Englert-type solutions of $d = 11$ supergravity

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Abstract

A family of geometries on $S^7$ arise as solutions of the classical equations of motion in 11 dimensions. In addition to the conventional riemannian geometry and the two exceptional Cartan-Schouten compact flat geometries with torsion, one can also obtain non-flat geometries with torsion. This torsion is given locally by the structure constants of a nonassociative geodesic loop in the affinely connected space.

1 Introduction

It is well known [1] that four-dimensional gravity, Yang-Mills interactions and matter fields may originate from a higher-dimensional theory of pure gravity. This prospect for unifying matter and the fundamental interactions including gravity is also offered by supergravity. This is even more so because supersymmetry puts restrictions on the number of possible space-time dimensions. The maximal dimension for which one can balance bosonic and fermionic degrees of freedom with highest spin two is eleven. Supergravity in 11 dimensions [2] is thus possible to study spontaneous compactifications of this theory, i.e. solutions of the 11-dimensional equations of motion for which the ground state corresponds to a product space of a 4-dimensional space-time and a compact 7-dimensional space.

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In the Bose sector of this theory the equations of motion (Einstein equations and equations for the antisymmetric gauge field strength) have the form

\[ R_{MN} - \frac{1}{2} g_{MN} R = 12 \left( 8 F_{MPQR} F_{N}^{PQR} - g_{MN} F_{SPQR} F^{SPQR} \right), \quad (1) \]

\[ F^{MNPQ}_{;M} = -\frac{\sqrt{2}}{24} \varepsilon^{NPQ M_1 \ldots M_8} F_{M_1 M_2 M_3 M_4} F_{M_5 M_6 M_7 M_8}, \quad (2) \]

where \( \varepsilon^{M_1 \ldots M_r} \) is a fully antisymmetric covariant constant tensor such that \( \varepsilon_{1 \ldots r} = \left| g \right|^{1/2} \). The Englert solution \([3]\) compactifies \( d = 11 \) spacetime into a Riemann product \( M = AdS_4 \times S^7 \) of two Einstein spaces, a 4-dimensional anti-de Sitter universe and the 7-sphere. The ansatz of Englert is to set

\[ F_{\mu\nu\sigma\lambda} = \rho \varepsilon_{\mu\nu\sigma\lambda}, \quad (3) \]

\[ F_{mnpq} = \lambda \partial[q S_{mnp}], \quad (4) \]

where \( \rho \) and \( \lambda \) are real constants and \( S_{mnp} = S_{[mnp]} \) is a suitable totally antisymmetric torsion tensor. Note that the connection between an antisymmetric gauge field strength and a torsion defined by \([1]\) has an universal character in the 11-dimensional supergravity. Bars and McDowell \([4]\) have shown that the \( g_{MN}/A_{MNP} \) gravity-matter system may be reinterpreted, in first-order formalism, as a pure gravity theory with torsion \( S_{MNP} \) such that

\[ A_{MNP} = \lambda S_{[MNP]}, \quad (5) \]

\[ F_{MNP S} = \partial[S A_{MNP}], \quad (6) \]

In addition, in Ref. \([5]\) was pointed out that the torsion tensor is fully antisymmetric, i.e. \( S_{MNP} = S_{[MNP]} \). In this sense the deformation \( \Gamma_{MNP} \rightarrow \Gamma_{MNP} + S_{MNP} \) of the Riemann connection converts \( M \) into an affinely connected space with torsion.

In this paper we shall derive the surprising result that there exists not only three Cartan-Schouten affine metric geometries on \( S^7 \), but a family of non-flat geometries on \( S^7 \) which emerge as solutions of 11-dimensional supergravity. In Sec. 2, we construct an one-parameter family of affine connections in the space of unit octonions. Then we find the torsion and curvature tensors of the connections. In Sec. 3, we show that each torsion satisfies the equations of motion of 11-dimensional supergravity.
2 Extended Cartan-Schouten construction

Although the notion of a space with an affine connection were initially created by Cartan and Schouten independently, in 1926 the joint papers [6] and [7] of both geometers were published. Both papers were concerned with Riemannian geometry, but in both cases, one way or another, the geometry of a space with an affine connection was discussed. In the first of these papers, the authors considered three affine connections associated with any Lie group. In the second paper, they considered the absolute parallelisms of the first and the second type in the 7-sphere of the unite octonions. Later, the theory of affine connections associated with any Lie group was developed by Akivis in his paper [8]. In this paper Akivis constructed, on the variety of any Lie group, an one-parameter affinely connected family, containing the three Cartan-Schouten connections. In the section we apply the Akivis method to construction of an one-parameter family of affine connections in $S^7$.

Let $a$ be an arbitrary point of $S^7$. In a neighborhood of this point, we introduce a binary operation as follows. Let $u$ and $v$ be two points belonging to this neighborhood. The parallel displacement of vectors of the first type on $S^7$ may be determined by the right translations $zR_x = zx$ in the 7-sphere of the unite octonions. Then the geodesic $\tilde{au}$ is translated into the geodesic $\tilde{vw}$, where $v = aR_x$ and $w = uR_x$. Therefore

\[ w = u(a^{-1}v) \equiv u \circ v. \]  

(7)

This equation defines a geodesic loop $G_a$ with the unity $a$. Obviously, the loop $G_a$ is nonassociative and it is locally isotopic to the Moufang loop $S^7$. Recall that the set of the unite octonions is closed relative to the multiplication in the algebra of octonions, and therefore it is an analytic Moufang loop. On the other hand [9, 10], every loop isotopic to a Moufang loop is isomorphic to its principal isotope and every principal isotope of $S^7$ is isomorphic to $S^7$. Therefore the loops $G_a$ and $S^7$ are locally isomorphic.

Now we define an one-parameter family of loops $G_a^\alpha$ with the multiplication

\[ w = v^\alpha \circ u \circ v^{1-\alpha} \equiv u * v, \]  

(8)

where $\alpha$ is a real constant. It follows from the isomorphism of the loops $G_a$ that the loops $G_a^\alpha$ are isomorphic to the loop $G^\alpha$ with the multiplication law

\[ w = v^\alpha uv^{1-\alpha} = (v^\alpha uv^{-\alpha})R_v. \]  

(9)
Let \( v \) be a fixed point of \( S^7 \) and let the point \( u \) runs its one-parameter subgroup \( g \). Then the point \( u' = v^\alpha uv^{-\alpha} \) runs an one-parameter subgroup \( g' \) and the point \( w \) runs a line obtained from \( g' \) by the translation \( R_v \). Hence if \( a \) and \( v \) are fixed points of \( S^7 \) and the point \( u \) describes the geodesic line \( \tilde{a}u \) generated by a family of one-parameter subgroups, then the point \( w \) describes the geodesic line \( \tilde{v}w \). Thus, all loops \( G^\alpha_a \) are geodesic loops of affine connections on \( S^7 \) which have the same system of geodesic lines generated by one-parameter subgroups of the loop of the unite octonions.

Now we shall find the torsion and curvature tensors of affine connections generated on \( S^7 \) by the geodesic loop \( G^\alpha_a \). To this end we represent the multiplication (9) in the coordinate form. Since the loop \( S^7 \) is di-associative (i.e. any two elements of \( S^7 \) generate an subgroup), it follows that in a neighborhood of the unity the multiplication operation in \( S^7 \) is expressed through the addition and multiplication operations in the tangent algebra by the usual Campbell-Hausdorff series

\[
(xy)^i = x^i + y^i + \frac{1}{2} \tilde{c}^i_{jk} x^j x^k + \frac{1}{12} \tilde{\tilde{c}}^i_{jk} m_{kl} (x^j x^k y^l + y^j y^k x^l) + \ldots,
\]

where \( \tilde{c}^i_{jk} \) are structure constants of the algebra of octonions. Twice using this formula, we find coordinates of the point \( w \) that defined by Eq. (9):

\[
w^i = u^i + v^i + \frac{1}{2} (1 - 2\alpha) \tilde{c}^i_{jk} u^j v^k \\
+ \frac{1}{12} \tilde{\tilde{c}}^i_{jk} m_{kl} \left[ u^j u^k v^l + (1 - 6\alpha + 6\alpha^2) v^j v^k u^l \right] + \ldots.
\]

On the other hand, if we introduce local coordinates in a neighborhood of unity of \( G^\alpha \), then the multiplication in \( S^7 \) is expressed by

\[
w^i = u^i + v^i + \lambda^i_{jk} u^j v^k + \frac{1}{2} \left( \mu^i_{jkl} u^j u^k v^l + \nu^i_{jkl} u^j v^k v^l \right) + \ldots.
\]

We define the tensors

\[
\alpha^i_{jk} = \lambda^i_{[jk]},
\]

\[
-2\beta^i_{jkl} = \mu^i_{j[k]} - \nu^i_{j[k]} + \lambda_j^m \lambda_k^m - \lambda_j^m \lambda_k^m.
\]

These tensors are called the fundamental tensors of the geodesic loop \( G^\alpha \). Note that these tensors are structure constants of the binary-ternary tangent
algebra of the geodesic loop [11] (see also [12]). Comparing (11) and (12), we get

\[ 2\alpha^i_{jk} = (1 - 2\alpha)\hat{c}^i_{jk}, \quad (15) \]
\[ -4\beta^i_{jkl} = \alpha(1 - \alpha)\hat{c}^i_{jm}\hat{c}^m_{kl} + (1 - 3\alpha + 3\alpha^2)\hat{c}^i_{m[j}\hat{c}^{m]}_{kl}. \quad (16) \]

It follows from (15) that the fundamental tensor \( \alpha^i_{jk} \) is defined by

\[ \alpha^i_{jk} = kc^i_{jk}, \quad (17) \]

where \( c^i_{jk} \) are standard structure constants of the algebra of octonions and \( k \) is a real number.

It is known [8] that for any geodesic loop constructed in a neighborhood of a point \( a \) of an affinely connected space, the fundamental tensors can be expressed using values of the torsion and curvature tensors in \( a \) by the formulas

\[ \alpha^i_{jk} = -S^i_{jk}, \quad (18) \]
\[ 4\beta^i_{jkl} = -2\nabla^l S^i_{jk} - R^i_{jkl}. \quad (19) \]

Note that we define all these tensors, as is done in ref. [5]. Since the tensor \( c_{ijk} \) is fully antisymmetric, it follows from (17) and (18) that the geodesic loops \( G^\alpha_a \) (the index \( \alpha \) is fixed) generate the metric-compatible affine connection on \( S^7 \):

\[ \Gamma_{ijk} = \hat{\Gamma}_{ijk} + S_{ijk}, \quad (20) \]

where \( \hat{\Gamma}_{ijk} \) is the riemannian symmetric connection and \( S_{ijk} \) is a fully antisymmetric torsion. Using the full skew-symmetry of \( S_{ijk} \), we can rewrite the tensor (19) in the form

\[ 4\beta_{ijkl} = -2S_{ijkl} + 6S^m_{[ij}S_{kl]m} - R_{ijkl}. \quad (21) \]

Let \( \alpha = 0 \). Then we have the remaining curvature-less Cartan-Schouten geometry of absolute parallelism on \( S^7 \). Using the cyclic identities for \( R_{ijkl} \) one easily obtains the conditions for such torsion, and hence for the parallelism:

\[ S_{ijkl} = S^m_{[ij}S_{kl]m}. \quad (22) \]
Now we consider the case of arbitrary $\alpha \neq \frac{1}{2}$. The case $\alpha = \frac{1}{2}$ corresponds to the torsionless riemannian geometry, and therefore it is not interesting. Using Eqs. (15) and (18), we get

$$S_{ijkl} = hS_{[ij}^m S_{kl]m}, \quad h = \frac{1}{1 - 2\alpha} \quad (23)$$

instead of (22). Using Eqs. (21) and (23), we obtain the curvature tensor of affine connection on $S^7$:

$$R_{ijkl} = 4\alpha(1 - \alpha)S_{ij}^m S_{klm} - 4\alpha(2 - 3\alpha)S_{[ij}^m S_{kl]m}. \quad (24)$$

Obviously, the curvature tensor is equal to zero if $\alpha = 0$ and it is fully antisymmetric if $\alpha = 1$. The corresponding geodesic loops $G_a^0$ and $G_a^1$ are locally isomorphic to the Moufang loop $S^7$. Solutions of the classical equations of motion of 11-dimensional supergravity connected with these loops was found in Refs. [3, 5]. In the next section we shall construct solutions of 11-dimensional supergravity connected with nonmoufang geodesic loops.

### 3 The Englert-type solutions

We consider the Bose sector of $d = 11$ supergravity as a pure gravity theory with torsion and suppose that the matter fields have non-vanishing components in the internal space $S^7$. Substituting the Freund-Rubin ansatz (3) in Eq. (2), we obtain

$$F_{mnpq,lm} = \sqrt{2} \rho \varepsilon^{n pqijkl} F_{ijkl}, \quad (25)$$

where $\varepsilon^{n pqijkl}$ is the fully antisymmetric covariant constant 7-tensor. It follows from (5) and (6) that solutions of Eq. (25) are related to torsion tensors by the relation (1). We shall therefore first derive analogs of the Cartan-Schouten-Englert equations [3, 7] which these tension tensors must satisfy. We shall need to the following seven-dimensional algebraic identities [5, 13]:

$$\alpha_{ijm}\alpha_{ijn} = 6k^2 \delta_m^n, \quad (26)$$

$$\alpha_{imj}^i \alpha_{ijn}^k \alpha_{kp}^i = 3k^2 \alpha_{mnp}, \quad (27)$$

$$\beta_{mijk}^i \beta_{nijkl} = 24k^4 \delta_m^n, \quad (28)$$

$$\varepsilon^{n pqijkl} \beta_{ijkl} = 24k \alpha^{npq}. \quad (29)$$
Using the identities (23), (26), and (27), we get
\[ S_{npqm} \equiv (2hk)^2 S_{npq} = 0. \] 
By taking into account the obtained identities, we rewrite the equations (25) as
\[ (2hk)^2 S_{npq} + \sqrt{2} \rho e^{npqijkl} S_{ijkl} = 0. \]
Substituting the ansatz (23) in (31) and using the relations of self-duality (29), we find the condition
\[ h = 6\sqrt{2} \rho k^{-1}, \]
in which Eq. (31) turn into an identity. Thus, if we choose the gauge field strength \( F_{mnpq} \) in the form (4), then we get solutions of the equations (25).

Now we consider the Einstein equations (1). It is obvious that these equations can be satisfied if \( F_{mnpq} F_{rnpq} \) is proportional to \( g_{mn} \). This is indeed the case as follows from Eqs. (23) and (28):
\[ F_{\mu\sigma\rho\lambda} F_{\nu}^{\sigma\rho\lambda} = -6 \rho^2 g_{\mu\nu}, \]
\[ F_{mnpq} F_{rnpq} = 24k^4 h^2 \lambda^2 g_{mn}. \]
Substituting these relations in the Einstein equations (1), we get
\[ \hat{R}_{mn} = 6(hk)^2 g_{mn}, \]
\[ \hat{R}_{\mu\nu} = -10(hk)^2 g_{\mu\nu}, \]
where the constant
\[ 2\lambda^2 = (12k)^{-2}. \]
It follows from Eqs. (32) and (37) that the constants \( k \) and \( h \) are determined by \( \rho \) and \( \lambda \) which still arbitrary. Thus we have obtained, in addition to the solutions with \( h = \pm 1 \) that was found in Refs. [3, 5], new solutions where the four-scalars take the values (3) and (4) with arbitrary \( \rho \) and \( \lambda \).

Note that in the paper [1] Englert constructed two independent solutions \( S_{mnp}^+ \) and \( S_{mnp}^- \) of the equations of motion. These solutions are connected with two types of parallelisms on \( S^7 \). The first of them coincides precisely with torsion generated on \( S^7 \) by the geodesics loops \( G_{a}^{0} \). In order that to obtain the second, we suppose \( h = (2\alpha - 1)^{-1} \) in Eq. (23). Then \( S_{mnp}^- \) will be coincided with torsion generated on \( S^7 \) by the loops \( G_{a}^{1} \). By repeating the previous reasoning, we get a new series of solutions of 11-dimensional supergravity.
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