Tensor Product of Polygonal Cell Complexes

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September 18, 2018

Abstract

We introduce the tensor product of polygonal cell complexes, which interacts nicely with the tensor product of link graphs of complexes. We also develop the unique factorization property of polygonal cell complexes with respect to the tensor product, and study the symmetries of tensor products of polygonal cell complexes.

1 Introduction

A polygonal cell complex is a 2-dimensional CW-complex with polygons as 2-cells, namely a graph with polygons attached. To be precise, we take a rather formal definition: a polygonal cell complex is a 2-dimensional CW-complex satisfying:

(1) Each 1-cell is an interval of length 1, and each 2-cell is a disc of positive integral circumference.

(2) For a 2-cell of circumference \( n \), the attaching map sends exactly \( n \) points evenly distributed on the boundary to the 0-skeleton.

(3) For each boundary segment between the points described in (2), the attaching map sends the segment isometrically onto an open 1-cell.

Intuitively speaking, we can think of each 2-cell as a regular polygon, and the attaching map glues vertices to vertices, and edges to edges. Those 2-cells act like faces of a polyhedron, and we will use the word face to denote a 2-cell alternatively. Note that the attaching map of a face may not be injective, and a polygonal cell complex can be quite different from polyhedra. A polygonal complex, which simulates polyhedra better, is a polygonal cell complex satisfying:

(i) The attaching map of each cell is injective.

(ii) The intersection of any two closed cell is either empty or exactly one closed cell.

Unless otherwise specified, when we use the word complex, it means polygonal cell complex, which may or may not be a polygonal complex.
Here is a concise way to describe the local structure of complexes. For a polygonal cell complex \( X \), the \textit{link} of \( X \) at a vertex \( v \) is a graph \( L(X, v) \) with vertices indexed by ends of edges attached to \( v \), and edges indexed by corners of faces attached to \( v \). Two vertices \( v_1 \) and \( v_2 \) in \( L(X, v) \) are joined by an edge \( e \) if and only if the corresponding ends of \( v_1 \) and \( v_2 \) are joined by the corresponding corner of \( e \). Basically a link describes the incidence relation of edges and faces at a vertex. Note that \( L(X, v) \) can also be identified as the set \( \{ x \in X | d(x, v) = \delta \} \), where \( d \) is the distance function in \( X \) and \( \delta \) is some positive number less than \( 1/2 \).

Take the dunce hat in Figure 1 as an example. Although there is only one edge in the complex, this edge has two ends attached to \( v \), and therefore contributes two vertices to the link at \( v \). Notice that the top corner of the face joins these two ends, and corresponds to an edge joining two vertices in the link at \( v \). The left corner of the face joins the same end of the edge, and hence corresponds to a loop in the link, while the right corner of the face also corresponds to a loop at the other vertex. Therefore the link at \( v \) is a graph with two vertices \( e_1 \) and \( e_2 \), one edge joining \( e_1 \) and \( e_2 \), and two loops at \( e_1 \) and \( e_2 \) respectively.

For polyhedra, a \textit{flag} is an incident triple of face, edge, and vertex. Such definition needs to be modified for polygonal cell complexes. Take Figure 1 as an example again. It has only one vertex, one edge, and one face, but we would like it to have six flags just as a usual triangle. In a polygon, each flag corresponds to a triangle in its barycentric subdivision. We can use this as an alternative definition of a flag, and this definition works for polygonal cell complexes as well. As Figure 1 shows, the shaded area is a flag of the dunce hat, and a dunce hat has six flags.

Highly symmetric polygonal complexes have been studied in \([1, 2, 9, 10]\). In particular, simply-connected flag-transitive polygonal complexes with complete graphs as links are classified in \([2]\). The main motivation of this paper is to use these flag-transitive complexes to generate more flag-transitive complexes. More specifically, we would like to develop a product of complexes which preserves flag-transitivity, and the link of the product is some graph product of the links of factors.

2 Graph Tensor Product

Suppose that \( \bullet \) is certain type of graph product such that \( V(\Gamma \bullet \Gamma') = V(\Gamma) \times V(\Gamma') \), and we want to define a complex product \( \ast \) with the following property: for any complexes \( X \) and \( X' \), and for any vertices \( v \in X \) and \( v' \in X' \), we have

\[
L(X, v) \bullet L(X', v') \cong L(X \ast X', (v, v')).
\]
Here we have already assumed that $V(X \ast X') = V(X) \times V(X')$. The above property provides sufficient information about how the complex product $\ast$ shall be defined. If we assume the 1-skeletons of $X$ and $X'$ are simple graphs, by considering the vertex sets of two link graphs in the equation, we have

\[
\{\text{neighbours of } v \text{ in } X\} \times \{\text{neighbours of } v' \text{ in } X'\} = \{\text{neighbours of } (v, v') \text{ in } X \ast X'\},
\]

which can be interpreted as two vertices $(v, v')$ and $(u, u')$ are adjacent in $X \ast X'$ if and only if $v$ is adjacent to $u$ in $X$ and $v'$ is adjacent to $u'$ in $X'$. This is essentially the definition of the direct product of simple graphs. Since the 1-skeletons of complexes are not necessarily simple, we shall generalize the direct product to suit arbitrary graphs.

**Definition 2.1.** Suppose that $\Gamma$ and $\Gamma'$ are two arbitrary graphs with edge sets $E(\Gamma) = \{e_\alpha \mid \alpha \in A\}$ and $E(\Gamma') = \{e_\beta \mid \beta \in B\}$. The tensor product of $\Gamma$ and $\Gamma'$, denoted by $\Gamma \otimes \Gamma'$, is a graph with vertex set $V(\Gamma \otimes \Gamma') = V(\Gamma) \times V(\Gamma')$, and edge set

\[
E(\Gamma \otimes \Gamma') = \{e^\delta_{\alpha,\beta} \mid \alpha \in A, \beta \in B, \delta \in \{0, 1\}\},
\]

where $e^\delta_{\alpha,\beta}$ is an edge joining $(v_0, v'_\delta)$ and $(v_1, v'_1 - \delta)$, given $e_\alpha$ joins $v_0$ and $v_1$ in $\Gamma$, and $e_\beta$ joins $v'_0$ and $v'_1$ in $\Gamma'$.

Note that for simple graphs, the tensor product defined above is exactly the direct product of graphs. Like direct product, each pair of edges from two factors generates two edges in the tensor product, even when loops are involved, as illustrated in Figures 2 and 3. In some literatures such as [1], direct product is defined over graphs without parallel edges but admitting loops. In such definition, a loop serves as the identity of direct product. In particular a loop times an edge is an edge, and a loop times a loop is again a loop, while in our definition a loop times an edge is two parallel edges, and a loop times a loop creates two loops around the same vertex. Since we will need such direct product later, we take a different name and symbol for our generalized product.

There are some reasons to define tensor product in this manner. First, note the number of vertices in $L(X, v)$ is exactly the valency of $v$ in $X$, where a loop at $v$ contributes 2 to the number. Assuming $L(X, v) \bullet L(X', v') \cong L(X \ast X', (v, v'))$, this implies

\[
d_X(v) \cdot d_{X'}(v') = d_{X \ast X'}((v, v')),
\]

which is true for the tensor product, but not for the direct product admitting loops. Secondly, when we glue a face along a loop, the orientation of gluing matters, and the
Definition 2.2. Assume the notation of Definition 2.1. The projection from \( \Gamma \otimes \Gamma' \) to \( \Gamma \), denoted by \( \pi_\Gamma \), is a continuous function such that \( \pi_\Gamma \) maps \( (v, v') \in V(\Gamma \otimes \Gamma') \) to \( v \in V(\Gamma) \), and \( e^{a,\beta}_\Gamma, e^{a,\beta}_\Gamma' \in E(\Gamma \otimes \Gamma') \) to \( e_a \in E(\Gamma) \) isometrically between endpoints. The projection \( \pi_{\Gamma'} \) from \( \Gamma \otimes \Gamma' \) to \( \Gamma' \) is likewise defined.

The projections defined above are graph homomorphisms in the following sense.

Definition 2.3. Let \( \Gamma \) and \( \Gamma' \) be two arbitrary graphs. A continuous function \( \varphi \) from \( \Gamma \) to \( \Gamma' \) is a homomorphism if \( \varphi \) maps each vertex of \( \Gamma \) to a vertex of \( \Gamma' \), and each open edge of \( \Gamma \) isometrically onto an open edge of \( \Gamma' \).

Remark. In the above definition, the continuity of \( \varphi \) is essentially saying that a homomorphism maps incident vertices and edges to incident vertices and edges. Meanwhile, the isometric condition helps to choose a representative from all homotopic maps.

Note that the composition of two graph homomorphisms is again a graph homomorphism. Together with the trivial automorphisms, the class of graphs forms a category. The following proposition shows that the tensor product defined above is actually the categorical product of this category.

Proposition 2.4. Let \( \Gamma \) and \( \Gamma' \) be two arbitrary graphs. Suppose that \( \Gamma_0 \) is a graph with two homomorphisms \( \varphi : \Gamma_0 \rightarrow \Gamma \) and \( \varphi' : \Gamma_0 \rightarrow \Gamma' \). Then there exists a unique homomorphism \( \psi : \Gamma_0 \rightarrow \Gamma \otimes \Gamma' \) such that \( \varphi = \pi_\Gamma \circ \psi \) and \( \varphi' = \pi_{\Gamma'} \circ \psi \). In other words, there exists a unique \( \psi \) such that the diagram in Figure 4 commutes.

Proof. Assume that there exists a continuous function \( \psi : \Gamma_0 \rightarrow \Gamma \otimes \Gamma' \) such that \( \varphi = \pi_\Gamma \circ \psi \) and \( \varphi' = \pi_{\Gamma'} \circ \psi \). Then \( \forall v \in V(\Gamma_0) \), we have \( \varphi(v) = \pi_\Gamma \circ \psi(v) \) and \( \varphi'(v) = \pi_{\Gamma'} \circ \psi(v) \). By Definition 2.2 we know that \( \psi(v) = (\varphi(v), \varphi'(v)) \).

Suppose that \( e \) is an open edge joining \( v \) and \( u \) in \( \Gamma_0 \), and we denote \( \varphi(e) \) and \( \varphi'(e) \) by \( e_a \) and \( e_\beta \) respectively. By the continuity of \( \psi, \psi(e) \) is an open path connecting \( (\varphi(v), \varphi'(v)) \) and \( (\varphi(u), \varphi'(u)) \). Notice that \( e_a = \varphi(e) = \pi_\Gamma \circ \psi(e) \) and \( e_\beta = \varphi'(e) = \pi_{\Gamma'} \circ \psi(e) \). By Definition 2.2 we know \( \psi(e) \) is either \( e^{a,\beta}_\Gamma \) or \( e^{a,\beta}_{\Gamma'} \), determined by endpoints \( (\varphi(v), \varphi'(v)) \) and \( (\varphi(u), \varphi'(u)) \). In case \( e_a \) or \( e_\beta \) is a loop, by keeping track of ends of the loop, \( \psi(e) \) is also uniquely determined. Moreover, the local isometry over open edges of \( \varphi \) and \( \pi_\Gamma \) forces \( \psi \) to map \( e \) isometrically to \( \psi(e) \). Note that we have explicitly constructed a continuous
ψ satisfying our initial assumption. We have also shown that ψ is uniquely determined, and actually a homomorphism, which finishes the proof.

For any two graphs Γ and Γ′, we denote the set of all homomorphisms from Γ to Γ′ by Hom(Γ, Γ′). We have the following corollary about the number of homomorphisms.

**Corollary 2.5.** For any graphs Γ, Γ₁, Γ₂, we have

|Hom(Γ, Γ₁ ⊗ Γ₂)| = |Hom(Γ, Γ₁)| · |Hom(Γ, Γ₂)|.

**Proof.** An immediate consequence of Proposition 2.4.

Note that for any graph Γ, there is a homomorphism from Γ to a loop. Since we distinguish the orientations when we map an edge to a loop, there are actually 2^n such homomorphisms, where n is the number of edges of Γ. In particular, a loop is not the terminal object in the category of arbitrary graphs.

**Corollary 2.6.** Let Γ and Γ′ be two graphs, P be a path in Γ of length n from v to u, and P′ be a path in Γ′ of length n from v′ to u′. Then in Γ ⊗ Γ′, there exists a unique path, denoted by (P, P′)⊗, from (v, v′) to (u, u′) such that πΓ((P, P′)⊗) = P and πΓ′((P, P′)⊗) = P′.

**Proof.** Let I be a graph which is a path of length n. We can give I a specific orientation from one end to the other. Then there is a natural homomorphism ϕ from I to P, as well as one ϕ′ from I to P′. By Proposition 2.3, there exists a unique homomorphism ψ : I → Γ ⊗ Γ′ such that ϕ = πΓ ◦ ψ and ϕ′ = πΓ′ ◦ ψ. Hence we have P = ϕ(I) = πΓ ◦ ψ(I) and P′ = ϕ′(I) = πΓ′ ◦ ψ(I). Note that ψ(I) satisfies the conditions of (P, P′)⊗, and the uniqueness of (P, P′)⊗ follows the uniqueness of ψ.

**Remark.** For simple graphs, this result is straightforward from the definition of tensor product. This corollary clarifies the case when P or P′ contains a loop, where the orientation going through the loop will determine the edge to choose in (P, P′)⊗.

### 3 Complex Tensor Product

To define our complex product more concisely, we would like to extend the notation ( , )⊗ above. Let Γ₁ and Γ₂ be two graphs, C₁ be a cycle of length n in Γ₁, and C₂ be a cycle of length m in Γ₂. Both C₁ and C₂ are assigned initial vertices and orientations.
Specifically, $C_2$ is $(v_0, e_0, v_1, e_1, \ldots, e_{m-1}, v_m = v_0)$, where $v_i \in V(\Gamma_2)$ and $e_j \in E(\Gamma_2)$. Then for $i \in \{0, 1, \ldots, m - 1\}$ we define

$$(C_1, C_2)^{\delta}_{\otimes} := \left( \frac{[n, m]}{n} C_1, \frac{[n, m]}{m} C_2^{\delta} \right)_{\otimes},$$

a cycle of length $[n, m]$ in $\Gamma_1 \otimes \Gamma_2$, where $[n, m]$ is the least common multiple of $n$ and $m$, $kC_1$ is the cycle repeating $C_1$ $k$ times, and $C_2^{\delta}$ is the same cycle as $C_2$, but starting at $v_i$, while $C_2^{\delta}$ is the reversed cycle of $C_2$ starting at $v_i$.

**Definition 3.1.** Let $X$ and $Y$ be two polygonal cell complexes with face sets $F(X) = \{f_\alpha \mid \alpha \in A\}$ and $F(Y) = \{f_\beta \mid \alpha \in B\}$. We denote the boundary length of $f_\alpha$ and $f_\beta$ by $n_\alpha$ and $n_\beta$ respectively, and let $(n_\alpha, n_\beta)$ be the greatest common divisor of $n_\alpha$ and $n_\beta$. The **tensor product** of $X$ and $Y$, denoted by $X \otimes Y$, is a polygonal cell complex with 1-skeleton $X^1 \otimes Y^1$, the tensor product of the 1-skeletons of $X$ and $Y$, and face set

$$F(X \otimes Y) = \{f_{\alpha, \beta}^{\delta} \mid \alpha \in A, \beta \in B, i \in \{0, 1, \ldots, (n_\alpha, n_\beta) - 1\}, \delta \in \{0, 1\}\},$$

where $f_{\alpha, \beta}^{\delta}$ is a face attached along $(C_\alpha, C_\beta)^{\delta}_{\otimes}$, while $C_\alpha$ is the cycle along which $f_\alpha$ is attached in $X$, and $C_\beta$ is the cycle along which $f_\beta$ is attached in $Y$.

**Remark.** We will use the jargon that $f_{\alpha, \beta}^{\delta}$ is generated by $f_\alpha$ and $f_\beta$, especially when faces are not clearly indexed. In the above definition, note that $(C_\alpha, C_\beta)^{\delta}_{\otimes}$ and $(C_\alpha, C_\beta)^{\delta + (n_\alpha, n_\beta)}_{\otimes}$ are identical cycles with different starting vertices. To get a pair of corners of $f_\alpha$ and $f_\beta$ contribute to exactly one face corner in $X \otimes Y$, we only choose $i \in \{0, 1, \ldots, (n_\alpha, n_\beta) - 1\}$. Here we discard repeated corner pairs, not faces in $X \otimes Y$ attached along the same cycle. For example, let $X$ and $Y$ be 15-gons wrapped around a cycle of length 3 and 5 respectively. Note that the tensor product of a triangle and a pentagon is not the same as $X \otimes Y$. The former has only $2 \cdot (3, 5) = 2$ faces, while $X \otimes Y$ has $2 \cdot (15, 15) = 30$ faces in two groups, each of which has 15 faces with cyclically identical attaching maps.

In the example of a triangle tensor a pentagon, the only two faces meet at every vertex in the product. In general, when $n_\alpha \neq n_\beta$, two faces $f_{\alpha, \beta}^{\delta}$ and $f_{\alpha, \beta}^{\delta'}$ meet at more than one vertex. Therefore the tensor product of two polygonal complexes is not necessarily polygonal. How about the case when $n_\alpha = n_\beta$? For $n_\alpha$ even, note that $f_{\alpha, \beta}^{\delta}$ and $f_{\alpha, \beta}^{\delta'}$ have two vertices $(0, 0)$ and $(\frac{n_\alpha}{2}, \frac{n_\beta}{2})$ in common, and the tensor product is not polygonal. For odd cases, we have the following result.

**Proposition 3.2.** Suppose that $X$ and $Y$ are polygonal complexes with all faces of the same odd length $n$. Then the tensor product $X \otimes Y$ is a polygonal complex.

**Proof.** Since $X$ and $Y$ are polygonal complexes, we know that $X^1$ and $Y^1$ are simple graphs, and hence the 1-skeleton of $X \otimes Y$, namely $X^1 \otimes Y^1$, is a simple graph as well. Consider the boundary of an arbitrary face $f_{\alpha, \beta}^{\delta}$ in $X \otimes Y$, namely $(C_\alpha, C_\beta)^{\delta}_{\otimes}$. Note that $C_\alpha$ and $C_\beta$ are both simple closed cycles of the same length $n$, as they are boundaries of faces of polygonal complexes. Therefore $(C_\alpha, C_\beta)^{\delta}_{\otimes}$ is a simple closed cycle of length $n$. In brief, every face of $X \otimes Y$ is attached along a simple closed cycle.

Now all we have to show is that the intersection of two faces in $X \otimes Y$ is either empty, a vertex, or an edge in $X \otimes Y$. Suppose that there exist two faces $f_{\alpha, \beta}^{\delta}$ and $f_{\alpha', \beta'}^{\delta'}$ in $X \otimes Y$.
such that the intersection of $f_{\alpha,\beta}^{i}$ and $f_{\alpha,\beta}^{j}$ is neither empty, a vertex, nor an edge. For the case of $n = 3$, it is not hard to see that $f_{\alpha,\beta}^{i}$ and $f_{\alpha,\beta}^{j}$ share the same boundary, and in fact are the same face by the polygonality of $X$ and $Y$. For the case of odd $n > 3$, note that $f_{\alpha,\beta}^{i}$ and $f_{\alpha,\beta}^{j}$ share two vertices which are not consecutive on the boundary of faces. By the polygonality of $X$ and $Y$, this implies that $f_{\alpha} = f_{\alpha}^{i}$ and $f_{\beta} = f_{\beta}^{j}$. Consider the boundaries of $f_{\alpha}^{i}$ and $f_{\alpha,\beta}^{j}$, namely $(C_{\alpha}, C_{\beta})_{\odot}^{i}$ and $(C_{\alpha}, C_{\beta})_{\odot}^{j}$. When $\delta = \delta'$ and $i \neq j$, $(C_{\alpha}, C_{\beta})_{\odot}^{i}$ and $(C_{\alpha}, C_{\beta})_{\odot}^{j}$ have no vertex in common. When $\delta \neq \delta'$, notice that a common vertex of $(C_{\alpha}, C_{\beta})_{\odot}^{i}$ and $(C_{\alpha}, C_{\beta})_{\odot}^{j}$ corresponds to an integer $m$ such that

$$j + m \equiv i - m \mod n \iff 2m = i - j \mod n,$$

which has a unique solution when $n$ is odd. In other words, when $\delta \neq \delta'$, $(C_{\alpha}, C_{\beta})_{\odot}^{i}$ and $(C_{\alpha}, C_{\beta})_{\odot}^{j}$ intersect at exactly one vertex. Since $f_{\alpha,\beta}^{i}$ and $f_{\alpha,\beta}^{j}$ share two vertices, we can conclude that $\delta = \delta'$ and $i = j$. This finishes the proof.

The complex tensor product does not preserve simple connectedness either.

**Proposition 3.3.** Let $X$ and $Y$ be an $n$-gon and $m$-gon respectively, where $n$ and $m$ are two positive integers. Then $X \otimes Y$ is simply-connected if and only if $n = m = 1$.

**Proof.** When $n = m = 1$, the 1-skeleton of $X \otimes Y$ is a vertex with two loops, as illustrated in Figure 3, and $X \otimes Y$ has two faces attached along these two loops respectively. In this case, $X \otimes Y$ is actually contractible, and of course simply-connected.

Now suppose that $n$ and $m$ are not both equal to 1. Without loss of generality, we can assume $n \geq 2$. Note that $X$ has $n$ vertices, $n$ edges, and 1 face, whereas $Y$ has $m$ vertices, $m$ edges, and 1 face. By Definition 3.1, the complex $X \otimes Y$ has $nm$ vertices, $2nm$ edges, and $2(n, m)$ faces. Therefore $X \otimes Y$ has Euler characteristic

$$\chi(X \otimes Y) = nm - 2nm + 2(n, m) = -nm + 2(n, m) \leq -2m + 2m = 0.$$  

By the following lemma, we know $X \otimes Y$ is not simply-connected. □

**Lemma 3.4.** Suppose $X$ is a finite simply-connected polygonal cell complex. Then the Euler characteristic of $X$ is at least 1.

**Proof.** Suppose $X$ has $v$ vertices, $e$ edges, and $f$ faces. First we find an arbitrary spanning tree $T$ for the 1-skeleton of $X$, and then contract $T$ to get a new complex $X'$, which is also simply-connected. Note that $T$ has $v - 1$ edges, and therefore $X'$ has 1 vertex, $e - v + 1$ edges, and $f$ faces. The fundamental group $\pi_1(X')$, a trivial group, can be presented as a group with $e - v + 1$ generators and $f$ relators. Consider the abelianization of $\pi_1(X')$, which is again trivial. Then the presentation can be expressed as $f$ homogeneous equations of $e - v + 1$ unknowns over $\mathbb{Z}$. To have only trivial solution, the number of equations needs to be at least the number of unknowns. So we have $f \geq e - v + 1$, and therefore $v - e + f \geq 1$. □

**Remark.** Let $X$ and $Y$ be two arbitrary complexes, and $C$ be a cycle along the 1-skeleton of $X \otimes Y$. This proposition shows that the contractibility of $\pi_X(C)$ and $\pi_Y(C)$ does not guarantee the contractibility of $C$. Conversely, when $C$ is contractible in $X \otimes Y$, we can
conclude that \( \pi_X(C) \) and \( \pi_Y(C) \) are contractible? The answer is positive. We can find a series \( \{C_j\} \) of homotopic cycles of \( C \) such that \( C_0 = C \), \( C_n \) is a vertex, and each \( C_j \) morphs through a single face \( f^\delta_{\alpha,\beta} \) to obtain \( C_{j+1} \). Note that \( \pi_X(C_j) \) can morph through a single face \( f_\alpha \) to obtain \( \pi_X(C_{j+1}) \), even when the length of \( f_\alpha \) properly divides the length of \( f^\delta_{\alpha,\beta} \). Therefore \( \pi_X(C) = \pi_X(C_0) \) is homotopic to \( \pi_X(C_n) \), which is a vertex.

In the above remark, we actually abuse the notation \( \pi_X \), as we have not yet defined projection maps for complex tensor products. To define such projection maps, first we introduce some terminology. Let \( X \) and \( Y \) be an \( n \)-gon and \( m \)-gon with centre \( O_X \) and \( O_Y \) respectively. A function \( \rho : X \rightarrow Y \) is radial if \( \rho \) sends \( O_X \) to \( O_Y \), \( \partial X \) to \( \partial Y \), and for every point \( P \in \partial X \), every real number \( t \in [0, 1] \), we have

\[
\rho(t \cdot O_X + (1-t)P) = t \cdot O_Y + (1-t)\rho(P).
\]

**Definition 3.5.** Assume the notation of Definition 3.1. The projection from \( X \otimes Y \) to \( X \), denoted by \( \pi_X \), is a continuous function such that \( \pi_X \) restricted to \( X^1 \otimes Y^1 \) is exactly \( \pi_X^1 \), the projection of the graph tensor product, and \( \pi_X \) maps \( f^\delta_{\alpha,\beta} \in F(X \otimes Y) \) radially to \( f_\alpha \in F(X) \). The projection \( \pi_Y \) from \( X \otimes Y \) to \( Y \) is likewise defined.

The projection maps defined above are complex homomorphisms in the following sense.

**Definition 3.6.** Let \( X \) and \( Y \) be two polygonal cell complexes. A continuous function \( \varphi \) from \( X \) to \( Y \) is a homomorphism if \( \varphi \) restricted to \( X^1 \) is a graph homomorphism to \( Y^1 \), and \( \varphi \) maps each face of \( X \) radially to a face of \( Y \) and each open face corner (ignoring the boundary) of \( X \) homeomorphically to an open face corner of \( Y \).

**Remark.** In the above definition, the continuity of \( \varphi \) is essentially saying that a complex homomorphism maps incident cells to incident cells. Similar to the isometric condition in graph homomorphism, the radial condition is imposed to rule out homotopic complex homomorphisms. Most important of all, the homeomorphic corner condition forces a face of \( X \) to wrap around a face \( f \) of \( Y \) along the direction of the attaching map of \( f \), possibly more than once. In particular, a face of length \( n \) can only be mapped to a face of length dividing \( n \). Figure 5 illustrates such phenomenon, where corners are mapped to a corner with the same label. The projection \( \pi_X \) of complex tensor product mapping \( f^\delta_{\alpha,\beta} \) to \( f_\alpha \) is also a typical example.

Note that the composition of two complex homomorphisms is again a complex homomorphism. Together with the trivial automorphisms, the class of polygonal cell complexes forms a category. The following proposition shows that the complex tensor product defined above is actually the categorical product of this category.
Proposition 3.7. Let $X$ and $Y$ be two polygonal cell complexes. Suppose that $Z$ is a complex with two homomorphisms $\varphi_X : Z \to X$ and $\varphi_Y : Z \to Y$. Then there exists a unique homomorphism $\psi : Z \to X \otimes Y$ such that $\varphi_X = \pi_X \circ \psi$ and $\varphi_Y = \pi_Y \circ \psi$. In other words, there exists a unique $\psi$ such that the diagram in Figure 6 commutes.

Proof. Assume that there exists a continuous function $\psi : Z \to X \otimes Y$ such that $\varphi_X = \pi_X \circ \psi$ and $\varphi_Y = \pi_Y \circ \psi$. Note that $\varphi_X$, $\varphi_Y$, $\pi_X$, and $\pi_Y$ restricted to the 1-skeletons of their domains are all graph homomorphisms. By Proposition 2.4, the restriction of $\psi$ to $Z^1$ is a uniquely determined graph homomorphism to $X^1 \otimes Y^1$.

Suppose that $f$ is a face in $Z$, $\varphi_X(f)$ wraps around a face $f_\alpha$ in $X$, and $\varphi_Y(f)$ wraps around a face $f_\beta$ in $X$. Then $\varphi_X(f) = \pi_X \circ \psi(f)$ wraps around $f_\alpha$, and $\varphi_Y(f) = \pi_Y \circ \psi(f)$ wraps around $f_\beta$. By Definition 3.5, $\psi(f)$ must wrap around $f_{\alpha,\beta}^i$ for some $i$ and $\delta$. Let $c$ be a corner of $f$. Then we must have $\varphi_X(c) = \pi_X \circ \psi(c)$ and $\varphi_Y(c) = \pi_Y \circ \psi(c)$. By the remark after Definition 3.1, this pair of corners $(\varphi_X(c), \varphi_Y(c))$, orientation included, appears in exactly one $f_{\alpha,\beta}^i$. Therefore $i$ and $\delta$ are uniquely determined, and $\psi(f)$ wraps around this $f_{\alpha,\beta}^i$. Moreover, the radiality of $\varphi_X$ and $\pi_X$ forces $\psi$ to map $f$ radially to $f_{\alpha,\beta}^i$. Note that we have explicitly constructed a continuous $\psi$ satisfying our initial assumption. We have also shown that $\psi$ is uniquely determined, and actually a complex homomorphism, which finishes the proof. 

Remark. For any two complexes $X$ and $Y$, we denote the set of all complex homomorphisms from $X$ to $Y$ by $\text{Hom}(X,Y)$. Similarly to Corollary 2.5 we have

$$|\text{Hom}(Z, X \otimes Y)| = |\text{Hom}(Z, X)| \cdot |\text{Hom}(Z, Y)|.$$

As we mentioned earlier, for any graph $\Gamma$, there is a homomorphism from $\Gamma$ to a loop. It is reasonable to ask the following question: for any complex $X$, is there always a homomorphism from $X$ to a 1-gon? The answer is negative. Take Figure 7 as an example. Once the image of the leftmost edge is determined, it determines the image of all other edges. If we identify the leftmost and the rightmost edges with a twist, i.e. making it a Mobius strip, then there is no way to have a homomorphism. Note that this question is not related to orientability. If the complex is a strip with 3 squares, then the Mobius case has a homomorphism, while the orientable case does not.

Proposition 3.8. Let $X$ and $Y$ be two polygonal cell complexes, and $\varphi : X \to Y$ be a complex homomorphism mapping a vertex $v \in V(X)$ to $u \in V(Y)$. Then $\varphi$ induces a graph homomorphism $L(\varphi)$ from $L(X,v)$ to $L(Y,u)$. Moreover, let $Z$ be another complex and $\rho : Y \to Z$ be a complex homomorphism mapping $u$ to $w \in V(Z)$. Then we have $L(\rho \circ \varphi) = L(\rho) \circ L(\varphi)$, as illustrated in Figure 8.
Figure 7: a homomorphism to a 1-gon

Figure 8: functoriality of $L$

**Proof.** By definition, $L(X, v)$ has vertices corresponding to edge ends around $v$ in $X$, and edges corresponding to face corners at $v$ in $X$. Since $\varphi$ restricted to $X^1$ is a graph homomorphism, $\varphi$ maps an edge end around $v$ in $X$ to an edge end around $u$ in $Y$. In addition, by the homeomorphic condition in Definition 3.6, $\varphi$ maps a face corner at $v$ joining two edge ends around $v$ homeomorphically to a face corner at $u$ joining two edge ends around $u$. Therefore $\varphi$ induces a graph homomorphism $L(\varphi)$ from $L(X, v)$ to $L(Y, u)$. Once these induced graph homomorphisms between link graphs are defined, the equality $L(\rho \circ \varphi) = L(\rho) \circ L(\varphi)$ follows immediately.

**Remark.** To each polygonal cell complex, we can assign a distinguished vertex to be the basepoint. Together with basepoint-preserving homomorphisms, the class of pointed polygonal cell complexes also forms a category. The above proposition is essentially saying that $L$ is a functor from this category to the category of graphs.

Now we move back to the main purpose of this chapter: to develop a complex product interacting nicely with some product of link graphs. From the above discussion, we know that the complex tensor product arises naturally in the category of polygonal cell complexes. Does this natural categorical product fulfill the main job? Yes, it does.

**Theorem 3.9.** Suppose that $X$ and $Y$ are two polygonal cell complexes, and $v$ and $u$ are two vertices in $X$ and $Y$ respectively. Then we have

$$L(X, v) \otimes L(Y, u) \cong L(X \otimes Y, (v, u)).$$

**Proof.** We can identify edge ends incident to a vertex as paths of length 1 leaving the vertex, since a loop contributes to two edge ends as well as two such paths, which we call 1-paths for short. By Corollary 2.6, there is a bijection between 1-paths leaving $(v, u)$ in $X \otimes Y$ and pairs of 1-path leaving $v$ in $X$ and 1-path leaving $u$ in $Y$. Therefore we can index 1-paths leaving $(v, u)$ in $X \otimes Y$ by such 1-path pairs in $X$ and $Y$. 

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Suppose that  \( f_\alpha \in F(X) \) has a corner  \( c_\alpha \) at  \( (e_{\alpha_1}, v, e_{\alpha_2}) \), and  \( f_\beta \in F(Y) \) has a corner  \( c_\beta \) at  \( (e_{\beta_1}, u, e_{\beta_2}) \), as illustrated in Figure 9. These  \( e_i \)'s should be understood as 1-paths. By the remark after Definition 3.1, the pairing of these two corners appears exactly once in  \( f_{i_0}^{\alpha,\beta} \) and  \( f_{i_1}^{\alpha,\beta} \) respectively, forming corners  \( ((e_{\alpha_1}, e_{\beta_1}), (v, u), (e_{\alpha_2}, e_{\beta_2})) \) and  \( ((e_{\alpha_1}, e_{\beta_2}), (v, u), (e_{\alpha_2}, e_{\beta_1})) \) in  \( X \otimes Y \). Note that by taking projection maps, we know that any face corner at  \((v, u)\) comes from some pairing of corners at  \( v \) and  \( u \).

Now we translate the above statements in terms of corresponding link graphs. First of all, we have  \( V(L(X, v)) \times V(L(Y, u)) \cong V(L(X \otimes Y, (v, u))) \). Secondly, the corner  \( c_\alpha \) is an edge joining vertices  \( e_{\alpha_1} \) and  \( e_{\alpha_2} \) in  \( L(X, v) \), and  \( c_\beta \) is an edge joining vertices  \( e_{\beta_1} \) and  \( e_{\beta_2} \) in  \( L(Y, u) \). Notice that the edge pair  \( (c_\alpha, c_\beta) \) contributes to one edge joining  \( (e_{\alpha_1}, e_{\beta_1}) \) and  \( (e_{\alpha_2}, e_{\beta_2}) \), and one edge joining  \( (e_{\alpha_1}, e_{\beta_2}) \) and  \( (e_{\alpha_2}, e_{\beta_1}) \) in  \( L(X \otimes Y, (v, u)) \). Meanwhile, taking all possible pairings of edges exhausts all edges in  \( L(X \otimes Y, (v, u)) \). By Definition 2.1 this is exactly saying that  \( L(X, v) \otimes L(Y, u) \cong L(X \otimes Y, (v, u)) \).

**Remark.** In the terminology of category theory, this theorem is essentially saying that the functor  \( L \) from the category of pointed complexes to the category of graphs preserves categorical products, which is not always true for an arbitrary functor.

As indicated in Propositions 3.2 and 3.3, the complex tensor product does not necessarily preserve polygonality and simple connectedness. Fortunately, complex tensor product does preserve the most important property for our purpose.

**Theorem 3.10.** Let  \( X \) and  \( Y \) be any two flag-transitive polygonal cell complexes. Then the complex tensor product  \( X \otimes Y \) is flag-transitive.

**Proof.** In case  \( X \) or  \( Y \) has no faces, then  \( X \otimes Y \) is simply a graph, and the flag-transitivity follows easily from the definition of graph tensor product. Hereafter we assume that both  \( X \) and  \( Y \) have at least one face.
Let \((e_{\alpha_1}, e_{\beta_1}), (v, u), (e_{\alpha_2}, e_{\beta_2})\) be a face corner in \(X \otimes Y\), which projects to a corner \((e_{\alpha_1}, e_{\alpha_2})\) in \(X\) and a corner \((e_{\beta_1}, u, e_{\beta_2})\) in \(Y\), as illustrated in Figure 9. Let \(((e'_{\alpha_1}, v'), (v', u'), (e'_{\alpha_2}, e'_{\beta_2}))\) be another face corner in \(X \otimes Y\), which projects to a corner \((e'_{\alpha_1}, v'), (e'_{\alpha_2})\) in \(X\) and a corner \((e'_{\beta_1}, u', e'_{\beta_2})\) in \(Y\), as illustrated in Figure 10. Since \(X\) and \(Y\) are flag-transitive, there exist \(\rho \in \text{Aut}(X)\) mapping \((e_{\alpha_1}, v, e_{\alpha_2})\) to \((e'_{\alpha_1}, v', e'_{\alpha_2})\) and \(\sigma \in \text{Aut}(Y)\) mapping \((e_{\beta_1}, u, e_{\beta_2})\) to \((e'_{\beta_1}, u', e'_{\beta_2})\). Comparing Figures 9 and 10, note that \((\rho, \sigma)\) gives an automorphism of \(X \otimes Y\) mapping \(((e_{\alpha_1}, e_{\beta_1}), (v, u), (e_{\alpha_2}, e_{\beta_2}))\) to \(((e'_{\alpha_1}, e'_{\beta_1}), (v', u'), (e'_{\alpha_2}, e'_{\beta_2}))\). The above discussion shows that \(\text{Aut}(X \otimes Y)\) acts transitively on face corners with orientations, and therefore transitively on half-corners. In other words, \(\text{Aut}(X \otimes Y)\) acts transitively on flags.

**Remark:** In Figure 9, flipping both corners in \(X\) and \(Y\) will flip both corners in \(X \otimes Y\), whereas flipping only one corner in either \(X\) or \(Y\) will swap two corners in \(X \otimes Y\).

## 4 Factorization and Symmetry

In the proof of Theorem 3.10, the key fact we used is the following relation:

\[
\text{Aut}(X) \times \text{Aut}(Y) \leq \text{Aut}(X \otimes Y).
\]

Is it possible that these two groups are actually isomorphic? When \(X\) and \(Y\) are isomorphic, we can swap \(X\) and \(Y\) to obtain an extra automorphism, since the complex tensor product is commutative up to isomorphism. In addition to swapping, the following proposition gives more extra automorphisms in a less obvious way.

**Proposition 4.1.** Let \(X\), \(Y\), and \(Z\) be polygonal cell complexes. Then we have

\[
(X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z).
\]

In other words, complex tensor product is associative up to isomorphism.

**Proof.** A categorical result of the universal property in Proposition 3.7. See [6].

The associativity of the complex tensor product complicates \(\text{Aut}(X \otimes Y)\). For example, if \(Y\) can be factorized into \(X \otimes Z\), then \(X \otimes Y \cong X \otimes (X \otimes Z)\) has an automorphism swapping the two copies of \(X\). Hence the symmetry of the product of complexes is also related to the factoring of complexes. In response to associativity, we modify the original question as follows: for complexes \(X_i\) which are irreducible with respect to complex tensor product, is the automorphism group \(\text{Aut}(\otimes X_i)\) generated by automorphisms of \(X_i\)’s, together with permutations of isomorphic factors? By a **Cartesian automorphism**, we mean an element in the subgroup of \(\text{Aut}(\otimes X_i)\) generated in the above manner.

There have been lots of studies about the symmetry of different products of graphs. One of the major goals of this chapter is to apply the theory of the graph direct product to the complex tensor product. Hence we first introduce related theorems about the graph direct product. The book [4] by Hammack, Imrich, and Klavzar offers a comprehensive survey of products of graphs, and we shall follow their approach and terminology here.

We briefly mentioned the direct product of graphs in Chapter 2. Here we give the definition again, with an emphasis on the possible presence of loops. We say that a graph \(\Gamma\) is a **simple graph with loops admitted** if for any \(u, v \in V(\Gamma)\), there is at most one
edge joining \( u \) and \( v \), including the case \( u = v \). In particular, there is at most one loop at a vertex. For convenience, we use \( \mathcal{S} \) to denote the class of simple graphs, and \( \mathcal{S}_0 \) to denote the class of simple graphs with loops admitted.

**Definition 4.2.** Let \( \Gamma \) and \( \Gamma' \) be two graphs in \( \mathcal{S}_0 \). The **direct product** of \( \Gamma \) and \( \Gamma' \), denoted by \( \Gamma \times \Gamma' \), is a graph in \( \mathcal{S}_0 \) with vertex set \( V(\Gamma \times \Gamma') = V(\Gamma) \times V(\Gamma') \). There is an edge joining two vertices \((v, v')\) and \((u, u')\) in \( \Gamma \times \Gamma' \) if and only if there is an edge joining \( v \) and \( u \) in \( \Gamma \), and there is an edge joining \( v' \) and \( u' \) in \( \Gamma' \).

Note in the above definition, \( v \) and \( v' \) could be the same vertex, as well as \( u \) and \( u' \). Figure 11 illustrates the direct product of two graphs in \( \mathcal{S}_0 \). Under this definition, notice that a loop \( L \) serves as the identity element of direct product of graphs. In other words, for any simple graph \( \Gamma \) with loops admitted, we always have

\[
L \times \Gamma \cong \Gamma \times L \cong \Gamma.
\]

Also note that the direct product of two edges is again two edges, laid out as a cross in the figure, which is part of the reason why graph theorists choose the symbol “\( \times \)” [4]. Therefore the direct product of two connected graphs is not necessarily connected. The following theorem is known as Weichsel’s Theorem [4].

**Theorem 4.3.** Suppose that \( \Gamma \) and \( \Gamma' \) are two connected simple graphs with at least two vertices. If \( \Gamma \) and \( \Gamma' \) are both bipartite, then \( \Gamma \times \Gamma' \) has exactly two components. If at least one of \( \Gamma \) and \( \Gamma' \) is not bipartite, then \( \Gamma \times \Gamma' \) is connected.

**Proof.** The first part of the theorem is straightforward. For the second part, note that a simple graph is not bipartite if and only if there is an odd cycle in the graph. By exploiting such a cycle properly, the second part of the theorem follows. For a detailed proof, please refer to Theorem 5.9 in [3].

A graph \( \Gamma \) is **prime** if \( \Gamma \) has more than one vertex, and \( \Gamma \cong \Gamma_1 \times \Gamma_2 \) implies that either \( \Gamma_1 \) or \( \Gamma_2 \) is a loop. Note that the idea of being prime depends on the class of graphs we are talking about. For example, let \( \Gamma \) be a path of length 3, which has 4 vertices. Then \( \Gamma \) is prime in \( \mathcal{S} \), as the only possible factoring is the product of two edges, which is the disjoint union of two edges. And the statement that \( \Gamma \cong \Gamma_1 \times \Gamma_2 \) implies either \( \Gamma_1 \) or \( \Gamma_2 \) is a loop is still logically true. However, \( \Gamma \) can be factorized in \( \mathcal{S}_0 \) as the graph on the left of Figure 11 times one edge in the bottom, and hence \( \Gamma \) is not prime in \( \mathcal{S}_0 \).
Consider the question of factoring a graph into the product of prime graphs. For a finite graph, such a prime factorization always exists, since the number of vertices of factors decreases as the factoring goes. However, such a prime factorization is not necessarily unique, and it depends on the graph itself and the class of graphs where we do the factoring. For example, a path of length 3 together with associativity can be used to create graphs with non-unique prime factorizations in $\mathcal{S}$. There are also graphs with non-unique prime factorizations in $\mathcal{S}_0$, an example of which can be found in [4]. The following theorem of unique prime factorization is due to McKenzie [8].

**Theorem 4.4.** Suppose that $\Gamma \in \mathcal{S}_0$ is a finite connected non-bipartite graph with more than one vertex. Then $\Gamma$ has a unique factorization into primes in $\mathcal{S}_0$.

The next question is about the automorphism group of direct product, which hopefully has only these Cartesian automorphisms with respect to the product. Note that a pair of vertices with the same set of neighbours creates pairs of vertices with the same set of neighbours in the direct product, and results in lots of non-Cartesian automorphisms. This phenomenon is illustrated in Figure 12, where a vertex with a loop should have itself as a neighbour. We say that a graph is $R$-thin if there are no vertices with the same set of neighbours. In addition to $R$-thinness, the disconnectedness due to Theorem 4.3 also creates non-Cartesian automorphisms. Even when the direct product is connected, there might still be some exotic automorphisms. The following theorem is due to Dörfler [3].

**Theorem 4.5.** Suppose that $\Gamma \in \mathcal{S}_0$ is a finite connected non-bipartite $R$-thin graph with a prime factorization $\Gamma = \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_n$ in $\mathcal{S}_0$. Then $\text{Aut}(\Gamma)$ is generated by automorphisms of prime factors and permutations of isomorphic factors.

We would like to use Theorems 4.4 and 4.5 to develop similar results for the complex tensor product. The first problem we immediately encounter is that, for the complex tensor product, we obtain the 1-skeleton of the product through the graph tensor product, which is not exactly the same as the direct product of graphs. Fortunately, such a difference does not really take place in graphs with higher symmetries.

**Proposition 4.6.** Let $\Gamma \in \mathcal{S}_0$ be a finite connected non-bipartite $R$-thin graph with more than one vertex, and $\Gamma = \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_n$ be the unique prime factorization in $\mathcal{S}_0$. If $\Gamma$ is edge-transitive, then $\Gamma$ and each prime factor $\Gamma_i$ are in $\mathcal{S}$.
Proof. Since $\Gamma$ has more than one vertex, the connectedness of $\Gamma$ implies that $\Gamma$ has a non-loop edge. By the edge-transitivity of $\Gamma$, we know $\Gamma$ has no loop, and hence is in $\mathcal{S}$. If each factor $\Gamma_i$ has a loop, then the product $\Gamma$ will have a loop, which is not true. If each factor $\Gamma_i$ is loop-free, then we have finished the proof. Hence we can assume there is at least one factor with a loop, and at least one factor without a loop.

Let $\Gamma_\alpha$ be the direct product of all factors with a loop, and $\Gamma_\beta$ be the direct product of all factors without a loop. Then we have $\Gamma = \Gamma_\alpha \times \Gamma_\beta$. Note that permuting isomorphic factors of $\Gamma$ does not involve permuting factors of $\Gamma_\alpha$ with factors of $\Gamma_\beta$. By Theorem 4.5, we have $\text{Aut}(\Gamma) = \text{Aut}(\Gamma_\alpha) \times \text{Aut}(\Gamma_\beta)$. Since a prime factor has more than one vertex, $\Gamma_\alpha$ and $\Gamma_\beta$ both have more than one vertex. Since $\Gamma$ is connected, $\Gamma_\alpha$ and $\Gamma_\beta$ are both connected. Hence $\Gamma_\alpha$ has a loop at some vertex $v$ and a non-loop edge joining two vertices $v_a$ and $v'_a$, while $\Gamma_\beta$ has a non-loop edge joining two vertices $v_\beta$ and $v'_\beta$. Then in $\Gamma = \Gamma_\alpha \times \Gamma_\beta$, there is an edge joining $(v, v_\beta)$ and $(v, v'_\beta)$, and another edge joining $(v_a, v_\beta)$ and $(v'_a, v'_\beta)$. Notice that $\text{Aut}(\Gamma) = \text{Aut}(\Gamma_\alpha) \times \text{Aut}(\Gamma_\beta)$ can not send the first edge to the second one, contradicting the assumption that $\Gamma$ is edge-transitive. \qed

Remark. To visually interpret the last few lines of the proof, it says that a Cartesian product cannot permute horizontal edges with slant edges in Figure 12.

Now we move on to the factorization of polygonal cell complexes. First consider the following example. Let $X$ and $Y$ be a triangle and a pentagon respectively, $X'$ be a cycle of length 3 with two triangles attached, and $Y'$ be a cycle of length 5 with two pentagons attached. Since the numbers of vertices of these complexes are prime, the only possible way to factorize them is to have a factor of one vertex with at least a loop and a face, which creates double edges in the product. Hence we know these complexes can not be factorized further, and we have non-unique factorizations $X \otimes Y' \cong X' \otimes Y$.

Here we give another example of non-unique factorization. Let $X$ be a triangle, and $Y'$ be a $(7 \cdot 5)$-gon wrapped around a cycle of length 5. By Definition 3.1, since 3 and 7·5 are coprime, $X \otimes Y'$ has two faces of length $3 \cdot 5 = 15$, wrapped around two cycles of length 3·5 for 7 rounds. Consider a $(7 \cdot 3)$-gon $X'$ wrapped around a cycle of length 3, and a pentagon $Y$. It is easy to see that $X \otimes Y' \cong X' \otimes Y$, and these complexes can not be factorized further. To avoid these non-uniquly factorized situations, we restrict our discussion to the factorization of simple complexes.

Definition 4.7. A polygonal cell complex $X$ is a simple complex if $X$ has at least one face, $X$ has no pairs of faces attached along the same cycle, and the attaching map of each face does not wrap around a cycle more than once. A polygonal cell complex $X$ is a prime complex if there do not exist complexes $X_1$ and $X_2$ such that $X = X_1 \otimes X_2$.

Remark. Figure 13 above is a simple complex with two 1-gons. If we add another 2-gon attached along two different loops, the resulting complex is still a simple complex, as the boundary cycles of theses faces are not exactly the same.

To factorize a complex $X$, our general setting is as follows. We assume that we know a factorization of the 1-skeleton $X^1 = \Gamma_1 \otimes \Gamma_2$, and try to find a complex factorization $X = X'_1 \otimes X'_2$ such that $X'_1 = \Gamma_1$ and $X'_2 = \Gamma_2$. A natural thought is to project the faces of $X$ down to $\Gamma_1$ and $\Gamma_2$ to be faces. Consider the complex tensor product of a triangle and a pentagon, which is a complex with two 15-gons. Note that when we project these two 15-gons back to the 1-skeletons of factors, what we obtain are 15-gons wrapped around cycles of length 3 and 5 respectively, not the original faces.
Definition 4.8. Let $X$ be a polygonal cell complex, $f$ be a face of $X$ attached along a cycle $C_f$, and $\Gamma_1$ and $\Gamma_2$ be two graphs such that $X^1 = \Gamma_1 \otimes \Gamma_2$. The reductive projection of $f$ to $\Gamma_i$, denoted by $\pi_{\Gamma_i}(f)$, is a face attached along the reduced cycle of $\pi_{\Gamma_i}(C_f)$ in $\Gamma_i$, namely the shortest cycle $C$ such that repeating $C$ gives $\pi_{\Gamma_i}(C_f)$.

Remark. In exactly the same way, we can define $\pi_{\Gamma_i}(f)$ for the case $X^1 = \otimes_{i=1}^n \Gamma_i$. Note that when $X^1 = \Gamma_1 \otimes \Gamma_2 \otimes \Gamma_3$, we have $\pi_{\Gamma_1}(f) = \pi_{\Gamma_1}(\pi_{\Gamma_1 \otimes \Gamma_3}(f)) = \pi_{\Gamma_1}(\pi_{\Gamma_1 \otimes \Gamma_2}(f))$.

Proposition 4.9. Let $X$ be a simple complex, and $\Gamma_1$ and $\Gamma_2$ be two graphs such that $X^1 = \Gamma_1 \otimes \Gamma_2$. If there exist two complexes $X_1$ and $X_2$ with 1-skeletons $\Gamma_1$ and $\Gamma_2$ respectively such that $X = X_1 \otimes X_2$, then $X_1$ and $X_2$ are simple complexes whose faces are precisely the reductive projections of faces of $X$.

Proof. Suppose that such complexes $X_1$ and $X_2$ exist. Let $f$ be a face of $X$ attached along a cycle $C_f$ of length $n$, and let $C_j$ of length $n_j$ be the reduced cycle of $\pi_{\Gamma_j}(C_f)$ in $X_j$ for $j \in \{1, 2\}$. Note that $f$ is generated by a face $f_1$ of $X_1$ attached along $m_1 C_1$, and by a face $f_2$ of $X_2$ attached along $m_2 C_2$, where $m_i C_i$ is the cycle made by repeating $C_i$ for $m_i$ times. By Definition 3.1, $f_1$ and $f_2$ generate faces attached along $(m_1 C_1, m_2 C_2)^{\delta}$, where $i \in \{0, 1, \ldots, (m_1 n_1, m_2 n_2) - 1\}$ and $\delta \in \{0, 1\}$. By the Euclidean algorithm, we can find an integer $\delta > 0$ such that $\delta \equiv 0 \mod n_1$ and $\delta \equiv (n_1, n_2) \mod n_2$. Note that in $k$ steps along $(m_1 C_1, m_2 C_2)^{\delta}$, we can walk from the starting vertex of $(m_1 C_1, m_2 C_2)^0$ to the starting vertex of $(m_1 C_1, m_2 C_2)^{(n_1, n_2)^{\delta}}$, so these two cycles are identical. Since $X$ is simple, there are no pairs of faces attached along the same cycle in $X$. Therefore we have $(n_1, n_2) \geq (m_1 n_1, m_2 n_2) \geq (n_1, n_2)$. Now consider the length of the face $f$, which is

$$n = [m_1 n_1, m_2 n_2] = \frac{m_1 n_1 \cdot m_2 n_2}{(m_1 n_1, m_2 n_2)} = \frac{m_1 m_2 \cdot n_1 n_2}{(n_1, n_2)} = m_1 m_2 \cdot [n_1, n_2].$$

This shows that $f$ is attached along some cycle $(C_1, C_2)^{\delta}$ of length $[n_1, n_2]$ for $m_1 m_2$ rounds, and the simplicity of $X$ implies that $m_1 = m_2 = 1$. In other words, $X_i$ must have the reductive projection $\pi_{\Gamma_i}(f)$ of $f$ as its face. Note that different faces of $X$ might have the same reductive projection in $X_i$, and we have to discard duplicated ones. Otherwise duplicated faces in $X_i$ will generate duplicated faces in $X$, violating the simplicity of $X$. Conversely, any faces $f_1$ of $X_1$ and $f_2$ of $X_2$ are the reductive projections of the faces in $X$ they generate. Hence $X_1$ and $X_2$ are the simple complexes with exactly those faces from the reductive projections of faces of $X$. □

Proposition 4.10. Let $X$, $X_1$, and $X_2$ be polygonal cell complexes such that $X = X_1 \otimes X_2$. Then $X$ is a simple complex if and only if $X_1$ and $X_2$ are simple complexes.

Proof. Proposition 4.9 takes care of the only if part, and here we prove the if part. Suppose that $X$ has an $n$-gon $f$ attached along a cycle for $m$ rounds. Since $X_1$ and $X_2$ are simple, $f$
must be generated by the reductive projections of $f$ to $X^1$ and $X^2$, which are of length $l_1$ and $l_2$ respectively. Note that $l_1$ and $l_2$ both divide $\frac{n}{m}$. Then the two reductive projections generate faces of length $n = [l_1, l_2] \leq \frac{n}{m}$. Hence we can conclude that $m = 1$. If there is another face $f'$ in $X$ attached along the same cycle with $f$, then $f'$ is also generated by the reductive projections of $f$. If we can show a face in $X_1$ and a face in $X_2$ do not generate duplicated faces in $X$, then this implies $X$ is a simple complex.

Suppose that a face $f_1$ of $X_1$ has vertices $v_0, v_1, \ldots, v_{p-1}, v_0$ in order, and a face $f_2$ of $X_2$ has vertices $u_0, u_1, \ldots, u_{q-1}, u_0$ in order. By the remark after Definition 3.11 every pair of corners of $f_1$ and $f_2$ appears exactly once in the faces generated by $f_1$ and $f_2$. If two faces generated by $f_1$ and $f_2$ are attached along the same cycle in $X$, there must be two pairs of corners of $f_1$ and $f_2$ forming the same corner in $X$. In particular, we can find $(v_i, u_{j'}) = (v_j, u_{j'})$ such that $i \neq j$ or $i' \neq j'$. When $i \neq j$, we have $v_i = v_j$ and $u_{i+k} = v_{j+k}$ for any integer $k$ mod $p$. This implies that $f_1$ wraps around a cycle more than once, violating the simplicity of $X_1$. Similarly $i' \neq j'$ contradicts the simplicity of $X_2$. The contradiction results from the assumption that two faces generated by $f_1$ and $f_2$ are attached along the same cycle in $X$. Hence we know that $f_1$ and $f_2$ do not generate duplicated faces, and the simplicity of $X$ follows.

**Proposition 4.11.** Let $X$ be a simple complex, and $\Gamma_1$ and $\Gamma_2$ be two graphs such that $X^1 = \Gamma_1 \otimes \Gamma_2$. Then the following two statements are equivalent:

1. There exist two complexes $X_1$ and $X_2$ such that $X_1 \otimes X_2$.
2. For any faces $f_1$ and $f_2$ of $X$, $X$ contains all faces generated by $\pi_{\Gamma_1}(f_1)$ and $\pi_{\Gamma_2}(f_2)$.

**Proof.** Assume (1). By Proposition 4.9, $X_1$ and $X_2$ are the simple complexes with exactly those reductive projections of $X$ as faces. For any faces $f_1$ and $f_2$ of $X$, $\pi_{\Gamma_1}(f_1)$ is a face of $X_1$, and $\pi_{\Gamma_2}(f_2)$ is a face of $X_2$. Since $X = X_1 \otimes X_2$, $X$ contains all faces generated by $\pi_{\Gamma_1}(f_1)$ and $\pi_{\Gamma_2}(f_2)$. Hence (1) implies (2).

Assume (2). First we show that a face $f$ of $X$ can be generated by $\pi_{\Gamma_1}(f)$ and $\pi_{\Gamma_2}(f)$. Let $C_f$, $C_1$, and $C_2$ be the boundary cycles of $f$, $\pi_{\Gamma_1}(f)$, and $\pi_{\Gamma_2}(f)$ respectively. By Definition 4.8, we can assume that $\pi_{\Gamma_j}(C_j) = n_j C_j$ for $j \in \{1, 2\}$, namely repeating $C_j$ for $n_j$ times gives $\pi_{\Gamma_j}(C_f)$. Note that $f$ is attached along some cycle $(n_1 C_1, n_2 C_2)^{\alpha, \beta}$, which can be rewritten as $(n_1, n_2) C(n_{m_1 n_{m_2}} C_1, n_{m_1 n_{m_2}} C_2)^{\alpha, \beta}$. Since the simple complex $X$ has no face attached around a cycle more than once, we know that $(n_1, n_2) = 1$, and therefore

$$\text{length } C_f = n_1 \cdot (\text{length } C_1) = n_2 \cdot (\text{length } C_2) = [\text{length } C_1, \text{length } C_2].$$

This shows that $\pi_{\Gamma_1}(f)$ and $\pi_{\Gamma_2}(f)$ can generate the face $f$. Now let $X_1$ and $X_2$ be the simple complexes with exactly those faces from the reductive projections of $X$. By Proposition 4.10, $X_1 \otimes X_2$ is a simple complex, and in particular $X_1 \otimes X_2$ has no duplicated faces. By the assumption of (2), $X$ contains all the faces of $X_1 \otimes X_2$. Conversely, any face $f$ of $X$ is a face of $X_1 \otimes X_2$, since $f$ can be generated by $\pi_{\Gamma_1}(f)$ and $\pi_{\Gamma_2}(f)$. Then we have $X = X_1 \otimes X_2$, and hence (2) implies (1).

Although we already know the associativity of complex tensor product through the universal property, it will be helpful to understand how faces are formed in the product of more than two complexes. First let us review the product of two complexes. Let $f_\alpha$ be a face of length $n_\alpha$ attached along a cycle $C_\alpha$ in $X$, and $f_\beta$ be a face of length $n_\beta$
Figure 14: a face generated by 3 faces in complex tensor product

attached along a cycle $C_β$ in $Y$. By Definition 3.1, $f_α$ and $f_β$ generate faces $f_{α,β}^δ$ of length $[n_α, n_β]$ attached along $(C_α, C_β)^{i}_δ$, $i \in \{0, 1, \ldots, (n_α, n_β) - 1\}$, $δ \in \{0, 1\}$. To explain the boundary cycle of $f_{i,δ}^α$ in plain language, basically we pick a pair of corners of $f_α$ and $f_β$ to start, and go around $C_α$ and $C_β$ in two coordinates respectively until we return to the starting pair of corners. Note that the index $i$ is chosen in such a way that each pair of corners appears exactly once among all faces generated by $f_α$ and $f_β$.

A good way to visualize this is a slot machine of two reels of length $[n_α, n_β]$, cyclically labeled by the vertices of $f_α$ and $f_β$ respectively. Faces generated by $f_α$ and $f_β$ have a one-to-one correspondence with different combinations of two reels, with flipping allowed for the second reel. From this aspect, it is easy to see that for face $f_j$ of length $n_j$ in complex $X_j$, $j \in \{1, 2, \ldots, m\}$, $f_1, f_2, \ldots, f_m$ generate faces in $\otimes_{j=1}^{m}X_j$ of length $[n_1, n_2, \ldots, n_m]$ such that each $m$-tuple of corners appears exactly once among all generated faces. Faces generated by $f_1, f_2, \ldots, f_m$ have a one-to-one correspondence with different combinations of $m$ reels of length $[n_1, n_2, \ldots, n_m]$, cyclically labeled by the vertices of $f_j$ respectively, with flipping allowed from the second reel on. Figure 14 illustrates how a face is generated by the complex tensor product of 3 faces from such an aspect.

**Theorem 4.12.** Let $X$ be a simple polygonal cell complex. If the 1-skeleton of $X$ is a finite simple connected non-bipartite $R$-thin edge-transitive graph with more than one vertex, then $X$ has a unique factorization into prime complexes.

**Proof.** By Theorem 4.4, since $X^1 \in \mathfrak{S} \subset \mathfrak{S}_0$ is a finite connected non-bipartite graph with more than one vertex, $X^1$ has a unique factorization $X^1 = \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_n$ into primes in $\mathfrak{S}_0$ with respect to direct product of graphs. By Proposition 4.6, the edge-transitivity of $X^1$ implies that each prime factor $\Gamma_i$ is in fact a simple graph. On the other hand, if we factorize $X^1$ with respect to graph tensor product, each factor would also be a simple graph with more than one vertex, because a loop creates double edges in the product, and a single vertex breaks the connectivity of the product. Note that direct product and graph tensor product coincide in $\mathfrak{S}$. Hence we know $X^1$ has a unique factorization $X^1 = \Gamma_1 \otimes \Gamma_2 \otimes \cdots \otimes \Gamma_n$ into primes in $\mathfrak{S}$ with respect to graph tensor product.

Now we consider the factorization of the complex $X$. Note that we can always obtain a prime factorization of $X$, since the number of vertices of factors decreases as the factoring goes. Suppose $X$ has two factorizations $A$ and $B$, and $X_0$ is a prime factor of $X$ in $A$ with 1-skeleton $Γ_1 \otimes Γ_2$. By Proposition 4.11, there exist two faces $f_1$ and $f_2$ such that $X_0$ lacks certain face generated by $π_{Γ_1}(f_1)$ and $π_{Γ_2}(f_2)$. In other words, there is certain
pair of corners of \( \pi_{\Gamma_1}(f_1) \) and \( \pi_{\Gamma_2}(f_2) \) missing in the faces of \( X_0 \), and hence such pair will be absent in the \( n \)-tuples representing face corners of \( X \). By Proposition 4.9 we can find faces \( f_1 \) and \( f_2 \) of \( X \) such that \( \pi_{\Gamma_1 \otimes \Gamma_2}(f_1) = f_1 \) and \( \pi_{\Gamma_1 \otimes \Gamma_2}(f_2) = f_2 \), and we have \( \pi_{\Gamma_1}(f_1) = \pi_{\Gamma_1}(f_1) \) and \( \pi_{\Gamma_2}(f_2) = \pi_{\Gamma_2}(f_2) \). If \( \Gamma_1 \) and \( \Gamma_2 \) belong to different prime factors \( X_1 \) and \( X_2 \) in \( B \), we can reductively project \( f_i \) to \( X_i \) to obtain a face \( f'_i \) of \( X_i \), \( i \in \{1, 2\} \). Then we have \( \pi_{\Gamma_1}(f'_1) = \pi_{\Gamma_1}(f_1) \) and \( \pi_{\Gamma_2}(f'_2) = \pi_{\Gamma_2}(f_2) = \pi_{\Gamma_2}(f_2) \). Notice that \( f'_1 \) and \( f'_2 \) generate all possible pairs of corners of \( \pi_{\Gamma_1}(f_1) \) and \( \pi_{\Gamma_2}(f_2) \) in \( X_1 \otimes X_2 \) and hence in \( X \), a contradiction. So \( \Gamma_1 \) and \( \Gamma_2 \) belong to the same prime factor in \( B \).

The above argument can be applied to the case when the 1-skeleton of \( X_0 \) is the graph tensor product of more than two prime graphs, simply by splitting prime graph factors into two groups. It follows that every prime 1-skeleton factor of \( X_0 \) belongs to the same prime complex \( X'_0 \) in \( B \). Conversely, every prime 1-skeleton factor of \( X'_0 \) belongs to \( X_0 \), and hence \( X_0 \) and \( X'_0 \) are actually the same. In case \( X_0 \) has a prime 1-skeleton \( \Gamma_j \), then \( \Gamma_j \) belongs to some \( X'_0 \) in \( B \) with a prime 1-skeleton, otherwise the prime 1-skeleton factors of \( X'_0 \) belong to at least two complexes in \( A \). In conclusion, we know two factorizations \( A \) and \( B \) are identical, and \( X \) has a unique factorization into prime complexes. \( \square \)

**Theorem 4.13**. Suppose that \( X \) is a simple polygonal cell complex, and its 1-skeleton is a finite simple connected non-bipartite edge-transitive \( R \)-thin graph with more than one vertex. Let \( X = X_1 \otimes X_2 \otimes \cdots \otimes X_n \) be a prime factorization of \( X \). Then \( \text{Aut}(X) \) is generated by automorphisms of prime factors and permutations of isomorphic factors.

**Proof.** Since \( X \) has no faces attached along the same cycle, an automorphism of \( X \) is completely determined by its action on the 1-skeleton \( X^1 \), and we can identify \( \text{Aut}(X) \) as a subgroup of \( \text{Aut}(X^1) \). To understand \( \text{Aut}(X^1) \), by the argument in the proof of Theorem 4.12, we know \( X^1 \) has a unique factorization \( X^1 = \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_m = \Gamma_1 \otimes \Gamma_2 \otimes \cdots \otimes \Gamma_m \) into primes in \( \mathcal{G} \). By Theorem 4.5, the extra \( R \)-thin condition on \( X^1 \) implies that \( \text{Aut}(X^1) \) is generated by automorphisms of \( \Gamma_i \)'s and permutations of isomorphic \( \Gamma_j \)'s.

Let \( \varphi \) be an arbitrary automorphism of \( X \), which can be represented as some \( \rho \in \times_{i=1}^m \text{Aut}(\Gamma_i) \) followed by a permutation of \( \Gamma_j \)'s. This implies that for any face \( f \) of \( X \)

\[
\varphi(\pi_{\bigotimes i \in I, I_1}(f)) = \pi_{\bigotimes i \in I, I_1}(\varphi(f)) = \pi_{\bigotimes i \in I, I_1}(\varphi_f(f)),
\]

where \( I \) is an arbitrary non-empty subset of \( \{1, 2, \ldots, m\} \). Suppose that \( X_1 \) has 1-skeleton \( X^1_1 = \bigotimes_{i \in I} \Gamma_i \) for some \( I \subset \{1, 2, \ldots, m\} \). We claim that \( \forall i \in I \), \( \varphi(\Gamma_i) \) belongs to the same prime factor \( X_k \) of \( X \). If not, then we can find \( I_1 \cup I_2 = I \) such that \( \forall i \in I_1, \forall j \in I_2, \varphi(\Gamma_i) \) and \( \varphi(\Gamma_j) \) belong to different prime factors of \( X \). Let \( \Gamma_a = \bigotimes_{i \in I_1} \Gamma_i \) and \( \Gamma_b = \bigotimes_{j \in I_2} \Gamma_j \), and hence we have \( X^1_1 = \Gamma_a \otimes \Gamma_b \). Since \( X_1 \) is prime, by Proposition 4.11 we can find faces \( f_1 \) and \( f_2 \) of \( X_1 \) such that \( X_1 \) lacks certain face generated by \( \pi_{\Gamma_a}(f_1) \) and \( \pi_{\Gamma_b}(f_2) \). By Proposition 4.9 we can find faces \( f_1 \) and \( f_2 \) of \( X \) such that \( \pi_{\Gamma_a \otimes \Gamma_b}(f_1) = f_1 \) and \( \pi_{\Gamma_a \otimes \Gamma_b}(f_2) = f_2 \). Then the complex \( X \) lacks certain corner combination of \( \pi_{\Gamma_a}(f_1) \) and \( \pi_{\Gamma_b}(f_2) \) in the \( m \)-tuples representing face corners of \( X \). By taking the automorphism \( \varphi \), the complex \( X \) lacks certain corner combination of \( \pi_{\bigotimes i \in I, I_1}(\varphi(\Gamma_i))(\varphi(f_1)) \) and \( \pi_{\bigotimes i \in I, I_1}(\varphi(\Gamma_i))(\varphi(f_2)) \), which is impossible because \( \varphi(\Gamma_i) \) and \( \varphi(\Gamma_j) \) belong to different prime factors of \( X \), and taking complex tensor product of these factors generates all the corner combinations.

Hence for every 1-skeleton factor \( \Gamma_i \) of \( X_1 \), \( \varphi(\Gamma_i) \) belongs to the same prime factor \( X_k \) of \( X \). By considering \( \varphi^{-1} \), we know that \( X_k \) has exactly these \( \varphi(\Gamma_i)'s \) as 1-skeleton.
Suppose that one of the prime factors is not flag-transitive, without loss of generality say \( \sigma \in X_k \). This shows that every \( \sigma \in \text{Aut}(X) \) can be represented as some \( \sigma \in \times_{i=1}^n \text{Aut}(X_i) \) followed by a permutation of \( X_j \)'s, and the theorem holds.

**Remark.** Let \( \tilde{X} \) be the disjoint union of prime factors of \( X \). Then the above theorem implies that \( \text{Aut}(X) \cong \text{Aut}(\tilde{X}) \), which is a convenient way to describe \( \text{Aut}(X) \).

The following corollary is a partial converse of Theorem 3.10.

**Corollary 4.14.** Suppose that \( X \) is a simple polygonal cell complex, and its 1-skeleton is a finite simple connected non-bipartite edge-transitive \( R \)-thin graph with more than one vertex. If \( X \) is flag-transitive, then any factor of \( X \) is flag-transitive.

**Proof.** Note that it suffices to show that any prime factor of \( X \) is flag-transitive. Then by Theorem 4.12 and Theorem 3.10 any factor of \( X \) is a complex tensor product of flag-transitive prime factors of \( X \), and hence is flag-transitive.

By Theorem 4.12 \( X \) has a unique prime factorization \( X = X_1 \otimes X_2 \otimes \cdots \otimes X_n \). Suppose that one of the prime factors is not flag-transitive, without loss of generality say \( X_1 \), and \( X_1 \) is isomorphic to \( X_i \) if and only of \( 1 \leq i \leq m \) for some integer \( m \leq n \). Since \( X_1 \) is not flag-transitive, there exist two oriented face corners \( (e_1^1, v_1, e_1^2) \) and \( (e_1^1, v_1', e_1^2) \) in \( X_1 \) such that \( \text{Aut}(X_1) \) can not map one corner to the other. For each \( j \) such that \( m + 1 \leq j \leq n \), we pick an arbitrary corner \( (e_j^j, v_j, e_j^2) \) of \( X_j \). Consider the following two corners of \( X \):

\[
\left( (e_1^1, \ldots, e_1^n), (v_1, \ldots, v_1, v_{m+1}, \ldots, v_n), (e_1^2, \ldots, e_1^2, e_{m+1}^2, \ldots, e_n^2) \right)
\]

and

\[
\left( (e_{1}^{1'}, \ldots, e_{1}^{n'}), (v_{1}', \ldots, v_{1}', v_{m+1}', \ldots, v_n'), (e_{1}^{2'}, \ldots, e_{1}^{2'}, e_{m+1}^{2'}, \ldots, e_n^{2'}) \right).
\]

By Theorem 4.13 \( \text{Aut}(X) \) is generated by automorphisms of prime factors and permutations of isomorphic factors. In particular, it is impossible for \( \text{Aut}(X) \) to map one of the above corners to the other, contradicting to the flag-transitivity of \( X \). Therefore we can conclude that any prime factor of \( X \) is flag-transitive.

The corollary below answers the question we posed in the beginning of the chapter.

**Corollary 4.15.** For \( i \in \{1, 2, \ldots, n\} \), let \( X_i \) be a simple prime complex with a finite simple connected non-bipartite symmetric \( R \)-thin 1-skeleton having more than one vertex. Then the complex tensor product \( X = \otimes_{i=1}^n X_i \) has automorphism group \( \text{Aut}(X) \) generated by \( \text{Aut}(X_i) \)'s and permutations of isomorphic \( X_j \)'s.

**Proof.** By Proposition 4.10 we know \( X \) is a simple complex. By the definition of graph tensor product, we know \( X^1 \) is a finite simple graph. Note that a simple graph is non-bipartite if and only if there is a cycle of odd length. Then the graph tensor product of two non-bipartite graphs contains a cycle of odd length and hence is non-bipartite. Induction shows that \( X^1 \) is non-bipartite, and by Theorem 4.13 we know that \( X^1 \) is connected. By the special case of Theorem 3.10 (complexes without faces), we know \( X^1 \) is symmetric and hence edge-transitive. Note that for two graphs \( \Gamma_1 \) and \( \Gamma_2 \), the set of neighbours of a vertex \( (u, v) \in V(\Gamma_1 \otimes \Gamma_2) \) is the direct product of the set of neighbours of \( u \) in \( \Gamma_1 \) with the set of neighbours of \( v \) in \( \Gamma_2 \). This implies the graph tensor product of \( R \)-thin graphs is a \( R \)-thin graph. To summarize, we know \( X \) is a simple complex with a prime factorization \( X = \otimes_{i=1}^n X_i \), and its 1-skeleton \( X^1 \) is a finite simple connected non-bipartite edge-transitive \( R \)-thin graph with more than one vertex. By Theorem 4.13 we know that \( \text{Aut}(X) \) is as described in the corollary.

\[\square\]
Remark. The tensor products of edge-transitive graphs are not necessarily edge-transitive. Therefore we require each $X_i^1$ to be symmetric to ensure the edge-transitivity of $X^1$.

Note that when a complex has a face of odd length, then the 1-skeleton of the complex is non-bipartite, and Corollary 4.15 has a chance to work. In the next chapter, we will investigate the automorphism group of the tensor product of complexes with only faces of even lengths from a different aspect.

5 Even Cases

In this chapter we investigate the tensor product of complexes with only faces of even lengths, and our goal is to develop results similar to Corollary 4.15 which basically says an automorphism of certain complex tensor products must be of Cartesian type. Note that when there is more than one bipartite factor, Theorem 4.3 implies that the complex tensor product is disconnected, and the product is likely to have non-Cartesian automorphisms from the direct product of automorphism groups of components. Hence in such a context, the proper question to pose should be as follows: for complexes $X_i$ with only faces of even lengths, is the automorphism group of a component of $\otimes X_i$ generated by automorphisms of $X_i$'s together with permutations of isomorphic factors?

For graph tensor products, the connectedness of the product does not guarantee the absence of non-Cartesian automorphisms. For complex tensor products, we hope that the extra face structure helps to eliminate non-Cartesian automorphisms. For example, let us look at the complex tensor product of two squares, which has two isomorphic components. We denote vertices of a square by $0, 1, 2, -1$ cyclically, and illustrate one component of the product in Figure 15. Note that the 1-skeleton of the component is actually a complete bipartite graph with $2 \cdot 4! \cdot 4!$ automorphisms, and not all of them give a complex automorphism due to the extra face structure.

Figure 15 also reveals an important fact of the tensor product of complexes with only faces of even lengths: a face is antipodally attached to another face generated by the same pair of faces, and through such antipodally attached relation we can find all other faces generated by the same pair of faces in that component. Such face blocks (defined in Definition 5.4) help to determine the Cartesian structure of a complex tensor product, and if we can show a generic face block has only Cartesian automorphisms, then we have a
chance to force a complex automorphism stabilizing a face block to be of Cartesian type. To simplify the problem, we restrict our discussion to the tensor product of complexes with faces of the same even length, and the first step is to establish the Cartesian result for the tensor product of $2n$-gons. The following lemma is a useful tool for this purpose.

**Lemma 5.1.** Suppose on a real line, someone wants to take $d$ steps to walk from an integer $d - 2k$ to 0, where $\left\lceil \frac{d}{2} \right\rceil \geq k \geq 0$ is an integer, and each step is either plus 1 or minus 1. Then there are $\binom{d-1}{k}$ ways to arrive from 1, and $\binom{d-1}{k-1}$ ways to arrive from $-1$. The ratio $\frac{\binom{d-1}{k}}{\binom{d-1}{k-1}}$ is greater than or equal to 1, with equality if and only if $d - 2k = 0$. Moreover, when $d$ is fixed and $k$ is increasing, the ratio is decreasing.

**Proof.** Suppose this person takes $x$ steps of minus 1 and $y$ steps of plus 1 to arrive at 0. Then we have $x + y = d$ and $-x + y = -d + 2k$, and therefore $x = d - k$ and $y = k$. By ordering two types of steps arbitrarily, we can obtain all different ways to arrive at 0. To arrive from 1, the last step must be minus 1, and there are $\binom{d-1}{k}$ such combinations. To arrive from $-1$, the last step must be plus 1, and there are $\binom{d-1}{k-1}$ such combinations.

When $d$ is odd, we have $\frac{d+1}{2} \geq k$ and hence $\binom{d+1}{k} > \binom{d+1}{k-1}$. When $d$ is even, we have $\frac{d}{2} \geq k$ which implies $\frac{d-k}{2} > k - 1$ and hence $\binom{d-1}{k} \geq \binom{d-1}{k-1}$, with equality if and only if $k + (k - 1) = d - 1$, namely $d - 2k = 0$. To show that the ratio $\binom{d-1}{k}/\binom{d-1}{k-1}$ decreases as $k$ increases, we simply have to verify the following inequality:

\[
\frac{(d-1)/(d-1)}{(k-1)/(k-1)} > \frac{(d-1)/(k+1)}{(d-1)/(k-1)}
\]

\[
\iff \frac{(d-1)(d-2)(d-3) \cdots (d-k)}{k!} > \frac{(d-1)(d-2)(d-3) \cdots (d-k-1)(d-k)}{(k-1)!}
\]

\[
\iff \frac{d-k}{k} > \frac{d-k-1}{k+1}
\]

\[
\iff \frac{d}{k} - 1 > \frac{d}{k+1} - 1
\]

\[
\iff \frac{d}{k} - 1 > \frac{d}{k+1} - 1
\]

\[
\iff \frac{d}{k} > \frac{d}{k+1}
\]

\[
\iff \frac{d}{k} > \frac{d}{k+1}
\]

\[
\iff k + 1 > k,
\]

which is obviously true. \qed

**Proposition 5.2.** For $i \in \{1, 2, \ldots, m\}$, let $C_i$ be a graph which is a cycle of length $2n$, where $n$ is an integer at least 3. Then the automorphism group of a component of $\otimes_{i=1}^{m} C_i$ can be generated by elements of $\text{Aut}(C_i)$’s together with permutations of $C_i$’s.

**Proof.** We denote vertices of $C_i$ by $0, 1, \ldots, n - 1, n, -(n - 1), -(n - 2), \ldots, -1$ cyclically, and let $\Gamma$ be the component of $\otimes_{i=1}^{m} C_i$ containing the vertex $v = (0, 0, \ldots, 0)$. Note that $\times_{i=1}^{m} \text{Aut}(C_i)$ acts transitively on vertices of $\otimes_{i=1}^{m} C_i$. Therefore to prove this proposition, it suffices to show that the $v$-stabilizer $G_v$ of $\text{Aut}(\Gamma)$ can be generated by elements of $\text{Aut}(C_i)$’s together with permutations of $C_i$’s. Notice that there are $2^m \cdot m!$ Cartesian automorphisms of $\Gamma$ fixing $v$, generated by the reflection fixing 0 in each $C_i$ and all permutations of $m$ factors. If we can show $|G_v| \leq 2^m \cdot m!$, then the proposition follows.

First we show that $\Gamma$ is a rigid graph. Namely we want to show that if $\varphi \in G_v$ fixes all neighbours of $v$, then $\varphi$ must be trivial. Note that two vertices $(b_1, b_2, \ldots, b_m)$ and $(c_1, c_2, \ldots, c_m)$ are adjacent if and only if $b_i - c_i \equiv \pm 1 \mod 2n$ for all $i$, and therefore

\[
V(\Gamma) \subseteq V^* = \{(a_1, a_2, \ldots, a_m) \in V(\otimes_{i=1}^{m} C_i) \mid a_1 \equiv a_2 \equiv \cdots \equiv a_m \mod 2\}.
\]
For each \( u = (a_1, a_2, \ldots, a_m) \in V^* \), there is a path of length \( d = \max\{|a_1|, |a_2|, \ldots, |a_m|\} \) from \( u \) to \( v \), because we can reach 0 in \( d \) steps in the coordinates with absolute value \( d \), and we can also reach 0 in \( d \) steps in the other coordinates by walking back and forth as each coordinate has the same parity. Hence \( V(\Gamma) = V^* \), and \( d(u, v) = d \) follows easily.

Note that the number of geodesics from \( u \) to \( v \) is the product of the number of ways in each coordinate to walk to 0 in \( d \) steps. Look at the \( i \)-th coordinate of \( v \). For now we assume that \( a_i \geq 0 \), and let \( k_i \) be the integer such that \( a_i = d - 2k_i \). If \( n > a_i > 0 \), we have \( \left\lfloor \frac{d}{2} \right\rfloor \geq k_i \geq 0 \), and walking to 0 in \( d \) steps is equivalent to the setting of Lemma 5.1. By the lemma, the ratio of numbers of \( u - v \) geodesics arriving from 1 and from \( -1 \) in the \( i \)-th coordinate is \( \frac{\binom{d-1}{k_i}}{\binom{d-1}{k_i-1}} > 1 \). Since the automorphism \( \varphi \) fixes \((\pm 1, \pm 1, \ldots, \pm 1)\) and preserves geodesics, this ratio does not change under \( \varphi \). Again by the Lemma, \( k_i \) must remain the same to keep this ratio, and hence the \( i \)-th coordinate of \( \varphi(u) \) must be \( a_i \). If \( a_i = 0 \), then \( u \) has a neighbour \( w \) with the \( i \)-th coordinate 1. Note that \( \varphi(u) \) is adjacent to \( \varphi(w) \) with the \( i \)-th coordinate 1, and the \( i \)-th coordinate of \( \varphi(u) \) is either 0 or 2. In the latter case, since \( n > 2 > 0 \), by taking \( \varphi^{-1} \) the above argument implies \( a_i = 2 \), a contradiction. Hence the \( i \)-th coordinate of \( \varphi(u) \) is 0. Similarly if \( a_i = n \), then the \( i \)-th coordinate of \( \varphi(u) \) is \( n \). For negative \( a_i \), by applying the mirror version of Lemma 5.1 we know that the \( i \)-th coordinate of \( \varphi(u) \) is \( a_i \). Note that the above result is true for every coordinate. Hence \( \varphi(u) = u \) for every \( u \in V(\Gamma) \), and \( \varphi \) is trivial.

Now look at the local structure around \( v \). Note that two neighbours of \( v \) taking different values in \( k \) coordinates have \( 2^{m-k} \) common neighbours. In particular, two neighbours of \( v \) differ in exactly one coordinate if and only of they have \( 2^{m-1} \) common neighbours. Hence among the neighbours of \( v \), the relation of differing in exactly one coordinate is preserved under \( G_v \). If we draw an auxiliary edge between any two such neighbours of \( v \), then the \( 2^m \) neighbours of \( v \) plus these auxiliary edges form a hypercube \( Q_m \) preserved under \( G_v \). Since \( \Gamma \) is rigid, an automorphism of \( G_v \) is completely determined by its action on the neighbours of \( v \), which also induces an automorphism of the auxiliary \( Q_m \). As a result, we have \( |G_v| \leq \text{Aut}(Q_m) = 2! \cdot m! \), which finishes the proof.

\[ \square \]

**Remark.** Let \( H \) be the subgroup of \( \times_{i=1}^m \mathbb{Z}_{2n} \) generated by \( S = \{ (\pm 1, \pm 1, \ldots, \pm 1) \} \). Note that the component \( \Gamma \) in the above proof is actually isomorphic to the Cayley graph of \( H \) with respect to the generating set \( S \).

**Corollary 5.3.** Suppose that \( X_i \) is a \( 2n \)-gon for \( i \in \{1, 2, \ldots, m\} \), where \( n \) is an integer at least 3. Then the automorphism group of a component of \( \otimes_{i=1}^m X_i \) can be generated by elements of \( \text{Aut}(X_i) \)'s together with permutations of \( X_i \)'s.

**Proof.** Note that a \( 2n \)-gon has the same automorphism group as its 1-skeleton, and \( \otimes_{i=1}^m X_i \) has the same Cartesian automorphisms as \( \otimes_{i=1}^m X_i^1 \). Hence a vertex stabilizer \( G_v \) of a component \( X \) of \( \otimes_{i=1}^m X_i \) has \( 2^m \cdot m! \) Cartesian automorphisms, and \( |G_v| \) is at most the cardinality of the stabilizer of \( v \) in \( X^1 \), which is \( 2^m \cdot m! \) by Proposition 5.2. \[ \square \]

**Remark.** We do need the condition \( n \geq 3 \) in Proposition 5.2 and Corollary 5.3. For \( n = 2 \), Figure 15 illustrates a component of the tensor product of two squares. Its 1-skeleton is the complete bipartite graph \( K_{4,4} \) with lots of non-Cartesian automorphisms. With the face structure, there are much fewer complex automorphisms, but swapping (0,2) and (2,0) still gives a non-Cartesian complex automorphism.
Figure 16: a non-elementary complex

Now we formally define the face blocks mentioned in the beginning of the chapter. An intuitive definition of a face block in a complex tensor product $\otimes_{i=1}^{m} X_i$ would be any connected component in $\otimes_{i=1}^{m} f_i$, where each $f_i$ is a face of $X_i$. Note that if each $f_i$ is an even gon attached injectively, then $\otimes_{i=1}^{m} f_i$ has $2^{m-1}$ components, and hence $2^{m-1}$ face blocks. If these $f_i$’s are attached non-injectively, then the above face blocks could have extra incidence relations, and we might end up having fewer components. We would like to define a face block regardless of attaching maps, so we take the following definition.

Definition 5.4. For $i \in \{1, 2, \ldots, m\}$, let $X_i$ be a polygonal cell complex with only faces of even length $\geq 2$. Let $f_i$ be a face of $X_i$ with corners labeled by $0, 1, \ldots, 2n - 1$ cyclically. A face block generated by $f_1, f_2, \ldots, f_m$ is a subcollection of faces generated by $f_1, f_2, \ldots, f_m$ such that two faces $f_a$ and $f_b$ are in the same face block if and only if a corner of $f_a$ with label $(a_1, a_2, \ldots, a_m)$ and a corner $f_b$ with label $(b_1, b_2, \ldots, b_m)$ have

$$a_1 - b_1 \equiv a_2 - b_2 \equiv \cdots \equiv a_m - b_m \mod 2.$$ 

Remark. It is easy to see that a face block is well-defined no matter how faces are cyclically labeled and no matter which corners are chosen to verify the above criterion. In general it is not obvious whether or not two faces are in the same face block of a complex tensor product without knowing the tensor product structure. In the tensor product of the following class of complexes, recognizing a face block is much easier.

Definition 5.5. A connected polygonal cell complex $X$ is an elementary complex if $X$ satisfies the following three conditions:

1. Every face of $X$ is of the same even length $\geq 2$.
2. No antipodal corners of a face are attached to the same vertex.
3. For any two vertices, there is at most one pair of antipodal face corners attached.

Remark. Condition (3) basically says no two faces can be attached antipodally, and in a face different pairs of antipodal corners are not attached to the same pair of vertices. For example, the complex in Figure 16 is not an elementary complex.

Proposition 5.6. For $i \in \{1, 2, \ldots, m\}$, let $X_i$ be an elementary complex with faces of even length $2n \geq 2$. Then in the complex tensor product $\otimes_{i=1}^{m} X_i$, for any antipodal vertices $u$ and $v$ of a face in $\otimes_{i=1}^{m} X_i$, there are exactly $2^{m-1}$ faces having $u$ and $v$ as antipodal vertices, and these faces are in the same face block. Moreover, for any two faces $f$ and $f'$ in the same face block, we can find a series of faces $f_0, f_1, \ldots, f_k$ such that $f_0 = f$, $f_k = f'$, $f_i$ and $f_{i+1}$ share antipodal vertices for $i \in \{0, 1, \ldots, k - 1\}$, and $k \leq n$. 

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Proof. In $\otimes_{i=1}^m X_i$, suppose that $u = (u_1, u_2, \ldots, u_m)$ and $v = (v_1, v_2, \ldots, v_m)$ are antipodal vertices of a face $f$ generated by $f_1, f_2, \ldots, f_m$, where $u_i$ and $v_i$ are vertices of $X_i$ and $f_i$ is a face of $X_i$ for $i \in \{1, 2, \ldots, m\}$. Note that for each $i \in \{1, 2, \ldots, m\}$, projecting $f$ to $X_i$ gives $f_i$, and $f_i$ has $u_i$ and $v_i$ as antipodal vertices. Since $X_i$ is elementary, $u_i$ and $v_i$ are not the same vertex, and $f_i$ is the only face of $X_i$ having $u_i$ and $v_i$ as antipodal vertices, with a unique pair of antipodal corners attached to $u_i$ and $v_i$. Hence any face in $\otimes_{i=1}^m X_i$ having $u$ and $v$ as antipodal vertices must be generated by $f_1, f_2, \ldots, f_m$ in such a way that the corresponding corners $c_i$ of the $f_i$’s at $u_i$ are combined together. With the corner $c_1$ of $f_1$ fixed, flipping $f_i$ at $c_i$ for $i \in \{1, 2, \ldots, m\}$ gives all $2^{m-1}$ faces having $u$ and $v$ as antipodal vertices, and these faces are in the same face block.

Now suppose that $f$ and $f'$ are two faces in the same face block $B$ generated by faces with corners labeled by $0, 1, \ldots, 2n - 1$ cyclically. Then we can label corners in $B$ according to such a corner labeling, and by following steps of $(\pm 1, \pm 1, \ldots, \pm 1)$, we can start from a vertex $v$ of $f$ to reach any other vertex in $B$ in $n$ steps. In particular, there is a unique vertex in $B$ such that we need $n$ steps to reach it from $v$. Since $f'$ has more than one vertex, we can start from $v$ to reach a vertex $u$ of $f'$ in $n - 1$ steps. By adding one step in $f$ and one step in $f'$ if necessary, we can find a path from $f$ to $f'$ of length at most $n + 1$ such that the first and the last steps are in $f$ and $f'$ respectively. Note that each $(\pm 1, \pm 1, \ldots, \pm 1)$ step determines a unique face in $B$, and hence the above path determines a series of faces $f_0, f_1, \ldots, f_k$ such that $f_0 = f, f_k = f'$, and $k \leq n$. If $f_i$ and $f_{i+1}$ are determined by the same $(\pm 1, \pm 1, \ldots, \pm 1)$ step, then $f_i$ and $f_{i+1}$ are actually the same face, and we can remove one of them from the sequence. If $f_i$ and $f_{i+1}$ are determined by different $(\pm 1, \pm 1, \ldots, \pm 1)$ steps, then $f_i$ and $f_{i+1}$ are two different faces with a common vertex with label $(a_1, a_2, \ldots, a_m)$. Note that

$$(a_1, a_2, \ldots, a_m) + n(\pm 1, \pm 1, \ldots, \pm 1) = (a_1 + n, a_2 + n, \ldots, a_m + n) \mod 2n,$$

which is also a common vertex of $f_i$ and $f_{i+1}$, and therefore $f_i$ and $f_{i+1}$ share antipodal vertices. The above argument is illustrated in Figure 15.

Proposition 5.6 allows us to easily recognize a face block in a complex tensor product. If we impose the following conditions on each factor, then we can read the Cartesian structure of a complex tensor product through the incidence relation of face blocks.

**Definition 5.7.** A connected polygonal cell complex $X$ is an **ordinary** complex if every face $f$ of $X$ is of the same even length $2n \geq 4$, and satisfies the following extra conditions:

1. If we label corners of $f$ cyclically from 1 to 2n, then any two corners with different parities are not attached to the same vertex.

2. For any face $f'$ incident to $f$, either $f$ has only one corner meeting $f'$, or $f$ has only two consecutive corners meeting $f'$.

**Remark.** If the 1-skeleton of $X$ is bipartite, then $X$ satisfies (1) automatically. Also note that a polygonal complex satisfies both (1) and (2). The reader might have noticed that (2) implies the condition (3) of an elementary complex. Since there are alternative conditions serving our purpose as effectively as (2), we avoid defining ordinary complexes as a subclass of elementary complexes.


**Proposition 5.8.** For \( i \in \{1, 2, \ldots, m\} \), suppose that \( X_i \) is an ordinary complex with faces of even length \( 2n \geq 4 \). Let \( B \) be a face block generated by \( f_1, f_2, \ldots, f_m \) and \( B' \) be a face block generated by \( f'_1, f'_2, \ldots, f'_m \), where \( f_i \) and \( f'_i \) are faces of \( X_i \). If \( B \) and \( B' \) are incident, then the following two statements are equivalent:

1. \( \exists j \) such that \( f_j \) is incident to \( f'_j \) in \( X_j \), and \( \forall i \neq j \) we have \( f_i = f'_i \).
2. Every face of \( B \) is incident to a face of \( B' \).

**Proof.** Assume (1). Without loss of generality, we can assume that \( j = 1 \). Since \( B \) and \( B' \) are incident, there is a face corner \( c \) of \( B \) meeting a face corner \( c' \) of \( B' \). Suppose that \( c \) is the combination of corners \( c_i \) of the \( f_i \)'s, and \( c' \) is the combination of corners \( c'_i \) of the \( f'_i \)'s. Note that \( c_1 \) of \( f_1 \) meets \( c'_1 \) of \( f'_1 \) in \( X_1 \). Also note that for \( i \neq 1 \), \( c_i \) and \( c'_i \) are in the same face \( f_i \), and they are either the same corner or different corners attached to the same vertex. In particular, by condition (1) of Definition 5.7, \( c_i \) and \( c'_i \) have the same parity under cyclic \( \mathbb{Z}_{2n} \) labeling for \( i \neq 1 \). Let \( f \) be an arbitrary face of \( B \) generated by combining \( c_1 \) of \( f_1 \) with corners \( c_i \) of the \( f_i \)'s for \( i \neq 1 \). By Definition 5.4, \( c_i \) has the same parity as \( c_i \), and therefore has the same parity as \( c'_i \). Then again by Definition 5.4 the face \( f' \) generated by combining \( c'_1 \) of \( f'_1 \) with \( c'_i \)'s of the \( f_i \)'s is a face of \( B' \). It is obvious that \( f \) is incident to \( f' \). To summarize, given an arbitrary face \( f \) of \( B \), we can find a face \( f' \) of \( B' \) incident to \( f \). Hence (1) implies (2).

Assume (2). If \( f_i \) and \( f'_i \) are disjoint, then \( B \) and \( B' \) are disjoint, which contradicts (2). Hence for each \( i \in \{1, 2, \ldots, m\} \), \( f_i \) and \( f'_i \) are either incident or actually the same. Suppose that there is more than one \( j \), say for \( j \in \{1, 2\} \), such that \( f_j \) and \( f'_j \) are incident. By condition (2) of Definition 5.7, \( f_1 \) and \( f_2 \) have either one corner or two consecutive corners meeting \( f'_1 \) and \( f'_2 \) respectively. Pick two consecutive corners of \( f_1 \) containing all corners meeting \( f'_1 \) and colour them blue. Similarly pick two consecutive corners of \( f_2 \) containing all corners meeting \( f'_2 \) and colour them red. Consider the faces generated by \( f_1, f_2, \ldots, f_m \) with the following corner combination: coloured corners of \( f_1 \) and \( f_2 \) are placed at the opposite positions, as illustrated in Figure 17. Note that these faces are disjoint with faces generated by \( f'_1, f'_2, \ldots, f'_m \). If \( B \) does not contain any of these faces, we can flip two red corners of \( f_2 \) to generate faces of \( B \), and the resulting faces are still disjoint with faces generated by \( f'_1, f'_2, \ldots, f'_m \). In other words, we can find a face of \( B \) incident to no face in \( B' \), a contradiction. So there is at most one \( j \) such that \( f_j \) and \( f'_j \) are incident. Moreover, condition (1) of Definition 5.7 implies that different face blocks generated by \( f_1, f_2, \ldots, f_m \) are disjoint. Since \( B \) and \( B' \) are incident, we know that there is exactly one \( j \) such that \( f_j \) and \( f'_j \) are incident. Hence (2) implies (1).
Remark. Note that condition (2) of Definition 5.7 is only used for the argument illustrated in Figure 17. It is not hard to have alternative conditions serving this purpose, especially when the length of faces is higher. We also want to point out that through finer examination of incidence relation between face blocks, it is possible to obtain more information such as how \( f_j \) meets \( f'_j \) in \( X_j \), perhaps under weaker conditions.

With Propositions 5.6 and 5.8, in a tensor product 
\[ X = X_1 \otimes X_2 \otimes \cdots \otimes X_m \]
where each \( X_i \) is an elementary ordinary complex with only faces of even length \( 2n \geq 4 \), we can recognize face blocks and the Cartesian structure of \( X \) through the incidence relation on faces, which is preserved under automorphisms of \( X \). Now we define a graph \( \Gamma_X \) to encode the Cartesian structure of \( X \). Let \( \Gamma_X \) be a simple graph with vertex set \( \times_{i=1}^m F(X_i) \), where a vertex \( (f_1, f_2, \ldots, f_m) \) represents all faces of \( X \) generated by \( f_1, f_2, \ldots, f_m \), such that two vertices are adjacent if and only if they take the same face in \( m-1 \) coordinates, and have incident faces in the remaining coordinate. Let \( \Gamma_{X_i} \) be a simple graph with vertex set \( F(X_i) \), such that two vertices are adjacent if and only if the corresponding faces are incident in \( X_i \). Notice that \( \Gamma_X = \Gamma_{X_1} \boxtimes \Gamma_{X_2} \boxtimes \cdots \boxtimes \Gamma_{X_m} \), where \( \boxtimes \) is the Cartesian product of graphs (see [4] for the definition). Figure 18 illustrates the case \( m = 2 \), where \( B^{i,j} = (f^i_1, f^j_2) \) represents all faces generated by \( f^i_1 \) and \( f^j_2 \). The following theorem due to Imrich [5] and Miller [7] restricts the automorphism group of \( \Gamma_X \).

**Theorem 5.9.** Suppose that \( \Gamma \) is a finite simple connected graph with a factorization \( \Gamma = \Gamma_1 \boxtimes \Gamma_2 \boxtimes \cdots \boxtimes \Gamma_m \), where each \( \Gamma_i \) is prime with respect to Cartesian product. Then the automorphism group of \( \Gamma \) is generated by automorphisms of prime factors and permutations of isomorphic factors.

We can not guarantee \( \Gamma_{X_i} \) is prime, but at least \( X_i \) is indeed a prime complex.

**Proposition 5.10.** Let \( Y \) be an elementary complex. Then \( Y \) is a prime with respect to complex tensor product, and \( Y \) is not a component of any complex tensor product.

**Proof.** Suppose that there exist complexes \( Y_1 \) and \( Y_2 \) such that \( Y \) is a component of \( Y_1 \otimes Y_2 \). Note that a face of \( Y \) is of even length, and must be generated by either two even faces or by one even and one odd face. In either case, by Definition 3.1 \( Y \) will have faces antipodally attached together, violating that \( Y \) is elementary.

Note that in Figure 18 each \( B^{i,j} \) actually contains two face blocks generated by \( f^i_1 \) and \( f^j_2 \), and in general each vertex of \( \Gamma_X \) defined above contains \( 2^{|m-1|} \) face blocks. Even if we have some control over the automorphism group of \( \Gamma_X \), having multiple face blocks

![Figure 18: Cartesian structure of face blocks](image-url)
at one vertex of $\Gamma_X$ could lead to non-Cartesian automorphisms of $X$. Let us look at the tensor product of a hexagon with a 3-hexagon necklace as illustrated in Figure 19, where $v_i$ is the vertex generated by $u_i$ and $v$, and coloured vertices in the product are generated by coloured $u_1$, $u_3$, and $u_5$. For brevity, half of the faces in the product are omitted. Consider the automorphism $\rho$ of the product induced by fixing $f$, $f^1$, and $f^3$ but flipping $f^2$ (swapping the top and the bottom edges) in two factors. Then $\rho$ fixes the four face blocks on the left and right, and permutes vertices in each of the two middle blocks. In particular, we can permute vertices in a block while its two incident blocks are fixed. Therefore we can permute vertices in one middle block and fix all other five blocks. This gives a non-Cartesian automorphism.

There are two main reasons why we have the above non-Cartesian automorphism. First, there is more than one face block generated by the same faces lying in the same component of the product. Secondly, factors are not rigid enough, so the action on one face block can not affect incident blocks, and can not be transmitted to blocks generated by the same faces. We suspect that if either of these two reasons is absent, then each component of the product might have only Cartesian automorphisms. In particular, if the 1-skeleton of each factor is bipartite, then face blocks generated by the same faces are in different components. Also note that if a complex is a surface, it is rigid enough that the action on one face completely determines the whole automorphism. So far we do not have a definite result yet, and hence we pose the following two conjectures. We hope to resolve these problems in the near future.

**Conjecture 5.11.** For $i \in \{1, 2, \ldots, m\}$, suppose that $X_i$ is an elementary ordinary complex with faces of the same even length $2n \geq 6$, and $X_i$ has bipartite 1-skeleton. Then for any component $X$ of the complex tensor product $\otimes_{i=1}^{m} X_i$, $\text{Aut}(X)$ can be generated by automorphisms of $X_i$’s together with permutations of isomorphic factors.

**Conjecture 5.12.** For $i \in \{1, 2, \ldots, m\}$, suppose that $X_i$ is an elementary ordinary complex with faces of the same even length $2n \geq 6$, and $X_i$ has surface structure. Then
for any component $X$ of the complex tensor product $\otimes_{i=1}^{m}X_i$, $\text{Aut}(X)$ can be generated by automorphisms of $X_i$’s together with permutations of isomorphic factors.

6 Acknowledgements

The author appreciates the support of National Center for Theoretical Science, Taiwan, and would like to thank Ian Leary for introducing this topic and valuable advice.

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