Universality of the Distribution Functions of Random Matrix Theory

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Dedicated to James B. McGuire on the occasion of his sixty-fifth birthday.

1 Introduction

Statistical mechanical lattice models are called solvable if their associated Boltzmann weights satisfy the factorization or star-triangle equations of McGuire [1], Yang [2] and Baxter [3]. For such models the free energy per site and the one-point correlations in the thermodynamic limit are expressible in closed form [4]. There exists a deep mathematical structure [4, 5] underlying these solvable models; and near critical points, a wider applicability than one would initially expect. This last phenomenon, called universality, has its mathematical roots in the strong law of large numbers and the central limit theorems of probability theory and its physical origins in critical phenomena and conformal field theory.
The exact computation of \( n \)-point correlation functions is generally an open problem for most “solvable” models. In the special case of the 2D Ising model, the \( n \)-point functions (in the scaling limit) are expressible in terms of solutions to integrable differential equations \([6, 7, 8]\). The Wigner-Dyson theory of random matrices \([9, 10]\) is a second class of statistical models where integrable differential equations and \( n \)-point correlations (or more precisely, level-spacing distributions) are related. This paper reviews some of these relationships.

In section 2 we define the basic objects of random matrix theory (RMT). In section 3 we recall the bulk scaling limit and the edge scaling limit and express various distribution functions in terms of Painlevé transcendents. In section 4 we discuss the universality of these distribution functions.

## 2 Random Matrix Models

In the Gaussian models \([9, 10]\), the probability density that the eigenvalues lie in infinitesimal intervals about the points \( x_1, \ldots, x_N \) is given by

\[
P_{\beta}(x_1, \ldots, x_N) = C_{N\beta} e^{-\frac{1}{2\beta} \sum x_j^2 \prod_{j<k} |x_j - x_k|^\beta},
\]

where \( C_{N\beta} \) is a normalization constant and

\[
\beta := \begin{cases} 
1 & \text{for GOE,} \\ 
2 & \text{for GUE,} \\ 
4 & \text{for GSE.} 
\end{cases}
\]

We recall that for \( \beta = 1 \) the matrices are \( N \times N \) real symmetric, for \( \beta = 2 \) the matrices are \( N \times N \) complex Hermitian, and for \( \beta = 4 \) the matrices are \( 2N \times 2N \) self-dual Hermitian matrices. (For \( \beta = 4 \) each eigenvalue has multiplicity two.) In each case the eigenvalues are real.

In RMT the probabilities

\[
E_{N\beta}(0; J) := \int_{x_j \notin J} \cdots \int P_{N\beta}(x_1, \ldots, x_N) \, dx_1 \cdots dx_N,
\]

are particularly interesting \([10]\). The simplest choices of \( J \) are \((a, b)\) and \((t, \infty)\). In the first instance the mixed second partial derivative \( E_{N\beta}(0; J) \) with respect to \( a \) and \( b \) gives the spacing distribution between consecutive eigenvalues; and in the second case, \( F_{N\beta}(t) := E_{N\beta}(0; (t, \infty)) \) is the distribution function for the largest eigenvalue.
The important fact 1, 2, 3 (see 4 for simplified proofs) is that $E_{N \beta}(0; J)$ (or its square for $\beta = 1$ or 4) is expressible as a Fredholm determinant. For $\beta = 2$ the kernel is a scalar kernel acting on $J$ whereas for $\beta = 1, 4$, $E_{N \beta}(0; J)^2$ equals a Fredholm determinant of $2 \times 2$ matrix kernel acting on $J$.

3 Painlevé Representations

The representation of $E_{N \beta}(0; J)$, or its square, as a Fredholm determinant of an operator with kernel of a special form permits the determinant to be expressed in terms of Painlevé functions 4, 5, 6, 7.

3.1 Bulk Scaling Limit

Denote the density of eigenvalues at the point $x_0$ by $\rho(x_0)$. It is customary in the limit $N \to \infty$ to scale distances so that the resulting density is one. Precisely, we define $\xi = \rho_N(x_0)(x-x_0)$, $x_0$ independent of $N$, and consider the limit $N \to \infty$, $x \to x_0$, such that $\xi$ is fixed. By requiring $x_0$ to be independent of $N$, we are choosing a point in the “bulk” of the spectrum and are examining the local statistics of the eigenvalues in some small neighborhood of the point $x_0$. In this limit, and for $\beta = 2$, we are led to the Fredholm determinant of the operator on $L^2(0, s)$ whose kernel is the famous sine-kernel 11, 10

$$K(\xi, \xi') := \frac{1}{\pi} \sin(\pi(\xi - \xi')).$$

Observe that the kernel is translationally invariant and independent of the point $x_0$.

It is a result of Jimbo et al. 15 that

$$\det (I - \lambda K) = \exp \left( \int_0^s \frac{\sigma(x; \lambda)}{x} \, dx \right),$$

where $\sigma$ satisfies the differential equation

$$(x\sigma'')^2 + 4(x\sigma' - \sigma) (x\sigma' - \sigma + (\sigma')^2) = 0$$

with the boundary condition $\sigma(x; \lambda) \sim -\frac{2}{x} \lambda$ as $x \to 0$. (Here $K$ is the operator whose kernel is the sine-kernel acting on $L^2(0, s)$. ) The differential equation is the “$\sigma$ representation” of the $P_V$ equation 13, 20. For other proofs of this result see 21, 22, 6.
If \( E_\beta(0; s) \) denotes the limiting value of \( E_{N\beta}(0; (-t, t)) \) in the bulk scaling limit with the scaled length of \( J \) set equal to \( s \), then \([10, 16, 19]\)

\[
E_1(0; s) = \det (I - K_+), \\
E_2(0; s) = \det (I - K), \\
E_4(0; s/2) = \frac{1}{2} (\det (I - K_+) + \det (I - K_-))
\]

where \( K_\pm \) are the operators with kernels \( K(x, y) \pm K(-x, y) \). The \( \det (I - K_\pm) \) can be expressed in terms of \( \det (I - K) \) \([15, 16]\):

\[
\det (I - K_\pm)^2 = \det (I - K) \exp \left( \mp \int_0^s \sqrt{-\frac{d^2}{dx^2} \log \det (I - K)} \, dx \right).
\]

These formulas for \( E_\beta(0; s) \) are well adapted for producing graphs of \( E_\beta(0; s) \) and the level-spacing densities \( p_\beta(s) = \frac{d^2 E_\beta(0; s)}{ds^2} \) once one knows the large \( s \) asymptotics of \( \sigma(s; 1) \) and \( E_\beta(0; s) \) \([24, 16, 25]\).

3.2 Edge Scaling Limit

The limiting law, called the Wigner semi-circle law, is well known

\[
\lim_{N_\beta \to \infty} \frac{1}{2\sigma\sqrt{N_\beta}} \rho_N \left( 2\sigma \sqrt{N_\beta} \, x \right) = \begin{cases} 
\frac{1}{\pi} \sqrt{1 - x^2} & |x| < 1, \\
0 & |x| > 1.
\end{cases}
\]

Here \( \sigma, \sigma/\sqrt{2}, \sigma/\sqrt{2} \) (for \( \beta = 1, 2, 4 \), respectively) is the standard deviation of the Gaussian distribution in the off-diagonal elements and

\[
N_\beta = \begin{cases} 
N, \ \beta = 1, \\
N, \ \beta = 2, \\
2N + 1, \ \beta = 4.
\end{cases}
\]

For the normalization here, \( \sigma = 1/\sqrt{2} \). Less known is that the distribution function of the largest eigenvalue satisfies \([26]\):

\[
F_{N_\beta} \left( 2\sigma \sqrt{N_\beta} + x \right) \to \begin{cases} 
0 & \text{if } x < 0, \\
1 & \text{if } x > 0,
\end{cases}
\]

as \( N \to \infty \). The edge scaling variable, \( s \), defined by

\[
t = 2\sigma \sqrt{N_\beta} + \frac{\sigma s}{N_\beta^{1/4}},
\]
Table 1: The mean ($\mu_\beta$), standard deviation ($\sigma_\beta$), skewness ($S_\beta$) and kurtosis ($K_\beta$) of $F_\beta$.

| $\beta$ | $\mu_\beta$ | $\sigma_\beta$ | $S_\beta$ | $K_\beta$ |
|---------|-------------|----------------|-----------|-----------|
| 1       | -1.20653    | 1.2680         | 0.293     | 0.165     |
| 2       | -1.77109    | 0.9018         | 0.224     | 0.093     |
| 4       | -2.30688    | 0.7195         | 0.166     | 0.050     |

gives the scale of the fluctuations at the edge of the spectrum [27, 28, 17]. As $N \to \infty$ with $s$ fixed [17, 18]

$$F_{N2}(t) \to \exp \left( - \int_s^\infty (x - s)q(x)^2 \, dx \right) =: F_2(s)$$

where $q$ is the solution to the $P_{1I}$ equation

$$q'' = sq + 2q^3$$

satisfying the condition

$$q(s) \sim Ai(s) \quad \text{as} \quad s \to \infty,$$

with $Ai$ the Airy function.

For $\beta = 1, 4$ the results are [11]

$$F_{N1}(t)^2 \to F_1(s)^2 = F_2(s) \exp \left( - \int_s^\infty q(x) \, dx \right)$$

and

$$F_{N4}(t)^2 \to F_4(s/\sqrt{2})^2 = F_2(s) \cosh^2 \left( \frac{1}{2} \int_s^\infty q(x) \, dx \right).$$

Table 1 gives some statistics of $F_\beta$ and Figure 1 shows the densities $f_\beta(s) = dF_\beta/ds.$

4 Universality

The universality of the distribution functions of RMT largely accounts for its success in a wide range of applications, see, e.g. [29]. However, this universality has only been proved in a small number of cases. The evidence for universality is largely numerical.
There are two types of universality results. The first, and the easier to establish rigorously, consists in modifying the random matrix model itself by, say, replacing the Gaussian $x^2/2$ appearing in (1) by an “arbitrary” potential $V(x)$. Changing the potential $V$ changes the density of eigenvalues $\rho$, but several authors have established that the bulk scaling limit results in the sine-kernel and at soft edges one generically obtains the Airy universality class. (Fine tuning at the edge can result in different universality classes, see, e.g. [27].)

The second type of universality, and the one first envisioned by Wigner, asserts [30] that for a classical, “fully” chaotic Hamiltonian the corresponding quantum system has a level spacing distribution equal to $p_\beta(s)$ in the bulk. (The symmetry class determines which ensemble.) A nice numerical example of this quantum chaos is Robnik’s work [31] on chaotic billiards.

Here we discuss three other examples of RMT universality.

4.1 Zeros of the Riemann Zeta Function

Work by Montgomery [32] followed by extensive numerical calculations by Odlyzko [33] on zeros of the Riemann zeta function have given convincing numerical evidence that the normalized consecutive spacings follow the GUE distribution, see Figure 2 (the GUE Hypothesis). Rudnick and Sarnak [34] have proved a restricted form of this hypothesis.

4.2 Eigenvalues of Adjacency Matrices of Quasiperiodic Tilings

The discovery of quasicrystals has made the study of statistical mechanical models whose underlying lattice is quasiperiodic of considerable interest to
physicists. In particular, in order to understand transport properties, tight binding models have been defined on various quasiperiodic lattices. One such study by Zhong et al. [35] defined a simplified tight binding model for the octagonal tiling of Ammann and Beenker. This quasiperiodic tiling consists of squares and rhombi with all edges of equal lengths (see, e.g., [35]) and has a $D_8$ symmetry around the central vertex. On this tiling the authors take as their Hamiltonian the adjacency matrix for the graph with free boundary conditions. The largest lattice they consider has 157,369 vertices. The matrix splits into ten blocks according to the irreducible representations of the dihedral group $D_8$. For each of these ten independent subspectra, they compare the empirical cumulative distribution of the normalized spacings of the consecutive eigenvalues with the integrated density $I_1(s) = \int_s^\infty p_1(x) \, dx$ where $p_1$ is the GOE level spacing density. In Figure 3 we have reproduced a small portion of their data for one such subspectrum together with $I_1$.

### 4.3 Distribution of the Length of the Longest Increasing Subsequence of Random Permutations

An old problem going back to Ulam asks for the limiting behavior of the length of the longest increasing subsequence of a random permutation. Precisely, if $\pi$ is a permutation of $\{1, 2, \ldots, N\}$, we say that $\pi(i_1), \cdots, \pi(i_k)$ is an increasing
subsequence in \( \pi \) if \( i_1 < \cdots < i_k \) and \( \pi(i_1) < \cdots < \pi(i_k) \). Let \( \ell_N(\pi) \) be the length of the longest increasing subsequence of \( \pi \). We take each permutation to be equally likely thus making \( \ell_N \) a random variable. The problem then is to understand the distribution of \( \ell_N \). This problem has a long history (see, e.g. [36]) but its connection with RMT is recent [37, 38]. In a recent paper, Baik, Deift and Johansson [36] have proved the following remarkable result:

**Theorem:** Let \( S_N \) be the group of all permutations of \( N \) numbers with uniform distribution and let \( \ell_N(\pi) \) be the length of the longest increasing subsequence of \( \pi \in S_N \). Let \( \chi \) be a random variable whose distribution function is \( F_2 \), the distribution function for the largest eigenvalue in the GUE in the edge scaling limit (2). Then, as \( N \to \infty \),

\[
\chi_N := \frac{\ell_N - 2\sqrt{N}}{N^{1/6}} \to \chi
\]

in distribution, i.e.

\[
\lim_{N \to \infty} \text{Prob} \left( \chi_N := \frac{\ell_N - 2\sqrt{N}}{N^{1/6}} \leq s \right) = F_2(s)
\]

for all \( s \in \mathbb{R} \).
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