A FAST ALGORITHM TO THE CONJUGACY PROBLEM ON GENERIC BRAIDS

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Abstract. Random braids that are formed by multiplying randomly chosen permutation braids are studied by analyzing their behavior under Garaside’s weighted decomposition and cycling. Using this analysis, we propose a polynomial-time algorithm to the conjugacy problem that is successful for random braids in overwhelming probability. As either the braid index or the number of permutation-braid factors increases, the success probability converges to 1 and so, contrary to the common belief, the distribution of hard instances for the conjugacy problem is getting sparser. We also prove a conjecture by Birman and González-Meneses that any pseudo-Anosov braid can be made to have a special weighted decomposition after taking power and cycling. Moreover we give polynomial upper bounds for the power and the number of iterated cyclings required.

1. Preliminaries and introduction

Recently the braid groups have become a potential source for cryptography, especially, for public-key cryptosystems (see [1, 10] for few). The braid groups have two important features that are useful for cryptography. Each word can be quickly put into a unique canonical form, which provides a fast algorithm not only for the word problem but also for the group operation (see [5, 13, 4]). On the other hand no polynomial-time solution to the conjugacy problem in the braid group is known, which provides many interesting one-way functions for public-key cryptosystems.

Before we discuss the history and the main result, we quickly introduce the terminologies and basic facts about braid groups. Artin who first studied braids systematically in the early 20th century proved that the group $B_n$ of $n$-strand braids can be given by the following presentation:

$$B_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_j \sigma_i \sigma_j = \sigma_i \sigma_j \sigma_i \text{ if } |i - j| > 1 \right\rangle.$$ 

The monoid given by the same presentation is denoted by $B_n^+$ whose elements will be called positive braids.

A partial order $\prec$ on $B_n^+$ can be given by saying $x \prec y$ for $x, y \in B_n^+$ if $x$ is a (left) subword of $y$, that is, $xz = y$ for some $z \in B_n^+$. Given $x, y \in B_n^+$, the (left) join $x \vee y$ of $x$ and $y$ is the minimal element with respect to $\prec$ among all $z$’s satisfying that $x \prec z$ and $y \prec z$, and the (left) meet $x \wedge y$ of $x$ and $y$ is the maximal element with respect to $\prec$ among all $z$’s satisfying that $z \prec x$ and $z \prec y$. Even though “left” is our default choice, we sometimes need the corresponding right versions:

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the partial order \( \prec_R \) of being a right subword, the right join \( \vee_R \), and the right meet \( \wedge_R \). For example, \( x \prec_R y \) if \( zx = y \) for some \( z \in B_n^+ \).

The fundamental braid \( \Delta = (\sigma_1 \cdots \sigma_{n-1})(\sigma_1 \cdots \sigma_{n-2}) \cdots (\sigma_1 \sigma_2)\sigma_1 \) plays an important role in the study of \( B_n \). Since it represents a half twist as a geometric braid, \( x\Delta = \Delta \tau(x) \) for any braid \( x \) where \( \tau \) denotes the involution of \( B_n \) sending \( \sigma_i \) to \( \sigma_{n-i} \). It also has the property that \( \sigma_i \prec \Delta \) for each \( i = 1, \ldots, n-1 \). Since the symmetric group \( \Sigma_n \) is obtained from \( B_n \) by adding the relations \( \sigma_i^2 = 1 \), there is a quotient homomorphism \( q : B_n \rightarrow \Sigma_n \). For \( S_n = \{ x \in B_n^+ \mid x \prec \Delta \} \), the restriction \( q : S_n \rightarrow \Sigma_n \) becomes a 1:1 correspondence and an element in \( S_n \) is called a permutation braid.

A product \( ab \) of a permutation braid \( a \) and a positive braid \( b \) is (left) weighted, written \( ab \), if \( a^* \wedge b = e \) where \( e \) denotes the empty word and \( a^* = a^{-1}\Delta \) is the right complement of \( a \). Each braid \( x \in B_n \) can be uniquely written as

\[
x = \Delta^n x_1 x_2 \cdots x_k
\]

where for each \( i = 1, \ldots, k \), \( x_i \in S_n \setminus \{ e, \Delta \} \) and \( x_i \mid x_{i+1} \). This decomposition is called the (left) weighted form of \( x \) for the first time.\[ 
\begin{align*}
c(x) &= \Delta^n x_2 \cdots x_k \tau^n(x_1) = \tau^n(H(x)^{-1})x \tau^n(H(x)), \\
d(x) &= \Delta^n \tau^n(x_k) x_1 \cdots x_{k-1} = T(x)xT(x)^{-1}.
\end{align*}
\]

A braid \( x \) is cyclically weighted if its weighted form \( x = \Delta^n x_1 x_2 \cdots x_k \) has the property that \( x_k \mid \tau^n(x_1) \). A braid \( x \) is weakly cyclically weighted if \( H(x \tau^n(x_1)) = x_1 = H(x) \). A cyclically weighted braid is clearly weakly cyclically weighted. When \( \ell(x) = 1 \), the two properties are equivalent and they require \( x_k \mid \tau^n(x_1) \).

Let \( \inf_c(x) \) and \( \sup_c(x) \) respectively denote the maximum of infimums and the minimum of supremums of all braids in the conjugacy class \( C(x) \) of \( x \). A typical solution to the conjugacy problem in the braid group \( B_n \) is to generate a finite set uniquely determined by a conjugacy class. Historically, the following four finite subsets of the conjugacy class \( C(x) \) of \( x \in B_n \) have been used in this purpose:

The sum set

\[
SS(x) = \{ y \in C(x) \mid \inf(y) = \inf_c(x) \}
\]

was used by Garside in \[6\] to solve the conjugacy problem in \( B_n \) for the first time.

The super sum set

\[
SSS(x) = \{ y \in C(x) \mid \inf(y) = \inf_c(x) \text{ and } \sup(y) = \sup_c(x) \}
\]

was used by El-Rifai and Morton in \[4\] to improve Garside’s solution. The reduced super sum set

\[
RSSS(x) = \{ y \in C(x) \mid c^M(y) = y = d^N(y) \text{ for some positive integers } M, N \}
\]
was used by Lee in his Ph.D. thesis [12] to give a polynomial-time solution to the conjugacy problem in $B_4$. Finally the ultra summit set

$$USS(x) = \{y \in SSS(x) \mid c^M(y) = y \text{ for some positive integer } M\}$$

was used by Gebhardt in [8] to propose a new algorithm together with experimental data demonstrating the efficiency of his algorithm. Clearly

$$RSSS(x) \subset USS(x) \subset SSS(x) \subset SS(x),$$

and $RSSS(x) = USS(x)$ if $x$ is cyclically weighted. All four invariant sets enjoy the property that if $a^{-1}ya \in P$ and $b^{-1}yb \in P$ for $y \in P$ and $a, b \in S_n$ then $(a \land b)^{-1}y(a \land b) \in P$ where $P$ denotes one of invariant sets. So for $y \in P$, there is a minimal element $a \in S_n$ such that $a^{-1}ya \in P$. Franco and González-Meneses [5] first proved this property for the super summit set and then Gebhardt [8] did it for the ultra summit set. Using this property, they were able to generate an invariant set more efficiently. Unfortunately there is no estimate for the sizes of the invariant sets and so we do not know the complexity of any algorithm based on the generation of an invariant set.

In this paper we survey a fast algorithm to the conjugacy problem on generic braids. In the algorithmic sense, generic braids means random braids that are built by multiplying randomly chosen permutation braids. In the dynamical sense, generic braids means pseudo-Anosov braids. In Section 2, we first give a combinatorial analysis on random braids to find out how quickly the head of a random braid becomes stable as the braid index or the canonical length increases. Then we show that a random braid is cyclically weighted up to cycling in an overwhelming probability so that its $RSSS(x)$ is predictable and small. Using this, we propose a polynomial-time algorithm to the conjugacy problem for random braids. Some of proofs are omitted or brief in this section and full proofs will appear elsewhere.

In Section 3, we show that some power of a pseudo-Anosov braid is always cyclically weighted up to cycling and we also give upper bounds for the necessary exponent and the necessary number of iterated cyclings. Our upper bounds are polynomial in canonical length so that there would be a polynomial-time solution to the conjugacy problem for pseudo-Anosov braids once the size of reduced super summit sets are known to be polynomial in canonical length. Finally we give an example of a cyclically weighted pseudo-Anosov braid whose reduced super summit set is relatively large to show there are still some more work required to give a good estimate of the size of reduced super summit sets.

2. A FAST ALGORITHM TO THE CONJUGACY PROBLEM FOR RANDOM BRAIDS

We assume that the permutation braids in $S_n$ are uniformly distributed so that each permutation braid can be chosen with an equal probability of $1/n!$. We consider random braids that are formed by multiplying $k$ factors, each of which is a permutation braid chosen randomly from $S_n$. Random braids need not be positive and nonpositive random braids are obtained by multiplying a random (negative) power of the fundamental braid $\Delta$. Since $\Delta$ commutes with any braid up to the involution $\tau$, a power of $\Delta$ can be ignored in most of the discussions so that random braids are assumed to be positive. In this section, we study the behavior of random braids with respect to two parameters $k$ and $n$. We reveal some unexpected facts regarding the braid index $n$. 
For integers $1 \leq i < j \leq n$, we say that a positive $n$-braid $x$ begins with an inversion $(i, j)$ if the head $H(x)$ exchanges $i$ and $j$ as a permutation. For permutation braids $x_1, x_2, \ldots, x_k$ chosen randomly from $S_n$, let $D(n, k, (i, j))$ denote the probability that $x_1x_2\cdots x_k$ begins with the inversion $(i, j)$. In particular an inversion $(i, i + 1)$ that a positive braid $x$ begins with is called a descent of $x$ and $D(x)$ denotes the set of all descents of $x$. We will write $D(n, k, (i, i + 1)) := D(n, k, i)$. Then $D(n, k) := \sum_{i=1}^{n-1} D(n, k, i)$ denotes the average number of descents of a random braid $x_1x_2\cdots x_k$. It is easy to see that $D(n, 1, (i, j)) = 1/2$ for $i = 1, \ldots, n - 1$ since the $i$-th and $j$-th strings in $x_1$ cross each other in the probability $1/2$. Thus $D(n, 1) = (n - 1)/2$. We need more delicate combinatorial analysis to obtain an estimate of $D(n, k)$ for $k \geq 2$ that is sharp enough to be useful. In fact we will give an estimate on how fast $d(n, k) := D(n, k) - D(n, k - 1)$, the average contribution to descents of the product $x_1x_2\cdots x_k$ by the last factor $x_k$, approaches to 0 as either $n$ or $k$ increases.

**Lemma 2.1.** We have $$D(n, 2, i) = \frac{1}{2} + \frac{1}{n(n - 1)} \sum_{k=0}^{n-2} (n - k - 1) \sum_{j=0}^{k} \binom{i-1}{j} \binom{n-i-1}{k-j} \frac{(n-2)!}{(j+2)!},$$ and $d(n, 2) \leq \frac{1}{n} \sum_{k=0}^{n-2} \frac{n - k - 1}{k + 2}$. In particular, $d(n, 2)$ is in $O(\log n)$ and is not a decreasing function.

**Proof.** For a braid $x \in B_n$, $\hat{x} : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ denotes the permutation $q(x) \in \Sigma_n$. In order that $\sigma_i$ is a descent of $x = x_1x_2$ contributed by $x_2$, all the following two conditions must hold.

1. $\hat{x}_1(i) < \hat{x}_1(i + 1)$;
2. If $\hat{x}_1^{-1}(a) \leq i$ and $\hat{x}_1^{-1}(b) \geq i + 1$ for $a, b$ with $\hat{x}_1(i) \leq a, b < \hat{x}_1(i + 1)$, then $\hat{x}_2(a) < \hat{x}_2(b)$.

The condition (i) contributes $1/2$. The number of choices for $\hat{x}_1(i)$ and $\hat{x}_1(i + 1)$ can be expressed in two distinct ways: $(\binom{n}{2}) = \sum_{k=0}^{n-2} (n - k - 1)$ where $k = \hat{x}_1(i + 1) - \hat{x}_1(i) - 1$. The number of choices for $k$ integers sent between $\hat{x}_1(i)$ and $\hat{x}_1(i + 1)$ by $\hat{x}_1$ can be also expressed in two ways: $(\binom{n-2}{k}) = \sum_{j=0}^{k} \binom{i-1}{j} \binom{n-i-1}{k-j}$. Then the $k + 2$ integers $\hat{x}_1(i), \hat{x}_1(i + 1), \ldots, \hat{x}_1(i + 1)$ are divided into two groups such that the first group consists of $j + 1$ integers whose preimage under $\hat{x}_1$ is less than or equal to $i$, and the remaining $k - j + 1$ integers have preimages greater than or equal to $i + 1$. The condition (ii) requires that $\hat{x}_2$ permutes the $k + 2$ numbers so that each image of the first group is larger than all images of the second group. The claimed formula for $D(n, 2, i)$ in Lemma should be clear now. The rest of proof is technical and omitted. □

In general, we have the following properties that are extremely useful to give an estimate for an upper bound of $D(n, k, i)$.

**Lemma 2.2.**

1. $D(n, k, i) = D(n, k, n - i)$ and $D(n, k, i) \geq D(n, k, j)$ for $1 \leq i < j \leq \lfloor n/2 \rfloor$.
2. $D(n, k, (i, j)) \leq D(n + i - j + 1, k, i)$ (The equality holds for $k = 1, 2$)

**Proof.** Since $D(x_1 \ldots x_k) = D(x_1, H(x_2 \ldots x_k))$, the argument for random braids of two factors in the previous lemma can similarly be applied to show (1). For a
Lemma 2.5. For randomly chosen permutation $-1$ braids made of $k$ permutation braids, let $x'$ be the $(n+i−j+1)$-braid obtained from $x$ by deleting $j−i−1$ strings from the $(i+1)$-th to $(j+1)$-th. If $x$ begins with an inversion $(i,j)$, then $x'$ must have the descent $\sigma_i$. The converse is also true for $k = 1, 2$. This proves (2).

Proof. Again since $D(x_1\ldots x_k) = D(x_1H(x_2\ldots x_k))$, a typical usage of induction on $k$ together with inequalities in Lemma 2.2 gives a proof. The details are omitted.

As a corollary, we have the following estimate for $d(n,3)$. This is rather surprising because the total number of descents of a random $n$-braid contributed by the third factor (and by all following factors) eventually decreases to 0 as the braid index increases. The maximum occurs at $n = 9$ and this means that 9-braids are the most well-mixed in their weighted forms, for example, when two braids are multiplied.

Corollary 2.4. We have

$$D(n,3,1) \leq \frac{1}{2} + \frac{\ln n}{n-1} + \frac{3(\ln n)^2}{n(n-1)}$$

and so asymptotically $d(n,3) \leq \frac{3(\ln n)^2}{n}$.

Proof. Omitted.

Even though a recursive upper bound is given in Theorem 2.3, it is difficult to describe an upper bound for $D(n,k,i)$ as a neat formula. Instead we use $(n−1)(D(n,k,1)−D(n,k−1,1))$ as an estimate for an upper bound of $d(k,n)$ and present a table for these upper bounds for some choices of $(n,k)$ that are relevant to Gebhardt’s experiment in [8]. The table shows that $d(n,k)$ converge to 0 as $k$ increases and moreover the larger the $n$ becomes the faster it converges to 0.

Since $d(n,k)$ quickly converges to 0 and $D(n,k)$ is much less than $n−1$, it is extremely difficult to produce $\Delta$ by multiplying randomly chosen permutation $n$-braids unless the number of chosen permutation $n$-braids is comparable to $n!$. Thus we assume in the rest of the article that $\inf(x_1\cdots x_k) = 0$. Since $\sup(x) = −\inf(x^{-1})$, we may assume that $\sup(x_1\cdots x_k) = k$ as well.

We now observe some of the properties that random braids enjoy with an overwhelming probabilities. We will use the notation $\Prob[S(x) : x]$ or simply $\Prob[S(x)]$ to denote the probability that the statement $S(x)$ is true for a random choice of $x$.

Lemma 2.5. For randomly chosen permutation $n$-braids $x_1,\ldots, x_k$, let $x = x_1\cdots x_k$. Then the probability that the following equivalent properties hold is greater than $1−\min\{d(n,k),1\}$:

1. For a randomly chosen permutation $n$-braid $a$, $H(x) = H(xa)$;

2. For a randomly chosen permutation $n$-braid $a$, $T(x) = T(ax)$. 

Corollary 2.7. For \( k \geq 3 \) and randomly chosen permutation \( n \)-braids \( x_1, \ldots, x_k \) and a randomly chosen integer \( u \), the probability that \( x = \Delta^u x_1 \cdots x_k \) is weakly cyclically weighted is greater than \( 1 - 4d(n,k) \).

Proof. For \( H(x) \neq H(xa) \), then \( D(x_2 \cdots x_k) \subseteq D(x_2 \cdots x_ka) \).

\[
\text{Prob} \{ D(x_2 \cdots x_k) \neq D(x_2 \cdots x_ka) : (x_2, \ldots, x_k, a) \leq \min \{ d(n,k), 1 \}
\]

because \( d(n,k) \) is the average contribution to descents by the \( k \)-th factor which is \( a \). Thus (1) follows.

Consider \( (ax)^{-1} = x_k^* \tau(x_{k-1}^*) \cdots \tau^{k-1}(x_1^*) \tau^k(a^*) \Delta^{-(k+1)} \). Then

\[
T(ax)^* = H(x_k^* \tau(x_{k-1}^*) \cdots \tau^{k-1}(x_1^*) \tau^k(a^*))
\]

and similarly \( T(x)^* = H(x_1^* \tau(x_{k-1}^*) \cdots \tau^{k-1}(x_1^*)) \). If \( x_1, \ldots, x_k, a \) are random, so are \( x_k^*, \tau(x_{k-1}^*), \ldots, \tau^{k-1}(x_1^*), \tau^k(a^*) \). Thus (2) follows since \( T(ax) = T(x) \) if and only if \( H(x_k^* \tau(x_{k-1}^*) \cdots \tau^{k-1}(x_1^*) \tau^k(a^*)) = H(x_1^* \tau(x_{k-1}^*) \cdots \tau^{k-1}(x_1^*)) \).

Lemma 2.6. For \( k \geq 3 \) and randomly chosen permutation \( n \)-braids \( x_1, \ldots, x_k \) and a randomly chosen integer \( u \), the probability that \( x = \Delta^u x_1 \cdots x_k \) is weakly cyclically weighted is greater than \( 1 - 2d(n,k) \). In particular \( \text{Prob} \{ x \in SS(x) : x \} \geq 1 - 2d(n,k) \).

Proof. For the simplicity of notation, we assume \( u = 0 \). Lemma 2.5 implies

\[
\text{Prob} \{ H(x_1 \cdots x_k) = H(x_1 \cdots x_k x_1 \cdots x_k) : (x_1, \ldots, x_k) \}
\]

\[
\geq 1 - \min \{ d(n,k) + \cdots + d(n,2k), 1 \}
\]

According to our estimate via Theorem 2.3, \( 1 > d(n,k) \geq 2d(n,k+1) \) for \( k \geq 3 \). Thus

\[
1 - \min \{ d(n,k) + \cdots + d(n,2k), 1 \} \geq 1 - 2d(n,k).
\]

If \( x \) is weakly cyclically weighted, \( \inf(x) = \inf(x^i(x)) \) for all \( i > 0 \) and so \( x \in SS(x) \).
Proof. By Lemma 2.6
\[ \text{Prob}[x \notin SSS(x)] = \text{Prob}[x \notin SS(x) \text{ or } x^{-1} \notin SS(x^{-1})] \leq \text{Prob}[x \notin SS(x)] + \text{Prob}[x^{-1} \notin SS(x^{-1})] = 2d(n,k) + 2d(n,k) \]
and so
\[ \text{Prob}[x \in SSS(x)] > 1 - 4d(n,k). \]

\[ \square \]

Lemma 2.8. For \( k \geq 3 \) and randomly chosen permutation \( n \)-braids \( x_1, \ldots, x_k \), let \( x = x_1 \cdots x_k \). Then the probability that the following equivalent properties hold is greater than \( 1 - 2d(n,k) \):

1. For any permutation braid \( a \), \( H(x) = H(xa) \) or \( \tau(H(xa)) \);
2. For any permutation braid \( a \), \( T(x) = T(ax) \).

Proof. The argument is similar to Lemma 2.5 but the difference is that we need to assume that the probability that a pair of strands has a crossing in \( a \) is 1 for (1) and 0 for (2). On the other hand, it was 1/2 in Lemma 2.5 Then this lemma becomes obvious.

\[ \square \]

Theorem 2.9. For \( k \geq 12 \) and randomly chosen permutation \( n \)-braids \( x_1, \ldots, x_k \) and a random integer \( u \), the probability that \( c^j(\Delta^ux_1 \cdots x_k) \) is cyclically weighted for some \( j \leq \lfloor \frac{k}{2} \rfloor \) is greater than \( 1 - 2d(n, \lfloor \frac{k}{2} \rfloor) \).

Proof. For the simplicity, we again assume \( u = 0 \). We also assume that \( x \in SSS(x) \) and this happens with the probability \( \geq 1 - 4d(n,k) \). Let \( y_1 \cdots y_k \) be the weighted form of \( x \). Let \( y_1 \cdots y_k \) be the weighted form of \( x \).

Set \( a = T(x_1 \cdots x_h)^* \land H(x_{h+1} \cdots x_k) \). Then we have
\[ H(y_{h+1} \cdots y_k y_{h+1} \cdots y_h) = H(y_{h+1} \cdots y_k y_{h+1}) = H(a^{-1}x_{h+1} \cdots x_k y_{h+1}) \]
and
\[ y_{h+1} = H(y_{h+1} \cdots y_k) = H(a^{-1}x_{h+1} \cdots x_k). \]
By Lemma 2.8
\[ \text{Prob}[H(x_{h+q+1} \cdots x_k) = H(x_{h+q+1} \cdots x_k y_{h+1})] \geq 1 - 2d(n,k - h - q) \]
and
\[ \text{Prob}[T(a^{-1}x_{h+1} \cdots x_{h+q}) = T(a^{-1}x_{h+1} \cdots x_{h+q})] \geq 1 - 2d(n,q). \]
So
\[ \text{Prob}[H(a^{-1}x_{h+1} \cdots x_k) = H(a^{-1}x_{h+1} \cdots x_k y_{h+1})] \geq 1 - 2d(n,q). \]

Thus
\[ \text{Prob}[H(y_{h+1} \cdots y_k y_{h+1} \cdots y_h) = y_{h+1}] \geq 1 - 2d(n,q). \]

Similarly, we have
\[ \text{Prob}[T(y_{h+1} \cdots y_k y_{h+1} \cdots y_h) = y_h] \geq 1 - 2d(n,q). \]

Since \( y_j \cdot y_{j+1} \), the probability that \( c^j(\Delta^ux_1 \cdots x_k) \) is cyclically weighted is greater than or equal to \( 1 - 2d(n, \lfloor \frac{k}{2} \rfloor) \). We note that \( 1 > 2d(n, \lfloor \frac{k}{2} \rfloor) > 4d(n,k) \) for \( k \geq 12 \) and so our assumption \( x \in SSS(x) \) makes no difference.

\[ \square \]
We now know from Theorem 2.9 that a random braid can be made cyclically weighted by a small number of iterated cyclings with an overwhelming probability. A cyclically weighted braid \( x \) already belongs to \( \text{USS}(x) = \text{RSSS}(x) \) and hence the conjugacy problem can be solved by generating \( \text{USS}(x) \). In the remaining of this section, we will show \( \text{USS}(x) \) is very small for a random braid \( x \), in fact \( |\text{USS}(x)| \leq 2\ell(x) \), with an overwhelming probability.

Let \( P \) denote one of the conjugacy invariant sets \( \text{S}, \text{SS}, \text{USS}, \text{RSSS} \) and let \( y \in P(x) \). If a nontrivial positive \( n \)-braid \( \gamma \) satisfies \( \gamma^{-1}y\gamma \in P(x) \), \( \gamma \) is called a \( P \)-conjugator of \( y \). A \( P \)-conjugator \( \gamma \) of \( y \) is minimal if either \( \gamma < \beta \) or \( \gamma \wedge \beta = e \) for each positive braid \( \beta \) with \( \beta^{-1}y\beta \in P(x) \). In fact it is not hard to see that a minimal \( P \)-conjugator satisfies \( \gamma \prec \tau^{\inf(y)}(H(y)) \) or \( \gamma \prec T(y)^* \) or both (For example, see [5]). A conjugator \( \gamma \) satisfying \( \gamma \prec \tau^{\inf(y)}(H(y)) \) (or \( \gamma \prec T(y)^* \), respectively) will be called a cut-head (or add-tail) conjugator. In particular, if \( y \) is cyclically weighted and \( \gamma \) is its \( P \)-conjugator then it can not be both cut-head and add-tail since \( T(y)[\tau^{\inf(y)}(H(y))] \), that is, \( T(y)^* \wedge \tau^{\inf(y)}(H(y)) = e \). If \( \gamma \) is a \( \text{USS} \)-conjugator of a cyclically weighted braid \( y \), it is also a \( \text{RSSS} \)-conjugator and \( \gamma^{-1}y\gamma \) is cyclically weighted. We note that if \( \gamma \) is an add-tail conjugator of \( y \), then \( \gamma \) is a cut-head conjugator of \( y^{-1} \).

**Theorem 2.10.** For \( k \geq 3 \) and randomly chosen permutation \( n \)-braids \( x_1, \ldots, x_k \) and a randomly chosen integer \( u \), assume that \( y \in \text{USS}(\Delta^u x_1 \cdots x_k) \) is cyclically weighted and \( t \) is a \( \text{USS} \)-minimal cut-head (or add-tail, respectively) conjugator of \( y \). Then the probability that \( t = \tau^u(H(y)) \) (or \( t = T(y) \)) is greater than \( 1 - 2d(n, k-1) \).

**Proof.** If \( T(t^{-1}y) = T(y) \) and \( t \not\preceq \tau^u(H(y)) \), \( t^{-1}yt \) can not be in \( \text{SSS}(\Delta^u x_1 \cdots x_k) \) since \( \sup(t^{-1}yt) = \sup(y) + 1 \). Now the conclusion is immediate from Lemma 2.8. □

**Corollary 2.11.** For \( k \geq 12 \) and randomly chosen permutation \( n \)-braids \( x_1, \ldots, x_k \) and an integer \( u \), let \( x = \Delta^u x_1 \cdots x_k \). Then the probability that \( \text{USS}(x) = O(y) \cup O(\tau(y)) \) for some \( y \in \text{USS}(x) \) is greater than \( 1 - 2d(n, \left\lfloor \frac{k}{4} \right\rfloor) \) where \( O(y) = \{c^i(y) \mid i > 0 \} \) is a finite set called the cycling orbit of \( y \in \text{USS}(x) \).

**Proof.** Immediate from Theorem 2.9 and Theorem 2.10 since \( 1 > 2d(n, \left\lfloor \frac{k}{4} \right\rfloor) > 2d(n, k-1) \) for \( k \geq 12 \). □

Given a random braid \( x \in B_n \), an algorithm to generate \( \text{USS}(x) \) is now extremely simple. In fact, it proceeds as follows:

1. Compute \( y = c^i(x) \) where \( j = \left\lceil \frac{\ell(x)}{2} \right\rceil \).
2. Output either \( \{c^i(y), \tau(c^i(y)) \mid 0 \leq i \leq \ell(x) - 1 \} \) if \( \inf(y) \) is even, or \( \{c^i(y) \mid 0 \leq i \leq 2\ell(x) - 1 \} \) if \( \inf(y) \) is odd.

Since the operation to build a left canonical form has running time \( O(k^2n \log n) \) (see [13]) and this algorithm requires \( k \) cycling operations, the overall running time is \( O(k^3n \log n) \). For a given \( n \)-braid \( x \) of \( k \) random permutations, it generates \( \text{USS}(x) \) successfully with probability greater than \( 1 - 2d(n, \left\lfloor \frac{k}{4} \right\rfloor) \) by Corollary 2.11.

### 3. Conjugacy problem for pseudo-Anosov braids

As far as Garside’s weighted decomposition is concerned, we will show that pseudo-Anosov braids behave similarly to random braids that we discussed. This is rather surprising because the dynamical notion of generic braids are seemingly
far from the combinatorial notion. On the other hand, this may be natural in the sense that a braid chosen randomly as a mapping class should be expected to be pseudo-Anosov. J. González-Meneses discovered a surprising phenomenon that some power of any pseudo-Anosov braid is cyclically weighted up to cyclings and J. Birman announced this phenomenon as a conjecture at the first East Asian School of Knot Theory and Related Topics in 2004. We verify this conjecture and give upper bounds for the exponent and the number of iterated cyclings required. Recently the proposers independently verified their conjecture in [2]. In short, we will prove that for any pseudo-Anosov braid \( x \), there are integers \( 1 \leq M \leq D^3 \), \( 1 \leq N \leq D^4 \ell(x) \) such that \( c^N(x^M) \) is cyclically weighted where \( D = \frac{n(n-1)}{2} \). In [2], we give a polynomial-time algorithm to decide the dynamical type of any given braid by using this special property and these bounds.

**Lemma 3.1.** Let \( x \) be an \( n \)-braid. Then there exists \( y \in C(x) \) and an integer \( 1 \leq M \leq D^3 \) such that \( y^M \) is weakly cyclically weighted.

**Proof.** It was shown in [11] that for any \( n \) braid \( x \), there exists \( y \in C(x) \) and an integer \( M_1 > 0 \) such that \( \inf((y^{M_1})^i) = \inf(y^{M_1}) \) for all \( i \geq 1 \). Let \( z = y^{M_1} \). Since any unexpected \( \Delta \) can not be produced by taking powers of \( z \), there exists \( 0 < M_2 < D \) such that \( H(z^{M_2}) = H(z^k) \) for all \( k \geq M_2 \) as argued in [3]. Thus \( z^{M_2} \) is weakly cyclically weighted.

**Lemma 3.2.** Let \( x \) be an \( n \)-braid. If \( x \) is cyclically weighted and \( \ell(x) \geq 2 \), then every braid in \( RSSS(x) \) is cyclically weighted.

**Proof.** Since \( x \) is cyclically weighted, \( x \in RSSS(x) \). It is enough to show that \( t^{-1}xt \) is cyclically weighted when \( t \) is minimal among conjugators such that \( t^{-1}xt \in RSSS(x) \). Let \( x = \Delta^u x_1 \cdots x_k \) be the weighted form that is cyclically weighted. Then either \( \tau^u(t) \prec x_1 \) or \( t \prec x_k \) since \( t \) is minimal. Suppose that \( \tau^u(t) \prec x_1 \). Then

\[
 t^{-1}xt = \Delta^u(t_{-1}^{-1}x_1t_2) \cdots (t_{-1}^{-1}x_k\tau^u(t_1))
\]

is the weighted form where \( t_1 = \tau^u(t) \) and \( t_i = x_i \wedge x_{i-1}\tau^u(t_{i-1}) \) for \( 1 < i \leq \ell \). If \( t^{-1}xt \) is not cyclically weighted, then \( (t_{-1}^{-1}x_k\tau^u(t_1))(\tau^u(t_{-1}^{-1}x_1t_2)) \) is not weighted. Thus

\[
 c(t^{-1}xt) = \Delta^u(t_2^{-1}x_2t_3) \cdots (t_{-1}^{-1}x_{i-1}t_\ell(t_{-1}^{-1}x_{i-1}\tau^u(t_1))) = \tau^u(t_{-1}^{-1}x_1t_2)
\]

is the weighted form and \( t_1 \prec t_2 \prec x_1 \) because

\[
 \tau^u(t_1)^{-1}x_1^*\tau^u(t_2) \wedge \tau^{-u}(x_1) = \tau^u(t_1)^{-1}x_1^* \neq e
\]

and so \( \tau^u(t_1) \neq x_1^*\tau^u(t_2) \). Since the last \( \ell - 2 \) factors of \( t^{-1}xtT(t^{-1}xt)^{-1} \) are equal to the first \( \ell - 2 \) factors of \( c(t^{-1}xt) \), \( c(t^{-1}xt)^{-1} = t_{-1}^{-1}x_1t_1 \) and \( \tau^u(t_1) t_1 \neq t_1 x_1 \). This implies that \( c^{2i}\tau^u(t^{-1}xt)^{-1} = s_{(i)}x t_{(i)} \) for \( i \geq 0 \), where \( s_{(i)} = t \) and \( s_{(i)} \neq s_{(i+1)} \). Thus \( s_{(i)} = \tau^u(x_1) \) for some \( j > 0 \) and so \( s_{(i)} = \tau^u(x_1) \) for \( i > j \). Since \( t^{-1}xt \in RSSS(x) \) and \( t^{-1}xt = c^{2j}\tau^u(t^{-1}xt) \) for some \( j > j \), \( c(t^{-1}xt) = \tau^u(x_1)^{-1}x \tau^{-u}(x_1) = c(x) \). But this is a contradiction since \( c(x) \) is cyclically weighted. Thus \( t^{-1}xt \) must be cyclically weighted. If \( t \prec x_1 \) then \( \tau^u(t) \prec H(x^{-1}) \). Since \( x^{-1} \in RSSS(x^{-1}) \) and \( x^{-1} \) is cyclically weighted, so is \( t^{-1}xt \) by the above. Thus \( t^{-1}xt \) is cyclically weighted.

\[\square\]
Theorem 3.3 (Birman-González-Meneses Conjecture). Let $x$ be a pseudo-Anosov $n$-braid. Then there is a positive integer $M$ such that every braid in $RSSS(x^M)$ is cyclically weighted.

Proof. By Lemma 3.1 there exists $y \in C(x)$ and an integer $K > 0$ such that $y^K$ is weakly cyclically weighted and $\ell(y^K) \geq 2$. Since $y^K \in SS(x^K)$ and $SS(x^K)$ is finite, $c^j(y^K) = c^{j+N}(y^K)$ for some $j$, $N > 0$. Set $z = c^j(y^K)$ and $c^i(z) = \Delta^u Z_i Q_i$, where $Z_i$ and $Q_i$ are the head and the rest of $c^i(z)$, respectively. Since $z = c^N(z) = \gamma^{-1} z \gamma$, $\gamma$ is an element of the centralizer of $z$, where $\gamma = \tau^{-u}(Z_0 \cdots Z_{N-1})$. It is well-known that the centralizer of $\gamma$ is generated by a pseudo-Anosov $n$-braid $\alpha$ and a periodic $n$-braid $\rho$ such that $\rho^\mu = \Delta^2$ (see [7]). Thus we can write $z = \rho^{\mu_1} \alpha^{\nu_1}$ and $\gamma = \rho^{u_2} \alpha^{v_2}$. Then $z^{v_2} = \rho^{u_1 v_2 - u_2 v_1} \gamma^{v_1}$ and so $z^{v_2 p} = \Delta^{2(u_1 v_2 - u_2 v_1)} \gamma^{v_1 p}$. On the other hand, $Z_i Z_{i+1}$ for $i > 0$ since $y^K$ is weakly cyclically weighted and so is $z$. Since $Z_{N-1}|z$ and $Z_N = Z_0$, $Z_{N-1}|\gamma$. Thus $\gamma$ is cyclically weighted and so $z^{v_2 p}$ is cyclically weighted. Since $\ell(z^{v_2 p}) \geq 2$, every braid in $RSSS(x^{v_2 p K})$ is cyclically weighted by Lemma 3.2.

Lemma 3.4. Suppose that $x$ is a braid such that $x \in SSS(x)$ and $\inf(x^i) = i \inf(x)$, $\sup(x^i) = i \sup(x)$ for $i \geq 1$. If $x^N$ is cyclically weighted for some $N \geq 1$ then $x$ itself is cyclically weighted.

Proof. Under the hypotheses, neither new $\Delta$’s can be formed nor factors can be merged by taking powers. Thus $\tau^{u(N-1)} (H(x)) \leq H(x^N)$ and $T(x) \geq T(x^N)$. Consequently $T(x^i) | \tau^{u(N-1)} (H(x^N))$ implies $T(x) | \tau^{u}(H(x))$. \qed

Corollary 3.5. Let $x$ be a pseudo-Anosov braid in $B_n$. Then $x^M$ is conjugate to a cyclically weighted braid for some $1 \leq M \leq D^2$. Moreover, every braid in $RSSS(x^M)$ is cyclically weighted for some $1 \leq M' \leq 2D^2$.

Proof. In Theorem 3.3 we have already proved the existence of such an $M$ and so we discuss the upper bound for $M$. By [11], there exists a positive integer $M \leq D^2$ and $y \in C(x)$ such that $\inf((y^M)^i) = i \inf(y^M)$ and $\sup((y^M)^i) = i \sup(y^M)$ for $i \geq 1$. Let $z = y^M$. Since $z$ is pseudo-Anosov, $z^M$ is conjugate to a cyclically weighted braid for some $M' \geq 1$. Hence, $z$ is conjugate to a cyclically weighted braid by Lemma 3.4.

On the other hand, every braid in $RSSS(z^2)$ is cyclically weighted by Lemma 3.2 since $\ell(z^2) \geq 2$. Thus $M' \leq 2M$ and so $1 \leq M' \leq 2D^2$. \qed

The next theorem tells us how fast we can obtain a cyclically weighted braid that is conjugate to a power of a given pseudo-Anosov braid. In the theorem, we assume that a braid is conjugate to a cyclically weighted braid instead of being pseudo-Anosov. This assumption is weaker because of Corollary 3.5.

Theorem 3.6. Let $y$ be an $n$-braid such that $\inf((y^i)^l) = i \inf(y)$ and $\sup((y^i)^l) = i \sup(y)$ for all $i \geq 1$ and $y \in SSS(y)$ and let $x = y^{2D}$ If $x$ is conjugate to a cyclically weighted braid, then a cyclically weighted braid must be obtained from $x$ by at most $n! \ell(x)$ iterated cyclings.

Proof. It was proved in Lemma 3.2 that if $RSSS(x)$ contains at least one cyclically weighted braid, then every braid in $RSSS(x)$ is cyclically weighted. Thus iterated cyclings on $x$ must produce a cyclically weighted braid. Let $y = c^N(x)$ be the cyclically weighted braid obtained from $x$ by the minimal number of iterated cyclings. Since $\inf(x)$ is even, we assume $\inf(x)$ for the sake of simplicity.
Let \( y = y_1 y_2 \cdots y_k \) be the weighted form. First one can prove by induction on \( i \geq 1 \) that for all \( 1 \leq i \leq N \)
\[
e^{-i}(x) = a_i y_{1-i} z_i
\]
for some permutation braid \( a_i \) satisfying \( y_{2-k-i} \cdots y_{1-i} y_{1-i} a_i^{-1} \) where \([m]\) denotes the integer between 1 and \( k \) that equals \( m \mod k \). Then \( e^{-i}(x) \) is completely determined by choosing a nontrivial permutation braid \( a_i \) satisfying
\[
a_i \prec_R y_{2-k-i} \cdots y_{1-i} y_{1-i} a_i^{-1}.
\]
For each \( 1 \leq i \leq \ell(x) \), there are at most \( n! \) such choices. Thus \( N \leq n! \ell(x) \). A complete proof appears in \cite{9}.

Given any pseudo-Anosov braid, we now know that we are able to generate a cyclically weighted braid that is conjugate to some power of the given braid in polynomial time. Thus a polynomial-time algorithm to solve the conjugacy problem for pseudo-Anosov braids will be completed as soon as we know how to generate the whole set \( \text{RSSS}(x) \) for a pseudo-Anosov and cyclically-weighted braid \( x \). If \( x \) is a pseudo-Anosov and cyclically-weighted braid obtained by iterated cyclings on a product of randomly chosen permutation braids, then \( \text{RSSS}(x) \) has at most two cycling orbits in an overwhelming probability. But there are plenty of pseudo-Anosov and cyclically weighted braids whose reduced super summit set are not so simple.

Consider the following permutation 7-braids:
\[
x_1 = \sigma_2 \sigma_1 \sigma_3 \sigma_2 \sigma_5, \quad x_2 = \sigma_2 \sigma_5 \sigma_6, \quad x_3 = \sigma_2 \sigma_6 \sigma_5, \quad x_4 = \sigma_2 \sigma_5 \sigma_4 \sigma_6 \sigma_5.
\]
Then \( x = x_1 x_2 x_3 x_4 \) is pseudo-Anosov and cyclically weighted. But \( \sigma_5^{-1} x \sigma_5 \) is again cyclically weighted and forms a new cycling orbit in \( \text{RSSS}(x) \). In fact, the number of cycling orbits in \( \text{RSSS}(x) \) is 10. We say that pseudo-Anosov braids of this kind are quasi-reducible because they are almost reducible and contain most of complications due to reducibility. Consequently we still need more study to estimate the size of \( \text{RSSS}(x) \) for a pseudo-Anosov and cyclically weighted braid \( x \).

**Figure 1.** A pseudo-Anosov braid that is “quasi-reducible”

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