INTEGRAL REPRESENTATIONS FOR $\zeta(3)$ WITH THE INVERSE SINE FUNCTION

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ABSTRACT. We show four new integral representations for $\zeta(3)$ as a reformulation of Ewell (1990) and Yue-Williams (1993) with the inverse sine function and Wallis integral. As a consequence, we also show a local integral representation for the trilogarithm function.

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1. INTRODUCTION

1.1. Apéry number. The Riemann zeta function is one of the important topics in number theory. For a complex number $s$ such that $\text{Re}(s) > 1$, we usually define

$$\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots.$$  

Leonhard Euler proved

$$\zeta(2n) = \frac{1}{2} \frac{(2\pi i)^{2n}}{(2n)!} B_{2n} \quad (n \geq 1)$$

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\( \{B_{2n}\} \) are Bernoulli numbers) while \( \{\zeta(2n + 1)\}_{n \geq 1} \) remain to be unknown. In this article, we particularly study
\[
\zeta(3) = 1.2020569 \ldots,
\]
the Apery number, as Apéry proved its irrationality [2] in 1979. Although its exact value is still unclear, there exist many representations for \( \zeta(3) \) such as
\[
\zeta(3) = \frac{2}{7} \pi^2 \log 2 + \frac{16}{7} \int_0^{\pi/2} x \log(\sin x) \, dx,
\]
\[
\zeta(3) = \frac{7}{180} \pi^3 - 2 \sum_{n=1}^{\infty} \frac{1}{n^3(e^{2\pi n} - 1)},
\]
\[
\zeta(3) = \frac{5}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3(2n)^n},
\]
due to Euler, Plouffe [7], Ramanujan, Apéry and doubtless others; see also Chen-Srivastava [4] and Nash-O’Connor [6] for other representations.

1.2. Main result. The main result of this article is to show the following integral representations for \( \zeta(3) \); all of these are new.

**Main Theorem.**
\[
\zeta(3) = \frac{8}{7} \int_0^1 \frac{\sin^{-1} x \cos^{-1} x}{x} \, dx
\]
\[
= \frac{8}{\pi} \int_0^1 \frac{(\sin^{-1} x)^2 \cos^{-1} x}{x} \, dx
\]
\[
= \frac{16}{5\pi} \int_0^1 \frac{\sin^{-1} x(\cos^{-1} x)^2}{x} \, dx
\]
\[
= \frac{32}{3\pi} \int_0^1 \frac{\sinh^{-1} x)^2 \cos^{-1} x}{x} \, dx.
\]

Before going into the proof of this theorem in the next section, we wish to mention work of Ewell (1990) and Yue-Williams (1993) and set up notation.

1.3. Results of Ewell and Yue-Williams.

**Fact 1.1** (Ewell [5], Yue-Williams [8, p.1582]).
\[
\zeta(3) = \frac{\pi^2}{7} \left( 1 - 4 \sum_{n=1}^{\infty} \frac{\zeta(2n)}{(2n + 1)(2n + 2)2^{2n}} \right),
\]
\[
\zeta(3) = -2\pi^2 \left( \sum_{n=0}^{\infty} \frac{\zeta(2n)}{(2n + 2)(2n + 3)2^{2n}} \right).
\]
These series are quite similar because they both derived some infinite sums related to $\zeta(3)$ from the Maclaurin series involving $\sin^{-1} x$ using Wallis integral; this method is modification of Boo Rim Choe’s elementary proof for $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ [3].

1.4. Notation. Throughout $n$ denotes a nonnegative integer. Let

$$(2n)!! = 2n(2n - 2) \cdots 4 \cdot 2,$$

$$(2n - 1)!! = (2n - 1)(2n - 3) \cdots 3 \cdot 1.$$

In particular, we understand that $(-1)!! = 0!! = 1$. Moreover, let

$$c_n = \frac{(n - 1)!!}{n!!}.$$ 

This number appears in the following integral:

**Fact 1.2** (Wallis integral).

$$\int_0^{\pi/2} \sin^n x \, dx = \begin{cases} \frac{\pi}{2} c_n & n \text{ even,} \\ c_n & n \text{ odd.} \end{cases}$$

By $\sin^{-1} x$ and $\cos^{-1} x$, we mean the real inverse sine and cosine functions ($\arcsin x, \arccos x$), that is,

$$y = \sin^{-1} x \iff x = \sin y, \quad -\frac{\pi}{2} \leq y \leq \frac{\pi}{2},$$

$$y = \cos^{-1} x \iff x = \cos y, \quad 0 \leq y \leq \pi.$$ 

**Fact 1.3**.

$$\sin^{-1} x = \sum_{n=0}^{\infty} c_{2n} \frac{x^{2n+1}}{2n+1}, \quad |x| \leq 1.$$ 

**Remark 1.4**. In the literature, some researchers exclude $|x| = \pm 1$. However, even for $|x| = \pm 1$, this equality indeed holds as Boo Rim Choe pointed out [3].

Further, $\sinh^{-1} x = \log(x + \sqrt{x^2 + 1}) \ (x \in \mathbb{R})$ denotes the inverse hyperbolic sine function (some authors write $\text{arsinh} x, \text{arcsinh} x$ or $\text{argsinh} x$ for this function).

2. **Proof of Main theorem**

In this section, we give a proof of equalities in Main Theorem one by one.

**Theorem 2.1**.

$$\zeta(3) = \frac{8}{7} \int_0^1 \frac{\sin^{-1} x \cos^{-1} x}{x} \, dx.$$
Proof. Let us start with the Maclaurin series
\[
\frac{\arcsin y}{y} = \sum_{n=0}^{\infty} c_{2n} \frac{y^{2n}}{2n + 1}, \quad |y| \leq 1.
\]

Let \( t, u \) be real variables such that \( 0 \leq t, u \leq 1 \). Recall from calculus that term-wise integration is possible for convergent power series. By integrating the series above from 0 to \( tu \), we have
\[
I(t, u) := \int_{0}^{tu} \frac{\arcsin y}{y} dy = \int_{0}^{tu} \left( \sum_{n=0}^{\infty} c_{2n} \frac{y^{2n}}{2n + 1} \right) dy
\]
\[
= \sum_{n=0}^{\infty} c_{2n} \frac{1}{2n + 1} \int_{0}^{tu} y^{2n} dy = \sum_{n=0}^{\infty} c_{2n} \frac{1}{(2n + 1)^2} t^{2n+1} u^{2n+1}.
\]

Now, let \( v = \arcsin u \) (\( 0 \leq v \leq \frac{\pi}{2} \)). Then \( u = \sin v \). Integrate \( I(t, \sin v) \) from 0 to \( \frac{\pi}{2} \) in \( v \). On one hand,
\[
J(t) := \int_{0}^{\pi/2} I(t, \sin v) dv = \sum_{n=0}^{\infty} c_{2n} \frac{t^{2n+1}}{(2n + 1)^2} \int_{0}^{\pi/2} \sin^{2n+1} v dv
\]
\[
= \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n + 1)^3}.
\]

On the other hand, exchanging order of the integrals yields
\[
J(t) = \int_{0}^{\pi/2} \int_{0}^{t \sin v} \frac{\arcsin y}{y} dy dv = \int_{0}^{t} \int_{0}^{\pi/2} \frac{\arcsin y}{y} d(\frac{\pi}{2} - \sin^{-1} \frac{y}{t}) dy
\]
\[
= \int_{0}^{t} \frac{\arcsin y}{y} \left( \frac{\pi}{2} - \sin^{-1} \frac{y}{t} \right) dy = \int_{0}^{1} \frac{\arcsin(tx) \cos^{-1} x}{x} dx.
\]

Thus,
\[
(1 - 2^{-3})\zeta(3) = \sum_{n=0}^{\infty} \frac{1}{(2n + 1)^3} = J(1) = \int_{0}^{1} \frac{\arcsin x \cos^{-1} x}{x} dx.
\]

Conclude that
\[
\zeta(3) = \frac{8}{7} \int_{0}^{1} \frac{\arcsin x \cos^{-1} x}{x} dx.
\]

\[\square\]

Corollary 2.2.
\[
\int_{0}^{1} \frac{\arcsinh x \cos^{-1} x}{x} dx = \frac{\pi^3}{32}.
\]
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Proof. Start with

$$\sinh^{-1} y = \sum_{n=0}^{\infty} (-1)^n c_{2n} \frac{y^{2n+1}}{2n + 1}, \quad |y| \leq 1$$

which easily follows from

$$\sin^{-1} y = \sum_{n=0}^{\infty} c_{2n} \frac{y^{2n+1}}{2n + 1}, \quad |y| \leq 1$$

and $\sinh^{-1} z = -i \sin^{-1}(iz)$ (for all $z \in \mathbb{C}$) [1, p.87]. Now consider the argument with replacing $\sin^{-1} y$ by $\sinh^{-1} y$ in Proof of Theorem 2.1 throughout. Then, the proof goes without any substantial changes. It leads us to

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1)^3} = \int_0^1 \frac{\sinh^{-1} x \cos^{-1} x}{x} dx.$$ 

The left hand side is $\frac{\pi^3}{32}$ [1, p.808]. □

Theorem 2.3.

$$\zeta(3) = \frac{8}{\pi^3} \int_0^1 \frac{(\sin^{-1} x)^2}{x} \cos^{-1} x \ dx.$$ 

Proof. Start with

$$\frac{(\sin^{-1} y)^2}{y} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{c_{2n}} \frac{y^{2n-1}}{n^2}, \quad |y| \leq 1.$$ 

Again, let $t, u$ be real variables such that $0 \leq t, u \leq 1$. Set

$$K(t, u) := \int_0^t \frac{(\sin^{-1} y)^2}{y} dy.$$ 

Then

$$K(t, u) = \int_0^t \left( \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{c_{2n}} \frac{y^{2n-1}}{n^2} \right) dy = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{c_{2n}} \frac{1}{n^2} \int_0^t y^{2n-1} dy$$

$$= \frac{1}{4} \sum_{n=1}^{\infty} \frac{\ell_{2n} u^{2n}}{c_{2n} \ n^3}.$$ 

Let $v = \sin^{-1} u$ ($0 \leq v \leq \frac{\pi}{2}$). Then $u = \sin v$. Integrate $K(t, \sin v)$ from 0 to $\frac{\pi}{2}$ in $v$:

$$L(t) := \int_0^{\pi/2} K(t, \sin v) dv = \frac{1}{4} \sum_{n=1}^{\infty} \frac{\ell_{2n}}{c_{2n} \ n^3} \int_0^{\pi/2} \sin^{2n} v \ dv$$

$$= \frac{\pi}{8} \sum_{n=1}^{\infty} \frac{\ell_{2n}}{n^3}.$$
On the other hand, exchanging order of the integrals yields
\[ L(t) = \int_0^{\pi/2} \int_0^{\sin t} \frac{(\sin^{-1} y)^2 y}{y} dydv = \int_0^{\pi/2} \frac{(\sin^{-1} y)^2 y}{y} dvdy \]
\[ = \int_0^1 \frac{(\sin^{-1} y)^2 y}{y} \left( \frac{\pi}{2} - \sin^{-1} \frac{y}{t} \right) dy \]
\[ = \int_0^1 \frac{(\sin^{-1}(tx))^2 \cos^{-1} x}{x} dx. \]

Hence
\[ \frac{\pi}{8} \zeta(3) = \frac{\pi}{8} \sum_{n=1}^{\infty} \frac{1}{n^3} = L(1) = \int_0^1 (\sin^{-1} x)^2 \cos^{-1} x \frac{1}{x} dx \]
and conclude that
\[ \zeta(3) = \frac{8}{\pi} \int_0^1 (\sin^{-1} x)^2 \cos^{-1} x \frac{1}{x} dx. \]

**Theorem 2.4.**
\[ \zeta(3) = \frac{32}{3\pi} \int_0^1 (\sinh^{-1} x)^2 \cos^{-1} x \frac{1}{x} dx. \]

**Proof.** With \( \sinh^{-1} z = -i \sin^{-1}(iz) \) (for \( z \in \mathbb{C} \)), we see that
\[ \frac{(\sinh^{-1} y)^2 y}{y} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} y^{2n-1}}{c_{2n} n^2}, \quad |y| \leq 1. \]

Again, consider the argument with replacing \( \sin^{-1} y \) by \( \sinh^{-1} y \) in Proof of Theorem 2.3 throughout. Then, the proof goes without any substantial changes. It leads us to
\[ \frac{\pi}{8} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3} = \int_0^1 (\sinh^{-1} x)^2 \cos^{-1} x \frac{1}{x} dx, \]
\[ \frac{\pi}{8} (1 - 2^{1-3}) \zeta(3) = \int_0^1 (\sinh^{-1} x)^2 \cos^{-1} x \frac{1}{x} dx, \]
and hence
\[ \zeta(3) = \frac{32}{3\pi} \int_0^1 (\sinh^{-1} x)^2 \cos^{-1} x \frac{1}{x} dx. \]

**Corollary 2.5.**
\[ \zeta(3) = \frac{16}{5\pi} \int_0^1 \sin^{-1} x (\cos^{-1} x)^2 \frac{1}{x} dx. \]
Proof. This is an easy consequence of Theorems 2.1 and 2.3 as follows. Recall that $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$ whenever $0 \leq x \leq 1$. Thus,

$$
\int_0^1 \frac{\sin^{-1} x \cos^{-1} x}{x} dx = \int_0^1 \frac{\sin^{-1} x \cos^{-1} x}{x} \left( \frac{\pi}{2} - \sin^{-1} x \right) dx
$$

$$
= \frac{\pi}{2} \int_0^1 \frac{\sin^{-1} x \cos^{-1} x}{x} dx - \int_0^1 \frac{(\sin^{-1} x)^2 \cos^{-1} x}{x} dx
$$

$$
= \frac{\pi}{2} \left( \frac{7}{8} \zeta(3) \right) - \frac{\pi}{8} \zeta(3) = \frac{5\pi}{16} \zeta(3).
$$

Hence

$$
\zeta(3) = \frac{16}{5\pi} \int_0^1 \frac{\sin^{-1} x \cos^{-1} x}{x} x dx.
$$

□

We completed the proof of Main Theorem.

3. FURTHER REMARKS

We end with some remarks for our future research.

[1] Main Theorem suggests us to think of the following family of integrals:

$$
I(m, n) = \int_0^1 \frac{(\sin^{-1} x)^m (\cos^{-1} x)^n}{x} dx, \quad m, n \geq 0, m + n \geq 1.
$$

Here, let us observe several examples for some interest. As is well-known,

$$
I(1, 0) = \int_0^1 \frac{\sin^{-1} x}{x} dx = \frac{1}{2} \pi \log 2
$$

while

$$
I(0, 1) = \int_0^1 \frac{\cos^{-1} x}{x} dx = \int_0^1 \left( \frac{\pi}{2} - 1 - \frac{x^2}{6} - \cdots \right) dx
$$

is divergent. For $m, n \geq 1$, the first three integrals are

$$
I(1, 1) = \frac{7}{8} \zeta(3), \quad I(2, 1) = \frac{\pi}{8} \zeta(3), \quad I(1, 2) = \frac{5\pi}{16} \zeta(3)
$$

as we have shown before. Now such integrals satisfy the relations

$$
I(m, n) = \frac{\pi}{2} I(m - 1, n) - I(m - 1, n + 1), \quad m \geq 1
$$

and similarly

$$
I(m, n) = \frac{\pi}{2} I(m, n - 1) - I(m + 1, n - 1), \quad n \geq 1
$$

since $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$ for $0 \leq x \leq 1$. We then have

$$
I(2, 0) = \frac{\pi}{2} I(1, 0) - I(1, 1) = \frac{\pi^2}{4} \log 2 - \frac{7}{8} \zeta(3),
$$
\[ I(3,0) = \frac{\pi}{2} I(2,0) - I(2,1) = \frac{\pi^3}{8} \log 2 - \frac{9}{16} \zeta(3) \]

and so on. Wolfram alpha [9] says that \( I(4,0) \) “is close to”

\[ \frac{1}{32} \left( -18\pi^2 \zeta(3) + 93 \zeta(5) + 2\pi^4 \log 2 \right) \]

which involves \( \zeta(3) \) and \( \zeta(5) \); actually, we computed this as

\[ I(4,0) = \int_0^1 \frac{(\sin^{-1} x)^4}{x} \, dx = \int_0^{\pi/2} u^4 \cot u \, du. \]

Once we can figure out all the coefficients of \( \frac{\sin^{-1} x}{x} \), then it may be possible to derive some series involving \( \zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3} \) and \( \zeta(5) = \sum_{n=1}^{\infty} \frac{1}{n^5} \) in a similar method.

[2] The polylogarithm function \( \text{Li}_s(t) = \sum_{n=1}^{\infty} \frac{t^n}{n^s} \) plays a significant role in many areas of number theory. As a byproduct of Proof of Theorem 2.3, we obtained a local integral representation of the trilogarithm function; that is, for \( 0 \leq t \leq 1 \), we have

\[ \text{Li}_3(t) = 8 \pi \int_0^1 \frac{(\sin^{-1} (\sqrt{t}x))^2 \cos^{-1} x}{x} \, dx. \]

This seems to be also new.

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