Nonalgebraic length dependence of transmission through a chain of barriers with a Lévy spacing distribution

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The recent realization of a “Lévy glass” (a three-dimensional optical material with a Lévy distribution of scattering lengths) has motivated us to analyze its one-dimensional analogue: A linear chain of barriers with independent spacings \( s \) that are Lévy distributed: \( p(s) \propto s^{1-\alpha} \) for \( s \to \infty \). The average spacing diverges for \( 0 < \alpha < 1 \). A random walk along such a sparse chain is not a Lévy walk because of the strong correlations of subsequent step sizes. We calculate all moments of conductance (or transmission), in the regime of incoherent sequential tunneling through the barriers. The average transmission from one barrier to a point at a distance \( L \) scales as \( L^{-\alpha} \ln L \) for \( 0 < \alpha < 1 \). The corresponding electronic shot noise has a Fano factor \((\langle S \rangle / \langle T \rangle)^2\) that diverges only as \(1/\ln L\). The average transmission diverges for \( \alpha < 1 \), with \( 0 < \alpha < 2 \), such that the second moment (and for \( \alpha < 1 \) also the first moment) diverges. The transmission of light through the Lévy glass was analyzed for photons. Because the randomness in the Lévy glass is frozen in time (“quenched” disorder), correlations exist between subsequent scattering events. Backscattering after a large step is likely to result in another large step. This is different from a Lévy walk, where subsequent steps are independently drawn from the Lévy distribution (“annealed” disorder). Numerical and analytical theories indicate that the difference between quenched and annealed disorder can be captured (at least approximately) by a renormalization of the Lévy walk exponent — from the annealed value \( \alpha \) to the quenched value \( \alpha' = \alpha + (2/d) \max(0, \alpha - d) \) in \( d \) dimensions. Qualitatively speaking, the correlations in a Lévy glass slow down the diffusion relative to what is expected for a Lévy walk, and the effect is the stronger the lower the dimension.

To analyze the effect of such correlations in a quantitative manner, we consider in this paper the one-dimensional analogue of a Lévy glass, which is a linear chain of barriers with independently Lévy distributed spacings \( s \). Such a system might be produced artificially, along the lines of Ref. [10], or it might arise naturally in a porous medium [7] or in a nanowire [8]. Earlier studies of this system [3, 11, 12, 13] have compared the dynamical properties with those of a Lévy walk. In particular, Barkai, Fleurov, and Klafter [11] found a superdiffusive mean-square displacement as a function of time \([x^2(t)] \propto t^{\gamma} \) with \( \gamma > 1 \) — reminiscent of a Lévy walk (where \( \gamma = 3 - \alpha \)). No precise correspondence to a Lévy walk is to be expected in one dimension, because subsequent step lengths are highly correlated: Backscattering after a step of length \( s \) to the right results in the same step length \( s \) to the left.

The simplicity of one-dimensional dynamics allows for an exact solution of the static transmission statistics, without having to assume a Lévy walk. In particular, this calculation here, and find significant differences with the \( L^{-\alpha/2} \) scaling of the average transmission expected [14, 15, 16] for a Lévy walk (annealed disorder) through a system of length \( L \). If the length of the system is measured from the first barrier, we find for the case of quenched disorder an average transmission \( \langle T \rangle \propto L^{-\alpha} \ln L \) for \( 0 < \alpha < 1 \) and \( \langle T \rangle \propto L^{-1} \) for \( \alpha > 1 \). Note that the nonalgebraic length dependence for \( 0 < \alpha < 1 \) goes beyond what can be captured by a renormalization of \( \alpha \).

In the electronic context the average conductance \( \langle G \rangle \) is proportional to \( \langle T \rangle \), in view of the Landauer formula. In that context it is also of interest to study the shot noise power \( S \), which quantifies the time dependent fluctuations of the current due to the granularity of the electron charge. We calculate the Fano factor \( F \propto \langle S \rangle / \langle G \rangle \), and find that \( F \) approaches the value \( 1/3 \) characteristic of normal diffusion [17, 18] with increasing \( L \) — but with relatively large corrections that decay only as \(1/\ln L\) for \( 0 < \alpha < 1 \).

II. FORMULATION OF THE PROBLEM

We consider a linear chain of tunnel barriers, see Fig. 1 with a distribution of spacings \( p(s) \) that decays for large \( s \) as \( 1/s^{\alpha+1} \). A normalizable distribution requires \( \alpha > 0 \).
In terms of the Fourier (or Laplace) transform

\[ f(\xi) = \int_0^\infty ds \, e^{i\xi s} p(s), \]  

We seek the scaling with \( L \) in the limit \( L \to \infty \) of the negative moments \( \langle R(L)^p \rangle \) (\( p = -1, -2, -3, \ldots \)) of the resistance. This information will give us the scaling of the positive moments of the conductance \( G = R^{-1} \) and transmission \( T = (h/Ne^2)^2R^{-1} \). It will also give us the average of the shot noise power \( S \), which for an arbitrary number of identical tunnel barriers in series is determined by the formula \[ S = \frac{2}{3} c|V| r^{-1}[(R/r)^{-1} + 2(R/r)^{-3}], \]  

where \( V \) is the applied voltage. From \( \langle S \rangle \) and \( \langle G \rangle \) we obtain the Fano factor \( F \), defined by

\[ F = \frac{\langle S \rangle}{2e|V|\langle G \rangle}. \]  

### III. ARBITRARY MOMENTS

The general expression for moments of the resistance is

\[ \langle R(L)^p \rangle = r^p \left\langle \left( 1 + \sum_{n=1}^\infty \theta(x_n)\theta(L-x_n) \right)^p \right\rangle, \]  

where the brackets \( \langle \cdots \rangle \) indicate the average over the spacings,

\[ \langle \cdots \rangle = \prod_{n=1}^\infty \int_{-\infty}^\infty dx_n \, p(x_n-x_{n-1}) \cdots, \]  

with the definitions \( x_0 = 0 \) and \( p(s) = 0 \) for \( s < 0 \). We work out the average,

\[ \langle R(L)^p \rangle = r^p \sum_{n=1}^\infty n^p \left( \prod_{i=1}^n \int_{-\infty}^\infty ds_i \, p(s_i) \right) \times \theta \left( \sum_{i=1}^n s_i - L \right) \theta \left( L - \sum_{i=1}^{n-1} s_i \right). \]  

It is more convenient to evaluate the derivative with respect to \( L \) of Eq. (3.3), which takes the form of a multiple convolution of the spacing distribution \( p(s) \).

\[ \frac{d}{dL} \langle R(L)^p \rangle = \frac{r^p}{2\pi} \int_{-\infty}^{\infty+i0^+} d\xi \, e^{-i\xi L} \sum_{n=1}^\infty [(n+1)^p - n^p] f(\xi)^n, \]  

the series (3.4) can be summed up,

\[ \frac{d}{dL} \langle R(L)^p \rangle = \frac{r^p}{2\pi} \int_{-\infty+i0^+}^{\infty+i0^+} d\xi \, e^{-i\xi L} \frac{1 - f(\xi)}{f(\xi)} \text{Li}_{-p}[f(\xi)]. \]
The function $\text{Li}(x)$ is the polylogarithm. The imaginary infinitesimal $i0^+$ added to $\xi$ regularizes the singularity of the integrand at $\xi = 0$. For negative $p$ this singularity is integrable, and the integral may be rewritten as an integral over the positive real axis,

$$\frac{d}{dL} \langle R^p \rangle = \frac{p^p}{\pi} \text{Re} \int_0^\infty d\xi \, e^{-i\xi L} \frac{1 - f(\xi)}{f(\xi)} \text{Li}_{-p}(f(\xi)).$$

(3.7)

IV. SCALING WITH LENGTH

A. Asymptotic expansions

In the limit $L \to \infty$ the integral over $\xi$ in Eq. (3.7) is governed by the $\xi \to 0$ limit of the Fourier transformed spacing distribution. Because $p(s)$ is normalized to unity one has $f(0) = 1$, while the large-$s$ scaling $p(s) \propto 1/s^{\alpha+1}$ implies

$$\lim_{\xi \to 0} f(\xi) = \begin{cases} 1 + c_\alpha(s_0 \xi)^\alpha, & 0 < \alpha < 1, \\ 1 + i\bar{s}\xi + c_\alpha(s_0 \xi)^\alpha, & 1 < \alpha < 2. \end{cases}$$

(4.1)

The characteristic length $s_0 > 0$, the mean spacing $\bar{s}$, as well as the numerical coefficient $c_\alpha$ are determined by the specific form of the spacing distribution.

The limiting behavior of the polylogarithm is governed by

$$\text{Li}_1(1 + \epsilon) = -\ln(-\epsilon),$$

$$\lim_{\epsilon \to 0} \text{Li}_2(1 + \epsilon) = \zeta(2) - \epsilon \ln(-\epsilon),$$

$$\lim_{\epsilon \to 0} \text{Li}_n(1 + \epsilon) = \zeta(n) + \zeta(n - 1)\epsilon, \quad n = 3, 4, \ldots$$

(4.2) \quad (4.3) \quad (4.4)

In combination with Eq. (4.1) we find, for $0 < \alpha < 1$, the following expansions of the integrand in Eq. (3.7):

$$\lim_{\xi \to 0} \frac{1 - f}{f} \text{Li}_{-p}(f) = c_\alpha(s_0 \xi)^\alpha \ln[-c_\alpha(s_0 \xi)^\alpha],$$

if $p = -1$,

$$\lim_{\xi \to 0} \frac{1 - f}{f} \text{Li}_{-p}(f) = -\zeta(-p)c_\alpha(s_0 \xi)^\alpha,$$

$p = -2, -3, \ldots$ \quad (4.5) \quad (4.6)

For $1 < \alpha < 2$ we should replace $c_\alpha(s_0 \xi)^\alpha$ by $i\bar{s}\xi + c_\alpha(s_0 \xi)^\alpha$.

B. Results

We substitute the expansions (4.5) and (4.6) into Eq. (3.7), and obtain the large-$L$ scaling of the moments of conductance with the help of the following Fourier integrals ($L > 0$, $\alpha > -1$):

$$\int_0^\infty d\xi \, e^{-i\xi L} \xi^\alpha \ln \xi = i\Gamma(1 + \alpha)e^{-i\pi\alpha/2}L^{-1-\alpha}$$

$$\times (\ln L + i\pi/2 + \gamma_E - H_a),$$

$$\int_0^\infty d\xi \, e^{-i\xi L} \xi^\alpha = -i\Gamma(1 + \alpha)e^{-i\pi\alpha/2}L^{-1-\alpha},$$

$$\text{Re} \int_0^\infty d\xi \, e^{-i\xi L} i\xi = 0,$$

$$\text{Re} \int_0^\infty d\xi \, e^{-i\xi L} i\xi \ln \xi = -\frac{1}{3}\pi L^{-2}.$$ \quad (4.7) \quad (4.8) \quad (4.9) \quad (4.10)

Here $\gamma_E$ is Euler’s constant and $H_a$ is the harmonic number. The resulting scaling laws are listed in Table I:

| $0 < \alpha < 1$ | $1 < \alpha < 2$ |
|-----------------|------------------|
| $\langle R^{-1} \rangle \equiv \langle G \rangle$ | $L^{-\alpha}$ |
| $\langle R^p \rangle \equiv \langle G^{-p} \rangle, p = -2, -3, \ldots$ | $L^{-\alpha}$ |

Two physical consequences of these scaling laws are:

- The Fano factor $F$ approaches $1/3$ in the limit $L \to \infty$, regardless of the value of $\alpha$, but for $0 < \alpha < 1$ the approach is very slow: $F = 1/3 \propto 1/L$. For $1 < \alpha < 2$ the approach is faster but still sub-linear, $F = 1/3 \propto 1/L^{\alpha-1}$.

- The root-mean-square fluctuations $\text{rms} G = \sqrt{\langle (G^2) \rangle - \langle G \rangle^2}$ of the conductance become much larger than the average conductance for large $L$, scaling as $\text{rms} G/\langle G \rangle \propto L^{\alpha/2}/\ln L$ for $0 < \alpha < 1$ and as $\text{rms} G/\langle G \rangle \propto L^{1-\alpha/2}$ for $1 < \alpha < 2$.

V. NUMERICAL TEST

To test the scaling derived in the previous sections, in particular to see how rapidly the asymptotic $L$-dependence is reached with increasing $L$, we have numerically generated a large number of random chains of tunnel barriers and calculated moments of conductance and the Fano factor from Eqs. (2.2)–(2.4).

For the spacing distribution in this numerical calculation we took the Lévy stable distribution [21] for $\alpha = 1/2$,

$$p_{1/2}(s) = (s_0/2\pi)^{1/2}s^{-3/2}e^{-s_0/2s}.$$ \quad (5.1)

Its Fourier transform is

$$f_{1/2}(\xi) = \exp(-\sqrt{-2i\bar{s}s_0}) \Rightarrow c_{1/2} = i - 1.$$ \quad (5.2)

Inserting the numerical coefficients, the large-$L$ scaling
of conductance moments for the distribution \( \alpha \) is
\[
\lim_{L \to \infty} \langle G \rangle = \frac{1}{r} (2\pi L/s_0)^{-1/2} [\ln(2L/s_0) + \gamma_E],
\]
\[
\lim_{L \to \infty} \langle G^p \rangle = 2\zeta(p) \frac{1}{p^r} (2\pi L/s_0)^{-1/2}, \quad p \geq 2.
\]
The resulting scaling of the conductance fluctuations and Fano factor is
\[
\left( \frac{\text{rms} G}{\langle G \rangle} \right)^2 = \frac{\langle G^2 \rangle}{\langle G \rangle^2} - 1 \approx \frac{(\pi^2/3)(2\pi L/s_0)^{1/2}}{[\ln(2L/s_0) + \gamma_E]^2} - 1,
\]
\[
F \approx \frac{1}{3} + \frac{(4/3)\zeta(3)}{\ln(2L/s_0) + \gamma_E}.
\]

In Fig. 2 we compare these analytical large-\( L \) formulas with the numerical data. The average conductance converges quite rapidly to the scaling \( \frac{5.3}{5.3} \), while the convergence for higher moments (which determine the conductance fluctuations and Fano factor) requires somewhat larger systems. We clearly see in Fig. 2 the relative growth of the conductance fluctuations with increasing system size and the slow decay of the Fano factor towards the diffusive \( 1/3 \) limit.

**VI. CONCLUSION AND OUTLOOK**

In conclusion, we have analyzed the statistics of transmission through a sparse chain of tunnel barriers. The average spacing of the barriers diverges for a Lévy spacing distribution \( p(s) \propto 1/s^{1+\alpha} \) with \( 0 < \alpha < 1 \). This causes an unusual scaling with system length \( L \) (measured from the first tunnel barrier) of the moments of transmission or conductance, as summarized in Table 1. A logarithmic correction to the power law scaling appears for the first moment. Higher moments of conductance all scale with the same power law, differing only in the numerical prefactor. As a consequence, sample-to-sample fluctuations of the transmission become larger than the average with increasing \( L \).

This theoretical study of a one-dimensional “Lévy glass” was motivated by a recent optical experiment on its three-dimensional analogue \( \text{[1]} \). The simplicity of a one-dimensional geometry has allowed us to account exactly for the correlations between subsequent step lengths, which distinguish the random walk through the sparse chain of barriers from a Lévy walk. We surmise that step length correlations will play a role in two and three dimensional sparse arrays as well, complicating a direct application of the theory of Lévy walks to the experiment. This is one line of investigation for the future.

A second line of investigation is the effect of wave interference on the transmission of electrons or photons through a sparse chain of tunnel barriers. Here we have considered the regime of incoherent sequential transmission, appropriate for a multi-mode chain with modemixing or for a single-mode chain with a short coherence length. The opposite, phase coherent regime was studied in Ref. 13. In a single-mode and phase coherent chain interference can lead to localization, producing an exponential decay of transmission. An investigation of localization in this system is of particular interest because the sparse chain belongs to the class of disordered systems with long-range disorder, to which the usual scaling theory of Anderson localization does not apply 22.

A third line of investigation concerns the question “what is the shot noise of anomalous diffusion”? Anomalous diffusion \( \text{[4]} \) is characterized by a mean square displacement \( \langle x^2 \rangle \propto t^\gamma \) with \( 0 < \gamma < 1 \) (subdiffusion) or \( \gamma > 1 \) (superdiffusion). The shot noise for normal diffusion (\( \gamma = 1 \)) has Fano factor \( 1/3 \) \( \text{[15, 18]} \), and Ref. 23 concluded that subdiffusion on a fractal also produces \( F = 1/3 \). Here we found a convergence, albeit a
logarithmically slow convergence, to the same $1/3$ Fano factor for a particular system with superdiffusive dynamics. We conjecture that $F = 1/3$ in the entire subballistic regime $0 < \gamma < 2$, with deviations appearing in the ballistic limit $\gamma \to 2$ — but we do not have a general theory to support this conjecture.

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