Analytic solutions for the energy eigenvalues are obtained from a confined potentials of the form $br$ in 3 dimensions. The confinement is effected by linear term which is a very important part in Cornell potential. The analytic eigenvalues and numerical solutions are exactly matched.

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I. INTRODUCTION AND THE FORMULA

This paper is concerned with the energy of system with the confined potential (linear) and obeying nonrelativistic quantum mechanics. The main result is that we obtain an exact formula for the energy eigenvalues. The confined potential (linear) is of interest in many quantum mechanics system, such as the Stark effect of atom. Furthermore, The Cornell potential (Coulomb-plus-linear potential) is very success in describing the spectrum of the quarkonium [1–4]. The eigenvalue of Cornell potential was studied for many years [5–17], and yet there is no exact result for the eigenvalues. References [5, 12, 16–18] gave some approach formulas. In this paper, we obtain an exact formula for eigenvalue of the linear part of Cornell potential which can describe the high excited states for the quarkonium [19, 20].

We consider a Schrödinger equation of the form ($-\frac{1}{2}\mu \Delta + V(r))\psi = E\psi$, where $V(r)$ is an attractive central potential with confining terms read:

$$ V(r) = br - \frac{\alpha}{r} - C. \quad (1.1) $$

If we now choose

$$ \psi_{nlm}(r) = \frac{R_{nl}(r)}{r} Y_{lm}(\theta, \phi), \quad (1.2) $$

the Schrödinger equation will have the form:

$$ \left[-\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} + 2\mu \left(br - \frac{\alpha}{r} - C\right)\right] R_{nl}(r) = 2\mu E R_{nl}(r). \quad (1.3) $$

When $\alpha = 0$ and $l(l+1) = 0$, it become an Airy equation and has the solution $N_a \text{Airy}((2\mu b)^{1/3}r + \text{AiryZero}(n+1))$, the eigenvalue will be $\lambda_{nl0} = -(2\mu b)^{2/3}\text{AiryZero}(n+1)$, where $N_a$ is the normalized factor and \text{AiryZero}(n) is the $n$-th zero point of Airy function.

Used the transformation of

$$ r = \xi/\sigma, a = \frac{2\mu \alpha}{\sigma}, \lambda_{nl} = \frac{2\mu(\xi + C)}{\sigma^2}, \sigma = (2\mu b)^{1/3}, \quad (1.4) $$

the Schrödinger equation in Eq. 1.3 reads

$$ \left[-\frac{d^2}{d\xi^2} + \frac{l(l+1)}{\xi^2} + \frac{2\mu \xi}{\xi} \right] R_{nl}(\xi) = \lambda_{nl} R_{nl}(\xi). \quad (1.5) $$
We can easily give the series solution of the equation 1.5

$$R_{al}(\xi) = \sum_{i} c_i \xi^{i+l+1}$$  \hspace{1cm} (1.6)

and the coefficient $c_i$ reads:

$$c_i = (-1)^i \frac{\text{Det} A}{\text{Det} B} c_{0i}, (i > 0),$$ \hspace{1cm} (1.7)

$$A_{pq} = \delta_{p+1}^q p(p + 1 + 2l) + \delta_p^q a + \delta_{p-1}^q \lambda_{nl} - \delta_{p-2}^q, B_{pq} = \delta_{p+1}^q p(p + 1 + 2l) + \delta_p^q a + \delta_{p-1}^q \lambda_{nl} - \delta_{p-3}^q. \hspace{1cm} (1.8)$$

And we can obtain that

$$\text{Det} A = NF[a + \frac{i}{i+2} a_s], \hspace{1cm} (1.9)$$

where NF representivs take numerator and

$$a_i = -\lambda(i - 1)(i + 2l), \hspace{1cm} (1.10)$$

$$a_{i-1} = -(i - 2)(2l + i - 1) \left( \frac{(i-1)(2l+i)}{a} + \lambda_{nl} \right), \hspace{1cm} (1.11)$$

$$a_{i-k} = -(i-k-1)(i-k+2l) \left( \frac{(i-k)(i-k+2l+1)}{K_{s=i-k+2} a + \lambda_{nl}} \right), \hspace{1cm} (1.12)$$

$$K_{s=i+1} a_i = \frac{a_1}{a + \frac{i}{i+1} a_s}. \hspace{1cm} (1.13)$$

and

$$\text{Det} B = i!\Pi_{j=1}^{l+1} (2l + j + 1), \hspace{1cm} (1.14)$$

where $\text{Det}(A)$ and $\text{Det}(B)$ are denoted by the determinants of matrixes $A$ and $B$ respectively. When we select a eigenvalue of Eq. 1.5, the series 1.6 is convergent.

We give an ansatz that the eigenvalue of Eq. 1.5, has the following form with $a = 0, \lambda_{nl} = -\frac{\delta_1 l + \frac{2\pi}{\sqrt{3}} \delta_2 n + 1}{\frac{2\pi}{\sqrt{3}} (\delta_1 l + \delta_2 n) + 1} \text{AiryZero}(l + n + 1). \hspace{1cm} (1.15)$

And we find that when $\delta_1 = 0.797533, \delta_2 = 0.797804, \delta_1 \approx \delta_2 \approx \sqrt{2/5} \approx 0.798 \approx 0.8 \approx \delta$, which is solved by the numerical eigenvalues of $\lambda_{11}, \lambda_{01} = 3.361254522, \lambda_{02} = 4.88445124014$ with $a = 0$ and $b = 1$. This eigenvalues formula and numerical solutions are exactly matched as illustrated in Tabs. I and II. For the $\lambda_{0s}$, the analytic eigenvalues and numerical solutions are exactly matched, and for the radial excitation, the relative error between analytic eigenvalues and numerical results below 0.0004. As shown in Tabs. I and II, the results of WKB approach is only well described the low $l$ excitation and our formula is consistent with the numerical eigenvalues of linear potential for all $n$ and $l$. We also calculate the $\lambda_{100}$ and $\lambda_{0100}$ numerically and by using the formula 1.15, and obtain 41.08626 and 41.08631, 80.6248 and 80.650, respectively.
II. APPLICATION

A. Regge trajectory and the WKB approach of linear potential

Using this eigenvalue formula in Eq. 1.6, one can obtain the eigenfunctions. In the other hand, this is the important step to reach the exact solution to Eq. 1.5. We can expand the Eq. 1.15 when \( n + l + 1 \to \infty \) with

\[
\lambda_{nl} = \left( \frac{12\pi}{4} \right)^{2/3} \left( l + n + \frac{1}{4} \right)^{2/3} \left( \delta \left( l + \frac{1}{2} \right) \left( l + n \right) + 1 \right)
\]

(2.1)

In fact, Eq. 2.1 has the high precision for all \( n \) and \( l \).

We can expand the Eq. 1.15 when \( n \gg l \) with

\[
\lambda_{n0}^3 \approx \left( \frac{3\pi}{2} \right)^2 \frac{n^2}{l},
\]

(2.2)
or

\[ \lambda_{00}^3 \approx \left( \frac{3\pi}{2} (n+1) \right)^2, \quad (2.3) \]

which indicate that when \( n \) or \( l \) is bigger enough, \( n+1 \) can be regard as the principle quantum number consist with the guess of Ref. [21, 22], and the similar formula was obtained in Ref. [23] from the relativistic flux tube model. When \( l \gg n \), the approximate will be

\[ \lambda_{00}^3 \approx \left( \frac{3/2}{2} l \right)^2. \quad (2.4) \]

or

\[ \lambda_{00}^3 \approx \left( \frac{3/2}{2} (l+n) \right)^2. \quad (2.5) \]

Eqs 2.2 and 2.4 are the famous results for the linear potential [24].

### B. The eigenvalue of the conell potential

In this section, we will give an explicit form for the Schrödinger equation 1.5. The eigenvalue of the Coulomb potential reads

\[ \lambda(a,n) = -\frac{a^2}{4(n+l+1)^2}. \quad (2.6) \]

So, we ansatz the eigenvalue of the conell potential has the form:

\[ \lambda_{nl} = -\frac{a^2}{4(n+l+1)^2} - \frac{\delta_1 l + \frac{\delta_2 n + 1}{\sqrt{3}}}{\frac{\delta_1 l + \delta_2 n + 1}{\sqrt{3}}} \text{AiryZero}(l + n + 1) f(a,n,l) + g(a), \quad (2.7) \]

where \( f(0,n,l) = 1, g(0) = 0 \). Then we try the following form:

\[ f(a,n,l) = 1 - \frac{(a_1 + a_2(a_{n+l})^{3/2} + a_3(a_{n+l})^3)}{a_{l}(a_{n+l})^{3/2} + a_3(a_{n+l})^3} \frac{b_1 + b_2 + b_3(a_{n+l})^{3/2} + b_4(a_{n+l})^3}{b_{n+l}(a_{n+l})^{3/2} + b_4(a_{n+l})^3} a_{2/5} \frac{a^{2/5}}{a_{n+l}^{3/5} + a_{n+l}^{3/5}} \frac{\text{AiryZero}(l + n + 1)}{\text{AiryZero}(l + n + 1)} \quad \text{(2.8)} \]

with

\[ \begin{align*}
    a_{l1} &= -0.0131096, a_{l2} = -0.0526298, a_{l3} = 0.000656495, a_{c} = 5.04779, a_{l} = 1.59813, \\
    b_{l1} &= 0.214579, b_{l2} = -0.0439848, b_{l3} = 0.00362465, b_{l4} = -0.22573, b_{l5} = 0.839639, \\
    c_{l1} &= -0.689677, c_{l2} = -1.911553, c_{l3} = -0.274089, c_{l4} = 4.19812, c_{l5} = 0.75866, \\
    d_{l1} &= 0.365051, d_{l2} = 0.148248, d_{l3} = 0.0362142, d_{l4} = 1.43633, a_{l5} = 0.621341, \\
    a_{m1} &= 0.254494, a_{m2} = 0.520001, b_{m1} = 1.29783, b_{m2} = 0.77, \\
    c_{m1} &= 0.4000669, c_{m2} = 1.61998, d_{m1} = 0.433161, d_{m2} = 1.92501, \\
    e_{m1} &= 0.4, e_{m2} = 0.520996, f_{m1} = 0.00685891, f_{m2} = 8.47589. 
\end{align*} \quad (2.9) \]

By using Eq. 2.7, we can obtain the eigenvalues of Cornell potential with the relative error is below 0.03 in the ranges of \( 0 \leq a \leq 5 \) and \( n, l \leq 10 \), below 0.001 for \( n=0 \).

In Tab. III, we use 2.7 to calculate the spectrum of Bottomonium(\( \Upsilon(nS) \)), and find that it is consistent with Ref. [15], numerical result and the experimental value [26] in the case of \( a=2.67 \).
TABLE III: Bottomonium($\Upsilon(nS_1)$) mass spectrum using formula 2.7. The parameters of Cornell potential are $b=0.18$ GeV$^2$, $C=0.29$ GeV, $\alpha=0.52$, and $\mu=\frac{4\pi}{\alpha}$. The mass has the unit of GeV.

| State | This work | Ref. [25] | Numerical | Experiment [26] |
|-------|------------|-----------|------------|-----------------|
| $0^3S_1$ | 9.4225 | 9.447 | 9.4222 | 9.4603 |
| $1^3S_1$ | 10.013 | 10.012 | 10.055 | 10.023 |
| $2^3S_1$ | 10.360 | 10.353 | 10.350 | 10.355 |
| $3^3S_1$ | 10.641 | 10.629 | 10.628 | 10.579 |
| $4^3S_1$ | 10.889 | – | 10.871 | 10.882 |
| $5^3S_1$ | 11.114 | – | 11.092 | 11.003 |

C. Conclusion

In this work, we obtain an analytic formula for the energy eigenvalues from a confined potentials of the form $br$ in 3 dimensions which is widely used to solve the spectrum of the heavy meson family. The analytic eigenvalues and numerical solutions are exact matched which is very close to the exact energy eigenvalues. This can encourage us to find the exact solution for the Schrödinger equation with confined potentials.

The analytic formula for the energy eigenvalues of linear potential can reproduce the Regge trajectory of its higher n-excitation and l-excitation.

By using the analytic formula for the energy eigenvalues of linear potential, we give an fit for the Cornell potential which is very well to describe its energy eigenvalues. Tested by the Bottomonium($\Upsilon(nS_1)$) mass spectrum, we find that it is very useful to analyse the spectrum problem.

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