Constructing Classical and Quantum Superconformal Algebras on the Boundary of AdS$_3$

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Abstract

Motivated by recent progress on the correspondence between string theory on anti-de Sitter space and conformal field theory, we address the question of constructing space-time $N$ extended superconformal algebras on the boundary of AdS$_3$. Based on a free field realization of an affine $SL(2|N/2)$ current superalgebra residing on the world sheet, we construct explicitly the Virasoro generators and the $N$ supercurrents. $N$ is even. The resulting superconformal algebra has an affine $SL(N/2) \otimes U(1)$ current algebra as an internal subalgebra. Though we do not complete the general superalgebra, we outline the underlying construction and present supporting evidence for its validity. Particular attention is paid to its BRST invariance. In the classical limit where the free field realization may be substituted by a differential operator realization, we discuss further classes of generators needed in the closure of the algebra. We find sets of half-integer spin fields, and for $N \geq 6$ these include generators of negative weights. An interesting property of the construction is that for $N \neq 2$ it treats the supercurrents in an asymmetric way. Thus, we are witnessing a new class of superconformal algebras not obtainable by conventional Hamiltonian reduction. The complete classical algebra is provided in the case $N = 4$ and is of a new and asymmetric form.

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1 Introduction

Recently, Maldacena proposed a duality between type IIB string theory and (super-)conformal field theory (CFT) on the boundary of anti-de Sitter space (AdS) \cite{1}, further elaborated on in Refs. \cite{2,3}. This remarkable conjecture has induced a tremendous activity in theoretical high energy physics. In the cases involving AdS$_3$, see e.g. Refs. \cite{4,5,6,7,8,9,10} and the review \cite{11}, the corresponding space-time CFT is two-dimensional and thus possesses many well known properties.

The boundary of AdS$_3$ enjoys conformal symmetry, as recognized by Brown and Henneaux \cite{12}. In the work \cite{6}, Giveon et al have constructed explicitly the generators of the space-time conformal algebra from the string theory on AdS$_3$. The construction starts from the world sheet $SL(2)$ current algebra. In terms of the Wakimoto free field realization \cite{13} of that, they have provided a general expression for the Virasoro generators and computed the central charge.

In Ref. \cite{14} Ito has succeeded in constructing superconformal algebras (SCAs) on the boundary of AdS$_3$ and again the construction starts from a world sheet current algebra. However, in order to obtain an extended conformal symmetry in space-time one needs to consider higher world sheet current (super-)algebras than $SL(2)$. Thus, Ito has found that the appropriate Lie superalgebras leading to $N = 1, 2$ and 4 superconformal algebras are osp(1|2), sl(2|1) and sl(2|2), respectively. The explicit constructions are based on generalized Wakimoto free field realizations of the associated affine current superalgebras \cite{15,16,17}. A related approach to construct $N = 1, 2$ and 4 SCAs is discussed in \cite{18} in which also one- and two-point functions and some unitary representations are considered. In Ref. \cite{19} space-time $N = 3$ superconformal theories are studied.

The objective of the present paper is to take a first step in the direction of classifying the SCAs that may be induced by string theory on AdS$_3$, thus generalizing the work by Ito \cite{14}. A main property of an appropriate world sheet current (super-)algebra is that the bosonic part may be decomposed as $G = SL(2) \otimes G'$. This is necessary as we want the free fields in the Wakimoto realization of the embedded $SL(2)$ to be considered as coordinates on AdS$_3$. As discussed in Ref. \cite{3}, a purely bosonic world sheet algebra with such a decomposition leads immediately to an affine Lie algebra on the boundary of AdS$_3$. The present paper is devoted to discussing the class of superconformal algebras that may be constructed starting from affine $SL(2|N/2)$ current superalgebras having as bosonic part $SL(2) \otimes SL(N/2) \otimes U(1)$. By construction, $N$ is even.

We construct explicitly the Virasoro generators, the $N$ supercurrents, and the generators of an internal $SL(N/2) \otimes U(1)$ Kac-Moody algebra. For $N = 2$, the $SL(N/2)$ and $U(1)$ current algebras collapse to a single $U(1)$ current algebra. This conventional $N = 2$ superconformal algebra has already been obtained by Ito \cite{14}. For higher $N$ we turn to a classical limit in which the generators may be substituted by first order linear differential operators. The resulting classical SCAs are center-less and are shown to include classes of primary generators of half-integer weights which are all smaller than 2. Some weights are negative for $N \geq 6$. SCAs based on free field realizations and with generically non-vanishing central charges are denoted quantum SCAs as opposed to such classical SCAs. Thus, our use of the notion quantum is not in the quantum group sense of $q$-deformations.
A new and important property of the construction is that for \( N \neq 2 \) it treats the supercurrents asymmetrically. This is illustrated in the case \( N = 4 \) where the classical SCA is completed and found to be of a new and asymmetric form. Thus, it is not included in the standard classification of \( N = 4 \) SCAs \( [21, 21, 22, 23, 24] \). In particular, it deviates essentially from the small \( N = 4 \) SCA which has otherwise been announced to be the result \( [14] \) of a construction similar to the one employed in the present paper. This is argued not to be the correct result. The full quantum \( N = 4 \) SCA with generic central charge will be presented elsewhere \( [25] \). There we shall also show that the small \( N = 4 \) SCA may be obtained by replacing the original world sheet \( SL(2|2) \) current superalgebra by the related \( SL(2|2)/U(1) \) current superalgebra.

A complete classification along the lines indicated is reached when the SCAs induced by any world sheet current superalgebra with \( SL(2) \otimes G' \) decomposable bosonic part have been constructed. We anticipate that the techniques employed in the present paper may be enhanced to cover the general case and hope to come back elsewhere with a discussion on this generalization.

As pointed out in Ref. \( [6] \), BRST invariance of the construction of the space-time conformal algebra is equivalent to requiring the Virasoro generators to be primary fields of weight one with respect to the world sheet energy-momentum tensor, ensuring that the integrated fields commute with the world sheet Virasoro algebra. This carries over to the superconformal case, and we shall verify that the generators of our SCAs meet the requirement of being primary of weight one with respect to the world sheet current superalgebra Sugawara tensor.

The algebras constructed in Ref. \( [18] \) are simpler than the ones by Ito \( [14] \) as they are based on smaller Lie superalgebras. For example, the \( N = 4 \) SCA is constructed from an \( sl(2|1) \) Lie superalgebra. However, the central charges are essentially fixed, and the algebras are in general not ensured to be BRST invariant\(^1\).

The remaining part of this paper is organized as follows. In Section 2 we review the construction of the Virasoro algebra and the immediate extension to an affine Lie algebra \( [3] \).

In Section 3 we introduce our notation for Lie superalgebras and their associated current superalgebras, and review the free field realizations of the latter obtained in Ref. \( [17] \).

In Section 4 we provide our explicit construction of the supercurrents, the Virasoro generators, and the generators of the internal \( SL(N/2) \otimes U(1) \) Kac-Moody algebra.

In Section 5 BRST invariance of the construction is addressed.

In Section 6 we discuss the classical \( N \) extended SCAs and write down the explicit result for \( N = 4 \).

Section 7 contains concluding remarks, while details on the Lie superalgebra \( sl(2|M) \) are given in Appendix A.

\(^1\)We thank O. Andreev for pointing out that the \( N = 4 \) SCA in Ref. \( [18] \) is nevertheless BRST invariant.
2 Virasoro Algebra

The standard Wakimoto free field realization of the affine $SL(2)$ current algebra with level $k^\vee$ is

\[ \begin{align*}
E &= \beta \\
H &= -2\gamma\beta + \sqrt{k^\vee + 2}\partial\varphi \\
F &= -\gamma^2\beta + \sqrt{k^\vee + 2}\gamma\partial\varphi + k^\vee\partial\gamma
\end{align*} \]

(1)

Here and throughout the paper, normal ordering is implicit. The operator product expansions (OPEs) of the ghost fields $\beta, \gamma$ and the bosonic scalar field $\varphi$ are

\[ \beta(z)\gamma(w) = \frac{1}{z-w}, \quad \varphi(z)\varphi(w) = 2\ln(z-w) \]

(2)

where regular terms have been omitted. In Ref. [6] it is shown that the world sheet $SL(2)$ current algebra with level $k^\vee$ induces the Virasoro algebra

\[ [L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0} \]

(3)

on the boundary of $AdS_3$. The generators are given by

\[ \begin{align*}
L_n &= \oint \frac{dz}{2\pi i} \mathcal{L}_n(z) \\
\mathcal{L}_n &= a_+(n)\gamma^{n+1}E + a_3(n)\gamma^nH + a_-(n)\gamma^{n-1}F
\end{align*} \]

(4)

with constants

\[ \begin{align*}
a_+(n) &= \frac{1}{2}(1-n)n \\
a_3(n) &= \frac{1}{2}(1-n)(1+n) \\
a_-(n) &= \frac{1}{2}n(1+n)
\end{align*} \]

(5)

The central charge is found to be

\[ c = -6k^\vee p \]

(6)

where $p$ is the integer winding number

\[ p = \oint \frac{dz}{2\pi i} \frac{\partial\gamma}{\gamma} \]

(7)

BRST invariance requires $\mathcal{L}_n$ to be conformal primary of weight 1 with respect to the world sheet Virasoro generator

\[ T = \partial\gamma\beta + \frac{1}{2}\partial\varphi \cdot \partial\varphi - \frac{1}{2\sqrt{k^\vee + 2}}\partial^2\varphi \]

(8)

\[ ^2 \text{Note that the conventions used here differ slightly from the ones used in Ref. [6], which is also reflected in a sign discrepancy in the central charge.} \]
This property is readily verified.

The Virasoro algebra is immediately extended to an affine Lie algebra if the world sheet affine \( SL(2) \) current algebra is replaced by an affine \( G = SL(2) \otimes G' \) current algebra, where \( G' \) is a Lie group. Indeed, let \( J_a \) denote the currents of the affine \( G' \) current algebra with central extension \( k' \), and define the generators

\[
I_{a;n} = \oint dz \frac{d}{2\pi i} \gamma^n(z) J_a(z)
\]

From the defining OPE

\[
J_a(z)J_b(w) = \frac{\kappa_{a,b}k'}{(z-w)^2} + \frac{f_{a,b}^c J_c(w)}{z-w}
\]

where \( \kappa_{a,b} \) and \( f_{a,b}^c \) are the Cartan-Killing form and the structure constants, respectively, of the underlying Lie algebra \( g' \), one finds

\[
\begin{align*}
\{L_n, I_{b;m}\} &= -mI_{b;n+m} \\
\{I_{a;n}, I_{b;m}\} &= f_{a,b}^c I_{c;n+m} + npk'\kappa_{a,b}\delta_{n+m,0}
\end{align*}
\]

Summation over “properly” repeated indices is implicit. The central extension \( k'p \) of the space-time affine Lie algebra generated by \( \{I_{a;n}\} \) is seen to be the one, \( k' \), of the world sheet \( G' \) current algebra multiplied by the winding number \( p \) of the embedded \( sl(2) \) subalgebra. As the currents \( J_a \) are spin one primary fields, the construction is readily seen to be BRST invariant due to the decomposition \( SL(2) \otimes G' \).

3 Affine Current Superalgebra

3.1 Lie Superalgebra

Let \( g^0 \) and \( g^1 \) denote the even and odd parts, respectively, of the Lie superalgebra \( g \) of rank \( r \), see Ref. [26] and references therein. \( \Delta = \Delta^0 \cup \Delta^1 \) is the set of roots \( \alpha \) of \( g \) where \( \Delta^0 \ (\Delta^1) \) is the set of even (odd) roots. The set of positive roots \( \alpha > 0 \) is \( \Delta_+ = \Delta^0_+ \cup \Delta^1_+ \). A choice of simple roots is written \( \{\alpha_i\}_{i=1,...,r} \). A distinguished representation is characterized by exactly one simple root being odd. Related to the triangular decomposition

\[
g = g_- \oplus h \oplus g_+
\]

the raising and lowering operators are denoted \( E_\alpha, J_\alpha \in g_+ \) and \( F_\alpha, J_{-\alpha} \in g_- \), respectively, with \( \alpha \in \Delta_+ \), while \( H_i, J_i \in h \) are the Cartan generators. Generic Lie superalgebra elements are denoted \( J_a \) and satisfy

\[
\{J_a, J_b\} = f_{a,b}^c J_c
\]

where \( [\cdot, \cdot] \) is an anti-commutator if both arguments are fermionic, and otherwise a commutator. The Jacobi identities read

\[
\{J_a, \{J_b, J_c\}\} = \{\{J_a, J_b\}, J_c\} + (-1)^{p(J_a)p(J_b)} \{J_b, \{J_a, J_c\}\}
\]
where the parity $p(J_a)$ is 1 (0) for $J_a$ an odd (even) generator. The Cartan-Killing form $\kappa_{a,b}$

$$2h^\vee \kappa_{a,b} = \text{str}(\text{ad}_{J_a}\text{ad}_{J_b})$$ (15)

and the Cartan matrix $A_{ij} = \alpha_j(H_i)$ are related as $\kappa_{i,j} = A_{ij}\kappa_{\alpha_j,-\alpha_j}$. $h^\vee$ is the dual Coxeter number. The Weyl vector

$$\rho = \rho^0 - \rho^1$$

$$\rho^0 = \frac{1}{2} \sum_{\alpha \in \Delta^0} \alpha, \quad \rho^1 = \frac{1}{2} \sum_{\alpha \in \Delta^1} \alpha$$ (16)

satisfies $\rho \cdot \alpha_i = \alpha_i^2/2$.

For each positive even or odd root $\alpha > 0$ we introduce a super-triangular coordinate denoted by $x^\alpha$ or $\theta^\alpha$, respectively, where $\theta^\alpha$ is Grassmann odd. In terms of the matrix

$$C^b_a(x, \theta) = - \sum_{\alpha \in \Delta^0} x^\alpha f_{a,a}^b - \sum_{\alpha \in \Delta^1} \theta^\alpha f_{a,a}^b$$ (17)

one may then realize the Lie superalgebra in terms of differential operators [17]

$$J_a(x, \theta, \partial, \Lambda) = \sum_{\alpha > 0} V^\alpha_a(x, \theta) \partial_{\alpha} + \sum_{j=1}^{r} P^j_a(x, \theta) \Lambda_j$$ (18)

where $\Lambda$ is the weight of the representation, and $\Lambda_j$ are the labels defined by

$$H_j|\Lambda\rangle = \Lambda(H_j)|\Lambda\rangle = \Lambda_j|\Lambda\rangle$$ (19)

$\partial_{\alpha}$ is differentiation with respect to $x^\alpha$ or $\theta^\alpha$ depending on the parity of $\alpha$, whereas $V$ and $P$ are finite dimensional polynomials:

$$V^\alpha_a(x, \theta) = [B(C(x, \theta))]^a_{\alpha}$$

$$V^\alpha_i(x, \theta) = - [C(x, \theta)]^a_{i}$$

$$V^\alpha_{-\alpha}(x, \theta) = \sum_{\alpha'' > 0} \left[ e^{C(x, \theta)} \right]^{\alpha''}_{-\alpha} [B(-C(x, \theta))]^{\alpha'}_{\alpha''}$$

$$P^j_a(x, \theta) = 0$$

$$P^j_i(x, \theta) = \delta^j_i$$

$$P^j_{-\alpha}(x, \theta) = \left[ e^{-C(x, \theta)} \right]^j_{-\alpha}$$ (20)

$B(u)$ is the generating function for the Bernoulli numbers $B_n$

$$B(u) = \frac{u}{e^u - 1} = \sum_{n \geq 0} \frac{B_n}{n!} u^n$$

$$(B(u))^{-1} = \frac{e^u - 1}{u} = \sum_{n \geq 0} \frac{1}{(n + 1)!} u^n$$ (21)
The formal power series expansions (20) all truncate and become polynomials due to the nilpotency of the matrix \( C \) (17). For later use, let us also introduce the notation \( V^+ \) for the first of the polynomials in (20)

\[
V^+(x, \theta) = [B(C(x, \theta))]^+ = B(C^+(x, \theta))
\]

where \( C^+ \) is the submatrix of \( C \) with both row and column indices positive (even or odd) roots. \( V^+ \) is immediately seen to be invertible

\[
(V^+(x, \theta))^{-1} = (B(C^+(x, \theta)))^{-1} = \sum_{n \geq 0} \frac{1}{(n+1)!} (C^+(x, \theta))^n
\]

Most Lie superalgebras with even subalgebra \( g^0 = sl(2) \oplus g' \) have the property that the embedding of \( sl(2) \) in \( g \) carried by \( g^1 \) is a spin 1/2 representation. This means that the space of odd roots may be divided into two parts

\[
\Delta^1 = \Delta^{1-} \cup \Delta^{1+}
\]

where the roots \( \alpha^\pm \in \Delta^{1\pm} \) are characterized by

\[
\frac{\alpha_{sl(2)} \cdot \alpha^\pm}{\alpha_{sl(2)}^2} = \pm \frac{1}{2}
\]

and we have the correspondence

\[
\Delta^{1+} = \alpha_{sl(2)} + \Delta^{1-}
\]

\( \alpha_{sl(2)} \) is the positive root associated to the embedded \( sl(2) \). In particular, the division (24) is present in the case of our main interest, namely the Lie superalgebra \( sl(2|M) \) which is considered in Section 4 and further in Appendix A.

### 3.2 Free Field Realization

Associated to a Lie superalgebra is an affine Lie superalgebra characterized by the central extension \( k \), and associated to an affine Lie superalgebra is an affine current superalgebra whose generators are conformal spin one primary fields and have the mutual operator product expansions\(^4\)

\[
J_a(z)J_b(w) = \frac{\kappa_{a,b} k}{(z-w)^2} + \frac{\delta_{a,b} \delta_c J_c(w)}{z-w}
\]

We use the same notation \( J, E, F, H \) for the currents as for the algebra generators. Hopefully, this will not lead to misunderstandings. The associated Sugawara construction

\[
T = \frac{1}{2(k + h^\vee)} \kappa_a J_a J_b
\]

\(^3\)This is true for all basic Lie superalgebras with even subalgebra \( g^0 = sl(2) \oplus g' \) except \( osp(3|2M) \) where the embedding is a spin 1 representation, see e.g. [27].

\(^4\)We note that the extension of the Virasoro algebra to include an affine Lie algebra (11) discussed in Section 2 may readily be generalized to an affine Lie superalgebra simply by substituting the Lie group \( G' \) with a Lie supergroup; the only change being that the commutator \( [I_{a,n}, I_{b,m}] \) becomes an anti-commutator for \( J_a \) and \( J_b \) both fermionic.
generates the Virasoro algebra with central charge
\[ c = \frac{k \cdot \text{sdim}(g)}{k + h^\vee} \quad (29) \]

The standard free field construction \cite{15, 16, 17} consists in introducing for every positive even root \( \alpha \in \Delta^0_+ \), a pair of free bosonic ghost fields \( (\beta_\alpha, \gamma^\alpha) \) of conformal weights \( (1,0) \) satisfying the OPE
\[ \beta_\alpha(z) \gamma^{\alpha'}(w) = \frac{\delta_\alpha \delta^{\alpha'}}{z - w} \quad (30) \]
The corresponding energy-momentum tensor is
\[ T_{\beta\gamma} = \sum_{\alpha \in \Delta^0_+} \partial \gamma^\alpha \beta_\alpha \quad (31) \]
with central charge
\[ c_{\beta\gamma} = 2|\Delta^0_+| = \text{dim}(g^0) - r \quad (32) \]
For every positive odd root \( \alpha \in \Delta^1_+ \) one introduces a pair of free fermionic ghost fields \( (b_\alpha, c^\alpha) \) of conformal weights \( (1,0) \) satisfying the OPE
\[ b_\alpha(z) c^{\alpha'}(w) = \frac{\delta_\alpha \delta^{\alpha'}}{z - w} \quad (33) \]
The corresponding energy-momentum tensor is
\[ T_{bc} = \sum_{\alpha \in \Delta^1_+} \partial c^\alpha b_\alpha \quad (34) \]
with central charge
\[ c_{bc} = -2|\Delta^1_+| = -\text{dim}(g^1) \quad (35) \]
For every Cartan index \( i = 1, ..., r \) one introduces a free scalar boson \( \varphi_i \), with contraction
\[ \varphi_i(z) \varphi_j(w) = \kappa_{i,j} \ln(z - w) \quad (36) \]
The corresponding energy-momentum tensor
\[ T_\varphi = \frac{1}{2} \partial \varphi \cdot \partial \varphi - \frac{1}{\sqrt{k + h^\vee}} \rho \cdot \partial^2 \varphi \quad (37) \]
has central charge
\[ c_\varphi = r - \frac{h^\vee \cdot \text{sdim}(g)}{k + h^\vee} \quad (38) \]
where the super-dimension \( \text{sdim}(g) \) of the Lie superalgebra \( g \) is defined as the difference \( \text{dim}(g^0) - \text{dim}(g^1) \). In obtaining (38) we have used Freudenthal-de Vries (super-)strange formula
\[ \rho^2 = \frac{h^\vee}{12} \cdot \text{sdim}(g) \quad (39) \]
In particular, for $\alpha$ and $\gamma$ in the differential operator realization and are given by obtained by the substitution $T$

Anomalous terms are only added to the lowering generators $F$

The total free field realization of the Sugawara energy-momentum tensor is $T = T_{\beta \gamma} + T_{bc} + T_\varphi$ and has indeed central charge $\langle 23 \rangle$.

The generalized Wakimoto free field realization of the affine current superalgebra is obtained by the substitution

\[
\partial_x^\alpha \rightarrow \beta_\alpha(z) , \quad x^\alpha \rightarrow \gamma^\alpha(z) , \quad \Lambda_i \rightarrow \sqrt{k + h^2 \partial \varphi}(z)
\]

in the differential operator realization $\{J_a(x, \theta, \partial, \Lambda)\}$ $\langle 18 \rangle$, $\langle 20 \rangle$, and a subsequent addition of anomalous terms linear in $\partial \gamma$ or $\partial c$:

\[
J_a(z) = \sum_{\alpha \in \Delta_0^+} V^\alpha_a(\gamma(z), c(z)) \beta_\alpha(z) + \sum_{\alpha \in \Delta_1^+} V^\alpha_a(\gamma(z), c(z)) b_\alpha(z)
\]

\[
+ \sqrt{k + h^2} \sum_{j=1}^r P^j_a(\gamma(z), c(z)) \partial \varphi_j(z) + J^{\text{anom}}_a(\gamma(z), c(z), \partial \gamma(z), \partial c(z)) \quad (41)
\]

Anomalous terms are only added to the lowering generators $F_a(z)$

\[
J^{\text{anom}}_a(\gamma(z), c(z), \partial \gamma(z), \partial c(z)) = \begin{cases} 
0 & \text{for } a = i, \alpha > 0 \\
\sum_{\alpha' \in \Delta_0^+} \partial \gamma^{\alpha'}(z) F_{a,\alpha'}(\gamma(z), c(z)) + \sum_{\alpha' \in \Delta_1^+} \partial c^{\alpha'}(z) F_{a,\alpha'}(\gamma(z), c(z)) & \text{for } a = \alpha < 0
\end{cases} \quad (42)
\]

and are given by

\[
F_{a,\alpha'}(\gamma, c) = k \sum_{\mu \in \Delta_+^+} \left[ (V^+_\mu(\gamma, c))^{-1} \right]^{\mu}_{\alpha'}_{\alpha} \kappa_{\mu, -\alpha}
\]

\[
+ \sum_{\mu, \sigma \in \Delta_0^+, \nu \in \Delta_+^+} \left[ (V^+_\mu(\gamma, c))^{-1} \right]^{\mu}_{\alpha'}_{\alpha} \partial_\sigma V^\nu_\mu(\gamma, c) \partial_\nu \nu_{\alpha}(\gamma, c)
\]

\[
- \sum_{\mu \in \Delta_0^+, \sigma, \nu \in \Delta_+^+} \left[ (V^+_\mu(\gamma, c))^{-1} \right]^{\mu}_{\alpha'}_{\alpha} \partial_\sigma V^\nu_\mu(\gamma, c) \partial_\nu \nu_{\alpha}(\gamma, c)
\]

\[
+ \sum_{\mu, \sigma \in \Delta_1^+, \nu \in \Delta_+^+} \left[ (V^+_\mu(\gamma, c))^{-1} \right]^{\mu}_{\alpha'}_{\alpha} \partial_\sigma V^\nu_\mu(\gamma, c) \partial_\nu \nu_{\alpha}(\gamma, c)
\]

\[
F_{a,\alpha'}(\gamma, c) = k \sum_{\mu \in \Delta_+^+} \left[ (V^+_\mu(\gamma, c))^{-1} \right]^{\mu}_{\alpha'}_{\alpha} \kappa_{\mu, -\alpha}
\]

\[
+ \sum_{\mu, \sigma \in \Delta_0^+, \nu \in \Delta_+^+} \left[ (V^+_\mu(\gamma, c))^{-1} \right]^{\mu}_{\alpha'}_{\alpha} \partial_\sigma V^\nu_\mu(\gamma, c) \partial_\nu \nu_{\alpha}(\gamma, c)
\]

\[
+ \sum_{\mu, \sigma \in \Delta_1^+, \nu \in \Delta_+^+} \left[ (V^+_\mu(\gamma, c))^{-1} \right]^{\mu}_{\alpha'}_{\alpha} \partial_\sigma V^\nu_\mu(\gamma, c) \partial_\nu \nu_{\alpha}(\gamma, c)
\]

\[
- \sum_{\mu, \sigma \in \Delta_1^+, \nu \in \Delta_+^+} \left[ (V^+_\mu(\gamma, c))^{-1} \right]^{\mu}_{\alpha'}_{\alpha} \partial_\sigma V^\nu_\mu(\gamma, c) \partial_\nu \nu_{\alpha}(\gamma, c)
\]

In particular, for $\alpha$ a simple root the anomalous term $F_{a,\alpha'}$ is a constant independent of $\gamma$ and $c$

\[
F_{a,\alpha'}(\gamma, c) = \frac{1}{2} \delta_{\alpha, \alpha'} (2k + h^2) \kappa_{\alpha, -\alpha} - A_{ii} \quad (43)
\]
This concludes the explicit free field realization of general affine current superalgebras obtained in Ref. [17], where the polynomials \( V, P \) and \((V_+)^{-1}\) are given in (20) and (23).

4 Generators of the Superconformal Algebra

In this section we shall construct the Virasoro generators and the supercurrents of the SCA in space-time that may be induced by the affine \( SL(2|N/2) \) current superalgebra on the world sheet. Some steps towards constructing the complete set of SCA generators are also taken. For simplicity, we consider the underlying Lie superalgebra \( sl(2|N/2) \) in the distinguished representation, see Appendix A. Let us introduce the abbreviations

\[
\gamma = \gamma_{\alpha_1}, \quad E_1 = E_{\alpha_1} \text{ etc for the objects related to the embedded } sl(2).
\]

Hopefully, no misunderstandings will arise, as we are also using \( \gamma \) to represent a general bosonic ghost field argument in the polynomials \( V \) and \( P \), though in general we will leave out the arguments.

Using the explicit polynomials

\[
V_{\alpha_1}(\gamma, c) = 1 \quad V_1(\gamma, c) = -2\gamma \quad V_1^{-\alpha_1}(\gamma, c) = -\gamma^2
\]

(45)

it is straightforward to verify that the Virasoro algebra (3) is generated by

\[
L_n = \oint \frac{dz}{2\pi i} \mathcal{L}_n(z) \quad \mathcal{L}_n = a_+(n)\gamma^{n+1}E_1 + a_3(n)\gamma^nH_1 + a_-(n)\gamma^{n-1}F_1
\]

(46)

and has central charge

\[
c = -6k_1^\gamma p_1
\]

(47)

\( p_1 \) is the winding number [7] for the ghost field \( \gamma_{\alpha_1} = \gamma \), and \( k_1^\gamma = \kappa_{\alpha_1,-\alpha_1}k \) is the level of the embedded \( sl(2) \) or the level in the direction \( \alpha_1 \).

As a preparation for constructing the algebra generators, let us introduce the generators

\[
J_{a;n} = \oint \frac{dz}{2\pi i} \gamma^n(z)J_a(z)
\]

(48)

and consider the OPE

\[
\mathcal{L}_n(z)J_a(w) = \frac{1}{z-w} \left\{ \left( a_+(n)\gamma^{n+1}f_{\alpha_1,a} + a_3(n)\gamma^n f_{1,a} \right) J_c - na_3(n)\gamma^{n-2} V_{a_1}(\gamma^2E_1 + \gamma H_1 - F_1) \right\}
\]

\[- \frac{1}{(z-w)^2} \left( a_3(n)\gamma^{n-2} (\gamma^2(z)V_{a_1}^\nu(z) + \gamma(z)V_{1}^\nu(z) - V_{-\alpha_1}(z)) \partial_\nu V_{a_1}^\alpha(w) \right) \]

(49)

Here and in the following summations over repeated indices are implicit. Summations over root indices are meant to be over all positive roots if not otherwise indicated. Actually,
this restriction is not necessary as the super-triangular coordinates are defined for positive roots only, i.e. $\partial_\nu$ exists only for $\nu > 0$. Now, due to the structure of the root space (see Appendix A) we immediately obtain

$$[L_n, J_{\alpha;0}] = 0 \quad \text{for} \quad \alpha \in \Delta^0 \setminus \{\pm \alpha_1\}$$

Likewise, it follows that

$$[L_n, J_{\alpha^-,0}] = \frac{1}{2} (n - 1) \gamma^n (n + 1) J_{\alpha^-,n} - n \gamma J_{\alpha^+,n+1})$$

$$[L_n, J_{\alpha^+,0}] = \frac{1}{2} (n + 1) \gamma^{n-1} (n J_{\alpha^-,n-1} - (n - 1) \gamma J_{\alpha^+,n})$$

(51)

The idea is to use the right hand sides in the construction of the supercurrents. To this end let us consider the general setting where a primary field $\Phi$ of weight $h$

$$[L_n, \Phi_{m+\eta}] = ((h - 1)n - m - \eta) \Phi_{n+m+\eta}$$

may be obtained as a commutator of the form

$$[L_m, \oint B] = b(m; \eta; h) \Phi_{n+m+\eta}$$

(53)

$\eta$ is a possible non-integer shift in the modes, whereas $b$ is some function. From the Jacobi identities this function satisfies the recursion relation

$$(n - m)b(n + m) = ((h - 1)n - m - \eta)b(m) - ((h - 1)m - n - \eta)b(n)$$

(54)

allowing the simple solution

$$b(m) = b_0 + b_1 m, \quad (1 - h)b_0 = \eta b_1$$

(55)

This is precisely of the form we have encountered in (51). Since we want to construct supercurrents of weight $3/2$ we should choose $\eta \in \mathbb{Z} + \frac{1}{2}$, and we define the generators

$$G_{\alpha^-,n+1/2} = \oint \frac{dz}{2\pi i} G_{\alpha^-,n+1/2}(z)$$

$$G_{\alpha^-,n+1/2} = (n + 1) \gamma^n J_{\alpha^-} - n \gamma^{n+1} J_{\alpha^+}$$

(56)

Of course, it still remains to verify that the supercurrents $G$ are indeed primary:

$$[L_n, G_{\alpha^-,m+1/2}] = \left(\frac{1}{2} n - m - \frac{1}{2}\right) G_{\alpha^-,n+m+1/2}$$

(57)

However, that follows immediately from the simple computation of the OPE $L_n G_{\alpha^-,m+1/2}$, and we have thus constructed half of the supercurrents. Note that both commutators in (51) lead to the same supercurrent as we have

$$G_{\alpha^-,n+1/2} = \begin{cases} \frac{2}{n-1} [L_n, J_{\alpha^-;0}] \\ \frac{2}{n+2} [L_{n+1}, J_{\alpha^+;0}] \end{cases}$$

(58)
This means that $G$ may be represented as a commutator for all integer modes $n$.

Let us now turn to the construction of the supercurrents $\overline{G}$. For $a = -\alpha \pm \in \Delta_{1}^{\pm}$ it follows from (59) that a situation like (58) occurs provided

$$V_{\alpha}^{\alpha} = -\gamma V_{\alpha}$$

This important relation is proven in Appendix A where also some identities involving $V_{\alpha}^{\alpha}$ are derived. We are led to define the supercurrents $\overline{G}$ by

$$\overline{G}_{-\alpha;m-1/2} = \oint \frac{dz}{2\pi i} \overline{G}_{-\alpha;m-1/2}$$

$$\overline{G}_{-\alpha;n-1/2} = (n-1)\gamma J_{-\alpha} + n\gamma^{-1}J_{-\alpha}$$

$$n(n-1)\gamma^{-2}V_{-\alpha}^{\alpha} \left(2E_{1} + \gamma H_{1} - F_{1}\right)$$

$$n(n-1)\gamma^{-2} \left(\gamma^{2}V_{2}^{\nu} + \gamma V_{1}^{\nu} - V_{-\alpha}^{\nu}\right) \partial_{\nu} V_{-\alpha}^{\nu} \partial_{\nu} V_{\alpha}$$

$$= (n-1)\gamma J_{-\alpha} + n\gamma^{-1}J_{-\alpha}$$

$$- n(n-1)\gamma^{-2} \left(\Gamma_{-\alpha}^{\nu} b_{\nu} - (k - 1 + h'2/2) \partial_{\nu} - \partial_{\nu} \Gamma_{\alpha}^{\nu} \partial_{\alpha}\right) V_{-\alpha}^{\nu}$$

where

$$\Gamma_{-\alpha}^{\nu} = \gamma^{2}V_{2}^{\nu} + \gamma V_{1}^{\nu} - V_{-\alpha}^{\nu}$$

As indicated in (60), one may show that the upper root index $\nu$ is always an odd (and positive) root. The analogue to (58) reads

$$\overline{G}_{-\alpha;m-1/2} = \left\{ \begin{array}{ll}
\frac{-2}{n+1} [L_{m}, J_{-\alpha}^{0}]
\frac{2}{n-2} [L_{n-1}, J_{-\alpha}^{0}]
\end{array} \right.$$

As in the case of the supercurrents $G$, there is a free overall scaling. However, in order to produce the conventional prefactor of plus one multiplying the Virasoro generator in the anti-commutator $\{G, \overline{G}\}$ (see (73)), the relative factor is fixed.$^5$

Before proving that $G$ is a primary field of weight 3/2

$$[L_{n}, \overline{G}_{\alpha;m-1/2}] = \left(\frac{1}{2}n - m + \frac{1}{2}\right) \overline{G}_{\alpha;m+n-1/2}$$

let us observe the following property of the construction. Consider the Jacobi identity

$$[J_{\delta}, [L_{n}, J_{\pm\alpha}^{0}]] + [L_{n}, [J_{\pm\alpha}^{0}, J_{\delta}^{0}]] = [J_{\pm\alpha}^{0}, [L_{n}, J_{\delta}^{0}]] = 0$$

where $\delta \in \Delta^{0} \{\pm\alpha\}$ is any even (positive or negative) root different from $\pm\alpha$. From the construction of $G$ and $\overline{G}$ it follows that

$$G_{\beta;m+1/2} = - \left[ J_{\beta}^{\alpha-m+1/2}, \right]$$

$$\overline{G}_{-\beta;n-1/2} = \left[ J_{-\beta}^{\alpha-n-1/2}, \right]$$

$^5$ A more commonly used convention is a prefactor of plus two. However, we have found it natural to define $G$ and $\overline{G}$ without introducing any powers of $\sqrt{2}$. To comply with the standard convention is straightforward, though.

$^6$Subtleties for $n = \pm 1, \pm 2$ are immediately resolved by the alternative commutator representations (58) and (62).
Here we have used that
\[ f_{\alpha^-,\beta^-} = -f_{\alpha^-,\beta^-} = 1, \text{ for } \beta^- - \alpha^- \in \Delta^0 \setminus \{\pm \alpha\} \quad (66) \]

Besides providing information on the underlying algebraic structure of our construction, the translational property \((65)\) may be used to reduce considerations for general supercurrents to similar ones for the supercurrents \(G_{\alpha_2;n+1/2}\) and in particular \(G_{-\alpha_2;n-1/2}\). \(\alpha_2\) is the only fermionic simple root, see Appendix A. Thus, as a first application we shall prove that \(G\) is primary. From the Jacobi identities we find
\[ [L_n, G_{-\alpha_2;n-1/2}] = [J_{\alpha_2-\alpha^-,0}, [L_n, G_{-\alpha_2;n-1/2}]] \quad (67) \]
leaving us with the task of proving that \(G_{-\alpha_2}\) is primary. To that end we work out
\[ V_{-\alpha_2} = \Gamma_{-\alpha_2} = \frac{1}{2} \gamma c + C \quad (68) \]
where \(c\) and \(C\) are the fermionic ghost fields associated to the odd roots \(\alpha_2\) and \(\alpha_1 + \alpha_2\), respectively, and the supercurrent becomes
\[ G_{-\alpha_2;m-1/2} = (m-1)\gamma^m J_{-\alpha_2} + m\gamma^{m-1} J_{-\alpha_1-\alpha_2} - m(m-1)\gamma^{m-2} V_{-\alpha_2} \left( \gamma^2 E_1 + \gamma H_1 - F_1 - \frac{1}{2} \partial \gamma \right) \quad (69) \]

Now, one may compute the OPE \(L_n G_{-\alpha_2;m-1/2}\) and reduce the result using
\[ V_{\alpha_1}^{\alpha_1+\alpha_2} = -\frac{1}{2} c, \quad V_{\alpha_1}^{\alpha_1+\alpha_2} = -\frac{1}{2} c V_{\alpha_2}^{\alpha_2}, \quad V_{\alpha_1}^{\alpha_1+\alpha_2} = -\frac{1}{2} \gamma V_{\alpha_2}^{\alpha_2} \quad (70) \]
to the desired commutator
\[ [L_n, G_{-\alpha_2;m-1/2}] = \left( \frac{1}{2} n - m + \frac{1}{2} \right) G_{-\alpha_2;n+m-1/2} \quad (71) \]
This concludes the proof of \((63)\) that \(G_{-\alpha^-}\) is primary of weight \(3/2\).

4.1 Affine \(SL(N/2) \otimes U(1)\) Current Subalgebra

In order to derive the entire set of generators of the SCA, one should first consider the anti-commutators \(\{G_{\alpha^-}, G_{\beta^-}\}, \{G_{\alpha^-}, G_{-\beta^-}\}\) and \(\{G_{-\alpha^-}, G_{-\beta^-}\}\). It is readily seen that
\[ \{G_{\alpha^-;n+1/2}, G_{\beta^-;m+1/2}\} = 0 \quad (72) \]
whereas a rather cumbersome but essentially straightforward computation reveals that
\[ \{G_{\alpha^-;n+1/2}, G_{-\beta^-;m-1/2}\} = \delta_{\alpha^-,-\beta^-} L_{n+m} + (n-m+1) K_{\alpha^-,-\beta^-;n+m} + \frac{1}{6} c m(n+1) \delta_{n+m,0} \delta_{\alpha^-,-\beta^-} \quad (73) \]
where the current $K$ is defined by

$$K_{\alpha^-;\beta^-;n} = \oint \frac{dz}{2\pi i} K_{\alpha^-;\beta^-;n}(z)$$

$$K_{\alpha^-;\beta^-;n} = n\gamma^{n-1}V_{\beta^-}^{\alpha^-} (\gamma J_{\alpha^+} - J_{\alpha^-}) - \gamma^n f_{\alpha^-;\beta^-} c J_c + \frac{1}{2} \delta_{\alpha^-;\beta^-} \gamma^{n-1} \left(n \left( \gamma^2 E_1 + \gamma H_1 - F_1 \right) - \gamma H_1 \right) + n\gamma^{n-1} (\gamma V_{\alpha^+}^{\alpha^-} - V_{\alpha^-}^{\alpha^+}) \partial_{\alpha^-} \partial_{\beta^-} \partial \gamma^\sigma$$

(74)

There are several ways of representing $K_{\alpha^-;\beta^-;n}$ of which the following two turn out to be useful

$$K_{\alpha^-;\beta^-;n} = \begin{cases} \frac{1}{n+1} \left( \{G_{\alpha^-;n+1/2};\overline{G}_{\beta^-;-1/2}\} - \delta_{\alpha^-;\beta^-} L_n \right) \\ \frac{1}{n-1} \left( \{G_{\alpha^-;1/2};\overline{G}_{\beta^-;n-1/2}\} - \delta_{\alpha^-;\beta^-} L_n \right) \end{cases}$$

(75)

In particular, they may be used in a straightforward verification that the current $K$ is primary of weight 1:

$$[L_n, K_{\alpha^-;\beta^-;m}] = -m K_{\alpha^-;\beta^-;n+m}$$

(76)

In section 6 we shall provide evidence from considering the classical counterpart, that $\{K_{\alpha^-;\beta^-;n}\}$ generate an affine $SL(N/2) \otimes U(1)$ current subalgebra. The number of generators is accordingly

$$|\Delta^{1-}|^2 = (N/2)^2 = \text{dim}(sl(N/2)) + 1$$

(77)

A novel feature of our construction is its asymmetry in the two sets of supercurrents $\{G\}$ and $\{\overline{G}\}$, originating in (72) and

$$\{\overline{G}_{\alpha^-;n-1/2};\overline{G}_{\beta^-;m-1/2}\} \neq 0, \quad \text{for} \quad \alpha^- \neq \beta^-, \ n \neq m, \ n + m \neq 1$$

(78)

A proof at the classical level is presented in Section 6, however it is obvious that a result as (78) at the classical level remains true at the quantum level. The right hand side of (78) involves new fields to be introduced in Section 6.

### 4.2 Underlying Lie Superalgebra

Here we shall express the underlying Lie superalgebra in terms of selected modes of the SCA generators. From the Virasoro generator we have

$$E_1 = -L_{-1}, \quad H_1 = 2L_0, \quad F_1 = L_1$$

(79)

while the supercurrents allow us to write

$$J_{\alpha^-} = G_{\alpha^-;1/2}, \quad J_{\alpha^+} = G_{\alpha^-;-1/2}$$

$$J_{-\alpha^-} = -\overline{G}_{-\alpha^-;-1/2}, \quad J_{-\alpha^+} = \overline{G}_{-\alpha^-;1/2}$$

(80)
As we have

\[ K_{\alpha^-,\beta^-;0} = -\frac{1}{2}\delta_{\alpha^-,\beta^-}H_1 - f_{\alpha^-,\beta^-}J_c \]

\[ \{J_{\epsilon_2-\delta_u}, J_{-(\epsilon_2-\delta_u)}\} = \delta_{u,v} \left(2H_2 - \sum_{i=u'}^{u} H_{u'+1}\right) + J_{\delta_u-\delta_u} \quad (81) \]

where \( J_{\delta_u-\delta_u} \) is defined only for \( v \neq u \) (see Appendix A), we find that the remaining \((N/2)^2\) Lie superalgebra generators are given by

\[ J_{\delta_u-\delta_u} = -K_{\epsilon_2-\delta_u;-(\epsilon_2-\delta_u)}, \quad \text{for } u \neq v \]
\[ H_2 = -K_{\alpha_2;0} - L_0 \]
\[ H_i = K_{\epsilon_2-\delta_{i-1};-(\epsilon_2-\delta_{i-1})}, \quad \text{for } i = 3, \ldots, N/2 + 1 \quad (82) \]

4.3 \( N = 2 \) Superconformal Algebra

For \( N = 2 \) the only positive \( \alpha^- \)-root is \( \alpha_2 \) (i.e. \( \Delta_+^{1^-} = \{\alpha_2\} \)) and we have the 4 generators

\[ L_n = a_+(n)\gamma^{n+1}E_1 + a_3(n)\gamma^nH_1 + a_-(-n)\gamma^{n+1}F_1 \]
\[ G_{n+1/2} = (n + 1)\gamma^nJ_{\alpha_2} - n\gamma^{n+1}J_\theta \]
\[ \overline{G}_{n-1/2} = (n - 1)\gamma^nJ_{-\alpha_2} + n\gamma^{n+1}J_\theta \]
\[ \mathcal{K}_n = n\gamma^{n-1}V_{-\alpha_2}(\gamma J_\theta - J_{\alpha_2}) - \frac{1}{2}n\gamma^n(H_1 - F_1 - \partial\gamma) \]
\[ + \frac{1}{2}n\gamma^{n-1}\left(\gamma^2E_1 + \gamma H_1 - F_1 - \partial\gamma\right) \quad (83) \]

where \( \theta = \alpha_1 + \alpha_2 \). Note that the contribution \(-\frac{1}{2}(k+1/2)n\gamma^{n-1}\partial\gamma\) to \( \mathcal{K}_n \) vanishes upon integration as \( n \int \frac{dz}{2\pi i} \gamma^{n-1}\partial\gamma = np\delta_{n,0} = 0 \). The \( \tilde{N} = 2 \) SCA becomes

\[ [L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3-n)\delta_{n+m,0} \]
\[ [L_n, A_m] = ((h(A) - 1)n - m)A_{n+m}, \quad A \in \{G, \overline{G}, K\} \]
\[ \{G_{n+1/2}, G_{m+1/2}\} = \{\overline{G}_{n-1/2}, \overline{G}_{m-1/2}\} = 0 \]
\[ \{G_{n+1/2}, \overline{G}_{m-1/2}\} = L_{n+m} + (n - m + 1)K_{n+m} + \frac{1}{6}cn(n+1)\delta_{n+m,0} \]
\[ [K_n, G_{m+1/2}] = \frac{1}{2}G_{n+m+1/2}, \quad [K_n, \overline{G}_{m-1/2}] = -\frac{1}{2}\overline{G}_{n+m-1/2} \]
\[ [K_n, K_m] = \frac{1}{12}cn\delta_{n+m,0} \quad (84) \]

and closure is seen to be ensured by the 4 generators \((83)\). This result has already been obtained by Ito \([14]\), though his construction is based on a slightly different but equivalent free field realization of the associated affine \( SL(2|1) \) current superalgebra.


5 BRST Invariance

Before turning to the classical SCA let us discuss an important property of our construction. As pointed out in Ref. [6], BRST invariance of the construction of the space-time conformal algebra from a world sheet \( SL(2) \) current algebra requires the Virasoro generators to be primary fields of weight one with respect to the world sheet energy-momentum tensor. This ensures that the integrated fields commute with the world sheet Virasoro algebra. This requirement carries over to the superconformal case, where all (super-)currents (the Virasoro generators \( L \), the supercurrents \( G \) and \( G \), the affine Lie algebra generators \( K \) etc) are primary of weight one with respect to the Sugawara energy-momentum tensor of the world sheet \( SL(2|N/2) \) current superalgebra. A naive inspection immediately tells that the four types of currents considered so far have weight one, so all we need to verify is that they are primary. This amounts to verifying that third and higher order poles in the OPEs with the Sugawara tensor \( T \) all vanish. From the free field realization of \( T \) it follows that no higher order poles than third order appears. Using that the affine currents \( J \) are primary fields, the BRST invariance of the supercurrents \( G \) is readily confirmed as

\[
0 = n(n - 1)\gamma^{n-1}V_{\alpha+}^{\alpha} + n(n - 1)\gamma^{n-2}V_{\alpha+}^{\alpha} - n(n - 1)\gamma^{n-1}V_{\alpha+}^{\alpha} + n(n - 2)\gamma^{n-3}V_{\alpha+}^{\alpha} - n(n - 1)\gamma^{n-2}V_{\alpha+}^{\alpha} - n(n - 1)\gamma^{n-1}V_{\alpha+}^{\alpha} - n(n - 2)\gamma^{n-3}V_{\alpha+}^{\alpha} - n(n - 1)\gamma^{n-2}V_{\alpha+}^{\alpha} - n(n - 1)\gamma^{n-1}V_{\alpha+}^{\alpha} - n(n - 2)\gamma^{n-3}V_{\alpha+}^{\alpha}
\]

(85)

The three lines vanish separately due to (59), (45) and (130), respectively. Likewise, BRST invariance of the affine Lie algebra generators \( K \) amounts to verifying

\[
0 = -n\gamma^{n-1}f_{\alpha+}^{\alpha+} + n\gamma^{n-1}f_{\alpha+}^{\alpha+} + n\gamma^{n-2}V_{\alpha+}^{\alpha} + n\gamma^{n-1}V_{\alpha+}^{\alpha} - n\gamma^{n-2}V_{\alpha+}^{\alpha} - n\gamma^{n-1}V_{\alpha+}^{\alpha} - n\gamma^{n-2}V_{\alpha+}^{\alpha} - n\gamma^{n-1}V_{\alpha+}^{\alpha}
\]

(86)

Here we have used that \( V_{\alpha+}^{\alpha} = 0 \), and the identity (86) follows from (135) and (130). Likewise, BRST invariance of the affine Lie algebra generators \( K \) amounts to verifying

\[
0 = n\gamma^{n-1}f_{\alpha+}^{\alpha+} + n\gamma^{n-1}f_{\alpha+}^{\alpha+} + n\gamma^{n-2}V_{\alpha+}^{\alpha} + n\gamma^{n-1}V_{\alpha+}^{\alpha} - n\gamma^{n-2}V_{\alpha+}^{\alpha} - n\gamma^{n-1}V_{\alpha+}^{\alpha} - n\gamma^{n-2}V_{\alpha+}^{\alpha} - n\gamma^{n-1}V_{\alpha+}^{\alpha} - n\gamma^{n-2}V_{\alpha+}^{\alpha}
\]

(86)

The three lines vanish separately due to (59), (45) and (130), respectively. Likewise, BRST invariance of the affine Lie algebra generators \( K \) amounts to verifying

\[
0 = n\gamma^{n-1}f_{\alpha+}^{\alpha+} + n\gamma^{n-1}f_{\alpha+}^{\alpha+} + n\gamma^{n-2}V_{\alpha+}^{\alpha} + n\gamma^{n-1}V_{\alpha+}^{\alpha} - n\gamma^{n-2}V_{\alpha+}^{\alpha} - n\gamma^{n-1}V_{\alpha+}^{\alpha} - n\gamma^{n-2}V_{\alpha+}^{\alpha} - n\gamma^{n-1}V_{\alpha+}^{\alpha} - n\gamma^{n-2}V_{\alpha+}^{\alpha}
\]

(86)

Here we have used that \( V_{\alpha+}^{\alpha} = 0 \), and the identity (86) follows from (135) and (130). One may take a more general point of view observing that the (anti-)commutator of two BRST invariant fields commutes with the world sheet Virasoro generators. This follows from the Jacobi identities. Thus, having established that all but one field appearing on the right hand side of a (anti-)commutator (of two BRST invariant fields) are BRST invariant, is sufficient to conclude that the final field is likewise BRST invariant. A trivial example is the alternative deduction that \( K \) is BRST invariant following from

\[
\{ J_{\alpha+}^{\alpha+}, V_{\alpha+}^{\alpha+} \} = 0
\]

6 Classical Superconformal Algebra

Here we shall distinguish between classical and quantum SCAs. Our use of the notion *quantum* is not in the quantum group sense of \( q \)-deformations but rather as opposed to *classical* as described in the following. Let us recall the situation for free field realizations of affine current superalgebras discussed in Section 3. In that case one may start with a
first order linear differential operator realization of the underlying Lie superalgebra. The free field realization of the associated current superalgebra is then obtained by substituting with (normal ordered products of) free fields and subsequently adding “quantum corrections”, “anomalous terms” or “normal ordering terms”. We shall denote the differential operator realization a classical limit or version of the associated “quantum” free field realization. A classical algebra thus defined has vanishing central extensions.

A similar situation may be expected in the present case. Thus, there should exist a classical counterpart of the full SCA which allows a differential operator realization (and accordingly has vanishing central extensions). Based on this assumption our program is to first work out the classical SCA for then to perform the appropriate substitutions and additions of anomalous terms in order to obtain the full (quantum) SCA. It should be stressed that to each mode of the generators of the quantum SCA, there is an associated differential operator. This results in an infinite dimensional algebra of differential operators contrary to the situation described above where the classical algebra is a standard (finite dimensional) Lie superalgebra.

Having identified the differential operator
\[ A(x, \theta, \partial, \Lambda) = \sum_a Y^a(x, \theta) J_a(x, \theta, \partial, \Lambda) \] (87)
as a generator of the classical SCA, the corresponding quantum generator
\[ A = \oint \frac{dz}{2\pi i} A(z) \] (88)
is obtained by performing the substitutions in \( A(x, \theta, \partial, \Lambda) \) and adding appropriate anomalous terms linear in derivatives of the spin 0 ghost fields in order to produce \( A(z) \):

\[
A(z) = \sum_a Y^a(\gamma(z), c(z)) J_a(z) + \sum_{\alpha \in \Delta_0} X_\alpha(\gamma(z), c(z)) \partial \gamma^\alpha(z) + \sum_{\alpha \in \Delta_1} X'_\alpha(\gamma(z), c(z)) \partial c^\alpha(z)
\] (89)

So with the ansatz, \( A(z) \) is linear in the affine currents \( J_a(z) \) with spin 0 ghost field dependent coefficients. Note that in the expression some anomalous terms are “hidden” in the definition of \( J_a(z) \), cf. (87).

The question of BRST invariance of \( A \) may be addressed even without explicit knowledge on the anomalous term. This follows from the fact that any term of the form \( \sum_\alpha Z_\alpha(\gamma, c) \partial \gamma^\alpha + \sum_\alpha Z'_\alpha(\gamma, c) \partial c^\alpha \) is primary of weight one. Thus, in order to establish that \( A \) is BRST invariant it suffices to consider the term linear in the affine (super-)currents. In the following we shall accordingly define a classical differential operator to be BRST invariant when its “naively quantized” form linear in the affine (super-)currents is BRST invariant.

\(^7\)It should be stressed that a non-vanishing central charge of a classical Virasoro algebra may well exist when classical is defined to denote single contractions only, as \( \partial^2 \varphi(z) \partial^2 \varphi(w) = -12/(z-w)^4 \). Here we have used the convention \( \varphi(z) \varphi(w) = 2 \ln(z-w) \). However, terms like \( \partial^2 \varphi \) are excluded in our “differential operator realization picture” employed in the present paper. Note that BRST invariance is used as an implicit guideline as \( \partial^2 \varphi \) has weight 2 with respect to the world sheet energy-momentum tensor.
Before continuing our program let us briefly justify it. It has turned out to be an immense technical task to complete the derivation of the SCA for general $N$. Even at the classical level, the computations are rather involved. A study of the center-less classical SCA seems therefore a natural first project to concentrate on, and one from which one may get structural insight into the full quantum SCA. In the following we shall present some essential steps in the direction of deriving the classical SCA. In the presumably most interesting case of $N = 4$, the classical SCA is completed. The full quantum $N = 4$ with generic central charge will be presented elsewhere [23].

### 6.1 Algebra Generators

In the remaining part of this Section all fields $A$ are represented by their classical differential operator analogues $A(x, \theta, \partial, \Lambda)$. To be explicit, let us summarize our findings for the classical generators:

\[
\begin{align*}
L_n &= a_+(n)x^{n+1}E_1 + a_-(n)x^nH_1 + a_+(n)x^{-1}F_1 \\
G_{\alpha^{-}n+1/2} &= (n + 1)x^nJ_{\alpha^{-}} - nx^{n+1}J_{\alpha^{+}} \\
\overline{G}_{-\alpha^{-}n-1/2} &= (n - 1)x^nJ_{-\alpha^{-}} + nx^{n-1}J_{-\alpha^{+}} \\
K_{\alpha^{-};\beta^{-};n} &= nx^{n-1}V_{\alpha^{-}}\left(xJ_{\alpha^{+}} - J_{\alpha^{-}} - x^nF_{\alpha^{-};\beta^{-}}\right) \\
&\quad + \frac{1}{2}\delta_{\alpha^{-};\beta^{-}}x^{n-1}\left(nx^2E_1 + (n - 1)xH_1 - nF_1\right) \\
h(L, G, \overline{G}, K) &= (2, 3/2, 3/2, 1)
\end{align*}
\]

$J_{\alpha}$ ($E$, $H$ or $F$) denotes the differential operator $J_{\alpha}(x, \theta, \partial, \Lambda)$ given in (18), (20) while $V_{\alpha^{-}}$ is the polynomial in the super-triangular coordinates $x$ and $\theta$ given in (20). Here and in the following $x$ may denote either $x^{\alpha^{+}}$ or a general triangular coordinate argument, though it should be clear from the context which it is. $h(A_m)$ indicates that $A_m$ is primary of weight $h$:

\[
[L_n, A_m] = ((h(A) - 1)n - m)A_{n+m}
\]

The generators respect among others the anti-commutators

\[
\begin{align*}
\left\{G_{\alpha^{-};n+1/2}, G_{\beta^{-};m+1/2}\right\} &= 0 \\
\left\{G_{\alpha^{-};n+1/2}, \overline{G}_{-\beta^{-};m-1/2}\right\} &= \delta_{\alpha^{-};\beta^{-}}L_{n+m} + (n - m + 1)K_{\alpha^{-};\beta^{-};n+m}
\end{align*}
\]

We shall now discuss the subalgebra generated by $\{K\}$. One finds straightforwardly

\[
[K_{\alpha^{-};\beta^{-};n}, K_{\mu^{-};\nu^{-};m}] = \delta_{\mu^{-};\beta^{-}}K_{\alpha^{-};\nu^{-};n+m} - \delta_{\alpha^{-};\nu^{-}}K_{\mu^{-};\beta^{-};n+m}
\]

In order to show explicitly that this has the affine structure

\[
SL(N/2) \otimes U(1)
\]

we introduce the following notation. From Appendix A we know that any root $\alpha^{-} \in \Delta^{1-}_{+}$ may be represented as $\epsilon_2 - \delta_u$ for some $u = 1, \ldots, N/2$, so abbreviate $K$ by

\[
K_{\epsilon_2;\delta_u} = K_{\epsilon_2;\delta_u \left(\epsilon_2;\delta_u\right) n}
\]
Note also that $\delta_v - \delta_u > 0$ for $u > v$. Define now

$$
\tilde{E}_{in} = K_{i+1;i:n} \\
\tilde{H}_{in} = K_{i+1;i+1:n} - K_{i;i:n} \\
\tilde{F}_{in} = K_{i;i+1:n}
$$

(96)

where $i = 1, \ldots, N/2 - 1$ by construction. One may then show that these correspond to the Chevalley generators of an (center-less) affine $SL(N/2)$ Lie algebra. In general, the currents $K_{u;v;n}$ correspond to raising operators for $u > v$, and to lowering operators for $u < v$. Furthermore, the generator

$$
U_n = \sum_{u=1}^{N/2} K_{u;u;n}
$$

(97)

is seen to commute with all ladder operators $K_{u;v;\neq u;m}$, with the Cartan generators $\tilde{H}_{i;m}$ and with $U_m$ itself. Thus, $U$ generates a (center-less) $U(1)$ current algebra and we have the decomposition (34).

Let us return to the anti-commutator $\{ \mathcal{G}_{-\alpha^+}, \mathcal{G}_{-\beta^+} \}$ and prove the classical counterpart of the assertion (18). The anti-commutator may be computed directly or obtained as a special case of a much more general consideration: Introduce the operator

$$
\mathcal{G}_{-\beta_1^-, \ldots, -\beta_k^-, m-1+k/2} = V\alpha_1^- \ldots V\alpha_1^- J_{-\beta_1^-}^- + mx_{m-1}J_{-\beta_k^-}^- \\
+ \ldots \\
+ \{(m - 2 + k)x_{m}J_{-\beta_1^-}^- + mx_{m-1}J_{-\beta_1^-}^- \} V\alpha_1^- \ldots V\alpha_1^- \\
- V\alpha_1^- \ldots V\alpha_1^- \{ (m + k - 1)(m + k - 2)x_{m}E_1 \\
+ (m + k - 2)m x_{m-1}H_1 - m(m - 1)x_{m-2}F_1 \}
$$

(98)

which is seen to reduce to $\mathcal{G}_{-\beta^-}$ for $k = 1$. $\mathcal{G}_{-\beta_1^-, \ldots, -\beta_k^-; m}$ is bosonic (fermionic) for $k$ even (odd). In the latter notation the mode $m$ is meant to be integer or half-integer depending on the parity of the generator, i.e. depending on $k$ being even or odd, respectively. Note that $\mathcal{G}_{-\beta_1^-, \ldots, -\beta_k^-}$ is anti-symmetric in its root indices. $J_{-\beta_j^\pm}$ is defined not to act on $V\alpha_1^- \ldots V\alpha_1^-$, and is only written to the left of the $V$-monomial for convenience of notation. Thus, within $\mathcal{G}_{-\beta_1^-, \ldots, -\beta_k^-}$ one has $J_{-\beta_j^\pm} V\alpha_1^- \ldots V\alpha_1^- = (-1)^{k-j} V\alpha_1^- \ldots V\alpha_1^- J_{-\beta_j^\pm}$.

This resembles normal ordering needed in the free field realization and may be regarded as a normal ordering of the differential operator. It is to be employed throughout this section. One may now work out the (anti-)commutator

$$
\left[ \mathcal{G}_{-\beta_1^-, \ldots, -\beta_k^-; m}, \mathcal{G}_{-\lambda_1^-, \ldots, -\lambda_l^-; m} \right] = ((k - 2)m - (l - 2)n) \mathcal{G}_{-\beta_1^-, \ldots, -\beta_k^-; -\lambda_1^- \ldots, -\lambda_l^-; m+n+m}
$$

(99)

We observe that $\{ \mathcal{G}_{-\beta_1^-, \ldots, -\beta_k^-} \}_{k=1}^{\Delta_1^-}$ generate a subalgebra of dimension $2^{\Delta_1^-} - 1 = 2^{N/2} - 1$ and that $2^{N/2-1}$ of the generators are fermionic. Note that the commutator for
\( k = l = 2 \) vanishes identically. One may also show that (classically) the generators are primary:

\[
\left[ L_n, \overline{G}_{-\beta_1^-, \ldots, -\beta_k^-} \right] = \left( 1 - k/2 \right) n - m \overline{G}_{-\beta_1^-, \ldots, -\beta_k^-} ; n + m , \quad h = 2 - k/2
\]

Thus, including \( L \) as a generator of the subalgebra it has dimension \( 2^{\Delta_k} \) and equal numbers of bosonic and fermionic generators. BRST invariance is readily verified, either directly or as a consequence of the recursive relation (99) and the general approach of Section 5.

We note that the generator (98) may be written in the following compact form

\[
\overline{G}_{-\beta_1^-, \ldots, -\beta_k^-} = x^{m-2} \left\{ \sum_{j=1}^{k} \left( (m+k-2)x^2 J_{-\beta_j^-} + mx J_{-\beta_j^+} \right) \frac{\partial}{\partial V_{-\beta_j^-}^{\alpha_1}} \right. \\
- (m+k-1)(m+k-2)x^2 E_1 - (m+k-2)mxH_1 \\
+ m(m-1)F_1 \left\} \left( V_{-\beta_1^-}^{\alpha_1} \ldots V_{-\beta_k^-}^{\alpha_1} \right)
\]

where we have defined

\[
\frac{\partial}{\partial V_{-\beta_j^-}^{\alpha_1}} V_{-\beta_j^-}^{\alpha_1} \equiv \delta_{ij}
\]

As \( \overline{G}_{-\beta_1^-, \ldots, -\beta_k^-} \) is a first order differential operator, \( \partial/\partial V_{-\beta_j^-}^{\alpha_1} \) is meant to act only on the explicitly written products of \( V \)'s within \( \overline{G}_{-\beta_1^-, \ldots, -\beta_k^-} \) itself. A special situation occurs for \( k = 2 \) since we then have

\[
\overline{G}_{-\beta_1^-, -\beta_2^-} = mS_{-\beta_1^-, -\beta_2^-}
\]

\[
S_{-\beta_1^-, -\beta_2^-} = x^{m-2} \left\{ \sum_{j=1}^{2} \left( x^2 J_{-\beta_j^-} + mx J_{-\beta_j^+} \right) \frac{\partial}{\partial V_{-\beta_j^-}^{\alpha_1}} \right. \\
- (m+1)x^2 E_1 - mxH_1 + (m-1)F_1 \left\} \left( V_{-\beta_1^-}^{\alpha_1} V_{-\beta_2^-}^{\alpha_1} \right)
\]

and \( \overline{G}_{-\beta_1^-, -\beta_2^-} \) is readily seen to have weight 0. \( \overline{G}_{-\beta_1^-, -\beta_2^-} \) may be interpreted as the derivative of the scalar \( S_{-\beta_1^-, -\beta_2^-} \).

The list of generators presented hitherto is by no means exhaustive. Let us consider generators which may be obtained by the adjoint action of \( \{ G_{\alpha^-} \} \) on \( \overline{G}_{-\beta_1^-, \ldots, -\beta_k^-} \). Firstly, we find the (anti-)commutator

\[
\left[ G_{\alpha^-} ; n+1/2, \overline{G}_{-\beta_1^-, \ldots, -\beta_k^-} \right] = \left( -m - (k-2)n \right) \Phi_{\alpha^-; -\beta_1^-, \ldots, -\beta_k^-} ; n + m + 1/2
\]

\[
+ \frac{k-2}{k-3} \sum_{j=1}^{k} (-1)^{j-1} \delta_{\alpha^- - \beta_j^-} \overline{G}_{-\beta_1^-, \ldots, -\beta_j^- \ldots, -\beta_k^-} ; n + m + 1/2
\]

where \( \Phi_{\alpha^-; -\beta_1^-, \ldots, -\beta_k^-} \) corresponds to the special case \( l = 1 \) in the following general expression

\[
\Phi_{\mu_1^-, \ldots, \mu_k^-; -\beta_1^-, \ldots, -\beta_k^-} ; n
\]
These generators are only defined for certain integer pairs \((l, k)\) to be discussed below. A hat over an object indicates that the object is left out. We observe that \(\Phi_{\mu_1^-, \ldots, \mu_l^-; -\beta_1^-, \ldots, -\beta_k^-}\) is bosonic (fermionic) for \(l + k\) even (odd), and that it is anti-symmetric in the positive root indices and in the negative root indices, separately. Using the polynomial relations listed at the end of Appendix A, one may show that \(\Phi\) satisfies the recursion relation

\[
\left[ G_{\nu^-; n+1/2}, \Phi_{\mu_1^-; \ldots; \mu_l^-; -\beta_1^-; \ldots; -\beta_k^-; m} \right] = \frac{k - 2}{k - l - 2} \sum_{j=1}^{k} (-1)^{j-1} \delta_{\nu^-; \beta_j^-} \Phi_{\mu_1^-; \ldots; \mu_l^-; -\beta_1^-; \ldots; -\beta_j^-; \ldots; -\beta_k^-; n+m+1/2} - \frac{l}{k - l - 2} \Phi_{\nu^-; \mu_1^-; \ldots; \mu_l^-; -\beta_1^-; \ldots; -\beta_k^-; n+m+1/2}
\]

(106)

In addition, \(\Phi_{\mu_1^-; \ldots; \mu_l^-; -\beta_1^-; \ldots; -\beta_k^-}\) may be shown to be primary of weight

\[
h(\Phi_{\mu_1^-; \ldots; \mu_l^-; -\beta_1^-; \ldots; -\beta_k^-}) = 1 - (k - l)/2
\]

(107)

As for \(G_{-\beta_1^-; \ldots; -\beta_k^-}\) \((18)\), BRST invariance of \(\Phi_{\mu_1^-; \ldots; \mu_l^-; -\beta_1^-; \ldots; -\beta_k^-}\) is verified straightforwardly, either directly or indirectly. There is a slight subtlety in \((106)\) for \(k = l = 1\), which is easily resolved, though, as \(\Phi_{\mu^-} \) \((103)\) may be interpreted as \(G_{\mu^-}\). One then has

\[
\left[ K_{\mu^-; -\nu^-; n+1/2}, G_{\nu^-; m+1/2} \right] = \delta_{\nu^-; \beta^-} G_{\mu^-; n+m+1/2} - \frac{1}{2} \delta_{\mu^-; \beta^-} G_{\nu^-; n+m+1/2}
\]

(108)

In the definition of \(\Phi_{\mu_1^-; \ldots; \mu_l^-; -\beta_1^-; \ldots; -\beta_k^-}\) we obviously have \(k - l - 2 \neq 0\), and as the expression is obtained by the adjoint action of \(G\) on \(G_{-\beta_1^-; \ldots; -\beta_k^-}\), \(1 \leq l < k - 2\) is seen to be a valid domain. It is relevant for the discussion below on \(N = 4\) that besides \((l, k) = (1, 1)\) which corresponds to the \(K\) generators, also \((l, k) \in \{(2, 1), (1, 2), (2, 2), (3, 2)\}\) may be reached by this adjoint action. It turns out that three of these four generators may be expressed in terms of simpler generators. First we note that up to permutations in the root indices, \(\Phi_{\mu^-; \beta_1^-; \beta_2^-; -\beta_1^-; -\beta_2^-}\) is the only non-vanishing generator of type \((3, 2)\) (provided \(\mu^- \neq \beta_1^-\) and \(\beta_1^- \neq \beta_2^-\), of course), while \(\Phi_{\mu^-; -\beta_1^-; -\beta^-}\) is the only non-vanishing generator of type \((2, 1)\) (provided \(\mu^- \neq \beta^-\)). It is easily shown that they satisfy

\[
\Phi_{\mu_1^-; \mu_2^-; \mu_3^-; -\beta_1^-; -\beta_2^-; m+1/2} = \left( \delta_{\mu_1^-; \beta_1^-} - \delta_{\mu_1^-; \beta_2^-} \right) G_{\mu_1^-; m+1/2}
\]
for the discussion on the evidence that our construction does produce a finitely generated and BRST invariant are all the BRST invariant generators at hand, above we have presented substantial 

(2) and that it may be reduced as

\begin{align}
\Phi_{\mu_1, \mu_2: -\beta_1, -\beta_2} \cdot m + 1/2 &= \delta_{\mu_2, \beta_1} G_{\mu_1} - \delta_{\mu_1, \beta_2} G_{\mu_2} \\
\Phi_{\mu_1, \mu_2: -\beta_1, -\beta_2} \cdot m &= -\delta_{\mu_1, -\beta_1} K_{\mu_2: -\beta_2} \cdot m + \delta_{\mu_1, \beta_2} K_{\mu_2: -\beta_1} \cdot m
\end{align}

For \( l > k + 1, k = 1, 2 \), the generator \( \Phi_{\mu_1, ..., \mu_l: -\beta_1, ..., -\beta_k} \) is readily seen to vanish. Relevant for the discussion on the \( N = 4 \) SCA in the following, is the result

\[
\left[ K_{-\mu_1: -\nu, \overline{G}_{-\alpha}: m - 1/2} \right] = \frac{1}{2} \delta_{\mu_1, -\nu} \overline{G}_{-\alpha: n + m - 1/2} - \delta_{-\mu_1, -\nu} \overline{G}_{-\alpha: n + m - 1/2} + n \Phi_{\mu_1, -\nu, -\alpha: n + m - 1/2}
\]

which one may work out explicitly.

Secondly, we observe that \( \Phi_{-\mu_1: -\alpha, -\beta} \) is the only non-vanishing generator of type (2, 2) and that it may be reduced as

\[
\Phi_{\mu_1, \mu_2: -\beta_1, -\beta_2} \cdot m = \delta_{\mu_1, -\beta_1} K_{\mu_2: -\beta_2} \cdot m - \delta_{\mu_2, -\beta_2} K_{\mu_1: -\beta_1} \cdot m
\]

Despite the fact that closure of the BRST invariant algebra is still not ensured, nor are all the BRST invariant generators at hand, above we have presented substantial evidence that our construction does produce a finitely generated and BRST invariant \( N \) extended SCA. Below we shall demonstrate that this is indeed the case for \( N = 4 \). We intend to come back elsewhere with further discussion on the BRST invariant SCA and its quantization as described above.

As \(|\Delta_{-1}^-| = N/2\), we observe that for \( N = 6, 8, \ldots \) the SCA contains primary generators of negative weight causing problems for the unitarity of the associated CFT, and in particular for its applications to string theory. From an algebraic point of view, however, we see no severe obstacles arising from the appearance of negative weights, and believe that further studies of these algebras are warranted. Their explicit realizations and linearity in the affine currents seem to make such investigations feasible.
## 6.2 Classical $N = 4$ Superconformal Algebra

We recall that $sl(2|2)$ has precisely two positive (fermionic) $\alpha^-$-roots, see Appendix A:

$$\Delta^+_1 = \{\alpha_2, \alpha_{2+3} \equiv \alpha_2 + \alpha_3\}, \quad \alpha_2 = \epsilon_2 - \delta_1, \quad \alpha_3 = \delta_1 - \delta_2$$

(113)

Using the results on (classical) SCA obtained above for general $N$ we may almost immediately complete the (classical) $N = 4$ SCA. We find that closure is ensured by the following 12 BRST invariant generators which may be characterized as:

| Virasoro generator | $L$ | $h = 2$ |
|--------------------|-----|---------|
| supercurrents      | $G_{\alpha_2}$, $G_{\alpha_{2+3}}$, $\overline{G}_{-\alpha_2}$, $\overline{G}_{-\alpha_{2+3}}$ | $h = 3/2$ |
| affine $SL(2)$     | $E = K_{\alpha_{2+3}; -\alpha_2}$, $\tilde{H} = K_{\alpha_{2+3}; -\alpha_{2+3}} - K_{\alpha_2; -\alpha_2}$, $\tilde{F} = K_{\alpha_2; -\alpha_{2+3}}$ | $h = 1$ |
| affine $U(1)$      | $U = K_{\alpha_2; -\alpha_2} + K_{\alpha_{2+3}; -\alpha_{2+3}}$ | $h = 1$ |
| fermions           | $\phi_{-\alpha_2} = \Phi_{\alpha_{2+3}; -\alpha_2}$, $\phi_{-\alpha_{2+3}} = \Phi_{\alpha_2; -\alpha_{2+3}}$ | $h = 1/2$ |
| scalar             | $S$ | $h = 0$ |

$$\{nS, G_{\alpha_{2+3}; n} \} = (nS_n = \overline{G}_{-\alpha_2; -\alpha_{2+3}; n} h = 1)$$

(114)

The non-trivial (anti-)commutators are:

$$\{L, A_m\} = ((h(A) - 1)n - m)A_{n+m}$$

$$\{G_{\alpha^{-}; n+1/2}, G_{\beta^{-}; m+1/2}\} = 0$$

$$\{G_{\alpha^{-}; n+1/2}, \overline{G}_{-\beta^{-}; m-1/2}\} = \delta_{\alpha^{-}, \beta^{-}}L_{n+m} + (n - m + 1)K_{\alpha^{-}; -\beta^{-}; n+m}$$

$$\{\overline{G}_{\alpha_{2}; n-1/2}, \overline{G}_{\alpha_{2+3}; m-1/2}\} = (n - m)(n + m - 1)S_{n+m-1}$$

$$\left[\tilde{H}_n, \tilde{E}_m\right] = 2\tilde{F}_n + m, \quad \left[\tilde{H}_n, \tilde{F}_m\right] = -2\tilde{E}_n + m$$

$$\left[\tilde{E}_n, G_{\alpha_{2+3}; m+1/2}\right] = G_{\alpha_{2+3}; n+1/2}, \quad \left[\tilde{F}_n, G_{\alpha_{2+3}; m+1/2}\right] = G_{\alpha_{2+3}; n+1/2}$$

$$\left[\tilde{H}_n, G_{\alpha_{2}; m+1/2}\right] = -G_{\alpha_{2}; n+1/2}, \quad \left[\tilde{H}_n, G_{\alpha_{2+3}; m+1/2}\right] = G_{\alpha_{2+3}; n+1/2}$$

$$\left[\tilde{E}_n, \overline{G}_{\alpha_{2+3}; m-1/2}\right] = -\overline{G}_{\alpha_{2+3}; n-m+1/2} - m\phi_{-\alpha_{2+3}; m-1/2}$$

$$\left[\tilde{H}_n, \overline{G}_{\alpha_{2}; m-1/2}\right] = -\overline{G}_{\alpha_{2}; n-m+1/2} + m\phi_{-\alpha_{2}; n-m+1/2}$$

$$\left[\tilde{H}_n, \overline{G}_{\alpha_{2+3}; m-1/2}\right] = -\overline{G}_{\alpha_{2+3}; n-m+1/2} - m\phi_{-\alpha_{2+3}; n-m+1/2}$$

$$\left[\tilde{F}_n, \overline{G}_{-\alpha_{2}; m-1/2}\right] = -\overline{G}_{-\alpha_{2}; n+1/2} - m\phi_{-\alpha_{2}; n+1/2}$$

$$\left[U_n, \overline{G}_{-\alpha_{2}; m-1/2}\right] = m\phi_{-\alpha_{2}; n+1/2}$$

$$\left[U_n, \overline{G}_{\alpha_{2+3}; m-1/2}\right] = n\phi_{-\alpha_{2}; n+1/2}$$

$$\left[S_n, G_{\alpha_{2}; m+1/2}\right] = \phi_{-\alpha_{2+3}; n-m+1/2}, \quad \left[S_n, G_{\alpha_{2+3}; m+1/2}\right] = -\phi_{-\alpha_{2}; n+1/2}$$

$$\{G_{\alpha_{2}; n+1/2}, \phi_{-\alpha_{2}; m-1/2}\} = U_{n+m}, \quad \{G_{\alpha_{2+3}; n+1/2}, \phi_{-\alpha_{2+3}; m-1/2}\} = U_{n+m}$$
\[ \{ \mathcal{G}_{-\alpha; n-1/2}, \phi_{-\alpha; n+1/2} \} = (n + m - 1)S_{n+m-1} \]
\[ \{ \mathcal{G}_{-\alpha; m-1/2}, \phi_{-\alpha; m+1/2} \} = -(n + m - 1)S_{n+m-1} \]
\[ [\bar{E}_n, \phi_{-\alpha; n+1/2}] = -\phi_{-\alpha; n+m-1/2}, \]
\[ [\bar{F}_n, \phi_{-\alpha; m+1/2}] = -\phi_{-\alpha; n+m-1/2}, \]
\[ [\bar{H}_n, \phi_{-\alpha; m+1/2}] = \phi_{-\alpha; n+m-1/2}. \]
\[ (115) \]

\( A_m \) denotes any of the 12 BRST invariant generators listed in (114). We observe that only the derivative of the scalar \( S \) appears on the right hand sides of (115). Thus, the zero mode of \( S \) decouples from the algebra. One may verify explicitly that the Jacobi identities are satisfied.

Finally, we note that this center-less \( N = 4 \) SCA is of a new and asymmetric form. In particular, it deviates essentially from the small \( N = 4 \) SCA announced in Ref. [14] to be obtained by a similar construction. Even though the free field realization of the associated affine \( SL(2|2) \) current superalgebra used in Ref. [14] is slightly different from ours, one may show that the result in Ref. [14] for the \( N = 4 \) SCA is incorrect. One way of reaching this conclusion is to consider the analogue to our (78) and specialize to the case \( n = 0 \). In the notation of Ref. [14], this corresponds to considering the anti-commutator \( \{ G_{-1/2}, G_{m-1/2} \} \) in which case \( G_{-1/2} \) reduces to \( \oint dz j_{a_1+a_2} \) (still in the notation of Ref. [14]). We find that the anti-commutator for generic \( m \) is non-vanishing in agreement with our result but contrary to the definition of the small \( N = 4 \) SCA. Nevertheless, the small \( N = 4 \) SCA can be obtained by a construction equivalent to the one employed above. One simply has to replace the original world sheet \( SL(2|2) \) current superalgebra by an \( SL(2|2)/U(1) \) current superalgebra, whereby the resulting space-time SCA reduces to the standard small \( N = 4 \) SCA. This will be discussed further in Ref. [23].

7 Conclusion

In the present paper a new class of two-dimensional \( N \) extended SCAs has been discussed. The algebras are induced by free field realizations of affine \( SL(2|N/2) \) current superalgebras, where \( N \) is even. In the framework of string theory on \( AdS_3 \) the affine \( SL(2|N/2) \) current superalgebra resides on the world sheet providing a space-time SCA on the boundary of \( AdS_3 \). The construction generalizes recent work by Ito [14]. The Virasoro generators, the \( N \) supercurrents, and the generators of an internal \( SL(N/2) \otimes U(1) \) Kac-Moody algebra have all been constructed explicitly. Reducing the considerations to a classical center-less limit has provided additional insight into the structure of the full SCA. BRST invariance has also been addressed. The classical \( N = 4 \) SCA is complete and of a new type. In particular, it differs from the small \( N = 4 \) SCA. It also illustrates the new and important property of the general construction that it treats the supercurrents asymmetrically.

The results presented here offer (“stringy”) representations of superconformal algebras which are linear in the currents. This suggests that they may be useful when discussing
representation theoretical questions, and in the computation of correlation functions. Many other applications may be envisaged.

Several classes of Lie supergroups enjoy decompositions of the bosonic part \( G = SL(2) \otimes G' \) as in the case of \( SL(2|N/2) \). Based on their associated current superalgebras, we anticipate that other classes of SCAs may be constructed along the lines employed in the present paper. This is currently being investigated.

In the classification of CFT with extended symmetries, the construction of SCAs in the present paper presents an alternative to conventional Hamiltonian reduction and otherwise constructed non-linearly extended SCAs \([28, 29, 30, 31, 27, 32, 33]\). Whole new classes of extended Virasoro algebras seem to be the result of it. There are strong indications that we are even able to produce new and purely bosonic (and linear) extensions of the Virasoro algebra. These will be the subject of a forthcoming publication.

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A Lie Superalgebra \( sl(2|M) \)

The root space of the Lie superalgebra \( sl(2|M) \) in the distinguished representation may be realized in terms of an orthonormal two-dimensional basis \( \{\epsilon_1, \epsilon_2\} \) and an orthonormal \( M \)-dimensional basis \( \{\delta_u\}_{u=1,\ldots,M} \) with metrics

\[
\epsilon_i \cdot \epsilon_i' = \delta_{i,i'} , \quad \delta_u \cdot \delta_u' = -\delta_{u,u'} , \quad \epsilon_i \cdot \delta_u = 0
\]

(116)

The \( \frac{1}{2}(M+1)(M+2) \) positive roots are then represented as

\[
\Delta^0_+ = \{\epsilon_1 - \epsilon_2\} \cup \{\delta_u - \delta_v \mid u < v\}
\]

\[
\Delta^1_+ = \{\epsilon_1 - \delta_u \mid u = 1, \ldots, M\}
\]

\[
\Delta^1_- = \{\epsilon_2 - \delta_u \mid u = 1, \ldots, M\}
\]

(117)

where the \( M + 1 \) simple roots \( \alpha_i \) are

\[
\alpha_1 = \epsilon_1 - \epsilon_2
\]

\[
\alpha_2 = \epsilon_2 - \delta_1
\]

\[
\alpha_{u+2} = \delta_u - \delta_{u+1}
\]

(118)

The associated ladder operators \( E_{\alpha}, F_{\alpha} \), and the Cartan generators \( H_i \) admit a standard oscillator realization (see e.g. \([34]\))

\[
E_{\epsilon_1-\epsilon_2} = a_1^\dagger a_2 , \quad E_{\epsilon_1 - \delta_u} = a_1^\dagger b_u , \quad E_{\delta_u - \delta_v} = b_u^\dagger b_v
\]

\[
F_{\epsilon_1-\epsilon_2} = a_2^\dagger a_1 , \quad F_{\epsilon_1 - \delta_u} = b_u^\dagger a_u , \quad F_{\delta_u - \delta_v} = b_u^\dagger b_u
\]

\[
H_1 = a_1^\dagger a_1 - a_2^\dagger a_2 , \quad H_2 = a_2^\dagger a_2 + b_1^\dagger b_1 , \quad H_{u+2} = b_u^\dagger b_u - b_{u+1}^\dagger b_{u+1}
\]

(119)
where $a_i^{(t)}$ and $b_u^{(t)}$ are fermionic and bosonic oscillators, respectively, satisfying

$$\{a_i, a_i\} = \delta_{i,i'}, \quad [b_u, b_i\dagger] = \delta_{u,v}, \quad [b_u^{(t)}, a_i^{(t)}] = 0 \quad (120)$$

It is not possible to design a root string from $\delta_u - \delta_v < 0$ to $\alpha_1$ implying that

$$V_{\delta_u - \delta_v}^{\alpha_1} = 0 \quad (121)$$

This fact is used in deriving (50). However, root strings from $-\alpha^\pm$ to $\alpha_1$ do exist. They are of the form

$$-\alpha^- + ... + (\epsilon_2 - \delta_u) + \alpha_1 = \alpha_1$$
$$-\alpha^- + ... + (\epsilon_1 - \delta_u) = \alpha_1 \quad (122)$$

where root strings from $-\alpha^+$ are obtained by "inserting" an additional $\alpha_1$:

$$-\alpha^+ + ... + \alpha_1 + ... + (\epsilon_2 - \delta_u) + \alpha_1 = \alpha_1$$
$$-\alpha^+ + ... + (\epsilon_2 - \delta_u) + \alpha_1 + \alpha_1 = \alpha_1$$
$$-\alpha^+ + ... + \alpha_1 + ... + (\epsilon_1 - \delta_u) = \alpha_1$$
$$-\alpha^+ + ... + (\epsilon_1 - \delta_u) + \alpha_1 = \alpha_1 \quad (123)$$

Possible additions of positive even roots $\delta_{i'} - \delta_v$ are indicated by "...". Let us consider the polynomials (20)

$$V_{\alpha_1, -\alpha_1}^{\alpha_1} = \left[ \sum_{n \geq 0} \frac{1}{n!} (-C)^n \right]^{\alpha_1}_{-\alpha_1} \quad (124)$$

and compare the two polynomials in order to derive the relation (59). This we do by considering $\frac{1}{n!} (-C)^n$ in $V_{\alpha_1, -\alpha_1}$ and $\frac{1}{(n+1)!} (-C)^{n+1}$ in $V_{\alpha_1, -\alpha_1}^{\alpha_1}$. There are two cases, as such terms involve either $c^{\epsilon_1 - \delta_u}$ or $c^{\epsilon_2 - \delta_u}$ for some $u$, and we will discuss them separately. In the first case the relevant root strings are the lower one in (122) and the two lower ones in (123). Their differences are characterized by the structure constants

$$f_{-(\epsilon_2 - \delta_u), (\epsilon_1 - \delta_u)}^{\alpha_1} = 1 \quad (125)$$

and

$$f_{-(\epsilon_1 - \delta_u), (\epsilon_1 - \delta_u)}^{-(\epsilon_2 - \delta_u)} f_{-(\epsilon_2 - \delta_u), (\epsilon_1 - \delta_u)}^{\alpha_1} = 1$$
$$f_{-(\epsilon_1 - \delta_u), (\epsilon_1 - \delta_u)}^{-(\epsilon_2 - \delta_u)} f_{j, \alpha_1}^{\alpha_1} = 1 \quad (126)$$

Now, each time the situation (123) occurs in $\frac{1}{n!} (-C)^n$ in $V_{\alpha_1, -\alpha_1}$, the situations (126) occur $n$ times and once, respectively, in $\frac{1}{(n+1)!} (-C)^{n+1}$ in $V_{\alpha_1, -\alpha_1}^{\alpha_1}$. This is in accordance with (59).

A similar analysis of the terms involving $c^{\epsilon_2 - \delta_u}$ leads to the characterizations

$$f_{-(\epsilon_2 - \delta_u), (\epsilon_2 - \delta_u)}^{\alpha_1} = 1$$
$$f_{-(\epsilon_1 - \delta_u), (\epsilon_2 - \delta_u)} f_{j, \alpha_1}^{\alpha_1} = -1 \quad (127)$$

and

$$f_{-(\epsilon_1 - \delta_u), (\epsilon_2 - \delta_u)} f_{-(\epsilon_2 - \delta_u), (\epsilon_2 - \delta_u)} f_{j, \alpha_1}^{\alpha_1} = -1$$
$$f_{-(\epsilon_1 - \delta_u), (\epsilon_2 - \delta_u)} f_{j, \alpha_1}^{\alpha_1} f_{j, \alpha_1}^{\alpha_1} = -2 \quad (128)$$

Each time the situation (127) occurs in $\frac{1}{n!} (-C)^n$ in $V_{\alpha_1, -\alpha_1}$, the situations (128) occur $n - 1$ times and once, respectively, in $\frac{1}{(n+1)!} (-C)^{n+1}$ in $V_{\alpha_1, -\alpha_1}^{\alpha_1}$. However, as the last situation contributes with a factor $-2$, the terms involving $c^{\epsilon_2 - \delta_u}$ also agree with the relation (59), which is thereby proven.
A.1 Polynomial Relations

Using that the polynomials $V$ enter in a differential operator realization of a Lie superalgebra leads to the polynomial relations

$$f_{a,b}^c V_\alpha^c = V_\alpha^b \partial_b V_\alpha^a - (-1)^{p(a)p(b)} V_\alpha^b \partial_b V_\alpha^a$$

(129)
as discussed in Ref. [17]. The parity $p(a)$ of the index $a$ is defined as 1 for $a$ an odd root and 0 otherwise. Particularly useful are the following relations:

$$V_\alpha^\nu \partial_\nu V_{-\alpha}^{\alpha_1} = 0$$

$$V_\alpha^\mu \partial_\mu V_{-\beta}^{\alpha_1} = -V_{-\alpha}^{\alpha_1}$$

$$V_{-\alpha}^\nu \partial_\nu V_{-\alpha}^{\alpha_1} = -\gamma V_{-\alpha}^{\alpha_1}$$

(130)

$$V_{-\alpha}^\mu \partial_\mu V_{-\beta}^{\alpha_1} = \delta_{\alpha,-\beta}$$

$$V_{-\alpha}^\nu \partial_\nu V_{-\beta}^{\alpha_1} = \gamma \delta_{\alpha,-\beta}$$

$$V_{-\alpha}^\mu \partial_\mu V_{-\beta}^{\alpha_1} = -V_{-\alpha}^{\alpha_1} V_{-\beta}^{\alpha_1}$$

$$V_{-\alpha}^\mu \partial_\mu V_{-\beta}^{\alpha_1} = 0$$

(131)

$$f_{\alpha,-\beta}^{\alpha_1} = \gamma \delta_{\alpha,-\beta}$$

$$f_{\alpha,-\beta}^{\alpha_1} = \delta_{\alpha,-\nu} V_{-\beta}^{\alpha_1}$$

(132)

$$f_{\alpha,+,-\beta}^{\alpha_1} J_c = \delta_{\alpha,-\beta} \epsilon E_1$$

$$f_{\alpha,+,-\beta}^{\alpha_1} J_c = \delta_{\alpha,-\beta} \epsilon H_1 + f_{\alpha,-\beta}^{\alpha_1} J_c$$

$$f_{\alpha,-\beta}^{\alpha_1} J_c = \delta_{\alpha,-\beta} \epsilon F_1$$

(133)

$$f_{\alpha,-\beta}^{\alpha_1} f_{c,\lambda}^d J_d = \delta_{\alpha,-\beta} \epsilon J_\lambda - \delta_{\beta,-\lambda} \epsilon J_\alpha$$

$$f_{\alpha,-\beta}^{\alpha_1} f_{c,\lambda}^d J_d = -\delta_{\beta,-\lambda} \epsilon J_\alpha$$

$$f_{\alpha,-\beta}^{\alpha_1} f_{c,\lambda}^d J_d = \delta_{\alpha,-\beta} \epsilon J_\lambda - \epsilon J_\lambda$$

$$f_{\alpha,-\beta}^{\alpha_1} f_{c,\lambda}^d J_d = \delta_{\alpha,-\beta} \epsilon J_\lambda - \epsilon J_\lambda$$

$$f_{\alpha,-\beta}^{\alpha_1} f_{c,\lambda}^d J_d = \delta_{\alpha,-\beta} \epsilon E_1$$

$$f_{\alpha,-\beta}^{\alpha_1} f_{c,\lambda}^d J_d = 0$$

$$f_{\alpha,-\beta}^{\alpha_1} f_{c,-\alpha_1}^d J_d = \delta_{\alpha,-\beta} \epsilon F_1$$

(134)

$$f_{\alpha,-\beta}^{\alpha_1} f_{\mu,-\nu}^d J_c = \delta_{\alpha,-\nu} \epsilon f_{\mu,-\beta}^{\alpha_1} J_c - \delta_{\mu,-\beta} \epsilon f_{\alpha,-\nu}^{\alpha_1} J_c$$

(135)

In deriving some of these relations we have made use of the explicit oscillator realization [19].
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