Discrete Polynomials and Discrete Holomorphic Approximation

Christian MERCAT
Technische Universität Berlin, Germany
merc@slb288.math.tu-berlin.de

November 14, 2018

Abstract

We use discrete holomorphic polynomials to prove that, given a refining sequence of critical maps of a Riemann surface, any holomorphic function can be approximated by a converging sequence of discrete holomorphic functions.

1 Introduction

The notion of discrete Riemann surfaces has been defined in [1]. We are interested in discrete surfaces given by a cellular decomposition $\diamond$ of dimension two, where all faces are quadrilaterals. We suppose $\diamond$ bipartite and it defines (away from the boundary) two dual cellular decompositions $\Gamma$ and $\Gamma^*$, edges $\Gamma_1^*$ are dual to edges $\Gamma_1$, faces $\Gamma_2^*$ are dual to vertices $\Gamma_0$ and vice-versa. Their union is denoted the double $\Lambda = \Gamma \sqcup \Gamma^*$. A discrete conformal structure on $\Lambda$ is a real positive function $\rho$ on the unoriented edges satisfying $\rho(e^*) = 1/\rho(e)$. It defines a genuine Riemann surface structure on the discrete surface: Choose a length $\delta$ and realize each quadrilateral by a lozenge whose diagonals have a length ratio given by $\rho$. Gluing them together provides a flat riemannian metric with conic singularities at the vertices, hence a conformal structure [2]. This data leads to a straightforward discrete version of the Cauchy-Riemann equation. A function on the vertices of $\diamond$ is discrete
holomorphic iff for every quadrilateral \((x, y, x', y') \in \Diamond_2,\)

\[ f(y') - f(y) = i \rho(x, x') (f(x') - f(x)). \] (1.1)

Figure 1: The discrete Cauchy-Riemann equation.

Given a fixed flat riemannian metric on a Riemann surface, with a finite number of conic singularities, we define a discrete conformal structure as critical if the flat riemannian metric it gives rise to is isometric to this fixed one. Let’s stress that this notion is useful mostly when considering a sequence of discrete conformal structures, all adapted to the same fixed flat metric. A refining sequence is such a sequence of critical maps where the common side lengths \(\delta_k\) of the diamonds in the map \(\Diamond_k\), form a sequence converging to zero, and where the lozenge angles are bounded away from zero (the faces don’t collapse).

In [1] we proved that 1) any Riemann surface accepts a refining sequence of critical maps and 2) a converging sequence of discrete holomorphic functions on a refining sequence of critical maps converges to a continuous holomorphic function. It is the main purpose of this article to prove the converse result:

**Theorem 1.1** Given a refining sequence of critical maps of a compact flat simply connected surface \(U\), any holomorphic function on \(U\) can be approximated by a converging sequence of discrete holomorphic functions.
The proof is based on series expansion. In Sec. 2 we define discrete integration at criticality and show that the sequence of primitives of converging discrete holomorphic functions converge to the continuous primitive. We define discrete polynomials and use them to prove the main theorem.

In Sec. 3 we define discrete derivation and present its differences with the continuous one. It allows none the less for the coefficients of a discrete polynomial to be computed by successive derivation. In the annex, we discuss different aspects of the discrete theory which differ or don’t appear in the continuous case, in Sec. A another notion of derivation, in Sec. B the analog of the Leibnitz formulae, in Sec. C a motivation of the derivation formula through a study of the minimal polynomial and in Sec. D a discussion of the problems arising when considering discrete series, with the example of the discrete exponential.

## 2 Integration at criticality

Away from a conic singularity, a critical map locally forms a flat partition of the plane by lozenges. The relevance of this kind of maps in the context of discrete holomorphy was first pointed out and put to use by Duffin [3]. He defined the discrete analogues of the integer powers of $\mathbb{Z}$ and the derivation formula that we are going to give.

The crucial point about criticality is the following remark: Given an isometric local map $Z : U \cap \Lambda \to \mathbb{C}$, where the image of the quadrilaterals are lozenges in $\mathbb{C}$, any holomorphic function $f \in \Omega(\Lambda)$ gives rise to an holomorphic 1-form $f dZ$ defined by the formula,

$$\int_{(x,y)} f dZ := \frac{f(x) + f(y)}{2} (Z(y) - Z(x)),$$  \hspace{1cm} (2.1)

where $(x, y) \in \triangle_1$ is an edge of a lozenge. It is averaged into a holomorphic 1-form on $\Lambda$:

$$\int_{(x,x')} f dZ = \frac{f(x) + f(y) + f(x') + f(y')}{4} (Z(x') - Z(x)).$$ \hspace{1cm} (2.2)

Alternatively, it provides a way to integrate a function by taking the primitive of $f dZ$. 


**Theorem 2.1** Given a sequence of discrete holomorphic functions \((f_k)\) on a refining sequence of critical maps, converging to a holomorphic function \(f\), the sequence of primitives \(\int f_k dZ\) converges to \(\int f(z) dz\). Moreover, in the compact case, if the convergence of the functions is of order \(O(\delta_k^2)\), it stays this way for the primitives.

**Proof 2.1.** We recall from [1] that we extend a discrete holomorphic function \(f_d\) from the vertices \(\diamondsuit_0\) linearly to the edges \(\diamondsuit_1\) and harmonically to the faces \(\diamondsuit_2\) to obtain a continuous piecewise harmonic function \(\hat{f}_d\) of the surface. We proved that, given a refining sequence \((\diamondsuit_k)\) of critical maps, the point-wise convergence of a sequence of discrete holomorphic functions \((f_k)\), restricted to the sequence of vertices \(\diamondsuit_0\), implies a uniform limit of the continuous extensions \(\hat{f}_k\) to a genuine (continuous) holomorphic function \(f\).

Let \(U\) be the flat simply connected patch under consideration. We suppose that we are given a sequence of flat vertices \(O_k \in \diamondsuit_k\) where the face containing the fixed flat origin \(O \in U\) is adjacent to \(O_k\). Let \(\hat{F}_k\) the extension of the primitive \(\int_{O_k} f_k dZ\) to the whole surface. We want to prove that the following sequence tends to zero

\[
\left| (\hat{F}_k(x) - \hat{F}_k(O)) - \int_{O}^{x} f(z) dz \right| \kappa_{\in \mathbb{N}}.
\tag{2.3}
\]

Let us fix a point \(x\), and for each integer \(k\) consider a vertex \(x_k \in \diamondsuit_0\) on the boundary of the face of \(\diamondsuit_2\) containing \(x\).

We decompose the difference (2.3) into three parts, inside the face containing the origin \(O\) and its neighbor \(O_k\), similarly for \(x\) and \(x_k\), and purely along the edges of the graph \(\diamondsuit^k\) itself.

\[
\left| (\hat{F}_k(x) - \hat{F}_k(O)) - \int_{O}^{x} f(z) dz \right| =
\]

\[
\left| (\hat{F}_k(x) - \hat{F}_k(x_k)) + \int_{O_k}^{x_k} f_k dZ + (\hat{F}_k(O) - \hat{F}_k(O_k)) - \int_{O}^{x} f(z) dz \right|
\leq \left| \hat{F}_k(x) - \hat{F}_k(x_k) - \int_{x_k}^{x} f(z) dz \right| + \left| \int_{O_k}^{x_k} f_k dZ - \int_{O_k}^{x_k} f(z) dz \right|
\]

\[
\leq \left| \hat{F}_k(O_k) - \hat{F}_k(O) - \int_{O}^{x_k} f(z) dz \right|.
\tag{2.4}
\]

On the face of \(\diamondsuit\) containing \(x\), the primitive \(x \mapsto \int_{x_k}^{x} f(z) dz\) is a holomorphic, hence harmonic function as well as \(x \mapsto \hat{F}_k(x)\). By the maximum principle,
the harmonic function \( x \mapsto \hat{F}_k(x) - \hat{F}_k(x_k) - \int_{x_k}^x f(z) \, dz \) reaches its maximum on that face, along its boundary. The difference of the discrete primitive along the edges of \( \diamond \) is equal to

\[
\hat{F}_k((1 - \lambda)x + \lambda y) - \hat{F}_k(x) = \lambda(y - x)\frac{f_k(x) + f_k(y)}{2} \tag{2.5}
\]

while because \( f \) is differentiable with a bounded derivative on \( U \),

\[
\int_x f(z) \, dz = \lambda(y - x)\frac{f(x) + f(y)}{2} + (y - x)^2 \frac{\lambda^2 f'(x) + (1 - \lambda)^2 f'(y)}{4} + o(\delta_k^3) \\
= \lambda(y - x)\frac{f(x) + f(y)}{2} + O(\delta_k^3) \tag{2.6}
\]

so that

\[
|\hat{F}_k(x) - \hat{F}_k(x_k) - \int_{x_k}^x f(z) \, dz| = O(\delta_k^2). \tag{2.7}
\]

Similarly for the term around the origin.

By definition of \( \hat{F}_k \), the 1-form \( \hat{f}_k(z) \, dz \) along edges of the graph \( \diamond \) is equal to the discrete form \( f_k dZ \) so that \( \int_{O_k} f_k dZ = \int_{O_k} \hat{f}_k(z) \, dz \) on a path along \( \diamond \) edges. Therefore the difference

\[
\left| \int_{O_k} f_k dZ - \int_{O_k} f(z) \, dz \right| \leq \int_{O_k} \left| \left( \hat{f}_k(z) - f(z) \right) \, dz \right| \tag{2.8}
\]

is of the same order as the difference \( |f_k(z) - f(z)| \) times the length \( \ell(\gamma_k) \) of a path on \( \diamond_k \) from \( O_k \) to \( x_k \). We showed in \[1\] that provided that the lozenge angles are in the interval \((\eta, 2\pi - \eta)\) with \( \eta > 0 \) (the faces don’t collapse), this length can be bounded as \( \ell(\gamma_k) \leq \frac{4}{\sin(\eta)} |x_k - O_k| \). Since we are interested in the compact case, this length is bounded uniformly and the difference (2.8) is of the same order as the point-wise difference. We conclude that the sequence of discrete primitives converges to the continuous primitive and if the limit for the functions was of order \( O(\delta^2) \), it remains of that order. \( \diamond \)

Following Duffin \[4, 3\], we define by inductive integration the discrete analogues of the integer power monomials \( z^k \), that we denote \( Z^{:k} \):

\[
Z^{:0} := 1, \tag{2.9}
\]

\[
Z^{:k} := \int_{O_k} Z^{:k-1} \, dZ, \tag{2.10}
\]

\( 5 \)
where $O$ is a fixed flat origin. The choice of this origin is relevant and we present in the Appendix the formulae for the change of basis of discrete polynomials from one base point to another. Although polynomials are translated into polynomials of same degree, the formulae are slightly more involved than the Leibnitz rule $(Z - a)^n = \sum_{k=0}^{n} \binom{n}{k} Z^k a^{n-k}$. When sequences of maps are involved, we suppose that we have chosen a flat origin for each map, forming a converging sequence to a flat point. For example we suppose that a fixed flat point $O$ on the surface is a vertex of each map.

The discrete polynomials of degree less than three agree point-wise with their continuous counterpart, $Z^2(x) = Z(x)^2$.

A simple induction then gives the following

**Corollary 2.2** The discrete polynomials converge to the continuous ones, the limit is of order $O(\delta_k^2)$.

Which implies the main theorem: Proof [1.4]. On the simply connected compact set $U$, a holomorphic function $f$ can be written, in a local map $z$ as a series,

$$f(z) = \sum_{k \in \mathbb{N}} a_k z^k. \quad (2.11)$$

Therefore, by a diagonal procedure, there exists an increasing integer sequence $(N(n))_{n \in \mathbb{N}}$ such that the sequence of discrete holomorphic polynomials converge to the continuous series.

$$\left( \sum_{k=0}^{N(n)} a_k Z^k; \right)_{n \in \mathbb{N}} \rightarrow f. \quad (2.12)$$

\diamondsuit

### 3 Derivation at criticality

We are interested in getting the coefficients of a polynomial by the same process as in the continuous case: successive derivation. We are going to see that this operation, while possible, is more difficult than in the continuous.

Let $\varepsilon$ be the biconstant $\varepsilon(\Gamma) = +1$, $\varepsilon(\Gamma^*) = -1$. The main issue is to separate this "mode" from the constant mode in order to define properly the discrete analog of the value of a function at the origin.
On a finite map, the dimension of the space of discrete holomorphic functions is equal to half of the number of boundary points plus one,

\[
\Omega(U) \sim \mathbb{C}^{|\partial U|/2 + 1},
\]  

(3.1)

for example the values on the boundary points \( \Gamma_0^* \cap \partial U \) on the graph \( \Gamma^* \) and in one point \( O \in \Gamma \). The dimension of the space of discrete polynomials can not be greater than this number. This implies the existence of a minimal polynomial

\[
P_Z = \sum_{k=1}^{n} a_k Z^k : \equiv 0.
\]  

(3.2)

For a holomorphic function \( f \), the equality \( f dZ \equiv 0 \) is equivalent to \( f = \lambda \varepsilon \) for some \( \lambda \in \mathbb{C} \), we can uniquely normalize \( P_Z \) so that \( \sum_{k=1}^{n} k a_k Z^{k-1} : = \varepsilon \), that is \( a_1 = 1 \).

We can therefore take as a basis of the discrete polynomials the set

\[
(1, Z, Z^2; \ldots, Z^{n-2}; \varepsilon).
\]  

(3.3)

We conjectured in [1] that when \( U \) is convex [5], (3.3) is a basis of the whole space of discrete holomorphic functions. In any case, one can supplement them by an orthogonal complement and we will denote \( \nu \in \mathbb{C}^{|\partial U|/2 + 1} \) the vector encoding the linear form yielding the coordinate along the constant function 1 in this basis:

\[
\nu \cdot \sum_{k=0}^{n-2} a_k Z^k : = a_0,
\]

\[
\nu \cdot \varepsilon = 0,
\]

\[
\nu \cdot f = 0 \quad \text{when} \quad \forall k, \ Z^k : \cdot f = 0.
\]  

(3.4)

Notice that since the change of base point for polynomials is triangular in the basis of monomials, the vector \( \nu \) does not depend on the map \( Z \) or on the base point but only on the discrete conformal structure. An explicit value for this vector would be desirable, especially for numerical purposes.

Following Duffin [4, 3] (see Sec.C for a motivation), we introduce the

**Definition 3.1** For a holomorphic function \( f \), define on a flat simply connected map \( U \) the holomorphic functions \( f^\dagger \), the dual of \( f \), and \( f' \), the derivative of \( f \), by the following formulae:

\[
f^\dagger(z) := \varepsilon(z) \bar{f}(z),
\]  

(3.5)
where \( \tilde{f} \) denotes the complex conjugate, \( \varepsilon = \pm 1 \) is the biconstant, and

\[
 f'(z) := \frac{4}{\delta^2} \left( \int_{O}^{z} f^\dagger dZ \right)^\dagger + \lambda \varepsilon, \tag{3.6}
\]

with \( \lambda \) determined by

\[
 \lambda = \nu \cdot f'. \tag{3.7}
\]

If, in the context of sequences, the function \( f \) can be Taylor expanded around the origin, \( f(x) = f(O) + \mu x + O(x^2) \), then \( \lambda = \mu \) and the identity (3.7) is equivalent in the \( O(x^2) \) sense to

\[
 \sum_k \rho(O, x_k) \left( \frac{f(y_{k+1}) - f(y_k)}{Z(y_{k+1}) - Z(y_k)} - \lambda \right) = 0, \tag{3.8}
\]

where \((O, y_k, x_k, y_{k+1}) \in \diamondsuit_2\) are the quadrilaterals adjacent to the origin. Therefore this condition can be used for numerical purposes in replacement of the exact derivation formula (3.7). It states that the derivative at the origin is the mean value of the nearby face derivatives (see Sec. A). For example, in the rectangular lattice \( \diamondsuit = \delta (Ze^{i\theta} + Ze^{-i\theta}) \), with horizontal parameter \( \rho = \tan \theta \) and vertical parameter its inverse, it provides the good choice for the first three and all the even degrees, \( \lambda_0 = 0, \lambda_1 = 1 \) and \( \lambda_{2k} = 0 \), but yields \( \lambda_k = k! \left( \frac{\delta}{2} \right)^{k-1} \cos(k - 1)\theta = O(\delta^{k-1}) \) for \( k \) odd and fixed, instead of 0 for \( k > 2 \).

**Proposition 3.2** The derivative \( f' \) fulfills

\[
 d_{\diamondsuit} f = f'dZ. \tag{3.9}
\]

The discrete monomials verify

\[
 \forall k < n - 1, \quad (Z^{k:})' = k Z^{k-1:]. \tag{3.10}
\]

Eq.(3.9) was proved in [1] and Eq.(3.10) follows from

\[
 f dZ \equiv 0 \iff f = \lambda \varepsilon. \quad \diamondsuit \tag{3.11}
\]
A Face derivation

Another derivation operator can be defined. By definition of holomorphy, for each face of the graph $\Diamond$, the 1-form $d_\Lambda f$ is proportional to $dZ$ along its pair of dual diagonals. This defines a linear operator

$$\frac{d}{dZ} : \Omega(\Lambda \cap U) \to C^2(\Diamond \cap U). \quad (A.1)$$

Given the local flat map $Z$, the space $C^2(\Diamond \cap U)$ can be seen as the space $C^{(1,0)}(\Lambda \cap U)$ of type $(1,0)$ forms on $\Lambda$, the $+i$-eigenvectors of the Hodge star $\ast$, that is to say the forms $\alpha$ such that, for dual edges $(y, y') = (x, x')^*$,

$$\int_{(y,y')} \alpha = i\rho(x, x') \int_{(x,x')} \alpha. \quad (A.2)$$

It defines

$$a_Z \in C^2(\Diamond \cap U) \quad \text{for} \quad \alpha = a_Z dZ \quad (A.3)$$

We can say that the face function $a_Z$ is holomorphic whenever the corresponding form $\alpha$ is closed, that is to say by imposing the Morera theorem. This idea was developed by Kenyon [6]. Notice that the 1-form $\alpha$ is then co-closed as well. If the quadrilateral graph $\Diamond$, along with its discrete conformal structure was isomorphic to its dual graph $\Diamond^*$, a choice of isomorphy could define a derivation endomorphism $\frac{d}{dZ} : \Omega(\Lambda \cap U) \to \Omega(\Lambda \cap U)$. But for this to happen, the vertices have to have degree four, so although important, the only case is $\mathbb{Z}^2$ with the choice of a translation by $(n + \frac{1}{2}, m + \frac{1}{2})$, $n, m \in \mathbb{Z}$.

It is immediate that given a $O(\delta^2)$-converging sequence of discrete holomorphic functions, this face derivation yields a sequence of functions, extended on each face as piece-wise constant functions, which converges to the continuous derivative.

B Change of basis

We are now going to consider the change of coordinate. Let $Z$ a critical map. If $\zeta$ is another critical map with $\zeta = a (Z - b)$ on their common definition set, the change of map for $Z^{;k}$ is not as simple as the Leibnitz rule $(a(z - b))^k = a^k \sum_{j=0}^{k} \binom{k}{j} (z - b)^{k-j}$ but is a deformation of it. The problem
is that pointwise product is not respected, $Z^{k+\ell}(z) \neq Z^k(z) \times Z^\ell(z)$. In particular the first is holomorphic and not the second. For each partition of $m = k + \ell$ into a sum of integers there is a corresponding monomial of degree $m$. An inference shows that the result is still a polynomial in $Z$:

**Proposition B.1** The powers $\zeta^k$ of the translated critical map $\zeta = a(Z-b)$ are given by

$$
\zeta^k = a^k \sum_{j=0}^{k} \binom{k}{j} (-1)^j Z^{k-j} B^j(b) 
$$

(B.1)

where $B^j(b)$ corresponding to $b^j$ is a sum over all the degree $j$ monomials in $b$, defined recursively by $B^0 = 1$ and

$$
B^k(b) := \sum_{j=0}^{k-1} \binom{k}{j} (-1)^{k+j+1} Z^{k-j} B^j(b) 
$$

(B.2)

**Proof** The calculation is easier to read using Young diagrams. Denote the point-wise product of monomials

$$
\prod_{i=1}^{n} \left( Z^{k_i}(z) \right)^{\ell_i} 
$$

with $k_1 > k_2 > \ldots > k_n$ as a Young diagram $Y$, coding columnwise the partition of the integer given by the total degree $k = \sum_{j=1}^{n} k_j \times \ell_j$ into the sum of $\ell = \sum_{j=1}^{n} \ell_j$ integers. For example the following monomial of degree $15 = 3 \times 2 + 2 \times 4 + 1$ is noted

$$
\left( Z^{3}(z) \right)^{2} \left( Z^{2}(z) \right)^{4} Z^{1}(z) 
$$

(B.3)

Then, $B^j(b) = \sum_{Y} c(Y) Y(b)$ where the sum is over all Young diagrams of total degree $j$, $c(Y)$ is an integer coefficient that we are going to define and $Y(b)$ is the pointwise product of the monomials coded by $Y$ at $b$. The coefficient of the Young diagram $Y$ above is given by the multinomials

$$
c(Y) = (-1)^{k+\ell} \frac{k!}{(k_1)!^{\ell_1} (k_2)!^{\ell_2} \cdots (k_n)!^{\ell_n}} \frac{\ell!}{\ell_1! \ell_2! \cdots \ell_n!}. 
$$

(B.5)
\begin{align*}
B^0 &= + 1 \\
B^1 &= + 1 \\
B^2 &= - 2 + 2 \\
B^3 &= + 6 - 6 \\
B^4 &= - 8 + 6 - 36 + 24 \\
B^5 &= + 10 - 20 + 60 + 90 - 240 + 120 \\
B^6 &= - 12 + 30 - 90 + 20 - 360 + 480 \\
&\quad - 90 + 1080 - 1800 + 720
\end{align*}

Table 1: The first analogs of $z^k$ needed in a change of basis.

For example, the first Young diagrams have the coefficients $c(\young(\cdot,\cdot)) = \frac{n!}{n! \cdot n!} = n!$, $c(\young(\cdot,\cdot,\cdot)) = (-1)^{n+1}$, $c(\young(\cdot,\cdot,\cdot,\cdot)) = (-1)^{n+1} \frac{(n+1)!}{n! \cdot n!} \cdot \frac{2!}{2!} = (-1)^{n+1} 2(n+1)$, and $c(\young(\cdot,\cdot,\cdot,\cdot,\cdot)) = -\frac{(n+1)!}{2!} \frac{n!}{n! \cdot (n-1)!} = -\frac{(n+1)! \cdot n}{2}$. The first few terms are listed explicitly in Table 1.

It is to be noted that the formula doesn’t involve the shape of the graph, the integer coefficients for each partition are universal constants and add up to 1 in each degree. As a consequence, since in the context of a refining sequence of critical maps,

\begin{equation}
Z^{k:z}(z) \mathbf{Z}^{\ell\mathbf{z}}(z) = Z^{k+\ell}(z) + O(\delta^2) \tag{B.6}
\end{equation}

with $k, \ell, z$ fixed, the usual Leibnitz rule is recovered in $O(\delta^2)$. Let’s stress again that these functions $B^k$ are discrete functions on the graph $\Lambda$ which are not discrete holomorphic.

C  Minimal polynomial

We give a motivation to Duffin’s derivation formula Eq. (B.6) by studying the minimal polynomial, its behavior regarding duality $\dagger$ and its derivatives. It shows that derivation in $\mathbf{Z}$ acts as integration in $\mathbf{Z}^\dagger$. 

11
Consider the minimal polynomial
\[ P_Z = \sum_{k=1}^{n} a_k Z^k : \equiv 0. \]  
(C.1)

with \(a_1 = 1\), or equivalently \(\sum_{k=1}^{n} k a_k Z^{k-1}: = \varepsilon = 1\). This implies that

\[ \sum_{k=2}^{n} k(k-1) a_k Z^{k-2} = \frac{4}{\delta^2} Z^\dagger + a_2 2 \varepsilon, \]  
(C.2)

\[ \sum_{k=3}^{n} \frac{k!}{(k-3)!} a_k Z^{k-3} = \left( \frac{4}{\delta^2} \right)^2 \frac{Z^{2\dagger}}{2} + a_2 \frac{4}{\delta^2} 2 Z^\dagger + a_3 3! 1^\dagger, \]

\[ \sum_{k=\ell}^{n} \frac{k!}{(k-\ell)!} a_k Z^{k-\ell} = \sum_{k=0}^{\ell-1} a_{\ell-k} \left( \frac{4}{\delta^2} \right)^k \frac{(\ell-k)!}{k!} Z^{k\dagger}, \]

\[ 0 = \sum_{k=0}^{n} a_{n+1-k} \left( \frac{4}{\delta^2} \right)^k \frac{(n+1-k)!}{k!} Z^{k\dagger}. \]

Since \(P_Z(Z)^\dagger = \tilde{P}_Z(Z^\dagger)\), a symmetry among the coefficients follows:

\[ a_k = \frac{(n+1-k)!}{n! k!} \tilde{a}_{n+1-k} \left( \frac{4}{\delta^2} \right)^{k-1} \]  
(C.3)

leading to

\[ k! |a_k| \left( \frac{4}{\delta^2} \right)^{n-2k+1} = (n+1-k)! |a_{n+1-k}| \left( \frac{4}{\delta^2} \right)^{n-2(n+1-k)+1}, \]  
(C.4)

in particular \(|a_n| = \frac{1}{(n!) \left( \frac{4}{\delta^2} \right)^{n-1}}\) and when \(n\) is odd, \(\arg (a_{n+1}) = \frac{1}{2} \arg (a_n)\).

\[ \text{D Series} \]

We present here some problems linked to discrete series. They can converge only when their coefficients decrease at least exponentially fast. And exponentially fast is enough since one can define a discrete exponential.

The \(O(\delta^2)\) convergence of discrete polynomials to continuous ones is in fact slower and slower as the degree of the polynomial grows. Consider for
\[

table
\]

| k  | 3          | 4          | 5          | 6          | 7          |
|----|------------|------------|------------|------------|------------|
| \(Z^k(x)\) | \(x^3 + \frac{x}{2n^2}\) | \(x^4 + \frac{2x^2}{n^2}\) | \(x^5 + \frac{5x^3}{n^2} + \frac{3x}{2n^2}\) | \(x^6 + \frac{10x^4}{n^2} + \frac{23x^2}{n^2}\) | \(x^7 + \frac{35x^5}{n^2} + \frac{49x^3}{n^2} + \frac{45x}{4n^2}\) |

Table 2: The first powers on the interval \([0, 1]/n\) for \(x = \ell/n\).

example ♦ containing the chain \(\{0, \frac{1}{n}, \frac{2}{n}, \ldots, 1\}\), the first \(Z^{k_i}(x)\) with \(x = \ell/n\) are listed in Table 2. And in general, a scaling argument shows that

\[
|Z^{k_i}(x) - x^k| \leq \lambda_k |x|^{k-2}\delta^2.
\] (D.1)

But the constant \(\lambda_k\) is growing exponentially fast with \(k\). It is for example not true that, at a point close enough to the origin, \(Z^{k_i}\) will tend to zero with increasing \(k\). On the contrary, if \(x = \delta e^{i\theta}\) is a neighbor of the origin with \((O, x) \in \Diamond_1\) and \(k \geq 1\), then

\[
Z^{k_i}(x) = \frac{k!}{2^{k-1}} x^k
\] (D.2)

in fact diverges with \(k\). If \(y\) is a next neighbor of the origin, with the rhombi \((O, x, y, x') \in \Diamond_2\) having a half angle \(\theta\) at the origin,

\[
Z^{k_i}(y) = \frac{k!}{2^{2k-2}} \frac{\sin k\theta}{\sin \theta \cos^{k-1}\theta} y^k
\] (D.3)

has the same diverging behavior, and so has every point at a given finite distance of the origin. It is only in the scaling limit with the proper balance given by criticality that one recovers the usual behavior \(|x| < 1 \implies |x^k| \underset{k \to \infty}{\longrightarrow} 0\).

Therefore, for its general term to converge to zero, the coefficients of a discrete series must decrease at least exponentially fast. The discrete exponential \(\Exp(\lambda; Z)\), for \(|\lambda| \neq 2/\delta\), is defined by

\[
\Exp(\lambda; O) = 1
\] (D.4)

\[
d\Exp(\lambda; Z) = \lambda \Exp(\lambda; Z) dZ
\] (D.5)

This discrete holomorphic function was first defined in [3] and put to a very interesting use in [4]. The discrete exponential is a rational fraction in \(\lambda\) at
every point,

\[
\text{Exp} (\lambda; x) = \prod_k \frac{1 + \frac{\delta}{2} e^{i \theta_k}}{1 - \frac{\delta}{2} e^{i \theta_k}}
\]  \hspace{1cm} (D.6)

where \((\theta_k)\) are the angles defining \((\delta e^{i \theta_k})\), the set of \((Z\text{-images of}) \Diamond\text{-edges between} \ x = \sum \delta e^{i \theta_k} \text{ and the origin.}\) Expanding \(\exp(\log(\text{Exp} (\lambda; x)))\) in \(O(\delta^2)\), for \(|\lambda| < 2/\delta\) and \(x\) fixed in a refining sequence of critical maps, we get that

\[
\text{Exp} (\lambda; x) = \exp(\lambda x) + O(\delta^2). \hspace{1cm} (D.7)
\]

For \(|\lambda| < 2/\delta\), the series \(\sum_{k=0}^{\infty} \frac{\lambda^k Z^{k:}}{k!} \) is equal to \((D.6)\) whenever the former is defined, as its exterior derivative fulfills the right equation \((D.5)\) and its value at the origin is 1. The great difference with the continuous case is that the series is absolutely convergent only for bounded parameters, \(|\lambda| < \frac{2}{\delta}\). This suggests that asking for a product such that

\[
\text{Exp} (\lambda; Z) \cdot \text{Exp} (\mu; Z) = \text{Exp} (\lambda + \mu; Z), \hspace{1cm} (D.8)
\]

or equivalently \(Z^{k:} \cdot Z^{\ell:} = Z^{k+\ell:}\) may not be the right choice. The symmetry \(\dagger\) is interesting as well,

\[
\text{Exp} (\lambda; Z) \dagger = \text{Exp} (\frac{1}{\lambda}; Z), \hspace{1cm} (D.9)
\]

in particular, \(\text{Exp} (\infty; Z) = \varepsilon\).

The general change of basis of a given series however possible in theory is nevertheless complicated and the information on the convergence of the new series is difficult to obtain. The exponential remains a particularly simple case, if \(\zeta = a(Z - b)\):

\[
\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \zeta^{k:} \propto \sum_{k=0}^{\infty} \frac{(a \lambda)^k}{k!} Z^{k:}. \hspace{1cm} (D.10)
\]

**Acknowledgements**

I thank Trevor Welsh for simplifying Eq. \((B.5)\). This research is supported by the Sonderforschungsbereich 288.
References

[1] Christian Mercat. Discrete Riemann surfaces and the Ising model. *Comm. Math. Phys.*, 218(1):177–216, 2001.

[2] Marc Troyanov. Les surfaces euclidiennes à singularités coniques. *Enseign. Math. (2)*, 32(1-2):79–94, 1986.

[3] R. J. Duffin. Potential theory on a rhombic lattice. *J. Combinatorial Theory*, 5:258–272, 1968.

[4] R. J. Duffin. Basic properties of discrete analytic functions. *Duke Math. J.*, 23:335–363, 1956.

[5] Christian Mercat. Discrete Period Matrices and Related Topics. *math-ph/0111043*.

[6] Richard Kenyon. The Laplacian and ∂ operators on critical planar graphs. *math-ph/0202018*. 