Ergodic Capacity Analysis of Amplify-and-Forward MIMO Dual-Hop Systems

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Abstract

This paper presents an analytical characterization of the ergodic capacity of amplify-and-forward (AF) MIMO dual-hop relay channels, assuming that the channel state information is available at the destination terminal only. In contrast to prior results, our expressions apply for arbitrary numbers of antennas and arbitrary relay configurations. We derive an expression for the exact ergodic capacity, simplified closed-form expressions for the high SNR regime, and tight closed-form upper and lower bounds. These results are made possible to employing recent tools from finite-dimensional random matrix theory to derive new closed-form expressions for various statistical properties of the equivalent AF MIMO dual-hop relay channel, such as the distribution of an unordered eigenvalue and certain random determinant properties. Based on the analytical capacity expressions, we investigate the impact of the system and channel characteristics, such as the antenna configuration and the relay power gain. We also demonstrate a number of interesting relationships between the dual-hop AF MIMO relay channel and conventional point-to-point MIMO channels in various asymptotic regimes.

Index Terms

Multiple-input multiple-output (MIMO), amplify-and-forward (AF), ergodic capacity.

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I. INTRODUCTION

The relay channel, first introduced in [1, 2], has been considered in recent years as a means to improve the coverage and reliability, and to reduce the interference in wireless networks [3–11]. Generally speaking, there are three main types of relaying protocols: decode-and-forward (DF), compress-and-forward (CF), and amplify-and-forward (AF). Of these protocols, the AF approach is the simplest scheme, in which case the sources transmit messages to the relays, which then simply scale their received signals according to a power constraint and forward the scaled signals onto the destinations.

Point-to-point multiple-input multiple-output (MIMO) communication systems have also been receiving considerable attention in the last decade due to their potential for providing linear capacity growth and significant performance improvements over conventional single-input single-output (SISO) systems [12, 13]. Recently, the application of MIMO techniques in conjunction with relaying protocols has become a topic of increasing interest as a means of achieving further performance improvements in wireless networks [14–18]

In this paper we investigate the ergodic capacity of AF MIMO dual-hop systems. This problem has been recently considered in various settings. In [19], the ergodic capacity of AF MIMO dual-hop systems was examined for a large numbers of relay antennas $K$, and was shown to scale with $\log K$. Asymptotic ergodic capacity results were also obtained in [20] by means of the replica method from statistical physics. In [21, 22], the asymptotic network capacity was examined as the number of source/desination antennas $M$ and relay antennas $K$ grew large with a fixed-ratio $K/M \rightarrow \beta$ using tools from large-dimensional random matrix theory. It was demonstrated that for $\beta \rightarrow \infty$, the relay network behaved equivalently to a point-to-point MIMO link. The results of [21, 22] were further elaborated in [23] where a general asymptotic ergodic capacity formula was presented for multi-level AF relay networks. Recently, the asymptotic mean and variance of the mutual information in correlated Rayleigh fading was studied in [24]. All of these prior capacity results, however, were derived by employing asymptotic methods (i.e. by letting the system dimensions grow to infinity). To the best of our knowledge, there appear to be no analytical ergodic capacity results which apply for AF MIMO dual hop systems with arbitrary finite antenna and relaying configurations.

In this paper we derive new exact analytical results, simple closed-form high SNR expressions, and tight closed-form upper and lower bounds on the ergodic capacity of AF MIMO dual-hop systems. In contrast to previous results, our expressions apply for any finite number of MIMO antennas and for arbitrary numbers of relay antennas. The results are based heavily on the theory of finite-dimensional random matrices. In particular, our exact ergodic capacity results are based on a new exact expression which we derive for the exact unordered eigenvalue distribution of a certain product of finite-dimensional
random matrices, corresponding to the equivalent cascaded AF MIMO relay channel. In prior work [22], an asymptotic expression was obtained for this unordered eigenvalue density. However, that asymptotic result, which serves as an approximation for finite-dimensional systems, was rather complicated and required the numerical computation of a certain fixed-point equation. Our result, in contrast, is a simple exact closed-form expression, involving only standard functions which can be easily and efficiently evaluated. In addition to the unordered eigenvalue distribution, we also present a number of new random determinant properties (such as the expected characteristic polynomial) of the equivalent cascaded AF MIMO relay channel. These results are subsequently employed to derive simplified closed-form expressions for the ergodic capacity in the high SNR regime, as well as tight upper and lower bounds. Again, these random determinant properties are exact closed-form analytical results which apply for arbitrary antenna and relaying configurations, and are expressed in terms of standard functions which are easy to compute. As a by-product of these derivations, we also present some new unified expressions for the expected characteristic polynomial and expected log-determinant of semi-correlated Wishart and pseudo-Wishart random matrices.

Based on our analytical expressions, we investigate the effect of the different system and channel parameters on the ergodic capacity. For example, we show that when either the number of source, destination, or relay antennas, or the the relay gain grows large, the AF MIMO dual-hop capacity admits a simple interpretation in terms of the ergodic capacity of conventional single-hop single-user MIMO channels. In the high SNR regime, we present simple closed-form expressions for the key performance parameters—the high SNR slope and the high SNR power offset—which reveal the intuitive result that the multiplexing gain is determined by the minimum of the number of antennas at the source, destination, and relay, whereas the power offset is a more intricate function which depends on all three. For example, we show that by adding more antennas at the destination, whilst keeping the number of source and destination antennas fixed, may lead to a significant improvement in the high SNR power offset; however the relative gain becomes less significant as the initial number of destination antennas is increased. Our analytical expressions also reveal the interesting result that the ergodic capacity of AF MIMO dual-hop channels is upper bounded by the capacity of a SISO additive white Gaussian noise (AWGN) channel.

The remainder of this paper is organized as follows. Section II presents the AF MIMO dual-hop system model under consideration. Section III presents our new random matrix theory contributions, which are subsequently used to derive the exact, high SNR, and upper and lower bound expressions for the ergodic capacity in Sections IV and V. Section VI summarizes the main results of the paper. All of the main mathematical proofs have been placed in the Appendices.
II. SYSTEM MODEL

We employ the same AF MIMO dual-hop system model as in [21, 22]. In particular, suppose that there are \( n_s \) source antennas, \( n_r \) relay antennas and \( n_d \) destination antennas, which we represent by the 3-tuple \((n_s, n_r, n_d)\). All terminals operate in half-duplex mode, and as such communication occurs from source to relay and from relay to destination in two separate time slots. It is assumed that there is no direct communication link between the source and destination, as sketched in Fig. 1. The end-to-end input-output relation of this channel is then given by

\[
y = H_2 F H_1 s + H_2 F n_r + n_d
\]

where \( s \) is the transmit symbol vector, \( n_r \) and \( n_d \) are the relay and destination noise vectors respectively, \( F = \sqrt{\alpha/(n_r (1 + \rho))} I_{n_r} \) (\( \alpha \) corresponds to the overall power gain of the relay terminal) is the forwarding matrix at the relay terminal which simply forwards scaled versions of its received signals, and \( H_1 \in \mathbb{C}^{n_r \times n_s} \) and \( H_2 \in \mathbb{C}^{n_d \times n_r} \) denote the channel matrices of the first hop and the second hop respectively, where their entries are assumed to be zero mean circular symmetric complex Gaussian (ZMCSCG) random variables of unit variance. The input symbols are chosen to be independent and identically distributed (i.i.d.) ZMCSCGs and the per antenna power is assumed to be \( \rho/n_s \), i.e., \( E \{ ss^\dagger \} = (\rho/n_s) I_{n_s} \). The additive noise at the relay and destination are assumed to be white in both space and time and are modeled as ZMCSCG with unit variance, i.e., \( E \{ n_r n_r^\dagger \} = I_{n_r} \) and \( E \{ n_d n_d^\dagger \} = I_{n_d} \). We assume that the source and relay have no channel state information (CSI), and that the destination has perfect knowledge of both \( H_2 \) and \( H_2 H_1 \).

The ergodic capacity (in b/s/Hz) of the AF MIMO dual-hop system described above can be written as [20–22]

\[
C = \frac{1}{2} E \left\{ \log_2 \det \left( I + R_s R_n^{-1} \right) \right\}
\]
where $\mathbf{R}_s$ and $\mathbf{R}_n$ are $n_d \times n_d$ matrices given by

$$
\mathbf{R}_s = \frac{\rho a}{n_s} \mathbf{H}_2 \mathbf{H}_1 \mathbf{H}_1^\dagger \mathbf{H}_2^\dagger
$$

(3)

and

$$
\mathbf{R}_n = \mathbf{I}_{n_d} + a \mathbf{H}_2 \mathbf{H}_2^\dagger
$$

(4)

respectively, with

$$
a = \frac{\alpha}{n_r (1 + \rho)}.
$$

(5)

Using the identity

$$
\det (\mathbf{I} + \mathbf{AB}) = \det (\mathbf{I} + \mathbf{BA}),
$$

(6)

(2) can be alternatively expressed as follows

$$
C (\rho) = \frac{1}{2} \mathbb{E} \left\{ \log_2 \det \left( \mathbf{I}_{n_r} + \frac{\rho a}{n_s} \mathbf{H}_1^\dagger \mathbf{H}_2 \mathbf{R}_n^{-1} \mathbf{H}_2^\dagger \mathbf{H}_1 \right) \right\}.
$$

(7)

Next, we utilize the singular value decomposition to write $\mathbf{H}_2 = \mathbf{U}_2 \mathbf{D}_2 \mathbf{V}_2^\dagger$, where

$$
\mathbf{D}_2 = \text{diag} \{ \lambda_1, \ldots, \lambda_{\min(n_d,n_r)} \}
$$

(8)

is an $n_d \times n_r$ diagonal matrix, with diagonal elements pertaining to the increasing ordered singular values, and $\mathbf{U}_2 \in \mathbb{C}^{n_d \times n_d}$ and $\mathbf{V}_2 \in \mathbb{C}^{n_r \times n_r}$ are unitary matrices containing the respective eigenvectors. Since $\mathbf{H}_1$ is invariant under left and right unitary transformation, the ergodic capacity in (7) can be further simplified as

$$
C (\rho) = \frac{1}{2} \mathbb{E} \left\{ \log_2 \det \left( \mathbf{I}_{n_r} + \frac{\rho a}{n_s} \mathbf{H}_1^\dagger \mathbf{H}_1 \mathbf{H}_1 \mathbf{H}_2 \mathbf{L} \mathbf{H}_1 \right) \right\}
$$

(9)

where

$$
\mathbf{\Psi} = \begin{cases} 
\text{diag} \left\{ \frac{\lambda_1^2}{1 + a \lambda_1^2}, \ldots, \frac{\lambda_{n_r}^2}{1 + a \lambda_{n_r}^2} \right\}, & n_r \leq n_d, \\
\text{diag} \left\{ \frac{\lambda_1^2}{1 + a \lambda_1^2}, \ldots, \frac{\lambda_{n_d}^2}{1 + a \lambda_{n_d}^2}, 0, \ldots, 0 \right\}, & n_r > n_d.
\end{cases}
$$

(10)

It is then easily established that

$$
C (\rho) = \frac{1}{2} \mathbb{E} \left\{ \log_2 \det \left( \mathbf{I}_{n_s} + \frac{\rho a}{n_s} \tilde{\mathbf{H}}_1^\dagger \mathbf{L} \tilde{\mathbf{H}}_1 \right) \right\}
$$

(11)
where $\tilde{H}_1^\dagger \sim CN_{n_s,q} (0, I_{n_s} \otimes I_q)$, with $q = \min (n_d, n_r)$, and

$$L = \text{diag} \left\{ \frac{\lambda_i^2}{1 + a\lambda_i^2} \right\}_{i=1}^q.$$  \hspace{1cm} (12)

Equivalently, we can now write

$$C (\rho) = \frac{s}{2} \int_0^\infty \log_2 \left( 1 + \frac{\rho a}{n_s} \frac{\lambda}{\rho a} \right) f_\lambda (\lambda) d\lambda$$  \hspace{1cm} (13)

where $s = \min (n_s, q)$, $\lambda$ denotes an unordered eigenvalue of the random matrix $\tilde{H}_1^\dagger L \tilde{H}_1$, and $f_\lambda (\cdot)$ denotes the corresponding probability density function (p.d.f.). Although the distribution of $\lambda$ has been well-studied in the asymptotic antenna regime \cite{21, 22}, currently there are no exact closed-form expressions for $f_\lambda (\cdot)$ which apply for arbitrary finite-antenna systems.

III. NEW RANDOM MATRIX THEORY RESULTS

In this section, we derive a new exact closed-form expression for the unordered eigenvalue distribution $f_\lambda (\cdot)$ of the random matrix $\tilde{H}_1^\dagger L \tilde{H}_1$. We also present a number of other key results, such as random determinant properties, which will prove useful in subsequent derivations. It is convenient to define the following notation: $\alpha_i = \lambda_i^2$, $\beta_i = \lambda_i^2 / (1 + a\lambda_i^2)$ ($i = 1, \ldots, q$), and $p = \max (n_d, n_r)$.

To derive the unordered eigenvalue distribution $f_\lambda (\cdot)$, we first need to establish some key preliminary results, as given below.

Lemma 1: The marginal p.d.f. of an unordered eigenvalue $\lambda$ of $\tilde{H}_1^\dagger L \tilde{H}_1$, conditioned on $L$, is given by

$$f_{\lambda|L} (\lambda) = \frac{1}{s \prod_{i<j} (\beta_j - \beta_i)} \sum_{l=1}^q \sum_{k=q+s+1}^q \frac{\lambda^{n_s+k-q-1} e^{-\lambda/\beta_l} \beta_l^{q-n_s-1}}{\Gamma (n_s - q + k)} D_{l,k}$$  \hspace{1cm} (14)

where $D_{l,k}$ is the $(l,k)$th cofactor of a $q \times q$ matrix $D$ whose $(m,n)$th entry is

$$\{D\}_{m,n} = \beta_m^{n-1}.$$  \hspace{1cm} (15)

Proof: See Appendix I-A.

This lemma presents a new expression for the unordered eigenvalue distribution of a complex semi-correlated central Wishart matrix. In prior work \cite{25}, two separate alternative expressions for this p.d.f. were obtained for the specific scenarios $n_s \leq q$ and $n_s > q$ respectively; the latter case\footnote{For this case ($n_s > q$), the random matrix $\tilde{H}_1^\dagger L \tilde{H}_1$ has reduced rank and the corresponding distribution, conditioned on $L$, is commonly referred to as pseudo-Wishart \cite{26}.} being a complicated expression in terms of determinants with entries depending on the inverse of a certain Vandermonde matrix.

Here, \textit{Lemma} presents a simpler and more computationally-efficient \textit{unified} expression, which applies for arbitrary $n_s$ and $q$.

To remove the conditioning on $L$ in \textit{Lemma} it is necessary to establish a closed-form expression for
the joint p.d.f. of $\beta_1, \cdots, \beta_q$. We will also require the p.d.f. of an arbitrarily selected $\beta \in \{\beta_1, \cdots, \beta_q\}$. These results are given in the following lemma.

**Lemma 2:** The joint p.d.f. of $\{0 \leq \beta_1 < \cdots < \beta_q \leq 1/a\}$ is given by

$$f(\beta_1, \ldots, \beta_q) = \mathcal{K} \prod_{i<j} (\beta_j - \beta_i)^2 \prod_{i=1}^q \frac{\beta_i^{-q} e^{-\frac{\beta_i}{1-a\beta_i}}}{(1-a\beta_i)^{p+q}}$$

(16)

where

$$\mathcal{K} = \left( \prod_{i=1}^q \Gamma (q-i+1) \Gamma (p-i+1) \right)^{-1}.$$  

(17)

The p.d.f. of an unordered (randomly-selected) $\beta \in \{\beta_1, \cdots, \beta_q\}$ is given by

$$f(\beta) = \frac{1}{q} \sum_{i=0}^{q-1} \sum_{j=0}^{i} \sum_{l=0}^{2j} \mathcal{A}(i,j,l,p,q) \beta_i^{p-q+l} \exp \left( -\frac{\beta}{1-a\beta} \right)$$

(18)

where

$$\mathcal{A}(i,j,l,\kappa_1,\kappa_2) = \frac{(-1)^l (\frac{2i-j}{i-j}) \Gamma(\frac{2i-2j}{2i-l}) (2j)!}{2^{2i-l} (\kappa_1 - \kappa_2 + j)! j!}.$$  

(19)

**Proof:** See Appendix I-B.

Having established the results in **Lemma 1** and **Lemma 2**, we are now ready to derive the desired unconditional unordered eigenvalue distribution $f_\lambda(\cdot)$, as given below.

**Theorem 1:** The marginal p.d.f. of an unordered eigenvalue $\lambda$ of $\tilde{H}_1^\dagger \tilde{H}_1$ is given by

$$f_\lambda(\lambda) = \frac{2e^{-\lambda a} \mathcal{K}}{s} \sum_{l=1}^q \sum_{k=q-s+1}^q \frac{(q+n_s-l+1)\Gamma(n_s-q+k)}{\Gamma(n_s-q+k)} \lambda^{(2n_s+2k+p-q-i-3)/2} K_{p+q-i-1} \left( 2\sqrt{\lambda} \right) G_{l,k}$$

(20)

where $K_v(\cdot)$ is the modified Bessel function of the second kind and $G_{l,k}$ is the $(l,k)$th cofactor of a $q \times q$ matrix $G$ whose $(m,n)$th entry is

$$\{G\}_{m,n} = a^{q-p-m-n+1} \Gamma(p-q+m+n-1) U(p-q+m+n-1,p+q,1/a)$$

(21)

with $U(\cdot,\cdot,\cdot)$ denoting the confluent hypergeometric function of the second kind [27, Eq. 9.211.4].

**Proof:** See Appendix I-C.

We note that an asymptotic expression for $f_\lambda(\cdot)$ has been considered previously in [22], based on large-dimensional random matrix theory. However, that asymptotic p.d.f. result, which serves as an approximation for finite-dimensional systems, is not in closed-form, requiring the numerical computation of a certain fixed-point equation. Indeed, to further facilitate computation of the asymptotic eigenvalue p.d.f. in [22], an algorithmic approach with certain heuristic elements was also presented. Our result in **Theorem 1** in contrast, gives the exact eigenvalue p.d.f. which applies for arbitrary finite system dimensions, and is presented in a simple closed-form involving only standard functions which can be easily and efficiently
evaluated. In the following section, this result will be employed to evaluate the ergodic capacity of AF MIMO dual-hop channels.

**Corollary 1:** For the special case $(1, 1, 1)$, the unordered eigenvalue p.d.f. \((20)\) reduces to

\[
f_{\lambda}^{(1,1,1)}(\lambda) = 2e^{-\frac{\lambda}{1+\rho}} \left[ \frac{\alpha}{1+\rho} \right] \sqrt{\lambda K_1 \left( 2\sqrt{\lambda} \right) + K_0 \left( 2\sqrt{\lambda} \right)}.
\]

**Proof:** The proof is straightforward and is omitted. \(\square\)

We note that this special case has also been derived previously in [28].

**Corollary 2:** Let \(\tilde{L} = \text{diag} \{ \lambda_i^2 \}_{i=1}^q\). Then, the marginal p.d.f. of an unordered eigenvalue \(\lambda\) of \(\tilde{H}_1^H \tilde{L} \tilde{H}_1\) is given by

\[
\tilde{f}_\lambda(\lambda) = \frac{2K}{s} \sum_{l=1}^{q} \sum_{k=q-s+1}^{q} \frac{\lambda^{(n_s+2k+p+l-2q-3)/2} K_{p-n_s+l-1} \left( 2\sqrt{\lambda} \right) \tilde{G}_{l,k}}{\Gamma(n_s-q+k)}
\]

where \(\tilde{G}_{l,k}\) is the \((l, k)\)th cofactor of a \(q \times q\) matrix \(\tilde{G}\) whose \((m, n)\)th entry is

\[
\{ \tilde{G} \}_{m,n} = \Gamma(p-q+m+n-1).
\]

**Proof:** The result is obtained by taking the limit as \(a \to 0\) in \((20)\). \(\square\)

This result will be used to study the capacity of AF MIMO dual-hop channels in the high SNR regime. It is also worth noting that \((23)\) can be applied to the ergodic capacity analysis of Rayleigh-product MIMO channels [29, 30].

Fig. 2 compares the analytical result presented in Theorem 1 with Monte Carlo simulations. We plot the p.d.f. of the unordered eigenvalue \(\lambda\) with system configuration \((2, 3, 4)\). The simulated p.d.f. curve is based on 100,000 channel realizations. The figure shows that the analytical result is in agreement with the simulations.

Fig. 3 shows the analytical result presented in Theorem 1 and Corollary 2. The curves corresponding to \(\rho = 0\) dB, \(\rho = 10\) dB, and \(\rho = 20\) dB are generated using \((23)\) while the “Rayleigh Product” curve is generated using \((24)\). We can see that the exact unordered eigenvalue distribution converges to the unordered eigenvalue distribution of the Rayleigh product channel as \(a \to \infty\), as expected.

Fig. 4 compares our exact unordered eigenvalue distribution, based on \((20)\), with the corresponding asymptotic eigenvalue distribution presented in [22], for the random matrix \(\tilde{H}_1^H \tilde{L} \tilde{H}_1/(n_s n_r)\) with different system configurations. We use the same simulation parameters as in [22, Fig. 5 (a)], setting \(a = 1/n_r\) and \(n_r/n_s = 1/2\). We clearly see the convergence of the exact and asymptotic p.d.f.s as the numbers of antennas become large (eg. the \((16, 8, 16)\) scenario), however when the systems dimensions are not so large (eg. the \((2, 1, 2)\) and \((4, 2, 4)\) scenarios), the asymptotic eigenvalue p.d.f. exhibits noticeable inaccuracies with respect to our new exact result in \((20)\).

The following theorems present new closed-form random determinant properties, involving the random
Fig. 2. Comparison of the analytical and Monte Carlo-simulated unordered eigenvalue p.d.f. of $\tilde{H}_1^\dagger L \tilde{H}_1$. Results are shown for $(2, 3, 4)$ system configuration, with $\alpha = 2$.

Fig. 3. Comparison of the analytical unordered eigenvalue p.d.f. of $\tilde{H}_1^\dagger L \tilde{H}_1$ and $\tilde{H}_1^\dagger \tilde{L} \tilde{H}_1$ for different $\rho$. Results are shown for a $(2, 3, 4)$ system configuration, with $\alpha = 2$. 
Fig. 4. Comparison of the analytical unordered eigenvalue p.d.f. of $\tilde{H}_1^\dagger L \tilde{H}_1/(n_sn_r)$ for different system configurations. Results are shown for $a = 1/n_r$ and $n_r/n_s = 1/2$.

matrix $\tilde{H}_1^\dagger L \tilde{H}_1$. These results will be applied to derive tight bounds on the ergodic capacity.

Lemma 3: The expected determinant of $I_{n_s} + (\rho a/n_s) \tilde{H}_1^\dagger L \tilde{H}_1$, conditioned on $L$, is given by

$$E \left\{ \det \left( I_{n_s} + \frac{\rho a}{n_s} \tilde{H}_1^\dagger L \tilde{H}_1 \right) \right\} = \frac{\det (\Delta)}{\prod_{j<i} (\beta_j - \beta_i)}$$

(25)

where $\Delta$ is a $q \times q$ matrix with entries

$$\{\Delta\}_{m,n} = \begin{cases} \beta_m^{n-1}, & n \leq q - n_s, \\ \beta_m^{n-1} \left( 1 + \frac{\rho a}{n_s} \beta_m (n_s - q + n) \right), & n > q - n_s. \end{cases}$$

(26)

Proof: See Appendix I-D.

This theorem presents a new expression for the expected characteristic polynomial of a complex semi-correlated central Wishart matrix. In prior work [31, 32], alternative expressions were obtained via a different approach (i.e. by exploiting a classical characteristic polynomial expansion for the determinant). Those results, however, involved summations over subsets of numbers, with each term involving determinants of partitioned matrices. In contrast, our result in Lemma 1 is more computationally-efficient, involving only a single determinant with simple entries. Moreover, it is more amenable to the further analysis in this paper, leading to the following important theorem.

When $q < n_s$, $\{\Delta\}_{m,n} = \beta_m^{n-1} \left( 1 + \frac{\rho a}{n_s} \beta_m (n_s - q + n) \right)$.
Theorem 2: The unconditional expected determinant of \( I_{n_s} + (\rho a/n_s) \tilde{H}_1^\dagger L \tilde{H}_1 \) is given by

\[
E \left\{ \det \left( I_{n_s} + \frac{\rho a}{n_s} \tilde{H}_1^\dagger L \tilde{H}_1 \right) \right\} = \mathcal{K} \det (\Xi)
\]  

(27)

where \( \Xi \) is a \( q \times q \) matrix with entries

\[
\{ \Xi \}_{m,n} = \begin{cases} 
  a^{1-\tau} \vartheta_{\tau-1}(a), & n \leq q - n_s \\
  a^{1-\tau} \left( \vartheta_{\tau-1}(a) + \frac{\varrho}{n_s} (n_s - q + n) \vartheta_{\tau}(a) \right), & n > q - n_s
\end{cases}
\]

(28)

with \( \tau = p - q + m + n \), and

\[ \vartheta_{\tau}(a) = \Gamma(\tau) U(\tau, p+q, 1/a) . \]

(29)

Proof: Utilizing Lemma 3 [33, Lemma 2] and (112) yields the desired result.

Lemma 4: Let

\[ \Phi = \begin{cases} 
  \tilde{H}_1^\dagger L \tilde{H}_1, & q \geq n_s, \\
  L \tilde{H}_1^\dagger, & q < n_s.
\end{cases} \]

(30)

The expected log-determinant of \( \Phi \), conditioned on \( L \), is given by

\[
E \{ \ln \det (\Phi) | L \} = \sum_{k=1}^{s} \psi (n_s - s + k) + \sum_{k=q-s+1}^{q} \det (Y_k) 
\]

(31)

where \( \psi (\cdot) \) is the digamma function [27], and \( Y_k \) is a \( q \times q \) matrix with entries

\[
\{ Y_k \}_{m,n} = \begin{cases} 
  \beta_{m}^{n-1}, & n \neq k, \\
  \beta_{m}^{n-1} \ln \beta_{m}, & n = k.
\end{cases}
\]

(32)

When \( q = s \), (31) reduces to

\[
E \{ \ln \det (\Phi) | L \} = \sum_{k=1}^{s} \psi (n_s - s + k) + \ln \det (L) .
\]

(33)

Proof: See Appendix I-E

We note that the above expected natural logarithm of the determinant for \( q \geq n_s \) has been investigated in [34], where the derived expression is rather complicated, involving summations of determinants whose elements are in terms of the inverse of a certain Vandermonde matrix. We also note the \( q < n_s \) and \( q = n_s = s \) cases have been considered in [32, 35]. Our result, in contrast, gives a simple unified expression which embodies all of these cases. Moreover, based on Lemma 4 we obtain the following important theorem.

Theorem 3: The unconditional expected log-determinant of \( \Phi \) is given by

\[
E \{ \ln \det (\Phi) \} = \sum_{k=1}^{s} \psi (n_s - s + k) + K \sum_{k=q-s+1}^{q} \det (W_k)
\]

(34)
where $W_k$ is a $q \times q$ matrix with entries

$$\{W_k\}_{m,n} = \begin{cases} 
a^{1-\tau}\vartheta_{\tau-1}(a), & n \neq k \\
\varsigma_{m+n}(a), & n = k \end{cases}$$  \quad (35)

where $\tau$ and $\vartheta_{\tau-1}(\cdot)$ are defined as in (29), and

$$\varsigma_t(a) = 2^{2q-t} \sum_{i=0}^{2q-t-2} a^{2q-t-i} \Gamma(p+q-i-1) \binom{2q-t}{i} \left( \psi(p+q-i-1) - \frac{\sum_{l=0}^{p+q-i-2} g_l \left( \frac{1}{a} \right) }{k} \right)$$  \quad (36)

where $g_l(\cdot)$ denotes the auxiliary function

$$g_l(x) = e^{x}E_{l+1}(x)$$  \quad (37)

with $E_{l+1}(\cdot)$ denoting the exponential integral function of order $l + 1$.

When $q = s$, (34) reduces to

$$E \{ \ln \det(\Phi) \} = \sum_{k=1}^{s} \psi(n_s - s + k) + \sum_{i=0}^{q-1} \sum_{j=0}^{2} \sum_{l=0}^{2q-2} \sum_{k=0}^{(2q-2)_k} \binom{2q-l-2}{k} A(i,j,l,p,q) \times \frac{a^{2q-l-2-k} \Gamma(p+q-k-1) \left( \psi(p+q-k-1) - \sum_{m=0}^{p+q-k-2} g_m \left( \frac{1}{a} \right) \right) }{k} \quad (38)$$

**Proof:** See Appendix I-F

### IV. Ergodic Capacity Analysis

In this section we present new analytical expressions for the ergodic capacity of AF MIMO dual-hop systems.

#### A. Exact Expression for Ergodic Capacity

Substituting (20) into (13) we obtain

$$C(\rho) = K \sum_{l=1}^{q} \sum_{k=q-s+1}^{q} \sum_{i=0}^{q} \binom{q+n_s-l}{i} a^{q+n_s-l-i} \frac{G_{l,k} \mathcal{J}_{i,k}}{\Gamma(n_s-q+k)} \quad (39)$$

where

$$\mathcal{J}_{i,k} = \int_{0}^{\infty} \log_2 \left( 1 + \frac{\rho a}{n_s} \lambda \right) e^{-\lambda a} \chi^{2(n_s+2k+p-q-i-3)/2} K_{p+q-i-1} \left( 2\sqrt{\lambda} \right) d\lambda \quad (40)$$

The integral in (40) can be evaluated either numerically, or can be expressed as an infinite series involving Meijer-G functions. These results are confirmed in Fig. 5, where we compare the exact analytical capacity of AF MIMO dual-hop systems, based on (39) and (40), with Monte-Carlo simulated curves for two different antenna and relay configurations. In both cases, there is exact agreement between the analysis and simulations, as expected.
1) Analogies with Single-Hop MIMO Ergodic Capacity: Let $C_{\text{SH-MIMO}}^{\text{-MIMO}}(n_s, n_d, \rho)$ denote the ergodic capacity of a conventional single-hop i.i.d. Rayleigh fading MIMO channel matrix $\mathbf{H} \in \mathbb{C}^{n_d \times n_s}$, with $n_s$ transmit and $n_d$ receive antennas, and average SNR $\rho$; i.e.

$$C_{\text{SH-MIMO}}^{\text{-MIMO}}(n_s, n_d, \rho) = \mathbb{E}\left\{ \log_2 \det \left( \mathbf{I}_{n_d} + \frac{\rho}{n_s} \mathbf{H} \mathbf{H}^\dagger \right) \right\}.$$  \hspace{1cm} (41)

Here, we demonstrate four particular cases for which the AF MIMO dual-hop channel relates directly to single-hop MIMO channels, in terms of ergodic capacity.

- As the number of relay antennas grows large, i.e. $n_r \to \infty$, the ergodic capacity of AF MIMO dual-hop systems becomes

$$\lim_{n_r \to \infty} C(\rho) = \frac{1}{2} C_{\text{SH-MIMO}}^{\text{-MIMO}}(n_s, n_d, \frac{\rho \alpha}{1 + \rho + \alpha}).$$ \hspace{1cm} (42)

A proof is presented in Appendix II-A. Note that a similar phenomenon has been derived in [19], for the special case $n_s = n_d$. Here, (42) generalizes that result for arbitrary source and destination antenna configurations.

- As the number of source antennas grows large, i.e. $n_s \to \infty$, the ergodic capacity of AF MIMO dual-hop systems becomes

$$\lim_{n_s \to \infty} C(\rho) = \frac{1}{2} C_{\text{SH-MIMO}}^{\text{-MIMO}}(n_r, n_d, \alpha) - \frac{1}{2} C_{\text{SH-MIMO}}^{\text{-MIMO}}(n_r, n_d, \frac{\alpha}{1 + \rho}).$$ \hspace{1cm} (43)

A proof is presented in Appendix II-B. Interestingly, we see that as $\rho$ grows large, the right-most term in (43) disappears, and the AF MIMO dual-hop capacity becomes equivalent to one half of the ergodic capacity of a single-hop MIMO channel with $n_r$ transmit antennas, $n_d$ receive antennas, and average SNR $\alpha$.

- As the number of destination antennas grows large, i.e. $n_d \to \infty$, the ergodic capacity of AF MIMO dual-hop systems becomes

$$\lim_{n_d \to \infty} C(\rho) = \frac{1}{2} C_{\text{SH-MIMO}}^{\text{-MIMO}}(n_s, n_r, \rho).$$ \hspace{1cm} (44)

The result is trivially obtained by directly taking $\lambda_1^2 \to \infty$ in (11). We see that the AF MIMO dual-hop capacity becomes equivalent to one half of the ergodic capacity of a single-hop MIMO channel with $n_s$ transmit antennas, $n_r$ receive antennas, and average SNR $\rho$.

- As the power gain of the relay grows large, i.e. $\alpha \to \infty$, the ergodic capacity of AF MIMO dual-hop systems becomes

$$\lim_{\alpha \to \infty} C(\rho) = \frac{1}{2} C_{\text{SH-MIMO}}^{\text{-MIMO}}(n_s, q, \rho).$$ \hspace{1cm} (45)
The result is trivially obtained by directly taking $\alpha \rightarrow \infty$ in (11). Thus we see the interesting result that even as the relay power gain becomes very large, the capacity of AF MIMO dual-hop channels remains bounded, and in fact becomes equivalent to one half of the ergodic capacity of a single-hop MIMO channel with $n_s$ transmit antennas, $q = \min(n_r, n_d)$ receive antennas, and average SNR $\rho$.

We note that for each of the cases (42)–(45), closed-form expressions can be obtained by directly invoking known results from the single-hop MIMO capacity literature (eg. see [31]).

In order to obtain further simplified closed-form results, it is useful to investigate the ergodic capacity in the high SNR regime. This is presented in the subsection below.

B. High SNR Capacity Analysis

For the high SNR regime, we consider two important scenarios; namely, one where the source and relay powers grow large proportionately, and one where the source power grows large but the relay power is kept fixed.

1) Large Source Power, Large Relay Power: Here we have $\alpha \rightarrow \infty$, $\rho \rightarrow \infty$, with $\alpha/\rho = \beta$, for some fixed $\beta$. Then $\rho a \rightarrow \frac{\alpha}{n_r}$ and $a \rightarrow \beta/n_r$, and the ergodic capacity at high SNR reduces to

$$C(\rho)|_{\alpha,\rho \rightarrow \infty, \alpha/\rho = \beta} = \frac{1}{2} E \left\{ \log_2 \det \left( I_{n_s} + \frac{\rho \beta}{n_s n_r} \tilde{H}_1^\dagger \tilde{H}_1 \right) \right\}$$

(46)

where $\tilde{L} = \text{diag} \{ \frac{\lambda_i^2}{1 + (\beta/n_r) \lambda_i^2} \}_{i=1}^q$. We can express (46) in the general form [34]

$$C(\rho)|_{\alpha,\rho \rightarrow \infty, \alpha/\rho = \beta} = S_{\infty} \left( \frac{\rho_{\text{dB}}}{3\text{dB}} - L_{\infty} \right) + o(1)$$

(47)

where $3\text{dB} = 10 \log_{10}(2)$. Here, the two key parameters are $S_{\infty}$, which denotes the high-SNR slope in bits/s/Hz/(3 dB) given by

$$S_{\infty} = \lim_{\alpha,\rho \rightarrow \infty} \frac{C(\rho)|_{\alpha,\rho \rightarrow \infty, \alpha/\rho = \beta}}{\log_2(\rho)}$$

(48)

and $L_{\infty}$, which represents the high-SNR power offset in 3 dB units given by

$$L_{\infty} = \lim_{\alpha,\rho \rightarrow \infty} \left( \log_2(\rho) - \frac{C(\rho)|_{\alpha,\rho \rightarrow \infty, \alpha/\rho = \beta}}{S_{\infty}} \right).$$

(49)

From (46), we can evaluate $S_{\infty}$ and $L_{\infty}$ in closed-form as follows.

**Theorem 4**: For the case $\alpha \rightarrow \infty$, $\rho \rightarrow \infty$, with $\alpha/\rho = \beta$, the high-SNR slope and high-SNR power offset of AF MIMO dual-hop systems are given by

$$S_{\infty} = \frac{s}{2} \text{ bit/s/Hz/(3dB)}$$

(50)
and

\[ \mathcal{L}_\infty(n_s, n_r, n_d) = \log_2 \left( \frac{n_s n_r}{\beta} \right) - \frac{1}{s \ln 2} \left[ \sum_{k=1}^{s} \psi(n_s + k - s) + \sum_{q=1}^{q} \det(\bar{W}_k) \right] \]

(51)

respectively, where \( \bar{W}_k \) is a \( q \times q \) matrix with entries

\[
\{ \bar{W}_k \}_{m,n} = \begin{cases} \left( \frac{\beta}{n_r} \right)^{1-r} \vartheta_{r-1} \left( \frac{\beta}{n_r} \right), & n \neq k, \\ s_{m+n} \left( \frac{\beta}{n_r} \right), & n = k. \end{cases}
\]

(52)

For the case \( q = s \) (i.e. corresponding to \( \min(n_s, n_r, n_d) = n_d \) or \( \min(n_s, n_r, n_d) = n_r \)), the high SNR power offset (51) admits the alternative form

\[
\mathcal{L}_\infty(n_s, n_r, n_d) = \log_2 \left( \frac{n_s n_r}{\beta} \right) - \frac{1}{s \ln 2} \left[ \sum_{k=1}^{s} \psi(n_s - s + k) + \sum_{i=0}^{q-1} \sum_{j=0}^{i+1} \sum_{k=0}^{2q-2} \binom{2q-1}{k} \psi(p+q-k-1) \right] \]

\[ \times \mathcal{A}(i, j, l, p, q) \left( \frac{\beta}{n_r} \right)^{2q-2-k} \Gamma(p+q-k-1) \left( \frac{n_s}{\beta} \right) \sum_{m=0}^{p+q-k-2} g_m \left( \frac{n_r}{\beta} \right) \]

(53)

**Proof:** See Appendix II-C.

Interestingly, we see that the high SNR slope depends only on the minimum system dimension, i.e. \( s = \min(n_s, n_r, n_d) \), whereas the high SNR power offset is a much more intricate function of \( n_s, n_r, \) and \( n_d \). Fig. 5 depicts the analytical high SNR capacity approximations for AF MIMO dual-hop systems, based on (50) and (51). These approximations are seen to converge to their respective exact capacity curves for quite moderate SNR levels (e.g. \( < 20 \) dB).

It is important to note that Theorem 4 presents an exact characterization of the key high SNR ergodic capacity parameters, \( S_\infty \) and \( \mathcal{L}_\infty(\cdot) \), for arbitrary numbers of antennas at the source, relay, and destination terminals. We now examine some particularizations of Theorem 4 in which these expressions reduce to simple forms.

**Corollary 3:** Let \( n_r = 1 \). Then \( S_\infty = 1/2 \), and \( \mathcal{L}_\infty(\cdot) \) reduces to

\[ \mathcal{L}_\infty(n_s, 1, n_d) = \log_2 \left( \frac{n_s}{\beta} \right) - \frac{1}{\ln 2} \left[ \psi(n_s) + \psi(n_d) - \sum_{m=0}^{n_d-1} g_m \left( \frac{1}{\beta} \right) \right]. \]

(54)

Note that, as \( n_s \) grows large, \( \psi(n_s) = \ln n_s + o(1) \) [36, Eq. 6.3.18.], where the \( o(1) \) term disappears as \( n_s \to \infty \), and as such we have

\[ \lim_{n_s \to \infty} \mathcal{L}_\infty(n_s, 1, n_d) = \log_2 \left( \frac{1}{\beta} \right) - \frac{1}{\ln 2} \left[ \psi(n_d) - \sum_{m=0}^{n_d-1} g_m \left( \frac{1}{\beta} \right) \right]. \]

(55)

Note that here we explicitly indicate the dependence of the high SNR power offset on \( n_s, n_r, \) and \( n_d \).
TABLE I. High SNR offset as function of $n_d$, where $n_s = 2$, $n_r = 3$ and $\beta = 2$.

| $n_d$ | 4  | 6  | 8  | 10 | 12 | 14 |
|-------|----|----|----|----|----|----|
| $L_\infty$ (dB) | 2.593 | 1.573 | 1.147 | 0.88 | 0.73 | 0.622 |

TABLE II. High SNR offset as function of $n_r$, where $n_s = 2$, $n_d = 4$ and $\beta = 2$.

| $n_r$ | 3  | 5  | 7  | 9  | 11 | 13 |
|-------|----|----|----|----|----|----|
| $L_\infty$ (dB) | 2.593 | 1.251 | 0.847 | 0.636 | 0.493 | 0.429 |

**Corollary 4:** Let $n_d = 1$. Then $S_\infty = 1/2$, and $L_\infty(\cdot)$ reduces to

$$L_\infty(n_s, n_r, 1) = \log_2 \left( \frac{n_s n_r}{\beta} \right) - \frac{1}{\ln 2} \left[ \psi(n_s) + \psi(n_r) - \sum_{m=0}^{n_r-1} g_m \left( \frac{n_r}{\beta} \right) \right].$$  \hfill (56)

In this case, as $n_s$ grows large we have

$$\lim_{{n_s \to \infty}} L_\infty(n_s, n_r, 1) = \log_2 \left( \frac{n_r}{\beta} \right) - \frac{1}{\ln 2} \left[ \psi(n_r) - \sum_{m=0}^{n_r-1} g_m \left( \frac{n_r}{\beta} \right) \right].$$  \hfill (57)

Based on these results, we can easily examine the effect of the relative power gain factor $\beta$ on the ergodic capacity. In particular, noting that $g_l(x)$ in (37) is a monotonically decreasing function of $x$ in the interval $[0, \infty)$, we see that increasing $\beta$, whilst having no effect on the high SNR capacity slope $S_\infty$, results in decreasing the high SNR power offset $L_\infty(\cdot)$, and therefore increasing the ergodic capacity in the high SNR regime.

**Corollary 5:** Let $n_s = n_r = 1$. Adding $k$ destination antennas, while not altering $S_\infty$, would reduce the high SNR power offset as

$$\delta(n_d, k) \triangleq L_\infty(1, 1, n_d + k) - L_\infty(1, 1, n_d)$$

$$= -\frac{1}{\ln 2} \sum_{l=n_d}^{n_d+k-1} \left( \frac{1}{\ell} + g_l \left( \frac{1}{\beta} \right) \right).$$  \hfill (58)

Note that, to obtain this result, we have invoked the definition of the digamma function [27]. Since $g_l(x) > 0$ for $x \in [0, \infty)$, it is clear that the high SNR power offset $L_\infty(\cdot)$ in (58) is a decreasing function of $k$, thereby confirming the intuitive notion that adding more antennas to the destination terminal has the effect of improving the ergodic capacity.

**Example 1:** With respect to $\beta = 1$,

$$L_\infty(1, 1, 2) = L_\infty(1, 1, 1) - 2.58 \text{ dB}$$  \hfill (59)

$$L_\infty(1, 1, 3) = L_\infty(1, 1, 1) - 3.46 \text{ dB}$$  \hfill (60)

$$L_\infty(1, 1, \infty) = L_\infty(1, 1, 1) - 5.08 \text{ dB}$$  \hfill (61)

where $L_\infty(1, 1, 1) = 7.57 \text{ dB}$.

4This conclusion is easily established by noting that $d/dx \left( g_l(x) \right) = e^x \left[ E_{l+1} (x) - E_l (x) \right]$, and using [36, Eq. 5.1.17].
Fig. 5. Comparison of exact analytical, high SNR analytical, and Monte Carlo simulation results for ergodic capacity of AF MIMO dual-hop systems with different antenna configurations. Results are shown for $\alpha/\rho = 2$.

Fig. 6. High SNR power offset shift, in decibels, obtained by adding either (a) one antenna to the destination, (b) two antennas to the destination, or (c) four antennas to the destination. Results are shown for $n_s = n_r = 1$ and $\alpha/\rho = 2$. 
Fig. 6 illustrates the relationship in Corollary 5, where the high SNR power offset shift $\delta(n_d, k)$ is plotted against $n_d$, for $k = 1$, $k = 2$, and $k = 4$. As expected, for a fixed value of $k$, $\delta(n_d, k)$ is an increasing function of $n_d$, approaching a limit of 0 dB as $n_d \to \infty$. Table I and Table II present the high SNR power offset as a function of $n_d$ and $n_r$ respectively, for $n_s = 2$. We see that when $n_d$ (resp. $n_r$) is small, then a small increase in $n_d$ (resp. $n_r$) yields a significant improvement in terms of the high SNR power offset. However, in agreement with Fig. 6, adding more and more antennas yields diminishing returns.

2) Large Source Power, Fixed Relay Power: Here we take $\rho \to \infty$ and keep $\alpha$ fixed. Then, noting that $\rho a|_{\rho \to \infty} \to \alpha/n_r$, the ergodic capacity reduces to

$$
\lim_{\rho \to \infty} C(\rho) = \frac{s}{2} E \left\{ \log_2 \left( 1 + \frac{\alpha}{n_s n_r} \tilde{\lambda} \right) \right\}
$$

where $\tilde{\lambda}$ denotes an unordered eigenvalue of $\tilde{H}_1^\dagger \tilde{L} \tilde{H}_1$. Using Corollary 2, we can evaluate this constant as

$$
\lim_{\rho \to \infty} C(\rho) = \frac{K}{\ln 2} \sum_{l=1}^{q} \sum_{k=q-s+1}^{q} \tilde{G}_{l,k} \tilde{F}_{l,k}
$$

where

$$
\tilde{F}_{l,k} = \int_0^\infty \ln \left( 1 + \frac{\alpha}{n_s n_r} y \right) y^{(n_s+2k+p+l-2q-3)/2} K_{p+l-n_s-1} (2\sqrt{y}) dy.
$$

To evaluate the remaining integral in (62), we first express the logarithm in terms of the Meijer G-function as [37, Eq. 8.4.6.5]

$$
\log_2 \left( 1 + \frac{\alpha}{n_s n_r} \tilde{\lambda} \right) = \frac{1}{\ln 2} G_{1,2}^{1,2} \left( \frac{\alpha \tilde{\lambda}}{n_s n_r} \right| 1, 1 \left| 1, 0 \right)
$$

and then apply the integral relationships [27, Eq. 7.821.3] and [27, Eq. 9.31.1]. This leads to the following closed-form expression for the ergodic capacity of AF MIMO dual-hop systems as the source power $\rho$ grows large for fixed relay power $\alpha$,

$$
\lim_{\rho \to \infty} C(\rho) = \frac{K}{2 \ln 2} \sum_{l=1}^{q} \sum_{k=q-s+1}^{q} \frac{\tilde{G}_{l,k}}{\Gamma(n_s - q + k)} \\
\times G_{1,4}^{4,1} \left( \frac{n_s n_r}{\alpha} \right| 0, 1 \left| k + p + l - q - 1, n_s + k - q, 0, 0 \right).
$$

This result shows that if we fix $\alpha$ and take $\rho$ large, then the ergodic capacity of AF MIMO dual-hop systems remains bounded (as a function of $\alpha$). This confirms the intuitive notion that the capacity is restricted by the weakest link in the relay network; in this case, the relay-destination link.
Fig. 7. Comparison of bounds, exact analytical, high SNR analytical, and Monte Carlo simulation results for ergodic capacity of AF MIMO dual-hop systems with different antenna configurations. Results are shown for $\alpha/\rho = 2$.

V. TIGHT BOUNDS ON THE ERGODIC CAPACITY

In order to obtain further simplified closed-form results, in this section we derive new upper and lower bounds on the ergodic capacity.

A. Upper Bound

The following theorem presents a new tight upper bound on the ergodic capacity of AF MIMO dual-hop systems.

**Theorem 5:** The ergodic capacity of AF MIMO dual-hop systems is upper bounded by

$$ C(\rho) \leq C_U(\rho) = \frac{1}{2} \log_2 \left( K \det(\bar{\Xi}) \right) $$

(67)

where $\bar{\Xi}$ is defined in (28).

**Proof:** Application of Jensen’s inequality gives

$$ C(\rho) \leq \frac{1}{2} \log_2 \mathbb{E} \left\{ \det \left( \mathbf{I}_{n_s} + \frac{\rho \alpha}{n_s} \tilde{\mathbf{H}}_1^\dagger \tilde{\mathbf{H}}_1 \right) \right\} $$

(68)

The result now follows by using *Theorem 2*.

Note that this inequality has also been applied in the ergodic capacity analysis of single-user single-hop MIMO systems (see eg. [32, 38, 39]).
Fig. 7 compares the closed-form upper bound (67) with the exact analytical ergodic capacity based on (39) and (40), for two different AF MIMO dual-hop system configurations. The results are shown as a function of SNR $\rho$, with $\alpha = 2\rho$. We see that the closed-form upper bound is very tight for all SNRs, for both system configurations considered. Moreover, we see that in the low SNR regime (e.g. $\rho \approx 5$ dB), the upper bound and exact capacity curves coincide.

The ensuing corollaries present some example scenarios for which the upper bound (67) reduces to simplified forms.

**Corollary 6:** For the case $n_s \to \infty$, $C_U(\rho)$ becomes

$$
\lim_{n_s \to \infty} C_U(\rho) = \frac{1}{2} \log_2 \left( \mathcal{K} \det(\tilde{\Xi}_1) \right)
$$

(69)

where $\tilde{\Xi}_1$ is a $q \times q$ matrix with entries

$$
\{\tilde{\Xi}_1\}_{m,n} = a^{1-\tau} \vartheta_{r-1}(a) + \rho a^{1-\tau} \vartheta_r(a).
$$

(70)

**Proof:** The proof is straightforward and is omitted.

This result shows that in AF MIMO dual-hop systems, when the numbers of antennas at both the relay and destination remain fixed, the ergodic capacity remains bounded as the number of source antennas grows large. This is in agreement with the results in Section IV-A.1.

Note that for the scenarios $n_r \to \infty$ and $n_d \to \infty$, simplified closed-form results can also be obtained by taking the corresponding limits in (69) or, alternatively, by using the equivalent single-hop MIMO capacity relations in (42) and (44), and applying known upper bounds for single-hop MIMO channels in [40]. We omit these expressions here for the sake of brevity.

**Corollary 7:** Let $n_r = 1$. Then, $C_U(\rho)$ reduces to

$$
C_{U}^{n_r=1}(\rho) = \frac{1}{2} \log_2 \left( 1 + \rho n_d e^{\frac{i\pi}{\alpha}} E_{n_d+1} \left( \frac{1 + \rho}{\alpha} \right) \right).
$$

(71)

When $n_d \to \infty$, $C_{U}^{n_r=1}(\rho)$ becomes

$$
\lim_{n_d \to \infty} C_{U}^{n_r=1}(\rho) = \frac{1}{2} \log_2 (1 + \rho).
$$

(72)

When $\alpha \to \infty$, $C_{U}^{n_r=1}(\rho)$ becomes

$$
\lim_{\alpha \to \infty} C_{U}^{n_r=1}(\rho) = \frac{1}{2} \log_2 (1 + \rho).
$$

(73)

**Proof:** See Appendix II-D.

This shows the interesting result that, if a single relay antenna is employed, then when either the number of destination antennas $n_d$ or the relay gain $\alpha$ grows large, the ergodic capacity is upper bounded by the capacity of an AWGN SISO channel.
Corollary 8: In the high SNR regime, (i.e. as $\rho \to \infty$) for fixed relay gain $\alpha$, $C_U(\rho)$ becomes

$$
\lim_{\rho \to \infty} C_U(\rho) = \frac{1}{2} \log_2 \left( \mathcal{K} \det(\tilde{\Xi}) \right)
$$

where $\tilde{\Xi}$ is a $q \times q$ matrix with entries

$$
\begin{cases}
\tilde{\Xi}_{m,n} = \\
\quad \Gamma (\tau - 1), & n \leq q - n_s, \\
\Gamma (\tau - 1) \left(1 + \frac{\alpha}{n_s n_r} (n_s - q + n) (\tau - 1)\right), & n > q - n_s.
\end{cases}
$$

Proof: See Appendix II-E.

This expression is clearly much simpler than the exact ergodic capacity expression given for this regime in (66).

B. Lower Bound

The following theorem presents a new tight lower bound on the ergodic capacity of AF MIMO dual-hop systems.

Theorem 6: The ergodic capacity of AF MIMO dual-hop systems is lower bounded by

$$
C(\rho) \geq C_L(\rho) = \frac{s}{2} \log_2 \left(1 + \frac{\rho a}{n_s} \exp \left(\frac{1}{s} \left[\sum_{k=1}^{s} \psi(n_s - s + k) + \mathcal{K} \sum_{k=q-s+1}^{q} \det(W_k)\right]\right)\right)
$$

where $W_k$ is defined as in (35).

Proof: See Appendix II-F.

In Fig. 7, this closed-form lower bound is compared with the exact ergodic capacity of AF MIMO dual-hop systems. Results are shown for different system configurations. The lower bound is clearly seen to be tight for the entire range of SNRs. Moreover, in the high SNR regime (e.g. $\rho \approx 15$ dB), we see that the lower bound and exact capacity curves coincide.

The ensuing corollaries present some example scenarios for which the lower bound (76) reduces to simplified forms.

Corollary 9: For the case $n_s \to \infty$, $C_L(\rho)$ reduces to

$$
\lim_{n_s \to \infty} C_L(\rho) = \frac{s}{2} \log_2 \left(1 + \rho a \exp \left(\frac{1}{s} \sum_{k=1}^{q} \det(W_k)\right)\right).
$$

Proof: See Appendix II-G.

Again, we note that for the scenarios $n_r \to \infty$ and $n_d \to \infty$, simplified closed-form results can also be obtained by taking the corresponding limits in (69) or, alternatively, by using (42) and (44), and applying known lower bounds for single-hop MIMO channels in [40].

Corollary 10: For the case $n_r = 1$, $C_L(\rho)$ reduces to

$$
C_{L_{n_r=1}}(\rho) = \frac{1}{2} \log_2 \left(1 + \frac{\rho c}{n_s (1 + \rho)} \exp \left(\psi(n_s) + \psi(n_d) - e^{(1+\rho) / \alpha} \sum_{l=0}^{n_d-1} E_{l+1} \left(\frac{1 + \rho}{\alpha}\right)\right)\right).
$$
Fig. 8. Comparison of capacity bounds, high $\alpha$ approximation, and exact analytical results for different relay gains. Results are shown for $n_r = 1$, $n_s = 2$, $n_d = 4$ and $\rho = 10$dB.

When $n_s \to \infty$, $C_{\nu_r=1}^{n_r=1}(\rho)$ becomes
\[
\lim_{n_s \to \infty} C_{\nu_r=1}^{n_r=1}(\rho) = \frac{1}{2} \log_2 \left( 1 + \frac{\rho \alpha}{1 + \rho} \exp \left( \psi(n_d) - e^{(1+\rho)/\alpha} \sum_{l=0}^{n_d-1} E_{l+1} \left( \frac{1 + \rho}{\alpha} \right) \right) \right) . \tag{79}
\]

When $n_d \to \infty$, $C_{\nu_r=1}^{n_r=1}(\rho)$ becomes
\[
\lim_{n_d \to \infty} C_{\nu_r=1}^{n_r=1}(\rho) = \frac{1}{2} \log_2 \left( 1 + \frac{\rho \alpha}{n_s (1 + \rho)} \exp \left( \psi(n_s) + \psi \left( \frac{1 + \rho}{\alpha} \right) \right) \right) . \tag{80}
\]

When $\alpha \to \infty$, $C_L(\rho)$ becomes
\[
\lim_{\alpha \to \infty} C_{\nu_r=1}^{n_r=1}(\rho) = \frac{1}{2} \log_2 \left( 1 + \frac{\rho}{n_s} \exp \left( \psi(n_s) \right) \right) . \tag{81}
\]

Proof: See Appendix II-H

As also observed from the upper bound in Corollary [7] this result shows that for a system with a single relay antenna, when the relay gain $\alpha$ grows large, the ergodic capacity of an AF MIMO dual-hop channel is lower bounded by the capacity of an AWGN SISO channel (with scaled average SNR).

Fig. 8 plots the closed-form upper bound (71), closed-form lower bound (78), and the exact analytical ergodic capacity based on (39) and (40), for an AF MIMO dual-hop system with $n_r = 1$. The results are presented as a function of the relay gain $\alpha$. We see that both the upper and lower bounds are quite tight for the entire range of $\alpha$ considered. The asymptotic approximations for the upper and lower bounds, based on (73) and (81) respectively, are also shown for further comparison, and are seen to converge for
Fig. 9. Comparison of capacity bounds, high SNR approximations, and exact analytical results. Results are shown for a system configuration $(3, 4, 2)$ and $\alpha = 2$.

Corollary 11: In the high SNR regime, (i.e. as $\rho \to \infty$) for fixed relay gain $\alpha$, $C_L(\rho)$ becomes

$$
\lim_{\rho \to \infty} C_L(\rho) = \frac{s}{2} \log_2 \left( 1 + \frac{\alpha}{n_r n_s} \exp \left( \frac{K}{s} \sum_{k=q-s+1}^{q} \det \left( \tilde{W}_k \right) \right) \right),
$$

where $\tilde{W}_k$ is a $q \times q$ matrix with entries

$$
\{ \tilde{W}_k \}_{m,n} = \begin{cases} 
\Gamma(\tau - 1), & n \neq k \\
\Gamma(\tau - 1) [\psi (n_s - q + n) + \psi (\tau - 1)], & n = k 
\end{cases}
$$

Proof: See Appendix II-I.

As for the high SNR upper bound presented in (74), this closed-form lower bound expression is simpler than the exact ergodic capacity expression given for this regime in (66).

Fig. 9 depicts the closed-form high SNR approximations for the exact ergodic capacity, as well as the respective upper and lower bounds, based on (65), (74), and (82) respectively. For comparison, curves are also presented for the upper bound (67), lower bound (76), and the exact analytical ergodic capacity based in (39) and (40). Results are shown for an AF MIMO dual-hop system with configuration $(3, 4, 2)$. Clearly, the analytical high SNR approximations are seen to be very accurate for even moderate SNR levels (e.g. $\rho \approx 20$ dB).
VI. CONCLUSIONS

This paper has presented an analytical characterization of the ergodic capacity of AF MIMO dual-hop relay channels under the common assumption that CSI is available at the destination terminal, but not at the relay or the source terminal. We derived a new exact expression for the ergodic capacity, as well as simplified and insightful closed-form expressions for the high SNR regime. Simplified closed-form upper and lower bounds were also presented, which were shown to be tight for all SNRs. The analytical results were made possible by first employing random matrix theory techniques to derive new expressions for the p.d.f. of an unordered eigenvalue, as well as random determinant results for the equivalent AF MIMO dual-hop relay channel, described by a certain product of finite-dimensional complex random matrices. The analytical results were validated through comparison with numerical simulations.

VII. ACKNOWLEDGEMENT

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APPENDIX I

PROOFS OF NEW RANDOM MATRIX THEORY RESULTS

A. Proof of Lemma [7]

To prove this lemma, it is convenient to give a separate treatment for the two cases, $q < n_s$ and $q \geq n_s$.

1) The $q < n_s$ Case: For this case, an expression for the p.d.f. $f_{\lambda|L}(\cdot)$ has been given previously as [25]

$$f_{\lambda|L}(\lambda) = \frac{\sum_{l=1}^{q} \sum_{k=1}^{q} \lambda^{n_s-q+k-1} e^{-\lambda/\beta_l} \tilde{D}_{l,k} \det(L)^{n_s-q+1} \prod_{i=1}^{q} \Gamma(n_s-i+1) \prod_{i<j}^q (\beta_j - \beta_i)}{q \det(L)^{n_s-q+1} \prod_{i=1}^{q} \Gamma(n_s-i+1) \prod_{i<j}^q (\beta_j - \beta_i)}$$

(84)

where $\tilde{D}_{l,k}$ is the $(l,k)$th cofactor of a $q \times q$ matrix with entries

$$\{\tilde{D}\}_{i,j} = \Gamma(n_s-i+j) \beta_i^{n_s-q+j}.$$  

(85)

After some basic manipulations, we can express this cofactor as

$$\tilde{D}_{l,k} = \frac{\prod_{j=1}^{q} \Gamma(n_s-j+1) \det(L)^{n_s-q+1} D_{l,k}}{\Gamma(n_s-q+k) \beta_l^{n_s-q+1}}.$$  

(86)

Substituting (86) into (84) yields the desired result.
2) The $q \geq n_s$ Case: For this case, we start by employing a result from [41, Eq. 11] to express the joint p.d.f. of the unordered eigenvalues $\gamma_1, \ldots, \gamma_{n_s}$ of $\hat{H}_1^\dagger L \hat{H}_1$, conditioned on $L$, as follows

$$f (\gamma_1, \ldots, \gamma_{n_s} | L) = \frac{\det (\Delta_1) \prod_{i<j}^n (\gamma_j - \gamma_i)}{n_s \prod_{i=1}^{n_s} \Gamma (n_s - i + 1) \prod_{i<j}^q (\beta_j - \beta_i)},$$

(87)

where $\Delta_1$ is the $q \times q$ matrix

$$\Delta_1 = \begin{bmatrix}
1 & \beta_1 & \cdots & \beta_1^{q-n_s} & \beta_1^{q-n_s-1} e^{-\frac{\lambda}{\beta_1}} & \cdots & \beta_1^{q-n_s-1} e^{-\frac{\lambda}{\beta_1}} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & \beta_q & \cdots & \beta_q^{q-n_s} & \beta_q^{q-n_s-1} e^{-\frac{\lambda}{\beta_q}} & \cdots & \beta_q^{q-n_s-1} e^{-\frac{\lambda}{\beta_q}}
\end{bmatrix}.$$

(88)

The p.d.f. of a single unordered eigenvalue $\lambda$ is found from (87) via

$$f_{\lambda | L} (\lambda) = \frac{1}{n_s \prod_{i=1}^{n_s} \Gamma (n_s - i + 1) \prod_{i<j}^q (\beta_j - \beta_i)} \int_0^\infty \cdots \int_0^\infty \det (\Delta_1) d (\gamma_1^{j-1}) d \gamma_1 \cdots d \gamma_{n_s-1} | \gamma_{n_s} = \lambda,$$

(89)

where we have used $\prod_{i<j}^n (\gamma_j - \gamma_i) = \det (\gamma_i^{j-1})$. To evaluate the $n_s - 1$ integrals, we expand $\det (\Delta_1)$ along its last column and $\det (\gamma_i^{j-1})$ along its last row, and then integrate term-by-term by virtue of [33, Lemma 2]. This yields

$$f_{\lambda | L} (\lambda) = \sum_{l=1}^q \sum_{k=q-n_s+1}^q \beta_l^{q-n_s-1} e^{-\frac{\lambda}{\beta_l}} \chi^{q-n_s+k-1} \bar{D}_{l,k}$$

(90)

$$\frac{n_s \prod_{i=1}^{n_s} \Gamma (n_s - i + 1) \prod_{i<j}^q (\beta_j - \beta_i)}$$

where $\bar{D}_{l,k}$ is the $(l,k)$th cofactor of a $q \times q$ matrix $\Xi = [A \ C]$, with entries

$$\{A\}_{m,n} = \beta_m^{n-1} \quad m = 1, \ldots, q, \quad n = 1, \ldots, q - n_s$$

(91)

and

$$\{C\}_{m,n} = \Gamma (n) \beta_m^{q-n_s+n-1} \quad m = 1, \ldots, q, \quad n = 1, \ldots, n_s$$

(92)

Then, it can be shown that

$$\sum_{l=1}^q \sum_{k=q-n_s+1}^q \beta_l^{q-n_s-1} e^{-\frac{\lambda}{\beta_l}} \chi^{q-n_s+k-1} \bar{D}_{l,k} = \sum_{k=q-n_s+1}^q \det (D_k),$$

(93)
where \( \mathbf{D}_k \) is a \( q \times q \) matrix with entries

\[
\{ \mathbf{D}_k \}_{m,n} = \begin{cases} 
\beta_m^{n-1}, & m = 1, \ldots, q, \quad n = 1, \ldots, q - n_s \\
\Gamma(n - q + n_s - 1) \beta_m^n, & m = 1, \ldots, q, \quad n = q - n_s + 1, \ldots, q, \quad n \neq k \\
\beta_m^{q-n_s-1} e^{-\lambda/\beta_m} \lambda^{n-q-n_s-1}, & m = 1, \ldots, q, \quad n = k 
\end{cases}
\]  

Hence, we can rewrite (90) as follows

\[
f_{\lambda|\mathbf{L}}(\lambda) = \frac{\sum_{k=q-n_s+1}^{q} \det(\mathbf{D}_k)}{n_s \prod_{i=1}^{n_s} \Gamma(n_s - i + 1) \prod_{i<j}^{q} (\beta_j - \beta_i)}.
\]  

After some basic manipulations, (95) can be further simplified as

\[
f_{\lambda|\mathbf{L}}(\lambda) = \frac{1}{n_s \prod_{i<j}^{q} (\beta_j - \beta_i)} \sum_{k=q-n_s+1}^{q} \lambda^{n_s-q+k-1} \frac{\det(\bar{\mathbf{D}}_k)}{\Gamma(n_s + q + k)}
\]

where \( \bar{\mathbf{D}}_k \) is a \( q \times q \) matrix with entries

\[
\{ \bar{\mathbf{D}}_k \}_{m,n} = \begin{cases} 
\beta_m^{n-1}, & n \neq k, \\
e^{-\lambda/\beta_m} \beta_m^{q-n_s+1}, & n = k.
\end{cases}
\]  

Finally, we apply Laplace’s expansion to (96) to yield the desired result.

**B. Proof of Lemma 2**

The joint p.d.f. of \( \mathbf{W}_1 = \text{diag} \{ \alpha_1, \ldots, \alpha_q \} \) is given by [42–44]

\[
f_{\mathbf{W}_1}(\alpha_1, \ldots, \alpha_q) = K e^{-\sum_{i=1}^{q} \alpha_i} \prod_{i=1}^{q} \alpha_i^{\beta_i - q} \prod_{i<j}^{q} (\alpha_j - \alpha_i)^2.
\]  

Recalling that

\[
\alpha_i = \frac{\beta_i}{1 - a \beta_i}
\]

we derive the joint p.d.f. of \( \mathbf{W}_2 = \text{diag} \{ \beta_1, \ldots, \beta_q \} \) from (98) by applying a vector transformation [45]

\[
f_{\mathbf{W}_2}(\beta_1, \ldots, \beta_q) = f_{\mathbf{W}_1} \left( \frac{\beta_1}{1 - a \beta_1}, \ldots, \frac{\beta_q}{1 - a \beta_q} \right) | \mathbf{J} ((\alpha_1, \ldots, \alpha_q) \rightarrow (\beta_1, \ldots, \beta_q)) |,
\]  

where

\[
\mathbf{J} ((\alpha_1, \ldots, \alpha_q) \rightarrow (\beta_1, \ldots, \beta_q)) = \det \begin{bmatrix} \frac{\partial \alpha_1}{\partial \beta_1} & \cdots & \frac{\partial \alpha_1}{\partial \beta_q} \\
\vdots & \ddots & \vdots \\
\frac{\partial \alpha_q}{\partial \beta_1} & \cdots & \frac{\partial \alpha_q}{\partial \beta_q} \end{bmatrix}.
\]
From (99), we have
\[
\frac{\partial \alpha_i}{\partial \beta_i} = \frac{1}{(1 - a \beta_i)^2},
\]
(102)
therefore the Jacobian transformation in (101) is evaluated as
\[
J((\alpha_1, \ldots, \alpha_q) \rightarrow (\beta_1, \ldots, \beta_q)) = \prod_{i=1}^{q} \frac{1}{(1 - a \beta_i)^2}.
\]
(103)
Substituting (98) and (103) into (100) yields
\[
f_{W_2}(\beta_1, \ldots, \beta_q) = K \prod_{i<j}^{q} \frac{\beta_j - \beta_i}{(1 - a \beta_j)(1 - a \beta_i)}^2.
\]
(104)
Finally, simplifying using
\[
\prod_{i<j}^{q} \left( \frac{\beta_j - \beta_i}{1 - a \beta_j} \right)^2 = \frac{\prod_{i<j}^{q} (\beta_j - \beta_i)^2}{\prod_{i=1}^{q} (1 - a \beta_i)^2(q-1)}
\]
(105)
yields the joint p.d.f. of \(L\).
We now derive the p.d.f. of an unordered eigenvalue \(\beta\) of the diagonal matrix \(L\). According to [31, Eq. 42], the unordered eigenvalue p.d.f. of \(H_2H_2^\dagger\) is given by
\[
f(\lambda) = \frac{1}{q} \sum_{i=0}^{q-1} \sum_{j=0}^{i} \sum_{l=0}^{2j} A(i, j, l, p, q) \lambda^{p-q+l} e^{-\lambda}.
\]
(106)
Recalling that \(\beta = \lambda / (1 + a \lambda)\), the result follows after applying a simple transformation.

C. Proof of Theorem 7

We start by re-expressing the conditional unordered eigenvalue p.d.f. \(f_{\lambda|L}(\cdot)\) in Lemma 7 as follows
\[
f_{\lambda|L}(\lambda) = \frac{1}{s} \prod_{i<j}^{q} (\beta_j - \beta_i) \sum_{k=q+s+1}^{q} \frac{\lambda^{n_s-q+j-1}}{\Gamma(n_s - q + j)} \det(\tilde{D}_k),
\]
(107)
where \(\tilde{D}_k\) is a \(q \times q\) matrix with entries
\[
\{\tilde{D}_k\}_{m,n} = \begin{cases} 
\beta_m^{n-1}, & n \neq k, \\
e^{-\lambda/\beta_m} \beta_m^{a-n_s-1}, & n = k.
\end{cases}
\]
(108)
Now, utilizing Lemma 7 we can evaluate the unconditional p.d.f. as follows
\[
f_\lambda(\lambda) = E_L[f_{\lambda|L}(\lambda)]
= \frac{K}{s} \sum_{k=q-s+1}^{q} \frac{\lambda^{n_s-q+k-1}}{\Gamma(n_s - q + k)} \bar{I}_k
\]
(109)
where
\[
\tilde{I}_k = \int_{0 \leq \beta_1 < \cdots < \beta_q \leq 1/a} \det(\tilde{D}_k) \prod_{i<j} (\beta_j - \beta_i) \prod_{l=1}^{q} \frac{\beta_l^{p-q} e^{-\frac{\beta_l}{1-a\beta_l}}}{(1-a\beta_l)^{p+q}} d\beta_1 \cdots d\beta_q
\]
\[
= \det(\tilde{Y}_k),
\]
where \(\tilde{Y}_k\) is a \(q \times q\) matrix with entries
\[
\{\tilde{Y}_k\}_{m,n} = \begin{cases} 
\int_0^{1/a} \frac{x^{p+q+m+n-2}}{(1-ax)^{p+q}} e^{-\frac{x}{1-ax}} dx, & n \neq k, \\
\int_0^{1/a} \frac{x^{p-n_s+m-2}}{(1-ax)^{p+q}} e^{-\frac{x}{1-ax}} e^{\lambda/x} dx, & n = k.
\end{cases}
\]

Let \(t = x/(1-ax)\). Utilizing [27, Eq. 3.383.5] and [27, Eq. 3.471.9], the integrals in (111) can be evaluated, respectively, as
\[
\int_0^{1/a} \frac{x^{p+q+m+n-2}}{(1-ax)^{p+q}} e^{-\frac{x}{1-ax}} dx = \int_0^{\infty} e^{p-q+m-n} (1+at)^{2q-m-n} e^{-t} dt = a^{q-p-m-n+1} \Gamma(p-q+m+n-1) U(p-q+m+n-1, p+q, 1/a)
\]
and
\[
\int_0^{1/a} \frac{x^{p-n_s+m-2}}{(1-ax)^{p+q}} e^{-\frac{x}{1-ax}} e^{\lambda/x} dx
\]
\[
= e^{-\lambda} \int_0^{\infty} t^{p-n_s+m-2} (1+at)^{q+n_s-m} e^{-t-\lambda} dt
\]
\[
= e^{-\lambda} \sum_{i=0}^{q+n_s-m} \binom{q+n_s-m}{i} a^{q+n_s-m-i} \int_0^{\infty} t^{p+q-i-2} e^{-t-\lambda} dt
\]
\[
= 2e^{-\lambda} \sum_{i=0}^{q+n_s-m} \binom{q+n_s-m}{i} a^{q+n_s-m-i} \lambda^{(p+q-i-1)/2} K_{p+q-i-1} \left(2\sqrt{\lambda}\right),
\]
where \(U(\cdot, \cdot, \cdot)\) is the confluent hypergeometric function of the second kind [27, Eq. 9.211.4].

Combining (109)–(113) and then applying Laplace’s expansion yields the desired result.

**D. Proof of Lemma 3**

We will prove the lemma by giving a separate treatment for the two cases, \(q < n_s\) and \(q \geq n_s\).

\(^6^\)Note that, by using the Binomial expansion, (112) can be alternatively expressed as
\[
\int_0^{\infty} t^{p-q+m+n-2} (1+at)^{2q-m-n} e^{-t} dt = \sum_{i=0}^{2q-m-n} a^i \Gamma(p-q+m+n+i-1).
\]
I) $q < n_s$ Case: In this case, we start by writing

$$E \left\{ \det \left( I_{n_s} + \frac{\rho a}{n_s} \tilde{H}_1^* L \tilde{H}_1 \right) \bigg| L \right\} = E \left\{ \det \left( I_q + \frac{\rho a}{n_s} L \tilde{H}_1^* \tilde{H}_1 \right) \bigg| L \right\}$$

$$= E \left\{ \prod_{i=1}^q \left( 1 + \frac{\rho a}{n_s} \gamma_i \right) \bigg| L \right\}$$

(114)

where $\gamma_1, \ldots, \gamma_q$ are the ordered eigenvalues of $L \tilde{H}_1^* \tilde{H}_1$. Conditioned on $L$, the joint p.d.f. of $\gamma_1, \ldots, \gamma_q$ is given in [46]. Using this result, we can express (114) as follows

$$E \left\{ \det \left( I_{n_s} + \frac{\rho a}{n_s} \tilde{H}_1^* L \tilde{H}_1 \right) \bigg| L \right\} = \frac{\int_{D_{ord}} \det \left( e^{-\gamma_i/\beta_i} \right) \prod_{i=1}^q \left( 1 + \frac{\rho a}{n_s} \gamma_i \right) \beta_i^{q-n_s-1} \gamma_i^{n_s-q} \det(\beta_j^{-1})d\gamma_1 \cdots d\gamma_q}{\prod_{i=1}^q \Gamma(n_s - i + 1) \prod_{i<j} (\beta_j - \beta_i)}$$

(115)

where the integrals are taken over the region $D_{ord} = \{ \infty \geq \gamma_1 \geq \cdots \gamma_q \geq 0 \}$. Applying [46, Corollary 2], (115) can be evaluated in closed-form as follows

$$E \left\{ \det \left( I_{n_s} + \frac{\rho a}{n_s} \tilde{H}_1^* L \tilde{H}_1 \right) \bigg| L \right\} = \frac{\prod_{i=1}^q \beta_i^{q-n_s-1} \det(\Xi_1)}{\prod_{i=1}^q \Gamma(n_s - i + 1) \prod_{i<j} (\beta_j - \beta_i)},$$

(116)

where $\Xi_1$ is a $q \times q$ matrix with entries

$$\{\Xi_1\}_{m,n} = \beta_m^{n_s-q+n} \left( \Gamma(n_s - q + n) + \frac{\rho a}{n_s} \beta_m \Gamma(n_s - q + n + 1) \right).$$

(117)

Extracting common factors from the determinant in (116) and simplifying yields the desired result.

2) $q \geq n_s$ Case: In this case, we use the joint eigenvalue p.d.f. (87) to obtain

$$E \left\{ \det \left( I_{n_s} + \frac{\rho a}{n_s} \tilde{H}_1^* L \tilde{H}_1 \right) \bigg| L \right\} = E \left\{ \prod_{i=1}^{n_s} \left( 1 + \frac{\rho a}{n_s} \gamma_i \right) \bigg| L \right\}$$

$$= \frac{\int_{D_{ord}} \prod_{i=1}^{n_s} \left( 1 + \frac{\rho a}{n_s} \gamma_i \right) \det(\Delta_1) \det(\gamma_i^{-1})d\gamma_1 \cdots d\gamma_{n_s}}{\prod_{i=1}^{n_s} \Gamma(n_s - i + 1) \prod_{i<j} (\beta_j - \beta_i)},$$

(118)

where $\gamma_1, \ldots, \gamma_{n_s}$ are the ordered eigenvalues of $\tilde{H}_1^* L \tilde{H}_1$, $\Delta_1$ is defined in (88), and the integration region is $D_{ord} = \{ \infty \geq \gamma_1 \geq \cdots \gamma_{n_s} \geq 0 \}$. Applying [33, Lemma 2], (118) can evaluated in closed-form as follows

$$E \left\{ \det \left( I_{n_s} + \frac{\rho a}{n_s} \tilde{H}_1^* L \tilde{H}_1 \right) \bigg| L \right\} = \frac{\det(\Xi_2)}{\prod_{i=1}^{n_s} \Gamma(n_s - i + 1) \prod_{i<j} (\beta_j - \beta_i)},$$

(119)

where $\Xi_2 = \begin{bmatrix} A_1 & C_1 \end{bmatrix}$ is a $q \times q$ matrix with entries

$$\{A_1\}_{m,n} = \beta_m^{n-1}, \quad n = 1, \ldots, q - n_s,$$

(120)
and
\[
\{C_1\}_{m,n} = \beta_m^{n+q-n_s-1} \Gamma (n) + (\rho a/n_s) \beta_m \Gamma (n+1), \quad n = 1, \ldots, n_s. \tag{121}
\]
Extracting common factors from \(\det(\Xi_2)\) and simplifying yields the desired result.

\textbf{E. Proof of Lemma 4}

To prove this lemma, it is convenient give a separate treatment for the two cases, \(q < n_s\) and \(q \geq n_s\).

1) \(q < n_s\) Case: Now we need to calculate the expectation \(E \left\{ \ln \det \left( L \tilde{H}_1 \tilde{H}_1^\dagger \right) \right\}\). The moment generating function (m.g.f.) of \(\ln \det \left( L \tilde{H}_1 \tilde{H}_1^\dagger \right)\), conditioned on \(L\), is given by

\[
M_1 (t \mid L) = E \left\{ \det \left( L \tilde{H}_1 \tilde{H}_1^\dagger \right)^t \mid L \right\}. \tag{122}
\]
Utilizing the joint p.d.f. of the eigenvalues \(\gamma_1, \ldots, \gamma_q\) of \(L \tilde{H}_1 \tilde{H}_1^\dagger\), presented in \([25, 46]\), we get

\[
M_1 (t \mid L) = \frac{\det (\Xi_3)}{\prod_{i=1}^q \Gamma (n_s - i + 1) \prod_{i<j}^q (\beta_j - \beta_i)} \tag{124}
\]
where \(\Xi_3\) is a \(q \times q\) matrix with entries

\[
\{\Xi_3\}_{m,n} = \beta_m^{n-q-n_s-1} \int_0^\infty e^{-y/\beta_m} y^{n_s-q+t+n-1} dy = \beta_t^{n+1} \Gamma (n_s - q + t + n). \tag{125}
\]
From \(M_1 (t \mid L)\), we get
\[
E \left\{ \ln \det \left( L \tilde{H}_1 \tilde{H}_1^\dagger \right) \mid L \right\} = \frac{d}{dt} M_1 (t \mid L) \bigg|_{t=0} = \sum_{k=1}^q \det (\Sigma_k) \prod_{i=1}^q \Gamma (n_s - i + 1) \prod_{i<j}^q (\beta_j - \beta_i)^{n-1} \tag{126}
\]
where \(\Sigma_k\) is a \(q \times q\) matrix whose entries are

\[
\{\Sigma_k\}_{m,n} = \begin{cases} 
\beta_m^{n-1} \Gamma (n_s - q + n), & n \neq k, \\
\beta_m^{n-1} \Gamma (n_s - q + n) [\psi (n_s - q + n) + \ln \beta_m], & n = k.
\end{cases} \tag{127}
\]
where \(\psi(\cdot)\) is the digamma function. Now, \(\det (\Sigma_k)\) can be further simplified as

\[
\det (\Sigma_k) = \det \left( \tilde{\Sigma}_k \right) \prod_{k=1}^q \Gamma (n_s - q + k) \tag{128}
\]
where $\tilde{\Sigma}_k$ is a $q \times q$ matrix with entries

$$\{\tilde{\Sigma}_k\}_{m,n} = \begin{cases} \beta_m^{n-1}, & n \neq k, \\ \beta_m^{n-1} \left[ \psi(n_s - q + n) + \ln \beta_m \right], & n = k. \end{cases}$$  (129)

By using the multi-linear property of determinants, along with some basic manipulations, we can write

$$\det \left( \tilde{\Sigma}_k \right) = \psi(n_s - q + k) \det \left( \beta_m^{j-1} \right) + \det \left( Y_k \right).$$  (130)

Substituting (128) and (130) into (126) and simplifying yields the desired result.

2) $q \geq n_s$ Case: We now evaluate the m.g.f. of $\ln \det \left( \tilde{H}_1^* \tilde{L} \tilde{H}_1 \right)$, conditioned on $L$, which is given by

$$M_2(t|L) = E \left\{ \det \left( \tilde{H}_1^* \tilde{L} \tilde{H}_1 \right)^t \bigg| L \right\}.$$  (131)

Utilizing (87), (131) can be expressed as

$$M_2(t|L) = \frac{1}{\prod_{i=1}^{n_s} \Gamma(n_s - i + 1) \prod_{i<j}^{q} (\beta_j - \beta_i)} \int_{D_{ord}} \prod_{i=1}^{n_s} \gamma_i^t \det(\Delta_2) \det(\gamma_i^{-1}) d\gamma_1, \ldots, d\gamma_{n_s},$$  (132)

where $D_{ord} = \{ \infty \geq \gamma_1 \geq \cdots \gamma_{n_s} \geq 0 \}$. Applying [33, Lemma 2] yields

$$M_2(t|L) = \frac{\det(\Xi_4)}{\prod_{i=1}^{n_s} \Gamma(n_s - i + 1) \prod_{i<j}^{q} (\beta_j - \beta_i)},$$  (133)

where $\Xi_4 = [A_2 \ C_2]$ is a $q \times q$ matrix with entries

$$\{A_2\}_{m,n} = \beta_m^{n-1}, \quad n = 1, \ldots, q - n_s$$  (134)

and

$$\{C_2\}_{m,n} = \Gamma(t + n) \beta_m^{q-n_s+t+n-1}, \quad n = 1, \ldots, n_s$$  (135)

From the m.g.f. (133), we can then obtain

$$E \left\{ \ln \det \left( \tilde{H}_1^* \tilde{L} \tilde{H}_1 \right) \bigg| L \right\} = \frac{d}{dt} \bigg|_{t=0} M_2(t|L)$$

$$= \sum_{k=q-n_s+1}^{q} \frac{\det(\Omega_k)}{\prod_{i=1}^{n_s} \Gamma(n_s - i + 1) \prod_{i<j}^{q} (\beta_j - \beta_i)}$$  (136)
where $\Omega_k$ is a $q \times q$ matrix with entries

$$
\{\Omega_k\}_{m,n} = \begin{cases} 
\beta_{m}^{n-1}, & n \neq k, \ n = 1, \ldots, q - n_s, \\
\Gamma (n_s - q + n) \beta_{m}^{n-1}, & n \neq k, \ n = q - n_s + 1, \ldots, q_s, \\
\beta_{m}^{n-1} \Gamma (n_s - q + n) \left[ \beta_n (n_s - q + n) + \ln \beta_m \right], & n = k.
\end{cases}
$$

(137)

By using the multi-linear property of determinants, along with some basic manipulations, we can obtain the desired result.

3) $q = s$ Case: In this case, starting with (31), we can write the determinant summation over $k$ as follows

$$
\sum_{k=1}^{q} \det (Y_k) = \sum_{k=1}^{q} \sum_{(\alpha)} \text{sgn}(\alpha) \left[ \prod_{i=1}^{q} \beta_{\alpha(i)}^{i-1} \right] \ln \beta_{\alpha(k)}
$$

(138)

where the second summation is over all permutations $\alpha = \{\alpha(1), \ldots, \alpha(q)\}$ of the set $\{1, \ldots, q\}$, with $\text{sgn}(\alpha)$ denoting the sign of the permutation. We can further write

$$
\sum_{k=1}^{q} \det (Y_k) = \sum_{\{\alpha\}} \text{sgn}(\alpha) \left[ \prod_{i=1}^{q} \beta_{\alpha(i)}^{i-1} \right] \sum_{k=1}^{q} \ln \beta_{\alpha(k)}
$$

$$
= \ln \det (\text{diag} \{\beta_i\}_{i=1}^{q}) \prod_{i<j}^{q} (\beta_j - \beta_i)
$$

(139)

Substituting (139) into (31) yields the final result.

F. Proof of Theorem 3

We start with Lemma 4 and remove the conditioning on $L$ by using Lemma 2 as follows

$$
E \{\ln \det (\Phi)\} = \sum_{k=1}^{s} \psi (n_s - s + k)
$$

$$
+ \mathcal{K} \int_{0<\beta_1<\cdots<\beta_q \leq 1/a} \det \left( \beta_i^{q-1} \right) \prod_{i=1}^{q} g(\beta_i) \sum_{k=q-n_s+1}^{q} \det (Y_k) d\beta_1 \cdots d\beta_q,
$$

(140)

where

$$
g(u) = \frac{u^{p-q} e^{-u/(1-au)}}{(1-au)^{p+q}}.
$$

(141)

Using [33, Lemma 2], these integrals can be simplified to give

$$
E \{\ln \det (\Phi)\} = \sum_{k=1}^{s} \psi (n_s - s + k) + \mathcal{K} \sum_{k=q-n_s+1}^{q} \det \left( \tilde{W}_k \right),
$$

(142)
where \( \tilde{W}_k \) is a \( q \times q \) matrix with entries

\[
\{ \tilde{W}_k \}_{m,n} = \begin{cases}
\int_0^{1/a} u^{p-q+m+n-2} (1-au)^{p+q-1} e^{-u/(1+au)} du, & n \neq k, \\
\int_0^{1/a} u^{p-q+m+n-2} (1-au)^{p+q-1} \ln u du, & n = k.
\end{cases}
\] (143)

For the case \( n \neq k \), a closed-form expression is given in (112). For the case \( n = k \), we utilize [27, Eq. 4.358.5] and [31, Eq. 47], to obtain

\[
\int_0^{1/a} u^{p-q+m+n-2} (1-au)^{p+q-1} e^{-u/(1+au)} \ln u du = \int_0^{\infty} t^{p-q+m+n-2} (1+at)^{p+q-i} e^{-t/(1+at)} dt = \sum_{i=0}^{2q-m-n} a^{2q-m-n-i} \left( \frac{2q-m-n}{i} \right) \Gamma (p+q-i) \times \left[ \psi (p+q-i-1) - e^{1/a} \sum_{l=0}^{p+q-i-2} E_{l+1} \left( \frac{1}{a} \right) \right].
\] (144)

Substituting (112) and (144) into (143) and (142) yields (33).

When \( q = s \), we start with (33) and remove the conditioning on \( L \) as follows

\[
E \{ \ln \det (\Phi) \} = \sum_{k=1}^{q} \psi (n_s - q + k) + q \int_0^{\infty} f (\bar{\beta}) \ln \bar{\beta} d\bar{\beta}
\] (145)

where \( f (\bar{\beta}) \) denotes the unordered eigenvalue p.d.f. of \( L \) (i.e. p.d.f. of a randomly-selected \( \bar{\beta} \in \{ \beta_1, \ldots, \beta_q \} \)). Substituting this p.d.f. from (18) and integrating using (144), we obtain the desired result.

**APPENDIX II**

**ERGODIC CAPACITY PROOFS**

A. Proof of Eq. (42)

When \( n_r \to \infty \), the ergodic capacity expression (11) can be expressed as follows

\[
\lim_{n_r \to \infty} C (\rho) = \frac{1}{2} E \left\{ \log_2 \det \left( I_{n_s} + \frac{\rho \alpha}{n_s} \tilde{H}_1 \tilde{L}_1 \tilde{H}_1^\dagger \right) \right\}
\] (146)

where \( \tilde{L}_1 = \text{diag} \left\{ \lambda_i^2 / (n_r (1 + a \lambda_i^2)) \right\} \). Noting that \( q = n_d \), by the Law of Large Numbers we have

\[
\lim_{n_r \to \infty} H_2 H_2^\dagger \frac{1}{n_r} = I_{n_d}
\] (147)

which implies that

\[
\lim_{n_r \to \infty} \frac{\lambda_i^2}{n_r} = 1, \quad i = 1, \ldots, n_d.
\] (148)
Recalling (5), application of (148) in (146) yields
\[ \lim_{n_r \to \infty} C(\rho) = \frac{1}{2} E \left\{ \log_2 \det \left( I_{n_s} + \frac{\rho \alpha}{n_s(1 + \rho + \alpha)} H^H H \right) \right\} , \] (149)
where \( H \) is an \( n_d \times n_s \) i.i.d. Rayleigh fading MIMO channel matrix. Applying the identity (6) to (149) yields the desired result.

B. Proof of Eq. (43)

Using (6), the ergodic capacity expression (11) can be alternatively written as
\[ C(\rho) = \frac{1}{2} E \left\{ \log_2 \det \left( I_q + \frac{\rho a}{n_s} \tilde{H}_1 \tilde{H}_1^\dagger L \right) \right\} . \] (150)
By the Law of Large Numbers we have
\[ \lim_{n_s \to \infty} \frac{\tilde{H}_1 \tilde{H}_1^\dagger}{n_s} \to I_q \] (151)
and hence (150) reduces to
\[ \lim_{n_s \to \infty} C(\rho) = \frac{1}{2} E \left\{ \log_2 \det (I_q + \rho a L) \right\} . \] (152)
Substituting (12) into (152), after some simple manipulations we easily obtain
\[ \lim_{n_s \to \infty} C(\rho) = \frac{1}{2} E \left\{ \log_2 \det (I_q + (\rho + 1) a H_2^H H_2) \right\} - \frac{1}{2} E \left\{ \log_2 \det (I_q + a H_2^H H_2) \right\} . \] (153)
Substituting (5) into (153) and applying the identity (6) yields the desired result.

C. Proof of Theorem 4

We will consider the following cases separately; namely, \( q < n_s \) and \( q \geq n_s \).

1) \( q < n_s \) Case: We start by applying the identity (6) to obtain the ergodic capacity, in the high SNR regime, as follows
\[ C(\rho) \big|_{\alpha, \rho \to \infty, \alpha/\rho = \beta} = \frac{1}{2} q \log_2 \rho - q \log_2 \left( \frac{\beta}{n_s n_r} \right) + E \left\{ \log_2 \det (LH_2 \tilde{H}_2^\dagger) \right\} . \] (154)
The high SNR slope can be calculated as
\[ S_{\infty} = \frac{q}{2} \text{ bit/s/Hz (3dB)} . \] (155)
Applying (49), the high SNR power offset is given by
\[ L_{\infty} = \frac{q}{2} \log_2 \left( \frac{\beta}{n_s n_r} \right) - \frac{1}{2} E \left\{ \log_2 \det (L \tilde{H}_2 \tilde{H}_2^\dagger) \right\} . \] (156)
Invoking Theorem 3 and simplifying yields the high SNR power offset for case \( q < n_s \).

The proof of (53) follows along similar lines to that used above, but in this case invoking Theorem 3 in place of Theorem 4.

2) \( q \geq n_s \) Case: In the high SNR regime, the ergodic capacity can be approximated as

\[
C (\rho) \big|_{\alpha, \rho \to \infty, \alpha / \rho = \beta} = \frac{1}{2} \left[ n_s \log_2 (\rho) - n_s \log_2 \left( \frac{\beta}{n_s n_r} \right) + E \{ \log_2 \det (\tilde{H}_1^\dagger \bar{L} \tilde{H}_1) \} \right].
\]  

(157)

In this case, the high SNR slope is

\[
S_\infty = \frac{n_s}{2} \ \text{bits/s/Hz (3dB)}
\]  

(158)

and the high SNR power offset can be obtained as

\[
L_\infty = \frac{n_s}{2} \log_2 \left( \frac{\beta}{n_s n_r} \right) - \frac{1}{2} E \{ \log_2 \det (\tilde{H}_1^\dagger \bar{L} \tilde{H}_1) \}.
\]  

(159)

The result follows by applying Theorem 3.

D. Proof of Corollary 7

Substituting \( n_r = 1 \) into (67) yields

\[
C_{U,1}^{n_r=1} (\rho) = \frac{1}{2} \log_2 \left( a^{-n_d} U \left( n_d, n_d + 1, \frac{1 + \rho}{\alpha} \right) + \rho n_d U \left( n_d + 1, n_d + 1, \frac{1 + \rho}{\alpha} \right) \right).
\]  

(160)

Using the following properties of the confluent hypergeometric function of the second kind [27]:

\[
U (a, a, z) = e^z z^{-a} E_a (z)
\]  

(161)

and

\[
U (a, a + 1, z) = z^{-a},
\]  

(162)

we get the final expression for \( C_{U,1}^{n_r=1} (\rho) \) in (71). Note that \( C_{U,1}^{n_r=1} (\rho) \) can be lower and upper bounded as

\[
C_{U,1}^{n_r=1} (\rho) < C_{U}^{n_r=1} (\rho) \leq C_{U,2}^{n_r=1} (\rho),
\]  

(163)

with

\[
C_{U,1}^{n_r=1} (\rho) = \frac{1}{2} \log_2 \left( 1 + \rho n_d \frac{1}{\alpha + n_d + 1} \right)
\]  

(164)

and

\[
C_{U,2}^{n_r=1} (\rho) = \frac{1}{2} \log_2 \left( 1 + \rho n_d \frac{1}{\alpha + n_d} \right),
\]  

(165)
where we have used the inequality [36, Eq. 5.1.19]. Taking \( n_d \to \infty \), we see that both (164) and (165) converge to the same limit in (72). Taking \( \alpha \to \infty \) and utilizing [36, Eq. 5.1.23], we obtain (73).

### E. Proof of Corollary 8

Note that when \( \rho \to \infty \), then \( a \to 0 \). Therefore, we apply the following asymptotic first-order expansion for the confluent hypergeometric function [36]

\[
U(c,b,z) = z^{-c} + o(1), \quad z \to \infty
\]

(166)
to yield the desired result.

### F. Proof of Theorem 6

We will use the lower bound derived in [40, Theorem 1] and consider the following cases separately; namely, \( q < n_s \) and \( q \geq n_s \).

1) \( q < n_s \) Case: Applying the (6) and [40, Theorem 1] to (11), we lower bound the ergodic capacity, conditioned on \( L \), as follows

\[
C(\rho) \geq q \log_2 \left( 1 + \frac{\rho \alpha}{n_sn_r} \exp \left( \frac{1}{q} E \left\{ \ln \det \left( L\tilde{H_1}\tilde{H}_1^* \right) \right\} \right) \right).
\]

(167)

Now, using Theorem 3 yields the desired result.

2) \( q \geq n_s \) Case: In this case, the lower bound can be written as

\[
C(\rho) \geq n_s \log_2 \left( 1 + \frac{\rho \alpha}{n_sn_r} \exp \left( \frac{1}{n_s} E \left\{ \ln \det \left( \tilde{H}_1L\tilde{H}_1^* \right) \right\} \right) \right).
\]

(168)

Again, we use Theorem 3 to obtain the desired result.

### G. Proof of Corollary 9

When \( n_s \to \infty \), \( \psi(n_s - q + k) \) can be approximated as [36, Eq. 6.3.18]

\[
\psi(n_s - q + k)_{n_s \to \infty} \approx \ln (n_s - q + k)
\]

\[
\approx \ln n_s.
\]

(169)

Substituting (169) into (76) yields the desired result.

### H. Proof of Corollary 10

Taking \( n_s \to \infty \) and using [36, Eq. 6.3.18], we get (79).
For the case $n_d \to \infty$, we first apply [36, Eq. 5.1.19] and [27, Eq. 8.365.3] to obtain the following approximation

$$\exp \left( \frac{1 + \rho}{\alpha} \right)^{n_d - 1} \sum_{l=1}^{n_d - 1} E_{l+1} \left( \frac{1 + \rho}{\alpha} \right) \approx \psi \left( n_d + \frac{1 + \rho}{\alpha} \right) - \psi \left( \frac{1 + \rho}{\alpha} \right).$$  \hfill (170)

Furthermore, substituting (170) into (78) and using [27, Eq. 8.365.5] and [36, Eq. 6.3.18] yields (80).

Now consider the case $\alpha \to \infty$. Utilizing the recurrence relation for the exponential integral [36, Eq. 5.1.14], the summation in (78) can be alternatively written as

$$\exp \left( \frac{1 + \rho}{\alpha} \right)^{n_d - 1} \sum_{l=1}^{n_d - 1} E_{l+1} \left( \frac{1 + \rho}{\alpha} \right) = \exp \left( \frac{1 + \rho}{\alpha} \right) F_{1} \left( \frac{1 + \rho}{\alpha} \right) - \sum_{l=1}^{n_d - 1} \left( \frac{1 + \rho}{\alpha l} \right) E_{l} \left( \frac{1 + \rho}{\alpha} \right) + \psi \left( n_d \right) + \gamma$$ \hfill (171)

where $\gamma = 0.577215\ldots$ is the Euler’s constant. Note that, in deriving (171), we have applied the definition of the digamma function [27, Eq. 8.365.4]. Using the series expansion given in [36, Eq. 5.1.11], when $\alpha \to \infty$, we get

$$E_{1} \left( \frac{1 + \rho}{\alpha} \right) \bigg|_{\alpha \to \infty} \to -\gamma - \ln \left( \frac{1 + \rho}{\alpha} \right)$$ \hfill (172)

and therefore

$$\sum_{l=1}^{n_d - 1} \left( \frac{1 + \rho}{\alpha l} \right) E_{l} \left( \frac{1 + \rho}{\alpha} \right) \bigg|_{\alpha \to \infty} \to 0.$$ \hfill (173)

Applying (171)–(173) in (78) yields the desired result.

I. Proof of Corollary [1]

Using the following approximation [36]

$$E_{v} (z) \approx \frac{1}{z} e^{-z} \left( 1 + o \left( \frac{1}{z} \right) \right) \quad |z| \to \infty,$$ \hfill (174)

$\varsigma_{m+n}(a)$ can be approximated as

$$\varsigma_{m+n}(a) \bigg|_{\rho \to \infty} \approx \Gamma (\tau - 1) \psi (\tau - 1),$$ \hfill (175)

which leads to the final result.

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