Regularity of Solutions to Kolmogorov Equations with Perturbed Drifts

Vladimir I. Bogachev\textsuperscript{1,2} \v•\ Egor D. Kosov\textsuperscript{1,2} \v•\ Alexander V. Shaposhnikov\textsuperscript{3}

Received: 6 April 2021 / Accepted: 8 September 2021 / Published online: 21 September 2021 © The Author(s), under exclusive licence to Springer Nature B.V. 2021

Abstract

We prove that a probability solution of the stationary Kolmogorov equation generated by a first order perturbation $\nu$ of the Ornstein–Uhlenbeck operator $L$ possesses a highly integrable density with respect to the Gaussian measure satisfying the non-perturbed equation provided that $\nu$ is sufficiently integrable. More generally, a similar estimate is proved for solutions to inequalities connected with Markov semigroup generators under the curvature condition $CD(\theta, \infty)$. For perturbations from $L^p$ an analog of the Log-Sobolev inequality is obtained. It is also proved in the Gaussian case that the gradient of the density is integrable to all powers. We obtain dimension-free bounds on the density and its gradient, which also covers the infinite-dimensional case.

Keywords Kolmogorov equation · Ornstein–Uhlenbeck operator · Curvature condition · Perturbation of drift · Integrability of density

Mathematics Subject Classification (2010) 35J15 · 46G12 · 60J60

1 Introduction

There is a vast literature on perturbations of Ornstein–Uhlenbeck operators and their generalizations by vector fields $\nu$ that can be regarded as small as compared to the main first
order term $-x$. Many authors studied Dirichlet forms obtained by such perturbations, diffusion processes with perturbed drifts, and the corresponding stationary distributions or solutions to the stationary Fokker–Planck–Kolmogorov equation, both in finite and infinite dimensions. In particular, Shigekawa [28] proved his celebrated result that, given a centered Gaussian measure $\gamma$ with the Cameron–Martin space $H$ on a separable Banach space $X$, for every bounded Borel $H$-valued vector field $v$ there is a Borel probability measure $\mu$ on $X$ absolutely continuous with respect to $\gamma$ and satisfying the perturbed Fokker–Planck–Kolmogorov equation

$$L^*\mu = 0,$$

where $L$ is the Ornstein–Uhlenbeck operator and

$$L\varphi = L\varphi + (v, \nabla \varphi)_H.$$ 

Moreover, $\mu = f \cdot \gamma$ with $f$ belonging to the Sobolev space $W^{2,1}(\gamma)$ over $\gamma$. In the finite-dimensional case

$$L\varphi(x) = \Delta \varphi(x) - \langle x, \nabla \varphi(x) \rangle + \langle v(x), \nabla \varphi(x) \rangle.$$ 

The uniqueness of invariant measures was established in [28] in the case of measures absolutely continuous with respect to $\gamma$. It was later shown in [11] (see also [13]) that all probability solutions to this equation are absolutely continuous with respect to $\gamma$ provided that $|v|_H \in L^2(\mu)$. Moreover, it is also true that for $f := d\mu/d\gamma$ one has $\sqrt{f} \in W^{2,1}$ (but not always $f \in W^{2,1}$) and

$$\int \frac{|\nabla f|_H^2}{f^2} \, d\mu \leq \int |v|_H^2 \, d\mu.$$ 

In particular, the uniqueness result holds in the class of all probability solutions if $v$ is bounded. For unbounded $v$ the uniqueness assertion can fail even in the one-dimensional case. The existence and uniqueness statements were reinforced by Hino [21] by replacing the boundedness with the condition $\exp(\theta |v|_H^2) \in L^1(\gamma)$, $\theta > 2$. In this case, one also has the inclusions $f \in W^{2,1}(\gamma)$ and $f \in L^p(\gamma)$ for all $p \in [1, (\theta/2)^{1/2})$. Further extensions to measure metric spaces have been obtained by Suzuki [31].

For a general Fokker–Planck–Kolmogorov equation

$$\Delta \mu - \text{div}(b\mu) = 0$$

with respect to a Borel probability measure $\mu$ on $\mathbb{R}^d$ with a Borel vector field $b$ such that $|b|$ is locally integrable with respect to $\mu$, which is understood as the identity

$$\int \left[ \Delta \varphi + \langle b, \nabla \varphi \rangle \right] \, d\mu = 0 \quad \forall \varphi \in C_0^{\infty},$$

i.e., the equality is interpreted in the sense of distributions, the following is known (see [6, 11], and surveys in [8, 9]):

- the measure $\mu$ is always absolutely continuous with respect to Lebesgue measure, i.e., is given by a density $\varrho$;
- if $|b|^p$ is locally Lebesgue integrable or locally $\mu$-integrable with some $p > d$, then the density $\varrho$ of $\mu$ belongs to the local Sobolev class $W^{p,1}_{loc}$; if $|b| \in L^p(\mu)$ with some $p > d$, then $\varrho \in W^{p,1}(\mathbb{R}^d)$ and $\|\varrho\|_\infty < \infty$;
Regularity of Solutions to Kolmogorov Equations with Perturbed Drifts

• if $|b| \in L^2(\mu)$, then $\sqrt{\varrho} \in W^{2,1}(\mathbb{R}^d)$ and

$$
\int \frac{|
abla \varrho|^2}{\varrho} \, dx \leq \int |b|^2 \, d\mu,
$$

where we set $|\nabla \varrho|^2/\varrho = 0$ on $\{ \varrho = 0 \}$.

It is not known whether there is an exact $L^p$-analog of the last result. However, it has been shown in [10] that

- the condition $|b| \in L^1(\mu)$ is not sufficient for the inclusion $|\nabla \varrho| \in L^1(\mathbb{R}^d)$, but gives the inclusion $\varrho \in H^{r,\alpha}(\mathbb{R}^d)$ for each exponent $s \in [1, d/(d-1)]$;
- if $|b| \in L^p(\mu)$ with some $p \in (1, d)$, then $\varrho \in W^{q,1}(\mathbb{R}^d)$ for each exponent $q < d/(d+1-p)$, hence $\varrho \in L^s(\mathbb{R}^d)$ for all $s < d/(d-p)$.

Although the condition $|b| \in L^1(\mu)$ is not enough for the membership of $\varrho$ in the Sobolev class $W^{1,1}(\mathbb{R}^d)$, it implies some weaker version of the logarithmic Sobolev inequality (see [15]). Sufficient conditions for the boundedness of $\varrho$ can be found in [7, 12, 20] along with some other bounds (see [9] for a survey). Other related questions have been studied in [5, 16, 19, 22–27], for the infinite-dimensional case see [17, 18], and [30].

In this paper, we complement this picture by the following three results:

(i) a higher integrability result for the density of the solution to the perturbed equation, in particular, some exponential integrability in the case of bounded $v$; the main result (Theorem 2.1) deals with first order perturbations of Markov generators satisfying the $CD(\theta, \infty)$ condition, so the aforementioned Gaussian case is a quite special example, and in this case we obtain that $\exp(\varepsilon \log(f \lor 1)^2) \in L^1(\gamma)$ for all numbers $\varepsilon < (4\pi^2 \|v\|_H^2\|\varrho\|_\infty^{-1})$, moreover, our estimates of the Orlicz norms of the density are dimension-free;

(ii) if $|v| \in L^p(\mu)$ with some $p > 2$, then $f \log^\alpha(1+f) \in L^1(\mu)$ whenever $\alpha < 2 \wedge \frac{p+2}{4}$ (Theorem 2.11);

(iii) in the Gaussian case, it is shown that the density of the solution with respect to the Gaussian measure belongs to the Gaussian Sobolev class $W^{p,1}(\gamma)$ for all $p \geq 1$ provided that the perturbation $v$ belongs to a suitable Orlicz class, e.g., is bounded, and a sufficient Orlicz integrability condition is given for the inclusion in $W^{p,1}(\gamma)$ with a given $p$, moreover, our estimates of the $L^p$-norms of the gradient are dimension-free (Theorem 3.1).

In particular, the cited results of Shigekava and Hino are reinforced and the first sufficient condition for the inclusion of the density in all $W^{p,1}(\gamma)$ is given in terms of the integrability of $v$; this result is new even for bounded $v$. Some further extensions are possible to the case of a non-constant diffusion term, but to keep the presentation less technical we confine ourselves to the unit diffusion matrix.

2 Integrability of Densities

In this section, we work in the framework of abstract Markov triples (see [2]). Let $(E, \mathcal{E}, \mu)$ be a probability space and let $\{T_t\}_{t \geq 0}$ be a strongly continuous semigroup of Markov operators on $L^2(\mu)$, i.e., $0 \leq T_t \varphi \leq 1$ whenever $0 \leq \varphi \leq 1$ and $T_t 1 = 1$. We also assume
that
\[ T_t \phi \to \int \phi \, d\mu \quad \text{as} \quad t \to \infty. \]

Let \( L \) be the generator of this semigroup and let \( \mathcal{D}(L) \) be its domain. We assume that \( L \) is symmetric:
\[ \int_E \phi L \psi \, d\mu = \int_E \psi L \phi \, d\mu \quad \forall \phi, \psi \in \mathcal{D}(L). \]

Suppose, in addition, that \( \mathcal{A} \) is an algebra of bounded measurable functions dense in \( \mathcal{D}(L) \) and in all \( L^p(\mu) \) with \( p < \infty \), contains 1, and is stable under \( L \) and under compositions with \( C^\infty \) functions of several variables, which means that \( F(f_1, \ldots, f_n) \in \mathcal{A} \) whenever \( F \in C^\infty(\mathbb{R}^n) \) and \( f_i \in \mathcal{A} \). Let
\[ \Gamma(\varphi, \psi) = \frac{1}{2} [L(\varphi \psi) - \varphi L \psi - \psi L \varphi] \]
for \( \varphi, \psi \in \mathcal{A} \) and let
\[ \Gamma(\varphi) = \Gamma(\varphi, \varphi). \]

One has the following integration by parts formula:
\[ \int_E \phi L \psi \, d\mu = -\int_E \Gamma(\varphi, \psi) \, d\mu, \quad \varphi, \psi \in \mathcal{A}. \]

Finally, we also assume that \( L \) is a diffusion operator: for all \( \Psi \in C^\infty(\mathbb{R}^k) \) and \( \psi_1, \ldots, \psi_k \in \mathcal{A} \) one has
\[ L\Psi(\psi_1, \ldots, \psi_k) = \sum_{j=1}^k \partial_j \Psi(\psi_1, \ldots, \psi_k) L \psi_j + \sum_{i,j=1}^k \partial^2_{i,j} \Psi(\psi_1, \ldots, \psi_k) \Gamma(\psi_i, \psi_j). \]

We say that the curvature-dimension condition \( CD(\theta, \infty) \) with some number \( \theta \) holds if
\[ \Gamma_2(\varphi) \geq \theta \Gamma(\varphi) \quad \forall \varphi \in \mathcal{A}, \]
where
\[ \Gamma_2(\varphi, \psi) := \frac{1}{2} [L \Gamma(\varphi, \psi) - \Gamma(\varphi, L \psi) - \Gamma(L \varphi, \psi)], \quad \Gamma_2(\varphi) := \Gamma_2(\varphi, \varphi). \]

It follows from our assumptions that
\[ L^* \mu = 0, \]
provided we define this equation as the identity
\[ \int L \varphi \, d\mu = 0 \quad \forall \varphi \in \mathcal{A}. \]

However, our main object will not be the solution \( \mu \), but probability solutions \( f \cdot \mu \) to some associated inequalities and equations \( L^*_\rho(f \cdot \mu) = 0 \) with certain perturbations of \( L \).

An important example is the case where \( \mu = e^{-W} \, dx \) for some \( W \in C^2(\mathbb{R}^d) \) and
\[ L \varphi = \Delta \varphi - \langle \nabla W, \nabla \varphi \rangle. \]

If \( W(x) = |x|^2/2 + C \), then \( \mu \) is the standard Gaussian measure on \( \mathbb{R}^d \) denoted by \( \gamma \).

One can also consider the Gaussian measure \( \gamma \) in the infinite dimensional setting. In this case \( L \) is the Ornstein–Uhlenbeck operator associated with \( \gamma \). For the above measure the curvature-dimension condition \( CD(\theta, \infty) \) holds if \( D^2 W \geq \theta I \), since
\[ \Gamma_2(\varphi) = \|D^2 \varphi\|_{HS}^2 + \langle D^2 W \cdot \nabla \varphi, \nabla \varphi \rangle, \]
where \( \| \cdot \|_{HS} \) denotes the Hilbert–Schmidt norm.
2.1 Orlicz-Integrable Drifts

Let $m \geq 1$ and let $\psi_m(t) := e^{mt} - 1$ for $t > 0$. Recall the definition of the Orlicz norm. Let $\mu$ be a probability measure on some space $E$. We say that a measurable function $w$ belongs to the Orlicz class $L_{\psi_m}(\mu)$ if

$$\int_E \psi_m(\lambda^{-1}|w|) \, d\mu < \infty \quad \text{for some } \lambda > 0.$$  

We denote by $\|w\|_{L_{\psi_m}(\mu)}$ the Orlicz norm of $w$ defined by

$$\|w\|_{L_{\psi_m}(\mu)} := \inf \left\{ \lambda : \int_E \psi_m(\lambda^{-1}|w|) \, d\mu \leq 1 \right\}.$$  

Here is our first main result.

**Theorem 2.1** Assume that the Markov triple $(E, \mu, \Gamma_1)$ satisfies the curvature-dimension condition $\text{CD}(\theta, \infty)$ with some $\theta > 0$. Suppose that we are given a probability density $f$ with respect to $\mu$ and a measurable function $w \geq 0$ such that $\lambda := \|w\|_{L_{\psi_m}(f \cdot \mu)} < \infty$ for some $m \in [2, +\infty)$. Let also

$$\int_E \frac{L(\varphi)f}{\varphi(f)} \, d\mu \leq \int_E \frac{w}{\sqrt{\Gamma_1(\varphi)}} \varphi(f) \, d\mu \quad \forall \varphi \in A.$$  

Set $\mu_f := f \cdot \mu$. The following assertions are true.

(i) If $m = 2$, then $\mu_f(A) \leq e^{2(\mu(A) - \sigma_2^2)}$, where $\sigma_2 := \exp(-\frac{2\pi}{\sqrt{\theta}} \lambda)$. In particular,

$$\mu(f \geq t) \leq e^{2(t - \frac{\ln t}{1 - \sigma_2})},$$  

and $f \in L^p(\mu)$ for all $p < \frac{1}{1 - \sigma_2}$.

(ii) If $m > 2$, then

$$\mu(f \geq t) \leq e^{2 \exp(-\sigma_m(\ln t)^{\frac{2}{1 + 2/m}})} \quad \forall t > 1$$  

and $e^{(\ln \max\{f, 1\})^{\frac{1}{1 + 2/m}}} \in L^1(\mu)$ for all $\varepsilon < \sigma_m$, where

$$\sigma_m := \frac{1 - 2/m}{1 + 2/m} \left( \frac{2\pi}{\sqrt{\theta}} \lambda (1 - 2/m) \right)^{-\frac{2}{1 + 2/m}}.$$  

(iii) If $w$ is bounded, then

$$\mu(f \geq t) \leq e^{2 \varepsilon e^{-\sigma_\infty(\ln t)^2}} \quad \forall t > 1$$  

and $e^{(\ln \max\{f, 1\})^2} \in L^1(\mu)$ for all $\varepsilon < \sigma_\infty$, where $\sigma_\infty := \left( \frac{2\pi}{\sqrt{\theta}} \|w\|_\infty \right)^{-2}$.

**Proof** Recall that for every positive function $\varphi \in A$ one has (see [2, Theorem 4.7.2 and Theorem 5.5.5])

$$T_t \varphi^2 - (T_t \varphi)^2 \geq \frac{2\theta}{2\theta - 1} \Gamma(T_t \varphi), \quad (T_t (T_t - s \varphi)^q)^{1/q} \leq (T_t \varphi^p)^{1/p},$$  

whenever $\frac{q - 1}{p - 1} = \frac{2\theta - 1}{2\theta - 1} > 1$.

Fix $\alpha \in (0, 1)$. Let $\varphi \in A$ be a function such that $\alpha \leq \varphi \leq e^{-2} < 1$. Let us consider the function

$$F(t) = \int_E T_t \varphi f \, d\mu.$$  

\(\text{ Springer}\)}
Note that $α ≤ F(t) ≤ e^{-2}$ and that for every $s ∈ (0, t)$ we have

$$F'(t) = ∫_E LT_t ϕ f dμ ≤ ∫_E w \sqrt{Γ(ϕ)} f dμ = ∫_E w \sqrt{Γ(T_s T_t−sϕ)} f dμ$$

$$≤ \frac{\sqrt{θ}}{\sqrt{e^{2θt}−1}} ∫_E w(T_s(T_t−sϕ)^2)^{1/2} f dμ ≤ \frac{\sqrt{θ}}{\sqrt{e^{2θt}−1}} ∫_E w(T_tϕ/(r−1))^{1−1/r} f dμ$$

$$= \frac{\sqrt{θ}}{\sqrt{e^{2θt}−1}} ∫_E w r^{1/r} F(t)^{1−1/r} F(t)$$

where $r − 1 = \frac{e^{2θt}−1}{e^{2θt}−1}$. We note that for all $a > 0$ one has $a^x e^{-a} ≤ x^x$, since this is true for $x ≥ a$ and for $x ≤ a$ one has $x \log a ≤ a + x \log x$ by differentiation in $a$. This implies the bound

$$\left(∫_E w r^{1/r} F(t)^{1−1/r} F(t)\right)^{1/r} ≤ 2^{1/r} λ r^{1/m}$$

by taking $a = λ r^{−m} w^{m}$ and $x = r/m$. Thus, for all $r ≥ 2$ we have

$$|F'(t)| ≤ \sqrt{2} λ \sqrt{θ} \frac{r}{e^{2θt}−1} F(t)^{1−1/r} F(t).$$

We now take $r = −\ln F(t)$ and, since $\sqrt{2} e ≤ 4$, obtain the bound

$$|F'(t)| ≤ 4λ \sqrt{θ} [−\ln F(t)]^{1/2+m} F(t).$$

First we consider the case $m = 2$. In this case, for all $a, b ∈ (0, +∞)$, one has

$$\ln[−\ln F(b)] − \ln[−\ln F(a)] = ∫_a^b \frac{−F'(t)}{−\ln F(t)} F(t) dt ≤ 4λ ∫_a^b \frac{\sqrt{θ}}{e^{2θt}−1} dt ≤ \frac{2π}{\sqrt{θ}} λ.$$

Thus,

$$F(a) ≤ [F(b)]^{σ_2},$$

where $σ_2 = \exp\left(-\frac{2π}{\sqrt{θ}} λ\right)$. Taking the limit as $a → 0$ and $b → +∞$, we obtain

$$∫_E ϕ f dμ = F(0) ≤ ∥ϕ∥_1^{σ_2}.$$

Therefore, for each $ϕ ∈ A$ such that $0 ≤ φ ≤ 1$ we have

$$∫_E ϕ f dμ ≤ e^{2−2σ_2} ∥ϕ∥_1^{σ_2}.$$

By approximation, for every $μ$-measurable set $A$ we obtain

$$∫_A f dμ ≤ e^{2−2σ_2} [μ(A)]^{σ_2}.$$
We now consider the case \( m > 2 \). In this case, for all \( a, b \in (0, +\infty) \), one has

\[
[- \ln F(b)]^{1/2-1/m} - [- \ln F(a)]^{1/2-1/m} = \frac{m-2}{2m} \int_a^b \frac{-F'(t)}{[- \ln F(t)]^{1/2+1/m} F(t)} dt \\
\leq \frac{2}{m} (m-2) \lambda \int_a^b \frac{\sqrt{\theta}}{\sqrt{e^{2\theta t} - 1}} dt \leq \frac{\pi}{m \sqrt{\theta}} (m-2) \lambda := K.
\]

Let \( m' > 2 \) be such that \( \frac{1}{\lambda} - \frac{1}{m} = \frac{1}{m'} \). Then, by convexity,

\[
- \ln F(b) \leq (K + [- \ln F(a)]^{1/m'})^{m'} \leq (1 + \varepsilon^{-1})^{m'-1} K^{m'} + (1 + \varepsilon)^{m'-1} [- \ln F(a)]
\]

for every \( \varepsilon > 0 \). Therefore,

\[
\ln F(a) \leq \varepsilon^{1-m'} K^{m'} + (1 + \varepsilon)^{1-m'} \ln F(b)
\]

and

\[
F(a) \leq [F(b)]^{(1+\varepsilon)^{1-m'}} e^{\varepsilon^{1-m'} K^{m'}}.
\]

Passing to the limit as \( a \to 0 \) and \( b \to +\infty \), we obtain

\[
\int_{E} \varphi f \, d\gamma = F(0) \leq \|\varphi\|_{1} (1+\varepsilon)^{1-m'} e^{\varepsilon^{1-m'} K^{m'}}
\]

for all \( \varepsilon > 0 \). Thus, for every \( \varphi \in A \) such that \( 0 \leq \varphi \leq 1 \) we have

\[
\int_{E} \varphi f \, d\mu \leq e^{2-2(1+\varepsilon)^{1-m'}} \|\varphi\|_{1} (1+\varepsilon)^{1-m'} e^{\varepsilon^{1-m'} K^{m'}}.
\]

By approximation, for every \( \mu \)-measurable set \( A \) we obtain

\[
\int_{A} f \, d\mu \leq e^{2-2(1+\varepsilon)^{1-m'}} [\mu(A)]^{(1+\varepsilon)^{1-m'}} e^{\varepsilon^{1-m'} K^{m'}}.
\]

We now take \( A := \{ f \geq t \} \) and get

\[
\mu(f \geq t) \leq e^{2} \left[ t^{-1} e^{\varepsilon^{1-m'} K^{m'}} \right] \frac{1}{1-(1+\varepsilon)^{1-m'}}.
\]

For \( t > 1 \) we take \( \varepsilon := \left( \frac{2K^{m'}}{\ln t} \right)^{1-m'} \), i.e., \( e^{1-m'} K^{m'} = 2^{-1} \ln t \), and get

\[
\mu(f \geq t) \leq e^{2} \left[ e^{-2^{-1} \ln t} \right] \frac{1}{1-(1+\varepsilon)^{1-m'}}.
\]

Since

\[
1 - (1 + \varepsilon)^{1-m'} = \int_{0}^{\varepsilon} (m' - 1)(1 + s)^{-m'} \, ds \leq (m' - 1) \varepsilon,
\]

we arrive at the estimate

\[
\mu(f \geq t) \leq e^{2} \left[ e^{-2^{-1} \ln t} \right] \frac{1}{(m' - 1) \varepsilon} = e^{2} e^{-\sigma_{m} \ln t} \frac{m'}{m'-1},
\]

where

\[
\sigma_{m} := \frac{1}{m'-1} (2K)^{-m'/m'-1} = \frac{1}{m'-1} \left( \frac{2\pi}{\sqrt{\theta}} \lambda (1 - 2/m) \right)^{-\frac{1}{m'-1}}.
\]

Thus, for every \( s > 1 \) we have

\[
\mu(e^{\ln \max\{f,1\}}^{m'/m'-1} \geq s) = \mu(f \geq e^{\ln s^{1/m'}}^{m'-1} \frac{m'-1}{m'}) \leq e^{2} e^{-\sigma_{m} s^{-1}}.
\]
Hence $e^{\varepsilon \ln \max\{f, 1\}} \in L^1(\mu)$ for all $\varepsilon < \sigma_m$.

In the case of bounded $w$ we have $\|w\|_{L^\infty} \leq 2^{1/m} \|w\|_\infty$ and the result follows by taking the limit as $m \to +\infty$. The theorem is proved. \qed

Remark 2.2 In the setting of the previous theorem for each $m \in (2, +\infty)$ (again $m = +\infty$ is understood as $\|w\|_\infty < \infty$) and for each $p \in (1, +\infty)$ there is a number $C(m, p)$ such that for all $t > 1$ one has

$$\mu(f \geq t) \leq t^{-p} \exp \left( 2 + C(m, p) \frac{\theta^{-1/2}}{1^{2/m}} \right)$$

and

$$\int_E |f|^p d\mu \leq 1 + \exp \left( 2 + C(m, p) \frac{\theta^{-1/2}}{1^{2/m}} \right) \leq 2 \exp \left( 2 + C(m, p) \frac{\theta^{-1/2}}{1^{2/m}} \right).$$

The abstract result above yields the following bound for solutions to Fokker–Planck–Kolmogorov equations.

Theorem 2.3 Let $\mu = e^{-W} dx$ with $W \in C^2(\mathbb{R}^d)$, $D^2 W \geq \theta \cdot 1$, $\theta > 0$ and let $f \in L^1(\mu)$ be a probability solution to the equation $L^{*} \Psi [f \cdot \mu] = 0$ with some $(f \cdot \mu)$-integrable vector field $v$, where $L_v \mu = \Delta u + (-\nabla W + v, \nabla u)$.

(i) If $v \in L^2(\mu)$, then

$$\mu(f \geq t) \leq e^{2t^{-1/2}}$$

and $f \in L^p(\mu)$ for all $p < \frac{1}{1-\sigma_2}$, where $\sigma_2 := \exp(-\frac{2\pi}{\sqrt{\theta}} \|L^{2}(f \cdot \mu)\|)$;

(ii) If $|v| \in L^m(\mu)$ with $m > 2$, then

$$\mu(f \geq t) \leq e^{2 \ln \max\{f, 1\} \frac{\theta}{1^{2/m}}} \forall t > 1$$

and $e^{\varepsilon \ln \max\{f, 1\} \frac{\theta}{1^{2/m}}} \in L^1(\mu)$ for all $\varepsilon < \sigma_m$, where

$$\sigma_m := \frac{1 - 2/m}{1 + 2/m} \left( \frac{2\pi}{\sqrt{\theta}} \|v\|_{L^\infty}(1 - 2/m) \right)^{-\frac{2}{1^{2/m}}}.$$

(iii) If $v$ is bounded, then

$$\mu(f \geq t) \leq e^{2e^{-\sigma_\infty \ln t}^2} \forall t > 1$$

and $e^{\varepsilon \ln \max\{f, 1\}^2} \in L^1(\mu)$ for all $\varepsilon < \sigma_\infty$, where $\sigma_\infty := \left( \frac{2\pi}{\sqrt{\theta}} \|v\|_\infty \right)^{-2}$.

Proof The previous theorem applies with $w = |v|$ and

$$L \varphi = \Delta \varphi - \langle \nabla W, \nabla \varphi \rangle,$$

because the equation reads as

$$\int L \varphi f \ d\mu = -\int \langle \nabla \varphi, v \rangle \ f \ d\mu, \quad \varphi \in C^\infty_0(\mathbb{R}^d),$$

where $|\langle \nabla \varphi, v \rangle| \leq |v| |\nabla \varphi|$ and $\Gamma(\varphi, \psi) = \langle \nabla \varphi, \nabla \psi \rangle$. \qed

We note that the above result is applicable to the standard Gaussian measure $\mu$ for which $\theta = 1$. \hfill \copyright Springer
Let us show that the result is nearly sharp even for the one-dimensional Ornstein–Uhlenbeck operator and a constant \( v \).

Example 2.4 Let \( \gamma \) be the standard Gaussian measure on the real line and let \( \mu = f \cdot \gamma \), where \( f(x) = \exp(vx) \) with some constant \( v \neq 0 \). Then \( \mu \) satisfies the FPK-equation with the drift \( b(x) = -x + v \). We have \( f \in L^p(\mu) \) for all \( p > 1 \), but \( \exp(\varepsilon f) \notin L^1(\gamma) \) for all \( \varepsilon > 0 \), one only has \( \exp(\varepsilon |\log f|^2) \in L^1(\gamma) \) for \( \varepsilon < (2|v|^2)^{-1} \).

Remark 2.5 In the situation of the previous theorem the operator

\[
L \varphi = \Delta \varphi + \langle -\nabla W, \nabla \varphi \rangle
\]

is symmetric on the domain \( C_0^\infty \) in \( L^2(\mu) \), which follows by the integration by parts formula:

\[
\int \psi L \varphi \, d\mu = -\int \langle \nabla \psi, \nabla \varphi \rangle \, d\mu.
\]

One can introduce the divergence \( \delta_\mu w \) of a locally Sobolev vector field \( w \) by

\[
\delta_\mu w = \text{div} \, w - \langle w, \nabla W \rangle.
\]

In particular, for the standard Gaussian measure \( \gamma \) we have

\[
\delta_\gamma w(x) = \text{div} \, w(x) - \langle w(x), x \rangle.
\]

For every \( \varphi \in C_0^\infty \) there holds the equality

\[
\int \langle \nabla \varphi, w \rangle \, d\mu = -\int \varphi \delta_\mu w \, d\mu,
\]

which can be used as the definition of divergence in the sense of distributions for locally integrable \( w \) that is not locally Sobolev.

If \( v \) is locally integrable to some power \( p > d \) with respect to Lebesgue measure, the equation \( L^*_v(f \cdot \mu) = 0 \) for a measure \( f \cdot \mu \) with a density \( f \in L^1(\mu) \) is equivalent to the equation

\[
L f = \delta_\mu w \tag{2.1}
\]

with the vector field

\[
w := f \cdot v,
\]

understood as the identity

\[
\int L \varphi f \, d\mu = -\int \langle \nabla \varphi, v \rangle f \, d\mu \quad \forall \varphi \in C_0^\infty.
\]

This follows by the integration by parts formula taking into account that \( f \in W_{loc}^{p,1} \). The same is true if in the condition of the local integrability of \( |v|^p \) Lebesgue measure is replaced by the solution \( f \cdot \mu \).

It is known in the Gaussian case (see [29, Section 4.2]) that if \( f \) and \( |w| \) belong to \( L^p(\gamma) \), \( p > 1 \), then \( \delta_\gamma w \) and \( \delta_\gamma w - f \) belong to the negative class \( W^{p,-1}(\gamma) \), i.e., \( Lf - f \in W^{p,-1}(\gamma) \), which yields the inclusion \( f \in W^{p,1}(\gamma) \), along with the estimate

\[
\|f\|_{p,1} \leq C(p)(\|w\|_p + \|f\|_p),
\]

where \( C(p) \) depend only on \( p \).

It is worth noting that the existence of a probability solution was part of our hypotheses. Some sufficient condition on the perturbation ensuring this can be found in [14].
Finally, we prove a result of independent interest, we owe its proof to Stanislav Shaposhnikov.

**Proposition 2.6** Let $\gamma$ be the standard Gaussian measure on $\mathbb{R}^d$. If a probability measure $f \cdot \gamma$ satisfies the equation $L_v^\ast(f \cdot \gamma) = 0$, where $v$ has compact support and is $f \cdot \gamma$-integrable, for example, is bounded, then $f$ is bounded.

**Proof** Let $\varrho$ be the standard Gaussian density. From the FPK equation we have

$$\text{div}(\varrho \nabla f - \varrho f v) = 0$$

and outside of some ball $Q$ we obtain

$$L_f = 0.$$

In particular, $f$ is smooth outside of $Q$. Let us observe that for $\delta \in (1/2, 1)$ close to $1$ such that $1 - \delta < 1/d$ there exists $C_\delta > 0$ for which

$$f(x)\varrho(x) \leq C_\delta e^{-(1-\delta)|x|^2/2}.$$

Indeed, the function $V(x) = e^{\beta|x|^2/2}$, where $0 < \beta < 1$ and $1 - \delta < \beta/d$, serves as a Lyapunov function for the operator $L_v$, that is,

$$L_v V(x) = \beta(d + \beta|x|^2) V(x) - \beta|x|^2 V(x) + \beta \langle x, v(x) \rangle V(x)$$

is majorized by $-2^{-1}\beta(1 - \beta)|x|^2 V(x)$ outside of $Q$. On the whole space we have the estimate $L_v V \leq F - V$, where $F$ is a $(f \cdot \gamma)$-integrable function with compact support (bounded in the case where $v$ is bounded). Therefore, $V$ is integrable against $f \cdot \gamma$ (see [9, Section 2.3]), i.e., the function $Vf \varrho$ is integrable on $\mathbb{R}^d$. Thus, the function $\Phi(x) = e^{(1-\delta)|x|^2/2}$ satisfies the hypotheses of [9, Theorem 3.3.1]. These hypotheses, in addition to some technical assumptions about the coefficients of the equation (which are trivially fulfilled in our case) require the inclusions $\Phi \in L^1(f \cdot \gamma)$, $|\nabla \Phi| \in L^\theta(f \cdot \gamma)$ with some $\theta > d$. These inclusions are ensured by the condition $d(1 - \delta) < \beta$ along with the integrability of $V$ established above. It follows by the cited theorem that

$$f(x) \leq C_\delta e^{\delta|x|^2/2}.$$

Now let $\delta < q < 1$. Outside of some ball (again denoted by $Q$) we have

$$L \Psi(x) \leq 0, \quad \Psi(x) = e^{q|x|^2/2}.$$

Let $M = \sup_Q f$. For every $\varepsilon > 0$ we have

$$L(f - \varepsilon \Psi) \geq 0$$

outside of $Q$. Since $f(x) - \varepsilon \Psi(x)$ tends to $-\infty$ as $|x| \to \infty$, by the maximum principle

$$f(x) - \varepsilon \Psi(x) \leq M \quad \forall x.$$

Letting $\varepsilon$ to 0, we conclude that $f \leq M$. Thus, $f$ is bounded.

Note that the assumption of compactness of the support of $v$ is important (as follows from the example above with constant $v$).
2.2 \textit{L}^p\textit{-Integrable Drifts}

In this subsection, we consider probability solutions to the equation

\[ L_v^* [f \cdot \mu] = 0 \]

with a vector field \( v \) belonging to \( L^p(f \cdot \mu) \), where

\[ L_v u = \Delta u + (-\nabla W + v, \nabla u), \quad W \in C^2(\mathbb{R}^d). \]

It is known in the case \( p = 2 \) that \( \sqrt{f} \in W^{2,1}(\mu) \) and the logarithmic Sobolev inequality (which holds under \( CD(\theta, \infty) \), see \cite[Proposition 5.7.1]{2}) yields the bound

\[
\int f \log f \, d\mu \leq \frac{1}{2\theta} \int \frac{\|\nabla f\|^2}{f} \, d\mu \leq \frac{1}{2\theta} \int |v|^2 f \, d\mu. \tag{2.2}
\]

In the case \( p = 1 \) the arguments analogous to \cite{15} provide the following estimate:

\[
\int f \log^{1/4-\varepsilon} (1 + f) \, d\mu \leq C(\theta, \varepsilon) \left( 1 + \|v\|_{L^1(f \cdot \mu)} \right) \log^{1/4-\varepsilon} \left( 1 + \|v\|_{L^1(f \cdot \mu)} \right),
\]

where \( \varepsilon > 0 \). The reasoning in \cite{15} is essentially based on the a priori estimates

\[
\int T_t f \log^{1/2} (1 + T_t f) \, d\mu \leq C t^{-1/2}, \quad t \in (0, 1), \tag{2.3}
\]

\[
\|T_t f - f\|_{L^1(\mu)} \leq C t^{1/2}, \tag{2.4}
\]

where \( C \) depends on \( \theta \) and the \( L^1(f \cdot \mu) \) norm of the drift \( v \). In this section, we show that for sufficiently large \( p \) one can obtain an improvement over estimate (2.2) with respect to the integrability of \( f \).

Let us recall the dual description of the Kantorovich metric \( W_p(\mu_1, \mu_2), \ p \geq 1 \):

\[
\frac{1}{p} W_p(\mu_1, \mu_2) = \sup \left( \int Q_s \varphi \, d\mu_1 - \int \varphi \, d\mu_2 \right),
\]

where the supremum is taken over all bounded continuous functions \( \varphi \) and

\[
Q_s \varphi(x) := \inf_y \left( \varphi(y) + \frac{|x - y|^p}{p s^{p-1}} \right), \quad s > 0.
\]

It is well-known that \( Q_s \varphi \) satisfies the Hamilton–Jacobi equation

\[
\frac{d}{ds} Q_s \varphi = -\frac{1}{q} |\nabla Q_s \varphi|^q
\]

with initial condition \( \varphi \), where \( \frac{1}{p} + \frac{1}{q} = 1 \).

\textbf{Lemma 2.7} Let \( f \cdot \mu \) be a probability solution to the equation

\[ L_v^* [f \cdot \mu] = 0 \]

with \( v \in L^1(f \cdot \mu) \) and let \( \psi \) be a bounded Lipschitz function on \( \mathbb{R}_+ \times \mathbb{R}^d \). Then for all \( u \geq t \) we have

\[
\int \psi(u, x) T_u f(x) \, d\mu - \int \psi(t, x) T_t f(x) \, d\mu = \int_{[t, u]} \left( \frac{\partial \psi}{\partial s} T_s f - \langle \nabla T_s \psi, v \rangle f \right) \, d\mu \, ds.
\]
The required equality trivially holds for smooth functions $\psi$ that are finite linear combinations of functions of the form $(t, x) \mapsto \psi_1(t)\psi_2(x)$. It remains to notice that the general case follows by the standard approximation arguments.

**Theorem 2.8** Let $f \cdot \mu$ be a probability solution to the equation

$$L^p_v [f \cdot \mu] = 0$$

with $v \in L^p(f \cdot \mu)$, $p > 1$. Then

$$W_p^p (T_{t+h} f \cdot \mu, T_t f \cdot \mu) \leq h^p \int |v|^p f \, d\mu,$$

(2.5)

$$W_p^p (f \cdot \mu, \mu) \leq \theta^{-p+1} \int |v|^p f \, d\mu.$$ (2.6)

**Proof** Let $0 < t < u < \infty$ and let $\eta$ be an increasing function on the real line such that $\eta(t) = 0$, $\eta(u) = 1$. The particular choice of $t$, $u$ and $\eta$ will be specified below. Let us fix $\varphi \in C_0^\infty$ and set

$$\psi(s, x) := Q_{\eta(s)} \varphi, \; s \in [t, u].$$

Applying Lemma 2.7 we obtain the equality

$$\int Q_1 \varphi T_u f \, d\mu - \int \varphi T_t f \, d\mu
= \int_{[t, u]} \left( - \frac{\eta'(s)}{q} |\nabla Q_{\eta(s)} \varphi|^q T_s f - \langle \nabla T_s Q_{\eta(s)} \varphi, v \rangle f \right) \, d\mu \, ds.$$

The gradient commutation inequality (see [2, Theorem 3.2.4]) provides the bound

$$|\nabla T_s g| \leq e^{-\theta s} T_s |\nabla g|.$$

Therefore, for each positive number $K$ that may depend on $s$ we have

$$|\langle \nabla T_s Q_{\eta(s)} \varphi, v \rangle f| \leq \frac{K}{q} |\nabla T_s Q_{\eta(s)} \varphi|^q f + \frac{K^{-p/q}}{p} |v|^p f$$

$$\leq \frac{K}{q} e^{-\theta s} T_s |\nabla Q_{\eta(s)} \varphi|^q f + \frac{K^{-p+1}}{p} |v|^p f,$$

where Young’s and Jensen’s inequalities have been used. Next, the numbers $t$, $u$, $\eta$ and $K$ will be picked in such a way that the term

$$\frac{K}{q} e^{-\theta s} T_s |\nabla Q_{\eta(s)} \varphi|^q f$$

will be canceled by

$$- \frac{\eta'(s)}{q} |\nabla Q_{\eta(s)} \varphi|^q T_s f.$$

To establish inequality (2.5) let us set

$$u := t + h, \; \eta(s) := \frac{s - t}{h}, \; K := 1/h.$$

In this case

$$\int Q_1 \varphi T_u f \, d\mu - \int \varphi T_t f \, d\mu \leq \int_{[t, t+h]} \int \frac{h^{p-1}}{p} |v|^p f \, d\mu \, ds = \frac{h^p}{p} \int |v|^p f \, d\mu.$$
Finally, to complete the proof of inequality (2.6) we set
\[ t := 0, \ u := \infty, \ \eta(s) := 1 - e^{-\theta s}, \ K := \theta e^{\theta(q-1)s}. \]
In this case
\[
\int Q_1 \varphi \, d\mu - \int \varphi f \, d\mu \leq \int_0^{\infty} \varphi \left( \frac{\theta^{-p+1} e^{-\theta(q-1)(p-1)s}}{p} |v|^p f \, d\mu \, ds \right) \leq \frac{\theta^{-p+1}}{p(q-1)(p-1)} \int |v|^p f \, d\mu,
\]
which gives our claim, since \((q - 1)(p - 1) = 1\).

The next proposition can be considered as an \(L^p\)-counterpart of the classical inequality
\[
\int T_t g \log T_t g \, d\mu \leq \frac{1}{4t} W^2_p(g \cdot \mu, \mu)
\]
that is known under the \(CD(0, \infty)\)-condition (see [3, inequality (11)]).

**Proposition 2.9** Assume that the curvature-dimension condition \(CD(0, \infty)\) holds. Then, for every probability measure \(g \cdot \mu\) with finite Kantrovich distance \(W_p(g \cdot \mu, \mu)\) of order \(p \geq 1\) and every \(t > 0\), we have
\[
\int T_t g \log T_t g \, d\mu \leq 6^{p/2+1} \log^{p/2}(c_p + 1) + t^{-p/2}(3/2)^{p/2} W^p_p(g \cdot \mu, \mu),
\]
where \(c_p := \max(1, e^{p/2-1})\).

**Proof** Let us recall that the condition \(CD(0, \infty)\) implies (see e.g. [2]) Wang’s Harnack inequality:
\[
T_t h^{1/2}(y) \leq (T_t h(x))^{1/2} e^{[x-y]^2/4t},
\]
where \(h\) is a nonnegative measurable function. Now let us consider the following auxiliary function:
\[
\varphi(z) := \log^{p/2}(z + c_p), \quad z \geq 0.
\]
Since
\[
\varphi'(z) = \frac{p}{2} \frac{\log^{p/2-1}(z + c_p)}{z + c_p},
\]
\[
\varphi''(z) = \frac{p}{2} \frac{\log^{p/2-2}(z + c_p)((p/2 - 1) - \log(z + c_p))}{(z + c_p)^2}
\]
it is easy to see that
\[
\varphi'(z) \geq 0, \quad \varphi''(z) \leq 0, \quad z \geq 0,
\]
i.e., \(\varphi\) is increasing and concave on \(\mathbb{R}_+\). Now let us apply \(\varphi\) to the both sides of (2.8) and take into account Jensen’s inequality:
\[
T_t \varphi(h^{1/2})(y) \leq \varphi(T_t h^{1/2}(y)) \leq \varphi((T_t h(x))^{1/2} e^{[x-y]^2/4t}),
\]
\[
T_t \log^{p/2}(c_p + h^{1/2})(y) \leq \log^{p/2}(c_p + (T_t h(x))^{1/2} e^{[x-y]^2/4t})
\]
Since \(c_p \geq 1\) and for \(z \geq 0, z_1 \geq 1\) one has
\[
(c_p + z)^{1/2} \leq c_p + z^{1/2}, \quad c_p + z^{1/2} z_1 \leq (c_p + 1)(c_p + z)^{1/2} z_1,
\]

\(\square\) Springer
we obtain
\[
T_t \log^{p/2}(c_p + h)(y) \leq \left(2 \log(c_p + 1) + \log(c_p + T_t h(x)) + \frac{|x - y|^2}{2t}\right)^{p/2},
\]
\[
T_t \log^{p/2}(c_p + h)(y) \leq 6^{p/2} \log^{p/2}(c_p + 1) + 3^{p/2} \log^{p/2}(c_p + T_t h(x)) + t^{-p/2} (3/2)^{p/2} |x - y|^p.
\]
Now let us pick \( h := T_t g \) and integrate this inequality over the optimal coupling between the measures \( g \cdot \mu \) and \( \mu \):
\[
\int g T_t \log^{p/2}(c_p + T_t g) \, d\mu \leq 6^{p/2} \log^{p/2}(c_p + 1) + 3^{p/2} \log^{p/2}(c_p + T_t g) + t - p/2 \left(\frac{3}{2}\right)^{p/2} W_p(g \cdot \mu, \mu),
\]
where Jensen’s inequality has been used to obtain the bound
\[
\int \log^{p/2}(c_p + T_t g) \, d\mu \leq \log^{p/2}(c_p + 1).
\]
Taking into account the symmetry of \( T_t \) we finally obtain (2.7).

**Proposition 2.10** Assume that the curvature-dimension condition \( CD(0, \infty) \) holds. Then for every nonnegative function \( f \) such that \( f^{1/2} \in W^{2,1}(\mu) \) one has
\[
\int |\nabla T_t f|^2 \, d\mu \leq \int |\nabla f|^2 \, d\mu.
\]

**Proof** This easily follows by the inequality
\[
\frac{|\nabla T_t f|^2}{T_t f} \leq T_t \left(\frac{|\nabla f|^2}{f}\right) \leq T_t \left(\frac{|\nabla f |^{1/2} f^{1/2}}{f}\right) \leq T_t \frac{|\nabla f|^2}{f}
\]
and integration with respect to \( \mu \). \( \square \)

Now we are ready to present the main theorem of this subsection.

**Theorem 2.11** Let us assume that the curvature condition \( CD(\theta, \infty) \) holds with some \( \theta > 0 \) and let \( f \cdot \mu \) be a probability solution to the equation
\[
L^* v[f \cdot \mu] = 0
\]
with some vector field \( v \) belonging to \( L^p(f \cdot \mu) \), \( p > 2 \). Then for every number \( \alpha < \min(2, \frac{p+2}{4}) \) there exists \( C > 0 \) depending on \( \theta, p, \alpha \) such that
\[
\int f \log^\alpha (1 + f) \, d\mu \leq C + C \int |v|^p f \, d\mu.
\]

**Proof** Let us set
\[
\Phi_R(z) := \int_{[0,z]} R \wedge \frac{\log^\alpha(1 + x)}{\alpha} \, dx, \ z \geq 0.
\]
It is clear that
\[
\Phi'_R(z) = R \wedge \frac{\log^\alpha(1 + z)}{\alpha}, \ \Phi''_R(z) = I_{[z<R']} \frac{\log^\alpha(1 + z)}{1 + z}
\]
for the appropriate value \( R' \). Let us fix \( \delta > 0 \) and set
\[
f_{\delta,t}(x) := \frac{1}{\delta} \int_{[t,t+\delta]} T_u f(x) \, du, \ f_{\delta} : t \mapsto f_{\delta,t} \in L^1(\mathbb{R}^d, W^{1,1}[0,1]).
\]
Taking into account Proposition 2.10 one can show that

\[ \int \frac{\left| \nabla f_{\delta,t} \right|^2}{f_{\delta,t}} \, d\mu \leq \int \frac{\left| \nabla f \right|^2}{f} \, d\mu \leq \| v \|^2_{L^2(f \cdot \mu)}. \]

Indeed, similarly to the proof of Proposition 2.10 this follows by the chain of inequalities

\[
\int \frac{\left| \nabla f_{\delta,t} \right|^2}{f_{\delta,t}} \, d\mu \leq \int \frac{1}{f_{\delta,t}} \left[ \frac{1}{\delta} \int_{[t,t+\delta]} \left| \nabla T_{u} f \right| \, d\mu \right]^2 \, d\mu \\
\leq \int \frac{1}{f_{\delta,t}} \left[ \frac{1}{\delta} \int_{[t,t+\delta]} \left| \nabla T_{u} f \right| \cdot T_{u}^{-1/2} f \cdot T_{u}^{1/2} f \, d\mu \right]^2 \, d\mu \\
\leq \frac{1}{\delta} \int \int_{[t,t+\delta]} \left| \nabla T_{u} f \right| \cdot \left[ \frac{1}{2} \int_{[t,t+\delta]} \left| \nabla T_{u} f \right| \, d\mu \right] \, d\mu \leq \int \frac{\left| \nabla f \right|^2}{f} \, d\mu \leq \| v \|^2_{L^2(f \cdot \mu)}. \]

Inequality (2.5) from Theorem 2.8 yields the bound

\[ W_p(T_t f \cdot \mu, T_{t+h} f \cdot \mu) \leq h \| v \|_{L^p(f \cdot \mu)}. \]

It is easy to see that this bound implies the estimate

\[ W_p(f_{\delta,t} \cdot \mu, f_{\delta,t+\delta} \cdot \mu) \leq h \| v \|_{L^p(f \cdot \mu)}. \]  

(2.9)

Now let us consider the curve of probability measures \( \{ \mu_{\delta,t} \}_{t \in [0,1]} \) given by

\[ t \mapsto \mu_{\delta,t} := f_{\delta,t} \cdot \mu. \]

By the Benamou–Brenier formula (see [1, Theorem 8.3.1]) there exists a time-dependent Borel vector field \( V_{\delta,t} \), such that

\[ \int |V_{\delta,t}|^p \, d\mu_{\delta,t} \leq \| v \|^p_{L^p(f \cdot \mu)}, \quad \frac{\partial}{\partial t} \mu_{\delta,t} + \text{div}(V_{\delta,t} \cdot \mu_{\delta,t}) = 0. \]

Since \( \Phi_R \) is Lipschitz and \( f_{\delta} \in L^1(\mathbb{R}^d, W^{1,1}[0,1]) \), the mapping

\[ t \mapsto \int \Phi_R(f_{\delta,t}) \, d\mu \]

is absolutely continuous and

\[
\left| \frac{d}{dt} \int \Phi_R(f_{\delta,t}) \, d\mu \right| = \left| \int \Phi'_R(f_{\delta,t}) \frac{\partial}{\partial t} f_{\delta,t} \, d\mu \right| \\
\leq \left| \Phi'_R(f_{\delta,t}) \right| \left| \langle \nabla f_{\delta,t}, \nabla \Phi_R(f_{\delta,t}) \rangle \right| f_{\delta,t} \, d\mu \\
\leq \left[ \int \log \frac{2^p(\alpha-1)(1 + f_{\delta,t}) f_{\delta,t}}{f_{\delta,t}} \, d\mu \right]^{1/2-1/p} \left[ \int \frac{\left| \nabla f_{\delta,t} \right|^2}{f_{\delta,t}} \, d\mu \right]^{1/2} \left[ \int |V_{\delta,t}|^p f_{\delta,t} \, d\mu \right]^{1/p} \\
\leq \left[ \int \log \frac{2^p(\alpha-1)(1 + f_{\delta,t}) f_{\delta,t}}{f_{\delta,t}} \, d\mu \right]^{1/2-1/p} \left[ \int \frac{\left| \nabla f \right|^2}{f} \, d\mu \right]^{1/2} \| v \|^p_{L^p(f \cdot \mu)}. \]

Applying Theorem 2.8 we obtain that there exists a constant \( C > 0 \) depending only on \( p, \theta \) such that

\[ W^p_p(f \cdot \mu, \mu) \leq C \| v \|^p_{L^p(f \cdot \mu)}. \]

Therefore, by Proposition 2.9

\[ \int T_t f \log^{p/2}(1 + T_t f) \, d\mu \leq C t^{-p/2} \| v \|^p_{L^p(f \cdot \mu)}, \quad t \in (0, 1]. \]
where the constant in the right-hand side depends only on \( p \). By Hölder’s inequality for all \( \beta \in (0, p/2] \) we obtain

\[
\int T_t f \log^\beta (1 + T_t f) \, d\mu \leq C t^{-\beta} \|v\|_{L^p(f \cdot \mu)}^{\beta}, \quad t \in (0, 1].
\]

Jensen’s inequality provides the bound

\[
\int f_\delta \log^\beta (1 + f_\delta) \, d\mu \leq C t^{-\beta} \|v\|_{L^p(f \cdot \mu)}^{\beta}, \quad \beta \in (0, p/2], \quad t \in (0, 1].
\]

Due to the assumptions \( \alpha < (p + 2)/4 \), \( 2 \leq p \) we have

\[
\frac{2p}{p - 2}(\alpha - 1) \leq \frac{p}{2}, \quad \alpha + 1 \leq p.
\]

Consequently,

\[
\left[ \int \log^{\frac{2p}{p - 2}(\alpha - 1)} (1 + T_t f) T_t f \, d\mu \right]^{1/2 - 1/p} \leq C t^{-\alpha + 1} \|v\|_{L^p(f \cdot \mu)}^{\alpha - 1}.
\]

Combining the established estimates we obtain

\[
\left| \frac{d}{dt} \int \Phi_R(f_\delta \cdot t) \, d\mu \right| \leq C |t|^{-\alpha + 1} \|v\|_{L^p(f \cdot \mu)}^{\alpha - 1} \|v\|_{L^p(f \cdot \mu)} \leq C |t|^{-\alpha + 1} \|v\|_{L^p(f \cdot \mu)}^{\alpha + 1}.
\]

Since \( \alpha < 2 \), so that \( t^{-\alpha + 1} \) is integrable at zero, we have

\[
\int \Phi_R(f_\delta \cdot t) \, d\mu \leq \left| \int_{[t, 1]} \frac{d}{du} \int \Phi_R(f_\delta \cdot u) \, d\mu \, du \right| + \int \Phi_R(f_\delta \cdot 1) \, d\mu \leq C(\alpha, p, \theta) \left[ 1 + \|v\|_{L^p(f \cdot \mu)}^p \right].
\]

Since the obtained bound does not depend on \( \delta, t, R \), applying Fatou’s lemma now it is easy to complete the proof. \( \square \)

### 3 Integrability of Gradients

In this section we consider the case of the standard Gaussian measure \( \gamma \) on \( \mathbb{R}^d \) and its infinite-dimensional analog and prove that the density \( f \) of the perturbed equation with respect to \( \gamma \) belongs to the Sobolev space \( W^{p, 1}(\gamma) \). It has already been noted in Remark 2.5 that if \( |f v| \in L^p(\gamma) \) with some \( p > 1 \), then \( f \in W^{p, 1}(\gamma) \). So it is necessary to study the integrability of \( f v \). For example, if \( f \) belongs to all \( L^r(\gamma) \), as it holds under the appropriate assumptions in Section 2, and \( |v| \in L^p(\gamma) \) or \( |v| \in L^p(f \cdot \gamma) \) for some \( p > 1 \), then we obtain the inclusion \( f \in W^{p - \varepsilon, 1}(\gamma) \) for each \( \varepsilon > 0 \). In the next theorem we use Orlicz norms to obtain sufficient conditions for the inclusion \( f \in W^{p, 1}(\gamma) \) in terms of integrability of \( |v| \).

Let us recall the Poincaré inequality

\[
\int |f - I(f)|^p \, d\gamma \leq C(p) \int |\nabla f|^p \, d\gamma, \quad f \in W^{p, 1}(\gamma),
\]

where \( I(f) \) is the integral of \( f \).

There is also the \( L^p \)-version of the logarithmic Sobolev inequality

\[
\int |f|^p \log |f| \, d\gamma \leq \frac{p}{2} \|\nabla f\|_p^2 \|f\|_{L^p}^{p - 2} + \|f\|_p^p \log \|f\|_p.
\]
which follows from the standard logarithmic Sobolev inequality

\[ \int |f|^2 \log |f| \, d\gamma \leq \|\nabla f\|_2^2 + \|f\|_2^2 \log \|f\|_2 \]

by considering \(|f|^{p/2}\) in place of \(f\).

It follows from these inequalities that for every \(\varepsilon > 0\) there is a number \(C(\varepsilon, p)\) such that

\[ \|f\|_p \leq \varepsilon \|\nabla f\|_p + C(\varepsilon, p) \|f\|_1 \quad \forall f \in W^{p,1}(\gamma). \]  

(3.1)

Indeed, suppose first that \(f\) has zero integral. If the claim is false, we can find functions \(f_n \in W^{p,1}(\gamma)\) with zero integrals such that \(\|\nabla f_n\|_p = 1\) and

\[ \|f_n\|_p \geq \varepsilon + n \|f_n\|_1. \]

By the Poincaré inequality \(\|f_n\|_p \leq C(p)\). Hence \(\|f_n\|_1 \to 0\). By the logarithmic Sobolev inequality the integrals of the functions \(|f_n|^p \log(1 + |f_n|)\) are uniformly bounded, so \(\|f_n\|_p \to 0\), which is a contradiction. Hence the desired constant exists for functions with zero integrals. In the general case we obtain

\[ \|f - I(f)\|_p \leq \varepsilon \|\nabla f\|_p + C \|f - I(f)\|_1. \]

Hence for \(f\) we obtain a similar bound with \(2C + 1\) in place of \(C\).

Suppose that \(\mu\) is a probability measure on \(\mathbb{R}^d\) satisfying the Fokker–Planck–Kolmogorov equation

\[ L^*_v\mu = 0 \]

with

\[ L_v\varphi(x) = \Delta \varphi(x) + \langle -x + v(x), \nabla \varphi(x) \rangle, \]

where \(v\) is a Borel vector field such that \(|v| \in L^1(\mu)\). Then \(\mu = f \cdot \gamma\). We already know that if \(|v|\) is sufficiently integrable, then \(f\) is integrable to all powers and even better. The condition \(|v| \in L^1(\mu)\) is not enough for the inclusion \(\|\nabla f\| \in L^1(\gamma)\). The next result gives sufficient conditions for integrability of \(|\nabla f|\) to high powers. If \(v\) is bounded, then we can use (3.1) to get the bound

\[ \|f\|_{p,1} \leq C(p) \|v\|_p \leq C(p) \|v\|_\infty (\varepsilon \|\nabla f\|_p + C(\varepsilon, p)), \]

which after taking \(\varepsilon = C(p)^{-1}/2\) leads to

\[ \|f\|_{p,1} \leq C'(p) \|v\|_\infty + C'(p) \quad \text{if} \quad \|v\|_\infty \leq 1. \]

For \(\|v\|_\infty \geq 1\) this gives a nonlinear bound \(\|f\|_{p,1} \leq C(p, \|v\|_\infty)\), but a more constructive estimate is obtained below. For unbounded \(v\) we estimate \(|v|f\) by means of suitable Orlicz norms.

**Theorem 3.1** Let \(\mu = f \cdot \gamma\) be a probability solution to the equation \(L^*_v\mu = 0\) with a vector field \(v\) such that

\[ |v| \in L_{\psi_m}(f \cdot \gamma) \quad \text{for some} \quad m \in (2, +\infty), \]

where \(m = +\infty\) is understood as \(\|v\|_\infty < \infty\). Then \(f \in W^{p,1}(\gamma)\) for every \(p > 1\) and for any such \(p\) there are numbers \(C_1 := C_1(p, m)\) and \(C_2 := C_2(p, m)\), depending only on \(m\) and \(p\), such that

\[ \|\nabla f\|_{L^p(\gamma)} \leq C_1 \|v\|_{L_{\psi_m}(f \cdot \gamma)} \exp(C_2 \|v\|_{L_{\psi_m}(f \cdot \gamma)}^{2/\gamma_m(f \cdot \gamma)}). \]

If \(|v| \in L_{\psi_2}(f \cdot \gamma)\), then \(f \in W^{p,1}(\gamma)\) for every \(p \in (1, p^*)\), where

\[ p^* = (1 - e^{-2\pi \|v\|_{L_{\psi_2}(f \cdot \gamma)}^2})^{-1}. \]
Proof As explained above, the integrability of \( f \) established in the previous section implies the inclusion to the Sobolev classes. We now study bounds on the norms of \(|\nabla f|\). If \( m > 2 \), we note that
\[
\|v \cdot f\|_p \leq \left( \int |v|^{2p} f \, d\gamma \right)^{1/(2p)} \left( \int f^{2p-1} \, d\gamma \right)^{1/(2p)}.
\]

The first term is bounded by \( 2^{(2p)/m} \|v\|_{Lp_m(f \cdot \gamma)} \) and the second term is estimated according to Remark 2.2.

If \( m = 2 \), then \( p \in (1, p^*) \) and for every \( q \in (1, p^* - 1) \) we have
\[
\|v \cdot f\|_p \leq \left( \int |v|^{pq'} f \, d\gamma \right)^{1/(pq')} \left( \int f^{(p-1)q+1} \, d\gamma \right)^{1/(pq)}.
\]

The first term again is bounded by \( 2^{(pq')/2} \|v\|_{L^2(f \cdot \gamma)} \) and the second one by
\[
(1 + e^{2(p^* - 1 - q(p-1))^{-1}})^{1/(pq)}.
\]

Thus, \( \|\nabla f\|_p < \infty \) for each \( p \in (1, p^*) \). \( \square \)

The constants above are independent of the dimension \( d \), so our finite-dimensional estimate extends to the infinite-dimensional case as follows.

The most transparent way of formulating an infinite-dimensional analog is to use the standard Gaussian measure \( \gamma \) on the space \( \mathbb{R}^\infty \) of all real sequences (the countable power of the real line) equipped with its natural Borel \( \sigma \)-algebra generated by the coordinated functions. This measure \( \gamma \) is just the countable power of the standard Gaussian measure on the real line. The Cameron–Martin space of \( \gamma \) is the usual Hilbert space \( l^2 \) with its natural norm \( |h|_H = \left( \sum_{n=1}^\infty h_n^2 \right)^{1/2} \) and the corresponding inner product \( (h, k)_H \).

The Ornstein–Uhlenbeck operator \( L \) is first defined on the space \( \mathcal{F}C \) of cylindrical functions of the form
\[
\varphi(x) = \varphi_0(x_1, \ldots, x_n), \quad \varphi_0 \in C^\infty_b(\mathbb{R}^n)
\]
by the finite-dimensional expression
\[
L\varphi(x) = \sum_{i=1}^n [\partial^2_{x_i} \varphi(x) - x_i \partial_{x_i} \varphi(x)].
\]

The Sobolev norms on such functions are defined by
\[
\|\varphi\|_{p,1} = \left( \int |\varphi|^p \, d\gamma \right)^{1/p} + \left( \int |\nabla_H \varphi|^p \, d\gamma \right)^{1/p},
\]
where \( \nabla_H \varphi = (\partial_{x_i} \varphi) \in H \). The Sobolev space \( W^{p,1}(\gamma) \) is the completion of \( \mathcal{F}C \) with respect to this norm.

There is a smaller convenient subclass in \( \mathcal{F}C \): the set \( \mathcal{F}C_0 \) of functions \( \varphi \) for which the corresponding function \( \varphi_0 \) can be taken with compact support. This set is not a linear subspace, since a function of one variable as a function of two variables has no compact support. Nevertheless, \( W^{p,1}(\gamma) \) equals the completion of this subset with respect to the metric generated by the Sobolev norm.

Given a Borel vector field \( v = (v_i) \) with values in \( H \), we introduce the perturbed operator
\[
L_v \varphi = L \varphi + (v, \nabla_H \varphi)_H, \quad \varphi \in \mathcal{F}C,
\]

\( \square \) Springer
and obtain the corresponding Fokker–Planck–Kolmogorov equation

\[ L^\mu v = 0 \]

with respect to Borel probability measures \( \mu \) on \( \mathbb{R}^\infty \) such that \( v_j \in L^1(\mu) \), understood as the identity

\[ \int L_0 \varphi \, d\mu = 0 \quad \forall \varphi \in \mathcal{F}C_0. \]  \hfill (3.2)

It is also possible to introduce a stronger form of this equation requiring the last identity for all \( \varphi \in \mathcal{F}C \), but for this we need in addition the integrability of \( x_i \) and \( v_j \) against \( \mu \). The integrability of the coordinate functions becomes important even if \( v \) is readily seen that \( \mu \) is bounded. An advantage of dealing with the nonlinear class \( \mathcal{F}C_0 \) of test functions is that the equation is meaningful if \( v_j \) are bounded.

Assuming the integrability of \( v_j \), it is readily seen that \( \mu \) satisfies the FPK precisely when its finite-dimensional projections \( \mu_n \) satisfy the equations on \( \mathbb{R}^n \) with the drifts obtained by perturbations of \( -x \) by the fields \( v^n = (E_n v_1, \ldots, E_n v_n) \), where \( E_n v_j \) is the conditional expectation of \( v_j \) with respect to the projection on \( \mathbb{R}^n \) and the measure \( \mu \). In particular, if \( |v|_H \leq C \), then also \( |v^n| \leq C \), and if \( |v|_H \in L^2(\mu) \), then \( |v^n| \in L^2(\mu_n) \).

From the finite-dimensional result we obtain the following conclusion.

**Corollary 3.2** If \( \mu \) be a probability measure satisfying equation (3.2) with \( |v|_H \in L^1(\mu) \). Then \( \mu = f \cdot \gamma \) and the following assertions are true.

(i) If \( v \in L_{\psi_2}(\mu) \), then

\[ \gamma(f \geq t) \leq e^{2t} - \frac{1}{\sigma_2} \]

and \( f \in L^p(\gamma) \) for all \( p < \frac{1}{\sigma_2} \), where \( \sigma_2 := \exp(-2\pi \|v\|_{L_{\psi_2}(\mu)}) \);

(ii) If \( |v|_H \in L_{\psi_m}(\mu) \) with \( m > 2 \), then

\[ \gamma(f \geq t) \leq e^{2t} \exp(-\sigma_m [\ln t]^{2/m}) \quad \forall t > 1 \]

and \( e^{\varepsilon[\ln max(f,1)]^{2/m}} \in L^1(\gamma) \) for all \( \varepsilon < \sigma_m \), where

\[ \sigma_m := \frac{1 - 2/m}{1 + 2/m} \left(2\pi \|v\|_{L_{\psi_2}(\mu)}(1 - 2/m)\right)^{-2/m}. \]

(iii) If \( |v|_H \) is bounded, then

\[ \gamma(f \geq t) \leq e^{2t} e^{-\sigma_\infty [\ln t]^2} \quad \forall t > 1 \]

and \( e^{\varepsilon[\ln max(f,1)]^2} \in L^1(\mu) \) for all \( \varepsilon < \sigma_\infty \), where \( \sigma_\infty := (2\pi \|v\|_\infty)^{-2} \).

**Proof** The measures \( \mu_n \) are given by densities \( f_n \) with respect to the standard Gaussian measures \( \gamma_n \) on \( \mathbb{R}^n \). The sequence \( \{f_n\} \) is a martingale with respect to the Gaussian measure \( \gamma \) and the sequence of \( \sigma \)-algebras generated by the projections to \( \mathbb{R}^n \). According to [15], this sequence is uniformly integrable, hence converges in \( L^1(\gamma) \) to some function \( f \in L^1(\gamma) \). It is readily seen that \( \mu = f \cdot \gamma \). Convergence also holds in all \( L^p(\gamma) \). Moreover, by Jensen’s inequality for conditional expectations there hold uniform bounds on the Orlicz norms of \( |v_n|_H \), which imply the corresponding bounds for \( f_n \) and consequently for \( f \).

**Corollary 3.3** If \( \mu \) is a probability measure satisfying the equation (3.2) and the hypotheses of Theorem 3.1 are fulfilled with \( |v|_H \) in place of \( |v| \), then the conclusion of that corollary is true.
Proof In the proof of the previous corollary we have $f_n \in W^{p,1}(\gamma_n)$ and there hold the stated bounds on $\nabla f_n$. By the known properties of Sobolev spaces the same bounds hold for $\nabla f$.

It is worth mentioning that it is not necessary to refer to the finite-dimensional case, because the reasoning applied in the previous section remains in force in the infinite-dimensional case once we have the integrability of $f$ used there.

Note again that the inclusion $f \in W^{2,1}(\gamma)$ follows from [28] and [11] and the inclusions $f \in L^p(\gamma)$ with $p$ from some interval follow from [21]. If $|v|_H \in L^2(\mu)$, then $\mu = f \cdot \gamma$ with $\sqrt{f} \in W^{2,1}(\gamma)$ according to [11], and if $|v|_H \in L^1(\mu)$, then $f$ exists, but can fail to be in $W^{1,1}(\gamma)$.

In the case of an abstract centered Radon Gaussian measure $\gamma$ on a locally convex space $X$, having the Cameron–Martin subspace $H$ (the subspace of vectors with finite Cameron–Martin norm $|h|_H = \sup \{l(h) : l \in X^*, \|l\|_{L^2(\gamma)} \leq 1\}$), the same conclusion holds with the following change in the formulation: in place of $v_i$ we consider the functions $l_i(v)$, where $\{l_i\}$ is an orthonormal base in the dual space $X^*$ considered as a subspace in $L^2(\gamma)$ (it is known that such a basis exists, see, e.g., [4]). The proof is the same, but it is not necessary to repeat the proof, using instead the following fact (Tsirelson’s theorem, see [4]): if $\gamma$ is not concentrated on a finite-dimensional subspace, then the mapping $T : x \mapsto \langle l_i(x) \rangle$ from $X$ to $\mathbb{R}^\infty$ takes $\gamma$ to the standard Gaussian measure on $\mathbb{R}^\infty$ and it is a Borel isomorphism between two Borel linear subspaces of full measure, in addition, its restriction is an isometry of the Cameron–Martin subspaces.

It is worth noting that the assertion from Proposition 2.6 about bounded densities for compactly supported perturbations does not extend to the infinite-dimensional case. Indeed, let us take functions $v_n$ on the real line such that $v_n(t) = 2^{-n}$ if $|t| \leq T_n$, where $T_n$ will be large enough. If $|t| > T_n$, we set $v_n(t) = 0$. Let $f_n$ be the density of the solution to the equation with the drift $-t + v_n(t)$ with respect to the standard Gaussian measure with density $\varrho$. Then

$$f_n(t) = \exp \left( \int_0^t v_n(s) \, ds + c_n \right),$$

where $c_n$ is the normalization constant. On $[-T_n, T_n]$ we have

$$f_n(t) = \exp(t2^{-n} + c_n).$$

Take $T_n > 4^n$ so large that the integral of $\exp(2^{-n}t)\varrho(t)$ over $[-T_n, T_n]$ is between $\exp(2^{-2n-1} - 1)$ and $\exp(2^{-2n-1} + 1)$, which is possible, since the integral of $\exp(2^{-n}t)\varrho(t)$ over $\mathbb{R}$ is $\exp(2^{-2n-1})$. Then $c_n \geq -2$, hence $f_n(4^n) \geq 2^n - 2$. Taking $v_n(x) = v_n(x_n)$ on $\mathbb{R}^\infty$, we obtain a vector field with $|v|_H \leq 1$ and compact support in $\mathbb{R}^\infty$, for which the corresponding probability solution has an unbounded density with respect to the standard Gaussian measure (it equals $\prod_{n=1}^\infty f_n(x_n)$). Both measures can be also regarded on the weighted Hilbert space of sequences with $\sum_{n=1}^\infty (2T_n)^2 x_n^2 < \infty$, in which $v$ also has compact support.

Remark 3.4 The same reasoning applies to more general measures on $\mathbb{R}^\infty$ in place of $\gamma$, namely, to any uniformly log-concave measure $\mu$, that is, a probability measure whose projections on the spaces $\mathbb{R}^n$ have densities $e^{-W_n}$ with convex functions $W_n$ such that $D^2W_n \geq \theta$. I with a common constant $\theta > 0$.

Acknowledgements We are very grateful to the anonymous referee for thorough reading and important corrections. This research is supported by the Russian Science Foundation Grant 17-11-01058 at Lomonosov.
Moscow State University (the results in Section 2.1 and Section 3). The second author is a winner of the “Young Russian Mathematics” contest and thanks its sponsors and jury. The work of A.V. Shaposhnikov (the results in Section 2.2) was performed at the Steklov International Mathematical Center and supported by the Ministry of Science and Higher Education of the Russian Federation (agreement no. 075-15-2019-1614).

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

References

1. Ambrosio, L., Gigli, N., Savare, G.: Gradient Flows in Metric Spaces and in the Space of Probability Measures. Lectures in Mathematics ETH Zürich. Basel, Birkhäuser (2008)
2. Bakry, D., Gentil, I., Ledoux, M.: Analysis and Geometry of Markov Diffusion Operators. Springer, Berlin (2013)
3. Bakry, D., Gentil, I., Ledoux, M.: On Harnack inequalities and optimal transportation. Annali Scu. Norm. Super. Pisa Cl. Sci. (5) 14, 705–727 (2015)
4. Bogachev, V.I.: Gaussian Measures. Amer. Math. Soc. Providence, Rhode Island (1998)
5. Bogachev, V.I., Da Prato, G., Röckner, M., Sobol, Z.: Gradient bounds for solutions of elliptic and parabolic equations. In: Da Prato, G., Tubaro, L. (eds.) Stochastic Partial Differential Equations and AppLiCatIoNs – VII, pp. 27–34. Chapman and Hall/CRC, Boca Raton (2006)
6. Bogachev, V.I., Krylov, N.V., Röckner, M.: On regularity of transition probabilities and invariant measures of singular diffusions under minimal conditions. Comm. Partial Differ. Eq. 26, 2037–2080 (2001)
7. Bogachev, V.I., Krylov, N.V., Röckner, M.: Elliptic equations for measures: regularity and global bounds of densities. J. Math. Pures Appl. 85, 743–757 (2006)
8. Bogachev, V.I., Krylov, N.V., Röckner, M.: Elliptic and parabolic equations for measures. Uspehi Matem. Nauk 64(6), 5–116 (2009) (in Russian); English transl. Russian Math. Surveys 64(6), 973–1078 (2009)
9. Bogachev, V.I., Krylov, N.V., Röckner, M., Shaposhnikov, S.V.: Fokker–Planck–Kolmogorov Equations. Amer. Math. Soc. Providence, Rhode Island (2015)
10. Bogachev, V.I., Popova, S.N., Shaposhnikov, S.V.: On Sobolev regularity of solutions to Fokker–Planck–Kolmogorov equations with drifts in $L^1$. Rendiconti Lincei – Matematica e Applicazioni 30(1), 205–221 (2019)
11. Bogachev, V.I., Röckner, M.: Regularity of invariant measures on finite and infinite dimensional spaces and applications. J. Funct. Anal. 133, 168–223 (1995)
12. Bogachev, V.I., Röckner, M., Shaposhnikov, S.V.: Estimates of densities of stationary distributions and transition probabilities of diffusion processes. Teor. Verojatn. i Primen. 52(2), 240–270 (2007). (in Russian); English transl.: Theory Probabl. Appl. 52(2): 209–236 2008
13. Bogachev, V.I., Röckner, M., Wang, F.-Y.: Elliptic equations for invariant measures on finite and infinite dimensional manifolds. J. Math. Pures Appl. 80, 177–221 (2001)
14. Bogachev, V.I., Röckner, M., Zhang, T.S.: Existence of invariant measures for diffusions with singular drifts. Appl. Math. Optim. 41, 87–109 (2000)
15. Bogachev, V.I., Shaposhnikov, A.V., Shaposhnikov, S.V.: Log-sobolev-type inequalities for solutions to stationary Fokker–Planck–Kolmogorov equations. Calc. Var. Partial Differ. Equ. 58(5), Article 176 (2019)
16. Carbonaro, A.: Functional calculus for some perturbations of the Ornstein–Uhlenbeck operator. Math. Z. 262(2), 313–347 (2009)
17. Da Prato, G.: Kolmogorov Equations for Stochastic PDEs. Basel, Birkhäuser (2004)
18. Da Prato, G., Zabczyk, J.: Second Order Partial Differential Equations in Hilbert Spaces. Cambridge University Press, Cambridge (2002)
19. Es-Sarhir, A., Farkas, B.: Invariant measures and regularity properties of perturbed Ornstein–Uhlenbeck semigroups. J. Differ. Eq. 233(1), 87–104 (2007)
20. Fornaro, S., Fusco, N., Metafune, G., Pallara, D.: Sharp upper bounds for the density of some invariant measures. Proc. Roy. Soc. Edinburgh Sect. A 139(6), 1145–1161 (2009)
21. Hino, M.: Existence of invariant measures for diffusion processes on a Wiener space. Osaka. J. Math. 35, 717–734 (1998)
22. Hino, M.: Exponential decay of positivity preserving semigroups on $L^p$. OsakaJ. Math. 37, 603–624 (2000). Correction: ibid. 39: 771(2002)
23. Hino, M., Matsuura, K.: An integrated version of Varadhan’s asymptotics for lower-order perturbations of strong local Dirichlet forms. Potential Anal. 48(3), 257–300 (2018)
24. Manca, L.: Differentiable perturbations of Ornstein–Uhlenbeck operators. Dynam. Systems Appl. 17, 435–443 (2008)
25. Metafune, G., Pallara, D., Rhandi, A.: Global regularity of invariant measures. J. Funct. Anal. 223, 396–424 (2005)
26. Metafune, G., Prüss, J., Schnaubelt, R., Rhandi, A.: $L^p$-regularity for elliptic operators with unbounded coefficients. Adv. Differ. Eq. 10(10), 1131–1164 (2005)
27. Metafune, G., Spina, C.: Elliptic operators with unbounded diffusion coefficients in $L^p$ spaces. Annali Scu. Norm. Super. Pisa Cl. Sci. (5) 11(2), 303–340 (2012)
28. Shigekawa, I.: Existence of invariant measures of diffusions on an abstract Wiener space. Osaka J. Math. 24(1), 37–59 (1987)
29. Shigekawa, I.: Stochastic Analysis. Amer. Math. Soc. Providence, Rhode Island (2004)
30. Shigekawa, I.: A non-symmetric diffusion process on the Wiener space. Math. J. Okayama Univ. 60, 137–153 (2018)
31. Suzuki, K.: Regularity and stability of invariant measures for diffusion processes under synthetic lower Ricci curvature bounds. arXiv:1812.00745v2, to appear in Annali Scu. Norm. Super. Pisa. Cl. Sci (2021)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.