Kinetic energy of the Langevin particle

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Abstract
We compute the kinetic energy of the Langevin particle using different approaches. We build stochastic differential equations that describe this physical quantity based on both the Itô and Stratonovich stochastic integrals. It is shown that the Itô equation possesses a unique solution, whereas the Stratonovich one possesses infinitely many, all but one absent of physical meaning. We discuss how this fact matches with the existent discussion on the Itô vs Stratonovich dilemma and the apparent preference toward the Stratonovich interpretation in the physical literature.

KEYWORDS
Itô vs Stratonovich dilemma, multiplicity of solutions, stochastic differential equations, uniqueness of solution

1 | INTRODUCTION

The position of the Langevin particle obeys the stochastic differential equation

\[ m \frac{d^2 X_t}{dt^2} = -\gamma \frac{dX_t}{dt} + \sigma \xi_t, \]

\[ X_t|_{t=0} = X_0, \]

\[ \frac{dX_t}{dt}|_{t=0} = V_0, \]

where \( \xi_t \) is Gaussian white noise and \( m, \gamma, \sigma > 0 \). Clearly, this is Newton second law for a particle subjected to both viscous damping and a random force. It is a classical model for the random
dispersal of a particle,\textsuperscript{1,2} that can be considered as a refined version of Brownian Motion, and hence the alternative name \textit{Physical Brownian Motion}~\textsuperscript{3}.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a filtered probability space completed with the $\mathbb{P}$–null sets in which a Wiener Process $\{W_t\}_{t \geq 0}$ is defined; moreover assume $\mathcal{F}_t \supset \sigma(\{W_s, 0 \leq s \leq t\})$. This equation can be written in the precise manner

\begin{equation}
m \, dV_t = -\gamma V_t \, dt + \sigma \, dW_t, \tag{1}
\end{equation}

\[
\frac{dX_t}{dt} = V_t, \\
V_t|_{t=0} = V_0, \\
X_t|_{t=0} = X_0,
\]

where $X_0, V_0 \in L^2(\Omega)$ are $\mathcal{F}_0$–measurable random variables. Obviously, $V_t$ is the velocity of the Langevin particle. It is straightforward to check that the classical theorem of existence and uniqueness of solution of stochastic differential equations applies to this model.\textsuperscript{4,5}

If we formally take the limit $m \searrow 0$, we arrive at the model

\[
\gamma \frac{dX_t}{dt} = \sigma \xi_t, \\
X_t|_{t=0} = X_0,
\]

which can be translated to the precise version

\[
dX_t = \frac{\sigma}{\gamma} dW_t, \\
X_t|_{t=0} = X_0,
\]

its solution reads

\[
X_t = X_0 + \frac{\sigma}{\gamma} W_t.
\]

Clearly, its derivative is not well defined, at least as a (function-valued) stochastic process and, moreover, for any $\Delta t > 0$, we find

\[
\frac{X_{t+\Delta t} - X_t}{\Delta t} = \frac{\sigma W_{t+\Delta t} - W_t}{\Delta t} \Rightarrow
\]

\[
\mathbb{E}\left[\left(\frac{X_{t+\Delta t} - X_t}{\Delta t}\right)^2\right] = \frac{\sigma^2}{\gamma^2} \mathbb{E}[\left(W_{t+\Delta t} - W_t\right)^2] = \frac{\sigma^2}{\gamma^2} \frac{1}{\Delta t} \xrightarrow{\Delta t \searrow 0} \infty,
\]
so the mean kinetic energy of the Brownian model is not well defined. In the next section, we will show how this deficiency of the Brownian model can be solved with the Langevin model.

2 | COMPUTATION OF THE KINETIC ENERGY

The kinetic energy of the Langevin particle is

$$K_t = \frac{1}{2} m V_t^2.$$  

To compute it, we can simple solve for $V_t$ to find

$$V_t = e^{-(\gamma/m)t} V_0 + \frac{\sigma}{m} \int_0^t e^{(\gamma/m)(s-t)} dW_s,$$

and therefore

$$K_t = \frac{1}{2} m \left[ e^{-(\gamma/m)t} V_0 + \frac{\sigma}{m} \int_0^t e^{(\gamma/m)(s-t)} dW_s \right]^2. \quad (2)$$

We can compute its mean value

$$\mathbb{E}(K_t) = \frac{1}{2} m \mathbb{E} \left\{ \left[ e^{-(\gamma/m)t} V_0 + \frac{\sigma}{m} \int_0^t e^{(\gamma/m)(s-t)} dW_s \right]^2 \right\}$$

$$= \frac{1}{2} m \mathbb{E} \left( e^{-2(\gamma/m)t} V_0^2 \right)$$

$$+ \sigma \mathbb{E} \left[ e^{-(\gamma/m)t} V_0 \int_0^t e^{(\gamma/m)(s-t)} dW_s \right]$$

$$+ \frac{1}{2} \frac{\sigma^2}{m} \mathbb{E} \left\{ \left[ \int_0^t e^{(\gamma/m)(s-t)} dW_s \right]^2 \right\}$$

$$= \frac{1}{2} m e^{-2(\gamma/m)t} \mathbb{E}(V_0^2)$$

$$+ \sigma e^{-(\gamma/m)t} \mathbb{E}(V_0) \mathbb{E} \left[ \int_0^t e^{(\gamma/m)(s-t)} dW_s \right]$$

$$+ \frac{1}{2} \frac{\sigma^2}{m} \int_0^t e^{2(\gamma/m)(s-t)} ds$$

$$= \frac{1}{2} m e^{-2(\gamma/m)t} \mathbb{E}(V_0^2) + \frac{\sigma^2}{4\gamma} \left[ 1 - e^{-2(\gamma/m)t} \right]$$
\[
\lim_{t \to \infty} \frac{\sigma^2}{4\gamma} > 0,
\]

where we have used, in the third equality, the independence of \( V_0 \) and the Wiener integral (by assumption on \( V_0 \)), and the Itô isometry, while the zero mean property of the Wiener integral has been used in the fourth.

Also note that

\[
\lim_{m \to 0} \mathbb{E}(K_t) = \frac{\sigma^2}{4\gamma},
\]

so the vanishing mass limit of the Langevin model allows to define a value of the mean kinetic energy for the Brownian model. Moreover, this value coincides with the long time limit of the mean kinetic energy of the Langevin particle. This coincidence matches well with previous analyses that showed the agreement of the long time limit of the Langevin model and the Brownian one with respect to the dispersal of the trajectories.

3 | DIRECT COMPUTATION

One could instead directly compute the stochastic differential equation obeyed by the stochastic process \( K_t \). But this equation cannot be interpreted samplewise as (1) and requires to incorporate a notion of stochastic integration (different from the Wiener one); herein we consider the stochastic integrals of \( \text{Itô} \)\(^7,8 \) and \( \text{Stratonovich} \)\(^9 \) as has been traditionally done in the physical literature, and what coincides with the historical development. From now on let us assume \( V_0 \in L^4(\Omega) \) is a \( \mathcal{F}_0 \)-measurable random variable. Using Itô calculus and following\(^{10} \) one arrives at

\[
dK_t = \frac{\sigma^2}{2m} dt - 2\frac{\gamma}{m} K_t dt + \sqrt{2 \sigma^2 m} K_t dW_t, \quad K_t \vert_{t=0} = \frac{1}{2} m V_0^2, \tag{3}
\]

alternatively, using Stratonovich calculus and following Ref.\(^{10} \), one gets

\[
dK_t = -2\frac{\gamma}{m} K_t dt + \sqrt{2 \sigma^2 m} K_t \circ dW_t, \quad K_t \vert_{t=0} = \frac{1}{2} m V_0^2. \tag{4}
\]

\textbf{Remark 1}. There is a well-known formula that connects Itô and Stratonovich stochastic differential equations by means of a drift redefinition.\(^{4,5} \) If one formally applied this formula to the present situation, one would be tempted to conclude that Equations (3) and (4) are equivalent. However, this formula requires the continuous differentiability of the diffusion of the given stochastic differential equations, a requirement that is not fulfilled by the square root diffusions of the present models. The lack of validity of this formula in such situations was demonstrated in Ref.\(^{11} \) and in the following it will become apparent again that it may lead to the creation or destruction of solutions when applied out of its range of validity.

We note that Equation (3) possesses a unique solution, which is both strong and global, by the Wanatabe-Yamada theorem;\(^{12} \) however, this theorem is not applicable to Equation (4) or, in general, to stochastic differential equations in Stratonovich form.\(^{11} \) It is a simple exercise of stochastic
calculus to check that formula (2) solves both Equations (3) and (4). On the other hand, consider Equation (4) subject to the initial condition

\[ K_t|_{t=0} = 0; \tag{5} \]

it is clear that \( K_t = 0 \) is an absorbing state for this equation and, given this initial condition, it is a global solution to it too. Nevertheless, it is not an absorbing state for Equation (3), which we remind possesses a unique solution. Clearly, the stochastic differential equation (4) subject to the initial condition (5) possesses at least two solutions: \( K_t = 0 \) and (2). Actually, it is easy to combine both to get the family of solutions

\[
K_t = \frac{\sigma^2}{2m} \left[ \int_{\lambda}^t e^{(\gamma/m)(s-t)} \, dW_s \right]^2 1_{t > \lambda},
\]

where \( \lambda \geq 0 \) is an arbitrary parameter; that is, Equation (4) subject to (5) admits an uncountable number of solutions. This fact, apparently, remained unseen before.\(^{10,13}\)

Now we can focus again on the general case (4). If we choose an \( \omega \in \Omega \) such that \( V_0(\omega) = 0 \), then the problem reduces to the previous case; so we consider instead those samples \( \omega \in \Omega \) such that \( V_0(\omega) \neq 0 \) and consequently \( V_0(\omega)^2 > 0 \). For such an \( \omega \), Equation (4) possesses a unique solution up to some stopping time \( T(\omega) \) that is positive almost surely; for such a time interval the solution is given by (2). Given that this equation falls under the assumptions of the classical existence and uniqueness theorem\(^{4,5}\) while \( K_t > 0 \), we conclude

\[ T(\omega) = \inf \{ t > 0 : K_t = 0 \} =: T_1(\omega). \]

Now we can construct at least the following family of solutions to (4):

\[
K_t = \frac{1}{2m} \left[ e^{-(\gamma/m)t} V_0 + \frac{\sigma}{m} \int_0^t e^{(\gamma/m)(s-t)} \, dW_s \right]^2 1_{t < T_1(\omega)} + \frac{\sigma^2}{2m} \left[ \int_{\lambda + T_1(\omega)}^t e^{(\gamma/m)(s-t)} \, dW_s \right]^2 1_{t > \lambda + T_1(\omega)},
\]

where \( \lambda \geq 0 \) is arbitrary.

Additionally define recursively the family of stopping times

\[ T_n := \inf \{ t > T_{n-1} + \lambda_{n-1} + \tau_{n-1} : K_t = 0 \}, \quad n = 2, 3, \ldots, \]

where \( \{\tau_n\}_{n=1}^{\infty} \) is an arbitrary sequence of almost surely positive, \( L^0(\Omega) \), and \( T_{T_n(\omega)} \)—measurable random variables. This allows to extend the family of solutions to

\[
K_t = \frac{1}{2m} \left[ e^{-(\gamma/m)t} V_0 + \frac{\sigma}{m} \int_0^t e^{(\gamma/m)(s-t)} \, dW_s \right]^2 1_{t < T_1(\omega)} + \frac{\sigma^2}{2m} \left[ \int_{\lambda + T_1(\omega)}^t e^{(\gamma/m)(s-t)} \, dW_s \right]^2 1_{t > \lambda + T_1(\omega)},
\]
\[
\begin{align*}
+ \frac{\sigma^2}{2m} & \left[ \int_{\lambda_1 + T_1(\omega)}^t e^{(\gamma/m)(s-t)} \, dW_s \right]^2 1_{\lambda_1 + T_1(\omega) < t < T_2(\omega)} \\
+ \frac{\sigma^2}{2m} & \left[ \int_{\lambda_2 + T_2(\omega)}^t e^{(\gamma/m)(s-t)} \, dW_s \right]^2 1_{\lambda_2 + T_2(\omega) < t < T_3(\omega)} \\
+ \cdots & \\
+ \frac{\sigma^2}{2m} & \left[ \int_{\lambda_n + T_n(\omega)}^t e^{(\gamma/m)(s-t)} \, dW_s \right]^2 1_{\lambda_n + T_n(\omega) < t < T_{n+1}(\omega)} \\
+ \cdots,
\end{align*}
\]

where \( \{\lambda_n\}_{n=1}^{\infty} \) is an arbitrary sequence of nonnegative real numbers.

Finally, we can build yet another extension of our set of solutions to

\[
K_t = \frac{1}{2m} \left[ e^{-(\gamma/m)t} V_0 + \frac{\sigma}{m} \int_0^t e^{(\gamma/m)(s-t)} \, dW_s \right]^2 1_{t < T_1(\omega)}
\]

\[
+ \sum_{n=1}^{\infty} \frac{\sigma^2}{2m} \left[ \int_{\lambda_n + T_n(\omega)}^t e^{(\gamma/m)(s-t)} \, dW_s \right]^2 1_{\lambda_n + T_n(\omega) < t < T_{n+1}(\omega)},
\]

if \( V_0(\omega) \neq 0 \), for any sequence \( \{\mu_n\}_{n=1}^{\infty} \) of almost surely nonnegative, \( L^0(\Omega) \), and \( F_{T_n(\omega)} \)-measurable random variables, and

\[
K_t = \frac{1}{2m} \left[ e^{-(\gamma/m)t} V_0 + \frac{\sigma}{m} \int_0^t e^{(\gamma/m)(s-t)} \, dW_s \right]^2 1_{t < T'_1(\omega)}
\]

\[
+ \sum_{n=1}^{N} \frac{\sigma^2}{2m} \left[ \int_{\mu_n + T'_n(\omega)}^t e^{(\gamma/m)(s-t)} \, dW_s \right]^2 1_{\mu_n + T'_n(\omega) < t < T'_{n+1}(\omega)},
\]

if \( V_0(\omega) \neq 0 \), for any finite sequence \( \{\mu_n\}_{n=1}^{N}, N = 1, 2, \ldots, \) of almost surely nonnegative, \( L^0(\Omega) \), and \( F_{T'_n(\omega)} \)-measurable random variables, where

\[
T'_1 := T_1, \quad T'_n := \inf \{t > T'_{n-1} + \mu_{n-1} + \tau'_{n-1} : K_t = 0 \}, \quad n = 2, 3, \ldots, N + 1,
\]

where \( \{\tau'_n\}_{n=1}^{N} \) is an arbitrary finite sequence of almost surely positive, \( L^0(\Omega) \), and \( F_{T'_n(\omega)} \)-measurable random variables. The solution

\[
K_t = \frac{1}{2m} \left[ e^{-(\gamma/m)t} V_0 + \frac{\sigma}{m} \int_0^t e^{(\gamma/m)(s-t)} \, dW_s \right]^2 1_{t < T_1(\omega)}.
\]
is also acceptable again if \( V_0(\omega) \neq 0 \). If \( V_0(\omega) = 0 \), we have the solutions:

\[
K_t = \frac{\sigma^2}{2m} \left[ \int_{\lambda_0}^{t} e^{(\gamma/m)(s-t)} \, dW_s \right]^2 \mathbb{1}_{\lambda_0 < t < S_1(\omega)} + \sum_{n=1}^{\infty} \frac{\sigma^2}{2m} \left[ \int_{\lambda_n + S_n(\omega)}^{t} e^{(\gamma/m)(s-t)} \, dW_s \right]^2 \mathbb{1}_{\lambda_n + S_n(\omega) < t < S_{n+1}(\omega)},
\]

and

\[
K_t = \frac{\sigma^2}{2m} \left[ \int_{\mu_0}^{t} e^{(\gamma/m)(s-t)} \, dW_s \right]^2 \mathbb{1}_{\mu_0 < t < S'_1(\omega)} + \sum_{n=1}^{N} \frac{\sigma^2}{2m} \left[ \int_{\mu_n + S'_n(\omega)}^{t} e^{(\gamma/m)(s-t)} \, dW_s \right]^2 \mathbb{1}_{\mu_n + S'_n(\omega) < t < S'_{n+1}(\omega)},
\]

for any sequences (finite the second) \( \{\lambda_n\}_{n=0}^{\infty} \) and \( \{\mu_n\}_{n=0}^{N} \) \( (N = 1, 2, \ldots) \) of nonnegative, \( L^0(\Omega) \), and, respectively, \( F_{S_n(\omega)} \)-measurable and \( F_{S'_n(\omega)} \)-measurable random variables, and where

\[
S_n := \inf\{t > S_{n-1} + \lambda_{n-1} + \tau_{n-1} : K_t = 0\}, \quad n = 1, 2, \ldots,
\]

\[
S'_n := \inf\{t > S'_{n-1} + \mu_{n-1} + \tau'_{n-1} : K_t = 0\}, \quad n = 1, 2, \ldots, N + 1,
\]

with \( S_0 = \lambda_{-1} + \tau_{-1} \) and \( S'_0 = \mu_{-1} + \tau'_{-1} \), where \( \{\lambda_n\}_{n=0}^{\infty} \) and \( \{\mu_n\}_{n=0}^{N} \) are arbitrary sequences (finite sequence the latter) of almost surely positive, \( L^0(\Omega) \), and, respectively, \( F_{S_n(\omega)} \)-measurable and \( F_{S'_n(\omega)} \)-measurable random variables; also, \( \lambda_{-1} \) and \( \mu_{-1} \) are two arbitrary almost surely nonnegative, \( L^0(\Omega) \), and \( F_0 \)-measurable random variables, and \( \tau_{-1} \) and \( \tau'_{-1} \) are almost surely positive, \( L^0(\Omega) \), and, respectively, \( F_{\lambda_{-1}} \)-measurable and \( F_{\mu_{-1}} \)-measurable random variables. Yet the solution \( K_t \equiv 0 \) is also acceptable.

### 4 TIME SCALE OF THE SPURIOUS SOLUTIONS

In this section, we show that the appearance of the spurious solutions has a well-defined time scale. As we are discussing the mathematical properties of a physical model, it is important to establish the observability of these solutions.

**Theorem 1.** The mean first passage time to zero as a function of the initial kinetic energy \( K \) is given by the formula:

\[
\mathcal{F}(K) = \frac{m \sqrt{\pi}}{\gamma} \exp \left( \frac{2\gamma}{\sigma^2} K \right) D_+ \left( \frac{\sqrt{2}\gamma}{\sigma} K^{1/2} \right) - \frac{2m}{\gamma} \int_{0}^{\sqrt{2}\gamma K^{1/2}} D_-(w) \, dw,
\]
where the Dawson integrals

\[ D_+(\cdot) := e^{-(\cdot)^2} \int_0^{(\cdot)} e^{u^2} du, \]
\[ D_-(\cdot) := e^{(\cdot)^2} \int_0^{(\cdot)} e^{-u^2} du. \]

Proof. Since

\[ K_t = \frac{1}{2} m V_t^2, \tag{9} \]

it is clear that \( K_t = 0 \Leftrightarrow V_t = 0 \), and \( K_t > 0 \Leftrightarrow V_t \neq 0 \). Then the kinetic energy and the velocity become zero simultaneously, and therefore it suffices to study the stopping time

\[ \mathcal{I} := \inf\{ t > 0 : V_t = 0 \}. \]

As \( V_t \) obeys the Ornstein-Uhlenbeck stochastic differential equation

\[ m \, dV_t = -\gamma V_t \, dt + \sigma \, dW_t, \]

then \( \mathcal{I}_M = \mathcal{I}_M(v) \) solves

\[ \frac{\sigma^2}{2m} \frac{d^2 \mathcal{I}_M}{dv^2} - \gamma v \frac{d \mathcal{I}_M}{dv} = -m, \]

subjected to \( \mathcal{I}_M(0) = \frac{\partial}{\partial v} \mathcal{I}_M(M) = 0 \) and either \( v \in [0, M] \) or \( v \in [M, 0] \) depending on the sign of \( M \), with \( M \neq 0 \). The solution reads

\[ \mathcal{I}_M(v) = \frac{2m}{\gamma} \int_0^{\sqrt{\frac{\gamma}{m}} |v|} \left[ \exp \left( w^2 - \frac{\gamma v^2}{m} M^2 \right) D_\left( \frac{\sqrt{\gamma m v}}{m} |M| \right) - D_-(w) \right] dw. \]

Now

\[ \mathcal{I}(v) = \lim_{M \to \pm \infty} \mathcal{I}_M(v) \]

\[ = \lim_{M \to \pm \infty} \frac{2m}{\gamma} \int_0^{\sqrt{\frac{\gamma}{m}} |v|} \left[ \exp \left( w^2 - \frac{\gamma v^2}{m} M^2 \right) D_\left( \frac{\sqrt{\gamma m v}}{m} |M| \right) - D_-(w) \right] dw \]

\[ = \lim_{M \to \pm \infty} \frac{m}{\gamma} \left\{ \sqrt{\pi} \left[ 2 \Phi \left( \frac{\sqrt{2 m \gamma v}}{\sigma |M|} \right) - 1 \right] \exp \left( \frac{\gamma v^2}{m} \right) D_+ \left( \frac{\sqrt{\gamma m v}}{m} |v| \right) - \frac{m \gamma v^2}{\sigma^2} 2F_2 \left( 1, 1; 3/2, 2; \frac{m \gamma v^2}{\sigma^2} \right) \right\} \]
\[
\sup_{M \in \mathbb{R}} \frac{m}{\gamma} \left\{ \sqrt{\pi} \left[ \Phi \left( \frac{\sqrt{2m\gamma}}{\sigma} |M| \right) - 1 \right] \exp \left( \frac{m\gamma}{\sigma^2} v^2 \right) D_+ \left( \frac{\sqrt{m\gamma}}{\sigma} |v| \right) \right. \\
- \frac{m\gamma}{\sigma^2} v^2 \mathbf{2}_F \left( 1, 1; 3/2, 2; \frac{m\gamma}{\sigma^2} v^2 \right) \left\} \\
= \frac{m\sqrt{\pi}}{\gamma} \exp \left( \frac{m\gamma}{\sigma^2} v^2 \right) D_+ \left( \frac{\sqrt{m\gamma}}{\sigma} |v| \right) - \frac{m^2\gamma}{\sigma^2 v^2} \mathbf{2}_F \left( 1, 1; 3/2, 2; \frac{m\gamma}{\sigma^2} v^2 \right) \\
= \frac{m\sqrt{\pi}}{\gamma} \exp \left( \frac{m\gamma}{\sigma^2} v^2 \right) D_+ \left( \frac{\sqrt{m\gamma}}{\sigma} |v| \right) - \frac{2m}{\gamma} \int_0^{\sqrt{m\gamma}/|v|} D_- (w) \, dw
\]

for all \( v \in \mathbb{R} \), where

\[
\Phi(\cdot) := \text{Prob}(X \leq \cdot) \quad \text{with} \quad X \sim \mathcal{N}(0,1),
\]

and \( \mathbf{2}_F (\cdot, \cdot; \cdot, \cdot; \cdot) \) is a generalized hypergeometric function.\(^{14}\) The statement follows from changing variables as in (9).

**Corollary 1.** Let the initial kinetic energy of a Langevin particle be positive. Then it becomes zero in finite mean time and, in particular, it becomes zero in finite time almost surely.

**Proof.** This corollary is directly implied by the proof of Theorem 1. \(\square\)

The explicit formula in the statement of Theorem 1 describes the time scale of observability of the spurious solutions as a function of the initial kinetic energy. We note that this quantity is not just always finite, as stated in Corollary 1, but it can also be arbitrarily small (depending on the initial condition), as an initial kinetic energy \( K = 0 \) leads to an immediate observation of them. Therefore the unphysical properties of the spurious solutions in Section 3 can be observed in a well-defined time scale, which can be made arbitrarily short by tuning the initial condition.

## 5 | Long Time Behavior of a Class of Spurious Solutions

In this section, we show that the spurious solutions we have constructed are indeed spurious as they do not obey important physical laws. For technical reasons, we limit our analysis to a subclass of the explicit solutions built in Section 3.

**Theorem 2.** Let \( K_t \) be as in (6), (7), (8), or \( K_t \equiv 0 \); moreover assume \( \{\tau'_n\}_{n=1}^N \), \( \{\mu_n\}_{n=1}^N \), \( \{\tau^p_n\}_{n=1}^N \), and \( \{\mu_n\}_{n=-1}^N \) are finite almost surely. Then

\[
\mathbb{E}(K_t) \xrightarrow{t/\infty} 0.
\]
Proof. The case \( K_i \equiv 0 \) is obvious. First note that

\[
K_i \xrightarrow{t/\infty} 0 \quad \text{almost surely}
\]

in all three cases (6), (7), and (8) as a consequence of the assumptions on \( \{\tau_n^N\}_{n=1}^N, \{\mu_n^N\}_{n=1}^N, \{\bar{\tau}_n^N\}_{n=-1}^N, \) and \( \{\bar{\mu}_n^N\}_{n=-1}^N, \) and Corollary 1.

We start proving that all of these families of solutions are bounded in \( L^1(\Omega) \) uniformly in \( t \). Let us begin with (6) by noting

\[
|K_t| = \left| \frac{1}{2} m \left[ e^{-(\gamma/m)t} V_0 + \frac{\sigma}{m} \int_0^t e^{(\gamma/m)(s-t)} dW_s \right]^2 \mathbb{1}_{t<T_n'(\omega)} \right|
\]

\[
+ \sum_{n=1}^N \frac{\sigma^2}{2m} \left[ \int_{\mu_n + T_n'(\omega)} e^{(\gamma/m)(s-t)} dW_s \right]^2 \mathbb{1}_{\mu_n + T_n'(\omega) \leq t < T_{n+1}'(\omega)}
\]

\[
\leq m e^{-2(\gamma/m)t} V_0^2 + \frac{\sigma^2}{m} \left[ \int_0^t e^{(\gamma/m)(s-t)} dW_s \right]^2
\]

\[
+ \sum_{n=1}^N \frac{\sigma^2}{2m} \left[ \int_{\mu_n + T_n'(\omega)} e^{(\gamma/m)(s-t)} dW_s \right]^2 \mathbb{1}_{\mu_n + T_n'(\omega) \leq t},
\]

by Young inequality. Then in consequence

\[
\mathbb{E}(|K_t|) \leq m e^{-2(\gamma/m)t} \mathbb{E}(V_0^2) + \frac{\sigma^2}{m} \mathbb{E} \left\{ \left[ \int_0^t e^{(\gamma/m)(s-t)} dW_s \right]^2 \right\}
\]

\[
+ \sum_{n=1}^N \frac{\sigma^2}{2m} \mathbb{E} \left\{ \left[ \int_{\mu_n + T_n'(\omega)} e^{(\gamma/m)(s-t)} dW_s \right]^2 \mathbb{1}_{\mu_n + T_n'(\omega) \leq t} \right\}
\]

\[
\leq m \mathbb{E}(V_0^2) + \frac{\sigma^2}{m} \int_0^t e^{2(\gamma/m)(s-t)} ds
\]

\[
+ \sum_{n=1}^N \frac{\sigma^2}{2m} \mathbb{E} \left\{ \left[ \int_{\mu_n + T_n'(\omega)} e^{(\gamma/m)(s-t)} dW_s \right]^2 \mathbb{1}_{\mu_n + T_n'(\omega) \leq t} \mathbb{1}_{t < T_{n+1}'(\omega)} \right\}
\]

\[
= m \mathbb{E}(V_0^2) + \frac{\sigma^2}{2\gamma} \left[ 1 - e^{-2(\gamma/m)t} \right]
\]

\[
+ \sum_{n=1}^N \frac{\sigma^2}{2m} \left\{ \int_{\mu_n + T_n'(\omega)} e^{2(\gamma/m)(s-t)} ds \mathbb{1}_{\mu_n + T_n'(\omega) \leq t} \right\}.
\]
\[
\begin{align*}
\leq m\mathbb{E}(V_0^2) + \frac{\sigma^2}{2\gamma} \\
+ \sum_{n=1}^{N} \frac{\sigma^2}{4\gamma} \mathbb{E}\left\{ \left[ 1 - e^{2(\gamma/m)(\mu_n + T_n'(\omega) - t)} \right] \mathbb{1}_{\mu_n + T_n'(\omega) < t} \right\} \\
\leq m\mathbb{E}(V_0^2) + \frac{\sigma^2}{2\gamma} \left( 1 + \frac{N}{2} \right) \\
< \infty,
\end{align*}
\]
by the Itô isometry and the tower property of conditional expectation. Arguing analogously for (7) we find

\[
\mathbb{E}(|K_t|) = \mathbb{E}\left\{ \frac{1}{2} m \left[ e^{-(\gamma/m)t} V_0 + \frac{\sigma}{m} \int_0^t e^{(\gamma/m)(s-t)} \, dW_s \right]^2 \mathbb{1}_{t < T_1(\omega)} \right\} \\
\leq \mathbb{E}\left\{ me^{-(\gamma/m)t} V_0^2 + \frac{\sigma^2}{m} \left[ \int_0^t e^{(\gamma/m)(s-t)} \, dW_s \right]^2 \right\} \\
= me^{-(\gamma/m)t} \mathbb{E}(V_0^2) + \frac{\sigma^2}{m} \int_0^t e^{2(\gamma/m)(s-t)} \, ds \\
\leq m\mathbb{E}(V_0^2) + \frac{\sigma^2}{2\gamma} \left[ 1 - e^{-(\gamma/m)t} \right] \\
\leq m\mathbb{E}(V_0^2) + \frac{\sigma^2}{2\gamma} \\
< \infty.
\]
And for (8) we get

\[
\mathbb{E}(|K_t|) = \frac{\sigma^2}{2m} \mathbb{E}\left\{ \left[ \int_{\bar{\mu}_0}^t e^{(\gamma/m)(s-t)} \, dW_s \right]^2 \mathbb{1}_{\bar{\mu}_0 < t < S'_1(\omega)} \right\} \\
+ \sum_{n=1}^{N} \frac{\sigma^2}{2m} \mathbb{E}\left\{ \left[ \int_{\bar{\mu}_n + S'_n(\omega)}^t e^{(\gamma/m)(s-t)} \, dW_s \right]^2 \mathbb{1}_{\bar{\mu}_n + S'_n(\omega) < t < S'_{n+1}(\omega)} \right\} \\
\leq \frac{\sigma^2}{2m} \mathbb{E}\left\{ \left[ \int_{\bar{\mu}_0}^t e^{(\gamma/m)(s-t)} \, dW_s \right]^2 \mathbb{1}_{\bar{\mu}_0 < t} \right\}
\]
\[ + \sum_{n=1}^{N} \frac{\sigma^2}{2m} \mathbb{E} \left\{ \left[ \int_{\bar{\mu}_n + T'_n(\omega)}^{t} e^{(\gamma/m)(s-t)} \, dW_s \right]^2 1_{\bar{\mu}_n + T'_n(\omega) < t} \right\} \]

\[ = \frac{\sigma^2}{2m} \mathbb{E} \left\{ \left[ \int_{\bar{\mu}_0}^{t} e^{(\gamma/m)(s-t)} \, dW_s \right]^2 1_{\bar{\mu}_0 < t} \right\} \]

\[ + \sum_{n=1}^{N} \frac{\sigma^2}{2m} \mathbb{E} \left\{ \left[ \int_{\bar{\mu}_n + S'_n(\omega)}^{t} e^{(\gamma/m)(s-t)} \, dW_s \right]^2 1_{\bar{\mu}_n + S'_n(\omega) < t} \right\} \]

\[ = \frac{\sigma^2}{2m} \mathbb{E} \left\{ \int_{\bar{\mu}_0}^{t} e^{2(\gamma/m)(s-t)} \, ds 1_{\bar{\mu}_0 < t} \right\} \]

\[ + \sum_{n=1}^{N} \frac{\sigma^2}{2m} \mathbb{E} \left\{ \int_{\bar{\mu}_n + S'_n(\omega)}^{t} e^{2(\gamma/m)(s-t)} \, ds 1_{\bar{\mu}_n + S'_n(\omega) < t} \right\} \]

\[ = \frac{\sigma^2}{4\gamma} \mathbb{E} \left\{ \left[ 1 - e^{2(\gamma/m)(\bar{\mu}_0 - t)} \right] 1_{\bar{\mu}_0 < t} \right\} \]

\[ + \sum_{n=1}^{N} \frac{\sigma^2}{4\gamma} \mathbb{E} \left\{ \left[ 1 - e^{2(\gamma/m)(\bar{\mu}_n + S'_n(\omega) - t)} \right] 1_{\bar{\mu}_n + S'_n(\omega) < t} \right\} \]

\[ \leq \frac{\sigma^2}{4\gamma} (N + 1) \]

\[ \leq \sigma^2 (N + 1) \]

\[ < \infty, \]

again by the use of the tower property and the Itô isometry.

For the next step, consider \( \mathcal{E} \in \mathcal{F} \) such that \( \text{Prob}(\mathcal{E}) = \delta > 0 \). In the case of (6), we have

\[ \mathbb{E}( |K_t| 1_{\mathcal{E}}) \leq m e^{-\frac{2(\gamma/m)t}{2}} \mathbb{E}(V_0^2 1_{\mathcal{E}}) + \frac{\sigma^2}{m} \mathbb{E} \left\{ \left[ \int_{0}^{t} e^{(\gamma/m)(s-t)} \, dW_s \right]^2 1_{\mathcal{E}} \right\} \]

\[ + \sum_{n=1}^{N} \frac{\sigma^2}{2m} \mathbb{E} \left\{ \left[ \int_{\bar{\mu}_n + T'_n(\omega)}^{t} e^{(\gamma/m)(s-t)} \, dW_s \right]^2 1_{\bar{\mu}_n + T'_n(\omega) < t} 1_{\mathcal{E}} \right\} \]

\[ \leq m \mathbb{E}(V_0^4)^{1/2} \text{Prob}(\mathcal{E})^{1/2} + \frac{\sigma^2}{m} \mathbb{E} \left\{ \left[ \int_{0}^{t} e^{(\gamma/m)(s-t)} \, dW_s \right]^4 \right\}^{1/2} \text{Prob}(\mathcal{E})^{1/2} \]
\[
\begin{align*}
+ \sum_{n=1}^{N} \frac{\sigma^2}{2m} & \left\{ \left[ \int_{\mu_n + T_n'(\omega)}^{t} e^{(\gamma/m)(s-t)} \, dW_s \right]^4 \right\}^{1/2} \right\} \text{Prob}(\mathcal{E})^{1/2} \\
= m \mathbb{E}\left(V_0^4\right)^{1/2} \delta^{1/2} + \frac{\sqrt{3}\sigma^2}{2\gamma} \left[ 1 - e^{-2(\gamma/m)t} \right] \delta^{1/2} + \sum_{n=1}^{N} \frac{\sigma^2}{2m} \times \\
\mathbb{E} \left\{ \left[ \int_{\mu_n + T_n'(\omega)}^{t} e^{(\gamma/m)(s-t)} \, dW_s \right]^4 \right\}^{1/2} \right\} \delta^{1/2} \\
\leq m \mathbb{E}\left(V_0^4\right)^{1/2} \delta^{1/2} + \frac{\sqrt{3}\sigma^2}{2\gamma} \delta^{1/2} + \sum_{n=1}^{N} \frac{\sqrt{3}\sigma^2}{4\gamma} \times \\
\mathbb{E} \left\{ \left[ 1 - e^{2(\gamma/m)(\mu_n + T_n'(\omega)-t)} \right]^2 \right\}^{1/2} \delta^{1/2} \\
\leq \left[ m \mathbb{E}\left(V_0^4\right)^{1/2} + \frac{3\sigma^2}{2\gamma} \left( \frac{N}{2} + 1 \right) \right] \delta^{1/2} \\
< \varepsilon
\end{align*}
\]

for any \( \varepsilon > 0 \) provided

\[
\delta < \frac{\varepsilon^2}{\left[ m \mathbb{E}\left(V_0^4\right)^{1/2} + \frac{3\sigma^2}{2\gamma} \left( \frac{N}{2} + 1 \right) \right]^2},
\]

where we have used (10), Hölder inequality, the tower property, and the normal distribution of the Wiener integrals

\[
\int_{0}^{t} e^{(\gamma/m)(s-t)} \, dW_s \sim \mathcal{N}\left( 0, \frac{m}{2\gamma} \left[ 1 - e^{-2(\gamma/m)t} \right] \right),
\]

\[
\int_{\mu_n + T_n'(\omega)}^{t} e^{(\gamma/m)(s-t)} \, dW_s \left|_{\mu_n + T_n'(\omega)} \right. \sim \mathcal{N}\left( 0, \frac{m}{2\gamma} \left[ 1 - e^{2(\gamma/m)(\mu_n + T_n'(\omega)-t)} \right] \right).
\]

Consequently, there follows uniform integrability, and by the Vitali convergence theorem we conclude

\[
\lim_{t \to \infty} \mathbb{E}(|K_t|) = \mathbb{E} \left( \lim_{t \to \infty} |K_t| \right) = 0.
\]
We employ an analogous argument for (7)

\[
\mathbb{E}(|K_t| \mathbbm{1}_\varepsilon) \leq \mathbb{E}\left\{ me^{-2(\gamma/m)t}V_0^2 \mathbbm{1}_\varepsilon + \frac{\sigma^2}{m} \left[ \int_0^t e^{(\gamma/m)(s-t)} \, dW_s \right]^2 \mathbbm{1}_\varepsilon \right\}
\]

\[
\leq me^{-2(\gamma/m)t} \mathbb{E}(V_0^4)^{1/2} \Pr(\mathcal{E})^{1/2} + \frac{\sigma^2}{m} \left[ \int_0^t e^{(\gamma/m)(s-t)} \, dW_s \right]^2 \Pr(\mathcal{E})^{1/2}
\]

\[
\leq m\mathbb{E}(V_0^4)^{1/2} \delta^{1/2} + \frac{\sqrt{3}\sigma^2}{2\gamma} \left| 1 - e^{-2(\gamma/m)t} \right| \delta^{1/2}
\]

\[
\leq m\mathbb{E}(V_0^4)^{1/2} + \frac{\sqrt{3}\sigma^2}{2\gamma} \delta^{1/2}
\]

\[
< \varepsilon,
\]

for any \( \varepsilon > 0 \) provided

\[
\delta < \frac{\varepsilon^2}{m\mathbb{E}(V_0^4)^{1/2} + \frac{\sqrt{3}\sigma^2}{2\gamma}}^{1/2}.
\]

By the same convergence theorem, we conclude

\[
\lim_{t \to \infty} \mathbb{E}(|K_t|) = 0.
\]

Finally, in the case of (8), we have the estimate

\[
\mathbb{E}(|K_t| \mathbbm{1}_\varepsilon) \leq \frac{\sigma^2}{2m} \mathbb{E}\left\{ \left[ \int_{\bar{\mu}_0}^t e^{(\gamma/m)(s-t)} \, dW_s \right]^2 \mathbbm{1}_{\bar{\mu}_0 < t} \mathbbm{1}_\varepsilon \right\}
\]

\[
+ \sum_{n=1}^N \frac{\sigma^2}{2m} \mathbb{E}\left\{ \left[ \int_{\bar{\mu}_n + S_n(\omega)}^t e^{(\gamma/m)(s-t)} \, dW_s \right]^2 \mathbbm{1}_{\bar{\mu}_n + S_n(\omega) < t} \mathbbm{1}_\varepsilon \right\}
\]

\[
\leq \frac{\sigma^2}{2m} \left[ \int_{\bar{\mu}_0}^t e^{(\gamma/m)(s-t)} \, dW_s \right]^4 \mathbbm{1}_{\bar{\mu}_0 < t} \Pr(\mathcal{E})^{1/2}
\]
\[
+ \sum_{n=1}^{N} \frac{\sigma^2}{2m} \left( \left[ \int_{\overline{\mu}_n + S'_n(\omega)}^{t} e^{(\gamma/m)(s-t)} \, dW_s \right]^{4} \right)^{1/2} \mathbb{P}(\mathcal{E})^{1/2}
\]

\[
= \frac{\sigma^2}{2m} \left( \int_{\overline{\mu}_0}^{t} e^{(\gamma/m)(s-t)} \, dW_s \right)^{4} \left( \int_{\overline{\mu}_n + S'_n(\omega)}^{t} e^{(\gamma/m)(s-t)} \, dW_s \right)^{4} \delta^{1/2} + \sum_{n=1}^{N} \frac{\sigma^2}{2m} \mathbb{I}_{\overline{\mu}_n + S'_n(\omega) < t}
\]

\[
= \frac{\sqrt{3} \sigma^2}{2m} \left( \int_{\overline{\mu}_0}^{t} e^{2(\gamma/m)(s-t)} \, ds \right)^{2} \left( \int_{\overline{\mu}_n + S'_n(\omega)}^{t} e^{2(\gamma/m)(s-t)} \, ds \right)^{2} \delta^{1/2}
\]

\[
+ \sum_{n=1}^{N} \frac{\sqrt{3} \sigma^2}{2m} \left( \int_{\overline{\mu}_n + S'_n(\omega)}^{t} e^{2(\gamma/m)(s-t)} \, ds \right)^{2} \mathbb{I}_{\overline{\mu}_n + S'_n(\omega) < t} \delta^{1/2}
\]

\[
= \frac{\sqrt{3} \sigma^2}{4\gamma} \left( 1 - e^{2(\gamma/m)(\overline{\mu}_0 - t)} \right)^{2} \mathbb{I}_{\overline{\mu}_0 < t} \delta^{1/2}
\]

\[
+ \sum_{n=1}^{N} \frac{\sqrt{3} \sigma^2}{4\gamma} \left( 1 - e^{2(\gamma/m)(\overline{\mu}_n + S'_n(\omega) - t)} \right)^{2} \mathbb{I}_{\overline{\mu}_n + S'_n(\omega) < t} \delta^{1/2}
\]

\[
\leq \frac{\sqrt{3} \sigma^2}{4\gamma} (N + 1) \delta^{1/2}
\]

\[
< \varepsilon,
\]

for any \( \varepsilon > 0 \) whenever

\[
\delta < \frac{16 \gamma^2 \varepsilon^2}{3 \sigma^4 (N + 1)^2},
\]

where we have employed Hölder inequality, the tower property, and the fact that the Wiener integrals have the Gaussian distribution

\[
\int_{\overline{\mu}_0}^{t} e^{(\gamma/m)(s-t)} \, dW_s \left\{ \overline{\mu}_0 < t \right\} \mathcal{N} \left( 0, \frac{m}{2\gamma} \left[ 1 - e^{2(\gamma/m)(\overline{\mu}_0 - t)} \right] \right),
\]

\[
\int_{\overline{\mu}_n + S'_n(\omega)}^{t} e^{(\gamma/m)(s-t)} \, dW_s \left\{ \overline{\mu}_n + S'_n(\omega) < t \right\} \mathcal{N} \left( 0, \frac{m}{2\gamma} \left[ 1 - e^{2(\gamma/m)(\overline{\mu}_n + S'_n(\omega) - t)} \right] \right).
Again by the Vitali convergence theorem, our conclusion is

\[
\lim_{t \to \infty} \mathbb{E}(|K_t|) = 0.
\]

Note on one hand that the long time behavior of the physical solutions was computed in Section 2, where we found

\[
\lim_{t \to \infty} \mathbb{E}(K_t) = \frac{\sigma^2}{4\gamma}.
\]

On the other hand, the equipartition theorem of classical statistical mechanics\[^{16}\] imposes

\[
\lim_{t \to \infty} \mathbb{E}(K_t) = \frac{k_B T}{2},
\]

where \(k_B\) is Boltzmann constant and \(T\) is the absolute temperature. The agreement between both is mediated by the fluctuation-dissipation relation\[^{6}\]

\[
\sigma^2 = 2k_B T \gamma.
\]

The asymptotic behavior described by Theorem 2, which is uniform in the parameter values, is, however, inconsistent with these. We thus conclude that the long-time behavior of this class of spurious solutions is not physical.

6 | CONCLUSIONS

We can summarize some of our results (precisely, some of the results of section 3) in the following two statements. The first one is concerned with the solutions to the Itô equation for the kinetic energy of the Langevin particle.

**Theorem 3.** The stochastic differential equation

\[
dK_t = \frac{\sigma^2}{2m} dt - 2 \frac{\gamma}{m} K_t dt + \sqrt{2 \frac{\sigma^2}{m}} K_t dW_t, \quad K_{t|t=0} = \frac{1}{2} m V_0^2,
\]

admits the unique solution

\[
K_t = \frac{1}{2} m \left[ e^{-(\gamma/m)t} V_0 + \frac{\sigma}{m} \int_0^t e^{(\gamma/m)(s-t)} dW_s \right]^2.
\]

While the second one refers to the solutions of the Stratonovich equation for the same quantity.
Theorem 4. The stochastic differential equation

\[ dK_t = -\frac{\gamma}{m} K_t \, dt + \sqrt{2 \frac{\sigma^2}{m}} K_t \, dW_t, \quad (K_t |_{t=0} = \frac{1}{2} m V_0^2, \]

admits infinitely many solutions, and the solution set includes (11) along with the family

\[ K_t = \frac{1}{2} m \left[ e^{-(\gamma/m)t} V_0 + \frac{\sigma}{m} \int_0^t e^{(\gamma/m)(s-t)} \, dW_s \right] \mathbb{1}_{t<T_1(\omega)} \]

\[ + \sum_{n=1}^{\infty} \frac{\sigma^2}{2m} \left[ \int_{\lambda_n+T_n(\omega)}^t e^{(\gamma/m)(s-t)} \, dW_s \right] \mathbb{1}_{\lambda_n+T_n(\omega)<t<T_{n+1}(\omega)}, \]

if \( V_0(\omega) \neq 0 \), for any sequence \( \{\lambda_n\}_{n=1}^{\infty} \) of almost surely nonnegative, \( L^0(\Omega) \), and \( \mathcal{F}_{T_n(\omega)} - \text{measurable} \) random variables, where

\[ T_1 := \inf \{ t > 0 : K_t = 0 \}, \quad T_n := \inf \{ t > T_{n-1} + \lambda_{n-1} + \tau_{n-1} : K_t = 0 \}, \quad n = 2, 3, \ldots, \]

with \( \{\tau_n\}_{n=0}^{\infty} \) an arbitrary sequence of almost surely positive, \( L^0(\Omega) \), and \( \mathcal{F}_{T_n(\omega)} - \text{measurable} \) random variables, and

\[ K_t = \frac{\sigma^2}{2m} \left[ \int_{\lambda_0}^t e^{(\gamma/m)(s-t)} \, dW_s \right] \mathbb{1}_{\lambda_0<t<S_1(\omega)} \]

\[ + \sum_{n=1}^{\infty} \frac{\sigma^2}{2m} \left[ \int_{\lambda_n+S_n(\omega)}^t e^{(\gamma/m)(s-t)} \, dW_s \right] \mathbb{1}_{\lambda_n+S_n(\omega)<t<S_{n+1}(\omega)}, \]

if \( V_0(\omega) = 0 \), for any sequence \( \{\lambda_n\}_{n=0}^{\infty} \) of nonnegative, \( L^0(\Omega) \), and \( \mathcal{F}_{S_n(\omega)} - \text{measurable} \) random variables, and where

\[ S_n := \inf \{ t > S_{n-1} + \tilde{\lambda}_{n-1} + \tilde{\tau}_{n-1} : K_t = 0 \}, \quad n = 1, 2, \ldots, \]

with \( S_0 = \tilde{\lambda}_0 + \tilde{\tau}_0 \), where \( \{\tilde{\tau}_n\}_{n=0}^{\infty} \) is an arbitrary sequence of almost surely positive, \( L^0(\Omega) \), and \( \mathcal{F}_{S_n(\omega)} - \text{measurable} \) random variables; also, \( \tilde{\lambda}_0 \) is an arbitrary almost surely nonnegative, \( L^0(\Omega) \), and \( \mathcal{F}_0 - \text{measurable} \) random variable, and \( \tilde{\tau}_0 \) is an almost surely positive, \( L^0(\Omega) \), and \( \mathcal{F}_{\lambda_0} - \text{measurable} \) random variable.

More solutions can be found in Section 3, among which let us focus on

\[ K_t = \frac{1}{2} m \left[ e^{-(\gamma/m)t} V_0 + \frac{\sigma}{m} \int_0^t e^{(\gamma/m)(s-t)} \, dW_s \right] \mathbb{1}_{t<T_1(\omega)} \]
that fulfils both

\[
\lim_{t \to \infty} E(K_t) = 0
\]

by Theorem 2, and

\[
\lim_{t \to \infty} K_t = 0 \quad \text{almost surely}
\]

by Corollary 1. The first mode of convergence shows the impossibility for this solution to replicate the results in Section 2; moreover it is discussed in Section 5 how this long-time behavior is inconsistent with the equipartition of energy and the fluctuation-dissipation relation. The second mode of convergence implies convergence in distribution, and therefore the violation of the Maxwell-Boltzmann distribution of the velocity, which in turn implies that the kinetic energy should be chi-squared distributed. It is then clear that this is a spurious rather than a physical solution. The same results were proven for a class of solutions in Section 5. In general, one can see that all of the solutions, except the unique solution of the Itô equation, imply that the Langevin particle is at rest, with null kinetic energy, during some time intervals, in spite of the presence of nonvanishing thermal fluctuations. Obviously, this is not a physical effect.

The formal stochastic differential equation

\[
dX_t \over dt = f(X_t) + g(X_t) \xi_t,
\]

where \(\xi_t\) is Gaussian white noise, is a \textit{pre-equation} following van Kampen, and it only becomes an actual equation when a suitable notion of stochastic integral is added. If this notion is not provided, then at best this pre-equation would admit multiple solutions, at least one for each possible interpretation of noise. However, the situation for the stochastic differential equation

\[
dK_t = -2 \frac{\gamma}{m} K_t dt + \sqrt{2 \frac{\sigma^2}{m}} K_t dW_t, \quad K_t|_{t=0} = \frac{1}{2} m V_0^2,
\]

is not absolutely different, as it admits infinitely many solutions, only one of which has physical meaning. Consequently, this is not a valid model to describe the kinetic energy of the Langevin particle, at least if some further prescription is not added in order to select the physical solution. Such a situation is not new in finance, where models with multiple solutions have been studied, and the right solution has been selected by the addition of a new requirement, such as the no-arbitrage assumption. In the present case, considering the Stratonovich stochastic differential equation along with the additional prescription to ensure the physical character of the unique (by prescription) solution would be equivalent to directly consider the Itô equation

\[
dK_t = \frac{\sigma^2}{2m} dt - 2 \frac{\gamma}{m} K_t dt + \sqrt{2 \frac{\sigma^2}{m}} K_t dW_t, \quad K_t|_{t=0} = \frac{1}{2} m V_0^2.
\]

In Ref. 10, van Kampen studied the direct computation of the kinetic energy of the Langevin particle using the same Itô and Stratonovich equations that have been considered herein. The discussion was based on a previous reference, which claimed the superiority of the Stratonovich over the Itô interpretation of noise to compute this quantity. On the other hand, van Kampen
claimed the equality of both approaches, but did not consider the spurious solutions. Herein we have observed a certain advantage of the use of the Itô interpretation, as it has not to be supplemented with additional conditions in order to assure the uniqueness of solution.

Overall, van Kampen in Ref. 10 concludes that, from a methodological viewpoint, one can use both the Itô and Stratonovich stochastic differential equations to model physical systems. From a physical viewpoint, however, he prefers the Stratonovich interpretation whenever the fluctuations are external. His conclusions were supported 30 years later in Ref. 18, where the authors claim that the Stratonovich interpretation should be preferred in the case of a continuous physical system. In this work, we have dealt with a continuous physical system influenced by external fluctuations; in fact a system studied in Ref. 10. We have shown that for this system the Stratonovich interpretation presents an infinite set of spurious solutions that are not present in the Itô case. Although this is not a fundamental difficulty, as one can add additional conditions in order to select the right physical solution in the case of the Stratonovich equation, it is a fact that makes somewhat simpler the Itô approach. The conclusions in Ref. 10 and Ref. 18 are useful as general guidelines for the modeler, but some of them have to be taken cum grano salis. Physical modeling is crucial in order to select the right interpretation of noise, but the final selection has to be done problemwise; all in all part of the charm of complex systems is that they rebel against general rules. And just as crucial as physical facts are stochastic analytical facts. In particular, when one chooses the interpretation of noise in a given problem, one should not disregard neither the validity of the Watanabe-Yamada theorem for Itô stochastic differential equations, nor the impossibility to extend it for the Stratonovich ones.

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