$

| n | $q$ | 1 | 1 |
|---|---|---|---|
| 6 | $q^3$ | . | 1 |
| 6 | $q^3$ | . | 1 |
| 31 | $q^7$ | . | 1 |
| 31 | $q^7$ | . | 1 |
| 31 | $q^7$ | . | 1 |
| 32 | $q^7$ | . | 1 |
| 32 | $q^7$ | . | 1 |
| 32 | $q^7$ | . | 1 |
| 21,3 | $q^{12}$ | . | 1 |
| 21,3 | $q^{12}$ | . | 1 |
| 3,2 | $q^{12}$ | . | 1 |
| 3,2 | $q^{12}$ | . | 1 |
| 3,2 | $q^{12}$ | . | 1 |
| 3,2 | $q^{12}$ | . | 1 |
| 21 | $q^{13}$ | . | 1 |
| 21 | $q^{13}$ | . | 1 |
| 21 | $q^{13}$ | . | 1 |
| 21 | $q^{13}$ | . | 1 |
| 21 | $q^{13}$ | . | 1 |
| 16 | $q^{16}$ | . | 1 |
| 16 | $q^{16}$ | . | 1 |
| 2 | $q^{21}$ | . | 1 |
| 2 | $q^{21}$ | . | 1 |
| 2 | $q^{21}$ | . | 1 |
| 2 | $q^{21}$ | . | 1 |
| 1 | $q^{42}$ | . | 1 |

Table 7. Brauer trees for $B_n(q)$ and $C_n(q)$ ($5 \leq n \leq 6$), $7 \leq \ell |(q^2 + q + 1)$

$B_5(q)$: $B_2: 3.\quad B_2: 21.\quad B_2: 1^3.\quad \bigcirc \quad B_2: 1^3\quad B_2: 21\quad B_2: 3$

$B_6(q)$: $B_2: 4.\quad B_2: 2^2.\quad B_2: 1^4.\quad \bigcirc \quad B_2: 1^3.\quad B_2: 21\quad B_2: 1.3$

$B_2: 3.1\quad B_2: 21.1\quad B_2: 1^3.1\quad \bigcirc \quad B_2: 1.3\quad B_2: 2^2\quad B_2: 4$

**Proof.** The arguments are exactly the same for $G = B_6(q)$ and $G = C_6(q)$. All columns and their respective Harish-Chandra series except those labelled $A_2^2$ are directly obtained by (HCi). Since the centraliser of a Sylow $\ell$-subgroup of $G$ is contained inside a Levi subgroup of type $A_5$, there are no cuspidal PIMs by (Csp). Thus the five missing
5. UNIPOTENT DECOMPOSITION MATRIX OF $F_4(q)$

Table 8. $B_6(q)$ and $C_6(q)$, $7 \leq \ell|q^2 + q + 1$

| $\ell$ | 1  |
|-------|----|
| 1.6   | 1  |
| 5.1   | 1  |
| 3.2   | 1  |
| 3.3   | 1  |
| 5.1   | 1  |
| 41^2  | 1  |
| 321   | 1  |
| 3.21  | 1  |
| 21.3  | 1  |
| 21.21 | 1  |
| 2^3   | 1  |
| 1^3.3 | 1  |
| 3^2   | 1  |
| 31^3  | 1  |
| .41   | 1  |
| .321  | 1  |
| 21.1^3| 1  |
| 1^2.21| 1  |
| 1^2.1^3| 1 |
| .2^3  | 1  |
| 21^1  | 1  |
| .31^3 | 1  |
| 1^6   | 1  |
| .21^4 | 1  |
| 1^6   | 1  |


PIMs belong to Brauer characters in the series of the cuspidal Steinberg character of a Levi subgroup of type $A_2^2$. With (HCi) we find projectives with unipotent parts

$$\Psi_{12} + \Psi_{21} + \Psi_{25}, \quad \Psi_{21} + \Psi_{22} + \Psi_{27}, \quad \Psi_{11} + \Psi_{12} + \Psi_{21} + \Psi_{22} \quad \text{and} \quad \Psi_{21} + \Psi_{25} + \Psi_{27}.$$ 

The only way these can decompose into sums of unipotent characters consistent with (HCr) is the one given in the table. The indecomposability of all listed projectives also readily follows with (HCr). \square

5. Unipotent decomposition matrix of $F_4(q)$

**Theorem 5.13.** Let $(q, 6) = 1$. Then the decomposition matrix for the principal $\ell$-block of $F_4(q)$ for primes $\ell \geq 7$ with $(q^2 + q + 1)\ell > 7$ is approximated as given in Table 8.

Here the unknown parameters satisfy $x_1 \geq 2$, $y_1, y_2 \geq 1$, $y_6 = 3 - 2y_1 - 2y_2 + y_3 + y_4 + 2y_5$ and $y_5 \leq y_1 + y_2 - 1$. 

### Table 9. $F_4(q)$, $(q^2 + q + 1)\ell > 7$, $(q, 6) = 1$

| $\phi_{1,0}$ | 1 | 1 |
| $\phi'_{2,4}$ | $\frac{1}{2}q$ | 1 | 1 |
| $\phi''_{2,4}$ | $\frac{1}{2}q$ | 1 | 1 |
| $\phi_{4,1}$ | $\frac{1}{2}q$ | . . | 1 |
| $\phi'_{8,3}$ | $q^3$ | . . | 1 | 1 |
| $\phi''_{8,3}$ | $q^3$ | . . | 1 | 1 |
| $F_4^I[1]$ | \(\frac{1}{2}q^4\) | . . . | 1 |
| $\phi'_{1,12}$ | \(\frac{1}{2}q^4\) | 1 | . . | . | 1 |
| $\phi''_{1,12}$ | \(\frac{1}{2}q^4\) | 1 | 1 | 1 | . . | . | 1 |
| $\phi_{4,8}$ | \(\frac{1}{2}q^4\) | . | 1 | . | . | . | 1 |
| $\phi'_{4,7}$ | \(\frac{1}{2}q^4\) | . | 1 | . | . | . | 1 |
| $\phi''_{4,7}$ | \(\frac{1}{2}q^4\) | . | 1 | . | . | . | 1 |
| $\phi_{16,5}$ | \(\frac{1}{2}q^4\) | . | 1 | 1 | 1 | . . | . | 1 |
| $F_4[\zeta_3]$ | \(\frac{1}{2}q^4\) | . . . | . . | . | 1 |
| $F_4[\zeta_3]^2$ | \(\frac{1}{2}q^4\) | . . . | . . | . | 1 |
| $\phi'_{8,9}$ | $q^9$ | . . | . | 1 | . | . | 1 | $y_1 \ y_1 \ 1$
| $\phi_{8,9}$ | $q^9$ | . . | . | 1 | 1 | $y_2 \ y_2$ | . | 1 |
| $\phi'_{2,16}$ | \(\frac{1}{2}q^{13}\) | 1 | . | . | . | 1 | 1 | $y_4 \ y_4$ | . | . | 1 |
| $\phi''_{2,16}$ | \(\frac{1}{2}q^{13}\) | . | 1 | . | . | 1 | 1 | $y_5 \ y_5 \ 1 \ 1 \ . \ 1 |
| $\phi'_{4,13}$ | \(\frac{1}{2}q^{13}\) | . | $x_1$ | . | . | 1 | 1 | $y_6 \ y_6$ | . | 1 | 2 | 1 |
| $\phi''_{4,13}$ | \(\frac{1}{2}q^{13}\) | . | . | . | 2$x_1-3$ | . | 1 | . | $y_6 \ y_6$ | . | 1 | 1 | 2 | 1 |

Here $\tilde{A}_2$ denotes the Harish-Chandra series above the Steinberg PIM of $A_2 < C_3$ in $F_4$.

**Proof.** The triangular shape of the decomposition matrix when $(q, 6) = 1$ was shown by Köhler [33, Tab. A.160]. He also proved that the ordinary cuspidal character $F_4^I[1]$ can only occur in the modular reduction of $\phi_{1,12}$ or $\phi_{1,24}$. We will denote their multiplicities as $x_1$ and $x_2$ respectively. When $(q^2 + q + 1)\ell > 7$ they satisfy $x_1 \geq 1$ and $3 + x_2 - 2x_1 \geq 0$. Similarly, he proved that the Brauer character for $\phi_{1,13}$ can only possibly occur in the modular reduction of the Steinberg character $\phi_{1,24}$, with a multiplicity $z$ that satisfies $z \geq 2$ whenever $(q^2 + q + 1)\ell > 7$. The multiplicities of unipotent characters in the PIMs corresponding to the two Galois conjugate cuspidal characters $F_4[\zeta_3]$ and $F_4[\zeta_3]^2$ will be denoted by $y_1, \ldots, y_6$, so that we have

$$\Psi_{14} = F_4[\zeta_3] + y_1\phi'_{8,9} + y_2\phi''_{8,9} + y_3\phi'_{2,16} + y_4\phi''_{2,16} + y_5\phi_{4,13} + y_6\phi_{1,24}$$

and

$$\Psi_{15} = F_4[\zeta_3]^2 + y_1\phi'_{8,9} + y_2\phi''_{8,9} + y_3\phi'_{2,16} + y_4\phi''_{2,16} + y_5\phi_{4,13} + y_6\phi_{1,24}.$$ 

Under the assumption on $\ell$, we can use (Red) in the case of a maximal torus to get the relation $-3 + 2y_1 + 2y_2 - y_3 - y_4 - 2y_5 + y_6 \geq 0$ which will be useful in the sequel.
Let \( w \) be a Coxeter element and \( R_w \) be the corresponding Deligne–Lusztig character. The multiplicities of \( \Psi_{20} \) and \( \Psi_{21} \) in \( R_w \) are respectively
\[
2X = -2 + 2y_1 + 2y_2 - 2y_5 \geq 0
\]
and
\[
Y = 2 + 2y_3 + 2y_4 - 2y_6 - zX \geq 0.
\]
Since \( z \geq 2 \) we have
\[
Y \geq 2 + 2y_3 + 2y_4 - 2y_6 - 2X = 6 - 4y_1 - 4y_2 + 2y_3 + 2y_4 + 4y_5 - 2y_6
\]
which is non-positive by (Red). Therefore it must be zero, and we must also have \( X = 0 \) or \( z = 2 \). In any case, the coefficient of \( \Psi_{21} \) is zero.

The coefficient of \( \Psi_{21} \) in \( R_w \) with \( w = s_1s_2s_3s_4s_2s_3 \) is \( 2 - z \), hence \( z = 2 \) by (DL). The coefficient on \( R_w \) with \( w = s_1s_2s_3s_4s_1s_2s_3s_4s_1s_2s_3s_4 \) is \( -3 + 2x_1 - x_2 \) and therefore \( x_2 = 2x_1 - 3 \) by (DL) and the previous inequality on the \( x_i \)'s.

Finally, using (Red) with a the trivial character of the 2-split Levi of type \( A_2(q^2+q+1) \) we get \( y_1 \geq 1 \) and \( y_2 \geq 1 \).

**Remark 5.14.** As before the virtual characters \( Q_w \) for \( w \in W \) provide conjectural upper bounds for the remaining unknown entries. We start with \( w = s_1s_2s_1s_4s_3s_2s_1 \), and we compute
\[
\langle Q_w; \varphi_{16} \rangle = 12 - 12y_1,
\]
\[
\langle Q_w; \varphi_{17} \rangle = 12 - 12y_2,
\]
\[
\langle Q_w; \varphi_{18} \rangle = 12 - 12y_3,
\]
\[
\langle Q_w; \varphi_{19} \rangle = 18 - 12y_4.
\]
If [16 Conj. 1.2] holds then this forces \( y_1, y_2, y_3, y_4 \leq 1 \). Using the relations in Theorem 5.13 we deduce that \( y_1 = y_2 = 1 \), \( y_5 \leq 1 \) and \( y_6 \leq 3 \). Furthermore, with \( w' = s_2s_3s_2s_1s_3s_2s_3s_4s_3s_2s_1s_3s_2s_4s_3s_2 \) we have \( \langle Q_w; \varphi_{20} \rangle = 176 - 8x_1 \) from which we get \( x_1 \leq 22 \).

We close this section by collecting in Table 10 some data on the \( \Phi_3 \)-modular Harish-Chandra series in the principal \( \ell \)-blocks \( B_0 \) considered above. Here \( W_G(B_0) \) denotes the relative Weyl group of a Sylow \( \Phi_3 \)-torus of the ambient group \( G \) (which contains a Sylow \( \ell \)-subgroup of \( G \) and hence a defect group of \( B_0 \)). This is known to be a complex reflection group; more concretely, here it is one of two imprimitive complex reflection groups of rank 2, denoted \( G(6,1,2) \) and \( G(6,2,2) \), respectively the primitive reflection groups \( G_5 \), \( G_{25} \) and \( G_{26} \).

**Table 10. Modular Harish-Chandra-series in \( \Phi_3 \)-blocks**

| \( G \) | \( W_G(B_0) \) | \( |\text{IBr } B_0| \) | \( ps \) | \( A_2 \) | \( A_2^2 \) | \( E_6 \) | \( c \) |
| --- | --- | --- | --- | --- | --- | --- | --- |
| \( D_6 \) | \( G(6,2,2) \) | 18 | 10 | 4 | 4 |
| \( B_6, C_6, D_7, 2D_7 \) | \( G(6,1,2) \) | 27 | 14 | 8 | 5 |
| \( E_6 \) | \( G_{25} \) | 24 | 10 | 5 | 4 | 5 |
| \( E_7 \) | \( G_{26} \) | 48 | 20 | 10 | 8 | 10 |
| \( F_4, 2E_6 \) | \( G_5 \) | 21 | 8 | 4 | 4 | 5 |
It can be observed from the decomposition matrices that the principal $\ell$-blocks of $D_7(q)$ and $^2D_7(q)$ are not Morita equivalent, even though their modular Harish-Chandra series distribution agrees. On the other hand, the decomposition matrices for the principal $\ell$-blocks of $^2D_7(q)$, $B_6(q)$ and $C_6(q)$ agree after a suitable simultaneous permutation of rows and columns of the one of $^2D_7(q)$, so these blocks might be Morita equivalent.
Decomposition matrices at $d_\ell(q) = 6$

Now, we consider unipotent decomposition matrices for groups $G = G(q)$ at primes $\ell$ with $d_\ell(q) = 6$, so $\ell|(q^2 - q + 1)$ and $\ell \geq 7$. For groups of classical type, such primes are what is called unitary for $G$, so the theory of $q$-Schur algebras does not apply and the decomposition matrices are not even understood theoretically.

While our assumption already implies that $\ell \geq 7$, we will often need to make a further assumption on $(q^2 - q + 1)$ in order to use (Red).

1. Even-dimensional split orthogonal groups

As in the previous sections we begin with groups of type $D_n$, $4 \leq n \leq 7$. The Brauer trees for the unipotent $\ell$-blocks of cyclic defect can easily be obtained by Harish-Chandra induction and are well-known \[20\]. For completeness and better reference we collect them here:

**Proposition 6.1.** Let $q$ be a prime power and $\ell$ a prime with $d_\ell(q) = 6$. The Brauer trees of the unipotent $\ell$-blocks of $D_n(q)$, $4 \leq n \leq 7$, with cyclic defect are as given in Table \[7\].

**Table 1.** Brauer trees for $D_n(q)$ ($4 \leq n \leq 7$), $7 \leq \ell|(q^2 - q + 1)$

| $D_4(q)$ | $A$ | $1.3$ | $1^2.2$ | $1^3.1$ | $1^4$ | $D_4$ |
|----------|-----|-------|---------|---------|-------|-------|
| $D_5(q)$ | 5   | 3.2   | $2.21$  | $1.21^2$| $2.1^3$| $D_4:2$ |
|          | .41 | 1.31  | $1^2.21$| $1^3.1^2$| $1^5$  | $D_4:1^2$ |
| $D_6(q)$ | .42 | 1.32  | $1^2.2^2$| $1^2.1^4$| $1.1^5$| $D_4:2.$ |
|          | 1.5 | 2.4   | $2.2^2$  | $1.2^3$ | $2.1^2$ | $D_4:1^2$ |
| $D_7(q)$ | .61 | 3.31  | $31.21$  | $31^2.1^2$| .31$^4$| $D_4:1.2$ |
|          | .51 | 2.31  | $21.21^2$| $21^2.1^3$| $21^5$ | $D_4:1^2.1$|
|          | 5.1 | 4.21  | $3.2^2$  | $1.2^3$ | $2.3^1$ | $D_4:1^3$ |
|          | .43 | 1.3$^2$| $1^3.2^2$| $1^4.21$| $1.5.2$ | $D_4:3.$ |

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Here and later on, “$1^4$” stands for the cuspidal unipotent $\ell$-modular Brauer character of a Levi subgroup of type $D_4$ labelled by the unordered bipartition $(-; 1^4)$, and “$D_4$” for the $\ell$-modular reduction of the cuspidal unipotent character of a Levi subgroup of type $D_4$.

To treat the blocks with non-cyclic defect, again we now first determine the parameters of certain relative Hecke algebras:

**Lemma 6.2.** Let $q$ be a prime power and $\ell | (q^2 - q + 1)$. The Hecke algebras of various $\ell$-modular cuspidal pairs $(L, \lambda)$ of Levi subgroups $L$ in $D_n(q)$ and their respective numbers of irreducible characters are as given in Table 3.

| $(L, \lambda)$ | $\mathcal{H}$ | $n = 6$ | $7$ | $8$ |
|----------------|-------------|--------|-----|-----|
| $(D_4, D_4)$   | $\mathcal{H}(B_{n-4}; q^4, q)$ | 2 | 4 | 10 |
| $(D_4, \varphi_{1^4})$ | $\mathcal{H}(B_{n-4}; q^2; q)$ | 2 | 4 | 10 |
| $(A_5, \varphi_{1^6})$ | $\mathcal{H}(A_1; q^3) \otimes \mathcal{H}(D_{n-6}; q)$ | 1 | 1 | 4 |
| $(D_6, \varphi_{1^6})$ | $\mathcal{H}(B_{n-6}; q; q)$ | 1 | 2 | 5 |

**Proof.** In the first three cases, the cuspidal Brauer character lies in a block with cyclic defect (see the Brauer tree in Table 1), and hence reduction stability follows from Example 2.5(a).

A Levi subgroup $L$ of type $D_4$ has relative Weyl group of type $B_{n-4}$ inside $D_n$. The cuspidal Brauer character $\varphi_{1^4}$ is a constituent of the $\ell$-modular reduction of an ordinary cuspidal character $\lambda$ of $L$ lying in the Lusztig series of an $\ell$-element $s$ with centraliser $(q^3 + 1)(q + 1)$. By Corollary 2.4, the Hecke algebra for $\varphi_{1^4}$ is the same as for $\lambda$. The minimal Levi overgroups are of types $D_5$ when $n \geq 5$, where $s$ has centraliser $2D_2(q)(q^3+1)$, and $D_4A_1$ when $n \geq 6$, which gives the parameters $q^2$ and $q$.

The relative Weyl group of a Levi subgroup of type $A_5$ in $D_n$ is of type $A_1D_{n-6}$, by [35] p. 72. As can be seen from the Brauer tree, the $\ell$-modular Steinberg character $\varphi_{1^6}$ of $A_5$ is liftable, so again the parameter $q^3$ for the Hecke algebra is determined inside the minimal Levi overgroups of types $D_6$ and $A_5A_1$.

Finally, the relative Weyl group of $D_6$ inside $D_7$ has type $A_1$. The modular Steinberg character $\varphi_{1^6}$ lifts to a cuspidal Deligne–Lusztig character, so it is reduction stable by Example 2.5(c), and its Hecke algebra is the $\ell$-modular reduction of an Iwahori–Hecke algebra in characteristic 0. The parameters are determined already inside Levi subgroups of types $D_7, D_6A_1$.

**Proposition 6.3.** Assume $(T_\ell)$. The decomposition matrix for the principal $\ell$-block of $D_6(q)$ for primes $\ell$ with $(q^2 - q + 1)\ell > 7$ is as given in Table 3.

**Proof.** Let us write $\Psi_i, 1 \leq i \leq 18$, for the linear combinations of unipotent characters given by the columns of Table 3. We need to show that these are restrictions to the principal block of $\ell$-modular PIMs of $G = D_6(q)$. Note that the unipotent decomposition matrices of all proper Levi subgroups are known, either by Proposition 6.1 or by 37.
Here we use that the graph automorphism of labelled 1. From uni-triangularity of the decomposition matrix, we deduce Lemma 6.2) has two simple modules in characteristic of \( G \) Hecke algebra, computed with the programme of N. Jacon [36]. As a Sylow \( \ell \)-subgroup of \( G \) is not contained in any proper Levi subgroup, by (St) the \( \ell \)-modular reduction of the Steinberg character contains a cuspidal Brauer character. The relative Hecke algebra \( \mathcal{H}(B_3; q^4; q) \) for the cuspidal Brauer character \( D_4 \) of a Levi subgroup of type \( D_4 \) (see Lemma [6.2]) has two simple modules in characteristic \( \ell \), so the corresponding modular Harish-Chandra series only contains two PIMs. Furthermore, the two projectives obtained by Harish-Chandra induction of \( \Psi_1 \) from \( D_5(q) \) are indecomposable by (HCr), as are those in the \( A_5 \)-series, so there are no further PIMs in those series. Hence the remaining PIM must be cuspidal. From uni-triangularity of the decomposition matrix, we deduce that it will have the form

\[
[D_4; .2] + a_1[1^3+] + a_1[1^3-] + a_2[.21^4] + a_3[.1^6].
\]

Here we use that the graph automorphism of \( G \) interchanges the two unipotent characters labelled \( 1^3+ \), \( 1^3- \) but fixes \( [D_4; .2] \) and hence the corresponding two entries in that column must agree. Note also that the family consisting of the unipotent character \([.31^3]\) is not comparable to the family containing \([D_4 : .2]\) which explains why it does not appear in the

| \(.6\) | \(1\) \(q^2\Phi_5\Phi_{10}\) \(1\) \(1\) \(1\) \(1\) |
| \(.51\) | \(q^3\Phi_5\Phi_{10}\Phi_{10}\) \(1\) \(1\)
| \(3^+\) | \(q^3\Phi_5\Phi_{10}\Phi_{10}\) \(1\) \(1\) |
| \(3^-\) | \(q^3\Phi_5\Phi_{10}\Phi_{10}\) \(1\) \(1\) |
| \(2.31\) | \(q^4\Phi_5^2\Phi_8\Phi_{10}\) \(1\) \(1\) \(1\) \(1\) |
| \(D_4; 1^2\) | \(q^4\Phi_5^2\Phi_8\Phi_{10}\) \(1\) \(1\) \(1\) \(1\) |
| \(4^2\) | \(q^5\Phi_8\Phi_{10}\) \(1\) \(1\) \(1\) \(1\) |
| \(1.3^2\) | \(q^7\Phi_5^2\Phi_8\Phi_{10}\) \(1\) \(1\) \(1\) \(1\) |
| \(D_4; 3^+\) | \(q^7\Phi_5^2\Phi_8\Phi_{10}\) \(1\) \(1\) \(1\) \(1\) |
| \(21^+\) | \(q^7\Phi_5^2\Phi_8\Phi_{10}\) \(1\) \(1\) \(1\) \(1\) |
| \(21^-\) | \(q^7\Phi_5^2\Phi_8\Phi_{10}\) \(1\) \(1\) \(1\) \(1\) |
| \(1^2.21^2\) | \(q^5\Phi_5^2\Phi_8\Phi_{10}\) \(1\) \(1\) \(1\) \(1\) |
| \(D_4; 2^3\) | \(q^5\Phi_5^2\Phi_8\Phi_{10}\) \(1\) \(1\) \(1\) \(1\) |
| \(31^3\) | \(q^{12}\Phi_8\Phi_{10}\) \(1\) \(1\) \(1\) \(1\) |
| \(1^3^+\) | \(q^{15}\Phi_5\Phi_{10}\) \(1\) \(1\) \(1\) \(1\) |
| \(1^3^-\) | \(q^{15}\Phi_5\Phi_{10}\) \(1\) \(1\) \(1\) \(1\) |
| \(.21^4\) | \(q^{20}\Phi_8\Phi_{10}\) \(1\) \(1\) \(1\) \(1\) |
| \(.1^6\) | \(q^{30}\Phi_8\Phi_{10}\) \(1\) \(1\) \(1\) \(1\) |

\( ps \) \( ps \) \( ps \) \( ps \) \( D_4 \) \( ps \) \( ps \) \( ps \) \( ps \) \( c.1^4 \) \( A_5 \) \( A_5^- \) \( .1^4 \) \( c \)
previous projective character. The coefficient of $\Psi_{18}$ in the Deligne–Lusztig character $R_w$ for $w$ a Coxeter element is $2 + 2a_1 + a_2 - a_3$, and therefore by (DL) it must be non-negative.

On the other hand we have $2 + 2a_1 + a_2 - a_3 \leq 0$ and therefore $a_3 = a_2 + 2a_1 + 2$ provided that we can use (Red), that is when there exists a semisimple regular $\ell$-element of $G$. By Example 1.7(b) such an element exists whenever $(q^2 - q + 1)_\ell > 12$ (in particular whenever $(q^2 - q + 1)_\ell > 7$ since we also assumed $\ell \geq 7$).

It will be a consequence of the determination of the decomposition matrices for $D_8(q)$ in Proposition 6.5 that in fact $a_1 = a_2 = 0$ and thus $a_3 = 2$, completing the proof.

Proposition 6.4. Assume $(T_i)$. The decomposition matrix for the principal $\ell$-block of $D_7(q)$ for primes $\ell$ with $(q^2 - q + 1)_\ell > 7$ is as given in Table 4.

Here $D^\circ_6$ denotes the Harish-Chandra series of the cuspidal $\ell$-modular constituent of the $\ell$-modular reduction of the unipotent character $[D_4:2]$ of $D_6(q)$.

Proof. As before, let’s denote by $\Psi_i$, $1 \leq i \leq 27$, the linear combinations of unipotent characters corresponding to the columns of Table 4. Those $\Psi_i$ with $i$ not equal to

1, 15, 16, 19, 22, 24, 26 and 27

are obtained by (HCi). Furthermore, by uni-triangularity the projectives $\Psi_{17} + \Psi_{19}$, $\Psi_{18} + \Psi_{19}$ yield $\Psi_{18}$, and $\Psi_{21} + \Psi_{24}$, $\Psi_{23} + \Psi_{24}$ yield $\Psi_{24}$. Since a Levi subgroup $L$ of type $D_6$ contains the centraliser of a Sylow $\ell$-subgroup of $D_7(q)$, there are no cuspidal Brauer characters by (Csp) and in particular the Harish-Chandra induction of the Steinberg PIM of $L$ splits off the Steinberg PIM of $G$; this yields $\Psi_{26}$ and $\Psi_{27}$.

The projective cover of the trivial character lies in the principal series and is determined by the decomposition matrix of the Hecke algebra $H(D_7(q))$; this yields $\Psi_1$ (HC-induction only gives $\Psi_1 + \Psi_2$), and similarly we obtain $\Psi_{15}$ (HC-induction only yields $\Psi_{15} + \Psi_{21}$). Now the Harish-Chandra induction to $G$ of the PIM of the $\ell$-modular cuspidal unipotent character $D^\circ_6$ of a Levi subgroup of type $D_6$ equals

$$\Psi = [D_4:3] + [D_4:21] + a_2[1.21^4] + a_2[2^21^3] + 2a_1[1^31^4] + a_3[1^6] + a_3[1^7].$$

As we have accounted for all other Harish-Chandra series, and there are no cuspidal unipotent Brauer characters, this projective character has to have two summands in that series, which by uni-triangularity must start at $[D_4:3]$ and $[D_4:21]$ respectively. The only possibility for splitting $\Psi$ into two summands compatible with (HCr) is

$$\tilde{\Psi}_{16} = [D_4:3] + (a_2 - z_2)[1.21^4] + z_2[2^21^3] + a_1[1^31^4] + (a_3 - z_3)[1^6] + z_3[1^7],$$

$$\tilde{\Psi}_{22} = [D_4:21] + z_2[1.21^4] + (a_2 - z_2)[2^21^3] + a_1[1^31^4] + z_3[1^6] + (a_3 - z_3)[1^7],$$

with suitable non-negative integers $z_2 \leq a_2$ and $z_3 \leq a_3$. It will be a consequence of the determination of the decomposition matrices for the unipotent blocks of $D_8(q)$ in Proposition 6.5 that in fact $a_1 = a_2 = z_2 = 0$ and thus $a_3 = 2$. This accounts for the last two missing columns $\Psi_{16}$ and $\Psi_{22}$.

Finally, the coefficient of $\Psi_{26}$ on $R_w$ when $w$ is a Coxeter element is $2z_4$. Since $\Psi_{26}$ does not occur in any $R_v$ for $v < w$, we deduce from (DL) that $z_4 \leq 0$ since $\ell(w) = 7$ is odd. Therefore $z_4 = 0$. Then (HCr) shows that all of the $\Psi_i$ are in fact indecomposable.
Table 4. $D_7(q)$, $(q^2 - q + 1)\ell > 7$

| $\ell$ | $D$\,7\,$(q)$, $(q^2 - q + 1)\ell > 7$ |
|--------|----------------------------------|
| .7     | $D$\,7\,$(q)$, $(q^2 - q + 1)\ell > 7$ |
| 1.6    | $q$ \quad 1 |
| 1.51   | $q^3$ \quad 1 \quad 1 |
| .52    | $q^3$ \quad 1 \quad 1 |
| 3.4    | $q^3$ \quad 1 \quad 1 \quad 1 |
| 2.41   | $q^4$ \quad 1 \quad 1 \quad 1 \quad 1 |
| $D_4$: 21 | $q^4$ \quad 1 \quad 1 \quad 1 \quad 1 |
| 2.32   | $q^6$ \quad 1 \quad 1 \quad 1 \quad 1 |
| $D_4$: 13 | $q^6$ \quad 1 \quad 1 \quad 1 \quad 1 |
| .421   | $q^7$ \quad 1 \quad 1 \quad 1 \quad 1 |
| 1.412  | $q^7$ \quad 1 \quad 1 \quad 1 \quad 1 |
| $D_4$: 21 | $q^7$ \quad 1 \quad 1 \quad 1 \quad 1 |
| 21.2$^2$ | $q^9$ \quad 1 \quad 1 \quad 1 \quad 1 |
| 1.321  | $q^9$ \quad 1 \quad 1 \quad 1 \quad 1 |
| 1.$^2$.21 | $q^{12}$ \quad 1 \quad 1 \quad 1 \quad 1 |
| $D_4$: 3 | $q^{12}$ \quad 1 \quad 1 \quad 1 \quad 1 |
| .41$^3$ | $q^{12}$ \quad 1 \quad 1 \quad 1 \quad 1 |
| 1.31$^3$ | $q^{13}$ \quad 1 \quad 1 \quad 1 \quad 1 |
| .32$^2$ | $q^{13}$ \quad 1 \quad 1 \quad 1 \quad 1 |
| $D_4$: 1.1$^2$ | $q^{13}$ \quad 1 \quad 1 \quad 1 \quad 1 |
| 1.$^2$.21$^3$ | $q^{16}$ \quad 1 \quad 1 \quad 1 \quad 1 |
| $D_4$: 21 | $q^{16}$ \quad 1 \quad 1 \quad 1 \quad 1 |
| 1.2$^4$ | $q^{21}$ \quad 1 \quad 1 \quad 1 \quad 1 |
| .2$^2$.1$^3$ | $q^{21}$ \quad 1 \quad 1 \quad 1 \quad 1 |
| 1.$^3$.1$^4$ | $q^{21}$ \quad 1 \quad 1 \quad 1 \quad 1 |
| 1.1$^6$ | $q^{31}$ \quad 1 \quad 1 \quad 1 \quad 1 |
| .1$^7$ | $q^{42}$ \quad 1 \quad 1 \quad 1 \quad 1 |

| $ps$ | $ps$ | $ps$ | $ps$ | $D_4$ | $ps$ | $D_4$ | $ps$ | $ps$ | $ps$ | $D_6$.1$^4$.ps.1$^4$ | $D_4$ | $ps$ | $D_6$.1$^4$.1$^4$.A$\,5$.1$^6$.1$^6$ |

**Proposition 6.5.** Assume $(T_\ell)$. The decomposition matrices for the three unipotent $\ell$-blocks of $D_8(q)$ of non-cyclic defect for primes $\ell$ with $(q^2 - q + 1)\ell > 7$ are as given in Tables 5 and 6.

Here the blocks are labelled by the 6-cuspidal unipotent characters of the centraliser $D_2(q).\Phi_6^2$ of a Sylow $\Phi_6$-torus of $D_8(q)$ (according to the parametrisation of $\ell$-blocks in [4]).
### 6. Decomposition Matrices at $d_e(q) = 6$

**Table 5.** $D_8(q)$, $(q^2 - q + 1)\ell > 7$, blocks $\binom{2}{0}$ and $\binom{1,2}{0,1}$

| $D_8$ | 1 |
|-------|---|
| 2.6   | 1 |
| 3.5   | 1 1 1 |
| 2.51  | 1 1 |
| 5.3   | 1 1 |
| $D_4$:31 | 1 1 |
| 1.512 | 1 1 1 |
| $D_4$:3 | 1 1 1 |
| 2.412 | 1 1 1 |
| 4.31 | 1 1 1 |
| 2.33 | 1 1 1 1 |
| 1.331 | 1 1 1 1 |
| $D_4$:13.1 | 1 1 1 1 |
| 5.12 | 1 1 1 |
| 2.12 | 1 1 1 |
| $D_4$:4 | 1 1 1 |
| 2.313 | 1 1 |
| 1.313 | 1 1 |
| $D_4$:12.12 | 1 1 |
| 12.213 | 1 1 |
| 21.213 | 1 1 |
| 12.412 | 1 1 |
| 14.412 | 1 1 |
| 4.412 | 1 1 |
| $D_4$:2 | 1 1 |
| 12.2 | 1 1 |
| 1.2 | 1 1 |
| $D_4$:1.3 | 1 1 |
| 12.313 | 1 1 |
| 32.31 | 1 1 |
| 22.31 | 1 1 |
| $D_4$:14 | 1 1 |
| 12.412 | 1 1 |
| 14.412 | 1 1 |
| 4.412 | 1 1 |
| $D_4$:13 | 1 1 |
| 12.13 | 1 1 |
| 1.13 | 1 1 |
| $D_4$:21 | 1 1 |
| 12.21 | 1 1 |
| 1.21 | 1 1 |
| $D_4$:14 | 1 1 |
| 12.412 | 1 1 |
| 14.412 | 1 1 |
| 4.412 | 1 1 |
| $D_4$:16 | 1 1 |
| 12.16 | 1 1 |
| 1.16 | 1 1 |

---

| $A_5$ | 1 |
|-------|---|
| 7.1   | 1 1 |
| 12.6  | 1 1 |
| 31.4  | 1 1 |
| $D_4$:2.2 | 1 1 |
| 12.51 | 1 1 |
| 21.41 | 1 1 |
| 3.23 | 1 1 |
| 22.31 | 1 1 |
| $D_4$:42 | 1 1 |
| 12.2 | 1 1 |
| 1.2 | 1 1 |
| $D_4$:1.3 | 1 1 |
| 12.313 | 1 1 |
| 32.31 | 1 1 |
| 22.31 | 1 1 |
| $D_4$:14 | 1 1 |
| 12.412 | 1 1 |
| 14.412 | 1 1 |
| 4.412 | 1 1 |
| $D_4$:13 | 1 1 |
| 12.13 | 1 1 |
| 1.13 | 1 1 |
| $D_4$:21 | 1 1 |
| 12.21 | 1 1 |
| 1.21 | 1 1 |
| $D_4$:14 | 1 1 |
| 12.412 | 1 1 |
| 14.412 | 1 1 |
| 4.412 | 1 1 |
| $D_4$:16 | 1 1 |
| 12.16 | 1 1 |
| 1.16 | 1 1 |

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| $A_5$ | 1 |
|-------|---|
| 7.1   | 1 1 |
| 12.6  | 1 1 |
| 31.4  | 1 1 |
| $D_4$:2.2 | 1 1 |
| 12.51 | 1 1 |
| 21.41 | 1 1 |
| 3.23 | 1 1 |
| 22.31 | 1 1 |
| $D_4$:42 | 1 1 |
| 12.2 | 1 1 |
| 1.2 | 1 1 |
| $D_4$:1.3 | 1 1 |
| 12.313 | 1 1 |
| 32.31 | 1 1 |
| 22.31 | 1 1 |
| $D_4$:14 | 1 1 |
| 12.412 | 1 1 |
| 14.412 | 1 1 |
| 4.412 | 1 1 |
| $D_4$:13 | 1 1 |
| 12.13 | 1 1 |
| 1.13 | 1 1 |
| $D_4$:21 | 1 1 |
| 12.21 | 1 1 |
| 1.21 | 1 1 |
| $D_4$:14 | 1 1 |
| 12.412 | 1 1 |
| 14.412 | 1 1 |
| 4.412 | 1 1 |
| $D_4$:16 | 1 1 |
| 12.16 | 1 1 |
| 1.16 | 1 1 |

---

| $A_5$ | 1 |
|-------|---|
| 7.1   | 1 1 |
| 12.6  | 1 1 |
| 31.4  | 1 1 |
| $D_4$:2.2 | 1 1 |
| 12.51 | 1 1 |
| 21.41 | 1 1 |
| 3.23 | 1 1 |
| 22.31 | 1 1 |
| $D_4$:42 | 1 1 |
| 12.2 | 1 1 |
| 1.2 | 1 1 |
| $D_4$:1.3 | 1 1 |
| 12.313 | 1 1 |
| 32.31 | 1 1 |
| 22.31 | 1 1 |
| $D_4$:14 | 1 1 |
| 12.412 | 1 1 |
| 14.412 | 1 1 |
| 4.412 | 1 1 |
| $D_4$:13 | 1 1 |
| 12.13 | 1 1 |
| 1.13 | 1 1 |
| $D_4$:21 | 1 1 |
| 12.21 | 1 1 |
| 1.21 | 1 1 |
| $D_4$:14 | 1 1 |
| 12.412 | 1 1 |
| 14.412 | 1 1 |
| 4.412 | 1 1 |
| $D_4$:16 | 1 1 |
| 12.16 | 1 1 |
| 1.16 | 1 1 |
1. EVEN-DIMENSIONAL SPLIT ORTHOGONAL GROUPS

Table 6. $D_8(q)$, $(q^2 - q + 1)\ell > 7$, block $\binom{1}{1}$

| 1.7 | 1 |
| 4+ | 1 | 1 |
| 4- | 1 | . | 1 |
| 1.52 | 1 | . | 1 |
| 2.42 | 1 | 1 | 1 | 1 | 1 |
| $D_4$·$2\ell^2$ | . | . | . | . | 1 |
| 1.421 | . | . | 1 | 1 | . | 1 |
| 2$^2+$ | . | 1 | . | 1 | . | 1 |
| 2$^2-$ | . | 1 | . | 1 | . | 1 |
| .421$^2$ | . | . | . | . | 1 | . | 1 |
| $D_6$·$2\ell^2$ | . | . | . | 1 | . | . | 1 |
| 1.321$^2$ | . | . | . | 1 | 1 | . | 1 |
| 1$^2$.2$^2$1$^2$ | . | . | . | 1 | . | 1 | 1 | 1 |
| $D_4$·3$^1$ | . | . | . | . | . | . | 1 | . | 1 |
| 1.2$^1$3$^1$ | . | . | . | . | . | 1 | 1 | 1 | 1 |
| 1$^4+$ | . | . | . | . | . | 1 | . | 1 | 1 |
| 1$^4-$ | . | . | . | . | . | 1 | . | 1 | 1 |
| 1.1$^7$ | . | . | . | . | . | 1 | 2 | 1 | 1 | 1 | 1 |

| ps | ps | ps | ps | $D_4$ps | ps | ps | $D_6^*$.1$^4$ $A_5.A_5$.1$^6$ |

**Proof.** In the principal block, labelled by the trivial character $\binom{2}{0}$ of $D_2(q)$, the columns $\Psi_i, i \neq 16, 23$, are obtained by Harish-Chandra inducing PIMs from Levi subgroups of type $D_7$ and $A_7$. (HCi) also yields projectives

\[
\tilde{\Psi}_{16} = [D_4:1^4] + a_1[1^4, 21^2] + (a_2 - z_2)[2, 21^4] + z_2[2^21^4] + (a_3 - z_3)[1^6, 2] + z_3[21^6],
\]
\[
\tilde{\Psi}_{23} = a_1[1^4, 21^2] + [D_4:2^2] + z_2[2, 21^4] + (a_2 - z_2)[2^21^4] + z_3[1^6, 2] + (a_3 - z_3)[21^6],
\]

with $a_1, z_i$ as in the proof of Proposition 6.4. By $(T_\ell)$, we must have that

\[
\tilde{\Psi}_{23} - a_1\tilde{\Psi}_{22} = [D_4:2^2] + z_2[2, 21^4] + (a_2 - z_2)[2^21^4] + (z_3 - a_1)[1^6, 2] + (a_3 - z_3 - a_1)[21^6]
\]

is a projective character. Now Harish-Chandra restriction of this to a Levi subgroup of type $D_6.A_1$ yields negative multiplicities in PIMs unless $a_1 = 0$.

For the block labelled $\binom{1}{1}$ again all columns except the 15th and 24th are obtained by Harish-Chandra inducing suitable PIMs from proper Levi subgroups. We also obtain

\[
\tilde{\Psi}_{15} = [D_4:1^3] + (a_2 - z_2)[1.31^4] + (a_2 - z_2)[1^2, 21^4] + z_2[2^21^4] + (a_3 - z_3)[1^2, 1^6] + z_3[1, 1^8],
\]
\[
\tilde{\Psi}_{24} = z_2[1.31^4] + z_2[1^2, 21^4] + (a_2 - z_2)[2^21^4] + [D_4:21^2] + z_3[1^2, 1^6] + (a_3 - z_3)[1^8].
\]

Triangularity shows that

\[
\tilde{\Psi}_{24} - z_2\tilde{\Psi}_{20} - (a_2 - z_2)\tilde{\Psi}_{23} = [D_4:21^2] + (z_3 - z_2)[1^2, 1^6] + (a_3 - z_3 - a_2 + z_2)[1^8]
\]

must be a projective character. Harish-Chandra restriction of this to a Levi subgroup of type $D_6.A_1$ yields negative multiplicities in PIMs unless $z_2 = a_2 = 0$. Using that
$a_3 = a_2 + 2a_1 + 2 = 2$ this completes the determination of the decomposition matrices for blocks 1 and 3, as well as for the principal blocks of $D_6(q)$ and $D_7(q)$.

For the second block of $D_8(q)$, labelled $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, all columns are obtained directly by (HCi). Finally, (HCr) shows that all projectives constructed in the three blocks are in fact indecomposable. $\Box$

2. Unipotent decomposition matrix of $E_6(q)$

For $E_6(q)$ and $d_\ell(q) = 6$, the triangular shape of the decomposition matrix and thus property $(T_\ell)$ for the unipotent blocks of $G$ and primes $\ell > 3$ has been shown by Geck–Hiss [25, Thm. 7.4] under the additional assumption that $q$ is a power of a good prime for $E_6$.

**Lemma 6.6.** Let $q$ be a prime power and $\ell | (q^2 - q + 1)$. The Hecke algebras of various $\ell$-modular cuspidal pairs $(L, \lambda)$ of Levi subgroups $L$ in $E_6(q)$ and their respective numbers of irreducible characters are as given in Table 7.

| $(L, \lambda)$ | $\mathcal{H}$ | $|\text{Irr } \mathcal{H}|$ |
|----------------|----------------|-----------------|
| $(D_4, D_4)$    | $\mathcal{H}(A_2; q^k)$ | 2               |
| $(D_4, \varphi_{14})$ | $\mathcal{H}(A_2; q^3)$ | 2               |
| $(A_5, \varphi_{16})$ | $\mathcal{H}(A_1; q^5)$ | 1               |

**Proof.** Reduction stability holds in all cases as Sylow $\ell$-subgroups of $L$ are cyclic. The proof is now as in the previous cases. For example, the $\ell$-modular Steinberg character $\varphi_{16}$ of $A_5(q)$ lifts to an ordinary cuspidal character in the Lusztig series of a regular $\ell$-element with centraliser a maximal torus of order $\Phi_2 \Phi_3 \Phi_6$. In $E_6(q)$ such an element has centraliser $2A_2(q) \Phi_3 \Phi_6$, whence we find the parameter $q^3$. $\Box$

The only unipotent block of positive $\ell$-defect of $E_6(q)$ is the principal block.

**Theorem 6.7.** Assume $(T_\ell)$. The decomposition matrix for the principal $\ell$-block of $E_6(q)$ for primes $\ell > 3$ with $(q^2 - q + 1)\ell > 13$ is approximated by Table 8. Here the unknown entries satisfy $a_5 \leq 1$, $b_4 \leq 2$,

$$a_7 = -1 - a_1 - a_2 + a_6, \quad \text{and} \quad a_8 = -2 - a_1 - a_2 - a_3 + a_4 + 2a_5 + a_6.$$

**Proof.** Let $\Psi_i$ denote the linear combinations corresponding to the columns of Table 8 (HCi) yields all $\Psi_i$ except for those with number

$$i \in \{1, 4, 8, 12, 13, 14, 17, 18, 20, 21\}.$$ Further, $\Psi_4 + \Psi_5$ and $\Psi_4 + \Psi_7$ yield $\Psi_4$, $\Psi_4 + \Psi_8$ and $\Psi_7 + \Psi_8$ yield $\Psi_8$. The two principal series PIMs $\Psi_1$ and $\Psi_{14}$ are obtained via the theorem of Dipper from the decomposition matrix of the Hecke algebra $\mathcal{H} = \mathcal{H}(E_6; q)$ which has been determined by Geck [23, Table D]. By a result of Geck–Müller [30, Thm. 3.10] the decomposition matrix of $\mathcal{H}$ does not depend on $\ell$ for all $\ell \geq 7$. 

To determine some of the remaining entries we use the combination of (DL) and (Red). We denote the three unknown entries below the diagonal in the 18th column by $b_1, b_2, b_3$, and the eight unknown entries below the diagonal in the 12th and 13th column by $a_1, \ldots, a_8$ (they agree in the two columns since the cuspidal characters $E_6[\zeta_3]$ and $E_6[\zeta_3^2]$ are Galois conjugate). Let us first note that explicit computations in Chevie \cite{chevie} show that there exists a regular $\ell$-element in $G^*$ whenever $(q^2 - q + 1)\ell > 13$. In addition, the condition $\ell > 3$ forces $\ell$ to be good and therefore by Example \cite{chevie}d) one can also use (Red) for centralisers of $\Phi_d$-tori of rank 1. Let $w$ be a Coxeter element and $R_w$ be the corresponding Deligne–Lusztig character. We consider the generalised 1-eigenspace of $F$ on $R_w$. The coefficient of the PIM $\Psi_{20}$ is $b_1 - b_2$, hence $b_1 \geq b_2$ by (DL). On the other hand, the cuspidal unipotent character of the 6-split Levi subgroup $^2A_2(q).\Phi_3 \Phi_6$ gives $b_2 - b_1 \geq 0$ by (Red), hence $b_2 = b_1$. The coefficient of the PIM $\Psi_{21}$ is $2 + b_1 - b_3$. This number is non-positive by (Red) applied to a maximal torus. Therefore $b_3 = 2 + b_1$.

We now turn to the generalised $q$-eigenspace of $F$ on $R_w$. The coefficient of the PIM $\Psi_{18}$ is $1 - a_5$, therefore $a_5 \leq 1$ by (DL). The coefficient of $\Psi_{20}$ is $-1 - a_1 - a_2 + a_6 - a_7$. It is non-negative by (DL) but also non-positive by (Red) applied to the cuspidal unipotent character of $^2A_2(q).\Phi_3 \Phi_6$. Therefore $a_7 = -1 - a_1 - a_2 + a_6$. Finally, the coefficient of $\Psi_{21}$
is \(-2 - a_1 - a_2 - a_3 + a_4 + 2a_5 + a_6 - a_8\). Again, it must be zero by the combination of (DL) and (Red) for a maximal torus. Consequently \(a_8 = -2 - a_1 - a_2 - a_3 + a_4 + 2a_5 + a_6\).

We finish with the Deligne–Lusztig character \(R_w\) for \(w = s_1s_2s_3s_1s_5s_4s_6s_5s_4s_2s_3s_4\). The coefficient of the PIM \(\Psi_{21}\) is \(48 - 24b_4\), which forces \(b_4 \leq 2\). □

### 3. A non-principal block of \(E_8(q)\)

**Theorem 6.8.** Assume \((T_\ell)\). The decomposition matrix for the unipotent \(\ell\)-block of \(E_8(q)\) for primes \(\ell > 3\) with \((q^2 - q + 1)\ell > 13\), of defect \((\Phi_6)^2\), is approximated by Table 9. Here the unknown entries \(a_1, \ldots, a_8, b_1, b_4\) are as in Theorem 6.7 and starred entries in the column \(E_6^b\) are understood to be the same as in the column for \(E_6^a\).

**Table 9.** \(E_8(q), \ell > 3, (q^2 - q + 1)\ell > 13\), block of defect \((\Phi_6)^2\)

| \(\phi\) | \(\Phi_i\) |
|-------|-------|
| \(\phi_{112,3}\) | 1 |
| \(\phi_{160,7}\) | 1 |
| \(\phi_{400,7}\) | 1 . 1 |
| \(\phi_{1344,8}\) | 1 1 . 1 |
| \(D_4: \phi_{8,3}\) | . . . . 1 |
| \(\phi_{2240,10}\) | 1 . 1 1 . 1 |
| \(\phi_{3360,13}\) | . 1 . 1 . . 1 |
| \(\phi_{3200,16}\) | . . . . . . . 1 |
| \(\phi_{7168,17}\) | . . . . . . . 1 |
| \(E_6[\zeta_3]: \phi_{2,2}\) | . . . . . . . 1 |
| \(E_6[\zeta_3^2]: \phi_{2,2}\) | . . . . . . . 1 |
| \(\phi_{1344,19}\) | . 1 . 1 . . . . 1 |
| \(D_4: \phi_{16,5}\) | . . . . . . . 1 |
| \(\phi_{3200,22}\) | . . . . . . 1 1 . 2a_2 - a_1 - a_3 + a_6 * . . . 1 |
| \(\phi_{3360,25}\) | . . . . . . 1 1 . 2a_3 - a_2 + a_4 + a_6 * . . . 1 |
| \(\phi_{2240,28}\) | . . . . . . 1 1 . 1 a_1 - a_2 + a_3 + a_4 + a_6 * 1 . . . . 1 |
| \(D_4: \phi_{8,9}\) | . . . . . . . 1 a_5 * . . . . 1 |
| \(\phi_{1344,38}\) | . . . . . . 1 a_1 - 4a_6 + a_7 + a_8 * . . . 1 1 1 3b_1 + 2 1 |
| \(\phi_{400,43}\) | . . . . . . a_3 + a_2 + 2a_4 + a_6 * 1 . . . . 1 . . 1 |
| \(\phi_{160,55}\) | . . . . . . a_6 + 2a_7 + a_8 * . . . . 1 3b_1 + 2 1 . 1 |
| \(\phi_{112,63}\) | . . . . . . a_6 + a_7 + 2a_8 * . . . . 1 3b_1 + 2 1 1 b_4 1 |

\(p p p p p p D_4 p p p p E_6^a E_6^b p D_4 .1^4 p p E_6^a .1^4 A_5 E_6^a E_6^a E_6^a\)

**Proof.** The principal series PIMs can be found in [29 Table 7.15]. Then (HCi) yields the other listed projectives. □
4. Even-dimensional non-split orthogonal groups

We continue with the unipotent blocks of twisted orthogonal groups $2D_n(q)$, $n \leq 7$, for primes $\ell$ with $d_\ell(q) = 6$. Again the Brauer trees in the case of cyclic defect were first described by Fong and Srinivasan [20] and are easily obtained:

**Proposition 6.9.** Let $q$ be a prime power and $\ell$ a prime with $d_\ell(q) = 6$. The Brauer trees of the unipotent $\ell$-blocks of $2D_n(q)$, $3 \leq n \leq 7$, with cyclic defect are as given in Table 10.

| $2D_3(q)$ | 2. 1.1 .1 2 |
| $2D_5(q)$ | 31. 1.2 .2 1 |
| $2D_7(q)$ | 42. 1.2.31 1.31 2 |
| | 31.2 21 2.2 2.1 |
| | ps ps .1 2 |

| $2D_4(q)$ | 3. 1.2 .21 2 1 |
| $2D_5(q)$ | 4. 1.3 .31 2.1 2 |
| | 3.1 2.2 .2 1 |
| | .1 2 1.3 1.2 |
| $2D_6(q)$ | 5. 1.4 .41 2.1 2 2.1 |
| | 41. 1.2 .3 1.31 2.121 1.2 2.1 |
| | 41. 2.3 .32 1.1 1.3 1.2 2.2 |
| | 31.2 1.3 2.1 2.2 |
| | .1 2 2.2 1.2 |
| | .1 5 1.4 1.2 |
| $2D_7(q)$ | 5.1 2.4 .42 2.1 21.2 2.1 |
| | 41.1 2.1 3. 3 2.1 2.1 |
| | 4.2 3.3 .3 2 |
| | ps ps .1 2 .1 2 ps ps |

Here, .1 2 denotes the Harish-Chandra series of the cuspidal $\ell$-modular Steinberg character of $2D_3(q)$. 

**Table 10.** Brauer trees for $2D_n(q)$ ($3 \leq n \leq 7$), $7 \leq \ell | (q^2 - q + 1)$
In Table 11 we have collected the number $|\text{Irr } \mathcal{H}|$ for some small rank modular Iwahori–Hecke $\mathcal{H}$ algebras occurring later. Again, they can be computed using the programme of Jacon [36].

### Table 11. $|\text{Irr } \mathcal{H}|$ for some modular Hecke algebras

| $n$ | 1 | 2 | 3 | 4 | 5 |
|-----|---|---|---|---|---|
| $\mathcal{H}(B_n; 1; q)$ | 2 | 5 | 10 | 18 | 32 |
| $\mathcal{H}(B_n; q; q)$ | 2 | 5 | 9 | 18 | 30 |
| $\mathcal{H}(B_n; q^2; q)$ | 2 | 4 | 8 | 15 | 26 |
| $\mathcal{H}(B_n; q^3; q)$ | 1 | 3 | 5 | 10 | 16 |
| $\mathcal{H}(B_n; q^4; q)$ | 2 | 4 | 8 | 15 | 26 |

**Lemma 6.10.** Let $q$ be a prime power and $\ell|(q^2 - q + 1)$. The Hecke algebras of various $\ell$-modular cuspidal pairs $(L, \lambda)$ of Levi subgroups $L$ in $2D_n(q)$, $n \geq 4$, and their respective numbers of irreducible characters are as given in Table 12.

### Table 12. Hecke algebras and $|\text{Irr } \mathcal{H}|$ in $2D_n(q)$ for $d_\ell(q) = 6$

| $(L, \lambda)$ | $\mathcal{H}$ | 6 | 7 | 8 | 9 |
|----------------|---------------|---|---|---|---|
| $(2D_3, \varphi_{12})$ | $\mathcal{H}(B_{n-3}; 1; q)$ | 10 | 18 | 32 | 54 |
| $(A_5, \varphi_{16})$ | $\mathcal{H}(A_1; q^3) \otimes \mathcal{H}(B_{n-7}; q^2; q)$ | – | 1 | 2 | 4 |
| $(2D_7, \varphi_{16})$ | $\mathcal{H}(B_{n-7}; q^4; q)$ | – | 1 | 2 | 4 |
| $(2D_7, \varphi_{16})$ | $\mathcal{H}(B_{n-7}; q^4; q)$ | – | 1 | 2 | 4 |

**Proof.** Note that $2D_n(q)$ has Weyl group of type $B_{n-1}$. Now the relative Weyl group of a Levi subgroup $B_2(q)$ inside $B_{n-1}(q)$ has type $B_{n-3}$, and the one of a Levi subgroup $A_5(q)$ has type $B_{n-6}$ by [35], p. 70]. The modular Steinberg character $\varphi_{12}$ of $2D_3(q)$ is liftable by the $\ell$-modular Brauer tree in Table 10 and the one of $A_5(q)$ by the Brauer tree for $\text{GL}_6(q)$, so we can argue as usual, with reduction stability following by Example 2.3 (a).

Next, the $\ell$-modular Steinberg character $\varphi_{16}$ of $2D_7(q)$ occurs with multiplicity 1 in the reduction of an ordinary Deligne–Lusztig character $R_\ell^G(\theta)$ from a maximal torus $T$ centralising a Sylow $\Phi_6$-torus as can be seen from the decomposition matrix in Table 13. Thus we obtain reduction stability with Example 2.3 (c).

Finally, again using the decomposition matrix one checks that $\varphi_{16}$ lifts to a non-unipotent character whose Jordan correspondent is the cuspidal unipotent character of $D_4(q)\Phi_2\Phi_6$. The latter is invariant under all automorphisms, and thus $\varphi_{16}$ is also reduction stable.

Observe that Sylow $\ell$-subgroups of $2D_n(q)$, $n \leq 6$, are cyclic for primes $\ell$ with $d_\ell(q) = 6$, so we only need to deal with the case $n \geq 7$.

**Proposition 6.11.** Assume $(T_7)$. The decomposition matrix for the principal $\ell$-block of $2D_7(q)$ for primes $\ell$ with $(q^2 - q + 1)\ell > 7$ is as given in Table 13. Here, the unknown entries satisfy $y_2 \leq 5$ and $y_3 \leq 2$. 

Proof. All columns in Table 13 except for the 23rd, the 25th and the 27th are obtained by Harish-Chandra induction of PIMs from Levi subgroups of types $2D_6$ and $A_5$. With Lemma 6.10 this accounts for all Harish-Chandra series from proper Levi subgroups, so the remaining three PIMs must be cuspidal. The ordinary Gelfand-Graev character gives the Steinberg PIM in column 27. The uni-triangular shape of the decomposition matrix shows that there must exist PIMs with unipotent part of the form

$$\Psi_{23} = [1^6] + y_1[.21^4] + y_2[.1^6], \quad \Psi_{25} = [2^3] + y_3[.21^4] + y_4[.1^6],$$

with suitable coefficients $y_i \geq 0$. Not that $[1^6]$ and $[2^3]$ lie in families which are not comparable, which explains why $[2^3]$ does not occur in $\Psi_{23}$. It will turn out in the determination of the decomposition matrices for $2D_8(q)$ in Proposition 6.12 that $y_1 = 0$. 

### Table 13. $2D_7(q)$, $(q^2 - q + 1)_\ell > 7$

| $\ell$ | $[2^3] + [1^6]$ | $[2^3] + [1^6]$ |
|---|---|---|
| 6 | $q$ | 1 |
| 51 | $q$ | 1 |
| .6 | $\frac{1}{q^3}$ | 1 |
| 4| $\frac{1}{q^3}$ | 1 |
| 3.2 | $\frac{1}{q^3}$ | 1 |
| 1.5 | $\frac{1}{q^4}$ | 1 |
| 32.1 | $\frac{1}{q^4}$ | 1 |
| 12.4 | $\frac{1}{q^6}$ | 1 |
| 22.2 | $\frac{1}{q^6}$ | 1 |
| 5.1 | $\frac{1}{q^7}$ | 1 |
| 31.12 | $\frac{1}{q^7}$ | 1 |
| 31.3 | $\frac{1}{q^7}$ | 1 |
| 21.21 | $\frac{1}{q^9}$ | 1 |
| 12.22 | $\frac{1}{q^9}$ | 1 |
| 14.2 | $\frac{1}{q^9}$ | 1 |
| 31 | $\frac{1}{q^{12}}$ | 1 |
| .41.2 | $\frac{1}{q^{12}}$ | 1 |
| 22.12 | $\frac{1}{q^{12}}$ | 1 |
| 21.4 | $\frac{1}{q^{12}}$ | 1 |
| 1.2 | $\frac{1}{q^{16}}$ | 1 |
| 1.1 | $\frac{1}{q^{16}}$ | 1 |
| 1 | $\frac{1}{q^{21}}$ | 1 |
| .31.3 | $\frac{1}{q^{21}}$ | 1 |
| .23 | $\frac{1}{q^{21}}$ | 1 |
| .21 | $\frac{1}{q^{21}}$ | 1 |
| .1.6 | $\frac{1}{q^{42}}$ | 1 |

| $\ell$ | $[1^6]$ | $[21^4]$ |
|---|---|---|
| 51 | $1 \times c$ | $1 \times c$ |
| 31 | $1 \times c$ | $1 \times c$ |
| 16 | $1 \times c$ | $1 \times c$ |

| $\ell$ | $[1^6]$ | $[21^4]$ |
|---|---|---|
| 51 | $1 \times c$ | $1 \times c$ |
| 31 | $1 \times c$ | $1 \times c$ |
| 16 | $1 \times c$ | $1 \times c$ |

| $\ell$ | $[1^6]$ | $[21^4]$ |
|---|---|---|
| 51 | $1 \times c$ | $1 \times c$ |
| 31 | $1 \times c$ | $1 \times c$ |
| 16 | $1 \times c$ | $1 \times c$ |

| $\ell$ | $[1^6]$ | $[21^4]$ |
|---|---|---|
| 51 | $1 \times c$ | $1 \times c$ |
| 31 | $1 \times c$ | $1 \times c$ |
| 16 | $1 \times c$ | $1 \times c$ |
It remains to compute $y_2$ and $y_4$. To this end we consider the decomposition of Deligne-Lusztig characters $R_w$. For $w$ being a Coxeter element the coefficient of $\Psi_{27}$ in $R_w$ is equal to $2 + y_3 - y_4$ which by (DL) forces $y_1 \leq 2 + y_3$. On the other hand, by Example 6.11(c) there exists a regular $\ell$-element of $G^*$ whenever $(q^2 - q + 1)_\ell > 20$ (in particular whenever $(q^2 - q + 1)_\ell > 12$ since we also assumed $\ell \geq 7$). Therefore one can use (Red) to show that $y_4 \geq 2 + y_3$, which proves that $y_4 = 2 + y_3$. Now let $w = s_2s_1s_3s_4s_5s_6s_7s_6s_5s_4s_3s_8s_1s_2s_3s_4s_5s_6s_7$. By explicit computations and using that $y_4 = 2 + y_3$, the PIM $\Psi_{27}$ does not occur in $R_v$ for $v < w$. In addition, the coefficient of $\Psi_{27}$ in $R_w$ is $60 - 12y_2$, which by (DL) gives $y_2 \leq 5$.

Proof. First consider $^2D_8(q)$. All columns $\Psi_i$ in the principal block apart from $\Psi_{23}$ and $\Psi_{25}$ are obtained by Harish-Chandra induction, and we find two further projectives with unipotent parts

$$\Psi_{23} = [32^2] + y_3[1.21^4] + y_4[1.1^6],$$

$$\Psi_{25} = [1^7] + y_1[1.21^4] + y_2[1.1^6].$$

Since $[1.21^4]$ and $[1^7]$ lie in families which are not comparable, the character $\tilde{\Psi}_{25}$ must involve $y_1$ copies of $\Psi_{26}$ by $(T_\ell)$. Then (HCr) shows that $y_1 = 0$.

The columns in the block labelled $(0\ 1\ 2)$ except for the 20th and 25th are again obtained by Harish-Chandra induction, as well as

$$\Psi_{20} = [21^5] + [1^6.1] + y_2[1^7],$$

$$\Psi_{25} = [2^31] + y_3[2^21^3] + y_4[1^7].$$

We now turn to $^2D_9(q)$. For the principal block with (HCi) we find all columns $\Psi_i$ but $\Psi_1, \Psi_{11}, \Psi_{13}, \Psi_{22}$ and $\Psi_{26}$, as well as the projectives

$$\Psi_{22} = [32^2] + y_3[1^2.21^4] + y_4[1^2.1^6],$$

$$\Psi_{26} = [1^8] + y_2[1^2.1^6].$$

Furthermore, we get $\Psi_1 + \Psi_{11}, \Psi_1 + \Psi_5$ and $\Psi_{11} + \Psi_{13}$, which yield $\Psi_1, \Psi_{11}$ and $\Psi_{13}$ by triangularity.

The block for $^2D_9(q)$ labelled $(0\ 1\ 2)$ is now obtained as usually. An application of (HCr) shows that all columns correspond to PIMs. This completes the proof of Propositions 6.12 and 6.11.

□
Table 14. $^2D_8(q), (q^2 - q + 1)^7 > 7$, blocks $^0^2$ and $^0^1^2$
6. DECOMPOSITION MATRICES AT $d_c(q) = 6$

Table 15. $^2D_9(q), (q^2 - q + 1)^\epsilon > 7$, blocks $^{(1\ 2)}$ and $^{(0\ 1\ 2)}$

\[
\begin{array}{c|cccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccc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5. Odd-dimensional orthogonal groups

Proposition 6.13. Let \( q \) be a prime power and \( \ell \) a prime with \( d(q) = 6 \). The Brauer trees of the unipotent \( \ell \)-blocks of \( B_n(q) \) and of \( C_n(q) \), \( 3 \leq n \leq 6 \), with cyclic defect are as given in Table 16.

Here \( B_2 \) denotes the Harish-Chandra series of the ordinary cuspidal unipotent character of \( B_2(q) \), and \( B_3^\ell \) the cuspidal \( \ell \)-modular constituent of the unipotent character \([B_2:.1]\) of \( B_3(q) \).

Table 16. Brauer trees for \( B_n(q) \) and \( C_n(q) \) (3 \( \leq n \leq 6 \)), 7 \( \leq \ell |(q^2 - q + 1)

\[
\begin{align*}
B_3(q) &: 3. \quad 2.1 \quad 1.1^2 \quad .1^3 \quad \bigcirc \quad B_2:.1 \quad B_2:1. \\
B_4(q) &: 4. \quad 2.2 \quad 1.21 \quad .21^2 \quad \bigcirc \quad B_2:.1^2 \quad B_2:1^2. \\
& \quad 31. \quad 21.1 \quad 1.1^2 \quad .1^4 \quad \bigcirc \quad B_2:.2 \quad B_2:2. \\
B_5(q) &: 5. \quad 2.3 \quad 1.31 \quad .31^2 \quad \bigcirc \quad B_2:1.1^2 \quad B_2:1.21. \\
& \quad 41. \quad 21.2 \quad 1.21 \quad .21^3 \quad \bigcirc \quad B_2:.21 \quad B_2:21. \\
& \quad 31^2 \quad 21^2.1 \quad 1.1^3 \quad .1^5 \quad \bigcirc \quad B_2:.12 \quad B_2:2.1. \\
& \quad 32. \quad 22.1 \quad 1.2.1 \quad 1.1^4 \quad \bigcirc \quad B_2:.3 \quad B_2:3. \\
& \quad 41. \quad 3.2 \quad 1.22 \quad .221 \quad \bigcirc \quad B_2:.13 \quad B_2:1^3. \\
B_6(q) &: 5.1 \quad 3.3 \quad 1.32 \quad .321 \quad \bigcirc \quad B_2:1.1^3 \quad B_2:1.31. \\
& \quad 42. \quad 2^2.2 \quad 1.21^2 \quad 1.21^3 \quad \bigcirc \quad B_2:.31 \quad B_2:31. \\
& \quad 33. \quad 2.1^2 \quad 21.1^3 \quad 2.1^4 \quad \bigcirc \quad B_2:.4 \quad B_2:4. \\
& \quad 41.1 \quad 31.2 \quad 1^2.2^2 \quad 2.21^2 \quad \bigcirc \quad B_2:.21^2 \quad B_2:2.1^2. \\
& \quad 4.1^2 \quad 3.21 \quad 2.2^2 \quad 2.3 \quad \bigcirc \quad B_2:.1^4 \quad B_2:1^4. \\
& \quad 321 \quad 2^2.1 \quad 1.1^3 \quad 1.1^5 \quad \bigcirc \quad B_2:.1^3 \quad B_2:.31. \\
& \quad \text{ps} \quad \text{ps} \quad \text{ps} \quad \text{ps} \quad .1^3 \quad B_3^\ell \quad B_2 \quad B_2:.31. \\
B_5(q) &: 5 \quad 1.4 \quad 1.2.3 \quad 1.3.2 \quad 1.4.1 \quad 1.5 \quad \bigcirc \\
& \quad \text{ps} \quad \text{ps} \quad \text{ps} \quad \text{ps} \quad \text{ps} \quad \text{ps} \quad \text{ps} \quad \text{ps} \quad 1^5. \\
\end{align*}
\]

We will encounter the following Hecke algebras:

Lemma 6.14. Let \( q \) be a prime power and \( \ell |(q^2 - q + 1) \). The Hecke algebras of various \( \ell \)-modular cuspidal pairs \((L, \lambda)\) of Levi subgroups \( L \) in \( B_n(q) \) and their respective numbers of irreducible characters are as given in Table 17.
6. DECOMPOSITION MATRICES AT $d_{\ell}(q) = 6$

| $(L, \lambda)$ | $\mathcal{H}$ | $n = 5$ | $6$ | $7$ | $8$ |
|----------------|----------------|--------|--------|--------|--------|
| $(B_2, B_2)$   | $\mathcal{H}(B_{n-2}; q^6; q)$ | $5 + 0$ | $6 + 2 + 2$ | $12 + 4$ | $13 + 15$ |
| $(B_3, \varphi_{1^3})$ | $\mathcal{H}(B_{n-3}; q; q)$ | $5$ | $6 + 3$ | $12 + 6$ | $10 + 20$ |
| $(B_3, B_3^*)$ | $\mathcal{H}(B_{n-3}; q; q)$ | $5$ | $6 + 3$ | $12 + 6$ | $10 + 20$ |
| $(A_5, \varphi_{1^3})$ | $\mathcal{H}(B_{n-5}; q^3; q)$ | $-1$ | $1$ | $1 + 2$ | $5$ |
| $(B_5, \varphi_{1^5})$ | $\mathcal{H}(B_{n-5}; q^3; q)$ | $1$ | $1$ | $1 + 2$ | $5$ |

**Table 17.** Hecke algebras and $|\text{Irr} \mathcal{H}|$ in $B_n(q)$ for $d_{\ell}(q) = 6$

**Proof.** In all cases, reduction stability for the cuspidal Brauer characters follows with Example 2.5(a) as Sylow $\ell$-subgroups of $L$ are cyclic. The cuspidal $\ell$-modular character $B_2$ is the $\ell$-modular reduction of the cuspidal unipotent character of $B_2(q)$, thus its Hecke algebra is the reduction of the one in characteristic 0. The Brauer tree in Table 16 shows that the cuspidal $\ell$-modular Brauer character $B_3^*$ of $B_2(q)$ is a constituent of the $\ell$-modular reduction of an ordinary cuspidal character $\lambda$ in the Lusztig series of an $\ell$-element with centraliser a torus of order $q^3 + 1$. The minimal Levi overgroups are of types $B_4$ and $B_3A_1$, hence the parameters of the Hecke algebra for $\lambda$ are as given.

Again, by the shape of the Brauer tree the cuspidal $\ell$-modular Brauer character $\varphi_{1^5}$ of $B_5(q)$ is liftable to an ordinary cuspidal character lying in the Lusztig series of an $\ell$-element with centraliser $B_2(q)(q^3 + 1)$, and with centraliser $B_2(q)(q^3 + 1)$ in $B_6(q)$. Thus its Hecke algebra in $B_6(q)$ is $\mathcal{H}(B_1; q^3)$. Similarly, the cuspidal $\ell$-modular Steinberg character $\varphi_{1^6}$ of $A_5(q)$ is liftable to an ordinary cuspidal character in the Lusztig series of an $\ell$-element with centraliser $(q^6 - 1)/(q - 1)$, and with centraliser $GU_2(q^3)$ in $B_6(q)$. Thus again its Hecke algebra in $B_6(q)$ is $\mathcal{H}(B_1; q^3)$.

**Proposition 6.15.** Assume $(T_\ell)$. The decomposition matrix for the principal $\ell$-block of $B_6(q)$ for primes $\ell$ with $(q^2 - q + 1)\ell > 7$ is as given in Table 18.

**Proof.** The Hecke algebras for the cuspidal $\ell$-modular Brauer character $1^5$ of $B_5(q)$ and the cuspidal $\ell$-modular Steinberg character of $A_5(q)$ were determined in Lemma 6.14. As neither is semisimple modulo $\ell$, there is just one PIM of $B_6(q)$ in either series.

All columns except for those numbered 14, 22, 26 and 27 are obtained by Harish-Chandra induction from a Levi subgroup of type $B_5$. According to the description of the Hecke algebras in Lemma 6.14, with this we have accounted for all PIMs from proper Harish-Chandra series, so the remaining four PIMs must be cuspidal. By trianguularity, their non-zero entries lie below the diagonal, and we denote them by $a_1, \ldots, a_{13}$ for $\Psi_{14}$, by $a_{14}, \ldots, a_{18}$ for $\Psi_{22}$ and by $a_{19}$ for $\Psi_{26}$. Using the order on families we actually have $a_i = 0$ for $i \in \{1, 2, 3, 14, 15\}$.

To get relations we use the combination of (DL) and (Red). Let $w$ be a Coxeter element. The coefficients of $R_w$ on $\Psi_{26}$ and $\Psi_{27}$ are equal to $-a_{17}$ and $2 + a_{16} - a_{18} + a_{17}a_{19}$ respectively. Therefore by (DL) we must have $a_{17} = 0$ and $a_{18} \leq 2 + a_{16}$. On the other hand, if $(q^2 - q + 1)\ell > 7$ and $\ell > 7$ then $(q^2 - q + 1)\ell > 12$. By Example 1.7(a) there exists a regular $\ell$-element of $G^*$. Therefore one can invoke (Red) to get $a_{18} \geq 2 + a_{16}$ hence $a_{18} = 2 + a_{16}$. We go on to the next Deligne–Lusztig character $R_w$ for which $\Psi_{26}$ and $\Psi_{27}$ can potentially occur. It corresponds to $w = s_1s_2s_3s_2s_1s_2s_3s_4s_5s_6$. By (DL), the
coefficients on these two PIMs give $2 - a_4 - a_6 + a_7 + a_9 - a_{12} \geq 0$ and

$$-(1 + a_{19}) \left(2 - a_4 - a_6 + a_7 + a_9 - a_{12}\right) \geq 0 \text{ by (DL)}$$

$$-\left(-2 + 2a_4 + 2a_5 + 2a_6 - 2a_7 - 2a_8 - a_9 + a_{10} - a_{11} + a_{12} + a_{13}\right) \geq 0.$$  

Therefore we must have $a_{12} = 2 - a_4 - a_6 + a_7 + a_9$ and $a_{13} = 2 - 2a_4 - 2a_5 - 2a_6 + 2a_7 + 2a_8 + a_9 - a_{10} + a_{11} - a_{12}$, hence $a_{13} = -a_4 - 2a_5 - a_6 + a_7 + 2a_8 - a_{10} - a_{11}$. Finally, the coefficient of $\Psi_{27}$ in $R_w$ for $w = s_1 s_2 s_3 s_4 s_5 s_6$ is $48 - 24a_{19}$. By (DL) this forces $a_{19} \leq 2$.

| Table 18. $B_6(q)$, $(q^2 - q + 1)_\ell > 7$ |
|---|---|
| 6  | 1  |
| .6 | $\frac{1}{q}$ | 1 |
| 51 | $\frac{1}{q^2}$ | 1  |
| 2.4 | $\frac{1}{q^3}$ | 1  |
| $B_2$: 2$^2$ | $\frac{1}{q^3}$ | . . .  |
| .51 | $\frac{1}{q^4}$ | . 1  |
| 41$^2$ | $\frac{1}{q^4}$ | . . .  |
| 1.41 | $\frac{1}{q^5}$ | . . .  |
| $B_2$: 21.1 | $\frac{1}{q^5}$ | . . .  |
| 21.3 | $q^5$ | . . .  |
| $1^2$ | $q^7$ | . . .  |
| $21^2$ | $q^7$ | . . .  |
| $B_2$: 12.2 | $q^7$ | . . .  |
| $B_6$ | $\frac{1}{q}$ | . . .  |
| 31$^3$ | $\frac{1}{q^2}$ | . 1  |
| $41^2$ | $\frac{1}{q^2}$ | . 1  |
| $B_2$: 21$^2$ | $\frac{1}{q^2}$ | . . .  |
| $21^3$.1 | $q^{11}$ | . . .  |
| $B_2$: 12.1 | $q^{11}$ | . . .  |
| $1^3$.21 | $q^{11}$ | . . .  |
| $1^4$.1$^2$ | $q^{13}$ | . . .  |
| $B_2$: 2$^2$ | $q^{13}$ | . . .  |
| $21^4$. | $q^{16}$ | . . .  |
| $.31^3$ | $\frac{1}{q^4}$ | . 1  |
| $.21^4$ | $\frac{1}{q^4}$ | . 1  |
| $1^6$. | $\frac{1}{q^5}$ | . . .  |
| $.1^6$ | $\frac{1}{q^6}$ | . . .  |
| $ps\ ps\ ps\ B_2\ ps\ ps\ B_2\ ps\ ps\ B_2^3\ c\ ps\ .1^3B_2^3\ ps\ B_3^6\ ps\ A_5\ c\ 1^5.1^4.1^3\ c\ c$ |
It will be shown in the proofs for \( B_7(q) \) and \( B_8(q) \) in Proposition \( 6.15 \) that \( a_i = 0 \) for \( i \in \{4, \ldots, 11, 16, 19\} \). All columns are indecomposable by \( (\text{HCr}) \), respectively by the decomposition numbers for the principal series Hecke algebras.

As already for \( ^2D_7(q) \) in order to obtain further information on the decomposition matrix for \( B_6(q) \) we will consider blocks of larger groups.

**Proposition 6.16.** Assume \((T_i)\). The decomposition matrices for the unipotent \( \ell \)-blocks of \( B_7(q) \) and \( B_8(q) \) of non-cyclic defect for primes \( \ell \) with \((q^2 - q + 1)\ell > 7 \) are as given in Tables \( 19, 20, 21 \) and \( 22 \) except that for the block labelled \( \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \end{pmatrix} \) the 9th column, labelled \( B_6 \), might not be indecomposable.

Here, \( a_i \) is as in Proposition \( 6.15 \) and we have \( b_3 = 2 - b_1 + b_2 \).

Here \( B_6^q \) denotes the Harish-Chandra series of the cuspidal \( \ell \)-modular constituent of the unipotent character \( B_5 \cdot 2^2 \) of \( B_6(q) \), \( B_2^a \) the series of the \( \ell \)-modular cuspidal unipotent character \( B_2 \boxtimes \varphi_{16} \) of \( B_2 A_5 \).

**Proof.** Let us first consider the blocks of \( B_7(q) \). In the principal block, \((\text{HCi}) \) yields all columns except for the 19th, and a projective character \( \Psi \) with unipotent part \( a_4[2^2 1^2 1] + a_5[B_2; 1^3 1] + [B_6; 1^2 2] \) plus further constituents with larger \( a \)-value. By our assumption of triangularity, \( \Psi - a_4 \Psi_{17} - a_5 \Psi_{18} \) must then also be a projective character. By \( (\text{HCr}) \) this is only the case when \( a_4 = a_5 = 0 \).

Next, consider the blocks of \( B_8(q) \). Here, the principal block shows that we must have \( a_6 = a_8 = 0 \), the block with label \( \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \end{pmatrix} \) gives \( a_{16} = 0 \), and the block with label \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) forces that \( a_9 = a_{10} = a_{19} = 0 \). Thus we have obtained all information that had been missing in the proof of Proposition \( 6.15 \). The correctness of the six printed decomposition matrices is now verified as in our previous proofs. The relation between the \( b_i \) in the block labelled \( \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \end{pmatrix} \) is obtained from (DL) using the Coxeter element.

Finally, to conclude that \( a_{11} = 0 \) (resp. \( a_7 = 0 \)) we have to go up to the unipotent block of \( B_9(q) \) (resp. \( B_{10}(q) \)) labelled by \( \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{pmatrix} \) (resp. by \( \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix} \)) and invoke \((T_i)\) (we do not print the corresponding decomposition matrices).

6. Symplectic groups

Finally, we consider the symplectic groups. The Brauer trees here are the same as for the odd-dimensional orthogonal groups and had already been given in Proposition \( 6.13 \). The arguments used to determine the decomposition matrices for blocks of defect 2 involve only Harish-Chandra induction/restriction, unipotent Deligne–Lusztig characters (which depend only on \((W, F)\)) and the existence of \( \ell \)-regular elements (which are similar for \( G \) and \( G^\ast \) when \( \ell \) is odd, see Remark \( 3.1 \)). Consequently, under our assumptions on \( \ell \), the unipotent part of the decomposition matrices of the unipotent \( \ell \)-blocks of \( C_n(q) \) and \( B_n(q) \) are identical.

**Corollary 6.17.** Assume \((T_i)\). The decomposition matrices for the unipotent \( \ell \)-blocks of \( C_n(q) \), \( n = 6, 7, 8 \), of non-cyclic defect for primes \( \ell \) with \((q^2 - q + 1)\ell > 7 \) are the same as for \( B_n(q) \) and hence as given in Tables \( 18, 22 \) except possibly for the values of the yet unknown entries \( b_1 \) and \( b_2 \) (which need not be the same as for type \( B_n \)).
6. SYMPLECTIC GROUPS

Table 19. $B_7(q)$, $(q^2 - q + 1)^\ell > 7$, blocks $^1$ and $^0^1$:

| $7$ | $52$ | $16$ | $31$ | $12^1$ | $241$ | $1^321^2$ | $1^332$ | $1^6$ |
|-----|-----|-----|-----|------|------|---------|--------|-----|
| $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $B_2$: | $32$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $1.51$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $B_6$: | $31.1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $25^1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $B_2$: | $12.3$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $92^1.2$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $B_2$: | $31^2$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $31^2$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $B_2$: | $12^1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $1^321^2$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $1^332$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $1^6$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |

| $7$ | $6.1$ | $5.1$ | $3.4$ | $31.3$ | $B_2$: $2^1$ | $52$ | $41^2$ | $1.42$ | $31^22$ | $B_2$: $21^2.1$ | $B_6$: $2$ | $1^22^1$ | $B_2$: $1^32$ | $A_4^2$ | $31^3$ | $1^32^2$ | $B_2$: $2^1$ | $31^4$ | $B_2$: $21^3$ | $32^2$ | $21^1$ | $B_2$: $1^221^2$ | $1^51^2$ | $B_2$: $3^2$ | $B_6^5$: $1^8$ | $B_2^5$: $1^8$ | $A_5$: $1^8$ | $B_6^6$: $1^8$ | $1^61^6$ |
|-----|-----|-----|-----|-----|------|-----|------|-----|-----|------|------|------|------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $B_2$: | $ps$ | $ps$ | $ps$ | $ps$ | $ps$ | $ps$ | $ps$ | $ps$ | $ps$ | $ps$ | $ps$ | $ps$ | $ps$ | $ps$ | $ps$ | $ps$ | $ps$ | $ps$ | $ps$ | $ps$ | $ps$ | $ps$ | $ps$ | $ps$ | $ps$ | $ps$ | $ps$ |
| $B_6$: | $ps$ | $ps$ | $ps$ | $ps$ | $ps$ | $ps$ | $ps$ | $ps$ | $ps$ | $ps$ | $ps$ | $ps$ | $ps$ | $ps$ | $ps$ | $ps$ | $ps$ | $ps$ | $ps$ | $ps$ | $ps$ | $ps$ | $ps$ | $ps$ | $ps$ | $ps$ | $ps$ |


Table 20. $B_8(q)$, $(q^2 - q + 1)\ell > 7$, blocks $(2)$ and $(0 \ 1 \ 2)$
Table 21. $B_8(q), (q^2 - q + 1) < 7$, blocks $\binom{0 \ 2}{1}$ and $\binom{1 \ 2}{0}$

|    | 1   |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
|----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 3.5 |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
| 7.1 |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
| 1.7 |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
| 32.1 | 1  | 1   |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
| $B_2$: 321 | . . | . . | 1   |     |     |     |     |     |     |     |     |     |     |     |     |     |
| 421.1 |     |     | 1   | 1 |     |     |     |     |     |     |     |     |     |     |     |     |     |
| 1.52 |     |     | 1   | 1 |     |     |     |     |     |     |     |     |     |     |     |     |     |
| $B_2$: 31^2.1 | . . | . . | 1   | 1 | . . |     |     |     |     |     |     |     |     |     |     |     |     |
| 32.3 |     |     | 1   | 1 | . . |     |     |     |     |     |     |     |     |     |     |     |     |
| 321.2 |     |     |     | 1 | 1 | . . |     |     |     |     |     |     |     |     |     |     |     |
| 521 |     |     |     | 1 | 1 |     |     |     |     |     |     |     |     |     |     |     |     |
| 321^2.1 |     |     |     | 1 | 1 | . . |     |     |     |     |     |     |     |     |     |     |     |
| 1.421 |     |     |     | 1 | 1 | . . |     |     |     |     |     |     |     |     |     |     |     |
| $B_2$: 31^3 |     |     |     | 1 | 1 | . . |     |     |     |     |     |     |     |     |     |     |     |
| 12.13 |     |     |     | . | . | . |     |     |     |     |     |     |     |     |     |     |     |
| 12^2.1 |     |     |     | . | . | . |     |     |     |     |     |     |     |     |     |     |     |
| $B_2$: 51 | . . | . . | . | . | . | . | . | . | . | . | . | . | . | . | . | . |
| 71. |     |     | . | . | . |     |     |     |     |     |     |     |     |     |     |     |     |
| 62. |     |     | . | . | . |     |     |     |     |     |     |     |     |     |     |     |     |
| $B_2$: 32^2 | . . | . . | . | . | . |     |     |     |     |     |     |     |     |     |     |     |     |
| 21.5 |     |     | . | . | . |     |     |     |     |     |     |     |     |     |     |     |     |
| 12.51 |     |     | . | . | . | 1 | 1 |     |     |     |     |     |     |     |     |     |     |
| 42^3 | . . | . . | . | . | . | . | . | . | . | . | . | . | . | . | . | . |
| $B_2$: 31.2 | . . | . . | . | . | . | . | . | . | . | . | . | . | . | . | . | . |
| 2^2.4 | . . | . . | . | . | . | . | . | . | . | . | . | . | . | . | . | . |
| $B_2$: 21.3 | . . | . . | . | . | . | . | . | . | . | . | . | . | . | . | . | . |
| $B_2$: 3.21 | . . | . . | . | . | . | . | . | . | . | . | . | . | . | . | . | . |
| 32.1^2 | . . | . . | . | . | . | . | . | . | . | . | . | . | . | . | . | . |
| 12.41^2 | . . | . . | . | . | . | . | . | . | . | . | . | . | . | . | . | . |
| $B_2$: 2.31 | . . | . . | . | . | . | . | . | . | . | . | . | . | . | . | . | . |
| 3^2 | . . | . . | . | . | . | . | . | . | . | . | . | . | . | . | . | . |
| 2^4.3 | . . | . . | . | . | . | . | . | . | . | . | . | . | . | . | . | . |
| $B_2$: 3.1 | . . | . . | . | . | . | . | . | . | . | . | . | . | . | . | . | . |
| 3^2.1 | . . | . . | . | . | . | . | . | . | . | . | . | . | . | . | . | . |
| 5^2 | . . | . . | . | . | . | . | . | . | . | . | . | . | . | . | . | . |
| 51^2 | . . | . . | . | . | . | . | . | . | . | . | . | . | . | . | . | . |
| 1.41^3 | . . | . . | . | . | . | . | . | . | . | . | . | . | . | . | . | . |
| 2^3.1 | . . | . . | . | . | . | . | . | . | . | . | . | . | . | . | . | . |
| $B_2$: 1.2 | . . | . . | . | . | . | . | . | . | . | . | . | . | . | . | . | . |
| $B_2$: 3^2 | . . | . . | . | . | . | . | . | . | . | . | . | . | . | . | . | . |
| $B_2$: 2.1^2 | . . | . . | . | . | . | . | . | . | . | . | . | . | . | . | . | . |
| 32.1^3 | . . | . . | . | . | . | . | . | . | . | . | . | . | . | . | . | . |
| 1.3^2 | . . | . . | . | . | . | . | . | . | . | . | . | . | . | . | . | . |
| $B_2$: 1.3^2 | . . | . . | . | . | . | . | . | . | . | . | . | . | . | . | . | . |
| 1.4^2 | . . | . . | . | . | . | . | . | . | . | . | . | . | . | . | . | . |
| $B_2$: 1.4^2 | . . | . . | . | . | . | . | . | . | . | . | . | . | . | . | . | . |
| 1.3^3 | . . | . . | . | . | . | . | . | . | . | . | . | . | . | . | . | . |
| $B_2$: 1.3^3 | . . | . . | . | . | . | . | . | . | . | . | . | . | . | . | . | . |
| 1.4^3 | . . | . . | . | . | . | . | . | . | . | . | . | . | . | . | . | . |
| $B_2$: 1.4^3 | . . | . . | . | . | . | . | . | . | . | . | . | . | . | . | . | . |
| $B_2$: 3^3 | . . | . . | . | . | . | . | . | . | . | . | . | . | . | . | . | . |
| $B_2$: 2.3^2 | . . | . . | . | . | . | . | . | . | . | . | . | . | . | . | . | . |
| $B_2$: 3.3^2 | . . | . . | . | . | . | . | . | . | . | . | . | . | . | . | . | . |
| $B_2$: 3.4^2 | . . | . . | . | . | . | . | . | . | . | . | . | . | . | . | . | . |

|     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
|----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
|    |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
|     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |

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### Table 22. \( B_8(q), (q^2 - q + 1)t > 7, \) blocks \( \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} \) and \( \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} \)

| \( B_2 \) | 1 |
| --- | --- |
| \( B_3 \) | 1, 1 |
| \( B_4 \) | \( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) |
| \( B_6 \) | \( \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \) |
| \( B_8 \) | \( \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \) |

| \( B_{12} \) | \( \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \) |
| \( B_{21} \) | \( \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \) |
| \( B_{32} \) | \( \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \) |

| \( B_{21} \) | \( \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \) |
| \( B_{32} \) | \( \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \) |
| \( B_{43} \) | \( \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \) |
| \( B_{54} \) | \( \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \) |

| \( B_{12} \) | \( \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \) |
| \( B_{21} \) | \( \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \) |
| \( B_{32} \) | \( \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \) |
| \( B_{43} \) | \( \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \) |

| \( B_{12} \) | \( \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \) |
| \( B_{21} \) | \( \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \) |
| \( B_{32} \) | \( \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \) |
| \( B_{43} \) | \( \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \) |

| \( B_{12} \) | \( \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \) |
| \( B_{21} \) | \( \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \) |
| \( B_{32} \) | \( \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \) |
| \( B_{43} \) | \( \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \) |

| \( B_{12} \) | \( \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \) |
| \( B_{21} \) | \( \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \) |
| \( B_{32} \) | \( \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \) |
| \( B_{43} \) | \( \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \) |

| \( B_{12} \) | \( \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \) |
| \( B_{21} \) | \( \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \) |
| \( B_{32} \) | \( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) |
| \( B_{43} \) | \( \begin{pmatrix} 1 \end{pmatrix} \) |

| \( B_{12} \) | \( \begin{pmatrix} 1 \end{pmatrix} \) |
| \( B_{21} \) | \( \begin{pmatrix} 1 \end{pmatrix} \) |
| \( B_{32} \) | \( \begin{pmatrix} 1 \end{pmatrix} \) |
| \( B_{43} \) | \( \begin{pmatrix} 1 \end{pmatrix} \) |

| \( B_{12} \) | \( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) |
| \( B_{21} \) | \( \begin{pmatrix} 1 \end{pmatrix} \) |
| \( B_{32} \) | \( \begin{pmatrix} 1 \end{pmatrix} \) |
| \( B_{43} \) | \( \begin{pmatrix} 1 \end{pmatrix} \) |

| \( B_{12} \) | \( \begin{pmatrix} 1 \end{pmatrix} \) |
| \( B_{21} \) | \( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) |
| \( B_{32} \) | \( \begin{pmatrix} 1 \end{pmatrix} \) |
| \( B_{43} \) | \( \begin{pmatrix} 1 \end{pmatrix} \) |

| \( B_{12} \) | \( \begin{pmatrix} 1 \end{pmatrix} \) |
| \( B_{21} \) | \( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) |
| \( B_{32} \) | \( \begin{pmatrix} 1 \end{pmatrix} \) |
| \( B_{43} \) | \( \begin{pmatrix} 1 \end{pmatrix} \) |
7. Unipotent decomposition matrix of $F_4(q)$

**Theorem 6.18.** Let $(q, 6) = 1$. The decomposition matrix for the principal $\ell$-block of $F_4(q)$ for primes $\ell \geq 7$ with $(q^2 - q + 1)\ell > 7$ is approximated by Table $23$.

Here, the unknown entries satisfy $y_6 = 1 + y_1 + y_2$, $z_3 \leq 1$ and $x_1 + 3(z_1 + z_2) \leq 5$ (in particular all the $z_i$ are equal to either $0$ or $1$).

**Table 23.** $F_4(q)$, $(q^2 - q + 1)\ell > 7$, $(q, 6) = 1$

|  | $q^4$ | 1 |
|---|---|---|
| $\phi_{1,0}$ | 1 | 1 |
| $\phi'_{2,4}$ | $\frac{1}{2}q^4\phi_4\phi_8\phi_{12}$ | 1 |
| $\phi''_{2,4}$ | $\frac{1}{2}q^4\phi_4\phi_8\phi_{12}$ | 1 |
| $B_2: 2^2$ | $\frac{1}{2}q^4\phi_4^2\phi_8^2\phi_{12}$ | 1 |
| $\phi'_{8,3}$ | $q^3\phi_4\phi_8\phi_{12}$ | 1 |
| $\phi''_{8,3}$ | $q^3\phi_4\phi_8\phi_{12}$ | 1 |
| $\phi'_{12,4}$ | $\frac{1}{2}q^4\phi_4\phi_8^4\phi_{12}$ | 1 |
| $\phi''_{9,6}$ | $\frac{1}{2}q^4\phi_3\phi_8\phi_{12}$ | 1 |
| $\phi''_{9,6}$ | $\frac{1}{2}q^4\phi_3\phi_8\phi_{12}$ | 1 |
| $F_4[1]$ | $\frac{1}{2}q^4\phi_4^2\phi_8^2\phi_{12}$ | 1 |
| $B_2: 2^2$ | $\frac{1}{2}q^4\phi_4^2\phi_8^2\phi_{12}$ | 1 |
| $B_2: 2^2$ | $\frac{1}{2}q^4\phi_4^2\phi_8^2\phi_{12}$ | 1 |
| $F_4[\zeta_3]$ | $\frac{1}{2}q^4\phi_4\phi_8\phi_{12}$ | 1 |
| $F_4[\zeta_3^2]$ | $\frac{1}{2}q^4\phi_4\phi_8\phi_{12}$ | 1 |
| $\phi'_{8,9}$ | $q^3\phi_4\phi_8\phi_{12}$ | 1 |
| $\phi''_{8,9}$ | $q^3\phi_4\phi_8\phi_{12}$ | 1 |
| $\phi'_{2,16}$ | $\frac{1}{2}q^4\phi_4^2\phi_8\phi_{12}$ | 1 |
| $\phi''_{2,16}$ | $\frac{1}{2}q^4\phi_4^2\phi_8\phi_{12}$ | 1 |
| $B_2: 2^2$ | $\frac{1}{2}q^4\phi_4^2\phi_8^2\phi_{12}$ | 1 |
| $\phi'_{12,24}$ | $q^4$ | 1 |

| $ps$ | $ps$ | $B_2$ | $ps$ | $ps$ | $ps$ | $ps$ | $ps$ | $B_3$ | $C_3$ | $c$ | $c$ | $c$ | $B_3$ | $C_3$ | $c$ | $c$ | $c$ | $c$ |

**Proof.** Köhler [38, Table A.162] has shown uni-triangularity and determined the $\ell$-modular Harish-Chandra series of the unipotent Brauer characters under the stated assumptions on $\ell$. He has also proved that the ordinary cuspidal characters $F_4[1]$ and $F_4[-1]$ can only occur in the reduction of the Steinberg character $\phi_{1,24}$. We denote these multiplicities by $x_1$ and $x_2$. Similarly, he showed that the cuspidal irreducible Brauer characters belonging to $\phi'_{2,16}$, $\phi''_{2,16}$ and $B_2: 2^2$ can occur only in the reduction of the Steinberg character, and we denote by $z_1, z_2, z_3$ their respective multiplicities. The projective characters corresponding to the Galois conjugate cuspidal characters $F_4[\zeta_3]$ and $F_4[\zeta_3^2]$ can have potentially several unipotent constituents. We will write the unipotent
part of these projective characters as

\[ \Psi_{14} = F_4^{[1]}[\zeta] + \psi_1^\prime \phi_8^\prime + \psi_2^\prime \phi_8^\prime + \psi_3 \phi_{2,16} + \psi_4 \phi_{2,16} + \psi_5 \phi_{2,16} + \psi_6 \phi_{1,24} \]

and

\[ \Psi_{15} = F_4^{[1]}[\zeta] + \psi_1 \phi_8 \phi_9 + \psi_2 \phi_8^\nu + \psi_3 \phi_{2,16} + \psi_4 \phi_{2,16} + \psi_5 \phi_{2,16} + \psi_6 \phi_{1,24}. \]

We now use (DL) to compute some of these coefficients. We start with the Deligne–Lusztig character \( R_w \) where \( w \) is a Coxeter element. The coefficients of \( \Psi_{18}, \Psi_{19} \) and \( \Psi_{20} \) in \( R_w \) are \( 2y_1 - 2y_3, 2y_2 - 2y_4 \) and \( 2 - 2y_5 \) respectively and are all non-negative by (DL). The coefficient on \( \Psi_{21} \) is

\[ X = 2 + 2y_1 + 2y_2 - 2y_6 - z_1(2y_1 - 2y_3) - z_2(2y_2 - 2y_4) - z_3(2 - 2y_5) \geq 0. \]

Now if \( (q^2 - q + 1)^2 > 7 \), we can use (Red) for the maximal torus of order \( (q^2 - q + 1)^2 \) to obtain \(-3 - 2y_1 - 2y_2 + y_3 + y_4 + 2y_5 + y_6 \geq 0\). Adding twice this non-negative number to \( X \) we get

\[-(1 + z_1)(2y_1 - 2y_3) - (1 + z_2)(2y_2 - 2y_4) - (2 + z_3)(2 - 2y_5) \geq 0\]

which forces \( 2y_1 - 2y_3 = 2y_2 - 2y_4 = 2 - 2y_5 = 0 \) and \( X = 0 \). This gives

\[ y_1 = y_3, \quad y_2 = y_4, \quad y_5 = 1 \quad \text{and} \quad y_6 = 1 + y_1 + y_2. \]

We continue with the Deligne–Lusztig character associated with \( w = s_1s_2s_3s_4s_2s_3 \). The coefficient of \( \Psi_{21} \) in \( R_w \) is \( 2 - x_2 \), hence \( x_2 \leq 2 \) by (DL). On the other hand, (Red) applied to the case of a torus gives \( x_2 \geq 2 \) hence \( x_2 = 2 \).

We finish with the Deligne–Lusztig character \( R_w \) with \( w = s_1s_2s_3s_4s_1s_2s_3s_4 \). The coefficient of the PIM \( \Psi_{21} \) is \( 25 - x_1 - 9z_1 - 9z_2 - 14z_3 \). Therefore by (DL) we must have \( z_3 \leq 1 \) and \( z_1 + z_2 \leq 2 \). If all of the \( z_i \) are zero then we can only deduce that \( x_1 \leq 25 \), which is not satisfactory. However, if one considers the generalised \( q^4 \)-eigenspace of \( F \) on \( R_w \), then the coefficient of \( \Psi_{21} \) is \( 5 - x_1 - 3z_1 - 3z_2 \), which gives \( x_1 \leq 5 \) if \( z_1 = z_2 = 0 \) or \( x_1 \leq 2 \) if \( z_1 \) or \( z_2 \) is non-zero.

Again, we collect data on the \( \Phi_6 \)-modular Harish-Chandra series in the blocks \( b \) considered in this section in Tables 24, 25 and 26. None of the decomposition matrices determined here agree after any permutations of rows and columns, so none of the blocks can be Morita equivalent.

### Table 24. Modular Harish-Chandra-series in \( \Phi_6 \)-blocks

| \( G \) | \( b \) | \( W_G(b) \) | \( |\text{Br} b| \) | \( ps \) | \( D_4 \) | \( .1^4 \) | \( A_5 \) | \( .1^6 \) | \( D_6^a \) | \( E_6 \) |
|---|---|---|---|---|---|---|---|---|---|---|
| \( D_6 \) | \( G(6,2,2) \) | 18 | 10 | 2 | 2 | 1 | 1 | | |
| \( D_8 \) | 2 | \( G(6,2,2) \) | 18 | 10 | 2 | 2 | 1 | 1 | | |
| \( D_7 \) | \( G(6,1,2) \) | 27 | 14 | 4 | 4 | 1 | 2 | 2 | | |
| \( D_8 \) | 1, 3 | \( G(6,1,2) \) | 27 | 14 | 4 | 4 | 1 | 2 | 2 | | |
| \( E_6 \) | \( G_5 \) | 21 | 11 | 2 | 2 | 1 | | | | |
| \( E_8 \) | 2 | \( G_5 \) | 21 | 11 | 2 | 2 | | | | |
| \( F_4 \) | | \( |\text{Br} b| \) | \( ps \) | \( B_2 \) | \( B_3 \) | \( C_3 \) | \( c \) |
| | | | | | | | | | | |
### Table 25. Modular Harish-Chandra-series in $\Phi_6$-blocks, twisted groups

| $G$   | $b$     | $W_G(b)$ | $|\text{IBr } b|$ | $ps$ | $1^2$ | $A_5$ | $1^6$ | $2^3$ | $1^6$ |
|-------|---------|----------|-------------------|------|------|-------|-------|------|-------|
| $2D_7$ | $(6, 1, 2)$ | 27       | 15 8 1 1 1 1 1 1 1 |
| $2D_8$ 1, 2 | $(6, 1, 2)$ | 27       | 15 8 1 1 1 1 1 1 1 |
| $2D_9$ 1, 2 | $(6, 1, 2)$ | 27       | 15 8 1 1 1 1 1 1 1 |

### Table 26. Modular Harish-Chandra-series in $\Phi_6$-blocks, type $B_n$

| $G$   | $b$     | $W_G(b)$ | $|\text{IBr } b|$ | $ps$ | $B_2$ | $1^3$ | $B_3^a$ | $A_5$ | $B_6^t$ | $1^5$ | $B_6$ | $1^6$ | $1^6$ | $B_2^a$ | $c$ |
|-------|---------|----------|-------------------|------|------|-------|---------|-------|---------|-------|-------|-------|-------|---------|-----|
| $B_6$, $B_7$ | $(6, 1, 2)$ | 27       | 13 2 3 3 1 1 1 1 1 1 |
| $B_8$ 1-5 | $(6, 1, 2)$ | 27       | 13 2 3 3 1 1 1 1 1 1 |
| $B_8$ 6 | $(6, 1, 2)$ | 27       | 9 5 5 5 0 0 0 0 0 1 2 |
CHAPTER 7

Decomposition matrices at \( d_\ell(q) = 5, 7, 8, 10, 12, 14 \)

In this final section we collect the decomposition matrices for primes \( \ell \) with \( d_\ell(q) \notin \{1, 2, 3, 4, 6\} \) for the classical groups considered in this work. (The Brauer trees for exceptional groups can be found in [10].) In all cases, the corresponding cyclotomic polynomial divides the order polynomial of the groups in question at most once, so all such blocks are of cyclic defect. We thus give their Brauer trees; they were all first determined by Fong and Srinivasan [20].

**Proposition 7.1.** The Brauer trees of the unipotent \( \ell \)-blocks, for \( 11 \leq \ell | \Phi_5(q) \), of \( B_5(q) \), \( D_n(q) \) with \( 5 \leq n \leq 7 \) and of \( ^2D_n(q) \) with \( 6 \leq n \leq 7 \), are as given in Table 1.

**Table 1.** Brauer trees for \( 11 \leq \ell | \Phi_5(q) \)

\[
\begin{align*}
D_5(q) & : \ .5 \ .41 \ .31^2 \ .21^3 \ .1^5 \ .\bigcirc \\
D_7(q) & : \ 1.6 \ 1.42 \ 1.321 \ .1.21^2 \ .1^2 \ .1.16 \ .\bigcirc \\
D_6(q) & : \ .6 \ .42 \ .321 \ .2^21^2 \ .1^6 \ .\bigcirc \ 1.1^5 \ 1.21^3 \ 1.31^2 \ 1.41 \ 1.5 \\
D_7(q) & : \ .7 \ .43 \ .3^2 \ .2^21^3 \ .21^5 \ .\bigcirc \ 1.5 \ 2 \ 2.21^3 \ 2.31^2 \ 2.41 \ 2.5 \\
B_5(q) & : \ 5. \ 41. \ .31^2 \ .21^3 \ .1^5 \ .\bigcirc \ .5 \ \ .1^5 \ 1.21^3 \ .31^2 \ .41 \ .5 \\
^2D_6(q) & : \ 5. \ 41. \ .31^2 \ .21^3 \ .1^5 \ .\bigcirc \ .5 \ \ .1^5 \ 1.21^3 \ .31^2 \ .41 \ .5 \\
^2D_7(q) & : \ 6. \ 42. \ .321. \ .21^2 \ .1^6 \ .\bigcirc \ 1.1^5 \ 1.21^3 \ 1.31^2 \ 1.41 \ 1.5 \\
\end{align*}
\]

Here, "1\(^4\)" denotes the cuspidal Steinberg PIM of \( A_4(q) \), and similarly in the subsequent Brauer trees, the Harish-Chandra series are labelled by names of unipotent characters in which the corresponding cuspidal Brauer character first appears.
Table 2. Brauer trees for $17 \leq \ell | \Phi_8(q)$

|       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| $B_4(q)$ | 4. | 3.1 | 2.1² | 1.1³ | .1⁴ | | | | | B₂: .1² | B₂: 1.1 | B₂: 2. |
| $B_5(q)$ | 5. | 3.2 | 2.21 | 1.21² | .2³ | .1⁴ | | | | | B₂: .1³ | B₂: 1² .1 | B₂: 21. |
|         |       | 31.1 | 21.1² | 1² .1³ | .1⁵ | | | | | B₂: .21 | B₂: 2.1 | B₂: 1.2 | B₂: 3. |
| $D_5(q)$ | .5 | 1.4 | 1² .3 | 1³ .2 | 1.1⁴ | .1⁵ | | | | | | | | | | |
| $D_6(q)$ | .6 | 2.4 | 21.3 | 21² .2 | 21³ .1 | 21⁴ | | | | | | | | | | |
|         |       | .51 | 1.41 | 1² .31 | 1³ .21 | 1⁴ .1² | 1⁶ | | | | | | | | | | |
| $D_7(q)$ | .7 | 3.4 | 31.3 | 31² .2 | 31³ .1 | 31⁴ | | | | | | | | | | |
|         |       | 1.6 | 2.5 | 2² .3 | 2¹ .2 | 2² .1² | 2¹ .2² | | | | | | | | | | |
|         |       | .61 | 2.41 | 21.31 | 21² .21 | 21³ .1² | 21⁵ | | | | | | | | | | |
|         |       | .52 | 1.42 | 1² .3² | 1³ .2² | 1⁵ .1² | 1⁵ .1 | | | | | | | | | | |
|         |       | .51² | 1.41² | 1² .31² | 1³ .21² | 1⁴ .1³ | 1⁷ | | | | | | | | | | |
| $^2D_5(q)$ | 4. | 2.2 | 1.21 | .2¹ | .2² | | | | | | | | | | | | |
| $^2D_6(q)$ | 5. | 2.3 | 1.31 | .3¹ | .2 | | | | | | | | | | | | |
|         |       | 4.1 | 3.2 | 1.2² | .2² | .1² | | | | | | | | | | | | |
| $^2D_7(q)$ | 6. | 2.4 | 1.41 | .4¹ | | | | | | | | | | | | | | |
|         |       | 5.1 | 21.3 | 1³ .31 | .3¹ | | | | | | | | | | | | | |
|         |       | 5.1 | 3.3 | 1.32 | .3² | | | | | | | | | | | | | |
|         |       | 41² | 21² .2 | 1³ .21 | .2¹ | | | | | | | | | | | | | |
|         |       | 4.1² | 3.21 | 2.2² | .2³ | | | | | | | | | | | | | |
| $^2D_4(q)$ | 3. | 2.1 | 1.1² | .1³ | | | | | | | | | | | | | | |
| $^2D_6(q)$ | 41. | 21.2 | 1² .21 | 21³ | | | | | | | | | | | | | | |

**Proposition 7.2.** The Brauer trees of the unipotent $\ell$-blocks, for $17 \leq \ell | \Phi_8(q)$, of $B_4(q)$, $B_5(q)$, $D_n(q)$ with $5 \leq n \leq 7$ and of $^2D_n(q)$ with $4 \leq n \leq 7$, are as given in Table 2.
7. DECOMPOSITION MATRICES AT $d_r(q) = 5, 7, 8, 10, 12, 14$

**Proposition 7.3.** The Brauer tree of the principal $\ell$-block of $D_7(q)$, for $29 \leq \ell|\Phi_r(q)$, is as given by

$$\begin{array}{cccccccccc}
.7 & .61 & .51^2 & .41^3 & .31^4 & .21^5 & .17 & \circ \\
ps & ps & ps & ps & ps & ps & c & c
\end{array}$$

**Proposition 7.4.** The Brauer tree of the principal $\ell$-block of $2D_7(q)$, for $29 \leq \ell|\Phi_{14}(q)$, is as given by

$$\begin{array}{cccccccccc}
6 & 5.1 & 4.1^2 & 3.1^3 & 2.1^4 & 1.1^5 & .1^6 & \circ \\
ps & ps & ps & ps & ps & ps & c & c
\end{array}$$

**Proposition 7.5.** The Brauer trees of the unipotent $\ell$-blocks, for $11 \leq \ell|\Phi_{10}(q)$, of $B_6(q)$, $D_n(q)$ with $6 \leq n \leq 7$ and of $2D_n(q)$ with $5 \leq n \leq 7$, are as given in Table 3.

**Table 3.** Brauer trees for $11 \leq \ell|\Phi_{10}(q)$

| $B_6(q)$ | 5 | 4.1 | 3.1^2 | 2.1^3 | 1.1^4 | .1^5 | \circ | B_2:1^3 | B_2:1.1^2 | B_2:2.1 | B_2:3. |
|-----------|---|-----|-------|-------|-------|-----|----|-------|--------|--------|--------|
| $D_6(q)$ | .6 | 1.5 | 1^2.4 | 1^3.3 | 1^4.2 | 1^5.1 | 1^6. | \circ | D_4:1^2 | D_4:1.1 | D_4:2. |
| $D_7(q)$ | .7 | 2.5 | 21.4 | 21^2.3 | 21^3.2 | 21^4.1 | 21^5 | \circ | D_4:1^3 | D_4:1.2 | D_4:2.1 |
|           | .61 | 1.51 | 1^2.41 | 1^3.31 | 1^4.21 | 1^5.1^2 | 1^6.1 | \circ | D_4:2.1 | D_4:1.2 | D_4:3. |

| $2B_6(q)$ | 5 | 3.2 | 2.21 | 1.21^2 | 21^3 | \circ | .1^5 | 1^2.1^3 | 21.1^2 | 31.1 | 41. |
| $2D_7(q)$ | 6 | 3.3 | 2.31 | 1.31^2 | .31^3 | \circ | 1.1^7 | 1^2.1^4 | 2^2.1^2 | 32.1 | 42. |
|           | 5.1 | 4.2 | 2.2^2 | 1.2^2.1 | 2.2^1^2 | \circ | .1^6 | 1^3.1^3 | 2^1.2^1 | 31.2 | 41. |
|           | ps | ps | ps | ps | ps | .1^4 | .1^4 | ps | ps | ps |

| $2D_6(q)$ | 4 | 3.1 | 2.1^2 | 1.1^3 | .1^4 | \circ |
| $2D_7(q)$ | 5 | 31.2 | 21.21 | 1^2.21 | 2^1.4 | \circ |

| ps | ps | ps | ps | .1^4 |

**Proposition 7.6.** The Brauer trees of the unipotent $\ell$-blocks, for $13 \leq \ell|\Phi_{12}(q)$, of $B_6(q)$, $D_7(q)$, $2D_6(q)$ and $2D_7(q)$ are as given in Table 4.
Table 4. Brauer trees for $13 \leq \ell | \Phi_{12}(q)$

\[
\begin{align*}
D_7(q) : & 7 \quad 16 \quad 1^2.5 \quad 1^3.4 \quad 1^4.3 \quad 1^5.2 \quad 1^6.1 \quad 1^7. \\
D_4 : & 3 \quad D_4 : 2.1 \quad D_4 : 1.1^2 \quad D_4 : 1^2 \quad \bigcirc \\
2D_6(q) : & 5 \quad 4.1 \quad 3.1^2 \quad 2.1^3 \quad 1.1^4 \quad .1^5 \quad \bigcirc \\
2D_7(q) : & 6 \quad 4.2 \quad 3.21 \quad 2.21^2 \quad 1.21^3 \quad .21^4 \quad \bigcirc \\
& 51 \quad 41.1 \quad 31.1^2 \quad 21.1^3 \quad 1^2.1^4 \quad .1^6
\end{align*}
\]
CHAPTER 8
On a conjecture of Craven

Let $G = G^F$ for $G$ connected reductive defined over $\mathbb{F}_q$. By $[40]$ for any unipotent character $\rho$ of $G$ there exists a degree polynomial $R_\rho \in \mathbb{Q}[x]$ such that, in particular, $\rho(1) = R_\rho(q)$. The degree (resp. the valuation at $x$) of $R_\rho$ is denoted $A_\rho$ (resp. $a_\rho$). By the main theorem of $[5]$, under suitable conditions the decomposition matrix of the unipotent blocks of $G$ is unitriangular with respect to the ordering on families. This induces a bijection $\rho \mapsto \varphi_\rho$ between the unipotent characters and the irreducible Brauer characters lying in unipotent blocks. Since both the $a$-function and $A$-function are decreasing with respect to the ordering on families, we have, for unipotent characters $\rho \neq \rho'$

$$\langle \rho, \Psi_{\varphi_\rho'} \rangle \neq 0 \implies a_\rho > a_{\rho'} \text{ and } A_\rho > A_{\rho'}.$$ 

In $[10]$, Craven predicts the existence of perverse equivalences between unipotent $\ell$-blocks of $G$ and their Brauer correspondents, for primes $\ell$ such that Sylow $\ell$-subgroups of $G$ are abelian. A consequence of the existence of such equivalences is that the decomposition matrix would be unitriangular with respect to the perversity function (see $[9]$ Prop. 8.1).

Let us recall how the perversity function of Craven’s conjectural perverse equivalence is defined. For a root of unity $\zeta \in \mu(\mathbb{C})$ we denote by $\text{Arg}(\zeta) \in [0, 2\pi]$ the argument of $\zeta$. We write $m(\zeta, R)$ for the multiplicity of $\zeta$ as a root of a polynomial $R$. For any positive integer $d$ and unipotent character $\rho$ of $G$ we then define

$$\pi_d(\rho) := \frac{A_\rho + a_\rho}{d} + \frac{m(1, R_\rho)}{2} + \sum_{\zeta \in \mu(\mathbb{C}) \setminus \{1\}, \text{Arg}(\zeta) < 2\pi/d} m(\zeta, R_\rho).$$

It is shown in $[10]$ that $\pi_d(\rho) - \pi_d(\rho')$ is an integer whenever $\rho$ and $\rho'$ lie in the same $d$-Harish-Chandra series of $G$. In particular $\pi_d(\rho)$ is an integer for every unipotent character $\rho$ in the principal $d$-Harish-Chandra series, as $\pi_d(1_G) = 0$.

**Conjecture 8.1** (Craven). Let $\ell$ be a prime such that Sylow $\ell$-subgroups of $G$ are abelian, and let $d = d_\ell(q)$ be the order of $q$ in $\mathbb{F}_\ell^\times$. Then for any two unipotent characters $\rho \neq \rho'$ of $G$ we have

$$\langle \rho, \Psi_{\varphi_\rho'} \rangle \neq 0 \implies \pi_d(\rho) > \pi_d(\rho').$$

When $d = 2$, every non-real root $\zeta$ of $R_\rho$ satisfies $\text{Arg}(\zeta, R_\rho) < 2\pi/d$ or $\text{Arg}(\zeta, R_\rho) < 2\pi/d$. Since $R_\rho$ has real coefficients, this shows that

$$\pi_2(\rho) = A_\rho - \frac{m(-1, R_\rho)}{2}.\]$$

In addition, for $\ell$ a good prime for $G$, $m(-1, R_\rho)$ is constant on the unipotent characters in a fixed unipotent $\ell$-block since they form a single $2$-Harish-Chandra series, therefore in this case Conjecture 8.1 follows from the result of $[5]$. 

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On the other hand, when \( d > 2 \) the order coming from \( \pi_d \) might be quite different from the one given by the families. As an example, let us consider the principal \( \Phi_6 \)-block of \( B_6(q) \) whose decomposition matrix is given in Table 18. The unipotent characters

\[
21^3.1, B_2: 1.21, 1^3.21, 1^4.1^2, B_2: .2^2, 21^4, .31^3, .21^4, 1^6, .1^6
\]

all lie in families which are smaller than the family containing the cuspidal unipotent character \( B_6 \). However the \( \pi_6 \)-function on these characters is given by

\[
\begin{array}{cccccc}
\rho & B_6 & 21^3.1 & B_2: 1.21 & 1^3.21 & 1^4.1^2 & B_2: .2^2 & 21^4, .31^3, .21^4, 1^6, .1^6 \\
p_{B_6}(\rho) & 11 & 10 & 10 & 11 & 11 & 10 & 11 & 12 & 12
\end{array}
\]

which predicts that the character of the PIM \( P_{B_6} \) can only have \( 1^6 \) and \( .1^6 \) as unipotent constituents.

We have checked that the non-zero entries in the decomposition matrices computed in the previous chapters are compatible with Conjecture 8.1. Furthermore, the conjecture predicts the following about yet unknown decomposition numbers in our tables:

**Proposition 8.2.** Assume that Craven’s Conjecture 8.1 holds. Then the following decomposition numbers in our tables vanish:

- for \( E_6(q) \) with \( d = 3 \): \( b_1 = 0 \),
- for \( E_6(q) \) and \( E_8(q) \) with \( d = 6 \): \( a_1 = a_2 = a_4 = b_1 = b_4 = 0 \),
- for \( 2D_7(q) \) with \( d = 6 \), \( \ell > 7 \): \( y_2 = y_3 = 0 \),
- for \( F_4(q) \) with \( d = 6 \): \( x_1 = y_1 = y_2 = z_1 = z_2 = z_3 = 0 \).

Moreover, several of our proofs would become easier given the conjecture. Observe that Sylow \( \ell \)-subgroups of \( G \) are abelian in all cases of the proposition.

In [18] decomposition matrices for \( d_4(q) = 4 \) were computed up to a few unknown entries. Craven’s conjecture also predicts that some of them vanish.

**Proposition 8.3.** Assume that Craven’s Conjecture 8.1 holds. Then the following decomposition numbers at \( d_4(q) = 4 \) computed in [18] vanish:

- for \( D_5(q) \): \( b_1 = b_3 = b_4 = b_5 = b_7 = c = d = 0 \),
- for \( 2D_5(q) \), \( 2D_7(q) \): \( a = 0 \),
- for \( 2E_6(q) \): \( c_1 = c_4 = c_6 = d_2 = 0 \),
- for \( C_4(q) \): \( a = 0 \).

**Remark 8.4.** Note that we did not use the assumption \((T_\ell)\) when computing the matrices in [18], but rather proved a weaker property for the matrices we considered. Using \((T_\ell)\) for larger groups and \((HCl)\), \((HCr)\), we can actually deduce that all the entries listed in Proposition 8.3 vanish except possibly for \( c_4 \) and \( c_6 \) in \( 2E_6(q) \).
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| 17    | Hecke algebras and $|\text{Irr } H|$ in $B_n(q)$ for $d_\ell(q) = 6$ |
| 18    | $B_6(q)$, $(q^2 - q + 1)\ell > 7$ |
| 19    | $B_7(q)$, $(q^2 - q + 1)\ell > 7$, blocks $\binom{1}{0}$ and $\binom{0}{1}$ |
| 20    | $B_8(q)$, $(q^2 - q + 1)\ell > 7$, blocks $\binom{2}{0}$ and $\binom{0}{1} \binom{1}{2}$ |
| 21    | $B_9(q)$, $(q^2 - q + 1)\ell > 7$, blocks $\binom{0}{1} \binom{2}{1}$ and $\binom{1}{2}$ |
| 22    | $B_9(q)$, $(q^2 - q + 1)\ell > 7$, blocks $\binom{0}{1}$ and $\binom{0}{1} \binom{1}{2}$ |
| 23    | $F_4(q)$, $(q^2 - q + 1)\ell > 7$, $(q,6) = 1$ |
| 24    | Modular Harish-Chandra-series in $\Phi_6$-blocks |
| 25    | Modular Harish-Chandra-series in $\Phi_6$-blocks, twisted groups |
| 26    | Modular Harish-Chandra-series in $\Phi_6$-blocks, type $B_n$ |
| 1     | Brauer trees for $11 \leq \ell | \Phi_5(q)$ |
| 2     | Brauer trees for $17 \leq \ell | \Phi_8(q)$ |
| 3     | Brauer trees for $11 \leq \ell | \Phi_{10}(q)$ |
| 4     | Brauer trees for $13 \leq \ell | \Phi_{12}(q)$ |
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