Constant Delay Lattice Train Schedules

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Abstract

The following geometric vehicle scheduling problem has been considered: given continuous curves \( f_1, \ldots, f_n : \mathbb{R} \to \mathbb{R}^2 \), find non-negative delays \( t_1, \ldots, t_n \) minimizing \( \max\{t_1, \ldots, t_n\} \) such that, for every distinct \( i \) and \( j \) and every time \( t \), \( |f_j(t - t_j) - f_i(t - t_i)| > \ell \), where \( \ell \) is a given safety distance.

We study a variant of this problem where we consider trains (rods) of fixed length \( \ell \) that move at constant speed and sets of train lines (tracks), each of which consisting of an axis-parallel line-segment with endpoints in the integer lattice \( \mathbb{Z}^d \) and of a direction of movement (towards \( \infty \) or \( -\infty \)). We are interested in upper bounds on the maximum delay we need to introduce on any line to avoid collisions, but more specifically on universal upper bounds that apply no matter the set of train lines.

We show small universal constant upper bounds for \( d = 2 \) and any given \( \ell \) and also for \( d = 3 \) and \( \ell = 1 \). Through clique searching, we are also able to show that several of these upper bounds are tight.

1 Introduction

Ajay et al. [1] considered the following problem: given a set \( S = \{f_1, \ldots, f_n\} \) of curves, find a maximum subset \( S' \subseteq S \) such that, for any two distinct \( f_i, f_j \in S' \), there is no time \( 0 \leq t \leq 1 \) when \( f_i(t) = f_j(t) \). Though there are some approximation algorithms, it is shown that most versions of these problems are \( \text{NP-hard} \). This is true even when the curves are same-sized L-shapes and trajectories have the same constant speed. During a talk by Sasanka Roy on this problem at Stony Brook University, Joseph S. B. Mitchell and Esther M. Arkin posed the following variation: given \( S = \{f_1, \ldots, f_n\} \) where \( f_i : \mathbb{R} \to \mathbb{R}^2 \) is a continuous function, we wish to assign delays \( t_i \geq 0 \) so that, for any two distinct \( f_i, f_j \in S \), there is no time \( t \in \mathbb{R} \) when \( |f_j(t - t_j) - f_i(t - t_i)| \leq \ell \); we also wish that the maximum delay is minimized.

Automated guided vehicles (AGVs) and automated vehicle routing are currently some of the most prolific fields in the domain of motor vehicle technology, enjoying a large body of research addressing a large number of related algorithmic and optimization problems. The field of automated

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guiding vehicles has a long and rich history and is closely related to our work. For an elaborate review, the reader is referred to a survey by Vis [15], though [3] is a more recent one. A central topic in this field is collision avoidance. Kim and Tanchoco [5] propose an algorithm based on Dijkstra’s that facilitates conflict-free routing of AGVs. It runs in $O(v^4n^2)$ time, where $v$ is the number of vehicles and $n$ is the number of nodes, i.e., path intersections. Arora et al. [14] used techniques based on game theory to design a methodology for AGV traffic control. Yan et al. [16] studied collision-free routing of AGVs on both unidirectional and bidirectional paths using digraphs. For more information on conflict-free routing of AGVs, the reader is referred to [6, 9, 7]. For an elaborate survey on routing and scheduling algorithms for AGVs, see [8]. Other related work in this area includes the study of One Way Road Networks (OWRNs) by Ajaykumar et al. [4], its motivation [12] and [10, 2, 13].

1.1 New Results

In this paper, we study a simple vehicle scheduling problem in which, many times, we are guaranteed schedules with at most a constant delay. We consider trains of fixed length $\ell$ that move at constant speed and sets of train lines (tracks), each of which consisting of an axis-parallel line-segment with endpoints in the integer lattice $\mathbb{Z}^d$ and of a direction of movement (towards $\infty$ or $-\infty$). For an example see Figure 1. We are interested in upper bounds on the maximum delay we need to introduce on any line to avoid collisions, but more specifically on universal upper bounds that apply no matter the set of train lines.

We show small universal constant upper bounds for $d = 2$ and any given $\ell$ (Theorem 3.2) and also for $d = 3$ and $\ell = 1$ (Theorem 3.3). Through clique searching, we are also able to show that several of these upper bounds are tight (Section 4). In Appendix, we provide a Python code that...
we have used for clique searching, and the results are summarized in Table 1.

2 Preliminaries

Definition 2.1. A (d-dimensional) train line consists of:
- A track, which is a line segment in $\mathbb{R}^d$ with distinct endpoints distinguished between a departure point and an arrival point;
- A train length, which is a positive real number; and
- A speed, which is also a positive real number.

Furthermore, a set of train lines with non-overlapping (but possibly crossing) tracks is called a train network.

Definition 2.2. For a set of real numbers $X$ and a real number $y$, we denote $\{x+y : x \in X\}$ by $X+y$.

![Figure 2: An illustration of the journey of a train.](image)

Definition 2.3. A (collision-free) schedule for a train network is an assignment of a delay, which is a non-negative real number, to each of its lines. It also must result in no collisions between the trains. More precisely, for any two lines whose tracks cross, the open intervals $(0; \ell/v) + t + \delta/v$ and $(0; \ell'/v') + t' + \delta'/v'$ must not intersect, where:
- $t$ and $t'$ are the respective delays assigned to the lines;
- $\delta$ and $\delta'$ are the respective distances between the departure points of the lines and the crossing point;
- $\ell$ and $\ell'$ are the respective train lengths of the lines; and
- $v$ and $v'$ are the respective speeds of the lines.
The delay of the schedule is the maximum delay it assigns. An integer schedule is one that only assigns integer delays.

An alternative way to understand schedules is to imagine each line’s train starts “underground” with the front at the departure point and moves towards the arrival point at the line’s speed, where it moves “underground” again (“underground” trains cannot collide with other trains); see Figure 2. A delay assignment is then a schedule if, and only if, the line segments that correspond to the “above-ground” parts of these trains never cross. For an example, see Figure 3.

**Definition 2.4.** A \( d \)-dimensional train network is regular if all its lines’ speeds are 1, all its train lengths are the same and integer, all its tracks are axis-parallel and all its departure and arrival points are in \([0, 2] \times [0, 2] \times [0, 2] \times [0, 2] \times [0, 2]\).

![Figure 3](image)

Figure 3: If unit speed trains of length 2 depart simultaneously from the circles in (a), there will be collisions at the marked crossings. However, if delays are introduced as in (b), no collisions will occur.

**Theorem 2.5.** All regular train networks admit an integer schedule of minimum delay (among all schedules, integer or otherwise).

**Proof.** It is enough to show that, for an arbitrary schedule, assigning to each line a delay equal to the floor of the original delay will not result in any collisions. Indeed, take two lines whose tracks cross and define \( t, t', \delta, \delta', v, v', \ell, \ell' \) as in Definition 2.3. We thus have \( v = v' = 1 \) and \( \ell = \ell' \in \mathbb{N}\{0\} \). The original schedule ensures that

\[
((0; \ell) + t + \delta) \cap ((0; \ell) + t' + \delta') = \emptyset,
\]

which is equivalent to

\[
(0; \ell) \cap ((0; \ell) + t' - t + \delta' - \delta) = \emptyset.
\]

Because \( t' - t \) and \( |t'| - |t| \) differ by less than 1, \( |t'| - |t| \) is either \( |t' - t| \) or \( |t' - t| \). Therefore, since \( \ell \) and \( \delta' - \delta \) are integers, the open interval \( (0; \ell) + |t'| - |t| + \delta' - \delta \) cannot intersect \((0; \ell)\), which implies our delay assignment is collision-free.

**Definition 2.6.** Consider a train line with an axis-parallel track that departs from \( p \in \mathbb{R}^d \) and arrives at \( p + \alpha e_i \), where \( e_i \) is the \( i \)-th vector of the canonical basis of \( \mathbb{R}^d \) (\( 0 \leq i < d \)) and \( \alpha \in \mathbb{R}\{0\} \). The axis of the line is the number \( i \). Furthermore, we say this line is positive if \( \alpha > 0 \) and negative if \( \alpha < 0 \), with its sign being 1 or \(-1\), respectively.

**Definition 2.7.** For any modulus \( k \in \mathbb{N}\{0\} \), we extend the modulo \( k \) function to real numbers as follows: if \( x \in \mathbb{R} \), then \( x \mod k = (\lfloor x \rfloor \mod k) + x - \lfloor x \rfloor \). Furthermore, for a set \( X \subseteq \mathbb{R} \), we denote \( \{ x \mod k : x \in X \} \) by \( X \mod k \).
3 Results

Theorem 3.1. Any regular \(d\)-dimensional train network with trains of length \(\ell\) and only positive lines has a schedule with delay at most \(d\ell - 1\).

Proof. Assign delay \(\ell a + \sum_{i=0}^{d-1} p_i \mod(d\ell)\) to lines departing from \(p = (p_0, \ldots, p_{d-1}) \in \mathbb{Z}^d\) with axis \(a \in \{0, \ldots, d-1\}\). Consider then a crossing between a line departing from \(p = (p_0, \ldots, p_{d-1}) \in \mathbb{Z}^d\) with axis \(a\) and another line departing from \(p' = (p'_0, \ldots, p'_{d-1}) \in \mathbb{Z}^d\) with axis \(a' \neq a\). In the notation of Definition 2.3 we have:

- \(t = (\ell a + \sum_{i=0}^{d-1} p_i \mod(d\ell))\);
- \(t' = (\ell a' + \sum_{i=0}^{d-1} p'_i \mod(d\ell))\);
- \(\delta = p'_a - p_a\); and
- \(\delta' = p_{a'} - p'_{a'}\).

Thus, modulo \(d\ell\),

\[
t' - t + \delta' - \delta \equiv \ell(a' - a) + \sum_{i=0}^{d-1} (p'_i - p_i) - (p'_{a'} - p_a) - (p'_a - p_a)
\]

\[
\equiv \ell(a' - a) + \sum_{i=0}^{d-1} (p'_i - p_i)_{i \notin \{a, a'\}}
\]

\[
\equiv \ell(a' - a)
\]

since, for axes \(i \notin \{a, a'\}\), we have \(p_i = p'_i\) from the fact that the lines cross. However, there cannot be an intersection between \(((0; \ell) + t' - t + \delta' - \delta) \mod(d\ell)\) and \((0; \ell) \mod(d\ell) = (0; \ell)\) because \(a \neq a'\). Therefore, \((0; \ell) \cap ((0; \ell) + t' - t + \delta' - \delta) = \{\}\) and there are no collisions.

\[\square\]

Theorem 3.2. Any regular 2-dimensional train network with trains of length \(\ell\) has a schedule with delay at most \(M - 1\), where

\[
M = \begin{cases} 
2, & \ell = 1 \\
8, & \ell = 2 \\
6\ell, & \ell \geq 3.
\end{cases}
\]

Proof. Assign delay

\[
\sigma(x + y + (1 - a)(-2(y \mod \ell) - \ell + 1) + a(-2(x \mod \ell) + 2\ell - 1)) \mod M
\]

to lines departing from \((x, y)\) with axis \(a \in \{0, 1\}\) and sign \(\sigma \in \{-1, 1\}\). Consider then a horizontal line departing from \((x, y)\) with sign \(\sigma\) that crosses a vertical line departing from \((x', y')\) with sign \(\sigma'\). In the notation of Definition 2.3 we have:

- \(t = \sigma(x + y - 2(y \mod \ell) - \ell + 1)\);
\begin{itemize}
  \item \( t' = \sigma'(x' + y' - 2(x' \mod \ell) + 2\ell - 1); \)
  \item \( \delta = \sigma(x' - x); \) and
  \item \( \delta' = -\sigma'(y' - y). \)
\end{itemize}

Note that \( M \geq 2\ell \) and so it is enough to show that
\[
(t' - t + \delta' - \delta) \mod M \in \{\ell, \ldots, M - \ell\}.
\]
Indeed, then \((0; \ell) \mod M = (0; \ell)\) does not intersect \((0; \ell) + t' - t + \delta' - \delta) \mod M\), which implies \((0; \ell) \cap (0; \ell) + t' - t + \delta' - \delta) = \{\}, \) i.e., a lack of collisions. We prove this by cases starting when \( \sigma = \sigma' = 1 \), which gives us, modulo \( M \),
\[
t' - t + \delta' - \delta \equiv x' + y' - 2(x' \mod \ell) + 2\ell - 1 - y' + y - x - y + 2(y \mod \ell) + \ell - 1 - x' + x
\equiv 2(y \mod \ell) - 2(x' \mod \ell) + 3\ell - 2.
\]
Because \((y \mod \ell), (x' \mod \ell) \in \{0, \ldots, \ell - 1\}\), we have that
\[
2(y \mod \ell) - 2(x' \mod \ell) + 3\ell - 2 \in \{\ell, \ldots, 5\ell - 4\}
\subseteq \{\ell, \ldots, M - \ell\}.
\]

In case \( \sigma = -1 \) and \( \sigma' = 1 \), we have, modulo \( M \),
\[
t' - t + \delta' - \delta \equiv x' + y' - 2(x' \mod \ell) + 2\ell - 1 - y' + y + x + y - 2(y \mod \ell) - \ell + 1 + x' - x
\equiv 2(x' - (x' \mod \ell) + y - (y \mod \ell)) + \ell.
\]
Because \( x' - (x' \mod \ell) \) and \( y - (y \mod \ell) \) are multiples of \( \ell \), we must have that \( t' - t + \delta' - \delta \) is an odd multiple of \( \ell \). Moreover, since \( M \) is an even multiple of \( \ell \), \((t' - t + \delta' - \delta) \mod M \) must indeed be in \( \{\ell, \ldots, M - \ell\}\).

To complete the remaining two cases, note that if \( \sigma \) and \( \sigma' \) were simultaneously multiplied by \(-1\), so would be \( t' - t + \delta' - \delta \). However, \( \{\ell, \ldots, M - \ell\} \) is closed under negation modulo \( M \).

\textbf{Theorem 3.3.} Any regular 3-dimensional train network with trains of length 1 has a schedule with delay at most 5.

\textbf{Proof.} We delay each line by the only integer in \( \{0, \ldots, 5\}\) that is equivalent to \( \sigma(p_0 + p_1 + p_2 + a) \mod 3 \) and equivalent to \( p_0 + p_1 + p_2 + (\sigma + 1)/2 \mod 2 \), where \( p = (p_0, p_1, p_2) \in \mathbb{Z}^3 \) is its departure point, \( a \in \{0, 1, 2\} \) is its axis and \( \sigma \in \{-1, 1\} \) is its sign (the existence and uniqueness of this integer is assured by the Chinese Remainder Theorem).

Consider then two crossing lines that:
\begin{itemize}
  \item Depart respectively from \( p = (p_0, p_1, p_2) \in \mathbb{Z}^3 \) and \( p' = (p_0', p_1', p_2') \in \mathbb{Z}^3; \)
  \item Have respective signs \( \sigma, \sigma' \in \{-1, 1\}; \) and
  \item Have \( a \in \{0, 1, 2\} \) and \( a + 1 \in \{0, 1, 2\} \) as their respective axes (axis arithmetic is done modulo 3).
\end{itemize}
Thus, in the notation of Definition 2.3, we have $\delta = \sigma(p_a' - p_a)$ and $\delta' = -\sigma'(p_{a+1}' - p_{a+1})$. Modulo 2, we also have $t \equiv p_0 + p_1 + p_2 + (\sigma + 1)/2$ and $t' \equiv p_0' + p_1' + p_2' + (\sigma' + 1)/2$. Therefore, in case $\sigma \neq \sigma'$, also modulo 2,

$$t' - t + \delta' - \delta \equiv \sum_{i=0}^{2} (p_i' - p_i) + (\sigma' + 1)/2 - (\sigma + 1)/2 - \sigma'(p_{a+1}' - p_{a+1}) - \sigma(p_a' - p_a)$$

$$\equiv \sum_{i=0}^{2} (p_i' - p_i) - (p_{a+1}' - p_{a+1}) - (p_a' - p_a) + (\sigma' - \sigma)/2$$

$$\equiv p_a' + 2 - p_{a+2} + (\sigma' - \sigma)/2.$$  

However, since the lines cross, $p_{a+2} = p_a'$ and, since $\sigma, \sigma' \in \{-1, 1\}$ and $\sigma \neq \sigma'$, it must be that $(\sigma' - \sigma)/2 \equiv 1$ modulo 2, so $(t' - t + \delta' - \delta) \mod 2 = 1$. The collision avoidance condition for these lines is $(0; 1) \cap ((0; 1) + t' - t + \delta' - \delta) = \emptyset$ and must be true since $(0; 1) \mod 2 = (0; 1)$ but $((0; 1) + t' - t + \delta' - \delta) \mod 2 = (1; 2)$.

On the other hand, if $\sigma = \sigma'$, then note that, modulo 3,

$$t \equiv \sigma(p_0 + p_1 + p_2 + a)$$

and

$$t' \equiv \sigma(p_0' + p_1' + p_2' + a + 1).$$

Therefore, again modulo 3,

$$t' - t + \delta' - \delta \equiv \sigma \left( \sum_{i=0}^{2} (p_i' - p_i) + a + 1 - a \right) - \sigma(p_a' - p_a) - \sigma(p_{a+1}' - p_{a+1})$$

$$\equiv \sigma(p_a' + 2 - p_{a+2} + 1)$$

$$\equiv \sigma$$

as, once more, $p_{a+2} = p_a'$. As before, $(0; 1) \mod 3 = (0; 1)$ but

$$(0; 1) + (t' - t + \delta' - \delta) \mod 3 = (0; 1) + (\sigma \mod 3),$$

which is either $(1; 2)$ or $(2; 3)$. No collisions are therefore possible.

\[
\square
\]

4 Clique Searching and Lower Bounds

Whether a train network admits an integer schedule with delay at most $D \in \mathbb{N}$ can be decided with a clique search: create a graph $G$ containing a vertex $v_{L,t}$ for each line $L$ and time $t \in \{0, \ldots, D\}$; for every pair of distinct lines $L$ and $L'$ and for each pair of times $t, t' \in \{0, \ldots, D\}$ which would not result in a collision if assigned as delays respectively to $L$ and $L'$, put an edge between $v_{L,t}$ and $v_{L',t'}$. Note that if $L$ and $L'$ do not cross, then $v_{L,t}(v_{L',t'})$ is an edge for all $t, t' \in \{0, \ldots, D\}$. Note also that an integer schedule with delay at most $D$ exists if, and only if, $G$ has a clique with as many vertices as there are lines. This is because two vertices associated with the same line cannot be selected and we encoded potential collisions in the edges between vertices associated with different lines. So, essentially, the clique is a delay assignment.
Due to Theorem 2.5, we can decide whether a regular train network has a schedule with delay at most a given number $D \in \mathbb{R}$. We have implemented this algorithm with the Cliquer clique solver \cite{11} and recorded some lower bounds for regular train networks in Table 1.

For greater exposition we will walk through a simple example of determining the lower bounds for a train network with only positive lines and a train length of 2. Referring to Theorem 3.1 we see that a delay of $2\ell - 1 = 4 - 1 = 3$ is possible. Therefore we will test a train network using the configuration in Figure 4a and a delay of 2. If we construct a graph as outlined above and cannot find a clique of size 4, then our configuration does not admit a network with a delay of 2, and by Theorem 3.1 and Theorem 2.5 our lower bound on the delay must then be 3. Although not necessary to our lower bound proof, for completeness we also show a graph built using a delay of 3 and illustrate a clique of size 4, which represents an assignment of delays that would result in no collisions.

We will express our train network in the following format:\footnote{This input format is a more readable version of the input we used to run our Python script, which built the graph in Cliquer readable format, and for which we refer the reader to the appendix.}

\begin{verbatim}
<label> <train_len> <axis><direction> <x> <y> <z>
\end{verbatim}

Table 1: A report on some useful regular train network classes including: the networks’ dimension $d$, the length $\ell$ of their trains, whether their lines are all positive or of unrestricted sign, a lower bound on the delay of schedules for some networks in the class, our knowledge on whether this is an upper bound on the delay required to schedule all networks in the class and a reference to a figure describing a network that, when input to our algorithm, will produce the lower bound.
where:

- `<label>` is the line’s label;
- `<train_len>` is the line’s train length, a positive integer;
- `<axis>` is the axis the track is parallel to (“x”, “y”, or “z”);
- `<direction>` is the line’s direction of movement (“-” or “+”);
- `<x>` `<y>` `<z>` are the line’s departure point.

Our train network in the above format based on the configuration of Figure 4a and illustrated in Figure 5a is then:

A 2 x+ 0 1 0
B 2 x+ 0 2 0
C 2 y+ 1 0 0
D 2 y+ 2 0 0

We will refer to the above train network as Network 1. We create a vertex for each combination of line and possible delay, then for each pair of vertices consisting of distinct lines, we connect them with an edge if the delay assignment does not result in a collision. See Figure 5b. We may then search this graph for cliques of size 4, the number of train lines. To facilitate this search, we built our graph using a Python script, and fed the output into the Cliquer clique solver [11]. Figure 6 shows the graph resulting from Network 1 with a maximum delay of 3, and the resulting clique which gives a delay assignment that will not result in any collisions.

![Graph](attachment:image.png)

(a) Network 1.

(b) The resulting graph for Network 1 and a maximum delay of 2. The reader may verify there are no cliques of size 4.

Figure 5: Network 1 and its associated graph with delay 2.
Figure 6: The resulting graph for Network 1 and a maximum delay of 3. Note the clique of size 4 which gives a delay assignment of A:3, B:0, C:1 and D:2.

5 Open Problems

The main problem that remains open is whether, for values of $\ell \geq 2$, every three-dimensional regular train network with trains of length $\ell$ admits a schedule with delay bounded by a constant. However, we are also interested in extending the lower bounds in Table 1 to more values of $\ell$.

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Appendix

We share our implementation, in Python 3, of a program to convert a train network with a parameter $D \in \mathbb{N}$ into a graph that has a clique with as many vertices as the network has lines if, and only if, there is an integer schedule for the network with delay at most $D$. If the script were saved as, for example, trains.py, then a call to:

```
./trains.py <D>
```

where `<D>` is the parameter $D$ would initiate the script. Network 1 from Section 4 would then be input from the command line as:

```
2 x+ 0 1 0
2 x+ 0 2 0
2 y+ 1 0 0
2 y+ 2 0 0
```

The output from this script is meant to be piped into (or read from a file) the cliquer. We will give a brief overview of the format, but for more details we refer the reader to the Cliquer clique solver [11] technical report. The first line of output would be:

```
p graph <V> 0
```

The character `p` indicates to the cliquer that this is the format line. The `graph` entry indicates the format, but it is there for consistency with older formats and is ignored by the cliquer. The `<V>` field is the number of vertices in the output graph. The `0` field indicates the number of edges, but the cliquer ignores this number as well, and determines the number of edges from the input. Each edge of the resulting graph would then be output on a separate line consisting of:

```
e <v_1> <v_2>
```

where $v_1$ and $v_2$ are integers representing a vertex. The above output should be piped into the cliquer, which would output the largest clique found and the vertices contained therein. The line number (starting with line 0) and delay assignment from a vertex $v$ on a network with $\ell$ lines total could be obtained with the following formulas:

- Line number: $(v - 1) \text{ div } \ell$.
- Delay assignment: $(v - 1) \text{ mod } \ell$.

Our Python script, given below, operates from the command line but could be easily modified to read from and write to text files.

```python
#!/usr/bin/env python3

import sys
import itertools

# Standard input: one train line per input line, each in the form
# <train_len> <axis><dir> <x> <y> <z>
# where:
# * <train_len> is the line's train length, a positive integer;
```
# * <axis> is the axis the track is parallel to ("x", "y", or "z");
# * <dir> is the line's direction of movement ("-" or "+"); and
# * <x>, <y> and <z> are the line's departure point.
#
# Command line argument: a non-negative integer D.
#
# Output: a graph in a Cliquer-compatible format that has a clique of size
# equal to the number of train lines (or input lines) if, and only if, the
# lines form a network that has an integer schedule with delay at most D.

class TrainLine:
    # train_len: a positive integer, the line's train length
    # start: a 3d point, the line's departure point
    # axis: the number of the axis the track is parallel to (0, 1 or 2)
    # dir: the line's direction of movement (-1 or 1)
    def __init__(l, train_len, start, axis, dir):
        l.train_len = train_len
        l.start = start
        l.axis = axis
        l.dir = dir

        # l[i] is the departure point's i-th coordinate (i is 0, 1 or 2)
    def __getitem__(l, i):
        return l.start[i]

    # Whether two line's tracks overlap
    def overlaps(l0, l1):
        if l0.axis != l1.axis:
            return False
        a = l0.axis
        if l0[(a+1)%3] != l1[(a+1)%3] or l0[(a+2)%3] != l1[(a+2)%3]:
            return False
        return l0.dir == l1.dir or l0[a]*l0.dir < l1[a]*l1.dir

    # Input: two train lines with non-overlapping tracks.
    # Output: None if the line's tracks do not cross and a pair with the
    # respective distances from the line's departure points to the
    # crossing point otherwise.
    def distances_to_crossing(l0, l1):
        other_axis = 0
        while other_axis == l0.axis or other_axis == l1.axis:
            other_axis+= 1
            if l0[other_axis] != l1[other_axis]:
                return None
            if l0[l1.axis]*l1.dir <= l1[l1.axis]*l1.dir:
                return None
            if l1[l0.axis]*l0.dir <= l0[l0.axis]*l0.dir:
                return None
        return (l1[l0.axis]-l0[l0.axis])*l0.dir,
(10[l1.axis]-l1[l1.axis])*l1.dir
)

# Reads a train line from a text line
@staticmethod
def read(text_line):
tokens = text_line.split()
train_len = int(tokens[0])
axis = {"x": 0, "y": 1, "z": 2}[tokens[1][0]]
dir = {"-": -1, "+": 1}[tokens[1][1]]
start = tuple(map(int, tokens[2:5]))
return TrainLine(train_len, start, axis, dir)

# Whether two open intervals intersect
def intervals_overlap(I0, I1):
    (a0, b0) = I0
    (a1, b1) = I1
    return a0 < b1 and a1 < b0

# Command line argument
D = int(sys.argv[1])

# Read standard input into a list of train lines
L = [TrainLine.read(input_line) for input_line in sys.stdin.readlines()]

# Check input tracks are non-overlapping
for l0, l1 in itertools.combinations(L, 2):
    if l0.overlaps(l1):
        raise Exception("Input contains overlapping tracks")

# Assign vertices to lines
for i in range(len(L)):
    L[i].vertices = range((D+1)*i, (D+1)*(i+1))

# Print the graph
print("p graph %d 0" % ((D+1)*len(L)))
for l0, l1 in itertools.combinations(L, 2):
    ds = l0.distances_to_crossing(l1)
    if ds is not None:
        d0, d1 = ds
        for t0, t1 in itertools.product(range(D+1), repeat=2):
            if ds is None or not intervals_overlap(
                (t0 + d0, t0 + d0 + l0.train_len),
                (t1 + d1, t1 + d1 + l1.train_len)
            )�:
                print("e %d %d" % (l0.vertices[t0]+1, l1.vertices[t1]+1))