A Logic of Expertise

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Abstract. In this paper we introduce a simple modal logic framework to reason about the expertise of an information source. In the framework, a source is an expert on a proposition \( p \) if they are able to correctly determine the truth value of \( p \) in any possible world. We also consider how information may be false, but true after accounting for the lack of expertise of the source. This is relevant for modelling situations in which information sources make claims beyond their domain of expertise. We use non-standard semantics for the language based on an expertise set with certain closure properties. It turns out there is a close connection between our semantics and S5 epistemic logic, so that expertise can be expressed in terms of knowledge at all possible states. We use this connection to obtain a sound and complete axiomatisation.

Keywords: Expertise · Modal logic · Information

1 Introduction

Many scenarios require handling information from non-expert sources. Such information can be false, even when sources are sincere, when sources make claims regarding topics on which they are not experts. Accordingly, the (lack of) expertise of the source must be taken into account when new information is received.

In this paper we introduce a modal logic formalism for reasoning about the expertise of information sources. Our logic includes operators for expertise (\( E\varphi \)) and soundness (\( S\varphi \)). Intuitively, a source \( s \) has expertise on \( \varphi \) if \( s \) is able to correctly determine the truth value of \( \varphi \) in any possible world. On the other hand, \( \varphi \) is sound if it true up to the limits of the expertise of \( s \). That is, if \( \varphi \) is logically weakened to ignore information beyond the expertise of \( s \), the resulting formula is true. This provides a crucial link between expertise and truthfulness of information, which allows some information to derived from false statements.

The related notion of trust has been well-studied from a logical perspective \([3,7,4,8,6]\). Despite some similarities, trustworthiness on \( \varphi \) is different from expertise on \( \varphi \). Firstly, whether \( s \) is trusted on \( \varphi \) is a property of the truster, not of \( s \). In contrast, we aim to model expertise objectively as a property of \( s \) alone. Secondly, we interpret expertise globally: whether or not \( s \) is an expert on \( \varphi \) does not depend on the truth value of \( \varphi \) in any particular state. In this sense expertise is a counterfactual notion, in that it refers to possible worlds other than the “actual” one. This is not necessarily so for trust; e.g. we may not want to trust the judgement of \( s \) on \( \varphi \) if we know \( \varphi \) to be false in the actual world.
Contribution and paper outline. Our main conceptual contribution is a modal logic framework for reasoning about expertise and soundness of information. This framework is motivated via an example in Section 2, after which the syntax and semantics are formally introduced. Section 3 goes on to establish a connection between our logic and S5 epistemic logic, which provides an alternative interpretation of our notion of expertise in terms of S5 knowledge. Our main technical result is a sound and complete axiomatisation, given in Section 4.

2 Expertise and Soundness

The core notions we aim to model are expertise and soundness of information. We illustrate both with a simplified example.

Example 1. Consider an economist reporting on the effects of COVID-19 vaccine rollout, who states that widespread vaccination will aid economic recovery ($r$), but that the vaccine can cause health problems ($p$). Assume that the economist is an expert on matters to do with the economy ($Er$), so that they only provide correct information on proposition $r$, but is not an expert on matters of health ($¬Ep$). For the sake of the example, suppose economic recovery will indeed follow, but there are no health problems associated with the vaccine. Then while the economist’s report of $r \land p$ is false, it is true on the propositions on which they are an expert. Consequently, if one ignores the parts of the report on which the economist has no expertise, the report becomes true. We say that $r \land p$ is sound, given the expertise of the source on $r$ but not $p$.

2.1 Syntax

We introduce the language of expertise and soundness. Let $\text{Prop}$ be a countable set of propositional variables. The language $\mathcal{L}$ is defined by the following grammar:

$$\varphi ::= p \mid \neg \varphi \mid \varphi \land \varphi \mid E\varphi \mid S\varphi \mid A\varphi$$

for $p \in \text{Prop}$. Note that formulas of $\mathcal{L}$ describe the expertise of a single source. The language can be easily extended to handle multiple sources by adding modalities $E_s$ and $S_s$ for each source $s$, but we do not do so here.

We read $E\varphi$ as “the source has expertise on $\varphi$”, and $S\varphi$ as “$\varphi$ is sound for the source to report”. We include the universal modality $A[5]$ for technical reasons to aid in the axiomatisation of Section 4; $A\varphi$ is to be read as “$\varphi$ holds in all states”. Other Boolean connectives ($\lor$, $\rightarrow$, $\leftrightarrow$) and truth values ($\top$, $\bot$) are introduced as abbreviations. We denote by $\bar{E}$, $\bar{S}$ and $\bar{A}$ the dual operators corresponding to $E$, $S$ and $A$ respectively (e.g. $\bar{E}\varphi$ stands for $\neg E\neg \varphi$).

2.2 Semantics

Formulas of $\mathcal{L}$ are interpreted via non-standard semantics on the basis of an expertise set.
Definition 1. An expertise frame is a pair \( F = (X, P) \), where \( X \) is a set of states and \( P \subseteq 2^X \) is an expertise set satisfying the following properties:

\[(P1) \ X \in P\]
\[(P2) \text{ If } A \in P \text{ then } X \setminus A \in P\]
\[(P3) \text{ If } \{A_i\}_{i \in I} \subseteq P, \text{ then } \bigcap_{i \in I} A_i \in P\]

An expertise model is a triple \( M = (X,P,v) \), where \((X,P)\) is an expertise frame and \( v : \text{Prop} \to 2^X \) is a valuation function.

Intuitively, \( A \in P \) means the source has the expertise to determine whether or not the “actual” state of the world, whatever that may be, lies in \( A \). Implicitly in this interpretation we assume that the expertise of the source does not depend on this “actual” state. This makes our semantics for expertise formulas a special case of the \textit{neighbourhood semantics} \cite{9}, where the neighbourhoods of all states in \( X \) are the same (i.e. \( N(x) \equiv P \)).

\((P1)\) simply says that the source is able to determine that the state lies in \( X \), i.e. that the source has expertise on tautologies. \((P2)\) says that \( P \) is closed under complements. This is a natural requirement, given our intended interpretation of \( P \): if the source can determine whether the actual state \( x \) lies inside \( A \) or not, the same clearly holds for \( X \setminus A \). Note that \((P1)\) and \((P2)\) together imply that \( \emptyset \in P \). Finally, \((P3)\) says that \( P \) is closed under (arbitrary) intersections, which implies expertise is closed under conjunctions. Together with \((P2)\), this implies \( P \) is also closed under (arbitrary) unions, and thus expertise is also closed under disjunctions. We come to the truth conditions for \( L \) formulas with respect to expertise models.

Definition 2. Let \( M = (X,P,v) \) be an expertise model. The satisfaction relation between points \( x \in X \) and formulas \( \varphi \in L \) is defined inductively as follows:

\[ M, x \models p \iff x \in v(p)\]
\[ M, x \models \neg \varphi \iff M, x \not\models \varphi\]
\[ M, x \models \varphi \land \psi \iff M, x \models \varphi \text{ and } M, x \models \psi\]
\[ M, x \models E \varphi \iff \|\varphi\|_M \in P\]
\[ M, x \models S \varphi \iff \text{ for all } A \in P, \|\varphi\|_M \subseteq A \text{ implies } x \in A\]
\[ M, x \models A \varphi \iff \text{ for all } y \in X, M, y \models \varphi\]

where \( \|\varphi\|_M = \{x \in X \mid M, x \models \varphi\} \). We write \( M \models \varphi \) if \( M, x \models \varphi \) for all \( x \in X \), and \( \models \varphi \) if \( M \models \varphi \) for all expertise models \( M \); we say \( \varphi \) is valid in this case.

Write \( \varphi \equiv \psi \) if \( \models \varphi \iff \models \psi \).

The clauses for propositional variables and propositional connectives are standard, and the clause for \( A \varphi \) is straightforward. The clause for \( E \varphi \) follows the intuition highlighted above: \( E \varphi \) is true iff the set of states \( \|\varphi\|_M \) at which \( \varphi \) is true lies in the expertise set. Note that the truth value of \( E \varphi \) does not depend on the state \( x \). For \( S \varphi \) to hold at \( x \in X \), we require that all supersets of \( \|\varphi\|_M \) on which the source is an expert must contain \( x \). That is, any logical weakening of \( \varphi \) is true at \( x \), whenever the source has expertise on the weaker formula. We illustrate the semantics by formalising Example 1, and conclude this section by noting some (in)validities that follow directly from Definition 2.
Example 2. Consider a model $M = (X, P, v)$, where $X = \{a, b, c, d\}$, $v(r) = \{a, c\}$ and $v(p) = \{a, b\}$, and $P = \emptyset, \{a, c\}, \{b, d\}, X$. Then $a$ satisfies $r \land p$, $b$ satisfies $\neg r \land p$, $c$ satisfies $r \land \neg p$, and $d$ satisfies $\neg r \land \neg p$.

In Example 1 we assumed the economist had expertise on $r$. Here we have $\parallel r \parallel_M = \{a, c\} \in P$, so $M, c \models E r$ as expected. We also claimed $r \land p$ was sound when $r$ is true and $p$ is false, i.e. $M, c \models S (r \land p)$. Indeed, $\parallel r \land p \parallel_M = \{a\}$, and the supersets of $\{a\}$ in $P$ are $\{a, c\}$ and $X$. Clearly both contain $c$, so $M, c \models S (r \land p)$. This situation is also depicted graphically in Fig. 1.

![Fig. 1.](image)

In this section we show that, despite the non-standard semantics for $E \varphi$ and $S \varphi$, expertise and soundness can be equivalently defined by the standard relational

1. $E \varphi \equiv E \neg \varphi \equiv \neg E \neg \varphi$
2. Either $M \models E \varphi$ or $M \models \neg E \varphi$
3. $\models E \top \land E \bot \land EE \varphi$
4. $\models (E \varphi \land E \psi) \rightarrow (E \varphi \rightarrow E \psi)$
5. The distribution axiom $E (\varphi \rightarrow \psi) \rightarrow (E \varphi \rightarrow E \psi)$ is not in general valid
6. $\models \varphi \rightarrow S \varphi$
7. If $\models \varphi \rightarrow \psi$ then $\models (S \varphi \land E \psi) \rightarrow \psi$

3 Connection with S5 Epistemic Logic

In this section we show that, despite the non-standard semantics for $E \varphi$ and $S \varphi$, expertise and soundness can be equivalently defined by the standard relational

$1$ For a counterexample, consider $X = \{a, b, c\}$, $P = \emptyset, \{a\}, \{b, c\}, X$ and $v$ such that $v(p) = \{a\}$, $v(q) = \{b\}$. Then $\parallel p \parallel_M = \{a\} \in P$, $\parallel q \parallel_M = \{b\} \notin P$ and $\parallel p \rightarrow q \parallel_M = \{b, c\} \in P$. Then we have $M \models E (p \rightarrow q) \land Ep \land \neg E q$. 
semantics [1] in the language $\mathcal{L}_{\mathbf{KA}}$ with knowledge and universal modalities $K$ and $A$. It will be seen that the accessibility relation for $K$ in the relational model $M^*$ corresponding to an expertise model $M$ is in fact an equivalence relation, so that $M^*$ is an $S5$ model [1, §4.1]. $S5$ represents an “ideal” form of knowledge, which satisfies the KT5 axioms: knowledge is closed under logical consequence ($K: K(\varphi \rightarrow \psi) \rightarrow (K\varphi \rightarrow K\psi)$), all that is known is true ($T: K\varphi \rightarrow \varphi$), and if $\varphi$ is not known, this is itself known ($5: \neg K\varphi \rightarrow K\neg K\varphi$). First, let us define the relational semantics for $\mathcal{L}_{\mathbf{KA}}$.

**Definition 3.** A relational model is a triple $M' = (X, R, v)$, where $X$ is a set of states, $R \subseteq X \times X$ is a binary relation on $X$, and $v: \text{Prop} \rightarrow 2^X$ is a valuation. Given a relational model $M'$, the satisfaction relation between points $x \in X$ and formulas $\varphi \in \mathcal{L}_{\mathbf{KA}}$ is defined inductively by

$$M', x \models K\varphi \iff \text{for all } y \in X, x Ry \implies M', y \models \varphi$$

$$M', x \models A\varphi \iff \text{for all } y \in X, M', y \models \varphi$$

where the clauses for propositional connectives are as in Definition 2.

Say a relational model $M' = (X, R, v)$ is an $S5$ model if $R$ is an equivalence relation. In the context of $S5$, $R$ is an epistemic accessibility relation: $xRy$ means the source considers $y$ as a possible state if the actual state is $x$. The source then ‘knows’ $\varphi$ at $x$ if $\varphi$ is true in every state accessible from $x$.

The following result shows how one can form a unique $S5$ model from an expertise model, and vice versa.

**Lemma 1.** Let $X$ be a set. Let $\mathcal{P}$ denote the set of expertise sets over $X$ – i.e. the set of all $P \subseteq X$ satisfying (P1), (P2) and (P3) – and let $\mathcal{E}$ denote the set of equivalence relations over $X$. Then there is a bijection $P \mapsto R_P$ from $\mathcal{P}$ into $\mathcal{E}$ such that, for all $A \subseteq X$,

$$A \in P \iff A \text{ is a union of equivalence classes of } R_P$$

(1)

**Proof.** Given an expertise set $P \in \mathcal{P}$ and $x \in X$, write

$$A_x = \bigcap \{ A \in P \mid x \in A \}$$

Note that $A_x \in P$ by (P3), so $A_x$ is the smallest set in $P$ containing $x$.\(^2\) Set $\Pi = \{ A_x \mid x \in X \} \subseteq P$. We claim that $\Pi$ is a partition of $X$. It is clear that $\Pi$ covers $X$, since each $x$ lies in $A_x$ by definition. We show that any distinct $A_x, A_y \in \Pi$ are disjoint. Without loss of generality, $A_x \nsubseteq A_y$. Hence $x \notin A_y$ (otherwise $A_y$ appears in the intersection defining $A_x$ and we get $A_x \subseteq A_y$). That is, $x \in A_x \setminus A_y = A_x \cap (X \setminus A_y)$. But this difference lies in $P$ by (P2) and (P3). Since $A_x$ is the smallest set in $P$ containing $x$, we get $A_x \subseteq A_x \setminus A_y$. In particular, $A_x \cap A_y = \emptyset$.

\(^2\) Also note that $X \in P$ by (P1), so there is at least one $A \in P$ containing $x$. 
Let \( R_P \) be the equivalence relation defined by the partition \( \Pi \), i.e. \( x R_P y \) iff \( A_x = A_y \). We show Eq. (1) holds. First suppose \( A \in P \). Then \( A = \bigcup_{x \in A} A_x \); the left-to-right inclusion is clear since \( x \in A_x \) for all \( x \), and the right-to-left inclusion holds since \( x \in A \) implies \( A_x \subseteq A \) for \( A \in P \). Since the \( A_x \) form the equivalence classes of \( R_P \), we are done.

Now suppose \( A \) is a union of equivalence classes of \( R_P \), i.e. \( A = \bigcup_{x \in B} A_x \) for some \( B \subseteq X \). Since each \( A_x \) lies in \( P \) and \( P \) is closed under unions by (P2) and (P3), we have \( A \in P \). Hence Eq. (1) is shown.

It only remains to show that the mapping \( P \mapsto R_P \) is bijective. Injectivity follows easily from Eq. (1), since \( P \) is fully determined by \( R_P \). For surjectivity, take any equivalence relation \( R \in \mathcal{E} \) on \( X \). For \( x \in X \), let \( [x]_R \) denote the equivalence class of \( X \). Let \( P \) consist of all unions of equivalence classes, i.e.

\[
P = \left\{ \bigcup_{x \in B} [x]_R \mid B \subseteq X \right\}
\]

We need to show that \( P \in \mathcal{P} \) — i.e. (P1), (P2) and (P3) hold — and that \( R_P = R \).

For (P1), taking \( B = X \) gives \( X \in P \). For (P2), suppose \( A = \bigcup_{x \in B} [x]_R \in P \).

It is easily verified that \( X \setminus A = \bigcup_{y \in X \setminus A} [y]_R \in P \), so (P2) holds. (P3) follows from (P2) and the fact that \( P \) is closed under unions, which is evident from the definition. Finally, it follows from the definition of \( P \) and Eq. (1) that a set \( A \subseteq X \) is a union of equivalence classes of \( R \) if and only if it is a union of equivalence classes of \( R_P \). Since distinct equivalence classes are disjoint, this implies that the equivalence classes of \( R \) coincide with those of \( R_P \), and \( R = R_P \) as required.

\[\Box\]

On the syntactic side, define a translation \( t : \mathcal{L} \rightarrow \mathcal{L}_{KA} \) inductively by \( t(p) = p \), \( t(\neg \varphi) = \neg t(\varphi) \), \( t(\varphi \land \psi) = t(\varphi) \land t(\psi) \), \( t(A \varphi) = Kt(\varphi) \), and

\[
t(\exists \varphi) = A(t(\varphi) \rightarrow Kt(\varphi)); \quad t(\forall \varphi) = \neg K \neg t(\varphi)
\]

We then have that \( \varphi \in \mathcal{L} \) is true in an expertise model exactly when \( t(\varphi) \) is true in the induced S5 model \( M^* \).

**Theorem 1.** Let \( M = (X, P, v) \) be an expertise model. Then \( M^* = (X, R_P, v) \) is an S5 model, and

\[M, x \models \varphi \iff M^*, x \models t(\varphi)\]

Before the proof, note that since the mapping \( P \mapsto R_P \) is a bijection into the set of equivalence relations on \( X \) (by Lemma 1), any S5 model \( M' = (X, R, v) \) has an expertise counterpart \( M = (X, P, v) \) such that \( M^* = M' \). In this sense, the converse of Theorem 1 also holds.

**Proof (Theorem 1).**

Let \( M = (X, P, v) \) be an expertise model. By Lemma 1, \( R_P \) is an equivalence relation and \( M^* \) is indeed an S5 model. Let \( \Pi \) denote the partition of \( X \) corresponding to \( R_P \), and as in Lemma 1, let \( A_x \in \Pi \) denote the cell of \( \Pi \) containing
x, i.e. the equivalence class of x in R_P. By Eq. (1) in Lemma 1, A ∈ P iff A is a union of cells from Π.

We show the desired semantic correspondence by induction on formulas. The cases for the Boolean connectives and A are straightforward. Suppose the result holds for ϕ and M, x ⊨ Eϕ. Then ||ϕ||_M ∈ P, so ||ϕ||_M = ∪A for some collection A ⊆ Π. Now suppose y ∈ X and M^*, y ⊨ t(ϕ). Suppose yR_Pz. By the inductive hypothesis, ||t(ϕ)||_{M^*} = ||ϕ||_M, so y ∈ ∪A. Since A_y is the unique set in Π containing y, we must have A_y ∈ A. Consequently, yR_Pz implies z ∈ A_y ⊆ ∪A = ||t(ϕ)||_{M^*}. That is, M^*, z ⊨ t(ϕ). This shows M^*, y ⊨ t(ϕ) → Kt(ϕ) for arbitrary y ∈ X, and so M^*, x ⊨ A(t(ϕ) → Kt(ϕ)), i.e. M^*, x ⊨ t(Eϕ).

Conversely, suppose M^*, x ⊨ A(t(ϕ) → Kt(ϕ)). We claim that ||ϕ||_M = ∪_{y ∈ ||ϕ||_M} A_y. The left-to-right inclusion is clear since y ∈ A_y for each y. For the reverse inclusion, let y ∈ ||ϕ||_M and z ∈ A_y. Then yR_Pz. By the inductive hypothesis, M^*, y ⊨ t(ϕ). Since t(ϕ) → Kt(ϕ) holds everywhere in M^* by assumption, we have M^*, z ⊨ t(ϕ). Hence M^*, z ⊨ t(ϕ), so M^*, z ⊨ ϕ and z ∈ ||ϕ||_M. This shows ||ϕ||_M = ∪_{y ∈ ||ϕ||_M} A_y, i.e. ||ϕ||_M is an union of cells of Π. Hence ||ϕ||_M ∈ P and M, x ⊨ Eϕ as required.

Next we take the Sϕ case. We prove both directions by contraposition. First suppose M, x ⊭ Sϕ. Then there is some A ∈ P with ||ϕ||_M ⊆ A and x /∈ A. Suppose xR_Py. Then A_x = A_y. If y ∈ A we would get A_x = A_y ⊆ A, since A_y is the smallest set in P containing y, but this contradicts x /∈ A. Hence y /∈ A. In particular, y /∈ ||ϕ||_M. By the inductive hypothesis, y /∈ ||t(ϕ)||_{M^*}, so M^*, y ⊨ ¬t(ϕ). This shows M^*, x ⊨ K¬t(ϕ), i.e. M^*, x ⊭ ¬K¬t(ϕ) as required.

Finally, suppose M^*, x ⊭ ¬K¬t(ϕ). Take A = ∪_{y ∈ ||ϕ||_M} A_y. Since each A_y is in P and P is closed under unions by (P2) and (P3), we have A ∈ P. Clearly ||ϕ||_M ⊆ A. Suppose for contradiction that M, x ⊨ Sϕ. Then x ∈ A, i.e. there is y ∈ ||ϕ||_M such that x ∈ A_y. Consequently xR_Py, and M^*, x ⊨ K¬t(ϕ) implies M^*, y ⊨ ¬t(ϕ). But this means y /∈ ||t(ϕ)||_{M^*} = ||ϕ||_M contradiction.

Note that, in the case of a propositional formula ϕ, the translation t takes Sϕ to ¬K¬ϕ, and Eϕ to A(ϕ → Kϕ). The semantic correspondence in Theorem 1 therefore shows that the soundness operator S is just the dual of an S5 knowledge operator: ϕ is sound if the source does not know ¬ϕ. Similarly, Eϕ holds iff for all possible states, if ϕ were true then the source would know it. Moreover, the equivalence relation used to interpret K is uniquely derived from the expertise model which interprets E and S, by Lemma 1. This gives a new interpretation of expertise and soundness which refers directly to the source’s epistemic state via the K operator.

Theorem 1 also allows Eϕ be expressed solely in terms of A and S:

\[ Eϕ \equiv A(Sϕ \rightarrow ϕ) \]

i.e. the source has expertise on ϕ iff, in every possible state, ϕ is sound only if it is in fact true. This can be seen by recalling that Eϕ is equivalent to E¬ϕ (by Proposition 1), and noting that t(E¬ϕ) is equivalent to t(A(Sϕ → ϕ)). Similarly,
we can lack of expertise in terms of $S$ and the dual operator $\hat{A}$:

$$-E\varphi \equiv \hat{A}(S\varphi \land \neg \varphi) \equiv \hat{A}(\varphi \land S\neg \varphi)$$

### 4 Axiomatisation

Theorem 1 demonstrates a close semantic link between the logic of expertise and $S5$. Accordingly, we can obtain a sound and complete axiomatisation of the validities of $L$ by adapting any axiomatisation of $S5$ (although some care is required to handle the universal modality). Let $L$ be the extension of the propositional calculus containing the axioms and inference rules shows in Table 1.

| Axiom | Rule |
|-------|------|
| $(K_S)$ | $S\varphi \land \neg S\psi \rightarrow S(\varphi \land \neg \psi)$ |
| $(T_S)$ | $\varphi \rightarrow S\varphi$ |
| $(5_S)$ | $S\neg S\varphi \rightarrow \neg S\varphi$ |
| $(K_A)$ | $A(\varphi \rightarrow \psi) \rightarrow (A\varphi \rightarrow A\psi)$ |
| $(T_A)$ | $A\varphi \rightarrow \varphi$ |
| $(5_A)$ | $\neg A\varphi \rightarrow A\neg A\varphi$ |
| $(ES)$ | $E\varphi \leftrightarrow A(S\varphi \rightarrow \varphi)$ |
| $(Inc)$ | $A\varphi \rightarrow \neg S\neg \varphi$ |
| $(MP)$ | From $\varphi$ and $\varphi \rightarrow \psi$ infer $\psi$ |
| $(Nec_A)$ | From $\varphi$ infer $A\varphi$ |
| $(R_S)$ | From $\varphi \leftrightarrow \psi$ infer $S\varphi \leftrightarrow S\psi$ |

Here $(K_A)$, $(T_A)$ and $(5_A)$ are the standard KT5 axioms for $A$, which characterise $S5$. $(K_S)$, $(T_S)$ and $(5_S)$ are reformulations of the KT5 axioms for the dual operator $\hat{S} = \neg S\neg$; we present them in terms of $S$ rather than $\hat{S}$ to aid readability and intuitive interpretation of the axioms. $(ES)$ captures the interaction between expertise and soundness; the validity of this axiom was already shown as a consequence of Theorem 1. Note that the necessitation rule $(Nec_A)$ for $A$ and $(Inc)$ imply necessitation for $\hat{S}$ by $(MP)$.

**Theorem 2.** $L$ is sound\(^3\) and complete with respect to expertise frames.

Soundness is immediate for most of the axioms and inference rules. We give details only for $(K_S)$ and $(5_S)$.

**Lemma 2.** Axioms $(K_S)$ and $(5_S)$ are valid in all expertise frames.

**Proof.** Let $M = (X, P, v)$ be an expertise model. For $(K_S)$, suppose $M, x \models S\varphi \land \neg S\psi$. Take any $A \in P$ with $\| \varphi \land \neg \psi \|_M \subseteq A$. Then $\| \varphi \|_M \setminus \| \psi \|_M \subseteq A$, so

\(^3\) Soundness of the logic $L$ for expertise frames should not be confused with the notion of soundness inside the language.
\[\|\varphi\|_M \subseteq A \cup \|\psi\|_M.\] Since \(M, x \not\models S \psi\), there is \(B \in P\) with \(\|\psi\|_M \subseteq B\) and \(x \notin B\). Now, \(\|\varphi\|_M \subseteq A \cup \|\psi\|_M \subseteq A \cup B\), and \(A \cup B \in P\) since \(P\) is closed under unions. Since \(M, x \models S \varphi\), any superset of \(\|\varphi\|_M\) in \(P\) must contain \(x\). Hence \(x \in A \cup B\). But \(x \notin B\), so we must have \(x \in A\). This shows \(M, x \models S (\varphi \land \neg \psi)\).

For \((5_S)\), suppose \(M, x \models S \neg S \varphi\). It can be seen from Definition 2 that \(\|\neg S \varphi\|_M = \bigcup \{X \setminus B \mid B \in P, \|\varphi\|_M \subseteq B\}\). It follows from \((P2)\) and \((P3)\) that \(\|\neg S \varphi\|_M \in P\). Consequently, \(\|\neg S \varphi\|_M\) is itself a set in \(P\) containing \(\|\neg S \varphi\|_M\). Since \(M, x \models S \neg S \varphi\), we get \(x \in \|\neg S \varphi\|_M\), i.e. \(M, x \models \neg S \varphi\) as required. \(\Box\)

The completeness proof requires some more machinery, and we use ideas found in [2,5]. Let \(L_{SA}\) denote the fragment of \(\mathcal{L}\) without the \(E\) modality, and let \(L_{SA}\) denote the logic of \(\mathcal{L}\) for \(L_{SA}\) without axiom (ES). For a frame \(F = (X, P)\), let \(R_P\) denote the corresponding equivalence relation on \(X\) from Lemma 1. An augmented expertise frame is obtained by adding to any frame \(F\) an equivalence relation \(R_A\) on \(X\) such that \(R_P \subseteq R_A\) (c.f. [2]). An augmented model \(N\) is an augmented frame equipped with a valuation \(v\). We define a satisfaction relation \(\models_{aug}\) between augmented models and \(L_{SA}\) formulas, where
\[
N, x \models_{aug} A \varphi \iff \text{for all } y \in X, x R_A y \text{ implies } N, y \models_{aug} \varphi
\]
and satisfaction for other formulas is as in Definition 2. That is, \(A \varphi\) is no longer the universal modality, and is instead interpreted via relational semantics.

**Lemma 3.** \(L_{SA}\) is complete for \(L_{SA}\) with respect to augmented frames.

**Proof (sketch).** First note that from \((K_S)\), \((T_S)\), \((5_S)\) and \((R_S)\), one can prove as theorems of \(L_{SA}\) the usual KT5 axioms for the dual operator \(\hat{S}\) — that is, \(\models_{L_{SA}} \hat{S}(\varphi \to \psi) \to (\hat{S} \varphi \to \hat{S} \psi), \models_{L_{SA}} \hat{S} \varphi \to \psi\) and \(\models_{L_{SA}} \neg \hat{S} \varphi \to \hat{S} \neg \varphi\) — where \(\hat{S}\) is an abbreviation for \(\neg S\). As remarked before, we also have the necessitation rule for both \(\hat{S}\) and \(A\) by \((\text{Nec}_{SA})\) and \((\text{Inc})\).

By the standard canonical model construction [1], we obtain the canonical relational model \((X, R_S, R_A, v)\), where \(X\) is the set of all maximally \(L_{SA}\)-consistent subsets \((\text{MCS})\) of \(L_{SA}\), \(R_S\) and \(R_A\) are accessibility relations for \(\hat{S}\) and \(A\) respectively, and \(\varphi \in X\) satisfies \(\varphi\) under the relational semantics iff \(\varphi \in \Delta\), for any \(\varphi \in L_{SA}\) (this fact is known as the truth lemma). Moreover, \(R_S\) and \(R_A\) are equivalence relations by the KT5 axioms for \(\hat{S}\) and \(A\) respectively, and \((\text{Inc})\) implies \(R_S \subseteq R_A\). By Lemma 1, there is an equivalence set \(P\) such that \(R_P = R_S\). Consequently, we obtain an augmented model \(N = (X, P, R_A, v)\). Applying the link between expertise-based and relational semantics established in Theorem 1, one can adapt the truth lemma to show that \(N, \Delta \models_{aug} \varphi\) iff \(\varphi \in \Delta\) for any MCS \(\Delta \in X\) and \(\varphi \in L_{SA}\). Completeness now follows by contraposition. If \(\varphi \in L_{SA}\) is not a theorem of \(L_{SA}\), then \(\{\neg \varphi\}\) is \(L_{SA}\)-consistent, and so there is a MCS \(\Delta\) containing \(\neg \varphi\) by Lindenbaum’s Lemma [1]. Consequently \(\varphi \notin \Delta\), so \(N, \Delta \not\models_{aug} \varphi\) and \(\varphi\) is not valid in augmented frames. \(\Box\)

Completeness of \(L_{SA}\) for (non-augmented) expertise frames follows by considering generated sub-frames of augmented frames.
Lemma 4. $L_{SA}$ is complete for $L_{SA}$ with respect to expertise frames.

Proof (sketch). Suppose $\varphi \in L_{SA}$ is not a theorem of $L_{SA}$. By Lemma 3, there is an augmented model $N = (X, P, RA, v)$ and a state $x \in X$ such that $N, x \not\models_{\text{aug}} \varphi$. Let $X' \subseteq X$ be the equivalence class of $x$ in $RA$. Consider the generated sub-model $N' = (X', P', R'_{A}, v')$, where $P' = \{ A \cap X' \mid A \in P \}$, $R'_{A} = R_{A} \cap (X' \times X')$ and $v'(p) = v(p) \cap X'$. It can be shown that for all $\psi \in L_{SA}$ and $y \in X'$, we have $N, y \models_{\text{aug}} \psi$ if $N', y \models_{\text{aug}} \psi$. Hence $N', x \not\models_{\text{aug}} \varphi$.

Now, note that the relation $R'_{A}$ was obtained by restricting $R_{A}$ to one of its equivalence classes $X'$. It follows that $R'_{A}$ is in fact the universal relation $X' \times X'$ on $X'$. Consequently, $N', y \models_{\text{aug}} A\psi$ if $N', z \models_{\text{aug}} \varphi$ for all $z \in X'$, i.e. $A$ is just the universal modality for $N'$.

Writing $M$ for the non-augmented model obtained from $N'$ by simply dropping the $R'_{A}$ component, we see that $M, y \models \psi$ if $N', y \models_{\text{aug}} \psi$, for all $y \in X'$ and $\psi \in L_{SA}$. In particular, $M, x \not\models \varphi$, so $\varphi$ is not valid in all expertise frames. \qed

The completeness of $L$ for the validities of the whole language $L$ now follows. Indeed, let $g : L \rightarrow L_{SA}$ be the natural embedding of $L$ in $L_{SA}$, where $g(E\varphi) = A(S(g(\varphi)) \rightarrow g(\varphi))$. In light of earlier remarks and axiom (ES), we have both $\varphi \equiv g(\varphi)$ and $\models L \varphi \iff g(\varphi)$ for all $\varphi \in L$. Consequently, $\models L \varphi$ implies $\models L g(\varphi)$ and thus $\models L_{SA} g(\varphi)$ by Lemma 4. By Lemma 4, there is $N, y \models_{\text{aug}} S\psi$ if $N', z \models_{\text{aug}} \varphi$ for all $z \in X'$, i.e. $A$ is just the universal modality for $N'$.

Writing $M$ for the non-augmented model obtained from $N'$ by simply dropping the $R'_{A}$ component, we see that $M, y \models \psi$ if $N', y \models_{\text{aug}} \psi$, for all $y \in X'$ and $\psi \in L_{SA}$. In particular, $M, x \not\models \varphi$, so $\varphi$ is not valid in all expertise frames. \qed

5 Conclusion

This paper introduced a simple modal language to reason about the expertise of an information source. We used the notion of “soundness” – when information is true after ignoring parts on which the source has no expertise – to establish a connection with S5 epistemic logic. This provided alternative interpretation of expertise, and led to a sound and complete axiomatisation.

There are many possible directions for future work. For instance, it may be unrealistic to expect the expertise set of a source is fully known up front. Methods for estimating the expertise, e.g. based on past reports [4], could be developed to reason about expertise approximately in practical settings. The “binary” notion of expertise we consider may also be unrealistic: either $E\varphi$ holds or $\neg E\varphi$ holds. Enriching the language and semantics to handle graded or probabilistic levels is a natural generalisation which would allow a more nuanced discussion of expertise.

One could also investigate the relation between expertise and trust. For example, can the trustworthiness of a source on $\varphi$ be derived from expertise on

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4 This is clear by induction on formulas, except for the case $S\psi$. Here we use the fact that $R_{P} \subseteq R_{A}$ to show $X' = \bigcup_{z \in X} [z]_{R_{P}}$ – where $[z]_{R_{P}}$ is the equivalence class of $z$ in $R_{P}$ – which implies $X' \in P$. Using the inductive hypothesis it is then straightforward to show that $N, y \models_{\text{aug}} S\psi$ iff $N', y \models_{\text{aug}} S\psi$.

5 ... and $g(p) = p, \neg g(\varphi) = \neg g(\varphi), g(\varphi \land \psi) = g(\varphi) \land g(\psi), g(S\varphi) = Sg(\varphi)$ and $g(A\varphi) = Ag(\varphi)$. 

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ϕ? The language $L$ could be extended with a trust operator $T$ to model this formally in future work.

Also note that in this paper we only consider static expertise. In reality, expertise may change over time as new evidence becomes available and as the epistemic state of the information source evolves. One could introduce dynamic operators, as is done in Dynamic Epistemic Logic, to model this change in expertise in response to evidence and other epistemic events. When it comes to the interaction between expertise and evidence specifically, evidence logics [11,10] may be highly relevant. These logics use neighbourhood semantics to interpret the evidence modalities, which is technically (and perhaps also conceptually) similar to our semantics for the expertise modality. We save the detailed analysis and comparison for future work.

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