1. Introduction

Let \( k \) be an algebraically closed field and \( K \) be a finitely generated \( k \)-field. In the first half of the 20-th century, Zariski defined a Riemann variety \( RZ_K(k) \) associated to \( K \) as the projective limit of all projective \( k \)-models of \( K \). Zariski showed that this topological space, which is now called a Riemann-Zariski (or Zariski-Riemann) space, possesses the following set-theoretic description: to give a point \( x \in RZ_K(k) \) is equivalent to give a valuation ring \( \mathcal{O}_x \) with fraction field \( K \) and such that \( k \subset \mathcal{O}_x \). The Riemann-Zariski space possesses a sheaf of rings \( \mathcal{O} \) whose stalks are valuation rings of \( K \) as above. Zariski made extensive use of these spaces in his desingularization works.

Let \( S \) be a scheme and \( U \) be a subset closed under generalizations, for example \( U = S_{\text{reg}} \) is the regular locus of \( S \), or \( U = \eta \) is a generic point of \( S \). In many birational problems one wants to consider only \( U \)-modifications \( S' \to S \), i.e. modifications which do not modify \( U \). Then it is natural to consider the projective limit \( S = \text{RZ}_U(S) \) of all \( U \)-modifications of \( S \). It was remarked in [Tem2, §3.3] that working with such relative Riemann-Zariski spaces one can extend the \( P \)-modification results of [Tem2] to the case of general \( U \) and \( S \), and this plan is realized in §2. In §2.2 we give a preliminary description of the space \( \mathcal{S} \), which is used in §2.3 to prove the first main result of the paper, the stable modification theorem 2.3.2 generalizing its analog from [Tem2]. Our improvement to the stable modification theorem [Tem2, 1.1] is in the control on the base change one has to perform in order to construct a stable modification of a relative curve \( C \to S \). Namely, we prove that in order to find a stable modification of a relative curve with semi-stable \( U \)-fibers it suffices to replace the base \( S \) with a \( U \)-etale covering.

Though a very rough study of relative RZ spaces suffices for the proof of theorem 2.3.2, it seems natural to investigate these spaces deeper. Furthermore, the definition of relative Riemann-Zariski spaces can be naturally generalized to the case of an arbitrary morphism \( f : Y \to X \), and the case when \( f \) is a dominant point was already applied in [Tem1], so it is natural to investigate the relative RZ spaces attached to a morphism \( f : Y \to X \). We will see that under a very mild assumption that \( f \) is a separated morphism between quasi-compact quasi-separated schemes, one obtains a very specific description of the space \( \text{RZ}_Y(X) \) which is similar to the classical case of \( \text{RZ}_K(k) \). Let us say that \( f \) is decomposable if it factors into a composition of an affine morphism \( Y \to Z \) and a proper morphism \( Z \to X \). Actually, in §2.2 we study \( \text{RZ}_Y(X) \) in the case of a general decomposable morphism because

\begin{quote}
I want to express my deep gratitude to B. Conrad for pointing out various gaps and mistakes in an earlier version of the article and to thank R. Huber for a useful discussion. Large part of the article was written during my stay at the Max Planck Institute for Mathematics at Bonn, and I want to thank the Institute for its hospitality.
\end{quote}
this case is not essentially easier than the case of an open immersion \( Y \rightarrow X \). We define a set \( \text{Val}_Y(X) \) whose points are certain \( X \)-valuations of \( Y \), and construct a surjection \( \psi : \text{Val}_Y(X) \rightarrow \text{RZ}_Y(X) \). It will require some additional work to prove in 3.4.7 that \( \psi \) is actually a bijection (and even a homeomorphism with respect to natural topologies defined in the paper). Now, a natural question to ask is if the decomposition assumption is essential. Slightly surprisingly, the answer is negative because the assumption is actually empty. A second main result of this paper is decomposition theorem 1.1.3 which states that a morphism of quasi-compact quasi-separated schemes is decomposable if and only if it is separated. Thus, the description of relative RZ spaces obtained in the decomposable case is actually the general one.

We give two proofs of the decomposition theorem in this paper. The first proof is based on Nagata compactification and Thomason approximation theorems. Actually, we prove in §1.1 that the decomposition theorem is essentially equivalent to the union of these two theorems. This accomplish the first proof. On the other hand, it turns out that a deeper study of relative RZ spaces leads to an independent proof of the decomposition theorem as explained in §3.5. In particular, we obtain new proofs of Nagata’s and Thomason’s theorems. Though there are few known proofs of Nagata’s theorem, see [Con] and [Lüt], the author hopes that the new proof might be better suited for attacking a (conjectural so far) generalization to the case of algebraic spaces.

Let us describe briefly the structure of the paper. In §1.1 we prove a slight generalization of Thomason’s theorem and show that the decomposition theorem is essentially equivalent to the union of Nagata’s and Thomason’s theorems. In §2 we start our study of relative RZ spaces and apply them to the strong stable modification theorem. Then, §3 is devoted to further study of the relative RZ spaces. In §3.1 we establish an interesting connection between Riemann-Zariski spaces and adic spaces of R. Huber; in particular, we obtain an intrinsic topology on \( \text{Val}_Y(X) \). However, it turns out that the notion of an open subdomain in the spaces \( \text{Val}_Y(X) \) is much finer than its analog in the adic spaces. It requires some work to prove in theorem 3.3.4 that open subdomains of the form \( \text{Val}_{\text{Spec}(B)}(\text{Spec}(A)) \) form a basis for the topology of \( \text{Val}_Y(X) \). In §3.4 we study \( Y \)-blow ups of \( X \), which are analogs of \( U \)-admissible or formal blow ups from Raynaud’s theory, see [BL].

As a corollary, we prove that \( \psi : \text{Val}_Y(X) \rightarrow \text{RZ}_Y(X) \) is a homeomorphism in the decomposable case. Finally, we prove in theorem 3.5.1 that any open quasi-compact subset of \( \text{Val}_Y(X) \) admits a scheme model of the form \( \text{Val}_{\text{Spec}(A)}(\text{Spec}(B)) \) with \( Y \) being \( X \)-affine. This result implies the decomposition theorem, and, therefore, leads to a new proof of Nagata’s theorem.

I want to mention that I was motivated by Raynaud’s theory in studying Riemann-Zariski spaces in the decomposable case, and the basic ideas are taken from [BL]. I give a simple illustration of those ideas in the proof of the generalized Thomason’s theorem.

When this paper was almost finished I was informed about a recent paper [FK] by Fujiwara and Kato, which contains a survey on a theory of generalized Riemann-Zariski spaces they are developing. The survey announces many exciting results, including Nagata compactification for algebraic spaces. It is clear that there is a certain overlap between that theory and the present paper which can be rather large, though it is difficult to make any conclusion on this subject until the actual
proofs are published. The generalized RZ spaces mentioned in [FK] are exactly the relative RZ spaces of open immersions $Y \hookrightarrow X$ (the same case which is used in the proof of the stable modification theorem).

1.1. On noetherian approximation and Nagata compactification. For shortness, a filtered projective family of schemes with affine transition morphisms will be called affine filtered family. Also, we abbreviate the words ”quasi-compact and quasi-separated” by the single “word” qcqs. In [TT, C.9], Thomason proved a very useful approximation theorem, which states that any qcqs scheme $Y$ over a ring $\Lambda$ is isomorphic to a scheme proj lim $Y_\alpha$, where $\{Y_\alpha\}_\alpha$ is an affine filtered family of $\Lambda$-schemes of finite presentation. Due to the following lemma, this theorem may be reformulated in a more laconic way as follows: $Y$ is affine over a $\Lambda$-scheme $Y_0$ of finite presentation.

**Lemma 1.1.1.** A morphism of qcqs schemes $f : Y \to X$ is affine if and only if $Y \sim \text{proj lim} Y_\alpha$, where $\{Y_\alpha\}_\alpha$ is a filtered family of $X$-affine finitely presented $X$-schemes.

**Proof.** If $Y \sim \text{proj lim} Y_\alpha$ is as in the lemma, then $Y_\alpha = \text{Spec}(\mathcal{E}_\alpha)$ for an $\mathcal{O}_X$-algebra $\mathcal{E}$, hence $Y = \text{Spec}(\mathcal{E})$ where $\mathcal{E} = \text{inj lim} \mathcal{E}_\alpha$. Conversely, suppose that $f$ is affine. By [EGA I, 6.9.16(iii)], $f_*(\mathcal{O}_Y) \sim \text{inj lim} \mathcal{E}_\alpha$, where $\{\mathcal{E}_\alpha\}$ is a direct filtered family of finitely presented $\mathcal{O}_X$-algebras. Hence $Y = \text{proj lim} \text{Spec}(\mathcal{E}_\alpha)$. □

We generalize Thomason’s theorem below. As a by-product, we obtain a simplified proof of the original theorem.

**Theorem 1.1.2.** Let $f : Y \to X$ be a (separated) morphism of qcqs schemes. Then $f$ can be factored into a composition of an affine morphism $Y \to Z$ and a (separated) morphism $Z \to X$ of finite presentation.

**Proof.** Step 1. Preliminary work. First we observe that if $f$ is separated and $Y \to Z \to X$ is a factorization as in the theorem, then $Y$ is the projective limit of schemes $Y_\alpha$ which are affine over $Z$ and of finite presentation. By [TT, C.7], already some $Y_\alpha$ is separated over $X$, hence replacing $Z$ with $Y_\alpha$, we achieve a factorization with $X$-separated $Z$. This allows us to deal only with the general (not necessarily separated) case in the sequel.

If $Y$ is affine and $f(Y)$ is contained in an open affine subscheme $X' \subset X$, then the claim is obvious. So, $Y$ admits a finite covering by open qcqs subschemes $Y_1, \ldots, Y_n$ such that the induced morphisms $Y_i \to X$ satisfy the conclusion of the theorem. It suffices to prove that one can decrease the natural number $n$ until it becomes 1, and, obviously, it suffices to deal only with the case of $n = 2$. Then the schemes $U := Y_1$ and $V := Y_2$ can be represented as $U = \text{proj lim} U_\beta$ and $V = \text{proj lim} V_\gamma$, where the limits are taken over affine filtered families of $X$-schemes of finite presentation.

Step 2. Affine domination. By [EGA IV, 8.2.11], for $\beta \geq \beta_0$ and $\gamma \geq \gamma_0$, the schemes $U_\beta$ and $V_\gamma$ contain open subschemes $U'_\beta$ and $V'_\gamma$, whose preimages in $U$ and $V$ coincide with $W := U \cap V$. By [EGA IV, 8.13.1], the morphism $W \to U'_{\beta_0}$ factors through $V'_\gamma$ for sufficiently large $\gamma$. Replace $\gamma_0$ by $\gamma$. By the same reason, the morphism $W \to V'_{\gamma_0}$ factors through some $U'_{\beta_\gamma}$ and the morphism $W \to U'_{\beta_\gamma}$ factors through some $V'_\gamma$. Let us denote the corresponding morphisms as $f_{\gamma, \beta} : V'_\gamma \to U'_\beta$, $f_{\beta, \gamma_0}$ and $f_{\gamma_0, \beta_0}$. Now comes an obvious but critical argument: $f_{\beta, \gamma_0}$ is separated because the composition $f_{\gamma_0, \beta_0} \circ f_{\beta, \gamma_0} : U'_\beta \to U'_{\beta_0}$ is separated (and even affine);
$f_{\gamma,\beta}$ is affine because its composition with a separated morphism $f_{\beta,\gamma_0}$ is affine. We gather the already defined objects in the left diagram below. Note that everything is defined over $X$, the horizontal arrows are open immersions, the vertical arrows are affine morphisms and the indexed schemes are of finite $X$-presentation.

\[
\begin{array}{c}
V & \xleftarrow{\phi'} & W' & \xrightarrow{\phi} & U \\
\downarrow & & \downarrow & & \downarrow \\
V_{\gamma} & \xleftarrow{\phi'} & V'_{\gamma} & \xrightarrow{\phi} & U_{\gamma} \\
U_{\beta}' & \downarrow & U_{\beta}' & \xrightarrow{f_{\gamma,\beta}} & U_{\beta} \\
\end{array}
\]

Step 3. **Affine extension.** The main task of this step is to produce the right diagram from the left one. It follows from the previous stage that $V' _{\gamma} = \text{Spec}(E')$, where $E'$ is a finitely presented $O_{U'_{\beta}}$-algebra. The morphism $\phi' : W \to V' _{\gamma}$ to a $U'_{\beta}$-affine scheme corresponds to a homomorphism $\varphi' : E' \to h' _{\gamma}(O_W)$, where $h' : W \to U'_{\beta}$ is the projection. Obviously $h_*(O_U)|_{U'_{\beta}} = h' _{\gamma}(O_W)$, where $h : U \to U_{\beta}$ is the projection. Hence we can apply [EGA I, 6.9.10.1], to find a finitely presented $O_{U_{\gamma}}$-algebra $E$ and a homomorphism $\varphi : E \to h_*(O_U)$ such that $E|_{U'_{\beta}} \cong E'$ and the restriction of $\varphi$ to $U' _{\beta}$ is $\varphi'$. Set $U_{\gamma} = \text{Spec}(E)$, then $U_{\gamma} \to U_{\beta}$ is an affine morphism whose restriction over $U'_{\beta}$ is $f_{\gamma,\beta}$, and $\varphi$ induces a morphism $\phi : U \to U_{\gamma}$. Finally, we glue $U_{\gamma}$ and $V_{\gamma}$ along $V' _{\gamma}$ obtaining a finitely presented $X$-scheme $Z$, and notice that the affine morphisms $U \to U_{\gamma}$ and $V \to V_{\gamma}$ glue to an affine morphism $Y \to Z$ over $X$. \qed

Our proof is a simple analog of Raynaud’s theory. Thomason used the first two steps (induction argument in the proof of C.9 and lemma C.6). Our simplification of his proof is due to the third step. The same arguments are used in Raynaud’s theory, for which we refer to [BL]: see the end of the proof of 4.1, statement (d) from that proof and lemma 2.6 (a). In our paper, they also appear in the proofs of lemmas 3.4.2 (i) and 3.4.4, and theorem 3.5.1.

Next, we recall Nagata compactification theorem, see [Nag]. A scheme theoretic proof of the theorem can be found in [Con] or [Lüt]. Recall that a morphism $f : Y \to X$ is called compactifiable if it can be factored as a composition of an open immersion $g : Y \to Z$ and a proper morphism $h : Z \to X$. Nagata proved that a finite type morphism $f : Y \to X$ of qcqs schemes is compactifiable if and only if it is separated. Actually, Nagata considered noetherian schemes, and the general case was proved by Conrad in [Con]. Let $\mathcal{I} \subset O_Z$ be an ideal with support $Z \setminus Y$, and $Z'$ be the blow up of $Z$ along $\mathcal{I}$. We can choose a finitely generated $\mathcal{I}$ because the morphism $Y \to Z$ is quasi-compact. The open immersion $g' : Y \to Z'$ is affine because $Z' \setminus Y$ is a locally principal divisor. It follows that $g$ is a composition of an affine morphism $g'$ of finite type and a proper morphism $Z' \to X$. Conversely, assume that $g : Y \to Z$ is affine of finite type and $Z \to X$ is proper. Then $Y$ is quasi-projective over $Z$, hence there is an open immersion of finite type $Y \to \overline{Y}$ with $Z$-projective and, therefore, $X$-proper $\overline{Y}$. Thus, Nagata’s theorem can be reformulated as follows: a finite type morphism is separated if and only if it can be represented as a composition of an affine morphism of finite type and a proper morphism. Now, one sees that a weak form of theorem 1.1.2 ($f$ is separated and
Z → X is of finite type) and Nagata’s theorem are together equivalent to the
following decomposition theorem, which will be also proved in §3.5 by a different
method.

**Theorem 1.1.3.** A morphism f : Y → X of quasi-compact quasi-separated schemes
is separated if and only if it can be factored as a composition of an affine morphism
Y → Z and a proper morphism Z → X.

2. Preliminary description of relative RZ spaces and applications

Throughout §2, f : Y → X denotes a separated morphism between qcqs schemes.

2.1. Valuations and projective limits. We are going to recall some notions
introduced in [Tem2, §3.2]. Consider a factorization of f into a composition of a
schematically dominant morphism f_i : Y → X_i and a proper morphism g_i : X_i →
X. We call the pair (f_i, g_i) a Y-modification of X, and usually it will be denoted
simply as X_i. Given two Y-modifications of X, we say that X_j dominates or refines
X_i, if there exists an X-morphism b_{ji} : X_j → X_i compatible with f_i and f_j. A
standard graph argument shows that if g_{ji} exists, then it is unique (one uses only
that f_j is schematically dominant and X_i is X-separated). The family \{X_i\}_{i∈I}
of all Y-modifications of X is filtered because any two Y-modifications X_i, X_j are
dominated by the scheme-theoretic image of Y in X_i ×_X X_j, and it has an initial
object corresponding to the schematic image of Y in X.

A relative Riemann-Zariski space X = RZ_Y(X) is defined as the projective limit
of the underlying topological spaces of Y-modifications of X. Note that if X is
integral and Y is its generic point, then one recovers the classical Riemann-Zariski
spaces, and a slightly more general case, when Y is a dominant point, was considered
in [Tem1, §1]. Let \pi_i : X → X_i be the projections and η : Y → X be the map
induced by f_i’s. We provide X with the sheaf MX = η∗(OX), which will be called
the sheaf of meromorphic functions, and with the sheaf OX = inj lim π_i−1(OX_i),
which will be called the sheaf of regular functions. The natural homomorphisms
α_i : π_i−1(OX_i) → MX induce a homomorphism α : OX → MX, and we will prove
later that η is injective and α is a monomorphism. Actually, we will give in corollary
3.5.2 a rather precise meaning to the ”claim” that MX is a sheaf of semi-fractions
of the sheaf OX.

**Remark 2.1.1.** For any filtered projective family of locally ringed spaces \{Y_j\}_{j∈J}
the projective limit \mathcal{Y} = proj lim_{j∈J} Y_j always exists and satisfies [\mathcal{Y}]_j := proj lim |Y_j|
and O\mathcal{Y} = inj lim π_j−1OY_j where π_j : \mathcal{Y} → Y_j’s are the projections. If Y_j’s are
schemes, then \mathcal{Y} is known to be a scheme when the transition morphisms are affine:
this situation is studied very extensively in [EGA IV, §8], and the obtained results
have a plenty of various very important applications. Though \mathcal{Y} does not have to
be a scheme in general, it is still a locally ringed space of a rather special form
which deserves a study. Our relative RZ spaces (X, O_X) provide a nice example
of such pro-schemes (while MX corresponds to an extra-structure related to Y),
and we will later obtain a very detailed description of such spaces (e.g. we will
decribe the stalks of O_X). Another interesting example of a pro-scheme which is
not a scheme but has a very nice realization is as follows: let X be a scheme with a
subset U closed under generalizations, when (U, O_X|U) is the projective limit of all
open neighborhoods of U which does not have to be a scheme: for example, take
U to be the set of all non-closed points on a surface X.
The classical absolute \( RZ \) spaces viewed either as topological spaces or, more generally, as locally ringed spaces admit two alternative descriptions: a projective limit of schemes; a space whose points are valuations. We defined the relative spaces \( RZ_Y(X) \) using projective limits, but they also admit a “valuative” description as spaces \( \text{Val}_Y(X) \). In §2 we only introduce the sets \( \text{Val}_Y(X) \) and establish a certain connection between \( RZ_Y(X) \) and \( \text{Val}_Y(X) \) which suffices for applications to the stable modification theorem 2.3.2. Throughout this paper by a valuation on a ring \( B \) we mean a commutative ordered group \( \Gamma \) with a multiplicative map \( | | : B \to \Gamma \cup \{0\} \) which satisfies the strong triangle inequality and sends 1 to 1. Recall that if \( B \) is a field, then \( R = \{x \in B \mid |x| \leq 1\} \) is a valuation ring of \( B \) (i.e. \( \text{Frac}(R) = B \)) which defines \( | | \) up to an equivalence. In general, a valuation is defined up to an equivalence by its kernel \( p \), which is a prime ideal, and by the induced valuation on the residue field \( \text{Frac}(B/p) \). By slight abuse of language, the point of \( \text{Spec}(B) \) given by \( p \) will be also called the kernel of \( | | \). Also, we will often identify equivalent valuations.

In this paragraph only we drop any assumptions on \( X, Y \) and \( f \). Let \( \text{Val}_Y(X) \) be the set of triples \( y = (y, R, \phi) \), where \( y \in Y \) is a point, \( R \) is a valuation ring of \( k(y) \) (in particular \( \text{Frac}(R) = k(y) \)) and \( \phi : S = \text{Spec}(R) \to X \) is a morphism compatible with \( y = \text{Spec}(k(y)) \to Y \) and such that the induced morphism \( y \to S \times_X Y \) is a closed immersion. Let \( O_Y \) denote the preimage of \( R \) in \( O_Y \) currently, it is just a ring attached to \( y \). We would like to axiomatize the properties of \( O_Y \) as follows.

By a semi-valuation ring we mean a ring \( O \) with a valuation \( | | \) such that any zero divisor of \( O \) lies in the kernel of \( | | \) and for any pair \( g, h \in O \) with \( |g| \leq |h| \) one has that \( h|g \). Two structures of a semi-valuation ring on \( O \) are equivalent if their valuations are equivalent. Let \( m \) be the kernel of \( | | \). Note that \( O \) embeds into \( A = O_m \) by our assumption on zero divisors, \( mA = m \) because the prime ideal \( m \) is \( (O \setminus m) \)-divisible, and \( R = O/m \) is the valuation ring of \( A/m \) corresponding to the valuation induced by \( | | \). Therefore, \( O \) is composed from the local ring \( A \) and the valuation ring \( R \subset A/m \) in the sense that \( O \) is the preimage of \( R \) in \( A \). We say that \( A \) is a semi-fraction ring of \( O \). Conversely, any ring composed from a local ring and a valuation ring is easily seen to be a semi-valuation ring. Semi-valuation rings play the same role in the theory of relative \( RZ \) spaces as valuation rings do in the theory of usual \( RZ \) spaces.

**Remark 2.1.2.** (i) The structure of a semi-valuation ring on an abstract local ring \( O \) is uniquely defined (up to an equivalence) by its embedding into the semi-fraction ring \( A \).

(ii) An abstract ring \( O \) can admit many semi-valuation ring structures. For example, if \( O \) is a valuation ring, then any its localization (i.e. a larger valuation ring in its field of fractions) can serve as its semi-fraction ring.

Here is a generalization of the classical criterion that an integral domain \( O \) is a valuation ring if and only if for any pair of elements \( f, g \in O \) either \( f|g \) or \( g|f \).

**Lemma 2.1.3.** Let \( O \subset A \) be two rings. Then the following conditions are equivalent: \( O \) admits a structure of a semi-valued ring such that \( A \) is \( O \)-isomorphic to the semi-fraction ring of \( O \); if \( f, g \in A \) are co-prime (i.e. \( fA + gA = A \)), then either \( f \in gO \) or \( g \in fO \).

**Proof.** We will prove only the converse implication since the direct one is obvious. We claim that \( A \) is a local ring. Indeed, if it is not local then \( A \setminus A^\times \) is not an ideal,
hence there exists non-invertible \(f, g\) with invertible \(f + g\). But by our assumption either \(f \in gA\) or \(g \in fA\), hence \(f + g\) is contained in a proper ideal equal to either \(f A\) or \(g A\), that is an absurd. Let \(m \in A\) be the maximal ideal, then taking \(f \in m\) and \(g = 1\) and observing that \(f\) does not divide 1 in \(\mathcal{O}\) (and even in \(A\)), we deduce that \(f \in \mathcal{O}\). Thus, we proved that \(m \subset \mathcal{O}\), in particular, \(\mathcal{O}\) is the preimage of the ring \(\mathcal{O}/m \subset A/m\) under the surjection \(A \to A/m\). It remains to show that \(\mathcal{O}/m\) is a valuation ring of \(A/m\). For pair of elements \(\bar{f}, \bar{g} \in \mathcal{O}/m\) choose liftings \(f, g \in \mathcal{O}\). Since either \(f|g\) or \(g|f\) in \(\mathcal{O}\), it follows that either \(f|\bar{g}\) of \(\bar{g}|f\). Hence \(\mathcal{O}/m\) is a valuation ring, and we are done. 

2.2. RZ space of a decomposable morphism. Let \(y = (y, R, \phi)\) be a point of \(\text{Val}_Y(X)\) and \(S = \text{Spec}(R)\). By the valuative criterion of openness, \(\phi\) factors uniquely through a morphism \(\phi_i: Y \to X_i\) for any \(Y\)-modification \(X_i \to X\). Since \(S \times_X Y\) is a closed subscheme of \(S \times_X Y\) by \(X\)-separatedness of \(X_i\), we obtain that the \(\phi_i\) induces a closed immersion \(y \to S \times_X Y\), and, in particular, \((y, R, \phi_i)\) is an element of \(\text{Val}_Y(X_i)\). It follows that the natural map \(\text{Val}_Y(X_i) \to \text{Val}_Y(X)\) is a bijection. So, \(\text{RZ}_Y(X)\) and \(\text{Val}_Y(X)\) depend on \(X\) and \(Y\) only up to replacing \(X\) with its \(Y\)-modification.

Now we will construct a map of sets \(\psi: \text{Val}_Y(X) \to \text{RZ}_Y(X)\). For any \(i \in I\), let \(x_i \in X_i\) be the center of \(R\) on \(X_i\), i.e. the image of the closed point of \(S\) under \(\phi_i\). Then the family of points \((x_i)\) defines a point \(x \in \mathfrak{X}\) and we obtain a map \(\psi\) as above. For any \(i, x_i\) specializes \(f_i(y)\), hence we obtain a homomorphism \(\mathcal{O}_{X_i, x_i} \to \mathcal{O}_{X_\infty, f_i(y)} \to \mathcal{O}_{Y, y} \to k(y)\) whose image lies in \(R\) because \(x_i\) is the center of \(R\) on \(X_i\). Therefore, the image of \(\mathcal{O}_{X_i, x_i}\) in \(\mathcal{O}_{Y, y}\) lies in \(\mathcal{O}_y\), and we obtain a natural homomorphism \(\mathcal{O}_{X, x} = \text{inj lim} \mathcal{O}_{X_i, x_i} \to \mathcal{O}_y\).

**Proposition 2.2.1.** Suppose that \(f\) is decomposable. Then any point \(x \in \mathfrak{X}\) possesses a preimage \(y = \lambda(x)\) in \(\text{Val}_Y(X)\) such that the homomorphism \(\mathcal{O}_{X, x} \to \mathcal{O}_y\) is an isomorphism. In particular, \(\lambda\) is a section of \(\psi\).

Actually, we will prove in §3 that \(\psi\) is a bijection (so \(\lambda\) is its inverse), but the proposition as it is already covers our applications in §2.

**Proof.** Factor \(f\) into a composition of an affine morphism \(Y \to Z\) and a proper morphism \(Z \to X\). After replacing \(X\) with the scheme-theoretic image of \(Y\) in \(Z\), we can assume that \(f\) is affine. Note that then for any \(Y\)-modification \(X_i \to X\), the morphism \(f_i: Y \to X_i\) is affine. Let \(x_i\) be the image of \(x\) in \(X_i\). Obviously, the schemes \(U_i = \text{Spec}(\mathcal{O}_{X_i, x_i}) \times_X Y\) are affine. In addition, on the level of sets each \(U_i\) consists of points \(y \in Y\) such that \(x_i\) is a specialization of \(f_i(y)\), the morphisms \(U_i \to Y\) are topological embeddings and \(\mathcal{O}_Y|_{U_i} \to \mathcal{O}_{U_i}\). Notice that the schemes \(U_i = \text{Spec}(B_i)\) form a filtered family, hence \(U_\infty := \text{proj lim} U_i = \text{Spec}(B_\infty)\), where \(B_\infty = \text{inj lim} B_i\). By [EGA IV, §8], \(U_\infty = \cap U_i\) set-theoretically. Since \(f_i: Y \to X_i\) is schematically dominant and the latter property is preserved under (possibly infinite) localizations on the base, the morphism \(U_i \to \text{Spec}(\mathcal{O}_{X_i, x_i})\) is schematically dominant too. So, for each \(i \in I\) we have that \(\mathcal{O}_{X_i, x_i} \to B_i\), and then an embedding of the direct limits \(\mathcal{O}_{X, x} \hookrightarrow B_\infty\) arises.

**Lemma 2.2.2.** Suppose that elements \(g, h \in B_\infty\) do not have common zeros on \(U_\infty\), then either \(g \in h\mathcal{O}_{X, x}\) or \(h \in g\mathcal{O}_{X, x}\).

**Proof.** Find \(i\) such that \(g\) and \(h\) are defined and do not have common zeros on \(U_i\). Note that \(U_i = \cap f^{-1}(V_j)\), where \(V_j\) runs over affine neighborhoods of \(x_i\).
Hence we can choose a neighborhood $X'_i = \text{Spec}(A)$ of $x_i$ such that $g,h \in B$ and $gB + hB = 1$, where $Y' = \text{Spec}(B)$ is the preimage of $X'_i$ in $Y$. To ease the notation we will write $X$ and $x$ instead of $X_i$ and $x_i$ (we can freely replace $X$ with $X_i$ because $RZ_Y(X)$ remains unchanged). Now, the pair $(g,h)$ induces a morphism $\alpha' : Y' \to P' \coloneqq \text{Proj}(A[T_g, T_h])$, whose scheme-theoretic image $\overline{X'}$ is a $Y'$-modification of $X'$. It would suffice to extend the $Y'$-modification $\alpha' : \overline{X'} \to X'$ to a $Y$-modification $\alpha : \overline{X} \to X$. Indeed, either $T_g \in T_hO_{\overline{X'},x'}$ or $T_h \in T_gO_{\overline{X'},x'}$, where $x' \in \overline{X'}$ is the image of $x$ on $\overline{X}$. So, existence of $\alpha$ would imply that $g/h$ or $h/g$ already in the image of $O_{\overline{X},x'}$ in $B_\infty$, which is by definition contained in $O_{X,x}$.

It can be difficult to extend $\alpha'$ (without applying Nagata compactification), but fortunately we can replace $\overline{X'}$ with any of its $Y'$-modification $\overline{X''}$ and it suffices to extend $\overline{X''} \to X'$ to a $Y$-modification of $X$. Choose $a,b$ such that $ag + bh = 1$, then there exists a natural morphism $\beta' : Y' \to P'' \coloneqq \text{Proj}(A[T_ag, T_{ah}, T-bg, T_{bh}])$ which takes $Y'$ to the affine chart on which $T_ag + T_{bh}$ is invertible. We define $\overline{X''}$ to be the scheme-theoretic image of $\beta'$. Since $\beta'$ factors through Segre embedding $\text{Proj}(A[T_g, T_h]) \times \text{Proj}(A[T_a, T_b]) \to P''$, we obtain that $\overline{X''}$ is a closed subscheme of the source which is mapped to $\overline{X'}$ by the projection onto the first factor. In particular, $\overline{X''}$ is a $Y'$-modification of $\overline{X'}$. We will show that $\overline{X''} \to X'$ extends to a $Y$-modification $\overline{X} \to X$.

First, we note that if $E \subset B$ is the $A$-submodule generated by $ag, ah, bg, bh$ (in particular, it contains $1$), then $\overline{X''} \to P \coloneqq \text{Proj}(\bigoplus_{n=0}^\infty E^n)$, where $E^n$ is the $n$-th power of $E$ in $B$ and $E^0$ is the image of $A$. Indeed, $\beta'$ obviously factors through $P_1 \coloneqq \text{Spec}(\bigcup_{n=0}^\infty E^n)$ (the union is taken inside $B$) which is actually the affine chart of $P$ on which the image of $1 \in E$ is invertible. Since $\bigcup_{n=0}^\infty E^n \subset B$, the morphism $Y \to P_1$ is schematically dominant, so the morphism $Y \to P$ is schematically dominant too. But clearly, $P$ is a closed subscheme in $P''$ containing the image of $Y$, hence $P$ is the schematic image of $Y$ in $P''$, i.e. $\overline{X''} = P$ as subschemes in $P''$. By [EGA I, 6.9.7], $E$ can be extended to a finitely generated $O_X$-submodule $E \subset f_*(O_Y)$, and replacing $E$ by $E + O_X$ we achieve in addition that $E$ contains the image of $O_X$ in $f_*(O_Y)$. Let $E^n$ be the $n$-th power of $E$ in the sheaf of $O_X$-algebras $f_*(O_Y)$ (so, $E^0$ is the image of $O_X$), then exactly the same computation as was used above shows that $\overline{X} \coloneqq \text{Proj}(\bigoplus_{n=0}^\infty E^n)$ is a $Y$-modification of $X$. Since $\overline{X} \to X$ obviously extends $\overline{X''} \to X'$, we are done. \hfill \square

The above lemma combined with lemma 2.1.3 provides $O_{X,x}$ with a semi-valuation ring structure such that $B_\infty$ is its semi-fraction ring. In particular, $B_\infty$ is a local ring, hence $U_\infty$ possesses a unique closed point $y$. So, $B_\infty = O_{Y,y}$, its subring $O_{X,x}$ contains $m_y$ and $R \coloneqq O_{X,x}/m_y$ is a valuation ring of $k(y)$. Define $\phi : S = \text{Spec}(R) \to X$ as the composition of the closed immersion $S \to \text{Spec}(O_{X,x})$ with the natural morphism $\text{Spec}(O_{X,x}) \to X$. For any $i$, $U_i = \text{Spec}(O_{X,i,x}) \times_X Y$ is a closed subscheme of $\text{Spec}(O_{X,i,x}) \times_X Y$, hence $U_\infty \to \text{proj lim}_{i \in I} U_i$ is a closed subscheme of $\text{Spec}(O_{X,x}) \times_X Y \to \text{proj lim}_{i \in I} \text{Spec}(O_{X,i,x}) \times_X Y$. Since $y$ is closed in $U_\infty$, we obtain that the morphism $y \to \text{Spec}(O_{X,x}) \times_X Y$ is a closed immersion. Hence the morphism from $y$ to a closed subscheme $S \times_X Y$ of $\text{Spec}(O_{X,x}) \times_X Y$ is a closed immersion too, and we obtain that $y = (y,R,\phi)$ is a required element of $\text{Val}_Y(X)$ with $\psi(y) = x$ and $O_{X,x} \to O_y$. \hfill \square
2.3. Applications. A preliminary description of relative Riemann-Zariski spaces obtained in the previous section, suffices for some applications. Assume we are given a qcqs scheme $S$ with a schematically dense quasi-compact subset $U$ (i.e. any neighborhood of $U$ is schematically dense) which is closed under generalizations. An $S$-scheme $X$ is called $U$-admissible if the preimage of $U$ in $X$ is schematically dense. By a $U$-etale covering we mean a separated morphism $\phi : S' \to S$ such that $\phi$ is etale over $U$, $S'$ is $U$-admissible, and for any valuation ring $R$ any morphism $\text{Spec}(R) \to S$ taking the generic point to $U$ lifts to $S'$ (actually those are $U$-etale $h$-coverings). Note that in [BLR] one considers a more restrictive class of $U$-etale coverings $S' \to S$, which split to a composition of a surjective flat $U$-etale morphism and a $U$-modification. However, it follows from the flattening theorem [RG, 5.2.2] of Raynaud-Gruson that the coverings of the latter type form a cofinal subclass.

Let $\mathcal{O}$ be a semi-valuation ring with semi-fraction ring $A$ and $m$ be the maximal ideal of $A$. Then $R := \mathcal{O}/m$ is the valuation ring in $K := A/m$, the scheme $S = \text{Spec}(\mathcal{O})$ is covered by its pro-open subscheme $U = \text{Spec}(A)$ (i.e. $U$ is the intersection of open subschemes; note also that $A$ is a (possibly infinite) localization of $\mathcal{O}$) and closed subscheme $T = \text{Spec}(R)$, and the intersection $U \cap T$ is a single point $\eta = \text{Spec}(K)$ which is the generic point of $T$ and the closed point of $U$.

Note that in some sense $S$ is glued from $U$ and $T$ along $\eta$, for example, there is a bi-Cartesian square

$$
\begin{array}{ccc}
\eta & \longrightarrow & U \\
\downarrow & & \downarrow \\
T & \longrightarrow & S
\end{array}
$$

Moreover, we will see that $U$-admissible $S$-schemes (resp. quasi-coherent $O_S$-modules) can be glued from $T$-schemes and $U$-schemes (resp. modules). Given a quasi-coherent $O_S$-module $M$, we identify with an $O$-module, set $M_U = M \otimes_O A$, $M_T = M \otimes_O R = M/mM$ and $M_\eta = M \otimes_O K$. We say that $M$ is $U$-admissible if the localization homomorphism $M \to M_U$ is injective. Note that any $O$-module $M$ defines a descent datum consisting of $M_U$, $M_T$ and an isomorphism $\phi_M : M_U \otimes_A K \cong M_T \otimes_R K$, and a similar claim holds for $S$-schemes. The corresponding categories of descent data are defined in an obvious way, and, naturally, we have the following gluing lemma.

**Lemma 2.3.1.** Keep the above notation.

(i) The natural functor from the category of $U$-admissible quasi-coherent $O_S$-modules $M$ to the category of descents data $(M_U, M_T, \phi_M)$ with quasi-coherent $M_U$ and quasi-coherent $\eta$-admissible $M_T$ is an equivalence of categories.

(ii) The natural functor from the category of qcqs $U$-admissible $S$-schemes $X$ to the category of descents data $(X_U, X_T, \phi_X)$ with qcqs $X_U$ and qcqs $\eta$-admissible $X_T$ is an equivalence of categories.

(iii) A qcqs $U$-admissible $S$-scheme $X$ is of finite type (resp. finite presentation) if and only if $X_U$ and $X_T$ are so.

**Proof:** The assertion (iii) of the lemma is exactly Step 1 from the proof of [Tem2, 2.4.3]. To prove (i) we note $mM_U = mM$, hence $M_T = M/mM$ embeds into $M_\eta = M_U/mM_U$. So, $M_T$ is $\eta$-admissible and the embedding $M \to M_U$ identifies $M$ with the preimage of $M_T$ under the projection $M_U \to M_\eta$. In particular, an exact sequence $0 \to M \to M_U \oplus M_T \to M_\eta \to 0$ arises. Conversely, given a descent...
datum as in (i), we can define an \(O\)-module \(M = \ker(M_U \oplus M_T \to M_s)\), and one easily sees that \(M_U\) and \(M_T\) are the base changes of this \(M\). We constructed maps from \(O_S\)-modules to descent data and vice versa, and one immediately sees that these maps extend to functors. Then it is obvious from the above that these functors are actually equivalence of categories which are essentially inverse one to another.

We proved (i) and similarly to the classical case one deduces quasi-affine descent rather automatically. Indeed, it follows obviously that the category of affine \(U\)-admissible \(S\)-schemes is equivalent to the category of affine descent data. In order to extend this equivalence to the categories of all qcqs schemes, the only non-obvious claim is effectiveness of descent. So, let assume that \(\eta \times_T X_T \cong \eta \times_U X_U\) is a descent datum as in the assertion of (ii). We know that the descent holds in the affine case, and the case of quasi-affine descent data follows because \(X_T\) (resp. \(X_U\)) is an open subscheme of its affine hull \(\overline{X_T} = \text{Spec}(\Gamma(O_{X_T}))\) (resp. \(\overline{X_U}\)) and one easily checks that \(\eta \times_T \overline{X_T} \cong \eta \times_U \overline{X_U}\). Hence affine hulls define an affine descent data which gives rise to an \(S\)-scheme \(\overline{X}\), and the desired scheme \(X\) is realized as an open subscheme in \(\overline{X}\). Finally, in our case the general descent follows from the quasi-affine one because one can easily construct open quasi-affine coverings \(X_T = \bigcup_{i=1}^{n} X_{T,i}\) and \(X_U = \bigcup_{i=1}^{n} X_{U,i}\) with \(\eta \times_T X_{T,i} \cong \eta \times_U X_{U,i}\) for each \(i\) (use that open subschemes of \(X_\eta\) extend to open subschemes in \(X_U\) and \(X_T\)). \(\square\)

We assume again that \(S\) is a qcqs scheme with a schematically dense quasi-compact subset \(U\) closed under generalizations. We will prove a stable modification theorem which strengthens its analog from [Tem2], and we refer to the introduction of loc.cit. for terminology. Our strengthening is in imposing natural restrictions on the base change required in order to construct a stable modification. It is reasonable to expect that in some sense one can preserve the locus \(U\) of \(S\) over which the given curve is already semi-stable. Since already when \(U\) is the generic point of an integral base scheme \(S\) one has to allow its finite etale coverings (i.e. one has to allow separable alterations rather than modifications), it seems that one cannot hope for more than admitting general \(U\)-etale coverings of the base.

**Theorem 2.3.2.** Let \(C\) be an \(S\)-multipointed curve with semi-stable \(U\)-fibers, then there exists a \(U\)-etale covering \(S' \to S\) such that the curve \(C \times_S S'\) admits a stable \(U\)-modification \(C_{st}\).

**Proof.** Step 1. Reduction to the case of \(S = \text{Spec}(O)\) and \(U = \text{Spec}(A)\), where \(O\) is a semi-valuation ring with the semi-fraction ring \(A\). Since \(C\) is semi-stable over an open subscheme of \(S\), we can enlarge \(U\) to an open schematically dense qcqs subscheme. Note that by noetherian approximation there exists a scheme \(S'\) of finite type over \(\mathbb{Z}\) with a morphism \(S \to S'\) such that \(U\) and \(C\) are induced from a schematically dense open subscheme \(U' \subseteq S'\) and a multipointed curve \(C' \to S'\). Then it suffices to solve our problem for \(S', U'\) and \(C'\), so we can assume that \(S\) is of finite type over \(\mathbb{Z}\). By [Tem2, 2.2.1], \(\mathcal{S} = RZ_U(S)\) is a qcqs topological space. For any point \(x = (\gamma, R, \phi) \in \mathcal{S}\), set \(S_x = \text{Spec}(O_{\mathcal{S}, x})\), \(U_x = \text{Spec}(O_{Y,y})\) and \(C_x = C \times_S S_x\). Since the embedding \(U \hookrightarrow S\) is obviously decomposable, proposition 2.2.1 implies that \(O_{\mathcal{S}, x}\) is a semi-valuation ring with the semi-fraction ring \(O_{Y,y}\). So, if the case mentioned in the formulation of the step is known, then there exists a \(U_x\)-etale covering \(S'_x \to S_x\) such that the \(S'_x\)-multipointed curve \(C_x \times S'_x \cong C \times_S S'_x\) admits a stable \(U_x\)-modification for \(U'_x = U_x \times_{S_x} S'_x\).
Consider the family \( \{ S_i \}_{i \in I} \) of all \( U \)-modifications of \( S \), and let \( x_i \) be the center of \( x \) on \( S_i \). Recall that \( \mathcal{O}_{S,x} = \text{inj lim} \mathcal{O}_{S_i,x_i} \). By approximation, there exists \( i = i(x) \) and a \( U \)-etale morphism \( h_x : S'_i \to S_i \) such that \( x_i \) lies in its image and \( C \times_S S'_i \) admits a stable \( U \)-modification. Furthermore, by the flattening theorem of Raynaud-Gruson [RG, 5.2.2], we can achieve that \( h_x \) is flat by replacing \( S_i \) by some its \( U \)-modification and replacing \( S'_i \) by its strict transform with respect to that modification. Then \( h_x(S'_i) \) is open in \( S_i \), hence its preimage in \( \mathcal{O} \) is an open neighborhood of \( x \). Note that in the sequel we can replace \( i \) by any larger index \( k \) simply by replacing \( h_x \) by its base change with respect to the \( U \)-modification \( S_k \to S_i \). Since \( \mathcal{O} \) is quasi-compact, we need only finitely many points \( x_1, \ldots, x_n \) to cover the whole \( \mathcal{O} \) by the preimages of the sets \( h_x(S'_i(x_i)) \). Then by the argument given above, we can enlarge all indexes \( i(1), \ldots, i(n) \) so that \( i := i(1) = \ldots i(n) \). Then the open subschemes \( h_x(S'_i(x_i)) \) cover \( S_i \) because their preimages cover \( \mathcal{O} \), and by flatness of \( h_x \)'s we have that \( \bigcup_{i=1}^n S'_i(x_i) \) is a required \( U \)-etale covering of \( S \).

Step 2. The theorem holds over a semi-valuation ring \( \mathcal{O} \). We assume that \( \mathcal{O} \) is composed from a local ring \((A,m)\) and a valuation ring \( R \subset K = A/m \), \( S = \text{Spec}(\mathcal{O}) \) and \( U = \text{Spec}(A) \). Set also \( T = \text{Spec}(R) \) and \( \eta = \text{Spec}(K) \). By [Tem2, 1.4], the theorem is known in the case of a valuation ring, i.e. the case when \( m = 0 \). Thus, there exists a finite separable extension \( K'/K \) with a valuation ring \( R' \) lying over \( R \) and such that \( C \times_S T' \) admits a stable modification, where \( T' = \text{Spec}(R') \). Let \( A' \) be an \( A \)-etale local ring such that \( A'/m = K' \), and let \( \mathcal{O}' \) be the semi-valuation ring composed from \( A' \) and \( R' \). Replacing \( \mathcal{O} \) by \( \mathcal{O}' \), we can assume that already \( C \times_S T \) admits a stable modification \( C_T \). By lemma 2.3.1(ii), we can glue \( C_U \) and \( C_T \) to a \( U \)-modification \( C' \to C \). It remains to show that \( C' \) is an \( S \)-curve, then it is clear that \( C' \) is a stable \( U \)-modification of \( C \). By lemma 2.3.1(iii), \( C' \) is of finite presentation over \( S \). Notice that \( S \) has no non-trivial \( U \)-modifications because \( T \) is the only modification of itself. But by the flattening theorem, the morphism \( C' \to S \) can be flattened by performing a \( U \)-modification (and even a \( U \)-admissible blow up) on \( S \) and replacing \( C' \) with its strict transform, hence we obtain that \( C' \) itself is \( S \)-flat.

A scheme version of the reduced fiber theorem of Bosch-Lütkebohmert-Raynaud [BLR, 2.1'], can be proved absolutely similarly.

**Theorem 2.3.3.** Let \( X \to S \) be a schematically dominant finitely presented morphism whose \( U \)-fibers are geometrically reduced. Then there exists a \( U \)-etale covering \( S' \to S \) and a finite \( U \)-modification \( X' \to X \times_S S' \) such that \( X' \) is flat, finitely presented and has reduced geometric fibers over \( S' \).

**Proof.** If \( S \) is the spectrum of a valuation ring and \( U \) is its generic point, then the theorem follows from [Tem2, 2.4.4] (actually it was the content of Steps 1–4 of the loc.cit.). Acting as in Step 2 of the previous proof, we deduce the case when \( S \) is the spectrum of a semi-valuation ring and \( U \) is the corresponding local scheme. Then it remains to repeat the argument of Step 1. □

### 3. Nagata compactification

Throughout §3, \( f : Y \to X \) is a morphism of schemes and \( \mathfrak{X} = \text{Val}_Y(X) \). Later we will also introduce topological spaces \( \text{Spa}(Y,X) \) and then we will use the
notation $\mathfrak{X} = \text{Spa}(Y, X)$. Sometimes we will consider another morphism of schemes $f' : Y' \to X'$ and then $\mathfrak{X}' = \text{Val}_Y(X')$, $\mathfrak{X} = \text{Spa}(Y', X')$.

3.1. **Connection to adic spaces.** Let $A$ be a ring and $B$ be an $A$-algebra. Huber considers in [Hub1] the set $\text{Spv}(B)$ of all equivalence classes of valuations on $B$ and provides it with the weakest topology in which the sets of the form $\{| | \in \text{Spv}(B)| |a| \leq |b| \neq 0\}$ are open for any $a, b \in B$. Huber proves in [Hub1, 2.2] that the resulting topological space is quasi-compact. Furthermore, he considers the quasi-compact subspace $\text{Spa}(B, A) \subset \text{Spv}(B)$ consisting of the valuations of $B$ with $|A| \leq 1$: see the definition on p. 467 in loc.cit., where one treats the generality because replacing $A$ by the integral closure of its image in $B$ has no impact on the topological space $\text{Spa}(B, A)$. Actually, the topological space $\text{Spa}(B, A)$ has a much finer structure of an adic space, but we will not use it.

Let us generalize the above paragraph to schemes. Note that a valuation on a ring $A$ is defined by its kernel $x \in \text{Spec}(A)$ and the induced valuation on $k(x)$. So, by a **valuation on a scheme $Y$** we mean a pair $y = (y, R)$, where $y \in Y$ is a point called the **kernel of $y$** and $R$ is a valuation ring in $k(y)$. One can define $y$ by giving a valuation $| |_y : O_{Y, y} \to \Gamma_y$ whose kernel is $m_y$. By $O_y$ we denote the subring of $O_{Y, y}$ given by the condition $| |_y \leq 1$, it is the preimage of $R$ in $O_{Y, y}$. Remark that $O_y$ is a semi-valuation ring with the semi-fraction ring $O_{Y, y}$. Often it is convenient to describe a valuation locally by choosing an affine neighborhood $\text{Spec}(A)$ of $y$ and giving a valuation $A \to O_{Y, y} \to \Gamma_y$ on $A$.

Furthermore, if $f : Y \to X$ is a morphism of schemes, then by an **$X$-valuation on $Y$** we mean a valuation $y = (y, R)$ provided with a morphism $\phi : S = \text{Spec}(R) \to X$ which is compatible with the natural morphism $\eta = \text{Spec}(k(y)) \to X$. Recall that in the valuative criteria of properness/separatedness one considers commutative diagrams of the form

$$\begin{matrix}
\eta & \xrightarrow{i} & Y \\
\downarrow & & \downarrow f \\
S & \xrightarrow{\phi} & X
\end{matrix} \quad (1)$$

where $S = \text{Spec}(R)$ is the spectrum of a valuation ring and $\eta = \text{Spec}(K)$ is its generic point, and studies liftings of $S$ to $Y$. It is easy to see (and will be proved in 3.2.1) that it suffices to consider only the case when $k(\tilde{y}) \subset K$ for $y = i(\eta)$ in the valuative criteria. In the latter particular case, diagrams 1 are exactly the diagrams which correspond to $X$-valuations of $Y$. Note also that an $X$-valuation $y = (y, R, \phi)$ gives rise to the following finer diagram

$$\begin{matrix}
\eta & \xrightarrow{i} & \text{Spec}(O_{Y, y}) & \xrightarrow{Y} & Y \\
\downarrow & \downarrow & \downarrow & \downarrow \\
S & \xrightarrow{\phi} & \text{Spec}(O_y) & \xrightarrow{X} & X
\end{matrix} \quad (2)$$

Indeed, the center $x \in X$ of $R$ specializes the image of $y$ and the induced homomorphism $O_{X, x} \to O_{Y, y} \to k(y)$ coincides with $O_{X, x} \to R \to \text{Frac}(R) \to k(y)$. Hence these homomorphisms factor through $O_y$ (it was noted earlier that the left square is bi-Cartesian, and we have just checked directly that it is co-Cartesian).
Let $\text{Spa}(Y, X)$ denote the set of all isomorphism classes of $X$-valuations of $Y$. We claim that $\text{Spa}(Y, X)$ depends functorially on $f$. Indeed, given a morphism $f' : Y' \to X'$ and a morphism $g : f' \to f$ consisting of a compatible pair of morphisms $g_Y : Y' \to Y$ and $g_X : X' \to X$, there is a natural map $\text{Spa}(g) : \text{Spa}(Y', X') \to \text{Spa}(Y, X)$ which to a point $(y', R', \phi')$ associates a point $(y, R, \phi)$, where $y = g_Y(y')$, $R = R' \cap k(y)$ and $\phi$ is defined as follows. The morphism $g_X \circ \phi' : \text{Spec}(R') \to X$ factors through $\text{Spec}(\mathcal{O}_{X,x})$, where $x$ is the image of the closed point of the source, hence we obtain a homomorphism $\alpha : \mathcal{O}_{X,x} \to R'$. Since the morphism $\text{Spec}(k(y')) \to X$ factors uniquely through $\text{Spec}(k(y))$, the image of $\alpha$ is contained in $R := R' \cap k(y)$. So, $g_X \circ \phi'$ factors uniquely through a morphism $\phi : \text{Spec}(R) \to X$ and the map $\text{Spa}(g)$ is constructed.

If $g_Y$ is an immersion and $g_X$ is separated, then $\text{Spa}(g)$ is injective. Indeed, if a point $y = (y, R, \phi) \in \overline{X} := \text{Spa}(Y, X)$ has a non-empty preimage in $\overline{X}' := \text{Spa}(Y', X')$, then $y \in Y'$ and any preimage of $y$ is given by a lifting of $\phi : \text{Spec}(R) \to X$ to $X'$, which is unique by the valuative criterion of separatedness. Note that if we are given another morphism between morphisms $h : (Y_1 \to X_1) \to (Y \to X)$ with the corresponding map $\text{Spa}(h) : \overline{X}_1 \to \overline{X}$, then $Y'_1 := (Y' \times_X Y_1)$ is a subscheme in $Y_1$ and $X'_1 := X' \times_X X_1$ is separated over $X_1$, hence $\overline{X}_1 := \text{Spa}(Y'_1, X'_1)$ embeds into $\overline{X}_1$. Furthermore, we say that $\overline{X}$ is an affine subset of $\overline{X}$ if $Y'$ is an open affine subscheme of $Y$, and $X'$ is affine and of finite type over $X$. We provide $\overline{X}$ with the weakest topology in which all affine subsets are open.

**Lemma 3.1.1.** Let $\text{Spa}(g) : \overline{X} \to \overline{X}$ and $\text{Spa}(h) : \overline{X}_1 \to \overline{X}$ be as above and assume that $g_Y$ is an immersion and $g_X$ is separated, then

(i) $\overline{X}_1$ is the preimage of $\overline{X}$ under $\text{Spa}(h)$;
(ii) if $Y, X$ are separated and $\overline{X}$ is affine then $\overline{X}_1$ is affine; in particular, if $Y$ and $X$ are separated, then the intersection of affine subsets in $\overline{X}$ is an affine subset;
(iii) affine subsets form a basis of the topology on $\overline{X}$, and if $Y$ and $X$ are qcqs then any intersection of two affine subsets is a finite union of affine subsets;
(iv) if $g_Y$ is an open immersion and $g_X$ is of finite type, then $\overline{X}$ is open in $\overline{X}$;
(v) the maps $\text{Spa}(h)$ are continuous.

**Proof.** The first claim is proved by a straightforward check. If $Y$ and $X$ are separated, then $Y' \times_Y Y_1$ and $X' \times_X X_1$ are affine, hence (i) implies (ii). Furthermore, in general (i) implies that the intersection of affine subsets in $\overline{X}$ is of the form $\text{Spa}(\overline{Y}, \overline{X})$. Since affine subset in $\text{Spa}(\overline{Y}, \overline{X})$ is also an affine subset in $\overline{X}$, to prove (iii) it suffices to show that any space $\text{Spa}(\overline{Y}, \overline{X})$ (resp. with qcqs $\overline{X}$ and $\overline{Y}$) is covered by (resp. finitely many) affine subsets. Find open affine (resp. finite) coverings $\overline{X} = \bigcup \overline{X}_j$ and $\overline{Y} = \bigcup \overline{Y}_j$ such that each $\overline{Y}_j$ is mapped to some $\overline{X}_i(j)$, then $\text{Spec}(\overline{Y}, \overline{X})$ is the union of affine subsets $\text{Spa}(\overline{Y}_j, \overline{X}_i(j))$. It proves (iii), and the same argument proves (iv). Finally, (v) follows from the fact the preimage of each affine subset is open due to (i) and (iv).

We claim that in the affine case the above topology agrees with the topology defined by Huber.

**Lemma 3.1.2.** If $X = \text{Spec}(A)$ and $Y = \text{Spec}(B)$ are affine, then the canonical bijection $\phi : \text{Spa}(Y, X) \to \text{Spa}(B, A)$ is a homeomorphism.
Proof. It follows from the definitions of the topologies that the map is continuous, so we have only to establish openness. Let $\mathfrak{X} = \text{Spa}(\text{Spec}(C), \text{Spec}(A'))$ be an affine subset in $\mathfrak{X}$, where $\text{Spec}(C)$ is an open subscheme of $\text{Spec}(B)$ and $A'$ is a finitely generated $A$-algebra. It suffices to prove that $\phi(\mathfrak{X})$ is a neighborhood of each point $z$ it contains. Replacing $A'$ with its image in $C$ we can assume that it is an $A$-subalgebra of $C$ generated by $h_1, \ldots, h_n \in C$. Note that if $\{U_i\}$ is an open covering of $\text{Spec}(C)$, then the sets $\text{Spa}(U_i, \text{Spec}(A'))$ cover $\mathfrak{X}$. Therefore, shrinking $\text{Spec}(C)$ we can assume that $C = B_b$ for an element $b \in B$. Then $h_i = h_i/b^m$ with $b_i \in B$ and $m \in \mathbb{N}$, and $\phi(\mathfrak{X})$ consists of all valuations of $B$ with $|b_i| \leq |b^m| \neq 0$ for any $i$. Thus, $\phi(\mathfrak{X})$ is open in $\text{Spa}(B, A)$, and we are done. \qed

Since Huber’s spaces $\text{Spa}(B, A)$ are qcqs, we obtain the following corollary.

Corollary 3.1.3. If $X$ and $Y$ are qcqs schemes, then the space $\text{Spa}(Y, X)$ is qcqs.

Let $B$ be a ring provided with a valuation $| | : B \to \Gamma \cup \{0\}$, and let $y \in \text{Spec}(B)$ be its kernel. We say that a convex subgroup $\Gamma' \subseteq \Gamma$ bounds $B$, if for any element $b \in B$, there exists an element $h \in \Gamma'$ with $|b| \leq h$. For any such subgroup we can define a valuation $| |' : B \to \Gamma'$ by the rule $|x'| = |x|$ if $|x| \in \Gamma'$ and $|x'| = 0$ otherwise. Obviously, the kernel $y'$ of $| |'$ specializes $y$. Recall that $| |'$ is called a primary specialization of $| |$, see [Hub1, 2.3]. Here are simple properties of primary specializations.

Remark 3.1.4. (i) Primary specialization is a transitive operation and the set $P$ of primary specializations of $| |$ is ordered.

(ii) The set $P$ possesses a minimal element corresponding to the intersection of all subgroups bounding $B$; it is called the minimal primary specialization.

(iii) A valuation on $AB$ is called minimal if it has no non-trivial primary specializations. For a valuation given by a point $y \in \text{Spec}(B)$ and a valuation ring $R \subset k(y)$ the following are equivalent: (a) $(y, R)$ is minimal; (b) $k(y)$ is generated by $R$ and the image of $B$; (c) the morphism $\text{Spec}(k(y)) \to \text{Spec}(R) \times \text{Spec}(B)$ is a closed immersion.

(iv) Let $| | : B \to \Gamma \cup \{0\}$ be a valuation with kernel $y$, $\Gamma' \subseteq \Gamma$ be a convex subgroup, and $R \subseteq R'$ be the valuation rings of $k(y)$ corresponding to the induced valuations $k(y) \to \Gamma$ and $k(y) \to \Gamma \to \Gamma/\Gamma'$. Then the following conditions are equivalent: (a) there exists a primary specialization $| |'$ corresponding to $\Gamma'$; (b) the image of $B$ in $k(y)$ is contained in $R'$; (c) the morphism $y \to \text{Spec}(B)$ extends to a morphism $\text{Spec}(R') \to \text{Spec}(B)$. Moreover, if the conditions are satisfied, then the kernel $y'$ of $| |'$ is the center of $R'$ on $\text{Spec}(B)$. The equivalences (a)$\Leftrightarrow$(b) and (b)$\Leftrightarrow$(c) are obvious. As for the additional claim, we note that the center of $R'$ corresponds to the kernel of the homomorphism $B \to R' \to R'/m_{\nu'}$, and the latter consists of the elements $b \in B$ with $|b| < \Gamma'$, i.e. coincides with the kernel of $| |'$.

Let, more generally, $y = (y, R)$ be a valuation on a scheme $Y$. By a primary specialization of $y$ we mean a valuation $\overline{y} = (\overline{y}, \overline{R})$ such that $\overline{y}$ specializes $y$ and the valuation $| |_{\overline{y}}$ on $\mathcal{O}_Y, \overline{y}$ is a primary specialization of the valuation induced from $y$ via the homomorphism $\mathcal{O}_Y, \overline{y} \to \mathcal{O}_{Y, \overline{y}}$. Equivalently, if $\text{Spec}(A)$ is an affine neighborhood of $y$ and $\overline{y}$, then the valuation on $A$ induced by $(\overline{y}, \overline{R})$ is a primary specialization of the valuation induced by $(y, R)$.

Lemma 3.1.5. Let $(y, R)$ be a valuation on a separated scheme $Y$. 


(i) The set of primary valuations of \((y, R)\) is totally ordered by specialization;
(ii) If \(Y\) is also quasi-compact then \((y, R)\) admits a minimal primary specialization.

Proof. We claim that (i) follows from 3.1.4(iv). Indeed, for any \(R_1\) with \(R \subseteq R_1 \subseteq k(y)\) there exists at most one possibility to extend \(y\) to a morphism \(\text{Spec}(R_1) \to Y\). So, if we have two primary specializations \((y', R')\) and \((y'', R'')\) corresponding to valuation rings \(R \subseteq R_1, R_2 \subseteq k(y)\), then without loss of generality we have that \(R_1 \subseteq R_2\) and the unique morphism \(\text{Spec}(R_2) \to Y\) is obtained by localizing the morphism \(\text{Spec}(R_1) \to Y\). So, \(y'\) specializes \(y''\), and everything reduces to affine theory of primary specializations on \(O_{Y,Y'}\), see 3.1.4(i). To prove (ii) we note that \((y, R)\) admits a minimal primary specialization because the set of all specializations \(\{(y_i, R_i)\}_{i \in I}\) is such that the set of points \(\{y_i\}_{i \in I}\) is ordered with respect to specialization. By quasi-compactness there exists a point \(\overline{y}\) specializing all \(y_i\)'s. So, the claim reduces to the affine theory on \(O_{Y,Y'}\), see 3.1.4(ii). \(\Box\)

Finally, taking a morphism \(f : Y \to X\) into account, by a primary specialization of an \(X\)-valuation \(y = (y, R, \phi)\) we mean an \(X\)-valuation \(\overline{y} = (\overline{y}, \overline{R}, \overline{\phi})\) such that \((\overline{y}, \overline{R})\) is a primary specialization of \((y, R)\) and the image of \(\overline{\phi}\) in \(X\) is contained in the image of \(\phi\) in \(X\). Primary specialization is a particular case of a specialization relation in \(\text{Spa}(Y, X)\). An \(X\)-valuation \((y, R, \phi)\) (resp. a valuation \((y, R)\)) on \(Y\) is called minimal if it has no non-trivial primary specializations.

**Lemma 3.1.6.** Let \((y, R, \phi)\) be an \(X\)-valuation on \(Y\). Then any primary specialization \((\overline{y}, \overline{R})\) of the valuation \((y, R)\) admits at most one extension to a primary specialization \((\overline{y}, \overline{R}, \overline{\phi})\) of \((y, R, \phi)\), and the extension exists if and only if \(f(\overline{\phi})\) belongs to the image of \(\phi\). The latter is automatically the case when \(X\) is separated.

**Proof.** Obviously, the assertion on \(f(\overline{\phi})\) is necessary for an extension to exist. Furthermore, by 3.1.4(iv) there exists a valuation ring \(R'\) with \(R \subseteq R' \subseteq k(y)\) such that \(y\) extends to a morphism \(\text{Spec}(R') \to Y\) with \(\overline{\phi}\) being the image of the closed point. If \(X\) is separated then the induced map \(\text{Spec}(R') \to X\) must coincide with the corresponding localization of \(\phi : \text{Spec}(R) \to X\), hence we obtain the last assertion of the lemma. The remaining claims are local at the center \(x \in X\) of \(R\) (i.e. the image of the closed point of \(\phi\)). So, we can replace \(X\) and \(Y\) with a neighborhood of \(x\) and its preimage achieving that the schemes become separated. The uniqueness is now clear. To establish existence we should check that the image of the homomorphism \(O_{X,x} \to O_{Y,Y'}\) is in \(\overline{R}\). The latter follows from the following two facts: by existence of \(\phi\) the image of \(O_{X,x}\) in \(k(y)\) is in \(R\); \(\overline{R}\) is induced from \(R\) in the sense that an element \(\overline{r} \in O_{Y,Y'}\) satisfies \(f(\overline{r}) \in \overline{R}\) if and only if \(f(y) \in R\). \(\Box\)

The lemma shows that we can actually ignore \(\phi\) when \(X\) is separated. In particular, minimality of \((y, R, \phi)\) is then equivalent to that of \((y, R)\).

**Corollary 3.1.7.** Let \(f : Y \to X\) be a separated morphism of qcqs schemes and \(y = (y, R, \phi)\) be an \(X\)-valuation on \(Y\), then:

(i) the set of primary specializations of \(y\) is totally ordered and contains a minimal element;
(ii) \(y\) is minimal if and only if the morphism \(h : \text{Spec}(k(y)) \to Y \times_X \text{Spec}(R)\) is a closed immersion.
Proof. The claim is local at the center of \( x \in X \) of \( R \) (with respect to \( \phi \)), hence we can assume that \( X \) and, hence, \( Y' \) are separated. Then primary specializations of \( y \) can be identified with primary specializations of the valuation \((y, R)\), hence (i) follows from lemma 3.1.5. To prove (ii) we note that as soon as \( X \) is separated, \( h \) is a closed immersion if and only if the morphism \( \text{Spec}(k(y)) \to Y \times \text{Spec}(R) \) is a closed immersion, hence the claim follows from remark 3.1.4(iii).

Until the end of §3, we assume that \( f : Y \to X \) is a separated morphism of qcqs schemes, unless the contrary is said explicitly. It follows from the second part of the lemma that the set \( \mathfrak{X} = \text{Val}_Y(X) \) defined in section 2.2 coincides with the subset of \( \mathfrak{X} \) consisting of minimal \( X \)-valuations. Note that in affine situation, such subsets were considered by Huber, see [Hub1], 2.6 and 2.7. We provide \( \mathfrak{X} \) with the induced topology. The following lemma follows easily from the valuative criterion of properness, so we omit the proof.

Lemma 3.1.8. (i) If \( X' \) is a \( Y \)-modification of \( X \), then there are natural homeomorphisms \( \text{Spa}(Y, X') \to \text{Spa}(Y, X) \) and \( \text{Val}(X') \to \text{Val}(X) \).

(ii) If \( X' \) is an open subscheme of \( X \), then its preimage in \( \text{Val}_Y(X) \) (resp. \( \text{Spa}(Y, X) \)) is canonically homeomorphic to \( \text{Val}_Y'(X') \) (resp. \( \text{Spa}(Y', X') \)), where \( Y' = X' \times_X Y \).

Remark 3.1.9. (i) If \( f' : X' \to Y' \) and \( f : Y \to X \) are separated morphisms of qcqs schemes, and \( g : f' \to f \) is a morphism such that \( gy \) is an open immersion and \( gx \) is separated and of finite type, then \( \text{Spa}(Y', X') \) maps homeomorphically onto an open subspace of \( \mathfrak{X} \). Beware, however, that it may (and usually does) happen that the image of \( \text{Val}_Y'(X') \) in \( \mathfrak{X} \) is not contained in \( \mathfrak{X} \). The problem originates from the fact that a minimal valuation on \( Y' \) may admit non-trivial primary specializations on \( Y \).

(ii) There exists a natural contraction \( \pi_\mathfrak{X} : \mathfrak{X} \to \mathfrak{X} \) which maps any valuation to its minimal primary specialization, but it is a difficult fact that \( \pi_\mathfrak{X} \) is continuous.

(iii) Using \( \pi_\mathfrak{X} \) we can extend \( \text{Val} \) to a functor by composing \( \text{Spa}(g) \) with the contraction \( \pi_\mathfrak{X} \) as \( \text{Val}(g) : \mathfrak{X}' \to \mathfrak{X} \to \mathfrak{X} \). However, we do not know that it is continuous until continuity of the \( \pi_\mathfrak{X} \) is established.

Actually, the above problems are very tightly connected, and we will solve them only in the end of §3.3.

Proposition 3.1.10. Assume that \( f : Y \to X \) is a separated morphism of qcqs schemes. Then the spaces \( \text{Val}_Y(X) \) and \( \text{RZ}_Y(X) \) are quasi-compact and the map \( \psi : \text{Val}_Y(X) \to \text{RZ}_Y(X) \) is continuous.

Proof. Let \( \{\mathfrak{X}_i\}_{i \in I} \) be an open covering of \( \mathfrak{X} \). Find open sets \( \mathfrak{X}_i \subset \mathfrak{X} \) such that \( \mathfrak{X}_i = \mathfrak{X} \cap \mathfrak{X}_i \). Since any point of \( \mathfrak{X} \) has a specialization in \( \mathfrak{X} \) by corollary 3.1.7, \( \{\mathfrak{X}_i\}_{i \in I} \) is a covering of \( \mathfrak{X} \). By quasi-compactness of \( \mathfrak{X} \), we can find a subcovering \( \{\mathfrak{X}_i\}_{i \in J} \) with a finite \( J \), and then \( \{\mathfrak{X}_i\}_{i \in J} \) is a finite covering of \( \mathfrak{X} \). Thus, \( \mathfrak{X} \) is quasi-compact.

We claim that for any \( Y \)-modification \( X' \to X \), the map \( \phi : \mathfrak{X} \to X' \) is continuous. Indeed, if \( U \subset X' \) is open, then its preimage in \( \mathfrak{X} \) is the open subspace \( \mathfrak{X} \to \text{Spa}(Y \times_X U, U) \). Therefore, the preimage of \( U \) in \( \mathfrak{X} \) is the open set \( \mathfrak{X} \cap \mathfrak{X} \), as required. The continuity of the maps \( \phi \) implies that the map \( \psi : \mathfrak{X} \to \text{RZ}_Y(X) \) is continuous. Since \( \mathfrak{X} \) is quasi-compact and \( \psi \) is surjective by proposition 2.2.1, the space \( \text{RZ}_Y(X) \) is quasi-compact. \( \square \)
3.2. Valuative criteria. In the sequel, we will need to strengthen the classical valuative criteria of separatedness and properness, [EGA II], 7.2.3 and 7.3.8. Our aim is to show that it suffices to consider valuative diagrams of specific types. We say that a morphism is compatible with a commutative diagram, if the diagram remains commutative after adjoining this morphism. Until end of §3.2, $f$ is not assumed to be separated.

**Lemma 3.2.1.** Keep the notation of diagram (1) and set $K' = k(i(\eta))$ and $R' = R \cap K$. Then diagram (1) completes uniquely to a commutative diagram

$$
\begin{array}{cccc}
\text{Spec}(K) & \longrightarrow & \text{Spec}(K') & \longrightarrow \ Y \\
\downarrow & & \downarrow & \\
\text{Spec}(R) & \longrightarrow & \text{Spec}(R') & \longrightarrow \ X
\end{array}
$$

and any morphism $h : \text{Spec}(R) \to \ Y$ compatible with the above diagram is induced from a morphism $h' : \text{Spec}(R') \to \ Y$ compatible with the diagram and determines $h'$ uniquely.

**Proof.** The morphism $\text{Spec}(K) \to \ Y$ obviously factors through $\text{Spec}(K')$. The morphism $\text{Spec}(R) \to \ X$ factors through $\text{Spec}(\mathcal{O}_{X,x})$, where $x$ is the image of the closed point of $\text{Spec}(R)$. The image of $\mathcal{O}_{X,x}$ in $R \subseteq K$ is contained in $K'$, hence the morphism $\text{Spec}(R) \to \ X$ factors through $\text{Spec}(R')$. By the same reasoning, a morphism $h : \text{Spec}(R) \to \ Y$ compatible with the diagram factors through $h' : \text{Spec}(R') \to \ Y$, and they both are determined uniquely by the image of the closed point in $Y$. □

**Lemma 3.2.2.** Keep the notation of diagram (1) and assume that $R \subseteq R' \subseteq K$ is such that the morphism $\text{Spec}(R') \to \ X$ admits a lifting $g : \text{Spec}(R') \to \ Y$ compatible with the diagram. Let $\tilde{K}$ be the residue field of $R'$ and $\tilde{R}$ be the image of $R$ in $\tilde{K}$.

Then any morphism $\tilde{h} : \text{Spec}(\tilde{R}) \to \ Y$ compatible with the above diagram is induced from a morphism $h : \text{Spec}(R) \to \ Y$ compatible with the diagram and determines $h$ uniquely.

**Proof.** Consider a morphism $\tilde{h} : \text{Spec}(\tilde{R}) \to \ Y$ compatible with the diagram. It suffices to show that it factors through $\text{Spec}(R)$, since the uniqueness is again trivial. Let $y$ be the image of the closed point of $\text{Spec}(\tilde{R})$, so $\tilde{h}$ induces a homomorphism $\mathcal{O}_{Y,y} \to \tilde{R}$. Since $y$ specializes the image $y'$ of the closed point of $\text{Spec}(R')$, we have also a homomorphism $\mathcal{O}_{Y,y} \to \mathcal{O}_{Y,y'} \to R'$. Then the compatibility implies that the image of $\mathcal{O}_{Y,y}$ in $\tilde{K} = R'/m_{R'}$ lies in $\tilde{R}$. Therefore, the image of $\mathcal{O}_{Y,y}$ in $R'$ lies in $R$ which is the preimage of $\tilde{R}$ under $R' \to \tilde{R}$, and we obtain that the homomorphism $\mathcal{O}_{Y,y} \to \tilde{R}$ factors through $R$. It gives the desired morphism $h : \text{Spec}(R) \to \ Y$. □
Note that lemma 3.2.1 implies that in the valuative criteria it suffices to consider only the case when $k((\eta)) \to K$ (i.e. to take valuative diagrams corresponding to the elements of $\text{Spa}(Y, X)$), and then lemma 3.2.2 and remark 3.1.4(iv) imply that it even suffices to consider only the valuative diagrams corresponding to the elements of $\text{Val}_Y(X)$. It is also well known that in the valuative criteria one can restrict to the case when the image of $\eta$ lies in a given dense subset which is closed under generalization (e.g. the generic point of an irreducible scheme), and such strengthening is the main issue of the following proposition.

**Proposition 3.2.3.** Let $h : Z \to Y$ and $f : Y \to X$ be morphisms of qcqs schemes. Then the natural map $\overline{\psi} : \text{Spa}(Z, Y) \to \text{Spa}(Z, X)$ induces a map $\psi : \text{Val}_Z(Y) \to \text{Val}_Z(X)$. Assume in addition that $h$ is dominant and $g = f \circ h$ is separated, then:

(i) The following conditions are equivalent: (a) the morphism $f$ is separated; (b) $\overline{\psi}$ is injective; (c) $\psi$ is injective.

(ii) Let us assume in addition that $f$ is of finite type, then the following conditions are equivalent: (a) $f$ is proper; (b) $\overline{\psi}$ is bijective; (c) $\psi$ is bijective.

*Proof.* The assertion that $\overline{\psi}$ takes $\text{Val}_Z(Y)$ to $\text{Val}_Z(X)$ (i.e. preserves minimality) follows from lemma 3.1.6. The implications (a)$\Rightarrow$(b) follow from the standard valuative criteria, and the implications (b)$\Rightarrow$(c) are obvious. To prove the implications (c)$\Rightarrow$(b) we have to relate the fibers of $\overline{\psi}$ and $\psi$. Consider any point $z \in \text{Spa}(Z, X)$ and let $z_0 \in \text{Val}_Z(X)$ be its minimal primary specialization. Then lemma 3.2.2 implies that the sets $\overline{\psi}^{-1}(z)$ and $\psi^{-1}(z_0)$ are naturally bijective.

It remains to establish the implications (b)$\Rightarrow$(a), and we prefer to show that if (a) fails, then (b) fails in both (i) and (ii). So, suppose that $f$ is of finite type, separated and not proper in (i) (resp. not separated in (i)). By the standard valuative criterion and lemma 3.2.1, there exists an element $y = (y, R_y, \phi_y) \in \text{Spa}(Y, X)$ such that the number of liftings of the morphism $\phi_y : \text{Spec}(R_y) \to X$ to $Y$ is zero (resp. at least two). Let $x$ denote the center of $R_y$ on $X$.

By [EGA I, 6.6.5], there exists a point $z \in Z$ with $h(z)$ generalizing $y$, so a homomorphism $\mathcal{O}_{Y,y} \to \mathcal{O}_{Z,z} \to k(z)$ arises. Let $R'$ be any valuation ring of $k(z)$ which dominates the image of $\mathcal{O}_{Y,y}$. It gives rise to an element $(z, R', \phi') \in \text{Spa}(Z, Y)$ centered on $y$. Choose a valuation ring $\tilde{R}$ of the residue field $\tilde{K}$ of $R'$ such that $\tilde{R}$ dominates the valuation ring $R_y$ of $k(y) \subset \tilde{K}$, and define a valuation ring $R$ of $k(z)$ as the composition of $R'$ and $\tilde{R}$. The compatible homomorphisms $\mathcal{O}_{X,x} \to \mathcal{O}_{Y,y} \to R'$ and $\mathcal{O}_{X,x} \to R_y \to \tilde{R}$ induce a homomorphism $\mathcal{O}_{X,x} \to R$, and we obtain the following commutative diagrams.

```
  Spec(k(z))  Z
  |      |      |
  v      v      v
  Spec($\tilde{K}$) Spec($R'$) Spec($R$) Spec($\tilde{R}$)
  |      |      |      |
  v      v      v      v
  Spec($\tilde{R}$) Spec($R'$) Spec($R$) Spec($\tilde{R}$)
```

Lemma 3.2.1 implies that there is a one-to-one correspondence between morphisms $\text{Spec}(R_y) \to Y$ and $\text{Spec}(\tilde{R}) \to Y$ compatible with the right diagram. By
lemma 3.2.2, the latter morphisms are in one-to-one correspondence with the morphisms \( \phi : \text{Spec}(R) \to Y \) compatible with the left diagram. Note that \( z = (z, R, \phi_z) \) is an element in \( \text{Spa}(Z, X) \), and any morphism \( \phi \) as above gives a preimage of \( z \) in \( \text{Spa}(Z, Y) \). We obtain that in the case (i), \( z \) has at least two preimages, so \( \psi \) is not injective. The same argument would prove (ii) if we also know that, conversely, any preimage of \( z \) in \( \text{Spa}(Z, Y) \) comes from \( \phi \). So it remains to note that \( Y \to X \) is separated by the already established case part (i), hence the morphism \( \phi' \) is uniquely determined by morphisms \( \text{Spec}(k(z)) \to Y \) and \( \text{Spec}(R') \to X \). \( \square \)

3.3. Affinoid domains. Let \( f' : Y' \to X' \) be another separated morphism of qcqs schemes and \( g : f' \to f \) be a morphism. Recall that we defined in §3.1 a continuous map \( \text{Spa}(g) : \overline{X} \to \overline{X} \) which was shown to be injective if \( g_Y \) is an immersion and \( g_X \) is separated. However, our definition of a map \( \text{Val}(g) : \overline{X} \to \overline{X} \) was rather cumbersome because even if \( \text{Spa}(g) \) is injective, it does not have to respect the subspaces \( \text{Val} \) in the spaces \( \text{Spa} \). The following proposition gives a criterion when \( \text{Spa}(g) \) does respect \( \text{Val} \)'s.

**Proposition 3.3.1.** Suppose that \( g_Y \) is an open immersion and \( g_X \) is separated. Then \( \text{Spa}(g)(\overline{X}') \subset \overline{X} \) if and only if the locally closed immersion \( (g_Y, f') : Y' \to Y \times_X X' \) is a closed immersion, in which case one actually has that \( \overline{X}' = \text{Spa}(g)^{-1}(\overline{X}) \).

**Proof.** Suppose that \( h := (g_Y, f') \) is a closed immersion. Let \( y' = (y', R', \phi) \in \overline{X}' \) be a point with \( \eta' = \text{Spec}(k(y')) \) and \( S' = \text{Spec}(R') \), and let \( y = (y, R, \phi) \) be its image in \( \overline{X} \). By lemma 3.1.7(ii), \( y' \) is minimal if and only if the natural morphism \( \eta' \to Y' \times_X S' \) is a closed immersion. By closedness of \( h \), the latter happens if and only if the composition morphism \( \eta' \to Y' \times_X S' \to (Y \times_X X') \times_X S' \to Y \times_X S' \) is a closed immersion. Finally, since \( R = k(y) \cap R' \), it follows that \( \eta' \to Y \times_X S' \) is a closed immersion if and only if \( \text{Spec}(k(y)) \to Y \times_X \text{Spec}(R) \) is a closed immersion, i.e. \( y \) is minimal. Thus, under our assumption on \( h \), minimality of \( y' \) is equivalent to minimality of its image. This establishes the inverse implication in the proposition, and the complement.

It remains to show that if \( h \) is not a closed immersion, then \( \text{Spa}(g) \) does not respect the subsets \( \text{Val} \). Note that \( h \) is a locally closed immersion because \( g_Y \) is an open immersion, and assume that \( h \) is not a closed immersion. Set \( Z = Y \times_X X' \) and find a \( Z \)-valuation \( y' = (y', R', \phi) \) of \( Y' \) such that the morphism \( \phi : \text{Spec}(R') \to Z \) cannot be lifted to a morphism \( \text{Spec}(R') \to Y' \). Replacing \( y' \) by its minimal primary specialization, we achieve that \( y' \) is minimal and \( R' \subseteq k(y') \). Clearly \( y' \) defines an \( X' \)-valuation \( y = (y', R', \phi') \) on \( Y' \) with \( \phi' = \text{pr}_X \circ \phi \), and \( y \) is minimal because any non-trivial primary specialization corresponds to a lifting \( \text{Spec}(R''') \to X' \) for some \( R' \subseteq R'' \subseteq k(y) \) and such a lifting would induce a lifting \( \text{Spec}(R'') \to Z \) corresponding to a non-trivial primary specialization of \( y' \). Thus, \( y \in \overline{X}' \), but \( \text{Spa}(g)(y) \) is not a minimal \( X \)-valuation on \( Y \) because the morphism \( \text{Spec}(R') \to X \) lifts to the morphism \( \text{pr}_Y \circ \phi : \text{Spec}(R') \to Y \). \( \square \)

Let us assume that \( g_Y \) is an open immersion and \( g_X \) is separated and of finite type. We saw that if \( h \) is a closed immersion, then \( \overline{X}' \) is naturally identified with a quasi-compact open subset of \( \overline{X} \) via \( \text{Spa}(g) \), and we say that \( \overline{X}' \) is an open subdomain of \( \overline{X} \). If, in addition, \( X' \) and \( Y' \) can be chosen to be affine, then we say that \( \overline{X}' \) is an affinoid subdomain of \( \overline{X} \). Note also that the situation described in the proposition appears in Deligne’s proof of Nagata compactification theorem under the name of
quasi-domination. (Recall that by a quasi-domination of $Y$ over $X'$ one means an open subscheme $Y' \subset Y$ and a morphism $Y' \to X'$ such that the morphism $Y' \to Y \times_X X'$ is a closed immersion, see [Con, §2].) The notion of quasi-domination plays a central role in Deligne’s proof. We list simple properties of open and affinoid subdomains in the following lemma. Note that it will be much more difficult to prove that open subdomains are preserved under taking finite unions (remark that this is a typical situation in algebraic geometry that preimages, intersections, projective limits, etc., are much easier for study than pushouts, images, direct limits, etc.).

**Lemma 3.3.2.** Open subdomains are transitive and are preserved by finite intersections. Moreover, the intersection of open subdomains $\text{Val}_Y(X_i)$ with $i \in \{1, 2\}$ is the open subdomain $\text{Val}_{Y_1 \cap Y_2}(X_1 \times_X X_2)$. In particular, if $X$ is separated and $X_i$'s are affinoid, then $X_{12}$ is affinoid.

**Proof.** This follows from the analogous lemma 3.1.1 concerning the spaces $\text{Spa}$. □

The following remark will not be used in the sequel.

**Remark 3.3.3.** (i) Our definition of RZ spaces is a straightforward generalization of the classical one. It is also possible to define RZ spaces directly as follows: an affinoid space is a topological space $X$ isomorphic to a compactification of an affinoid space $X$. This follows from the analogous lemma 3.1.1 concerning the spaces $\text{Spa}$.

(ii) The following example illustrates a difference between adic and Riemann-Zariski spaces. Let $k$ be a field, $A = B = A' = k[T], B' = k[T, T^{-1}]$ and $X, X', X, X'$ are as above. Then $X'$ is a rational subdomain in $X$ in the sense of [Hub2]. From other side, $X'$ is not an affinoid domain in $X$. One can show that the map $X' \to X$ is bijective, but the sheaves $\mathcal{M}_X$ and $\mathcal{M}_{X'}$ are not isomorphic.

**Theorem 3.3.4.** The affinoid subdomains of $X$ form a basis of its topology.

**Proof.** It follows from lemma 3.3.2 that we should prove that for any affine subset $X_0 = \text{Spa}(B_0, A_0)$ in $X$ and a point $y = (y, R, \phi) \in X \cap X_0$ there exists an affinoid subdomain $\text{Val}_Y(X)$ containing $y$ and contained in $X_0$. Moreover, we can assume that $X = \text{Spec}(A)$ is affine because $X$ is covered by affinoid subdomains of the form $\text{Val}_{Y'}(X')$, where $X' = \text{Spec}(A)$ is an open subscheme of $X$ and $Y' = X' \times_X Y$. In order to construct $\text{Val}_Y(X)$ as required we will extend diagram (2) to the following one, where $\overline{Y} = \text{Spec}(\overline{R})$ and $\overline{X} = \text{Spec}(\overline{A})$ will be finally defined in the end of the proof. Recall that $\mathcal{O}_Y$ is a semi-valuation ring with semi-fraction ring $\mathcal{O}_{Y,y}$ and such that $\mathcal{O}_y/m_y = R$.

\[
\begin{array}{c}
\text{Spec}(k(y)) \quad \text{Spec}(\mathcal{O}_{Y,y}) \quad Y \\
\downarrow \quad \downarrow \\
\text{Spec}(R) \quad \text{Spec}(\mathcal{O}_y) \quad \overline{X} \quad \overline{X}
\end{array}
\]

Since $\text{Spec}(R) \times_X Y$ is closed in $\text{Spec}(R) \times Y$ by separatedness of $X$, lemma 3.1.7(ii) implies that the morphism $h : \text{Spec}(k(y)) \to \text{Spec}(R) \times Y$ is a closed immersion. To explain the strategy of the proof we remark that the morphism $\text{Spec}(\mathcal{O}_{Y,y}) \to \text{Spec}(\mathcal{O}_y) \times Y$ is a closed immersion (actually it can be proved by the same argument as we use below), and our strategy will be to approximate $\mathcal{O}_y$.
and \( \mathcal{O}_{Y,y} \) by \( A \)-rings \( \mathcal{A} \) and \( \mathcal{B} \) so that \( \mathcal{A} \) is finitely generated over \( A \), \( \overline{Y} = \text{Spec}(\mathcal{B}) \) is a neighborhood of \( y \) and \( \overline{Y} \rightarrow \overline{X} \times Y \) is a closed immersion.

It will be more convenient to work with affine schemes and \( Y \) is the only non-affine scheme in our consideration, so let us cover \( Y \) with open affine subschemes \( Y_i = \text{Spec}(B_i) \), \( Z_j = \text{Spec}(C_j) \), where \( 1 \leq i \leq n, 1 \leq j \leq m, y \in Y_i \) and \( y \notin Z_j \). Since \( \text{Spec}(B_0) \) contains \( y \) by our assumptions, we also set \( Y_0 = \text{Spec}(B_0) \). For each \( i \), \( h \) factors through closed immersion \( \text{Spec}(k(y)) \rightarrow \text{Spec}(R) \times Y_i \), hence the images of \( R \) and \( B_i \) generate \( k(y) \). Now, we will find a neighborhood \( \overline{Y} = \text{Spec}(\mathcal{B}) \) of \( y \) which is contained in all \( Y_i \)'s and satisfies the following condition: for each \( i \), \( \mathcal{B} \) is a localization of the form \( (B_i)_f \), and, the most important, we have that \( f_i(y) \notin m_R \).

Let us (until the end of this paragraph only) call \( R\)-localization for localization of an affine neighborhood \( \text{Spec}(C) \) of \( y \) at an element \( f \) such that \( f(y) \notin m_R \). Obviously, \( R \)-localizations are transitive and we claim that the family of \( R \)-localizations of each \( Y_i \) form a basis of neighborhoods of \( y \). Indeed, for any element \( f \in B_i \) with \( f(y) \neq 0 \) we can find \( g \in B_i \) with \( f(y)g(y) \notin m_R \) (we use that \( B_i(y) \) generates \( k(y) \) over \( R \), so it contains elements of arbitrary large valuation). Thus, \( (B_i)_f \) is an \( R \)-localization of \( B_i \) where \( f \) is inverted and we obtain that the maximal (finite) \( R \)-localization of \( B_i \) is actually \( \mathcal{O}_{Y,y} \). Now, set \( \text{Spec}(B) = \cap_{i=1}^n Y_i \) and find \( R \)-localizations \( Y'_i = \text{Spec}(B_i)_y \) contained in \( \text{Spec}(B) \), and let \( \overline{Y} = \text{Spec}(\mathcal{B}) \) be an \( R \)-localization of \( \text{Spec}(B) \) contained in all \( Y'_i \). Then \( \overline{Y} \) is an \( R \)-localization of each \( Y'_i \), hence an \( R \)-localization of each \( Y_i \) too. So, \( \overline{B} = (B_i)_f \) is as required.

Let \( \mathcal{A} \) be the preimage of \( R \) under the character \( \overline{B} \rightarrow k(y) \) corresponding to \( y \). Clearly \( \mathcal{A} \) contains each element \( f_i^{-1} \), hence the ring \( \overline{B}(y) = B_i(y)[f_i^{-1}(y)] \) is generated by \( \overline{A}(y) \) and \( B_i(y) \). So, we obtain epimorphisms \( \mathcal{A} \otimes B_i \rightarrow \overline{B} \) and then the homomorphisms \( h_i : \mathcal{A} \otimes B_i \rightarrow \mathcal{B} \) are also surjective because \( \mathcal{A} \) contains the kernel \( p_y \) of \( \overline{B} \rightarrow k(y) \). In particular, each morphism \( \overline{Y} \rightarrow \overline{X} \times Y_i \) is a closed immersion. We claim that actually, \( \alpha : \overline{Y} \rightarrow \overline{X} \times Y \) is a closed immersion, and to check this we should show that the morphism \( \alpha_j : \overline{Y} \times Y Z_j \rightarrow \overline{X} \times Z_j \) with \( 1 \leq j \leq m \) are so. By separatedness of \( Y \) the source is affine, hence \( \overline{Y} \times Y Z_j = \text{Spec}(\mathcal{C}_j) \) where \( \mathcal{C}_j \) is generated by the images of \( c_j : C_j \rightarrow \mathcal{C}_j \) and \( b_j : B_j \rightarrow C_j \). Since our claim about \( \alpha \) would follow if we prove that the homomorphisms \( h_j' : \mathcal{A} \otimes C_j \rightarrow \mathcal{C}_j \) are surjective, it remains only to prove that for each \( j \) the image of \( h_j' \) contains the image of \( b_j \). Since \( y \in \overline{Y} \) and \( y \notin Z_j \) we have that \( b_j(p_y)\mathcal{C}_j = \mathcal{C}_j \), and hence the equality \( \mathcal{C}_j = b_j(p_y)\mathcal{C}_j \) can be strengthened as \( \mathcal{C}_j = b_j(p_y)c_j(C_j) \), i.e. \( \mathcal{C}_j \) is actually generated by \( b_j(p_y) \) and \( c_j(C_j) \). Since \( p_y \subset \mathcal{A} \) by the definition of \( \mathcal{A} \), we obtain that \( h_j' \) is onto, as claimed.

Now, the morphism \( \overline{Y} \rightarrow \overline{X} \) is almost as required: \( \overline{Y} \) is open in \( Y \) and \( \alpha \) is a closed immersion. In addition, since \( y \subset \overline{X}_0 \), the image of \( A_0 \) under the homomorphism \( A_0 \rightarrow B_0 \rightarrow \overline{B} \rightarrow \mathcal{B}(y) \) is contained in \( R \), and hence the image of \( A_0 \) in \( \overline{B} \) is actually contained in \( \mathcal{A} \). So, it only remains to decrease the \( A \)-subalgebra \( \mathcal{A} \subset \mathcal{B} \) so that \( \overline{X} = \text{Spec}(\mathcal{A}) \) becomes of finite type over \( X \), but all good properties are preserved: \( \alpha \) is still a closed immersion, and \( \mathcal{A} \) contains the image of \( A_0 \) in \( \mathcal{B} \). As we saw, \( \alpha \) being a closed immersion is equivalent to surjectivity of the homomorphisms \( h_i : \mathcal{A} \otimes B_i \rightarrow \mathcal{B} \) and \( h_j' : \mathcal{A} \otimes C_j \rightarrow \mathcal{C}_j \). Since the homomorphisms \( B_i \rightarrow \mathcal{B} \) and \( C_j \rightarrow \mathcal{C}_j \) are of finite type, all we need for surjectivity of \( h_i \)’s and \( h_j' \)’s is a finite subset \( S \subset \mathcal{A} \). So, replacing \( \mathcal{A} \) with its \( A_0 \)-subalgebra generated by \( S \) we obtain \( \overline{X} \) as required. Obviously, \( \text{Val}(\overline{Y}(\overline{X})) \) is an affinoid domain containing \( y \), and \( \text{Val}(\overline{Y}(\overline{X})) \) is
Corollary 3.3.5. The space $\mathcal{X}$ is qcqs.

Proof. Any open subdomain is quasi-compact by 3.1.10, hence it suffices to prove that the intersection of open subdomains is quasi-compact. The latter follows from 3.3.2. \qed

Recall that we defined in remark 3.1.9 the contraction $\pi_\mathcal{X} : \mathcal{X} \to \mathcal{X}$ and used it to define the maps $\text{Val}(g) : \mathcal{X}' \to \mathcal{X}$ for $g : f' \to f$.

Corollary 3.3.6. The contraction $\pi_\mathcal{X}$ is continuous. In particular, the maps $\text{Val}(g)$ are continuous.

Proof. Since open subdomains $\mathcal{X}' = \text{Val}_Y(X')$ form a basis of the topology of $\mathcal{X}$ by theorem 3.3.4, it suffices to prove that the preimage of $\mathcal{X}'$ in $\text{Spa}(Y, X)$ is open. Since the minimality condition in $\text{Spa}(Y, X)$ and $\text{Spa}(Y', X')$ agree, $\pi^{-1}(\mathcal{X}')$ coincides with the open affine subset $\text{Spa}(Y', X')$. \qed

3.4. $Y$-blow ups of $X$. In this section we assume that $f$ is affine. Then we will show that there exists a large family of projective $Y$-modifications of $X$ having good functorial properties. Using these morphisms we will be able to describe the set $\text{Val}_Y(X)$ very concretely.

Definition 3.4.1. A $Y$-modification $g_i : X_i \to X$ is called a $Y$-blow up of $X$ if there exists a $g_i$-ample $O_{X_i}$-module $\mathcal{L}$ provided with a homomorphism $\varepsilon : O_{X_i} \to \mathcal{L}$ such that $f_i^*(\varepsilon) : O_{Y,g_i^{-1}f_i^*(\mathcal{L})}$. We call $\varepsilon$ a $Y$-trivialization of $\mathcal{L}$; actually it is a section of $\mathcal{L}$ that is invertible on the image of $Y$.

It will be more convenient to say $X$-ample instead of $g_i$-ample in the sequel.

Lemma 3.4.2. The $Y$-blow ups satisfy the following properties.

(i) Suppose that $X_j \to X_i$ and $X_i \to X$ are $Y$-modifications such that $X_j$ is a $Y$-blow up of $X_i$, then $X_j$ is a $Y$-blow up of $X_i$.

(ii) The family of $Y$-blow ups of $X$ is filtered.

(iii) The composition of $Y$-blow ups $g_{ij} : X_j \to X_i$ and $g_i : X_i \to X$ is a $Y$-blow up.

Proof. The first statement is obvious because any $X$-ample $O_{X_i}$-module $\mathcal{L}$ is $X_i$-ample, and the notion of $Y$-trivialization of $\mathcal{L}$ depends only on the morphism $Y \to X_j$.

(iii) Let $X_i, X_j$ be two $Y$-blow ups of $X$. Find $X$-ample sheaves $\mathcal{L}_i, \mathcal{L}_j$ with $Y$-trivializations $\varepsilon_i, \varepsilon_j$. Then the $X$-proper scheme $X_{ij} = X_i \times_X X_j$ possesses an $X$-ample sheaf $\mathcal{L} = p_1^*(\mathcal{L}_1) \otimes p_2^*(\mathcal{L}_2)$, where $p_i, p_j$ are the projections. The natural isomorphism $O_{X_{ij}} \xrightarrow{\sim} O_{X_{ij}} \otimes O_{X_{ij}}$ followed by $f_i^*(\varepsilon_i) \otimes f_j^*(\varepsilon_j) : O_{X_{ij}} \otimes O_{X_{ij}} \xrightarrow{\sim} \mathcal{L}$ provides a $Y$-trivialization of $\mathcal{L}$. Consider the scheme-theoretic image $X'$ of $Y$ in $X_{ij}$, and let $\mathcal{L}'$ and $\varepsilon'$ be the pull backs of $\mathcal{L}$ and $\varepsilon$. Then $(X', \mathcal{L}', \varepsilon')$ is a $Y$-blow up of $X$ which dominates $X_i$ and $X_j$.

(iii) Choose an $X$-ample $O_{X_i}$-sheaf $\mathcal{L}_i$ and an $X_i$-ample $O_{X_j}$-sheaf $\mathcal{L}_j$ with $Y$-trivializations $\varepsilon_i$ and $\varepsilon_j$. By [EGA II, 4.6.13(ii)], the sheaf $\mathcal{L}_j \otimes g_i^*(\mathcal{L}_i^{\otimes n})$ is $X$-ample for sufficiently large $n$. It remains to notice that the composition of $O_{X_j} \xrightarrow{\sim} O_{X_j} \otimes O_{X_j}^{\otimes n}$ with $\varepsilon_j \otimes g_i^*(\varepsilon_i^{\otimes n})$ is a $Y$-trivialization. \qed
We will need an explicit description of $Y$-blow ups. Let $E \subset f_* (O_Y)$ be a finitely generated $O_X$-submodule containing the image of $O_X$, and let $E^n \subset f_* (O_Y)$ denote the $O_X$-modules which are powers of $E$ with respect to the natural multiplication on $f_* (O_Y)$ (so $E^0$ is the image of $O_X$). We claim that $X_E := \text{Proj} (\bigoplus_{n=0}^{\infty} E^n)$ is a $Y$-modification of $X$. Clearly, $X_{E}$ is $X$-projective and there is a natural morphism $g_E : Y = \text{Spec} (f_* O_Y) \to \text{Spec} (\bigcup_{n=0}^{\infty} E^n)$ where the union is taken inside $f_* (O_Y)$. The target of $g_E$ is the $X$-affine chart of $X_E$ defined by non-vanishing of the section $s \in \Gamma (E)$ which comes from the unit section of $O_X$, in particular, a map $Y \to X_E$ naturally arises. In addition, the very ample sheaf $O_{X_E} (1)$ on $X_E$ has a $Y$-trivialization $O_{X_E} \to O_{X_E} (1)$ induced by $s$. So, among all properties of $Y$-blow ups it remains to check that $g_E$ is schematically dominant. The latter can be checked locally over $X$, so assume that $X = \text{Spec} (A), Y = \text{Spec} (B)$ and $E \subset B$ is an $A$-module containing $1$. Then $X_E = \text{Proj} (\bigoplus_{n=0}^{\infty} E^n)$ is glued from affine charts $(X_E)_b$ given by non-vanishing of elements $b \in E$, so it suffices to show that the morphism $\alpha : Y \times_{X_E} (X_E)_b \to (X_E)_b$ is schematically dominant. Note that the source is the localization of $Y$ at $b$, hence is isomorphic to $\text{Spec} (B_b)$, and the target is $\text{Spec} (C)$ where $C$ is the zeroth graded component of $(\bigoplus_{n=0}^{\infty} E^n)_b$. But $C = \text{inj lim}_n b^{-n} E^n/I_n$ where $I_n$ is the submodule of elements killed by a power of $b$, hence the natural homomorphism $C \to B_b$ is injective (all $b$-torsion of $\bigoplus_{n=0}^{\infty} E^n$ is killed already in $C$). Thus, $\alpha$ is schematically dominant.

**Lemma 3.4.3.** Any $Y$-blow up of $X$ is isomorphic to some $X_E$ as a $Y$-blow up of $X$.

**Proof.** Let $g_i : X_i \to X$ be a $Y$-blow up. Find an $X$-ample $O_{X_i}$-module $L$ with a $Y$-trivialization $\varepsilon : O_{X_i} \to L$. Then there is a closed immersion of $X$-schemes $h : X_i \to P := \text{Proj} (\bigoplus_{n=0}^{\infty} (g_i)_* L^\otimes n)$ and the morphism $h \circ f_i : Y \to X_i \to P$ factors through the chart of $P$ given by non-vanishing of the section $s \in \Gamma ((g_i)_* L)$ corresponding to $\varepsilon$. The latter chart is of the form $\text{Spec} (A)$ where $A$ is the zeroth graded component of the localization $(\bigoplus_{n=0}^{\infty} (g_i)_* L^\otimes n)_s$, hence the $O_X$-homomorphism $A \to f_* (O_Y)$ corresponding to $f_i$ induces a homomorphism $(g_i)_* L \to f_* (O_Y)$ taking $s$ to the unit section. Now we can define $E$ to be the image of $(g_i)_* L$ in $f_* (O_Y)$, and we claim that actually $X_i \sim X_E$ as a $Y$-modification of $X$. Indeed, the obvious epimorphism $\bigoplus_{n=0}^{\infty} (g_i)_* L^\otimes n \to \bigoplus_{n=0}^{\infty} E^n$ corresponds to a closed immersion $X_E \to P$ which agrees with the morphisms $Y \to X_E$ and $Y \to P$. Since, the first morphism is schematically dominant, $X_E$ is the schematic image of $Y$ in $P$, hence it must coincide with $X_i$ as the closed subscheme of $P$. 

**Corollary 3.4.4.** Assume that $X'$ is an open subscheme of $X$ and $Y' = f^{-1} (X')$. Then any $Y'$-blow up $X'_i \to X'$ extends to a $Y$-blow up $X_i \to X$.

**Proof.** Let $f' : Y' \to X'$ be the restriction of $f$, so $f'_* (O_{Y'})$ is the restriction of $f_* (O_Y)$ on $X'$. By the lemma, a $Y'$-blow up of $X'$ is determined by a finitely generated $O_{X'}$-submodule $E' \subset f'_* (O_{Y'})$ containing the image of $O_{X'}$. By [EGA I, 6.9.7], one can extend $E'$ to a finitely generated $O_X$-submodule $E \subset f_* (O_Y)$. Replacing $E$ by $E + O_X$, if necessary, we can achieve that $E$ contains the image of $O_X$. Now, $E$ defines a required extension of the blow up.

**Remark 3.4.5.** Lemma 3.4.3 indicates that the notion of $Y$-blow up is in some sense a generalization of the notion of $U$-admissible blow up, where $i : U \hookrightarrow X$ is a schematically dense open subscheme, to the case of an arbitrary affine morphism.
\(Y \to X\). Indeed, there is much similarity, but the notions are not equivalent in general: both \(U\)-admissible blow ups and \(U\)-blow ups are of the form \(\text{Proj}(\oplus_{n=0}^{\infty} C^n)\), but in the first case \(\mathcal{E}\) is an \(\mathcal{O}_X\)-submodule of \(\mathcal{O}_X\) which is trivial over \(U\), and in the second one \(\mathcal{E}\) is an \(\mathcal{O}_X\)-submodule of \(i_*\mathcal{O}_U\) that contains \(\mathcal{O}_X\) (so, it is trivial over \(U\) as well). Basic facts concerning compositions, extensions, etc., (see the above lemmas) hold for both families of \(U\)-modifications, but a slight advantage of our definition is that the proofs seem to be easier. For example, compare with [Con, 1.2] where one proves that \(U\)-admissible blow ups are preserved by compositions.

The following lemma is an analog of [BL, 4.4].

**Lemma 3.4.6.** Given a quasi-compact open subset \(\mathcal{U} \subset \mathcal{X} = \text{Val}_Y(X)\), there exists a \(Y\)-modification \(X' \to X\) and an open subscheme \(U \subset X'\) such that \(\mathcal{U}\) is the preimage of \(U\) in \(X\).

**Proof.** If \(X_1, \ldots, X_n\) form a finite open affine covering of \(X\) and \(Y_i = f^{-1}(X_i)\), then \(\mathcal{X}_i = \text{Val}_Y(X_i)\) form an open covering of \(\mathcal{X}\) by lemma 3.1.8. It suffices to solve our problem for each \(\mathcal{X}_i\), separately because any \(Y_i\)-blow up of \(X_i\) extends to a \(Y\)-blow up of \(X\), and \(Y\)-blow ups of \(X\) form a filtered family. Thus, we can assume that \(X = \text{Spec}(A)\), and then \(Y = \text{Spec}(B)\). We can furthermore assume that \(\mathcal{U} = \mathcal{X} \cap \text{Spa}(B, A[1/1, \ldots, a_n/b])\) with \(a_i, b \in B\) because as we saw in the proof of lemma 3.1.2, the sets \(\text{Spa}(B, A[1/1, \ldots, a_n/b])\) form a basis of the topology of \(\text{Spa}(B, A)\). Now, the morphism \(Y \to \text{Proj}(A[T_1, T_2, \ldots, T_n, T_b])\) defined by \((1, a_1, \ldots, a_n, b)\) determines a required \(Y\)-blow up \(X' \to X\) with \(U\) given by the condition \(T_b \neq 0\). □

**Corollary 3.4.7.** The map \(\psi : \text{Val}_Y(X) \to \text{RZ}_Y(X)\) is a homeomorphism.

**Proof.** Recall that \(\psi\) is surjective and continuous by propositions 2.2.1 and 3.1.10, respectively. From other side, the lemma implies that \(\psi\) is injective and open. Indeed, any open quasi-compact \(\mathcal{U} \subset \mathcal{X}\) is the full preimage of some \(U \subset X'\) for a \(Y\)-modification \(X' \to X\), hence \(\psi(\mathcal{U})\), which is the full preimage of \(U\) in \(\text{RZ}_Y(X)\), is open. In addition, since any pair of different points of \(\mathcal{X}\) is distinguished by some open quasi-compact set \(\mathcal{U} \subset \mathcal{X}\), their images in an appropriate \(X'\) do not coincide. □

We use the corollary to identify \(\mathcal{X}\) with \(\text{RZ}_Y(X)\) when \(f\) is decomposable. In particular, it provides \(\mathcal{X}\) with a sheaf \(\mathcal{O}_X\) of regular functions which was earlier defined on \(\text{RZ}_Y(X)\), and for any point \(x \in \mathcal{X}\), thanks to proposition 2.2.1, the semi-valuation ring \(\mathcal{O}_x\) obtains a new interpretation as the stalk of \(\mathcal{O}_X\) at \(x\). As another corollary, we obtain the following version of Chow lemma.

**Corollary 3.4.8.** Any \(Y\)-modification \(\overline{\mathcal{X}} \to X\) is dominated by a \(Y\)-blow up of \(X\).

**Proof.** Let \(\overline{U}_1, \ldots, \overline{U}_n\) be an affine covering of \(\overline{\mathcal{X}}\), and let \(U_i\) and \(\mathcal{U}_i\) denote the preimages of \(\overline{U}_i\) in \(Y\) and \(\mathcal{X}\), respectively. By the lemma, we can find a \(Y\)-blow up \(X' \to X\) and a covering \(\{U'_i\}\) of \(X'\), whose preimage in \(\mathcal{X}\) coincides with \(\{\mathcal{U}_i\}\). Note that the scheme-theoretic image \(X''\) of \(Y\) in \(\overline{\mathcal{X}} \times_X \overline{X}\) is a \(Y\)-modification of both \(X'\) and \(\overline{\mathcal{X}}\). Since the preimages of \(\overline{U}_i\) and \(U'_i\) in \(\overline{\mathcal{X}}\) coincide, their preimages \(U''_i \to X''\) coincide too. Consider the induced \(Y\)-modification \(h : X'' \to X'\) with restrictions \(h_i : U''_i \to U'_i\). For any \(1 \leq i \leq n\), the proper morphism \(h_i\) is affine because the morphism \(\overline{U}_i \to X\) is affine (we use that \(U''_i\) is closed in \(U'_i \times_X \overline{U}_i\)). Thus, \(h_i\) is finite, and therefore \(h\) is finite.
We claim that finiteness of $h$ implies that it is a $Y$-blow up (this claim is an analog of [BL, 4.5]). Indeed, $\mathcal{O}_{X''}$ is very ample relatively to $h$ because $h$ is affine, and the identity homomorphism gives its $Y$-trivialization. By lemma 3.4.2 (iii), $X''$ is a $Y$-blow up of $X$, and obviously $X''$ dominates $\mathcal{X}$. □

3.5. Decomposable morphisms. In this section we will complete a basic description of the relative Riemann-Zariski space $\mathcal{X}$ attached to a separated morphism $f: Y \to X$ between qcqs schemes by proving that the finite union of open domains is an open domain, and any open domain in $X$ is of the form $\text{Val}_Y(X)$ where the morphism $Y \to X$ is affine and schematically dominant. The first claim actually means that any quasi-compact open subset is an open domain, i.e. admits a model by a morphism of schemes, and the second claim states that this model can be chosen especially nice. In particular, applying the second claim to $X$ itself we obtain a bijection $RZ_Y(X) \cong RZ_X(X)$ with $Y = Y$ and affine morphism $Y \to X$, but then $X$ is proper over $X$ by the valuative criterion 3.2.3. So, $X$ admits a $Y$-modification $\mathcal{X}$ such that the morphism $Y \to \mathcal{X}$ is affine, i.e. the morphism $f: Y \to X$ is decomposable. This gives a new proof of theorem 1.1.3, and one obtains, in particular, new proofs of Nagata compactification and Thomason approximation theorems.

**Theorem 3.5.1.** Let $f: Y \to X$ be a separated morphism between qcqs schemes and $\mathcal{X} = RZ_Y(X)$, then:

(i) open domains in $\mathcal{X}$ are closed under finite unions;

(ii) any open domain $\mathcal{X}'$ is of the form $\text{Val}_Y(X)$, where the morphism $Y \to \mathcal{X}$ is affine and schematically dominant.

**Proof.** Note that any affinoid domain satisfies the assertion of (ii) (since schematical dominance is achieved by simply replacing $X$ with the schematic image of $Y$), and by theorem 3.3.4 and corollary 3.3.5, $\mathcal{X}'$ admits a finite affinoid covering. Therefore, both (i) and (ii) would follow if we prove the following claim: the union of two domains satisfying the assertion of (ii) is an open domain that satisfies the assertion of (ii). So, we assume that $\mathcal{X}' = X_1 \cup X_2$ where $X_i = \text{Val}_{Y_i}(X_i)$ with $i \in \{1, 2\}$ are open subdomains with affine morphisms $Y_i \to X_i$.

Set $X_{12} = X_1 \cap X_2$ and $Y_{12} = Y_1 \cap Y_2$. In the sequel, we will act as in Step 3 of the proof of 1.1.2, and the main difference is that we will use $Y_i$-blow ups instead of affine morphisms. For reader’s convenience, we give a commutative diagram containing the main objects which were and will be introduced.

Since $Y_i$’s are $X_i$-affine, lemma 3.4.6 implies that we can replace $X_i$’s by their $Y_i$-blow ups such that each $X_i$ contains an open subscheme $X_i'$, whose preimage in
\( \mathcal{X}_i \) coincides with \( \mathcal{X}_{12} \). Then the preimage of \( X'_i \) in \( Y \) is, obviously, \( Y_{12} \). It can be impossible to glue \( X'_i \)’s along \( X''_i \)’s, but by lemma 3.1.8(ii), we at least know that \( \text{Val}_{Y_{12}}(X''_i) \rightarrow \mathcal{X}_{12} \) for \( i = 1, 2 \). Let \( T \) be the scheme-theoretic image of \( Y_{12} \) in \( X' \times_X X'' \). Then \( \text{Val}_{Y_{12}}(T) = \text{Val}_{Y_{12}}(X'_1) \cap \text{Val}_{Y_{12}}(X'_2) = \mathcal{X}_{12} \) by lemma 3.3.2, and, therefore, \( T \) is a \( Y_{12} \)-modification of \( X'_i \)’s by the valuative criterion 3.2.3.

By corollary 3.4.8, we can find a \( Y_{12} \)-blow up \( T' \rightarrow X'_1 \), which dominates \( T \). It still can happen that \( T' \) is not a \( Y_{12} \)-blow up of \( X''_2 \), but it is dominated by a \( Y_{12} \)-blow up \( Z_{12} \rightarrow X''_2 \). Then \( Z_{12} \rightarrow T' \) is a \( Y_{12} \)-blow up by lemma 3.4.2 (i), hence \( Z_{12} \rightarrow X'_1 \) is a \( Y_{12} \)-blow up by lemma 3.4.2 (iii). By lemma 3.4.4, we can extend the \( Y_{12} \)-blow ups \( Z_{12} \rightarrow X'_1 \) to \( Y_i \)-blow ups \( Z_i \rightarrow X_i \). Then, the finite type \( X \)-schemes \( Z_i \) can be glued along the subschemes \( X \)-isomorphic to \( Z_{12} \) to a single \( X \)-scheme \( \overline{\mathcal{X}} \) of finite type, and the schematically dominant affine morphisms

\[ Y_i \rightarrow Z_i \rightarrow X_i \]  

are dominated by a single schematically dominant affine morphism \( \overline{Y} \rightarrow \overline{X} \). Note that \( \text{Val}_Y(Z_i) = \mathcal{X}_i \) is the preimage of \( Z_i \) in \( \text{Val}_X(X) \), in particular, the latter is covered by its open subdomains \( \mathcal{X}_i, \ i \in \{1, 2\} \). Now, it remains to show that \( \text{Val}_X(X) \) is an open subdomain in \( \mathcal{X} \), since this would immediately imply that \( \text{Val}_X(X) \) is a required model of \( \overline{\mathcal{X}} \). The morphism \( \alpha : \overline{Y} \rightarrow \overline{X} \times_X Y \) is glued from the morphisms \( \alpha_i : Y_i \rightarrow Z_i \times_X Y \) because \( Y_i \) is the preimage of \( Z_i \) in \( Y \), but \( \alpha_i \)’s are closed immersions by the construction (they are models of \( \mathcal{X}_i \)’s). So, \( \alpha \) is a closed immersion as well, and we are done.

Corollary 3.5.2. The map \( \eta : Y \rightarrow \mathcal{X} := \text{RZ}_X(X) \) is injective, each point \( x \in \text{RZ}_X(X) \) possesses a unique minimal generalization in \( \eta(Y) \), and the stalk \( \mathcal{M}_{X,x} \) is the semi-fraction ring of the semi-valuation ring \( \mathcal{O}_{X,x} \). In particular, \( \mathcal{O}_X \) is a subsheaf of \( \mathcal{M}_X \).

Proof. By theorem 3.5.1 and corollary 3.4.7, we can identify \( \mathcal{X} \) with \( \text{Val}_Y(X) \). So, a point \( x \) can be interpreted as an \( X \)-valuation \( (y, R, \phi) \) on \( Y \). Then it is clear that the map \( \eta \) sends \( y \in Y \) to a trivial valuation \( (y, k(y), f\mid_y) \) (with the obvious morphism \( f\mid_y : \text{Spec}(k(y)) \rightarrow X \)), and for an arbitrary \( x = (y, R, \phi) \) its minimal generalization in \( \eta(Y) \) is \( (y, k(y), f\mid_y) \). Uniqueness of minimal generalization implies that the stalk of \( \mathcal{M}_X = \eta_*(\mathcal{O}_Y) \) at \( x \) is simply \( \mathcal{O}_{X,Y} \), so it remains to recall that the latter is the semi-fraction field of the semi-valuation ring \( \mathcal{O}_X \) defined in \( \S 2.1 \), which coincides with the stalk \( \mathcal{O}_{X,x} \) by proposition 2.2.1.

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