DIMENSION-FREE ESTIMATES FOR DISCRETE HARDY–LITTLEWOOD AVERAGING OPERATORS OVER THE CUBES IN $\mathbb{Z}^d$

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Abstract. Dimension-free bounds will be provided in maximal and $r$-variational inequalities on $\ell^p(\mathbb{Z}^d)$ corresponding to the discrete Hardy–Littlewood averaging operators defined over the cubes in $\mathbb{Z}^d$. We will also construct an example of a symmetric convex body in $\mathbb{Z}^d$ for which maximal dimension-free bounds fail on $\ell^p(\mathbb{Z}^d)$ for all $p \in (1, \infty)$. Finally, some applications in ergodic theory will be discussed.

1. Introduction and notation

In the 1980s dimension-free estimates for the Hardy–Littlewood maximal functions over symmetric convex bodies had begun to be studied and gone through a period of considerable changes and developments [2, 3, 7, 15, 17, 18]. We refer also to more recent results [1, 4, 5] and the survey article [8] for a very careful and exhaustive exposition of the subject. However at that time the discrete analogues of these dimension-free estimates had not been investigated, and only recently has the dimension-free role of $r$-variations been broached [5].

In this article we initiate systematic studies of the estimates independent of the dimension for the Hardy–Littlewood averaging operators in the discrete setup. On the one hand, we give a counterexample that shows that the phenomenon of dimension-free estimates in the discrete setting cannot be as broad as in the continuous setting. On the other hand, for the discrete Hardy–Littlewood averaging operators over the cubes in $\mathbb{Z}^d$ some positive results will be proved here. We will also discuss dimension-free $r$-variational estimates and their applications to ergodic theory.

Let $G$ be a bounded, closed and symmetric convex subset of $\mathbb{R}^d$ with non-empty interior. Throughout the paper such a set $G$ will be called a symmetric convex body. We remark that usually in the literature a symmetric convex body $G$ is assumed to be open. In fact, when averaging operators over convex sets in $\mathbb{R}^d$ are considered there is no difference whether we assume $G$ is closed or open, since the boundary of a convex set has Lebesgue measure zero. However, in the discrete case in order to avoid some technicalities, we will assume that a symmetric convex body $G$ is always closed.

For every $x \in \mathbb{Z}^d$ and $t > 0$ and for every function $f \in \ell^1(\mathbb{Z}^d)$ let

$$\mathcal{M}^G_t f(x) = \frac{1}{|G_t \cap \mathbb{Z}^d|} \sum_{y \in G_t \cap \mathbb{Z}^d} f(x - y)$$

be the discrete Hardy–Littlewood averaging operator over $G_t \cap \mathbb{Z}^d$, where $G_t = \{y \in \mathbb{R}^d : t^{-1}y \in G\}$.

The operator $\mathcal{M}^G_t$ is a convolution operator with the kernel

$$\mathcal{K}^G_t(x) = \frac{1}{|G_t \cap \mathbb{Z}^d|} \sum_{m \in G_t \cap \mathbb{Z}^d} \delta_m(x),$$

where $\delta_m$ stands for the Dirac’s delta at $m \in \mathbb{Z}^d$.

It is natural that $\mathcal{M}^G_t$ can be thought of as a discrete analogue of the integral Hardy–Littlewood averaging operator

$$\mathcal{M}^G_t f(x) = \frac{1}{|G_t|} \int_{G_t} f(x - y) \, dt,$$

defined for every $f \in L^1_{\text{loc}}(\mathbb{R}^d)$.

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1.1. Maximal estimates. We know from [3, 7] that for every $p \in (3/2, \infty]$, there is $C_p > 0$ independent of the dimension such that for every convex symmetric body $G \subset \mathbb{R}^d$ and for every $f \in L^p(\mathbb{R}^d)$ we have
\[
\left\| \sup_{t > 0} |M_G^t f| \right\|_{L^p} \leq C_p \|f\|_{L^p}.
\] (1.3)

For the dyadic/lacunary variant of $M_G^t$ the range of $p$’s can be extended and one can show that for every $p \in (1, \infty]$, there is $C_p > 0$ independent of the dimension such that for every convex symmetric body $G \subset \mathbb{R}^d$ and for every $f \in L^p(\mathbb{R}^d)$ we have
\[
\left\| \sup_{n \in \mathbb{Z}} |M_G^{2n} f| \right\|_{L^p} \leq C_p \|f\|_{L^p}.
\] (1.4)

It is conjectured that the inequality in (1.3) holds for all $p \in (1, \infty]$ and for all convex symmetric bodies $G \subset \mathbb{R}^d$ with the implied constant independent of the dimension. Therefore, only parameters $p \in (1, \infty)$ will matter.

At first glance one thinks that it should be possible, in view of (1.3), to deduce bounds in (1.6) that are independent of $d$ on $L^p(\mathbb{Z}^d)$ from the dimension-free results on $L^p(\mathbb{R}^d)$ by comparison of the maximal function corresponding to $M_G^t$ on $\mathbb{Z}^d$ with the maximal function corresponding to $M_G^t$ on $\mathbb{R}^d$.

This idea is efficient, and gives a satisfactory answer, but additional assumptions are required. Namely, in Proposition 2.1 we show that for every symmetric convex body $G \subset \mathbb{R}^d$ there exists $t_G > 0$ with the property that the norm of the discrete maximal function $\sup_{t \leq t_G} |M_G^t f|$ is controlled by a constant multiple of the norm of its continuous counterpart, and the implied constant is independent of the dimension. So this simple comparison argument will allow us to deduce dimension-free estimates for those discrete maximal functions whose supremum is taken over $t > t_G$ as long as the corresponding dimension-free bounds are available for their continuous analogues. At this stage, the whole difficulty lies in estimating $\sup_{0 < t \leq t_G} |M_G^t f|$, and here the things are getting more complicated.

We shall show that the dimension-free estimates in the discrete case are not as broad as in the continuous setup by constructing an example of a symmetric convex body in $\mathbb{Z}^d$ for which maximal estimates on $L^p(\mathbb{Z}^d)$ for every $p \in (1, \infty)$ involve constants which are unbounded as $d \to \infty$.

Namely, let $1 \leq \lambda_1 < \cdots < \lambda_d < \sqrt{2}$ be a fixed sequence and define the ellipsoid
\[
E = \left\{ x \in \mathbb{R}^d : \sum_{j=1}^d \lambda_j^2 x_j^2 \leq 1 \right\}.
\] (1.7)

Then on the one hand, in view of the comparison principle described in Proposition 2.1 and inequality (1.3) with $G = E \subset \mathbb{R}^d$, one is able to show that for every $p \in (3/2, \infty]$ there is $C_p > 0$ independent of $d \in \mathbb{N}$ such that the following estimate
\[
\left\| \sup_{t > d^{1/2}} |M_E^t f| \right\|_{L^p} \leq C_p \|f\|_{L^p}
\] (1.8)
holds for every $f \in L^p(\mathbb{Z}^d)$.
On the other hand, Theorem 1 shows that (1.8) is not true if the full maximal function corresponding to \( M_t^E \) is considered. Namely, we have the following result.

**Theorem 1.** For every \( p \in (1, \infty) \) and \( d \in \mathbb{N} \) the maximal inequality from (1.6) with \( G = E \subset \mathbb{R}^d \) involves the smallest constant \( C_p(d) > 0 \) unbounded in \( d \). In fact, there is \( C_p > 0 \) such that for every \( d \in \mathbb{N} \) one has

\[
C_p(d) \geq C_p \cdot (\log d)^{1/p}.
\]

Theorem 1 shows that the question about the dimension-free estimates in the discrete setting for the Hardy-Littlewood maximal functions is much more delicate and there is no obvious conjecture to prove. So, it is interesting to know whether we can expect bounds independent of the dimension on \( \ell^p(\mathbb{Z}^d) \) with \( p \in (1, \infty) \) for the discrete maximal function \( \sup_{t>0} \| M_t^B f \|_{\ell^p} \), where \( B_t \) is a ball as in (1.5) with \( q \in [1, \infty) \). This question is considerably harder due to the lack of reasonable estimates for the number of lattice points in the sets \( B_t \) and definitely new methods must be invented. Therefore, even the \( \ell^2 \) theory is very intriguing. In the the ongoing project [6] we initiated investigations in this direction and the context of the discrete Euclidean balls \( B^2 \) is studied.

However, if \( q = \infty \) then \( B_t^\infty = [-t,t]^d \) is a cube and the number of lattice points is not a problem any more. The product structure of the cubes allows us to count the number of lattice points in \( B_t^\infty \) and we get \( |B_t^\infty \cap \mathbb{Z}^d| = (2|t| + 1)^d \). This property distinguish the cubes from the \( B_t^2 \) balls for \( q \in [1, \infty) \) and in some sense encourages us to think that the inequality (1.10) may hold with the bound independent of the dimension for a certain range of \( p \)’s.

Form now on, for simplicity of the notation we will write \( Q_t = [-t,t]^d \) for \( t > 0 \) and \( Q = [-1,1]^d \). We shall provide analogues of inequalities (1.3) and (1.4) for the discrete operators \( M_t^Q \) over the cubes \( Q_t \cap \mathbb{Z}^d \). One of the main theorems of this paper is the following maximal result.

**Theorem 2.** For every \( p \in (3/2, \infty) \) there exists a constant \( C_p > 0 \) such that for every \( d \in \mathbb{N} \) and every \( f \in \ell^p(\mathbb{Z}^d) \) we have

\[
\| \sup_{t>0} |M_t^Q f| \|_{\ell^p} \leq C_p \| f \|_{\ell^p}.
\]  

(1.9)

If we restrict the supremum in (1.9) to the dyadic times, i.e. \( t \in \{2^n : n \in \mathbb{N}_0\} \), where \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \), then the range of \( p \)’s can be improved.

**Theorem 3.** For every \( p \in (1, \infty] \) there exists a constant \( C_p > 0 \) such that for every \( d \in \mathbb{N} \) and every \( f \in \ell^p(\mathbb{Z}^d) \) we have

\[
\| \sup_{n \in \mathbb{N}_0} |M_{2^n} f| \|_{\ell^p} \leq C_p \| f \|_{\ell^p}.
\]  

(1.10)

In fact, in Section 4 we prove a stronger result and we show that the maximal inequality from (1.10) holds with any \( a_n \) in place of \( 2^n \), where \( (a_n : n \in \mathbb{N}_0) \) is a lacunary sequence\(^1\) in \((0, \infty)\).

In the second part of the paper our aim will be to strengthen the maximal estimates from (1.9) and (1.10) and provide \( r \)-variational estimates independent of the dimension for the operators \( M_t^Q \).

1. **\( r \)-variational estimates.** Very recently, in [5] we studied estimates independent of the dimension for the averaging operators (1.3) in the context of \( r \)-variational seminorms. Recall that for \( r \in [1, \infty) \) the \( r \)-variation seminorm \( V_r \) of a complex-valued function \((0, \infty) \times X \ni (t, x) \mapsto a_t(x) \) on some measure space \((X, \mathcal{B}(X), \mu)\) is defined by setting

\[
V_r(a_n(x) : t \in \mathbb{N}_0) = \sup_{t_0 < \ldots < t_J} \left( \left| \sum_{j=0}^J |a_{t_{j+1}}(x) - a_{t_j}(x)|^r \right|^{1/r} \right),
\]

where \( Z \) is a subset of \((0, \infty)\) and the supremum is taken over all finite increasing sequences in \( Z \). If \( Z \) is the dyadic set \((2^n : n \in \mathbb{Z})\) then the \( r \)-variation \( V_r \) is often called the \( r \)-variation seminorm.

In what follows we will assume that \((0, \infty) \ni t \mapsto a_t(x)\) is a continuous function for every \( x \in X \) or that \( Z \) is countable, then there is no problem with the measurability of \( V_r(a_n(x) : t \in \mathbb{Z}) \). In the discrete setup the function \((0, \infty) \ni t \mapsto |G_t \cap \mathbb{Z}^d| \) takes only countably many values, so the parameter \( t \) will be always restricted to a countable subset of \((0, \infty)\). In the case of the discrete cubes \( Q_t \cap \mathbb{Z}^d \) we will have \( Z = \mathbb{N} \).

\[^1\)A sequence \((a_n : n \in \mathbb{N}_0) \subseteq (0, \infty)\) is called lacunary, if \( \inf_{n \in \mathbb{N}_0} \frac{a_{n+1}}{a_n} > 1 \).
The $r$-variational seminorm is a very useful tool in pointwise convergence problems. If for some $r \in [1, \infty)$ and $x \in X$ we have
\[ V_r(a_t(x) : t > 0) < \infty \]
then the limits $\lim_{t \to 0} a_t(x)$ and $\lim_{t \to \infty} a_t(x)$ exist. So we do not need to establish pointwise convergence on a dense class as it is usually done in the classical approach. This is very important while pointwise convergence problems are discussed in the ergodic context and there is no easy way to find a candidate for such a dense class. However, $V_r$ is more difficult to bound than the maximal function, since it dominates the supremum norm, i.e. for any $t_0 > 0$ we have
\[ \sup_{t > 0} |a_t(x)| \leq |a_{t_0}(x)| + 2V_r(a_t(x) : t > 0). \]
There is an extensive literature about the $r$-variational estimates. For the purposes of this article the most relevant will be [9], [10] and [13], see also the references given there.

In [5] we proved that for every $p \in (3/2, 4)$ and for every $r \in (2, \infty)$ there exists a constant $C_{p,r} > 0$ independent of the dimension $d \in \mathbb{N}$ such that for every symmetric convex body $G \subset \mathbb{R}^d$ and for all $f \in L^p(\mathbb{R}^d)$ we have
\[ \|V_r(M^G_r f : t > 0)\|_{L^p} \leq C_{p,r} \|f\|_{L^p}. \] (1.11)
The range for the parameter $p$ in (1.11) can be improved if we consider only long r-variations. Namely, for all $p \in (1, \infty)$ and $r \in (2, \infty)$ we have
\[ \|V_r(M^G_r f : n \in \mathbb{Z})\|_{L^p} \leq C_{p,r} \|f\|_{L^p}. \] (1.12)
Moreover, if $G = B^q$ for $q \in [1, \infty]$ and $B^q$ is a ball as in (1.6), then the inequality (1.11) holds for all $p \in (1, \infty)$ and $r \in (2, \infty)$ with a constant $C_{p,q,r} > 0$ independent of the dimension.

The results have been encouraging enough to merit further investigation, especially in the discrete setup. Therefore, in the second part of the paper we will be concerned with estimating $r$-variations for the discrete operators $M^G_d$ over the cubes with bounds independent of the dimension as in (1.11) and (1.12). The next theorem is a variational counterpart of Theorem 2.

**Theorem 4.** Let $p \in (3/2, 4)$ and $r \in (2, \infty)$. Then there exists a constant $C_{p,r} > 0$ independent of the dimension $d \in \mathbb{N}$ such that for every $f \in \ell^p(\mathbb{Z}^d)$ we have
\[ \|V_r(M^G_r f : t > 0)\|_{\ell^p} \leq C_{p,r} \|f\|_{\ell^p}. \] (1.13)

**Theorem 5.** Let $p \in (1, \infty)$ and $r \in (2, \infty)$. Then there exists a constant $C_{p,r} > 0$ independent of the dimension $d \in \mathbb{N}$ such that for every $f \in \ell^p(\mathbb{Z}^d)$ we have
\[ \|V_r(M^G_r f : n \in \mathbb{Z}^d)\|_{\ell^p} \leq C_{p,r} \|f\|_{\ell^p}. \] (1.14)

The range for parameter $r \in (2, \infty)$ in Theorem 4 and 5 is sharp, see for instance [10]. Dimension dependent versions of Theorem 4 and 5 with sharp ranges of parameters $p \in (1, \infty)$ and $r \in (2, \infty)$, may be easily proven using the methods of the paper.

Finally some applications of Theorem 4 and Theorem 5 will be discussed. These $r$-variational results have a natural ergodic theoretical interpretation and will be discussed in the next paragraph.

### 1.3. Applications.

Let $(X, \mathcal{B}(X), \mu)$ be a $\sigma$-finite measure space with a family of commuting and invertible measure-preserving transformations $T_1, T_2, \ldots, T_d$ which map $X$ to itself. For every $f \in L^1(X)$ we define the ergodic Hardy–Littlewood averaging operator by setting
\[ A^Q_f(x) = \frac{1}{|G_t \cap \mathbb{Z}^d|} \sum_{y \in G_t \cap \mathbb{Z}^d} f(T_1^{y_1} \circ T_2^{y_2} \circ \cdots \circ T_d^{y_d} x). \] (1.15)

The operator $A^Q_f$ can be thought of as an ergodic counterpart of $M^G_d$. Indeed, it suffices to take $X = \mathbb{Z}^d$, $\mathcal{B}(\mathbb{Z}^d)$ the $\sigma$-algebra of all subsets of $\mathbb{Z}^d$, $\mu = \cdot | t$ be the counting measure on $\mathbb{Z}^d$ and $S^d_j : \mathbb{Z}^d \to \mathbb{Z}^d$ the shift operator acting of $j$-th coordinate, i.e. $S^d_j(x) = x - ye_j$ for all $j \in \{1, \ldots, d\}$ and $y \in \mathbb{Z}$, where $e_j$ is the $j$-th basis vector from the standard basis in $\mathbb{Z}^d$.

For the operators $A^Q_N$ defined over the cubes we also have dimension-free $r$-variational estimates.
**Theorem 6.** Let $p \in (3/2, 4)$ and $r \in (2, \infty)$. Then there exists a constant $C_{p,r} > 0$ independent of the dimension $d \in \mathbb{N}$ such that for all $f \in L^p(X)$ the following inequality holds

$$
\left\| V_r(A^Q_f : N \in \mathbb{N}) \right\|_{L^p} \leq C_{p,r} \| f \|_{L^p}.
$$

(1.16)

Moreover, if we consider only long variations, then (1.16) remains true for all $p \in (1, \infty)$ and $r \in (2, \infty)$ and we have

$$
\left\| V_r(A^Q_f : n \in \mathbb{N}_0) \right\|_{L^p} \leq C_{p,r} \| f \|_{L^p}.
$$

(1.17)

In Proposition 5.2 we provide a transference principle, which allows us to derive inequalities (1.16), (1.17) from the corresponding estimates in (1.13) and (1.14) respectively. Now two remarks are in order. Firstly, the remarkable feature of Theorem 5 is that the implied bounds in (1.16), (1.17) are independent of the number of underlying transformations $T_1, \ldots, T_k$. Secondly, for the operators $A^Q_f$, which are defined on an abstract measure space, there is no obvious way how to find a candidate for a dense class to establish pointwise convergence. Fortunately, due to the properties of $r$-variation seminorm we immediately know that the limit $\lim_{t \to \infty} A^Q_f(x)$ exists almost everywhere on $X$ for every $f \in L^p(X)$ and the desired conclusion follows directly.

### 1.4. Overview of the methods

We shall briefly outline the strategy for proving our main results. The first step in the proofs of Theorem 2 and Theorem 3 will rely to a large extent on an adaptation of Carbery’s almost orthogonality principle [7 Theorem 2], which is stated as Proposition 5.2 in the paper. In the second step we are reduced to verify the assumptions of Proposition 4.2. In order to do this we have to construct a suitable symmetric diffusion semigroup $P_t$, provide dimension-free estimates for the multiplier $m^Q_t$ corresponding to the operator $\mathcal{M}^Q_t$ and finally we have to control the maximal function $\sup_{2^n \leq t < 2^{n+1}} \| \mathcal{M}^Q_t f \|_{L^p(S^d)}$ in a certain range of $p$’s. However, due to the discrete nature of our questions, the methods employed in the continuous setting in [4], [7], and [15] for verifying underlying assumptions do not easily adapt to the discrete setting.

Fortunately, for the operators $\mathcal{M}^Q_t$ over the cubes in $\mathbb{Z}^d$ we will be able to obtain the desired conclusions. We begin by constructing a suitable symmetric diffusion semigroup $P_t$ introduced in Section 3. The semigroup $P_t$ in our case corresponds to the discrete Laplacian on $\mathbb{Z}^d$, and provides maximal and $r$-variational estimates and the Littlewood–Paley theory with bounds independent of the dimension, which one obtains by appealing to the general theory of symmetric diffusion semigroups in the sense of [16 Chapter III].

Further, we have to understand the behavior of the multiplier $m^Q_t$ associated with the operator $\mathcal{M}^Q_t$. This in turn is an exponential sum, which is the product of one dimensional Dirichlet’s kernels. The explicit formula for $m^Q_t$ in terms of the Dirichlet kernels is essential for the further calculations and allows us to furnish the bounds independent of the dimension as described in [4, 6]. The inequalities in (4.1) are based on elementary estimates, which are interesting in its own right. For this reason our method does not extend to discrete convex bodies other than $Q$. This is the second place where we extend the operators $\mathcal{M}^Q_t$ over the cubes apart the operators $\mathcal{M}^B_t$ for $q \in [1, \infty)$, where $B^2$ is a ball as in (1.2). The multiplier $m^B_t$ associated with the operator $\mathcal{M}^B_t$ is again an exponential sum, however the absence of the product of the underlying semigroup makes the estimates incomparably harder. The estimates for $m^B_t$, which are a part of the ongoing project [6], are based on delicate combinatorial arguments, which differ completely from the methods of estimates for $m^Q_t$ provided in Section 3.

The crucial new ingredient we shall use is an elementary numerical inequality, as in [4], which asserts that for every $n \in \mathbb{N}_0$ and for every function $a : [2^n, 2^{n+1}] \cap \mathbb{N} \to \mathbb{C}$ and $r \geq 1$ we have

$$
\sup_{2^n \leq t < 2^{n+1}} |a(t) - a(2^n)| \leq V_r \left( \mathbb{A}^Q_t : t \in [2^n, 2^{n+1}] \right)
$$

\begin{equation}
\leq 2^{1-1/r} \sum_{0 \leq k \leq n} \left( \sum_{k=0}^{2^k} |a(2^n + 2^{n-l}(k+1)) - a(2^n + 2^{n-l}k)|^r \right)^{1/r}.
\end{equation}

(1.18)

Inequality (1.18) replaces the fractional integration argument from [7] (as it is not clear if this argument is available in the discrete setting) and allows us to obtain (1.9) for $p \in (3/2, 2]$. A variant of this inequality was proven by Lewko–Lewko [11] Lemma 13 in the context of variational Rademacher–Menshov type results for orthonormal systems and it was also obtained independently by the second author and Trojan [14] Lemma 1 in the context of variational estimates for discrete Radon transforms, see also [13]. Inequality (1.18) reduces estimates for a supremum or an $r$-variation restricted to a dyadic block to the
situation of certain square functions, where the division intervals over which differences are taken (in these square functions) are all of the same size. Inequality (1.18), combined with the estimates from (1.11), is an invaluable tool in establishing the following maximal bound

\[ \sup_{n \in \mathbb{N}_0} \sup_{2^n \leq t < 2^{n+1}} |M^t f| \leq C_p f \quad (1.19) \]

for all \( f \in \ell^p(\mathbb{Z}^d) \) and \( p \in (3/2, \infty) \) with some constant \( C_p > 0 \), which does depend on the dimension.

Gathering now all together and invoking Proposition (1.2) and dimension-free Littlewood–Paley inequality from (1.7) we may extend inequality (1.19) to the full maximal inequality (1.10) for all \( p \in (3/2, \infty) \), with the implied bound which does not depend on \( d \in \mathbb{N} \). In the dyadic case we do not need to prove inequality (1.19) and this is loosely speaking the reason why we obtain (1.10) for all \( p \in (1, \infty) \). It is worth emphasizing that the method described above can be used to obtain (1.3) and (1.4) without appealing to the fractional integration method.

The approach undertaken in this paper is robust enough to provide \( r \)-variational dimension-free estimates for the operators \( M_t^Q \). We now briefly outline the key steps for proving Theorem [4] and [5].

We first split the consideration into long and short variations as in (5.2). The long variations (5.3) are handled in Theorem [5] by invoking the dimension-free estimates for \( r \)-variations of the semigroup \( P_t \). We refer to [1] Theorem 3.3 or [4] inequality (2.30) for more details. To establish Theorem [5] it remains to control the error term, which is handled by the square function methods, and the Littlewood–Paley theory, see (5.5).

The analysis of short variations (5.10) breaks into two cases, whether \( p \in [2, 4) \) or \( p \in (3/2, 2] \). In the first case for \( p \in [2, 4) \) we use the square function methods, and the Littlewood–Paley theory and reduce the estimates basically to Theorem [3]. In the second case for \( p \in (3/2, 2] \) we proceed actually very much in the spirit of the proof of Theorem [2]. Namely, we rely on the numerical inequality (1.15) and adapt the methods of the proof of Proposition (1.2) to the \( r \)-variational case, in fact with \( r = 2 \), which is suited to an application of the Fourier transform techniques with estimates from (3.1).

There is a natural question which now arises. Is it possible to extend the range of \( p \)-s in Theorem [2] to \( p \in (1, 3/2] \)? For the maximal function associated with the operators \( M_t^Q \) over the cubes in \( \mathbb{R}^d \) given by (1.2) this was accomplished in [4]. However, how to do this for \( p \in (1, 3/2] \) in the discrete case is not obvious. There are two ingredients, which were employed in [1], that seem to fail in the discrete case. Firstly, it is not clear if there is a satisfactory counterpart of the theory of fractional integration in the discrete setup. The idea of fractional integration was very fruitful and strongly exploited in [7], [15] and [3]. Secondly, in [15] and [5] one of the key points is based on the dimension-free estimates for the Riesz transforms. However, the discrete Riesz transforms, which naturally arise in the context of the discrete Laplacian on \( \mathbb{Z}^d \), do not have dimension-free bounds on \( \ell^p(\mathbb{Z}^d) \) for \( p \in (1, 2) \), see the Appendix for a counterexample of Lust–Piquard [12].

A similar question concerns the estimates of \( r \)-variations for the operators \( M_t^Q \). We would like to know whether inequality (1.11) can be extended to \( p \in (1, 3/2] \) or \( p \in [4, \infty) \). Here the situation is even more complicated since we cannot interpolate with \( p = \infty \) as we did in the case of maximal estimates, so the case for \( p \in [4, \infty) \) must be treated separately. However, we know [5] that the operators \( M_t^Q \) over the cubes in \( \mathbb{R}^d \) given by (1.2) do have the dimension-free estimates for \( r \)-variations on \( L^p(\mathbb{R}^d) \) for all \( p \in (1, \infty) \) and \( r \in (2, \infty) \).

The results of [4] and [5], Theorem [1] and the counterexample of Lust–Piquard [12] are certainly encouraging to understand the situation better and continue further study of \( M_t^Q \), which together with \( M_t^{BP^2} \), is the most natural setting for the discrete Hardy–Littlewood maximal functions. We hope to return to these questions in the near future.

1.5. Notation.

- Throughout the whole paper \( d \in \mathbb{N} \) will denote the dimension and \( C > 0 \) will be an absolute constant which does not depend on the dimension, however it may change from line to line.
- For two real numbers \( A, B \) we will write \( A \lesssim B \) \( (A \lesssim_\delta B) \) to say that there is an absolute constant \( C_\delta > 0 \) (which possibly depends on \( \delta > 0 \)) such that \( A \leq C_\delta B \) \( (A \geq C_\delta B) \). We will write \( A \simeq B \) when \( A \lesssim B \) and \( A \gtrsim B \) hold simultaneously.
- Let \( \mathbb{N} := \{1, 2, \ldots\} \) be the set of positive integers and \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \).
• The Euclidean space $\mathbb{R}^d$ is endowed with the standard inner product
  \[ x \cdot \xi := \sum_{k=1}^d x_k \xi_k \]
  for every $x = (x_1, \ldots, x_d)$ and $\xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d$. Sometimes we will also write $\langle x, \xi \rangle := x \cdot \xi$.

• We will consider two norms on $\mathbb{R}^d$. For every $x \in \mathbb{R}^d$
  \[ |x| = |x|_2 := \sqrt{x \cdot x} \text{ and } |x|_\infty := \max_{1 \leq k \leq d} |x_k|. \]

• For a countable set $\mathcal{Z}$ endowed with the counting measure we will write for any $p \in [1, \infty)$ that
  \[ \ell^p(\mathcal{Z}) := \{ f : \mathcal{Z} \to \mathbb{C} : \| f \|_{\ell^p(\mathcal{Z})} < \infty \}, \]
  where for any $p \in [1, \infty)$ we have
  \[ \| f \|_{\ell^p(\mathcal{Z})} := \left( \sum_{m \in \mathcal{Z}} |f(m)|^p \right)^{1/p} \text{ and } \| f \|_{\ell^\infty(\mathcal{Z})} := \sup_{m \in \mathcal{Z}} |f(m)|. \]
  In our case usually $\mathcal{Z} = \mathbb{Z}^d$.

• Let $(X, B(X), \mu)$ be a $\sigma$-finite measure space. Let $p \in [1, \infty]$ and suppose that $(T_t : t \in \mathcal{Z})$ is a family of linear operators such that $T_t$ maps $L^p(X)$ to itself for every $t \in \mathcal{Z} \subseteq (0, \infty)$. Then the corresponding maximal function will be denoted by
  \[ T_* f := \sup_{t \in \mathcal{Z}} |T_t f| \text{ for every } f \in L^p(X). \]
  We will abbreviate $T_{*, \mathcal{Z}}$ to $T_*$ if $\mathcal{Z} = (0, \infty)$. We use the convention that $T_{*, \emptyset} = 0$.

• Let $(B_1, \| \cdot \|_{B_1})$ and $(B_2, \| \cdot \|_{B_2})$ be Banach spaces. For a linear or sub-linear operator $T : B_1 \to B_2$ its norm is defined by
  \[ \| T \|_{B_1 \to B_2} := \sup_{\| f \|_{B_1} \leq 1} \| T(f) \|_{B_2}. \]

• Let $\mathcal{F}$ denote the Fourier transform on $\mathbb{R}^d$ defined for any function $f \in L^1(\mathbb{R}^d)$ and for any $\xi \in \mathbb{R}^d$ as
  \[ \mathcal{F} f(\xi) := \int_{\mathbb{R}^d} f(x) e^{2\pi i \xi \cdot x} dx. \]
  If $f \in L^1(\mathbb{T}^d)$ we define the discrete Fourier transform by setting
  \[ \hat{f}(\xi) := \sum_{x \in \mathbb{Z}^d} f(x) e^{2\pi i \xi \cdot x}. \]
  for any $\xi \in \mathbb{T}^d$, where $\mathbb{T}^d$ denote $d$-dimensional torus which will be identified with $[-1/2, 1/2]^d$.

• To simplify notation we denote by $\mathcal{F}^{-1}$ the inverse Fourier transform on $\mathbb{R}^d$ or the inverse Fourier transform (Fourier coefficient) on the torus $\mathbb{T}^d$. It will cause no confusions and the meaning will be always clear from the context.

2. A COUNTEREXAMPLE WITH DISCRETE ELLIPSOIDS: PROOF OF THEOREM [41]

In this section we prove Theorem [41] which shows that the phenomenon of dimension-free estimates in the discrete setting may be completely different from that which we have seen so far in the continuous setting. However, we begin with the observation which shows that the dimension-free estimates for the discrete Hardy–Littlewood maximal functions are only interesting if the supremum is taken over small scales. The case when the supremum is taken over large scales can be easily deduced from the corresponding continuous estimates by a comparison principle described in Proposition [44].

2.1. Comparison principle. For a closed symmetric convex body $G \subset \mathbb{R}^d$ we define the constant
\[ c(G) := \inf \{ t > 0 : Q_{1/2} \subseteq tG \}, \]
where $Q_{1/2} = [-1/2, 1/2]^d$. A transfer of dimension-free estimates (for large scales) between discrete and continuous settings will be deduced from the following result.

**Proposition 2.1.** For every $p \in (1, \infty)$ the following inequality
\[ \| \sup_{t \geq c(G)} |M_t G f| \|_{L^p} \leq c \| |M_t G f| \|_{L^p(\mathbb{R}^d)} \| f \|_{L^p} \]
holds for all $f \in \ell^p(\mathbb{Z}^d)$.
Proof. For any \( f \in \mathbb{Z}^d \) we define its extension \( F \) on \( \mathbb{R}^d \) by setting
\[
F(x) := \sum_{n \in \mathbb{Z}^d} f(n) \mathbb{I}_{Q_{1/2}}(x - n)
\]
for every \( x \in \mathbb{R}^d \). Observe that \( F(n) = f(n) \) for \( n \in \mathbb{Z}^d \) and \( F \in L^p(\mathbb{R}^d) \) if and only if \( f \in \ell^p(\mathbb{Z}^d) \) with \( \| F \|_{L^p(\mathbb{R}^d)} = \| f \|_{\ell^p(\mathbb{Z}^d)} \) for every \( p \geq 1 \).

Without loss of generality we assume that \( f \geq 0 \), hence \( F \geq 0 \). We show that for every \( t \geq c(G)d, d \geq 2 \) and every \( x \in n + Q_{1/2} \) we have
\[
\mathcal{M}_t^G f(n) \leq \left( 1 + \frac{6}{d^2} \right)^d M_{t+2c(G)}^G F(x).
\]
(2.3)

Clearly, this establishes (2.1). We now focus on (2.3). For \( x \geq n \) we denote by \( |x|_G \) the Minkowski norm corresponding to \( G \subset \mathbb{R}^d \), i.e.
\[
|x|_G := \inf \{ t > 0 : t^{-1}x \in G \}.
\]

Then the formula (2.1) may be rephrased as
\[
c(G) = \sup_{s \in Q_{1/2}} |s|_G.
\]

Assume that \( x \in n + Q_{1/2} \), then we have
\[
\mathcal{M}_t^G f(n) = \frac{1}{|G_t \cap \mathbb{Z}^d|} \sum_{m \in \mathbb{Z}^d : |n - m|_G \leq t} f(m) = \frac{1}{|G_t \cap \mathbb{Z}^d|} \sum_{m \in \mathbb{Z}^d : |n - m|_G \leq t} \int_{m+Q_{1/2}} F(s)ds \leq \frac{1}{|G_t \cap \mathbb{Z}^d|} \int_{|x-s|_G \leq t+2c(G)} F(s)ds,
\]
(2.4)
since, if \( |n - m|_G \leq t \) and \( |s - m|_G \leq 1/2 \), then
\[
|x - s|_G \leq |n - s|_G + |x - n|_G \leq |n - m|_G + |s - m|_G + |x - n|_G \leq t + 2c(G).
\]

We claim, for \( t > c(G) \), that
\[
|G_{t-c(G)}| \leq |G_t \cap \mathbb{Z}^d|.
\]
(2.5)

Indeed, if \( |s|_G \leq t - c(G) \) and \( |s - n|_G \leq 1/2 \), then
\[
|n|_G \leq |s|_G + |s - n|_G \leq t
\]
and consequently, we have
\[
|G_{t-c(G)}| = \sum_{n \in \mathbb{Z}^d} \int_{n+Q_{1/2}} \mathbb{I}_{G_{t-c(G)}}(s)ds \leq \sum_{n \in \mathbb{Z}^d : |n|_G \leq t} \int_{n+Q_{1/2}} 1ds \leq |G_t \cap \mathbb{Z}^d|.
\]

Hence, using (2.3) and (2.4) we obtain for \( n \in \mathbb{Z}^d \) and \( x \in n + Q_{1/2} \) that
\[
\mathcal{M}_t^G f(n) \leq \frac{1}{|G_{t-c(G)}|} \int_{|x-s|_G \leq t+2c(G)} F(s)ds = \left( \frac{t+2c(G)}{t-c(G)} \right)^d M_{t+2c(G)}^G F(x) = \left( 1 + \frac{3c(G)}{t-c(G)} \right)^d M_{t+2c(G)}^G F(x) \leq \left( 1 + \frac{3}{d-1} \right)^d M_{t+2c(G)}^G F(x).
\]
The above implies (2.3), hence (2.3) is proved. \( \square \)

As a corollary of Proposition 2.1 we obtain dimension-free estimates for the maximal functions over large scales associated with the Hardy–Littlewood averaging operators \( \mathcal{M}_t^B \) for \( q \in [1, \infty] \), where \( B^q \) is a ball as in (1.3).
Our aim will be to construct, for every \( p \in \mathbb{R}^d \), the ellipsoid as in (1.7). We note that

\[
C \quad \text{for some constant } C \quad \text{and the implied constant } C_{p,q} \text{ is independent of the dimension } d. 
\]

Proof. By [15] and [4] we know that for all \( p \in (1, \infty) \) and \( q \in [1, \infty] \)

\[
\| M_{p,q}^* f \|_{L^p} \leq C_{p,q} \| f \|_{L^p},
\]

and for some constant \( C_{p,q} > 0 \), which is independent of the dimension. Moreover, a simple calculation shows that

\[
2e(B^q) = d^{1/q}. 
\]

Therefore, invoking Proposition 2.1 and arguing in a similar way as in the proof of Corollary 2.2 we obtain

\[
\| M_{p,q}^* f \|_{L^p} \leq C_{p,q} \| f \|_{L^p}, 
\]

for all \( p \in (1, \infty) \). This completes the proof of the corollary. \( \square \)

2.2. Proof of Theorem 1 We fix a sequence \( 1 \leq \lambda_1 < \ldots < \lambda_d < \sqrt{2} \) and recall that

\[
E = \left\{ x \in \mathbb{R}^d : \sum_{j=1}^d \lambda_j^2 x_j^2 \leq 1 \right\}
\]

is the ellipsoid as in (1.7). We note that \( \sqrt{2} B^2 \subseteq E \subseteq B^2 \), hence

\[
\frac{1}{2} \leq c(E) \leq d^{1/2}. 
\]

Therefore, invoking Proposition 2.1 and arguing in a similar way as in the proof of Corollary 2.2 we obtain that for every \( p \in (3/2, \infty) \) there is a constant \( C_p > 0 \) such that the following inequality

\[
\| M_{p,q}^* f \|_{L^p} \leq C_p \| f \|_{L^p},
\]

holds for all \( f \in \ell^p(\mathbb{Z}^d) \), since by 3 and 4 we know, for \( p \in (3/2, \infty) \), that

\[
\| M_{p,q}^* \|_{L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)} \leq C_p',
\]

for some constant \( C_p' > 0 \), which is independent of the dimension.

On the other hand we shall show that for all \( p \in (1, \infty) \) and for all \( f \in \ell^p(\mathbb{Z}^d) \) the full maximal inequality

\[
\| \sup_{t \geq 0} |M_{p,q}^* f| \|_{L^p} \leq C_p(d) \| f \|_{L^p},
\]

involves the smallest constant \( C_p(d) > 0 \) unbounded in \( d \). In fact, as claimed in Theorem 1 one has

\[
C_p(d) \gtrsim (\log d)^{1/p}
\]

with the implicit constant, which does not depend on \( d \in \mathbb{N} \).

Proof of Theorem 3 Let \( e_i \) be the i-th basis vector from the standard basis in \( \mathbb{R}^d \) and note that for every \( j \in \{1, \ldots, d\} \) we have

\[
\Omega_j := \lambda_j E \cap \mathbb{Z}^d = \{0, \pm e_1, \ldots, \pm e_j\}. 
\]

For every \( j \in \{1, \ldots, d\} \) and \( x \in \mathbb{Z}^d \), let

\[
\mathcal{K}_{\Omega_j}(x) := \mathcal{K}_{\mathcal{V}_j}(x) = \frac{1}{|\Omega_j|} \mathbf{1}_{\Omega_j}(x).
\]

Our aim will be to construct, for every \( p \in (1, \infty) \), a non-zero function \( f \in \ell^p(\mathbb{Z}^d) \) such that

\[
\| \sup_{t \geq 0} |K_{\mathcal{V},q}^* f| \|_{L^p} \geq \| \max_{1 \leq j \leq d} |K_{\Omega_j}^* f| \|_{L^p} \geq C_p(d)^{1/p} \| f \|_{L^p},
\]

for some constant \( C_p > 0 \) which depends only on \( p \). For this purpose let \( r \in \mathbb{N}_0 \) be such that

\[
2^{r+1} - 1 \leq d < 2^{r+2} - 1.
\]

With this choice of \( r \), since \( \sum_{n=0}^r 2^n = 2^{r+1} - 1 \), we decompose \( \mathbb{Z}^d \) as follows

\[
\mathbb{Z}^d = \left( \prod_{n=0}^r \mathbb{Z}^{l_i} \right) \times \mathbb{Z}^{a(r,d)},
\]

where \( l_i \) and \( a(r,d) \) are determined by the dimension of \( \mathbb{Z}^d \) and the number of basis vectors \( d \) of the standard basis in \( \mathbb{R}^d \).
where \( I_s = \{2^s - 1, 2^s, \ldots, 2^{s+1} - 2\} \) and \( a(r, d) = d - 2^{r+1} + 1 \). For \( s \in \{0, \ldots, r\} \) we set
\[
A_s := \left\{ y \in \mathbb{Z}^{I_s} : \forall i \in I_s \ |y_i| \leq 2^d \text{ and } \sum_{i \in I_s} y_i \text{ is odd} \right\}.
\]
Note that \( |I_s| = 2^s \), hence
\[
(2^{d+1} + 1)^{2^{r-s}} \cdot 2^d \leq |A_s \cap \mathbb{Z}^{I_s}| \leq (2^{d+1} + 1)^{2^{r-s}}
\]
and thus
\[
\frac{1}{3} (2^{d+1} + 1)^{2^{r-s}} \leq |A_s \cap \mathbb{Z}^{I_s}| \leq (2^{d+1} + 1)^{2^{r-s}}.
\]
Now for each \( x \in \mathbb{Z}^d \) we take
\[
f(x) = \mathbb{1}_{A_0} \cdots \mathbb{1}_{A_r} \otimes \delta_0(x),
\]
where \( \delta_0 \) stands for the Dirac delta at zero in \( \mathbb{Z}^{n(r,d)} \).

Therefore, for all \( x \in \mathbb{Z}^d \) we have
\[
\max_{1 \leq j \leq d} |K_{\Omega_j} \ast f(x)| \geq \max_{0 \leq s \leq r} |K_{\Omega_{2^s}} \ast f(x)| \geq \max_{0 \leq s \leq r} \left( \mathbb{1}_{A_0} \cdots \mathbb{1}_{A_{s-1}} \otimes \left( \frac{1}{|\Omega_{2^s}|} \sum_{j \in I_s} \mathbb{1}_{A_r} \pm \varepsilon_j \right) \otimes \mathbb{1}_{A_{s+1} \times \ldots \times A_r} \otimes \delta_0 \right)(x).
\]
(2.6)

For every \( s \in \{0, \ldots, r\} \) let
\[
A'_s := \left\{ y \in \mathbb{Z}^{I_s} : \forall i \in I_s \ |y_i| \leq 2^d \text{ and } \sum_{i \in I_s} y_i \text{ is even} \right\}
\]
and observe that for all \( j \in I_s \) we have \( A'_s \subseteq A_s \otimes \varepsilon_j \), and
\[
\frac{1}{|\Omega_{2^s}|} \sum_{j \in I_s} \mathbb{1}_{A_s \pm \varepsilon_j}(x) = \frac{1}{2^{s+1} + 1} \sum_{j \in I_s} \mathbb{1}_{A_s \pm \varepsilon_j}(x) \geq \frac{1}{3} \mathbb{1}_{A'_s}(x);
\]
(2.7)
as well as
\[
A'_s \cap A_s = \emptyset \text{ and } \frac{1}{3} (2^{d+1} + 1)^{2^{r-s}} \leq |A'_s \cap \mathbb{Z}^{I_s}| \leq (2^{d+1} + 1)^{2^{r-s}}.
\]
In particular, for \( s \in \{0, \ldots, r\} \), the sets
\[
B_s := A_0 \times \ldots \times A_{s-1} \times A'_s \times A_{s+1} \times \ldots \times A_r,
\]
are pairwise disjoint subsets of \( \prod_{s=0}^r Z^{I_s} = \mathbb{Z}^{2^{r+1} - 1} \) such that
\[
|B_s \cap \mathbb{Z}^{2^{r+1} - 1}| \geq \frac{1}{3} \prod_{s=0}^r |A_s \cap \mathbb{Z}^{I_s}| = \frac{1}{3} \|f\|_{\ell_p}.
\]
(2.8)

Having defined \( B_s \) and using (2.7) it follows, for all \( x \in \mathbb{Z}^d \), that
\[
\max_{0 \leq s \leq r} \left( \mathbb{1}_{A_0} \cdots \mathbb{1}_{A_{s-1}} \otimes \left( \frac{1}{|\Omega_{2^s}|} \sum_{j \in I_s} \mathbb{1}_{A_s} \pm \varepsilon_j \right) \otimes \mathbb{1}_{A_{s+1} \times \ldots \times A_r} \otimes \delta_0 \right)(x)
\]
\[
\geq \frac{1}{3} \max_{0 \leq s \leq r} \left( \mathbb{1}_{B_s} \otimes \delta_0 \right)(x) = \frac{1}{3} \left( \sum_{s=0}^r \left( \mathbb{1}_{B_s} \otimes \delta_0 \right)(x) \right)^{1/p}.
\]
Thus, by (2.6) and (2.8) we obtain
\[
\| \max_{1 \leq j \leq d} |K_{\Omega_j} \ast f(x)| \|_{\ell_p} \geq 3^{-1/p} \| f \|_{\ell_p} \geq C_p \| \log d \|_p \| f \|_{\ell_p}
\]
for some constant \( C_p > 0 \) which depends only on \( p \). This completes the proof of Theorem. []
3. Fourier transform estimates

In this section we turn to the main positive results of this paper and only treat the case of cubes. We supply estimates independent of the dimension for the Fourier multipliers \( m^Q_t = K^Q_t \) corresponding to the operators \( M^Q_t \) defined in \([1]\) with \( G = Q \), where \( Q = [-1, 1]^d \). In what follows, the product structure of the cubes \( Q_t \cap \mathbb{Z}^d \) for \( t > 0 \) will be crucial. It allows us to prove the key inequalities in Proposition 3.1 which are very reminiscent of corresponding inequalities for the continuous case in \([2]\), \([3]\) and \([4]\).

From now on, we will only be working with the cubes, so we shall abbreviate

\[
M_t = M^Q_t, \quad K_t = K^Q_t, \quad m_t = m^Q_t.
\]

Note that \( |Q_t \cap \mathbb{Z}^d| = |Q_{\lfloor t \rfloor} \cap \mathbb{Z}^d| \) and \( Q_t \cap \mathbb{Z}^d = Q_{\lfloor t \rfloor} \cap \mathbb{Z}^d \) for all \( t \in (0, \infty) \). Thus

\[
K_t(x) = \lfloor t \rfloor (2\lfloor t \rfloor + 1)^d \sum_{m \in Q_{\lfloor t \rfloor}} \delta_m(x) \quad \text{for} \quad x \in \mathbb{Z}^d,
\]

and

\[
m_t(x) = m_{\lfloor t \rfloor}(x) = \frac{1}{(2\lfloor t \rfloor + 1)^d} \sum_{m \in Q_{\lfloor t \rfloor}} e^{2\pi im \cdot x} \quad \text{for} \quad x \in \mathbb{T}^d.
\]

For \( \xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d \), by a simple calculation, we have

\[
m_t(\xi) = \frac{1}{(2\lfloor t \rfloor + 1)^d} \sum_{m \in Q_{\lfloor t \rfloor}} e^{2\pi im \cdot \xi} = \prod_{k=1}^d \frac{\sin((2\lfloor t \rfloor + 1)\pi \xi_k)}{(2\lfloor t \rfloor + 1)\sin(\pi \xi_k)}.
\]

Remark 3.1. The torus \( \mathbb{T}^d \) is a priori endowed with the periodic norm

\[
\|\xi\| := \left( \sum_{k=1}^d \|\xi_k\|^2 \right)^{1/2} \quad \text{for} \quad \xi = (\xi_1, \ldots, \xi_d) \in \mathbb{T}^d,
\]

where \( \|\xi_k\| := \text{dist}(\xi_k, Z) \) for all \( \xi_k \in \mathbb{T} \) and \( k \in \{1, \ldots, d\} \). However, we identify \( \mathbb{T}^d \) with \([-1/2, 1/2]^d\), hence the norm \( \|\cdot\| \) coincides with the Euclidean norm \( \|\cdot\| \) restricted to \([-1/2, 1/2]^d\). Therefore, throughout this section, unless otherwise stated, all estimates will be provided in terms of the Euclidean norm \( |\xi| = \left( \sum_{k=1}^d |\xi_k|^2 \right)^{1/2} \) for all \( \xi \in \mathbb{T}^d \).

The main results of this section are gathered in the proposition below.

Proposition 3.1. There exists a universal constant \( C > 0 \) such that for every \( d \in \mathbb{N} \), \( t, t_1, t_2 \geq 1 \), and for every \( \xi \in \mathbb{T}^d \) we have

\[
|m_t(\xi)| \leq \frac{C}{|\xi|},
\]

\[
|m_t(\xi) - 1| \leq C|\xi|,
\]

\[
|m_{t_1}(\xi) - m_{t_2}(\xi)| \leq C\|t_1 - t_2\| \max \{t_1^{-1}, t_2^{-1}\}.
\]

The first estimate in (3.1) will follow from Lemma 3.2 and the remaining two estimates will be a consequence of Lemma 3.3.

Lemma 3.2. There exists a constant \( C > 0 \) such that for every \( d, N \in \mathbb{N} \) and for every \( \xi \in \mathbb{T}^d \) we have

\[
|m_N(\xi)| = \left| \prod_{k=1}^d \frac{\sin((2N + 1)\pi \xi_k)}{(2N + 1)\sin(\pi \xi_k)} \right| \leq \frac{C}{N|\xi|}.
\]

Proof. The proof will be completed if we show equivalently that there is a constant \( C > 0 \) such that for every \( d, N \in \mathbb{N} \) and \( 0 < \xi_k \leq 1/2 \) for \( k \in \{1, \ldots, d\} \) we have

\[
\sum_{k=1}^d (2N + 1)^2 \xi_k^2 \cdot \left( \prod_{j=1}^d \left( \frac{\sin((2N + 1)\pi \xi_j)}{(2N + 1)\sin(\pi \xi_j)} \right)^2 \right) \leq C.
\]

For \( 0 < |x| \leq \pi/2 \) we know that

\[
\frac{2}{\pi} \leq \frac{\sin x}{x} \leq 1.
\]
Thus instead of (3.3) it suffices to show
\[
\sum_{k=1}^{d} ((2N + 1) \sin(\pi \xi_k))^2 \cdot \left( \prod_{j=1}^{d} \frac{(\sin((2N + 1)\pi \xi_j))^2}{((2N + 1) \sin(\pi \xi_j))^2} \right) \leq C.
\]

For this purpose set
\[
A = \{ k \in [1, d] \cap \mathbb{Z} : ((2N + 1) \sin(\pi \xi_k))^2 \geq 2 \}
\]
and note that
\[
\sum_{k=1}^{d} ((2N + 1) \sin(\pi \xi_k))^2 \cdot \left( \prod_{j=1}^{d} \frac{(\sin((2N + 1)\pi \xi_j))^2}{((2N + 1) \sin(\pi \xi_j))^2} \right)
\]
\[
\leq \sum_{k \in A} ((2N + 1) \sin(\pi \xi_k))^2 \cdot \left( \prod_{j \in A} \frac{(\sin((2N + 1)\pi \xi_j))^2}{((2N + 1) \sin(\pi \xi_j))^2} \right) \quad (3.5)
\]
\[
+ \sum_{k \in A^c} ((2N + 1) \sin(\pi \xi_k))^2 \cdot \left( \prod_{j \in A^c} \frac{(\sin((2N + 1)\pi \xi_j))^2}{((2N + 1) \sin(\pi \xi_j))^2} \right).
\]

We shall estimate the sums from (3.5) separately.

For the first sum let \( M = \max_{k \in A} (2N + 1) \sin(\pi \xi_k) \). Then
\[
\sum_{k \in A} ((2N + 1) \sin(\pi \xi_k))^2 \cdot \left( \prod_{j \in A} \frac{(\sin((2N + 1)\pi \xi_j))^2}{((2N + 1) \sin(\pi \xi_j))^2} \right)
\]
\[
\leq \sum_{k \in A} ((2N + 1) \sin(\pi \xi_k))^2 \cdot \left( \prod_{j \in A} \frac{(\sin((2N + 1)\pi \xi_j))^2}{((2N + 1) \sin(\pi \xi_j))^2} \right) \frac{1}{M^{22|A| - 2}}
\]
\[
\leq \sum_{k \in A} \frac{M^2}{M^{22|A| - 2}} \leq 4|A| \cdot 4^{-|A|} \leq C.
\]

For the second sum in (3.5) we may assume, without loss of generality, that \( A^c = \{ 1, \ldots, d \} \). Then it suffices to prove that for any \( 0 < \xi_k \leq \pi/2 \) with \( k \in \{ 1, \ldots, d \} \) we have
\[
\sum_{k=1}^{d} ((2N + 1) \sin \xi_k)^2 \cdot \left( \prod_{j=1}^{d} \frac{(\sin((2N + 1)\xi_j))^2}{((2N + 1) \sin \xi_j)^2} \right) \leq C, \quad (3.6)
\]
provided that for all \( k \in \{ 1, \ldots, d \} \) we have \( ((2N + 1) \sin \xi_k)^2 \leq 2 \). For every \( x > 0 \) we know that
\[
x - \frac{x^3}{3!} < \sin x < x - \frac{x^3}{3!} + \frac{x^5}{5!}.
\]
This in turn implies that for \( 0 \leq x \leq 2 \) we get
\[
x - \frac{x^3}{6} \leq \sin x \leq x - \frac{x^3}{8}. \quad (3.7)
\]
Invoking (3.7) twice we obtain
\[
\sin((2N + 1)\xi_k) \leq (2N + 1)\xi_k - \frac{(2N + 1)\xi_k^3}{8}
\]
\[
\leq (2N + 1)\sin \xi_k + \frac{(2N + 1)\xi_k^3}{6} - \frac{(2N + 1)\xi_k^3}{8}
\]
\[
= (2N + 1)\sin \xi_k - (2N + 1)\xi_k \left( \frac{(2N + 1)^2}{8} - \frac{1}{6} \right)
\]
\[
\leq (2N + 1)\sin \xi_k - \frac{(2N + 1)\xi_k^3}{10},
\]
since
\[
\frac{(2N + 1)^2}{8} - \frac{1}{6} = \frac{(2N + 1)^2}{10} \iff (2N + 1)^2 \geq \frac{20}{3} \iff N \geq 1.
\]
Moreover, since \( \nu \) is a constant \( C > 0 \), we have
\[
\frac{\sin((2N + 1)\xi_k)}{(2N + 1)\sin \xi_k} \leq 1 - \frac{(2N + 1)\xi_k^3}{10(2N + 1)\sin \xi_k}
\]
Using (3.9) we can dominate the left hand side of (3.6) and obtain
\[
\int_Q \sum_{k=1}^d (2N + 1)\sin \xi_k \leq 10 \prod_{j=1}^d \left(1 - \frac{(2N + 1)\sin \xi_j^3}{10(2N + 1)\sin \xi_j}\right),
\]
provided that for all \( k \in \{1, \ldots, d\} \) we have \( (2N + 1)\sin \xi_k^3 \leq 2 \). Changing the variables in (3.10) by taking \( a_k = (2N + 1)\sin \xi_k^3 \) we have to show that there is a constant \( C > 0 \) such that for any \( d \in \mathbb{N} \) one has
\[
F(a_1, \ldots, a_d) = (a_1 + \ldots + a_d) \prod_{j=1}^d \left(1 - \frac{a_j}{10}\right) \leq C
\]
for all \( 0 \leq a_k \leq 2 \) with \( k \in \{1, \ldots, d\} \). To obtain (3.11) we note that the estimates
\[
1 - u \leq e^{-u}, \quad u e^{-u} \leq 1,
\]
which are valid for \( u \geq 0 \), lead to
\[
F(a_1, \ldots, a_d) \leq (a_1 + \ldots + a_d) \exp \left(-\frac{a_1 + \ldots + a_d}{10}\right) \leq 10.
\]
This completes the proof of (3.11) and the proof of Lemma 3.2.

The inequality (3.12) immediately implies the first inequality in (3.1). We now provide the remaining two inequalities in (3.1). For this purpose we will need some portion of notations and facts from [2].

For every \( t \geq 0 \) and \( \xi \in \mathbb{R}^d \), we introduce
\[
\nu_t(\xi) = \frac{1}{(2t + 1)^d} \int_{[-t-1/2,t+1/2]^d} e^{2\pi i \xi \cdot x} dx.
\]
Changing the variables one obtains
\[
\nu_t(\xi) = \int_{Q_{1/2}} e^{2\pi i (2t+1)\xi \cdot x} dx = \prod_{k=1}^d \frac{\sin((2t + 1)\pi \xi_k)}{(2t + 1)\pi \xi_k}.
\]
Observe that \( |Q_{1/2}| = 1 \) and that the cube \( Q_{1/2} \) is in the isotropic position, i.e.
\[
\int_{Q_{1/2}} (x \cdot \xi)^2 dx = L(Q_{1/2}) \cdot |\xi|^2 \quad \text{for all} \quad \xi \in \mathbb{R}^d,
\]
with the isotropic constant \( L(Q_{1/2}) = 1/12 \). Therefore, it follows from [2] (see also [3] p. 63) that there is a constant \( C > 0 \) such that for every \( d \in \mathbb{N}, t \geq 0 \) and for every \( \xi \in \mathbb{R}^d \) we have
\[
|\nu_t(\xi)| \leq C \min \{1, (t|\xi|)^{-1}\}, \quad |\nu_t(\xi) - 1| \leq C(2t + 1)|\xi|, \quad |\xi, \nabla \nu_t(\xi)| \leq C. \tag{3.12}
\]
Moreover, since \( \nu_t(\xi) = \nu_0((2t + 1)\xi) \) the estimate \( |\xi, \nabla \nu_0(\xi)| \leq C \) and the mean value theorem give
\[
|\nu_{N_1}(\xi) - \nu_{N_2}(\xi)| \leq C |N_1 - N_2| \max \{N_1^{-1}, N_2^{-1}\} \tag{3.13}
\]
for every \( N_1, N_2 \in \mathbb{N} \) and \( \xi \in \mathbb{R}^d \).

We now prove Lemma 3.3, which will imply the second and the third inequality in (3.1).
Lemma 3.3. There exists a constant $C > 0$ such that for every $d, N, N_1, N_2 \in \mathbb{N}$ and for every $\xi \in \mathbb{T}^d$ we have
\[ |m_N(\xi) - 1| \leq C N|\xi|, \]  
and
\[ |m_{N_1}(\xi) - m_{N_2}(\xi)| \leq CN_1 - N_2| \max \{N_1^{-1}, N_2^{-1}\}. \]  
Proof. By (3.12) with $t = 0$ we have
\[ |1 - \prod_{k=1}^d \sin(\pi \xi_k)\pi \xi_k| = |1 - \nu_0(\xi)| \leq C|\xi|. \]  
Therefore using (3.12) we obtain
\[ |m_N(\xi) - \nu_N(\xi)| = \left| 1 - \prod_{k=1}^d \sin(\pi \xi_k)\pi \xi_k \right| \leq C \min \left\{ \left| \frac{1}{N} \right|, \left| \frac{1}{N} \right| \right\} |\xi|. \]  
Hence (3.14) follows, since by (3.16) and (3.12) with $t = N$ we get
\[ |m_N(\xi) - 1| \leq |m_N(\xi) - \nu_N(\xi)| + |\nu_N(\xi) - 1| \leq CN|\xi|. \]  
To prove (3.15) we will use (3.13) and (3.16). We may assume that $N_1 \neq N_2$, otherwise there is nothing to do. Then
\[ |m_{N_1}(\xi) - m_{N_2}(\xi)| \leq |m_{N_1}(\xi) - \nu_{N_1}(\xi)| + |m_{N_2}(\xi) - \nu_{N_2}(\xi)| + |\nu_{N_1}(\xi) - \nu_{N_2}(\xi)| \leq C \sum_{j=1}^2 \min \{1, (N_j|\xi|)^{-1}\} |\xi| + C|N_1 - N_2| \max \{N_1^{-1}, N_2^{-1}\} \leq 3C|N_1 - N_2| \max \{N_1^{-1}, N_2^{-1}\}. \]  
and the proof of Lemma 3.3 is completed. \qed

4. Maximal estimates: proofs of Theorem 2 and Theorem 3

In this section we will be concerned with proving Theorem 2 and Theorem 3. Both of the theorems will be a consequence of a variant of an almost orthogonality principle, which was used by Carbery [7, Theorem 2] to prove (1.5) for $p \in (3/2, \infty]$. In Proposition 4.2 we will adjust the concept from [7, Theorem 2] to the discrete setup, nevertheless the main idea remains the same. These ideas will also be employed in the next section to estimate $r$-variations. We begin with the proof of Theorem 3 which is simpler. In fact, we prove a stronger result which will work for any lacunary sequence.\footnote{A sequence $(a_n : n \in \mathbb{N}_0) \subseteq (0, \infty)$ is called lacunary, if $a := \inf_{n \in \mathbb{N}_0} \frac{a_{n+1}}{a_n} > 1$.}

Theorem 7. Let $(a_n : n \in \mathbb{N}_0) \subseteq (0, \infty)$ be a lacunary sequence. Then for every $p \in (1, \infty]$ there exists a constant $C_p > 0$ such that for every $d \in \mathbb{N}$ and every $f \in L^p(\mathbb{T}^d)$ we have
\[ \| \sup_{n \in \mathbb{N}_0} |\mathcal{M}_{a_n}f| \|_{L^p} = \| \sup_{n \in \mathbb{N}_0} |\mathcal{M}_{a_n}^Qf| \|_{L^p} \leq C_p \| f \|_{L^p}. \]  
The implied constant $C_p$ may also depend on the quantity $a := \inf_{n \in \mathbb{N}_0} \frac{a_{n+1}}{a_n} > 1$, which corresponds to the lacunary sequence $(a_n : n \in \mathbb{N}_0)$.

We note that by passing to a denser sequence, by a suitable completion of gaps in the underlying sequence, we can assume that the lacunary sequence $(a_n : n \in \mathbb{N}_0)$ in Theorem 7 satisfies additionally an upper bound and is defined on $\mathbb{Z}$. Namely, in the rest of the paper, we will assume for all $n \in \mathbb{Z}$ that
\[ 1 < a \leq \frac{a_{n+1}}{a_n} \leq a^2. \]  
\[ (1.2) \]
In order to prove our maximal and variational results we have to construct a suitable semigroup on $\mathbb{Z}^d$, which will be adjusted to our problems. This will be provided in the next paragraph.

### 4.1. A diffusion semigroup and corresponding Littlewood–Paley theory.

For every $t \geq 0$ let $P_t$ be the Poisson semigroup on $\mathbb{Z}^d$, which is a convolution operator defined on the Fourier transform side by the multiplier

$$p_t(\xi) = e^{-t|\xi|^2}$$

for every $\xi \in \mathbb{T}^d$, where

$$|\xi_{\sin}| = \left(\sum_{k=1}^{d}(\sin(\pi \xi_k))^2\right)^{1/2}.$$

By (3.4), for every $\xi \in \mathbb{T}^d \equiv [-1/2, 1/2]^d$, we have

$$|\xi| \leq |\xi_{\sin}| \leq \pi|\xi|. \quad (4.3)$$

Let $\{e_1, \ldots, e_d\}$ be the standard basis in $\mathbb{Z}^d$. For every $k \in \{1, \ldots, d\}$ and $x \in \mathbb{Z}^d$ let

$$\Delta_k f(x) = f(x) - f(x + e_k)$$

be the discrete partial derivative on $\mathbb{Z}^d$, and let

$$L_k f(x) = \frac{1}{4} \Delta_k^* \Delta_k f(x)$$

be the corresponding discrete partial Laplacian. Then we see that

$$L_k f(x) = \frac{1}{2} f(x) - \frac{1}{4} (f(x + e_k) + f(x - e_k)),$$

and for every $\xi \in \mathbb{T}^d$ we obtain

$$\langle L_k f \rangle(\xi) = \frac{1 - \cos(2\pi \xi_k)}{2} \hat{f}(\xi) = (\sin(\pi \xi_k))^2 \hat{f}(\xi).$$

For every $x \in \mathbb{Z}^d$ we introduce the maximal function

$$P_* f(x) = \sup_{t > 0} P_t |f|(x)$$

and the square function

$$g(f)(x) = \left(\int_{0}^{\infty} t \left| \frac{d}{dt} P_t f(x) \right|^2 \, dt \right)^{1/2}$$

associated with the Poisson semigroup $P_t$.

**Lemma 4.1.** For every $p \in (1, \infty)$ there exists a constant $C_p > 0$, which does not depend on $d \in \mathbb{N}$, such that for every $f \in \ell^p(\mathbb{Z}^d)$ we have

$$\|P_* f\|_{\ell^p} \leq C_p \|f\|_{\ell^p}. \quad (4.4)$$

and

$$\|g(f)\|_{\ell^p} \leq C_p \|f\|_{\ell^p}. \quad (4.5)$$

For the proof we will have to check that $(P_t : t \geq 0)$ is a symmetric diffusion semigroup in the sense of [10], Chapter III. For the convenience of the reader we recall the definition of a symmetric diffusion semigroup from [10] Chapter III, p.65. Let $(X, \mathcal{B}(X), \mu)$ be a $\sigma$-finite measure space. Let $(T_t : t \geq 0)$ be a strongly continuous semigroup on $L^p(X)$ which maps $\bigcup_{1 \leq p < \infty} L^p(X)$ to $\bigcup_{1 \leq p < \infty} L^p(X)$ for every $t \geq 0$. We say that $(T_t : t \geq 0)$ is a symmetric diffusion semigroup, if it satisfies for all $t \geq 0$ the following conditions:

1. **Contraction property:** for all $p \in [1, \infty]$ and $f \in L^p(X)$ we have $\|T_t f\|_{L^p(X)} \leq \|f\|_{L^p(X)}$.
2. **Symmetry property:** each $T_t$ is a self-adjoint operator on $L^2(X)$.
3. **Positivity property:** $T_t f \geq 0$ if $f \geq 0$.
4. **Conservation property:** $T_t 1 = 1$. 

Proof of Lemma 4.1. By definition $P_t$ satisfies the semigroup property on $\ell^2$. Moreover, it is easy to check, working on the Fourier transform side, that for every $f \in \ell^2(\mathbb{Z}^d)$ we have

$$\lim_{t \to 0} \|P_tf - f\|_{\ell^2} = 0.$$ 

We shall now justify that $(P_t : t \geq 0)$ satisfies conditions (1)-(4) (in particular the contraction property (1) will ensure that $P_tf \in \ell^p(\mathbb{Z}^d)$ if $f \in \ell^p(\mathbb{Z}^d)$). Then, using the general theory of semigroups from [16, Chapter III, Section 3, p.73] together with the pairwise commuting we have

$$e^{-tL} \text{ is self-adjoint on } \ell^2(\mathbb{Z}^d) \text{ and }$$

$$e^{-tL}f = e^{-t/2} \sum_{n=0}^{\infty} \frac{(t/2)^n}{n!} (P_k)^n f \quad \text{for all } t \geq 0.$$ 

This formula obviously yields that $e^{-tL}$ is self-adjoint on $\ell^2(\mathbb{Z}^d)$, that $e^{-tL}$ is positive, and that $e^{-tL}1 = 1$. Finally, we also deduce the contraction property for $e^{-t\mathcal{L}_k}$, since

$$\|e^{-\mathcal{L}_k}f\|_{\ell^p} \leq e^{-t/2} \sum_{n=0}^{\infty} \frac{(t/2)^n}{n!} \|P_k^n f\|_{\ell^p} \leq \|f\|_{\ell^p}.$$ 

Summarizing $(e^{-t\mathcal{L}_k} : t \geq 0)$ is a symmetric diffusion semigroup. Since the operators $\mathcal{L}_1, \ldots, \mathcal{L}_d$ are pairwise commuting we have

$$\exp(-t(\mathcal{L}_1 + \ldots + \mathcal{L}_d)) = \exp(-t\mathcal{L}_1) \circ \ldots \circ \exp(-t\mathcal{L}_d) \quad \text{for all } t \geq 0.$$ 

Thus $\mathcal{L} = \mathcal{L}_1 + \ldots + \mathcal{L}_d$ generates a symmetric diffusion semigroup. Using the subordination formula

$$e^{-t\mathcal{L}^{1/2}} = \int_0^\infty e^{-tL \gamma} (\pi s)^{-1/2} e^{-s} \, ds \quad \text{for all } t \geq 0.$$ 

we see that $(e^{-t\mathcal{L}^{1/2}} : t \geq 0)$ is a symmetric diffusion semigroup as well.

It suffices to note that

$$P_tf = e^{-t\mathcal{L}^{1/2}} f,$$

since for every $\xi \in \mathbb{T}^d$ we have

$$\langle e^{-t\mathcal{L}^{1/2}} f \rangle(\xi) = e^{-t|\xi|^2} \hat{f}(\xi) = \langle P_t f \rangle(\xi).$$

Applying now the maximal theorem for semigroups [16, Chapter III, Section 3, p.73] together with the Littlewood–Paley theory for semigroups [16, Chapter IV, Section 5, p.111] we obtain (4.4) and (4.5) respectively. This completes the proof of Lemma 4.1. \qed

Finally, we will need a discrete variant of the Littlewood–Paley inequality. For the lacunary sequence $(a_n : n \in \mathbb{Z})$ as in (4.2) we define the Poisson projections $S_n$ by setting

$$S_n = P_{a_n} - P_{a_{n-1}}$$

for every $n \in \mathbb{Z}$. Then, clearly for every $f \in \ell^2(\mathbb{Z}^d)$, we have

$$f = \sum_{n \in \mathbb{Z}} S_n f.$$ 

Using (4.3) we show that for every $� \in (1, \infty)$ there is $C_{\�} > 0$ independent of $d \in \mathbb{N}$ such that for every $f \in \ell^p(\mathbb{Z}^d)$ we have the following Littlewood–Paley estimates

$$\left\| \left( \sum_{n \in \mathbb{Z}} |S_n f|^2 \right)^{1/2} \right\|_{\ell^p} \leq C_{\�} a \|f\|_{\ell^p}$$ 

(4.7)

with $a > 1$ as in (4.3). In order to establish (4.7) it suffices to observe that

$$S_n f(x) = \int_{a_{n-1}}^{a_n} \frac{d}{dt} P_t f(x) \, dt.$$
Thus by the Cauchy–Schwarz inequality we obtain for every \( n \in \mathbb{Z} \) and \( x \in \mathbb{Z}^d \) the following bound
\[
|S_n f(x)|^2 \leq \left( \int_{a_n}^{a_{n-1}} \left| \frac{d}{dt} P_t f(x) \right| \, dt \right)^2 \leq (a_n - a_{n-1}) \left( \int_{a_n}^{a_{n-1}} \left| \frac{d}{dt} P_t f(x) \right|^2 \, dt \right) \leq (a^2 - 1) a_{n-1} \int_{a_n}^{a_{n-1}} \left| \frac{d}{dt} P_t f(x) \right|^2 \, dt.
\]
Now summing over \( n \in \mathbb{Z} \) and invoking (4.5) we obtain (4.7) and we are done.

4.2. An almost orthogonality principle. We now adapt an almost orthogonality principle from [7] for our purposes. The proofs of Theorem 7 and Theorem 2 will be based on Proposition 4.2.

**Proposition 4.2.** Let \( (T_t : t \in U) \) be a family of linear operators defined on \( \bigcup_{1 \leq p \leq \infty} L^p(\mathbb{Z}^d) \) for some index set \( U \subseteq (0, \infty) \). Suppose that \( T_t = M_t - H_t \) for each \( t \in U \), where \( M_t, H_t \) are positive linear operators. Assume that the following conditions are satisfied.

- For every \( p \in (1, 2] \) we have
  \[
  \|H_{n,t}\|_{L^p \to L^p} < \infty. \tag{4.8}
  \]
- There is \( p_0 \in (1, 2) \) with the property that for every \( p \in (p_0, 2] \) we have
  \[
  \sup_{n \in \mathbb{Z}} \|T_{n,t}u\|_{L^p \to L^p} < \infty, \tag{4.9}
  \]
where \( U_n = [a_n, a_{n+1}] \cap U \) and \((a_n : n \in \mathbb{Z}) \subseteq (0, \infty)\) is a lacunary sequence obeying (4.2).
- There exists a sequence \((b_j : j \in \mathbb{Z})\) of positive numbers so that \( \sum_{j \in \mathbb{Z}} b_j^p = \rho < \infty \) for every \( \rho > 0 \). Moreover, for every \( j \in \mathbb{Z} \) we have
  \[
  \sup_{\|f\|_{L^p} \leq 1} \left\| \left( \sum_{n \in \mathbb{Z}} \sup_{t \in U_n} |T_{t,S_j+n}f|^2 \right)^{1/2} \right\|_{\ell^2} \leq b_j, \tag{4.10}
  \]
where \((S_n : n \in \mathbb{Z})\) is the resolution of identity satisfying (4.6) and (4.7) for all \( p \in (1, \infty) \).

Then for every \( p \in (p_0, 2] \), there exists a constant \( C_p > 0 \) such that
\[
\sup_{\|f\|_{L^p} \leq 1} \left\| \left( \sum_{n \in \mathbb{Z}} \sup_{t \in U_n} |T_{t,f}|^2 \right)^{1/2} \right\|_{L^p} \leq C_p. \tag{4.11}
\]
The implied constant \( C_p \) depends on the parameters \( p, p_0, \rho, B_p, \) the quantities (4.8) and (4.9) and the constants \( a > 1 \) and \( C_p > 0 \) as in (4.11). Therefore, \( C_p \) is independent of the dimension as long as the underlying parameters do not depend on \( d \in \mathbb{N} \). In particular,
\[
\|T_{n,t}u\|_{L^p \to L^p} \leq C_p. \tag{4.12}
\]

**Proof.** Fix \( p \in (p_0, 2) \). We note that (4.11) immediately implies (4.12). To prove (4.12) let \( \Sigma \) be a family of all possible sequences \( t = (t_n : n \in \mathbb{Z}) \) such that each component is a function \( t_n : \mathbb{Z}^d \to U_n \). For each \( N \in \mathbb{N}_0 \) and each \( t \in \Sigma \) we define a linear operator \( R^N_{t} : L^p \to L^p(\mathbb{Z}^d) \) by setting
\[
R^N_{t} f = \begin{cases} \quad t_n f, & \text{if } n \in [-N,N] \cap \mathbb{Z}, \\ \quad 0, & \text{otherwise.} \end{cases}
\]
We observe that \( \|R^N_{t} f\|_{L^p \to L^p(\mathbb{Z}^d)} \leq (2N + 1) \sup_{|n| \leq N} \|T_{n,t} u\|_{L^p \to L^p} < \infty \) for all \( p \in (p_0, 2] \), by (4.9). Our aim will be to show that there is a constant \( C_p > 0 \) such that
\[
\sup_{N \in \mathbb{N}_0} \sup_{t \in \Sigma} \|R^N_{t} f\|_{L^p \to L^p(\mathbb{Z}^d)} \leq C_p. \tag{4.13}
\]
Assuming momentarily (4.13) we pick a sequence \( t_f = (t_{n,f} : n \in \mathbb{Z}) \) where each component \( t_{n,f} : X \to U_n \) is a function such that \( T_{n,f}(x) = \sup_{t \in U_n} |T_{t,f}(x)| \) and obtain for every \( N \in \mathbb{N}_0 \) that
\[
\left\| \left( \sum_{|n| \leq N} \sup_{t \in U_n} |T_{t,f}|^2 \right)^{1/2} \right\|_{L^p} \leq \left\| \left( \sum_{|n| \leq N} |T_{n,f}|^2 \right)^{1/2} \right\|_{L^p} \leq \sup_{t \in \Sigma} \|R^N_{t} f\|_{L^p \to L^p(\mathbb{Z}^d)} \leq C_p \|f\|_{L^p},
\]
\[\overset{3}{\text{We say that a linear operator } T \text{ is positive if } Tf \geq 0 \text{ for every function } f \geq 0.} \]
where the last inequality follows from (4.13). Then invoking the monotone convergence theorem we obtain the claim in (4.11). To prove (4.11) we fix \( t = (t_n : n \in \mathbb{Z}) \in \mathcal{T} \) and for \( s \in (p_0, 2] \) and \( r \in [1, \infty] \) let \( A_N(s, r) \) be the best constant in the following inequality

\[
\left\| \left( \sum_{|n| \leq N} |T_{t_n}g_n|^r \right)^{1/r} \right\|_{\ell^s} \leq A_N(s, r) \left\| \left( \sum_{|n| \leq N} |g_n|^r \right)^{1/r} \right\|_{\ell^s}.
\]

(4.14)

Using (4.13) it is easy to see that \( A_N(s, r) < \infty \). We pick a real number \( q \) such that \( p_0 < q < p < 2 \) and define \( \theta \in (0, 1) \) by setting

\[
\frac{1}{2} = \frac{1 - \theta}{q} + \frac{\theta}{\infty}.
\]

This in turn implies that \( \theta = 1 - q/2 \) and consequently determines \( u \in (q, p) \) such that

\[
\frac{1}{u} = \frac{1 - \theta}{q} + \frac{\theta}{p}.
\]

Using complex interpolation we obtain

\[
A_N(u, 2) \leq A_N(q, q)^{1-\theta} A_N(p, \infty)^{\theta}.
\]

Invoking (4.13) we have \( A_N(q, q) \leq \sup_{n \in \mathbb{Z}} \|T_{t_n}u_n\|_{\ell^q \rightarrow \ell^q} \), since

\[
\left\| \left( \sum_{|n| \leq N} |T_{t_n}g_n|^q \right)^{1/q} \right\|_{\ell^q} \leq \left( \sum_{|n| \leq N} \sup_{t \in U_n} |T_{t}g_n|^q \right)^{1/q} \leq \sup_{n \in \mathbb{Z}} \|T_{t_n}u_n\|_{\ell^q \rightarrow \ell^q}\left( \sum_{|n| \leq N} |g_n|^q \right)^{1/q} \right\|_{\ell^q}.
\]

Invoking (4.13) we obtain \( A_N(p, \infty) \leq \|R_N^{1}||_{\ell^p \rightarrow \ell^p(\mathbb{Z})} + 2\|H_{s, U}\|_{\ell^p \rightarrow \ell^p} \). Indeed, let \( g = \sup_{|n| \leq N} |g_n| \) and recall that \( M_t = T_t + H_t \), then

\[
\left\| \sup_{|n| \leq N} |T_{t_n}g_n| \right\|_{\ell^p} \leq \left\| \sup_{|n| \leq N} M_{t_n}g \right\|_{\ell^p} + \left\| \sup_{t \in U} H_tg \right\|_{\ell^p}
\]

\[
\leq \left\| \left( \sum_{|n| \leq N} |T_{t_n}g|^2 \right)^{1/2} \right\|_{\ell^p} + 2\left\| \sup_{t \in U} H_tg \right\|_{\ell^p}
\]

\[
\leq \left( \|R_N^{1}||_{\ell^p \rightarrow \ell^p(\mathbb{Z})} + 2\|H_{s, U}\|_{\ell^p \rightarrow \ell^p} \right)\left\| g \right\|_{\ell^p}.
\]

Moreover, (4.10) implies that

\[
\left\| \left( \sum_{|n| \leq N} |T_{t_n}S_{j+n}f|^2 \right)^{1/2} \right\|_{\ell^q} \leq b_j \|f\|_{\ell^2}.
\]

(4.15)

By (4.14) and (4.7) we get

\[
\left\| \left( \sum_{|n| \leq N} |T_{t_n}S_{j+n}f|^2 \right)^{1/2} \right\|_{\ell^q} \leq A_N(u, 2) \left\| \left( \sum_{|n| \leq N} |S_{j+n}f|^2 \right)^{1/2} \right\|_{\ell^q}
\]

\[
\leq C_{\theta} a A_N(u, 2) \|f\|_{\ell^p}.
\]

(4.16)

We now take \( \rho \in (0, 1) \) satisfying

\[
\frac{1}{p} = \frac{1 - \rho}{u} + \frac{\rho}{2}
\]

and interpolate (4.15) with (4.16), then

\[
\left\| \left( \sum_{|n| \leq N} |T_{t_n}S_{j+n}f|^2 \right)^{1/2} \right\|_{\ell^p} \leq (C_{\theta} a A_N(u, 2))^{1-\rho} b_j \|f\|_{\ell^p}.
\]

(4.17)

Summing (4.17) over \( j \in \mathbb{Z} \) it is easy to see that

\[
\|R_N^{1}||_{\ell^p \rightarrow \ell^p(\mathbb{Z})} \leq \left( C_{\theta} a \sup_{n \in \mathbb{Z}} \|T_{t_n}u_n\|_{\ell^q \rightarrow \ell^q} \left( \|R_N^{1}||_{\ell^p \rightarrow \ell^p(\mathbb{Z})} + 2\|H_{s, U}\|_{\ell^p \rightarrow \ell^p} \right)^{\rho} \right)^{1-\rho} B_{\rho}.
\]

Thus there exists \( 0 < C_{\rho} < \infty \), such that (4.13) holds. This completes the proof of Proposition 4.2 \( \Box \)

We now prove Theorem 7, which immediately implies Theorem 3 by taking \( a_n = 2^n \) for all \( n \in \mathbb{N}_0 \).
4.3. Proof of Theorem 7 To prove Theorem 7 we shall exploit Proposition 4.2 and Proposition 5.1 and properties of the Poisson semigroup $P_t$ from Lemma 4.4 and the Littlewood–Paley inequality (4.1).

Proof of Theorem 7. Observe first that since $M_{a_n} f = f$ if $a_n < 1$ we can assume without loss of generality that our lacunary sequence is such that $a_0 > 1$.

Inequality (4.10) ensures that

$$
\| \sup_{n \in \mathbb{N}_0} |M_{a_n} f| \|_{L^p} \leq \| P_t f \|_{L^p} + \| \sup_{n \in \mathbb{N}_0} |(M_{a_n} - P_{a_n}) f| \|_{L^p} \leq C \| f \|_{L^p} + \| \sup_{n \in \mathbb{N}_0} |(M_{a_n} - P_{a_n}) f| \|_{L^p}.
$$

We only have to handle the second maximal function. For this purpose we will appeal to Proposition 4.2 with the parameter $p_0 = 1$, the set $U = \{ a_n : n \in \mathbb{N}_0 \}$ (so that $U \subseteq [1, \infty)$), the operators $M_t = M_{a_n}$ and $H_t = P_t$, where $P_t$ is the Poisson semigroup and a sequence $b_j \lesssim a^{-j/2}$, where $a > 1$ is the parameter from (4.2).

In this case (4.18) and (4.19) are obvious, since $U_n = [a_n, a_{n+1}) \cap U = \{ a_n \}$ if $n \geq 0$ and $U_n = \emptyset$ if $n < 0$ (recall our convention that $T_s \emptyset = 0$). Thus it only remains to verify condition (4.11). For every $t > 0$ and $\xi \in \mathbb{T}^d$ let

$$
n_t(\xi) = m_t(\xi) - p_t(\xi) = m_t(\xi) - e^{-t|\xi|_{\text{min}}},
$$

be the multiplier associated with the operator $T_t = M_t - H_t = M_t - P_t$. Observe that by Proposition 5.1 the inequality (4.12) and the properties of $p_t(\xi)$ there exists a constant $C > 0$ independent of the dimension such that for $t \geq 1$ it holds

$$
|n_t(\xi)| \leq |m_t(\xi) - 1| + |p_t(\xi) - 1| \leq C t|\xi|, \quad \text{and} \quad |n_t(\xi)| \leq C(t|\xi|)^{-1}.
$$

Therefore, by (4.11), the Plancherel theorem and (4.13) we obtain

$$
\left\| \left( \sum_{n \in \mathbb{Z}} \sup_{t \in U_n} |T_t S_j + n f|^2 \right)^{1/2} \right\|_{L^2} = \left\| \left( \sum_{n \in \mathbb{N}_0} |T_{a_n} S_{j+n} f|^2 \right)^{1/2} \right\|_{L^2}
= \left( \int_{\mathbb{T}^d} \sum_{n \in \mathbb{N}_0} |n_{a_n}(\xi)(e^{-a_n + j|\xi|_{\text{min}}} - e^{-a_n - j|\xi|_{\text{min}}})|^2 |f(\xi)|^2 d\xi \right)^{1/2}
\lesssim \left( \int_{\mathbb{T}^d} \sum_{n \in \mathbb{N}_0} E_{n,j}(\xi)^2 |f(\xi)|^2 d\xi \right)^{1/2},
$$

where

$$
E_{n,j}(\xi) := \min \{ \{ a_n|\xi|, (a_n|\xi|)^{-1} \} \{ e^{-a_n + j|\xi|_{\text{min}}} - e^{-a_n - j|\xi|_{\text{min}}} \}. \}
$$

We claim that

$$
E_{n,j}(\xi) \lesssim a^{-j/2} \min \{ (a_n|\xi|)^{1/2}, (a_n|\xi|)^{-1/2} \}.
$$

Indeed, if $j \geq 0$, then

$$
E_{n,j}(\xi) \lesssim \min \{ (a_n|\xi|, (a_n|\xi|)^{-1}) \cdot e^{-a_n - j|\xi|_{\text{min}}}
\lesssim \min \{ (a_n|\xi|)^{1/2}, (a_n|\xi|)^{-1/2} \} \cdot e^{-a_n - j|\xi|_{\text{min}}}
\lesssim a^{-j/2} \min \{ (a_n|\xi|)^{1/2}, (a_n|\xi|)^{-1/2} \}.
$$

If $j < 0$, then

$$
E_{n,j}(\xi) \lesssim \min \{ (a_n|\xi|, (a_n|\xi|)^{-1}) \min \{ a_{n+j}|\xi|, e^{-a_{n+j} - j|\xi|_{\text{min}}} \}
\lesssim \min \{ (a_n|\xi|)^{1/2}, (a_n|\xi|)^{-1/2} \} \cdot (a_n|\xi|)^{-j/2} \cdot (a_{n+j}|\xi|)^{1/2}
\lesssim a^{j/2} \min \{ (a_n|\xi|)^{1/2}, (a_n|\xi|)^{-1/2} \}.
$$

We use (4.21) to estimate (4.20) and obtain

$$
\left\| \left( \sum_{n \in \mathbb{N}_0} \sup_{t \in U_n} |T_t S_{j+n} f|^2 \right)^{1/2} \right\|_{L^2} \lesssim a^{-j/2} \left( \int_{\mathbb{T}^d} \sum_{n \in \mathbb{N}_0} \min \{ a_n|\xi|, (a_n|\xi|)^{-1} \} |f(\xi)|^2 d\xi \right)^{1/2}
\lesssim a^{-j/2} \| f \|_{L^2}.
$$

Note that in (4.21) and (4.22) only the lower bound from (4.12) is required. The inequality (4.22) implies condition (4.10) in Proposition 4.2. Hence, the proof of Theorem 7 is completed. □
4.4. Proof of Theorem 3 We shall demonstrate how to use Proposition 4.2 and Proposition 3.1 to deduce Theorem 2. The new ingredient will be inequality (4.23), which is invaluable here. The proof of Lemma 4.3 immediately follows from [5, Lemma 2.1] for r = ∞, hence we omit it here.

**Lemma 4.3.** For every n ∈ N₀ and every function a : [2ⁿ, 2ⁿ⁺¹] → ℂ satisfying a(t) = a([t]) we have

\[
\sup_{2^n ≤ t < 2^{n+1}} |a(t) - a(2^n)| ≤ 2^{1-1/r} \sum_{0 ≤ k ≤ n} \left( \sum_{k=0}^{2^k-1} \left| a(2^n + 2^{n-i}(k+1)) - a(2^n + 2^{n-i}k) \right|^r \right)^{1/r}.
\]

**Proof of Theorem 3** Observe that Mₜ = I for 0 < t < 1. Therefore it suffices to bound supₜ≥1 |Mₜ|f|

In a similar way as in Theorem 7 we will use Proposition 4.2 with the parameter p₀ = 3/2, the sequence aₙ = 2ⁿ, the set \( U = [1, \infty) \), the operators \( Mₜ = Mₜ \) and \( Hₜ = M₂ⁿ \) for every \( t ∈ Uₙ = [2ⁿ, 2ⁿ⁺¹] \cap [1, \infty) \), and a sequence \( b_j \approx 2^{-ε\|j\|/4} \) for some \( ε ∈ (0, 1) \).

Theorem 3 ensures that condition (4.3) holds for the operators \( Hₜ \). It remains to prove (4.9) for all \( p ∈ (3/2, 2] \) and verify condition (4.10) for the operators \( Tₜ = Mₜ - Hₜ \). First we prove (4.9), for this purpose we use Lemma 4.3 and obtain, for \( n ≥ 0 \),

\[
\left\| \sup_{t ∈ Uₙ} |Tₜ|f \right\|_{l₁} = \left\| \sup_{2^n ≤ t < 2^{n+1}} \| (Mₜ - M₂ⁿ) f \|_{l₁} \right\|_{l₁} ≤ \sum_{0 ≤ k ≤ n} \left( \sum_{k=0}^{2^k-1} \left| (M₂ⁿ⁺2ⁿ⁻¹(k+1) - M₂ⁿ⁺2ⁿ⁻¹k) f \right|^2 \right)^{1/2} \leq 2^{1/2\|f\|_{l₁}}.
\]

The last inequality in (4.24) will follow if we show that for every \( p ∈ (3/2, 2] \), there is \( δ_p > 0 \) such that

\[
\left\| \left( \sum_{k=0}^{2^k-1} \left| (M₂ⁿ⁺2ⁿ⁻¹(k+1) - M₂ⁿ⁺2ⁿ⁻¹k) f \right|^2 \right)^{1/2} \right\|_{l₁} ≤ 2^{-1/2\|f\|_{l₁}}.
\]

This in turn will follow by interpolation between (4.25) and (4.26), since for \( p = 1 \) it is easy to see that

\[
\| |(M₂ⁿ⁺2ⁿ⁻¹(k+1) - M₂ⁿ⁺2ⁿ⁻¹k) f |² \|_{l₁}^{1/2} ≤ 2^{1/2\|f\|_{l₁}},
\]

and for \( p = 2 \) we are going to show that

\[
\left\| \left( \sum_{k=0}^{2^k-1} \left| (M₂ⁿ⁺2ⁿ⁻¹(k+1) - M₂ⁿ⁺2ⁿ⁻¹k) f \right|^2 \right)^{1/2} \right\|_{l₂} ≤ 2^{-1/2\|f\|_{l₂}}.
\]

We have reduced the matter to estimate (4.26), which will be based on inequality (4.27). For every \( ε ∈ [0, 1) \) we have by (4.13) that

\[
\sum_{k=0}^{2^k-1} \left| m₂ⁿ⁺2ⁿ⁻¹(k+1) - m₂ⁿ⁺2ⁿ⁻¹k \right|^2 \leq \sum_{k=0}^{2^k-1} \left( \frac{2^{n-l}}{2^n + 2^{n-l}k} \right)^{2-ε} \leq \frac{1}{2(1-ε)^l}.
\]

Plancherel’s theorem and inequality (4.27) with \( ε = 0 \) yield (4.26), which completes the proof of (4.24).

We now verify condition (4.10). As before we apply Lemma 4.3 and for every \( ε ∈ (0, 1) \) we get

\[
\left\| \left( \sum_{n∈N₀} \left\| T_{(n)} S_{(n)} f \right\|^2 \right)^{1/2} \right\|_{l₂} = \left\| \left( \sum_{n∈N₀} \left\| T_{(n)} S_{(n)} f \right\|^2 \right)^{1/2} \right\|_{l₂} \leq \left\| \sum_{n∈N₀} \left( \sum_{k=0}^{2^k-1} \left| (M₂ⁿ⁺2ⁿ⁻¹(k+1) - M₂ⁿ⁺2ⁿ⁻¹k) S_{(n)} f \right|^2 \right)^{1/2} \right\|_{l₂} \leq 2^{-1/2\|f\|_{l₂}}.
\]

The last inequality follows from the following inequality

\[
\left\| \left( \sum_{n∈N₀} \left( \sum_{k=0}^{2^k-1} \left| (M₂ⁿ⁺2ⁿ⁻¹(k+1) - M₂ⁿ⁺2ⁿ⁻¹k) S_{(n)} f \right|^2 \right)^{1/2} \right) \right\|_{l₂} \leq 2^{-1/2\|f\|_{l₂}}.
\]
The proof of (4.29) will be very much in spirit of (4.26). Indeed, \((2^n + 2^n - k) \approx 2^n\) for every \(0 \leq k \leq 2^n\), hence due to (4.30), (5.2), and (5.11) we obtain
\[
\left| m_{2^n+2^n-k}(\xi) - m_{2^n+2^n-k}(\xi) \right|^2 \leq \min \left\{ \left| 2^n\xi, |2^n\xi|^{-1} \right| \right\} e^{-2^n|\xi|_{\text{min}}} - e^{-2^n|\xi|_{\text{min}}}^2
\]
\[
\lesssim 2^{-\epsilon|\xi|/2} \min \left\{ \left| 2^n\xi, |2^n\xi|^{-1} \right| \right\}^{\epsilon/2},
\]
where the last inequality follows from (4.21). Finally, (4.30) combined with (4.27) yields
\[
\sum_{k=0}^{2^n-1} \left| m_{2^n+2^n-k}(\xi) - m_{2^n+2^n-k}(\xi) \right|^2 \lesssim 2^{-\epsilon|\xi|/2} 2^{-n(1-\epsilon)} \min \left\{ \left| 2^n\xi, |2^n\xi|^{-1} \right| \right\}^{\epsilon/2}.
\]
Therefore, (4.31) with the Plancherel theorem establish (4.30), since
\[
\left\| \left( \sum_{n \geq 1} \sum_{k=0}^{2^n-1} (M_{2^n+2^n-k}(\xi) - M_{2^n+2^n-k}(\xi))^2 \right)^{1/2} \right\|_{L^2} \lesssim 2^{-\epsilon|\xi|/2} 2^{-n(1-\epsilon)} \int_{\{x = 0\} \cup \{x = 1\}} \min \left\{ \left| 2^n\xi, |2^n\xi|^{-1} \right| \right\}^{\epsilon/2} \sqrt{f(\xi)} d\xi \lesssim 2^{-\epsilon|\xi|/2} 2^{-n(1-\epsilon)} \|f\|_{L^2}.
\]
This justifies (4.28) and completes the proof of Theorem 2.

5. \(r\)-variational estimates: proofs of Theorem 5 and Theorem 4

We begin with some remarks on \(r\)-variation seminorms. For \(r \in [1, \infty)\) the \(r\)-variation seminorm \(V_r\) of a complex-valued function \((0, \infty) \ni t \to a_t\) is defined by
\[
V_r(a_t : t \in Z) = \sup_{t_0 < \cdots < t_j < t} \left( \sum_{j=0}^{j} |a_{t_{j+1}} - a_{t_j}|^r \right)^{1/r},
\]
where the supremum is taken over all finite increasing sequences in \(Z \subseteq (0, \infty)\).

- If \(1 \leq r_1 \leq r_2 < \infty\) then \(V_{r_2}(a_t : t \in Z) \leq V_{r_1}(a_t : t \in Z)\).
- If \(Z_1 \subseteq Z_2\) then \(V_r(a_t : t \in Z_1) \leq V_r(a_t : t \in Z_2)\).
- If \(Z\) is a disjoint sum of \(Z_1\) and \(Z_2\) then \(V_r(a_t : t \in Z) \leq V_r(a_t : t \in Z_1) + V_r(a_t : t \in Z_2) + 2 \sup_{t \in Z} |a_t|\).

For every \(t_0 \in Z\) we have
\[
\sup_{t \in Z} |a_t| \leq |a_{t_0}| + 2V_r(a_t : t \in Z).
\]
- If \(Z\) is a countable subset of \((0, \infty)\) then \(V_r(a_t : t \in Z) \leq 2 \left( \sum_{t \in Z} |a_t|^r \right)^{1/r}\).

Finally, for every \(r \in [1, \infty)\) there exists \(C_r > 0\) such that
\[
V_r(a_t : t \in Z) \leq C_r V_r(a_t : t \in Z \cap D) + C_r \left( \sum_{n \in Z} V_r((a_n - a_{2^n}) : t \in [2^n, 2^{n+1}) \cap Z)^r \right)^{1/r},
\]
where \(D = \{2^n : n \in Z\}\). The first quantity on the right side in (5.1) is called the long variation seminorm, whereas the second is called the short variation seminorm. This is a very useful inequality which, in view of Theorem 5, will allow us to reduce the proof of Theorem 4 to the estimates of short variations associated with \(M_r\).
5.1. Long variations: proof of Theorem [5] We have shown in Section 4 that the semigroup $P_t$ is a symmetric diffusion semigroup. Therefore from [9] Theorem 3.3 (see also [8] inequality (2.30)) we conclude that for every $p \in (1, \infty)$ and for every $r \in (2, \infty)$ there is a constant $C_{p,r} > 0$ independent of the dimension such that for every $f \in L^p(\mathbb{Z}^d)$ we have
\[
\|V_r(P_tf : t>0)\|_{L^p} \leq C_{p,r} \|f\|_{L^p}. \tag{5.3}
\]
For every $f \in L^p(\mathbb{Z}^d)$ we obtain
\[
\|V_r(M_{2^n}f : n \in \mathbb{N}_0)\|_{L^p} \leq \|V_r(P_{2^n}f : n \in \mathbb{N}_0)\|_{L^p} + \left( \sum_{n \in \mathbb{N}_0} |(M_{2^n} - P_{2^n})f|^2 \right)^{1/2} \tag{5.4}
\]
The first term in (5.4) is bounded on $L^p(\mathbb{Z}^d)$ by (5.3). Therefore, it remains to obtain $L^p(\mathbb{Z}^d)$ bounds for the square function in (5.4). For this purpose we will use (4.6) with $\alpha = 2^n$ (so that $a = 2$). Indeed, observe that
\[
\left\| \left( \sum_{n \in \mathbb{N}_0} |(M_{2^n} - P_{2^n})f|^2 \right)^{1/2} \right\|_{L^p} = \left\| \left( \sum_{n \in \mathbb{N}_0} |\sum_{j \in \mathbb{Z}} (M_{2^n} - P_{2^n})S_{j+n}f^2 \right)^{1/2} \right\|_{L^p}
\leq \sum_{j \in \mathbb{Z}} \left\| \left( \sum_{n \in \mathbb{N}_0} |(M_{2^n} - P_{2^n})S_{j+n}f^2 \right)^{1/2} \right\|_{L^p} \lesssim \sum_{j \in \mathbb{Z}} 2^{-dj} \|f\|_{L^p} \lesssim \|f\|_{L^p}. \tag{5.5}
\]
In order to justify the last but one inequality in (5.5) it suffices to show, for each $j \in \mathbb{Z}$, that
\[
\left\| \left( \sum_{n \in \mathbb{N}_0} |M_{2^n}S_{j+n}f^2 \right)^{1/2} \right\|_{L^p} + \left\| \left( \sum_{n \in \mathbb{N}_0} |P_{2^n}S_{j+n}f^2 \right)^{1/2} \right\|_{L^p} \lesssim \|f\|_{L^p}, \tag{5.6}
\]
and
\[
\left\| \left( \sum_{n \in \mathbb{N}_0} |(M_{2^n} - P_{2^n})S_{j+n}f^2 \right)^{1/2} \right\|_{L^p} \lesssim 2^{-dj/2} \|f\|_{L^p} \tag{5.7}
\]
then interpolation does the job. To prove (5.6) we first show the following dimension-free vector-valued bounds
\[
\left\| \left( \sum_{n \in \mathbb{N}_0} |M_{2^n}g_n^2 \right)^{1/2} \right\|_{L^p} \lesssim \left\| \left( \sum_{n \in \mathbb{N}_0} |g_n^2 \right)^{1/2} \right\|_{L^p}, \tag{5.8}
\]
and
\[
\left\| \left( \sum_{n \in \mathbb{N}_0} |P_{2^n}g_n^2 \right)^{1/2} \right\|_{L^p} \lesssim \left\| \left( \sum_{n \in \mathbb{N}_0} |g_n^2 \right)^{1/2} \right\|_{L^p}, \tag{5.9}
\]
for all $p \in (1, \infty)$. Then in view of (4.7) we conclude
\[
\left\| \left( \sum_{n \in \mathbb{N}_0} |M_{2^n}S_{j+n}f^2 \right)^{1/2} \right\|_{L^p} + \left\| \left( \sum_{n \in \mathbb{N}_0} |P_{2^n}S_{j+n}f^2 \right)^{1/2} \right\|_{L^p} \lesssim \left\| \left( \sum_{n \in \mathbb{N}_0} |S_{j+n}f^2 \right)^{1/2} \right\|_{L^p} \lesssim \|f\|_{L^p},
\]
which proves (5.6).

The proof of (5.8) and (5.9) follows respectively from (4.1) (with $a_n = 2^n$) and (4.4) and a vector-valued interpolation. We only demonstrate (5.8), the estimate in (5.9) will be obtained similarly. Indeed, for $p \in (1, \infty)$ and $s \in [1, \infty]$, let $A(p, s)$ be the best constant in the following inequality
\[
\left\| \left( \sum_{n \in \mathbb{N}_0} |M_{2^n}g_n^s \right)^{1/s} \right\|_{L^p} \leq A(p, s) \left\| \left( \sum_{n \in \mathbb{N}_0} |g_n|^s \right)^{1/s} \right\|_{L^p}.
\]
Then interpolation, duality ($A(p, s) = A(p', s')$), and (4.1) yield (5.8), since
\[
A(p, 2) \leq A(p, 1)^{1/2}A(p, \infty)^{1/2} = A(p', \infty)^{1/2}A(p, \infty)^{1/2} \leq C_{p', \infty}^{1/2}C_{p, \infty}^{1/2}.
\]
By Plancherel’s theorem to prove (5.7) we need to estimate
\[
\left\| \left( \sum_{n \in \mathbb{N}_0} |(M_{2^n} - P_{2^n})S_{j+n}f^2 \right)^{1/2} \right\|_{L^2} = \left( \int_{\mathbb{T}^d} \sum_{n \in \mathbb{N}_0} |n_{2^n}(\xi)(e^{-2^{n+j} |\xi|_{\infty}} - e^{-2^{n+j-1} |\xi|_{\infty}})^2 |f(\xi)|^2 d\xi \right)^{1/2},
\]
with \( n_2^n \) defined in 4.13. This has been already done in 4.20 (for \( a_n = 2^n \) and \( a = 2 \)) and thus 5.7 holds. This completes the proof of 5.3 and hence also the proof of Theorem 5.

5.2. Short variations: proof of Theorem 4. In view of inequalities (5.11) (with \( Z_1 = (0, 1) \) and \( Z_2 = [1, \infty) \)) and (5.2) (with \( Z = [1, \infty) \)), and Theorem 2 and Theorem 3 the proof of Theorem 4 will be completed if we show that for every \( p \in (3/2, 4) \) there is a constant \( C_p > 0 \) such that for every \( f \in \ell^p(\mathbb{Z}^d) \) we have

\[
\left\| \left( \sum_{n \in N_0} V_2(\mathcal{M}_t f : t \in [2^n, 2^{n+1}])^2 \right)^{1/2} \right\|_{\ell^p} \leq C_p \| f \|_{\ell^p}.
\]  

(5.10)

As in 5 the essential tool will be inequality (5.11) from Lemma 5.1.

Lemma 5.1. For every \( n \in N_0 \), for every \( r \geq 1 \) and for every function \( a : [2^n, 2^{n+1}] \to \mathbb{C} \) satisfying \( a(t) = a([t]) \) we have

\[
V_r(\{a_t : t \in [2^n, 2^{n+1}]\}) \leq 2^{1-1/r} \sum_{0 \leq l \leq n} \left( \sum_{k=0}^{2^l-1} |a_{2^n+2^{l-(k+1)}-a_{2^n+2^{l-k}}}|^{r} \right)^{1/r}.
\]  

(5.11)

We refer to 5 Lemma 2.1 for the proof. The advantage of this inequality is that the variational seminorm on a dyadic block is controlled by a sum of suitable square functions, which are better adjusted to investigations on \( \ell^p(\mathbb{Z}^d) \) spaces. In order to prove (5.10) it suffices to show that there are \( \delta_p, \varepsilon_p \in (0, 1) \) such that for every \( l \in N, \) for every \( j \in \mathbb{Z} \) and for every \( f \in \ell^p(\mathbb{Z}^d) \) we have

\[
\left\| \left( \sum_{n \geq l} \sum_{k=0}^{2^l-1} \left( |(M_{2^n+2^{l-(k+1)}} - M_{2^n+2^{l-k}})S_{j+n+f}| \right)^2 \right)^{1/2} \right\|_{\ell^p} \leq 2^{-\delta_p l^2 \varepsilon_p |j|} \| f \|_{\ell^p}.
\]  

(5.12)

Once (5.12) is established we appeal to (4.6), (5.11) and (5.12) and obtain

\[
\left\| \left( \sum_{n \in N_0} V_2(\mathcal{M}_t f : t \in [2^n, 2^{n+1}])^2 \right)^{1/2} \right\|_{\ell^p} \leq \sum_{j \in \mathbb{Z}} \sum_{l \geq 0} 2^{-\delta_p l^2 \varepsilon_p |j|} \| f \|_{\ell^p} \leq \| f \|_{\ell^p}.
\]

The proof of inequality (5.12) will be given in the next two paragraphs.

5.2.1. Proof of inequality (5.12) for \( p \in (3/2, 2] \) Fix \( N \in \mathbb{N} \) and for \( s \in (3/2, 2] \) and \( r \in [1, \infty) \) let \( A_N(s, r) \) be the smallest constant in the following inequality

\[
\left\| \left( \sum_{l \leq s \leq N} \sum_{k=0}^{2^l-1} \left( |(M_{2^n+2^{l-(k+1)}} - M_{2^n+2^{l-k}})g_n| \right)^2 \right)^{1/2} \right\|_{\ell^s} \leq A_N(s, r) \left\| \left( \sum_{l \leq s \leq N} |g_n^s| \right)^{1/r} \right\|_{\ell^s}.
\]  

(5.13)

It is easy to see that \( A_N(s, r) < \infty \). Let \( u \in (1, p) \) be such that

\[
\frac{1}{u} = \frac{1}{2} + \frac{1}{2p}.
\]

Now it is not difficult to see that \( A_N(1, 1) \leq 2^{1+1} \), since \( \| M_t f \|_{\ell^p} \leq \| f \|_{\ell^p} \). Moreover, by Theorem 2 if \( g = \sup_{l \leq s \leq N} |g_n| \) then

\[
\left\| \sup_{l \leq s \leq N} \sup_{0 \leq k < 2^l} \left( |(M_{2^n+2^{l-(k+1)}} - M_{2^n+2^{l-k}})g_n| \right) \right\|_{\ell^p} \leq 2 \left\| \sup_{t \geq 0} M_t g \right\|_{\ell^p} \leq 2C_p \| g \|_{\ell^p}.
\]

Hence by the complex interpolation we obtain

\[
A_N(u, 2) \leq A_N(1, 1)^{1/2} A_N(p, \infty)^{1/2} \leq 2^{1/2}.
\]

Then by (5.13) and (4.7) (with \( a_n = 2^n \)) we get

\[
\left\| \left( \sum_{l \leq s \leq N} \sum_{k=0}^{2^l-1} \left( |(M_{2^n+2^{l-(k+1)}} - M_{2^n+2^{l-k}})S_{j+n+f}| \right)^2 \right)^{1/2} \right\|_{\ell^u} \leq A_N(u, 2) \left\| \left( \sum_{n \in \mathbb{Z}} |S_{j+n+f}|^2 \right)^{1/2} \right\|_{\ell^u} \leq 2^{1/2} \| f \|_{\ell^u}.
\]  

(5.14)

We now take \( \rho \in (0, 1] \) satisfying

\[
\frac{1}{p} = \frac{1}{u} + \frac{\rho}{2}.
\]
then $\rho = p - 1$ and $1 - \rho = 2 - p$. Interpolation between (5.14) and (4.29) (with $0 < \varepsilon < 2 - 1/(p-1)$) yields

$$\left\| \left( \sum_{l \leq n \leq N} \sum_{k=0}^{2^l-1} \left| (M_{2^{n+2^{-l}}(k+1)} - M_{2^{n+2^{-l}}-ik})S_{j+n}f \right|^2 \right)^{1/2} \right\|_{\ell^p} \lesssim 2^{(1-\rho)/2}2^{-\rho\varepsilon/2}2^{-\varepsilon/2}2^{-\varepsilon j/2}f_{\ell^p} \lesssim 2^{(2-\rho)/2}2^{-\rho\varepsilon/2}2^{-\varepsilon j/2}(p-1)/4f_{\ell^p} \lesssim 2^{-\delta_p j - \varepsilon_p j}||f||_{\ell^p},$$

where $\delta_p = \frac{1-\varepsilon(p-1)}{2} > 0$, if $p \in (3/2, 2]$ and $\varepsilon_p = \frac{2(p-1)}{p}$. This completes the proof.

5.2.2. Proof of inequality (5.14) for $p \in (2, 4)$. To this end, we show that, for $p \in [2, \infty)$, we have

$$\left\| \left( \sum_{n \geq 1} \sum_{k=0}^{2^l-1} \left| (M_{2^{n+2^{-l}}(k+1)} - M_{2^{n+2^{-l}}-ik})S_{j+n}f \right|^2 \right)^{1/2} \right\|_{\ell^p} \lesssim 2^{l/2}||f||_{\ell^p}.$$  

(5.15)

Then interpolation of (5.15) with (4.29) does the job and we obtain (5.14) for all $p \in [2, 4)$.

Thus we focus on proving (5.15). Since $p \geq 2$ we estimate

$$\left\| \left( \sum_{n \geq 1} \sum_{k=0}^{2^l-1} \left| (M_{2^{n+2^{-l}}(k+1)} - M_{2^{n+2^{-l}}-ik})S_{j+n}f \right|^2 \right)^{1/2} \right\|_{\ell^p} \leq 2^l \max_{0 \leq k < 2^l} \left\| \left( \sum_{n \geq 1} \left| (M_{2^{n+2^{-l}}(k+1)} - M_{2^{n+2^{-l}}-ik})S_{j+n}f \right|^2 \right)^{1/2} \right\|_{\ell^{p/2}} \lesssim 2^{l} \max_{0 \leq k < 2^l} \left\| \left( \sum_{n \geq 1} \left| (M_{2^{n+2^{-l}}-ik})S_{j+n}f \right|^2 \right)^{1/2} \right\|_{\ell^p} \lesssim 2^l \|f\|_{\ell^p},$$

where the last inequality follows from (4.7) (with $a_n = 2^n$) and

$$\sup_{l \geq 0} \max_{0 \leq k < 2^l} \left\| \left( \sum_{n \geq 1} \left| (M_{2^{n+2^{-l}}-ik})g_n \right|^2 \right)^{1/2} \right\|_{\ell^p} \lesssim \left\| \left( \sum_{n \in \mathbb{Z}} \left| g_n \right|^2 \right)^{1/2} \right\|_{\ell^p},$$  

(5.16)

which holds for all $p \in (1, \infty)$ and the implicit constant independent of the dimension. To prove (5.16) we follow the argument used to justify (6.8). This is feasible, since for every $p \in (1, \infty)$ and for every $f \in L^p(\mathbb{Z}^d)$ we have the following lacunary estimate

$$\sup_{l \geq 0} \max_{0 \leq k < 2^l} \left\| \left( \sum_{n \geq 1} \left| (M_{2^{n+2^{-l}}-ik})f \right|^2 \right)^{1/2} \right\|_{\ell^p} \leq C_{p, \infty} \|f\|_{\ell^p}.$$  

The above is a consequence of Theorem 4 with the lacunary sequence $a_n = (1 + 2^{-1})2^n$ and $a = 2$.

5.3. Transference principle to the ergodic setting: proof of Theorem 6. Recall that $(X, B(X), \mu)$ is a $\sigma$-finite measure space with a family of commuting and invertible measure-preserving transformations $T_1, \ldots, T_d$, which map $X$ to itself. In Proposition 5.2 we prove the transference principle, which will allow us to deduce estimates for $r$-variations on $L^p(X)$ for the operator $A^G_1$ defined in (1.15) from the corresponding bounds for $M^G_1$ on $L^p(\mathbb{Z}^d)$. Theorem 6 will follow directly from Proposition 5.2 combined with Theorem 3 and Theorem 5.

**Proposition 5.2.** Given $p \in (1, \infty)$ and $r \in (2, \infty]$ suppose that there is a constant $C_{p, r} > 0$ such that for a symmetric convex body $G \subset \mathbb{R}^d$ the following estimate

$$\left\| V_r \left( M^G_t f : t \in Z \right) \right\|_{L^p(\mathbb{Z}^d)} \leq C_{p, r} \|f\|_{L^p(\mathbb{Z}^d)}$$  

(5.17)

holds for all $f \in L^p(\mathbb{Z}^d)$ with the implied constant independent of $d \in \mathbb{N}$, where $Z \subseteq (0, \infty)$. Let $A^G_t$ be the ergodic counterpart of $M^G_t$. Then for every $h \in L^p(X)$ the inequality

$$\left\| V_r \left( A^G_t h : t \in Z \right) \right\|_{L^p(X)} \leq C_{p, r} \|h\|_{L^p(X)}$$  

(5.18)

holds with the parameters $p$, $r$, and the constant $C_{p, r}$ as in (5.17).
Proof. For any convex symmetric body \( G \subseteq \mathbb{R}^d \) there is a constant \( c = c_G \in \mathbb{N} \) such that \( G_t \subseteq Q_{ct} \) for every \( t > 0 \). We fix \( f \in L^p(X) \), \( \varepsilon > 0 \), and \( R \in \mathbb{N} \). Let us define for every \( x \in X \) the function
\[
\phi_x(y) = \begin{cases} \int f(T_1^{g_1} \circ \ldots \circ T_d^{g_d} x) \, d, & \text{if } |y| \leq cR(1 + \varepsilon/d), \\
0, & \text{otherwise.}
\end{cases}
\]
Observe that for every \( z \in Q_{\varepsilon R} \) and \( t < R\varepsilon/d \), we have
\[
A_t^G \{f(T_1^{g_1} \circ \ldots \circ T_d^{g_d} x) = \frac{1}{|G_t \cap \mathbb{R}^d|} \sum_{y \in G_t \cap \mathbb{R}^d} f \left( T_1^{g_1-y_1} \circ \ldots \circ T_d^{g_d-y_d} x \right) = \frac{1}{|G_t \cap \mathbb{R}^d|} \sum_{y \in G_t \cap \mathbb{R}^d} \phi_x(z-y) = M_t^G \phi_x(z),
\]

since \( |z-y| \leq cR(1 + \varepsilon/d) \), whenever \( z \in Q_{\varepsilon R} \) and \( y \in G_t \). Hence, by (5.17) and (5.19) we get
\[
\sum_{z \in Q_{\varepsilon R} \cap \mathbb{Z}^d} |V_r (A_t^G \{f(T_1^{g_1} \circ \ldots \circ T_d^{g_d} x) : t \in Z \cap (0, R\varepsilon/d)]|^p \leq \sum_{z \in Q_{\varepsilon R} \cap \mathbb{Z}^d} |V_r (M_t^G \phi_x(z) : t \in Z \cap (0, R\varepsilon/d)]|^p \leq \|V_r (M_t^G \phi_x : t \in Z)\|_{L^p(\mathbb{Z}^d)} \leq C_{p,r} \|\phi_x\|_{L^p(\mathbb{Z}^d)}.
\]

Averaging (5.20) over \( x \in X \) we obtain
\[
\sum_{z \in Q_{\varepsilon R} \cap \mathbb{Z}^d} \|V_r (A_t^G \{f(T_1^{g_1} \circ \ldots \circ T_d^{g_d} x) : t \in Z \cap (0, R\varepsilon/d)]\|_{L^p(X)}^p \leq C_{p,r} \sum_{z \in Q_{\varepsilon R} \cap \mathbb{Z}^d} \|f(T_1^{g_1} \circ \ldots \circ T_d^{g_d} x)\|_{L^p(\mathbb{Z}^d)}^p \leq C_{p,r} \|f\|_{L^p(\mathbb{Z}^d)}.
\]
Proposition A.1. For every $q \in (1, 2)$ and $\varepsilon > 0$ there is $C_{q, \varepsilon} > 0$ independent of $d \in \mathbb{N}$ and such that for every $f \in \ell^q(\mathbb{Z}^d)$ we have

$$\left\| \left( \sum_{j=1}^d \| R_j f \|^2 + \| R_j^* f \|^2 \right)^{1/2} \right\|_{\ell^q} \leq C_{q, \varepsilon} d^{1/q - 1/2 + \varepsilon} \| f \|_{\ell^q}. \quad (A.2)$$

The bound in (A.2) is essentially sharp. Namely, there exists $C_q > 0$ such that for all $d \in \mathbb{N}$ we have

$$\sup_{0 < \| f \|_{\ell^q} \leq 1} \left\| \left( \sum_{j=1}^d \| R_j f \|^2 + \| R_j^* f \|^2 \right)^{1/2} \right\|_{\ell^q} \geq C_q d^{1/q - 1/2}. \quad (A.3)$$

Proof. We first demonstrate (A.2). By Khintchine’s inequality and Fubini’s theorem we have

$$\left\| \left( \sum_{j=1}^d \| R_j f \|^2 + \| R_j^* f \|^2 \right)^{1/2} \right\|_{\ell^q} \simeq_q \mathbb{E} \left\| \sum_{j=1}^d \epsilon_j R_j f \right\| q_{\ell^q} + \mathbb{E} \left\| \sum_{j=1}^d \epsilon_j R_j^* f \right\| q_{\ell^q},$$

where $\epsilon_j \in \{-1, 1\}$ are independent and identically distributed Rademacher variables. We note that

$$\left\| \sum_{j=1}^d \epsilon_j R_j f \right\| q_{\ell^q} = \sup_{\| g \|_{\ell^q} \leq 1} \left| \sum_{x \in \mathbb{Z}^d} f(x) \sum_{j=1}^d \epsilon_j R_j^* g(x) \right| \text{ by duality}$$

$$\leq \| f \|_{\ell^q} \sup_{\| g \|_{\ell^q} \leq 1} \left\| \sum_{j=1}^d \epsilon_j R_j^* g \right\|_{\ell^{q'}} \text{ by Hölder’s inequality}$$

$$\leq d^{1/2} \| f \|_{\ell^q} \sup_{\| g \|_{\ell^q} \leq 1} \left\| \sum_{j=1}^d \| R_j^* g \|^2 \right\|_{\ell^{q'}} \text{ by Cauchy–Schwarz inequality}$$

$$\leq C_q d^{1/2} \| f \|_{\ell^q} \quad \text{ by Theorem 8}$$

The same inequality holds with $R_j^*$ in place of $R_j$ and we conclude that

$$\left\| \left( \sum_{j=1}^d \| R_j f \|^2 + \| R_j^* f \|^2 \right)^{1/2} \right\|_{\ell^q} \leq C_q d^{1/q} \| f \|_{\ell^q}.$$

Interpolating the last bound for $q > 1$ (which is close to 1) with (A.1) (for $p = 2$) we obtain (A.2) with $\varepsilon$ loss for arbitrary small $\varepsilon > 0$.

We now demonstrate (A.3). Here we follow [12, Proposition 2.9] but we are keen on keeping the dependence on $q$ and $d$. Let $g = \delta_0$ be the Dirac delta at zero in $\mathbb{Z}$ and consider

$$G(x) = \prod_{k=1}^d g(x_k) \quad \text{ for } x \in \mathbb{Z}^d.$$

Then

$$\| g \|_{\ell^q(\mathbb{Z})} = \| G \|_{\ell^q(\mathbb{Z}^d)} = 1.$$

Let $\Delta$ denote the discrete derivative on $\mathbb{Z}$, i.e.

$$\Delta g(y) = g(y) - g(y+1) \quad \text{ for } y \in \mathbb{Z}.$$

Then for every $j \in \{1, \ldots, d\}$, with $\Delta_j$ as in Section 3 we have

$$\Delta_j G(x) = \Delta g(x_j) \prod_{k \neq j} g(x_k).$$

For $j \in \{1, \ldots, d\}$ we define

$$E_j = \{0\} \times \cdots \times \{0\}^{c} \times \cdots \times \{0\},$$

where $c$ is such that $c + \varepsilon > 1$. Then

$$\| g \|_{\ell^q(\mathbb{Z}^d)} \| G \|_{\ell^q(\mathbb{Z}^d)} \leq \| \Delta g \|_{\ell^q(\mathbb{Z})} \| G \|_{\ell^q(\mathbb{Z}^d)} \leq \| \Delta g \|_{\ell^q(\mathbb{Z})} \| G \|_{\ell^q(\mathbb{Z}^d)}.$$
where \( \{0\}^c \) occurs in the \( j \)-th factor. Then the sets \( E_j \) are disjoint. We note that

\[
\left\| \left( \sum_{j=1}^{d} \Delta_j |G|^2 \right)^{1/2} \right\|_{L^q(\mathbb{Z}^d)} \geq \left\| \left( \sum_{j=1}^{d} 1_{E_j} \right)^{1/2} \left( \sum_{j=1}^{d} |\Delta_j G|^2 \right)^{1/2} \right\|_{L^q(\mathbb{Z}^d)} \quad \text{since } 1_{\mathbb{Z}^d} \geq \sum_{j=1}^{d} 1_{E_j}
\]

\[
\geq \left\| \sum_{j=1}^{d} 1_{E_j} |\Delta_j G| \right\|_{L^q(\mathbb{Z}^d)} \quad \text{by Cauchy–Schwarz inequality}
\]

\[
= \left( \sum_{j=1}^{d} \left( 1_{\{0\}} \Delta g(x_j) \prod_{k \neq j} g(x_k) \right)^q \right)^{1/q} \quad \text{disjointness of } E_j \text{'s}
\]

\[
= d^{1/q} \left( 1_{\{0\}} |\Delta g| \right)^{1/q} \quad \text{(A.4)}
\]

We will use the following inequality

\[
\| \mathcal{L}^{1/2} G \|_{L^q(\mathbb{Z}^d)} \leq 2 \| \mathcal{L} G \|_{L^q(\mathbb{Z}^d)}^{1/2} \| G \|_{L^q(\mathbb{Z}^d)}^{1/2}, \quad \text{(A.5)}
\]

Indeed, by the Taylor formula with integral reminder we have

\[
e^{-t\mathcal{L}^{1/2}} = I - t\mathcal{L}^{1/2} + \int_0^t (t-u) e^{-u\mathcal{L}^{1/2}} \mathcal{L} du \quad \text{for } t > 0.
\]

This implies that

\[
\| \mathcal{L}^{1/2} G \|_{L^q(\mathbb{Z}^d)} \leq t^{-1} \| G - e^{-t\mathcal{L}^{1/2}} G \|_{L^q(\mathbb{Z}^d)} + t^{-1} \int_0^t (t-u) \| e^{-u\mathcal{L}^{1/2}} \mathcal{L} G \|_{L^q(\mathbb{Z}^d)} du,
\]

which together with the contractivity of \( e^{-t\mathcal{L}^{1/2}} \) on \( L^q(\mathbb{Z}^d) \) gives

\[
\| \mathcal{L}^{1/2} G \|_{L^q(\mathbb{Z}^d)} \leq \frac{2}{t} \| G \|_{L^q(\mathbb{Z}^d)} + \frac{t}{2} \| \mathcal{L} G \|_{L^q(\mathbb{Z}^d)}.
\]

Optimizing over \( t > 0 \) we obtain (A.5).

We now observe

\[
\mathcal{L} G(x) = \frac{1}{4} \sum_{j=1}^{d} \Delta_j \Delta_j^* G(x) = \frac{1}{4} \sum_{j=1}^{d} \Delta \Delta^* g(x_j) \prod_{k \neq j} g(x_k)
\]

and consequently obtain

\[
\| \mathcal{L} G \|_{L^q(\mathbb{Z}^d)} \leq \frac{d}{4} \| \Delta \Delta^* g \|_{L^q(\mathbb{Z}^d)},
\]

which combined with (A.3) implies

\[
\| \mathcal{L}^{1/2} G \|_{L^q(\mathbb{Z}^d)} \leq 2 \| \mathcal{L} G \|_{L^q(\mathbb{Z}^d)}^{1/2} \| G \|_{L^q(\mathbb{Z}^d)}^{1/2} \leq d^{1/2} \| \Delta \Delta^* g \|_{L^q(\mathbb{Z}^d)}^{1/2}, \quad \text{(A.6)}
\]

Combining (A.3) with (A.4) we see that

\[
\left\| \left( \sum_{j=1}^{d} |\Delta_j G|^2 \right)^{1/2} \right\|_{L^q(\mathbb{Z}^d)} \| \mathcal{L}^{1/2} G \|_{L^q(\mathbb{Z}^d)}^{-1} \geq B d^{1/q-1/2}, \quad \text{(A.7)}
\]

where

\[
B := \frac{\| 1_{\{0\}} |\Delta g| \|_{L^q(\mathbb{Z}^d)} \| \Delta \Delta^* g \|_{L^q(\mathbb{Z}^d)}^{1/2}}{\| \Delta \Delta^* g \|_{L^q(\mathbb{Z}^d)}^{1/2}}.
\]

It is not difficult to see that \( B \neq 0 \), since \( \Delta g(-1) = -1 \), even though \(-1\) is not in the support of \( g \).

To complete the proof of (A.3) we assume for a contradiction that for all \( C_q > 0 \) there is \( d \in \mathbb{N} \) such that for all \( f \in \ell^q(\mathbb{Z}^d) \) we have

\[
\left\| \left( \sum_{j=1}^{d} |\mathcal{R}_j f|^2 \right)^{1/2} \right\|_{L^q(\mathbb{Z}^d)} \| f \|_{L^q(\mathbb{Z}^d)}^{-1} \leq C_q d^{1/q-1/2}.
\]

But this contradicts (A.7) by taking \( C_q = B/4 \) and \( f = \mathcal{L}^{1/2} G \), since \( \mathcal{R}_j f = 1/2 \Delta_j G \). \( \square \)
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