On the vacuum wavefunction and string tension of Yang-Mills theories in (2+1) dimensions

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Abstract

We present an analytical continuum calculation, starting from first principles, of the vacuum wavefunction and string tension for pure Yang-Mills theories in (2+1) dimensions, extending our previous analysis using gauge-invariant matrix variables. The vacuum wavefunction is consistent with what is expected at both high and low momentum regimes. The value of the string tension is in very good agreement with recent lattice Monte Carlo evaluations.
In recent papers we have done a Hamiltonian analysis of non-Abelian gauge theories in two spatial dimensions [1,2]. The analysis was facilitated by a special matrix parametrization for the gauge potentials and the use of some results from conformal field theory. We obtained results regarding the mass gap and wavefunctions as well as a reduction of the Hamiltonian to gauge-invariant degrees of freedom. In this paper, we shall extend our analysis with a more exact calculation of the vacuum wavefunction and the string tension. Our results are in very good agreement with recent Monte Carlo simulations of (2+1)-dimensional gauge theories. It should be emphasized that our work is an analytical calculation directly in the continuum and based on first principles.

We shall begin by a brief recapitulation of the main results. In our previous papers, we have used the $A$-diagonal representation. In this paper, we give a reduction of the Hamiltonian to the gauge-invariant degrees of freedom in a representation independent way, i.e., valid for the $E$-representation as well as the $A$-representation, before specializing to the $A$-representation and recovering the previous result. As far as the kinetic energy operator is concerned, the vacuum wavefunction is trivially obtained. The effect of the potential energy is included in a systematic perturbation expansion. The expansion parameter is $k/m$, where $m = \frac{e^2 c A}{2\pi}$ is the mass parameter which emerges from our analysis and $k$ is the characteristic momentum. From the vacuum wavefunction, with first order corrections due to the potential energy, we calculate the expectation value of the Wilson loop operator. This obeys the area law and gives the value of the string tension. By summing up sequences of terms in the $1/m$-expansion, the vacuum wavefunction is reexpressed in terms of a series in $J$, where $J$ is a current to be introduced below. The terms in this series interpolate smoothly between low and high momentum (standard perturbative) regimes. Similar analysis for the low energy excitation spectrum of the Hamiltonian is outlined.

We consider the Hamiltonian version of an SU($N$)-gauge theory in the $A_0 = 0$ gauge. The gauge potentials are $A_i = -it^a A_i^a$, $i = 1, 2$, where $t^a$ are hermitian ($N \times N$)-matrices which form a basis of the Lie algebra of SU($N$) with $[t^a, t^b] = if^{abc} t^c$, $\text{Tr}(t^a t^b) = \frac{1}{2} \delta^{ab}$. The Hamiltonian can be written as

$$H = T + V, \quad T = \frac{e^2}{2} \int E_i^a E_i^a, \quad V = \frac{1}{2e^2} \int B^a B^a$$

(1)
where $e$ is the coupling constant, $E_i^a$ is the electric field and $B^a = \frac{1}{2} \epsilon_{jk}(\partial_j A_k - \partial_k A_j + [A_j, A_k])^a$ is the magnetic field. We shall use complex coordinates $z = x_1 - ix_2$, $\bar{z} = x_1 + ix_2$ with the corresponding components $A = A_z = \frac{1}{2}(A_1 + iA_2)$, $\bar{A} = \frac{1}{2}(A_1 - iA_2)$, $E = \frac{1}{2}(E_1 + iE_2)$, $\bar{E} = \frac{1}{2}(E_1 - iE_2)$. In (2+1) dimensions, $e^2$ has the dimension of mass.

The parametrization of the gauge potentials we have used in our analysis is

$$A = -\partial M M^{-1}, \quad \bar{A} = M^\dagger \bar{\partial} M^\dagger$$

(2)

where $M$ is a complex $SL(N, \mathbb{C})$-matrix. In terms of this parametrization, the volume element on the space $\mathcal{C}$ of gauge-invariant configurations can be explicitly calculated as

$$d\mu(\mathcal{C}) = d\mu(H) \ e^{2c_A S(H)}$$

(3)

where $H = M^\dagger M$ and $d\mu(H) = \prod_x \det [d\varphi^a]$ is the Haar measure for the hermitian matrix-valued field $H$ [1-4]. Here we parametrize $H$ in terms of real fields $\varphi^a(x)$ with

$$H^{-1} dH = d\varphi^a r_{ak}(\varphi) t_k. \ c_A is the quadratic Casimir of the adjoint representation, c_A \delta^{ab} = f^{amn} f^{bmn} and is equal to $N$ for an SU($N$)-gauge theory. $S(H)$ is the Wess-Zumino-Witten (WZW) action for the hermitian matrix field $H$ given by

$$S(H) = \frac{1}{2\pi} \int \text{Tr}(\partial H \bar{\partial} H^{-1}) + \frac{i}{12\pi} \int \epsilon^{\mu\nu\alpha} \text{Tr}(H^{-1} \partial_\mu H H^{-1} \partial_\nu H H^{-1} \partial_\alpha H)$$

(4)

The inner product for the gauge-invariant physical states is given by

$$\langle 1 | 2 \rangle = \int d\mu(H) e^{2c_A S(H)} \Psi_1^*(H) \Psi_2(H)$$

(5)

This reduces matrix elements of the (2 + 1)-dimensional gauge theory to Euclidean correlators of a hermitian WZW model. This is the essence of the simplification of calculations in the gauge theory. Expressions for the Hamiltonian, discussions of states and detailed analysis of regulators are all given in references [1,2].

We now consider the reduction to gauge-invariant degrees of freedom directly in the operator language, without choosing a representation. The Gauss law operator $I^a(x)$ is given by

$$I^a(x) = 2(D\bar{E} + \bar{D}E)^a(x)$$

(6)
where \((D, \bar{D})\) are covariant derivatives, \(Dh = \partial h + [A, h], \bar{D}h = \bar{\partial} h + [\bar{A}, h]\). We shall consider \((\bar{E}^a, I^a)\) as the independent variables writing

\[ E(\vec{x}) = \int_y \bar{G}(\vec{x}, \vec{y})(\frac{1}{2} I - D\bar{E})(\vec{y}) \]  

(7)

where \((\bar{D}_x \bar{G}(\vec{x}, \vec{y}))^{ab} = \delta^{ab} \delta(\vec{x} - \vec{y})\). The basic commutation rules are

\[ [E^a(\vec{x}), \bar{A}^b(\vec{y})] = \frac{i}{2} \delta^{ab} \delta(\vec{x} - \vec{y}) \]

\[ [I^a(\vec{x}), A^b(\vec{y})] = -i D_x^{ab} \delta(\vec{x} - \vec{y}) \]

(8)

Eq.(7) is consistent with these commutation rules and hence is valid as an operator identity.

The kinetic energy operator can now be written as

\[ T = 2e^2 \int x E^a(\vec{x})\bar{E}^a(\vec{x}) = 2e^2 \int_{x,y} \left[ \bar{G}^{ab}(\vec{x}, \vec{y}) \left( \frac{1}{2} I - D\bar{E} \right)^b(\vec{y}) \right] \bar{E}^a(\vec{x}) \]  

(9)

We now move the Gauss law operator to the right end of this expression. We find

\[ \frac{1}{2} \int_y \bar{G}^{ab}(\vec{x}, \vec{y}) I^b(\vec{y}) \bar{E}^a(\vec{x}) = \frac{1}{2} \int_y \bar{G}^{ab}(\vec{x}, \vec{y}) \bar{E}^a(\vec{x}) I^b(\vec{y}) - \frac{i}{2} \int_y \bar{G}^{ab}(\vec{x}, \vec{y}) f^{abc} \bar{E}^c(\vec{y}) \delta(\vec{x} - \vec{y}) \]

\[ = \frac{1}{2} \int_y \bar{G}^{ab}(\vec{x}, \vec{y}) \bar{E}^a(\vec{x}) I^b(\vec{y}) - \frac{1}{2} \text{Tr} \left[ T^c \bar{G}(\vec{x}, \vec{y}) \right] \bar{y} \to \bar{x} \bar{E}^c(\vec{x}) \]  

(10)

where \(T^c_{ab} = -if^{abc}\) is the adjoint representation of \(t^c\). The coincident point limit of the Green’s function has to be evaluated with a gauge-invariant regulator and, as we have noted before, it is equivalent to an anomaly calculation in two Euclidean dimensions. (The Green’s function \(\bar{G}(\vec{x}, \vec{y})\) can be considered as the propagator for a chiral fermion in two Euclidean dimensions. The coincident point limit we need is the fermionic current in a background field \(A, \bar{A}\). The covariant divergence of the current is the standard gauge anomaly and hence we can obtain the current by integration of the anomaly. Regularization issues have been discussed in detail in reference [2].) The result is

\[ -\frac{1}{2} \text{Tr} \left[ T^a \bar{G}(\vec{x}, \vec{y}) \right] \bar{y} \to \bar{x} = \frac{icA}{2\pi} (A - M^\dagger - 1 \partial M^\dagger)^a \]  

(11)

(In terms of integrating the anomaly, this equation should read

\[ -\frac{1}{2} \text{Tr} \left[ T^a \bar{G}(\vec{x}, \vec{y}) \right] \bar{y} \to \bar{x} = \frac{icA}{2\pi} \int_y \bar{G}(\vec{x}, \vec{y})^{ab} (\bar{\partial} A - \partial \bar{A} + [\bar{A}, A])^b(\vec{y}) \]  

(12)
A partial integration then leads to Eq.(11).)

Using Eqs.(10,11), the kinetic energy operator becomes
\[
T = 2im \int_x (A - M\dagger^{-1}\partial M\dagger)(\vec{x})\tilde{E}^a(\vec{x}) - 2e^2 \int_{x,y} (\tilde{G}(\vec{x}, \vec{y})D\tilde{E}(\vec{y}))^a\tilde{E}^a(\vec{x})
\]
\[
+ e^2 \int_{x,y} \tilde{G}^{ab}(\vec{x}, \vec{y})\tilde{E}^a(\vec{x})I^b(\vec{y})
\]
(13)
where \(m = e^2c_A/2\pi\). On physical states which are annihilated by the Gauss law operator \(I^a\), the last term gives zero. The first term carries information about the mass gap.

Eq.(13) gives an expression for \(T\) which is valid in both \(E\)- and \(A\)-representations. In the \(E\)-representation, the quantity \(M\dagger^{-1}\partial M\dagger\) is a very nonlocal operator involving differentiations with respect to \(E^a\). In the \(A\)-representation, we can simplify expression (13) further. The parametrization (2) for the \(A\)'s can be written as
\[
A = M\dagger^{-1}(-\partial H H^{-1})M\dagger + M\dagger^{-1}\partial M\dagger
\]
\[
\bar{A} = M\dagger^{-1}\bar{\partial} M\dagger
\]
(14)
where \(H = M\dagger M\). Thus \((A, \bar{A})\) is a complex \(SL(N, \mathbb{C})\)-gauge transform of \((-\partial H H^{-1}, 0)\).

Eventhough this involves a complex gauge transformation, it is possible to use this information to simplify \(T\). In the \(A\)-representation, the wavefunction \(\Psi(A, \bar{A})\) may be taken to be a function of \(J^a = (c_A/\pi)\partial H H^{-1}\) and \(M\dagger\), as seen from (14). A change of \(M\dagger\) is equivalent to a gauge transformation, but with complex gauge parameters. Thus we may write, for infinitesimal \(\theta\),
\[
\Psi(M\dagger e^{\theta}, J) \approx \Psi(M\dagger, J) + \int \theta^a I^a \Psi(M\dagger, J)
\]
(15)
\(I^a\) may be thought of as a functional differential operator on functions of \(J, M\dagger\). Eventhough \(\theta\) is complex in general (and not purely imaginary as for a unitary transformation), the condition \(I^a\Psi = 0\) is sufficient to write
\[
\Psi(M\dagger e^{\theta}, J) = \Psi(M\dagger, J)
\]
(16)
for physical states. By a sequence of such transformations, we may set \(M\dagger\) to 1, i.e., \(\Psi\) may be taken to be purely a function of \(J\). (Notice that, in two dimensions, all configurations \(M\dagger\)
can be connected to the identity, i.e., are homotopic to the identity, since $\Pi_2(SL(N, C)) = 0$.) In this case, we may replace $A$ by $-\partial H^{-1}$, $\tilde{A}$ by zero and Eq.(13) for $T$ then becomes

$$T = m \left[ \int_u J^a(\vec{u}) \frac{\delta}{\delta J^a(\vec{u})} + \int \Omega_{ab}(\vec{u}, \vec{v}) \frac{\delta}{\delta J^a(\vec{u})} \frac{\delta}{\delta J^b(\vec{v})} \right]$$

$$\Omega_{ab}(\vec{u}, \vec{v}) = \frac{c_A}{\pi^2} \frac{\delta_{ab}}{(u - v)^2} - i \frac{f_{abc} J^c(\vec{v})}{\pi(u - v)}$$

(17)

Essentially the first term in $T$ provides a mass gap $\sim nm$ for a state composed out of $n$ $J$'s. (Of course, this value may be modified by extra contributions from the second term, see reference [2].)

In principle, one may also obtain the measure of integration for the inner product of the wavefunctions by requiring self-adjointness of the above expression. This will coincide with the converse calculation in reference [2], where we have checked that this expression is self-adjoint with the inner product as given in Eq.(5).

Eq.(17) may be taken as the starting point for analyzing the physical spectrum of the theory. Here we have outlined a particular way to derive this expression. There are many other ways to arrive at Eq.(17), some of which are discussed in references [1,2].

In terms of the collective field variable $J$ the potential energy term is written as

$$V = \frac{1}{2e^2} \int B^2(\vec{x}) = \frac{\pi}{mc_A} \int \partial J_a(\vec{x}) \partial J_a(\vec{x})$$

(18)

Notice that, for momentum modes $k \ll e^2 \sim m$, the potential energy term gives contributions of the order $k^2/m$. For momenta of this order, $V$ can be treated perturbatively. So in this regime, which can be thought of as a strong coupling regime, one can in principle analyze the spectrum of the full theory by studying the spectrum of the kinetic energy operator, Eq.(17), and including the perturbative corrections from the potential energy term. This is the approach we are going to follow in order to derive an expression for the vacuum wavefunction of the theory.

As far as the kinetic energy operator is concerned, $\Psi_0 = 1$ may be taken as the vacuum wavefunction. Trivial as it may seem, it is important that $\Psi_0 = 1$ is normalizable with the inner product (5). For low momentum modes, $k \ll m$, the inclusion of the potential energy term leads to a modified vacuum wavefunction which can be written as

$$\Psi = e^P \Psi_0$$

(19)
where $P$ is a functional of the $J$'s which can be expanded in powers of $1/m$. The various terms in this expansion can be determined from the Schrödinger equation for the vacuum wavefunction

$$\mathcal{H}\Psi = (T + V)\Psi = 0$$  \hspace{1cm} (20a)

or equivalently

$$\tilde{\mathcal{H}}\Psi_0 = e^{-P}(T + V)e^P\Psi_0 = 0$$  \hspace{1cm} (20b)

Further, since $T$ contains at most two derivatives with respect to $J$'s,

$$\tilde{\mathcal{H}} = e^{-P}(T + V)e^P = T + V + [T, P] + \frac{1}{2}[[T, P], P]$$  \hspace{1cm} (21)

Using Eqs.(17), (20) and (21), we can, in principle, calculate the full $1/m$-expansion of the vacuum wavefunction. The first few terms are given as

$$P = -\frac{\pi}{m^2c_A}\text{Tr} \int :\bar{\partial}J\partial J :$$

$$- \left(\frac{\pi}{m^2c_A}\right)^2 \text{Tr} \int [:\bar{\partial}J(D\partial)\partial J + \frac{1}{3}\bar{\partial}J[J, \partial^2 J] :]$$

$$- 2 \left(\frac{\pi}{m^2c_A}\right)^3 \text{Tr} \int [:\bar{\partial}J(D\partial)^2\partial J + \frac{2}{9}[D\partial J, \bar{\partial}J][\bar{\partial}^2 J] + \frac{8}{9}[D\bar{\partial}^2 J, J]\partial^2 J$$

$$- \frac{1}{6}[J, \bar{\partial}J][\bar{\partial}J, \partial^2 J] - \frac{2}{9}[J, \bar{\partial}J][J, \bar{\partial}^3 J] :] + \mathcal{O}(\frac{1}{m^8})$$  \hspace{1cm} (22)

where $Dh = \frac{\epsilon_A}{\pi}\partial h - [J, h]$. The normal ordering of various terms in Eq.(22) is necessary for $P$ to satisfy Eq.(20). The second derivative in Eq.(17) can give singularities when acting on composite operators. The normal ordering subtracts out precisely these singularities.

There are several interesting points regarding the expansion in Eq.(22). The leading order term for the vacuum wavefunction is

$$\Psi \approx \exp \left[-\frac{\pi}{m^2c_A}\text{Tr} \int :\bar{\partial}J\partial J :\right] = \exp \left[-\frac{1}{2me^2}\text{Tr} \int B^2 \right]$$  \hspace{1cm} (23)

The calculation of expectation values involves averaging with the factor $\Psi^*\Psi \approx e^{-S}$, where $S$, as seen from the above equation, is the action of a Euclidean two-dimensional Yang-Mills theory of coupling constant $g^2 = me^2 = e^4c_A/2\pi$. Thus, retaining only the leading term
in $\Psi$, the expectation value of the Wilson loop operator in the fundamental representation is given by
\[
\langle W_F(C) \rangle = \exp \left[ -\frac{e^4 c_A c_F}{4\pi} A_C \right]
\]
(24)
where $A_C$ is the area of the loop $C$ and $c_F$ is the quadratic Casimir of the fundamental representation [5]. The expectation value of the Wilson loop exhibits an area law behavior, as expected for a confining theory. From Eq.(24) we can easily identify the expression for the string tension $\sigma$ as
\[
\sigma = \frac{e^4 c_A c_F}{4\pi} = e^4 \left( \frac{N^2 - 1}{8\pi} \right)
\]
(25)
Recent Monte Carlo calculations of the string tension give the values $\sqrt{\sigma}/e^2 = 0.335, 0.553, 0.758, 0.966$ for the gauge groups $SU(2)$, $SU(3)$, $SU(4)$ and $SU(5)$ respectively [6]. The corresponding values calculated from Eq.(25) are $0.345, 0.564, 0.772, 0.977$. We see that there is excellent agreement (upto $\sim 3\%$) between Eq.(25) and the Monte Carlo results. It is further interesting to notice that our analytic expression for the string tension (25) has the appropriate $N$-dependence as expected from large-$N$ calculations.

Eq.(23) is roughly in agreement with conjectures on the form of the vacuum wavefunction proposed by Greensite and others [7, 8]. It was suggested there that, for long wavelength configurations, the vacuum wavefunction admits an expansion in terms of local gauge-invariant quantities of the form
\[
\ln \Psi = \int \frac{b_1}{e^4} B^2 + \frac{b_2}{e^8} (D_i B)^2 + ...
\]
(26)
where $D_i$ is the covariant derivative, $D_i = \partial_i - [A_i, ]$. Our analytical expansion (22) does not quite agree with this conjecture. Our expansion is local in terms of the gauge-invariant variables $J$, but not local in terms of $B$. One can easily work out the following relations between various derivatives of $J$ and $B$.
\[
\bar{\partial}^n J = -\frac{c_A}{2\pi} M^\dagger (\bar{D}^{n-1} B) M^{\dagger-1}
\]
\[
(D\bar{\partial})^n \bar{\partial} J = \frac{1}{2} \left( \frac{c_A}{\pi} \right)^{n+1} M^\dagger (D\bar{D})^n B M^{\dagger-1}
\]
(27)
Using these relations one can easily check that all the expressions which involve only derivatives of $J$ are local expressions in terms of $B$. The nonlocality appears in terms
which contain bare $J$’s. These terms involve the expression $M^\dagger^{-1}\partial M^\dagger - A$, which is nonlocal in terms of $B$ since $M^\dagger^{-1}\partial M^\dagger - A = -\frac{1}{2}\bar{D}^{-1}B$. For example, the term of order $1/m^4$ in Eq.(22) can be written in terms of $B$ as

$$-rac{c_A}{m^4\pi} \text{Tr} \int \left[ \frac{1}{4} BDD\bar{B} + \frac{1}{24} [\bar{D}^{-1}B, B]\bar{D}\bar{B} \right]$$

(28)

The first term in the above expression is local in $B$ and is the same term that appears in Eq.(26), but the second term is nonlocal in $B$. Similarly the term of order $1/m^6$ in Eq.(22) can be written in terms of $B$ as

$$-rac{c_A}{2m^6\pi} \text{Tr} \int \left[ B(D\bar{D})^2 B - \frac{1}{9}[DB, B]\bar{D}\bar{B} - \frac{4}{9}[DD\bar{B}, \bar{D}^{-1}B]\bar{D}\bar{B} - \frac{1}{24}[\bar{D}^{-1}B, B][B, \bar{D}\bar{B}] \
- \frac{1}{18}[\bar{D}^{-1}B, B][\bar{D}^{-1}B, \bar{D}^2B] \right]$$

(29)

There have been several attempts to numerically estimate the coefficients $b_1, b_2$ using Monte Carlo simulations of the corresponding lattice gauge theory [8]. In these calculations a local expansion as in Eq.(26) has been assumed. It is interesting to investigate whether one could incorporate the nonlocal terms in lattice calculations.

The approximation of the vacuum wavefunction by the first few terms in Eq.(22) makes sense only in the low momentum regime $k \ll m$. On the other hand if we were able to sum up the whole series we could get information on the vacuum wavefunction away from the low momentum region. In fact, we can now show that Eq.(22) can be used to reconstruct the vacuum wavefunction for short distances, $k \gg e^2$, which can be thought of as a weak coupling regime. The terms in Eq.(22) can be naturally rearranged into terms with two $J$’s, terms with three $J$’s, etc. This way we convert the $1/m$ expansion into a series expansion in $J$’s. The series of terms with only two $J$’s can be summed up to give

$$P = -\frac{1}{2e^2} \int_{x,y} B_a(x) \left[ \frac{1}{(m + \sqrt{m^2 - \nabla^2})} \right]_{x,y} B_a(y)$$

(30)

In the weak coupling or high momentum regime $k \gg e^2$, the leading order term in Eq.(30) is

$$P = -\frac{1}{2e^2} \int_{x,y} B_a(x) \left[ (-\nabla^2)^{-\frac{1}{2}} \right]_{x,y} B_a(y)$$

(31)
This is the vacuum wavefunction for an Abelian theory as expected. Of course, in the low momentum regime \( k \ll e^2 \) the leading order term in Eq.(30) will reproduce Eq.(23). (This wavefunction is similar to, but not quite the same, as the trial function suggested in reference [9].)

We now turn to the contribution of the \( 3J \)-terms. This can be, in principle, derived by resumming all the \( 3J \)-terms in Eq.(22). An easier way is to postulate a series expansion in \( J \)'s and solve the recursion relations on the coefficients which follow from Eqs.(20,21).

This gives

\[
P = -\frac{2}{e^2} \left[ \frac{\pi^2}{cA^2} \int \partial J_a \left( \frac{1}{m + \sqrt{m^2 - \nabla^2}} \right) \partial J_a + f_{abc} \int f^{(3)}(\vec{x}, \vec{y}, \vec{z})J_a(\vec{x})J_b(\vec{y})J_c(\vec{z}) \right]
\]

where \( f^{(3)} \) is given, in momentum space, as

\[
f^{(3)}(\vec{k}, \vec{p}, \vec{q}) = (2\pi)^2 \delta(\vec{k} + \vec{p} + \vec{q}) \frac{1}{8} \left( \frac{\pi}{cA} \right)^3 \frac{(E_k - m)(E_p - m)}{E_k + E_p + E_q} \frac{\vec{k} - \vec{p}}{kp}
\]

with \( E_k = \sqrt{m^2 + \vec{k}^2} \), etc. The momenta in the denominator in the above expression are the holomorphic components, \( k = \frac{1}{2}(k_1 + ik_2) \), etc. Some, but not all, of the non-Abelian terms involving the structure constants are just what is needed to covariantize the derivatives in the first term of Eq.(32), so that \( \nabla^2 \) is appropriately changed to \( 4D\partial \), or equivalently \( \nabla^2 \) in Eq.(30) is changed to the gauge-covariant Laplacian.

The low momentum expansion of the terms in Eq.(32) reproduces Eq.(22) to the appropriate order. At high momenta, we see from Eq.(33) that the \( 3J \)-term is subdominant compared to the leading \( 2J \)-term in \( P \). This is consistent with what is expected from perturbation theory. (In comparing with perturbation theory, recall that \( eA \sim (\pi/c_A)J \), where \( A \) is the gauge potential. Therefore the \( 3J \)-terms involve one power of \( e \) and an \( f_{abc} \)-factor.) This shows that the \( 3J \)-term contribution is subdominant compared to the \( 2J \)-term in Eq.(32) for both the low and high momentum regimes. Similar arguments hold, based on dimensional analysis, for the higher \( J \)-terms. The analytic expansion in Eq.(32), as a series expansion in \( J \)'s is thus consistent with both the low and high momentum regimes.

So far we have discussed the structure of the vacuum wavefunction. One could, in principle, extend the above analysis for the excitation spectrum of the Schrödinger equation.
For example, in the absence of the potential term $V$, the current $J$ is an eigenstate of the kinetic energy operator $T$, Eq.(17), with eigenvalue $m$. One could now ask what the corresponding modified eigenstate and eigenvalue should be, once the potential term is included. We would expect to find an expression of the form $\tilde{J}e^{P}$, where $\tilde{J} = J + O(J^2)$. The higher $J$-terms can, in principle, be calculated by solving the Schrödinger equation for $\tilde{H}$, where $\tilde{H}$ is given by Eq.(21). If we neglect the higher $J$-terms and keep only the $2J$-term in the expression for $P$ we find

$$\tilde{H} = \int \left[ \sqrt{m^2 - \nabla^2} J_a(\vec{x}) \right] \frac{\delta}{\delta J_a(\vec{x})} + m \int \delta_{ab}(\vec{x}, \vec{y}) \frac{\delta}{\delta J^a(\vec{x})} \frac{\delta}{\delta J^b(\vec{y})}$$

As expected in a relativistic theory, the mass $m$ gets corrected to its relativistic expression $\sqrt{m^2 + \vec{k}^2}$. This is very similar to what happens with solitons in a weak coupling expansion [10]. We are currently investigating how the Hamiltonian gets modified by the inclusion of the higher $J$-terms in Eq.(32). As in the case of the vacuum wavefunction we expect these terms to be subdominant in both the low and high energy regimes.

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