D-brane Couplings, RR Fields
and Clifford Multiplication

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Abstract

Non-trivial configurations of Yang-Mills fields and gravitational backgrounds induce charges on Dp-branes that couple them to lower and higher RR potentials. We show that these couplings can be described in a systematic and coordinate independent way by using Clifford multiplication. In the minimal formulation, D-brane charges and RR potentials combine into bispinors of an $SO(1,9)$ which is defined with a flat metric and does not coincide with the space-time Lorentz group. In a non-minimal formulation, the RR potentials combine into $SO(10,10)$ spinors while the space of charges is formally enlarged to construct $SO(10,10)$ bispinors. The formalism suggests that the general form of the gravitational contribution to the D-brane charges is not modified, though the replacement of wedge product by Clifford multiplication gives rise to new couplings, consistent with T-duality.

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1 Introduction

Couplings on D-branes to RR potentials played an important role in the developments of the last few years. Since the early days of D-branes, it has been known that by virtue of non-trivial gauge field configurations on the brane worldvolume, a Dp-brane couples not only to a \( p + 1 \)-form Ramond-Ramond potential, but also to lower Ramond-Ramond potentials enabling it to carry charges corresponding to smaller branes \[1\]. It was later found that non-trivial gravitational backgrounds can also induce charges on a Dp-brane worldvolume coupling it to lower Ramond-Ramond potentials \[2, 3\]. Finally, it was shown in \[4, 5\] that topologically non-trivial normal bundles also induce RR charges. The existence of these couplings can be checked both by anomaly cancellation arguments, as well as by microscopic string computations.

We start with a very brief review of D-brane couplings. Let \( f : \mathcal{W} \hookrightarrow X \) be the embedding of the Dp-brane worldvolume into space-time (See footnote 4 for the notation). We will denote the tangent bundles by \( T\mathcal{W} \) and \( TX \), respectively. The normal bundle \( \mathcal{N}(\mathcal{W}, X) \) is then defined by

\[
0 \to T\mathcal{W} \xrightarrow{f^*} TX \to \mathcal{N}(\mathcal{W}, X) \to 0 .
\]

The (anomalous) coupling on the worldvolume of \( N \) coincident D-branes may be written in the form

\[
I^{(p)}_{\text{WZ}} = \int_{\mathcal{W}} C \wedge Y, \quad \text{(1.2)}
\]

where \( \mathcal{W} \) is the \( p + 1 \)-dimensional worldvolume of the brane, \( C = f^*C \) is the pullback of the total RR potential \( C \), \( \mathcal{M} = \mathcal{F} - f^*B \) where \( B \) is the NS-NS 2-form potential, \( \mathcal{F} \) is the field strength of the \( U(N) \) gauge field on the brane, and \( g \) is the restriction of the space-time metric to the brane. Defining \( \text{ch}(E) = \text{tr}_N \exp \left( \frac{M}{2\pi} \right) \), the final formula for \( Y \) is of the form

\[
Y = \text{ch}(E)e^{\frac{1}{2\pi} \sqrt{\hat{A}(\mathcal{W}) \hat{A}(\mathcal{N})}}.
\]

Here \( d \) is a degree two class defining a \( \text{Spin}^c \) structure on \( \mathcal{W} \) and in case of (almost) complex manifolds \( d = -c_1(\mathcal{N}) \).

One of the important features of (1.2) is that it can be written in terms of the bulk data (via the so-called \( K \)-theoretic Gysin map \( f_! \)) and be brought to a form

\[
\int_X C \wedge Y = \int_X C \wedge \text{ch}(f_!E) \sqrt{\hat{A}(TX)}, \quad \text{(1.4)}
\]

thus leading to an interpretation of the RR charges \( \text{ch}(f_!E) \sqrt{\hat{A}(TX)} \) as elements \( (x = f_!E) \) of \( K \)-theory of space-time \( K(X) \) \[5, 6\]. Moreover as turns out similar formulae and thus similar interpretations can be possible for the RR potentials themselves \[7\].

As mentioned already, one of the outcomes of the physics of WZ couplings (starting from refs \[1, 2\] has been the observation that the higher brane physics is influenced by the lower
branes. This logic has an obvious drawback of not being T-duality invariant, under which lower and higher branes can be easily interchanged. Indeed, consider a Dp-brane carrying a Dp'-brane charge for \( p > p' \). Since T-duality can interchange \( p \) and \( p' \), one expects that a Dp-brane should also carry charges corresponding to higher D-branes. The form of the Dp-brane coupling to higher RR forms, when the WZ term in the worldvolume action does not contain gravitational interactions, was obtained in [8, 9] in the static gauge. The higher brane charges are induced by the transverse non-Abelian scalars on the D-brane worldvolume which are T-dual to the worldvolume gauge fields. The presence of these interactions, which are required by T-duality can also be checked by Matrix theory calculations [8] as well as by microscopic string calculations [10]. It is expected that analogous couplings should also be induced by gravitational backgrounds.

A generic formula that contains both the standard gauge WZ terms (wedge products) as well as terms involving contractions (inner products) was given in [9] in the static gauge and is of the form

\[
I_{WZ}^{(p)} = \text{Str} \int_W (\exp \iota \Phi \iota \Phi C) \wedge \text{ch}(E),
\]

where \( \Phi^i \) (\( i = p + 1, \ldots, 9 \)) are the transverse scalars and the usual pull-backs \( \partial_\mu X^i \) are replaced by the gauge covariant derivatives \( D_\mu \Phi^i \) (as we will see later, this should also include a normal bundle connection). \( \iota \Phi \) denotes a contraction with \( \Phi^i \) and \( \iota \Phi)^n C = \frac{1}{n!} \Phi^{i_1} \ldots \Phi^{i_n} C_{i_1 \ldots i_n j k l \ldots} \). This coupling serves a dual purpose of restoring T-duality covariance and incorporating the D-brane scalars. One can however ask questions about the inclusion of gravitational contributions and the possibility of a systematic and coordinate independent description of the new D-brane couplings to RR backgrounds, on par with the case when the inner product couplings are absent. We will concentrate on these issues in this paper. This is, in particular, important for understanding the influence of the new interactions on the K-theory interpretation of D-brane charges.

The structure of the paper is as follows: In section 2, we use T-duality to obtain the general form of the D-brane couplings to RR potentials and write down a constraint on the form of the generalized brane charges. We also point out the appearance of a normal bundle connection and describe how a flat \( SO(1,9) \) metric enters in the calculations. In section 3 we show that the couplings can be described in a systematic and coordinate independent way by Clifford multiplication with respect to a flat metric. The formulation is based on a flat \( SO(1,9) \) Clifford algebra. We also discuss the from of the gravitational contributions to the generalized charges and show that, besides reproducing the known couplings, the formalism predicts a host of new couplings not considered before, but expected from T-duality in the presence of gravity. In section 4 we describe an alternative formulation based on the \( SO(10,10) \) Clifford algebra which leads to the same results. In section 5, as an application, we consider a simple example of the new couplings which gives rise to the gravitational version of Myer’s dielectric effect [11]. Section 6 contains some comments and a summary of the results.
2 Dp-brane coupling to higher RR potentials from T-duality

In this section, following [8, 9], we start with the WZ term in the non-Abelian D9-brane action and use T-duality to write down the generic form of Dp-brane couplings to RR potentials. One of the main results in these references was the realization that, besides coupling to $C^{(p+1)}$ and lower forms, Dp-branes also couple to higher form RR potentials through charges induced on the worldvolume by non-trivial configurations of the transverse scalars. The treatment below is general and applies to non-trivially embedded branes and to D-brane charges with both Yang-Mills as well as gravitational origins, though the explicit form of the coupling to transverse gravitational fields will not be discussed in detail. The action obtained in this way is in the static gauge and contains terms which are no-longer in the standard WZ form. In the next section we give a systematic, coordinate independent formulation of the D-brane couplings in terms of Clifford algebra valued RR-potentials and D-brane charges. This formulation is also valid in cases where T-duality can no longer be used to obtain the associated Dp-branes from D9-branes, that is, in the absence of isometries in the transverse directions. An important feature of our formalism is that the Clifford algebra in which the brane charges take their values is always defined with respect a flat metric, even when the D-brane is placed in curved space. Though at first sight this may look puzzling, a closer examination shows that the flat-space Clifford algebra arises naturally in the T-duality transformations of RR potentials to which the D-brane charges couple. We will briefly describe this below, before considering the couplings required by T-duality.

Since the appearance of a flat-space Clifford algebra is central to our formulation of the D-brane couplings, we begin by describing how it arises in the context of T-duality transformations of the RR potentials, using some results in [11]. Denoting the RR field strengths by $F^{(n)}$, there exist two alternative sets of RR potentials $C^{(n)}$ and $C'^{(n)}$ defined by

$$F = dC - H \wedge C, \quad C' = C \wedge e^{-B}. \quad (2.1)$$

Here, $H = dB$ and we use the notation $F = \sum_n F^{(n)}$, and similarly for the potentials. The potentials $C^{(n)}$ are invariant under $B_{MN}$ gauge transformations while $C'^{(n)}$ are not. From the form of the WZ term in the D-brane worldvolume action one can see that it is the latter set of potentials to which D-brane charges couple. Let us now consider the non-trivial elements of the T-duality group $O(d, d)$ which are contained in a subgroup $O(d) \times O(d)/O(d)_{\text{diag}}$. The action of these elements on the RR potentials was studied in [11] where it was shown that the potentials $C^{(n)}$ transform as a Lorentz spinor and their transformation depends on the NS-NS backgrounds $G_{MN}$ and $B_{MN}$. Furthermore, it was shown that the transformation of $C'^{(n)}$ is independent of $G_{MN}$ and $B_{MN}$ and can be obtained from that of $C^{(n)}$ by substituting a flat metric $\hat{\eta}_{MN}$ for $G_{MN}$ and setting $B_{MN}$ to zero in the transformation equations\footnote{We use a hat to distinguish the flat metric $\hat{\eta}$, which appears in the definition of the duality group $O(d, d)$, from the flat space-time metric $\eta$.} [11]. In\footnote{This independence is manifest in an alternative approach where components of $C'^{(n)}$ transform as spinors of the T-duality group $O(d, d)$.}
other words, since the T-duality rotations of $C^{(n)}$ do not depend on the NS-NS backgrounds, they have the same form in curved space as in flat space (defined by the flat metric $\eta_{MN}$ and vanishing $B_{MN}$ field). This is the origin of the flat-space Clifford algebra appearing in the next section. Though the transformation can be directly read off from [12, 11], we give a simple derivation below to make the discussion self-contained. The reader not interested in the details can skip to equation (2.4).

As stated above, to obtain the transformation of $C^{(n)}$ under the action of non-trivial elements of the T-duality group, it suffices to consider the flat space case. The non-trivial elements of the T-duality group can be parametrized by a matrix $R \in O(d) \subset O(1,9)$ which acts (in flat space) on the left-moving parts of space-time coordinates and worldsheet fermions as $\partial_+ \tilde{X} = R \partial_+ X$, $\tilde{\psi}_+ = R \psi_+$; leaving the right-moving parts unchanged. In the Ramond sector, the zero modes of $\sqrt{2} \psi^M_+$ and $\sqrt{2} \psi^M_-$ satisfy the Clifford algebra $\{ \Gamma^M, \Gamma^N \} = 2 \eta^{MN}$ and the transformation of the zero modes of $\psi^M_+$ can be re-written as $R^M_N \psi^N_+ = \Omega^{-1} \psi^M_+ \Omega$, where $\Omega$ is the spinor representation of the rotation matrix $R$. This relative rotation of the two Clifford algebras originating in the left- and right-moving sectors of the worldsheet can be absorbed by the corresponding spin fields $S_+$ and $S_-$(which generate the Ramond ground states from the corresponding Neveu-Schwarz ground states), giving rise to their T-duality transformation,

$$\tilde{S}_+ = \Omega S_+ , \quad \tilde{S}_- = S_- .$$

Let us now restrict ourselves to the case of a single discrete T-duality transformation, say, along the $X^1$ direction for which $\tilde{\psi}_+^1 = - \psi^1_+$. In this case one can easily construct the spinor representation as $\Omega = a_{(i-f)} \Gamma_{11} \Gamma_1$. Here, $a_{(i-f)}$ denotes a sign ambiguity that could depend$^3$ on the initial theory $i$ and the final theory $f$. For a T-duality from IIA to IIB, we choose $a_{(A-B)} = +1$ and for one from IIB to IIA, we set $a_{(B-A)} = -1$. The relative sign between these two cases is fixed by the requirement that T-duality squares to +1 on the Ramond sector, and the overall sign is fixed such that the WZ term in the worldvolume action does not change sign under T-duality.

Let us now consider the RR vertex operator in flat space, $V_{RR} = \bar{S}_{++} \ F \ F_S S'$, where $F$ denotes the RR bosonic field. For the sake of definiteness, we take $S_-$ to have negative chirality in both IIA and IIB theories. Invariance of $V_{RR}$ under T-duality leads to the transformation of $\bar{F}$ which, after taking the above sign conventions into account, takes the form $\bar{F} = \Gamma_1 \ F$. In flat space the RR potentials $C^{(n)}$ and $C'^{(n)}$ both reduce to $C^{(n)}_o$, given by $F = dC_o$, the transformation of which under the above T-duality can be easily worked out as $\bar{C}_o = - \Gamma_1 \ C_o$. Here, $C_o$ stands for the bispinor constructed out of the flat-space RR potentials. Using the result for a single T-duality, the transformation of $C^{(n)}_o$ under $d$ T-dualities along $X^1, X^2, \ldots, X^d$ (in that order) can be easily written as

$$\bar{C}_o = (-1)^{d(d+1)/2} \Gamma_1 \cdots \Gamma_d \ C_o .$$

The transformation of the components $C^{(n)}_o$ can be worked out by using the expression $\bar{C}_o = \sum_m ((-1)^m/m!) C^{(n)}_o \Gamma_{M_1 \cdots M_m} \Gamma^{M_1 \cdots M_m}$ in (2.3) and observing that the product $\Gamma_1 \cdots \Gamma_d$ implements Hodge duality in $d$ dimensions. Now, if the NS-NS backgrounds $G_{MN}$ and $B_{MN}$

$^3$We would like to thank A. Sen for a discussion on this issue.
are switched on, the transformation of $C^{(n)}$ develops a dependence on the backgrounds, while that of $C'^{(n)}$ retains the flat-space form above. Thus we can simply replace $C^{(n)}_o$ by $C'^{(n)}$ in the component form of the transformation and obtain

$$
\tilde{C}'_{i_{r+1} \cdots i_{d} \mu_1 \cdots \mu_m}^{(m+d-r)} = \frac{(-1)^{r(r+1)/2 + d(d-1)/2 + rd}}{r!} C'^{(m+r)}_{i_1 \cdots i_r \mu_1 \cdots \mu_m} \epsilon^{i_1 \cdots i_r \cdots i_{r+1} \cdots i_d}.
$$

Note that though this equation is valid in curved space, the indices on the epsilon-tensor are raised and lowered using a flat metric $\hat{\eta}$ that appears in the definition of the T-duality group $O(d, d) \subset O(10, 10)$, and not the space-time metric $G_{MN}$. In fact, the transformation of $C'^{(n)}$ in curved space can still be written in the form (2.3) provided the $\Gamma$-matrices used are no longer associated with the space-time Lorentz group, but are defined with respect to an auxiliary flat metric $\hat{\eta}$. This applies not only to the discrete T-dualities discussed above, but to all non-trivial elements of the T-duality group. Though equation (2.4) could also be obtained in other ways, the derivation presented above emphasizes the existence of an associated $SO(1, 9)$ Clifford algebra defined with a flat metric. This feature will be used in the next section.

We now get back to the derivation of Dp-brane couplings to n-form RR potentials using T-duality. Following the approach in [8, 9], we start with the WZ term in the non-Abelian D9-brane action which has no room for coupling to higher branes:

$$
I^{(9)}_{WZ} = T^{(9)} \text{Str} \int_{W^{(10)}} C' \wedge \mathcal{Y}(\mathcal{F}) = T^{(9)} \text{Str} \sum_n \frac{n!}{(10-n)!} \int_{W^{(10)}} C'^{(n)} \mathcal{Y}^{(10-n)} \wedge \cdots \wedge \wedge d\xi^{\alpha_1} \wedge \cdots \wedge d\xi^{\alpha_{10}}.
$$

The coordinates $\xi^\alpha$ parameterize the D-brane worldvolume and $C' = \sum_n C'^{(n)}$ contains the pull-backs to the worldvolume of RR potentials $C'^{(n)}$ defined in (2.3). $\mathcal{Y} = \sum \mathcal{Y}^{(m)}$ denotes the lower brane charges\(^4\) induced on the D9-brane by worldvolume gauge fields $A_\alpha$ and background gravitational fields $G_{MN}$ and $B_{MN}$. Explicitly, in the absence of non-trivial gravitational contributions, $\mathcal{Y}(\mathcal{F}) = e^F$, where $\mathcal{F}$ is the non-Abelian gauge field strength [1]. In the presence of gravitational interactions, but with $H = dB = 0$, the induced charges are given by $\mathcal{Y} = e^F \wedge \sqrt{A(TW)}$ [3]. This expression receives $H$ dependent contributions the exact forms of which are not known. By absorbing the usual $e^{-B}$ factor in the RR potentials, we have lost the manifestly gauge invariant combination $\mathcal{F} - \mathcal{B}$, but have gained on other fronts: First, note that now there is a clear split between the worldvolume and space-time quantities; while $C'^{(n)}$ are intrinsically space-time forms pulled back to the brane worldvolume, $\mathcal{Y}^{(m)}$ are forms that live on the brane worldvolume. Second, note that the $C'^{(n)}$ used above are pure Ramond-Ramond forms in the sense that under the T-duality group they do not mix with the NS-NS fields. This is not the case with the alternative RR potentials $C^{(n)}$. This property will play an important role in the next section.

\(^4\)In general, unless otherwise stated, we will use calligraphic symbols $C, \mathcal{Y}, A, \mathcal{F}$, etc. to denote worldvolume quantities, including pull-backs, while the straight symbols $C, Y, A, F$, etc. are reserved for the bulk quantities.

\(^5\)Of course, the change from $C$ to $C'$ affects (1.2). In order not to clutter the notation too much, we do not introduce $\mathcal{Y}'$; hopefully this will not lead to a confusion.
Let us split the 10-dimensional space-time coordinates $X^M$ into the $d$ coordinates $X^i$ ($i = 1, \cdots, d$) and the remaining $10-d$ coordinates $X^\mu$. Assuming that the fields are independent of the coordinates $X^i$, we perform T-dualities along these directions to obtain the general form of the $(p = 9 - d)$-brane action, including its couplings to higher RR potentials. Choosing the static gauge, the D9-brane WZ term can be written in form suitable for T-duality as

$$I_{WZ}^{(9)} = T^{(9)} \Str \sum_n \sum_r \frac{(-1)^{(n-r)(d-r)}}{n! (10-n)!} nC_r 10-nC_{d-r} \int_{V^{(10)}} \gamma^{(10-n)}_{n\mu \cdots \mu_{n-r+1} \cdots \mu_{10-d}}$$

$$\times C_{i_1 \cdots i_d}^r e^{i_1 \cdots i_d} dX^1 \wedge \cdots \wedge dX^d \wedge dX^{\mu_1} \wedge \cdots \wedge dX^{\mu_{10-d}},$$

(2.6)

where $nC_r$ is the combinatorial factor $n!/r!(n-r)!$. T-duality along $d$ directions converts the D9-brane worldvolume action to a D$(9-d)$-brane action and replaces all fields by their duals. Using (2.4) in (2.6), and integrating over the transverse space (assumed to be compact of volume $V_{(d)}$), one can easily obtain the generalized WZ term for the D$(p=9-d)$-brane as

$$I_{WZ}^{(p)} = T^{(p)} \Str \int_{V^{(p+1)}} \gamma^{(p+1)}_{\mu_1 \cdots \mu_{s} \cdots \mu_{p+1}} dX^{\mu_1} \wedge \cdots \wedge dX^{\mu_{p+1}},$$

(2.7)

where $T^{(p)} = T^{(9)} V_{(d)}$ and $u = p + 1 - t$. It should be emphasized that the indices $i_1 \cdots i_s$ on $\gamma$ are raised using the flat metric $\eta_{MN}$, and not the space-time metric $G_{MN}$. The $s = 0$ term is the standard WZ Lagrangian which is responsible for a Dp-brane carrying Yang-Mills and gravity induced charges corresponding to smaller branes. The important feature of the action, however, is the presence of $s \neq 0$ terms, noted in [3, 4] for the pure Yang-Mills case, that couple Dp-branes to higher RR potentials. Note that the contractions run only over the directions transverse to the brane which, in general, need not be compact.

Below we will discuss some features of the above action in more detail. Let $\lambda^a$ denote the set of variables on which $\gamma$ could depend, for example, the gauge fields, the scalars, the metric, the NS-NS 2-form, etc. In the above we have assumed that $\gamma$ is the same as $\gamma$ but now expressed in terms of the dual variables,

$$\check{\gamma}(\lambda^a) = \gamma(\lambda^a(\lambda)),$$

(2.8)

This is a constraint on the form of $\gamma$ imposed by the requirement of compatibility of the worldvolume action with T-duality. Though this constraint is automatically satisfied in the pure Yang-Mills case, it can be used to obtain $B$-dependent corrections in the gravitational couplings.

Let us first consider the pure Yang-Mills case $\gamma_{i_1 \cdots i_{n-s} \mu_{n-s} \cdots \mu_n} = (e^F)_{i_1 \cdots i_{n-s} \mu_{n-s} \cdots \mu_n}$. This case has been extensively discussed in the literature [3, 4]. Here we emphasize a subtlety regarding the identification of the transverse scalars, which is often ignored though it is crucial for the validity of our coordinate independent formulation of the brane couplings in the next sections. If we start with gauge fields $A_i$ with constant background values, then after T-duality we can identify the transverse scalars on the resulting D$(9-d)$-brane as $\Phi^j = A_i \check{\eta}^j$. Strictly speaking, the discussion in the literature mostly applies to this case. However, if the $A_i$ depend on $X^\mu$ before T-duality, then the resulting D$(9-d)$-brane after duality is non-trivially
embedded in space-time and the directions spanned by the coordinates \( X^i \) are not globally transverse to it. Therefore, the transverse scalars have to be defined locally as sections of the normal bundle. Let \( a_M^I (I = 9 - d, \cdots 9) \) span a frame in the normal bundle. Then, the transverse scalars are defined as \( \tilde{\Phi}^I = A_M \hat{\eta}^{MN} a_N^I \) and the \( F_{\mu i} \hat{\eta}^{ij} \) in \( Y \) gives

\[
D_\mu A_i \hat{\eta}^{ij} = \nabla_\mu \Phi^I a_j^I \equiv \left( \partial_\mu \tilde{\Phi}^I + \Theta^I_{\mu J} \tilde{\Phi}^J + [\tilde{A}_\mu, \tilde{\Phi}^I] \right) a_j^I ,
\]

where \( \Theta^I_{\mu J} = a_N^I \partial_\mu a_J^N \) is the connection on the normal bundle to the brane. The appearance of this connection, which is required by the covariant formulation of the next section, can also be checked by applying T-duality to the DBI part of the action and constructing the kinetic energy terms for \( \tilde{\Phi}^I \). Perhaps the easiest way of seeing the above is from point of view of supersymmetry and appearance of the connection on the normal bundle in the context of the gauged \( R \)-symmetry. Similar covariantizations were used for description of normal bundles both for M5, M2 and D-branes [14].

Now, roughly speaking, the factors of \( F_{\mu \nu} = \bar{F}_{\mu \nu} \) in \( Y \) will combine into \( e^{\bar{\Phi}} \), while \( \bar{F}_{ij} \) will give rise to \( [\tilde{\Phi}^I, \tilde{\Phi}^J] \) which will contract the indices on \( C'^{(n)} \) through the \( a_I^j \). The factors \( F_{\mu i} \hat{\eta}^{ij} \) will couple to \( C'^{(n)} \) as a generalized “pull-back” which corresponds to replacing the ordinary pull-back \( \partial_\mu X^i \) by \( \nabla_\mu \tilde{\Phi}^I a_j^I \), as in [9, 10, 15], though now with the connection \( \Theta \) included. Although the appearance of the generalized pull-back in the theory may look appealing, it seems to suffer from two drawbacks: 1) By construction the expression is obtained in the static gauge and is not covariant, 2) The geometric meaning of the pull-back, in terms of the embedding of the brane in space-time, is lost. The covariant formulation in the next sections cures both these problems and yields the generalized pull-back in the static gauge.

While equation (2.8) can be easily verified for the Yang-Mills case, it can be used as a constraint for the gravitational part of the interaction, to find correction to (1.3). First, since the metric mixes with the antisymmetric tensor field under T-duality, it is clear that this form should be modified when \( dB \neq 0 \). Second, in branes smaller than D9-brane, the action also contains couplings which involve contractions between the RR potentials and the transverse components of gravitational fields, which can give raise to the gravitational version of the dielectric effect discussed in [9]. We will discuss this in an example in section 5. In principle, once one finds the correct \( B \)-dependence for the gravitational interactions of the D9-brane, one can find the form of the corresponding terms for the lower branes by T-duality. We will not address this issue here, but hope to return to it in a future publication.

3 D-brane charges as \( SO(1, 9) \) bispinors - Minimal formalism

We have seen that, besides coupling to \( C'^{(q+1)} \) forms for \( q \leq p \), a Dp-brane also couples to \( C'^{(q+1)} \) forms for \( q > p \) through the excitations of the transverse scalars on the worldvolume, as well as through gravitational fields transverse to the brane. The expression for these new couplings, obtained by T-duality, is always in the static gauge. However, since these
couplings should survive even when the Dp-brane is not related to a D9-brane by T-duality, it is clear that these interactions should have a more general coordinate invariant form. Furthermore, the new interactions can no longer be expressed neatly as a wedge product between RR potentials $C'$ and the generalized brane charges $Y$. It is desirable to find a formulation which allows us to again express the couplings as some kind of product between the RR potentials and the complete set of brane charges. We show that both these problems can be resolved by considering the brane charges as taking values in a Clifford algebra and replacing the wedge product by Clifford multiplication (by which here we simply mean the multiplication of $\Gamma$-matrices).

The presence of both anti-symmetrized products as well as inner products in (2.7) suggests the use of Clifford multiplication to couple $C'$ and $Y$, however, the metric with respect to which the algebra is defined is not the most obvious one. The Clifford algebra that naturally appears in the theory is the one associated with the $SO(1,9)$ space-time Lorentz group which is defined with respect to the space-time metric $G_{MN}$: $\{\hat{\Gamma}_M, \hat{\Gamma}_N\} = 2\hat{\eta}_{MN}$. In string theory, the RR field strengths $F^{(n)}$ and potentials $C^{(n)}$ come as bispinors of this Clifford algebra. However, one can easily check that this is not the correct candidate (see footnote 6 below). The Clifford algebra that appears in the minimal formulation of the D-brane couplings is an $SO(1,9)$ Clifford algebra defined with a flat metric $\hat{\eta}_{MN}$, $\hat{\eta}_{MN} = \epsilon_{MN}$. (3.1)

We use a hat to emphasize the fact that we are using flat space quantities even in curved space. Though, at first, this may look un-natural, we noted that a result of [11], described in the previous section, shows that such an algebra arises naturally in the T-duality transformations of $C' = C \wedge e^{-B}$, which is the quantity that appears in the D-brane worldvolume action. We will see below that this structure survives in the general situation even when T-duality is not applicable. In the next section we will describe a non-minimal formulation of the D-brane couplings in terms of an $SO(10,10)$ Clifford algebra. The minimal formalism of this section can also be understood as a "gauge-fixed" version of this non-minimal formalism.

Let us introduce flat $SO(1,9)$ bispinors $\mathcal{C}'$ and $\mathcal{Y}$ corresponding to RR potentials $C'^{(n)}$ and a space-time version (the meaning of which will be made more precise below) of the generalized D-brane charges $Y^{(n)}$,

\[
\mathcal{C}' = \sum_m \frac{(-1)^m}{m!} C'_{M_1\cdots M_m} \hat{\Gamma}^{M_1\cdots M_m},
\]

\[
\mathcal{Y} = \sum_n \frac{(-1)^n}{n!} Y^{(n)}_{N_1\cdots N_n} \hat{\Gamma}^{N_1\cdots N_n}.
\]

The space-time quantities $Y^{(n)}$ are introduced in order to obtain a coordinate independent description of the couplings and are defined such that their restriction to the brane worldvolume gives the generalized D-brane charges. For example, the worldvolume gauge fields $A_\alpha(\xi)$ and the transverse scalars $\Phi^I(\xi)$ can be combined into a ten dimensional space-time field $A_M(X)$ such that

\[
A_\alpha(\xi) = A_M(X(\xi)) \partial X^M / \partial \xi^\alpha, \quad \Phi^I(\xi) = A_M(X(\xi)) \hat{\eta}^{MN} a^I_N.
\]
Here, \( X^M(\xi) \) give the embedding of the brane in space-time and \( \partial X^M / \partial \xi^\alpha \) and \( a_I^M \) span, respectively, the tangent and normal bundles to the worldvolume. In general, on the worldvolume, \( Y_{M_1 \ldots M_n}^{(n)} \) restricts to \( Y^{(n)} I_1 \cdots I_{n_a} \), though we will not give an explicit construction for the gravitational contributions. We will return to this point later in this section. This space-time picture is consistent both with T-duality as well as with the \( K \)-theory interpretation of the D-brane charges (in the absence of the inner product terms in the action) and gives a covariant way of defining the generalized charges.

The product \( (C' Y) \) can now be easily evaluated using the formula for the product of \( \Gamma \)-matrices given in the appendix, which is essentially what we refer to as Clifford multiplication. To restrict the resulting formula to a Dp-brane worldvolume and eliminate the \( \Gamma \)-matrices in it, we should find the correct projection operator. To this end, note that the dimension of a D-brane worldvolume decreases (increases) under T-dualities performed along (transverse to) the brane. This is very similar to how RR forms pick up or lose components under the transformation, showing that the Dp-brane worldvolume forms transform very similar to \( C'^{(n)} \) in (2.4). From this we infer that the Dp-brane worldvolume forms, for all \( p \), could also be combined into a flat \( SO(1,9) \) bispinor,

\[
\mathcal{Y} = \sum_q \mathcal{Y}^{(q)} = \sum_q \frac{(-1)^q}{(q)!} T_{(q-1)} \, dX^{L_1} \wedge \cdots \wedge dX^{L_q} \hat{\Gamma}_{L_1 \cdots L_q}.
\]

When restricted to a given Dp-brane worldvolume defined by the embedding \( X^M(\xi^\alpha) \), the term \( \mathcal{Y}^{(p+1)} \) takes the form

\[
\mathcal{Y}^{(p+1)} = \frac{(-1)^{p+1}}{(p+1)!} T_{(p)} \left( \frac{\partial X^{L_1}}{\partial \xi^{\alpha_1}} \cdots \frac{\partial X^{L_{p+1}}}{\partial \xi^{\alpha_{p+1}}} \, d\xi^{\alpha_1} \wedge \cdots \wedge d\xi^{\alpha_{p+1}} \right) \hat{\Gamma}_{L_1 \cdots L_{p+1}}.
\]

For convenience we have inserted the couplings \( T_{(p)} \) as part of the volume bispinor. It is now easy to write the general covariant form of a Dp-brane coupling to all RR potentials (including the higher ones) in terms of a Clifford product,

\[
I_{WZ}^{(p+1)} = -\text{Str} \int \mathcal{W}^{(p+1)} \, \text{Tr} \left( \Gamma_{11} \nabla_{11} C' \mathcal{Y} \right),
\]

where \( \text{Str} \) is the symmetrized gauge trace and \( \text{Tr} \) is a trace over the spinor index normalized to unity. \( \nabla = \Gamma^0 \mathcal{Y}^T \Gamma^0 \) is the Dirac conjugate of \( \mathcal{Y} \) and, along with the trace, converts bispinors to forms. The factors of \( \Gamma_{11} \) are inserted to get the correct relative sign in IIA and IIB theories and are not unexpected since we are dealing with Majorana-Weyl bispinors. The integral over \( \mathcal{W}^{(p+1)} \) restricts the expression to the Dp-brane worldvolume and picks up the contribution from the term \( \mathcal{Y}^{(p+1)} \) of (3.5) alone.

The \( \Gamma \)-matrix multiplication and tracing can be carried out using the formulae in the appendix leading to the component form for the generalized WZ action,

\[
I_{WZ}^{(p+1)} = T_{(p)} \text{Str} \sum_{s,t} \frac{1}{s! t! u!} \int \mathcal{W}^{(p+1)} C_{L_1 \cdots L_t N_1 \cdots N_u}^{(t+s)} Y_{L_{t+1} \cdots L_{p+1}}^{(s+u) N_1 \cdots N_u} \, d\xi^{\alpha_1} \wedge \cdots \wedge d\xi^{\alpha_{p+1}},
\]

where \( C_{L_1 \cdots L_t N_1 \cdots N_u}^{(t+s)} \) is the \( \Gamma \)-matrix multiplication and tracing can be carried out using the formulae in the appendix leading to the component form for the generalized WZ action,
where $u = p + 1 - t$. The indices on $Y$ are raised using the flat metric $\hat{\eta}$. Since the indices $L_1, \cdots, L_{p+1}$ are pulled-back to the brane worldvolume, antisymmetry in the indices implies that $N_1, \cdots, N_s$ are automatically restricted to the directions transverse to the brane. Of course, for this to happen, it is crucial that $C'$ and $Y$ are contracted by the flat metric $\hat{\Gamma}$. In the static gauge, $X^\mu = \xi^\mu$ and $X^i = X^i(\xi)$, (3.8) reduces to (2.7). Note that the interactions in (3.8) explicitly preserve the geometric nature of the embedding of the brane worldvolume in space-time through the appearance of the pull-backs $\partial X^M/\partial \xi^a$. The generalized pull-backs discussed in section 2 appear as part of the restriction of $D_M A_N$ to the worldvolume which, using (3.4), contains

$$\frac{\partial X^M}{\partial \xi^a} (D_M A_N) \hat{\eta}^{NP} a_P = \partial_\alpha \Phi^I + \Theta_{\alpha J}^I \Phi^J + [A_\alpha, \Phi^J].$$

(3.9)

The appearance of the normal bundle connection $\Theta_{\alpha J}^I = a_M^I \partial_\alpha a^M_J$ here is a reflection of the covariance of the formalism.

To summarize, the product ($\mathcal{C}' Y$) yields a series of even (odd) forms encoding certain couplings for IIB (IIA) theory, while multiplication by $Y^{(p+1)}$ and tracing restricts these to a Dp-brane worldvolume. Clifford multiplication can also be formulated abstractly, independent of $\Gamma$-matrices, as an operation on differential forms. Given a vector $v$ we can define a contraction $\iota_v$, using again the flat metric, and thus an isomorphism $\sigma_v = v \wedge - \iota_v : \Lambda^{even} X \rightarrow \Lambda^{odd} X$ corresponding to $\phi : S^+ \otimes S^- \oplus S^+ \rightarrow S^+ \otimes S^- \oplus S^-$. This is a Clifford multiplication by a vector $v$, and can be straightforwardly generalized to any form $C'$ with the combinatorial factors simply given by those of the $\Gamma$-matrix multiplication (see appendix). In this notation, (3.7) can be written in a more conventional form as,

$$L_{WZ}^{(p+1)} = \sigma_{C'}(Y)\Big|_{W_{p+1}}.$$  

(3.10)

Note that $(\sigma_{C'}(Y))_{10} = (C' \wedge Y)_{10}$ and as expected there are no contractions for D9-branes. Of course, by design (3.7) or (3.10) correctly reduce to the covariant form of the coupling (1.3) in the case of flat space, including the normal bundle contribution to the covariant derivatives. However, up to now we have not really specified the “generalized brane charge” $Y$ for the curved space, other than by outlining the generic constraints imposed on it by T-duality. When the NS-NS $B$-field is set to zero, choosing $Y$ in the form

$$Y(dB = 0) = \text{ch}(x) \sqrt{A(T X)},$$  

(3.11)

ensures that at least the part $C' \wedge Y$ of (3.11) restricts to the expected coupling (1.2) as given by (3.3) following the calculations of [3]. Thus we conjecture that this form of $Y$ should not change and taking into account the (non-Abelian) dynamics of D-branes amounts to replacing the wedge product $C' \wedge Y$ by Clifford product $\sigma_{C'}(Y)$. This form also predicts

6If one uses the curved space $\Gamma$-matrices $\Gamma_M$ instead of the flat-space ones $\hat{\Gamma}_M$, one still gets (3.8) but now with $C'$ and $Y$ contracted by the space-time metric $G_{MN}$. This is of course incorrect since the indices $N_1, \cdots, N_s$ are no longer restricted to the transverse space. This is a reflection of the fact that the natural bispinors appearing in the theory are the ones constructed out of $C' = C \wedge e^{-B}$ and $\hat{\Gamma}_M$ (as seen in the previous section) or $C$ and $\Gamma_M$. A bispinor containing $C'$ and $\Gamma_M$ does not arise in the theory.
a series of new couplings containing the curvature of the space transverse to the D-brane. It is not easy to check these predictions explicitly for arbitrary worldvolume $W_q$ and arbitrary embedding, since many of the couplings have not been calculated on the worldvolume yet. However, for simple embeddings it shows that the transverse curvature form can enter \( \text{via} \) contractions with the transverse components of RR fields, quite similar to those involving the scalars, namely terms of the form

\[
\int_{W} \text{Tr} R_{pq} R_{rs} C^{tprqs} \cdots .
\]  

(3.12)

Note that the flat metric is used only for the contractions, and the (transverse) space is curved. This formula is somewhat symbolic and appropriate restrictions to the worldvolume are assumed here. Furthermore, there are no contractions between the transverse scalars and the transverse components of the curvatures. We will return to this in section 5, where the existence of such interactions will be demonstrated more explicitly in a case of trivial embedding, and consider a simple example of D-brane polarization where non-trivial gravitational background and thus (3.12) play some role.

4 D-brane charges as $SO(10, 10)$ bispinors - Non minimal formalism

The formalism in the previous section is suggested by the fact that the action of non-trivial elements $O(d) \times O(d)/O(d)_{\text{diag}}$ of the T-duality group on the RR fields can be written in terms of $SO(1, 9)$ spinors \([11, 12]\). As we saw, the generalized D-brane charges are then forms in a 10 dimensional space. One could also try to use an alternative formalism based on the full T-duality group $O(d, d) \subset O(10, 10)$. As we will see below, a formulation in terms of $SO(10, 10)$ spinors is possible provided the space of charges is formally enlarged to a 20 dimensional space, though the physical content, of course, remains 10 dimensional. For this reason we refer to this as the non-minimal formalism.

To understand the origin of the formal enlargement of the space of charges, we consider the Yang-Mills sector in flat space. The charges are related to the $p + 1$ gauge fields $A_\alpha$ and the $9 - p$ transverse scalars $\Phi^I$, which were combined into a 10 dimensional space-time vector $A_M$ \([3, 4]\). In the open string picture of a Dp-brane, the worldvolume gauge fields are associated with the $p + 1$ canonical momenta along the brane, while the transverse scalars are associated with the $9 - p$ variables, $\partial_\sigma X$, that one could refer to as “windings” or “length densities”. However, in the bulk closed string theory the phase space is spanned by 10 canonical momenta $P_M$ and 10 string “length densities” $\partial_\sigma X^M$. Thus one may formally promote both the worldvolume gauge fields as well as the transverse scalars to 10 component fields $A_M$ and $\Phi^M$ (which can further combined into a 20-dimensional vector) such that the Dirichlet boundary conditions defining the D-brane projects these onto $A_\alpha$ and $\Phi^I$ that carry all the physical content. This, in fact, is an alternative way of defining the D-brane charges in a coordinate independent way.
It has been known that, in toroidal compactifications, the components of RR potentials $C^{(n)}$ can be arranged into spinors of the T-duality group [16, 17, 11]. As in the previous section, one can now go beyond T-duality and develop a formalism based on the Clifford algebra associated with the $SO(10, 10)$ group which, in the off-diagonal basis, is written as

$$\{\gamma^m, \gamma^n\} = 2J^{mn}, \quad J = \begin{pmatrix} 0 & \hat{\eta} \\ \hat{\eta} & 0 \end{pmatrix},$$ (4.1)

where $m, n = 1 \cdots 20$. Let us denote the first 10 values of the index by $\hat{m}$ and the remaining 10 by $\check{m}$. Then,

$$\{\gamma^{\hat{m}}, \gamma^{\check{m}}\} = 2\hat{\eta}^{\hat{m}\check{m}}.$$ (4.2)

One can see that $\gamma^{\hat{m}}/\sqrt{2}$ and $\gamma^{\check{m}}/\sqrt{2}$ (where, $\gamma^{\check{m}} = \hat{\eta}^{\check{m}\hat{m}}\gamma^{\hat{m}}$) satisfy the Heisenberg algebra and can be used to construct the spinor representation of the algebra (we will not distinguish between the raising and lowering operators and their matrix representations). Now we combine the the RR potentials into an $SO(10, 10)$ Majorana-Weyl spinors and the volume forms into the adjoint spinor,

$$|C'\rangle = \sum_n \frac{(-1)^n}{n!} C^{(n)}_{\hat{m}_1 \cdots \hat{m}_n} \gamma^{\hat{m}_1 \cdots \hat{m}_n} |0\rangle,$$ (4.3)

$$\langle V| = \sum_p \frac{(-1)^{p+1}}{(p+1)!} \langle 0| \gamma^{\check{m}_{p+1} \cdots \hat{m}_1} dX^{\hat{m}_1} \wedge \cdots \wedge dX^{\check{m}_p+1} |\psi(p+1) T(p)\rangle,$$ (4.4)

where, $dX^{\hat{m}}|\psi = \partial X^{\hat{m}}/\partial \xi^\alpha d\xi^\alpha$. Note that the indices on $C^{(n)}$ and the space-time coordinates always run from 1 to 10 and are not affected by the enlargement of the space of charges. Furthermore, the D-brane charges can be combined into an $SO(10, 10)$ bispinor,

$$Y = \sum_q \frac{(-1)^q}{q!} Y_{m_1 \cdots m_q} \gamma^{m_1 \cdots m_q}.$$ (4.5)

$$= \sum_q \sum_r \frac{(-1)^q}{q!} q C_r Y_{m_1 \cdots m_r, m_{r+1} \cdots m_q} \gamma^{m_1 \cdots m_r, m_{r+1} \cdots m_q}.$$ (4.6)

For example, in terms of differential forms, in the pure Yang-Mills case we have $Y = e^{\mathcal{F}}$ with $F_{\hat{m}\check{m}} = D[\hat{m} A_{\check{m}}], \ F_{\hat{m}\hat{m}} = \{\Phi^{\hat{m}}, \Phi_{\check{m}}\}$, and $F_{\check{m}\check{m}} = D[\hat{m} \Phi_{\check{m}}]$. The generalized WZ term can now be written as

$$I_{WZ}^{(p)} = \text{Str} \int W(p+1) \langle V|Y|C'\rangle.$$ (4.7)

Only the $p$-form term in (4.4) contributes to the integral and on identifying $\hat{m}$ with the space-time coordinate label $M$, the component form of this expression reduces to equation (3.8).

5 An application: D0-D6 system

We present here the simplest case when the non-trivial gravitational background influences the D-brane polarization. In [9], D0-branes in the background of $C_3$ potential with a constant
field strength were considered and it was shown that due to (1.3), a system of D0’s expands into a fuzzy two-sphere. Let’s consider a minimal extension of the D2-D0 system that includes gravity. Since in essence our discussion closely follows that of [9], we will be rather brief here.

We consider here a system of D0-branes in the background of a seven-form potential $C_7$ with a constant field strength and a curved four-dimensional manifold $X$, let say $X = K3$ for sake of concreteness. We take the field strength of the $C_7$ to be of the form

$$F_{ijkpqrs} = \epsilon_{ijk} \epsilon_{pqrs}$$

(5.1)

with $i, j, k = 1, 2, 3$ and the indices $p, q, r, s = 1, 2, 3, 4$ spanning $X$. Following the general discussion of section 3 (see (3.12)), we can consider a coupling of D0-branes to the RR seven-form potential

$$\text{Tr} \left( \Phi^i \Phi^j \Phi^k \right) \text{Tr} R^p R^q C_{ijkpqrs}$$

(5.2)

Note that the embedding (the normal bundle) is trivial and we are not distinguishing between the pull-backed and bulk quantities; we are also dropping the primes on RR since B-field is turned off. These couplings can be seen explicitly by starting from a D6-brane with a worldvolume $W_7 = W_3 \times X$ in a presence of a RR three-form field. Since this is the only RR filed switched on, there is just a WZ coupling on the brane

$$\int_W C_3 \wedge \text{Tr} R^2$$

that can be T-dualized along $X$ to a more familiar D2-brane transverse to $X$ (even if, like for $X = K3$, there are no isometries, we can follow our rules for T-duality). Recalling that the action on RR fields amounts to Hodge duality along $X$, and since $C_3$ is transverse to $X$ it gets rotated into the seven-form (5.1), while the wedge product is replaced by contractions. Following [9] for the gauge part we arrive at (5.2).

The rest of the story directly follows [9], and performing the non-Abelian Taylor series expansion of the RR potential looking at the minima of the potential we arrive at the condition

$$[[\Phi^i, \Phi^j], \Phi^j] + \text{Tr} R_{pq} R_{rs} \epsilon_{pqrs} \epsilon^{ijk} [\Phi^j, \Phi^k] = 0.$$  

(5.3)

This has a solution in the form

$$[\Phi^i, \Phi^j] = P_X \epsilon^{ijk} \Phi^k$$

where $P_X = \text{Tr} R_{pq} R_{rs} \epsilon^{pqrs}$. Thus as expected we get and equation for a fuzzy two-sphere, however we see that its radius depends on the curvature forms of the transverse manifold $X$. Following the work [9], further polarization effects have been discussed in literature. We believe it is of some interest to expand these discussions to the cases involving non-trivial backgrounds and embeddings.
6 Conclusions and Discussion

It is well known that worldvolume gauge field configurations and gravitational backgrounds induce charges on a Dp-brane corresponding to lower dimensional branes. These charges couple to the corresponding RR potentials by wedge products. If the background gravity contribution to the brane charges is neglected, then T-duality arguments, as well as Matrix theory calculations and microscopic string calculations show that Dp-branes can also carry charges corresponding to higher dimensional branes by virtue of non-trivial configurations of the transverse scalars on the brane worldvolume.

We consider generic D-brane charges, which may contain both Yang-Mills and gravitational contributions, and employ the construction based on T-duality to write down the form of their couplings to RR potentials. Though the full gravitational contributions to the charges are not explicitly constructed, their forms are strongly constrained by T-duality covariance. It also shows that, similar to the transverse scalars, background gravity can induce charges on the worldvolume that couple a Dp-brane to higher RR potentials.

The generalized interactions obtained in this way have three main features: 1) the couplings can no-longer be described in a systematic way by a wedge product of D-brane charges and RR potentials, 2) the couplings are obtained in the static gauge and do not have a coordinate independent form, 3) the usual pull-backs to the worldvolume are replaced by the covariant derivatives of the transverse scalars. Though this is an appealing way of encoding some worldvolume interactions, the generalized pull-back does not have a geometric meaning in terms of the embedding of the brane. For non-trivial embeddings of the brane, the covariant derivative of the scalars also contains a normal bundle connection that is often missed in the discussions of D-brane T-duality.

We show that the generalized D-brane couplings to RR potentials can be formulated in a covariant and systematic way, essentially by replacing the wedge product by Clifford multiplication. There are two alternative formulations based on $SO(1,9)$ and $SO(10,10)$ Clifford algebras. The minimal formulation is based on $SO(1,9)$ which is defined with respect to a flat metric and does not coincide with the space-time Lorentz group, except in the flat space. In this formalism, the generalized D-brane charges, the RR potentials and the brane volume forms are all combined into Majorana-Weyl bispinors of the flat $SO(1,9)$ Clifford algebra. In the non-minimal $SO(10,10)$ formulation, the space of charges is formally enlarged so that they fit into bispinors of the $SO(10,10)$ Clifford algebra while the RR potentials and volume forms, each combine into $SO(10,10)$ spinors. The appearance of both Clifford algebras can be understood in terms of the T-duality group, though our results are independent of T-duality and remain valid even in the absence of isometries. From the T-duality group point of view, the minimal formalism can be regarded as a gauge fixed version of the non-minimal formalism.

Both formalisms lead to the same coordinate invariant form for the generalized WZ
interactions on Dp-brane worldvolume,
\[ L_{WZ}^{(p+1)} = \sigma_{C'}(Y) \big|_{W_{p+1}}, \tag{6.1} \]
where \( \sigma_{C'} \) denotes Clifford multiplication by \( C' \), defined as an operation on forms, with respect to a flat metric. \( Y \) is a space-time version of the D-brane charges, the form of which is explicitly known in the absence of gravitational contributions. The restriction of the Clifford product to a Dp-brane worldvolume then reproduces the correct covariant form of the known couplings, including the effective generalized pull-backs with respect to the covariant derivatives of the transverse scalars. The covariant derivatives now contain both the gauge as well as the normal bundle connection.

The full complete gravitational contribution to \( Y \) is not known though, its form is constrained by T-duality. However, it reduces to the known form \( [5] \) when the NS-NS 3-form field strength is set to zero,
\[ Y(dB = 0) = \text{ch}(x)\sqrt{\hat{A}(\mathcal{T}X)}. \tag{6.2} \]
All the novelty then simply comes down to replacing the wedge product in (1.4) by Clifford multiplication of \( Y \) by RR form \( C' \). Then (6.1) correctly reproduces, at least formally, the known gravitational couplings (1.2) while encoding many interactions that have not yet been tested directly. We have not explicitly constructed these new interactions in the general case, though some special situations have been discussed. Of course, it is important to verify some of the new worldvolume terms by microscopic calculations. Obtaining a T-duality invariant form of \( Y \) with a non-trivial \( B \)-field is another direction of further research. Our results should also be useful for obtaining a full non-Abelian \( \kappa \)-symmetric action for D-branes.

We would like to conclude this section with a comment on RR charges which, in principle can be derived from the equations of motion, in a way manifestly consistent with our formulation. We do not pursue this here, however, in view of the formalism developed here, it is clear that one arrives at a formula similar to the one in the introduction compatible with the interpretation of RR charges as element of \( K \)-theory of space-time.

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**A \( \Gamma \)-matrix multiplication**

We use the metric signature \( \{-1, +1, \ldots, +1\} \) and define the \( \Gamma \)-matrices such that,
\[ \{\Gamma^a, \Gamma^b\} = 2\eta^{ab}, \tag{A.1} \]
In the Majorana-Weyl representation all $\Gamma^a$ are real, with $\Gamma^0$ antisymmetric and others symmetric. To fuse products of $\Gamma$-matrices into antisymmetrized ones, we use the identity (See for example, [18])

$$\Gamma_{a_1\cdots a_i} \Gamma^{b_1\cdots b_j} = \sum_{k=|i-j|}^{i+j} \frac{i! j!}{s! t! u!} \delta_{[a_1}^{b_1} \cdots \delta_{a_{i+1}}^{b_{s+1}} \Gamma_{a_1\cdots a_i} \Gamma^{b_1\cdots b_j]} ,$$ (A.2)

with

$$s = \frac{1}{2}(i + j - k) , \quad t = \frac{1}{2}(i - j + k) , \quad u = \frac{1}{2}(-i + j + k) .$$

In the summation, only those values of $k$ appear for which $s$, $t$ and $u$ are integers, i.e., $k = |i - j|, |i - j| + 2, \ldots, i + j - 2, i + j$. The trace of products of $\Gamma$-matrices is given by

$$\text{Tr} (\Gamma_{a_1\cdots a_l} \Gamma^{b_1\cdots b_k}) = 2^5 \delta_{kl} (-1)^{k(k-1)/2} k! \delta_{[a_1}^{b_1} \cdots \delta_{a_k]}^{b_k]} .$$ (A.3)

All antisymmetrizations are with unit weight. This trace is not normalized to unity.

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