ADJOINT SPACES AND FLAG VARIETIES OF \( p \)-COMPACT GROUPS

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Abstract. For a compact Lie group \( G \) with maximal torus \( T \), Pittie and Smith showed that the flag variety \( G/T \) is always a stably framed boundary. We generalize this to the category of \( p \)-compact groups, where the geometric argument is replaced by a homotopy theoretic argument showing that the class in the stable homotopy groups of spheres represented by \( G/T \) is trivial, even \( G \)-equivariantly. As an application, we consider an unstable construction of a \( G \)-space mimicking the adjoint representation sphere of \( G \) inspired by work of the second author and Kitchloo. This construction stably and \( G \)-equivariantly splits off its top cell, which is then shown to be a dualizing spectrum for \( G \).

1. Introduction

Let \( G \) be a compact, connected Lie group of dimension \( d \) and rank \( r \) with maximal torus \( T \). Left translation by elements of \( G \) on the tangent space \( g = T_e G \) induces a framing of \( G \). By the Pontryagin-Thom construction, \( G \) with this framing represents an element \([G]\) in the stable homotopy groups of spheres; this has been extensively studied for example in \([\text{Sm}74, \text{Wo}76, \text{Kna}78, \text{Oss}82]\).

The following classical argument shows that the flag variety \( G/T \), while not necessarily framed, is still a stably framed manifold: since every element in a compact Lie group is conjugate to an element in the maximal torus, the conjugation map \( G \times T \to G \), \((g,t) \mapsto gtg^{-1}\) is surjective, and furthermore, it factors through \( c: G/T \times T \to G \). An element \( s \in T \) is called regular if the centralizer \( C_G(s) \supseteq T \), or, equivalently, if \( c|_{G/T \times \{s\}} \) is an embedding; it is a fact from Lie theory that the set of irregular elements has positive codimension in \( T \). Thus there is a regular element \( s \in T \) such that the derivative of \( c \) has full rank along \( G/T \times \{s\} \), and by the tubular neighborhood theorem, it induces an embedding of \( G/T \times U \), where \( U \) is a contractible neighborhood of \( s \) in \( T \). Thus the framing of \( G \) can be pulled back to a stable framing of \( G/T \).

Pittie and Smith showed in \([\text{Pi}75, \text{PS}75]\) that \( G/T \) is always the boundary of another framed manifold \( M \), and moreover, that \( M \) has a \( G \)-action which agrees with the standard \( G \)-action on \( G/T \) on the boundary. In terms of homotopy theory, this is saying that the class \([G/T]\) \( \in \pi^s_{d-r} \) induced by the Pontryagin-Thom construction is trivial.

The first main result of this paper generalizes this fact to \( p \)-compact groups.

**Theorem 1.1.** Let \( G \) be a \( \mathbb{Z}/p \)-local, \( p \)-finite group with maximal torus \( T \) such that \( \text{dim}(G) > \text{dim}(T) \). Then the Pontryagin-Thom construction \([G/T]: S_G \to S_T \) is \( G \)-equivariantly null-homotopic, with \( G \) acting trivially on \( S_T \).

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The statement of this theorem requires some explanation. A \( p \)-compact group \([\text{DW94}]\) is a triple \((G, BG, e)\) such that

- \( G \) is \( p \)-finite, i.e., \( H_*(G; F_p) \) is finite;
- \( BG \) is \( \mathbb{Z}/p \)-local, i.e. whenever \( f : X \to Y \) is a mod-\( p \) homology equivalence of CW-complexes, then \( f^* : [Y, BG] \to [X, BG] \) is an isomorphism;
- \( e : G \to \Omega BG \) is a homotopy equivalence.

Clearly, \( G \) and \( e \) are determined by \( BG \) up to homotopy, making this definition somewhat redundant. Although a priori \( G \) is only a loop space, we will henceforth assume we have chosen a rigidification such that \( G \) is actually a topological group. This is always possible, for example by using the geometric realization of Kan’s group model of the loops on a simplicial set \([\text{Kan56}]\).

A \( \mathbb{Z}/p \)-local, \( p \)-finite loop space is only slightly more general than a \( p \)-compact group in that the latter also requires \( \pi_0(G) \) to be a \( p \)-group. We will have no need to assume this in Theorem 1.1.

By \([\text{DW94}]\), every \( p \)-compact group has a maximal torus \( T \); that is, there is a monomorphism \( T \to G \) with \( T \simeq L_p(S^1)^r \) and \( r \) is maximal with this property. By definition, a monomorphism of \( p \)-compact groups is a group monomorphism \( H \to G \) such that \( G/H \) is \( p \)-finite (see \([\text{Bau04}]\) for this slightly nonstandard point of view). Dwyer and Wilkerson show that \( T \) is essentially unique. Since a maximal torus is always contained in the identity component of a \( p \)-compact group, the same works for \( \mathbb{Z}/p \)-local, \( p \)-finite groups.

Denote by \( S^0[X] \) the suspension spectrum of a space \( X \) with a disjoint base point added.

**Definition** (\([\text{Kle01}]\)). Let \( G \) be a topological group. Define \( S_G \), the dualizing spectrum of \( G \), to be the spectrum of homotopy fixed points of the right action of \( G \) on its own suspension spectrum. That is, \( S_G = (S^0[G])^{h\mathbb{G}^{op}} \) as left \( G \)-spectra.

In \([\text{Bau04}]\), the first author showed that for a connected, \( d \)-dimensional \( p \)-compact group \( G \), \( S_G \) is always homotopy equivalent to a \( \mathbb{Z}/p \)-local sphere of dimension \( d \). Furthermore, there is a \( G \)-equivariant logarithm map \( S^0[G] \to S_G \), where \( G \) acts on the left by conjugation. If \( G \) is the \( \mathbb{Z}/p \)-localization of a connected Lie group, then \( S_G \) is canonically identified with the suspension spectrum of the one-point compactification of the Lie algebra of \( G \). Thus we may call \( S_G \) the adjoint (stable) sphere of \( G \).

In a spectacular case of shortsightedness, \([\text{Bau04}]\) restricts its scope to connected \( p \)-compact groups where everything would have worked for \( \mathbb{Z}/p \)-local, \( p \)-finite groups \( G \) as well. In this case, \( S_G \) has the mod-\( p \) homology of a \( d \)-dimensional sphere. Similarly, the proof of the following was given in \([\text{Bau04}]\) Cor. 24] for connected groups, but immediately generalizes.

Let \( DM \) be the Spanier-Whitehead dual of a finite CW-spectrum \( M \).

**Lemma 1.2.** Let \( H < G \) be a monomorphism of \( \mathbb{Z}/p \)-local, \( p \)-finite groups. Then there is a relative \( G \)-equivariant duality weak equivalence

\[
G_+ \wedge_H S_H \simeq D \left(S^0[G/H]\right) \wedge S_G.
\]

For any space \( X \), there is a canonical map \( e : S^0[X] \to S^0 \) given by applying the functor \( S^0[-] \) to \( X \to * \). If \( T < G \) is a sub-torus in a \( \mathbb{Z}/p \)-local, \( p \)-finite group then
there is a stable $G$-equivariant map

\[ (1.3) \quad [G/T] : S_G \xrightarrow{\text{Id} \wedge D\epsilon} S_G \wedge D\left(S^0[G/T]\right) \]

\[ \simeq_{\text{Lemma 1.2}} G_+ \wedge_T S_T \simeq S^0[G/T] \wedge S_T \xrightarrow{\epsilon} S_T \]

where the homotopy equivalence on the right hand side holds because $S_T$ has a homotopy trivial $T$-action as $T$ is homotopy abelian. The first map is studied in [Bau04]. This is the map referred to in Theorem 1.1; it generalizes the Pontryagin-Thom construction.

In the second part of this paper, as an application of Theorem 1.1, we study the relationship between two notions of adjoint objects of $p$-compact groups. It is an interesting question to ask whether the action of $G$ on $S_G$ actually comes from an unstable action of $G$ on $S^d$. We will not be able to answer this question in this paper. However, there is an alternative, unstable construction of an adjoint object for a connected $p$-compact group $G$ inspired by the following:

**Theorem 1.4** ([CK02], [Mit88]). Let $G$ be a semisimple, connected Lie group of rank $r$. There exist subgroups $G_I < G$ for every $I \subsetneq \{1, \ldots, r\}$ and a homeomorphism of $G$-spaces

\[ A_G := \Sigma \text{hocolim}_{I \subseteq \{1, \ldots, r\}} G/G_I \to g \cup \{\infty\} \]

\[ \text{to the one-point compactification of the Lie algebra } g \text{ of } G. \]

In the second part of this paper, we define a $G$-space $A_G$ for every connected $p$-compact group $G$ and show:

**Theorem 1.5.** For any connected $p$-compact group $G$, there is a $G$-equivariant splitting

\[ S^0[A_G] \simeq S_G \vee R \]

for some finite $G$-spectrum $R$.

This result links the two notions of adjoint objects together. Thus stably, the adjoint sphere is a wedge summand of the adjoint space.

Unfortunately, $A_G$ is in general not a sphere.

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## 2. The Stable $p$-Complete Splitting of Complex Projective Space

### 2.1. Stable splittings from homotopy idempotents

Let $p$ be a prime. We denote by $L_p$ the localization functor on topological spaces with respect to mod-$p$ homology, which coincides with $p$-completion on nilpotent spaces [BK72]. Let $S = L_p S^1$ be the $p$-complete 1-sphere, and set $P = S^0[BS]$. It is a classical result that

\[ P \simeq \bigvee_{s=0}^{p-2} P_s \]

for certain $(2i-1)$-connected spectra $P_s$. In this section, we will investigate this splitting and its compatibility with certain transfer maps.

Let $X$ be a spectrum, $e \in [X, X]$ and define

\[ eX = \text{hocolim}\{X \xrightarrow{\epsilon} X \xrightarrow{\epsilon} \cdots\}. \]
If \( e \) is idempotent, this is a homotopy theoretic analog of the image of \( e \). Any such idempotent \( e \) yields a stable splitting \( X \cong eX \vee (1-e)X \). If \( \{e_1, \ldots, e_n\} \) are a complete set of orthogonal idempotents (this means that each \( e_j \) is idempotent, \( e_ie_j \cong \star \), and \( \text{id}_X \cong e_1+\cdots+e_n \)), then they induce a splitting \( X \cong e_1X \vee \cdots \vee e_nX \).

**Example 2.2.** Let \( p \) be an odd prime. Denote by \( \psi: P \to P \) the map induced by multiplication with a \((p-1)\)st root of unity \( \zeta \). Define \( e_\zeta: P \to P \) by

\[
e_\zeta = \frac{1}{p-1} \left( \sum_{i=0}^{l-1} \zeta^{-i} \psi^i \right).
\]

It is straightforward to check that \( \{e_0, \ldots, e_{p-2}\} \) are a complete set of orthogonal idempotents in \([P, P]\). They induce the splitting \( (2.1) \) by defining \( P_s = e_\zeta P \).

Setting \( H_s(P) = \mathbb{Z}_p\{x_j\} \) with \( |x_j| = 2j \), we have that \( (e_j)_*: H_s(P) \to H_s(P) \) is given by

\[
(e_j)_*(x_j) = \begin{cases} x_j; & j \equiv i \pmod{p-1} \\ 0; & \text{otherwise.} \end{cases}
\]

2.2. **Transfers as splittings.** Let \( 1 \to H \xrightarrow{i} G \to W \to 1 \) be an extension of compact Lie groups. Then associated to the fibration \( W \to BH \to BG \) there are two versions of functorial stable transfer maps \([BG75, BG76]\):

1. The Becker-Gottlieb transfer \( \tau: S^0[BG] \to S^0[BH] \)
2. The stable Umkehr map \( \tau: BG^\theta \to BH^\theta \) of Thom spaces of the adjoint representation of the Lie groups.

Both versions can be generalized to a setting where the groups involved are not Lie groups but only \( \mathbb{Z}/p \)-local and \( p \)-finite \([Dwy96, Bau04]\). For such a group \( G \), \( BG^\theta \) is defined to be the homotopy orbit spectrum of \( G \) acting on the dualizing spectrum \( S_G \); since \( H_s(S_G) = H_s(S^\theta_l; \mathbb{Z}_p) \), we have a (possibly twisted) Thom isomorphism between \( H_s(BG) \) and \( H_s(BG^\theta) \).

Note that \( \tau \) factors through \( \tau' \) in the following way:

\[
S^0[BG] \xrightarrow{\tau'} BH^\nu \xrightarrow{\text{comult.}} BH^\nu \wedge_{BG} S^0[BH] \xrightarrow{\text{id} \wedge \Delta} BH^\nu \wedge_{BG} S^0[BH] \wedge_{BG} S^0[BH] \xrightarrow{\text{eval} \wedge \text{id}_0} S^0[BH]
\]

where \( \nu = h - i'g \) is the normal fibration along the fibers of \( BH \to BG \), \( \tau' \) is \( \tau \) twisted by \( -g \), and the right hand side evaluation map is defined by identifying \( BH^\nu \) with the fiberwise Spanier-Whitehead dual of \( BH \) over \( BG \).

**Proposition 2.5.** Let \( W = C_l \) be a finite cyclic group acting freely on \( S \), with \( l \mid p - 1 \). Denote by \( N = S \rtimes W \) the semidirect product with respect to this action. Then the Becker-Gottlieb transfer map \( \tau: S^0[BN] \to P \) factors through \( fP \to P \) for some idempotent \( f: P \to P \) which induces the same map in homology as \( e_0 + e_1 + \cdots + e_{p-1-l} \), and the induced map \( S^0[BN] \to fP \) is a mod-\( p \) homology equivalence.

**Proof.** Since \( p \mid |W| \), the Serre spectral sequence associated to the group extension \( S \xrightarrow{i} N \to W \) is concentrated on the vertical axis and shows that

\[
H^*(BN; \mathbb{Z}_p) \cong H^*(BS; \mathbb{Z}_p)^W \cong \mathbb{Z}_p[z^l] \to \mathbb{Z}_p[z] \to H^*(BS; \mathbb{Z}_p).
\]
In this case, the Becker-Gottlieb transfer is nothing but the usual transfer for finite coverings, therefore \( i \circ \tau \) is multiplication by \( |W| = l \in \mathbb{Z}_p \). Setting \( I = l^{-1}i: P \rightarrow S^0[L_pBN] \), we thus get orthogonal idempotents in \([P,P]\):

\[
f = \tau \circ I \quad \text{and} \quad e = \text{id}_P - f.
\]

Clearly, \( e \circ \tau \simeq * \), thus \( \tau \) factors through \( fP \) and induces an isomorphism \( S^0[L_pBN] \rightarrow fP \), in particular a mod-\( p \) homology isomorphism between \( S^0[BN] \) and \( fP \). The computation of the homology of \( BN \) together with (2.3) implies that \( f_* = (e_0 + e_1 + \cdots + e_{p-1-l})_* \).

**Corollary 2.6.** Let \( S, N, W \) be as above. Then the stable Umkehr map

\[ BN^n \rightarrow BS^s \simeq \Sigma P \]

factors through \( \Sigma fP \rightarrow \Sigma P \) for some \( f: P \rightarrow P \) which induces the same morphism in homology as \( \sum_{i=0}^{p-1} e_{(i+1)l-1} \). The induced map \( BN^n \rightarrow \Sigma fP \) is a mod-\( p \) homology equivalence.

**Proof.** This follows from a similarly simple homological consideration. The \( S \)-fibration \( n \) is not orientable, thus we have a twisted Thom isomorphism

\[ H^{n+1}(BN^n) \cong H^n(BN; H^1(S; \mathbb{Z}_p)) \]

where \( \pi_1(BN) = \mathbb{Z}/l \) acts on \( H^1(S; \mathbb{Z}_p) \cong \mathbb{Z}_p \) by multiplication by an \( l \)-th root of unity. Thus

\[ H^i(BN^n; \mathbb{Z}_p) = \begin{cases} \mathbb{Z}_p; & i \equiv -1 \pmod{l} \\ 0; & \text{otherwise} \end{cases} \]

The factorization (2.4) of \( \tau \) through \( \tau \)

\[
S^0[BN] \xrightarrow{\tau} BS^v \xrightarrow{\text{mult}} BS^v \wedge_{BN} S^0[BS] \xrightarrow{\text{id} \wedge \text{id}} BS^v \wedge_{BN} S^0[BS] \wedge_{BN} S^0[BS] \xrightarrow{\text{eval} \wedge \text{id}} S^0[BS]
\]

simplifies considerably since \( i^*v \) is the trivial 1-dimensional fibration over \( BS \), and the composition of the three right hand side maps is an equivalence.

In Prop. 2.5 it was shown that \( I \circ \tau = \text{id}_{[L_pBN]} \), thus the same holds after twisting with \( n \):

\[ \text{id}_{BN^n}: L_pBN^n \xrightarrow{\tau} BS^s \rightarrow BS^{i^n} \xrightarrow{I^n} L_pBN^n. \]

If we denote the composition \( BS^s \rightarrow BS^{i^n} \xrightarrow{I^n} L_pBN^n \) by \( I \), overriding its previous meaning, the argument now proceeds as in Prop. 2.5. Using the computation of \( H^i(BN^n; \mathbb{Z}_p) \), we find that \( L_pBN^n \simeq (\tau \circ I)P \), and

\[ (\tau \circ I)_* = \sum_{i=0}^{p-1} (e_{(i+1)l-1})_* \]

\[ \square \]
3. Framing $p$-Compact Flag Varieties

Before proving Theorem 1.1, we need an alternative description of the Pontryagin-Thom construction (1.3) on $G/T$.

**Lemma 3.1.** The map $|G/T|$ is $G$-equivariantly homotopic to the map

$$S_G \xrightarrow{\text{incl}} BG^\theta \xrightarrow{\tau} BT^t \simeq \Sigma^r S^0[BT] \xrightarrow{\Sigma^r \epsilon} S^r,$$

where $BG^\theta$, $BT^t$, and $\tau$ are as in Section 2.2 and all spectra except $S_G$ have a trivial $G$-action.

**Proof.** Applying homotopy $G$-orbits to (1.3), we get a $G$-equivariant diagram

$$S_G \xrightarrow{\text{incl}} S_G \wedge D(S^0[G/T]) \xrightarrow{\sim} S^0[G/T] \wedge S_T \xrightarrow{\sim} S_T$$

which is commutative by the definition of $\tau$ [Bau04, Def. 25]. □

In the proof of Theorem 1.1, certain special subgroups will play an important role. In order to define them we need to recall certain facts about the Weyl group of a $p$-compact group.

Dwyer and Wilkerson showed in their ground-breaking paper [DW94] that given any connected $p$-compact group $G$ with maximal torus $T$, there is an associated Weyl group $W(G)$, which is defined as the group of components of the homotopy discrete space of automorphisms of the fibration $BT \to BG$. This generalizes the notion of Weyl groups of compact Lie groups; they are canonically subgroups of $\text{GL}(H_1(T)) = \text{GL}_r(\mathbb{Z}_p)$, and they are so-called finite complex reflection groups. This means that they are generated by elements (called reflections or, more classically, pseudo-reflections) that fix hyperplanes in $\mathbb{Z}_p^r$. The complete classification of complex reflection groups over $\mathbb{C}$ is classical and due to Shephard and Todd [ST54], the refinement to the $p$-adics is due to Clark and Ewing [CE74].

Call a reflection $s \in W$ **primitive** if there is no reflection $s' \in W$ of strictly larger order such that $s = (s')^k$ for some $k$.

Denote by $s \in W$ a primitive reflection of minimal order $l > 1$. Let $T^s < T$ be the fixed point subtorus under $s$. Since $s$ is primitive,

$$\langle s \rangle = \{ w \in W \mid w|_{T^s} = \text{id}_{T^s} \}.$$

**Definition.** Given connected $p$-compact group $G$ and a primitive reflection $s \in W(G)$ of minimal order $l > 1$, define $C_s$ to be the centralizer of $T^s$ in $G$.

Since $G$ is connected, so is the subgroup $C_s$ [DW95, Lemma 7.8]. Furthermore, $C_s$ has maximal rank because $T < C_s$ by definition, and the inclusion $C_s < G$ induces the inclusion of Weyl groups $\langle s \rangle < W$ [DW95 Thm. 7.6]. Since the Weyl group of $C_s$ is $\mathbb{Z}/l$, the quotient of $C_s$ by its $p$-compact center, $C_s/Z(C_s)$, can have rank at most 1. By the (almost trivial) classification of rank-1 $p$-compact groups, we find that its rank is equal to 1 and

$$C_s \cong \left( L_p(S^1)^{r-1} \times L_pS^{2l-1} \right) / \Gamma,$$
where \( L_pS^{2l-1} \) is simply \( L_p SU(2) \) for \( l = 2 \), and the Sullivan group given by

\[
L_pS^{2l-1} = \Omega L_p \left( L_p(\mathbb{S}^1hZ/l) \right)
\]

for \( p \) odd, and \( \Gamma \) is a finite central subgroup.

**Proof of Thm. 1.1.** By Lemma 3.1 showing equivariant null-homotopy is equivalent to showing that the map

\[
h(G/T) : BG^\# \overset{\tau}{\to} BT^t \xrightarrow{\text{proj}} S'
\]

is null. Note that for any given subgroup \( H < G \) of maximal rank, there is a factorization of \( \tau \) through \( BH^h \). In particular, we may assume that \( G \) is connected. By the dimension hypothesis of the theorem, \( W(G) \) is nontrivial. If \( H = C_\tau \) is the subgroup associated to a primitive reflection \( s \in W(G) \) of minimal order \( l > 1 \), then the map \( h(C_\tau/T) \) is the \((r-1)\)-fold suspension of \( h(L_pS^{2l-1}/S) \) by \((3.2)\). Therefore, it is enough to prove the theorem for those \( p \)-compact groups \( C_\tau \).

We distinguish two cases. First suppose that \( l = 2 \). By the classification of complex reflection groups \((ST54), \) and with the terminology of that paper, this is always the case except when \( W \) is a product of any number of groups from the list

\[
\{G_4, G_5, G_{16}, G_{18}, G_{20}, G_{25}, G_{32}\}.
\]

This comment is only meant to intimidate the reader and is insubstantial for what follows.

In this case, the map \( h(L_p SU(2)/S) \) is null by Pittie \((Pit75, PS75)\) since the spaces involved are Lie groups, thus \( h(G/T) \simeq * \).

Now suppose that \( l > 2 \). This forces \( p > 2 \) as well, and since \( \langle s \rangle \) acts faithfully on some line in \( H^1(T; Z_p) \) while fixing the complementary hyperplane, we must have that it acts by an \( l \)th root of unity, and thus \( l \mid p - 1 \). The proof is finished if we can show that

\[
h(L_pS^{2l-1}/S) = 0
\]

where \( S \) is the 1-dimensional maximal torus in the Sullivan group \( G' := L_pS^{2l-1}. \)

To see this, note that the inclusion \( S \rightarrow G' \) factors through the maximal torus normalizer \( N_G(S) \simeq S \times Z/l \), and thus

\[
h(G'/S) : BG'/\tau \overset{\tau}{\to} BN^\tau \xrightarrow{\tau} \sum S^0[BS] \to S^1.
\]

If \( P \simeq \bigvee_{i=0}^{p-2} e_iP \) is any stable splitting of the \( p \)-completed complex projective space \( P = S^0[BS] \) induced by idempotents \( e_i \); as in the previous section, then the rightmost projection map clearly factors through \( e_0P \), which is the part containing the bottom cell. Since \( p > 2 \), Corollary \((2.6)\) shows that there is an idempotent \( f \in [P, P] \) such that \( \tau \simeq f \circ \tau \) and \( \tau \circ e_0 = e_0 \circ \tau = 0 \), proving the theorem.

4. **The Adjoint Representation**

Let \( G \) be a \( d \)-dimensional connected \( p \)-compact group with maximal torus \( T \) of rank \( r \). Choose a set \( \{s_1, \ldots, s_v\} \) of generating reflections of \( W = W(G) \) with \( r' \) minimal. The classification of pseudo-reflection groups \((ST54, CE74)\) implies that for \( G \) semisimple, most of the time \( r = r' \), but there are cases where \( r' = r + 1 \).
Example 4.1 (The group no. 7). Let \( p = 1 \pmod{12} \). Let \( G_7 \) be the finite group generated by the reflection \( s \) of order 2 and the two reflections \( t, u \) of order 3, where \( s, t, u \in GL_2(\mathbb{Z}_p) \) are given by

\[
s = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad t = \frac{1}{\sqrt{2}} \begin{pmatrix} -\zeta & \zeta^7 \\ \zeta & -\zeta^7 \end{pmatrix}, \quad u = \frac{1}{\sqrt{2}} \begin{pmatrix} -\zeta^7 & -\zeta^2 \\ \zeta & -\zeta \end{pmatrix}.
\]

Here \( \zeta \) is a 24th primitive root of unity. Note that although possibly \( \zeta \notin \mathbb{Z}_p \), \( \frac{1}{\sqrt{2}} \zeta \in \mathbb{Z}_p \). In Shephard and Todd’s classification, this is the restriction to \( \mathbb{Z}_p \) of the complex pseudo-reflection group no. 7. They show that even over \( \mathbb{C} \), \( G_7 \) cannot be generated by two reflections. The associated \( p \)-compact group is given by

\[
\Omega L_p((BT^2)_{hG_7}).
\]

If \( G \) is not semisimple (i.e. it contains a nontrivial normal torus subgroup), then \( r' \) may be smaller than \( r \). Set \( \kappa = r + 1 - r' \geq 0 \).

Let \( \mathcal{I}_r \) be the set of proper subsets of \( \{1, \ldots, r'\} \), and for \( I \subseteq \{1, \ldots, r'\} \), let \( T_I \) be the fixed point subtorus \( T(s_{i \in I}) \) and \( C_I = C_G(T_I) \) be the centralizer in \( G \), which is connected by [DW95, Lemma 7.8].

**Definition.** Let \( G \) be a connected \( p \)-compact group. Define the **adjoint space** \( A_G \) by the homotopy colimit

\[
A_G = \Sigma^k \hocolim_{I \in \mathcal{I}_r} G / C_I
\]

with the induced left \( G \)-action, and the trivial \( G \)-action on the suspension coordinates.

Theorem 1.4 shows that if \( G \) is a the \( p \)-completion of a connected, semisimple Lie group (in this case \( r = r' \) and \( \kappa = 1 \)), then \( A_G \) is a \( d \)-dimensional sphere \( G \)-equivariantly homotopy equivalent to \( g \cup \{ \infty \} \). This holds more generally: if \( G \) is a connected, compact Lie group with maximal normal torus \( T^k \) then

\[
A_G \cong \Sigma^k A_{G / T^k} = (t \cup \{ \infty \}) \wedge (g / t u \{ \infty \}) = g \cup \{ \infty \}.
\]

**Lemma 4.2.** Let \( \mathcal{I}_r \) be the poset category of proper subsets of \( \{1, \ldots, r\} \) and

\[
F, G : \mathcal{I}_r \to \{ \text{finite CW-complexes or finite CW-spectra} \}
\]

be two functors. Then

1. If \( F \) is has the property that \( \dim F(\emptyset) > \dim F(I) \) for every \( I \neq \emptyset \), then
   \[
   \dim \hocolim F = \dim F(\emptyset) + k - 1.
   \]

2. If \( f : F \to G \) is a natural transformation of two such functors such that
   \[
   f_*(\emptyset) : H_{\dim F(\emptyset)}(F(\emptyset)) \xrightarrow{\cong} H_{\dim G(\emptyset)}(G(\emptyset)),
   \]
   then \( f \) induces an isomorphism
   \[
   \hocolim f_* : H_{\dim \hocolim F}(\hocolim F) \to H_{\dim \hocolim G}(\hocolim G).
   \]

3. Let \( F : \mathcal{I}_r \to \text{Top} \) be the functor given by \( F(\emptyset) = S^n, F(I) = * \) for \( I \neq \emptyset \). Then
   \[
   \hocolim_{\mathcal{I}_r} F \simeq S^{n+r-1}.
   \]
The first two assertions follow from the Mayer-Vietoris spectral sequence \([\text{BK72}, \text{Chapter XII.5}]\),

\[
E^1_{p,q} = \bigoplus_{l \in I_k, \, |l| = k-1-p} H_q(F(I)) \Longrightarrow H_{p+q}(\hocolim F),
\]

along with the observation that under the dimension assumptions of (1), \(E^1_{p,q} = 0\) for \(q \geq \dim F(\emptyset)\) except for \(E^1_{k-1,\dim F(\emptyset)} = H_{\dim F(\emptyset)}(F(\emptyset))\). In particular, this group cannot support a nonzero differential and thus

\[
H_i(F(\emptyset)) \cong H_{i+k-1}(\hocolim F) \quad \text{for } i \geq \dim F(\emptyset).
\]

The third one is an immediate consequence of the Mayer-Vietoris spectral sequence. \(\square\)

**Corollary 4.3.** For any connected \(p\)-compact group \(G\), \(A_G\) is a \(d\)-dimensional \(G\)-space.

**Proof.** This follows from Lemma 4.2. Indeed, since any \(C_I\) \((I \neq \emptyset)\) is connected and has the nontrivial Weyl group \(W_I\), its dimension is greater than \(\dim T\). So the condition

\[
\dim F(\emptyset) = \dim G/T > \dim F(I)
\]

is satisfied, and

\[
\dim \hocolim F = d - r + r' - 1 = d - \kappa.
\]

As mentioned at the end of the introduction, for \(p\)-compact groups \(G\), \(A_G\) is not usually a sphere, as the following example illustrates.

**Example 4.4.** Let \(p \geq 5\) be a prime, and let \(G = S^{2p-3}\) be the Sullivan sphere, whose group structure is given by \(BG = L_p(\text{BS}_h,\mathbb{C}_{p-1})\), where \(\mathbb{C}_{p-1} \subseteq \mathbb{Z}_p^\times\) acts on \(BS = K(\mathbb{Z}_p, 2)\) by multiplication on \(\mathbb{Z}_p\). Clearly, \(G\) has rank 1, and \(I_1\) consists only of a point, thus \(A_G = \Sigma G/T \simeq L_p \Sigma \mathbb{C}P^{p-2}\). Since \(p \geq 5\), this is not a sphere.

For the proof of Theorem 1.5 we need a preparatory result.

**Proposition 4.5.** Let \(P\) be a \(p\)-compact subgroup of maximal rank in a \(p\)-compact group \(G\). Denote by \(T\) a maximal torus of \(P\) (and thus also of \(G\)). Then the following composition is \(G\)-equivariantly null-homotopic:

\[
f_{G,P} : S_G \land DS_T \to S^0[G/T] \to S^0[G/P].
\]

The second map is the canonical projection, whereas the first map is given by using the duality isomorphism

\[
S_G \land DS_T \to S_G \land D(S^0[G/T]) \land DS_T \simeq G_+ \land T S_T \land DS_T \\
\simeq S^0[G/T] \land S_T \land DS_T \xrightarrow{id \land \text{ev}} S^0[G/T].
\]

**Proof.** In [Bau04, Cor. 24] it was shown that the relative duality isomorphism from Lemma 1.2 is natural in the sense that the following diagram commutes:

\[
\begin{array}{ccc}
S^0[G]^{h\mathbb{Z}_p} & \xrightarrow{\text{res}} & G_+ \land P S_P \\
\downarrow \text{res} & & \downarrow \text{D(proj)/id} \\
S^0[G]^{h\mathbb{Z}_p} & \xrightarrow{\text{res}} & G_+ \land T S_T \\
\end{array}
\]

\[
\xrightarrow{\text{D(proj)/id}} D(S^0[G/P]) \land S_G \\
\xrightarrow{id} D(S^0[G/T]) \land S_G
\]
Taking duals and smashing with $DS_G$, we find that the map of the proposition is the left hand column in the diagram

\[
\begin{array}{ccc}
S_G \land D(G_+ \land p S_p) & \sim & S^0[G/P] \\
S_G \land D(G_+ \land T S_T) & \sim & S^0[G/T] \\
S_G \land D(S^0[G/T]) \land DS_T & \sim & S_G \land DS_T \\
\end{array}
\]

Thus we need to show that the composition

\[G_+ \land p S_p \to G_+ \land T S_T \simeq S^0[G/T] \land S_T \to S_T\]

is $G$-equivariantly trivial, or equivalently, that

\[S_p \to P_+ \land T S_T \to S_T\]

is $P$-equivariantly trivial. But this map is exactly the homotopy class represented by $[P/T]$, thus the assertion follows from Theorem 1.1.

**Proof of Thm. 1.5.** Let $G$ be a connected $p$-compact group whose Weyl group is generated by a minimal set of $r'$ reflections. Let $F, A : I_{p'} \to \text{Top}$ be the functors given by $F(I) = S_G \land DS_T$, $F(I) = *$ for $I \neq \emptyset$, and $A(I) = G/C_I$. Note that, since $G$ is connected, $C_G(T) = T$ [DW94, Proposition 9.1] and $A(\emptyset) = G/T$. There is a map $\Phi : F \to A$ of $I_{p'}$-diagrams in the homotopy category of $G$-spectra which is fully described by defining

\[\Phi(I) = f_{G,T} : F(I) = S_G \land DS_T \to S^0[G/T]\]

as the map given in Prop. 1.5. The strategy of the proof is to obtain a functor $F : I_{p'} \to \text{Top}$ such that $F(\emptyset) = S_G \land DS_T$, $F(I) \simeq *$ for $I \neq \emptyset$, and a map of $I_{p'}$-diagrams $\Phi : F \to A$ in the category of $G$-spectra such that $\Phi(\emptyset) = f_{G,T}$. From this we get a $G$-equivariant map

\[S_G \simeq S^x \land \Sigma^{r'-1}S_G \land DS_T \simeq S^x \land \text{hocolim}_{I_{p'}} F \to \Sigma^x \text{hocolim}_{I_{p'}} S^0[G/C_I] \simeq S^0[A_G],\]

which will give us the splitting.

We will proceed by induction on the number of generating reflections $r'$. If $r' = 1$ then $A_G$ is $S^x \land G/T$ and $\Phi(\emptyset) = S^x \land f_{G,T}$. We can construct the functor $F$ and the natural transformation $\Phi$ step by step. Fix a subset $I$ of cardinality $k$, and assume that $F$ and $\Phi$ have been defined for all vertices in the diagram corresponding to $I'$ with $|I'| < k$.

Let $P(I)$ be the poset category of all proper subsets of $I$. Since $F$ and $\Phi$ are defined over $P(I)$ by induction hypothesis, we can consider $\text{hocolim}_{P(I)} F \simeq \Sigma^{k-1}S_G \land DS_T \to S^0[G/C_I]$. It is enough to show that this map is $G$-equivariantly nullhomotopic. Then, we can fix a null-homotopy and extend the map to the cone of $\text{hocolim}_{P(I)} F$. Finally, we define $F(I) = \tilde{C}(\text{hocolim}_{P(I)} F)$ and $\Phi(I)$ is the corresponding extension.
Note that $S_G \wedge DS_T \to S^0[G/T]$ factors through $G_+ \wedge_{C_J} S_{C_J} \wedge DS_T$. By induction, we know there is a map

$$\Sigma^{k-1}S_{C_J} \wedge DS_T \to S^0[\text{hocolim}_{J \in I_k} C_I/C_J],$$

which splits the top cell. We get a factorization

$$\Sigma^{k-1}S_G \wedge DS_T \to \Sigma^{k-1}G_+ \wedge_{C_J} S_{C_J} \wedge DS_T \to G_+ \wedge_{C_J} S^0[\text{hocolim}_{J \in I_k} C_I/C_J] \to G_+ \wedge_{C_J} S^0.$$

It thus suffices to show that in the $I_k$-diagram

$$
\begin{array}{ccc}
S_{C_J} \wedge DS_T & \xrightarrow{\text{hocolim}} & \Sigma^{k-1}S_{C_J} \wedge DS_T \\
\downarrow & & \downarrow \\
S^0[C_I/T] & \xrightarrow{\text{hocolim}} & S^0[\text{hocolim}_{J \in I_k} C_I/C_J] \\
\downarrow & & \downarrow \\
S^0 & \xrightarrow{\text{hocolim}} & S^0
\end{array}
$$

the right hand side composition $\Sigma^{k-1}S_{C_J} \wedge DS_T \to S^0$ is $C_I$-equivariantly null-homotopic. In the latter diagram, it makes no difference whether the centralizers are taken in $C_J$ or in $G$. But by Theorem 1.1 the left hand column is already null-homotopic, thus, as a colimit of null-homotopic maps over a contractible diagram, so is the right hand column. 

Conclusion and questions. In this paper, we have compared two imperfect notions of adjoint representations of a $p$-compact group $G$. One ($S_G$) is a sphere, but has a $G$-action only stably; the other ($A_G$) is an unstable $G$-space, but fails to be a sphere. The question remains whether there is an unstable $G$-sphere whose suspension spectrum is $S_G$. It might even be true that $A_G$ splits off its top cell after only one suspension, yielding a solution to this problem in the cases where the Weyl group of the rank-$r$ group $G$ is generated by $r$ reflections.

There are also a number of interesting open questions about the flag variety $G/T$ of a $p$-compact groups:

- By the classification of $p$-compact groups, $H^*(G/T; Z_p)$ is torsion free and generated in degree 2. Can this be seen directly?
- Is there a manifold $M$ such that $L_p M \simeq G/T$, analogous to smoothings of $G$ [BKNP04, BDP06]? Is it a boundary of a manifold?
- If such a manifold $M$ exists, can it be given a complex structure?

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