ON JOINS AND INTERSECTIONS OF SUBGROUPS IN FREE GROUPS

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Abstract. We study graphs of (generalized) joins and intersections of finitely generated subgroups of a free group. We show how to disprove a lemma of Imrich and Müller on these graphs and how to repair this lemma.

1. Introduction

Suppose that $F$ is a free group of finite rank, $\text{r}(F)$ denotes the rank of $F$, and $\bar{\text{r}}(F) := \max(\text{r}(F) - 1, 0)$ is the reduced rank of $F$. Let $H, K$ be finitely generated subgroups of $F$ and let $\langle H, K \rangle$ denote the subgroup generated by $H, K$, called the join of $H, K$. Hanna Neumann [17] proved that $\bar{\text{r}}(H \cap K) \leq 2\bar{\text{r}}(H)\bar{\text{r}}(K)$ and conjectured that $\bar{\text{r}}(H \cap K) \leq \bar{\text{r}}(H)\bar{\text{r}}(K)$. This problem, known as the Hanna Neumann conjecture on subgroups of free groups, was solved in the affirmative by Friedman [7] and Mineyev [15], see also Dicks’s proof [3]. Relevant results and generalizations of this conjecture can be found in [2], [5], [6], [9], [10], [11], [18], [20].

Imrich and Müller [8] introduced the reduced rank $\bar{\text{r}}(\langle H, K \rangle)$ of the join $\langle H, K \rangle$ in this context and attempted to prove the following inequality

$$\bar{\text{r}}(H \cap K) \leq 2\bar{\text{r}}(H)\bar{\text{r}}(K) - \bar{\text{r}}(\langle H, K \rangle) \min(\bar{\text{r}}(H), \bar{\text{r}}(K)) \quad (1.1)$$

under the assumption that if $H^*, K^*$ are free factors of $H, K$, resp., then the equality $H^* \cap K^* = H \cap K$ implies that $H = H^*$ and $K = K^*$. We remark that this inequality provides a stronger than Hanna Neumann conjecture’s bound for $\bar{\text{r}}(H \cap K)$ whenever $\bar{\text{r}}(\langle H, K \rangle) > \max(\bar{\text{r}}(H), \bar{\text{r}}(K))$ and this looks quite remarkable. We also note that (1.1) was an improvement of an earlier result of Burns [1], see also [16], [19], stating that

$$\bar{\text{r}}(H \cap K) \leq 2\bar{\text{r}}(H)\bar{\text{r}}(K) - \min(\bar{\text{r}}(H), \bar{\text{r}}(K)).$$

Later Kent [14] discovered a serious gap in the proof of a key lemma of Imrich–Müller [8] p. 195 and gave his own proof to the inequality (1.1) under the weakened assumption that $H \cap K \neq \{1\}$. Kent [14] p. 312 remarks that the lemma in [8] “would be quite useful, and though its proof is incorrect, we do not know if the lemma actually fails”. In this note we give an example that shows that the lemma of [8] is indeed false. On positive side, we suggest a repair for this lemma so that the arguments of Imrich–Müller [8] could be saved. Our approach seems to be of independent interest and can be outlined as follows. Given subgroups $H, K$ of a free group $F$ with $H \cap K \neq \{1\}$, we modify $H, K, F$ by certain deformations of Stallings graphs of $H, K, F$ so that the modified subgroups $\tilde{H}, \tilde{K}$ would satisfy $\bar{\text{r}}(\tilde{H}) = \bar{\text{r}}(H), \bar{\text{r}}(\tilde{K}) = \bar{\text{r}}(K), \bar{\text{r}}(\tilde{H} \cap \tilde{K}) = \bar{\text{r}}(H \cap K)$, and $\bar{\text{r}}(\langle \tilde{H}, \tilde{K} \rangle) \geq \bar{\text{r}}(\langle H, K \rangle)$.

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Furthermore, our modification is done so that Stallings graphs of subgroups \( \langle H, K \rangle \), \( H, K \) have vertices of degree 2 or 3 only and every vertex of the graph of \( \langle H, K \rangle \) of degree 3 has a preimage of degree 3 in the graph of \( \tilde{H} \) or \( \tilde{K} \). These properties mean that the lemma of Imrich–Müller [8, p. 195] holds in a modified setting and the arguments of Imrich–Müller are retained otherwise. As an application of this strategy developed in Sect. 3, we will prove Theorem 4.1 in Sect. 4. Theorem 5.1 in Sect. 5 deals with a natural question related to our main construction used in proofs of key Lemmas 3.4–3.5.

2. A Counterexample to Imrich–Müller Lemma

Similarly to Stallings [20], see also [2], [14], [13], we consider finite graphs associated with finitely generated subgroups of a free group \( F \). We consider the ambient free group \( F \) as the fundamental group of a finite connected graph \( U, F = \pi_1(U) \), without vertices of degree 1. Let \( H, K \) be finitely generated subgroups of \( F \) and let \( X, Y \) denote finite graphs associated with \( H, K \), resp. Recall that there are locally injective graph maps \( \varphi_X : X \to U, \varphi_Y : Y \to U \). Conjugating \( H, K \) if necessary, we may assume that the graphs \( X, Y \) have no vertices of degree 1, i.e., core(\( X \)) = \( X \) and core(\( Y \)) = \( Y \), where core(\( \Gamma \)) is the subgraph of a graph \( \Gamma \) consisting of all edges that can be included into circuits of \( \Gamma \).

Let \( S(H, K) \) denote a set of representatives of those double cosets \( HtK \subseteq F, t \in F \), that have the property \( HtKt^{-1} \neq \{1\} \). Recall that the connected components of the core \( W := \text{core}(X \times_U Y) \) of the pullback \( X \times_U Y \) of graph maps \( \varphi_X : X \to U, \varphi_Y : Y \to U \) are in bijective correspondence with elements of the set \( S(H, K) \), see [2], [13], [18]. Hence, we can write

\[
W = \bigvee_{t \in S(H, K)} W_t,
\]

where \( \bigvee \) denotes a disjoint union. In addition, if \( W_t \) is a connected component of \( W \) that corresponds to \( t \in S(H, K) \), then

\[
\bar{r}(H \cap tKt^{-1}) = |EW_t| - |VW_t|,
\]

where \( ET \) is the set of (nonoriented) edges of a graph \( \Gamma \), \( VT \) is the set of vertices of \( \Gamma \) and \( |A| \) is the cardinality of a set \( A \). Using notation \( \bar{r}(\Gamma) := |ET| - |VT| \), we have

\[
\bar{r}(W) = \sum_{t \in S(H, K)} \bar{r}(W_t) = \sum_{t \in S(H, K)} \bar{r}(H \cap tKt^{-1}).
\]

For \( S_1 \subseteq S(H, K), S_1 \neq \emptyset \), denoting \( W(S_1) := \bigvee_{s \in S_1} W_s \), we obtain

\[
\bar{r}(W(S_1)) = \sum_{s \in S_1} \bar{r}(W_s) = \sum_{s \in S_1} \bar{r}(H \cap sKs^{-1}) = \bar{r}(H, K, S_1).
\tag{2.1}
\]

Let \( \alpha_X : W(S_1) \to X, \alpha_Y : W(S_1) \to Y \) denote the restrictions on \( W(S_1) \subseteq X \times_U Y \) of the pullback projection maps

\[
\bar{\alpha}_X : X \times_U Y \to X, \quad \bar{\alpha}_Y : X \times_U Y \to Y,
\]

resp. Consider the pushout \( P \) of these maps \( \alpha_X : W(S_1) \to X, \alpha_Y : W(S_1) \to Y \). The corresponding pushout maps are denoted \( \beta_X : X \to P, \beta_Y : Y \to P \). We also consider the Stallings graph \( Z \) corresponding to the subgroup \( \langle H, K, S_1 \rangle \) of \( F \). It is clear from the definitions that there are graph maps \( \gamma : P \to Z \) and \( \delta : Z \to U \) such that the diagram depicted in Fig. 1 is commutative.
As was found out by Kent [14, Fig. 1], one has to distinguish between $P$ and $Z$ as $P \neq Z$ in general. Since the map $\delta : Z \to U$ is locally injective and $P \neq Z$, we see that the map $\gamma : P \to Z$ need not be locally injective, hence, $\gamma$ factors out through a sequence of edge foldings and $\bar{r}(P) \geq \bar{r}(Z)$. According to Kent [14], the erroneous identification $P = Z$ is the source of a mistake in a key lemma of Imrich–Müller [8, p. 195]. Recall that this lemma claims, in our terminology, that if $Z$ is an almost trivalent graph, i.e., every vertex of $Z$ has degree 3 or 2, then every vertex of $Z$ of degree 3, has a preimage of degree 3 in $X$ or $Y$. Kent [14] explains in detail a mistake in the proof of this lemma and comments that, while the proof of lemma of [8] can be somewhat corrected if one replaces $Z$ by $P$, the subsequent arguments of Imrich–Müller rely on the property that $Z = U$ is trivalent, the property that the graph $P$ does not possess. Furthermore, Kent [14, p. 312] points out that the lemma in [8] “would be quite useful, and though its proof is incorrect, we do not know if the lemma actually fails”.

We now present an example that shows that the lemma of [8] does fail. Consider subgroups

$$H = \langle b_1 b_2 a_2 b_2^{-1} b_1^{-1}, a_1^2, a_1 b_1 b_3 a_3 (a_1 b_1 b_3)^{-1} \rangle,$$  \hspace{1cm} (2.2)

$$K = \langle b_1 b_2 a_2 b_2^{-1} b_1^{-1}, a_1^2, a_1 b_1 b_3 a_3 (a_1 b_1 b_3)^{-1} \rangle$$  \hspace{1cm} (2.3)

of a free group $F = \pi_1(U)$ whose graphs $X, Y, Z$, resp., are depicted in Fig. 2, where the base vertices of $X, Y, Z$ are dashed circled.

Figure 1

Figure 2
It is easy to check that $S(H, K)$ has a single element, say $S(H, K) = \{1\}$, and
\[ H \cap K = \langle b_1b_2b_2^{-1}b_1^{-1}, a_1, a_1b_1b_3(a_1b_1b_3)^{-1} \rangle. \]

Furthermore, the graphs $W, P$ look like those in Fig. 3. Hence, we can see that $Z$ is a trivalent graph that has a vertex of degree 3, which is the center of $Z$, without a preimage of degree 3 in $X \vee Y$, as desired.

![Figure 3](image)

### 3. Fixing the Lemma of Imrich and Müller

We now discuss how to do certain deformations over the graphs $W(S), X, Y, P, Z, U$ to achieve the situation when $P = Z = U$, $U$ is almost trivalent and lemma of [S] p. 195 would hold for $Z$.

Our idea could be illustrated by the remark that, when studying graphs $W(S), X, Y, P, Z, U$, or corresponding subgroups, we can replace the ambient group $F = \pi_1(U)$ by $\tilde{F} = \pi_1(Z)$, i.e., we can replace the graph $U$ by $Z$. We can go further and replace the group $F = \pi_1(U)$ with $\tilde{F} = \pi_1(P)$ by using the pushout graph $P$ in place of $U$.

Either of these replacements $U \to Z, U \to P$ results in obvious cosmetic changes to subgroups $H, K$, to sets $S(H, K), S_1 \subseteq S(H, K)$, and to subgroups $H \cap sKs^{-1}$, $s \in S_1, \langle H, K, S_1 \rangle$.

Either of these replacements $U \to Z, U \to P$ preserves the graphs $W(S), X, Y, P$ and the maps $\alpha_X, \alpha_Y, \beta_X, \beta_Y$. The replacement $U \to Z$ turns $\delta$ into the identity map and the replacement $U \to P$ turns both $\delta, \gamma$ into the identity maps. Otherwise, the structure of the diagram depicted on Fig. 1 is retained. In particular, the ranks $\bar{r}(H), \bar{r}(K), \bar{r}(W(S))$ do not change but the rank $\bar{r}(P)$ could increase as $\bar{r}(P) \geq \bar{r}(Z)$ originally.

For future references, we record this replacement in the following.

**Lemma 3.1.** Replacing the graph $U$ in the diagram depicted in Fig. 1 by $P$ and changing the maps $\gamma : P \to Z, \delta : Z \to U$ by $\gamma = \delta = id_U$, resp., and changing the maps $\varphi_X : X \to U, \varphi_Y : Y \to U$ by $\beta_X : X \to P, \beta_Y : Y \to P$, resp., preserve the properties that $W(S_1) = \bigvee_{s \in S_1} W_s$ consists of connected components of core($X \times_U Y$) and that $P = X \vee_{W(S_1)} Y$. In particular, this replacement does not change the ranks $\bar{r}(X), \bar{r}(Y), \bar{r}(W(S_1))$ but could increase $\bar{r}(Z)$.

**Proof.** If $(e_1, e_2)$ is an edge of $W(S_1) \subseteq \text{core}(X \times_U Y)$, where $\varphi_X \alpha_X((e_1, e_2)) = e_1$ is an edge of $X$ and $\varphi_Y \alpha_Y((e_1, e_2)) = e_2$ is an edge of $Y$, then the edges $(e_1, e_2)$ are identified in $P$, whence, $(e_1, e_2)$ is also an edge of $X \times_P Y$. Since $\text{core}(W(S_1)) = W(S_1)$, it follows that $\text{core}(X \times_P Y)$ contains all edges $(e_1, e_2)$ of $W(S_1)$, hence, $\text{core}(X \times_P Y)$ will contain the graphs naturally isomorphic to connected components
of the graph \( W(S_1) = \bigvee_{s \in S_1} W_s \). Therefore, the original pushout \( P = X \vee_{W(S_1)} Y \) will also be preserved.

Since \( \tilde{r}(P) \geq \tilde{r}(Z) \) for the original graphs \( P, Z \) and since \( P = Z = U \) after the replacement \( U \to P \), we see that the rank \( \tilde{r}(Z) \) could increase after the replacement.

\[ \Box \]

Suppose that \( v \) is a vertex of \( U \). Subdividing the edges incident to \( v \) if necessary, we may assume that every oriented edge \( e \) of \( U \) whose terminal vertex, denoted \( e_+ \), is \( e_+ = v \) has the initial vertex, denoted \( e_- \), of degree 2, \( \deg e_- = 2 \). Hence, we may assume that \( U \) contains a subgraph \( \text{St}(v) \) isomorphic to a star with deg \( v \) rays whose center is \( v \). Let \( T \) be a tree whose vertices of degree 1 are in bijective correspondence with vertices of degree 1 of \( \text{St}(v) \) whose set we denote by \( V_1 \text{St}(v) \). Taking \( \text{St}(v) \) out of \( U \) and putting the tree \( T \) in place of \( \text{St}(v) \), using the bijective correspondence to identify vertices of degree 1 of \( (U \setminus \text{St}(v)) \cup V_1 \text{St}(v) \) and those of \( T \), results in a transformation of \( U \) which we call an elementary deformation of \( U \) around \( v \) by means of \( T \). It is clear that the obtained graph \( U_T := (U \setminus \text{St}(v)) \cup T \) is homotopically equivalent to \( U \) and \( \pi_1(U_T) \) is isomorphic to \( F = \pi_1(U) \). We lift this elementary deformation of the star around \( v \) in \( U \) to all the graphs \( X, Y, W, Z \) by replacement of stars around preimages of the vertex \( v \) by trees isomorphic to suitable subtrees of \( T \) and change, accordingly, the maps \( \alpha_X, \alpha_Y, \beta_X, \beta_Y, \varphi_X, \varphi_Y \). The new graphs and maps obtained this way we denote by \( X_T, Y_T, W(S_1)_T, P_T, Z_T, \alpha_X, \alpha_Y, \beta_X, \beta_Y, \varphi_X, \varphi_Y, \gamma_T, \delta_T \).

Clearly, \( \tilde{r}(Q_T) = \tilde{r}(Q) \), where \( Q \in \{X, Y, W, Z, U\} \). As far as the new pushout graph \( P_T \) is concerned, we can only claim that \( \tilde{r}(P_T) \geq \tilde{r}(P) \) which would be analogous to the following.

**Lemma 3.2.** Let \( P = Z = U \), let \( \gamma, \delta \) be identity maps and let the graphs \( X_T, \ldots, U_T \) be obtained from \( X, \ldots, U \) by an elementary deformation around a vertex \( v \in VU \) by means of a tree \( T \). Then the map \( \delta_T : Z_T \to U_T \) is an isomorphism, i.e., the graphs \( Z_T, U_T \) are naturally isomorphic. Furthermore, the restriction of the map \( \delta_T : P_T \to U_T \) is bijective on the set \( \gamma_T^{-1}\delta_T^{-1}(U_T \setminus T) \) and the subgraph \( \tilde{T} := \gamma_T^{-1}\delta_T^{-1}(T) \) of \( P_T \) is connected. In particular, \( \tilde{r}(P_T) \geq \tilde{r}(P) \) and the equality holds if and only if \( \tilde{T} \) is a tree.

**Proof.** It follows from the definitions that if we collapse edges of \( T \subset U_T \) into a point then we obtain the graph \( U \) back. Similarly, collapsing lifts of the edges of \( T \) in \( Q_T, Q \in \{X, Y, W(S_1), P\} \), into points, we obtain the original graph \( Q \). This observation, together with the local injectivity of the map \( \delta_T \), implies that \( Z_T = U_T \) and that the graph \( \tilde{T} := \gamma_T^{-1}\delta_T^{-1}(T) \) is connected.

Furthermore, the graph \( P_T \) consists of a subgraph isomorphic to \( (U_T \setminus T) \cup V_1T \), where \( V_1T \) is the set of vertices of \( T \) of degree 1, along with the graph \( \tilde{T} = \gamma_T^{-1}\delta_T^{-1}(T) \) which is mapped by \( \delta_T \gamma_T \) to \( T \). Surjectivity of the restriction of \( \delta_T \gamma_T \) on \( \tilde{T} \) follows from connectedness of \( \tilde{T} \).

The graph \( \tilde{T} \) of Lemma 3.2 can be regarded as a “blow-up” of the vertex \( v \in VU \). If \( \tilde{T} \) turns out to be a tree, then our attempt to nontrivially “blow up” the vertex \( v \) is unsuccessful. On the other hand, if \( \tilde{T} \) is not a tree, then we can increase \( \tilde{r}(U) \) by invoking Lemma 5.1 and picking \( P_T \) in place of \( U \).
It is of interest to note that even when \( P = Z = U \) and \( \bar{r}(P_T) = \bar{r}(P) \), i.e., \( \hat{T} \) is a tree, the tree \( \hat{T} \) might look different from \( T \), in this connection, see Lemma 3.4 and its proof.

The following is due to Kent [14]. We provide a proof for completeness.

**Lemma 3.3.** Every vertex of \( P \) of degree at least 3 has a preimage in \( X \lor Y \) of degree at least 3.

**Proof.** Arguing on the contrary, assume that \( u \in VP \) has degree \( \deg u > 2 \) and every lift \( v \in VX \lor VY \) of \( u \) has degree 2. Let \( v, v' \in VX \lor VY \) be two arbitrary lifts of \( u \). It follows from the definitions of \( P, W(S_t) \) that there is a sequence of vertices \( v_1 = v, \ldots, v_k = v' \) in \( VX \lor VY \) with the following properties:

(a) The vertices \( v_1 = v, \ldots, v_k = v' \) are mapped by \( \beta_X, \beta_Y \) to \( u \) and have degree 2;

(b) For every \( i = 1, \ldots, k - 1 \), the vertices \( v_i, v_{i+1} \) belong to distinct connected components of \( X \lor Y \);

(c) For every \( i = 1, \ldots, k - 1 \), there is a vertex \( w_i \in VW \) of degree 2 such that

either \( \alpha_X(w_i) = v_i, \alpha_Y(w_i) = v_{i+1} \) if \( v_i \in VX \) or \( \alpha_Y(w_i) = v_i, \alpha_X(w_i) = v_{i+1} \) if \( v_i \in VY \).

It is clear from these properties and from the definitions of the graphs \( P, W(S_t) \) that the vertices \( v_1 = v, \ldots, v_k = v' \in VX \lor VY \) will be identified in the pushout \( P \) so that the resulting vertex will have degree 2. Since \( v, v' \in VX \lor VY \) were chosen arbitrarily, it follows that the degree of \( u \in VP \) is also 2. This contradiction to \( \deg u > 2 \) proves our claim. \( \square \)

We say that \( T \) is a trivalent tree if every vertex of \( T \) has degree 1 or 3.

**Lemma 3.4.** Suppose that \( P = Z = U \) and \( v_0 \) is a vertex of \( U \) of degree at least 4. Then there is an elementary deformation of \( U \) around \( v_0 \) by means of a trivalent tree \( T \) such that either \( \bar{r}(P_T) > \bar{r}(P) \) or the following are true: \( \bar{r}(P_T) = \bar{r}(P) \), the subgraph \( \hat{T} = \gamma_T^{-1} \delta_T^{-1}(T) \) of \( P_T \) is a tree and

\[
\sum_{v \in VP} \max(\deg v - 3, 0) < \sum_{v \in VP} \max(\deg v - 3, 0).
\]

**Proof.** Denote \( \deg v_0 = m_0 > 3 \). Let \( e_1, \ldots, e_{m_0} \) be all of the oriented edges of \( U \) that end in \( v_0 \). By Lemma 3.3 there is a vertex \( v_1 \in VX \lor VY \), say \( v_1 \in VX \), such that \( \beta_X(v_1) = v_0 \) and \( \deg v_1 = m_1 \), where \( 3 \leq m_1 \leq m_0 \). Reindexing if necessary, we may assume that if \( f_1, \ldots, f_{m_1} \) are all of the edges of \( X \) that end in \( v_1 \), then \( \beta_X(f_i) = e_i, i = 1, \ldots, m_1 \).

Let \( T \) be a trivalent tree with \( m_0 \) vertices of degree 1. Let \( U_T \) denote the graph obtained from \( U \) by an elementary deformation around \( v_0 \) by means of \( T \).

Let \( u_i \) denote the vertex of \( T \) which becomes \( (e_i)_- \) in \( U_T, i = 1, \ldots, m_0 \). Here we are assuming that \( \deg(e_i)_- = 2 \) for every \( i \) as in the definition of an elementary deformation around \( v \). It follows from Lemma 3.2 that if the subgraph \( \hat{T} = \gamma_T^{-1} \delta_T^{-1}(T) \) is not a tree for some \( T \), that is, if \( \bar{r}(\hat{T}) \geq 0 \), then \( \bar{r}(\hat{T}) > \bar{r}(\hat{Z}) \) and our lemma is proven. Hence, we may assume that \( \hat{T} \) is a tree for every trivalent tree \( T \).

Suppose that \( \hat{T} \) contains no vertex of degree \( m_0 \) for some \( T \). Then, obviously,

\[
\sum_{v \in VU_T} \max(\deg v - 3, 0) < \sum_{v \in VU} \max(\deg v - 3, 0)
\]

and our lemma is true.
Thus we may suppose that, for every trivalent tree $T$, $\hat{T}$ is a tree which contains a vertex $\hat{u}$ of degree $m_0$, i.e., $\hat{T}$ is homeomorphic to a star.

It follows from the definitions of the graphs $P_T, \hat{T}$ that the minimal subtree $S$ of $T$ that contains the vertices $u_1, \ldots, u_m$ (recall $u_i = (e_i)_{-}$ in $P_T$) is isomorphic to a subtree $\hat{S}$ of $\hat{T}$. Indeed, a copy of $S$ will show up in $X_T$ in place of a star around the vertex $v_1$ of $X$, hence, a copy $\hat{S}$ of $S$ will also show up in $P_T$ as a subgraph of $\hat{T}$. Since $T$ is trivalent and $\hat{T}$ is a tree that has $m_0$ vertices of degree 1, one vertex $\hat{u}$ of degree $m_0$ and other vertices of degree 2, it follows that $\hat{S}$ contains a single vertex of degree $\geq 3$ which implies $m_1 = 3$. Since $T$ is an arbitrary trivalent tree, we may pick a tree $T = T_{12}$ that has adjacent vertices $w_{12}, w_3$ of degree 3 such that $w_{12}$ is adjacent to both $u_1, u_2$ and $w_3$ is adjacent to $u_3$, see Fig. 4(a). Note that, in this case, $S$ is the subtree of $T = T_{12}$ that contains vertices $u_1, u_2, u_3, w_{12}, w_3$ and has 4 edges that connect these vertices, see Fig. 4(a).

![Figure 4(a)](image)

Since the image $\hat{w}_{12}$ of $w_{12}$ in $\hat{S}$ has degree 3, we have $\deg \hat{w}_{12} = m_0$ in $\hat{T}_{12}$. This, in particular, means that every preimage of $w_3$ in $\hat{T}_{12}$ has degree 2 and adjacent to a preimage of $w_{12}$.

Consider a vertex $u \in VQ$, $Q \in \{X, Y\}$, such that $\beta_Q(u) = v_0$. Let $g_1, \ldots, g_k$ be all of the oriented edges of $Q$ that end in $u$. We claim that the set $\{\beta_Q(g_1), \ldots, \beta_Q(g_k)\}$ may not contain both $e_j$ and $e_j$, where $j > 3$. Arguing on the contrary, assume that

$$\{e_3, e_j\} \subseteq \{\beta_Q(g_1), \ldots, \beta_Q(g_k)\}. \quad (3.1)$$

Then a star neighborhood of $u$ in $Q$ would turn in $QT_{12}$ into a tree isomorphic to a subtree $S_u$ of $T_{12}$ which contains vertices $w_3, u_3, u_j$, see Fig. 4(a). Clearly, either $\deg w_3 = 3$ in $S_u$ or $\deg w_3 = 2$ in $S_u$ and then $S_u$ contains no vertex $w_{12}$. In either case, a copy of $S_u$ is present in $\hat{T}_{12}$ which is impossible because, as we saw above, every preimage of $w_3$ in $\hat{T}_{12}$ has degree 2 and adjacent to a preimage of $w_{12}$. Thus the inclusion (3.1) is impossible.

Recall that we can pick any trivalent tree $T$, in particular, we can pick a tree $T = T_{13}$ that has adjacent vertices $w_{13}, w_2$ of degree 3 such that $w_{13}$ is adjacent to both $u_1, u_3$ and $w_2$ is adjacent to $u_2$, see Fig. 4(b).

Repeating the above argument with indices 2 and 3 switched, we can show that

$$\{e_2, e_j\} \not\subseteq \{\beta_Q(g_1), \ldots, \beta_Q(g_k)\}$$

for every $j > 3$.

Similarly, switching indices 1 and 3, we prove that

$$\{e_1, e_j\} \not\subseteq \{\beta_Q(g_1), \ldots, \beta_Q(g_k)\}$$

for every $j > 3$. 

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**Figure 4(a)**

**Figure 4(b)**
Now we see that for every vertex \( u \in V_X \cup V_Y \) such that \( \beta_Q(u) = v_0 \), where \( Q \in \{ X, Y \} \), the edges \( g_1, \ldots, g_k \) of \( Q \) that end in \( u \) have the following property: either

\[
\{ \beta_Q(g_1), \ldots, \beta_Q(g_k) \} \subseteq \{ e_1, e_2, e_3 \} \quad \text{or} \quad \{ \beta_Q(g_1), \ldots, \beta_Q(g_k) \} \subseteq \{ e_4, \ldots, e_m \}.
\]

Since the edges of \( X \cup Y \) are identified in the pushout \( P = X \cup W(S_1) \cup Y \) if and only if their \( \beta_X, \beta_Y \)-images in \( P \) are equal, it follows that the terminal vertex of \( e_1, e_2, e_3 \) must be different in \( P \) from the terminal vertex of an edge \( e_j \), where \( j \geq 4 \). This contradiction to the definition of the edges \( e_1, \ldots, e_m \) of \( U \) completes the proof.

\[ \square \]

We are now ready to prove our key lemma.

**Lemma 3.5.** Suppose that the graphs \( X, Y, W(S_1), P, Z, U \) and corresponding maps \( \alpha_X, \ldots, \delta \) are defined as in Fig. 1. Then there exists a finite sequence \( \tau \) of alternating replacements as in Lemma 3.2 and elementary deformations as in Lemma 3.4 that result in graphs \( X^\tau, Y^\tau, W(S_1)^\tau, P^\tau, Z^\tau, U^\tau \) that have the following properties:

- \( \bar{r}(X^\tau) = \bar{r}(X) \)
- \( \bar{r}(Y^\tau) = \bar{r}(Y) \)
- \( \bar{r}(W(S_1)^\tau) = \bar{r}(W(S_1)) \)
- \( P^\tau = Z^\tau = U^\tau \)
- \( \bar{r}(P^\tau) \geq \bar{r}(P) \)

Every vertex of \( U^\tau \) has degree 3 or 2 and every vertex of degree 3 of \( U^\tau \) has a preimage of degree 3 in \( X^\tau \) or \( Y^\tau \).

**Proof.** Applying Lemma 3.1 we may assume that \( P = Z = U \). If every vertex of \( U \) has degree 2 or 3, then our claim holds true as follows from Lemma 3.3. Hence, we may assume that \( U \) contains a vertex \( v_0 \) of degree at least 4. In view of Lemma 3.4 by checking all possible elementary deformations of \( U \) around \( v_0 \), we can find an elementary deformation by means of a trivalent tree \( T \) such that either \( \bar{r}(P_T) > \bar{r}(P) \) or \( \bar{r}(P_T) = \bar{r}(P) \) and

\[
\sum_{v \in V_{P_T}} \max(\deg v - 3, 0) < \sum_{v \in V_P} \max(\deg v - 3, 0).
\]

Invoking Lemma 3.1 we replace \( U_T \) with \( P_T \) and, completing one cycle of changes in graphs \( X, Y, W(S_1), P, Z, U \), we rename \( Q := Q_T \), where \( Q \in \{ X, Y, W(S_1), P, Z, U \} \), and start over.

Observe that the rank \( \bar{r}(Z) = \bar{r}(\langle H, K, S_1 \rangle) \) is bounded above by the number of generators \( r(H) + r(K) + |S_1| \) and this bound does not change as we perform cycles of changes of graphs \( X, Y, W(S_1), P, Z, U \), because \( |S_1| \) is equal to the number of connected components of \( W(S_1) \) and \( \bar{r}(H) = \bar{r}(X), \bar{r}(K) = \bar{r}(Y) \). This implies that the number

\[
\sum_{v \in V_Z} \max(\deg v - 3, 0)
\]

is bounded above by \( 2 \bar{r}(Z) \leq 2(\bar{r}(H) + \bar{r}(K) + |S_1|) \). Thus the total number of cycles will not exceed \( 2(\bar{r}(H) + \bar{r}(K) + |S_1|)^2 \) and our lemma is proved.

Note that the arguments of the proof of Lemma 3.5 provide an algorithm that deterministically constructs the desired graphs \( X^\tau, Y^\tau, W(S_1)^\tau, P^\tau, Z^\tau, U^\tau \) in polynomial space (of size of input which are graphs \( X, Y, W(S_1), P, Z, U \) along with associated maps \( \alpha_X, \ldots, \delta \)).

**Lemma 3.6.** \( \bar{r}(Z) \leq \bar{r}(X) + \bar{r}(Y) \).
Proof. In view of Lemma 3.3 we may assume that Stallings graphs $X, Y, Z$ of subgroups $H, K, (H, K, S_1)$ satisfy the conclusion of Lemma 3.3. Then all of the vertices of $X, Y, Z$ have degree 2 or 3. Hence, $2\bar{r}(U) = |V_3 U|$, where $V_3 U$ is the set of vertices of degree 3 of $U$. Similarly, we have $2\bar{r}(X) = |V_3 X|$ and $2\bar{r}(Y) = |V_3 Y|$. Now application of Lemma 3.3 yields $|V_3 Z| \leq |V_3 X| + |V_3 Y|$ which implies the desired inequality $\bar{r}(Z) \leq \bar{r}(X) + \bar{r}(Y)$.

4. Applications

As an application of Lemma 3.3 we state and prove a couple of specific inequalities for reduced ranks of the generalized intersections and joins of subgroups in free groups.

**Theorem 4.1.** Let $H, K$ be finitely generated subgroups of a free group $F$. Let $S(H, K) \subseteq F$ denote a set of representatives of those double cosets $HtK \subseteq F$, $t \in F$, that have the property $HtKt^{-1} \neq \{1\}$, and let $S_1 \subseteq S(H, K)$ be nonempty. Then $\bar{r}(x_{H, K, S_1}) \leq \bar{r}(H) + \bar{r}(K)$ and

$$
\bar{r}(H, K, S_1) := \sum_{s \in S_1} \bar{r}(H \cap sKs^{-1}) \leq 2\bar{r}(H)\bar{r}(K) - \bar{r}(\langle H, K, S_1 \rangle) \min(\bar{r}(H), \bar{r}(K)). \tag{4.1}
$$

Moreover,

$$
\bar{r}(H, K, S_1) \leq \frac{1}{2}(\bar{r}(H) + \bar{r}(K) - \bar{r}(\langle H, K, S_1 \rangle))(\bar{r}(H) + \bar{r}(K) - \bar{r}(\langle H, K, S_1 \rangle) + 1). \tag{4.2}
$$

Note that the inequality (4.1) is a strengthened version of (1.1) and this strengthening is analogous to a strengthened version of the Hanna Neumann conjecture introduced by Walter Neumann [18]. We also remark that the inequality (4.1) in the cases when $S_0 = \{1\}$ and $S_0 = S(H, K)$ is due to Kent [14] and the inequality (4.2) is shown by Dicks [3] who also obtained other inequalities.

It is worthwhile to mention that the natural question on the existence of a bound that would be a stronger version of (1.1), in which the first term would have the coefficient 1 in place of 2 and the second negative term would contain the factor $\bar{r}(\langle H, K, S_1 \rangle)$ with some coefficient, has a negative solution. Indeed, according to [12], there are finitely generated subgroups $H, K$ of a free group $F$ such that $\bar{r}(H, K, S(H, K)) = \bar{r}(H)\bar{r}(K) > 0$ and $\bar{r}(\langle H, K, S(H, K) \rangle) > C$ for any constant $C > 0$.

**Proof.** In view of Lemma 3.3 we may assume that Stallings graphs $X, Y, Z$ of subgroups $H, K, (H, K, S_1)$, resp., and graphs $W(S_1)^+, P^*$ satisfy the conclusion of Lemma 3.3. Recall that $\bar{r}(H) = \bar{r}(X), \bar{r}(K) = \bar{r}(Y)$ and $\bar{r}(\langle H, K, S_1 \rangle) \leq \bar{r}(Z) = \bar{r}(U)$.

As before, if $Q$ is a graph whose vertices have degree 2 or 3, then $V_3 Q$ denotes the set of vertices of $Q$ of degree 3. Recall that $2\bar{r}(Q) = |V_3 Q|$.

For every $v \in VU$, denote

$$
k_v := |V_3 X \cap \beta_X^{-1}(v)|, \quad \ell_v := |V_3 Y \cap \beta_Y^{-1}(v)|, \quad m := |V_3 W(S_1)|, \quad n := |V_3 U|.
$$

We also let $V_3 U := V_3 U \cup V_3 U \cup V_3 U$, where $V_3 U$ is the set of vertices of $U$ that have preimages of degree 3 in both $X$ and $Y, V_3 X$ is the set of vertices of $U$ that have preimages of degree 3 in $X$ only, and $V_3 Y$ is the set of vertices of $U$ that have preimages of degree 3 in $Y$ only. Denote $n_B := |V_3 U|, n_X := |V_3 U|, n_Y := |V_3 U|$. We let $V_3 X = V_3 X \cup V_3 X$, where $V_3 X := \beta_X^{-1}(V_3 X)$ and $V_3 X := \beta_X^{-1}(V_3 X)$, and $V_3 Y := \beta_Y^{-1}(V_3 Y)$ and $V_3 Y := \beta_Y^{-1}(V_3 Y)$. Then $2\bar{r}(Q) = |V_3 Q|$.
\( V_B U \). Similarly, we let \( V_2 Y = V_{31} Y \lor V_{32} Y \), where \( V_{31} Y := \beta^{-1}_Y (V_U Y) \) and \( V_{32} Y := \beta^{-1}_Y (V_B U) \). Denote

\[ k := |V_3 X|, \quad k_1 := |V_{31} X|, \quad k_2 := |V_{32} X|, \quad \ell := |V_3 Y|, \quad \ell_1 := |V_{31} Y|, \quad \ell_2 := |V_{32} Y|. \]

Clearly, \( k = k_1 + k_2, \quad \ell = \ell_1 + \ell_2 \), \( n = n_B + n_X + n_Y \), \( k_1 \geq n_X, \quad \ell_1 \geq n_Y \), and

\[ m \leq \sum_{v \in V_U} k_v \ell_v = \sum_{v \in V_B U} k_v \ell_v. \]

We now continue with arguments similar to those of Imrich-Müller [8] that follow the proof of their lemma. Since \( \sum_{v \in V_B U} \ell_v = \ell_2 \) and \( \ell = \ell_1 + \ell_2 \), we obtain

\[
m \leq \sum_{v \in V_B U} k_v \ell_v = k \ell - \sum_{v \in V_B U} (k - k_v) \ell_v - k \ell_1
\leq k \ell - \sum_{v \in V_B U} (k - k_v) \ell_1 = k \ell - kn_B + k_2 - k \ell_1
\leq k \ell - k(n_B + n_Y - 1). \tag{4.3}
\]

Switching \( X \) and \( Y \), we analogously obtain

\[
m \leq k \ell - \ell(n_B + n_X - 1). \tag{4.4}
\]

Assume that \( n_B = 0 \). Then \( \bar{r}(W(S_1)) = 0 \) and the inequality \((4.1)\) is equivalent to

\[ 0 \leq 2\bar{r}(X)\bar{r}(Y) - \min(\bar{r}(X), \bar{r}(Y))\bar{r}(Z) \]

which is equivalent to \( \bar{r}(Z) \leq 2 \max(\bar{r}(X), \bar{r}(Y)) \). Since \( \bar{r}(Z) \leq \bar{r}(X) + \bar{r}(Y) \) by Lemma 3.6 it follows that \((4.1)\) holds true. A reference to Lemma 3.6 also proves that

\[ \bar{r}(\langle H, K, S_1 \rangle) \leq \bar{r}(Z) \leq \bar{r}(X) + \bar{r}(Y) = \bar{r}(H) + \bar{r}(K). \]

Suppose that \( n_B \geq 1 \). Since \( n_B + n_X + n_Y = 2\bar{r}(U) \), it follows that

\[ n_X + n_Y \leq 2\bar{r}(U) - 1. \]

Hence, \( \min(n_X, n_Y) \leq \bar{r}(U) - 1 \) and \( \max(n_B + n_X - 1, n_B + n_Y) \geq \bar{r}(U) \). Therefore, the inequalities \((4.3)\)–\((4.4)\) imply that

\[ 2\bar{r}(W(S_1)) = m \leq 4\bar{r}(X)\bar{r}(Y) - 2 \min(\bar{r}(X), \bar{r}(Y))\bar{r}(Z). \]

Dividing by \( 2 \), we obtain the required bound \((4.1)\).

Observe that Lemma 3.6 together with the foregoing classification of vertices of degree 3 in graphs \( W(S), X, Y, P = Z = U \) make it possible to produce other inequalities for \( \bar{r}(H, K, S_1) = \bar{r}(W(S_1)) \) that would involve \( \bar{r}(\langle H, K, S_1 \rangle) = \bar{r}(Z) \). For example, when maximizing the sum \( \sum_{v \in V_B U} k_v \ell_v \) which gives an upper bound for \( m = 2\bar{r}(W(S_1)) \), we may assume that there is a single vertex \( v \) such that both \( k_v, \ell_v > 0 \), whence \( n_B = 1, k_2 = k_v \) and \( \ell_2 = \ell_v \). We may further assume that \( k_1 = n_X, \ell_1 = n_Y \). Indeed, if, say \( k_1 > n_X \), then we could increase \( k_v \) by \( k_1 - n_X \) and decrease \( k_1 \) by \( k_1 - n_X \) making thereby \( k_2 \ell_2 = k_v \ell_v \) greater. Hence, to get an upper bound for \( m \), we can maximize the product \( k_2 \ell_2 = (k - k_1)(\ell - \ell_1) \) subject to \( k_1 + \ell_1 = 1 = n \). Since \( k, \ell, n \) are fixed positive even integers, equal to \( 2\bar{r}(X), 2\bar{r}(Y), 2\bar{r}(Z) \), resp., it follows that the product \( (k - k_1)(\ell - n + 1 - k_1) \) has the maximum at \( k_1 = \frac{1}{2}(k - \ell + n) \). Since \( k_1 \) is an integer, it follows that \( (k - k_1)(\ell - n + 1 - k_1) \)
has a maximal value for integer \( k_1 \) when \( k_1 = \frac{1}{2}(k - \ell + n) \) or \( k_1 = \frac{1}{2}(k - \ell + n) - 1 \). This means that, unconditionally, we have

\[
m \leq (k - k_1)(\ell - n + 1 - k_1) \leq \frac{1}{2}(k + \ell - n + 2) \cdot \frac{1}{2}(k + \ell - n).
\]

Equivalently,

\[
2\bar{r}(W(S_1)) = m \leq (\bar{r}(X) + \bar{r}(Y) - \bar{r}(Z) + 1)(\bar{r}(X) + \bar{r}(Y) - \bar{r}(Z))
\]

which implies the bound (4.2). Theorem 4.1 is proved. \( \square \)

5. One More Question

One might wonder what would be the conditions that guarantee the existence of a nontrivial “blow-up” of a vertex \( v_0 \) of the graph \( P = Z = U \) by an elementary deformation around \( v_0 \), as defined in Sect. 3. Here we present a result that provides a criterion for the existence of such a “blow-up” of a vertex \( v_0 \) of \( U \), i.e., the existence of an elementary deformation around a vertex \( v_0 \) by means of a tree \( T \) such that \( \bar{r}(P_T) > \bar{r}(P) \).

Assume that \( P = Z = U \) and the maps \( \gamma, \delta \) in Fig. 1 are identity maps on \( U \). Let \( v_0 \) be a fixed vertex of \( U \) and let \( D \) denote the set of all oriented edges of \( U \) that end in \( v_0 \).

A vertex \( v \) of one of the graphs \( W(S_1), X, Y \) is called a \( v_0 \)-vertex if the image of \( v \) under \( \beta_X\alpha_X, \beta_X, \beta_Y \), resp., is \( v_0 \).

For every \( \bar{v}_0 \)-vertex \( v \in VQ \), where \( Q \in \{X, Y, W(S_1)\} \), define the set \( D(v) \subseteq D \) so that \( e \in D(v) \) if and only if there is an edge \( f \in \bar{E}Q \) so that the terminal vertex \( f_+ \) of \( f \) is \( v \) and the image of the edge \( f \) in \( U \) is \( e \).

Consider a partition \( D = A \lor B \) of the set \( D \) into two nonempty subsets \( A, B \).

A \( \bar{v}_0 \)-vertex \( v \in VQ \), where \( Q \in \{X, Y, W(S_1)\} \), is said to be of type \( \{A, B\} \) if both intersections \( D(v) \cap A \) and \( D(v) \cap B \) are nonempty. The set of all vertices of type \( \{A, B\} \) of a graph \( Q \), where \( Q \in \{X, Y, W(S_1)\} \), is denoted \( V_{\{A, B\}}Q \).

Consider a bipartite graph \( \Psi(\{A, B\}) \) whose set of vertices

\[
V\Psi(\{A, B\}) = V_X\Psi(\{A, B\}) \lor V_Y\Psi(\{A, B\})
\]

consists of two disjoint parts

\[
V_X\Psi(\{A, B\}) := V_{\{A,B\}}X \quad \text{and} \quad V_Y\Psi(\{A, B\}) := V_{\{A,B\}}Y.
\]

Two vertices \( v_X \in V_X\Psi(\{A, B\}) \) and \( v_Y \in V_Y\Psi(\{A, B\}) \) are connected in \( \Psi(\{A, B\}) \) by an edge if and only if there is a vertex \( v \in W(S_1) \) such that \( \alpha_X(v) = v_X \), \( \alpha_Y(v) = v_Y \) and \( v \in V_{\{A,B\}}W(S_1) \).

Theorem 5.1. Suppose \( P = Z = U \), \( v_0 \) is a vertex of \( U \). The equality \( \bar{r}(P_T) = \bar{r}(U_T) = \bar{r}(U) \) holds for every elementary deformation around \( v_0 \) by means of a tree \( T \) if and only if the following conditions hold. For every partition \( D = A \lor B \) with nonempty \( A, B \) if the graph \( \Psi(\{A, B\}) \) contains \( k \) connected components, then there exists a partition \( D = \bigvee_{i=1}^{k+1} C_i \) such that, for every \( i, C_i \) is nonempty and either \( C_i \subseteq A \) or \( C_i \subseteq B \). Next, for every vertex \( v \in VQ \), where \( Q \in \{X, Y\} \), such that \( \beta_Q(v) = v_0 \), the intersection \( D(v) \cap A \) is either empty or \( D(v) \cap A \subseteq C_{i_vA} \) for some \( i_vA, i_vA = 1, \ldots, k + 1 \), and the intersection \( D(v) \cap B \) is either empty or \( D(v) \cap B \subseteq C_{i_vB} \) for some \( i_vB, i_vB = 1, \ldots, k + 1 \).
Proof. First we introduce the notation we will need below. As above, let $T$ be a tree whose set of vertices of degree 1 is $D'$. If $H' \subseteq D'$ is a nonempty subset of vertices of degree 1 in $T$, we let $M_T(H')$ denote the minimal subtree of $T$ that contains $H'$. Clearly, the set of vertices of degree 1 of $M_T(H')$ is $H'$. Note that $M_T(H') = H'$ if $H'$ consists of a single vertex. If $E \subseteq D$, then $E' \subseteq D'$ denotes the image of $E$ under the map $d \to \varphi'$ for every $d \in D$.

For every vertex $v \in VQ$, where $Q \in \{X, Y, W(S_1)\}$, consider a tree $M_T(D(v))$ which is isomorphic to the subtree $M_T(D(v'))$ of $T$ and let

$$\zeta_v : M_T(D(v)) \to T \quad (5.1)$$

denote the natural monomorphism whose image is $M_T(D(v'))$.

If $w$ is a vertex of $W(S_1)$, then the tree $M_T(D(\alpha_X(w)))$ contains a subtree isomorphic to $M_T(D(w))$ which we denote $M_{T,X}(D(w))$.

Similarly, the tree $M_T(D(\alpha_Y(w)))$ contains a subtree isomorphic to $M_T(D(w))$, denoted $M_{T,Y}(D(w))$.

It follows from the definitions that the graph $\widehat{T} = \gamma^{-1}_T \delta^{-1}_T(T)$ can be described as follows. $\widehat{T}$ is the union of graphs $M_T(D(v))$ over all $v \in VQ$, $Q \in \{X, Y\}$, which are identified along their subgraphs of the form $M_{T,X}(D(v)) = M_{T,Y}(D(w))$ over all $w \in VW(S_1)$.

Now assume that the equality $\bar{r}(P_T) = \bar{r}(U_T)$ holds for every tree $T$ or, equivalently, see Lemma 3.2, the graph $\widehat{T}$ is a tree for every $T$. Note that $\widehat{T}$ being a tree need not imply that $\widehat{T}$ is naturally isomorphic to $T$.

Let $D = A \lor B$ be a partition of $D$ with nonempty $A, B$. Our goal is to find a partition $D = \bigvee_{i=1}^{k+1} C_i$ with the properties described in Theorem 5.1.

Consider a trivalent tree $T = T(\{A, B\})$ such that $T$ contains a nonoriented edge $e$ so that the graph $T \setminus \{e\}$ consists of two connected components $T_{eA}, T_{eB}$ such that $A' \subseteq VT_{eA}$ and $B' \subseteq VT_{eB}$.

Note that the tree $M_T(D(v))$, where $v \in VQ$, $Q \in \{X, Y, W(S_1)\}$, contains an edge $f$ with $\zeta_v(f) = e$, where $\zeta_v$ is defined by (5.1), if and only if $v$ has type $\{A, B\}$. Hence, the set $\gamma^{-1}_T \delta^{-1}_T(e)$ consists of images of such edges $f$ of $M_T(D(v))$ with $\zeta_v(f) = e$.

Furthermore, let $v, v' \in V_{\{A,B\}} X \lor V_{\{A,B\}} Y$ be two vertices that belong to the same connected component of $\Psi(\{A, B\})$. Then there exists a sequence $v = v_1, v_2, \ldots, v_\ell = v'$ of vertices in the graph $\Psi(\{A, B\})$ so that $v_i, v_{i+1}$ are connected by an edge. It follows from the definitions that the edges $f_1, f_2, \ldots, f_\ell$ such that $\zeta_{v_i}(f_i) = e$, $i = 1, \ldots, \ell$, are identified in $\widehat{T}$. Conversely, it follows from the definitions in a similar fashion that if $v, v' \in V_{\{A,B\}} X \lor V_{\{A,B\}} Y$ and the edges $f, f'$ such that $\zeta_v(f) = e = \zeta_{v'}(f')$ are identified in $\widehat{T}$, then $v, v'$ belong to the same connected component of $\Psi(\{A, B\})$. These remarks prove that there is a bijection between the set $\gamma^{-1}_T \delta^{-1}_T(e) = \{f_1, \ldots, f_\ell\} \subseteq \widehat{T}$ and the set of connected components of the graph $\Psi(\{A, B\})$.

Since $\widehat{T}$ is a tree with $|D|$ vertices of degree 1 whose set we denote $D''$, it follows that the graph $\widehat{T} - (\gamma^{-1}_T \delta^{-1}_T(e))$ splits into $k + 1$ connected components which induce a partition $D'' = \bigvee_{i=1}^{k+1} C_i''$, where for every $i$, $C_i''$ is nonempty and either $C_i'' \subseteq A''$ or $C_i'' \subseteq B''$. Then $D = \bigvee_{i=1}^{k+1} C_i$, where $C_i''$ is the image of $C_i$ under the map $d \to d''$ for every $d \in D$, provides a desired partition for $D$ because, for every $v_0$-vertex $v \in VQ$, where $Q \in \{X, Y\}$, the vertices of $A'' \lor D(v)'' \subseteq D''$ and those of $B'' \lor D(v)''$ are connected in $\widehat{T} - (\gamma^{-1}_T \delta^{-1}_T(e))$ by images of edges of
$M_T(D(v)) - \zeta_{v}^{-1}(\{v\})$ of $\hat{T}$. Therefore, $D(v) \cap A \subseteq C_{i_v A}$ for some $i_v A$ whenever the intersection $D(v) \cap A$ is nonempty and $D(v) \cap B \subseteq C_{i_v B}$ for some $i_v B$ whenever the intersection $D(v) \cap B$ is nonempty.

Now we will prove the converse. Assume that for every partition $D = A \lor B$ there exists a partition $D = \bigvee_{i=1}^{k+1} C_i$, with the properties of Theorem 5.1. Arguing on the contrary, suppose that there is a tree $T$ for which the graph $\hat{T}$ is not a tree. Pick a shortest circuit $p$ in $\hat{T}$ of positive length. Let $e$ be a nonoriented edge of $T$ such that $e \in \delta_T \gamma_T(p)$. The graph $T - \{e\}$ consists of two connected components $T_{e A}, T_{e B}$ which define a partition $D' = A' \lor B'$, where $A' \subseteq T_{e A}, B' \subseteq T_{e B}$, and both $A', B'$ are nonempty. Let $k$ denote the number of connected components of the graph $\Psi(\{A, B\})$. It is clear from Lemma 3.2 and the definitions that $k \geq 1$. According to our assumption, there exists a partition $D = \bigvee_{i=1}^{k+1} C_i$, where, for every $i$, $C_i$ is nonempty, $C_i \subseteq A$ or $C_i \subseteq B$, and, for every $v_0$-vertex $v \in \bigvee Q$, where $Q \in \{X, Y\}$, we have that $D(v) \cap A \subseteq C_{i_v A}$ for some $i_v A$ whenever $D(v) \cap A$ is nonempty and we have that $D(v) \cap B \subseteq C_{i_v B}$ for some $i_v B$ whenever $D(v) \cap B$ is nonempty.

As above, we observe that if $v, v' \in V_{\{A, B\}} X \lor V_{\{A, B\}} Y$ are two vertices, then the edges $\zeta_{v}^{-1}(e), \zeta_{v'}^{-1}(e)$ are identified in $\hat{T}$ if and only if $v, v'$ belong to the same connected component of the graph $\Psi(\{A, B\})$. This implies that the edges $\gamma_T^{-1} \delta_T^{-1}(e) \subseteq \hat{T}$ are in bijective correspondence with connected components of $\Psi(\{A, B\})$. Hence, there are $k$ edges in $\gamma_T^{-1} \delta_T^{-1}(e)$. Denote $\gamma_T^{-1} \delta_T^{-1}(e) = \{f_1, \ldots, f_k\}$. Since identification of subgraphs

\begin{equation*}
M_{T,X}(D(w)) \subseteq M_T(D(\alpha_X(w))) \quad \text{and} \quad M_{T,Y}(D(w)) \subseteq M_T(D(\alpha_Y(w)))
\end{equation*}

in process of construction of $\hat{T}$ respects the partition $D' = \bigvee_{i=1}^{k+1} C'_i$, it follows that the graph $\hat{T}$ consists of pairwise disjoint subgraphs $S_1, \ldots, S_{k+1}$, where

\begin{equation*}
\delta_T \gamma_T(S_j) \cap D' = C'_j, \quad j = 1, \ldots, k + 1,
\end{equation*}

which are joined by $k$ nonoriented edges of $\gamma_T^{-1} \delta_T^{-1}(e)$. Collapsing subgraphs $S_1, \ldots, S_{k+1}$ into points, we see that $\hat{T}$ turns into a tree with $k$ nonoriented edges and $k + 1$ vertices. Consequently, a shortest circuit in $\hat{T}$ of positive length may not contain any edge of $\gamma_T^{-1} \delta_T^{-1}(e)$. This contradiction to the choice of the path $p$ and the edge $e$ in $\delta_T \gamma_T(p)$ completes the proof of Theorem 5.1.

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