Some new operations on $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic Hadamard matrices

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December 20, 2011

Abstract

Following the ideas of [3] about $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic Hadamard matrices, we introduce the notion of diagram, which visually represents any set of coboundaries. Diagrams are a very useful tool for the description and the study of paths and intersections, as described in [3]. Then, we will study four different operations on $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic matrices. These operations will be defined on the set of coboundaries defining the matrix, preserve the Hadamard character of the cocyclic matrices, and allow us to obtain new Hadamard matrices from old ones. We split the set of Hadamard matrices into disjoint orbits, define representatives for them and take advantage of this fact to compute them in an easier way than the usual purely exhaustive way.

1 Introduction

Hadamard matrices are $n \times n$ square matrices $H$ with entries in $\{1, -1\}$ such that every pair of rows (respectively, columns) are orthogonal, that is, $HH^T = nI_n$.

It is easy to check that if $H$ is a Hadamard matrix, then $n$ is 1, 2 or a multiple of 4. The Hadamard Conjecture asserts that there exist Hadamard matrices for every size $n$ multiple of 4.

A cocyclic matrix $M_f$ over a group $G$ is a matrix $M_f = (f(g, h))$, for $f$ being a 2-cocycle over $G$, that is, a function $f : G \times G \rightarrow \{1, -1\}$ such that for every $a, b, c \in G$, $f(a, b) \cdot f(ab, c) \cdot f(a, bc) \cdot f(b, c) = 1$. They were first noticed in [7, 8], and since then, many Hadamard matrices have been checked to be cocyclic. For more on cocyclic matrices, the interested reader is referred to [6] and the references there cited.

*All authors are partially supported by FEDER funds via the research projects FQM-016 and P07-FQM-02980 from JJAA and MTM2008-06578 from MICINN (Spain).

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In [3], a characterization of $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic Hadamard matrices was described, depending on the notions of distributions, ingredients and recipes, which we do not reproduce here, since they are far from the scope of the present paper. In the process, the $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic Hadamard matrices were expressed as the product of some elementary coboundaries and a matrix $R$ coming from representative cocycles. The use of the complete set of elementary coboundaries instead of just a basis for coboundaries implied that the same $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic Hadamard matrix could be expressed in 8 different ways (see [3] for details). This way, the full set of $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic Hadamard matrices was determined, for $3 \leq t \leq 13$.

In this paper we will introduce what we call diagrams, a visual representation of the coboundaries which define a $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic matrix.

Then, we will study four different operations on the set of coboundary matrices over $\mathbb{Z}_t \times \mathbb{Z}_2^2$: complements, rotations, swappings and dilatations. In particular, these operations extend to operations over $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic matrices, which will be proved to preserve the Hadamard character of the matrices. For commodity, these operations will be defined on the full set of elementary coboundaries over $\mathbb{Z}_t \times \mathbb{Z}_2^2$ (not just a basis), so that after applying some of them, the $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic matrices obtained so far might be expressed in any of the eight possible ways described in [3]. Anyway, there is no imprecision, as we will see. These operations partition the set of $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic matrices into disjoint orbits, which can be easily computed once one element is known. Among all the elements of an orbit, a representative can be chosen, in a standard way that will be precised.

Finally, by applying these ideas to the task of searching for cocyclic Hadamard matrices over $\mathbb{Z}_t \times \mathbb{Z}_2^2$, we have been able to extend the table in p. 132 in [4] for values of $t$ in the range $3 \leq t \leq 23$, and to explain the fact that the set of symmetric Williamson type Hadamard matrices is in proportion $\frac{1}{t}$ with respect to the full set of $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic Hadamard matrices.

A preliminary version of this work can be found in [5].

We organize the paper as follows. Section II is dedicated to the description of diagrams, after a discussion about the convenience of using the whole set of coboundaries instead of the basis, in order to represent $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic matrices. Section III defines the four Hadamard preserving operations, which have a nice interpretation in terms of diagrams. These operations split the set of Hadamard matrices into disjoint orbits, which can be generated from any of their elements (for instance by its representative). Last section includes some final remarks and further work.

2 On basis, generators and diagrams

In this section we introduce some elementary notations, and give the notion of diagram, a very useful presentation of a $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic matrix. In particular, the notion of symmetry in a diagram (see Definition [2]) will lead to a fast Hadamard test for $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic matrices, which we state in Theorem [1].
In [2] a basis for 2-cocycles over \( \mathbb{Z}_t \times \mathbb{Z}_2^2 \) is described. This basis \( B = \{ \partial_2, \ldots, \partial_{4t-2}, \beta_1, \beta_2, \gamma \} \) for 2-cocycles consists of \( 4t - 3 \) coboundaries \( \partial_k \), two cocycles \( \beta_i \) coming from inflation and one cocycle \( \gamma \) coming from transgression.

In these circumstances, every 2-cocycle over \( G \) admits an unique representation as a product of the generators in \( B \), \( f = f_1^\alpha_1 \cdots f_k^\alpha_k \), \( \alpha_i \in \{0, 1\} \). The tuple \( (\alpha_1, \ldots, \alpha_k)_B \) defines the coordinates of \( f \) with regards to \( B \). Accordingly, every cocyclic matrix \( M_f = (f(g_i, g_j)) \) for \( f = (\alpha_1, \ldots, \alpha_k)_B \) admits an unique decomposition as the Hadamard (pointwise) product \( M_f = M_{\alpha_1} \cdots M_{\alpha_k} \).

As usual, \( \partial_i \) refers to the coboundary associated to the \( i \)-th element in \( G \). Let us recall that, once we fixed the basis \( B \) of coboundaries, \( \{M_{\partial_1^1}, M_{\partial_1^2}, \ldots, M_{\partial_t^1}, M_{\partial_t^2}\} \), each of the three dropped coboundaries \( \{M_{\partial_1^3}, M_{\partial_1^4}, \ldots, M_{\partial_t^3}, M_{\partial_t^4}\} \) can be expressed in terms of the basis, namely:

\[
\begin{align*}
M_{\partial_1} &= -\prod_{i=1}^{t-1} M_{\partial_{4i+1}} \prod_{i=0}^{t-1} M_{\partial_{4i+2}}. \\
M_{\partial_{t-1}} &= \prod_{i=0}^{t-2} M_{\partial_{4i+1}} \prod_{i=0}^{t-1} M_{\partial_{4i+2}}. \\
M_{\partial_t} &= \prod_{i=1}^{t-1} M_{\partial_{4i}} \prod_{i=0}^{t-1} M_{\partial_{4i+2}}.
\end{align*}
\]

From now on, we will denote the coboundary matrix corresponding to the element \( \partial_i \) by its number \( i \), and assume that every cocyclic matrix \( M \) is obtained as a pointwise product of some coboundaries and the representative matrix \( R = M_{\beta_1} M_{\beta_2} M_{\gamma} \) (see [4] for details).

Depending on whether we use (or not) each of these three coboundaries, we will have eight different expressions for every cocyclic matrix (these eight expressions are related by means of substitutions).

For \( t = 7 \), Table 1 shows the eight different expressions of the same Hadamard matrix. First of them is the expression with respect to the basis, not using any of the three dropped coboundaries.

Cocyclic matrices over \( \mathbb{Z}_t \times \mathbb{Z}_2^2 \) can be visualized by diagrams which represent the coboundaries which appear in the expression of the matrix.

**Definition 1** Given a pointwise product of coboundaries \( M_{\partial_{d_1}} \ldots M_{\partial_{d_k}} \) defining a cocyclic matrix over \( \mathbb{Z}_t \times \mathbb{Z}_2^2 \), its diagram is a \( 4 \times t \) matrix, such that \( a_{ij} \), \( 1 \leq i \leq 4, 1 \leq j \leq t \) is \( \times \), \( (a_{ij} = \times) \) if \( 4t - 4(j - 1) - 3 + i \mod 4t \in \{d_1, \ldots, d_k\} \) and empty elsewhere.

**Remark 1** The definition of diagram has to do with the expression of the matrix in terms of the coboundaries, so every cocyclic matrix over \( \mathbb{Z}_t \times \mathbb{Z}_2^2 \) has eight different diagrams.
Table 1: $t = 7$. Eight expressions for the same Hadamard matrix

\[
\begin{array}{c}
\{\{14, 10, 6\}, \{11\}, \{20, 12, 4\}, \{25, 21, 9\}\} \\
\{\{26, 22, 18, 2\}, \{27, 23, 19, 15, 7, 3\}, \{20, 12, 4\}, \{25, 21, 9\}\} \\
\{\{26, 22, 18, 2\}, \{11\}, \{28, 24, 16, 8\}, \{25, 21, 9\}\} \\
\{\{26, 22, 18, 2\}, \{11\}, \{20, 12, 4\}, \{17, 13, 5, 1\}\} \\
\{\{14, 10, 6\}, \{27, 23, 19, 15, 7, 3\}, \{28, 24, 16, 8\}, \{25, 21, 9\}\} \\
\{\{14, 10, 6\}, \{27, 23, 19, 15, 7, 3\}, \{20, 12, 4\}, \{17, 13, 5, 1\}\} \\
\{\{14, 10, 6\}, \{11\}, \{28, 24, 16, 8\}, \{17, 13, 5, 1\}\} \\
\{\{26, 22, 18, 2\}, \{27, 23, 19, 15, 7, 3\}, \{28, 24, 16, 8\}, \{17, 13, 5, 1\}\}
\end{array}
\]

For instance, one Hadamard matrix for $t = 7$ is given by the following coboundaries:

\[
\{\{14, 10, 6\}, \{11\}, \{20, 12, 4\}, \{25, 21, 9\}\}
\]

A presentation of this subset of coboundaries as a $4 \times t$ matrix, such that coboundaries mod $i$ are placed at row $i - 1$ is:

\[
\begin{array}{ccccccc}
26 & 22 & 18 & 14 & 10 & 6 & 2 \\
27 & 23 & 19 & 15 & 11 & 7 & 3 \\
28 & 24 & 20 & 16 & 12 & 8 & 4 \\
25 & 21 & 17 & 13 & 9 & 5 & 1
\end{array}
\]

In short, using $\times$ instead of the subindexes of the coboundaries, we get the diagram

\[
\begin{array}{c}
- & - & - & \times & \times & \times & - \\
- & - & - & - & - & - & - \\
- & - & \times & - & \times & - & \times \\
\times & \times & - & - & \times & - & -
\end{array}
\]

Diagrams are a very useful tool. This presentation of the coboundaries allows us to read easily the adjacency conditions, the number of paths, their length and productivity, as introduced in [3].

A set $\{M_{\partial_j} : 1 \leq j \leq w\}$ of generalized coboundary matrices defines a $n$-walk if these matrices may be ordered in a sequence $(M_1, \ldots, M_w)$ so that consecutive matrices $M_i$ and $M_{i+1}$ share a negative entry at the $n^{th}$ row, precisely at the position $(n, l_{i+1})$, $1 \leq i \leq w - 1$. Such a walk is called a path if the initial (equivalently, the final) matrix shares a $-1$ entry with a generalized coboundary matrix which is not in the walk itself, and a cycle otherwise.

With this notation at hand, the summation of the $n^{th}$-row of a matrix $M_{\partial_1} \ldots M_{\partial_w} \cdot R$ is zero if and only if $2c_n + r_n - 2I_n = 2t$, where $c_n$ is the number of maximal $n$-paths in $\{M_{\partial_1}, \ldots, M_{\partial_w}\}$, $r_n$ is the number of $-1$s in the $n^{th}$-row of $R$ and $I_n$ is the number of positions in which $R$ and $M_{\partial_1} \ldots M_{\partial_w}$ share a common $-1$ in their $n^{th}$-row (obviously, $0 \leq I_n \leq r_n$).
At this point, it is important to notice that paths and intersections give a Hadamard test for cocyclic matrices, equivalent to the usual cocyclic Hadamard test of [4], which claims that a cocyclic matrix is Hadamard if and only if the summation of each row (but the first) is zero. When this test is applied to $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic matrices, it suffices to check whether the summation of each of the rows from 5 to $2t + 2$ is 0 (see [1, 3] for details).

**Corollary 1** In particular, for $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic matrices:

1. the summation of a row $n \equiv 1 \mod 4$ is zero if and only if
   \[ c_n = t, \]  
2. the summation of a row $n \equiv 0, 2, 3 \mod 4$ is zero if and only if
   \[ c_n = I_n. \]

Every row of a diagram represents one 5-cycle. Actually, the diagram is not a matrix, but a cylinder, the first and last columns being adjacent. The paths in rows not congruent to 1 can be read by alternating pair of rows in the diagram. The productivity of the even paths comes from the congruency class module 4 of its first (or last) coboundary. This can be easily checked in any diagram.

**Definition 2** A diagram associated to a pointwise product of coboundaries $M = M_{b_1} \ldots M_{b_k} R$ is called symmetric if the $\times$ are symmetric with respect to a column. If one of the diagrams representing a cocyclic matrix is symmetric, so are all other diagrams (they are obtained by complementing some of the rows, and symmetry is preserved). We will say, by extension, that the cocyclic matrix presents symmetry (the matrix is not symmetric, the diagram is).

**Theorem 1** If a cocyclic matrix over $\mathbb{Z}_t \times \mathbb{Z}_2^2$ is represented by symmetric diagrams and has exactly $t$ paths in rows congruent to 1 mod 4, then it is Hadamard.

**Proof.**

Taking into account Corollary 1 we have to prove that $c = I$ for rows 6, 7, 8 and congruent. Odd paths give one intersection, and the only thing to prove is that even paths are equally distributed between productive and impro- ductive. Even paths come in pairs (they do not use any of the coboundaries on the symmetry axis) and the productivity of each path is the opposite of its symmetric.

\[ \square \]

3 Operations

In this section we will study four different operations on the set of coboundary matrices over $\mathbb{Z}_t \times \mathbb{Z}_2^2$: complements, rotations, swappings and dilatations. In
particular, these operations extend to operations over $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic matrices, which will be proved to preserve the Hadamard character of the matrices. These operations partition the set of $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic matrices into disjoint orbits, which can be easily computed once one element is known. Among all the elements of an orbit, a representative can be chosen, in a standard way that will be precised.

### 3.1 Complements. The group $\mathbb{Z}_2$

Given a set of coboundaries, $t$, and one of the four congruency classes mod 4, for instance $i$-th, $1 \leq i \leq 4$, we can consider the cocyclic matrix defined by the set of coboundaries obtained by substituting the subset of coboundaries belonging to the $i$-th congruency by its complement.

**Definition 3** Let $\{\{c_{ij_2}\}, \{c_{ij_3}\}, \{c_{ij_4}\}, \{c_{ij_1}\}\}$ with $j_k \subseteq [k]4$, be a set of coboundaries. The complement in the $i$-th component, $1 \leq i \leq 4$, denoted $C_i(\{\{c_{ij_2}\}, \{c_{ij_3}\}, \{c_{ij_4}\}, \{c_{ij_1}\}\})$, is the union of the complement of $\{c_{ij_i}\}$ in the $i$-th component and the rest of the initial coboundaries.

For instance, if we choose $i = 2$:

$$C_2(\{\{14, 10, 6\}, \{11\}, \{28, 12, 4\}, \{25, 21, 9\}\}) =$$

$$= \{\{26, 22, 18\}, \{11\}, \{28, 12, 4\}, \{25, 21, 9\}\}$$

In terms of diagrams:

$$C_2 = \begin{array}{cccc}
- & - & - & x \\
- & - & - & x \\
- & - & x & x \\
- & x & - & x \\
\end{array}$$

**Lemma 1** If we express the matrices with respect to the basis of coboundaries, then the only complement to compute is the complement with respect to the component congruent to 2.

**Proof.**

If we consider any other complement, there will appear one of the coboundaries 1, $4t - 1$ or $t$. Substitution of the expression of this coboundary gives us the complement with respect to the congruency 2.

If two different sets of coboundaries give the same cocyclic matrix their complements define the same matrix. It suffices to compute the image of the different
expressions for a cocyclic matrix and check that we get the different expressions for its image. So, there is no imprecision when we say the complement matrix of a cocyclic matrix over $\mathbb{Z}_t \times \mathbb{Z}_2^2$.

**Theorem 2** If a set a coboundaries define a Hadamard matrix, so does its complement.

**Proof.**

It suffices to check that the hypotheses of Theorem 1 are satisfied.

The complement of a set of paths in a cycle is another set with the same number of paths. So the Hadamard condition in rows congruent to 1 is preserved.

For the other rows, just observe that the complement preserves symmetry in the diagram.

□

**Remark 2** Note that the operation complement modifies the number of coboundaries, substituting the $k_0$ coboundaries congruent to 2 mod 4 by $t - k_0$ coboundaries.

This operation can be observed as an action of the group $\mathbb{Z}_2$ over the set of $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic matrices. For every Hadamard cocyclic matrix over $\mathbb{Z}_t \times \mathbb{Z}_2^2$, we can consider its orbit. This orbit has exactly 2 elements.

### 3.2 Rotations. The group $\mathbb{Z}_t$

Given that the Hadamard condition for a cocyclic matrix over $\mathbb{Z}_t \times \mathbb{Z}_2^2$ depends on the conditions of position and adjacency of the coboundaries defining such matrix, one natural idea is rotating the positions of the coboundaries.

**Definition 4** Let $\{c_{2j_2}, \{c_{3j_3}, \{c_{4j_4}, \{c_{1j_1}\}}\}\}$ with $j_k \subseteq [k]_4$, be a set of coboundaries. The $i$-rotated set, $0 \leq i \leq t - 1$ of this set of coboundaries, denoted by $T_i(\{c_{2j_2}, \{c_{3j_3}, \{c_{4j_4}, \{c_{1j_1}\}}\}\})$ is the set:

$$\{c_{2j_2-4i}, c_{3j_3-4i}, c_{4j_4-4i}, c_{1j_1-4i}\}$$

where additions are mod 4$t$.

For instance, for $i = 2$

$$T_2(\{\{14, 10, 6\}, \{11\}, \{20, 12, 4\}, \{25, 21, 9\}\}) =$$

$$= \{\{6, 2, 26\}, \{3\}, \{12, 4, 24\}, \{17, 13, 1\}\}$$

In diagrams:

$$\begin{array}{cccccc}
- & - & - & x & x & x \\
- & - & - & x & x & x \\
- & - & x & - & x & - \\
x & x & - & - & x & - \\
\end{array} =$$

$$\begin{array}{cccccc}
- & - & - & x & x & x \\
- & - & - & x & x & x \\
- & - & x & - & x & - \\
x & x & - & - & x & - \\
\end{array}$$
The $i$-rotation operation moves the marked positions $i$ places to the right.

As in the complement operation, $i$-rotation over each of the eight expressions for a Hadamard matrix gives each of the eight expressions of the same cocyclic Hadamard matrix, so we are rigorous when speaking of the $i$-rotated of a matrix.

This operation can be observed as an action of the group $\mathbb{Z}_t$ over the set of $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic matrices. The element $i$, $0 \leq i \leq t - 1$ acts on every cocyclic matrix by subtracting $4i \mod 4t$ to each of the coboundaries defining the matrix.

**Theorem 3** The $i$-rotated set of a set defining a cocyclic Hadamard matrix over $\mathbb{Z}_t \times \mathbb{Z}_2^2$ defines a Hadamard matrix too.

**Proof.**
Rotations preserve the relative positions of the coboundaries at every row and, thus, the number, length and productivity of the paths.

□

**Remark 3** There is no need to consider symmetry condition for this proof.

For every Hadamard cocyclic matrix over $\mathbb{Z}_t \times \mathbb{Z}_2^2$, we can consider its orbit. This orbit has exactly $t$ elements.

**Remark 4** Notice that the complement and rotation operations commute.

### 3.3 Swappings. The group $S_4$

The word *swapping* explains clearly the effect of this operation on a set of coboundaries. Adjacency conditions are permuted for some of the four components.

We will denote the set of coboundaries:

$$\{\{c_{2j_2}\}, \{c_{3j_3}\}, \{c_{4j_4}\}, \{c_{1j_1}\}\}$$

by $\{c_2, c_3, c_4, c_1\}$, and the expression $c_j + k$ denotes the set of coboundaries obtained by adding $k$ to each of $c_j$.

**Definition 5** For the set of coboundaries $\{c_2, c_3, c_4, c_1\}$ we define the swapping operations:

- $s_{23}(\{c_2, c_3, c_4, c_1\}) = \{c_3 - 1, c_2 + 1, c_4, c_1\}$.
- $s_{24}(\{c_2, c_3, c_4, c_1\}) = \{c_4 - 2, c_3, c_2 + 2, c_1\}$.
- $s_{21}(\{c_2, c_3, c_4, c_1\}) = \{c_1 + 1, c_3, c_4, c_2 - 1\}$. 

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• $s_{34}(\{c_2, c_3, c_4, c_1\}) = \{c_2, c_4 - 1, c_3 + 1, c_1\}$.
• $s_{31}(\{c_2, c_3, c_4, c_1\}) = \{c_2, c_1 + 2, c_4, c_3 - 2\}$.
• $s_{41}(\{c_2, c_3, c_4, c_1\}) = \{c_2, c_3, c_1 + 3, c_4 - 3\}$.

Although we have defined six operations, any composition of them can be considered, obtaining one of the possible 24 permutations on the rows of the diagram.

Theorem 4 If $\{c_2, c_3, c_4, c_1\}$ is a set of coboundaries defining a Hadamard cocyclic matrix over $\mathbb{Z}_t \times \mathbb{Z}_2^2$, then $s_{ij}(\{c_2, c_3, c_4, c_1\})$, $1 \leq i < j \leq 4$ is also a Hadamard cocyclic matrix over $\mathbb{Z}_t \times \mathbb{Z}_2^2$.

Proof.
Once again, it suffices to check whether the hypotheses of Theorem 1 are satisfied.

On one hand, any swapping preserves the number of paths in rows congruent to 1.

On the other hand, any permutation of the rows in the diagram preserves symmetry.

□

Remark 5 This result can be proved without using the symmetry condition, one only need to assume that the productive and improductive paths are equally distributed among the two subsets of components defined in each row, attending to the congruency mod 4 of the coboundaries.

This gives us an action of $S_4$ on diagrams, which commute with complement and rotations. Depending on the diagram (i.e. on the size of the permutation group of its rows), the orbit can have less than 24 elements (actually, 1,4,6,12 or 24).

3.4 Dilatations. The group $\mathbb{Z}_t^*$

As we have seen, the Hadamard character of a cocyclic matrix over $\mathbb{Z}_t \times \mathbb{Z}_2^2$ is determined by the position and adjacency of the coboundaries which define it. This is the reason because of rotating the rows of a diagram preserves the Hadamard character of the corresponding $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic matrix. Now we are interested in other kind of geometric transformations: homothecies.

Definition 6 Fixed $t$ and given a set of coboundaries $S$ determining a diagram with columns $(C_t, \cdots, C_1)$, the $r$-th dilatation applied to $S$, with $r \in \mathbb{Z}/\mathbb{Z}_t^*$, denoted by $V_r(S)$, is the set corresponding to the image of the diagram under the homothecy with ratio $r$ and center the column placed at the right hand-side of the diagram.
For instance, for $i = 2$

$$V_2(\{\{14, 10, 6\}, \{11\}, \{20, 12, 4\}, \{25, 21, 9\}\}) =$$

$$= \{\{26, 18, 10\}, \{19\}, \{20, 8, 4\}, \{21, 17, 13\}\}$$

In diagrams:

As the column located at the right-hand side corresponds to the coboundaries $2, 3, 4, 1$, a formula for $V_r$ is given by $V_r(\partial_k) = \partial_h$, for

$$h = 4 \left\lfloor \frac{k - [k]_4}{4} \cdot r \right\rfloor + [k]_4. \quad (3)$$

Here we adopt the notation $[m]_n$ for $m \mod n$ for brevity.

**Remark 6** Notice that a dilatation $V_r$ defines a bijection over the set of $\mathbb{Z}_t \times \mathbb{Z}_2^2$-coboundaries if and only if $r \perp t$, that is, $r \in \mathbb{Z}_t^*$.  

**Theorem 5** Dilatations $V_r$ with $r \perp t$ preserve the Hadamard character of a $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic matrix.

**Proof.**

It suffices to check whether the hypotheses of Theorem 1 are satisfied. Dilatations define some permutation on the number of paths defined in each row congruent to 1, so the total number of paths remains equal to $t$. Since the image diagram is symmetric as well, the result holds.

**Remark 7** The size of the orbit of a symmetric $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic matrix under the action of dilatations is, at most, $\phi(t)$, being $\phi$ the Euler function.

**Remark 8** Depending on the diagrams, the image under dilatations can coincide with the image under some of the previously defined operations; for instance the action on the previous diagram of the dilatation $V_2$ coincides with the action of the composition $s_{14}s_{24}T_5$. 
3.5 Orbits and representatives

**Definition 7** The total orbit of a cocyclic Hadamard matrix over $\mathbb{Z}_t \times \mathbb{Z}_2^2$ is the union of all orbits under the action of complement, rotations, swappings and dilatations.

**Definition 8** A representative of a total orbit is a set of coboundaries associated to a diagram, with minimal number of coboundaries, symmetric with respect to the central column, and with an increasing number of coboundaries on each row of the diagram.

Table 2 gives us the description of all Hadamard matrices obtained in Table III of [3], split in orbits, which can be generated by its representatives under the action of the four operations previously defined. The second column shows the size of the orbit under complement/rotation/swapping/dilatations, when providing new matrices.

4 Application to computations

There are two special repartitions of coboundaries attending to their congruency, for every Hadamard matrix. One is the repartition given by the expression of the matrix with respect to the basis. And the other is the minimal repartition (this last one can correspond to the matrix, or to its complement).

When performing calculations we can restrict to the minimal repartition, in order to simplify the search, avoiding 15 of the 16 possible repartitions for each distribution (see [3] for details on repartitions and distributions, existence and computation).

The rotation, swapping and dilatations preserve the number of coboundaries involved. On the other hand, the image of the Hadamard matrix does not depend on the chosen representation (from the possible eight). Thus we can obtain the whole orbit of the matrix under all these operations by only knowing the expression of the matrix with the minimal repartition, so we can restrict the searching to this easier case.

Once this is done, we can compute the complement of each of the obtained matrices, to complete the orbit under all operations.

By following this idea, we have computed the set of cocyclic Hadamard matrices over $\mathbb{Z}_t \times \mathbb{Z}_2^2$, having a symmetric diagram for $15 \leq t \leq 23$, which are listed in Tables 3 and 4 in terms of their representatives and the size of their total orbit under complement/rotation/swapping/dilatations, when providing new matrices.

4.1 Williamson matrices

The first row of Table 5 shows the number of Williamson matrices obtained in [4], together with the total number of cocyclic Hadamard matrices over $\mathbb{Z}_t \times \mathbb{Z}_2^2$ which we have computed.
Table 2: Orbits of $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic Hadamard matrices

| $t$ | orbit | coboundaries |
|-----|-------|--------------|
| 3   | $2 \times 3 \times 4 \times 1$ | $\{\}, \{7\}, \{8\}, \{5\}$ |
| 5   | $2 \times 5 \times 12 \times 1$ | $\{10\}, \{11\}, \{8, 16\}, \{1, 17\}$ |
| 7   | $2 \times 7 \times 24 \times 1 = 336$ | $\{14\}, \{11, 15, 19\}, \{8, 16, 24\}, \{1, 13, 25\}$ |
|     | $2 \times 7 \times 12 \times 3 = 504$ | $\{10, 18\}, \{11, 19\}, \{4, 28\}, \{1, 13, 25\}$ |
| 9   | $2 \times 9 \times 12 \times 3 = 648$ | $\{14, 22\}, \{15, 19, 23\}, \{4, 16, 24, 36\}, \{1, 13, 21, 33\}$ |
|     | $2 \times 9 \times 24 \times 3 = 1296$ | $\{14, 22\}, \{3, 19, 35\}, \{12, 16, 24, 28\}, \{1, 9, 25, 33\}$ |
|     | $2 \times 9 \times 24 \times 1 = 432$ | $\{14, 18, 22\}, \{11, 19, 27\}, \{8, 20, 32\}, \{1, 17, 33\}$ |
| 11  | $2 \times 11 \times 24 \times 5 = 2640$ | $\{18, 22, 26\}, \{7, 15, 31, 39\}, \{4, 16, 32, 44\}, \{9, 13, 21, 29, 33\}$ |
| 13  | $2 \times 13 \times 24 \times 3 = 3744$ | $\{22, 26, 30\}, \{3, 15, 23, 31, 39, 51\}, \{8, 16, 20, 36, 40, 48\}, \{1, 5, 17, 33, 45, 49\}$ |
|     | $2 \times 13 \times 24 \times 3 = 1872$ | $\{18, 22, 30, 34\}, \{7, 15, 39, 47\}, \{8, 12, 24, 32, 44, 48\}, \{1, 5, 21, 29, 45, 49\}$ |
|     | $2 \times 13 \times 24 \times 3 = 1872$ | $\{14, 18, 34, 38\}, \{15, 23, 27, 31, 39\}, \{8, 24, 28, 32, 48\}, \{5, 17, 25, 33, 45\}$ |
|     | $2 \times 13 \times 12 \times 3 = 936$ | $\{14, 18, 34, 38\}, \{15, 19, 27, 35, 39\}, \{4, 12, 28, 44, 52\}, \{1, 9, 25, 41, 49\}$ |

It can be observed that the set of symmetric Williamson type Hadamard matrices is in proportion $\frac{1}{t}$ with respect to the full set of $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic Hadamard matrices.

By Lemma 3.4 in [4], any Williamson matrix is Hadamard equivalent to a cocyclic matrix over $\mathbb{Z}_t \times \mathbb{Z}_2^2$, which is $t \times t$ back-circulant by blocks $4 \times 4$,

$$
\begin{pmatrix}
    W_1 & \ldots & \ldots & W_1 \\
    W_2 & W_2 & W_2 & \vdots \\
    \vdots & \ddots & \ddots & \ddots \\
    W_t & W_1 & \ldots & W_{t-1}
\end{pmatrix}
$$

(4)
with $W_{i+1} = W_{t-i+1}$ for $1 \leq i \leq t - 1$, and $W_i$ being

$$
\begin{pmatrix}
  n_i & x_i & y_i & z_i \\
  x_i & -n_i & z_i & -y_i \\
  y_i & -z_i & -n_i & x_i \\
  z_i & y_i & -x_i & -n_i
\end{pmatrix}.
$$

This matrix (4) is a pointwise product of the representative matrix $R$ and a certain set of generalized coboundaries $\{M_{\partial_1}, \ldots, M_{\partial_t}\}$. The additional condition $W_{i+1} = W_{t-i+1}$ for $1 \leq i \leq t - 1$, leaving $W_1$ alone means that the symmetry column in the diagram representing the matrix (4) is, precisely, the last one to the right, which is associated to the coboundaries $\partial_2, \partial_3, \partial_4, \partial_1$. This fact leads us to conclude that:

**Proposition 1** Let $\mathcal{H}$ the set of cocyclic Hadamard matrices over $\mathbb{Z}_t \times \mathbb{Z}_2^2$, having a symmetric diagram. Then, the set of Williamson matrices of type (4) is a subset of $\mathcal{H}$ of size $|\mathcal{H}|_t$. Moreover, for each element $H \in \mathcal{H}$, one and only one of the rotated matrices $T_iH$, $1 \leq i \leq t$, is a Williamson matrix.

**Proof.**

The condition needed tell us that the diagram has to be symmetric with respect to the last column, which is only possible for one of the rotated.

So, the predicted number of Williamson matrices for $t \in [17, 23]$ will be $13056/17 = 768$, $34200/19 = 1800$, $31248/21 = 1488$ and $12144/23 = 528$, which is indeed the case (see [9]).

## 5 Further questions

After an observation process, some questions come into our minds:

- All $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic Hadamard matrices obtained so far have a symmetric diagram up to $t = 15$, and we have taken advantage of it when computing them for $t \in [17, 23]$ by restricting ourselves to the symmetric case. Can this symmetry assumption be proved?

- The result about Williamson matrices allows us to go beyond $t = 23$. Actually, taking into account the results in [9], we can express every Williamson matrix as a pointwise product of coboundaries and compute the whole orbit of matrices for $t \in [25, 39]$. We show the results obtained in Table 6 after identifying every matrix obtained in terms of coboundaries, and computing the size of its orbit under the action of complement/rotation/swapping/dilatations, when providing new matrices.

- Now that we have proved that our operations preserve the Hadamard character, arises the question: Do these operations preserve Hadamard
equivalece? Rotations, dilatations and some of the swappings do, because they can be expressed in terms of the shifting operation defined by Horadam in [6] (this fact will be detailed elsewhere). In addition, the number of orbits coincide with the number of non equivalent Williamson matrices computed in [9], so the result is probably true in general.

References

[1] V. Álvarez, J.A. Armario, M.D. Frau and P. Real, "A system of equations for describing cocyclic Hadamard matrices", J. Comb. Des., vol. 16, pp. 276–290, 2008.

[2] V. Álvarez, J.A. Armario, M.D. Frau and P. Real, "The homological reduction method for computing cocyclic Hadamard matrices", J. Symb. Comput., vol. 44, pp. 558-570, 2009.

[3] V. Álvarez, F. Gudiel and M.B. Güemes, "On \( \mathbb{Z}_t \times \mathbb{Z}_2^2 \)-cocyclic Hadamard matrices". Preprint. Submitted to journal.

[4] A. Baliga and K.J. Horadam, "Cocyclic Hadamard matrices over \( \mathbb{Z}_t \times \mathbb{Z}_2^2 \), Australas. J. Combin., vol. 11, pp. 123–134, 1995.

[5] M. B. Güemes, “Cocyclic Hadamard matrices over \( \mathbb{Z}_t \times \mathbb{Z}_2^2 \). Description, classification and search. PhD thesis (English translation), University of Seville, 2011.

[6] K.J. Horadam, Hadamard matrices and their applications. Princeton: Princeton University Press, 2007.

[7] K.J. Horadam and W. de Launey, "Cocyclic development of designs", J. Algebraic Combin., vol. 2, no. 3, pp. 267–290, 1993. Erratum, J. Algebraic Combin., vol. 3, pp. 129, 1994.

[8] K.J. Horadam and W. de Launey, "Generation of cocyclic Hadamard matrices", in: Proc. Computational Algebra and Number Theory, Sydney, 1992, in Math. Appl., vol. 325, Kluwer Acad. Publ., Dordrecht, 1995, pp. 279–290.

[9] C. Koukouvinos, web page, http://www.math.ntua.gr/~ckoukov/
Table 3: $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic Hadamard matrices [15-19]

| $t/\text{orbit}/\#H./\text{coboundaries}$ | 4 | 12 | 16 |
|------------------------------------------|---|----|----|
| 15 2 $\times$ 15 $\times$ 24 $\times$ 4 | 2880 | \{18, 22, 38, 42\}, \{7, 19, 27, 35, 43, 55\}, \{16, 20, 28, 32, 36, 44, 48\}, \{1, 9, 25, 29, 33, 49, 57\} | |
| 15 2 $\times$ 15 $\times$ 24 $\times$ 2 | 1440 | \{14, 26, 34, 46\}, \{11, 15, 27, 35, 47, 51\}, \{16, 20, 28, 32, 36, 44, 48\}, \{1, 5, 21, 29, 37, 53, 57\} | |
| 15 2 $\times$ 15 $\times$ 12 $\times$ 4 | 1440 | \{14, 26, 30, 34, 46\}, \{15, 27, 31, 35, 47\}, \{12, 16, 24, 40, 48, 52\}, \{9, 13, 21, 29, 37, 49\} | |
| 17 2 $\times$ 17 $\times$ 24 $\times$ 4 | 3264 | \{18, 30, 34, 38, 50\}, \{3, 23, 31, 35, 39, 47, 67\}, \{16, 20, 28, 36, 44, 52, 56\}, \{1, 3, 21, 25, 41, 45, 53, 65\} | |
| 17 2 $\times$ 17 $\times$ 24 $\times$ 4 | 3264 | \{18, 30, 34, 38, 50\}, \{3, 23, 27, 35, 43, 47, 67\}, \{4, 16, 28, 36, 44, 56, 68\}, \{13, 17, 21, 29, 37, 49, 53\} | |
| 17 2 $\times$ 17 $\times$ 24 $\times$ 4 | 3264 | \{18, 30, 34, 38, 50\}, \{7, 15, 31, 35, 39, 55, 63\}, \{12, 20, 24, 36, 48, 52, 60\}, \{5, 9, 13, 21, 45, 53, 57, 61\} | |
| 17 2 $\times$ 17 $\times$ 24 $\times$ 4 | 3264 | \{18, 26, 30, 38, 42, 50\}, \{3, 19, 31, 39, 51, 67\}, \{8, 24, 32, 36, 40, 48, 64\}, \{9, 13, 17, 33, 49, 53, 57\} | |
| 19 2 $\times$ 19 $\times$ 24 $\times$ 9 | 8208 | \{18, 30, 34, 42, 46, 58\}, \{15, 19, 23, 39, 55, 59, 63\}, \{8, 16, 28, 36, 40, 44, 52, 64, 72\}, \{5, 21, 25, 29, 37, 45, 49, 53, 69\} | |
| 19 2 $\times$ 19 $\times$ 12 $\times$ 9 | 4104 | \{10, 30, 34, 42, 46, 66\}, \{11, 31, 35, 39, 43, 47, 67\}, \{4, 12, 16, 32, 40, 48, 64, 68, 76\}, \{1, 9, 13, 29, 37, 45, 61, 65, 73\} | |
| 19 2 $\times$ 19 $\times$ 24 $\times$ 9 | 8208 | \{10, 30, 34, 42, 46, 66\}, \{15, 27, 35, 39, 43, 51, 63\}, \{8, 16, 32, 36, 40, 44, 48, 64, 72\}, \{5, 9, 25, 33, 37, 41, 49, 65, 69\} | |
| 19 2 $\times$ 19 $\times$ 24 $\times$ 9 | 2376 | \{6, 10, 34, 42, 66, 70\}, \{11, 23, 27, 35, 43, 51, 55, 67\}, \{8, 16, 20, 36, 44, 60, 64, 72\}, \{1, 5, 9, 29, 45, 65, 69, 73\} | |
| 19 2 $\times$ 19 $\times$ 24 $\times$ 3 | 2736 | \{6, 10, 34, 42, 66, 70\}, \{7, 15, 31, 35, 43, 47, 63, 71\}, \{12, 20, 24, 36, 44, 56, 60, 68\}, \{1, 5, 9, 25, 49, 65, 69, 73\} | |
| 19 2 $\times$ 19 $\times$ 24 $\times$ 9 | 8208 | | |
Table 4: $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic Hadamard matrices [21-23]

| $t$ | orbit/ $\# H.$/coboundaries |
|-----|--------------------------|
| 21  | $2 \times 21 \times 24 \times 2$ | 2016 |
|     | $\{22, 26, 38, 46, 58, 62\}, \{3, 15, 19, 27, 39, 47, 59, 67, 71, 83\}$, |   |
|     | $\{8, 12, 16, 24, 40, 48, 64, 72, 76, 80\}, \{13, 21, 25, 29, 33, 49, 53, 27, 61, 69\}$ |   |
| 21  | $2 \times 21 \times 24 \times 6$ | 6048 |
|     | $\{10, 34, 38, 42, 46, 50, 74\}, \{3, 19, 23, 31, 55, 63, 67, 83\}$, |   |
|     | $\{12, 24, 28, 36, 44, 52, 60, 64, 76\}, \{5, 9, 13, 21, 33, 49, 61, 69, 73, 77\}$ |   |
| 21  | $2 \times 21 \times 24 \times 6$ | 6048 |
|     | $\{18, 26, 38, 42, 46, 58, 66\}, \{19, 23, 31, 35, 51, 55, 63, 67\}$, |   |
|     | $\{12, 16, 32, 40, 44, 48, 56, 72, 76\}, \{1, 5, 17, 29, 37, 45, 53, 65, 77, 81\}$ |   |
| 21  | $2 \times 21 \times 24 \times 6$ | 6048 |
|     | $\{18, 26, 38, 42, 46, 58, 66\}, \{11, 23, 27, 35, 51, 59, 63, 75\}$, |   |
|     | $\{12, 16, 32, 40, 44, 48, 56, 72, 76\}, \{1, 5, 9, 25, 37, 45, 57, 73, 77, 81\}$ |   |
| 21  | $2 \times 21 \times 12 \times 6$ | 3024 |
|     | $\{6, 30, 34, 38, 46, 50, 54, 78\}, \{7, 31, 35, 39, 47, 51, 55, 79\}$, |   |
|     | $\{16, 20, 36, 52, 68, 72, 80\}, \{13, 17, 33, 41, 49, 65, 69, 77\}$ |   |
| 21  | $2 \times 21 \times 24 \times 6$ | 6048 |
|     | $\{18, 30, 34, 38, 46, 50, 54, 66\}, \{3, 7, 31, 39, 47, 55, 79, 83\}$, |   |
|     | $\{8, 24, 32, 36, 52, 56, 64, 80\}, \{1, 17, 21, 29, 41, 43, 61, 65, 81\}$ |   |
| 21  | $2 \times 21 \times 24 \times 2$ | 2016 |
|     | $\{14, 18, 30, 38, 46, 54, 66, 70\}, \{7, 15, 27, 31, 55, 59, 71, 79\}$, |   |
|     | $\{8, 12, 20, 24, 64, 68, 72, 80\}, \{13, 21, 25, 37, 41, 45, 57, 61, 69\}$ |   |
| 23  | $2 \times 23 \times 24 \times 11$ | 12144 |
|     | $\{22, 30, 38, 42, 50, 54, 62, 70\}, \{15, 19, 23, 35, 47, 59, 71, 75, 79\}$, |   |
|     | $\{12, 20, 28, 40, 44, 52, 56, 68, 76, 84\}, \{9, 13, 25, 29, 33, 37, 57, 61, 65, 77, 81\}$ |   |

Table 5: Williamson / $\mathbb{Z}_t \times \mathbb{Z}_2^2$-cocyclic Hadamard matrices

| $t$ | 3 | 5 | 7 | 9 | 11 | 13 | 15 | 17 | 19 |
|-----|---|---|---|---|----|----|----|----|----|
| $\# Will.$ | 8 | 24 | 120 | 264 | 240 | 648 | 576 | – | – |
| $\# H.$ | 24 | 120 | 840 | 2376 | 2640 | 8424 | 8640 | 13056 | 34200 |
| $t$ | $4t = A^2 + B^2 + C^2 + D^2$ | $\mathbb{Z}(\text{comp/rot/swap/dil})$ |
|-----|--------------------------------|--------------------------------------|
| 25  | $100 = 9^2 + 3^2 + 3^2 + 1^2$ | $2 \times 25 \times 24 \times 10 = 12000$ |
|     | $100 = 7^2 + 7^2 + 1^2 + 1^2$ | $2 \times 25 \times 12 \times 5 = 3000$ |
|     |                                 | $2 \times 25 \times 24 \times 5 = 6000$ |
|     | $100 = 7^2 + 5^2 + 5^2 + 1^2$ | $2 \times 25 \times 24 \times 10 = 12000$ |
|     |                                 | $2 \times 25 \times 24 \times 5 = 6000$ |
|     | $100 = 5^2 + 5^2 + 5^2 + 5^2$ | $2 \times 25 \times 24 \times 5 = 6000$ |
| 27  | $108 = 9^2 + 5^2 + 1^2 + 1^2$ | $2 \times 27 \times 24 \times 9 = 11664$ |
|     |                                 | $2 \times 27 \times 24 \times 9 = 11664$ |
|     | $108 = 7^2 + 7^2 + 3^2 + 1^2$ | $2 \times 27 \times 12 \times 9 = 5832$ |
|     |                                 | $2 \times 27 \times 24 \times 9 = 11664$ |
|     | $108 = 7^2 + 5^2 + 5^2 + 3^2$ | $2 \times 27 \times 24 \times 9 = 11664$ |
| 29  | $116 = 9^2 + 5^2 + 3^2 + 1^2$ | $2 \times 29 \times 24 \times 14 = 19488$ |
| 31  | $124 = 7^2 + 7^2 + 5^2 + 1^2$ | $2 \times 31 \times 24 \times 15 = 22320$ |
|     | $124 = 7^2 + 5^2 + 5^2 + 5^2$ | $2 \times 31 \times 12 \times 15 = 11160$ |
| 33  | $132 = 11^2 + 3^2 + 1^2 + 1^2$ | $2 \times 33 \times 24 \times 10 = 15840$ |
|     | $132 = 9^2 + 7^2 + 1^2 + 1^2$ | $2 \times 33 \times 24 \times 10 = 15840$ |
|     | $132 = 9^2 + 5^2 + 5^2 + 1^2$ | $2 \times 33 \times 24 \times 10 = 15840$ |
|     | $132 = 7^2 + 7^2 + 5^2 + 3^2$ | $2 \times 33 \times 24 \times 10 = 15840$ |
| 37  | $148 = 11^2 + 3^2 + 3^2 + 3^2$ | $2 \times 37 \times 24 \times 3 = 5328$ |
|     | $148 = 9^2 + 7^2 + 3^2 + 3^2$ | $2 \times 37 \times 12 \times 18 = 15984$ |
|     | $148 = 7^2 + 7^2 + 5^2 + 5^2$ | $2 \times 37 \times 24 \times 9 = 15984$ |
|     |                                 | $2 \times 37 \times 24 \times 9 = 15984$ |
| 39  | $156 = 9^2 + 5^2 + 5^2 + 5^2$ | $2 \times 39 \times 24 \times 4 = 7488$ |
| 41  | $164 = 9^2 + 9^2 + 1^2 + 1^2$ | $2 \times 41 \times 12 \times 5 = 4920$ |
| 43  | $172 = 7^2 + 7^2 + 7^2 + 5^2$ | $2 \times 43 \times 24 \times 7 = 14448$ |
| 45  | $180 = 9^2 + 7^2 + 5^2 + 5^2$ | $2 \times 45 \times 12 \times 12 = 12960$ |
| 49  | $196 = 9^2 + 9^2 + 5^2 + 3^2$ | $2 \times 49 \times 12 \times 21 = 24696$ |
| 51  | $204 = 11^2 + 9^2 + 1^2 + 1^2$ | $2 \times 51 \times 12 \times 16 = 19584$ |
|     | $204 = 11^2 + 7^2 + 5^2 + 8^2$ | $2 \times 51 \times 24 \times 16 = 39168$ |
| 55  | $220 = 11^2 + 9^2 + 3^2 + 3^2$ | $2 \times 55 \times 12 \times 20 = 26400$ |
| 57  | $228 = 9^2 + 7^2 + 7^2 + 7^2$ | $2 \times 57 \times 12 \times 18 = 24624$ |
| 61  | $244 = 11^2 + 11^2 + 1^2 + 1^2$ | $2 \times 61 \times 12 \times 15 = 21960$ |
| 63  | $252 = 11^2 + 11^2 + 3^2 + 1^2$ | $2 \times 63 \times 12 \times 6 = 9072$ |