QCD effective action with dressing functions – consistency checks in the perturbative regime

Gorazd Cvetić, Igor Kondrashuk and Iván Schmidt

Dept. of Physics, Universidad Técnica Federico Santa María, Valparaíso, Chile

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In a previous paper, we presented solution to the Slavnov–Taylor identity for the QCD effective action, and argued that the action terms containing (anti)ghost fields are unique. These terms have the same form as those in the classical action, but the gluon and (anti)ghost effective fields are convoluted with gluon and ghost dressing functions \( G_A \) and \( G_c \), the latter containing perturbative and nonperturbative effects (but not including the soliton-like vacuum effects). In the present work we show how the perturbative QCD (pQCD) can be incorporated into the framework of this action, and we present explicit one-loop pQCD expressions for \( G_A \) and \( G_c \). We then go on to check the consistency of the obtained results by considering an antighost Dyson–Schwinger equation (DSE). By solving the relations that result from the Legendre transformation leading to the effective action, we obtain the effective fields as power expansions of sources. We check explicitly that the aforementioned one-loop functions \( G_A \) and \( G_c \) fulfil the antighost DSE at the linear source level. We further explicitly check that these one-loop \( G_A \) and \( G_c \) have the regularization-scale and momentum dependence consistent with the antighost DSE at the quadratic source level. These checks suggest that the the effective action with dressing functions represents a consistent framework for treating QCD, at least at the one-loop level.

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I. INTRODUCTION

Nonperturbative methods are essential in order to rigorously prove well accepted QCD phenomena such as confinement and chiral symmetry breaking. In principle the effective action contains all these effects, allowing for the calculation of all proper (one-particle-irreducible) \( n \)-point Green functions. Furthermore, its minimum gives the true quantum mechanical vacuum of the theory. One way to obtain information about the effective action is through the Slavnov–Taylor (ST) identity \([1, 2, 3]\), which is a consequence of the invariance of the classical (tree level) action with respect to the BRST transformation \([4, 5]\). The ST identity for the effective action relates different functional derivatives of this action with respect to the effective fields. In the \( N = 1 \) four-dimensional supersymmetric theory without matter, the form of the solution to the ST has been found in Ref. \([6]\). In Ref. \([7]\), we presented a solution to the ST identity for the effective action of nonsupersymmetric non-Abelian gauge theory.

QCD is an example of nonsupersymmetric non-Abelian gauge theory. The solution has the form of the classical (QCD) action, but the fields entering it are convolutions of the corresponding effective fields with pertaining dressing functions. These functions are supposed to contain the quantum effects of the theory – perturbative and nonperturbative, with the possible exception of soliton-like vacuum effects. Further, in that work we argued that the presented terms of the solution which contain the ghost (and antighost) fields are the only ones possible. Stated differently, the terms in the effective action containing (anti)ghost terms have the classical form and all the quantum effects connected with Green functions with at least one (anti)ghost leg must be contained in the ghost and gluon dressing functions \( G_A \) and \( G_c \).

We also assume that those terms in the effective action solution which contain no (anti)ghost fields have essentially the classical form, but with the fields convoluted with the dressing functions. Further, the coupling parameter is running. In this case, we have not presented arguments that such a solution to the ST identity for the effective action is the only possible one. The main attractive feature of this effective action is that all the quantum effects are parametrized with a finite number of dressing functions and with the running function of the coupling.

A natural question appearing is whether the known QCD results can be consistently incorporated into the aforementioned effective action, i.e., into the gluon, ghost and quark dressing functions. In this work we show how the known one-loop perturbative QCD (pQCD) results are reflected in this effective action. In fact, we find explicit
one-loop pQCD expressions for the ghost and gluon dressing functions $G_A$ and $G_c$ by requiring that the the correct known one-loop gluon and ghost propagators be reproduced starting from our effective action. It is not clear in advance whether our QCD effective action framework with the obtained dressing functions gives a self-consistent description of QCD in the high-momentum (perturbative) regime. We address this question of self-consistency in the following way. We first write down an antighost Dyson–Schwinger equation (DSE) for QCD. Then we express the effective action fields as a power series in the (external) sources, by solving the relations that follow from the Legendre transformation leading from the path integral to the effective action. These power series involve the aforementioned dressing functions. We then insert these expansions in the antighost DSE, with the one-loop dressing functions $G_A$ and $G_c$. We explicitly check that the regularization-scale and the momentum dependence of these $G_A$ and $G_c$ are consistent with the antighost DSE at the linear source and the quadratic source level,\(^1\) while at the linear source level we even check the consistency of the constant (“finite”) parts of the one-loop $G_A$ and $G_c$. All this strongly suggests that our effective action framework is consistent with pQCD in the corresponding (high) momentum region. It further keeps open the realistic possibility that the dressing functions of our effective action represent a sufficient framework for including not just all the (high momentum) perturbative, but also (lower momentum) nonperturbative quantum effects.

In Sec. II we recapitulate the results of Ref. [7] and present the aforementioned antighost DSE. In Sec. III we deduce the one-loop gluon and ghost dressing functions $G_A$ and $G_c$ from requiring the correct one-loop gluon and ghost propagators. In Sec. IV we write down the relations arising from the Legendre transformation leading to the effective action, and obtain the effective fields as power expansions in sources. In Sec. V we explicitly check that the obtained one-loop dressing functions $G_A$ and $G_c$ are consistent with the antighost DSE at the linear level of sources. At the quadratic level of sources, we explicitly check that the regularization-scale and momentum dependence of the one-loop $G_A$ and $G_c$ is consistent with the antighost DSE. In Sec. VI we summarize our results.

II. THE EFFECTIVE ACTION AND THE ANTIGHOST DYSON–SCHWINGER EQUATION

In order to fix notations, we first write the well-known classical action of QCD, in the Lorentz gauge

$$S_{\text{QCD}}[A, b, c, q, ar{q}] = \int d^4x \left\{ -\frac{1}{2g^2} \text{Tr} \left[ F_{\mu\nu}(A(x))F^{\mu\nu}(A(x)) \right] + \bar{q}(j) \gamma^\mu \nabla_\mu (A(q(j)) \right\} .$$ (1)

Here, $A$, $b$, $c$, and $q_j$ are the gluon, antighost, ghost, and quark ($j$'th flavor) fields, and $\alpha$ is the gauge parameter. For simplicity we omit the quark masses. For all the fields $X$ in the adjoint representation ($X = A, b, c$) we use the notation $X = X^a T^a$, where $T^a = \lambda^a/2$ are the gauge group generators ($N_c \times N_c$ hermitian matrices in the color space) with the normalization

$$\text{Tr} \left( T^a T^b \right) = \frac{1}{2} \delta^{ab}, \quad (T^a)^\dagger = T^a ,$$

$$[T^b, T^c] = if^{abc} T^a .$$ (2a)

The antighost and ghost fields are hermitian Grassmann numbers $b^\dagger = b$, $c^\dagger = c$. The covariant derivatives in the ghost part in (1) are

$$\nabla_\mu (A) c = \partial_\mu c + i [A_\mu, c] ,$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu] ,$$

$$\nabla_\mu (A) q(j) = \partial_\mu q(j) + i A_\mu q(j) .$$ (3a)

The fields $X^a$, $q(j)$, the coupling $g$, and the gauge-fixing parameter $\alpha$ in the action (1) are bare, i.e., the theory has a large (but finite) UV momentum cutoff $\Lambda$

$$X \equiv X^{(\Lambda)} , \quad q(j) \equiv q^{(\Lambda)}(j) , \quad g \equiv g^{(\Lambda)} , \quad \alpha \equiv \alpha^{(\Lambda)} = g^2(\Lambda)\xi^{(\Lambda)} ,$$ (4)

and the cutoff is implemented through a gauge invariant regularization, e.g., dimensional regularization.

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\(^1\) The two DSE’s involve indirectly three- and four-point connected Green functions.
A solution to the Slavnov–Taylor (ST) identity for the effective action $\Gamma$ has a similar form as the classical action $\Gamma_{QCD} \equiv \int d^4x \left\{ \mathcal{L}^\text{eff}_1(\bar{A}(x)) + \mathcal{L}^\text{eff}_2(\tilde{q}_{(j)}(x), \bar{q}_{(j)}(x), \bar{A}(x)) \right\} - \frac{1}{\alpha} \text{Tr} \left[ (\partial_\mu A^\mu(x))^2 \right] - 2 \text{Tr} \left[ i \bar{b}(x) \partial^\mu \nabla_\mu (\bar{A}(x)) \bar{c}(x) \right]$. \hfill (5)

Here, $\mathcal{L}^\text{eff}_1(\bar{A}(x))$ is a gauge invariant combination of the gluon field $\bar{A}$. The modified fields $\bar{X}$ appearing in (3) are the original fields convoluted with the corresponding dressing functions $G_X$

\[ \bar{A}_\mu^a(x) = (G_A^{-1} \circ A_\mu^a)(x) = \int d^4x' G_A^{-1}(x - x') A_\mu^a(x') , \] \hfill (6a)

\[ \bar{c}^a(x) = (G_c^{-1} \circ c^a)(x) = \int d^4x' G_c^{-1}(x - x') c^a(x') , \] \hfill (6b)

\[ \tilde{q}_{(j)}(x) = (G_q^{-1} \circ q_{(j)})(x) = \int d^4x' G_q^{-1}(x - x') \xi \xi q_{(j)}(x') , \] \hfill (6c)

\[ \tilde{b}^a(x) = (G_A \circ b^a)(x) = \int d^4x' G_c(x - x') b^a(x') . \] \hfill (6d)

Notice that the dressing functions $G_X$ appearing in relations (6a) and (6b) are the same ($G_A$). In (2), the spinor structure is reflected in indices $\xi, \xi^\prime = 1, 2, 3, 4$. The spacetime structure of the dressing functions is $\langle x | G_X | x' \rangle = G_X(x - x')$ due to translational invariance. These functions are real for $X = A, c$, and are in general complex $4 \times 4$ spinor matrices for $X = q_{(j)}$. Furthermore, the Fourier transforms of the dressing functions $G_A$ and $G_c$

\[ G_X(k^2) = \int d^4x \exp(ik \cdot x) G_X(x) \] \hfill (7)

are functions of $k^2$ due to Lorentz invariance. As a consequence, $G_X(x) = G_X(-x)$ ($X = A, c$). Apart from the aforementioned invariance properties of the dressing functions, the ST-identity for the effective action does not tell us anything about these functions.

We have argued in Ref. 3 that the terms containing the (anti)ghost fields in the solution (3) of the ST identity have a unique form as given in Eq. (4). This was a consequence of the assumption of Ref. 3 that the term $\sim L c^2$ of the effective action fulfills separately the ST identity for the effective action. Here, $L$ is an auxiliary (nonpropagating) background field which couples at the tree level to the BRST transformation of the ghost field $c$ in the form $2 \text{Tr}[L(x)c(x)^2]$ in the (classical) Lagrangian density. The aforementioned assumption was then shown to imply the following structure of the $\sim L c^2$ term of the effective action:

\[ \int d^4x \ 2 \text{Tr} \left[ (G_c \circ L)(x)(G_c^{-1} \circ c)(x)(G_c^{-1} \circ c)(x) \right] , \] \hfill (8)

where $G_c(x)$ is, a priori, an arbitrary dressing function. An argument was given in Ref. 4 in favor of the aforementioned assumption. Furthermore, arguments were presented there which suggest the structure (8) as the only consistent $\sim L c^2$ structure even when the mentioned assumption is not adopted. The structure of this correlator was then shown in Ref. 3 to restrict the gluon-ghost-antighost term to the one written in Eq. (4), and the physical part of the effective action was shown to be gauge-invariant in terms of the effective fields convoluted with the dressing functions (3) $X = G_X^{-1} \circ X$.

The gauge-fixing term in (4) is also unique, due to its known insensitivity to quantum corrections. In (3), the expression $L^\text{eff}_2$ which contains the quark fields is gauge-invariant combination of $\tilde{q}_{(j)}$, $\bar{q}_{(j)}$ and $\bar{A}$. In our further considerations in this paper, the precise form of these terms will not matter as we will concentrate on the question of consistency of the (anti)ghost and gluon sectors of the effective action (3) in the perturbative region.

The pure gluon term $L^\text{eff}_1(\bar{A}(x))$ in the effective action has a form somewhat different from the form of the pure gluon term in the classical action (4). This should be expected because the latter term is the only one which contains,

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2 The exception is the gauge-fixing part which is unchanged under quantum corrections.
in addition to the field(s), the coupling parameter \(g\). This parameter is running. Thus, it is natural to expect that the latter parameter in the effective action is not just normalized to a specific value \(g(\mu_0)\), where \(\mu_0\) is a specific, in principle arbitrary, renormalization scale. Rather, when going into the Fourier-transformed (momentum) space, we should associate to each \(k\)-mode of the dressed gluon field \(\tilde{A}\) the value of \(g(k)\). Specifically, while the pure gluon sector in the classical action (\(\Box\)) has the following form in the momentum space:

\[
S_{gl}[A] = -\frac{1}{2g^2} \int \frac{d^4k}{(2\pi)^4} \text{Tr} \left[ F_{\mu\nu}(A)(k) F^{\mu\nu}(A)(-k) \right],
\]

where \(F_{\mu\nu}(A)(k)\) denotes the Fourier transform of \(F_{\mu\nu}(A)(x) \equiv F_{\mu\nu}(A(x))\), the pure gluon sector in the effective action (\(\hat{\Box}\)) contains the analogous term

\[
\Gamma_{gl,1}[A] = -\frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \frac{1}{g^2(-k^2)} \text{Tr} \left[ F_{\mu\nu}(\tilde{A})(k) F^{\mu\nu}(\tilde{A})(-k) \right].
\]

However, in contrast to (\(\Box\)), the expression (\(\hat{\Box}\)) is not gauge invariant when \(\tilde{A} \to A\), although the full gluon sector of the effective action (\(\Box\)) must have this property as a consequence of the ST-identity. Thus, gluon contributions additional to those in (\(\hat{\Box}\)) must appear. In order to illustrate how to obtain these additional gluon terms, we will find them explicitly in the one-loop perturbative approximation. In this case, we have the relation

\[
\frac{1}{g^2(-k^2)} = \frac{1}{g^2(\mu_0^2)} + \frac{\beta_0}{4\pi^2} \ln \left( \frac{-k^2}{\mu_0^2} \right),
\]

where \(\beta_0 = \frac{1}{12}(11N_c - 2n_f)\).

Here, \(\beta_0\) is the one-loop coefficient for the beta-function for \(\alpha_s = g^2/(4\pi)\), \(N_c = 3\) is the number of colors, \(n_f\) is the number of active quark flavors. Using the one-loop evolution form (\(11a\)), in (\(10\)), we can rewrite the latter in the form of the following spacetime integrals:

\[
\Gamma_{gl,1}^{1\text{-loop}}[A] = -\frac{1}{2g^2(\mu_0)} \int d^4x \text{Tr} \left[ F_{\mu\nu}(\tilde{A})(x) F^{\mu\nu}(\tilde{A})(x) \right] - \frac{1}{2} \frac{\beta_0}{4\pi^2} \int d^4x \text{Tr} \left[ F_{\mu\nu}(\tilde{A})(x) \ln \left( \frac{\partial^2}{\mu_0^2} \right) F^{\mu\nu}(\tilde{A})(x) \right].
\]

The integrand in the second term is not gauge invariant when \(\tilde{A} \to A\). However, the minimal extension of the latter integrand to recover the aforementioned property is to replace \(\partial_\mu \to \nabla_\mu(\tilde{A})\)

\[
\Gamma_{gl}^{1\text{-loop}}[A] = -\frac{1}{2g^2(\mu_0)} \int d^4x \text{Tr} \left[ F_{\mu\nu}(\tilde{A})(x) F^{\mu\nu}(\tilde{A})(x) \right] - \frac{1}{2} \frac{\beta_0}{4\pi^2} \int d^4x \text{Tr} \left[ F_{\mu\nu}(\tilde{A})(x) \ln \left( \frac{\nabla^2(\tilde{A})(x)}{\mu_0^2} \right) F^{\mu\nu}(\tilde{A})(x) \right].
\]

By expanding the new logarithm in (\(13\)) around its value at \(A = 0\), we can immediately see that the new terms, i.e., the difference of the integrands of the second integrals in (\(12\)) \(\leftrightarrow (10)\) and (\(13\)), are terms at least cubic in gluon fields and of the type \(1/\partial\), i.e., in momentum space these are power-suppressed terms of the type \(1/k\). Such terms will not be relevant in our perturbative considerations, only the terms that depend logarithmically on the momenta will be. Therefore, in the perturbative considerations, we can consider the form (\(10\)) to represent the entire gluonic sector of the effective action.

In Appendix \(A\) we show explicitly how Eq. (\(12\)) follows from Eqs. (\(11\)) and (\(10\)), present clear expressions for the logarithmic operators appearing in the integrals in Eqs. (\(12\)) and (\(13\)), and show (tilde-)gauge invariance of the expression (\(13\)). Further, in that Appendix we argue how to proceed in the general (nonperturbative) case, i.e., when \(1/g(-k^2)\) is a general function of \(k^2\), in order to obtain from (\(14\)) the (tilde-)gauge invariant version of the gluonic effective action.

For further use in the manuscript, we will write here a general Dyson–Schwinger-type equation (DSE) which the QCD path integral has to fulfill. The QCD path integral \(Z \equiv \exp(-iW)\) is defined as

\[
\exp(-iW[J, \rho, \eta, \bar{j}, j]) = \int [dA][db][d\ell][dq_1][dq_2] \exp \left( iS_{\text{QCD}}[A, b, c, q, \bar{q}] + i\Xi[A, b, c, q, \bar{q}; J, \rho, \eta, \bar{j}, j] \right),
\]

(14a)
with
\[
\Xi[A, b, c, q, \bar{q}; J, \rho, \eta, j, j] = 2 \text{Tr} \left( \int d^4x \; J_\mu A^\mu + i \int d^4x \; \rho(x)b(x) + i \int d^4x \; \eta(x)c(x) \right) + \int d^4x \; (\bar{q}(j)j(j) + \bar{j}(j)q(j)) ,
\]
(14b)
where \( J, \rho, \eta, j, j \) are the (external) sources for the corresponding fields. The path integral \( Z \) is just a function of sources, with the fields integrated out. Therefore, in particular, \( Z \) is invariant under the (infinitesimal) shift of the antighost fields \( b(x) \rightarrow b(x) + c(x) \). This invariance can be rewritten, after some straightforward algebraic manipulations,\(^3\) in the form
\[
\partial^2 \rho^a(x) - f^{abc} \partial^a \left[ A^b_\mu(x)c^c(x) + \frac{\delta^2 W}{\delta J^\mu_b(x)} \delta \eta(x) \right] = -\rho^a(x) .
\]
(15)
Moreover, due to the relation \( \delta W/\delta \eta^c(x) = -ie^c(x) \) which follows from the Legendre transformation that connects \( W \) and \( \Gamma_{\text{QCD}} \), this relation can be rewritten as
\[
\partial^2 \rho^a(x) - f^{abc} \partial^a \left[ A^b_\mu(x)c^c(x) - i \frac{\delta \rho^c(x)}{\delta J^\mu_c(x)} \right] = -\rho^a(x) .
\]
(16)
Here it is understood that the effective fields \( c \) and \( A \) are functions of sources. We will call the identity (16) the antighost Dyson–Schwinger equation (b-DSE). The usual b-DSE equation is obtained by applying to (16) the variation \( \delta \rho^a(y) \) and then setting all the sources and fields equal to zero. In (19), the fields and the sources are in general nonzero.

### III. PQCD PREDICTIONS FOR THE DRESSING FUNCTIONS

We will here deduce some implications of perturbative QCD (pQCD) for the hitherto unknown dressing functions appearing in (19). We will write explicit expressions for them as predicted by one-loop pQCD.

The pQCD prediction for the full gluon propagator is
\[
\left( \frac{\delta^2 W}{\delta J^{a_1a_1} \delta J^{a_2a_2}} \right)(k) = \left( \frac{\delta^2 \Gamma}{\delta A^{a_1}_{\mu_1} \delta A^{a_2}_{\mu_2}} \right)^{-1}(k) = g^2(\Lambda) \; Z_3(\Lambda, K) \; D^{a_1a_2(K)}(k) ,
\]
(17)
where \( Z_3(\Lambda, K) \) appears as a factor in the gluon field rescaling function for the theory with cutoff \( K (K^2 = -k^2) \)
\[
A^{(K)}_{\mu} = Z_3(\Lambda, K) \; Z_3^{1/2}(\Lambda, K) \; A^{(K)}_{\mu} ,
\]
(18)
and \( Z_3(\Lambda, K) \) is the rescaling (“running”) function for the QCD coupling parameter
\[
g(\Lambda) = Z_3(\Lambda, K) \; g(K) .
\]
(19)
Here we use a simplified notation for the running \( g(\Lambda) \equiv g(\Lambda^2) \). The rescaling functions \( Z_3 \) and \( Z_g \) are calculable in pQCD. In (17), \( D^{a_1a_2(K)}_{\mu_1\mu_2}(k) \) is the usual gluon propagator in the theory with cutoff \( K (K^2 = -k^2) \), i.e., the usual tree level propagator
\[
D^{a_1a_2(K)}_{\mu_1\mu_2}(k) = \delta_{a_1a_2} \left( \frac{-1}{k^2} \right) g_{\mu_1\mu_2} - \frac{k_{\mu_1}k_{\mu_2}}{k^2} + \xi(K) \frac{k_{\mu_1}k_{\mu_2}}{k^2} ,
\]
(20)
and \( \xi(K) \equiv a(K)/g^2(K) (\sim g^0) \) is the usual gauge parameter. On the other hand, the gluon propagator as predicted by the effective action (19), with the gluon part (10), is
\[
\left( \frac{\delta^2 \Gamma}{\delta A^{a_1}_{\mu_1} \delta A^{a_2}_{\mu_2}} \right)^{-1}(k) = \delta_{a_1a_2} \left( \frac{-1}{k^2} \right) g^2(K) \; (G_A(k^2))^2 \left( g_{\mu_1\mu_2} - \frac{k_{\mu_1}k_{\mu_2}}{k^2} \right) + \alpha \frac{k_{\mu_1}k_{\mu_2}}{k^2} .
\]
(21)

\(^3\) Care should be taken about the Grassmannian anticommuting character of \( \rho \) and \( b \).
We stress here that this is the full gluon propagator in our framework. The additional nonperturbative terms in the gluonic sector of the effective action which appeared in Eq. (13) as a consequence of the substitution $\partial \to \nabla (A(x))$ in the logarithm, are terms at least cubic in the gluon fields ($\sim (A(k))^3/k$) and thus do not affect the gluon propagator which is determined solely by the $\sim (A(k))^2$ terms of the gluon part (11). Comparing the transversal parts of the pQCD propagator (17) and of our propagator (21), we conclude

$$G_A(k^2) = \left( g(\Lambda) \right) \frac{g(K)}{\bar{g}(\Lambda)} Z_3^{1/2}(\Lambda, K) = Z_g(\Lambda, K) Z_3^{1/2}(\Lambda, K) .$$

(22)

The longitudinal parts of the pQCD propagator (17) and of our propagator (21) agree now automatically, because of the relation (13) for the gauge parameter $\alpha$, and the known relation

$$\xi^{(A)} = Z_3(\Lambda, K) \xi^{(K)} .$$

(23)

Now we analogously consider the ghost propagator. The pQCD prediction for the full ghost propagator is

$$\left( \frac{\delta^2 W}{\delta \eta^a \delta \bar{\eta}^a} \right) (k) = \left( \frac{\delta^2 \Gamma}{\delta \bar{\rho}^a \delta \rho^a} \right)^{-1} (k) = \delta_{a\bar{a}2} Z_2^{(c)}(\Lambda, K) \frac{i}{k^2} ,$$

(24)

where $Z_2^{(c)}(\Lambda, K)$ is the ghost and antighost field rescaling function for the theory with cutoff $K (K^2 = -k^2)$

$$c^{(A)} = Z_2^{(c)1/2}(\Lambda, K) c^{(K)} , \quad b^{(A)} = Z_2^{(c)1/2}(\Lambda, K) b^{(K)} .$$

(25)

On the other hand, our effective action (3) predicts

$$\left( \frac{\delta^2 \Gamma}{\delta \bar{\rho}^a \delta \rho^a} \right)^{-1} (k) = \delta_{a\bar{a}2} \left( \frac{G_c(k^2)}{G_A(k^2)} \right) \frac{i}{k^2} .$$

(26)

Comparing this with the pQCD result (24) gives us

$$G_c(k^2) = Z_2^{(c)}(\Lambda, K) \frac{G_A(k^2)}{Z_3^{1/2}(\Lambda, K) Z_g(\Lambda, K) .}$$

(27)

Although we will not use the explicit form of the quark dressing function (matrix) $G_q$, we mention here that completely analogous considerations involving the quark propagator give us a possible (but not unique) pQCD form for $G_q$

$$G_q(k^2) = Z_2^{1/2}(\Lambda, K) \exp \left( i\theta(k^2) \right) \delta_{\xi\xi'} .$$

(28)

where $\theta(k^2)$ is a momentum-dependent phase, and $Z_2(\Lambda, K)$ is the quark field rescaling function ($K^2 = -k^2)$

$$q_{(ij)}^{(A)} = Z_2^{1/2}(\Lambda, K) q_{(ij)}^{(K)} .$$

(29)

The rescaling (“running”) functions $Z_3, Z_2^{(c)}, Z_2$ and $Z_g$ are all computable in pQCD, and at the one-loop level are given by

$$Z_3(\Lambda, K) = 1 + \frac{1}{\pi \alpha_s} \left[ \kappa_3 \ln \left( \frac{\Lambda^2}{K^2} \right) + \bar{\kappa}_3 \right] + \mathcal{O}(\alpha_s^2) ,$$

(30a)

$$Z_2^{(c)}(\Lambda, K) = 1 + \frac{1}{\pi \alpha_s} \left[ \kappa_2^{(c)} \ln \left( \frac{\Lambda^2}{K^2} \right) + \bar{\kappa}_2^{(c)} \right] + \mathcal{O}(\alpha_s^2) ,$$

(30b)

$$Z_2(\Lambda, K) = 1 + \frac{1}{\pi \alpha_s} \left[ \kappa_2 \ln \left( \frac{\Lambda^2}{K^2} \right) + \bar{\kappa}_2 \right] + \mathcal{O}(\alpha_s^2) ,$$

(30c)

$$Z_g(\Lambda, K) = 1 + \frac{1}{\pi \alpha_s} \left[ \kappa_g \ln \left( \frac{\Lambda^2}{K^2} \right) + \bar{\kappa}_g \right] + \mathcal{O}(\alpha_s^2) ,$$

(30d)

where $\alpha_s \equiv g^2/(4\pi)$ and the coefficients $\kappa_X$ are well known

$$\kappa_3 = \frac{1}{24} \left[ (13 - 3\xi)N_c - 4n_f \right] ,$$

(31a)

$$\kappa_2^{(c)} = \frac{1}{16} (3 - \xi)N_c ,$$

(31b)

$$\kappa_2 = - \frac{\xi}{4} \left( N_c^2 - 1 \right) \frac{1}{2N_c} ,$$

(31c)

$$\kappa_g = - \frac{1}{2} \beta_0 = - \frac{1}{24} (11N_c - 2n_f) .$$

(31d)
Here, $N_c = 3$ is the number of colors, $n_f$ is the number of active quark flavors, and $\beta_0$ is from Eq. (11). Further, the “finite” one-loop parts, which are $\Lambda$- and $k$-independent, are known as well. For example, in the $\overline{\text{MS}}$ renormalization scheme, using dimensional regularization, we have

$$\tilde{\kappa}_3 = \frac{1}{4} \left\{ \frac{31}{9} - (1 - \xi) + \frac{1}{4} (1 - \xi)^2 \right\} N_c - \frac{10}{9} n_f \right\} , \quad (32a)$$

$$\tilde{\kappa}_2 (c) = \frac{1}{4} N_c \, . \quad (32b)$$

The constants $\tilde{\kappa}_2$ and $\tilde{\kappa}_g$ are also known, but we will not need them at the level of our analysis. The “finite” part constant (32) is written for the case when the active quarks are massless. The two constants (32) can be obtained by evaluating the dimensionally regularized gluon and ghost self-energy parts – see, for example, the book [8] [Eqs. (A.21) and (A.23) there]. In the above formulas, we denote $\Lambda$ the $\overline{\text{MS}}$ UV cutoff which is obtained by the formal substitution $2/(4 - D) - \gamma + \ln(4\pi) \to \ln(\Lambda^2)$, where $D (\to 4)$ is the dimension in the regularization. The coupling $\alpha_s$ is, in principle, the “bare” $\overline{\text{MS}}$ coupling $\alpha_s = \alpha_s(\Lambda^2; \overline{\text{MS}})$ here. However, the momentum dependence of $\alpha_s$ in expressions (32) is irrelevant at the $\sim \alpha_s$ level as it affects only $\sim \alpha_s^2$ terms.

Inserting the known one-loop results (30)–(32) into (22) and (27), we obtain the one-loop pQCD predictions for the gluon and ghost dressing functions $G_A$ and $G_c$

$$G_X (k^2) = \left[ 1 + \frac{1}{2\pi} \alpha_s \left( \tilde{\kappa}_X \ln \left( \frac{\Lambda^2}{-k^2} \right) + \tilde{\kappa}_X \right) + \mathcal{O}(\alpha_s^2) \right], \quad (X = A, c) \, , \quad (33a)$$

$$\kappa_A = (\tilde{\kappa}_3 - \beta_0) = -\frac{1}{8} (3 + \xi) N_c \, , \quad (33b)$$

$$\kappa_c = (2\kappa_2 (c) + \tilde{\kappa}_3 - \beta_0) = -\frac{1}{4} \xi N_c \, , \quad (33c)$$

$$\tilde{\kappa}_A = \tilde{\kappa}_3 + 2 \tilde{\kappa}_g \, , \quad (33d)$$

$$\tilde{\kappa}_c - \tilde{\kappa}_A = 2 \kappa_2 (c) = \frac{1}{2} N_c \, . \quad (33e)$$

The freedom of choice of the UV regularization scale $\Lambda$ is reflected in the dressing functions $G_X$ which depend on $\Lambda$

$$G_X (k^2) = G_X^{(A)} (k^2) \, . \quad (34)$$

Up until now, we considered the renormalization scale in the dressing functions to be a very high UV cutoff scale $\Lambda \gg \Lambda_{\text{QCD}}$. But the arbitrariness of $\Lambda$ implies that, in the Wilsonian sense, we could have started from the classical action with a lower cutoff $\mu < \Lambda$, where $\mu$ is a renormalization scale. Then, in all our formulas, we would have to make the simple substitution $\Lambda \to \mu$. The resulting rescaling of the dressing functions under the change $\Lambda \to \mu$ is then

$$G_A^{(A)} = Z_g (\Lambda, \mu) Z_3^{1/2} (\Lambda, \mu) G_A^{(\mu)} \, , \quad (35a)$$

$$G_c^{(A)} = Z_2^{(c)} (\Lambda, \mu) Z_3^{1/2} (\Lambda, \mu) Z_g (\Lambda, \mu) G_c^{(\mu)} \, . \quad (35b)$$

It is then straightforward to see that the field $\bar{A}$ and the combination $\bar{b} \bar{c}$, appearing in the effective action (8), are independent of the choice of the renormalization scale $\mu$. This can be seen via Eqs. (33), (18) and (22), in the latter two replacing $K \to \mu$. Analogously, it can be shown that the combination $\tilde{q}_{(j)} \bar{q}_{(j)}$ appearing in the effective action is independent of $\mu$.

**IV. FIELDS AS POWER EXPANSIONS IN SOURCES**

In this Section we will write down the transformations associated with the Legendre transformation which relates the path integral function $W$ [14a] with the effective action $\Gamma_{\text{QCD}}$ [4]. We will then solve them, thus obtaining the effective fields $c$ and $A$ as power expansions in sources. We recall that the following Legendre transformation must relate $W$ with $\Gamma_{\text{QCD}}$

$$\Gamma[A, b, c, q, \bar{q}] = -W[J, \rho, \eta, \bar{j}, j] - \Xi[A, b, c, q, \bar{q}; J, \rho, \eta, \bar{j}, j] \, , \quad (36)$$
where $\Xi$ is given in Eq. (14). From here follows the $b$(antighost)-relation

$$\frac{\delta \Gamma}{\delta b^a(x)} \left( = \frac{\delta \hat{\delta}^b}{\delta b^a} \circ \frac{\delta \Gamma}{\delta \hat{\delta}^b} (x) \right) = i \rho^a(x) . \tag{37}$$

By Eqs. (6) and (8) we have

$$\frac{\delta \dot{b}^c(y)}{\delta b^a(x)} = \delta_{ab} G_A(y - x) , \quad \frac{\delta \Gamma}{\delta \dot{b}^c(y)} = -i \left( \partial^\mu \nabla_\mu (\hat{A}) \dot{c}(y) \right)^b . \tag{38}$$

Applying to (37) the convolution $i G_A^{-1}$ on the left, and using the aforementioned symmetry $G_A^{-1}(-x) = G_A^{-1}(x)$, we then obtain

$$\partial^2 \tilde{c}(x) - f^{abc} \partial^\mu \left( \tilde{A}_\mu^b (x) \tilde{c}^c(x) \right) = - \int d^4x' G_A^{-1}(x - x') \rho^a(x'). \tag{39}$$

This is the Legendre-related $b$-relation in spacetime coordinates. Applying the Fourier transformation of type (7) to this relation, we obtain the following Legendre-related $b$(antighost)-relation in four-momentum coordinates, involving the corresponding Fourier transformed functions:

$$-k^2 \frac{c^a(k)}{G_A(k^2)} + i f^{abc} k^\mu \int \frac{d^4k_1 A_\mu^b(k_1)}{(2\pi)^4 G_A(k_1^2)} \frac{c^c(k - k_1)}{G_A((k - k_1)^2)} = - \frac{\rho^a(k)}{G_A(k^2)} . \tag{40}$$

Completely analogously, we obtain the following Legendre-related $c$(ghost)-relation in four-momentum space:

$$-k^2 G_A(k^2) b^a(k) + i f^{abc} \int \frac{d^4k_1 G_A((k - k_1)^2)}{(2\pi)^4 G_A(k_1^2)} (k - k_1)^\mu A_\mu^b(k_1)b^c(k - k_1) = G_c(k^2) \eta^a(k) . \tag{41}$$

Very analogous, but algebraically more involved manipulations of the Legendre-related $A$(gluon)-relation lead to the following explicit form of the Legendre-related $A$(gluon)-relation in four-momentum coordinates:

$$-\frac{1}{g^2(K)G_A(k^2)} \left\{ k^\mu \left[ \left( g^{\mu_1\mu_2} - \frac{k_1^\mu_1 k_2^\mu_2}{k^2} \right) + \frac{g^2(K)}{\alpha} (G_A(k^2)) \frac{2 k_1^\mu_1 k_2^\mu_2}{k^2} \right] A_{\mu_2}^a(k) \right\}$$

$$+ i f^{a_1a_2a_3} G_A(k^2) \int \frac{d^4k_1}{(2\pi)^4 G_A(k_1^2) G_A((k - k_1)^2)}$$

$$\times \left[ (k_1^\mu_1 A_{\mu_2}^a (k_1) - k_2^\mu_2 A_{\mu_2}^a (k_1)) A_{\mu_2}^a (k - k_1) - k_1^\mu_2 A_{\mu_2}^a (k_1) A_{\mu_2}^a (k - k_1) \right]$$

$$+ f^{a_1 a_2 b} f^{a_3 a_4 b} G_A(k^2) \int \frac{d^4k_1 d^4k_2}{(2\pi)^8 G_A(k_1^2) G_A((k - k_1)^2)} \frac{1}{G_A(k_2^2)G_A((k - k_1 - k_2)^2)} A^{a_2 a_3} (k_1) A^{a_1 a_2} (k_2) A_{\mu_2}^a (k_1 - k - k_2) \right\}$$

$$+ f^{a_1 a_2 a_3} \int \frac{d^4k_1}{(2\pi)^4 G_A(k_1^2)} \frac{G_A(k_2^2)}{G_A((k - k_1)^2)} k_1^\mu_2 b_{a_3}^a (k_1) c^{a_3} (k - k_1)$$

$$+ \int \frac{d^4k_1}{(2\pi)^4} \left( \bar{q}_{(j)} (k_1) \gamma^0 G_{q_1}^{-1} (k_1) \gamma^0 \right) \gamma^{a_2} \Gamma_A^{-1} \left( G_{q_1}^{-1} (k - k_1) q (k - k_1) \right) = - G_a(k^2) J_{a_1}^a (k) . \tag{43}$$

The $\bar{q}$-relation from the Legendre transformation would involve, analogously, terms $\sim q_{(j)}$ and $\sim A q_{(j)}$ on the left-hand side and the quark current $j_{(j)}$ on the right-hand side. In the above relations, and in the rest of the article, we omit for simplicity any notational reference to the dependence of the effective fields, the dressing functions, and the gauge parameter $\alpha$ on the UV cutoff $\Lambda$.

The Legendre-related relations (10), (41), and (43) connect the effective fields with the sources. They allow us to obtain the effective fields as expansions in powers of sources

$$X = X^{(1)} + X^{(2)} + X^{(3)} + \cdots \tag{44}$$
Here we denote by \( X^{(n)} \) the part of the field \( X \) of power \( n \) in the sources \((J^a_i, \rho, \eta, \bar{\eta}(i), j(j))\). We assume that the fields go to zero when the sources go to zero \((X^{(0)} = 0)\), i.e., we assume that there are no soliton-like vacuum effects. When inserting these expansions in the Legendre-related relations, and requiring that each coefficient in the expansion fulfill the equation, we obtain an infinite series of Legendre-related equations for \( X^{(n)} \)'s which can be solved successively. For example, the \( b \)-relation (43) thus gives the series of equations

\[
- \frac{k^2}{G_c(k^2)} c^a(k)^{(1)} = - \frac{1}{G_A(k^2)} \rho^a(k) ,
\]

\[
- \frac{k^2}{G_c(k^2)} c^a(k)^{(2)} + i f^{abc} k^c \int \frac{d^4k_1}{(2\pi)^4} \frac{1}{G_A(k_1^2)G_c((k - k_1)^2)} A^b_{\mu}(k_1)(1) c^c(k - k_1)^{(1)} = 0 ,
\]

\[
- \frac{k^2}{G_c(k^2)} c^a(k)^{(3)} + i f^{abc} k^c \int \frac{d^4k_1}{(2\pi)^4} \left[ A^b_{\mu}(k_1)(1) c^c(k - k_1)^{(2)} + A^b_{\mu}(k_1)(2) c^c(k - k_1)^{(1)} \right] = 0 ,
\]

e etc. Solving these equations successively for \( c^{(1)}, c^{(2)}, c^{(3)}, \ldots \), we obtain

\[
c^a(k)^{(1)} = \frac{G_c(k^2)}{G_A(k^2)} \frac{1}{k^2} \rho^a(k) ,
\]

\[
c^a(k)^{(2)} = i f^{abc} \frac{G_c(k^2)}{k^2} \int \frac{d^4k_1}{(2\pi)^4} \frac{1}{G_A(k_1^2)G_c((k - k_1)^2)} A^b_{\mu}(k_1)(1) \rho^c(k - k_1) ,
\]

\[
c^a(k)^{(3)} = i f^{abc} \frac{G_c(k^2)}{k^2} \int \frac{d^4k_1}{(2\pi)^4} \left[ A^b_{\mu}(k_1)(1) c^c(k - k_1)^{(2)} + A^b_{\mu}(k_1)(2) c^c(k - k_1)^{(1)} \right] ,
\]

e etc. In an analogous, though algebraically more involved way, by inserting expansions (44) into the Legendre-related \( A \)-relation (43) we obtain expressions for \( A^{(n)} \)

\[
A^{(1)}_{\mu_1}(k) = g^2(K) \left( G_A(k^2) \right)^2 \frac{1}{k^2} P_{\mu_1,\mu_2}(k) J^{a^2}_{\mu_2}(k) ,
\]

\[
A^{(2)}_{\mu_1}(k) = i f^{a_1 a_2 a_3} g^4(K) \frac{G_c(k^2)}{k^2} P_{\mu_1,\mu_2}(k) \int \frac{d^4k_1}{(2\pi)^4} \frac{G_A(k_1^2)G_A((k - k_1)^2)}{k_1^2(k - k_1)^2} \times \left[ (k_1^2 P_{\mu_1 \mu_4}(k_1) - (k_1 + k_\mu_2) P_{\mu_2 \mu_4}(k_1)) P^{a^1}_{\mu_3}(k - k_1) \right] J^{a^2}_{\mu_2}(k_1)J^{a_3}_{\mu_3}(k - k_1) + O(\eta \rho, \bar{\eta}(i), j(j)) ,
\]

e etc., where we denoted by \( P_{\mu_1} \) the function proportional to the gluon propagator (21)

\[
P_{\mu_1,\mu_2}(k) = \left[ \left( g_{\mu_1,\mu_2} - \frac{k_{\mu_1} k_{\mu_2}}{k^2} \right) + \frac{\alpha}{g^2(K)} \frac{1}{G_A(k^2)} \frac{k_{\mu_1} k_{\mu_2}}{k^2} \right] = \left[ \left( g_{\mu_1,\mu_2} - \frac{k_{\mu_1} k_{\mu_2}}{k^2} \right) + \frac{\xi(k) \mu_1 k_{\mu_2}}{k^2} \right] .
\]

In the analysis of the next Section, we will not need the explicit terms \( O(\eta \rho, \bar{\eta}(i), j(j)) \) in (174). They originate from terms \( \sim c^{(1)}(k)^{(1)} \) and \( \tilde{q}^{(1)}(k)^{(1)} \). We can obtain the coefficient functions \( b^{(n)} \), \( q^{(n)} \), \( \tilde{q}^{(n)} \) in a similar way, but we will not need them in the following analysis. The cubic-in-sources part \( c^{(3)} \) of \( c \) can be obtained by inserting in the expression (46) the explicit solutions (46a)-(46b) and (47a)-(47b).

V. CHECKING THE ANTIGHOST DSE EQUATION

In this Section we will perform a consistency check of the effective action framework (5) for the high momenta (pQCD) region. In Section II we deduced the dressing functions for the region of high momenta by requiring that they, in the framework of (5), should give us the known pQCD two-point Green functions. In particular, the known one-loop pQCD expressions for the gluon and ghost propagators resulted in the high-momentum behavior (33) for the dressing functions \( G_A \) and \( G_c \). On the other hand, in the expansion in sources various Dyson–Schwinger equations (DSE’s) can give us (infinite) series of equations for the dressing functions. For example, the antighost DSE (b-DSE)
\[ \partial^2 c^{(a)}(x)^{(1)} + i f^{abc} \partial^u_{(x)} \left( \frac{\delta c^c(\mu)^{(2)}(x)}{\delta J^c_\mu(x)} \right) = -\rho^a(x), \] (49a)

\[ \partial^2 c^{(a)}(x)^{(2)} - f^{abc} \partial^\mu_{(x)} \left( A^b_\mu(x)^{(1)} c^c(x)^{(1)} + i f^{abc} \partial^\mu_{(x)} \left( \frac{\delta c^c(\mu)^{(3)}(x)}{\delta J^c_\mu(x)} \right) = 0, \] (49b)

etc. In part A of this Section, we will check explicitly that the aforementioned one-loop pQCD solutions (33) fulfill Eq. (49a). In part B we will check explicitly that their dependence on the UV cutoff \( \Lambda \) and on the momentum \( k \) is compatible with Eq. (49a).

We first apply the Fourier transformations to these relations, obtaining relations for the corresponding Fourier-transformed functions

\[ -k^2 \rho^a(k)^{(1)} + f^{abc} k^\mu \left( \frac{\delta c^c(\mu)^{(2)}(k)}{\delta J^c_\mu(k)} \right) = -\rho^a(k), \] (50a)

\[ -k^2 \rho^a(k)^{(2)} + i f^{abc} k^\mu \int \frac{d^4 k_1}{(2\pi)^4} A^b_\mu(k_1)^{(1)} c^c(k-k_1)^{(1)} + f^{abc} k^\mu \left( \frac{\delta c^c(\mu)^{(3)}(k)}{\delta J^c_\mu(k)} \right) = 0, \] (50b)

etc. The variational derivatives of the type \( (\delta c^{(n)}/\delta J)(k) \) in Eqs. (50) denote simply the Fourier transforms, with respect to \( k \), of the functions \( (\delta c(x)^{(n)}/\delta J(x)) \).

We note that the term \( \delta c^{(2)}/\delta J \) in the linear-in-sources DSE (49) contains implicitly the three-point connected Green function \( \delta^2 c^{(2)}/(\delta \rho \delta J) = i \delta^2 W/(\delta \rho \delta \eta \delta J) \sim \langle bcA \rangle \). Analogously, the term \( \delta c^{(3)}/\delta J \) in the quadratic-in-sources DSE (49) contains implicitly the four-point connected Green function \( \delta^3 W/(\delta \rho \delta \eta \delta J) \sim \langle bcAA \rangle \).

### A. Linear-in-sources DS equation

First we will show, in our effective action framework (11) and (16), that the one-loop dressing functions (33) are consistent with the linear-in-sources DS equation (50a). When we use in Eq. (50a) the expressions (46a), (46b), and (47a), the common factor \( \rho^a(k) \) factorizes out and we end up with the equivalent relation (we multiply with \( -G_A(k^2) \))

\[ G_c(k^2) + i g^2(K) N_c \frac{1}{k^2} \int \frac{d^4 k_1}{(2\pi)^4} G_c(k_1^2) G_A((k-k_1)^2) \delta^{(4)}(k-k_1) \delta^{(4)}(k_1) = G_A(k^2). \] (51)

All the squared momenta in the denominators, here and in the rest of the article, are understood to have \(+i\epsilon\) added to ensure causality. The integral must be understood as having been regularized with a gauge-invariant cutoff \( \Lambda \), which can be implemented with dimensional regularization (see Sec. 11 and Appendix B).

Further, we insert in (51) the one-loop results (33), and thus are allowed to keep only terms up to order \( O(\alpha_s^4) \). This means that we replace in the integral \( G_c(k_1^2) G_A((k-k_1)^2) \rightarrow 1 \) and \( \xi^{(K-K_1)} \rightarrow \xi^{(A)} \) and obtain the equivalent (one-loop) relation

\[ G_A(k^2) - G_c(k^2) = 4 i \pi \alpha_s(K) N_c \frac{1}{k^2} \int \frac{d^4 k_1}{(2\pi)^4} \frac{1}{k_1^2(k-k_1)^2 k^{\mu_1}} \left[ g_{\mu_1 \mu_2} + (\xi^{(A)} - 1) \frac{(k-k_1)_\mu (k-k_1)_\mu}{(k-k_1)^2} \right] k^{\mu_2}. \] (52)

In Appendix B, we explicitly evaluated the integral on the right-hand side of DSE (52), with the result given in Eq. (B1). On the other hand, the one-loop expression results (33) for the dressing functions give for the left-hand side of DSE (52)

\[ G_A(k^2) - G_c(k^2) = -\alpha_s(K) N_c \frac{1}{16\pi} \left[ (3 - \xi^{(A)}) \ln \left( \frac{\Lambda^2}{k^2} \right) + 4 \right] + O(\alpha_s^5). \] (53)

This is just the same as the result (B1) for the right-hand side of DSE (52), at the one-loop level at which we are working. This shows explicitly, in our effective action framework, that the one-loop dressing functions (33) are consistent with the DSE (51), or equivalently with the first (linear in sources) DSE (50a) in the series of the antighost DSE’s (50a), (50b), etc.
B. Quadratic-in-sources DS equation

We now proceed to show, in our effective action framework (5) and (10), that the $\Lambda^2$- and $k^2$-dependent parts of the one-loop dressing functions $G_X$ (33) are consistent with the quadratic-in-sources antighost DSE (50). The explicit form of the first two terms in this DSE, in terms of the dressing functions and sources, can be obtained by straightforward insertion of the expressions (46a), (46b) and (47a)

$$-k^2 e^{\alpha}(k)^{(2)} + i f^{\alpha bc} k^\mu \int^{(A)} \frac{d^4 k_1}{(2\pi)^4} A^b_\mu(k_1)^{(1)} c^e(k - k_1)^{(1)} =$$

$$-i g^2(K) k^\mu \int^{(A)} \frac{d^4 k_1}{(2\pi)^4} 1 \left[ \frac{G_c(k_1)}{G_c((k - k_1)^2)G_A(k_1^2)} - 1 \right] \left( f^{abc} J^b_\mu(k_1) \rho^c(k - k_1) \right).$$

(54)

We recall that in our convention, $A \sim g$, $J \sim g^{-1}$, $\rho \sim g^0$. But the expression in brackets in the integral is $\sim g^2$, thus the expression (54) is $\sim g^3 \sim \alpha_s^{3/2}$. Using the one-loop expressions (33) for the dressing functions, the expression in brackets is

$$\left[ \frac{G_c(k_1)}{G_c((k - k_1)^2)G_A(k_1^2)} - 1 \right] = -\frac{\kappa_A}{2\pi} \alpha_s(K) \left[ \ln \left( \frac{\Lambda^2}{k^2} \right) + \text{finite} \right] + O(\alpha_s^2).$$

(55)

Thus, using the explicit form (33b) for $\kappa_A$, expression (54) can be rewritten as

$$-k^2 e^{\alpha}(k)^{(2)} + i f^{\alpha bc} k^\mu \int^{(A)} \frac{d^4 k_1}{(2\pi)^4} A^b_\mu(k_1)^{(1)} c^e(k - k_1)^{(1)} =$$

$$-i \left( 3 + \xi^{(A)} \right) \frac{N_c \alpha_s^2}{\kappa_A} \left( -k^2 \right) k^\mu \int^{(A)} \frac{d^4 k_1}{(2\pi)^4} \left[ \ln \left( \frac{\Lambda^2}{k^2} \right) + \text{finite} \right] \left( f^{abc} J^b_\mu(k_1) \rho^c(k - k_1) \right) + O(\alpha_s^{5/2}).$$

(56)

The calculation of the last term in DSE (50), i.e., its reduction to a form of the type (56), is more involved since it entails evaluation of (a variational derivative of) the cubic-in-sources function $e^{(3)}$. The calculation is given in Appendix C in the Feynman gauge ($\xi^{(A)} = 1$), with the result given in Eq. (C8). Since the sum in the first brackets there is one, we see that the third term result (C8) is just the negative of the result (54) for the first two terms in the Feynman gauge. This explicitly shows that the $\Lambda^2$- and $k^2$-dependence of the one-loop dressing functions (33) is consistent with the quadratic-in-sources antighost DSE (50) in our effective action framework.

VI. SUMMARY

Based on arguments presented in our previous work Ref. [1], we propose a specific form (5) of the QCD effective action which basically repeats the structure of the classical QCD action, but where all the fields $X$ are replaced by their convolutions with dressing functions $G_X$: $\tilde{X} = G_X^{-1} \circ X$. The proposed pure gluon part is given in Eq. (10), and we argue that it should be supplemented by terms which restore gauge invariance in the dressed gluon field $\tilde{A}$ and that these supplementary pure gluon terms do not contribute to perturbative effects. In general, the proposed effective action should have all perturbative and nonperturbative effects (except for possible soliton-like vacuum effects) encoded in the dressing functions $G_X$.

We then investigated the consistency of the proposed framework in the regime of application of perturbative QCD (pQCD). First we deduced the gluon and ghost dressing functions $G_A$ and $G_e$ from the requirement that the full gluon and antighost-ghost propagators of the framework should agree with those known from pQCD (with an UV regularization scale $\Lambda$), and obtained explicit pQCD expressions for $G_A$ and $G_e$ at the one-loop level. Stated differently, we fixed the two dressing functions of the framework by requiring the agreement of two two-point Green functions calculated in this framework with those of pQCD. But the two dressing functions appear also in many other (three-point, four-point, ...) Green functions. These Green functions can be related with the two-point ones via Dyson-Schwinger equations (DSE’s). A priori, it is far from clear whether a very limited number of dressing functions fulfills many (infinite number of) DSE’s. We performed explicit checks of the antighost DSE (b-DSE) at the linear-

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4 The "finite", i.e. $\Lambda$-independent parts, in addition to $\ln(\Lambda^2/(-k^2))$, may also appear in Eqs. (C8) and (56). The algebra of including them becomes very involved, and they are not considered in the present analysis.
and at the quadratic-in-sources level, with the aforementioned one-loop dressing functions $G_A$ and $G_c$. These two DSE’s involve implicitly the connected three-point $\langle bcA \rangle$ and four-point $\langle bcAA \rangle$ Green functions. Thus we presented nontrivial checks of the consistency of the proposed QCD framework in the perturbative (high momentum) region.

However, in the quadratic-in-sources DSE we considered only the large logarithms containing the regularization scale $\ln(\Lambda^2/(-k^2))$ in the dressing functions and in the DSE’s, leaving the algebraically involved problem of the cancelation of the $\Lambda$-independent (“finite”) terms in the DSE’s for future investigation. Subsequently, we plan to investigate, in the proposed QCD effective action framework, the behavior of the quark sector in the high momentum region, and subsequently the behavior of all sectors in the low momentum region.

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APPENDIX A: LOGARITHM OF THE (COVARIANT) D’ALAMBERTIAN

In this Appendix we write down explicit expressions for the logarithm of d’alambertian $\partial^2 \equiv \partial_\mu \partial^\mu$ and the logarithm of its (tilde-)covariant version $\nabla^2(\tilde{A}) \equiv \nabla_\mu(\tilde{A})\nabla^\mu(\tilde{A})$, and clarify why expression (13) is not gauge invariant and expression (12) is.

For any function $B(x)$ in spacetime, the function $\ln(\partial^2/\mu^2_0)B(x)$ can be written in terms of the Taylor expansion of the logarithm around the point $\ln(\mu^2_0/\mu^2_0) = 1$

$$\ln \left( \frac{\partial^2}{\mu^2_0} \right) B(x) = \left[ \frac{1}{\mu^2_0}(\partial^2 - \mu^2_0) - \frac{1}{2\mu^2_0}(\partial^2 - \mu^2_0)^2 + \frac{1}{3\mu^2_0}(\partial^2 - \mu^2_0)^3 + \cdots \right] B(x) . \quad (A1)$$

The Fourier transformation (7) of this function is obtained by integration by parts which replaces $\partial^2 \mapsto -k^2$

$$\ln \left( \frac{\partial^2}{\mu^2_0} \right) B(k) = \left[ -\frac{1}{\mu^2_0}(k^2 + \mu^2_0) - \frac{1}{2\mu^2_0}(k^2 + \mu^2_0)^2 - \frac{1}{3\mu^2_0}(k^2 + \mu^2_0)^3 + \cdots \right] B(k)$$

$$\ln \left( \frac{-k^2}{\mu^2_0} \right) B(k) . \quad (A2)$$

This shows that the one-loop relation (12) inserted in expression (14) gives expression (13). Further, since $\partial^2 F^{\mu\nu}(\tilde{A})(x)$ is not gauge covariant $[F^{\mu\nu}(\tilde{A})(x)$ is gauge covariant], expression $\text{Tr}[F^{\mu\nu}(\tilde{A})(x)\partial^2 F^{\mu\nu}(\tilde{A})(x)]$ is not gauge invariant, and then Eq. (A1) implies that the second integrand on the right-hand side of Eq. (13) is not gauge invariant.

Unlike $\partial^2 F^{\mu\nu}(\tilde{A})(x)$, expression $\nabla^2(\tilde{A})F^{\mu\nu}(\tilde{A})(x)$ is gauge covariant. Therefore, $\text{Tr}[F^{\mu\nu}(\tilde{A})(x)\nabla^2(\tilde{A})F^{\mu\nu}(\tilde{A})(x)]$ is gauge invariant. The logarithmic version can be written in complete analogy with Eq. (A1)

$$\ln \left( \frac{\nabla^2(\tilde{A})}{\mu^2_0} \right) B(x) = \left[ \frac{1}{\mu^2_0}(\nabla^2(\tilde{A}) - \mu^2_0) - \frac{1}{2\mu^2_0}(\nabla^2(\tilde{A}) - \mu^2_0)^2 + \frac{1}{3\mu^2_0}(\nabla^2(\tilde{A}) - \mu^2_0)^3 + \cdots \right] B(x) . \quad (A3)$$

From here we see that, when $B(x) = F^{\mu\nu}(\tilde{A})(x)$, this expression is also gauge covariant, and thus the trace representing the second integrand on the right-hand side of Eq. (13) is gauge invariant.

From this procedure, we can also see how to proceed in the general (nonperturbative) case, i.e., when $1/g(-k^2) \equiv \mathcal{G}(-k^2)$ is a general function of $k^2$, in order to obtain from (14) the (tilde-)gauge invariant version

$$\Gamma_{\text{gl,1}}[A] = \int \frac{d^4 k}{(2\pi)^4} \mathcal{G}(-k^2) \text{Tr} \left[ F^{\mu\nu}(\tilde{A})(k) F^{\mu\nu}(\tilde{A})(-k) \right] \quad (A4a)$$

$$\Rightarrow \Gamma_{\text{gl}}[A] = \int \frac{1}{2} \int d^4 x \text{Tr} \left[ F^{\mu\nu}(\tilde{A})(x) \mathcal{G} \left( \nabla^2(\tilde{A}(x)) \right) F^{\mu\nu}(\tilde{A})(x) \right] , \quad (A4b)$$

where $\mathcal{G}(-k^2)$ should be represented as a power expansion.
APPENDIX B: REGULARIZED INTEGRALS

In this article, we need the regularized form of the following types of integrals:

\[
I(q_1)^{(1; \mu_1)} = \int \frac{d^4k_1}{(2\pi)^4} \frac{1}{k_1^2(k_1 + q_1)^2} \{1; k_1^{\mu_1}\}, \tag{B1a}
\]

\[
I(q_1, q_2)^{(1; \mu_1 \mu_2; \mu_1 \mu_2 \mu_3)} = \int \frac{d^4k_1}{(2\pi)^4} \frac{1}{k_1^2(k_1 + q_1)^2(k_1 + q_2)^2} \{1; k_1^{\mu_1}; k_1^{\mu_2}; k_1^{\mu_3}; k_1^{\mu_2} k_1^{\mu_3}\}, \tag{B1b}
\]

where the various options are indicated in the curly brackets. The factors in the denominators are understood to have \(+i\,\varepsilon\) added to ensure causality. The UV regularization is denoted in (B1) by \(\Lambda\) above the integration symbol. It is implemented by dimensional regularization, i.e., in \(D\) dimensions, thus replacing \(d^4k_1/(2\pi)^4 \rightarrow d^Dk_1/(2\pi)^D\) and \(g_{\mu\nu}g^{\mu\nu} = 4 \rightarrow D\). The UV cutoff \(\Lambda\) is introduced in the final results, after \(D \rightarrow 4\), according to the substitution

\[
\frac{2}{4-D} - \gamma + \ln(4\pi) = \ln(\Lambda^2), \tag{B2}
\]

representing thus the effective “bare” UV cutoff in \(\overline{\text{MS}}\) scheme. The relevant formulas leading to the evaluation of integrals (B1) can be found, for example, in Refs. [8, 9].

The product of denominators is transformed into one denominator via the method of Feynman parameters

\[
\frac{1}{A_1 A_2 \ldots A_n} = \int_0^1 dx_1 \ldots dx_n \delta \left( \sum_1^n x_j - 1 \right) \frac{(n - 1)!}{[x_1 A_1 + \ldots + x_n A_n]^n}. \tag{B3}
\]

Specifically,

\[
\frac{1}{k_1^2(k_1 + q_1)^2} = \int_0^1 \int_0^1 dx_1 dx_2 \delta(x_1 + x_2 - 1) \frac{1}{(\ell^2 - \Delta^{(2)})^2}, \tag{B4a}
\]

\[
\ell = k_1 + x_2 q_1, \quad \Delta^{(2)} = x_2(1 - x_2)(-q_1^2); \tag{B4b}
\]

and

\[
\frac{1}{k_1^2(k_1 + q_1)^2(k_1 + q_2)^2} = 2 \int_0^1 \int_0^1 \int_0^1 dx_1 dx_2 dx_3 \delta(x_1 + x_2 + x_3 - 1) \frac{1}{(\ell^2 - \Delta^{(3)})^3}, \tag{B5a}
\]

\[
\ell = k_1 + x_2 q_1 + x_3 q_2, \quad \Delta^{(3)} = -x_1 x_2 q_1^2 - x_1 x_3 q_2^2 - x_2 x_3 (q_1 - q_2)^2. \tag{B5b}
\]

Introducing the integration variable \(\ell\) according to (B4), (B5), the integrals (B3) can be rewritten as

\[
I(q_1)^{(1; \mu_1)} = \lim_{D \rightarrow 4} \int_0^1 \int_0^1 dx_1 dx_2 \delta(x_1 + x_2 - 1) \int \frac{d^D\ell}{(2\pi)^D} \frac{1}{(\ell^2 - \Delta^{(2)})^2} \times \{1; \ell^{\mu_1} - x_2 q_1^{\mu_1}\}, \tag{B6a}
\]

\[
I(q_1, q_2)^{(1; \mu_1 \mu_2; \mu_1 \mu_2 \mu_3)} = 2 \lim_{D \rightarrow 4} \int_0^1 \int_0^1 \int_0^1 dx_1 dx_2 dx_3 \delta(x_1 + x_2 + x_3 - 1) \times \int \frac{d^D\ell}{(2\pi)^D} \frac{1}{(\ell^2 - \Delta^{(3)})^3} \{1; \ell^{\mu_1}; q_1^{\mu_1} q_2^{\mu_2}; q_1^{\mu_1} q_2^{\mu_2} q_3^{\mu_3}\} \big|_{q_1 \equiv -x_2 q_1 - x_3 q_2}. \tag{B6b}
\]

The integration over \(\ell\) can be performed after Wick-rotation, using dimensional regularization, and introducing \(\epsilon \equiv (4 - D)/2\)

\[
\int \frac{d^D\ell}{(2\pi)^D} \frac{1}{(\ell^2 - \Delta)^2} = \frac{i}{(4\pi)^2} \left[ \left( \frac{1}{\epsilon} - \gamma + \ln(4\pi) - \ln(\Delta) \right) + \mathcal{O}(\epsilon) \right], \tag{B7a}
\]

\[
\int \frac{d^D\ell}{(2\pi)^D} \frac{\ell^2}{(\ell^2 - \Delta)^3} = \frac{i}{(4\pi)^2} \left( 1 - \frac{\epsilon}{2} \right) \left[ \left( \frac{1}{\epsilon} - \gamma + \ln(4\pi) - \ln(\Delta) \right) + \mathcal{O}(\epsilon) \right], \tag{B7b}
\]

\[
\int \frac{d^D\ell}{(2\pi)^D} \frac{1}{(\ell^2 - \Delta)^3} = -\frac{i}{(4\pi)^2} \frac{1}{2} \frac{1}{\Delta} + \mathcal{O}(\epsilon) \quad \text{(finite)}. \tag{B7c}
\]
Further, by symmetry considerations the following identities hold:

\[
\int \frac{d^D\ell}{(2\pi)^D} \frac{\{\ell_{\mu_1}, \ell_{\mu_2}, \ell_{\mu_3}\}}{(\ell^2 - \Delta)^4} = 0 ,
\]

\[
\int \frac{d^D\ell}{(2\pi)^D} \frac{\ell_{\mu_1} \ell_{\mu_2}}{(\ell^2 - \Delta)^4} = \frac{1}{D} g_{\mu_1 \mu_2} \int \frac{d^4\ell}{(2\pi)^4} \frac{\ell^2}{(\ell^2 - \Delta)^4} .
\]

(B8a)

(B8b)

Applying the identities (B7)–(B8) to (B9), and at the end performing the limit \( \epsilon \equiv (4-D)/2 \to 0 \), and then introducing the cutoff notation (B2), we obtain the regularized integrals (B3)

\[
I(q) = \frac{i}{(4\pi)^2} \left[ \ln \left( \frac{\Lambda^2}{-q^2} \right) + 2 \right] ,
\]

(B9a)

\[
I(q)^\mu = -g^\mu \frac{i}{2(4\pi)^2} \left[ \ln \left( \frac{\Lambda^2}{-q^2} \right) + 2 \right] ,
\]

(B9b)

\[
I(q_1, q_2) = -\frac{i}{(4\pi)^2} \int_0^1 dx_3 \int_0^{1-x_3} dx_2 \frac{1}{F(q_1, q_2; x_2, x_3)} ,
\]

(B9c)

\[
I(q_1, q_2)^\mu = \frac{i}{(4\pi)^2} \int_0^1 dx_3 \int_0^{1-x_3} dx_2 \frac{(x_2 q_1 + x_3 q_2)^\mu}{F(q_1, q_2; x_2, x_3)} ,
\]

(B9d)

\[
I(q_1, q_2)^{\mu_1 \mu_2} = \frac{i}{(4\pi)^2} \left[ \frac{1}{4} g^{\mu_1 \mu_2} \ln \left( \frac{\Lambda^2}{-q^2} \right) - \frac{1}{2} g^{\mu_1 \mu_2} \int_0^1 dx_3 \int_0^{1-x_3} dx_2 \ln \left[ \frac{F(q_1, q_2; x_2, x_3)}{-q^2} \right] 
\]

\[
- \left. \frac{1}{6} g^{\mu_1 \mu_2} (x_2 q_1 + x_3 q_2)^{\mu_1} (x_2 q_1 + x_3 q_2)^{\mu_2} g^{\mu_3, \mu_3} \right] ,
\]

(B9e)

\[
I(q_1, q_2)^{\mu_1 \mu_2 \mu_3} = \frac{i}{(4\pi)^2} \left[ -\frac{1}{12} [g^{\mu_1 \mu_2} (q_1 + q_2)^{\mu_3} + g^{\mu_2 \mu_3} (q_1 + q_2)^{\mu_1} + g^{\mu_3 \mu_1} (q_1 + q_2)^{\mu_2}] \ln \left( \frac{\Lambda^2}{-q^2} \right) 
\]

\[
+ \frac{1}{2} \left. \int_0^1 dx_3 \int_0^{1-x_3} dx_2 \left[ g^{\mu_1 \mu_2} (x_2 q_1 + x_3 q_2)^{\mu_3} + g^{\mu_2 \mu_3} (x_2 q_1 + x_3 q_2)^{\mu_1} + g^{\mu_3 \mu_1} (x_2 q_1 + x_3 q_2)^{\mu_2} \right] \ln \left[ \frac{F(q_1, q_2; x_2, x_3)}{-q^2} \right] 
\]

\[
\right. + \int_0^1 dx_3 \int_0^{1-x_3} dx_2 \frac{(x_2 q_1 + x_3 q_2)^{\mu_1} (x_2 q_1 + x_3 q_2)^{\mu_2} (x_2 q_1 + x_3 q_2)^{\mu_3}}{F(q_1, q_2; x_2, x_3)} \right] ,
\]

(B9f)

where we denoted

\[
F(q_1, q_2; x_2, x_3) \equiv -x_2(1-x_2)q_1^2 - x_3(1-x_3)q_2^2 + 2x_2x_3(q_1+q_2) ,
\]

(B10)

and the squared scale \((-q^2)\) appearing in Eqs. (B9e)–(B9f) is arbitrary, e.g., \(-q^2 = q_1^2\) or \(-q^2 = q_2^2\) [the results (B9e)–(B9f) are independent of \((-q^2)\)]. In the simpler cases when only one fixed scale appears in the integrals, say \(q_1 = 0\) and \(q_2 = q\), we obtain from the above results immediately

\[
I(0, q) = \frac{i}{(4\pi)^2} \frac{1}{4q^2} \int_0^1 dx_3 \frac{(\eta \to +0)}{x_3} ,
\]

(B11a)

\[
I(0, q)^\mu = -\frac{i}{(4\pi)^2} \frac{g^\mu}{q^2} ,
\]

(B11b)

\[
I(0, q)^{\mu_1 \mu_2} = \frac{i}{2(4\pi)^2} \left[ \frac{1}{2} g^{\mu_1 \mu_2} \ln \left( \frac{\Lambda^2}{-q^2} \right) + g^{\mu_1 \mu_2} + \frac{q^{\mu_1} q^{\mu_2}}{q^2} \right] ,
\]

(B11c)

\[
I(0, q)^{\mu_1 \mu_2 \mu_3} = -\frac{i}{3(4\pi)^2} \left( \frac{1}{4} g^{\mu_1 \mu_2} q^{\mu_3} + g^{\mu_2 \mu_3} q^{\mu_1} + g^{\mu_3 \mu_1} q^{\mu_2} \right) \ln \left( \frac{\Lambda^2}{-q^2} \right) + \frac{5}{3} + \frac{q^{\mu_1} q^{\mu_2} q^{\mu_3}}{q^2} \right) .
\]

(B11d)
As an application, using the results \([B9], \; [B11]\), we obtain the integrals on the right-hand side of DSE \((B2)\):

\[
\begin{align*}
\int^{(A)} &\frac{d^4k_1}{(2\pi)^4} \frac{1}{k_1^2(k_1-k_2)^2} k^\mu_1 g_{\mu_1\mu_2} k^\nu_2 = I(-k)^{\mu_2} k_{\mu_2} = \frac{i}{2(4\pi)^2} k^2 \left[ \ln \left( \frac{\Lambda^2}{-k^2} \right) + 2 \right], \\
\int^{(A)} &\frac{d^4k_1}{(2\pi)^4} \frac{k^\mu_1}{k_1^2(k_1-k_2)^2} \frac{(k-k_1)_{\mu_1}(k-k_2)_{\mu_2}}{k^\nu_2} = k^\mu_1 k^\nu_2 I(0,k)_{\mu_1\mu_2} + k^\mu_1 I(0,k)_{\mu_1\mu_2} \\
&= k^\mu_1 k^\nu_2 I(0,k)_{\mu_1\mu_2} + k^\mu_1 I(0,k)_{\mu_1\mu_2} \\
&= \frac{i}{4(4\pi)^2} k^2 \left\{ \left[ \frac{1}{4\pi} \ln \left( \frac{\Lambda^2}{-k^2} \right) + 1 \right] - \frac{1}{3} \left[ \frac{1}{4\pi} (D - 4 + 6) \left( \ln \left( \frac{\Lambda^2}{-k^2} \right) + \frac{5}{3} \right) + 1 \right] \right\} \\
&= -\frac{i}{4(4\pi)^2} k^2 \ln \left( \frac{\Lambda^2}{-k^2} \right).
\end{align*}
\]  

The step from Eq. \((B13)\) to Eq. \((B14)\) follows by using the identity \((B2)\).

Combining \((B13)\) and \((B14)\), we obtain the result for the regularized expression on the right-hand side of the DSE \((B2)\):

\[
\begin{align*}
i &4\pi\alpha_s(K) N_c \frac{1}{k^2} \int^{(A)} \frac{d^4k_1}{(2\pi)^4} \frac{1}{k_1^2(k_1-k_2)^2} k^\mu_1 \left[ g_{\mu_1\mu_2} + (\xi^{(A)} - 1) \frac{(k-k_1)_{\mu_1}(k-k_2)_{\mu_2}}{(k_1-k_2)^2} \right] k^\nu_2 = \\
&-\alpha_s(K) N_c \frac{1}{16\pi} \left[ (3 - \xi^{(A)}) \ln \left( \frac{\Lambda^2}{-k^2} \right) + 4 \right].
\end{align*}
\]  

APPENDIX C: EVALUATION OF THE VARIATIONAL DERIVATIVE OF \(c^{(3)}\)

In this Appendix, we evaluate, i.e., reduce to the form of the type \((C9)\), the last term in the quadratic-in-sources antighost DSE \((C2)\). This term can be first rewritten as an integral over two momenta, by using the expression \((C6)\):
The cubic terms in the structure constants can be reduced due to the following identities:

\[ f^{ac_2c_3} f^{c_3c_2b_3} = -N_c \delta_{ab_3}, \quad \] (C3)

\[ f^{ac_2c_3} f^{c_3b_2b_3} f^{b_3c_2d_3} = -\frac{1}{2} N_c f^{ac_2b_3}, \quad \] (C4)

Identity (C3) is well known, and identities (C4) are direct consequences of the Jacobi identity. We use identities (C3) and (C4) in expressions (C2); further, we rename in the second double integral \( b_2 \leftrightarrow d_2 \) and \( \mu_2 \leftrightarrow \mu_3 \); in the third double integral, \( c_3 \leftrightarrow d_2, \ b_3 \leftrightarrow d_3, \mu_4 \leftrightarrow \mu_5 \) and \( k_2(\text{new}) = k_2 - k_1 \). This allows us to write expressions (C2) in a somewhat more compact form

\[ f^{ac_2c_3} k^{\mu_1} \left( \frac{\delta c^{c_3(3)}}{\delta J^{c_2}} \right)(k) = \]

\[ g^2(K) N_c \int^{(A)} d^4k_1 d^4k_2 \frac{G_A(k_1^2)G_A((k_1 + k_2)^2)G_A(k_2^2)}{G_A((k_2 - k)^2)} \left\{ \frac{1}{k^2} k^{\mu_1} P_{\mu_1 \mu_2}(k_1) (k_1 + k)^{\mu_2} k^{\mu_3} P_{\mu_3 \mu_4}(k_2) \right\} \]

\[ + \frac{1}{2} k^{\mu_1} P_{\mu_1 \mu_2}(k_1) (k_1 + k - k_2)^{\mu_2} (k_1 + k)^{\mu_3} P_{\mu_3 \mu_4}(k_2) \]

\[ + \frac{1}{2} k_{1}^{\mu_1} (k_1 + k)^{\mu_3} P_{\mu_3 \mu_4}(k_1) P_{\mu_4 \mu_5}(k_2) \]

\[ + (2k_2 + k_1)^{\mu_5} P_{\mu_5 \mu_1}(k_1) P_{\mu_1 \mu_4}(k_2) - (2k_2 + k_1)^{\mu_5} P_{\mu_5 \mu_1}(k_1) P_{\mu_1 \mu_4}(k_2) \right\} \times \left( f^{ad_2d_3} J^{a_1}_d (k_2) \rho^{d_3}(k - k_2) \right). \] (C5)

The next step is to perform in Eq. (C5) the integration over \( k_1 \). Since we work in the one-loop approximation, we need to obtain only the leading order term of expression (C5), i.e., the terms \( \sim g^3 \sim \alpha_s^{3/2} \) (note that \( J \sim g^{-1} \)), because the one-loop expressions for the dressing functions predict only the term \( \sim \alpha_s^{3/2} \) in Eq. (50) for the first two terms of the quadratic-in-sources antighost DSE (50). Therefore, we replace in Eq. (C5) all the dressing functions \( G_A(k_1^2) \rightarrow 1 \).

Nonetheless, the integration over \( k_1 \) is still involved, because of the rather complicated momentum dependence of the gluon propagators \( P_{\mu_1 \mu}(k_1) \), cf. Eq. (18). However, in the Feynman gauge, \( \xi^{(A)} = 1 \), and therefore \( \xi^{(K)} = 1 + \mathcal{O}(\alpha_s) \) by relation (23). So we can set \( \xi^{(K)} \rightarrow 1 \) in the Feynman gauge propagators in Eq. (C7) at the considered order, i.e., \( P_{\mu_1 \mu}(k) \rightarrow g_{\mu_1 \mu} \). Thus, we obtain

\[ f^{ac_2c_3} k^{\mu_1} \left( \frac{\delta c^{c_3(3)}}{\delta J^{c_2}} \right)(k) \left. \right|_{\xi^{(A)} = 1} = (16\pi^2) \alpha_s^4(K) N_c \int^{(A)} d^4k_1 d^4k_2 \left(\frac{1}{2\pi} \right)^8 \]

\[ \times \left\{ \frac{1}{k^2} (k \cdot (k_1 + k))^\mu_1 + \frac{1}{2} (k \cdot (k_1 + k - k_2))^\mu_1 (k_1 + k - k_2)^\mu_2 \right\} \]

\[ + \frac{1}{2} \left\{ (k_1 + k - k_2) \cdot (k_1 + k)^\mu_1 + (k \cdot (k_1 + k + 2k_2)) (k_1 + k)^\mu_1 - (k \cdot (k_1 + k))^\mu_1 (2k_1 + 2k_2)^\mu_1 \right\} \]

\[ \times (f^{ad_2d_3} J^{a_1}_d (k_2) \rho^{d_3}(k - k_2)) + \mathcal{O}(\alpha_s^{5/2}) \]. (C6)

We should first perform the integration over \( k_1 \), which is complicated. Here we only take into account the cutoff-dependent terms \( \sim \ln(\Lambda^2/(-q^2)) \). Then the integration of each term over \( k_1 \) becomes less difficult, using the results (B3a)–(B3d) [the terms \( \sim \ln(\Lambda^2/(-q^2)) \) there ]. The integrals needed for the first, second, and third term of Eq. (C6)
are respectively

\[
\int^{(A)} \frac{d^4 k_1}{(2\pi)^4} \frac{(k^2 + k^\mu_1 k_{\mu_1} k_{\mu_1}) k_{\mu}}{k_1^2 (k_1 + k)^2} = I(k)k^2 k_{\mu} + I(k)^{\mu_1} k_{\mu_1} = k^2 k_{\mu} \frac{i}{2} \left( \frac{\Lambda^2}{-k^2} \right)^\infty + \text{finite} \quad ,
\]

\[
\int^{(A)} \frac{d^4 k_1}{(2\pi)^4} \frac{(k_{\mu_1} k_{\mu_1} + k \cdot (k - k_2)) (k_{\mu_1} + k_{\mu})}{k_1^2 (k_1 + k)^2 (k_1 + k - k_2)^2}
\]

\[
= I(k, k - k_2)^{\mu_1} k_{\mu_1} + \text{finite} = k_{\mu} \frac{i}{4} \left( \frac{\Lambda^2}{-k^2} \right)^\infty + \text{finite} \quad ,
\]

\[
\int^{(A)} \frac{d^4 k_1}{(2\pi)^4} \frac{(k_{\mu_1} k_{\mu_1} + k_{\mu_1} k_{\mu_1} - 2 k_{\mu_1} k_{\mu_1} k_{\mu_1})}{k_1^2 (k_1 + k)^2 (k_1 + k - k_2)^2}
\]

\[
= I(k, k_2)^{\mu_1} g_{\mu_1 \mu_2} k_{\mu_2} - I(k, k_2)^{\mu_1} k_{\mu_1} + \text{finite} = k_{\mu} \frac{3}{4} \left( \frac{\Lambda^2}{-k^2} \right)^\infty + \text{finite} \quad .
\]

Using the results of $k_1$-integration (C7), we can rewrite Eq. (C8) as

\[
\left[ f_{a_{5_{c3}} k_{\mu_1}} \left( \frac{\delta c_{3}}{\delta J_{c_2}} \right)(k) \right] \bigg|_{\xi (\alpha) = 1} = (16\pi^2) \alpha_s^2(K) N_c \int^{(A)} \frac{d^4 k_2}{(2\pi)^4} \frac{1}{k_2^2 (k_2 - k_2)^2} \frac{k_{\mu}}{4\pi^2} \left( f_{a_{5_{d3}} J_{d_2}^\mu (k_2)} \rho_{d_3}^\mu (k_2) - O(\alpha_s^5/2) \right) .
\]

The sum in the first brackets is one. This is the result for the third term of the quadratic-in-sources antighost DSE \cite{20}, in the Feynman gauge and at the leading ($\sim \alpha_s^3/2$) level.

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