The Theory of Pure Algebraic (Co)Homology

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Abstract  Polynomial: algebra is essential in commutative algebra since it can serve as a fundamental model for differentiation. For module differentials and Loday’s differential commutative graded algebra, simplified homology for polynomial algebra was defined. In this article, the definitions of the simplicial, the cyclic, and the dihedral homology of pure algebra are presented. The definition of the simplicial and the cyclic homology is presented in the Algebra of Polynomials and Laurent’s Polynomials. The long exact sequence of both cyclic homology and simplicial homology is presented. The Morita invariance property of cyclic homology was submitted. The relationship

\[ \mathcal{H}^\mathcal{C} n(P) \cong \mathcal{H}_\mathcal{D}^\alpha n(P) \oplus \mathcal{H}_\mathcal{D}^{-\alpha} n(P), \alpha = \pm 1 \]

was introduced, representing the relationship between dihedral and cyclic (co)homology in polynomial algebra. Besides, a relationship

\[ \mathcal{H}^\mathcal{C} n(P[p, p^{-1}]) \cong \mathcal{H}_\mathcal{D}^\alpha n(P[p,p^{-1}]) \oplus \mathcal{H}_\mathcal{D}^{-\alpha} n(P[p,p^{-1}]), \alpha = \pm 1 \]

was examined, defining the relationship between dihedral and cyclic (co)homology in polynomial algebra. Furthermore, the Morita invariance property of dihedral homology in polynomial algebra was investigated. Also, the Morita property of dihedral homology in Laurent polynomials was studied. For the dihedral homology, the long exact sequence

\[ \cdots \rightarrow \mathcal{H}_\mathcal{D} n(P) \rightarrow \mathcal{H}_\mathcal{D} n(P') \rightarrow \mathcal{H}_\mathcal{D} n(P'') \rightarrow \mathcal{H}_\mathcal{D} n-1(P) \rightarrow \mathcal{H}_\mathcal{D} n-1(P') \rightarrow \mathcal{H}_\mathcal{D} n-1(P'') \rightarrow \cdots \]

was obtained of the short sequence

\[ 0 \rightarrow P \rightarrow P' \rightarrow P'' \rightarrow 0. \]

The long exact sequence of the short sequence

\[ 0 \rightarrow P \rightarrow P' \rightarrow P'' \rightarrow 0 \]

was obtained from the reflexive (co)homology of polynomial algebra. Studying polynomial algebra helps calculate COVID-19 vaccines.

Keywords  Homology Theory, Pure Algebras, Exact Sequence, Polynomial Algebra, Dihedral Homology

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1. Introduction

The homology theory of algebras over a field refers to the Hochschild (co)homology in the mathematical field. Hochschild introduced simplicial cohomology for algebras [1], and Henri and Samuel expanded it over rings in [2].

Certain (co)homology theories for associative algebras in disciplines of mathematics and non-commutative geometry that generalise de Rham homology and cohomology of manifolds are referred to as cyclic (co)homology. Boris Tsygan [3] and Alain Connes [4] pioneered the concepts of homology and cohomology independently. Many older branches of mathematics, such as de Rham’s speculation, simplicial (co)homology, group (co)homology, and K -theory, have intriguing relationships with these invariants.

The dihedral (co)homology, proposed independently and demonstrated in various algebras, is the hermitian equivalent of a cyclic (co)homology [4]. The dihedral homology of algebras over the field is defined as the algebraic homology of the dihedral group [5]. The dihedral (co)homology is denoted as (co)homology with group symmetry [6]. There are two types of (co)homology theory: in-discrete and discrete. The Hochschild mentioned in reference [1] is related to Hochschild in the discrete field's (co)homology of algebra with id. The first nontrivial (co)homology group was introduced by Tsygan [3] and Connes [4]. In 1987, the involutive unital
algebra's reflexive and dihedral (co)homology was investigated, and the remaining (co)homology group was studied in 1989. The analogue simplified cohomology of operator algebras was studied [7]. The Banach cyclic (co)homology has been explored [8] [9] [10]. The group Banach dihedral cohomology and its relation to cyclic cohomology were investigated [12]. The dihedral cohomology groups of certain operator algebras were investigated [13]. Calculating operator algebras' group symmetry, bisymmetry, and Weil (co)homology have his proof of the Zeta function investigated [13].

The dihedral Banach dihedral cohomology and its relation to cyclic homology were investigated [12]. The dihedral cohomology groups of certain operator algebras were studied [13]. The Banach cyclic homology will be presented. In the second part, the exact sequences of simplicial homology and cyclic homology in polynomial algebra and Laurent polynomials will be obtained that is not analysable since the polynomial's infinite algebras of polynomial growth were defined [11].

In 1985 and 1987, cyclic homology of algebra was determined when the characteristic is 0. In 1991, if \( F \) is polynomial and \( K \) is a ring with unital, the cyclic homology of algebra \( A = K[\{X\}]/(F) \) is calculated. The dimensions of simplicial (co)homology of periodic infinite algebras of polynomial growth were defined [11]. For applying this, a nonstandard periodic representation was obtained that is not analysable since the polynomial's infinite algebra does not derive from scalar algebra.

The first section will outline the definitions of Hochschild homology and cyclic homology. The long exact sequences of simplicial homology and cyclic homology will be presented. In the second part, the algebraic definitions of polynomials and Laurentian polynomials will be reviewed. The simplicial and cyclic homology will be identified first, followed by a discussion of their len exact sequences. In part three, the dihedral homology of polynomial algebra will be examined, discussing the long exact between cyclic, dihedral homology in polynomial algebra and Laurent polynomials algebra. The Morita of dihedral homology property and Morita's invariance for dihedral homology will be examined in polynomial algebra and Laurent polynomials algebra, implying that the trace map is inverse to the inclusion map.

The following section contains definitions for the homology theorem. A Hochschild homology description of pure algebra will be presented. The concept of cyclic homology will be expanded, defining both reflexive and dihedral algebraic homology.

2. Homology Theory of Algebras

The cyclic and dihedral homologies for pure algebra were introduced to define Hochschild homology. Let \( \mathcal{A} \) be a pure algebra over \( K \) and \( \mathcal{M} \) bimodule with involution \( * : \mathcal{A} \to \mathcal{A}; a \to a^* \forall a \in \mathcal{A} \). First, the Hochschild homology was determined by defining the chain complex

\[
\mathcal{C}(\mathcal{A}) := \cdots \to \mathcal{A}_0 \xrightarrow{d_0} \mathcal{A}_1 \xrightarrow{d_1} \mathcal{A}_2 \to \cdots
\]

using the operator

\[
d_n(a_0, a_1, \ldots, a_n) = \\
\sum_{i=0}^{n} (-1)^i (a_0, \ldots, a_i, a_{i+1}, \ldots, a_n) + \\
(-1)^n (a_n, a_0, a_1, \ldots, a_{n-1}),
\]

where \( d^2 = d_n d_{n+1} = 0 \).

If \( Z(C_n(\mathcal{A})) = \ker(d_n), B(C_n(\mathcal{A})) = \text{im}(d_{n+1}) \) and \( \text{im}(d_{n+1}) \subseteq \ker(d_n) \), then the (co)homology of the upper complex is

\[
H_n(C_n(\mathcal{A})) = \frac{Z(C_n(\mathcal{A}))}{B(C_n(\mathcal{A}))} = \frac{\ker(d_n)}{\text{im}(d_{n+1})}.
\]

It is defined as \( \mathcal{A} \)-simplicial algebra's homology and is denoted by \( \mathcal{H}H_n(\mathcal{A}) \). Another definition of Hochschild (co)homology was reported [16]. If \( \mathcal{A} \) is a tensor product \( \mathcal{A}^e = \mathcal{A} \otimes \mathcal{A}^{op} \), it can be defined by \((\text{Tor}^n)\) and \((\text{Ext}^n)\) as following:

\[
\mathcal{H}H_n(\mathcal{A}, \mathcal{M}) = \text{Tor}^n(\mathcal{A}, \mathcal{M}), \\
\mathcal{H}H^n(\mathcal{A}, \mathcal{M}) = \text{Ext}^n(\mathcal{A}, \mathcal{M}).
\]

Before defining periodic homology, the cyclic operator \( t_n : \mathcal{C}_n(\mathcal{A}) \to \mathcal{C}_n(\mathcal{A}) \) must be defined, where

\[
t_n(a_0, \ldots, a_{n-1}, a_n) = (-1)^n (a_n, a_0, \ldots, a_{n-1}). \tag{1}
\]

If the next complex called the subcomplex of \( \mathcal{C}_n(\mathcal{A}) \),

\[
\begin{array}{cccccccccc}
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
C_2 & \xleftarrow{1-t} & C_2 & \xrightarrow{N} & C_2 & \xleftarrow{1-t} & C_2 & \xrightarrow{N} & \cdots \\
\end{array}
\]

\( \mathcal{C}_n(\mathcal{A}) \): \( b \uparrow \leftarrow -b' \downarrow \)

\[
\begin{array}{cccccccccc}
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
C_1 & \xleftarrow{1-t} & C_1 & \xrightarrow{N} & C_1 & \xleftarrow{1-t} & C_1 & \xrightarrow{N} & \cdots \\
\end{array}
\]

\( \mathcal{C}_n(\mathcal{A}) \): \( b \uparrow \leftarrow -b' \downarrow \)

\[
\begin{array}{cccccccccc}
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
C_0 & \xleftarrow{1-t} & C_0 & \xrightarrow{N} & C_0 & \xleftarrow{1-t} & C_0 & \xrightarrow{N} & \cdots \\
\end{array}
\]

where \( b = \sum_{i=0}^{n} (-1)^i d_i, b' = \sum_{i=0}^{n} (-1)^i d_i \) and \( N := 1 + t + \cdots + t^n \), then its homology called cyclic homology and given as follows ([17], [18]):

\[
\mathcal{H}C_n(\mathcal{A}, \mathcal{M}) = \ker(d_n) / \text{im}(d_{n+1} - d_n).
\]
Homology of Polynomial Algebras

Define $\mathcal{P}$ as algebra and $[\mathcal{P}]$ as class of polynomials;

$$\mathcal{K}[\mathcal{P}] = \sum_{i=0}^{n} a_i p^i = (a_0, a_1 p, a_2 p^2, ..., a_n p^n),$$

where $\mathcal{K}[\mathcal{P}]$ is a polynomial algebra with the multiplication of coefficient of polynomials. A Laurent polynomial in a field $\mathcal{K}[p, p^{-1}]$ with coefficients is defined as follows:

$$\mathcal{P} = \sum_i a_i p^i, \forall a_i \in \mathcal{K}[p, p^{-1}].$$

If the coefficients of two Laurent polynomials are the same, they are equal, achieving

$$\left(\sum_i a_i x^i\right) + \left(\sum_i b_i x^i\right) = \sum_i (a_i + b_i) x^i,$$

and

$$\left(\sum_i a_i x^i\right) \cdot \left(\sum_i b_i x^i\right) = \sum_i (\sum_{i+j=n} a_i b_j) x^n.$$

Suppose that $\mathcal{P}[p]$ is a polynomial algebra over the $\mathcal{K}$ ring with an involution $*: \mathcal{P} \to \mathcal{P}$; $p \mapsto p^*$ for all $p \in \mathcal{P}$ [12]. We define a complex $(\mathcal{C}(\mathcal{P}[p])) = (\mathcal{C}_n(\mathcal{P}), b_n)$, since $\mathcal{C}_n(\mathcal{P}) = p \oplus (n+1)$, $b_n; \mathcal{C}_n(\mathcal{P}) \to \mathcal{C}_{n-1}(\mathcal{P}), n \geq 0$ is the boundary operator

$$b_n(p_0, p_1, ..., p_n) = \sum_{i=0}^{n-1} (-1)^i (p_0, ..., p_i, p_{i+1}, ..., p_n)$$

$$+ (-1)^n(p_0 p_0, p_1, ..., p_{n-1}).$$

It is well known that $b_n b_{n+1} = 0$. The simplicial homology of algebra $\mathcal{P}$, denoted by $\mathcal{H}_n(\mathcal{P})$, is the homology of this complex;

$$\mathcal{H}_n(\mathcal{P}[p]) = \mathcal{H}_n(\mathcal{C}(\mathcal{P}[p])) = \mathcal{H}_n(\mathcal{P}[p]) \oplus \mathcal{H}_n(\mathcal{K}[p]).$$

Since $\mathcal{P}[p, p^{-1}] = \mathcal{K}[p, p^{-1}]$ is Laurent polynomial algebra over the filed $\mathcal{K}[p, p^{-1}]$, the simplicial homology of $\mathcal{P}[p, p^{-1}]$ algebra is

$$\mathcal{H}_n(\mathcal{P}[p, p^{-1}]) = \mathcal{H}_n(\mathcal{P}[p]) \oplus \mathcal{H}_n(\mathcal{K}[p, p^{-1}]).$$

Concerning $\mathcal{P}[p]$ polynomial algebra, the cyclic homology of this is given by

$$\mathcal{H}_n(\mathcal{P}[p]) = \mathcal{H}_n(\mathcal{C}(\mathcal{P}[p])) \oplus \mathcal{H}_n(\mathcal{K}[p]).$$

The cyclic homology of Laurent polynomial algebra $\mathcal{P}[p, p^{-1}]$ is

$$\mathcal{H}_n(\mathcal{K}[p, p^{-1}]) = \mathcal{H}_n(\mathcal{K}[p]) \oplus (\mathcal{K}[p] \oplus \mathcal{K}[p, p^{-1}]) \oplus \mathcal{H}_n(\mathcal{P}).$$

If $k$ contains $\mathbb{Q}$,

$$\mathcal{H}_n(\mathcal{P}[p, p^{-1}]) = \mathcal{H}_n(\mathcal{P}) \oplus \mathcal{H}_n(\mathcal{P}[p]) \oplus \mathcal{H}_n(\mathcal{P}).$$

The polynomial can be expressed using the matrix if all its coefficients are matrix and called polynomial matrices [19]. It can be mathematically expressed as follows:

$$\mathcal{P} = \sum_{n=0}^{\mathcal{P}} A(n) x^n$$

$$= A(0) + A(1)x + A(2)x^2 + ... + A(z)x^z,$$

where $(n)$ denotes a matrix of constant coefficients. Let $\mathcal{P}$ be an associative unital polynomial algebra over $\mathcal{K}$, and let $M_m(\mathcal{P})$ be matrices algebra with coefficients in $\mathcal{P}$ of order $m$. $M_m(\mathcal{P})$ undergoes an involution:

$$y \mapsto y^*, \quad y = (y_{ij}), \quad y^* = (y_{ij}^*) \in M_m(\mathcal{A}),$$

where the $\mathcal{K}$-module homomorphism
\[ Tr_n: M_n(P) \otimes (n-1) \rightarrow P \otimes (n-1). \]

**Theorem 2-1:**

Let \( P' \subseteq P \) be a polynomial algebra. The long exact sequence is then obtained as follows:

\[ \ldots \rightarrow \mathcal{H} \mathcal{H}_n(P') \rightarrow \mathcal{H} \mathcal{H}_n(P) \rightarrow \mathcal{H} \mathcal{H}_n(P/P') \rightarrow \mathcal{H} \mathcal{H}_{n-1}(P') \rightarrow \ldots. \]

**Proof:** See [14].

**Theorem 2-2:**

If \( P' \subseteq P \), long exact sequence for cyclic (co)homology of polynomial algebra is

\[ \ldots \rightarrow \mathcal{H} \mathcal{C}_n(P') \rightarrow \mathcal{H} \mathcal{C}_n(P) \rightarrow \mathcal{H} \mathcal{C}_n(P/P') \rightarrow \mathcal{H} \mathcal{C}_{n-1}(P') \rightarrow \ldots. \]

**Proof:** See [20].

**Theory 2-3:**

The long exact sequences are known as Connes' exact periodicity sequences;

\[ \ldots \rightarrow \mathcal{H} \mathcal{C}_{n-1}(P) \xrightarrow{B} \mathcal{H} \mathcal{H}_n(P) \xrightarrow{I} \mathcal{H} \mathcal{H}_n(P) \xrightarrow{S} \mathcal{H} \mathcal{C}_n(P) \rightarrow \ldots \]

Where; \( I \) is an inclusion map, \( B \) is a boundary map and \( S \) is the periodicity map.

**Proof:** See [14].

**Theorem 2-4:**

If \( I \) is a unital \( k \)-algebra, then the algebra of matrices \( M_r(I) \) is unital, where

\[ tr: \mathcal{H} \mathcal{H}_n(M_r(I), M_r(P)) \rightarrow \mathcal{H} \mathcal{H}_n(I, P) \]

and \( inc: \mathcal{H} \mathcal{H}_n(I, P) \rightarrow \mathcal{H} \mathcal{H}_n(M_r(I), M_r(P)) \)

are isomorphism and inverse to each other.

**Proof:** See [3].

**Theorem 2-5:**

For any \( H \)-unital \( k \)-algebra \( P \), a map \( tr: \mathcal{H} \mathcal{C}_n(M_r(I), M_r(P)) \rightarrow \mathcal{H} \mathcal{C}_n(I, P) \) is an isomorphism and is the inverse of

\[ inc: \mathcal{H} \mathcal{C}_n(I, P) \rightarrow \mathcal{H} \mathcal{C}_n(M_r(I), M_r(P)). \]

**Proof:** See [3].

The following section will examine the dihedral homology of polynomial algebra. The relation between dihedral and cyclic homology will be introduced for polynomial algebra and Laurent polynomials algebra. Morita's property of dihedral homology and Morita's invariance for dihedral homology in polynomial algebra and Laurent polynomials algebra will be examined, which states the trace map is inverse to the inclusion map. Finally, the long sequence of the dihedral homology will be studied.

**4. Main Results**

Let \( P = K[P] \) be the dihedral submodule of the dihedral module \( P \) generated by polynomial \( p_1, \ldots, p_2 \in P \), with involution \( \sum_i \delta_i(y_i') = \sum_i \delta_i y_i', \delta_i \in P \). The dihedral homology and cohomology of the group can be defined as

\[ aH \mathcal{D}_n(P) = Tor_n(k^1[P], E^P_{\alpha}), \]

\[ aH \mathcal{C}_n(P) = Ext_n(k^1[P], E^P_{\alpha}). \]

If \( P \) is an involutive Polynomial algebra, then \( H \mathcal{D}_n(P) = H_n(Tot(CC^+(P))) \) is the dihedral homology of \( P \), and the skew-dihedral homology of \( P \) is

\[ H \mathcal{D}_n(P) = H_n(TotCC^-(P)). \]

There is a canonical splitting of cyclic homology, which follows directly from the preceding definition:

\[ H \mathcal{C}_n(P) = H \mathcal{D}_n(P) \oplus H \mathcal{D}_n(P) \forall n. \]

The direct sum of the following two exact sequences breaks up Connes' periodicity exact sequence naturally:

\[ \ldots \rightarrow H \mathcal{C}_n \rightarrow H \mathcal{D}_n \rightarrow H \mathcal{D}_{n-1} \rightarrow H \mathcal{C}_n \rightarrow \ldots, \]

\[ \ldots \rightarrow H \mathcal{D}_n \rightarrow H \mathcal{D}_{n-1} \rightarrow H \mathcal{D}_{n-2} \rightarrow H \mathcal{C}_n \rightarrow \ldots. \]

when \( K \) contains \( \mathbb{Q} \):

\[ H \mathcal{D}_n = H \mathcal{C}_n^{(1)} \oplus H \mathcal{C}_n^{(2)} \oplus \ldots, \]

and \( H \mathcal{D}_n = H \mathcal{C}_n^{(0)} \oplus H \mathcal{C}_n^{(2)} \oplus \ldots. \)

If \( P \) is Laurent polynomial algebra \( K[p, p^{-1}] \), with the natural involution:

\[ \sum_i \alpha_i y_i + \sum_j \beta_j y_j^{-1} = \sum_i \alpha_i y_i + \sum_j \beta_j y_j^{-1} \forall \alpha_i, \beta_j \in P. \]

Then we can define the dihedral of Laurent polynomial algebra by a formula:

\[ aH \mathcal{D}_n(P[p, p^{-1}]) = \]

\[ aH \mathcal{D}_n(P) \otimes aH \mathcal{D}_{n-1}(P) \otimes \mathcal{R}_{\alpha}^\circ(P). \]

If \( K \) contains \( \mathbb{Q} \), the last relation given as;

\[ H \mathcal{C}_n(P) \cong aH \mathcal{D}_n(P) \oplus -aH \mathcal{D}_n(P), \alpha = \pm 1. \]

will be shown in the following theory.

**Theorem 3-1:**

If \( P \) is a polynomial algebra, then there is an isomorphism dihedral and cyclic homology;

\[ H \mathcal{C}_n(P) \cong H \mathcal{D}_n(P) \oplus -aH \mathcal{D}_n(P), \alpha = \pm 1. \]

**Proof:**

If \( K \) is a commutative ring with unital, then the long
exact sequence for dihedral and cyclic homology groups is
\[ \cdots \to -\alpha \mathcal{H} \mathcal{D}_n(P) \xrightarrow{g_*} \mathcal{H} C_n(P) \xrightarrow{i_*} \alpha \mathcal{H} \mathcal{D}_n(P) \to \cdots, \]
where \( g_* \) is homomorphism. When \( \frac{1}{2} \in \mathcal{K} \) is used, the above exact sequence is broken down into short exact sequences:
\[ 0 \to -\alpha \mathcal{H} \mathcal{D}_n(P) \xrightarrow{g_*} \mathcal{H} C_n(P) \xrightarrow{i_*} \alpha \mathcal{H} \mathcal{D}_n(P) \to 0. \]
We have the following natural isomorphisms hold:
\[ \mathcal{H} C_n(P) \cong \mathcal{H} n_n(P) \oplus \mathcal{H} C_n-1(P) \oplus \cdots, \]
where
\[ \mathcal{H} \mathcal{D}_n(P) \cong \mathcal{H} n_n(P) \oplus \mathcal{H} n_{-4}(P) \mathcal{K} \oplus \mathcal{H} n_{-8}(P) \mathcal{K} \oplus \cdots. \]

**Theorem 3-2:**

If \( \mathcal{P} \) is a Laurent polynomial algebra \( \mathcal{K}[p, p^{-1}] \), then we have:
\[ \mathcal{H} \mathcal{C}_n(P[p, p^{-1}]) \cong \mathcal{H} \mathcal{D}_n(P[p, p^{-1}]) \oplus -\alpha \mathcal{H} \mathcal{D}_n(P[p, p^{-1}]), \alpha = \pm 1. \]

**Proof:**

From the definition and we take \( \alpha = \pm 1 \) then the following hold:
\[ \mathcal{H} \mathcal{D}_n(P[p, p^{-1}]) \cong \mathcal{H} n_n(P[p, p^{-1}]) \oplus \mathcal{H} n_{-4}(P[p, p^{-1}]) \oplus \mathcal{H} n_{-8}(P[p, p^{-1}]) \oplus \cdots. \]
\[
\begin{align*}
&\rightarrow \mathcal{H}_{n}(\mathbb{Z}/2, a\mathcal{R}(M_m(\mathcal{P}))) \\
&\rightarrow \cdots \rightarrow \mathcal{H}_{n}(\mathbb{Z}/2, \text{Tot}^a\Lambda(M_m(\mathcal{P}))) \\
&\rightarrow \mathcal{H}_{n}(\mathbb{Z}/2, a\mathcal{R}(M_m(\mathcal{P}))) \\
&\rightarrow \cdots.
\end{align*}
\]

If \( Tr \) is a homomorphism between \( \mathcal{H}_{n}(\text{Tot}^a\Lambda(M_m(\mathcal{P}))) \) and \( \mathcal{H}_{n}(\text{Tot}^a\Lambda(\mathcal{P})) \), then

\[
\mathcal{H}_{n}(\text{Tot}^a\Lambda(M_m(\mathcal{P}))) = \mathcal{H}_{C_n}(M_m(\mathcal{P})).
\]

Applying \( Tr: M_m(\mathcal{P}) \rightarrow \mathcal{P} \) on the cyclic homology, we get

\[
Tr: \mathcal{H}_{C_n}(M_m(\mathcal{P})) \rightarrow \mathcal{H}_{C_n}(\mathcal{P}).
\]

If we apply \( Tr: M_m(\mathcal{P}) \rightarrow \mathcal{P} \) on the dihedral homology with consideration;

\[
\mathcal{H}_{D_n}(M_m(\mathcal{P})) = \mathcal{H}_{n}(\mathbb{Z}/2, a\Lambda(M_m(\mathcal{P}))),
\]

we get

\[
\mathcal{H}_{D_n}(M_m(\mathcal{P})) = \mathcal{H}_{n}(\mathbb{Z}/2, a\Lambda(\mathcal{P})),
\]

we get

\[
Tr: \mathcal{H}_{D_n}(M_m(\mathcal{P})) \rightarrow \mathcal{H}_{D_n}(\mathcal{P}).
\]

In the following theorem, we prove the inclusion map of the Laurent polynomial algebra.

**Theorem 3-4:**

If \( \mathcal{P} \) is Laurent polynomial algebra over \( \mathcal{K}[p, p^{-1}] \) and \( M_m(\mathcal{P}[p, p^{-1}]) \) is the Laurent polynomial algebra of matrices, then the map

\[
inc: \mathcal{H}_{D_n}(\mathcal{P}[p, p^{-1}]) \rightarrow \mathcal{H}_{D_n}(M_m(\mathcal{P}[p, p^{-1}]))
\]

is an isomorphism for all \( m \geq 1 \) and \( n \geq 0 \).

**Proof:**

Suppose that bicomplex \( \Lambda(\mathcal{P}) = \Lambda(\mathcal{P}) \) has a fixed action of the group \( \mathbb{Z}/2 \).

If \( i: a\mathcal{R}(M_m(\mathcal{P}[p, p^{-1}])) \rightarrow \text{Tot}^a\Lambda(M_m(\mathcal{P}[p, p^{-1}])) \)

\[
j: \text{Tot}^a\Lambda(M_m(\mathcal{P}[p, p^{-1}])) \rightarrow \text{Tot}^a\Lambda(M_m(\mathcal{P}[p, p^{-1}])),
\]

and \( a\mathcal{R}(M_m(\mathcal{P}[p, p^{-1}]))) = \ker j \), where \( j \) natural projection, then the short sequence of \( \mathbb{Z}/2 \)-complexes after that is

\[
0 \rightarrow a\mathcal{R}(M_m(\mathcal{P}[p, p^{-1}])) \rightarrow \text{Tot}^a\Lambda(M_m(\mathcal{P}[p, p^{-1}])) \\
\rightarrow \text{Tot}^a\Lambda(M_m(\mathcal{P}[p, p^{-1}])), \quad [-2] \rightarrow 0.
\]

The hyper-homology of the upper short exact sequence gives a long sequence;

\[
\cdots \rightarrow \mathcal{H}_{n}(\mathbb{Z}/2, a\mathcal{R}(M_m(\mathcal{P}[p, p^{-1}]))) \\
i \rightarrow \mathcal{H}_{n}(\mathbb{Z}/2, \text{Tot}^a\Lambda(M_m(\mathcal{P}[p, p^{-1}]))) \\
\rightarrow \mathcal{H}_{n}(\mathbb{Z}/2, a\mathcal{R}(M_m(\mathcal{P}[p, p^{-1}]))) \\
\rightarrow \cdots.
\]

If \( inc \) is a homomorphism between

\[
\mathcal{H}_{n}(\text{Tot}^a\Lambda(\mathcal{P}[p, p^{-1}])),
\]

then

\[
\mathcal{H}_{n}(\text{Tot}^a\Lambda(M_m(\mathcal{P}[p, p^{-1}])))
\]

we get

\[
inc: \mathcal{H}_{C_n}(M_m(\mathcal{P}[p, p^{-1}])) \rightarrow \mathcal{H}_{C_n}(\mathcal{P}[p, p^{-1}]).
\]

If we take the cyclic of;

\[
inc: \mathcal{H}_{C_n}(\mathcal{P}[p, p^{-1}]) \rightarrow \mathcal{H}_{C_n}(M_m(\mathcal{P}[p, p^{-1}]))
\]

we get

\[
inc: \mathcal{H}_{D_n}(\mathcal{P}[p, p^{-1}]) \rightarrow \mathcal{H}_{D_n}(M_m(\mathcal{P}[p, p^{-1}])).
\]

In the following theory, we will demonstrate that the trace map of dihedral is inverse of the inclusion map.

**Theorem 3-5:**

Consider that \( \mathcal{P} \) is polynomial algebra and \( I \) is \( H \)-unital over \( \mathcal{K} \). Assume

\[
tr: a\mathcal{H}_{D_n}(\mathcal{P} \otimes I) \rightarrow a\mathcal{H}_{D_n}(M_m(\mathcal{P}) \otimes M_m(I))
\]

and

\[
inc: a\mathcal{H}_{D_n}(M_m(\mathcal{P}) \otimes M_m(I)) \rightarrow a\mathcal{H}_{D_n}(\mathcal{P} \otimes I)
\]

are used to define these maps \( \alpha = \begin{pmatrix} a & 0 \\ 0 & \alpha \end{pmatrix} \) and

\[
inc(\alpha_{ij} \otimes \delta_{ij}) = \sum_{i,j} a_{ij} \otimes \delta_{ji}, \text{ respectively.}
\]

Accordingly, the map

\[
a\mathcal{H}_{D_n}(M_m(\mathcal{P}) \otimes M_m(I)) \xrightarrow{inc} a\mathcal{H}_{D_n}(\mathcal{P} \otimes I)
\]

inverses the map

\[
a\mathcal{H}_{D_n}(\mathcal{P} \otimes I) \xrightarrow{tr} a\mathcal{H}_{D_n}(M_m(\mathcal{P}) \otimes M_m(I)).
\]
Proof:

The following commutative diagram is the extension morphism;

\[
\begin{array}{cccccc}
0 & \to & I & \to & I_+ & \to & \mathcal{K} & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & M_m(I) & \to & M_m(I_+) & \to & M_m(\mathcal{K}) & \to & 0,
\end{array}
\]

where the vertical and horizontal maps are executed. Isomorphism can be seen in the left vertical arrow by extension property of polynomial algebra, and the rows are accurate. If we consider the upper diagram's dihedral homology, then we obtain

\[
0 \to a\mathcal{H}_D(n(I)) \to a\mathcal{H}_D(n(I_+)) \to a\mathcal{H}_D(\mathcal{K}) \to 0
\]

Consequently, the right vertical arrow is isomorphic, and we get a commutative diagram;

\[
\begin{array}{c}
\mathcal{H}_R(n(\mathcal{P}) \otimes M_m(I)) \cong a\mathcal{H}_D(n(\mathcal{P}) \otimes M_m(I_+)) \\
\mathcal{H}_R(n(I)) \cong a\mathcal{H}_D(n(\mathcal{P}) \otimes I_+)
\end{array}
\]

If the right \( tr^* \) is an isomorphism according to Morita invariance for unital polynomial algebra, then

\[
tr^* : a\mathcal{H}_D(n(\mathcal{P}) \otimes M_m(I)) \to a\mathcal{H}_D(n(\mathcal{P} \otimes I))
\]

is an isomorphism. Thus, \( inc \) is a right inverse of \( tr \).

The theorem for dihedral (co)homology is introduced and proved in the next theory.

**Theorem 3-6:**

In Dihedral Homology, if we consider the exact short sequence \( 0 \to \mathcal{P} \to \mathcal{P}' \to \mathcal{P}'' \to 0 \) of polynomials algebras over a field, we obtain the following long exact sequence;

\[
\cdots \to \mathcal{H}_D(n(\mathcal{P})) \to \mathcal{H}_D(n(\mathcal{P}')) \to \mathcal{H}_D(n(\mathcal{P}'')) \to \mathcal{H}_D(n-1(\mathcal{P})) \to \mathcal{H}_D(n-1(\mathcal{P}')) \to \mathcal{H}_D(n-1(\mathcal{P}'')) \to \cdots,
\]

where \( \mathcal{P}'' \subset \mathcal{P}' \subset \mathcal{P} \).

Proof:

For the algebra \( \mathcal{P}' \), we define the next short exact sequence;

\[
0 \to \mathcal{P}' \to \mathcal{P}'' \to \mathcal{P}'' \to 0,
\]

where \( \mathcal{P}' \) is the algebra of non-unital involution polynomials over \( \mathcal{P} \) and \( \mathcal{P}'' \) is ideal in a unital polynomials algebra \( \mathcal{P}' \), then a long exact sequence is

\[
\cdots \to \mathcal{H}_D(n(\mathcal{P}')) \to \mathcal{H}_D(n(\mathcal{P}'')) \to \mathcal{H}_D(n(\mathcal{P}')) \to \mathcal{H}_D(n-1(\mathcal{P}')) \to \mathcal{H}_D(n-1(\mathcal{P}')) \to \cdots. \tag{12}
\]

If \( \mathcal{K} \) be a kernel of the following sequence

\[
0 \to \mathcal{K} \to \mathcal{P}' \to \mathcal{P}'' \to \mathcal{P}'' \to 0
\]

\[
\cdots \to \mathcal{H}_D(n(\mathcal{K})) \to \mathcal{H}_D(n(\mathcal{P}')) \to \mathcal{H}_D(n(\mathcal{P}'')) \to \mathcal{H}_D(n-1(\mathcal{K})) \to \mathcal{H}_D(n-1(\mathcal{P}')) \to \mathcal{H}_D(n-1(\mathcal{P}')) \to \cdots, \tag{13}
\]

where \( \mathcal{K} \) is the free algebra of unital polynomials \( \mathcal{P}' \), then \( \mathcal{P}'' \) and \( \mathcal{K} \) are roughly \( H \)-unital from a long exact sequence for the sequences (12, 13).

We have

\[
\mathcal{H}_D(n)(\mathcal{P}') \cong \mathcal{H}_D(n)(\mathcal{P}'') \& \mathcal{H}_D(n-1)(\mathcal{K}) \cong \mathcal{H}_D(n-1)(\mathcal{P}'').
\]

Suppose that the next short sequence is

\[
0 \to \mathcal{P}'' \to \mathcal{K} \to \mathcal{P} \to 0
\]

\[
\cdots \to \mathcal{H}_D(n(\mathcal{P}')) \to \mathcal{H}_D(n(\mathcal{P})) \to \mathcal{H}_D(n-1(\mathcal{P}')) \to \mathcal{H}_D(n-1(\mathcal{K})) \to \cdots. \tag{14}
\]

The proof of our theorem comes from the long exact sequence.

The theorem for reflexive (co)homology is introduced and demonstrated in the next theory.

**Theorem 3-7:**

If there is an exact short sequence \( 0 \to \mathcal{P} \to \mathcal{P}' \to \mathcal{P}'' \to 0 \) of polynomials algebras over the field, then we have the next long exact sequence in hyperhomology

\[
\cdots \to \mathcal{H}_R(n(\mathcal{P})) \to \mathcal{H}_R(n(\mathcal{P}')) \to \mathcal{H}_R(n(\mathcal{P}'')) \to \mathcal{H}_R(n-1(\mathcal{P}')) \to \mathcal{H}_R(n-1(\mathcal{P}')) \to \cdots,
\]

where \( \mathcal{P}'' \subset \mathcal{P}' \subset \mathcal{P} \).

Proof:

Let \( \mathcal{P}' \) be the algebra of non-unital involution polynomials over \( \mathcal{P} \), \( \mathcal{P}'' \) be the ideal in the unital polynomials algebra \( \mathcal{P}' \), and \( \mathcal{K} \) be the kernel.

For the short exact sequence \( 0 \to \mathcal{P}'' \to \mathcal{P}' \to \mathcal{P}'' \to 0 \), we obtain long exact sequence as follows:

\[
\cdots \to \mathcal{H}_R(n(\mathcal{P}')) \to \mathcal{H}_R(n(\mathcal{P}')) \to \mathcal{H}_R(n(\mathcal{P}')) \to \mathcal{H}_R(n-1(\mathcal{P}')) \to \mathcal{H}_R(n-1(\mathcal{P}')) \to \cdots. \tag{15}
\]

We get the long exact sequence for \( 0 \to \mathcal{K} \to \mathcal{P}' \to \mathcal{P}'' \to 0 \) as

\[
\cdots \to \mathcal{H}_R(n(\mathcal{K})) \to \mathcal{H}_R(n(\mathcal{P}')) \to \mathcal{H}_R(n(\mathcal{P}'')) \to \mathcal{H}_R(n-1(\mathcal{K})) \to \mathcal{H}_R(n-1(\mathcal{P}')) \to \cdots. \tag{16}
\]

From equations (15) and (16), we get

\[
\mathcal{H}_R(n(\mathcal{P}'')) \cong \mathcal{H}_R(n-1(\mathcal{P}'))
\]

and \( \mathcal{H}_R(n(\mathcal{P}'')) \cong \mathcal{H}_R(n-1(\mathcal{K})). \tag{17} \)

If we take the short sequence \( 0 \to \mathcal{P}'' \to \mathcal{K} \to \mathcal{P} \to 0 \), then the long exact sequence is as follows:

\[
\cdots \to \mathcal{H}_R(n(\mathcal{P}')) \to \mathcal{H}_R(n(\mathcal{K})) \to \mathcal{H}_R(n(\mathcal{P}')) \to \mathcal{H}_R(n-1(\mathcal{K})) \to \mathcal{H}_R(n-1(\mathcal{P}')) \to \cdots. \tag{18}
\]

By applying the isomorphism in equation (17) to equation (18), we get the proof of our theory.
5. Conclusions

We demonstrate the dihedral and reflexive homology of polynomial algebra. The relationship between dihedral and cyclic (co)homology for polynomial algebra was introduced, which is $\mathcal{H}_n(P) \cong a\mathcal{D}_n(P) \oplus -^a\mathcal{D}_n(P), \alpha = \pm 1$. For the algebra of Laurent polynomials, the relationship for the dihedral (co)homology and cyclic homology was demonstrated, which is $\mathcal{H}_n(P[p, p^{-1}]) \cong a\mathcal{D}_n(P[p, p^{-1}]) \oplus -^a\mathcal{D}_n(P[p, p^{-1}]), \alpha = \pm 1$. The Morita of dihedral homology was studied. The trace and inclusion maps of the dihedral homology were indicated as $tr: a\mathcal{D}_n(P \otimes I) \to a\mathcal{D}_n(M_n(P) \otimes M_n(I))$, and $inc: a\mathcal{D}_n(M_n(P) \otimes M_n(I)) \to a\mathcal{D}_n(P \otimes I)$, respectively.

From the sequence $0 \to P \to P' \to P'' \to 0$, the exact long sequence of the dihedral homology and hyperhomology of polynomial algebra were obtained as follows:

\[
\cdots \to \mathcal{H}_n(P) \to \mathcal{H}_n(P') \to \mathcal{H}_n(P'') \to \mathcal{H}_n(P') \to \mathcal{H}_n(P'') \to \cdots,
\]

\[
\cdots \to \mathcal{R}_n(P) \to \mathcal{R}_n(P') \to \mathcal{R}_n(P'') \to \mathcal{R}_n(P') \to \mathcal{R}_n(P'') \to \cdots,\]

respectively.

We can apply this result with ([22], [23]) and give use more generalizations.

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