A simplified model for monopole catalysis of nucleon decay

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Abstract

We present a simple model where a nucleon is treated as the Skyrmion, both monopole and Skyrmion are smooth solutions of the classical field equations, the baryon number non-conservation is explicitly built in and is due to the classical analogue of the triangle anomaly, and there is no necessity to consider leptons. We show by numerical analysis that there are no static classical solutions with non-zero baryon number on top of the monopole, i.e., that the monopole catalysis of Skyrmion decay indeed proceeds classically.
1 Introduction

An interesting feature of magnetic monopoles in gauge theories is that they can catalyse nucleon decay at strong interaction rate [1, 2]. The actual computation of the monopole catalysis cross section remains, however, an open problem. This computation would involve long-distance physics of two different kinds — quark-monopole interactions and QCD phenomena — and also short-distance physics responsible for baryon number non-conservation. Most of the relevant effects are non-perturbative, and this makes even a semi-qualitative calculation difficult.

An approach to this problem was suggested in Ref. [3]. It was proposed to invoke the Skyrme model of a nucleon [4] which indeed works reasonably well in describing low energy properties of baryons [5, 6]. In the Skyrme model, the nucleon is essentially a classical field theory object, a soliton. Given that the ’t Hooft–Polyakov monopole is also a soliton, the monopole-nucleon interactions are then possible to describe at the level of classical field theory.

In the proposal of Ref. [3], the monopole is treated as point-like. The non-conservation of baryon number comes in through the boundary condition imposed on the Skyrme field at the monopole centre, i.e., at the singularity. It is not entirely obvious whether concrete mechanisms of baryon number non-conservation indeed produce, in the limit of vanishing monopole size, the conjectured boundary condition. Also, dealing with the singularity and the boundary condition imposed at the singularity may be inconvenient for numerical computations. One more complication inherent in the proposal of Ref. [3] stems from the fact that the spherically symmetric Skyrme field in the monopole background corresponds to an electrically charged object (“proton”), so one has to explicitly introduce leptons to make baryon number violation consistent with electric charge conservation.

In this paper we present a simplified model where both monopole and Skyrmion are smooth solutions of the classical field equations, the baryon number non-conservation is explicitly built in and is due to the classical analogue [7, 8] of the triangle anomaly, and there is no necessity to consider leptons. The price to pay is that the Skyrme field in the presence of the monopole does not carry net electric charge, so our model mimicks the monopole catalysed decay of a neutron, rather than a proton.

We shall concentrate on two aspects of our model. First, we study whether or not the baryon number non-conservation proceeds without suppression at the classical level. For this we shall have to consider a region near the monopole core. Of course, it is not realistic to use the non-linear sigma model for describing physics near the core, but this step is inevitable in the classical calculation of monopole catalysis. By carrying out numerical analysis, we show that there are no states with non-zero baryon number on top of the ’t Hooft–Polyakov monopole, irrespective of the size of the monopole core, i.e., monopole catalysis of nucleon decay indeed takes place.

Second, we find that the regime of “point-like” monopole sets in when the two scales, responsible for the sizes and masses of the monopole and Skyrmion, respectively, are not yet widely different: our results are stable when the ratio of monopole
and Skyrmion sizes is below 0.03, and the mass ratio is below 1. This is a good sign for future numerical analysis of the Skyrmion decay in Skyrmion-monopole collisions. Hence, our model may be a reasonable starting point for a semi-realistic calculation of the monopole catalysis cross section.

This paper is organized as follows. We present our model in Section 2. To set the stage, we recapitulate the properties of the monopole and the Skyrmion in Section 3. Section 4 contains our main results, both analytical and numerical. We present concluding remarks in Section 5.

2 The model

To motivate our model, we recall first that the simplest theory incorporating Skyrmions has $SU(2)_L \times SU(2)_R$ global symmetry and contains the sigma field $U(x)$ taking values in $SU(2)$. We wish to introduce baryon number non-conservation through the classical anomaly in the baryonic current. The baryon number will be anomalous if $SU(2)_L$ or $SU(2)_R$ or both are gauged. The model should also possess magnetic monopoles; if we do not extend the symmetry group, the monopoles are to be associated with the gauged (part of) $SU(2)_L \times SU(2)_R$. Gauging $SU(2)_L$ alone is not sufficient: in that case the Skyrme field would break the gauge group completely, no electromagnetic $U(1)$ would be left unbroken, and there would be no magnetic monopoles. So, we come to fully gauged $SU(2)_L \times SU(2)_R$ with gauge potentials

$$SU(2)_L : \quad A_\mu$$
$$SU(2)_R : \quad B_\mu$$

Realistic monopoles have very small sizes, so we have to break the non-Abelian symmetry down to an Abelian subgroup at a high energy scale. Thus, we introduce two Higgs triplets $\Phi_A$ and $\Phi_B$ in $(3,1)$ and $(1,3)$ representations, respectively. These break $SU(2)_L \times SU(2)_R$ down to $U(1)_L \times U(1)_R$. The electromagnetic group is the vectorial subgroup of this $U(1)_L \times U(1)_R$, so we have to break axial $U(1)$ at some point. For simplicity, we consider the minimal option, i.e., we do not introduce extra Higgs fields. Then the axial $U(1)$ is broken by the Skyrme field itself. The existence of a relatively long-ranged axial gauge field is an unrealistic feature of our model; however, in this paper we study the dynamics near the monopole core, so it will not be important whether or not the axial $U(1)$ is truly long ranged. It is straightforward to extend our model by introducing yet another Higgs field which would break the axial $U(1)$, but not the electromagnetic subgroup.

To summarise, our model has $SU(2)_L \times SU(2)_R$ gauge symmetry, with the Skyrme field and two Higgs fields in the following representations

$$U \in SU(2), \quad U : \quad (2, \bar{2})$$
$$\Phi_A : \quad (3, 1)$$
$$\Phi_B : \quad (1, 3)$$

(1)
To simplify the calculations, we take the gauge couplings, Higgs self-couplings and vacuum expectation values of the Higgs fields the same in the left and right sectors,

\[ g_A = g_B \equiv g, \quad \lambda_A = \lambda_B \equiv \lambda, \quad v_A = v_B \equiv v \]

The action for this model is

\[
S = \int d^4x \left[ -\frac{1}{2g^2} \text{Tr}(F_{\mu\nu}^2) - \frac{1}{2g^2} \text{Tr}(G_{\mu\nu}^2) \right] \\
- \int d^4x \left[ \frac{1}{2} \text{Tr}(D_\mu \Phi_A)^2 + \frac{1}{2} \text{Tr}(D_\mu \Phi_B)^2 \right] \\
- \int d^4x \left[ \frac{\lambda}{8} \left( \frac{1}{2} \text{Tr}(\Phi_A^2) + v^2 \right)^2 + \frac{\lambda}{8} \left( \frac{1}{2} \text{Tr}(\Phi_B^2) + v^2 \right)^2 \right] \\
+ \int d^4x \left[ -\frac{F_\pi^2}{16} \text{Tr}(U^\dagger D_\mu U)^2 + \frac{1}{32e^2} \text{Tr}([U^\dagger D_\mu U, U^\dagger D_\nu U]^2) \right] \\
+ \Gamma_{WZW} \tag{2}
\]

where \( F_{\mu\nu} \) and \( G_{\mu\nu} \) are the field strengths of \( A_\mu \) and \( B_\mu \), the fields \( \Phi_A \) and \( \Phi_B \) are \( 2 \times 2 \) matrices from the algebras of \( SU(2)_L \) and \( SU(2)_R \), respectively,

\[
D_\mu U = \partial_\mu U + A_\mu U - B_\mu \\
D_\mu \Phi_A = \partial_\mu \Phi_A + [A_\mu, \Phi_A] \\
D_\mu \Phi_B = \partial_\mu \Phi_B + [B_\mu, \Phi_B] \tag{3}
\]

and \( F_\pi \) and \( e \) are the pion decay constant and the Skyrme constant, respectively. The last term in eq. (2) is introduced to reflect anomalies \([7]\). Unlike in Ref. \([3]\), this term has almost no effect in our model. In most of this paper we neglect this term, and discuss its effect towards the end.

In complete analogy to Refs. \([7, 8]\), the gauge-invariant baryonic current is

\[
j_\mu = \frac{1}{24\pi^2} \varepsilon_{\mu\nu\rho\sigma} \text{Tr}(U^\dagger D_\nu U D_\rho U^\dagger D_\sigma U) \\
+ \frac{1}{16\pi^2} \varepsilon_{\mu\nu\rho\sigma} \text{Tr}(F_{\rho\sigma} D_\nu U U^\dagger + G_{\rho\sigma} U^\dagger D_\nu U) \tag{4}
\]

In the absence of the gauge fields, \( A_\mu = B_\mu = 0 \), this current reduces to the topological current,

\[
k_\mu = \frac{1}{24\pi^2} \varepsilon_{\mu\nu\rho\sigma} \text{Tr}(U^\dagger \partial_\nu U \partial_\rho U^\dagger \partial_\sigma U)
\]

while in the presence of gauge fields it has an anomaly,

\[
\partial_\mu j_\mu = \frac{1}{16\pi^2} \text{Tr} F_{\mu\nu} F_{\mu\nu} - G_{\mu\nu} G_{\mu\nu}
\]

Thus, topologically non-trivial gauge fields lead to the non-conservation of baryon number; in particular, they can unwind the Skyrmion (cf. Refs. \([3, 4, 10]\)).
3 Monopole and Skyrmion

If it were not for the Skyrme field $U$, the left and right sectors of this model would decouple, the model would reduce to two identical copies of the Georgi–Glashow $SU(2)$ model and there would exist two types of ’t Hooft–Polyakov monopoles. In the presence of the Skyrme field, the axial $U(1)$ is broken, and there exists only one type of monopole. The monopole solution with zero baryon number has $U(x) = 1$ everywhere in space, whereas $A_i(x) = B_i(x)$, $\Phi_A(x) = \Phi_B(x)$ are precisely the monopole fields of the Georgi–Glashow model. The monopole size is determined by the mass of the vector bosons; at $v \gg F_\pi$ one has $m_V = 2gv$, so that

$$ r_{\text{mon}} = \frac{1}{2gv} \tag{5} $$

At large $v$, the monopole mass is twice the mass of the $SU(2)$ monopole, i.e.,

$$ M_{\text{mon}} = 4\pi D_{\text{mon}} \frac{v}{g} \tag{6} $$

where the numerical constant $D_{\text{mon}}$ depends slightly on $\lambda/g^2$ and at $\lambda/g^2 = 0.5$ is equal to $[11]$

$$ D_{\text{mon}}(\lambda/g^2 = 0.5) \approx 2.4 \tag{7} $$

The estimates (5), and (6) and (7) will serve as reference points for our study of the monopole–Skyrmion system.

In the regime of weak gauge coupling, the model possesses the Skyrmion. Once the gauge fields are ignored, this is the standard Skyrmion of the $SU(2)$ sigma model. Its radius, as defined in Ref. [5], is

$$ r_{\text{Sk}} = \frac{2.1}{eF_\pi} \tag{8} $$

and its mass is equal to

$$ M_{\text{Sk}} = 4\pi D_{\text{Sk}} \cdot \frac{F_\pi}{e}, \quad D_{\text{Sk}} \approx 2.9 \tag{9} $$

To mimic the situation existing in Grand Unified Theories, in what follows we shall mostly consider the range of parameters in which the ratios $r_{\text{mon}}/r_{\text{Sk}}$ and $M_{\text{Sk}}/M_{\text{mon}}$ are small.

4 Disappearing baryon number

Our main purpose is to consider configurations with non-vanishing baryon number sitting on top of the monopole. We shall see that the baryon number need not be integer, and that there are no static classical solutions with non-zero baryon number. The strategy to deal with such a situation is standard [12, 10, 13]: we shall impose a constraint ensuring that the baryon number takes on a non-vanishing value, find a static classical solution of the constrained system and study the behaviour of energy as a function of the baryon number.
4.1 Spherically symmetric Ansatz

We shall seek spherically symmetric solutions of the constrained system. To write down the Ansatz, we note that the action (2) is invariant under the spatial reflection, \( x^0 \to x^0, \ x \to -x \) supplemented by the interchange of the left and right \( SU(2) \), i.e.,

\[
\begin{align*}
A_0 & \leftrightarrow B_0, \quad A_i \to -B_i, \quad B_i \to -A_i \\
\Phi_A & \to -\Phi_B, \quad \Phi_B \to -\Phi_A \\
U & \to U^\dagger
\end{align*}
\]

In the case of static fields with \( A_0 = B_0 = 0 \), the most general spherically symmetric Ansatz consistent with this discrete symmetry \(^1\) is

\[
\begin{align*}
A_i & = -\frac{i}{2} \left[ \left( \frac{a_1(r) - 1}{r} \right) \varepsilon_{ijk} \sigma_j \hat{x}_k + \left( \frac{a_2(r)}{r} \right) (\sigma_i - \hat{x}_i \hat{x} \cdot \sigma) + \left( \frac{a_3(r)}{r} \right) \hat{x}_i \hat{x} \cdot \sigma \right] \\
B_i & = -\frac{i}{2} \left[ \left( \frac{a_1(r) - 1}{r} \right) \varepsilon_{ijk} \sigma_j \hat{x}_k - \left( \frac{a_2(r)}{r} \right) (\sigma_i - \hat{x}_i \hat{x} \cdot \sigma) - \left( \frac{a_3(r)}{r} \right) \hat{x}_i \hat{x} \cdot \sigma \right] \\
\Phi_A & = \Phi_B = ivh(r) \hat{x} \cdot \sigma \\
U & = \cos f(r) + i \hat{x} \cdot \sigma \sin f(r)
\end{align*}
\]

where \( \hat{x} \) is the unit radius-vector. With this Ansatz, regularity of the Skyrme field at the origin requires that \( f(0) \) is an integer multiple of \( \pi \); without loss of generality we set

\[ f(0) = \pi \]

Other conditions, ensuring that the fields are regular at the origin, are

\[ a_1(0) = 1, \quad a_2(0) = a_3(0) = 0, \quad h(0) = 0 \]

It is convenient to introduce the dimensionless coordinate,

\[ \rho = gvr \]

With this coordinate, the distances are measured essentially in units of the size of the monopole core. The static energy functional is then written as follows (neglecting the WZW term in eq. (2))

\[
H = \frac{4\pi v}{g} \int d\rho \ E(\rho)
\]

\[
E = \left[ \left( a'_1 + \frac{a_2 a_3}{\rho} \right)^2 + \left( a'_2 - \frac{a_1 a_3}{\rho} \right)^2 + \frac{1}{2\rho^2} (a_1^2 + a_2^2 - 1)^2 \right]
\]

\[
+ 2 \left[ \rho^2 (h')^2 + 2(a_1^2 + a_2^2) h^2 + \frac{\lambda}{4g^2} \rho^2 (h^2 - 1)^2 \right]
\]

\[
+ \kappa_1 (X^2 + 2Y^2) + \frac{4\kappa_2}{\rho^2} Y^2 (2X^2 + Y^2)
\]

\(^1\) In our numerical study we used the most general spherically symmetric Ansatz without imposing the symmetry \([11]\). We have found, however, that the solutions always have this symmetry.
where the prime denotes the derivative with respect to $\rho$,
\[
X = \rho f' - a_3 \\
Y = a_1 \sin f - a_2 \cos f
\]
and we introduced the dimensionless parameters
\[
\kappa_1 = \frac{F_x^2}{8v^2} \\
\kappa_2 = \frac{g^2}{64e^2}
\]
(14)

It is worth noting that in terms of these parameters, the ratios of the sizes and masses of the monopole and Skyrmion, as given by eqs. (5) – (9), are expressed as
\[
\frac{r_{\text{mon}}}{r_{\text{Sk}}} = 0.084 \cdot \sqrt{\frac{\kappa_1}{\kappa_2}} \\
\frac{M_{\text{Sk}}}{M_{\text{mon}}} = 27 \cdot \sqrt{\kappa_1 \kappa_2}
\]
(15)
(16)

where we take $\lambda = 0.5g^2$ for definiteness. These ratios must be small in the limit of point-like monopole.

The energy functional (12) is still invariant under radial gauge transformations which shift $f(\rho)$ and $a_3(\rho)$,
\[
f(\rho) \rightarrow f(\rho) + \alpha(\rho) \\
a_3(\rho) \rightarrow a_3(\rho) + \rho \partial_\rho \alpha(\rho)
\]
(17)

and rotate $(a_1, a_2)$. This gauge symmetry is the remnant of the axial $U(1)$ (we have fixed the vectorial $U(1)$ by imposing the symmetry (10)). We shall fix the remaining gauge freedom shortly.

It follows immediately from eqs. (12) and (13) that in the absence of the monopole, i.e., when $a_1(\infty) = 1$, $a_2(\infty) = 0$, the convergence of the term proportional to $Y^2$ in the energy integral requires that $f(\infty)$ is an integer multiple of $\pi$. This means that the winding number of the Skyrme field is integer, and hence is conserved. On the other hand, in the presence of the monopole, one has $a_1(\infty) = a_2(\infty) = 0$, and the convergence of energy does not impose any constraints on $f(\infty)$. In the presence of the monopole, the winding number of the Skyrme field is arbitrary.

For spherically symmetric fields, the gauge-invariant baryon number takes the form
\[
B = \int_0^\infty d\rho \ b(\rho)
\]
\[
b(\rho) = \frac{1}{\pi} \frac{d}{d\rho} \left[ - f + \frac{1}{2}(a_1^2 - a_2^2) \sin 2f - a_1 a_2 \cos 2f \right] \\
+ \frac{1}{\pi} \left[ (a_1 a_2' - a_1' a_2) - \frac{a_2}{\rho} (a_1^2 + a_2^2 - 1) \right]
\]
(18)

Like the winding number of the Skyrme field, the baryon number too need not be integer.
4.2 The constrained system

To consider configurations with non-vanishing baryon number, it would seem natural to impose a constraint involving the baryon number, e.g.,

$$\int_0^\infty d\rho \ b(\rho) = \text{fixed}$$

However, we have chosen a slightly different constraint which simplifies the equations. We note first that one of the Euler–Lagrange equations following from eq. (12) does not contain the second derivatives. This equation is

$$\rho \delta H \delta a_3 = 0 \quad (19)$$

In fact, this equation is not completely independent: its derivative with respect to \( \rho \) is a consequence of the other Euler–Lagrange equations (this is analogous to the time derivative of Gauss’ law). If \( \delta H/\delta a_3 \) is zero at one value of \( \rho \), then it is zero everywhere, provided the second-order Euler–Lagrange equations are satisfied. The idea is to impose a constraint which would modify the resulting equations in such a way that the left hand side of eq. (19) be allowed to be a non-vanishing constant, while the second order equations remain intact. The suitable form of this constraint is

$$-\frac{1}{\pi} \int_0^\infty d\rho \left( f' - \frac{a_3}{\rho} \right) = \text{fixed} \quad (20)$$

Indeed, upon adding this constraint with the Lagrange multiplier \( \Lambda \) to the Hamiltonian (12), one finds that the only change in the resulting equations is that eq. (19) now becomes

$$\rho \delta H \delta a_3 + \frac{\Lambda}{\pi} = 0 \quad (21)$$

The procedure is then to disregard the latter equation altogether, solve the original second order Euler–Lagrange equations and find \( \Lambda \) from eq. (21) (the last step is actually unnecessary).

We note that the constraint (20) is invariant under gauge transformations (17). In fact, this constraint can be written in fully gauge-invariant form, since

$$\left( f' - \frac{a_3}{\rho} \right) \propto \hat{x}_i \text{Tr} \left( \frac{\Phi_A}{|\Phi_A|} D_i U U^\dagger \right)$$

Note also that the expression on the left hand side of eq. (20) coincides with the expression for the baryon number, eq. (18), except for terms in the latter containing \( a_1 \) and \( a_2 \). This indicates that the constraint (20) should give rise to non-vanishing baryon number (indeed, we shall see below that the integral (20) approximates the baryon number very well).

The fact that one no longer has to consider the variations of \( a_3 \) simplifies the analysis considerably. We can now impose the gauge condition

$$a_3 = 0$$
directly in the Hamiltonian. Thus, the problem reduces to solving the Euler–Lagrange equations following from the functional

\[ \tilde{H} \equiv H(a_3 = 0) = \frac{4\pi v}{g} \int d\rho \tilde{E}(\rho) \]

\[
\tilde{E} = \left[ (a_1')^2 + (a_2')^2 + \frac{1}{2\rho^2}(a_1^2 + a_2^2 - 1)^2 \right] \\
+ 2 \left[ \rho^2(h')^2 + 2(a_1^2 + a_2^2)h^2 + \frac{\lambda}{4g^2} \rho^2(h^2 - 1)^2 \right] \\
+ \kappa_1(\rho^2(f')^2 + 2Y^2) + \frac{4\kappa_2}{\rho^2} Y^2 (2\rho^2(f')^2 + Y^2) \]  

(22)

where \( Y \) is defined in eq. (13). All these equations are of second order; their form is not very illuminating, and we do not reproduce all of them here. We only write down the equation which is obtained by varying \( \tilde{H} \) with respect to \( f \),

\[
\frac{\partial}{\partial \rho} \left[ (\kappa_1 \rho^2 + 8\kappa_2 Y^2) f' \right] = 2 \left[ \kappa_1 + 4\kappa_2(f')^2 + 4\frac{\kappa_2}{\rho^2} Y^2 \right] (a_1 \cos f + a_2 \sin f) Y 
\]

(23)

It is straightforward to see that \( a_1 \) and \( a_2 \) decay exponentially at large \( \rho \),

\[ a_1, a_2 \propto Ce^{-2\rho} + C' e^{-\sqrt{4+2\kappa_1}\rho} \]

Hence, away from the monopole core, equation (23) becomes

\[ (\rho^2 f')' = 0 \]

and its solution is

\[ f(\rho) = f_\infty + \frac{c_f}{\rho} \]

(24)

where \( f_\infty \) and \( c_f \) are constants. In fact, only the asymptotic value \( f_\infty \) is a free parameter (for given \( \kappa_1 \) and \( \kappa_2 \), as the constant \( c_f \) is a function of \( f_\infty \) for the solutions of the complete system.

Thus, the system of Euler–Lagrange equations, corresponding to the functional (22), admits a one-parameter family of solutions parametrized by the asymptotic value \( f_\infty \). The baryon number and the energy for the solutions are determined by \( f_\infty \).

### 4.3 Numerical results

We have obtained these families of solutions numerically for various values of \( \kappa_1 \) and \( \kappa_2 \) and computed their energies and baryon numbers. We shall present our results for a fixed value of the Higgs self-coupling,

\[ \lambda/g^2 = 0.5 \]
To a good accuracy the baryon number is a linear function of \( f_\infty \): according to eq. (18), in the gauge \( a_3 = 0 \) we have

\[
B = \frac{\pi - f_\infty}{\pi} + \frac{1}{\pi} \int_0^\infty d\rho \left( a_1 a'_2 - a'_1 a_2 \right)
\]

The last integral here is small; we shall discuss this property later on.

Our first result is that the energy is always a monotonic function of the baryon number. This is shown in Fig. 1 for three sets of parameters \( (\kappa_1, \kappa_2) \). This means that there are no static solutions with non-vanishing baryon number on top of the monopole.

We note that the third set of parameters, \( \kappa_1 = 10^{-4}, \kappa_2 = 0.1 \) corresponds to a very small size of the monopole core as compared to the Skyrmion size, \( r_{\text{mon}}/r_{\text{Sk}} = 2.7 \cdot 10^{-3} \). It is natural to expect that in this case, the regime of “point-like” monopole is realized. Let us discuss this in some detail, with the purpose of estimating the range of parameters at which this regime sets in.

The situation of interest is when the monopole is small and heavy,

\[
\frac{r_{\text{mon}}}{r_{\text{Sk}}} \ll 1, \quad \frac{M_{\text{mon}}}{M_{\text{Sk}}} \gg 1
\]

This situation occurs at small \( \kappa_1/\kappa_2 \) and \( \kappa_1 \cdot \kappa_2 \), see eqs. (15), (16). In this case the monopole core is unaffected by the Skyrme field, i.e., the profiles of \( a_1(\rho) \) and \( h(\rho) \) coincide with the undistorted monopole profiles (and hence are independent of the baryon number), whereas \( a_2(\rho) \) is small. The first two terms in eq. (22) are then irrelevant, and the properties of the radial Skyrme field \( f \) are determined by the Hamiltonian

\[
(E - M_{\text{mon}})[f] = 4\pi \frac{F_\pi}{\sqrt{512}} \int d\rho \left( \tilde{\mathcal{E}} - \tilde{\mathcal{E}}_{\text{mon}} \right)(\rho)
\]

\[
\tilde{\mathcal{E}} - \tilde{\mathcal{E}}_{\text{mon}} = \sqrt{\frac{\kappa_1}{\kappa_2}} \left[ \rho^2 (f')^2 + 2Y^2 \right] + 4 \sqrt{\frac{\kappa_2}{\kappa_1}} \frac{Y^2}{\rho^2} \left[ 2 \rho^2 (f')^2 + Y^2 \right]
\]

where we made use of eq. (14). The fields \( a_1 \) and \( h \) entering this expression are spectators (they correspond to the undistorted monopole). This is what we mean by the regime of “point-like” monopole. In fact, the monopole is in some sense never point-like, as its core affects the asymptotics of the radial Skyrme field (this feature we shall discuss later on), so it is more appropriate to speak of a spectator monopole.

There are several properties of the spectator monopole regime that may be used to estimate the range of parameters at which it is actually realised:

(i) the profiles of \( a_1 \) and \( h \) must coincide with the undistorted monopole profiles;

(ii) the profile \( f(\rho) \) should depend on the ratio \( \kappa_1/\kappa_2 \), and not on the overall magnitude of \( \kappa_1, \kappa_2 \); in particular, for fixed \( f_\infty \), the second asymptotic coefficient \( c_f \) in eq. (24) should depend on \( \kappa_1/\kappa_2 \) only (this follows from eq. (23));

(iii) the baryon number should be equal to \( (\pi - f_\infty)/\pi \) (this follows from eq. (23) and \( a_2 = 0 \));
(iv) The energy, referenced from the monopole mass and expressed in units of the Skyrmion mass, must depend only on the baryon number and ratio \( \kappa_1/\kappa_2 \),

\[
\frac{E - M_{\text{mon}}}{M_{\text{Sk}}} = \Delta E \left( B, \frac{\kappa_1}{\kappa_2} \right)
\]

(this again follows from eq. (26)).

The properties (ii) – (iv) are illustrated in Figs. 2 – 4.

The property (i) is particularly useful for establishing at what values of \( \kappa_1 \) (i.e., \( F_\pi/v \)) the spectator monopole regime sets in for given \( \kappa_2 \) (i.e., for given \( g^2/e^2 \)). We present the corresponding plots in Fig. 5.

Figures 2 – 5 show that the spectator monopole regime sets in at fairly large \( \kappa_1 \) and \( \kappa_2 \), as large as \( \kappa_1 = 0.01, \kappa_2 = 0.1 \). These values of parameters correspond to \( r_{\text{mon}} \sim 0.03r_{\text{Sk}} \) and \( M_{\text{Sk}} \sim M_{\text{mon}} \). A “point-like” monopole need not be really very small and very heavy.

### 4.4 Radial Skyrme field

Let us now discuss how the asymptotics of the radial Skyrme function, namely, the constant \( c_f \) at given \( f_\infty \), depends on the properties of the monopole core in the spectator monopole regime. At small \( \kappa_1/\kappa_2 \), the behaviour (24) terminates at \( \rho > 1 \) where the monopole function \( a_1 \) is already exponentially decaying (i.e., somewhat outside the core). Indeed, the term \( 8\kappa_2Y^2 \) on the left hand side of eq. (23) becomes comparable to \( \kappa_1\rho^2 \) at

\[
a_1 \sin f \sim \sqrt{\frac{\kappa_1}{\kappa_2} \rho}
\]

This implies that the regime (24) terminates when \( a_1 \) is small, and \( \rho \sim |\ln a_1|/2 \) is larger than 1. For a crude estimate we take into account this effect only, set \( \sin f \sim 1 \) on the left hand side and \( \rho \sim 1 \) on the right hand side of eq. (27) and obtain that the asymptotic regime (24) terminates at

\[
\rho \sim \rho_c = \frac{1}{4}|\ln \left( \frac{\kappa_1}{\kappa_2} \right)|
\]

At this value of \( \rho \), the function \( f(\rho) \) should be reasonably close to its value at \( \rho = 0 \), i.e., \( f(\rho_c) \sim \pi \). Hence

\[
\frac{c_f}{\rho_c} \sim (\pi - f_\infty)
\]

that is

\[
c_f \sim \frac{1}{4}(\pi - f_\infty)|\ln \left( \frac{\kappa_1}{\kappa_2} \right)|
\]

up to terms varying slower than logarithmically. This shows that \( c_f \) depends on the ratio \( \kappa_1/\kappa_2 \) logarithmically. We confirmed this expectation by numerical calculations,
though instead of the coefficient $\pi/4$ at $f_\infty = 0$, we obtained that in the spectator monopole regime

$$c_f = 0.61 \cdot |\ln \left( \frac{\kappa_1}{\kappa_2} \right) | + \text{const}, \quad f_\infty = 0$$

Thus, in terms of the physical coordinate $r$, the radial Skyrme function outside the core behaves as

$$f(r) = f_\infty + \text{const} \cdot \frac{r_{\text{mon}} |\ln(r_{\text{mon}}/r_{\text{Sk}})|}{r} \quad (29)$$

Everywhere in space, except for a region close to the monopole core, the least energy configuration with given baryon number is the $r$-independent hedgehog field, $f = \cos f_\infty + i \hat{x} \cdot \vec{\sigma} \sin f_\infty$, while $f(r)$ changes from $\pi$ to $f_\infty$ near $\rho \sim \rho_c$.

A crude estimate for the energy of configurations with, say, $B = 1$ is obtained by integrating the energy density, $(F_\pi^2/8) \cdot r^2 (f')^2$, over the region $r \geq r_c \equiv g v \rho_c$. One finds, using the asymptotics (24)

$$(E - M_{\text{mon}}) \sim \frac{F_\pi^2 c_f^2}{g v \rho_c}$$

i.e., up to logarithms,

$$E - M_{\text{mon}} \sim \frac{F_\pi^2}{g v} \quad (30)$$

Hence, the Skyrmion loses most of its mass when approaching the monopole. This is illustrated in Fig. 6.

We note in passing that if the baryon number were conserved, there would exist Skyrmion–monopole bound states with binding energy almost equal to the mass of the Skyrmion. This is in accord with Refs. [3, 14, 15].

4.5 Effect of Wess–Zumino–Witten term

Let us now discuss the Wess–Zumino–Witten term in the action, the last term in eq. (2). In our context, its main effect is to generate charge densities induced by the Skyrme and gauge fields [16]. A suitable gauge for calculating this effect is $U = 1$. In this gauge, the WZW term is $[7]$ (we write the terms which do not vanish in the $SU(2)$ case)

$$\Gamma_{\text{WZW}} = \int d^4 x \ e^{i \mu \lambda \rho} Z_{\mu \nu \lambda \rho}$$

$Z_{\mu \nu \lambda \rho} = -\frac{n}{48 \pi^2} \text{Tr} \left[ A_\mu A_\nu \partial_\lambda B_\rho + B_\mu B_\nu \partial_\lambda A_\rho + \frac{1}{2} (A_\mu A_\nu B_\lambda B_\rho + B_\mu A_\nu B_\lambda A_\rho) \right]$ where $n$ is an integer (number of colours in QCD). A straightforward calculation gives for general spherically symmetric fields

$$\int d^3 x \ e^{i \mu \lambda \rho} Z_{\mu \nu \lambda \rho} = \frac{n}{48 \pi^2} \int 4 \pi r^2 dr \ Z$$
\[ Z = \frac{\tilde{a}_0}{r^2} \left[ \frac{b_3}{r} (\tilde{b}_1 \tilde{a}_2 - \tilde{a}_1 \tilde{b}_2) - \tilde{b}_2' (\tilde{a}_2 - \tilde{b}_2) - \tilde{b}_1' (\tilde{a}_1 - \tilde{b}_1) \right] + (\tilde{\alpha} \leftrightarrow \tilde{\beta}) \]

where \( \tilde{a}_i, \tilde{b}_i \) are the radial fields in the gauge \( U = 1 \) (in writing this formula we did not assume the symmetry (11); the field \( B_i \) has the same form as \( A_i \) in (11) but with \( b_i \) substituted for \( a_i \)), and

\[ \tilde{a}_0 = i \text{Tr}(\hat{x} \cdot \hat{\sigma} A_0), \quad \tilde{b}_0 = i \text{Tr}(\hat{x} \cdot \hat{\sigma} B_0) \]

Making use of the symmetry (11), we find

\[ Z = \frac{2(\tilde{a}_0 + \tilde{b}_0)}{r^2} \left( \frac{\tilde{a}_2 \tilde{a}_2'}{r} - \frac{\tilde{a}_1 \tilde{a}_2 \tilde{a}_3}{r} \right) \]  

(31)

Hence, the WZW term induces (vectorial) electric charge density in the vicinity of the monopole.

We have to check that the Coulomb self-interaction due to this effect does not spoil our previous analysis. Let us estimate the Coulomb energy, making use of the spectator monopole approximation. Returning to the original gauge \( a_3 = b_3 = 0 \), and setting \( a_2 = b_2 = 0 \), we obtain

\[ Z = \frac{2(a_0 + b_0)}{r^2} a_1 a_1' \sin^2 f \]

The electric charge density is induced in the region where \( f \) is substantially different from \( \pi \). This occurs near and above \( \rho \sim \rho_c \), where \( a_1 \) is small. Making use of eq. (27) we find that the total induced charge is, up to logarithms,

\[ Q \sim [a_1(\rho_c) \sin f(\rho_c)]^2 \sim \frac{\kappa_1}{\kappa_2} \sim \frac{e^2 F_\pi^2}{g^2 v^2} \]

Since \( r_c \sim r_{\text{mon}} \sim (gv)^{-1} \), up to logarithms, we obtain an estimate for the Coulomb energy

\[ E_{\text{Coul}} \sim \frac{g^2 Q^2}{r_c} \sim \frac{F_\pi^2}{gv} \left( \frac{e^2 F_\pi}{v} \right)^2 \]

which is very small compared to eq. (30).

Hence, in our model, the Skyrmion on top of the monopole is (nearly) neutral, and the Coulomb energy is negligible. The WZW term in the action may be safely ignored.

5 Discussion

For fixed parameters of the model, \( \kappa_1 \) and \( \kappa_2 \), we have constructed the family of constrained solutions along which the baryon number monotonically decreases to zero and energy monotonically decreases to the monopole mass. These solutions, however,
have different behaviours of the Skyrme field at spatial infinity, i.e., different values of the radial Skyrme function at $r = \infty$. One may doubt that the system can actually travel along this path in configuration space: in the gauge $A_0 = B_0 = 0$, the kinetic energy contains a term

$$\frac{\pi F}{2} \int r^2 dr \dot{f}^2$$

which diverges if $f_\infty$ changes in time. Also, the baryonic current (4) has spatial components whose asymptotics at large $r$ (again in the gauge $A_0 = B_0 = 0$) are

$$j_i = \frac{\dot{x}_i}{4\pi^2 r^2} \dot{f}$$

(32)

For $\dot{f}_\infty \neq 0$, there is the baryonic flux at $r \to \infty$,

$$F = \int j_i d\Sigma^i = \frac{1}{\pi} \dot{f}_\infty$$

This may seem to indicate that the baryon number leaks to spatial infinity.

Certainly, an evolution with infinite kinetic energy is impossible, so in the gauge $A_0 = B_0 = 0$ the asymptotic value $f_\infty$ cannot change in time. To see how, in this gauge, the system can travel along the path with varying baryon number (say, leading from $B = 1$ to $B = 0$), one has first to transform each constrained solution of our family to a gauge in which $f(r)$ rapidly tends to zero as $r \to \infty$. According to eqs. (17), (24), each configuration of the new family will have non-vanishing $a_3$, so that at large $\rho$

$$a_3 = -\frac{c_f}{\rho}$$

(33)

Because of gauge invariance, the baryon numbers and energies of old and new configurations will be the same.

Travelling along the new path in the configuration space does not cost infinite kinetic energy. Indeed, in the gauge $A_0 = B_0 = 0$, the electric field at large $r$ behaves as

$$F_{0i} \propto \frac{\dot{a}_3}{r} \propto \frac{\dot{c}_f}{r^2}$$

and the integral of $F_{0i}^2$ converges. Also, the baryonic current (32) vanishes as $r \to \infty$, so the baryon number does not leak to spatial infinity. It merely disappears.

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Figure captions

Fig 1. Energy (referenced from the monopole mass and expressed in units of the Skyrmion mass) versus baryon number for $\kappa_2 = 0.1$, and $\kappa_1 = 0.01$ (solid line) $\kappa_1 = 0.001$, (dashed line), $\kappa_1 = 0.0001$ (dotted line).

Figure 2. The asymptotic coefficient $c_f$ (see eq. (24)) at $f_\infty = 0$ as a function of $\kappa_1$ for fixed ratio $\kappa_1/\kappa_2 = 0.1$ (squares) and $\kappa_1/\kappa_2 = 0.01$ (bullets).

Figure 3. The baryon number as a function of $\kappa_1$ for two values of the ratio $\kappa_1/\kappa_2$:
(a) $(B - 1)$ at $f_\infty = 0$;
(b) $(B - 0.5)$ at $f_\infty = \pi/2$.

Figure 4. Energy as a function of baryon number at different $\kappa_1$ and fixed ratio $\kappa_1/\kappa_2$: (a) $\kappa_1/\kappa_2 = 0.1$; (b) $\kappa_1/\kappa_2 = 0.01$. The curves for $\kappa_1 \leq 0.0001$ are indistinguishable.

Figure 5. Profiles of the functions $a_1$, $a_2$, $h$ and $f$ for $\kappa_2 = 1$ and $\kappa_1 = 1$ (dotted line), $\kappa_1 = 0.1$ (dashed line), $\kappa_1 = 0.01$ (solid line). For $\kappa_1 \leq 0.01$ the profiles of $a_1$, $a_2$ and $h$ are indistinguishable and coincide with the undistorted monopole profiles.

Figure 6. Energy (referenced from the monopole mass and expressed in units of the Skyrmion mass) for the configurations with baryon number $B = 1$ as a function of $\kappa_1$ for a fixed value of $\kappa_2$. 
Figure 1

\[ \Delta E \]

- \[ \kappa_2 = 0.1 \]
- \[ \kappa_1 = 0.01 \]
- \[ \kappa_1 = 0.001 \]
- \[ \kappa_1 = 0.0001 \]
Figure 2

\[ \frac{\kappa_1}{\kappa_2} = 0.01 \]

\[ \frac{\kappa_1}{\kappa_2} = 0.1 \]
Figure 4a

\[ \frac{\kappa_1}{\kappa_2} = 0.1 \]

\[ \kappa_1 = 0.1 \]
\[ \kappa_1 = 0.01 \]
\[ \kappa_1 = 0.001 \]
\[ \kappa_1 = 0.0001 \]

Figure 4b

\[ \frac{\kappa_1}{\kappa_2} = 0.01 \]

\[ \kappa_1 = 0.05 \]
\[ \kappa_1 = 0.01 \]
\[ \kappa_1 = 0.001 \]
\[ \kappa_1 = 0.0001 \]
Figure 5

\[ \rho \]

\[ \kappa_1 = 0.01 \]
\[ \kappa_1 = 0.1 \]
\[ \kappa_1 = 1.0 \]
Figure 6

$k_2 = 0.1$

$\Delta E$ vs. $\log(k_1)$