Linear Connections on Graphs

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Abstract

In recent years, discrete spaces such as graphs attract much attention as models for physical spacetime or as models for testing the spirit of non-commutative geometry. In this work, we construct the differential algebras for graphs by extending the work of Dimakis et al and discuss linear connections and curvatures on graphs. Especially, we calculate connections and curvatures explicitly for the general nonzero torsion case. There is a metric, but no metric-compatible connection in general except the complete symmetric graph with two vertices.

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I. INTRODUCTION

In the last few years, there has been a rapid increase of interest in non-commutative geometry. Non-commutative geometry is the geometry of quantum spaces, which are generalized spaces replacing classical smooth manifolds\(^1\). A quantum space is an associative algebra, usually non-commutative, with possibly more structures on it\(^1,2\). Even though non-commutative geometry is developed usually for non-commutative algebras, however, there can be interesting commutative quantum spaces such as discrete sets, which we are mainly concerned with in this work. In fact, discrete spaces attract much attention in recent years as models for physical spacetime or as models for testing the spirit of quantum spaces\(^3−6\). Differential calculi on commutative algebras have already been investigated by some authors\(^6−8\).

One of the motivations for studying quantum spaces is to understand the small scale structure of spacetime and quantum gravity. There are some efforts to try to understand gravity in the framework of the non-commutative version of Riemannian geometry (see, e.g., Ref. 9, 10). However, non-commutative Riemannian geometry is not a straightforward extension of the ordinary one and several definitions of a connection have been considered. In particular, definitions of a linear connection which make use of the bimodule structure of differential forms have been suggested for projective modules\(^11\) and for differential calculi based on derivations\(^12,13\). For discrete sets, one can define a linear connection using a group structure supplied to the discrete sets\(^14\). Recently, a more general definition has been proposed by Mourad\(^15\) of a linear connection, which is also our main concern in this work.

In this work, we shall extend the formulation of Dimakis et al\(^6\) on graphs in order to make them suitable for calculation of linear connections proposed by Mourad and curvatures. In sec. II, we shall have a review of the universal differential algebra necessary in the sequel. And differential calculi on a graph\(^6\) is extended toward the construction of the differential algebra for the graph. In sec. III, we shall calculate explicitly Mourad’s linear connections and curvatures on graphs. Nonzero torsion connections, bilinear curvatures and metrics shall be discussed.

II. DIFFERENTIAL ALGEBRAS FOR GRAPHS

A. Universal differential algebra

Let \( A \) be an associative algebra with 1 over the field \( \mathbb{C} \) of complex numbers. Let a direct sum of vector spaces \( \Omega = \bigoplus_{n=0}^{\infty} \Omega^n \) be a differential complex so that there are homomorphisms \( d \)'s

\[
\cdots \rightarrow \Omega^{n-1} \xrightarrow{d} \Omega^n \xrightarrow{d} \Omega^{n+1} \rightarrow \cdots
\]

such that \( d^2 = 0 \). The homomorphism \( d \) is usually called the differential operator of
the complex $\Omega$. If there is a gradation-respecting multiplication $\cdot$ in $\Omega$ so that $\Omega$ is an algebra over $C$ and the homomorphisms $d$'s satisfy the Leibniz rule

$$d(\omega \cdot \omega') = (d\omega) \cdot \omega' + (-1)^n \omega \cdot d\omega'$$

(2)

where $\omega \in \Omega^n$, then the algebra is called a differential algebra over $C$.

There is an important example for a differential algebra, which is constructed from an associative algebra $A$ with 1 over $C$ as follows (See, e.g., Ref. 16, 17). Let $\Gamma$ be an $A$-bimodule. Let $d : A \to \Gamma$ be a linear map such that for any $a, b \in A$, $d(ab) = (da)b + a db$. If every element of $\Gamma$ is of the form $\sum_k a_k db_k$ where $a_k, b_k \in A$, then $(\Gamma, d)$ is said to be a 1st order differential calculus over $A$. We have a special 1st order differential calculus over $A$. Let $m : A \otimes_A A \to A$ be the multiplication map in $A$ such that $m(a \otimes b) = ab$. Let $\Omega^1 \equiv \ker m$ and let $d : A \to \Omega^1$ be a map defined by $da \equiv 1 \otimes a - a \otimes 1$. Then $\Omega^1$ is an $A$-bimodule and $(\Omega^1, d)$ is a 1st order differential calculus over $A$. Since $A \otimes_A A$ also carries an $A$-bimodule structure and the multiplication map $m : A \otimes_A A \to A$ is a bimodule homomorphism, we have an exact sequence of bimodule homomorphisms

$$0 \to \Omega^1 \to A \otimes_A A \xrightarrow{m} A \to 0.$$  

(3)

It is well known that the 1st order differential algebra $(\Omega^1, d)$ over $A$ is universal, i.e. if there is another 1st order differential calculus $(\Gamma, \delta)$ over $A$, then there exists a unique $A$-bimodule homomorphism $\phi : \Omega^1 \to \Gamma$ such that

$$\delta = \phi \circ d.$$  

(4)

Every 1st order differential calculus $(\Gamma, \delta)$ over $A$ is isomorphic, as $A$-bimodules, to a quotient of $\Omega^1$ by the $A$-submodule $\ker \phi$.

We can extend the 1st order differential calculus $\Omega^1$ to higher orders: Let $\Omega^n \equiv \{ \rho \in A \otimes_C \cdots \otimes_C A \equiv A^{\otimes_C(n+1)} \mid m_i \rho = 0 \text{ for all } i = 1, \cdots, n \}$ where $m_i$ is the multiplication acting in the $i, (i + 1)$th place. Then $\Omega^n$ is an $A$-bimodule and $\Omega^n = \text{span}\{a_0 da_1 \otimes \cdots \otimes da_n \mid a_i \in A\}$ $\equiv \Omega^1 \otimes_A \cdots \otimes_A \Omega^1 \equiv (\Omega^1)^{\otimes_A n}$. From these $\Omega^n$’s, we can construct a differential algebra. Let $\Omega(A) \equiv \oplus_{n=0}^{\infty} \Omega^n$ where $\Omega^0 \equiv A$. And we define a multiplication $\cdot$ in $\Omega(A)$ as follows:

(i) Let $a_0 da_1 \otimes da_2 \cdots \otimes da_n \equiv (a_0, a_1, a_2, \cdots, a_n)$ and define a multiplication of two elements in $\Omega^n$ and $\Omega^{m-1}$ to get an element in $\Omega^{n+m-1}$ by

$$(a_0, a_1, \cdots, a_n) \cdot (a_{n+1}, \cdots, a_{n+m})$$

$$= \sum_{i=0}^{n} (-1)^{n-i} (a_0, a_1, \cdots, a_i a_{i+1}, \cdots, a_n, a_{n+1}, \cdots, a_{n+m}).$$  

(5)

1) We shall use the same notation $\otimes$ for the two kinds of tensor products $\otimes_C, \otimes_A$ if there is no confusion.
(ii) Extend this multiplication to the whole of $\Omega(A)$ using the distributive rule in an obvious manner.

(iii) Extend $d$ to the whole space $\Omega(A)$ as follows:

$$d(a_0, a_1, \ldots, a_n) \equiv (1, a_0, a_1, \ldots, a_n)$$ (6)

and

$$d(1, a_0, a_1, \ldots, a_n) \equiv 0.$$ (7)

Then $\Omega(A)$ is a differential algebra. This differential algebra is universal.

It is straightforward to see that if $a_i a_{i+1} = 0$ for $i = 0, 1, \ldots, n - 1$, then

$$(a_0, a_1, \ldots, a_n) = a_0 \otimes a_1 \otimes \cdots \otimes a_n,$$ (8)

and

$$(a_0, a_1, \ldots, a_n) \cdot (a_{n+1}, \ldots, a_{n+m}) = (a_0, \ldots, a_{n-1}, a_n a_{n+1}, a_{n+2}, \ldots, a_{n+m}).$$ (9)

and

$$d(a_0, a_1, \ldots, a_n) = \sum_{q=0}^{n+1} (-1)^q a_0 \otimes a_1 \otimes \cdots \otimes a_{q-1} \otimes 1 \otimes a_q \otimes \cdots \otimes a_n.$$ (10)

B. Differential algebras on graphs

Let $V$ be a set of $N$ points $x_1, \ldots, x_N$ ($N < \infty$). As in Ref. 6, let $A$ be the algebra of complex functions on $V$ with $(fg)(x_i) = f(x_i)g(x_i)$. Let $e_i \in A$ be defined by

$$e_i(x_j) = \delta_{ij}.$$ (11)

Then it follows that

$$e_i e_j = \delta_{ij} e_i, \quad \sum_i e_i = 1$$ (12)

and each $f \in A$ can be written as $f = \sum_i f(i) e_i$ where $f(i) = f(x_i) \in \mathbb{C}$. It is obvious that $A$ is not only a commutative algebra with 1 but also an $N$-dimensional complex vector space.

Now let us introduce the universal differential algebra $\Omega(A)$ and the differential operator $d$ as in the previous section. Then the differentials satisfy the following relations.

$$e_i de_j = -(de_i)e_j + \delta_{ij}de_i$$ (13)

and

$$\sum_i de_i = 0, \quad d1 = 0.$$ (14)

The universal 1st order differential calculus $\Omega^1$ is generated by $B \equiv \{e_i de_j \mid i, j = 1, 2, \ldots, N(i \neq j)\}$ as an $A$-bimodule. In this work, we note that $\Omega^1$ is a finite-dimensional complex vector space with having the generators $B$ as a basis.
Similarly, for \( n \geq 2 \), \( \Omega^n \) is not only an \( A \)-bimodule generated by \( \{ e_{i_1}de_{i_2} \otimes \cdots \otimes de_{i_{n+1}} \mid i_k = 1, 2, \ldots, N \ (i_k \neq i_{k+1}) \} \) but also a complex vector space with having the generators as a basis. In the universal differential algebra \( \Omega(A) \) of \( A \), the multiplication \( \cdot \) in Eq.(8) yields

\[
(e_{i_1}de_{i_2} \otimes \cdots \otimes de_{i_n}) \cdot (e_{j_1}de_{j_2} \otimes \cdots \otimes de_{j_s}) = e_{i_1}de_{i_2} \otimes \cdots \otimes d(e_{i_1}e_{j_1}) \otimes \cdots \otimes de_{j_s} = \delta_{i_1j_1}e_{i_1}de_{i_2} \otimes \cdots \otimes de_{i_{r-1}} \otimes de_{j_1} \otimes de_{j_2} \otimes \cdots \otimes de_{j_s}.
\]

Thus, effectively, the multiplication \( \cdot \) is the same as the tensor product \( \otimes_A \), which is crucial in the sequel together with the fact that the differential operator \( d \) satisfies Eq. (10).

A graph is a set of vertices which are interconnected by a set of edges\(^{18}\). Graphs are assumed to be connected, i.e. two arbitrary vertices can be connected by a sequence of consecutive edges. A complete graph is a graph for which every pair of distinct vertices is connected by one edge. In some applications, it is natural to assign a direction to each edge of a graph. Diagrammatically, the direction of each edge is represented by an arrow. A directed graph (or digraph) is a graph augmented in this way. A symmetric digraph is a digraph in which any connected pair of distinct vertices is connected in both directions. In this work, we shall be concerned only with digraphs and hence every graph is a digraph unless otherwise stated.

Now let us regard the points \( x_i \) of \( V \) as vertices and a generator \( e_i de_j (i \neq j) \) in \( B \) as an arrow from \( x_i \) to \( x_j \). Then we can associate a graph with a set \( V \) of points and a subset \( S \) of \( B \). From now on, we shall denote a graph by \( (A, K^1) \), where \( K^1 \) is an \( A \)-bimodule generated by the subset \( S \). Accordingly, \( (A, \Omega^1) \) represents a complete symmetric graph since the generators in \( B \) correspond to all arrows connecting any two vertices. In this case, the graph \( (A, K^1) \) is said to be a subgraph of \( (A, \Omega^1) \). A subgraph \( (A, K^1) \) is obtained by deleting some of the arrows in a complete symmetric graph \( (A, \Omega^1) \). It is obvious that \( \Omega^1 \) is the direct sum of \( K^1 \) and its complement which is an \( A \)-bimodule generated by the set \( B - S \). We define an \( A \)-bimodule homomorphism \( \phi_1 : \Omega^1 \rightarrow K^1 \) to be the projection map. If we define a map \( \delta : A \rightarrow K^1 \) by \( \delta = \phi_1 \circ d \), then it is straightforward to see that \((K^1, \delta)\) is a 1st order differential calculus over \( A \) with an observation \( e_i de_j = \phi_1(e_i de_j) = e_i \phi_1(de_j) = e_i \delta e_j \) for \( e_i de_j \in K^1 \).

Now we shall construct a differential algebra from \( K^1 \). We define \( K^n \) to be the quotient space:

\[
K^n \equiv \frac{\Omega^n}{< d(\ker\phi_{n-1}) >}
\]

for \( n = 2, 3, 4, \ldots \), where \( \phi_{n-1} \) is the projection map from \( \Omega^{n-1} \) to \( K^{n-1} \) and the bracket \( < X > \) means the \( A \)-bimodule generated by the set \( X \). So \( < d(\ker\phi_{n-1}) > \) is the \( A \)-bimodule generated by the subspace \( d(\ker\phi_{n-1}) \) of \( \Omega^n \), and \( \ker\phi_n = < d(\ker\phi_{n-1}) > \).
We define \( \delta : K^n \rightarrow K^{n+1} \) by \( x + < d(\ker \phi_{n-1}) > \mapsto \phi_{n+1}(dx) \) for \( x \in \Omega^n \) and for \( n = 1, 2, \cdots \). This is well-defined since if \( x + < d(\ker \phi_{n-1}) > = y + < d(\ker \phi_{n-1}) > \), then \( d(x - y) \in < d(\ker \phi) > \) and hence \( \delta(x + < d(\ker \phi_{n-1}) >) = \delta(y + < d(\ker \phi_{n-1}) >) \).

Now let us investigate the form of generators for \( \ker \phi_n \) \( (n \geq 2) \) since it is an essential ingredient in the sequel. Let \( e_i de_j \in \ker \phi_1 \). From Eq. (10), we have

\[
d(e_i de_j) = 1 \otimes e_i \otimes e_j - e_i \otimes 1 \otimes e_j + e_i \otimes e_j \otimes 1.
\]

By multiplying \( e_k \) \( (k \neq i) \) to the left of \( d(e_i de_j) \), we obtain generators \( e_k \otimes e_i \otimes e_j = e_k de_i \otimes de_j \). Similarly, by multiplying \( e_k \) \( (k \neq j) \) to the right of \( d(e_i de_j) \), we obtain generators \( e_i \otimes e_j \otimes e_k = e_i de_j \otimes de_k \). Thus the remaining term \( e_i \otimes (1 - e_i - e_j) \otimes e_j \) of \( d(e_i de_j) \), which is of the form \( e_i de_1 \otimes de_j + e_i de_2 \otimes de_j + \cdots + e_i de_N \otimes de_j \) with two terms \( e_i de_i \otimes de_j \) and \( e_i de_j \otimes de_j \) being deleted, is a generator of \( \ker \phi_2 \). At present, we do not have to worry about the possibility of decomposing the generator further into simpler ones. We note that each term of the generator begins with \( e_i \) and ends with \( de_j \). It is straightforward to proceed further in a similar manner. This leads us to the following lemma.

**Lemma 1:** For \( n \geq 2 \), \( \ker \phi_n (\neq \emptyset) \) can be generated by either single elements \( e_{i_1} de_{i_2} \otimes \cdots \otimes de_{i_{n+1}} \) or elements of the form \( e_i e_{i_1} de_{i_2} \otimes \cdots \otimes de_{i_{n+1}} + e_j e_{j_1} de_{j_2} \otimes \cdots \otimes de_{j_{n+1}} + \cdots + e_K e_{k_1} de_{k_2} \otimes \cdots \otimes de_{k_{n+1}} \) with \( e_i = e_j = \cdots = e_k \) and \( e_{i_1} = e_{j_1} = \cdots = e_{k_1} \) and \( e_{i_1} = e_{j_1} = \cdots = e_{k_1} \). Let \( \epsilon_I, \epsilon_{i_1}, \cdots, \epsilon_K = \pm 1 \).

The coefficients \( \pm 1 \) originate from the alternating sign in Eq. (10).

**Lemma 2:** Let \( x \in \Omega^m \) and let \( u \in \ker \phi_n \) for \( m \geq 0, n \geq 1 \). Then \( x \otimes u, u \otimes x \in \ker \phi_{n+m} \).

**Proof:** It is trivial for \( m = 0 \). Let \( m \neq 0 \) and let \( u_\alpha \) be either a single element generator \( e_{i_1} de_{i_2} \otimes \cdots \otimes de_{i_{n+1}} \) or a generator of the form \( e_I e_{i_1} de_{i_2} \otimes \cdots \otimes de_{i_{n+1}} + e_J e_{j_1} de_{j_2} \otimes \cdots \otimes de_{j_{n+1}} + \cdots + e_k e_{k_1} de_{k_2} \otimes \cdots \otimes de_{k_{n+1}} \) for \( \ker \phi_n \) \( (u_\alpha \) is a single element generator for \( n = 1) \). Then for any \( e_i de_j \in \Omega^1, e_i de_j \otimes u_\alpha = \delta_{ji} e_i du_\alpha \in \ker \phi_{n+1} \) and \( u_\alpha \otimes e_i de_j = (-1)^m \delta_{i_1 i_{n+1}} (du_\alpha) e_j \in \ker \phi_{n+1} \). Since \( \Omega^m = (\Omega^1)^{\otimes n} \), the proof is completed by induction. QED

The existence of generators for \( \ker \phi_n \) with such specific forms as in Lemma 1 gives an \( A \)-bimodule which is a complement of \( \ker \phi_n \) \( (n \geq 2) \). Let \( \{e_{i_1} de_{i_2} \otimes \cdots \otimes de_{i_{n+1}} | i_k = 1, 2, \cdots, N \ (i_k \neq i_{k+1}) \} \) be a basis for \( \Omega^n \). Let \( S_{ij} \) be the \( A \)-subbimodule of \( \Omega^n \) generated by all basis elements of \( \Omega^n \) with \( i_1 = i \) and \( i_{n+1} = j \). Then we have

\[
\Omega^n = \oplus_{i,j} S_{ij}.
\]
By Lemma 1, we have generators of $\ker \phi_n$, each of which belongs to $S_{ij}$ for some $i, j$. Let $S_{ij}^{(1)}$ be the subspace of $S_{ij}$ spanned by the generators of $\ker \phi_n$ in $S_{ij}$. Then it is clear that

$$\ker \phi_n = \bigoplus_{ij} S_{ij}^{(1)}.$$  

(18)

Now let $S_{ij}^{(2)}$ be a complement of $S_{ij}^{(1)}$ in $S_{ij}$ and define

$$Q^n = \bigoplus_{ij} S_{ij}^{(2)}.$$  

(19)

Then $Q^n$ is not only a complement of $\ker \phi_n$ in $\Omega^n$ but also an $A$-bimodule. In fact, if we let $v \in Q^n$, $v$ may be written as $v = \sum_{ij} v_{ij}$, where $v_{ij}$ is in $S_{ij}^{(2)}$. Let $f \in A$. Since $f(i)$ is a complex number, both $f v = \sum_{ij} f(i) v_{ij}$ and $vf = \sum_{ij} f(j) v_{ij}$ belong to $Q^n$.

By construction, we have a splitting exact sequence of $A$-bimodules

$$0 \rightarrow \ker \phi_1 \rightarrow \Omega^1 \xrightarrow{\phi_1} K^1 \rightarrow 0.$$  

(20)

The splitting map $\phi_1 : K^1 \rightarrow \Omega^1$ is just the inclusion map. Now we can also have a splitting exact sequence of $A$-bimodules for $n \geq 2$. It is clear that the map $\phi_n$, restricted to $Q^n$ is an $A$-bimodule isomorphism. Thus if we define a splitting map $\phi_n : K^n \rightarrow \Omega^n$ to be the inverse $(\phi_n |_{Q^n})^{-1}$, we have the following lemma.

**LEMMA 3:** For $n \geq 1$, the exact sequence of $A$-bimodules

$$0 \rightarrow \ker \phi_n \rightarrow \Omega^n \xrightarrow{\phi_n} K^n \rightarrow 0.$$  

is split.

**Proof:** It is enough to show that $\phi_n$ is an $A$-bimodule homomorphism for $n \geq 2$. Let $\xi = e_{i_1} de_{i_2} \otimes \cdots \otimes de_{i_{n+1}}$ be a generator of $\Omega^n$. Since $S_{i_1 i_{n+1}} = S_{i_1 i_{n+1}}^{(1)} \oplus S_{i_1 i_{n+1}}^{(2)}$, $\xi$ is expressed uniquely as $u + v$ for $u \in S_{i_1 i_{n+1}}^{(1)}$ and $v \in S_{i_1 i_{n+1}}^{(2)}$. Thus since $j_n(\xi + \ker \phi_n) = v \in S_{i_1 i_{n+1}}^{(2)}$, it follows that for any $f \in A$, $j_n(f(\xi + \ker \phi_n)) = j_n(f(i_1)\xi + \ker \phi_n) = f(i_1) j_n(\xi + \ker \phi_n) = f j_n(\xi + \ker \phi_n)$. Similarly, $j_n((\xi + \ker \phi_n)f) = j_n(\xi + \ker \phi_n)f(i_{n+1}) = j_n(\xi + \ker \phi_n)f$. QED

Let us put $K^0 = A$ and $\phi_0 = j_0 = id$.

**PROPOSITION 1:** The following diagram

$$\begin{array}{cccccccc}
A \xrightarrow{d} & \Omega^1 \xrightarrow{d} & \cdots \xrightarrow{d} & \Omega^{n-1} \xrightarrow{d} & \Omega^n \xrightarrow{d} & \Omega^{n+1} \xrightarrow{d} & \cdots \\
\downarrow \phi_0 & \downarrow \phi_1 & & \downarrow \phi_{n-1} & \downarrow \phi_n & & \downarrow \phi_{n+1} \\
A \xrightarrow{\delta} & K^1 \xrightarrow{\delta} & \cdots \xrightarrow{\delta} & K^{n-1} \xrightarrow{\delta} & K^n \xrightarrow{\delta} & K^{n+1} \xrightarrow{\delta} & \cdots 
\end{array}$$
commutes and $\delta^2 = 0$.

Proof: The commutativity of the diagram is a consequence of the definition of $\delta$. For any $n = 1, 2, \cdots$, let $u \in K^{n-1}$. Then $d(j_{n-1}u) = j_n v + w$ for some $v \in K^n$ and $w \in \ker \phi_n$. Now $d(w) \in d(ker \phi_n) \subset ker \phi_{n+1}$, and $d(j_n v) = j_{n+1}v_1 + v_2$ for some $v_1 \in K^{n+1}$ and $v_2 \in ker \phi_{n+1}$. Thus from $d^2 j_{n-1} = 0$, it follows that $j_{n+1} v_1 + v_2 + dw = 0$. But $j_{n+1} v_1 = -v_2 - dw = 0$ since $j_{n+1} K^{n+1} \cap ker \phi_{n+1} = \{0\}$. Hence $\delta^2 u = v_1 = 0$. QED

Let us define $P_{n,m} : K^n \otimes_A K^m \rightarrow K^{n+m}$, $\omega \otimes \omega' \mapsto \omega \cdot \omega'$, to be $P_{n,m} \equiv \phi_{n+m} \circ (j_n \otimes j_m)$ for $n, m \geq 0$. Note that the multiplication in $\Omega(A)$ is implicitly incorporated from Eq. (15). The map $P_{n,m}$ is well-defined. In fact, it is trivial for the cases where $n = 0$ and $m = 0$. Otherwise, let us take any splitting maps $j_n$ and $j_m$. Then $j_n \omega \otimes j_m \omega' - j_n \omega \otimes j_m \omega' = j_n \omega \otimes (j_m \omega' - j_m \omega') + (j_n \omega - j_n \omega) \otimes j_m \omega'$. Since $j_m \omega' - j_m \omega' \in ker \phi_m$ and $j_n \omega \in \Omega^n$ etc., $j_n \omega \otimes j_m \omega' - j_n \omega \otimes j_m \omega'$ belongs to $ker \phi_{n+m}$ by Lemma 2. Thus $\phi_{n+m}((j_n \omega \otimes j_m \omega')) = \phi_{n+m}(j_n \omega \otimes j_m \omega')$. Also, by the $A$-bilinearity of the splitting maps, $P_{n,m}(\omega f \otimes \omega') = P_{n,m}(\omega \otimes f \omega')$ for any $f \in A$.

Moreover, $P_{n,m}$ is associative.

Lemma 4: Let $\omega \in K^n$, $\omega' \in K^m$ and $\omega'' \in K^l$ for $n, m, l \geq 0$. Then

$$(\omega \cdot \omega') \cdot \omega'' = \omega \cdot (\omega' \cdot \omega'').$$

Proof: For $n = m = l = 0$, it is just the associativity of $A$. Otherwise, we have

$$(\omega \cdot \omega') \cdot \omega'' = P_{n+m,l}((\omega \cdot \omega') \otimes \omega'')
= \phi_{n+m+l}[j_{n+m}(\phi_{n+m}(j_n \omega \otimes j_m \omega')) \otimes j_l \omega'']
= \phi_{n+m+l}[j_n \omega \otimes j_m \omega' \otimes j_l \omega'']
= P_{n+m+l}(\omega \otimes (\omega' \cdot \omega'')) = \omega \cdot (\omega' \cdot \omega'').$$

The third equality comes from the fact that

$$j_{n+m}(\phi_{n+m}(j_n \omega \otimes j_m \omega')) \otimes j_l \omega'' - j_n \omega \otimes j_m \omega' \otimes j_l \omega''
= [j_{n+m}(\phi_{n+m}(j_n \omega \otimes j_m \omega')) - j_n \omega \otimes j_m \omega'] \otimes j_l \omega''
- j_n \omega \otimes [j_m \omega' \otimes j_l \omega'']
- [j_{n+m}(\phi_{n+m}(j_n \omega \otimes j_m \omega')) - j_n \omega \otimes j_m \omega'] \otimes j_l \omega''],$$

which is in $ker \phi_{n+m+l}$ by Lemma 2 since $j_{n+m}(\phi_{n+m}(j_n \omega \otimes j_m \omega')) - j_n \omega \otimes j_m \omega' \in ker \phi_{n+m}$ and $j_l \omega'' \in \Omega^l$ etc.. QED

Lemma 5: $\phi_{n+m} = P_{n,m}(\phi_n \otimes \phi_m)$ for $n, m \geq 0$. 

8
Proof: It is trivial for \( n = m = 0 \). Otherwise, let \( \alpha \otimes \beta \in \Omega^n \otimes_A \Omega^m = \Omega^{n+m} \).

\[
P_{n,m}(\phi_n \otimes \phi_m)(\alpha \otimes \beta) = (\phi_{n+m} \circ (J_n \otimes J_m))(\phi_n \alpha \otimes \phi_m \beta) = \phi_{n+m}(J_n \phi_n \alpha \otimes J_m \phi_m \beta).
\]

Now \( J_n \phi_n \alpha \otimes J_m \phi_m \beta - \alpha \otimes \beta = J_n \phi_n \alpha \otimes (J_m \phi_m \beta - \beta) + (J_n \phi_n \alpha - \alpha) \otimes \beta \), which is in \( \ker \phi_{n+m} \) by Lemma 2. Hence \( \phi_{n+m}(J_n \phi_n \alpha \otimes J_m \phi_m \beta) = \phi_{n+m}(\alpha \otimes \beta) \). QED

Now we are ready to show the fact that the operator \( \delta \) satisfies the Leibniz rule as \( d \) in Eq.(2).

**PROPOSITION 2:** Let \( \omega \in K^n \) and \( \omega' \in K^m \) for \( n, m \geq 0 \). Then

\[
\delta(\omega \cdot \omega') = (\delta \omega) \cdot \omega' + (-1)^n \omega \cdot \delta \omega'.
\]

**Proof:** For \( n, m \geq 0 \), it follows that

\[
\delta(\omega \cdot \omega') = \delta P_{n,m}(\omega \otimes \omega') = \delta \phi_{n+m}(J_n \otimes J_m)(\omega \otimes \omega')
\]

\[
= \phi_{n+m+1}(d J_n \omega \otimes J_m \omega') = \phi_{n+m+1}(d J_n \omega \otimes d J_m \omega')
\]

\[
= P_{n+1,m}(\phi_n d J_n \omega \otimes \phi_m J_m \omega') + (-1)^n P_{n,m+1}(\phi_n J_n \omega \otimes \phi_{m+1} d J_m \omega')
\]

\[
= (\delta \omega) \cdot \omega' + (-1)^n \omega \cdot \delta \omega',
\]

where the fifth equality comes from Lemma 5. QED

Thus we have proved the following,

**PROPOSITION 3:** \( K(A) = A \oplus K^1 \oplus K^2 \oplus \cdots \) is a differential algebra.

It is well known\(^6\) that \( < d \Omega^n > = \Omega^{n+1} \). The same relation holds for \( \delta \). In fact, \( K^{n+1} \supset \phi_{n+1}(< d J_n K^n >) \). To show \( K^{n+1} \subset \phi_{n+1}(< d J_n K^n >) \), we first observe that any element of \( \Omega^{n+1} \) is expressed (not necessarily uniquely) as a sum of an element in \( < d \ker \phi_n > \) and an element in \( < d J_n K^n > \) since \( \Omega^{n+1} = < d \Omega^n > \). Thus if we let \( v \in K^{n+1} \), we have \( J_{n+1}(v) = v_1 + v_2 \) for some \( v_1 \in < d \ker \phi_n > \) and \( v_2 \in < d J_n K^n > \).

Then \( v = \phi_{n+1}(v_1 + v_2) = \phi_{n+1}(v_1) + \phi_{n+1}(v_2) \in \phi_{n+1}(< d J_n K^n >) \). Thus \( K^{n+1} = \phi_{n+1}(< d J_n K^n >) \). On the other hand, \( \phi_{n+1}(< d J_n K^n >) = < \phi_{n+1} d J_n K^n > = < d K^n > \) since \( \phi_{n+1} \) is an \( A \)-bimodule homomorphism. Thus we have \( < d K^n > = K^{n+1} \).

From now on, we shall write \( e_{i_1 i_2 \cdots i_n} \) for \( e_{i_1} d e_{i_2} \otimes \cdots \otimes d e_{i_n} \) for simplicity, according to Ref. 6, with the convention that \( e_{i_1 i_2 \cdots i_{n-1}} = 0 \) if \( i_k = i_{k+1} \) for some \( k \). Also we shall often write \( v \) for \( x + \ker \phi_n \) of \( K^n \) if \( J_n(x + \ker \phi_n) = v \) for a given splitting map \( j_n \).
Example 1: Let \( V \) be a set \( \{x_1, x_2, x_3\} \) of three points. If we let \( K^1 \) be the space generated by \( \{e_{12}, e_{23}, e_{13}\} \), then \( \ker \phi_1 = < \{e_{21}, e_{32}, e_{31}\} > \). From this, it follows that \( \ker \phi_2 \) is generated by \( \Omega^2 = < \{e_{123}\} > \), and \( K^2 = < e_{123} > \) with the obvious splitting map \( j_2 \). Moreover, \( K^n = 0 \) for \( n \geq 3 \).

Similarly, if we let \( K^1 \) be the space generated by \( \{e_{12}, e_{23}, e_{32}\} \), then \( \ker \phi_1 = < \{e_{13}, e_{31}\} > \) and \( \ker \phi_2 \) is generated by \( \Omega^2 = < \{e_{121}, e_{212}, e_{232}, e_{323}\} > \). Also \( K^2 = < \{e_{121}, e_{212}, e_{232}, e_{323}\} >, \text{ etc..} \)

Example 2: Let \( V \) be a set \( \{x_1, x_2, x_3, x_4\} \) of four points and let \( K^1 \) be the space generated by \( \Omega^1 = < \{e_{14}, e_{41}\} > \). Then \( \ker \phi_1 = < \{e_{14}, e_{41}\} > \) and \( \ker \phi_2 \) is generated by the following 12 elements

\[ e_{141}, e_{142}, e_{143}, e_{241}, e_{243}, e_{341}, e_{342}, e_{412}, e_{413}, e_{414}, e_{124}, e_{134}, e_{214}, e_{234}, e_{314}, e_{324}, e_{421}, e_{431}. \]

Note that the two elements \( e_{124} + \ker \phi_2 \) and \( e_{134} + \ker \phi_2 \) are not linearly independent in \( K^2 \) since \( e_{124} + e_{134} \) belongs to \( \ker \phi_2 \). Rather, \( e_{124} = -e_{134} \) modulo \( \ker \phi_2 \).

In the case of the universal differential algebra \( \Omega(A) \), it is known that the sequence

\[ A \xrightarrow{\delta} \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n \xrightarrow{d} \cdots \]  

is exact\(^6\). However, this is not true for \( \delta \). The counter-example can be seen in the following.

Example 3: Let \( V \) be a set \( \{x_1, x_2, x_3\} \) of three points and let \( K^1 \) be the space generated by \( \{e_{12}, e_{23}, e_{31}\} \). Then \( \ker \phi_1 = < \{e_{21}, e_{32}, e_{31}\} > \). It is straightforward to see that \( \ker \phi_2 = \Omega^2 \). Hence \( K^n = 0 \) for \( n \geq 2 \). We note that there is an element, say \( e_{12} \otimes e_{23} = e_{123} \), which is in \( K^1 \otimes_A K^1 \), but not in \( K^2 \).

From the fact that the dimensions of both \( A \) and \( K^1 \) are the same in the sequence

\[ A \xrightarrow{\delta} K^1 \xrightarrow{\delta} K^2 = 0, \]  

the map \( \delta : A \rightarrow K^1 \) is an isomorphism if the sequence is exact. But this contradicts the fact that \( \delta 1 = 0 \). Thus the sequence is not exact.

III. LINEAR CONNECTIONS ON GRAPHS

A. Linear connections

Let \( A \) be an associative algebra and \( E \) be an \( A \)-bimodule. One may impose a reality condition once \( A \) is given as a \( * \)-algebra\(^12\). Let \( (\Omega^*, d) \) be a differential calculus over \( A \). If \( \Omega^1 \) is an \( A \)-bimodule, we can define a left and a right connection on \( E \). A left connection on \( E \) is defined to be a linear map

\[ D : E \rightarrow \Omega^1 \otimes_A E \]  

(22)
\[ D(f \omega) = df \otimes \omega + f D\omega \]  \hspace{1cm} (23)

for any \( f \in A \) and \( \omega \in E \). One can also define a right connection on \( E \) to be a linear map
\[ D : E \longrightarrow E \otimes_A \Omega^1 \]  \hspace{1cm} (24)

satisfying
\[ D(\omega f) = (D\omega)f + \omega \otimes df. \]  \hspace{1cm} (25)

In Ref. 15, a definition of a bimodule connection is proposed. A bimodule connection on \( E \) is a left connection \( D \) such that for any \( f \in A \) and \( \omega \in E \), \( D(\omega f) \) is of the form
\[ \sigma(\omega \otimes D\omega'). \]  \hspace{1cm} (26)

where \( \sigma \) is a map from \( E \otimes_A \Omega^1 \to \Omega^1 \otimes_A \Omega^1 \) generalizing the permutation. In particular, if we take \( E = \Omega^1 \), the bimodule connection is called a linear connection, which we are mainly concerned with in this work.

Let \( \Omega^2 \) be the \( A \)-bimodule of two-forms and let \( \pi : \Omega^1 \otimes_A \Omega^1 \longrightarrow \Omega^2 \) be a linear map satisfying \( \pi(\omega \otimes \omega') = \omega \cdot \omega' \) where \( \cdot \) is the multiplication map between forms. For the consistency of the definition of a linear connection, the map \( \sigma : \Omega^1 \otimes_A \Omega^1 \longrightarrow \Omega^1 \otimes_A \Omega^1 \) is assumed to be \( A \)-bilinear, i.e. for \( f \in A \) and \( \omega, \omega' \in \Omega^1 \),
\[ \sigma(f \omega \otimes \omega') = f \sigma(\omega \otimes \omega'), \quad \sigma(\omega \otimes \omega' f) = \sigma(\omega \otimes \omega') f. \]  \hspace{1cm} (27)

Moreover, \( \sigma \) is assumed to satisfy the following
\[ \pi \circ (\sigma + 1) = 0. \]  \hspace{1cm} (28)

The relation in Eq. (28) is a necessary and sufficient condition for the torsion \( T \) of the connection \( D \), defined by \( T = d - \pi \circ D \), to be \( A \)-bilinear (more precisely, right \( A \)-linear).

A linear connection \( D \) can be extended to two linear maps \( D_1 : \Omega^1 \otimes_A \Omega^1 \longrightarrow \Omega^2 \otimes_A \Omega^1 \), and \( D_2 : \Omega^1 \otimes_A \Omega^1 \longrightarrow \Omega^1 \otimes_A \Omega^2 \otimes_A \Omega^1 \), respectively, satisfying
\[ D_1(\omega \otimes \omega') = d\omega \otimes \omega' - \pi_{12}(\omega \otimes D\omega') \]  \hspace{1cm} (29)

and
\[ D_2(\omega \otimes \omega') = D\omega \otimes \omega' + \sigma_{12}(\omega \otimes D\omega') \]  \hspace{1cm} (30)

for \( \omega, \omega' \in \Omega^1 \), where \( \pi_{12} = \pi \otimes 1 \) and \( \sigma_{12} = \sigma \otimes 1 \).

It is easy to see that the linear map \( D_1 \circ D \) is a left \( A \)-linear. Also \( \pi_{12} D_2 \circ D \) is a left \( A \)-linear if the torsion \( T = 0 \). For the right \( A \)-linearity of \( D_1 \circ D \) and \( \pi_{12} D_2 \circ D \), there is not yet a widely accepted prescription even though it seems to be an essential property for the concept of a curvature. However, one prescription to obtain an \( A \)-bilinear curvature
has been proposed recently\textsuperscript{19}, which we shall use in this work. Especially, since $\Omega^1$ is free for graphs, one can construct the curvature invariants from the linear connection\textsuperscript{19}. In the next subsections, we shall calculate linear connections and curvatures explicitly on graphs with respect to the natural basis for the general nonzero torsion case. There are some other models\textsuperscript{20−23} for which linear connections and curvatures are calculated mostly without torsion.

B. Linear connections on complete symmetric graphs

Let $A$ be the associative algebra of complex functions on a set $V$ of $N$ points and let $\Omega(A)$ be the universal differential algebra, which corresponds to a complete symmetric graph, introduced in Sec. II. B. Since $\pi = 1$ for the universal differential algebra $\Omega(A)$, we take $\sigma = -1$ from Eq. (28) for the $A$-bilinearity of the torsion $T$. Thus a linear connection is given by a linear map $D : \Omega^1 \rightarrow \Omega^1 \otimes_A \Omega^1$ satisfying

$$D(f\omega) = df \otimes \omega + fD\omega, \quad D(\omega f) = (D\omega)f - \omega \otimes df$$

for any $f \in A$ and $\omega \in \Omega^1$. Moreover, $D_1$ and $D_2$ satisfy

$$D_1(\omega \otimes \omega') = d\omega \otimes \omega' - \omega \otimes D\omega', \quad D_2(\omega \otimes \omega') = D\omega \otimes \omega' - \omega \otimes D\omega'.$$

In this case, we have $D_1 = D_2 + T \otimes 1$.

Since $\Omega^1 \otimes_A \Omega^1 = \Omega^2$ is a vector space with a basis $\{e_{ijk} \mid i, j, k = 1, 2, \cdots, N \ (i \neq j, j \neq k)\}$, we may put

$$D(e_{ij}) = \sum_{k,l,m} \Gamma^{klm}_{ij} e_{klm}$$

for some numbers $\Gamma^{klm}_{ij}$'s. Here let us put $\Gamma^{klm}_{ij} = 0$ if $i = j$ or $k = l$ or $l = m$.

Now from the following two expressions of $D(e_{ij})$

$$D(e_{ij}) = D(e_i de_j) = de_i \otimes de_j + e_i D(de_j)$$
$$= \sum_a (e_{aij} - e_{iaj} + e_{ija}) + \sum_{m,b,c} (\Gamma^{ibc}_{mj} - \Gamma^{ibc}_{jm}) e_{ibc}$$

and

$$D(e_{ij}) = D(e_i de_j) = -D(de_i e_j) = -D(de_i) e_j + de_i \otimes de_j$$
$$= \sum_a (e_{aij} - e_{iaj} + e_{ija}) - \sum_{m,a,b} (\Gamma^{abj}_{mi} - \Gamma^{abj}_{im}) e_{abj},$$

we have $\Gamma^{abc}_{ij} = 0$ except for

$$\Gamma^{ija}_{ij} = 1 \ (a \neq j),$$
$$\Gamma^{aiz}_{ij} = 1 \ (a \neq i)$$

(36)
with $\Gamma_{ij}^{iaj}$ ($a \neq i, j$) undetermined. From these values of $\Gamma_{ij}^{klm}$’s, we obtain the following lemma.

**Lemma 6:** For any $i, j$ ($i \neq j$), $D(e_{ij})$ and $T(e_{ij})$ are of the following forms

$$D(e_{ij}) = de_i \otimes de_j + \sum_a (1 + \Gamma_{ij}^{iaj})e_{iaj},$$

$$T(e_{ij}) = -\sum_a (1 + \Gamma_{ij}^{iaj})e_{iaj}.$$ 

Thus it is obvious that the torsion $T = d - D$ is 0 if and only if $\Gamma_{ij}^{iaj} = -1$ for all $i, j$($i \neq j$) and $a(\neq i, j)$. Moreover, if the torsion $T$ is 0, the curvatures $D_1 \circ D = D_2 \circ D = 0$ in a complete symmetric graph, which is already known (see, e.g., Ref. 23). However, we shall calculate curvatures for the nonvanishing torsion $T$.

We can have a general form of the curvature $D_1 \circ D$ for any linear connection $D$ obeying Eq.(33):

$$D_1 De_{ij} = -\sum_{k,l,m} D_1 (\Gamma_{ij}^{klm} de_k \otimes e_l de_m)$$

$$= \sum_{k,l,m} \Gamma_{ij}^{klm} de_k \otimes D(e_l de_m)$$

$$= \sum_{a,b,c,d} \sum_{l,m} (\Gamma_{ij}^{blm} \Gamma_{lm}^{bcd} - \Gamma_{ij}^{alm} \Gamma_{lm}^{bcd}) e_{abcd}$$

$$= \sum_{a,b,c,d} \Omega_{abcd}^{ij} e_{abcd},$$

where

$$\Omega_{abcd}^{ij} = \sum_{l,m} (\Gamma_{ij}^{blm} \Gamma_{lm}^{abcd} - \Gamma_{ij}^{alm} \Gamma_{lm}^{abcd}).$$

Using $\Gamma_{ij}^{klm} = \delta_i^k \delta_j^l + \delta_i^l \delta_j^m + \Gamma_{ij}^{klm} \delta_i^k \delta_j^m$ in Eq. (36), we obtain $D_1 De_{ij}$. Also, $D_2 De_{ij}$ can be obtained from Lemma 6 and the following observations

$$D(de_j) = \sum_a \sum_{m \neq j} [(1 + \Gamma_{mj}^{ma})e_{maj} - (1 + \Gamma_{jm}^{jam})e_{jam}],$$

and for $a \neq i, j$,

$$de_i \otimes de_a \otimes de_j = \sum_l (e_{liaj} - e_{ilaj} + e_{ialj} - e_{iajl}).$$

The results are as follows.
Proposition 4:

\[ D_1 De_{ij} = -\sum_{l \neq i} (1 + \Gamma_{ij}^{il})e_{ijl} + \sum_{l \neq j} (1 - \Gamma_{ij}^{il})e_{dilj} \]

\[ -\sum_{l \neq j} \sum_{m \neq i} (\Gamma_{ij}^{im})e_{ilmj} + \sum_{l \neq j} (1 - \Gamma_{ij}^{il})e_{iljm}, \]

and

\[ D_2 De_{ij} = -\sum_{l \neq i} (1 - \Gamma_{ij}^{il})e_{ijl} + \sum_{l \neq j} (1 - \Gamma_{ij}^{il})e_{dilj} \]

\[ +\sum_{l \neq j} \sum_{m \neq i} (\Gamma_{ij}^{im})e_{ilmj} - \sum_{l \neq j} (1 - \Gamma_{ij}^{il})e_{iljm} + \sum_{l \neq i} \sum_{m \neq j} (1 + \Gamma_{lj}^{im})e_{ilmj}. \]

If \( D_1 \circ D \) is \( A \)-bilinear, the torsion \( T \) should vanish, i.e. all \( \Gamma_{jm}^{im} = -1 \) from the vanishment of the \( e_{ijlm} \) or \( e_{iljm} \) term in \( D_1 De_{ij} \). Similarly for \( D_2 \circ D \). We thus have

Corollary 1: Let the torsion \( T \) be \( A \)-bilinear on a complete symmetric graph. Then the necessary and sufficient condition for \( D_1 \circ D \) to be \( A \)-bilinear is \( T = 0 \). Similarly for \( D_2 \circ D \).

One prescription to get an \( A \)-bilinear curvature for the nonzero torsion \( T \) is to factor out all those elements that do not satisfy the desired condition. We refer to Ref. 19 for a recent discussion about this prescription. Thus if we factor out the terms \( e_{ijlm} \) and \( e_{iljm} \) of \( D_1 De_{ij} \), we have an \( A \)-bilinear curvature, denoted by \( Curv_1 \), since the remaining terms belong to the \( A \)-bimodule \( S_{ij} \). Similarly, we have an \( A \)-bilinear curvature denoted by \( Curv_2 \) from \( \pi_{12}D_2 De_{ij} = D_2 De_{ij} \).

Corollary 2: For all \( i, j \) \((i \neq j)\),

\[ Curv_1(e_{ij}) = -\sum_{l \neq i} (1 + \Gamma_{ij}^{il})e_{ijl} + \sum_{l \neq j} (1 - \Gamma_{ij}^{il})e_{dilj} \]

\[ -\sum_{l \neq j} \sum_{m \neq i} (\Gamma_{ij}^{im})e_{ilmj} + \sum_{l \neq j} (1 + \Gamma_{ij}^{il})e_{iljm}, \]

and

\[ Curv_2(e_{ij}) = -\sum_{l \neq i} (1 - \Gamma_{ij}^{il})e_{ijl} + \sum_{l \neq j} (1 - \Gamma_{ij}^{il})e_{dilj} \]

\[ +\sum_{l \neq j} \sum_{m \neq i} (\Gamma_{ij}^{im})e_{ilmj} - \sum_{l \neq j} (1 + \Gamma_{lj}^{im})e_{ilmj}. \]
Now let us consider an interesting special case where

\[ \Gamma_{ij}^{klm} = \Gamma_{\hat{g}(k)\hat{g}(l)\hat{g}(m)} \]

(41)

for any permutation \( \hat{g} \) on the set \( V \) of \( N \) points. This is motivated by the fact that a complete symmetric graph \( (A, \Omega^1) \) is invariant under the permutations \( \hat{g} \) on \( V \) in the sense that the transformed bases \( \{e_{\hat{g}(i)}\} \) for \( A \) and \( \{e_{\hat{g}(i)\hat{g}(j)}\} \) for \( \Omega^1 \) are equivalent respectively to the original ones since the graph \( (A, \Omega^1) \) is complete and symmetric. In this special case, let us put \( 1 + \Gamma_{ij}^{aj} = \gamma \) for all \( i, j \) \( (i \neq j) \) and \( a(\neq i, j) \). Then the curvatures in the above can be written immediately as follows.

**Corollary 3:** For all \( i, j \) \( (i \neq j) \),

\[
D_1 D e_{ij} = \gamma \{ - \sum_{l \neq i} e_{ijkl} + (2 - \gamma) \sum_{l \neq j} e_{iij} + (1 - \gamma) \sum_{l \neq j} \sum_{m \neq i} e_{ilmj} \\
- \sum_{l} \sum_{m \neq j} e_{ijlm} - \sum_{l} \sum_{m} e_{ilmj} \},
\]

\[
Curv_1(e_{ij}) = \gamma \{ - \sum_{l \neq i} e_{ijkl} + (2 - \gamma) \sum_{l \neq j} e_{iij} + (1 - \gamma) \sum_{l \neq j} \sum_{m \neq i} e_{ilmj} \},
\]

and

\[
D_2 D e_{ij} = \gamma \{ (\gamma - 2) \sum_{l \neq i} e_{ijkl} + (2 - \gamma) \sum_{l \neq j} e_{iij} - \sum_{l} \sum_{m \neq j} e_{ijlm} \\
+ \sum_{l \neq i} \sum_{m} e_{ilmj} \},
\]

\[
Curv_2(e_{ij}) = \gamma (\gamma - 2) \{ \sum_{l \neq i} e_{ijkl} - \sum_{l \neq j} e_{iij} \}.
\]

The parameter \( \gamma \) determines a connection and hence curvatures. These one-parameter families of connections and curvatures on a graph are closely analogous to those on matrix geometries\(^{20}\) and those on the ordinary quantum plane\(^{23}\). However, the torsion is not zero in general in this work. In fact, the torsion also depends on the parameter since \( \Theta e_{ij} = -\gamma \sum_{a} e_{iaj} \) from Lemma 6. A surprising result that \( Curv_1 = Curv_2 \) arises when \( \gamma = 1 \) for complete symmetric graphs.

We define a metric \( g : \Omega^1 \otimes_{A} \Omega^1 \rightarrow A \) on \( (A, \Omega^1) \) to be an \( A \)-bilinear nondegenerate map. By nondegenerate, we mean \( g(\omega \otimes \omega') = 0 \) for all \( \omega \in \Omega^1 \) implies \( \omega' = 0 \) and \( g(\omega \otimes \omega') = 0 \) for all \( \omega' \in \Omega^1 \) implies \( \omega = 0 \). Then we have \( g(e_{ij} \otimes e_{jk}) = \mu_i e_{i} \delta_{ik} \) for some constant \( \mu_i \). From the same motivation as that for Eq. (44), we assume that \( \mu_i \)'s are the same, say \( \mu \). Now if we define a metric-compatible connection to be a linear connection satisfying \( d \circ g = (1 \otimes g) \circ D_2 \) as usual\(^{20-23}\), then it is straightforward to see that \( dg(e_{ij}) = \mu de_i \) and \( (1 \otimes g)D_2(e_{ij}) = \mu (de_i e_i - e_i de_j) \). Hence there is no
metric-compatible connection in general except the \( N = 2 \) case. This fact tells us that a metric in this sense is not so useful for graphs.

From the connections and curvatures of a complete symmetric graph, we can induce those of subgraphs. We shall do this in the next subsection.

C. Linear connections on graphs

Let \((A, K^1)\) be a subgraph of \((A, \Omega^1)\). Now we define a linear connection \(\nabla : K^1 \rightarrow K^1 \otimes_A K^1\) on a graph \((A, K^1)\) by the composite map \(\nabla \equiv (\phi_1 \otimes \phi_1) \circ D \circ j_1\) where \(j_1 : K^1 \rightarrow \Omega^1\) is the splitting map and \(\phi_1 \otimes \phi_1 : \Omega^1 \otimes_A \Omega^1 \rightarrow K^1 \otimes_A K^1\) is the projection map. We also define a generalized permutation \(\tau : K^1 \otimes_A K^1 \rightarrow K^1 \otimes_A K^1\) by \(\tau \equiv (\phi_1 \otimes \phi_1) \circ \sigma \circ (j_1 \otimes j_1)\). Let \(\Delta \equiv \Omega^1 \otimes_A \ker \phi_1 + \ker \phi_1 \otimes_A \Omega^1\). Then we have

**Proposition 5:** For \(f \in A\) and \(\omega \in K^1\),

1. \(\nabla(f \omega) = \delta f \otimes \omega + f \nabla \omega\)

2. \(\nabla(\omega f) = (\nabla \omega)f + \tau(\omega \otimes \delta f)\) if \(\sigma\) preserves \(\Delta\), i.e. \(\sigma(\Delta) \subset \Delta\).

**Proof:**

1. \(\nabla(f \omega) = (\phi_1 \otimes \phi_1)(df \otimes j_1 \omega + fDj_1 \omega) = \phi_1 df \otimes \phi_1(j_1 \omega) + f(\phi_1 \otimes \phi_1)(Dj_1 \omega) = \delta f \otimes \omega + f \nabla \omega\).

2. \(\nabla(\omega f) = (\phi_1 \otimes \phi_1)((Dj_1 \omega)f + \sigma(j_1 \omega \otimes df)) = (\nabla \omega)f + (\phi_1 \otimes \phi_1)\sigma(j_1 \omega \otimes j_1 \delta f + j_1 \omega \otimes u)\) for some \(u \in \ker \phi_1\) since \(\phi_1(df - j_1 \delta f) = 0\).

By assumption, \((\phi_1 \otimes \phi_1)\sigma(j_1 \omega \otimes u) = 0\). QED

Let \(p : K^1 \otimes K^1 \rightarrow K^2\) be the multiplication map \(P_{1,1} = \phi_2 \circ (j_1 \otimes j_1)\) and \(p_{12} = p \otimes 1\). We extend \(\nabla\) to \(\nabla_1\) and \(\nabla_2\), respectively, by defining \(\nabla_1 \equiv (\phi_2 \otimes \phi_1) \circ D_1 \circ (j_1 \otimes j_1)\) and \(\nabla_2 \equiv (\phi_1 \otimes \phi_1 \otimes \phi_1) \circ D_2 \circ (j_1 \otimes j_1)\).

**Proposition 6:** Let \(f \in A\), \(\omega, \omega' \in K^1\), and \(\tau_{12} = \tau \otimes 1\). Then

1. \(\nabla_1(\omega \otimes \omega') = \delta \omega \otimes \omega' - p_{12}(\omega \otimes \nabla \omega')\)

2. \(\nabla_2(\omega \otimes \omega') = \nabla \omega \otimes \omega' + \tau_{12}(\omega \otimes \nabla \omega')\), if \(\sigma\) preserves \(\Delta\).

**Proof:**

1. First, we observe that \(\phi_2 = p(\phi_1 \otimes \phi_1)\) from Lemma 5 and \(\pi = 1\) for the universal differential algebra \(\Omega(A)\).

\[
\nabla_1(\omega \otimes \omega') &= (\phi_2 \otimes \phi_1)(d j_1 \omega \otimes j_1 \omega') - (\phi_2 \otimes \phi_1)(j_1 \omega \otimes D j_1 \omega') \\
&= \delta \omega \otimes \omega' - p_{12}(\phi_1 \otimes \phi_1 \otimes \phi_1)(j_1 \omega \otimes D j_1 \omega') \\
&= \delta \omega \otimes \omega' - p_{12}(\omega \otimes \nabla \omega').
\]

2. \(\nabla_2(\omega \otimes \omega') = (\phi_1 \otimes \phi_1 \otimes \phi_1)D_2(j_1 \otimes j_1)(\omega \otimes \omega')

= (\phi_1 \otimes \phi_1 \otimes \phi_1)(D_1 j_1 \omega \otimes j_1 \omega' + \sigma_{12}(j_1 \omega \otimes D j_1 \omega'))

= \nabla \omega \otimes \omega' + (\phi_1 \otimes \phi_1 \otimes \phi_1)\sigma_{12}(j_1 \omega \otimes D j_1 \omega')

= \nabla \omega \otimes \omega' + (\phi_1 \otimes \phi_1 \otimes \phi_1)\sigma_{12}(j_1 \omega \otimes (j_1 \otimes j_1) \nabla \omega' + j_1 \omega \otimes u)

16
for some $u \in \Omega^1 \otimes_A \Omega^1$ such that $(\phi_i \otimes \phi_i)(u) = 0$ since $(\phi_i \otimes \phi_i)(Dj_1 \omega' - (j_1 \otimes j_1) \nabla \omega') = 0$. Now let us show that $u \in \Delta$. Let $u = \sum i x_i \otimes y_i$ for $x_i, y_i \in \Omega^1$. Now we can write $x_i$ uniquely as $x_i = x_i^{(1)} + j_1 x_i^{(2)}$ for $x_i^{(1)} \in \ker \phi_1$ and $x_i^{(2)} \in K^1$. Similarly for $y_i$. Then $u = \sum i j_1 x_i^{(2)} \otimes j_1 y_i^{(2)} + \alpha$ for some $\alpha \in \Delta$. From $(\phi_i \otimes \phi_i)(u) = 0$, $\sum i x_i^{(2)} \otimes y_i^{(2)} = 0$. Hence $u = \alpha \in \Delta$. Now that $\sigma(\Delta) \subset \Delta$, $\sigma_{12}(\Omega^1 \otimes \Delta) \subset \Omega^1 \otimes_A \Omega^1 \otimes_A \ker \phi_1 + \Delta \otimes_A \Omega^1$. Thus $(\phi_i \otimes \phi_i \otimes \phi_i) \sigma_{12}(j_1 \omega \otimes u) = 0$. QED

If we define the torsion of $\nabla$ by $T_\nabla = \delta - p \circ \nabla$, it is easy to see that the necessary and sufficient condition for $T_\nabla$ to be $A$-bilinear is that $p(1 + \tau) = 0$ in the case where $\sigma$ preserves $\Delta$. From now on, let us keep the $A$-bilinearity of $T$ as in the previous subsection. Thus $\tau = -1$. Then $\tau = -1$ and $T_\nabla$ is also $A$-bilinear. In this case, we have $\nabla_1 = p_{12} \nabla_2 + T_\nabla \otimes 1$. If $T = 0$, $T_\nabla = \phi_2 \circ d \circ j_1 - p \circ (\phi_1 \otimes \phi_1) \circ D \circ j_1 = (\phi_2 - p \circ (\phi_1 \otimes \phi_1)) \circ d \circ j_1 = 0$. Now let us calculate explicitly connections $\nabla$ and curvatures $\nabla_1 \circ \nabla, \nabla_2 \circ \nabla$ on graphs for the general nonzero $A$-bilinear torsion case.

As $D_1 \circ D$ for complete symmetric graphs, $\nabla_1 \circ \nabla$ is left $A$-linear and not right $A$-linear. We note that $D_1 e_{ij} = de_i \otimes de_j + \sum a (1 + \Gamma_{ij}^{ia}) e_{iaj}$ for $e_{ij} \in K^1$ from Lemma 6. Then it follows that

$$\nabla e_{ij} = (\phi_i \otimes \phi_i) Dj_1 e_{ij} = (\phi_i \otimes \phi_i)(de_i \otimes de_j + \sum a (1 + \Gamma_{ij}^{ia}) e_{iaj})$$

$$= \{ \sum_k \phi_i(e_{ki} - e_{ik}) + \sum_{a \neq j} (1 + \Gamma_{ij}^{ia}) \phi_i(e_{ia}) \} \otimes \sum_l \phi_i(e_{lj} - e_{jl}).$$

For simplicity, we shall consider examples for the case where $1 + \Gamma_{ij}^{iaj} = \gamma$ for all $i, j$ ($i \neq j$) and all $a(\neq i, j)$. Thus

$$\nabla e_{ij} = \{ \sum_k \phi_i(e_{ki} - e_{ik}) + \sum_{a \neq j} \phi_i(e_{ia}) \} \otimes \sum_l \phi_i(e_{lj} - e_{jl}).$$

**Example 4:** Let $\mathcal{V}$ be a set $\{x_1, x_2\}$ of two points as in the Connes-Lott’s model. If $K^1 = \Omega^1$, $D_1 D = D_2 D = 0$ from Corollary 3. If $K^1$ is the space generated by $\{e_{12}\}$, $K^1 \otimes_A K^1 = 0$ since $e_{12} \otimes e_{12} = 0$. Thus $\nabla_1 \nabla = \nabla_2 \nabla = 0$. The graph of two points seems to be too simple to have a nonzero curvature.

**Example 5:** Let $\mathcal{V}$ be a set $\{x_1, x_2, x_3\}$ of three points and let $K^1$ be the space generated by $\{e_{12}, e_{21}, e_{23}, e_{32}\}$. Then $ker \phi_1 = \{e_{13}, e_{31}\}$, $ker \phi_2 = \{e_{123}, e_{131}, e_{132}, e_{213}, e_{231}, e_{312}, e_{313}, e_{321}\}$ and $K^2 = \{e_{121}, e_{212}, e_{232}, e_{322}\}$ with the obvious splitting map. We note that $\delta e_{12} = \phi_2 d_1 e_{12} = e_{121} + e_{212} = \delta e_{21}$, and $\delta e_{23} = e_{232} + e_{323} = \delta e_{32}$. Moreover,

$$\nabla e_{12} = (e_{21} - e_{12}) \otimes (e_{12} + e_{32} - e_{21} - e_{23}) = e_{121} + e_{123} + e_{212}$$

$$\nabla e_{21} = (e_{12} + e_{32} - e_{21} - e_{23} + \gamma e_{23}) \otimes (e_{21} - e_{12}) = e_{121} + e_{212} + e_{321}$$

17
We can also calculate $\nabla e_{23}, \nabla e_{32}$ in the same way. $\nabla e_{23} = e_{123} + e_{232} + e_{323}$ and $\nabla e_{32} = e_{232} + e_{321} + e_{332}$.

Now we have

$$\nabla_1 \nabla e_{12} = \nabla_1 (e_{12} \otimes e_{21} + e_{21} \otimes e_{12} + e_{12} \otimes e_{23})$$
$$= \delta e_{12} \otimes e_{21} - p_{12}(e_{12} \otimes \nabla e_{21}) + \delta e_{21} \otimes e_{12} - p_{12}(e_{21} \otimes \nabla e_{12})$$
$$+ \delta e_{12} \otimes e_{23} - p_{12}(e_{12} \otimes \nabla e_{23})$$
$$= (e_{2121} - e_{1212}) + (e_{1212} - e_{2121} - e_{2123}) + e_{2123} = 0,$$

and

$$\nabla_1 \nabla e_{21} = \nabla_1 (e_{12} \otimes e_{21} + e_{21} \otimes e_{12} + e_{23} \otimes e_{21})$$
$$= \delta e_{12} \otimes e_{21} - p_{12}(e_{12} \otimes \nabla e_{21}) + \delta e_{21} \otimes e_{12} - p_{12}(e_{21} \otimes \nabla e_{12})$$
$$+ \delta e_{23} \otimes e_{21} - p_{12}(e_{23} \otimes \nabla e_{21})$$
$$= (e_{2121} - e_{1212}) + (e_{1212} - e_{2121} - e_{2123}) + e_{2321} = e_{2321} - e_{2123}.$$

On the other hand, $\nabla_1 \nabla e_{23} = e_{2123} - e_{3231}$ and $\nabla_1 \nabla e_{32} = 0$. If we factor out the second terms from $\nabla_1 \nabla e_{21}$ and $\nabla_1 \nabla e_{23}$, we obtain an $A$-bilinear curvature $\text{Curv}_1(e_{21}) = e_{2321}$ and $\text{Curv}_1(e_{23}) = e_{2123}$ with $\text{Curv}_1(e_{12}) = \text{Curv}_1(e_{32}) = 0$.

Now we make use of the following equation

$$\nabla_2(e_{ijk}) = \nabla e_{ij} \otimes e_{jk} - e_{ij} \otimes \nabla e_{jk}$$

to obtain the curvature $\nabla_2 \circ \nabla$:

$$\nabla_2 \nabla e_{12} = -e_{1232} + e_{3212} = -\nabla_2 \nabla e_{32},$$
$$\nabla_2 \nabla e_{21} = -e_{2123} + e_{2321} = -\nabla_2 \nabla e_{23}.$$
\[ K^2 = \{ e_{121}, e_{212}, e_{231}, e_{232}, e_{312}, e_{321}, e_{323} \} >. \] We note that
\[
\begin{align*}
\delta e_{12} &= e_{121} + e_{212} + e_{312}, \\
\delta e_{21} &= e_{121} + e_{321} - e_{231} + e_{212}, \\
\delta e_{23} &= e_{323} + e_{231} + e_{232}, \\
\delta e_{31} &= e_{231} - e_{321} + e_{312}, \\
\delta e_{32} &= e_{232} - e_{321} + e_{312} + e_{323},
\end{align*}
\]

and
\[
\begin{align*}
\nabla e_{12} &= e_{212} + e_{312} + e_{121} + e_{123}, \\
\nabla e_{21} &= e_{121} + e_{321} + e_{212} - e_{231} + \gamma e_{231}, \\
\nabla e_{23} &= e_{123} + e_{323} + e_{232} + e_{231}, \\
\nabla e_{31} &= e_{231} - e_{321} + e_{312} + \gamma e_{321}, \\
\nabla e_{32} &= e_{232} + e_{321} + e_{323} - e_{312} + \gamma e_{312}.
\end{align*}
\]

Then it follows that
\[
\begin{align*}
\nabla_1 \nabla e_{12} &= 0, \\
\nabla_1 \nabla e_{21} &= -e_{2123} - \gamma e_{2312} + \gamma(2 - \gamma)e_{2321}, \\
\nabla_1 \nabla e_{23} &= e_{2123} - \gamma(e_{2312} + e_{2321}), \\
\nabla_1 \nabla e_{31} &= -e_{3123} - \gamma e_{3121} + \gamma(2 - \gamma)e_{3231} - \gamma e_{3212}, \\
\nabla_1 \nabla e_{32} &= -\gamma(e_{3231} + e_{3121} + e_{3123}).
\end{align*}
\]

Thus by factoring out the unwanted terms, we obtain an A-bilinear curvature.
\[
\begin{align*}
\text{Curv}_1(e_{12}) &= 0, \\
\text{Curv}_1(e_{21}) &= \gamma(2 - \gamma)e_{2321}, \\
\text{Curv}_1(e_{23}) &= e_{1213}, \\
\text{Curv}_1(e_{31}) &= -\gamma e_{3121} + \gamma(2 - \gamma)e_{3231}, \\
\text{Curv}_1(e_{32}) &= -\gamma e_{3212}.
\end{align*}
\]

Similarly, we have
\[
\begin{align*}
\nabla_2 \nabla e_{12} &= -e_{1232} + \gamma(e_{2312} + e_{3212} - e_{1231}), \\
\nabla_2 \nabla e_{21} &= -e_{2123} + \gamma e_{3121} + \gamma(2 - \gamma)e_{2321}, \\
\nabla_2 \nabla e_{23} &= e_{2123} + \gamma(e_{3123} - e_{2312} - e_{2321}), \\
\nabla_2 \nabla e_{31} &= e_{1231} - e_{3123} + \gamma(\gamma - 2)e_{3121} + \gamma(2 - \gamma)e_{3231}, \\
\nabla_2 \nabla e_{32} &= e_{1232} - \gamma e_{3231} - \gamma e_{3212}.
\end{align*}
\]
and

\begin{align*}
Curv_2(e_{12}) &= 0, \\
Curv_2(e_{21}) &= \gamma(2 - \gamma)e_{2321}, \\
Curv_2(e_{23}) &= e_{2123}, \\
Curv_2(e_{31}) &= \gamma(\gamma - 2)e_{3121} + \gamma(2 - \gamma)e_{3231}, \\
Curv_2(e_{32}) &= -\gamma e_{3212}.
\end{align*}

We have one-parameter family of connections even on a subgraph of a complete symmetric graph. It is worthy to notice that $Curv_1 = Curv_2$ for $\gamma = 1$ in the above examples as in complete symmetric graphs. We note that even though the torsion $T$ is free on a complete symmetric graph (and hence $T\nabla = 0$), curvatures do not vanish in general on its subgraphs as expected.

IV. CONCLUSIONS

Quantum spaces arise as models for the description of the small scale structure of spacetime. In this work, we have treated graphs as quantum spaces and constructed their differential algebras by extending the formulation of Dimakis et al\textsuperscript{6} in such a manner that the calculation of connections and curvatures can be done. We have shown explicitly the form of linear connections and curvatures of a given complete symmetric graph for the general nonzero torsion case. Also $A$-bilinear curvatures have been obtained for graphs by factoring out the unwanted terms from curvatures which are not $A$-bilinear. An interesting question for further study is whether or not $Curv_1 = Curv_2$ whenever $\gamma = 1$ for any graphs.

There is one-parameter family of connections. This fact parallels those for other models such as quantum planes and matrix geometries. There is a metric, but no metric-compatible connection in general except the complete symmetric graph with two vertices. Even though the torsion vanishes and hence the curvature is zero on a complete symmetric graph, a subgraph obtained from it by deleting some arrows can have a nonzero curvature in general as expected.

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