We consider null bosonic $p$-branes in curved space-times. Some exact solutions of the classical equations of motion and of the constraints for the null membrane in general stationary, axially symmetrical, four dimensional, gravity background are found.

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1 Introduction

The null (tensionless) $p$-branes correspond to usual $p$-branes with their tension $T_p$ taken to be zero. This relationship between null $p$-branes and the tensionful ones may be regarded as a generalization of the massless-massive particles correspondence. The dynamics of the null branes in curved backgrounds is interesting also as a generalization of the motion of null strings in such backgrounds [1], [2], [3], [4], [5].

A model for null $p$-branes living in Friedmann-Robertson-Walker space-time (with

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\end{footnotesize}
$k = 0$) was proposed in [3]. The motion equations were solved and it was shown there that an ideal fluid of null $p$-branes may be considered as a source of gravity.

Here we investigate the classical evolution of the null branes in a curved background. In Sec. 2 we give the corresponding Lagrangian formulation. In Sec. 3 the BRST-BFV approach in its Hamiltonian version is applied to the considered dynamical system. In Sec. 4 some exact solutions of the equations of motion and of the constraints for the null membrane in general stationary, axially symmetrical, four dimensional, gravity background are found. The examples of Minkowski, de Sitter, Schwarzschild, Taub-NUT and Kerr space-times are considered in Sec. 5. Sec. 6 is devoted to comments and conclusions.

# 2 Lagrangian formulation

The action for the bosonic null $p$-brane in a D-dimensional curved space-time with metric tensor $g_{\mu\nu}(x)$ can be written in the form:

$$S = \int d^{p+1} \xi L, \quad L = V^J V^K \partial_J x^\mu \partial_K x^\nu g_{\mu\nu}(x),$$

$$\partial_J = \partial / \partial \xi^J, \quad \xi^J = (\xi^0, \xi^j) = (\tau, \sigma^j),$$

$$J, K = 0, 1, ..., p, \quad j, k = 1, ..., p, \quad \mu, \nu = 0, 1, ..., D - 1.$$  

It is an obvious generalization of the flat space-time action given in [3].

To prove the invariance of the action under infinitesimal diffeomorphisms on the world volume (reparametrizations), we first write down the corresponding transforma-
tion law for the \((r,s)\)-type tensor density of weight \(a\)

\[
\delta \varepsilon T_{K_1 \ldots K_s}^{J_1 \ldots J_r}[a] = L \varepsilon T_{K_1 \ldots K_s}^{J_1 \ldots J_r}[a] = \varepsilon^L \partial_L T_{K_1 \ldots K_s}^{J_1 \ldots J_r}[a] \\
+ T_{K_2 \ldots K_s}^{J_1 \ldots J_r}[a] \partial_{K_1} \varepsilon^K + \ldots + T_{K_1 \ldots K_{s-1} K_s}^{J_1 \ldots J_r}[a] \partial_{K_s} \varepsilon^K \\
- T_{K_1 \ldots K_s}^{J_1 \ldots J_r-1 J_s}[a] \partial_{J_s} \varepsilon^{J_1} - \ldots - T_{K_1 \ldots K_s}^{J_1 \ldots J_r-1 J_s}[a] \partial_{J_s} \varepsilon^{J_r} \\
+ a T_{K_1 \ldots K_s}^{J_1 \ldots J_r}[a] \partial_L \varepsilon^L,
\]

(2)

where \(L \varepsilon\) is the Lie derivative along the vector field \(\varepsilon\). Using (2), one verifies that if \(x^\mu(\xi), g_{\mu\nu}(\xi)\) are world-volume scalars \((a = 0)\) and \(V^J(\xi)\) is a world-volume \((1,0)\)-type tensor density of weight \(a = 1/2\), then \(\partial_J x^\nu\) is a \((0,1)\)-type tensor, \(\partial_J x^\mu \partial_K x^\nu g_{\mu\nu}\) is a \((0,2)\)-type tensor and \(L\) is a scalar density of weight \(a = 1\). Therefore,

\[
\delta \varepsilon S = \int d^{p+1} \xi \partial_J (\varepsilon^J L)
\]

and this variation vanishes under suitable boundary conditions.

The equations of motion following from (1) are:

\[
\partial_J \left( V^J V^K \partial_K x^\lambda \right) + \Gamma^\lambda_{\mu\nu} V^J V^K \partial_J x^\mu \partial_K x^\nu = 0,
\]

\[
V^J \partial_J x^\mu \partial_K x^\nu g_{\mu\nu}(x) = 0,
\]

where \(\Gamma^\lambda_{\mu\nu}\) is the connection compatible with the metric \(g_{\mu\nu}(x)\):

\[
\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} (\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}).
\]

For the transition to Hamiltonian picture it is convenient to rewrite the Lagrangian density (1) in the form \((\partial_\tau = \partial/\partial \tau, \partial_j = \partial/\partial \sigma^j)\):

\[
L = \frac{1}{4 \mu^0} g_{\mu\nu}(x) (\partial_\tau - \mu^j \partial_j) x^\mu (\partial_\tau - \mu^k \partial_k) x^\nu,
\]

(3)
where

\[ V^J = (V^0, V^j) = \left( -\frac{1}{2\sqrt{\mu^\sigma}}, \frac{\mu^j}{2\sqrt{\mu^0}} \right). \]

Now the equation of motion for \( x^\nu \) takes the form:

\[
\begin{align*}
\partial_\tau \left[ \frac{1}{2\mu^0} (\partial_\tau - \mu^k \partial_k) x^\lambda \right] - \partial_j \left[ \frac{\mu^j}{2\mu^0} (\partial_\tau - \mu^k \partial_k) x^\lambda \right] \\
+ \frac{1}{2\mu^0} \Gamma^\lambda_{\mu\nu} (\partial_\tau - \mu^j \partial_j) x^\mu (\partial_\tau - \mu^k \partial_k) x^\nu = 0.
\end{align*}
\]  

(4)

The equations of motion for the Lagrange multipliers \( \mu^0 \) and \( \mu^j \) which follow from (3) give the constraints:

\[
g_{\mu\nu}(x)(\partial_\tau - \mu^j \partial_j) x^\mu (\partial_\tau - \mu^k \partial_k) x^\nu = 0, \tag{5}
\]

\[
g_{\mu\nu}(x)(\partial_\tau - \mu^k \partial_k) x^\mu \partial_j x^\nu = 0.
\]

In terms of \( x^\nu \) and the conjugated momentum \( p_\nu \) they read:

\[
T_0 = g^{\mu\nu}(x)p_\mu p_\nu = 0, \quad T_j = p_\nu \partial_j x^\nu = 0. \tag{6}
\]

3 Hamiltonian formulation

The Hamiltonian which corresponds to the Lagrangian density (3) is a linear combination of the constraints (3) :

\[
H_0 = \int d^p \sigma (\mu^0 T_0 + \mu^j T_j).
\]

They satisfy the following (equal \( \tau \)) Poisson bracket algebra

\[
\{ T_0(\mathcal{G}_1), T_0(\mathcal{G}_2) \} = 0,
\]
\[ \{T_0(\sigma_1), T_j(\sigma_2)\} = [T_0(\sigma_1) + T_0(\sigma_2)]\partial_j\delta^p(\sigma_1 - \sigma_2), \]  
\[ \{T_j(\sigma_1), T_k(\sigma_2)\} = [\delta^j_l T_k(\sigma_1) + \delta^j_k T_j(\sigma_2)]\partial_l\delta^p(\sigma_1 - \sigma_2), \]

\[ \sigma = (\sigma^1, ..., \sigma^p). \]

The equalities (7) show that the constraint algebra is the same for flat and for curved backgrounds. Having in mind the above algebra, one can construct the corresponding BRST charge \( \Omega \) (*=complex conjugation)

\[ \Omega = \Omega^{\min} + \pi_j \bar{P}^j, \quad \{\Omega, \Omega\} = 0, \quad \Omega^* = \Omega. \]  

\( \Omega^{\min} \) in (8) can be written as

\[ \Omega^{\min} = \int d^p \sigma \{T_0\eta^0 + T_j\eta^j + P_0[(\partial_j\eta^j)\eta^0 + (\partial_j\eta^0)\eta^j] + P_k(\partial_j\eta^k)\eta^j\}, \]

and can be represented also in the form

\[ \Omega^{\min} = \int d^p \sigma [(T_0 + \frac{1}{2}T_0^{gh})\eta^0 + (T_j + \frac{1}{2}T_j^{gh})\eta^j] + \int d^p \sigma \partial_j\left(\frac{1}{2}P_k\eta^k\eta^j\right). \]

Here a superscript \( gh \) is used for the ghost part of the total gauge generators

\[ T^\text{tot}_j = \{\Omega, P_j\} = \{\Omega^{\min}, P_j\} = T_j + T_j^{gh}. \]

We recall that the Poisson bracket algebras of \( T^\text{tot}_j \) and \( T_j \) coincide for first rank systems which is the case under consideration. The manifest expressions for \( T^{gh}_j \) are:

\[ T^{gh}_0 = 2P_0\partial_j\eta^j + (\partial_j P_0)\eta^j, \]

\[ T^{gh}_j = 2P_0\partial_j\eta^0 + (\partial_j P_0)\eta^0 + P_j\partial_k\eta^k + P_k\partial_j\eta^k + (\partial_k P_j)\eta^k. \]

Up to now, we introduced canonically conjugated ghosts \((\eta^j, P_j), (\bar{\eta}^j, \bar{P}^j)\) and momenta \( \pi_j \) for the Lagrange multipliers \( \mu^j \) in the Hamiltonian. They have Poisson
brackets and Grassmann parity as follows ($\epsilon_J$ is the Grassmann parity of the corresponding constraint):

$$\{\eta^J, \mathcal{P}_K\} = \delta^J_K, \quad \epsilon(\eta^J) = \epsilon(\mathcal{P}_J) = \epsilon_J + 1,$$

$$\{\bar{\eta}_J, \bar{\mathcal{P}}^K\} = -(-1)^{\epsilon_J\epsilon_K} \delta^J_K, \quad \epsilon(\bar{\eta}_J) = \epsilon(\bar{\mathcal{P}}^J) = \epsilon_J + 1,$$

$$\{\mu^J, \pi_K\} = \delta^J_K, \quad \epsilon(\mu^J) = \epsilon(\pi_J) = \epsilon_J.$$

The BRST-invariant Hamiltonian is

$$H_{\tilde{\chi}} = H_{\text{min}} + \{\tilde{\chi}, \Omega\} = \{\tilde{\chi}, \Omega\}, \quad (9)$$

because from $H_{\text{canonical}} = 0$ it follows $H_{\text{min}} = 0$. In this formula $\tilde{\chi}$ stands for the gauge fixing fermion ($\tilde{\chi}^* = -\tilde{\chi}$). We use the following representation for the latter

$$\tilde{\chi} = \chi_{\text{min}} + \bar{\eta}_J(\chi^J + \frac{1}{2}\rho(J)\pi^J), \quad \chi_{\text{min}} = \mu^J\mathcal{P}_J,$$

where $\rho(J)$ are scalar parameters and we have separated the $\pi^J$-dependence from $\chi^J$. If we adopt that $\chi^J$ does not depend on the ghosts ($\eta^J, \mathcal{P}_J$) and $(\bar{\eta}_J, \bar{\mathcal{P}}^J)$, the Hamiltonian $H_{\tilde{\chi}}$ from (9) takes the form

$$H_{\tilde{\chi}} = H_{\chi_{\text{min}}} + \mathcal{P}_J\bar{\mathcal{P}}^J - \pi_J(\chi^J + \frac{1}{2}\rho(J)\pi^J) +$$

$$+ \bar{\eta}_J\{\chi^J, T_K\}\eta^K, \quad (10)$$

where

$$H_{\chi_{\text{min}}} = \{\chi_{\text{min}}, \Omega_{\text{min}}\}.$$

One can use the representation (10) for $H_{\tilde{\chi}}$ to obtain the corresponding BRST invariant Lagrangian density

$$L_{\tilde{\chi}} = L + L_{GH} + L_{GF}.$$
Here $L$ is given in (3), $L_{GH}$ stands for the ghost part and $L_{GF}$ - for the gauge fixing part of the Lagrangian density. The manifest expressions for $L_{GH}$ and $L_{GF}$ are [9]:

\[
L_{GH} = -\partial_{\tau}\bar{\eta}_0\partial_{\tau}\eta^0 - \partial_{\tau}\bar{\eta}_j\partial_{\tau}\eta^j + \mu^0[2\partial_{\tau}\bar{\eta}_0\partial_{\tau}\eta^j + (\partial_j\partial_{\tau}\bar{\eta}_0)\eta^j]
+ \mu^j[2\partial_{\tau}\bar{\eta}_0\partial_{\tau}\eta^0 + (\partial_j\partial_{\tau}\bar{\eta}_0)\eta^0 + \partial_{\tau}\bar{\eta}_k\partial_{\tau}\eta^k + \partial_{\tau}\bar{\eta}_j\partial_{\tau}\eta^k + (\partial_k\partial_{\tau}\bar{\eta}_j)\eta^k]
+ \int d^p\sigma'\{\bar{\eta}_0(\sigma')\{T_0, \chi^0(\sigma')\}\eta^0 + \{T_j, \chi^0(\sigma')\}\eta^j\}
+ \bar{\eta}_j(\sigma')\{T_0, \chi^j(\sigma')\}\eta^0 + \{T_k, \chi^j(\sigma')\}\eta^k\},
\]

\[
L_{GF} = \frac{1}{2\rho(0)}(\partial_{\tau}\mu^0 - \chi^0)(\partial_{\tau}\mu_0 - \chi_0) + \frac{1}{2\rho(j)}(\partial_{\tau}\mu^j - \chi^j)(\partial_{\tau}\mu_j - \chi_j).
\]

If one does not intend to pass to the Lagrangian formalism, one may restrict oneself to the minimal sector ($\Omega_{\text{min}}, \chi_{\text{min}}, H_{\chi_{\text{min}}}$). In particular, this means that Lagrange multipliers are not considered as dynamical variables anymore. With this particular gauge choice, $H_{\chi_{\text{min}}}$ is a linear combination of the total constraints

\[
H_{\chi_{\text{min}}} = \int d^p\sigma\left[\Lambda^0 T_0^{\text{tot}}(\sigma') + \Lambda^j T_j^{\text{tot}}(\sigma')\right],
\]

and we can treat here the Lagrange multipliers $\Lambda^0, \Lambda^j$ as constants. Of course, this does not fix the gauge completely.

### 4 Null membranes in D=4

Here we confine ourselves to the case of membranes moving in a four dimensional, stationary, axially symmetrical, gravity background of the type

\[
ds^2 = g_{00}(dx^0)^2 + g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + g_{33}(dx^3)^2 + 2g_{03}dx^0dx^3,
\]

\[
g_{\mu\nu} = g_{\mu\nu}(x^1, x^2).
\]
We will work in the gauge $\mu^0, \mu^j = \text{constants}$, in which the equations of motion (4) and constraints (5) for the membrane ($j, k = 1, 2$) have the form:

\[
(\partial_\tau - \mu^j \partial_j)(\partial_\tau - \mu^k \partial_k) x^\lambda + \Gamma^\lambda_{\mu \nu}(\partial_\tau - \mu^j \partial_j)x^\mu(\partial_\tau - \mu^k \partial_k)x^\nu = 0.
\] (12)

\[
g_{\mu \nu}(x)(\partial_\tau - \mu^j \partial_j)x^\mu(\partial_\tau - \mu^k \partial_k)x^\nu = 0,
\] (13)

\[
g_{\mu \nu}(x)(\partial_\tau - \mu^k \partial_k)x^\mu \partial_j x^\nu = 0.
\]

To establish the correspondence with the null geodesics we note that if we introduce the quantities

\[
u^\nu(x) = (\partial_\tau - \mu^j \partial_j)x^\nu,
\] (14)

the equations of motion (12) can be rewritten as

\[
u^\nu (\partial_\nu \nu^\lambda + \Gamma^\lambda_{\mu \nu} \nu^\mu) = 0.
\]

Then it follows from here that $\nu^2$ do not depend on $x^\nu$. In this notations, the constraints are:

\[
g_{\mu \nu} \nu^\mu \nu^\nu = 0, \quad g_{\mu \nu} \partial_\nu x^\nu\partial_j x^\nu = 0.
\]

Taking into account the metric (11), one can write the equations of motion (12) and the constraints (13) in the form:

\[
(\partial_\tau - \mu^j \partial_j)u^0 + 2\left(\Gamma^0_{01} u^0 + \Gamma^0_{13} u^3\right)(\partial_\tau - \mu^j \partial_j)x^1 +
+ 2\left(\Gamma^0_{02} u^0 + \Gamma^0_{23} u^3\right)(\partial_\tau - \mu^j \partial_j)x^2 = 0,
\]

\[
(\partial_\tau - \mu^j \partial_j)(\partial_\tau - \mu^k \partial_k)x^1 + \Gamma^1_{11}(\partial_\tau - \mu^j \partial_j)x^1(\partial_\tau - \mu^k \partial_k)x^1 +
+ 2\Gamma^1_{12}(\partial_\tau - \mu^j \partial_j)x^1(\partial_\tau - \mu^k \partial_k)x^2 + \Gamma^1_{22}(\partial_\tau - \mu^j \partial_j)x^2(\partial_\tau - \mu^k \partial_k)x^2 +
\]
\[ + \Gamma_{00}^1 (u^0)^2 + 2 \Gamma_{03}^1 u^0 u^3 + \Gamma_{33}^1 (u^3)^2 = 0, \]

\[
(\partial_\tau - \mu^j \partial_j) (\partial_\tau - \mu^k \partial_k) x^2 + \Gamma_{11}^2 (\partial_\tau - \mu^j \partial_j) x^1 (\partial_\tau - \mu^k \partial_k) x^1 + \]

\[ + 2 \Gamma_{12}^2 (\partial_\tau - \mu^j \partial_j) x^1 (\partial_\tau - \mu^k \partial_k) x^2 + \Gamma_{22}^2 (\partial_\tau - \mu^j \partial_j) x^2 (\partial_\tau - \mu^k \partial_k) x^2 + \]

\[ + \Gamma_{00}^2 (u^0)^2 + 2 \Gamma_{03}^2 u^0 u^3 + \Gamma_{33}^2 (u^3)^2 = 0, \]

\[ (\partial_\tau - \mu^j \partial_j) u^3 + 2 \left( \Gamma_{01}^3 u^0 + \Gamma_{13}^3 u^3 \right) (\partial_\tau - \mu^j \partial_j) x^1 + \]

\[ + 2 \left( \Gamma_{02}^3 u^0 + \Gamma_{23}^3 u^3 \right) (\partial_\tau - \mu^j \partial_j) x^2 = 0, \]

\[ g_{11} (\partial_\tau - \mu^j \partial_j) x^1 (\partial_\tau - \mu^k \partial_k) x^1 + g_{22} (\partial_\tau - \mu^j \partial_j) x^2 (\partial_\tau - \mu^k \partial_k) x^2 + \]

\[ + g_{00} (u^0)^2 + 2 g_{03} u^0 u^3 + g_{33} (u^3)^2 = 0, \]

\[ g_{11} (\partial_\tau - \mu^k \partial_k) x^1 \partial_j x^1 + g_{22} (\partial_\tau - \mu^k \partial_k) x^2 \partial_j x^2 + \]

\[ + (g_{00} \partial_j x^0 + g_{03} \partial_j x^3) u^0 + (g_{03} \partial_j x^0 + g_{33} \partial_j x^3) u^3 = 0, \]

where the notation introduced in (14) is used. To simplify these equations, we make the ansatz

\[
\begin{align*}
x^0(\tau, \sigma) &= f^0(z^1, z^2) + t(\tau), \\
x^1(\tau, \sigma) &= r(\tau), \quad x^2(\tau, \sigma) = \theta(\tau), \\
x^3(\tau, \sigma) &= f^3(z^1, z^2) + \varphi(\tau),
\end{align*}
\]

where \( f^0, f^3 \) are arbitrary functions of their arguments.

After substituting (16) in (15), we receive (the dot is used for differentiation with respect to the affine parameter \( \tau \)):

\[
\dot{u}^0 + \left[ (g^{00} \partial_1 g_{00} + g^{03} \partial_1 g_{03}) u^0 + (g^{00} \partial_1 g_{03} + g^{03} \partial_1 g_{33}) u^3 \right] \dot{\tau} = 0
\]
\[ + \left[ \left( g^{00} \partial_2 g_{00} + g^{03} \partial_2 g_{03} \right) u^0 + \left( g^{00} \partial_2 g_{03} + g^{03} \partial_2 g_{33} \right) u^3 \right] \dot{\theta} = 0, \]
\[ 2g_{11} \ddot{r} + \partial_1 g_{11} \dot{r}^2 + 2\partial_2 g_{11} \dot{r} \dot{\theta} - \partial_1 g_{22} \dot{\theta}^2 \]
\[ - \left[ \partial_1 g_{00} (u^0)^2 + 2\partial_1 g_{03} u^0 u^3 + \partial_1 g_{33} (u^3)^2 \right] = 0, \]
\[ 2g_{22} \ddot{\theta} + \partial_2 g_{22} \dot{\theta}^2 + 2\partial_1 g_{22} \dot{r} \dot{\theta} - \partial_2 g_{11} \dot{r}^2 \]
\[ - \left[ \partial_2 g_{00} (u^0)^2 + 2\partial_2 g_{03} u^0 u^3 + \partial_2 g_{33} (u^3)^2 \right] = 0, \]
\[ u^3 + \left[ \left( g^{33} \partial_1 g_{03} + g^{03} \partial_1 g_{00} \right) u^0 + \left( g^{33} \partial_1 g_{33} + g^{03} \partial_1 g_{03} \right) u^3 \right] \dot{r} \]
\[ + \left[ \left( g^{03} \partial_2 g_{03} + g^{03} \partial_2 g_{00} \right) u^0 + \left( g^{03} \partial_2 g_{33} + g^{03} \partial_2 g_{03} \right) u^3 \right] \dot{\theta} = 0, \]
\[ g_{11} \ddot{r}^2 + g_{22} \ddot{\theta}^2 + g_{00} (u^0)^2 + 2g_{03} u^0 u^3 + g_{33} (u^3)^2 = 0, \]
\[ (g_{00} \partial_2 f^0 + g_{03} \partial_2 f^3) u^0 + (g_{03} \partial_2 f^0 + g_{33} \partial_2 f^3) u^3 = 0. \]

If we choose
\[ f^0(z^1, z^2) = f^0(w), \quad f^3(z^1, z^2) = f^3(w), \]
where \( w = w(z^1, z^2) \) is an arbitrary function of \( z^1 \) and \( z^2 \), then the system of equations (22) reduces to the single equation
\[ \left( g_{00} \frac{df^0}{dw} + g_{03} \frac{df^3}{dw} \right) u^0 + \left( g_{03} \frac{df^0}{dw} + g_{33} \frac{df^3}{dw} \right) u^3 = 0. \]

To be able to separate the variables \( u^0, u^3 \) in the system of differential equations (17), (20) with the help of (23), we impose the following condition on \( f^0(w) \) and \( f^3(w) \)
\[ f^0(w) = C^0 f[w(z^1, z^2)], \quad f^3(w) = C^3 f[w(z^1, z^2)], \]
where \( C^0, C^3 \) are constants, and \( f(w) \) is an arbitrary function of \( w \). Then the solution of (17), (20) and (23) is (C1 = const):
\[ u^0(\tau) = -C_1 \left( C^0 g_{03} + C^3 g_{33} \right) \exp(-H), \]
\[ u^3(\tau) = +C_1 \left( C^0g_{00} + C^3g_{03} \right) \exp(-H), \]  

\[ H = \int \left( g^{00}dg_{00} + 2g^{03}dg_{03} + g^{33}dg_{33} \right). \]

The condition for the compatibility of (24) with (18), (19) and (21) is:

\[
\begin{align*}
    u^0(\tau) &= -C_1 \left( C^0g_{03} + C^3g_{33} \right) h^{-1} \\
    &= -C_1 \left( C^3g^{00} - C^0g^{03} \right) = \dot{i}(\tau), \\
    u^3(\tau) &= +C_1 \left( C^0g_{00} + C^3g_{03} \right) h^{-1} \\
    &= -C_1 \left( C^3g^{03} - C^0g^{33} \right) = \dot{\varphi}(\tau), \\
    h &= g_{00}g_{33} - g_{03}^2.
\end{align*}
\]

On the other hand, from (19) and (21) one has:

\[
\begin{align*}
    \dot{r}^2 &= -g^{11} \left[ C_1^2G + g^{22}(g_{22}^2\dot{\theta}^2) \right] \\
    &= g^{11} \left\{ C_1^2 \left( 2C^0C^3g^{03} - (C^3)^2g^{00} - (C^0)^2g^{33} \right) - g^{22}(g_{22}^2\dot{\theta}^2) \right\}, \\
    g_{22}^2\dot{\theta}^2 &= C_2 + C_1^2 \int \frac{d\theta}{h} \left[ g_{22}G_\theta + h \frac{\partial g_{22}G_\theta}{\partial \theta} \right], \\
    G &= (C^0)^2g_{00} + 2C^0C^3g_{03} + (C^3)^2g_{33}.
\end{align*}
\]

In obtaining (27), we have used the gauge freedom in the metric (11), to impose the condition (10):

\[ \partial_2 \left( \frac{g_{22}}{g_{11}} \right) = 0. \]

As a final result we have

\[
\begin{align*}
    x^0 &= C^0f[w(z^1, z^2)] + t(\tau), \\
    x^1 &= r(\tau),
\end{align*}
\]
\[ x^2 = \theta(\tau), \]
\[ x^3 = C^3 f[w(z^1, z^2)] + \varphi(\tau), \]

where \( i(\tau), i(\tau), \dot{\theta}(\tau), \dot{\varphi}(\tau) \) are given by (25), (26), and (27).

In the particular case when \( x^2 = \theta = \theta_0 = \text{const} \), one can integrate to obtain the following exact solution of the equations of motion and constraints for the null membrane in the gravity background (11):

\begin{align*}
   x^0(\tau, \sigma^1, \sigma^2) &= C^0 f[w(z^1, z^2)] + t_0 \\
   &\quad \pm \int_{r_0}^{r} dr \left( C^3 g^{00} - C^0 g^{03} \right) W^{-1/2}, \\
   x^3(\tau, \sigma^1, \sigma^2) &= C^3 f[w(z^1, z^2)] + \varphi_0 \\
   &\quad \pm \int_{r_0}^{r} dr \left( C^3 g^{03} - C^0 g^{33} \right) W^{-1/2}, \\
   C_1(\tau - \tau_0) &= \pm \int_{r_0}^{r} dr W^{-1/2}, \\
   W &= g^{11} \left[ 2C^0 C^3 g^{03} - \left( C^3 \right)^2 g^{00} - \left( C^0 \right)^2 g^{33} \right],
\end{align*}

\[ t_0, r_0, \varphi_0, \tau_0 \quad - \quad \text{constants}. \]

5 Examples

Here we give some examples of solutions of the type received in the previous section. To begin with, let us start with the simplest case of Minkowski space-time. The metric is

\[ g_{00} = -1, \quad g_{11} = 1, \quad g_{22} = r^2, \quad g_{33} = r^2 \sin^2 \theta, \]
and equalities (25), (26), (27) take the form:

\[
\begin{align*}
\dot{t} &= C_1 C^3, \\
\dot{r}^2 &= (C_1 C^3)^2 - \frac{C_2}{r^2}, \\
\dot{r}^4 \dot{\theta}^2 &= C_2 - \frac{(C_1 C^0)^2}{\sin^2 \theta}, \\
\dot{\phi} &= \frac{C_1 C^0}{r^2 \sin^2 \theta}.
\end{align*}
\]

When \( \theta = \theta_0 = \text{const} \), the solution (28) is:

\[
\begin{align*}
x_0^0(\tau, \sigma_1, \sigma_2) &= C_0 f[w(z_1, z_2)] + t_0 \mp C_3 \int_{r_0}^{r} \frac{dr}{[(C^3)^2 - (C^0)^2 r^{-2} \sin^{-2} \theta_0]^{1/2}}, \\
x_0^3(\tau, \sigma_1, \sigma_2) &= C^3 f[w(z_1, z_2)] + \varphi_0 \mp \frac{C_0}{\sin^2 \theta_0} \int_{r_0}^{r} \frac{dr}{r^2 [(C^3)^2 - (C^0)^2 r^{-2} \sin^{-2} \theta_0]^{1/2}}, \\
C_1(\tau - \tau_0) &= \pm \int_{r_0}^{r} \frac{dr}{[(C^3)^2 - (C^0)^2 r^{-2} \sin^{-2} \theta_0]^{1/2}}.
\end{align*}
\]

Our next example is the de Sitter space-time. We take the metric in the form

\[
g_{00} = -\left(1 - kr^2\right), \ g_{11} = \left(1 - kr^2\right)^{-1}, \ g_{22} = r^2, \ g_{33} = r^2 \sin^2 \theta,
\]

where \( k \) is the constant curvature. Now we have

\[
\begin{align*}
\dot{t} &= \frac{C_1 C^3}{1 - kr^2}, \\
\dot{r}^2 &= (C_1 C^3)^2 + C_2(k - r^{-2}), \\
\dot{r}^4 \dot{\theta}^2 &= C_2 - \frac{(C_1 C^0)^2}{\sin^2 \theta}, \\
\dot{\phi} &= \frac{C_1 C^0}{r^2 \sin^2 \theta},
\end{align*}
\]

and the corresponding solution (28) is:

\[
x^0(\tau, \sigma^1, \sigma^2) = C^0 f[w(z_1, z_2)] + t_0
\]
\[ x^3(\tau, \sigma^1, \sigma^2) = C^3 f[w(z^1, z^2)] + \varphi_0 \]
\[ \pm C^3 \int_{r_0}^{r} \frac{dr}{(1 - kr^2)((C^3)^2 + (C^0)^2(k - r^2)\sin^{-2}\theta_0)^{1/2}}, \]

Now let us turn to the case of Schwarzschild space-time. The corresponding metric may be written as

\[ g_{00} = -(1 - 2Mr^{-1}) \quad , \quad g_{11} = (1 - 2Mr^{-1})^{-1}, \]
\[ g_{22} = r^2 \quad , \quad g_{33} = r^2\sin^2\theta, \]

where \( M \) is the Schwarzschild mass. The equalities (25), (26) and (27) read

\[ \dot{t} = \frac{C_1C_3}{1 - 2Mr^{-1}}, \]
\[ \dot{r}^2 = (C_1C_3)^2 - \frac{C_2}{r^2}(1 - 2Mr^{-1}), \]
\[ r^4\dot{\theta}^2 = C_2 - \frac{(C_1C_0)^2}{\sin^2\theta}, \]
\[ \dot{\varphi} = \frac{C_1C_0}{r^2\sin^2\theta}. \]

When \( \theta = \theta_0 = const \), one obtains from (28)

\[ x^0(\tau, \sigma^1, \sigma^2) = C^0 f[w(z^1, z^2)] + t_0 \]
\[ \pm C^3 \int_{r_0}^{r} \frac{dr}{(1 - 2Mr^{-1})((C^3)^2 - (C^0)^2r^{-2}(1 - 2Mr^{-1})\sin^{-2}\theta_0)^{1/2}}, \]
\[ x^3(\tau, \sigma^1, \sigma^2) = C^3 f[w(z^1, z^2)] + \varphi_0 \]
\[ \pm \frac{C^0}{\sin^2\theta_0} \int_{r_0}^{r} \frac{dr}{r^2((C^3)^2 - (C^0)^2r^{-2}(1 - 2Mr^{-1})\sin^{-2}\theta_0)^{1/2}}, \]
\[ C_1(\tau - \tau_0) = \pm \int_{r_0}^{r} \frac{dr}{[(C^3)^2 - (C^0)^2r^{-2}(1 - 2Mr^{-1})\sin^{-2}\theta_0]^{1/2}}. \]
For the Taub-NUT space-time we take the metric as

\[ g_{00} = -\frac{\delta}{R^2}, \quad g_{11} = \frac{R^2}{\delta}, \quad g_{22} = R^2, \]

\[ g_{33} = R^2 \sin^2 \theta - 4l^2 \frac{\delta \cos^2 \theta}{R^2}, \quad g_{03} = -2l \delta \cos \theta R^2, \]

\[ \delta(r) = r^2 - 2Mr - l^2, \quad R^2(r) = r^2 + l^2, \]

where \( M \) is the mass and \( l \) is the NUT-parameter. Now we have from (25), (26) and (27):

\[ \dot{t} = C_1 \frac{R^2}{R^4} \left[ C_3 \left( \frac{R^4}{\delta} + 4l^2 \right) - \frac{2l}{\sin^2 \theta} \left( C_0 \cos \theta + 2C^3 l \right) \right], \]

\[ R^4 r^2 = \left( C_1 C^3 \right)^2 \left( R^4 + 4l^2 \delta \right) - C_2 \delta, \]

\[ R^4 \dot{\theta}^2 = C_2 - \frac{C_1^2}{2} \left[ (C_0)^2 + 4C_0 C^3 l \cos \theta \right], \]

\[ \dot{\varphi} = \frac{C_1}{R^2 \sin^2 \theta} \left( C_0 + 2C^3 l \cos \theta \right). \]

In the Taub-NUT metric the solution (28) is

\[ x^0(\tau, \sigma^1, \sigma^2) = C^0 f[w(z^1, z^2)] + t_0 \]

\[ \pm \int_{r_0}^r dr \left[ C^3 \left( R^4 \delta^{-1} + 4l^2 \right) - 2l \sin^{-2} \theta_0 \left( C_0 \cos \theta_0 + 2C^3 l \right) \right] U^{-1/2}(r), \]

\[ x^3(\tau, \sigma^1, \sigma^2) = C^3 f[w(z^1, z^2)] + \varphi_0 \]

\[ \pm \frac{1}{\sin^2 \theta_0} \int_{r_0}^r dr \left( C^0 + 2C^3 l \cos \theta_0 \right) U^{-1/2}(r), \]

\[ C_1(\tau - \tau_0) = \pm \int_{r_0}^r dr R^2 U^{-1/2}(r), \]

where

\[ U(r) = \left( C^0 \right)^2 \left( R^4 - 4l^2 \delta \cot^2 \theta_0 \right) - \delta \sin^{-2} \theta_0 \left[ (C_0)^2 + 4C_0 C^3 l \cos \theta_0 \right]. \]
Finally, we consider the *Kerr space-time* with metric taken in the form

\[
\begin{align*}
g_{00} &= -\left(1 - \frac{2Mr}{\rho^2}\right), \quad g_{11} = \frac{\rho^2}{\Delta}, \\
g_{22} &= \rho^2, \quad g_{33} = \left(r^2 + a^2 + \frac{2Ma^2 r \sin^2 \theta}{\rho^2}\right) \sin^2 \theta, \\
g_{03} &= -\frac{2Mar \sin^2 \theta}{\rho^2},
\end{align*}
\]

where

\[
\rho^2 = r^2 + a^2 \cos^2 \theta, \quad \Delta = r^2 - 2Mr + a^2,
\]

\(M\) is the mass and \(a\) is the angular momentum per unit mass of the Kerr black hole.

With this input in equations (25), (26) and (27), we have:

\[
\begin{align*}
i &= \frac{C_1}{\Delta \rho^2} \left[C^3(r^2 + a^2)^2 - C^3 a^2 \Delta \sin^2 \theta - 2C^0 Mar\right], \\
\rho^4 \dot{r}^2 &= C_1^2 \left[(C^3)^2(r^2 + a^2)^2 - 4C^0 C^3 Mar + (C^0)^2 a^2\right] - C_2 \Delta, \\
\rho^4 \dot{\theta}^2 &= C_2 - C_1^2 \left(\frac{(C^0)^2}{\sin^2 \theta} + (C^3)^2 a^2 \sin^2 \theta\right), \\
\dot{\varphi} &= \frac{C_1}{\Delta \rho^2} \left(C^0 \Delta \sin^2 \theta + 2C^3 Mar - C^0 a^2\right).
\end{align*}
\]

In this case, the exact solution (28) takes the form:

\[
\begin{align*}
x^0(\tau, \sigma^1, \sigma^2) &= C^0 f[w(z^1, z^2)] + t_0 \\
&\pm \int_{r_0}^{r} dr \Delta^{-1} \left\{2C^0 Mar - C^3 \left[(r^2 + a^2)\rho_0^2 + 2Ma^2 r \sin^2 \theta_0\right]\right\} V^{-1/2}(r), \\
x^3(\tau, \sigma^1, \sigma^2) &= C^3 f[w(z^1, z^2)] + \varphi_0 \\
&\pm \frac{1}{\sin^2 \theta_0} \int_{r_0}^{r} dr \Delta^{-1} \left[C^0 \left(2Mr - \rho_0^2\right) - 2C^3 Mar \sin^2 \theta_0 \right] V^{-1/2}(r), \\
C_1(\tau - \tau_0) &= \pm \int_{r_0}^{r} dr \rho_0^2 V^{-1/2}(r), \\
V(r) &= \left(C^0\right)^2 \left(a^2 - \Delta \sin^{-2} \theta_0\right) - 4C^0 C^3 Mar.
\end{align*}
\]
\[ + (C^3)^2 \left[ (r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta_0 \right], \]
\[ \rho_0^2 = r^2 + a^2 \cos^2 \theta_0. \]

6 Comments and conclusions

In the previous section we restrict ourselves to some particular cases of the generic solution (28) and do not pay attention to the existing possibilities for obtaining solutions in the case \( \theta \neq \text{const} \) in the considered examples.

Obviously, the examples given in Sec.5 do not exhaust all possibilities contained in the metric (11) [11]. On the other hand, in different particular cases of this type of metric, there exist more general brane solutions. They will be published elsewhere.

Here we only mention that in the gauge \( \mu^k = \text{const} \),

\[ x^\nu(\tau, \sigma) = x^\nu(\mu^k \tau + \sigma^k) \]

is an obvious nontrivial solution of the equations of motion and of the constraints (4), (5) depending on \( D \) arbitrary functions of \( p \) variables for the null \( p \)-brane in arbitrary \( D \)-dimensional gravity background.

From the results of the previous sections, it is easy to extract the corresponding formulas for the null string case simply by putting \( \sigma^1 = \sigma, \mu^1 = \mu, \sigma^2 = \mu^2 = 0 \). For example, our equalities (29) coincide with the ones obtained in [3] for the null string moving in Schwarzschild space-time after identification:

\[ E = C_1 C^3, \quad L = C_1 C^0, \quad L^2 + K = C_2. \]

Moreover, our solution (30) in the case \( p = 1 \), generalizes the solution given in [1]. The
latter corresponds to fixing the arbitrary function \( f(w) \) to a linear one and fixing the
gauge to \( \mu = 0 \), i.e.

\[
f[w(\mu^1 \tau + \sigma^1, \mu^2 \tau + \sigma^2)] \mapsto f(\mu \tau + \sigma) = \mu \tau + \sigma, \quad \text{with} \quad \mu = 0.
\]

In this paper we perform some investigation on the classical dynamics of the null
bosonic branes in curved background. In the second section we give the action, prove
its reparametrization invariance and give the equations of motion and constraints in an
arbitrary gauge. Then we construct the corresponding Hamiltonian and compute the
constraint algebra. Following [8], we obtain the manifest expressions for the classical
BRST charge, the total constraints, the BRST invariant Hamiltonian and the BRST
invariant Lagrangian. All this gives the possibility for quantization of the null \( p \)-
branes in a curved space-time. In the fourth section we consider the dynamics of the
null membranes (\( p = 2 \)) in a four dimensional, stationary, axially symmetrical, gravity
background. Some exact solutions of the equations of motion and of the constraints are
found there. The next section is devoted to examples of such solutions in Minkowski,
de Sitter, Schwarzschild, Taub-NUT and Kerr space-times.

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