The $L^r$-Variational Integral

Francesco Tulone and Paul Musial

Abstract. We define the $L^r$-variational integral and we prove that it is equivalent to the $HK^r$-integral defined in 2004 by P. Musial and Y. Sagher in the Studia Mathematica paper The $L^r$-Henstock–Kurzweil integral. We prove also the continuity of $L^r$-variation function.

Mathematics Subject Classification. 26A39, 28C99.

Keywords. $L^r$-Variational Integral, $HK^r$ Integral, Non-absolute integral.

1. History and Aim

At the beginning of the 1900s, Denjoy and Perron developed descriptive processes for recovering a function from its derivative that solved known problems of classical Riemann and Lebesgue integrals. Many years later, an equivalent constructive Riemann-type integral process was developed by Henstock and Kurzweil. Both integration processes were generalized quite recently for many different spaces (see [1,11] and [12]) solving the problem of recovering Fourier coefficients in Haar, Walsh and Vilenkin systems (see [9,10,14,15] and [16]). Many properties of these non-absolute integrals were investigated, for example, the Hake property was studied with an abstract differential basis in a topological spaces, in terms of variational measure and in Riesz spaces (see [13,17] and [2]).

To establish pointwise estimates for solutions of elliptic partial differential equations, in 1961 Calderon and Zygmund introduced the $L^r$-derivative (see [3]) and in 1968 L. Gordon described a Perron-type integral, the $P^r$-integral, that recovers a function from its $L^r$-derivative (see [4]). In 2004, Musial and Sagher extended the $P^r$-integral to the $L^r$-Henstock–Kurzweil integral, the $HK^r$-integral, that recovers also a function from its $L^r$-derivative (see [6]). Quite recently the integration by parts formula for the $HK^r$-integral was investigated by Musial and Tulone (see [7]) and the same authors described a norm on the space of $HK^r$-integrable functions and studied the dual and completion of this space (see [8]).
It is well known that the Henstock–Kurzweil integral is equivalent to the variational integral (see [5]). In this paper, we define the $L^r$-variational integral and we prove that it is equivalent to the $HKr$-integral.

2. Introduction

We will assume that $r \geq 1$ and we will consider the case of the closed interval $[a, b]$.

Definition 2.1. A function $f : [a, b] \rightarrow \mathbb{R}$ is $L^r$-variational integrable on $[a, b]$ if there exists a function $F \in L^r [a, b]$ with the following property: for each $\varepsilon > 0$ there exist a non-decreasing function $\phi$ defined on $[a, b]$ and a gauge $\delta$, i.e., a positive function, defined on $[a, b]$ such that $\phi(b) - \phi(a) < \varepsilon$ and for any $\delta$-fine tagged interval $(x, [c, d])$, where $[c, d] \subseteq [a, b]$,

$$
\left( \frac{1}{d-c} \int_c^d |F(y) - F(x) - f(x)(y-x)|^r \, dy \right)^{1/r} < \phi(d) - \phi(c). \tag{2.1}
$$

We will use the following definition given in [6]

Definition 2.2. A function $f : [a, b] \rightarrow \mathbb{R}$ is $L^r$-Henstock–Kurzweil integrable on $[a, b]$ if there exists a function $F \in L^r [a, b]$ so that for any $\varepsilon > 0$ there exists a gauge $\delta$ so that for any finite collection of nonoverlapping $\delta$-fine tagged intervals $Q = \{(x_i, [c_i, d_i]) : 1 \leq i \leq q\}$, we have

$$
\sum_{i=1}^q \left( \frac{1}{d_i - c_i} \int_{c_i}^{d_i} |F(y) - F(x_i) - f(x_i)(y-x_i)|^r \, dy \right)^{1/r} < \varepsilon.
$$

By Theorem 5 in [6], the function $F$ in the Definition 2.2 is unique up to an additive constant, so we can state that for each $x \in (a, b]$

$$
F(x) = (HKr) \int_a^x f.
$$

We need the following definition in a later theorem.

Definition 2.3. Let $F \in L^r [a, b]$. For $x \in [a, b]$ we say that $F$ is $L^r$-continuous at $x$ if

$$
\lim_{h \to 0} \left( \frac{1}{2h} \int_{x-h}^{x+h} |F(y) - F(x)|^r \, dy \right)^{1/r} = 0.
$$

If $F$ is $L^r$-continuous for all $x \in E$, we say that $F$ is $L^r$-continuous on $E$.

The Henstock–Kurzweil integral primitive is continuous in the usual sense. In [6] is proved an equivalent result for $L^r$-Henstock–Kurzweil indefinite integral.

Theorem 2.4. The function $F$ in the definition of the $L^r$-Henstock–Kurzweil is $L^r$-continuous on $[a, b]$. 

**Definition 2.5.** Let $\Phi$ be a function defined on the subintervals of $[a,b]$. The function $\Phi$ is superadditive if

$$\Phi([u,v]) + \Phi([v,w]) \leq \Phi([u,w]),$$

whenever $a \leq u < v < w \leq b$. The function $\Phi$ is continuous if for each $c \in (a,b)$,

$$\lim_{x \to c^{-}} \Phi([x,c]) = 0 = \lim_{x \to c^{+}} \Phi([c,x])$$

and

$$\lim_{x \to b^{-}} \Phi([x,b]) = 0 = \lim_{x \to a^{+}} \Phi([a,x]).$$

**Remark 2.6.** Throughout this paper, if an interval function is said to be continuous, it is to be considered continuous in the sense of Definition 2.5.

**Definition 2.7.** Let $\delta$ be a gauge and let

$$P = \{(x_{i},[c_{i},d_{i}]), 1 \leq i \leq n\}$$

be a $\delta$-fine partition of $[a,b]$. Let

$$W(P) = \sum_{i=1}^{q} \left( \frac{1}{d_{i} - c_{i}} \int_{c_{i}}^{d_{i}} |F(y) - F(x_{i}) - f(x_{i})(y - x_{i})|^{r} \, dy \right)^{1/r}. \quad (2.2)$$

The main tool we need to get the $L^{r}$-variational integral is the following definition of $L^{r}$-variation function.

**Definition 2.8.** For each subinterval $[c,d] \subseteq [a,b]$ define

$$\Phi([c,d]) = \Phi(F,\delta,[c,d]) = \sup \{W(P)\}, \quad (2.3)$$

where the supremum is taken over all $\delta$-fine partitions $P$ of $[c,d]$.

**Theorem 2.9.** The function $\Phi$ is superadditive.

**Proof.** Let $u, v$ and $w$ be such that $a \leq u < v < w \leq b$ and let $\varepsilon > 0$. If either $\Phi([u,v]) = \infty$ or $\Phi([v,w]) = \infty$ then surely $\Phi([u,w]) = \infty$ and the assertion holds. Otherwise let $P_{1}$ be a partition of $[u,v]$ such that $W(P_{1}) > \Phi([u,v]) - \varepsilon$ and let $P_{2}$ be a partition of $[v,w]$ such that $W(P_{2}) > \Phi([v,w]) - \varepsilon$. Let $P = P_{1} \cup P_{2}$, and clearly $W(P) = W(P_{1}) + W(P_{2})$. But $W(P) \leq \Phi([u,w])$. Therefore,

$$\Phi([u,v]) + \Phi([v,w]) - 2\varepsilon < W(P_{1}) + W(P_{2}) \leq \Phi([u,w]).$$

Now we can prove the following theorem that extends Theorem 11.9 in [5]
3. Main Results

**Theorem 3.1.** A function \( f : [a, b] \to \mathbb{R} \) is \( L^r \)-Henstock–Kurzweil integrable on \([a, b]\) if and only if there exists a function \( F : [a, b] \to \mathbb{R} \) with the following property: for each \( \varepsilon > 0 \) there exists a superadditive interval function \( \Phi \) defined on the subintervals of \([a, b]\) and a gauge \( \delta \) defined on \([a, b]\) such that \( \Phi ([a, b]) < \varepsilon \) and for any \( \delta \)-fine tagged interval \((x, [c, d])\), where \([c, d] \subseteq [a, b] \),

\[
\left( \frac{1}{d - c} \int_c^d |F(y) - F(x) - f(x)(y - x)|^r dy \right)^{1/r} < \Phi ([c, d]).
\]

Proof. Suppose there exists a function \( F \) with the property stated in the theorem. Let \( \varepsilon > 0 \) and choose \( \Phi \) and \( \delta \) according to the hypotheses. If \( P := \{(x_i, [c_i, d_i]), 1 \leq i \leq n\} \) is a \( \delta \)-fine tagged partition of \([a, b]\), then

\[
\sum_{i=1}^{n} \left( \frac{1}{d_i - c_i} \int_{c_i}^{d_i} |F(y) - F(x_i) - f(x_i)(y - x_i)|^r dy \right)^{1/r} \leq \sum_{i=1}^{n} \Phi ([c_i, d_i]) \leq \Phi ([a, b]), < \varepsilon
\]

and so \( f \) is \( L^r \)-Henstock–Kurzweil integrable on \([a, b]\).

Now suppose that \( f \) is \( L^r \)-Henstock–Kurzweil integrable on \([a, b]\) and let

\[
F(x) = (HK_r) \int_a^x f,
\]

for each \( x \in (a, b]. \) Let \( \varepsilon > 0 \). By hypothesis, there exists a gauge \( \delta \) on \([a, b]\) such that

\[
\sum_{i=1}^{n} \left( \frac{1}{d_i - c_i} \int_{c_i}^{d_i} |F(y) - F(x_i) - f(x_i)(y - x_i)|^r dy \right)^{1/r} < \varepsilon / 2,
\]

whenever \( P \) is a \( \delta \)-fine tagged partition of \([a, b]\). Let

\[
\mathcal{P} = \{(x_i, [c_i, d_i]), 1 \leq i \leq n\}
\]

and let \( W(\mathcal{P}) \) be defined as in (2.2) and let \( \Phi \) be defined on the subintervals of \([a, b]\) as in (2.3). By Theorem 2.9, \( \Phi \) is superadditive. Also,

\[
\Phi ([a, b]) \leq \varepsilon / 2 < \varepsilon.
\]

Finally, by the definition of \( \Phi \), if \((x, [c, d])\) is a \( \delta \)-fine tagged interval such that \([c, d] \subseteq [a, b]\),

\[
\left( \frac{1}{d - c} \int_c^d |F(y) - F(x) - f(x)(y - x)|^r dy \right)^{1/r} < \Phi ([c, d]).
\]

This completes the proof. \( \square \)

**Theorem 3.2.** A function \( f : [a, b] \to \mathbb{R} \) is \( L^r \)-Henstock–Kurzweil integrable on \([a, b]\) if and only if \( f \) is \( L^r \)-variational integrable on \([a, b]\).
Proof. Suppose first that $f$ is $L^r$-variational integrable on $[a, b]$. Let $\varepsilon > 0$ and let $F$, $\delta$ and $\phi$ satisfy the conditions in Definition 2.1. If $P = \{(x_i, [c_i, d_i]), 1 \leq i \leq n\}$ is a $\delta$-fine tagged partition of $[a, b]$, then

$$\sum_{i=1}^{n} \left( \frac{1}{d_i - c_i} \int_{c_i}^{d_i} |F(y) - F(x_i) - f(x_i)(y - x_i)|^r \, dy \right)^{1/r}$$

$$\leq \sum_{i=1}^{n} (\phi(d_i) - \phi(c_i)) = \phi(b) - \phi(a) < \varepsilon$$

and so $f$ is $L^r$-Henstock–Kurzweil integrable on $[a, b]$ and

$$(HK_r) \int_a^b f = F(b) - F(a).$$

Now suppose that $f$ is $L^r$-Henstock–Kurzweil integrable on $[a, b]$ and that for each $x \in (a, b]$,

$$F(x) = (HK_r) \int_a^x f.$$

Let $\varepsilon > 0$. By Theorem 3.1 there exists a superadditive interval function $\Phi$ defined on $[a, b]$ such that $\Phi([a, b]) < \varepsilon$ and

$$\left( \frac{1}{d-c} \int_c^d |F(y) - F(x) - f(x)(y-x)|^r \, dy \right)^{1/r} < \Phi([c,d]),$$

whenever $(x, [c, d])$ is a $\delta$-fine tagged interval such that $[c, d] \subseteq [a, b]$. Define $\phi : [a, b] \rightarrow \mathbb{R}$ by $\phi(a) = 0$ and $\phi(x) = \Phi([a, x])$ for all $x \in (a, b]$. If $a \leq c < d \leq b$, then

$$\phi(d) - \phi(c) = \Phi([a, d]) - \Phi([a, c]) \geq \Phi([c, d]) \geq 0$$

and so $\phi$ is non-decreasing. In addition,

$$\phi(b) - \phi(a) = \Phi([a, b]) < \varepsilon.$$

Suppose that $(x, [c, d])$ is a $\delta$-fine tagged interval such that $[c, d] \subseteq [a, b]$. Then,

$$\left( \frac{1}{d-c} \int_c^d |F(y) - F(x) - f(x)(y-x)|^r \, dy \right)^{1/r}$$

$$\leq \Phi([c,d]) \leq \phi(d) - \phi(c).$$

Hence, the function $f$ is $L^r$-variational integrable on $[a, b]$. This completes the proof. \qed

**Corollary 3.3.** If $f$ is $L^r$-variational integrable on $[a, b]$, then the function $F$ which satisfies the conditions of Definition 2.1 is unique up to an additive constant.

We now prove the continuity of the interval function $\Phi$. 
Proposition 3.4. Let $f$ be $L^r$-variational integrable on $[a, b]$ and let $F$ be a function that satisfies (2.1). Let $\delta$ be a gauge, $\Phi = \Phi (\delta, F)$ be as in (2.3), and assume that $\Phi ([a, b])$ is finite. Then, $\Phi$ is continuous.

Proof. We will prove that $\lim_{x \to c^-} \Phi ([x, c]) = 0$ for each $c \in (a, b]$; the proof for right-handed limits is similar. Suppose by way of contradiction that $\lim_{x \to c^-} \Phi ([x, c])$ either fails to exist or exists and is not equal to zero. Since $\Phi$ is nonnegative, there exists $\eta > 0$ such that $\limsup_{x \to c^-} \Phi ([x, c]) > \eta$. Let us see that for every $\xi \in [a, c)$, $\Phi ([\xi, c]) > \eta$. Fix $\xi$, there exists $\xi < \zeta < c$ such that $\Phi ([\zeta, c]) > \eta$. Since $\Phi$ is superadditive, we have that

$$\Phi ([\xi, c]) \geq \Phi ([\xi, \zeta]) + \Phi ([\zeta, c]) \geq \Phi ([\zeta, c]) > \eta.$$

Consequently, for each $\xi \in [a, c)$, there exists $P_{\xi}$, a $\delta$-fine tagged partition of $[\xi, c]$. We now prove that we can make the following three assumptions about $P_x$:

1. $P_x$ contains at least two tagged intervals,
2. $c$ is a tag of $P_x$, and
3. the interval containing $c$ is arbitrarily small.

Fix $x$ and $\varepsilon > 0$. Choose $y \in (\max (x, c - \varepsilon), c)$. By Cousin’s Lemma there exists $Q$, a $\delta$-fine tagged partition of $[x, y]$. Define $P_x = Q \cup P_y$. We then have

$$W (P_x) = W (Q) + W (P_y) \geq W (P_y) > \eta.$$

If $c$ is the tag of its interval, then $P_x$ has the desired properties.

Now suppose that $c$ is not the tag of its interval. Let $s$ and $t$ be such that $(t, [s, c])$ is the tagged interval which contains $c$. It is possible that $s = t$ but we assume that $t < c$.

It suffices to show that

$$\lim_{u \to c^-} W \left( \{(t, [s, u]) , (c, [u, c])\} \right) = W \left( \{(t, [s, c])\} \right).$$

Note that

$$W \left( \{(t, [s, u]) , (c, [u, c])\} \right) = \left( \frac{1}{u - s} \int_s^u |F(y) - F(t) - f(t)(y - t)|^r dy \right)^{1/r} + \left( \frac{1}{c - u} \int_u^c |F(y) - F(c) - f(c)(y - c)|^r dy \right)^{1/r}.$$
Using Minkowski’s inequality, we have
\[
\left( \frac{1}{c-u} \int_u^c |F(y) - F(c) - f(c) (y-c)|^r \, dy \right)^{1/r}
\]
\[
= \left( \frac{1}{c-u} \right)^{1/r} \left( \int_u^c |F(y) - F(c) - f(c) (y-c)|^r \, dy \right)^{1/r}
\]
\[
\leq \left( \frac{1}{c-u} \right)^{1/r} \left( \int_u^c |F(y) - F(c)|^r \, dy \right)^{1/r}
+ \left( \frac{1}{c-u} \right)^{1/r} \left( \int_u^c |f(c) (y-c)|^r \, dy \right)^{1/r}
\]
\[
\leq \left( \frac{1}{c-u} \int_u^c |F(y) - F(c)|^r \, dy \right)^{1/r}
+ |f(c)| \left( \frac{1}{c-u} \right)^{1/r} \left( \int_u^c |(c-u)|^r \, dy \right)^{1/r}
\]
\[
= \left( \frac{1}{c-u} \int_u^c |F(y) - F(c)|^r \, dy \right)^{1/r} + |f(c)| (c-u).
\]

By Theorem 2.4 the function $F$ is $L^r$-continuous at each point of $[a,b]$, and so we have that
\[
\lim_{u \to c^-} \left( \frac{1}{c-u} \int_u^c |F(y) - F(c) - f(c) (y-c)|^r \, dy \right)^{1/r}
\]
\[
\leq \lim_{u \to c^-} \left[ \left( \frac{1}{c-u} \int_u^c |F(y) - F(c)|^r \, dy \right)^{1/r} + |f(c)| (c-u) \right] = 0 (3.1)
\]

We also have that
\[
\lim_{u \to c^-} \left( \frac{1}{u-s} \int_s^u |F(y) - F(t) - f(t) (y-t)|^r \, dy \right)^{1/r}
\]
\[
= \left( \frac{1}{c-s} \int_s^c |F(y) - F(t) - f(t) (y-t)|^r \, dy \right)^{1/r}.
\]

It follows that
\[
\lim_{u \to c^-} W(\{(t,[s,u]),(c,[u,c])\})
\]
\[
= \lim_{u \to c^-} W(\{(t,[s,u])\}) = W(\{(t,[s,c])\}) .
\]

We now prove the proposition. Set $x_1 = a$ and write
\[
P_{x_1} = Q_1 \cup (c,[x_2,c])
\]
\[
P_{x_2} = Q_2 \cup (c,[x_3,c])
\]
\[
\vdots
\]
\[
P_{x_k} = Q_k \cup (c,[x_{k+1},c]).
\]

By the result proved above, we may assume that for each $k, c-x_k < 1/k$ and, therefore, that $x_k \to c$. 
For each \( n \), the collection

\[ P'_n = \bigcup_{k=1}^{n} Q_k \]

is a \( \delta \)-fine tagged partition of \([a, x_{n+1}]\). Hence,

\[
W(P'_n) = \sum_{k=1}^{n} W(Q_k) \leq \Phi([a, x_{n+1}]) \leq \Phi([a, b]) < \infty.
\]

This shows that the series

\[
\sum_{k=1}^{\infty} W(Q_k)
\]

converges and hence

\[
\lim_{k \to \infty} W(Q_k) = 0.
\]

We then have for each \( k \),

\[
\eta < W(P_{x_k})
\]

\[
= W(Q_k) + \left( \frac{1}{c-x_{k+1}} \int_{x_{k+1}}^{c} |F(y) - F(c) - f(c)(y-c)|^r \, dy \right)^{1/r}.
\]

By (3.1), the term on the right goes to zero; therefore, the entire right side of the equality goes to zero. This contradiction completes the proof.

\[ \square \]

**Funding** Open access funding provided by Università degli Studi di Palermo within the CRUI-CARE Agreement.

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Received: January 12, 2021.  
Revised: April 24, 2021.  
Accepted: December 7, 2021.