We study the renormalized energy-momentum tensor of cosmological scalar fluctuations during the slow-rollover regime for power-law inflation and find that it is characterized by a negative energy density at the leading order, with the same time behaviour as the background energy. The average expansion rate appears decreased by the back-reaction of the effective energy of cosmological fluctuations, but this value is comparable with the energy of background only if inflation starts at a Planckian energy. We also find that, for this particular model, the first and second order inflaton fluctuations are decoupled and satisfy the same equation of motion. To conclude, the fourth order adiabatic expansion for the inflaton scalar field is evaluated for a general potential $V(\phi)$.

I. INTRODUCTION

Inflationary models (see [1, 2] for a textbook review), born to solve the problems associated with the standard big-bang theory, have their greatest success in the generation of the initial condition for the large scale structure of the Universe through the amplification of the quantum cosmological fluctuations during their accelerated era [3]. Using the last cosmic microwave background (CMB) data [4] and considering the transition from inflation to the standard Big Bang cosmology the constraints on the amplitude and spectrum of CMB fluctuation become constraints on the different models of inflation. All these constraints depend on the linear treatment of cosmological perturbations, so, on using the most recent data, we are close to measuring the details of the spectrum of fluctuations. What is still not completely understood is their energy content and the eventual back-reaction on the inflationary expansion responsible for their amplification.

Among the interesting effects, appreciable only beyond the linear order, one notes, for example, the effect of the non-Gaussianities in the matter power spectrum and in CMB anisotropies (see [5] for a review). However, from the theoretical point of view, we feel that the back-reaction of gravitational fluctuations on the geometry is one of the most interesting issues [6]. Within the inflationary context, this problem has been tackled in [6, 7, 8] with the intriguing result that the energy-momentum tensor (EMT henceforth) of fluctuations may slow down inflation. Thanks to this result in the last decade there has been a renewed interest in the subject of back-reaction [9, 10, 11, 12, 13, 14, 15].

The aim of this paper is to compute the renormalized EMT of cosmological fluctuations for a power-law model of inflation, according to the adiabatic regularization scheme [17] already used in previous papers [13, 18]. While the adiabatic vacuum for a cosmological scalar fluctuations, and its associated EMT, can be computed for generic spacetimes, the unrenormalized EMT can be calculated analytically only if exact analytic solutions for the field Fourier modes are available. This is the case for the power-law inflation [19] where analytic solutions for a scalar field are available. The main theoretical problem which arises is that inflation never comes to an end because of its slow-roll parameter $\epsilon$ is constant.

Power-law solutions can also be applied in different backgrounds as, for example, the alternative cosmological scenario called Ekpyrotic Universe [20], and in the Pre-Big Bang scenario (for a review see [21]) in string cosmology.

The plan of the paper is as follow. In Sec. II we present the linear cosmological perturbations in the uniform curvature gauge (UCG henceforth). In Sec. III we extend the UCG to second order and using the Einstein equations to second order we give the equation of motion for the second order inflaton fluctuation. In Sec. IV we give the solutions for first order fluctuations for the particular case of power-law inflation. In Sec. V we compute the renormalized value for the correlator. We discuss the EMT and the back-reaction on the geometry in Sec. VI and we give our conclusions in Sec. VII. In the Appendix we exhibit the fourth order adiabatic expansion for a general potential $V(\phi)$.
We consider inflation in a flat universe driven by a classical minimally coupled scalar field with a general potential \( V(\phi) \). The action is:

\[
S \equiv \int d^4x \mathcal{L} = \int d^4x \sqrt{-g} \left[ \frac{R}{16\pi G} - \frac{1}{2} g^\mu_\nu \partial_\mu \phi \partial_\nu \phi - V(\phi) \right],
\]

where \( \mathcal{L} \) is the Lagrangian density.

Let us study the fluctuations of the scalar field \( \phi(t, x) \) around its homogeneous classical value \( \phi(t) \) and include metric perturbations. For the homogeneous case we have

\[
\dddot{\phi} + 3H \ddot{\phi} + \frac{1}{2} a^2 \nabla^2 \phi \phi - V(\phi) = 0,
\]

where \( H = \dot{a}/a \) is the Hubble parameter and \( a \) is the scale factor, using the notation \( M^2_{\text{pl}} = 1/(8\pi G) \) for the (reduced) Planck mass definition.

The scalar perturbations around a spatially flat Robertson-Walker metric are

\[
g_{00} = -1 - 2\alpha \]
\[
g_{0i} = -\frac{a}{2} \beta, i \]
\[
g_{ij} = a^2 [\delta_{ij}(1 - 2\psi) + D_{ij} E],
\]

where \( D_{ij} = \partial_i \partial_j - \frac{1}{3} \nabla^2 \delta_{ij} \). In the above we neglect vector and tensor perturbations. If we choose to work in the UCG one obtains:

\[
ds^2 = -(1 + 2\alpha) dt^2 - a \beta, i dt dx^i + a^2 \delta_{ij} dx^i dx^j.
\]

We note that this choice fixes uniquely the gauge, just as the more frequently used longitudinal gauge (for a review of cosmological perturbations in this gauge see [22]). The equation of motion for the first order scalar field fluctuations in the UCG is given by [13]:

\[
\dddot{\phi} + 3H \ddot{\phi} - \frac{1}{2} a^2 \nabla^2 \phi \phi + \frac{2}{a^2} (3H + \dot{H}) \phi = 0.
\]

The Fourier transform modes of \( v = a\phi \) satisfy:

\[
v''_k + (k^2 - \frac{z''}{z}) v_k = 0, \quad z = a \frac{\dot{\phi}}{H},
\]

where \( \dot{} \) denotes a derivative with respect to the conformal time \( \eta \), \( d\eta = dt/a \). On comparing the last equation with Eq. (12) of [22] it is immediate to see that \( \phi \) satisfies the same equation as the Mukhanov variable \( Q \). Therefore, the uniform curvature gauge has the advantage of singling out the true dynamical degrees of freedom (the matter ones), even if it has the disadvantage of being non diagonal in the metric perturbations.
III. BEYOND THE LINEAR ORDER

To second order we consider a metric having the following coefficients:

\[
\begin{align*}
g_{00} & = -1 - 2\alpha - 2\alpha^{(2)} \\
g_{0i} & = -\frac{a}{2} (\beta_i + \beta_i^{(2)}) \\
g_{ij} & = a^2 \left[ \delta_{ij} + \frac{1}{2} \left( \partial_i \chi_j^{(2)} + \partial_j \chi_i^{(2)} + h_{ij}^{(2)} \right) \right].
\end{align*}
\]

(11)

The above metric is the extension of the uniform curvature gauge to second order: \(\alpha^{(2)}\) and \(\beta^{(2)}\) are the scalar perturbations to second order. To second order, scalar, vector and tensor perturbations do not evolve independently as is the case in first order. For this reason we take into account second order vector and tensor perturbations, represented by the divergenceless vector \(\chi_j^{(2)}\) and by the transverse and traceless tensor \(h_{ij}^{(2)}\), respectively. In the above we have omitted first order vector perturbations (which die away kinematically) and tensor perturbations (which satisfy the usual equation \(\ddot{h} + 3H\dot{h} - \nabla^2 h / a^2 = 0\)). With this approximation we are neglecting the EMT of vector and tensor perturbations, and their correlations with the scalar perturbations. We finally note that the choice in Eq. (11) (including vector and tensor metric elements to first order) fixes the gauge completely to second order.

Using the energy and momentum constraints up to second order we can obtain the equations of motion for \(\alpha^{(2)}\), \(\beta^{(2)}\) and \(\varphi^{(2)}\), in particular one obtains for the second order field fluctuation the following equation (see [13]):

\[
\begin{align*}
\dot{\varphi}^{(2)} & + 3H\varphi^{(2)} - \frac{1}{a^2} \nabla^2 \varphi^{(2)} + \left[ V_{\varphi\varphi} + 2 \left( 3H + \frac{\dot{H}}{H} \right) \right] \varphi^{(2)} - \frac{1}{2} V_{\varphi\varphi}\varphi^2 + \frac{1}{M^2_{pl}} \frac{\dot{\varphi}}{2H} \left[ \left( -\frac{5}{2} V_{\varphi\varphi} - 9\dot{H} \right) \frac{\dot{H}}{H} \right. \\
& + \frac{\dot{H}^2}{H^2} - 2\frac{\ddot{H}}{H} \right] \varphi^2 - \frac{1}{a^2} \varphi^2 + \left( -2\frac{\dot{H}}{H} + \frac{\ddot{H}}{H} \right) \frac{\phi \dot{\varphi}}{2} + \frac{1}{a} \frac{\dot{H}}{H} \nabla^2 \varphi + \frac{1}{2a^2} \nabla \varphi^2 - \frac{\phi}{16H a^2} \left[ \beta_{ij} \beta^{ij} - (\nabla^2 \beta)^2 \right] \\
& - \frac{H}{a} \nabla \beta : \nabla \phi - \frac{1}{a} \nabla \beta : \nabla \phi - \frac{1}{a} \nabla \beta : \nabla \phi - \phi \left( -3H + \frac{\dot{H}}{H} - \frac{\ddot{H}}{H} \right) \tilde{s} + \phi \tilde{s}
\end{align*}
\]

(12)

where \(\tilde{s}\) is a non-local spatial contribution given by

\[
\tilde{s} = \frac{1}{\nabla^2} \left[ \frac{1}{2M^2_{pl} H} \nabla \cdot \left( \frac{\dot{\varphi}}{\nabla \varphi} \right) + \frac{1}{a} \frac{1}{M^2_{pl} 8H^2} \frac{\dot{\varphi}}{a} \left( \varphi^{(k)} \beta_{kj} - \varphi^{(k)} \beta_{kj}^{(2)} \right) \right].
\]

(13)

IV. POWER-LAW SOLUTION FOR LINEAR PERTURBATION

We now want restrict ourselves to the case of power-law inflation [19] where \(a(t) \sim t^p\) with \(p > 1\). In this case we have an exponential potential

\[
V(\phi) = V_0 \exp \left[ -\frac{\lambda}{M_{pl}} (\phi - \phi_i) \right]
\]

(14)

with \(V_0 = M^2_{pl} p(3p - 1)\) and \(\lambda = \left( \frac{\phi_i}{p} \right)^{1/2}\). The scale factor and the homogeneous solution are given by

\[
a(t) = \left( \frac{t}{t_i} \right)^p,
\]

\[
\phi(t) = \phi_i + M_{pl} (2p)^{1/2} \log \frac{t}{t_i}
\]

respectively, and the slow-roll parameters by

\[
\epsilon \equiv \frac{M^2_{pl}}{2} \left( \frac{V_{\phi\phi}}{V} \right)^2 = \frac{1}{p}, \quad \eta \equiv \frac{M^2_{pl}}{\nabla V_{\phi\phi}} = \frac{2}{p}.
\]
The equation of motion (9) has an exact solution for this particular case (19). In Fourier space, substituting the effective mass squared $V_{\phi \phi} = \frac{1}{2} (3p - 1)$ and the Hubble factor $H = p/t$, one obtains

$$\ddot{\phi}_k + 3H \dot{\phi}_k + \frac{k^2}{a^2} \phi_k = 0.$$  \hspace{1cm} (15)

Introducing $\psi_k = a^{3/2} \phi_k$ and the conformal time $\eta$ the equation becomes

$$\psi''_k - \frac{p}{1 - p \eta} \psi'_k + \left( k^2 + \left[ \frac{9}{4} p^2 + \frac{3}{2} \right] \frac{1}{(1 - p^2) \eta^2} \right) \psi_k = 0$$  \hspace{1cm} (16)

which gives the general solution

$$\phi_k = \frac{1}{a^{3/2}} (-k\eta) \frac{1}{1 + \frac{p}{1 - p \eta}} \left[ A H^{(1)}_\nu (-k\eta) + B H^{(2)}_\nu (-k\eta) \right]$$  \hspace{1cm} (17)

with $\nu = \frac{3}{2} + \frac{1}{p - 1}$. We now consider quantized fluctuations of the inflaton:

$${\hat{\phi}}(\eta, \mathbf{x}) = \frac{1}{(2\pi)^{3/2}} \int \frac{dk}{\ell} \left[ \phi_k(\eta) e^{i\mathbf{k} \cdot \mathbf{x}} + \phi_k^*(\eta) e^{-i\mathbf{k} \cdot \mathbf{x}} \right]$$  \hspace{1cm} (18)

where the $\hat{b}_k$ are time-independent Heisenberg operators. In order to have the usual commutation relations among the $\hat{b}_k$:

$$[\hat{b}_k, \hat{b}_{k'}] = [\hat{b}_k^\dagger, \hat{b}_{k'}^\dagger] = 0 \quad [\hat{b}_k, \hat{b}_{k'}^\dagger] = \delta^{(3)}(\mathbf{k} - \mathbf{k'})$$  \hspace{1cm} (19)

one must normalize the solution to the equations of motion through the Wronskian condition:

$$\phi_k \phi_k^* - \phi_k' \phi_k'^* = \frac{i}{a^2}.$$  \hspace{1cm} (20)

This normalization condition yields the following relation among the coefficients $A, B$ of Eq. (17):

$$|A|^2 - |B|^2 = -(-k\eta) \frac{1}{1 + \frac{p}{1 - p \eta}} a^{3\nu} \frac{\pi \eta}{4}.$$  \hspace{1cm} (21)

The solution associated with the adiabatic vacuum for $k \to \infty$ corresponds to choosing $A = (-k\eta) \frac{\pi}{4(1 - p)} a^{1/2} \left( -\frac{\pi \eta}{4} \right)^{1/2} H^{(1)}_\nu (-k\eta)$ and $B = 0$. Thus one obtains the following adiabatic solution:

$$\phi_k = \frac{1}{a} \left( -\frac{\pi \eta}{4} \right)^{1/2} H^{(1)}_\nu (-k\eta)$$  \hspace{1cm} (22)

which, in the proper time, becomes

$$\phi_k = \frac{1}{a^{3/2}} \left( \frac{\pi}{4H} \right)^{1/2} \left( \frac{p}{p - 1} \right)^{1/2} H^{(1)}_\nu \left( \frac{p}{p - 1} \frac{k}{aH} \right).$$  \hspace{1cm} (23)

**V. ADIABATIC SUBTRACTION**

We now compute the integrals obtained on taking the vacuum expectation values of the relevant operators. Let us consider the correlator $\langle \phi^2 \rangle$ (we will link this to the energy-momentum tensor in the next section)

$$\langle \phi^2 \rangle = \frac{1}{(2\pi)^3} \int_{|k| > \ell} dk |\phi_k|^2$$  \hspace{1cm} (24)

where $\ell = CH_i$ is an infrared cut-off related to the beginning of inflation (24, 25). Using solution (23) we obtain a UV divergence, so we employ, as in the previous papers (13, 18), dimensional regularization to treat the UV behaviour, therefore the integrands will be in 3 dimensions and the integration measure analytically continued in $d$ dimensions. Subsequently an adiabatic subtraction is performed in order to obtain the renormalized quantities. The $\langle \phi^2 \rangle$ becomes

$$\langle \phi^2 \rangle = \frac{1}{(2\pi)^d \Gamma(d/2)} \int_{\ell}^{+\infty} dk \, k^{d-1} \frac{\pi}{4H} \frac{1}{a^3} \frac{p}{p - 1} \left[ J_\nu \left( \frac{p}{p - 1} \frac{k}{aH} \right)^2 + N_\nu \left( \frac{p}{p - 1} \frac{k}{aH} \right)^2 \right]$$  \hspace{1cm} (25)
On using (B.3) of Appendix B of [18] with \( \alpha = d - 1 \) one obtains:

\[
\langle \varphi^2 \rangle = \frac{1}{16\pi^2} H^2 \left\{ 2p \left( \frac{H_i}{H} \right)^2 \left( \frac{\Gamma(\nu)}{\Gamma(3/2)} \right)^2 (1 - \frac{1}{p})^{2\nu} \left( \frac{\ell}{2H_i} \right)^{\frac{\pi}{2}} + \frac{2p - 1}{p} \pi \cot \left( \frac{\pi}{p - 1} \right) + \left( \frac{1}{2\pi^{1/2}} \right)^{d-3} \left( \frac{p - 1}{p} \right)^{d-3} \right. \\
\left. \frac{(aH)^{d-3}}{\Gamma(d/2)} \frac{\Gamma(3/2)}{\Gamma(3/2 + \nu)} \frac{\Gamma(3/2 + \nu)}{\Gamma(1 - 1/2 + \nu)} \right\} \\
\right. + O(d - 3) 
\]

where to avoid singularities in the analytic continuation we have to consider \( p \gg 3 \), this is not a problem because, for power-law inflation, the range \( p < 60 \) is disfavored at 2 \( \sigma \) (see, for example, [26]).

The adiabatic fourth order is (see Appendix and [18]):

\[
\langle \varphi^2 \rangle_{(4)} = \frac{1}{16\pi^2} H^2 \left\{ \frac{17}{10} \frac{1}{p^2} \left( \frac{131}{30} p + 31 + 12 - \frac{17}{60} p^2 - 1 + \left( \frac{1}{2\pi^{1/2}} \right)^{d-3} (aV_{\phi})^{d-3} \right. \\
\left. \left( -2 + \frac{1}{p} \right) \Gamma \left( \frac{1}{2} - \frac{d}{2} \right) + O \left( \frac{1}{a^3} \right) \right\} + O(d - 3) 
\]

and the resulting renormalized quantity is

\[
\langle \varphi^2 \rangle_{\text{REN}} = \lim_{d \to 3} \left( \langle \varphi^2 \rangle - \langle \varphi^2 \rangle_{(4)} \right) 
\]

\[
= \frac{1}{16\pi^2} H^2 \left\{ 2p \left( \frac{H_i}{H} \right)^2 \left( \frac{\Gamma(\nu)}{\Gamma(3/2)} \right)^2 (1 - \frac{1}{p})^{2\nu} \left( \frac{\ell}{2H_i} \right)^{\frac{\pi}{2}} + \frac{2p - 1}{p} \pi \cot \left( \frac{\pi}{p - 1} \right) - \frac{17}{10} \frac{1}{p^2} + \frac{131}{30} p - \frac{31}{60} \\
\left. - \frac{5}{12} p + \frac{17}{60} p^2 - 1 + \left[ \log \left( \frac{p - 1}{2(3p - 1)^{1/2}} \right) + \frac{1}{2} \left( \frac{p - 1}{2p - 1} + \frac{p - 1}{p} + 2\Psi \left( 1 + \frac{1}{p - 1} \right) \right) \right] \right\} \\
\left. \left( -4 + \frac{2}{p} \right) + O \left( \frac{1}{a^2} \right) \right\} . 
\]

So the leading behaviour of the renormalized correlator (which is essentially given by the infrared contribution of \( \langle \varphi^2 \rangle \), namely the first term of Eq. (26)), during the inflationary era, with \( p >> 1 \), is:

\[
\langle \varphi^2 \rangle_{\text{REN}} \sim \frac{p}{8\pi^2} H_i^2 , 
\]

the same result is obtained in [27] for a massless minimally coupled test scalar field in a power-law model of inflation. In fact this test scalar field satisfies the same equation of motion of the field fluctuation \( \varphi \) (see Eq. (15)).

Let us comment on the possible gauge dependence of this result. Starting from the general metric [4] and considering the first order gauge transformation (see, for example, [28]) the gauge invariant Mukhanov variable \( Q \) is given by \( \varphi + \frac{i}{2} \dot{\psi} \), and in the UCG this gauge invariant variable, as already stated, coincides with the scalar field fluctuation \( \varphi \). Thus, in this particular gauge, \( \varphi^2 \) coincides with \( Q^2 \) which is a gauge invariant quantity. If we choose a different gauge such as, for example, the longitudinal one, the Mukhanov variable is no longer equal to the scalar field fluctuation \( \varphi \) and, while \( \langle Q^2 \rangle \) takes the same value as in the UCG, \( \langle \varphi^2 \rangle \) takes a different value. More generally for any quantity, in a given gauge, one can always choose a gauge invariant quantity which coincides with it in that particular gauge. The difficulty is that, except for some particular cases (such as \( \varphi^2 \) in the UCG), it may be difficult to understand the physical meaning of a general gauge invariant quantity. This appears to be the case for the EMT, which is considered in the next section. A general gauge invariant variable connected with it in the UCG may not coincide with the value of the EMT in another gauge. So for the EMT the problem of the gauge dependence of its value does not appear to be solved.

VI. APPROACHES TO THE BACK-REACTION

One may follow different approaches in order to study the backreaction effects due to cosmological fluctuations. The main two methods in order to tackle this issue can be described in the following way.
The first consists of considering only first order perturbations, then imposing, to first order, the energy and momentum constraints and finally defining an effective EMT by including all the quadratic terms present in the Einstein equations. Subsequently one averages everything over the quantum vacuum and first order quantities disappear. One finds that the effective EMT which appears in the averaged Einstein equations is modified by the back-reaction.

The second approach is related to the standard perturbation analysis of the Einstein equations up to second order. In this case we impose the energy and momentum constraints and study the inflaton equation of motion perturbatively up to second order. One does not define any modified EMT but directly studies any observable averaged over the quantum vacuum in this framework.

A. The Energy-Momentum Tensor of Cosmological Fluctuations

On following the first approach the effective EMT of cosmological fluctuations is given by

$$\tau^\mu_\nu \equiv T^\mu_\nu \text{quadratic} - M^2_{\text{pl}} G^\mu_\nu \text{quadratic}.$$  \hfill (30)

This method of considering the EMT of gravitational fluctuations is treated in textbooks [29] and has also been previously used in [7, 8, 13, 30]. In this scheme one considers the modified Einstein equations

$$M^2_{\text{pl}} G^\mu_\nu (0) = T^\mu_\nu (0) + \langle \tau^\mu_\nu \rangle$$

which therefore includes back-reaction effects.

For a generic potential, with an effective mass $V_{\phi\phi}$ different from zero, the leading terms in the slow-roll parameter of the energy density are (see [13]):

$$\langle \varphi^2 \rangle \sim -\frac{V_{\phi\phi}}{2} \langle \varphi^2 \rangle - 6\dot{H} \langle \varphi^2 \rangle$$

$$\equiv -\varepsilon \sim -\frac{V_{\phi\phi}}{2} \langle \varphi^2 \rangle \left(1 - \frac{4\varepsilon}{\eta}\right).$$  \hfill (31)

Analogously the average pressure is:

$$\langle \tau^i_j \rangle \equiv p \delta^i_j \sim \delta^i_j \left(-\frac{V_{\phi\phi}}{2} \langle \varphi^2 \rangle - 6\dot{H} \langle \varphi^2 \rangle \right) \sim -\varepsilon \delta^i_j.$$  \hfill (32)

We now restrict the analysis to a power-law potential where $\eta = 2\varepsilon$.

The leading behaviour of the renormalized correlator is given by the infrared part of $\langle \varphi^2 \rangle$ (the adiabatic part does not contribute to this leading value) so on using Eq. (31) we can obtain, by Eq. (29), the leading value of the energy, without repeating the adiabatic subtraction, as

$$\varepsilon_{\text{REN}} \sim -\frac{3}{8\pi^2} H^2 H_i^2.$$  \hfill (33)

Such a value leads to a negative contribution in the right hand side of the Einstein equation of order of $H^2 H_i^2 / M^2_{\text{pl}}$, so we have that the average expansion rate appears to be decreased by the back-reaction of cosmological fluctuations. The importance of back-reaction is therefore related to the ratio $H^2 / M^2_{\text{pl}}$. If inflation starts at a Planckian energy then back-reaction during inflation cannot be neglected as expected. We feel, however, that it is important to underline that for energy near to the Planck value the semi-classical approach to the quantum gravity may be questionable.

We want to stress that, in this UCG, the back-reaction effects for a model of power-law inflation are very different from the effects for a chaotic model $\frac{1}{2} \dot{\phi}^2$ found in [13] and given by a negative effective energy of the order of $H^2 / H^2$. While in the case under consideration the effective energy has the same time behaviour as the background energy $3M^2_{\text{pl}} H^2$, in the chaotic case it grows with respect to the background value and a negligible back-reaction at the beginning of inflation can become important before the end of inflation (see [13]). This is in agreement with what Abramo and Woodard state in [11] on using the longitudinal gauge for the calculation of the back reaction in a power-law model of inflation, namely that the shape of the inflation potential can have an enormous impact on back-reaction.

For the sake of completeness we have to repeat that the problem of the gauge dependence of this effective energy as yet does not appear to be solved.
B. Back-Reaction on the Geometry

In the perturbative approach to the Einstein equations any back-reaction effect is analyzed by evaluating perturbatively quantities which characterize the geometry (as, for example, the expansion scalar $\Theta$). As stated in Sec. IV the leading effect in the renormalized quantities, for models with an effective mass different from zero, is given by the infrared part of the non renormalized integrals, so we shall use the long-wavelength approximation in the treatment of the second order scalar perturbations (see [13, 28]). In the long-wavelength limit, from the first order equation of motion Eqs. (9) and (10), one obtains $\phi \sim C \frac{H}{\dot{H}}$, where $C$ is a constant in time, thus in this limit the non-local spatial contribution $\tilde{s}$ (Eq. (13)) has an ordinary behavior given, for the isotropic case, by $\phi \dot{\phi}/(4M_p^2H)$. In general on using this limit on the right hand side of Eq. (12) one obtains the following result (valid to all orders in the slow-roll parameters):

$$\varphi^2 \left\{ \frac{1}{M_p^2} \left[ \frac{1}{2} \frac{d^2}{dt^2} \left( \frac{\dot{\phi}}{H} \right) + \frac{3}{4} \frac{H}{\phi} \frac{d}{dt} \left( \frac{\dot{\phi}}{H} \right) + \frac{1}{2} \frac{H}{\phi} \left( \frac{d}{dt} \left( \frac{\dot{\phi}}{H} \right) \right)^2 \right] + \frac{H}{2\phi^2} \frac{d}{dt^3} \left( \frac{\dot{\phi}}{H} \right) + \frac{3}{2} \frac{H^2}{\phi^2} \frac{d^2}{dt^2} \left( \frac{\dot{\phi}}{H} \right) - \frac{H^2}{2\phi^3} \frac{d}{dt} \left( \frac{\dot{\phi}}{H} \right) \frac{d^2}{dt^2} \left( \frac{\dot{\phi}}{H} \right) - \frac{3}{2} \frac{H^3}{\phi^3} \left[ \frac{d}{dt} \left( \frac{\dot{\phi}}{H} \right) \right]^2 \right\}. \quad (34)$$

For the case under consideration, namely power-law inflation, $\frac{\dot{H}}{H}$ is a constant and, consequently, the inhomogeneous term of Eq. (12) is equal to zero in the long-wavelength limit. A coupling between the modes of different orders in perturbation theory could then occur only in the ultraviolet. Thus, in the long-wavelength approximation, the equation of motion for the second order field fluctuation $\varphi^{(2)}$ is the same as that for the first order one, and one can write in the infrared an equation for a total fluctuation $\tilde{\varphi} = \varphi + \varphi^{(2)}$ which is to be constrained by the initial condition. This is equivalent to saying that since the first and the second order scalar perturbations $\varphi$ and $\varphi^{(2)}$ are decoupled, the second order contribution does not give us any new information, with respect to the first order, and it can be seen as a renormalization of the first order one. This is the only useful information which can be gained by the perturbative approach. We feel that the significance of the above considerations and results deserves further investigations.

VII. DISCUSSION AND CONCLUSIONS

The renormalized EMT of cosmological fluctuations for a power-law model of inflation has been studied. As in our previous work [13] we have self-consistently taken into account the gravitational fluctuations accompanying the scalar field fluctuations in a UCG.

We find that the renormalized EMT of cosmological fluctuations during slow-rollover carries negative energy density (due to the inclusion of gravitational fluctuations [13]) with a de Sitter like equation of state to leading order. The average expansion rate appears to be decreased by the back-reaction of cosmological fluctuations and the leading value of the effective energy is of the order of $H^2H_s^2$, so the back-reaction effects cannot be neglected if inflation started at Planckian energies, i.e., at $H_s \sim M_p$.

We stress that the back-reaction effects for this model are very different from the effects for a chaotic model $\frac{m^2}{2} \varphi^2$ ([13]) given by a negative effective energy of the order of $H_s^2/H^2$. While in the case under consideration the effective energy has the same time behaviour as the background energy $3M_p^2H^2$, in the chaotic case it grows with respect to the background value and also a negligible back-reaction at the beginning of inflation can become important before the end of inflation (see [13]). The back-reaction effects in inflation are strongly dependent on the shape of the potential (see [11] for similar consideration in longitudinal gauge).

In this model of power-law inflation and in the long-wavelength approximation we also find that the first and second order scalar perturbations $\varphi$ and $\varphi^{(2)}$ are decoupled and they have the same equation of motion, the second order can then be seen just as a renormalization of the first order with no other physical information.

To conclude we find in the Appendix the fourth order adiabatic expansion for a general potential $V(\phi)$. Let us note that to obtain this general fourth order adiabatic expansion a crucial point is to distinguish between the different adiabatic orders, the effective mass $V_{\phi\phi}$ obtained by a double field derivative is of adiabatic order 0 while the quantities obtained by deriving n-times with respect to the time are of adiabatic order n, independently of their value.

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VIII. APPENDIX: THE FOURTH ORDER ADIABATIC EXPANSION

In order to remove the divergences which appear in the integrated quantities as poles in the $\Gamma$ functions, we shall employ the method of adiabatic subtraction \cite{31, 32}. Such a method consists in replacing our functions with an expansion in powers of derivatives of the logarithm of the scale factor. This expansion coincides with the adiabatic expansion introduced by Lewis in \cite{33} for a time dependent oscillator.

Usually it is more convenient to formulate the adiabatic expansion by using the modulus of mode functions $x_k = |\varphi_k/\sqrt{2}|$ and the conformal time $\eta$ \cite{32} ($d\eta = dt/a$). We follow this procedure and write an expansion in derivatives with respect to the conformal time (denoted by $'$) for $x_k$. We then go back to cosmic time and insert the expansion in the expectation values we wish to compute. Adiabatic expansions in cosmic time and conformal time lead to equivalent results, because of the explicit covariance under time reparametrization \cite{34}.

The variable $x_k$ satisfies the following Pinney equation:

$$\ddot{x}_k + 3H\dot{x}_k + \left[\frac{k^2}{a^2}V_{\phi\phi} + 2 \left(3H + \frac{\dot{H}}{H}\right)\right] x_k = \frac{1}{a^6 x_k^4}. \quad (35)$$

Following \cite{13, 18} we rewrite Eq. (35) in conformal time in the following way:

$$(ax_k)'' + \Omega_k^2 (ax_k) = \frac{1}{(ax_k)^3} \quad (36)$$

where

$$\Omega_k^2 = k^2 + a^2V_{\phi\phi} - \frac{1}{6}a^2\tilde{R} \quad (37)$$

and $\tilde{R}$ is:

$$\tilde{R} = R - 6 \left(-\frac{a''}{a} - \frac{6}{a^2a'^2} + \frac{2}{a^2a''} \right), \quad (38)$$

with $R = 6\Omega''$ the Ricci curvature. The key point for this calculation is that $V_{\phi\phi}$ is of adiabatic order 0, while $\tilde{R}$ is of adiabatic order 2, independently of their value. From Eqs. (36, 37, 38) one obtains the expansion for $x_k$ up to the fourth adiabatic order:

$$x_k^{(4)} = \frac{1}{a} \frac{1}{\Omega_k^{1/2}} \left(1 - \frac{1}{4}\epsilon_2 + \frac{5}{32}\epsilon_4 - \frac{1}{4}\epsilon_4\right) \quad (39)$$

where $\Omega_k$ is defined in Eq. (37) and $\epsilon_2, \epsilon_4$ are given by:

$$\epsilon_2 = -\frac{1}{2} \frac{\Omega_k''}{\Omega_k^3} + \frac{3}{4} \frac{\Omega_k^2}{\Omega_k^4}$$

$$\epsilon_4 = \frac{1}{4} \frac{\Omega_k'}{\Omega_k} \epsilon_2 - \frac{1}{4} \frac{1}{\Omega_k} \epsilon_2 \epsilon_2 \quad (40)$$

The solution in Eq. (39) must be expanded again since $\tilde{R}$ is of adiabatic order 2. Therefore $x_k^{(4)}$, for a general potential, is:

$$x_k^{(4)} = \frac{1}{c^{1/2}} \frac{1}{\Sigma_k^{1/2}} \left(1 + \frac{1}{4} \frac{\tilde{R}}{6} \frac{1}{\Sigma_k} + \frac{5}{32} \frac{\tilde{R}^2}{36} \frac{1}{\Sigma_k} \right)$$

$$+ \frac{1}{16} \frac{1}{\Sigma_k} \left[ c''V_{\phi\phi} + 2c' V_{\phi\phi} + cV_{\phi\phi} - c' \frac{\tilde{R}}{6} - 2c' \frac{\tilde{R}'}{6} - c \frac{\tilde{R}''}{6} \right]$$
\[-\frac{5}{64} \sum_k \left[ \left( c' V_{\phi\phi} + c V' \right)^2 + 2 \left( c' V_{\phi\phi} + c V' \right) \left( -\frac{c' \tilde{R}}{6} - \frac{\tilde{R}'}{6} \right) \right] \]
\[+ \frac{9}{64} \sum_k \frac{c \tilde{R}}{6} \left( c' V_{\phi\phi} + 2 c' V_{\phi\phi} + c V' \phi \phi \right) - \frac{65}{256} \sum_k \frac{c \tilde{R}}{6} \left( c' V_{\phi\phi} + c V' \phi \phi \right)^2 + \frac{5}{32} \epsilon_{2s}^2 - \frac{1}{4} \epsilon_{4s} \}
\] (41)

where \( c = a^2 \) and

\[
\Sigma_k = (k^2 + a^2 V_{\phi\phi})^{1/2}
\]
\[
c_{2s} = \frac{1}{2} \frac{k'}{\Sigma_k} + 3 \frac{\Sigma_k'}{4 \Sigma_k}
\]
\[
c_{4s} = \frac{1}{4} \frac{\Sigma_k'}{\Sigma_k^2} c_{2s} - \frac{1}{4} \frac{\Sigma_k'}{\Sigma_k^2} c_{2s}'.
\] (42)