Quasi exactly solvable operators and Lie superalgebras

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(November 29, 2018)

Abstract

Linear operators preserving the direct sum of polynomial rings $\mathcal{P}(m) \oplus \mathcal{P}(n)$ are constructed. In the case $|m - n| = 1$ they correspond to atypical representations of the superalgebra osp(2,2). For $|m - n| = 2$ the generic, finite dimensional representations of the superalgebra q(2) are recovered. An example of a Hamiltonian possessing such a hidden algebra is analyzed.

PACS numbers: 02.20.Sv, 03.65.Fd
I. INTRODUCTION

Quasi exactly solvable (QES) operators \[1\] refer to linear differential operators which preserve a finite dimensional vector space of smooth functions. They can be used in the framework of quantum mechanics to construct Hamiltonians which possess a finite number of algebraic eigenfunctions. The classification of QES operators constitutes an interesting mathematical problem which generalizes the Bochner problem \[2\].

Having chosen the vector space \( V \) of functions to be left invariant, the first trial is to construct a set of basic operators preserving \( V \) from which all others can be generated in the sense of an enveloping algebra. The fact of having a set of normal ordering rules for the basic operators is also crucial at this stage.

However, one of the striking aspects of QES operators is their close relation to the theory of representations of Lie algebras and superalgebras. In particular the realisations of Lie algebras in terms of differential operators play a crucial role \[3\].

Of course, the fact that QES operators are particular realisations of Lie algebras is not a necessary requirement, as shown in \[4\], but it constitutes an advantage because, if so, representation theory helps constructing families of invariant vector spaces. In this paper, we put emphasis on the realisations of the algebra \( q(2) \) in terms of QES operators. We discuss some generalities in Section II and give our construction in Section III. An example of a QES Schrödinger equation related to these operators is analyzed in Section IV. We end with concluding remarks and an outlook in Section V.

II. GENERALITIES

Let us denote by \( \mathcal{P}(n) \) the ring of polynomials of degree less or equal to \( n \) in a real variable \( x \). The main result concerning the construction of QES operators \[1\] is the discovery that the linear operators which preserve \( \mathcal{P}(n) \) constitute the enveloping algebra of the Lie algebra \( \text{sl}(2) \) in the representation

\[
\begin{align*}
  j^+_n &= x(D - n), & j^0_n &= D - \frac{n}{2}, & j^- &= \frac{d}{dx} \quad \text{with} \quad D \equiv x \frac{d}{dx}.
\end{align*}
\]

Extending this result to the case of linear operators which preserve the vector space \( \mathcal{P}(n - \Delta) \oplus \mathcal{P}(n) \) it was shown in \[1\] that these operators can be assembled from \( 2\Delta + 6 \) basic operators which can be chosen according to

\[
\begin{align*}
  T^+ &= \begin{pmatrix} j^+_n & 0 \\ 0 & j^+_n \end{pmatrix}, & T^0 &= \begin{pmatrix} j^0_n & 0 \\ 0 & j^0_n \end{pmatrix}, & T^- &= \begin{pmatrix} j^- & 0 \\ 0 & j^- \end{pmatrix}
\end{align*}
\]

\[
J = \frac{1}{2} \begin{pmatrix} n + \Delta & 0 \\ 0 & n \end{pmatrix}
\]

The three operators \( T^{\pm,0} \) obey the \( \text{sl}(2) \) algebra:

\[
[T^+, T^-] = -2T^0, \quad [T^\pm, T^0] = \mp T^\pm
\]

\[4\]
and $J$ commutes with all $T$’s. In the following these operators play the role of bosonic operators. They have to be completed by $2(\Delta + 1)$ off-diagonal ones:

$$Q_\alpha = x^{\alpha-1}\sigma_-, \quad \alpha = 1, \cdots, \Delta + 1 \quad (5)$$

$$\bar{Q}_\alpha = \bar{q}_{\alpha,(n)}\sigma_+, \quad \alpha = 1, \cdots, \Delta + 1 \quad (6)$$

with the definition

$$\bar{q}_{\alpha,(n)} = \left(\prod_{j=0}^{\Delta-\alpha} (D - (n + 1 - \Delta) - j) \right) \left(\frac{d}{dx}\right)^{\alpha-1} \quad (7)$$

and $\sigma_\pm = (\sigma_1 \pm i \sigma_2)/2$.

### A. Commutation relations

After an algebra, the commutation relations between the diagonal operators and the off-diagonal ones can be obtained \[4\] :

$$[T^+, Q_\alpha] = -(1 - \alpha + \Delta)Q_{\alpha+1} \quad (8)$$

$$[T^0, Q_\alpha] = -(1 - \alpha + \frac{\Delta}{2})Q_\alpha \quad (9)$$

$$[T^-, Q_\alpha] = -(1 - \alpha)Q_{\alpha-1} \quad (10)$$

$$[T^+, \bar{Q}_\alpha] = (1 - \alpha)\bar{Q}_{1-\alpha} \quad (11)$$

$$[T^0, \bar{Q}_\alpha] = (1 - \alpha + \frac{\Delta}{2})\bar{Q}_\alpha \quad (12)$$

$$[T^-, \bar{Q}_\alpha] = (1 - \alpha + \Delta)\bar{Q}_{\alpha+1} \quad (13)$$

Using the representations of $\text{sl}(2)$, these formulae reveal that the set of operators $Q$ (and independently $\bar{Q}$) transforms according to the representation of spin $s \equiv \frac{\Delta}{2}$ under the adjoint action of $T$’s (they are also called tensorial operators). In fact both $\bar{Q}_\alpha$ and $P_\alpha \equiv Q_{\Delta+2-\alpha}$ behave exactly the same under the $T$’s. In terms of Young diagrams, the representation generated by the $Q$’s (or the $P$’s) is characterized by the Young diagram with one line of $\Delta$ boxes.

On the other hand, $J$ behaves as a grading operator :

$$[J, Q_\alpha] = -\frac{\Delta}{2}Q_\alpha \quad , \quad [J, \bar{Q}_\alpha] = \frac{\Delta}{2}\bar{Q}_\alpha \quad (14)$$

One can then convince oneself that the most natural set of ordering rules is obtained when anticommutators between the $Q$’s and the $\bar{Q}$’s are chosen, which makes the interpretation of the $Q$’s and $\bar{Q}$’s as fermionic operators natural. The unpleasant feature about the algebraic structure generated by $T, J, Q, \bar{Q}$ is that the anticommutators $\{Q_\alpha, \bar{Q}_\beta\}$ are generically polynomials of degree $\Delta$ in the bosonic operators. In the case $\Delta = 1$ the operators constitute an atypical representation of the superalgebra osp(2,2) \[3\]. For $\Delta > 1$ the operators $T, Q, \bar{Q}$ do not seem to be related to Lie superalgebras. However, we will show in the following section that it is possible to find a relation for $\Delta = 2$. 

3
III. THE CASE $\Delta = 2$

We now consider the case $\Delta = 2$. This case has the peculiarity that, under the $T$'s, the operators $Q$ and the $P$ transform according to the adjoint representation, in other words they transform as triplets of $\text{sl}(2)$. Let us characterize a triplet $V_1, V_2, V_3$ by

$$[T^+, V_\alpha] = (1 - \alpha)V_{\alpha-1}$$
$$[T^0, V_\alpha] = (2 - \alpha)V_\alpha$$
$$[T^-, V_\alpha] = (3 - \alpha)V_{\alpha+1}$$

with $\alpha = 1, 2, 3$. The following sets of operators

$$\bar{Q}_\alpha \equiv (\bar{Q}_1, \bar{Q}_2, \bar{Q}_3)$$
$$P_\alpha \equiv (Q_3, Q_2, Q_1)$$
$$T_\alpha \equiv (T^+, T^0, T^-) \equiv (T_1, T_2, T_3)$$

therefore transform as triplets under the $T$'s. Obviously any linear combination of these triplets (with coefficients being operators commuting with the $T$'s) also constitutes a triplet.

The case $\Delta = 2$ in this sense in special since for $\Delta > 2$ no match between the fermionic and the bosonic operators seems to exist. This is also true for QES operators depending on many variables [5]. In this case the diagonal operators obey the commutation relations of $\text{sl}(V+1)$ with $V$ being the number of independent variables. The counterpart of the $Q$'s then corresponds to the completely symmetric representation of $\text{sl}(V+1)$ with a Young diagram containing one line with $\Delta$ boxes. In contrast, the adjoint representation of $\text{sl}(V+1)$ is characterized by a diagram with $V-1$ lines, the first line with two boxes, the others with one box. The case $V = 1, \Delta = 2$ is therefore very peculiar.

To proceed further we remind that in the case studied here the anticommutators of $Q$ with $\bar{Q}$ lead to quadratic polynomials in the $T$'s and $J$, the details of which are given in [4]. In order to obtain a more conventional algebra, we take advantage of the coexistence of three independent triplets of operators and try to simplify the algebra of operators preserving $\mathcal{P}(n-2) \oplus \mathcal{P}(n)$. For this purpose, we define a new triplet of operators, $F_\alpha$ :

$$F_\alpha \equiv \bar{Q}_\alpha + cP_\alpha + DT_\alpha \quad , \quad \alpha = 1, 2, 3$$

where $c$ is a constant and $D$ is a constant diagonal matrix. If we choose $D^2 = 1_2$ and $c = -1$ we obtain

$$\{F_\alpha, F_\beta\} = n^2 g_{\alpha\beta}$$

where $g_{\alpha\beta}$ is the Cartan metric of $\text{sl}(2)$. In the representation used, the non-zero elements of this metric are given by

$$g_{\alpha\beta} = 1 \quad \text{if} \quad \alpha, \beta = 1, 3 \text{ or } 3, 1$$
$$= -\frac{1}{2} \quad \text{if} \quad \alpha = \beta = 2$$

From (21) it is apparent that the anticommutators $\{F_\alpha, F_\beta\}$ are now linear combinations of the bosonic operators.
To establish this result, the identities
\[ j_{\beta,n}(n)p_\alpha - p_\alpha j_{\beta,n-2} = (\beta - \alpha)p_{\alpha+\beta-2} \] (23)
\[ j_{\beta,n}(n-2)\bar{q}_\alpha(n) - \bar{q}_\alpha(n)j_{\beta,n} = (\beta - \alpha)\bar{q}_{\alpha+\beta-2,n} \] (24)
have to be used, and we introduced the notations \( p_\alpha \equiv x^{3-\alpha} \) and \( j_{1,n} \equiv j^+, j_{2,n} \equiv j^- \).

As an alternative to the operators (2)-(6) above, we propose the set given by \( T_1, T_2, T_3 \) and completed by \( \hat{F}_\alpha \equiv \frac{1}{\sqrt{n}}F_\alpha, \alpha = 1, 2, 3, h_0 = n \mathbf{1}_2, h_1 = \sqrt{n}\sigma_3 \). (25)

Now, \( T^{\pm,0}, h_0 \) are the bosonic operators and \( \hat{F}_\alpha, h_1 \) the fermionic operators. They fulfill the (anti)-commutation relations of the superalgebra \( q(2) \) \cite{[citation]}.

This latter set of operators therefore constitutes a series of realisations of the superalgebra \( q(2) \) by QES operators. This series is labelled by an integer \( n \) and the \( 2n \) dimensional vector space preserved is \( \mathcal{P}(n-2) \oplus \mathcal{P}(n) \). Notice that this result was presented in \cite{[citation]}, but the calculation was done by “brute force”. Our derivation uses the representation structure of the operators \( Q \) and \( \overline{Q} \) and moreover demonstrates the importance of the case \( \Delta = 2 \).

IV. EXAMPLE OF A QES HAMILTONIAN FOR \( \Delta = 2 \)

In this section, we discuss examples of QES Schrödinger operators preserving the vector space \( \mathcal{P}(n) \oplus \mathcal{P}(n-2) \) and construct the eigenvalues for \( n = 2 \) and \( n = 3 \). The Hamiltonian is given by:
\[ H = -\frac{d^2}{dy^2}1_2 + y^61_2 + (1 - 4n)y^21_2 - 4y^2\sigma_3 - 4nk_0\sigma_1 \] (29)
where \( \sigma_1, \sigma_3 \) are the Pauli matrices, \( k_0 \) is an arbitrary constant and \( n \) an integer. To our knowledge, this Hamiltonian is the only possible QES matrix Hamiltonian with polynomial potential \cite{[citation]} (which can however be generalised through the inclusion of a \( y^4 \) term). We introduce the following change of basis and variable:
\[ \hat{H} = U^{-1}HU \quad \text{with} \quad U = e^{-\frac{x^2}{2}} \begin{pmatrix} 1 & 0 \\ k_0\frac{d}{dx} & 1 \end{pmatrix}, \quad x = y^2. \] (30)

Then, the Hamiltonian reads:
\[ \hat{H} = -(4x\frac{d^2}{dx^2} + 2\frac{d}{dx})1_2 - 4nk_0^2\frac{d}{dx}\sigma_3 + 4 \begin{pmatrix} x^2\frac{d}{dx} - nx & 0 \\ 0 & x^2\frac{d}{dx} - (n-2)x \end{pmatrix} 
+ 4k_0 \begin{pmatrix} 0 & -n \\ (1 + k_0^2n)\frac{d^2}{dx^2} & 0 \end{pmatrix} \] (31)
which obviously preserves $\mathcal{P}(n) \oplus \mathcal{P}(n - 2)$.

As a consequence, $2n$ algebraic eigenvectors of $\hat{H}$ can be constructed. In the limit $k_0 = 0$ two decoupled sextic QES Hamiltonians are recovered. In the following, we discuss the algebraic spectrum of $\hat{H}$ for $n = 2$ and $n = 3$.

For $n = 2$ the four eigenvalues are given by

$$ E_2 = 32 + c^2 \pm 4\sqrt{64 + 2c^2} \quad , \quad c \equiv -4nk_0 $$  

(32)

and the corresponding eigenvectors are

$$ \phi \propto \left( y^4 - E_2 y^2 + \frac{E_2 - c^2 - 48}{E_2 - c^2 - 64} \right) e^{-\frac{c^2}{4}} $$

(33)

The degeneracy of the energy levels for $c = 0$ with $E = -8, 0, 0, 8$ is lifted for $c > 0$. In the limit $c \to 0$, two of the eigenvectors converge to linear combinations of the two eigenvectors of the decoupled system with $E = 0$. The ground state energy corresponds to $E_0 = \pm \sqrt{32 + c^2 + 4\sqrt{64 + 2c^2}}$ and both components of the corresponding eigenvector $\phi$ have no nodes. Denoting the four algebraic energy levels by $E_{(a)}$, $a = 0, 1, 2, 3$, in increasing order and by $k_{1(a)}$, $k_{2(a)}$ the number of nodes of the two components of the corresponding eigenvector $\phi_{(a)}$, we obtain:

| $a$  | $k_{1(a)}$ | $k_{2(a)}$ |
|------|------------|------------|
| 0    | 0          | 0          |
| 1    | 2          | 2          |
| 2    | 2          | 0          |
| 3    | 4          | 2          |

Table 1

For $n = 3$, the six algebraic eigenvalues are the solutions of the equation

$$ E^6 - (248 + 3c^2)E^4 + (4800 + 240c^2 + 3c^4)E^2 - (23040 - 1344c^2 - 8c^4 + c^6) = 0 \quad . $$

(34)

As for the case $n = 2$ it is apparent that the spectrum of the equation is invariant under the reflection $E \to -E$. The three values of $|E|$ are plotted as functions of $c$ in Fig. 1. This demonstrates in particular that a level degeneracy occurs at $c \approx 5$.

The reflection symmetry of the energy eigenvalues $E \to -E$ can be demonstrated for arbitrary values of $n$ by using a similar argument as pointed out in [9] for scalar equations. In the case of (29) the relevant symmetry of the eigenvalue equation is:

$$ y \to iy \quad , \quad \phi(E) \to \sigma_3 \phi(-E) \quad . $$

(35)

Of course, as in the case of [9], the inclusion of the $y^4$-term in the potential would spoil this reflection symmetry.

To our knowledge, no QES matrix Schrödinger operator is known which preserves $\mathcal{P}(n) \oplus \mathcal{P}(n - \Delta)$ with $\Delta \geq 3$. 
V. CONCLUDING REMARKS AND OUTLOOK

In this paper we have given a new construction of the realisations of the super Lie algebra \( q(2) \) by means of QES operators. This demonstrates that the set of operators preserving the vector space \( \mathcal{P}(n-2) \oplus \mathcal{P}(n) \) is just isomorphic to the enveloping algebra of \( q(2) \). The quadratic algebra called \( \mathcal{A}(2) \) in [3] can be replaced by \( q(2) \).

The natural extension of these results would be to study \( \Delta > 2 \). Unfortunately, for generic values of \( \Delta \) the combination (19) is limited to \( F_\alpha = \overline{Q}_\alpha + cQ_\alpha, (\alpha = 1, \ldots, \Delta + 1) \) and no simplification occurs. For even values of \( \Delta \), multiplets having the same tensorial structure as the \( Q \)'s can be constructed out of the \( T \)'s, e.g. for \( \Delta = 4 \) we find:

\[
S_\alpha = \left( (T^+)^2, \frac{1}{2}\{T^+, T^0\}, \frac{1}{3}(2(T^0)^2 + \frac{1}{2}\{T^+, T^-\}), \frac{1}{2}\{T^0, T^-\}, (T^-)^2 \right)
\]

which behaves as a 5-plet (similarly to the \( \overline{Q}_\alpha \)'s under the \( T \)'s). Considering combinations analog to (19)

\[
F_\alpha = \overline{Q}_\alpha + cP_\alpha + DS_\alpha, \quad \alpha = 1, \ldots, 5
\]

we have shown that no values of \( c \) and \( D \) can be constructed such that the anti-commutators \( \{F_\alpha, F_\beta\} \) are polynomials in the bosonic operators.

Another possible extension of these considerations would be the construction of finite difference operators preserving \( \mathcal{P}(n-2) \oplus \mathcal{P}(n) \). It could be checked then, whether such operators obey some deformation of \( q(2) \) in a similar way as in [3] where the relevant structure is the so called commutator deformation of the Lie superalgebra \( \text{osp}(2,2) \).

Acknowledgements Y. B. gratefully acknowledges the Belgian F.N.R.S. for financial support. B.H. was supported by an EPSRC grant.
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FIG. 1. The modulus of the energy eigenvalues is shown for the QES Hamiltonian presented in Section IV with $n = 3$. 