DEFORMATIONS OF POISSON ALGEBRAS

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Abstract. We study $\mathbb{Z}_2$-graded Poisson structures defined on $\mathbb{Z}_2$-graded commutative polynomial algebras. In small dimensional cases, we exhibit classifications of such Poisson structures, obtain the associated Poisson $\mathbb{Z}_2$-graded cohomology and in some cases, deformations of these Poisson brackets and $P_\infty$-algebra structures. We highlight differences and analogies between this $\mathbb{Z}_2$-graded context and the non graded context, by studying for example the links between Poisson cohomology and singularities.

1. Introduction

Let $V = \langle x_1, \ldots, x_m, \theta_1, \ldots, \theta_n \rangle$ be an $m|n$-dimensional $\mathbb{Z}_2$-graded vector space, over a field $\mathbb{K}$, which for simplicity, we will assume to be of characteristic zero, and sometimes will assume to be algebraically closed. The polynomial algebra $A = \mathbb{K}[x_1, \ldots, x_m, \theta_1, \ldots, \theta_n]$ is just the symmetric algebra $S(V)$, which is $\mathbb{Z}_2$-graded commutative.

The parity of a monomial $ax_{i_1} \cdots x_{i_l} \theta_{j_1} \cdots \theta_{j_l}$ is $l \mod 2$. A polynomial $f$ is said to be homogeneous if it is a sum of monomials of the same parity, which we call the parity of $f$ and denote by $|f|$. If $f$ and $g$ are homogeneous elements in $A$, then $fg$ is homogeneous of parity $|fg| = |f| + |g|$, and $gf = (-1)^{|f||g|}fg$, where, $(-1)^{|f||g|}$ is a shorthand notation for $(-1)^{|f||g|}$.

Let $\partial_{x_1}, \ldots, \partial_{x_m}, \partial_{\theta_1}, \ldots, \partial_{\theta_n}$ be the dual basis of $V^*$. An element of $V^*$ extends uniquely to a derivation of $A$. Moreover, we can identify the space $\text{Der}(A)$ of $\mathbb{Z}_2$-graded derivations of $A$ with $A \otimes V^*$.

The space $A \otimes \Lambda(V^*)$ is naturally identified with the space $\text{MDer}(A)$ of multiderivations of $A$, that is the linear maps $\Lambda A \to A$ which are derivations in each argument. In fact, if $\alpha = f \partial_{u_1} \cdots \partial_{u_k} \in A \otimes \Lambda^k(V^*)$, 

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where \( f \in A \) and \( u_i \in V^* \), then if \( g_1 \cdots g_k \in \bigwedge^k(A) \), we have

\[
\alpha(g_1 \cdots g_k) = f \sum_{\sigma \in S_k} (-1)^{\sigma} \epsilon(\sigma) (-1)^{\sigma^*} \partial_{u_1}(g_{\sigma(1)}) \cdots \partial_{u_k}(g_{\sigma(k)}).
\]

Here \((-1)^{\sigma}\) is the signature of the permutation \( \sigma \), \( \epsilon(\sigma) \) is a sign depending on \( \sigma \) and \( g_1, \cdots, g_k \), which is determined by

\[
\epsilon(\sigma)g_{\sigma(1)} \cdots g_{\sigma(k)} = g_1 \cdots g_k,
\]

and \((-1)^{\sigma^*}\) is the sign given by

\[
(1) \quad g \cdot \sigma = u_2g_{\sigma(1)} + u_3(g_{\sigma(1)} + g_{\sigma(2)}) + \cdots + u_k(g_{\sigma(1)} + \cdots + g_{\sigma(k-1)}),
\]

which results from applying the tensor \( \partial_{u_2} \otimes \cdots \otimes \partial_{u_k} \) to \( g_{\sigma(1)} \otimes \cdots \otimes g_{\sigma(k)} \).

The bidegree of \( \alpha \) is \((\text{deg}(\alpha), |\alpha|)\) where \( \text{deg}(\alpha) = k - 1 \) is the exterior degree of \( \alpha \), that is, its degree in terms of the \( \mathbb{Z} \)-grading of \( \bigwedge(V^*) \), and \(|\alpha| = |f| + |u_1| + \cdots + |u_k|\) is the parity of \( \alpha \) as a \( \mathbb{Z}_2 \)-graded map. We regard \( \bigwedge A \) as a coalgebra with coproduct \( \Delta \) given by

\[
\Delta(g_1 \cdots g_n) = \sum_{k=0}^n \sum_{\sigma \in \text{Sh}(k,n-k)} (-1)^{\sigma} \epsilon(\sigma) g_{\sigma(1)} \cdots g_{\sigma(k)} \otimes g_{\sigma(k+1)} \cdots g_{\sigma(n)},
\]

where \( \text{Sh}(k,n-k) \) is the set of unshuffles of type \((k,n-k)\), that is, permutations in \( S_n \) which are increasing on \( 1, \cdots, k \) and on \( k+1, \cdots, n \).

The multiderivation \( \alpha \) extends to a bigraded coderivation of \( \bigwedge A \) by

\[
\alpha(g_1 \cdots g_n) = \sum_{\sigma \in \text{Sh}(k,n-k)} (-1)^{\sigma} \epsilon(\sigma) \alpha(g_{\sigma(1)} \cdots g_{\sigma(k)}) g_{\sigma(k+1)} \cdots g_{\sigma(n)}.
\]

The space of multiderivations \( \text{MDer}(A) \) is a Lie subalgebra of the algebra of coderivations on \( \bigwedge A \), under the usual bracket of coderivations. Denote \( C^k = \bigwedge^k(\text{Der}(A)) \), so that \( \text{MDer}(A) = \prod_{k=0}^{\infty} C^k \).

If \( \delta = \delta_1 \cdots \delta_k \in C^k \) and \( \mu = \mu_1 \cdots \mu_l \in C^l \), then

\[
[\delta, \mu] = \sum_{i=1}^k \sum_{j=1}^l (-1)^{i+j+*} [\delta_i, \mu_j] \delta_1 \cdots \hat{\delta}_i \cdots \delta_k \mu_1 \cdots \hat{\mu}_j \cdots \mu_l,
\]

where

\[
* = \delta_i (\delta_1 + \cdots + \delta_{i-1}) + \mu_j (\delta_1 + \cdots + \hat{\delta}_i + \cdots + \delta_k + \mu_1 + \cdots + \mu_{j-1}),
\]

and

\[
[\delta_i, \mu_j] = \delta_i \circ \mu_j - (-1)^{\delta_i \mu_j} \mu_j \circ \delta_i
\]

is the usual bracket of \( \delta_i \) and \( \mu_j \) as derivations of \( A \), which coincides with their bracket as coderivations of \( \bigwedge(A) \). In particular,

\[
[f \partial_u, g \partial_v] = f \partial_u(g) \partial_v + (-1)^{(u+f)(V+g)+1} g \partial_v(f) \partial_u.
\]
As a consequence, we obtain that
\[
[f \partial_{u_1} \cdots \partial_{u_m}, g \partial_{v_1} \cdots \partial_{v_n}] = \sum_{i=1}^{m} (-1)^{m-i+*} f \partial_{u_i} (g) \partial_{u_1} \cdots \partial_{u_{i-1}} \partial_{u_{i+1}} \cdots \partial_{u_m} \partial_{v_1} \cdots \partial_{v_n}
\]
\[
+ \sum_{j=1}^{n} (-1)^{j+**} g \partial_{v_j} (f) \partial_{u_1} \cdots \partial_{u_m} \partial_{v_1} \cdots \partial_{v_{j-1}} \partial_{v_j} \cdots \partial_{v_n}
\]
where
\[
* = u_i (u_1 + \cdots + u_{i-1}) + g (u_1 + \cdots + \hat{u}_i + \cdots + u_m)
\]
\[
** = (f + u_1 + \cdots + u_m) (v_j + g) + v_j (v_1 + \cdots + v_{j-1}).
\]

The bracket above is known as the Schouten bracket, and it equips the \(\mathbb{Z}_2\)-graded multiderivations of \(\mathcal{A}\) with the structure of a \(\mathbb{Z} \times \mathbb{Z}_2\)-graded Lie algebra. This type of grading is unsatisfactory from the point of view of deformation theory, and there is a standard method to convert it to a \(\mathbb{Z}_2\)-grading. To do this, we introduce the modified Schouten bracket, given by
\[
\{\alpha, \beta\} = (-1)^{\deg(\alpha) \cdot |\beta|} [\alpha, \beta]
\]
if \(\alpha\) and \(\beta\) are homogeneous in bidegree. This bracket equips \(C = \text{MDer}(\mathcal{A})\) with the structure of a \(\mathbb{Z}_2\)-graded Lie algebra, rather than bigraded algebra, where the \(\mathbb{Z}_2\)-graded degree of \(\alpha\) becomes \(\deg(\alpha) + |\alpha|\) (mod 2). About this modified bracket, see [10], [5] and [4]. Denote by \(C_e\) and \(C_o\) the subspaces of \(C\) consisting of even and odd multiderivations, respectively, and similarly for \(C^k_e\) and \(C^k_o\).

A codifferential on a \(\mathbb{Z}_2\)-graded Lie algebra is an element \(d \in C_o\) such that \([d, d] = 0\). A \(P_\infty\) structure, or Poisson infinity structure, on an algebra \(\mathcal{A}\) is a codifferential in \(\text{MDer}(\mathcal{A})\) with respect to the modified Schouten bracket. In other words, \(\psi = \psi_1 + \cdots\) where \(\psi^k \in C_o^k\) is a \(P_\infty\)-structure provided that
\[
\sum_{k+l=n+1} \{\psi^k, \psi^l\} = 0, \quad n = 1, \ldots
\]
If \(\psi \in C^2_o\), then we say that \(\psi\) determines a Poisson algebra structure on \(\mathcal{A}\). In fact, if we define a bracket on \(\mathcal{A}\) by \([a, b] = \psi(a, b)\), then \(\psi\) is a Poisson structure on \(\mathcal{A}\) precisely when the following conditions occur:
\[
[a, [b, c]] = [[a, b], c] + (-1)^{ab}[b, [a, c]], \quad \text{The graded Jacobi Identity}
\]
\[
[a, bc] = [a, b][c] + (-1)^{ab}b[a, c], \quad \text{The derivation law}
\]
The formulas above make sense when \(\mathcal{A}\) is any associative algebra. In this paper, we will be studying cases where \(\mathcal{A}\) is a \(\mathbb{Z}_2\)-graded commutative algebra, but it is interesting to note that the definition of a Poisson
algebra given above does not depend on the \( \mathbb{Z}_2 \)-graded commutativity of the associative algebra structure on \( \mathcal{A} \).

Now suppose that \( \varphi \in C^k_e \) is an even (in the bigraded sense) \( k \)-derivation. Then \( \exp(\varphi) : \bigwedge \mathcal{A} \to \bigwedge \mathcal{A} \) is an automorphism of the tensor co-algebra of \( \mathcal{A} \), which induces an automorphism \( \exp(\varphi)^* \) of the coderivations of \( \bigwedge \mathcal{A} \), which is given by the formula

\[
\exp(\varphi)^*(\alpha) = \exp(-\varphi) \circ \alpha \circ \exp(\varphi).
\]

We have \( \exp(\varphi)^* = \exp(-\text{ad}_\varphi) \), where \( \text{ad}_\varphi \) is defined in terms of the modified bracket. In other words, we have

\[
\exp(\varphi)^*(\alpha) = \alpha + \{\alpha, \varphi\} + \frac{1}{2} \{\{\alpha, \varphi\}, \varphi\} + \cdots,
\]

from which it follows that \( \exp(\varphi)^*(\text{MDer}) \subseteq \text{MDer} \). Note that if \( \varphi \in C^1_e \), then \( \exp(\varphi) \) may not be well defined, but if it is, then it is an automorphism of \( \mathcal{A} \). We call such an automorphism a linear automorphism. If \( \varphi \in C^k_e \) for some \( k > 1 \), then \( \exp(\varphi) \), which is always well defined, is called a higher order automorphism.

Every automorphism \( g \) of \( \bigwedge \mathcal{A} \) is of the form \( g = \lambda \circ \prod_{k=2}^{\infty} \exp(\varphi^k) \) where \( \lambda \) is a linear automorphism and \( \varphi^k \in C^k_e \) for \( k > 1 \). Note that there is no problem with convergence of this infinite product. When \( \lambda \in \text{Aut}(\mathcal{A}) \) and \( \varphi^k \in \text{MDer}(\mathcal{A}) \), then we call \( g \) a multi-automorphism of \( \mathcal{A} \). Note that \( g^* = (\prod_{k=1}^{\infty} \exp(-\text{ad}_\varphi)) \circ \lambda^* \), and that \( g^* \) is an automorphism of \( \text{MDer}(\mathcal{A}) \).

If \( \psi \) and \( \tilde{\psi} \) are codifferentials, then we say that \( \psi \sim \tilde{\psi} \) if there is some automorphism \( g \) of \( \bigwedge \mathcal{A} \) such that \( g^*(\psi) = \tilde{\psi} \). In this case, we say that \( \psi \) and \( \tilde{\psi} \) are equivalent codifferentials, or that they give isomorphic \( P_\infty \)-algebra structures on \( \mathcal{A} \). If \( \psi = \psi^k + \text{ho} \) and \( \tilde{\psi} = \tilde{\psi}^l + \text{ho} \), where \( \psi^k \) is the leading term of \( \psi \) (i.e., the first nonvanishing term), and \( \tilde{\psi}^l \) is the leading term of \( \tilde{\psi} \), then if \( \psi \sim \tilde{\psi} \), we must have \( k = l \) and \( \psi^k \) and \( \tilde{\psi}^k \) must be linearly equivalent codifferentials (that is, equivalent by means of a linear automorphism). Note that \( \psi \) and \( \tilde{\psi} \) need not be linearly equivalent.

Suppose that \( \psi = \psi_k + \cdots + \psi_m \) is a codifferential and \( \alpha = \psi_m + \text{ho} \), where \( r \geq 1 \) and \( \alpha \in C_o \). Then \( d = \psi + \alpha \) is a codifferential iff the Maurer-Cartan equation

\[
D(\alpha) + \frac{1}{2} \{\alpha, \alpha\} = 0
\]

is satisfied, where \( D(\varphi) = \{\psi, \varphi\} \) is the coboundary operator associated to \( \psi \). Since \( D^2 = 0 \), we can define the cohomology \( H(\psi) \) determined by \( \psi \) by

\[
H(\psi) = \ker(D) / \text{Im}(D).
\]
The cohomology inherits a natural grading, so we have a decomposition
\[ H(\psi) = H_o(\psi) \oplus H_e(\psi). \]

Now suppose that the leading term of \( \alpha \) has degree \( m + r \), and is
the leading term of a coboundary; \( \text{i.e.}, \alpha + D(\beta) \) has order at least
\( m + r + 1 \). Let us also assume that we can choose \( \beta \) to have order at
least 2. Then \( \exp(-\text{ad}_\beta)(d) = \psi + ho \) where the higher order terms
have degree at least \( m + r + 1 \).

Let \( D_k \) be defined by \( D_k(\phi) = \{\psi_k, \phi\} \). Since \( \{\psi_k, \psi_k\} = 0, D_k^2 = 0 \).
Since the lowest order term in \( D(\alpha) + \frac{1}{2}\{\alpha, \alpha\} \) is \( D_k(\psi_{m+r}) \), it
follows that \( \psi_{m+r} \) is a \( D_k \)-coboundary. If \( \psi_{m+r} \) is a \( D_k \)-coboundary, then
the leading term of \( \alpha \) is automatically the leading term of a \( D \)-coboundary.
In particular, if \( H^n(D_k) = 0 \) for all \( n > m \), then this condition is
automatically satisfied. Note that
\[ H^n(D_k) = \ker(D_k : C^m \to C^{m+k-1}) / \text{Im}(D_k : C^{n-k+1} \to C^n) \]
is well defined because \( D_k \) is given by a codifferential consisting of a
single term.

This may seem an unlikely scenario, but we will use this construction later to compute some concrete examples. We will especially con-
sider small dimensional cases and in these cases, we will study classi-
fications of Poisson structures, associated Poisson \( \mathbb{Z}_2 \)-graded cohomol-
ogy, deformations of these Poisson brackets and \( P_\infty \)-algebra structures.
In non-graded and small dimensional cases, computations of Poisson
(co)homology have been done for example for the case of dimension
two in [2] (in a germified context) and in [9] (in an algebraic context),
for a family of Poisson structures in dimension three in [7] (and for the
formal deformations of these Poisson structures, see [8]), and for the
Sklyanin’s case in dimension four in [3], ...

2. Poisson Structures on a 0|1-Dimensional Polynomial
   Algebra

Poisson structures on a 1|0-dimensional algebra \( \mathbb{K}[x] \) are all trivial,
so not interesting. The same is not true of Poisson structures on a
0|1-dimensional algebra, which arises as the algebra of a supermanifold
over a singleton point.

Let \( \mathcal{A} = \mathbb{K}[\theta] = \mathbb{K} \oplus \mathbb{K}\theta \) be the 0|1-dimensional polynomial algebra.
Let
\[ \varphi^n = \theta \partial_\theta^n \]
\[ \psi^n = \partial_\theta^n \]
Then $C^n = \langle \psi^n \rangle$ and $C^n_i = \langle \varphi^n \rangle$ are the subspaces of $C^n$ of odd and even elements respectively. It is straightforward to show that

\[
[\psi^m, \psi^n] = 0, \\
[\psi^m, \varphi^n] = m\psi^{m+n-1}, \\
[\varphi^m, \varphi^n] = (m-n)\varphi^{m+n-1}.
\]

As a consequence, if $\psi = \psi^1 a_1 + \psi^2 a_2 + \cdots$ is any odd element in MDer, then $[\psi, \psi] = 0$, and thus $\psi$ determines a $P_\infty$-algebra structure on $A$. (It is also possible to include a term $\psi^0 a_0$ from $C_0^0$ in the definition of a $P_\infty$-algebra, but it is less conventional to do so.) One computes that

\[
\exp(c\varphi^1)(a + b\theta) = a + \exp(c) b \theta \\
\exp(c\varphi^1)^*(\psi^n) = \exp(cn) \psi^n \\
\exp(c\varphi^1)^*(\varphi^n) = \exp(c(n-1)) \varphi^n.
\]

Suppose that $\psi = \psi^k a_k + \psi^l a_l + ho$, where $k < l$, is a $P_\infty$-algebra structure of order $k$; in other words, $a_k \neq 0$. Then

\[
\exp(c\varphi^{l-k+1})(\psi) = \psi^k a_k + \psi^l (a_l + kca_k) + ho,
\]

so if we choose $c = -\frac{a_l}{ka_k}$, we can eliminate the $\psi^l$ term. It follows that $\psi \sim \psi^k a_k$. Then, applying a linear equivalence to $\psi^k a_k$, we see that $\psi^k a_k \sim \psi^k$. It follows that up to equivalence, the structures $\psi^k$, for $k = 1 \ldots$ give rise to all $P_\infty$-algebra structures on $A$. (We do not consider $P_\infty$ structures with a nonzero term in $C^0$ in this paper.)

Let $\psi = \psi^k$ be a codifferential on $A$. Then it is easy to see that

\[
H^n(\psi) = \begin{cases} 
\langle \varphi^0, \psi^0 \rangle, & n = 0 \\
\langle \psi^n \rangle, & 0 < n < k \\
0, & n \geq k.
\end{cases}
\]

This gives us another way to see that any codifferential is equivalent to one of the form $\psi^k$, because it is easy to see that if $H^n_0(\psi) = 0$ for all $n > k$, then every extension of a codifferential $\psi$ of degree $k$ to a codifferential of leading term $\psi$ by adding higher order terms must be equivalent to $\psi$.

Moreover, the above calculation shows that the codifferential $\psi^k$ has deformations to all $\psi^n$ for $1 \leq n < k$, and this gives the complete deformation picture on the space of all $P_\infty$-algebra structures on $A$. 
3. POISSON STRUCTURES ON A 1|1-DIMENSIONAL POLYNOMIAL ALGEBRA

We consider structures on $\mathcal{A} = \mathbb{K}[x, \theta]$, the 1|1-dimensional polynomial algebra. Let

$$\psi^k = \psi^k(f_k, g_k) = f_k(x)\theta \partial_x \partial_\theta^{k-1} + g_k(x)\partial_\theta^k$$

$$\varphi^k = \varphi^k(a_k, b_k) = a_k(x)\partial_x \partial_\theta^{k-1} + b_k(x)\theta \partial_\theta^k.$$  

Then

$$C^k_o = \langle \psi^k(f_k, g_k) | f_k, g_k \in \mathbb{K}[x] \rangle$$
$$C^k_e = \langle \varphi^k(a_k, b_k) | a_k, b_k \in \mathbb{K}[x] \rangle.$$  

One can check that

$$[\psi^k, \psi^l] = (-1)^{k-1}((k f_l g_k + l f_k g_l) \partial_x \partial_\theta^{k+l-2} + (f_k g_l + f_l g_k) \theta \partial_\theta^{k+l-1})$$

$$[\psi^k, \varphi^l] = (f_k a'_l - a_l f'_k + (k - l - 1) f_k b_l) \theta \partial_\theta^{k+l+1}$$
$$+ (k g_k b_l - a_l g'_k) \partial_\theta^{k+l-1}$$

$$[\varphi^k, \varphi^l] = (a_k a'_l - a_l a'_k + (k - 1) a_k b_l - (l - 1) a_l b_k) \partial_x \partial_\theta^{k+l-2}$$
$$+ (a_k b'_l - a_l b'_k + (k - l) b_k b_l) \theta \partial_\theta^{k+l-1}.$$  

If $\psi = \sum_{k=1}^\infty \psi^k$, then

$$\{\psi, \psi\} = \sum_{n=1}^{k+l=n+1} (-1)^{(k-1)(l-1)}((k f_l g_k + l f_k g_l) \partial_x \partial_\theta^{k+l-2} + (f_k g_l + f_l g_k) \theta \partial_\theta^{k+l-1}).$$  

It is easy to verify that $\{\psi, \psi\} = 0$ precisely when either all the $f_k$ or all the $g_k$ vanish. We shall call a coderivation with all the $f_k$ vanishing a codifferential of the first kind, and one with vanishing $g_k$ a codifferential of the second kind.

A linear equivalence is given by the exponential of a coderivation of bidegree $(0,0)$, in other words, by an even derivation of the algebra $\mathcal{A}$. In general, an even derivation of $\mathcal{A}$ is given by $\varphi = a(x)\partial_x + c(x)\theta \partial_\theta$. However, it is not hard to see that $\exp(\varphi)$ is well defined only when $a(x) = ax + b$ is linear and $c(x) = c$ is constant. (When we consider formal automorphisms, this restriction will be relaxed.)

We compute

$$\exp(c \theta \partial_\theta)(f(x) + g(x)\theta) = f(x) + \exp(c) g(x)\theta$$
$$\exp((ax + b) \partial_x)(f(x) + g(x)\theta) = f(rx + s) + g(rx + s)\theta$$
where } r = e^{-a} \text{ and } s = \frac{e^{-a} - b}{a}. \text{ (When } a = 0, \text{ we have } r = 1 \text{ and } s = -b.) \text{ It is easy to see that any linear automorphism is given by an exponential of a derivation. Since } \{ c \partial \partial_b, (ax + b) \partial_x \} = 0, \text{ we can factor}
\exp \left( (ax + b) \partial_x + c \partial \partial_b \right) = \exp ((ax + b) \partial_x) \exp (c \partial \partial_b),
\text{so the above calculations are sufficient to determine the form of any linear automorphism.}

3.1. Codifferentials of the first kind. Suppose that } \psi = g_k(x) \partial^k_b \text{ is a nontrivial codifferential of the first kind. Let us compute the action of the coboundary operator } D, \text{ given by } D(\phi) = \{ \psi, \phi \} \text{ on cochains. We can use the } \varphi_l \text{ as a basis of the even cochains, but need to introduce a notation for an odd cochain } \alpha^l, \text{ given by } \alpha^l(s_l, t_l) = s_l(x) \partial_x \partial^l_b + t_l(x) \partial^l_b. \text{ We compute}

\begin{align*}
[\psi^k, \alpha^l] &= (-1)^{-1} k s_l g_k \partial_x \partial_b^{k+l-2} + (-1)^{k-1} s_l g'_k \partial \partial_b^{k+l-1}.
\end{align*}

\text{It follows that an odd cochain } \alpha = c' \alpha_l \text{ is a } D\text{-cocycle precisely when } s_l = 0 \text{ for all } l, \text{ in other words, when } \alpha \text{ is of the first kind. Moreover, we have}

\begin{align*}
[\psi^k, \varphi^l] &= (k g_k b_l - a_l g'_k) \partial^k_b + (-1)^{k-1} s_l g'_k \partial \partial^l_b.
\end{align*}

\text{so if } \varphi = \varphi'(a_l, b_l) \text{ is an even cochain, then the condition for } \varphi \text{ to be a cocycle is}

\begin{align*}
\sum_{k+l=n+1} (-1)^{(k-1)(l-1)}(k g_k b_l - a_l g'_k) = 0 \quad n = 0 \ldots.
\end{align*}

\text{Note that } \varphi^0 = b_0(x) \partial; \text{ i.e., } a_0 = 0. \text{ If } m \text{ is the least integer such that } g_m \neq 0, \text{ then applying the equation above with } n = m - 1, \text{ we obtain that } b_0 = 0 \text{ for any even cocycle } \varphi. \text{ Also, } a_0 = t_0(x), \text{ so that automatically, } \alpha_0 \text{ is always a cocycle for any odd cochain } \alpha. \text{ Accordingly, we have } H^0(\psi) = \mathbb{K}[x].

\text{For a general codifferential } \psi, \text{ the definition of } H^n \text{ is a bit complex, because } D \text{ does not respect degrees of codifferentials. Rather, we have to consider } C(A) = \prod_{k=0}^{\infty} C^k(A) \text{ as a filtered complex, with } FC^n = \prod_{k=n}^{\infty} C^k; \text{ and then it is true that } D : FC^n \rightarrow FC^{n+1}. \text{ Usually, we are interested in computing } H(\psi^k), \text{ where } \psi^k \text{ is the first nonvanishing term in } \psi.

\text{Note that } \psi^k \text{ is itself a codifferential, and that the cohomology } H(\psi^k) \text{ governs extensions of } \psi^k \text{ to a codifferential with higher order terms. The cohomology } H(\psi^k) \text{ has a decomposition in the form}
\[ H = \prod_{n=0}^{\infty} (H^n) \text{ where } H^n = Z^n/B^n \text{ with} \]
\[ Z^n = \ker(D : C^n \to C^{n+k-1}) \]
\[ B^n = \text{Im}(D : C^{n-k+1} \to C^n). \]

Suppose \( \psi = g(x)\partial_y^k \). Let \( h(x) = \gcd(g(x), g'(x)) \). Then \( h \) measures the singularity of the codifferential \( \psi \). We first compute \( H_0^n \), the odd part of the cohomology. Now the odd cocycles \( Z_0^n \) and the odd coboundaries \( B_0^n \) are given by
\[ Z_0^n = \langle t(x)\partial_y^n | t(x) \in \mathbb{K}[x] \rangle \]
\[ B_0^n = \begin{cases} 0 & n < k - 1 \\ \langle kg(x)b(x)\partial_y^{k-1} | b(x) \in \mathbb{K}[x] \rangle & n = k - 1 \\ \langle (kg(x)b(x) - a(x)g'(x))\partial_y^n | a(x), b(x) \in \mathbb{K}[x] \rangle & n \geq k \end{cases} \]
It follows that
\[ B_0^n = \langle m(x)h(x)\partial_y^n | m(x) \in \mathbb{K}[x] \rangle, \quad n \geq k, \]
so we have
\[ H_0^n(\psi) = \begin{cases} \langle \mathbb{K}[x]\partial_y^n \rangle & n < k - 1 \\ \langle \mathbb{K}[x]/(g(x))\partial_y^{k-1} \rangle & n = k - 1 \\ \langle \mathbb{K}[x]/(h(x))\partial_y^n \rangle & n \geq k \end{cases}. \]

For deformation theory, we normally do not include the odd 0-cochains, because the interpretation of \( H_0 \) in terms of equivalence classes of infinitesimal deformations depends on an interpretation of coboundaries as generating infinitesimal automorphisms. If \( \varphi \) is a 0-cochain this interpretation fails. As a consequence, it is natural to interpret \( H_0^{k-1} = Z_0^{k-1} \). From this point of view, we obtain \( H_0^{k-1} = \langle \mathbb{K}[x]\partial_y^{k-1} \rangle \).

To calculate the even part of the cohomology, suppose
\[ \varphi^n = a(x)\partial_x\partial_y^{n-1} + b(x)\theta\partial_y^n \]
is an \( n \)-cocycle. The cocycle condition for \( \varphi^n \) is \( kb(x)g(x) = a(x)g'(x) \). Express \( g(x) = p(x)h(x) \) and \( g'(x) = q(x)h(x) \), so \( p(x) \) and \( q(x) \) are relatively prime. When \( n > 0 \), the cocycle condition reduces to
\[ a(x) = kp(x)m(x) \]
\[ b(x) = q(x)m(x), \]
for an arbitrary \( m(x) \in \mathbb{K}[x] \). Now, if
\[ \alpha_{n-k+1} = s(x)\theta\partial_x\partial_y^{n-k} + t(x)\partial_y^{n-k+1}, \]
then
\[
D(\alpha_{n-k+1}) = (-1)^{k-1}ks(x)g(x)\partial_x \partial^a_{\theta} + (-1)^{k-1}s(x)g'(x)\theta \partial^m_{\theta}.
\]
If we express \(m(x) = u(x)h(x) + r(x)\) where \(\deg(r(x)) < \deg(h(x))\), then we can express
\[
a(x) = kg(x)u(x) + kp(x)r(x) \\
b(x) = g'(x)u(x) + q(x)r(x)
\]
gives a decomposition of \(a(x)\) and \(b(x)\) into terms coming from trivial and nontrivial cocycles. When \(0 < n < k\), there are no \(n\)-coboundaries. When \(n = 0\), we must have \(a(x) = 0\), so the cocycle condition becomes \(b(x) = 0\). Thus there are no even \(0\)-cocycles. Thus we have
\[
H^n_e(\psi) = \begin{cases} 
0 & n = 0 \\
\langle \mathbb{K}[x](kp(x)\partial_x \partial^a_{\theta} + q(x)\theta \partial^m_{\theta}) \rangle & 0 < n < k \\
\langle \mathbb{K}[x]/(h(x))(kp(x)\partial_x \partial^a_{\theta} + q(x)\theta \partial^m_{\theta}) \rangle & n \geq k.
\end{cases}
\]
To understand the moduli space of degree \(k\) codifferentials, we need to examine when two such codifferentials are linearly equivalent. The action of a linear automorphism on \(\psi^k\) is given by We compute
\[
\exp(c\theta \partial_{\theta})^*(g(x)\partial^k_{\theta}) = \exp(ck)g(x)\partial^k_{\theta} \\
\exp((ax + b)\partial_x)^*(g(x)\partial^k_{\theta}) = g(rx + s)\partial^k_{\theta}
\]
where \(r = e^{-a}\) and \(s = \frac{e^{-a}b}{a}\). Therefore \(\tilde{\psi}^k = \tilde{g}(x)\partial^1_{\theta}\) is equivalent to \(\psi^k\) iff \(\tilde{g}(x) = Cg(Ax + B)\) for some constants \(A, B\) and \(C\), where \(A\) and \(C\) don’t vanish.

It follows that the singularity of \(\tilde{\psi}\) is given by \(\tilde{h}(x) = Ch(Ax + B)\). Note that in the absence of a \(c\theta \partial_{\theta}\) term in \(\varphi^1\), we only obtain automorphisms of the form \(g(x) \mapsto g(Ax + B)\). Thus the \(\mathbb{Z}_2\)-grading introduces a new kind of automorphism of the Poisson structures, which we do not see in the nongraded case. Note that the ideal \((\tilde{h}(x))\) generated by the transformed singularity is just \((h(Ax + B)) = \exp((ax + b)\partial_x)(h(x))\), so we don’t see anything new on the ideal level. This says that the nature of the singularity of the codifferential remains unchanged under automorphisms, in other words, equivalent codifferentials have equivalent singularities. There is a natural isomorphism between the quotients \(\mathbb{K}[x]/(h(x))\) and \(\mathbb{K}[x]/(\tilde{h}(x))\), in other words, equivalent codifferentials have the same cohomology.

Now we study some cases where \(\psi = g_1\partial^1_{\theta} + \cdots\) is a sum of several terms.
Example 1. Let $\psi = x^2 \partial_\theta$, so that $h_1(x) = x$. It follows that $H^n_\theta(\psi) = \langle \mathbb{K}[x]/(x) \rangle$ for $n \geq 1$. As a consequence we can extend $\psi$ nontrivially by adding $\psi^k = c \partial_\theta^k$ for any $k > 1$ and any nonzero constant $c$.

For example, let $\psi' = x^2 \partial_\theta + \partial_\theta^2$. Then if $\phi^k = a_k(x) \partial_x \partial_\theta^{k-1} + b_k(x) \partial_\theta^k$, we have

$$\{\psi', \sum \phi^k\} = x^2 b_0 + \sum_{k \geq 0} (x^2 b_{k+1}(x) - 2 x a_{k+1}(x) + (-1)^{k+1} 2 b_k(x)) \partial_\theta^{k+1}.$$ 

From this relation, we see that $B^0_\theta(\psi') = x^2 \mathbb{K}[x]$, $B^1_\theta(\psi') = x \mathbb{K}[x] \partial_\theta$, while $B^k_\theta(\psi') = \mathbb{K}[x] \partial_\theta^k$ for $k > 1$. This last condition follows from the fact that if $k > 1$, then setting $a_k(x) = \frac{1}{2} x b_k(x)$, we obtain that $\{\psi, \phi^k\} = (-1)^{k+1} 2 x a_k(x) \partial_\theta^{k+1}$, so every codifferential of the second kind of degree greater than 1 is a coboundary. This means that $H^k = 0$ for $k > 1$. As a consequence, every extension of $\psi'$ is equivalent to $\psi'$.

On the other hand, suppose that $\psi' = x^2 \partial_\theta + x \partial_\theta^2$. Note that we have added a 2-coboundary term to $\psi$, so it may seem that we ought to obtain something equivalent to $\psi$. In fact, this statement is true up to higher order terms, because if $\phi^2 = \frac{1}{2} \partial_x \partial_\theta$ we apply $\exp(\phi^2 \ast)$ to $\psi'$ we add $-x \partial_\theta^2$ plus higher order terms, so we see that $\psi'$ is equivalent to $\psi$ up to terms of order 3.

Note that we cannot apply the same reasoning to $\psi' = (x^2 + x) \partial_\theta$, because an exponential of a first order term $\phi^1$ contributes an infinite number of terms of the same degree. In fact, we already computed the effect of such an exponential, and it can only change $x^2$ into a polynomial of the form $a(x + b)^2$, where $a$ is a nonzero constant, and therefore we cannot obtain $x^2 + x$ as the coefficient of $\partial_\theta$ term in the exponential of $\psi$.

Example 2. Let $\psi = x^3 \partial_\theta$. Then $H^n_\theta(\psi) = \langle \mathbb{K}[x]/(x^3 \partial_\theta^n) \rangle$ for $n > 1$. Let $\psi' = x^3 \partial_\theta + x \partial_\theta^2$, so that $\psi'$ is a nontrivial extension of $\psi$. One computes that $H^n(\psi') = \langle \mathbb{K}[x]/(x^3 \partial_\theta^n) \rangle$ for $n \geq 2$. Thus we have a nontrivial extension $\psi'' = x^3 \partial_\theta + x \partial_\theta^2 + \partial_\theta^3$. We will show that that $H^n(\psi'') = 0$ for $n \geq 3$, which means that $\psi''$ has no nontrivial extensions. To do this, we will show that $f(x) \partial_\theta^{k+2} + h_0$ is a coboundary if $k \geq 1$ for any $f(x) \in \mathbb{K}[x]$. We want to find $\phi^k$, $\phi^{k+1}$ and $\phi^{k+2}$ such that

$$0 = \{x^3 \partial_\theta, \phi^k\} = (x^3 b_k - 3 x^2 a_k) \partial_\theta^k$$

$$0 = \{x^3 \partial_\theta, \phi^{k+1}\} + \{x \partial_\theta^2, \phi^k\}$$

$$= (x^3 b_{k+1} - 3 x^2 a_{k+1} + (-1)^{k-1} (2 x b_k - a_k)) \partial_\theta^{k+1}$$

$$f(x) \partial_\theta^n = \{x^3 \partial_\theta, \phi^{k+2}\} + \{x \partial_\theta^2, \phi^{k+1}\} + \{\partial_\theta^3, \phi^k\}$$

$$= (x^3 b_{k+2} - 3 x^2 a_{k+2} + (-1)^k (2 x b_{k+1} - a_{k+1}) + 3 b_k) \partial_\theta^{k+2}.$$
The first condition yields \( a_k = x b_k / 3 \). Plugging this into the second relation yields \( b_k = (-1)^{k+2} \left( \frac{2}{3} x^2 b_{k+1} - 3 x a_{k+1} \right) \). Finally, we obtain that

\[
f(x) = x^3 b_{k+2} - 3x^2 a_{k+2} + (-1)^k \left( \frac{9}{2} x^2 + 2x \right) b_{k+1} - \left( \frac{27}{5} x + 1 \right) a_{k+1}.
\]

This equation is solvable for any \( f(x) \), so the result follows.

In general, one can show that if we have a codifferential of the form \( \psi = \psi_{k_1} + \cdots + \psi_{k_n} \), where \( \psi_{k_{m+1}} \) is a nontrivial codifferential in \( H^{m+1} \left( \psi_{k_1} + \cdots + \psi_{k_n} \right) \), then there is an upper bound on \( n \). Thus any codifferential is equivalent to one with a finite number of terms.

The versal deformation of the \( P_\infty \) algebra determined by \( \psi \) coincides with the infinitesimal deformation, because the brackets of odd coderivations of the first kind with each other always vanishes, so there is no obstruction to the deformation. Recall that the odd cocycles are simply the codifferentials of the first kind.

3.2. Codifferentials of the second kind. Let us suppose that \( \psi = f_k(x) \partial_x \partial_\theta^{k-1} \) is a nontrivial codifferential of the second kind. If \( \alpha^l(s_l, t_l) = s_l(x) \partial_x \partial_\theta^{l-1} + t_l(x) \partial_\theta^l \), then

\[
[\psi^k, \alpha^l] = (-1)^{k-1-l} f_k t_l \partial_x \partial_\theta^{k+l-2} + (-1)^{k-1} f_k t_l' \partial_\theta^{k+l-1}.
\]

An odd cochain \( \alpha = c^l \alpha_l \) is a \( D \)-cocycle precisely when \( t_l = 0 \) for all \( l > 0 \), and \( t_0 \) is a constant. in other words, if \( \alpha \) contains no degree zero term then \( \alpha \) is of the second kind. As a consequence, we have \( H_0^\theta = \langle \mathbb{K} \rangle \). Moreover, we have

\[
[\psi^k, \varphi^l] = (f_k a_l' - f_k' a_l + (k - l - 1) f_k b_l) \partial_x \partial_\theta^{k+l-2},
\]

so if \( \varphi = \varphi^l(a_l, b_l) \) is an even cochain, then the condition for \( \varphi \) to be a cocycle is

\[
\sum_{k+l=n+1} (-1)^{(k-1)(l-1)}(f_k a_l' - f_k' a_l + (k - l - 1) f_k b_l) = 0 \quad n = 0, 1, \ldots
\]

If \( m \) is the least integer such that \( f_m \neq 0 \), then applying the equation above with \( n = m - 1 \), we obtain that \( b_0 = 0 \) for any even cocycle \( \varphi \). Thus \( H_e^0 = 0 \) if \( m > 1 \). When \( m = 1 \), then if \( f_k = 0 \) for all \( k > 1 \), in other words, if \( \psi = \psi_1 \), then \( H_e^0 = \mathbb{K}[x] \theta \).

Consider a single term codifferential \( \psi = f(x) \theta \partial_\theta \partial_\theta^{k-1} \). Let \( h(x) = \gcd(f(x), f'(x)) \). Then as in the case of codifferentials of the first kind, we say that \( h \) measures the singularity of the codifferential \( \psi \).
We first compute $H^1_o$, the odd part of the cohomology. Now, if $n \geq 1$, the odd cocycles $Z^n_o$ and the odd coboundaries $B^n_o$ are given by

$$Z^n_o = \langle s(x) \theta \partial_{\theta} \partial_{\theta}^{n-1} | s(x) \in \mathbb{K}[x] \rangle$$

$$B^n_o = \begin{cases} 
0 & n < k - 1 \\
\langle (k - 1) fb \theta \partial_{\theta} \partial_{\theta}^{k-1} \rangle & n = k - 1 \\
\langle (fa' - f'a) \theta \partial_{\theta} \partial_{\theta}^{2k-3} \rangle & n = 2(k - 1) \\
\langle (fa' - f'a + (2(k - 1) - n) fb) \theta \partial_{\theta} \partial_{\theta} \rangle & \text{otherwise}
\end{cases}$$

for $a, b \in \mathbb{K}[x]$. It follows that

$$B^n_o = \langle m(x) h(x) \theta \partial_{\theta} \partial_{\theta}^{n-1} | m(x) \in \mathbb{K}[x] \rangle, \quad n \geq k, \quad n \neq 2k - 2.$$ 

so we have

$$H^n_o(\psi) = \begin{cases} 
\langle \mathbb{K}[x] \theta \partial_{\theta} \partial_{\theta}^{n-1} \rangle & n < k - 1 \\
\langle \mathbb{K}[x]/(f(x)) \theta \partial_{\theta} \partial_{\theta}^{k-2} \rangle & n = k - 1 \\
\langle \mathbb{K}[x]/(h(x)) \theta \partial_{\theta} \partial_{\theta}^{n-1} \rangle & n \geq k, \quad n \neq 2(k - 1)
\end{cases}.$$

As in the case of codifferentials of the first kind, we should omit the coboundaries of 0-cochains, so for deformation purposes we have $H^{k-1} = \langle \mathbb{K}[x] \theta \partial_{\theta} \partial_{\theta}^{k-2} \rangle$.

However, $B^{2(k-1)}$ is not an ideal, unless $f = ax^m$ is a single term polynomial, so the cohomology of codifferentials is very different for codifferentials of the second kind. In fact, if we consider a codifferential of degree $k = 2$, which is the Poisson case, then $2(k - 1) = 2$, which means that $H^2$ is very difficult to describe, and in fact, is infinite dimensional in most cases.

To calculate the even part of the cohomology, suppose

$$\varphi^n = a(x) \partial_{\theta} \partial_{\theta}^{n-1} + b(x) \theta \partial_{\theta}^{n}$$

is an $n$-cocycle. The cocycle condition for $\varphi^n$ is $f(x)(a(x)' + (k - n - 1)b(x) = f'(x)a(x)$. Express $f(x) = p(x) h(x)$ and $f'(x) = q(x) h(x)$, so $p(x)$ and $q(x)$ are relatively prime. When $n \neq k - 1$, the cocycle condition reduces to

$$a(x) = m(x) p(x)$$

$$a'(x) + (k - n - 1)b(x) = m(x) q(x),$$

for an arbitrary $m(x) \in \mathbb{K}[x]$. Now, if

$$\alpha_{n-k+1} = s(x) \theta \partial_{\theta} \partial_{\theta}^{n-k} + t(x) \theta \partial_{\theta}^{n-k+1},$$

then

$$D(\alpha_{n-k+1}) = (-1)^{k-1}(2(k-1)-n)f(x)t(x)\partial_{\theta} \partial_{\theta}^{n-1} + (-1)^{k-1}f(x)t'(x)\theta \partial_{\theta}^{n}.$$
If \( n \neq 2(k-1) \), then we can express \( m(x) = (2(k-1)-n)u(x)h(x)+r(x) \) where \( \deg(r(x)) < \deg(h(x)) \).

\[
    a(x) = (n-k+1)f(x)u(x) + p(x)r(x)
\]

\[
    a'(x) + (k-n-1)b(x) = (n-k+1)f'(x)u(x) + q(x)r(x)
\]
gives a natural decomposition of \( a(x) \) into a part coming from a trivial and a nontrivial cocycle. Now we express

\[
    b(x) = f(x)u'(x) + \frac{1}{k-n-1}(r(x)q(x) - r'(x)p(x) - r(x)p'(x))
\]

which gives the corresponding decomposition of \( b(x) \) into trivial and nontrivial parts. Unlike the case for codifferentials of the first kind, we cannot express the cohomology in terms of products of a single generator. However, there is an isomorphism \( H^n(\psi) \) with \( \mathbb{K}[x]/(h(x)) \) because the nontrivial parts of the decomposition above are determined by \( r(x) \).

4. Poisson structures on a 2|1-dimensional polynomial algebra

Let \( \mathcal{A} = \mathbb{K}[x, y, \theta] \), the 2|1-dimensional polynomial algebra. In this section, we explain what are the conditions for an odd biderivation of \( \mathcal{A} \) to be a Poisson structure. We then give explicit families of Poisson structures that satisfy the property of admitting a nontrivial even or odd Casimir. We then finally completely determine the Poisson cohomology associated to one of these families of Poisson structure. To simplify the notations, we will often denote \( \mathbb{K}[x, y] \) by \( \mathcal{A}' \).

**Remark 1.** Let us first consider the de Rham complex, associated to the algebra \( \mathcal{A}' = \mathbb{K}[x, y] \). We denote by

\[
    \Omega^1(\mathcal{A}') = \{ f(x, y) \, d\, x + g(x, y) \, d\, y \mid (f, g) \in \mathcal{A}'^2 \}
\]

and by

\[
    \Omega^2(\mathcal{A}') = \{ f(x, y) \, d\, x \wedge d\, y \mid f \in \mathcal{A}' \}
\]

the spaces of 1 and 2-(Kähler) forms of the algebra \( \mathcal{A}' \). For \( (f, g) \in \mathcal{A}'^2 \), we have the de Rham differential \( d \) defined by:

\[
    d \, f = f_x \, d\, x + f_y \, d\, y,
\]

\[
    d(f \, d\, x + g \, d\, y) = d(f) \wedge d\, x + d(g) \wedge d\, y = (g_x - f_y) \, d\, x \wedge d\, y,
\]

\[
    d(f \, d\, x \wedge d\, y) = 0.
\]
We identify an element of $\Omega^1(A')$ with an element of $A'^2$ and an element of $\Omega^2(A')$ with an element of $A'$, by the following maps:

$$\begin{align*}
\Omega^1(A') & \to A'^2 \\
f \, dx + g \, dy & \mapsto (f, g)
\end{align*}$$

and

$$\begin{align*}
\Omega^2(A') & \to A' \\
f \, dx \wedge dy & \mapsto f.
\end{align*}$$

Then, using these identifications, we can write the de Rham complex as follows:

$$(2) \quad \{0\} \to A' \xrightarrow{\nabla} A'^2 \xrightarrow{\text{Div}} A'$$

where the gradient and divergence operators, $\nabla$ and $\text{Div}$, are defined as follows:

$$\nabla : A' \to A'^2, \quad f \mapsto \left( \frac{f_x}{f_y} \right), \quad \text{Div} : A'^2 \to A', \quad (f, g) \mapsto g_x - f_y.$$

Notice that for every $f \in A'$, we have the following identity:

$$\text{Div}(\nabla(f)) = 0.$$

Moreover, the de Rham complex (2) is exact. Indeed, if $f \in A'$ satisfies $\nabla f = (0)$, then of course, $f \in \mathbb{K}$. Next, let us assume that $(\frac{f}{g}) \in A'^2$ satisfies $\text{Div}((\frac{f}{g})) = 0$, this means that $g_x = f_y$. It suffices to show the result for $f$ and $g$, two homogeneous polynomials of the same degree $n \in \mathbb{N}$. Let $h = \frac{1}{n+1}(xf + yg)$. We then have

$$\nabla h = \frac{1}{n+1} \left( f + x f_x + y g_x \right) = \frac{1}{n+1} \left( f + x f_x + y f_y \right) = \left( \frac{g}{f} \right),$$

where we have used the Euler’s formula $xf_x + yf_y = nf$. This proves that the complex (2) is exact.

In the following, we will also use the cross product $\times : A'^2 \to A'$ given, for $(\frac{f}{g}, \frac{h}{k}) \in A'^2$, by:

$$(\frac{f}{g}) \times \left( \frac{h}{k} \right) = fk - gh.$$

An odd biderivation $\psi$ must be of the form

$$\psi = f(x, y) \partial_x \partial_y + g(x, y) \theta \partial_x \partial_\theta + h(x, y) \theta \partial_y \partial_\theta + k(x, y) \partial_\theta^2.$$

We have

$$\frac{1}{2} [\psi, \psi] = -( f g_x + f_x g - f h_y + f_y h) \theta \partial_x \partial_y \partial_\theta + (f k_y - 2 k g) \partial_x \partial_\theta^2$$

$$- (f k_x + 2 h k) \partial_y \partial_\theta^2 - (g k_x + h k_y) \partial_\theta^3.$$
The codifferential condition \([\psi, \psi] = 0\) is equivalent to the three conditions
\[
\begin{aligned}
&f \nabla k + 2k \left( \frac{h}{g} \right) = 0, \\
&\left( \frac{h}{g} \right) \times \nabla k = 0, \\
&-\left( \frac{h}{g} \right) \times \nabla f - f \text{Div} \left( \frac{h}{g} \right) = 0.
\end{aligned}
\]

It is here easy to see that the second condition follows from the first one (by applying \(\times \nabla k\) and because \(\nabla k \times \nabla k = 0\)).

By studying the previous equations, we are able to give a list of different families of Poisson structures that admit Casimirs. A Casimir of \(\psi\) is a cocycle in \(C^0\), in other words, an \(\alpha\) element of \(A\) such that \(\psi(\alpha, \beta) = 0\) for all \(\beta \in A\). Let us consider the conditions for an even element \(\alpha = a(x, y) \theta\) to be a Casimir for the Poisson structure.

\[
[\psi, \alpha] = (fa_y - ga)\theta \partial_x + (fa_x - ah)\theta \partial_y - 2ka \partial \theta.
\]

It follows that there are nonzero even Casimirs only when \(k(x, y) = 0\). Also, a special case of a Poisson structure which has a nontrivial even Casimir \(a(x, y) \theta\) is given by the following:

\[
\psi = a(x, y)\partial_x \partial_y - a_y(x, y)\theta \partial_x \partial \theta + a_x(x, y)\theta \partial_y \partial \theta, \text{ with } a \in \mathbb{K}[x, y],
\]

\((\text{i.e. } f = a, k = 0, g = -a_y, h = a_x, a(x, y) \in \mathbb{K}[x, y]).\)

On the other hand, suppose that \(\beta = b(x, y)\) is an odd element of \(C^0\). Then

\[
[\psi, \beta] = -(fb_y \partial_x - fb_x \partial_y + (gb_x + h h) \theta \partial \theta).
\]

Every even constant function is an odd Casimir. For a nonconstant Casimir \(\beta\), it follows that \(f(x, y) = 0\). A special case of a Poisson structure with a nonconstant odd Casimir \(b(x, y)\) is given by the following:

\[
\psi = b_y(x, y)\theta \partial_x \partial \theta - b_x(x, y)\theta \partial_y \partial \theta, \text{ with } b \in \mathbb{K}[x, y],
\]

\((\text{i.e. } f = k = 0, g = b_y, h = -b_x, b(x, y) \in \mathbb{K}[x, y]).\)

Another case where there are nontrivial odd Casimirs is given by the following.

\[
\psi = k(x, y) \partial \theta^2, \text{ with } k \in \mathbb{K}[x, y],
\]

\((\text{i.e. } f = g = h = 0, k(x, y) \in \mathbb{K}[x, y]).\)

For any such Poisson structure, every function \(b(x, y) \in \mathbb{K}[x, y]\) is an odd Casimir.
Another interesting family of Poisson structures is given by the following.

\[ \psi = -2k(x,y)\partial_x\partial_y - k_y(x,y)\theta\partial_x\partial_\theta + k_x(x,y)\theta\partial_y\partial_\theta + k(x,y)\partial_\theta^2, \]

with \( k \in \mathbb{K}[x,y], \)

(i.e. \( f = -2k, \quad g = -k_y, \quad h = k_x, \quad k(x,y) \in \mathbb{K}[x,y] \)).

When \( k \neq 0 \), the only Casimirs for this type of Poisson structure are the constant functions \( \beta = c \in \mathbb{K} \).

In order to write the Poisson cohomology complex associated to a Poisson structure

\[ \psi = f\partial_x\partial_y + g\theta\partial_x\partial_\theta + h\theta\partial_y\partial_\theta + k\partial_\theta^2, \]

in terms of the operators \( \times, \text{Div} \) and \( \vec{\nabla} \), we identify the cochains to elements of the spaces \( A' = \mathbb{K}[x,y], \ A' \times A'^2 \) or \( A' \times A' \times A'^2 \), as follows. First, the space of odd 0-cochains \( C^0_o = \{ c(x,y) \in A' \} \) is equal to \( A' \), while the space of even 0-cochains \( C^0_e = \{ a(x,y)\theta \mid a \in A' \} \) can be identified to \( A' \), by the following map:

\[ C^0_o \rightarrow A' \]

\[ a(x,y)\theta \mapsto a(x,y). \]

Next, we consider the space of odd 1-cochains \( C^1_o = \{ p\partial_x + q\partial_y + r\partial_\theta \mid (p,q,r) \in A'^3 \} \) and the space of even 1-cochains \( C^1_e = \{ a\partial_x + b\partial_y + c\theta\partial_\theta \mid (a,b,c) \in A'^3 \} \), which will be identified with the space \( A' \times A'^2 \) by the following maps:

\[ C^1_o \rightarrow A' \times A'^2 \]

\[ p\partial_x + q\partial_y + r\partial_\theta \mapsto (r, (\frac{q}{p})), \]

\[ C^1_e \rightarrow A' \times A'^2 \]

\[ a\partial_x + b\partial_y + c\theta\partial_\theta \mapsto (c, (\frac{b}{a})). \]

Finally, for every \( n \geq 2 \), the space of odd \( n \)-cochains \( C^n_o = \{ p\partial_x\partial_y\partial_\theta^{n-2} + q\partial_x\partial_y\partial_\theta^{n-1} + r\partial_y\partial_\theta^{n-1} + s\partial_\theta^n \mid (p,q,r,s) \in A'^4 \} \) and the space of even \( n \)-cochains \( C^n_e = \{ a\partial_x\partial_y\partial_\theta^{n-2} + b\partial_x\partial_y\partial_\theta^{n-1} + c\partial_y\partial_\theta^{n-1} + d\partial_\theta^n \mid (a,b,c,d) \in A'^4 \} \) will be identified with the space \( A' \times A' \times A'^2 \), by the following maps:

\[ C^n_o \rightarrow A' \times A' \times A'^2 \]

\[ p\partial_x\partial_y\partial_\theta^{n-2} + q\partial_x\partial_y\partial_\theta^{n-1} + r\partial_y\partial_\theta^{n-1} + s\partial_\theta^n \mapsto (p,s, (\frac{r}{q})), \]

\[ C^n_e \rightarrow A' \times A' \times A'^2 \]

\[ a\partial_x\partial_y\partial_\theta^{n-2} + b\partial_x\partial_y\partial_\theta^{n-1} + c\partial_y\partial_\theta^{n-1} + d\partial_\theta^n \mapsto (a,d, (\frac{c}{b})). \]
Also in the following an element \((f_g)\) of \(\mathcal{A}^2\) will often be denoted by a capital letter with an arrow: \(\vec{F} := (f_g)\). The element \((0_0)\) in \(\mathcal{A}^2\) will also be denoted by \(\vec{0}\).

We now want to determine the (odd and even) Poisson cohomology of a Poisson structure on \(\mathcal{A}\), of the form

\[
\psi_b := b_y \theta \partial_x \partial_\theta - b_x \theta \partial_y \partial_\theta,
\]

where \(b \in \mathcal{A}'\) is a polynomial.

Let us first point out that in this case, the singular locus of the codifferential \(\psi_b\) is defined as being the affine variety

\[
\{b_x = b_y = 0\} \subseteq \mathbb{K}^2,
\]

and because \(b\) is supposed to be homogeneous, this singular locus coincide with the singularities of the surface

\[
\mathcal{F}_b := \{(x, y) \in \mathbb{K}^2 \mid b(x, y) = 0\} \subseteq \mathbb{K}^2.
\]

From now, we denote by \(D_{\psi_b}\) the Poisson coboundary operator \(D_{\psi_b} := [\psi_b, \cdot]\) and we rather write cochains as elements in \(\mathcal{A}' = \mathbb{K}[x, y], \mathcal{A}' \times \mathcal{A}'^2\) or \(\mathcal{A}' \times \mathcal{A}' \times \mathcal{A}'^2\), as explained above. Let us write the values taken by this operator, under these identifications.

For \(\alpha_0 = c(x, y) \in C^0(\mathcal{A}) = \mathcal{A}'\):

\[
D_{\psi_b}(\alpha_0) = - \left( \vec{\nabla} b \times \vec{\nabla} c, \vec{0} \right) \in \mathcal{A}' \times \mathcal{A}'^2 \simeq C^1_\epsilon(\mathcal{A});
\]

for \(\varphi_0 = a(x, y) \in \mathcal{A}' \simeq C^0_\epsilon(\mathcal{A})\),

\[
D_{\psi_b}(\varphi_0) = \left( 0, a \vec{\nabla} b \right) \in \mathcal{A}' \times \mathcal{A}'^2 \simeq C^1_\epsilon(\mathcal{A});
\]

for \(\alpha_1 = (r, \vec{Q}) \in \mathcal{A}' \times \mathcal{A}'^2 \simeq C^1_\epsilon(\mathcal{A})\),

\[
D_{\psi_b}(\alpha_1) = \left( -\vec{\nabla} b \times \vec{Q}, \vec{\nabla} b \times \vec{\nabla} r, r \vec{\nabla} b \right) \in \mathcal{A}' \times \mathcal{A}' \times \mathcal{A}'^2 \simeq C^2_\epsilon(\mathcal{A});
\]

for \(\varphi_1 = (r, \vec{Q}) \in \mathcal{A}' \times \mathcal{A}'^2 \simeq C^1_\epsilon(\mathcal{A})\),

\[
D_{\psi_b}(\varphi_1) = \left( 0, 0, \vec{\nabla} (\vec{Q} \times \vec{\nabla} b) + \text{Div}(\vec{Q}) \vec{\nabla} b \right) \in \mathcal{A}' \times \mathcal{A}' \times \mathcal{A}'^2 \simeq C^2_\epsilon(\mathcal{A}).
\]

And for all \(n \geq 2\), for \(\alpha_n = (p, s, \vec{R}) \in \mathcal{A}' \times \mathcal{A}' \times \mathcal{A}'^2 \simeq C^m_\epsilon(\mathcal{A})\),

\[
D_{\psi_b}(\alpha_n) = \left( \vec{\nabla} b \times \vec{\nabla} p - (n - 2)\vec{R} \times \vec{\nabla} b, \vec{\nabla} b \times \vec{\nabla} s, ns \vec{\nabla} b \right) \in \mathcal{A}' \times \mathcal{A}' \times \mathcal{A}'^2 \simeq C^{m+1}_\epsilon(\mathcal{A});
\]
for $\varphi_n = (a, d, \vec{C}) \in \mathcal{A}' \times \mathcal{A}' \times \mathcal{A}^2 \simeq C^n_e(\mathcal{A})$, 

$$D_{\psi_b}(\varphi_n) = \left( (n-1)\vec{\nabla}b \times \vec{C}, 0, (n-1)d\vec{\nabla}b + \vec{\nabla}\left( \vec{C} \times \vec{\nabla}b \right) + \text{Div}(\vec{C})\vec{\nabla}b \right) \in \mathcal{A}' \times \mathcal{A}' \times \mathcal{A}^2 \simeq C^{n+1}_e(\mathcal{A}).$$

In order to determine the Poisson cohomology associated to the Poisson structure $\psi_b$, we will assume that $b(x, y) \in \mathcal{A}'$ is a non constant, homogeneous and square-free polynomial. These hypotheses imply in particular that the following Koszul complex is exact:

$$0 \rightarrow \mathcal{A}' \xrightarrow{a\vec{\nabla}b} \mathcal{A}^2 \xrightarrow{\vec{C} \times \vec{\nabla}b^\times} \mathcal{A}'$$

where the first map, from $\mathcal{A}'$ to $\mathcal{A}^2$, maps an element $a \in \mathcal{A}'$ to $a\vec{\nabla}b$ while the second, from $\mathcal{A}^2$ to $\mathcal{A}'$, maps $\vec{C} \in \mathcal{A}^2$ to $\vec{C} \times \vec{\nabla}b$. To prove that the above diagram is exact, we use the fact that, as $b$ is homogeneous, it satisfies the Euler’s identity: $\deg(b) b = xb_x + yb_y$ and because $b$ is non constant and square-free, $b_x$ and $b_y$ are coprime. Now if $\vec{C} = \left( \begin{array}{c} f \\ g \end{array} \right) \in \mathcal{A}^2$ satisfies $\vec{C} \times \vec{\nabla}b = 0$, then one has $fb_y - gb_x = 0$. This implies that $b_x$ divides $f$ in $\mathcal{A}'$, so that there exists $a \in \mathcal{A}'$ such that $f = ab_x$. This permits to conclude that $\vec{C} = a\vec{\nabla}b$.

Notice moreover that, given a non constant homogeneous polynomial $b \in \mathbb{K}[x, y]$, $b$ is square-free if and only if the quotient vector space

$$\mathcal{A}'_{\text{sing}}(b) := \frac{\mathcal{A}'}{\langle b_x, b_y \rangle} \simeq \frac{\mathcal{A}'}{\{\vec{\nabla}b \times \vec{C} \mid \vec{C} \in \mathcal{A}^2 \}}$$

is of finite dimension (see [6] for a proof of this fact), and in this case, one says that the surface $\mathcal{F}_b$ has an isolated singularity at the origin. The algebra $\mathcal{A}'_{\text{sing}}(b)$ is called the Milnor algebra and its dimension (as a $\mathbb{K}$-vector space) is denoted by $\mu$ and called the Milnor number of the singularity of $b$.

**Remark 2.** This Milnor number and the Milnor algebra $\mathcal{A}'_{\text{sing}}(b)$ give information about the singularity of the surface $\mathcal{F}_b$ (i.e., the singularity of the Poisson structure $\psi_b$), as its multiplicity (see [1]). We will see that the algebra $\mathcal{A}'_{\text{sing}}(b)$ appear in the Poisson cohomology spaces associated to $\psi_b$, so that this Poisson cohomology is linked to the type of the singularity of $\psi_b$. These results have to be compared with analogous results obtained in [7], where the studied Poisson structures are non graded Poisson structures, in dimension three (i.e., defined on $\mathbb{C}[x, y, z]$), of the form: $\varphi_x \partial_x \wedge \partial_y + \varphi_y \partial_y \wedge \partial_z + \varphi_z \partial_z \wedge \partial_x$, where
\[ \varphi \in \mathbb{C}[x, y, z] \] is a (weight-)homogeneous polynomial with an isolated singularity at the origin.

We denote by \( u_0 = 1, u_1, \ldots, u_{\mu - 1} \in \mathcal{A}' \) homogeneous polynomials in \( \mathcal{A}' \) such that their images in the quotient \( \frac{\mathcal{A}'}{\langle b_x, b_y \rangle} \) gives a \( \mathbb{K} \)-basis of this quotient vector space. One can then write:

\[
(3) \quad \mathcal{A}' = \mathbb{K}u_0 \oplus \mathbb{K}u_1 \oplus \cdots \oplus \mathbb{K}u_{\mu - 1} \oplus \{ \nabla b \times \vec{G} \mid \vec{G} \in \mathcal{A}^2 \}.
\]

Note that in the particular case where the degree of \( b \) is 1, then \( \mu = 0 \) and strictly speaking, if we demand that \( \mathcal{F}_b \) has a singularity at the origin, we should probably suppose that the degree of \( b \) is greater or equal to 2.

**Proposition 4.1.** Let \( b(x, y) \in \mathcal{A}' \) be a non-constant homogeneous polynomial. Let \( \psi_b \) be the Poisson structure given by the following formula:

\[ \psi_b = b_y \theta \partial_x \partial_\theta - b_x \theta \partial_y \partial_\theta. \]

If \( b \) is square-free then a basis of the odd Poisson cohomology 0-space is given by the following:

\[ H^0_o(\mathcal{A}, \psi_b) = \mathbb{K}[b]. \]

**Proof.** First, recall that one can write, under the identifications given above:

\[ H^0_o(\psi_b) = \{ c(x, y) \in \mathcal{A}' \mid \nabla c \times \nabla b = 0 \}. \]

Then let \( c \in \mathcal{A}' \) such that \( \nabla c \times \nabla b = 0 \). Because of the exactness of the Koszul complex, there exists \( a(x, y) \in \mathcal{A}' \) such that \( \nabla c = a \nabla b \).

Assume that \( c \) is a homogeneous polynomial then, using the Euler’s formula we obtain:

\[ \deg(c) c = xc_x + yc_y = a(xb_x + yb_y) = \deg(b)ab. \]

This implies that either \( \deg(c) = 0 \) or \( b \) divides the polynomial \( c \) in \( \mathcal{A}' \).

We then write \( c = b^r h \) with \( r \in \mathbb{N} \) and \( h \in \mathcal{A}' \), with \( b \) non dividing the polynomial \( h \) in \( \mathcal{A}' \). Then,

\[ \nabla c = rb^{r-1}h \nabla b + b^r \nabla h \quad \text{and} \quad 0 = \nabla c \times \nabla b = b^r \nabla h \times \nabla b. \]

From the above, we obtain that \( \deg(h) = 0 \) and \( c \in \mathbb{K}b^r \). \( \square \)

**Proposition 4.2.** Let \( b(x, y) \in \mathcal{A}' \) be a non-constant homogeneous polynomial. Let \( \psi_b \) be the Poisson structure given by the formula:

\[ \psi_b = b_y \theta \partial_x \partial_\theta - b_x \theta \partial_y \partial_\theta. \]

If \( b \) is square-free then the odd Poisson cohomology 1-space vanishes:

\[ H^1_o(\mathcal{A}, \psi_b) \simeq \{ 0 \}. \]
Proof. First, we have:

\[ H_0^1(\psi_b) \cong \left\{ (r, \bar{Q}) \in \mathcal{A}' \times \mathcal{A}'^2 \mid \vec{\nabla}\bar{b} \times \bar{Q} = 0; r\vec{\nabla}b = \vec{0}; \nabla b \times \nabla r = 0 \right\} \]

\[ \{ (0, a\vec{\nabla}b) \in \mathcal{A}' \times \mathcal{A}'^2 \mid a \in \mathcal{A}' \}. \]

Let then \((r, \bar{Q}) \in Z_0^1(\psi_b)\). Then, because \(b\) is non-constant and because \(r\vec{\nabla}b = \vec{0}\), we have that \(r = 0\). Moreover, using the exactness of the Koszul complex the condition \(\vec{\nabla}\bar{b} \times \bar{Q} = 0\) implies that there exists \(a \in \mathcal{A}'\) such that \(\bar{Q} = a\vec{\nabla}b\), which permits to conclude that \((r, \bar{Q}) \in B_0^1(\psi_b)\) and that \(H_0^1(\mathcal{A}, \psi_b) \cong \{0\} \).

\[ \square \]

**Proposition 4.3.** Let \(b(x, y) \in \mathcal{A}'\) be a non-constant homogeneous polynomial. Let \(n \in \mathbb{N}\) satisfying \(n \geq 3\). Let \(\psi_b\) be the Poisson structure given by the following formula:

\[ \psi_b = b_y\theta \partial_x \partial_{\theta} - b_x\theta \partial_y \partial_{\theta}. \]

If \(b\) is square-free then a basis of the odd Poisson cohomology \(n\)-space is given by the following:

\[ H_0^n(\psi_b) \cong \bigoplus_{i=0}^{\mu-1} \mathbb{K} \left( (n-2)u_i, 0, -\nabla u_i \right) \]

\[ \cong \bigoplus_{i=0}^{\mu-1} \mathbb{K} \left( (n-2)u_i \partial_x \partial_y \partial_{\theta}^{n-2} + (u_i)_y \theta \partial_x \partial_{\theta}^{n-1} - (u_i)_x \theta \partial_y \partial_{\theta}^{n-1} \right). \]

**Remark 3.** With this result, one easily sees that \(H_0^n(\psi_b) \cong \mathcal{A}'_{\text{sing}}(b)\), when \(n \geq 3\).

**Proof.** Let us recall that one can write:

\[ H_0^n(\psi_b) \cong \]

\[ \left\{ (p, s, \bar{R}) \in \mathcal{A}' \times \mathcal{A}' \times \mathcal{A}'^2 \mid \vec{\nabla}b \times \vec{\nabla}p + (n-2)\vec{\nabla}b \times \bar{R} = 0; ns\vec{\nabla}b = \vec{0}; \nabla b \times \nabla s = 0 \right\} \]

\[ \bigg\{ (n-2)\vec{\nabla}b \times \bar{C}, 0, \vec{\nabla} \left( \bar{C} \times \nabla b \right) + \text{Div}(\bar{C}) \vec{\nabla}b + (n-2)\text{Div}(\bar{C}) \bigg\} \]

\[ \mid (a, d, \bar{C}) \in \mathcal{A}' \times \mathcal{A}' \times \mathcal{A}'^2 \bigg\}. \]

Let us then consider an element \((p, s, \bar{R}) \in Z_0^n(\psi_b)\). As \(ns\vec{\nabla}b = \vec{0}\) and \(b\) is an non-constant polynomial, one necessarily obtains that \(s = 0\). The cocycle condition now becomes

\[ \vec{\nabla}b \times \vec{\nabla}p + (n-2)\vec{\nabla}b \times \bar{R} = 0, \text{ i.e., } \vec{\nabla}b \times \left( \vec{\nabla}p + (n-2)\bar{R} \right) = 0. \]

Because the Koszul complex is exact, this implies the existence of an element \(f \in \mathcal{A}'\) satisfying

\[ (4) \quad \vec{\nabla}p + (n-2)\bar{R} = (n-2)f\vec{\nabla}b. \]
Now, (3) implies that there exist $\lambda_0, \lambda_1, \ldots, \lambda_{\mu-1} \in \mathbb{K}$ and $\vec{C} \in \mathcal{A}^2$ such that:

$$p = (n - 2)\vec{\nabla} b \times \vec{C} + \sum_{i=0}^{\mu-1} \lambda_i (n - 2) u_i.$$ 

Thus,

$$\vec{\nabla} p = (n - 2)\vec{\nabla} \left( \vec{\nabla} b \times \vec{C} \right) + \sum_{i=0}^{\mu-1} \lambda_i (n - 2) \vec{\nabla} u_i,$$

and (4) becomes:

$$\vec{R} = -\vec{\nabla} \left( \vec{\nabla} b \times \vec{C} \right) - \sum_{i=0}^{\mu-1} \lambda_i \vec{\nabla} u_i + f \vec{\nabla} b.$$

Let $d := \frac{1}{(n-2)} \left( f - \text{Div}(\vec{C}) \right)$, then:

$$\vec{R} = -\vec{\nabla} \left( \vec{\nabla} b \times \vec{C} \right) + (n - 2)d \vec{\nabla} b + \text{Div}(\vec{C}) \vec{\nabla} b - \sum_{i=0}^{\mu-1} \lambda_i \vec{\nabla} u_i.$$

Finally,

$$\langle p, s, \vec{R} \rangle = \left( (n - 2)\vec{\nabla} b \times \vec{C}, 0, -\vec{\nabla} \left( \vec{\nabla} b \times \vec{C} \right) + (n - 2)d \vec{\nabla} b + \text{Div}(\vec{C}) \vec{\nabla} b \right)$$

$$+ \sum_{i=0}^{\mu-1} \lambda_i \left( (n - 2) u_i, 0, -\vec{\nabla} u_i \right)$$

$$\in B^a_0(\psi_b) + \sum_{i=0}^{\mu-1} \mathbb{K} \left( (n - 2) u_i, 0, -\vec{\nabla} u_i \right).$$

which implies that:

$$H^n_0(\psi_b) = \sum_{i=0}^{\mu-1} \mathbb{K} \left( (n - 2) u_i, 0, -\vec{\nabla} u_i \right).$$

It now remains to show that this sum is a direct one, by considering $\lambda_0, \lambda_1, \ldots, \lambda_{\mu-1} \in \mathbb{K}$ and $(d, \vec{C}) \in \mathcal{A}' \times \mathcal{A}^2$ such that

$$\sum_{i=0}^{\mu-1} \lambda_i \left( (n - 2) u_i, 0, -\vec{\nabla}(u_i) \right) =$$

$$\left( (n - 2)\vec{\nabla} b \times \vec{C}, 0, -\vec{\nabla} \left( \vec{\nabla} b \times \vec{C} \right) + \text{Div}(\vec{C}) \vec{\nabla} b + (n - 2)d \vec{\nabla} b \right).$$

But then, $\sum_{i=0}^{\mu-1} \lambda_i u_i = \vec{\nabla} b \times \vec{C} \in \langle b_x, b_y \rangle$, so that, by definition of the $u_i$, we conclude that $\lambda_i = 0$ for all $i = 0, \ldots, \mu - 1$. We finally have obtained the desired result. $\square$
The difficult part of the computation of the odd Poisson cohomology associated to the Poisson structure \( \psi_b = b_y \theta \partial_x \partial_\theta - b_x \theta \partial_y \partial_\theta \) lies in the second Poisson cohomology space, which we give here.

**Proposition 4.4.** Let \( b(x, y) \in \mathcal{A}' \) be a non-constant homogeneous polynomial. Let \( \psi_b \) be the Poisson structure given above. If \( b \) is square-free then a basis of the odd Poisson cohomology 2nd-space is given by:

\[
H^2_o(\psi_b) \cong \mathbb{K}[b] (1, 0, 0) \oplus \bigoplus_{i=0, \ldots, \mu-1} \mathbb{K}[b] u_i (0, 0, \nabla b)
\]

\[
\bigoplus_{j=0}^{\mu-1} \mathbb{K}[b] u_j (0, 0, \vec{E}) \bigoplus \bigoplus_{i=0, \ldots, \mu-1} \mathbb{K}[b] (0, 0, \nabla u_i)
\]

\[
\bigoplus_{j=1, \ldots, \mu-1} \mathbb{K} (0, 0, \nabla u_j),
\]

where \( \vec{E} := (\frac{y}{-x}) \in \mathcal{A}^2 \). This can be written more explicitly as

\[
H^2_o(\psi_b) \cong \mathbb{K}[b] \partial_x \partial_y \bigoplus \bigoplus_{i=0, \ldots, \mu-1} \mathbb{K}[b] u_i \psi_b
\]

\[
\bigoplus_{k=0}^{\mu-1} \mathbb{K}[b] u_j (x \theta \partial_x \partial_\theta + y \theta \partial_y \partial_\theta)
\]

\[
\bigoplus_{i=0, \ldots, \mu-1} \mathbb{K}[b] ((u_i)_y \theta \partial_x \partial_\theta - (u_i)_x \theta \partial_y \partial_\theta)
\]

\[
\bigoplus_{j=1, \ldots, \mu-1} \mathbb{K} ((u_j)_y \theta \partial_x \partial_\theta - (u_j)_x \theta \partial_y \partial_\theta).
\]

In order to be able to prove this proposition, we need the following

**Lemma 4.5.** Let \( b(x, y) \in \mathcal{A}' \) be a non-constant homogeneous polynomial. If \( b \) is square-free then, we have:

\[
\mathcal{A}' = \bigoplus_{i=0}^{\mu-1} \mathbb{K}[b] u_i \oplus \{ \nabla h \times \nabla b \mid h \in \mathcal{A}' \}.
\]

**Proof of Lemma 4.5.** We first prove that \( \mathcal{A}' = \bigoplus_{i=0}^{\mu-1} \mathbb{K}[b] u_i \oplus \{ \nabla h \times \nabla b \mid h \in \mathcal{A}' \} \). To do this, let \( f \in \mathcal{A}' \) a homogeneous polynomial in \( \mathbb{K}[x, y] \). According to (3), there exist \( \lambda_0, \lambda_1, \ldots, \lambda_{\mu-1} \in \mathbb{K} \) and \( \vec{F} = \)
\[(f_1, f_2) \in \mathcal{A}^2\] such that
\[
f = \vec{F} \times \vec{\nabla} b + \sum_{i=0}^{\mu-1} \lambda_i u_i,
\]
and such that \(f_1\) and \(f_2\) are two homogeneous polynomials of \(\mathbb{K}[x, y]\), of degree \(\deg(f_1) = \deg(f_2) = \deg(f) - \deg(b) + 1\). We will now proceed by induction on \(\deg(f)\).

First, if \(\deg(f) \leq \deg(b) - 1\), then \(f_1 = a \in \mathbb{K}\) and \(f_2 = b \in \mathbb{K}\), so that one can write \(\vec{F} = \vec{\nabla} h\), with \(h := ax + by \in \mathcal{A}'\) and \(f = \vec{\nabla} h \times \vec{\nabla} b + \sum_{i=0}^{\mu-1} \lambda_i u_i\).

Secondly, let \(d = \deg(f)\) and suppose that \(d \geq \deg(b)\). We also suppose that, for any homogeneous polynomial \(g \in \mathcal{A}'\) of degree less or equal to \(d - 1\), we have \(g \in \sum_{i=0}^{\mu-1} \mathbb{K}[b]u_i \oplus \{ \vec{\nabla} h \times \vec{\nabla} b \mid h \in \mathcal{A}'\}\). Because of Euler’s formula, for any homogeneous polynomial \(k \in \mathcal{A}'\), we have:
\[
\text{Div}(\text{Div}(\vec{F}) \vec{E} + (\deg(f) - \deg(b) + 2) \vec{F}) = 0.
\]
Now, using the exactness of the de Rham complex, we obtain the existence of a homogeneous polynomial \(k \in \mathcal{A}'\) such that:
\[
(7) \quad \vec{F} = \frac{-1}{(\deg(f) - \deg(b) + 2)} \text{Div}(\vec{F}) \vec{E} - \vec{\nabla} k.
\]
Moreover, \(\deg(\text{Div}(\vec{F})) = \deg(f) - \deg(b) < \deg(f)\) (by hypothesis, \(b\) is non-constant), so that we can apply the induction hypothesis on \(\text{Div}(\vec{F})\) and obtain the existence of \(\ell \in \mathcal{A}'\) and \(\alpha_{i,j} \in \mathbb{K}\), for all \(j \in \mathbb{N}\) and \(i = 0, \ldots, \mu - 1\) such that:
\[
\text{Div}(\vec{F}) = \vec{\nabla} \ell \times \vec{\nabla} b + \sum_{i=0}^{\mu-1} \sum_{j \in \mathbb{N}} \alpha_{i,j} b^j u_i,
\]
where of course, for each \(i \in \{0, \ldots, \mu - 1\}\), only a finite number of \(\alpha_{i,j}\) are non-zero (so that the previous sum is well-defined). Then, by (7), we have:
\[
\vec{F} = \frac{-1}{(\deg(f) - \deg(b) + 2)} \left( \vec{\nabla} \ell \times \vec{\nabla} b + \sum_{i=0}^{\mu-1} \sum_{j \in \mathbb{N}} \alpha_{i,j} b^j u_i \right) \vec{E} - \vec{\nabla} k,
\]
and by (6), we obtain:

\[ f = \frac{-1}{(\deg(f) - \deg(b) + 2)} \left( \bar{\nabla} \ell \times \bar{\nabla} b + \sum_{i=0}^{\mu-1} \sum_{j \in \mathbb{N}} \alpha_{i,j} b^j u_i \right) \bar{E} \times \bar{\nabla} b \]

\[ -\bar{\nabla} k \times \bar{\nabla} b + \sum_{i=0}^{\mu-1} \lambda_i u_i. \]

Now, by Euler’s formula, we compute \( \bar{E} \times \bar{\nabla} b = \deg(b) b \) and because \( \bar{\nabla} b \times \bar{\nabla} b = 0 \), we also write \( \left( \bar{\nabla} \ell \times \bar{\nabla} b \right) b = \left( \bar{\nabla} (\ell b) \times \bar{\nabla} b \right) \). Thus,

\[ f = \frac{-\deg(b)}{(\deg(f) - \deg(b) + 2)} \left( \bar{\nabla} (\ell b) \times \bar{\nabla} b \right) \]

\[ + \sum_{i=0}^{\mu-1} \sum_{j \in \mathbb{N}} \left( \frac{\deg(b)}{(\deg(f) - \deg(b) + 2)} \alpha_{i,j} \right) b^{j+1} u_i \]

\[ -\bar{\nabla} k \times \bar{\nabla} b + \sum_{i=0}^{\mu-1} \lambda_i u_i \]

\[ \in \{ \bar{\nabla} h \times \bar{\nabla} b \mid h \in \mathcal{A}' \} + \sum_{i=0}^{\mu-1} \mathbb{K}[b] u_i. \]

We then have shown that \( \mathcal{A}' = \sum_{i=0}^{\mu-1} \mathbb{K}[b] u_i + \{ \bar{\nabla} h \times \bar{\nabla} b \mid h \in \mathcal{A}' \} \), and it remains to show that this sum is a direct one. To do this, we suppose on the contrary that this sum is not direct. Then we define \( j_0 \) as being the smaller integer such that there exists \( 0 \leq i_0 \leq \mu - 1 \), a family of constants \( \gamma_{i,j} \in \mathbb{K} \), where \( j \in \mathbb{N}, i = 0, \ldots, \mu - 1 \) and \( \gamma_{i_0,j_0} \neq 0 \) and \( p \in \mathcal{A}' \) satisfying an equation of the form:

\[ \sum_{i=0}^{\mu-1} \sum_{j \in \mathbb{N}} \gamma_{i,j} b^j u_i = \bar{\nabla} p \times \bar{\nabla} b. \]

(8)

Now, if \( j_0 = 0 \), then there exist a family of constants \( \gamma_{i,j} \in \mathbb{K} \), where \( j \in \mathbb{N}, i = 0, \ldots, \mu - 1 \) and \( \gamma_{i_0,0} \neq 0 \) and \( p \in \mathcal{A}' \) satisfying:

\[ \sum_{i=0}^{\mu-1} \gamma_{i,0} u_i = -\sum_{i=0}^{\mu-1} \sum_{j \in \mathbb{N}^*} \gamma_{i,j} b^j u_i + \bar{\nabla} p \times \bar{\nabla} b. \]
As \( b = \deg(b)(xb_x + yb_y) \), this leads to:

\[
\sum_{i=0}^{\mu-1} \gamma_{i,0} u_i \in \langle b_x, b_y \rangle,
\]

which implies, regarding the definition of the \( u_i \), that \( \gamma_{i,0} = 0 \), for all \( 0 \leq i \leq \mu - 1 \). We obtain a contradiction with the definition of \( j_0 \).

Now, assuming that \( j_0 \geq 1 \) and using once more \( b = \frac{1}{\deg(b)} \vec{E} \times \vec{\nabla} b \) in (8),

\[
\sum_{i=0}^{\mu-1} \sum_{j \geq j_0} \frac{\gamma_{i,j}}{\deg(b)} b_j^{i-1} u_i \vec{E} \times \vec{\nabla} b = \vec{\nabla} p \times \vec{\nabla} b.
\]

(Recall that, according to the definition of \( j_0 \), for all \( 0 \leq i \leq \mu - 1 \) and all \( j \leq j_0 - 1 \), one has \( \gamma_{i,j} = 0 \).) As the Koszul complex is exact, there exists \( d \in A' \) satisfying:

\[
\sum_{i=0}^{\mu-1} \sum_{j \geq j_0} \frac{\gamma_{i,j}}{\deg(b)} b_j^{i-1} u_i \vec{E} = \vec{\nabla} p + d \vec{\nabla} b.
\]

Computing the divergence of this,

\[
\sum_{i=0}^{\mu-1} \sum_{j \geq j_0} \frac{\gamma_{i,j}}{\deg(b)} (\deg(b)(j-1) + \deg(u_i) + 2) b_j^{i-1} u_i = -\vec{\nabla} d \times \vec{\nabla} b.
\]

Denoting by \( \tilde{\gamma}_{i,j} \) the constant \( \tilde{\gamma}_{i,j} := \frac{\gamma_{i,j+1}}{\deg(b)} (\deg(b)j + \deg(u_i) + 2) \), we obtain the equation:

\[
\sum_{i=0}^{\mu-1} \sum_{j' \geq j_0 - 1} \tilde{\gamma}_{i,j'} b_j^{i} u_i = -\vec{\nabla} d \times \vec{\nabla} b,
\]

with \( \tilde{\gamma}_{i_0,j_0-1} = \frac{\gamma_{i_0,j_0}}{\deg(b)} (\deg(b)(j_0 - 1) + \deg(u_{i_0}) + 2) \neq 0 \). We obtain a contradiction with the definition of \( j_0 \). Finally, we have shown the fact that the previous sum is direct and the lemma is proved.

We now prove Proposition 4.4. To do this, we denote by \( D' : A^2 \to A^2 \) the operator given, for \( \vec{Q} \in A^2 \) by \( D'(\vec{Q}) := -\vec{\nabla} \left( \vec{\nabla} b \times \vec{Q} \right) + \text{Div}(\vec{Q}) \vec{\nabla} b \).

**Remark 4.** Using the Euler’s formula, we obtain, for every homogeneous polynomial \( h \in A' \),

\[
D'(h \vec{E}) = \deg(b) b \vec{\nabla} h + (\deg(b) - \deg(h) - 2) h \vec{\nabla} b.
\]
Secondly, we compute
\[
\left(\nabla h \times \nabla b\right) \tilde{E} = \text{deg}(b) b \nabla h - \text{deg}(h) h \nabla b.
\]
These two equalities imply:
\[
D'(h \tilde{E}) = \left(\nabla h \times \nabla b\right) \tilde{E} + (\text{deg}(b) - 2) h \nabla b.
\]

**Remark 5.** According to the proposition 4.4, we see that the Poisson structure \(\psi_b\) is an odd 2-coboundary for the Poisson cohomology associated to \(\psi_b\) itself if and only if \(\text{deg}(b) \neq 2\) and this is due to the equality \(D'(\tilde{E}) = (\text{deg}(b) - 2) \nabla b\).

**Proof of the proposition 4.4**
Recall that, with the help of the identifications given previously, we have
\[
H^2_o(\psi_b) \cong \left\{(p, s, \tilde{R}) \in \mathcal{A}' \times \mathcal{A}' \times \mathcal{A}^2 \mid \nabla b \times \nabla p = 0; 2s \nabla b = 0; \nabla b \times \nabla s = 0\right\}
\]
\[
\left\{(0, 0, D'(\tilde{C}) = -\nabla \left(\nabla b \times \tilde{C}\right) + \text{Div}(\tilde{C}) \nabla b) \mid \tilde{C} \in \mathcal{A}^2\right\}
\]
Let \((p, s, \tilde{R}) \in Z^2_o(\psi_b).\) As \(s \nabla b = 0\), we have \(s = 0\). Moreover, the equation \(\nabla b \times \nabla p = 0\) and the proposition 4.1 imply that \(p \in \mathbb{K}[b]\). This shows that one can write
\[
H^2_o(\psi_b) \cong \mathbb{K}[b] \partial_x \partial_y \oplus \left\{\tilde{R} \in \mathcal{A}^2\right\}
\]
\[
\left\{-\nabla \left(\nabla b \times \tilde{C}\right) + \text{Div}(\tilde{C}) \nabla b) \mid \tilde{C} \in \mathcal{A}^2\right\}
\]
Let then \(\tilde{F} \in \mathcal{A}^2\) be a homogeneous element, that is \(\tilde{F} = (f_1, f_2)\) such that \(f_1\) and \(f_2\) are homogeneous polynomials of the same degree \(\text{deg}(f_1) = \text{deg}(f_2) = d \in \mathbb{N}\).

Because \(\text{Div} \left(\text{Div}(\tilde{F}) \tilde{E}\right) = -(d + 1) \text{Div}(\tilde{F})\) and because the de Rham complex is exact, there exists \(k \in \mathcal{A}'\) such that
\[
\tilde{F} = \frac{-1}{d + 1} \text{Div}(\tilde{F}) \tilde{E} + \nabla k.
\]
According to the lemma 4.5, there exist \(h \in \mathcal{A}'\) and \(\lambda_{i,j} \in \mathbb{K}\), for \(i \in \mathbb{N}\) and \(0 \leq j \leq \mu - 1\), satisfying
\[
k = \nabla h \times \nabla b + \sum_{j=0}^{\mu-1} \sum_{i \in \mathbb{N}} \lambda_{i,j} b^i u_j,
\]
where for each 0 \leq j \leq \mu - 1, only a finite number of \lambda_{i,j} are non-zero. Then,

\[ \hat{\nabla} k = \hat{\nabla} (\hat{\nabla} h \times \hat{\nabla} b) + \sum_{j=1}^{\mu-1} \sum_{i \in \mathbb{N}} \lambda_{i,j} b^i \hat{\nabla} u_j + \sum_{j=0}^{\mu-1} \sum_{i \in \mathbb{N}^*} \lambda_{i,j} i b^{i-1} u_j \hat{\nabla} b, \]

\[ = D'(\hat{\nabla} h) + \sum_{j=1}^{\mu-1} \sum_{i \in \mathbb{N}} \lambda_{i,j} b^i \hat{\nabla} u_j + \sum_{j=0}^{\mu-1} \sum_{i \in \mathbb{N}^*} \lambda_{i,j} i b^{i-1} u_j \hat{\nabla} b. \]

Moreover, applying successively two times the lemma 4.5, we obtain the existence of \( h', \ell \in \mathcal{A}' \) and some constants \( \alpha_{i,j}, \gamma_{i,j} \in \mathbb{K}, \) for \( i \in \mathbb{N} \) and 0 \leq j \leq \mu - 1 (with, for each \( j \), only a finite number of non-zero \( \alpha_{i,j} \) and \( \gamma_{i,j} \)) such that:

\[ \text{Div}(\vec{F}) = \hat{\nabla} h' \times \hat{\nabla} b + \sum_{j=0}^{\mu-1} \sum_{i \in \mathbb{N}} \alpha_{i,j} b^i u_j, \]

\[ h' = \hat{\nabla} \ell \times \hat{\nabla} b + \sum_{j=0}^{\mu-1} \sum_{i \in \mathbb{N}} \gamma_{i,j} b^i u_j. \]

We compute \( D'((\ell \hat{\nabla} b)) = \text{Div}(\ell \hat{\nabla} b) \hat{\nabla} b = (\hat{\nabla} \ell \times \hat{\nabla} b) \hat{\nabla} b, \) so that

\[ \ell \hat{\nabla} b = D'((\ell \hat{\nabla} b)) + \sum_{j=0}^{\mu-1} \sum_{i \in \mathbb{N}} \gamma_{i,j} b^i u_j \hat{\nabla} b. \]

According to (10), \( (\hat{\nabla} h' \times \hat{\nabla} b) \vec{E} = D'\left(h' \vec{E}\right) - (\deg(b) - 2)h' \hat{\nabla} b. \) This permits us to write:

\[ \text{Div}(\vec{F}) \vec{E} = D'\left(h' \vec{E}\right) - (\deg(b) - 2)D'\left((\ell \hat{\nabla} b)\right) \]

\[ - \sum_{j=0}^{\mu-1} \sum_{i \in \mathbb{N}} (\deg(b) - 2)\gamma_{i,j} b^i u_j \hat{\nabla} b \]

\[ + \sum_{j=0}^{\mu-1} \sum_{i \in \mathbb{N}} \alpha_{i,j} b^i u_j \vec{E}. \]
We finally obtain:

\[
\vec{F} = \frac{-1}{d+1} \text{Div}\left(\vec{F}\cdot\vec{E}\right) + \vec{\nabla}k
\]

\[
= D'\left(\frac{-1}{d+1}h'\vec{E} + \frac{(\deg(b) - 2)}{d+1}\ell\vec{b} + \vec{\nabla} h\right)
\]

\[
+ \sum_{j=0}^{\mu-1} \sum_{i \in \mathbb{N}} \frac{(\deg(b) - 2)}{d+1} \gamma_{i,j} b^i u_j \vec{\nabla}b + \sum_{j=0}^{\mu-1} \sum_{i \in \mathbb{N}} \frac{-1}{d+1} \alpha_{i,j} b^i u_j \vec{E}
\]

\[
+ \sum_{j=1}^{\mu-1} \sum_{i \in \mathbb{N}} \lambda_{i,j} b^i \vec{\nabla}u_j + \sum_{j=0}^{\mu-1} \sum_{i \in \mathbb{N}} \lambda_{i,j} b^{i-1} u_j \vec{\nabla}b
\]

\[
\in \text{Im}(D') + \sum_{j=0}^{\mu-1} \mathbb{K}[b] u_j \vec{\nabla}b + \sum_{j=0}^{\mu-1} \mathbb{K}[b] u_j \vec{E} + \sum_{j=1}^{\mu-1} \mathbb{K}[b] \vec{\nabla}u_j.
\]

Now, using equation (9) (with \(h = u_j\)) and the fact that \(D'(b\vec{Q}) = bD'(\vec{Q})\), for all \(\vec{Q} \in A^2\), we obtain

\[
b^i \vec{\nabla}u_j = D'\left(\frac{b^{i-1}}{\deg(b)} u_j \vec{E}\right), \quad \text{if } i \geq 1 \text{ and if } \deg(u_j) = \deg(b) - 2,
\]

hence

\[
\sum_{j=1}^{\mu-1} \mathbb{K}[b] \vec{\nabla}u_j \in \text{Im}(D') + \sum_{\deg(u_j) = \deg(b) - 2} \mathbb{K}[b] \vec{\nabla}u_j
\]

\[
+ \sum_{\deg(u_j) \neq \deg(b) - 2} \mathbb{K}[b] \vec{\nabla}u_j.
\]

Moreover, if \(\deg(u_j) \neq \deg(b) - 2\), (9) implies

\[
u_j \vec{\nabla}b = \frac{1}{\deg(b) - \deg(u_j) - 2} D'\left(u_j \vec{E}\right) - \frac{\deg(b)}{\deg(b) - \deg(u_j) - 2} b \vec{\nabla}u_j,
\]

hence,

\[
\sum_{j=0}^{\mu-1} \mathbb{K}[b] u_j \vec{\nabla}b \in \text{Im}(D') + \sum_{\deg(u_j) = \deg(b) - 2} \mathbb{K}[b] u_j \vec{\nabla}b
\]

\[
+ \sum_{\deg(u_j) \neq \deg(b) - 2} \mathbb{K}[b] \vec{\nabla}u_j.
\]
This leads to:

\[ \tilde{F}' \in \text{Im}(D') + \sum_{0 \leq j \leq \mu - 1 \atop \deg(u_j) = \deg(b) - 2} \mathbb{K}[b] u_j \nabla b \]

\[ + \sum_{0 \leq j \leq \mu - 1 \atop \deg(u_j) \neq \deg(b) - 2} \mathbb{K}[b] \nabla u_j + \sum_{0 \leq j \leq \mu - 1} \mathbb{K}[b] u_j \tilde{E} \]

\[ + \sum_{1 \leq j \leq \mu - 1 \atop \deg(u_j) = \deg(b) - 2} \mathbb{K} \nabla (u_j) \]

and

\[ \mathcal{A}^{2} \left\{ D'(\tilde{C}) \mid \tilde{C} \in \mathcal{A}^{2} \right\} \simeq \sum_{0 \leq j \leq \mu - 1 \atop \deg(u_j) = \deg(b) - 2} \mathbb{K}[b] u_j \nabla b \]

\[ + \sum_{0 \leq j \leq \mu - 1 \atop \deg(u_j) \neq \deg(b) - 2} \mathbb{K}[b] \nabla u_j + \sum_{0 \leq j \leq \mu - 1} \mathbb{K}[b] u_j \tilde{E} \]

\[ + \sum_{1 \leq j \leq \mu - 1 \atop \deg(u_j) = \deg(b) - 2} \mathbb{K} \nabla (u_j) \]

If \( \deg(b) = 1 \), then \( \mu = 0 \) and this vector space is \{0\}. It now remains to show that this sum is a direct one, in the case \( \deg(b) \geq 2 \). To do this, let us suppose that there exist some constants \( e_{k,\ell}, m_{s,t}, c_{r,j}, a_i \in \mathbb{K} \), for \( 0 \leq \ell, j \leq \mu - 1, 1 \leq t, i \leq \mu - 1, k, s, r \in \mathbb{N} \), and there exists \( \tilde{Q} \in \mathcal{A}^{2} \), satisfying:

\[ \text{Div}(\tilde{Q})\nabla b - \nabla \left( \nabla b \times \tilde{Q} \right) = \]

\[ \sum_{k \in \mathbb{N}, 0 \leq k \leq \mu - 1 \atop \deg(u_k) = \deg(b) - 2} e_{k,\ell} b^k u_\ell \nabla b + \sum_{s \in \mathbb{N}, 1 \leq t \leq \mu - 1 \atop \deg(u_t) \neq \deg(b) - 2} m_{s,t} b^r \nabla u_t \]

\[ + \sum_{r \in \mathbb{N}, 0 \leq j \leq \mu - 1 \atop \deg(u_j) = \deg(b) - 2} c_{r,j} b^r u_j \tilde{E} + \sum_{1 \leq i \leq \mu - 1 \atop \deg(u_i) = \deg(b) - 2} a_i \nabla u_i, \]

(11)

(where, as usual, all the sums are supposed to be finite). By computing the divergence of these terms and because, for every \( k, \ell \in \mathcal{A}' \), one has \( \text{Div}(k \nabla \ell) = \nabla k \times \nabla \ell \), one obtains that:

\[ \sum_{r \in \mathbb{N}, 0 \leq j \leq \mu - 1} c_{r,j} (r \deg(b) + \deg(u_j) + 2) b^r u_j \in \{ \nabla h \times \nabla b \mid h \in \mathcal{A}' \}. \]

Together with (9), this implies that \( c_{r,j} = 0 \), for all \( r, j \).
Now, by computing the cross product of $\mathbf{E}$ and $\mathbf{E}$, we obtain that:

$$\sum_{1 \leq t \leq \mu - 1} m_{0,t} \deg(u_t) u_t + \sum_{\deg(u_t) = \deg(b) - 2} \deg(u_t) a_i u_i \in \langle b_x, b_y \rangle,$$

so that, by definition of the $u_i$, this equality implies that $m_{0,t} = 0$ and $a_i = 0$, for all $t$ and $i$. It now remains:

$$\text{Div}(\vec{Q}) \vec{\nabla} b - \vec{\nabla} \left( \vec{\nabla} b \times \vec{Q} \right) = \sum_{k \in \mathbb{N}} e_{k} u_{k} b_{k} \vec{\nabla} b + \sum_{s \in \mathbb{N}^*} m_{s,t} b^s \vec{\nabla} u_t$$

This implies

$$\vec{\nabla} \left( \sum_{s \in \mathbb{N}^*, 1 \leq t \leq \mu - 1, \deg(u_t) \neq \deg(b) - 2} m_{s,t} b^s u_t + \left( \vec{\nabla} b \times \vec{Q} \right) \right)$$

$$= \left( - \sum_{k \in \mathbb{N}} e_{k} u_{k} b_{k} + \sum_{s \in \mathbb{N}^*, 1 \leq t \leq \mu - 1, \deg(u_t) \neq \deg(b) - 2} s m_{s,t} u_t b^{s-1} \right) \vec{\nabla} b,$$

so that the element $\sum_{s \in \mathbb{N}^*, 1 \leq t \leq \mu - 1, \deg(u_t) \neq \deg(b) - 2} m_{s,t} b^s u_t + \left( \vec{\nabla} b \times \vec{Q} \right)$ is a Casimir (element of $Z_0^0(\psi_b)$) and, according to the proposition 4.1, there exist some constants $\alpha_v \in \mathbb{K}$, for $v \in \mathbb{N}$, such that

$$(12) \sum_{s \in \mathbb{N}^*, 1 \leq t \leq \mu - 1, \deg(u_t) \neq \deg(b) - 2} m_{s,t} b^s u_t + \left( \vec{\nabla} b \times \vec{Q} \right) = \sum_{v \in \mathbb{N}^*} \alpha_v b^v.$$
(here $\alpha_0 = 0$, because $\deg(b) \geq 2$ and for example using (3)) and

$$- \sum_{k \in \mathbb{N}, 0 \leq k \leq \mu - 1}^{\deg(u_k) = \deg(b) - 2} e_{k, \ell} b^k u_\ell + \sum_{s \in \mathbb{N}^*, 1 \leq s \leq \mu - 1}^{\deg(u_s) \neq \deg(b) - 2} s m_{s, t} u_t b^{s-1} + \text{Div}(\vec{Q})$$

\begin{equation}
\sum_{s \in \mathbb{N}^*, 1 \leq s \leq \mu - 1}^{\deg(u_s) \neq \deg(b) - 2} m_{s, t} \frac{b^{s-1}}{\deg(b)} u_t \vec{E} - \vec{Q} = \sum_{v \in \mathbb{N}^*}^{\deg(b)} \frac{\alpha_v}{\deg(b)} b^{v-1} \vec{E} + d\vec{\nabla}b.
\end{equation}

Computing the divergence in this last equation leads to

\begin{equation}
\sum_{s \in \mathbb{N}^*, 1 \leq s \leq \mu - 1}^{\deg(u_s) \neq \deg(b) - 2} m_{s, t} \frac{b^{s-1}}{\deg(b)} u_t ((s - 1) \deg(b) + \deg(u_t) + 2) b^{s-1} u_t + \text{Div}(\vec{Q}) = \sum_{v \in \mathbb{N}^*}^{\deg(b)} \frac{\alpha_v}{\deg(b)} ((v - 1) \deg(b) + 2) b^{v-1} - \vec{\nabla}d \times \vec{\nabla}b.
\end{equation}

Now, (13), together with (14), give

\[ \text{Div}(\vec{Q}) = \]

\[ - \sum_{k \in \mathbb{N}, 0 \leq k \leq \mu - 1}^{\deg(u_k) = \deg(b) - 2} e_{k, \ell} b^k u_\ell + \sum_{s \in \mathbb{N}^*, 1 \leq s \leq \mu - 1}^{\deg(u_s) \neq \deg(b) - 2} s m_{s, t} u_t b^{s-1} + \sum_{v \in \mathbb{N}^*}^{\deg(b)} v \alpha_v b^{v-1}, \]
which gives us:

$$\sum_{s \in \mathbb{N}^*, 1 \leq t \leq \mu - 1} \frac{m_{s,t}}{\deg(b)} (-\deg(b) + \deg(u_t) + 2) \ b^{s-1} \ u_t$$

$$+ \sum_{k \in \mathbb{N}, 0 \leq \ell \leq \mu - 1} e_{k,\ell} \ b^k \ u_{\ell} - \sum_{v \in \mathbb{N}^*} \frac{\alpha_v}{\deg(b)} (-\deg(b) + 2) \ b^{v-1}$$

$$= -\vec{\nabla} d \times \vec{\nabla} b.$$ 

Now, by lemma [4.5] this leads to $e_{k,\ell} = 0$, $m_{s,t} = 0$ and $\alpha_v = 0$, for all $k, \ell, s, t$ and $v$. This permits us to conclude that the sum is a direct sum and permits us to obtain the desired result. \(\square\)

Let us now determine the even Poisson cohomology associated to the Poisson structure

$$\psi_b = b_y \theta \partial_x \partial_\theta - b_x \theta \partial_y \partial_\theta.$$ 

First of all, we give the 0-th even Poisson cohomology space.

**Proposition 4.6.** Let $b \in \mathcal{A}'$ be an non-constant polynomial of $\mathcal{A}' = \mathbb{K}[x,y]$. The 0-th even Poisson cohomology space associated to $\psi_b$ is zero:

$$H^0_e(\psi_b) \simeq \{0\}.$$ 

**Proof.** Under the identifications of the cochain spaces we can write:

$$H^0_e(\psi_b) \simeq \{a \in \mathcal{A}' \mid a \vec{\nabla} b = \vec{0}\}.$$ 

As $b$ is supposed to be non-constant, this gives $H^0_e(\psi_b) \simeq \{0\}$. \(\square\)

**Proposition 4.7.** Let $b \in \mathcal{A}'$ be a homogeneous and non-constant polynomial of $\mathcal{A}' = \mathbb{K}[x,y]$. If $b$ is square-free then a basis of the first
even Poisson cohomology space associated to \( \psi_b \) is given by:

\[
H^1_e(\psi_b) \simeq \begin{cases} 
(0, \mathbb{K}[b]^2) & \text{if } \deg(b) = 1, \\
\bigoplus_{i=0}^{\mu-1} \mathbb{K}[b] \left( u_i, \vec{0} \right) \oplus \mathbb{K}[b] \left( 0, \vec{\nabla} b \right) & \text{if } \deg(b) = 2, \\
\bigoplus_{i=0}^{\mu-1} \mathbb{K}[b] \left( u_i, \vec{0} \right) \oplus \mathbb{K}[b] \left( 0, \vec{\nabla} b \right) & \text{if } \deg(b) > 2.
\end{cases}
\]

**Proof.** Let us recall that one can write

\[
H^1_e(\psi_b) \simeq \left\{ (r, \vec{Q}) \in \mathcal{A}^1 \times \mathcal{A}^2 \mid \vec{\nabla} \left( \vec{Q} \times \vec{\nabla} b \right) + \text{Div}(\vec{Q}) \vec{\nabla} b = \vec{0} \right\}.
\]

Let \((r, \vec{Q}) \in Z^1_e(\psi_b)\) be an even 1-cocycle. The element \(\vec{Q}\) then satisfies the equation:

\[
(15) \quad \vec{\nabla} \left( \vec{Q} \times \vec{\nabla} b \right) + \text{Div}(\vec{Q}) \vec{\nabla} b = \vec{0}.
\]

First of all, suppose that \(\deg(b) = 1\). In this case, writing \(\vec{Q} = (\vec{Q}_1, \vec{Q}_2)\), one computes that

\[
\vec{0} = \vec{\nabla} \left( \vec{Q} \times \vec{\nabla} b \right) + \text{Div}(\vec{Q}) \vec{\nabla} b = - \left( \vec{\nabla} b \times \vec{\nabla} Q_1 \right).
\]

According to proposition 4.11, this is equivalent to \(Q_1 \in \mathbb{K}[b]\) and \(Q_2 \in \mathbb{K}[b]\). Finally, because \(\deg(b) = 1\), one has \(\mu = 0\), so that lemma 4.5 proves that \(H^1_e(\psi_b) \simeq (0, \mathbb{K}[b]^2)\) if \(\deg(b) = 1\).

Now, suppose \(\deg(b) \geq 2\). Because the operator \(\vec{Q} \mapsto \vec{\nabla} \left( \vec{Q} \times \vec{\nabla} b \right) + \text{Div}(\vec{Q}) \vec{\nabla} b\) is homogeneous, we can suppose that \(\vec{Q}\) is homogeneous, which means that \(\vec{Q}\) is given by two homogeneous polynomials of the same degree. Equation (15) implies that the element \(\vec{Q} \times \vec{\nabla} b \in \mathcal{A}'\) is an odd 0-cocycle, so that, according to the proposition 4.11, there exists a constant \(\alpha \in \mathbb{K}\) and \(v \in \mathbb{N}\) such that

\[
(16) \quad \vec{Q} \times \vec{\nabla} b = \alpha b^v = \frac{\alpha}{\deg(b)} b^{\deg(b)-1} \vec{E} \times \vec{\nabla} b,
\]
(notice that $v \neq 0$, because $\deg(b) \geq 2$ and because of (3) for example). Because the Koszul complex is exact, this implies that there exists an element $q \in \mathcal{A}'$ satisfying

$$\vec{Q} = \frac{\alpha}{\deg(b)} b^{v-1} \vec{E} + q \vec{\nabla}b.$$ 

Now, we compute

$$\text{Div}(\vec{Q}) = -\frac{\alpha((v - 1) \deg(b) + 2)}{\deg(b)} b^{v-1} + \vec{\nabla}q \times \vec{\nabla}b,$$

which, together with (15) and (16), lead to:

$$(17) \quad -\frac{\alpha(- \deg(b) + 2)}{\deg(b)} b^{v-1} + \vec{\nabla}q \times \vec{\nabla}b = 0.$$ 

Now, two cases have to be studied: whether $\deg(b) = 2$ or $\deg(b) \neq 2$.

We first assume that $b$ is a polynomial of degree 2. Then, equation (17) becomes simply $\vec{\nabla}q \times \vec{\nabla}b = 0$, which, in view of proposition 4.1 implies that $q \in \mathbb{K}[b]$ and in this case $\vec{Q} = \frac{\alpha}{\deg(b)} b^{v-1} \vec{E} + q \vec{\nabla}b \in \mathbb{K}[b] \vec{E} + \mathbb{K}[b] \vec{\nabla}b$, so that we have shown that, if $\deg(b) = 2$, then

$$H^1_e(\psi_b) \subseteq \mathcal{A}' \cap \{\vec{\nabla}b \times \vec{\nabla}c \mid c \in \mathcal{A}'\} \quad (1, \vec{0}) \oplus \left(\mathbb{K}[b](0, \vec{E}) \oplus \mathbb{K}[b](0, \vec{\nabla}b)\right).$$

Conversely, it is easy to see that $\mathbb{K}[b](0, \vec{E}) \oplus \mathbb{K}[b](0, \vec{\nabla}b) \subseteq Z^1_e(\psi_b)$ in the case $\deg(b) = 2$ and that the sum is a direct one, so that, according to the lemma 4.5, we can conclude that

$$H^1_e(\psi_b) \simeq \bigoplus_{i=0}^{\mu-1} \mathbb{K}[b] \ u_i (1, \vec{0}) \oplus \mathbb{K}[b](0, \vec{E}) \oplus \mathbb{K}[b](0, \vec{\nabla}b), \quad \text{if } \deg(b) = 2.$$ 

Let us now consider the last case where $\deg(b) \neq 2$. In this case, the equation $\frac{\alpha(2 - \deg(b))}{\deg(b)} b^{v-1} = \vec{\nabla}q \times \vec{\nabla}b$, together with the lemma 4.5 imply in particular (because $1 = u_0$) that $\alpha = 0$ and $\vec{\nabla}b \times \vec{\nabla}q = 0$, which as above leads to $q \in \mathbb{K}[b]$. In this case, we then have shown that:

$$H^1_e(\psi_b) \simeq \bigoplus_{i=0}^{\mu-1} \mathbb{K}[b] \ u_i (1, \vec{0}) \oplus \mathbb{K}[b](0, \vec{\nabla}b), \quad \text{if } \deg(b) \neq 2.$$ 

□
Proposition 4.8. Let $b \in \mathcal{A}'$ be a homogeneous and non-constant polynomial of $\mathcal{A}' = \mathbb{K}[x, y]$ and $\psi_b$ be the Poisson structure as above. If $b$ is square-free then a basis of the 2-nd even Poisson cohomology space associated to $\psi_b$ is given by:

$$H^2_e(\psi_b) \simeq \bigoplus_{i=0}^{\mu-1} \mathbb{K} (u_i, 0, 0) \quad (\simeq \mathcal{A}'_{\text{sing}}(b))$$

$$\simeq \bigoplus_{i=0}^{\mu-1} \mathbb{K} u_i \partial_x \partial_y.$$  

Proof. Recall that one can write

$$H^2_e(\psi_b) \simeq \left\{ (a, d, \vec{C}) \in \mathcal{A}' \times \mathcal{A}' \times \mathcal{A}^2 \mid \nabla b \times \vec{C} = 0; \quad \begin{array}{c} d \nabla b + \nabla \left( \vec{C} \times \nabla b \right) \\ + \text{Div}(\vec{C}) \nabla b = 0 \end{array} \right\}.$$

Let now $(a, d, \vec{C}) \in Z^2_e(\psi_b)$ be an even 2-cocycle. Because $\nabla b \times \vec{C} = 0$ and because the Koszul complex is exact, there exists $f \in \mathcal{A}'$ satisfying $\vec{C} = f \nabla b$. Then, the other cocycle condition becomes $d \nabla b + \left( \nabla f \times \nabla b \right) \nabla b = 0$, so that $d = -\nabla f \times \nabla b$. Now, according to (3), there exist $\vec{Q} \in \mathcal{A}^2$ and some constants $\lambda_i \in \mathbb{K}$, for $0 \leq i \leq \mu - 1$, such that

$$a = \vec{Q} \times \nabla b + \sum_{i=0}^{\mu-1} \lambda_i u_i.$$

We finally have:

$$(a, d, \vec{C}) = \left( \vec{Q} \times \nabla b, -\nabla f \times \nabla b, f \nabla b \right) + \sum_{i=0}^{\mu-1} \lambda_i (u_i, 0, 0)$$

$$\quad \in B^2_e(\psi_b) \oplus \bigoplus_{i=0}^{\mu-1} \mathbb{K} (u_i, 0, 0),$$

which gives the result. \qed

Proposition 4.9. Let $b \in \mathcal{A}'$ be a homogeneous and non-constant polynomial of $\mathcal{A}' = \mathbb{K}[x, y]$ and $\psi_b$ be the Poisson structure defined as previously. If $b$ is square-free then a basis of the third even Poisson
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cohomology space associated to $\psi_b$ is given by:

$$H^3_e(\psi_b) \cong \bigoplus_{i=0}^{\mu-1} \mathbb{K}[b] \left( u_i, 0, 0 \right) \cong \mathcal{A}'_{\text{sing}}(b)$$

$$\cong \bigoplus_{i=0}^{\mu-1} \mathbb{K}[b] u_i \theta \partial_x \partial_y \partial_\theta.$$

Proof. We have:

$$H^3_e(\psi_b) \cong \left\{ (a, d, \vec{C}) \in \mathcal{A}' \times \mathcal{A}' \times \mathcal{A}'^2 \mid \vec{\nabla} b \times \vec{C} = 0; \right. \left. 2d\vec{\nabla} b + \vec{\nabla} \left( \vec{C} \times \vec{\nabla} b \right) \right. \left. + \text{Div}(\vec{C})\vec{\nabla} b = 0 \right\}$$

Assume that an element $(a, d, \vec{C}) \in Z^3_e(\psi_b)$ is an even 3-cocycle. According to the lemma 4.5, there exist $p \in \mathcal{A}'$ satisfying $a \in \vec{\nabla} b \times \vec{\nabla} p + \sum_{j=0}^{\mu-1} \mathbb{K}[b] u_j$.

Moreover, as we have $\vec{\nabla} b \times \vec{C} = 0$ and because of the exactness of the Koszul complex, there exists $f \in \mathcal{A}'$ such that $\vec{C} = 2f\vec{\nabla} b$. The other cocycle condition now becomes:

$$2d\vec{\nabla} b + 2 \left( \vec{\nabla} f \times \vec{\nabla} b \right) \vec{\nabla} b = 0,$$

i.e., $d = -\vec{\nabla} f \times \vec{\nabla} b$.

Finally, this leads to

$$(a, d, \vec{C}) \in \left( \vec{\nabla} b \times \vec{\nabla} p, -\vec{\nabla} f \times \vec{\nabla} b, 2f\vec{\nabla} b \right) + \sum_{j=0}^{\mu-1} \mathbb{K}[b] (u_j, 0, 0)$$

$$\in \ B^3_e(\psi_b) \oplus \bigoplus_{j=0}^{\mu-1} \mathbb{K}[b] (u_j, 0, 0),$$

\[ \blacksquare \]

Finally, we give the $n$-th Poisson cohomology associated to $\psi_b$, for $n \geq 4$.

Proposition 4.10. Let $b \in \mathcal{A}'$ be a homogeneous and non-constant polynomial of $\mathcal{A}' = \mathbb{K}[x, y]$ and let $n \in \mathbb{N}$ such that $n \geq 4$. Let $\psi_b$ be the Poisson structure defined by the following

$$\psi_b = b_y \theta \partial_x \partial_\theta - b_x \theta \partial_y \partial_\theta.$$
If \( b \) is square-free then a basis of the \( n \)-th even Poisson cohomology space associated to \( \psi_b \) is given by:

\[
H^e_n(\psi_b) \simeq \bigoplus_{i=0}^{\mu-1} \mathbb{K} \begin{pmatrix} u_i, 0, 0 \end{pmatrix} \quad (\simeq A'_{\text{sing}}(b))
\]

\[
\simeq \bigoplus_{i=0}^{\mu-1} \mathbb{K} u_i \theta \partial_x \partial_y \partial_\theta^{n-2}.
\]

**Proof.** As previously, we write:

\[
H^e_n(\psi_b) \simeq \left\{ \begin{array}{l}
\vec{\nabla} b \times \vec{C} = 0; \\
(a, d, \vec{C}) \in A' \times A' \times A^2 \mid (n-1)d\vec{\nabla} b + \vec{\nabla} (\vec{C} \times \vec{\nabla} b) \\
+ \text{Div}(\vec{C}) \vec{\nabla} b = \vec{0}
\end{array} \right\}
\]

\[
H^e_n(\psi_b) \simeq \left\{ \begin{array}{l}
\vec{\nabla} b \times \vec{\nabla} p - (n-3)\vec{R} \times \vec{\nabla} b, \vec{\nabla} b \times \vec{\nabla} s, (n-1)s\vec{\nabla} b \\
\mid (p, s, \vec{R}) \in A' \times A^2
\end{array} \right\}.
\]

Now, to determine \( H^e_n(\psi_b) \), one uses the same arguments (the exactness of the Koszul complex and the equation \([3]\)) and a very similar reasoning as for the computation of the space \( H^e_2(\psi_b) \). \qed

**Remark 6.** Notice that the determination of the odd and even Poisson cohomology spaces associated to \( \psi_b \) could have been done with the hypothesis of \( b \) being homogeneous replaced by \( b \) being a weight-homogeneous polynomial (i.e., where the two variables \( x \) and \( y \) are equipped with weights which are not necessarily equal to 1, see \([7]\)).

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