Efficient numerical solvers for the nonlinear beam
and wave equations

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Abstract.
In this paper we give the main ideas about the numerical temporal resolution of second order
in time partial differential equations with high order symmetric multistep cosine methods. These
methods are efficient numerical solvers when integrating second-order in time partial differential
problems subject to periodic boundary conditions, like the nonlinear wave or the beam equations.

1. Introduction
High order symmetric multistep cosine methods \cite{2} are known to be efficient numerical solvers
when integrating second-order in time partial differential problems subject to periodic boundary
conditions, like

\[
\begin{align*}
\frac{du}{dt}(t,x) &= Au(t,x) + g(t,u(t,x)), \quad x \in [a,b], \quad t_0 < t < T, \\
u(t_0, x) &= u_0(x), \quad u_t(t_0, x) = v_0(x), \\
u(t, a) &= u(t, b), \quad u_x(t, a) = u_x(t, b),
\end{align*}
\]

(1)

where \( g, u_0 \) and \( v_0 \) are smooth enough functions and \( A \) is a differential linear operator with
negative eigenvalues. For the nonlinear wave equation we have \( Au = u_{xx} \) and in the case of the
nonlinear beam (or Euler-Bernoulli) equation we have \( Au = -u_{xxxx} \). Here, we are assuming
hypotheses of enough regularity of the solution, which is interesting in practice, for example,
when the solutions are solitons or balanced geophysical flows \cite{5}.

Multistep cosine methods can be considered as a subtype of ‘exponential integrators’
\cite{6, 7, 8, 9, 10}, which integrate the linear and stiff part of a problem in an exact way. Because of
this, it is possible to obtain methods which are explicit and stable at the same time for linearly
stiff problems. In order to construct these methods, we have taken profit of the existence in
the literature \cite{11} of standard explicit symmetric multistep methods of high order for second-
order systems, which have been optimized with respect to their interval of stability. These
methods are used when integrating planetary orbits. In \cite{4} it is shown the detailed construction
of methods of orders 8 (SMC8) and 10 (SMC10), which are the ones we are using here in
the numerical experiments to numerically solve the beam equation. These methods present a
good long-term behaviour when integrating Hamiltonian systems because of its symmetry (see [1] for the study about the conservation of invariants when using these methods to integrate Hamiltonian systems).

The structure of the paper is as follows. In Section 2 we give the main ideas about the spatial discretization we are using. In Section 3 we introduce multistep cosine methods SMC8 and SMC10 and finally, we give some numerical results in Section 4.

2. Problem to solve
In order to discretize in space problem (1), we take, for a natural number \( M \), as an approximation of the solution, \( u_M(t,x) = \sum_{j=-M}^{M-1} a_j(t)e^{2\pi ij\frac{x}{T}} \). Then, we obtain

\[
\begin{align*}
\dot{Y}_M(t) &= -\Omega^2 M Y_M(t) + G_M(t,Y_M(t)), \\
Y_M(t_0) &= Y_{M,0}, \quad \dot{Y}_M(t_0) = V_{M,0},
\end{align*}
\]

where \( Y_M(t) = [a_{-M}(t), \ldots, a_{M-1}(t)]^T, \quad \Omega_M = \text{diag}(\lambda_j)_{j=-M}^{M-1}, \quad \text{with} \quad \lambda_j = \left(\frac{2\pi j}{T}\right)^2 \) for the nonlinear wave equation and \( \lambda_j = \left(\frac{2\pi j}{T}\right)^2 \) for the beam equation. Furthermore,

\[
G_M(t,Y_M(t)) = \frac{1}{2M} \text{shift}\left(F_{2M}g(t,2MF_{2M}^{-1}\text{shift}(Y_M(t)))\right),
\]

where \( F_{2M} \) is the \( 2M \)-dimensional discrete Fourier transform, with shift an application such that for every vector \( \vec{v} = [v_0, \ldots, v_{2M-1}] \), shift(\( \vec{v} \)) = [\( v_M, \ldots, v_{2M-1}, v_0, \ldots, v_{M-1} \)]. Finally, \( g(t, \vec{v}) \) denotes the vector whose components are \( (g(t,v_j))_{j=0}^{2M-1} \) and \( Y_{M,0} \) and \( V_{M,0} \) contain the initial conditions. This spatial discretization is very accurate, so the values \( Y_M(t) \) and \( G_M(t,Y_M(t)) \) approximate the Fourier coefficients of the regular functions \( u(t,x) \) and \( g(t,u(t,x)) \), respectively.

In what follows, in order to simplify the notation, we will omit the subscript \( M \).

3. Multistep cosine methods
Multistep cosine methods are of the form

\[
\begin{align*}
Y_{n+2k} + \delta_{k-1}(h\Omega)(Y_{n+2k-1} + Y_{n+1}) + \ldots + \delta_1(h\Omega)(Y_{n+k+1} + Y_{n+k-1}) + Y_n \\
-2[\cos(kh\Omega)+\delta_{k-1}(h\Omega)\cos((k-1)h\Omega)+\ldots+\delta_2(h\Omega)\cos(2h\Omega)+\delta_1(h\Omega)\cos(h\Omega)]Y_{n+k} \\
= h^2[\gamma_1(h\Omega)(G_{n+2k-1}+G_{n+1})+\ldots+\gamma_{k-1}(h\Omega)(G_{n+k+1}+G_{n+k-1})+\gamma_k(h\Omega)G_{n+k}],
\end{align*}
\]

where \( Y_n \) is the numerical approximation to \( Y(t_n) \) and \( G_{n+j} \) means \( G(t_{n+j},Y_{n+j}) \), \( j = 1, \ldots, 2k-1 \). Besides, \( \delta_j(h\Omega) \), for \( j = 1, \ldots, k-1 \) and \( \gamma_i(h\Omega) \), for \( i = 1, \ldots, k \) are real functions. The method is uniquely determined by functions \( \delta_j(h\Omega) \), for \( j = 1, \ldots, k-1 \) and have the desired order of consistency with the only condition of their being bounded on the real axis. In [4] we suggest how to choose them in order to assure stability and we give a detailed and very technical study for general multistep cosine methods. Once they have been calculated, functions \( \gamma_i(\epsilon), \quad i = 1, \ldots, k \) are given by

\[
\gamma_i(\epsilon) = \int_0^k \epsilon^{-1} \sin((k-u)\epsilon)I_{k-l}(u)du + \sum_{j=1}^{k-1} \delta_j(\epsilon) \int_0^j \epsilon^{-1} \sin((j-u)\epsilon)I_{k-l}(u)du,
\]

with \( I_l(u) \) the polynomial functions of degree \( 2k-2 \) in \( u \), \( l = 0, \ldots, k-1 \), such that \( I_l(-l) = \delta_l(l) = 1 \) and \( I_l(j) = 0 \), for \( j = -(k-1), \ldots, k-1, \) with \( j \neq l \).
3.1. Resonances
When we are integrating a problem like (2), sometimes we can observe that the behaviour of the numerical solution is not the expected one and that the error is much more greater than the desirable one. These bad results are caused because there are some particular values of the time step size for which the error is greater than for other values which are well apart from them. This is called as resonances and it is related with the double roots of the first characteristic polynomial.

In [3, 4] resonances are deeply studied. There, it is suggested to considered a filter function

$$\phi(\epsilon) = \begin{cases} 
1 & \text{if } |\epsilon| < \delta, \\
0 & \text{if } |\epsilon| \geq \delta.
\end{cases}$$  \hspace{1cm} (4)

This value is related with the first value of $\epsilon$ for which double roots of the first characteristic polynomial are met. The filter method then reads as (2) but by substituting $G(t_{n+j}, Y_{n+j})$ by $G(t_{n+j}, \phi(h\Omega)Y_{n+j})$, $j = 1, \ldots, 2k - 1$.

3.2. Optimal suggested method of eighth order
In this case, we take $k = 4$. Therefore, the method is uniquely determined when functions $\delta_l(h\Omega)$, $l = 1, 2, 3$ are given. For method SMC8 we have the following ones [4]:

$$\delta_1(\epsilon) = 1 - 2\cos(\epsilon), \quad \delta_2(\epsilon) = 2, \quad \delta_3(\epsilon) = -2\cos(\epsilon).$$

Functions $\gamma_j(\epsilon)$, $j = 1, \ldots, 5$ have been calculated with Mathematica® by using formulae (3). These functions are given by

$$\gamma_1(\epsilon) = \frac{360 - 840\epsilon^2 + 721\epsilon^4 - 288\epsilon^6 + \cos(\epsilon)(-360 + 660\epsilon^2 - 376\epsilon^4 + 72\epsilon^6)}{36\epsilon^8},$$

$$\gamma_2(\epsilon) = \frac{-120 + 260\epsilon^2 - 192\epsilon^4 + 52\epsilon^6 + \cos(\epsilon)(120 - 200\epsilon^2 + 87\epsilon^4)}{2\epsilon^8},$$

$$\gamma_3(\epsilon) = \frac{-600 + 1240\epsilon^2 - 855\epsilon^4 + 228\epsilon^6 + \cos(\epsilon)(600 - 940\epsilon^2 + 360\epsilon^4 - 8\epsilon^6)}{4\epsilon^8},$$

$$\gamma_4(\epsilon) = \frac{1800 - 3660\epsilon^2 + 2480\epsilon^4 - 612\epsilon^6 + \cos(\epsilon)(-1800 + 2760\epsilon^2 - 1025\epsilon^4 - 18\epsilon^6)}{9\epsilon^8}.$$

For this method we have taken $\delta = 1.2$ in (4).

3.3. Optimal suggested method of tenth order
In order to construct the method of tenth order, we have taken $k = 5$. For this method, we have obtained

$$\delta_1(\epsilon) = 5 - 4\cos(\epsilon), \quad \delta_2(\epsilon) = 3 - 4\cos(\epsilon), \quad \delta_3(\epsilon) = 3 - 2\cos(\epsilon), \quad \delta_4(\epsilon) = 1 - 2\cos(\epsilon).$$

For this method, by using again (3) and Mathematica® we have

$$\gamma_1(\epsilon) = \frac{15120 - 39060\epsilon^2 + 39165\epsilon^4 - 19811\epsilon^6 + 5544\epsilon^8 + \cos(\epsilon)(-15120 + 31500\epsilon^2 - 22785\epsilon^4 + 7085\epsilon^6 - 1008\epsilon^8)}{504\epsilon^{10}},$$

$$\gamma_2(\epsilon) = \frac{60480 - 148680\epsilon^2 + 136500\epsilon^4 - 58013\epsilon^6 + 10584\epsilon^8 + \cos(\epsilon)(-60480 + 118440\epsilon^2 - 74760\epsilon^4 + 15614\epsilon^6 + 504\epsilon^8)}{252\epsilon^{10}},$$

$$\gamma_3(\epsilon) = \frac{15120 - 35820\epsilon^2 + 31065\epsilon^4 - 12197\epsilon^6 + 2214\epsilon^8 + \cos(\epsilon)(-15120 + 28260\epsilon^2 - 16305\epsilon^4 + 2846\epsilon^6 - 72\epsilon^8)}{18\epsilon^{10}},$$

$$\gamma_4(\epsilon) = \frac{60480 - 140040\epsilon^2 + 117780\epsilon^4 - 44549\epsilon^6 + 7380\epsilon^8 + \cos(\epsilon)(-60480 + 109800\epsilon^2 - 60360\epsilon^4 + 9710\epsilon^6 + 144\epsilon^8)}{36\epsilon^{10}},$$

$$\gamma_5(\epsilon) = \frac{75600 - 173700\epsilon^2 + 144705\epsilon^4 - 54055\epsilon^6 + 9216\epsilon^8 + \cos(\epsilon)(-75600 + 135900\epsilon^2 - 73605\epsilon^4 + 11485\epsilon^6 - 216\epsilon^8)}{36\epsilon^{10}}.$$

In this case, we have $\delta = 1$ in (4).
4. Numerical experiments
Here, we integrate the following equation, which has periodic boundary conditions

\[ u_{tt}(t, x) = -u_{xxxx}(t, x) - 0.4u^3(t, x), \quad x \in [0, 2\pi], \]

\[ u(0, x) = \frac{5}{2}e^{-\frac{\cos(x)}{5}} - \frac{5}{2}, \quad u_t(0, x) = -\frac{5}{2}e^{-\frac{\sin(x)}{5}} + \frac{5}{2}, \quad x \in [0, 2\pi], \]

\[ u(t, 0) = u(t, 2\pi), \quad u_x(t, 0) = u_x(t, 2\pi), \quad u_{xx}(t, 0) = u_{xx}(t, 2\pi), \quad u_{xxx}(t, 0) = u_{xxx}(t, 2\pi). \]

We have integrated this problem with the spatial discretization described in Section 2, with 32 nodes in space. After this, we have integrated in time, by considering both methods with the filter function (4). In Table (1) we can observe the discrete global \(L^2\)-error until time \(T = 1\) committed for different values of \(h\) and the order achieved by SMC8 method in the left part. In the right part of this table, we have the results corresponding to SMC10 method.

| \(h\) | \(L^2\)-error | Order | \(h\) | \(L^2\)-error | Order |
|------|--------------|-------|------|--------------|-------|
| \(\frac{1}{40}\) | 6.61590E-12 |       | \(\frac{1}{40}\) | 9.05268E-13 |       |
| \(\frac{1}{30}\) | 2.01536E-14 | 8.35876 | \(\frac{1}{30}\) | 3.48342E-16 | 11.3436 |
| \(\frac{1}{20}\) | 7.49850E-17 | 8.07022 | \(\frac{1}{20}\) | 2.36999E-19 | 10.5214 |
| \(\frac{1}{10}\) | 2.89651E-19 | 8.01614 | \(\frac{1}{10}\) | 2.22237E-22 | 10.0586 |
| \(\frac{1}{6}\) | 1.12577E-21 | 8.00726 | \(\frac{1}{6}\) | 2.14850E-25 | 10.0146 |
| \(\frac{1}{4}\) | 4.38926E-24 | 8.00272 | \(\frac{1}{2}\) | 2.11144E-28 | 9.99089 |

SMC8 method | SMC10 method

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