BDG inequalities for model-free continuous price paths with instant enforcement

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Abstract

Shafer and Vovk introduce in their book \cite{6} the notion of instant enforcement and instantly blockable properties. In this paper we introduce an outer measure on the space of continuous paths which assigns zero value exactly to those sets (properties) of pairs of time \( t \) and elementary event \( \omega \) which are instantly blockable. Next, for the introduced measure we prove Itô’s isometry and BDG inequalities, and use them to define Itô-type integral. Additionally, we prove few properties for the quadratic variation of model-free continuous paths which hold with instant enforcement.

1 Introduction

Since the last subprime mortgage financial crisis there is growing interest in the robust financial models, usually models with minimal, widely accepted non-arbitrage assumptions. Such assumptions together with game-theoretic considerations allow to establish properties which characterise trajectories of prices of financial assets which exclude possibility of arbitrage. In a series of papers, among others in \cite{7, 8, 10, 9, 11}, Vovk introduced and considered outer measures on the spaces of continuous or more general, càdlàg trajectories, which assign zero value to the sets of trajectories of prices of financial assets, which allow for arbitrage. "Typical" (not leading to arbitrage) trajectories possess quadratic variation and model-free, Itô-type integration with respect to such trajectories may be established \cite{5, 12, 3}.

The investigations in game-theoretic approach to model-free, financial models of continuous price paths culminated in publishing by Glenn Shafer and Vladimir Vovk their book \cite{6}. In their book Shafer and Vovk introduce the notion of instant enforcement but they do not characterise it using any outer measure. Informally, property \( E \) is instantly enforceable if there exists a trading strategy making a trader using this strategy infinitely rich as soon as the property \( E \) ceases to hold. In this paper we introduce an outer measure on the space of continuous paths, which assigns zero value exactly to those sets (properties) of pairs of time \( t \) and elementary event \( \omega \), complements of which are instantly enforceable. Second, we establish Itô’s isometry and BDG inequalities for this measure. Third, using the obtained BDG inequalities, we define Itô-type integral, which allows to integrate more general (not necessarily continuous) integrands than those considered in \cite{6}. Finally, we present sequence of processes, which do not depend on any partitions, which tend locally uniformly and with instant enforcement to the quadratic variations of continuous paths.

1.1 Definitions and notation

Now we outline a general setting in which we will work and which follows closely \cite{6} Chapt. 14]. For simplicity, we consider only finite families of basic martingales. We will work with a martingale space which is a quintuple

\[
\left( \Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0}, J = \{1, 2, \ldots, d\}, \{S^j, j \in J\} \right)
\]

of the following objects: \( \Omega \) is a space of possible outcomes of reality, \( \mathcal{F} \) is a \( \sigma \)-field of the subsets of \( \Omega \) which we call events, \( \mathbb{F} = (\mathcal{F}_t)_{t \geq 0} \) is a filtration and \( \{S^j, j \in J\} = \{S^1, S^2, \ldots, S^d\} \) is a family of basic continuous martingales, that is for any \( t \in [0, +\infty) \) and \( j \in J \), \( S^j_t \) is a \( (\mathcal{F}_t, \mathcal{B}(\mathbb{R})) \)-measurable real variable \( S^j_t : \Omega \rightarrow \mathbb{R} \).
such that for each $\omega \in \Omega$ the trajectory $[0, +\infty) \ni t \mapsto S^j_t(\omega)$ is continuous ($\mathcal{B}(\mathbb{R})$ denotes the $\sigma$-field of Borel subsets of $\mathbb{R}$). Throughout the paper the filtration $\mathbb{F}$ is fixed, moreover, we assume that $\mathcal{F}_0$ is trivial, $\mathcal{F}_0 = \{\emptyset, \Omega\}$, thus all $(\mathcal{F}_0, \mathcal{B}(\mathbb{R}))$-measurable variables $S^j_0$, $j \in J$, are deterministic.

A real process $X : [0, +\infty) \times \Omega \to \mathbb{R}$ is a collection of real variables $X_t : \Omega \to \mathbb{R}$, $t \in [0, +\infty)$, such that $X_t$ is $(\mathcal{F}_t, \mathcal{B}(\mathbb{R}))$-measurable, thus all processes which we consider are adapted to $\mathbb{F}$.

A process $Y : [0, +\infty) \times \Omega \to \mathbb{R} \cup \{-\infty, +\infty\} = [-\infty, +\infty]$, is a collection of extended variables $Y_t : \Omega \to [-\infty, +\infty]$, $t \in [0, +\infty)$, such that $Y_t$ is $(\mathcal{F}_t, \mathcal{B}([-\infty, +\infty]))$-measurable (any set in $\mathcal{B}([-\infty, +\infty])$ is of the form $A$, $A \cup \{-\infty\}$, $A \cup \{+\infty\}$ or $A \cup \{-\infty, +\infty\}$, where $A \in \mathcal{B}(\mathbb{R})$).

A generalized process is any function $Y : [0, +\infty) \times \Omega \to [-\infty, +\infty]$. For any generalized process $Y$ we define its supremum process $Y^*$, which is a generalized process defined as

$$Y^*_t(\omega) := \sup_{0 \leq s \leq t} |Y_s(\omega)|,$$

where we denote $Y_t(\omega) := Y(t, \omega)$.

A generalized process $Y$ is globally bounded if

$$\sup_{(t, \omega) \in [0, +\infty) \times \Omega} |Y_t(\omega)| = \sup_{(t, \omega) \in [0, +\infty) \times \Omega} Y^*_t(\omega) < +\infty.$$ 

A real random variable $X : \Omega \to \mathbb{R}$ is globally bounded if $\sup_{\omega \in \Omega} |X(\omega)| < +\infty$.

Throughout the whole paper we apply the following convention. A sequence of real numbers $d_n$, where $n = 0, 1, 2, \ldots$, is denoted by $(d_n)$ or $(d_n)_n$ and a sequence of real numbers $d^n$, where $n = 0, 1, 2, \ldots$, is denoted by $(d^n)$ or $(d^n)_n$ (without indication that $n$ ranges over the set of nonnegative integers $\mathbb{N}$). A similar convention will be applied to infinite sequences of stopping times, variables etc.

Since almost all reasonings in this article are pathwise, we will often omit the argument $\omega \in \Omega$ in formulas even if the quantities appearing in these formulas depend on it.

Now let us introduce sequences of stopping times which we will work with.

A sequence of $\mathbb{F}$-stopping times $(\tau_n)$ is called non-decreasing if for all $n \in \mathbb{N}$ and each $\omega \in \Omega$, $\tau_{n+1}(\omega) \geq \tau_n(\omega)$.

A sequence of $\mathbb{F}$-stopping times $(\tau_n)$ is called proper if it is non-decreasing, $\tau_0 \equiv 0$ and for each $\omega \in \Omega$ the sequence $(\tau_n(\omega))$ is divergent to $+\infty$ or there exists some $n \in \mathbb{N}$ such that $\tau_n(\omega) = \tau_{n+1}(\omega) = \ldots = 0, +\infty$.

A simple trading strategy is a triplet $G = (c, (\tau_n), (g_n))$ which consists of the initial capital $c \in \mathbb{R}$, a proper sequence of $\mathbb{F}$-stopping times $(\tau_n)$ and a sequence of $(\mathcal{F}_{\tau_n}, \mathcal{B}(\mathbb{R}))$-measurable, globally bounded real variables $g_n : \Omega \to \mathbb{R}$, $n = 0, 1, \ldots$, such that $g_n(\omega) = 0$ whenever $\tau_n(\omega) = +\infty$.

A step process $G$ is a real process which may be represented as

$$G_t(\omega) = \sum_{n=1}^{+\infty} g_{n-1}(\omega) \mathbb{1}_{[\tau_{n-1}(\omega), \tau_n(\omega))}(t)$$

where $G = (c, (\tau_n), (g_n))$ is a simple trading strategy.

For a real process $X : [0, +\infty) \times \Omega \to \mathbb{R}$ and a simple trading strategy $G = (c, (\tau_n), (g_n))$ we define

$$(G \cdot X)_t(\omega) := c + \sum_{n=1}^{+\infty} g_{n-1}(\omega) \left( X_{\tau_n(\omega) \wedge t} - X_{\tau_{n-1}(\omega) \wedge t} \right).$$

(For two numbers $a, b \in [-\infty, +\infty]$ we define $a \wedge b = \min\{a, b\}$.) Let us note that since the sequence $(\tau_n)$ is proper, there is only finite number of non-zero summands in the sum $\sum_{n=1}^{+\infty} g_{n-1}(\omega) \left( X_{\tau_n(\omega) \wedge t} - X_{\tau_{n-1}(\omega) \wedge t} \right)$ appearing in the definition of $(G \cdot X)_t(\omega)$.

We define the simple capital process or simple integral corresponding to the vector $G = (G^j)_{j \in J}$ of simple trading strategies $G^j$, $j \in J$, as

$$(G \cdot S)_t(\omega) := \sum_{j \in J} (G^j \cdot S^j)_t(\omega).$$

The simple capital process has a very natural interpretation – it is the capital accumulated till time $t$ by the application of the simple trading strategy $G^j$ to the asset whose price is represented by the basic martingale $S^j$, $j \in J$. 

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Remark 1.1. If \( G = (c, (\tau_n), (g_n)) \) is a trading strategy and for some \( \omega \in \Omega \) and \( n \in \mathbb{N} \), \( \tau_n(\omega) = \tau_{n+1}(\omega) = \ldots \in [0, +\infty) \) then the process \( G \cdot X \) for the strategy \( G \) and a real process \( X \), is the same for \( t \geq \tau_n(\omega) \) as if the trading was ceased at \( \tau_n(\omega) \), even though \( g_n(\omega) \neq 0 \). Thus for all trading strategies we could add requirement that \( g_n(\omega) = 0 \) if \( \tau_{n+1}(\omega) = \tau_n(\omega) \) for some \( n \in \mathbb{N} \) (it is possible to verify this condition at the moment \( \tau_n(\omega) \)), or for a given trading strategy always modify it so it satisfies this condition.

Requirement on the sequence \( (\tau^n) \) in the definition of a simple trading strategy to be proper together with the condition \( g_n(\omega) = 0 \) if \( \tau_{n+1}(\omega) = \tau_n(\omega) \) for some \( n \in \mathbb{N} \) guarantees that the trading never occurs with infinite frequency till any finite time; but to avoid dealing with too many technical details we do not add this requirement.

2 Nonnegative supermartingales, instantly enforceable properties, outer measure of properties related to the instant enforcement, martingales

The class \( \mathcal{C} \) of nonnegative supermartingales is defined as the smallest \( \lim \inf \)-closed class containing all simple capital processes which are non-negative, that is \( \mathcal{C} \) contains all nonnegative simple capital processes and for any sequence \( (X^n) \) such that \( X^n \in \mathcal{C} \) for \( n \in \mathbb{N} \), we have that \( X := \lim \inf_{n \to +\infty} X^n \) also belongs to \( \mathcal{C} \).

In [6, Sect. 14.1] there is defined a notion of instant enforcement of a subset \( E \subseteq [0, +\infty) \times \Omega \) (also called a property of \( t \) and \( \omega \)). Informally, property \( E \) is instantly enforceable if there exists a trading strategy making a trader using this strategy infinitely rich as soon as the property \( E \) ceases to hold. A formal definition is the followong: a property \( E \subseteq [0, +\infty) \times \Omega \) is instantly enforceable, or holds with instant enforcement, w.r.t. in short, if there exists a nonnegative supermartingale \( X \) such that \( X_0 = 1 \) and

\[
(t, \omega) \notin E \implies X_t(\omega) = +\infty.
\]

Complements of instantly enforceable properties (sets) are called instantly blockable.

The main result of this section is that it is possible to introduce an outer measure \( \overline{\mathbb{P}} \) on the subsets of \( [0, +\infty) \times \Omega \) (similarly as in the case of the notion of null events, where one can introduce Vovk’s outer probability, cf. [7]) such that \( B \subseteq [0, +\infty) \times \Omega \) is instantly blockable iff \( \overline{\mathbb{P}}(B) = 0 \).

For \( A \subseteq [0, +\infty) \times \Omega \) we define

\[
\overline{\mathbb{P}}(A) := \inf \{ \lambda \in \mathbb{R} : \exists X \in \mathcal{C} \text{ such that } \forall (t, \omega) \in [0, +\infty) \times \Omega, X_0(\omega) \leq \lambda \\
\text{and } X_t(\omega) \geq 1_A(t, \omega) \}.
\]

We have the following lemma.

**Lemma 2.1.** The set \( B \subseteq [0, +\infty) \times \Omega \) is instantly blockable iff

\[
\overline{\mathbb{P}}(B) = 0.
\]

**Proof.** If \( B \) is instantly blockable then there exists \( X \in \mathcal{C} \) such that \( X_0 = 1 \) and

\[
(t, \omega) \in B \implies X_t(\omega) = +\infty.
\]

Thus, taking arbitrary \( \varepsilon > 0 \) we have \( (\varepsilon X)_0 = \varepsilon \) and

\[
(t, \omega) \in B \implies (\varepsilon X)_t(\omega) = +\infty > 1_B(t, \omega);
\]

\[
(t, \omega) \notin B \implies (\varepsilon X)_t(\omega) \geq 0 = 1_B(t, \omega);
\]

and since \( \varepsilon X \in \mathcal{C} \) we get

\[
\overline{\mathbb{P}}(B) \leq \varepsilon.
\]

Since \( \varepsilon \) may be as close to 0 as we wish, \( \overline{\mathbb{P}}(B) \leq 0 \) and thus \( \overline{\mathbb{P}}(B) = 0 \) (the opposite inequality \( \overline{\mathbb{P}}(B) \geq 0 \) holds since for any \( X \in \mathcal{C} \), \( X_0 \geq 0 \)).
Assume now that \( \mathbb{P}(B) = 0 \). For \( n = 1, 2, \ldots \), there exists \( X^n \in \mathcal{C} \) such that \( X^n_0(\omega) \leq 2^{-n} \) and for all \( (t, \omega) \in [0, +\infty) \times \Omega \),

\[
(t, \omega) \in B \implies X^n_t(\omega) \geq 1_B(t, \omega) = 1.
\]

Taking \( X = \left(1 - \sum_{n=1}^{+\infty} X^n_0\right) + \sum_{n=1}^{+\infty} X^n \) we get \( X \in \mathcal{C} \), \( X_0 = 1 \) (notice that \( \sum_{n=1}^{+\infty} X^n_0 \leq 1 \)) and

\[
(t, \omega) \in B \implies X_t(\omega) \geq \sum_{n=1}^{+\infty} X^n_t(\omega) \geq \sum_{n=1}^{+\infty} 1 = +\infty.
\]

\[\square\]

Hence, since \( X \in \mathcal{C} \),

\[
\mathbb{P}(X) = \inf \{ \lambda \in \mathbb{R} : \exists X \in \mathcal{C} \text{ such that } \forall (t, \omega) \in [0, +\infty) \times \Omega, X_0(\omega) \leq \lambda \text{ and } X_t(\omega) \geq Y_t(\omega) \}.
\]

For \( A \subseteq [0, +\infty) \times \Omega \) we have

\[
\mathbb{P}(A) = \mathbb{P}{1_A}.
\]

For two generalised processes \( X \) and \( Y \) we say that \( X \) dominates \( Y \) if they satisfy the condition

\[
\forall (t, \omega) \in [0, +\infty) \times \Omega, \quad X_t(\omega) \geq Y_t(\omega).
\]

For two generalised processes \( X \) and \( Y \) we say that \( X \) dominates \( Y \) with instant enforcement (w.i.e.) if the set of \( (t, \omega) \) where the inequality \( X_t(\omega) \geq Y_t(\omega) \) holds is instantly enforceable.

Below we list properties of \( \mathbb{E} \) which imply that \( \mathbb{P} \) is an outer measure:

1) non-negativity: for any generalised process \( Y \), \( \mathbb{E}Y \geq 0 \);

2) monotonicity with respect to domination of generalised processes w.i.e.:

if \( Z \) dominates \( Y \) w.i.e. and \( \forall (t, \omega) \in [0, +\infty) \times \Omega, Z_t(\omega) > -\infty \) then

\[
\mathbb{E}Y \leq \mathbb{E}Z;
\]

3) positive homogeneity for non-negative generalised processes:

if \( \forall (t, \omega) \in [0, +\infty) \times \Omega, Y_t(\omega) \geq 0 \) and \( \alpha \geq 0 \) then

\[
\mathbb{E}(\alpha Y) = \alpha \mathbb{E}Y,
\]

where we apply the convention that \( 0 \cdot (+\infty) = 0 \);

4) countable subadditivity for non-negative generalised processes:

if \( \forall (t, \omega) \in [0, +\infty) \times \Omega, Y^1_t(\omega), Y^2_t(\omega), \ldots \geq 0 \) then

\[
\mathbb{E}\left(\sum_{k=1}^{+\infty} Y^k\right) \leq \sum_{k=1}^{+\infty} \mathbb{E}Y^k.
\]

Also, almost immediate consequence of the definition of \( \mathbb{E} \) is the Fatou lemma.

**Fact 2.2** (Fatou’s lemma). If \( \{Y^n\} \) is a sequence of generalised processes then

\[
\mathbb{E}\liminf_{n \to +\infty} Y^n \leq \liminf_{n \to +\infty} \mathbb{E}Y^n.
\]

*Proof.* Let \( X^n \in \mathcal{C}, n = 1, 2, \ldots \), be such that \( X^n_0 \leq \mathbb{E}Y^n + 1/n \) and \( \forall (t, \omega) \in [0, +\infty) \times \Omega, X^n_t(\omega) \geq Y^n_t(\omega) \). Denote \( X = \liminf_{n \to +\infty} X^n \) then

\[
\forall (t, \omega) \in [0, +\infty) \times \Omega \quad X_t(\omega) = \liminf_{n \to +\infty} X^n_t(\omega) \geq \liminf_{n \to +\infty} Y^n_t(\omega).
\]

Hence, since \( X \in \mathcal{C} \),

\[
\mathbb{E}\liminf_{n \to +\infty} Y^n \leq X_0 = \liminf_{n \to +\infty} X^n_0 \leq \liminf_{n \to +\infty} \mathbb{E}Y^n.
\]

\[\square\]
Next to the class of nonnegative supermartingales, other important class of processes which we will work
with is the family of martingales. The class of martingales \( \mathcal{M} \) is defined as the smallest \( \text{lim}\text{-}closed \) class of
real process such that it contains all simple capital processes. By the fact that \( \mathcal{M} \) is \( \text{lim}\text{-}closed \) we mean
that whenever \( X^n \in \mathcal{M} \), \( n \in \mathbb{N} \), and \( X \) is a real process such that for any \( (t, \omega) \in [0, +\infty) \times \Omega \),
\[
\lim_{n \to +\infty} \sup_{s \in [0, t]} \left| X_s(\omega) - X^n_s(\omega) \right| = 0 \text{ w.i.e.}
\]
then also \( X \in \mathcal{M} \). Condition (2.1) guarantees that the limit process \( X \) is continuous w.i.e.

If \( G \cdot S \) is a simple capital process, \( G = (G^j)_{j \in J}, G^j = (e^j, (\tau^j_1, \tau^j_2), (g^j_k)) \), \( j \in J \), and \( H = (d, (\sigma_n), (h_n)) \)
is another simple trading strategy then \( H \cdot (G \cdot S) \) is again a simple capital process. This may be seen by considering the sum
\[
(H \cdot (G \cdot S))_t = d + \sum_{j \in J} \sum_{m=1}^{+\infty} \tilde{h}_{m-1} \left( \sum_{j \in J} (G^j \cdot S^j)_{\tau_m \wedge t} - \sum_{j \in J} (G^j \cdot S^j)_{\tau_{m-1} \wedge t} \right),
\]

where \( (\tau_m) \) is non-decreasing rearrangement of stopping times from the sequences \( (\sigma_n) \) and \( (\tau^j_k) \), \( j \in J \), (with redundancies deleted) and for \( m \in \mathbb{N} \), \( \tilde{h}_m := h_n \), where \( n = \max \{ l \in \mathbb{N} : \sigma_l \leq \tau_m \} \). Defining for \( m \in \mathbb{N} \) and \( j \in J \), \( \tilde{g}^j_m := g^j_k \), where \( n = \max \{ l \in \mathbb{N} : \tau^j_l \leq \tau_m \} \), we further get
\[
(H \cdot (G \cdot S))_t = d + \sum_{j \in J} \sum_{m=1}^{+\infty} \tilde{h}_{m-1} \left\{ \sum_{k=1}^{+\infty} \tilde{g}^j_{k-1} \left( S^j_{\tau_k \wedge \tau_m \wedge t} - S^j_{\tau_{k-1} \wedge \tau_m \wedge t} \right) \right\},
\]

where the last equality follows from consideration of separate cases \( k > m, k < m \) and \( k = m \).

**Remark 2.3.** By transfinite induction it is possible to prove (see [6, Sect. 14.2]) that whenever \( X \) is a
martingale and \( G \) is a simple trading strategy then \( G \cdot X \) is again a martingale. We will use this in the
sequel.

**Remark 2.4.** To deal with the improper (or non-existent) limits of sequences of processes, Shafer and Vovk
introduce in [6, Sect. 14.1] also a ‘cemetary’ state \( \partial \), which may be attained by martingales since some
moment in time, but in this article we will deal with real-valued martingales only.

### 3 Simple quadratic variation - definition, Itô’s isometry and BDG inequalities

Let \( X \) be a martingale and \( \tau = (\tau_n) \) be a proper sequence of stopping times. We define the simple quadratic
variation process of \( X \) along \( \tau \) as
\[
[X]^\tau_t := \sum_{n=1}^{+\infty} \left( X_{\tau_n \wedge t} - X_{\tau_{n-1} \wedge t} \right)^2, \quad t \in [0, +\infty).
\]

**Lemma 3.1.** Let \( X \) be a martingale and \( \tau = (\tau_n) \) be a proper sequence of stopping times. The process
\[
Y_t := (X_t - X_0)^2 - [X]^\tau_t, \quad t \in [0, +\infty),
\]
is a martingale.
Proof. For $M \in (0, +\infty)$ let $\sigma(M) = \sigma(X, M)$ denote the stopping time defined as

$$\sigma(X, M) := \inf \{ t \in [0, +\infty) : |X_t| \geq M \}. \quad (3.1)$$

Now let us define the simple trading strategy $G^M = (0, (\tau_n), (g_n^M))$ with $g_0 : \mathbb{R} \to \mathbb{R}$, $g_0(\omega) := 0$ and $g_n : \mathbb{R} \to \mathbb{R}$,

$$g_n^M(\omega) := \begin{cases} 2(X_{\tau_n}(\omega) - X_0(\omega)) & \text{if } n \in \mathbb{N} \text{ and } \tau_n < \sigma(M), \\ 0 & \text{if } n \in \mathbb{N} \text{ and } \tau_n \geq \sigma(M). \end{cases}$$

All variables $g_n$, $n = 0, 1, 2, \ldots$, are bounded, thus $G^M$ is indeed a simple trading strategy. A direct calculation for any $t \in [0, +\infty)$ gives

$$Y_t^M := Y_{t \wedge \sigma(M)} = (G^M \cdot X)_t$$

and thus (recall Remark 2.3) $Y^M$ is a martingale. Moreover, for $t \in [0, +\infty)$ and $M > \sup_{s \in [0,t]} |X_s(\omega)|$ we have $Y_s^M(\omega) = Y_s(\omega)$ for $s \in [0, t]$ so

$$\lim_{M \to +\infty} \sup_{s \in [0,t]} |Y_s^M(\omega) - Y_s(\omega)| = 0$$

and hence $Y$ is a martingale. \qed

Fact 3.2 (Itô’s isometry for simple quadratic variation). Let $X$ be a martingale and $\tau = (\tau_n)$ be a proper sequence of stopping times. We have

$$\mathbb{E}(X - X_0)^2 = \mathbb{E}[X]^\tau.$$

Proof. Let $\varepsilon > 0$ and $Y$ be a non-negative supermartingale such that $Y_0 \leq \mathbb{E}(X - X_0)^2 + \varepsilon$ and $\forall (t, \omega) \in \mathbb{R} \times \Omega$, $Y_t(\omega) \geq (X_t(\omega) - X_0(\omega))^2$. We have

$$\forall (t, \omega) \in [0, +\infty) \times \Omega \quad Y_t(\omega) - (X_t(\omega) - X_0)^2 + |X|^\tau_t(\omega) \geq 0.$$

By Lemma 3.1 $(X - X_0)^2 - |X|^\tau$ is a martingale thus $Y - \left((X - X_0)^2 - |X|^\tau\right)$ is a non-negative supermartingale that dominates $|X|^\tau$ and the following estimates follow

$$\mathbb{E}[X]^\tau \leq Y_0 - \left((X_0 - X_0)^2 - |X|^\tau\right) = Y_0 \leq \mathbb{E}(X - X_0)^2 + \varepsilon.$$

Since $\varepsilon$ may be as close to 0 as we wish, we have

$$\mathbb{E}[X]^\tau \leq \mathbb{E}(X - X_0)^2.$$

The opposite inequality follows by similar reasoning – if $Y$ is a non-negative supermartingale that dominates $|X|^\tau$ and such that $Y_0 \leq \mathbb{E}[X]^\tau + \varepsilon$ then $Y - |X|^\tau + (X - X_0)^2$ is a non-negative supermartingale that dominates $(X - X_0)^2$. \qed

Remark 3.3. From the proof of Fact 3.2 we see that more general statement holds: if there are two non-negative, real processes $X$ and $Y$ whose difference is a martingale then

$$\mathbb{E}X = \mathbb{E}Y.$$

Now we proceed to the Burkholder-Davis-Gundy inequalities for the simple quadratic variation along some proper sequence of stopping times. As it is one of the main ingredients in the proof of the next fact, let us briefly recall the pathwise version of the Burkholder-Davis-Gundy inequalities (BDG inequalities in short) of Beiglooeck and Siorpaes [1]. Let $x_k$, $k \in \mathbb{N}$, be a sequence of real numbers and for $k \in \mathbb{N}$ define

$$x_k^* := \max_{l=0,1,\ldots,k} |x_l|, \quad [x]_k := x_0^2 + \sum_{l=1}^{k} (x_l - x_{l-1})^2.$$
Moreover, by (3.5), if
\[ (h \cdot x)_k = \sum_{l=1}^{k} h_{l-1} (x_l - x_{l-1}) \text{ with } h_l = \frac{x_l}{\sqrt{|x|_l + x_l}} \] (3.3)
and we apply the convention that \( \frac{0}{0} = 0 \). Inequalities (3.2) may be viewed as a pathwise version of the BDG inequalities for \( p = 1 \). To formulate a pathwise version of the BDG inequalities for \( p > 1 \), for \( k, l \in \mathbb{N}, k \geq l \), we introduce
\[ e_k^{(i)} := \frac{x_k - x_{l-1}}{\sqrt{|x|_k} - |x|_{l-1} + \max_{m \leq k} (x_m - x_{l-1})^2} \]
\[ f_k := p^2 \sum_{l=0}^{k} \left( \sqrt{|x|_{l}^{p-1}} - \sqrt{|x|_{l-1}^{p-1}} \right) e_k^{(i)}, \]
\[ g_k := p^2 \sum_{l=0}^{k} \left( (x_{l}^{p-1} - x_{l-1}^{p-1}) e_k^{(i)} \right) \] (3.4)
where together with the convention \( \frac{0}{0} = 0 \) we also use \( x_{-1} = x_{-1}^* = |x|_{-1} = 0 \). With the just defined quantities and \( (f \cdot x)_k, (g \cdot x)_k \) defined similarly as \( (h \cdot x)_k \) one has the following pathwise versions of the BDG inequalities for \( p > 1 \): if \( C_p = 6^p (p-1)^{-p} \) then for \( k \in \mathbb{N} \)
\[ (x_k^{p}) \leq C_p \sqrt{|x|_k^{p}} + 2 (g \cdot x)_k \text{ and } \sqrt{|x|_k^{p}} \leq C_p (x_k^{p}) - (f \cdot x)_k. \] (3.5)

Now, for a generalized process \( Y \) and a proper sequence of stopping times \( \tau = (\tau_n) \) we define a process
\[ Y_t^{\tau, *} := \max_{n \in \mathbb{N}} |Y_{\tau_n \wedge t}|, \quad t \in [0, +\infty) \]
(the maximum \( \max_{n \in \mathbb{N}} |Y_{\tau_n \wedge t}| \) is well defined since \( \tau \) is proper).

**Fact 3.4** (BDG inequality for simple quadratic variation). Let \( X \) be a martingale and \( \tau = (\tau_n) \) be a proper sequence of stopping times. For any \( p \geq 1 \) there exist finite, positive constants \( c_p \) and \( C_p \) such that
\[ c_p \mathbb{E}((X^{\tau})^{p/2}) \leq \mathbb{E}((X - X_0)^{\tau, *})^{p/2} \leq C_p \mathbb{E}((X^{\tau})^{p/2}). \]

In the case \( p > 1 \) one may take \( C_p = 6^p (p-1)^{-p} \) and \( c_p = 1/C_p \), while in the case \( p = 1 \) one may take \( C_p = 6 \) and \( c_p = 1/3 \).

**Proof.** The proof is almost straightforward application of pathwise versions of the BDG inequalities.

If \( p > 1 \) we fix \( M > 0 \), recall the stopping time \( \sigma(M) = \sigma(M, X) \) defined in (3.1) and define a simple strategy \( G^M = (0, \tau_n, (g_n^M)) \) in the following way: for \( \omega \in \Omega \) and \( n \in \mathbb{N} \) we define \( x_n = X_{\tau_n}(\omega) - X_0 \) and
\[ g_n^M(\omega) := \begin{cases} g_n & \text{if } n \in \mathbb{N} \text{ and } \tau_n < \sigma(M), \\ 0 & \text{if } n \in \mathbb{N} \text{ and } \tau_n \geq \sigma(M), \end{cases} \]
where \( g_n \) for the given sequence \( (x_n) \) is defined as in (3.3). Functions \( g_n^M \) are globally bounded (by constants depending on \( n \) and \( M \)) and \( \mathcal{F}_{\tau_n} \)-measurable. The pathwise limit \( G^X_t(\omega) := \lim_{M \to +\infty} (G^M \cdot X)_t(\omega), \quad (t, \omega) \in [0, +\infty) \times \Omega, \) is a martingale since \( (G^M \cdot X)_s(\omega), s \in [0, t], \) is the same for all \( M > \sup_{s \in [0, t]} |X_s(\omega)| \).

Moreover, by (3.3), if \( Y \) is a nonnegative supermartingale dominating \((X^{\tau})^{p/2}\) then \( C_p Y + 2G^X \) is a nonnegative supermartingale dominating \((X - X_0)^{\tau, *})^{p} \) (this follows from the application of (3.5) to the sequence \( \hat{x}_k = X_{\tau_n \wedge t} - X_0, k \in \mathbb{N} \) thus
\[ \mathbb{E}((X - X_0)^{\tau, *})^{p} \leq C_p \mathbb{E}((X^{\tau})^{p/2}). \]
The inequality
\[ \mathbb{E}((X - X_0)^{\tau, *})^{p} \geq c_p \mathbb{E}((X^{\tau})^{p/2}) \] with \( c_p = 1/C_p \) may be proven similarly, with help of the sequence \( (f_n) \).

The case \( p = 1 \) is even easier since one does not need to use stopping time \( \sigma(M) \) to define appropriate trading strategies, since \( h_l, l \in \mathbb{N}, \) in (3.2) always belong to the interval \([-1, 1]\).
4 Quadratic variation – existence, Itô’s isometry and BDG inequalities

4.1 Quadratic variation – existence

In this section we will prove that the simple quadratic variations of a martingale along sequences of stopping times satisfying some condition converge w.i.e. To formulate this condition we need to define a fine cover of a real process.

A non-decreasing sequence of \( \mathbb{F} \)-stopping times \((\tau_n)\) is called a fine cover of the real process \( X \) with accuracy \( \delta > 0 \) (or: \((\tau_n)\) finely covers the real process \( X \) with accuracy \( \delta > 0 \)) if \( \tau_0 \equiv 0 \), for any \( \omega \in \Omega \) the sequence \((\tau_n(\omega))\) is divergent to \(+\infty\), and for any \( n \in \mathbb{N} \)

\[
\sup_{t \in [\tau_n(\omega), \tau_{n+1}(\omega)]} X_t - \inf_{t \in [\tau_n(\omega), \tau_{n+1}(\omega)]} X_t \leq \delta.
\]

Naturally, any sequence of \( \mathbb{F} \)-stopping times \((\tau_n)\) which is a fine cover of the real process \( X \) (with positive accuracy) is also a proper sequence since it is divergent to \(+\infty\).

Using ideas from \[6\], which may be attributed already to Kolmogorov, we first prove the following lemma.

**Lemma 4.1.** Let \( X \) be a martingale, \( \sigma = (\sigma_n) \) be a fine cover of \( X \) with accuracy \( \delta > 0 \), \( \tau = (\tau_n) \) be a proper sequence of \( \mathbb{F} \)-stopping times such that for any \( \omega \in \Omega \) the sequence \((\tau_n(\omega))\) is divergent to \(+\infty\) and let \( \upsilon \) be the non-decreasing rearrangement of the stopping times from both sequences \( \sigma \) and \( \tau \), \( \upsilon = (\upsilon_n) \), with redundancies deleted. Then for any \( t \in [0, +\infty) \)

\[
[X]^{\sigma} - [X]^{\upsilon} \leq 4\delta^2 [X]^{\upsilon}_t.
\] (4.1)

**Proof.** We have

\[
[X]^{\sigma} - [X]^{\upsilon} = \sum_{n=1}^{+\infty} \left( [X]^{\sigma}_{\tau_n \wedge t} - [X]^{\upsilon}_{\tau_n \wedge t} - [X]^{\upsilon}_{\tau_{n-1} \wedge t} + [X]^{\upsilon}_{\tau_{n-1} \wedge t} \right)^2.
\]

Denoting

\[
n(t) := \max \{ n \in \mathbb{N} : \upsilon_n \leq t \}, \quad t \in [0, +\infty),
\]

we further estimate

\[
[X]^{\sigma} - [X]^{\upsilon} \leq \sum_{n=1}^{n(t)} \left( [X]^{\sigma}_{\upsilon_n} - [X]^{\sigma}_{\upsilon_{n-1}} - (X_{\upsilon_n} - X_{\upsilon_{n-1}})^2 \right) + \left( [X]^{\sigma}_t - [X]^{\sigma}_{\upsilon_{n(t)}} - (X_t - X_{\upsilon_{n(t)}})^2 \right)^2.
\] (4.2)

Next, denoting

\[
m(n) := \max \{ m \in \mathbb{N} : \sigma_m \leq \upsilon_n \}, \quad n \in \mathbb{N},
\]

for \( n \in \mathbb{N} \setminus \{0\} \) we have

\[
[X]^{\sigma}_{\upsilon_n} - [X]^{\sigma}_{\upsilon_{n-1}} = (X_{\upsilon_n} - X_{\sigma_{m(n)-1}})^2 - (X_{\upsilon_{n-1}} - X_{\sigma_{m(n)-1}})^2
\]

\[
= (X_{\upsilon_n} - X_{\upsilon_{n-1}}) (X_{\upsilon_n} + X_{\upsilon_{n-1}} - 2X_{\sigma_{m(n)-1}})
\]

(this may be proven by considering two possible cases: \( \upsilon_n = \sigma_{m(n)} > \upsilon_{n-1} \geq \sigma_{m(n-1)} \) and \( \upsilon_n \geq \upsilon_{n-1} \geq \sigma_{m(n)} = \sigma_{m(n-1)} \) so

\[
[X]^{\sigma}_{\upsilon_n} - [X]^{\sigma}_{\upsilon_{n-1}} - (X_{\upsilon_n} - X_{\upsilon_{n-1}})^2 = (X_{\upsilon_n} - X_{\upsilon_{n-1}}) (2X_{\upsilon_{n-1}} - 2X_{\sigma_{m(n)-1}}).
\] (4.3)

Similarly,

\[
[X]^{\sigma}_t - [X]^{\sigma}_{\upsilon_{n(t)}} - (X_t - X_{\upsilon_{n(t)}})^2 = (X_t - X_{\upsilon_{n(t)}}) (2X_{\upsilon_{n(t)}} - 2X_{\sigma_{m(n(t))}}).
\] (4.4)

Plugging in (4.2) equalities (4.3) and (4.4), and using the estimates

\[
2X_{\upsilon_{n-1}} - 2X_{\sigma_{m(n)-1}} \leq 2\delta, \quad 2X_{\upsilon_{n(t)}} - 2X_{\sigma_{m(n(t))}} \leq 2\delta,
\]

which stem from the fact that \( \sigma \) is a fine cover of \( X \) with accuracy \( \delta \), we get (4.1). \( \square \)
For a positive real number $d$ and $r \in [0, d]$ let us consider the net $d \cdot Z + r = \{d \cdot n + r : n \in \mathbb{Z}\}$. For a real process $X$ let now $\tau(X, d, r) = (\tau_n(X, d, r))$ be a sequence of stopping times $\tau_n = \tau_n(X, d, r)$ defined as: $\tau_0 \equiv 0$ and for $n = 1, 2, \ldots$

$$\tau_n = \inf \{t > \tau_{n-1} : X_t \in (d \cdot Z + r) \setminus \{X_{\tau_{n-1}}\}\}.$$\n
We will call $\tau(X, d, r)$ the Lebesgue sequence of stopping times for $X$ and the net $d \cdot Z + r$.

To state the next proposition we need two more definitions.

**Definition 4.2.** Let $\sigma$ be a stopping time. By the uniform convergence of the sequence of processes $(Y^m)$ on the random interval $[0, \sigma \setminus \{+\infty\}$, with instant enforcement, to the process $Y$, we mean that for any $\varepsilon > 0$ there exists a nonnegative supermartingale $Z^\varepsilon$ such that $\sup_{s \leq t} Y^m(s) - Y(s) \to 0$ whenever $\sup_{s \leq t} |Y^m(s) - Y(s)| \to 0$.

By locally uniform convergence of the sequence of processes $(Y^m)$ with instant enforcement to the process $Y$ we mean that for any $\varepsilon > 0$ there exists a nonnegative supermartingale $Z^\varepsilon$ such that $\sup_{s \leq t} Y^m(s) - Y(s) \to 0$ whenever $\sup_{s \leq t} |Y^m(s) - Y(s)| \to 0$.

Since the condition $\sup_{s \leq t} |Y^m(s) - Y(s)| \to 0$ implies the condition $\sup_{s \leq t} |Y^m(s) - Y(s)| \to 0$, the uniform convergence on the random interval $[0, \sigma \setminus \{+\infty\}$, with instant enforcement, implies locally uniform convergence with instant enforcement.

Now we are ready to state and prove a proposition on existence of quadratic variation.

**Proposition 4.3.** Let $X$ be a martingale, $M$ be a positive real and $\sigma(M) = \sigma(X, M)$ be defined by (3.1). Let $(\sigma^m)$ be a sequence of stopping times that $(\sigma^m)$ is a fine cover of $X$ with accuracy $\delta_m$ and let $(\tau^m) = (\tau(X, 2^{-m}, 0))$ be the sequence of the Lebesgue sequences of stopping times for $X$ and the nets $2^{-m} \cdot Z$. If $\sum_{m=0}^{+\infty} \delta_m < +\infty$ then there exists a real continuous process $X^\tau$ and $|X^\tau|_t$ converge uniformly on the random interval $[0, \sigma(M)) \setminus \{+\infty\}$ with instant enforcement to the process $X$.

**Proof.** Let us fix $M \in (0, +\infty)$. Since $\tau^{m+1}$ is the same as the non-decreasing rearrangement of $\tau^m$ and $\tau^{m+1}$ (all stopping times from the sequence $\tau^m$ also appear in the sequence $\tau^{m+1}$) and since $\tau^m$ is a fine cover of $X$ with accuracy $2^{-m}$, by Lemma 4.1 we have, for any $t \in [0, +\infty)$,

$$\left|[X]^m - [X]^{m+1}\right|_t \leq 4 \cdot 2^{-2m} |X|^{m+1}_t.$$ \hspace{1cm} (4.5)

By Fact 3.3 the difference

$$Y_t^m := [X]^{m+1}_{t \wedge \sigma(M)} - [X]^m_{t \wedge \sigma(M)} = (X_{t \wedge \sigma(M)} - X_0)^2 - [X]^{m+1}_{t \wedge \sigma(M)} - (X_{t \wedge \sigma(M)} - X_0)^2 - [X]^m_{t \wedge \sigma(M)}.$$\n
is a difference of two martingales stopped at $\sigma(M)$, thus a martingale. Recall a definition of the supremum process (of a generalised process) and consider $(Y^m)_t^\tau$. Now, since $\tau^{m+1}$, $m \in \mathbb{N}$, is a fine cover of $X$ with accuracy $2^{-m-1}$ we have

$$(Y^m)_t^\tau \leq (Y^m)^{m+1}_t + 2 \cdot 2^{-2m-2}$$

and by this and Fact 3.3 (discrete BDG inequality) we have

$$\mathbb{E} (Y^m)^* \leq \mathbb{E} (Y^m)^{m+1} + 2^{-2m-2} \leq 6 \mathbb{E} |X|^{m+1} + 2^{-2m-1}.$$\n
Further, using (4.3), the elementary estimate $\sqrt{x} \leq \frac{1}{2} + \frac{1}{2}x$ ($x \geq 0$) and the Itô isometry (Fact 3.2) we have

$$\mathbb{E} (Y^m)^* \leq 6 \mathbb{E} 4 \cdot 2^{-2m} |X|^{m+1}_{t \wedge \sigma(M)} + 2^{-2m-1} \leq 6 \cdot 2^{-m} \left(1 + \mathbb{E} |X|^{m+1}_{t \wedge \sigma(M)}\right) + 2^{-2m-1} \leq 6 \cdot 2^{-m} \left(1 + 4M^2\right) + 2^{-2m-1} \leq 7 \left(1 + 4M^2\right) 2^{-m}.$$ \hspace{1cm} (4.6)
Now let $B \subseteq [0, +\infty) \times \Omega$ be the set where the sequence of processes $[X]^{\tau_m}$, $m \in \mathbb{N}$, does not converge uniformly on $[0, \sigma(M)] \setminus \{+\infty\}$. Let us fix $\varepsilon > 0$. For each $(t, \omega) \in B$ we have

$$
\varepsilon \sum_{m=0}^{+\infty} (Y^{\tau_m})_t^* (\omega) = +\infty \geq 1_B(t, \omega).
$$

By (4.6) there exists a non-negative supermartingale $Z^{\tau_m}$ such that $Z^{\tau_m}_0 \leq 8 \left(1 + 4M^2\right) 2^{-m}$ and $Z^{\tau_m}_t(\omega) \geq (Y^{\tau_m})_t^* (\omega)$ for each $(t, \omega) \in [0, +\infty) \times \Omega$. Hence

$$U^\varepsilon := \varepsilon \cdot \sum_{m=0}^{+\infty} Z^{\tau_m}
$$

is a non-negative supermartingale such that $U^\varepsilon_0 \leq 8 \left(1 + 4M^2\right) \varepsilon \sum_{m=0}^{+\infty} 2^{-m} = 16 \left(1 + 4M^2\right) \varepsilon$ and for each $(t, \omega) \in B$

$$U^\varepsilon_t(\omega) = +\infty > 1_B(t, \omega).
$$

Since $\varepsilon$ may be as close to 0 as we wish, we get that the set $B$ is instantly blockable.

Let $[X]$ denotes any real continuous process to which $[X]^{\tau_m}$ converges uniformly on $[0, \sigma(M)] \setminus \{+\infty\}$ with instant enforcement for all $M = 1, 2, \ldots$ (we may take for example $[X]_t(\omega) := \lim_{m \to +\infty} [X]^{\tau_m}_t(\omega)$ if the limit exists and $[X]_t(\omega) := 0$ if the limit does not exist).

For $m \in \mathbb{N}$ let $\nu^m$ be the non-decreasing rearrangement of the stopping times from both sequences $\sigma^m$ and $\tau^m$ with redundancies deleted. $\nu^m$ is also a proper sequence of stopping times. Reasoning similarly as for $\tau^m$ and $\tau^{m+1}$ we infer that for the differences

$$R^m := [X]^{\nu^m}_\sigma(M) - [X]^{\nu^m}_\tau(M), \quad V^m := [X]^{\nu^m}_\tau(M) - [X]^{\nu^m}_\sigma(M)
$$

one has

$$
\mathbb{E}(R^m)^* \leq 6\delta_m \left(1 + 4M^2\right) + 2\delta^2_m
$$

and

$$
\mathbb{E}(V^m)^* \leq 6 \cdot 2^{-m} \left(1 + 4M^2\right) + 2 \cdot 2^{-2m}.
$$

Now, if $D \subseteq [0, +\infty) \times \Omega$ is the set where the sequence of processes $[X]^{\tau^m}$, $m \in \mathbb{N}$, does not converge uniformly to $[X]$ on $[0, \sigma(M)] \setminus \{+\infty\}$ then for each $\varepsilon > 0$ and $(t, \omega) \in D$

$$
\varepsilon \sum_{m=0}^{+\infty} (R^m)_t^* (\omega) + \varepsilon \sum_{m=0}^{+\infty} (V^m)_t^* (\omega) = +\infty \geq 1_D(t, \omega).
$$

Inequalities (4.7), (4.8) and (4.9) imply existence of a nonnegative supermartingale which starts from the initial capital smaller than

$$6\varepsilon \left(1 + 4M^2\right) \sum_{m=0}^{+\infty} \left(\delta_m + 2^{-m}\right) + 2\varepsilon \sum_{m=0}^{+\infty} \left(\delta^2_m + 2^{-2m}\right)
$$

and attains value $+\infty$ on the set $D$. Thus, since $\varepsilon$ may be arbitrary close to 0, $D$ is instantly blockable. \[ \blacksquare \]

**Definition 4.4.** By quadratic variation of a martingale $X$ we will mean any real continuous process $[X]$ which satisfies the thesis of Proposition 4.3.

### 4.1.1 Quadratic covariation

Now, let $X$ and $Y$ be two martingales. Naturally, $X + Y$ and $X - Y$ are also martingales. Let $(\tau(X, 2^{-m}, 0))$ and $(\tau(Y, 2^{-m}, 0))$ be two sequences of the Lebesgue sequences of stopping times for $X$ and $Y$ respectively (and the nets $2^{-m} \mathbb{Z}$). Let $\nu^m$ be the non-decreasing rearrangement of the stopping times from both sequences $\tau(X, 2^{-m}, 0)$ and $\tau(Y, 2^{-m}, 0)$, with redundancies deleted. $\nu^m$ is also a proper sequence of stopping times and finely covers both $X + Y$ and $X - Y$ with accuracy $2^{-m+1}$. By Proposition 4.3 we get that $[X + Y]^{\nu^m}$ and
\[ [X + Y]^\epsilon \] converge locally uniformly w.r.t. to the quadratic variations \([X + Y]\) and \([X - Y]\) respectively. The difference
\[ [X, Y] := \frac{1}{4}[X + Y] - \frac{1}{4}[X - Y] \]
is called \textit{quadratic covariation}. Substituting \(a = X_{m,t} - X_{m-1,t}\), \(b = Y_{m,t} - Y_{m-1,t}\) in the identity
\[ \frac{1}{4}(a + b)^2 - \frac{1}{4}(a - b)^2 = a \cdot b \]
we get that \([X, Y]\) is the limit of the \textit{simple quadratic covariation processes along} \((\nu_m)\):
\[ [X, Y]_{t}^{\nu_m} = \sum_{n=1}^{+\infty} \left( X_{n,t}^{\nu_m} - X_{n-1,t}^{\nu_m} \right) \left( Y_{n,t}^{\nu_m} - Y_{n-1,t}^{\nu_m} \right), \quad t \in [0, +\infty), \]
which converge to \([X, Y]\) locally uniformly w.r.t.

4.2 \textbf{Itô’s isometry}

Using Fact 3.2 (Itô’s isometry for simple quadratic variation) and the just proven Proposition 4.3 we can obtain Itô’s isometry for quadratic variation.

\textbf{Fact 4.5} (Itô’s isometry for quadratic variation). \textit{Let} \(X\) \textit{be a martingale and} \([X]\) \textit{its quadratic variation. We have}
\[ \mathbb{E}(X - X_0)^2 = \mathbb{E}[X]. \]

\textit{Proof.} Let \((\tau^m) = (\tau(X, 2^{-m}, 0))\) be the sequence of the Lebesgue sequences of stopping times for \(X\) and the nets \(2^{-m} \cdot \mathbb{Z}\). For any \(m = 0, 1, 2, \ldots\), by the Itô isometry for simple quadratic variation (Fact 3.2) we have \(\mathbb{E}(X - X_0)^2 = \mathbb{E}[X]^\epsilon\) which yields
\[ \mathbb{E}(X - X_0)^2 = \liminf_{m \to +\infty} \mathbb{E}[X]^\epsilon. \quad (4.10) \]

Since \(\liminf_{m \to +\infty} [X]^\epsilon = [X] \) w.r.t. (understood for all \((t, \omega) \in [0, +\infty) \times \Omega\) as the property that \(\liminf_{m \to +\infty} [X]^\epsilon_t^m(\omega) = [X]_t(\omega)\) we have \(\mathbb{E}[X] = \mathbb{E}\liminf_{m \to +\infty} [X]^\epsilon m\) and by (4.10) and the Fatou lemma (Lemma 2.2) we get
\[ \mathbb{E}(X - X_0)^2 \geq \mathbb{E}\liminf_{m \to +\infty} [X]^\epsilon = \mathbb{E}[X]. \]

To prove the upper bound let us fix \(\varepsilon, M > 0\). Let \(\sigma(M) = \sigma(X, M)\) be defined by \(3.1\) and
\[ \rho^m(\varepsilon) := \inf \left\{ t \geq 0 : \left| [X]_t - [X]_t^\epsilon \right| \geq \varepsilon \right\}, \]
where we take \([X] = \liminf_{m \to +\infty} [X]^\epsilon m\). Similarly as in the proof of the Itô isometry for simple quadratic variation we define a trading strategy \(G^{\varepsilon, M} = (\varrho^m, \left\{ g^{\varepsilon, M}_m \right\})\) in the following way:
\[ g^{\varepsilon, M}_n(\omega) := \begin{cases} 2(X_n(\omega) - X_0(\omega)) & \text{if} \ n \in \mathbb{N} \text{ and} \ \tau_n < \sigma(M) \land \rho^m(\varepsilon), \\ 0 & \text{if} \ n \in \mathbb{N} \text{ and} \ \tau_n \geq \sigma(M) \land \rho^m(\varepsilon). \end{cases} \]

Functions \(g^{\varepsilon, M}_n\) are globally bounded (by 2M) and \(\mathcal{F}_{\tau^m}\)-measurable. The pathwise limit
\[ G^{\varrho, X}_t(\omega) := \lim_{M \to +\infty} G^{\varepsilon, M} \cdot X_t(\omega), \quad (t, \omega) \in [0, +\infty) \times \Omega, \]
is a martingale since \((G^{\varepsilon, M} \cdot X)_s(\omega), s \in [0, t]\), is the same for all \(M > \sup_{\varepsilon \in [0, t]} |X_s(\omega)|\). Moreover, if \(Y\) is a nonnegative supermartingale dominating \([X]\) and such that \(Y_0 \leq \varepsilon + \mathbb{E}[X]\) then by the definition of the stopping time \(\rho^m(\varepsilon), \varepsilon + Y_{\vee \rho^m(\varepsilon)}\) is a nonnegative supermartingale dominating \([X]_{\vee \rho^m(\varepsilon)}\). Next, \(\varepsilon + Y_{\vee \rho^m(\varepsilon)} + G^{\varepsilon, X}\) is a nonnegative supermartingale dominating \((X - X_0)^2_{\vee \rho^m(\varepsilon)}\). Proceeding to the lower limit we have that
\[ H^{\varepsilon, X} := \liminf_{m \to +\infty} \left\{ \varepsilon + Y_{\vee \rho^m(\varepsilon)} + 2G^{\varepsilon, X} \right\} \]
is a nonnegative supermartingale dominating
\[ \liminf_{m \to +\infty} (X - X_0)^2 = (X - X_0)^2 \]
w.i.e. (since \([X]_t^m\) converges locally uniformly w.i.e. on the random interval \([0, \sigma(M)] \setminus \{+\infty\}, \rho^m(\varepsilon) \to +\infty as \ m \to +\infty\) w.i.e.). Hence
\[ \mathbb{E}(X - X_0)^2 \leq H_0^{\varepsilon,X} = \varepsilon + Y_0 \leq 2\varepsilon + \mathbb{E}[X] \]
which gives the desired bound by letting \(\varepsilon \to 0+\).

\[\square\]

### 4.3 BDG inequalities

Using Fact 3.3 (BDG inequalities for simple quadratic variation) and Proposition 4.3 we obtain the following proposition.

**Proposition 4.6** (BDG inequalities for quadratic variation). Let \(X\) be a martingale and \([X]\) be its quadratic variation. For any \(p \geq 1\) there exist finite, positive constants \(c_p\) and \(C_p\) such that
\[ c_p \mathbb{E}[X]^{p/2} \leq \mathbb{E}((X - X_0)^p) \leq C_p \mathbb{E}[X]^{p/2}. \]
In the case \(p > 1\) one may take \(C_p = 6^p(p - 1)^{p-1}\) and \(c_p = 1/C_p\), while in the case \(p = 1\) one may take \(C_p = 6\) and \(c_p = 1/3\).

**Proof.** The proof is similar to the proof of Itô’s isometry for quadratic variation. Let \((\tau^m) = (\tau(X, 2^{-m}, 0))\) be the sequence of the Lebesgue sequences of stopping times for \(X\) and the nets \(2^{-m} \cdot \mathbb{Z}\). For any \(m = 0, 1, 2, \ldots\), by the BDG inequality for simple quadratic variation (Fact 3.3), we have
\[ \mathbb{E}((X - X_0)^p) \geq \mathbb{E}((X - X_0)^{\tau^m,\ast}) \geq c_p \mathbb{E}([X]^{\tau^m})^{p/2} \]
which yields
\[ \mathbb{E}((X - X_0)^p) \geq c_p \liminf_{m \to +\infty} \mathbb{E}([X]^{\tau^m})^{p/2}. \quad (4.11) \]
Since \(\liminf_{m \to +\infty} [X]^{\tau^m} = [X]\) w.i.e. (understood for all \((t, \omega) \in [0, +\infty) \times \Omega\) as the property that \(\liminf_{m \to +\infty} [X]^{\tau^m}_t(\omega) = [X]_t(\omega)\)) we have \(\mathbb{E}([X])^{p/2} = \mathbb{E}\liminf_{m \to +\infty} ([X]^{\tau^m})^{p/2}\) and by (4.11) and the Fatou lemma (Lemma 2.2) we get
\[ \mathbb{E}((X - X_0)^p) \geq c_p \mathbb{E}\liminf_{m \to +\infty} ([X]^{\tau^m})^{p/2} = c_p \mathbb{E}([X])^{p/2}. \]

To prove the upper bound let us fix \(\varepsilon, M > 0\). Let \(\sigma(M) = \sigma(X, M)\) be defined by 3.1 and
\[ \rho^m(\varepsilon) := \inf \left\{ t \geq 0 : \left| ([X]_t^{\tau^m})^{p/2} - ([X]^{\tau^m}_t)^{p/2} \right| \geq \varepsilon \right\}, \]
where we take \([X] = \liminf_{m \to +\infty} [X]^{\tau^m}\). Similarly as in the proof of the BDG inequality for simple quadratic variation we define a trading strategy \(G^{m,\varepsilon,M}(\omega) = (\varepsilon, \tau^m, (g^{m,\varepsilon,M}_n))\) in the following way: \(x_n = X_{\tau^m_n}(\omega) - X_0\) and
\[ g^{m,\varepsilon,M}_n(\omega) := \begin{cases} g_n & \text{if } n \in \mathbb{N} \text{ and } \tau^m_n < \sigma(M) \wedge \rho^m(\varepsilon), \\ 0 & \text{if } n \in \mathbb{N} \text{ and } \tau^m_n \geq \sigma(M) \wedge \rho^m(\varepsilon), \end{cases} \]
where \(g_n\) for the given sequence \((x_n)\) is defined as in 3.4. Functions \(g^{m,\varepsilon,M}_n\) are globally bounded (by constants depending on \(m, n\) and \(M\)) and \(\mathcal{F}_{\tau^m_n}\)-measurable. The pathwise limit
\[ G^{m,\varepsilon,X}_t(\omega) := \lim_{M \to +\infty} (G^{m,\varepsilon,M}(\omega), (t, \omega) \in [0, +\infty) \times \Omega, \)
is a martingale since \((G^{m,\varepsilon,M} \cdot X)(\omega)\) is the same for all \(M > \sup_{s \in [0,t]} |X_s(\omega)|\). Moreover, if \(Y\) is a nonnegative supermartingale dominating \((|X|)^{p/2}\) and such that \(Y_0 \leq \varepsilon + \mathbb{E}(|X|)^{p/2}\) then by the definition of the stopping time \(\rho^m(\varepsilon)\), \(\varepsilon + Y_{\wedge \rho^m(\varepsilon)}\) is a nonnegative supermartingale dominating \((|X|_{\wedge \rho^m(\varepsilon)})^{p/2}\). Next, by (3.3), \(C_p \left(\varepsilon + Y_{\wedge \rho^m(\varepsilon)} + 2G^{m,\varepsilon,X}\right)\) is a nonnegative supermartingale dominating \((X - X_0)_{\wedge \rho^m(\varepsilon)}^{p} \) (this follows from the application of (3.5) to the sequence \(\tilde{x}_k = X_{e^k \wedge t} - X_0, k \in \mathbb{N}\)). Proceeding to the lower limit we have that

\[ H_{\varepsilon,X}^t := \liminf_{m \to +\infty} \{C_p \left(\varepsilon + Y_{\wedge \rho^m(\varepsilon)} + 2G^{m,\varepsilon,X}\right) \}
\]

is a nonnegative supermartingale dominating

\[ \liminf_{m \to +\infty} \left((X - X_0)_{\wedge \rho^m(\varepsilon)}^{p} \right) \]

w.i.e. (let us notice that since \(\tau^m\) finely covers \(X\) with accuracy \(2^{-m}\), we have

\[ (X - X_0)_{\wedge \rho^m(\varepsilon)}^{p} \leq (X - X_0)^{p} \leq (X - X_0)^{p} + 2^{-m} \]

and since \(|X|_{\wedge \rho^m(\varepsilon)}^{p}\) converges locally uniformly w.i.e. on the random interval \([0, \sigma(M)] \setminus \{+\infty\}, \rho^m(\varepsilon) \to +\infty\) as \(m \to +\infty\) w.i.e.) hence

\[ \mathbb{E}(X - X_0)^p \leq H_{\varepsilon,X}^t = C_p(\varepsilon + Y_0) \leq C_p(2\varepsilon + \mathbb{E}(|X|)^{p/2}) \]

which gives the desired bound by letting \(\varepsilon \to 0+\).

\[ \square \]

5. Model-free stochastic integral – definition and its quadratic variation

5.1 Model-free stochastic integral – definition

If \(H = (d,(\sigma_n),(h_n))\) is a trading strategy and \(X\) is a martingale then \(H \cdot X\) is again a martingale. Almost immediate consequence of Proposition 4.3 is the following fact.

**Fact 5.1.** Let \(H = (d,(\sigma_n),(h_n))\) be a simple trading strategy and \(X\) be a martingale then the quadratic variation of the martingale \(H \cdot X\) reads as

\[ [H \cdot X]_t = \sum_{n=1}^{+\infty} h_{n-1}^2 ([X]_{\sigma_{n-1} \wedge t} - [X]_{\sigma_{n-1} \wedge t}) = \int_0^t H_s^2 \cdot d[X]_s \text{ w.i.e.,} \]

where \(H_t := \sum_{n=1}^{+\infty} h_{n-1} \mathbf{1}_{[\sigma_{n-1}, \sigma_n]}(t)\) is the step process corresponding to \(H\) and the integral \(\int_0^t H_s^2 \cdot d[X]_s\) is understood as the (pathwise) Lebesgue-Stieltjes integral.

**Sketch of a proof.** Let \(\tilde{H} = (d,(\tilde{\sigma}_n), (\tilde{h}_n))\) be a modification of the trading strategy \(H\) obtained in the following way. If for some \(\omega \in \Omega\) and \(n \in \mathbb{N}\), \(\sigma_n(\omega) = \sigma_{n+1}(\omega) = \ldots\) then we set

\[ \tilde{\sigma}_n = \tilde{\sigma}_{n+1} = \ldots = +\infty, \quad \tilde{h}_n = \tilde{h}_{n+1} = \ldots = 0, \]

otherwise nor \(\sigma_n\) neither \(h_n\) are changed. This way we get a trading strategy satisfying for all \((t, \omega) \in [0, +\infty) \times \Omega\), \((\tilde{H} \cdot X)_t(\omega) = (H \cdot X)_t(\omega)\) and such that for all \(\omega \in \Omega\), \(\tau_n(\omega) \to +\infty\).

Let us consider \(m \in \mathbb{N}\), let \((\tau_n^m)_n = (\tau(X, 2^{-m}, 0))\) be the Lebesgue sequence of stopping times for \(X\) and the net \(2^{-m} \cdot \mathbb{Z}\), let \((\tilde{\tau}_n^m)_n = \left(\tilde{\tau}(\tilde{H} \cdot X, 2^{-m}, 0)\right)\) be the Lebesgue sequence of stopping times for \(\tilde{H} \cdot X\) and
observe that the paths $\tilde{s}$ tends locally uniformly w.r.e. Moreover the limit is equal to the process

$$[H \cdot X]_t^{m} = \left[ \tilde{H} \cdot X \right]_t^{m} = \sum_{t=1}^{\infty} H_{\tilde{s}^{m}_t}^{2} \left( [X]_{\tilde{s}^{m}_t \wedge t} - [X]_{s_t^{m}_t \wedge t} \right)$$

tends locally uniformly w.r.e. Moreover the limit is equal to the process

$$\sum_{n=1}^{\infty} H_{\tilde{s}^{m}_n}^{2} \left( [X]_{\tilde{s}^{m}_n \wedge t} - [X]_{s^{m}_{n-1} \wedge t} \right) = \sum_{n=1}^{\infty} H_{s^{m}_{n-1}}^{2} \left( [X]_{s^{m}_{n} \wedge t} - [X]_{s^{m}_{n-1} \wedge t} \right)
= \int_{0}^{t} H_{s}^{2} \cdot d[X]_{s}$$

since the (random) function $t \mapsto H_{t}$ is constant on each interval of the form $[\tilde{s}^{m}_{n-1}, \tilde{s}^{m}_{n}]$ (we apply the convention that $[+\infty, +\infty) = \emptyset$).

Now, having at hand Fact 5.1, Remark 2.3 and BDG inequalities we are going to extend the definition of the integral with the martingale integrator $X$.

Similarly as in [3] we equip the family of simple trading strategies with the following pseudo-distance. For two simple trading strategies $G = (c, (\tilde{\tau}_{n}), (g_{n}))$ and $H = (d, (\sigma_{n}), (h_{n}))$ we define

$$d_{QV,X,loc}(G, H) := \sum_{N=1}^{\infty} 2^{-N} \mathbb{E} \left( \int_{0}^{\sigma(X,N)} (G_{s} - H_{s})^{2} d[X]_{s} \right)^{1/2},$$

where $G_{t} := \sum_{n=1}^{\infty} g_{n-1} 1_{[\tilde{\tau}_{n-1}, \tilde{\tau}_{n}]}(t)$, $H_{t} := \sum_{n=1}^{\infty} h_{n-1} 1_{[\sigma_{n-1}, \sigma_{n}]}(t)$, $\sigma(X,N)$ is defined by (3.1) and the integral $\int_{0}^{\sigma(X,N)} (G_{s} - H_{s})^{2} d[X]_{s}$ is understood as the (pathwise) Lebesgue-Stieltjes integral.

Similarly one can also define $d_{QV,X,loc}(G, H)$ for any two generalized processes $G$ and $H$ (for the differences $G_{s} - H_{s}$ we apply the convention that $[+\infty, -\infty] = 0$ and $[-\infty, -(-\infty)] = 0$). It is not difficult to observe that $d_{QV,X,loc}$ satisfies the triangle inequality. We call $d_{QV,X,loc}$ a pseudo-distance since, for example, the paths $s \mapsto G_{s}$ and $s \mapsto H_{s}$, $s \in [0, +\infty)$, may differ on the intervals where the martingale $X$ is constant, but still $d_{QV,X,loc}(G, H) = 0$, it may also attain value $+\infty$.

Next, for two generalized processes $Y$ and $Z$ we define

$$d_{\infty,X,loc}(Y, Z) := \sum_{N=1}^{\infty} 2^{-N} \mathbb{E} (Y - Z)^{**}_{\wedge \sigma(X,N)},$$

Now we will deal with relationship between $d_{\infty,X,loc}(G \cdot X, H \cdot X)$ and $d_{QV,X,loc}(G, H)$ when $G$ and $H$ are step processes. We have

$$\int_{0}^{\sigma(X,N)} (G_{s} - H_{s})^{2} d[X]_{s} = \left[ (G - H) \cdot X,_{\wedge \sigma(X,N)} \right],$$

and, by Remark 2.3, $(G - H) \cdot X,_{\wedge \sigma(X,N)}$ is a martingale. Now, applying BDG inequality we obtain the estimate

$$\mathbb{E} \left( \int_{0}^{\wedge \sigma(X,N)} (G_{s} - H_{s})^{2} d[X]_{s} \right)^{1/2} = \mathbb{E} \left[ (G - H) \cdot X,_{\wedge \sigma(X,N)} \right]^{1/2} \geq C_{1}^{-1} \mathbb{E} (G \cdot X - H \cdot X)^{**}_{\wedge \sigma(X,N)},$$

which yields

$$d_{\infty,X,loc}(G \cdot X, H \cdot X) \leq C_{1} \cdot d_{QV,X,loc}(G, H). \quad (5.1)$$
Lemma 5.2. For any martingale $X$ the function $d_{\infty,X,loc}$ defines a metric (possibly attaining also value $+\infty$) on the space of (equivalence classes of) generalized processes $G$, which, equipped with this metric, is complete. Moreover, if $M := \sum_{n=1}^{+\infty} d_{\infty,X,loc}(Y^n, Y) < +\infty$ then the sequence of generalized processes $Y^n$ converges locally uniformly with instant enforcement. Moreover, if $Y^n$ are processes then their limit is also a process.

Proof. The proof that $d_{\infty,X,loc}$ defines a metric is omitted. To prove the completeness let $(Y^n)$ be a Cauchy sequence with respect to $d_{\infty,X,loc}$. Let $(d_k)$ be any sequence of positive reals such that $\sum_{k=1}^{+\infty} d_k < +\infty$. There exists a subsequence $(Y^{n_k})$ such that for $n \geq n_k$, $n, k = 1, 2, \ldots$ one has $d_{\infty,X,loc}(Y^n, Y^{n_k}) \leq d_k$. Taking $Y := \lim_{t \to +\infty} Y^n$, for $n \geq n_k$ we get

$$d_{\infty,X,loc}(Y^n, Y) \leq d_{\infty,X,loc}(Y^n, Y^{n_k}) + \sum_{l=k}^{+\infty} d_{\infty,X,loc}(Y^{n_k}, Y^{n_{l+1}}) \leq d_k + \sum_{l=k}^{+\infty} d_l,$$

thus $Y$ is the limit of the sequence $(Y^n)$. (As a limit one can also take $\limsup_{t \to +\infty} Y^n$).

To prove the second statement of the thesis let $B \subseteq [0, +\infty) \times \Omega$ be the set where the sequence of generalized processes $Y^n$ does not converge locally uniformly to $Y$, that is $(Y^n(\omega) - Y(\omega))_{\sigma(X(\omega),N)}^* \not\to 0$. Let $(t, \omega) \in B$ and $N$ be a natural number such that $N > \sup_{0 \leq s \leq t} |X_s(\omega)|$. This implies $t < \sigma(X(\omega), N)$ and

$$\sum_{n=1}^{+\infty} (Y^n(\omega) - Y(\omega))_{\sigma(X(\omega),N)}^* \geq \sum_{n=1}^{+\infty} (Y^n(\omega) - Y(\omega))_{t}^* = +\infty.$$

As a result, for any $\varepsilon > 0$, we get

$$\varepsilon \sum_{n=1}^{+\infty} \sum_{N=1}^{+\infty} 2^{-N} (Y^n(\omega) - Y(\omega))_{\sigma(X(\omega),N)}^* = +\infty.$$

On the other hand, since

$$\mathbb{E} \sum_{n=1}^{+\infty} \sum_{N=1}^{+\infty} 2^{-N} (Y^n(\omega) - Y(\omega))_{\sigma(X(\omega),N)}^* \leq \sum_{n=1}^{+\infty} \mathbb{E} \sum_{N=1}^{+\infty} 2^{-N} (Y^n(\omega) - Y(\omega))_{\sigma(X(\omega),N)}^* \leq \sum_{n=1}^{+\infty} d_{\infty,X,loc}(Y^n, Y) =: M < +\infty,$$

we know that there exist a non-negative supermartingale which starts from a capital no greater than $\varepsilon M$ and attains value $+\infty$ on $B$. Since $\varepsilon$ is arbitrary positive real, $B$ is instantly blockable.

The fact that if $Y^n$ are processes then $Y$ is also a process follows from the proof of completeness, more precisely from the fact that as the limit $Y$ one may take $\liminf$ of some subsequence of $(Y^n)$.

What will be important to us is that for any adapted, real process $F$ with càdlàg trajectories, which is globally bounded (sup$_{(t, \omega) \in [0, +\infty) \times \Omega} |F_t(\omega)| < +\infty$), we are able to construct a sequence of simple trading strategies $\hat{F}^m = (0, (\tau^m_n), (f^m_n))$ such that the sequence $(\hat{F}^m)$ of step processes

$$F^m_t := \sum_{n=1}^{+\infty} F^m_{n-1} 1_{[\tau^m_{n-1}, \tau^m_n)}(t)$$

converges in $d_{QV,loc}$ to $F$, $\lim_{m \to +\infty} d_{QV,loc}(\hat{F}^m, F) = 0$. For example, we can define $\tau^m_0 := 0$,

$$\tau^m_n := \inf\left\{ t > \tau^m_{n-1} : |F_t - F_{\tau^m_{n-1}}| \geq 2^{-m} \right\}, \quad n = 1, 2, \ldots$$

and $f^m_n = F_{\tau^m_n}$. For any $t \in [0, +\infty)$ we naturally have $|F_t - F^m_t| \leq 2^{-m}.$
For a simple trading strategy $\tilde{F}$ and its corresponding step process $F$, instead of $\tilde{F} \cdot X$ we will often write $F \cdot X$.

Now, using Itô’s isometry we estimate

$$d_{QV,X,loc}(F^m, F) = \sum_{N=1}^{\infty} 2^{-N} E \left( \int_0^{\sigma(N)} (F^m_s - F_s)^2 \, d[X]_s \right)^{1/2}$$

$$\leq \sum_{N=1}^{\infty} 2^{-N} \mathbb{E} 2^{-m} |X|_{\sigma(N)}^{1/2}$$

$$\leq 2^{-m} \sum_{N=1}^{\infty} 2^{-N} \mathbb{E} \left( \frac{1}{2} |X|_{\sigma(N)} + \frac{1}{2} \right)$$

$$\leq 2^{-m} \sum_{N=1}^{\infty} 2^{-N} \left( \frac{1}{2} 4N^2 + \frac{1}{2} \right) = 12.5 \cdot 2^{-m}. \quad (5.2)$$

Using (5.1), (5.2) and the fact that $d_{QV,X,loc}(F^m, F) \leq d_{QV,X,loc}(F^m, F) + d_{QV,X,loc}(F^n, F)$ we obtain that $(F^m, X)$ is a Cauchy sequence in the space of (equivalence classes of) generalized processes $G$, equipped with the metric $d_{\infty,X,loc}$.

Now, using Lemma 5.2 we are able to extend the definition of the the integral $F \cdot X$ at least for any adapted, globally bounded, real process $F$ with càdlàg trajectories.

**Definition 5.3.** For any adapted, globally bounded, real process $F$ with càdlàg trajectories by the model-free integral $F \cdot X$ we will mean any process which is (a representative of) the limit of the integrals $F^m \cdot X$, $m = 1, 2, \ldots$, where $F^m_t := \sum_{n=1}^{\infty} f^m_{n-1} 1_{[\tau^m_{n-1}, \tau^m_n)}(t)$ and $\tau^m_0 := 0$,

$$\tau^m_n := \inf \left\{ t > \tau^m_{n-1} : |F_1 - F^m_{\tau^m_{n-1}}| \geq 2^{-m} \right\}, \quad n = 1, 2, \ldots,$$

and $f^m_n = F^m_{\tau^m_n}$, in the space of (equivalence classes of) generalized processes $G$, equipped with the metric $d_{\infty,X,loc}$.

Similarly, for any real process $F$, for which there exists a sequence of step processes $(F^n)$ such that $\lim_{m \to +\infty} d_{QV,X,loc}(F^m, F) = 0$ we define the model-free integral $F \cdot X$ as any process which is (a representative of) the limit of the integrals $F^m \cdot X$, $m = 1, 2, \ldots$ in the above mentioned space.

To extend this definition to any adapted, real process $F$ with càdlàg trajectories, which is not necessarily globally bounded, we may use the following ‘split’ of $F$:

$$F_1 = \sum_{N=1}^{+\infty} F \cdot 1_{[\sigma(F,N-1), \sigma(F,N))}(t),$$

where $\sigma(F,N)$ is defined similarly as $\sigma(X, N)$ defined by (3.1). Each of the processes

$$F^{(N)} := F \cdot 1_{[\sigma(F,N-1), \sigma(F,N))}(t), \quad N = 1, 2, \ldots,$$

is globally bounded, adapted and has càdlàg trajectories. Now we can define $F \cdot X$ as

$$F \cdot X := \sum_{N=1}^{+\infty} F^{(N)} \cdot X, \quad (5.3)$$

since for each $(t, \omega) \in [0, +\infty) \times \Omega$ there are w.i.e. only finitely many non-zero summands in the sum on the right side of Eq. (5.3) (more precisely - as $F \cdot X$ we take the limit of sums of martingales which are representatives of $F^{(N)} \cdot X$ as elements of $G$). Moreover, for $t \in [0, +\infty)$ and $M > \sup_{s \in [0, t]} |F_s(\omega)|$, $M \in \mathbb{N}$, we have $(F \cdot X)_s(\omega) = \sum_{N=1}^{M} \left( F^{(N)} \cdot X \right)_s(\omega)$ for $s \in [0, t]$ so

$$\lim_{M \to +\infty} \sup_{s \in [0, t]} \left| \sum_{N=1}^{M} \left( F^{(N)} \cdot X \right)_s(\omega) - (F \cdot X)_s(\omega) \right| = 0 \text{ w.i.e.}$$

and hence $F \cdot X$ is a martingale as the limit of sums of martingales.
5.2 Quadratic covariation of model-free integrals

Now we are to prove the following fact

**Lemma 5.4.** Let \( G = (c, (\tau_n), (g_n)), H = (d, (\sigma_n), (h_n)) \) be simple trading strategies and \( X, Y \) be martingales. Then the quadratic covariation of the martingales \( G \cdot X \) and \( H \cdot Y \) equals

\[
[G \cdot X, H \cdot Y]_t = \int_0^t (G_s \cdot H_s) \cdot d[X, Y]_s \text{ w.i.e.,}
\]

where \( G_t := \sum_{n=1}^{+\infty} g_{n-1} \mathbf{1}_{[\tau_{n-1}, \tau_n)}(t), H_t := \sum_{n=1}^{+\infty} h_{n-1} \mathbf{1}_{[\sigma_{n-1}, \sigma_n)}(t) \) and the integral \( \int_0^t G_s \cdot H_s \cdot d[X, Y]_s \) is understood as the (pathwise) Lebesgue-Stieltjes integral.

**Sketch of a proof.** Let \( \tilde{G} = (c, (\tilde{\tau}_n), (\tilde{g}_n)), \tilde{H} = (d, (\tilde{\sigma}_n), (\tilde{h}_n)) \) be modifications of the trading strategies \( G, H \) respectively, obtained in the same way as in the proof of Fact 5.1.

Let us consider \( m \in \mathbb{N} \), let \( \tau(X, 2^{-m}, 0), \tau(Y, 2^{-m}, 0), \tau(\tilde{G} \cdot X, 2^{-m}, 0) \) and \( \tau(\tilde{H} \cdot Y, 2^{-m}, 0) \) be the Lebesgue sequences of stopping times for \( X, Y, \tilde{G} \cdot X \) and \( \tilde{H} \cdot Y \) respectively (and the net \( 2^{-m} \cdot Z \)), and let \( v^m \) be the non-decreasing rearrangement of the stopping times from these sequences and \( (\tilde{\tau}_n)_m, (\tilde{\sigma}_n)_m \) with redundancies deleted. Since \( v^m \) is a proper sequence of stopping times and finely covers both \( -X + Y \) and \( X - Y \) with accuracy \( 2^{-m+1} \), we easily see that by Proposition 4.3 (and reasoning in subsubsection 4.4.1)

\[
[G \cdot X, H \cdot Y]_{t}^{v^m} = [\tilde{G} \cdot X, \tilde{H} \cdot Y]_{t}^{v^m} = \sum_{l=1}^{+\infty} G_{v^m_{l-1}} \cdot H_{v^m_{l-1}} \left( |X|_{v^m_{l-1}} \wedge t - |X|_{v^m_{l-1}} \wedge t \right) \left( |Y|_{v^m_{l-1}} \wedge t - |Y|_{v^m_{l-1}} \wedge t \right)
\]

tends locally uniformly w.i.e. Moreover the limit is equal to the process

\[
\int_0^t (G_s \cdot H_s) \cdot d[X, Y]_s
\]

since the (random) functions \( t \mapsto G_t, t \mapsto H_t \) are constant on intervals of the form \([\tilde{\tau}_{n-1}, \tilde{\tau}_n), \tilde{\sigma}_{n-1}, \tilde{\sigma}_n) \) respectively (we apply the convention that \([+\infty, +\infty) = \emptyset\)).

**Fact 5.5.** Let \( G \) and \( H \) be adapted, real processes with càdlàg trajectories and \( X, Y \) be martingales. Then the quadratic covariation of the martingales \( G \cdot X \) and \( H \cdot Y \) equals

\[
[G \cdot X, H \cdot Y]_t = \int_0^t (G_s \cdot H_s) \cdot d[X, Y]_s \text{ w.i.e.}
\]

(the integral \( \int_0^t (G_s \cdot H_s) \cdot d[X, Y]_s \) is understood as the (pathwise) Lebesgue-Stieltjes integral).

**Sketch of a proof.** Let us consider \( n \in \mathbb{N}, N > 0 \), and let \( G^{n,N} \) and \( H^{n,N} \) be step processes such that \(|G_t - G^{n,N}_t| \leq 2^{-n} \) and \(|H_t - H^{n,N}_t| \leq 2^{-n} \) on the interval \([0, \sigma(X, N))\) and \( G^{n,N}_t, H^{n,N}_t = 0 \) for \( t \geq \sigma(X, N) \).

Let \( \tau(X, 2^{-m}, 0), \tau(Y, 2^{-m}, 0), \tau(G \cdot X, 2^{-m}, 0) \) and \( \tau(H \cdot Y, 2^{-m}, 0) \) be the Lebesgue sequences of stopping times for \( X, Y, G \cdot X \) and \( H \cdot Y \) respectively (and the net \( 2^{-m} \cdot Z \)), and let \( v^m \) be the non-decreasing rearrangement of the stopping times from these sequences with redundancies deleted. From inequality \( ab \leq (a^2 + b^2) / 2 \) it follows that for \( t < \sigma(X, N) \) the differences between

\[
[G \cdot X, H \cdot Y]_{t}^{v^m} \text{ and } [G^{n,N} \cdot X, H \cdot Y]_{t}^{v^m}, \ n \in \mathbb{N},
\]

are no greater than

\[
\frac{1}{2} 2^{-m} \sup_{t \in [0, \sigma(X, N))]} (|H_t| + 2^{-n}) \left( |X|_{t}^{v^m} + |Y|_{t}^{v^m} \right).
\]

Similarly, the differences between

\[
[G^{n,N} \cdot X, H \cdot Y]_{t}^{v^m} \text{ and } [G^{n,N} \cdot X, H^{n,N} \cdot Y]_{t}^{v^m}, \ n \in \mathbb{N},
\]
are no greater than
\[ \frac{1}{2} 2^{-n} \sup_{t \in [0, \sigma(X,N))] \left( |G_t| + 2^{-n} \left( |X|_{t}^{\epsilon} + |Y|_{t}^{\epsilon} \right) \right). \]

Thus, for fixed \( n \in \mathbb{N} \), sending \( m \) to \(+\infty\) we have that for \( t < \sigma(X,N) \), w.i.e.,
\[ |[G \cdot X, H \cdot Y]_{t} - [G^{n,N} \cdot X, H^{n,N} \cdot Y]_{t}| \leq 2^{-n} \sup_{t \in [0, \sigma(X,N))] \left( |G_t| + |H_t| + 2^{-n} \right( |X|_{t} + |Y|_{t} \right) . \]

Since
\[ [G^{n,N} \cdot X, H^{n,N} \cdot Y]_{t} = \int_{0}^{t} \left( G^{n,N}_{s} \cdot H^{n,N}_{s} \right) \cdot d[X,Y]_{s} \]
and we have locally uniform convergence
\[ \int_{0}^{t} \left( G^{n,N}_{s} \cdot H^{n,N}_{s} \right) \cdot d[X,Y]_{s} \to \int_{0}^{t \wedge \sigma(X,N)} \left( G_{s} \cdot H_{s} \right) \cdot d[X,Y]_{s} \]
as \( n \to +\infty \), we get the assertion. \( \square \)

6 Quadratic variation expressed via limit of truncated variations

In this section, for any martingale \( X \) we present another sequence of processes which tend locally uniformly w.i.e. to the quadratic variation of \( X \). To define these processes we introduce truncated variation of a càdlàg function \( x : [0, +\infty) \to \mathbb{R} \). The truncated variation of \( f \) over the interval \([a,b]\) \((-\infty < a < b < +\infty)\) with the truncation parameter \( c > 0 \) is defined as
\[ TV^{c}(x, [a,b]) := \sup_{n} \sup_{a \leq t_{0} < t_{1} < \ldots < t_{n} \leq b} \sum_{i=1}^{n} \max \{|x(t_{i}) - x(t_{i-1})| - c, 0\} . \]

Notice that \( TV^{c}(x, [a,b]) \) does not depend on any partition, since it is the supremum over all partitions of the interval \([a,b]\).

**Proposition 6.1.** Let \( X \) be a martingale and \((c_{n})\) a sequence of positive reals tending to 0. The processes \( t \mapsto c_{n} \cdot TV^{c_{n}}(X, [0, t]) \) tend locally uniformly and w.i.e. to \([X]\) as \( n \to +\infty \).

**Sketch of a proof.** First we will prove the thesis for the sequence of processes \( t \mapsto m^{-2} \cdot TV^{m^{-2}}(X, [0, t]) \), \( m = 1, 2, \ldots \). Let \( M \) be a positive real and \( \sigma(M) = \sigma(X,M) \) be defined by (3.1). Let \( \tau^{m,k} = (\tau^{m,k}_{n})_{n} := \tau(X, m^{-2}, k \cdot m^{-3}) \), \( m \in \mathbb{N} \setminus \{0, 1\} \), \( k \in \{0, 1, 2, \ldots, m - 1\} \), be the Lebesgue sequence of stopping times for \( X \) and the net \( m^{-2} \cdot Z + k \cdot m^{-3} \). We define \( \tau^{m,k} \wedge \sigma(M) \) as the sequence \((\tau^{m,k}_{n} \wedge \sigma(M))_{n}\). For \( m \in \mathbb{N} \setminus \{0\} \), \( k \in \{0, 1, 2, \ldots, m - 1\} \), let \( \nu^{m,k} \) be the non-decreasing rearrangement of the stopping times from both sequences \( \tau^{m,k} \) and \( \tau^{m,0} \) with redundancies deleted. \( \nu^{m,k} = \left( \nu^{m,k}_{i} \right)_{i} \) is a proper sequence of stopping times and we define \( \nu^{m,k} \wedge \sigma(M) \) as the sequence \((\nu^{m,k}_{i} \wedge \sigma(M))_{i}\). Similarly as in the proof of Fact 4.3 (inequalities 4.7 and 4.8), we infer that
\[ \mathbb{E} \left( |X|^{\tau^{m,k} \wedge \sigma(M)} - |X|^{\nu^{m,k} \wedge \sigma(M)} \right)^{2} \leq 6m^{-2} (1 + 4M^{2}) + 2m^{-4} \]
and
\[ \mathbb{E} \left( |X|^{\tau^{m,0} \wedge \sigma(M)} - |X|^{\nu^{m,k} \wedge \sigma(M)} \right)^{2} \leq 6m^{-2} (1 + 4M^{2}) + 2m^{-4} \]
thus
\[ \mathbb{E} \left( |X|^{\tau^{m,k} \wedge \sigma(M)} - |X|^{\nu^{m,0} \wedge \sigma(M)} \right)^{2} \leq 12m^{-2} (1 + 4M^{2}) + 4m^{-4}. \quad (6.1) \]
Summing both sides of (6.1) over \( k \in \{0, 1, 2, \ldots, m - 1\} \) and dividing by \( m \) we get that
\[
\mathbb{E}\left( \frac{1}{m} \sum_{k=0}^{m-1} [X] \tau_{m,k}^{m,k \wedge \sigma(M)} - [X] \tau_{m,0}^{m,0 \wedge \sigma(M)} \right)^* \leq 12m^{-2} (1 + 4M^2) + 4m^{-4}
\]
which yields
\[
\sum_{m=1}^{+\infty} \mathbb{E}\left( \frac{1}{m} \sum_{k=0}^{m-1} [X] \tau_{m,k}^{m,k \wedge \sigma(M)} - [X] \tau_{m,0}^{m,0 \wedge \sigma(M)} \right)^* < +\infty.
\] (6.2)

Notice that on the set where
\[
\left( \frac{1}{m} \sum_{k=0}^{m-1} [X] \tau_{m,k}^{m,k \wedge \sigma(M)} - [X] \tau_{m,0}^{m,0 \wedge \sigma(M)} \right)^* \rightarrow m \rightarrow +\infty 0
\]
one has
\[
\sum_{m=1}^{+\infty} \left( \frac{1}{m} \sum_{k=0}^{m-1} [X] \tau_{m,k}^{m,k \wedge \sigma(M)} - [X] \tau_{m,0}^{m,0 \wedge \sigma(M)} \right)^* = +\infty.
\]
This and (6.2) imply that \( \frac{1}{m} \sum_{k=0}^{m-1} [X] \tau_{m,k}^{m,k} \) tends locally uniformly and w.i.e. to the same limit as \( [X] \tau_{m,0}^{m,0} \), that is to \( [X] \).

The next ingredient of the proof which we need is the following identity
\[
TV^{m-2}(X, [0, \sigma(M) \wedge t]) = \int_{\mathbb{R}} n^{z,m-2}(X, [0, \sigma(M) \wedge t])dz,
\] (6.3)
where for a càdlàg function \( x : [0, +\infty) \rightarrow \mathbb{R} \) and real numbers \( 0 \leq a < b < +\infty, c > 0 \), \( n^{z,c}(x, [a, b]) \) denotes the number of crossings by \( x \) the value interval \([z-c/2, z+c/2]\) on the interval \([a, b]\). For precise definitions of \( n^{z,c}(x, [a, b]) \) see [4, Subsect. 2.4] and for the proof of (6.3) see [2].

Next, let us notice that for \( t > 0, m \in \mathbb{N} \setminus \{0, 1\} \) and \( k \in \{0, 1, 2, \ldots, m - 1\} \)
\[
[X]_t^{m,k \wedge \sigma(M)} = \sum_{p \in \mathbb{Z}} m^{-4/2} m^{-2/2} m^{-2/2} m^{-2/2} (X, [0, t \wedge \sigma(M)])
\] (6.4)
since \( X \) has continuous trajectories and the Lebesgue stopping times are hitting times of consecutive levels of the net \( m^{-2} \cdot \mathbb{Z} + k \cdot m^{-3} \).

For \( p \in \mathbb{Z}, m \in \mathbb{N} \setminus \{0, 1, 2\}, k \in \{0, 1, 2, \ldots, m - 2\} \) and
\[
z \in \left[ \frac{p}{(m-1)^2} + \frac{k}{(m-1)^3}, \frac{p}{(m-1)^2} + \frac{k+1}{(m-1)^3} \right]
\] (6.5)
we have
\[
\frac{p}{(m-1)^2} + \frac{k}{(m-1)^3} + \frac{1}{m^2} \leq z + \frac{1}{m^2} < \frac{p}{(m-1)^2} + \frac{k+1}{(m-1)^3} + \frac{1}{m^2} < \frac{p+1}{(m-1)^2} + \frac{k}{(m-1)^3},
\]
which follows from the estimate
\[
\frac{p+1}{(m-1)^2} + \frac{k}{(m-1)^3} - \frac{p}{(m-1)^2} - \frac{k+1}{(m-1)^3} - \frac{1}{m^2} = \frac{1}{(m-1)^2} - \frac{1}{(m-1)^3} - \frac{1}{m^2} = \frac{m^2 - 3m + 1}{m^2(m-1)^2} > 0,
\]
valid for \( m \geq 3 \). Thus, each crossing of the interval
\[
\left[ \frac{p}{(m-1)^2} + \frac{k}{(m-1)^3}, \frac{p+1}{(m-1)^2} + \frac{k}{(m-1)^3} \right]
\]
implies crossing of the interval $[z, z + m^{-2}]$, whenever $z$ satisfies (6.5). This implies that

$$
\int_{\frac{p}{(m-1)^2} + \frac{k+1}{(m-1)^3}}^{\frac{p}{(m-1)^2} + \frac{k+1}{(m-1)^3}} n^{z+m^{-2}/2,m^{-2}} (X, [0, t \wedge \sigma(M)]) dz \\
\leq \int_{\frac{p}{(m-1)^2} + \frac{k+1}{(m-1)^3}}^{\frac{p}{(m-1)^2} + \frac{k+1}{(m-1)^3}} n^{p(m-1)^{-2}+k(m-1)^{-3}+(m-1)^{-2}/2,(m-1)^{-2}} (X, [0, t \wedge \sigma(M)]) dz \\
= \frac{1}{(m-1)^3} n^{p(m-1)^{-2}+k(m-1)^{-3}+(m-1)^{-2}/2,(m-1)^{-2}} (X, [0, t \wedge \sigma(M)]).
$$

Now, summing over $p \in \mathbb{Z}$ and $k \in \{0, 1, 2, \ldots, m - 2\}$, and using (6.4) (with $m$ replaced by $m - 1$) we get

$$
TV^{m^{-2}} (X, [0, \sigma(M) \land t]) \\
= \int_{\mathbb{R}} n^{z+m^{-2}/2,m^{-2}} (X, [0, t \wedge \sigma(M)]) dz \\
= \sum_{p \in \mathbb{Z}} \sum_{k=0}^{m-2} \int_{\frac{p}{(m-1)^2} + \frac{k+1}{(m-1)^3}}^{\frac{p}{(m-1)^2} + \frac{k+1}{(m-1)^3}} n^{z+m^{-2}/2,m^{-2}} (X, [0, t \wedge \sigma(M)]) dz \\
\leq \frac{1}{(m-1)^3} \sum_{k=0}^{m-2} \sum_{p \in \mathbb{Z}} n^{p(m-1)^{-2}+k(m-1)^{-3}+(m-1)^{-2}/2,(m-1)^{-2}} (X, [0, t \wedge \sigma(M)]) \\
= (m-1) \sum_{k=0}^{m-2} [X]^n_{t}^{m^{-1}, k \wedge \sigma(M)}. \tag{6.6}
$$

On the other hand, for $p \in \mathbb{Z}$, $m \in \mathbb{N} \setminus \{0\}$, $k \in \{0, 1, 2, \ldots, m\}$ and

$$
z \in \left( \frac{p}{(m+1)^2} + \frac{k}{(m+1)^3}, \frac{p}{(m+1)^2} + \frac{k+1}{(m+1)^3} \right) \tag{6.7}
$$

we have

$$
\frac{p+1}{(m+1)^2} + \frac{k+1}{(m+1)^3} < \frac{p}{(m+1)^2} + \frac{k}{(m+1)^3} + \frac{1}{m^2}, \quad z + \frac{1}{m^2} < \frac{p}{(m+1)^2} + \frac{k+1}{(m+1)^3} + \frac{1}{m^2}
$$

since

$$
\frac{p}{(m+1)^2} + \frac{k}{(m+1)^3} + \frac{1}{m^2} - \frac{p+1}{(m+1)^2} - \frac{k+1}{(m+1)^3} \\
= \frac{1}{m^2} - \frac{1}{(m+1)^2} - \frac{1}{(m+1)^3} = \frac{m^2 + 3m + 1}{m^2(m+1)^3} > 0.
$$

Thus, each crossing of the interval $[z, z + m^{-2}]$ implies crossing of the interval

$$
\left[ \frac{p}{(m-1)^2} + \frac{k+1}{(m-1)^3}, \frac{p}{(m-1)^2} + \frac{k+1}{(m-1)^3} \right]
$$

whenever $z$ satisfies (6.6). This implies analogous inequality to (6.5), but in opposite direction:

$$
TV^{m^{-2}} (X, [0, \sigma(M) \land t]) \geq (m+1) \sum_{k=0}^{m} [X]^n_{t}^{m^{-1}, k \wedge \sigma(M)}. \tag{6.8}
$$
and \( (6.8) \) give bounds

\[
\frac{m+1}{m^2} \sum_{k=0}^{m} [X]_t^{m+1,k \wedge \sigma(M)} \leq \frac{1}{m^2} TV^{m-2}(X, [0, \sigma(M) \wedge t]) \leq \frac{m-1}{m^2} \sum_{k=0}^{m} [X]_t^{m-1,k \wedge \sigma(M)},
\]

which imply that \( m^{-2}TV^{m-2}(X, [0, \cdot]) \) tends locally uniformly and w.i.e. to the same limit as \( \frac{1}{m} \sum_{k=0}^{m} [X]^{m,k} \), that is to \( [X] \).

Finally, the convergence of \( c_n \cdot TV^{c_n}(X, [0, \cdot]) \) for any sequence \( c_n \to 0+ \) follows from the estimates

\[
\frac{1}{[1/c_n]^2} \frac{1}{[1/c_n]+1} \cdot TV^{1/[1/\sqrt{c_n}]^2}(X, [0, t]) \leq c_n \cdot TV^{c_n}(X, [0, t]) \leq \frac{1}{\sqrt{c_n}} \cdot TV^{1/[1/\sqrt{c_n}]^2}(X, [0, t])
\]

valid for \( c_n < 1 \), which stem directly from inequalities

\[
\frac{1}{[1/c_n]+1} \leq c_n \leq \frac{1}{[1/c_n]-1} \quad \text{and} \quad \frac{1}{[1/c_n]^2} \leq c_n \leq \frac{1}{[1/c_n]^2}
\]

(valid for \( c_n < 1 \)), and the fact that the function \((0, +\infty) \ni c \mapsto TV^c(X, [0, t])\) is non-increasing. □

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