Abelian Categories from Triangulated Categories via Nakaoka–Palu’s Localization

Yasuaki Ogawa

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Abstract

The aim of this paper is to provide an expansion of Abe–Nakaoka’s heart construction to the following two different realizations of the module category over the endomorphism ring of a rigid object in a triangulated category: Buan–Marsh’s localization and Iyama–Yoshino’s subfactor. Our method depends on a modification of Nakaoka–Palu’s HTCP localization, a Gabriel–Zisman localization of extriangulated categories which is also realized as a subfactor of the original ones. Besides of the heart construction, our generalized HTCP localization involves the following phenomena: (1) stable category with respect to a class of objects; (2) recollement of triangulated categories; (3) recollement of abelian categories under a certain assumption.

Keywords Gabriel–Zisman localization · Extriangulated category · Cotorsion pair · Heart

Mathematics Subject Classification Primary 18E35; Secondary 18G80, 18E10

1 Introduction

In many fields of mathematics containing representation theory, it is basic to study how to construct related abelian categories from a given triangulated category \( C \). There are many researches in this context, for example \([1,3,4,6,7,13,18,19,24–26,30]\). Our study has its origin in the following result in \([19, \text{Thm. 3.3}]\): If \( T \) is a cluster-tilting object in a triangulated category \( C \), then the additive quotient \( C/\text{add}\ T \) is equivalent to \( \text{mod} \ \text{End}_C(T)^{\text{op}} \) the module category of finitely presented right \( \text{End}_C(T)^{\text{op}} \)-modules (see also \([18]\)).

Afterwards, Buan and Marsh pointed out that, as long as \( T \) is a rigid object in \( C \), there exists a localization functor from \( C \) to \( \text{mod} \ \text{End}_C(T)^{\text{op}} \), generalizing Koenig–Zhu’s set-up. More precisely, in \([6]\), they firstly found a class \( S \) of morphisms with an explicit description such that the associated localization \( L_S : C \to C[S^{-1}] \) induces an equivalence \( C[S^{-1}] \simeq \)
mod \text{End} \, \mathcal{C}(T)^{\text{op}}. Following this, in [7], they factorized the above localization \( L_{\mathcal{S}} \) as the composition of two localizations:

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\sigma} & \text{mod} \, \text{End} \, \mathcal{C}(T)^{\text{op}} \\
\text{Loc} & \xrightarrow{[6]} & \mathcal{C}/(T^{\perp_{-1}})
\end{array}
\]

where \( \sigma \) denotes an additive quotient. Under additional assumptions on \( \mathcal{C} \), there are many advantages of this factorization, since \( \mathcal{C}/(T^{\perp_{-1}}) \) is preabelian and the second one \( \text{Loc} \) is a Gabriel–Zisman localization admitting a calculus of left and right fractions. Their construction of preabelian categories was improved by many authors, e.g. [4,13,23,24,26]. On the other hand, before [6,7], the category \( \text{mod} \, \text{End} \, \mathcal{C}(T)^{\text{op}} \) has already been realized as a subfactor of \( \mathcal{C} \) in [17], namely, there exists an equivalence

\[
\frac{\text{add}(T[-1]) \ast \text{add}T}{\text{add}T} \sim \text{mod} \, \text{End} \, \mathcal{C}(T)^{\text{op}},
\]

where \([-1]\) is a desuspension of \( \mathcal{C} \).

As another generalization of Koenig–Zhu’s construction, Abe and Nakaoka provided a new construction of a related abelian category \( \mathcal{H}/\mathcal{W} \) from a given triangulated category \( \mathcal{C} \) equipped with a cotorsion pair \( (\mathcal{S}, \mathcal{V}) \) in [1]. Their method also generalizes the construction of the heart of a \( t \)-structure proved in [3] (see [25, Prop. 2.6] for details). So their abelian category \( \mathcal{H}/\mathcal{W} \) is still called the heart of \( (\mathcal{S}, \mathcal{V}) \). As we will see in Sect. 4.2, Abe–Nakaoka’s heart construction is a generalization of Iyama–Yoshino’s subfactor (1.2). So it is natural to ask whether the heart \( \mathcal{H}/\mathcal{W} \) can be realized as a nice Gabriel–Zisman localization of \( \mathcal{C} \). This question has already arisen in [6, Section 6] and [7, Section 6], and some answers were obtained (see Sect. 5).

The aim of this article is to give a more complete answer to Buan-Marsh’s question in connection with Iyama–Yoshino’s subfactor, which unifies and improves some related results (e.g. [26, Thm. 6.3], [13, Thm. 5.6]), where the method completely differs from them. Our method is a modification of Nakaoka–Palu’s localization via Hovey twin cotorsion pair (HTCP localization) which was inspired from Hovey triple [11,15] and introduced in [27,28]. They firstly introduced a notion of extriangulated category which is a simultaneous generalization of exact category and triangulated one. The HTCP localization turns out to be a Gabriel–Zisman localization of an extriangulated category which can be realized as a subfactor of the original one, and covers many important phenomena, e.g. recollement of triangulated categories, Happel’s (projectively) stable category of a Frobenius category [12] and Iyama–Yoshino’s triangulated structure via mutation pairs [17]. Our generalized HTCP localization is still a Gabriel–Zisman localization of an extriangulated category which is equivalent to a subfactor. It covers a wider class of localizations containing stable category with respect to a class of objects and recollement of abelian categories under a certain assumption.

**Theorem A** (Theorem 3.10) Let \( ((\mathcal{S}, T), (\mathcal{U}, \mathcal{V})) \) be a generalized HTCP in an extriangulated category \( \mathcal{C} \). Then, there exists a class \( \mathcal{V} \) of morphisms in \( \mathcal{C} \) such that the associated Gabriel–Zisman localization \( L_{\mathcal{V}} : \mathcal{C} \to \mathcal{C}[\mathcal{V}^{-1}] \) induces an equivalence

\[
\Phi : \frac{T \cap \mathcal{U}}{S \cap \mathcal{V}} \sim \mathcal{C}[\mathcal{V}^{-1}].
\]
Note that there are two similar pictures below deduced from different set-ups in (1.1), (1.2) and Theorem A:

\[
\begin{array}{c}
\text{add}T[-1] \xrightarrow{\sim} \text{add}T \xrightarrow{\sim} C[S^{-1}] \\
\text{add}T \xrightarrow{\sim} C[S^{-1}] \\
(T: \text{rigid}) \\
\text{Buan-Marsh} \\
\text{Iyama–Yoshino}
\end{array}
\]

\[
\begin{array}{c}
((S, T), (U, V)) \\
\text{add}T \xrightarrow{\sim} C[V^{-1}] \\
U \cap S \xrightarrow{\sim} C[V^{-1}] \\
(V - 1)
\end{array}
\]

This article was motivated by the above analogy. Our second result shows that the heart construction can be regarded as a generalized HTCP localization.

**Theorem B** (Corollary 4.7) Let \( C \) be a triangulated category equipped with a cotorsion pair \((S, V)\). Then, there exists a class \( \mathcal{V} \) of morphisms in \( C \) such that the associated localization \( L_{\mathcal{V}} : C \to C[\mathcal{V}^{-1}] \) induces an equivalence \( \Phi : \mathcal{H}/\mathcal{W} \xrightarrow{\sim} C[\mathcal{V}^{-1}] \).

Furthermore, we factorize the localization \( L_{\mathcal{V}} \) into nice localizations, following Buan–Marsh’s investigation.

**Theorem C** (Theorem 4.9) Let \( C \) be the above. Assume that \( S \ast V \) is functorially finite. Then,

1. The additive quotient \( C/\text{add}(S \ast V) \) is preabelian and the class \( R \) of regular morphisms in it admits a calculus of left and right fractions;
2. The localization \( L_{\mathcal{V}} : C \to \mathcal{H}/\mathcal{W} \) is factored as the composition of the additive quotient \( \varpi : C \to C/\text{add}(S \ast V) \) and the Gabriel–Zisman localization \( L_{R} : C/\text{add}(S \ast V) \to \mathcal{H}/\mathcal{W} \) with respect to the class \( R \).

This article is organized as follows: Sect. 2 will be devoted to prepare basic results on extriangulated category, Gabriel–Zisman localization and preabelian category. Sect. 3 contains the first main result of this article. Here, we formulate a generalized HTCP localization and prove Theorem A. In Sect. 4, using the generalized HTCP, we investigate two different aspects of the heart in a triangulated category \( C \), i.e., a subfactor and a localization of \( C \), and then Theorems B and C are proved. Finally, in Sect. 5, we mention another related approach.

**Notation and Convention**

The symbol \( C \) always denotes a category, and the set of morphisms \( X \to Y \) in \( C \) is denoted by \( C(X, Y) \) or simply denoted by \( (X, Y) \) if there is no confusion. The class of objects (resp. morphisms) in \( C \) is denoted by \( \text{Ob}C \) (resp. \( \text{Mor}C \)). We denote by \( C^{\text{op}} \) the opposite category. If there exists a fully faithful functor \( N \hookrightarrow C \), we often regard \( N \) as a full subcategory of \( C \). If a given category \( C \) is additive, its subcategory \( U \) is always assumed to be full, additive and closed under isomorphisms, and we only consider additive functors between them. For \( X \in C \), if \( C(U, X) = 0 \) for any \( U \in U \), we write abbreviately \( C(U, X) = 0 \). Similar notations will be used in many places. For an additive functor \( F : C \to D \), we define the image and kernel of \( F \) as the full subcategories

\[
\text{Im } F := \{ Y \in D \mid \exists X \in C, \ F(X) \cong Y \} \quad \text{and} \quad \text{Ker } F := \{ X \in C \mid F(X) = 0 \},
\]

respectively. For a full subcategory \( U \) in \( C \), the symbol \( F|_U \) denotes the restriction of \( F \) on \( U \).
2 Preliminary

2.1 Extriangulated Category

In this section, we recall results necessary for our purpose and terminology on extriangulated categories. An extriangulated category is defined to be a triple \((\mathcal{C}, \mathcal{E}, \mathcal{s})\) of

- an additive category \(\mathcal{C}\);
- an additive bifunctor \(\mathcal{E} : \mathcal{C}^{\text{op}} \times \mathcal{C} \to \text{Ab}\), where \(\text{Ab}\) is the category of abelian groups;
- a correspondence \(\mathcal{s}\) which associates each equivalence class of a sequence of the form \(X \to Y \to Z\) in \(\mathcal{C}\) to an element in \(\mathcal{E}(Z, X)\) for any \(Z, X \in \mathcal{C}\),

which satisfies some ‘additivity’ and ‘compatibility’. It is simply denoted by \(\mathcal{C}\) if there is no confusion. We refer to [28, Section 2] for its detailed definition. An extriangulated category was introduced to unify triangulated category and exact one. More precisely, by putting \(\mathcal{E} := \mathcal{C}(\cdot, \cdot[1])\), a triangulated category \((\mathcal{C}, [1], \Delta)\) can be regarded as an extriangulated category [28, Prop. 3.22]. By putting \(\mathcal{E} := \text{Ext}^1_\mathcal{C}(\cdot, \cdot)\), an exact category \(\mathcal{C}\) can be regarded as an extriangulated category [28, Example 2.13]. We shall use the following terminology in many places.

**Definition 2.1** Let \((\mathcal{C}, \mathcal{E}, \mathcal{s})\) be an extriangulated category.

1. We call an element \(\delta \in \mathcal{E}(Z, X)\) an \(\mathcal{E}\)-extension, for any \(X, Z \in \mathcal{C}\);
2. A sequence \(X \xrightarrow{f} Y \xrightarrow{g} Z\) corresponding to an \(\mathcal{E}\)-extension \(\delta \in \mathcal{E}(Z, X)\) is called a conflation. In addition, \(f\) and \(g\) are called an inflation and a deflation, respectively.
3. An object \(P \in \mathcal{C}\) is said to be projective if for any deflation \(g : Y \to Z\), the induced morphism \(\mathcal{C}(P, g) : \mathcal{C}(P, Y) \to \mathcal{C}(P, Z)\) is surjective. We denote by \(\text{P}(\mathcal{C})\) the subcategory of projectives in \(\mathcal{C}\). An injective object and \(\text{l}(\mathcal{C})\) are defined dually.
4. We say that \(\mathcal{C}\) has enough projectives if for any \(Z \in \mathcal{C}\), there exists a conflation \(X \to P \to Z\) with \(P\) projective. Having enough injectives is defined dually.

Keeping in mind the triangulated case, we introduce the notions cone and cocone.

**Proposition 2.2** Let \(\mathcal{C}\) be an extriangulated category. For an inflation \(f \in \mathcal{C}(X, Y)\), take a conflation \(X \xrightarrow{f} Y \xrightarrow{g} Z\), and denote this \(Z\) by \(\text{Cone}(f)\). We call \(\text{Cone}(f)\) a cone of \(f\). Similarly, we denote the object \(X\) by \(\text{CoCone}(g)\) and call it a cocone of \(g\). For a given inflation \(f\) (resp. deflation \(g\)), \(\text{Cone}(f)\) (resp. \(\text{CoCone}(g)\)) is uniquely determined up to isomorphism.

Furthermore, for any subcategories \(\mathcal{U}\) and \(\mathcal{V}\) in \(\mathcal{C}\), we define a full subcategory \(\text{Cone}(\mathcal{V}, \mathcal{U})\) to be the one consisting of objects \(X\) appearing in a conflation \(V \to U \to X\) with \(U \in \mathcal{U}\) and \(V \in \mathcal{V}\). A subcategory \(\text{CoCone}(\mathcal{V}, \mathcal{U})\) is defined dually. Next, we recall the notion of weak pullback in extriangulated categories. Consider a conflation \(X \xrightarrow{f} Y \xrightarrow{g} Z\) corresponding to \(\mathcal{E}\)-extension \(\delta \in \mathcal{E}(Z, X)\) and a morphism \(z : Z' \to Z\). Put \(\delta' := \mathcal{E}(z, X)(\delta)\) and consider a corresponding conflation \(X \to E \to Z'\). Then, there exists a commutative diagram of the following shape

\[
\begin{array}{ccc}
X & \longrightarrow & E \\
\downarrow & & \downarrow_{(Pb)} \\
X & \longrightarrow & Z' \\
\end{array}
\]

\[
\begin{array}{ccc}
Z & \longrightarrow & Z' \\
\downarrow & & \downarrow \\
Y & \longrightarrow & Z \\
\end{array}
\]
The commutative square (Pb) is called a weak pullback of \( g \) along \( z \) which is a generalization of the pullback in exact categories and the homotopy pullback in triangulated categories. The dual notion weak pushout exists and it will be denoted by (Po) (see [28, Cor. 3.16] for details).

We end the section by mentioning that the class of extriangulated categories is closed under certain operations.

**Proposition 2.3** [28, Rem. 2.18, Prop. 3.30] Let \( C \) be an extriangulated category.

1. Any extension-closed subcategory admits an extriangulated structure induced from that of \( C \).
2. Let \( I \) be a full additive subcategory, closed under isomorphisms which satisfies \( I \subseteq P(C) \cap I(C) \), then the additive quotient \( C/I \) has an extriangulated structure, induced from that of \( C \) (see Example 2.6 for the definition of additive quotient).

### 2.2 Gabriel–Zisman Localization

Since we are interested in Gabriel–Zisman localizations of extriangulated categories, we recall its definition, following [9] (see also [10]).

**Definition 2.4** Let \( C \) and \( D \) be categories and \( S \) a class of morphisms in \( C \). A functor \( L_S : C \to D \) is called a Gabriel–Zisman localization of \( C \) with respect to \( S \) if the following universality holds:

1. \( L_S(s) \) is an isomorphism in \( D \) for any \( s \in S \);
2. For any functor \( F : C \to D' \) which sends each morphism in \( S \) to an isomorphism in \( D' \), there uniquely, up to isomorphism, exists a functor \( F' : D \to D' \) such that \( F \cong F' \circ L_S \).

In this case, we denote by \( L_S : C \to C[S^{-1}] \).

The Gabriel–Zisman localization \( C[S^{-1}] \) always exists provided there are no set-theoretic obstructions. So, whenever we consider the Gabriel–Zisman localization of \( C \), we assume that \( C \) is skeletally small. Morphisms in the new category \( C[S^{-1}] \) can be regarded as compositions of the original morphisms and the formal inverses. We refer to [9, Thm. 2.1] for an explicit construction of \( C[S^{-1}] \). If the class \( S \) satisfies the following conditions and its dual ones, any morphism in \( C[S^{-1}] \) has a very nice description, see [10, Section I.2] for details.

**Definition 2.5** Let \( S \) be a class of morphisms in \( C \).

1. (RF1) The identity morphisms of \( C \) lie in \( S \) and \( S \) is closed under composition.
2. (RF2) Any diagram of the form:

\[
\begin{array}{ccc}
B & \xrightarrow{f} & D \\
\downarrow & & \\
C & \xrightarrow{s} & D
\end{array}
\]

with \( s \in S \) can be completed in a commutative square of the form:

\[
\begin{array}{ccc}
A & \xrightarrow{s'} & B \\
\downarrow & & \downarrow \\
C & \xrightarrow{s} & D
\end{array}
\]

with \( s' \in S \).
(RF3) If \( s : Y \to Y' \) in \( S \) and \( f , f' : X \to Y \) are morphisms such that \( sf = sf' \), then there exists \( s' : X' \to X \) in \( S \) such that \( fs' = f's' \).

In this situation, we say that \( S \) or \( L_S \) admits a calculus of left and right fractions.

As is well-known, the Verdier localization of a triangulated category \( C \) with respect to its thick subcategory \( N \) is defined to be a Gabriel–Zisman localization which admits a calculus of left and right fractions. More precisely, it is the Gabriel–Zisman localization of \( C \) with respect to the class of morphisms whose cones belong to \( N \) (e.g. [29]). Similarly, the Serre localization of an abelian category is also an example of such Gabriel–Zisman localizations (e.g. [31]). The following example will play an important role in Sect. 3.

**Example 2.6** Let \( C \) be an additive category and \( \mathcal{I} \) its full subcategory closed under direct summands. We define the stable category \( C/\mathcal{I} \) of \( C \) with respect to \( \mathcal{I} \) as the ideal quotient of \( C \) modulo the (two-sided) ideal in \( C \) consisting of all morphisms having a factorization through an object in \( \mathcal{I} \). This is also called the additive quotient of \( C \) with respect to \( \mathcal{I} \). Consider the class \( S \) of all sections whose cokernels belonging to \( \mathcal{I} \), i.e., the morphisms \( s \) appearing in some splitting short exact sequence of the form \( 0 \to X \xrightarrow{s} X \oplus I \to I \to 0 \) with \( I \in \mathcal{I} \). Then, \( C/\mathcal{I} \) is equivalent to the Gabriel–Zisman localization \( C[S^{-1}] \).

**Proof** This is well-known for experts, but we can not find proper references. So we include a detailed proof here.

It suffices to show that the additive quotient \( \pi : C \to C/\mathcal{I} \) has the same universality as that of \( L_S : C \to C[S^{-1}] \). To this end, we consider a functor \( F : C \to D \) which sends any morphisms of \( S \) to isomorphisms in \( D \). Note that we do not assume that \( F \) is additive.

1. It is obvious that \( \pi(s) \) is an isomorphism in \( C/\mathcal{I} \) whenever \( s \) belongs to \( S \).
2. We shall show the existence of a functor \( F' : C/\mathcal{I} \to D \) with \( F \cong F' \circ \pi \) and its uniqueness up to isomorphism. Define such a functor \( F' \) as follows: For any \( X \in \text{Ob}(C/\mathcal{I}) = \text{Ob} C \), we set \( F'(X) := F(X) \); For any morphism \( \pi(f) : X \to Y \) in \( C/\mathcal{I} \), we set \( F'(\pi(f)) := F(f) \). To check the well-definedness, we assume that \( \pi(f) = \pi(g) \) where \( f , g : X \to Y \) are morphisms in \( C \), that is, the morphism \( f - g \) factors through an object \( I \in \mathcal{I} \) as \( f - g : X \xrightarrow{a} I \xrightarrow{b} Y \). Then we get \( f = (g b) (1_a) \) and consider the following diagram in \( C \):

\[
\begin{array}{ccc}
X & \xrightarrow{(1)} & X \oplus I & \xrightarrow{(g \ b)} & Y \\
& \downarrow{(1)} & \downarrow{g} & & \\
X & & & & \\
\end{array}
\]

Notice that the left half of the above diagram is not commutative. However, applying \( F \) makes it commute. In fact, the projection \( (1 \circ) : X \oplus I \to X \) is a left inverse of \( (1_a) \) and \( (1_0) \). Since both \( F(1_a) \) and \( F(1_0) \) are isomorphisms in \( D \), \( F(1_0) \) is an inverse of them. It guarantees \( F(1_a) = F(1_0) \) in \( D \). Hence we have a desired equality

\[
F(f) = F(g \ b) \circ F(1_a) = F(g \ b) \circ F(1_0) = F(g).
\]
The commutativity \( F = F' \circ \pi \) automatically holds.

Since \( \pi \) is full and dense, the uniqueness of \( F' \) is satisfied. This finishes the proof. \( \square \)

### 2.3 Preabelian Category

We recall some basic properties of preabelian categories which will be used in Sect. 4.3, following [33] (see also [6]). An additive category \( A \) is called a preabelian category if any morphism has a kernel and a cokernel. A morphism is said to be regular if it is both an epimorphism and a monomorphism. Note that pullbacks and pushouts always exist in a preabelian category \( A \). In fact, for morphisms \( C \xrightarrow{d} D \xleftarrow{c} B \), we take the kernel sequence

\[
A \rightarrow B \oplus C \xrightarrow{(c-d)} D
\]

(2.1) to obtain the pullback of \( d \) along \( c \):

\[
\begin{array}{c}
A \xrightarrow{a} B \\
\downarrow b \quad \quad \quad \quad \downarrow \text{(Pb)} \quad \downarrow c \\
C \xrightarrow{d} D
\end{array}
\]

**Definition 2.7** A preabelian category \( A \) is said to be left integral if for any pullback (2.1), \( a \) is an epimorphism whenever \( d \) is an epimorphism. The right integrality is defined dually.

An integral category is defined as a preabelian category which is both left integral and right integral.

The following provides a nice connection between integral categories and abelian categories.

**Proposition 2.8** [33, p. 173] For an integral category \( A \), the following hold.

1. The class \( R \) of regular morphisms admits a calculus of left and right fractions.
2. The localization functor \( L_R : A \rightarrow A[\mathbb{R}^{-1}] \) is additive.
3. The category \( A[\mathbb{R}^{-1}] \) is abelian.

### 3 Localization via Generalized Hovey Twin Cotorsion Pair (gHTCP)

#### 3.1 Definition of gHTCP

Throughout this section, the symbol \( C = (\mathcal{C}, \mathcal{E}, \mathcal{S}) \) is an extriangulated category. For a subcategory \( \mathcal{U} \subseteq \mathcal{C} \) and an object \( X \in \mathcal{C} \), a morphism \( U \rightarrow X \) from an object \( U \in \mathcal{U} \) is called a right \( \mathcal{U} \)-approximation of \( X \) if the induced morphism \( \mathcal{C}(U', U) \rightarrow \mathcal{C}(U', X) \) is an epimorphism for any \( U' \in \mathcal{U} \). The dual notion left \( \mathcal{U} \)-approximation exists.

Let \( (\mathcal{U}, \mathcal{V}) \) be a pair of full subcategories in \( \mathcal{C} \) which are closed under isomorphisms and direct summands. We introduce the notion of right/left cotorsion pair as follows.

**Definition 3.1** Let \( X \) be an object in \( \mathcal{C} \). A right \( \mathcal{U} \)-approximation \( p_X : U_X \rightarrow X \) of \( X \) is said to be a \( \mathcal{U} \)-deflation of \( X \) if it is a deflation and \( \text{CoCone}(p_X) \in \mathcal{V} \). Dually, a left \( \mathcal{V} \)-approximation \( t_X : X \rightarrow V_X \) of \( X \) is said to be a \( \mathcal{V} \)-inflation of \( X \) if it is an inflation and \( \text{Cone}(t_X) \in \mathcal{U} \).

**Definition 3.2** (1) The pair \( (\mathcal{U}, \mathcal{V}) \) of subcategories in \( \mathcal{C} \) is called a right cotorsion pair if it satisfies the following conditions:
(a) For any $X \in C$, there exists a conflation

$$V_X \to U_X \xrightarrow{p_X} X$$

where $p_X$ is a $U$-deflation of $X$. In this case, we say that $X$ is resolved by $(U, V)$;

(b) Any $f \in C(U, V)$ with $Y \in U$ and $V \in V$ factors through an object in $U \cap V$.

For a right cotorsion pair $(U, V)$, we denote by $\mathcal{U}$-def the class of $\mathcal{U}$-deflations $p_X$ for all $X \in C$.

(2) The pair $(S, T)$ of subcategories in $C$ is called a left cotorsion pair if it satisfies the following conditions:

(a) For any $X \in C$, there exists a conflation

$$X \xrightarrow{\iota_X} T^X \to S^X$$

where $\iota_X$ is a $T$-inflation of $X$. In this case, we also say that $X$ is resolved by $(S, T)$;

(b) Any $f \in C(S, T)$ with $S \in S$ and $T \in T$ factors through an object in $S \cap T$.

For a left cotorsion pair $(S, T)$, we denote by $\mathcal{T}$-inf the class of $\mathcal{T}$-inflations $\iota_X$ for all $X \in C$.

**Lemma 3.3** For a right cotorsion pair $(U, V)$ with $\mathcal{I} := U \cap V$, an assignment $R : X \mapsto U_X$ gives rise to a right adjoint $R : C/\mathcal{I} \to U/\mathcal{I}$ of the canonical inclusion $U/\mathcal{I} \hookrightarrow C/\mathcal{I}$. Dually, for a left cotorsion pair $(S, T)$ with $\mathcal{I} := S \cap T$, an assignment $L : X \mapsto T^X$ gives rise to a left adjoint $L : C/\mathcal{I} \to T/\mathcal{I}$ of the inclusion $T/\mathcal{I} \hookrightarrow C/\mathcal{I}$.

**Proof** Fix a conflation $V_X \to U_X \xrightarrow{p_X} X$ with $p_X$ being a $U$-deflation for each $X \in C$ and consider an assignment $R : X \mapsto U_X$. Due to the condition (b) in Definition 3.2, for any morphism $f \in C(X, Y)$, we have a unique morphism $Rf : RX \to RY$ which makes the following diagram commutative in $C/\mathcal{I}$:

$$
\begin{array}{ccc}
V_X & \longrightarrow & RX \\
\downarrow{Rf} & & \downarrow{f} \\
V_Y & \longrightarrow & RY
\end{array}
$$

Thus, it is easily checked that the assignment $R$ gives rise to a functor $C/\mathcal{I} \to U/\mathcal{I}$. The $\mathcal{U}$-deflation $p_X$ induces a surjective morphism $C/\mathcal{I}(U, RX) \xrightarrow{p_X \circ} C/\mathcal{I}(U, X)$ for any $U \in U$. Again, due to the condition (b), the above morphism $p_X \circ -$ is injective. Hence the functor $R : C/\mathcal{I} \to U/\mathcal{I}$ is a right adjoint of the inclusion $U/\mathcal{I} \hookrightarrow C/\mathcal{I}$. The assertion for a left cotorsion pair can be proved dually. $\square$

Obviously a right/left cotorsion pair is a weaker notion of a cotorsion pair which are firstly defined in abelian categories [34] and recently in extriangulated categories [28].

**Definition 3.4** A pair $(U, V)$ is called a cotorsion pair if it satisfies the following conditions:

1. Both $U$ and $V$ are closed under direct summands, isomorphisms and extensions;
2. $\mathcal{E}(U, V) = 0$;
3. $\text{Cone}(V, U) = \mathcal{C} = \text{CoCone}(V, U)$.

Note that, since we do not require $\mathcal{E}$-orthogonality, a left and right cotorsion pair does not form a cotorsion pair in general. In fact, $(\mathcal{C}, \mathcal{C})$ is a left and right cotorsion pair. Moreover, we
should remark that some authors use the terminology left/right cotorsion pair in a different sense [5,11].

Let \((S, T)\) and \((U, V)\) be a left cotorsion pair and a right cotorsion pair, respectively, and denote by \(\mathcal{P} := ((S, T), (U, V))\) the pair of them. We consider the following two conditions:

(Hov1) \(S \subseteq U\) and \(T \supseteq V\);
(Hov2) \(S \cap T = U \cap V\).

In the rest of this article, let \(\mathcal{P}\) denote the above pair satisfying (Hov1) and (Hov2). The following notions will be used in many places: (1) \(\mathcal{I} := S \cap V\); (2) \(\mathcal{Z} := T \cap U\); (3) \(\mathcal{V} := U\)-def \(\circ T\)-inf,

where \(U\)-def \(\circ T\)-inf denotes the class of morphisms \(f\) which has a factorization \(f = f_2 \circ f_1\) with \(f_1 \in T\)-inf and \(f_2 \in U\)-def. Note that \(\mathcal{I} = U \cap V\) holds.

**Definition 3.5** If \(\mathcal{P}\) satisfies the following two conditions, it is called a Hovey twin cotorsion pair (HTCP) introduced in [28]:

(Hov3) Both \((S, T)\) and \((U, V)\) are cotorsion pairs;
(Hov4) \(\text{Cone}(V, S) = \text{CoCone}(V, S)\).

We shall study basic properties of \(\mathcal{P}\) in this section. Denote by \(\pi : C \to C/\mathcal{I}\) the additive quotient. First, we consider two functors \(QLR := LR\pi\) and \(QRL := RL\pi : C \to C/\mathcal{I}\) by composing a right adjoint \(R : C/\mathcal{I} \to U/\mathcal{I}\) and a left adjoint \(L : C/\mathcal{I} \to T/\mathcal{I}\). We also consider a Gabriel–Zisman localization \(L_\mathcal{V} : C \to C[\mathcal{V}^{-1}]\) with respect to the class \(\mathcal{V}\). The following proposition draws a comparison between \(Z/\mathcal{I}\) and \(C[\mathcal{V}^{-1}]\), which is instrumental to formulate our main theorem.

**Proposition 3.6** We assume that both \(\text{Im } QLR\) and \(\text{Im } QRL\) are contained in \(Z/\mathcal{I}\). Then, there uniquely exists a functor \(\Phi : Z/\mathcal{I} \to C[\mathcal{V}^{-1}]\) which makes the following diagram commutative up to isomorphism:

\[
\begin{array}{ccc}
C & \xrightarrow{QLR} & Z/\mathcal{I} \\
\downarrow L_\mathcal{V} & & \downarrow \Phi \\
C[\mathcal{V}^{-1}]. & \xleftarrow{QRL} & Z/\mathcal{I}
\end{array}
\]

Before proving Proposition 3.6, we check the next easy lemma.

**Lemma 3.7** Let \(\mathcal{V}_Z\) be the class of morphisms in \(\mathcal{Z}\) which consists of sections \(f\) with \(\text{Cok } f \in \mathcal{I}\). Then, a containment \(\mathcal{V}_Z \subseteq \mathcal{V} \cap \text{Mor } \mathcal{Z}\) is true.

**Proof** Let \(X \xrightarrow{f} Y \to I\) be a splitting short exact sequence in \(\mathcal{Z}\) with \(I \in \mathcal{I}\). The morphism \(f\) is obviously a \(T\)-inflation. Thus \(\mathcal{V}_Z \subseteq T\)-inf. \(\square\)

**Proof of Proposition 3.6** Let us recall from Example 2.6 that the additive quotient \(\pi : \mathcal{Z} \to \mathcal{Z}/\mathcal{I}\) can be regarded as a Gabriel–Zisman localization of \(\mathcal{Z}\) with respect to \(\mathcal{V}_Z\). First, we consider the functor \(L_\mathcal{V}|_Z : \mathcal{Z} \to C[\mathcal{V}^{-1}]\) restricted onto \(\mathcal{Z}\). Since \(\mathcal{V}_Z \subseteq \mathcal{V}\), by the universality of \(\pi\), we have a unique functor \(\Phi : \mathcal{Z}/\mathcal{I} \to C[\mathcal{V}^{-1}]\) with \(L_\mathcal{V}|_Z \cong \Phi \circ \pi\). Thus we have the following diagram commutative up to isomorphism:

\[
\begin{array}{ccc}
\mathcal{Z} & \xrightarrow{\text{inc}} & C \\
\downarrow \pi & & \downarrow L_\mathcal{V} \\
\mathcal{Z}/\mathcal{I} & \xleftarrow{\Phi} & C[\mathcal{V}^{-1}].
\end{array}
\] (3.1)

where inc means the canonical inclusion.
We shall show that the obtained functor \( \Phi \) satisfies \( L \cong \Phi \circ Q_{LR} \). Let \( X \in C \). Associated to the definition of \( Q_{LR} \), we have a \( U \)-deflation \( p_X : RX \to X \) and \( T \)-inflation \( \iota_{RX} : RX \to LRX \) in \( C \), as depicted in the diagram below:

\[
\begin{array}{c}
V_X \\
\downarrow^{\iota_{RX}} \\
LRX \\
\downarrow^{S^{RX}} \\
X
\end{array}
\]

where the row and the column are conflations and \( V_X \in V \) and \( S^{RX} \in S \). Since \( L \) sends \( V \) to a class of isomorphisms, we have isomorphisms in \( C[V^{-1}] \):

\[
\begin{array}{c}
L_V(X) \\
\downarrow^{L_V(p_X)} \\
L_V(RX) \\
\downarrow^{L_V(\iota_{RX})} \\
L_V(LRX)
\end{array}
\]

Since \( LRX \in Z \), the diagram (3.1) shows a natural isomorphism \( L_V(LRX) \cong \Phi \circ \pi(LRX) \).

By the assumption, we notice \( \pi(LRX) = Q_{LR}(X) \) in \( Z/I \) and obtain a desired natural isomorphism \( L_V(X) \cong \Phi \circ Q_{LR}(X) \) in \( X \). By a similar argument, we have an isomorphism \( L_V \cong \Phi \circ Q_{RL} \).

It remains to show the uniqueness of \( \Phi \). To this end, we assume that there exists a functor \( \Phi' \) with \( L \cong \Phi' \circ Q_{RL} \). Since \( Q_{RL} \) is an identity functor on \( Z/I \), \( \pi \) equalize both \( \Phi \) and \( \Phi' \), precisely, \( \Phi' \circ \pi \cong \Phi \circ \pi \cong L_V|_Z \). However, by the universality of \( \pi \), such functors are uniquely determined up to isomorphism. Hence \( \Phi \cong \Phi' \).

We deduce the following result from Proposition 3.6.

**Lemma 3.8** If there exists a functorial isomorphism \( \eta : Q_{LR} \cong Q_{RL} \), the functors \( Q_{LR} \cong Q_{RL} \) send \( V \) to a class of isomorphisms in \( C/I \). Moreover, we have restricted functors \( Q_{LR} \cong Q_{RL} : C \to Z/I \). This situation can be depicted below:

\[
\begin{array}{c}
Z/I \\
\downarrow^L \\
C/I \\
\downarrow^R \\
C
\end{array}
\]

**Proof** By definition, \( Q_{LR}(U\text{-def}) \) and \( Q_{RL}(T\text{-inf}) \) form classes of isomorphisms. Thus the first assertion holds. Since \( \text{Im } Q_{LR} \subseteq T/I \) and \( \text{Im } Q_{RL} \subseteq U/I \), the existence of \( \eta \) forces \( \text{Im } Q_{RL} \in Z/I \).

A commutativity of \( R \) and \( L \) is a key property to prove many assertions in this article. So we introduce the following terminology.

**Definition 3.9** Let \( \mathcal{P} \) be a pair of a left cotorsion pair and a right cotorsion pair satisfying (Hov1) and (Hov2). If there exists a functorial isomorphism \( \eta : Q_{LR} \cong Q_{RL} \), then the pair \( \mathcal{P} \) is called a generalized Hovey twin cotorsion pair \( \text{(gHTCP)} \). Moreover, the associated localization \( L_V : C \to C[V^{-1}] \) is called the gHTCP localization with respect to \( \mathcal{P} \).

The following is our main result.
Theorem 3.10 Let $\mathcal{P}$ be a pair satisfying (Hov1) and (Hov2). The following are equivalent:

(i) There exists a natural isomorphism $\eta : Q_{LR} \to Q_{RL}$;
(ii) $R(T/I) \subseteq Z/I$ holds and the functor $L : U/I \to Z/I$ sends $R\pi(T\text{-inf})$ to isomorphisms;
(iii) $L(U/I) \subseteq Z/I$ holds and the functor $R : T/I \to Z/I$ sends $L\pi(U\text{-def})$ to isomorphisms;
(iv) Both $\text{Im } Q_{LR}$ and $\text{Im } Q_{RL}$ are contained in $Z/I$ and the functor $\Phi : Z/I \to \mathcal{C}[\mathbb{V}^{-1}]$ is an equivalence.

Proof (i) $\Rightarrow$ (ii), (iii): These implications directly follow from Lemma 3.8.

(ii) $\Rightarrow$ (i): For $X \in \mathcal{C}$, we consider a $T$-inflation $X \xrightarrow{\iota_X} LX, U$-deflations $p_X : RX \to X$ and $p_{LX} : RLX \to LX$. Then we get a morphism $R_{\iota_X} : RX \to RLX$ such that $\iota_X \circ p_X = p_{LX} \circ R_{\iota_X}$. In addition, taking $T$-inflation starting from $RX$ yields the following commutative diagram in $\mathcal{C}$:

$$
\begin{array}{ccc}
X & \xrightarrow{\iota_X} & LX \\
\downarrow{p_X} & & \uparrow{p_{LX}} \\
RX & \xrightarrow{R_{\iota_X}} & RLX \\
\downarrow{\iota_{RX}} & & \\
LRX & & \\
\end{array}
$$

Since $RLX \in Z$ and $\iota_{RX}$ is a left $T$-approximation, we have a morphism $\eta_X : LRX \to RLX$ with $R_{\iota_X} = \eta_X \circ \iota_{RX}$. By the assumption, we have isomorphisms $L\pi(\iota_{RX})$ and $L\pi(R_{\iota_X})$. Hence we conclude that $L\pi(\eta_X)$ is also an isomorphism in $Z/I$. Due to $Q_{LR}(X) \cong L\pi(RRX)$ and $Q_{RL}(X) \cong L\pi(RLX)$, the morphism $\eta_X$ gives rise to a desired natural isomorphism.

(iii) $\Rightarrow$ (i): It can be checked by the dual argument.

(iv) $\Rightarrow$ (ii), (iii): Since $L_Y$ sends $\mathbb{V}$ to isomorphisms, the assertions (ii) and (iii) hold.

(i) $\Rightarrow$ (iv): We shall show that the functor $Q_{LR} : \mathcal{C} \to Z/I$ has the same universality as that of $L_Y$. Due to Lemma 3.8, it follows that $Q_{LR}$ sends $\mathbb{V}$ to isomorphisms. To check the universality of $Q_{LR}$, let $F : \mathcal{C} \to \mathcal{D}$ be a functor which sends $\mathbb{V}$ to a class of isomorphisms in a category $\mathcal{D}$. We shall construct a functor $F' : Z/I \to \mathcal{D}$ such that $F \cong F' \circ Q_{LR}$ as follows: For an object $X \in \text{Ob}(Z/I) \subseteq \text{Ob}(C)$, we set $F'(X) := F(X)$; For a morphism $f : X \to Y$ in $Z$, we set $F'(Q_{LR}(f)) := F(f)$. This assignment can give rise to a desired functor $F'$. To show the well-definedness of this assignment: Let $f, g : X \to Y$ be morphisms in $Z$ with $Q_{LR}(f) = Q_{LR}(g)$ in $Z/I$, namely, $f - g$ is factored as $X \xrightarrow{a} I \xrightarrow{b} Y$ with $I \in I$. Then we get $f = (g \ b) \left( \begin{smallmatrix} 1 \\ a \end{smallmatrix} \right)$. Since the remaining argument to check the well-definedness is completely same as the proof of Example 2.6, we skip the details.

To verify the commutativity up to isomorphism, namely, the existence of an isomorphism $F \cong F' \circ Q_{LR}$, we let $f : X \to Y$ be a morphism in $\mathcal{C}$ and consider the following commutative diagram in $\mathcal{C}$:

$$
\begin{array}{ccc}
LRX & \xrightarrow{\iota_{RX}} & RX \xrightarrow{p_X} X \\
\downarrow{LRf} & & \downarrow{Rf} \\
LRY & \xleftarrow{\iota_{RY}} & RY \xrightarrow{p_Y} Y \\
\end{array}
$$
where \( R_* \in \mathcal{U}, LR_* \in \mathcal{Z}, p_* \in \mathcal{U}\text{-}\text{def} \) and \( \iota_* \in \mathcal{T}\text{-}\text{inf} \) for \( * = X,Y \). Note that, due to an isomorphism \( \eta : Q_{\mathcal{L}R} \sim Q_{\mathcal{R}L} \), we get isomorphisms \( Q_{\mathcal{L}R}(\iota_*) \) and \( Q_{\mathcal{L}R}(p_*) \) for \( * = X,Y \). Since \( LRf \) is a morphism in \( \mathcal{Z}/\mathcal{T} \), by definition, we have an equality \( F' \circ Q_{\mathcal{L}R}(LRf) = F(LRf) \). The functors \( F \) and \( Q_{\mathcal{L}R} \) send the above horizontal arrows to isomorphisms in \( \mathcal{D} \) and \( \mathcal{Z}/\mathcal{T} \), respectively. Hence we have a desired natural isomorphism

\[
F(X) \xrightarrow{F' \circ Q_{\mathcal{L}R}(p_X)} F' \circ Q_{\mathcal{L}R}(X)
\]
in \( X \in \mathcal{C} \).

Finally, to show the uniqueness of a desired functor \( F' \), we assume that there exists a functor \( F'' \) with \( F \cong F'' \circ Q_{\mathcal{L}R} \). Then, the functors restricts on \( \mathcal{Z} \) to have isomorphisms \( F|_{\mathcal{Z}} \cong F' \circ Q_{\mathcal{L}R}|_{\mathcal{Z}} \cong F'' \circ Q_{\mathcal{L}R}|_{\mathcal{Z}} \). Note that \( F' \circ Q_{\mathcal{L}R}|_{\mathcal{Z}} \cong F'|_{\mathcal{Z}} \) and \( F'' \circ Q_{\mathcal{L}R}|_{\mathcal{Z}} \cong F''|_{\mathcal{Z}} \).

Since the ideal quotient \( \mathcal{Z} \to \mathcal{Z}/\mathcal{I} \) is a localization with respect to \( \mathcal{V}_{\mathcal{Z}} \) (which is a subclass of \( \mathcal{V} \)), by the universality, we have \( F' \cong F'' \).

Although the \( g\text{HTCP} \) localization \( L_{\mathcal{V}} \) does not admit a calculus of left and right fractions in general, the morphisms in \( C[\mathcal{V}^{-1}] \) admit the following nice descriptions.

**Corollary 3.11** Suppose that a pair \( \mathcal{P} \) forms a \( g\text{HTCP} \). For any morphism \( \overline{\alpha} : L_{\mathcal{V}}(X) \to L_{\mathcal{V}}(Y) \) in \( C[\mathcal{V}^{-1}] \), there exist morphisms \( s : RX \to X, t : Y \to LY \) in \( \mathcal{V} \) and \( \alpha : RX \to LY \) in \( \mathcal{C} \) such that \( \overline{\alpha} = L_{\mathcal{V}}(t)^{-1} \circ L_{\mathcal{V}}(\alpha) \circ L_{\mathcal{V}}(s)^{-1} \).

**Proof** Due to the equivalence \( \Phi : \mathcal{Z}/\mathcal{T} \sim C[\mathcal{V}^{-1}] \), we have a morphism \( \overline{f} : Q_{\mathcal{L}R}(X) \to Q_{\mathcal{L}R}(Y) \) in \( \mathcal{Z}/\mathcal{T} \) such that \( \Phi(\overline{f}) = \overline{\alpha} \). Regarding \( Q_{\mathcal{L}R}(X), Q_{\mathcal{R}L}(Y) \in \mathcal{Z} \), we get a morphism \( f : Q_{\mathcal{L}R}(X) \to Q_{\mathcal{R}L}(Y) \) with \( \pi(f) = \overline{f} \). For \( X \in \mathcal{C} \), there exists a diagram \( RX \xrightarrow{\iota_{RX}} Q_{\mathcal{L}R}X \) with \( p_X \in \mathcal{U}\text{-}\text{def} \) and \( \iota_{RX} \in \mathcal{T}\text{-}\text{inf} \). Similarly, for \( Y \in \mathcal{C} \), we have a diagram \( Y \xrightarrow{\iota_Y} LY \xleftarrow{p_{LY}} Q_{\mathcal{R}L}Y \) with \( p_{LY} \in \mathcal{U}\text{-}\text{def} \) and \( \iota_Y \in \mathcal{T}\text{-}\text{inf} \). Our situation can be depicted as:

\[
\begin{array}{ccc}
V_X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \iota_Y \\
RX & \xrightarrow{\iota_{RX}} & Q_{\mathcal{L}R}X \\
\downarrow p_X & \quad & \downarrow f \\
X & \xrightarrow{f} & Y \\
\downarrow & \quad & \downarrow \\
V_{LY} & \xrightarrow{p_{LY}} & LY \\
\downarrow & \quad & \downarrow \\
& \xrightarrow{\iota_Y} & \\
\end{array}
\]

Setting \( \alpha := p_{LY} \circ f \circ \iota_{RX} \), we have obtained \( \overline{\alpha} = L_{\mathcal{V}}(\iota_Y)^{-1} \circ L_{\mathcal{V}}(\alpha) \circ L_{\mathcal{V}}(p_X)^{-1} \).

### 3.2 Comparison Between \( g\text{HTCP} \) and \( \text{HTCP} \)

Throughout this subsection, we always suppose that the extriangulated category \( \mathcal{C} \) satisfies the following condition (WIC):

**Condition 3.12** (WIC) For a given extriangulated category \( \mathcal{C} \), we consider the following conditions.

1. For a composed morphism \( g \circ f \) in \( \mathcal{C} \), if \( g \circ f \) is an inflation, then so is \( f \).
(2) For a composed morphism \( g \circ f \) in \( C \), if \( g \circ f \) is an deflation, then so is \( g \).

To bridge the gap between gHTCP and HTCP, we consider some additional conditions on the gHTCP. Keeping in mind Nakaoka–Palu’s correspondence theorem between HTCP’s and admissible model structures on \( C \) [28, Section 5], for a pair \( \mathcal{P} \) of a left cotorsion pair and a right one, we consider the following classes of morphisms:

- \( \text{wFib} := \) the class of deflations \( f \) with \( \text{CoCone}(f) \in \mathcal{V} \);
- \( \text{wCof} := \) the class of inflations \( f \) with \( \text{Cone}(f) \in \mathcal{S} \);
- \( \mathcal{W} := \text{wFib} \circ \text{wCof} \).

where \( \text{wFib} \circ \text{wCof} \) denotes the class of morphisms \( f \) which has a factorization \( f = f_2 \circ f_1 \) with \( f_1 \in \text{wCof} \) and \( f_2 \in \text{wFib} \). Obviously, we have \( \mathcal{U}\text{-def} \subseteq \text{wFib} \), \( \mathcal{T}\text{-inf} \subseteq \text{wCof} \) and \( \mathcal{V} \subseteq \mathcal{W} \). Recall the definition of HTCP localization.

**Theorem 3.13** [28, Cor. 5.25] Let \( \mathcal{P} \) be an HTCP. Then, the Gabriel–Zisman localization \( L_{\mathcal{W}} : C \to C[\mathcal{W}^{-1}] \) induces an equivalence \( \Psi : \mathcal{Z}/\mathcal{I} \to C[\mathcal{W}^{-1}] \), unique up to isomorphism, which is depicted as follows:

\[
\begin{array}{c}
\mathcal{Z} \\
\mathcal{Z}/\mathcal{I} \end{array} \xrightarrow[]{\text{inc}} C \xrightarrow[]{L_{\mathcal{W}}} C[\mathcal{W}^{-1}] \xrightarrow[]{\Psi} \mathcal{Z}/\mathcal{I}
\]

We call this localization the HTCP localization of \( C \) with respect to \( \mathcal{P} \).

The aim of this subsection is to show that gHTCP is a generalization of HTCP in the following sense:

**Theorem 3.14** Let \( \mathcal{P} = ((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V})) \) be an HTCP. Then \( \mathcal{P} \) is also a gHTCP. Moreover, the HTCP localization \( C[\mathcal{W}^{-1}] \) is equivalent to the gHTCP localization \( C[\mathcal{V}^{-1}] \).

To prove the theorem, we shall use the following easy lemmas.

**Lemma 3.15** Let \( (\mathcal{U}, \mathcal{V}) \) be a right cotorsion pair in an extriangulated category \( C \). Assume that \( \mathcal{V} \) is extension-closed and \( \mathcal{E}(\mathcal{U}, \mathcal{V}) = 0 \). Then \( R\pi : C \to \mathcal{U}/\mathcal{I} \) sends \( \text{wFib} \) to a class of isomorphisms.

**Proof** Take a morphism \( f \in \text{wFib} \) together with a conflation \( V \to X \xrightarrow{f} Y \) with \( V \in \mathcal{V} \). We resolve \( X \) by \( (\mathcal{U}, \mathcal{V}) \) to get a conflation \( V_X \to U_X \xrightarrow{p_X} X \) with \( V_X \in \mathcal{V} \) and \( U_X \in \mathcal{U} \). Thanks to the following commutative diagram and extension-closedness of \( \mathcal{V} \), we have that the composed morphism \( f \circ p_X \) is in \( \text{wFib} \):

\[
\begin{array}{cccc}
V_X & \xrightarrow{f \circ p_X} & Y \\
\downarrow & & \downarrow \\
V' & \xrightarrow{p_X} & U_X & \xrightarrow{f} Y \\
\downarrow & & \downarrow \\
V & \xrightarrow{f} & X & \xrightarrow{f} Y
\end{array}
\]

Since \( \mathcal{E}(\mathcal{U}, \mathcal{V}) = 0 \), we get \( f \circ p_X \in \mathcal{U}\text{-def} \). Hence we conclude \( RX = U_X \cong RY \) in \( \mathcal{U}/\mathcal{I} \). \( \square \)
Lemma 3.16 Let $\mathcal{P}$ be a pair of left cotorsion pair and a right cotorsion pair satisfying (Hov1) and (Hov2). If both $\mathcal{T}$ and $\mathcal{U}$ are extension-closed, then the images of $Q_{LR}$ and $Q_{RL}$ are contained in $\mathbb{Z}/\mathcal{I}$.

Proof Let $X \in \mathcal{C}$ and consider the following commutative diagram associated to the definition of $Q_{LR}$:

\[
\begin{array}{c}
V_X \\ \downarrow^{RX} \\ LR_X \\
\downarrow^{SR_X} \\
X
\end{array}
\]

which is same as (3.2). Since $\mathcal{U}$ is extension-closed, $RX, SR_X \in \mathcal{U}$ forces $LR_X \in \mathbb{Z}$. This says $\text{Im } Q_{LR} \subseteq \mathbb{Z}/\mathcal{I}$.

The assertion for $Q_{RL}$ can be checked dually. $\square$

The following proposition shows the latter statement in Theorem 3.14.

Proposition 3.17 Let $\mathcal{P}$ be an HTCP. Then, there exists an equivalence $F : \mathcal{C}[\mathcal{V}^{-1}] \sim \mathcal{C}[\mathcal{W}^{-1}]$.

Proof Since $\mathcal{V} \subseteq \mathcal{W}$, there exists a natural functor $F : \mathcal{C}[\mathcal{V}^{-1}] \rightarrow \mathcal{C}[\mathcal{W}^{-1}]$ such that $L_{\mathcal{W}} \cong F \circ L_{\mathcal{V}}$. Denote by $\mathcal{V}$ the class of morphisms $f \in \text{Mor} \mathcal{C}$ such that $L_{\mathcal{V}}(f)$ is an isomorphism. We also use the symbol $\mathcal{V}$ in an obvious meaning. A basic property of the model category shows $\mathcal{V} = \mathcal{W}$ (see [28, Remark 7.8]). By Lemma 3.15 and the dual, we conclude $\mathcal{W} \subseteq \mathcal{V} = \mathcal{W}$ which forces $F$ to be an equivalence. $\square$

Proof of Theorem 3.14 By Proposition 3.6 and Lemma 3.16, there exists a functor $\Phi : \mathbb{Z}/\mathcal{I} \rightarrow \mathcal{C}[\mathcal{V}^{-1}]$. We only have to show that $\Phi$ is an equivalence. Thanks to Theorem 3.13, we have the following diagram commutative up to isomorphism:

\[
\begin{array}{c}
\mathbb{Z}/\mathcal{I} \\
\downarrow^{\Phi} \\
\mathcal{C}[\mathcal{V}^{-1}]
\end{array}
\]

The commutativity $\Psi \cong F \circ \Phi$ follows from the uniqueness of $\Psi$. Since $F$ and $\Psi$ are equivalences, so is $\Phi$. This finishes the proof. $\square$

Remark 3.18 We remark that the existence of the above isomorphism $\eta$ follows from the model structure corresponding to the HTCP via the general theory of model category, e.g. [8, Ch. 1.1], [14, Thm. 1.2.10].

3.3 Examples: Recollement of Abelian/Triangulated Categories

In this section, as applications of Theorem 3.10, we interpret some important phenomena via gHTCP localizations.
3.3.1 Additive Quotient

Let $\mathcal{C}$ be an extriangulated category and $\mathcal{D}$ its full subcategory which is closed under isomorphisms and direct summands. Then, $\mathcal{P} := ((\mathcal{D}, \mathcal{C}), (\mathcal{C}, \mathcal{D}))$ obviously forms a gHTCP. The following is straightforward.

**Corollary 3.19** The following hold for the above $\mathcal{P}$.

1. $\mathcal{I} = \mathcal{D}$ and $\mathcal{Z} = \mathcal{C}$.
2. The class $\mathcal{U}$-def consists of retractions with the cocones belonging to $\mathcal{D}$.
3. The class $\mathcal{T}$-inf consists of sections with the cones belonging to $\mathcal{D}$.
4. The functors $L\pi, R\pi : \mathcal{C}/\mathcal{D} \to \mathcal{C}/\mathcal{D}$ are isomorphic to the identity functors.

In particular, we have an equivalence $\Phi_1 : \mathcal{C}/\mathcal{D} \xrightarrow{\sim} \mathcal{C}[V^{-1}]$.

This equivalence $\Phi_1$ is nothing other than the equivalence $\mathcal{C}/\mathcal{I} \xrightarrow{\sim} \mathcal{C}[S^{-1}]$ appearing in Example 2.6. Note that the above construction of $\mathcal{C}[V^{-1}]$ does not depend on extriangulated structures on $\mathcal{C}$. Since an additive category always has a splitting exact structure, any additive quotient can be considered as a gHTCP localization.

3.3.2 Recollement of Triangulated Categories

We shall explain that a recollement of triangulated categories gives an example of HTCP localizations. More generally, we shall investigate the following Iyama–Yang’s realization of Verdier quotients as subfactors via HTCP localizations (see also [21]). We introduce the following notions.

**Definition 3.20** Let $\mathcal{C}$ be an extriangulated category and $\mathcal{N}$ its full subcategory. We define the full subcategories

- $\mathcal{N}^{\perp 1} := \{ X \in \mathcal{C} \mid E(\mathcal{N}, X) = 0 \}$;
- $\mathcal{N}^\perp := \{ X \in \mathcal{C} \mid E(\mathcal{N}, X) = 0 = C(\mathcal{N}, X) \}$.

The subcategories $\mathcal{N}^{\perp 1}$ and $\mathcal{N}^\perp$ are defined dually.

**Corollary 3.21** Let $\mathcal{C} = (\mathcal{C}, [1], \Delta)$ be a triangulated category and $\mathcal{N}$ its thick subcategory and denote by $\mathcal{C}_N$ the Verdier quotient of $\mathcal{C}$ by $\mathcal{N}$. We suppose that $\mathcal{N}$ has a cotorsion pair $(S, V)$; $\mathcal{C}$ has cotorsion pairs $(S, S^{\perp 1})$, $(S^\perp V, V)$. Then, $\mathcal{P} := ((S, S^{\perp 1}), (S^\perp V, V))$ forms an HTCP. Moreover, the canonical functor $\mathcal{Z} \subseteq \mathcal{C} \to \mathcal{C}_N$ induces an equivalence $\mathcal{Z}/\mathcal{I} \simeq \mathcal{C}_N$. Here we use the symbols $S^{\perp 1}$ and $S^\perp V$ regarding $S$ and $V$ as subcategories in $\mathcal{C}$.

**Proof** Since $\mathcal{P}$ forms an HTCP, we have an equivalence $\mathcal{Z}/\mathcal{I} \simeq \mathcal{C}[W^{-1}]$. Let $S$ be the multiplicative system associated to the thick subcategory $\mathcal{N}$, namely, $S := \{ f \in \text{Mor}(\mathcal{C}) \mid \text{Cone}(f) \in \mathcal{N} \}$. We identify $\mathcal{C}_N$ with $\mathcal{C}[S^{-1}]$. It suffices to show $W = S$. By definition, we get $W \subseteq S$. To show the converse, consider a triangle $X \xrightarrow{f} Y \to N \to X[1]$ with $N \in \mathcal{N}$. Resolving $N$ by the cotorsion pair $(S, V)$, we obtain the following commutative diagram:

$$
\begin{array}{ccc}
V_N & \xrightarrow{V} & V_N \\
\downarrow & & \downarrow \\
X & \xrightarrow{f_1} & Y' \\
\downarrow & & \downarrow \\
Y & \xrightarrow{f_2} & S_N \\
\downarrow & & \downarrow \\
X & \xrightarrow{g} & Y \\
\downarrow & & \downarrow \\
& & N \\
\end{array}
$$

(Pb)
where all rows and columns are triangles and \( S_N \in S, V_N \in V \). Thus we get a factorization \( f = f_2 \circ f_1 \) with \( f_1 \in \text{wCof} \) and \( f_2 \in \text{wFib} \). Hence \( f \in \mathbb{W} \).

**Remark 3.22** The equivalence \( \mathcal{Z}/\mathcal{I} \simeq \mathcal{C}_N \) is obtained in [16, Thm. 1.1] with a more explicit construction.

The following result can be regarded as a special case of Corollary 3.21.

**Corollary 3.23** [27, Cor. 6.20] Let \((\mathcal{N}, C, C_N)\) be a recollement of triangulated categories and \((S, V)\) a cotorsion pair in \(\mathcal{N}\). Then, the pair \(P := ((S, S^{\perp}), (\perp^1 V, V))\) forms an HTCP in \(C\). Moreover, the canonical functor \( \mathcal{Z} \subseteq C \to C_N \) induces an equivalence \( \mathcal{Z}/\mathcal{I} \simeq \mathcal{C}_N \).

Let us remark that, in Corollary 3.23, the thick subcategory \(\mathcal{N}\) always has a cotorsion pair, for example, by setting \((S, V) := (\mathcal{N}, 0)\).

### 3.3.3 Recollement of Abelian Categories

It is pointed out in [28, Rem. 5.27] that a recollement of abelian categories can be rarely regarded as an HTCP localization. However, it is still a gHTCP localization under a certain assumption. Recall the definition of recollement and its needed properties.

**Definition-Proposition 3.24** Let \(C, N\) and \(C_N\) be abelian categories. A recollement of abelian categories is a diagram of functors between abelian categories of the form

\[
\begin{array}{c}
N \\
\downarrow \text{i} \downarrow \text{p} \\
C \\
\downarrow \text{e} \\
C_N \\
\downarrow \text{r} \\
\end{array}
\]

with the conditions:

- \((q, i, p)\) and \((l, e, r)\) form adjoint triples;
- The functors \(i, l\) and \(r\) are fully faithful;
- \(\text{Im } i = \text{Ker } e\).

We denote the recollement by \((N, C, C_N)\) for short. In this case,\n
1. \(\text{Im } r = N^{\perp} \) and \(\text{Im } l = \perp^1 N\) hold;
2. for any \(X \in C\), there exist the following exact sequences

\[
0 \to N \to le(X) \to X \to iq(X) \to 0 \tag{3.4}
\]

\[
0 \to ip(X) \to X \to re(X) \to N' \to 0 \tag{3.5}
\]

with \(N, N' \in \mathcal{N}\).

We refer to [32, Section. 2] for details. A recollement can be considered as a special case of Serre quotients in the following sense.

**Proposition 3.25** [32, Rem. 2.3] Consider the Serre quotient \(C_N\) of an abelian category \(C\) with respect to its Serre subcategory \(\mathcal{N}\). If the quotient functor \(e : C \to C_N\) admits a right adjoint \(r\) and a left adjoint \(l\), then the inclusion \(i : \mathcal{N} \to C\) also admits a right adjoint \(p\) and a left adjoint \(q\). Moreover, these six functors form a recollement.

The following is an abelian version of Corollary 3.23.
Corollary 3.26 Let \((\mathcal{N}, \mathcal{C}, \mathcal{C}_\mathcal{N})\) be a recollement of abelian categories. Assume that \(\mathcal{C}\) has enough projectives and enough injectives, and \(\mathcal{N}\) has a cotorsion pair \((S, V)\) with \(\text{Ext}^2_{\mathcal{C}}(S, V) = 0\). Then, the pair \(((S, S^{\perp_{-1}}), (\perp_{-1}V, V))\) forms a gHTCP in \(\mathcal{C}\). Moreover, the canonical functor \(Z \subseteq \mathcal{C} \to \mathcal{C}_\mathcal{N}\) induces an equivalence \(Z/I \simeq \mathcal{C}_\mathcal{N}\). Here we use the symbols \(S^{\perp_{-1}}\) and \(\perp_{-1}V\) regarding \(S\) and \(V\) as subcategories in \(\mathcal{C}\).

To show Corollary 3.26, we include more investigations on an extriangulated category with a gHTCP.

Proposition 3.27 Let \(\mathcal{C}\) be an extriangulated category with a gHTCP \(P\). Assume that both \(S\) and \(V\) are extension-closed and \(E(S, T) = E(U, V) = 0\). Then the functor \(L_V : \mathcal{C} \to \mathcal{C}[V^{-1}]\) sends \(W\) to a class of isomorphisms. Moreover, there exists an equivalence \(\Psi : \mathcal{C}[V^{-1}] \sim \mathcal{C}[W^{-1}]\), uniquely up to isomorphism, such that \(L_W \sim= \Psi \circ L_V\).

Proof By Lemma 3.15 and the dual, we know that \(Q_{LR}(\text{wFib})\) and \(Q_{RL}(\text{wCof})\) form classes of isomorphisms in \(Z/I\). Since \(P\) is a gHTCP, the functor \(Q_{LR} \sim= Q_{RL} \sim= L_V\) sends \(W\) to a class of isomorphisms. Since \(V \subseteq W\), the functor \(L_W : \mathcal{C} \to \mathcal{C}[W^{-1}]\) sends \(V\) to isomorphisms. The second assertion follows from the universality.

We are in position to prove Corollary 3.26.

Proof (A) The first step shows that the pair \(((S, S^{\perp_{-1}}), (\perp_{-1}V, V))\) is a gHTCP.

(1) We shall show that \((\perp_{-1}V, V)\) is a right cotorsion pair. Let \(X\) be an object in \(\mathcal{C}\) and consider the first syzygy of \(X\), namely, an exact sequence
\[
0 \to \Omega X \xrightarrow{a} P(X) \to X \to 0 \quad (3.6)
\]
in \(\mathcal{C}\) with \(P(X) \in P(\mathcal{C})\). Next, we consider an exact sequence (3.4) of \(\Omega X\) in Proposition 3.24
\[
0 \to N \to \text{le}(\Omega X) \to \Omega X \xrightarrow{f} \text{iq}(\Omega X) \to 0 \quad (3.7)
\]
with \(N, \text{iq}(\Omega X) \in \mathcal{N}\) and \(\text{le}(\Omega X) \in \perp_{-1}\mathcal{N}\). Resolve \(\text{iq}(\Omega X)\) by the cotorsion pair \((S, V)\), and get an exact sequence \(0 \to \text{iq}(\Omega X) \xrightarrow{g} V \to S \to 0\) with \(V \in \mathcal{V}\) and \(S \in \mathcal{S}\). Put \(h := g \circ f : \Omega X \to V\). Since \(f\) is surjective and \(g\) is injective, we have \(\text{Cok} h \simeq S\) and \(\text{Ker} h = \text{Ker} f\). By taking a pushout of \(a : \Omega X \to P(X)\) along \(h\), we obtain the following commutative diagram:
in which all rows and columns are exact. The second row is a desired one. It suffices to show \( X' \in \perp \perp \mathcal{U} \). Since \( \text{Ker} \ h \) is a factor of \( \text{le}(\Omega X) \in \perp \mathcal{N} \), we get \( \text{Ker} \ h \in \perp \perp \mathcal{N} \). We deduce from the second column and the above diagram an exact sequence \( 0 \to \text{Ker} \ h \to P(X) \to \text{Ker} \ b \to 0 \). The fact \( \text{Ker} \ h \in \perp \perp \mathcal{N} \) and \( P(X) \in \perp \perp \mathcal{V} \) forces \( \text{Ker} \ b \in \perp \perp \mathcal{V} \). The exact sequence \( 0 \to \text{Ker} \ b \to X' \to S \to 0 \) shows \( X' \in \perp \perp \mathcal{V} \). Therefore, the morphism \( c : X' \to X \) is a \( \perp \perp \mathcal{V} \)-deflation associated to the right cotorsion pair \((\perp \perp \mathcal{V}, \mathcal{V})\). By the dual argument, we can verify that \((\mathcal{S}, \mathcal{S}^{\perp 1})\) forms a left cotorsion pair.

(2) The condition \((\text{Hov2})\) is obvious.

(3) The condition \((\text{Hov2})\) follows from \( \mathcal{S} \cap \mathcal{S}^{\perp 1} = \mathcal{S} \cap \mathcal{V} = \perp \perp \mathcal{V} \cap \mathcal{V} \).

(4) We shall show that \( \text{Im} \ Q_{LR} \) is contained in \( \mathcal{Z}/\mathcal{I} \). Take an object \( X \in \mathcal{C} \). Following the definition of \( Q_{LR} \), we consider the following commutative diagram

\[
\begin{array}{ccccccccc}
0 & \to & V_X & \to & RX & \to & X & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & V_X & \to & LRX & \to & X' & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
S^{RX} & \to & S^{RX} & \to & S^{RX} & \to & 0 & \to & 0 \\
\end{array}
\]

(3.8)

where all rows and columns are exact, \( p_X \) is a \( \mathcal{U} \)-deflation of \( X \) and \( \iota_{RX} \) is a \( T \)-inflation of \( RX \). Since \( RX, S^{RX} \in \perp \perp \mathcal{V} \), the second column shows \( LRX \in \perp \perp \mathcal{V} \cap \mathcal{S}^{\perp 1} = \mathcal{Z} \). Dually, we get \( \text{Im} \ Q_{LR} \subseteq \mathcal{Z}/\mathcal{I} \).

(5) We shall show that, if \( \text{Ext}^2_{\mathcal{C}}(\mathcal{S}, \mathcal{V}) = 0 \), there exists an isomorphism \( \eta : Q_{LR} \to Q_{RL} \).

It is enough to show \( X' \in \mathcal{S}^{\perp 1} \) in the diagram (3.8). Applying \( \mathcal{C}(\mathcal{S}, -) \) to the middle row in (3.8), we have an exact sequence \( \text{Ext}^1_{\mathcal{C}}(\mathcal{S}, LRX) \to \text{Ext}^1_{\mathcal{C}}(\mathcal{S}, X') \to \text{Ext}^2_{\mathcal{C}}(\mathcal{S}, V_X) \) in which the both sides are zero. Thus we have \( X' \in \mathcal{S}^{\perp 1} \). Hence \( \iota \) is a \( T \)-inflation of \( X \) and \( p \) is a \( \mathcal{U} \)-deflation of \( X' \). We conclude that the pair \((\mathcal{S}, \mathcal{S}^{\perp 1}), (\perp \perp \mathcal{V}, \mathcal{V})\) forms a gHTTCP.

(B) We shall show that there exists an equivalence \( \mathcal{Z}/\mathcal{I} \simeq \mathcal{C}_N \). Let \( \mathcal{S} \) be the class of morphisms whose kernels and cokernels are contained in \( \mathcal{N} \). An equality \( \mathcal{W} = \mathcal{S} \) can be checked by the same method as in the proof of Corollary 3.21. Therefore, we identify the Serre localization with \( \text{L}_{\mathcal{W}} : \mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}] \). By Propositon 3.27, we obtain a desired equivalence \( \mathcal{C}[\mathcal{V}^{-1}] \simeq \mathcal{C}[\mathcal{W}^{-1}] \).}

Let us add a short argument on an extriangulated structure on \( \mathcal{Z}/\mathcal{I} \) in Corollary 3.26. Note that, if both \( \mathcal{T} \) and \( \mathcal{U} \) are extension-closed, so is \( \mathcal{Z} \). Furthermore, as stated below, the subfactor \( \mathcal{Z}/\mathcal{I} \) also has a natural extriangulated structure.

**Lemma 3.28** Let \( \mathcal{C} \) be an extriangulated category with a gHTTCP \( \mathcal{P} \). Suppose that \( \mathcal{T} \) and \( \mathcal{U} \) are extension-closed and \( \mathbb{E}(\mathcal{S}, \mathcal{T}) = 0 = \mathbb{E}(\mathcal{U}, \mathcal{V}) \) holds. Then, \( \mathcal{Z} \) is extension-closed and \( \mathbb{E}(\mathcal{I}, \mathcal{Z}) = 0 = \mathbb{E}(\mathcal{Z}, \mathcal{I}) \) holds. Moreover, \( \mathcal{Z}/\mathcal{I} \) is an extriangulated category.

**Proof** Since \( \mathcal{T} \) and \( \mathcal{U} \) are extension-closed, so is \( \mathcal{Z} \). By Proposition 2.3(1), \( \mathcal{Z} \) is an extriangulated category. The equations directly follow from \( \mathbb{E}(\mathcal{S}, \mathcal{T}) = 0 = \mathbb{E}(\mathcal{U}, \mathcal{V}) \). Thus, \( \mathcal{I} \) is a class of projective-injective objects in \( \mathcal{Z} \). Due to Propositon 2.3(2), we have the latter assertion. \( \square \)
We focus the equivalence $C_{\mathcal{N}} \xrightarrow{\sim} \mathcal{Z}/\mathcal{I}$ in Corollary 3.26. It is natural to compare the abelian exact structure on $C_{\mathcal{N}}$ and the above extriangulated structure on $\mathcal{Z}/\mathcal{I}$. The following example indicates that they do not coincide.

**Example 3.29** We consider the Auslander algebra $B$ of the path algebra $A$ determined by the quiver $[\bullet \rightarrow \bullet \rightarrow \bullet]$. The algebra $B$ can be considered as the one defined by the quiver with relations:

$$
\begin{array}{cccccc}
\bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow \\
2 & \longrightarrow & 5 & \longrightarrow & 3 & \longrightarrow \\
1 & \longrightarrow & 4 & \longrightarrow & 6 & \\
\end{array}
$$

where the dotted line stands for the natural mesh relation. The Auslander-Reiten quiver of the category $\mathcal{C} := \text{mod } B$ of finite dimensional modules is the following:

$$
\begin{array}{ccccccc}
\bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \\
2 & \longrightarrow & 5 & \longrightarrow & 6 & \longrightarrow & 1 \\
\bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \\
3 & \longrightarrow & 4 & \longrightarrow & 5 & \longrightarrow & 6 \\
\bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \\
5 & \longrightarrow & 3 & \longrightarrow & 2 & \longrightarrow & 1 \\
\end{array}
$$

where the dotted arrows denote the Auslander-Reiten translation. We denote by "o" in a quiver the objects belonging to a subcategory and by "·" the objects do not. We consider the (injectively) stable Auslander algebra $\overline{B}$ and the associated inclusion $\mathcal{N} := \text{mod } \overline{B} \hookrightarrow \text{mod } B$. Then, it is well-known that $\text{mod } \overline{B}$ is a Serre subcategory in $\text{mod } B$:

There exists a cotorsion pair $(\mathcal{S}, \mathcal{V}) := (\text{mod } \overline{B}, \text{inj } \overline{B})$ in $\text{mod } \overline{B}$ with $\text{Ext}_B^2(\mathcal{S}, \mathcal{V}) = 0$, where $\text{inj } \overline{B} := \mathcal{I}(\text{mod } \overline{B})$. In fact, we have $\text{add}(4 \oplus 5 \oplus 6) = \text{inj } \overline{B}$ and the following calculations:

$$
\begin{align*}
\text{Ext}^2_{\mathcal{C}}(\mathcal{S}, 4) & \cong \text{Ext}^1_{\mathcal{C}}(\mathcal{S}, 2) = 0 \\
\text{Ext}^2_{\mathcal{C}}(\mathcal{S}, 5) & \cong \text{Ext}^1_{\mathcal{C}}(\mathcal{S}, 3) = 0 \\
\text{Ext}^2_{\mathcal{C}}(\mathcal{S}, 6) & \cong \text{Ext}^1_{\mathcal{C}}(\mathcal{S}, 3) = 0
\end{align*}
$$

Corollary 3.26 guarantees that the pair

$$
\mathcal{P} := ((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V})) := ((\text{mod } \overline{B}, (\text{mod } \overline{B})^{-1}), (\mathcal{I}(\text{inj } \overline{B}), \text{inj } \overline{B}))
$$
forms a gHTCP in \( \text{mod} \, B \). We calculate to get that the associated subcategory \( Z = T \cap U \) consisting of \( \circ \) and \( \bullet \):

\[
\begin{array}{cccccccc}
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\end{array}
\]

The symbols \( \bullet \) denotes the subcategory \( \mathcal{I} \). Since \( \text{Ext}^1_B(Z, Z) = 0 \), the natural extriangulated structure of \( Z/\mathcal{I} \) is splitting, more precisely it is a splitting exact structure. On the other hand, known as Auslander’s formula [2,20], the Serre localization \( \text{mod} \, B/\text{mod} \, B \) is equivalent to \( \text{mod} \, A \). Hence, the equivalence \( C_N \sim \rightarrow \) \( Z/\mathcal{I} \) obtained in Corollary 3.26 is not necessarily exact.

4 Aspects of the Heart Construction

4.1 Basic Properties of the Heart

We recall the definition and some basic properties of the heart. Throughout this section, we fix a triangulated category \( \mathcal{C} \) equipped with a cotorsion pair \((S, V)\). For two classes \( \mathcal{U} \) and \( \mathcal{V} \) of objects in \( \mathcal{T} \), we denote by \( \mathcal{U} \ast \mathcal{V} \) the class of objects \( X \) occurring in a triangle \( U \to X \to V \to U[1] \) with \( U \in \mathcal{U} \) and \( V \in \mathcal{V} \).

Definition 4.1 Let \( \mathcal{C} \) be a triangulated category equipped with a cotorsion pair in \( \mathcal{C} \), and put \( \mathcal{W} := S \cap V \). We define the following associated subcategories:

\[
\mathcal{C}^- := S[-1] \ast \mathcal{W}; \quad \mathcal{C}^+ := \mathcal{W} \ast V[1]; \quad \mathcal{H} := \mathcal{C}^+ \cap \mathcal{C}^-.
\]

The additive quotient \( \mathcal{H}/\mathcal{W} \) is called the heart of \((S, V)\). Abe and Nakaoka showed the following assertions.

Lemma 4.2 [1, Def. 3.5] For any \( X \in \mathcal{C} \), there exists the following commutative diagram

\[
\begin{array}{cccc}
V_X & \longrightarrow & U_X & \longrightarrow & X & \longrightarrow & V_X[1] \\
& & & \downarrow{\alpha} & & \downarrow{V_X'} & \\
& & & V_X' & & & \\
\end{array}
\] (4.1)

where \( V_X, V_X' \in \mathcal{V} \), \( \alpha \) is a left \((\mathcal{C}^-)\)-approximation of \( X \) and the first row is a triangle. This triangle is called a coreflection triangle of \( X \).

By a closer look at the above and its dual, we conclude the above left \((\mathcal{C}^-)\)-approximation \( \alpha \) gives rise to a functor as below.

Lemma 4.3 [1, Lem. 4.2] The canonical inclusion \( \mathcal{C}^-/\mathcal{W} \hookrightarrow \mathcal{C}/\mathcal{W} \) has a left adjoint \( L \) which restricts to the functor \( L : \mathcal{C}^+/\mathcal{W} \to \mathcal{H}/\mathcal{W} \). Dually, the canonical inclusion \( \mathcal{C}^+/\mathcal{W} \hookrightarrow \mathcal{C}/\mathcal{W} \) has a right adjoint \( R \) which restricts to the functor \( R : \mathcal{C}^-/\mathcal{W} \to \mathcal{H}/\mathcal{W} \). Furthermore, there exists a natural isomorphism \( \eta : LR \sim \rightarrow RL \).

The following is their main result.
Theorem 4.4 [25, Thm. 6.4] [1, Thm. 5.7] The heart $\mathcal{H}/\mathcal{W}$ is abelian. Moreover, the functor $\text{coh} := LR\pi : \mathcal{C} \to \mathcal{H}/\mathcal{W}$ is cohomological.

Note that the cotorsion class $S$ admits an extriangulated structure. For later use, we also recall some consequences of the assumption that $S$ has enough projectives. In this case, the heart $\mathcal{H}/\mathcal{W}$ and the cohomological functor $\text{coh}$ have nicer descriptions.

Lemma 4.5 [24, Cor. 3.8] The kernel of cohomological functor $\text{coh} : \mathcal{C} \to \mathcal{H}/\mathcal{W}$ is $\text{add}S \ast V$, the additive closure of $S \ast V$. In particular, $\text{add}S \ast V$ is extension-closed in $\mathcal{C}$.

The subcategory $S \ast V$ will play important roles in the rest.

Proposition 4.6 [24, Thm. 4.10, Prop. 4.15] The following are equivalent:

(i) $S$ has enough projectives;
(ii) $(\mathcal{P}(S), \text{add}(S \ast V))$ forms a cotorsion pair of $\mathcal{C}$.

Under the above equivalent conditions, there exists an equivalence $\Psi : \mathcal{H}/\mathcal{W} \sim \text{mod}(\mathcal{P}(S[-1]))$ which makes the following diagram commutative up to isomorphisms

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\text{coh}} & \mathcal{H}/\mathcal{W} \\
\text{Hom}(\mathcal{P}(S[-1]), -) & \downarrow{\Psi} & \\
\text{mod}(\mathcal{P}(S[-1])) & & \\
\end{array}
$$

Note that this explains why the heart $\mathcal{H}/\mathcal{W}$ is abelian and the associated functor $\text{coh}$ is cohomological.

4.2 From Cotorsion Pair to gHTCP

We shall show that the heart in triangulated categories can be obtained as a gHTCP localization. We consider the pair $\mathcal{P} := ((S, \mathcal{C}^+), (\mathcal{C}^-, \mathcal{V}))$. Our result shows that $\mathcal{P}$ is a gHTCP and the heart construction is nothing but the Gabriel–Zisman localization of $\mathcal{C}$ with respect to the class $\mathcal{V}$ associated to $\mathcal{P}$.

Corollary 4.7 The pair $\mathcal{P}$ forms a gHTCP. Moreover, the following hold.

(1) We have $\mathcal{Z} = \mathcal{H}$, $\mathcal{I} = \mathcal{W}$ and $\mathcal{Z}/\mathcal{I} = \mathcal{H}/\mathcal{W}$.
(2) There exists a class $\mathcal{V}$ of morphisms in $\mathcal{C}$ such that the associated localization $L_\mathcal{V} : \mathcal{C} \to \mathcal{C}[\mathcal{V}^{-1}]$ induces an equivalence $\Phi : \mathcal{H}/\mathcal{W} \sim \mathcal{C}[\mathcal{V}^{-1}]$.
(3) We have $L_\mathcal{V} \cong \Phi \circ \text{coh}$.

Proof We shall verify that the pair $\mathcal{P}$ forms a gHTCP. We firstly show that $\mathcal{C}^-$ is closed under direct summands. Let $T$ be an object in $\mathcal{C}^-$ together with a decomposition $T \cong T_1 \oplus T_2$. By definition, there exists a triangle $S[-1] \xrightarrow{a} T \xrightarrow{b} I \to S$ with $S \in S$ and $I \in I$. A canonical projection $p_i : T \to T_i$ induces a triangle $S[-1] \xrightarrow{p_i a} T_i \xrightarrow{d} X_i \to S$ and a morphism $c : I \to X_i$ with $cb = dp_i$ for $i = 1, 2$. Next, we resolve $X_i$ by $(S, \mathcal{V})$ to get a triangle $S'[−1] \xrightarrow{e} V \to S'$ with $S' \in S$ and $V \in \mathcal{V}$. We obtain the following commutative diagram from the above triangles:
where all rows and columns are triangles. Note that $Y \in \mathcal{S}$. If $V \in \mathcal{I}$, this forces $T_i \in \mathcal{C}^-$. It suffices to check $V \in \mathcal{S}$. To this end, we consider a morphism $f : V \to V'[1] \in \mathcal{V}[1]$. The composition $f ec : I \to V'[1]$ is zero because of $(\mathcal{I}, \mathcal{V}[1]) = 0$. Thus we have $f ecb = fedp_i = 0$ and hence $f e = 0$ which shows $f e$ factors through $S \in \mathcal{S}$. Moreover, $(\mathcal{S}, \mathcal{V}[1]) = 0$ forces $f e = 0$ and $f$ factors through $S' \in \mathcal{S}$. We thus conclude $f = 0$ and $V \in \mathcal{S}$. Dually we can confirm that $\mathcal{C}^+$ is also closed under direct summands.

Thanks to Lemma 4.2 and the dual, it follows that $(\mathcal{S}, \mathcal{C}^+)$ and $(\mathcal{C}^-, \mathcal{V})$ are a left cotorsion pair and a right one, respectively.

(Hov1): Let $S$ be an object in $\mathcal{S}$. Since $(\mathcal{C}^-, \mathcal{V})$ is a right cotorsion pair, we have a conflation $V_S \to S^- \to S$ with $S^- \in \mathcal{C}^-$ and $V_S \in \mathcal{V}$. The condition $\text{Ext}_{\mathcal{C}}^1(S, V) = 0$ shows that $S$ is a direct summand of $S^-$. Dually we have $\mathcal{C}^+ \supseteq \mathcal{V}$.

(Hov2): Consider $X \in \mathcal{S} \cap \mathcal{C}^+$ together with a conflation $V_X \to I_X \to X$ with $V_X \in \mathcal{V}$ and $I_X \in \mathcal{I}$. Since $\text{Ext}_C^1(X, V_X) = 0$, $X$ belongs to $\mathcal{I}$. Similarly, we have $\mathcal{I} = \mathcal{C}^- \cap \mathcal{V}$.

Due to Lemma 4.3, we conclude that $\mathcal{P}$ is a gHTCP. The remaining assertions are obvious. 

We can obtain from Corollary 4.7 the following picture which is a special case of the right one in (1.3).

\[
\begin{array}{c}
\text{Corollary 4.7} \\
\vdash \quad \vdash \\
\vdash \quad \vdash \\
\vdash \quad \vdash \\
\vdash \quad \vdash
\end{array}
\]

Combining Corollary 4.7 and Proposition 4.6, we recover the following result which can be deduced from [6, Thm 4.4] and [17, Prop 6.2(3)].

**Corollary 4.8** Let $k$ be a field. We consider a $k$-linear Hom-finite Krull–Schmidt triangulated category $\mathcal{C}$ with a rigid object $T$.

1. There exists a class $\mathcal{V}$ of morphisms and an equivalence $\mathcal{C}[-1]\mathcal{V} \to \text{mod } \text{End}_{\mathcal{C}}(T)^{\text{op}}$.
2. There exists an equivalence $\frac{\text{add}T[-1] + \text{add}T}{\text{add}T} \to \text{mod } \text{End}_{\mathcal{C}}(T)$.

In particular, we have an equivalence $\Phi : \frac{\text{add}T[-1] + \text{add}T}{\text{add}T} \to \mathcal{C}[-1]$.

**Proof** Due to the Hom-finiteness of $\mathcal{C}$ and the rigidity of $T$, we have a cotorsion pair $(\text{add}T, (\text{add}T)^{-1})$ of $\mathcal{C}$. Applying Corollary 4.7 to the cotorsion pair, we have a gHTCP $((\text{add}T, C), (\text{add}T[-1] \ast \text{add}T, (\text{add}T)^{-1}))$ and a desired equivalence $\Phi$. 

\[\square\]
4.3 A Factorization Through a Preabelian Category

Keeping in mind Buan–Marsh’s localizations, we shall subsequently show that the above Gabriel–Zisman localization $L_V : C \to \mathcal{H}/\mathcal{W}$ is not far from one admitting calculus of left and right fractions. The following our main result is formulated under functorial finiteness of $S \ast V$.

**Theorem 4.9** Let $C$ be a triangulated category equipped with a cotorsion pair $(S, \mathcal{V})$. Assume that $S \ast \mathcal{V}$ is functorially finite in $C$. Then,

1. the additive quotient $C/(S \ast \mathcal{V})$ is preabelian;
2. the class $\mathcal{R}$ of regular morphisms in $C/(S \ast \mathcal{V})$ admits a calculus of left and right fractions;
3. there exists the following commutative diagram up to isomorphism and a factorization of $L_V \cong L_R \circ \varpi$.

$$
\begin{array}{ccc}
\mathcal{H}/\mathcal{W} & \xrightarrow{\phi} & C[\mathcal{V}^{-1}], \\
\downarrow{\text{coh}} & & \downarrow{L_V} \\
C & \xrightarrow{L_V} & C/(S \ast \mathcal{V}) \\
\downarrow{\varpi} & & \downarrow{L_R} \\
C/(S \ast \mathcal{V}) & \to & \\
\end{array}
$$

where $\varpi$ is the natural additive quotient functor;

4. the localization functor $L_R : C/(S \ast \mathcal{V}) \to (C/(S \ast \mathcal{V}))[\mathcal{R}^{-1}]$ induces an equivalence

$$(C/(S \ast \mathcal{V}))[\mathcal{R}^{-1}] \xrightarrow{\sim} \mathcal{H}/\mathcal{W}.$$ 

The rest of this section will be occupied to prove the theorem. Our method is similar to Buan–Marsh’s one which strongly depends on Rump’s localization theory of preabelian category. To show Theorem 4.9, we firstly show that $C/(S \ast \mathcal{V})$ is an integral preabelian category. Put $\mathcal{K} := \text{add} S \ast \mathcal{V}$ and identify $C/(S \ast \mathcal{V})$ with $C/\mathcal{K}$.

**Proposition 4.10** The additive quotient $C/(S \ast \mathcal{V})$ is preabelian.

To show the above, we prepare the following auxiliary triangle to construct cokernels in $C/\mathcal{K}$.

**Lemma 4.11** For any object $X \in C$, there exists a triangle

$$K'[−1] \xrightarrow{b} X \xrightarrow{i} \tilde{K} \to K'$$

in $C$ such that $\tilde{K}, K' \in \mathcal{K}$ and $i$ is a left $\mathcal{K}$-approximation of $X$.

**Proof** Resolve $X$ by the cotorsion pair $(S, \mathcal{V})$ to get a triangle $S[−1] \xrightarrow{a} X \xrightarrow{v} V \to S$ with $S \in S$ and $V \in \mathcal{V}$. Since $\mathcal{K}$ is functorially finite, there exists a left $\mathcal{K}$-approximation $S[−1] \xrightarrow{b} K$ of $S[−1]$ which completes a triangle $K[−1] \to K'[−1] \xrightarrow{a} S[−1] \xrightarrow{b} K$. Since Lemma 4.5 says $\mathcal{K}$ is extension-closed, $K'$ belongs to $\mathcal{K}$. By the octahedral axiom, the
triangles appearing so far induce the following commutative diagram:

$$
\begin{array}{c}
K \\
p \\
K'[−1] \xrightarrow{ac} X \xrightarrow{d} \tilde{K} \xrightarrow{c[1]} K' \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \\
S[−1] \xrightarrow{a} X \xrightarrow{V} S \\
\downarrow \quad \quad \quad \downarrow b[1] \\
K[1] \xrightarrow{\iota[1]} \tilde{K}[1]
\end{array}
$$

with all rows and columns are triangles. Again, since $\mathcal{K}$ is extension-closed, we have $\tilde{K} \in \mathcal{K}$. The second row is a desired one. In fact, for any morphism $\alpha : X \to L \in \mathcal{K}$, the composition $\alpha \iota$ factors through the left $\mathcal{K}$-approximation $b$. It follows that $\alpha \iota = 0$ and $\alpha$ factors through $d$. \qed

**Proof of Proposition 4.10** Let $X \xrightarrow{f} Y \xrightarrow{g} Z \to X[1]$ be a triangle in $\mathcal{C}$. We shall provide a construction of the cokernel of $\varpi(f)$ in $\mathcal{C}/\mathcal{K}$. For the object $X$, we consider an auxiliary triangle (4.2). The octahedral axiom gives the following commutative diagram:

$$
\begin{array}{c}
\tilde{K} \\
\downarrow \\
\tilde{K} \\
\downarrow \\
\tilde{K}[1]
\end{array}
$$

The image of the sequence $X \xrightarrow{f} Y \xrightarrow{h} Z'$ is a cokernel sequence in $\mathcal{C}/\mathcal{K}$. To verify this, we take a morphism $g_0 : Y \to M$ such that $g_0 f$ factors through an object in $\mathcal{K}$. Since $\iota : X \to \tilde{K}$ is a left $\mathcal{K}$-approximation, $g_0 f$ factors through $\iota$. Therefore we have $g_0 f p = 0$ and that there exists a morphism $g_1 : Z' \to M$ together with $g_1 h = g_0$. To show the uniqueness of $g_1$, we consider a morphism $g_2 : Z' \to M$ together with $\varpi(g_2 h) = \varpi(g_0)$. Then $(g_1 - g_2)h$ factors through an object $K_0 \in \mathcal{K}$. More precisely, there exists morphisms $Y \xrightarrow{X} K_0 \xrightarrow{y} Z$ with $yx = (g_1 - g_2)h$. Taking a weak pushout of $h$ along $x$ we get the following commutative diagram made of triangles.

$$
\begin{array}{c}
K'[−1] \xrightarrow{fp} Y \xrightarrow{h} Z' \xrightarrow{K'} \\
\downarrow \quad \quad \quad \downarrow \\
K'[−1] \xrightarrow{Po} K_0 \xrightarrow{K_1} K'
\end{array}
$$

Note that $K_1$ also belongs to $\mathcal{K}$. Hence the commutative square of $yx = (g_1 - g_2)h$ shows that $g_1 - g_2$ factors through $K_1$. We have thus concluded that $\mathcal{C}/\mathcal{K}$ has cokernels. Dually we can construct the kernel. \qed

By using this construction of the cokernel, we characterize epimorphisms in $\mathcal{C}/\mathcal{K}$.

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**Lemma 4.12** Let $X \xrightarrow{f} Y \xrightarrow{g} Z \rightarrow X[1]$ be a triangle in $C$. Then, the morphism $\sigma(f)$ is an epimorphism in $C/K$ if and only if the morphism $g : Y \rightarrow Z$ factors through an object in $K$.

**Proof** [A] Firstly, we consider the case of $Z \in K$. We shall show that $\sigma(f)$ is epic in $C/K$. To construct a cokernel of $\sigma(f)$, we take an auxiliary triangle (4.2) $K[-1] \xrightarrow{p} X \xrightarrow{\iota} \tilde{K} \rightarrow K$ and complete the following commutative diagram by the octahedral axiom:

$$
\begin{array}{cccccc}
\tilde{K} & \xrightarrow{=} & \tilde{K} \\
\downarrow & & \downarrow \\
K[-1] & \xrightarrow{fp} & Y & \xrightarrow{g'} & Z' & \rightarrow & K \\
\downarrow & & \downarrow f & & \downarrow & & \downarrow h \\
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \rightarrow & X[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow d[1] \\
\tilde{K}[1] & \xrightarrow{=} & \tilde{K}[1]
\end{array}
$$

(4.3)

Note that $Z' \in K$. Since the image of the sequence $X \xrightarrow{f} Y \xrightarrow{g'} Z' \rightarrow 0$ is a cokernel sequence in $C/K$, $\sigma(Z') = 0$ forces the morphism $\sigma(f)$ to be epic.

[B] Assume that $\sigma(f)$ is an epimorphism. To construct the cokernel of $\sigma(f)$, we take an auxiliary triangle (4.2) and complete the same diagram (4.3). Thus we have a cokernel sequence $X \xrightarrow{\sigma(f)} Y \xrightarrow{\sigma(g')} Z' \rightarrow 0$ in $C/K$. Since $\sigma(f)$ is epic, we get $Z' \in K$. Hence $g : Y \rightarrow Z$ factors through $Z' \in K$.

To show the converse, we consider a triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \rightarrow X[1]$ where $g$ factors through an object in $K$. To construct the cokernel of $\sigma(f)$, we consider the same diagram as (4.3). Since $g$ factors through an object in $K$, we have $\sigma(g) = \sigma(hg') = 0$. Since $\text{CoCone}(h) \cong \tilde{K} \in K$, by the dual of [1], we have that $\sigma(h)$ is a monomorphism. Thus $\sigma(g') = 0$ which forces $\sigma(f)$ is an epimorphism. $\square$

We now verify a nicer property.

**Proposition 4.13** The preabelian category $C/\mathcal{S} \ast \mathcal{V}$ is integral.

**Proof** We consider the following pullback diagram in $C/K$

$$
\begin{array}{ccc}
A & \xrightarrow{\sigma(a)} & B \\
\sigma(b) \downarrow & & \downarrow \sigma(c) \\
C & \xrightarrow{\sigma(d)} & D
\end{array}
$$

with $\sigma(d) : C \rightarrow D$ epic. We shall show that $\sigma(a)$ is epic. Complete a triangle $C \xrightarrow{d} D \xrightarrow{e} E \rightarrow C[1]$ in $C$. By Lemma 4.12, the morphism $e$ is factored as $e : D \xrightarrow{e_2} L \xrightarrow{e_1} E$ with $L \in K$. For the object $B \in C$, take a triangle $K[-1] \xrightarrow{p} B \xrightarrow{\iota} \tilde{K} \rightarrow K$ (4.2). Since $\iota : B \rightarrow \tilde{K}$ is a left $K$-approximation, we have a morphism $f : \tilde{K} \rightarrow L$ which makes the following diagram in $C$ commutative.

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Then we get the above dotted arrow \( g : K[-1] \to C \). By the universality of the pullback, we have a further commutative diagram in \( \mathcal{C}/(S \ast V) \):

\[
\begin{array}{ccc}
K[-1] & \xrightarrow{g} & C \\
\downarrow & & \downarrow \\
K & \xrightarrow{f} & D \\
\downarrow & & \downarrow \\
K & \xrightarrow{p} & B \\
\end{array}
\]

Since \( \sigma(p) \) is an epimorphism by Lemma 4.12, so is \( \sigma(a) \).

Thanks to Proposition 2.8, we have the following localization admits a calculus of left and right fractions.

**Corollary 4.14** The class \( R \) of regular morphisms admits a calculus of left and right fractions. The Gabriel–Zisman localization \((\mathcal{C}/K)[R^{-1}]\) of \((\mathcal{C}/K)\) with respect to \( R \) is abelian.

Next we shall show that the abelian category \((\mathcal{C}/K)[R^{-1}]\) is equivalent to the heart.

**Proposition 4.15** There exists an equivalence \( F : \mathcal{C}[V^{-1}] \xrightarrow{\sim} (\mathcal{C}/K)[R^{-1}] \) which makes the following diagram commutative up to isomorphism

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\sigma} & \mathcal{C}/K \\
\downarrow & & \downarrow \\
\mathcal{C}[V^{-1}] & \xrightarrow{L_{\mathcal{V}}} & \mathcal{C}/K[R^{-1}] \\
\end{array}
\]

**Proof** To show the existence of \( F \), we shall confirm \( L_{\mathcal{R}} \circ \sigma(V) \) is a class of isomorphisms in \((\mathcal{C}/K)[R^{-1}]\). Consider a morphism \( f \in \mathcal{U}\)-def together with a triangle \( V \to U \to X \xrightarrow{\delta} V[1] \) where \( V \in \mathcal{V} \) and \( f \) is a right \( \mathcal{U}\)-approximation of \( X \). By comparing the above triangle and the coreflection triangle (4.1), we have that \( g \) factors through \( V_X \in \mathcal{V} \). Due to Lemma 4.12 and its dual, we conclude that \( \sigma(f) \) is a regular morphism in \( \mathcal{C}/K \). Similarly, \( \sigma(T\text{-inf}) \) forms a class of regular morphisms in \( \mathcal{C}/K \). By the universality of \( \mathcal{C}[V^{-1}] \), the existence and uniqueness of \( F \) follow.

Next, we shall construct the inverse of \( F \). Firstly we recall \( R\pi(V) = L\pi(S) = 0 \) in \( \mathcal{Z}/\mathcal{I} \). Hence, since \( L_{\mathcal{V}} \) is cohomological, we also have \( L_{\mathcal{V}}(K) = 0 \). Therefore we have an additive functor \( G' : \mathcal{C}/K \to \mathcal{C}[V^{-1}] \) such that \( L_{\mathcal{V}} \cong G' \circ \sigma \). It suffices to show that \( G' \) sends any regular morphism to an isomorphism in \( \mathcal{C}[V^{-1}] \). Due to Lemma 4.12, again since \( L_{\mathcal{V}} \) is cohomological, \( G'(R) \) is a class of isomorphisms. By the universality, the existence of a desired functor \( G \) follows and \( F \) is an equivalence.

Now we are ready to prove Theorem 4.9.
Proof of Theorem 4.9  The assertions (1) and (2) follow from Proposition 4.10 and Corollary 4.14. The others (3) and (4) follow from Proposition 4.15 and Corollary 4.7.

As an application of Theorem 4.9, we conclude that the cohomological functor \( \text{coh} : C \to \mathcal{H}/\mathcal{W} \) has a universality in the following sense.

Corollary 4.16  Assume that \( S \ast V \) is functorially finite in \( C \). Let \( H \) be a cohomological functor from \( C \) to an abelian category \( A \). If \( H(K) = 0 \), then there, uniquely up to isomorphism, exists an exact functor \( H' \) such that \( H \cong H' \circ \text{coh} \).

Proof  By \( H(K) = 0 \), we have a functor \( H'' : C/K \to A \) which makes the following diagram commutative up to isomorphism
\[
\begin{array}{ccc}
C & \xrightarrow{\pi} & C/K \\
\downarrow{H} & & \downarrow{H''} \\
A & \xrightarrow{H'} & (C/K)[R^{-1}]
\end{array}
\]
Since \( H \) is cohomological, \( H'' \) sends \( R \) to a class of isomorphisms in \( A \). Thus we have a desired functor \( H' \). The exactness directly follows from the construction of cokernels and kernels in \( (C/K)[R^{-1}] \) (see the proof of Proposition 4.10).

5 Comments on Another Related Work

In this section, we give some comments on results closely related to Theorem 4.9 which can be easily deduced from [13,26] (see [23] for exact case). Their method depends on a use of the twin cotorsion pair \( Q := ((S, T), (U, V)) \) which is defined to be a pair of two cotorsion pairs with \( S \subseteq U \).

Remark 5.1  (1) The twin cotorsion pair \( Q = ((S, T), (U, V)) \) does not require that \((S, V)\) forms a cotorsion pair. If this is the case, it forces \( S = U \) and \( T = V \).

(2) Note that our gHTCP is not a twin cotorsion pair in general.

The heart \( \mathcal{H}_Q \) of twin cotorsion pair \( Q \) was introduced as a generalization of the heart of a cotorsion pair [26, Def. 2.8]. Their results show that the heart \( \mathcal{H}_Q \) of some special twin cotorsion pair is an integral category, and its localization \( \mathcal{H}_Q[R^{-1}] \) with respect to the class of regular morphisms is equivalent to the heart \( \mathcal{H}_{(S, T)} \) of the cotorsion pair \((S, T)\).

By Proposition 4.6 and its dual, we have the following corollary which shows that, under some assumption, a cotorsion pair \((S, V)\) gives rise to a twin cotorsion pair.

Corollary 5.2  Let \( C \) be a triangulated category equipped with a cotorsion pair \((S, V)\). If \( S \) has enough projectives \( P(S) \) and \( V \) has enough injectives \( I(V) \), then we have a twin cotorsion pair
\[
Q := ((P(S), K), (K, I(V))),
\]
where \( K = \text{add}(S \ast V) \). Moreover, there exists an equivalence between the heart \( \mathcal{H}/\mathcal{W} \) of \((S, V)\) and that of \((P(S), K)\).
In this case, it directly follows from [26, Thm. 6.3] and [13, Thm. 5.6] that there exists a commutative diagram which is a special case of Theorem 4.9(3) as below:

\[
\begin{array}{ccc}
C & \xrightarrow{\text{coh}} & \mathcal{H}/W \\
& & \\
& \downarrow_{L_R} & \\
& \mathcal{H}_Q & 
\end{array}
\]

where $L_R$ denotes the localization with respect to the class $\mathbb{R}$ of regular morphisms.

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