Nonlinear Saturation of the Weibel Instability in a Dense Fermi Plasma

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We present an investigation for the generation of intense magnetic fields in dense plasmas with an anisotropic electron Fermi-Dirac distribution. For this purpose, we use a new linear dispersion relation for transverse waves in the Wigner-Maxwell dense quantum plasma system. Numerical analysis of the dispersion relation reveals the scaling of the growth rate as a function of the Fermi energy and the temperature anisotropy. The nonlinear saturation level of the magnetic fields is found through fully kinetic simulations, which indicates that the final amplitudes of the magnetic fields are proportional to the linear growth rate of the instability. The present results are important for understanding the origin of intense magnetic fields in dense Fermionic plasmas, such as those in the next generation intense laser-solid density plasma experiments.

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I. INTRODUCTION

The existence of feeble magnetic fields of several microgauss in our galaxies [1], as well as of gigagauss in intense laser-plasma interaction experiments [2] and of billions of gauss in compact astrophysical objects [3] (e.g. super dense white dwarfs, neutron stars/magnetars, degenerate stars, supernovae) is well known. The generation mechanisms for seed magnetic fields in cosmic/astrophysical environments are still debated, while the spontaneous generation of magnetic fields in laser-produced plasmas is attributed to the Biermann battery [4] (also referred to as the baroclinic vector containing non-parallel electron density and electron temperature gradients) and to the return electron current from the solid target. Computer simulations of laser-fusion plasmas have shown evidence of localized anisotropic electron heating by resonant absorption, which in turn can drive a Weibel-like instability resulting in megagauss magnetic fields [5]. There have also been observations of the Weibel instability in high intensity laser-solid interaction experiments [6]. Furthermore, a purely growing Weibel instability [7], arising from the electron temperature anisotropy (a bi-Maxwellian electron distribution function) is also capable of generating magnetic fields and associated shocks [8].

However, plasmas in the next generation intense laser-solid density plasma experiments [9] would be very dense. Here the equilibrium electron distribution function may assume the form of a deformed Fermi-Dirac distribution due to the electron heating by intense laser beams. It then turns out that in such dense Fermi plasmas, quantum mechanical effects (e.g. the electron tunneling and wave-packet spreading) would play a significant role [10]. The importance of quantum mechanical effects at nanometer scales has been recognized in the context of quantum diodes [11] and ultra-small semiconductor devices [12]. Also, recently there have been several developments on fermionic quantum plasmas, involving the addition of a dynamical spin force [13, 14, 15, 16], turbulence or coherent structures in degenerate Fermi systems [17, 18], as well as the coupling between nonlinear Langmuir waves and electron holes in quantum plasmas [19]. The quantum Weibel or filamentational instability for non-degenerate systems has been treated in [20, 21].

In this work, we present an investigation of linear and nonlinear aspects of a novel instability that is driven by equilibrium Fermi-Dirac electron temperature anisotropic distribution function in a nonrelativistic dense Fermi plasma. Specifically, we show that the free energy stored in electron temperature anisotropy is coupled to purely growing electromagnetic modes. First, we take the Wigner-Maxwell system [22] with an anisotropic Fermi-Dirac distribution for the analysis of the linearly growing electromagnetic perturbations as a function of the physical parameters. Second, we use a fully kinetic simulation to assess the saturation level of the magnetic fields as a function of the growth rate. The treatment is restricted to transverse waves, since the latter are associated with the largest Weibel instability growth rates. The nonlinear saturation of the Weibel instability for classical, non-degenerate plasmas has been considered elsewhere [23].

II. BASIC EQUATIONS

It is well known [24] that a dense Fermi plasma with isotropic equilibrium distributions does not admit any purely growing linear modes. This can be verified, for instance, from the expression for the imaginary part of the transverse dielectric function, as derived by Lindhard [25], for a fully degenerate non-relativistic Fermi plasma. It can be proven (see Eq. (30) of [26]) that the only exception would be for extremely small wavelengths, so that $k > 2k_F$, where $k$ is the wave number and $k_F$ the...
characteristic Fermi wave number of the system. However, in this situation the wave would be super-luminal. On the other hand, in a classical Vlasov-Maxwell plasma containing anisotropic electron distribution function, we have a purely growing Weibel instability via which electron temperature anisotropy may develop due to an anisotropic electron heating by intense laser beams [6], where there is a signature of the Weibel instability as well. In the next generation intense laser-solid density plasma experiments, it is likely that the electrons would be degenerate and that electron temperature anisotropy arises due to the heating of the plasma by laser beams [6], where there is a signature of the Weibel instability [7], via which anisotropic electron distribution function that is appropriate for the Fermi plasma.

Proceeding with the time evolution equation for the distribution widely used in the random phase approximation [23], we shall derive a modified dispersion relation accounting for transverse waves of the Wigner-Maxwell system [21, 29], one then obtains the general dispersion relation [21, 29] for the transverse waves of the Wigner-Maxwell system

\[
\omega^2 - c^2 k^2 - \omega_p^2 + \frac{m_0 c^2}{2 n_0 \hbar} \int d\mathbf{v} \left( \frac{v_x^2 + v_y^2}{\omega - kv_z} \right) \times \left( f_0(v_x, v_y, v_z + \frac{\hbar k}{2m}) - f_0(v_x, v_y, v_z - \frac{\hbar k}{2m}) \right) = 0,
\]

where \( \omega \) is the frequency, \( c \) is the speed of light in vacuum, \( \hbar \) is the Planck constant divided by \( 2\pi \), \( m \) is the rest electron mass, \( n_0 \) is the unperturbed plasma number density, \( \omega_p \) is the plasma frequency, \( \mathbf{v} = (v_x, v_y, v_z) \) is the velocity vector, and \( f_0(v_x, v_y, v_z) \) is the equilibrium Wigner function associated to Fermi systems.

For spin 1/2 particles, the equilibrium pseudo distribution function is in the form of a Fermi-Dirac function. Here we allow for velocity anisotropy and express

\[
f_0 = \frac{\alpha}{\exp \left( \frac{v_x^2 + v_y^2}{\kappa_B T_\perp} + \frac{v_z^2}{\kappa_B T_\parallel} - \beta \mu \right) + 1},
\]

where \( \mu \) is the chemical potential, \( \kappa_B \) the Boltzmann constant, and the normalization constant is

\[
\alpha = -\frac{n_0}{L_3^{3/2}(-e^{\beta \mu})} \left( \frac{m \beta}{2\pi} \right)^{3/2} = 2 \left( \frac{m}{2\pi \hbar} \right)^3.
\]

Here \( L_3^{3/2} \) is a polylogarithm function [31]. Also, \( \beta = 1/|\kappa_B(T_e^2T_i^1)^{1/3}] \), where \( T_\perp \) and \( T_i \) are related to velocity dispersion in the direction perpendicular and parallel to the axis, respectively. In the special case when \( T_\perp = T_i \), the usual Fermi-Dirac equilibrium is recovered. The chemical potential is determined by solving the normalization condition \( f_0(v) = 1/\exp[-(v - H/2)^2 + \beta \mu] \), yielding, in particular, \( \mu = E_F \) in the limit of zero temperature, where \( E_F = (3\pi^2 n_0)^{2/3} \hbar^2/(2m) \) is the Fermi energy. Also, the Fermi-Dirac distribution \( f_0(v) \), where \( \mathbf{K} \) is the appropriate wave vector in momentum space, is related to the equilibrium Wigner function \( f_0(v) \) by \( f_0(v) = (1/2\pi \hbar/m)^3 f_0(v) \), with the factor 2 coming from spin [33]. However, these previous works refer to the cases where there is no temperature anisotropy. Notice that it has been suggested [34] that in laser plasmas the Fermi instability is responsible for further increase of \( T_i \) with time.

Inserting (2) into (1) and integrating over the perpendicular velocity components, we obtain

\[
\omega^2 - c^2 k^2 - \omega_p^2 \left(1 + \frac{T_i}{T_\perp} W_Q\right) = 0,
\]

where

\[
W_Q = \frac{1}{2\sqrt{\pi} H L_3^{3/2}(-e^{\beta \mu})} \int d\nu \frac{\nu - \xi}{\nu - \xi} \times \left[ -\text{Li}_2 \left( -\exp \left[-(\nu - H/2)^2 + \beta \mu \right] \right) \right] \text{Li}_2 \left( -\exp \left[-(\nu - H/2)^2 + \beta \mu \right] \right) = 0.
\]

In [4], \( \text{Li}_2 \) is the dilogarithm function [30, 31]. \( H = \hbar k/(mv_{\parallel}) \) is a characteristic parameter representing the quantum diffusion effect, \( \xi = \omega/(kv_{\parallel}) \), and \( \nu = \nu_x/v_{\parallel} \), with \( v_x = (2\kappa_B T_i/m)^{1/2} \). In the simultaneous limit of a small quantum diffusion effect \( (H \ll 1) \) and a dilute system \( (e^{\beta \mu} \ll 1) \), it can be shown that \( W_Q \approx -1 - \xi Z(\xi) \), where \( Z \) is the standard plasma dispersion function [33]. It is important to notice that either \( \omega < 0 \) or \( \omega > 0 \) reproduces the transverse dielectric function calculated from the random phase approximation for a fully degenerate quantum plasma [23], in the case of an isotropic system. The simple way to verify this equivalence is to put \( T_i = T_\parallel \) in [1] and then take the limit of zero temperature, so that \( f_0 = 3n_0/(4\pi v_F^3) \) for \( |\mathbf{v}| < v_F \), and \( f_0 = 0 \) otherwise, where \( v_F = (2E_F/m)^{1/2} \) is the Fermi velocity. However, to the best of our knowledge, there is no corresponding calculation for an anisotropic Fermi equilibrium, as necessary in laser-solid interaction experiments with an anisotropic electron heating due to resonant absorption. Also notice that in this Letter we
are mainly interested in the real part of the transverse response function, since we are looking for purely growing instabilities ($\omega^2 < 0$), so that the contribution from the poles at $i\omega_\perp$ is not relevant.

III. NUMERICAL RESULTS

![Graph showing the growth rate for the Weibel instability as a function of $k c / \omega_p$.](image)

**FIG. 1:** The growth rate for the Weibel instability of a dense Fermionic plasma with $n_0 = 10^{23}$ m$^{-3}$ ($\omega_p = 1.8 \times 10^{18}$ s$^{-1}$) and $\beta \mu = 5$, relevant for the next generation inertially compressed material in intense laser-solid density plasma interaction experiments. The temperature anisotropies are $T_\perp / T_\parallel = 3$ (dashed line), $T_\perp / T_\parallel = 2$ (solid line) and $T_\perp / T_\parallel = 1.5$ (dotted line), yielding, respectively, $T_\parallel = 3.9 \times 10^6$ K, $T_\parallel = 5.2 \times 10^6$ K and $T_\parallel = 6.3 \times 10^6$ K.

![Graph showing the growth rate for the Weibel instability as a function of $k c / \omega_p$.](image)

**FIG. 2:** The growth rate for the Weibel instability of a dense Fermionic plasma with $n_0 = 10^{23}$ m$^{-3}$ ($\omega_p = 1.8 \times 10^{18}$ s$^{-1}$). Here the temperature anisotropy is $T_\perp / T_\parallel = 2$. We used $\beta \mu = 1$ (dashed line), $\beta \mu = 5$ (solid line) and $\beta \mu = 10$ (dotted line), yielding $T_\parallel = 1.6 \times 10^7$ K, $T_\parallel = 5.2 \times 10^7$ K and $T_\parallel = 2.6 \times 10^8$ K, respectively.

We next solve our new dispersion relation (4) for a set of parameters that are representative of the next generation laser-solid density plasma interaction experiments. The normalization condition (3) can also be written as $-\text{Li}_{3/2}[-\exp(\beta \mu)] = (4/3 \sqrt{\pi})(\beta \varepsilon_F)^{3/2}$, which is formally the same relation holding for isotropic Fermi-Dirac equilibria [36]. For a given value on the product $\beta \mu$ and the density, this relation yields the value $\beta$, from which the temperatures $T_\parallel$ and $T_\parallel$ can be calculated, if we know $T_\perp / T_\parallel$. Consider only purely growing modes. From the definition (4), one can show that $\omega_\perp \rightarrow -1$ when $\omega = i\gamma \rightarrow 0$ for a finite wavenumber $k$. From (4) we then obtain the maximum wavenumber for instability as $k_{\text{max}} = (\omega_p / \gamma) \sqrt{T_\perp / T_\parallel - 1}$. When $T_\perp / T_\parallel \rightarrow 1$, the range of unstable wavenumbers shrinks to zero. In Figs. 1 and 2, we have used the electron number density $n_0 = 10^{23}$ m$^{-3}$, which can be obtained in laser-driven compression schemes. The growth rate for different values on $T_\perp / T_\parallel$ is displayed in Fig. 1. We see that the maximum unstable wavenumber is $k_{\text{max}} = (\omega_p / \gamma) \sqrt{T_\perp / T_\parallel - 1}$, as predicted, and that the maximum growth rate occurs at $k \approx k_{\text{max}} / 2$. Figure 1 also reveals that the maximum growth rate of the instability is almost linearly proportional to $T_\perp / T_\parallel - 1$. In Fig. 2, we have varied the product $\beta \mu$, which is a measure of the degeneracy of the quantum plasma. We see that for $\beta \mu$ larger than 5, the instability reaches a limiting value, which is independent of the temperature, while thermal effects start to play an important role for $\beta \mu$ of the order unity.

![Image showing the magnetic field components $B_\parallel$ (top panel) and $B_\perp$ (bottom panel) as a function of space and time, for $\beta \mu = 5$ and $T_\perp / T_\parallel = 2$. The magnetic field has been normalized by $\omega_p m / e$. We see a nonlinear saturation of the magnetic field components at an amplitude of $\sim 0.01$.](image)

**FIG. 3:** The magnetic field components $B_\parallel$ (top panel) and $B_\perp$ (bottom panel) as a function of space and time, for $\beta \mu = 5$ and $T_\perp / T_\parallel = 2$. The magnetic field has been normalized by $\omega_p m / e$. We see a nonlinear saturation of the magnetic field components at an amplitude of $\sim 0.01$.

From several numerical solutions of the linear dispersion relation, we have been able to deduce an approximate scaling law for the instability as $\gamma_{\text{max}} / \omega_p = \text{constant} \times n_0^{1/3} (T_\perp / T_\parallel - 1)$, where the constant is approximately $8.5 \times 10^{-14}$ m$^{-1}$. Using that $n_0 = (2m \varepsilon_F / h^2)^{3/2} / (3\pi^2) \approx 1.67 \times 10^{36} (\varepsilon_F / mc^2)^{3/2}$, we have

$$\gamma_{\text{max}} / \omega_p = 0.10 \left( \varepsilon_F / mc^2 \right)^{1/2} \left( T_\perp / T_\parallel - 1 \right),$$

for the maximum growth rate of the Weibel instability in a degenerate Fermi plasma. This scaling law, where the growth rate depends on the Fermi energy and the temperature anisotropy, should be compared to that of a classical plasma [36, 37], where the growth rate depends on the thermal energy and the temperature anisotropy.

For a Maxwellian plasma, it has been found [36] that the Weibel instability saturates nonlinearly once the magnetic bounce frequency $\omega_c = eB / m$ has increased to
3 shows the magnetic field components as a function of space and time. We see that the magnetic field initially grows, and saturates to steady state magnetic field fluctuations with an amplitude of $eB/m\omega_p \approx 0.008$. The maximum amplitude of the magnetic field over the simulation box as a function of time is shown in Fig. 4, where we see that the magnetic field saturates at $eB/m\omega_p \approx 0.0082$, while the linear growth rate of the most unstable mode is $\gamma_{\text{max}}/\omega_p \approx 0.009$. Similar to the classical Maxwellian plasma case [38], we can thus estimate the magnetic field (in Tesla) as

$$B = \frac{m\gamma_{\text{max}}}{e},$$

for a degenerate Fermi plasma. For our parameters relevant for intense laser-solid interaction experiments, we will thus have magnetic fields of the order $10^9$ Tesla (one gigagauss).

**IV. CONCLUSION**

In conclusion, we have demonstrated the existence of the Weibel instability for a Wigner-Maxwell dense quantum plasma, taking into account an anisotropic Fermi-Dirac equilibrium distribution function and the quantum diffraction effect. Numerically solving the dispersion relation for transverse waves, we found the dependence of the growth rate on the Fermi energy and the temperature anisotropy. The nonlinear saturation level of the magnetic field was found by means of kinetic simulations, which show a linear dependence between the growth rate and the saturated magnetic field. The present results may account for intense magnetic fields in dense quantum plasmas, such as those in the next generation of intense laser-solid density plasma interaction experiments.

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