Polymer quantization and symmetries

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Abstract
Polymer quantization was rediscovered during the construction of loop quantum cosmology. For the simplest quantum theory of one degree of freedom, the implications for dynamics were studied for the harmonic oscillator as well as some other potentials. For more degrees of freedom, the possibility of continuous, kinematic symmetries arises. While these are realized on the Hilbert space of polymer quantum mechanics, their infinitesimal versions are not supported. For an invariant Hamiltonian, these symmetry realizations imply infinite degeneracy suggesting that the symmetry should be spontaneously or explicitly broken. The estimation of symmetry violations in some cases have been analysed before. Here, we explore the alternative of shifting the arena to the distributional states. We discuss both the polymer quantum mechanics case and the polymer-quantized scalar field.

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1. Introduction

In loop quantum gravity (LQG) [1], the twin demands of SU(2) gauge invariance and diffeomorphism covariance are met by the use of the holonomies as basic variables and an inner product defined using the Haar measure on SU(2). The Hilbert space one obtains is unique up to unitary equivalence [2]. This also has the unusual feature that while the diffeomorphisms have a well-defined unitary action, their infinitesimal versions cannot be defined as operators [3]. When specialized to the mini-superspace models, an analogous procedure leads to the so-called polymer quantization [4]. Its Hilbert space is non-separable and has the same feature of finite translations being well defined but not the infinitesimal generators—the momenta variables. The Stone–von Neumann theorem is evaded by relaxing the requirement of (weak) continuity of the representation of the Weyl–Heisenberg algebra. One can however introduce an approximate version of momenta and construct corresponding non-relativistic dynamics. This necessarily introduces a fundamental scale and modifies the energy spectra. Nevertheless for certain systems, it can be seen explicitly that the deviations from the usual Schrödinger quantized model are essentially indistinguishable observationally. Such generic conclusions have been obtained for the one-dimensional harmonic oscillator [4] and inverse power potentials in (effectively) one dimension [5]. For particles moving in more

1 Please see acknowledgements.
dimensions, we have the possibilities of rotationally invariant systems and a natural question is to ask how the symmetry can be incorporated. This question, in the more general context of Galilean symmetries, has been addressed by Chiou [6]. He also noted that while finite group actions are well defined, the infinitesimal ones are not. He then explored the ‘approximated forms’ of the usual generators (which do not form a closed algebra) and concluded that the deviations are small within the domain of validity of the non-relativistic model.

We would like to ask if this technical feature of non-existence of infinitesimal generators has any physically relevant consequences? Is this necessarily an undesirable feature? If yes, how is the role of the polymer representation to be understood? After all in the LQG context, the analogue of polymer representation is very much physically well motivated but forms only an intermediate step due to the constrained nature of the system.

The short answers are that the non-existence of infinitesimal generators of rotations in the polymer representation also implies infinite degeneracy for any rotationally invariant Hamiltonian and this is a physically undesirable feature. One option then is to break the symmetry either explicitly or spontaneously. Yet another possibility is to note that we could view the polymer Hilbert space as part of a Gelfand-like triple, $\text{Cyl} \subset H_{\text{poly}} \subset \text{Cyl}^*$ as in the case of LQG and define infinitesimal generators on a suitable subspace of $\text{Cyl}^*$. The usual Schrodinger quantization can then be recovered, albeit trivially. The corresponding steps in the context of a polymer-quantized scalar field reveal further possibilities. It is the second alternative that is explored in this work.

In section 2, we briefly present the basic definitions of polymer quantization in terms of a triple, as well as specify the action of rotations. We point out how rotationally invariant Hamiltonians can be constructed and show that the spectra of such Hamiltonians are infinitely degenerate.

In section 3, we show how to define infinitesimal generators on a suitable subspace of the dual $\text{Cyl}^*$. A new inner product can be naturally defined on this subspace which makes these generators self-adjoint and also makes the completion unitarily equivalent to the Schrodinger quantization. Although recovering Schrodinger quantization is hardly the aim, we view this as an illustration of a multi-step quantization procedure which could be needed in more complex systems.

With this in view, a polymer-quantized scalar field is considered in section 4. Although it shares the features seen in polymer quantum mechanics, there seem to be many more possibilities for a quantum theory admitting infinitesimal symmetries. In section 5, we give a summary and conclude with a discussion.

We would like to emphasize the viewpoint taken in this work. While the polymer quantization, especially in the field theory context, is naturally adapted to diffeomorphism covariance, nothing prevents us from using it in the context of a fixed background geometry and coordinates. The background structures limit the diffeomorphisms to isometries of the background geometry and now become symmetries (transformations among physical states leaving the dynamics invariant). We are concerned with the representations of these symmetries in the Hilbert space of polymer quantization. The potential violations, if any, refer to these symmetries and never to any local invariances (gauge invariances). Keeping this in mind, we work with a background geometry which is Euclidean and explore the implications of polymer quantization with regard to symmetry implementation.

2. Rotational invariance in polymer quantum mechanics

Consider a non-relativistic particle moving in three dimensions. Classically, it is described by the configuration space, $\mathbb{R}^3$ coordinatized by $\vec{q} \leftrightarrow q^i, i = 1, 2, 3$. To construct polymer
quantization, choose a countable set \( \gamma \) of three-dimensional vectors \( \vec{k}_i \) and define a set \( \text{Cyl}_\gamma \) of linear combinations of functions of \( \vec{q} \) of the form \( \text{Cyl}_\gamma := \{ \sum_j f_j e^{i \vec{k}_j \cdot \vec{q}}, f_j \in \mathbb{C} \}. \) Here, the coefficients \( f_j \) satisfy certain regularity conditions \([4]\) which do not concern us here. Next, define the set of functions of \( \vec{q} \), \( \text{Cyl} := \bigcup \gamma \text{Cyl}_\gamma \). On this set, define the inner product\(^2\):

\[
\langle \psi | \psi' \rangle := \lim_{R \to \infty} \frac{3}{4 \pi R^3} \int_0^R q^2 dq \sin \theta d \theta d \phi \; \psi^* (\vec{q}) \psi' (\vec{q}) \Rightarrow (1)
\]

\[
\langle \vec{k} | \vec{k}' \rangle := \lim_{R \to \infty} \frac{3}{4 \pi R^3} \int_0^R q^2 dq \sin \theta d \theta d \phi \; e^{i (\vec{k} - \vec{k}') \cdot \vec{q}} = \delta_{\vec{k}, \vec{k}'} \; \forall \vec{k}, \vec{k}' \in \mathbb{R}^3. \tag{2}
\]

Clearly, \( \{ e^{i \vec{q} / \hbar} \vec{k} \in \mathbb{R}^3 \} \) form an uncountable, orthonormal set and we denote them as the kets \( | \vec{k} \rangle \). We denote the completion of \( \text{Cyl} \) w.r.t. this inner product, as \( H_{\text{poly}} := \text{Cyl} \). The Hilbert space is non-separable and we also have the natural triple, \( \text{Cyl} \subset H_{\text{poly}} \subset \text{Cyl}^* \), where \( \text{Cyl}^* \) denote the algebraic dual of \( \text{Cyl} \). Note that the integration measure is invariant under three-dimensional rotations and preserves the orthonormality in the second equation above. This will permit the unitary representation of rotation group (equation (5)).

It is clear that \( \vec{q} \) cannot be represented on the polymer Hilbert space as a multiplicative operator since \( q^3 \) acting on a basis element does not produce a countable linear combination of the basis elements (exponentials). The exponentials of the form \( e^{i \vec{q} / \hbar} \) however do form multiplicative (and unitary) operators. The derivatives too act invariantly on \( \text{Cyl} \) and \( p_i := -i \hbar \frac{\partial}{\partial q^i} \) are self-adjoint operators representing the momenta. The exponentials are the eigenfunctions of the momenta \( \hat{p}_i | \vec{k} \rangle = \hbar k_i | \vec{k} \rangle \).

That the self-adjoint position operators \( \vec{q} \) do not exist can be seen more formally as well. Consider a one-parameter family of unitary operators, defined by \( U(\alpha, \vec{m}) | \vec{k} \rangle := | \vec{k} + \alpha \vec{m} \rangle \; \forall \vec{k} \in \mathbb{R}^3 \). For any vector \( \vec{c} \), \( (\vec{c} | U(\alpha, \vec{m}) | \vec{l} \rangle) \vec{c} = (\vec{c} + \alpha \vec{m}) \delta_{\vec{a}, \vec{b}} \), as implied by the orthonormality. Hence, the family of unitary operators is not weakly continuous at \( \alpha = 0 \). If a self-adjoint operator of the form \( \vec{m} \cdot \vec{q} \) existed, then we could define a one-parameter family of unitary operators \( V(\alpha, \vec{m}) := e^{i \alpha \vec{m} \cdot \vec{q}} \) which is continuous at \( \alpha = 0 \) and precisely matches the \( U(\alpha, \vec{m}) \) family, thus reaching a contradiction. Hence, on the polymer Hilbert space, the momenta and exponentials of positions are well-defined operators but there are no self-adjoint operators representing positions. This feature of polymer quantization has profound implications for the implementation of continuous, non-Abelian symmetries.

Recall that any group of symmetries is represented in a quantum theory by unitary operators\(^3\), with the states transforming as \( | \psi \rangle \to U(\lambda) | \psi \rangle \) and the operators transforming as \( A \to A_{\lambda} := U(\lambda) A U(\lambda)^\dagger \) for each group element \( g \in \mathcal{G} \). The specific unitary operators representing a specific symmetry operation can be determined by stipulating how the basic observables transform. For example, with \( q^i, p_i \) being the basic observables in the usual quantization, the unitary operators corresponding to rotations are determined by

\[
q^i_{\lambda} := U(\Lambda) q^i U(\Lambda)^\dagger, \quad p^i_{\lambda} := U(\Lambda) p_i U(\Lambda)^\dagger = \Lambda^i_j p_j, \quad \Lambda^i_{\mu} \Lambda^\mu_{\nu} \delta^{\nu\mu} = \delta^{ij}. \tag{3}
\]

For infinitesimal rotations, \( \Lambda^i_j := \delta^i_j + \epsilon^i_j, \; U(1 + \epsilon) := 1 - \frac{i}{\hbar} \epsilon \cdot \hat{J} \), we obtain

\[
- \frac{i}{\hbar} [\epsilon \cdot \hat{J}, q^i] = \epsilon^i_j q^j, \quad - \frac{i}{\hbar} [\epsilon \cdot \hat{J}, p_i] = \epsilon^i_j p_j. \tag{4}
\]

\(^2\) Strictly, it is not necessary to give an explicit expression for the inner product. In fact, an analogous expression cannot be given when one wants to realize Lorentz symmetry. It is enough to stipulate the orthonormal set, as in equation (2).

\(^3\) We will not be considering time reversal or charge conjugation symmetries, so we will not consider anti-unitary operators.
With the identifications $\epsilon^i_j := \epsilon^i_k \epsilon^{kj}_l$, $\epsilon \cdot \hat{J} := \epsilon_{kl} \hat{J}^k$, we deduce $\hat{J}^k := \epsilon^{nk} q^n p_k$ as the operators representing the infinitesimal generators.

Alternatively, the operators $U(\Lambda)$ could also be determined by specifying their action on wavefunctions—explicit functions on the configuration space (say), e.g., $\Psi(\vec{q}) := \Psi(\Lambda \vec{q})$.

For the polymer quantization, the defining stipulations for the action of rotations are as follows:

\[ (e^{i \vec{q}}) : U(\Lambda)(e^{i \vec{q}}) U(\Lambda)^\dagger = (e^{i \Lambda \vec{q}}), \quad p^k := U(\Lambda) p_i U(\Lambda)^\dagger = \Lambda^i_j p_j. \tag{5} \]

Noting that $|\vec{k}\rangle$ are eigenstates of $\hat{p}_i$, it follows that

\[ U^\dagger(\Lambda) \hat{p}_i U(\Lambda) |\vec{k}\rangle = (\Lambda^{-1})^j_i |\vec{k}\rangle \Rightarrow (\Lambda^{-1} \hat{p}_j |\vec{k}\rangle \Rightarrow \hat{p}_i [U(\Lambda) |\vec{k}\rangle] = [(\Lambda^{-1})^j_i] [U(\Lambda) |\vec{k}\rangle] \]

\[ \therefore U(\Lambda) |\vec{k}\rangle = [\Lambda^{-1} |\vec{k}\rangle]. \tag{6} \]

Evidently, this action of rotation group on the polymer Hilbert space is reducible, with the orbit through any $\vec{k}$ being spanned by the orthonormal kets $\{|\vec{i}\rangle\}$ with $\vec{i}$ lying on the 2-sphere through $\vec{k}$. The subspace spanned by $\{|\vec{k}\rangle, \vec{k} \vec{k} = \text{constant}\}$ forms an irreducible representation and is clearly infinite dimensional.

This may come as a surprise as one recalls the theorem that all unitary, irreducible representations of the rotation group (indeed any compact group) are finite dimensional. However, it is to be noted that the theorem is proved only for continuous representations of the group (which arise from and also induce representations of the corresponding Lie algebra).

It is also a theorem that if $G$ is a locally compact topological group whose every irreducible representation on a Hilbert space is continuous, then the group itself is discrete [7]. Since the rotation group is locally compact and is not a discrete group, it must have discontinuous representations as well and these do not have to be finite dimensional. What we have is an explicit example of just such a representation whose discontinuous nature is shown below.

The above action of the rotation group, coupled with the fact that the kets $|\vec{k}\rangle$ are orthonormalized, implies that $U(\Lambda)$ also cannot be weakly continuous. Unlike the one-dimensional case where the group action necessarily transformed a basis vector to another basis vector, here we have the possibility that $\vec{k}$ could be along the axis of rotation represented by $U(\Lambda)$ and hence invariant under $U(\Lambda)$. To show discontinuity, consider any one-parameter subgroup of rotations. All these rotations will leave some particular axis invariant. Choose any $\vec{k}$, orthogonal to this axis. Now, the subgroup action transforms a basis ket to another distinct basis ket. The lack of weak continuity for every one-parameter subgroup follows as before and we cannot write $U(\Lambda) = 1 - \frac{1}{h} \epsilon \cdot \hat{J}$. Note that it is not the case that every one-parameter family of unitary operators is necessarily non-continuous. For continuity to be possible, members of the unitary family must not map any basis vector to another basis vector.

So, while we do not have a representation of infinitesimal action, finite rotations are perfectly well defined. However for rotations to be a symmetry, their action must also preserve the dynamics. Classically, we have three ‘elementary’ rotational invariants: $p \cdot p$, $q \cdot q$, and $p \cdot q$. Only first of these can be promoted to operator on the polymer Hilbert space. A Hamiltonian which is a function of $p^2$ alone will describe only a ‘free’ dynamics. Is this the only possible rotationally invariant dynamics supported by the polymer Hilbert space? Not quite. As noted in the context of the ‘improved quantization’ of LQC, exponentials of arbitrary functions of momenta times $q^i$ (i.e., functions linear in $q^i$) can also be promoted to well-defined operators. This is because the Hamiltonian vector field $X_q$ generates translations along $p_i$ and any function of $\vec{p}$ multiplying $X_q$ generates more general infinitesimal transformations, also along $p_i$. While

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4 We thank Alok Laddha for pointing this out.
$X_\ell$ cannot be promoted to an operator, its exponential which generates \textit{finite diffeomorphisms} can be! Incorporating rotational invariance, we can thus have unitary operators of the form $e^{i\theta (p^2)_{\nu\rho} q^\rho}$ . From these, the corresponding sin and cos self-adjoint operators can be defined. A candidate rotationally invariant Hamiltonian will be a function of $p^2$ and the sin, cos operators. There is no corresponding trick to use the $q \cdot q$ invariant.

To compute the action of finite diffeomorphism, say by a unit parameter, consider the integral curves by

$$\frac{dp}{d\lambda} = f(p \cdot p) p \implies \frac{dp \cdot p}{d\lambda} = 2(p \cdot p) f(p \cdot p)$$

$$\int_0^1 d\lambda = \frac{1}{2} \int_{p_{\text{final}}}^{p_{\text{initial}}} \frac{dp^2}{p^2 f(p^2)} \tag{7}$$

This defines the change in the $p \cdot p$ for unit change in the parameter. Note that the vector field is \textit{radial}, and therefore the integral curves are in the radial direction (in $p'$ space) and for unit change in the parameter, connect two spheres of radii $p_{\text{initial}}$ and $p_{\text{final}} := \xi^2 p_{\text{initial}}$. The corresponding unitary operator is then defined by

$$e^{-i\theta (p^2)_{\nu\rho} q^\rho} := |\vec{k}' = \xi \vec{k}\rangle,$$

the scale $\xi$ being determined by equation (7).

Thus, we \textit{can} have non-trivial rotationally invariant dynamics. However, there is now a different problem. As noted before, the unitary representation of SO(3) on the polymer Hilbert space is reducible with irreducible representations carried by $\mathcal{H}_\sigma := \text{span}[|\vec{k}\rangle, k \cdot k = \sigma^2 > 0]$. Each of these is infinite dimensional. Each eigenspace of any invariant Hamiltonian will carry a representation of SO(3) which has to be infinite dimensional, being made up of some of the irreducible representations together possibly with the trivial representation ($\sigma = 0$). Thus, we face the problem of \textit{infinite degeneracy} which is physically untenable: the partition function of such a system will be undefined. We now have two possibilities: (a) rotations cease to be a symmetry (explicit breaking of symmetry) and (b) \textit{spontaneous breaking} of rotational symmetry.

To see both possibilities, we first seek an approximate substitute for the position operators. The operators, $e^{i\hat{q}_j}$, allow us to define families of self-adjoint operators. For instance, choosing $\hat{k}_j := \delta \hat{e}_j$, $\hat{e}_j$ as a unit vector, we can define $\sin_{\hat{k}_0} := (2i)^{-1} (e^{i\hat{q}_0} - e^{-i\hat{q}_0})$ and a cos operator analogously\textsuperscript{5}. We could choose several triplets of linearly independent unit vectors $\hat{e}_j$ and also choose many different parameters $\ell$ (equivalently, finitely many $\hat{k}_j$). If we collect finitely many of such sets and restrict ourselves to observables which are functions of these (and the momenta) operators, then from any given $|\vec{k}_0\rangle$, we will generate a collection of basis vectors, $\{|\vec{k}_0 + \sum_j n_j \vec{k}_j\rangle, n_j \in \mathbb{Z}\}$. The closed subspace generated by this set will be a \textit{proper subspace} of the polymer Hilbert space and is clearly \textit{separable}. If we also include operators which are exponentials in $p \cdot q$, discussed above, then the lattice generated will also involve scaling determined by the choices for $f(p^2)$. As long as the number of such operators is finite, we will continue to have separable sectors. The chosen set of observables will act invariantly on each of these subspaces and will provide \textit{superselection sectors}. Observe that among the chosen class of observables, we can also have an invariant Hamiltonian. The action of rotations however mixes different sectors and we have \textit{spontaneous breaking of rotational invariance}. If we chose a Hamiltonian involving the approximated position operators, we have \textit{explicit breaking} of rotations controlled by the $\delta$-parameter(s). The example of spherically symmetric harmonic oscillator in three dimensions illustrates this. For an economical parametrization of violation,

\textsuperscript{5} These operators however do not suffice to represent the Lie algebra of rotations [6].
we can choose a single common δ. For sufficiently small values of this, at a certain level of
observational precision, it is, of course, possible to have the illusion of rotational invariance.

To summarize, having made a choice of the polymer Hilbert space $H_{\text{poly}}$, we can have exact
rotational symmetry with a somewhat restricted form of dynamics (no $q$-$q$ dependence) but with
uncountably infinite degeneracy. To avoid the problem of infinite degeneracy, the symmetry
must be broken—either explicitly or spontaneously. By introducing separable sectors, we can
see both possibilities.

One can however also view the polymer quantization as an intermediate step in
constructing a quantum theory, much as the kinematical Hilbert space of LQG is. Using
the triple, we can try to select a suitable subspace of $\text{Cyl}^*$ on which infinitesimal generators
can be defined. With a suitable choice of a new inner product, we can obtain a ‘physical’
Hilbert space with a rotationally invariant dynamics.

3. Infinitesimal generators

The possibility of looking to $\text{Cyl}^*$ for a home to a suitable quantum theory is inspired by
analogous steps taken in the context of LQG. In LQG, the step is motivated for a very different
reason. The kinematical quantization is essentially forced upon us by the demand of SU(2)
invariance and diffeomorphism covariance. Since there are constraints whose kernels are in
general distributional, an appropriate diffeo-invariant subspace of corresponding $\text{Cyl}^*$ is a
natural arena. In our case, the polymer quantization itself is not a compulsion, but is a useful
illustration of a multi-step construction of a quantum theory.

Recall that the construction of $H_{\text{poly}}$ naturally gave us the triple $\text{Cyl} \subset H_{\text{poly}} \subset \text{Cyl}^*$. This
structure provides us with a convenient representation of the elements $(\Psi)$ of $\text{Cyl}^*$ by complex-valued linear functions $\psi^*(\vec{k}) := (\Psi|\vec{k})$. No smoothness properties are assumed at this stage
for these functions. Furthermore, for every operator $A : \text{Cyl} \rightarrow \text{Cyl}$, we can define an operator
$A : \text{Cyl}^* \rightarrow \text{Cyl}^*$ by the ‘dual action’, e.g., $(A|\Psi|f) := (\Psi|Af)$, $\forall f \in \text{Cyl}$, $\forall (\Psi) \in \text{Cyl}^*$. Conversely, given an operator $A$ defined on all of $\text{Cyl}^*$, we can define an operator $A$ on $\text{Cyl}$ by the same equation as above (read backwards). In particular, this means that we have the operators $\hat{U}(A)$ defined on $\text{Cyl}^*$. We will use these to define infinitesimal generators on $\text{Cyl}^*$. We will also define the position operators.

We begin with infinitesimal rotation generators.

$$
(\Psi|U(1 + \epsilon) - U(1 - \epsilon)|\vec{k}) = (\Psi|\vec{k} + \vec{\epsilon}k) - (\Psi|\vec{k} - \vec{\epsilon}k) \\
\approx 2\epsilon_i e^{\epsilon_i j} k^j \frac{\partial \psi^*}{\partial k^i}
$$

$$
\therefore \lim_{\epsilon_i \rightarrow 0} 2\epsilon_i \frac{U(1 + \epsilon) - U(1 - \epsilon)}{2\epsilon_i} |\vec{k}\rangle = \mathcal{E}^{ij} k^j \frac{\partial \psi^*}{\partial k^i}
$$

$$
\therefore (\mathcal{J}\psi|\vec{k}) := -i\hbar e^{\epsilon_i j} k^j \frac{\partial \psi^*}{\partial k^i}.
$$

Note that these operators are defined only on a subspace of $\text{Cyl}^*$, consisting of those $(\Psi)$
whose corresponding $\psi^*(\vec{k})$ are differentiable functions. Hence, by the dual action, we cannot
define the corresponding operators on $\text{Cyl}$.

Next, recall the $\sin_{\delta_{ij}}$ operators defined in the previous section. For each orthonormal
triad, $\hat{e}_j$, $j = 1, 2, 3$, $\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$ and a small parameter $\delta$, we have $U_{\hat{e}_j}(\vec{q}) := e^{i\hat{e}_j \cdot \vec{q}}$ and $
\sin_{\delta_{ij}} := (2i)^{-1}(U_{\hat{e}_j}(\vec{q}) - U_{-\hat{e}_j}(\vec{q}))$. Now,

$$
2i(\Psi|\sin_{\delta_{ij}}|\vec{k}) = (\Psi|\vec{k} + \delta \hat{e}_j) - (\Psi|\vec{k} - \delta \hat{e}_j) \\
= \psi^*(\vec{k} + \delta \hat{e}_j) - \psi^*(\vec{k} - \delta \hat{e}_j)
$$
\[ \approx 2\delta \hat{\epsilon}_j \cdot \frac{\partial \psi^*}{\partial \vec{k}} \]
\[ \therefore \lim_{\delta \to 0} \left( \sin_{\delta_{i}} \right) \frac{\partial \psi^*}{\partial \vec{k}} = -i \hat{\epsilon}_j \cdot \hat{\nabla}_\epsilon \psi^*. \] (10)

Thus, by restricting to functions \( \psi^* \) which are at least differentiable, we can define a position operator on a subspace of Cyl\(^*\) via the dual action:
\[ (\hat{\epsilon}_j \cdot \hat{q} \psi|\vec{k} \rangle) := -i \hat{\epsilon}_j \cdot \hat{\nabla}_\epsilon \psi^*, \forall (\psi| \in \text{Cyl}^* \text{ such that } \psi^* (\vec{k}) \text{ is differentiable}. \] (11)

It is easy to see that the position operators defined above and the momentum operators defined by the dual action also satisfy
\[ (i\hbar \hat{\epsilon}_m \cdot \hat{\epsilon}_n \cdot \hat{p}) \psi|\vec{k} \rangle = (i\hbar \hat{\epsilon}_m \cdot \hat{\epsilon}_n) \psi|\vec{k} \rangle. \]

So far, we have not specified any subspace of Cyl\(^*\) except to say that it consists of, at least, differentiable functions. The space of all differentiable functions is too large a subspace to choose. We are guided in our choice of a subspace by the requirement that the ‘position’ and the ‘momentum’ operators be self-adjoint with respect to a suitable inner product and satisfy the canonical commutation relation on an invariant, common dense domain. Representations of the canonical commutation relations are usually analysed by going to the bounded, unitary operators (exponentials of the positions and momenta), satisfying the Weyl–Heisenberg relations. The Stone–von Neumann theorem then guarantees a unique continuous representation of the Weyl–Heisenberg relations and the corresponding canonical commutation relations. This representation corresponds to the choice of Schwartz space as the subspace of Cyl\(^*\) and the usual inner product with the Lebesgue measure. The Hilbert space is then obtained by completing Schwartz space in the \( L_2 \) norm. Making this choice, we just get back the usual Schrodinger quantization using functions of ‘momenta’, \( \vec{k} \) instead of functions of ‘positions’, \( \vec{q} \). The intermediate polymer quantization has only led us to the Heisenberg representation instead of the Schrodinger representation. The measure being invariant under rotations also admits the (unitary) representation of infinitesimal rotations. It is interesting to note that one can also choose a subspace which is larger, e.g., space of \( \psi^* (\vec{k}) \) which is normalizable with respect to a Sobolev norm, and choose the Lorentz-invariant measure \( \frac{d^4 k}{2 \sqrt{k + m^2}} \) to construct Hilbert space of a free, relativistic particle of mass \( m \) [8]. This is not our primary concern though.

This is obviously a roundabout way of arriving at the usual quantization. But it shows that (a) not every choice of quantization may be flexible enough for physical modelling and (b) we can reach a satisfactory quantum theory by modifying the quantization algorithm. In principle, if the quantum theory constructed from a subspace of Cyl\(^*\) were not satisfactory, we could repeat the process forming a new triple. This is further discussed in the last section. In the next section, we discuss the case of a scalar field theory.

4. The case of a scalar field theory

Can rotational invariance be supported in a ‘polymerized scalar field theory’? Consider the example of a scalar field \( \phi (\vec{x}) \) defined on \( \mathbb{R}^3 \). The rotations act on the space which in turn induces a transformation on the field: \( \phi' (\vec{x}) = \phi (\Sigma \vec{z} \cdot \vec{x}) \). The polymer quantization of the scalar field is done as follows [9].

Define a vertex set \( V = (\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n) \), of finitely many, distinct points. For non-zero real numbers \( \lambda_j, j = 1, \ldots, n \), define the functions \( N_{\lambda_j} (\phi) := e^{i \sum \lambda_j \phi (\vec{x}_j)} \). For each fixed set \( V \), let Cyl\(_V\) denote the set of finite, complex linear combinations of these functions. Let Cyl
Thus, every element of Cyl is a function of \( \phi \) which is a \emph{finite} linear combination of functions \( N_{V_{\lambda_{i}}} \) for some vertex set \( V \) and some choice of \( \lambda \). Define an inner product
\[
\langle \psi | \psi' \rangle := \int d\mu(\phi) \psi^* (\phi) \psi' (\phi) = \int d\mu(\phi) \sum_{V, \lambda} C_{V, \lambda}^* C_{V', \lambda'} e^{i \sum \lambda_j \phi(x_j) - i \sum \lambda'_j \phi'(x'_j)}. \tag{12}
\]

Observe that each term in the summand is again of the form \( N_{U_{\lambda_{i}}} \), except that all vertices in the union \( V \cup V' \) are not necessarily distinct. If \( x'_k = x_j \), then the exponent would be \( (\lambda'_k - \lambda_j) \phi(x'_k) \). If \( \lambda'_s \) are equal, then the exponent is identically zero and the integral contributes to the sum. Otherwise, the integral gives zero. It follows that \( N_{V_{\lambda_{i}}} \) and \( N_{V'_{\lambda'_i}} \) are orthogonal unless the two sets of vertices coincide and their corresponding \( \lambda \)'s are equal.

The Hilbert space \( H_{\text{poly}} \) is obtained as the Cauchy completion of Cyl with respect to this inner product. The functions \( N_{V_{\lambda_{i}}} (\phi) \), with every \( \lambda \neq 0 \), form an orthonormal basis for the polymer Hilbert space. The constant function corresponding to the empty vertex set, \( N (\phi) = 1 \), is also included in the basis.

The action of rotations on Cyl is defined by \( [U_{\Lambda} | \psi \rangle := | \psi \rangle (\Lambda \circ \phi) \). Evaluating it on the elementary functions leads to
\[
N_{V_{\lambda_{i}}} (\phi) \rightarrow N_{V'_{\lambda'}_{i}} (\phi) := N_{V_{\lambda_{i}}} (\phi') = N_{V_{\lambda_{i}}} (\phi) \tag{13}
\]

The middle equality is the definition of the action \( \phi' = \Lambda \circ \phi \) and we have used the scalar nature of \( \phi, \phi' (\vec{x}) = \phi(\Lambda \vec{x}) \), in the last equality.

Observe that under the action of rotation \( \Lambda \), a vertex set \( V = (\vec{x}_1, \ldots, \vec{x}_n) \) changes to a new vertex set \( V' := (\Lambda \vec{x}_1, \ldots, \Lambda \vec{x}_n) \). \( \lambda \)'s are unchanged and the field is evaluated at the transformed points. Since \( \lambda \)'s do not change and the inner product depends only on them, the inner product among elementary functions is invariant under the action of the rotations and therefore rotations are represented unitarily on the Hilbert space.

That this unitary action is also non-weakly continuous can be seen easily. For a non-trivial rotation, a diagonal matrix element between basis states is zero while for the identity rotation, the matrix element is 1. Thus, infinitesimal generators have no representation on the polymer Hilbert space.

The momenta variables are defined as
\[
P_{\vec{x}} := \int d^3 x g(\vec{x}) \pi_{\phi}(x) = -i h \int d^3 x g(\vec{x}) \frac{\delta}{\delta \phi(\vec{x})} \tag{14}
\]

Here, \( g(\vec{x}) \) is a ‘suitably smooth’ function (\( \pi_{\phi} \) has density weight 1, though not relevant here). It is easy to see that
\[
P_{\vec{x}} N_{V_{\lambda_{i}}} = \left[ h \sum_{j} \lambda_{j} g(\vec{x}_j) \right] N_{V_{\lambda_{i}}}, \quad [P_{\vec{x}}, P_{\vec{y}}] = 0, P_{\vec{x}}^{\dagger} = P_{\vec{x}} \tag{15}
\]

Thus, the momentum representation exists and the elementary functions \( N_{V_{\lambda_{i}}} \) are simultaneous eigenstates of the momenta variables \( P_{\vec{x}} \). Under the action of rotation, \( U(\Lambda) \), the momentum variables transform as
\[
U_{\Lambda} P_{\vec{x}} (\tau) U_{\Lambda}^{\dagger} := P_{\vec{x}} (\Lambda \circ \tau) \quad \text{(from the definition)} \Rightarrow \nonumber \tag{16}
U_{\Lambda} P_{\vec{x}} (\tau) = P_{\Lambda^{-1} \varphi} (\tau) U_{\Lambda}.
\]

This is consistent with (13). Let us use the notation \( | V, \vec{x} \rangle \leftrightarrow N_{V_{\lambda_{i}}} (\phi) \).

Observe that \( e^{i \phi(\vec{x})} \), a ‘point holonomy operator’, clearly acts as a multiplication operator:
\[
e^{i \phi(\vec{x})} | V, \vec{x} \rangle := \begin{cases} | \vec{x}_1, \ldots, \vec{x}_n, \vec{x}; \lambda_1, \ldots, \lambda_n, \lambda \rangle, & \text{if } \vec{x} \neq \vec{x}_i \text{ for any } i, \\ | \vec{x}_1, \ldots, \vec{x}_i \vec{x}_i \ldots \vec{x}_n; \lambda_1, \ldots, \lambda_k, \lambda_k + \lambda_i, \ldots, \lambda_n \rangle, & \text{if } \vec{x} = \vec{x}_i, \lambda + \lambda_k \neq 0, \\ | \vec{x}_1, \ldots, \vec{x}_n; \lambda_1, \ldots, \lambda_n \rangle, & \text{if } \vec{x} = \vec{x}_i, \lambda + \lambda_k = 0. \nonumber \tag{17} \end{cases}
\]

In the last equation, the \( \vec{x}_i, \lambda_k \) labels are missing on the right-hand side.
What about the scalar field operator itself? It does not exist since the point holonomy operators are not weakly continuous, exactly as in the point-particle case. In the usual Schrodinger-type representation too, a scalar field operator exists only as an operator-valued distribution. This has to do with the presence of Dirac delta in the canonical commutation relations. In the polymer representation, it does not exist even as an operator-valued distribution.

Now consider an element \((\Psi) \in \text{Cyl}^*\). Its action on an elementary function \(\mathcal{N}_{V, \tilde{\lambda}}(\phi)\) is given by

\[
(\Psi|V, \tilde{x}) := \psi^*(\tilde{x}_1, \ldots, \tilde{x}_n, \lambda_1, \ldots, \lambda_n), \quad \text{distinct } \tilde{x}'s \text{ and non-zero } \lambda's.
\]

Under the action of rotations, the arguments of the elementary function change: \(|V, \tilde{x}) \rightarrow |V', \tilde{x}'.\) Thus, if we choose the functions \(\psi^{*}\)'s to be differentiable, we can define infinitesimal rotations as operators on a subspace of \(\text{Cyl}^*\), exactly as before. Explicitly,

\[
(\Psi|U(1 + \epsilon) - U(1 - \epsilon)|V, \tilde{x}) = (\Psi|V', \tilde{x}) - (\Psi|V'', \tilde{x})
\]

\[
= \psi^*(\tilde{x}_1 + \epsilon \tilde{x}_1, \ldots, \tilde{x}_n + \epsilon \tilde{x}_n, \lambda_1)
- \psi^*(\tilde{x}_1 - \epsilon \tilde{x}_1, \ldots, \tilde{x}_n - \epsilon \tilde{x}_n, \lambda_1)
\]

\[
\approx 2\epsilon \mathcal{E}^k \sum_{m=1}^{n} x_m \frac{\partial \psi^*}{\partial x_m}.
\]

\[
\lim_{\epsilon \rightarrow 0} \frac{U(1 + \epsilon) - U(1 - \epsilon)}{2\epsilon}|V, \tilde{x}) = \mathcal{E}^k \sum_{m=1}^{n} x_m \frac{\partial \psi^*}{\partial x_m}.
\]

Thus, by restricting to a subspace of \(\text{Cyl}^*\), corresponding to suitably differentiable functions \(\psi^*(V, \tilde{x})\), we can define the generator of the infinitesimal rotations.

Likewise, to define a smeared operator scalar field on \(\text{Cyl}^*\), consider

\[
(\Psi|\phi^0_j|V, \tilde{x}) := \int d^3x f(\tilde{x}) \left( \frac{e^{i\phi_j(\tilde{x})}}{2i\delta} - \frac{e^{-i\phi_j(\til{x})}}{2i\delta} \right)|V, \tilde{x})
\]

\[
= \int d^3x f(\tilde{x}) \left( (\Psi|V, \tilde{x}, \tilde{x}, \delta) - (\Psi|V, \tilde{x}, \tilde{x}, -\delta) \right).
\]

For a generic \(\tilde{x}\), assuming differentiability of \(\psi^*\), we will obtain a function of the vertices of \(V\) and the corresponding \(\lambda's\) together with the additional point \(\tilde{x}\) and the corresponding \(\delta = 0\). This function cannot come from any element of \(\text{Cyl}^*\) acting on \(|V, \tilde{x})\). Hence, we should avoid obtaining a contribution from a generic \(\tilde{x}\). If, however, \(\tilde{x}\) coincides with one of the vertices in \(V\), then the resultant function (derivative) is a function of \((V, \tilde{x})\) and we can interpret the right-hand side as a new element of \(\text{Cyl}^*\) evaluated on the basis element \(|V, \tilde{x})\). This can be made more precise by employing the commonly used procedure of defining the integral by introducing a cell decomposition adapted to the ‘graph’ (vertices of \(V\) and \(demanding\)

\[
\frac{\partial \psi^*}{\partial \lambda_j}(\tilde{x}_1, \ldots, \tilde{x}_n, \lambda_1, \ldots, \lambda_n)|_{\lambda_j=0} = 0, \forall j = 1, 2, \ldots, n.
\]

This condition ensures that there is no contribution from cells that do not contain a vertex of \(V\) and we are led to the definition

\[
(\phi^*_j|V, \tilde{x}) := \lim_{\delta \rightarrow 0} (\Psi|\phi^*_j|V, \tilde{x}) := -i \sum_j f(\tilde{x}_j) \frac{\partial \psi^*(\tilde{x}_1, \ldots, \tilde{x}_n, \lambda_1, \ldots, \lambda_n)}{\partial \lambda_j}.
\]
It is now easy to verify that
\[
([\phi_f, P_g])|V, \tilde{x}\rangle := (\tilde{P}_g \phi_f \Psi|V, \tilde{x}\rangle - (\tilde{P}_g \phi_f \Psi|V, \tilde{x}\rangle)
\]
\[
= -i\hbar \left( \sum_{j=1}^{n} f(\tilde{x}_j)g(\tilde{x}_j) \right) (\Psi|V, \tilde{x}\rangle)
\]
\[
= \left\{ +i\hbar \left( \sum_{j=1}^{n} f(\tilde{x}_j)g(\tilde{x}_j) \right) \right\} \Psi|V, \tilde{x}\rangle.
\]
(22)

We have thus succeeded in defining the smeared versions of the field operators \(\phi_f, P_g\) in a subspace of \(\text{Cyl}^*\).

We can also verify that the infinitesimal generators \(J^k\) induce expected actions on the smeared field operators.

\[
([J^k, \phi_f])|V, \tilde{x}\rangle = (\tilde{J}_k \phi_f \Psi|V, \tilde{x}\rangle - (\tilde{J}_k \phi_f \Psi|V, \tilde{x}\rangle)
\]
\[
= -i \sum_{m=1}^{N} f(\tilde{x}_m) \frac{\partial \psi^*_{\phi_f}(V, \tilde{x})}{\partial \lambda_m} + (i\hbar)E^{ki}_{m} \sum_{n=1}^{N} x_n^j \frac{\partial \psi^*_{\phi_f}(V, \tilde{x})}{\partial x_n^j}
\]
\[
= \hbar \left[ -i \sum_{m=1}^{N} \left( E^{ki}_{m} \frac{\partial f(\tilde{x}_m)}{\partial x_n^j} \frac{\partial \psi^*_{\phi_f}(V, \tilde{x})}{\partial \lambda_m} \right) \right]
\]
\[
= i\hbar (\tilde{\phi}_{\tilde{C}_k} \phi_f |V, \tilde{x}\rangle, \quad L_k f(\tilde{x}) := E^{ki}_{m} \frac{\partial f(\tilde{x}_m)}{\partial x_n^j}.
\]
(23)

Similar computation can be done for the commutator of \([J^k, P_n]\).

We have now identified the minimal conditions, namely differentiability in all arguments and the condition of equation (20), on functions \(\psi^*(\tilde{x}_1, \ldots, \tilde{x}_n, \lambda_1, \ldots, \lambda_n)\) in order that the smeared field operators and the infinitesimal rotation actions are well defined. Since such an element of \(\text{Cyl}^*\) can be viewed as a sequence of differentiable, complex functions defined on \((\mathbb{R}^3 - \text{diagonal}) \times (\mathbb{R}^n - 0)\) where the diagonal is the subset of \(\mathbb{R}^3\) with two or more points coinciding, we are restricted to a subspace of \(\text{Cyl}^*\). The next step is to choose a suitable inner product on this subspace, possibly restricted further with additional conditions. Let us denote such a subspace by \(\text{Cyl}_1\). Here, we initiate first steps. For notational simplicity, let us denote elements of \(\text{Cyl}^*\) generically by underlined letters such as \(\tilde{\psi}, \tilde{\phi}, [\tilde{V}, \tilde{x}]\), etc.

Heuristically, we can represent each element of \(\text{Cyl}_1\) and a yet-to-be-defined inner product as follows:

\[
\Psi := \sum_{V, \tilde{x}} \psi^*(V, \tilde{x})|V, \tilde{x}\rangle,
\]
(24)

\[
\langle \Psi, \Phi \rangle := \sum_{V, \tilde{x}} \sum_{V', \tilde{x}'} \psi(V, \tilde{x})\phi^*(V', \tilde{x}')\langle |V, \tilde{x}\rangle, [V', \tilde{x}']\rangle
\]
\[
= \sum_{V, \tilde{x}} \sum_{V', \tilde{x}'} \psi(V, \tilde{x})\phi^*(V', \tilde{x}')G(V, \tilde{x}; V', \tilde{x}').
\]
(25)

The coefficients \(\psi^*(V, \tilde{x})\) in the first line contain the information about the subspace \(\text{Cyl}_1\). \(G\) denotes the inner product between ‘basis’ elements.

For example, \(\text{Cyl}\) is a subspace of \(\text{Cyl}^*\) through the natural embedding \(|V, \tilde{x}\rangle \in \text{Cyl} \rightarrow [V, \tilde{x}] \in \text{Cyl}^*\). If \(\text{Cyl}_1\) were to be this subspace, then \(\psi^*(V, \tilde{x})\) in equation (24) would
be non-zero only for finitely many \((V, \vec{\lambda})\) sets and \(G(V, \vec{\lambda}; V', \vec{\lambda}')\) would equal \(\delta_{V,V'}\delta_{\vec{\lambda},\vec{\lambda}'}\). The double summation would then collapse to a finite sum over \((V, \vec{\lambda})\) (compare equation (12)). Likewise, if \(\text{Cyl}_1\) were to echo the Hilbert space of the \(r\)-Fock construction [9, 10], \(G(V, \vec{\lambda}; V', \vec{\lambda}')\) would be \(\sim \exp[-\frac{1}{4} \sum_{ij} G_{ij}(\vec{\chi}_i, \vec{\chi}_j)\lambda_i\lambda_j]\), where the sum over \((i, j)\) is over the vertices of \(V \cup V'\) and we use the notation of [9]. The double sum will be a finite sum since \(\psi^*\) is non-zero for finitely many \((V, \vec{\lambda})\) sets.

More generally, we could have uncountably many non-zero \(\psi^*\) and then each \(\Psi\) can be thought of as a potentially infinite sequence of functions, \(\psi_n\), on \(\sim \mathbb{R}^{4n}\). If we choose an inner product so that the ‘basis states’ are orthonormal \((G \propto \delta_{V,V'}\delta_{\vec{\lambda},\vec{\lambda}'})\), then we may write the inner product as follows:

\[
\langle \Psi, \Phi \rangle := \sum_{V,\vec{\lambda}} \psi(V, \vec{\lambda})\psi^*(V', \vec{\lambda}')
\]

\[
\approx \sum_{n=0}^{\infty} \int_{\mathbb{R}^{3n}} d^{3n} \vec{x} \int_{\mathbb{R}^{3n}} d^n \lambda \psi(\vec{x}_1, \ldots, \vec{x}_n, \lambda_1, \ldots, \lambda_n)\psi^*(\vec{x}_1, \ldots, \vec{x}_n, \lambda_1, \ldots, \lambda_n).
\]

\(\sim\) indicates that the integration measures need to be defined and we need to put conditions to ensure the convergence of the sum.

Assuming that we can choose suitable weights in the sum and measures in the integrations, what further conditions we need to put on \(\psi_n\) so that our basic operators and generators are self-adjoint? It is easy to see that we need only the usual fall-off conditions on these so that surface terms resulting from the partial integrations drop out. Roughly, we make each member \(\psi_n\) as an element of \(L_2(\mathbb{R}^{3n})\). This indicates that it is, at least heuristically, conceivably to choose suitable definitions to construct a new Hilbert space.

Many more issues have to be addressed. Even for the point-particle case, self-adjointness and even commutation relations were not enough to lead to a unique choice, the Weyl–Heisenberg relations are needed to be invoked. For a field theory, it is known that even after invoking the Weyl–Heisenberg relations, there are infinitely many inequivalent representations of the canonical commutation relations. In the usual case, Poincare invariance is additionally invoked to uniquely single out the Fock representation [11]. A detailed analysis of the possibilities is beyond the scope of this work.

5. Summary and discussion

We began by exploring the symmetries and their violations in polymer-quantized systems. Specifically, we focused on three-dimensional rotations and explored the polymer-quantized particle in three dimensions and a scalar field defined on \(\mathbb{R}^3\). It is certainly possible to have a unitary representation of \(SO(3)\) on the polymer Hilbert space but the representation is discontinuous and consequently does not admit the representation of its Lie algebra. The non-availability of configuration space operators—position operators—severely restricts the possible invariant Hamiltonians and every one of these has infinitely degenerate eigenvalues. In effect, physically acceptable dynamics on the polymer Hilbert space must necessarily violate rotational symmetry, either explicitly or spontaneously. In the case of explicit breaking, one can then look for economical parametrization of symmetry violations and put bounds on the parameters. As noted in the introduction, this route has already been followed in [4, 6, 5]. We explored another route to see if acceptable quantization, with infinitesimal symmetries, can be arrived at viewing polymer quantization as an intermediate step. This was done by looking for suitable subspace(s) of the dual member of the Gelfand triple with a hope of defining a new inner product and a new Hilbert space. For the point-particle case, we verified that it is possible to construct a new Hilbert space which carries continuous representations
of the rotation group as well as continuous representations of the Heisenberg group. By the Stone–von Neumann theorem, this is, of course, the usual Schrodinger representation which supports the usual rotationally invariant non-trivial Hamiltonians. The case of scalar field revealed greater richness. There can be infinitely many choices of inner products, all of which can support infinitesimal rotations as well as elementary smeared field.

In principle, neither of the two routes is unnatural. It is not certain that continuous symmetries need be realized exactly in nature even if observations support their existence to excellent approximation, e.g., Lorentz symmetry. Symmetries help to exercise tighter control over theoretical frameworks but the physical system may not exactly respect the implicit idealization. The in-built, non-invariant dynamics of a polymer-quantized system suggests a particular parametrization of symmetry violation, e.g., the use of the ‘trigonometric’ operators to build the Hamiltonian. At least in the cases explored, such violations are viable.

The second alternative is anyway needed in the context of theories with first class constraints. It could well be thought of as a multi-step quantization procedure. Just as in a classical theory, specified by an action, the variables we begin with need not represent the physical states (e.g., when there are constraints). However, following a systematic procedure—the Dirac algorithm of constraint analysis—we can arrive at a formulation which is either a theory with a first class constraint algebra or a theory without any constraints. Likewise, one could begin with a set of basic functions on the configuration space forming $\text{Cyl}_0$, choose an inner product $\langle | \rangle_0$, obtain $\text{Cyl}_0^*$ as well as a Hilbert space $H_0$ forming a triple: $\text{Cyl}_0 \subset H_0 \subset \text{Cyl}_0^*$. If the model is satisfactory, we are done. If not, look for a subspace $\text{Cyl}_1 \subset \text{Cyl}_0^*$, define a new inner product $\langle | \rangle_1$ and obtain a new triple $\text{Cyl}_1 \subset H_1 \subset \text{Cyl}_1^*$. Hopefully, the process would terminate after a finite number of iterations. This procedure offers a flexibility to refine the class of observables we wish to support on the quantum state space. It is constructive and could help keep the focus on physical observables. This possibility needs to be examined further to see its viability/utility.

We have considered the scalar field theory with ‘point holonomies’ as basic functions generating the commutative $\mathbb{C}^*$ algebra. Fermions are similar to point holonomies as far as the label sets are concerned. For gauge fields, we will have $H_{\text{poly}} := H_{\text{kin}}$ with the basis labelled by discrete labels. Hence, the analogues of $(V, \lambda)$ will now have embedded graphs and representation labels of the gauge group. One will have to impart a ‘manifold structure’ for these spaces of labels to attempt a definition of infinitesimal generators in the manner discussed above.

We would like to end by drawing a parallel with the recent work on the polymer quantization of parametrized field theory (PFT) [12]. PFTs are field theories with a background geometry which however are presented in a diffeomorphism-covariant form by promoting the background coordinates to fields. The diffeomorphism covariance introduces constraints and the physical sector of the theory is the old theory with a background. Consider for definiteness a free-field theory on the flat Minkowski spacetime and its parametrized form. In the non-parametrized form, isometries of the Minkowski metric are the symmetries of the theory. One may choose the usual Fock quantization and see the representations of the infinitesimal symmetries. In the parametrized form however, the diffeomorphism covariance would suggest the polymer quantization, not just for the embedding variables but also for the scalar field. One can now ask how the isometries are represented in such a quantization.

If we insist that the Dirac quantization of the PFT should produce a physical sector which is the same as the quantized non-parametrized theory and it is possible to realize this, then the quantization of the matter sector chosen in the non-parametrized form will already determine the symmetry realization, regardless of its parametrized version. However, it is conceivable that there is a (different) Dirac quantization of the polymer-quantized PFT such
that the physical states carry the usual Fock representation. There is no definitive statement available on this as yet. Such a possibility could be quite relevant for LQG, at least in a ‘semiclassical approximation’. This is one context in which the discussion of this work, especially the Cyl alternative, could be directly relevant.

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References

[1] Rovelli C 2004 Quantum Gravity (Cambridge: Cambridge University Press)

[2] Thiemann T 2007 Introduction to modern canonical quantum general relativity arXiv:gr-qc/0110034

[3] Lewandowski J, Okolow A, Sahlmann H and Thiemann T 2006 Uniqueness of diffeomorphism invariant states on holonomy-flux algebras Commun. Math. Phys. 267 703–33 (arXiv:gr-qc/0504147)

[4] Fleischhack C 2009 Representations of the Weyl algebra in quantum geometry Commun. Math. Phys. 285 67–140 (arXiv:math-ph/0407006)

[5] Ashtekar A, Lewandowski J, Marolf D, Mourao J and Thiemann T 1995 Quantization of diffeomorphism invariant theories of connections with local degrees of freedom J. Math. Phys. 36 6456–93 (arXiv:gr-qc/9504018)

[6] Ashtekar A, Fairhurst S and Willis J L 2003 Quantum gravity, shadow states, and quantum mechanics Class. Quantum Grav. 20 1031–62 (arXiv:gr-qc/0207106)

[7] Thirring W and Narnhofer N 1992 Covariant QED without indefinite metric Rev. Math. Phys. 4 197–211

[8] Halvorson H 2004 Complementarity of representations in quantum mechanics Stud. Hist. Phil. Mod. Phys. 35 45–56 (arXiv:quant-ph/0110102)

[9] Husain V, Louko J and Winkler O 2007 Quantum gravity and the Coulomb potential Phys. Rev. D 76 084002 (arXiv:0707.0273)

[10] Kunstatter G, Louko J and Ziprick J 2009 Polymer quantization, singularity resolution and the 1/r^2 potential Phys. Rev. A 79 032104 (arXiv:0809.5098)

[11] Chiou D-W 2007 Galileo symmetries in polymer particle representation Class. Quantum Grav. 24 2603–20 (arXiv:gr-qc/0612155)

[12] Barut A O and Raczka R 1986 Theory of Group Representations and Applications (Singapore: World Scientific) p 197 theorem II

[13] Jaffe A 2005 Introduction to Quantum Field Theory www.arthurjaffe.com/Assets/pdf/IntroQFT.pdf

[14] Ashtekar A, Lewandowski J and Sahlmann H 2003 Polymer and Fock representations for a scalar field Class. Quantum Grav. 20 L11–1 (arXiv:gr-qc/0210102)

[15] Varadarajan M 2000 Fock representations from U(1) holonomy algebras Phys. Rev. D 61 104001 (arXiv:gr-qc/0001050)

[16] Ashtekar A and Lewandowski J 2001 Relation between polymer and Fock excitations Class. Quantum Grav. 18 L117–28 (arXiv:gr-qc/0107043)

[17] Bogolubov N N, Logunov A A, Oksak A I and Todorov I 1990 General Principles of Quantum Field Theory (Dordrecht: Kluwer)

[18] Laddha A and Varadarajan M 2010 Polymer quantization of the free scalar field and its classical limit Class. Quantum Grav. 27 175010 (arXiv:1001.3505)

[19] Thiemann T 2010 Lessons for loop quantum gravity from parametrised field theory arXiv:1010.2426

[20] Laddha A and Varadarajan M 2011 The Hamiltonian constraint in polymer parametrized field theory Phys. Rev. D 83 025019 (arXiv:1011.2463)

6 In the two-dimensional spacetime, R × S^1, the work of Laddha and Madhavan, the third paper of [12], obtains the isometry group being spontaneously broken to its discrete subgroup.