Uncertain External Pressure Act on Thick Cylinder in Fuzzy Context

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Abstract. In this paper, we study a thick cylinder with uncertain external pressure. The traditional model was extended in fuzzy context. We use the governing equation which was rewritten as delta operator equation. The interpretations of external pressure are based on triangular, trapezoidal and Gaussian fuzzy numbers. By Adomian Decomposition Method gives an approximate solution. The maximal and minimal radial stresses are obtained, corresponding to the level cut. Finally, we get the methodology for simulating uncertain problems in fuzzy context and utility of each membership function.

1. Introduction

Many theoretical mechanics are interpreted by concept of calculus. In solid mechanics, calculus is used to describe deformation of solid. The equilibrium of forces can be written as governing equation with fact conditions, called initial value problems or boundary valued problems. But, most of models are based certain system.

In fact, we avoid the uncertainty in nature by design and control our experiments. The other method, we may use probability theorem to understand the randomness of existence of events. In addition to both previous concepts, considering the possibility of the fact is an alternative way. It is well known as fuzzy systems.

Mass flow in tube is one of the problem, which concerns uncertainty of system and environment [1], [2]. The governing equation of the thick cylinder with radius \( r \in [a,b] \) subjects to internal pressure \( p_i \) and external pressure \( p_0 \) is form of differential equation: 

\[ u''(r) + u'/r - u/r^2 = 0 \]

subjects to boundary conditions, \( \sigma(u(a)) = -p_i \) and \( \sigma(u(b)) = -p_0 \), where \( u(r) \) is displacement and \( \sigma(r) \) is stress at the radius \( r \). If at least one of parameters in previous problem is vague and we solve it in fuzzy context. This problem becomes fuzzy differential equation.

Fuzzy differential equation is a concept of differential equation on fuzzy space. The quantities are indicated by a grade of membership function in sense of Zadah’s fuzzy set and logic [3]. The researchers still continue to develop theory and methods of fuzzy differential equations.

Recently, the governing equation of thick cylinder was developed from derivative to delta operator, defined by \( \delta u(r) = ru'(r) \). In this paper, we organize concepts and methods for simulating thick cylinder with uncertain external pressure by delta operator in Fuzzy Context. In the next section, we recall delta operator model and its computational method, then we
extend the problem in real space to fuzzy space under triangular, trapezoidal and Gaussian membership functions. Their simulation results are presented in section 4. Finally, we conclude and discuss our results in this work.

2. Mathematical model and its computation

Throughout this section, the mathematical and theoretical framework are omitted. We refer some results which is used to investigate the simulation, see [4] for more details.

2.1. Delta operator model
Let \( u : [a, b] \rightarrow R \) and its second order delta derivative exists. The stress distribution in thick cylinder can be written as following delta operator equation:

\[
\delta^2 u(r) = u(r)
\]

subject to

\[
du(a) - \left(\frac{\nu}{1 - \nu}\right) u(a) = -k_1 \quad \text{and} \quad du(b) - \left(\frac{\nu}{1 - \nu}\right) u(b) = -k_2,
\]

where,

\[
k_1 = ap_o(1 + \nu)(1 - 2\nu)/(E(1 - \nu)), \quad k_2 = bp_o(1 + \nu)(1 - 2\nu)/(E(1 - \nu)).
\]

2.2. n-terms ADM approximation

The closed-form of the \( n \)-terms ADM approximated solution of 1, given by

\[
u^n = D_1^n \left( \sum_{k=0}^{n} \frac{\ln^{2k+1}(r/a)}{(2k+1)!} \right) + D_2^n \left( \sum_{k=0}^{n} \frac{\ln^{2k}(r/a)}{(2k)!} \right) = D_1^n S_\Omega(r) + D_2^n S_E(r).
\]

Moreover, 2-terms ADM approximations are close to the exact solution. Here, we use \( u^2 \approx u(r) \).

3. Fuzzy Model

In this section, we assume that the external pressure is uncertain quantity (\( p_2 \in E \)). Note that \( E \) is fuzzy set. We describe this vague quantity into 3 types of fuzzy number: 1) Triangular Fuzzy Number (TFN), 2) Trapezoidal Fuzzy number (TrFS) and 3) Gaussian Fuzzy Number (GFN)[5]. Let \( u : [a, b] \rightarrow E \) be a \( d \) monotone fuzzy function and its second order delta derivative exists. The stress distribution in thick cylinder with uncertain external pressure is represented as follow:

\[
\delta^2 u(r) = u(r)
\]

subject to

\[
du(a) - \left(\frac{\nu}{1 - \nu}\right) u(a) = -k_1 \in E \quad \text{and} \quad du(b) - \left(\frac{\nu}{1 - \nu}\right) u(b) = -k_2 \in E.
\]

Applying \( t \)-cut representation, the fuzzy solution is in form of \( u(r) = [\bar{u}(r), \underline{u}(r)] \approx [u^2, \bar{u}^2] \) where,

\[
\bar{u} \approx u^2 = D_1^2 \left( \frac{\ln(r/a)}{3!} + \frac{\ln^3(r/a)}{3!} + \frac{\ln^5(r/a)}{5!} \right) + D_2^2 \left( 1 + \frac{\ln^2(r/a)}{2!} + \frac{\ln^4(r/a)}{4!} \right).
\]

Its arbitrary constants \( D_1^2 \) and \( D_2^2 \) can be updated by solving linear system which is replaced by the boundary conditions. Note that \( D_i^2 = [\bar{D}_i^2, \underline{D}_i^2] \) corresponding to \( [\bar{k}_i, \underline{k}_i] \) for \( i \in \{1, 2\} \). The approximation of \( \bar{u}(r) \) is obtained in the same manner.
3.1. Triangular Fuzzy Number
In a case of triangular fuzzy number, let $p_2 = [p_m, \bar{p}, p_M]$. Then $t$-cut representation of $p_2$ is given by

$$[p_2, \bar{p}_2] = [p_m + (\bar{p} - p_m) t, p_M - (p_M - \bar{p}) t]$$ (4)

Suppose that $p_2 = (100, 200, 300)$ MPa, the grade of membership function of $p_2$ is shown in Fig. 1.

![Figure 1. Triangular Fuzzy Number (100,200,300)](image)

3.2. Trapezoidal Fuzzy Number
The trapezoidal fuzzy number $p_2 = [p_m, p_u, p_U, p_M]$. The $t$-cut approach

$$[p_2, \bar{p}_2] = [a + (p_u - p_m) t, p_M - (p_M - p_U) t]$$ (5)

Fig. 2 is the trapezoidal fuzzy number of external pressure (100, 175, 225, 300) MPa

3.3. Gaussian Fuzzy Number
The external pressure $p_2$ is defined by Gaussian Fuzzy Number, $[\bar{p}, \sigma_l, \sigma_r]$. Its $t$-cut represents

$$[p_2, \bar{p}_2] = \left[ \bar{p} - \sqrt{-\frac{\ln t}{\lambda}}, \bar{p} + \sqrt{-\frac{\ln t}{\lambda}} \right] ,$$

where $\lambda = 1/(2\sigma^2)$ Assume that the external pressure distribute as Gaussian fuzzy number (200, 50, 100) MPa, see its shape in Fig 3.
Figure 2. Trapezoidal Fuzzy Number (100,175,225,300)

Figure 3. Gaussian Fuzzy Number (200,50,100)
4. Numerical results and simulations

We illustrated numerical solutions and simulations of the cylinder which is 50 cm mean diameter with 15 cm thick. Suppose that internal pressure ($p_i = 35$ MPa) and allowed outside pressure is decided by $t$-cut of fuzzy number in Fig. 1, Fig. 2 and Fig. 3. Young modulus and poisson’s ratio of material are $210 \text{MN/m}^2$ and 0.3 respectively.

Table 1 shows numerical results of minimal and maximal stress belonging to $t$-cut at the radius $r$. The Fig. 4 is radial stress simulation of 3 and 4 when $t = 0, 0.5, 1.0$ and similarly for trapezoidal shape 5. Its simulation is presented in Fig. 5. But $t = 0$ cannot be evaluated in case of GFN. Then we set $t = 0.025, 0.5, 1.0$ and plots Gaussian fuzzy solution in Fig. 6.

![Radial stress](image)

**Figure 4.** Triangular fuzzy solutions $[t = 0, '–'], [t = 0.5, '–'], [t = 1.0, 'o']$

5. Conclusion and discussion

In this work, we studied the thick cylinder subjects to internal and external pressures such that inside is certainty but not outside. We used concept of fuzzy set to indicate the grade of membership function. Consequently, the traditional deformation become fuzzy deformation.

Three types of fuzzy number are demonstrated and investigated simulations of radial stress distribution. Firstly, $t = 0$ of triangular fuzzy number is the support and $t = 1$ is core of TFN. For $t \in [0, 1]$, TFN yields minimal and maximal radial stresses while TrFN yields radial stress interval for all $t \in [0, 1]$. GFN is the last one that we used to interpret external pressure. Since the support of GFN is $(-\infty, \infty)$, its shape should be established from density and distribution of external pressure.

We observed that if the quantity is certain, the fuzzy interval is singleton set, see internal pressure and core of TFN. This mean that TFN is suitable when the best selection of external pressure is fixed. On the other hand, TrFN is the better if best selection still vague. But if we want to decide the possibility from data, determine Gaussian fuzzy shape and apply GFN.
Figure 5. Trapezoidal fuzzy solutions \([t = 0, '-', [t = 0.5, '–'], [t = 1.0, 'o']]\)

Figure 6. Gaussian Fuzzy solutions \([t = 0.025, '-', [t = 0.5, '–'], [t = 1.0, 'o']]\)
Table 1. Approximation of radial stress intervals by 2-terms ADM

| t-cut | Radial stress(MPa) |
|-------|--------------------|
|       | r = 10             | r = 15             | r = 20             | r = 25             |
| TFN   | [-35.35, -35]      | [-78.30,-211.4]   | [-93.32,-272.6]   | [-100,-300]        |
|       | [-35.35, -35]      | [-111.6,-178.1]   | [-138.2,-227.8]   | [-150,-250]        |
|       | [-35.35, -35]      | [-144.8,-144.8]   | [-183,-183]       | [-200,-200]        |
| TrFN  | [-35.35, -35]      | [-78.30,-211.4]   | [-93.32,-272.6]   | [-100,-300]        |
|       | [-35.35, -35]      | [-103.3,-186.4]   | [-126.9,-239]     | [-137.5,-262.5]    |
|       | [-35.35, -35]      | [-128.2,-161.5]   | [-160.6,-205.4]   | [-175,-225]        |
| GFN   | [-35.35, -35]      | [-54.47,-325.6]   | [-61.21,-426.5]   | [-64.19,-471.6]    |
|       | [-35.35, -35]      | [-105.7,-186.4]   | [-130.2,-239]     | [-141.1,-262.5]    |
|       | [-35.35, -35]      | [-144.8,-144.8]   | [-183,-183]       | [-200,-200]        |

Hence the utility of fuzzy differential equation can support decision making for planing and control under uncertainty of static and dynamics problems with soft computing. Our concepts maybe develop and apply to some engineering problems such as a pipeline subjects to internal pressure (water) and external pressure (traffic and earth)[6].

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6. References

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