Coding for Sequence Reconstruction for Single Edits

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Abstract—The sequence reconstruction problem, introduced by Levenshtein in 2001, considers a communication scenario where the sender transmits a codeword from some codebook and the receiver obtains multiple noisy reads of the codeword. The common setup assumes the codebook to be the entire space and the problem is to determine the minimum number of distinct reads that is required to reconstruct the transmitted codeword.

Motivated by modern storage devices, we study a variant of the problem where the number of noisy reads \( N \) is fixed. Specifically, we design reconstruction codes that reconstruct a codeword from \( N \) distinct noisy reads. We focus on channels that introduce single edit error (i.e. a single substitution, insertion, or deletion) and their variants, and design reconstruction codes for all values of \( N \). In particular, for the case of a single edit, we show that as the number of noisy reads increases, the number of redundant bits required can be gracefully reduced from \( \log n + O(1) \) to \( \log \log n + O(1) \), and then to \( O(1) \), where \( n \) denotes the length of a codeword. We also show that the redundancy of certain reconstruction codes is within one bit of optimality.

I. INTRODUCTION

As our data needs surge, new technologies emerge to store these huge datasets. Interestingly, besides promising ultra-high storage density, certain emerging storage media rely on technologies that provide users with multiple cheap, albeit noisy, reads. In this paper, we leverage on these multiple reads to increase storage density, certain emerging storage media rely on technologies that provide users with multiple cheap, albeit noisy, reads. The common setup assumes the codebook to be the entire space and the problem is to determine the minimum number of distinct reads that is required to reconstruct the transmitted codeword.

Motivated by modern storage devices, we study a variant of the problem where the number of noisy reads \( N \) is fixed. Specifically, we design reconstruction codes that reconstruct a codeword from \( N \) distinct noisy reads. We focus on channels that introduce single edit error (i.e. a single substitution, insertion, or deletion) and their variants, and design reconstruction codes for all values of \( N \). In particular, for the case of a single edit, we show that as the number of noisy reads increases, the number of redundant bits required can be gracefully reduced from \( \log n + O(1) \) to \( \log \log n + O(1) \), and then to \( O(1) \), where \( n \) denotes the length of a codeword. We also show that the redundancy of certain reconstruction codes is within one bit of optimality.

II. PROBLEM STATEMENT AND CONTRIBUTIONS

Consider a data storage scenario described by an error-ball function. Formally, given an input space \( \mathcal{X} \) and output space \( \mathcal{Y} \), an error-ball function \( B \) maps a word \( x \in \mathcal{X} \) to a subset of noisy reads \( B(x) \subseteq \mathcal{Y} \). Given a code \( \mathcal{C} \subseteq \mathcal{X} \), we define the read coverage of \( \mathcal{C} \), denoted by \( \nu(\mathcal{C}; B) \), to be the quantity

\[
\nu(\mathcal{C}; B) = \max \{ |B(x) \cap B(y)| : x, y \in \mathcal{C}, x \neq y \}. \tag{1}
\]

In other words, \( \nu(\mathcal{C}; B) \) is the maximum intersection between the error-balls of any two codewords in \( \mathcal{C} \). The quantity \( \nu(\mathcal{C}; B) \) was introduced by Levenshtein [9], where he showed that the number of reads required to reconstruct a codeword from \( \mathcal{C} \) is at least \( \nu(\mathcal{C}; B) + 1 \). The problem to determine \( \nu(\mathcal{C}; B) \) is referred to as the sequence reconstruction problem.

The sequence reconstruction problem was studied in a variety of storage and communication scenarios [5], [8], [10], [11]. In a tape-like strip and are separated by domain walls [6], [7]. The magnetization of a domain is programmed to store a single bit value, which can be read by sensing its magnetization direction. The reading mechanism is operated by a read-only port, called a head, together with a reference domain. Since the head is fixed, a shift operation is required in order to read all the domains and this is accomplished by applying shift current which moves the domain walls in one direction. Multiple heads can also be used in order to significantly reduce the read access latency of the memory.

When these heads read overlapping segments, we have multiple noisy reads. Recently, Chee et al. [8] leveraged on these noisy reads to correct shift errors in racetrack memories. They designed an arrangement of heads and devised a corresponding coding strategy to correct such errors with a constant number of redundant bits.

Motivated by these applications, we study the following coding problem in a general setting. Consider a data storage scenario where \( N \) distinct noisy reads are provided. Our task is to design a codebook such that every codeword can be uniquely reconstructed from any \( N \) distinct noisy reads. Hence, our fundamental problem is then: how large can this codebook be? Or equivalently, what is the minimum number of redundancy?

In this paper, we study in detail the case where the reads are affected by a single edit (a substitution, deletion, or insertion) and its variants. In particular, for the case of a single edit, we show that as the number of noisy reads increases, the number of redundant bits required can be gracefully reduced from \( \log n + O(1) \) to \( \log \log n + O(1) \), and then to \( O(1) \), where \( n \) denotes the length of a codeword.

(a) DNA-based data storage. In these data systems [1]–[3], digital information is stored in native or synthetic DNA strands and to read the information, a user typically employs a sequencing platform like the popular Illumina sequencer or more recently, a nanopore sequencer. In most sequencing platforms, a DNA strand undergoes polymerase chain reaction (PCR) and multiple copies of the same strand are created. The sequencer then reads all copies and provides multiple (possibly) erroneous reads to the user (see Figure 1). In nanopore sequencers, these reads are often inaccurate and high-complexity read-alignment and consensus algorithms are required to reconstruct the original DNA strand from these noisy reads.

To reduce the read-alignment complexity and improve the read accuracy, one may employ various coding strategies to design DNA information strands. Yazdi et al. [4] proposed a simple coding strategy and verified it experimentally. Later, Cheraghchi et al. [5] provided a marker-based coding strategy that has provable reconstruction guarantees.

(b) Racetrack memories. Based on spintronic technology, a racetrack memory, also known as domain wall memory, is composed of cells, also called domains, which are positioned...
these cases, $C$ is usually assumed to be the entire space (all binary words of some fixed length) or a classical error-correcting code.

However, in most storage scenarios, the number of noisy reads $N$ is a fixed system parameter and when $N$ is at most $\nu(C; B)$, we are unable to uniquely reconstruct the codeword. This work looks at this regime where we design codes whose read coverage is strictly less than $N$. Specifically, we say that $C$ is an $(n, N; B)$-reconstruction code if $C \subseteq \{0, 1\}^n$ and $\nu(C; B) < N$.

This gives rise to a new quantity of interest that measures the trade-off between codebook redundancy and read coverage. Specifically, given $N$ and an error-ball function $B$, we study the quantity

$$\rho(n, N; B) \triangleq \min \left\{ n - \log |C| : C \subseteq \{0, 1\}^n, \nu(C; B) < N \right\}.$$  

(2)

Note that the case $N = 1$ is the classical model which has been studied for years in the design of error-correcting codes. Thus, we see the framework studied in this work as a natural extension of this classical model.

For a word $x \in \{0, 1\}^n$, we consider the following error ball functions. Let $B^i(x), B^D(x)$, and $B^S(x), B^I(x)$, and $B^D(x)$ denote the set of all words obtained from $x$ via one insertion, deletion, and at most one substitution, respectively. In this work, we study in detail the following error balls:

$$B^{SD}(x) \triangleq B^S(x) \cup B^D(x), \quad B^{BI}(x) \triangleq B^B(x) \cup B^I(x), \quad B^{edi}(x) \triangleq B^S(x) \cup B^I(x) \cup B^D(x).$$

**Example 1.** We consider the single-deletion error-ball $B^D$ and two different codebooks. First, let $C_{all} = \{0, 1\}^n$. Levenshtein in his seminal work showed that $\nu(C_{all}; B^D) = 2$. In other words, three distinct noisy versions of $x$ allow us to uniquely reconstruct $x$. Hence, $\rho(n, N; B^D) = 0$ for $N \geq 3$.

In contrast, to correct a single deletion, we have the classical Varshamov-Tenengolts (VT) code $VT(n; a)$ whose redundancy is at most $\log(n + 1)$ [15] (see also Theorem 3). In this case, $\nu(VT(n; a); B^D) = 0$ and one noisy read is sufficient to recover a codeword. Furthermore, it can be shown that $VT(n; a)$ is asymptotically optimal, or, $\rho(n, 1; B^D) = \log n + \Theta(1)$ (see Theorem 2).

A natural question is then: how should we design the codebook when we have only two noisy reads? Or, what is the value of $\rho(n, 2; B^D)$?

Recently, Chee et al. constructed a $(n, 2; B^D)$-reconstruction code with $\log n + O(1)$ redundant bits [8]. Hence, $\rho(n, 2; B^D) \leq \log n + O(1)$. In other words, even though there are only two noisy reads, it is possible to employ a coding strategy that encodes approximately $\log n - \log \log n$ bits of information more than that of the VT code $VT(n; a)$.

In this paper, we extend this analysis and design such reconstruction codes for other error-balls.

### A. Related Work

We first review previous work related to our problem when there is only one noisy read, i.e. $N = 1$. In this case, we recover the usual notion of error-correcting codes. For the error-ball functions studied in this paper, we have the following classical results.

**Theorem 2.** For $n > 0$,

(i) $\log(n + 1) \leq \rho(n, 1; B^S) \leq \lfloor \log(n + 1) \rfloor$ [16];

(ii) $\log(n - 1) \leq \rho(n, 1; B^D) = \rho(n, 1; B^I) \leq \log(n + 1) + 1$ [15].

(iii) $\log(n + 1) \leq \rho(n, 1; B^{edi}) \leq 1 + \log n$ [15].

Therefore, for $B \in \{B^S, B^D, B^I, B^{edi}\}$, we have that $\rho(n, 1; B^S) = \log n + \Theta(1)$.

For completeness, we present the families of single error-correcting codes provided by Levenshtein [15]. Crucial to these constructions is the concept of *syndrome*.

**Definition 3.** The VT syndrome of a binary sequence $x \in \{0, 1\}^n$ is defined to be $\text{Syn}(x) \triangleq \sum_{i=1}^{n} ix_i$.

**Theorem 4** (Levenshtein [15]).

(i) For $a \in \mathbb{Z}_{n+1}$, let

$$VT(n; a) \triangleq \{x \in \{0, 1\}^n : \text{Syn}(x) = a \mod (n + 1)\}. \quad (3)$$

Then, the code $VT(n; a)$ is an $(n, 1; B)$-reconstruction code for $B \in \{B^S, B^D, B^I\}$.

(ii) For $a \in \mathbb{Z}_{2n}$, let

$$L(n; a) \triangleq \{x \in \{0, 1\}^n : \text{Syn}(x) = a \mod (2n)\}. \quad (4)$$

Then, the code $L(n; a)$ is an $(n, 1; B)$-reconstruction code for $B \in \{B^S, B^{SD}, B^{edi}\}$.

When there is more than one noisy read, previous works usually focus on determining the maximum intersection size between two error balls. When $C = \{0, 1\}^n$ and the error-balls involve insertions only, deletions only and substitutions only, the value of $\nu(C; B)$ was first determined by Levenshtein [9]. Later, Levenshtein’s results were extended in [10] for the case where the error-ball involves deletions only and $C$ is a single-deletion error-correcting code. Recently, the authors of [18] investigated the case where errors are combinations of single substitution and single insertion. Furthermore, they also
simplified the reconstruction algorithm when the number of noisy copies exceeds the minimum required, i.e. $N > \nu(\xi; B) + 1$.

Another recent variant of the Levenshtein sequence reconstruction problem was studied by the authors in [19]. Similar to our model, the authors consider the scenario where the number of reads is not sufficient to reconstruct a unique codeword. As with classical list-decoding, they determined the size of the list of possible codewords.

As mentioned above, the sequence reconstruction problem has been studied by Levenshtein and others for several error channels and distances. In many cases, such as for substitutions, the size of the set $B(x) \cap B(y)$ does not depend on the specific choice of $x$ and $y$, but only on their distance. In the substitutions case, for any length-$n$ code $C$ of minimum Hamming distance $d$ and $B = B_C^2$, which is the radius-$t$ substitution ball, it holds that [9]

$$\nu(\xi; B_C^2) = N_n(t, d) \triangleq \binom{n - d}{t} \sum_{h=d-t+1}^{t-i} \binom{d}{h}.$$  \hspace{1cm} (5)

Therefore, studying the quantity $\rho(n, N; B_C^2)$ can be directly solved by finding the minimum Hamming distance of any valid code for this case. Denote by $A(n, d)$ the size of the largest length-$n$ code of minimum Hamming distance $d$. The following theorem holds.

**Theorem 5.** For all $N \geq 1$, it holds that

$$\rho(n, N; B_C^2) = n - \log(A(n, d)),$$

where $d$ is smallest integer such that $N_n^S(t, d) < N$.

For odd values of $d$ it holds that $N_n^S(t, d) = N_n^S(t, d-1)$ [10] and therefore it is enough to consider only odd values of $d$. Furthermore, for $d = 2t - 1$ it holds that $N_n^S(t, d = 2t - 1) = \binom{2t}{t}$ [9], which implies that for all $t \geq 1$,

$$\rho\left(n, N = \binom{2t}{t} + 1; B_C^2\right) = n - \log(A(n, 2t - 1)),$$

and for all $1 \leq N \leq \binom{2t}{t}$, $\rho\left(n, N; B_C^2\right) = n - \log(A(n, 2t+1))$.

**B. Main Contributions**

In this work, we focus on the case where $2 \leq N \leq \nu(\xi; B)$, where $B \in \{B^S, B^1, B^D, B^{SD}, B^{SI}, B^{ID}, B^{edit}\}$. When $N = 2$ and $B = B^D$, we have a recent code construction by Chee et al. [8] (see Example 1 and Theorem 16). Specifically, in Section IV we make suitable modifications to this code construction and show that the resulting codes are $(n, N; B)$-reconstruction code for the error-balls of interest.

To do so, in Section III we study in detail the intersection of certain error-balls and derive the necessary and sufficient conditions for the size of an intersection. Using these characterizations, we not only design the desired reconstruction codes, but also obtain asymptotically tight lower bounds in Section V. We summarize our results in Table II and observe that most values of $\rho(n, N; B)$ have been determined up to one bit of redundancy.

**III. COMBINATORIAL CHARACTERIZATION OF THE INTERSECTION OF ERROR BALLS**

In this section, we set $C = \{0, 1\}^n$ and compute the read coverage $\nu(\xi; B)$ for the error-ball function $B \in \{B^S, B^D, B^1, B^{SI}, B^{SD}, B^{ID}, B^{edit}\}$. In addition to determining the read coverage or maximum intersection size, we also characterize when the error-balls of a pair of words have intersection of a certain size. This combinatorial characterization will be crucial in the code construction in Section IV.

First, we recall the following result from Levenshtein’s seminal work.

**Theorem 6** (Levenshtein [9, Theorem 1, Eq. (26)]). Let $B \in \{B^S, B^1, B^D\}$. If $x$ and $y$ are distinct binary words of length $n$, then $|B(x) \cap B(y)| \leq 2$. Therefore, if we set $C = \{0, 1\}^n$, we have that $\nu(\xi; B)$ and $\rho(n, N; B) = 0$ for $N \geq 3$.

Next, we characterize when the error-balls of a pair of words have intersection of size two. The case for single substitution is straightforward consequence of (5).

**Lemma 7** (Levenshtein [9]). Let $x$ and $y$ be distinct binary words of length $n$. We have that

(i) $|B^S(x) \cap B^S(y)| = 2$ if and only if the Hamming distance of $x$ and $y$ is at most two.

(ii) $|B^S(x) \cap B^S(y)| = 0$ if and only if the Hamming distance of $x$ and $y$ is at least three.

For the case $B \in \{B^D, B^1\}$, we define the following notion of confusability.

**Definition 8.** Two words $x$ and $y$ of length $n$ are said to be Type-A-confusable if there exists subwords $a$, $b$, and $c$ such that the following holds.

(A1) $x = a cb$ and $y = a cb$, where $c$ is the complement of $c$.

(A2) $c$ is one of the following forms: $(01)^m$, $(01)^m 0$, $(10)^m 1$, $(10)^m 1$ for some $m \geq 1$.

**Proposition 9.** Let $B \in \{B^D, B^1\}$ and $x$ and $y$ be distinct binary words of length $n$. If the Hamming distance of $x$ and $y$ is at least two, we have that $|B(x) \cap B(y)| = 2$ if and only if $x$ and $y$ are Type-A-confusables. On the other hand, if the Hamming distance of $x$ and $y$ is less than or equal to one, it holds that $|B(x) \cap B(y)| = 0$.
distance of \( x \) and \( y \) is one, we have \(|B^D(x) \cap B^D(y)| = 1\) while \(|B^I(x) \cap B^I(y)| = 2\).

**Proof.** We first consider the case that the Hamming distance \( x \) and \( y \) is at least two and show that \(|B(x) \cap B(y)| = 2\) if and only if \( x \) and \( y \) are Type-A-confusable. We prove for the case where \( B = B^D \) and the case for \( B = B^I \) can be similarly proved. Let \( x = x_1x_2 \cdots x_n, y = y_1y_2 \cdots y_n \) and \(|B^D(x) \cap B^D(y)| = \{z, z'\} \). Now, the Hamming distance \( x \) and \( y \) is at least two. Otherwise, \(|B^D(x) \cap B^D(y)| = 1\). Since \( x \) and \( y \) are distinct, we set \( i \) and \( j \) be the smallest and largest indices, respectively, where the two words differ.

We first consider \( z \in B^D(x) \cap B^D(y) \). Let \( z \) be obtained from \( x \) by deleting index \( k \) and from \( y \) by deleting index \( \ell \). We first claim that either \( k \leq i \) or \( k \geq j \). Suppose otherwise that \( i < k < j \) and we have two cases.

- When \( \ell < j \), we consider the \((j-1)\)th index of \( z \). On one hand, since \( k < j \) and we delete \( x_k \) from \( x \), the \((j-1)\)th index of \( z \) is \( x_j \). On the other hand, since we delete \( y_k \) from \( y \), the \((j-1)\)th index of \( z \) is \( y_j \). Hence, \( x_j = y_j \), yielding a contradiction.
- When \( \ell > j \), we consider the \(i\)th index of \( z \). Proceeding as before, we conclude that \( x_i = y_i \), which is not possible.

Without loss of generality, we assume that \( k \leq i \). A similar argument shows that \( \ell \geq j \). Therefore, we have that \( y_i = x_{i+1} \) for \( k \leq i \leq \ell - 1 \). Recall that \( x_i = y_i \) whenever \( t \leq i - 1 \) or \( t \geq j +1 \). Hence, we have that \( x_k = x_{k+1} = \cdots = x_{i-1}, y_k = y_{k+1} = \cdots = y_{j-1}, x_{j+1} = x_{j+2} = \cdots = x_{\ell}, y_j = y_{j+1} = \cdots = y_{\ell} \).

In summary, if we set \( a = x_1x_2 \cdots x_{i-1} = y_1y_2 \cdots y_{i-1} \) and \( b = x_{j+1}x_{j+2} \cdots x_n = y_{j+1}y_{j+2} \cdots y_n \), then \( z = ax_{i+1}x_{i+2} \cdots x_jb = ay_{i+1}y_{i+2} \cdots y_{j-1}b \).

Now, consider \( z' \), the other word in \( B^D(x) \cap B^D(y) \). Since \( z' \) is distinct from \( z \) and proceeding as before, we have that \( z' = ax_{i+1}x_{i+2} \cdots x_jb = ay_{i+1}y_{i+2} \cdots y_{j-1}b \).

Hence, \( x_{i+1} = x_{i+2} \) for \( i \leq t \leq j - 2 \). Since \( z \neq z' \), we have that \( x_i = x_{i+1} \) and \( y_i = y_{i+1} \). Therefore, \( x \) and \( y \) satisfy conditions (A1) and (A2) and are Type-A-confusable.

Conversely, suppose that \( x \) and \( y \) are Type-A-confusable. Hence, there exist subwords \( a, b, c \) that satisfy conditions (A1) and (A2). We further set \( c_1 \) and \( c_2 \) to be words obtained by deleting the first and last index from \( c \), respectively. Then \( ac_1b \) and \( ac_2b \) are two distinct subwords that belong to \( B^D(x) \cap B^D(y) \). Since \(|B^D(x) \cap B^D(y)| \leq 2 \), we have that the intersection size must be exactly two.

When the Hamming distance of \( x \) and \( y \) is one, i.e. \( x = aab \) and \( y = a\alpha a \) where \( \alpha \neq \beta \), we have \(|B^D(x) \cap B^D(y)| = \{ab\} \) and \(|B^I(x) \cap B^I(y)| = \{a\alpha \beta b, a\beta \alpha b\} \).

**Proof.** From \([15]\), we have that for all \( x, y \),

\[
B^D(x) \cap B^D(y) = \emptyset \text{ if and only if } B^I(x) \cap B^I(y) = \emptyset.
\]

Hence, since \(|B^D(x) \cap B^D(y)| = (B^D(x) \cap B^D(y)) \cup (B^I(x) \cap B^I(y)) \) follows from Theorem 3 that \(|B^D(x) \cap B^D(y)| \in \{0, 2, 3, 4\} \).

Furthermore, when \(|B^D(x) \cap B^D(y)| = 4 \), we have that \(|B^D(x) \cap B^D(y)| = 2 \). Proposition 9 then implies that \( x \) and \( y \) are Type-A-confusable.

When the error-balls include single substitutions, we require the following notion of confusability.

**Definition 11.** Two words \( x \) and \( y \) of length \( n \) are said to be Type-B-confusable if there exists subwords \( a, b, c \) such that the following hold:

- \((B1) x = acb \) and \( y = ac'b \);
- \((B2) \{c, c'\} \) is one of the following forms: \(\{0^m, 1^m0\}, \{1^m, 0^m1\}\) for some \( m \geq 1 \).

**Proposition 12** (Single SD/SI). Let \( B \in \{B^SD, B^SI\} \) and \( x \) and \( y \) be distinct binary words of length \( n \). Then \(|B(x) \cap B(y)| \leq 4 \).

Furthermore, we have the following characterizations.

- If the Hamming distance of \( x \) and \( y \) is two, then
  - \((i) |B(x) \cap B(y)| = 4 \) if and only if \( x \) and \( y \) are Type-B-confusable with condition \((B2)\) satisfied with \( m = 1 \);
  - \((ii) |B(x) \cap B(y)| = 3 \) if and only if \( x \) and \( y \) are Type-B-confusable with condition \((B2)\) satisfied with \( m \geq 2 \).

- If the Hamming distance of \( x \) and \( y \) is one, then \(|B^SI(x) \cap B^SI(y)| = 4 \) while \(|B^SD(x) \cap B^SD(y)| = 3 \).

Therefore, when \( C = \{0, 1\}^n \), we have \( \nu(C; B) = 4 \) and \( \rho(n, N; B) = 0 \) for \( N \geq 5 \).

**Proof.** We consider the case \( B = B^SD \) and the case for \( B^SI \) can be similarly proved. Now, for all \( x, y \),

\[
|B^SD(x) \cap B^SD(y)| = |B^SI(x) \cap B^SI(y)| + |B^D(x) \cap B^D(y)|
\]

Hence, \(|B^SD(x) \cap B^SD(y)| \leq 4 \) follows from Theorem 6.

When \(|B^SD(x) \cap B^SD(y)| \geq 3 \), it follows that \(|B^SI(x) \cap B^SI(y)| = 2 \). Lemma 7 then implies that the Hamming distance of \( x \) and \( y \) is at most two. When the Hamming distance of \( x \) and \( y \) is one, we have that \(|B^D(x) \cap B^D(y)| = 1 \) while \(|B^I(x) \cap B^I(y)| = 2 \). So, \(|B^SD(x) \cap B^SD(y)| = 3 \) and \(|B^SI(x) \cap B^SI(y)| = 4 \).

When the Hamming distance of \( x \) and \( y \) is two, we set \( i \) and \( j \) (\( i < j \)) to be the two indices where \( x \) and \( y \) differ. We consider two cases.

- \(|B^SD(x) \cap B^SD(y)| = 3 \) and \(|B^D(x) \cap B^D(y)| = 1 \). Set \( B(x) \cap B^D(y) = \{z\}, a = x_1x_2 \cdots x_{i-1} = y_1y_2 \cdots y_{i-1} \) and \( b = x_{j+1}x_{j+2} \cdots x_n = y_{j+1}y_{j+2} \cdots y_n \). Then proceeding as the proof in Proposition 9, we have that \( z = ax_{i+1}x_{i+2} \cdots x_jb = ay_{i+1}y_{i+2} \cdots y_{j-1}b \).

Hence, we have \( y_i = x_{i+1} \) for \( i + 1 \leq t \leq j - 1 \) and therefore, \( x \) and \( y \) satisfy conditions (B1) and (B2) with \( m \geq 2 \).

- \(|B^SD(x) \cap B^SD(y)| = 4 \) and \(|B^D(x) \cap B^D(y)| = 2 \). Then Proposition 9 implies that \( x \) and \( y \) are Type-A-confusable. Since \( x \) and \( y \) differ at exactly two coordinates, we have that they satisfy conditions (B1) and (B2) with \( m = 1 \).
Finally, we characterize the intersection sizes when the error-balls arise from single edits.

**Proposition 13 (Single Edit).** Let $x$ and $y$ be distinct binary words of length $n$. Then $|B_{\text{edit}}(x) \cap B_{\text{edit}}(y)| \leq 6$. Furthermore, we have the following characterizations.

(i) $|B_{\text{edit}}(x) \cap B_{\text{edit}}(y)| = 6$ if and only if $x$ and $y$ are Type-B-confusable with condition (B2) satisfied with $m = 1$.

(ii) $|B_{\text{edit}}(x) \cap B_{\text{edit}}(y)| = 5$ if and only if the Hamming distance of $x$ and $y$ is one.

(iii) $|B_{\text{edit}}(x) \cap B_{\text{edit}}(y)| = 4$ if and only if $x$ and $y$ are Type-A-confusable with $|c| \geq 3$ or Type-B-confusable with condition (B2) satisfied with $m \geq 2$.

Therefore, when $c = \{0,1\}^n$, we have $\nu(c; B_{\text{edit}}) = 6$ and $\rho(n, N; B_{\text{edit}}) = 0$ for $N \geq 7$.

**Proof.** For all $x, y$,

$$|B_{\text{edit}}(x) \cap B_{\text{edit}}(y)| = |B_S(x) \cap B_S(y)| + |B_D(x) \cap B_D(y)| + |B_I(x) \cap B_I(y)| \leq 6.$$  

We have the following cases.

(i) When $|B_{\text{edit}}(x) \cap B_{\text{edit}}(y)| = 6$, we have that $|B_S(x) \cap B_S(y)| = |B_D(x) \cap B_D(y)| = 2$ or $|B_{SD}(x) \cap B_{SD}(y)| = 4$. Proposition 12 then states that $x$ and $y$ are Type-B-confusable with condition (B2) satisfied with $m = 1$.

(ii) When $|B_{\text{edit}}(x) \cap B_{\text{edit}}(y)| = 5$, we consider the following subcases.

- If the Hamming distance of $x$ and $y$ is one, Proposition 12 states that $|B_S(x) \cap B_S(y)| = |B_D(x) \cap B_D(y)| = 2$ while $|B_D(x) \cap B_D(y)| = 1$.

- If the Hamming distance of $x$ and $y$ is two. We then have $|B_S(x) \cap B_S(y)| = |B_I(x) \cap B_I(y)| = 2$ while $|B_D(x) \cap B_D(y)| = 1$, which implies $|B_{SD}(x) \cap B_{SD}(y)| = 4$. Proposition 12 states that $x$ and $y$ are Type-B-confusable with condition (B2) satisfied with $m = 1$. However, this implies $|B_D(x) \cap B_D(y)| = 2$, we get a contradiction.

- If the Hamming distance of $x$ and $y$ is at least three, then it follows from Lemma 7 that $|B_S(x) \cap B_S(y)| = 0$. Since $|B_S(x) \cap B_S(y)| + |B_D(x) \cap B_D(y)| \leq 4$, we get a contradiction.

(iii) When $|B_{\text{edit}}(x) \cap B_{\text{edit}}(y)| = 4$, we have two cases.

- $|B_S(x) \cap B_S(y)| = 2$ and $|B_D(x) \cap B_D(y)| = |B_I(x) \cap B_I(y)| = 1$. Hence, $|B_{SD}(x) \cap B_{SD}(y)| = |B_{SI}(x) \cap B_{SI}(y)| = 3$ and Proposition 12 implies that $x$ and $y$ are Type-B-confusable with condition (B2) satisfied with $m \geq 2$.

- $|B_D(x) \cap B_D(y)| = |B_I(x) \cap B_I(y)| = 2$ and $|B_S(x) \cap B_S(y)| = 0$. Then Proposition 12 implies that $x$ and $y$ are Type-A-confusable. Since the Hamming distance of $x$ and $y$ is at three, we require $|c| \geq 3$.

IV. RECONSTRUCTION CODES WITH $o(\log n)$ REDUNDANCY

Trivially, an $(n, N; B)$-reconstruction code is also an $(n, N'; B)$-reconstruction code for $N' > N$. Hence, it follows from Theorem 2 that there exists an $(n, N; B)$-reconstruction code with $\log n + O(1)$ redundant bits for all $B \in \{B_D, B_I, B_{ID}, B_{SD}, B_{SI}, B_{edit}\}$ and $N > 1$. In this section, we provide reconstruction codes with redundancy $o(\log n)$ when $N > 1$.

To do so, we recall a recent construction of an $(n, 2; B_D)$-reconstruction code provided by Chee et al. [8] in the context of racetrack memories. Crucial to this construction is the notion of period.

**Definition 14.** Let $\ell$ and $t$ be two positive integers where $\ell < t$. Then the word $u = u_1 u_2 \cdots u_t \in \{0,1\}^t$ is said to have period $\ell$ if $u_i = u_{i+\ell}$ for all $1 \leq i \leq t - \ell$. We use $\mathcal{R}(n, \ell, t)$ to denote the set of all words $c$ of length $n$ such that the length of any subword of $c$ with period $\ell$ is at most $t$.

In [8], the following lower bound on the size of $\mathcal{R}(n, \ell, t)$ is given.

**Proposition 15 ([8]).** For $\ell \in \{1, 2\}$, if $t \geq \lceil \log n \rceil + \ell$, we have that the size of $\mathcal{R}(n, \ell, t)$ is at least $2^{n-1}$.

We are now ready to present the following construction of an $(n, 2; B_D)$-reconstruction code from [8]. Here, we demonstrate its correctness for completeness and also because the key ideas are crucial to the constructions in Theorems 13 and 23. Furthermore, we improve the construction from [8] and reduce the redundancy by approximately one bit.

**Theorem 16 (Single Deletion, $N = 2$ [8]).** For $n, P > 0$, let $c \in \{1,2\}$ and $d \in \{2\}$. Define $\mathcal{C}_D(n; c, d)$ to be the set of all words $x = x_1 x_2 \cdots x_n$ such that the following holds.

(i) $\operatorname{Syn}(x) = c \pmod{1 + P/2}$.

(ii) $\sum_{i=1}^{n} x_i = d \pmod{2}$.

(iii) $x$ belongs to $\mathcal{R}(n, 2, P)$.

Then $\mathcal{C}_D(n; c, d)$ is an $(n, 2; B_D)$-reconstruction code. Furthermore, if we set $P = \lfloor \log n \rfloor + 2$, the code $\mathcal{C}_D(n; c, d)$ has redundancy $1 + \log(\lfloor \log n \rfloor + 4) = \log(\log n + O(1))$ for some choice of $c$ and $d$.

**Proof.** We prove by contradiction. Suppose that $x$ and $y$ are two distinct words in $\mathcal{C}_D(n; c, d)$ with $|B_D(x) \cap B_D(y)| = 2$. Then Proposition 2 states that $x$ and $y$ are Type-A-confusable. In other words, there exist substrings $a, b, c$ such that $x = abc, y = acb$ and $c$ has period at most two.

Note that since the weights of $x$ and $y$ have the same parity, we have $c \in \{(01)^m, (10)^m\}$ for some $m \geq 1$. First, suppose that $c = (01)^m$. Then by construction,

$$\operatorname{Syn}(x) - \operatorname{Syn}(y) = 0 \pmod{1 + P/2}. \tag{7}$$

On the other hand, since $x = abc$ and $y = acb$, the left-hand side of (7) evaluates to $m$. However, since $c$ is a subword of $x$ with period at most two, we have that $2m \leq P$, and so, $m \neq 0 \pmod{1 + P/2}$, arriving at a contradiction. Similarly, when $c = (10)^m$, the left-hand side of (7) evaluates to $-m \neq 0 \pmod{1 + P/2}$.

In a similar manner, we show that the code $\mathcal{C}_D(n; c, d)$ is capable of reconstructing codewords from noisy reads affected by single insertions or deletions.

**Corollary 17 (Single Insertion/Deletion, $N \in \{3, 4\}$).** Let $\mathcal{C}_D(n; c, d)$ as be defined in Theorem 12. Then $\mathcal{C}_D(n; c, d)$ is an $(n, N; B_{ID})$-reconstruction code for $N \in \{3, 4\}$.

**Proof.** If two distinct words $x$ and $y$ have $|B_{ID}(x) \cap B_{ID}(y)| \geq 3$, then $|B_{D}(x) \cap B_{D}(y)| \geq 2$ or $|B_{I}(x) \cap B_{I}(y)| \geq 2$. \hfill $\blacksquare$
Suppose that $|B^1(x) \cap B^1(y)| = 2$. Since the Hamming distance of $x$ and $y$ is at least two, Proposition 10 states that $x$ and $y$ are Type-A-confusable, and hence, $|B^D(x) \cap B^D(y)| = 2$, contradicting Theorem 16. Thus, $E_D(n;c,d)$ is an $(n,N;B^{ID})$-reconstruction code for $N \in \{3,4\}$.

When $B \in \{B^{SD}, B^{SI}\}$, we make suitable modifications to the code $E_D(n;c,d)$ to correct (possibly) a single substitution.

**Theorem 18** (Single Substitution/Deletion, $N = 3$). For $n, P > 0$, let $c \in \mathbb{Z}_{1+P}$ and $d \in \mathbb{Z}_2$. Define $E_{SD}(n;c,d)$ to be the set of all words $x = x_1 x_2 \cdots x_n$ such that the following holds.

(i) $\text{Syn}(x) = c \pmod{1 + P}$.
(ii) $\sum_{i=1}^n x_i = d \pmod{2}$.
(iii) $x$ belongs to $\mathcal{R}(n,1,P)$.

Then $E_{SD}(n;c,d)$ is an $(n,3,B)$-reconstruction code for $B \in \{B^{SD}, B^{SI}\}$. Furthermore, if we set $P = \lfloor \log n \rfloor + 1$, the code $E_D(n;c,d)$ has redundancy $2 + \log(\lfloor \log n \rfloor + 1) = \log n + O(1)$ for some choice of $c$ and $d$.

**Proof.** We prove for the error-ball function $B^{SD}$ and prove by contradiction. Suppose that $x$ and $y$ are two distinct words in $E_{SD}(n;c,d)$ with $|B_{SD}(x) \cap B_{SD}(y)| \geq 3$. Since $x$ and $y$ have the same parity, the Hamming distance of $x$ and $y$ is at least two. Then Proposition 12 states that $x$ and $y$ are Type-B-confusable. Without loss of generality, let $x = a01^m b, y = a10^m b$.

As before, we have

$$\text{Syn}(x) - \text{Syn}(y) = 0 \pmod{1 + P}.$$  

Now, the left-hand side of [8] evaluates to $m$. However, since $x$ belongs to $\mathcal{R}(n,1,P)$, we have that $m \leq P$, arriving at a contradiction.

To correct a single edit with three or four reads, we make a small modification to $E_{SD}(n;c,d)$.

**Corollary 19** (Single Edit, $N \in \{3,4\}$). For $n, P > 0$, let $c \in \mathbb{Z}_{1+P}$ and $d \in \mathbb{Z}_2$. Define $E_{edit}(n;c,d)$ to be the set of all words $x = x_1 x_2 \cdots x_n$ such that the following holds.

(i) $\text{Syn}(x) = c \pmod{1 + P}$.
(ii) $\sum_{i=1}^n x_i = d \pmod{2}$.
(iii) $x$ belongs to $\mathcal{R}(n,2,P)$.

Then $E_{edit}(n;c,d)$ is an $(n,N;B^{edit})$-reconstruction code for $N \in \{3,4\}$. Furthermore, if we set $P = \lfloor \log n \rfloor + 2$, the code $E_{edit}(n;c,d)$ has redundancy $2 + \log(\lfloor \log n \rfloor + 2) = \log n + O(1)$ for some choice of $c$ and $d$.

**Proof.** Observe from [13]. $B^D(x) \cap B^D(y) \neq \emptyset$ if and only if $B^1(x) \cap B^1(y) \neq \emptyset$, therefore if two distinct words $x$ and $y$ have $|B_{edit}(x) \cap B_{edit}(y)| \geq 3$, then $|B_{edit}(x) \cap B_{edit}(y)| \geq 4$. Hence, it suffices to show for the case $N = 4$.

We prove by contradiction and assume that there are two distinct words in $E_{edit}(n;c,d)$ with $|B_{edit}(x) \cap B_{edit}(y)| \geq 4$. Then we have two possibilities.

- $|B_{SD}(x) \cap B_{SD}(y)| \geq 3$. Note that since $x \in \mathcal{R}(n,2,P)$, we have $x \in \mathcal{R}(n,1,P)$. Hence, following the proof of Theorem 18, we obtain a contradiction.
- $|B^{ID}(x) \cap B^{ID}(y)| = |B^1(x) \cap B^1(y)| = 2$. Since $1 + P/2 < 1 + P$, we can follow the proof of Theorem 16 to obtain a contradiction.

Our final code constructions introduce one and two bits of redundancy, respectively. Instead of taking the parity bit of all coordinates, we take the parity of all even coordinates.

**Theorem 20** (Single Substitution/Deletion, $N = 4$). Let $E_1$ be the set of all words $x = x_1 x_2 \cdots x_n$ such that $\sum_{i=1}^n |x_{2i}| = 0 \pmod{2}$. Then $E_1$ is an $(n,4;B^{SD})$-reconstruction code with one redundant bit.

**Proof.** We prove by contradiction. Suppose that $x$ and $y$ are two distinct words in $E_{SD}(n;c,d)$ with $|B_{SD}(x) \cap B_{SD}(y)| = 4$. Then Proposition 12 states that $x$ and $y$ are Type-B-confusable with $m = 1$. Without loss of generality, let $x = a01b, y = a10b$.

Then $\sum_{i=1}^n x_{2i} - y_{2i} = 1 \neq 0 \pmod{2}$, a contradiction.

The next construction takes another bit of redundancy, that is, the parity bit of all coordinates.

**Theorem 21** (Single Substitution/Insertion, $N = 4$). Let $E_2$ be the set of all words $x = x_1 x_2 \cdots x_n$ such that $\sum_{i=1}^n |x_{2i}| = 0 \pmod{2}$ and $\sum_{i=1}^n x_i = 0 \pmod{2}$. Then $E_2$ is an $(n,4;B^{SI})$-reconstruction code with two redundant bits.

Finally, we show that $E_1$ can correct a single edit whenever we have at least five noisy reads.

**Corollary 22** (Single Edit, $N \in \{5,6\}$). Let $E_1$ and $E_2$ be as defined in Theorems 20 and 21 respectively. Then $E_1$ is an $(n,5;B^{edit})$-reconstruction code and $E_2$ is an $(n,6;B^{edit})$-reconstruction code.

**Proof.** Suppose that there are two distinct words in $E_1$ with $|B_{edit}(x) \cap B_{edit}(y)| = 5$. Then it is necessary that $|B_{SD}(x) \cap B_{SD}(y)| = 4$ and the corollary follows from Theorem 20.

On the other hand, if there are two distinct words in with $|B_{edit}(x) \cap B_{edit}(y)| = 5$, we can show that $|B_{SI}(x) \cap B_{SI}(y)| = 4$ and the corollary follows from Theorem 21.

V. LOWER BOUNDS FOR THE REDUNDANCY OF RECONSTRUCTION CODES

In this section, we provide some straightforward lower bounds for the redundancy of reconstruction codes when $N = 2$. The following theorem demonstrates that the codes from Theorem 2 are essentially optimal.

**Proposition 23.** Let $n > 0$.

(i) $\rho(n,2;B^{S}) = \rho(n,1;B^{S})$.
(ii) $\rho(n,2;B^{ID}) = \rho(n,1;B^{ID})$.
(iii) $\rho(n,2;B^{SD}) \geq \rho(n,1;B^{SD})$.
(iv) $\rho(n,2;B^{SI}) \geq \rho(n,1;B^{SI})$.
(v) $\rho(n,2;B^{edit}) \geq \rho(n,1;B^{edit})$.

Therefore, we have that $\rho(n,2;B) = \log n + \Theta(1)$ for $B \in \{B^{S}, B^{ID}, B^{SD}, B^{SI}, B^{edit}\}$.

**Proof.** We establish the lower bounds via the following claims.

(i) We claim that $E$ is an $(n,1;B^{S})$-reconstruction code if and only if $E$ is an $(n,2;B^{S})$-reconstruction code. The forward implication is clear and we demonstrate the converse. Suppose on the contrary that $E$ is an $(n,2;B^{S})$-reconstruction code, but not an $(n,1;B^{S})$-reconstruction code. Then $|B^{S}(x) \cap B^{S}(x)| \geq 1$ for a pair of distinct words $x,y \in E$. However, Lemma 7 implies that $|B^{S}(x) \cap B^{S}(x)| = 2$, contradicting our assumption that $E$ is an $(n,2;B^{S})$-reconstruction code.
We claim that $\mathcal{C}$ is an $(n, 1; B^{D})$-reconstruction code if and only if $\mathcal{C}$ is an $(n, 2; B^{D})$-reconstruction code. This follows directly from (6).

We claim that if $\mathcal{C}$ is an $(n, 2; B^{SD})$-reconstruction code, then $\mathcal{C}$ is an $(n, 1; B^{S})$-reconstruction code. Suppose on the contrary that $\mathcal{C}$ is not an $(n, 1; B^{S})$-reconstruction code. Then $|B^{S}(x) \cap B^{S}(y)| \geq 1$ for a pair of distinct words $x, y \in \mathcal{C}$. However, Lemma 7 implies that $|B^{S}(x) \cap B^{S}(y)| = 2$ and hence, $|B^{SD}(x) \cap B^{SD}(y)| \geq 2$, a contradiction.

The next proposition shows that we need at least one bit of redundancy for certain instances.

**Proposition 24.** Let $n > 0$.

(i) $\rho(n, 4; B^{SI}) \geq 1$.

(ii) $\rho(n, 5; B^{edit}) \geq 1$.

**Proof.** We prove for (i) and the proof for (ii) is similar. Suppose that there exists an $(n, 4; B^{SI})$-reconstruction code with less than one bit of redundancy. Then there exists distinct codewords $x$ and $y$ whose Hamming distance is one. Then Proposition 12 states that $|B^{SI}(x) \cap B^{SI}(y)| = 4$, yielding a contradiction.

**VI. Conclusion**

We studied the sequence reconstruction problem in the context when the number of noisy reads $N$ is fixed. Specifically, for a variety of error-balls $B$, we designed $(n, N; B)$-reconstruction codes for $2 \leq N \leq \nu(\mathcal{C}; B)$ and derived their corresponding lower bounds. Of significance, our code constructions use $o(\log n)$ bits of redundancy and in certain cases are within one bit of optimality. Our results for $\rho(n, N; B)$ are summarized in Table [I].

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