UNIFORM ATTRACTORS FOR A PHASE TRANSITION MODEL COUPLING MOMENTUM BALANCE AND PHASE DYNAMICS

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Abstract. In this paper we are concerned with the uniform attractor for a nonautonomous dynamical system related to the Frémond thermo-mechanical model of shape memory alloys. The dynamical system consists of a diffusive equation for the phase proportions coupled with the hyperbolic momentum balance equation, in the case when a damping term is considered in the latter and the temperature field is prescribed. We prove that the solution to the related initial-boundary value problem yields a semiprocess which is continuous on the proper phase space and satisfies a dissipativity property. Then we show the existence of a unique compact and connected uniform attractor for the system.

1. Introduction. Let us fix a bounded and regular subset $\Omega$ of $\mathbb{R}^3$ and consider the following system (VSMA) of partial differential equations and relations in terms of the unknown functions $\chi_1, \chi_2$ and $u$, in the space-time domain $Q = \Omega \times (0, +\infty)$

$$k \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} - \eta \begin{pmatrix} \Delta \chi_1 \\ \Delta \chi_2 \end{pmatrix} + \begin{pmatrix} \frac{\partial}{\partial \vartheta} (\vartheta - \vartheta^*) \\ \alpha (\vartheta) \text{div} u \end{pmatrix} + \partial I_C (\chi_1, \chi_2) \ni \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad (1.1)$$

$$u_{tt} + c u_t - \text{div} \left( (-\nu \Delta (\text{div} u) + \lambda \text{div} u) \mathbf{I} + 2\mu \varepsilon (u) + \alpha (\vartheta) \chi_2 \mathbf{I} \right) = G, \quad (1.2)$$

$$\chi_1 (\cdot, 0) = \chi_1^0, \quad \chi_2 (\cdot, 0) = \chi_2^0, \quad u (\cdot, 0) = u_0, \quad u_t (\cdot, 0) = v_0 \text{ in } \Omega, \quad (1.3)$$

$$\partial_n \chi_j = 0 \text{ on } \partial \Omega \times (0, +\infty), \quad j = 1, 2, \quad (1.4)$$

$$u = 0 \text{ on } \partial \Omega \times (0, +\infty), \quad (1.5)$$

$$\partial_n (\nu \text{div} u) = 0 \text{ on } \partial \Omega \times (0, +\infty). \quad (1.6)$$

Here, $k, \eta, \ell, \vartheta^*, c, \lambda, \mu$ are strictly positive parameters, while the coefficient $\nu$ is allowed to be greater than or equal to 0. Note that $u = (u_1, u_2, u_3) \in \mathbb{R}^3$, $\varepsilon (u) := (\varepsilon_{ij})$ is the strain tensor with

$$\varepsilon_{ij} := \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad i, j = 1, 2, 3.$$
and $I$ denotes the identity matrix in $\mathbb{R}^3$. Moreover, $\vartheta$, $\alpha(\vartheta)$ and $G$ are given functions with some properties to be specified later, and $K$ is the triangle

$$K := \{(\gamma_1, \gamma_2) \in \mathbb{R}^2 \text{ such that } |\gamma_2| \leq \gamma_1 \leq 1\}. \quad (1.7)$$

The symbol $I_K$ in (1.1) denotes the indicator function of the convex set $K$, namely $I_K(v) = 0$ if $v$ belongs to $K$ and $I_K(v) = +\infty$ elsewhere, while $\partial I_K : \mathbb{R}^2 \to 2^{\mathbb{R}^2}$ stands for its subdifferential, namely $y \in \partial I_K(x)$ iff $x \in K$ and $(y, x - w) \geq 0$ $\forall w \in K$, where $(\cdot, \cdot)$ denotes the scalar product in $\mathbb{R}^2$. In particular, $\partial I_K$ turns out to be a multivalued maximal monotone operator (see, e.g., [6, p. 25]). Finally, $\partial_n$ denotes the outward normal derivative to the boundary $\partial \Omega$.

The system (VSMA) arises in the study of the behaviour of a viscoelastic shape memory body subjected to mechanical deformations when the temperature field is prescribed. A *shape memory* material is a metallic alloy which exhibits this peculiar and surprising behaviour: it can be permanently deformed (avoiding fractures) up to 8% of its strain and subsequently forced to recover its original shape just by thermal means. This unusual property is used nowadays in a variety of engineering applications. In particular, the field of applications of shape memory technologies ranges from bio-engineering, to structures-engineering and aerospace sciences (see [9]). The shape memory phenomenon has been interpreted as the effect of a thermo-elastic solid-solid phase transition between two different crystallographic configurations, the *austenite*, which is stable at higher temperatures and variants of *martensite*, stable at lower temperatures (see [2] and [16]). Here, we are interested in the macroscopic modelling proposed by Frémond in [16]. In this connection, $\vartheta$ has to be regarded as the absolute temperature (assumed to be known) of the shape memory sample, while $u$ stands for the vector of small displacements. Hence, the 2-tensor $\varepsilon(u)$ represents the linearised strain tensor. Finally, $\chi_1, \chi_2$ are quantities related to the pointwise proportions of the phases. In particular, if $\beta_1, \beta_2, \beta_3$ denote the local proportions of the two martensitic variants and of the austenite, respectively, we point out that the following conditions have to be fulfilled

$$0 \leq \beta_i \leq 1 \quad \text{for } i = 1, 2, 3, \quad \text{and } \beta_1 + \beta_2 + \beta_3 = 1. \quad (1.8)$$

We stress that the latter condition forces the phases to attain only meaningful values, that is we are requiring that neither voids nor overlapping zones appear between the phases. Then, by simply eliminating $\beta_3$ in the equation above and letting $\chi_1 = \beta_1 + \beta_2$ and $\chi_2 = \beta_1 - \beta_2$, it turns out that (1.8) reduces to (see (1.7))

$$(\chi_1, \chi_2) \in K. \quad (1.9)$$

In particular, the set $\{\chi_1 = 1\}$ corresponds to the situation in which no austenite is present, and the set $\{\chi_1 = \chi_2\}$ (resp. $\{\chi_1 = -\chi_2\}$) corresponds to the region where only the first (resp. second) variant of martensite is present.

Finally, $G$ stands for the density of the body forces while $\alpha$ is a rather smooth function related to the thermal expansion of the system (see [16] for further details): in fact, let us refer to [16] for the physical derivation of the model and the related comments. However, we point out that, although the full Frémond’s model comprises the evolution of an unknown temperature field as well, our setting, in which $\vartheta$ is a given datum, is physically justified and interesting for applications. We recall that also the positive damping $\alpha u_t$ (which is not present in the original Frémond’s model) in (1.2) has a physical motivation. In particular, this element can be understood as a friction term, hence it serves as a dissipation mechanism.
The mathematical analysis of Frémond’s model was initiated in [12] and then extended in various directions. In particular, the reader is referred to [4] and [5] (and references therein) for an updated and minute presentation of the analytical results concerning Frémond’s model. We note however, that [12], [4], [5] and most of the quoted references deal with the quasistationary situation, in which the macroscopic accelerations are not taken into account. On the other hand, the case in which the macroscopic accelerations are retained in the momentum balance has been studied in [10] (in the simple setting of a linearized energy balance and without diffusive effects for the phase variables) and, more recently, in [26] (for the system (1.1)-(1.6) when $c = 0$). Let us point out that, to our knowledge, the initial-boundary value problem for the full Frémond’s model with macroscopic accelerations and the energy balance equation looks rather difficult and is still open even for the existence of solutions.

The problem of the long-time behaviour of the solutions to Frémond’s model has been first considered in [14]. In that paper, the authors investigate the convergence to the steady state solutions for the full model (but without the inertial term in the momentum balance equation) in the one-dimensional situation. The structure of the $\omega$-limit set has been further analysed in [30]. There, still in the 1-D case it is shown how, in a prescribed temperature range, the set of solutions to the stationary problem contains elements that present a deeply structured alternance of martensitic variants. This fact is in complete agreement with experimental evidence. The study of the asymptotic stability from the point of view of global attractors has been tackled in [15] in the one-dimensional setting and then extended to the three-dimensional situation in [11]. The analysis of [15] relies on the crucial observation that, in the absence of inertial terms, the momentum balance equation (i.e., our (1.2) without the part $u_{tt} + cu_t$), along with the boundary conditions (1.5) and (1.6), allows to completely determine the displacement $u$ in terms of data of the problem and other unknowns. Thus, the original system for three unknowns $(x_1, x_2, u, \theta)$ reduces to a system for the two unknowns $(x_1, x_2, \theta)$, in which $u$ (that is now a function of $(x_1, x_2, \theta)$) plays the role of a driving force depending on time. Consequently, the system is intrinsically nonautonomous. The long-time dynamics of the related dynamical system has been characterised with the aid of the study of a proper limiting autonomous system. In [11] the same type of result has been extended to the three-dimensional situation.

In this paper, we aim to analyse the long-time behaviour in terms of global attractors for (1.1)-(1.2). Our situation differs from the one studied in [15] and [11], since we retain the macroscopic acceleration in the momentum balance equation (1.2), thus obtaining a hyperbolic equation for the vector $u$. Moreover, we assume that the evolution of the temperature field $\theta$ is known. In this regard, the function $\theta$ becomes a forcing term depending on time. In particular, our system (VSMA) is nonautonomous. The problem of existence, uniqueness and continuous dependence of solutions to the full system (1.1)-(1.6) has been investigated in [26] when $c = 0$ in (1.2). The presence of this damping term is however mandatory from the point of view of the long-time behaviour: in fact, it provides some dissipation to the system. The existence and uniqueness analysis of [26] refers to the situation in which $c = 0$, but let us note that all the results established in [26] extend to the case $c \neq 0$. In this paper, we rely on the concept of uniform attractor to handle the fact that the system is nonautonomous (see Subsection 2.2). In particular, we will prove that the
solution operator to (a suitable weak formulation of) (1.1)-(1.6) is a semiprocess which is continuous on the proper phase space (see Theorem 2.8). Then, we show the dissipativity of the system in Theorem 3.1 and finally the existence of a unique compact and connected uniform attractor in Theorem 3.5. The crucial step in proving the existence of the uniform attractor relies on the proof of some form of compactness for the solution operator. The simplest and, by the way, the strongest form of compactness one could expect is that the solution operator itself becomes a compact operator after some finite time. Unfortunately, this form of compactness is not usually available for hyperbolic equations (as our (1.2)). Thus, we rely here on the concept of uniform asymptotic compactness, which is well known for autonomous and also nonautonomous systems (see [24]). In the simpler autonomous setting, the uniform asymptotic compactness (there called asymptotic compactness) means, roughly speaking, that \( \{u_n(t_n)\} \) is convergent (up to a subsequence) with respect to the topology of the phase space, when \( t_n \to +\infty \) and \( \{u_n\} \) is a sequence of solutions corresponding to initial data bounded in the phase space. Now, if the underlying phase space is a Hilbert space (but a uniformly convex Banach space would be sufficient), one can try to prove the abovementioned convergence by first checking the weak convergence of the sequence and then showing the convergence of the corresponding norms, by means of the energy identity provided by the equation. Hence, the so-called energy method introduced by J.M. Ball in [3] provides an efficient and elegant way to establish the asymptotic compactness.

The same strategy works also in the nonautonomous case. In fact, following [24], once the concept of uniform asymptotic compactness has been introduced, one can prove the uniform asymptotic compactness of the process by using a proper extension of the energy method. In the present paper, we follow exactly this approach. However, it is worthwhile noting that there exists at least another method to prove the asymptotic compactness of the system. As in the autonomous case, one can try to decompose the solution operator into two parts: a (uniformly) compact part and a part which decays to zero as time goes to infinity (see, e.g., [7], [17], [18]). This method could be in principle successfully applied to our system (1.1)-(1.6), under some extra regularity with respect to time for the forcing function \( G \) than the one we use to prove the mere existence of solutions and the continuity of the semiprocess (cf. Theorems 2.6 and 2.8). In this concern, we can say that the energy method, essentially relying on the standard energy estimate for hyperbolic problems, is optimal with respect to the regularity of the data.

This is the plan of the paper. In Section 2 we introduce the weak formulation of (1.1)-(1.6). Moreover, we summarize some preliminary machinery on the long-time behaviour of nonautonomous dynamical systems. Section 2.3 contains the results on the well-posedness of the weak formulation of (1.1)-(1.6). Finally, in Section 3 we prove the existence of the uniform attractor for (VSMA).

2. Mathematical setting and preliminaries.

2.1. Function spaces and weak formulation. We first introduce some notation. We set

\[
H := L^2(\Omega), \quad V := H^1(\Omega),
\]

\[
H := (L^2(\Omega))^3, \quad V := \{v \in (H^1_0(\Omega))^3 : \nu \div v \in V\},
\]
where the coefficient $\nu$ in the definition of $V$ allows to consider at the same time both the $\nu = 0$ case and the $\nu > 0$ situation. As usual, we identify $H$ and $H'$ with their respective dual spaces $H'$ and $H^*$, so that $V \subset H \subset V'$ and $V \subset H \subset V'$ may be regarded as classical Hilbert triplets. The spaces $H, V, H$ will be endowed with usual norms, while for $V$ we prescribe the equivalent norm
\[
\|v\|_V := \left\{ \sum_{i=1}^{3} \|\nabla v_i\|_H^2 + \nu \|\nabla (\text{div } v)\|_H^2 \right\}^{1/2}.
\] (2.1)

In the sequel, we denote by $(\cdot, \cdot)_\Omega$ the scalar product in $H$ or in $H'$, by $(\cdot, \cdot)$ the duality pairing between $V'$ and $V$ or between $V'$ and $V$. The symbol $\| \cdot \|_E$ will indicate the norm in a generic normed vector space $E$. In addition, we introduce the continuous and symmetric bilinear form $a(\cdot, \cdot)$ defined for all $v_1, v_2$ in $V$ by
\[
a(v_1, v_2) := \nu \int_\Omega \nabla (\text{div } v_1) \cdot \nabla (\text{div } v_2) + \lambda \int_\Omega \text{div } v_1 \text{div } v_2 + 2\mu \sum_{i,j=1}^3 \int_\Omega \varepsilon_{ij}(v_1) \varepsilon_{ij}(v_2) \quad (2.2)
\]

Recalling the well-known Korn inequality, the following property holds
\[
a(v, v) \geq c_a \|v\|_V^2 \quad \forall v \in V \quad (2.3)
\]
for some positive constant $c_a$. Moreover, we have that
\[
\exists \bar{c}_a : \ |a(v_1, v_2)| \leq \bar{c}_a \|v_1\|_V \|v_2\|_V \quad \text{for all } v_1, v_2 \in V. \quad (2.4)
\]

Since the special triangular form of $K$ in (1.7) is not needed in our analysis, we let $K$ stand for any bounded, convex and closed subset of $\mathbb{R}^2$ such that $(0, 0) \in K$. Consequently, we denote by $K := \{(\gamma_1, \gamma_2) \in H \times H : (\gamma_1, \gamma_2) \in K \ a.e. \ in \ \Omega\}$ the realization of $K$ in $H \times H$, which is clearly bounded, convex and closed. In particular, it is straightforward to find a positive constant $c_K$ such that
\[
\{|\gamma_1|^2 + |\gamma_2|^2\}^{1/2} \leq c_K, \quad (2.5)
\]
for all $(\gamma_1, \gamma_2) \in K$ and almost everywhere in $\Omega$. The symbol $I_K$ will clearly indicate the indicator function of $K$, while $\partial I_K$ stands for its subdifferential, which is now a maximal monotone operator in $H \times H$.

In order to describe the asymptotic behaviour of solutions, we need to introduce the Banach space of $L_{loc}^p$-translation bounded functions with values in a Banach space $B$. More precisely, for $p \geq 1$ we set
\[
L_{loc}^p(0, +\infty; B) := \{v : (0, +\infty) \rightarrow B \text{ measurable}, v \in L^p(0, T; B) \ \forall T > 0\} \quad (2.6)
\]
(note that this is not the standard position) and consequently define
\[
T_p(B) := \left\{ v \in L_{loc}^p(0, +\infty; B) : \|v\|_{T_p(B)} = \sup_{t \geq 0} \int_t^{t+1} \|v(s)\|^p_B \, ds < +\infty \right\}. \quad (2.7)
\]

We prescribe the following assumptions on data
\[
(\chi_1^0, \chi_2^0) \in K \cap (V \times V), \quad (2.8)
\]
\[
u_0 \in V, \quad v_0 \in H, \quad (2.9)
\]
\[
\alpha \in W^{1,\infty}(\mathbb{R}), \quad (2.10)
\]
\[
G \in L_{loc}^2(0, +\infty; H), \quad (2.11)
\]
\[
\vartheta \in L_{loc}^2(0, +\infty; W^{1,3}(\Omega)). \quad (2.12)
\]
Assumptions (2.8) and (2.9) suggest to investigate the long-time behaviour of solutions in the complete phase space $\mathcal{X} := (K \cap V^2) \times V \times H$, with the metric

$$d_\mathcal{X}((\xi_1, \xi_2, \xi_3, \xi_4), (\zeta_1, \zeta_2, \zeta_3, \zeta_4)) := \sum_{j=1}^{2} ||\xi_j - \zeta_j||_V + ||\xi_3 - \zeta_3||_V + ||\xi_4 - \zeta_4||_H$$

(2.13)

We consider now the weak formulation of (1.1)-(1.6). For our convenience, we also introduce the space

$$W := \{ v \in H^2(\Omega): \partial_n v = 0 \text{ in } \partial\Omega \},$$

(2.14)

which takes into account Neumann homogeneous boundary conditions.

**Problem 2.1.** Under the assumptions (2.8)-(2.12), find $x_1, x_2, h_1, h_2, u$ satisfying

$$x_1, x_2 \in H^1(0, T; H) \cap C^0([0, T]; V) \cap L^2(0, T; W),$$

(2.15)

$$h_1, h_2 \in L^2(0, T; H),$$

(2.16)

$$u \in H^2(0, T; V') \cap C^1([0, T]; H) \cap C^0([0, T]; V)$$

(2.17)

for all $T > 0$,

$$\langle x_1(t), x_2(t) \rangle \in K$$

(2.18)

for every $t \geq 0$, solving almost everywhere in the time interval $(0, +\infty)$

$$k \frac{\partial x_1}{\partial t} - \eta \frac{\Delta x_1}{\Delta t} + \frac{b}{\alpha}(\vartheta - \vartheta^*) + \left(\begin{array}{c} h_1 \\ h_2 \end{array}\right) = \left(\begin{array}{c} 0 \\ 0 \end{array}\right), \text{ a.e. in } \Omega,$$

(2.19)

$$\left(\begin{array}{c} h_1 \\ h_2 \end{array}\right) \in \partial I_K(x_1, x_2), \text{ a.e. in } \Omega,$$

(2.20)

$$\langle u_{tt}, v \rangle + c\langle u_t, v \rangle + a(u, v) + (\varphi(\vartheta)x_2, \text{div } v)_{\Omega} = \langle G, v \rangle \text{ for all } v \in V,$$

(2.21)

and such that

$$u(0) = u_0 \text{ in } V, \quad u_t(0) = v_0 \text{ in } H,$$

$$\chi_1(0) = \chi_0 \text{ in } H, \quad \chi_2(0) = \chi_2 \text{ in } H.$$

(2.22)

### 2.2. Long-time behaviour of nonautonomous evolution systems: the abstract setting.

In this subsection, we present some known results on the long-time behaviour of nonautonomous systems, especially in connection with the construction of the so-called **uniform attractor**. The reader is referred to the seminal references [28, 29, 19, 7] for the related proofs and further remarks.

The basic concept in studying the long-time behaviour of a nonautonomous system is the notion of **semiprocess**. Let $\mathcal{X}$ and $\Sigma$ be two complete metric spaces. We say that the set $\{U_\sigma(t, \tau)\}_{t \geq \tau \geq 0}, \sigma \in \Sigma$ is a family of semiprocesses in $\mathcal{X}$ if the following properties are satisfied

$$U_\sigma(t, \tau) : \mathcal{X} \rightarrow \mathcal{X} \text{ for any } t \geq \tau \geq 0;$$

(2.23)

$$U_\sigma(\tau, \tau) \text{ is the identity map on } \mathcal{X} \text{ for any } \tau \geq 0;$$

(2.24)

$$U_\sigma(t, s)U_\sigma(s, \tau) = U_\sigma(t, \tau) \text{ for any } t \geq s \geq \tau \geq 0.$$ 

(2.25)

for each $\sigma \in \Sigma$. $\Sigma$ is the so-called **symbol space**. As we shall see in the concrete case of system (2.19)-(2.21), the symbol space $\Sigma$ will be a space of time-dependent functions, which collects all the forcing terms that depend on time. Let $\{T_h\}_{h \geq 0}$
be a semigroup of translations in $\Sigma$, that is $(T_h(\sigma))(t) := \sigma(t + h)$, and assume the following translation invariance condition

$$U_{T_h(\sigma)}(t, \tau) = U_\sigma(t + s, \tau + s), \quad \forall \sigma \in \Sigma, \quad \forall t \geq \tau \geq 0, \quad \forall s \geq 0.$$  \hspace{1cm} (2.26)

The parameter $\sigma$ is then termed the time symbol of the semiprocess $U_\sigma(t, \tau)$. The class of translation compact forcing functions will be of interest for us: we say that $\sigma$ is translation compact if

$$\text{the hull } H(\sigma) := [T_h(\sigma), h \in [0, +\infty)]_\Sigma \text{ is compact in } \Sigma,$$  \hspace{1cm} (2.27)

where $[\cdot]_\Sigma$ denotes the closure in $\Sigma$. A useful criterion to check if a given function $\sigma \in L^p_{\text{loc}}(0, +\infty; B)$ is translation-compact is as follows (see [8, Proposition V.3.3]).

**Proposition 2.2.** A function $\sigma$ is translation-compact in $L^p_{\text{loc}}(0, +\infty; B)$ if and only if

1. for any $h \geq 0$ the set $\left\{ \int_t^{t+h} \sigma(s)ds : t \geq 0 \right\}$ is precompact in $B$;
2. there exists a function $\lambda$, with $\lambda(s) \searrow 0$ as $s \searrow 0$, such that

$$\int_t^{t+1} \|\sigma(s + h) - \sigma(s)\|^p_B ds \leq \lambda(h)$$  \hspace{1cm} (2.28)

for all $t \geq 0$ and $h \geq 0$.

The class of translation compact functions is quite large. For example, it contains the constant $B$-valued functions and the periodic, quasiperiodic and the almost periodic functions (in the Bochner-Amerio sense, see [1]).

The family of semiprocesses is said to be $X \times \Sigma$-continuous if, for any $t, \tau$ with $t \geq \tau \geq 0$, the map $(v, \sigma) \mapsto U_\sigma(t, \tau)v$ is continuous from $X \times \Sigma$ to $X$.

Now, let us recall the notions of absorbing set and attractor for the family of semiprocesses $\{U_\sigma(t, \tau)\}_{t \geq \tau \geq 0, \sigma \in \Sigma}$. We say that $B \subset X$ is a uniformly absorbing set if for any $\tau \geq 0$ and any $B \subset X$ bounded, there exists a time $T = T(\tau, B) \geq \tau$ such that $U_\sigma(t, \tau)B \subset B$ for any $t \geq T$ and for all $\sigma \in \Sigma$. Then, we say that $C \subset X$ is uniformly attracting for $\{U_\sigma(t, \tau)\}$ if

$$\lim_{t \to +\infty} \sup_{\sigma \in \Sigma} \text{dist}_X(U_\sigma(t, \tau)B, C) = 0, \quad \forall \tau \geq 0, \quad \forall B \subset X \text{ bounded},$$  \hspace{1cm} (2.29)

where

$$\text{dist}_X(A, B) := \sup_{a \in A} \inf_{b \in B} d_X(a, b)$$

denotes the semidistance of two sets $A, B \subset X$. Finally, we say that $A$ is the uniform attractor for the family $\{U_\sigma(t, \tau)\}$ if it is at the same time uniformly attracting and contained in every closed uniformly attracting set (minimality property). Then, it is unique by construction.

Now, we quote a general abstract criterion providing sufficient conditions for the existence of the uniform attractor (see [24, Theorem 2.3]).

**Theorem 2.3.** Let $\{U_f(t, \tau)\}_{t \geq \tau \geq 0}$ be a continuous family of semiprocesses in $X$ with $f \in H(\sigma)$ and $\sigma$ being translation compact. Let us assume that

1. $\{U_f(t, \tau)\}$ possesses a bounded uniformly absorbing set $B$ (dissipativity);
2. $\{U_f(t, \tau)\}$ is uniformly asymptotically compact, i.e.

$$\begin{align*}
\{z_n\}_{n \in \mathbb{N}} & \text{ bounded in } X \\
\{f_n\}_{n \in \mathbb{N}} & \subset H(\sigma) \\
t_n & \nearrow +\infty
\end{align*} \implies \{U_{f_n}(t_n, 0)z_n\}_{n \in \mathbb{N}} \text{ precompact in } X.$$  \hspace{1cm} (2.30)
Then, \{U_f(t, \tau)\} possesses a compact uniform attractor.

Note that the above definition of uniform asymptotic compactness (taken from [24]) is different from the one given by Haraux [19]. More precisely, in [19] a semiprocess is said to be \textit{uniformly asymptotically compact} if it possesses a compact \textit{uniformly attracting set} in the sense of (2.29). However, it is not difficult to prove that if a semiprocess is \textit{uniformly asymptotically compact} in the sense of (2.30) and possesses a bounded uniformly absorbing set, then it is \textit{uniformly asymptotically compact} in the sense of Haraux. Furthermore, we can note that the notion of uniform asymptotic compactness introduced in [24] and used in this paper, is also completely in agreement with the corresponding definition for semigroups (see [21]) and seems easier to be verified using the energy method.

Starting from the semiprocess \(U_f(t, \tau), f \in \mathcal{H}(\sigma)\), we can define a semigroup \(S_t\) acting on the extended phase space \(X \times H\) as follows

\[
S_t(z_0, f) := (U_f(t, 0)z_0, T_t(f)), \quad S_t : X \times H(\sigma) \to X \times H(\sigma).
\]  

This construction is well known (see, e.g., [23]). It is also well known that the uniform attractor \(\mathcal{A}\) could be equivalently defined in terms of the global attractor \(\mathcal{A} \subset X \times H(\sigma)\) of the semigroup \(S_t\), that is

\[
\mathcal{A} = \Pi_1 \mathcal{A}, \quad \text{where } \Pi_1 \text{ is the projection on the first component}. 
\]  

This construction will help us in proving the connectedness of the \textit{uniform attractor} for our system (VSMA).

We conclude this subsection by recalling two technical results which will be useful in the course of our analysis. We start with the so-called Uniform Gronwall Lemma (for the proof see [32, Lemma III.1.1]).

**Lemma 2.4.** Let \(y, a, b \in L^1_{\text{loc}}(0, +\infty)\) be three non-negative functions such that \(y' \in L^1_{\text{loc}}(0, +\infty)\) and

\[
y'(t) \leq a(t)y(t) + b(t) \quad \text{for a.e. } t > 0, 
\]

and let \(a_1, a_2, a_3\) be three non-negative constants such that

\[
\|a\|_{T_1} \leq a_1, \quad \|b\|_{T_1} \leq a_2, \quad \|y\|_{T_1} \leq a_3.
\]

Then, we have

\[
y(t + 1) \leq (a_2 + a_3) \exp^{a_1} \quad \text{for all } t > 0.
\]

Next, we prepare a Gronwall-type lemma prompted by [25].

**Lemma 2.5.** Let \(\varphi, m_1\) and \(m_2\) be three non-negative locally summable functions on \([\tau, +\infty)\) which satisfy, for some \(\varepsilon > 0\), the differential inequality

\[
\frac{d}{dt} \varphi^2(t) + \varepsilon \varphi^2(t) \leq m_1(t)\varphi(t) + m_2(t) \quad \text{for a.e. } t \in [\tau, +\infty). 
\]  

Then, there holds

\[
\varphi^2(t) \leq 2\varphi^2(\tau)e^{-\varepsilon(t-\tau)}
\]

\[
+ \left( \int_{\tau}^{t} m_1(s)e^{-\varepsilon(t-s)/2} ds \right)^2 + 2 \int_{\tau}^{t} m_2(s)e^{-\varepsilon(t-s)} ds 
\]  

(2.34)

for any \(t \in [\tau, +\infty)\). Moreover, the inequality

\[
\int_{\tau}^{t} m(s)e^{-\varepsilon(t-s)} ds \leq \frac{1}{1 - e^{-\varepsilon}} \sup_{r \geq \tau} \int_{r}^{r+1} m(s) ds 
\]  

(2.35)
holds for every non-negative locally summable function $m$ on $[\tau, +\infty)$ and every $\varepsilon > 0$.

**Proof.** Observe that (2.33) entails
\[
\frac{d}{dt} \left( e^{\varepsilon(t-\tau)} \varphi^2(t) \right) \leq m_1(t) e^{\varepsilon(t-\tau)} \varphi(t) + m_2(t) e^{\varepsilon(t-\tau)}.
\]

Therefore, setting $\psi(t) = e^{\varepsilon(t-\tau)/2} \varphi(t)$ and integrating, it turns out that
\[
\frac{1}{2} \psi^2(t) \leq \frac{1}{2} \psi^2(\tau) + \frac{1}{2} \int_{\tau}^t m_2(s) e^{\varepsilon(s-\tau)} ds + \int_{\tau}^t \frac{m_1(s)}{2} e^{\varepsilon(s-\tau)/2} \psi(s) ds
\]
for any $t \geq \tau$. Now, by applying, e.g., [6, Lemme A.5, p. 157] we infer that
\[
\psi(t) \leq \left( \psi^2(\tau) + \int_{\tau}^t m_2(s) e^{\varepsilon(s-\tau)} ds \right)^{1/2} + \int_{\tau}^t \frac{m_1(s)}{2} e^{\varepsilon(s-\tau)/2} ds,
\]
whence (2.34) follows easily. Let us now check (2.35). Denoting by $j$ the integer part of $(t-\tau)$, we have
\[
\int_{\tau}^t m(s) e^{-\varepsilon(t-s)} ds \leq \int_0^{t-\tau} m(t-s) e^{-\varepsilon s} ds \\
\leq \sum_{n=0}^{j-1} \left( e^{-\varepsilon n} \int_n^{n+1} m(t-s) ds \right) + e^{-\varepsilon j} \int_{\tau}^{\tau+1} m(s) ds \\
\leq \sup_{\tau \leq t} \int_{\tau}^{\tau+1} m(s) ds \sum_{n=0}^{j} e^{-\varepsilon n} \\
\leq \frac{1}{1 - e^{-\varepsilon}} \sup_{\tau \leq t} \int_{\tau}^{\tau+1} m(s) ds
\]
and this concludes the proof. \hfill \Box

### 2.3. Well-posedness.

The well-posedness of (the Cauchy problem for) (2.19)-(2.21) has been proved in [26, Theorems 2.2 and 2.4] for the case $c = 0$ in (2.21). The argument of the proof relies on a time discretization – a priori estimates – passage to the limit procedure. Moreover, [27] contains the error estimates for the time discretization scheme approximating (2.19)-(2.21). The situation in which $c > 0$ can be analysed similarly, by simply adapting the proofs of [26] and [27]. The following statement holds.

**Theorem 2.6** (Well-posedness). Under the assumptions (2.8)-(2.12), there exists a quintuple $(\chi_1, \chi_2, h_1, h_2, u)$ uniquely solving Problem 2.1 and continuously depending on data. Namely, letting $\mathcal{F}_i = \{ u_{0i}, v_{0i}, (\chi_{1i}^0, \chi_{2i}^0), \theta_i, G_i \}, i = 1, 2$, be two families of data that satisfy (2.8)-(2.12), and denoting by $(u_1, \chi_{11}, \chi_{21}), (u_2, \chi_{12}, \chi_{22})$ the corresponding solution components, then there is a positive constant $\Lambda$, which depends only on the quantities $k, \eta, \ell, \varphi^*, c_k, c, c_a, T, \Omega, \| \alpha \|_{W^{1,\infty}(\mathbb{R})}$, and
\[
\max_{i=1,2} \left\{ \| \nabla \theta_i \|_{L^2(0,T;L^2(\Omega)^3)} \right\},
\]

...
such that
\[ \|u_1 - u_2\|_{C^1([0,T];H)}^2 + C_0^0([0,T];V) + \sum_{j=1}^2 \|\chi_j - \chi_{j2}\|_{C^0([0,T];H)\cap L^2(0,T;V)}^2 \]
\[ \leq \Lambda \left( \|w_{01} - w_{02}\|_H^2 + \|u_{01} - u_{02}\|_V^2 \right) \]
\[ + \sum_{j=1}^2 \|\chi_j^0 - \chi_{j2}\|_H^2 + \|G_1 - G_2\|_{L^2(0,T;H)}^2 \]
\[ + \|\vartheta_1 - \vartheta_2\|_{L^2(0,T;H)}^2 + \|\alpha(\vartheta_1) - \alpha(\vartheta_2)\|_{L^2(0,T;V)}^2 \right). \]

Finally, if \((u,\chi_1,\chi_2)\) yields a solution to (2.19)-(2.21) and we set
\[ \mathcal{E}(u,\varphi)(t) := \frac{1}{2}\|u(t)\|_H^2 + \frac{c}{2}(u(t),\varphi) + \frac{1}{2}\varphi(u(t),u(t)), \] (2.37)
the following identity
\[ \mathcal{E}(u,\varphi)(M) = e^{-cM}\mathcal{E}(u,\varphi)(0) + \int_0^M e^{c(t-M)}(G(t),u(t) + \frac{c}{2}u(t))_V dt \]
\[ + \int_0^M e^{c(t-M)}(\nabla(\alpha(\vartheta))\chi_2(t),u(t) + \frac{c}{2}u(t))_V dt \] (2.38)
is satisfied for all \(M > 0\).

Proof. The existence, uniqueness and continuous dependence result is essentially proved in [26]. Here, we give a proof of the energy equality (2.38) that will be of fundamental importance in proving the \textit{uniform asymptotic compactness} of the system. To show (2.38) we rely on an approximation argument similar to the one devised in [13, Appendix]. If \(u \in H^1_{\text{loc}}(0, +\infty, V') \cap C^1([0, +\infty), H) \cap C^0([0, +\infty); V)\) solves the hyperbolic equation (2.21), for any \(\varepsilon > 0\) we let \(u^\varepsilon\) be the unique solution of
\[ \langle u^\varepsilon, \varphi \rangle + \varepsilon^2 a(u^\varepsilon, \varphi) = (u, \varphi), \text{ a.e. in } (0, +\infty), \forall \varphi \in V. \] (2.39)
The behaviour of \(u^\varepsilon\) as \(\varepsilon \searrow 0\) is well known (cf., e.g., [22]). In particular, for all \(T \in [0, +\infty)\) we have that
\[ u^\varepsilon(T) \rightarrow u(T) \text{ in } V, \] (2.40)
\[ u^\varepsilon(T) \rightarrow u(T) \text{ in } H, \] (2.41)
\[ \varepsilon u^\varepsilon(T) \rightarrow 0 \text{ in } V, \] (2.42)
\[ u^\varepsilon(T) \rightarrow u(T) \text{ in } L^2(0, T; V') \] (2.43)
as well as other convergences that can be inferred from the regularity of \(u\) and (2.39). Now, putting \(v = u^\varepsilon + (c/2)u^\varepsilon\) in (2.21) and using also a Green formula, we obtain
\begin{align*}
\frac{d}{dt}\mathcal{E}(u^\varepsilon, v)(t) + c\mathcal{E}(u^\varepsilon, v)(t)
&+ \langle u_{tt}(t) - u^\varepsilon(t), u^\varepsilon(t) \rangle + c(u_t(t) - u^\varepsilon(t), u^\varepsilon(t))
&+ \langle u_{tt}(t) - u^\varepsilon(t), \frac{c}{2}u^\varepsilon(t) \rangle + c(u_t(t) - u^\varepsilon(t), \frac{c}{2}u^\varepsilon(t))
&+ a(u(t) - u^\varepsilon(t), u^\varepsilon(t)) + \frac{c}{2}a(u(t) - u^\varepsilon(t), u^\varepsilon(t))
&= (G(t), u^\varepsilon(t)) + \frac{c}{2}(u^\varepsilon(t))_\Omega + (\nabla(\alpha(\vartheta))\chi_2(t), u^\varepsilon(t)) + \frac{c}{2}(u^\varepsilon(t))_\Omega
\end{align*} (2.44)
for a.e. \( t \in (0, +\infty) \). Next, we multiply (2.44) by \( e^{c(t-M)} \), with \( M > 0 \), and integrate between 0 and \( M \). We infer that

\[
\mathcal{E}(u_\varepsilon^t, u_\varepsilon^\varepsilon)(M) = e^{-cM}\mathcal{E}(u_\varepsilon^t, u_\varepsilon^\varepsilon)(0) - \sum_{i=1}^{6} J_i^\varepsilon(M) \\
+ \int_0^M e^{c(t-M)}(G(t), u_\varepsilon^t(t) + \frac{c}{2}u_\varepsilon^\varepsilon(t))dt \\
+ \int_0^M e^{c(t-M)}(\nabla(\alpha(\theta(t)))\chi_2(t)), u_\varepsilon^t(t) + \frac{c}{2}u_\varepsilon^\varepsilon(t))dt,
\]

(2.45)

where the residual terms \( \{J_i^\varepsilon(M)\}_{i=1}^{6} \) are

\[
J_1^\varepsilon(M) = \int_0^M e^{c(t-M)}(u_{tt}(t) - u_\varepsilon^t(t), u_\varepsilon^\varepsilon(t))dt, \\
J_2^\varepsilon(M) = \int_0^M c e^{c(t-M)}(u_{t}(t) - u_\varepsilon^t(t), u_\varepsilon^\varepsilon(t))dt, \\
J_3^\varepsilon(M) = \int_0^M e^{c(t-M)}(u_{tt}(t) - u_\varepsilon^t(t), \frac{c}{2}u_\varepsilon^\varepsilon(t))dt, \\
J_4^\varepsilon(M) = \int_0^M c e^{c(t-M)}(u_{t}(t) - u_\varepsilon^t(t), \frac{c}{2}u_\varepsilon^\varepsilon(t))dt, \\
J_5^\varepsilon(M) = \int_0^M e^{c(t-M)}(u(t) - u_\varepsilon^t(t), u_\varepsilon^\varepsilon(t))dt, \\
J_6^\varepsilon(M) = \int_0^M \frac{c}{2} e^{c(t-M)}(u(t) - u_\varepsilon^t(t), u_\varepsilon^\varepsilon(t))dt.
\]

(2.46)

(2.47)

(2.48)

(2.49)

(2.50)

(2.51)

The goal is plainly to prove that all of these \( J_i^\varepsilon(M) \)-terms tend to 0 as \( \varepsilon \searrow 0 \). Indeed, from (2.40)- (2.43) and the regularity of \( u \) it is clear that the left-hand side and the first three terms in the right-hand side of (2.45) converge to their respective limits in (2.38). On the other hand, the convergence to 0 of the residual terms \( \{J_i^\varepsilon(M)\}_{i=1}^{6} \) can be shown using the methods employed in [13, Appendix], to which we refer for getting the right hints on how to manage things. Just for helping the reader a bit, let us develop the computation for

\[
J_1^\varepsilon(M) = \int_0^M e^{c(t-M)}(u_{tt}(t) - u_\varepsilon^t(t), u_\varepsilon^\varepsilon(t))dt \\
= \int_0^M e^{c(t-M)}(u_\varepsilon^t(t), u_\varepsilon^\varepsilon(t))dt \\
= \frac{1}{2} a(\varepsilon u_\varepsilon^t(M), \varepsilon u_\varepsilon^\varepsilon(M)) - \frac{1}{2} e^{-cM} a(\varepsilon u_\varepsilon^t(0), \varepsilon u_\varepsilon^\varepsilon(0)) \\
- \int_0^M \frac{c}{2} e^{c(t-M)} a(u_\varepsilon^t(t), u_\varepsilon^\varepsilon(t))dt \\
= \frac{1}{2} a(\varepsilon u_\varepsilon^t(M), \varepsilon u_\varepsilon^\varepsilon(M)) - \frac{1}{2} e^{-cM} a(\varepsilon u_\varepsilon^t(0), \varepsilon u_\varepsilon^\varepsilon(0)) - \frac{1}{2} J_2^\varepsilon(M)
\]

and note that the last line tends to 0 as \( \varepsilon \searrow 0 \) because of (2.41)-(2.42) and (2.4).

\[\Box\]

**Definition 2.7.** For \( t \geq \tau \geq 0 \) we denote by \( U_\sigma(t, \tau)z_0 \) the triplet

\[((\chi_1, \chi_2)(t), u(t), u_t(t))\]

related to the solution of Problem 2.1 and precisely
• satisfying (2.18) for every $t \geq \tau$;
• solving (2.19)-(2.21) almost everywhere in $(\tau, +\infty)$, for some selection pair $(h_1, h_2)$, with source term
  \[ \sigma = (G, \vartheta) \in L^2_{\text{loc}}(0, +\infty; H) \times L^2_{\text{loc}}(0, +\infty; W^{1,3}(\Omega)); \]
• assuming the initial value
  \[ z_0 = ((\chi^0_1, \chi^0_2), u_0, v_0) \in (K \cap V^2) \times V \times H \]
at time $\tau$, that is $U_\sigma(\tau, \tau)z_0 = z_0$.

As an immediate consequence of Theorem 2.6, we have that the collection of solving operators $\{U_\sigma(t, \tau)\}$ yields a family of semiprocesses in $\mathcal{X}$ (see (2.13)) with (time) symbol space
\[ \Sigma = L^2_{\text{loc}}(0, +\infty; H) \times L^2_{\text{loc}}(0, +\infty; W^{1,3}(\Omega)). \] (2.52)
In fact, it satisfies (2.23)-(2.25). Moreover, the solution operator $U_\sigma(t, \tau)$ enjoys also the translation invariance condition (2.26). Also, note that the space $\Sigma$, equipped with the topology of the local $L^2-$convergence in all intervals $(0, T)$, $T > 0$ (cf. (2.6)), is metrizable and the corresponding metric space is complete (for more details on such spaces see, e.g., [8, Chapter V]).

Observe however that we cannot immediately conclude from (2.36) that the the mapping $(z_0, \sigma) \mapsto U_\sigma(t, \tau)z_0$ is continuous from $\mathcal{X} \times \Sigma$ to $\mathcal{X}$. Indeed, the metric on $\mathcal{X}$ (cf. (2.13)) involves the gradients of the phase variable $(\chi_1, \chi_2)$ and the continuous dependence estimate (2.36) entails no pointwise control in time for the gradients of $(\chi_1, \chi_2)$. Nonetheless, in the next theorem we will actually check such continuity property.

**Theorem 2.8.** Assume (2.8)-(2.12). Then the family of semiprocesses
\[ \{U_\sigma(t, \tau)\}_{t \geq \tau \geq 0}, \sigma \in \Sigma \text{ is } \mathcal{X} \times \Sigma - \text{continuous}. \]

**Proof.** Assume that \( \{(\chi^0_{1n}, \chi^0_{2n}, u_{0n}, v_{0n}) \in \mathcal{X} \} \) is a sequence of initial data converging to \((\chi^0_1, \chi^0_2), u_0, v_0) \in \mathcal{X}\) in the metric (2.13) and take a sequence \((G_n, \vartheta_n) \in \Sigma\) converging to \((G, \vartheta)\) in $\Sigma$. Moreover, let us fix $t$ and denote by
\[ ((\chi_{1n}(t), \chi_{2n}(t)), u_n(t), v_n(t)) \quad (\text{resp. } ((\chi_1(t), \chi_2(t)), u(t), v(t))) \]
the solution to (2.19)-(2.21) at time $t$ starting from
\[ ((\chi^0_{1n}, \chi^0_{2n}), u_{0n}, v_{0n}) \quad (\text{resp. } ((\chi^0_1, \chi^0_2), u_0, v_0)) \]
at time $\tau$ and with forcing terms \((G_n, \vartheta_n)\) (resp. \((G, \vartheta)\)). The existence and the uniqueness for both \((\chi_{1n}(t), \chi_{2n}(t)), u_n(t), v_n(t))\) and \((\chi_1(t), \chi_2(t)), u(t), v(t))\) are obviously guaranteed by Theorem 2.6. Moreover, thanks to (2.36), we have that, for any $t \geq \tau \geq 0$,
\begin{align*}
  u_n & \rightarrow u \text{ in } C^0([\tau, t]; V), \quad (2.53) \\
  u_{nt} & \rightarrow u_t \text{ in } C^0([\tau, t]; H), \quad (2.54) \\
  \chi_{jn} & \rightarrow \chi_j \text{ in } C^0([\tau, t]; H), \quad (2.55)
\end{align*}
since, in particular, it turns out that (see [23] or, e.g., [20, Théorème 16.7]) $\alpha(\vartheta_n) \rightarrow \alpha(\vartheta)$ in $L^2(\tau, t; V)$ for any $t \geq \tau \geq 0$. Thus, in view of (2.13) we only have to prove that
\[ \sum_{j=1}^{2} ||\nabla(\chi_{jn}(t) - \chi_j(t))||^2_H \rightarrow 0 \text{ for all } t > 0. \] (2.56)
Standard energy estimates (see [26] for details) entail the boundedness of the sequences

\[ \{\chi_{1n}\}, \{\chi_{2n}\} \text{ in } H^1(\tau, t; H) \cap L^\infty(\tau, t; V) \cap L^2(\tau, t; W), \]
\[ \{u_n\} \text{ in } H^2(\tau, t; V') \cap L^\infty(\tau, t; V), \]
\[ \text{and } \{u_{nt}\} \text{ in } H^1(\tau, t; V') \cap L^\infty(\tau, t; H) \]

(cf. the regularity in (2.15) and (2.17)) uniformly with respect to \( n \), for all \( t \geq \tau \geq 0 \). Thus, by well-known compactness arguments, the related weak or weak star convergences to \( \chi_1, \chi_2, u, u_t \) hold. Note that the whole sequences converge since the limits are perfectly identified and, in particular, let us point out the following convergence

\[ \chi_{jn} \to \chi_j \text{ in } H^1(\tau, t; H) \text{ for all } t \geq \tau \geq 0 \text{ and for } j = 1, 2. \] (2.57)

Now, in order to show (2.56), we exploit the following semicontinuity comparison argument. Formally test (2.19) at level \( n \) by the vector of components \( \chi_{1nt}, \chi_{2nt} \) and then integrate over \( (\tau, t) \), with \( t \geq \tau, \tau \geq 0 \). We get

\[
\frac{\eta}{2} \sum_{j=1}^{2} \|\nabla \chi_{jn}(t)\|_H^2 + I_K(\chi_{1n}(t), \chi_{2n}(t)) \\
= \frac{\eta}{2} \sum_{j=1}^{2} \|\nabla \chi_{jn}^0\|_H^2 + I_K(\chi_{1n}^0, \chi_{2n}^0) - k \sum_{j=1}^{2} \int_{\tau}^{t} \|\partial_t \chi_{jn}(s)\|_H^2 ds \\
- \int_{\tau}^{t} \frac{\ell}{\vartheta^*}(\vartheta_n - \vartheta^*, \partial_t \chi_{1n}(s)) ds - \int_{\tau}^{t} (\alpha(\vartheta_n) \text{ div } u_n, \partial_t \chi_{2n}(s)) ds ds.
\] (2.58)

Observe that the same identity follows rigorously from [6, Théorème 3.6, pp. 72-73]. Taking the lim sup as \( n \to +\infty \) of both sides of (2.58), our aim is clearly to verify that the terms on the right-hand side actually pass to the limit. This is the case. First, note that

\[ \alpha(\vartheta_n) \text{ div } u_n - \alpha(\vartheta) \text{ div } u = \alpha(\vartheta_n)(\text{div } u_n - \text{ div } u) + (\alpha(\vartheta_n) - \alpha(\vartheta)) \text{ div } u \]

tends to 0 strongly in \( L^2(\tau, t; H) \) due to (2.10), (2.53), \( \alpha(\vartheta_n) \to \alpha(\vartheta) \) a.e. in \( \Omega \times (\tau, t) \) for a subsequence, and the Lebesgue dominated convergence theorem. Then, thanks to (2.57) and the lower semicontinuity of norms with respect to weak convergence, from (2.53) and the convergence of \( \vartheta_n \to \vartheta \) it follows that

\[
\limsup_{n \to +\infty} \left( \int_{\tau}^{t} -\frac{\ell}{\vartheta^*}(\vartheta_n - \vartheta^*, \partial_t \chi_{1n})_{\Omega} + (\alpha(\vartheta_n) \text{ div } u_n, \partial_t \chi_{2n})_{\Omega} (s) ds \\
+ k \sum_{j=1}^{2} \int_{\tau}^{t} \|\partial_t \chi_{jn}(s)\|_H^2 ds \right) \\
\leq \left( \int_{\tau}^{t} -\frac{\ell}{\vartheta^*}(\vartheta - \vartheta^*, \partial_t \chi_{1})_{\Omega} + (\alpha(\vartheta) \text{ div } u, \partial_t \chi_{2})_{\Omega} (s) ds \\
+ k \sum_{j=1}^{2} \int_{\tau}^{t} \|\partial_t \chi_{j}(s)\|_H^2 ds \right). \] (2.59)
Then, by recovering the identity analogous to (2.58) for the limiting pair \((\chi_1, \chi_2)\), a comparison with (2.58) yields

\[
\limsup_{n \rightarrow +\infty} \left( \frac{n}{2} \sum_{j=1}^{2} \left\| \nabla \chi_{jn}(t) \right\|_H^2 + I_K(\chi_{1n}(t), \chi_{2n}(t)) \right) \\
\leq \frac{n}{2} \sum_{j=1}^{2} \left\| \nabla \chi_j(t) \right\|_H^2 + I_K(\chi_1(t), \chi_2(t))
\]  

(2.60)

for all \( t \geq \tau \geq 0 \). Since the functions \( t \mapsto \frac{n}{2} \sum_{j=1}^{2} \left\| \nabla \chi_{jn}(t) \right\|_H^2 + I_K(\chi_{1n}(t), \chi_{2n}(t)) \) and \( t \mapsto \frac{n}{2} \sum_{j=1}^{2} \left\| \nabla \chi_j(t) \right\|_H^2 + I_K(\chi_1(t), \chi_2(t)) \) are absolutely continuous [6, Théorème 3.6, pp. 72-73] and (2.18) holds, (2.60) reduces to

\[
\limsup_{n \rightarrow +\infty} \sum_{j=1}^{2} \left\| \nabla \chi_{jn}(t) \right\|_H^2 \leq \sum_{j=1}^{2} \left\| \nabla \chi_j(t) \right\|_H^2 \quad \text{for all } t \geq \tau \geq 0.
\]  

(2.61)

The converse \( \text{lim inf} \) inequality clearly follows from the weak lower semicontinuity of norms and from the fact that \( \chi_{jn}(t) \rightarrow \chi_j(t), j = 1, 2, \) strongly in \( H \) and weakly in \( V \) for all \( t \geq \tau \geq 0 \), due to (2.55) and the boundedness of \( \{\chi_{jn}\}, j = 1, 2, \) in \( L^\infty(\tau, T; V) \) for all \( T > 0 \). Thus, we have that \( \left\| \nabla \chi_{jn}(t) \right\|_H^2 \rightarrow \left\| \nabla \chi_j(t) \right\|_H^2 \) for \( j = 1, 2 \) and any \( t \geq \tau \). This convergence, combined with the abovementioned weak convergence, plainly leads to the strong convergence of \( \chi_{jn}(t) \) to \( \chi_j(t) \) in \( V \) for \( j = 1, 2 \) and any \( t \geq \tau \geq 0 \). Then, recalling again (2.53)-(2.54), it turns out that the theorem is completely proved.

3. Uniform attractor for our system. In this section, we prove that the system (2.19)-(2.21) possesses the compact uniform attractor \( A \). We advise the reader that in the sequel we will make often use of some formal estimates, which can be rigorously justified by adopting some, by now standard, approximation argument. The occurrence of these formal estimates will be however pointed out to the reader. To simplify the notation, from now on we denote by \( C \) (or \( C_i, i = 1, 2, \ldots \)) some possibly different constants depending on the data of the problem. Moreover, we let \( c = 1 \) in (2.21).

We start with the proof of the dissipativity of the system and state the following result.

**Theorem 3.1** (Uniformly absorbing set). Under the same conditions as in Definition 2.7, let the triplet

\[ (G, \vartheta, \vartheta_i) \]

lie in a bounded subset \( F \) of \( T_2(H) \times T_2(H) \times T_2(H) \).

Then, there exists a constant \( D > 0 \) depending on the quantity

\[
M_F := \sup_{(G, \vartheta, \vartheta_i) \in F} \left( \left\| G \right\|_{T_2(H)}^2 + \left\| \vartheta \right\|_{T_2(H)}^2 + \left\| \vartheta_i \right\|_{T_2(H)}^2 \right)
\]

such that the \( X \)-ball with radius \( D \) turns out to be a uniform absorbing set for the family \( \{U_{(G, \vartheta)}(t, \tau), (G, \vartheta, \vartheta_i) \in F \} \). Moreover, for any \( R > 0 \) there is a constant
\( C, \text{ which depends on } M_F \text{ as well, such that for any } \tau \geq 0 \text{ there holds} \)
\[
\sup_{t \geq \tau} \int_{t}^{t+1} \sum_{j=1}^{2} \| x_j(t) \|_{H}^2 \, dt \leq C \tag{3.1}
\]
whenever \( d_X((x_0^1, x_0^2), u_0, v_0, 0) \leq R \) and \((x_1(t), x_2(t))\) stands the first component of \( U_{(G, \varphi)}(t, \tau)((x_0^1, x_0^2), u_0, v_0)\).

**Proof.** The notation of Definition 2.7 being in force, we test (2.19) by \( \left( \begin{array}{c} x_{1t} + x_1 \\ x_{2t} + x_2 \end{array} \right) \).

Then, we obtain
\[
\sum_{j=1}^{2} \left( k \| x_j(t) \|_{H}^2 + \eta \| \nabla x_j(t) \|_{H}^2 \right) + I_K(x_1(t), x_2(t)) \\
+ \frac{d}{dt} \left( \sum_{j=1}^{2} \left( k \| x_j(t) \|_{H}^2 + \eta \| \nabla x_j(t) \|_{H}^2 \right) + I_K(x_1(t), x_2(t)) \right) \\
\leq I_K(0, 0) + \int_{\Omega} (\vartheta - \vartheta^*) x_1 t, x_1 \bigg| - (\alpha(\vartheta) \text{div} u, x_2)_{\Omega}(t)
\]
for a.e. \( t \in (\tau, +\infty) \). Using (2.18), (2.5), and the elementary Young inequality, we infer that
\[
\sum_{j=1}^{2} \left( k \| x_j(t) \|_{H}^2 + \eta \| \nabla x_j(t) \|_{H}^2 \right) + \frac{d}{dt} \sum_{j=1}^{2} \left( k \| x_j(t) \|_{H}^2 + \eta \| \nabla x_j(t) \|_{H}^2 \right) \\
\leq C \left( 1 + \| \vartheta(t) \|_{H}^2 \right) - (\alpha(\vartheta) \text{div} u, x_2)_{\Omega}(t) 
\]
for a.e. \( t \in (\tau, +\infty) \). Now, we take \( v = u_t + \delta u \) as test function in (2.21), with \( \delta \in [0, 1/2] \) to be chosen later. Note that this procedure is formal since \( u_t \notin V \), however one can argue rigorously as in Theorem 2.6. Anyway, by the computation we are led to the equality
\[
\frac{d}{dt} \left( \frac{1}{2} \| u_t(t) \|_{H}^2 + \frac{1}{2} a(u(t), u(t)) + \delta (u_t(t), u(t)) + (\alpha(\vartheta) \chi_2, \text{div} u)_{\Omega}(t) \right) \\
+ (1 - \delta) \| u_t(t) \|_{H}^2 + \delta a(u(t), u(t)) \\
+ \delta^2 (u_t(t), u(t)) + (\alpha(\vartheta) \chi_2, \text{div} u)_{\Omega} \\
= (G, u_t + \delta u)(t) + (\delta^2 - \delta) (u_t(t), u(t)) \\
+ (\alpha(\vartheta) \partial_t \chi_2, \text{div} u)_{\Omega}(t) + (\alpha(\vartheta) \chi_2, \text{div} u)_{\Omega}(t).
\tag{3.3}
\]

At this point, in view of (2.3) and (2.10) we deduce that
\[
(G, u_t + \delta u)(t) \leq C \| G(t) \|_{H}^2 + \frac{1}{8} \| u_t(t) \|_{H}^2 + \frac{\delta}{8} a(u(t), u(t)),
\]
\[
(\delta^2 - \delta) (u_t(t), u(t)) \leq C \| u_t(t) \|_{H}^2 + \frac{\delta}{8} a(u(t), u(t)),
\]
\[
(\alpha(\vartheta) \partial_t \chi_2, \text{div} u)_{\Omega}(t) \leq \frac{C}{\delta} \| \partial_t \chi_2 \|_{H}^2 + \frac{\delta}{8} a(u(t), u(t)).
\]

Hence, by introducing the function
\[
\Psi(t) := \frac{1}{2} \| u_t(t) \|_{H}^2 + \frac{1}{2} a(u(t), u(t)) + \delta (u_t(t), u(t)) + (\alpha(\vartheta) \chi_2, \text{div} u)_{\Omega}(t),
\tag{3.4}
\]
provided $\delta$ is sufficiently small we find out that

$$
\frac{d}{dt} \Psi(t) + \delta \Psi(t) \leq C \|G(t)\|_{H^2}^2 + \frac{C}{\delta} \|\vartheta(t)\|_H^2 + (\alpha(\vartheta)\chi_{2t}, \Div u)_{\Omega}(t). \tag{3.5}
$$

On the other hand, with the help of (2.3), (2.5) and (2.10) we can determine two positive constants $C_1, C_2$ such that

$$
\Psi(t) + C_1 \geq C_2 \left( \|u_t(t)\|_{H^2}^2 + \|u(t)\|_{V}^2 \right), \tag{3.6}
$$

again for $\delta$ small enough. Now, we set

$$
\Phi(t) := \sum_{j=1}^{2} \left( k \frac{3}{2} \|\chi_j(t)\|_H^2 + \frac{\eta}{2} \|\nabla \chi_j(t)\|_{H^2}^2 \right) + \Psi(t) + C_1 \tag{3.7}
$$

and sum (3.2) and (3.5), noting that two terms cancel out we obtain

$$
\frac{k}{2} \sum_{j=1}^{2} \|\chi_j(t)\|_H^2 + \frac{d}{dt} \Phi(t) + \delta \Phi(t)
\leq \frac{k}{2} \sum_{j=1}^{2} \|\chi_j(t)\|_H^2 + \frac{d}{dt} \Phi(t) + \delta \Phi(t)
\leq C \left( 1 + \|\vartheta(t)\|_{H^2}^2 + \|\vartheta_t(t)\|_{H^2}^2 + \|G(t)\|_{H^2}^2 \right)
$$

for a.e. $t \in (\tau, +\infty)$. Using again (2.5), (2.10) and the Young inequality, eventually we derive

$$
\frac{k}{2} \sum_{j=1}^{2} \|\chi_j(t)\|_H^2 + \frac{d}{dt} \Phi(t) + \delta \Phi(t)
\leq C \left( 1 + \|\vartheta(t)\|_{H^2}^2 + \|\vartheta_t(t)\|_{H^2}^2 + \|G(t)\|_{H^2}^2 \right)
$$

in which $\delta$ is finally fixed and the constant $C$ on the right-hand side depends also on $\delta$. Thus, Lemma 2.5 applies with $\varphi(t) = \sqrt{\Phi(t)}$, $\varepsilon = \delta/2$, $m_1(t) = 0$, and an obvious position for $m_2(t)$. Then, we obtain the fundamental inequality

$$
\Phi(t) \leq 2\Phi(\tau)e^{-\delta(t-\tau)/2} + C_3 \left( 1 + \|\vartheta\|_{T_{2\tau}(H)}^2 + \|\vartheta_t\|_{T_{2\tau}(H)}^2 + \|G\|_{T_{2\tau}(H)}^2 \right) \tag{3.9}
$$

for every $t \geq \tau$, $\tau \geq 0$, and for some positive constant $C_3$. Then, letting $M_F$ be as in the statement and fixing a constant $C_4 > C_3(1 + M_F)$, it turns out that we can always determine a suitable time $T$, depending on $\tau$ and on the radius of the $X$-ball in which $((\chi_1, \chi_2), u(\tau), u_t(\tau)) = ((\chi_1^0, \chi_2^0), u_0, v_0)$ lives, such that

$$
2\Phi(\tau)e^{-\delta(T-\tau)/2} \leq C_4 - C_3(1 + M_F)
$$

and consequently $\Phi(t) \leq C_4$ for all $t \geq T$. Thus, in view of (3.6)-(3.7), it is straightforward to find the desired radius $D$. Finally, to obtain (3.1), we only need to integrate (3.8) between $t$ and $t+1$ and use (3.9).

The following lemmata will be crucial in the course of our investigation. In fact, in the first one we will prove a weak continuity result for the solution semiprocess, while in the second one we will show some smoothing property in finite time for the $(\chi_1, \chi_2)$-component of the solution operator $U_{(G, \vartheta)}(t, \tau)$.

**Lemma 3.2.** Let

$$
\sigma_n = (G_n, \vartheta_n) \rightarrow \sigma := (G, \vartheta) \text{ in } L^2_{\text{loc}}(0, +\infty; H) \times L^2_{\text{loc}}(0, +\infty; W^{1,3}(\Omega))
$$
and let \( z_0 = (\chi_1^0, \chi_2^0, u_0, v_0) \) denote a sequence in \( \mathcal{X} \) that weakly converges to \( z_0 := (\chi_1^0, \chi_2^0, u_0, v_0) \) in \( V^2 \times V \times H \) as \( n \to +\infty \). Then

\[
U_{\sigma_n}(t, \tau)z_0 \to U_{\sigma}(t, \tau)z_0 \quad \text{weakly in } V^2 \times V \times H \quad \text{for all } t \geq \tau \geq 0.
\]

**Proof.** As in the proof of Theorem 2.8, we agree to set

\[
((\chi_{1n}(t), \chi_{2n}(t)), u_n(t), u_{nt}(t)) = U_{(G_n, \vartheta_n)}(t, \tau)z_0.
\]

Now, we apply standard estimates, that is, perform on (2.19) the same estimate as in (2.58), and formally test (2.21) by \( u_{nt} \). Note that

\[
- \int_{\tau}^{t} (\alpha(\vartheta_n) \chi_{2n}, \text{div} u_{nt})_{\Omega(s)} ds = \int_{\tau}^{t} (\chi_{2n} \alpha'(\vartheta_n) \nabla \vartheta_n + \alpha(\vartheta_n) \nabla \chi_{2n}, u_{nt})_{\Omega(s)} ds.
\]

where the product \( \alpha'(\vartheta_n) \nabla \vartheta_n \) has to be understood properly. Then, summing the resulting inequalities and so on, with the help of (2.3), (2.5), (2.10) and the Gronwall lemma it is not difficult to obtain the following bound

\[
\sum_{j=1}^{2} \| \chi_{jn} \|_{H^1(\tau, t; H)} + \| u_{nt} \|_{L^\infty(\tau, t; H)} + \| u_n \|_{L^\infty(\tau, t; H)} \leq C,
\]

where the constant \( C \) depends on data and on \( t \), but is independent of \( n \) due to the convergences \( z_{0n} \rightharpoonup z_0 \) and \( (G_n, \vartheta_n) \to (G, \vartheta) \) which we have assumed. Consequently, a formal test of (2.19) by the vector of components \( h_{1n}, h_{2n} \) and a subsequent comparison argument, along with well-known regularity results, yield (cf. (2.15))

\[
\sum_{j=1}^{2} \left( \| h_{jn} \|_{L^2(\tau, t; H)} + \| \chi_{jn} \|_{L^2(\tau, t; V)} \right) \leq C,
\]

while a comparison of terms in (2.21) leads to the estimate

\[
\| u_{nt} \|_{L^2(\tau, t; V')} \leq C,
\]

where, again, \( C \) is independent of \( n \). Thus, we have (up to a subsequence not relabeled)

\[
\chi_{jn} \rightharpoonup \chi_j \quad \text{in } H^1(\tau, t; H) \cap L^\infty(\tau, t; V) \cap L^2(\tau, t; W), \quad j = 1, 2,
\]

\[
h_{jn} \to h_j \quad \text{in } L^2(\tau, t; H), \quad j = 1, 2,
\]

\[
u_n \rightharpoonup u \quad \text{in } L^\infty(\tau, t; V),
\]

\[
u_{nt} \rightharpoonup u_{tt} \quad \text{in } L^\infty(\tau, t; H),
\]

\[
u_{ntt} \to u_{tt} \quad \text{in } L^2(\tau, t; V')
\]

as \( n \to +\infty \), for any \( t \geq \tau \geq 0 \) and for some limit functions \( \chi_1, \chi_2, h_1, h_2, u \). Note that known compactness results (cf., e.g., [31, Corollary 4, p. 84]) imply

\[
\chi_{jn} \to \chi_j \quad \text{in } C^0([\tau, t]; H) \cap L^2(\tau, t; V), \quad j = 1, 2.
\]

Convergences (3.15)-(3.20) are enough to conclude that the limit triplet \( (\chi_1, \chi_2, u) \) yields indeed the unique solution to (2.19)-(2.21) starting from \( z_0 \) and with forcing terms \( (G, \vartheta) \). In fact, the only delicate point consists in showing the identification

\[
(h_1, h_2) \in \partial I_K(\chi_1, \chi_2) \quad \text{in } \Omega \times (\tau, t).
\]
However, this is a direct consequence of the strong-weak closedness of maximal monotone operators (see, e.g., [6, pp. 24-27]) and of the convergences (3.16) and \( \chi_j \to \chi \in L^2(\tau; t; H) \). Therefore, we have that
\[
((\chi_1(t), \chi_2(t)), u(t), u_t(t))) = U_{(G, \vartheta)}(t, \tau) \varepsilon_0. \tag{3.21}
\]
Notice that, as the limiting solution is unique, the convergences in (3.15)-(3.19) hold for the whole sequence of \( n \) and not only for a proper subsequence. It remains to show (3.10). Actually, by (3.15)-(3.19) and the generalized Ascoli theorem we also infer (see again [31, Corollary 4, p. 84])
\[
u_n \to \nu \text{ in } C^0([\tau, t]; H), \quad u_{nt} \to u_t \text{ in } C^0([\tau, t]; V') \tag{3.22}
\]
for all \( t \geq \tau \geq 0 \). At this point, it is not difficult to obtain (3.10) from (3.20), (3.22) and the uniform estimates in (3.12).

**Lemma 3.3.**

Under the assumption of Theorem 3.1, let moreover \((G, \vartheta, \vartheta_t) \in \mathcal{T}_2(H) \times \mathcal{T}_2(W^{1,\lambda}(\Omega)) \times \mathcal{T}_2(L^2(\Omega)).\)

Then, for any \( R > 0 \) there exists a constant \( \Lambda \) such that for any \( \tau \geq 0 \) there holds
\[
\sup_{t \geq \tau+1} \sum_{j=1}^{2} \left( \| \chi_{j\epsilon}(t) \|_{H}^2 + \| \chi_{j\epsilon}(t) \|_{W}^2 \right) \leq \Lambda \tag{3.23}
\]
whenever \( d_X((\chi_1, \chi_2), (u_0, v_0), 0) \leq R \) and
\[
\| G \|_{\mathcal{T}_2(\mathcal{H})} + \| \vartheta \|_{\mathcal{T}_2(W^{1,\lambda}(\Omega))} + \| \vartheta_t \|_{\mathcal{T}_2(L^2(\Omega))} \leq R,
\]
(\( \chi_1(t), \chi_2(t) \)) denoting the first component of \( U_{(G, \vartheta)}(t, \tau)((\chi_0^1, \chi_0^2), (u_0, v_0)) \).

**Proof.** Adopting the usual notation as in Definition 2.7, take \( R > 0 \) such that \( d_X((\chi_1^0, \chi_2^0), (u_0, v_0), 0) \leq R \) and \( \| G \|_{\mathcal{T}_2(\mathcal{H})} + \| \vartheta \|_{\mathcal{T}_2(W^{1,\lambda}(\Omega))} + \| \vartheta_t \|_{\mathcal{T}_2(L^2(\Omega))} \leq R \).

Let us differentiate (2.19) with respect to time and then test by the vector of components \( \chi_{1\epsilon}, \chi_{2\epsilon} \). This procedure is only formal. However, it might be made rigorous by working on a regularized level and then passing to the limit. Since this technique is quite standard, we prefer to avoid such details and proceed in a formal way. Thus, by the monotonicity of \( \partial I_K \) we get
\[
\sum_{j=1}^{2} \left( \frac{k}{2} \frac{d}{dt} \| \chi_{j\epsilon}(t) \|_{H}^2 + \eta \| \nabla \chi_{j\epsilon}(t) \|_{H}^2 \right) \leq -\frac{\ell}{\vartheta} (\vartheta_t, \chi_{1\epsilon}) \Omega(t) - \langle \text{div } u_t, \alpha(\vartheta) \chi_{2\epsilon} \rangle \Omega(t) - \langle \text{div } u_t, \alpha'(\vartheta) \vartheta_t \chi_{2\epsilon} \rangle \Omega(t). \tag{3.24}
\]

Now, we sum \( \sum_{j=1}^{2} \eta \| \chi_{j\epsilon}(t) \|_{H}^2 \) to both sides of (3.24) in order to get the full \( V^2 \)-norm in the left-hand side of (3.24). Subsequently, taking advantage of (2.10), the Hölder inequality, the continuous embedding \( H^1(\Omega) \subset L^6(\Omega) \) and the Young inequality in the form \( ab \leq \frac{1}{2} a^2 + \frac{\varepsilon}{2} b^2 \) for all \( \varepsilon > 0 \) and \( a, b \in \mathbb{R} \), we get
\[
-\frac{\ell}{\vartheta} (\vartheta_t, \chi_{1\epsilon}) \Omega(t) \leq C \left( \| \vartheta_t \|_{H}^2 + \| \chi_{1\epsilon}(t) \|_{H}^2 \right),
\]
\[
-\langle \text{div } u_t, \alpha(\vartheta) \chi_{2\epsilon} \rangle \Omega(t) \leq C \| u_t(t) \|_{H} \left( \| \chi_{2\epsilon}(t) \|_{V} + \| \nabla \vartheta(t) \|_{L^2(\Omega)} \right),
\]
\[
\leq \frac{\eta}{4} \| \chi_{2\epsilon}(t) \|_{V}^2 + C \| u_t(t) \|_{H}^2 \left( 1 + \| \vartheta(t) \|_{W^{1,\lambda}(\Omega)} \right),
\]
\[-(\div u, \alpha'(/)\partial_t x_{2\ell}(\tau)) \leq C\|u(\tau)\|_V\|\partial_t(\tau)\|_{L^2(\Omega)}\|x_{2\ell}(\tau)\|_{L^4(\Omega)} \leq \frac{\eta}{4}\|x_{2\ell}(\tau)\|_V^2 + C\|u(\tau)\|_V^2\|\partial_t(\tau)\|_{L^2(\Omega)}^2.\]

Recalling Theorem 3.1 (cf., in particular, (3.6)-(3.7) and (3.9)), it turns out that

\[
\sum_{j=1}^2 \left( \frac{d}{dt}\|x_{j}\|_H^2 + \|x_{j}(t)\|_{W_1,3}^2 \right) \leq C \left( 1 + \|\partial_t(t)\|_{W_1,3}^2 + \|\partial_t(t)\|_{W_1,3}^2 + \sum_{j=1}^2 \|x_{j}\|_H^2 \right),
\]

for some constant \(C\) depending especially on \(R\). Thus, recalling (3.1) we are in the position of applying the Uniform Gronwall Lemma 2.4 to obtain a uniform (in time) bound for \(\|x_{j}(t)\|_H^2, j = 1, 2\). Next, by Theorem 3.1 and a simple comparison argument in (2.19) we infer

\[
\sup_{\tau \geq 0} \sup_{t \geq \tau + 1} \sum_{j=1}^2 \| - \Delta x_{j}(t) + h_{j}(t)\|^2_H \leq C.
\]

Then, the monotonicity of \(\partial_t\) and standard elliptic regularity results allow us to conclude that (cf. also (2.15))

\[
\sum_{j=1}^2 \left( \|x_{j}\|_W^2 + \|h_{j}(t)\|_H^2 \right) \leq \sum_{j=1}^2 \| - \Delta x_{j}(t) + h_{j}(t)\|^2_H \leq C
\]

for all \(\tau \geq 0\) and \(t \geq \tau + 1\), whence (3.23) is completely proved.

We now prove the uniform asymptotic compactness of the system. To this end, fix a translation compact function \((\tilde{\sigma}_1, \tilde{\sigma}_2) \in L^2_{\text{loc}}(0, +\infty; H) \times L^2_{\text{loc}}(0, +\infty; W^{1,3}(\Omega)).\) Then, we allow \((G, \vartheta)\) to vary in \(\mathcal{H}(\tilde{\sigma}_1, \tilde{\sigma}_2)\). First of all, note that (see, e.g., [8, Proposition V.3.4])

\[
\|(G, \vartheta)\|_{T_2(H \times W^{1,3}(\Omega))} \leq \|(\tilde{\sigma}_1, \tilde{\sigma}_2)\|_{T_2(H \times W^{1,3}(\Omega))} < +\infty
\]

for any \((G, \vartheta) \in H(\tilde{\sigma}_1, \tilde{\sigma}_2)\).

**Theorem 3.4** (Uniform asymptotic compactness). *Within the framework of Definition 2.7, assume in addition that*
\[(\tilde{\sigma}_1, \tilde{\sigma}_2) \in L^2_{\text{loc}}(0, +\infty; H) \times L^2_{\text{loc}}(0, +\infty; W^{1,3}(\Omega)) \text{ is translation compact (3.28) and there exists } R > 0 \text{ such that } \|(\tilde{\sigma}_2)\|_{T_2(\mathcal{L}(\Omega))} \leq R.\]

Then, the family \(\{U_{(G, \vartheta)}(t, \tau), (G, \vartheta) \in H(\tilde{\sigma}_1, \tilde{\sigma}_2)\}\) is uniformly asymptotically compact in \(\mathcal{X}\).

**Proof.** Recalling Definition 2.7, we let
\[
((X_{1n}(t), X_{2n}(t), u_n(t), u_n(t)) := U_{(G_n, \vartheta_n)}(t, 0)z_{0n}
\]

denote the solutions emanating from the \(\mathcal{X}\)-bounded sequence
\[z_{0n} = ((\chi^0_{1n}, \chi^0_{2n}), (u_{0n}, v_{0n})\]
at initial time 0, with forcing terms \((G_n, \vartheta_n)\) in \(H(\tilde{\sigma}_1, \tilde{\sigma}_2)\). Moreover, take an arbitrary time sequence \(t_n \to +\infty\). Owing to Theorem 3.1, it turns out that
(3.27)-(3.29) and the boundedness of \( \{z_0n\} \) in \( (X, d_X) \) entail the following uniform estimate
\[
\sum_{j=1}^{2} \|x_{jn}(t)\|_V + \|u_n(t)\|_V + \|u_{nt}(t)\|_H \leq C \quad \forall t \geq 0. \tag{3.31}
\]
Thus, (3.31) holds in particular for \( t = t_n \) and, up to the extraction of a subsequence of \( n \), it results that
\[
U_{(G_n, \vartheta_n)}(t_n, 0)z_{0n} = ((x_{1n}(t_n), x_{2n}(t_n)), u_n(t_n), u_{nt}(t_n))
\rightarrow ((x_{1\infty}, x_{2\infty}), u_\infty, v_\infty) =: z_\infty \text{ in } V^2 \times V \times H. \tag{3.32}
\]
On the other hand, since \( \mathcal{H}(\hat{\sigma}_1, \hat{\sigma}_2) \) is compact, in view of [8, Proposition V.3.4] we have that
\[
T_{t_n}G_n \rightarrow G_\infty \text{ in } L^2_{\text{loc}}(0, +\infty; H),
\]
\[
T_{t_n}\vartheta_n \rightarrow \vartheta_\infty \text{ in } L^2_{\text{loc}}(0, +\infty; W^{1,3}(\Omega)) \tag{3.33}
\]
as \( n \not\rightarrow +\infty \), still up to a subsequence, for some pair \( (G_\infty, \vartheta_\infty) \in \mathcal{H}(\hat{\sigma}_1, \hat{\sigma}_2) \). Moreover, possibly by a diagonal procedure one can select another subsequence of \( n \) such that for all \( M \in \mathbb{N} \) there holds
\[
U_{(G_n, \vartheta_n)}(t_n - M, 0)z_{0n} \text{ (extended with } z_{0n} \text{ value for } t_n \leq M)
\]
weakly converges in \( V^2 \times V \times H \) to some element \( z_M := ((x_{1M}, x_{2M}), u_M, v_M) \)
as well as
\[
T_{t_n - M}G_n \rightarrow G_M \text{ in } L^2_{\text{loc}}(0, +\infty; H),
\]
\[
T_{t_n - M}\vartheta_n \rightarrow \vartheta_M \text{ in } L^2_{\text{loc}}(0, +\infty; W^{1,3}(\Omega)) \tag{3.36}
\]
as \( n \not\rightarrow +\infty \), with the limits \( (G_M, \vartheta_M) \in \mathcal{H}(\hat{\sigma}_1, \hat{\sigma}_2) \). Incidentally, note that
\[
G_M(s) := G_\infty(s - M), \quad \vartheta_M(s) := \vartheta_\infty(s - M) \quad \text{for } s > M.
\]
Then, from the translation invariance condition (2.26) we have that
\[
((x_{1n}, x_{2n}), u_n, u_{nt})(t_n)
= U_{(G_n, \vartheta_n)}(t_n, 0)z_{0n}
= U_{(G_n, \vartheta_n)}(t_n - M + M, t_n - M)U_{(G_n, \vartheta_n)}(t_n - M, 0)z_{0n}
= U_{T_{t_n - M}(G_n, \vartheta_n)}(M, 0)U_{(G_n, \vartheta_n)}(t_n - M, 0)z_{0n}
= U_{T_{t_n - M}(G_n, \vartheta_n)}(M, 0)((x_{1n}, x_{2n}), u_n, u_{nt})(t_n - M)
\]
if \( t_n \geq M \). Thus, by the weak continuity property stated in Lemma 3.2 and by (3.32), (3.34) we deduce that
\[
U_{(G_n, \vartheta_n)}(t_n, 0)z_{0n} \rightarrow U_{(G_M, \vartheta_M)}(M, 0)z_M \equiv z_\infty \text{ in } V^2 \times V \times H. \tag{3.38}
\]
Let us now recall (2.30) and observe that, in order to complete the proof, we should check that actually the strong convergence holds as well in (3.38). Let us analyse separately the components of \( U_{(G_n, \vartheta_n)}(t_n, 0)z_{0n} \). From Lemma 3.3 (cf., in particular, estimate (3.23)) we immediately conclude that
\[
(x_{1n}(t_n), x_{2n}(t_n)) \rightarrow (x_{1\infty}, x_{2\infty}) \in V^2, \tag{3.39}
\]
due to the compact embedding of \( W \) into \( V \). Next, setting
\[
((\xi^M_{1n}, \xi^M_{2n}), w^M_n, w^M_{nt}) := U_{T_{t_n - M}(G_n, \vartheta_n)}(t_n, 0)((x_{1n}, x_{2n}), u_n, u_{nt})(t_n - M),
\]
\[
((\xi^M_1, \xi^M_2), w^M, w^M_t) := U_{(G_M, \vartheta_M)}(t_n, 0)((x_{1M}, x_{2M}), u_M, v_M)
\]
and recalling (3.35)-(3.36) and (3.34), Lemma 3.2 and especially (3.20), (3.17), (3.18) ensure that
\[
\xi^M_{jn} \to \xi^M_j \quad \text{in} \quad C^0([0,M]; H) \cap L^2(0,M; V), \quad j = 1, 2, \tag{3.40}
\]
\[
w^M_n \rightharpoonup w^M \quad \text{in} \quad L^\infty(0,M; V), \tag{3.41}
\]
\[
w^M_{nt} \rightharpoonup w^M_t \quad \text{in} \quad L^\infty(0,M; H). \tag{3.42}
\]
To conclude the proof of the asymptotic compactness of the semiprocess, we use the \textit{energy functional} introduced in (2.37). We remind that \( c = 1 \) throughout this section. We now write the energy identity (2.38) for \(((\xi^M_1, \xi^M_2), w^M, w^M_{nt})\) in the time interval \([0,M]\). Thus, by (3.37) we have
\[
\mathcal{E}(u_{nt}, u_n)(t_n) - e^{-M} \mathcal{E}(u_{nt}, u_n)(t_n - M) = \mathcal{E}(w^M_n, w^M_n)(M) - e^{-M} \mathcal{E}(w^M_{nt}, w^M_n)(0)
\]
\[
= \int_0^M e^{-M} ((T_{t_0-M}G_n)(t), w^M_{nt}(t)) + \frac{1}{2} w^M_n(t)) dt \tag{3.43}
\]
\[
+ \int_0^M e^{-M} (\nabla(\alpha((T_{t_0-M}\vartheta)(t))\xi^M_2(t)), w^M_{nt}(t) + \frac{1}{2} w^M_n(t)) dt.
\]
Owing to (3.35)-(3.36), it turns out that (see [23] or, e.g., [20, Théorème 16.7])
\[
\alpha(T_{t_0-M}\vartheta)(t) \to \alpha(\vartheta_M) \quad \text{in} \quad L^2(0,M; V).
\]
Hence, with the help of (3.40), (2.5), (2.10) and possibly using the Lebesgue dominated convergence theorem, one can directly check that
\[
\nabla(\alpha((T_{t_0-M}\vartheta)(t))\xi^M_2) \to \nabla(\alpha(\vartheta_M)\xi_2^M) \quad \text{in} \quad L^2(0,M; H).
\]
Then, thanks to (3.40)-(3.42) we can pass to the limit in the right-hand side of (3.43) by virtue of the strong-weak (star) convergences and find out that
\[
\lim_{n \to +\infty} \mathcal{E}(u_{nt}, u_n)(t_n) - e^{-M} \mathcal{E}(u_{nt}, u_n)(t_n - M)
\]
\[
= \int_0^M e^{-M} (G_M(t), w^M_t(t)) + \frac{1}{2} w^M(t)) dt \tag{3.43}
\]
\[
+ \int_0^M e^{-M} (\nabla(\alpha(\vartheta_M)\xi^M_2(t)), w^M_{nt}(t) + \frac{1}{2} w^M_n(t)) dt,
\]
which is nothing but
\[
\mathcal{E}(w^M, w^M)(M) - e^{-M} \mathcal{E}(w^M, w^M)(0) = \mathcal{E}(v_\infty, u_\infty) - e^{-M} \mathcal{E}(v_M, u_M)
\]
thanks to (3.37)-(3.38) and to identity (2.38) applied to \(((\xi^M_1, \xi^M_2), w^M, w^M_{nt})\). Thus, due to the uniform bound in (3.31) and in view of (3.34) one can easily deduce that
\[
\limsup_{n \to +\infty} \mathcal{E}(u_{nt}, u_n)(t_n) \leq \lim_{n \to +\infty} \mathcal{E}(u_{nt}, u_n)(t_n - M) + C e^{-M} \tag{3.44}
\]
\[
\leq \mathcal{E}(v_\infty, u_\infty) + 2C e^{-M}.
\]
Now, since \(\mathcal{E}\) is weakly lower semicontinuous in the energy space \(H \times V\), by letting \(M \to +\infty\) in (3.44) we conclude that
\[
\limsup_{n \to +\infty} \mathcal{E}(u_{nt}, u_n)(t_n) \leq \mathcal{E}(v_\infty, u_\infty) \leq \lim_{n \to +\infty} \mathcal{E}(u_{nt}, u_n)(t_n), \tag{3.45}
\]
whence $E(u_n, u_n)(t_n) \to E(v_\infty, u_\infty)$. At this point, it is not difficult to check that (3.45) entails the strong convergence $(u_n, u_n)(t_n) \to (u_\infty, v_\infty)$ in $V \times H$. Hence, recalling (3.38) and (3.39), the desired uniform asymptotic compactness for the family $\{U_{(G, \vartheta)}(t, \tau), (G, \vartheta) \in \mathcal{H}(\hat{\sigma}_1, \hat{\sigma}_2)\}$ follows.

Recalling Definition 2.7, we can now state and prove the main result concerning the long-time behaviour of the solutions to our system.

**Theorem 3.5** (Uniform attractor). Under the assumptions of Theorem 3.4, the family of semiprocesses $\{U_{(G, \vartheta)}(t, \tau), (G, \vartheta) \in \mathcal{H}(\hat{\sigma}_1, \hat{\sigma}_2)\}$ possesses a uniform attractor $A$ in the phase space $X$. Moreover, the uniform attractor $A$ is connected.

**Proof.** From Theorem 2.8 we infer that the family of semiprocesses

$$\{U_{(G, \vartheta)}(t, \tau), (G, \vartheta) \in \mathcal{H}(\hat{\sigma}_1, \hat{\sigma}_2)\}$$

is continuous from $X \times \mathcal{H}(\hat{\sigma}_1, \hat{\sigma}_2)$ to $X$ for any $t \geq \tau \geq 0$. Theorem 3.1 implies that $\{U_{(G, \vartheta)}(t, \tau), (G, \vartheta) \in \mathcal{H}(\hat{\sigma}_1, \hat{\sigma}_2)\}$ has a bounded uniformly absorbing set and finally, with help of Theorem 3.4, we conclude that such a family is also uniformly asymptotically compact. The existence of the compact uniform attractor is thus a consequence of the abstract result in Theorem 2.3. We now give a direct proof of the connectedness of the uniform attractor $A$. To this end, note that the set $A \times \mathcal{H}(\hat{\sigma}_1, \hat{\sigma}_2)$ yields the global attractor for the semigroup $S_1$ introduced in (2.31). Now, the latter is connected. In fact, it turns out that the set $B \times \mathcal{H}(\hat{\sigma}_1, \hat{\sigma}_2)$, where $B$ is a uniformly absorbing ball in $X$ (whose existence has been assured by Theorem 3.1), is a bounded absorbing set for the semigroup $S_1$. Moreover, since $\mathcal{H}(\hat{\sigma}_1, \hat{\sigma}_2)$ is connected (we recall the general definition (2.27)), then $B \times \mathcal{H}(\hat{\sigma}_1, \hat{\sigma}_2)$ is connected as well. Thus, standard results on semigroups (see, e.g., [19, Proposition 5.2.7]) show that $A \times \mathcal{H}(\hat{\sigma}_1, \hat{\sigma}_2)$ is connected, too. This means that $A = \Pi_1(A \times \mathcal{H}(\hat{\sigma}_1, \hat{\sigma}_2))$ is connected.

**Remark 3.6.** Our definition of solution $U_{(G, \vartheta)}(t, \tau)z_0$ works for $\tau \geq 0$ and for $(G, \vartheta) \in L^2_{\text{loc}}(0, +\infty; H) \times L^2_{\text{loc}}(0, +\infty; W^{1,3}(\Omega))$. However, let us point out that one can extend the notion of solution and semiprocess to values $\tau \in \mathbb{R}$ and to forcing terms

$$(G, \vartheta) \in L^2_{\text{loc}}(\mathbb{R}; H) \times L^2_{\text{loc}}(\mathbb{R}; W^{1,3}(\Omega)).$$

In this case, the uniform attractor $A$ can be represented as

$$A = \left\{z(0) : \begin{array}{l}
z(t) \text{ is any bounded complete trajectory of } U_{(G, \vartheta)}(t, \tau), \\
\text{that is, } U_{(G, \vartheta)}(t, \tau)z(\tau) = z(t) \quad \forall t \geq \tau, \ \tau \in \mathbb{R}, \\
\text{for some } (G, \vartheta) \in \mathcal{H}(\hat{\sigma}_1, \hat{\sigma}_2) \end{array} \right\}, (3.46)$$

where $(\hat{\sigma}_1, \hat{\sigma}_2)$ will be now translation compact in $L^2_{\text{loc}}(\mathbb{R}; H) \times L^2_{\text{loc}}(\mathbb{R}; W^{1,3}(\Omega))$. The precise structure of the attractor given in (3.46) is a direct consequence of the known results on uniform attractors (see, e.g., [8, Theorem IV.5.1]).

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