1. Introduction

Let $\text{DM}_{gm}(k)$ be the triangulated category of mixed motives over a perfect base field $k$, constructed by Hanamura, Levine, and Voevodsky, and we here adopt Voevodsky’s category of geometric motives; see [34, Lect. 14], [49]. This category yields various exciting developments.

The category $\text{DM}_{gm}(k,\mathbb{Q})$ with rational coefficients contains the category of mixed Tate motives as a symmetric monoidal triangulated subcategory. As for the case of mixed Tate motives, many pleasant properties are well-understood. In particular, if the base field is a number field, there is an expected motivic $t$-structure. In contrast with this, the structures of the ambient category $\text{DM}_{gm}(k,\mathbb{Q})$ is very mysterious but important and challenging to study.

The first purpose of this paper is to investigate a symmetric monoidal triangulated subcategory generated by the motif $M(X)$ of an abelian variety $X$ (more generally, we consider abelian schemes). Let $\text{DM}_{gm,X}(k,\mathbb{Q})$ be the smallest symmetric monoidal triangulated thick subcategory of $\text{DM}_{gm}(k,\mathbb{Q})$ which contains $M(X)$ and is closed under taking duals (cf. Definition 5.1, Remark 5.2). We call this category the triangulated category of mixed abelian motives generated by $X$. In a nutshell the first main result describes $\text{DM}_{gm,X}(k,\mathbb{Q})$ as a certain module category.

For many operations and constructions we would like to perform in this paper, the machinery of triangulated category causes a lot of troubles. The theory of $\infty$-categories provides a natural formulation from both conceptual and technical perspectives. The triangulated categories $\text{DM}_{gm}(k,\mathbb{Q})$ and $\text{DM}_{gm,X}(k,\mathbb{Q})$ have natural $\infty$-categorical enhancements which we denote by $\text{DM}^\otimes$ and $\text{DM}^\otimes_X$ respectively; the homotopy category of the full subcategory of $\text{DM}^\otimes_X$ (resp. $\text{DM}^\otimes$) spanned by compact objects is equivalent to $\text{DM}_{gm,X}(k,\mathbb{Q})$ (resp. $\text{DM}_{gm}(k,\mathbb{Q})$) as symmetric monoidal triangulated categories. In this introduction, by an $\infty$-category we mean the notion of $(\infty,1)$-categories in the sense of [6], and many $\infty$-categories are defined only informally. We employ the theory of quasi-categories, extensively developed by Joyal [25] and Lurie [31], from the next Section.

To describe the result, consider an abelian variety $X$ of dimension $g$ defined over a perfect field $k$. Let $\text{Rep}^\otimes_{\text{GL}_{2g}(\mathbb{Q})}$ be the symmetric monoidal $\infty$-category which consists of (not necessarily bounded) complexes of $\mathbb{Q}$-vector spaces equipped with action of the general linear group $\text{GL}_{2g}(\mathbb{Q})$; see Section 4.4 for the precise definition. We will construct a commutative differential graded algebra $A_X$ equipped with action of $\text{GL}_{2g}(\mathbb{Q})$, that is, a commutative algebra object lying in $\text{CAlg}(\text{Rep}^\otimes_{\text{GL}_{2g}(\mathbb{Q})})$ which we call the motivic algebra of $X$. Let $\text{Mod}^\otimes_A(\text{Rep}^\otimes_{\text{GL}_{2g}(\mathbb{Q})})$ be the symmetric monoidal $\infty$-category of $A_X$-module objects in $\text{Rep}_{\text{GL}_{2g}(\mathbb{Q})}$. Roughly speaking, an object in $\text{Mod}^\otimes_A(\text{Rep}^\otimes_{\text{GL}_{2g}(\mathbb{Q})})$ amounts to $C \in \text{Rep}_{\text{PGL}_{2g}(\mathbb{Q})}$ endowed with a module structure $A_X \otimes C \to C$ whose homotopy coherence is encoded by means of $\infty$-operads; cf. Section 2, [32, 4.4]. The first main result of this paper can be informally stated as follows (cf. Theorem 5.3):

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Theorem 1.1. There exists a natural equivalence of symmetric monoidal ∞-categories
\[ \text{Mod}_{A_X}^\otimes(\text{Rep}_\mathbb{Q} GL_2) \overset{\sim}{\to} \text{DM}_X^\otimes. \]

In addition, \( A_X \) can be expressed by using motivic cohomology; cf. Proposition 5.6. Moreover, the relative version, i.e. the result for abelian schemes, also holds (despite in this Introduction, we discuss abelian varieties).

This theorem can be viewed as a generalization of the well-known representation theorem of the category of mixed Tate motives due to Spitzweck and Levine, cf. [44], [29], [30], which plays one of central roles in the theory of mixed Tate motives.

Our proof of Theorem 1.1 is quite different from [44], [29], [30]. Our approach comes from our tannakian viewpoint in \( \infty \)-categories [13], [22], [23], and the non-commutative algebraic geometry (a la Bondal, Kapranov, Kontsevich, Van den Bergh,...) by which we mean the world where one identifies the derived category \( D(Y) \) of quasi-coherent sheaves of an algebraic variety \( Y \) with the triangulated category of modules over an endomorphism differential graded algebra of a compact generator in \( D(Y) \).

We provide the general notion of perfect symmetric monoidal functors which satisfies a simple set of conditions; this notion appears in Definition 3.1. Our result about perfect symmetric monoidal functors is informally the following (cf. Proposition 3.2):

Proposition 1.2. Let \( D^\otimes \) and \( C^\otimes \) be symmetric monoidal stable presentable \( \infty \)-categories whose tensor products preserve small colimits separately in each variable. Let \( F : D^\otimes \to C^\otimes \) be a symmetric monoidal colimit-preserving functor. If \( F : D^\otimes \to C^\otimes \) is perfect, then one can construct a commutative algebra object \( B \in \text{CAlg}(D^\otimes) \) and an equivalence \( F' : \text{Mod}_B^\otimes(D^\otimes) \overset{\sim}{\to} C^\otimes \) of symmetric monoidal \( \infty \)-categories.

Though we hope that there are applications of Proposition 1.2 to various directions, the notion of perfect symmetric monoidal functors is motivated by the conjectural relation between numerical motives and mixed motives. Our idea here is to apply Proposition 1.2 to numerical motives and \( \infty \)-categories of mixed motives and prove Theorem 1.1. For this, the difficult part is a construction of an appropriate symmetric monoidal functor from the symmetric monoidal derived \( \infty \)-category of numerical motives to \( \text{DM}^\otimes \). Contrary to the natural fully faithful embedding of the homological Chow motives into \( \text{DM}_{\text{km}}(k, \mathbb{Q}) \), an expected functor from the category of numerical motives is a rather non-trivial functor (cf. [1] 21.1.5 for a conjectural description). To get a feeling for this, let us contemplate the symmetric monoidal (tannakian) category \( NM(E)^\otimes \) of numerical motives generated by an elliptic curve \( E \) (see Section 4.1.1). Let \( \text{Ind}(NM(E))^\otimes \) be the symmetric monoidal abelian category of \( \text{Ind} \)-objects and \( \text{D(Ind}(NM(E))^\otimes \) the derived \( \infty \)-category of \( \text{Ind}(NM(E)) \). To apply Proposition 1.2 to our setting, we need to construct a symmetric monoidal exact functor \( \text{D(Ind}(NM(E))^\otimes \to \text{DM}^\otimes \) of symmetric monoidal \( \infty \)-categories. It should send the numerical motives of \( E \) in \( NM(E) \) to \( M_2(E)[-2] \oplus M_1(E)[-1] \oplus M_0(E) \) in an appropriately functorial way, where \( M(E) \simeq M_0(E) \oplus M_1(E) \oplus M_2(E) \) is a (homological) Chow-Künneth decomposition of the motif \( M(E) \) of \( E \) in \( \text{DM}^\otimes \) (cf. Section 4.1.2 and 4.1.3). Moreover, we need to construct \( \text{D(Ind}(NM(E))^\otimes \to \text{DM}^\otimes \) at the level of symmetric monoidal \( \infty \)-categories. Combining with various methods and ideas we construct a symmetric monoidal functor by a step-by-step construction. From an intuitive point of view, we prove the following:

Theorem 1.3. There exists a symmetric monoidal exact functor from the symmetric monoidal derived \( \infty \)-category of numerical motives generated by an abelian varieties \( X \) to \( \text{DM}^\otimes \) which carries \( h_1(X) \) to \( M_1(X)[-1] \). See Section 4.1.2 for the notation \( h_1(X) \).

See Section 4, Theorem 4.11 Lemma 4.11 for the details. Using Proposition 1.2 and Theorem 1.3 we prove Theorem 1.1. We should remark that we introduce the notion of “rigidified”
numerical motives for an abelian variety $X$, called framed numerical/pure motives (cf. Section 5.1.5), and actually we construct a symmetric monoidal exact functor from the symmetric monoidal derived $\infty$-category of framed numerical motives (generated by $X$) to $\mathbf{DM}^\otimes$.

Theorem 1.1 can be viewed as a tannakian type result. In the framework of derive algebraic geometry [33, 39, 48], one can work with the derived affine scheme $\text{Spec} A_X$ endowed with action of $\text{GL}_{2g}(\mathbb{Q})$. We then identify $\text{Mod}_{A_X}^\otimes(\text{Rep}_{\text{GL}_{2g}(\mathbb{Q})})$ (and $\mathbf{DM}_X^\otimes$) with the $\infty$-category of quasi-coherent complexes on the quotient derived stack $[\text{Spec} A_X/\text{GL}_{2g}(\mathbb{Q})]$. In a sense one can think of the quotient $[\text{Spec} A_X/\text{GL}_{2g}(\mathbb{Q})]$ as a “higher classifying stack” whose higher data can be captured by $A_X$. Thus Theorem 1.1 gives a tannakian presentation for $\mathbf{DM}_X^\otimes$. This tannakian and geometric viewpoint is relevant to the second purpose of this paper.

The second purpose of this paper is the study of motivic Galois groups of $\mathbf{DM}_X^\otimes$. It is one major motivation of Theorem 1.1. We refer the reader to e.g. [1, 29, 43] for topics on motivations and structures of motivic Galois groups. In the case of mixed Tate motives, as a consequence of the existence of a motivic $t$-structure on the triangulated category, one has the motivic Galois group of mixed Tate motives by means of classical Tannaka duality. But we do not have a motivic $t$-structure on $\mathbf{DM}_{gm}(k, \mathbb{Q})$ (whose existence is eventually related with Beilinson-Soulé vanishing conjecture, Bloch-Beilinson-Murre filtration and standard conjectures). In order to extract motivic Galois groups directly from $\mathbf{DM}^\otimes$, in [22, 23] we introduced the approach: constructing derived group schemes which represent automorphism functors from $\mathbf{DM}^\otimes$ (the representability is one important reason for the need to work in the framework of $\infty$-categories).

Let us recall the constructions of derived affine group schemes from [22, 23] (here derived affine group schemes are analogues of affine group schemes in derived algebraic geometry). Let $\mathbf{DM}_X^\otimes$ be the symmetric monoidal stable subcategory of $\mathbf{DM}^\otimes$ spanned by dualizable (or compact) objects. The homotopy category of $\mathbf{DM}_X^\otimes$ is equivalent to $\mathbf{DM}_{gm}(k, \mathbb{Q})$ as triangulated categories. Let $\mathbf{DM}^\otimes_{sub}$ be an arbitrary symmetric monoidal stable idempotent complete subcategory of $\mathbf{DM}^\otimes$ which is closed under taking dual objects. Let $K$ be a field of characteristic zero and $\mathcal{D}(K)^\otimes \simeq \text{Mod}^\otimes_{H K}$ the unbounded derived $\infty$-category of $\text{of } K$-vector spaces. Here $\text{Mod}^\otimes_{H K}$ denotes a symmetric monoidal stable $\infty$-category of module spectra over the Eilenberg-Maclane spectrum $H K$. Let $R : \mathbf{DM}^\otimes \to \mathcal{D}(K)^\otimes$ be a homological realization functor associated to Weil cohomology theory with $K$-coefficients, which is a symmetric monoidal exact functor. For example, there are étale, singular (Betti), de Rham realizations; see Section 7, [22, Section 5] based on mixed Weil theories. Let $\text{CAlg}_{H K}$ be the $\infty$-category of commutative ring spectra over $H K$ (in characteristic zero one can take a commutative differential graded algebra as a model). Let $\text{Grp}(S)$ be the $\infty$-category of group objects in the $\infty$-category of $\infty$-groupoids, that is, the $\infty$-category of so-called group-like $A_{\infty}$-spaces. Let $\mathbf{R}_{sub}$ be the composite $\mathbf{DM}^\otimes_{sub} \hookrightarrow \mathbf{DM}^\otimes \overset{R}{\to} \mathcal{D}(K)^\otimes$. Let $\text{Aut}(\mathbf{R}_{sub}) : \text{CAlg}_{H K} \to \text{Grp}(S)$ be a functor which informally sends $A \in \text{CAlg}_{H K}$ to the “automorphism group space” of $\mathbf{DM}^\otimes_{sub} \to \text{Mod}^\otimes_{H K} \to \text{Mod}^\otimes_{A}$ where the second functor is the base change by $H K \to A$. In [22] we have constructed $A_{sub} \subset \text{CAlg}_{H K}$ and a derived affine group scheme $\text{MG}(\mathbf{DM}^\otimes_{sub}, R) := \text{Spec} A_{sub} \text{ over } H K$ that represents $\text{Aut}(\mathbf{R}_{sub})$ when we regard it as a functor $\text{CAlg}_{H K} \to \text{Grp}(S)$. By this representability, we shall refer to $\text{MG}(\mathbf{DM}^\otimes_{sub}, R)$ to the derived motivic Galois group for $\mathbf{DM}^\otimes_{sub}$ with respect to the realization $R$. Then we can pass $\text{Spec} A_{sub}$ to a pro-algebraic group scheme $\text{MG}(\mathbf{DM}^\otimes_{sub}, R)$. The pro-algebraic group scheme $\text{MG}(\mathbf{DM}^\otimes_{sub}, R)$ may be viewed as the coarse moduli space of $\text{MG}(\mathbf{DM}^\otimes_{sub}, R)$ because of a certain universality and its coarse representability; there are functorial isomorphisms of groups $\overline{\text{Aut}}(\mathbf{R}_{sub})(K) \simeq \text{MG}(\mathbf{DM}^\otimes_{sub}, R)(K)$ for any $K$-field $K$ where $\overline{\text{Aut}}(\mathbf{R}_{sub})(K)$ denotes the group of the equivalence classes of objects in $\text{Aut}(\mathbf{R}_{sub})(K)$. We refer to $\text{MG}(\mathbf{DM}^\otimes_{sub}, R)$ as the motivic Galois group for $\mathbf{DM}^\otimes_{sub}$ with respect to $R$. 
The guiding principle behind this construction is that $\text{DM}^\otimes$ equipped with a realization functor should forms a “tannakian $\infty$-category”, and the coarse moduli space of the Tannaka dual of $\text{DM}^\otimes$ should be the Tannaka dual of the heart of a motivic $t$-structure; especially we focus on automorphism groups of fiber functors. If $\text{DM}^\otimes_{\text{sub}}$ is the stable $\infty$-category of mixed Tate motives, then $\text{MG}(\text{DM}^\otimes_{\text{sub}}, \mathbb{R})$ coincides with the conventional motivic Galois group arising from the abelian category of mixed Tate motives [23]. The following table shows the principle of correspondences:

| Category          | Group                      |
|-------------------|----------------------------|
| $\text{DM}^\otimes_{\text{sub}}$ | $\text{MG}(\text{DM}^\otimes_{\text{sub}}, \mathbb{R})$ |
| A conjectural heart | $\text{MG}(\text{DM}^\otimes_{\text{sub}}, \mathbb{R})$ |

Let $\text{MG}$ be the motivic Galois group for $\text{DM}^\otimes$ with respect to a realization functor. It has been conjectured that the structure of motivic Galois group for mixed motives is given by $\text{MG} \simeq U\text{G} \rtimes \text{MG}_{\text{pure}}$ such that $\text{MG}_{\text{pure}}$ is a pro-reductive group scheme which is a Tannaka dual of the (conjecturally tannakian) category of Grothendieck’s numerical motives, and $U\text{G}$ is a pro-unipotent group scheme which is described in terms of motivic complexes; see [11, 1.3.3], [29, 5.3.1]. When $\text{DM}^\otimes_{\text{sub}} = \text{DM}^\otimes_X$ and $R_{\text{et}}$ is the $l$-adic étale realization functor, we write $\text{MG}_{\text{et}}(X)$ for $\text{MG}(\text{DM}^\otimes_X, R_{\text{et}})$. The following results prove this in the case of mixed abelian motives (cf. Theorem 6.8, Corollary 6.9):

**Theorem 1.4.** Let $X$ be an abelian variety over a number field $k$. Let $\text{MG}_{\text{pure}}(X)_{\mathbb{Q}_l}$ be the reductive algebraic group over $\mathbb{Q}_l$ which is the Tannaka dual of the (conjecturally abelian) conjectural category of numerical motives generated by the motives of $X$ (see Section 4.1.2, 4.1.4). Suppose that $X$ satisfies either (i) $\text{End}(X \otimes k \bar{k}) = \mathbb{Z}$ or (ii) $X$ is one dimensional, or (iii) $X$ is a simple CM abelian variety of prime dimension (with some more conditions, see Theorem 6.8 for the precise formulation). Then there exists an exact sequence of pro-algebraic group schemes $1 \to U\text{G}_{\text{et}}(X) \to \text{MG}_{\text{et}}(X) \to \text{MG}_{\text{pure}}(X)_{\mathbb{Q}_l} \to 1$.

Moreover, $U\text{G}_{\text{et}}(X)$ is a connected pro-unipotent group scheme over $\mathbb{Q}_l$ which is constructed from the motivic algebra $A_X$. We refer the reader to Section 6.2 for further details.

**Corollary 1.5.**

(i) There is an isomorphism of affine group schemes $\text{MG}_{\text{et}}(X) \simeq U\text{G}_{\text{et}}(X) \rtimes \text{MG}_{\text{pure}}(X)_{\mathbb{Q}_l}$.

(ii) $U\text{G}_{\text{et}}(X)$ is the unipotent radical of $\text{MG}_{\text{et}}(X)$, i.e., the maximal normal unipotent closed subgroup.

(iii) $\text{MG}_{\text{pure}}(X)_{\mathbb{Q}_l}$ is the reductive quotient.

Recall that there is a conjectural deep and fundamental connection between Grothendieck’s numerical motives and the category $\text{DM}^\otimes$ of mixed motives; the semi-simple portion of the heart of a conjectural motivic $t$-structure on $\text{DM}^\otimes$ should be the category of numerical motives; see [3], [11], [35]. That is, it is conjectured that the full subcategory of semi-simple objects of its conjectural heart is the category of numerical motives and there is a (weight) filtration of every object whose graded quotients are semi-simple objects. It predicts the above conjecture on the structures of motivic Galois groups. Our results provide an affirmative answer to the case of a large important class of mixed abelian motives (in an unconditional way). Corollary 1.5 especially says that one can extract the Tannaka dual of numerical abelian motives from $\text{DM}^\otimes_X$ in the group-theoretic fashion: (i) take a derived affine group scheme which represents $\text{Aut}(R_{\text{et}})$, (ii) then take its coarse moduli space $\text{MG}_{\text{et}}(X)$, (iii) finally, the reductive quotient of $\text{MG}_{\text{et}}(X)$,
i.e., the quotient by the unipotent radical is the Tannaka dual of numerical motives generated by $X$.

The proof uses Theorem [1.1] and theorems on Galois representations as key ingredients. Although derived affine group schemes do not appear in the statement of Theorem [1.4], derived techniques along with these key ingredients play pivotal roles in the proof. For example, it is our feeling that the derived affine group scheme $U_{\text{et}}(X)$ obtained as a bar construction of $A_X$ appearing Section 6 is a more natural object than $UG_{\text{et}}(X)$, which are much well-behaved from various respects (it should be called the “derived unipotent radical”).

**Relation to other works:** Consider the case when $X = E$ is an elliptic curve with no complex multiplication. In [36], [26], Patashnik and Kimura-Terasoma have constructed categories of mixed elliptic motives. Both constructions have built upon the construction of Bloch-Kriz [7] and give delicate (and clever) handmade constructions of differential graded algebras (in [26], a quasi-DGA has been constructed). On the other hand, (quasi-differential graded, triangulated or abelian) categories constructed in [36] and [26] were not compared with $\text{DM}_{\text{gm},E}(k, \mathbb{Q})$ (they seem by no means easy to compare), whereas in the introduction of [26] the authors hope that the their triangulated category coincides with $\text{DM}_{\text{gm},E}(k, \mathbb{Q})$. The comparison between these constructions and ours might be interesting. The construction of $A_E$ here is somewhat abstract since we use $\infty$-categorical setting, but it has an explicit description using motivic cohomology (see Proposition [5.6], Remark [5.7]) and simultaneously it is adequate for homotopical operations.

Let us give some instructions to the reader. In Section 2 we recall some generalities concerning $\infty$-categories, $\infty$-operads, and spectra, etc. In Section 3 we introduce the notion of perfect symmetric monoidal functors and prove Proposition [1.2]. In Section 4, we construct a symmetric monoidal exact functor from the derived $\infty$-category of numerical abelian motives to mixed motives; see Theorem [4.1]. In Section 5 we then construct the motivic algebra $A_X$ of an abelian variety $X$ and prove Theorem [5.3] that appeared as Theorem [1.1] in this introduction. Also, we study $A_X$ explicitly by means of Weyl’s construction. In Section 6 we study the motivic Galois group for $\text{DM}^\otimes$. That is to say, the main results in Section 6 are Theorem [1.4] and Corollary [1.5] in this introduction.

In the final Section, we construct an $l$-adic realization functor from the symmetric monoidal $\infty$-category of mixed motives with $\mathbb{Z}$-coefficients to the derived $\infty$-category of $\mathbb{Z}_l$-modules, which is endowed with Galois action; see Proposition [7.1], Remark [7.2]. This is used in Section 6. But the construction of a realization functor has a different nature from the main objectives of this paper. Thus we treat this issue in the final Section. The author would like to thank S. Yasuda and S. Mochizuki for enlightening questions and valuable comments on the case of abelian schemes.

2. **Notation and Convention**

We fix some notation and convention.

2.1. **$\infty$-categories.** In this paper we use the theory of quasi-categories in a critical way. A quasi-category is a simplicial set which satisfies the weak Kan condition of Boardman-Vogt. The theory of quasi-categories from higher categorical viewpoint has been extensively developed by Joyal and Lurie. Following [31] we shall refer to quasi-categories as $\infty$-categories. Our main references are [31] and [32] (see also [25], [33]). We often refer to a map $S \to T$ of $\infty$-categories as a functor. We call a vertex in an $\infty$-category $S$ (resp. an edge) an object (resp. a morphism). When $S$ is an $\infty$-category, by $s \in S$ we mean that $s$ is an object of $S$. For the rapid introduction to $\infty$-categories, we refer to [31] Chapter 1, [16], [13] Section 2. We remark also that there are several alternative theories such as Segal categories, complete Segal spaces, simplicial categories, relative categories,... etc. We list some of notation concerning $\infty$-categories:
• \( \Delta \): the category of linearly ordered finite sets (consisting of \([0], [1], \ldots, [n] = \{0, \ldots, n\} \ldots \)
• \( \Delta^n \): the standard \( n \)-simplex
• \( N \): the simplicial nerve functor (cf. [31 1.1.5]). We do not often distinguish notationally between ordinary categories and their nerves.
• \( C^{op} \): the opposite \( \infty \)-category of an \( \infty \)-category \( C \)
• Let \( C \) be an \( \infty \)-category and suppose that we are given an object \( c \). Then \( C_{c/} \) and \( C_{c/} \) denote the undercategory and overcategory respectively (cf. [31 1.2.9]).
• \( \text{Cat}_{\infty} \): the \( \infty \)-category of small \( \infty \)-categories in a fixed Grothendieck universe \( U \) (cf. [31 3.0.0.1]). We employ the ZFC-axiom together with the universe axiom of Grothendieck.

2.2. Stable \( \infty \)-categories. We shall employ the theory of stable \( \infty \)-categories developed in [32]. The homotopy category of a stable \( \infty \)-category forms a triangulated category in a natural way, and stable \( \infty \)-categories are thought of as the \( \infty \)-categorically enhanced versions of triangulated categories. For generalities we refer to [32 Chap. 1]. We denote the suspension functor and the loop functor by \( \Sigma \) and \( \Omega \) respectively. For a stable \( \infty \)-category \( C \) and two objects \( C, C' \in C \), we write \( \text{Ext}_{C}(C, C') \) for \( \pi_0(\text{Map}_{C}(C, C')) \). If no confusion seems to arise, we also use the shift \([-] \) instead of \( \Sigma \) and \( \Omega \) when we treat (co)chain complexes.

2.3. Symmetric monoidal \( \infty \)-categories. We use the theory of symmetric monoidal \( \infty \)-categories developed in [32]. We refer to [32] for its generalities. Let \( \text{Fin} \) be the category of marked finite sets (our notation is slightly different from [32]). Namely, objects are marked finite sets and a morphism from \( (n) := \{1 < \cdots < n\} \sqcup \{\ast\} \rightarrow (m) := \{1 < \cdots < m\} \sqcup \{\ast\} \) is a (not necessarily order-preserving) map of finite sets which preserves the distinguished points \( \ast \). Let \( \alpha^{i,n} : (n) \rightarrow (1) \) be a map such that \( \alpha^{i,n}(i) = 1 \) and \( \alpha^{i,n}(j) = \ast \) if \( i \neq j \in (n) \). A symmetric monoidal category is a coCartesian fibration \( p : \mathcal{M}^{\otimes} \rightarrow \text{N}(\text{Fin}_n) \) (cf. [31 2.4]) such that for any \( n \geq 0 \), \( \alpha^{1,n} \ldots \alpha^{n,n} \) induce an equivalence \( \mathcal{M}^{\otimes}_n \rightarrow (\mathcal{M}^{\otimes}_1)^\times n \) where \( \mathcal{M}^{\otimes}_n \) and \( \mathcal{M}^{\otimes}_1 \) are fibers of \( p \) over \((n) \) and \((1) \) respectively. A symmetric monoidal functor is a map \( \mathcal{M}^{\otimes} \rightarrow \mathcal{M}'^{\otimes} \) of coCartesian fibrations over \( \text{N}(\text{Fin}_n) \), which carries coCartesian edges to coCartesian edges.

• \( \text{Mod}_A \): the stable \( \infty \)-category of \( A \)-module spectra for a commutative ring spectrum \( A \). We usually write \( \text{Mod}^{\otimes}_A \) for \( \text{Mod}_A \) equipped with the symmetric monoidal structure given by smash product \( (-) \otimes_A (-) \). See [32 4.4].
• \( \text{CAlg}(\mathcal{M}^{\otimes}) \): \( \infty \)-category of commutative algebra objects in a symmetric monoidal \( \infty \)-category \( \mathcal{M}^{\otimes} \rightarrow \text{N}(\text{Fin}_n) \). See [32 2.13].
• \( \text{CAlg}^R \): \( \infty \)-category of commutative algebra objects in the symmetric monoidal \( \infty \)-category \( \text{Mod}^{\otimes}_R \) where \( R \) is a commutative ring spectrum. We write \( \text{CAlg} \) for the \( \infty \)-category of commutative algebra objects in \( \text{Mod}^{\otimes} \) where \( S \) is the sphere spectrum. The \( \infty \)-category \( \text{CAlg}_R \) is equivalent to the undercategory \( \text{CAlg}^{\text{dis}}_R \) as an \( \infty \)-category. If \( A \) is an ordinary commutative ring, then we denote by \( HA \) the Eilenberg-MacLane spectrum that belongs to \( \text{CAlg} \). We remark that the full subcategory \( \text{CAlg}^{\text{dis}}_{HA} \) of \( \text{CAlg}^{\text{dis}}_{HA} \) spanned by discrete objects \( M \) (i.e., \( \pi_n(M) = 0 \) when \( n \neq 0 \)) is naturally categorical equivalent.
to the nerve of the category of commutative $A$-algebras. The inclusion is given by the Eilenberg-MacLane functor $A \mapsto HA$.

- $\text{Mod}_{A}^{\otimes}(\mathcal{M}^{\otimes}) \rightarrow \text{N}(\text{Fin}_{*})$: symmetric monoidal infinite category of $A$-module objects, where $\mathcal{M}^{\otimes}$ is a symmetric monoidal infinite category such that (1) the underlying infinite category admits a colimit for any simplicial diagram, and (2) its tensor product functor $\mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M}$ preserves colimits of simplicial diagrams separately in each variable. Here $A$ belongs to $\text{CAlg}(\mathcal{M}^{\otimes})$ (cf. [32, 3.3.3, 4.4.2]).

Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal infinite category. We usually denote, dropping the superscript $\otimes$, by $\mathcal{C}$ its underlying infinite category. If no confusion likely arises, we omit the superscript $(-)^{\otimes}$.

We say that an object $X$ in $\mathcal{C}$ is dualizable if there exist an object $X^{\vee}$ and two morphisms $e : X \otimes X^{\vee} \rightarrow 1$ and $c : 1 \rightarrow X \otimes X^{\vee}$ with $1$ a unit such that both compositions

$$X \xrightarrow{id_{X} \otimes c} X \otimes X^{\vee} \otimes X \xrightarrow{e \otimes id_{X}} X, \quad X^{\vee} \xrightarrow{c \otimes id_{X^{\vee}}} X^{\vee} \otimes X \otimes X^{\vee} \xrightarrow{id_{X^{\vee}} \otimes e} X^{\vee}$$

are equivalent to $id_{X}$ and $id_{X^{\vee}}$ respectively. For example, for $R \in \text{CAlg}$, compact and dualizable objects coincide in the symmetric monoidal infinite category $\text{Mod}_{R}^{\otimes}$ (see e.g. [5]). We shall write $\text{PMod}_{R}^{\otimes}$ for symmetric monoidal full subcategory of $\text{Mod}_{R}^{\otimes}$ spanned by dualizable objects. The symmetric monoidal structure of $\mathcal{C}$ induces that of the homotopy category $\text{h}(\mathcal{C})$. If we consider $X$ to be an object also in $\text{h}(\mathcal{C})$, then $X$ is dualizable in $\mathcal{C}$ if and only if $X$ is dualizable in $\text{h}(\mathcal{C})$.

2.4. Model categories and infinite-categories. Our references of model categories are [19] and [31, Appendix]. Let $\mathbb{M}$ be a simplicial model category. Let $\mathbb{M}^{\otimes}$ be the full subcategory of cofibrant-fibrant objects. Then every simplicial hom complex in $\mathbb{M}^{\otimes}$ is a Kan complex and $\text{N}(\mathbb{M}^{\otimes})$ is an infinite category. There is another construction of infinite-categories from model categories. Let $\mathbb{M}$ be an arbitrary combinatorial model category. We regard the full subcategory $\mathbb{M}^{c}$ spanned by cofibrant objects as a usual category and take its simplicial nerve $\text{N}(\mathbb{M}^{c})$. Then there exist an infinite-category $\text{N}_{W}(\mathbb{M}^{c})$ and a functor $\text{N}(\mathbb{M}^{c}) \rightarrow \text{N}_{W}(\mathbb{M}^{c})$ such that for any infinite-category $\mathcal{D}$ the composition induces a fully faithful functor

$$\text{Fun}(\text{N}_{W}(\mathbb{M}^{c}), \mathcal{D}) \rightarrow \text{Fun}(\text{N}(\mathbb{M}^{c}), \mathcal{D})$$

whose essential image is spanned by those functors $\text{N}(\mathbb{M}^{c}) \rightarrow \mathcal{D}$ which carry weak equivalences in $\mathbb{M}^{c}$ to equivalences in $\mathcal{D}$. We refer to $\text{N}_{W}(\mathbb{M}^{c})$ as the infinite category obtained from $\mathbb{M}$ by inverting weak equivalences. One can obtain the infinite-category $\text{N}_{W}(\mathbb{M}^{c})$ by taking a fibrant replacement of the marked simplicial set $(\text{N}(\mathbb{M}^{c}), W^{c})$ in the model category of marked simplicial sets (cf. [31, 3.1.3.7]), where $W^{c}$ is the collection of weak equivalences in $\mathbb{M}^{c}$. This construction can be viewed as Lurie’s version of Dwyer-Kan localization. The restriction to the full subcategory of cofibrant objects is not essential; for example, if $W$ is the collection of weak equivalences in $\mathbb{M}$, then a fibrant replacement of $(\text{N}(\mathbb{M}), W)$ is naturally equivalent to $\text{N}_{W}(\mathbb{M}^{c})$, see [32, 1.3.4.16]. When $\mathbb{M}$ is a simplicial model category, we have $\text{N}(\mathbb{M}^{\otimes}) \simeq \text{N}_{W}(\mathbb{M}^{c})$ (cf. [32, 1.3.4.20]). If $\mathbb{M}$ is a symmetric monoidal model category, $\text{N}_{W}(\mathbb{M}^{c})$ inherits a symmetric monoidal structure (cf. [32, 4.1.3]). See [32, 1.3.4, 4.1.3] for details.

3. Perfect adjoint pair

3.1. Let $\mathcal{C}^{\otimes}$ and $\mathcal{D}^{\otimes}$ be symmetric monoidal stable presentable infinite-categories whose tensor product preserve colimits separately in each variable. Let

$$F : \mathcal{D}^{\otimes} \rightarrow \mathcal{C}^{\otimes}$$

be a symmetric monoidal functor which preserves small colimits. Under this situation, by virtue of the relative version of adjoint functor theorem [32, 8.3.2.6], there exists a right adjoint $G : \mathcal{D}^{\otimes} \rightarrow \mathcal{C}^{\otimes}$ of $F$, which is a lax symmetric monoidal functor. Let $1_{\mathcal{C}}$ be a unit object in $\mathcal{C}^{\otimes}$.
Set $B = G(1_C)$. Then the lax symmetric monoidal functor induces $\text{CAlg}(\mathcal{C}^\otimes) \to \text{CAlg}(\mathcal{D}^\otimes)$ and thus we regard $B$ as an object in $\text{CAlg}(\mathcal{D}^\otimes)$. Let

$$R : \mathcal{D} \rightleftharpoons \text{Mod}_B(\mathcal{D}^\otimes) : U$$

be a adjoint pair where $U$ is the forgetful functor and $R$ assigns a free left $B$-module $B \otimes M$ to any $M \in \mathcal{D}^\otimes$. The both $\mathcal{D}$ and $\text{Mod}_B(\mathcal{D}^\otimes)$ are stable, and $R$ and $U$ are exact (moreover, $U$ preserves small limits and colimits). The functor $R$ can be naturally promoted to a symmetric monoidal functor, and $U$ is promoted to a lax symmetric monoidal functor. The symmetric monoidal functor $F$ induces the composite of symmetric monoidal colimit-preserving functors

$$F' : \text{Mod}_B^\otimes(\mathcal{D}^\otimes) \to \text{Mod}_{F(B)}^\otimes(\mathcal{C}^\otimes) \to \text{Mod}_{1_C}^\otimes(\mathcal{C}^\otimes) \simeq \mathcal{C}^\otimes$$

where the second functor is the base change by the counit map $F \circ G(1_C) \to 1_C$. The composite $F' \circ R : \mathcal{D}^\otimes \to \mathcal{C}^\otimes$ is equivalent to $F$ as symmetric monoidal functors. Let $G' : \mathcal{C}^\otimes \to \text{Mod}_B^\otimes(\mathcal{D}^\otimes)$ be a lax symmetric monoidal functor which is a right adjoint functor of $F'$. The existence of the right adjoint functor follows from the relative version of adjoint functor theorem (see [32 8.3.2.6]). Therefore we have a diagram

All functors are exact. Let $\mathcal{E}$ be a stable presentable $\infty$-categories. We say that a (small) set of objects $\{M_\lambda\}_{\lambda \in \Lambda}$ in $\mathcal{E}$ is a set of compact generators if every $M_\lambda$ is compact in $\mathcal{E}$, and for any $N \in \mathcal{E}$, $N$ is zero whenever $\text{Ext}^n_{\mathcal{E}}(M_\lambda, N) = 0$ for every $\lambda \in \Lambda$ and every integer $n \in \mathbb{Z}$. If $\mathcal{E}$ has a set of compact generators, $\mathcal{E}$ is said to be compactly generated. A family $\{M_\lambda\}_{\lambda \in \Lambda}$ is a set of compact generators if and only if $\mathcal{E}$ is the smallest stable subcategory which has small colimits (which are compatible with colimits in the whole category $\mathcal{E}$) and contains $\{M_\lambda\}_{\lambda \in \Lambda}$. See [31 5.5.7] and [32 1.4.4] for more details.

**Definition 3.1.** We say that a symmetric monoidal colimit-preserving functor $F : \mathcal{D}^\otimes \to \mathcal{C}^\otimes$ between stable presentable $\infty$-categories is perfect if the following conditions hold:

(i) there exists a set of compact generators $\{I_\lambda\}_{\lambda \in \Lambda}$ of $\mathcal{D}$, such that

(ii) each $I_\lambda$ is dualizable,

(iii) $\{F(I_\lambda)\}_{\lambda \in \Lambda}$ is a set of compact generators of $\mathcal{C}$.

If $F : \mathcal{D}^\otimes \rightleftharpoons \mathcal{C}^\otimes : G$ is an adjoint pair such that $F$ is a perfect symmetric monoidal functor, we call it a perfect adjoint pair.

**Proposition 3.2.** If $F : \mathcal{D}^\otimes \to \mathcal{C}^\otimes$ is perfect, then $F' : \text{Mod}_B^\otimes(\mathcal{D}^\otimes) \to \mathcal{C}^\otimes$ gives an equivalence of symmetric monoidal $\infty$-categories.

**Lemma 3.3.** Suppose that $\{I_\lambda\}_{\lambda \in \Lambda}$ is a set of compact generators in $\mathcal{D}$. Then $\{R(I_\lambda)\}_{\lambda \in \Lambda}$ is a set of compact generator in $\text{Mod}_B(\mathcal{D}^\otimes)$.

**Proof.** We first show that $R(I_\lambda)$ is compact. Let $\lim N_i$ is a filtered colimit in $\text{Mod}_B(\mathcal{D}^\otimes)$. Note that by [32 4.2.3.5] $U$ preserves colimits and thus $\lim U(N_i) \simeq U(\lim N_i)$. Then we have natural equivalences

$$\text{Map}_{\text{Mod}_B(\mathcal{D}^\otimes)}(R(I_\lambda), \lim N_i) \simeq \text{Map}_{\mathcal{D}}(I_\lambda, U(\lim N_i))$$

$$\simeq \text{Map}_{\mathcal{D}}(I_\lambda, \lim U(N_i))$$

$$\simeq \lim \text{Map}_{\mathcal{D}}(I_\lambda, U(N_i))$$

$$\simeq \lim \text{Map}_{\text{Mod}_B(\mathcal{D}^\otimes)}(R(I_\lambda), N_i)$$
in $S$. Notice that the third equivalence follows from the compactness of $I_\lambda$. By these equivalences, we conclude that $R(I_\lambda)$ is compact. It remains to prove that if $\text{Ext}_{\text{Mod}_B(D^\otimes)}^n(R(I_\lambda), N) = 0$ for any $\lambda \in \Lambda$ and any integer $n \in \mathbb{Z}$, then $N$ is zero. Since

$$\text{Ext}_{\text{Mod}_B(D^\otimes)}^n(R(I_\lambda), N) \simeq \text{Ext}_{\mathcal{D}}^n(I_\lambda, U(N)) \simeq \text{Ext}_{\mathcal{D}}^n(I_\lambda, U(N)) = 0,$$

our claim follows from the fact that $\{I_\lambda\}_{\lambda \in \Lambda}$ is a compact generator and $\text{Mod}_B(D^\otimes) \to \mathcal{D}$ is conservative. \hfill $\square$

**Proof of Proposition 3.2.** If $F'$ is fully faithful, $F'$ is also essentially surjective. In fact, if $F'$ is fully faithful, the essential image of $F'$ is the smallest stable subcategory of $\mathcal{C}$ which has colimits and contains $F(I_\lambda)$ for all $\lambda \in \Lambda$. By the condition (iii) in Definition 3.1, the essential image of $F'$ coincides with $\mathcal{C}$. Hence we will prove that $F'$ is fully faithful. For this purpose, since $F'$ is an exact functor between stable $\infty$-categories $\text{Mod}_B(D^\otimes)$ and $\mathcal{C}$, by Lemma 5.8 it will suffice to show that $F'$ induces a fully faithful functor between their homotopy categories. We will prove that $F'$ induces a bijection

$$\alpha : \text{Hom}_{\text{Mod}_B(D^\otimes)}(R(I_\lambda), R(\Sigma^n I_\mu)) \to \text{Hom}_C(F'(R(I_\lambda)), F'(R(\Sigma^n I_\mu)))$$

where $\text{Hom}(\cdot, \cdot)$ indicates $\pi_0(\text{Map}(\cdot, \cdot))$ and $n$ is an integer. Note that by adjunction, we have natural bijections

$$\text{Hom}_{\text{Mod}_B(D^\otimes)}(R(I_\lambda), R(\Sigma^n I_\mu)) \simeq \text{Hom}_{\mathcal{D}}(I_\lambda, U(R(\Sigma^n I_\mu)))$$

$$\simeq \text{Hom}_{\mathcal{D}}(I_\lambda \otimes (\Sigma^n I_\mu)^\vee, G(1_\mathcal{C})).$$

Here $(\Sigma^n I_\mu)^\vee$ is the dual of $\Sigma^n I_\mu$. On the other hand, we have natural bijections

$$\text{Hom}_C(F'(R(I_\lambda)), F'(R(\Sigma^n I_\mu))) \simeq \text{Hom}_C(F(I_\lambda), F(\Sigma^n I_\mu))$$

$$\simeq \text{Hom}_C(F(I_\lambda \otimes (\Sigma^n I_\mu)^\vee, 1_\mathcal{C}).$$

Also, by adjunction there is a bijection

$$\beta : \text{Hom}_{\mathcal{D}}(I_\lambda \otimes (\Sigma^n I_\mu)^\vee, G(1_\mathcal{C})) \to \text{Hom}_C(F(I_\lambda \otimes (\Sigma^n I_\mu)^\vee, 1_\mathcal{C})$$

which carries $f : I_\lambda \otimes (\Sigma^n I_\mu)^\vee \to G(1_\mathcal{C})$ to $F(I_\lambda \otimes (\Sigma^n I_\mu)^\vee) \xrightarrow{F(I_\lambda \otimes (\Sigma^n I_\mu)^\vee) \xrightarrow{f}} F(G(1_\mathcal{C})) \to 1_\mathcal{C}$ where the second morphism is the counit map. Therefore, it is enough to identify $\alpha$ with $\beta$ through the natural bijections. Since $F'$ is symmetric monoidal, by replacing $I_\lambda$ and $\Sigma^n I_\mu$ by $I_\lambda \otimes (\Sigma^n I_\mu)^\vee$ and $1_\mathcal{D}$ respectively, we may and will assume that $\Sigma^n I_\mu = 1_\mathcal{D}$. According to the definition, $\alpha$ carries $f : R(I_\lambda) = G(1_\mathcal{C} \otimes I_\lambda) \to R(1_\mathcal{D}) = B$ to

$$1_\mathcal{C} \otimes F(G(1_\mathcal{C})) F \circ U(f) : 1_\mathcal{C} \otimes F(G(1_\mathcal{C})) F(G(1_\mathcal{C}) \otimes I_\lambda) \to 1_\mathcal{C} \otimes F(G(1_\mathcal{C})) F(G(1_\mathcal{C}) \otimes 1_\mathcal{C}) \simeq 1_\mathcal{C}.$$ 

Unwinding the definitions, $\beta$ sends $f : R(I_\lambda) \to R(1_\mathcal{D})$ to the composite

$$F(I_\lambda) \to F \circ U \circ R(I_\lambda) \to F \circ U \circ R(1_\mathcal{D}) = F(G(1_\mathcal{C})) \to 1_\mathcal{C} = F(1_\mathcal{D})$$

where the first functor is induced by the unit $\text{id} \to U \circ R$ together with $F$, the second functor is $F \circ U(f)$, and the third functor is induced by the counit $F \circ G \to \text{id}$. Note that $F(I_\lambda) \to F \circ U \circ R(I_\lambda)$ can be identified with $1_\mathcal{C} \otimes F(G(1_\mathcal{C})) F(G(1_\mathcal{C}) \otimes I_\lambda) \to F(G(1_\mathcal{C}) \otimes I_\lambda)$ induced by the unit $1_\mathcal{C} \to F(G(1_\mathcal{C}))$ of $F(G(1_\mathcal{C})) \in \text{CAlg}(\mathcal{C})$. Now the desired identification with $\beta$ follows from the fact that $F(I_\lambda) \to F \circ U \circ R(I_\lambda) \to 1_\mathcal{C} \otimes F(G(1_\mathcal{C})) F(G(1_\mathcal{C}) \otimes I_\lambda) \simeq F(I_\lambda)$ is the identity (note that $1_\mathcal{C} \to F(G(1_\mathcal{C})) \to 1_\mathcal{C}$ is the identity).

Next we then apply the bijection $\alpha$ to conclude that $F'$ is fully faithful. Since $F'$ preserves colimits (in particular, exact), we see that if $N, M \in \text{Mod}_B(D^\otimes)$ belongs to the smallest stable subcategory $\mathcal{E}$ which contains $\{R(I_\lambda)\}_{\lambda \in \Lambda}$, then $F'$ induces a bijection

$$\text{Hom}_{\text{Mod}_B(D^\otimes)}(N, M) \to \text{Hom}_C(F'(N), F'(M)).$$
There is a categorical equivalence \( \text{Ind}(E) \simeq \text{Mod}_D(D^\otimes) \) which follows from Lemma 3.3 and [31, 5.3.5.11]. Again by [31, 5.3.5.11] and the fact that \( F'(E) \) is compact for any \( E \in E \) (by condition (iii)), a left Kan extension \( \text{Ind}(E) \to C \) induced by \( F' : E \to C \) (cf. [31, 5.3.5.10], [32, 6.3.1.13]) is fully faithful. This implies that \( F' \) is fully faithful.

\[ \square \]

3.2. Let \( A \) be a Grothendieck semisimple abelian category. Here we say that an abelian category is semisimple if any object is a small coproduct of simple objects. Let \( \text{Comp}(A) \) be the category of (unbounded) complexes of objects in \( A \). We suppose that \( \text{Comp}(A) \) is endowed with a combinatorial stable model structure in which weak equivalences are quasi-isomorphisms. Let \( D(A) = N_W(\text{Comp}(A)^c) \) that is a stable presentable \( \infty \)-category. (This assumption is sufficient for our purpose.) Set \( D = D(A) \). Let \( K \) be a field and \( HK \) its Eilenberg-MacLane spectrum. Suppose that \( D \) and \( C \) are \( HK \)-linear stable presentable \( \infty \)-categories. Let \( F : D \to C \) be a \( HK \)-linear colimit-preserving functor in the sense of [23]. (Recall that an \( HK \)-structure is a promotion to an object \( \text{Mod}_{\text{Mod}_{HK}^\otimes}(\text{Cat}_\infty^\otimes) \) such that the action \( \text{Mod}_{HK}^\otimes \times D \to D \) preserves small colimits separately in each variable, where the symmetric monoidal structure of \( \text{Cat}_\infty^\otimes \) is the Cartesian symmetric monoidal structure [32, 2.4.1] and we view \( \text{Mod}_{\text{Mod}_{HK}^\otimes}(\text{Cat}_\infty^\otimes) \) as an algebra object in \( \text{Cat}_\infty^\otimes \). An \( HK \)-linear functor is a morphism in \( \text{Mod}_{\text{Mod}_{HK}^\otimes}(\text{Cat}_\infty^\otimes) \).)

Let \( h(D) \) and \( h(C) \) denote homotopy categories of \( D \) and \( C \) respectively. These are automatically triangulated categories. Furthermore, \( h(D) \) and \( h(C) \) are endowed with the structures of \( K \)-linear categories. That is to say, \( h(D) \) and \( h(C) \) are promoted to categories enriched over the monoidal category of \( K \)-vector spaces (see [31] for enriched categories). Moreover, \( h(F) : h(D) \to h(C) \) is a \( K \)-linear functor.

We suppose that there exists a set of objects \( \{I_\lambda\}_{\lambda \in \Lambda} \) of \( A \) such that \( \{I_\lambda\}_{\lambda \in \Lambda} \) forms a set of compact generator in \( D \), and it satisfies the condition (ii) and (iii) in Definition 3.1. Furthermore, we assume \( \text{Hom}_{h(D)}(I_\lambda, I_\mu) = K \) if \( I_\lambda \simeq I_\mu \), and \( \text{Hom}_{h(D)}(I_\lambda, I_\mu) = 0 \) if otherwise.

Let us recall the definition of hom complexes. Since \( C \) is \( HK \)-linear, we have the tensor structure \( \otimes : \text{Mod}_{HK} \times C \to C \). For \( C \in C \), it gives rise to the colimit-preserving functor \( \text{Mod}_{HK} \simeq \text{Mod}_{HK} \times \{C\} \to C \). By adjoint functor theorem [31, 5.5.2.9], this functor has a right adjoint functor, which we shall denote by \( \text{Hom}_C(C, -) \). By the construction, there is a natural equivalence

\[ \text{Map}_C(M \otimes C, C') \simeq \text{Map}_{\text{Mod}_{HK}}(M, \text{Hom}_C(C, C')) \]

in \( S \). The following Proposition tells us an explicit structure of \( B \) in Proposition 3.2.

**Proposition 3.4.** The right adjoint functor \( h(G) : h(C) \to h(D) \) of \( h(F) \) is given by

\[ C \mapsto \prod_{\lambda \in \Lambda} I_\lambda \otimes \text{Hom}_C(F(I_\lambda), C) \simeq \bigoplus_{\lambda \in \Lambda} I_\lambda \otimes \text{Hom}_C(F(I_\lambda), C) \]

**Proof.** We have natural homotopy equivalences

\[ \text{Map}_D(I_\mu, \prod_{\lambda \in \Lambda} I_\lambda \otimes \text{Hom}_C(F(I_\lambda), C)) \simeq \prod_{\lambda \in \Lambda} \text{Map}_D(I_\mu, I_\lambda \otimes \text{Hom}_C(F(I_\lambda), C)) \]

\[ \simeq \text{Map}_D(I_\mu, \bigoplus_{\lambda \in \Lambda} I_\lambda \otimes \text{Hom}_C(F(I_\lambda), C)) \]

\[ \simeq \text{Map}_{\text{Mod}_{HK}}(HK, \text{Hom}_C(F(I_\mu), C)) \]

\[ \simeq \text{Map}_C(F(I_\mu), C). \]

The set \( \{I_\lambda\}_{\lambda \in \Lambda} \) is a set of compact generators in \( D \), and it follows that \( h(G)(C) \) is equivalent to \( \bigcap_{\lambda \in \Lambda} I_\lambda \otimes \text{Hom}_C(F(I_\lambda), C) \).
To see the right equivalence, it is enough to check that for any $I_\alpha$ and any $n \in \mathbb{Z}$, the natural morphism $\bigoplus_{\lambda \in \Lambda} \lambda \otimes \text{Hom}_C(F(I_\lambda), C) \to \bigcap_{\lambda \in \Lambda} \lambda \otimes \text{Hom}_C(F(I_\lambda), C)$ induces a bijective map

$$\text{Hom}(I_\alpha[n], \bigoplus_{\lambda \in \Lambda} \lambda \otimes \text{Hom}_C(F(I_\lambda), C)) \to \text{Hom}(I_\alpha[n], \bigcap_{\lambda \in \Lambda} \lambda \otimes \text{Hom}_C(F(I_\lambda), C)).$$

of hom sets in $h(\mathcal{D})$. Since $I_\alpha[n]$ is compact, the left hand side is $H^{-n}(\text{Hom}_C(F(I_\alpha), C))$. The right hand side is also $H^{-n}(\text{Hom}_C(F(I_\alpha), C))$, and the induced map is an isomorphism. □

4. Mixed abelian motives

This Section is devoted to the construction of a symmetric monoidal functor from the category of numerical motives generated by an abelian scheme to the category of mixed motives; see Theorem 4.4.1.4.

4.1. Pure and mixed abelian motives.

4.1.1. Homological Chow and numerical motives. We first recall the category of homological Chow motives. Our reference is [11], whereas in loc. cit. the cohomological theory is presented. But we shall adopt the homological theory. For this reason we here give definitions and fix convention for the reader’s convenience. Let $\text{CHM}'$ be the $\mathbb{Q}$-linear category whose objects are formal symbols $(X, i)$ where $X$ is a (not necessarily connected) smooth projective variety over a perfect field $k$ and $i$ is an integer. In this paper, by a smooth variety over $k$ we mean a scheme that is separated of finite type and smooth over $k$. Any object in $\text{CHM}'$ is purely $i$-dimensional. If $Y = \sqcup Y_s$ where each $Y_s$ is connected, then $\text{Hom}_{\text{CHM}'}((X, i), (Y, j)) = (\oplus \text{Hom}_{\text{CHM}}((X, i), (Y_s, j))$. Composition is defined by

$$\text{Hom}_{\text{CHM}'}((X_1, i_1), (X_2, i_2))_\mathbb{Q} \times \text{Hom}_{\text{CHM}'}((X_2, i_2), (X_3, i_3))_\mathbb{Q} \to \text{Hom}_{\text{CHM}'}((X_1, i_1), (X_3, i_3))_\mathbb{Q}$$

which carries $(U, V)$ to $(p_{1,3} (p_{1,2}^* U \cdot p_{2,3}^* V))$ where $p_{i,j} : X_1 \times_k X_2 \times_k X_3 \to X_i \times_k X_j$ is the natural projection. Define $\text{CHM}$ to be the idempotent completion of $\text{CHM}'$. Explicitly, we define $\text{CHM}$ to be its Karoubi envelope. Objects in the Karoubi envelope are pairs $((X, i), p : (X, i) \to (X, i))$ such that $p \circ p = p$. The hom set $\text{Hom}_{\text{CHM}}((X, i), (Y, j), q))$ is the set of morphisms $\phi : (X, i) \to (Y, j)$ such that $\phi \circ p = \phi = q \circ \phi$. This can be identified with $q \circ \text{Hom}_{\text{CHM}'}((X, i), (Y, j)) \circ p$. The composition is defined in the obvious way. The natural functor $\text{CHM}' \to \text{CHM}$ given by $(X, i) \mapsto ((X, i), \text{id})$ is fully faithful. Let $\eta_{X,Y} : ((X \times_k Y, i + j) \to (Y \times_k X, j + i))$ be the isomorphism determined by the flip $X \times_k Y \to Y \times_k X$. The symmetric monoidal structure of $\text{CHM}$ is given by $((X, i), p) \otimes ((Y, j), q) = ((X \times_k Y, i + j), p \otimes q)$ and the commutative constraint $((X, i) \otimes (Y, j), p \otimes q) \to ((Y, j) \otimes (X, i), q \otimes p)$ is induced by $q \otimes p \circ \eta_{X,Y} \circ p \otimes q$. Here $q \otimes p = \eta_{X,Y} \circ p \otimes q \circ \eta_{X,Y}$. If $\text{SmPr}_{/k}$ denotes the category of smooth projective varieties over $k$, there is a natural functor $h : \text{SmPr}_{/k} \to \text{CHM} \to \text{CHM} ; X \mapsto h(X) = (X, 0)$ (any morphism $f : X \to Y$ induces the graph $\Gamma f$ in $X \times_k Y$). We usually regard objects in $\text{CHM}'$ as objects in $\text{CHM}$. We put $L = (\text{Spec } k, 1)$ and let $L^{-1}$ be $(\text{Spec } k, -1)$. We will denote $L^n \otimes (L^{-1})^\otimes m$ by $L^{n-m}$. Any object in $\text{CHM}$ is a direct summand of $(X, i) = h(X) \otimes L^i$ where $i$ is an integer. The symmetric monoidal category $\text{CHM}$ is rigid, that is, every object is dualizable. For a smooth $d$-dimensional projective variety $X$, $h(X) \otimes L^{-d}$ is a dual of $h(X)$. As in [11, Section 6.1], if one replaces smooth projective varieties over the base field $k$ by smooth projective morphisms over a smooth quasi-projective base $k$-scheme $S$ in the above procedure, one obtains the category $\text{CHM}_S$ of relative Chow motives over $S$.

In the above construction (over the base field $k$), if we replace the Chow group $\text{CH}^\dim Y(X \times_k Y)_\mathbb{Q}$ by $\text{CH}^\dim Y(X \times_k Y)_\mathbb{Q}/ \sim_{\text{num}}$, we obtain another symmetric monoidal category $\text{NM}^{\otimes}$ and
the natural symmetric monoidal functor $CHM^\otimes \to NM^\otimes$ where $\sim_{\text{num}}$ indicates the numerical equivalence. We refer to $NM$ as the category of numerical motives or pure motives. We use both “numerical” and “pure” interchangeably. By the theorem of Jannsen [24], $NM$ is a semisimple abelian category. Let $\mathcal{L}$ and $\mathcal{L}^{-1}$ be the image of $L$ and $L^{-1}$ in $NM$.

Let $X$ be a smooth projective variety over $k$ and $h(X)$ the image in $NM$. Let $NM(X)^\otimes$ be the smallest symmetric monoidal abelian subcategory of $NM^\otimes$ which contains $h(X)$ and $h(X)^\vee$ (consequently, it is closed under duals).

4.1.2. Decompositions and symmetric monoidal structures. Let $X$ be an abelian scheme over a smooth quasi-projective $k$-scheme $S$.

Let us recall that if $\Delta \simeq X$ denotes the diagonal in $X \times S X$, then there is a decomposition $[\Delta] = \Sigma_{i=0}^{2g} \pi_i$ in $\text{CH}^0(X \times S X)_Q$ such that $\pi_i \circ \pi_i = \pi_i$ for any $i$, and $\pi_i \circ \pi_j = 0$ for $i \neq j$, due to Manin-Sernescu, Deninger-Murre, and Künemann (see e.g., [28] Section 3). Let $h_i(X)$ be the direct summand of $h(X)$ in $\text{CHM}_S$ corresponding to the idempotent morphism $\pi_i$. We have a natural isomorphism $h(X) \simeq \oplus_{i=0}^{2g} h_i(X)$ in $\text{CHM}_S$. If $[\times n] : X \to X$ denotes the multiplication by $n$, then $[n]$ acts on $h_i(X)$ as the multiplication by $n^i$. The idempotent morphisms $\pi_0$ and $\pi_2g$ are determined by $X \times S e(S)$ and $e(S) \times S X$ respectively, where $e : S \to X$ is a unit of the abelian scheme. In $\text{CHM}_S$, $h_0(X)$ is a unit object, and $h_{2g}(X)$ is isomorphic to $L^g$. When the base scheme $S$ is Spec $k$, this decomposition induces the decomposition $h_i(X) \simeq \oplus_{i=0}^{2g} h_i(X)$ in $NM$, which we call the motivic decomposition. We here remark that $h_i(X)$ is a direct summand of $h_1(X)^\otimes$.

Suppose that $X$ is an abelian variety over $k$. In what follows we will modify the symmetric monoidal structure of $NM(X)^\otimes$ defined above. We change only the commutative constraint. In $NM$, the structure morphism of the commutative constraint

$$\iota : h_1(X)^\otimes \times h_1(X)^\otimes \to h_1(X)^\otimes \otimes h_1(X)^\otimes$$

is induced by the flip $X \times X \to X \times X ; (a, b) \mapsto (b, a)$. We let $(-1)ij_k : h_1(X)^\otimes \otimes h_1(X)^\otimes \to h_1(X)^\otimes \otimes h_1(X)^\otimes$ be a modified commutative constraint. This modification is extended to retracts of $h_1(X)^\otimes$ ($i \in \mathbb{Z}$) in the obvious way. Unless otherwise stated, from now on we equip $NM(X)$ with this modified symmetric monoidal structure. When $NM(X)$ is equipped with this (modified) symmetric monoidal structure, there is a natural isomorphism $h_i(X) \simeq \wedge^i h_1(X)$, where the latter denotes the wedge product. We shall refer to $NM(X)^\otimes$ as the symmetric monoidal abelian category of numerical/pure abelian motives (numerical/pure elliptic motives if $X$ is an elliptic curve) generated by $X$. In this case, the underlying abelian category is generated (as an abelian subcategory) by the set of objects $\{h(X)^\otimes_n \otimes \mathcal{L}^m\}_{n \geq 0, m \in \mathbb{Z}}$ (notice that by the polarization of an abelian variety, $h_1(X)^\otimes_2$ contains $\mathcal{L}$ as a direct summand).

4.1.3. Mixed motives. Let $S$ be a smooth quasi-projective scheme over a perfect field $k$. Let $\text{DM}^\otimes$ be the symmetric monoidal stable presentable $\infty$-category of mixed motives over the base scheme $S$ with rational coefficients. We recall the construction of $\text{DM}^\otimes$ also in subsection 4.3 (see also [22] Section 5) when the base scheme is Spec $k$. There is a natural functor from $M : \text{Sm}_{/S} \to \text{DM}$ where $\text{Sm}_{/S}$ is the nerve of the category of smooth separated $S$-schemes of finite type. For smooth projective $S$-schemes $X, Y$, there is a natural isomorphism

$$\text{Hom}_{h^\otimes(\text{DM})}(M(X), M(Y)) \simeq \text{CH}^d(X \times_S Y)_Q$$

where $Y$ is relatively $d$-dimensional (cf. [9, 11.3.8]). Through this comparison the composition $\text{Hom}_{h^\otimes(\text{DM})}(M(X), M(Y)) \times \text{Hom}_{h^\otimes(\text{DM})}(M(Y), M(Z)) \to \text{Hom}_{h^\otimes(\text{DM})}(M(X), M(Z))$ can be identified with the composition in relative Chow motives since the comparison commutes with flat pullbacks, intersection product [27], and proper push-forwards. Let $X$ be an abelian scheme over $S$ of relative dimension $g$. The decomposition $[\Delta] = \Sigma_{i=0}^{2g} \pi_i$ in $\text{CH}^g(X \times S X)_Q$ described
in Section 4.1.3 induces the decomposition
\[ M(X) \simeq \bigoplus_{i=0}^{2g} M_i(X) \]
such that the multiplication \([\times n] : X \to X\) acts on \(M_i(X)\) as the multiplication by \(n^i\). By [28, 3.3.1], we have a natural equivalence
\[ \Lambda^i(M_1(X)[-1]) \simeq M_i(X)[-i] \]
for any \(i \geq 0\). The 0-th and 2\(g\)-th components \(M_0(X)\) and \(M_{2g}(X)[-2g]\) are isomorphic to the unit and the Tate object \(\mathbb{Q}(g)\) respectively (notice that an isomorphism \(M_{2g}(X) \simeq \mathbb{Q}(g)[2g]\) amounts to an isomorphism \(h_{2g}(X) \simeq L^g\)).

Let \(\text{DM}^\otimes\) be a symmetric monoidal full subcategory of \(\text{DM}^\otimes\) spanned by dualizable objects. When the base scheme \(S\) is \(\text{Spec} \ k\), compact objects and dualizable objects coincides in \(\text{DM}^\otimes\) (since we work with rational coefficients and alteration).

4.1.4. Suppose that \(k\) is a subfield of \(\mathbb{C}\) and we assume that \(X\) is the abelian variety of dimension \(g\) over \(k\). The category \(\text{NM}(X)^\otimes\) is equipped with the realization functor of Betti cohomology which is a symmetric monoidal exact functor \(R_B : \text{NM}(X)^\otimes \to \text{Vect}_\mathbb{Q}^\otimes\), where \(\text{Vect}_\mathbb{Q}^\otimes\) is the symmetric monoidal category of \(\mathbb{Q}\)-vector spaces. The functor \(R_B\) sends \(\mathfrak{h}(X)\) to the dual of the vector space \(H^*(X(\mathbb{C}), \mathbb{Q})\) of the singular cohomology of the complex manifold \(X(\mathbb{C})\). This functor is faithful and it makes \(\text{NM}(X)^\otimes\) a \(\mathbb{Q}\)-linear neutral tannakian category (cf. [1], in this case all Grothendieck standard conjectures hold). Its Tannaka dual \(MG_{pure}(X)\) is a reductive algebraic subgroup in \(\text{GL}_{2g}(\mathbb{Q})\). For a tannakian category equipped with a fiber functor, by its Tannaka dual we mean the pro-algebraic (affine) group scheme that represents the automorphism group of the fiber functor. We have \(MG_{pure}(X) \simeq \text{Aut}(R_B)\) and the closed immersion \(MG_{pure}(X) \hookrightarrow \text{GL}_{2g}(\mathbb{Q})\) is determined by the action of \(MG_{pure}(X) \simeq \text{Aut}(R_B)\) on \(R_B(\mathfrak{h}_1(X)) \simeq \mathbb{Q}^{\otimes 2g}\). For example, if \(X\) is an elliptic curve with no complex multiplication (i.e., \(\text{End}(X) \otimes_k \mathbb{C} = \mathbb{Z}\)), its Tannaka dual is \(\text{GL}_{2}(\mathbb{Q})\). The object \(\mathfrak{h}_1(X)\) corresponds to the natural action on \(\text{GL}_2(\mathbb{Q}) = \text{GL}(H^1(X(\mathbb{C}), \mathbb{Q})^\vee)\) on \(H^1(X(\mathbb{C}), \mathbb{Q})^\vee\).

4.1.5. Framed numerical motives. To illustrate the idea of frames numerical motives, suppose that the base scheme \(S\) is Spec \(k\) such that \(k\) is a subfield of \(\mathbb{C}\). We fix an isomorphism \(R_B(\mathfrak{h}_1(X)) \simeq \mathbb{Q}^{\otimes 2g}\). It naturally gives rise to a closed immersion \(G := MG_{pure}(X) \hookrightarrow \text{GL}_{2g}(\mathbb{Q})\). If \(\text{fVect}_\mathbb{Q}^\otimes(G)\) denotes the symmetric monoidal abelian category of finite dimensional \(\mathbb{Q}\)-linear representations of \(G\), then \(R_B : \text{NM}(X)^\otimes \to \text{Vect}_\mathbb{Q}^\otimes\) naturally factors into an equivalence \(\text{NM}(X)^\otimes \simeq \text{fVect}_\mathbb{Q}^\otimes(G)\) followed by the forgetful functor \(\text{fVect}_\mathbb{Q}^\otimes(G) \to \text{Vect}_\mathbb{Q}^\otimes\). By \(\text{NM}(X)^\otimes \simeq \text{fVect}_\mathbb{Q}^\otimes(G)\) one can identify an object \(M \in \text{NM}(X)\) with \(R_B(M)\) endowed with action of \(G\). Consider a pair \((V, \rho)\) where \(V\) is a finite dimensional \(\mathbb{Q}\)-vector space and \(\rho : \text{GL}_{2g}(\mathbb{Q}) \to \text{Aut}_\mathbb{Q}(V)\) is a homomorphism, that is, an action of \(\text{GL}_{2g}(\mathbb{Q})\) on \(V\). Namely, \((V, \rho)\) belongs to \(\text{fVect}_\mathbb{Q}^\otimes(\text{GL}_{2g}(\mathbb{Q}))\). By the inclusion \(i : G \subset \text{GL}_{2g}(\mathbb{Q})\), we have \((V, \rho \circ i) \in \text{fVect}_\mathbb{Q}^\otimes(G) \simeq \text{NM}(X)\). Consequently, we may say that \((V, \rho)\) is an object in \(\text{NM}(X)\) equipped with an additional structure. There is a natural symmetric monoidal functor \(\text{fVect}_\mathbb{Q}^\otimes(\text{GL}_{2g}(\mathbb{Q})) \to \text{NM}(X)^\otimes\). The standard representation of \(\text{GL}_{2g}(\mathbb{Q})\) (i.e., the action of \(\text{GL}_{2g}(\mathbb{Q})\) on \(\mathbb{Q}^{\otimes 2g}\) determined by the identity \(\text{GL}_{2g}(\mathbb{Q}) \to \text{GL}_{2g}(\mathbb{Q})\)) maps to \(\mathfrak{h}_1(X)\). From this perspective, let us call \(\text{fVect}_\mathbb{Q}^\otimes(\text{GL}_{2g}(\mathbb{Q}))\) the symmetric monoidal abelian category of framed numerical (or pure) motives generated by \(X\).

More generally, for any abelian scheme \(X\) of relative dimension \(g\) over a smooth base variety \(S\) we define the symmetric monoidal abelian category of framed numerical motives of \(X\) to be \(\text{fVect}_\mathbb{Q}^\otimes(\text{GL}_{2g}(\mathbb{Q}))\).

Let \(\text{Ind}(\text{NM}_{fr}(X))^\otimes\) be the symmetric monoidal category of Ind-objects in \(\text{NM}_{fr}(X)\) (cf. [28, 6.3.1.13]). Let \(\mathcal{D}^- (\text{Ind}(\text{NM}_{fr}(X)))^\otimes\) be the symmetric monoidal stable \(\infty\)-category of right
bounded complexes of objects in $\text{Ind}(NM_{fr}(X))^\otimes$. We postpone the definition of $\mathcal{D}^-(\text{Ind}(NM_{fr}(X)))^\otimes$ until Section 4.4.

**Theorem 4.1.** Let $X$ be an abelian scheme of relative dimension $g$ over a smooth quasi-projective $k$-scheme $S$. Then there is a symmetric monoidal exact functor

$$\mathcal{D}^-(\text{Ind}(NM_{fr}(X)))^\otimes \to \text{DM}^\otimes$$

which carries the standard representation $V$ of $\text{GL}_{2g}(\mathbb{Q})$ to $M_1(X)[-1]$.

4.2. **Absolute functor.** In Section 4.2–4.4, we will construct a symmetric monoidal exact functor $\mathcal{D}^-(\text{Ind}(NM_{fr}(X)))^\otimes \to \text{DM}^\otimes$.

Before proceeding into technical details, it seems appropriate to describe some observations. Recall that by Jannsen we know that $NM$ is a symmetric monoidal semisimple abelian category. Suppose that $X$ is an abelian variety over $S = \text{Spec} \ k$. We wish to construct $NM(X)^\otimes \to \text{DM}^\otimes$. For simplicity, suppose that $NM(X)$ is a $k$-linear category and for every simple object $Q$ in $NM(X)$, the endomorphism algebra of $Q$ is the field $k$. Moreover, assume that for every simple object $Q \in NM$ which is a direct summand of $h(X) \otimes L^\otimes m$, there is a direct summand $Q_M$ of $M(X) \otimes k(m)[i]$ in $DM$ which should “correspond to $Q$” (where $[i]$ is some shift). A $k$-linear map $\text{Hom}_{NM}(Q, Q) \simeq k \to \text{Map}_{DM}(Q_M, Q_M)$ is uniquely determined by the condition that it sends the identity morphism to the identity morphism (here $\text{Map}_{DM}(Q_M, Q_M)$ is a $k$-module spectrum). Then we may have a functor $NM(X) \to DM$ which carries $Q$ to $Q_M$. This idea is quite simple, but naive and not sufficient (e.g., invertible objects should map to invertible objects). We need to give a construction in a functorial fashion at the level of symmetric monoidal $\infty$-categories. Moreover, to handle all simple objects in $NM(X)$, we should know the Tannaka dual of $NM(X)$ and its representation theory; in general it would be hopeless. We can remedy this situation by framed numerical motives and additional categories. We will construct an expected composite

$$\mathcal{D}^-(\text{Ind}(NM_{fr}(X)))^\otimes \to \mathcal{D}^-(\text{Ind}(NM(X)))^\otimes \to \text{DM}^\otimes$$

by a step-by-step construction.

4.2.1. **Finite correspondences.** Let $S$ be a smooth quasi-projective scheme over a perfect field $k$. We first recall a $\mathbb{Q}$-linear category $\text{Cor}$, see [34 Lec. 1], [9]. Objects in $\text{Cor}$ are smooth separated $S$-schemes of finite type which we regard as formal symbols. We denote by $\mathcal{L}(X)$ the object in $\text{Cor}$ corresponding to a smooth scheme $X$. For $X$ and $Y$, we let $\text{Hom}_{\text{Cor}}(\mathcal{L}(X), \mathcal{L}(Y))$ be the $\mathbb{Q}$-vector space $c_0(X \times_S Y/X)$ of finite $S$-correspondences (see [9 9.1.2]). The composition is determined by intersection product (see [34 page 4], [9 9.1]). Let $\text{Sm}/S$ be the category of (not necessarily connected) smooth separated $S$-schemes of finite type. Then there is a functor $\mathcal{L} : \text{Sm}/S \to \text{Cor}$ which carries $X$ to $\mathcal{L}(X)$ and sends $f : X \to Y$ to the graph $\Gamma_f \in \text{Hom}_{\text{Cor}}(\mathcal{L}(X), \mathcal{L}(Y))$. Let $\mathcal{L}(X) \otimes \mathcal{L}(Y) = \mathcal{L}(X \times_S Y)$ and define $\gamma_{X,Y} : \mathcal{L}(X) \otimes \mathcal{L}(Y) \to \mathcal{L}(Y) \otimes \mathcal{L}(X)$ to be the isomorphism induced by the graph of the flip $X \times_S Y \to Y \times_S X$. These data makes $\text{Cor}$ a symmetric monoidal category, which shall call as the $\mathbb{Q}$-linear category of finite $S$-correspondences. Next let $\text{Cor}$ be the nerve of functor category consisting of $\mathbb{Q}$-linear functors from $\text{Cor}^{op}$ to $\text{Vect}_G$; $\text{Fun}_G(\text{Cor}^{op}, \text{Vect}_G)$. By enriched Yoneda’s lemma, $\text{Cor}$ can be viewed as the full subcategory of $\widehat{\text{Cor}}$, and every object of $\text{Cor}$ is compact in $\widehat{\text{Cor}}$. Since $\widehat{\text{Cor}}$ admits small colimits, by [34 5.3.5.11] we have a fully faithful left Kan extension $\text{Ind}(\text{Cor}) \to \widehat{\text{Cor}}$. Since $\text{Ind}(\text{Cor})$ admits filtered colimits, it is idempotent complete. Day convolution product defines the symmetric monoidal structure on $\text{Ind}(\text{Cor})$ whose tensor product preserves filtered colimits separately in each variable, and $\mathcal{L}(X) \otimes \mathcal{L}(Y) = \mathcal{L}(X \times_S Y)$ (cf. [32 6.3.1.13]).

Let $\text{Cor}'$ be the Karoubi envelope of $\text{Cor}$ as in the construction in section 4.1. In explicit terms, recall that objects in $\text{Cor}'$ are pairs $(\mathcal{L}(X), p : \mathcal{L}(X) \to \mathcal{L}(X))$ such that $p \circ p = p$. The hom set $\text{Hom}_{\text{Cor}'}((\mathcal{L}(X), p), (\mathcal{L}(Y), q))$ is the set of morphisms $\phi : \mathcal{L}(X) \to \mathcal{L}(Y)$ such...
that $\phi \circ p = \phi = q \circ \phi$. The natural functor $\text{Cor} \to \text{Cor}'$ given by $\mathcal{L}(X) \mapsto (\mathcal{L}(X), \text{id})$ is fully faithful. The symmetric monoidal structure of $\text{Cor}$ inherits that of $\text{Cor}'$; the tensor product is given by $((\mathcal{L}(X), p) \otimes (\mathcal{L}(Y), q) = (\mathcal{L}(X \times_S Y), p \otimes q)$ and the commutative constraint $(\mathcal{L}(X) \otimes \mathcal{L}(Y), p \otimes q) \to (\mathcal{L}(Y) \otimes \mathcal{L}(X), q \otimes p)$ is induced by $q \otimes p \circ \gamma_{X,Y} \circ p \otimes q$. We have a sequence of fully faithful symmetric monoidal functors $\text{Cor} \to \text{Cor}' \to \text{Ind}(\text{Cor})$, since $\text{Ind}(\text{Cor})$ is idempotent complete.

4.2.2. Now we begin to construct a functor.

We say that a $\mathbb{Q}$-linear functor $F : \text{Cor}^{op} \to \text{Vect}_{\mathbb{Q}}$ if a Nisnevich sheaf is the composite $\text{Sm}^{op}_{/ S} \xrightarrow{\mathcal{L}} \text{Cor}^{op} \xrightarrow{F} \text{Vect}_{\mathbb{Q}}$ is a sheaf with respect to Nisnevich topology on $\text{Sm}_{/ S}$ (see e.g. [34] for the definition). For example, $\mathcal{L}(X)$ and its direct summands are sheaves if $X \in \text{Sm}_{/ S}$. Let $\text{Sh} \subset \text{Cor}$ be the full subcategory of sheaves. It is a Grothendieck abelian category (cf. [8, 2.4]). We refer to an object in $\text{Sh}$ as a sheaf with transfers. Let $\text{Comp}(\text{Sh})$ be the category of (not necessarily bounded) complexes of objects in $\text{Sh}$. We continue to assume that $X$ is the abelian $S$-scheme of relative dimension $g$. Let $\mathcal{L}_1(X)$ be a cofibrant object in $\text{Comp}(\text{Sh})$ that represents $M_1(X)$. We here employ the $\mathcal{G}_{\text{Sh}}$-model structure on $\text{Comp}(\text{Sh})$ explained in Section 13.2 but at this point we do not need to know the definition. We shall write $h_1(X)$ for $\mathcal{L}_1(X)$. Consider $h_1(X)[-1]^{\otimes n}$ to be a functor $\alpha : \text{Cor}^{op} \to \text{Comp}(\text{Vect}_{\mathbb{Q}}) ; \mathcal{L}(Y) \mapsto h_1(X)[-1]^{\otimes n}(Y)$. The symmetric group $\mathcal{G}_n$ gives permutation action on $h_1(X)[-1]^{\otimes n}$, and it yields an action on $h_1(X)[-1]^{\otimes n}(Y)$ for each $Y$ in the obvious functorial fashion. Let $\mathbb{Q} [\mathcal{G}_n]$ denote the group algebra. Let $W$ be a left $\mathbb{Q}[\mathcal{G}_n]$-module. Then it gives rise to a functor $\text{Cor}^{op} \to \text{Comp}(\text{Vect}_{\mathbb{Q}})$ given by $\mathcal{L}(Y) \mapsto h_1(X)[-1]^{\otimes n}(Y) \otimes_{\mathbb{Q}[\mathcal{G}_n]} W$.

We denote it by $h_1(X)[-1]^{\otimes n} \otimes_{\mathbb{Q}[\mathcal{G}_n]} W$. When $W$ is a retract (direct summand) of $\mathbb{Q}[\mathcal{G}_n]$, $h_1(X)^{\otimes n} \otimes_{\mathbb{Q}[\mathcal{G}_n]} W$ is a retract of $h_1(X)^{\otimes n}$. Explicitly, the tensor product

\[ (h_1(X)[-1]^{\otimes n} \otimes_{\mathbb{Q}[\mathcal{G}_n]} W) \otimes (h_1(X)[-1]^{\otimes m} \otimes_{\mathbb{Q}[\mathcal{G}_m]} W') \]

is

\[ h_1(X)[-1]^{\otimes n+m} \otimes_{\mathbb{Q}[\mathcal{G}_{n+m}]} \mathbb{Q}[\mathcal{G}_{n+m}] \otimes_{\mathbb{Q}[\mathcal{G}_n \times \mathcal{G}_m]} (W \otimes W') \]

where $\mathbb{Q}[\mathcal{G}_{n+m}]$ is the right $\mathbb{Q}[\mathcal{G}_n \times \mathcal{G}_m]$-module determined by the natural inclusion $\mathcal{G}_n \times \mathcal{G}_m \subset \mathcal{G}_{n+m}$.

We introduce some categories. Let $\mathcal{C}$ be a symmetric monoidal category whose objects are $\mathbb{Q}[\mathcal{G}_n]$ with $n \geq -1$ (regarded as formal symbols). The hom sets form $\mathbb{Q}$-linear vector spaces, and compositions are bilinear. We let $\text{Hom}_{\mathcal{C}}(\mathbb{Q}[\mathcal{G}_n], \mathbb{Q}[\mathcal{G}_m]) = 0$ for $n \neq m$ or $n = -1$ or $m = -1$. For $n = m$, let $\text{Hom}_{\mathcal{C}}(\mathbb{Q}[\mathcal{G}_n], \mathbb{Q}[\mathcal{G}_n])$ be the abelian group of homomorphisms $\mathbb{Q}[\mathcal{G}_n] \to \mathbb{Q}[\mathcal{G}_n]$ as (left) $\mathbb{Q}[\mathcal{G}_n]$-modules. Here by convention $\text{Hom}_{\mathcal{C}}(\mathbb{Q}[\mathcal{G}_0], \mathbb{Q}[\mathcal{G}_0]) = \mathbb{Q}$. Note that $\mathbb{Q}[\mathcal{G}_{-1}]$ is a zero object (thus we may consider $\mathbb{Q}[\mathcal{G}_{-1}]$ to be 0). It is important to distinguish $\mathbb{Q}[\mathcal{G}_1]$ from $\mathbb{Q}[\mathcal{G}_0]$. The tensor product is given by $\mathbb{Q}[\mathcal{G}_n] \otimes \mathbb{Q}[\mathcal{G}_m] := \mathbb{Q}[\mathcal{G}_{n+m}]$ for $n, m \geq 0$. If either $n$ or $m$ is $-1$, then $\mathbb{Q}[\mathcal{G}_n] \otimes \mathbb{Q}[\mathcal{G}_m] = \mathbb{Q}[\mathcal{G}_{-1}]$. For $f : \mathbb{Q}[\mathcal{G}_n] \to \mathbb{Q}[\mathcal{G}_m]$ and $g : \mathbb{Q}[\mathcal{G}_m] \to \mathbb{Q}[\mathcal{G}_n]$, the induced morphism $\mathbb{Q}[\mathcal{G}_n] \otimes \mathbb{Q}[\mathcal{G}_m] \to \mathbb{Q}[\mathcal{G}_n] \otimes \mathbb{Q}[\mathcal{G}_m]$ is given by $\mathbb{Q}[\mathcal{G}_{n+m}] \otimes \mathbb{Q}[\mathcal{G}_n \times \mathcal{G}_m] (f \otimes g)$. The commutative constraint $\mathbb{Q}[\mathcal{G}_n] \otimes \mathbb{Q}[\mathcal{G}_m] = \mathbb{Q}[\mathcal{G}_{n+m}] \to \mathbb{Q}[\mathcal{G}_{n+m}] = \mathbb{Q}[\mathcal{G}_n] \otimes \mathbb{Q}[\mathcal{G}_m]$ is given by the right multiplication by $g : (1, \ldots, n, 1, \ldots, n + m) \mapsto (n + 1, \ldots, n + m, 1, \ldots, n) \in \mathcal{G}_{n+m}$. The unit is $\mathbb{Q}[\mathcal{G}_0]$. Let $\mathcal{C}_1$ be the Karoubi envelope of $\mathcal{C}$ (cf. Section 11.11). An object of $\mathcal{C}_1$ is of the form $(\mathbb{Q}[\mathcal{G}_n], f : \mathbb{Q}[\mathcal{G}_n] \to \mathbb{Q}[\mathcal{G}_n])$, where $f$ is an idempotent homomorphism $\mathbb{Q}[\mathcal{G}_n] \to f(\mathbb{Q}[\mathcal{G}_n]) = W \hookrightarrow \mathbb{Q}[\mathcal{G}_n]$ with a direct summand $W$ as a $\mathbb{Q}[\mathcal{G}_n]$-module. By convention $\mathbb{Q}[\mathcal{G}_{-1}]$, $\mathbb{Q}[\mathcal{G}_0]$ and $\mathbb{Q}[\mathcal{G}_1]$ have no non-trivial direct summand. Let $\mathcal{C} := \text{Ind}(\mathcal{C}_0)$ be the $\text{Ind}$-category which is endowed with the symmetric monoidal structure (cf. Section 27). The tensor product preserves (small) colimits in each variable. The $\text{Ind}(\mathcal{C}_0)$ is a presentable (and Grothendieck) abelian category, and by
semi-simplicity of the representation of symmetric groups (in characteristic zero) an object in $C$ is a (possibly infinite) coproduct of objects in $C_1$. We have the sequence of fully faithful symmetric monoidal functors $C_0 \subset C_1 \subset C$. Roughly speaking, these categories are $\mathbb{Q}$-linear envelopes (in suitable senses) of the free symmetric monoidal category generated by the discrete category having one object. In the next subsection, they will be related with the category of representations in $\text{GL}_2(\mathbb{Q})$, which appears in an interesting way.

Let $\{W_{n,i}\}_i$ be the set of irreducible representations of $\mathfrak{S}_n$. Using the representation theory of symmetric groups (see e.g. [15, Lecture 4]) one can take $W_{n,i}$ to be $\mathbb{Q}[\mathfrak{S}_n]c_{\lambda_i}$ where $c_{\lambda_i} \in \mathbb{Q}[\mathfrak{S}_n]$ is the Young symmetrizer associated to a Young tableau $\lambda_i$. The Young symmetrizer $c_{\lambda_i}$ satisfies $c_{\lambda_i}c_{\lambda_i} = (n!/ \dim W_{n,i})c_{\lambda_i}$ and

$$Q[\mathfrak{S}_n] \to \mathbb{Q}[\mathfrak{S}_n]c_{\lambda_i} \hookrightarrow \mathbb{Q}[\mathfrak{S}_n]$$

is an idempotent morphism, where the first map is given by the multiplication by $(n!/ \dim W_{n,i})c_{\lambda_i}$. For each $n \geq 0$ we fix a set of Young tableaux $\{\lambda_i^{(n)}\}_i$ such that $W_{n,i} = \mathbb{Q}[\mathfrak{S}_n]c_{\lambda_i^{(n)}}$, and we usually omit the superscript $(n)$. Then there is the natural isomorphism $\mathbb{Q}[\mathfrak{S}_n] \cong \bigoplus_i \text{End}(W_{n,i})$ which carries $\Sigma g \in \mathfrak{S}_n, a_gg$ to $\Sigma \rho_i g \rho_i(g)$ where $\rho_i : \mathfrak{S}_n \to \text{End}(W_{n,i})$ denotes the representation. Hence for any $n \geq 0$ we have natural isomorphisms in $C$,

$$\mathbb{Q}[\mathfrak{S}_n] \cong \bigoplus_i \text{End}(W_{n,i}) \cong \bigoplus_i W_{n,i} \otimes W_{n,i}^\vee \cong \bigoplus_i W_{n,i} \otimes \dim W_{n,i}$$

where $W_{n,i}^\vee$ is the dual of the underlying vector space of $W_{n,i}$. We will henceforth identify $\mathbb{Q}[\mathfrak{S}_n]$ with the right hand side through this isomorphism. In addition, we abuse notation and write $0$ for $\mathbb{Q}[\mathfrak{S}_{-1}]$.

Now we construct a symmetric monoidal functor $\Phi : C \to \text{Comp(Sh)}$. We let $\Phi(W_{n,i})$ be $h_1(X)[-1]^{\otimes n} \otimes_{\mathbb{Q}[\mathfrak{S}_n]} W_{n,i}$. We set $\Phi(\mathbb{Q}[\mathfrak{S}_{-1}]) = 0$. Let $\text{Vect}_{\mathbb{Q}[\mathfrak{S}_n]}$ be the category of left $\mathbb{Q}[\mathfrak{S}_n]$-modules. Let

$$\text{Hom}_{\text{Vect}_{\mathbb{Q}[\mathfrak{S}_n]}}(W_{n,i}, W_{n,i}) \quad \Phi_{n,i} := h_1(X)[-1]^{\otimes n} \otimes_{\mathbb{Q}[\mathfrak{S}_n]}(-)$$

$$\text{Hom}_{\text{Comp(Sh)}}(h_1(X)[-1]^{\otimes n} \otimes_{\mathbb{Q}[\mathfrak{S}_n]} W_{n,i}, h_1(X)[-1]^{\otimes n} \otimes_{\mathbb{Q}[\mathfrak{S}_n]} W_{n,i})$$

be a linear map which carries $f : W_{n,i} \to W_{n,i}$ to

$$h_1(X)[-1]^{\otimes n} \otimes_{\mathbb{Q}[\mathfrak{S}_n]} W_{n,i} \to h_1(X)[-1]^{\otimes n} \otimes_{\mathbb{Q}[\mathfrak{S}_n]} W_{n,i}$$

induced by $\text{id} \otimes f$. We remark that $\text{Hom}_{\text{Vect}_{\mathbb{Q}[\mathfrak{S}_n]}}(W_{n,i}, W_{n,i})$ is isomorphic to $\mathbb{Q}$. Since $\text{Hom}_C(\mathbb{Q}[\mathfrak{S}_n], \mathbb{Q}[\mathfrak{S}_n])$ is

$$\bigoplus_i \text{Hom}_C(W_{n,i}, W_{n,i}) \otimes W_{n,i} \otimes W_{n,i}^\vee,$$

$\Phi_{n,i} \otimes W_{n,i} \otimes W_{n,i}^\vee$ defines a linear map

$$\text{Hom}_C(\mathbb{Q}[\mathfrak{S}_n], \mathbb{Q}[\mathfrak{S}_n]) \to \text{Hom}_{\text{Comp(Sh)}}(h_1(X)[-1]^{\otimes n}, h_1(X)[-1]^{\otimes n})$$

where $h_1(X)[-1]^{\otimes n}$ is identified with $\bigoplus_i (h_1(X)[-1]^{\otimes n} \otimes_{\mathbb{Q}[\mathfrak{S}_n]} W_{n,i}) \otimes (W_{n,i})^\vee$. We note that if $n \neq m$ or $n = -1$ of $m = -1$, $\text{Hom}_C(\mathbb{Q}[\mathfrak{S}_n], \mathbb{Q}[\mathfrak{S}_m])$ is zero. Consequently, these data $\{\Phi(-), \Phi_{n,i}\}$ defines a functor $C_1 \to \text{Comp(Sh)}$ which carries $\mathbb{Q}[\mathfrak{S}_n]$ to $h_1(X)[-1]^{\otimes n}$. By a left Kan extension (cf. [31, 5.3.5.10]), it is extended to $C \to \text{Comp(Sh)}$. We shall denote this functor by $\Phi$.

**Lemma 4.2.** The functor $\Phi : C \to \text{Comp(Sh)}$ is (promoted to) a symmetric monoidal functor.
Proof. It will suffice to prove that the restriction of $\Phi$ to $C_0$ is promoted to a symmetric monoidal functor (the extension to the Karoubi envelope $C_1$ and to $C$ is automatic by left Kan extensions). By construction, $\Phi(Q[E_n] \otimes Q[E_m]) = h_1(X)[-1]^{\otimes n+m} = \Phi(Q[E_n] \otimes \Phi(E_m))$, and thus we see that $\Phi$ is monoidal. It remain to show the compatibility of commutative constraints. Let $\gamma_{n,m} : Q[E_n] \otimes Q[E_m] \rightarrow Q[E_{n+m}]$ be the commutative constraint in $C$. Let $\alpha_{n,m} : h_1(X)[-1]^{\otimes n} \otimes h_1(X)[-1]^{\otimes m} \rightarrow h_1(X)[-1]^{\otimes n+m} \otimes h_1(X)[-1]^{\otimes m}$ be the commutative constraint in $\text{Comp}(\text{Sh})$. We need to show that $\Phi(\gamma_{n,m}) = \alpha_{n,m}$. Note that $\gamma_{n,m}$ is

$$
\oplus_i (Q[S_n] \otimes S_m) W_{n+m,i} \otimes W_{n+m,i}^\vee 
$$

induced by $\oplus_i (Q[S_n] \otimes S_m) \rho_i(g) \otimes W_{n+m,i}^\vee$ where $\rho_i(g) : W_{n+m,i} \rightarrow W_{n+m,i}$ is determined by the action on $W_{n+m,i}$ of $g : (1, \ldots, n, n+1, \ldots, n+m) \mapsto (n+1, \ldots, n+m, 1, \ldots, n) \in S_{n+m}$. Therefore $\Phi(\gamma_{n,m})$ is

$$
\oplus_i (h_1(X)[-1]^{\otimes n+m} \otimes Q[S_{n+m}] W_{n+m,i}) \otimes (W_{n+m,i})^\vee 
$$

induced by $\oplus_i (h_1(X)[-1]^{\otimes n+m} \otimes Q[S_{n+m}] (\rho_i(g))) \otimes W_{n+m,i}^\vee$. On the other hand, by the definition $\alpha_{n,m}$ is

$$
\oplus_i (h_1(X)[-1]^{\otimes n+m} \otimes Q[S_{n+m}] W_{n+m,i}) \otimes (W_{n+m,i})^\vee 
$$

induced by $\oplus_i (h_1(X)[-1]^{\otimes n+m} \otimes Q[S_{n+m}] (\rho_i(g))) \otimes (W_{n+m,i})^\vee$. Hence $\Phi(\gamma_{n,m}) = \alpha_{n,m}$. \hfill \Box

By Lemma 4.2 we have

**Corollary 4.3.** There is a symmetric monoidal functor

$$
\Phi : C \rightarrow \text{Comp}(\text{Sh})
$$

which carries $Q[E_1]$ to $h_1(X)[-1]$ and preserves coproducts.

**Remark 4.4.** It seems difficult to construct a symmetric monoidal functor directly from $NM_{fr}(X)$ to $\text{Comp}(\text{Sh})$. It is because $\wedge^r h_1(X)[-1]$ is not necessarily zero in $\text{Comp}(\text{Sh})$ for $r > 2g$ whereas $\wedge^r V$ is zero for $r > 2g$ where $V$ is the standard representation of $GL_{2g}(Q)$. The category $C$ remedies this problem. We note that in Corollary 4.3 the condition that $X$ is an abelian scheme is not used.

### 4.3. Localizations and the functor from pure motives

In this subsection, using Corollary 4.3 we construct a symmetric monoidal functor from the category of framed numerical motives to $\text{DM}$.

#### 4.3.1. We shall introduce some notation. Let $\hat{h}_1(X)$ denote the standard representation of $GL_{2g}(Q)$ that belongs to $NM_{fr}(X) = i\text{Vect}_Q(GL_{2g}(Q))$. (By this notation, we emphasize that the standard representation is regarded as the framed numerical motive.) Let $\mathcal{N}_{s,f}$ be the symmetric monoidal full subcategory of $NM_{fr}(X)$ spanned by finite coproducts of objects of the form

$$
\hat{h}_1(X)^{\otimes n} \otimes Q[E_n] W
$$

where $W$ is a direct summand of $Q[E_n]$ (as a $Q[E_n]$-module). The $\hat{h}_1(X)^{\otimes n} \otimes Q[E_n] W$ can be viewed as the direct summand of $\hat{h}_1(X)^{\otimes n}$ induced by an idempotent map $Q[E_n] \rightarrow W \hookrightarrow Q[E_n]$. Let $\mathcal{N}_s := \text{Ind}(\mathcal{N}_{s,f})$. Every object in $\mathcal{N}_s$ is a coproduct of (simple) objects in $\mathcal{N}_{s,f}$. It is a Grothendieck abelian category. The category $\mathcal{N}_s$ is naturally embedded into $\mathcal{N} := \text{Vect}(GL_{2g}(Q))$ as a symmetric monoidal full subcategory. Here $\text{Vect}_Q(G)$ is the category of (not necessarily finite dimensional) $Q$-linear representations of an algebraic group $G$. There
is a natural equivalence \( \text{Ind}(N M_{fr}(X)) \simeq \text{Vect}_Q(\text{GL}_2(Q)) \) of symmetric monoidal categories. We let \( N_f \) be the full subcategory of \( N \) spanned by finite coproducts of simple objects in \( N \). The \( N_f \) coincides with \( N M_{fr}(X) \).

4.3.2. Let us recall from \[8, 4.3\] the descent structure of \( \text{Sh} \). Let \( \mathcal{G}_{\text{Sh}} \) be the set of \( \{ \mathcal{L}(X) \}_{X \in \text{Sm}/S} \). Let \( \mathcal{H}_{\text{Sh}} \) be the set of complexes obtained as cones of \( \mathcal{L}(X) \to \mathcal{L}(X) \) where \( X \in \text{Sm}/S \) and \( X \to X \) is any Nisnevich hypercovering of \( X \). Then \( \mathcal{G}_{\text{Sh}}, \mathcal{H}_{\text{Sh}} \) is weakly flat descent structure (see \[8, 3.3\]). Then \( \text{Comp}(\text{Sh}) \) has a symmetric monoidal model structure, described in \[8, 2.5, 3.2\], in which weak equivalences are quasi-isomorphisms, and cofibrations are \( \mathcal{G}_{\text{Sh}} \)-cofibrations. We call this model structure the \( \mathcal{G}_{\text{Sh}} \)-model structure. Next put \( \mathcal{T} \) be the set of complexes of sheaves with transfers obtained as cones of \( p_\ast : \mathcal{L}(X \times_S A^1_S) \to \mathcal{L}(X) \) for any \( X \in \text{Sm}/S \), where \( A^1_S \) is the affine line over \( S \), and \( p : X \times_S A^1_S \to X \) is the natural projection. Invoking \[8, 4.3, 4.12\] we take the left Bousfield localization of the above model structure on \( \text{Comp}(\text{Sh}) \) with respect to \( \mathcal{T} \), in which weak equivalences are called \( A^1 \)-local equivalences, and fibrations are called \( A^1 \)-local fibrations. Let \( \mathcal{Q}(1) \) be \( \text{Ker}(\mathcal{L}(\mathcal{G}_{m,S}) \to \mathcal{L}(S))[-1] \) where \( \mathcal{G}_{m,S} = \text{Spec} \mathcal{O}_S[t, t^{-1}] \). Consider the symmetric monoidal category \( \text{Sp}_{\mathcal{Q}(1)}^{\Sigma}(\text{Comp}(\text{Sh})) \) of symmetric \( \mathcal{Q}(1) \)-spectra (we shall refer the reader to \[20\] for the generalities of symmetric spectra). By \[8, 7.9\] (see also \[20\]), \( \text{Sp}_{\mathcal{Q}(1)}^{\Sigma}(\text{Comp}(\text{Sh})) \) has a symmetric monoidal model structure such that weak equivalences (resp. fibrations) are termwise \( A^1 \)-local equivalences (resp. \( A^1 \)-local fibrations). We refer to this model structure as the \( A^1 \)-local projective model structure. Following \[8, 7.13\] and \[20\] we define the stable model structure on \( \text{Sp}_{\mathcal{Q}(1)}^{\Sigma}(\text{Comp}(\text{Sh})) \) to be the left Bousfield localization with respect to \( \{ s_n^{\mathcal{Q}(X)} : F_{n+1}(\mathcal{L}(X) \otimes \mathcal{Q}(1)) \to F_n(\mathcal{L}(X)) \}_{X \in \text{Sm}/S} \) of the \( A^1 \)-local projective model structure (see \[20, 7.7\] for the notation \( s_n^{\mathcal{L}(X)}, F_n, \text{etc.} \)). We refer to a weak equivalence (resp. fibration) in the stable model structure as a stable equivalence (resp. stable fibration). We let \( \text{Sp}_{\mathcal{Q}(1)}^{\Sigma}(\text{Comp}(\text{Sh}))^{\otimes} \) be the symmetric monoidal \( \infty \)-category obtained from (the full subcategory of cofibrant objects of) \( \text{Sp}_{\mathcal{Q}(1)}^{\Sigma}(\text{Comp}(\text{Sh})) \) by inverting stable equivalences. Set \( \text{DM}^{\otimes} := \text{Sp}_{\mathcal{Q}(1)}^{\Sigma}(\text{Comp}(\text{Sh}))^{\otimes} \) (we use the notation \( \text{DM}^{\otimes}(k) \) in \[22\]). We abuse notation and often write \( \mathcal{Q}(1) \) also for the image of \( \mathcal{Q}(1) \) in \( \text{DM}^{\otimes} \) (and its homotopy category).

Let \( N_{W}^{\otimes}(\text{Comp}(\text{Sh})^{\circ}) \) be the symmetric monoidal \( \infty \)-category obtained from \( \text{Comp}(\text{Sh})^{\circ} \) by inverting \( A^1 \)-local equivalences. Put \( \text{DM}^{\text{eff},\otimes} := N_{W}^{\otimes}(\text{Comp}(\text{Sh})^{\circ}) \). There is a natural symmetric monoidal functor \( \Sigma^\infty : \text{DM}^{\text{eff},\otimes} \to \text{DM}^{\otimes} \) obtained from the left Quillen adjoint functor \( \text{Comp}(\text{Sh}) \to \text{Sp}_{\mathcal{Q}(1)}^{\Sigma}(\text{Comp}(\text{Sh})) \).

4.3.3. Let \( W_{2g} = \{ Z \to 0 \}_{Z \in \Lambda} \) be the subset of morphisms in \( \mathcal{C} \) such that \( \Lambda \) is the set of objects \( Z \) of the form: a simple \( \mathbb{Q}[\mathcal{S}_n] \)-module which corresponds to a Young diagram having \( r \) rows with \( r > 2g \), that is, there is a Young tableau \( \lambda \) having \( r \) rows and \( n \) boxes with \( r > 2g \) and \( Z \cong \mathbb{Q}[\mathcal{S}_n] c_\lambda \) (we shall refer to such a simple object as a \( 2g \)-acyclic object). According to \[31, 5.5.4.15\] we obtain the localization \( \mathcal{C} = \mathcal{C}_{W_{2g}} \) (cf. \[31, 5.2.7.2\]), where \( \mathcal{C}_{W_{2g}} \) can be identified with the full subcategory of \( \mathcal{C} \) spanned by \( W_{2g} \)-local objects (see \[31, 5.5.4.1\] for local objects) through the fully faithful right adjoint \( \mathcal{C}_{W_{2g}} \to \mathcal{C} \). In this case, \( W_{2g} \)-local objects in \( \mathcal{C} \) are objects which have no \( 2g \)-acyclic object as a direct summand.

4.3.4. Let us briefly recall the representation theory of \( \text{GL}_n(Q) \) by means of Schur-Weyl’s construction. Our reference is \[15, \text{Lecture 6, 15.5}\]. For a Young tableau \( \lambda \) with \( l \) boxes and the associated Young stabilizer \( c_\lambda \in \mathbb{Q}[\mathcal{S}_l] \) (see \[15, \text{Lecture 4.1}\]), \( S_\lambda V := V^{\otimes \lambda} \otimes_{Q[\mathcal{S}_l]} Q[c_\lambda] \) is an irreducible representation of \( \text{GL}_n(Q) \) (via the left multiplication), where \( V \) is the standard representation of \( \text{GL}_n(Q) \), i.e., the natural action of \( \text{GL}_n(Q) \) on \( V = Q^{\otimes n} \). The isomorphism class of \( S_\lambda V \) as a representation does not depend on the choice of Young tableaux but only
on the underlying Young diagram. The dimension of $S_\lambda V$ can be calculated in terms of the Young diagram; [15] Theorem 6.3. The representation $S_\lambda V$ is zero if and only if the number of rows of $\lambda$ is strictly larger than $n$. If both $\lambda$ and another Young tableau $\mu$ have at most $n$ rows, then $S_\lambda V \simeq S_\mu V$ (as representations of $GL_n(\mathbb{Q})$) if and only if both underlying Young diagrams coincide. An irreducible representation is not necessarily of this form, but every irreducible representation has (up to isomorphism) the form $S_\lambda V \otimes ((\Lambda^n V)^*)^s$ with $s \geq 0$. Here $(-)^*$ indicates a dual representation. The tensor product $S_\lambda V \otimes S_\mu V$ of two irreducible representations is decomposed into a coproduct of irreducible representations. One can compute the decomposition in terms of Young diagrams by the Littlewood-Richardson rule. There is the Littlewood-Richardson rule for the tensor product $Q[\mathfrak{S}_1+m] \otimes Q[\mathfrak{S}_n] \otimes Q[\mathfrak{S}_m]_{c\mu}$, and it implies the Littlewood-Richardson rule for $S_\lambda V \otimes S_\mu V$; see [21], [14, Section 7.3], [15, Lecture 4.3, 6].

4.3.5.

**Lemma 4.5.** The functor $\Phi : C \to \text{Comp(Sh)}$ sends objects to cofibrants. Moreover, it carries $W_2g$ to $A^1$-local equivalences.

**Proof.** Note that any object $C_0$ maps to a cofibrant object since any tensor power $h_1(X)[-1]^*n$ is cofibrant. Thus any object in $C$ maps to a coproduct of retracts of cofibrant objects. This implies the first assertion. To prove the second assertion, it will suffice to show that for any 2g-acyclic object $Z$, $\Phi(Z) \to 0$ is an $A^1$-local equivalence. Let $Z = W_{n,i}$ be a 2g-acyclic object where $\lambda_i^{(n)}$ is a Young tableau with $r(> 2g)$ rows and $W_{n,i} = Q[\mathfrak{S}_n]C_{\lambda_i^{(n)}}$. Let $W_\alpha$ be the simple $Q[\mathfrak{S}_\alpha]$-module corresponding to the Young diagram $\alpha$ having $r$ rows and $r$ boxes, i.e., it amounts to the partition $(1, 1, \ldots, 1)$. Let $W_\beta$ be the simple $Q[\mathfrak{S}_\beta]$-module corresponding to the Young diagram $\beta := \lambda_i^{(n)} - \alpha$ rearranged in left-justified rows. In virtue of Littlewood-Richardson rule, we see that $W_{n,i}$ is (isomorphic to) a direct summand of $W_\alpha \otimes W_\beta$. For $r > 2g$, $\Lambda^r(M_1(X)[-1]) \simeq 0$ in $DM_{eff}$. It follows that $\Phi(W_\alpha) = \Phi(\Lambda^r Q[\mathfrak{S}_n]) = \Lambda^r(h_1(X)[-1]) \simeq 0$ in $DM_{eff}$. Hence $\Phi(W_{n,i}) \simeq 0$ in $DM_{eff}$. \hfill \Box

If $Z$ is 2g-acyclic and $Z'$ is another simple $Q[\mathfrak{S}_m]$-module, by Littlewood-Richardson rule $Z \otimes Z'$ is the coproduct of 2g-acyclic objects. Thus applying [32] 4.1.3.4 we see that $C_{W_{2g}}$ is endowed with a symmetric monoidal structure and $C \to C_{W_{2g}}$ is promoted to a symmetric monoidal functor. According to Lemma 4.5 and the universality of $C_{W_{2g}}$; [32] 4.1.3.4, the symmetric monoidal functor $C \otimes \to N_{W_{2g}}(\text{Comp(Sh)}^r) = DM_{eff, \otimes}$ induced by $\Phi$ gives rise to a symmetric monoidal functor $C_{W_{2g}} \otimes \to DM_{eff, \otimes}$. Here we will omit the nerve of $C$ and $C_{W_{2g}}$.

**Proposition 4.6.** There is a natural equivalence $C_{W_{2g}} \otimes \to N_{s}^\otimes$ as symmetric monoidal categories.

**Proof.** We first construct a symmetric monoidal functor $C_{W_{2g}} \to N_{s}$. To this end, we construct a symmetric monoidal functor $F : C \to N_{s}$. For a retract $W$ of $Q[\mathfrak{S}_n]$ for $n \geq 0$, we define $F(W)$ to be $\hat{h}_1(X)^{\otimes n} \otimes Q[\mathfrak{S}_n] W$. Set $F(Q[\mathfrak{S}_1]) = 0$. As in the proof of Lemma 4.2 we see that $C_1 \to N_{s}$ is promoted to a symmetric monoidal functor. By a left Kan extension, it is extended to a symmetric monoidal functor $F : C \otimes \to N_{s}^\otimes$ which preserves filtered colimits. Note here that according to Schur-Weyl construction (cf. Section 3.4.3), for any 2g-acyclic simple object $W$, $F(W) = \hat{h}_1(X)^{\otimes n} \otimes Q[\mathfrak{S}_n] W$ is zero. Consequently, by the universality of $C_{W_{2g}}$ (cf. [32] 1.3.4, 4.1.3) it gives rise to a symmetric monoidal functor $F' : C_{W_{2g}} \to N_{s}^\otimes$ which preserves coproducts. Observe that $F'$ is essentially surjective. To see this, note that by Schur-Weyl construction and the semi-simplicity of representations of $GL_2(\mathbb{Q})$, any object in $N_{s}$ is (isomorphic to) a coproduct of objects of the form $\hat{h}_1(X)^{\otimes n} \otimes Q[\mathfrak{S}_n] W_{n,i}$. It follows that $F'$
is essentially surjective. Finally, we show that $F'$ is fully faithful, that is, $\text{Hom}_{\mathcal{W}_{2g}}(W, W') \xrightarrow{\sim} \text{Hom}_{\mathcal{N}_s}(F'(W), F'(W'))$. Taking account of the semi-simplicity of the symmetric groups and general linear groups and the compactness of simple objects in $\mathcal{C}$ and $\mathcal{N}_s$, we may and will assume that both $W$ and $W'$ are simple, and $W = W' = W_{n,i} = \mathbb{Q}[\mathfrak{S}_n]c_{\chi_i}^{(n)}$ for some $n$ and $\lambda_i^{(n)}$ (cf. Section 4.3.4). Then $\text{Hom}_{\mathcal{N}_s}(F'(W), F'(W)) \simeq \mathbb{Q}$, and $\mathbb{Q} \simeq \text{Hom}_{\mathcal{W}_{2g}}(W, W) \rightarrow \text{Hom}_{\mathcal{N}_s}(F'(W), F'(W)) \simeq \mathbb{Q}$ is a non-trivial $\mathbb{Q}$-linear map, i.e., an isomorphism.

Thanks to Proposition 4.6, we obtain a symmetric monoidal functor
\[ \Phi_1 : \mathcal{C}_{\mathcal{W}_{2g}} \simeq \mathcal{N}_s \rightarrow \text{DM}^{\text{eff}, \infty} \rightarrow \text{DM}^\otimes, \]
which preserves filtered colimits. We remark that it does not preserve pushouts. Set $K = \wedge^g_{\mathfrak{h}_1}(X)$. Then $\Phi_1(K)$ is $\wedge^g(M_1(X)[-1]) \simeq M_{2g}(X)[-2g] \simeq \mathbb{Q}(g)$ in $\text{DM}^\otimes$.

If $\mathcal{A}^\otimes$ and $\mathcal{B}^\otimes$ are symmetric monoidal small $\infty$-categories, we let $\text{Map}^\otimes(\mathcal{A}^\otimes, \mathcal{B}^\otimes)$ be the mapping space of symmetric monoidal functors. By [40, Prop. 4.1, 4.2] there is a symmetric monoidal small $\infty$-category $\mathcal{N}_{s,f}[K^{-1}]$ equipped with a symmetric monoidal functor $\mathcal{N}_{s,f} \rightarrow \mathcal{N}_{s,f}[K^{-1}]$ having the universal property: for any symmetric monoidal small $\infty$-category $\mathcal{D}^\otimes$, the composition induces a fully faithful functor
\[ \text{Map}^\otimes(\mathcal{N}_{s,f}[K^{-1}], \mathcal{D}^\otimes) \rightarrow \text{Map}^\otimes(\mathcal{N}_{s,f}, \mathcal{D}^\otimes) \]
whose essential image is spanned by those functors $F : \mathcal{N}_{s,f} \rightarrow \mathcal{D}^\otimes$ such that $(-) \otimes F(K) : \mathcal{D} \rightarrow \mathcal{D}$ induces an equivalence of $\mathcal{D}$.

Since $(-) \otimes \mathbb{Q}(g) : \text{DM} \rightarrow \text{DM}$ is an equivalence, the restriction $\Phi_1|_{\mathcal{N}_{s,f}} : \mathcal{N}_{s,f} \rightarrow \text{DM}^\otimes$ induces $\Phi_2 : \mathcal{N}_{s,f}[K^{-1}] \rightarrow \text{DM}^\otimes$ (note that DM is not small, but the essential image of $\Phi_1|_{\mathcal{N}_{s,f}}$ is contained in an essentially small symmetric monoidal full subcategory).

**Proposition 4.7.** There is a natural equivalence $\mathcal{N}_{s,f}[K^{-1}] \rightarrow \mathcal{N}_{f}$ as symmetric monoidal $\infty$-categories.

**Proof.** The functor $(-) \otimes \wedge^g_{\mathfrak{h}_1}(X) : \mathcal{N}_{f} \rightarrow \mathcal{N}_{f}$ is an equivalence. Thus we obtain $\mathcal{N}_{s,f}[K^{-1}] \rightarrow \mathcal{N}_{f}$ from $\mathcal{N}_{s,f} \rightarrow \mathcal{N}_{f}^\otimes$. According to [40] 4.21, 4.24 (the proof of 4.24 in loc. cit. which treats the presentable setting can apply mutatis-mutandis in this situation), the underlying $\infty$-category $\mathcal{N}_{s,f}[K^{-1}]$ is a colimit of the sequence
\[ \mathcal{N}_{s,f} \xrightarrow{\otimes K} \mathcal{N}_{s,f} \xrightarrow{\otimes K} \mathcal{N}_{s,f} \xrightarrow{\otimes K} \ldots \]
in the $\infty$-category of small $\infty$-categories. The natural functor $\mathcal{N}_{s,f}[K^{-1}] \rightarrow \mathcal{N}_{f}$ is given by the presentation of $\mathcal{N}_{f}$ as a colimit of $\mathcal{N}_{f} \xrightarrow{\otimes K} \mathcal{N}_{f} \xrightarrow{\otimes K} \ldots$ and the inclusion $\mathcal{N}_{s,f} \hookrightarrow \mathcal{N}_{f}$ which commutes with $(-) \otimes K$. Note again that $K$ is (isomorphic to) the character of the determinant of $\text{GL}_{2g}(\mathbb{Q})$. Thus $\mathcal{N}_{s,f} \xrightarrow{\otimes K} \mathcal{N}_{s,f}$ is fully faithful. The colimit $\lim_{\mathcal{N}_{s,f}} \mathcal{N}_{s,f} \xrightarrow{\otimes K} \mathcal{N}_{s,f} \xrightarrow{\otimes K} \ldots$ is the “union of all $\mathcal{N}_{s,f}$”. That is, an object in $\lim_{\mathcal{N}_{s,f}}$ is a pair $(X, -n)$ where $X \in \mathcal{N}_{s,f}$ and an integer $n \geq 0$, which we think of as an object $\bar{X}$ belonging to the $n$-th $\mathcal{N}_{s}$. The hom set is
\[ \text{Hom}_{\lim_{\mathcal{N}_{s,f}}}((X, -n), (Y, -m)) \colonequals \lim_{k \gg 0} \text{Hom}_{\mathcal{N}_{s,f}}(X \otimes K^{\otimes k-n}, Y \otimes K^{\otimes k-m}). \]
Then we have a natural functor $h : \lim_{\mathcal{N}_{s,f}} \mathcal{N}_{s,f} \rightarrow \mathcal{N}_{f}$ which extends $\mathcal{N}_{f} \rightarrow \mathcal{N}_{f}$. It carries $(X, -n)$ to $X \otimes K^{\otimes -n}$. It suffices to prove that $h : \lim_{\mathcal{N}_{s,f}} \mathcal{N}_{s,f} \rightarrow \mathcal{N}_{f}$ is a categorical equivalence. Here $K^{\otimes -1}$ is the dual of $K$ in $\mathcal{N}$. To see that $h$ is essential surjective, it suffices to observe that all irreducible representations of $\text{GL}_{2g}(\mathbb{Q})$ arise as $V \otimes K^{\otimes m}$ where $m \in \mathbb{Z}$ and $V \in \mathcal{N}_{s,f}$ (cf. Section 4.3.4). Finally, noting that $(-) \otimes K : \mathcal{N}_{s,f} \rightarrow \mathcal{N}_{s,f}$ is fully faithful, we easily see that $h : \lim_{\mathcal{N}_{s,f}} \mathcal{N}_{f} \rightarrow \mathcal{N}_{f}$ is fully faithful. \qed
4.3.6. By Proposition 4.4, we obtain a symmetric monoidal functor
\[ \Phi_{2,f} : \mathcal{N}^\otimes_{s,f}[K^{-1}] \simeq \mathcal{N}^\otimes_f \rightarrow \mathcal{D}^\otimes. \]
which preserves finite coproducts. By a left Kan extension, \( \Phi_{2,f} \) is extended to a symmetric monoidal functor
\[ \Phi_2 : \text{Ind}(\mathcal{N}^\otimes_f) \simeq \mathcal{N}^\otimes \rightarrow \mathcal{D}^\otimes \]
which preserves filtered colimits.

4.4. **Extension to the derived \( \infty \)-category of pure motives.** For any two symmetric monoidal \( \infty \)-categories \( \mathcal{C}^\otimes \) and \( \mathcal{D}^\otimes \), we denote by \( \text{Map}^\otimes(\mathcal{C}^\otimes, \mathcal{D}^\otimes) \) the mapping space of symmetric monoidal functors. We will prove Proposition 4.8 and complete the proof of Theorem 4.1.

Let \( \mathcal{A} \) be a closed symmetric monoidal Grothendieck abelian category endowed with a weakly flat descent structure \( (\mathcal{G}, \mathcal{H}) \) in the sense of [8]. We equip \( \text{Comp}(\mathcal{A}) \) with the symmetric monoidal model structure given in [8 Proposition 3.2] (weak equivalences are defined to be quasi-isomorphisms). Let \( \mathcal{D}(\mathcal{A})^\otimes \) be the symmetric monoidal stable presentable \( \infty \)-category obtained from \( \text{Comp}(\mathcal{A})^\circ \) by inverting quasi-isomorphisms. We shall refer to \( \mathcal{D}(\mathcal{A})^\otimes \) as the derived \( \infty \)-category of \( \mathcal{A} \). Our definition of \( \mathcal{D}(\mathcal{A}) \) is apparently different from that of [32], but there is a natural equivalence between them; see [32, 1.3.5.3, 1.3.5.8, 1.3.5.15]. Let \( \mathcal{D}(\mathcal{A})^\otimes_{\geq n} \) (resp. \( \mathcal{D}(\mathcal{A})^\otimes_{< n} \)) be the full subcategory of \( \mathcal{D}(\mathcal{A})^\otimes \) spanned by objects \( M \) such that \( H_r(M) = 0 \) for \( r < n \) (resp. \( r > n \)). The pair \( (\mathcal{D}(\mathcal{A})^\otimes_{\geq 0}, \mathcal{D}(\mathcal{A})^\otimes_{< 0}) \) determines a \( t \)-structure (cf. [32, 1.3.5.21]). We write \( \mathcal{D}^-(\mathcal{A}) \) for the full subcategory \( \bigcup_{n \leq 0} \mathcal{D}(\mathcal{A})^\otimes_{\geq n} \).

For example, our principal interest lies in the following: Let \( \text{Comp}(\mathcal{N}) \) be the presentable category of complexes of objects in \( \mathcal{N} = \text{Vect}_\mathbb{Q}(\text{GL}_{2g}(\mathbb{Q})) \). Let \( \mathcal{G} \) be the set of objects in \( \mathcal{N} \) which correspond to finite coproducts of simple objects in \( \mathcal{N} \). A weak equivalence in \( \text{Comp}(\mathcal{N}) \) is a quasi-isomorphism. A cofibration in \( \text{Comp}(\mathcal{N}) \) is a \( \mathcal{G} \)-cofibration in the sense of [8], that is, it is contained in the weakly saturated class \( \mathcal{A}^\circ \) generated by the complexes concentrated in cohomological degree \( n \) with \( (S^n E)^{n+1} = E \) (resp. in degrees \( n \) and \( n + 1 \) with \( (D^n E)^n = (D^n E)^{n+1} = E \) and only a nontrivial differential by the identity of \( E \)). Let \( \mathcal{H} \) be the set of objects which are \( \mathcal{G} \)-cofibrant. Then \( (\mathcal{G}, \mathcal{H}) \) is a (weakly) flat descent structure on \( \mathcal{N} \) in the sense of [8], and hence by [8, 2.5, 3.2] there is a proper cellular and combinatorial symmetric monoidal model category structure on \( \text{Comp}(\mathcal{N}) \) with these weak equivalences and cofibrations. Moreover, every object is fibrant, and every object in \( \mathcal{N} \) is cofibrant.

**Proposition 4.8.** Let \( \mathcal{A} \) be a closed symmetric monoidal Grothendieck abelian category endowed with a weakly flat descent structure. Suppose that \( \mathcal{A} \) has enough projective objects, and the full subcategory \( \mathcal{A}^{pr} \) of projective objects in \( \mathcal{A} \) is stable under tensor product and forms a symmetric monoidal full subcategory. Assume further that \( \mathcal{D}^-(\mathcal{A}) \) and \( \mathcal{D}_{\geq 0}(\mathcal{A}) \) are stable under the tensor product in \( \mathcal{D}(\mathcal{A}) \). Let \( \mathcal{C}^\otimes \) be a symmetric monoidal stable presentable \( \infty \)-category whose tensor product preserves small colimits separately in each variable. Then we have the followings:

(i) Let \( \text{Map}^\otimes_{\circ}(\mathcal{D}_{\geq 0}(\mathcal{A})^\circ, \mathcal{C}^\otimes) \) be the full subcategory of \( \text{Map}^\otimes(\mathcal{D}_{\geq 0}(\mathcal{A})^\circ, \mathcal{C}^\otimes) \) spanned by those symmetric monoidal functors which preserve geometric realizations of simplicial diagrams and finite coproducts. Let \( \text{Map}^\otimes_{\circ}(\mathcal{A}^{pr})^\circ, \mathcal{C}^\otimes) \) be the full subcategory of \( \text{Map}^\otimes(\mathcal{A}^{pr})^\circ, \mathcal{C}^\otimes) \) spanned by those symmetric monoidal functors which preserve finite coproducts. Then the embedding \( \mathcal{A}^{pr} \subset \mathcal{D}_{\geq 0}(\mathcal{A})^\circ \) induces a homotopy equivalence
\[ \text{Map}^\otimes_{\circ}(\mathcal{D}_{\geq 0}(\mathcal{A})^\circ, \mathcal{C}^\otimes) \rightarrow \text{Map}^\otimes(\mathcal{A}^{pr})^\circ, \mathcal{C}^\otimes). \]

(ii) Let \( \text{Map}^\otimes_{\circ}(\mathcal{D}^-(\mathcal{A})^\circ, \mathcal{C}^\otimes) \) be the full subcategory of \( \text{Map}^\otimes(\mathcal{D}^-(\mathcal{A})^\circ, \mathcal{C}^\otimes) \) spanned by those symmetric monoidal functors which preserve finite colimits and whose restrictions to
\(D_{\geq 0}(A)\) belong to \(\text{Map}_{\Sigma}^{\otimes}(D_{\geq 0}(A)^{\otimes}, C^{\otimes})\). Then the embedding \(D_{\geq 0}(A)^{\otimes} \subset D^{-}(A)^{\otimes}\) induces a homotopy equivalence
\[
\text{Map}^{\otimes}_{\Sigma}(D^{-}(A)^{\otimes}, C^{\otimes}) \to \text{Map}^{\otimes}_{\Sigma}(D_{\geq 0}(A)^{\otimes}, C^{\otimes}).
\]

The proof of this Proposition needs several lemmata. The following lemma is a “plane version” of Proposition \[4.8\] that is proved in \[32\] 1.3.3.3.

**Lemma 4.9.** We adopt notation similar to Proposition \[4.8\]. The subscripts \(\odot\), \(\Sigma\) and \(\oplus\) indicate the full subcategories spanned by functors with the corresponding property respectively. Let \(C\) be an \(\infty\)-category which has geometric realizations of simplicial diagrams and finite coproducts. Then the followings hold:

(i) The embedding \(A^{pr} \subset D_{\geq 0}(A)\) induces a categorical equivalence
\[
\text{Fun}_{\Sigma}(D_{\geq 0}(A), C) \to \text{Fun}_{\oplus}(A^{pr}, C).
\]

(ii) Suppose further that \(C\) is stable. The embedding \(D_{\geq 0}(A) \subset D^{-}(A)\) induces a categorical equivalence
\[
\text{Fun}_{\odot}(D^{-}(A), C) \to \text{Fun}_{\Sigma}(D_{\geq 0}(A), C).
\]

Proof. The claim (i) is proved in \[32\] 1.3.3.8. The claim (ii) follows from the argument, explained in the proof of \[32\] 1.3.3.11 (1), together with \[32\] 1.3.3.10 (2). \(\square\)

**Lemma 4.10.** We adopt notation similar to Proposition \[4.8\]. The subscripts \(\odot\), \(\Sigma\) and \(\oplus\) indicate the full subcategories spanned by functors which preserves colimits of respective kinds separately in each variable. Let \(C\) be an \(\infty\)-category which has geometric realizations of simplicial diagrams and finite coproducts. Let \(n \geq 0\). Then the followings hold:

(i) The embedding \((A^{pr})^{\times n} \subset D_{\geq 0}(A)^{\times n}\) induces a categorical equivalence
\[
\text{Fun}_{\Sigma}(D_{\geq 0}(A)^{\times n}, C) \to \text{Fun}_{\oplus}((A^{pr})^{\times n}, C).
\]

(ii) Suppose further that \(C\) is stable. The embedding \(D_{\geq 0}(A)^{\times n} \subset D^{-}(A)^{\times n}\) induces a categorical equivalence
\[
\text{Fun}_{\odot}(D^{-}(A)^{\times n}, C) \to \text{Fun}_{\Sigma}(D_{\geq 0}(A), C).
\]

Proof. We first observe that \(\text{Fun}_{\Sigma}(D_{\geq 0}(A), C), \text{Fun}_{\oplus}(A^{pr}, C),\) and \(\text{Fun}_{\odot}(D^{-}(A), C)\) have geometric realizations of simplicial diagrams and finite coproducts. We treat the case of \(\text{Fun}_{\Sigma}(D_{\geq 0}(A), C)\). The other cases are similar. Recall that \(I^{\circ} \to \text{Fun}(D_{\geq 0}(A), C)\) is a colimit diagram, that is, the cone point corresponds to a colimit of the restriction to \(I\), if and only if for each \(A\) in \(D_{\geq 0}(A)\), the restriction to \(A\) induces a colimit diagram \(I^{\circ} \to C\) (cf. \[31\] 5.1.2.3). Here \(I^{\circ}\) is the right cone (see \[31\] 1.2.8.4]). Thus \(\text{Fun}(D_{\geq 0}(A), C)\) has geometric realizations of simplicial diagrams and finite coproducts since \(C\) does. We deduce that for any simplicial diagram or coproduct diagram \(I \to \text{Fun}_{\Sigma}(D_{\geq 0}(A), C)\), the colimit in \(\text{Fun}(D_{\geq 0}(A), C)\) belongs to \(\text{Fun}_{\Sigma}(D_{\geq 0}(A), C)\). Hence our claim follows from the fact that the left adjoint functor \(\text{Fun}(I, C) \to C\) which carries \(I \to C\) to its colimit preserves geometric realizations and finite coproducts.

Next we will prove (i). The case of \(n = 1\) follows from the previous Lemma. Now suppose that our assertion holds for the case \(n = l\). By the above fact and the assumption, we have natural equivalences
\[
\text{Fun}_{\Sigma}(D_{\geq 0}(A)^{\times l+1}, C) \simeq \text{Fun}_{\Sigma}(D_{\geq 0}(A)^{\times l}, \text{Fun}_{\Sigma}(D_{\geq 0}(A), C))
\]
\[
\simeq \text{Fun}_{\oplus}((A^{pr})^{\times l}, \text{Fun}_{\Sigma}(D_{\geq 0}(A), C))
\]
\[
\simeq \text{Fun}_{\oplus}((A^{pr})^{\times l}, \text{Fun}_{\oplus}(A^{pr}, C))
\]
\[
\simeq \text{Fun}_{\oplus}((A^{pr})^{\times l+1}, C).
\]
By induction, we see (i). The same argument shows (ii). \qed

Proof of Proposition 4.8 If \( \mathcal{A} \) is a zero category, then our claim is clear, and thus we assume that \( \mathcal{A} \neq 0 \). We abuse notation and denote by \( p : \mathcal{D}_{\geq 0}(\mathcal{A})^\otimes \to N(\text{Fin}_*) \) a coCartesian fibration corresponding to the symmetric monoidal \( \infty \)-category (see Section 2 for the notation Fin*). According to [31, 3.2.0.1, 4.2.4.4], we have categorical equivalences

\[
\text{Fun}(N(\text{Fin}_*), \mathcal{C}_{\infty}) \simeq N(\text{Fun}(\text{Fin}_*, \mathcal{S}_{\Delta}^+)^\circ) \simeq N((\mathcal{S}_{\Delta}^+)^\circ / N(\text{Fin}_*))
\]

where \( \mathcal{S}_{\Delta}^+ \) is the category of (not necessarily small) marked simplicial sets, \( \text{Fun}(\text{Fin}_*, \mathcal{S}_{\Delta}^+) \) is endowed with the projective simplicial model structure, and \( (\mathcal{S}_{\Delta}^+)^\circ / N(\text{Fin}_*) \) is equipped with the coCartesian model structure (see [31, 3.1, A.2.8, 3.1.3.9]). Here \((-)^\circ\) indicates the full subcategory of cofibrant-fibrant objects. Through these equivalences, \( p \) gives rise to a cofibrant-fibrant functor \( u : \text{Fin}_* \to \mathcal{S}_{\Delta}^+ \) such that \( u(n) \) is an \( \infty \)-category which is equivalent to the \( n \)-fold product \( \mathcal{D}_{\geq 0}(\mathcal{A})^\otimes \times n \) (the set of marked edges is the set of equivalences). Put \( \mathcal{D}_{\geq 0}(n) := u(\langle n \rangle) \). Similarly, the symmetric monoidal category \( (\mathcal{A}_{pr})^\otimes \) gives rise to a cofibrant-fibrant functor \( v : \text{Fin}_* \to \mathcal{S}_{\Delta}^+ \) such that \( v(\langle n \rangle) \) is an \( \infty \)-category which is equivalent to the \( n \)-fold product \( (\mathcal{A}_{pr})^\otimes \times n \). Put \( \mathcal{A}(n) = v(\langle n \rangle) \). Let \( w : \text{Fin}_* \to \mathcal{S}_{\Delta}^+ \) be a cofibrant-fibrant functor corresponding to \( \mathcal{A}^\circ \). Put \( \mathcal{C}(n) = w(\langle n \rangle) \). By the above equivalence, we have the homotopy equivalence

\[
\text{Map}_{N(\text{Fun}(\text{Fin}_*, \mathcal{S}_{\Delta}^+)\circ)}(u, w) \simeq \text{Map}(\mathcal{D}_{\geq 0}(\mathcal{A})^\otimes, \mathcal{C}^\circ).
\]

Also, the symmetric monoidal functor \( (\mathcal{A}_{pr})^\otimes \hookrightarrow \mathcal{D}_{\geq 0}(\mathcal{A})^\otimes \) induces \( \alpha : v \to u \). By decomposing \( \alpha \) into a cofibration followed by a trivial fibration, we may and will assume that \( \alpha \) is cofibration. We denote by \( \alpha_n : \mathcal{A}(n) \to \mathcal{D}_{\geq 0}(n) \) the monomorphism of \( \infty \)-categories determined by \( \alpha \). We will define two simplicial categories \( \mathcal{P} \) and \( \mathcal{P}' \). The set of objects of \( \mathcal{P} \) (resp. \( \mathcal{P}' \)) is \( \{\mathcal{D}_{\geq 0}(0), \mathcal{D}_{\geq 0}(1), \ldots\} \sqcup \{\mathcal{C}(0), \mathcal{C}(1), \ldots\} \) (resp. \( \{\mathcal{A}(0), \mathcal{A}(1), \ldots\} \sqcup \{\mathcal{C}(0), \mathcal{C}(1), \ldots\} \)). Hom simplicial sets are defined as follows:

- For any \( n, m \geq 0 \), the simplicial set \( \text{Hom}_{\mathcal{P}}(\mathcal{A}(n), \mathcal{A}(m)) \) (resp. \( \text{Hom}_{\mathcal{P}}(\mathcal{D}_{\geq 0}(n), \mathcal{D}_{\geq 0}(m)) \)) is the subcomplex of \( \text{Map}(\mathcal{A}(n), \mathcal{A}(m)) \) (resp. \( \text{Map}(\mathcal{D}_{\geq 0}(n), \mathcal{D}_{\geq 0}(m)) \)) generated by the edges of the image of \( \text{Hom}_{\text{Fin}_*}(\langle n \rangle, \langle m \rangle) \) by \( v \) (resp. \( u \)). (We note that any \( n \)-simplex in \( \text{Hom}_{\mathcal{P}}(\mathcal{A}(n), \mathcal{A}(m)) \) or \( \text{Hom}_{\mathcal{P}}(\mathcal{D}_{\geq 0}(n), \mathcal{D}_{\geq 0}(m)) \) is degenerate.)
- For any \( n, m \geq 0 \), the hom simplicial set \( \text{Hom}_{\mathcal{P}}(\mathcal{C}(n), \mathcal{C}(m)) = \text{Hom}_{\mathcal{P}}(\mathcal{C}(n), \mathcal{C}(m)) \) is the full subcategory of \( \text{Hom}_{\mathcal{P}}(\mathcal{C}(n), \mathcal{C}(m)) \) spanned by the image of \( \text{Hom}_{\text{Fin}_*}(\langle n \rangle, \langle m \rangle) \) by \( w \).
- Let \( p^{i,n} : \mathcal{D}_{\geq 0}(n) \to \mathcal{D}_{\geq 0}(1) \) be the functor determined by \( \alpha^{i,n} \) (see Section 2 for the notation \( \alpha^{i,n} \)). The product \( p^{i,n} \times \cdots \times p^{n,n} : \mathcal{D}_{\geq 0}(n) \to \mathcal{D}_{\geq 0}(1)^{\times n} \) is a categorical equivalence, and we let \( q^n \) be its quasi-inverse. For \( d := (d_1, \ldots, d_{i-1}, d_{i+1}, \ldots, d_n) \in \mathcal{D}_{\geq 0}(1)^{\times n-1} \), let \( d_i : \mathcal{D}_{\geq 0}(1) \to \mathcal{D}_{\geq 0}(1)^{\times n} \) be the functor given by

\[
c \mapsto (d_1, \ldots, d_{i-1}, c, d_{i+1}, \ldots, d_n).
\]

For any \( n, m \geq 0 \), the hom simplicial set \( \text{Hom}_{\mathcal{P}}(\mathcal{D}_{\geq 0}(n), \mathcal{C}(m)) \) is the full subcategory of \( \text{Map}(\mathcal{D}_{\geq 0}(n), \mathcal{C}(m)) \) spanned by functors \( f \) such that (i) \( f \circ q^n \) preserves geometric realizations of simplicial diagrams and finite coproducts separately in each variable, and (ii) for any \( 1 \leq k \leq n \) and any \( d = (d_1, \ldots, d_{k-1}, d_{k+1}, \ldots, d_n) \), there is at most one \( 1 \leq i \leq m \) such that \( w(\alpha^{i,m}) \circ f \circ q^n \circ d_i : \mathcal{D}_{\geq 0}(1) \to \mathcal{C}(1) \) is not equivalent to a constant functor. We refer to (ii) as the good property.
- For any \( n, m \geq 0 \), the hom simplicial set \( \text{Hom}_{\mathcal{P}}(\mathcal{A}(n), \mathcal{C}(m)) \) is the full subcategory of \( \text{Map}(\mathcal{A}(n), \mathcal{C}(m)) \) spanned by functors \( f \) such that (i) \( f \circ q^n_A \) preserves finite coproducts separately in each variable (here \( q^n_A \) is a quasi-inverse \( A(1)^{\times n} \to A(n) \) of the natural
equivalence $\mathcal{A}(n) \to \mathcal{A}(1)^\times n$, (ii) $f$ has the good property (this property is defined as above).

- For any $n, m \geq 0$, $\text{Hom}_{\mathcal{P}'}(\mathcal{C}(m), \mathcal{A}(n))$ and $\text{Hom}_{\mathcal{P}}(\mathcal{C}(m), D_{\geq 0}(n))$ are empty sets.

The tensor product $\mathcal{A}^p \times \mathcal{A}^p \to \mathcal{A}^p$ (resp. $D_{\geq 0}(\mathcal{A}) \times D_{\geq 0}(\mathcal{A}) \to D_{\geq 0}(\mathcal{A})$) preserves finite coproducts (resp. geometric realizations and finite coproducts), and thus the compositions are well-defined so that $\mathcal{P}$ and $\mathcal{P}'$ form simplicial categories. Next we define a simplicial functor $\phi : \mathcal{P} \to \mathcal{P}'$. The functor $\phi$ sends $D_{\geq 0}(n)$ and $\mathcal{C}(n)$ to $\mathcal{A}(n)$ and $\mathcal{C}(n)$ respectively. The maps of hom simplicial sets are defined as follows.

- The map $\text{Hom}_{\mathcal{P}}(\mathcal{C}(n), \mathcal{C}(m)) \to \text{Hom}_{\mathcal{P}'}(\mathcal{C}(n), \mathcal{C}(m))$ is the identity.
- The map $\text{Hom}_{\mathcal{P}}(D_{\geq 0}(n), \mathcal{C}(m)) \to \text{Hom}_{\mathcal{P}'}(\mathcal{A}(n), \mathcal{C}(m))$ is induced by the composition with $\alpha_n : \mathcal{A}(n) \to D_{\geq 0}(n)$.
- $\text{res}^{n,m} : \text{Hom}_{\mathcal{P}}(D_{\geq 0}(n), D_{\geq 0}(m)) \to \text{Hom}_{\mathcal{P}'}(\mathcal{A}(n), \mathcal{A}(m))$ is induced by the restriction. Note that $u$ and $v$ induces isomorphisms $N(\text{Hom}_{\text{Fin}_{\mathcal{A}}}(n, m)) \simeq \text{Hom}_{\mathcal{P}}(D_{\geq 0}(n), D_{\geq 0}(m))$ and $N(\text{Hom}_{\text{Fin}_{\mathcal{A}}}(n, m)) \simeq \text{Hom}_{\mathcal{P}'}(\mathcal{A}(n), \mathcal{A}(m))$ by the assumption $\mathcal{A} \neq 0$, and thus $\text{res}^{n,m}$ is an isomorphism.

These data define a simplicial functor $\phi : \mathcal{P} \to \mathcal{P}'$. Then applying Lemma 4.10 to $\phi : \mathcal{P} \to \mathcal{P}'$ we deduce that that $\phi$ is a Dwyer-Kan equivalence (we easily observe that the extension of $(\mathcal{A}^p)^\times n \simeq \mathcal{A}(n) \to \mathcal{C}(m)$ to $D_{\geq 0}(n)$ in Lemma 4.10 preserves the good property). There are natural functors $u : N(\text{Fin}_{\mathcal{A}}) \to N(\mathcal{P})$, $w : N(\text{Fin}_{\mathcal{A}}) \to N(\mathcal{P})$, $w : N(\text{Fin}_{\mathcal{A}}) \to N(\mathcal{P}')$ and $w : N(\text{Fin}_{\mathcal{A}}) \to N(\mathcal{P}')$ induced by $u$, $v$ and $w$ (here we abuse notation). Note the homotopy equivalences

$$\text{Map}_{\mathcal{P}}(N(\text{Fin}_{\mathcal{A}}), N(\mathcal{P}))(u, w) \simeq \text{Map}^\otimes(D_{\geq 0}(\mathcal{A})^\otimes, \mathcal{C}^\otimes)$$

and

$$\text{Map}_{\mathcal{P}}(N(\text{Fin}_{\mathcal{A}}), N(\mathcal{P}'))(v, w) \simeq \text{Map}^\otimes((\mathcal{A}^p)^\otimes, \mathcal{C}^\otimes).$$

Moreover, $\text{Map}^\otimes(D_{\geq 0}(\mathcal{A})^\otimes, \mathcal{C}^\otimes) \to \text{Map}^\otimes((\mathcal{A}^p)^\otimes, \mathcal{C}^\otimes)$ can be identified with

$$\text{Map}_{\mathcal{P}}(N(\text{Fin}_{\mathcal{A}}), N(\mathcal{P}))(u, w) \to \text{Map}_{\mathcal{P}}(N(\text{Fin}_{\mathcal{A}}), N(\mathcal{P}'))(v, w).$$

Now the equivalence $N(\mathcal{P}) \simeq N(\mathcal{P}')$ implies our claim (i). The proof of (ii) is similar.

**Proof of Theorem 4.4** At the end of Section 4.3 we have constructed the symmetric monoidal functor $\Phi_2 : N^\otimes \to \mathcal{D}^\otimes$. Applying Proposition 4.3 to it we obtain a symmetric monoidal exact functor

$$\mathcal{D}^{-}(N)^\otimes \to \mathcal{D}^\otimes,$$

which carries $\tilde{h}_1(X)$ to $M_1(X)[-1]$. 

4.5. We conclude this Section with technical results; Lemma 4.11, 4.13 which will be used in the other Sections. Let $G$ be a reductive algebraic group over a field $K$ of characteristic zero. Let $\mathcal{M} = \text{Vect}_K(G)$. It is a Grothendieck semisimple abelian category. Let $\mathcal{G}_\mathcal{M}$ be the set of finite coproducts of irreducible representations of $G$. Let $\mathcal{H}_\mathcal{M} = \{0\}$. Then we easily see that the pair $(\mathcal{G}_\mathcal{M}, \mathcal{H}_\mathcal{M})$ is a flat descent structure in the sense of [3]. We equip $\text{Comp}(\mathcal{M})$ with the $\mathcal{G}_\mathcal{M}$-model structure; see loc. cit.. Let $\mathcal{D}(\mathcal{M})^\otimes$ be the symmetric monoidal $\infty$-category obtained from the full subcategory of cofibrant objects of $\text{Comp}(\mathcal{M})$ by inverting weak equivalences. Let $\mathcal{D}_v(\mathcal{M})^\otimes$ denote the full subcategory of dualizable objects in $\mathcal{D}(\mathcal{M})^\otimes$. In [22] Section A.6 we define a symmetric monoidal stable presentable $\infty$-category $\text{Rep}_G^\otimes$. Intuitively speaking, $\text{Rep}_G^\otimes$ is a symmetric monoidal $\infty$-category which consists of complexes of $K$-vector spaces endowed with action of $G$. We here recall the definition of $\text{Rep}_G^\otimes$ by using model categories. Put $\text{Spec} B = G$. The group structure of $G$ gives rise to a cosimplicial diagram $\{B^\otimes n\}_{n \in \Delta}$ of commutative $K$-algebras whose $n$-th term is $B^\otimes n$, i.e., it comes from the $\check{C}$ech nerve of the natural projection to the classifying stack $\pi : \text{Spec} K \to BG$; see [31] 6.1.2 for $\check{C}$ech nerves.
For a commutative algebra $A$, we let $\text{Comp}(A)$ be the category of (not necessarily bounded) complexes of $A$-modules. We here equip $\text{Comp}(A)$ with the projective model structure (cf. [19, 2.3.3]). The cosimplicial diagram $\{B^\otimes n\}_{[n] \in \Delta}$ yields a cosimplicial diagram of (symmetric monoidal) categories $\{\text{Comp}(B^\otimes n)\}_{[n] \in \Delta}$ in which each $\text{Comp}(B^\otimes n) \to \text{Comp}(B^\otimes m)$ is the base change by $B^\otimes n \to B^\otimes m$, that is a left Quillen adjoint functor. Then it gives rise to a cosimplicial diagram $\{\text{Comp}(B^\otimes n)^c\}_{[n] \in \Delta}$ of symmetric monoidal categories consisting of cofibrant objects.

Inverting quasi-isomorphisms in each category $\text{Comp}(B^\otimes n)^c$ we obtain a cosimplicial diagram $\{\text{NW}(\text{Comp}(B^\otimes n)^c)\}_{[n] \in \Delta}$ of symmetric monoidal stable presentable $\infty$-categories. We define $\text{Rep}^\otimes_G$ to be a limit of $\{\text{NW}(\text{Comp}(B^\otimes n)^c)\}_{[n] \in \Delta}$ among symmetric monoidal $\infty$-categories.

The limit $\text{Rep}^\otimes_G$ is also stable and presentable. Let $\text{PRep}^\otimes_G \subseteq \text{Rep}^\otimes_G$ be the full subcategory of dualizable objects. The following Lemma gives a relation between $\text{PRep}^\otimes_G$ and $\mathcal{D}(\mathcal{M})^\otimes$.

**Lemma 4.11.** There exists an equivalence $\mathcal{D}(\mathcal{M})^\otimes \simeq \text{Rep}^\otimes_G$ of symmetric monoidal $\infty$-categories.

**Proof.** We first construct a symmetric monoidal functor $\mathcal{D}(\mathcal{M})^\otimes \to \text{Rep}^\otimes_G$ which preserves small colimits. Define $\text{Comp}(\text{Vect}_K(G)) \to \text{Comp}(B^\otimes n)$ to be the constant cosimplicial diagram. These induce a map of cosimplicial diagrams $\{\text{Comp}(M)\}_{[n] \in \Delta} \to \{\text{Comp}(B^\otimes n)\}_{[n] \in \Delta}$. Note that each functor $\text{Comp}(M) \to \text{Comp}(B^\otimes n)$ preserves cofibrant objects since it preserves small colimits and the generating cofibration $\{S^{n+1}E \to D^nE\}_{n \in \mathbb{Z}, E \in \mathcal{M}}$ maps to cofibrations in $\text{Comp}(B^\otimes n)$. We then have the map $\text{Comp}(M)^c \to \{\text{Comp}(B^\otimes n)^c\}_{[n] \in \Delta}$. By inverting weak equivalences, it gives rise to a map of cosimplicial symmetric monoidal $\infty$-categories

$$\text{NW}(\text{Comp}(M)^c) \to \{\text{NW}(\text{Comp}(B^\otimes n)^c)\}_{[n] \in \Delta}.$$ 

Since $\text{Rep}^\otimes_G$ is the limit of $\{\text{NW}(\text{Comp}(B^\otimes n)^c)\}_{[n] \in \Delta}$, we obtain a symmetric monoidal colimit-preserving functor $\mathcal{D}(\mathcal{M})^\otimes \to \text{Rep}^\otimes_G$.

Next we define a $t$-structure on $\text{Rep}_G$. Let $\text{Rep}_{G,0}$ (resp. $\text{Rep}_{G,0}$) be the inverse image of $\text{Mod}_{H_K,0}$ (resp. $\text{Mod}_{H_K,0}^\perp$) under the forgetful functor $p : \text{Rep}_G \to \text{Mod}_{H_K}$. Here $C \in \text{Mod}_{H_K}$ belongs to $\text{Mod}_{H_K,0}$ (resp. $\text{Mod}_{H_K,0}^\perp$) if and only if $pi_i(C) = 0$ for any $i < 0$ (resp. $i > 0$). The comonad $T : \text{Mod}_{H_K} \to \text{Mod}_{H_K}$ of the adjoint pair

$$p : \text{Rep}_G \rightleftarrows \text{Mod}_{H_K} : q$$

is given by $C \mapsto B \otimes C$. Here $q$ is a right adjoint of $p$. Identifying $\text{Rep}_G$ with the $\infty$-category of $T$-comodules by [22, 6.2.4.1], we conclude by [23, VII, 6.20] that $(\text{Rep}_{G,0}, \text{Rep}_{G,0}^\perp)$ defines a both left and right complete $t$-structure. Let $\text{Rep}_{G}^b$ (resp. $\text{Rep}_{G}^+$) denote the full subcategory of $\text{Rep}_G$ spanned by bounded objects (resp. left bounded objects) with respect to this $t$-structure. Since $p$ is symmetric monoidal, they are stable under tensor product.

We claim that $\mathcal{D}(\mathcal{M}) \to \text{Rep}_G$ induces a categorical equivalence $\mathcal{D}^+(\mathcal{M}) \to \text{Rep}_G^+$. We first prove that the induced functor $w : \mathcal{D}^+(\mathcal{M}) \to \text{Rep}_G^+$ is fully faithful. Let $C$ and $C'$ be objects in $\mathcal{D}^+(\mathcal{M})$. We need to show that the induced map $w_{C,C'} : \text{Map}_{\mathcal{D}(\mathcal{M})}(C, C') \to \text{Map}_{\text{Rep}_G^+}(w(C), w(C'))$ is an equivalence in $S$. Since $\mathcal{D}(\mathcal{M}) \to \text{Rep}_G$ preserves small colimits, the $t$-structure on $\mathcal{D}(\mathcal{M})$ is right complete and $C$ is a colimit of bounded objects, thus we may assume that $C$ lies in $\mathcal{D}^b(\mathcal{M})$. The full subcategory of $\mathcal{D}^b(\mathcal{M})$ spanned by those objects $C$ such that $w_{C,C'}$ is an equivalence for any $C' \in \mathcal{D}^+(\mathcal{M})$, is a stable subcategory. Hence we may and will assume that $C$ belongs to the heart $\mathcal{M}$. To compute $\text{Ext}^n_{\mathcal{D}(\mathcal{M})}(C, C')$, we use the injective model structure on $\text{Comp}(\mathcal{M})$ in which weak equivalences are quasi-isomorphisms, and cofibrations are monomorphisms (cf. [4, 8, 22, 1.3.5]). Since $\mathcal{M}$ has enough injective objects we suppose that $C'$ is a left bounded complex of the form

$$\cdots \to O \to 0 \to C' \to C'+1 \to \cdots$$
where $C^i$ is an injective object $M$ for any $i \in \mathbb{Z}$. It is a fibrant object. To compute $\text{Map}_{D(M)}(C, C')$ and $\text{Map}_{\text{Rep}_G}(w(C), w(C'))$, since $C$ lies in the heart and $w$ is $t$-exact, we may and will suppose that $C^i = 0$ for $i > 1$. Let $I$ be the full subcategory spanned by finite-length complexes of injective objects. We claim that for any $C' \in I$ the map $\theta^n_{C,C'} : \text{Ext}^n_{D(M)}(C, C') \to \text{Ext}^n_{\text{Rep}_G}(w(C), w(C'))$ is an isomorphism for any $n \in \mathbb{Z}$. If $P$ is the full subcategory of $D^b(M)$ spanned by finite-length complexes $C'$ of injective objects such that $\theta^n_{C,C'}$ is an isomorphism for any $n \in \mathbb{Z}$, then $P$ is stable under shifts and cones. Thus we may and will assume that $C'$ is an injective object in the heart $M$. When $n \leq 0$, clearly it is an isomorphism. When $n > 0$, we will prove that $\text{Ext}^n_{D(M)}(C, C') = \text{Ext}^n_{\text{Rep}_G}(w(C), w(C')) = 0$. To see $\text{Ext}^n_{\text{Rep}_G}(w(C), w(C')) = 0$ for $n > 0$, let $I = w(C')$ be an injective object in the heart $\text{vect}_K(G)$ of $\text{Rep}_G$ and let $p(I) \to J$ be an injective resolution, that is, $J$ is an injective object in the heart $\text{vect}_K$ of $\text{Mod}_H(K)$. Then $q(J)$ is injective, and $I \to q(p(I)) \to q(J)$ is a monomorphism since $p(I) \to p(q(J))$ is the monomorphism where $p(q(J)) \to J$ and $I \to q(p(I))$ are a counit map and a unit map respectively. Notice that $I$ is injective, thus $I$ is a retract of $q(J)$. Consequently, it will suffice to show that $\text{Ext}^n_{\text{Rep}_G}(w(C), q(J)) = 0$ for $n > 0$. It follows from the adjunction that $\text{Ext}^n_{\text{Rep}_G}(w(C), q(J)) = \text{Ext}^n_{\text{Mod}_H(K)}(p(w(C)), J) = 0$ for $n > 0$. Since $C$ is cofibrant and $C'$ is fibrant in any $\text{Comp}(M)$ endowed with the injective model structure, $\text{Ext}^n_{D(M)}(C, C') = 0$ for any $n > 0$.

Next we will prove that $w$ is essentially surjective. Let $D \in \text{Rep}^+_G$. We must show that there is $C \in D^+(M)$ such that $w(C) \simeq D$. Since $\text{Rep}_G$ is right complete and $D(M) \to \text{Rep}_G$ preserves small colimits, thus by the fully faithfulness proved above, we may and will suppose that $D$ belongs to $\text{Rep}^+_G$. Let $I$ be the amplitude of $D$. We proceed by induction on $I$. The case of $I = 1$ is obvious (in this case $D$ is a shift of an object in the heart). Using $t$-structure one can take a distinguished triangle

$$D_1 \to D \to D_2 \to D_1[1]$$

such that the amplitude of $D_1$ is equal or less than $I − 1$, and the amplitude of $D_2$ is equal or less than 1. By the inductive assumption, we have $C_1$ and $C_2$ such that $w(C_1) \simeq D_1$ and $w(C_2) \simeq D_2$. Moreover, the fully faithfulness implies that there exists $C_2 \to C_1[1]$ such that $w(C_2) \to w(C_1[1])$ represents the homotopy class of $D_2 \to D_1[1]$. Note that $D$ is a fibre of $D_2 \to D_1[1]$. Let $C$ be a fibre of $C_2 \to C_1[1]$. By the exactness of $w$, we conclude that $w(C) \simeq D$.

It remains to show how one can derive an equivalence $D^\otimes(M) \simeq \text{Rep}^\otimes_G$ from $D^+(M) \simeq \text{Rep}^+_G$. We have constructed the symmetric monoidal functor $D^\otimes(M) \to \text{Rep}^\otimes_G$, and thus it suffices to prove that the underlying functor $D(M) \to \text{Rep}_G$ is a categorical equivalence. The equivalence $D^+(M) \simeq \text{Rep}^+_G$ induces an equivalence $D(M) \simeq \text{PRep}_G$, where $D(M)$ denotes the full subcategory spanned by dualizable objects. Note that by the assumption that $G$ is a reductive algebraic group over a field of characteristic zero, $D(M)$ is compactly generated, and the set of (finite dimensional) irreducible representations is a set of compact generators. Thus $D(M) \simeq \text{Ind}(D(M))$. Moreover, if $\text{PMod}_G$ denotes the full subcategory of $\text{Rep}_G$ spanned by dualizable objects, then $\text{Ind}(\text{PRep}_G) \simeq \text{Rep}_G$; see [3] 3.22. Hence we obtain $D(M) \simeq \text{Ind}(D(M)) \simeq \text{Ind}(\text{PRep}_G) \simeq \text{Rep}_G$. Finally, we remark another way to deduce $D(M) \simeq \text{Rep}_G$. Since $\text{Rep}_G$ is left complete and $D^+(M) \simeq \text{Rep}^+_G$, the functor $D(M) \to \text{Rep}_G$ can be viewed as a left completion [32, 1.2.1.17] of $D(M)$. By using the semi-simplicity of $M$ we can easily check that $D(M)$ is left complete.

\[ \square \]

Remark 4.12. As shown in the proof, we have

$$D(M) \simeq \text{Ind}(D(M)) \simeq \text{Ind}(\text{PRep}_G) \simeq \text{Rep}_G.$$
Let $\mathcal{M}$ be a left proper combinatorial model category. Let $S$ be a small set of morphisms in $\mathcal{M}$. Then we have a new model structure of $\mathcal{M}$: a left Bousfield localization of $\mathcal{M}$ with respect to $S$ (see e.g. [2], [31 A. 3.7.3]), where (new) weak equivalences are called $S$-equivalences. We then obtain an $\infty$-category $N_{W}(\mathcal{M}[S^{-1}])$ by inverting $S$-equivalences. On the other hand, we have the $\infty$-category $N_{W}(\mathcal{M})$ obtained form $\mathcal{M}$ by inverting weak equivalences. By using the localization theory at the level of $\infty$-category [31 5.5.4], one can take the localization $N_{W}(\mathcal{M}) \rightarrow N_{W}(\mathcal{M})[S^{-1}]$ (see [31 5.5.4.15]). Then the universality of the localization $L : N_{W}(\mathcal{M}) \rightarrow N_{W}(\mathcal{M})[S^{-1}]$ [31 5.5.4.20] induces a functor $F : N_{W}(\mathcal{M})[S^{-1}] \rightarrow N_{W}(\mathcal{M}[S^{-1}])$.

Lemma 4.13. The functor $N_{W}(\mathcal{M})[S^{-1}] \rightarrow N_{W}(\mathcal{M}[S^{-1}])$ is a categorical equivalence.

Proof. We have the commutative diagram

\[
\begin{array}{ccc}
N_{W}(\mathcal{M}) & \xrightarrow{L} & N_{W}(\mathcal{M})[S^{-1}] \\
| & & | \\
W(N_{W}(\mathcal{M})) & \xrightarrow{\varphi} & W(N_{W}(\mathcal{M})[S^{-1}])
\end{array}
\]

that consists of left adjoint functors of presentable $\infty$-categories. Here $\mathcal{L}$ is the “localization functor” that comes from the left Quillen functor. Note that the right adjoint functors of $\mathcal{L}$ and $\mathcal{L}'$ are fully faithful. We denote them by $i$ and $i'$ respectively. Moreover, the essential image of $i$ consists of $S$-local objects, that is, those objects $Z$ such that $\text{Map}_{W(N_{W}(\mathcal{M}))}(X, Z) \rightarrow \text{Map}_{N_{W}(\mathcal{M})}(X, Z)$ is a weak homotopy equivalence for any $X \rightarrow Y \in S$. Similarly, the essential image of $i'$ consists of “model theoretic $S$-local objects”, that is, those objects $Z$ such that $\text{Map}_{W}(Y, Z) \rightarrow \text{Map}_{W}(X, Z)$ is a weak homotopy equivalence for any $X \rightarrow Y \in S$. Here we slightly abuse notation and $\text{Map}_{W}(\cdot, \cdot)$ denotes the mapping space in $\mathcal{M}$ given by machinery of simplicial and cosimplicial frames [19 5.4] or hammock localization of Dwyer-Kan (we implicitly assume suitable cofibrant or fibrant replacements). Thus it will suffice to prove both mapping spaces coincide. If $\mathcal{M}$ is a simplicial model category, then our assertion follows from [32 1.3.4.20]. In the general case, our assertion follows from the simplicial case and a theorem of Dugger [12] which says that every combinatorial model category is Quillen equivalent to a left proper simplicial combinatorial model category. \hfill $\square$

5. Representation theorem of mixed abelian motives

The objective of this Section is to show that Thereom 1.1. Namely, we will construct the motivic algebra $A_{X}$ of an abelian scheme $X$ of relative dimension $g$ and prove an equivalence between the $\infty$-category of “$A_{X}$-modules equipped with action of $GL_{2g}(\mathbb{Q})$” and the $\infty$-category of mixed abelian motives generated by $X$ (in DM). We also study the complex $A_{X}$ and the case of mixed Tate motives.

5.1.

Definition 5.1. Let $S$ be a smooth quasi-projective scheme over a perfect field $k$. Let $X$ be an abelian scheme of relative dimension $g$ over $S$. Let $\mathcal{D}_{M}$ be the smallest stable subcategory of $DM$, which contains $M(X)$ and are closed under small coproducts, tensor products and dual objects. We refer to $\mathcal{D}_{M}$ as the $\infty$-category of mixed abelian motives generated by $X$. Let $\mathcal{D}_{mM}_{X}$ be the smallest stable subcategory of $DM$, which contains $M(X)$ and are closed under retracts, tensor products and dual objects.

Remark 5.2. We note that since $\mathcal{D}_{M}$ has small colimits, it is closed under retracts. By the definition and duality theorem (i.e. $M(X)^{\vee} \simeq M(X) \otimes_{\mathcal{O}} \mathbb{Q}(-g)[-2g]$), $\mathcal{D}_{M}$ is the smallest stable subcategory of $DM$ which contains $M(X)^{\otimes n} \otimes \mathbb{Q}(g)^{\otimes m}$ for any $n \geq 0$ and any $m \in \mathbb{Z}$ and admits small coproducts. The stable subcategory $\mathcal{D}_{M}$ inherits a symmetric monoidal
structure from $\text{DM}$. Also, we remark that when $S = \text{Spec} \ k(1)$ belongs to $\text{DM}_X$. A polarization $\phi : h_1(X) \otimes h_1(X) \rightarrow \mathcal{L}$ in $NM$ corresponds to an element $\text{CH}^1(X \times_k X)_Q/\sim_{\text{num}}$ arising from a symmetric invertible sheaf $\mathcal{L}$ on $X \times_k X$ such that the pullback to the diagonal $\Delta_X \mathcal{L}$ is ample. Since $NM$ is semisimple, $\mathcal{L}$ is a direct summand of $h_1(X)^{\otimes 2}$ and we have a sequence $\mathcal{L} \xrightarrow{\text{Ker}(\phi)} \mathcal{L} \simeq h_1(X)^{\otimes 2}$ preserving $\mathcal{L}$. This sequence is promoted to a sequence $Q(1) \rightarrow M_1(X)[-1] \otimes M_1(X)[-1] \rightarrow Q(1)$ whose composite is non-tirivial (here $\text{CHM}$ is identified with a full subcategory of the homotopy category of $\text{DM}$). Thus $Q(1)$ is a direct summand of $M_1(X)[-1]^{\otimes 2}$. Moreover, in this case the homotopy category of $\text{DM}^\otimes_X$ forms a symmetric monoidal triangulated category, which is equivalent to the symmetric monoidal triangulated category $\text{DM}_{\text{gm}}(k, Q)$ of Voevodsky’s geometric motives with rational coefficients \cite{28}. Let $\text{DM}_{\text{gm}, X}(k, Q)$ be the smallest triangulated subcategory which contains $M(X)^{\otimes n} \otimes Q(1)^{\otimes m}$ for any $n \geq 0$ and any $m \in \mathbb{Z}$ and is closed under retracts. This triangulated subcategory is closed under tensor product. The homotopy category of $\text{DM}_{\text{gm}, X}$ is equivalent to $\text{DM}_{\text{gm}, X}(k, Q)$.

We will construct the motivic algebra of $X$. Let $\mathcal{D}_V(N)$ be the full subcategory of $\mathcal{D}(N)$ spanned by dualizable objects. It is contained in $\mathcal{D}^b(N)$. By virtue of Theorem 4.11 we have a symmetric monoidal exact functor $\mathcal{D}_V(N)^{\otimes} \rightarrow \text{DM}^{\otimes}_X$ which is extended to a symmetric monoidal colimit-preserving functor $\chi : \text{Ind}(\mathcal{D}_V(N))^{\otimes} \rightarrow \text{DM}^{\otimes}_X$.

Using Theorem 4.11 Lemma 4.11 and Remark 4.12, we have a symmetric monoidal colimit-preserving functor of stable presentable $\infty$-categories

$$\chi : \mathcal{D}^{\otimes}(N) \simeq \text{Rep}^{\otimes}_{\text{GL}_{2g}(Q)} \rightarrow \text{DM}^{\otimes}_X.$$  

Let $\omega : \text{DM}^{\otimes}_X \rightarrow \text{Rep}^{\otimes}_{\text{GL}_{2g}(Q)}$ be the right adjoint of $\chi$, which is a lax symmetric monoidal functor. This right adjoint exists by the relative version of adjoint functor theorem (see \cite{32} 8.3.2.6)). Let $1_{\text{DM}^+_X}$ be a unit of $\text{DM}^+_X$. Put

$$A_X = \omega(1_{\text{DM}^+_X}).$$

Since $1_{\text{DM}^+_X}$ is a commutative algebra object and $\omega$ is lax symmetric monoidal, thus $A_X$ is a commutative algebra object in $\text{Rep}^{\otimes}_{\text{GL}_{2g}(Q)}$. Namely, $A_X$ lies in $\text{CAlg}(\text{Rep}^{\otimes}_{\text{GL}_{2g}(Q)}) \simeq \text{CAlg}(\mathcal{D}(N)^{\otimes})$. If $U : \text{Rep}^{\otimes}_{\text{GL}_{2g}(Q)} \rightarrow \text{Mod}^H_{\text{HQ}}$ denotes the forgetful functor, then $U(A_X)$ belongs to $\text{CAlg}(\text{Mod}^H_{\text{HQ}}) = \text{CAlg}_{\text{HQ}}$, that is, it is a commutative differential graded algebra over $Q$; by \cite{32} 8.1.4.11 $\text{CAlg}_{\text{HQ}}$ is equivalent to the nerve of model category of commutative differential graded algebras over $Q$, as $\infty$-categories (cf. \cite{17,32} 8.1.4]). We refer to $A_X \in \text{CAlg}(\text{Rep}^{\otimes}_{\text{GL}_{2g}(Q)})$ as the motivic algebra of $X$. Intuitively, $A_X$ can be viewed as a commutative differential graded algebra endowed with action of $\text{GL}_{2g}(Q)$. We abuse notation and write $A_X$ also for $U(A_X)$.

**Theorem 5.3.** Let $\text{Mod}^\otimes_{A_X}(\text{Rep}^{\otimes}_{\text{GL}_{2g}(Q)})$ be the symmetric monoidal stable presentable $\infty$-category of $A_X$-module objects in $\text{Rep}_{\text{GL}_{2g}(Q)}$ (see Section 2, \cite{32} 4.4). Then there exists a natural equivalence of symmetric monoidal $\infty$-categories

$$\text{Mod}^\otimes_{A_X}(\text{Rep}^{\otimes}_{\text{GL}_{2g}(Q)}) \xrightarrow{\sim} \text{DM}^{\otimes}_X.$$  

**Proof.** Observe first that $\chi : \text{Rep}^{\otimes}_{\text{GL}_{2g}(Q)} \rightarrow \text{DM}^{\otimes}_X$ is perfect; see Definition 3.1. We must check the conditions (i), (ii), (iii) in Definition 3.1. The set of the irreducible representations of $\text{GL}_{2g}(Q)$ is a set of compact generators of $\text{Rep}_{\text{GL}_{2g}(Q)}$. Moreover, each irreducible representation is dualizable. Since $\chi$ carries $h_1(X)$ to $M_1(X)[-1]$, thus for any $n \geq 0$ and any $m \in \mathbb{Z}$,
\{\chi(\hat{h}_1(X)^\otimes n \otimes K^\otimes m)\}_{n \geq 0, m \in \mathbb{Z}}\) forms a set of compact generators in \(\text{DM}_X\) (here we should note also that \(M(X) \simeq \bigoplus_{i=0}^{2g} M_i(X)\) and \(\wedge^i(M_1(X)[-1]) \simeq M_i(X)[i]\) for each \(i \geq 0\). Thus \(\chi\) is perfect, and our assertion follows from Proposition 3.2. 

We will reformulate Theorem 5.3 in terms of quasi-coherent complexes on a quotient derived stack. The geometric perspective gives us a clear picture in the next Section. Let \(\text{Sh}(\text{CAlg}^{\text{et}}_{\hat{H}Q})\) be the full subcategory of \(\text{Fun}(\text{CAlg}_{\hat{H}Q}, \hat{S})\) spanned by sheaves with respect to \(\text{étale}\) topology; see [22 A.1]. Here \(\hat{S}\) is the \(\infty\)-category of spaces in the enlarged universe \(V\). The Yoneda embedding \(\text{Aff}_{\hat{H}Q} := \text{CAlg}_{\hat{H}Q}^{\text{op}} \subset \text{Fun}(\text{CAlg}_{\hat{H}Q}, \hat{S})\) is contained in \(\text{Sh}(\text{CAlg}^{\text{et}}_{\hat{H}Q})\). For \(A \in \text{CAlg}^{\text{et}}_{\hat{H}Q}\) we denote by \(\text{Spec } A\) the corresponding object in \(\text{Fun}(\text{CAlg}_{\hat{H}Q}, \hat{S})\). We shall refer to a sheaf of this form as a derived affine scheme over \(H_Q\). Another type of derived stacks we treat in this paper are the quotients of derived affine schemes by action of an algebraic group scheme \(G\). The quotient of a derived affine scheme \(\text{Spec } A\) by action of an algebraic group scheme \(G\) is defined to be a colimit (in \(\text{Sh}(\text{CAlg}^{\text{et}}_{\hat{H}Q})\)) of a simplicial diagram \(N(\Delta)^{op} \to \text{Aff}_{\hat{H}Q}\) of derived affine schemes that satisfies a certain condition described in [23 Example 4.1].

Let \(\text{Spec } B = \text{GL}_{2g}(Q)\). Let

\[
\text{CAlg}(\text{Rep}^{\otimes}_{\text{GL}_{2g}(Q)}) \to \lim_{[n] \in \Delta} \text{CAlg}(\text{W}(\text{Comp}(B^{\otimes n})^{\otimes})) \simeq \lim_{[n] \in \Delta} \text{CAlg}_{H_B^{\otimes n}}
\]

be the natural functor associated to the \(\check{C}\)ech nerve associated to the natural projection \(\text{Spec } HQ \to B\text{GL}_{2g}(Q)\). If \(A^*_X\) denotes the image of \(\text{Spec } X\) in \(\lim_{[n] \in \Delta} \text{CAlg}_{H_B^{\otimes n}},\) then it gives rise to a quotient derived stack \(\text{Spec } A^*_X/\text{GL}_{2g}(Q)\). The construction of the quotient derived stack is as follows: The cosimplicial diagram \(\{B^{\otimes n}\}_{[n] \in \Delta}\) of ordinary commutative \(Q\)-algebras has a natural map from the constant simplicial diagram \(\{Q\}\). Both cosimplicial diagrams naturally induce \(c_Q, c_B : N(\Delta) \to \text{Cat}_\infty\) such that \(c_Q\) is the constant diagram of \(\text{CAlg}_{\hat{H}Q}\), and \(c_B([n]) = \text{CAlg}_{H_B^{\otimes n}}\) and \([n] \to [m]\) maps to \(\text{CAlg}_{H_B^{\otimes n}} \to \text{CAlg}_{H_B^{\otimes m}}: R \mapsto H_B^{\otimes m} \otimes H^{\otimes n} R\). By [31 3.2.0.1, 4.2.4.4] the cosimplicial diagrams \(c_Q\) and \(c_B\) give rise to coCartesian fibrations \(pr_2 : \text{CAlg}_{\hat{H}Q} \times N(\Delta) \to N(\Delta)\) and \(\text{CAlg}_{H_B^{\bullet}} \to N(\Delta)\) respectively. The morphism \(c_Q \to c_B\) induced by \(\{Q\} \to \{B^{\otimes n}\}_{[n] \in \Delta}\) gives rise to a morphism of coCartesian fibrations \(\alpha : \text{CAlg}_{\hat{H}Q} \times N(\Delta) \to \text{CAlg}_{H_B^{\bullet}}\) over \(N(\Delta)\) that preserves coCartesian edges. By [22 8.3.2.7], there is a right adjoint \(\beta\) of \(\alpha\) relative to \(N(\Delta)\). Let \(s : N(\Delta) \to \text{CAlg}_{H_B^{\bullet}}\) be a section that corresponds to \(A^*_X\) (cf. [31 3.3.3.2]). Then the composite

\[
\xi : N(\Delta) \xrightarrow{s} \text{CAlg}_{H_B^{\bullet}} \xrightarrow{\beta} \text{CAlg}_{\hat{H}Q} \times N(\Delta) \xrightarrow{pr_2} \text{CAlg}_{\hat{H}Q}
\]

gives rise to a simplicial diagram \(\xi^{op} : N(\Delta)^{op} \to \text{Aff}_{\hat{H}Q}\). We define [\(\text{Spec } A^*_X/\text{GL}_{2g}(Q)\)] to be a colimit (geometric realization) of the simplicial diagram \(\xi^{op}\) in \(\text{Sh}(\text{CAlg}^{\text{et}}_{\hat{H}Q})\) (in [23], a colimit is taken in \(\text{Fun}(\text{CAlg}_{\hat{H}Q}, \hat{S})\), but this difference is not relevant to us since it does not change the \(\infty\)-category of quasi-coherent complexes). The symmetric monoidal \(\infty\)-category \(\text{Mod}^{\otimes}_{\text{Spec } A^*_X} (\text{Rep}^{\otimes}_{\text{GL}_{2g}(Q)})\) is equivalent to the symmetric monoidal \(\infty\)-category \(\text{Mod}^{\otimes}_{\text{Spec } A^*_X/\text{GL}_{2g}(Q)}\) of quasi-coherent complexes defined in [33 VIII, 2.7, 2.7.9]. If \(\{A_X(n)\}_{[n] \in \Delta}\) is the cosimplicial diagram \(\xi\), then \(\text{Mod}^{\otimes}_{\text{Spec } A^*_X/\text{GL}_{2g}(Q)} \simeq \lim_{\Delta} \text{Mod}^{\otimes}_{A_X(n)}(A_X(n) \simeq A_X \otimes B^{\otimes n}\text{ where } A_X\) is the underlying object lying in \(\text{CAlg}_{\hat{H}Q}\)). We have:

**Corollary 5.4.** There exists an equivalence of symmetric monoidal \(\infty\)-categories

\[
\text{Mod}^{\otimes}_{\text{Spec } A^*_X/\text{GL}_{2g}(Q)} \simeq \text{DM}^{\otimes}_{X^*}.
\]
Corollary 5.5. Suppose that $S = \text{Spec} k$. Let $\text{PMOD}_{[\text{Spec} \, A_X/GL_{2g}(Q)]}$ be the full subcategory of $\text{Mod}_{[\text{Spec} \, A_X/GL_{2g}(Q)]}$ spanned by dualizable objects. Then the homotopy category of $\text{PMOD}_{[\text{Spec} \, A_X/GL_{2g}(Q)]}$ is equivalent to $\text{DM}_{\text{gm}, X(k, Q)^\otimes}$ as symmetric monoidal triangulated categories.

5.2. Let $X^d$ denote the $d$-fold product of $X$ over the base scheme $S$. Let $H^i(X^d, Q(j))$ be the motivic cohomology with rational coefficients. Namely, we define $H^i(X^d, Q(j))$ to be $\text{Ext}^i_{DM}(M(X)^\otimes, Q(j))$. Let $H^\pm_0(X^d, Q(j))$ be the subspace of $H^i(X^d, Q(j))$ such that for each $1 \leq k \leq d$ and each $n \in \mathbb{Z}$, the multiplication by $n$ of the $k$-th factor $\text{id} \times \ldots \times \text{id} : X^d = X^{d-k} \times X \times X^d \rightarrow X^d = X^{d-k} \times X \times X^d$ acts on it by the multiplication by $n$. It can be identified with $\text{Ext}^i_{DM}(M_1(X)^\otimes, Q(j))$. For a Young tableau $\lambda$ of size $d$ and its Young symmetrizer $c_\lambda \in Q[\mathcal{S}_d]$, we let $S^\lambda_+H^\pm_0(X^d, Q(j))$ be the subspace of $H^\pm(X^d, Q(j))$ which corresponds to $\text{Ext}^i_{DM}(M_1(X)^\otimes \otimes Q[\mathcal{S}_d])c_\lambda, Q(j))$. Here if $c_\lambda = \sum_{g \in \mathcal{S}_d}a_gg$ with $a_g \in Q$, then we let $c_\lambda'$ be $\sum_{g \in \mathcal{S}_d}a_g\text{sign}(g)g$, that naturally acts on $M_1(X)^\otimes$ as an idempotent morphism (since $c_\lambda' \cdot c_\lambda' = (|d|/\dim_Q Q[\mathcal{S}_d])c_\lambda \cdot c_\lambda'$). The $S^\lambda_+H^\pm_0(X^d, Q(j))$ is the direct summand of $H^\pm(X^d, Q(j))$ determined by action of $c_\lambda'$ on $M_1(X)^\otimes$. The structure of $A_X$ in $\text{Rep}_{GL_{2g}(Q)}$ is described as follows:

Proposition 5.6. Let $Z$ be the set of isomorphism classes of all (finite-dimensional) irreducible representations of $GL_{2g}(Q)$. For $z \in Z$, we denote by $V_z \in \text{Vect}_Q(GL_{2g}(Q))$ the corresponding irreducible representation. Let $1_{DM}$ be a unit of $DM^\otimes$. Then there exist equivalences

$$A_X \simeq \prod_{z \in Z} V_z \otimes \text{Hom}_{DM}(\chi(V_z), 1_{DM}) \simeq \bigoplus_{z \in Z} V_z \otimes \text{Hom}_{DM}(\chi(V_z), 1_{DM})$$

in $\text{Rep}_{GL_{2g}(Q)}$. Here $\text{Hom}_{DM}(-, -) \in \text{Mod}_{\text{DH}_Q}$ is the hom complex in $\text{DM}$; see Section 3.2. The action of $GL_{2g}(Q)$ on the right hand side (and the middle) is given by the action on $V_z$ and the trivial action on the hom complexes.

Let $V = Q^\otimes_{2g} = \mathfrak{h}_1(X)$ be the standard representation of $GL_{2g}(Q)$, i.e., an natural action of $GL_{2g}(Q)$ on $Q^\otimes_{2g}$. If $V_z$ is of the form $(V^\otimes \otimes Q[\mathcal{S}_d])c_\lambda \otimes (\wedge^2 V^\otimes)^\otimes_j$ associated to a Young tableau $\lambda$ and $j \in \mathbb{Z}$, then

$$H^i(\text{Hom}_{DM}(\chi(V_z), 1_{DM})) = S^\lambda_+H^i_+(X^d, Q(jg)).$$

Remark 5.7. Every irreducible representation of $GL_{2g}(Q)$ is isomorphic to

$$(V^\otimes \otimes Q[\mathcal{S}_d])c_\lambda \otimes (\wedge^2 V^\otimes)^\otimes_j$$

for some $\lambda$ and $j \geq 0$. By taking highest weights into account, the set $Z$ can be identified with the set $\{(\lambda_1, \ldots, \lambda_{2g}) \in \mathbb{Z}^{2g}; \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{2g}\}$: for an irreducible representation $W$ of $GL_{2g}(Q)$, the highest weight is defined as the natural action of the diagonal torus subgroup $T \simeq \mathbb{G}_m^{2g}$ in $GL_{2g}(Q)$ on the (one dimensional) invariant subspace $W_U$, where $U$ is the subgroup scheme of the upper triangular invertible matrices with 1’s on the diagonal. When $\lambda_{2g} \geq 0$, $(\lambda_1, \ldots, \lambda_{2g})$ corresponds to the irreducible representation $V^\otimes \otimes Q[\mathcal{S}_d])c_\lambda$ where $\lambda$ has the underlying Young diagram associated to $(\lambda_1, \ldots, \lambda_{2g})$. When $0 > \lambda_{2g}$, $(\lambda_1, \ldots, \lambda_{2g})$ corresponds to the irreducible representation $(V^\otimes \otimes Q[\mathcal{S}_d])c_\lambda \otimes (\wedge^2 V^\otimes)^\otimes_{(-\lambda_{2g})}$ where $\lambda$ has the underlying Young diagram $(\lambda_1 - \lambda_{2g}, \lambda_2 - \lambda_{2g}, \ldots, \lambda_{2g} - \lambda_{2g})$.

Proof. The first assertion follows from Proposition 3.4. The second assertion follows from

$$\text{Ext}^i_{DM}(\chi(V_z), 1_{DM}) = \text{Ext}^i_{DM}((M_1(X)[-1])^\otimes \otimes Q[\mathcal{S}_d])c_\lambda \otimes Q(-jg), 1_{DM})$$

where the right hand side is $S^\lambda_+H^i_+(X^d, Q(jg))$ (note the sign issue due to the shift $[-1]$). \hfill \Box
5.3. We will explain how one can derive the representation theorem of mixed Tate motives from our result. Suppose that $S = \text{Spec } k$ and $X = E$ is an elliptic curve over $S = \text{Spec } k$, that is, $g = 1$. Let $\det : \text{GL}_2(\mathbb{Q}) \to \mathbb{G}_m = \text{Spec } \mathbb{Q}[x^{\pm 1}]$ be the homomorphism of algebraic groups given by the determinant. It gives rise to a symmetric monoidal colimit-preserving functor $\det^* : \text{Rep}^\otimes_{\mathbb{G}_m} \to \text{Rep}^\otimes_{\text{GL}_2(\mathbb{Q})}$, which is determined by a pull-back by $B\text{GL}_2(\mathbb{Q}) \to B\mathbb{G}_m$. Thus we have the composite

$$\chi_{\text{Tate}} : \text{Rep}^\otimes_{\mathbb{G}_m} \xrightarrow{\det^*} \text{Rep}^\otimes_{\text{GL}_2(\mathbb{Q})} \xrightarrow{\chi} \text{DM}^\otimes$$

and its right adjoint $\omega_{\text{Tate}} : \text{DM}^\otimes \to \text{Rep}^\otimes_{\mathbb{G}_m}$. Let $\text{DTM}$ be the smallest stable subcategory of $\text{DM}$, which contains $\mathbb{Q}(1)$ and is closed under small coproducts, tensor products, and duals. The stable (presentable) category $\text{DTM}$ inherits a symmetric monoidal structure in the obvious way. We call this category stable $\infty$-category of mixed Tate motives over $k$. Note that $\text{DTM}$ is contained in $\text{DM}_E$ for any elliptic curve $E$. Let $A_{\text{Tate}} = \omega_{\text{Tate}}(1_{\text{DTM}}) \in \text{CAlg}(\text{Rep}^\otimes_{\mathbb{G}_m})$. The functor $\chi_{\text{Tate}}$ sends the (one dimensional) irreducible representation $\text{id} : \mathbb{G}_m \to \text{Aut}(\mathbb{Q}) = \mathbb{G}_m$ to $\mathbb{Q}(1)$. Applying Proposition 5.2 to $\chi_{\text{Tate}} : \text{Rep}^\otimes(\mathbb{G}_m) \to \text{DTM}^\otimes$ as in the case of elliptic motives we deduce the well-known representation theorem of mixed Tate motives proved by Spitzweck, Levine [29, 30, 44]:

**Corollary 5.8** (Spitzweck, Levine). We have the natural equivalences of symmetric monoidal $\infty$-categories

$$\text{Mod}^\otimes_{A_{\text{Tate}}}(\text{Rep}^\otimes_{\mathbb{G}_m}) \simeq \text{Mod}^\otimes_{\text{Spec } A_{\text{Tate}}/\mathbb{G}_m} \simeq \text{DTM}^\otimes.$$

**Remark 5.9.** Recall that the abelian category of pure Tate motives is equivalent to the category of finite dimensional representations of $\mathbb{G}_m$. We can view $\chi_{\text{Tate}} : \text{Rep}^\otimes_{\mathbb{G}_m} \to \text{DM}_{\text{Tate}}$ as the symmetric monoidal functor from derived $\infty$-category of pure Tate motives to the stable $\infty$-category of mixed Tate motives.

We can easily see that the underlying complex of $\omega_{\text{Tate}}$ sends the (one dimensional) irreducible representation $\text{id} : \mathbb{G}_m \to \text{Aut}(\mathbb{Q}) = \mathbb{G}_m$ to $\mathbb{Q}(1)$. Applying Proposition 5.2 to $\chi_{\text{Tate}} : \text{Rep}^\otimes(\mathbb{G}_m) \to \text{DTM}^\otimes$ as in the case of elliptic motives we deduce the well-known representation theorem of mixed Tate motives proved by Spitzweck, Levine [29, 30, 44].

6. **Motivic Galois group of mixed abelian motives**

In this Section, applying the results in Section 4, 5 we will construct the derived motivic Galois group and (underived) motivic Galois group for $\text{DM}_X$ by applying Theorem 5.3 and based loop constructions. Based loop construction can be considered to be the equivariant bar construction. We then prove Theorem 1.4 and Corollary 1.5.

In this Section, the base scheme is $S = \text{Spec } k$.

6.1. Let $R : \text{DM}^\otimes_X \to \text{Mod}^\otimes_{H\mathbb{Q}}$ be the realization functor associated to Betti cohomology. It is the composite $\text{DM}^\otimes_X \hookrightarrow \text{DM}^\otimes \to \text{Mod}^\otimes_{H\mathbb{Q}}$, where the second functor is the realization functor associated to Betti cohomology constructed in [22, 5.12]. It is a symmetric monoidal colimit-preserving functor. Consider the composite

$$\text{Rep}^\otimes_{\text{GL}_2(\mathbb{Q})} \xrightarrow{\chi} \text{DM}^\otimes_X \xrightarrow{R} \text{Mod}^\otimes_{H\mathbb{Q}}.$$

By the relative version of adjoint functor theorem, we have a lax symmetric monoidal right adjoint functor $U$. Let $1_{\text{Mod}_{H\mathbb{Q}}}$ be a unit of $\text{Mod}^\otimes_{H\mathbb{Q}}$ and $A' = U(1_{\text{Mod}_{H\mathbb{Q}}}) \in \text{CAlg}(\text{Rep}^\otimes_{\text{GL}_2(\mathbb{Q})})$.

**Lemma 6.1.** The stack $[\text{Spec } A'/\text{GL}_2(\mathbb{Q})]$ is isomorphic to $\text{Spec } H\mathbb{Q}$.

**Proof.** Let $\text{GL}_2(\mathbb{Q}) = \text{Spec } B$. We will show that $A'$ has the underlying commutative algebra object $B$ equipped with the natural action of $\text{GL}_2(\mathbb{Q})$ that arises from the multiplication of $\text{GL}_2(\mathbb{Q})$. It suffices to prove that the composite $\text{Rep}^\otimes_{\text{GL}_2(\mathbb{Q})} \to \text{Mod}^\otimes_{H\mathbb{Q}}$ is a
forgetful functor. By Lemma 4.11 and the uniqueness of the left Kan extension, it is enough to show that \( \mathcal{D}_v(N)^\otimes \to \text{Mod}^\otimes_{H_Q} \) is (equivalent to) a forgetful functor. Again noting the uniqueness of a symmetric monoidal exact functor \( \mathcal{D}^-(N)^\otimes \to \text{Mod}^\otimes_{H_Q} \) that extends the composite \( N^\otimes \to \mathcal{D}_v(N)^\otimes \to \text{Mod}^\otimes_{H_Q} \) (see Proposition 4.8) we are reduced to proving that the restriction \( N^\otimes \to \text{Vect}_Q^\otimes \) is a forgetful functor. Thanks to the (classical) tannaka duality, any symmetric monoidal functor \( \rho : N^\otimes \to \text{Vect}_Q^\otimes \) uniquely corresponds to a morphism \( f : \text{Spec} Q \to B\text{GL}_{2g}(Q) \) with \( \rho \simeq f^* \). A morphism \( f \) amounts to a vector bundle of rank \( 2g \) on \( \text{Spec} Q \), which is unique up to isomorphisms. Thus the claim follows. □

GL\(_{2g}(Q)\)-equivariant bar construction. According to Lemma 6.1 there exists a morphism

\[ \text{Spec} H \mathbb{Q} \simeq \left[ \text{Spec} A' / \text{GL}_{2g}(Q) \right] \to \left[ \text{Spec} A_X / \text{GL}_{2g}(Q) \right]. \]

We refer this morphism as the Betti point. Then if \( \Delta_+ \) denotes the category of finite (possibly empty) linearly ordered sets, the Čech nerve \( N(\Delta_+)^{op} \to \text{Sh}(\text{CAlg}^\otimes_{H_Q}) \) associated to the Betti point gives rise to a derived affine group scheme \( \text{MG}(X) \) over \( H \mathbb{Q} \) as the group object \( N(\Delta)^{op} \to N(\Delta_+)^{op} \to \text{Sh}(\text{CAlg}^\otimes_{H_Q}) \). In fact, it is a group object of derived affine schemes. This construction of \( \text{MG}(X) \) is a \( \text{GL}_{2g}(Q)\)-equivariant version of bar construction. For the notion of group objects, Čech nerves and derived group schemes, we refer the reader to [31, 6.1.2, 7.2.2.1], [22, Appendix], [23, Section 4]. The underlying derived affine scheme of \( \text{MG}(X) \) is the fiber product

\[ \text{Spec} H \mathbb{Q} \times_{\left[ \text{Spec} A_X / \text{GL}_{2g}(Q) \right]} \text{Spec} H \mathbb{Q} \]

associated to the Betti point. Let \( \text{Grp}(\mathcal{S}) \) be the \( \infty \)-category of group object of \( \hat{\mathcal{S}} \). From the functorial point of view, as explained in [23, Appendix], the derived affine group scheme \( \text{MG}(X) \) can be viewed as a functor \( \text{CAlg}^\otimes_{H_Q} \to \text{Grp}(\mathcal{S}) \). It is easy to see that \( \text{MG}(X) \) is an derived affine group scheme; we put \( \text{MG}_X = \text{Spec} B_X \). We call this derived group scheme the derived motivic Galois group for \( \text{DM}^\otimes_X \) with respect to the Betti realization.

Automorphism group of the realization functor. We abuse notation and write \( R : \text{DM}^\otimes_{\text{gm}, X} \to \text{PMod}^\otimes_{H_Q} \) for the restriction of \( R : \text{DM}^\otimes_X \to \text{Mod}^\otimes_{H_Q} \). The automorphism group functor \( \text{Aut}(R) \) of \( R : \text{DM}^\otimes_{\text{gm}, X} \to \text{PMod}^\otimes_{H_Q} \) is a functor \( \text{CAlg}^\otimes_{H_Q} \to \text{Grp}(\mathcal{S}) \) which is informally given by \( R \to \text{Aut}(f_R \circ R) \), where \( f_R : \text{PMod}^\otimes_{H_Q} \to \text{PMod}^\otimes_{\text{gm}, X} \) is the base change \((-) \otimes_{H_Q} R \), and \( \text{Aut}(f_R \circ R) \) is \( \text{Map}_{\text{PMod}^\otimes_{\text{gm}, X}}(\text{PMod}^\otimes_{H_Q}, f_R \circ R, f_R \circ R) \). Roughly speaking, the group structure of \( \text{Aut}(f_R \circ R) \in \text{Grp}(\mathcal{S}) \) is determined by the composition of symmetric monoidal natural equivalences. We shall refer the reader to [22, Section 3] for the precise definition.

Theorem 6.2. The automorphism group functor \( \text{Aut}(R) \) of \( R : \text{DM}^\otimes_{\text{gm}, X} \to \text{PMod}^\otimes_{H_Q} \) is representable by the derived affine group scheme \( \text{MG}(X) \).

Proof. We apply [23, 4.6, 4.9] and Corollary 5.4 to \( \text{DM}^\otimes_{\text{gm}, X} \to \text{PMod}^\otimes_{H_Q} \). □

Remark 6.3. In [22] we have constructed derived motivic Galois groups which represent the automorphism group functors of the realization functors in a much more general situation by the abstract machinery of tannakization. Here we give an explicit construction of \( \text{MG}(X) \) by means of equivariant bar constructions. Consequently, Theorem 6.2 reveals the structure of derived motivic Galois groups by means of \( \text{GL}_{2g}(Q)\)-equivariant bar constructions in the case of mixed abelian motives.

The natural morphism \( \text{Spec} H \mathbb{Q} \to \text{Spec} H \mathbb{Q} \times_{B\text{GL}_{2g}(Q)} \left[ \text{Spec} A_X / \text{GL}_{2g}(Q) \right] \simeq \text{Spec} A_X \) gives rise to its Čech nerve as in the case of \( \text{Spec} H \mathbb{Q} \to \left[ \text{Spec} A_X / \text{GL}_{2g}(Q) \right] \), and thus we have a derived affine group scheme \( \text{UG}(X) \) whose underlying derived scheme is given by \( \text{Spec}(H \mathbb{Q} \otimes_{A_X} H \mathbb{Q}) \).
Proposition 6.4. There is a pullback square of derived group schemes

\[
\begin{array}{c}
\text{UG}(X) \\
\downarrow \\
\text{Spec } HQ \\
\downarrow \\
\text{MG}(X) \\
\downarrow \\
\text{GL}_{2g}(Q).
\end{array}
\]

Proof. There is a pullback square of derived stacks

\[
\begin{array}{c}
\text{Spec } A_X \\
\downarrow \\
\text{Spec } [\text{Spec } A_X/GL_{2g}(Q)] \\
\downarrow \\
\text{Spec } HQ \\
\downarrow \\
BGL_{2g}(Q).
\end{array}
\]

All Spec $A_X$, $[\text{Spec } A_X/GL_{2g}(Q)]$ and $BGL_{2g}(Q)$ are pointed; they are equipped with morphisms from the final object Spec $HQ$. The above pullback square can be viewed as a pullback square of pointed derived stacks. Applying the based loop functor to this pullback square, we have the desired pullback square of derived group schemes. \(\square\)

Remark 6.5. Let $H_*(A_X)$ be the $\mathbb{Z}$-graded algebra associated to $A_X$. Using the adjunction and Proposition 5.5, one can compute the graded algebra structure of $H_*(A_X)$ in terms of the cup product on motivic cohomology. It is straightforward and left to the reader. There is a convergent spectral sequence

\[
E_2^{p,q} := \text{Tor}_{p}^{H_*(A_X)}(Q, Q)_q \Rightarrow H_{p+q}(HQ \otimes_{A_X} HQ)
\]

where $Q$ is placed in degree zero, cf. \[32\].

For any commutative differential graded algebra $A$ let $\tau A$ be the quotient of $A$ by the differential graded ideal generated by elements of negative cohomological degrees. As explained in \[22\] Section 5, the rule $A \mapsto \tau A$ determines a functor $\text{CAlg}_{HQ} \rightarrow \text{CAlg}_{HQ}$ which we denote by $\tau$ again. (In explicit terms, $\text{CAlg}_{HQ} \rightarrow \text{CAlg}_{HQ}$ sends $A$ to $\tau A'$ where $A'$ is a cofibrant model of $A$ in the model category of commutative differential graded algebras \[17\].) We then derive a pro-algebraic group $MG(X) = \text{Spec } H^0(\tau B_X)$ over $Q$ from $MG(X)$. The pro-algebraic group $MG(X)$ is the \emph{coarse moduli space} of $MG(X)$ in the sense that the natural morphism $\text{Spec } \tau B_X \rightarrow MG(X)$ is universal among morphisms to pro-algebraic groups. (Remark that $\text{Spec } \tau B_X \rightarrow MG(X)$ can be viewed as the morphism $MG(X) \rightarrow MG(X)$ when we regard them as objects in $\text{Fun}(\text{CAlg}_{HQ}^{\text{dis}}, \text{Grp}(\mathcal{S}))$ since every $\text{Spec } HR \rightarrow MG(X)$ factors though $\text{Spec } \tau B_X$ uniquely up to the contractible space of choice.) We refer to $MG(X)$ as the \emph{motivic Galois group} for $\text{DM}_X^{\otimes}$ with respect to the Betti realization. By the same argument as in \[22\] Section 5, Theorem 5.17, we have

Proposition 6.6. Let $K$ be a $\mathbb{Q}$-field. Let $\text{Aut}(R)(K)$ be the group of isomorphism class of automorphisms of $R$, that is, $\pi_0(\text{Aut}(R)(HK))$. Let $MG(X)(K)$ be the group of $K$-valued points of $MG(X)$. Then there is a natural isomorphism of groups

\[
MG(X)(K) \simeq \text{Aut}(R)(K).
\]

The isomorphisms are functorial among $\mathbb{Q}$-fields in the natural way.

Remark 6.7. Suppose that $\text{DM}_{gm,X}(k, \mathbb{Q})$ admits a motivic $t$-structure, that is, a non-degenerate $t$-structure such that the realization functor is $t$-exact, and the tensor operation $\text{DM}_{gm,X}(k, \mathbb{Q}) \times \text{DM}_{gm,X}(k, \mathbb{Q}) \rightarrow \text{DM}_{gm,X}(k, \mathbb{Q})$ is $t$-exact. Then according to \[23\] 7.12, 7.14, the heart of the motivic $t$-structure is equivalent to the category of finite dimensional representations of $MG(X)$ as a symmetric monoidal abelian category (we will not use this in this paper).
6.2. We will study the structure of the pro-algebraic group scheme $MG(X)$. We slightly change the situation. We replace the Betti realization by the $l$-adic étale realization $R_{\et, \mathbb{Q}_l} : \text{DM}^\otimes_{g_m, X} \subset \text{DM}^\otimes \to \text{Mod}^\otimes_{H^{\mathbb{Q}_l}}$ constructed in Section 7; see Proposition 7.1. By the same construction as in Lemma 6.1, we have a base point called the $l$-adic étale point

$$\text{Spec } H^{\mathbb{Q}_l} \to \text{[Spec } A_X / \text{GL}_{2g}(\mathbb{Q})]$$

associated to the $l$-adic étale realization and the derived affine group scheme $MG_{\et}(X) = \text{Spec } B_X$ over $\text{Spec } H^{\mathbb{Q}_l}$ representing the automorphism group functor $\text{Aut}(R_{\et, \mathbb{Q}_l}) : \text{CAlg}_{H^{\mathbb{Q}_l}} \to \text{Grp}(\mathcal{S})$. Also, we obtain the version $UG_{\et}(X) = \text{Spec}(H^{\mathbb{Q}_l} \otimes_{A_X} H^{\mathbb{Q}_l})$ of $UG(X)$ associated to the base point $\text{Spec } H^{\mathbb{Q}_l} \to \text{Spec } A_X \simeq \text{Spec } H^{\mathbb{Q}_l} \times_{\text{BGL}_{2g}} [\text{Spec } A_X / \text{GL}_{2g}(\mathbb{Q})]$. Let us denote by $MG_{\et}(X)$ and $UG_{\et}(X)$ the pro-algebraic group schemes over $\mathbb{Q}_l$ which are associated to $MG_{\et}(X)$ and $UG_{\et}(X)$ respectively. Explicitly, $UG_{\et}(X) = \text{Spec } H^0(\tau(H^{\mathbb{Q}_l} \otimes_{A_X} H^{\mathbb{Q}_l}))$ and $MG_{\et}(X) = \text{Spec } H^0(\tau B_X)$. Let $MG_{\text{pure}}(X)$ be the Tannaka dual of $NM(X)^\otimes$ and put $MG_{\text{pure}}(X)_{\mathbb{Q}_l} = MG_{\text{pure}}(X) \otimes_{\mathbb{Q}} \mathbb{Q}_l$. Our main results are stated as follows:

**Theorem 6.8.** Let $X$ be an abelian variety of dimension $g$ over a number field $k$. Suppose either (i) or (ii) or (iii):

1. $\text{End}(X \times_k \bar{k}) = \mathbb{Z}$. Assume neither that (a) $2g$ is a $n$-th power for any odd number $n > 1$, nor (b) $2g$ is of the form $\binom{2n}{n}$ for any odd number $n > 1$.
2. $X$ is an elliptic curve. If $X$ has complex multiplication, we suppose that the complex multiplication is defined over $k$.
3. $X$ is a simple abelian variety of prime dimension $\dim X = p$ (including 1) which has complex multiplication, i.e., $\text{End}(X) \otimes \mathbb{Q}$ is a CM-field of degree $2p$. Suppose further that $X$ is absolutely simple, i.e., it is also simple after the base change to an algebraic closure.

Then there exists an exact sequence of pro-algebraic group schemes

$$1 \to UG_{\et}(X) \to MG_{\et}(X) \to MG_{\text{pure}}(X)_{\mathbb{Q}_l} \to 1.$$  

Moreover, $UG_{\et}(X)$ is a connected pro-unipotent group scheme over $\mathbb{Q}_l$.

**Corollary 6.9.**

1. There is an isomorphism of affine group schemes

$$MG_{\et}(X) \simeq UG_{\et}(X) \rtimes MG_{\text{pure}}(X)_{\mathbb{Q}_l}.$$  

2. $UG_{\et}(X)$ is the unipotent radical of $MG_{\et}(X)$, i.e., the maximal normal unipotent closed subgroup.
3. $MG_{\text{pure}}(X)_{\mathbb{Q}_l}$ is the reductive quotient.

**Proof.** By Levi decomposition [18] (see also [28] Section 1]), there is a section of $MG_{\et}(X) \to MG_{\text{pure}}(X)_{\mathbb{Q}_l}$ which is unique up to conjugation by $UG_{\et}(X)$. It gives rise to $MG_{\et}(X) \simeq UG_{\et}(X) \rtimes MG_{\text{pure}}(X)_{\mathbb{Q}_l}$. Other claims are clear. 

We will prove Theorem 6.8. From now on, for simplicity we write $MG$, $UG$, $M_G$, $MG_{\text{pure}}$, and $\text{GL}_{2g}$ for $MG_{\et}(X)$, $UG_{\et}(X)$, $MG_{\et}(X)$, $UG_{\et}(X)$ $MG_{\text{pure}}(X)$ and $\text{GL}_{2g}(\mathbb{Q}_l)$ respectively.

**Proof of the exactness in Theorem 6.8.** The natural sequence $UG \to MG \to \text{GL}_{2g}$ in Proposition 6.3 induces $UG \xrightarrow{\delta} MG \xrightarrow{\iota} \text{GL}_{2g}$. We first claim that for any $\mathbb{Q}_l$-field $K$, the sequence of the groups of $K$-valued points

$$1 \to UG(K) \to MG(K) \to \text{GL}_{2g}(K)$$

is exact. To see this, consider the sequence of mapping spaces

$$\text{Map}(\text{Spec } HK, \text{Spec } A_X) \to \text{Map}(\text{Spec } HK, [\text{Spec } A_X / \text{GL}_{2g}]) \to \text{Map}(\text{Spec } HK, \text{BGL}_{2g}),$$

where $HK$ is a $K$-field.

The map $\text{Map}(\text{Spec } HK, [\text{Spec } A_X / \text{GL}_{2g}]) \to \text{Map}(\text{Spec } HK, \text{BGL}_{2g})$ is an isomorphism by the base change property of the $	ext{Hom}$-functor. The sequence is exact if and only if $\text{Map}(\text{Spec } HK, \text{BGL}_{2g})$ is exact, which follows from the projectivity of $\text{BGL}_{2g}$.
where Spec $A_X \to [\text{Spec } A_X / GL_{2g}] \to BGL_{2g}$ is given in the proof in Proposition 6.3, and the mapping spaces are taken in Fun(CAlg$_{H\bar{Q}_l}$, $\mathcal{S}$). Let Spec $H\bar{Q}_l \to [\text{Spec } A_X / GL_{2g}]$ be the morphism arising from the $l$-adic étale realization functor. For each Spec $H\bar{K} \to \text{Spec } H\bar{Q}_l$, the composition determines $K$-valued base points of $[\text{Spec } A_X / GL_{2g}]$, Spec $A_X$ and $BGL_{2g}$. Again by Proposition 6.3 and the homotopy exact sequence, we have an exact sequence

$$
\pi_2(\text{Map}(\text{Spec } HK, BGL_{2g})) \to \pi_1(\text{Map}(\text{Spec } HK, \text{Spec } A_X)) \simeq \pi_0(UG(K))
$$

$$
\to \pi_1(\text{Map}(\text{Spec } HK, [\text{Spec } A_X / GL_{2g}]))) \simeq \pi_0(MG(K)) \to \pi_1(\text{Map}(\text{Spec } HK, BGL_{2g}))
$$

where the base points of the homotopy groups are defined above. Note that

$$\pi_1(\text{Map}(\text{Spec } K, BGL_{2g})) \simeq GL_{2g}(K)
$$

and $\pi_2(\text{Map}(\text{Spec } HK, BGL_{2g})) \simeq 1$. Proposition 6.6 implies that $\pi_0(MG(K)) \simeq MG(K)$. The similar argument (cf. [22, Theorem 5.18]) also shows $\pi_0(UG(K)) \simeq UG(K)$. Thus we obtain the desired exact sequence.

Next we will prove that a sequence of pro-algebraic group schemes $1 \to UG \xrightarrow{\delta} MG \xrightarrow{t} GL_{2g}$ is exact. We first observe that $\pi_3$ is injective. Let $\text{Ker}(s) = \text{Spec } P$ be the affine (pro-algebraic) group scheme of the kernel of $s$. Then $P$ is of the form of a filtered colimit $\lim_{\lambda} P_\lambda$ where each $P_\lambda$ is a finitely generated commutative Hopf subalgebra of $P$. Since we work over the field $\bar{Q}_l$ of characteristic zero, each algebraic group Spec $P_\lambda$ is reduced, and thus so is Spec $P$. As proved above, the group Ker$(s)(K)$ of $K$-valued points is trivial for any $\bar{Q}_l$-field $K$. Hence we conclude that the unit morphism Spec $\bar{Q}_l \to \text{Ker}(s)$ is an isomorphism, that is, Ker$(s)$ is trivial. Then we prove that the injective homomorphism $UG \to \text{Ker}(t)$ is a surjective morphism of affine group schemes, where Ker$(t)$ is the kernel of $t$. Suppose that $UG \subset \text{Ker}(t)$ is a proper closed subgroup scheme. Since Ker$(t)$ is also reduced, it gives rise to a contradiction that $UG(K) \to \ker(t)(K)$ is bijective for any $\bar{Q}_l$-field $K$. Thus we see that $UG \simeq \text{Ker}(t)$.

Finally, we will prove that the image of $t : MG \to GL_{2g}$ is isomorphic $MG_{\text{pure}}$. The morphism $t$ factors into a sequence of homomorphisms $MG \xrightarrow{t'} G \xrightarrow{t''} GL_{2g}$ of affine group schemes such that $t'$ is surjective and $t''$ is a closed immersion. It suffices to show that $G \simeq MG_{\text{pure}}$. For this purpose, consider the action of the absolute Galois group $\Gamma = \text{Gal}(k/k)$ on $R_{et, \bar{Q}_l} : \text{DM}^{gm,X} \rightarrow \text{PMod}^{H\bar{Q}_l}$. According to Proposition 6.6 and Proposition 7.1 it gives rise to a homomorphism

$$
\Gamma \rightarrow MG(\bar{Q}_l) \simeq \pi_0(\text{Aut}(R_{et, \bar{Q}_l})(H\bar{Q}_l)).
$$

Let $\iota : \text{fVect}^\otimes_{\bar{Q}}(GL_{2g}) \rightarrow \text{DM}^{gm,X}$ be the symmetric monoidal functor defined as the composite $\text{fVect}^\otimes_{\bar{Q}}(GL_{2g}) = \text{NM}_{fr}(X)^\otimes \subset \mathcal{D}^-(\text{Ind}(\text{NM}_{fr}(X)))^\otimes \rightarrow \text{DM}^\otimes$ where the right functor is given in Theorem 5.1 and $\text{NM}_{fr}(X)^\otimes \subset \mathcal{D}^-(\text{Ind}(\text{NM}_{fr}(X)))^\otimes$ is the natural inclusion to the heart. Here $\text{fVect}^\otimes_{\bar{Q}}(GL_{2g})$ is the full subcategory of $\text{Vect}^\otimes_{\bar{Q}}(GL_{2g})$ spanned by finite dimensional representations. Then the composition with $\text{fVect}^\otimes_{\bar{Q}}(GL_{2g}) \xrightarrow{\iota} \text{DM}^{gm,X} \xrightarrow{R_{et, \bar{Q}_l}} \text{PMod}^{H\bar{Q}_l}$ induces

$$
\Gamma \rightarrow MG(\bar{Q}_l) \rightarrow GL_{2g}(\bar{Q}_l).
$$

Here $GL_{2g}(\bar{Q}_l)$ denotes the group of $\bar{Q}_l$-valued points on $GL_{2g}$, which we identify with the automorphism group of the composite $\text{fVect}^\otimes_{\bar{Q}}(GL_{2g}) \rightarrow \text{fVect}^\otimes_{\bar{Q}_l} \subset \text{PMod}^{H\bar{Q}_l}$, where $\text{fVect}^\otimes_{\bar{Q}_l}$ is the symmetric monoidal category of finite dimensional $\bar{Q}_l$-vector spaces (note here that by the classical tannaka duality $\text{fVect}^\otimes_{\bar{Q}}(GL_{2g}) \rightarrow \text{fVect}^\otimes_{\bar{Q}_l}$ is the composition of the forgetful functor and the base change). The standard representation of $GL_{2g}$ maps to $M_1(X)[-1]$ (cf. Theorem 5.1) and the realization functor sends it to $R_{et, \bar{Q}_l}(M_1(X)[-1]) \simeq \bar{Q}_l^\otimes$ by Proposition 7.1, where $\bar{Q}_l^\otimes$ is considered to be a complex placed in degree zero. The image of $g \in MG(\bar{Q}_l)$ in $GL_{2g}$ can be viewed as the action of $g$ on $R_{et, \bar{Q}_l}(M_1(X)[-1]) \simeq \bar{Q}_l^\otimes$. 
Let $A$ be a commutative differential graded $K$-algebra $A$ such that $H^i(A) = 0$ for $i < 0$, which we regard as an object in $\text{CAlg}_{H_K}$. We first recall that the natural morphism $H^0(A) \to A$ in $\text{CAlg}_{H_K}$ is decomposed into a sequence

$$H^0(A) = A(0) \to A(1) \to A(2) \to \cdots \to A(i) \to \cdots$$

in $\text{CAlg}_{H_K}$ such that the natural morphism $\lim_{\to} A(i) \to A$ is an equivalence (here $H^0(A)$ is viewed as a discrete commutative algebra object which belongs to $\text{CAlg}_{HK}$). Moreover, for each $i \geq 0$ the morphism $A(i) \to A(i+1)$ fits in a pushout diagram of the form

$$\begin{array}{ccc}
\text{Sym}^*(C) & \longrightarrow & A(i) \\
\downarrow & & \downarrow \\
K & \longrightarrow & A(i+1)
\end{array}$$

in $\text{CAlg}_K$, where $\text{Sym}^*(C)$ is a free commutative algebra associated to some $C \in \text{Mod}_{HK}$. This is the well-known fact in the theory of commutative differential graded algebras and rational homotopy theory (see e.g. [17]). Here we refer the reader to the proof of [33, VIII, 4.1.4]...
(strictly speaking, in loc. cit. the case of $H^0(A) = K$ is treated, but the argument reveals that our claim holds). The base point $\text{Spec} H \mathbb{Q}_l \to \text{Spec} A$ induces the base points of derived affine schemes in the sequence

$$\text{Spec} A \to \cdots \to Z(i) = \text{Spec} A(i) \to \cdots \to Z(1) = \text{Spec} A(1) \to Z(0) = \text{Spec} A(0)$$

by the composition.

Let $\pi_n(\text{Spec} A_X) : \text{CAlg}^\text{dis}_{HK} \to \text{Grp}$ be a group-valued functor which to any $R \in \text{CAlg}^\text{dis}_{HK}$ assigns the homotopy group $\pi_n$ by the composition.

such that the functor $V$ (strictly speaking, in schemes in the sequence is discrete, i.e., 0-truncated. Suppose that our claim holds for the case $i = 0$. To this end, we define $\pi_n(Z(i))$ in a similar way. We first prove that $\pi_n(Z(i))$ is represented by a pro-unipotent group scheme over $\mathbb{Q}_l$. We use the induction on $i$. The case $i = 0$ is obvious since the mapping space $\text{Map} \text{Aff}_{HK}(\text{Spec} R, Z(0))$ is discrete, i.e., 0-truncated. Suppose that our claim holds for the case $i \leq r$. Consider the pullback diagram

$$\begin{array}{ccc}
Z(r+1) & \to & \text{Spec} K \\
\downarrow & & \downarrow \\
Z(r) & \to & \text{Spec} \text{Sym}^n C
\end{array}$$

in $\text{Aff}_{HK}$. By homotopy exact sequence, we have an exact sequence

$$\pi_{n+1}(Z(r))(R) \xrightarrow{\eta} \pi_{n+1}(\text{Spec} \text{Sym}^n C)(R) \xrightarrow{\mu} \pi_n(Z(r+1))(R) \to \pi_n(Z(r))(R) \to \pi_n(\text{Spec} \text{Sym}^n C)(R).$$

Note that by the assumption $\pi_{n+1}(Z(r))$ and $\pi_n(Z(r))$ are pro-unipotent group schemes over $K$. Since $\pi_{n+1}(Z(r))(R)$ is commutative and $K$ is characteristic zero, there is a $K$-vector space $V$ such that the functor $\pi_{n+1}(Z(r))$ is given by $R \mapsto \text{Hom}_K(V, R)$. (Namely, in characteristic zero every commutative pro-unipotent group scheme is of the form $\mathbb{G}_a = \text{Spec} K[I]$ for some set $I$.) Here $\text{Hom}_K(\cdot, \cdot)$ indicates the group of homomorphism of $K$-vector spaces. Unwinding the definitions, $\pi_{n+1}(\text{Spec} \text{Sym}^n C)$ is a commutative pro-unipotent group scheme given by $R \mapsto \text{Hom}_K(H^{n+1}(C), R)$ as a group-valued functor. The homomorphisms $\text{Hom}_K(V, R) \to \text{Hom}_K(H^{r+1}(C), R)$ which are functorial with respect to $R$ determines a homomorphism $\xi : H^{r+1}(C) \to V$ of $K$-vector spaces. The cokernel $\text{Coker}(\eta)$ of $\pi_{n+1}(Z(r)) \to \pi_{n+1}(\text{Spec} \text{Sym}^n C)$ is a commutative pro-unipotent group scheme given by $R \mapsto \text{Hom}_K(\text{Ker}(\xi), R)$. We will show that the kernel $\text{Ker}(\mu)$ of $\mu : \pi_n(Z(r)) \to \pi_n(\text{Spec} \text{Sym}^n C)$ is a pro-unipotent group scheme. Note that $\pi_n(\text{Spec} \text{Sym}^n C)$ is also a pro-algebraic group scheme; we put $\pi_n(\text{Spec} \text{Sym}^n C) = \text{Spec} D$ (moreover, it is commutative and pro-unipotent). Put $\text{Spec} Q = \text{Spec} \lim_{\lambda \in \Lambda} Q_{\lambda} = \pi_n(Z(r))$ where $\{Q_{\lambda} \}_{\lambda \in \Lambda}$ is a filtered diagram of finitely generated commutative Hopf subalgebras in $Q$ such that each $\text{Spec} Q_{\lambda}$ is a unipotent algebraic group scheme. The kernel $\text{Ker}(\mu)$ is $\text{Spec} K \times_{\text{Spec} D} \text{Spec} Q = \text{Spec} Q/I$ which is the fiber over the unit $\text{Spec} K \to \text{Spec} D$. Hence $\text{Ker}(\mu) = \lim_{\lambda \in \Lambda} \text{Spec} Q_{\lambda}/(Q_{\lambda} \cap I)$. Here $\text{Spec} Q_{\lambda}/(Q_{\lambda} \cap I)$ is a closed subgroup scheme in $\text{Spec} Q_{\lambda}$ which is the image of $\text{Ker}(\mu)$ in $\text{Spec} Q_{\lambda}$. Note that a closed subgroup scheme of a unipotent algebraic group is unipotent. We then conclude that $\text{Ker}(\mu)$ is a pro-unipotent group scheme. Using the surjective map $\pi_n(Z(r+1))(I/Q) \to \text{Ker}(\mu)(Q/I)$ we have its section $\text{Ker}(\mu) \to \pi_n(Z(r+1))$. Thus $\pi_n(Z(r+1)) \simeq \text{Coker}(\eta) \times \text{Ker}(\mu)$ as a set-valued functor. It follows that $\pi_n(Z(r+1))$ is an affine group scheme. Moreover, it is an extension of pro-unipotent group scheme by a pro-unipotent group scheme. Consequently, $\pi_n(Z(r+1))$ is represented by a pro-unipotent group scheme over $K$. Finally, it remains to prove that the natural map
\[ \pi_n(\text{Spec } A) \to \lim \pi_n(Z(i)) \] is an equivalence. Taking account of Milnor exact sequence associated to \( \text{Spec } A \sim \lim_i Z(i) \) we are reduced to showing that \( \lim_1 \pi_n(Z(i))(R) = 0 \) for \( n \geq 2 \). Since \( \pi_n(Z(i)) \) is commutative and unipotent, we have \( \pi_n(Z(i))(R) \sim \text{Hom}_K(V(i), R) \) for some \( K \)-vector space \( V(i) \) (\( \pi_n(Z(i)) \) is given by \( \text{Hom}_K(V(i), -) \)). It is enough to show that \( \lim_1 \text{Hom}_K(V(i), R) = 0 \) for any \( K \)-algebra \( R \). By definitions, \( \lim_1 \text{Hom}_K(V(i), R) \sim \pi_0(\lim_i \text{Map}_{\text{Mod}_{\text{HK}}}(V(i), R[1])) = \pi_0(\text{Map}_{\text{Mod}_{\text{HK}}}(\lim_i V(i), R[1])) = \text{Ext}^1_{\text{lim}_i}(V(i), R) \). Since \( \lim_i V(i) \) is obviously a \( K \)-vector space, \( \text{Ext}^1_{\text{lim}_i}(V(i), R) \) is trivial. Then the isomorphism \( \pi_n(\text{Spec } A) \sim \lim_i \pi_n(Z(i)) \) shows that \( \pi_n(\text{Spec } A) \) is represented by a pro-unipotent group scheme over \( K \).

Now we will show that there is a natural equivalence \( \pi_1(\text{Spec } \tau A) \simeq \text{UG} \). Note first the natural equivalences \( \pi_1(\text{Spec } \tau A) \simeq \pi_1(\text{Spec } A) \simeq \pi_0(\text{Spec } H\mathbb{Q}_l \otimes A_X H\mathbb{Q}_l) = \pi_0(\text{UG}) \) as group-valued functors \( \text{CAlg}_{H\mathbb{Q}_l}^\text{dis} \to \text{Grp} \). By compositions with the natural inclusions \( \text{Grp} \to \text{Grp}(\widehat{S}) \) and \( \text{CAlg}_{H\mathbb{Q}_l}^\text{dis} \to \text{CAlg}_{H\mathbb{Q}_l} \), let us consider \( \pi_1(\text{Spec } A) \) and \( \text{UG} \) to be functors \( \text{CAlg}_{H\mathbb{Q}_l}^\text{dis} \to \text{Grp}(\widehat{S}) \). Thus the natural morphism \( \text{UG} \to \pi_1(\text{Spec } A) \) given by \( \text{UG}(R) \to \pi_0(\text{UG}(R)) \) is universal among morphisms functors taking values in discrete groups. On the other hand, \( \text{UG} \to \text{UG} \) is universal among morphisms to affine group schemes over \( \mathbb{Q}_l \) in \( \text{CAlg}_{H\mathbb{Q}_l}^\text{dis} \to \text{Grp}(\widehat{S}) \). Since \( \pi_1(\text{Spec } \tau A) \) is an affine group scheme, we see that \( \pi_1(\text{Spec } \tau A) \simeq \text{UG} \). This implies that \( \text{UG} \) is a pro-unipotent group scheme over \( \mathbb{Q}_l \).

Finally, we remark that \( \text{UG} \) is connected. If we set \( \text{UG} = \text{Spec } R \), then \( R \) is a filtered colimit \( \lim_\lambda R_\lambda \) such that \( R_\lambda \) is a commutative Hopf subalgebra and \( \text{Spec } R_\lambda \) is a unipotent algebraic group. Since we work over the field of characteristic zero, thus \( \text{Spec } R_\lambda \) is connected (if otherwise, the (nontrivial) quotient by the identity component is unipotent and we have a contradiction). Therefore \( R \) has non idempotent element. This means \( \text{UG} \) is connected. \( \square \)

**Remark 6.10.** In the light of Mumford-Tate conjecture it is desirable to have an argument that uses Hodge realization.

### 7. Étale Realization

#### 7.1. In this Section, we construct an \( l \)-adic étale realization functor from \( \text{DM}^\otimes \) at the level of symmetric monoidal \( \infty \)-categories, which equips with an action of the absolute Galois group.

We fix some notation. Some are different from those used in Section 4–6 since we here use various coefficients. Let \( R \) be a commutative ring. Unlike in Section 4.2.1 where we consider the \( \mathbb{Q} \)-linear category and the general base \( S \), we here suppose that the base scheme \( S \) is the Zariski spectrum of a perfect field \( k \) and let \( \text{Cor} \) be the \( \mathbb{Z} \)-linear category of finite \( S \)-correspondences. Let \( \text{Sh}_{\text{ét}}(\text{Cor}, R) \) be the abelian category of étale sheaves with transfers with \( R \)-coefficients. Namely, an object in \( \text{Sh}_{\text{ét}}(\text{Cor}, R) \) is an additive functor \( F : \text{Cor}^{\text{op}} \to R \)-mod such that \( \text{Sm}^{\text{op}}_k \to \text{Cor}^{\text{op}} \xrightarrow{F} R \)-mod is an étale sheaf (cf. [34], Lec. 6]). Here \( k \) is a perfect field and \( R \)-mod is the abelian category of \( R \)-modules. If \( \iota : \mathcal{L}(\text{Spec } k) \to \mathcal{L}(\mathbb{G}_m) \) is a morphism induced by the unit morphism \( \text{Spec } k \to \mathbb{G}_m \), we let \( R(1) = \text{Coker}(\iota)[-1] \) in \( \text{Comp}(\text{Sh}_{\text{ét}}(\text{Cor}, R)) \). We denote by \( R(X) \in \text{Comp}(\text{Sh}_{\text{ét}}(\text{Cor}, R)) \) the image of \( X \in \text{Sm}_{/k} \) under \( \mathcal{L} \), which we regard as a complex placed in degree zero. As in Section 4.3.2 consider the symmetric monoidal presentable category \( \text{Sp}^\otimes_{R(1)}(\text{Comp}(\text{Sh}_{\text{ét}}(\text{Cor}, R))) \). For simplicity, we shall write \( \text{MSP}^\otimes_{\text{ét}}(R) \) for \( \text{Sp}^\otimes_{R(1)}(\text{Comp}(\text{Sh}_{\text{ét}}(\text{Cor}, R))) \), but we often omit the superscript \( \otimes \). We equip \( \text{MSP}^\otimes_{\text{ét}}(R) \) with the stable symmetric monoidal model structure described in [3] Example 7.15 or Section 4.3.2 (but we here use the localization by étale hypercoverings instead of Nisnevich hypercoverings). We abuse notation and write \( R(X) \) (resp. \( R(1) \)) also for the images of \( R(X) \) (resp. \( R(1) \)) in \( \text{MSP}^\otimes_{\text{ét}}(R) \) and the associated \( \infty \)-category. Let \( \text{Sh}(\text{Cor}, R) \) be the Nisnevich version of \( \text{Sh}_{\text{ét}}(\text{Cor}, R) \), that is, it consists of Nisnevich sheaves of \( R \)-modules.
Proposition 7.1. Let $l$ be a prime number which is different from the characteristic of $k$. Then there exists a symmetric monoidal exact functor

$$R_{\et, l} : \text{DM}^\otimes_{\gm}(Z) \to \text{Mod}^\otimes_{H\et}$$

which has the following properties:

1. for any $X \in \text{SmPr}_{/k}$ and $m \in \mathbb{Z}$, $n \in \mathbb{Z}$, there is a natural isomorphism $H^n(R_{\et, l}(Z(X)^{\vee} \otimes Z(m))) \simeq H^n(\overline{X}(Z, l(m)))$, where $Z(X)^{\vee}$ is a dual of $Z(X)$, $\overline{X}$ is $X \times k$ with an algebraic closure $k$ of $k$, and $H^n_{\et}(\overline{X}, l(m))$ is the $l$-adic étale cohomology $\lim_{\rightarrow j} H^n_{\et}(X, \mu_l^{\otimes m})$.

2. there is an action of $\Gamma = \text{Gal}(\overline{k}/k)$ on $R_{\et, l}$, that is a morphism $\Gamma \to \text{Aut}(R_{\et, l})$ as objects in $\text{Grp}(\mathcal{S})$, such that it induces the action on $H^n(R_{\et, l}(Z(X)^{\vee} \otimes Z(m)))$ which coincides with the Galois action on $H^n_{\et}(\overline{X}, l(m))$.

Moreover, there is also a symmetric monoidal exact functor $R_{\et, Q_l} : \text{DM}^\otimes_{\gm}(Q) \to \text{PMod}^\otimes_{H\et}$ satisfying the same properties; $H^n(R_{\et, Q_l}(Q(X)^{\vee} \otimes Q(m))) = H^n_{\et}(\overline{X}, Q_l(m))$ and the Galois actions coincide.

Remark 7.2. For any projective smooth scheme $X$ over $k$, $Z(X)$ admits a dual $Z(X)^{\vee}$ (cf. [9, 11.3.4 (4)]. $\text{Aut}(R_{\et, l})$ is the automorphism group space of $R_{\et, l}$, that is, a group object in $\mathcal{S}$ which is $\text{Aut}(R_{\et, l})(H_{Q_l})$ in Section 6; see Section 7.4. The essential image of $R_{\et, Q_l}$ is contained in $\text{PMod}_{H\et}$ since every object in $\text{DM}^\otimes_{\gm}(Q)$ is dualizable.

Using the machinery of mixed Weil theory (see [9, 17.2]) one can construct an étale realization functor with $Q_l$-coefficients at the level of symmetric monoidal ∞-categories (as we did in [22]). New pleasant features here are (1) the realization functor comes equipped with Galois action, and (2) we can work with $Z_l$-coefficients. Our construction makes use of the rigidity theorem due to Suslin-Voevodsky [31], [32], and the derived generalization of Grothendieck’s existence theorem by Lurie [23, XII].

7.2. We will begin by constructing some symmetric monoidal Quillen functors. Let $f : R \to R'$ be a homomorphism of commutative rings. Then it gives rise to a symmetric monoidal colimit-preserving functor $\hat{f} : \text{Comp}(\text{Sh}_{\et}(\text{Cor}, R)) \to \text{Comp}(\text{Sh}_{\et}(\text{Cor}, R'))$ which carries $R(X)$ to $R'(X)$. Thus we have a symmetric monoidal colimit-preserving functor $\text{Sp}(\hat{f}) : \text{MSP}_{\et}(R) \to \text{MSP}_{\et}(R')$ which carries the symmetric spectrum $(E_n, \sigma_n)_{\geq 0}$ (here each $\sigma_n$ denotes the structure map $E_n \otimes R(1) \to E_{n+1}$) to $(\hat{f}(E_n))_{\geq 0}$ with structure maps $\hat{f}(E_n) \otimes R'(1) \simeq \hat{f}(E_n \otimes R(1)) \to \hat{f}(E_{n+1})$. Moreover, we see the following:

Lemma 7.3. $\text{Sp}(\hat{f}) : \text{MSP}_{\et}(R) \to \text{MSP}_{\et}(R')$ is a left Quillen adjoint functor.

Proof. The functor $\text{Comp}(\text{Sh}_{\et}(\text{Cor}, R)) \to \text{Comp}(\text{Sh}_{\et}(\text{Cor}, R'))$ is a left adjoint by adjoint functor theorem. Let $\overline{f}$ be a right adjoint of $\hat{f}$. Moreover, $\overline{f}$ is a left Quillen functor with respect to the model structure given in [3, Example 4.12] since it preserves generating cofibrations and generating trivial cofibrations, and $R(X \times A^1) \to R(X)$ induced by the projection maps to $R'(X \times A^1) \to R'(X)$; see [3, Proposition 4.9]. Observe that $\text{Sp}(\hat{f})$ is a left Quillen adjoint.
Explicitly, its right adjoint carries \((F_n, \tau_n)_{n \geq 0}\) to \((\tilde{g}(F_n))_{n \geq 0}\) with structure maps given by the composite
\[
\tilde{f}(\tilde{g}(F_n) \otimes R(1)) \simeq \tilde{f}(\tilde{g}(F_n)) \otimes R'(1) \rightarrow F_0 \otimes R'(1) \overset{\tau_n}{\rightarrow} F_{n+1}.
\]
Therefore, the right adjoint preserves termwise \(A^1\)-local fibrations and termwise \(A^1\)-local equivalences, and thus
\[
\mathrm{MSp}_{\mathfrak{Et}}(R) \rightleftarrows \mathrm{MSp}_{\mathfrak{Et}}(R')
\]
is a Quillen adjunction with respect to the \(A^1\)-local projective model structures (cf. Section 4.3.2). Then according to [20, Theorem 2.2], to see that
\[
\mathrm{MSp}_{\mathfrak{Et}}(R) \rightleftarrows \mathrm{MSp}_{\mathfrak{Et}}(R')
\]
is a Quillen adjunction with respect to the stable model structures, it is enough to observe that
\[
\mathrm{Sp}((\tilde{f})'(s_n^C)) \quad \text{is a stable equivalence whenever} \quad n \geq 0 \quad \text{and} \quad C \quad \text{is either a domain or codomain of a generating cofibration in} \quad \mathrm{Comp}(\mathrm{Sh}_{\mathfrak{Et}}(\mathrm{Cor}, R)) \quad \text{(see [20, 7.7] for the notation).}
\]
Note that
\[
\mathrm{Sp}((\tilde{f})'(s_n^C)) = s_n^C.
\]
Hence our claim is clear. \(\Box\)

Let \(l\) be a prime number which is different from the characteristic of \(k\). For each \(n \geq 1\) the natural projection \(Z \rightarrow Z/l^n\) induces a symmetric monoidal left Quillen adjoint functor
\[
\mathrm{MSp}_{\mathfrak{Et}}(Z) \rightarrow \mathrm{MSp}_{\mathfrak{Et}}(Z/l^n)
\]
which we denote by \(p_n\). Moreover, the projective system
\[
\cdots \rightarrow Z/l^n \rightarrow \cdots \rightarrow Z/l^2 \rightarrow Z/l\quad \text{induces a diagram} \quad (\bigwedge)
\]
of stable symmetric monoidal model categories
\[
\mathrm{MSp}_{\mathfrak{Et}}(Z) \rightarrow \cdots \rightarrow \mathrm{MSp}_{\mathfrak{Et}}(Z/l^n) \rightarrow \cdots \rightarrow \mathrm{MSp}_{\mathfrak{Et}}(Z/l^2) \rightarrow \mathrm{MSp}_{\mathfrak{Et}}(Z/l)
\]
in which all arrows are symmetric monoidal left Quillen functors.

Consider the Quillen adjoint pair
\[
\Sigma^\infty : \mathrm{Comp}(\mathrm{Sh}_{\mathfrak{Et}}(\mathrm{Cor}, Z/l^n)) \rightleftarrows \mathrm{MSp}_{\mathfrak{Et}}(Z/l^nZ) : \Omega^\infty.
\]
We here equip \(\mathrm{Comp}(\mathrm{Sh}_{\mathfrak{Et}}(\mathrm{Cor}, Z/l^n))\) with the model structure given in [3] 4.12] (or Section 4.3.2) such that weak equivalences (resp. fibrations) are exactly \(A^1\)-local equivalences (resp. \(A^1\)-local fibrations). We refer to it as the \(A^1\)-local model structure. For ease of notation, we write \(M_{\mathfrak{Et}}(Z/l^n)\) for \(\mathrm{Comp}(\mathrm{Sh}_{\mathfrak{Et}}(\mathrm{Cor}, Z/l^n))\). Observe that it is a Quillen equivalence. According to [31, 8.19], tensoring with \(Z/l^n(1)\) is invertible in the homotopy category of \(M_{\mathfrak{Et}}(Z/l^n)\), thus we deduce from [20, Theorem 8.1] that the pair \((\Sigma^\infty, \Omega^\infty)\) is a Quillen equivalence.

Let \(\mathrm{Sh}_{\mathfrak{Et}}(\mathfrak{Et}/k, Z/l^n)\) be the Grothendieck abelian category of sheaves of \(Z/l^n\)-modules on the étale site \(\mathfrak{Et}/k\) which consists of étale morphisms of finite type to \(\text{Spec} \, k\). Let \(X \in \mathfrak{Et}/k\).

Let \(Z/l^n[X]\) be the étale sheaf \(\mathfrak{Et}^{op}/k \rightarrow Z/l^n\)-mod associated to the presheaf \(Y \mapsto Z/l^n \cdot \mathrm{Hom}_k(Y, X)\) where \(Z/l^n\). \(\mathrm{Hom}_k(Y, X)\) is the free \(Z/l^n\)-module generated by the set of \(k\)-morphisms \(\mathrm{Hom}_k(Y, X)\). This sheaf is the restriction of \(Z/l^n(X)\) in \(\mathrm{Sh}_{\mathfrak{Et}}(\mathrm{Cor}, Z/l^n)\) to \(\mathfrak{Et}^{op}/k\) (by the classical Galois theory). The abelian category \(\mathrm{Sh}_{\mathfrak{Et}}(\mathfrak{Et}/k, Z/l^n)\) has a set of generators \(\{Z/l^n[X]\}_{X \in \mathfrak{Et}/k}\). Recall that the symmetric monoidal category \(\mathrm{Sh}_{\mathfrak{Et}}(\mathfrak{Et}/k, Z/l^n)\) is equivalent to the symmetric monoidal category \(Z/l^n\text{-mod}_\Gamma\) of discrete \(Z/l^n\)-modules with action of \(\Gamma\), where \(\Gamma\) is the absolute Galois group \(\mathrm{Gal}(\bar{k}/k)\) (\(k\) is an algebraic closure of \(k\)). In the symmetric monoidal category \(\mathrm{Sh}_{\mathfrak{Et}}(\mathfrak{Et}/k, Z/l^n)\), \(Z/l^n[X] \otimes Z/l^n[Y] = Z/l^n[X \times_k Y]\) and the commutative constraint is determined by the flip \(X \times_k Y \simeq Y \times_k X\). For a \(k\)-field \(L\) the equivalences \(\mathrm{Sh}_{\mathfrak{Et}}(\mathfrak{Et}/k, Z/l^n) \simeq Z/l^n\text{-mod}_\Gamma\) send \(Z/l^n[\text{Spec} \, L]\) to \(Z/l^n \cdot \mathrm{Hom}_k(\text{Spec} \, k, \text{Spec} \, L)\) with action of \(\Gamma\) induced by composition. Here \(Z/l^n \cdot \mathrm{Hom}_k(\text{Spec} \, k, \text{Spec} \, L)\) is a free \(Z/l^n\)-module generated by the set of morphisms from \(\text{Spec} \, k\) to \(\text{Spec} \, L\) over \(k\). It carries \(Z/l^n[\text{Spec} \, L] \otimes Z/l^n[\text{Spec} \, L'] = Z/l^n[\text{Spec} \, L \times_k \text{Spec} \, L']\) to
\[
Z/l^n \cdot \mathrm{Hom}_k(\text{Spec} \, k, \text{Spec} \, L) \otimes Z/l^n \cdot \mathrm{Hom}_k(\text{Spec} \, k, \text{Spec} \, L')
\]
equipped with the action of \(\Gamma\) (by the tensor operation). Since \(\{Z/l^n[X]\}_{X \in \mathfrak{Et}/k}\) is the set of compact generators, and the tensor operations preserves colimits in each variable, thus we see that the equivalences \(\mathrm{Sh}_{\mathfrak{Et}}(\mathfrak{Et}/k, Z/l^n) \simeq Z/l^n\text{-mod}_\Gamma\) are extended to symmetric monoidal equivalences.
Let $\text{Ét}_{/k}$ be the étale site over $\text{Spec} \bar{k}$. The geometric point $q : \text{Spec} \bar{k} \to \text{Spec} k$ given by one inclusion $k \subset \bar{k}$ determines the exact pullback functor
\[
q_n^* : \text{Sh}_\text{ét}(\text{Ét}_{/k}, \mathbb{Z}/l^n) \to \text{Sh}_\text{ét}(\text{Ét}_{/k}, \mathbb{Z}/l^n).
\]
This sends $\mathbb{Z}/l^n[X]$ to the sheaf $\mathbb{Z}/l^n[X \times_k \text{Spec} \bar{k}]$, that is given by $Y \mapsto \mathbb{Z}/l^n \cdot \text{Hom}_{\text{Spec} \bar{k}}(Y, X \times_k \text{Spec} \bar{k})$. Note that there is a symmetric monoidal equivalence $\text{Sh}_\text{ét}(\text{Ét}_{/k}, \mathbb{Z}/l^n) \simeq \mathbb{Z}/l^n\text{-mod}$ which carries $F$ to $F(\text{Spec} \bar{k})$. If one identifies $\text{Sh}_\text{ét}(\text{Ét}_{/k}, \mathbb{Z}/l^n)$ with $\mathbb{Z}/l^n\text{-mod}$, then $q_n^*$ is equivalent to the forgetful functor $\mathbb{Z}/l^n\text{-mod} \to \mathbb{Z}/l^n\text{-mod}$ as symmetric monoidal functors.

We equip $\text{Comp}(\text{Sh}_\text{ét}(\text{Ét}_{/k}, \mathbb{Z}/l^n))$ and $\text{Comp}(\text{Sh}_\text{ét}(\text{Ét}_{/k}, \mathbb{Z}/l^n))$ with the symmetric monoidal model structures given in [8, Proposition 3.2, Example 2.3], in which weak equivalences are exactly quasi-isomorphisms.

Let $v_n : \text{Sh}_\text{ét}(\text{Ét}_{/k}, \mathbb{Z}/l^n) \to \text{Sh}_\text{ét}(\text{Cor}, \mathbb{Z}/l^n)$ be a left adjoint exact functor, given in [33, 6.7–6.11], which carries $F : \text{Ét}^{\text{op}}_{/k} \to \mathbb{Z}/l^n\text{-mod}$ to a unique sheaf with transfers $\overline{F} : \text{Cor}^{\text{op}} \to \mathbb{Z}/l^n\text{-mod}$ such that the composite $\text{Ét}^{\text{op}}_{/k} \to \text{Cor}^{\text{op}} \to \mathbb{Z}/l^n\text{-mod}$ is $F$. The right adjoint is determined by the composition with $\text{Ét}^{\text{op}}_{/k} \to \text{Cor}^{\text{op}}$. For any $X \in \text{Ét}_{/k}$, $\mathbb{Z}/l^n[X] \in \text{Sh}_\text{ét}(\text{Ét}_{/k}, \mathbb{Z}/l^n)$ maps to $\mathbb{Z}/l^n(X)$ as an object $\text{Sh}_\text{ét}(\text{Cor}, \mathbb{Z}/l^n)$. Let $\text{Sh}_\text{ét}^{\text{rep}}(\text{Ét}_{/k}, \mathbb{Z}/l^n)$ be the full subcategory of $\text{Sh}_\text{ét}(\text{Ét}_{/k}, \mathbb{Z}/l^n)$ spanned by representable objects $\{\mathbb{Z}/l^n[X]\}_{X \in \text{Ét}_{/k}}$. For $S \in \text{Sh}_\text{ét}(\text{Ét}_{/k}, \mathbb{Z}/l^n)$, we have $S \simeq \lim_{\longleftarrow} \mathbb{Z}/l^n[X] \to S \mathbb{Z}/l^n[X]$ where $\mathbb{Z}/l^n[X] \to S$ run over the small overcategory $\text{Sh}_\text{ét}^{\text{rep}}(\text{Ét}_{/k}, \mathbb{Z}/l^n)/S$. Since $v_n$ preserves small colimits, there are natural equivalences
\[
v_n(S) \otimes v_n(T) \simeq (\lim_{\longrightarrow} v_n(\mathbb{Z}/l^n[X]) \otimes (\lim_{\longrightarrow} v_n(\mathbb{Z}/l^n[Y])))
\]
\[
\simeq (\lim_{\longrightarrow} (\lim_{\longrightarrow} (\mathbb{Z}/l^n(X) \otimes \mathbb{Z}/l^n(Y))))
\]
\[
\simeq v_n((\lim_{\longrightarrow} (\lim_{\longrightarrow} (\mathbb{Z}/l^n[X] \otimes \mathbb{Z}/l^n[Y])))
\]
\[
\simeq v_n((\lim_{\longrightarrow} \mathbb{Z}/l^n[X] \otimes (\lim_{\longrightarrow} \mathbb{Z}/l^n[Y]))
\]
\[
\simeq v_n((\lim_{\longrightarrow} (\mathbb{Z}/l^n[X]) \otimes (\lim_{\longrightarrow} \mathbb{Z}/l^n[Y]))
\]
where $\lim'$ stands for the colimit in $\text{Sh}_\text{ét}(\text{Cor}, \mathbb{Z}/l^n)$. Similarly, the commutative constraint $i : S \otimes T \simeq T \otimes S$ commutes with $v_n(S) \otimes v_n(T) \simeq v_n(T) \otimes v_n(S)$. Moreover, $v_n(\mathbb{Z}/l^n[\text{Spec} k]) = \mathbb{Z}/l^n(\text{Spec} k)$. Hence we easily see that $v_n$ is (extended to) a symmetric monoidal functor. Thus we have an adjoint pair
\[
v_n : \text{Comp}(\text{Sh}_\text{ét}(\text{Ét}_{/k}, \mathbb{Z}/l^n)) \rightleftharpoons \text{Comp}(\text{Sh}_\text{ét}(\text{Cor}, \mathbb{Z}/l^n)) : \text{res}
\]
where the left adjoint $v_n$ is symmetric monoidal, and $\text{res}$ is induced by the composition with the natural functor $\text{Ét}^{\text{op}}_{/k} \to \text{Cor}^{\text{op}}$.

**Lemma 7.4.** We abuse notation and write
\[
q_n^* : \text{Comp}(\text{Sh}_\text{ét}(\text{Ét}_{/k}, \mathbb{Z}/l^n)) \to \text{Comp}(\text{Sh}_\text{ét}(\text{Ét}_{/k}, \mathbb{Z}/l^n))
\]
and
\[
v_n : \text{Comp}(\text{Sh}_\text{ét}(\text{Ét}_{/k}, \mathbb{Z}/l^n)) \to \text{Comp}(\text{Sh}_\text{ét}(\text{Cor}, \mathbb{Z}/l^n))
\]
for symmetric monoidal functors induced by $q_n^*$ : $\text{Sh}_\text{ét}(\text{Ét}_{/k}, \mathbb{Z}/l^n) \to \text{Sh}_\text{ét}(\text{Ét}_{/k}, \mathbb{Z}/l^n)$ and $v_n : \text{Sh}_\text{ét}(\text{Ét}_{/k}, \mathbb{Z}/l^n) \to \text{Sh}_\text{ét}(\text{Cor}, \mathbb{Z}/l^n)$ respectively. Then both $q_n^*$ and $v_n$ are left Quillen adjoint functors.
Proof. According to the definitions of generating (trivial) cofibrations and [8, Theorem 2.14], we see that \( q_n^* \) is a left Quillen adjoint. Next we will show that \( v_n \) is a left Quillen adjoint. We equip \( \text{Comp}(\text{Sh}_{\text{ét}}(\text{Cor}, \mathbb{Z}/l^n)) \) with the symmetric monoidal model structure given in [8, Example 2.4], in which weak equivalences are exactly quasi-isomorphisms. Then the \( \mathbb{A}^1 \)-local model structure is its left Bousfield localization with respect to \( \{ \mathbb{Z}/l^n(X \times \mathbb{A}^1) \to \mathbb{Z}/l^n(X) \}_{X \in \text{Sm}_{/k}} \). Thus it is enough to prove that \( v_n \) is left Quillen when \( \text{Comp}(\text{Sh}_{\text{ét}}(\text{Cor}, \mathbb{Z}/l^n)) \) is endowed with the model structure given in [8, Example 2.4]. To this end, note that \( v_n \) preserves quasi-isomorphisms. It remains to show that \( v_n \) preserves generating cofibrations. But it is clear from the definitions of generating cofibrations (see [8, Definition 2.2]) and the fact that \( v_n(\mathbb{Z}/l^n[X]) = \mathbb{Z}/l^n(X) \).

We obtain the diagram of symmetric monoidal left Quillen functors

\[
\begin{array}{ccc}
\text{MSp}_{\text{ét}}(\mathbb{Z}) & \xrightarrow{\Sigma^\infty} & \text{MSp}_{\text{ét}}(\mathbb{Z}/l^n) \\
\downarrow v_n & & \downarrow q_n^* \\
\text{Comp}(\text{Sh}_{\text{ét}}(\text{Cor}, \mathbb{Z}/l^n)) & \xrightarrow{\text{Comp}(\text{Sh}_{\text{ét}}(\text{Ét}_{/k}, \mathbb{Z}/l^n))} & \text{Comp}(\text{Sh}_{\text{ét}}(\text{Ét}_{/k}, \mathbb{Z}/l^n))
\end{array}
\]

where \( (\Sigma^\infty, \Omega^\infty) \) is a left Quillen equivalence.

7.3. Next we consider an action of the absolute Galois group on functors. There is a commutative diagram \( (\bigtriangledown) \) of symmetric monoidal model categories

\[
\begin{array}{ccc}
\cdots & \xrightarrow{q_2^*} & \text{Comp}(\text{Sh}_{\text{ét}}(\text{Ét}_{/k}, \mathbb{Z}/l^2)) \\
\downarrow q_1^* & & \downarrow \text{Comp}(\text{Sh}_{\text{ét}}(\text{Ét}_{/k}, \mathbb{Z}/l)) \\
\cdots & \xrightarrow{q_n^*} & \text{Comp}(\text{Sh}_{\text{ét}}(\text{Ét}_{/k}, \mathbb{Z}/l^n))
\end{array}
\]

in which all arrows are symmetric monoidal left Quillen functors (as in the proof of Lemma 7.3 for any \( n \geq 1 \), \( \text{Comp}(\text{Sh}_{\text{ét}}(\text{Ét}_{/k}, \mathbb{Z}/l^{n+1}Z)) \to \text{Comp}(\text{Sh}_{\text{ét}}(\text{Ét}_{/k}, \mathbb{Z}/l^nZ)) \) is a left Quillen functor since it preserves small colimits and generating (trivial) cofibrations, see [8, 2.4]). For each \( n \geq 1 \), we have the symmetric monoidal functor

\( q_n^*: \text{Comp}(\text{Sh}_{\text{ét}}(\text{Ét}_{/k}, \mathbb{Z}/l^n)) \simeq \mathbb{Z}/l^n\text{-mod}_{\Gamma} \to \mathbb{Z}/l^n\text{-mod} \simeq \text{Comp}(\text{Sh}_{\text{ét}}(\text{Ét}_{/k}, \mathbb{Z}/l^n)) \)

where the middle functor is the forgetful functor. Then \( \Gamma \) acts on the forgetful functor, i.e., \( q_n^* \). That is, if \( \text{Aut}(q_n^*) \) denotes the group of symmetric monoidal natural equivalences of \( q_n^* \), then we have the homomorphism \( \Gamma \to \text{Aut}(q_n^*) \) of groups which carries \( g \in \Gamma \) to the symmetric monoidal natural equivalence given by morphisms \( g: q_n^*(C) \to q_n^*(C) \) induced by the action of \( \Gamma \) on \( C \in \mathbb{Z}/l^n\text{-mod}_I \). This action commutes with the diagram in the following sense: For any pair \( m, n \in \mathbb{N} \) with \( m \geq n \), the action on \( q_n^* \) and the vertical composition with \( \text{Comp}(\text{Sh}_{\text{ét}}(\text{Ét}_{/k}, \mathbb{Z}/l^nZ)) \to \text{Comp}(\text{Sh}_{\text{ét}}(\text{Ét}_{/k}, \mathbb{Z}/l^n)) \) determines an action of \( \Gamma \) on \( \text{Comp}(\text{Sh}_{\text{ét}}(\text{Ét}_{/k}, \mathbb{Z}/l^nZ)) \to \text{Comp}(\text{Sh}_{\text{ét}}(\text{Ét}_{/k}, \mathbb{Z}/l^n)) \). On the other hand, the action on \( q_n^* \) and the vertical composition with \( \text{Comp}(\text{Sh}_{\text{ét}}(\text{Ét}_{/k}, \mathbb{Z}/l^nZ)) \to \text{Comp}(\text{Sh}_{\text{ét}}(\text{Ét}_{/k}, \mathbb{Z}/l^n)) \) also determines another action of \( \Gamma \). Then two actions coincide for any \( n, m \) with \( m \geq n \).

Consider the category \( I \) of the form

\[ \cdots \to n \to n-1 \to \cdots \to 2 \to 1. \]

Namely, objects of \( I \) are natural numbers, and the homset \( \text{Hom}_I(n, m) \) consists of one point if \( n \geq m \) and \( \text{Hom}_I(n, m) \) is the empty if otherwise. We abuse notation and write \( I \) also for the nerve of \( I \). Let \( \text{WCat}_\infty \) be the \( \infty \)-category which consists of pairs \( (\mathcal{C}, W) \) where \( \mathcal{C} \) is an
$\infty$-category and $W$ is a subset of edges of $C$, called a system, which are stable under homotopy, composition and contains all weak equivalences. Here $C$ belongs to an enlarged universe which contains model categories we treat. The mapping space $\text{Map}_{\text{WCat}_\infty}(C, W, (C', W'))$ is equivalent to the summands spanned by $f : C \to C'$ such that $f(W) \subset W'$; see [32, 4.1.3.1]. Moreover, we equip $\text{WCat}_\infty$ with the Cartesian monoidal structure and write $\text{CAlg}(\text{WCat}_\infty)$ for the $\infty$-category of commutative algebra objects with respect to this monoidal structure. Then the diagram (\text{\bigstar}) induces $\alpha : \Delta^1 \times I \to \text{CAlg}(\text{WCat}_\infty)$ such that the restriction $\alpha_0 : \{0\} \times I \to \text{CAlg}(\text{WCat}_\infty)$ is given by the sequence of symmetric monoidal full subcategories spanned by cofibrant objects:

$$\cdots \to \text{Comp}(\text{Sh}_{\text{et}}(\text{Ét}_{/k}, \mathbb{Z}/l^2))^c \to \text{Comp}(\text{Sh}_{\text{et}}(\text{Ét}_{/k}, \mathbb{Z}/l))^c,$$

and the restriction $\alpha_1 : \{1\} \times I \to \text{CAlg}(\text{WCat}_\infty)$ is given by

$$\cdots \to \text{Comp}(\text{Sh}_{\text{et}}(\text{Ét}_{/k}, \mathbb{Z}/l^2))^c \to \text{Comp}(\text{Sh}_{\text{et}}(\text{Ét}_{/k}, \mathbb{Z}/l))^c.$$

Let us denote by $D_k$ and $\bar{D}_k$ the objects in $\text{Fun}(I, \text{CAlg}(\text{WCat}_\infty))$ corresponding to $\alpha_0$ and $\alpha_1$ respectively. The mapping space $\text{Map}(D_k, \bar{D}_k)$ from $D_k$ to $\bar{D}_k$ in $\text{Fun}(I, \text{CAlg}(\text{WCat}_\infty))$ is described by the Kan complex

$$\{D_k \times_{\text{Fun}([0, \text{Fun}(I, \text{CAlg}(\text{WCat}_\infty))])} \text{Fun}(\Delta^1, \text{Fun}(I, \text{CAlg}(\text{WCat}_\infty))) \times_{\text{Fun}([1, \text{Fun}(I, \text{CAlg}(\text{WCat}_\infty))])} \{D_k\}.$$

Clearly, this mapping space is 1-truncated since $D_k$ and $\bar{D}_k$ are sequences of symmetric monoidal 1-categories equipped with systems. Let $\text{FGal}(\bar{k}/k)$ be the nerve of the category which consists of one object $\{\ast\}$ and $\text{Hom}_{\text{FGal}(\bar{k}/k)}(\ast, \ast) = \text{Gal}(\bar{k}/k) = \Gamma$ equipped with the composition determined by the multiplication. The action of $\Gamma$ on $\{q_n\}_{n \geq 1}$ induces the action of $\Gamma$ on the morphism $f : D_k \to \bar{D}_k$ corresponding to $\alpha$ in $\text{Fun}(I, \text{WCat}_\infty)$, which is described by a functor $\text{FGal}(\bar{k}/k) \to \text{Map}(D_k, \bar{D}_k)$ sending $\ast$ to $f \in \text{Map}(D_k, \bar{D}_k)$.

Now we will construct symmetric monoidal $\infty$-categories by inverting weak equivalences in symmetric monoidal model categories (cf. Section 2.4). There is a natural fully faithful functor $\text{Cat}_\infty \to \text{WCat}_\infty$ which carries an $\infty$-category $C$ to $\text{C}_{\text{W}(C)}$ where $\text{C}_{\text{W}(C)}$ is the collection of edges of $C$. It also induces a fully faithful functor $\text{CAlg}(\text{Cat}_\infty) \to \text{CAlg}(\text{WCat}_\infty)$, where $\text{Cat}_\infty$ is endowed with the Cartesian monoidal structure; see [32, 4.1.3]. According to [32, 4.1.3.4] this functor admits a left adjoint $L : \text{CAlg}(\text{WCat}_\infty) \to \text{CAlg}(\text{Cat}_\infty)$. By composition with this adjoint pair, we have an localization adjoint pair (see [31, 5.2.7.2])

$$L^1 : \text{Fun}(I, \text{CAlg}(\text{WCat}_\infty)) \Rightarrow \text{Fun}(I, \text{CAlg}(\text{Cat}_\infty))$$

by [31, 5.2.7.4]. Let $\alpha' : \Delta^1 \times I \to \text{CAlg}(\text{WCat}_\infty) \overset{L}{\to} \text{CAlg}(\text{Cat}_\infty)$ be the composite, and let $\alpha'_0$ and $\alpha'_1$ be the restrictions to $\{0\} \times I$ and $\{1\} \times I$ respectively. Let $D'_k$ and $\bar{D}'_k$ be the objects in $\text{Fun}(I, \text{CAlg}(\text{Cat}_\infty))$ corresponding to $\alpha'_0$ and $\alpha'_1$ respectively. The functor $\alpha'$ is informally depicted as

$$\cdots \to \mathcal{D}^\otimes(\text{Sh}_{\text{et}}(\text{Ét}_{/k}, \mathbb{Z}/l^2)) \to \mathcal{D}^\otimes(\text{Sh}_{\text{et}}(\text{Ét}_{/k}, \mathbb{Z}/l)) \to \cdots \to \mathcal{D}^\otimes(\mathbb{Z}/l^2) \to \mathcal{D}^\otimes(\mathbb{Z}/l)$$

where $\mathcal{D}^\otimes(\text{Sh}_{\text{et}}(\text{Ét}_{/k}, \mathbb{Z}/l^p))$ and $\mathcal{D}^\otimes(\mathbb{Z}/l^p))$ are symmetric monoidal stable presentable $\infty$-categories obtained from $\text{Comp}^\otimes(\text{Sh}_{\text{et}}(\text{Ét}_{/k}, \mathbb{Z}/l^p))$ and $\text{Comp}^\otimes(\text{Sh}_{\text{et}}(\text{Ét}_{/k}, \mathbb{Z}/l^p))$ respectively, by inverting weak equivalences (we often omit the subscript $\otimes$). The functor $L^1$ induces

$$L^\Delta^1 \times I : \text{Fun}(\Delta^1, \text{Fun}(I, \text{CAlg}(\text{WCat}_\infty))) \to \text{Fun}(\Delta^1, \text{Fun}(I, \text{CAlg}(\text{Cat}_\infty))).$$
By the description of Map($D_k, D_k$), $L^{\Delta^1 \times I}$ induces Map($D_k, D_k$) $\rightarrow$ Map($D'_k, D'_k$), where Map($D'_k, D'_k$) is given by

\[
\{ D'_k \} \times_{\text{Fun}(\{0\}, \text{Fun}(I, \text{CAlg}(\text{Cat}_\infty)))} \text{Fun}(\Delta^1, \text{Fun}(I, \text{CAlg}(\text{Cat}_\infty))) \times_{\text{Fun}(\{1\}, \text{Fun}(I, \text{CAlg}(\text{Cat}_\infty)))} \{ D'_k \}.
\]

By composition we have $t : \text{FGal}((\bar{k}/k) \rightarrow \text{Map}(D_k, D_k) \rightarrow \text{Map}(D'_k, D'_k)$ carrying $*$ to $f'$ which is the image of $f$ in Map($D'_k, D'_k$).

Let $\hat{D}^\otimes(\text{Sh}(\text{Et}_{/k}, Z_d))$ (resp. $\hat{D}^\otimes(Z_d)$) be a symmetric monoidal stable presentable $\infty$-category which is defined to be the limit of $\alpha'_0 : I \simeq \{0\} \times I \rightarrow \text{CAlg}(\text{Cat}_\infty)$ (resp. $\alpha'_1 : I \simeq \{1\} \times I \rightarrow \text{CAlg}(\text{Cat}_\infty)$). Then $\alpha'$ determines a symmetric monoidal colimit-preserving functor $\hat{D}^\otimes((\text{Sh}(\text{Et}_{/k}, Z_d)) \rightarrow \hat{D}^\otimes(Z_d)$.

Let $I \rightarrow \text{CAlg}(\text{WCat}_\infty)$ be the functor corresponding to $\cdots \rightarrow \text{MSp}^\otimes(\mathbb{Z}/l^2)^c \rightarrow \text{MSp}^\otimes(\mathbb{Z}/l)^c$. Composing with $L^I$ we have $I \rightarrow \text{CAlg}(\text{Cat}_\infty)$ which we described as $\cdots \rightarrow \text{DM}^\otimes(\mathbb{Z}/l^2) \rightarrow \text{DM}^\otimes(\mathbb{Z}/l)$. Let $\hat{\text{DM}}^\otimes(Z_d)$ be its limit. Similarly, from $\cdots \rightarrow \text{Comp}^\otimes(\text{Sh}(\text{Et}_{/k}, Z_d/\mathbb{Z}/l^2))^c \rightarrow \text{Comp}^\otimes(\text{Sh}(\text{Et}_{/k}, Z_d))^c$ we have a sequence $\cdots \rightarrow (\hat{\text{DM}}^\otimes)^\otimes(\mathbb{Z}/l^2) \rightarrow (\hat{\text{DM}}^\otimes)^\otimes(\mathbb{Z}/l)$. Let $(\hat{\text{DM}}^\otimes)^\otimes(Z_d)$ be its limit. Let $\text{DM}^\otimes(Z)$ be the symmetric monoidal stable presentable $\infty$-category obtained from $\text{MSp}^\otimes(\mathbb{Z})^c$ by inverting weak equivalences. Then the diagram (4) induces $\text{DM}^\otimes(Z) \rightarrow \hat{\text{DM}}^\otimes(Z_d)$. Consider the symmetric monoidal left Quillen functors

$\text{Comp}^\otimes(\text{Sh}(\text{Et}_{/k}, Z_d)) \quad \hat{\Delta} \rightarrow \text{Comp}^\otimes(\text{Sh}(\text{Cor}, Z_d)) \rightarrow \text{MSp}^\otimes(\mathbb{Z}/l^n)$,

we obtain $\hat{\Delta}^\otimes((\text{Sh}(\text{Et}_{/k}, Z_d)) \rightarrow (\hat{\text{DM}}^\otimes)^\otimes(Z_d) \simeq \hat{\text{DM}}^\otimes(Z_d)$ where the right equivalence follows from the Quillen equivalences $\text{Comp}^\otimes(\text{Sh}(\text{Cor}, Z_d)) \rightarrow \text{MSp}^\otimes(Z_d)$ and $\text{MSp}^\otimes(Z_d)$. Moreover, the rigidity theorem due to Suslin-Voevodsky [4, 19, 34, 9.35, 7.20] implies:

Lemma 7.5. The symmetric monoidal functor $\hat{\Delta}$ is an equivalence.

Proof. Since $\nu_n$ determines a symmetric monoidal exact functor between symmetric monoidal stable $\infty$-categories, we are reduced to proving that $\hat{\Delta}((\text{Sh}(\text{Et}_{/k}, Z_d)) \rightarrow \text{DM}^\otimes(Z/d)$ induces an equivalence of their homotopy categories (cf. [22 Lemma 5.8]). Thus our claim is a consequence of the rigidity theorem [34, 9.35, 7.20].

We define some stable subcategories. Let $\text{DM}^\otimes(Z/d)$ be the smallest stable idempotent complete subcategory of $\text{DM}^\otimes(Z/d)$ which consists of $\{Z/d(X)\}_{X \in \text{Sm}_{/k}}$. Let $\hat{\text{DM}}^\otimes(Z_d)$ be the limit $\lim n \text{DM}^\otimes(Z/d)$. We define $\hat{\Delta}_{gm}(\text{Sh}(\text{Et}_{/k}, Z_d))$ to be the stable subcategory of $\hat{\Delta}(\text{Sh}(\text{Et}_{/k}, Z_d))$ that corresponds to $\text{DM}^\otimes(Z_d)$ through the equivalence $\hat{\Delta}$. Both categories naturally inherit symmetric monoidal structures.

7.4. We will construct realization functors. Let $\text{Comp}^\otimes(Z_d)$ be the symmetric monoidal category of chain complexes of $Z_d$-modules. We equip $\text{Comp}^\otimes(Z_d)$ with the (symmetric monoidal) projective model structure (see e.g. [19, 2.3.11] or [32, 8.1.2.8, 8.1.4.3]), in which weak equivalences (resp. fibrations) are exactly quasi-isomorphisms (resp. termwise surjective maps). Comparing the set of generating cofibrations (see [19, 2.3.3]), we see that the model structures on $\text{Comp}(\text{Sh}(\text{Et}_{/k}, Z_d)) \simeq \text{Comp}(Z/d)$-mod is the projective model structure. Let $D^\otimes(Z_d)$ and $D^\otimes(Z/d)$ be the symmetric monoidal $\infty$-categories obtained from $\text{Comp}(Z_d)$ and $\text{Comp}(Z/d)$ by inverting weak equivalences. According to [32, 8.1.3.13], there are natural equivalences $\text{Mod}^\otimes_{HZ_d} \simeq D^\otimes(Z_d)$ and $\text{Mod}^\otimes_{HZ_d/d} \simeq D^\otimes(Z/d)$ of symmetric monoidal $\infty$-categories. The base change functor $- \otimes_{Z_d} Z/d : \text{Mod}^\otimes_{HZ_d} \rightarrow \text{Mod}^\otimes_{HZ_d/d}$
gives rise to $\mathcal{D}^\otimes(Z_l) \simeq \text{Mod}^\otimes_{HZ_l} \to \lim_n \text{Mod}^\otimes_{HZ_l/n} \simeq \widehat{\mathcal{D}}(Z_l)$. Let $R$ be a noetherian commutative ring. Let $P$ be an object in $\mathcal{D}^\otimes(Z_l) \simeq \text{Mod}^\otimes_{HR}$. We say that $P$ is almost perfect if $H^n(P) = 0$ for $n > 0$ and $H^1(P)$ is a finitely presented (generated) $R$-module for each $i \in \mathbb{Z}$; see [32, 8.2.5.10, 8.2.5.11, 8.2.5.17, [33, VIII, 2.7.20]. Let $\text{AMod}^\otimes_{HR}$ denote the stable subcategory of $\text{Mod}^\otimes_{HR}$ spanned by almost perfect objects. We remark that there is a sequence of fully faithful embeddings $\text{PMod}_{HR} \subset \text{AMod}_{HR} \subset \text{Mod}_{HR}$. We easily see that every almost perfect object $K$ in $\mathcal{D}^\otimes(Z_l/n) \simeq \text{Mod}^\otimes_{HZ_l/n}$ (regarded as a chain complex) has a quasi-isomorphism $P \to K$ such that $P$ is a right bounded complex of free $Z/l^n$-modules of finite rank. Therefore $\text{AMod}_{HZ_l/n} \subset \text{Mod}_{HZ_l/n}$ is closed under tensor operation. Similarly, $\text{AMod}_{HZ_l}$ is closed under tensor operation. Thus these come equip with symmetric monoidal structures. Let $\lim_n \text{AMod}_{HZ_l/n}$ be the full subcategory of $\lim_n \text{Mod}_{HZ_l/n} \simeq \widehat{\mathcal{D}}(Z_l)$ spanned by compatible systems $\{C(n) \in \text{Mod}_{HZ_l/n}\}_{n \leq 1}$ such that each $C(n)$ is almost perfect. Then thanks to the derived version of Grothendieck existence theorem [33, XII, 5.3, 5.1.17], $\text{Mod}_{HZ_l} \to \lim_n \text{Mod}_{HZ_l/n}$ induces an equivalence of symmetric monoidal $\infty$-categories $\widehat{\pi} : \text{AMod}_{HZ_l} \simeq \lim_n \text{AMod}_{HZ_l/n}$. Let $\text{DM}^\otimes_{\text{et}}(Z)$ be the smallest stable idempotent complete subcategory of $\text{DM}^\otimes_{\text{et}}(Z)$ which consists of $\{Z(X) \otimes Z(n)\}_{x \in \text{sm}_{Z/k}}$. By Corollary 7.8 combined with the finiteness of étale cohomology and cohomological dimension [10], we see that the essential image of $\text{DM}^\otimes_{\text{et}}(Z) \to \widehat{\mathcal{D}}(Z_l) \simeq \lim_n \text{Mod}_{HZ_l/n}$ is contained in $\lim_n \text{AMod}_{HZ_l/n}$. Hence combining with Lemma 7.5 we obtain a symmetric monoidal exact functor as the composite

$$\mathcal{R}_{\text{et},Z_l} : \text{DM}^\otimes_{\text{et}}(Z) \to (\text{DM}^\otimes_{\text{et}}(Z_l))^{(\Sigma^\infty)^{-1}} \simeq (\text{DM}^\otimes_{\text{et}}(Z_l)) \supset (\text{DM}^\otimes_{\text{et}}(Z_l)) \simeq \text{AMod}^\otimes_{HZ_l} \subset \mathcal{D}^\otimes(Z_l)$$

where the essential image of the first line belongs to $\text{DM}^\otimes_{\text{et}}(Z_l)$. Étale sheafification induces a symmetric monoidal exact functor $\text{Sh}(\text{Cor}, Z) \to \text{Sh}(\text{Cor}, Z)$ giving rising to a symmetric monoidal left adjoint functor $\text{MSp}(Z) \to \text{MSp}_{et}(Z)$. The right adjoint of $\text{MSp}(Z) \to \text{MSp}_{et}(Z)$ is the forgetful functor $\text{MSp}_{et}(Z) \to \text{MSp}(Z)$. As in the case of $\text{MSp}_{et}(Z)$ we equip $\text{MSp}(Z)$ with a stable model structure [8, Example 7.15]. Then repeating the argument of Lemma 7.3 we see that $\text{MSp}_{et}(Z) \to \text{MSp}_{et}(Z)$ is a symmetric monoidal left Quillen adjoint functor. Let $\text{DM}^\otimes(Z)$ be the symmetric monoidal stable presentable $\infty$-category obtained from $\text{MSp}(Z)$ by inverting weak equivalences. Then $\text{MSp}(Z) \to \text{MSp}_{et}(Z)$ determines a symmetric monoidal colimit-preserving functor $\eta : \text{DM}^\otimes(Z) \to \text{DM}^\otimes_{et}(Z)$. Consider the composite of symmetric monoidal exact functors

$$\mathcal{R}_{\text{et},Z_l} : \text{DM}^\otimes_{\text{et}}(Z) \xrightarrow{\eta} \text{DM}^\otimes_{et}(Z) \xrightarrow{\mathcal{R}_{\text{et},Z_l}} \text{AMod}^\otimes_{HZ_l}.$$

We shall refer to $\mathcal{R}_{\text{et},Z_l}$ as the $l$-adic étale realization functor. Let $\text{DM}^\otimes_{\text{et}}(Q)$ be the $Q$-coefficient version of $\text{DM}^\otimes_{\text{et}}(Z)$. By Lemma 7.6 below, the composite $\mathcal{R}_{\text{et},Z_l} \otimes Q_l : \text{DM}^\otimes_{\text{et}}(Z) \xrightarrow{\mathcal{R}_{\text{et},Z_l}} \text{AMod}^\otimes_{HZ_l} \xrightarrow{\eta} \text{AMod}^\otimes_{HZ_l} \otimes Q_l$ induces a symmetric monoidal exact functor $\mathcal{R}_{\text{et},Q_l} : \text{DM}^\otimes_{\text{et}}(Q) \to \text{AMod}^\otimes_{HZ_l} \otimes Q_l$, uniquely up to a contractible space of choice, such that $\text{DM}^\otimes_{\text{et}}(Z) \to \text{DM}^\otimes_{\text{et}}(Q) \xrightarrow{\mathcal{R}_{\text{et},Q_l}} \text{AMod}^\otimes_{HZ_l} \otimes Q_l$.

Next we will define Galois actions on $\mathcal{R}_{\text{et},Z_l}$ and $\mathcal{R}_{\text{et},Q_l}$. Since $\widehat{\mathcal{D}}^\otimes(\text{Sh}(\text{Et}_{/k}, Z_l))$ and $\widehat{\mathcal{D}}^\otimes(Z_l)$ are limits of $D_k^\prime$ and $D_k^\prime$ respectively, thus by taking their limits we have a natural map of mapping spaces $\text{Map}(D_k^\prime, D_k^\prime) \to \text{Map}_{\text{PAlg}(\text{Cat}_{\infty})}(\widehat{\mathcal{D}}^\otimes(\text{Sh}(\text{Et}_{/k}, Z_l)), \widehat{\mathcal{D}}^\otimes(Z_l))$. The composition
with $t : \text{FGal}(\bar{k}/k) \to \text{Map}(D_k', D'_k)$ constructed in \[7.3\] gives rise to

$$\text{FGal}(\bar{k}/k) \to \text{Map}_{\text{CAAlg(Cat}_{\infty}))}(\hat{D}^\diamond(\text{Sh}_{\text{et}}(\bar{\text{Et}}/k, \mathcal{Z}_d)), \hat{D}^\diamond(\mathcal{Z}_d)).$$

If $F : \hat{D}^\diamond(\text{Sh}_{\text{et}}(\bar{\text{Et}}/k, \mathcal{Z}_d)) \to \hat{D}^\diamond(\mathcal{Z}_d)$ denotes the symmetric monoidal functor determined by the limit of $f' : D_k' \to D'_k$, the based loop space induces $\Omega_* \text{FGal}(\bar{k}/k) = \text{Gal}(\bar{k}/k) \to \text{Aut}(F) := \Omega_* \text{Map}_{\text{CAAlg(Cat}_{\infty}))}(\hat{D}^\diamond(\text{Sh}_{\text{et}}(\bar{\text{Et}}/k, \mathcal{Z}_d)), \hat{D}^\diamond(\mathcal{Z}_d))$, where the target is the based loop space with respect to $F$ that is a group object in $S$. Recall that $R_{\text{et}, \mathcal{Z}_d} : \text{DM}_{\text{gm}}(\mathcal{Z}) \to \text{AMod}_{\mathcal{Z}_d}$ factors through $F : \hat{D}^\diamond(\text{Sh}_{\text{et}}(\bar{\text{Et}}/k, \mathcal{Z}_d)) \to \text{lim}_n \text{AMod}^\diamond_{\mathcal{H}Z/(n)} \subset \hat{D}^\diamond(\mathcal{Z}_d)$. Thus if $\text{Aut}(R_{\text{et}, \mathcal{Z}_d})$ is the based loop space of $\text{Map}_{\text{CAAlg(Cat}_{\infty}))}(\text{DM}_{\text{gm}}^\diamond(\mathcal{Z}), \text{AMod}^\diamond_{\mathcal{H}Z/(n)})$ with respect to $R_{\text{et}, \mathcal{Z}_d}$, then the vertical compositions with $\text{Gal}(\bar{k}/k) \to \text{Aut}(F)$ induces a map of group objects $\text{Gal}(\bar{k}/k) \to \text{Aut}(R_{\text{et}, \mathcal{Z}_d})$.

If we define $\text{Aut}(R_{\text{et}, \mathcal{Q}_1})$ in a similar way, then we also have $\text{Gal}(\bar{k}/k) \to \text{Aut}(R_{\text{et}, \mathcal{Q}_1})$. We will refer to $\text{Gal}(\bar{k}/k) \to \text{Aut}(R_{\text{et}, \mathcal{Z}_d})$ and $\text{Gal}(\bar{k}/k) \to \text{Aut}(R_{\text{et}, \mathcal{Q}_1})$ as the Galois action (or the action of $\Gamma$) on $R_{\text{et}, \mathcal{Z}_d}$ and $R_{\text{et}, \mathcal{Q}_1}$, respectively. We have constructed the $l$-adic étale realization functor $R_{\text{et}, \mathcal{Z}_d}$ which is endowed with the Galois action $\Gamma \to \text{Aut}(R_{\text{et}, \mathcal{Z}_d})$. Furthermore, there is its rational version $R_{\text{et}, \mathcal{Q}_1}$. The following lemmata complete the proof of Proposition \[7.4\]

**Lemma 7.6.** The composition with $\text{DM}_{\text{gm}}^\diamond(\mathcal{Z}) \to \text{DM}_{\text{gm}}^\diamond(\mathcal{Q})$ induces a categorical equivalence

$$\text{Map}_{\text{ex}}(\text{DM}_{\text{gm}}^\diamond(\mathcal{Z}), \text{Mod}_{\mathcal{H}Q_1}^\diamond) \to \text{Map}_{\text{ex}}(\text{DM}_{\text{gm}}^\diamond(\mathcal{Q}), \text{Mod}_{\mathcal{H}Q_1}^\diamond),$$

where $\text{Map}_{\text{ex}}(-, -)$ denotes the full subcategory of $\text{Map}_{\text{CAAlg(Cat}_{\infty}))}(-, -)$, spanned by those functors which preserve finite colimits, i.e., exact functors.

**Proof.** The objects in $\text{DM}_{\text{gm}}^\diamond(\mathcal{Z})$ forms a set of compact generators, and thus we have an equivalence $\text{Map}_{\text{ex}}(\text{DM}_{\text{gm}}^\diamond(\mathcal{Z}), \text{Mod}_{\mathcal{H}Q_1}^\diamond) \simeq \text{Map}_L(\text{DM}^\diamond(\mathcal{Z}), \text{Mod}_{\mathcal{H}Q_1}^\diamond)$ which is given by composition with the inclusion $\text{DM}_{\text{gm}}^\diamond(\mathcal{Z}) \subset \text{DM}^\diamond(\mathcal{Z})$. Here $\text{Map}_{\text{ex}}^\diamond(-, -)$ is the full subcategory of $\text{Map}_{\text{CAAlg(Cat}_{\infty}))}(-, -)$ spanned by those functors which preserve small colimits.

Let $\text{DM}(\mathcal{Z})[\mathcal{Z}^{-1}]$ be the stable presentable $\infty$-category obtained from $\text{MSp}(\mathcal{Z})$, endowed with the model structure of left Bousfield localization with respect to $S = \{m : \text{F}_a(\mathcal{Z}(X)[n]) \to F_a(\mathcal{Z}(X)[n]); X \in \text{Sm}_{k/n}, n \in \mathcal{Z}, a \geq 0, m \in \mathbb{N}\}$, by inverting $S$-equivalences. Here $m$ means the multiplication by $m$. See \[24\] Definition 6.3 for the notation $F_a$. The class of $S$-equivalences is closed under tensoring with cofibrant objects. Indeed, to see this, it will suffice to show that for any cofibrant object $C \in \text{MSp}(\mathcal{Z})$ and the cone $T$ of $m : \mathcal{Z}(X)[n] \to \mathcal{Z}(X)[n]$, $C \otimes F_a(T)$ is $S$-equivalent to zero. We may assume that $C$ is a relative $I$-cell complex in the sense of \[19\], where $I$ is the set of generating cofibrations. Thus it is enough to see that $F_a(T) \otimes S$ is $S$-equivalent to zero where $D$ is either a domain or target of generating cofibrations; it follows from a direct calculation. Therefore, according to Lemma \[14\], $\text{DM}^\diamond(\mathcal{Z})[\mathcal{Z}^{-1}]$ is equivalent to the symmetric monoidal $\infty$-category obtained from $\text{DM}(\mathcal{Z})$ as the localization with respect to $S$; see \[32\] 4.1.3.4. By the universality of localization (cf. \[31\] 5.5.4.20, \[32\] 4.1.3.4), there is a natural equivalence

$$\text{Map}_L(\text{DM}^\diamond(\mathcal{Z})[\mathcal{Z}^{-1}], \text{Mod}_{\mathcal{H}Q_1}^\diamond) \simeq \text{Map}_L(\text{DM}^\diamond(\mathcal{Z}), \text{Mod}_{\mathcal{H}Q_1}^\diamond).$$

Let $\text{DM}_{\text{gm}}(\mathcal{Z})[\mathcal{Z}^{-1}]$ be the smallest stable idempotent complete subcategory of $\text{DM}(\mathcal{Z})[\mathcal{Z}^{-1}]$ which consists of the image of $\{\mathcal{Z}(X) \otimes \mathcal{Z}(n)\}_{X \in \text{Sm}_{k/n}, n \in \mathcal{Z}}$, that forms a set of compact generators of $\text{DM}(\mathcal{Z})[\mathcal{Z}^{-1}]$. Then $\text{Map}_{\text{ex}}(\text{DM}_{\text{gm}}^\diamond(\mathcal{Z})[\mathcal{Z}^{-1}], \text{Mod}_{\mathcal{H}Q_1}^\diamond) \simeq \text{Map}_L(\text{DM}^\diamond(\mathcal{Z})[\mathcal{Z}^{-1}], \text{Mod}_{\mathcal{H}Q_1}^\diamond)$. Thus it will suffice to prove that the natural symmetric monoidal functor $\text{DM}_{\text{gm}}^\diamond(\mathcal{Z})[\mathcal{Z}^{-1}] \to \text{DM}_{\text{gm}}^\diamond(\mathcal{Q})$ is an equivalence. It is enough to prove that a categorical equivalence $\text{DM}_{\text{gm}}^\diamond(\mathcal{Z}[\mathcal{Z}^{-1}] \to \text{DM}_{\text{gm}}^\diamond(\mathcal{Q})$. To this end, we let $\text{DM}^\text{eff}(\mathcal{Z})[\mathcal{Z}^{-1}]$ be the localization of $\text{DM}^\text{eff}(\mathcal{Z})$ (Nisnevich version of $\text{DM}_{\text{et}}^\text{eff}(\mathcal{Z})$) with respect to $T = \{m : \mathcal{Z}(X)[n] \to \mathcal{Z}(X)[n]; X \in \text{Sm}_{k/n}, n \in \mathcal{Z}, m \in \mathbb{N}\}$. Using the argument above or \[5\] Corollary 4.11 we see that the collection of $T$-equivalences is
closed under tensoring with cofibrant objects. Thus by \textbf{[32] 4.1.3.4} \( \text{DM}^{\text{eff}}(\mathbb{Z})[\mathbb{Z}^{-1}] \) is equipped with a symmetric monoidal structure. Observe that there is an equivalence of symmetric monoidal \( \infty \)-categories \((\text{DM}^{\text{eff}})^{\otimes}(\mathbb{Z})[\mathbb{Z}^{-1}] \simeq (\text{DM}^{\text{eff}})^{\otimes}(\mathbb{Q}). \) For this, consider the adjoint pair \( R : \text{Comp}(\text{Sh}(\text{Cor}, \mathbb{Z})) \rightleftarrows \text{Comp}(\text{Sh}(\text{Cor}, \mathbb{Q})) : U, \) where \( U \) is the forgetful functor, and \( R \) is induced by the rationalization functor \( \text{Z-mod} \to \text{Q-mod} \) that is the left adjoint of the forgetful functor \( \text{Q-mod} \to \text{Z-mod}. \) If one equips \( \text{Comp}(\text{Sh}(\text{Cor}, \mathbb{Z})) \) and \( \text{Comp}(\text{Sh}(\text{Cor}, \mathbb{Q})) \) with the model structure in which weak equivalences (resp. cofibrations) are quasi-isomorphisms (resp. termwise monomorphisms) (cf. \textbf{[4], [8] Theorem 2.1]), then \((R, U)\) is a Quillen adjunction since the rationalization functor is exact. Moreover, \( U \) induces a fully faithful right derived functor: \( \mathcal{D}(\text{Sh}(\text{Cor}, \mathbb{Q})) \to \mathcal{D}(\text{Sh}(\text{Cor}, \mathbb{Z})). \) Note that the adjunction \( \mathcal{D}(\text{Sh}(\text{Cor}, \mathbb{Z})) \rightleftarrows \mathcal{D}(\text{Sh}(\text{Cor}, \mathbb{Q})) \) is the localization with respect to \( T. \) Unwinding the definition \( C \in \mathcal{D}(\text{Sh}(\text{Cor}, \mathbb{Z})) \) is a \( T \)-local object if and only if \( \text{Ext}^{n}_{\text{Sh}(\text{Cor}, \mathbb{Z})}(\mathbb{Z}(X), C) \) is a \( \mathbb{Q} \)-vector space for any \( n \in \mathbb{Z} \) and any \( X \in \text{Sm}_{/k}. \) Clearly, the essential image of \( \mathcal{D}(\text{Sh}(\text{Cor}, \mathbb{Q})) \to \mathcal{D}(\text{Sh}(\text{Cor}, \mathbb{Z})). \) lies in the full subcategory of \( T \)-local objects. Conversely, let \( D \) be a cofibrant-fibrant \( T \)-local object in \( \text{Comp}(\text{Sh}(\text{Cor}, \mathbb{Z})). \) The rationalization functor \( \text{Z-mod} \to \text{Q-mod} \) is exact, and the presheaf \( X \mapsto H^{n}(D(X)) \) is the same as \( X \mapsto H^{n}(R(D(X))). \) If one denotes by \( R(D)' \) the fibrant replacement of \( R(D), \) then the natural map \( D \to U(R(D)') \) is a quasi-isomorphism. It follows that \( D \) lies in the essential image of \( \mathcal{D}(\text{Sh}(\text{Cor}, \mathbb{Q})). \) Hence we see that \((R, U)\) is the localization with respect to \( T. \) Consequently, \( \text{DM}^{\text{eff}}(\mathbb{Q}) \) is the localization of \( \mathcal{D}(\text{Sh}(\text{Cor}, \mathbb{Z})) \) with respect to \( T \cup \{ \mathbb{Z}(X \times \mathbb{A}^{1}) \to \mathbb{Z}(X) \mid X \in \text{Sm}_{/k} \}. \) We then have \((\text{DM}^{\text{eff}})^{\otimes}(\mathbb{Z})[\mathbb{Z}^{-1}] \simeq (\text{DM}^{\text{eff}})^{\otimes}(\mathbb{Q}). \) Next observe that \( \text{DM}^{\text{eff}}(\mathbb{Z})[\mathbb{Z}^{-1}] \to \text{DM}(\mathbb{Z})[\mathbb{Z}^{-1}] \) induced by \( \Sigma^{\infty} : \text{DM}^{\text{eff}}(\mathbb{Z}) \to \text{DM}(\mathbb{Z}) \) is fully faithful. To see this, it is enough to show that \( \Sigma^{\infty} \) sends \( T \)-local objects to \( S \)-local objects. We here remark that by Voevodsky’s cancellation theorem \( \Sigma^{\infty} \) is fully faithful. Let \( C \) be a \( T \)-local object. To check that \( \Sigma^{\infty}(C) \) is \( S \)-local, it will suffice to prove that \( C \otimes \mathbb{Z}(1) \) is \( T \)-local. Let \( C \subset \mathcal{D}(\text{Sh}(\text{Cor}, \mathbb{Q})) \) be the stable subcategory that consists of those objects \( C \) such that \( C \otimes \mathbb{Z}(1) \) lies in \( \text{Sh}(\text{Cor}, \mathbb{Q}), \) that is, \( T \)-local. Then \( \mathbb{Q}(X) \simeq \mathbb{Q}(\mathbb{Z}(G_{m}) \simeq \mathbb{Q}(X \times G_{m}) \) in \( \text{Comp}(\text{Sh}(\text{Cor}, \mathbb{Z}), \) and the Suslin complex \( C_{\ast}(\mathbb{Q}(X \times G_{m})) \) belongs to \( \text{Comp}(\text{Sh}(\text{Cor}, \mathbb{Q})) \) (see \textbf{[31] 2.14} for Suslin complexes). Thus we deduce that \( \mathbb{Q}(X) \in C \) for any \( X \in \text{Sm}_{/k}. \) Moreover, \( C \) has small coproducts such that \( C \to \mathcal{D}(\text{Sh}(\text{Cor}, \mathbb{Q})) \) preserves small coproducts. Hence \( C = \mathcal{D}(\text{Sh}(\text{Cor}, \mathbb{Q})), \) and \( \Sigma^{\infty} \) sends \( T \)-local objects to \( S \)-local objects. On the other hand, the composite \( (\text{DM}^{\text{eff}}(\mathbb{Z})[\mathbb{Z}^{-1}] \to \text{DM}(\mathbb{Z})[\mathbb{Z}^{-1}] \to \text{DM}(\mathbb{Q}) \) is fully faithful since it can be identified with \( \text{DM}^{\text{eff}}(\mathbb{Z})[\mathbb{Z}^{-1}] \simeq (\text{DM}^{\text{eff}}(\mathbb{Q}) \to \text{DM}(\mathbb{Q}) \) by \( \Sigma^{\infty}. \) Since \( \text{DM}^{\text{eff}}(\mathbb{Z})[\mathbb{Z}^{-1}] \to \text{DM}(\mathbb{Z})[\mathbb{Z}^{-1}] \) is fully faithful when one restricts the domain to the essential image of \( \text{DM}^{\text{eff}}(\mathbb{Z})[\mathbb{Z}^{-1}] \). Let \( \text{DM}^{\text{eff}}(\mathbb{Z})[\mathbb{Z}^{-1}] \otimes \mathbb{Z}(n) \) be the full subcategory of \( \text{DM}(\mathbb{Z})[\mathbb{Z}^{-1}] \) spanned by \( C \otimes \mathbb{Z}(n) \) such that \( C \) lies in the essential image of \( \text{DM}^{\text{eff}}(\mathbb{Z})[\mathbb{Z}^{-1}] \). We define \( \text{DM}^{\text{eff}}(\mathbb{Q}) \otimes \mathbb{Q}(n) \) in a similar way. Then we have \( \cup_{n \geq 0} \text{DM}^{\text{eff}}(\mathbb{Z})[\mathbb{Z}^{-1}] \otimes \mathbb{Z}(-n) \simeq \cup_{n \geq 0} \text{DM}^{\text{eff}}(\mathbb{Q}) \otimes \mathbb{Q}(-n). \) Since \( \text{DM}_{\ast}(\mathbb{Z})[\mathbb{Z}^{-1}] \subset \cup_{n \geq 0} \text{DM}^{\text{eff}}(\mathbb{Z})[\mathbb{Z}^{-1}] \otimes \mathbb{Z}(-n) \), we have \( \text{DM}_{\ast}(\mathbb{Z})[\mathbb{Z}^{-1}] \simeq \text{DM}_{\ast}(\mathbb{Q}). \)
Comp(Sh_{et}(\mathbb{E}t_{/k}, \mathbb{Z}/l^n)) as a complex of discrete \(\Gamma\)-modules as follows: Put \(\bar{k} = \lim_k k_i\) where the right hand side is a filtered colimit of finite Galois extensions of \(k\). Then the filtered colimit \(\lim_k \res(Hom(\mathbb{Z}/l^n(X), \mathbb{Z}/l^n))(k_i)\) is a discrete \(\mathbb{Z}/l^n\)-modules with action of \(\Gamma\) which is determined by the natural actions of \(Gal(k_i/k)\) on \(\res(Hom(\mathbb{Z}/l^n(X), \mathbb{Z}/l^n))(k_i)\). It represents the image of \(\mathbb{Z}(X)^{\vee}\) in \(D(\mathbb{Z}/l^n)\). Note that \(C(k_i) := \res(Hom(\mathbb{Z}/l^n(X), \mathbb{Z}/l^n))(k_i)\) can be identified with

\[
\Ext^s(\mathbb{Z}/l^n(Spec(k_i)), Hom(\mathbb{Z}/l^n(X), \mathbb{Z}/l^n)) \simeq \Ext^s(\mathbb{Z}/l^n(X \times_k k_i), \mathbb{Z}/l^n) \\
\simeq \mathbb{H}^s_{\text{et}}(X \times_k k_i, \mathbb{Z}/l^n)
\]

where \(\Ext^s(-, -)\) is \(\pi_0(\text{Map}_{DM_{et}}(\mathbb{Z}/l^n)(-,-[s]))\). The second isomorphism follows from the equivalence in Lemma [7.3] see also [33] Theorem 10.2. Through isomorphisms, the action of \(Gal(k_i/k)\) on \(C(k_i)\) coincides with the action on \(\mathbb{H}^s_{\text{et}}(X \times_k k_i, \mathbb{Z}/l^n)\). (Here \(Gal(k_i/k)\) acts on the \(k\)-scheme \(X \times_k k_i\) in the obvious way, and it gives rise to action on \(\mathbb{H}^s_{\text{et}}(X \times_k k_i, \mathbb{Z}/l^n)\).

Since a filtered colimit commutes with taking cohomology groups, we have isomorphisms of \(\Gamma\)-modules

\[
\mathbb{H}^s(\lim_i C(k_i)) \simeq \lim_i \mathbb{H}^s(C(k_i)) \simeq \lim_i \mathbb{H}^s_{\text{et}}(X \times_k k_i, \mathbb{Z}/l^n) \simeq \mathbb{H}^s_{\text{et}}(\mathcal{X}, \mathbb{Z}/l^n).
\]

Taking account of the \(t\)-exactness of the equivalence \(A\text{Mod}_{H^{et}_Z} \simeq \lim_n A\text{Mod}_{H^{et}_{Z/l^n}}\) (see [33] 5.3.1, 5.2.12), we conclude that \(H^s(R(X, 0)) \simeq \lim_n H^s_{\text{et}}(\mathcal{X}, \mathbb{Z}/l^n)\). (More explicitly, through the equivalence, \((M_n)_{n \geq 1} \in \lim_n A\text{Mod}_{H^{et}_{Z/l^n}}\) with \(M_n \in A\text{Mod}_{H^{et}_{Z/l^n}}\) corresponds to the filtered limit \(\lim_n U_n(M_n)\) in \(\text{Mod}_{H^{et}_Z}\) where \(U_n : \text{Mod}_{H^{et}_{Z/l^n}} \to \text{Mod}_{H^{et}_Z}\) is naturally induced by \(\mathbb{Z}/l \to \mathbb{Z}/l^n\), and thus \(H^s(R(X, 0)) \simeq \lim_n H^s_{\text{et}}(\mathcal{X}, \mathbb{Z}/l^n)\) follows from Milnor exact sequence, Mittag-Leffler condition and the finiteness of étale cohomology.) We have \(H^s(R(X, 0)) \simeq H^s_{\text{et}}(\mathcal{X}, \mathbb{Z}/l^n)\).

Similarly, the image of \(\mathbb{Z}(m)\) in \(D(\text{Sh}_{et}(\mathbb{E}t_{/k}, \mathbb{Z}/l^n))\) is \(\mathbb{Z}/l^n(m)\), that is, the object corresponding to \(\mathbb{Z}/l^n(m)\) in \(\text{DM}_{et}(\mathbb{Z}/l^n)\). By [34] 10.6, 10.2 there is the natural equivalence \(\mathbb{Z}/l^n(m) \simeq \mu_{l^n}^{\otimes m}\) in \(D(\text{Sh}_{et}(\mathbb{E}t_{/k}, \mathbb{Z}/l^n))\), which corresponds to \(\mathbb{Z}/l^n\)-module \(\mu_{l^n}^{\otimes m}(\bar{k})\) placed in degree zero, which is endowed with the natural action \(\Gamma = Gal(\bar{k}/k)\). Here \(\mu_l\) is the sheaf given by \(L \mapsto \{a \in L\mid a^l = 1\}\). Thus we see that \(R_{et,Z[l]}(\mathbb{Z}(m))\) is equivalent to \(\lim_n \mu_{l^n}^{\otimes m}(\bar{k})\) placed in degree zero. Finally, the isomorphism \(H^s_{\text{et}}(\mathcal{X}, \mathbb{Z}(m)) \simeq H^s_{\text{et}}(\mathcal{X}, \mathbb{Z}/l^n) \otimes \mathbb{Z}(m)\) implies also the case of \(m \neq 0\).

**Corollary 7.8.** Let \(X\) be a smooth scheme over \(k\), and let \(R_{et,Z[l]}(\mathbb{Z}(X))(n)\) denote the image of \(\mathbb{Z}(X)\) in \(\text{Mod}_{H^{et}_{Z/l^n}}\). Then there is an isomorphism of \(\mathbb{Z}/l^n\)-modules

\[
\text{Hom}_{\mathbb{Z}/l^n}(H_s(R_{et,Z[l]}(\mathbb{Z}(X))(n)), \mathbb{Z}/l^n) \simeq H^s_{\text{et}}(\mathcal{X}, \mathbb{Z}/l^n).
\]

**Proof.** The same argument as in Lemma [7.7] shows that \(H_s(R_{et,Z[l]}(\mathbb{Z}(X))(n))\) is the algebraic singular homology \(H^*_\text{sing}(\mathcal{X}, \mathbb{Z}/l^n)\) [34] 10.8 of \(\mathcal{X}\). Hence our assertion follows from [34] 10.11.

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