Inductive constructions for frameworks on a
two-dimensional fixed torus

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Abstract

In this paper we find necessary and sufficient conditions for the minim-
imal rigidity of graphs on the two-dimensional fixed torus, \( \mathbb{T}_0^2 \). We use
these periodic orbit frameworks (gain graphs) as models of infinite peri-
odic graphs, and the rigidity of the gain graphs on the torus correspond
to the generic rigidity of the periodic framework under forced periodicity.
Here it is shown that every minimally rigid periodic orbit framework on
\( \mathbb{T}_0^2 \) can be constructed from smaller graphs through a series of inductive
constructions. This is a periodic version of Henneberg’s theorem about
finite graphs. We also describe a characterization of the generic rigidity
of a two-dimensional periodic framework through a consideration of the
gain assignment on the corresponding periodic orbit framework. This can
be viewed as a periodic analogue of Laman’s theorem about finite graphs.

MSC: 52C25

Key words: infinitesimal rigidity, generic rigidity, periodic frameworks,
inductive constructions, gain graphs

1 Introduction

The study of the rigidity of periodic frameworks has witnessed an explosion of
interest in recent years [19, 24, 13, 18, 2, 3, 21]. This is due in part to questions
raised by the materials science community about the rigidity or flexibility of
zeolites, a type of mineral with crystalline structure characterized by a repetitive
or periodic porous pattern [25]. Rigidity theory may provide some tools to
facilitate the study of these compounds.

In this paper we consider infinite periodic frameworks as frameworks on
a torus, where we use the torus as a “fundamental region” for a tiling of the
plane. In particular, we study frameworks on a flat torus of fixed size and shape.
Experimental evidence has suggested that for some molecular compounds, the

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time scales of lattice movement (e.g. deformations of the torus) may be several orders of magnitude slower than the molecular deformations within the lattice, making the “fixed torus” a more relevant setting for the study of rigidity than it may at first appear \[29\]. In addition, when we allow the lattice (torus) to deform, then any periodic framework that is not rigid will have the property that the velocities of the vertices that are “far away from the centre” will become arbitrarily large.

Our approach to periodic frameworks consists of regarding an infinite periodic framework as a finite labeled graph \(\langle G, m \rangle\) on the torus, where \(m : E^+ \to \mathbb{Z}^2\). In a previous paper we found necessary conditions for the rigidity of a framework on a fixed \(d\)-dimensional torus. This paper provides sufficient conditions for the generic rigidity of a 2-dimensional framework on a fixed torus, which depends in part on the labelling of the edges \(m\):

**Periodic Laman Theorem.** Let \(\langle G, m \rangle\) be a periodic orbit graph. Then \(\langle G, m \rangle\) is generically minimally rigid on the fixed torus if and only if \(\langle G, m \rangle\) satisfies

\[
(i) \ |E| = 2|V| - 2, \text{ and } |E'| \leq 2|V'| - 2 \text{ for all subgraphs } G' \subset G
\]

\(\quad\) (ii) \(m\) is a constructive gain assignment.

This can be viewed as a version of Laman’s theorem for finite frameworks (see Theorem 2.2), which characterizes the generic rigidity of finite frameworks in the plane. We also prove a periodic-adapted version of Henneberg’s theorem (Theorem 2.1), which is a constructive method of creating larger rigid frameworks from smaller ones:

**Periodic Henneberg Theorem.** A periodic orbit graph \(\langle G, m \rangle\) on the fixed torus is generically minimally rigid if and only if it can be constructed from a single vertex on the fixed torus by a sequence of periodic Henneberg moves.

We use inductive techniques, building up minimally rigid frameworks from smaller minimally rigid frameworks by carefully adding vertices and edges to the underlying graph. This work was completed in early 2009 \[20\], prior to the appearance of the work of Borcea and Streinu \[4\] or Malestein and Theran \[13\]. The statements in the applicable cases are the same as in \[13\], as are the algorithms that are naturally obtained from these statements. However, our techniques are inductive, adding to a vocabulary of methods that may be applied to a broad class of problems concerning periodic frameworks.

Inductive techniques are both general and widely used. They easily adapt to \(d\)-dimensional frameworks, and have been used to generate special classes of three-dimensional rigid structures. Inductive techniques have also played a key role in the development of global rigidity, the study of graphs with unique realizations. Furthermore, inductive methods also appear in the study of special classes of frameworks, for example Schulze’s work on symmetric frameworks \[26\], and Nixon, Owen and Power’s recent exploration of frameworks supported on surfaces embedded in \(\mathbb{R}^3\) \[16\]. For these reasons we believe that inductive techniques in the periodic setting are valuable.
Recently, a characterization of generic rigidity on the variable torus has appeared \[13\]. The authors note that it is possible to obtain the Laman type result described here through a restriction of their result. It is not clear, however, that inductive methods such as those presented here can be used to characterize rigidity on a variable torus. The inductive techniques presented here are truly "local" moves, in the sense that they could be viewed as usual Henneberg moves from finite rigidity theory performed on each cell of a periodic framework simultaneously. The development of inductive techniques for the variable torus is a challenging and interesting problem (see also Section 6 for further discussion).

1.1 Outline

In Section 2 we outline some background from (finite) rigidity theory, and summarize the key notions of rigidity theory for periodic frameworks as finite graphs on a fixed torus. In Section 3 we state our main result, which will be proved in several pieces which comprise Section 4 (periodic Henneberg theorem) and Section 5 (periodic Laman theorem). Section 6 concludes by connecting this work with some existing extensions and some areas for further work.

2 Background

The full background for the present work is recorded in an earlier paper \[22\]. In this section we summarize only the most essential definitions and results, including several from finite (i.e. not periodic) rigidity theory.

2.1 Rigidity theory for finite frameworks

A framework in \(\mathbb{R}^d\) is a pair \((G, p)\) consisting of a finite simple graph \(G = (V, E)\), together with an assignment \(p = (p_1, \ldots, p_{|V|})\) of point \(p_i \in \mathbb{R}^d\) to the vertices \(i\) of \(V\), such that \(p_i \neq p_j\) whenever \(\{i, j\} \in E\).

An infinitesimal motion of \((G, p)\) is a function \(u : V \rightarrow \mathbb{R}^d\) such that
\[
(p_i - p_j) \cdot (u_i - u_j) = 0 \quad \text{for all} \quad \{i, j\} \in E.
\]

An infinitesimal motion can be thought of as an assignment of infinitesimal velocities to the vertices of the framework in such a way that the lengths of the bars are instantaneously preserved. An infinitesimal motion is called a trivial infinitesimal motion if
\[
(p_i - p_j) \cdot (u_i - u_j) = 0 \quad \text{for all pairs} \quad \{i, j\}.
\]

The framework \((G, p)\) is called infinitesimally rigid if all of its infinitesimal motions are trivial, otherwise \((G, p)\) is infinitesimally flexible.

Maxwell’s rule gives us necessary conditions for the number of vertices and edges that a graph must have in order to possibly be rigid. We say that \((G, p)\)
is isostatic if it is infinitesimally rigid, and satisfies

$$|E| = d|V| - \binom{d+1}{2}.$$  

We also refer to this state as minimal rigidity, since deleting any edge will result in an infinitesimally flexible framework.

A framework is called generic if it does not have any special geometry. We will precisely define generic for periodic frameworks in the subsequent section. The essential idea is that the rigidity of generic frameworks (i.e. frameworks whose vertices are in generic position) is characterized by the properties of the underlying graph $G$, not the configuration $p$. When $d = 2$, generic rigidity is characterized combinatorially (graph-theoretically), as the theorems of Henneberg and Laman will demonstrate. No such results exist for $d > 2$.

Henneberg used inductive graph constructions known as vertex additions and edge splits in two dimensions to give the following characterization of generically isostatic frameworks in $\mathbb{R}^2$. We do not provide details for these Henneberg moves here, because we will define the periodic-adapted versions in detail in Section 4.1.

**Theorem 2.1** (Henneberg’s Theorem (1911), [11]). A framework $(G, p)$ is generically isostatic in $\mathbb{R}^2$ if and only if it may be constructed from a single edge by a sequence of vertex additions and edge splits.

**Theorem 2.2** (Laman’s Theorem (1970), [12]). The framework $(G, p)$ with $|V| \geq 2$ is generically isostatic in $\mathbb{R}^2$ if and only if

1. $|E| = 2|V| - 3$, and
2. $|E'| \leq 2|V'| - 3$ for all subgraphs $G' \subseteq G$ with $|V'| \geq 2$.

Proofs of Laman’s theorem can be found in [7, 27], among others.

### 2.2 Periodic orbit frameworks on $T^2_0$

Periodic frameworks in the plane are locally finite infinite graphs which are symmetric with respect to the free action of $\mathbb{Z}^2$. In particular, this means that the framework has a finite number of vertex and edge orbits under the action of $\mathbb{Z}^2$. Further details on periodic frameworks can be found in the work of Borcea and Streinu, [2, 3].

Throughout this paper our approach will be to consider periodic frameworks as orbit frameworks on a torus. The 2-dimensional topological torus can naturally be considered a fundamental region for a tiling of the plane. In this way, we will consider the rigidity properties of frameworks on the torus as a model of the periodic rigidity of frameworks in the plane.

In particular, we will consider frameworks on the fixed torus. Let $L_0$ be a $2 \times 2$ matrix whose rows are independent vectors of the form $(x, 0), (y_1, y_2)$, $x, y_1, y_2 \in \mathbb{R}$. Let $L_0 \mathbb{Z}^2$ denote the group generated by the rows of $L_0$, viewed as
translations of \( \mathbb{R}^2 \). We call \( L_0 \mathbb{Z}^2 \) the \textit{fixed lattice}, and \( L_0 \) is the \textit{lattice matrix}. The quotient space \( \mathbb{R}^2 / L_0 \mathbb{Z}^2 \) will be called the \textit{fixed torus}, and we denote it by \( T_0^2 \). Pairs of translations (2-vectors) may generate the ‘same’ torus, up to rotation at the origin.

In [22] it is shown that lower triangular \( 2 \times 2 \) matrices are representatives of the equivalence classes of pairs of translations which generate the ‘same’ torus up to rotation.

A \textit{periodic orbit framework} is a pair \((G, m)\), where \((G, m)\) consists of a directed multigraph \( G = (V, E) \), together with a labeling \( m : E^+ \rightarrow \mathbb{Z}^2 \) of the edges by the elements of the 2-dimensional integer lattice. The map \( p : V \rightarrow T_0^2 \) describes the position of the vertices of \((G, m)\) on the fixed torus \( T_0^2 \). The graph \((G, m)\) will be called a \textit{periodic orbit graph}. Note that \((G, m)\) is actually a \textit{gain graph} or \textit{voltage graph} [9, 32], that is, a graph whose edges are labeled invertibly by elements of a group \( G \).

The edges of \((G, m)\) (denoted \( E(G, m) \)) are recorded as follows: \( e = \{v_i, v_j; m_e\} \), where \( m_e \in \mathbb{Z}^2 \). Since \( m \) labels the edges of \( G \) invertibly, it follows that we can equivalently write

\[ e = \{v_i, v_j; m_e\} = \{v_j, v_i; -m_e\}. \]

An example of a periodic orbit graph \((G, m)\) is shown in Figure 1(a). From the periodic orbit graph \((G, m)\) we can define the \textit{derived graph} \( G^m \), shown in Figure 1(b). The derived graph \( G^m \) has vertex set \( V^m \) and \( E^m \) where \( V^m \) is the Cartesian product \( V \times \mathbb{Z}^2 \), and \( E^m = E \times \mathbb{Z}^2 \). Vertices of \( V^m \) have the form \((v_i, a)\), where \( v_i \in V \), and \( a \in \mathbb{Z}^2 \). Edges of \( E^m \) are denoted similarly. If \( e \) is the directed edge connecting vertex \( v_i \) to \( v_j \) in \((G, m)\), and \( b \) is the gain assigned to the edge \( e \), then the edge \((e, a) = \{(v_i, a), (v_j, a + b)\} \) of \( G^m \) connects vertex \((v_i, a)\) to \((v_j, a + b)\). In this way, the derived graph is a graph whose automorphism group contains \( \mathbb{Z}^2 \). Furthermore, the periodic orbit graph \((G, m)\) can be seen as a kind of ‘recipe’ for the infinite derived graph \( G^m \).

In a similar way, from the periodic orbit framework \((G, m, p)\) we can define the \textit{derived periodic framework} \((\langle G^m, L_0 \rangle, p^m)\), where \( p^m \) is given by

\[ p^m(v, z) = p(v) + zL_0. \]

### 2.3 Rigidity theory for periodic orbit frameworks on \( T_0^2 \)

#### 2.3.1 Infinitesimal motions of periodic orbit frameworks

An \textit{infinitesimal motion} of a periodic orbit framework \((G, m, p)\) on \( T_0^2 \) is an assignment of velocities to each of the vertices, \( u : V \rightarrow \mathbb{R}^2 \), with \( u(v_i) = u_i \) such that

\[ (u_i - u_j) \cdot (p_i - p_j - m_eL_0) = 0 \]

for each edge \( e = \{v_i, v_j; m_e\} \in E(G, m) \). Such an infinitesimal motion preserves the lengths of any of the bars of the framework.
1.4 External Deadlines

March 28, 2011
Last date for FGS to receive a Recommendation for Oral Examination form, from the Graduate Program Director for students who expect to fulfill all Doctor of Philosophy degree requirements for June 2011 Convocation

April 1, 2011
Deadline to apply to graduate

April 25, 2011
Last date for the Faculty of Graduate Studies to receive from Graduate Program Directors the favourable decisions of thesis and dissertation examining committees for students who expect to fulfill all Master's and Doctoral of Philosophy degree requirements for June 2011 convocation

April 29, 2011
Last date for FGS to receive three unbound copies

Dissertation Colloquium: at least one month before defense

Tentative defense date: April 15, 2011

May 31, 2011
Last day for full refund of summer fees.

Figure 1: A gain graph \( \langle G,m \rangle \), where \( m : E \rightarrow \mathbb{Z}^2 \), and its derived graph \( G^m \).

We use graphs with vertex labels as in (a) to depict gain graphs, and graphs without such vertex labels will record derived graphs, or graphs that are realized in \( \mathbb{R}^d \).

A trivial infinitesimal motion of \((\langle G,m \rangle,p)\) on \( T_0^2 \) is an infinitesimal motion that preserves the distance between all pairs of vertices. That is,

\[
(u_i - u_j) \cdot (p_i - p_j - m_e L_0) = 0
\]

for all triples \( \{v_i, v_j; m_e\} \), \( m_e \in \mathbb{Z}^2 \). For any periodic orbit framework \((\langle G,m \rangle,p)\) on \( T_0^2 \), there will always be a 2-dimensional space of trivial infinitesimal motions of the whole framework, namely the space of infinitesimal translations. Rotation is not a trivial motion for periodic orbit frameworks on \( T_0^2 \), since we have fixed our representation of the lattice matrix \( L_0 \) under rotation.

If the only infinitesimal motions of a framework \((\langle G,m \rangle,p)\) on \( T_0^d \) are trivial (i.e. infinitesimal translations), then it is infinitesimally rigid. Otherwise, the framework is infinitesimally flexible.

2.3.2 The fixed torus rigidity matrix

The fixed torus rigidity matrix \( R_0(\langle G,m \rangle,p) \) is the \( |E| \times 2|V| \) matrix that records equations for the space of possible infinitesimal motions of the periodic orbit framework \((\langle G,m \rangle,p)\). It has one row for each edge \( e = \{v_i, v_j; m_e\} \) of \( \langle G,m \rangle \) as follows:

\[
\begin{pmatrix}
0 & \cdots & 0 & p_i - (p_j + m_e L_0) & 0 & \cdots & 0 & (p_j + m_e L_0) - p_i & 0 & \cdots & 0
\end{pmatrix},
\]

where each entry is actually a 2-tuple, and the non-zero entries occur in the columns corresponding to vertices \( v_i \) and \( v_j \). The kernel of this matrix is the space of infinitesimal motions of \((\langle G,m \rangle,p)\) on \( T_0^2 \), and we may write

\[
R_0((\langle G,m \rangle,p) \cdot u^T = 0
\]
where \( u = (u_1, u_2, \ldots, u_{|V|}) \), and \( u_i \in \mathbb{R}^2 \). That is, \( u \) is an infinitesimal motion of \( (\langle G, m \rangle, p) \) on \( T_0^2 \).

Since a framework on \( T_0^2 \) always has a two-dimensional space of trivial motions (translations), it follows that the kernel of the rigidity matrix always has dimension at least 2. Furthermore, because a framework is infinitesimally rigid on \( T_0^2 \) if and only if the only infinitesimal motions are translations, it follows that

\[
\text{Theorem 2.3 (22).} \quad \text{A periodic orbit framework} \ (\langle G, m \rangle, p) \ \text{is infinitesimally rigid on the fixed torus} \ T_0^2 \ \text{if and only if the rigidity matrix} \ R_0(\langle G, m \rangle, p) \ \text{has rank} \ 2|V| - 2.
\]

It follows that a periodic orbit framework with \( |E| < 2|V| - 2 \) cannot be infinitesimally rigid on \( T_0^2 \).

Example 2.4. Consider the periodic orbit graph \( \langle G, m \rangle \) shown in Figure 1. Let \( L_0 \) be the matrix generating the torus \( T_0^2 \). The rigidity matrix \( R_0(\langle G, m \rangle, p) \) will have have six rows, and eight columns (two columns corresponding to the two coordinates of each vertex), as follows:

\[
\begin{bmatrix}
  p_1 & p_2 & p_3 & p_4 \\
  p_1 - p_2 & p_2 - p_1 & 0 & 0 \\
  0 & p_2 - p_3 & p_3 - p_2 & 0 \\
  0 & 0 & p_3 - p_4 & p_4 - p_3 \\
  p_1 - p_4 & 0 & 0 & p_4 - p_1 \\
  p_1 - p_3 + (1, 0)L_0 & 0 & p_3 - p_1 - (1, 0)L_0 & 0 \\
  p_1 - p_4 - (0, 1)L_0 & 0 & 0 & p_4 - p_1 + (0, 1)L_0
\end{bmatrix}
\]

A collection of edges \( E' \subset E \) of the periodic orbit framework \( (\langle G, m \rangle, p) \) is called independent (resp. dependent) if the corresponding rows of the rigidity matrix are linearly independent (resp. linearly dependent). For example, any loop edge is dependent on \( T_0^2 \), and no more than two copies of an edge of \( G \) may be independent. We may also refer to a framework \( (\langle G, m \rangle, p) \) as being independent or dependent, and for clarity we will at times write dependent on \( T_0^2 \) to differentiate this setting from the finite case (frameworks which are not on a torus). We say a framework with \( |E| > 2|V| - 2 \) is over-counted, meaning that it is always dependent.

A periodic orbit framework \( (\langle G, m \rangle, p) \) whose underlying gain graph satisfies \( |E| = 2|V| - 2 \) and is infinitesimally rigid on \( T_0^2 \) will be called minimally rigid. In other words, a minimally rigid framework on \( T_0^2 \) is one that is both infinitesimally rigid and independent. In fact, such a framework is maximally independent in the sense that adding any new edge will introduce a dependence among the edges. If a periodic orbit framework is minimally rigid, then the removal of any edge will result in a framework that is not infinitesimally rigid. This is analogous to the definition of “isostatic” for finite frameworks, although...
we avoid that terminology. There are essential differences between the rigidity of finite and infinite frameworks (namely an infinite dimensional rigidity matrix), which eliminates the key relationships between independence and infinitesimal rigidity for infinite graphs. See [10] for further details.

From Theorem 2.3 we obtain the following simple corollary:

**Corollary 2.5.** Let \((G, m, p)\) be a minimally rigid periodic orbit framework on \(T_0^2\). Then

1. \(|E| = 2|V| - 2\), and
2. for all subgraphs \(G' \subseteq G\), \(|E'| \leq 2|V'| - 2\).

The rows of \(R_0((G, m, p))\) corresponding to edges with zero gains are identical to rows in the rigidity matrix of a finite framework, as described in any introduction to rigidity; see [8] or [31], for example. Since at most \(2|V| - 3\) rows can be independent in the finite rigidity matrix, we have the following result:

**Proposition 2.6.** Let \((G, m)\) be a periodic orbit graph with all edges having zero gains, \(m = 0\). If \(|E| > 2|V| - 3\), then the edges of \((G, m, p)\) are dependent for any configuration \(p\).

### 2.3.3 The unit torus and affine transformations

It is possible to model all two-dimensional periodic frameworks by considering only graphs on the torus where the lattice matrix is the identity matrix. This follows from the fact that rigidity on the torus is invariant under affine transformations. When \(L_0 = I_{2 \times 2}\) we call \(\mathbb{R}^2/\mathbb{Z}^2\) the unit torus, and denote it by \(U^2\). That is, affine transformations preserve the rank of the fixed torus rigidity matrix \(R_0\). This result was shown in [21], and also appeared independently in [3]. As a consequence of this fact, for the remainder of this paper we assume that \(L_0 = I_{2 \times 2}\).

### 2.3.4 Generic periodic orbit frameworks

Since our goal in the remainder of this paper will be to characterize the rigidity properties of periodic orbit frameworks based on their periodic orbit graphs, we need a notion of a generic configuration on the torus. Roughly speaking, this will be a configuration without any geometric degeneracies. In addition, since we will be building up larger frameworks from smaller ones, we need a notion of generic which will continue to be generic as we add edges and vertices.

Let \(V\) be a finite set of vertices, and let \(p\) be a realization of these vertices on the \(d\)-dimensional unit torus \(T_0^d = [0, 1)^d\). Let \(k \in \mathbb{Z}_+\) be given, and let \(K\) be the set of all edges between pairs of vertices of \(V\) with gains \(m_e = (m_{e,1}, m_{e,2})\) where \(|m_{e,i}| \leq k\) for \(i = 1, 2\). Then \(K\) is the set of all edges with bounded gains.

Consider a set of edges \(E \subseteq K\) such that, for some realization \(p\), the rows of \(R_0\) corresponding to \(E\) are independent. By taking the \(p_i\)’s as variables, the determinants of the \(|E| \times |E|\) submatrices of these rows will either be identically
zero or will define an algebraic variety in $\mathbb{R}^{2|V|}$. The collection of all such varieties, corresponding to all such subsets $E$ will define a closed set of measure zero, as a finite union of closed sets of measure zero. Let this set be denoted $\mathcal{X}_k$. The complement of $\mathcal{X}_k$ in $\mathbb{R}^{2|V|}$ is an open dense set in $\mathbb{R}^{2|V|}$, and hence its restriction to the subspace of realizations $p$ of the vertices $V$ on the unit torus, $[0,1)^2$ is also open and dense.

Any realization $p$ of the vertex set $V$ where $p \notin \mathcal{X}_k$ will be called $k$-generic (recall that $k$ was the upper bound on the gain assignments). More generally, we may consider graphs that are $k$-generic for any $k$. By the Baire Category Theorem, the countable intersection

$$\bigcap_{k \in \mathbb{Z}} (\mathbb{R}^{2|V|} - \mathcal{X}_k)$$

is dense in $\mathbb{R}^{2|V|}$, as the intersection of open dense sets in the Baire space $\mathbb{R}^{2|V|}$. We refer to a realization in this set as simply generic, and it is this definition that we use throughout the remainder of this paper. All generic frameworks $\langle G, m, p \rangle$ with the same underlying periodic orbit graph $\langle G, m \rangle$ will have the same rigidity properties, a fact captured by the following result.

**Lemma 2.7 (Special Position Lemma).** Let $\langle G, m \rangle$ be a periodic orbit graph, and suppose that for some realization $p_0$ of $\langle G, m \rangle$ on $T^2_0$ the framework $\langle G, m, p_0 \rangle$ is infinitesimally rigid. Then for all generic realizations $p$ of $\langle G, m \rangle$ on $T^2_0$, the framework $\langle G, m, p \rangle$ is infinitesimally rigid.

This lemma justifies the use of the phrase “$\langle G, m \rangle$ is generically minimally rigid”.

### 2.4 $T$-gain procedure

Let $\langle G, m \rangle$ be a periodic orbit graph. The cycle space of $\langle G, m \rangle$ is the cycle space of $G$, namely the vector space generated by the set of all simple cycles of $G$. Elements of the cycle space $\mathcal{C}(G)$ are either simple cycles, or the disjoint union of simple cycles. The cycle space can be described as the vector space corresponding to a fundamental system of cycles [5]. For any simple cycle $C$ of $G$, we define the net gain on the cycle $C$ to be the sum of the gains on the edges of the cycle, with sign taken according to the order of traversal of the edges. Similar to the cycle space, we define the gain space of $\langle G, m \rangle$ to be the vector space (over $\mathbb{Z}$) of net gains on the cycles of the cycle space of $G$.

The $T$-gain procedure is a procedure that can be used to easily identify the net gains on the cycles of a periodic orbit graph $\langle G, m \rangle$. In particular, the $T$-gain procedure will identify the net gains on a fundamental system of cycles of $\langle G, m \rangle$, which form a basis for the gain space of $\langle G, m \rangle$. As we will soon see (Section 5), the rigidity of frameworks on $T^2_0$ is generically characterized by the net gains on the cycles of the periodic orbit graph. The $T$-gain procedure will thus be an essential proof technique in the rest of the paper. The $T$-gain procedure appears in [9] for general gain graphs, and we outline it here for
graphs whose gain group is \( \mathbb{Z}^2 \). More details can also be found in [22] or [21]. See Figure 2 for a worked example.

![Figure 2: A gain graph \( \langle G, m \rangle \) in (a), with identified tree \( T \) (in red), root \( u \), and \( T \)-potentials in (b). The resulting \( T \)-gain graph \( \langle G, m_T \rangle \) is shown in (c). The gain space is now seen to be generated by the elements \((4, 0)\) and \((2, 2)\), hence the gain space is \( 2\mathbb{Z} \times 2\mathbb{Z} \).](image)

**T-gain Procedure**

1. Select an arbitrary spanning tree \( T \) of \( G \), and choose a vertex \( u \) to be the root vertex. Such a spanning tree is known to exist, as we assumed \( G \) was connected.

2. For every vertex \( v \) in \( G \), there is a unique path in the tree \( T \) from the root \( u \) to \( v \). Denote the net gain along that path by \( m(v, T) \), and we call this the \( T \)-potential of \( v \). Compute the \( T \)-potential of every vertex \( v \) of \( G \).

3. Let \( e \) be a plus-directed edge of \( G \) with initial vertex \( v \) and terminal vertex \( w \). Define the \( T \)-gain of \( e \), \( m_T(e) \) to be
   \[
   m_T(e) = m(v, T) + m(e) - m(w, T).
   \]
   Compute the \( T \)-gain of every edge in \( G \). Note that the \( T \)-gain of every edge of the spanning tree will be zero.

4. Contract the graph along the spanning tree to obtain \( |E| - (|V| - 1) \) loops at the root vertex \( u \) (there are \( |V| - 1 \) edges as part of the spanning tree). The gains on these loops will generate the gain space. In other words, the gains on all of the edges of the graph that are not contained in \( T \) will generate the gain space.

**Theorem 2.8** ([9]). Let \( \langle G, m \rangle \) be a periodic orbit graph, and let \( \langle G, m_T \rangle \) be the same periodic orbit graph after the \( T \)-gain procedure. Then \( \langle G, m \rangle \) and \( \langle G, m_T \rangle \) have the same gain space.

**Proof.** The \( T \)-gain procedure preserves the net gains on cycles. \( \square \)
It is also true that the corresponding derived graphs are isomorphic.

**Theorem 2.9** ([9]). Let \( \langle G, m \rangle \) be a gain graph, let \( u \) be any vertex of \( G \), and let \( T \) be any spanning tree of \( G \). Then the derived graph \( G^{m,T} \) corresponding to \( \langle G, m_T \rangle \) is isomorphic to the derived graph \( G^m \).

We say that the graphs \( \langle G, m \rangle \) and \( \langle G, m_T \rangle \) are \( T \)-gain related and we write \( \langle G, m \rangle \sim \langle G, m_T \rangle \). More broadly, we say that \( \langle G, m \rangle \) and \( \langle G, m' \rangle \) are \( T \)-gain equivalent if \( \langle G, m \rangle \sim \langle G, m_T \rangle \) and \( \langle G, m' \rangle \sim \langle G, m_T \rangle \) for some choice of spanning tree \( T \). In fact, if this is true for one spanning tree, it must be true for all choices of spanning tree, since the \( T \)-gain procedure preserves the net gains on all cycles. \( T \)-gain equivalence can easily be shown to be an equivalence relation on the set of all gain assignments on a graph \( G \).

Most important to the remainder of the paper is the following result, which is proved in [22]. It states that \( T \)-gain equivalent periodic orbit graphs have the same generic rigidity properties.

**Theorem 2.10.** The periodic orbit graph \( \langle G, m \rangle \) is generically rigid on \( T_0^2 \) if and only if \( \langle G, m_T \rangle \) is generically rigid on \( T_0^2 \).

Further details on the \( T \)-gain procedure and the related notions of the fundamental group of a graph can be found in [22].

### 2.5 Toward a characterization of generic rigidity of periodic orbit frameworks on \( T_0^2 \)

The following theorem says that given a graph \( G \) with certain combinatorial properties, we can always find an appropriate gain assignment \( m \) and geometric realization \( p \) to yield a minimally rigid framework on \( T_0^d \).

**Theorem 2.11** (Whiteley, [30]). For a multigraph \( G \), the following are equivalent:

(i) \( G \) satisfies \( |E| = d|V| - d \), and every subgraph \( G' \subseteq G \) satisfies \( |E'| \leq d|V'| - d \),

(ii) \( G \) is the union of \( d \) edge-disjoint spanning trees,

(iii) For some gain assignment \( m \) and some realization \( p \), the framework \( (\langle G, m \rangle, p) \) is minimally rigid on \( T_0^d \).

Note that \( (iii) \Rightarrow (i) \) is a direct consequence of Theorem 2.3 and the equivalence of \( (i) \) and \( (ii) \) is the well known Nash-Williams correspondence [15]. The remainder of this paper will be aimed at strengthening and broadening the scope of Theorem 2.11. In particular, we will answer the question “for what gain assignments \( m \) is \( \langle G, m \rangle \) generically minimally rigid on \( T_0^2 \)?”
3 Statement of main result

The goal of the rest of this paper is to prove the following theorem, which builds on an earlier result of Whiteley (Theorem 2.11), and which characterizes minimal rigidity on $T_0^2$.

**Theorem 3.1.** For a multigraph $G = (V,E)$, the following are equivalent:

(i) $G$ is the union of 2 edge-disjoint spanning trees;

(ii) $G$ satisfies $|E| = 2|V| - 2$ and every subgraph $G' \subset G$ satisfies $|E'| \leq 2|V| - 2$;

(iii) If $\langle G, m \rangle$ is generically minimally rigid on $T_0^2$, then it can be constructed from a single vertex by a sequence of periodic vertex additions and edge splits;

(iv) for all constructive gain assignments $m$, $\langle G, m \rangle$ is generically minimally rigid on $T_0^2$;

(v) for some gain assignment $m$ and some realization $p$, the framework $\langle (G,m),p \rangle$ is minimally rigid on $T_0^2$.

This clearly builds on Theorem 2.11, but is extended in two key ways:

- (iii) is a periodic version of Henneberg’s Theorem, and is the subject of Section 4
- (iv) is a periodic version of Laman’s Theorem, and is the subject of Section 5

**Proof.** The theorem is proved as follows:

(i) $\leftrightarrow$ (ii) $\rightarrow$ (iv) $\hookleftarrow$ (v) $\leftrightarrow$ (iii)

The equivalence of (i), (ii) and (v) is the content of Theorem 2.11. We will show:

(iii) $\iff$ (v) is the periodic Henneberg theorem, Theorem 4.7

(iv) $\implies$ (v) is immediate.

(ii) $\implies$ (iv) is the content of Theorem 5.6

4 Generating minimally rigid frameworks on $T_0^2$

We now describe methods for generating infinitesimal rigid frameworks on the fixed two dimensional torus $T_0^2$, namely inductive constructions.
4.1 Inductive constructions

Let \((G, m, p)\) be an infinitesimally rigid periodic orbit framework. It is possible to construct other infinitesimally rigid frameworks from \((G, m, p)\) using periodic vertex additions and edge splits. We present here the details of these periodic inductive constructions, which we will also call periodic Henneberg moves after their finite counterparts which were developed by Henneberg [11]. Details about the finite versions of these moves can be found in [28] and [30]. The following arguments are based on the rigidity matrix. In particular, we will show that the periodic inductive constructions preserve the independence of the rows of \(R_0\).

4.1.1 Vertex addition

Given a periodic orbit graph \(\langle G, m \rangle\), a periodic vertex addition is the addition of a single new vertex \(v_0\) to \(V(G) = \langle G, m \rangle\), and the edges \(\{v_0, v_1; m_{01}\}\) and \(\{v_0, v_2; m_{02}\}\) to \(E(G, m)\), such that \(m_{01} \neq m_{02}\) whenever \(v_1 = v_2\) (see Figure 3). Provided that \(v_1 \neq v_2\), by definition, \(m_{01}\) and \(m_{02}\) may always taken to be \((0, 0)\), since this is simply the usual finite vertex addition. Examples are shown in Figure 4.

Proposition 4.1 (Periodic Vertex Addition). Let \(\langle G, m \rangle\) be a periodic orbit graph, and let \(\langle G', m' \rangle\) be the graph created by performing a vertex addition
on \( (G,m) \), adding the vertex \( v_0 \) to \( G \). For a generic choice of \( p : V \to T_0^2 \), and with \( p_0 \) chosen generically with respect to \( p \), the rows of \( R_0((G,m),p) \) are independent if and only if the rows of \( R_0((G',m'),p') \) are independent, where \( p' = p \cup p_0 \).

**Proof.** Suppose that the vertex \( v_0 \) is connected to the vertices \( v_{i_1} \) and \( v_{i_2} \) by the edges \( \{v_0,v_{i_1};m_{01}\} \) and \( \{v_0,v_{i_2};m_{02}\} \), where \( v_{i_1} \) and \( v_{i_2} \) may or may not be the same vertex. The rigidity matrix of \( (G',m') \) is

\[
R_0((G',m'),p') = \begin{pmatrix}
\{v_0,v_{i_1};m_{01}\} & p_0 - p_{i_1} - m_{01} & \cdots \\
\{v_0,v_{i_2};m_{02}\} & p_0 - p_{i_2} - m_{02} & \cdots \\
\vdots & \vdots & \ddots
\end{pmatrix}
\]

Suppose, toward a contradiction, that the rows of \( R_0((G',m'),p') \) are dependent. Then the columns of \( R_0((G',m'),p') \) corresponding to \( v_0 \) provide the relationship:

\[
\omega_{01}(p_0 - p_{i_1} - m_{01}) + \omega_{02}(p_0 - p_{i_2} - m_{02}) = 0
\]

for some \( \omega_{01}, \omega_{02} \in \mathbb{R} \). However, \((p_0 - p_{i_1} - m_{01})\) and \((p_0 - p_{i_2} - m_{02})\) are linearly independent (as vectors in \( \mathbb{R}^2 \)) if and only if the points \( p_0, p_{i_1} + m_{01} \) and \( p_{i_2} + m_{02} \) are not collinear.

If \( v_{i_1} \neq v_{i_2} \), or if \( v_{i_1} = v_{i_2} \) but \( m_{01} \neq m_{02} \), then these points are not collinear, since we chose \( p_0 \) generically with respect to \( p \). Hence \( \omega_{01} = \omega_{02} = 0 \), which leaves a dependence among the rows of \( R_0((G,m),p) \) and contradicts our assumption that the rows of \( R_0((G,m),p) \) were independent. The argument reverses for the converse. (Assume the rows of \( R((G',m'),p') \) are dependent and proceed from there.) \( \square \)

Note that Proposition 4.1 also has a geometric meaning. In fact, the proof of that result was geometric in nature, in the sense that we chose \( p \) so that the points \( p_0, p_{i_1} + m_{01} \) and \( p_{i_2} + m_{02} \) were not collinear in \( \mathbb{R}^2 \) (in fact we chose \( p \) to be generic, and the non-collinearity followed). This observation is summarized in the following corollary:

**Corollary 4.2.** Let \((G,m),p)\) be a periodic orbit framework, and let \((G',m')\) be the graph created by performing a vertex addition on \((G,m)\), adding the vertex \( v_0 \) to \( G \). If \( p_0 \) is not collinear with \( p_{i_1} + m_{01} \) and \( p_{i_2} + m_{02} \) in \( \mathbb{R}^2 \), then the rows of \( R_0((G,m),p) \) are independent if and only if the rows of \( R_0((G',m'),p') \) are independent, where \( p' = p \cup p_0 \).

### 4.1.2 Edge splitting

Let \((G,m)\) be a periodic orbit graph, and let \( e = \{v_{i_1},v_{i_2};m_e\} \) be an edge of \((G,m)\). A periodic edge split \((G',m')\) of \((G,m)\) is a graph with vertex set
\( V \cup \{v_0\} \) and edge set consisting of all of the edges of \( E(G, m) \) except \( e \), together with the edges
\[
\{v_0, v_i; (0, 0)\}, \{v_0, v_i_2; m_e\}, \{v_0, v_i_3; m_03\}
\]
where \( v_i_1 \neq v_i_3 \), and \( m_{03} \neq m_e \) if \( v_i_2 = v_i_3 \) (see Figure 5).

Figure 5: Periodic edge split.

Periodic edge splits, and reverse periodic edge splits, preserve infinitesimal rigidity. We will show this in two parts, by first showing that the periodic edge split preserves independence of the rows of the rigidity matrix.

**Proposition 4.3 (Periodic Edge Split).** Let \( (G, m) \) be a periodic orbit graph, and let \( (G', m') \) be an edge split of it. Let \( p : V \to T_0^2 \) be a generic realization of \( V(G) \) on \( T_0^2 \), and let \( p_0 \) be chosen generically with respect to \( p \). If the rows of \( R_0((G, m), p) \) are independent, then the rows of \( R_0((G', m'), p') \) are also independent, where \( p' = p \cup p_0 \).

**Proof.** Suppose that \( p \) is a generic realization of the vertices of \( G \) on \( T_0^2 \), with no vertex on the boundary of \( T_0^2 \), and place \( p_0 \) on the edge connecting the vertices \( v_i_1 \) and \( v_i_3 + m_e \), where the segment containing \( v_i_1 \) and \( p_0 \) lies in \([0, 1]^d \). Without loss of generality, suppose that \( e_1 \) is the split edge. Let \( R_0((G, m), p) - e_1 \) denote the rigidity matrix of \((G, m), p)\) without the row corresponding to the edge \( e_1 \).

Figure 6: Periodic edge splits on the torus. Parts (a) and (b) depict the case where \( v_0 \) is adjacent to three distinct vertices in \( (G, m) \), while (c) and (d) illustrate the case of only two distinct neighbours.
The rigidity matrix $R_0((G', m'), p')$ is:

$$
\begin{pmatrix}
\omega_0 & \omega_1 & \omega_2 & \cdots & \omega_{|V|} \\
0 & p_0 - p_{i_1} & p_{i_1} - p_0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\{v_0, v_{i_2}; m_e\} & p_0 - p_{i_2} - m_e & 0 & p_{i_2} - p_0 + m_e & \cdots \\
\{v_0, v_{i_3}; m_{03}\} & p_0 - p_{i_3} - m_{03} & 0 & \cdots & \cdots \\
\end{pmatrix}
$$

Suppose toward a contradiction, that there is a non-trivial dependence among the rows of $R_0((G', m'), p')$. That is, suppose that

$$\omega \cdot R_0((G', m'), p') = 0$$

for $\omega \neq 0$ where $\omega = [\omega_0 \omega_1 \cdots \omega_{|E|} \omega_{01} \omega_{02} \omega_{03}]$.

The vector equation describing the first two columns of this expression (the columns corresponding to $v_0$) becomes:

$$\omega_{01}(p_0 - p_{i_1}) + \omega_{02}(p_0 - p_{i_2} - m_e) + \omega_{03}(p_0 - p_{i_3} - m_{03}) = 0$$

with not all of $\omega_{01}, \omega_{02}, \omega_{03}$ being 0 (otherwise we would immediately have a nontrivial dependence among the rows of $R_0((G, m), p)$, contradicting our hypothesis).

Because we placed $p_0$ along the edge connecting $v_{i_1}$ and $v_{i_2} + m_e$, the vectors $(p_0 - p_{i_1})$ and $(p_0 - p_{i_2} - m_e)$ are parallel, and $(p_0 - p_{i_3} - m_{03})$ is in a distinct direction, therefore $\omega_{03} = 0$. Since both of these vectors are again parallel to the deleted edge, we have

$$\omega_{01}(p_0 - p_{i_1}) = -\omega_{02}(p_0 - p_{i_2} - m_e) = \omega_{12}(p_{i_1} - p_{i_2} - m_e)$$

for some scalar $\omega_{12} \neq 0$.

But then the coefficients of the rows of $R_0((G', m'), p')$ corresponding to the edges in $E \cap E'$, together with $\omega_{12}$ form a set of scalars that provide a dependence among the rows of $R_0((G, m), p)$, which contradicts our hypothesis.

By the Special Position Lemma (Lemma 2.7), we conclude that the edges of $(G', m'), p')$ are generically independent, since the edges are independent for a special position of $p_0$.

The reverse periodic edge split will delete a 3-valent vertex, and add an edge between two of the vertices formerly adjacent to that vertex (Figure 7). In particular, if $v_0$ is the 3-valent vertex adjacent to the vertices $v_{i_1}, v_{i_2}$ and $v_{i_3}$, where at most two of $v_{i_1}, v_{i_2}$ and $v_{i_3}$ may be the same, then a reverse edge split will add one of the edges

$$\{v_{i_1}, v_{i_2}; m_{02} - m_{01}\}, \{v_{i_2}, v_{i_3}; m_{03} - m_{02}\}, \{v_{i_3}, v_{i_1}; m_{01} - m_{03}\}.$$
Proposition 4.4 (Reverse Periodic Edge Split). If a 3-valent vertex \(v_0\) is deleted from a generically independent periodic orbit graph \(\langle G', m' \rangle\), then a single edge may be added between one pair of vertices formerly adjacent to \(v_0\) so that the resulting graph \(\langle G, m \rangle\) is also a generically independent periodic orbit graph.

Proof. Suppose that the rows of \(R_0((G', m'), p')\) are independent for some \(p' = p \cup p_0\), and suppose that the vertex \(v_0\) is connected to vertices \(v_{i_1}, v_{i_2}\) and \(v_{i_3}\), where at most two of these vertices are the same. Let \(E^*\) be the edge set created by deleting vertex \(v_0\) and its adjacent edges. Let \(G_{12}, G_{23}\) and \(G_{31}\) be the graphs with vertex set \(V \setminus \{v_0\}\), and edge sets \(E_{12} = E^* \cup \{v_{i_1}, v_{i_2}; m_{02} - m_{01}\}\) and similarly for \(E_{23}\) and \(E_{31}\). If any of these graphs is independent at \(p\) then we are done.

Assume to the contrary that no such graph is independent. Then the rows of the matrices corresponding to each of these frameworks are dependent. Writing \(R_e\) as the row of the rigidity matrix corresponding to the edge \(e\), we have

\[
\alpha_{12} R_{12} = \sum_{e \in E^*} -\alpha_e R_e \quad \text{with} \quad \alpha_{12} \neq 0,
\]

\[
\beta_{23} R_{23} = \sum_{e \in E^*} -\beta_e R_e \quad \text{with} \quad \beta_{23} \neq 0,
\]

\[
\gamma_{31} R_{31} = \sum_{e \in E^*} -\gamma_e R_e \quad \text{with} \quad \gamma_{31} \neq 0.
\]

We now have two cases:

1. The vertices \(v_{i_1}, v_{i_2}\) and \(v_{i_3}\) are distinct,

2. The vertices \(v_{i_1}, v_{i_2}\) and \(v_{i_3}\) are not distinct.

In Case 1, consider the graph on the vertices \(\{v_0, v_{i_1}, v_{i_2}, v_{i_3}\}\) with all of the candidate edges (see Figure 8). This has \(|E| = 2|V| - 2\). Note that the net gain on any closed path in the graph is \((0, 0)\), and hence this graph is \(T\)-gain equivalent to a graph with all gains identically zero. By Lemma 2.6 and Theorem 2.10 this graph is dependent.
Therefore, we have
\[ \omega_{01} R_{01} + \omega_{02} R_{02} + \omega_{03} R_{03} + \omega_{12} R_{12} + \omega_{23} R_{23} + \omega_{31} R_{31} = 0. \]
Scaling and substituting the expressions above, we obtain
\[ \omega_{01} R_{01} + \omega_{02} R_{02} + \omega_{03} R_{03} + \sum_{e \in E^*} - (\alpha'_e + \beta'_e + \gamma'_e) R_e = 0, \]
which is a dependence on the rows of \( R_{0}((G', m'), p') \), a contradiction. Therefore, at least one of the graphs \( G_{12}, G_{23}, G_{31} \) must be independent.

For Case 2, assume without loss of generality that \( v_{i2} = v_{i3} \). We consider the graph on the vertices \( \{v_0, v_{i1}, v_{i2}\} \) with all of the candidate edges (see Figure 9). This graph has \( |E| = 2|V| - 1 \), and hence is dependent. The proof of this case now follows the proof of the previous case.

Figure 9: This graph, corresponding to Case 2 of Proposition 4.4 satisfies \( |E| = 2|V| - 1 \), therefore a dependence exists among the edges.

Both the vertex addition and the edge split preserve the relationship between the number of edges and the number of vertices in the \( d \)-periodic orbit graph. If \( |E| = 2|V| - 2 \) then \( |E'| = 2|V'| - 2 \) as well. We have the following corollary to the previous propositions:

**Corollary 4.5.** Periodic vertex additions and edge splits, and their reverse operations, preserve generic minimal rigidity of periodic orbit graphs \( \langle G, m \rangle \) on \( \mathbb{T}^2_0 \).

The process of deleting a three-valent vertex from \( \langle G, m \rangle \) by a reverse edge split, and then performing an edge split will not usually produce a graph that is
identical to the original (see Figure 10). However, we can ensure that we always produce a graph with an isomorphic space of infinitesimal motions, using the following lemma:

**Lemma 4.6.** Let $(G, m)$ be a periodic orbit graph, and let $(G', m')$ be a reverse edge split of $(G, m)$. Then for some edge split $(G, m)$ of $(G', m')$ with $G = G'$, the resulting graph $(G, m)$ is $T$-gain equivalent to $(G, m)$.

**Proof.** Let $v_0$ be a 3-valent vertex of $(G, m)$, adjacent to vertices $v_1, v_2, v_3$ (see Figure 10). After deleting $v_0$, suppose without loss of generality that the edge $e = \{v_1, v_2; m_{02} - m_{01}\}$ was added to form the graph $(G', m')$. We perform an edge split on this edge to obtain a graph that differs from our original orbit graph, but which has a rigidity matrix with the same rank. In particular, we add to $(G', m')$ the vertex $v_0$ and the three edges:

$\{v_0, v_1; (0, 0)\}, \{v_0, v_2; m_{02} - m_{01}\}, \{v_0, v_3; m_{03} - m_{01}\}$.

Let the resulting infinitesimally rigid graph be denoted $(G, m)$. Note that the gains on the first two edges are determined by the reverse edge split, but the gain on the third edge is a 'free' choice.

Now let $T'$ be a spanning tree in $G'$ with root $u = v_1$ that does not include the edge $e = \{v_1, v_2; m_{02} - m_{01}\}$ (which has gain $m_{02} - m_{01}$ in $(G', m')$). It is always possible to select such a tree, since deleting this edge will not disconnect the graph. Let $T$ be the spanning tree of $G$ created by adding the edge $\{v_0, v_1\}$ to $T'$. This edge has gain $m_{01}$ in $(G, m)$, and gain $(0, 0)$ in $(G, m)$. Performing the $T$-gain procedure on $(G, m)$ and $(G, m)$ with $T$, we obtain identical periodic orbit graphs. For example, the edge $e_2 = \{v_0, v_2, m_{02}\} \in (G, m)$ has $T$-gain

$$m_T(e_2) = m(v_0, T) + m_{02} - m(v_2, T)$$
$$= -m_{01} + m_{02} - m(v_2, T)$$
$$= (0, 0) + (m_{02} - m_{01}) - m(v_2, T)$$
$$= (0, 0) + (m_{02} - m_{01}) - \overline{m}(v_2, T)$$
$$= \overline{m}_T(e_2).$$

The same is true of the other edges added in the edge split, and since $T = T'$
for all of the edges of \( \langle G', m' \rangle \), the orbit graphs are \( T \)-gain equivalent. That is,
\[
\langle G, m_T \rangle = \langle G, \overline{m_T} \rangle.
\]

\[\square\]

### 4.2 Periodic Henneberg Theorem

**Theorem 4.7** (Periodic Henneberg Theorem). A periodic orbit graph \( \langle G, m \rangle \) on \( T_{0}^{2} \) is generically minimally rigid if and only if it can be constructed from a single vertex on \( T_{0}^{2} \) by a sequence of periodic vertex additions and edge splits.

The proof of this result follows the proof of Henneberg’s result appearing in, for example, [8], and we simply sketch the main idea. The proof proceeds by induction on the number of vertices, with the observation that the graph consisting of a single vertex (no edges) is minimally rigid on \( T_{0}^{2} \). In addition, since \(|E| = 2|V| - 2\), we may always find a vertex of degree 2 or 3. This vertex is deleted by a reverse periodic Henneberg move, at which point the inductive hypothesis applies.

For a periodic orbit graph \( \langle G, m \rangle \), we call the sequence of orbit graphs
\[
\langle G_1, m_1 \rangle, \langle G_2, m_2 \rangle, \ldots, \langle G_n, m_n \rangle = \langle G, m \rangle
\]
beginning with a single vertex \(|V_1| = 1\) and ending with \( \langle G, m \rangle \) (\(|V_n| = n = |V|\)) the (periodic) Henneberg sequence for \( \langle G, m \rangle \). We observe that given a Henneberg sequence for a periodic orbit graph \( \langle G, m \rangle \), beginning with a single vertex and concluding with \( \langle G, m \rangle \), it can be checked in linear time that \( \langle G, m \rangle \) is generically rigid on \( T_{0}^{2} \) (with one step per vertex). An example of a Henneberg sequence is shown in Figure 11.

### 5 Gain assignments determine rigidity on \( T_{0}^{2} \)

In this section, we characterize the generic rigidity properties of a framework on the two-dimensional fixed torus \( T_{0}^{2} \) by its gain assignment. In Section 5.1 we show that only graphs with constructive gain assignments can be rigid, and Section 5.2 will demonstrate that all such periodic orbit graphs are generically rigid.

#### 5.1 Constructive gain assignments

Let \( \langle G, m \rangle \) be a periodic orbit graph. Let \( C \) be a closed oriented cycle with no repeated vertices, starting and ending at a vertex \( u \) in the multigraph. Recall that the net cycle gain is the sum \( m_C \) of the gain assignments of the edges of the cycle, where the signs of the edges are determined by the traversal order specified by the orientation. We say the net gain on the cycle is non-zero or non-trivial if it is non-zero on at least one of the coordinates of \( m_C \in \mathbb{Z}^2 \).
Figure 11: An example of a periodic Henneberg sequence. The single vertex (a) becomes a single cycle through a vertex addition (b). Adding a third vertex in (c), then splitting off the edge \{1,3; (1,1)\} and adding the fourth vertex (d). The final graph is shown in (e).
Let $G = (V, E)$ be a multigraph with $|E| = 2|V| - 2$ edges, and where every
subgraph $G' \subseteq G$ satisfies $|E'| = 2|V'| - 2$. A **constructive gain assignment**
on $G$ is a map $m : E \to \mathbb{Z}^2$ such that every subgraph $G' \subseteq G$ with $G' = (V', E')$and $|E| = 2|V| - 2$ contains some cycle of vertices and edges with a non-zero
net gain. A cycle $C$ with a non-zero net gain will be called a **constructive cycle**.
If $\langle H, m_H \rangle$ is a graph with $|E(H)| > 2|V(H)| - 2$, we say that $\langle H, m_H \rangle$ has a
constructive gain assignment if there is some subgraph $G \subseteq H$ such that $m_H|G$
is constructive on $G$.

**Proposition 5.1.** Let $\langle G, m \rangle$ be a periodic orbit graph with $|E| = 2|V| - 2$, and
$|E'| \leq 2|V'| - 2$ for all subgraphs $G' \subseteq G$. If $(\langle G, m \rangle, p)$ is infinitesimally rigid
on $T_0^2$ for some realization $p$, then $m$ is constructive.

**Proof.** We will show the contrapositive. Suppose that $m$ is not constructive,
and therefore there exists a subgraph $\langle G', m' \rangle \subseteq \langle G, m \rangle$ with $|E'| = 2|V'| - 2$
and no constructive cycles. Let $T'$ be a spanning tree in $G'$, and expand $T'$
to a spanning tree $T$ of all of $G$. This is always possible, since $G$ is connected.

Perform the $T$-gain procedure on $\langle G, m \rangle$. Every edge in $T$ and therefore in
$T'$ will have zero gains, and hence no other edge in $E'$ may have non-zero gain,
since the $T$-gain procedure preserves net cycle gains.

Hence $\langle G', m' \rangle$ consists of $2|V'| - 2$ edges with zero gains, which correspond
to dependent rows in the rigidity matrix, since at most $2|V'| - 3$ edges without
 gains can be independent in the rigidity matrix, by Lemma 2.6 Therefore,

$$\text{rank}_{R_0}(\langle G, m \rangle, p) < 2|V| - 2,$$

and $(\langle G, m \rangle, p)$ is infinitesimally flexible on $T_0^2$. \hfill \square

**Remark 5.2.** Malestein and Theran [13] independently use a similar idea, and
define the $\mathbb{Z}^2$-rank of a periodic orbit graph $(\langle G, m \rangle)$ to be the number of linearly
independent vectors among the cycle gains of the set of all simple cycles of the
graph. They use the word ‘coloured graphs’ to describe our gain graphs. \hfill \square

**Remark 5.3.** A constructive cycle in $(\langle G, m \rangle, p)$ corresponds to an infinite path
in the derived periodic framework $(G^m, p^m)$. Let $u$ be a vertex of $(\langle G, m \rangle, p)$,
and suppose $C$ is a cycle beginning and ending at $u$ with net gain $(z_1, z_2) \in \mathbb{Z}^2$.
Then the edges of $C$ correspond to a finite path connecting the vertices $(u, (0, 0))$
and $(u, (z_1, z_2))$ in $(G^m, p^m)$. Repeating the argument we find that all vertices
of the form $(u, c(z_1, z_2)), c \in \mathbb{Z}$ are connected along a single (infinite) path.
In contrast, cycles which are not constructive in $\langle G, m \rangle$ correspond to simple
(finite) cycles in $G^m$. \hfill \square

The following section will demonstrate that constructive gain assignments
are also sufficient for infinitesimal rigidity on $T_0^2$.

### 5.2 Periodic Laman Theorem on $T_0^2$

The main result of this section is the following:
Theorem 5.4 (Periodic Laman Theorem). Let \( \langle G, m \rangle \) be a periodic orbit graph. Then \( \langle G, m \rangle \) is generically minimally rigid on \( T_0^2 \) if and only if \( \langle G, m \rangle \) satisfies

(i) \(|E| = 2|V| - 2\), and \(|E'| \leq 2|V'| - 2\) for all subgraphs \( G' \subset G \)

(ii) \( m \) is a constructive gain assignment.

Remark 5.5. In [1], graphs satisfying (i) and (ii) are called “Ross graphs”. We avoid this terminology here for obvious reasons.

Since we have already established that (i) is necessary for minimal rigidity on \( T_0^2 \) (Corollary 2.5), we will prove the following:

Theorem 5.6. Let \( \langle G, m \rangle \) be a periodic orbit graph, with \(|E| = 2|V| - 2\), and \(|E'| \leq 2|V'| - 2\) for all subgraphs \( G' \subset G \). Then \( \langle (G, m), p \rangle \) is generically minimally rigid on \( T_0^2 \) if and only if \( m \) is a constructive gain assignment.

The ‘only if’ part was Proposition 5.1. The proof of the ‘if’ part of this theorem will require a number of technical results, which follow. In particular, we will show

Proposition 5.7. Let \( \langle G, m \rangle \) be a periodic orbit graph on \( T_0^2 \) satisfying

(i) \(|E| - 2|V| - 2\), and every subgraph \( G' \subset G \) satisfies \(|E'| \leq 2|V'| - 2\).

(ii) \( m \) is constructive

Then it is always possible to delete any 2-valent vertex \( v_0 \), or perform a reverse edge split on any 3-valent vertex \( v_0 \) such that the resulting graph \( \langle G_0, m_0 \rangle \) also satisfies the properties (i) – (ii) above.

Note that the graph \( \langle G, m \rangle \) in Proposition 5.7 is not assumed to be rigid, which distinguishes this result from the fact that vertex-deletions and reverse edge splits preserve infinitesimal rigidity on \( T_0^2 \) (Propositions 4.1 and 4.4).

We delay the proof of Proposition 5.7 until after the proof of our main result.

Proof of Theorem 5.6. The proof proceeds by induction on the number of vertices, \( n = |V| \).

First notice that the hypothesis is true in the case \(|V| = |E| = 2\). By the proof of the Periodic Henneberg Theorem (Theorem 4.7), any periodic orbit graph \( \langle G, m \rangle \) with a constructive gain assignment with 2 vertices can be obtained as a 2-addition to a single vertex (which is minimally rigid on \( T_0^2 \)).

Now let \(|V| \) be at least 3, and assume the claim holds for all graphs \( G = (V, E) \) with \(|V| < n \). That is, for a graph \( G \) satisfying (i), we assume that for all constructive gain assignments \( m \) the framework \( \langle (G, m), p \rangle \) is generically minimally rigid on \( T_0^2 \).

Let \( G = (V, E) \) be a graph with \(|V| \geq 3\), and suppose \( m \) is any constructive gain assignment of the edges. By Proposition 5.7 we can always delete a 2- or 3-valent vertex in a way that leaves a graph \( \langle G', m' \rangle \) satisfying (i) and with \( m' \)
constructive. Then $|V'| = n - 1$, hence the inductive hypothesis applies, and $(G', m')$ is generically minimally rigid on $\mathcal{T}_0^2$.

To obtain the original orbit graph under consideration, $(G, m)$, we simply perform the appropriate periodic Henneberg move on the graph $(G', m')$ as follows:

1. If a 2-valent vertex was deleted, simply add back the same edges that were deleted.

2. If a 3-valent vertex was deleted, then by Lemma 4.6 we can edge split the added edge to obtain the orbit graph $(G, \overline{m})$, which is $T$-gain equivalent to $(G, m)$.

In either case, $(G, m)$ is generically minimally rigid on $\mathcal{T}_0^2$. In the second case, $(G, m)$ is minimally rigid because $(G, \overline{m})$ is minimally rigid.

Proposition 5.7 can be broken into the following two propositions, which deal with the two cases of deleting 2- and 3-valent vertices respectively. The proof of Proposition 5.8 is straightforward. The remainder of this section is devoted to the proof of Proposition 5.9.

**Proposition 5.8.** Let $(G, m)$ be an orbit graph on $\mathcal{T}_0^2$ satisfying

(i) $|E| - 2|V| - 2$, and every subgraph $G' \subset G$ satisfies $|E'| \leq 2|V'| - 2$.

(ii) $m$ is constructive

Then it is always possible to delete any 2-valent vertex $v_0$ such that the resulting graph $(G_0, m_0)$ also satisfies the properties (i) and (ii) above.

**Proof.** Deleting the 2-valent vertex $v_0$ leaves a graph $G'$ which is a subgraph of the original graph $G$ with $|E'| = 2|V'| - 2$. Since $m$ was constructive, this subgraph $(G', m')$ also has a constructive gain assignment.

**Proposition 5.9.** Let $(G, m)$ be a graph on $\mathcal{T}_0^2$ satisfying

(i) $|E| - 2|V| - 2$, and every subgraph $G' \subset G$ satisfies $|E'| \leq 2|V'| - 2$.

(ii) $m$ is constructive

Then it is always possible to perform a reverse edge split on any 3-valent vertex $v_0$ such that the resulting graph $(G_0, m_0)$ also satisfies the properties (i) – (ii) above.

**Proof.** We have two cases, either $v_0$ is adjacent to two distinct vertices (Case 1), or $v_0$ is adjacent to three distinct vertices (Case 2).

**Case 1.** Suppose $v_0$ is adjacent to the vertices $v_1$ and $v_2$, and that there are two copies of the edge connecting $v_0$ to $v_1$, with gain assignments $m_a$ and $m_b$. Let the gain assignment of the edge connecting $v_0$ and $v_2$ be $m_{02}$. Then the two candidates for edges to insert are $\{v_1, v_2; m_{02} - m_a\}$, or $\{v_1, v_2; m_{02} - m_b\}$.
Lemma 5.13 will prove that it is always possible to add one of these two candidate edges, while preserving properties (i) and (ii).

Case 2. Suppose \( v_0 \) is adjacent to vertices \( v_1, v_2, v_3 \). Suppose the edge connecting \( v_0 \) with \( v_i \) has gain assignment \( m_i \). Then the three candidates for reverse edge split are: \( \{v_1, v_2; m_{02} - m_{01}\}, \{v_2, v_3; m_{03} - m_{02}\}, \{v_3, v_1; m_{01} - m_{03}\} \).

Our goal is to prove that, in both cases, there is always at least one edge that can be added while maintaining properties (i) and (ii). In particular, we will consider subgraphs \( G_{ij} \subseteq G \) where \( v_i, v_j \in V_{ij} \), for \( i, j \in \{1, 2, 3\} \). Such a subgraph could prevent the addition of the edge \( e = \{v_i, v_j; m_{0j} - m_{0i}\} \) for one of two reasons. Either the resulting graph would be over-counted (that is, \( |E_{ij}| = 2|V_{ij}| - 2 \) already), or adding the candidate edge would create a subgraph of \( G \) that did not have a constructive gain assignment. Lemmas 5.10 \( \langle \text{Lattice Lemma} \rangle \) 5.17 and 5.18 will cover all of the possible cases, and demonstrate that it is always possible to add at least one of the candidate edges.

The remainder of this section builds up the necessary pieces for the proof of Proposition 5.9. Lemma 5.10 is a straightforward combinatorial result. We obtain several simple and useful corollaries (5.11 and 5.12). Finally, Lemmas 5.13 \( \langle \text{Lemma} \rangle \) 5.18 cover all of the cases in the proof of Proposition 5.9.

Lemma 5.10 (Lattice Lemma). Let \( (G, m) \) be a periodic orbit graph satisfying \( |E| = 2|V| - 2 \) and \( |E'| \leq 2|V'| - 2 \) for all subgraphs \( G' \subseteq G \). Let \( v_0 \) be some vertex of the graph. Let \( G \) be the set of all subgraphs \( G' \subseteq G \) that contain \( v_0 \) and satisfy \( |E'| = 2|V'| - 2 \). Then \( G \) is a lattice.

The proof of this result is straightforward, and it omitted.

Corollary 5.11. There is a smallest (largest) subgraph \( G' \subseteq G \) \( (G'' \subseteq G) \) containing \( v_0 \) with \( |E'| = 2|V'| - 2 \) \( (|E''| = 2|V''| - 2) \).

Corollary 5.12. Let \( (G, m) \) be a graph as in Lemma 5.10. The set \( G'' \) of all subgraphs \( G'' \subseteq G \) with \( |E''| = 2|V''| - 2 \) and containing a finite number \( \{v_1, \ldots, v_k\} \) of vertices of \( G \) is also a lattice.

The following lemma attends to the case where \( v_0 \) is adjacent to only two distinct vertices.

Lemma 5.13. Let \( (G, m) \) be a periodic orbit graph satisfying (i) and (ii) of Proposition 5.9 where \( v_0 \) is a 3-valent vertex adjacent to vertices \( v_1 \) and \( v_2 \) only. After deleting \( v_0 \) it is always possible to add one of the edges \( \{v_1, v_2; m_{02} - m_{01}\} \) or \( \{v_1, v_2; m_{03} - m_{01}\} \) so that the resulting graph also satisfies (i) and (ii) of Proposition 5.9.

Proof. First notice that we cannot have a subgraph \( G' \subseteq G \) satisfying \( |E'| = 2|V'| - 2 \), \( v_0 \notin V' \), and \( v_1, v_2 \in V' \), since this would mean that after adding \( v_0 \) and its three adjacent edges, the resulting graph would be overcounted. Therefore, any subgraph \( G' \) containing \( v_1 \) and \( v_2 \) but not \( v_0 \) must satisfy \( |E'| \leq 2|V'| - 3 \).

We now address the question of whether it is possible that after adding one of the candidate edges, a subgraph \( G'' \) is created with \( |E''| = 2|V''| - 2 \) but that has no constructive cycles.
Select one of the candidate edges to add, say \( e = \{v_1, v_2; m_{02} - m_{01}\} \). Let \( \langle G_a, m_a \rangle \subset \langle G, m \rangle \) be a subgraph satisfying the following:

1. \(|E_a| = 2|V_a| - 2\)
2. \(v_0 \notin V_a\)
3. \(v_1, v_2 \in V_a\) (and therefore \( e \in E_a\))
4. all directed paths connecting \( v_1 \) to \( v_2 \) have net gain \( m_{02} - m_{01} \).

Let \( \langle G_b, m_b \rangle \subset \langle G, m \rangle \) be another subgraph satisfying 1 - 3, and also

4’. all directed paths connecting \( v_1 \) to \( v_2 \) (with the exception of the edge \( e \)) have net gain \( m_{03} - m_{01} \).

Then \( \langle G_a, m_a \rangle \) appears to be a subgraph of \( G \) with no constructive cycles. However, since \( G_a \) and \( G_b \) both have \( 2|V| - 2 \) edges, the intersection of these graphs must also have \( 2|V| - 2 \) edges. Therefore, there is at least one other edge in \( E_a \cap E_b \) in addition to the edge \( e \). In particular, there is some path from \( v_1 \) to \( v_2 \) that is distinct from \( e \). Because this path is in \( G_a \), it must have net gain \( m_{02} - m_{01} \). But because the path is also in \( G_b \), it must also have net gain \( m_{03} - m_{01} \). This is only possible if \( m_{03} = m_{02} \), which contradicts the fact that \( m \) is constructive.

Therefore, subgraphs \( \langle G_a, m_a \rangle \) and \( \langle G_b, m_b \rangle \) cannot both exist, and hence it is always possible to add one of the candidate edges. \( \square \)

The remainder of this section will be devoted to proving Case 2 of Proposition 5.9. Let \( v_0 \) be a three-valent vertex adjacent to the edges \( \{v_0, v_1; m_{01}\}, \{v_0, v_2; m_{02}\} \) and \( \{v_0, v_3; m_{03}\} \).

Lemma 5.14. Let \( \langle G, m \rangle \) be a periodic orbit graph satisfying the hypotheses of Proposition 5.9. Let \( i, j, k \) be assigned distinct values from the set \( \{1, 2, 3\} \). Let \( G_{ij} \subset G \), be a subgraph satisfying:

(i) \(|E_{ij}| = 2|V_{ij}| - 3\)
(ii) \(v_i, v_j \in V_{ij}\), and \( v_k, v_0 \notin V_{ij}\)
(iii) \( G_{ij} \) contains no constructive cycles

Let \( G_{ik} \) be defined analogously. Then either

1. \(|V_{ij} \cap V_{ik}| = 1\) and \(|E_{ij} \cup E_{ik}| = 2|V_{ij} \cup V_{ik}| - 4\) or
2. \(|V_{ij} \cap V_{ik}| > 1\) and \(|E_{ij} \cup E_{ik}| = 2|V_{ij} \cup V_{ik}| - 3\) and \(|E_{ij} \cap E_{ik}| = 2|V_{ij} \cap V_{ik}| - 3\)
Proof.

\[
|E_{ij} \cup E_{ik}| + |E_{ij} \cap E_{ik}| = |E_{ij}| + |E_{ik}|
\]
\[
= 2(|V_{ij}| + |V_{ik}|) - 6
\]
\[
= (2|V_{ij} \cup V_{ik}| - 3) + (2|V_{ij} \cap V_{ik}| - 3). \tag{3}
\]

We now have two cases. Either

**Case 1.** \(|V_{ij} \cap V_{ik}| = 1\). In this case \(|E_{ij} \cap E_{ik}| = 0\), since the intersection graph is a subgraph. From (3) we get

\[
|E_{ij} \cap E_{ik}| = 2|V_{ij} \cap V_{ik}| - 4.
\]

**Case 2.** \(|V_{ij} \cap V_{ik}| > 1\). First note that we must have

\[
|E_{ij} \cup E_{ik}| \leq 2|V_{ij} \cup V_{ik}| - 3 \tag{4}
\]

because this is a subgraph containing all three vertices \(v_1, v_2, v_3\) but not \(v_0\). We must also have

\[
|E_{ij} \cap E_{ik}| \leq 2|V_{ij} \cap V_{ik}| - 3 \tag{5}
\]

because this is a subgraph of both \(G_{ij}\) and \(G_{ik}\), neither of which possess a constructive cycle. Rewriting (3) we obtain:

\[
|E_{ij} \cup E_{ik}| - (2|V_{ij} \cup V_{ik}| - 3) = \{(2|V_{ij} \cap V_{ik}| - 3) - |E_{ij} \cap E_{ik}|\}. \tag{6}
\]

By (4) we obtain

\[
\{(2|V_{ij} \cap V_{ik}| - 3) - |E_{ij} \cap E_{ik}|\} \leq 0.
\]

It follows that in fact \(2|V_{ij} \cap V_{ik}| - 3 \leq |E_{ij} \cap E_{ik}|\), which, together with (5) shows that we have equality in this case. The relationship \(|E_{ij} \cup E_{ik}| = 2|V_{ij} \cup V_{ik}| - 3\) then follows from (3).

**Corollary 5.15.** The graph \((V_{ij} \cap V_{ik}, E_{ij} \cap E_{ik})\) is connected.

The following lemma shows that there is at most one choice of edges that will create a subgraph that fails combinatorially (that is, adding an edge would create a subgraph with \(|E'| > 2|V'| - 2\)). This result follows from Fekete and Szegő [6].

**Lemma 5.16.** Let \(\langle G, m \rangle\) be a periodic orbit graph satisfying the hypotheses of Proposition [5.3]. Then \(G\) has at most one subgraph \(G' \subset G\) that satisfies:

(i) \(|E'| = 2|V'| - 2\),
(ii) \(v_0 \notin V'\), and
(iii) \(V'\) contains at least two vertices from the set \(\{v_1, v_2, v_3\}\).
The following lemma states that if there is one choice of edge whose addition would cause a combinatorial failure, then there is at most one choice of edge that would cause a failure to have a constructive gain assignment.

**Lemma 5.17.** Let \( \langle G, m \rangle \) be a periodic orbit graph satisfying the hypotheses of Proposition 5.9, and that contains a subgraph \( G' \subset G \) that satisfies (a) – (c) of the previous lemma. Then there is at most one pair \( v_i, v_j \) of vertices from the set \( \{v_1, v_2, v_3\} \) and distinct from the pair contained in \( G' \) that are contained in a subgraph \( G'' \subset G \) satisfying:

1. \( v_0 \notin V'' \),
2. \(|E''| = 2|V''| - 3\), and
3. \( G'' \) contains no cycle with non-trivial net gain.

**Proof.** Let \( G_{i,j} \) be the graph with \( v_i, v_j \in V_{i,j} \), but \( v_k \notin V_{i,j} \); \( i, j, k \in \{1, 2, 3\} \). Suppose, toward a contradiction, that there are two subgraphs satisfying (i) – (iii). Without loss of generality suppose that \( G_{23} \) and \( G_{31} \) are these subgraphs, and that \( G_{12} \) satisfies \( |E_{12}| = 2|V_{12}| - 2 \).

First consider the intersection of \( G_{12} \) with some subgraph \( G^* \) satisfying (i) – (iii). In this case,

\[
|E^* \cup E_{12}| + |E^* \cap E_{12}| = |E^*| + |E_{12}|
= (2|V^*| - 3) + (2|V_{12}| - 2)
= 2(|V^*| + |V_{12}|) - 5
= 2|V^* \cup V_{12}| - 2 + 2|V^* \cap V_{12}| - 3. \tag{7}
\]

We now have two cases:

- **Case A:** \( |V^* \cap V_{12}| > 1 \).

Then, because \( G^* \) satisfies property (iii), we have \( |E^* \cap E_{12}| \leq 2|V^* \cap V_{12}| - 3 \), and hence (7) becomes \(|E^* \cup E_{12}| \geq 2|V^* \cup V_{12}| - 2 \). In fact, since the reverse inequality always holds, we have equality \(|E^* \cup E_{12}| = 2|V^* \cup V_{12}| - 2 \).

- **Case B:** \( |V^* \cap V_{12}| = 1 \).

Then \(|E^* \cap E_{12}| = 0 \) and hence \(|E^* \cup E_{12}| = 2|V^* \cup V_{12}| - 3 \).

Let \( \overline{G} \subset G \) be the subgraph of \( G \) on the vertices \( V_{23} \cup V_{31} \). Then \(|\overline{E}| \geq |E_{23} \cup E_{31}| \), since there could be edges in \( \overline{E} \) that were not part of either \( E_{23} \) or \( E_{31} \), but that connect vertices in \( V_{23} \cup V_{31} \). By Lemma 5.14 we know that either

- **Case 1.** \(|V_{23} \cap V_{31}| = 1 \) and \(|E_{23} \cup E_{31}| = 2|V_{23} \cup V_{31}| - 4 \) or
- **Case 2.** \(|V_{23} \cap V_{31}| > 1 \) and \(|E_{23} \cup E_{31}| = 2|V_{23} \cup V_{31}| - 3 \).

We will deal with **Case 2** first. In this case \(|\overline{E}| = |E_{23} \cup E_{31}| = 2|V_{23} \cup V_{31}| - 3 \). In other words, there can’t be any edges in \( \overline{E} \) that aren’t also in \( E_{23} \cup E_{31} \), otherwise the graph \((\overline{V}, \overline{E})\) would be over-counted. We now consider the intersection graph \( \overline{G} \cap G_{12} = (\overline{V} \cap V_{12}, \overline{E} \cap E_{12}) \). Note that \(|\overline{V} \cap V_{12}| > 1 \), since \( v_1 \) and \( v_2 \) are in both vertex sets. Hence by **Case A** above, \(|\overline{E} \cup E_{12}| = 2|\overline{V} \cup V_{12}| - 2 \). But this
is a contradiction, because this graph contains all three of the vertices $v_1, v_2, v_3$, which means that adding $v_0$ will produce an over-counted subgraph.

We now return to Case 1. Here $|E| \geq |E_{23} \cup E_{31}| = 2|V_{23} \cup V_{31}| - 4$. If $|E| > |E_{23} \cup E_{31}|$, then we are in the situation of Case 2. Hence we may assume that $|E| = |E_{23} \cup E_{31}|$.

Notice that the intersection of $V_{12}$ with either of the other graphs $V_{23}$ or $V_{31}$ may only consist of one element, otherwise we have, by Case A, an over-counted subgraph. So the three subgraphs must intersect pair-wise in one of the vertices $v_1, v_2, v_3$, and it follows that the intersection of all three of these subgraphs is empty.

$$|E_12| + |E \cap E_{12}| = |E| + |E_{12}|$$
$$= (2|V| - 4) + (2|V_{12}| - 2)$$
$$= 2(|V| + |V_{12}|) - 6$$
$$= (2|V \cup V_{12}| - 3) + (2|V \cap V_{12}| - 3). \quad (8)$$

But we know that $|V \cap V_{12}| = 2$, and it must be the case that $|E \cap E_{12}| = 0$, since the intersection of the three graphs is empty. Hence equation (8) becomes $|E_12| = 2|V \cup V_{12}| - 2$ which is a contradiction, since $v_1, v_2, v_3 \in V \cup V_{12}$. Adding $v_0$ would violate the subgraph property of $G$.

This final lemma shows that if there are no bad choices of edges for combinatorial reasons, there is always at least one choice of edge that will produce a constructive gain assignment.

**Lemma 5.18.** Let $(G, m)$ be a periodic orbit graph satisfying the hypotheses of Proposition 5.7. Then there are at most two distinct pairs of vertices from the set $\{v_1, v_2, v_3\}$ that are contained in subgraphs $G' \subset G$ satisfying the following:

(i) $v_0 \notin V'$

(ii) $|E'| = 2|V'| - 3$

(iii) $G'$ contains no cycle with non-trivial net gain

(iv) every path through $G'$ connecting $v_i$ with $v_j$ has net gain $m_{0j} - m_{0i}$.

In other words, there are at most two minimal subgraphs each containing a distinct pair of vertices from $v_1, v_2, v_3$, and having properties (i) - (iv).

**Proof.** Toward a contradiction, suppose that there are three such graphs $G_{12}, G_{23}, G_{31}$, with $v_i, v_j \in V_{ij}$. It will be presently be shown that the union of these graphs, $G' \subset G$ will always satisfy:

(a) $|E'| = 2|V'| - 3$ and

(b) $G'$ contains no cycle with non-trivial net gain

(c) every path through $G'$ connecting $v_i$ with $v_j$ has net gain $m_{ij} - m_i$. 

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We do this in two cases:

**Case 1.** \(V_{ij} \cap V_{jk} = \{v_j\} \text{ for } j \in \{1, 2, 3\}\)

In other words, each pair of subgraphs intersects in a single vertex. Here

\[
|E_{12} \cup E_{23} \cup E_{31}| = |E_{12}| + |E_{23}| + |E_{31}|
= 2(|V_{12}| + |V_{23}| + |V_{31}|) - 9
= 2|V_{12} \cup V_{23} \cup V_{31}| - 3
\]

since

\[
|V_{12} \cup V_{23} \cup V_{31}| = |V_{12}| + |V_{23}| + |V_{31}| - |V_{12} \cap V_{23}| - |V_{23} \cap V_{31}|
= |V_{12}| + |V_{23}| + |V_{31}| - 3.
\]

It is evident in this case that \(G'\) contains no non-trivial cycle, since any such cycle would pass through \(v_1, v_2\) and \(v_3\). This would contradict property (iv). It is evident that \(G'\) satisfies (c) in this case.

**Case 2.** \(|V_{ij} \cap V_{jk}| > 1\text{ for at least one } j \in \{1, 2, 3\}\).

By a repeated application of Lemma 5.14 we find that the union of these three graphs satisfies \(|E'| = 2|V'| - 3\). (Let \(G' = G_{12} \cup G_{23}\). Assuming that \(|V_{12} \cap V_{23}| > 1\), apply Lemma 5.14 to see that \(|E'| = 2|V'| - 3\). Now it must be the case that \(|V' \cap V_{31}| > 1\) as well, since \(v_1, v_3\) are in both vertex sets. Another application of Lemma 5.14 gives the result.) Note further that the intersection of \(G^*\) and \(G_{31}\) contains at least two vertices \((v_1\text{ and } v_3)\), and satisfies \(|E^* \cap E_{31}| = 2|V^* \cap V_{31}|\) by Lemma 5.14. Furthermore, this intersection is non-empty. Equivalently, the intersection \(V_{12} \cap V_{23} \cap V_{31}\) is non-empty.

We now demonstrate that \(G'\) contains no non-trivial net gain. We assume that there is a non-trivial net gain in \(G'\), and we will obtain a contradiction to condition (iii). We do this in two parts, first by showing that there are no constructive cycles in the union of any pair of subgraphs (a), and next showing that there are no non-trivial cycles in the union of all three (b).

**Case 2a.** Suppose without loss of generality, that there is a non-trivial cycle in the graph \((V_{12} \cup V_{23}, E_{12} \cup E_{23})\). Suppose that \(|V_{12} \cap V_{23}| > 1\), and that the non-trivial cycle passes through vertices \(x\) and \(y\), where \(x, y \in V_{12} \cap V_{23}\). See Figure 12

Let the non-trivial cycle through \(x\) and \(y\) be broken into two parts \(xAyByx\), where \(A\) and \(B\) are paths through \(G_{12}\) and \(G_{23}\) respectively that make up the non-trivial cycle. Let \(m_A\) and \(m_B\) be the net gain of paths \(A\) and \(B\) respectively. Then \(m_A + m_B \neq 0\) by assumption. By Corollary 5.15 the graph \((V_{12} \cap V_{23}, E_{12} \cap E_{23})\) is connected. Hence there exists a path through this graph that connects \(x\) with \(y\). Let the net gain of this path be \(m_C\). Then

\[
m_A - m_C = 0
\]

\[
\Rightarrow m_A = m_C
\]

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Figure 12: Two subgraphs satisfying (i) – (iv) of Lemma 5.18 whose intersection contains more than one vertex.

Figure 13: Three subgraphs satisfying (i) – (iv) of Lemma 5.18 that intersect in a vertex $x$. 
⇒ \( m_C + m_B \neq 0 \).

But this is a non-trivial net gain in \( G_{23} \), a contradiction.

**Case 2b.** Now assume that there is a non-trivial cycle in the subgraph of \( G \) on the vertices \( V_{12} \cup V_{23} \cup V_{31} \). See Figure 13. By a similar argument to the previous case, suppose that the non-trivial cycle is written as the sum of three paths, one through each of the graphs. That is, let \( x_1 \in V_{31} \cap V_{12}, x_2 \in V_{12} \cap V_{23}, \) and \( x_3 \in V_{23} \cap V_{31} \). If any of the vertices \( x_1, x_2, x_3 \) is in the intersection of all three graphs, then we are in the situation described above. So we assume that this is not the case. Let the nontrivial cycle be written \( x_1 Ax_2 Bx_3 Cx_1 \), where \( A \in G_{12}, B \in G_{23}, C \in G_{31} \). Let these paths have cycle gains \( m_A, m_B, m_C \) respectively, and our assumption is that \( m_A + m_B + m_C \neq 0 \).

Let \( x \in V_{12} \cap V_{23} \cap V_{31} \). Each intersection \( V_{ij} \cap V_{jk} \) is connected, hence for the vertex \( x_i \in V_{ki} \cap V_{ij} \) there is a path connecting \( x \) to \( x_i \). Let this path have net gain \( m_i \). Similarly we have paths connecting vertices \( x_j \) and \( x_k \) respectively to the vertex \( x \). Then

\[
\begin{align*}
    m_A - m_{01} + m_{03} &= 0 \\
    m_B - m_{02} + m_{01} &= 0 \\
    m_C - m_{03} + m_{02} &= 0.
\end{align*}
\]

But summing these three expressions gives \( m_A + m_B + m_C = 0 \), which contradicts our assumption. As in the previous case, the union of the three graphs can have no non-trivial cycle.

To see that \( G' \) also satisfies property (c), we consider without loss of generality, all paths \( P \) from \( v_1 \) to \( v_2 \) through \( G' \). If each vertex of the path is in \( V_{12} \) then it has net gain \( m_{02} - m_{01} \) by hypothesis. If some vertex in \( P \) is not in \( V_{12} \), then suppose \( P \) has net gain \( m_P \). Then \( m_P - (m_{02} - m_{01}) = 0 \), since \( G' \) has no trivial cycles, by (b). Hence \( m_P = m_{02} - m_{01} \), as desired.

In both Case 1 and Case 2, we have a subgraph \( G' \subset G \) that contains \( v_1, v_2, v_3 \) but not \( v_0 \), and satisfies \( |E'| = 2|V'| - 3 \). Furthermore, this graph contains no cycle with non-trivial net gain, and all paths connecting \( v_i \) to \( v_j \) have net gain \( m_{0j} - m_{0i} \). Let \( V_0 = V' \cup \{v_0\} \), and consider the graph \( G_0 = (V_0, E_0) \). \( E_0 \) will be \( E' \) augmented by the three edges connecting \( v_0 \) with \( v_1, v_2, v_3 \). Then \( |E_0| = 2|V_0| - 2 \), and hence this graph must be constructive. But we know that \( G' \) contains no cycle with non-trivial net gain, which means that the non-trivial net gain in \( G_0 \) must pass through \( v_0 \). Hence it must contain two of the edges adjacent to \( v_0 \). But any such cycle will have net gain zero, a contradiction.

The proofs of these technical results prove Theorem 5.6, which in turn establishes the Periodic Laman Theorem (Theorem 5.4) and completes the proof of the summary result, Theorem 3.1.
6 Further work

6.1 Algorithms

An algorithm for determining the rigidity of a periodic orbit framework on $T^2_0$ appears in [21]. It was also recently independently considered by several authors [1]. Both presentations are based on the pebble game algorithm for finite frameworks, with the key idea being to run the $(2,3)$- and $(2,2)$-pebble games simultaneously.

6.2 Higher dimensions

In [22] we presented necessary conditions for rigidity on the $d$-dimensional fixed torus $T^d_0$. Unfortunately, finding sufficient conditions for generic rigidity on the $d$-dimensional fixed torus rests on solving finite $d$-dimensional rigidity (finite rigidity is combinatorially characterized for $d = 1, 2$ but not for higher dimensions). For example, it is possible to embed the well-known “double bananas” example in a three-dimensional periodic framework. See [22] for further details.

6.3 Body-bar frameworks on the fixed torus

In contrast to the situation for bar-joint frameworks, there may be more hope for finding characterizations of body-bar frameworks in higher dimensions. In fact, we recently established a combinatorial characterization of body-bar frameworks on the fixed torus in three dimensions ($d = 1, 2$ follow from the bar-joint characterizations), and we conjecture that a $d$-dimensional version is possible [23]. The conjecture is based on the following sparsity condition which depends on the dimension of the gain space $G_C$:

**Conjecture 6.1.** $(H, m)$ is a periodic orbit graph corresponding to a generically minimally rigid body-bar periodic framework in $\mathbb{R}^d$ if and only if $|E(H)| = \binom{d+1}{2}|V(H)| - d$ and for all non-empty subsets $Y \subseteq E(H)$ of edges

$$|Y| \leq \left(\binom{d+1}{2}|V(Y)| - \binom{d+1}{2}\right) + \sum_{i=1}^{\frac{|G_C(Y)|}{2}} (d - i).$$

The conjecture has been verified for $d = 1, 2, 3$. Periodic body-bar frameworks have also been studied on the flexible torus [4].

6.4 Inductive constructions on the flexible torus

In [21], a characterization was established of the generic rigidity of periodic frameworks on a partially variable torus (allowing one degree of flexibility). Together with Anthony Nixon, we have recently outlined an inductive proof of this result [17].
Theorem 6.2 (Nixon and Ross [17]). A framework $(\langle G, m \rangle, p)$ is generically minimally rigid on the partially variable torus (with one degree of freedom) if and only if it can be constructed from a single loop by a sequence of gain-preserving Henneberg operations.

The operations referred to in the theorem above contain the periodic vertex addition and edge split operations described in this paper. We also require an additional move to deal with a special class of graphs for which the existing moves are insufficient. There is yet more subtlety. Any graph which is generically minimally rigid on the partially variable torus (one degree of freedom) can be shown to be the disjoint union of a spanning tree and a connected spanning map graph (a graph with exactly one cycle per connected component). In this way, the class of graphs we wish to inductively generate is strictly smaller than the set of graphs $G$ satisfying $|E| = 2|V| - 1$, and $|E'| \leq 2|V'| - 1$.

Further challenges await when attempting to apply inductive techniques to the fully variable torus (three degrees of freedom), or even the partially variable torus with two degrees of freedom. In particular, there are examples of graphs on such tori where all vertices are at least four-valent. This necessitates the use of further Henneberg moves, such as $X$-replacement or $V$-replacement. However, it is not obvious that we can define these moves to preserve the gains on cycles as we have in the present work. Of course, a complete combinatorial description of frameworks on the fully variable torus exists [13], but we believe that an inductive characterization would be a rich addition to our knowledge of these structures.

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