ON THE BOUNDEDNESS OF GENERALIZED CESÀRO OPERATORS
ON SOBOLEV SPACES

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Abstract. For \( \beta > 0 \) and \( p \geq 1 \), the generalized Cesàro operator
\[
C^\beta f(t) := \frac{\beta}{t^\beta} \int_0^t (t - s)^{\beta - 1} f(s) \, ds
\]
and its companion operator \( C^*_\beta \) defined on Sobolev spaces \( T^{(\alpha)}_p(t^\alpha) \) and \( T^{(\alpha)}_p(|t|^\alpha) \) (where \( \alpha \geq 0 \) is the fractional order of derivation and are embedded in \( L^p(\mathbb{R}^+) \) and \( L^p(\mathbb{R}) \) respectively) are studied. We prove that if \( p > 1 \), then \( C^\beta \) and \( C^*_\beta \) are bounded operators and commute on \( T^{(\alpha)}_p(t^\alpha) \) and \( T^{(\alpha)}_p(|t|^\alpha) \). We show explicitly the spectra \( \sigma(C^\beta) \) and \( \sigma(C^*_\beta) \) and its operator norms (which depend on \( p \)). For \( 1 < p \leq 2 \), we prove that \( \hat{C}^\beta f = \hat{C}^*_\beta \hat{f} \) and \( \hat{C}^*_\beta \hat{f} = \hat{C}^\beta \hat{f} \) where \( \hat{f} \) is the Fourier transform of a function \( f \in L^p(\mathbb{R}) \).

1. Introduction

Given \( 1 \leq p < \infty \), let \( L^p(\mathbb{R}^+) \) be the set of Lebesgue \( p \)-integrable functions, that is, \( f \) is a measurable function and
\[
||f||_p := \left( \int_0^\infty |f(t)|^p \, dt \right)^{1/p} < \infty.
\]

The classical Hardy inequality (see [10, p. 245]) establishes that
\[
\left( \int_0^\infty \left| \frac{1}{t} \int_0^t f(s) \, ds \right|^p \, dt \right)^{1/p} \leq \frac{p}{p - 1} ||f||_p, \quad f \in L^p(\mathbb{R}^+),
\]
for \( 1 < p < \infty \) and therefore the so-called Cesàro transformation \( \mathcal{C} \), defined by
\[
\mathcal{C}(f)(t) = \frac{1}{t} \int_0^t f(s) \, ds, \quad t > 0,
\]
is a bounded operator on \(L^p(\mathbb{R}^+)\) with \(\|C\| \leq \frac{p}{p-1}\) for \(1 < p < \infty\). In fact, it is also known that if \(\beta > 0\)

\[
\left( \int_0^\infty \left| \frac{\beta}{t^\beta} \int_0^t (t-s)^{\beta-1} f(s)ds \right|^p dt \right)^{1/p} \leq \frac{\Gamma(\beta + 1) \Gamma(1 - \frac{1}{p})}{\Gamma(\beta + 1 - \frac{1}{p})} \|f\|_p, \quad f \in L^p(\mathbb{R}^+),
\]

for \(1 < p < \infty\) and the constant \(\frac{\Gamma(\beta+1)\Gamma(1-\frac{1}{p})}{\Gamma(\beta+1-\frac{1}{p})}\) is optimal in this inequality, see [10] Theorem 329. A closer (and dual) inequality is the following

\[
\left( \int_0^\infty \beta \int_x^\infty \frac{(t-x)^{\beta-1}}{t^\alpha} f(t)dt \right)^{\frac{1}{p'}} \leq \frac{\Gamma(\alpha + 1) \Gamma\left(\frac{1}{p'}\right)}{\Gamma\left(\alpha + \frac{1}{p'}\right)} \|f\|_p.
\]

Also the constant \(\frac{\Gamma(\alpha+1)\Gamma\left(\frac{1}{p'}\right)}{\Gamma\left(\alpha + \frac{1}{p'}\right)}\) is optimal in the above inequality ([10] Theorem 329, p.245]).

Note that inequalities (1.2) and (1.3) show that operators \(C_\beta, C_\beta^*\) where

\[
C_\beta f(t) := \frac{\beta}{t^\beta} \int_0^t (t-s)^{\beta-1} f(s)ds, \quad C_\beta^* f(s) := \beta \int_s^\infty \frac{(t-s)^{\beta-1}}{t^\alpha} f(t)dt,
\]

define bounded operators on \(L^p(\mathbb{R}^+)\), \(C_1 = C\) and \(C_1^* = C^*\). By Fubini theorem, the dual operator of \(C_\beta\) on \(L^p(\mathbb{R}^+)\) is \(C_\beta^*\) on \(L^{p'}(\mathbb{R}^+)\), i.e,

\[
\int_0^\infty C_\beta f(t)g(t)dt = \int_0^\infty f(s)C_\beta^* g(s)ds, \quad f \in L^p(\mathbb{R}^+), \quad g \in L^{p'}(\mathbb{R}^+),
\]

where \(1 < p, p' < \infty\) and \(\frac{1}{p} + \frac{1}{p'} = 1\). See other properties about some of these operators in [5 6 17].

Recently, A. Arvanitidis and A. Siskakis [3] showed that the half-plane versions of Cesàro operators on the Hardy space \(H_p(\mathbb{U})\), defined on \(\mathbb{U} := \{ z \in \mathbb{C} : \text{Im}(z) > 0 \}\) by

\[
C(F)(z) := \frac{1}{z} \int_0^z F(s)ds, \quad C^*(F)(z) := \int_z^\infty \frac{F(s)}{s}ds, \quad F \in H^p(\mathbb{U}),
\]

define bounded operators on \(H_p(\mathbb{U})\) when \(p > 1\). Both operators \(C\) and \(C^*\) can be obtained as resolvent operators of generators of some appropriate strongly continuous \(C_0\)-semigroups on \(H_p(\mathbb{U})\).

Similarly, W. Arendt and B. de Pagter [2] studied the Cesàro operator (1.1) defined in an interpolation space \(E\) of \((L^1, L^\infty)\) on \(\mathbb{R}^+\). When \(E = L^p(\mathbb{R}^+)\), the authors obtain a representation of \(C\) in terms of an appropriate resolvent operator, see [2] Corollaries 2.2, 4.3.

In [12], Sobolev subspaces \(\mathcal{T}^{(\alpha)}_1(t^\alpha)\) and \(\mathcal{T}^{(\alpha)}_1(|t|^\alpha)\) (contained in \(L^1(\mathbb{R}^+)\) and \(L^1(\mathbb{R})\) respectively and where \(\alpha \geq 0\) is the fractional order of derivation) were introduced. In fact, these subspaces are sub-algebras for the convolution products given by

\[
f \ast g(t) = \int_0^t f(t-s)g(s)ds, \quad t \geq 0,
\]
and
\begin{equation}
  f \ast g(t) = \int_{-\infty}^{\infty} f(t-s)g(s)ds, \quad t \in \mathbb{R},
\end{equation}
respectively. These algebras are canonical to define some algebra homomorphisms (defined by integral representations) into $\mathcal{B}(X)$, the set of all linear and bounded operators on a Banach space $X$. See further details in [12].

Further, in [19] Sobolev subspaces $\mathcal{T}^{(\alpha)}_p(t^\alpha)$ contained in Lebesgue spaces $L^p(\mathbb{R}^+)$ ($p \geq 1$) were introduced and studied in detail. Some remarkable results were proved (see Proposition 2.2 below). In particular, the subspace $\mathcal{T}^{(\alpha)}_p(t^\alpha)$ is a module for the algebra $\mathcal{T}^{(\beta)}_1(t^\alpha)$ for the convolution product $\ast$ given by (1.5).

Hence, it is natural to ask in what extent the boundedness property of the operators $C_\beta$ and $C^*_\beta$ remain valid in the above described Sobolev spaces.

The main aim of this paper is to study boundedness, representation and spectral properties for the generalized Cesàro operators $C_\beta$ and $C^*_\beta$ on Sobolev subspaces of fractional order $\alpha \geq 0$ embedded in $L^p(\mathbb{R}^+)$ and $L^p(\mathbb{R})$ (which are denoted by $\mathcal{T}^{(\alpha)}_p(t^\alpha)$ and $\mathcal{T}^{(\alpha)}_p(|t|^\alpha)$ respectively).

The outline of the paper is as follows: In the second section we remind some basic properties of the Sobolev spaces $\mathcal{T}^{(\alpha)}_p(t^\alpha)$ (where $\mathcal{T}^{(\alpha)}_p(t^\alpha) \hookrightarrow L^p(\mathbb{R}^+)$). We also prove new results, see for example Proposition 2.4. The main tool of this section (and in the rest of the paper) are the $C_0$-group of isometries on $\mathcal{T}^{(\alpha)}_p(t^\alpha)$, $(T_{t,p})_{t \in \mathbb{R}}$ given by
\begin{equation}
  T_{t,p}f(s) := e^{-t/p}f(e^{-t}s), \quad f \in \mathcal{T}^{(\alpha)}_p(t^\alpha).
\end{equation}
In the Theorem 2.5 it is identified its infinitesimal generator and, its spectrum, in Proposition 2.6. We note that this strategy has been pursued by other authors. We mention here [2, 3, 7, 23].

In the third section, we study the generalized Cesàro operators $C_\beta$ and $C^*_\beta$ defined on Sobolev spaces $\mathcal{T}^{(\alpha)}_p(t^\alpha)$. We first show that both operators are bounded operators and commute for $p > 1$. In fact, we have
\begin{equation}
  \|C_\beta\| = \frac{\Gamma(\beta+1)\Gamma(1/p')}{\Gamma(\beta+1/p')} \quad \text{and} \quad \|C^*_\beta\| = \frac{\Gamma(\beta+1)\Gamma(1/p)}{\Gamma(\beta+1/p)},
\end{equation}
for $\alpha \geq 0, p > 1, \beta > 0, \frac{1}{p} + \frac{1}{p'} = 1$. It is remarkable that the composition $C_\alpha C^*_\beta$ may be described explicitly involving the Gaussian hypergeometric function $2F_1$ (see Theorem 3.12) as follows:
\begin{equation}
  (C_\alpha C^*_\beta)f(t) = \alpha \int_0^t f(r) \frac{1}{t-r} \left( \frac{t-r}{t} \right)^{\alpha+\beta} 2F_1(\alpha+\beta, \beta; \beta+1; \frac{r}{t})dr
  + \beta \int_t^\infty f(r) \frac{1}{r-t} \left( \frac{r-t}{t} \right)^{\alpha+\beta} 2F_1(\alpha+\beta, \alpha; \alpha+1; \frac{t}{r})dr,
\end{equation}
for $\alpha, \beta > 0$. 
Using the description of $C_\beta$ and $C_\beta^*$ in terms of the $C_0$-semigroups (Theorem 3.3 and Theorem 3.7), we are able to determine the spectra, $\sigma(C_\beta)$ and $\sigma(C_\beta^*)$ (Theorem 3.5 and 3.9) as:

$$\sigma(C_\beta) = \Gamma(\beta + 1) \left\{ \frac{\Gamma\left(\frac{1}{p} + it\right)}{\Gamma\left(\beta + \frac{1}{p} + it\right)} : t \in \mathbb{R} \right\};$$

and

$$\sigma(C_\beta^*) = \Gamma(\beta + 1) \left\{ \frac{\Gamma\left(\frac{1}{p} + it\right)}{\Gamma\left(\beta + \frac{1}{p} + it\right)} : t \in \mathbb{R} \right\},$$

where $1/p + 1/p' = 1$. In particular, the operators $C_1$ and $C_1^*$ can be obtained as the resolvent operator of appropriate $C_0$-semigroups, namely $(T_{t,p})_{t \geq 0}$ and $(T_{-t,p})_{t \geq 0}$, respectively.

We remark that in case $\beta = 1$ we obtain:

$$\sigma(C_1^*) = \left\{ w \in \mathbb{C} : \left| w - \frac{p}{2} \right| = \frac{p}{2} \right\}.$$

This gives a proof of a conjecture posed by F. Móricz on $L^p(\mathbb{R}^+)$ [17] Section 2] and new proofs of some results given in [5, 6].

In Section 4, we introduce and give some basic properties of the Sobolev spaces $T_p^{(\alpha)}(|t|^\alpha)$ (here $T_p^{(\alpha)}(|t|^\alpha) \leftrightharpoons L^p(\mathbb{R})$). We also prove that the space $T_p^{(\alpha)}(|t|^\alpha)$ is a module for the algebra $T_1^{(\alpha)}(|t|^\alpha)$ and the *-convolution product given by (1.6). Moreover, the following interesting inequality holds:

$$|||f * g|||_{\alpha,p} \leq C_{\alpha,p}|||f|||_{\alpha,p}|||g|||_{\alpha,1}, \quad f \in T_p^{(\alpha)}(|t|^\alpha), \quad g \in T_1^{(\alpha)}(|t|^\alpha).$$

In section 5, we study boundedness, representation and spectral properties of generalized Cesàro operators on $\mathbb{R}$. Again, it is relevant to mention that the $C_0$-group of isometries on $T_p^{(\alpha)}(|t|^\alpha)$, $(T_{t,p})_{t \in \mathbb{R}}$ given by

$$T_{t,p}f(s) := e^{-\frac{t}{p} s} f(e^{-t} s), \quad f \in T_p^{(\alpha)}(|t|^\alpha),$$

(Theorem 4.3) is the main tool to prove the main results in this section. The generalized Cesàro operators $C_\beta$ and $C_\beta^*$ defined on Sobolev spaces $T_p^{(\alpha)}(|t|^\alpha)$ are described in terms of the $C_0$-group of isometries $(T_{t,p})_{t \in \mathbb{R}}$. Similar results shown in the case $T_p^{(\alpha)}(\ell^\alpha)$ hold in this case, see Theorem 5.2 and 5.3 below.

In the last section we show that $C_\beta(f) = C_\beta^*(\hat{f})$ and $C_\beta^*(\hat{f}) = C_\beta(\hat{f})$ where $\hat{f}$ is the Fourier transform of a function $f \in L^p(\mathbb{R})$ and $1 < p \leq 2$, see Theorem 6.4. We notice that our studies in this section extends and complement the main result in [18].

2. COMPOSITION GROUPS ON SOBOLEV SPACES DEFINED ON $\mathbb{R}^+$. Let $\mathcal{D}_+$ be the class of $C^\infty$-functions with compact support on $[0, \infty)$ and $\mathcal{S}_+$ the Schwartz class on $[0, \infty)$. For a function $f \in \mathcal{S}_+$ and $\alpha > 0$, the Weyl fractional integral
of order $\alpha$, $W^{\alpha}_{+} f$, is defined by
\[
W^{\alpha}_{+} f(t) := \frac{1}{\Gamma(\alpha)} \int_{t}^{\infty} (s-t)^{\alpha-1} f(s) ds, \quad t \in \mathbb{R}^+.
\]
The Weyl fractional derivative $W^\alpha_+ f$ of order $\alpha$ is defined by
\[
W^\alpha_+ f(t) := (-1)^n \frac{d^n}{dt^n} W^{(n-\alpha)}_{+} f(t), \quad t \in \mathbb{R}^+
\]
where $n = [\alpha] + 1$, and $[\alpha]$ denotes the integer part of $\alpha$. It is proved that $W^{\alpha+\beta}_{+} = W^\alpha_{+}(W^\beta_{+})$ for any $\alpha, \beta \in \mathbb{R}$, where $W^0_{+} = I$ is the identity operator and $(-1)^n W^n_{+} = \frac{d^n}{dt^n}$ holds with $n \in \mathbb{N}$, see more details in [15] and [20].

Take $\lambda > 0$ and $f_\lambda$ defined by $f_\lambda(r) := f(\lambda r)$ for $r > 0$ and $f \in \mathcal{S}_+$. It is direct to check that
\[
W^\alpha_{+} f_\lambda = \lambda^\alpha (W^\alpha_{+} f)_\lambda, \quad f \in \mathcal{S}_+,
\]
for $\alpha \in \mathbb{R}$.

Now we introduce a family of subspaces $\mathcal{T}^p_+^{(\alpha)}(t^\alpha)$ which are contained in $L^p(\mathbb{R}^+)$. 

**Definition 2.1.** For $\alpha > 0$ let be the Banach space $\mathcal{T}^p_+^{(\alpha)}(t^\alpha)$ defined as the completion of the Schwartz class $\mathcal{S}_+$ in the norm
\[
\|f\|_{p, \alpha} := \frac{1}{\Gamma(\alpha + 1)} \left( \int_0^\infty |W^\alpha_{+} f(t)|^{p} t^{\alpha p} dt \right)^{\frac{1}{p}}.
\]
We understand that $\mathcal{T}^p_+^{(0)}(t^0) = L^p(\mathbb{R}^+)$ and $\| \cdot \|_{0, p} = \| \cdot \|_p$. The case $p = 1$ and $\alpha \in \mathbb{N}$ where introduced in [11] and for $\alpha > 0$ in [12].

In the next proposition we collect some results about these family of spaces $\mathcal{T}^p_+^{(\alpha)}(t^\alpha)$ which we may be found in [19].

**Proposition 2.2.** Take $p \geq 1$ and $\beta > \alpha > 0$. Then
(i) $\mathcal{T}^p_+^{(\beta)}(t^\beta) \hookrightarrow \mathcal{T}^p_+^{(\alpha)}(t^\alpha) \hookrightarrow L^p(\mathbb{R}^+)$.
(ii) $\mathcal{T}^p_+^{(\alpha)}(t^\alpha) \ast \mathcal{T}_1^{(\alpha)}(t^\alpha) \hookrightarrow \mathcal{T}^p_+^{(\alpha)}(t^\alpha)$ for $1 \leq p < \infty$, where
\[
\langle f \ast g, h \rangle = \int_0^t f(t-s) g(s) ds, \quad t \geq 0, \quad f \in \mathcal{T}^p_+^{(\alpha)}(t^\alpha), \quad g \in \mathcal{T}_1^{(\alpha)}(t^\alpha).
\]
(iii) The operator $D^\alpha_{+} : \mathcal{T}^p_+^{(\alpha)}(t^\alpha) \rightarrow L^p(\mathbb{R}^+)$ defined by
\[
f \mapsto D^\alpha_{+} f(t) = \frac{1}{\Gamma(\alpha + 1)} t^\alpha W^\alpha_{+} f(t), \quad t \geq 0, \quad f \in \mathcal{T}^p_+^{(\alpha)}(t^\alpha).
\]
is an isometry.
(iv) If $p > 1$ and $p'$ satisfies $\frac{1}{p} + \frac{1}{p'} = 1$, then the dual of $\mathcal{T}^p_+^{(\alpha)}(t^\alpha)$ is $\mathcal{T}^{p'}_+^{(\alpha)}(t^\alpha)$, where the duality is given by
\[
\langle f, g \rangle_\alpha = \frac{1}{\Gamma(\alpha + 1)^2} \int_0^\infty W^\alpha_{+} f(t) W^\alpha_{+} g(t) t^{2\alpha} dt.
\]
Lemma 2.3. If we see for example [9, p. 201]. With this formula, it is easy to check that

\[ \|f\|_{\alpha,p} = \|D_{+}^{\alpha}f\|_{p}, \quad \langle f, g \rangle_{\alpha} = \langle D_{+}^{\alpha}f, D_{+}^{\alpha}g \rangle_{0}, \]

for \( f \in \mathcal{T}_{p}^{(\alpha)}(t^{\alpha})\), \( g \in \mathcal{T}_{p'}^{(\alpha)}(t^{\alpha})\).

Note that, in fact,

\[ (2.3) \]

Thus for \( f \in \mathcal{T}_{p}^{(\alpha)}(t^{\alpha})\) and \( g \in \mathcal{T}_{p'}^{(\alpha)}(t^{\alpha})\) with \( \frac{1}{p} + \frac{1}{p'} = 1 \).

In the next lemma, we consider some functions which belong (or not) to \( \mathcal{T}_{p}^{(\alpha)}(t^{\alpha}) \) for \( p \geq 1 \).

**Lemma 2.3.** If \( \alpha, a > 0 \) and \( p \geq 1 \), then

(i) \( t^{\beta} \notin \mathcal{T}_{p}^{(\alpha)}(t^{\alpha}) \) for \( \beta \in \mathbb{C} \).

(ii) \( (a + t)^{-\beta} \in \mathcal{T}_{p}^{(\alpha)}(t^{\alpha}) \) for \( \Re \beta > 1/p \).

**Proof.** (i) It suffices to note that \( t^{\beta} \) does not belong to \( L^{p}(\mathbb{R}^{+}) \).

(ii) For \( 0 < \Re \gamma < \Re \delta \) and \( a > 0 \) it is well known that \( W_{+}^{\gamma}(a + t)^{-\delta} = \frac{\Gamma(\delta - \gamma)}{\Gamma(\delta)}(t + a)^{\gamma - \delta} \), see for example [9, p. 201]. With this formula, it is easy to check that

\[ W_{+}^{\alpha}(a + t)^{-\beta} = \frac{\Gamma(\alpha + \beta)}{\Gamma(\beta)}(t + a)^{-(\alpha + \beta)}. \]

Thus for \( f(t) = (a + t)^{-\beta} \) we obtain

\[ \left\| f \right\|_{\alpha,p}^{p} = \frac{1}{\Gamma(\alpha + 1)^{p}} \int_{0}^{\infty} \left| W_{+}^{\alpha}(t) \right|^{p} t^{p} dt = \left( \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \right)^{p} \int_{0}^{\infty} \frac{t^{p - \beta}}{(t + a)^{(\alpha + \beta)p}} dt. \]

and we conclude the proof. \( \blacksquare \)

Given \( f \in \mathcal{T}_{p}^{(\alpha)}(t^{\alpha}) \), as next result shows, we obtain that the function \( f \in C(\mathbb{R}^{+}) \) for \( p, \alpha \geq 1 \).

**Proposition 2.4.** Take \( p, \alpha \geq 1 \) and \( f \in \mathcal{T}_{p}^{(\alpha)}(t^{\alpha}) \). Then \( f \in C(\mathbb{R}^{+}) \), \( \lim_{t \to \infty} f(t) = 0 \) and

\[ \sup_{t > 0} t^{p}|f(t)| \leq C_{\alpha,p} \left\| f \right\|_{\alpha,p}, \quad f \in \mathcal{T}_{p}^{(\alpha)}(t^{\alpha}), \]

where \( C_{\alpha,p} \) is independent of \( f \).

**Proof.** By Proposition 2.2 (i), it is enough to check for \( \alpha = 1 \). Take \( t > s > 0 \), and we get that

\[ |f(t) - f(s)| \leq \int_{s}^{t} |f'(u)| du \leq \frac{1}{s} \int_{s}^{t} |f'(u)| u du. \]

For \( p = 1 \), it is clear that \( f \) is continuous and for \( p > 1 \), we apply the Hölder inequality to obtain

\[ |f(t) - f(s)| \leq \left\| f \right\|_{1,p} (t - s)^{\frac{1}{p'}}, \quad \frac{1}{p} + \frac{1}{p'} = 1. \]
Then $f$ is continuous in $\mathbb{R}^+$. Take $f \in T^{(α)}_1(t^α)$, and
\[
|f(t)| \leq \int_1^∞ |f'(u)| du \leq \frac{1}{t} \int_1^∞ u|f'(u)| du, \quad t > 0,
\]
and we conclude that $\lim_{t \to ∞} f(t) = 0$. Similarly take $f \in T^{(α)}_p(t^α)$ with $1 < p < ∞$. Then we have that
\[
|f(t)| \leq \int_1^∞ |f'(u)| du \leq \left( \int_1^∞ u^p |f'(u)|^p du \right)^{\frac{1}{p}} \left( \int_1^∞ \frac{1}{u^p} du \right)^{\frac{1}{p}} \leq \left( \frac{1}{p(p-1)} \right)^{\frac{1}{p}} \|f\|_{1,p}
\]
where we conclude that $\sup_{t > 0} t^p |f(t)| \leq \left( \frac{1}{p} \right)^{\frac{1}{p}} \|f\|_{1,p}$ and the proof is finished. \hfill \blacksquare

The following is the main result of this section. It will be the key in the study of spectral properties of the generalized Cesàro operators $C_β$ and $C^*_β$ defined on Sobolev spaces.

**Theorem 2.5.** For $1 \leq p$ and $α ≥ 0$, the family of operators $(T_{t,p})_{t ∈ \mathbb{R}}$ defined by
\[
T_{t,p}f(s) := e^{-\frac{s}{t}} f(e^{-t}s), \quad f ∈ T^{(α)}_p(t^α),
\]
is a $C_0$-group of isometries on $T^{(α)}_p(t^α)$ whose infinitesimal generator $Λ$ is given by
\[
(Λf)(s) := -sf'(s) - \frac{1}{p} f(s)
\]
with domain $D(Λ) = T^{(α+1)}_p(t^{α+1})$.

**Proof.** We check that the operators $(T_{t,p})_{t ∈ \mathbb{R}}$ are isometries:
\[
||T_{t,p}f||_{α,p}^p = \frac{1}{\Gamma(α+1)p} \int_0^∞ |W^α_+ T_{t,p}f(s)|^ps^{αp} ds = \frac{e^{-t}}{\Gamma(α+1)p} \int_0^∞ |W^α_+ f(e^{-t}s)|^ps^{αp} ds
\]
\[
= \frac{e^{-t}}{\Gamma(α+1)p} \int_0^∞ e^{t} |W^α_+ f(u)|^p(e^{αt}u^α)^p du = ||f||_{α,p}^p,
\]
where we have applied the equality (2.1).

Using some known properties for fractional derivative (p. 96) it can be shown that the family of operators $(T_{t,p})_{t ∈ \mathbb{R}}$ are strongly continuous, see similar ideas in [3] Proposition 2.1 and [2] Section 2. It is straightforward to check that the family $(T_{t,p})_{t ∈ \mathbb{R}}$ is a group of operators.

On $T^{(α)}_p(t^α)$ define $\{S_t\}_{t ≥ 0}$ by $S_t(f)(s) := f(e^{-t}s)$. Then, an easy computation shows that the generator $A$ of $\{S_t\}_{t ≥ 0}$ with domain $\{f ∈ T^{(α)}_p(t^α) : t f' ∈ T^{(α)}_p(t^α)\}$ is given by $Af(s) = -sf'(s)$. Therefore, the rescaled semigroup $(T_{t,p})_{t ≥ 0}$ has domain $\{f ∈ T^{(α)}_p(t^α) : t f' ∈ T^{(α)}_p(t^α)\}$ and his generator is $(Λf)(s) = -sf'(s) - \frac{1}{p} f(s)$. See [8] p. 60 for more details.

Finally, we prove that $D(Λ) = T^{(α+1)}_p(t^{α+1})$. In fact, let $f ∈ T^{(α+1)}_p(t^{α+1})$. Since $T^{(α+1)}_p(t^{α+1}) → T^{(α)}_p(t^α)$, we have $f ∈ T^{(α)}_p(t^α)$. From [15] p. 246 it is easy to show that $W^α_+(tf'(t)) = αW^α_+f(t) + tW^{α+1}_+f(t)$. Thus, $tf' ∈ T^{(α)}_p(t^α)$ and therefore $f ∈ D(Λ)$. 

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Conversely, if \( f \in D(\Lambda) \), then \( f \in \mathcal{T}_p^{(\alpha)}(t^\alpha) \) and \( tf' \in \mathcal{T}_p^{(\alpha)}(t^\alpha) \). The same above identity, implies that \( t^{\alpha+1}W_{-}^{\alpha+1}f(t) = t^{\alpha}W_{-}^{\alpha}(tf'(t)) - \alpha t^{\alpha}W_{-}^{\alpha}f(t) \), and therefore \( f \in \mathcal{T}_p^{(\alpha+1)}(t^{\alpha+1}) \).  

The proof of the following result is inspired in [3 Proposition 2.3]. We denote by \( \sigma(\Lambda) \) the usual spectrum of the operator \( \Lambda \) and by \( \sigma_p(\Lambda) \) the point spectrum of the operator \( \Lambda \).

**Proposition 2.6.** For \( 1 \leq p < \infty \) we have

\[
\begin{align*}
(i) \quad & \sigma_p(\Lambda) = \emptyset; \\
(ii) \quad & \sigma(\Lambda) = i\mathbb{R}.
\end{align*}
\]

**Proof.** (i) Let \( \lambda \in \mathbb{C} \) and \( f \in \mathcal{T}_p^{(\alpha)}(t^\alpha) \) such that \( \Lambda(f) = \lambda f \). Then, \( f \) is solution of the differential equation

\[
sf'(s) + (\lambda + \frac{1}{p})f(s) = 0.
\]

The nonzero solutions to this equation have the form \( f(t) = ct^{-\lambda/p} \) with \( c \neq 0 \). But by Lemma 2.3 these solutions are not in \( \mathcal{T}_p^{(\alpha)}(t^\alpha) \). Therefore \( \sigma_p(\Lambda) = \emptyset \).

(ii) Since each \( T_{t,p} \) is an invertible isometry its spectrum satisfies

\[
\sigma(T_{t,p}) \subseteq \{ z \in \mathbb{C} : |z| = 1 \}.
\]

By the spectral mapping theorem (see Theorem 8 IV.3.6), we have that

\[
e^{t\sigma(\Lambda)} \subseteq \sigma(T_{t,p}).
\]

Therefore, if \( w \in \sigma(\Lambda) \), then \( e^{tw} \in \{ z \in \mathbb{C} : |z| = 1 \} \). Thus, we obtain that \( \sigma(\Lambda) \subseteq i\mathbb{R} \).

Conversely, let \( \mu \in i\mathbb{R} \) and assume that \( \mu \in \rho(\Lambda) \). Let \( \lambda = \mu + \frac{1}{p} \). By Lemma 2.3 the function \( f \) defined by \( f(t) := (1 + t)^{-\lambda - 1} \in \mathcal{T}_p^{(\alpha)}(t^\alpha) \). Since \( R(\mu, \Lambda) \) is a bounded operator, the function \( g(t) := R(\mu, \Lambda)f(t) \) belongs to \( \mathcal{T}_p^{(\alpha)}(t^\alpha) \). Therefore, \( g \) is solution of equation

\[
\lambda g(t) + tg'(t) = f(t).
\]

An easy computation shows that the solution of this equation is \( G(t) := ct^{-\lambda} + \lambda^{-1}(1+t)^{-\lambda} \), where \( c \) is a constant. However, as in Lemma 2.3 one can check that \( G \not\in \mathcal{T}_p^{(\alpha)}(t^\alpha) \). Therefore, \( \mu \in \sigma(\Lambda) \).  

Now, consider the negative part \( \{ T_{-t,p}, t \geq 0 \} \) of the group \( \{ T_{t,p}, t \in \mathbb{R} \} \): that is, for \( f \in \mathcal{T}_p^{(\alpha)}(t^\alpha) \),

\[
T_{-t,p}f(s) = e^{\frac{t}{p}}f(e^{\frac{1}{p}}s), \quad t \geq 0.
\]

Obviously, \( \{ T_{-t,p}, t \geq 0 \} \) is a \( C_0 \)-semigroup on \( \mathcal{T}_p^{(\alpha)}(t^\alpha) \) of isometries whose generator is \( -\Lambda \).

We finish this section, establishing the relationship between the semigroups \( \{ T_{t,p}, t \geq 0 \} \) and \( \{ T_{-t,p'}, t \geq 0 \} \) with \( \frac{1}{p} + \frac{1}{p'} = 1 \).

**Proposition 2.7.** The semigroups \( \{ T_{t,p}, t \geq 0 \} \) and \( \{ T_{-t,p'}, t \geq 0 \} \) are dual operators of each other acting on \( \mathcal{T}_p^{(\alpha)}(t^\alpha) \) and \( \mathcal{T}_{p'}^{(\alpha)}(t^\alpha) \) with \( \frac{1}{p} + \frac{1}{p'} = 1 \).
Proof. By the Proposition 2.2 (iv) the dual of \( T^p_{\alpha}(t^\alpha) \) is \( T^p_{\alpha'}(t^{\alpha'}) \) (with \( \frac{1}{p} + \frac{1}{p'} = 1 \)).

Take \( f \in T^p_{\alpha}(t^\alpha) \) and \( g \in T^p_{\alpha'}(t^{\alpha'}) \). Then

\[
\begin{align*}
\langle T_{t,p}f, g \rangle_{\alpha} &= \frac{1}{\Gamma(\alpha+1)^2} \int_0^\infty (W^\alpha_{t,p}f)(s)W^\alpha_{t,p}g(s)s^{2\alpha}ds \\
&= \frac{1}{\Gamma(\alpha+1)^2} \int_0^\infty e^{-\frac{s}{t}}e^{-\alpha}(W^\alpha_{t,p}f)(e^{-t}s)W^\alpha_{t,p}g(s)s^{2\alpha}ds \\
&= \frac{1}{\Gamma(\alpha+1)^2} \int_0^\infty e^{-\frac{s}{t}}e^{t}(W^\alpha_{t,p}f)(u)e^{\alpha t}(W^\alpha_{t,p}g)(ue^t)u^{2\alpha}du \\
&= \frac{1}{\Gamma(\alpha+1)^2} \int_0^\infty (W^\alpha_{t,p}f)(u)(W^\alpha_{t,p}g)(u)u^{2\alpha}du = \langle f, T_{t,p}g \rangle_{\alpha},
\end{align*}
\]

where we change the variable and we conclude the result.

\[ \square \]

3. Generalized Cesàro operators on Sobolev spaces defined on \( \mathbb{R}^+ \).

For \( \beta > 0 \) the generalized Cesàro operator on \( T^p_{\alpha}(t^\alpha) \) is defined by

\[
C_\beta f(t) := \frac{\beta}{t^\beta} \int_0^t (t-s)^{\beta-1}f(s)ds = \beta \int_0^1 (1-r)^{\beta-1}f(tr)dr, \quad t > 0.
\]

Defining the function

\[
g_\beta(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}, \quad t > 0,
\]

we obtain the also equivalent formulation of the generalized Cesàro operator in terms of finite convolution as follows:

\[
C_\beta f(t) := \frac{1}{g_{\beta+1}(t)} \int_0^t g_\beta(t-s)f(s)ds, \quad t > 0.
\]

We remark that for certain classes of vector-valued functions \( f \), the asymptotic behavior as \( t \to \infty \) of \( C_\beta f(t) \) in the above representation has been studied in [13].

Note that we may calculate \( C_\beta(f) \) for some particular functions:

Example 3.1. (i) Functions \( g_\gamma \) are eigenfunctions of \( C_\beta \) with eigenvalue \( \frac{\Gamma(\beta+1)\Gamma(\gamma)}{\Gamma(\beta+\gamma)} \):

\[
C_\beta(g_\gamma)(t) = \frac{\beta}{\Gamma(\gamma)t^{\beta-1}} \int_0^t (t-s)^{\beta-1}s^{\gamma-1}ds = \frac{\Gamma(\beta+1)\Gamma(\gamma)}{\Gamma(\beta+\gamma)} g_\gamma(t), \quad t > 0.
\]

(ii) Take \( e_\lambda(t) := e^{-\lambda t} \) for \( t > 0 \) and \( \lambda \in \mathbb{C}^+ \). Then

\[
C_1(e_\lambda)(t) = \frac{1}{\lambda t}(1-e^{-\lambda t}), \quad C_2(e_\lambda)(t) = \frac{2}{\lambda t}(e^{-\lambda t} - 1 + \lambda t), \quad t > 0.
\]

Since \( C_1^2(e_\lambda)(t) = \frac{1}{t\lambda} \int_0^t \frac{1-e^{-\lambda s}}{s}ds \) for \( t > 0 \), we conclude that \( C_1^2(e_\lambda) \neq C_2(e_\lambda) \) and then \( C_1^2 \neq C_2 \).
(iii) More generally, take \( f_\lambda(t) := E_{\beta,1}(\lambda t^\beta) \) the Mittag-Leffler function, for \( t > 0 \) and \( \lambda \in \mathbb{C}^+ \). Then

\[
C_\beta(f_\lambda)(t) = \frac{1}{\lambda g_{\beta+1}(t)}(1 - f_\lambda(t)), \quad t > 0.
\]

The relationship between these generalized Cesàro operators and fractional evolution equations of order \( \alpha \) can be also observed in [13].

The next Lemma shows a key commutativity property.

**Lemma 3.2.** Take \( \alpha \geq 0 \) and \( \beta > 0 \). Then

\[
D_\alpha^\alpha(C_\beta(f))(t) = C_\beta(D_\alpha^\alpha(f)), \quad f \in \mathcal{S}_+,
\]

where \( D_\alpha^\alpha(t) = \frac{1}{\Gamma(\alpha + 1)} t^\alpha W_+^\alpha f(t) \) for \( f \in \mathcal{S}_+ \).

**Proof.** By the equality (2.1), we have that

\[
C_\beta(D_\alpha^\alpha(f))(t) = \beta \int_0^1 (1 - r)^{\beta-1}(tr)^\alpha W_+^\alpha f(tr)dr
\]

\[
= t^\alpha W_+^\alpha \left( \beta \int_0^1 (1 - r)^{\beta-1}f(r)dr \right) = D_\alpha^\alpha(C_\beta(f))(t)
\]

for \( f \in \mathcal{S}_+ \) and we conclude the proof.

The first main result in this section is the following theorem.

**Theorem 3.3.** The operator \( C_\beta \) is a bounded operator on \( T_\alpha^p(t^\alpha) \) and

\[
||C_\beta|| = \frac{\Gamma(\beta + 1)\Gamma(1 - 1/p)}{\Gamma(\beta + 1 - 1/p)}
\]

for \( \alpha \geq 0, p > 1 \) and \( \beta > 0 \). If \( f \in T_\alpha^p(t^\alpha) \), then

\[
C_\beta f(t) = \beta \int_0^\infty (1 - e^{-r})^{\beta-1}e^{-r(1-1/p)}T_{r,p}f(t)dr, \quad t \geq 0,
\]

where the semigroup \( (T_{r,p})_{t \geq 0} \) is defined in Theorem 2.5.

**Proof.** We apply the change of variable \( s = te^{-r} \) to get that

\[
C_\beta f(t) = \frac{\beta}{t^\beta} \int_0^t (t-s)^{\beta-1}f(s)ds = \beta \int_0^\infty (1 - e^{-r})^{\beta-1}e^{-r}f(te^{-r})dr,
\]

and the equality (3.1) is proved. Observe that by this equality, \( C_\beta \) is well defined and is a bounded operator on \( T_\alpha^p(t^\alpha) \) for \( p > 1 \) : taken \( f \in T_\alpha^p(t^\alpha) \), then

\[
||C_\beta f||_{\alpha,p} \leq \beta \int_0^\infty (1 - e^{-r})^{\beta-1}e^{-r(1-1/p)}||T_{r,p}f||_{\alpha,\beta}dr
\]

\[
= \beta ||f||_{\alpha,p} \int_0^\infty (1 - e^{-r})^{\beta-1}e^{-r(1-1/p)}dr = ||f||_{\alpha,p} \frac{\Gamma(\beta + 1)\Gamma(1 - 1/p)}{\Gamma(\beta + 1 - 1/p)}.
\]
To check the exact value of $||C_\beta||$, note that by the Lemma 3.2 and the boundedness of $C_\beta$ on $L^p(\mathbb{R}^+)$ (see the Introduction) we get that

$$||C_\beta f||_{\alpha,p} = ||D_+^\alpha \circ C_\beta f||_p = \frac{\Gamma(\beta + 1)\Gamma(1 - 1/p)}{\Gamma(\beta + 1 - 1/p)} ||f||_{\alpha,p},$$

where we have applied Proposition 2.2 (iii).

**Remark 3.4.**

(i) Recall that the Beta function, also called the Euler integral of the first kind, is defined by:

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt, \quad x > 0, \quad y > 0,$$

and satisfies the property $B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$. Hence, the obtained value for the norm of $C_\beta$ can be rewritten as

$$||C_\beta|| = \beta B(\beta, 1 - 1/p), \quad \beta > 0, \quad p > 1.$$

(ii) In the case $p = 1$ we remark that $C_\beta$ does not take $T_1^{(\alpha)}(t^\alpha)$ in $T_1^{(\alpha)}(t^\alpha)$. In fact, from Lemma 2.3 it follows that, for $\beta > 0$, $h_\beta(t) := (1 + t)^{-(\beta+1)}$ belongs to $T_1^{(\alpha)}(t^\alpha)$. By [20, Formula 2, p.173] and [16, p. 38], we have

$$C_\beta h_\beta(t) = \frac{\beta}{t^\beta} \int_0^t \frac{(t-s)^{\beta-1}}{(1+s)^{\beta+1}} ds = 2F_1(1, \beta + 1; \beta + 1; -t) = (1 + t)^{-1},$$

where $2F_1$ denotes the Gaussian hypergeometric function,

$$2F_1(a, b; c; z) := \frac{\Gamma(c)}{\Gamma(b)\Gamma(a)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n) z^n}{\Gamma(c+n) n!}.$$

Since $C_\beta h_\beta$ does not belong to $L^1(\mathbb{R}^+)$ and $T_1^{(\alpha)}(t^\alpha) \hookrightarrow L^1(\mathbb{R}^+)$ (see Proposition 2.2 (i)), we obtain $C_\beta h_\beta \notin T_1^{(\alpha)}(t^\alpha)$.

(iii) Take $\beta = 1$ and $f \in T_p^{(\alpha)}(t^\alpha)$. Then

$$C_1 f(t) = \int_0^\infty e^{-r(1-1/p)} T_{r,p} f(t) dr = R(\lambda_p, \Lambda) f(t), \quad \lambda_p = 1 - 1/p > 0,$$

and by the spectral theorem for resolvent operators (see for example [8, Theorem IV.1.13]) we get that

$$\sigma(C_1) = \left\{ \lambda \in \mathbb{C} : \left| \lambda - \frac{p}{2(p-1)} \right| = \frac{p}{2(p-1)} \right\},$$

see [17, Theorem 2] and similar results in [3, Theorem 3.1], and [2, Corollary 2.2]. Here, $R(\cdot, \Lambda)$ denotes the resolvent operator of $\Lambda$.
Note that in case $\beta = 2$ we obtain
\[ C_2f(t) = 2 \int_0^\infty e^{-r(1-1/p)}(1 - e^{-r})T_{r,p}f(t)dr = 2R(\lambda_p, \Lambda)f(t) - 2R(\lambda_p + 1, \Lambda)f(t), \]
and, more generally, for $\beta = n + 1$,
\begin{equation}
C_{n+1}f(t) = (n + 1) \sum_{k=0}^{n} \binom{n}{k} (-1)^k R(\lambda_p + k, \Lambda)f(t), \quad n \in \mathbb{Z}_.
\end{equation}

In the next result, we are able to describe $\sigma(C_\beta)$ for $\beta > 0$.

**Theorem 3.5.** Let $1 < p < \infty$, and $C_\beta : T_\alpha(t^\alpha) \to T_\alpha(t^\alpha)$ the generalized Cesàro operator. Then
\[ \sigma(C_\beta) = \beta B(\beta, 1 - 1/p + i\mathbb{R}) := \Gamma(\beta + 1) \left\{ \frac{\Gamma(1 - \frac{1}{p} + it)}{\Gamma(\beta + 1 - \frac{1}{p} + it)} : t \in \mathbb{R} \right\}. \]

**Proof.** As $(T_{t,p})_{t \in \mathbb{R}}$ is an uniformly bounded $C_0$-group (Theorem 2.7) whose infinitesimal generator is $(\Lambda, D(\Lambda))$ and $C_\beta = \hat{f}_{\beta,p}(\Lambda)$, i.e.,
\[ C_\beta f = \beta \int_0^\infty (1 - e^{-r})^{\beta-1}e^{-r(1-1/p)}T_{r,p}fdr = \int_{-\infty}^\infty f_{\beta,p}(r)T_{r,p}fdr, \]
where $f_{\beta,p}(r) = \chi_{[0,\infty)}(r)(1 - e^{-r})^{\beta-1}e^{-r(1-1/p)}$ for $r \in \mathbb{R}$, see Theorem 3.3. By [21, Theorem 3.1], we obtain
\[ \sigma(C_\beta) = \hat{f}_{\beta,p}(\sigma(i\Lambda)) \]
where $\hat{f}_{\beta,p}$ is the Fourier transform of the function $f_{\beta,p}$. As $\sigma(i\Lambda) = \mathbb{R}$ (see Proposition 2.6 (ii)) and $\hat{f}_{\beta,p}(t) = \mathcal{L}(f_{\beta,p})(it)$ we use that
\[ \mathcal{L}(f_{\beta,p})(z) = \beta \int_0^\infty e^{-zr}(1 - e^{-r})^{\beta-1}e^{-r(1-1/p)}dr = \frac{\Gamma(\beta + 1)\Gamma(1 - \frac{1}{p} + z)}{\Gamma(\beta + 1 - \frac{1}{p} + z)}, \quad z \in \mathbb{C}^+. \]
to conclude the result. \[ \square \]

**Remark 3.6.** In the case that $n \in \mathbb{N}$, we obtain that
\[ \sigma(C_n) = \left\{ \frac{n!p^n}{((n+it)p-1)\ldots((1+it)p-1)} : t \in \mathbb{R} \right\} \cup \{0\}, \]
and for $n = 1$
\[ \sigma(C_1) = \left\{ \frac{p}{(1+it)p-1} : t \in \mathbb{R} \right\} \cup \{0\} = \left\{ w \in \mathbb{C} : \left| w - \frac{p}{2(p-1)} \right| = \frac{p}{2(p-1)} \right\}. \]
Now we consider the generalized dual Cesàro operator $C^*_β$ on $T_p^{(α)}(t^α)$ defined by

$$C^*_βf(t) := \beta \int_t^∞ \frac{(s-t)^{β-1}}{s^β}f(s)ds = \beta \int_1^∞ \frac{(r-1)^{β-1}}{r^β}f(tr)dr, \quad t > 0.$$  

For $0 < γ < 1$, functions $g_γ$ are eigenfunctions of $C^*_β$ with eigenvalue $\Gamma(β+1)Γ(1−γ)/Γ(β−γ+1)$:

$$C^*_β(g_γ)(t) = \frac{β}{Γ(γ)}\int_t^∞ \frac{(s-t)^{β-1}s^{γ-1}}{s^β}ds = \frac{Γ(β+1)Γ(1−γ)}{Γ(β−γ+1)}g_γ(t),$$

for $t > 0$.

Using similar ideas to Lemma 3.2, we obtain

$$(3.5) \quad D_+^α \circ C^*_β(f) = C^*_β \circ D_+^α(f), \quad f ∈ S_+$$

where $D_+^αf(t) = \frac{1}{Γ(α+1)}t^αW_+^αf(t)$ for $f ∈ S_+$ and $t ≥ 0$. Hence we are ready to prove our main dual result.

**Theorem 3.7.** The operator $C^*_β$ is a bounded operator on $T_p^{(α)}(t^α)$ and

$$||C^*_β|| = \frac{Γ(β+1)Γ(1/p)}{Γ(β+1/p)},$$

for $α ≥ 0$, $p > 1$ and $β > 0$. The dual operator of $C_β$ on $T_p^{(α)}(t^α)$ is $C^*_β$ on $T_p^{(α)}(t^α)$, i.e.

$$\{C_βf, g\}_α = \{f, C^*_βg\}_α, \quad f ∈ T_p^{(α)}(t^α), \quad g ∈ T_p^{(α)}(t^α),$$

where $\{ , \}_α$ is given in Proposition 2.2 (iv) and $\frac{1}{p} + \frac{1}{p'} = 1$.

If $f ∈ T_p^{(α)}(t^α)$, then

$$C^*_βf(t) = β \int_{−∞}^0 (e^{−r}−1)^{β−1}e^{−r(1−(1/p−β))}T_{r,p}f(t)dr, \quad t ≥ 0,$$

where the $C_0$-group $(T_{r,p})_{t∈R}$ is defined in Theorem 2.2.

**Proof.** We change the variable $s = te^r$ to obtain

$$C^*_βf(t) = β \int_t^∞ \frac{(s-t)^{β-1}}{s^β}f(s)ds = β \int_0^∞ (e^r−1)^{β−1}e^{r(1−β)}f(te^r)dr$$

$$= β \int_0^∞ (e^r−1)^{β−1}e^{r(1−β)}e^{\frac{r}{p'}}f(te^r)dr$$

$$= β \int_{−∞}^0 (e^{−r}−1)^{β−1}e^{−r(1−(1/p−β))}T_{r,p}f(t)dr$$

for $f ∈ T_p^{(α)}(t^α)$, $p ≥ 1$ and $t ≥ 0$. To check $||C^*_β||$, note that by the Lemma 3.2 and the boundedness of $C_β$ on $L^p(\mathbb{R}^+)$ (see the Introduction) we get that

$$||C^*_β||_{α,p} = ||D_+^α \circ C^*_βf||_p = ||C^*_β \circ D_+^αf||_p$$

$$= \frac{Γ(β+1)Γ(1/p)}{Γ(β+1/p)}||D_+^αf||_p = \frac{Γ(β+1)Γ(1/p)}{Γ(β+1/p)}||f||_{α,p}.$$
where we have applied the Proposition 2.2 (iii) and the equality (3.3).
To conclude the proof, take \( f \in \mathcal{T}_{p}^{(\alpha)}(t^{\alpha}) \), and \( g \in \mathcal{T}_{p}^{(\alpha)}(t^{\alpha}) \). Then
\[
\langle C_{\beta}f,g \rangle_{\alpha} = \langle D_{+}^{\alpha}(C_{\beta}f), D_{+}^{\alpha}g \rangle_{0} = \langle C_{\beta}(D_{+}^{\alpha}f), D_{+}^{\alpha}g \rangle_{0} = \langle D_{+}^{\alpha}f, C_{\beta}^{\ast}(D_{+}^{\alpha}g) \rangle_{0} = \langle f, C_{\beta}^{\ast}g \rangle_{\alpha},
\]
where we have used Lemma 3.2, \((C_{\beta})^{\ast} = C_{\beta}^{\ast}\) and equality \((3.3)\). 

**Remark 3.8.** Take \( \beta = 1 \) and \( f \in \mathcal{T}_{p}^{(\alpha)}(t^{\alpha}) \). Then
\[
C_{\beta}^{\ast}f(t) = \int_{-\infty}^{0} e^{-\frac{z}{t^{\alpha}}} T_{-\alpha}f(t) dr ds = R(1/p,-\Lambda)f(t), \quad t \geq 0.
\]
and by the spectral theorem for the resolvent operator, see \[8\] Theorem IV.1.13, we obtain
\[
\sigma(C_{\beta}^{\ast}) = \left\{ w \in \mathbb{C} : \left| w - \frac{p}{2} \right| = \frac{p}{2} \right\}.
\]
This gives a proof of a conjecture posed by F. Móricz in \[17\] Section 2. See a similar result in \[3\] Theorem 3.2.

In the following theorem we describe \( \sigma(C_{\beta}^{\ast}) \) for \( \beta > 0 \).

**Theorem 3.9.** Let \( \beta > 0 \), \( 1 \leq p < \infty \), and \( C_{\beta}^{\ast} : \mathcal{T}_{p}^{(\alpha)}(t^{\alpha}) \to \mathcal{T}_{p}^{(\alpha)}(t^{\alpha}) \) the generalized dual Cesàro operator. Then
\[
\sigma(C_{\beta}^{\ast}) = \beta B(\beta,1/p + i\mathbb{R}) := \Gamma(\beta + 1) \left\{ \frac{\Gamma\left(\frac{1}{p} + it\right)}{\Gamma(\beta + \frac{1}{p} + it)} : t \in \mathbb{R} \right\}.
\]

**Proof.** We remind that \( (T_{t})_{t \in \mathbb{R}} \) is a uniformly bounded \( C_{0} \)-group whose infinitesimal generator is \( (\Lambda, D(\Lambda)) \) and \( C_{\beta}^{\ast} = \overline{g}_{\beta,p}(\Lambda) \), i.e,
\[
C_{\beta}^{\ast}f = \beta \int_{-\infty}^{0} (e^{-r} - 1)^{\beta - 1} e^{-r(1-1/p-\beta)} T_{r}f dr = \int_{-\infty}^{\infty} g_{\beta,p}(r) T_{r}f dr,
\]
where \( g_{\beta,p}(r) = \chi_{(-\infty,0]}(r) \beta(e^{-r} - 1)^{\beta - 1} e^{-r(1-1/p-\beta)} \) for \( r \in \mathbb{R} \), see Theorem 3.7 By \[21\] Theorem 3.1, we obtain
\[
\sigma(C_{\beta}^{\ast}) = \overline{g}_{\beta,p}(\sigma(i\Lambda))
\]
where \( \overline{g}_{\beta,p} \) is the Fourier transform of the function \( g_{\beta,p} \). As \( \sigma(i\Lambda) = \mathbb{R} \), and \( \overline{g}_{\beta,p}(t) = \mathcal{L}(\overline{g}_{\beta,p})(-it) \) (where \( \overline{g}_{\beta,p}(\xi) := g_{\beta,p}(-\xi) \) for \( \xi \geq 0 \)) we use that
\[
\mathcal{L}(\overline{g}_{\beta,p})(z) = \beta \int_{0}^{\infty} e^{-zr}(e^{-r} - 1)^{\beta - 1} e^{-r(1-1/p-\beta)} dr = \frac{\Gamma(\beta + 1)\Gamma\left(\frac{1}{p} + z\right)}{\Gamma(\beta + \frac{1}{p} + z)}, \quad z \in \mathbb{C}^{+},
\]
to conclude the result. 

\[\square\]
Remark 3.10. In the case that $n \in \mathbb{N}$, we obtain that
\[
\sigma(C_n^*) = \left\{ \frac{n!p^n}{((n-1)p + 1 + it) \ldots (p + 1 + it)(1 + it)} : t \in \mathbb{R} \right\} \cup \{0\},
\]
and for $n = 1$
\[
\sigma(C_1^*) = \left\{ \frac{p}{1 + it} : t \in \mathbb{R} \right\} \cup \{0\} = \left\{ w \in \mathbb{C} : \left| w - \frac{p}{2} \right| = \frac{p}{2} \right\}.
\]

Remark 3.11. In the case that $p = 2$ we have $\sigma(C_\beta) = \sigma(C_\beta^*)$ for all $\beta > 0$. Note that in case $p \neq 2$ the spectrum of $C_\beta$ and $C_\beta^*$ are dual in the sense that $\sigma(C_\beta)$, with $C_\beta$ defined on $T_p^{(\alpha)}(t^\omega)$, is identical to $\sigma(C_\beta^*)$, with $C_\beta^*$ defined on $T_p^{(\alpha)}(t^\omega)$, and where $\frac{1}{p} + \frac{1}{p'} = 1$.

To finish this section we prove the remarkable fact that $C_\alpha$ and $C_\beta^*$ commute on $L^p(\mathbb{R}^+)$ (and then on $T_p^{(\alpha)}(t^\omega)$). We also give explicitly the value of $C_\alpha C_\beta^*$ in terms of the the Gaussian hypergeometric function $2F_1$. This theorem includes [17, Lemma 2] for $\alpha = \beta = 1$.

Theorem 3.12. Let $C_\alpha$ and $C_\beta^*$ the generalized Cesàro operators on $L^p(\mathbb{R}^+)$ for $p > 1$. Then $C_\alpha C_\beta^* = C_\beta^* C_\alpha$ for $\alpha, \beta > 0$ and
\[
(C_\alpha C_\beta^*) f(t) = \alpha \int_0^t f(r) \frac{1}{t-r} \left( \frac{t-r}{t} \right)^{\alpha+\beta} 2F_1(\alpha + \beta, \beta + 1; 1; \frac{r}{t}) dr
+ \beta \int_t^\infty f(r) \frac{1}{r-t} \left( \frac{r-t}{r} \right)^{\alpha+\beta} 2F_1(\alpha + \beta, \alpha + 1; 1; \frac{t}{r}) dr,
\]
in particular
\[
(C_1 C_\beta^*) f(t) = C_1 f(t) + \beta \int_t^\infty f(r) \frac{(r-t)^\beta}{r^{\beta+1}} 2F_1(\beta + 1, 2; 1; \frac{r}{t}) dr,
\]
\[
(C_\alpha C_1^*) f(t) = \alpha \int_0^t f(r) \frac{(t-r)^\alpha}{r^{\alpha+1}} 2F_1(\alpha + 1, 2; 1; \frac{t}{r}) dr + C_1^* f(t),
\]
\[
(C_1 C_1^*) f = C_1 f + C_1^* f = (C_1^* C_1) f,
\]
for $f \in L^p(\mathbb{R}^+)$ and $t$ almost everywhere on $\mathbb{R}^+$.

Proof. By the integral representations (3.1) and (3.6), and since $T_{t,p}$ commutes with $T_{r,p}$ for any $t, r \in \mathbb{R}$, we conclude that $C_\alpha C_\beta^* = C_\beta^* C_\alpha$ for $\alpha, \beta > 0$. Take $f \in L^p(\mathbb{R}^+)$ and we apply the Fubini theorem to get that
\[
C_\beta^* C_\alpha f(t) = \beta \alpha \int_0^\infty \int_0^\infty \frac{x-t)^{\beta-1}}{x^{\beta+\alpha}} \int_0^x (x-r)^{\alpha-1} f(r) dr dx
+ \beta \alpha \int_0^{\infty} f(r) \int_{\max(t,r)}^{\infty} \frac{(x-t)^{\beta-1}(x-r)^{\alpha-1}}{x^{\beta+\alpha}} dx dr.
\]
for \( t \) almost everywhere on \( \mathbb{R}^+ \). For \( 0 < r < t \), this equality
\[
\int_0^\infty \frac{(x-t)^{\beta-1}(x-r)^{\alpha-1}}{x^{\beta+\alpha}}dx = \frac{1}{\beta(t-r)} \left( \frac{t-r}{t} \right)^{\alpha+\beta} 2F_1(\alpha+\beta; \beta+1; \frac{r}{t})
\]
holds, see for example [16, p. 314, 3197(1)].

Now take \( \alpha = 1 \). Since
\[
(1-z)^\alpha 2F_1(a,b;c;z) = 2F_1(a,c-b;c;\frac{z}{z-1}),
\]
(see for example [16, p. 47]), we get that
\[
\frac{1}{t-r} \left( \frac{t-r}{t} \right)^{1+\beta} 2F_1(1+\beta; \beta+1; \frac{r}{t}) = \frac{1}{t-r} 2F_1(1+\beta; 1+\beta; \frac{-r}{t-r}) = \frac{1}{t}
\]
where we apply that \( 2F_1(-a,b;b;-z) = (1+z)^a \), ([16, p. 38]). Similarly we prove the case \( \beta = 1 \).

\[\Box\]

4. Composition groups on Sobolev spaces defined on \( \mathbb{R} \).

In this section we introduce the subspaces \( \mathcal{T}_p^{(\alpha)}(||t||^a) \) which are contained in \( L^p(\mathbb{R}) \), similarly to \( \mathcal{T}_p^{(\alpha)}(t^n) \) are in \( L^p(\mathbb{R}^+) \). Let \( \mathcal{S} \) be the Schwartz class on \( \mathbb{R} \) and we set
\[
W^{-\alpha}_a f(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^x (x-t)^{\alpha-1} f(t) dt,
\]
\[
W^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_{-\infty}^x (x-t)^{n-\alpha-1} f(t) dt,
\]
and \( W^0 f = f \), for \( x \in \mathbb{R} \) and a natural number \( n > \alpha \). Putting \( \tilde{f}(x) = f(-x) \), it is readily seen that \( W^\alpha_a f(x) = W^\alpha_a \tilde{f}(-x) \) for all \( \alpha \in \mathbb{R} \), \( f \in \mathcal{S} \) and \( x \in \mathbb{R} \). Equalities \( W^{-\alpha+\beta}_a = W^\alpha_a W^\beta_a \) and \( W^\alpha f = f^{(\alpha)} \) hold for each natural number \( n \) and \( \alpha, \beta \in \mathbb{R} \).

For \( f \in \mathcal{S} \), put
\[
W^{0}_a f(t) := \begin{cases} W^\alpha_a f(t), & t < 0, \\ e^{i\pi \alpha} W^\alpha_a f(t), & t > 0. \end{cases}
\]
For \( \lambda > 0 \), we have that \( W^{\alpha}_a (f_\lambda) = \lambda^{\alpha} (W^\alpha_0 f)_\lambda \), where \( f_\lambda(t) = f(\lambda t) \) for \( t \in \mathbb{R} \).

**Definition 4.1.** Let \( 1 \leq p < \infty \). The Banach space \( \mathcal{T}_p^{(\alpha)}(||t||^a) \) is defined as the completion of the Schwartz class on \( \mathbb{R} \) in the norm
\[
||f||_{\alpha,p} := \frac{1}{\Gamma(\alpha+1)} \left( \int_{-\infty}^{\infty} (|W^\alpha_a f(t)| ||t||^a)^p dt \right)^{\frac{1}{p}}.
\]

Similar properties to \( \mathcal{T}_p^{(\alpha)}(t^n) \) hold in \( \mathcal{T}_p^{(\alpha)}(||t||^a) \). The proof of next proposition is similar to the proof of Proposition 2.2 and we skip it.

**Proposition 4.2.** Take \( p \geq 1 \) and \( \beta > \alpha > 0 \). Then
\(\begin{align*}
(\text{i}) \quad \mathcal{T}_p^{(\beta)}(||t||^\beta) \hookrightarrow \mathcal{T}_p^{(\alpha)}(||t||^\alpha) \hookrightarrow L^p(\mathbb{R}).
\end{align*}\)
(ii) The operator $D_0^\alpha : \mathcal{T}_p^{(\alpha)}(|t|^\alpha) \to L^p(\mathbb{R})$ defined by

$$f \mapsto D_0^\alpha f(t) := \frac{1}{\Gamma(\alpha + 1)} |t|^\alpha W_0^\alpha f(t), \quad t \in \mathbb{R}, \quad f \in \mathcal{T}_p^{(\alpha)}(|t|^\alpha),$$

is an isometry.

(iii) If $p > 1$ and $p'$ satisfies $\frac{1}{p} + \frac{1}{p'} = 1$, then the dual of $\mathcal{T}_p^{(\alpha)}(|t|^\alpha)$ is $\mathcal{T}_{p'}^{(\alpha)}(|t|^\alpha)$, where the duality is given by

$$\langle f, g \rangle_\alpha = \frac{1}{\Gamma(\alpha + 1)^2} \int_{-\infty}^{\infty} W_0^\alpha f(t) W_0^\alpha g(t) |t|^{2\alpha} dt,$$

for $f \in \mathcal{T}_p^{(\alpha)}(|t|^\alpha)$, $g \in \mathcal{T}_{p'}^{(\alpha)}(|t|^\alpha)$.

For $p = 1$, the subspace $\mathcal{T}_1^{(\alpha)}(|t|^\alpha)$ was introduced in [12, Definition 1.9]. In fact $\mathcal{T}_1^{(\alpha)}(|t|^\alpha)$ is a subalgebra of $L^1(\mathbb{R})$ for the convolution product

$$(4.1) \quad f \ast g(t) = \int_{-\infty}^{\infty} f(t - s) g(s) ds, \quad t \in \mathbb{R}, \quad f, g \in \mathcal{T}_1^{(\alpha)}(|t|^\alpha),$$

see [12, Theorem 1.8] and also [13, Theorem 2] for some more details.

**Theorem 4.3.** Let $1 < p < \infty$. The Banach space $\mathcal{T}_p^{(\alpha)}(|t|^\alpha)$ is a module for the algebra $\mathcal{T}_1^{(\alpha)}(|t|^\alpha)$ and

$$|||f \ast g|||_{p,\alpha} \leq C_{\alpha,p} ||f|||_{p,\alpha} ||g|||_{1,\alpha}, \quad f \in \mathcal{T}_p^{(\alpha)}(|t|^\alpha), \quad g \in \mathcal{T}_1^{(\alpha)}(|t|^\alpha).$$

**Proof.** Take $f, g \in \mathcal{S}$. We write $f_+ := f\chi_{(0,\infty)}$ and $f_- := f\chi_{(-\infty,0]}$. By considering the decomposition $f \ast g = (f_+ \ast g_+) + (f_+ \ast g_-) + (f_- \ast g_+) + (f_- \ast g_-)$ on $\mathbb{R}$, and we apply [12, Lemma 1.6] and the fact that $f_- \ast g_- = 0$ on $(0, \infty)$ to obtain that

$$W_+^\alpha(f \ast g)_+(t) = W_+^\alpha(f_+ \ast g_+)(t) + (W_+^\alpha f_+ \ast g_-)(t) + (W_+^\alpha g_+ \ast f_-)(t), \quad t > 0.$$

Now, first,

$$||f_+ \ast g_+|||_{\alpha,p} \leq C_{\alpha,p} ||f_+|||_{\alpha,p} ||g_+|||_{1,\alpha} \leq C_{\alpha,p} ||f|||_{\alpha,p} ||g|||_{1,\alpha}$$

by Proposition 2.2(ii).

On the other hand, $\mathcal{T}_1^{(\alpha)}(t^\alpha) \subset L^1(\mathbb{R}^+)$, and we apply the Minkowski inequality to get that

$$\left( \int_0^\infty |W_+^\alpha f_+ \ast g_-(t)|^p t^{\alpha p} dt \right)^{\frac{1}{p}} \leq \left( \int_0^\infty \left( \int_0^\infty |W_+^\alpha f_+(s + t)||g_-(s)| ds \right)^p t^{\alpha p} dt \right)^{\frac{1}{p}}$$

$$= \int_0^\infty |g_-(s)| \left( \int_0^\infty |W_+^\alpha f_+(t + s)|^p t^{\alpha p} dt \right)^{\frac{1}{p}} ds \leq \int_0^\infty |g_-(s)| \left( \int_s^\infty |W_+^\alpha f_+(u)|^p u^{\alpha p} du \right)^{\frac{1}{p}} ds$$

$$\leq \Gamma(\alpha + 1)||g|||_{0,1} ||f_+|||_{\alpha,p} \leq \Gamma(\alpha + 1)||g|||_{1,\alpha} ||f|||_{\alpha,p}.$$
As $\mathcal{T}_p^{(\alpha)}(\alpha) \subset L^p(\mathbb{R}^+)$ for $p > 1$, and we apply again the Minkowski inequality to obtain that

$$
\left( \int_0^\infty |(W_+^{\alpha}g_+ * f_+)(t)(t)|^p t^{\alpha p} dt \right)^{\frac{1}{p}} \leq \left( \int_0^\infty \left( \int_{t}^{\infty} |W_+^{\alpha}g_+(s)||f_+(-t + s)| ds \right)^p t^{\alpha p} dt \right)^{\frac{1}{p}}
$$

\[\begin{align*}
&= \int_0^\infty |W_+^{\alpha}g_+(s)| \left( \int_s^\infty |f_+(-t + s)|^p t^{\alpha p} dt \right)^{\frac{1}{p}} ds \\
&\leq \Gamma(\alpha + 1)||f||_{\alpha,p} ||g_+||_{\alpha,1} \leq \Gamma(\alpha + 1)||f||_{\alpha,p} ||g||_{\alpha,1}.
\end{align*}\]

Combining these estimates obtained, we get

$$
\frac{1}{\Gamma(\alpha + 1)} \left( \int_0^\infty |W_+^{\alpha}(f * g)(t)|^p t^{\alpha p} dt \right)^{\frac{1}{p}} \leq C||f||_{\alpha,p} ||g||_{\alpha,1}.
$$

Finally, because $W_+^{\alpha}(f * g)(t) = W_+^{\alpha}(\hat{f} * \hat{g})(-t)$ if $t < 0$ using the inclusion $\mathcal{T}_p^{(\alpha)}(\alpha) \subset L^p(\mathbb{R}^+)$ as above for $p > 1$, we have that

$$
\frac{1}{\Gamma(\alpha + 1)} \left( \int_{-\infty}^0 |W_+^{\alpha}(f * g)(t)|^p t^{\alpha p} dt \right)^{\frac{1}{p}} \leq C||f||_{\alpha,p} ||g||_{\alpha,1}.
$$

The result follows.

We remark that, as in the case of $\mathcal{T}_p^{(\alpha)}(\alpha)$, it is easy to verify that $(T_{t,p})_{t \in \mathbb{R}}$ is a $C_0$-group of isometries on $\mathcal{T}_p^{(\alpha)}(|t|^\alpha)$ as the next theorem shows. The proof runs parallel to the proofs of Theorem 2.5, Proposition 2.6 and Proposition 2.7 and hence we omit it.

**Theorem 4.4.** Let $1 \leq p$ and $\alpha \geq 0$. We define the family of operators $(T_{t,p})_{t \in \mathbb{R}}$ by

$$
T_{t,p}f(s) := e^{-\frac{t}{p}}f(e^{-t}s), \quad f \in \mathcal{T}_p^{(\alpha)}(|t|^\alpha).
$$

(i) Then $(T_{t,p})_{t \in \mathbb{R}}$ is a $C_0$-group of isometries on $\mathcal{T}_p^{(\alpha)}(|t|^\alpha)$ whose infinitesimal generator $\Lambda$ is given by

$$
(\Lambda f)(s) := -sf'(s) - \frac{1}{p}f(s)
$$

with domain $D(\Lambda) = \mathcal{T}_p^{(\alpha+1)}(|t|^{\alpha+1})$.

(ii) $\sigma_p(\Lambda) = \emptyset$ and $\sigma(\Lambda) = i\mathbb{R}$ (here $\sigma_p$ denotes the point spectrum).

(iii) The semigroups $(T_{t,p})_{t \geq 0}$ and $(T_{-t,p})_{t \geq 0}$ are dual operators of each other acting on $\mathcal{T}_p^{(\alpha)}(|t|^\alpha)$ and $\mathcal{T}_p^{(\alpha)}(|t|^{\alpha})$ with $\frac{1}{p} + \frac{1}{p'} = 1$ for $p > 1$. 

5. The generalized Cesàro operators on \( \mathbb{R} \).

For \( \beta > 0 \) we define the generalized Cesàro operator by

\[
C_\beta f(t) := \begin{cases} 
\frac{\beta}{|t|^\beta} \int_t^0 (s-t)^{\beta-1} f(s) ds, & t < 0, \\
f(0), & t = 0, \\
\frac{\beta}{t^\beta} \int_0^t (t-s)^{\beta-1} f(s) ds, & t > 0,
\end{cases}
\]

for \( f \in \mathcal{S} \). We are interested in the extension of \( C_\beta \) on \( T^{(\alpha)}_p(|t|^\alpha) \). Note that we may write

\[
C_\beta f(t) = \beta \int_0^1 (1-r)^{\beta-1} f(tr) dr, \quad t \in \mathbb{R}, \; f \in \mathcal{S}.
\]

We use this integral representation to prove the next lemma.

**Lemma 5.1.** Take \( \alpha \geq 0 \) and \( \beta > 0 \). Then \( D_0^\alpha \circ C_\beta = C_\beta \circ D_0^\alpha \), i.e.,

\[
D_0^\alpha(C_\beta(f)) = C_\beta(D_0^\alpha(f)), \quad f \in \mathcal{S},
\]

where \( D_0^\alpha f(t) = \frac{1}{\Gamma(\alpha+1)} |t|^\alpha W_0^\alpha f(t) \) for \( f \in \mathcal{S} \).

**Proof.** Since for \( \lambda > 0 \), we have that \( W_0^\alpha(f_\lambda) = \lambda^\alpha(W_0^\alpha f)_\lambda \), where \( f_\lambda(t) = f(\lambda t) \) for \( t \in \mathbb{R} \), the proof follows similarly to Lemma 3.2. \( \blacksquare \)

Similar results of \( C_\beta \) on \( T^{(\alpha)}_p(|t|^\alpha) \) hold for \( C_\beta \) on \( T^{(\alpha)}(\mathbb{R}) \). The proof of next result is analogous to the proof of Theorem 3.3 and Theorem 3.5.

**Theorem 5.2.** Let \( \alpha \geq 0, \; \beta > 0, \; 1 < p < \infty \) and the generalized Cesàro operator \( C_\beta \) on \( T^{(\alpha)}_p(|t|^\alpha) \). Then

(i) The operator \( C_\beta \) is bounded on \( T^{(\alpha)}_p(|t|^\alpha) \) and

\[
||C_\beta|| = \frac{\Gamma(\beta+1)\Gamma(1-1/p)}{\Gamma(\beta+1-1/p)}.
\]

(ii) If \( f \in T^{(\alpha)}_p(|t|^\alpha) \), then

\[
C_\beta f(t) = \beta \int_0^\infty (1-e^{-r})^{\beta-1} e^{-r(1-1/p)} T_{r,p} f(t) dr, \quad t \in \mathbb{R},
\]

where the \( C_0 \)-group \( (T_{r,p})_{r \in \mathbb{R}} \) is defined in Theorem 4.4.

(iii)

\[
\sigma(C_\beta) = \Gamma(\beta+1) \left\{ \frac{\Gamma(1-\frac{1}{p}+it)}{\Gamma(\beta+1-\frac{1}{p}+it)} : t \in \mathbb{R} \right\}.
\]
Now we consider the generalized dual Cesàro operator $C^*_\beta$ defined for $\beta > 0$ by

$$C^*_\beta f(t) := \begin{cases} 
\beta \int_{-\infty}^{t} \frac{(t-s)^{-1}}{|s|^\beta} f(s) ds, & t < 0, \\
0, & t = 0, \\
\beta \int_{t}^{\infty} \frac{(s-t)^{-1}}{|s|^\beta} f(s) ds, & t > 0, 
\end{cases}$$

and $D_0^\alpha \circ C^*_\beta = C^*_\beta \circ D_0^\alpha$, where $D_0^\alpha f(t) = \frac{1}{\Gamma(\alpha+1)} |t|^\alpha W_0^\alpha f(t)$ for $f \in S$ and $t \in \mathbb{R}$.

Note that we may write

$$C^*_\beta f(t) = \beta \int_{1}^{\infty} \frac{(s-1)^{-1}}{|s|^\beta} f(t s) ds, \quad t \neq 0,$$

for $f \in S$. The proof of next result runs parallel to the proof of Theorem 3.7 and 3.9.

**Theorem 5.3.** Let $\alpha \geq 0$, $\beta > 0$, $1 \leq p < \infty$ and the generalized dual Cesàro operator $C^*_\beta$ on $T_p^{(\alpha)}(|t|^\alpha)$. Then

(i) The operator $C^*_\beta$ is bounded on $T_p^{(\alpha)}(|t|^\alpha)$ and

$$||C^*_\beta|| = \frac{\Gamma(\beta + 1) \Gamma(1/p)}{\Gamma(\beta + 1/p)}.$$

(ii) The dual operator of $C_\beta$ on $T_{p'}^{(\alpha)}(|t|^\alpha)$ is $C^*_\beta$ on $T_{p'}^{(\alpha)}(|t|^\alpha)$, i.e.

$$\langle C_\beta f, g \rangle_\alpha = \langle f, C^*_\beta g \rangle_\alpha, \quad f \in T_p^{(\alpha)}(|t|^\alpha), \quad g \in T_{p'}^{(\alpha)}(|t|^\alpha),$$

where $\langle \cdot , \cdot \rangle_\alpha$ is given in Proposition 4.2 (iii).

(iii) If $f \in T_p^{(\alpha)}(|t|^\alpha)$, then

$$C^*_\beta f(t) = \beta \int_{-\infty}^{0} (e^{-r} - 1)^{-1} e^{-r(1-1/p-\beta)} T_{r,p} f(t) dr, \quad t \in \mathbb{R},$$

where the $C_0$-group $(T_{r,p})_{r \in \mathbb{R}}$ is defined in Theorem 4.4.

(iv)

$$\sigma(C^*_\beta) = \Gamma(\beta + 1) \left\{ \begin{array}{c} \Gamma(\frac{1}{p} + it) \\
\Gamma(\beta + \frac{1}{p} + it) \\
: \quad t \in \mathbb{R} \end{array} \right\}.$$

**Remark 5.4.** Note that for $t = 0$, by the integral representation (5.1)

$$C^*_\beta f(0) = f(0) \beta \int_{0}^{\infty} (1 - e^{-r})^{-1} dr = \infty, \quad f \in S.$$
6. FOURIER TRANSFORM AND CESÁRO GENERALIZED OPERATOR

We remind the reader that the Fourier transform of a function \( f \) in \( L^1(\mathbb{R}) \) is defined by

\[
\hat{f}(t) := \int_{-\infty}^{\infty} e^{-ixt} f(x) dx, \quad t \in \mathbb{R}.
\]

It is well-known that \( \hat{f} \) is continuous on \( \mathbb{R} \) and \( \hat{f}(t) \to 0 \) when \( |t| \to \infty \) (the Riemann-Lebesgue lemma). In the case that \( f \in L^p(\mathbb{R}) \) for some \( 1 < p \leq 2 \), the Fourier transform of \( f \) is defined in terms of a limit in the norm of \( L^p(\mathbb{R}) \) of truncated integrals:

\[
\hat{f} := \lim_{R \to \infty} f \chi_{(-R,R)}(t), \quad f \chi_{(-R,R)}(t) = \int_{-R}^{R} e^{-ixt} f(x) dx, \quad t \in \mathbb{R},
\]

i.e., \( \hat{f} \in L^{p'}(\mathbb{R}) \) and \( \lim_{R \to \infty} \|f \chi_{(-R,R)}\|_{p'} = 0 \) where \( \frac{1}{p} + \frac{1}{p'} = 1 \) and \( \chi_{(-R,R)} \) is the characteristic function of the interval \((-R,R)\), see for example [24 Vol 2, p.254]. Then the existence of \( \hat{f}(t) \) is guaranteed only at almost every \( t \) and \( \hat{f} \) may be non continuous and the Riemann-Lebesgue lemma could not hold (unlike the case when \( f \in L^1(\mathbb{R}) \)).

In case that \( f \in L^p(\mathbb{R}) \) for some \( 2 < p < \infty \), the Fourier transform \( \hat{f} \) cannot be defined as an ordinary function although \( \hat{f} \) can be defined as a tempered distribution, see for example [22 pp 19-30].

In the next theorem, we consider the Fourier transform on the Sobolev space \( \mathcal{T}^{(n)}_p(\mathbb{R}) \).

**Theorem 6.1.** Take \( 1 \leq p \leq 2 \) and \( n \in \mathbb{N} \). Then \( \hat{f} \in \mathcal{T}^{(n)}_p(\mathbb{R}) \) for \( f \in \mathcal{T}^{(n)}_p(\mathbb{R}) \) and \( \frac{1}{p} + \frac{1}{p'} = 1 \).

**Proof.** Take \( f \in \mathcal{T}^{(n)}_p(\mathbb{R}) \). Since \( \mathcal{T}^{(n)}_p(\mathbb{R}) \subset \mathcal{T}^{(j)}_p(\mathbb{R}) \), we have that \( x^j f^{(j)} \in L^p(\mathbb{R}) \) for \( 0 \leq j \leq n \). As

\[
(it)^n (\hat{f})^{(n)}(t) = \sum_{j=0}^{n} (-1)^n \binom{n}{j} \frac{n!}{j!} x^j f^{(j)}(t), \quad n \in \mathbb{N}, t \text{ a.e. on } \mathbb{R},
\]

(see for example [24]), we conclude that \( (it)^n (\hat{f})^{(n)} \in L^{p'}(\mathbb{R}) \) and then \( \hat{f} \in \mathcal{T}^{(n)}_p(\mathbb{R}) \). \( \square \)

In what follows, we show that

\[
\mathcal{C}_\beta(f) = \mathcal{C}_\beta^*(\hat{f}), \quad \text{ and } \quad \mathcal{C}_\beta^*(f) = \mathcal{C}_\beta(\hat{f}), \quad f \in L^p(\mathbb{R}),
\]

for \( 1 < p \leq 2 \) (Theorem 6.4). This theorem extends the case \( \beta = 1 \) formulated in [1] and proved in [13]. Our approach looks like to be new and is based in the integral representations of \( \mathcal{C}_\beta(f) \) and \( \mathcal{C}_\beta^*(f) \) given in Section 3.

**Lemma 6.2.** Let \( 1 \leq p \leq 2 \) and the family of operators \( (T_{t,p})_{t \in \mathbb{R}} \) defined by \( T_{t,p}(f) := e^{-\frac{t}{p}} f(e^{-t} \cdot) \), for \( f \in L^p(\mathbb{R}) \). Then

\[
T_{t,p}(f) = T_{-t,p^*}(\hat{f}), \quad f \in L^p(\mathbb{R}), \quad \frac{1}{p} + \frac{1}{p'} = 1.
\]
Proof. Consider $1 < p \leq 2$, $R > 0$ and $f \in L^p(\mathbb{R})$. We write by $f_R := f\chi_{(-R,R)}$. Taking $\alpha = 0$ in Theorem 4.4 we conclude that $T_{t,p}(f) \in L^p(\mathbb{R})$ and then $\widehat{T_{t,p}(f)} = \lim_{R \to \infty} (T_{t,p}(f))_R \in L^p(\mathbb{R})$. Note that
\[
\left(\widehat{T_{t,p}(f)}\right)_R(r) = e^{\frac{1}{r}} \int_{-R}^R e^{-ix} f(e^{-t}x)dx = e^{\frac{1}{r}(1-\frac{1}{p})} \int_{-e^{-t}R}^{e^{-t}R} e^{-iy} f(y)dy = T_{-t,p^*} (f_{-t,R}).
\]
Since $\lim_{R \to \infty} f_{-t,R} = \hat{f}$ in $L^p(\mathbb{R})$, and $T_{t,p}^*$ is continuous on $L^p(\mathbb{R})$, we conclude the result.

The case $p = 1$ may be considered directly taking $R = \infty$ in the integral expressions. 

Remark 6.3. Since $T_p^\alpha([t]^\alpha) \to L^p(\mathbb{R})$ (Proposition 4.2 (i)), the equality $\widehat{T_{t,p}(f)} = T_{t,p^*}(\hat{f})$ holds for $f \in T_p^\alpha([t]^\alpha)$ for $\alpha \geq 0$ and $1 \leq p \leq 2$.

Finally, we are ready to prove the main result in this section.

Theorem 6.4. Let $\beta > 0$.

(i) If $f \in L^p(\mathbb{R})$ for some $1 < p \leq 2$, then $C_\beta(f) = C_\beta^*(\hat{f})$.

(ii) If $f \in L^p(\mathbb{R})$ for some $1 \leq p \leq 2$, then $C_\beta^*(f) = C_\beta(f)$.

Proof. (i) Take $f \in L^p(\mathbb{R})$ for some $1 < p \leq 2$. By Theorem 5.2 (ii) and Lemma 6.2 we have that
\[
C_\beta(f)(x) = \beta \int_0^\infty (1-e^{-r})\beta^{-1} e^{-r(1-\frac{1}{p})} T_{r,p} \hat{f}(x) dr
\]
\[
= \beta \int_{e^{-t}R}^{-e^{-t}R} (e^{-r} - 1)\beta^{-1} e^{-r(1-\frac{1}{p})} T_{r,p} \hat{f}(x) dr
\]
\[
= \beta \int_{-\infty}^{\infty} (e^{-r} - 1)\beta^{-1} e^{-r(1-\frac{1}{p})} T_{r,p} \hat{f}(x) dr = C_\beta^*(\hat{f})(x)
\]
for almost every $x$ on $\mathbb{R}$ and we use the formula (5.1).

(ii) Now take $f \in L^p(\mathbb{R})$ for some $1 \leq p \leq 2$. By the integral representation (5.1) of $C_\beta^*$ and Lemma 6.2 we have that
\[
C_\beta^*(f)(x) = \beta \int_{-\infty}^{\infty} (1-e^{-r})\beta^{-1} e^{-r(1-\frac{1}{p})} T_{r,p} \hat{f}(x) dr
\]
\[
= \beta \int_{0}^{\infty} (1-e^{-r})\beta^{-1} e^{-r(1-\frac{1}{p})} T_{r,p} \hat{f}(x) dr
\]
\[
= \beta \int_{0}^{\infty} (1-e^{-r})\beta^{-1} e^{-r(1-\frac{1}{p})} T_{r,p} \hat{f}(x) dr = C_\beta(f)(x)
\]
for almost every $x$ on $\mathbb{R}$ and we use the Theorem 5.2 (ii). 

Remark 6.5. By the Proposition 2.4 we get that $C_\beta(f)(t) = C_\beta^*(\hat{f})(t)$ and $C_\beta(f)(t) = C_\beta(\hat{f})(t)$ for $t \neq 0$ and $f \in T_p^\alpha([t]^\alpha)$, $1 < p \leq 2$ and $\alpha \geq 1$. 

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