ANGULAR SIZE AND EMISSION TIMESCALES OF RELATIVISTIC FIREBALLS

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ABSTRACT

The detection of delayed X-ray, optical, and radio emission, “afterglow,” associated with γ-ray bursts (GRBs) is consistent with models in which the bursts are produced by relativistic expanding blast waves, driven by expanding fireballs at cosmological distances. In particular, the timescales over which radiation is observed at different wavebands agree with model predictions. It had recently been claimed that the commonly used relation between observation time \( t \) and blast wave radius \( r \), \( t = r/2\gamma^2(r)c \), where \( \gamma \) is the fluid Lorentz factor, should be replaced with \( t = r/16\gamma^2(r)c \) because of blast wave deceleration. Applying the suggested deceleration modification would make it difficult to reconcile observed timescales with model predictions. It would also imply an apparent source size too large to allow attributing observed radio variability to diffractive scintillation. We present a detailed analysis of the implications of the relativistic hydrodynamics of expanding blast waves to the observed afterglow. We find that modifications due to shock deceleration are small, therefore allowing for both the observed afterglow timescales and for diffractive scintillation. We show that at time \( t \) the fireball appears on the sky as a narrow ring of radius \( h = r\gamma(r) \) and width \( \Delta hlh \sim 0.1 \), where \( r \) and \( t \) are related by \( t = r/2\gamma^2(r)c \).

Subject heading: gamma rays: bursts

1. INTRODUCTION

The availability of accurate positions for GRBs from the BeppoSAX satellite (Costa et al. 1997a; Feroci et al. 1997; Heise et al. 1997) allowed for the first time to detect delayed emission associated with GRBs in X-ray (Costa et al. 1997a, 1997b; Feroci et al. 1997; Piro et al. 1997a, 1997b), optical (Groot et al. 1997; Sahu et al. 1997; van Paradijs et al. 1997; Bond 1997; Djorgovski et al. 1997), and radio (Frail & Kulkarni 1997) wavebands. The detection of absorption lines in the optical afterglow of GRB 970508 provided the first direct estimate of source distance, constraining the redshift of GRB 970508 to \( 0.8 < z < 2.3 \) (Metzger et al. 1997). Observed X-ray to radio afterglows are broadly consistent with models based on relativistic blast waves at cosmological distances (Paczynski & Rhoads 1993; Katz 1994; Mészáros & Rees 1997; Vietri 1997; Waxman 1997a, 1997b; Wijers, Rees, & Mészáros 1997). Using these models, combined radio and optical data allowed for the first time to directly estimate the total GRB energy, implying an energy of \( \sim 10^{52} \) ergs for GRB 970508 (Waxman 1997b).

In fireball models of GRB afterglow, the energy released by an explosion, \( \sim 10^{52} \) ergs, is converted to kinetic energy of a thin baryon shell expanding at ultrarelativistic speed. After producing the GRB, the shell impacts on surrounding gas, driving an ultrarelativistic shock into the ambient medium. After a short transition phase, the expanding blast wave approaches a self-similar behavior where the expansion Lorentz factor decreases with radius as \( \gamma \propto r^{-3/2} \). The expanding shock continuously heats fresh gas and accelerates relativistic electrons, which produce the observed radiation through synchrotron emission.

Photons emitted at radius \( r \) with frequency \( \nu \), measured in the shell rest frame, are observed over a wide frequency range, \( 2\gamma(r)\nu \) to \( \nu/2\gamma(r) \), and wide time range, since photons emitted further from the source-observer line of sight arrive at a later time and are observed to have lower energy. There is no unique relation, therefore, between the radius \( r \) and the time or frequency at which radiation emitted at \( r \) is observed. However, since most of the emission from radius \( r \) detected by a distant observer originates from a disk of radius \( \sim r\gamma(r) \) around the fireball appears on the sky as a narrow ring of radius \( h = r\gamma(r) \) and width \( \Delta hlh \sim 0.1 \), where \( r \) and \( t \) are related by \( t = r/2\gamma^2(r)c \).

In § 2 we present a detailed analysis of the implications of the relativistic hydrodynamics of expanding blast waves to the observed afterglow. We find that the modification due to shock deceleration of the numerical coefficient in the expression \( t = r/2\gamma^2(r)c \), giving the time at which the flux peaks at observed frequency \( \nu' = \gamma(r)\nu(r) \), is small. It should be emphasized that the exact value of the numerical coefficient depends...
not only on the hydrodynamics, but also on the details of the electron energy distribution and on the magnetic field distribution behind the shock. Due to the lack of a theory describing these distributions, the exact value of the numerical coefficient cannot be determined. However, it can be shown that the effect of deceleration is small. This is due to the fact that the relation \( t = r/16\gamma^2(r)c \) gives the delay of photons emitted from the fireball at radius \( r \) from a point on the shock front on the line of sight, while most photons suffer longer delays, since they are emitted from a shell of finite thickness behind the shock, and from positions off the line of sight. We show that at time \( t = r/2\gamma^2(r)c \) the fireball appears on the sky as a narrow ring of radius \( r/\gamma(r) \). (This conclusion is independent of the details of electron and magnetic field distribution). Our conclusions and their implications are summarized in § 3.

2. EMISSION FROM A RELATIVISTIC EXPANDING BLAST WAVE

Let us consider a strong, spherical, ultrarelativistic shock wave expanding into an ambient medium of uniform density. For a shock propagating with a Lorentz factor \( \Gamma \gg 1 \), the Lorentz factor, number density and energy density of shocked fluid at the shock discontinuity are \( \gamma = \Gamma / \sqrt{2} \), \( n = 4\gamma n_i \), and \( e = 4\gamma^2 n_i c^2 \) respectively, where \( n_i \) is the number density ahead of the shock (e.g., Blandford & McKee 1976). Observations indicate that the fraction of blast wave energy carried by electrons is not large, \( \sim 0.1 \), and that the electron cooling time is long compared to the dynamical time, i.e., to the blast wave expansion time (Waxman 1997a, 1997b). This implies that the energy lost to radiation is small and that the blast wave energy is approximately constant (so called “adiabatic blast wave”). Since the total energy in shocked particles is proportional to \( n m c^2 \), conservation of energy implies (defining \( \gamma_0 \equiv \gamma(r = r_0) \))

\[
\gamma = \gamma_0 \left( \frac{r}{r_0} \right)^{-3/2}.
\]

In fireball models of GRB afterglow it is assumed that the fractions of energy carried by magnetic field and by electrons are time independent. Under this assumption the magnetic field \( B \) and the characteristic electron Lorentz factor \( \gamma_e \) (in the shell frame) scale as \( B \propto \gamma \propto \gamma_e \). The observed radiation is produced by synchrotron emission of the shock accelerated electrons. The characteristic synchrotron frequency in the shell frame scales as \( \nu \propto \gamma^2 B \propto \gamma^3 \). The number of photons emitted as the shock propagates a distance \( dr \) may be obtained as follows. The number of radiating electrons scales as \( r^3 \), and the number of photons each electron emits per unit time in the shell frame is proportional to the magnetic field \( B \). The time in the shell frame over which the shock propagates a distance \( dr \) is \( dr/\gamma c \). Thus, the number of photons emitted scales as \( dN/\gamma c \propto r^3 \). Using equation (1) we have

\[
\nu = \nu_0 \left( \frac{r}{r_0} \right)^{-3/2}, \quad \frac{dN}{dr} = L^{-1} \left( \frac{r}{r_0} \right)^3,
\]

where \( L \) is a constant with dimensions of length.

The shock-heated gas expands relativistically in its rest frame. Since the time, measured in the rest frame, for the shock to expand to radius \( r \) is \( \sim r/\gamma c \), relativistic expansion in the shell frame implies that the rest frame thickness of the shell of shocked gas is \( \sim r/\gamma c \). Thus, in the observer frame most of the shocked gas, and most of the blast wave energy, are concentrated in a shell of thickness \( \Delta r = \xi r/\gamma^2 \), where \( \xi \) is some constant. We can obtain an estimate of \( \xi \) by assuming that the density in the shocked shell is uniform. In this case conservation of particle number implies \( 4\pi r^2 \gamma_0 \Delta r \gamma_0 n = 4\pi r^2 n/3 \), i.e., \( \xi = 1/12 \). In the self-similar solutions of Blandford & McKee (1976), which give the spatial dependence of the hydrodynamic variables, 90% of the energy is concentrated in a shell of thickness corresponding to \( \xi = 1/7 \). It is implicitly assumed in fireball afterglow models that the fractions of energy carried by electrons and magnetic field vary behind the shock over a length scale comparable to the scale for changes in the hydrodynamic variables, \( \Delta r \). This is a reasonable assumption, since the synchrotron cooling time of the electrons is longer than the dynamical time, i.e., the characteristic expansion time. If the energy fractions vary behind the shock on a scale much shorter than \( \Delta r \), due to some nonhydrodynamic process that is not accounted for, a new length scale would be introduced into the problem and the scalings (eq. [2]), on which the fireball afterglow model relies, will no longer hold. Since the details of the spatial dependence of the electron and magnetic field energy fractions are not known, we will assume below that radiation is produced within a homogeneous shell of width \( \Delta r \) behind the shock, and we will derive results for different values of \( \xi \). For clarity we first assume that at a fixed time all emitted photons have the same energy (in the fireball frame). We then discuss modifications expected due to a finite frequency range of emitted photons.

In the shell frame photons are emitted isotropically. Consider, therefore, the appearance of an isotropic distribution of photons of frequency \( \nu \) in a frame moving with Lorentz factor \( \gamma \) with respect to the frame in which the distribution is isotropic. We refer to the frame where the distribution is isotropic as the “rest frame.” Denoting with primes quantities measured in the moving frame, the rest frame and moving frame frequencies are related by

\[
\nu' = \gamma(1 + \beta \cos \theta) \nu,
\]

where \( \beta = (1 - 1/\gamma^2)^{1/2} \) and \( \theta \) is the (rest frame) angle between the photon momentum and the direction of motion of the moving frame. The angle measured in the moving frame is

\[
\tan \theta' = \frac{\sin \theta}{\gamma(\beta + \cos \theta)}.
\]

The fraction \( df' \) of photons with frequencies in the range \( \nu' \) to \( \nu' + d\nu' \) is

\[
\frac{df'}{d\nu'} = \frac{1}{2} \sin \theta \frac{d\theta}{d\nu'} = \frac{1}{2\gamma\beta} \quad \text{for} \quad \nu_{\min}' < \nu' < \nu_{\max}'.
\]

Here, \( \nu_{\min}' = \gamma(1 - \beta)\nu, \nu_{\max}' = \gamma(1 + \beta)\nu \).

Let us now consider photons observed by a distant observer in a frequency range \( \nu' \) to \( \nu' + d\nu' \). The number of photons produced by the fireball shell at radius \( r \), with observed frequency in the range \( \nu' \) to \( \nu' + d\nu' \), is given by equations (2) and (5):

\[
\frac{d^2N}{d\nu' d\nu'} = \frac{1}{2L\gamma_0 \nu_0} \left( \frac{r}{r_0} \right)^3,
\]

where we have approximated \( \beta = 1 \) in equation (5), since we
are interested in the limit $\gamma \gg 1$. Any photon that arrives at a distant observer must be emitted from the fireball in a direction parallel to the source-observer line of sight. A photon emitted at radius $r$, with frequency $\nu(r)$ in the shell frame, is observed with frequency $\nu'$ provided, therefore, it is emitted from the fireball on a line emerging from the explosion center and making an angle $\theta'$, given by equations (4) and (5), with the source-observer line of sight. Such a photon is delayed with respect to photons emitted on the line of sight by

$$\Delta t_{\theta} = (1 - \cos \theta') \frac{L}{c} \quad (7)$$

The total delay, i.e., the delay with respect to photons emitted from the center of the explosion $r = 0$, is given by the sum of $\Delta t_{\theta}$ and $\Delta t = t = r/c$, the difference between light travel time to radius $r$ and shock expansion time to radius $r$. Thus, the delay of photons emitted from the shock front at radius $r$, and observed with frequency $\nu'$, with respect to photons emitted from the center of the explosion $r = 0$, is

$$t(\nu', r) = \Delta t_{\theta}(\nu', r) + r/16\gamma^2 c. \quad (8)$$

Since photons are emitted uniformly from a shell of finite thickness $\delta(r\gamma^2 c)$ behind the shock, the arrival times of photons emitted at radius $r$ and observed with frequency $\nu'$ are uniformly distributed (for $\gamma \gg 1$) over the range $t = t(\nu', r)$ to $t = t(\nu', r) + \tau_r$, where $\tau_r = r/16\gamma^2 c$.

Using equations (6)–(8) we have numerically calculated the intensity (energy flux per unit frequency) $f_\nu = \nu d^2N/d\nu dt$ as a function of time. Results are shown in Figure 1 for different values of the shell thickness, $\xi = 1/4$, $1/16$, and $1/64$. Figure 1 presents the time dependent flux at $\nu' = \gamma_0 \nu_0$. The intensity at other frequencies is obtained using the scaling

$$f_\nu(\nu', t) = f_\nu(\nu'_0, t) \left[ \frac{\nu'_0}{\nu'} \right]^{-2/3} t. \quad (9)$$

For the shell width expected from fireball hydrodynamics, $\xi \sim 1/10$, the intensity peaks at frequency $\nu' = \gamma(r)\nu(r)$ at a time $t = r/4\gamma^2 c$. The numeric coefficient in this relation, $1/4$, is somewhat smaller than the commonly used value, $1/2$. However, since the spatial dependence of the fractions of energy carried by electrons and magnetic field is not known, the effective shell width $\xi$, and therefore the value of the numeric coefficient in the relation $t = r/4\gamma^2 c$, cannot be determined exactly.

It is nevertheless clear that the intensity peaks at $\nu' = \gamma(r)\nu(r)$ at a delay significantly larger than $t = r/16\gamma^2 c$. The results presented in Figure 1 were obtained under the assumption that at a fixed shell radius all emitted photons have the same frequency $\nu(r)$ (in the fireball frame). In fireball afterglow models the emission peaks at $\nu(r)$ and extends to high frequency $\nu \gg \nu(r)$ because of the power-law energy distribution of electrons, and to low-energy tail of synchrotron emission, $f(\nu)/f(\nu(r)) \sim [\nu/\nu(r)]^{\delta/2}$. These tails dominate the intensity at observed frequency $\nu' = \gamma(r)\nu(r)$ at delays much smaller and much larger than the time at which the intensity peaks at $\nu(r)$, $t \ll r/16\gamma^2 c$ and $t \gg r/16\gamma^2 c$. However, they are not important near the peak, $t \sim r/16\gamma^2 c$. The details of the frequency dependence of the emission at $\nu \sim \nu(r)$ will affect the behavior at $t \sim r/16\gamma^2 c$. Let us assume that photons are emitted at the shell frame at two different frequencies, $\nu(r)$ as given in equation (2) and $\bar{\nu}(r) = \nu(r)$. It is straightforward to show that the intensity $f_\nu(\nu', t)$ due to emission at $\bar{\nu}(r)$ is related to that of the intensity $f_\nu(\nu', t)$ due to emission at $\nu(r)$ by $f_\nu(\nu', t) \propto f_\nu(\nu', x^{2/3} t)$. Thus, if most of the energy is emitted in the shell frame over a frequency range $\delta\nu(r)/\nu(r) \sim 1$, the frequency spread would introduce a spread in arrival time of photons of frequency $\nu'$, $\Delta t_{\nu} \sim 1$, where $t_{\nu}$ is the time at which the intensity peaks under the assumption that all photons are emitted at the same frequency $\nu(r)$. From Figure 1, the spread in arrival time due to the relativistic expansion is $\Delta t_{\nu} \sim 2$. Thus, the effect of emission over a finite frequency range would not be significant, provided most of the energy is emitted over a frequency range $\delta\nu(r)/\nu(r) \lesssim 1$.

The flux observed at a fixed time originates from points at a range of distances transverse to the line of sight. Using the equations derived above we have numerically calculated the fraction of flux contributed from rings of radii $h - dh$ around the line of sight, as function of $h$. The results are shown in Figure 2 for several $\xi$-values. At time $t = r/2\gamma^2 c$ the flux originates from a ring of outer radius $h = r/\gamma_0$ and width $\Delta h \sim 0.1$. Note that this result is independent of the spectral distribution of the emission in the shell frame. The appearance of the fireball as a narrow ring can be qualitatively understood from Figure 3. Radiation produced at $r_0$ and observed at $t_{\gamma} = r_0/2\gamma^2 c$ originates from an angle $\sim 1/\gamma_0$, corresponding to $h = r_0/\gamma_0$. Contribution to the emission at smaller $h$ is due to emission from $r < r_0$ and $r > r_0$. However, the contribution from smaller radii is suppressed due to the fact that at smaller radii the radiation seen at a given time originates from larger angles, and the relativistic beaming suppresses the emission from angles greater than $1/\gamma$. The contribution from larger radii is small since the emissivity decreases as the fireball decelerates.
3. CONCLUSIONS

We have shown that fireball afterglow intensity peaks at observed frequency $\nu' = \gamma(r)\nu(r)$, where $\nu(r)$ is the frequency at which the emission peaks in the fireball frame at radius $r$, at time $t = r/4\gamma^2(r)c$. The exact value of the numerical coefficient depends on the effective thickness of the radiating shell, $\gamma(r)\gamma^2$. The value 1/4 is obtained for $\gamma \sim 1/10$, the value implied by fireball hydrodynamics (see Fig. 1). The numerical coefficient, 1/4, is somewhat smaller than the commonly used value, 1/2. However, it should be kept in mind that since the spatial dependence of the fractions of energy carried by electrons and magnetic field is not known, the effective shell width $\gamma$ cannot be determined exactly and the numerical coefficient may be somewhat larger or smaller than 1/4. For the normalization chosen in Figure 1, the amplitude of the peak intensity estimated following Waxman (1997a, 1997b) is 2/3, close to the numerical results obtained here. The agreement is due to the fact that most photons of frequency $\nu' = \gamma(r)\nu(r)$ arrive over a timescale $t = r/2\gamma^2(r)c$, as assumed in Waxman (1997a, 1997b).

At time $t$ the fireball appears on the sky as a narrow ring of radius $h = r/\gamma(r)$ and width $\Delta h/h \sim 0.1$, where $r$ and $t$ are related by $t = r/2\gamma^2(r)c$ (see Fig. 3). The apparent size $h = r/\gamma(r)$ is smaller by a factor of $(16/2)^{1/8} \sim 4$ compared to that obtained by using the relation $t = r/16\gamma^2(r)c$ (this follows from eq. [1]). The smaller size implies that diffractive scintillation is likely to modulate the afterglow radio flux (Goodman 1997), and that significant modification of the optical light curve due to microlensing is possible on day timescales (Loeb & Perna 1998). The narrowness of the emission ring would affect the predictions for both microlensing and scintillation.

The results presented here are valid for highly relativistic fireballs, $\gamma \gg 1$. For Lorentz factors $\gamma \sim 1$, the emission ring would be wider. Finally, we note that although we have implicitly assumed spherical symmetry throughout the paper, our results are valid for a fireball that is a cone of finite opening angle $\theta$ as long as $\gamma > 1/\theta$. This is due to the fact that most of the observed emission originates from a cone of opening angle $1/\gamma$.

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Fig. 2.—Fraction of flux observed at time $t = r/2\gamma^2(r)c$, which originates from different distances $h$ transverse to the line of sight, for various fireball shell widths $\gamma(r)\gamma^2(r)$.

Fig. 3.—Fraction of flux observed at time $t = r/2\gamma^2(r)c$ and produced over a radii range $dr$, $df/dr$, plotted as a function of $r$ with $h(r)$, the distance transverse to the line of sight from which radiation emitted at $r$ arrives at the observer at time $t$, for a shell of thickness $\gamma \sim 0$. 

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