Virtual Differential Passivity based Control for Tracking of Flexible-joints Robots

Rodolfo Reyes-Báez∗,∗∗ Arjan van der Schaft∗,∗∗
Bayu Jayawardhana∗,∗∗∗

∗ Jan C. Willems Center for Systems and Control, University of Groningen, The Netherlands
** Johann Bernoulli Institute for Mathematics and Computer Science, University of Groningen, P.O. Box 407, 9700 AK, Groningen, The Netherlands ({r.reyes-baez, a.j.van.der.schaft}@rug.nl)
∗∗∗ Engineering and Technology Institute Groningen (ENTEG), University of Groningen, Nijenborgh 4, 9747AG, The Netherlands (b.jayawardhana@rug.nl)

Abstract: Based on recent advances in contraction methods in systems and control, in this paper we present the virtual differential passivity based control (v-dPBC) technique. This is a constructive design method that combines the concept of virtual systems and of differential passivity. We apply the method to the tracking control problem of flexible-joints robots (FJRs) which are formulated in the port-Hamiltonian (pH) framework. Simulations on a single flexible joint link are presented for showing the performance of a controller obtained with this approach.

Keywords: Flexible-joints robots, port-Hamiltonian systems, differential passivity, virtual systems, contraction analysis.

1. INTRODUCTION

The problem of control of rigid robots has been widely studied since they are instrumental in modern manufacturing systems. However, the elasticity phenomena in the joints can not be neglected for accurate position tracking as reviewed in Nicosia and Tomei (1995). For every joint that is actuated by a motor, we need two degrees of freedom joints instead of one. Such FJRs are therefore underactuated. In Spong (1987) two state feedback control laws based on feedback linearization and singular perturbation are presented for a simplified model. Similarly in de Wit et al. (2012) a dynamic feedback controller for a more detailed model is presented. In Ailon and Ortega (1993) and Brogliato et al. (1995) passivity-based control (PBC) schemes are proposed. The first one is an observer-based controller which requires only motor position measurements. In the latter one a PBC controller is designed and compared with backstepping and decoupling techniques. For further details on PBC of FJRs we refer to Ortega et al. (1998) and references therein. In Astolfi and Ortega (2003), a global tracking controller based on the I&I method is introduced. From a more practical point of view, in Albu-Schäffer et al. (2007), a torque feedback is embedded into the passivity-based control approach, leading to a full state feedback controller; with this acceleration and jerk measurements are not required. In a recent work of Avila-Becerril et al. (2016), they design a dynamic controller which solves the global position tracking problem of FJRs based only on measurements of link and joint positions. All controllers mentioned above are for the second order Euler-Lagrange (EL) systems. Most of these schemes are based on the selection of a suitable storage function that together with the dissipativity of the closed-loop system, ensures the convergence state trajectories to the desired solution.

As an alternative to the EL formalism, the pH framework has been introduced in van der Schaft and Maschke (1995). The main characteristics of the pH framework are the existence of a Dirac structure (connects geometry with analysis), port-based network modeling and a clear physical energy interpretation. For the latter part, the energy function can directly be used to show the dissipativity and stability property of the systems. Some set-point controllers have been proposed for FJRs modeled as pH systems. For instance in Ortega and Borja (2014) the EL-controller for FJRs in Ortega et al. (1998) is adapted and interpreted in terms of Control by Interconnection (CbI). In Zhang et al. (2014), they propose an Interconnection and Damping Assignment PBC (IDA-PBC) scheme, where the controller is designed with respect to the pH representation of the EL-model in Albu-Schäffer et al. (2007). For the tracking control problem of FJRs in the pH framework, to the best of our knowledge, the only result is the one in Jardón-Kojakhmetov et al. (2016), where a singular perturbation approach is considered. The key result is that, both, the slow and fast dynamics are fully-actuated pH systems, so that we can apply directly rigid robots controllers.

In this work we extend our previous results in Reyes-Báez et al. (2017a,b), on v-dPBC of fully-actuated mechanical 1 We refer interested readers on CbI to Ortega et al. (2008).
2 For IDA-PBC technique see also Ortega et al. (2002).
port-Hamiltonian systems, to solve the tracking problem of FJRs viewed as pH systems. This method relies on the contraction properties of the so-called virtual systems, Forni and Sepulchre (2014); van der Schaft (2013); Pavlov and van de Wouw (2017); Lohmiller and Slotine (1998); Sontag (2010); Wang and Slotine (2005). Roughly speaking, the method \(^3\) consists in designing a control law for a virtual system associated to a FJR, such that it is differentially passive in the closed-loop, and has a desired steady-state behavior. Then, the FJR in closed-loop with above controller tracks the virtual system’s steady-state.

The paper is organized as follows: In Section 2, the theoretical preliminaries on differential incremental methods, their relation with virtual systems and the v-dPBC methodology are presented. Section 3 deals with some structural properties of mechanical pH systems and the explicit pH model of FJRs, together with its associated virtual mechanical system. A trajectory tracking v-dPBC scheme for FJRs is presented in section 4. In order to show the performance of a controller obtained with the proposed method, simulation results are presented. Finally, in Section 5 conclusions and future research are stated.

2. CONTRACTION, DIFFERENTIAL PASSIVITY AND VIRTUAL SYSTEMS

In this paper, we adopt the differential Lyapunov framework for contraction analysis as in the paper Forni and Sepulchre (2014), which unifies different approaches. Some arguments will be omitted due to space limitation.

Let \( \Sigma \) be a nonlinear control system with state space \( \mathcal{X} \) be the state-space of dimension \( N \), affine in the input \( u \),

\[
\Sigma: \begin{cases}
\dot{x} = f(x,t) + \sum_{i=1}^{n} g_i(x,t)u_i, \\
y_i = h_i(x,t), \\
& i \in \{1, \cdots, n\},
\end{cases}
\]

where \( x \in \mathcal{X}, u \in \mathcal{U} \subset \mathbb{R}^n \) and \( y \in \mathcal{Y} \). The vector fields \( f, g_i: \mathcal{X} \times \mathbb{R}^n \to \mathcal{T X} \) are assumed to be smooth and \( h_i: \mathcal{X} \to \mathbb{R}, \) for \( i \in \{1, \cdots, n\} \). The input space \( \mathcal{U} \) and output space \( \mathcal{Y} \) are assumed to be open subsets of \( \mathbb{R}^n \).

Given two initial states \( x(t_1) = x_{0}\) and \( x(t_2) = x_{20} \), take any forward invariant coordinate neighborhood \( \mathcal{O} \) of \( \mathcal{X} \), containing \( x(t_0) = x_{10} \) and \( x_{20} \). Consider a regular smooth curve \( \gamma: [0,1] \rightarrow \mathcal{O}, \) such that \( \gamma(0) = x_{10} \) and \( \gamma(1) = x_{20} \). Let \( t \in [t_0,T] \Rightarrow x(t) = \psi^{t}_{0}(t, \gamma(s)) \) be the solution to (1) from the initial condition \( \gamma(s) \), at time \( t_0 \), corresponding to the family of input functions \( t \in [t_0,T] \Rightarrow u(t) = u_{0}(t,s), \) for \( s \in I \). The differential in direction \( \frac{\partial}{\partial t} \) at a fixed \( s \) represents the time derivative. The time derivative of \( \psi^{t}_{0}(t, \gamma(s)) \) satisfies,

\[
\frac{\partial \psi^{t}_{0}(t, \gamma(s))}{\partial t} = f(\psi^{t}_{0}(t, \gamma(s)), t) + \sum_{i=1}^{n} g_i(\psi^{t}_{0}(t, \gamma(s)), t)u_i, \]

\[
y_i(t) = h_i(\psi^{t}_{0}(t, \gamma(s)), t), \quad i \in \{1, \cdots, n\}
\]

for all \( t \geq t_0 \), and all \( s \in I \). The differential in direction \( \frac{\partial}{\partial s} \) at a fixed \( t \) is a variation with respect to \( s \). For the input-state-output solution \((x, u, y)\), the variations are

\[
\delta u = \frac{\partial \psi^{t}_{0}}{\partial s}(t, s); \quad \delta x = \frac{\partial \psi^{t}_{0}}{\partial s}(t, \gamma(s)); \quad \delta y = \frac{\partial h}{\partial s}(\psi^{t}_{s}, t),
\]

which are nothing, but tangent vectors to \( \psi^{t}_{0}(t, s), \psi^{t}_{i}(t, \gamma(s)), \) and \( h_i(\psi^{t}_{s}, t, \gamma(s)), \) for \( i \in \{1, \cdots, n\} \), respectively; i.e. \( \delta u \in T \mathcal{U}, \delta x \in T \mathcal{X}, \) and \( \delta y \in T \mathcal{Y} \).

Let us introduce the concept of prolongation to the tangent bundle of a given system a long the trajectory \((u, x, y)(t) = (\psi^{t}_{0}(t, s), \psi^{t}_{i}(t, \gamma(s)), h_i(\psi^{t}_{s}, t, \gamma(s)), t))\).

Definition 1 (Crouch and van der Schaft (1987)). A prolonged system of system (1) corresponds to the original system together with its variational system, that is

\[
\dot{x} = f(x,t) + \sum_{i=1}^{n} g_i(x,t)u_i, \\
y = h(x,t), \\
\dot{\delta x} = \frac{\partial f}{\partial x}(x,t)\delta x + \sum_{i=1}^{n} \frac{\partial u_i}{\partial x}(x,t)\delta x + \sum_{i=1}^{n} g_i(x,t)\delta u, \\
\dot{\delta y} = \frac{\partial h}{\partial x}(x,t)\delta x.
\]

with \((u, \delta u) \in T \mathcal{U}, (x, \delta x) \in T \mathcal{X}, \) and \((y, \delta y) \in T \mathcal{Y} \).

System (1) in closed-loop with the uniformly smooth static feedback control law \( u = \eta(x,t) \) will be denoted by

\[
\dot{x} = F(x,t).
\]

2.1 Contraction and differential Lyapunov theory

Definition 2. (Forni and Sepulchre (2014)). A function \( V: \mathcal{X} \times \mathbb{R}^n \rightarrow \mathbb{R} \geq 0 \) is a candidate differential or Finsler-Lyapunov function if it satisfies uniformly the bounds

\[
c_1 F(\delta x, t)^p \leq V(x, \delta x, t) \leq c_2 F(\delta x, t)^p,
\]

where \( c_1, c_2 \in \mathbb{R} \geq 0, p \) is some positive integer and \( F(x, \cdot, t) := \| \cdot \|_{x,t} \) is, uniformly in \( t \), a Finsler structure.

The relation between a candidate differential Lyapunov function and the Finsler structure in (6) is a key property for incremental stability analysis. That is, a well-defined distance on \( \mathcal{X} \) via integration as defined below.

Definition 3. (Finsler distance). Consider a candidate differential Lyapunov function on \( \mathcal{X} \) and the associated Finsler structure \( F \). Let \( \Gamma(x_1,x_2) \) be the collection of piecewise \( C^1 \) curves \( \gamma: [0,1] \rightarrow \mathcal{X} \) connecting \( x_1 \) and \( x_2 \) such that \( \gamma(0) = x_1 \) and \( \gamma(1) = x_2 \). The Finsler distance \( d: \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R} \geq 0 \) induced by \( F \) is defined by

\[
d(x_1, x_2) := \inf_{\gamma \in \Gamma(x_1,x_2)} \int_{0}^{1} \left( \gamma(s) \right) ds
\]

The following result gives a sufficient condition for contraction by using differential Lyapunov functions.

Theorem 1. (Forni and Sepulchre (2014)). Consider the prolonged system of (5), a connected and forward invariant set \( \mathcal{D} \subseteq \mathcal{X} \), and a strictly increasing function \( \alpha: \mathcal{X} \rightarrow \mathbb{R} \geq 0 \). Let \( V \) be a candidate differential Lyapunov function satisfying

\[
V(\delta x, t) \leq -\alpha(V(\delta x, t))
\]

for each \((x, \delta x) \in T \mathcal{X} \) uniformly in \( t \). Then, system (5) contracts \( V \) in \( \mathcal{D} \). The function \( V \) is called the contraction measure, and \( \mathcal{D} \) is the contraction region.
Corollary 1. System (5) is incrementally
• stable on $D$ if $\alpha(s) = 0$ for each $s \geq 0$;
• asymptotically stable on $D$ if $\alpha(s)$ is a strictly increasing;
• exponentially stable on $D$ if $\alpha(s) = \beta s, \forall s > 0$.

Remark 1. Under hypothesis of Theorem 1, if the contraction region $D \subseteq \mathcal{X}$ is a compact set. Then, system (5) is convergent, Rüffer et al. (2013). In this case, (8) could be seen as a generalization of the Demidovich condition Pavlov and van de Wouw (2017).

2.2 Differential passivity

Definition 4. (Forni et al. (2013); van der Schaft (2013)). Consider system (4). Then, system (1) is called differentially passive if the prolonged system (4) is dissipative with respect to the supply rate $\delta y^\top \delta u$, that is, if there exist a differential storage function function $W : T\mathcal{X} \rightarrow \mathbb{R}_{\geq 0}$ satisfying
\[
\frac{dW}{dt}(x, \delta x) \leq \delta y^\top \delta u,
\]
for all $x, \delta x, u, \delta u$. Furthermore, system (1) is called differentially loss-less if (9) holds with equality.

If additionally, the differential storage function is required to be a differential Lyapunov function, then differential passivity implies contraction when the input is $u = 0$, for all $i \in \{1, \ldots, n\}$. For further details and a differential geometric characterization see van der Schaft (2013).

The following result will be extensively used in the paper.

Lemma 1. Consider system (1). Suppose that the control $u = \eta(x, t) + \omega$ is designed such that, via the differential transformation $\delta \tilde{x} = \Theta(x, t)\delta x$, the variational dynamics of the closed-loop system takes the form
\[
\dot{\delta \tilde{x}} = [\Xi(\tilde{x}, t) - \Upsilon(\tilde{x}, t)]\Pi(\tilde{x}, t)\delta \tilde{x} + \Psi(\tilde{x}, t)\delta \omega,
\]
where $\omega$ is an auxiliary input, $\Pi(\tilde{x}, t) > 0$ is a Riemannian metric, $\Xi(\tilde{x}, t) = -\Xi^\top(\tilde{x}, t)$ and $\Upsilon(\tilde{x}, t) = \Upsilon^\top(\tilde{x}, t)$ satisfying $\delta \tilde{x}^\top \Pi(\tilde{x}, t) - 2\Pi(\tilde{x}, t)\Upsilon(\tilde{x}, t)\Pi(\tilde{x}, t) \delta \tilde{x} \leq 0$. Then, the closed-loop system is differentially passive from $\delta \omega$ to $\delta \tilde{y} = \Psi(\tilde{x}, t)^\top \Pi(\tilde{x}, t)\delta \tilde{x}$ and differential storage function
\[
V(\tilde{x}, \delta \tilde{x}) = \frac{1}{2} \delta \tilde{x}^\top \Pi(\tilde{x}, t)\delta \tilde{x}.
\]

2.3 Contraction and differential passivity of virtual systems

A generalization of contraction was first introduced in Wang and Slotine (2005) and revisited in Jouffrout and Fossen (2010); Forni and Sepulchre (2014), with the name of partial contraction, which is based on the contraction behavior of the so-called virtual systems.

Definition 5. A virtual system associated to (5) is defined as a system
\[
\dot{x}_v = \Phi(x_v, x, t),
\]
in the state $x_v \in \mathcal{C}_v$ and parametrized by $x \in \mathcal{C}_x$, where $\mathcal{C}_v \subseteq \mathcal{X}$ and $\mathcal{C}_x \subseteq \mathcal{X}$ are connected and forward invariant, $\Phi : \mathcal{X} \times \mathcal{X} \times \mathbb{R}_{\geq 0} \rightarrow T\mathcal{X}$ is a smooth vector field satisfying $\Phi(x, x, t) = F(x, t), \quad \forall t \geq t_0$.

Furthermore, a virtual control system for the system with inputs (1), in the state $x_v \in \mathcal{X}$, is similarly defined as
\[
\begin{align*}
\dot{x}_v &= \Gamma(x_v, x, u, t), \\
y_v &= h_v(x_v, x, t)
\end{align*}
\]
parametrized by the variable $x \in \mathcal{X}$, the output $y_v \in \mathcal{Y}$, with smooth vector fields $h_v : \mathcal{X} \times \mathcal{X} \times \mathbb{R}_{\geq 0} \rightarrow \mathcal{Y}$ and $\Gamma : \mathcal{X} \times \mathcal{X} \times \mathcal{U} \times \mathbb{R}_{\geq 0} \rightarrow T\mathcal{X}$ satisfying
\[
\Gamma(x, x, u, t) = f(x, t) + G(x, t)u,
\]
\[
h_v(x_v, x, t) = h(x_v, t), \quad \forall u, \forall t \geq t_0.
\]

If it follows that any solution $x(t) = \psi^{\omega}(x_0, t)$ starting at $x_0 \in \mathcal{C}_x$ of the actual system (5), generates the solution $x_v(t) = \psi^{\omega}(x_0, t)$ to system (12) for all $t \geq 0$. In the same manner, any solution of (1) $x = \psi^{\omega}(x_0, t_0)$, for a certain input $u = \tau \in \mathcal{U}$, generates a solution $x_v(t) = \psi^{\omega}(x_0, t)$ to the virtual control system (14). However, not every solution $x_v(t)$ of the virtual system corresponds to a solution of the actual system. Thus, for any curve $x(t)$, we may consider the time-varying system with state $x_v$.

The convergence behavior of (5) can be induced from the contraction properties of an associated virtual system Wang and Slotine (2005); Forni and Sepulchre (2014).

Theorem 2. Consider two connected and forward invariant sets $\mathcal{C}_v \subseteq \mathcal{X}$ and $\mathcal{C}_x \subseteq \mathcal{X}$ for systems (5) and (12) respectively. Suppose that system (12) is uniformly contracting with respect to $x_v$. Then, for any given initial conditions $x_0 \in \mathcal{C}_v$, and $x_0 \in \mathcal{C}_x$, each solution to (12) converges asymptotically to the solution of (5).

If this holds, the actual system (5) is said to be virtually contracting to the virtual system (12). This does not imply that the actual system is contracting, but all its trajectories converge to the steady state solution of the virtual system. Thus, if a system is virtually contracting, then it is convergent as in Pavlov and van de Wouw (2017).

Furthermore, if there exist a virtual control system for (1), such that it is differentially passive, then the actual control system is said to be virtually differentially passive.

2.4 Virtual differential passivity based control

We propose a constructive control design method for system (1), that we shall call virtual differential passivity based control ($v$-dPBC), such that the closed-loop system is convergent to a desired behavior. The design procedure is divided in three main steps:

1. Design the virtual control system (14) for system (1).
2. Design the feedback $u = \eta(x, x, t) + \omega$ for (14) such that the closed-loop virtual system is differentially passive for the input-output pair $(\delta y_v, \omega)$ and has a desired trajectory $x_d(t)$ as steady-state solution.
3. Define the controller for system (1) as $u = \eta(x, x, t)$.

If we are able to design a controller following the above steps, then we call closed-loop system trajectories will converge to $x_d(t)$, for the external input $\omega = 0$.

3. MECHANICAL PORT-HAMILTONIAN SYSTEMS

Ideas in the previous section will be applied mechanical systems in the pH framework, which are described below.
where \( g(x) \) is a \( N \times m \) input matrix, and \( J(x) \), \( R(x) \) are the interconnection and dissipation \( N \times N \) matrices which satisfy \( J(x) = -J^T(x) \) and \( R(x) = R^T(x) \geq 0 \).

In the case of a standard mechanical system with generalized coordinates \( q \) on the configuration space \( Q \) of dimension \( n \) and velocity \( \dot{q} \in T_qQ \), the Hamiltonian function is given by the total energy

\[
H(x) = \frac{1}{2}D(q)\dot{q} - \omega(q, \dot{q}),
\]

where \( x = (q, p) \in T^*Q := X \) is the phase state, \( p := \dot{q} \) is the generalized momentum and the inertia matrix \( M(q) \) is symmetric and positive definite. Finally, the interconnection, dissipation and input matrices in (16) are

\[
J(x) = \begin{bmatrix} 0_n & I_n \\ -I_n & 0_n \end{bmatrix}; \quad R(x) = \begin{bmatrix} 0_n & 0_n \\ 0_n & D(q) \end{bmatrix}; \quad g(x) = \begin{bmatrix} 0_n \\ B(q) \end{bmatrix},
\]

with matrix \( D(q) = D^T(q) \geq 0_n \) being the damping matrix and \( I_n, 0_n \) the \( n \times n \) identity, respectively, zero matrices. The input force matrix \( B(q) \) has rank \( m \leq n \).

In Arimoto (1996), it was shown that the identity

\[
\frac{1}{2}D(q)\dot{q} = \frac{1}{2}M(q)\dot{q} \quad \implies \quad \text{existence of skew-symmetric matrix } S_L(q, \dot{q}) \text{ that satisfies the relation}
\]

\[
\frac{\partial}{\partial q} \left( \frac{1}{2}D(q)\dot{q} \right) = \left[ S_L(q, \dot{q}) - \frac{1}{2}M(q) \right] \dot{q} = \dot{q}.
\]

In order to express the relation (20) in the Hamiltonian framework, consider the generalized momentum and the Legendre transformation of the quadratic form in the brackets of the left hand side (Van der Schaft (2017)). Then, the following holds

\[
\frac{\partial}{\partial q} \left( \frac{1}{2}p^T M^{-1}(q) p \right) = -\frac{\partial}{\partial q} \left( \frac{1}{2} \dot{q}^T M(q) \dot{q} \right) \quad \text{for } (q, p).
\]

With this, (20) can be rewritten in terms of \( (q, p) \) as

\[
\frac{\partial}{\partial q} \left( \frac{1}{2}p^T M^{-1}(q) p \right) = \left[ S_L(q, p) - \frac{1}{2}M(q) \right] M^{-1}(q) p \quad \text{where the matrix } S_L(x) := M(q) S_H(x) M(q), \text{ which is nothing but } S_L(q, \dot{q}) \text{ in coordinates } x \in T^*X.
\]

Then, system (16)-(18) can be rewritten as

\[
\begin{align*}
\dot{q} &= \begin{bmatrix} 0_n \\ -I_n \end{bmatrix} + \begin{bmatrix} I_n \end{bmatrix} + \begin{bmatrix} q \end{bmatrix} + g(q)u, \\
y &= \begin{bmatrix} B^T(q) \end{bmatrix} \begin{bmatrix} q \end{bmatrix},
\end{align*}
\]

with

\[
E(q, p) := S_H(q, p) - \frac{1}{2}M(q).
\]

The structure of the matrix \( E(q, p) \) and the conservation of energy tell us that the associated forces are \textit{workless}.

Based on the above description, we can define a pH-like virtual control system as in (14) associated to system (16)-(18), with the state \( x_v = (q_v, p_v) \in X \) and parametrized by \( x = (q, p) \), as follows

\[
\begin{align*}
\dot{x}_v &= [J_v(x_v) - R_v(x_v)] \frac{\partial H_v}{\partial x_v}(x_v, x) + g(x)u, \\
y_v &= g^T(x_v) \frac{\partial H_v}{\partial x_v}(x_v, x),
\end{align*}
\]

with matrices \( J_v = -J_v^T \) and \( R_v = R_v^T \) defined by

\[
\begin{align*}
J_v &= \begin{bmatrix} 0_n & I_n \\ -I_n & -S_H \end{bmatrix}, \quad R_v := \begin{bmatrix} 0_n \\ 0_n \\ 0_n \end{bmatrix} \left( D - \frac{1}{2}M \right).
\end{align*}
\]

and virtual Hamiltonian function

\[
H_v(x_v, x) = \frac{1}{2}p_v^T M^{-1}(q_v) p_v + P(q_v).
\]

Remark 2. Matrix \( J_v(x) \) qualifies as interconnection structure of the virtual system (25). However, the matrix \( R_v(x) \) is not necessarily positive definite. This is the reason why the virtual system (25) is called a \textit{pH-like system}.

The variational virtual system of (25) is given by

\[
\begin{align*}
\delta \dot{x}_v &= [J_v(x_v) - R_v(x_v)] \frac{\partial^2 H_v}{\partial x_v^2}(x_v, x) \delta x_v + g(x) \delta u, \\
\delta y_v &= g^T(x_v) \frac{\partial^2 H_v}{\partial x_v^2}(x_v, x) \delta x_v.
\end{align*}
\]

System (28) has the form (10) with \( \Xi = J_v, \quad \Upsilon = R_v \) and \( \Pi(x_v, x) = \frac{\partial^2 H_v}{\partial x_v^2}(x_v, x) \). If \( \frac{\partial^2 H_v}{\partial x_v^2} > 0 \) is such that hypotheses in Lemma 1 are satisfied, then the pH-like virtual system (25) is \textit{differentially passive} with differential storage function

\[
V(x_v, \delta x_v, x) = \frac{1}{2} \delta x_v^T \Pi(x_v, x) \delta x_v.
\]

The presents a controller for fully-actuated system using v-dPBC i.e., \( n = m \). This controller will be used in next section. For notational purposes we add the subscript \( \ell \) in all the terms that define systems (18) and (25), e.g., \( x = x_\ell, \quad x_v = x_{\ell v}, \quad u = u_\ell, \) etc.

Lemma 2. (Reyes-Báez et al. (2017a)). Consider a desired smooth trajectory \( x_{\ell d} = (q_{\ell d}, p_{\ell d}) \in T^*Q_\ell \), with \( n_\ell = \text{dim}Q_\ell \). Let us introduce the following change coordinates

\[
\tilde{x}_{\ell v} := \begin{bmatrix} q_{\ell v} \\ p_{\ell v} \end{bmatrix} = \begin{bmatrix} q_{\ell d} - q_d \quad p_{\ell d} - p_d \end{bmatrix},
\]

and define the auxiliary momentum reference as

\[
p_{\ell r} := M_\ell(q_d)(q_{\ell d} - \phi_d(q_{\ell d})),
\]
with $\phi_t: \mathbb{Q} \to T_q^* Q_t$ and a positive definite Riemannian metric $\Pi_t: \mathbb{Q}_t \times \mathbb{R}_{\geq 0} \to \mathbb{R}^{r \times n_c}$ satisfying the inequality
\begin{equation}
\Pi_t(\tilde{q}_t, t) - \Pi_t(\tilde{q}_t, t, \frac{\partial \phi_t}{\partial \tilde{q}_t} - \frac{\partial \phi_t^T}{\partial \tilde{q}_t}(\tilde{q}_t)) \times \Pi_t(\tilde{q}_t, t) \leq -2\beta_t(\tilde{q}_t, t)\Pi_t(\tilde{q}_t, t),
\end{equation}
with $\beta_t(\tilde{q}_t, t) > 0$, uniformly. Consider also system (18), its virtual system (25) and the corresponding control law given by
\begin{equation}
u_t(\tilde{x}_t, x_t, t) := \nu_{\text{eff}} + \nu_{\text{fb}} \quad \text{with}
\begin{align*}
\nu_{\text{eff}} &= \tilde{p}_t + \frac{\partial \Pi_t}{\partial \tilde{q}_t} + [E_t + D_t]^{-1}(q(t))p_t,
\nu_{\text{fb}} &= -\int_{0_n}^{\tilde{q}_t} \Pi_t(\tilde{q}_t, t) d\tilde{q}_t - M_t^{-1}\sigma_{\text{eff}} + \omega_t,
\end{align*}
where $K_{\text{eff}} > 0$ and $\omega_t$ is an external input. Then, virtual system (25) in closed-loop with (33) is differentially passive for the input-output pair $(\delta \omega, \delta y_{\sigma_{\text{eff}}})$, with $\delta y_{\sigma_{\text{eff}}} = B_{\text{eff}}M_t^{-1}\delta \sigma_{\text{eff}}$ and differential storage function
\begin{equation}
V_t(\tilde{x}_t, \tilde{\delta}x_t, t) = \frac{1}{2}\delta \tilde{q}_t^{T}\Pi_t(\tilde{q}_t, t)\delta \tilde{q}_t + \frac{1}{2}\delta \tilde{q}_t^{T}[M_t^{-1}\delta \sigma_{\text{eff}} + \omega_t].
\end{equation}

4. TRAJECTORY TRACKING CONTROLLER OF FLEXIBLE-JOINT ROBOTS

4.1 Flexible-joints Robots as port-Hamiltonian systems

Flexible rotational joints robots are a particular class of mechanical systems (18), where the generalized position is split as $q = [\tilde{q}_t, q_m]^{T} \in \mathbb{Q} = Q_{n_t} \times Q_{n_m}$, where $\tilde{q}_t$ and $q_m$ are the $n_t$-links and the $n_m$-motors generalized positions, respectively; with $\dim \mathbb{Q} = n_t + n_m$. The inertia and damping matrices are partitioned into $M(q) = \text{diag}(M_t(q_t), M_m(q_m))$ and $D(q) = \text{diag}(D_t(q_t), D_m(q_m))$, where $M_t(q_t)$ and $M_m(q_m)$ are the link and motors inertias; similarly $D_t(q_t)$ and $D_m(q_m)$ are the link and motor damping matrices, respectively. The potential energy is
\begin{equation}
P(q) = P_t(q_t) + \frac{1}{2}q^{T} K_t \zeta,
\end{equation}
which is the sum of the links potential energy $P_t(q_t)$ and the joints potential energy $P_m(q_m)$, with $\zeta := q_m - q_t$ and $K \in \mathbb{R}^{n \times n}$ a symmetric, positive definite matrix of stiffness coefficients. The input acts only in the motor state, meaning that the flexible-joints robot is an underactuated system and $\text{rank}(B(q)) = n_m$.

Assumption 1. (Spong (1987)). The following standard assumptions on the system physical structure are made:
- The relative displacement $\zeta$ (deflection) at each joint is small, such that the spring’s dynamics is linear.
- The $i$-th motor, which drives the $i$-th link, is mounted at the $(i-1)$-th link.
- The center of mass of each motor is on the rotation axes.
- The angular velocity of the motors is due only to their own spinning.

Thus, a flexible-joints robot can be modeled as an underactuated pH system of the form (18), given by
\begin{equation}
\begin{bmatrix}
\dot{q}_t \\
\dot{p}_t \\
\dot{p}_m
\end{bmatrix}
= \begin{bmatrix}
0_{n_t} & 0_{n_m} & I_{n_t} \\
0_{n_t} & 0_{n_m} & 0_{n_m} & I_{n_m} \\
0_{n_t} & -I_{n_m} & -D_t & 0_{n_m} & -M_t
\end{bmatrix}
\frac{\partial H}{\partial x} + \begin{bmatrix}
0_{n_t} \\
0_{n_m} \\
0_{n_t}
\end{bmatrix}
\nu_t + \begin{bmatrix}
B_{\text{eff}}(q_m) & B_{\text{eff}}(q_m) & \ldots & B_{\text{eff}}(q_m)
\end{bmatrix}
u_t,
\end{equation}
where $\nu_t$ and $p_m$ are the links and motors momenta, $p = [p_t^T, p_m^T]^T$ and $B_{\text{eff}}(q_m)$ is the input matrix associated to the motors. System (36) can be rewritten as (23), with
\begin{equation}
E(x) = \begin{bmatrix}
S_t(q_t, p_t) - \frac{1}{2}M_t & 0_{n_m} \\
0_{n_t} & S_m(q_m, p_m) - \frac{1}{2}M_m
\end{bmatrix},
\end{equation}
with $S_t^T = -S_t$ and $S_m^T = -S_m$. With this specification, the virtual system (25) corresponding to (36) is
\begin{equation}\n\begin{bmatrix}
\dot{x}_v \\
\dot{y}_v
\end{bmatrix}
= \begin{bmatrix}
0_{n_t} & 0_{n_m} & I_{n_t} \\
0_{n_t} & 0_{n_m} & 0_{n_m} & I_{n_m} \\
0_{n_t} & -I_{n_m} & -M_t & -E_{22} + D_t
\end{bmatrix}
\frac{\partial H_v}{\partial x_v} + g(x)u,
\end{equation}
where $S_t^T = -S_t$ and $S_m^T = -S_m$. With this specification, the virtual system (25) corresponding to (36) is
\begin{equation}
\begin{bmatrix}
\dot{x}_v \\
\dot{y}_v
\end{bmatrix}
= \begin{bmatrix}
0_{n_t} & 0_{n_m} & I_{n_t} \\
0_{n_t} & 0_{n_m} & 0_{n_m} & I_{n_m} \\
0_{n_t} & -I_{n_m} & -M_t & -E_{22} + D_t
\end{bmatrix}
\frac{\partial H_v}{\partial x_v} + g(x)u.
\end{equation}

4.2 Tracking controller design

In this section we extend the controller of Lemma 2 to the underactuated pH system (36), using the v-dPBC technique described in Subsection 2.4 with respect to the virtual system (38). Through a recursive construction of differential storage functions, we will implicitly design the differential transformation $\Theta(x_t, t)$ such that the closed-loop variational virtual system satisfies Lemma 1.

Proposition 1. Consider the virtual system of FJRs in (38). Suppose that the hypotheses and controller in Lemma 2 hold for the link dynamics with the controller $u_t$ given by (33). Let the motor reference state be given by $x_m = (q_{m,t}, p_{m,t}) \in T^* Q_m$, with $q_m = \tilde{q} + K^{-1}\tilde{u}$ and $n_m = \dim Q_m$. Consider the following change of coordinates
\begin{equation}
\tilde{x}_{\text{mv}} := \begin{bmatrix}
\tilde{q}_{\text{mv}} \\
\sigma_{x_{\text{mv}}}
\end{bmatrix},
\end{equation}
define the auxiliary motor momentum reference as $p_{\text{mr}} := M_m(q_m(t))(\tilde{q}_{\text{mv}} - \phi_m(q_{\text{mv}}) - \Pi_m^{-1}K_m^T \sigma_{x_{\text{mv}}})$, (40) where $\phi_m: Q_m \to T_{q_m} Q_m$ and a positive definite Riemannian metric $\Pi_m: Q_m \times \mathbb{R}_{\geq 0} \to \mathbb{R}^{n_m \times n_m}$ satisfying
\begin{equation}
\Pi_m(\tilde{q}_{\text{mv}}, t) = \frac{\Pi_m(q_{\text{mv}}, t)}{\Pi_m(\tilde{q}_{\text{mv}}, t) \frac{\partial \phi_m}{\partial \tilde{q}_{\text{mv}}}(\tilde{q}_{\text{mv}}) - \frac{\partial \phi_m^T}{\partial \tilde{q}_{\text{mv}}}(\tilde{q}_{\text{mv}})} \times \Pi_m(\tilde{q}_{\text{mv}}, t) \leq -2\beta\Pi_m(\tilde{q}_{\text{mv}}, t)\Pi_m(\tilde{q}_{\text{mv}}, t),
\end{equation}
with $\beta\Pi_m(\tilde{q}_{\text{mv}}, t) > 0$. Consider also system (36), its corresponding virtual system (38) and the control law given by $u(x, x, t) := u_{\text{eff}} + u_{\text{fb}}$, with
\begin{equation}
u_{\text{eff}} := \frac{\partial \Pi_t}{\partial \tilde{q}_t} + [E_t + D_t]^{-1}(q(t))p_t,
\end{equation}
and
\begin{equation}
u_{\text{fb}} = -\int_{0_n}^{\tilde{q}_t} \Pi_t(\tilde{q}_t, t) d\tilde{q}_t - M_t^{-1}\sigma_{\text{eff}} + \omega_t,
\end{equation}
where $K_m > 0$ and $\omega$ and external input. Then, the closed-loop virtual system (38) is differentially pas-
sive with respect to input-output pair \((\delta_x, \delta_y)\), with \(\delta y_{\sigma_m} = B_m^{\top} M_m^{-1} \delta x_m\) and differential storage function
\[
V(\delta x, \delta y, t) = \frac{1}{2} \delta x_v^\top \left[ \Pi(\delta y_v, t) \right] \delta x_v, \tag{43}
\]
with \(\Pi = \text{diag}(\Pi_1(\delta y_v), \Pi_m(\delta y_m))\). Furthermore, the closed-loop variational dynamics of (38) preserves the structure of (28) with
\[
\frac{\partial^2 H_v}{\partial \delta x_v} = \text{diag}(\Pi, M^{-1}(q)),
\]
\[
R_v(\delta x_v) = \text{diag}\left\{ \frac{\partial \Pi}{\partial \delta y_v} \right\},
\]
\[
J_v(\delta x_v) = \begin{bmatrix}
0_{n_x} & 0_{n_m} & -I_{n_x} & 0_{n_m} \\
0_{n_x} & -I_{n_m} & 0_{n_x} & 0_{n_m}
\end{bmatrix},
\]
where \(\phi(\delta y_v) = [\phi_1^\top, \phi_m^\top] \) and \(K_d = \text{diag}(K_t, K_m)\).

Notice that from (44), the closed-loop variational dynamics of the differentially passive system (38) can be seen as the feedback interconnection between the variational pH-like link error dynamics
\[
\delta x_v = \left[ \frac{\partial \Pi}{\partial \delta y_v} \right] \begin{bmatrix}
I_{n_x} \\
-I_{n_m}
\end{bmatrix}
\frac{\partial V_v}{\partial \delta x_v} + \begin{bmatrix}
0_{n_x} \\
0_{n_m}
\end{bmatrix}
\frac{\partial V_v}{\partial \delta y_v} \delta y_v, \tag{45}
\]
and the variational pH-like motor link error dynamics
\[
\delta y_v = \begin{bmatrix}
0_{n_x} & 0_{n_m} & \frac{\partial V_v}{\partial \delta x_v}
\end{bmatrix}
\end{bmatrix} \delta x_v, \tag{46}
\]
with "differential Hamiltonian" function
\[
V_v = \frac{1}{2} \delta x_v^\top \begin{bmatrix}
\Pi_1 & 0_{n_x} & M_1^{-1}
\end{bmatrix} \delta x_v, \tag{47}
\]
and the variational pH-like motor link error dynamics
\[
\begin{bmatrix}
\delta y_{\sigma_m} \\
\delta y_{\sigma_m}
\end{bmatrix} = \begin{bmatrix}
0_{n_x} & 0_{n_m} & 0_{n_m} \\
0_{n_x} & 0_{n_m} & -I_{n_m} \Pi_m \theta_{\sigma_m} \\
0_{n_x} & 0_{n_m} & 0_{n_m}
\end{bmatrix}
\end{bmatrix} \delta x_v, \tag{48}
\]
through the interconnection law
\[
\begin{bmatrix}
\delta y_{\sigma_1} \\
\delta y_{\sigma_m}
\end{bmatrix} = \begin{bmatrix}
0_{n_x} & 0_{n_m} & 0_{n_m} \\
0_{n_x} & 0_{n_m} & -I_{n_m} \Pi_m \theta_{\sigma_m} \\
0_{n_x} & 0_{n_m} & 0_{n_m}
\end{bmatrix}
\end{bmatrix} \delta y_v + \begin{bmatrix}
0_{n_x} \\
0_{n_m} \\
I_{n_m}
\end{bmatrix} \delta \omega. \tag{49}
\]

Hence, the feedback interconnection between the differentially passive links and motor error dynamics is a differentially passive closed-loop system with differential storage function \(V = V_v + V_m\), as proved in van der Schaft (2013).

Corollary 2. (Trajectory tracking controller). Consider the controller (42). Then, all solutions of the flexible-joints robot (36) in closed-loop with the controller \(u(x, x, t)\) converges exponentially to the desired trajectory \(x_d(t)\) with rate
\[
\beta = \min\{\beta_1, \beta_m\}, \lambda_{\min}\{D + K_d\} \lambda_{\min}\{M^{-1}\}. \tag{50}
\]

5. EXAMPLE: A FLEXIBLE-JOINT ROBOT

In this numerical example, we consider the FJR with one flexible joint i.e., \(n_f = n_m = 1\) (36). For the simulation, the parameters of the system and the controller as in (33) and (42) are given in Table 1. Here, we consider the same setting of FJR as the one used in Ghorbel et al. (1989)

| Parameter | Value |
|----------|------|
| Link inertia, \(M_1\) | 0.031 kg \cdot m^2 |
| Rotor inertia, \(M_m\) | 0.004 kg \cdot m^2 |
| Rotor friction, \(D_t\) | 0.2 N \cdot m \cdot sec/rad |
| Rotor friction, \(D_m\) | 0.007 N \cdot m \cdot sec/rad |
| Nominal load, \(M_g\) | 0.8 N \cdot m |
| Controller \(u_1\) | Controller \(u_m\) |
| \(\sigma_1 = \lambda_1 q_1\) | \(\sigma_m = \lambda_m q_m\) |
| \(\Lambda_1 = 10\) | \(\Lambda_1 = 15\) |
| \(K_1 = 0.6\) | \(K_1 = 0.3\) |
| \(n_2 = 2\alpha_f\) | \(n_1 = 4\Lambda_m\) |

Table 1. Flexible joint link parameters and control laws specifications

The closed-loop performance for the stiffness constant \(k = 31\) is shown in Figure 1. After a short transient time, both, the position and momentum error converge to zero. The overshoot in the controller \(u\) is due to the high gain.

If the stiffness constant changes to \(k = 3.1\), the closed-loop system keeps convergent to the desired steady-state behavior as shown in Figure 2. Notice that the transient time is also maintained. However, the control effort has a considerably bigger overshoot than in the previous case.

6. CONCLUSION

In this paper, we propose a virtual differential passivity based control which is applied to the tracking control of FJRs. Firstly, we introduced a virtual system associated to the FJR in the pH framework, which inherits structural properties of the actual system. This system is used for the control design procedure such that the closed-loop virtual system is made strictly differentially passive with a prescribed steady state solution. Furthermore, the closed-loop virtual system preserves the variational dynamics structure in (28). We show that the closed-loop virtual system can be seen as the feedback interconnection of two differentially passive subsystems. The controller \(u(x, x, t)\) solves the tracking problem in FJRs. Simulations confirm the theoretical results. A major implementation drawback of our controller presented here is that we require acceleration and jerk measurements. This is an open problem left for future research.

ACKNOWLEDGEMENTS

The first author thanks to Dr. H. Jardón-Kojakhmetov for the fruitful discussions that motivated the research on FJRs and to L. Pan for his help in implementing the numerical simulations. The first author is also grateful with CONACyT-Government of the State of Puebla for the scholarship assigned to CVU 386575.
REFERENCES

Ailon, A. and Ortega, R. (1993). An observer-based set-point controller for robot manipulators with flexible joints. Systems & Control Letters, 21(4), 329 – 335.

Albu-Schäffer, A., Ott, C., and Hirzinger, G. (2007). A unified passivity-based control framework for position, torque and impedance control of flexible joint robots. The international journal of robotics research, 26(1).

Arimoto, S. (1996). Control Theory of Nonlinear Mechanical Systems: A Passivity-Based and Circuit-Theoretic Approach. Oxford University Press.

Astolfi, A. and Ortega, R. (2003). Immersion and invariance: A new tool for stabilization and adaptive control of nonlinear systems. IEEE Transactions on Automatic control, 48(4), 590–606.

Avila-Becerril, S., Lora, A., and Panteley, E. (2016). Global position-feedback tracking control of flexible-joint robots. In 2016 American Control Conference (ACC), 3008–3013. doi:10.1109/ACC.2016.7525377.

Brogliato, B., Ortega, R., and Lozano, R. (1995). Global tracking controllers for flexible-joint manipulators: A comparative study. Automatica, 31(7), 941–956.

Crouch, P.E. and van der Schaft, A. (1987). Variational and hamiltonian control systems. Springer-Verlag.

de Wit, C.C., Siciliano, B., and Bastin, G. (2012). Theory of robot control. Springer Science & Business Media.

Forni, F. and Sepulchre, R. (2014). A differential lyapunov framework for contraction analysis. IEEE Transactions on Automatic Control.

Forni, F., Sepulchre, R., and Van Der Schaft, A. (2013). On differential passivity of physical systems. In Decision and Control (CDC), 2013 IEEE 52nd Annual Conference on, 6580–6585. IEEE.

Ghorbel, F., Hung, J.Y., and Spong, M.W. (1989). Adaptive control of flexible-joint manipulators. IEEE Control Systems Magazine, 9(7), 9–13.

Jardón-Kojakhmetov, H., Munoz-Arias, M., and Scherpen, J.M. (2016). Model reduction of a flexible-joint robot: a port-hamiltonian approach. IFAC-PapersOnLine, 49(18), 832 – 837. 10th IFAC Symposium on Nonlinear Control Systems NOLCOS 2016.

Jouffroy, J. and Fossen, I. (2010). A tutorial on incremental stability analysis using contraction theory. Modeling, Identification and control, 31(3), 93–106.

Lohmiller, W. and Slotine, J.J.E. (1998). On contraction analysis for non-linear systems. Automatica.

Loria, A. and Ortega, R. (1995). On tracking control of rigid and flexible joints robots. Appl. Math. Comput. Sci, 5(2), 101–113.

Manchester, I.R., Tang, J.Z., and Slotine, J.J.E. (2015). Unifying classical and optimization-based methods for...
robot tracking control with control contraction metrics. In *International Symposium on Robotics Research (ISRR)*, 1–16.

Nicosia, S. and Tomei, P. (1995). A tracking controller for flexible joint robots using only link position feedback. *IEEE Transactions on Automatic Control*, 40(5).

Ortega, R. and Borja, L.P. (2014). New results on control by interconnection and energy-balancing passivity-based control of port-hamiltonian systems. In *Decision and Control (CDC), 2014 IEEE 53rd Annual Conference on*, 2346–2351. IEEE.

Ortega, R., Perez, J.A.L., Nicklasson, P.J., and Sira-Ramirez, H. (1998). *Passivity-based control of Euler-Lagrange systems*. Springer Science & Business Media.

Ortega, R., Van Der Schaft, A., Castaños, F., and Astolfi, A. (2008). Control by interconnection and standard passivity-based control of port-hamiltonian systems. *IEEE Transactions on Automatic Control*, 53.

Ortega, R., Van Der Schaft, A., Maschke, B., and Escobar, G. (2002). Interconnection and damping assignment passivity-based control of port-controlled hamiltonian systems. *Automatica*, 38(4), 585–596.

Pavlov, A. and van de Wouw, N. (2017). Convergent systems: nonlinear simplicity. In *Nonlinear Systems*, 51–77. Springer.

Reyes-Báez, R., Van der Schaft, A.J., and Jayawardhana, B. (2017a). Differential passivity based control of virtual systems for trajectory tracking of mechanical port-hamiltonian systems. In *preparation*.

Reyes-Báez, R., Van der Schaft, A.J., and Jayawardhana, B. (2017b). Tracking control of fully-actuated port-hamiltonian mechanical systems via sliding manifolds and contraction analysis. In *20th IFAC World Congress*.

Rüffer, B.S., van de Wouw, N., and Mueller, M. (2013). Convergent systems vs. incremental stability. *Systems & Control Letters*, 62(3), 277–285.

Sontag, E.D. (2010). Contractive systems with inputs. In *Perspectives in Mathematical System Theory, Control, and Signal Processing*, 217–228. Springer.

Spong, M.W. (1987). Modeling and control of elastic joint robots. *Journal of dynamic systems, measurement, and control*, 109(4), 310–319.

van der Schaft, A. and Maschke, B. (1995). The hamiltonian formulation of energy conserving physical systems with external ports. *Archiv für Elektronik und bertragungstechnik*, 49.

van der Schaft, A.J. (2013). On differential passivity. *IFAC Proceedings Volumes*, 46(23), 21–25.

Van der Schaft, A.J. (2017). *L2-Gain and Passivity Techniques in Nonlinear Control*. Springer International Publishing.

Wang, W. and Slotine, J.J.E. (2005). On partial contraction analysis for coupled nonlinear oscillators. *Biological cybernetics*, 92(1).

Zhang, Q., Xie, Z., Kui, S., Yang, H., Minghe, J., and Cai, H. (2014). Interconnection and damping assignment passivity-based control for flexible joint robot. In *Intelligent Control and Automation (WCICA), 2014 11th World Congress on*, 4242–4249. IEEE.