Point counting on curves using a gonality preserving lift

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Abstract

We study the problem of lifting curves from finite fields to number fields in a genus and gonality preserving way. More precisely, we sketch how this can be done efficiently for curves of gonality at most four, with an in-depth treatment of curves of genus at most five over finite fields of odd characteristic, including an implementation in Magma. We then use such a lift as input to an algorithm due to the second author for computing zeta functions of curves over finite fields using $p$-adic cohomology.

1 Introduction

This article is about efficiently lifting algebraic curves over finite fields to characteristic zero, in a genus and gonality preserving way, with an application to $p$-adic point counting. Throughout, our curves are always understood to be geometrically irreducible, but not necessarily non-singular and/or complete. By the genus of a curve we mean its geometric genus, unless otherwise stated. As for the gonality of a curve over a field $k$, we make a distinction between two notions: by its $k$-gonality we mean the minimal degree of a non-constant $k$-rational map to the projective line, while by its geometric gonality we mean the $\bar{k}$-gonality, where $\bar{k}$ denotes an algebraic closure of $k$. We also make a notational distinction between projective, affine or toric (= affine minus coordinate hyperplanes) $n$-space in characteristic zero, in which case we write $\mathbb{P}^n, \mathbb{A}^n, \mathbb{T}^n$, and their finite characteristic counterparts, where we opt for $\mathbb{F}^n, \mathbb{A}^n, \mathbb{T}^n$. Apart from that we avoid reference to the base field, which should always be clear from the context. Similarly we write $\mathbb{Q}$ for the field of rational numbers and $\mathbb{F}_q$ for the finite field with $q$ elements, where $q$ is a power of a prime number $p$. For each such $q$ we fix a degree $\log_p q$ extension $K \supset \mathbb{Q}$ in which $p$ is inert, and let $\mathcal{O}_K$ denote its ring of integers. We then identify $\mathbb{F}_q$ with the residue field $\mathcal{O}_K/(p)$. Our lifting problem is as follows:

**Problem 1.** Given a curve $\mathcal{C}$ over $\mathbb{F}_q$, find an efficient algorithmic way of producing a polynomial $f \in \mathcal{O}_K[x,y]$ such that

(i) its reduction mod $p$ defines a curve that is birationally equivalent to $\mathcal{C}$,

(ii) the curve $C \subset \mathbb{A}^2$ it defines has the same genus as $\mathcal{C}$,

(iii) its degree in $y$ equals the $\mathbb{F}_q$-gonality of $\mathcal{C}$.

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Note that these conditions imply that the $K$-gonality of $C$ equals the $\mathbb{F}_q$-gonality of $\overline{C}$, because the gonality cannot increase under reduction mod $p$; see e.g. [21, Thm. 2.5]. We are unaware of whether an $f$ satisfying (i-iii) exists in general. Grothendieck’s existence theorem [36] implies that in theory one can achieve (i) and (ii) over the ring of integers $\mathbb{Z}_q$ of the $p$-adic completion $\mathbb{Q}_q$ of $K$, but, firstly, it is not clear that we can always take $f$ to be defined over $\mathcal{O}_K$ and, secondly, we do not know whether it is always possible to incorporate (iii), let alone in an effective way. To give a concrete open case, we did not succeed in dealing with Problem 1 for curves of genus four having $\mathbb{F}_q$-gonality five, which can only exist if $q \leq 7$. (However, as we will see, among all curves of genus at most five, the only cases that we cannot handle are pathological examples of the foregoing kind.)

We are intentionally vague about what it means to be given a curve $C$ over $\mathbb{F}_q$. It could mean that we are considering the affine plane curve defined by a given absolutely irreducible polynomial $f \in \mathbb{F}_q[x, y]$. Or it could mean that we are considering the affine/projective curve defined by a given more general system of equations over $\mathbb{F}_q$. In all cases we will ignore the cost of computing the genus $g$ of $C$. Moreover, in case $g = 0$ we assume that it is easy to realize $C$ as a plane conic (using the anticanonical embedding) and if $g = 1$ we ignore the cost of finding a plane Weierstrass model. By the Hasse-Weil bound every genus one curve over $\mathbb{F}_q$ is elliptic, so this is indeed possible. If $g \geq 2$ then we assume that one can easily decide whether $C$ is hyperelliptic or not (note that over finite fields, curves are hyperelliptic iff they are geometrically hyperelliptic, so there is no ambiguity here). If it is then we suppose that it is easy to find a generalized Weierstrass model. If not then it is assumed that one can effectively compute a canonical embedding

$$\kappa : C \hookrightarrow \mathbb{P}^{g-1}$$

along with a minimal set of generators for the ideal of its image. The latter will usually be our starting point. Most of the foregoing tasks are tantamount to computing certain Riemann-Roch spaces. There is extensive literature on this functionality, which has been implemented in several computer algebra packages, such as Magma [7] and Macaulay2 [28].

The idea is then to use the output polynomial $f$ as input to a recent algorithm due to the second author [53, 54] for computing the Hasse-Weil zeta function of $\overline{C}$. This algorithm uses $p$-adic cohomology, which it represents through the map $\pi : C \to \mathbb{P}^1 : (x, y) \mapsto x$. The algorithm only works if $C$ and $\pi$ have appropriate reduction modulo $p$, in a rather subtle sense for the precise description of which we refer to [54, Ass. 1]. This condition is needed to be able to apply a comparison theorem between the (relative) $p$-adic cohomology of $\overline{C}$ and the (relative) de Rham cohomology of $C \otimes \mathbb{Q}_q$, which is where the actual computations are done. For such a theorem to hold, by dimension arguments it is necessary that $C$ and $\overline{C}$ have the same genus, whence our condition (ii). This may be insufficient, in which case $f$ will be rejected, but for $p > 2$ our experiments show that this is rarely a concern as soon as $q$ is sufficiently large. Moreover, in many cases below, our construction leaves enough freedom to retry in the event of a failure.

The algorithm from [53, 54] has a running time that is sextic in $\deg \pi$, which equals the degree in $y$ of $f$, so it is important to keep this value within reason. Because the $\mathbb{F}_q$-gonality of $\overline{C}$ is an innate lower bound, it is natural to try to meet this value, whence
our condition (iii). At the benefit of other parameters affecting the complexity, one could imagine it being useful to allow input polynomials whose degree in \( y \) exceeds the \( \mathbb{F}_q \)-gonality of \( C \), but in all cases that we studied the best performance results were indeed obtained using a gonality-preserving lift. At the same time, looking for such a lift is a theoretically neat problem.

**Remark 2.** For the purpose of point counting, it is natural to wonder why we lift to \( \mathcal{O}_K \), and not to the ring \( \mathbb{Z}_q \), which is a priori easier. In fact, most computations in the algorithm from [53, 54] are carried out to some finite \( p \)-adic precision \( N \), so it would even be sufficient to lift to \( \mathcal{O}_K/(p^N) = \mathbb{Z}_q/(p^N) \). A first reason for lifting to \( \mathcal{O}_K \) is simply that this turns out to be possible in the cases that we studied, without additional difficulties. A second more practical reason is that at the start of the algorithm from [53, 54] some integral bases have to be computed in the function field of the curve. Over a number field \( K \) this is standard and implemented in Magma, but to finite \( p \)-adic precision it is not clear how to do this, and in particular no implementation is available. Therefore, the integral bases are currently computed to exact precision, and we need \( f \) to be defined over \( \mathcal{O}_K \).

**Contributions** As explained in Section 2 the cases where \( C \) is rational, elliptic or hyperelliptic are straightforward. In this article we give a recipe for tackling Problem 1 in the case of curves of \( \mathbb{F}_q \)-gonality 3 and 4. Because of their practical relevance, our focus lies on curves having genus at most five, which is large enough for the main trigonal and tetragonal phenomena to be present. The details can be found in Section 3; more precisely in Sections 3.1, 3.2 and 3.3 we attack Problem 1 for curves of genus three, four and five, respectively, where we restrict ourselves to finite fields \( \mathbb{F}_q \) having odd characteristic. Each of these sections is organized in a stand-alone way, as follows:

- In a first part we classify curves by their \( \mathbb{F}_q \)-gonality \( \gamma \) and solve Problem 1 in its basic version (except for some pathological cases such as pentagonal curves in genus four or hexagonal curves in genus five, which are irrelevant for point counting because these can only exist over extremely small fields). If the reader is interested in such a basic solution only, he/she can skip the other parts, which are more technical.

- Next, in an optimization part we take into account the fact that the actual input to the algorithm from [53, 54] must be monic when considered as a polynomial in \( y \). This is easily achieved: if we write
  \[
  f = f_0(x)y^\gamma + f_1(x)y^{\gamma-1} + \cdots + f_{\gamma-1}(x)y + f_{\gamma}(x),
  \]
  then the birational transformation \( y \leftarrow y/f_0(x) \) gives
  \[
  y^\gamma + f_1(x)y^{\gamma-1} + \cdots + f_{\gamma-1}(x)f_0(x)^{\gamma-2}y + f_{\gamma}(x)f_0(x)^{\gamma-1},
  \]
  which still satisfies (i), (ii) and (iii). But one sees that the degree in \( x \) inflates, and this affects the running time. We discuss how our basic solution to Problem 1 can be enhanced such that (1) becomes a more compact expression.
• We have implemented the algorithms from this paper in the computer algebra system Magma. The resulting package is called goodmodels and can be found at the webpage http://perswww.kuleuven.be/jan_tuitman. In a third part we report on this implementation and on how it performs in composition with the algorithm from [53, 54] for computing Hasse-Weil zeta functions. We give concrete runtimes, memory usage and failure rates, but avoid a detailed complexity analysis, because in any case the lifting step is heavily dominated by the point counting step. All computations were carried out with Magma v2.22 on a single Intel Core i7-3770 CPU running at 3.40 GHz. The code used to generate the tables with running times, memory usage and failure rates can be found in the subdirectory ./profiling of goodmodels.

As we will see, the case of trigonal curves of genus five provides a natural transition to the study of general curves of \( \mathbb{F}_q \)-gonality 3 and 4. These are discussed in Section 4, albeit in a more sketchy way.

Consequences The main consequences of our work are that

• computing Hasse-Weil zeta functions using \( p \)-adic cohomology has now become practical on virtually all curves of genus at most five over finite fields \( \mathbb{F}_q \) of (small) odd characteristic,

• the same conclusion for curves of \( \mathbb{F}_q \)-gonality at most four looms around the corner, even though some hurdles remain, as explained in Section 4,

• we have a better understanding of which \( \mathbb{F}_q \)-gonalities can occur for curves of genus at most five, see the end of Section 2.1 for a summarizing table.

We stress that the general genus five curve, let alone the general tetragonal curve of any given genus, cannot be tackled using any of the previous Kedlaya-style point counting algorithms, that were designed to deal with elliptic curves [43], hyperelliptic curves [19, 29, 31, 35, 37], superelliptic curves [26, 40], \( C_{ab} \) curves [12, 20, 55] and nondegenerate curves [11, 52], in increasing order of generality. We refer to [13] for a discussion of which classes of curves do admit a nondegenerate model.

A reference problem (†) At sporadic places in this article, we refer to a paper that develops its theory over \( \mathbb{C} \) only, while in fact we need it over other fields, such as \( \mathbb{F}_q \). This concern mainly applies to the theory of genus five curves due to Arbarello, Cornalba, Griffiths and Harris [2, VI.§4.F]. We are convinced that most of the time this is not an issue (the more because we rule out even characteristic) but we did not sift every one of these references to the bottom to double-check this: we content ourselves with the fact that things work well in practice. In our concluding Section 4 on trigonal and tetragonal curves, the field characteristic becomes a more serious issue, for instance in the Lie algebra method developed by de Graaf, Harrison, Příhodová and Schicho [18]. More comments on this will be given there. Each time we cite a \( \mathbb{C} \)-only (or characteristic zero
only) reference whose statement(s) we carry over to finite characteristic without having verified the details, we will indicate this using the dagger symbol †.

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## 2 Background

### 2.1 First facts on the gonality

Let \( k \) be a field and let \( C \) be a curve over \( k \). The geometric gonality \( \gamma_{\text{geom}} \) of \( C \) is a classical invariant. It is 1 if and only if the genus of \( C \) equals \( g = 0 \), while for curves of genus \( g \geq 1 \), by Brill-Noether theory \( \gamma_{\text{geom}} \) lies in the range

\[
2, \ldots, \left\lceil \frac{g}{2} \right\rceil + 1.
\]

For a generic curve the upper bound \( \left\lceil \frac{g}{2} \right\rceil + 1 \) is met [15], but in fact each of the foregoing values can occur: inside the moduli space of curves of genus \( g \geq 2 \) the corresponding locus has dimension \( \min\{2g - 5 + 2\gamma_{\text{geom}}, 3g - 3\} \); see [1, §8]†. From a practical point of view, determining the geometric gonality of a given curve is usually a non-trivial computational task, although in theory it can be computed using so-called scrollar syzygies [44].

In the arithmetic (= non-geometric) case the gonality has seen much less study, even for classical fields such as the reals [16]. Of course \( \gamma_{\text{geom}} \) is always less than or equal to the \( k \)-gonality \( \gamma \), but the inequality may be strict. In particular the Brill-Noether upper bound \( \left\lceil \frac{g}{2} \right\rceil + 1 \) is no longer valid. For curves of genus \( g = 1 \) over certain fields \( \gamma \) can even be arbitrarily large [14]. As for the other genera, using the canonical or anticanonical linear system one finds

- if \( g = 0 \) then \( \gamma \leq 2 \),
- if \( g \geq 2 \) then \( \gamma \leq 2g - 2 \).

These bounds can be met. We refer to [41, Prop. 1.1] and the references therein for precise statements, along with some additional first facts.

If \( k = K \) is a number field then the notion of \( K \)-gonality has enjoyed more attention, both from a computational [21, 22] and a theoretical [41] point of view, especially in the case where \( C \) is a modular curve. This is due to potential applications towards effective versions of the uniform boundedness conjecture; see [49] for an overview. In the non-modular case not much literature seems available, but our rash guess would be that
almost all (in any honest sense) curves of genus \( g \geq 2 \) over \( K \) meet the upper bound \( \gamma \leq 2g - 2 \). This is distantly supported by the Franchetta conjecture; see again [41, Prop. 1.1] and the references therein for a more extended discussion.

Over finite fields \( k = \mathbb{F}_q \) the notion has attracted the attention of coding theorists in the context of Goppa codes [51]. They proved the following result:

Lemma 3. If the \( C \) is a curve over a finite field \( \mathbb{F}_q \) then its \( \mathbb{F}_q \)-gonality is at most \( g + 1 \). Moreover, if equality holds then \( g \leq 10 \) and \( q \leq 31 \).

Proof. See [51, §4.2].

In [51, §4.2] it is stated as an open problem to find tighter bounds for the \( \mathbb{F}_q \)-gonality. In fact we expect the sharpest possible upper bound to be \( \lfloor g/2 \rfloor + 1 + \varepsilon \) for some small \( \varepsilon \); maybe \( \varepsilon \leq 1 \) is sufficient as soon as \( q \) is large enough. A byproduct of this paper is a better understanding of which \( \mathbb{F}_q \)-gonalities can occur for curves of genus at most five, in the cases where \( q \) is odd (the cases where \( q \) is even should be analyzable in a similar way). The following table summarizes this.

| \( g \) | Brill-Noether upper bound | possible \( \mathbb{F}_q \)-gonalities (union over all odd \( q \)) | possible \( \mathbb{F}_q \)-gonalities (for a given odd \( q > B \)) | \( B \) |
|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 |
| 1 | 2 | 2 | 2 | 1 |
| 2 | 2 | 2 | 1 |
| 3 | 3 | 2, 3, 4 | 2, 3 | 29 |
| 4 | 3 | 2, 3, 4, 5 | 2, 3, 4 | 7 |
| 5 | 4 | 2, 3, 4, 5, 6 | 2, 3, 4, 5 | 3 |

For background we refer to Section 2.3 (for \( g \leq 2 \)), Lemma 7 (for \( g = 3 \)), Lemma 11 (for \( g = 4 \)), and Lemma 18, Remark 19 and Remark 20 (for \( g = 5 \)). The question mark indicates that over \( \mathbb{F}_3 \) there might exist curves of genus \( g = 5 \) having \( \mathbb{F}_3 \)-gonality 6, but there also might not exist such curves, see Remark 20.

2.2 Baker’s bound

Throughout a large part of this paper we will use the convenient language of Newton polygons. Let

\[
f = \sum_{(i,j) \in \mathbb{Z}^2_{\geq 0}} c_{i,j} x^i y^j \in k[x, y]
\]

be an irreducible polynomial over a field \( k \). Then its Newton polygon \( \Delta(f) \) is defined as \( \text{conv} \{ (i,j) \in \mathbb{Z}^2_{\geq 0} \mid c_{i,j} \neq 0 \} \subset \mathbb{R}^2 \). Note that \( \Delta(f) \) lies in the first quadrant and meets the coordinate axes in at least one point each, by the irreducibility of \( f \). Let \( C \) be the affine curve that is cut out by \( f \). Then one has the following bounds on the genus and the gonality of \( C \), purely in terms of the combinatorics of \( \Delta(f) \).
Genus  The genus of $C$ is at most the number of points in the interior of $\Delta(f)$ having integer coordinates: this is Baker’s theorem. See [5, Thm. 2.4] for an elementary proof and [17, §10.5] for a more conceptual version (using adjunction theory on toric surfaces). If one fixes the Newton polygon then Baker’s bound on the genus is generically attained, i.e. meeting the bound is a non-empty Zariski-open condition; this result is essentially due to Khovanskii [38]. An explicit sufficient generic condition is that $f$ is nondegenerate with respect to its Newton polygon [11, Prop. 2.3, Cor. 2.8].

Gonality  The $k$-gonality is at most the lattice width $lw(\Delta(f))$ of $\Delta(f)$. By definition, the lattice width is the minimal height $d$ of a horizontal strip $$\{ (a, b) \in \mathbb{R}^2 \mid 0 \leq b \leq d \}$$ inside which $\Delta(f)$ can be mapped using a unimodular transformation, i.e. an affine transformation of $\mathbb{R}^2$ with linear part in $GL_2(\mathbb{Z})$ and translation part in $\mathbb{Z}^2$.

This is discussed in [8, §2], but briefly the argument goes as follows. By applying the same transformation to the exponents, which is a $k$-rational birational change of variables, our polynomial $f$ can be transformed along with its Newton polygon. When orienting $f$ in this way one obtains $\text{deg}_y f = lw(\Delta(f))$, and the gonality bound follows by considering the $k$-rational map $(x, y) \mapsto x$. If a unimodular transformation can be used to transform $\Delta(f)$ into

for $d \geq 2$, then the geometric gonality enjoys the sharper bound $lw(\Delta(f)) - 1$ (amounting to 3 resp. $d - 1$); see [8, Thm.3]. If one fixes the Newton polygon then the sharpest applicable foregoing upper bound on the geometric gonality, i.e.

- $lw(\Delta(f)) - 1$ in the exceptional cases $2\Upsilon, d\Sigma \ (d \geq 2)$,
- $lw(\Delta(f))$ in the non-exceptional cases,

is generically met, and again nondegeneracy is a sufficient condition [10, Cor. 6.2]. In fact, the slightly weaker condition of meeting Baker’s genus bound is already sufficient [10, §4].
Remark 4. The results from [10] are presented in characteristic zero only, but [10, Cor. 6.2] holds in finite characteristic too, as can be seen as follows. Assume for simplicity that $\Delta(f)$ is not of the form $2\Upsilon$ or $d\Sigma$ for some $d \geq 2$, these cases are easy to deal with separately. Suppose that $C$ meets Baker’s bound but that the gonality of $C$ is strictly less than $\text{lw}(\Delta(f))$, say realized by a map $\pi : C \to \mathbb{P}^1$. We split this map in the usual way into a purely inseparable and a separable part

$$C \xrightarrow{\pi_s} C_{F_q} \xrightarrow{\pi_s} \mathbb{P}^1,$$

where $F_q$ denotes an appropriate Frobenius power and $C_{F_q}$ is the curve defined by $f^{F_q}$, the polynomial obtained by applying $F_q$ to each coefficient of $f$. Note that $\Delta(f) = \Delta(f^{F_q})$, so one sees that $C_{F_q}$ also meets Baker’s bound because Frobenius preserves the genus [30, Prop. IV.2.5]. Clearly $\deg \pi_s < \text{lw}(\Delta(f^{F_q}))$. Now the crucial ingredient in the proof of [10, Cor. 6.2] is a theorem due to Serrano on the possibility of extending morphisms from curves to ambient surfaces, which assumes $\text{char } k = 0$. However as Serrano points out [47, Rmk. 3.12] his theorem also holds in finite characteristic, provided that the morphism is separable, the ambient surface $S$ is rational, and $h^0(\mathcal{O}_S(C))$ is large enough compared to the degree of the morphism to be extended. The reader can verify that these conditions are satisfied when applying the proof of [10, Thm. 6.1] to $\pi_s$, leading to the conclusion that it is necessarily of the form $(x, y) \mapsto x^a y^b$ for some pair of coprime integers $a, b$. This contradicts that $\deg \pi_s < \text{lw}(\Delta(f^{F_q}))$.

Summing up in the non-geometric case, if we are not in the exceptional cases $2\Upsilon, d\Sigma$ ($d \geq 2$) then meeting Baker’s bound is sufficient for the $k$-gonality to equal $\text{lw}(\Delta(f))$. In the exceptional cases the $k$-gonality is either $\text{lw}(\Delta(f))$ or $\text{lw}(\Delta(f)) - 1$.

This yields a large class of defining polynomials $\overline{f} \in \mathbb{F}_q[x, y]$ for which finding an $f \in \mathcal{O}_K[x, y]$ satisfying (i), (ii) and (iii) is easy. Indeed, by semi-continuity the genus cannot increase under reduction modulo $p$. Therefore if $\overline{f}$ attains Baker’s upper bound on the genus, then it suffices to pick any $f \in \mathcal{O}_K[x, y]$ that reduces to $\overline{f}$ mod $p$, in such a way that $\Delta(f) = \Delta(\overline{f})$: the corresponding curve $C/K$ necessarily attains Baker’s upper bound, too. If moreover we are not in the exceptional cases $2\Upsilon$ and $d\Sigma$ ($d \geq 2$), then from the foregoing discussion we know that both the $\mathbb{F}_q$-gonality of $\overline{C}$ and the $K$-gonality of $C$ are equal to $\text{lw}(\Delta(\overline{f})) = \text{lw}(\Delta(f))$. A unimodular transformation then ensures that $\deg y f = \text{lw}(\Delta(f))$ as desired; such a transformation is computationally easy to find [24].

It is therefore justifiable to say that conditions (i), (ii) and (iii) are easy to deal with for almost all polynomials $\overline{f} \in \mathbb{F}_q[x, y]$. But be cautious: this does not mean that almost all curves $\overline{C}/\mathbb{F}_q$ are defined by such a polynomial. In terms of moduli, the locus of curves for which this is true has dimension $2g + 1$, except if $g = 7$ where it is 16; see [13, Thm. 12.1]. Recall that the moduli space of curves of genus $g$ has dimension $3g - 3$, so as soon as $g \geq 5$ the defining polynomial $\overline{f}$ of a plane model of a generic curve $\overline{C}/\mathbb{F}_q$ of genus $g$ can never attain Baker’s bound. For such curves, the foregoing discussion becomes counterproductive: if we take a naive coefficient-wise lift $f \in \mathcal{O}_K[x, y]$ of $\overline{f}$, then it is very likely to satisfy Baker’s bound, causing an increase of genus. This shows that $f$ has to be constructed with more care, which is somehow the main point of this article.
2.3 Preliminary discussion

We will attack Problem 1 in the cases where the genus $g$ of $\mathcal{C}$ is at most five (in Section 3) or the $\mathbb{F}_q$-gonality $\gamma$ of $\mathcal{C}$ is at most four (in Section 4), where we recall our overall assumption that $q$ is odd. In this section we quickly discuss the cases where $g$ and/or $\gamma$ are at most 2.

Remark 5. Note that for the purpose of computing the Hasse-Weil zeta function using the algorithm from [53, 54], the characteristic $p$ of $\mathbb{F}_q$ should moreover not be too large: this restriction is common to all $p$-adic point counting algorithms. For the lifting methods described in the current paper, the size of $p$ does not play a role.

If $\mathcal{C}$ is a curve of genus $g = 0$ then we can assume that $\mathcal{C} = \mathbb{P}^1$, because every plane conic carries at least one $\mathbb{F}_q$-point, and projection from that point gives an isomorphism to the line. In particular $\gamma = 1$ if and only if $g = 0$, in which case Problem 1 can be addressed by simply outputting $f = y$.

Next, if $g = 1$ then we can assume that $\mathcal{C}$ is defined by a polynomial $f \in \mathbb{F}_q[x,y]$ in Weierstrass form, i.e. $f = y^2 - h(x)$ for some squarefree cubic $h(x) \in \mathbb{F}_q[x]$. In this case $\gamma = 2$, and any $f \in \mathcal{O}_K[x,y]$ for which $\Delta(f) = \Delta(f)$ will address Problem 1 (for instance because Baker’s bound is attained, or because a non-zero discriminant must lift to a non-zero discriminant).

Finally, if $g \geq 2$ then $\mathcal{C}$ is geometrically hyperelliptic if and only if $\kappa$ realizes $\mathcal{C}$ as a degree 2 cover of a curve of genus zero [30, IV.5.2-3]. By the foregoing discussion the latter is isomorphic to $\mathbb{P}^1$, and therefore every geometrically hyperelliptic curve $\mathcal{C}/\mathbb{F}_q$ admits an $\mathbb{F}_q$-rational degree 2 map to $\mathbb{P}^1$. In particular, one can unambiguously talk about hyperelliptic curves over $\mathbb{F}_q$. In this case it is standard how to produce a defining polynomial $f \in \mathbb{F}_q[x,y]$ that is in Weierstrass form, i.e. $f = y^2 - h(x)$ for some squarefree $h(x) \in \mathbb{F}_q[x]$. Then again any $f \in \mathcal{O}_K[x,y]$ for which $\Delta(f) = \Delta(f)$ will address Problem 1.

Remark 6. Let $g_d^1$ be a complete base-point free $\mathbb{F}_q$-rational linear pencil of degree $d$ on a non-singular projective curve $\mathcal{C}/\mathbb{F}_q$. Then from standard arguments in Galois cohomology (that are specific to finite fields) it follows that this $g_d^1$ automatically contains an $\mathbb{F}_q$-rational effective divisor, which can be used to construct an $\mathbb{F}_q$-rational map to $\mathbb{P}^1$ of degree $d$. See for instance the proof of [27, Lem. 6.5.3]. This gives another way of seeing that a geometrically hyperelliptic curve over $\mathbb{F}_q$ is automatically $\mathbb{F}_q$-hyperelliptic, because the hyperelliptic pencil $g_d^1$ is unique, hence indeed defined over $\mathbb{F}_q$. The advantage of this argument is that it is more flexible: for instance it also shows that a geometrically trigonal curve $\mathcal{C}/\mathbb{F}_q$ of genus $g \geq 5$ always admits an $\mathbb{F}_q$-rational degree 3 map to $\mathbb{P}^1$, again because the $g_d^1$ on such a curve is unique. So we can unambiguously talk about trigonal curves from genus five on.

Summing up, throughout the paper, it suffices to consider curves of $\mathbb{F}_q$-gonality $\gamma > 2$, so that the canonical map $\kappa : \mathcal{C} \to \mathbb{P}^{g-1}$ is an embedding. In particular we have $g \geq 3$. From the $p$-adic point counting viewpoint, all omitted cases are covered by the algorithms of Satoh [43] and Kedlaya [29, 37].
3 Curves of low genus

3.1 Curves of genus three

3.1.1 Lifting curves of genus three

Solving Problem 1 in genus three in its basic version is not hard, so we consider this as a warm-up discussion. We first analyze which $\mathbb{F}_q$-gonalities can occur:

**Lemma 7.** Let $\overline{C}/\mathbb{F}_q$ be a non-hyperelliptic curve of genus 3 and $\mathbb{F}_q$-gonality $\gamma$, and assume that $q$ is odd. If $\#C(\mathbb{F}_q) = 0$ then $\gamma = 4$, while if $\#\overline{C}(\mathbb{F}_q) > 0$ (which is guaranteed if $q > 29$) then $\gamma = 3$.

**Proof.** Using the canonical embedding we can assume that $\overline{C}$ is a smooth plane quartic. It is classical that such curves have geometric gonality 3, and that each gonal map arises as projection from a point on the curve. For a proof see [47, Prop. 3.13], where things are formulated in characteristic zero, but the same argument works in positive characteristic; alternatively one can consult [33]. In particular if there is no $\mathbb{F}_q$-point then there is no rational gonal map and $\gamma > 3$. But then a degree 4 map can be found by projection from an $\mathbb{F}_q$-point outside the curve. By [34, Thm. 3(2)] there exist pointless non-hyperelliptic curves of genus three over $\mathbb{F}_q$ if and only if $q \leq 23$ or $q = 29$. 

We can now address Problem 1 as follows. As in the proof we assume that $\overline{C}$ is given as a smooth quartic in $\mathbb{P}^2$. First suppose that $\#C(\mathbb{F}_q) = 0$. Because this is possible for $q \leq 29$ only, the occurrence of this event can be verified exhaustively. In this case the Newton polygon of the defining polynomial $\overline{f} \in \mathbb{F}_q[x, y]$ of the affine part of $\overline{C}$ equals:

\[
\begin{array}{c}
\text{(\emph{\Delta}^{0,0})}
\end{array}
\]

In particular Baker’s bound is attained, and a naive Newton polygon preserving lift $f \in \mathcal{O}_K[x, y]$ automatically addresses (i), (ii) and (iii). If $\#\overline{C}(\mathbb{F}_q) > 0$ then one picks a random $\mathbb{F}_q$-point $P$ (which can be found quickly) and one applies a projective transformation that maps $P$ to $(0 : 1 : 0)$. After doing so the Newton polygon of $\overline{f} \in \mathbb{F}_q[x, y]$ becomes contained in (and typically equals):

\[
\begin{array}{c}
\text{(\emph{\Delta}^{1,0})}
\end{array}
\]

Again Baker’s bound is attained, and a naive Newton polygon preserving lift $f \in \mathcal{O}_K[x, y]$ satisfies (i), (ii) and (iii).
It is important to transform the curve before lifting to characteristic 0. Indeed, if one would immediately lift our input quartic to a curve $C \subset \mathbb{P}^2$ then it is highly likely that $C(K) = \emptyset$, and therefore that the $K$-gonality equals 4 (by the same proof as above). This type of reasoning plays an important role throughout the paper, often in a more subtle way than here.

Remark 8 (purely notational). The indices $i,j$ in $\Delta_{i,j}^3$ refer to the multiplicities of intersection of $C$ with the line at infinity at the coordinate points $(0 : 1 : 0)$ and $(1 : 0 : 0)$, assuming that it is defined by a polynomial having Newton polygon $\Delta_{i,j}^3$. Note that $\Delta_{0,0}^0$ is just another way of writing $3\Sigma$.

Algorithm 9. Lifting curves of genus 3: basic solution

**Input:** non-hyperelliptic genus 3 curve $C$ over $\mathbb{F}_q$  
**Output:** lift $f \in \mathcal{O}_K[x,y]$ satisfying (i), (ii), (iii) that is supported
- on $\Delta_{0,0}^0$ if $C(\mathbb{F}_q) = \emptyset$, or else
- on $\Delta_{2,0}^3$

1: $\overline{C} \leftarrow \text{CanonicalImage}(C)$ in $\mathbb{P}^2 = \text{Proj} \mathbb{F}_q[X,Y,Z]$  
2: if $q > 29$ or $\overline{C}(\mathbb{F}_q) \neq \emptyset$ (verified exhaustively) then  
3: \hspace{1em} $P := \text{Random}(\overline{C}(\mathbb{F}_q))$  
4: \hspace{1em} apply automorphism of $\mathbb{P}^2$ transforming $T_P(\overline{C})$ into $Z = 0$  
5: \hspace{1em} and $P$ into $(0 : 1 : 0)$  
6: return NaiveLift($\text{Dehomogenization}_Z(\text{DefiningPolynomial}(\overline{C}))$)

3.1.2 Optimizations

For point counting purposes we can of course assume that $q > 29$, so that $\gamma = 3$. By applying (1) to a polynomial with Newton polygon $\Delta_{1,0}^3$ one ends up with a polynomial that is monic in $y$ and that has degree $4 + (\gamma - 1) = 6$ in $x$. This can be improved: in addition to mapping $P$ to $(0 : 1 : 0)$, we can have its tangent line $T_P(\overline{C})$ sent to the line at infinity. If we then lift $\overline{f}$ to $\mathcal{O}_K[x,y]$ we find an $f$ whose Newton polygon is contained in (and typically equals):

$$(\Delta_{3,0}^2)$$

In particular $f$ is monic (up to a scalar) and $\deg_x f \leq 4$. We can in fact achieve $\deg_x f = 3$ in all cases of practical interest. Indeed, with an asymptotic chance of $1/2$ our tangent line $T_P(\overline{C})$ intersects $\overline{C}$ in two other rational points. The above construction leaves enough freedom to position one of those points $Q$ at $(1 : 0 : 0)$. The resulting lift $f$ then becomes contained in (and typically equals)
In the case of failure we retry with another $P$. If $q > 59$ (say) then there are enough $F_q$-points $P \in C$ for this approach to work with near certainty, although there might exist sporadic counterexamples well beyond that point.

**Remark 10 (non-generic optimizations).** For large values of $q$ one might want to pursue a further compactification of the Newton polygon. Namely, if one manages to choose $P \in \overline{C}(F_q)$ such that it is an ordinary flex or such that $T_P(C)$ is a bitangent, then $T_P(C)$ meets $\overline{C}$ in a unique other point $Q$, which is necessarily defined over $F_q$. By proceeding as before one respectively ends up inside the first and second polygon below. If one manages to let $P \in \overline{C}(F_q)$ be a non-ordinary flex, i.e. a hyperflex, then positioning it at $(0 : 1 : 0)$ results in a polygon of the third form:

![Newton polygons](image)

Heuristically, as $q \to \infty$ we expect to be able to realize the first two polygons with probabilities $1 - 1/e$ and $1 - 1/\sqrt{e}$, respectively; more background can be found in an arXiv version of our paper (1605.02162v2). In contrast the hyperflex case $\Delta_{3}^{4,0}$ is very exceptional, but we included it in the discussion because it corresponds to the well-known class of $C_{3,4}$ curves: even though $\deg_x f = 4$ here, the corresponding point count is slightly faster.

### 3.1.3 Implementation

We now report on timings, memory usage and failure rates of our implementation of the algorithms in this section for various values of $p$ and $q = p^n$. The first column in each table contains the time used to compute the lift to characteristic 0 averaged over 1000 random examples. Then the second column gives the time used by the point counting code `pcc` from [53, 54] averaged over 10 different random examples. Next, the third column contains the total memory used in the computation. Finally, the last column gives the number of examples out of the 1000 where we did not find a lift satisfying [54, Ass.1], which each time turned out to be 0, i.e. we always found a good lift.

| $p$ | time lift(s) | time `pcc`(s) | space (Mb) | fails /1000 | $q$ | time lift(s) | time `pcc`(s) | space (Mb) | fails /1000 | $q$ | time lift(s) | time `pcc`(s) | space (Mb) | fails /1000 |
|-----|--------------|---------------|------------|-------------|-----|--------------|---------------|------------|-------------|-----|--------------|---------------|------------|-------------|
| 11  | 0.2          | 0.2           | 32         | 0           | 5   | 0.4          | 2.3           | 64         | 0           | 3   | 0.8          | 15            | 76         | 0           |
| 521 | 0.2          | 0.6           | 32         | 0           | 75  | 0.4          | 6.6           | 64         | 0           | 7   | 0.6          | 40            | 118        | 0           |
| 4099| 0.2          | 4.2           | 64         | 0           | 17  | 0.4          | 12           | 76         | 0           | 17  | 0.7          | 82            | 241        | 0           |
| 32771| 0.2         | 590           | 1124       | 0           | 79  | 0.4          | 66           | 241        | 0           | 79  | 0.8          | 473           | 831        | 0           |

12
Alternatively, without using the methods from this section, we can just make any plane quartic monic using (1), then lift naively to characteristic 0 and try to use this lift as input for \( pcc \). This way, we obtain the following three tables.

| \( p \) | time \( \text{ppc(s)} \) (Mb) /1000 | space | fails | \( q \) | time \( \text{ppc(s)} \) (Mb) /1000 | space | fails | \( q \) | time \( \text{ppc(s)} \) (Mb) /1000 | space | fails |
|------|----------------|-------|------|------|----------------|-------|------|------|----------------|-------|------|
| 11   | 0.4            | 32    | 225  | 31   | 6.1            | 32    | 13   | 310   | 42              | 76    | 0    |
| 67   | 1.3            | 32    | 52   | 75   | 14             | 32    | 0    | 710   | 94              | 124   | 0    |
| 521  | 8.7            | 76    | 5    | 175  | 32             | 80    | 0    | 1710  | 248             | 320   | 0    |
| 4099 | 8.3            | 307   | 1    | 375  | 71             | 156   | 0    | 3710  | 524             | 589   | 0    |
| 32771| 1153           | 2086  | 0    | 795  | 161            | 288   | 0    | 7910  | 1296            | 1311  | 0    |

Comparing the different tables, we see that the approach described in this section saves a factor of about 3 in runtime and a factor of about 2 in memory usage. Moreover, for small fields the naive lift of a plane quartic sometimes does not satisfy \([54, \text{Ass. 1}]\), while this never seems to be the case for the lift constructed using our methods.

### 3.2 Curves of genus four

#### 3.2.1 Lifting curves of genus four

By \([30, \text{Ex. IV.5.2.2}]\) the ideal of a canonical model \( \overline{C} \subset \mathbb{P}^3 = \text{Proj} \mathbb{F}_q[X,Y,Z,W] \) of a non-hyperelliptic genus \( g = 4 \) curve is generated by a cubic \( S_3 \) and a unique quadric \( S_2 \). Since \( q \) is assumed odd, the latter can be written as

\[
(X \ Y Z W) \cdot \overline{M} \cdot (X \ Y Z W)^t, \quad \overline{M} \in \mathbb{F}_q^{4 \times 4}, \quad \overline{M}^t = \overline{M}.
\]

Let \( \chi_2 : \mathbb{F}_q \to \{0, \pm 1\} \) denote the quadratic character on \( \mathbb{F}_q \). Then \( \chi_2(\det \overline{M}) \) is an invariant of \( \overline{C} \), which is called the discriminant.

If we let \( S_2, S_3 \in \mathcal{O}_K[X,Y,Z,W] \) be homogeneous polynomials that reduce to \( \overline{S}_2 \) and \( \overline{S}_3 \) modulo \( p \), then by \([30, \text{Ex. IV.5.2.2}]\) these define a genus 4 curve \( C \subset \mathbb{P}^3 \) over \( K \), thereby addressing (i) and (ii). However as mentioned in Section 2.1 we expect the \( K \)-gonality of \( C \) to be typically \( 2g - 2 = 6 \). This exceeds the \( \mathbb{F}_q \)-gonality of \( \overline{C} \):

**Lemma 11.** Let \( \overline{C}/\mathbb{F}_q \) be a non-hyperelliptic curve of genus 4 and \( \mathbb{F}_q \)-gonality \( \gamma \), and assume that \( q \) is odd. If the discriminant of \( \overline{C} \) is 0 or 1 then \( \gamma = 3 \). If it is \( -1 \) and \( \# \overline{C}(\mathbb{F}_q^2) > 0 \) (which is guaranteed if \( q > 7 \)) then \( \gamma = 4 \). Finally, if it is \( -1 \) and \( \# \overline{C}(\mathbb{F}_q^2) = 0 \) then \( \gamma = 5 \).

**Proof.** By \([30, \text{Ex. IV.5.5.2}]\) our curve carries one or two geometric \( g^1_3 \)'s, depending on whether the quadric \( \overline{S}_2 \) is singular (discriminant 0) or not. In the former case the quadric is a cone, and the \( g^1_3 \) corresponds to projection from the top. This is automatically defined over \( \mathbb{F}_q \). In the latter case the quadric is \( \mathbb{F}_q^2 \)-isomorphic to the hyperboloid \( \mathbb{F}^1 \times \mathbb{F}^1 \subset \mathbb{P}^3 \) and the \( g^1_3 \)'s correspond to the two rulings of the latter. If the isomorphism can be defined over \( \mathbb{F}_q \) (discriminant 1) then the \( g^1_3 \)'s are \( \mathbb{F}_q \)-rational. In the other case (discriminant \( -1 \)) the smallest field of definition is \( \mathbb{F}_q^2 \). So we can assume that the discriminant of \( \overline{C} \) is \( -1 \), and therefore that \( \gamma > 3 \). Now suppose that \( \# \overline{C}(\mathbb{F}_q^2) > 0 \), which is guaranteed if
if there is an \( F_q \)-point then let \( \ell \) be the tangent line to \( \mathcal{C} \) at it. In the other case we can find two conjugate \( F_{q^2} \)-points, and we let \( \ell \) be the line connecting both. In both cases \( \ell \) is defined over \( F_q \), and the pencil of planes through \( \ell \) cuts out a \( g_1^3 \), as wanted. The argument can be reversed: if there exists a \( g_1^3 \) containing an effective \( F_q \)-rational divisor \( D \), then by Riemann-Roch we find that \( \#(\mathcal{C}(\mathbb{F}_{q^2})) > 0 \). Now note that \( \#(\mathcal{C}(\mathbb{F}_q)) > 0 \) by the Weil bound. So \( \mathcal{C} \) carries an effective \( F_q \)-rational divisor of degree \( 5 \). The linear system \( |K-D| \) must be empty, for otherwise there would exist an \( F_q \)-point on \( \mathcal{C} \). But then Riemann-Roch implies that \( \dim |D| = 1 \), i.e. our curve carries an \( F_q \)-rational \( g_1^5 \).

Remark 12. An example of a genus four curve \( \mathcal{C}/\mathbb{F}_3 \) having \( \mathbb{F}_3 \)-gonality five can be found in an arXiv version of our paper (1605.02162v2).

To address Problem 1 in the non-hyperelliptic genus 4 case we make a case-by-case analysis.

\[ \chi_2(\det \mathcal{M}_2) = 0 \] In this case \( \mathcal{S}_2 \) is a cone over a conic. A linear change of variables takes \( \mathcal{S}_2 \) to the form \( ZW - X^2 \), which we note is one of the standard realizations inside \( \mathbb{P}^3 \) of the weighted projective plane \( \mathbb{P}(1,2,1) \). It is classical how to find such a linear change of variables (diagonalization, essentially). Projecting from \( (0 : 0 : 0 : 1) \) on the \( XYZ \)-plane amounts to eliminating the variable \( W \), to obtain

\[ Z^3S_3(X,Y,Z,X^2) = S_3(XZ,YZ,Z^2,XY). \] (2)

After dehomogenizing with respect to \( Z \), renaming \( X \leftarrow x \) and \( Y \leftarrow y \) and rescaling if needed, we obtain an affine equation \( \mathcal{f} = y^3 + f_2(x)y^2 + f_4(x)y + f_6(x) \) with all \( f_i \in \mathbb{F}_q[x] \) of degree at most \( i \). Its Newton polygon is contained in (and typically equals):

So Baker’s bound is attained and we take for \( f \in \mathcal{O}_K[x,y] \) a naive coefficient-wise lift.

\[ \chi_2(\det \mathcal{M}_2) = 1 \] In this case \( \mathcal{S}_2 \) is a hyperboloid. A linear change of variables takes \( \mathcal{S}_2 \) to the standard form \( XY - ZW \), which we note is the image of \( \mathbb{P}^1 \times \mathbb{P}^1 \) in \( \mathbb{P}^3 \) under the Segre embedding. Projection from \( (0 : 0 : 0 : 1) \) on the \( XYZ \)-plane amounts to eliminating the variable \( W \), to obtain

\[ Z^3S_3(X,Y,Z,\frac{XY}{Z}) = \overline{S}_3(XZ,YZ,Z^2,XY). \]

After dehomogenizing with respect to \( Z \) and renaming \( X \leftarrow x \) and \( Y \leftarrow y \) we obtain an affine equation \( \mathcal{f} = f_0(x)y^3 + f_1(x)y^2 + f_2(x)y + f_3(x) \) with all \( f_i \in \mathbb{F}_q[x] \) of degree at most 3. Its Newton polygon is contained in (and typically equals)
So Baker’s bound is attained and we can take for \( f \in \mathcal{O}_K[x,y] \) a coefficient-wise lift of \( \overline{f} \).

\[
\chi_2(\det \overline{M}_2) = -1
\]

This is our first case where in general no plane model can be found for which Baker’s bound is attained \([13, \S 6]\). If \( \overline{C}(\mathbb{F}_q^2) = \emptyset \), or in other words if \( \gamma = 5 \), then unfortunately we do not know how to address Problem 1. We therefore assume that \( \overline{C}(\mathbb{F}_q^2) \neq \emptyset \) and hence that \( \gamma = 4 \). This is guaranteed if \( q > 7 \), so for point counting purposes this is amply sufficient. We follow the proof of Lemma 11: by exhaustive search we find a point \( P \in \overline{C}(\mathbb{F}_q^2) \) along with its Galois conjugate \( P' \) and consider the line \( \ell \) connecting both (tangent line if \( P = P' \)). This line is defined over \( \mathbb{F}_q \), so that modulo a projective transformation we can assume that \( \ell : X = Z = 0 \).

When plugging in \( X = Z = 0 \) in \( S_2 \) we find a non-zero quadratic expression in \( Y \) and \( W \). Indeed:

\[
S_3(0,Y,0,W) = (aY + bW)S_2(0,Y,0,W)
\]

for certain \( a, b \in \mathbb{F}_q \) that are possibly zero. Lift \( S_2 \) coefficient-wise to a homogeneous quadric \( S_2 \in \mathcal{O}_K[X,Y,Z,W] \) and let \( a, b \in \mathcal{O}_K \) reduce to \( a, b \mod p \). We now construct \( S_3 \in \mathcal{O}_K[X,Y,Z,W] \) as follows: for the coefficients at \( Y^3, Y^2W, W^2, W^3 \) we make the unique choice for which

\[
S_3(0,Y,0,W) = (aY + bW)S_2(0,Y,0,W),
\]

while the other coefficients are randomly chosen lifts of the corresponding coefficients of \( S_3 \). Then the genus 4 curve \( C \subset \mathbb{P}^3 \) defined by \( S_2 \) and \( S_3 \) is of gonality 4. Indeed, it is constructed such that the line \( \ell : X = Z = 0 \) intersects the curve in two points (possibly over a quadratic extension), and the pencil of planes through this line cuts out a \( g_1^1 \).

Now we project our lift \( C \subset \mathbb{P}^3 \) from \( (0:0:1) \) to a curve in \( \mathbb{P}^2 \). This amounts to eliminating \( W \) from \( S_2 \) and \( S_3 \). By dehomogenizing the resulting sextic with respect to \( Z \), and by renaming \( X \leftarrow x \) and \( Y \leftarrow y \) we end up with a polynomial \( f \in \mathcal{O}_K[x,y] \) whose Newton polygon is contained in \( (\Delta_{6,-1}) \) (and typically equals):

\[
(\Delta_{6,-1})
\]

Geometrically, what happens is that the points of \( C \) on \( \ell \) are both mapped to \( (0:1:0) \) under projection from \( (0:0:0:1) \), creating a singularity there, which in terms of the
Newton polygon results in $6\Sigma$ with its top chopped off. The polynomial $f$ satisfies (i), (ii) and (iii) from Problem 1. Note that Baker’s bound is usually not attained here: it gives an upper bound of 9, while $C$ has genus 4. So it is crucial to lift the equations to $O_K$ before projecting on the plane.

**Algorithm 13.** Lifting curves of genus 4: basic solution

**Input:** non-hyperelliptic genus 4 curve $\overline{C}/\mathbb{F}_q$ of $\mathbb{F}_q$-gonality $\gamma \leq 4$

**Output:** lift $f \in O_K[x,y]$ satisfying (i), (ii), (iii) that is supported

- on $\Delta^0_{4,0}$ if the discriminant is 0, or else
- on $\Delta^0_{4,1}$ if the discriminant is 1, or else
- on $\Delta^6_{4,-1}$

1: $\overline{C} \leftarrow \text{CanonicalImage}(\overline{C})$ in $\mathbb{P}^3 = \text{Proj} \mathbb{F}_q[X, Y, Z, W]$
2: $\overline{S}_2 \leftarrow$ unique quadric in Ideal($\overline{C}$); $\overline{M}_2 \leftarrow \text{Matrix}(\overline{S}_2)$; $\chi \leftarrow \chi_2(\text{det} \overline{M}_2)$
3: $S_3 \leftarrow$ cubic that along with $\overline{S}_2$ generates Ideal($\overline{C}$)
4: if $\chi = 0$ then
5: return NaiveLift(Dehomogenization$_Z(S_3(XZ, YZ, Z^2, X^2)))$
6: else if $\chi = 1$ then
7: return NaiveLift(Dehomogenization$_Z(S_3(XZ, YZ, Z^2, XY)))$
8: else
9: $P \leftarrow \text{Random}(\overline{C}(\mathbb{F}_q^2)); P' \leftarrow \text{Conjugate}(P)$
10: $\overline{l} \leftarrow$ line through $P$ and $P'$ (tangent line if $P = P'$)
11: apply automorphism of $\mathbb{P}^3$ transforming $\overline{S}_2 = 0$ into $ZW - X^2 = 0$
12: $S_2 \leftarrow \text{NaiveLift}(\overline{S}_2)$
13: $S_3 \leftarrow$ lift of $\overline{S}_3$ satisfying $S_3(0, Y, 0, W) = (aY + bW)S_2(0, Y, 0, W)$ for $a, b \in O_K$
14: return Dehomogenization$_Z(\text{res}_W(S_2, S_3))$

### 3.2.2 Optimizations

$\chi_2(\text{det} \overline{M}_2) = 0$ By applying (1) to a polynomial with Newton polygon $\Delta^0_{4,0}$ one ends up with a polynomial that is monic in $y$ and that has degree 6 in $x$. This can be improved as soon as $\overline{C}(\mathbb{F}_q) \neq 0$, which is guaranteed if $q > 49$ by [34, Thm. 2]. Namely we can view (2) as the defining equation of a smooth curve in the weighted projective plane $\mathbb{P}(1, 2, 1)$. Using an automorphism of the latter we can position a given $\mathbb{F}_q$-rational point $P$ at $(1 : 0 : 0)$ and the corresponding tangent line at $X = 0$, in order to end up with a Newton polygon that is contained in (and typically equals):

![Diagram](image)
See Remark 14 below for how to do this in practice. So we find \( \deg_x f = 4 \), which is optimal because the \( g_1^1 \) is unique in the case of a singular \( S_2 \). There is a caveat here, in that the tangent line at \( P \) might exceptionally be vertical, i.e. \( P \) might be a ramification point of our degree 3 map \((x, y) \mapsto x\). In this case it is impossible to position this line at \( X = 0 \), but in practice one can simply retry with another \( P \). But in fact having a vertical tangent line is an even slightly better situation, as explained in Remark 15 below.

**Remark 14.** The automorphisms of \( \mathbb{P}(1, 2, 1) \) can be applied directly to \( \mathcal{F} \). They correspond to

- substituting \( y \leftarrow \bar{a}y + \bar{b}x^2 + \bar{c}x + \bar{d} \) and \( x \leftarrow \bar{a}'x + \bar{b}' \) in \( \mathcal{F} \) for some \( \bar{a}, \bar{a}' \in \mathbb{F}_q^* \) and \( \bar{b}, \bar{b}', \bar{c}, \bar{d} \in \mathbb{F}_q \),
- exchanging the line at infinity for the \( y \)-axis by replacing \( \mathcal{F} \) by \( x^6 \mathcal{F}(x^{-1}, x^{-2}y) \),

or to a composition of both. For instance imagine that an affine point \( P = (a, b) \) was found with a non-vertical tangent line. Then \( \mathcal{F} \leftarrow \mathcal{F}(x + \bar{a}, y + \bar{b}) \) translates this point to the origin, at which the tangent line becomes of the form \( y = \bar{c}x \). Substituting \( \mathcal{F} \leftarrow \mathcal{F}(x, y + \bar{c}x) \) positions this line horizontally, and finally replacing \( \mathcal{F} \) by \( x^6 \mathcal{F}(x^{-1}, x^{-2}y) \) results in a polynomial with Newton polygon contained in \( \Delta_{4,0}^{1} \).

**Remark 15 (non-generic optimizations).** If \( P \) has a vertical tangent line then positioning it at \((1 : 0 : 0)\) results in a Newton polygon that is contained in (and typically equals) the first polygon below:

\[
\Delta_{4,0}^{2} \hspace{1cm} \Delta_{4,0}^{3}
\]

Even though \( \deg_x f = 5 \) here, this results in a slightly faster point count. Such a \( P \) will exist if and only if the ramification scheme of \((x, y) \mapsto x\) has an \( \mathbb{F}_q \)-rational point. Following the same heuristic as in Remark 10 we expect that this works in about \( 1 - 1/e \) of the cases. If there exists a point of ramification index 3 then one can even end up inside the second polygon. This event is highly exceptional, but we include it in our discussion because this corresponds to the well-known class of \( C_{3,5} \) curves.

\[\chi_2(\det \mathcal{M}_2) = 1\] By applying (1) to a polynomial with Newton polygon \( \Delta_{4,1}^{0} \) one ends up with a polynomial that is monic in \( y \) and that has degree \( 3 + (\gamma - 1)3 = 9 \) in \( x \). This can be improved as soon as \( \overline{C}(\mathbb{F}_q) \neq 0 \), which is guaranteed if \( q > 49 \) by [34, Thm. 2]. Assume as before that \( S_2 \) is in the standard form \( XY - ZW \). So it is the image of the Segre embedding

\[
\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^2 : ((X_0 : Z_0), (Y_0 : W_0)) \mapsto (X_0W_0 : Y_0Z_0 : Z_0W_0 : X_0Y_0).
\]

That is: we can view \( C \) as the curve in \( \mathbb{P}^1 \times \mathbb{P}^1 \) defined by the bihomogeneous polynomial

\[
\overline{S}_3(X_0W_0, Y_0Z_0, Z_0W_0, X_0Y_0)
\]

17
of bidegree \((3, 3)\). Remark that if we dehomogenize with respect to both \(Z_0\) and \(W_0\) and rename \(X_0 \leftarrow x\) and \(Y_0 \leftarrow y\) then we get the polynomial \(\overline{f}\) from before. Now if our curve has a rational point \(P\), by applying an appropriate projective transformation in each component we can arrange that \(P = ((1 : 0), (1 : 0))\). If we then dehomogenize we end up with a Newton polygon that is contained in (and typically equals):

\[
\Delta_{1,1}
\]

So Baker’s bound is attained and we take for \(f \in \mathcal{O}_K[x, y]\) a naive coefficient-wise lift. Now applying (1) typically results in a polynomial of degree \(3 + (\gamma - 1)2 = 7\) in \(x\).

**Remark 16.** The automorphisms of \(\mathbb{P}^1 \times \mathbb{P}^1\) can again be applied directly to \(\overline{f}\). They correspond to

- substituting \(y \leftarrow \overline{a}y + \overline{b}\) and \(x \leftarrow \overline{a}'x + \overline{b}'\) in \(\overline{f}\) for some \(\overline{a}, \overline{a}' \in \mathbb{F}_q^*\) and \(\overline{b}, \overline{b}' \in \mathbb{F}_q\),
- exchanging the \(x\)-axis for the horizontal line at infinity by replacing \(\overline{f}\) by \(y^2 \overline{f}(x, y^{-1})\),
- exchanging the \(y\)-axis for the vertical line at infinity by replacing \(\overline{f}\) by \(x^2 \overline{f}(x^{-1}, y)\),

or to a composition of these. For instance imagine that an affine point \(P = (\overline{a}, \overline{b})\) was found, then \(\overline{f} \leftarrow \overline{f}(x + \overline{a}, y + \overline{b})\) translates this point to the origin, and subsequently replacing \(\overline{f}\) by \(x^3y^3 \overline{f}(x^{-1}, y^{-1})\) results in a polynomial with Newton polygon contained in \(\Delta_{1,1}^1\).

**Remark 17 (non-generic optimizations).** If one manages to let \(P\) be a point with a horizontal tangent line, i.e. if \(P\) is a ramification point of the projection map from \(\overline{C}\) onto the second component of \(\mathbb{P}^1 \times \mathbb{P}^1\), then the Newton polygon even becomes contained in (and typically equals):

\[
\Delta_{1,1}^2
\]

This eventually results in a polynomial \(f \in \mathcal{O}_K[x, y]\) of degree \(3 + (\gamma - 1)1 = 5\) in \(x\). As in the discriminant 0 case, we heuristically expect the probability of success to be about \(1 - 1/e\). However, it is also fine to find a ramification point of the projection of \(\overline{C}\) onto the first component of \(\mathbb{P}^1 \times \mathbb{P}^1\), because we can change the role of \((X_0, Z_0)\) and \((Y_0, W_0)\) if wanted. Assuming independence of events, the percentage of non-hyperelliptic genus 4 curves with discriminant 1 that admit a Newton polygon of the form \(\Delta_{1,1}^2\) should be approximately \(1 - 1/e^2\).
\(\chi_2(\det \overline{M}_2) = -1\) By applying (1) to a polynomial with Newton polygon \(\Delta^4_{1,-1}\) we end up with a polynomial that is monic in \(y\) and that has degree \(3 + (\gamma - 1)2 = 9\). This can be improved as soon as \(C(\overline{\mathbb{F}_q}) \neq 0\), which is guaranteed if \(q > 49\) by [34, Thm. 2]. In this case we redo the construction with \(\overline{\ell}\) the tangent line to a point \(P \in C(\overline{\mathbb{F}_q})\). As before we apply a projective transformation to obtain \(\overline{\ell} : X = Z = 0\), but in addition we make sure that \(\overline{P} = (0 : 0 : 0 : 1)\). This implies that \(\overline{S}_2(0, Y, Z, W) = Y^2\), possibly after multiplication by a scalar. We now proceed as before, to find lifts \(S_2, S_3 \in \mathcal{O}_K[X, Y, Z, W]\) that cut out a genus 4 curve \(C \subset \mathbb{P}^3\), still satisfying the property of containing \((0 : 0 : 0 : 1)\) with corresponding tangent line \(\overline{\ell} : X = Z = 0\). If we then project from \((0 : 0 : 0 : 1)\) we end up with a quintic in \(\mathbb{P}^2\), rather than a sextic. The quintic still passes through the point \((0 : 1 : 0)\), which is now non-singular: otherwise the pencil of lines through that point would cut out a \(K\)-rational \(g_1^3\). We can therefore apply a projective transformation over \(K\) that maps the corresponding tangent line to infinity, while keeping the point at \((0 : 1 : 0)\). After having done so, we dehomogenize to find a polynomial \(f \in \mathcal{O}_K[x, y]\) whose Newton polygon is contained in (and typically equals) \((\Delta^5_{4,-1})\) It still satisfies (i), (ii) and (iii), while here \(\deg_x f \leq 5\).

### 3.2.3 Implementation

The tables below contain timings, memory usage and failure rates for \(\chi_2 = 0, 1, -1\) and various values of \(p\) and \(q = p^n\). For the precise meaning of the various entries in the table see Section 3.1.3.

\(\chi_2 = 0\)

| \(p\) | time (s) | time (scc) | space (Mb) | fails /1000 | \(q\) | time (s) | time (scc) | space (Mb) | fails /1000 |
|---|---|---|---|---|---|---|---|---|---|
| 11 | 0.01 | 0.3 | 32 | 159 | 3\(^2\) | 0.04 | 6.6 | 64 | 2 | 3\(10\) | 0.3 | 34 | 112 | 0 |
| 67 | 0.01 | 1.4 | 32 | 2 | 7\(3\) | 0.05 | 13 | 73 | 0 | 7\(10\) | 0.4 | 76 | 156 | 0 |
| 521 | 0.01 | 13 | 73 | 2 | 17\(2\) | 0.1 | 32 | 118 | 0 | 17\(10\) | 0.6 | 205 | 320 | 0 |
| 4099 | 0.01 | 189 | 323 | 0 | 37\(5\) | 0.1 | 73 | 197 | 0 | 37\(10\) | 0.7 | 537 | 653 | 0 |
| 32771 | 0.01 | 2848 | 2396 | 0 | 79\(5\) | 0.1 | 183 | 371 | 0 | 79\(10\) | 0.9 | 1392 | 1410 | 0 |

\(\chi_2 = 1\)

| \(p\) | time (s) | time (scc) | space (Mb) | fails /1000 | \(q\) | time (s) | time (scc) | space (Mb) | fails /1000 |
|---|---|---|---|---|---|---|---|---|---|
| 11 | 0.01 | 0.4 | 32 | 169 | 3\(^3\) | 0.1 | 7.5 | 64 | 0 | 3\(10\) | 0.7 | 41 | 150 | 0 |
| 67 | 0.02 | 1.8 | 32 | 1 | 7\(5\) | 0.1 | 16 | 112 | 0 | 7\(10\) | 1.2 | 102 | 320 | 0 |
| 521 | 0.02 | 14 | 76 | 0 | 17\(5\) | 0.2 | 41 | 197 | 0 | 17\(10\) | 2.1 | 276 | 556 | 0 |
| 4099 | 0.02 | 230 | 508 | 0 | 37\(5\) | 0.2 | 94 | 320 | 0 | 37\(10\) | 2.8 | 736 | 1070 | 0 |
| 32771 | 0.02 | 2614 | 3616 | 0 | 79\(5\) | 0.2 | 241 | 589 | 0 | 79\(10\) | 3.9 | 1904 | 2016 | 0 |
Contrary to the genus 3 case, we see that for very small $p$ or $q = p^n$, sometimes we do not find a lift satisfying \([54, \text{Ass. 1}]\). However, in these cases we can usually compute the zeta function by counting points naively, so not much is lost here in practice. Note that the point counting is considerably slower for $\chi_2 = -1$ than for $\chi_2 = 0, 1$ which is due to the map from the curve to $\mathbb{P}^1$ having degree 4 instead of 3 in this case.

### 3.3 Curves of genus five

#### 3.3.1 Lifting curves of genus five

By Petri’s theorem \([42]\) a minimal set of generators for the ideal of a canonical model

\[
\mathcal{C} \subset \mathbb{P}^4 = \text{Proj} \mathbb{F}_q[X, Y, Z, W, V]
\]

of a non-hyperelliptic genus 5 curve consists of

- three quadrics $\mathfrak{S}_{2,1}, \mathfrak{S}_{2,2}, \mathfrak{S}_{2,3}$ and two cubics $\mathfrak{S}_{3,1}, \mathfrak{S}_{3,2}$ in the trigonal case,
- just three quadrics $\mathfrak{S}_{2,1}, \mathfrak{S}_{2,2}, \mathfrak{S}_{2,3}$ in the non-trigonal case.

So given such a minimal set of generators, it is straightforward to decide trigonality. We denote the space of quadrics in the ideal of $\mathcal{C}$ by $\mathcal{I}_2(\mathcal{C})$. Then in both settings $\mathcal{I}_2(\mathcal{C})$ is a three-dimensional $\mathbb{F}_q$-vector space of which $\mathfrak{S}_{2,1}, \mathfrak{S}_{2,2}, \mathfrak{S}_{2,3}$ form a basis.

**Trigonal case** Here Petri’s theorem moreover tells us that $\mathcal{I}_2(\mathcal{C})$ cuts out a smooth irreducible surface $\mathfrak{S}$ that is a rational normal surface scroll of type $(1, 2)$. This means that up to a linear change of variables, it is the image $\mathfrak{S}(1, 2)$ of

\[
\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^4 : ((s : t), (u : v)) \mapsto (vst : ut^2 : us : vs^2),
\]

i.e. it is the ruled surface obtained by simultaneously parameterizing a line in the $YW$-plane (called the directrix) and a conic in the $XZV$-plane, each time drawing the rule through the points under consideration (each of these rules intersects our trigonal curve in three points, counting multiplicities). In other words, modulo a linear change of variables the space $\mathcal{I}_2(\mathcal{C})$ admits the basis

\[
X^2 − ZV, \quad XY − ZW, \quad XW − YV.
\]
Note that these are (up to sign) the $2 \times 2$ minors of
\[
\begin{pmatrix}
X & V & W \\
Z & X & Y
\end{pmatrix}.
\]

It is not trivial to find such a linear change of variables. A general method using Lie algebras for rewriting Severi-Brauer surfaces in standard form was developed by de Graaf, Harrison, Pínkivá and Schicho [18], and a Magma function `ParametrizeScroll` for carrying out this procedure in the case of rational normal surface scrolls was written by Schicho. Unfortunately this was intended to work over fields of characteristic zero only, and indeed the function always seems to crash when invoked over fields of characteristic three; see also Remark 25 below. We do not know how fundamental this flaw is, or to what extent it is an artefact of the implementation, but to resolve this issue we have implemented an ad hoc method that is specific to scrolls of type $(1, 2)$. It can be found in `convertscroll.m`; more background on the underlying reasoning can be read in an arXiv version of this paper (1605.02162v2).

Once our quadrics $S_{2,1}, S_{2,2}, S_{2,3}$ are given by (4) we project from the line $X = Y = Z = 0$, which amounts to eliminating the variables $V$ and $W$, in order to obtain the polynomials
\[
S_{3,i}^{pr} = Z^3 S_{3,i}(X, Y, Z, \frac{X^2}{Z}, \frac{XY}{Z}) = S_{3,i}(XZ, YZ, Z^2, X^2, XY)
\]
for $i = 1, 2$. Dehomogenizing with respect to $Z$ and renaming $X \leftarrow x$ and $Y \leftarrow y$ we obtain two polynomials $f_1, f_2 \in \mathbb{F}_q[x,y]$, whose zero loci intersect in the curve defined by $f = \gcd(f_1, f_2)$. The Newton polygon of $f$ is contained in (and typically equals):
\[
\Delta_{5, \text{trig}}^0
\]
Note that in particular $f$ attains Baker’s bound, and a naive Newton polygon preserving lift $f \in \mathcal{O}_K[x,y]$ satisfies (i), (ii) and (iii). An alternative (namely, toric) viewpoint on our construction of $f$, along with more background on the claims above, is given in Section 4.1.

**Non-trigonal case** In the non-trigonal case, let us write the quadrics as
\[
S_{2,i} = (X \ Y \ Z \ W \ V) \cdot M_i \cdot (X \ Y \ Z \ W \ V)^t, \quad M_i \in \mathbb{F}_q^{5 \times 5}, \ M_i^t = M_i.
\]
The curve $\mathcal{D}(\overline{C})$ in $\mathbb{P}^2 = \text{Proj} \ \mathbb{F}_q[\lambda_1, \lambda_2, \lambda_3]$ defined by
\[
\det(\lambda_1 M_1 + \lambda_2 M_2 + \lambda_3 M_3) = 0
\]
parameterizes the singular members of $\mathcal{I}_2(\overline{C})$. It is a possibly reducible curve called the discriminant curve of $\overline{C}$, known to be of degree 5 and having at most nodes as singularities.
[2]. The non-singular points correspond to quadrics of rank 4, while the nodes correspond to quadrics of rank 3. For a point \( P \in \mathcal{D}(\overline{C})(\mathbb{F}_q) \), let us denote by \( \mathcal{M}_P \) the corresponding \((5 \times 5)\)-matrix and by \( \mathcal{S}_P \) the corresponding quadric, both of which are well-defined up to a scalar. We define
\[
\chi : \mathcal{D}(\overline{C})(\mathbb{F}_q) \to \{0, \pm 1\} : P \mapsto \begin{cases} 
\chi_2(\text{pdet}(\mathcal{M}_P)) & \text{if } P \text{ is non-singular}, \\
0 & \text{if } P \text{ is singular},
\end{cases}
\]
where \( \text{pdet} \) denotes the pseudo-determinant, i.e. the product of the non-zero eigenvalues.

If we let \( S_{2,i} \in \mathcal{O}_K[X, Y, Z, W, V] \) be homogeneous polynomials that reduce to \( \mathcal{S}_{2,i} \) modulo \( p \), then by [30, Ex. IV.5.5.3] these define a genus 5 curve \( C \subset \mathbb{P}^4 \) over \( K \), thereby addressing (i) and (ii). But as mentioned in Section 2.1 we expect the \( K \)-gonality of \( C \) to be typically \( 2g - 2 = 8 \), which exceeds the \( \mathbb{F}_q \)-gonality of \( \overline{C} \):

**Lemma 18.** Let \( \overline{C}/\mathbb{F}_q \) be a non-hyperelliptic non-trigonal curve of genus 5 and \( \mathbb{F}_q \)-gonality \( \gamma \), and assume that \( q \) is odd. If there is a point \( P \in \mathcal{D}(\overline{C})(\mathbb{F}_q) \) for which \( \chi(P) \in \{0, 1\} \) then \( \gamma = 4 \). If there does not exist such a point and \( \#\overline{C}(\mathbb{F}_q) > 0 \) (which is guaranteed if \( q > 3 \)) then \( \gamma = 5 \). If there does not exist such a point and \( \#\overline{C}(\mathbb{F}_q) = 0 \) then \( \gamma = 6 \).

**Proof.** By [2, VI. Ex. F] the geometric \( g_1^4 \)'s are in correspondence with the singular quadrics containing \( \overline{C} \). More precisely:

- Each rank 4 quadric is a cone over \( \mathbb{P}^1 \times \mathbb{P}^1 \). By taking its span with the top, each line on \( \mathbb{P}^1 \times \mathbb{P}^1 \) gives rise to a plane intersecting the curve in 4 points. By varying the line we obtain two \( g_1^4 \)'s, one for each ruling of \( \mathbb{P}^1 \times \mathbb{P}^1 \).
- Each rank 3 quadric is a cone with a 1-dimensional top over a conic. By taking its span with the top, every point of the conic gives rise to a plane intersecting the curve in 4 points. By varying the point we obtain a \( g_1^4 \).

There are no other geometric \( g_1^4 \)'s. Over \( \mathbb{F}_q \), we see that there exists a rational \( g_1^4 \) precisely:

- when there is a rank 4 quadric that is defined over \( \mathbb{F}_q \), such that the base of the corresponding cone is \( \mathbb{F}_q \)-isomorphic to \( \mathbb{P}^1 \times \mathbb{P}^1 \), or

- when there is a rank 3 quadric that is defined over \( \mathbb{F}_q \).

In terms of the discriminant, this amounts to the existence of a \( P \in \mathcal{D}(\overline{C}) \) for which \( \chi(P) \in \{0, 1\} \). So let us assume that \( \gamma > 4 \). If \( \#\overline{C}(\mathbb{F}_q) > 0 \), which by the Serre-Weil bound is guaranteed for \( q > 3 \), then there exists an effective \( \mathbb{F}_q \)-rational degree 3 divisor \( D \) on \( \overline{C} \). Because our curve is non-trigonal we find \( \dim|D| = 0 \), so by the Riemann-Roch theorem we have that \( \dim|K - D| = 1 \), and because \( \deg(K - D) = 5 \) we conclude that there exists a rational \( g_1^5 \) on \( \overline{C} \). (Remark: geometrically, this \( g_1^5 \) is cut out by the pencil of hyperplanes through the plane spanned by the support of \( D \), taking into account multiplicities.) The argument can be reversed: if there exists a \( g_1^5 \subset D \) for some \( \mathbb{F}_q \)-rational divisor \( D \) on \( \overline{C} \), then Riemann-Roch implies that \( |K - D| \) is non-empty, yielding an effective divisor of degree 3, and in particular \( \#\overline{C}(\mathbb{F}_q) > 0 \). So it remains to prove that if \( \#\overline{C}(\mathbb{F}_q) = 0 \) then there exists a rational \( g_1^5 \). We make a case distinction:
• If \( \#\overline{C}(\mathbb{F}_{q^2}) > 0 \) then there exists a rational effective divisor \( D \) of degree 2, and Riemann-Roch implies that \( \dim |K - D| = 2 \), yielding the requested rational \( g_6^1 \) (even a \( g_6^2 \), in fact).

• If \( \#\overline{C}(\mathbb{F}_{q^2}) = 0 \) then at least \( \#\overline{C}(\mathbb{F}_{q^3}) > 0 \) by the Weil bound, so there exists a rational effective divisor \( D \) of degree 6. Then \( K - D \) is of degree 2 and by our assumption \( |K - D| \) is empty. But then Riemann-Roch asserts that \( \dim |D| = 1 \), and we have our rational \( g_6^1 \).

This ends the proof. \( \square \)

Remark 19. If \( q \) is large enough then it is very likely that \( \mathcal{D}(\overline{C})/(\mathbb{F}_q) \) will contain a point \( P \) with \( \chi(P) \in \{0, 1\} \), and therefore that \( \gamma = 4 \); a more precise discussion is given below. There do however exist counterexamples for every value of \( q \), as is shown by a construction explained in an arXiv version of this paper (1605.02162v2).

Remark 20. We do not know whether gonality 6 actually occurs or not. For this one needs to verify the existence of a non-trigonal genus five curve over \( \mathbb{F}_3 \) which is pointless over \( \mathbb{F}_{27} \) and whose discriminant curve has no \( \mathbb{F}_3 \)-rational points \( P \) for which \( \chi(P) \in \{0, 1\} \). We ran a naive brute-force search for such curves, but did not manage to find one.

If \( q \) is large enough and \( \mathcal{D}(\overline{C}) \) has at least one (geometrically) irreducible component that is defined over \( \mathbb{F}_q \), then a point \( P \in \mathcal{D}(\overline{C})/(\mathbb{F}_q) \) with \( \chi(P) \in \{0, 1\} \) exists and therefore \( \overline{C} \) has \( \mathbb{F}_q \)-gonality 4. To state a precise bound on \( q \), let us analyze the (generic) setting where \( \mathcal{D}(\overline{C}) \) is a non-singular plane quintic. In this case the 'good' points \( P \) are in a natural correspondence with pairs of \( \mathbb{F}_q \)-points on an unramified double cover of \( \mathcal{D}(\overline{C}) \); we refer to [4, §2(c)] and the references therein for more background. By Riemann-Hurwitz this cover is of genus 11, for which the lower Serre-Weil bound is positive from \( q > 467 \) on. The presence of singularities or of absolutely irreducible \( \mathbb{F}_q \)-components of lower degree can be studied in a similar way and leads to smaller bounds.

There are two possible ways in which \( \mathcal{D}(\overline{C}) \) does not have an absolutely irreducible \( \mathbb{F}_q \)-component: either it could decompose into two conjugate lines over \( \mathbb{F}_{q^2} \) and three conjugate lines over \( \mathbb{F}_{q^3} \), or it could decompose into five conjugate lines over \( \mathbb{F}_{q^5} \). But in the former case the \( \mathbb{F}_q \)-rational point \( P \) of intersection of the two \( \mathbb{F}_q \)-lines satisfies \( \chi(P) = 0 \), so here too our curve \( \overline{C} \) has \( \mathbb{F}_q \)-gonality 4. Thus the only remaining case is that of five conjugate lines over \( \mathbb{F}_{q^5} \), which can occur for every value of \( q \).

Let us now address Problem 1. First assume that \( \gamma = 4 \), i.e. that there exists a point \( P \in \mathcal{D}(\overline{C})/(\mathbb{F}_q) \) with \( \chi(P) \in \{0, 1\} \). This can be decided quickly: if \( q \leq 467 \) then one can proceed by exhaustive search, while if \( q > 467 \) it is sufficient to verify whether or not \( \mathcal{D}(\overline{C}) \) decomposes into five conjugate lines. To find such a point, we first look for \( \mathbb{F}_q \)-rational singularities of \( \mathcal{D}(\overline{C}) \): these are exactly the points \( P \) for which \( \chi(P) = 0 \). If no such singularities exist then we look for a point \( P \in \mathcal{D}(\overline{C})/(\mathbb{F}_q) \) for which \( \chi(P) = 1 \) by trial and error. Once our point has been found, we proceed as follows.

\[ \chi(P) = 0 \quad \text{In this case } P \text{ corresponds to a rank 3 quadric, which using a linear change of variables we can assume to be in the standard form } S = ZW - X^2. \text{ Choose homogeneous} \]
Let \( \mathcal{S}_2, \mathcal{S}_2' \in \mathcal{O}_K[X,Y,Z,W,V] \) be quadrics that reduce to \( \overline{\mathcal{S}}_2, \overline{\mathcal{S}}_2' \) modulo \( p \). Along with 
\[
S = ZW - X^2 \in \mathcal{O}_K[X,Y,Z,W,V]
\]
these cut out a canonical genus 5 curve \( C \subset \mathbb{P}^4 \). We view the quadric defined by \( S \) as a cone over the weighted projective plane \( \mathbb{P}(1,2,1) \) with top \( (0 : 0 : 0 : 0 : 1) \). Our curve is then an intersection of two quadrics inside this cone, and by projecting from the top we obtain a curve \( C_{pr} \) in \( \mathbb{P}(1,2,1) \). In terms of equations this amounts to eliminating \( V \) from \( S_2 \) and \( S_2' \) by taking the resultant \( S^{pr}_2 := \text{res}_V(S_2, S_2') \), which is a homogeneous quartic. Now as in (2) we further eliminate the variable \( W \) to end up with \( S^{pr}_2(XZ, YZ, Z^2, X^2) \). After dehomogenizing with respect to \( Z \), renaming \( X \leftarrow x \) and \( Y \leftarrow y \) and rescaling if needed, we obtain an affine equation 
\[
f = y^4 + f_2(x)y^3 + f_4(x)y^2 + f_6(x)y + f_8(x), \quad f_i \in \mathcal{O}_K[x] \text{ of degree at most } i.
\] Its Newton polygon is contained in (and typically equals):

\[
(\Delta_{0,0}^0)
\]

Note that Baker’s genus bound reads 9, so this exceeds the geometric genus by 4. Thus it was important to lift \( \mathcal{S}_2, \mathcal{S}_2' \) before projecting.

\( \chi(P) = 1 \) In this case \( P \) corresponds to a rank 4 quadric whose pseudo-determinant is a square. Using a linear change of variables we can assume it to be in the standard form \( \mathcal{S} = XY - ZW \), which is a cone over \( \mathbb{P}^1 \times \mathbb{P}^1 \) with top \( (0 : 0 : 0 : 0 : 1) \). Choose homogeneous quadratic polynomials 
\[
\mathcal{S}_2, \mathcal{S}_2' \in \mathbb{F}_q[X,Y,Z,W,V]
\]
that along with \( \mathcal{S} \) form a basis of \( \mathcal{I}_2(\overline{C}) \). (In practice one can usually take \( \mathcal{S}_2 = \overline{\mathcal{S}}_{2,1} \) and \( \mathcal{S}_2' = \overline{\mathcal{S}}_{2,2} \).) Let \( S_2, S_2' \in \mathcal{O}_K[X,Y,Z,W,V] \) be quadrics that reduce to \( \mathcal{S}_2, \mathcal{S}_2' \) modulo \( p \). Along with 
\[
S = XY - ZW \in \mathcal{O}_K[X,Y,Z,W,V]
\]
these cut out a canonical genus 5 curve \( C \subset \mathbb{P}^4 \), which can be viewed as an intersection of two quadrics inside a cone over \( \mathbb{P}^1 \times \mathbb{P}^1 \) with top \( (0 : 0 : 0 : 0 : 1) \). We first project from
In particular deg\(_{y} f = 4\), as wanted. Here again Baker’s bound reads 9, which exceeds the geometric genus by 4.

\[\forall P \in \mathcal{D}(\mathcal{C}(\mathbb{F}_q)) : \chi(P) = -1\] This case is very rare, so we will be rather sketchy here. If \(\gamma = 6\) then we do not know how to address Problem 1, which for point counting purposes is not an issue because this could only occur when \(q = 3\). If \(\gamma = 5\) then one can try to address Problem 1 by following the proof of Lemma 18, similar to the way we treated the \(\chi(\det M_2) = -1\) case in genus four. For instance this works as follows if \(\overline{\mathcal{C}}(\mathbb{F}_q)\) has at least three non-collinear points, which is guaranteed as soon as \(\# \mathcal{U}(\mathbb{F}_q) \geq 4\), which in turn is guaranteed if \(q > 101\) by the Serre-Weil bound. Apply a transformation of \(\mathbb{P}^4\) to position these points at \((0 : 1 : 0 : 0 : 0)\), \((0 : 0 : 0 : 1 : 0)\) and \((0 : 0 : 0 : 0 : 1)\), so that the plane they span is \(X = Z = 0\). This implies that the defining quadrics have no terms in \(Y^2, W^2\) and \(V^2\), a property which is of course easily preserved when lifting to \(\mathcal{O}_K[X,Y,Z,W,V]\), resulting in a curve \(C \subset \mathbb{P}^4\) again passing through \((0 : 1 : 0 : 0 : 0)\), \((0 : 0 : 0 : 1 : 0)\) and \((0 : 0 : 0 : 0 : 1)\). Eliminating \(W\) and \(V\), which geometrically amounts to projecting from the line \(X = Y = Z = 0\), results in a sextic in \(\mathbb{P}^2 = \text{Proj} K[X,Y,Z]\) passing through \((0 : 1 : 0)\) in a non-singular way (otherwise the pencil of lines through that point would cut out a \(K\)-rational \(g_4^1\)). We can therefore apply a projective transformation that maps the corresponding tangent line to infinity, while keeping the point at \((0 : 1 : 0)\). Then by dehomogenizing with respect to \(Z\) and renaming \(X \leftarrow x\) and \(Y \leftarrow y\) we end up with a polynomial \(f \in \mathcal{O}_K[x,y]\) whose Newton polygon is contained in (and typically equals):
Input: non-hyperelliptic genus 5 curve $\overline{C}/\mathbb{F}_q$ of $\mathbb{F}_q$-gonality $\gamma \leq 5$

Output: lift $f \in \mathcal{O}_K[x, y]$ satisfying (i), (ii), (iii) that is supported
- on $\Delta^0_{5, \text{trig}}$ if $\overline{C}$ is trigonal, or else
- on $\Delta^0_{5, 0}$ if $\exists P \in \mathcal{D}(\overline{C}) : \chi(P) = 0$, or else
- on $\Delta^0_{5, 1}$ if $\exists P \in \mathcal{D}(\overline{C}) : \chi(P) = 1$, or else
- on $\Delta_5^3$

We omit a further discussion.

Algorithm 21. Lifting curves of genus 5: basic solution

\begin{itemize}
  \item Input: non-hyperelliptic genus 5 curve $\overline{C}/\mathbb{F}_q$ of $\mathbb{F}_q$-gonality $\gamma \leq 5$
  \item Output: lift $f \in \mathcal{O}_K[x, y]$ satisfying (i), (ii), (iii) that is supported
  \begin{itemize}
    \item on $\Delta^0_{5, \text{trig}}$ if $\overline{C}$ is trigonal, or else
    \item on $\Delta^0_{5, 0}$ if $\exists P \in \mathcal{D}(\overline{C}) : \chi(P) = 0$, or else
    \item on $\Delta^0_{5, 1}$ if $\exists P \in \mathcal{D}(\overline{C}) : \chi(P) = 1$, or else
    \item on $\Delta_5^3$
  \end{itemize}
\end{itemize}

1: \(\overline{C} \leftarrow \text{CanonicalImage}(\overline{C})\) in \(\mathbb{P}^4 = \text{Proj} \mathbb{F}_q[X, Y, Z, W, V]\)
2: if Ideal(\(\overline{C}\)) is generated by quadrics then
3: \(\mathcal{S}_{2, 1}, \mathcal{S}_{2, 2}, \mathcal{S}_{2, 3} \leftarrow\) quadrics that generate Ideal(\(\overline{C}\))
4: \(\overline{M}_i \leftarrow \text{Matrix}(\mathcal{S}_{2, i})\) \((i = 1, 2, 3)\)
5: \(\mathcal{D}(\overline{C}) \leftarrow\) curve in \(\mathbb{P}^2 = \text{Proj} \mathbb{F}_q[\lambda_1, \lambda_2, \lambda_3]\) defined by \(\det(\lambda_1 \overline{M}_1 + \lambda_2 \overline{M}_2 + \lambda_3 \overline{M}_3)\)
6: if \(q \leq 467\) and \(\forall P \in \mathcal{D}(\overline{C})(\mathbb{F}_q) : \chi(P) = -1\) (verified exhaustively) then
7: \(\text{goodpoints} \leftarrow \text{false}\)
8: else
9: \(\text{goodpoints} \leftarrow \text{true}\)
10: if \(q > 467\) and \(\mathcal{D}(\overline{C})\) decomposes into five conjugate lines then
11: \(\text{goodpoints} \leftarrow \text{true}\)
12: if \(\mathcal{D}(\overline{C})\) has $\mathbb{F}_q$-rational singular point $P$ then
13: \(\mathcal{S}_2, \mathcal{S}'_2 \leftarrow\) quadrics such that \((\mathcal{S}_P, \mathcal{S}_2, \mathcal{S}'_2)_{\mathbb{F}_q} = (\mathcal{S}_{2, 1}, \mathcal{S}_{2, 2}, \mathcal{S}_{2, 3})_{\mathbb{F}_q}\)
14: apply automorphism of $\mathbb{P}^4$ transforming $\mathcal{S}_P$ into $WZ - X^2$
15: \(\mathcal{S}_2 \leftarrow \text{NaiveLift}(\mathcal{S}_2); \mathcal{S}'_2 \leftarrow \text{NaiveLift}(\mathcal{S}'_2); \mathcal{S}'_{2w} \leftarrow \text{res}_V(\mathcal{S}_2, \mathcal{S}'_2)\)
16: return \(\text{Dehomogenization}_Z(\mathcal{S}'_{2w}(XZ, YZ, Z^2, X^2))\)
17: else
18: repeat \(P \leftarrow \text{Random}(\mathcal{D}(\overline{C})(\mathbb{F}_q))\) until \(\chi(P) = 1\)
19: \(\mathcal{S}_2, \mathcal{S}'_2 \leftarrow\) quadrics such that \((\mathcal{S}_P, \mathcal{S}_2, \mathcal{S}'_2)_{\mathbb{F}_q} = (\mathcal{S}_{2, 1}, \mathcal{S}_{2, 2}, \mathcal{S}_{2, 3})_{\mathbb{F}_q}\)
20: apply automorphism of $\mathbb{P}^4$ transforming $\mathcal{S}_P$ into $XY - ZW$
21: \(\mathcal{S}_2 \leftarrow \text{NaiveLift}(\mathcal{S}_2); \mathcal{S}'_2 \leftarrow \text{NaiveLift}(\mathcal{S}'_2); \mathcal{S}'_{2w} \leftarrow \text{res}_V(\mathcal{S}_2, \mathcal{S}'_2)\)
22: return \(\text{Dehomogenization}_Z(\mathcal{S}'_{2w}(XZ, YZ, Z^2, XY))\)
23: else
24: \(P_1, P_2, P_3 \leftarrow\) distinct random points of $\overline{C}(\mathbb{F}_q)$

26
apply automorphism of $\mathbb{P}^4$ sending $P_1, P_2, P_3$ to $(0 : 1 : 0 : 0), (0 : 0 : 1 : 0), (0 : 0 : 0 : 1)$

\( S_{2,i} \leftarrow \text{NaiveLift}(\mathbb{S}_{2,i}) \) \((i = 1, 2, 3)\)

\( C^{pr} \leftarrow \text{res}_{W,V}(S_{2,1}, S_{2,2}, S_{2,3}) \)

apply automorphism of $\mathbb{P}^2$ transforming $T_{(0:1:0)}(C^{pr})$ into $Z = 0$

return Dehomogenization$_Z(C^{pr})$

else

apply automorphism of $\mathbb{P}^4$ transforming space of quadrics in $\text{Ideal}(\mathbb{C})$ to $\langle X^2 - ZV, XY - ZW, XW - YV \rangle_{\mathbb{F}_q}$

\( \mathbb{S}_{3,1}, \mathbb{S}_{3,2} \leftarrow \) cubics that along with quadrics generate $\text{Ideal}(\mathbb{C})$

\( \mathcal{F}_i \leftarrow \text{Dehomogenization}_Z(\mathbb{S}_{3,i}(XZ, YZ, Z^2, X^2, XY)) \) \((i = 1, 2)\)

return NaiveLift($\gcd(\mathcal{F}_1, \mathcal{F}_2)$)

3.3.2 Optimizations

**Trigonal case** By applying (1) to a polynomial with Newton polygon $\triangle^{0,0}_{\text{trig}}$ we end up with a polynomial $f \in \mathcal{O}_K[x, y]$ that is monic in $y$ and that has degree $5 + (\gamma - 1)2 = 9$ in $x$. This can be improved as soon as our curve $\overline{C}/\mathbb{F}_q$ has a rational point $P$, which is guaranteed if $q > 89$ by the Serre-Weil bound (probably this bound is not optimal).

The treatment below is very similar to the genus four case where $\chi_2(\det M_2) = 0$, as elaborated in Section 3.2.2. The role of $\mathbb{P}(1, 2, 1)$ is now played by our scroll $\mathbb{S}(1, 2)$. Recall that the latter is a ruled surface spanned by a line (the directrix) and a conic that are being parameterized simultaneously. Using an automorphism of $\mathbb{S}(1, 2)$ we can position $P$ at the point at infinity of the spanning conic, in such a way that the curve and the conic meet at $P$ with multiplicity at least two. This results in a Newton polygon that is contained in (and typically equals):

\[
(\triangle^{0,1}_{5,\text{trig}})
\]

See Remark 22 below for how this can be done in practice. Here an application of (1) typically results in $\deg_x f = 3 + (\gamma - 1)2 = 7$. There are two caveats here: our curve might exceptionally be tangent at $P$ to a rule of the scroll, in which case it is impossible to make it tangent to the conic at that point. Or worse: our point $P$ might lie on the directrix, in which case it is just impossible to move it to the spanning conic. In these cases one can most likely just retry with another $P$. But in fact these two situations are better, as explained in Remark 23 below.

**Remark 22.** The automorphisms of $\mathbb{S}(1, 2)$ can be applied directly to $\mathcal{F}$. They correspond to

- substituting $y \leftarrow \pi y + \bar{b} x + \bar{c}$ and $x \leftarrow \pi' x + \bar{b}'$ in $\mathcal{F}$ for some $\pi, \pi' \in \mathbb{F}_q^*$ and $\bar{b}, \bar{b}', \bar{c} \in \mathbb{F}_q$. 

---

27
• exchanging the rule at infinity for the $y$-axis by replacing $\overline{f}$ by $x^5 \overline{f}(x^{-1}, x^{-1}y)$, or to a composition of both. For instance imagine that an affine point $P = (\pi, \delta)$ was found with a non-vertical tangent line. Then $\overline{f} \leftarrow \overline{f}(x+\pi, y+\delta)$ translates this point to the origin, at which the tangent line becomes of the form $y = \pi x$. Substituting $\overline{f} \leftarrow \overline{f}(x, y + \pi x)$ positions this line horizontally, and finally replacing $\overline{f}$ by $x^5 \overline{f}(x^{-1}, x^{-1}y)$ results in a polynomial with Newton polygon contained in $\Delta^{0,1}_{5,\text{trig}}$.

Remark 23 (non-generic optimizations). As for the first caveat, if $C$ turns out to be tangent at $P$ to one of the rules of the scroll then moving $P$ to the point at infinity of the spanning conic results in a Newton polygon that is contained in (and typically equals):

$$\Delta^{0,2}_{5,\text{trig}}$$

Even though this yields $\deg_x f = 4 + (\gamma - 1)2 = 8$, the corresponding point count is slightly faster. Such a $P$ will exist if and only if the ramification scheme of $(x, y) \mapsto x$ has an $\mathbb{F}_q$-rational point. Following the heuristics from Remark 10 we expect that this works in about $1 - 1/e$ of the cases. As for the second caveat, if $P$ is a point on the directrix of the scroll, we can move it to its point at infinity. This results in a Newton polygon that is contained in (and typically equals) the left polygon below.

$$\Delta^{1,0}_{5,\text{trig}}$$

This again gives us $\deg_x f = 5 + (\gamma - 1)1 = 7$, but here too the corresponding point count is faster. As explained in an arXiv version of our paper (1605.02162v2), the probability of being able to realize this polygon is about $1/2$, and one can even end up inside the right polygon with a probability of about $3/8$, yielding $\deg_x f = 4 + (\gamma - 1)1 = 6$.

Non-trigonal case For point counting purposes it is advantageous to give preference to the case $\chi(P) = 0$, i.e. to use a singular point $P \in D(C)(\mathbb{F}_q)$ if it exists. Some optimizations over the corresponding discussion in Section 3.3.2 are possible, for instance generically one can replace $\Delta^{0}_{5,0}$ with the left polygon below:

$$\Delta^{1}_{5,0}$$

$$\Delta^{0}_{5,0}$$
With an estimated probability of about $1 - (3/8)^\rho$ one can even end up inside the right polygon. Here $10 \geq \rho \geq 1$ denotes the number of singular points $P \in \mathcal{D}(\overline{C})(\mathbb{F}_q)$. We will spend a few more words on this in Remark 24 below, after having discussed the $\chi(P) = 1$ case. However usually such a singular $\mathbb{F}_q$-point $P$ does not exist, i.e. $\rho = 0$. More precisely we expect that the proportion of curves for which $\mathcal{D}(\overline{C})$ is a smooth plane quintic tends to 1 as $q \to \infty$. Indeed, in terms of moduli the locus of (non-hyperelliptic, non-trigonal) genus five curves having a singular point on its discriminant curve has codimension one; see [50, 25]. For this reason we will focus our attention on the case $\chi(P) = 1$, and leave it to the interested reader to elaborate the remaining details.

As for the case $\chi(P) = 1$, note that by applying (1) to a polynomial with Newton polygon $\Delta_{5,1}$ one ends up with a polynomial that is monic in $y$ and that has degree $4 + (\gamma - 1)4 = 16$ in $x$. With near certainty this can be reduced to 10, as we will explain now. The idea is to exploit the fact that in practice the discriminant curve $\mathcal{D}(\overline{C})$ contains enough $\mathbb{F}_q$-rational points for there to be considerable freedom in choosing a $P$ for which $\chi(P) = 1$. We want to select a suited such $P$, by which we mean the following.

As before, assume that an automorphism of $\mathbb{P}^4$ has been applied such that $\overline{S}_P = \overline{S} = XY - ZW$ and let $\overline{S}_2, \overline{S}_2' \in \mathbb{F}_q[X, Y, Z, W, V]$ be quadrics that along with $\overline{S}$ cut out our curve $\overline{C}$. Now suppose that we would have projected $\overline{C}$ from the point $(0 : 0 : 0 : 0 : 1)$ before lifting to characteristic 0. Then we would have ended up with a curve $\overline{C}_{pr}$ in

$$\mathbb{P}^1 \times \mathbb{P}^1 : \overline{S} = 0 \quad \text{in} \quad \mathbb{P}^3 = \text{Proj} \mathbb{F}_q[X, Y, Z, W].$$

This curve has arithmetic genus 9, because in fact that is what Baker’s bound measures. Since the excess in genus is $9 - 5 = 4$ we typically expect there to be 4 nodes. Our point $P$ is ‘suited’ as soon as one of the singular points $Q$ of $\overline{C}_{pr}$ is $\mathbb{F}_q$-rational. If $P$ is not suited, i.e. if there is no such $\mathbb{F}_q$-rational singularity, then we retry with another $P \in \mathcal{D}(\overline{C})(\mathbb{F}_q)$ for which $\chi(P) = 1$. Heuristically we estimate the probability of success to be about $5/8$. In particular if there are enough candidates for $P$ available, we should end up being successful very quickly with overwhelming probability.

Given such a singular point $Q \in \overline{C}_{pr}(\mathbb{F}_q) \subset \mathbb{P}^1 \times \mathbb{P}^1$ we can move it to the point $((1 : 0), (1 : 0))$, similar to what we did in the genus 4 case where $\chi_2(\det \overline{M}_2) = 1$. In terms of the coordinates $X, Y, Z, W$ of the ambient space $\mathbb{P}^3$ this means moving the point to $(0 : 0 : 0 : 1)$. Let’s say this amounts to the change of variables

$$\begin{pmatrix} X \\ Y \\ Z \\ W \end{pmatrix} \leftarrow A \begin{pmatrix} X \\ Y \\ Z \\ W \end{pmatrix}$$

where $A \in \mathbb{F}_q^{4 \times 4}$. Then we can apply the change of variables

$$\begin{pmatrix} X \\ Y \\ Z \\ W \\ V \end{pmatrix} \leftarrow \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \\ W \\ V \end{pmatrix}.$$
directly to the defining polynomials \( \mathcal{S}, \mathcal{S}_1, \mathcal{S}_2 \) of \( \mathcal{C} \) to obtain the curve \( \mathcal{C}_{tr} \) cut out by

\[
\mathcal{S} = XY - ZW, \quad \mathcal{S}_{2, tr}, \quad \mathcal{S}_{2', tr} \in \mathbb{F}_q[X, Y, Z, W, V].
\]

Indeed the transformation affects \( \mathcal{S} \) at most through multiplication by a non-zero scalar.

If we would now project from \((0 : 0 : 0 : 0 : 1)\) as before, we would end up with a curve \( \mathcal{C}_{tr}^{pr} \subset \mathbb{P}^1 \times \mathbb{P}^1 \) having a singularity at \(((1 : 0), (1 : 0))\), which is at \((0 : 0 : 0 : 1)\) in the coordinates \(X, Y, Z, W\).

Recall that inside \( \mathbb{P}^4 \) we view \( \mathcal{S} \) as the defining equation of a cone over \( \mathbb{P}^1 \times \mathbb{P}^1 \) with top \((0 : 0 : 0 : 0 : 1)\). The fact that the projected curve has a singularity at \((0 : 0 : 0 : 1)\) implies that the line \(X = Y = Z = 0\) meets the curve at least twice, counting multiplicities (these points of intersection need not be \(\mathbb{F}_q\)-rational). Thus after multiplying \( \mathcal{S}_{2, tr} \) by a scalar if needed we find that

\[
\mathcal{S}_{2, tr}'(0, 0, 0, W, V) = aW^2 + bWV + cV^2
\]

for elements \( a, b, c \in \mathcal{O}_K \) that reduce to \( \bar{a}, \bar{b}, \bar{c} \) modulo \( p \). If we then proceed as before, we end up with a curve \( \mathcal{C}_{tr}^{pr} \subset \mathbb{P}^1 \times \mathbb{P}^1 \) having a singularity at \(((1 : 0), (1 : 0))\). This eventually results in a defining polynomial \( f \in \mathcal{O}_K[x, y] \) whose Newton polygon is contained in (and typically equals):

\[
(\Delta_{2, 5}^2)
\]

Applying (1) to \( f \) results in a polynomial having degree at most \( 4 + (\gamma - 1)2 = 10 \) in \( x \), as announced.

**Remark 24.** The same ideas apply to the case \( \chi(P) = 0 \), with the role of \( \mathbb{P}^1 \times \mathbb{P}^1 \) replaced by \( \mathbb{P}(1, 2, 1) \). If the projection \( \mathcal{C}_{tr}^{pr} \) of \( \mathcal{C} \) to \( \mathbb{P}(1, 2, 1) \) has an \( \mathbb{F}_q \)-rational singular point, then it can be arranged that the resulting curve \( \mathcal{C}_{tr}^{pr} \subset \mathbb{P}(1, 2, 1) \) has a singularity at \((1 : 0: 1)\), eventually yielding a polynomial \( f \in \mathcal{O}_K[x, y] \) whose Newton polygon is contained in \( \Delta_{2, 5}^2 \).

As in the \( \chi(P) = 1 \) case we expect that the probability that this works out for a given \( P \) is about \( 5/8 \). But unlike the \( \chi(P) = 1 \) case there is not much freedom to retry in the case of failure: we have \( \rho \) chances only. This explains our expected probability of \( 1 - (3/8)^\rho \) to be able to realize \( \Delta_{2, 5}^2 \).

If the foregoing fails every time then we can play the same game with a non-singular \( \mathbb{F}_q \)-rational point \( Q \) on \( \mathcal{C}_{tr}^{pr} \) (guaranteed to exist if \( q > 89 \) because then \( \mathcal{C} \) has an \( \mathbb{F}_q \)-rational point by the Serre-Weil bound). The result is a curve \( \mathcal{C}_{tr}^{pr} \subset \mathbb{P}(1, 2, 1) \) containing
the point \((1 : 0 : 0)\). We can then use an automorphism of \(\mathbb{P}(1, 2, 1)\) to make \(\mathcal{C}^{\text{pr}}\) tangent to \(X = 0\) at that point (unless the tangent line is vertical, in which case we simply retry with another \(Q\)). This is done similarly to the way we handled the case \(\chi_2(\det \mathcal{M}) = 0\) in Section 3.2.2: see in particular Remark 14. In this way one ends up in \(\Delta_{5,0}^1\).

### 3.3.3 Implementation

The tables below contain timings, memory usage and failure rates for the trigonal and non-trigonal case and various values of \(p\) and \(q = p^n\). For the precise meaning of the various entries in the tables see Section 3.1.3.

#### Trigonal

| \(p\) | time (s) | time pcc(s) | space (Mb) | fails | time (s) | time pcc(s) | space (Mb) | fails | time (s) | time pcc(s) | space (Mb) | fails |
|------|---------|-------------|------------|-------|---------|-------------|------------|-------|---------|-------------|------------|-------|
| 11   | 0.02    | 0.6         | 96         | 206   | 3\(^2\) | 0.1         | 17         | 108   | 6      | 3\(^3\)     | 1.2         | 82   |
| 67   | 0.02    | 2.4         | 96         | 45    | 7\(^2\) | 0.1         | 33         | 150   | 0      | 7\(^4\)     | 2.0         | 214  |
| 521  | 0.02    | 23          | 112        | 4     | 17\(^3\) | 0.2         | 76         | 556   | 0      | 17\(^4\)   | 3.6         | 587  |
| 4099 | 0.02    | 358         | 548        | 1     | 37\(^3\) | 0.2         | 186        | 1070  | 0      | 37\(^4\)   | 4.5         | 1584 |
| 32771| 0.02    | 4977        | 3982       | 0     | 79\(^3\) | 0.3         | 452        | 1716  | 0      | 79\(^4\)   | 6.3         | 4039 |

#### Non-trigonal

| \(p\) | time (s) | time pcc(s) | space (Mb) | fails | time (s) | time pcc(s) | space (Mb) | fails | time (s) | time pcc(s) | space (Mb) | fails |
|------|---------|-------------|------------|-------|---------|-------------|------------|-------|---------|-------------|------------|-------|
| 11   | 0.1     | 2.0         | 64         | 14    | 3\(^2\) | 2.5         | 59         | 229   | 0      | 3\(^3\)     | 16          | 504  |
| 67   | 0.1     | 7.2         | 76         | 0     | 7\(^3\) | 5.3         | 114        | 352   | 0      | 7\(^4\)     | 40          | 1191 |
| 521  | 0.2     | 65          | 165        | 0     | 17\(^3\) | 10          | 261        | 556   | 0      | 17\(^4\)   | 89          | 2946 |
| 4099 | 0.2     | 1326        | 1326       | 0     | 37\(^3\) | 14          | 662        | 919   | 0      | 37\(^4\)   | 128         | 7032 |
| 32771| 0.2    | 21974       | 10329      | 0     | 79\(^3\) | 19          | 1532       | 1494  | 0      | 79\(^4\)   | 193         | 15729 |

### 4 Curves of low gonality

#### 4.1 Trigonal curves

Recall from Remark 6 that from genus five on a curve \(\overline{\mathcal{C}}/\mathbb{F}_q\) is trigonal iff it is geometrically trigonal. It is known [42] that a minimal set of generators for the ideal of a canonical model \(\overline{\mathcal{C}} \subset \mathbb{P}^{g - 1} = \text{Proj} \mathbb{F}_q[X_1, X_2, \ldots, X_g]\) of a non-hyperelliptic curve of genus \(g \geq 4\) over \(\mathbb{F}_q\) consists of

- \((g - 2)(g - 3)/2\) quadrics
  \[
  S_{2,1}, S_{2,2}, \ldots, S_{2,(g-2)(g-3)/2}
  \]

  and \(g - 3\) cubics
  \[
  S_{3,1}, S_{3,2}, \ldots, S_{3,g-3}
  \]

  if \(\overline{\mathcal{C}}\) is trigonal or \(\mathbb{F}_q\)-isomorphic to a smooth curve in \(\mathbb{P}^2\) of degree five,

- just \((g - 2)(g - 3)/2\) quadrics in the other cases.
So given such a minimal set of generators, it is straightforward to decide trigonality, unless \( g = 6 \) in which case one might want to check whether \( C \) is isomorphic to a smooth plane quintic or not. See Remark 27 below for how to do this.

From now on assume that we are given a trigonal curve \( C / F_\mathbb{q} \) in the above canonical form. Then the quadrics \( S_{2,i} \) spanning \( I_2(C) \) are known to define a rational normal surface scroll \( S \) of type \((a, b)\), where \( a, b \) are non-negative integers satisfying

\[ a \leq b, \quad a + b = g - 2, \quad b \leq \frac{(2g - 2)}{3}, \] (6)

called the Maroni invariants\(^1\) of \( C \). This means that up to a linear change of variables, it is the image \( \overline{S}(a, b) \) of

\[ \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^{g-1}: ((s : t), (u : v)) \mapsto (ut^a : ut^{a-1}s : \cdots : us^a : vt^b : vt^{b-1}s : \cdots : vs^b), \]
i.e. it is the ruled surface obtained by simultaneously parameterizing

- a rational normal curve of degree \( a \) in the \( \mathbb{P}^a \) corresponding to \( X_1, X_2, \ldots, X_{a+1} \), and
- a rational normal curve of degree \( b \) in the \( \mathbb{P}^b \) corresponding to \( X'_1, X'_2, \ldots, X'_{b+1} \),

where \( X'_i \) denotes the variable \( X_{a+1+i} \), each time drawing the rule through the points under consideration (each of these rules intersects our trigonal curve in three points, counting multiplicities).

As a consequence, modulo a linear change of variables, the space \( I_2(C) \) admits the \( 2 \times 2 \) minors of

\[ \begin{pmatrix}
X_1 & X_2 & \ldots & X_a & X'_1 & X'_2 & \ldots & X'_b \\
X_2 & X_3 & \ldots & X_{a+1} & X'_2 & X'_3 & \ldots & X'_{b+1}
\end{pmatrix} \] (7)
as a basis, for some \( a, b \) satisfying (6). We assume that we have a function \texttt{ConvertScroll} at our disposal that upon input of \( I_2(C) \) and a pair \((a, b)\) satisfying (6), either finds such a linear change of variables, or outputs ‘wrong type’ in case the surface cut out by \( I_2(C) \) is not a scroll of type \((a, b)\).

**Remark 25.** If \( g = 5 \) then \((1, 2)\) is the only pair of integers satisfying (6), and one can use our ad hoc method from mentioned in Section 3.3.1 to find the requested linear change of variables as above. For higher genus we have written an experimental version of \texttt{ConvertScroll} in Magma, which can be found in the file \texttt{convertscroll.m}. It blindly relies on Schicho’s function \texttt{ParametrizeScroll}, which implements the Lie algebra method from [18]. Unfortunately the latter is only guaranteed to work in characteristic zero, and indeed one runs into trouble when naively applying \texttt{ParametrizeScroll} over finite fields of very small characteristic; empirically however, we found that \( p > g \) suffices for a slight modification of \texttt{ParametrizeScroll} to work consistently. We remark that it is an easy linear algebra problem to verify the correctness of the output, in case it is returned. In any case further research is needed to turn this into a more rigorous step.

\(^1\)The existing literature is ambiguous on this terminology. Some authors talk about the Maroni invariant of a trigonal curve, in which case they could mean either \( a = \min(a, b) \), or \( b - a \).
Remark 26. If ‘wrong type’ is returned then one retries with another pair \((a, b)\) satisfying (6). From a moduli theoretic point of view \([48]\)† the most likely case is \(a = b = (g - 2)/2\) if \(g\) is even, and \(a + 1 = b = (g - 1)/2\) if \(g\) is odd, so it is wise to try that pair first, and then to let \(a\) decrease gradually. According to \([45]\)† the Lie algebra method implicitly computes the Maroni invariants, so it should in fact be possible to get rid of this trial-and-error part; recall that we just use the function \texttt{ConvertScroll} as a black box.

Remark 27 \((g = 6)\). If ‘wrong type’ is returned on input \((2, 2)\) as well as on input \((1, 3)\), then we are in the smooth plane quintic case and therefore \(C\) is not trigonal. Here \(I_2(C)\) cuts out a Veronese surface in \(\mathbb{P}^5\), rather than a scroll. We will revisit this case at the end of the section.

Once our quadrics \(S_{2,i}\) are given by the minors of (7), we restrict our curve \(C\) to the embedded torus

\[ T^2 \hookrightarrow \mathbb{P}^{g-1}: (x, y) \mapsto (y : xy : \cdots : x^a y : 1 : x : \cdots : x^b) \]

by simply substituting

\[ X_1 \leftarrow y, \quad X_2 \leftarrow xy, \quad \ldots, \quad X_{a+1} \leftarrow x^a y \quad \text{and} \quad X'_1 \leftarrow 1, \quad X'_2 \leftarrow x, \quad \ldots, \quad X'_{b+1} \leftarrow x^b. \]

This makes the quadrics vanish identically, while the cubics become

\[ f_1, f_2, \ldots, f_{g-3} \in \mathbb{F}_q[x, y]. \]

The ideal generated by these polynomials is principal, i.e. of the form \((\overline{f})\), where the Newton polygon of \(\overline{f} = \gcd(f_1, f_2, \ldots, f_{g-3})\) is contained in (and typically equals):

\[
\begin{array}{c}
(0, 0) \\
(0, 3) \\
(2b + 2 - a, 0) \\
(2a + 2 - b, 3)
\end{array}
\]

The correctness of these claims follows for instance from \([9, \S 3]\). Note that in particular \(\overline{f}\) attains Baker’s bound, so a naive Newton polygon preserving lift \(f \in \mathcal{O}_K[x, y]\) satisfies (i), (ii) and (iii).

Remark 28. It should be clear that the above is a generalization of the corresponding method from Section 3.3.1, where we dealt with trigonal curves of genus five. But the method also generalizes the genus four cases \(\chi_2(\det \overline{M}_2) = 0\) and \(\chi_2(\det \overline{M}_2) = 1\) from Section 3.2.1, where the scrolls are \(\overline{S}(0, 2) = \mathbb{P}(1, 2, 1)\) and \(\overline{S}(1, 1) = \mathbb{P}^1 \times \mathbb{P}^1\), respectively.

Remark 29. Here too one could try to compress the Newton polygon by clipping off boundary points, similar to what we did in Section 3.3.2. But as the genus grows the resulting speed-ups become less and less significant, and we omit a further discussion.
Example  Let us carry out the foregoing procedure for the curve defined by
\[(x^3 + x + 1)y^3 + 42(2x^4 + x^3 + 3x^2 + 3x + 1)y^2 + (x + 1)(x^4 + 2x^2 + x + 1)y + 42(x^2 + 1) = 0\]
over \(\mathbb{F}_{43}\). This is the reduction mod 43 of the modular curve \(X_0^+(164)\), or rather an affine
model of it, whose equation we took from [32]. It is of genus 6, while we note that Baker’s
bound reads 7, so it is not met here. Using the intrinsic CanonicalMap one computes that
\[
X_1^2X_2 + 42X_1^2X_5 + 40X_1X_2X_6 + 40X_1X_2X_6 + X_1X_2^2 + 2X_1X_3X_6 + 42X_1X_2^2 + 40X_1X_4X_5 + X_1X_4X_6
+ 6X_1X_5X_6 + 7X_1X_2^2 + 42X_2X_3X_6 + 41X_2X_2^2 + 42X_3^2 + 40X_3X_6^2 + 2X_3^2X_5 + 4X_3X_6^2
\]
\[
X_2X_3 + 42X_2X_5 + 39X_3X_2 + X_2X_2^2 + 36X_1X_3X_6 + 42X_1X_4X_2 + X_1X_2X_3 + 7X_1X_2^2 + 2X_2X_3^2
+ 41X_1X_5X_6 + 8X_3X_2^2 + 42X_2X_3^2 + 4X_1X_2^2 + X_2X_2^2 + 5X_1X_3X_6 + X_1X_4X_2 + 40X_3X_2^2 + 37X_2^2
\]
\[
42X_1X_5 + 1X_2X_2 + 32X_2X_6 + 39X_1X_5X_6 + 42X_1X_4X_5 + 42X_1X_6X_2 + 6X_1X_6^2 + 8X_2X_5^2 + 39X_2X_5X_6
+ 7X_2X_5^3 + 42X_2^2X_5 + 5X_3X_2^2 + 42X_5X_6 + 5X_4X_5X_6 + 41X_4X_6^2 + X_5X_2^2 + 36X_2^2
\]
is a minimal set of generators for the ideal \(I(\mathcal{C})\) of a canonical model \(\mathcal{C} \subset \mathbb{P}^5\). We are
clearly in the trigonal case, so the six quadrics must cut out a rational normal surface
scroll. According to (6) the type of the latter is either (1, 3) or (2, 2). Following Remark 26
we first try (2, 2), so we search for a linear change of variables taking \(I_2(\mathcal{C})\) to the space
of quadrics spanned by the \(2 \times 2\) minors of
\[
\begin{pmatrix}
X_1 & X_2 & X_4 \\
X_2 & X_3 & X_5 \\
X_4 & X_5 & X_6
\end{pmatrix}
\]
Our experimental version of the function ConvertScroll turns out to work here, and the
type (2, 2) was a correct guess: the change of variables returned by Magma reads
\[
\begin{pmatrix}
X_1 \\
X_2 \\
X_3 \\
X_4 \\
X_5 \\
X_6
\end{pmatrix} \leftrightarrow \begin{pmatrix}
40 & 3 & 42 & 0 & 30 & 33 \\
38 & 0 & 12 & 35 & 40 & 42 \\
0 & 9 & 4 & 30 & 29 & 42 \\
20 & 37 & 5 & 2 & 8 & 22 \\
22 & 19 & 11 & 28 & 32 & 14 \\
32 & 27 & 19 & 6 & 17 & 36
\end{pmatrix}
\begin{pmatrix}
X_1 \\
X_2 \\
X_3 \\
X_4 \\
X_5 \\
X_6
\end{pmatrix}
\]
Applying this transformation to our generators of \(I(\mathcal{C})\) and then substituting
\[
X_1 \leftrightarrow y, \quad X_2 \leftrightarrow xy, \quad X_3 \leftrightarrow x^2y, \quad X_4 \leftrightarrow 1, \quad X_5 \leftrightarrow x, \quad X_6 \leftrightarrow x^2
\]
annihilates the quadrics, while the cubics become
\[
6(x + 27)(x + 32)\overline{T}, \quad 39(x + 13)(x + 20)\overline{T}, \quad 2(x + 13)^2\overline{T}
\]
respectively, where
\[
\overline{T} = x^4y^3 + 8x^4y^2 + 31x^4y + 29x^3 + 37x^3y^3 + 23x^2y^2 + 16x^2y + x^3 + 12x^2y^3 + 18x^2y^2
+ 12x^2y + 25x + 10xy^2 + 7xy^2 + 30xy + 11x + 13y^3 + 36y^2 + 3y + 2
\]

34
For this polynomial Baker’s bound is attained, so a naive lift to \( f \in \mathcal{O}_K[x, y] \) satisfies (i), (ii), (iii). After making \( f \) monic using (1) it can be fed to the algorithm from [53, 54] to find the numerator

\[
43^6 T^{12} + 43^5 \cdot 8 T^{11} + 43^4 \cdot 154 T^{10} + 43^3 \cdot 1032 T^9 + 43^2 \cdot 9911 T^8 + 43 \cdot 62496 T^7 \\
+ 444940 T^6 + 62496 T^5 + 9911 T^4 + 1032 T^3 + 154 T^2 + 8 T + 1
\]

of the zeta function \( Z_{C/F_{43}}(T) \) in a couple of seconds.

**Point counting timings** Despite the lack of a well-working function ConvertScroll, we can tell how the point counting algorithm from [53, 54] should perform in composition with the above method, by simply assuming that \( C \) is given as the genus \( g \) curve defined by a suitably generic polynomial \( f \in \mathbb{F}_q[x, y] \) supported on \( \text{conv}\{(0, 0), (2b + 2 - a, 0), (2a + 2 - b, 3), (0, 3)\} \). Then we can immediately lift to \( \mathcal{O}_K[x, y] \). The tables below give point counting timings and memory usage for randomly chosen such polynomials in genera \( g = 6, 7 \), where for the sake of conciseness we restrict to the generic Maroni invariants \( a = \lfloor (g - 2)/2 \rfloor \) and \( b = \lceil (g - 2)/2 \rceil \); the other Maroni invariants give rise to faster point counts.

\[
\begin{array}{c|c|c|c|c|c|c|c}
\hline
\text{\( p \)} & \text{time (s)} & \text{space (Mb)} & \text{\( q \)} & \text{time (s)} & \text{space (Mb)} & \text{\( q \)} & \text{time (s)} & \text{space (Mb)} \\
\hline
11 & 0.9 & 32 & 3^3^1 & 33 & 7^6 & 183 & 188 \\
67 & 6.0 & 32 & 7^5 & 64 & 80 & 7^10 & 503 & 320 \\
521 & 70 & 118 & 7^5 & 176 & 197 & 17^10 & 1490 & 749 \\
4099 & 769 & 824 & 37^5 & 415 & 371 & 37^10 & 3970 & 1663 \\
32771 & 8865 & 6829 & 79^6 & 1035 & 791 & 79^10 & 10945 & 3716 \\
\hline
\end{array}
\]

\[
\begin{array}{c|c|c|c|c|c|c|c}
\hline
\text{\( p \)} & \text{time (s)} & \text{space (Mb)} & \text{\( q \)} & \text{time (s)} & \text{space (Mb)} & \text{\( q \)} & \text{time (s)} & \text{space (Mb)} \\
\hline
11 & 1.5 & 42 & 3^7 & 45 & 76 & 3^10 & 283 & 197 \\
67 & 6.5 & 32 & 7^5 & 91 & 118 & 7^10 & 777 & 371 \\
521 & 88 & 118 & 7^5 & 257 & 241 & 17^10 & 2384 & 919 \\
4099 & 955 & 857 & 37^5 & 602 & 460 & 37^10 & 6706 & 2212 \\
32771 & 13279 & 6983 & 79^6 & 1561 & 983 & 79^10 & 18321 & 4682 \\
\hline
\end{array}
\]

**Smooth plane quintics** We end this section with a brief discussion of the genus 6 case where our canonical curve \( \overline{C} \subset \mathbb{P}^5 \) is \( \mathbb{F}_q \)-isomorphic to a smooth plane quintic. Such curves are never trigonal: using a variant of Lemma 7 one verifies that the \( \mathbb{F}_q \)-gonality is 4 if and only if \( \# C(\mathbb{F}_q) > 0 \), which is guaranteed if \( q > 137 \) by the Serre-Weil bound. In the other cases it is 5. Nevertheless from the point of view of the canonical embedding, smooth plane quintics behave ‘as if they were trigonal’, which is why we include them here. (The appropriate unifying statement reads that trigonal curves and smooth plane quintics are exactly the curves having Clifford index 1.) Here our main task towards tackling Problem 1 is to find a linear change of variables transforming the space \( \mathbb{I}_2(\overline{C}) \)
whose zero locus is the Veronese surface in ‘standard form’, i.e. the closure of the image of
\[ T^2 \hookrightarrow \mathbb{P}^5 : (x, y) \mapsto (x^2 : xy : x : y^2 : y : 1). \]
In order to achieve this, we simply assume that we have a function \texttt{ConvertVeronese} at our disposal. One could again try to use Schicho’s function \texttt{ParametrizeScroll} for this, but here too we expect problems because of the characteristic being finite (although we did not carry out the experiment). Once this standard form is attained, an easy substitution
\[ X_1 \leftarrow x^2, \ X_2 \leftarrow xy, \ X_3 \leftarrow x, \ X_4 \leftarrow y^2, \ X_5 \leftarrow y, \ X_6 \leftarrow 1 \]
makes the quadrics vanish identically, while the cubics have a gcd whose homogenization defines the desired smooth plane quintic. From here one proceeds as in the smooth plane quartic case described in Section 3.1.1.

4.2 Tetragonal curves

We conclude this article with some thoughts on how the foregoing material can be adapted to the tetragonal case. A full elaboration of the steps below (or even a rigorous verification of some corresponding claims) lies beyond our current scope. In particular we have not implemented anything of what follows. The main aim of this section is twofold: to illustrate how our treatment of non-trigonal curves of genus five from Section 3.3.1 naturally fits within a larger framework, and to propose a track for future research, involving mathematics that was developed mainly by Schreyer in [46, §6] and Schicho, Schreyer and Weimann in [44, §5].

Let \( \mathcal{C} \subset \mathbb{P}^{g-1} = \text{Proj} \overline{R} \), \( \overline{R} = \mathbb{F}_q[X_1, X_2, \ldots, X_g] \) be the canonical model of a genus \( g \geq 5 \) curve that is non-hyperelliptic, non-trigonal, and not isomorphic to a smooth plane quintic, so that a minimal set of generators of \( \mathcal{I}(\mathcal{C}) \subset \overline{R} \) consists of \( \beta_{12} := (g-2)(g-3)/2 \) quadrics
\[ \mathcal{S}_{2,1}, \mathcal{S}_{2,2}, \ldots, \mathcal{S}_{2,\beta_{12}}. \]
The notation \( \beta_{12} \) refers to the corresponding entry in the graded Betti table of the homogeneous coordinate ring of \( \mathcal{C} \), to which we will make a brief reference at the end of this section. Assume that the \( \mathbb{F}_q \)-gonality of \( \mathcal{C} \) is four, and consider a corresponding \( \mathbb{F}_q \)-rational map \( \pi : \mathcal{C} \to \mathbb{P}^1 \). We note that unlike the trigonal case this map may not be uniquely determined modulo automorphisms of \( \mathbb{P}^1 \), even for \( g \) arbitrarily large. The linear spans of the fibers of \( \pi \) form a one-dimensional family of planes in \( \mathbb{P}^{g-1} \) that cut out a rational normal threefold scroll \( \mathcal{S} \). Similar to before, up to a linear change of variables, such a scroll is obtained by simultaneously parameterizing
• a rational normal curve of degree $a$ in the $\mathbb{P}^a$ corresponding to $X_1, X_2, \ldots, X_{a+1}$,
• a rational normal curve of degree $b$ in the $\mathbb{P}^b$ corresponding to $X'_1, X'_2, \ldots, X'_{b+1}$, where $X'_i$ denotes the variable $X_{a+1+i}$, and
• a rational normal curve of degree $c$ in the $\mathbb{P}^c$ corresponding to $X''_1, X''_2, \ldots, X''_{c+1}$, where $X''_i$ denotes the variable $X_{a+b+2+i}$,

each time taking the plane connecting the points under consideration (each of these planes intersects our trigonal curve in four points, counting multiplicities). Again this concerns a determinantal variety, defined by the $2 \times 2$ minors of
\[
\begin{pmatrix}
X_1 & X_2 & \cdots & X_a & X'_1 & X'_2 & \cdots & X'_b \\
X_2 & X_3 & \cdots & X_{a+1} & X'_2 & X'_3 & \cdots & X'_{b+1} \\
\end{pmatrix}
\]
(8)

Alternatively our scroll can be thought of as the Zariski closure of the image of
\[
\mathbb{T}^3 \hookrightarrow \mathbb{P}^{g-1} : (x, y, z) \mapsto (z : xz : \cdots : x^az : y : \cdots : x^by : 1 : x : \cdots : x^c),
\]
or if one prefers, as the toric threefold associated to the polytope
\[
\begin{array}{c}
(0, 1, 0) \\
(0, 0, 0) \\
(0, 0, 1) \\
(a, 0, 1) \\
(b, 1, 0) \\
(c, 0, 0)
\end{array}

\] 

Let us denote this ‘standard’ scroll in $\mathbb{P}^{g-1}$ by $S(a, b, c)$. The non-negative integers $(a, b, c)$ are called the scrollar invariants of $C$ with respect to $\pi$ and can be chosen to satisfy
\[
a \leq b \leq c, \quad a + b + c = g - 3, \quad c \leq (2g - 2)/4,
\]
where the last inequality follows from Riemann-Roch.

Inside the scroll $S$ our curve $C$ arises as a complete intersection of two hypersurfaces $Y$ and $Z$ that are ‘quadratic’. More precisely the Picard group of $S$ is generated by the class $[H]$ of a hyperplane section and the class $[\Pi]$ of a ruling (i.e. of the linear span of a fiber of $\pi$), and $Y$ and $Z$ can be chosen such that
\[
Y \in 2[H] - b_1[\Pi], \quad Z \in 2[H] - b_2[\Pi]
\]
for non-negative integers $b_1 \geq b_2$ satisfying $b_1 + b_2 = g - 5$. These integers are invariants of the curve, that is, they do not depend on the choice of $\pi$. If $b_2 < b_1$ then also the surface $Y$ is uniquely determined by $C$. This is automatic when $g$ is even.

Let us now assume that $S$ is given in the standard form $S(a, b, c)$, which we consider along with the embedded torus $\mathbb{T}^3$. Then for $Y$ to be in the class $2[H] - b_1[\Pi]$ it means that $Y \cap \mathbb{T}^3$ is defined by an irreducible polynomial $f_Y \in \mathbb{F}_q[x, y, z]$ whose support is contained in

37
or more precisely\(^2\) in
\[
\text{conv}\{(0, 0, 0), (2c - b_1, 0, 0), (0, 2, 0), (2b - b_1, 2, 0), (0, 0, 2), (2a - b_1, 0, 2)\} \cap \mathbb{R}^3_{\geq 0}.
\]
In other words this is the polytope obtained from \(2\Delta_{(a,b,c)}\) by shifting its right-most face leftwards over a distance \(b_1\). Moreover \(b_1\) is the maximal integer for which this containment holds. The same applies to \(Z\), leading to a polynomial \(f_Z \in \mathcal{O}_K[x,y,z]\) whose support is contained in \(\Delta_{(a,b,c),b_2}\), which is the polytope obtained from \(2\Delta_{a,b,c}\) by shifting the right-most face inwards over a distance \(b_2\).

The main observation of this section is that \(f_Y, f_Z \in \mathcal{O}_K[x,y,z]\) is a pair of polynomials meeting a version of Baker’s bound for complete intersections, again due to Khovanovskii \cite{khovanovskii2007}. In the case of two trivariate polynomials supported on polytopes \(\Delta_1\) and \(\Delta_2\) the bound reads
\[
g \leq \# \text{(interior points of } \Delta_1 + \Delta_2) - \# \text{(interior points of } \Delta_1) - \# \text{(interior points of } \Delta_2) - 2.
\]
In our case where \(\Delta_1 = \Delta_{(a,b,c),b_1}\) and \(\Delta_2 = \Delta_{(a,b,c),b_2}\), this indeed evaluates to \(g - 0 - 0 = g\). Thus the strategy would be similar: lift these polynomials in a Newton polytope preserving way to polynomials \(f_Y, f_Z \in \mathcal{O}_K[x,y,z]\), and a polynomial \(f \in \mathcal{O}_K[x,y]\) satisfying (i)-(iii) can be found by taking the resultant of \(f_Y\) and \(f_Z\) with respect to \(z\) (or with respect to \(y\)).

Genus 5 curves revisited  Let us revisit our treatment of tetragonal curves of genus five \(C \subset \mathbb{P}^4 = \text{Proj} \mathbb{F}_q[X,Y,Z,W,V]\) from Section 3.3.1.

1. Our first step was to look for a point \(P \in D(C)(\mathbb{F}_q)\) for which \(\chi(P) = 0\) or \(\chi(P) = 1\). The corresponding quadrics were described as cones over \(\mathbb{P}(1,2,1)\) and \(\mathbb{P}^1 \times \mathbb{P}^1\), respectively. But in the current language these are just rational normal threefold scrolls of type \((0,0,2)\) resp. \((0,1,1)\). Note that this shows that the scroll \(S\) may indeed depend on the choice of \(\pi\).

2. For ease of exposition let us restrict to the case \(\chi(P) = 1\). Then the second step was to transform the quadric into \(XY = ZW\), whose zero locus is the Zariski closure of \(T^3 \hookrightarrow \mathbb{P}^4: (x,y,z) \mapsto (1 : xy : x : y : z)\), i.e. the transformation takes the scroll \(S(0,1,1)\) into ‘standard form’.

\(^2\)Indeed, the coordinate \(2a - b_1\) might be negative; an example of such behaviour can be found in an arXiv version of this paper (1605.02162v2).
3. The other quadrics $S_2, S_2'$ are instances of the surfaces $Y$ and $Z$. They are both in the class $2[H]$, i.e. $b_1 = b_2 = 0$. Viewing $Y$ and $Z$ inside the torus $T^3$ amounts to evaluating them at $(1, xy, x, y, z)$, resulting in polynomials that are supported on

$$
\begin{align*}
(0, 2, 0) & \\
(0, 0, 0) & \\
(2, 2, 0) & \\
(2, 0, 0) & \\
(0, 0, 2) &
\end{align*}
$$

as predicted. With the present approach we naively lift these polynomials to $f_Y, f_Z \in \mathcal{O}_K[x, y, z]$. In Section 3.3.1 we applied this naive lift directly to $S_2, S_2'$, which was fine there, but in higher genus it is more convenient to work in $T^3$, since $Y, Z \subset S$ will no longer be cut out by a single quadratic hypersurface of $\mathbb{P}^{g-1}$.

4. The last step was to project this lifted curve from $(0 : 0 : 0 : 0 : 1)$, which in our case amounts to taking the resultant of $f_Y, f_Z$ with respect to $z$.

**General recipe**  If we want to turn the above into a rigorous recipe for lifting tetragonal curves, three questions show up naturally. We share some brief first thoughts, but further research is needed regarding each of these.

1. How do we decide whether the input curve has $F_q$-gonality 4 or not, and how do we extract from $I_2(C)$ the equations of a corresponding rational normal threefold scroll $S$?

In genus five we used the discriminant curve for this, but in general the desired information should be traceable from (the first few steps of) a minimal free resolution

$$
\mathcal{R}(-4)^{\beta_{24}} \oplus \mathcal{R}(-5)^{\beta_{35}} \to \mathcal{R}(-3)^{\beta_{23}} \oplus \mathcal{R}(-4)^{\beta_{24}} \to \mathcal{R}(-2)^{\beta_{12}} \to \mathcal{R} \to \mathcal{R}/(S_2, \ldots, S_2, \beta_{12})
$$

of the homogeneous coordinate ring of $C$ as a graded $\mathcal{R}$-module, thanks to a proven part of Green’s canonical syzygy conjecture [44, Thm. 2.5], namely that $\beta_{24} \neq 0$ if and only if $C$ is $F_q$-tetragonal or $F_q$-isomorphic to a smooth plane sextic, which in turn holds if and only if $C$ has Clifford index 2. (The dimensions $\beta_{ij}$ are usually gathered in the so-called graded Betti table of $C$, and in general Green’s conjecture predicts that the Clifford index equals the number of leading zeroes on the cubic strand, i.e. the minimal $i$ for which $\beta_{i,i+2} \neq 0$.)

If $g \geq 7$ then a sufficiently generic geometrically tetragonal curve satisfies $\beta_{24} = g-4$. This is what Schicho, Schreyer and Weimann [44, Ex. 4.2] refer to as the *goneric* case; see also [23, Thm. 0.3]†. It implies that our curve admits a unique $g^1_4$, hence it is $F_q$-tetragonal, and that the ideal of the corresponding scroll $S$ can be computed as the annihilator of the cokernel of the map

$$
\mathcal{R}(-5)^{\beta_{35}} \to \mathcal{R}(-4)^{\beta_{24}}.
$$

39
See [44, Prop. 4.11].

In the non-goneric cases one has $\beta_{24} = (g-1)(g-4)/2$ and a finer analysis is needed. Some further useful statements can be found in [44] and [29]†.

2. How do we find the type $(a, b, c)$ of the scroll $\mathcal{S}$, along with a linear change of variables taking it into the standard form $\mathcal{S}(a, b, c)$ cut out by the minors of (8)? We encountered an analogous hurdle in the trigonal case. Here too it would be natural to try the Lie algebra method from [18], but as mentioned this was designed to work over fields of characteristic zero, and it is not clear to us how easily the method carries over to small finite characteristic.

3. How do we find the invariants $b_1, b_2$ along with hypersurfaces $\mathcal{Y} \in \mathbb{F}_q[x, y, z]$ and $\mathcal{Z} \in \mathbb{F}_q[x, y, z]$ that inside $\mathcal{S}(a, b, c)$ cut out our curve $C$?

By evaluating the generators of $I(C)$ in $(z, xz, \ldots, x^a z, y, xy, \ldots, x^b y, 1, x, \ldots, x^c)$ one easily finds a set of generators for the ideal of $\mathcal{C} \cap \mathbb{T}^3$. The challenge is now to replace this set by two polynomials that are supported on polytopes of the form

$$\Delta_{(a,b,c),b_1} \text{ and } \Delta_{(a,b,c),b_2},$$

with $b_1, b_2$ satisfying $b_1 + b_2 = g - 5$. Here our approach would be to use a Euclidean type of algorithm to find generators whose Newton polytopes are as small as possible.

**Point counting timings** We have not implemented anything of the foregoing recipe, but we can predict how its output should perform in composition with the point counting algorithm from [53, 54], by simply starting from a sufficiently generic pair of polynomials $f_\mathcal{Y}, f_\mathcal{Z} \in \mathbb{F}_q[x, y, z]$ that are supported on $\Delta_{(a,b,c),b_1}$ and $\Delta_{(a,b,c),b_2}$ for non-negative integers $a, b, c$ satisfying (9) and $b_1 + b_2 = g - 5$. Then one can naively lift to $\mathcal{O}_K[x, y, z]$, take the resultant with respect to $z$, make the outcome monic using (1), and feed the result to the point counting algorithm. The tables below contain point counting timings and memory usage for randomly chosen such pairs in genera $g = 6, 7$. For the sake of conciseness it makes sense to restrict to the case where the scrollar invariants $a, b, c$ and the tetragonal invariants $b_1, b_2$ are as balanced as possible, meaning that $c - a \leq 1$ and $b_1 - b_2 \leq 1$, because this is the generic case [3, 6]†. We expect the other cases to run faster.

### $g = 6$

| $p$ | time (Mb) | space (Mb) | $q$ | time (Mb) | space (Mb) |
|-----|-----------|------------|-----|-----------|------------|
| 11  | 8.5       | 32         | $7^2$| 266       | 214        |
| 67  | 34.7      | 64         | $7^2$| 549       | 325        |
| 521 | 445       | 379        | $3^{10}$| 2750 | 6072       |
| 4099| 4748      | 2504       | $7^{10}$| 6407 | 9814       |

### $g = 7$

| $p$ | time (Mb) | space (Mb) | $q$ | time (Mb) | space (Mb) |
|-----|-----------|------------|-----|-----------|------------|
| 11  | 11        | 32         | $7^2$| 294       | 156        |
| 67  | 46        | 80         | $7^2$| 550       | 241        |
| 521 | 445       | 347        | $3^{10}$| 2347 | 3606       |
| 4099| 4350      | 2441       | $7^{10}$| 5819 | 5724       |

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