ANALYTIC LATTICE COHOMOLOGY OF SURFACE SINGULARITIES, II
(THE EQUIVARIANT CASE)

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ABSTRACT. We construct the equivariant analytic lattice cohomology associated with the analytic type of a complex normal surface singularity whenever the link is a rational homology sphere. It is the categorification of the equivariant geometric genus of the germ. This is the analytic analogue of the topological lattice cohomology, associated with the link of the germ, and indexed by the spin$c$–structures of the link (which is a categorification of the Seiberg–Witten invariant and conjecturally it is isomorphic with the Heegaard Floer cohomology).

1. INTRODUCTION

1.1. Let us fix a complex normal surface singularity $(X, o)$ whose link is a rational homology sphere. In \cite{25,27,28} the (topological) lattice cohomologies and graded roots were introduced (using the combinatorics of the dual graph of any good resolution). Let us recall some of its main properties.

It has a rather different structure than any cohomology theory associated with analytic spaces by complex analytic or algebraic geometry. It has several gradings: first of all, it has a direct sum decomposition according to the spin$c$–structures $\sigma$ of $M$. (Recall that Spin$c(M)$ is an $H_1(M, \mathbb{Z})$ torsor, hence the cardinality of Spin$c(M)$ is the order of $H_1(M, \mathbb{Z})$.) Then each summand $\mathbb{H}_*^{top}(M, \sigma)$ has a decomposition $\bigoplus_{q \geq 0} \mathbb{H}^q(M, \sigma)$, where each $\mathbb{H}_*^{top}(M, \sigma)$ is a $\mathbb{Z}$–graded $\mathbb{Z}[U]$–module. Probably the presence of this additional $U$–action is the most outstanding property compared with the usual cohomology theories.

Conjecturally (see \cite{28}) $\mathbb{H}^*_*^{top}(M)$ is isomorphic to the Heegaard Floer cohomology $HF^+$ of Ozsváth and Szabó (which is defined for any 3–manifold), for $HF$–theory see their long list of article, e.g. \cite{38,39}. This conjecture was verified for several families of plumbed 3–manifolds (associated with negative definite connected graphs), cf. \cite{25,40}, but the general case is still open. (In fact, the Heegaard Floer theory is isomorphic with several other theories: with the Monopole Floer Homology of Kronheimer and Mrowka, or with the Embedded Contact Homology of Hutchings. They are based on different geometrical aspects of the 3–manifold $M$.) $\mathbb{H}^*_*^{top}$ is the categorification of the Seiberg–Witten invariant (similarly as $HF^+$ is). (This means that the Euler characteristic of $\mathbb{H}^*_*^{top}$ is the Seiberg–Witten invariant.) For several properties and application in singularity theory see \cite{25,26,27,30,31}. For its connection with the classification projective rational plane cuspidal curves (via superisolated surface singularities) see \cite{26,6,7,8,9,10}. It provides sharp topological bounds for certain sheaf cohomologies (e.g. for $p_g$), see e.g. \cite{34,35}. An improvement of $\mathbb{H}^0_\text{top}$ is the set of graded roots parametrized by the spin$c$–structures of $M$ \cite{25,27} (they have no analogues for general arbitrary 3–manifolds). The graded root is a special tree with $\mathbb{Z}$–graded vertices, it provides a very visual presentation of $\mathbb{H}^0_\text{top}$ (e.g., the $U$–action is coded in the edges). Hence, in particular it visualizes $HF^+$ too, when the Heegaard Floer homology is known to be isomorphic to $\mathbb{H}^0_\text{top}$ (see e.g. \cite{25}). In such cases the use of graded roots is significantly more convenient than any other method, see e.g. \cite{14,17,18,19}.

2010 Mathematics Subject Classification. Primary. 32S05, 32S25, 32S50, 57M27 Secondary. 14Bxx, 14J80.

Key words and phrases. normal surface singularity, resolution graph, rational homology sphere, Hilbert series, Laufer duality, lattice cohomology, cohomology of line bundles, graded roots, deformation of singularities.

The authors are partially supported by NKFIH Grant “Élvalon (Frontier)” KKP 126683.
1.2. In a series of articles we wish to develop the theory of analytic lattice cohomologies: they are associated with the analytic type of isolated singularities of any dimension, see \([1\, 2\, 3]\).

In \([1]\) we considered the case of a normal surface singularities, when we constructed the analytic lattice cohomology associated with the canonical spin\(^{c}\)–structure. The case of other spin\(^{c}\)–structures (under the assumption that the link is a rational homology sphere) is treated in the present note. For this general part, we need to generalize the constructions of \([1]\) to the level of the universal abelian covering of \((X, o)\) and we also need to use several technical parts regarding ‘natural line bundles’ of a resolution. This motivates that this equivariant discussion is separated in the present note.

The analytic lattice cohomology \(\mathbb{H}^*_an(X, o)\) has a very similar structure as the topological one. It decomposes into a direct sum, where the summands are indexed by the elements of \(H_1(M, \mathbb{Z})\) (hence, equivalently, by \(\text{Spin}^c(M)\)), and each summand is a double graded \(\mathbb{Z}[U]\)–module. The cohomology theory is the categorification of equivariant geometric genus. We also show that it admits a graded \(\mathbb{Z}[U]\)–module morphism \(\mathbb{H}^*_an(X, o) \rightarrow \mathbb{H}^*_top(M)\). We also present a reduction theorem similar to the non-equivariant case (and comparable with the topological case \([20]\)).

1.3. The structure of the article is the following.

In section 2 we recall the general definition of lattice cohomology (and graded root) associated with a weight function. For this construction we need a free module \(\mathbb{Z}'\) (with fixed basis) and a weight function \(w : \mathbb{Z}' \rightarrow \mathbb{Z}\). In both topological and analytical cases the lattice \(\mathbb{Z}'\) is given by \(H_2(\bar{X}, \mathbb{Z})\) of a good resolution \(\bar{X} \rightarrow X\). However, in the topological case, the weight function is determined topologically, and in the analytic case it is analytic: it is the difference of the coefficient of the Hilbert function and the dimension of a sheaf cohomology.

In section 3 we prove combinatorial theorems regarding the Euler characteristic of a lattice cohomology associated with a weight function with certain ‘nice’ properties.

In section 4 we review properties of the topological lattice cohomology.

In section 5 we collected certain needed terminologies, analytic results and constructions (universal abelian covering, equivariant geometric genus, natural line bundles, equivariant multivariable Hilbert series, vanishing and duality theorems, and cohomological cycle associated with a line bundle).

Section 6 contains the definition of the analytic lattice cohomology using a resolution. Here we also prove its independence of the choice of the resolution and we determine its Euler characteristic.

In section 7 we construct a graded \(\mathbb{Z}[U]\)–module morphism \(\mathcal{H}_h^* : \mathbb{H}^*_am, h(X, o) \rightarrow \mathbb{H}^*_top, h(M)\).

In section 8 we review the topological reduction theorem (reduction to a smaller rank lattice associated with the set of ‘bad’ vertices). Section 9 contains the analytic version of this.

2. Preliminaries. Basic properties of lattice cohomology

This is a short review of the lattice cohomology and graded roots associated with a weight function. Though this material was presented in many different articles, still is worth to recall the notations and basic results in order to make the next sections readable. This section is rather similar with section 2 of \([1]\).

2.1. The lattice cohomology associated with a weight function. \([25\, 28]\)

2.1.1. Weight function. We consider a free \(\mathbb{Z}\)-module, with a fixed basis \(\{E_v\}_{v \in \mathcal{V}}\), denoted by \(\mathbb{Z}'\), \(s := |\mathcal{V}|\). Additionally, we consider a weigh function \(w_0 : \mathbb{Z}' \rightarrow \mathbb{Z}\) with the property

\[
(2.1.2) \quad \text{for any integer } n \in \mathbb{Z}, \text{ the set } w_0^{-1}((-\infty, n]) \text{ is finite.}
\]

2.1.3. The weighted cubes. The space \(\mathbb{Z}' \otimes \mathbb{R}\) has a natural cellular decomposition into cubes. The set of zero-dimensional cubes is provided by the lattice points \(\mathbb{Z}'\). Any \(l \in \mathbb{Z}'\) and subset \(I \subset \mathcal{V}\) of cardinality \(q\)
defines a $q$-dimensional cube $\Box_q = (l, I)$, which has its vertices in the lattice points $(l + \sum_{i \in I} E_i) p$, where $I'$ runs over all subsets of $I$. The set of $q$-dimensional cubes is denoted by $\mathcal{D}_q (0 \leq q \leq s)$.

Using $w_0$ we define $w_q : \mathcal{D}_q \to \mathbb{Z}$ (0 $\leq q \leq s$) by $w_q (\Box_q) := \max \{ w_0 (l) : l$ is a vertex of $\Box_q \}$. For each $n \in \mathbb{Z}$ we define $S_n = S_n (w) \subset \mathbb{R}^s$ as the union of all the cubes $\Box_q$ (of any dimension) with $w(\Box_q) \leq n$. Clearly, $S_n = \emptyset$, whenever $n < m_w := \min \{ w_0 \}$. For any $q \geq 0$, set

$$H^q (\mathbb{R}^s, w) := \oplus_{n \geq m_w} H^q (S_n, \mathbb{Z}) \quad \text{and} \quad \tilde{H}^q (\mathbb{R}^s, w) := \oplus_{n \geq m_w} \tilde{H}^q (S_n, \mathbb{Z}).$$

Then $\mathbb{H}^q$ is $\mathbb{Z}$ (in fact, $2\mathbb{Z}$)-graded, the $2n$-homogeneous elements $\mathbb{H}^q_{2n}$ consist of $H^q (S_n, \mathbb{Z})$. Also, $\mathbb{H}^q$ is a $\mathbb{Z}[U]$-module; the $U$-action is given by the restriction map $r_{n+1} : H^q (S_{n+1}, \mathbb{Z}) \to H^q (S_n, \mathbb{Z})$. Namely, $U \ast (\alpha_n) = (r_{n+1} \alpha_{n+1})_n$. The same is true for $\tilde{H}^q_{red}$. Moreover, for $q = 0$, the fixed base-point $l_w \in S_n$ provides an augmentation (splitting) $H^0 (S_n, \mathbb{Z}) = \mathbb{Z} \oplus \tilde{H}^0 (S_n, \mathbb{Z})$, hence an augmentation of the graded $\mathbb{Z}[U]$-modules (where $\mathcal{Z}_{2m} = \mathbb{Z} (U^{-m}, U^{-m-1}, \ldots )$ as a $\mathbb{Z}$-module with its natural $U$-action)

$$\mathbb{H}^q \simeq \mathcal{Z}_m \oplus \tilde{\mathcal{H}}^q (S_n, \mathbb{Z}) \quad \text{and} \quad \tilde{\mathbb{H}}^q \simeq \mathcal{Z}_m \oplus \tilde{\mathbb{H}}^q_{red}.$$

Though $\mathbb{H}^q_{red} (\mathbb{R}^s, w)$ has finite $\mathbb{Z}$-rank in any fixed homogeneous degree, in general, without certain additional properties, is it not finitely generated over $\mathbb{Z}$, in fact, not even over $\mathbb{Z}[U]$.

**2.1.4. Restrictions.** Assume that $T \subset \mathbb{R}^s$ is a subspace of $\mathbb{R}^s$ consisting of a union of some cubes (from $\mathcal{D}_s$). For any $q \geq 0$ define $H^q (T, w)$ as $\oplus_{n \geq \min w_0} H^q (S_n \cap T, \mathbb{Z})$. It has a natural graded $\mathbb{Z}[U]$-module structure.

The restriction map induces a natural graded $\mathbb{Z}[U]$-module homogeneous homomorphism

$$r^q : \mathbb{H}^q (\mathbb{R}^s, w) \to H^q (T, w) \quad \text{(of degree zero)}.$$ 

In our applications to follow, $T$ (besides the trivial $T = \mathbb{R}^s$ case) will be one of the following: (i) the first quadrant $(\mathbb{R}_{\geq 0})^s$, (ii) the rectangle $[0, c] = \{ x \in \mathbb{R}^s : 0 \leq x \leq c \}$ for some lattice point $c \geq 0$, or (iii) a path of composed edges in the lattice, cf. [2.2].

**2.1.5. The ‘Euler characteristic’ of $\mathbb{H}^q$.** Fix $T$ as in [2.1.4] and we will assume that each $\mathbb{H}^q_{red} (T, w)$ has finite $\mathbb{Z}$-rank. The Euler characteristic of $\mathbb{H}^q (T, w)$ is defined as

$$eu (\mathbb{H}^q (T, w)) := - \min \{ w (l) : l \in T \cap \mathbb{Z}^s \} + \sum_q (-1)^q \text{rank}_Z (\tilde{\mathbb{H}}^q_{red} (T, w)).$$

**Lemma 2.1.6.** [29] If $T = [0, c]$ for a lattice point $c \geq 0$, then

$$\sum_{\Box_q \subset T} (-1)^{q+1} w_0 (\Box_q) = eu (\mathbb{H}^q (T, w)).$$

**2.2. Path lattice cohomology.** [28]

**2.2.1.** Fix $\mathbb{Z}$ as in [2.1] and fix also a compatible weight functions $\{ w_q \}_q$ as in [2.1.2]. Consider also a sequence $\gamma := \{ x_i \}_{i=0}^t$ so that $x_0 = 0$, $x_i \neq x_j$ for $i \neq j$, and $x_{i+1} = x_i + E_{i+1} (0 \leq i < t)$. We write $\gamma_T$ for the union of 0-cubes marked by the points $x_i$ and of the segments of type $[x_i, x_{i+1}]$. Then, by [2.1.4] we get a graded $\mathbb{Z}[U]$-module $\mathbb{H}^q (\gamma, w)$, which is called the path lattice cohomology associated with the ‘path’ $\gamma$ and weights $\{ w_q \}_{q=0}^1$. It is denoted by $\mathbb{H}^q (\gamma, w)$. It has an augmentation with $\mathcal{Z}_{2m}^+$, where $m_\gamma := \min \{ w_0 (x_i) \}$, and one gets the reduced path lattice cohomology $\mathbb{H}^0 (\gamma, w)$ with

$$\mathbb{H}^0 (\gamma, w) \simeq \mathcal{Z}_{2m}^+ \oplus \tilde{\mathbb{H}}^0_{red} (\gamma, w).$$

It turns out that $\mathbb{H}^q (\gamma, w) = 0$ for $q \geq 1$, hence its ‘Euler characteristic’ can be defined as (cf. [2.1.5])

$$eu (\mathbb{H}^q (\gamma, w)) := - m_\gamma + \text{rank}_Z (\tilde{\mathbb{H}}^0_{red} (\gamma, w)).$$
Lemma 2.2.3. One has the following expression of $eu(H^i(\gamma, w))$ in terms of the values of $w$:

\begin{equation}
(2.2.4) \quad eu(H^i(\gamma, w)) = -w_0(0) + \sum_{i=0}^{t-1} \max\{0, w_0(x_i) - w_0(x_{i+1})\}.
\end{equation}

2.3. Graded roots and their cohomologies. \cite{25, 27}

Definition 2.3.1. Let $\mathcal{R}$ be an infinite tree with vertices $\mathcal{V}$ and edges $\mathcal{E}$. We denote by $[u, v]$ the edge with end-vertices $u$ and $v$. We say that $\mathcal{R}$ is a graded root with grading $\tau : \mathcal{V} \to \mathbb{Z}$ if

- (a) $\tau(u) - \tau(v) = \pm 1$ for any $[u, v] \in \mathcal{E}$;
- (b) $\tau(u) > \min\{\tau(v), \tau(w)\}$ for any $[u, v], [u, w] \in \mathcal{E}$, $v \neq w$;
- (c) $\tau$ is bounded below, $\tau^{-1}(n)$ is finite for any $n \in \mathbb{Z}$, and $|\tau^{-1}(n)| = 1$ if $n \gg 0$.

An isomorphism of graded roots is a graph isomorphism, which preserves the gradings.

Definition 2.3.2. The $\mathbb{Z}[U]$-modules associated with a graded root. Let us identify a graded root $(\mathcal{R}, \tau)$ with its topological realization provided by vertices ($0$-cubes) and segments ($1$-cubes). Define $w_0(v) = \tau(v)$, and $w_i([u, v]) = \max\{\tau(u), \tau(v)\}$ and let $S_n$ be the union of all cubes with weight $\leq n$. Then we might set (as above) $H^i(\mathcal{R}, \mathcal{X}) = \bigoplus_{n \geq \min\tau} H^i(S_n, \mathbb{Z})$. However, at this time $H^2(\mathcal{R}, \mathcal{X}) = 0$; we set $H(\mathcal{R}, \mathcal{X}) := H^0(\mathcal{R}, \mathcal{X})$.

Similarly, one defines $H_{\text{red}}(\mathcal{R}, \mathcal{X})$ using the reduced cohomology, hence $H(\mathcal{R}, \mathcal{X}) \simeq J^{\min}_{\text{red}} \oplus H_{\text{red}}(\mathcal{R}, \mathcal{X})$.

For a detailed concrete description of $H(\mathcal{R})$ in terms of the combinatorics of the root see \cite{25}.

2.3.3. The graded root associated with a weight function. Fix a free $\mathbb{Z}$-module and a system of weights $\{w_q\}_q$. Consider the sequence of topological spaces (finite cubical complexes) $\{S_n\}_{n \geq m_w}$ with $S_n \subset S_{n+1}$, cf. \ref{2.1.3}

Let $\pi_0(S_n) = \{\mathcal{E}_n^i, \ldots, \mathcal{E}_n^{p_i}\}$ be the set of connected components of $S_n$.

Then we define the graded graph $(\mathcal{R}_w, \tau_w)$ as follows. The vertex set $\mathcal{V}(\mathcal{R}_w) = \bigcup_{n \in \mathbb{Z}} \pi_0(S_n)$. The grading $\tau_w : \mathcal{V}(\mathcal{R}_w) \to \mathbb{Z}$ is $\tau_w(\mathcal{E}_n^i) = n$, that is, $\tau_w|\pi_0(S_n) = n$. Furthermore, if $\mathcal{E}_n^i \subset \mathcal{E}_n^{i+1}$ for some $n, i$ and $j$, then we introduce an edge $[\mathcal{E}_n^i, \mathcal{E}_n^{i+1}]$. All the edges of $\mathcal{R}_w$ are obtained in this way.

Lemma 2.3.4. $(\mathcal{R}_w, \tau_w)$ satisfies all the required properties of the definition of a graded root, except maybe the last one: $|\tau^{-1}_w(n)| = 1$ whenever $n \gg 0$.

The property $|\tau^{-1}_w(n)| = 1$ for $n \gg 0$ is not always satisfied. However, the graded roots associated with connected negative definite plumbing graphs (see below) satisfies this condition as well.

Proposition 2.3.5. If $\mathcal{R}$ is a graded root associated with $(T, w)$ and $|\tau^{-1}_w(n)| = 1$ for all $n \gg 0$ then $H(\mathcal{R}) = H^0(T, w)$.

3. Combinatorial lattice cohomology

3.1. In this section we review several combinatorial statements regarding the lattice cohomology associated with any weight function with certain combinatorial properties. We follow \cite{1}.

3.1.1. Fix $\mathbb{Z}$ with a fixed basis $\{E_c\}_{c \in \mathcal{Y}}$. Write $E_l = \sum_{c \in \mathcal{Y}} E_c$ for $l \subset \mathcal{Y}$ and $E = E_{\mathcal{Y}}$. Fix also an element $c \in \mathbb{Z}$, $c \geq E$. Consider the lattice points $R = R(0, c) := \{l \in \mathbb{Z} : 0 \leq l \leq c\}$, and assume that to each $l \in R$ we assign

- (i) an integer $h(l)$ such that $h(0) = 0$ and $h(l + E) \geq h(l)$ for any $v$,
- (ii) an integer $h^v(l)$ such that $h^v(l + E_c) \leq h^v(l)$ for any $v$.

Once $h$ is fixed with (i), a possible choice for $h^v$ is $h^{\min}(l) = h(c - l)$. Clearly, it depends on $c$.

3.1.2. We say that the $h$-function satisfies the ‘matroid inequality’ if

\begin{equation}
(3.1.3) \quad h(l_1) + h(l_2) \geq h(\min\{l_1, l_2\}) + h(\max\{l_1, l_2\}), \quad l_1, l_2 \in R.
\end{equation}
This implies the ‘stability property’, valid for any \( \bar{t} \geq 0 \) with \( |\bar{t}| \not\in E_r \)

\[
(3.1.4) \quad h(l) = h(l + E_v) \Rightarrow h(l + \bar{t}) = h(l + \bar{t} + E_v).
\]

If \( h \) is given by a filtration (see below) then it automatically satisfies the matroid inequality.

3.1.5. We consider the set of cubes \( \{ \mathcal{Q}_q \}_{q \geq 0} \) of \( R \) as in \[\ref{2.1.1}\] and the weight function

\[
w_0 : \mathcal{Q}_0 \rightarrow \mathbb{Z} \text{ by } w_0(l) = h(l) + h^o(l) - h^o(0).
\]

Clearly \( w_0(0) = 0 \). Furthermore, we define \( w_q : \mathcal{Q}_q \rightarrow \mathbb{Z} \) by \( w_q(\square_q) = \max \{ w_0(l) : l \text{ is a vertex of } \square_q \} \). We will use the symbol \( w \) for the system \( \{ w_q \}_q \). The compatible weight functions define the lattice cohomology \( \mathbb{H}^*(R, w) \). Moreover, for any increasing path \( \gamma \) connecting 0 and \( c \) we also have a path lattice cohomology \( \mathbb{H}^0(\gamma, w) \) as in \[\ref{2.2.1}\]. Accordingly, we have the numerical Euler characteristics \( eu(\mathbb{H}^*(R, w)), eu(\mathbb{H}^0(\gamma, w)) \) and \( \min_{\gamma} eu(\mathbb{H}^0(\gamma, w)) \) too.

Lemma 3.1.6. \[\ref{1}\] We have \( 0 \leq eu(\mathbb{H}^0(\gamma, w)) \leq h^o(0) - h^o(c) \) for any increasing path \( \gamma \) connecting 0 to \( c \). The equality \( eu(\mathbb{H}^0(\gamma, w)) = h^o(0) - h^o(c) \) holds if and only if for any \( i \) the differences \( h(x_{i+1}) - h(x_i) \) and \( h^o(x_i) - h^o(x_{i+1}) \) simultaneously are not nonzero.

Definition 3.1.7. Fix \( (h, h^o, R) \) as in \[\ref{3.1.1}\]. We say that the pair \( h \) and \( h^o \) satisfy the ‘Combinatorial Duality Property’ (CDP) if \( h(l + E_v) - h(l) \) and \( h^o(l + E_v) - h^o(l) \) simultaneously cannot be nonzero for \( l, l + E_v \in R \). Furthermore, we say that \( h \) satisfies the CDP if the pair \( (h, h^{sym}) \) satisfies it.

Definition 3.1.8. We say that the pair \( (h, h^o) \) satisfy the

(a) ‘path eu-coincidence’ if \( eu(\mathbb{H}^0(\gamma, w)) = h^o(0) - h^o(c) \) for any increasing path \( \gamma \).

(b) ‘eu-coincidence’ if \( eu(\mathbb{H}^*(R, w)) = h^o(0) - h^o(c) \).

Remark 3.1.9. Example 4.3.3 of \[\ref{1}\] shows the following two facts.

Even if \( h \) satisfies the path eu-coincidence (and \( h^o = h^{sym} \)), in general it is not true that \( \mathbb{H}^0(\gamma, w) \) is independent of the choice of the increasing path. (This statement remains valid even if we consider only the symmetric increasing paths, where a path \( \gamma = \{ x_t \}_{t=0}^l \) is symmetric if \( x_{t-1} = c - x_t \) for any \( t \).

Even if \( h \) satisfies both the path eu-coincidence and the eu-coincidence, in general it is not true that \( \mathbb{H}^*(R, w) \) equals any of the path lattice cohomologies \( \mathbb{H}^0(\gamma, w) \) associated with a certain increasing path. (E.g., in the mentioned Example 4.3.3 we have \( \mathbb{H}^1(R, w) \neq 0 \), a fact which does not hold for any path lattice cohomology.) However, amazingly, all the Euler characteristics agree.

Theorem 3.1.10. Assume that \( h \) satisfies the stability property, and the pair \( (h, h^o) \) satisfies the Combinatorial Duality Property. Then the following facts hold.

(a) \( (h, h^o) \) satisfies both the path eu- and the eu-coincidence properties: for any increasing \( \gamma \) we have

\[
eu(\mathbb{H}^0(\gamma, w)) = eu(\mathbb{H}^*(R, w)) = h^o(0) - h^o(c).
\]

(b) \[
\sum_{l \geq 0} \sum_{I} (-1)^{|I|+1} w((l, I)) t^I = \sum_{l \geq 0} \sum_{I} (-1)^{|I|+1} h(l + E_I) t^I.
\]

4. Surface singularities and the topological lattice cohomology

4.1. The combinatorics of a resolution. \[\ref{24} \[\ref{25} \[\ref{27}\]

4.1.1. Let \((X, o)\) be the germ of a complex analytic normal surface singularity with link \( M \). Let \( \phi : \tilde{X} \rightarrow X \) be a good resolution of \((X, o)\) with exceptional curve \( E := \phi^{-1}(0) \), and let \( \cup_{v \in \mathcal{E}} E_v \) be the irreducible decomposition of \( E \). Let \( \Gamma \) be the dual resolution graph of \( \phi \). Note that \( \partial \tilde{X} \simeq M \).
The lattice \( L := H_2(\widetilde{X}, \mathbb{Z}) \) is endowed with the natural negative definite intersection form \( (\cdot, \cdot) \). It is freely generated by the classes of \( \{E_v\}_{v \in \mathcal{Y}} \). The dual lattice is \( L' = \text{Hom}_\mathbb{Z}(L, \mathbb{Z}) \simeq \{l' \in L \otimes \mathbb{Q} : (l', L) \in \mathbb{Z}\} \). It is generated by the (anti)dual classes \( \{E_v^*\}_{v \in \mathcal{Y}} \) defined by \((E_v^*, E_w) = -\delta_{vw}\) (where \( \delta_{vw} \) stays for the Kronecker symbol). \( L' \) is also identified with \( H^2(\widetilde{X}, \mathbb{Z}) \).

We define the Lipman cone as \( \mathcal{L}' := \{l' \in L' : (l', E_v) \leq 0 \text{ for all } v\} \), and we also set \( \mathcal{L} := \mathcal{L}' \cap L \). If \( s' \in \mathcal{L}' \setminus \{0\} \) then all its \( E_v \)-coordinates are strict positive.

The intersection form embeds \( L \) into \( L' \) with \( L'/L \simeq \text{Tors}(H_1(M, \mathbb{Z})) \), which is abridged by \( H \). The class of \( l' \) in \( H \) is denoted by \( [l'] \).

There is a natural partial ordering of \( L' \) and \( L \): we write \( l'_1 \geq l'_2 \) if \( l'_1 - l'_2 = \sum v_r E_v \) with every \( r_v \geq 0 \). We set \( L_{\geq 0} = \{l \in L : l \geq 0\} \) and \( L_{>0} = L_{\geq 0} \setminus \{0\} \). The support of a cycle \( l = \sum n_v E_v \) is defined as \( |l| = \bigcup_v n_v \neq 0 E_v \).

The (anti)canonical cycle \( Z_K \in L' \) is defined by the adjunction formulae \( (Z_K, E_v) = (E_v, E_v) + 2 - 2g_v \) for all \( v \in \mathcal{Y} \), where \( g_v \) denotes the genus of \( E_v \). The cycle \(-Z_K \) is the first Chern class of the line bundle \( \Omega^2_{\widetilde{X}} \).

We write \( \chi : L' \to \mathbb{Q} \) for the (Riemann–Roch) expression \( \chi(l') = -(l', l' - Z_K)/2 \).

If \( H_1(M, \mathbb{Q}) = 0 \) then each \( E_v \) is rational, and the dual graph of any good resolution is a tree. In this case \( H_1(M, \mathbb{Z}) = H \) is finite. In this case we denote the Pontrjagin dual Hom\((H, S^1)\) of \( H \) by \( \tilde{H} \). Let \( \theta : H \to \tilde{H} \) be the isomorphism \( [l'] \mapsto e^{2\pi i(l', \cdot)} \) of \( H \) with \( \tilde{H} \).

**Definition 4.1.2.** The set of characteristic elements are defined as

\[
\text{Char} = \text{Char}(L) = \{k \in L' : (l, l + k) \in 2\mathbb{Z} \text{ for any } l \in L\}.
\]

Note that \(-Z_K \in \text{Char} \) and \( \text{Char} = -Z_K + 2L' \) (and \( \text{Char} \) is an \( L' \) torsor by the action \( l' \ast k = k + 2l' \)). The RR-expression \( \chi \) has an analogue for any \( k \in \text{Char} \), namely one defines \( \chi_k : L \to \mathbb{Z} \) by \( \chi_k(l) := -(l, l + k)/2 \).

**4.1.4. Canonical representatives and spinc-structures.** For any \( h \in H \) there exists a unique element \( r_h = \sum r_v E_v \in L' \) with \( [r_h] = h \) such that each \( r_v \in [0, 1) \). Similarly, for any \( h \in H \) there is a unique minimal element of \( \{l' \in L' : [l'] = h\} \cap \mathcal{L}' \). It will be denoted by \( s_h \). For \( h = 0 \) we have \( r_h = s_h = 0 \). One has \( s_h \geq r_h \); in general, \( s_h \neq r_h \).

Assume that the link is a rational homology sphere. Then \( \text{Spin}^c(\widetilde{X}) \), the set of spin\(^c\)-structures on \( \widetilde{X} \), is identified with the set of characteristic elements on \( L' \). Moreover, any spin\(^c\)-structure on \( \partial \widetilde{X} = M \) is the restriction of a spin\(^c\)-structure of \( \widetilde{X} \) and if \( k \) and \( k' \) induces the same spin\(^c\)-structure on the link then \( k' = k + 2l \) for a certain \( l \in L \). This is an equivalence relation on \( \text{Char} \), the classes are denoted by \([k] \). If \( k' = k + 2l \) for some \( l \in L \) then \( \chi_k(x - l) = \chi_k(x) - \chi_k(l) \) for any \( x \in L \), hence the two functions \( \chi_k \) and \( \chi_{k'} \) can be easily compared, and they have identical qualitative properties. Therefore, for each class \([k] \) \( k = k + 2L \) (that is, for each spin\(^c\)-structure \( \sigma[k] \) of \( M \)), we might choose a representative of \([k] \). Since the set of classes is indexed by \( H \); we define the set of representatives by \( k_h := -Z_K + 2s_h \), for each \( h \in H \). Since \( s_0 = 0 \), for the trivial class \( h = 0 \) we get \( \chi_{k_0} = \chi \). (This choice will produce several pleasant consequences, e.g. 4.1.2.4.)

**4.2. The topological lattice cohomology associated with \( \phi : \widetilde{X} \to X \).** \( \{25, 28\} \)

**4.2.1.** We consider a good resolution \( \phi \) as above and we assume that the link \( M \) is a rational homology sphere. We write \( s := |\mathcal{Y}| \). We also fix a characteristic element \( k \in \text{Char} \).

Then we automatically have a free \( \mathbb{Z} \)-module \( L = \mathbb{Z}^s \) with a fixed bases \( \{E_v\}_v \), and \( k \) defines a set of compatible weight functions \( w \) by \( w_k(\square_q) = \max \{\chi_k(v) : v \text{ is a vertex of } \square_q\} \).

**Definition 4.2.2.** The \( \mathbb{Z}[U] \)-modules \( \text{H}^*(\mathbb{R}^s, w) \) and \( \text{H}_r^*(\mathbb{R}^s, w) \) obtained by these weight functions are called the **lattice cohomologies** associated with the pair \( (\phi, k) \) and are denoted by \( \text{H}^*(\Gamma, k) \), respectively \( \text{H}_r^*(\Gamma, k) \).

The graded root associated with \((\mathbb{Z}^s, w_k)\) will be denoted by \( \mathfrak{R}(\Gamma, k) \).
Proposition 4.2.3. \cite{25,27,20}

(a) \( H_\text{red}^*(\Gamma, k) \) is finitely generated over \( \mathbb{Z} \).

(b) The set \( H^\ast (\Gamma, k_r) \) (indexed by the spin\(^r\)-structures of \( M \)) depends only on \( M \) and is independent of the choice of the good resolution \( \phi \). They are called the topological lattice cohomologies of the singularity \( (X, o) \), or of the link \( M \). In the sequel we might also refer to is as \( H^\ast (M, k_r) \).

(c) The restriction \( H^\ast (\Gamma, k_r) \to H^\ast ((\mathbb{R}_{\geq 0})^s, k_r) \) induced by the inclusion \( (\mathbb{R}_{\geq 0})^s \to \mathbb{R}^s \) is an isomorphism of graded \( \mathbb{Z} [U] \) modules.

There are similar statements for \( \mathcal{B}(\Gamma, k_r) \) instead of \( \mathbb{H}^\ast (\Gamma, k_r) \), which will also be denoted by \( \mathcal{B}(M, k_r) \).

4.2.4. The Euler characteristic and the Seiberg–Witten invariant. The Seiberg–Witten invariant \( \text{Spin}^c(M) \to \mathbb{Q} \) associates a rational number \( s_{\text{w}}(M) \) to each spin\(^c\)-structure \( \sigma \) of the link. Recall also that \( \text{Spin}^c(M) \) is an \( H \)-torsor, and it can be parametrized by the classes \( [k] \in \text{Char}/2L \), or by the representatives \( \{k_r\} \).

Theorem 4.2.5. \cite{29} Let \( \sigma[k_r] \) be the spin\(^c\)-structure associated with \( k_r \). Then\footnote{In other words, the topological lattice cohomology is the categorification of the Seiberg–Witten invariant (normalized by \( (k_r^2 + |\gamma'|)/8 \)).}

\[ eu(H^\ast(M, k_r)) = s_{\text{w}}(\sigma[k_r]) \cdot (M) - \frac{k_r^2 + |\gamma'|}{8}. \]

Remark 4.2.6. Consider the topological lattice cohomologies associated with characteristic elements \( -Z_k + 2r_h \) and \( -Z_k + 2s_h \), and with cubes from \( \mathbb{R}^s \) and \( \mathbb{R}^s_{\geq 0} \). We claim that there also exists a graded \( \mathbb{Z} [U] \)-module isomorphism (the analogue of Proposition 4.2.3(c)):

\[ H^\ast(\mathbb{R}^s, -Z_k + 2r_h) \simeq H^\ast(\mathbb{R}^s_{\geq 0}, -Z_k + 2r_h). \]

Indeed, write \( s_h = r_h + \Delta_h \) for some \( \Delta_h \in L_{\geq 0} \). Then, for any \( l \in L_{\geq 0} \)

\[ \chi_{-Z_k + 2s_h}(l - \Delta_h) = \chi_{-Z_k + 2r_h}(l) - \chi_{-Z_k + 2r_h}(\Delta_h). \]

Therefore, up to a shift \( \chi_{-Z_k + 2r_h}(\Delta_h) \), we have the isomorphisms

\[ H^\ast(\mathbb{R}^s, -Z_k + 2r_h) \simeq H^\ast(\mathbb{R}^s, -Z_k + 2s_h), \quad \text{and } H^\ast(\mathbb{R}^s_{\geq 0}, -Z_k + 2r_h) \simeq H^\ast(\mathbb{R}^s_{\geq 0}, -Z_k + 2s_h). \]

But the contraction which realizes Proposition 4.2.3(c) (which contracts \( \mathbb{R}^s \) onto \( \mathbb{R}^s_{\geq 0} \) compatible with the weights) restricted to \( \mathbb{R}^s_{\geq 0} - \Delta_h \), realizes an isomorphism (cf. \cite{20})

\[ H^\ast(\mathbb{R}^s_{\geq 0} - \Delta_h, -Z_k + 2s_h) \simeq H^\ast(\mathbb{R}^s_{\geq 0}, -Z_k + 2s_h). \]

Therefore \[ H^\ast(\mathbb{R}^s_{\geq 0} - \Delta_h, -Z_k + 2s_h) \simeq H^\ast(\mathbb{R}^s_{\geq 0}, -Z_k + 2s_h). \]

Then use these identities together with Proposition 4.2.3(c).

5. Preliminaries regarding analytic invariants

5.1. Natural line bundles. Fix a complex normal surface singularity \( (X, o) \) and in subsections 5.1 and 5.2 we assume that the link is a rational homology sphere.

By duality, \( L' \) is isomorphic to \( H^2(\tilde{X}, \mathbb{Z}) \) and it is the target of the first Chern class \( c_1 : \text{Pic}(\tilde{X}) \to H^2(\tilde{X}, \mathbb{Z}) \). This morphism appears in the exact sequence (induced by the exponential exact sequence of sheaves):

\[ 0 \to H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) \to \text{Pic}(\tilde{X}) \xrightarrow{c_1} H^2(\tilde{X}, \mathbb{Z}) \to 0. \]

In this exact sequence \( c_1 \) admits a natural group section \( s_L \) over the integral cycles \( L \subset L' \). Namely, for any \( l \in L \) one takes \( \mathcal{O}_{\tilde{X}}(l) \in \text{Pic}(\tilde{X}) \) with \( c_1(\mathcal{O}(l)) = l \). By \cite{27} \( s_L \) can be extended in a unique way to a natural group section \( s : L' \to \text{Pic}(\tilde{X}) \). Its existence basically is guaranteed by the facts that \( H = L'/L \) is finite, while \( \text{Pic}^0(\tilde{X}) := H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) \) is torsion free.
Definition 5.1.2. The line bundles $s(l')$, indexed by $l' \in L'$, and denoted also by $\mathcal{O}_X(l') := s(l')$, will be called natural line bundles.

In fact, a line bundle $\mathcal{L} \in \text{Pic}(\tilde{X})$ is natural if and only if some power of it has the form $\mathcal{O}_X(l)$ for an integral cycle $l \in L$.

5.1.3. The universal abelian covering. Let $c : (X_a, o) \to (X, o)$ be the universal abelian covering of $(X, o)$: $(X_a, o)$ is the unique normal singular germ such that $X_a \setminus \{o\}$ is the regular covering of $X \setminus \{o\}$ associated with $\pi_1(X \setminus \{o\}) \to H$.

Since $\tilde{X} \setminus E \approx X \setminus \{o\}$, $\pi_1(\tilde{X} \setminus E) = \pi_1(X \setminus \{o\}) \to H$ defines a regular Galois covering of $\tilde{X} \setminus E$ as well. This has a unique extension $\tilde{c} : Z \to \tilde{X}$ with $Z$ normal and $\tilde{c}$ finite. (In other words, $\tilde{c} : Z \to \tilde{X}$ is the normalized pullback of $c$ via $\phi$.) The (reduced) branch locus of $\tilde{c}$ is included in $E$, and the Galois action of $H$ extends to $Z$ as well. Since $E$ is a normal crossing divisor, the only singularities what $Z$ might have are cyclic quotient singularities. Let $r : \tilde{Z} \to Z$ be a resolution of these singular points such that $(\tilde{c} \circ r)^{-1}(E)$ is a normal crossing divisor. Set $p := \tilde{c} \circ r$.

$$\begin{align*}
\tilde{Z} & \xrightarrow{r} Z \\
\downarrow \tilde{c} & \downarrow c \\
(\tilde{X}, E) & \xrightarrow{\phi} (X, o)
\end{align*}$$

(5.1.4)

Theorem 5.1.5. [27] [36] [37] $\tilde{c}_* \mathcal{O}_Z$ is a vector bundle and its $H$-eigensheaf decomposition has the form:

$$\tilde{c}_* \mathcal{O}_Z \simeq \bigoplus_{a \in H} \mathcal{L}_a,$$

(5.1.6)

where $\mathcal{L}_h = \mathcal{O}_X(-r_h)$ for any $h \in H$. In particular, $\tilde{c}_* \mathcal{O}_Z \simeq \bigoplus_{l' \in \mathbb{Q}} \mathcal{O}_X(-l')$.

More generally, for any $l' \in L'$ one has

$$\tilde{c}_* \mathcal{O}_Z(-\tilde{c}^*(l')) \simeq \bigoplus_{h \in H} \mathcal{O}_X(-r_h + \lfloor r_h - l' \rfloor).$$

(5.1.7)

5.1.8. The geometric genus of the universal abelian covering. In general (even if $H_1(M, \mathbb{Q}) \neq 0$), the geometric genus of $(X, o)$ is defined as $p_g(X, o) = h^1(\tilde{X}, \mathcal{O}_\tilde{X})$. It is independent of the resolution.

Assume that the link of $(X, o)$ is a rational homology sphere. In this situation one defines the equivariant geometric genera (indexed by $H$) as follows.

Let $(X_a, o) \to (X, o)$ be the universal abelian covering of $(X, o)$, and consider the notations of the diagram (5.1.4). By definition, the geometric genus $p_g(X_a, o)$ of $(X_a, o)$ is $h^1(\tilde{Z}, \mathcal{O}_\tilde{Z})$. Since $r : \tilde{Z} \to Z$ is the resolution of the cyclic quotient singularities of $Z$, we have $p_g(X_a, o) = h^1(\mathcal{O}_Z)$. Since $\tilde{c}$ is finite $h^1(\mathcal{O}_Z)$ equals $\dim(R^1\tilde{c}_* \mathcal{O}_Z)_o$, and it has an eigenspace decomposition $\bigoplus_{h \in H}(R^1\tilde{c}_* \mathcal{O}_Z)_o \theta(h)$. By Theorem 5.1.5 the dimension of the $\theta(h)$-eigenspace is

$$p_g(X_a, o)_{\theta(h)} := \dim(R^1\tilde{c}_* \mathcal{O}_Z)_o, \theta(h) = h^1(\tilde{X}, \mathcal{O}_\tilde{X}(-r_h)).$$

By summation:

$$p_g(X_a, o) = \sum_{h \in H} h^1(\tilde{X}, \mathcal{O}_\tilde{X}(-r_h)).$$

Clearly, for $h = 0$ we get $p_g(X_a, o)_{\theta(0)} = p_g(X, o)$.

Definition 5.1.9. If $H_1(M, \mathbb{Q}) = 0$ we define the equivariant geometric genus of $(X, o)$ associated with $h \in H$ by $p_g(X, o)_{\theta(h)} = h^1(\tilde{X}, \mathcal{O}_\tilde{X}(-r_h))$. Sometimes we abridge it by $p_{g,h} = p_{g,h}(X, o)$. 
5.2. Multivariable filtrations and series. Notations. \[11\] \[12\] \[29\]

5.2.1. The module $\mathbb{Z}[[L']]$. Once a resolution is fixed, hence the natural basis $\{E_i\}$, of $L$ is fixed too, $\mathbb{Z}[[L]]$ is identified with $\mathbb{Z}[t^\pm 1] = \mathbb{Z}[t_1^\pm, \ldots, t_s^\pm]$. It is contained in the larger module $\mathbb{Z}[t_1^\pm/d, \ldots, t_s^\pm/d]$, the module of formal (Laurent) power series in variables $t_i^{\pm/d}$, where $d := |H|$. $\mathbb{Z}[[L']]$ embeds into $\mathbb{Z}[[t^{\pm/d}]]$ as a submodule: it consists of the $\mathbb{Z}$-linear combinations of monomials of type

$$t^{l'} = t_1^{l'_1} \cdots t_s^{l'_s}, \text{ where } l' = \sum r_i E_i \in L'.$$

**Definition 5.2.2.** Any series $S(t) = \sum_{h \in H} S_h$ decomposes in a unique way as

$$S = \sum_{h \in H} S_h, \text{ where } S_h = \sum_{l'|h} a_{l'} t^{l'}.$$

$S_h$ is called the $h$-component of $S$.

5.2.4. Consider the diagram from (5.1.4) and set $\phi_a = \psi_a \circ r$ and $p = \tilde{c} \circ r$. One verifies that $p^*(l')$ is an integral cycle for any $l' \in L'$.

**Definition 5.2.5.** The $L'$-filtration on the local ring of holomorphic functions $\mathcal{O}_{X_a,o}$ is defined as follows. For any $l' \in L'$, we set

$$\mathcal{F}(l') := \{ f \in \mathcal{O}_{X_a,o} \mid \text{div}(f \circ \phi_a) \geq p^*(l') \}.$$

Notice that the natural action of $H$ on $(X_a,o)$ induces an action on $\mathcal{O}_{X_a,o}$, which keeps $\mathcal{F}(l')$ invariant. Therefore, $H$ acts on $\mathcal{O}_{X_a,o}/\mathcal{F}(l')$ as well. For any $l' \in L'$, let $b(l')$ be the dimension of the $\theta([l'])$-eigenspace $(\mathcal{O}_{X_a,o}/\mathcal{F}(l'))_{\theta([l'])}$. Then one defines the Hilbert series $H(t)$ by

$$H(t) := \sum_{l' \in L'} b(l') \cdot t^{l'} \in \mathbb{Z}[[L']].$$

By (5.2.5), for any $l' \in L'$ there exists a unique minimal $s(l') \in \mathcal{F}$ such that $l' \leq s(l')$ and $[l'] = [s(l')]$. Since for any $f \in \mathcal{O}_{X_a,o}$, that part of $\text{div}(f \circ \phi_a)$, which is supported by the exceptional divisor of $\phi_0$, is in the Lipman’s cone of $\tilde{Z}$, we get

$$\mathcal{F}(l') = \mathcal{F}(s(l')).$$

5.2.9. For a fixed $l'$ we write $[l'] = h$. If $l' > 0$ one has the exact sequence

$$0 \to \mathcal{O}_{\tilde{X}}(-p^*(l')) \to \mathcal{O}_{\tilde{X}} \to \mathcal{O}_{p^*(l')} \to 0.$$

The $\theta(h)$-eigenspaces form the exact sequence, cf. (5.1.7).

$$0 \to \mathcal{O}_{\tilde{X}}(-l') = \mathcal{O}_{\tilde{X}}(-r_h) \to \mathcal{O}_{l'-r_h}(-r_h) \to 0.$$

In particular, for $l' > 0$,

$$\tilde{b}(l') = \dim \left( \frac{H^0(\tilde{X},\mathcal{O}_{\tilde{X}})}{H^0(\tilde{X},\mathcal{O}_{\tilde{X}}(-p^*(l'))) \cdot \theta(h)} \right) = \dim \frac{H^0(\tilde{X},\mathcal{O}_{\tilde{X}}(-r_h))}{H^0(\tilde{X},\mathcal{O}_{\tilde{X}}(-l'))}.$$

**Example 5.2.13.** In (5.2.12) if $l' \in L$ then $r_h = 0$. Hence the $0$-component of $H(t)$ is

$$H_0(t) = \sum_{l' \in L} \dim \left( \frac{\mathcal{O}_{X_a,o}}{\{ f \in \mathcal{O}_{X_a,o} : \text{div}(f \circ \phi) \geq l \}} \right) \cdot t^{l'}.$$

This is the Hilbert series of $\mathcal{O}_{X_a,o}$ associated with the divisorial filtration $L \ni l \mapsto \mathcal{F}(l) = \{ f \in \mathcal{O}_{X_a,o} : \text{div}(f \circ \phi) \geq l \}$ of all irreducible exceptional divisors of $\phi$. 

Analytic lattice cohomology

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Proposition 5.4.1. Let $\phi$ be a normal surface singularity (without any restriction regarding its link) and we fix a good resolution $\phi$. Let $K_X$ be a canonical divisor on $X$, that is, $\Omega_X^1 \cong \mathcal{O}_X(K_X)$.

5.3. Vanishing theorems, dualities. Let $(X, o)$ be a normal surface singularity (without any restriction regarding its link) and we fix a good resolution $\phi$. Let $K_X$ be a canonical divisor on $X$, that is, $\Omega_X^1 \cong \mathcal{O}_X(K_X)$.

Theorem 5.3.1. Generalized Grauert–Riemenschneider Theorem. [16][21][41][33] Consider a line bundle $\mathcal{L} \in \text{Pic}(\tilde{X})$ such that $c_1(\mathcal{L}(Z_K)) \in \Delta - \mathcal{Q}$ for some $\Delta \in \mathcal{L}'$ with $|\Delta| = 0$. Then $h^1(l, \mathcal{L}|) = 0$ for any $l \in L_{>0}$. In particular, $h^1(\tilde{X}, \mathcal{L}) = 0$ too. (Here $\mathcal{Q}$ denotes the rational cone generated by $\mathcal{L}$.)

In particular, if $\mathcal{L} \in \text{Pic}(\tilde{X})$ and $l \in L_{>0}$ satisfies $l \in c_1(\mathcal{L}) + Z_K + \mathcal{L}$, then $H^1(\tilde{X}, \mathcal{L}) = H^1(l, \mathcal{L}|)$. As above, we denote by $\{Z_K\}$ the integral part of $Z_K$, and by $\{Z_K\}_+$ its effective part. The above statements imply the following. If $|Z_K|_+ = 0$ then $p_g = 0$. If $|Z_K|_+ > 0$ then for any $Z \geq |Z_K|_+, Z \in L$, $p_g = h^1(\mathcal{O}_Z)$.

Furthermore, if $l \in \mathcal{L}$ and $n \in \mathbb{Z}_{>0}$ such that $nl + |Z_K| > 0$ then by the above vanishing theorem we have $H^1(\tilde{X}, \mathcal{O}_X(-|Z_K| - nl - s_h)) = 0$, hence

\[
\dim \frac{H^0(\mathcal{O}_X(-s_h))}{H^0(\mathcal{O}_X(-|Z_K| - nl - s_h))} = \chi(|Z_K| + nl) - (s_h, |Z_K| + nl) + h^1(\mathcal{O}_X(-s_h)).
\]

This implies that for any $l \in \mathcal{L} \setminus \{0\}$ and $n > 0$, and $l_h'$ either $r_h$ or $s_h$ we have

\[
\dim \frac{H^0(\mathcal{O}_X(-l_h'))}{H^0(\mathcal{O}_X(-nl - l_h'))} = -\frac{n^2l^2}{2} + \text{lower order terms in } n.
\]

For certain cycles the Grauert-Riemenschneider Theorem [5.3.1] can be improved.

Proposition 5.3.3. Lipman’s Vanishing Theorem. [23] Theorem 11.1, [33] Take $l \in L_{>0}$ with $h^1(\mathcal{L}_i) = 0$ and $\mathcal{L} \in \text{Pic}(\tilde{X})$ for which $(c_1(\mathcal{L}, E_v) \geq 0$ for any $E_v$ in the support of $l$. Then $h^1(l, \mathcal{L}) = 0$.

5.3.4. By Serre duality $H^1(l, \mathcal{L}) = H^1(-l, \mathcal{L}^{-1}(K_X + l))^*$ for any $l \in L_{>0}$, $\mathcal{L} \in \text{Pic}(\tilde{X})$ and $i = 0, 1$.

5.3.5. Laufer’s Duality. [21], [22], p. 1281] We can identify the dual space $H^1(\tilde{X}, \mathcal{O}_X)^*$ with the space of global holomorphic 2-forms on $X \setminus E$ up to the subspace of those forms which can be extended holomorphically over $\tilde{X}$: $H^1(\tilde{X}, \mathcal{O}_X)^* \simeq H^0(\tilde{X} \setminus E, \Omega^2_X)/(H^0(\tilde{X}, \Omega^2_X)^*)$. Here $H^0(\tilde{X} \setminus E, \Omega^2_X)$ can be replaced by $H^0(\tilde{X}, \Omega^2_X(Z))$ for any $Z > 0$ with $h^1(\mathcal{O}_Z) = p_g$. Indeed, for any $Z > 0$, from the exact sequence of sheaves $0 \rightarrow \Omega^2_X \rightarrow \Omega^2_X(Z) \rightarrow \mathcal{O}_Z(Z + K_X) \rightarrow 0$ and from the vanishing $h^1(\Omega^2_X) = 0$ and Serre duality

\[
H^0(\Omega^2_X(Z))/H^0(\Omega^2_X) = H^0(\mathcal{O}_Z(Z + K_X)) \simeq H^1(\mathcal{O}_Z)^*.
\]

If $H^1(\mathcal{O}_Z) \simeq H^1(\mathcal{O}_X)$ then the inclusion $H^0(\Omega^2_X(Z))/H^0(\Omega^2_X) \rightarrow H^0(\mathcal{O}_Z(Z + K_X))/H^0(\Omega^2_X)$ is an isomorphism.

5.4. Cohomological cycles. [42], 4.8] Assume that $p_g > 0$. The set $L_{p_g} := \{l \in L_{>0} : h^1(\mathcal{L}_i) = p_g\}$ has a unique minimal element, denoted by $Z_{\text{coh}}$, and called the cohomological cycle of $\phi$. It has the property that $h^1(\mathcal{L}_i) < p_g$ for any $i \geq Z_{\text{coh}} (l > 0)$. By the consequences of Theorem 5.3.1 we obtain that $Z_{\text{coh}} \leq |Z_K|_+$. If $p_g = 0$ then we set $Z_{\text{coh}} := 0$ by definition. More generally, we have the following results.

Proposition 5.4.1. Fix a line bundle $\mathcal{L} \in \text{Pic}(\tilde{X})$.

(a) Assume that $h^1(\tilde{X}, \mathcal{L}) > 0$. The set $L_{\mathcal{L}} := \{l \in L_{>0} : h^1(\mathcal{L}) = h^1(\tilde{X}, \mathcal{L})\}$ has a unique minimal element, denoted by $Z_{\text{coh}}(\mathcal{L})$, called the cohomological cycle of $\mathcal{L}$ (and of $\phi$). It has the property that $h^1(\mathcal{L}) < h^1(\tilde{X}, \mathcal{L})$ for any $l \geq Z_{\text{coh}}(\mathcal{L}) (l > 0)$. 

(b) Let \( l_1, l_2 \in L_{>0} \) be effective cycles, and set \( l = \min \{ l_1, l_2 \} \) and \( \overline{l} = \max \{ l_1, l_2 \} \). Then

\[
h^1(\mathcal{I}, \mathcal{L}) + h^1(l, \mathcal{L}) \geq h^1(l_1, \mathcal{L}) + h^1(l_2, \mathcal{L}).
\]

We will refer to this inequality as the ‘opposite’ matroid rank inequality of \( h^1(\mathcal{L}) \).

(c) In particular, for any \( l \in L_{>0} \) we have \( h^1(l, \mathcal{L}) = h^1(\min \{ l, Z_{\text{coh}}(\mathcal{L}) \}, \mathcal{L}) \).

**Proof.** In (b) we can assume that \( a_i = l_i - l > 0 \), \( i = 1, 2 \). Consider the diagram with exact rows and columns.

\[
\begin{array}{ccc}
H^1(\mathcal{L}(l_1)_{a_2}) & \rightarrow & H^1(\mathcal{L}(l)_{a_2}) \\
\downarrow & & \downarrow \\
H^1(\mathcal{L}(l_2)_{a_1}) & \rightarrow & H^1(\mathcal{L}(l)_1) & \rightarrow & H^1(\mathcal{L}(l)_2) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \\
H^1(\mathcal{L}(l_1)_{a_1}) & \rightarrow & H^1(\mathcal{L}(l)_1) & \rightarrow & H^1(\mathcal{L}(l)_1) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0 & & 0 \\
\end{array}
\]

The exactness of the first row follows from the exact sequence \( \mathcal{L}(l_1-a_1)_{a_2} \rightarrow \mathcal{L}(l)_{a_2} \rightarrow \mathcal{L}(l_1)_{a_1} \rightarrow 0 \), where the support of \( \mathcal{L}(l_1)_{a_2} \) is 0–dimensional. From the diagram one gets that

\[
H^1(\mathcal{L}(l)_1) \rightarrow H^1(\mathcal{L}(l)_1) \oplus H^1(\mathcal{L}(l)_2) \rightarrow H^1(\mathcal{L}(l)_1) \rightarrow 0
\]

is exact, hence (b) follows.

Assume that \( h^1(\mathcal{L}(l)_1) = h^1(\mathcal{L}(l)_2) = h^1(\mathcal{L}) \) for \( l_1 \neq l_2, l_1, l_2 \in L_{>0} \). Set \( l = \min \{ l_1, l_2 \} \). If \( l = 0 \) then there is an exact sequence \( 0 \rightarrow \mathcal{L}(l_1)_1 \rightarrow \mathcal{L}(l_1)_1 \oplus \mathcal{L}(l)_2 \rightarrow A \rightarrow 0 \), where \( A \) has zero–dimensional support, hence \( H^1(\mathcal{L}(l_1)_1) \rightarrow H^1(\mathcal{L}(l)_1) \oplus H^1(\mathcal{L}(l)_2) = C^{\dim(\mathcal{L})} \) surjective, a fact which cannot happen. Hence \( l \neq 0 \). Then (5.4.2) implies \( H^1(\mathcal{L}(l)_1) = h^1(\mathcal{L}) \) too. Hence, whenever \( l_1, l_2 \in L_{\mathcal{L}} \) one also has \( \min \{ l_1, l_2 \} \in L_{\mathcal{L}} \).

This implies (a). Finally, (a) and (b) implies (c). \( \square \)

If \( h^1(\mathcal{L}) = 0 \) then we define \( Z_{\text{coh}}(\mathcal{L}) := 0 \).

### 6. The analytic lattice cohomology of \((X,o)\)

#### 6.1. Definition and independence of the choice of the rectangle.

**6.1.1.** Our goal is to construct the **analytic lattice cohomology** of a normal surface singularity \((X,o)\) under the assumption that the link is a rational homology sphere. In particular, for any spin\(^c\)–structures of the link, or for any representative \([k] \in \text{Char}/2L\), we wish to define a graded \( Z[U] \)–module.

We fix a good resolution \( g \) and \( h \in H \). Write \( Z_{\text{coh}} \) for \( Z_{\text{coh}}(\mathcal{O}_X(-r_h)) \).

For any \( c \in L, c \geq Z_{\text{coh}}, \) we consider the rectangle \( R(0,c) = \{ l \in L : 0 \leq l \leq c \} \). By definition of \( Z_{\text{coh}} \)

\[
p_{g,h} = h^1(\mathcal{L}, \mathcal{O}_X(-r_h)) = h^1(c, \mathcal{O}_X(-r_h)).
\]

Here we might consider the \( c = \infty \) case too, in this case \( R(0,c) = L_{>0} \).

**6.1.3.** The weight function. We consider the multivariable Hilbert function \( \bar{h} \), cf. (5.2.12), and

\[
h : R(0,c) \rightarrow \mathbb{Z}, \quad h(l) := \bar{h}(l + r_h) = \dim \left( H^0(\mathcal{O}_X(-r_h)) / H^0(\mathcal{O}_X(-l - r_h)) \right)
\]

associated with the divisorial filtration of \( \mathcal{O}_{X_o} \) and the resolution \( g \), cf. (5.2.12). Clearly \( h \) is increasing (that is, \( h(l_1) \geq h(l_2) \) whenever \( l_1 \geq l_2 \)) and \( h(0) = 0 \). Next, for any \( l \in R(0,c) \), we consider the function

\[
h^c(l) = p_{g,h} - h^1(\mathcal{O}_X(-r_h)),
\]
where $h^1(\mathcal{O}_{l=0}(-r_h))$, by definition, is 0. Then $h^6$ is decreasing, $h^6(0) = p_{g,h}$ and $h^6(c) = 0$, cf. (6.1.2). We have the following reinterpretation in terms of (twisted) 2–forms. For any $\bar{I} \geq 0$ consider the exact sequence

$$0 \rightarrow \Omega^2_X(r_h) \rightarrow \Omega^2_X(r_h + \bar{I}) \rightarrow \Omega^2_X(r_h + \bar{I})|_{\bar{I}} \rightarrow 0.$$  

Since $H^1(\Omega^2_X(r_h)) = 0$ (cf. Theorem 5.3.1) for any $\bar{I} \geq 0$ we obtain (using Serre duality too)

$$H^0(\Omega^2_X(r_h + \bar{I})) = H^0(\bar{I}, \Omega^2_X(r_h + \bar{I})) \simeq H^1(\mathcal{O}_{\bar{I}}(-r_h))^*.$$  

This applied for $\bar{I} = c$ and $\bar{I} = l$ gives

$$\dim \frac{H^0(\tilde{X}, \Omega^2_X(c + r_h))}{H^0(\Omega^2_X(l + r_h))} = p_{g,h} - h^1(\mathcal{O}_{\bar{I}}(-r_h)) = h^5(l).$$  

### 6.1.6. The lattice cohomology

We consider the natural cube-decomposition of $R(0,c)$ (where the 0-cubes are the lattice points) and the set of cubes $\{ \mathcal{Q}_q \}_{q \geq 0}$ of $R(0,c)$ as in 2.1.3. Then we define the weight function

$$w_0 : \mathcal{Q}_q \rightarrow \mathbb{Z}, \quad w_0(l) = h(l) + h^5(l) - h^5(0) = h(l) - h^1(\mathcal{O}_{\bar{I}}(-r_h)).$$

Clearly, $w_0(0) = 0$. Let us list some properties of $w_0$.

First of all, note that $0 \leq h^5(l) \leq p_{g,h}$ for every $l$, hence when $c = \infty$ then $h$ and $w_0$ have comparable asymptotic behaviours for $l \gg 1$. Using the monotony of $h$, we compute a formula shows that $w_0$ satisfies the requirement (2.1.2) (a), namely, $w_0^{-1}(\{\infty, n\})$ is finite for any $n \in \mathbb{Z}$.

Next, since $h$ is induced by a filtration, it satisfies the matroid rank inequality $h(l_1) + h(l_2) \geq h(l_1 + l_2)$, where $l = \min\{l_1, l_2\}$ and $\bar{l} = \max\{l_1, l_2\}$. On the other hand, $h^5$ satisfies the ‘opposite’ matroid rank inequality, see 5.4. Therefore, $w_0$ satisfies the matroid rank inequality (where $l_1, l_2 \geq 0$)

$$w_0(l_1) + w_0(l_2) \geq w_0(l_1 + l_2).$$

Furthermore, similarly as in 3.2.1 we define $w_q : \mathcal{Q}_q \rightarrow \mathbb{Z}$ by $w_q(\mathcal{Q}_q) = \max\{w_0(l) : l \text{ is any vertex of } \mathcal{Q}_q\}$. In the sequel we write $w$ for the system $\{w_q\}_{q}$ if there is no confusion. The weight functions $\{w_q\}_{q}$ define the lattice cohomology $\tilde{H}^\ast(R(0,c), w)$ and the graded root $\mathfrak{R}(R(0,c), w)$ associated with $R(0,c)$ and $w$.

**Lemma 6.1.9.** $\tilde{H}^\ast(R(0,c), w)$ and $\mathfrak{R}(R(0,c), w)$ are independent on the choice of $c \geq Z_{coh,h}$.

**Proof.** Fix some $c \geq Z_{coh,h}$ and choose $E_v \subset [c - Z_{coh,h}]$. Then for any $l \in R(0,c)$ with $l_v = c_v$ we have $\min\{l, Z_{coh,h}\} = \min\{l - E_v, Z_{coh,h}\}$. Therefore, by 5.4 $h^1(\mathcal{O}_{l-E_v}(-r_h)) = h^1(\mathcal{O}_{\bar{I}}(-r_h))$, thus $w_0(l - E_v) \leq w_0(l)$. Then for any $n \in \mathbb{Z}$, a strong deformation retract in the direction $E_v$ realizes a homotopy equivalence between the spaces $S_n \cap R(0,c)$ and $S_n \cap R(0,c - E_v)$, thus $w_0(l - E_v) \leq w_0(l)$. Therefore, $w_0(l_1) + w_0(l_2) \geq w_0(l_1 + l_2)$. The strong deformation retract is defined similarly.

**Corollary 6.1.10.** (a) The graded root $\mathfrak{R}(R(0,c), w)$ satisfies $|\tau^{-1}(n)| = 1$ for any $n \gg 0$.

(b) $\tilde{H}^\ast_{red}(R(0,c), w)$ is a finitely generated $\mathbb{Z}$-module (for any finite or infinite $c \geq Z_{coh}$).

**Proof.** For any $n \gg 0$ we have $R(0,c) = S_n$, hence $S_n$ is contractible for such $n$.  

6.2. Independence of $\phi$. Rewrite the $c$–independent module $\mathbb{H}^n(R(0,c),w)$ as $\mathbb{H}^n_{an,h}(\phi)$, and the garded root as $\mathcal{R}_{an,h}(\phi)$.

**Theorem 6.2.1.** The graded $\mathbb{Z}[U]$–module $\mathbb{H}^n_{an,h}(\phi)$ and the graded root $\mathcal{R}_{an,h}(\phi)$ are independent of the choice of the resolution $\phi$.

**Proof.** We need to verify that $\mathbb{H}^n_{an,h}(\phi)$ and $\mathcal{R}_{an,h}(\phi)$ are stable with respect to blow up of a point. We discuss two cases according to the position of the point with the singular locus of $E$.

**Case A.** We fix a resolution $\phi$, and denote the blow up of a point of $E \setminus \cup_{w \neq w_0} E_w$ by $\pi$, and set $\phi' := \phi \circ \pi$. Let $\Gamma$ and $\Gamma'$ be the corresponding graphs, $L(\Gamma)$, $L(\Gamma')$ the lattices and $(x), (x')$ the intersection forms.

We denote the new $(-1)$–vertex of $\Gamma'$ by $E_{new}$. In our notations we identify $E_v \in L$ with its strict transform in $L(\Gamma')$. We have the next natural morphisms: $\pi_* : L(\Gamma) \to L(\Gamma')$ defined by $\pi_* (\sum x_v E_v + x_{new} E_{new}) = \sum x_v E_v$, and $\pi^* : L(\Gamma') \to L(\Gamma)$ defined by $\pi^* (\sum x_v E_v) = \sum x_v E_v + x_{new} E_{new}$. They can be extended by similar formulae to rational cycles too, and $\pi^*(L(\Gamma) \subset L(\Gamma'))$. They satisfy the ‘projection formula’ $(\pi^*x, x') = (x, \pi_* x)$. This shows that $(\pi^* x, \pi^* y)' = (x, y)$ and $(\pi^* x, E_{new})' = 0$ for any $x, y \in L(\Gamma')$. Associated with $\phi$, let $h, h^\circ$ be the functions defined above, $w_0$ the analytic weight and $S_0(\phi) = \bigcup \{ \Box : w(\Box) \leq n \}$. We use similar notations $h^ ', (h')', w'_0$ and $S_0(\phi')$ for $\phi'$. Let also $r_h \in L(\Gamma)$ and $r'_h \in L(\Gamma')$ be the universal cycles associated with $h \in H$.

**Lemma 6.2.2.** $\pi^* (r_h) = r'_h$.

**Proof.** The composition $\phi_{\tilde{X}} : \text{Div}(\tilde{X}) \to \text{Pic}(\tilde{X}) \xrightarrow{\pi_*} L' / L = H$ is realized by $D \mapsto [D \cap \partial \tilde{X}]$ (for $\tilde{X}$ conveniently small and $\partial \tilde{X} = M$). If $D' \in \text{Div}(\tilde{X}')$ is the strict transform of $D \in \text{Div}(\tilde{X})$ then $\phi_{\tilde{X}'}(D') = \phi_{\tilde{X}}(D)$ in $H$. Therefore, if we chose $x \in L(\Gamma)$ and $x' \in L(\Gamma')$ such that $D + x$ and $D' + x'$ are numerically trivial in $H_1(\tilde{X}, \partial \tilde{X}, \mathbb{Q})$ (i.e. $D + x, E_v \tilde{X} = 0$ for all $v \in \mathcal{V}$, and similarly for $D' + x'$) then $x' = \pi^* x$. Hence, in the two resolutions, $x \in L(\Gamma)$ and $\pi^* x \in L(\Gamma')$ have the same class in $H$. On the other hand, clearly, all the $E_v$–entries of $\pi^* r_h$ are in $[0, 1]$.

**Lemma 6.2.3.** $H^*(\tilde{X}, \pi^* \mathcal{L}) = H^*(\tilde{X}, \mathcal{L})$ and $H^*(\pi^* x, \pi^* \mathcal{L}) = H^*(x, \mathcal{L})$ for any line bundle $\mathcal{L} \in \text{Pic}(\tilde{X})$ and $x \in L(\Gamma)$.

**Proof.** The first identity follows from Leray spectral sequence, the second one from the first by exact sequences of type $0 \to \mathcal{L}(-x) \to \mathcal{L} \to \mathcal{L} |_x \to 0$.

**6.2.4.** For $a \leq 0$ and $x \in R$ we claim that $H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(-\pi^* x - \pi^* r_h - aE_{new})) = H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(-\pi^* x - \pi^* r_h))$. Indeed, take the exact sequence of sheaves

$$0 \to \mathcal{O}_{\tilde{X}}(-\pi^* x - \pi^* r_h) \to \mathcal{O}_{\tilde{X}}(-\pi^* x - \pi^* r_h - aE_{new}) \to \mathcal{O}_{-aE_{new}}(-\pi^* x - \pi^* r_h - aE_{new}) \to 0$$

and use that $h^0(\mathcal{O}_i(l) \otimes \mathcal{L}) = 0$ for any $l > 0$ and line bundle $\mathcal{L}$ with $(c_1, E_v) = 0$ for any $E_v \in |l|$. This last vanishing follows from the Grauert–Riemenschneider Theorem via Serre duality. Therefore (using Lemma 6.2.3 too) $h^0(\pi^* x + aE_{new})$ equals

$$\dim \frac{H^0(\mathcal{O}_{\tilde{X}}(-\pi^* r_h))}{H^0(\mathcal{O}_{\tilde{X}}(-\pi^* x - \pi^* r_h - aE_{new}))} = \dim \frac{H^0(\mathcal{O}_{\tilde{X}}(-\pi^* r_h))}{H^0(\mathcal{O}_{\tilde{X}}(-\pi^* x + r_h))} = \dim \frac{H^0(\mathcal{O}_{\tilde{X}}(-r_h))}{H^0(\mathcal{O}_{\tilde{X}}(-x - r_h))} = h(x).$$

Hence

$$h^0(\pi^* x + aE_{new}) = \begin{cases} h(x) & \text{for any } a \leq 0 \\ \text{is increasing for } a \geq 0. \end{cases}$$

**6.2.5.**

**6.2.6.** Using the exact sequence

$$0 \to \mathcal{O}_{aE_{new}}(-\pi^* x - \pi^* r_h) \to \mathcal{O}_{\pi^* x + aE_{new}}(-r'_h) \to \mathcal{O}_{\pi^* x}(-r'_h) \to 0$$
and Lipman’s vanishing $h^1(\mathcal{O}_{E^0}(\pi^* x - \pi^* r_h)) = 0$ from (6.2.3) we get that $h^1(\mathcal{O}_{E^0}(\pi^* x + E_{\text{new}}(\pi^* r_h)) = h^1(\mathcal{O}_{E^0}(\pi^* x - r_h))$ for any $a \geq 0$. Furthermore, from

$$0 \to \mathcal{O}_{E_{\text{new}}}(\pi^* x - \pi^* r_h - \pi^* x + E_{\text{new}}) \to \mathcal{O}_{E^0}(\pi^* x - r_h) \to \mathcal{O}_{E^0}(\pi^* x - E_{\text{new}}(\pi^* r_h)) \to 0$$

we get that $h^1(\mathcal{O}_{E^0}(\pi^* x - E_{\text{new}}(\pi^* r_h))) = h^1(\mathcal{O}_{E^0}(\pi^* x - r_h))$ too. On the other hand, since $\pi^*(\pi^* r_h) = r_h$, by Lemma (6.2.3), $h^1(\mathcal{O}_{E^0}(\pi^* x - r_h)) = h^1(\mathcal{O}_{\pi^* x}(\pi^* x - r_h))$. Therefore,

\begin{equation}
(6.2.7) \quad h^1(\mathcal{O}_{E^0}(\pi^* x + E_{\text{new}}(\pi^* r_h))) = \begin{cases}
\text{is increasing for } a \leq -1, \\
\text{is decreasing for } a \geq 0.
\end{cases}
\end{equation}

These combined provide

\begin{equation}
(6.2.8) \quad a \mapsto w_0'(\pi^* x + aE_{\text{new}}) = \begin{cases}
w_0(x) & \text{for } a = -1 \text{ and } a = 0, \\
\text{is increasing for } a \geq 0.
\end{cases}
\end{equation}

Recall that we can compute $\mathbb{H}^n_{\text{an}, h}(\phi)$ using the cube $R(0, c)$ with $c \geq Z_{\text{coh}, h}(\phi)$. By Lemma (6.2.3) we obtain that $\pi^* c \geq Z_{\text{coh}, h}(\phi')$, hence $\mathbb{H}^n_{\text{an}, h}(\phi')$ can be computed in $R(0, \pi^* c)$. But we can take $c = \infty$ as well.

Furthermore, if $w_0'(\pi^* x + aE_{\text{new}}) \leq n$, then $w_0(x) \leq n$ too. In particular, the projection $\pi_0$ in the direction of $E_{\text{new}}$ induces a well-defined map $\pi_0 : S_n(\phi') \to S_n(\phi)$. We claim that this is a homotopy equivalence (with all fibers non-empty and contractible).

6.2.9. We proceed in two steps. First we prove that $\pi_0 : S_n(\phi') \to S_n(\phi)$ is onto.

Consider a zero dimensional cube (i.e. lattice point) $x \in S_n(\phi)$. Then $w_0(x) \leq n$. But then $w_0'(\pi^* x) = w_0(x) \leq n$ too, hence $\pi^* x \in S_n(\phi')$ and $x = \pi_0(\pi^* x) \in \text{im}(\pi_0)$.

Next, take a cube $(x, I) \subset S_n(\phi') (I \subset \ell')$. This means that $w_0(x + E_{\ell'}) \leq n$ for any $I' \subset I$. But

\begin{equation}
(6.2.10) \quad \pi^*(x + E_{\ell'}) = \pi^* x + E_{\ell'} + \varepsilon \cdot E_{\text{new}},
\end{equation}

where $\varepsilon = 0$ if $v_0 \not\in I'$ and $\varepsilon = 1$ otherwise. Hence

\begin{equation}
(6.2.11) \quad w_0'(\pi^* x + E_{\ell'}) = w_0'(\pi^* x + E_{\ell'}) - \varepsilon E_{\text{new}} \leq w_0(x + E_{\ell'}) \leq n.
\end{equation}

Therefore $(\pi^* x, I) \in S_n(\phi')$ and $\pi_0$ projects $(\pi^* x, I)$ isomorphically onto $(x, I)$.

Next, we show that $\pi_0$ is in fact a homotopy equivalence. In order to prove this fact it is enough to verify that if $\square \subset S_n(\phi)$ and $\square'$ denotes its relative interior, then $\pi_0^{-1}(\square') \cap S_n(\phi')$ is contractible.

Let us start again with a lattice point $x \in S_n(\phi)$. Then $\pi_0^{-1}(x) \cap S_n(\phi')$ is a real interval (whose end-points are lattice points, considered in the real line of the $E_{\text{new}}$ coordinate). Let us denote it by $\mathcal{I}(x)$. Now, if $\square = (x, I)$, then we have to show that all the intervals $\mathcal{I}(x + E_{\ell'})$ associated with all the subsets $I' \subset I$ have a common lattice point. But this is exactly what we verified above: the $E_{\text{new}}$ coordinate of $\pi^*(x)$ is such a common point. Therefore, $\pi_0^{-1}(\square') \cap S_n(\phi')$ has a deformation retract (in the $E_{\text{new}}$ direction) to $(\pi^* x, I')$.

For any $I \subset L$ let $N(l) \subset \mathbb{R}^t$ denote the union of all cubes which have $l$ as one of their vertices. Let $U(l)$ be its interior. Write $U_n(l) = U(l) \cap S_n(\phi)$. If $I \in S_n(\phi)$ then $U_n(l)$ is a contractible neighbourhood of $l$ in $S_n(\phi)$. Also, $S_n(\phi)$ is covered by $\{U_n(l)\}$. Moreover, $\pi_0^{-1}(U_n(l))$ has the homotopy type of $\pi_0^{-1}(l)$, hence it is contractible. More generally, for any cube $\square$,

$$\pi_0^{-1}(\text{some vertex of } \square U_n(l)) \sim \pi_0^{-1}(\square')$$

which is contractible by the above discussion. Since all the intersections of $U_n(l)$’s are of this type, we get that the inverse image of any intersection is contractible. Hence by Čech covering (or Leray spectral sequence) argument, $\pi_0$ induces an isomorphism $H^*(S_n(\phi'), \mathbb{Z}) = H^*(S_n(\phi), \mathbb{Z})$. In fact, this already shows that $\mathbb{H}^n_{\text{an}, h}(\phi') = \mathbb{H}^n_{\text{an}, h}(\phi)$. By the identification of the connected components of $S_n(\phi)$ and $S_n(\phi')$ we also
have $\mathcal{R}_{au,h}(\phi') = \mathcal{R}_{au,h}(\phi)$. Note that compatibility with the $U$-action also follows from the corresponding inclusions of the $S_n$-spaces.

In order to prove the homotopy equivalence, one can use quasification, defined in [15]; see also [13], e.g. the relevant Theorem 6.1.5. Since $\pi_n : S_n(\phi) \to S_n(\phi)$ is a quasification, and all the fibers are contractible, the homotopy equivalence follows.

6.2.12. Case B. Assume that we blow up an intersection point $E_0 \cap E_1$. The proof starts very similarly, however at some point there are two major differences, hence we need several additional arguments.

With very similar notation, in this case we have (define) $\pi^*(\sum x_i E_i) = \sum x_i E_i + (x_0 + x_1)E_{new}$. Then the strategy is the same as above in Case A, but two differences appear: the first one is related with $\pi^*h$: Lemma 6.2.2 is not always true. The second one is related with $\pi^*E_{new}$ in 6.2.10.

Let us analyse the analogue of Lemma 6.2.2 By the very same proof we have the following

Lemma 6.2.13. Write $r_h$ as $\sum a_i E_i$ for some $a_i \in [0, 1)$. Then $r_h' = \pi^*r_h$ if and only if $a_i + a_{r(i)} < 1$. Otherwise $r_h' = \pi^*r_h - E_{new}$.

We divide the proof of Case B in two parts, according to the two cases of Lemma 6.2.13

Case B.1. Assume that $r_h' = \pi^*r_h$.

Then all the statements of Case A from 6.2.4 and 6.2.6 remain valid (including the key (6.2.8)). However, 6.2.9 should be modified. The modifications start in 6.2.10. Indeed, in this case

$$\pi^*(x + E_{new}) = \pi^*x + E_{new} + \varepsilon E_{new},$$

where $\varepsilon$ is the cardinality of $I' \cap \{v_0, v_1\}$, This can be 0, 1 or 2. Therefore, if $\{v_0, v_1\} \not\subset I$, then $\varepsilon \in \{0, 1\}$ for any $I'$, hence for such cubes $(x, I)$ all the arguments of 6.2.9 work.

6.2.15. Assume in the sequel that $\{v_0, v_1\} \subset I$. Write $J = I \setminus \{v_0, v_1\}$.

There are two cube–candidates of $L'(\Gamma) \otimes \mathbb{R}$ which might cover the cube $(x, I) \in S_n(\phi)$. One of them is $(\pi^*x, I)$ (as above). However, by 6.2.8 the lattice points $\pi^*x + E_j$ are in $S_n(\phi)$, but the vertex $\pi^*x + E_j$ of $(\pi^*x, I)$ is not necessarily in $S_n(\phi)$. Another candidate is $(\pi^*x + E_{new}, I)$, but here again $\pi^*x$ and $\pi^*x - E_{new}$ are in $S_n(\phi')$ but $\pi^*x + E_{new}$ might be not. So both cubes a priori are obstructed if we apply merely 6.2.8.

Next we analyze these obstructions with more details and we show that one of the candidate cubes works.

6.2.16. Case 1. Assume that $w_0(\pi^*x) = w_0(\pi^*x + E_{new})$. Then by 6.2.5 and 6.2.7 we obtain that $h'(\pi^*x) = h'(\pi^*x + E_{new})$. By the matroid rank inequality of $h'$ we get that $h'(\pi^*x + E_{new}) = h'(\pi^*x + E_{new})$ for any $I' \subset J$. This again via 6.2.5 and 6.2.7 shows that $w_0(\pi^*x + E_{new}) = w_0(\pi^*x + E_{new})$. In particular,

$$w_0(\pi^*x + E_{new}) = w_0(\pi^*x + E_{new}) = w_0(\pi^*x + E_{new}) = w_0(\pi^*x + E_{new}) \leq n.$$ 

That is, the vertices of type $\pi^*x + E_{new}$ of $(\pi^*x + E_{new}, I)$ are in $S_n(\phi')$. For all other vertices we already know this fact (use 6.2.8). Hence $(\pi^*x + E_{new}, I)$ is in $S_n(\phi')$ and it projects via $\pi_n$ bijectively to $(x, I)$. Furthermore, $\pi_n^{-1}(x, I) \cap S_n(\phi')$ admits a deformation retract to $(\pi^*x + E_{new}, I)^0$, hence it is contractible.

6.2.17. Case 2. Assume that $w_0(\pi^*x + E_{new}) = w_0(\pi^*x + E_{new})$. Then by 6.2.5 and 6.2.7 we obtain that $h^1(\mathcal{O}(\pi^*x + E_{new} - r_h')) = h^1(\mathcal{O}(\pi^*x + E_{new} - r_h'))$. By the opposite matroid rank inequality of $h^1(\mathcal{O}(\pi^*x + E_{new} - r_h'))$ and 6.2.5 and 6.2.7 again we obtain that $w_0'(\pi^*x + E_{new} - E_{new}) = w_0'(\pi^*x + E_{new} - E_{new}) \leq n$.
That is, the vertices of type $\pi^*x + E_I - E_P$ of $(\pi^*x, I)$ are in $S_n(\phi')$. For all other vertices we already know this fact (use (6.2.3)). Hence $(\pi^*x, I)$ is in $S_n(\phi')$ and it projects via $\pi_{\mathbb{R}}$ bijectively to $(x, I)$. Furthermore, $\pi_{\mathbb{R}}^{-1}(x, I)^o \cap S_n(\phi')$ admits a deformation retract to $(\pi^*x, I)^0$, hence it is contractible.

6.2.18. Case 3. Assume that the assumptions from Case 1 and Case 2 do not hold. This means that
\[
\begin{cases}
    b'(\pi^*x) < b'(\pi^*x + E_{\text{new}}), \\
    h^1(\mathcal{O}_{\pi^*x+E_I}(-r_h')) < h^1(\mathcal{O}_{\pi^*x+E_{\text{new}}}(-r_h')).
\end{cases}
\]

This reads as follows (cf. (5.3.6))
\[
\begin{align*}
    (a) & \quad H^0(\mathcal{O}_{\tilde{X}}(-\pi^*x - r_h' - E_{\text{new}})) \subseteq H^0(\mathcal{O}_{\tilde{X}}(-\pi^*x - r_h'), \\
    (b) & \quad H^0(\tilde{X}', \Omega_{\tilde{X}}^2(\pi^*x + r_h' + E_I)) \subseteq H^0(\tilde{X}', \Omega_{\tilde{X}}^2(\pi^*x + r_h' + E_I + E_{\text{new}})).
\end{align*}
\]

Part (a) means the following: there exists a global section $s_1 \in H^0(\tilde{X}', \mathcal{O}_{\tilde{X}}(-r_h'))$ such that $\text{div}_{E_I}(s_1) \geq \pi^*x$, and in this inequality the $E_{\text{new}}$-coordinate entries are equal. By part (b), there exists a global section $s_2 \in H^0(\tilde{X}', \Omega_{\tilde{X}}^2(r_h'))$ such that $\text{div}_{E_I}(s_2) \geq -\pi^*x - E_I - E_{\text{new}}$ and the $E_{\text{new}}$-coordinate entries are equal.

Therefore, the global section $s_1s_2 \in H^0(\tilde{X}', \Omega_{\tilde{X}}^2(r_h'))$ has the property that $\text{div}_{E_I}(s_1s_2) \geq -E_I - E_{\text{new}}$ with equality at the $E_{\text{new}}$ coordinate. In particular, by duality (5.3.6) we obtain that in $\tilde{X}'$ the following strict inequality holds:
\[(6.2.19) \quad h^1(\mathcal{O}_{E_I+E_{\text{new}}}) > h^1(\mathcal{O}_{E_I}) \quad (\forall' = \forall' \cup \{\text{new}\}, I \subset \forall').\]

But if the link is a rational homology sphere then both left and right hand sides are zero, i.e. this strict inequality cannot happen.

6.2.20. In particular, for any $I \subset \forall'$ either $\{v_0, v_1\} \not\subset I$, or in the opposite case either Case 1 or Case 2 applies. Hence, in any case, $\pi_{\mathbb{R}}^{-1}(x, I)^o \cap S_n(\phi')$ is contractible. Therefore, $S_n(\phi)$ and $S_n(\phi')$ have the same homotopy type by the argument from the end of 6.2.9.

Case B.II. Assume that $r_h' = \pi^*r_h - E_{\text{new}}$.

In turns out that this case is very similar to the case B.I: compared with that case all the $E_{\text{new}}$-coefficients should be shifted by one. However, we have to go through all the verifications step by step.

Firstly, for $a \leq 1$,
\[
b'(\pi^*x + aE_{\text{new}}) = \dim \frac{H^0(\mathcal{O}_{\tilde{X}}(-\pi^*r_h + E_{\text{new}}))}{H^0(\mathcal{O}_{\tilde{X}}(-\pi^*x - \pi^*r_h + aE_{\text{new}} + E_{\text{new}}))}.
\]

Since $H^0(\mathcal{O}_{\tilde{X}}(-\pi^*r_h + E_{\text{new}})) = H^0(\mathcal{O}_{\tilde{X}}(-\pi^*r_h))$, and for $a \leq 1$ (by (6.2.4))
\[
H^0(\mathcal{O}_{\tilde{X}}(-\pi^*x - \pi^*r_h - aE_{\text{new}} + E_{\text{new}})) = H^0(\mathcal{O}_{\tilde{X}}(-\pi^*x - \pi^*r_h))
\]

we get
\[(6.2.21) \quad b'(\pi^*x + aE_{\text{new}}) = b(x) \quad \text{for any } a \leq 1, \quad \text{is increasing for } a \geq 1.\]

Next, for $a \geq 0$, in the cohomology exact sequence of
\[
0 \to \mathcal{O}_{E_{\text{new}}}(-\pi^*x - \pi^*r_h + E_{\text{new}}) \to \mathcal{O}_{\pi^*x+aE_{\text{new}}}(-r_h') \to \mathcal{O}_{\pi^*x}(-r_h') \to 0
\]
one has $h^1(\mathcal{O}_{aE_{\text{new}}}(-\pi^*x - \pi^*r_h + E_{\text{new}})) = 0$. Indeed, since $\text{Pic}^0(aE_{\text{new}}) = 0$, $h^1(\mathcal{O}_{aE_{\text{new}}}(-\pi^*x - \pi^*r_h + E_{\text{new}})) = h^1(\mathcal{O}_{aE_{\text{new}}}(E_{\text{new}}))$, whose vanishing follows by induction on $a$. Therefore, for $a \geq 0$,
\[(6.2.22) \quad h^1(\mathcal{O}_{\pi^*x+aE_{\text{new}}}(-r_h')) = h^1(\mathcal{O}_{\pi^*x}(-r_h')).\]
On the other hand, from the exact sequence

$0 \to \mathcal{O}_{\pi^*x}(-\pi^*r_h) \to \mathcal{O}_{\pi^*x+E_{new}}(-\pi^*r_h+E_{new}) \to \mathcal{O}_{E_{new}}(-\pi^*r_h+E_{new}) \to 0$

we obtain $h^i(\mathcal{O}_{\pi^*x}(-\pi^*r_h)) = h^i(\mathcal{O}_{\pi^*x+E_{new}}(-\pi^*r_h+E_{new}))$, which equals $h^i(\mathcal{O}_{\pi^*x}(-r'_h))$ by (6.2.22). Hence

(6.2.23) $h^i(\mathcal{O}_{\pi^*x+aE_{new}}(-r'_h))$

These combined provide

(6.2.24) $a \mapsto w'_0(\pi^*x + aE_{new})$

Here it is convenient to take $c = \infty$, hence we compare the two infinite rectangles (first quadrants).

Again, if $w'_0(\pi^*x + aE_{new}) \leq n$, then $w_0(x) \leq n$ too. Hence the projection $\pi_R$ in the direction of $E_{new}$ induces a map $\pi_R : S_n(\phi') \to S_n(\phi)$. We need to prove that this is a homotopy equivalence with all fibers non-empty and contractible.

First we verify that $\pi_R : S_n(\phi') \to S_n(\phi)$ is onto.

If $x \in S_n(\phi)$ then $w_0(x) \leq n$, hence by (6.2.24) $w'_0(\pi^*x) = w_0(x) \leq n$ too, hence $x \in \operatorname{im}(\pi_R)$.

If $(x,I) \subset S_n(\phi)$ then $w_0(x + E_I) \leq n$ for any $I \subset I$. For such $I'$ we have the identity (6.2.14) with $\varepsilon = |I' \cap \{v_0,v_1\}| \subset \{0,1,2\}$.

Assume that $\{v_0,v_1\} \subset I$. Then we claim that $(\pi^*x + E_{new},I)$ is in $S_n(\phi')$ and it projects isomorphically onto $(x,I)$. Indeed, in this case $\varepsilon \in \{0,1\}$ and by (6.2.24)

$$w'_0(\pi^*x + E_{new} + E_{I'}) = w'_0(\pi^*x + E_{I'} - \varepsilon E_{new} + E_{new}) = w_0(x + E_{I'}) \leq n.$$ 

Hence in the sequel we assume that $\{v_0,v_1\} \subset I$. Then we proceed as in (6.2.15). Again, there are two cube-candidates to lift $(x,I)$.

One of them is $(\pi^*x + E_{new},I)$. However, though $\pi^*x + E_I + 2E_{new}$ and $\pi^*x + E_I + 3E_{new}$ are in $S_n(\phi')$ but the vertex $\pi^*x + E_I + E_{new}$ of $(\pi^*x + E_{new},I)$ might not be part of $S_n(\phi')$.

The second candidate is $(\pi^*x + 2E_{new},I)$, but this case is also obstructed: $\pi^*x$ and $\pi^*x + E_{new}$ are in $S_n(\phi')$ but the vertex $\pi^*x + 2E_{new}$ of $(\pi^*x + 2E_{new},I)$ not necessarily.

Hence, again we have to analyse three case, the analogues of (6.2.16) and (6.2.17) and (6.2.18)

Case 1. We assume that $w'_0(\pi^*x + E_{new}) = w'_0(\pi^*x + 2E_{new})$. Then similarly as in (6.2.16) one can show that $(\pi^*x + 2E_{new},I) \subset S_n(\phi')$.

Case 2. We assume that $w'_0(\pi^*x + E_I + E_{new}) = w'_0(\pi^*x + E_I + 2E_{new})$. Then similarly as in (6.2.17) one can show that $(\pi^*x + E_{new},I) \subset S_n(\phi')$.

Case 3. Finally we show that either Case 1 or Case 2 must hold. Indeed, if not, that is, if

$$h'(\pi^*x + E_{new}) < h'(\pi^*x + 2E_{new}), \quad h^i(\mathcal{O}_{\pi^*x+E_I+E_{new}}(-r'_h)) < h^i(\mathcal{O}_{\pi^*x+E_I+2E_{new}}(-r'_h)),$$

then we get a contradiction similarly as in (6.2.18).

Having the surjectivity $\pi_R : S_n(\phi') \to S_n(\phi)$, the homotopy equivalence is proved as in the previous cases. 

\[\square\]

**Definition 6.2.25.** In the sequel we will use for $\mathbb{H}^n_{an,h}(\phi)$ the notation $\mathbb{H}^n_{an,h}(X,o)$ as well. It is called the analytic lattice cohomology of $(X,o)$ associated with $h \in H$. We also set $\mathbb{H}^n_{an}(X,o) := \oplus_{h \in H} \mathbb{H}^n_{an,h}(X,o)$. It is called the equivariant analytic lattice cohomology of $(X,o)$.

We adopt the notation $\mathfrak{Y}_{an,h}(X,o)$ for the graded root as well.
Remark 6.2.26. In order to run the equivariant version (indexed by $H$) we need the existence of the universal abelian covering, hence we need the finiteness of $H_1(M, \mathbb{Z})$, i.e. we need to require that the link is a rational homology sphere. On the other hand, if we wish to study only the analytic lattice cohomology associated with $h = 0$ (that is, with $\mathcal{O}_{X,o}$), then we do not need to consider the universal abelian covering. In that case, as the above proof shows, in order to prove the stability of $\mathbb{H}^\omega_{an,h=0}(\phi)$ it is enough to assume that $\Gamma$ is a tree (this is enough to conclude that (6.2.19) cannot happen). For details for the non-equivariant case see [1].

6.3. The ‘Combinatorial Duality Property’ of the pair $(h, h^\omega)$. The following property is needed in the Euler characteristic computation.

Lemma 6.3.1. Assume that the link is a rational homology sphere. Then there exists no $l \in L_{>0}$ and $v \in \mathcal{V}$ such that the differences $h(l + E_v) - h(l)$ and $h^\omega(l) - h^\omega(l + E_v)$ are simultaneously strict positive.

Proof. If $h(l + E_v) > h(l)$ then the inclusion $H^0(\mathcal{O}_X(-l - r_h - E_v)) \subset H^0(\mathcal{O}_X(-l - r_h))$ is strict. This means that there exists a section $s_1 \in H^0(\mathcal{O}_X(-r_h))$ with $div_E(s_1) \geq l$, where the $E_v$-coordinate is $(div_E(s_1))_v = l_v$.

Similarly, if $h^\omega(l) > h^\omega(l + E_v)$ then the inclusion $H^0(\mathcal{O}_X^\omega(l + r_h)) \subset H^0(\mathcal{O}_X^\omega(l + r_h + E_v))$ is strict, that is, there exists a section $s_2 \in H^0(\mathcal{O}_X^\omega(r_h))$ with $div_E(s_2) \geq -l - E_v$ and the $E_v$-coordinate is $(div_E(s_2))_v = -l_v + 1$.

Therefore, the section $s_1 s_2 \in H^0(\mathcal{O}_X^\omega)$ satisfies $div_E(s_1 s_2) \geq -E_v$ and $(div_E(s_1 s_2))_v = -1$. This implies $\mathcal{H}^0(\mathcal{O}_X^\omega(E_v))/\mathcal{H}^0(\mathcal{O}_X^\omega) \neq 0$, or, by (5.3.5), $h^1(\mathcal{O}_{E_v}) \neq 0$. This last fact contradicts $\mathcal{H}^1(M, \mathbb{Q}) = 0$. \hfill $\square$

6.4. The Euler characteristic $eu(\mathbb{H}^\omega_{an,h}(X, o))$.

6.4.1. Lemma 6.3.1 will allow us to determine the Euler characteristic $eu(\mathbb{H}^\omega_{an,h}(X, o))$ of the analytic lattice cohomology by a combinatorial argument. Surprisingly, this Euler characteristic automatically equals the Euler characteristic of path cohomologies associated with any increasing path (this equality definitely does not hold in the topological versions of the corresponding lattice cohomologies).

First, let us fix the notations. In the sequel we will also consider for any increasing path $\gamma$ connecting 0 and $c$ (that is, $\mathcal{V} = \{x_i\}_{i=0}^l, x_{i+1} = x_i + E_{v(i)}, x_0 = 0$ and $x_l = c, c \geq z_{coh,c}$) the path lattice cohomology $\mathbb{H}^\omega(\gamma, w(h))$ as in [2.2.1] associated with the weight function (depending on $h \in H$). Accordingly, we have the numerical Euler characteristic $eu(\mathbb{H}^\omega(\gamma, w(h)))$ as well.

Then Theorem 5.1.10 implies the following.

Theorem 6.4.2. Assume that the link is a QHS$^3$. Then $eu(\mathbb{H}^\omega_{an,h}(X, o)) = p_{g,h}(X, o)$ for any $h \in H$. Furthermore, for any increasing path $\gamma$ connecting 0 and $c$ (where $c \geq z_{coh,c}$) we also have $eu(\mathbb{H}^\omega_{an}(\gamma, w(h))) = p_{g,h}$.

Proof. We claim that the assumptions of Theorem 5.1.10 are satisfied. Indeed, the CDP was verified in 6.3.1 while the stability property of $h$ follows since it is associated with a filtration. \hfill $\square$

This in particular means that $\mathbb{H}^\omega_{an,h}(X, o)$ is a categorification of the equivariant geometric genus, that is, it is a graded cohomology module whose Euler characteristic is $p_{g,h}$.

6.5. Weighted cubes and the Poincaré series $P(t)$. Assume that $c = \infty$, i.e. $R(0, c) = L_{>0}$. Let us denote the weight function associated with $h \in H$ by $w_{an,h}$, in order to emphasise the $h$-dependence.

The reader is invited to review the definition of the Poincaré series $P(t)$ from (5.2.15). That identity together with part (h) of Theorem 5.1.10 show that the analytic Poincaré series associated with the divisorial filtration of the local ring $\mathcal{O}_{X,o}$ has the following interpretation in terms of the (analytic) weighted cubes:

$$P(t) = \sum_{h \in H} \sum_{l \geq 0} \sum_{I \in \mathcal{V}} (-1)^{|l|+1} w_{an,h}(l, I) t^{l + r_h}.$$ 

The above formula can be compared with its topological analogue. One defines a topological zeta (Poincaré) series $Z(t)$ from $\Gamma$, and there is an identical formula for $Z(t)$, where $w_{an}$ is replaced by $w_{top}$, cf. [29].
and a morphism of graded roots

For a fixed topological type find all the possible graded

Problem 7.1.2.

Furthermore, for every \( h \in H \) we have the analytic lattice cohomology \( \mathbb{H}^*_an(h)(X, o) \) of \( (X, o) \). \( \mathbb{H}^*_an(h)(X, o) \) is the categorification of \( p_h(X, o) \).

Is there a relationship between \( \mathbb{H}^*_an(0)(X, o) \) and the collection \( \{ \mathbb{H}^*_an(h)(X, o) \}_h \)?

7. COMPARISON OF \( \mathbb{H}^*_an(X, o) \) WITH \( \mathbb{H}^*_top(M) \)

7.1. Above, for every \( h \in H \), we defined the analytic lattice cohomology \( \mathbb{H}^*_an(h)(X, o) \) associated with \( L \geq 0 \) and the weight function \( w_{an,h} : L \geq 0 \to \mathbb{Z} \).

Similarly, for any \( h \in H \) we can consider the characteristic element \( k = -Z_k + 2r_h \) and the topological lattice cohomology associated with \( k = -Z_k + 2r_h \) and \( L \) via the weight function \( l \mapsto -(l, l-Z_k + r_h)/2 \). Let us denote it by \( \mathbb{H}^*_top,h(M) \). On the other hand, in \([4,5,6]\) we proved that \( \mathbb{H}^*_top,h(M) \simeq \mathbb{H}^*_top,h(M, L \geq 0) \), where the second cohomology is associated with the same \( k \) but with lattice points only on \( L \geq 0 \). The advantage of \( \mathbb{H}^*_top,h(M, L \geq 0) \) is that it is defined on the same set of lattice points as the analytic \( \mathbb{H}^*_an(h)(X, o) \).

Let us compare these two objects. First, we compare the analytic and topological weight functions (both defined on \( L \geq 0 \)). Consider the exact sequence

\[
0 \to \mathcal{O}_X(-l - r_h) \to \mathcal{O}_X(-r_h) \to \mathcal{O}_I(-r_h) \to 0,
\]

and in its cohomology long exact sequence the morphism \( \alpha_l(l) : H^0(\mathcal{O}_X(-r_h)) \to H^0(\mathcal{O}_I(-r_h)) \). Then,

\[
\mathcal{H}^0(\mathcal{O}_X(-r_h)) = \mathcal{H}^0(\mathcal{O}_I(-r_h)),
\]

or, \( w_{an,h} = \mathcal{H}(\mathcal{O}_I(-r_h)) \).

But \( \mathcal{H}(\mathcal{O}_I(-r_h)) = \mathcal{H}(\mathcal{O}_I(-r_h)) \).

Hence

\[
w_{an,h}(l) = w_{top,h}(l) - \text{coker}(\alpha_l(l)) \quad \text{for any } l \in L \geq 0.
\]

Corollary 7.1.1. If \( \alpha_l(l) \) is surjective for every \( l \in L \geq 0 \) then \( \mathbb{H}^*_an,h(X, o) \) and \( \mathbb{H}^*_top,h(M) \) are isomorphic as graded \( \mathbb{Z}[U] \)-modules. In particular, in such a case their Euler characteristics also coincide:

\[
p_{g,h} = sw_{\alpha}(M) - (k^2 + |Y|)/8, \quad \text{where } k = -Z_k + 2r_h.\]

In general, \( w_{an,h} \leq w_{top,h} \).

Recall that \( S_{an,h,n} = \cup \{ \square : w_{an,h}(\square) \leq n \} \) and \( S_{top,h,n} = \cup \{ \square : w_{top,h}(\square) \leq n \} \).

Therefore \( S_{top,h,n} \subseteq S_{an,h,n} \) for any \( n \in \mathbb{Z} \). In particular, we have a graded \( \mathbb{Z}[U] \)-module morphism

\[
\delta_h : \mathbb{H}^*_an,h(X, o) \to \mathbb{H}^*_top,h(M)
\]

and a morphism of graded roots

\[
v_h : \mathcal{R}_{top,h}(X, o) \to \mathcal{R}_{an,h}(M).
\]

Problem 7.1.2. (a) For a fixed topological type find all the possible graded \( \mathbb{Z}[U] \)-modules \( \{ \mathbb{H}^*_an \}_{an,h} \) associated with all the possible analytic structures supported on that topological type.

(b) For a fixed topological type (hence for a fixed \( \mathbb{H}^*_top,h(M) \)) and analytic type \( (X, o) \) supported on it find special properties of \( \mathbb{H}^*_an,h(X, o) \) (and of the morphism \( \mathbb{H}^*_an,h(X, o) \to \mathbb{H}^*_top,h(M) \)), which might characterize the classification from part (a).

8. PREPARATION FOR THE REDUCTION THEOREM. THE TOPOLOGICAL REDUCTION.

8.1. **What is the aim of a Reduction Theorem?** The definition of a lattice cohomology \( \mathbb{H}^*(T, w) \) is based on the choice of the following objects: a lattice \( L = \mathbb{Z}^s \), a convenient union of cubes \( T \subseteq \mathbb{R}^s \), a weight function \( w : T \cap \mathbb{Z}^s \to \mathbb{Z} \). In general, \( s \), the rank of \( L \), can be large, and the direct computations are very hard.

By Reduction Theorem we replace these starting objects by a new collection \( (L, \bar{T}, \bar{w}) \) such that \( \text{rank}(L) < \text{rank}(L) \) and \( \mathbb{H}^*(T, w) = \mathbb{H}^*(\bar{T}, \bar{w}) \).
The Reduction Theorems associated with the topological lattice cohomology are based on the following observation: the reduced cohomologies are vanishing if and only if \( M \) is the link of a rational singularity. Rationality can be characterized by properties of graphs (see below). In general, we wish to ‘eliminate’ parts/subgraphs, which behave like rational graphs. Technically, the procedure runs as follows: we choose \( s' \) vertices (the bad vertices) such that by the modification of their Euler numbers we get a rational graph. Then there is a reduction to rank \( s' \).

### 8.1.1. Rational graphs

Recall that \((X, o)\) is called rational if \( p_2 = 0 \). By a result of Artin [4, 5] \( p_2 = 0 \) if and only if \( \chi(l) \geq 1 \) for all \( l \in L_{> 0} \) (hence it is a topological property of \( M \) readable from \( \Gamma \)). The links of any rational singularity is a rational homology sphere. The class of rational graphs is closed while taking subgraphs or/and decreasing the Euler numbers.

### 8.2. Measure of non-rationality. ‘Bad’ vertices

Recall that decreasing all the Euler numbers of a tree, with all the vertices decorated by \( g_v = 0 \), we obtain a rational graph. The next definition aims to identify those vertices where such a decrease is really necessary.

**Definition 8.2.1.** Let \( \Gamma \) be a resolution graph such that \( M \) is a rational homology sphere.

A subset of vertices \( \overline{\mathcal{T}} = \{v_1, \ldots, v_s\} \subset \mathcal{Y} \) is called \( B-\)set, if by replacing the Euler numbers \( e_v = E_v^\mathcal{L} \) indexed by \( v \in \overline{\mathcal{T}} \) by some more negative integers \( e'_v \leq e_v \), we get a rational graph.

A graph is called AR-graph (‘almost rational graph’) if it admits a \( B-\)set of cardinality \( s \), indexed by some more negative integers \( e'_v \leq e_v \) we get a rational graph.

**Example 8.2.2.** (a) A possible \( B-\)set can be chosen in many different ways, usually it is not determined uniquely even if it is minimal with this property. Usually we allow non-minimal \( B-\)sets as well.

(b) If \( H_1(M, \mathbb{Q}) = 0 \) then the set of nodes is a \( B-\)set. Hence any star-shaped graph (with \( \mathcal{Y} \) is the link of a rational singularity.

(c) The class of AR graphs is closed while taking subgraphs or/and decreasing the Euler numbers.

### 8.2.3. The definition of the lattice points \( x(\overline{l}) \)

Assume that \( \overline{\mathcal{T}} := \{v_k\}_{k=1}^s \) is a subset of \( \mathcal{Y} \). Then we split the set of vertices \( \mathcal{Y} \) into the disjoint union \( \overline{\mathcal{T}} \cup \mathcal{Y}^* \). Let \( \{m_\ell(x)\}_\mathcal{Y} \), denote the coefficients of a cycle \( x \in L \otimes \mathbb{Q} \), that is \( x = \sum_{v \in \mathcal{Y}} m_v(x) E_v \). We also fix \( h \in H \) and the representative \( s_h \in L' \).

Our goal is to define some universal cycles \( x(\overline{l}) \in L \) associated with \( \overline{l} \in L(\overline{\mathcal{T}}) \) and \( h \in H \).

**Proposition 8.2.4.** [25, Lemma 7.6], [20] For any \( \overline{l} := \sum_{v \in \overline{\mathcal{T}}} \ell_v E_v \in L(\overline{\mathcal{T}}) \) there exists a unique cycle \( x(\overline{l}) \in L \) (depending also on \( h \)) satisfying the next properties:

- (a) \( m_\ell(x(\overline{l})) = \ell_v \) for any distinguished vertex \( v \in \overline{\mathcal{T}} \);
- (b) \( (x(\overline{l}) + s_h, E_v) \leq 0 \) for every ‘non-distinguished vertex’ \( v \in \mathcal{Y}^* \);
- (c) \( x(\overline{l}) \) is minimal with the two previous properties (with respect to \( \leq \)).

### 8.2.5. Note that the definition of an \( B-\)set does not involve any \( k_r \). In this section we fix such an \( B-\)set \( \overline{\mathcal{T}} \subset \mathcal{Y} \) as in 8.2.1 with cardinality \( \bar{s} \) and any \( k_r \in \text{Char} \). Then, for each \( \overline{l} := \sum_{v \in \overline{\mathcal{T}}} \ell_v E_v \in L(\overline{\mathcal{T}}) \), with every \( \ell_v \geq 0 \), we define the universal cycle \( x(\overline{l}) \) associated with \( \overline{l} \) and \( s_h \) (where \( k_r = -Z_k + 2s_h \) as in 8.2.4).

Our goal is to replace the cubes of the lattice \( \mathbb{R}^s \) (or from \( (\mathbb{R}_{>0})^s \) with cubes from \((\mathbb{R}_{>0})^s \). In particular, we need to define the new weights. Define the function \( \overline{\varpi}_0 : (\mathbb{Z}_{>0})^s \to \mathbb{Z} \) by \( \overline{\varpi}_0(\overline{l}) := x_0(x(\overline{l})) \). Then \( \overline{\varpi}_0 \) defines a set \( \{\overline{\varpi}_q\}_{q=0}^s \) of compatible weight functions as in 4.2.1. \( \overline{\varpi}_0(\square) = \max\{\overline{\varpi}_0(v) : v \text{ is a vertex of } \square\} \). This system is denoted by \( \overline{\varpi}[k_r] \). Let us denote the associated lattice cohomology by \( H^*(((\mathbb{R}_{>0})^s, \overline{\varpi}[k_r]) \).
Theorem 8.2.6. (Topological Reduction Theorem) Assume that $\overline{\mathcal{V}}$ is an $B$–set. Then there exists a graded $\mathbb{Z}[U]$–module isomorphism

\[(8.2.7) \quad \mathbb{H}^*\bigl((\mathbb{R}_{\geq 0})^d, k_d\bigr) \cong \mathbb{H}^*\bigl((\mathbb{R}_{\geq 0})^d, \mathbb{P}^d[k_d]\bigr).\]

9. Analytic Reduction Theorem

9.1. Analytic reduction theorem.

9.1.1. Our next goal is to prove a ‘Reduction Theorem’, the analogue of the topological Theorem 8.2.6. Via such a result, the rectangle $R = R(0, c)$ can be replaced by another rectangle sitting in a lattice of smaller rank. The procedure starts with identification of a set of ‘bad’ vertices, see 8.2. In the topological context the possible choice of $\overline{\mathcal{V}}$ was dictated by combinatorial properties of $\chi$ with a special focus on the topological characterization of rational germs. In the present context we start with certain analytic properties of 2-forms (which reflects the dominance of $\overline{\mathcal{V}}$ over $\mathcal{V}^*$). (Note that $p_g = 0$ if and only if $H^0(\overline{\mathcal{V}}) = H^0(\mathcal{V}, \Omega_{\overline{\mathcal{V}}})$.)

In this section we assume that the link is a rational homology sphere.

Definition 9.1.2. We say that $\overline{\mathcal{V}}$ is an $B_{an}$–set if it satisfy the following property: if some differential form $\omega \in H^0(\overline{\mathcal{V}}, \Omega^{\mathcal{V}})$ satisfies $\langle \text{div}_E \omega \rangle_{\overline{\mathcal{V}}} \geq - E_{\overline{\mathcal{V}}}$ then necessarily $\omega \in H^0(\mathcal{V}, \Omega^{\mathcal{V}})$. By (5.3.5) this is equivalent with the vanishing $h^1(\mathcal{O}_Z) = 0$ for any $Z = E_{\overline{\mathcal{V}}} + l^*$, where $l^* \geq 0$ and it is supported on $\mathcal{V}^*$. 

Lemma 9.1.3. Any $B$–set is a $B_{an}$–set.

Example 9.1.4. By the above lemma, the set $\overline{\mathcal{V}} = \mathcal{N}$ of nodes is an $B_{an}$–set. Moreover, if $\{\mathcal{V}\}$ is the $B$–set of an AR graph, then it is an $B_{an}$–set as well.

9.1.5. Associated with a disjoint decomposition $\mathcal{V} = \overline{\mathcal{V}} \sqcup \mathcal{V}^*$, we write any $l \in L$ as $\overline{l} + l^*$, or $(\overline{l}, l^*)$, where $\overline{l}$ and $l^*$ are supported on $\overline{\mathcal{V}}$ and $\mathcal{V}^*$ respectively. Fix any $c \geq Z_{\text{coh}, h}$ and set $R = R(0, c)$ as above. We also write $\overline{R}$ for the rectangle $R(0, \overline{\mathcal{V}})$, the $\overline{\mathcal{V}}$–projection of $R$. For any $\overline{l} \in \overline{R}$ define the weight function

$$w_0(\overline{l}) = h(\overline{l}) + h^*(\overline{l} + c^*) - p_{g,h} = h(\overline{l}) - h^1(\mathcal{O}_{\overline{l} + c^*}(-r_h)).$$

Consider all the cubes of $\overline{R}$ and the weight function $w_q : Q(\overline{R}) \to \mathbb{Z}$ defined by $w_q(\square_q) = \max\{w_0(\overline{l}) : \overline{l} \text{ is any vertex of } \square_q\}$.

Theorem 9.1.6. Reduction theorem for the analytic lattice cohomology. If $\overline{\mathcal{V}}$ is an $B_{an}$–set then there exists a graded $\mathbb{Z}[U]$–module isomorphism

$$\mathbb{H}^*_\text{an}(R, w) \cong \mathbb{H}^*_\text{an}(\overline{R}, w).$$

Proof. For any $\mathcal{I} \subset \mathcal{V}$ write $c_{\mathcal{I}}$ for the $\mathcal{I}$–projection of $c$.

We proceed by induction, the proof will be given in $|\mathcal{V}^*|$ steps. For any $\mathcal{I} \subset \mathcal{J} \subset \mathcal{V}$ we create the inductive setup. We write $\mathcal{I}^* = \mathcal{V} \setminus \mathcal{I}$, and according to the disjoint union $\mathcal{I} \sqcup \mathcal{I}^* = \mathcal{V}$ we consider the coordinate decomposition $l = (l_{\mathcal{J}}, l_{\mathcal{I}^*})$. We also set $R_{\mathcal{J}} = R(0, c_{\mathcal{J}})$ and the weight function

$$w_{\mathcal{J}}(l_{\mathcal{J}}) = h(l_{\mathcal{J}}) + h^*(l_{\mathcal{I}^*} + c_{\mathcal{I}^*}) - p_{g,h}.$$  

Then for $\overline{\mathcal{V}} \subset \mathcal{J} \subset \mathcal{I} \subset \mathcal{V}$, $\mathcal{J} = \mathcal{J} \cup \{v_0\}$ (without $\mathcal{J}$), we wish to prove that $\mathbb{H}^*(R_{\mathcal{J}}, w_{\mathcal{J}}) = \mathbb{H}^*(R_{\mathcal{J}} \setminus \mathcal{J}, w_{\mathcal{J}})$. For this consider the projection $p_{\mathcal{J}} : R_{\mathcal{J}} \to R_{\mathcal{J}}$.

For any fixed $y \in R_{\mathcal{J}}$ consider the fiber $\{y + tE_{v_0}\}_{0 \leq t \leq c_{v_0}, t \in \mathbb{Z}}$.

Note that $t \mapsto h(y + tE_{v_0})$ is increasing. Let $t_0 = t_0(y)$ be the smallest value $t$ for which $h(y + tE_{v_0}) < h(y + (t + 1)E_{v_0})$. If $t \mapsto h(y + tE_{v_0})$ is constant then we take $t_0 = c_{v_0}$. If $t_0 < c_{v_0}$, then $t_0$ is characterized by the existence of a global section

$$s_1 \in H^0(\mathcal{O}_{\overline{\mathcal{V}}}(\mathcal{I})(-r_h)) \text{ with } (\text{div}_E s_1)|_{\mathcal{I}} \geq y, \quad (\text{div}_E s_1)|_{v_0} = t_0.$$
Symmetrically, \( t \mapsto \hat{h}(y + c \varphi + tE_v) \) is decreasing. Let \( t_0^v = t_0^v(y) \) be the smallest value \( t \) for which \( \hat{h}(y + c \varphi + tE_v) = \hat{h}(y + c \varphi + (t + 1)E_v) \). The value \( t_0^v \) is characterized by the existence of a section

\[
s_2 \in H^0\left( \bar{\mathcal{X}} \setminus E, \Omega^2_X(r_h) \right) \quad \text{with} \quad (\text{div}_{E}s_2)|_{\varphi} \geq -y, \quad (\text{div}_{E}s_2)|_{0} = -t_0^v.
\]

This shows that there exist a form \( \omega = s_1s_2 \in H^0(\bar{\mathcal{X}} \setminus E, \Omega^2_X) \) such that \((\text{div}_{E}\omega)|_{\varphi} \geq 0 \) and \((\text{div}_{E}\omega)|_{0} = t_0 - t_0^v \).

By the \( B_{an} \) property we necessarily must have \( t_0 - t_0^v \geq 0 \). Therefore, the weight \( t \mapsto w_{E}(y + tE_v) = \hat{h}(y + tE_v) + \hat{h}(y + tE_v + c \varphi) - p_{E,h} \) is decreasing for \( t \leq t_0^v \), is increasing for \( t \geq t_0 \). Moreover, for \( t_0^v \leq t \leq t_0 \) it takes the constant value \( \hat{h}(y) + \hat{h}(y + cE_v + c \varphi) - p_{E,h} = w_{E}(y) \).

Next we fix \( y \in R_{\varphi} \) and some \( I \subset \varphi \) (hence a cube \( (y,I) \) in \( R_{\varphi} \)). We wish to compare the intervals \([t_0^v(y + E_P), t_0(y + E_P)]\) for all subsets \( I' \subset I \). We claim that they have at least one common element (in fact, it turns out that \( t_0(y) \) works).

Note that \( \hat{h}(y + tE_v) = \hat{h}(y + (t + 1)E_v) = \hat{h}(y + (t + 1)E_v + E_P) \) for any \( I' \), hence \( t_0(y) \leq t_0(y + E_P) \). In particular, we need to prove that \( t_0(y) \geq t_0^v(y + E_P) \). Similarly as above, the value \( t_0^v(y + E_P) \) is characterized by the existence of a form

\[
s_P \in H^0\left( \bar{\mathcal{X}} \setminus E, \Omega^2_X(r_h) \right) \quad \text{with} \quad (\text{div}_{E}s_P)|_{\varphi} \geq -y - E_P, \quad (\text{div}_{E}s_P)|_{0} = -t_0^v(y + E_P).
\]

Hence the from \( \omega_P = s_1s_P \in H^0(\bar{\mathcal{X}} \setminus E, \Omega^2_X) \) satisfies \( \text{div}_{E}\omega_P|_{\varphi} \geq -E_P \) and \((\text{div}_{E}\omega_P)|_{0} = t_0(y) - t_0^v(y + E_P) \).

By the \( B_{an} \) property we must have \( t_0(y) - t_0^v(y + E_P) \geq 0 \).

Set \( S_{\varphi,n} \) and \( S_{\varphi,n} \) for the lattice spaces defined by \( w_{\varphi} \) and \( w_{\varphi} \). If \( y + tE_v \in S_{\varphi,n} \) then \( w_{\varphi}(y + tE_v) \leq n \), hence by the above discussion \( w_{\varphi}(y) \leq n \) too. In particular, the projection \( \pi_{\mathbb{R}} : R_{\varphi} \to R_{\varphi} \) induces a map \( S_{\varphi,n} \to S_{\varphi,n} \). We claim that it is a homotopy equivalence. The argument is similar to the proof from \( \S 2.1 \) via the above preparations. \( \square \)

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