Normal geodesics connecting
two non–necessarily spacelike submanifolds
in a stationary spacetime *

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Abstract
In this paper we state an existence result for normal geodesics joining two given
submanifolds in a globally hyperbolic stationary spacetime $M$. The proof is based
on both variational and geometric arguments involving the causal structure of
$M$, the completeness of suitable Finsler metrics associated to it and some basic
properties of a submersion. By this interaction, unlike previous results on the
topic, also non–spacelike submanifolds can be handled.

Key words. Stationary spacetime, normal geodesic, global hyperbolicity, Finsler metric, submersion.

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1 Introduction and Background Tools

The aim of this paper is to state some results about the existence of normal
geodesics connecting two rather general submanifolds in a stationary spacetime
using an intrinsic approach developed for the study of the geodesic connected-
ness.

Let us recall the basic notions related to this topic (cf. [5] for the background
material on Lorentzian geometry used throughout the paper).

A Lorentzian manifold $(M, g)$ is a smooth connected finite dimensional manifold
equipped with a symmetric non–degenerate tensor field $g$ of type $(0, 2)$
having index 1.

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A **geodesic** of \((M, g)\) is a smooth curve \(z : [a, b] \subset \mathbb{R} \to M\) satisfying the equation
\[
\nabla_s \dot{z} = 0,
\]
where \(\nabla_s\) is the covariant derivative along \(z\) associated to the Levi–Civita connection of the metric \(g\). Without loss of generality, we can reduce our studies to geodesics defined in the same interval \([0, 1]\); furthermore, it is well known that geodesics satisfy the conservation law
\[
g(z)[\dot{z}, \dot{z}] \equiv E_z.
\]
Thus, they are classified according to their **causal character**, that is according to the sign of \(E_z\): \(z\) is said **timelike** if \(E_z < 0\), **lightlike** if \(E_z = 0\), **spacelike** if \(E_z > 0\) or \(\dot{z} \equiv 0\).

The same terminology is used also for any vector and for any vector field if it has the same causal character at each point, for any piecewise smooth curve according to the causal character of its velocity vector field and for submanifolds. In particular, a submanifold \(P\) of \(M\) is **spacelike** if, for each \(p \in P\), \(g\) restricted to \(T_p P\) is positive definite.

A **spacetime** is a Lorentzian manifold with a prescribed time–orientation, that is with a continuous choice of a causal cone at each point of \(M\). In such a case a piecewise smooth curve on \(M\) is said **future–pointing** (resp. **past–pointing**) if its velocity vector field belongs to the cones labelled as future ones at any point where it is defined.

A vector field \(K\) on \(M\) is **Killing** if one of the following equivalent assertions holds true (see [18, Propositions 9.23 and 9.25]):

(i) the stages of its local flow consist of isometries;

(ii) the Lie derivative of \(g\) in its direction is 0;

(iii) \(g[\nabla X K, Y] = -g[\nabla Y K, X]\) for each pair of vector fields \(X, Y\).

It is easy to see that if \(K\) is a Killing vector field and \(z\) is a geodesic, then a constant \(C_z \in \mathbb{R}\) exists such that
\[
g(z)[\dot{z}, K] \equiv C_z. \tag{1.1}
\]

The existence of a timelike Killing vector field gives some important information on the structure of the manifold (e.g., cf. [20]). Moreover, observers travelling on integral curves of timelike Killing vector fields see a constant metric.

A spacetime \((M, g)\) is called **stationary** if it admits a timelike Killing vector field and it is **globally hyperbolic** if it admits a (smooth) spacelike Cauchy hypersurface, i.e. a subset crossed exactly once by any inextendible timelike curve.

If a globally hyperbolic stationary spacetime \(M\) admits at least one complete Killing vector field \(K\) (i.e., the integral curves of \(K\) are defined on the whole real line), then it is **standard stationary** (see [2, Theorem 2.3]), that is \(M\) splits
as a product $\mathcal{M}_0 \times \mathbb{R}$, where the connected finite dimensional manifold $\mathcal{M}_0$ is endowed with a Riemannian metric $g_0$ and the metric $g$ is given by

$$g(x,t)[(y,\tau), (y,\tau)] = g_0(x)[y, y] + 2g_0(x)[\delta(x), y]\tau - \beta(x)\tau^2 \quad (1.2)$$

for any $(x,t) \in \mathcal{M}_0 \times \mathbb{R}$, $(y,\tau) \in T_x\mathcal{M}_0 \times \mathbb{R}$, with $\delta$ vector field and $\beta$ positive function, both on $\mathcal{M}_0$; in this case $K = \partial_t$. It is well known that locally any stationary spacetime looks like a standard one.

In [7] it is proved that a globally hyperbolic stationary spacetime, endowed with a complete timelike Killing vector field $K$ (and, thus, standard stationary) and with a complete spacelike Cauchy hypersurface $S$, is geodesically connected. Remarkably enough, there are counterexamples to geodesic connectedness if one of the assumptions in [7] is dropped.

Already in [15] stationary (non–necessarily standard) spacetimes are considered in relation to the problem of the existence of geodesics connecting two given points of the spacetime. In that paper the authors prove a variational principle for geodesics based on the natural constraint in (1.1) and on a compactness assumption, called pseudocoercivity, which makes classical critical point theorems applicable (see Section 3). Such assumption implies global hyperbolicity but it is rather difficult to verify if it holds or not and, indeed, in order to furnish an example of a stationary spacetime satisfying pseudocoercivity, a standard stationary one is chosen. In [7] the authors essentially show that their intrinsic geometric assumptions, involving the causal structure of the spacetime, are equivalent to pseudocoercivity. For a given complete spacelike smooth Cauchy hypersurface $S$ they consider the manifold $S \times \mathbb{R}$ which, by the flow of $K$, is diffeomorphic to $\mathcal{M}$ and isometric to $(\mathcal{M}, g)$ if endowed with a metric as in (1.2) such that

$$S \equiv \mathcal{M}_0, \quad K \equiv \partial_t,$$

$$x = \pi_S(z) \quad \text{with} \quad \pi_S \text{ canonical projection on } S,$$

$$g(z)[K(z), K(z)] = -\beta(x), \quad \delta(x) \text{ is the orthogonal projection of } K(z) \text{ on } T_z S, \text{ for any } z \in S. \quad (1.3)$$

Even if, in general, the global splitting is not unique and not canonically associated to $\mathcal{M}$, the result obtained is independent of the chosen $K$ and $S$ and no growth assumption on the coefficients of the metric needs (unlike in [11] and references therein).

Here, fixing two submanifolds $P$ and $Q$ of a Lorentzian manifold $\mathcal{M}$, our aim is to state an existence result for normal geodesics joining $P$ to $Q$, i.e. geodesics $z : [0,1] \to \mathcal{M}$ such that

$$\left\{ \begin{array}{l} z(0) \in P, \quad z(1) \in Q, \\ \dot{z}(0) \in T_{z(0)}P^\perp, \quad \dot{z}(1) \in T_{z(1)}Q^\perp. \end{array} \right. \quad (1.4)$$

\footnote{See [17] for a characterization in terms of the causal properties that a stationary Lorentzian manifold with a complete timelike Killing vector field has to satisfy in order to be standard stationary.}
We assume that the spacetime $M$ satisfies the assumptions in [7], which means that $M$ is indeed standard stationary. In [11] it is showed that to this kind of spacetimes two Finsler metrics of Randers type, named Fermat metrics, can be associated (see Section 2). Such metrics are related to the Fermat principle for future-pointing and past-pointing lightlike geodesics and their completeness is linked to the global hyperbolicity of $M$ (see [11, Theorem 4.8]).

We also remark that a standard stationary spacetime can be seen as the total space of a Lorentzian submersion $\pi: M \to M_0$ where the one-dimensional fibers are the flow lines of the timelike Killing vector field $\partial_t$ (see [12, Example 3.2]). The use of the Fermat metrics and the properties of a submersion seem to be very convenient for handling our problem. In fact, in some steps of the proof performed in [7] some technical difficulties arise when the initial and the ending points, namely $z(0) = p$ and $z(1) = q$ with $p, q \in M$, change into the boundary data \(1.4\) (cf. the proof of [7, Lemma 5.5]).

It is worth to stress that in Theorem 1.1 below we deal with rather general submanifolds which do not appear in related papers on the topic. Indeed, we consider a stationary Lorentzian manifold $M$ endowed with a complete timelike Killing vector field and a spacelike Cauchy hypersurface $S$; then denoting by $\Psi: \mathbb{R} \times M \to M$ the flow of $K$, we consider two disjoint (connected) closed submanifolds $P$ and $Q$ of $M$ satisfying one of the following conditions:

\[(H_1)\] $P$ is compact and 
\[
\sup_{q \in Q} |s_q| = D < +\infty, \tag{1.5}
\]

where for each $q \in Q$ the scalar $s_q$ is such that \(\Psi(s_q, q) = \Psi(\mathbb{R} \times \{q\}) \cap S;\)

\[(H_2)\] two submanifolds $P_S$ and $Q_S$ of $S$ exist, such that one of them is compact, $P = \Psi(\mathbb{R} \times P_S)$ and $Q = \Psi(\mathbb{R} \times Q_S)$. 

In previous literature on this subject, which is also mainly concerned to a fixed a priori splitting, much more restrictive assumptions are imposed on the submanifolds $P$ and $Q$ in order to apply variational methods; such assumptions make impossible to handle submanifolds as in $(H_1)$ or $(H_2)$. For example, in [6] it is considered a standard static spacetime (i.e., a standard stationary one with $\delta = 0$) and the submanifolds $P$ and $Q$ are given as $P = S_1 \times \{t_p\}, Q = S_2 \times \{t_q\}$, with $S_1, S_2$ submanifolds of $M_0$, $t_p, t_q \in \mathbb{R}$. Moreover, some results are obtained in standard stationary spacetimes if $P = S_1 \times \{t_p\}$ and $Q = S_2 \times \mathbb{R}$ (see [8] for lightlike geodesics and [7] for spacelike ones). Starting from [10], throughout the previous literature on this topic the submanifolds $S_1, S_2$ are assumed to be closed and at least one of them has to be compact, although it is also possible to consider more general cases, up to suitable additional assumptions involving them (cf. [9] and references therein).

Following the ideas developed in [16], in [2] it is stated a result for geodesics joining spacelike submanifolds of a stationary spacetime, and again in the standard case (cf. [2, Appendix B]) such submanifolds turn out to be as in the above cited works. Up to our knowledge, this is the only intrinsic result on this subject but, as that in [16], it suffers of limitations in relation with the pseudocoercivity.
Now, we are ready to state our main result which is proved in Section 3.

**Theorem 1.1** Let \((M, g)\) be a stationary Lorentzian manifold endowed with a complete timelike Killing vector field \(K\) and a complete (smooth, spacelike) Cauchy hypersurface \(S\). Denoting by \(\Psi: \mathbb{R} \times M \to M\) the flow of \(K\), let \(P\) and \(Q\) be two disjoint (connected) closed submanifolds of \(M\) which satisfy either condition \((H_1)\) or \((H_2)\). Then, there exists at least one normal geodesic joining \(P\) to \(Q\) in \(M\).

At the end of Section 3 under suitable additional assumptions, we briefly discuss multiplicity results for geodesics connecting two submanifolds when \((H_1)\) or \((H_2)\) holds.

## 2 Fermat Metrics and Submersions

Before proving our main result, we need some notions from Finsler geometry and some basic properties of a submersion in relation with stationary spacetimes. In particular, we recall the Fermat metrics of a standard stationary spacetime, as introduced in [11].

**Definition 2.1** A **Finsler manifold** is a couple \((M, F)\) such that \(M\) is a smooth finite dimensional manifold and \(F: TM \to [0, +\infty)\) is a Finsler structure on \(M\), i.e. a function such that

(i) it is continuous on \(TM\), \(C^\infty\) on \(TM \setminus 0\) and vanishing only on the zero section;

(ii) it is fiberwise positively homogeneous of degree one, i.e. \(F(x, \lambda y) = \lambda F(x, y)\) for all \(x \in M\), \(y \in T_x M\) and \(\lambda > 0\);

(iii) it has fiberwise strictly convex square, i.e. the matrix

\[
g_{ij}(x, y) = \begin{bmatrix} \frac{1}{2} \frac{\partial^2 (F^2)}{\partial y^i \partial y^j}(x, y) \end{bmatrix}
\]

is positively defined for all \((x, y) \in TM \setminus 0\).

If \((M, F)\) is a Finsler manifold, the **length** of a piecewise smooth curve \(\gamma: [a, b] \subset \mathbb{R} \to M\) with respect to the Finsler structure \(F\) is defined by

\[
\ell(\gamma) = \int_a^b F(\gamma(s), \dot{\gamma}(s)) \, ds.
\]

Hence, the **distance** between two arbitrary points \(p, q \in M\) is given by

\[
\text{dist}(p, q) = \inf_{\gamma \in \mathcal{P}(p, q)} \ell(\gamma),
\]

where \(\mathcal{P}(p, q)\) is the set of all piecewise smooth curves \(\gamma: [a, b] \to M\) with \(\gamma(a) = p\) and \(\gamma(b) = q\).
Let us point out that, even if the distance function with respect to a Finsler structure $F$ is non-negative and satisfies the triangle inequality, it is not symmetric as, in general, $F$ is non-reversible. Thus, one has to distinguish between the notions of forward and backward metric balls, Cauchy sequences and completeness (see [4 §6.2] for more details). At any case, the topologies generated by the forward and the backward metric balls coincide with the underlying manifold topology and a suitable version of Hopf–Rinow Theorem holds (see [4 Theorem 6.6.1]).

**Theorem 2.2 (Finslerian Hopf–Rinow Theorem)** Let $(M, F)$ be a Finsler manifold. The following statements are equivalent:

(i) the associated Finsler metric is forward (or backward) complete;

(ii) the closed and forward (or backward) bounded subsets of $M$ are compact.

Moreover, if (i) or (ii) holds, then any pair of points in $M$ is connected by a geodesic minimizing the Finslerian distance, i.e. $M$ is convex.

A Finsler metric on $M$ is said of Randers type if

$$F(x, y) = \sqrt{h(x)[y, y]} + \omega(x)[y],$$

where $h$ is a Riemannian metric on $M$ and $\omega$ is a one–form such that $\|\omega\|_x < 1$, where $\|\omega\|_x = \sup_{y \in T_xM \setminus \{0\}} \frac{|\omega(x)[y]|}{\sqrt{h(x)[y, y]}}$.

Now, let $(\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}, g)$ be a standard stationary Lorentzian manifold, with $g$ as in (1.2). If $z(s) = (x(s), t(s)) \in \mathcal{M}$, $s \in [0, 1]$, is a piecewise smooth future–pointing or past–pointing lightlike curve, then it satisfies

$$g_0(x)[\dot{x}, \dot{x}] + 2g_0(x)[\delta(x), \dot{x}] \dot{t} - \beta(x) \dot{t}^2 = 0.$$  

(2.2)

Solving equation (2.2) with respect to $\dot{t}$ and integrating over interval $[0, 1]$, we get that the difference $T_\pm(x)$ between the arrival time $t(1)$ and the starting time $t(0)$ of the lightlike curve $z$ depends only on $x$ and is given by

$$T_+(x) = \int_0^1 (\tilde{g}_0(x)[\delta(x), \dot{x}] + \sqrt{(\tilde{g}_0(x)[\delta(x), \dot{x}])^2 + \tilde{g}_0(x)[\dot{x}, \dot{x}]} \) ds$$

(2.3)

if $z$ is future–pointing and by

$$T_-(x) = \int_0^1 (\tilde{g}_0(x)[\delta(x), \dot{x}] - \sqrt{(\tilde{g}_0(x)[\delta(x), \dot{x}])^2 + \tilde{g}_0(x)[\dot{x}, \dot{x}]} \) ds$$

(2.4)

if it is past–pointing. Above by $\tilde{g}_0$ we denoted the conformal metric $g_0/\beta$. 

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**Definition 2.3** The Fermat metrics associated to \((\mathcal{M}, g)\) are the Randers metrics \(F_+\) and \(F_-\) on \(\mathcal{M}_0\) given by

\[
F_+(x, y) = \tilde{g}_0(x)[\delta(x), y] + \sqrt{(\tilde{g}_0(x)[\delta(x), y])^2 + \tilde{g}_0(x)[y, y]},
\]
\[
F_-(x, y) = -\tilde{g}_0(x)[\delta(x), y] + \sqrt{(\tilde{g}_0(x)[\delta(x), y])^2 + \tilde{g}_0(x)[y, y]},
\] (2.5)

for every \((x, y) \in T\mathcal{M}_0\), where the associated Riemannian metric \(h\) in (2.1) is given by

\[
h(x)[y, y] = (\tilde{g}_0(x)[\delta(x), y])^2 + \tilde{g}_0(x)[y, y],
\]

and

\[
\omega_+(x) = \tilde{g}_0(x)[\delta(x), \dot{x}] \quad \text{is the one–form related to } F_+,
\]
\[
\omega_-(x) = -\tilde{g}_0(x)[\delta(x), \dot{x}] \quad \text{is the one–form related to } F_-.
\]

Thus, according to (2.3)-(2.5), if \(z = (x, t)\) is a lightlike curve in \(\mathcal{M}\) we have

\[
\Delta_z = t(1) - t(0) = T_{\pm}(x) = \pm \int_0^1 F_\pm(x(s), \dot{x}(s))ds
\]

with + if \(z\) is future–pointing, resp. − if \(z\) is past–pointing; hence, \(T_{\pm}(x)\) is ± the length of the spatial projection \(x\) with respect to the Fermat metric \(F_\pm\).

Observe that \(F_-\) can be obtained by \(F_+\) reversing the sign of \(\delta\). Moreover, if \(F_+\) is forward (resp. backward) complete, \(F_-\) is backward (resp. forward) complete and vice versa.

Let us recall the following proposition (cf. [11, Theorem 4.8]):

**Proposition 2.4** If \((\mathcal{M}, g)\) is a standard stationary Lorentzian manifold and \(\bar{t} \in \mathbb{R}\), then:

1. if the Fermat metrics on \(\mathcal{M}_0\) defined at (2.5) are forward or backward complete, then \((\mathcal{M}, g)\) is globally hyperbolic;
2. if \((\mathcal{M}, g)\) is globally hyperbolic with Cauchy surface \(S = \mathcal{M}_0 \times \{\bar{t}\}\), then both \(F_+\) and \(F_-\) are forward and backward complete.

Moreover, the following result holds (cf. [13, Theorem 4.4]):

**Proposition 2.5** Let \((\mathcal{M}, g)\) be a standard stationary spacetime and fix \(\bar{t} \in \mathbb{R}\). If \(F_+\), or equivalently \(F_-\), is forward and backward complete, then \(S = \mathcal{M}_0 \times \{\bar{t}\}\) is a Cauchy surface.

Hence, the quoted result in [7] can be stated as follows: a standard stationary Lorentzian manifold \((S \times \mathbb{R}, g)\) with complete Riemannian component \((S, g_0)\) and forward and backward complete Fermat metric \((S, F_+)\) is geodesically connected.

Furthermore, observe that by [19, Corollary 3.4] sufficient conditions for the global hyperbolicity of a standard stationary spacetime \(\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}\) are the completeness of the Riemannian part \((\mathcal{M}_0, g_0)\) and some growth assumptions.
on the coefficients $\delta, \beta$ of the metric (1.2). Such conditions are exactly the optimal ones assumed in paper [1] in order to get geodesic connectedness on standard stationary spacetimes. Indeed, in [1] it is furnished an example which seems to exhaust the classical variational technique, and the presumable lack of connectedness by geodesics can be explained by a geometric viewpoint by the fact that the associated Fermat metrics are not forward and backward complete.

We conclude this section introducing the basic notions on semi–Riemannian submersions needed in the proof of Theorem 1.1 in the hypothesis $(H_2)$ (e.g., cf. [18]).

Let $(M, g_1)$ and $(B, h_1)$ be two semi–Riemannian manifolds. A semi–Riemannian submersion between $M$ and $B$ is a smooth map $\pi: M \to B$ such that for any $p \in M$:

- its differential $d\pi(p): T_pM \to T_{\pi(p)}B$ is surjective;
- the fiber $\pi^{-1}(\pi(p))$ is a nondegenerate submanifold of $M$;
- $d\pi(p): \mathcal{H}T_pM \to T_{\pi(p)}B$ is an isometry.

Here, $\mathcal{H}T_pM$ is the horizontal subspace of $T_pM$, i.e. the orthogonal subspace to the vertical one $\mathcal{V}T_pM = \text{Ker}(d\pi(p))$.

Giving a $C^1$ curve $\gamma: [a, b] \to B$, a horizontal lift of $\gamma$ is a curve $\alpha: [a, c] \subset [a, b] \to M$ such that $\pi \circ \alpha = \gamma$ and $\dot{\alpha}(s)$ is horizontal, i.e. $\dot{\alpha}(s) \in \mathcal{H}T_{\alpha(s)}M$, for all $s \in [a, c]$.

If $(M = M_0 \times \mathbb{R}, g)$ is a standard stationary spacetime, then the canonical projection $\pi_{M_0}$ on $M_0$ is a Lorentzian submersion between $(M, g)$ and $(M_0, h_1)$, where $h_1$ is the Riemannian metric defined as

$$h_1(x)[v, v] = g_0(x)[v, v] + \frac{1}{\beta(x)}(g_0(x)[\delta(x), v])^2.$$  

(2.6)

In fact, writing the metric $g$ as

$$g(x, t)[(v, \tau), (v, \tau)] = g_0(x)[v, v] + \frac{1}{\beta(x)}(g_0(x)[\delta(x), v]^2)^2$$

$$- \left( \frac{1}{\sqrt{\beta(x)}} g_0(x)[\delta(x), v] - \sqrt{\beta(x)} \tau \right)^2$$

and considering $\mathcal{H}T_{(x, t)}M$, the orthogonal subspace to the one–dimensional subspace $[\partial_t|_{(x, t)}]$ for all $(x, t) \in M_0 \times \mathbb{R}$, the map $d\pi_{M_0}(x, t): \mathcal{H}T_{(x, t)}M \to T_xM_0$ is an isometry with respect to the restriction of $g(x, t)$ to $\mathcal{H}T_{(x, t)}M$ and $h_1(x)$.

### 3 Proof of Theorem 1.1

Throughout this section, let $(M, g)$ be a stationary spacetime endowed with a timelike Killing vector field $K$ and let $P$ and $Q$ be two disjoint (connected) closed submanifolds of $M$. 

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which are pointwise collinear to $K$ stored in [15] and is based on the fact that, for any $a$ as the direct sum of $T$ belongs to $W$.

Firstly, let us point out that $\mathcal{M}$ can be equipped with a Riemannian metric defined as follows:

$$g_R(p)[v_1, v_2] = g(p)[v_1, v_2] - 2 \frac{g(p)[v_1, K(p)] g(p)[v_2, K(p)]}{g(p)[K(p), K(p)]}$$

for all $p \in \mathcal{M}$, $v_1, v_2 \in T_p\mathcal{M}$.

Furthermore, by standard arguments it can be proved that normal geodesics joining $P$ to $Q$ are the critical points of the functional

$$f(z) = \frac{1}{2} \int_0^1 g(z)[\dot{z}, \dot{z}] \, ds$$

(3.1)

defined on the Hilbert manifold $\Omega(P,Q)$ of the $H^1$-curves connecting $P$ to $Q$, that is

$$\Omega(P,Q) = \{ z : [0,1] \to \mathcal{M} : z \text{ is absolutely continuous},$$

$$z(0) \in P, z(1) \in Q, \int_0^1 g_R(z)[\dot{z}, \dot{z}] \, ds < +\infty \},$$

so for each $z \in \Omega(P,Q)$ the tangent space $T_z\Omega(P,Q)$ is given by the $H^1$-vector fields $\zeta : [0,1] \to T\mathcal{M}$ along $z$ such that $\zeta(0) \in T_z(0)P$ and $\zeta(1) \in T_{z(1)}Q$.

In our setting, geodesics satisfy the conservation law (1.1), hence the critical points of the functional $f$ belong to the subset

$$\Omega_K(P,Q) = \{ z \in \Omega(P,Q) : \exists C_z \in \mathbb{R} \text{ s.t. } g(z)[\dot{z}, K(z)] \equiv C_z \text{ a.e. on } [0,1] \}.$$

We point out that $\Omega_K(P,Q)$ may be empty. In order to guarantee $\Omega_K(P,Q) \neq \emptyset$, a sufficient condition is to assume that $K$ is complete (see [15] Lemma 5.7 and [2] Proposition 3.1) or to consider two submanifolds $P$ and $Q$ causally related (see [2] Appendix A).

Furthermore, it can be proved not only that $\Omega_K(P,Q)$ has a smooth manifold structure (see [2] Proposition 3.1) but also that a “fine” variational principle holds on it:

**Theorem 3.1** Let $f$ be as in (3.1). A curve $z \in \Omega(P,Q)$ is a critical point of $f$ on $\Omega(P,Q)$ (hence, a normal geodesic connecting $P$ to $Q$) if and only if it is a critical point of $f$ on $\Omega_K(P,Q)$.

**Proof.** Obviously, from (1.1) if $z \in \Omega(P,Q)$ is a critical point of $f$ on $\Omega(P,Q)$, then it is a critical point of $f$ on $\Omega_K(P,Q)$.

Now, assume that $z \in \Omega_K(P,Q)$ is a critical point of $f$ on $\Omega_K(P,Q)$, i.e.

$$df(z)[\zeta] = 0 \text{ for all } \zeta \in T_z\Omega_K(P,Q).$$

The proof in the particular case $P = \{p\}$ and $Q = \{q\}$, with $p, q \in \mathcal{M}$, is contained in [15] and is based on the fact that, for any $z \in \Omega_K(p,q)$, $T_z\Omega(p,q)$ splits as the direct sum of $T_z\Omega_K(p,q)$ plus the space $\mathcal{W}_z$ of the vector fields in $T_z\Omega(p,q)$ which are pointwise collinear to $K(z)$. Clearly, a vector field $\zeta \in T_z\Omega(p,q)$ belongs to $\mathcal{W}_z$ if and only if there exists a function $\mu \in H_0^1([0,1], \mathbb{R})$ such that
\[ \zeta = \mu K(z). \] Thus, by straightforward computations, we get that \( df(z)[\zeta] = 0 \) for all \( \zeta \in \mathcal{W}_z. \)

Later on, in [2] these results were extended to the more general case of two spacelike submanifolds \( P \) and \( Q. \) Also in this setting, it is

\[ T_z \Omega(P, Q) = \mathcal{W}_z \oplus T_z \Omega_K(P, Q), \quad (3.2) \]

where

\[ T_z \Omega_K(P, Q) = \{ \zeta \in T_z \Omega(P, Q) : \exists C \zeta \in \mathbb{R} \text{ s.t.} \]
\[ g(z)[\nabla_s \zeta, K(z)] - g(z)[\zeta, \nabla_s K(z)] = C \zeta \text{ a.e. on } [0, 1] \} \]

and again

\[ \mathcal{W}_z = \{ \zeta \in T_z \Omega(P, Q) : \zeta = \mu K(z) \text{ with } \mu \in H^1_0([0, 1], \mathbb{R}) \}. \]

In general, if \( P \) and \( Q \) are not spacelike submanifolds, the direct sum (3.2) does not hold. In fact, if \( K(z(0)) \in T_{z(0)}P \) and \( K(z(1)) \in T_{z(1)}Q, \) then any vector field \( \zeta = \mu K(z), \) with

\[ \mu(s) = \mu_0 + \int_0^s \frac{C}{g(z)[K(z), K(z)]} \, d\tau, \quad \mu_0, C \in \mathbb{R}, \]

belongs to \( T_z \Omega_K(P, Q) \cap \mathcal{W}_z. \)

Anyway, taking any \( \zeta \in T_z \Omega(P, Q), \) a vector field \( \tilde{\zeta} \in T_z \Omega_K(P, Q) \) and an \( H^1_0 \) function \( \mu \) exist such that

\[ \zeta = \tilde{\zeta} + \mu K(z), \quad (3.3) \]

since the proof of such a decomposition does not rely on the boundary conditions that \( \zeta \) and \( \tilde{\zeta} \) have to satisfy (see the proof of [2 Proposition 3.3]). On the other hand, reasoning as in [2 Proposition 2.2], it can be proved that \( df(z)[\zeta] = 0 \) for all \( \zeta \in \mathcal{W}_z; \) hence, (3.3) implies \( df(z)[\zeta] = 0 \) for all \( \zeta \in T_z \Omega(P, Q). \)

Henceforth, let us denote with \( \mathcal{J} : \Omega_K(P, Q) \to \mathbb{R} \) the restriction of \( f \) to \( \Omega_K(P, Q). \)

**Definition 3.2** Taking \( c \in \mathbb{R}, \) the set \( \Omega_K(P, Q) \) is said to be \( c-\text{precompact} \) if every sequence \( (z_n)_n \subset \Omega_K(P, Q) \) with \( \mathcal{J}(z_n) \leq c \) has a uniformly convergent subsequence. The functional \( \mathcal{J} \) is **pseudocoercive** if \( \Omega_K(P, Q) \) is \( c-\text{precompact} \) for any \( c \geq \inf \mathcal{J}. \)

The pseudocoercivity is the crucial property that \( \mathcal{J} \) has to satisfy in order to obtain the Palais–Smale condition at any level \( c \in \mathbb{R}, \) thus existence and (if the topology is rich enough) multiplicity results for normal geodesics between \( P \) and \( Q \) can be proved (cf. [2 Theorem 1.5]).

From now on, let us assume that the hypotheses of Theorem 1.1 hold. Hence, as already remarked in Section 1 as \( S \) is a complete Cauchy hypersurface, \( \mathcal{M} \)
is a standard stationary spacetime which globally splits as $\mathcal{M} = S \times \mathbb{R}$ and the
metric $g$ is as in (1.2) with the identifications in (1.3). Hence, we have
\[
g(z)[\dot{z}, K(z)] = g_0(x)[\delta(x), \dot{x}] - \beta(x)\dot{t}
\]
for any absolutely continuous curve $z = (x, t) : [0, 1] \to \mathcal{M}$.
Thus, if $z \in \Omega_K(P, Q)$, a constant $C_z$ exists such that
\[
\dot{t} = \tilde{g}_0(x)[\delta(x), \dot{x}] - \frac{C_z}{\beta(x)}, \quad \text{(3.4)}
\]
where, as in Section 2, we set $\tilde{g}_0 = g_0/\beta$. Integrating both hand sides of (3.4)
in $[0, 1]$, we get
\[
C_z = \left( \int_0^1 \tilde{g}_0(x)[\delta(x), \dot{x}] ds - \Delta_z \right) \left( \int_0^1 \frac{1}{\beta(x)} ds \right)^{-1}, \quad \text{(3.5)}
\]
with $\Delta_z = t(1) - t(0)$. Recalling the definition of $J$, replacing (1.2) in (3.1) and
substituting (3.5) in (3.4), we can express $J$ as a functional depending only on
the $x$ component of the curve $z \in \Omega_K(P, Q)$ and on $\Delta_z$, that is
\[
J(z) = \frac{1}{2} \int_0^1 g_0(x)[\dot{x}, \dot{x}] ds + \frac{1}{2} \int_0^1 \tilde{g}_0(x)[\delta(x), \dot{x}] \ g_0(x)[\delta(x), \dot{x}] ds
- \frac{1}{2} \left( \int_0^1 \tilde{g}_0(x)[\delta(x), \dot{x}] ds - \Delta_z \right)^2 \left( \int_0^1 \frac{1}{\beta(x)} ds \right)^{-1}.
\quad \text{(3.6)}
\]
Next, we state a pair of remarks useful in the proof of the pseudocoercivity
of $J$.

**Remark 3.3** Note that under assumption $(H_1)$ of Theorem 1.1, $P$ is compact
and, in the splitting $\mathcal{M} = S \times \mathbb{R}$, assumption (1.5) holds, then
$|\Delta_z|$ is bounded on $\Omega_K(P, Q)$.

**Remark 3.4** In Section 2 associated to any piecewise smooth lightlike curve
$s \mapsto z(s) = (x(s), t(s)) \in S \times \mathbb{R}$ we introduced the quantities $T_\pm(x)$, depending
only on $x$ (see (2.3) and (2.4)), so that $\Delta_z = T_+(x)$ if $z$ is future–pointing,
$\Delta_z = T_-(x)$ if $z$ is past–pointing.
On the other hand, consider
\[
z = (x, t) \in \Omega_K(P, Q) \quad \text{such that } J(z) = 0, \quad \text{(3.7)}
\]
then it has to be
\[
\Delta_z = \int_0^1 \tilde{g}_0(x)[\delta(x), \dot{x}] ds
+ \sqrt{\left( ||\dot{x}||^2 + \int_0^1 \tilde{g}_0(x)[\delta(x), \dot{x}] g_0(x)[\delta(x), \dot{x}] ds \right) \int_0^1 \frac{1}{\beta(x)} ds}
\]
if $\Delta z > 0$, while
\[
\Delta z = \int_0^1 \tilde{g}_0(x)[\delta(x), \dot{x}] ds \\
- \sqrt{\left(\|\dot{x}\|^2 + \int_0^1 \tilde{g}_0(x)[\delta(x), \dot{x}] g_0(x)[\delta(x), \dot{x}] ds\right) \int_0^1 \frac{1}{\beta(x)} ds}
\]
if $\Delta z < 0$, with $\|\dot{x}\|^2 = \int_0^1 g_0(x)[\dot{x}, \dot{x}] ds$. Thus, depending only on $x$, we can define
\[
\tilde{T}_+ (x) = \int_0^1 \tilde{g}_0(x)[\delta(x), \dot{x}] ds \\
+ \sqrt{\left(\|\dot{x}\|^2 + \int_0^1 \tilde{g}_0(x)[\delta(x), \dot{x}] g_0(x)[\delta(x), \dot{x}] ds\right) \int_0^1 \frac{1}{\beta(x)} ds},
\]
\[
\tilde{T}_- (x) = \int_0^1 \tilde{g}_0(x)[\delta(x), \dot{x}] ds \\
- \sqrt{\left(\|\dot{x}\|^2 + \int_0^1 \tilde{g}_0(x)[\delta(x), \dot{x}] g_0(x)[\delta(x), \dot{x}] ds\right) \int_0^1 \frac{1}{\beta(x)} ds},
\]
so that, if (3.7) holds, we have $\Delta z = \tilde{T}_\pm (x)$ according to the sign of $\Delta z$.
Hence, taking any $z = (x, t) \in \Omega_K(P, Q)$, by comparing (2.3) with (3.8) and (2.4) with (3.9), the definition of $\tilde{g}_0$ and the Cauchy–Schwarz inequality imply
\[ T_+(x) \leq \tilde{T}_+(x), \quad \tilde{T}_-(x) \leq T_-(x). \]

**Theorem 3.5** Under the hypothesis $(H_1)$ of Theorem 1.1, the functional $J$ is pseudocoercive.

**Proof.** Let $(z_n)_n \subset \Omega_K(P, Q)$ be such that
\[ (J(z_n))_n \text{ is bounded from above.} \]
Considering the splitting $S \times \mathbb{R}$ and $z_n = (x_n, t_n)$, we claim that
\[ (\|\dot{x}_n\|)_n \text{ is bounded.} \]
In fact, arguing by contradiction, let us assume that, up to subsequences, it is
\[ \|\dot{x}_n\| \xrightarrow{\text{n}} +\infty. \]
From (3.6) and the Cauchy–Schwarz inequality it follows
\[
2J(z_n) \geq \|\dot{x}_n\|^2 \\
- \Delta z_n \left(\Delta z_n - 2 \int_0^1 \tilde{g}_0(x_n)[\delta(x_n), \dot{x}_n] ds\right) \left(\int_0^1 \frac{1}{\beta(x_n)} ds\right)^{-1},
\]
where, by Remark 3.3 we have

\[ (|\Delta z_n|)_n \text{ is bounded.} \] (3.15)

Hence, we can rule out both the fact that \((\Delta z_n)_n\) is definitively equal to 0 and the existence of a compact subset of \(S\) containing all the supports of the curves \(x_n, n \in \mathbb{N}\). In fact, otherwise, from (3.13) and (3.14) it follows

\[ \mathcal{J}(x_n) \xrightarrow{n} +\infty \] (3.16)

in contradiction with (3.11).

Thus, assume that no compact subset of \(S\) contains the images of all the curves \(x_n\) and, up to subsequences, \(\Delta z_n > 0\), resp. \(\Delta z_n < 0\), for all \(n \in \mathbb{N}\). Then, a subsequence exists such that

\[ T_+(x_n) \xrightarrow{n} +\infty \text{ if } \Delta z_n > 0, \quad \text{resp. } T_-(x_n) \xrightarrow{n} -\infty \text{ if } \Delta z_n < 0 \] (3.17)

(recall (2.3) and (2.4)). In fact, since \(S\) is a Cauchy hypersurface, from Proposition 2.4 (2), the Fermat metrics defined in (2.5) on \(S\) are forward and backward complete, then by the already recalled Finslerian Hopf–Rinow Theorem (see Theorem 2.2), if \((T_+(x_n))_n\) is bounded from above (resp. \((T_-(x_n))_n\) is bounded from below), a compact subset of \(S\) must contain all the images of the curves \(x_n\), which is a contradiction.

Moreover, if \(\Delta z_n > 0\), resp. \(\Delta z_n < 0\), according to Remark 3.4, related to each \(z_n\) it can be considered \(\tilde{T}_+(x_n)\), resp. \(\tilde{T}_-(x_n)\), and from (3.10), (3.17) it follows

\[ \tilde{T}_+(x_n) \xrightarrow{n} +\infty \text{ if } \Delta z_n > 0, \quad \text{resp. } \tilde{T}_-(x_n) \xrightarrow{n} -\infty \text{ if } \Delta z_n < 0 \] (3.18)

But from (3.6) and (3.8) it follows

\[ 2\mathcal{J}(z_n) = (\tilde{T}_+(x_n) - \Delta z_n) (\tilde{T}_-(x_n) + \Delta z_n) \]
\[ - 2 \int_0^1 \tilde{g}_0(x)[\delta(x), \dot{\delta}(x)] ds \left( \int_0^1 \beta(x) ds \right)^{-1}, \]

with + if \(\Delta z_n > 0\), resp. − if \(\Delta z_n < 0\); so, following [7, Lemma 5.6] by (3.13), (3.15) and (3.18) we get (3.16) in contradiction with (3.11).

Therefore, claim (3.12) is proved and, as \(S\) is complete with respect to the metric \(g_0\), all the supports of the curves \(x_n\) lie in a compact subset of \(S\). Hence, by the Ascoli–Arzelà Theorem a uniformly convergent subsequence of \((x_n)_n\) exists. Furthermore, by (3.5) also the sequence \((C_{z_n})_n\) is bounded and by (3.4) so is \((|\dot{t}_n|)_n\). Then, again by the Ascoli–Arzelà Theorem there exists also a subsequence of \((t_n)_n\) which uniformly converges.

□

Proof of Theorem 1.1. Under assumption \((H_1)\), as recalled before, the pseudo-coercivity allows one to prove that \(\mathcal{J}\) is bounded from below, it has complete sublevels and satisfies the Palais–Smale condition at any level \(c\). In fact, even when \(P\) and \(Q\) are not spacelike submanifolds, as they are closed and \(\mathcal{M}\) has a
global splitting, we can repeat the proofs of Lemma 4.1, Proposition 4.2, Theorem 5.1 and Proposition 5.2 in [2]. Then, the existence of a minimum of $J$ follows by a standard application of the Deformation Lemma. By Theorem 3.1 such a minimum is a normal geodesic connecting $P$ and $Q$.

In case $(H_2)$, recalling that the canonical projection $\pi_S : (S \times \mathbb{R}, g) \to (S, h_1)$, where $h_1$ is the metric defined in (2.4), is a Lorentzian submersion, we can use the fact that the horizontal lift of any geodesic in the base of a semi–Riemannian submersion is a geodesic of the total space (see [15 Corollary 7.46]). Since $g_0$ is complete, also $h_1$ is complete. From a theorem of K. Grove [16, Theorem 2.6], at least one normal geodesic $x : [0,1] \to S$ in $(S, h_1)$ connecting $P_S$ and $Q_S$ exists. Hence, a horizontal lift of such a geodesic provides a normal geodesic of $(\mathcal{M}, g)$ connecting $P$ to $Q$ (observe that the $t$ component of its horizontal lift is given by $t(s) = t_0 + \int_0^s \frac{1}{\beta(x)} g_0 (x)[\delta(x), \dot{x}] d\tau$).

Observe that under assumption $(H_2)$ of Theorem 1.1, the normal geodesic connecting $P$ and $Q$, being horizontal, is spacelike. Moreover, changing the initial point of the geodesic on the fiber we obtain infinitely many spacelike normal geodesics connecting $P$ to $Q$ which all project on the same geodesic on $(S, h_1)$.

A more interesting multiplicity result can be obtained minimizing the energy functional of the Riemannian manifold $(S, h_1)$ on homotopy classes of curves from $P_S$ to $Q_S$ or assuming that $S$ is not contractible and $P_S, Q_S$ are contractible in $S$. In this last case the Lusternik–Schnirelmann category of $\Omega(P_S, Q_S)$ is infinite (cf. [10 and 14]) and then infinitely many normal geodesics in $(S, h_1)$ connecting $P_S$ and $Q_S$ exist (see [16 Theorem 2.6]); therefore, there exist infinitely many spacelike normal geodesics in $(\mathcal{M}, g)$ from $P$ to $Q$, having different projections on $S$ (up to be the iterates of a closed prime geodesic of $(S, h_1)$ crossing orthogonally $P_S$ and $Q_S$).

Analogous multiplicity results can be also obtained under the assumption $(H_1)$. Indeed, if the Killing vector field $K$ is complete the manifold $\Omega_K(P, Q)$ is homotopically equivalent to $\Omega(P, Q)$ (see [2 Proposition 5.5]).

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