We study numerically the Anderson model on partially disordered random regular graphs considered as the toy model for a Hilbert space of interacting disordered many-body system. The protected subsector of zero-energy states in a many-body system corresponds to clean nodes in random regular graphs ensemble. Using adjacent gap ratio statistics and inverse participation ratio we find the sharp mobility edge in the spectrum of one-particle Anderson model above some critical density of clean nodes. Its position in the spectrum is almost independent on the disorder strength. The possible application of our result for the controversial issue of mobility edge in the many-body localized phase is discussed.

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1. INTRODUCTION

Recently new mechanisms of ergodicity breaking in the complicated interacting many-body systems have been uncovered. The combination of interaction and strong enough disorder amounts to the emergent many-body localization (MBL phase with full ergodicity breaking [1—5]. The Anderson model on random regular graph (RRG serves as the toy model for the identification of the MBL phase in the physical space, see [6] for the recent review. The many-body localization in the physical space presumably gets mapped into the one-particle localization in a Hilbert space [7].

The localization in the Anderson model on the Bethe tree at some critical disorder has been established long time ago [8] and more recently the similar localization on the RRG has been derived analytically [9] and numerically. This issue has been analyzed in some details in [10, 12—17] and for flat disorder $W$ the critical value has been established in RRG for degree $p = 3$ with a high accuracy $W_{cr} = 18.17$.

The mechanisms for the destruction of localization are interesting both from the theoretical and practical viewpoints. In the context of the destruction of the MBL phase the main mechanism considered involves a inclusion of thermal bubbles [18, 19] which presumably yield the many-body mobility edge (MBME) in interacting many-body systems via the avalanche mechanism. This issue is quite controversial and there are discussions concerning the very existence of MBME [18—23]. The density of special states influences the correlations in corresponding Hilbert space hopping model [24] and the relation of special states with the MBME was discussed if the energy [25] or the density [20] are considered as the control parameters. It was conjectured that the density of thermal bubbles is related to fractal dimension of the localized part of the spectrum [26].

Another quite generic mechanism concerns the effects of the special topologically protected degenerated modes. For instance if we assume that there is no diagonal disorder at all and disorder is present only in the off-diagonal hopping terms there is the single delocalized mode at zero energy which provides the conductivity [27, 28]. Among the real systems enjoying such mechanism one could mention the disordered integer QHE in $2d$ when almost all modes are localized and a conductivity is solely due to zero energy topological modes [29, 30]. Similarly the topological modes influence the localization in $1d$ quantum wires [31, 32]. In both cases the two-coupling renormalization group (RG) flow governs the evolution of a system from ultraviolet to infrared. One coupling measures the strength of disorder while the second one measures the amount of topological modes estimated in some manner. The RG flow diagram is
quite nontrivial and a delocalized regime emerges near the boundaries between phases.

Recently the effect of zero modes has been discussed for the finite spin systems [33–36] in the context of MBL. It was argued that several systems host such exponentially degenerated symmetry protected modes. In this work, we discuss the combined effect of interaction, disorder and zero-modes on localization in a real space via the particular toy model in the Hilbert space. We mimic the presence of the symmetry protected degenerated zero-modes in a real space by the clean degenerated nodes in RRG ensemble. In such toy model localization in the interacting many-body system is described via localization in the Anderson model on partially disordered RRG. The density of the clean zero-energy nodes \( \beta \) in a Hilbert space is the second parameter which quantified the “topological” sector of Hilbert space. It turns out that there is the critical value of a disorder density \( \beta_{cr} \) and at \( \beta < \beta_{cr} \) the one-particle mobility edge in the Anderson model on the partially disordered RRG can be identified unambiguously. It will be confirmed numerically by the evaluation of the \( r \)-value statistics and inverse participation ratio (IPR) for the corresponding eigenmodes.

We conjecture that a one-particle mobility edge in the Hilbert space hopping model we found is related with the MBME in a real space hence providing the arguments that zero-mode degenerated states play the important role in the MBME formation in disordered system. At strong disorder the finite fraction of the protected zero modes produces the delocalized states in MBL phase in the center of energy band and finally forms stable MBME at large enough density of protected states.

2. MODEL

We study non-interacting spinless fermions hopping over RRG with connectivity \( p = 3 \) in a potential disorder described by Hamiltonian

\[
H = \sum_{\langle i,j \rangle} (c_i^+ c_j + c_j^+ c_i) + \sum_{i=1}^{N} \varepsilon_i c_i^+ c_i, \tag{1}
\]

where the first sum runs over the nearest-neighbor sites of the RRG, the second sum runs over \( \beta N \) nodes with potential disorder. The energies \( \varepsilon_i \) are independent random variables sampled from a uniform distribution on \([−W/2, W/2]\). We consider gaps between adjacent levels, \( \delta_i = E_{i+1} - E_i \), where the eigenvalues of a given realization of the Hamiltonian for a given total number of particles, \( E_i \), are listed in ascending order. The dimensionless quantity we have chosen to characterize the correlations between adjacent gaps in the spectrum is the ratio of two consecutive gaps:

\[
r_i = \frac{\min(\delta_i, \delta_{i+1})}{\max(\delta_i, \delta_{i+1})}. \tag{2}
\]

For uncorrelated Poisson spectrum the probability distribution of this ratio \( r \) satisfy by \( P(r) = 2/(1 + r)^3 \), with the mean value \( \langle r \rangle_p = 2 \ln 2 - 1 \approx 0.386 \). For large Gaussian orthogonal ensemble, the mean value \( \langle r \rangle_{\text{GOE}} = 0.53 \).

In turn, a direct measure of the (de)localization of the eigenfunctions is obtained by the IPR,

\[
\text{IPR}(i) = \frac{1}{N} \sum_{n} |\psi_n^{(i)}|^4, \quad \text{where } \psi_n^{(i)} \text{ is the } i\text{th eigenstate of the matrix and } n \text{ is the basis state index.}
\]

The IPR of a chaotic eigenstate is \( N \)-dependent, in contrast to a localized one.

3. NUMERICAL RESULTS

We study numerically the dependence of the single-particle localization on \( \beta \) and \( W \) using for diagnostics the adjacent gap ratio statistics and IPR.

Figure 1 shows the numerical results for RRG with \( N = 16000 \) and the degree \( p = 3 \) and different disorder parameter \( \beta \): the left column corresponds to \( \beta = 0.5 \), the right column corresponds to \( \beta = 1.0 \). We analyze the ratio \( \langle r \rangle \) and IPR for different parts of the spectrum. We divide the sorting spectrum into \( k = 100 \) equal parts and average the ratio \( \langle r \rangle \) and IPR over each window. The ordinate \( \alpha = i/(N - 1) \) in Fig. 1 corresponds the normalized level position with \( i = 0, 1, \ldots, N - 1 \), the energy level, the ordinate window respectively is \( \Delta \alpha = 1/k \). Figures 1a–1d present the maps of (a, b) the ratio \( \langle r \rangle \) and (c, d) log(IPR) on the plane of the disorder value \( W \) and the spectrum part \( \alpha \). Figures 1e and 1f show the dependences of log(IPR) on the spectrum position \( \alpha \) for different \( W \) values.

The heat maps in Figs. 1a and 1c explicitly demonstrate, that there is the mobility edge \( \lambda_m \) separating sharply the spectrum into two different regimes for RRG with partial disorder in vertices. For \( |\lambda| > \lambda_m \) we observe localization state with the ratio \( \langle r \rangle \) close to \( \langle r \rangle_p \) and independence of IPR on \( N \), while for central spectrum part with \( |\lambda| \leq \lambda_m \) the ratio \( \langle r \rangle \) and IPP indicate on the delocalized state. Note, that the mobility edge \( \lambda_m \) weakly depends on the disorder \( W \) and is observed even for small \( W \). Moreover, we do not observe the phase transition at large \( W \) to completely localized phase which is familiar for completely disor-
ordered RRG (see Figs. 1b, 1d, and 1f with the same plots for $\beta = 1.0$).

Figure 2 demonstrate the numerical results for fixed disorder parameter: the left column corresponds to subcritical disorder parameter $W < W_{cr}$, while the right column corresponds to supercritical value $W > W_{cr}$. In subcritical regime there are the mobility edges for all values $\beta$, while in supercritical regime we observe the existence of the mobility edge just for $\beta < \beta_{cr}$ with $\beta_{cr} = 0.8$, while for $\beta > \beta_{cr}$ the dependences correspond the localization state as for $\beta = 1.0$. We have checked numerically that position of the mobility edge is stable when we increase the disorder up to $W = 100$.

We analyze how eigenfunctions are distributed on network vertices. We calculate squared $i$th eigenvector component averaging for the vertices with disorder, i.e., $\langle \psi^2(i) \rangle = \frac{1}{\beta N} \sum_{m \in V_d} |\psi^{(i)}_m|^2$ and for the zero potential vertices: $\langle \psi^2(i) \rangle = \frac{1}{(1 - \beta)N} \sum_{m \in V_0} |\psi^{(i)}_m|^2$, where we denote $V_d$ and $V_0$ the vertices with disorder and with zero potential. Figure 3 shows the numerical results for the disorder parameter $\beta = 0.5$, we plot the value $\langle \psi^2(\alpha) \rangle$, calculated as the average in each of $k$ windows. Let us emphasize once again that the graphs clearly demonstrate the existence of mobility edge, regardless of the disorder level. In addition, we remark that in the localized phase, the eigenfunctions are distributed mainly at the vertices with disorder, while in the delocalized phase, mainly at the clean vertices.

**4. ON THE MOBILITY EDGE IN THE MBL PHASE**

In this study, we elaborate the combined effect of interaction, disorder and the protected states for the many-body system in the toy model of partially disordered RRG. Let us recall the relation between the MBL in the real space and one-particle localization in the Hilbert space [7]. The nodes in the graph correspond to the states in Fock space in the theory without interaction. The number of nodes equals to the dimension of the Hilbert space and, for instance, for $L$ spins equals to $2^L$. The matrix elements of the interaction yield the links between nodes. It was conjectured that the one-particle localization in a Hilbert space corresponds to the MBL phase in the physical space.

This attractive idea works well qualitatively and provides the explanations for the several phenomena.
For instance, there are strong finite-size effects shifting the apparent transition point, exponential growth of the correlation volume when approaching the transition point, ergodicity of delocalized phase. On the other hand, there are clear-cut arguments which explain that the relation between the localization in physical and Hilbert space cannot be exact. For example, the tree-like local structure of RRG does not fit completely the Hilbert space of the many-body system.

Having all these reservations in mind one could ask the question if the one-particle mobility edge in the Hilbert space we have found corresponds to a kind of MBME or another physically meaningful phenomenon in the real space.
To answer this question, recall the recent scenario concerning the physics in the vicinity of the mobility edge in MBL. It was suggested in [18, 19] that there are local regions—“thermal bubbles” with energy density above the mobility edge which destroy the localization in a whole system via a resonant tunneling. On the other hand, it was suggested in [20] that MBME is stable with respect to the local regions with high particle density which provide the thermal bubbles from [18, 19] and no spreading via the tunneling was observed. The expansion of such bubbles into the whole space was argued to be more probable origin of delocalization.

Our study suggests that there can be another scenario of MBME formation. Instead of thermal bubbles the zero-modes could do the job and provide the conductivity in the mid of the spectrum. We cannot add much to the question of origin of such protected modes and assume that they are protected or by the topology either by some unbroken symmetry of interacting many-body system. If they survive in the thermodynamic limit we should discover a large number of such modes. Their number is governed by the index theorem for the topologically protected modes or by the dimension of representation for the symmetry protected modes. Both of them in principle could be large enough. Nevertheless, we postpone these important questions for further studies.

We do not expect that zero-modes are completely protected under the disorder and interaction. More relevant scenario involves the formation of non-vanishing density of the quasi-zero modes in the mid of the spectrum somewhat similar to the situation in quantum chromodynamics (QCD) although one has to have in mind a lot of reservations concerning this analogy. Indeed, we have found numerically that the states in the delocalized mid part of the spectrum involve not only the clean modes but also the disorder dependent admixture of the dirty modes. Therefore, they propagate mainly along the clean nodes with some disorder dependent contribution of regions with dirty states.

In QCD the localization properties of the Dirac operator eigenfunctions in 4D Euclidean space-time are considered. In the confinement phase the physical many-body spectrum of QCD involves the composite mesons and baryons only. It is well-known that the density of quasi-zero modes $\rho(0)$ of the Dirac operator yields the breakdown of the chiral symmetry in the confinement phase via formation of the chiral condensate according to the Casher–Banks relation $\langle \Psi \Psi \rangle = V^{-1} \pi \rho(0)$, where $V$ is the 4-volume. The Dirac operator enjoys the discrete symmetry $[H_\rho, \gamma_5] = 0$ hence its spectrum is symmetric under $E \to -E$. All eigenmodes of the Dirac operator are delocalized in confined phase [37, 38] which usually is explained via the instanton liquid model for QCD ground state. The delocalization is attributed to the overlap of the topological fermionic zero modes localized on the instantons and anti-instantons. Moreover, the lattice studies indicate that a delocalization occurs not in full 4D Euclidean space but only along some lower dimensional sub-manifolds somewhat similar to the fraction picture.

When we increase the temperature the deconfinement phase transition occurs at some critical $T_c$. At $T > T_c$ the spectrum of QCD involves the interacting quarks and gluons. Remarkably exactly at $T = T_c$ the mobility edge in the Dirac operator spectrum gets emerged [37, 38]. It can be related to the formation of black hole horizon in the holographic approach [39]. In the deconfined phase two essential phenomena happen. First, the disordered condensate of the Polyakov lines gets formed providing the strong enough disorder. Second, the density of the topological defects (instantons) gets decreased due to the asymptotic freedom since it is proportional to $\exp \left(-\frac{1}{g_{YM}(T)}\right)$.

The mapping of the QCD situation to our model goes as follows. The instantons and anti-instantons in QCD ground state represent the nodes of the effective network and the links correspond to the overlap of the fermionic zero-modes. There is no effective disorder at $T < T_c$ however at $T > T_c$ the disordered condensate of Polyakov lines emerges and we have instantons and calorons in the ground state configurations now. The caloron involves the Polyakov line attached to instanton and it approximately corresponds to the “dirty” node in our model. The density of calorons according to the numerical studies is finite in the instanton–caloron ensemble hence we effectively have the finite fraction of the “dirty” nodes.

Hence indeed like in our model we have three characteristics which influence a localization: the strength of interaction, the density of topological defects and the strength of disorder. Like in our model when the topological defects in the QCD confinement phase dominate we have complete delocalization in one-particle spectrum while if the disorder becomes strong enough the mobility edge gets emerged in the deconfined phase.

5. DISCUSSION

In this work, we consider partially disordered RRG as the toy model of a Hilbert space for some interacting disordered many-body system with the topologically protected subsector. The nodes of RRG free from disorder correspond to zero-energy protected states in many-body system. To some extend our model probes the effects of disorder on zero-mode states.

It is found that at some density of clean nodes in partially disordered RRG the sharp mobility edge emerges in the spectrum of Anderson model and exists
up to arbitrarily large diagonal flat disorder \( W \). We have studied the distribution of the eigenfunctions in RRG and have found that localized states are distributed almost solely within the dirty nodes while the delocalized part of the spectrum mainly involves the clean nodes with small disorder dependent contribution of the dirty nodes.

The model certainly oversimplifies the issue nevertheless it can be considered as the indication that a one-particle mobility edge in the hopping model in the Hilbert space and a mobility edge in the MBL phase could be related. Indeed, if the mechanism behind the mobility edge in MBL involves density of highly degenerate zero-modes in the physical space then the density of clean nodes in the Hilbert space is its relevant counterpart. It would be important to get the analytic description of the observed phenomena using approach developed in [9, 40].

The mobility edge emerges in RRG ensemble also at another occasion when a chemical potential for the 3-cycles is added. At some critical value of the chemical potential the RRG with node degree \( p \) gets fragmented into the \( \frac{N}{p} \) clusters which are almost complete graphs [41]. The corresponding spectral density develops the second non-perturbative band filled by eigenvalue instantons corresponding to clusters. The emerging mobility edge separates the continuum filled by the delocalized modes and the second non-perturbative band with localized modes obeying Poisson statistics [42]. The formation of clusters is suppressed by factors \( e^{-N} \) however they do not “evaporate” completely at large \( N \) yielding the star-like hubs [43]. The similar fragmentation of the RRG takes place also if the chemical potential for 4-cycles is added [44, 45]. In this case the emerging bipartiteness of the fragments is new peculiar feature of the model [45].

It is natural to compare this pattern with other mechanisms of Hilbert space fragmentation which influence the localization, in particular quantum many-body scars (QMBS), see [46–48]. The scars are related to the symmetries of the many-body Hamiltonian and are protected algebraically when the interaction does not ruin the symmetry completely [49–54]. Note that QMBS are not degenerated enough to fill the finite part of the Hilbert space. It is assumed that scars are quantum counterpart of the peculiar unstable semi-classical orbits in classically chaotic systems. Somewhat similarly the \( k \)-cycles in the Hilbert space which induce fragmentation of RRG presumably correspond to the \( k \)-resonances which also are quantum counterpart of the peculiar semi-classical orbits [55]. It would be interesting to discuss their possible relations. It would be also interesting to combine the effects of a diagonal disorder and chemical potentials for \( k \)-cycles in RRG representation of the Hilbert space together. It could be the toy model for a interplay of scars with disorder discussed in [56].

We have found the mobility edge for the partially disordered RRG for the fixed number of nodes. However, the more extended analysis shows that it exists at all available values of \( N \) up to \( N = 10^6 \). This will be presented elsewhere.

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CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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