Abstract. A graph is vertex-transitive if its automorphism group acts transitively on vertices of the graph. A vertex-transitive graph is a Cayley graph if its automorphism group contains a subgroup acting regularly on its vertices. In this paper, the tetravalent vertex-transitive non-Cayley graphs of order $6p$ are classified for each prime $p$.

1. Introduction

In this paper we consider undirected finite connected graphs without loops or multiple edges. For a graph $X$ we use $V(X)$, $E(X)$, $A(X)$ and $\text{Aut}(X)$ to denote its vertex set, edge set, arc set and its full automorphism group, respectively. For $u, v \in V(X)$, $u \sim v$ represents that $u$ is adjacent to $v$, and is denoted by $\{u, v\}$ the edge incident to $u$ and $v$ in $X$, and $N_X(u)$ is the neighborhood of $u$ in $X$, that is, the set of vertices adjacent to $u$ in $X$. A graph $X$ is said to be $G$-vertex-transitive, $G$-edge-transitive and $G$-arc-transitive (or $G$-symmetric) if $G \leq \text{Aut}(X)$ acts transitively on $V(X)$, $E(X)$ and $A(X)$, respectively. In the special case, if $G = \text{Aut}(X)$ then $X$ is said to be vertex-transitive, edge-transitive and arc-transitive (or symmetric). An $s$-arc in a graph $X$ is an ordered $(s + 1)$-tuple $(v_0, v_1, \cdots, v_s)$ of vertices of $X$ such that $v_{i−1}$ is adjacent to $v_i$ for $1 \leq i \leq s$, and $v_{i−1} \neq v_{i+1}$ for $1 \leq i \leq s$; in other words, a directed walk of length $s$ which never includes a backtracking. A graph $X$ is said to be $s$-arc-transitive if $\text{Aut}(X)$ is transitive on the set of $s$-arcs in $X$. A subgroup of the automorphism group of a graph $X$ is said to be $s$-regular if it acts regularly on the set of $s$-arcs of $X$. Recall that a permutation group $G$ acting on a set $\Omega$ is called semiregular if the stabilizer of $\alpha \in G$, $G_\alpha = 1$ for all $\alpha \in G$ and is called regular if it is semiregular and transitive.

Let $G$ be a finite group and $S$ be a subset of $G$ such that $1 \notin S$ and $S = S^{-1}$ where $S^{-1} = \{s^{-1} \mid s \in S\}$. The Cayley graph $X = \text{Cay}(G, S)$ on $G$ with respect to $S$ is defined as the graph with vertex set $V(X) = G$ and edge set $E(X) = \{\{g, sg\} \mid g \in G, s \in S\}$. The automorphism group $\text{Aut}(X)$ of $X$ contains the right regular representation $R(G)$ of $G$, the acting group of $G$ by right multiplication, as a subgroup, and $R(G)$ is regular on $V(X)$. A Cayley graph, $\text{Cay}(G, S)$ is said to be normal if $R(G)$ is normal in $\text{Aut}(\text{Cay}(G, S))$. By \[36\] Proposition 3.1, $N_{\text{Aut}(\text{Cay}(G, S))}(R(G)) = R(G)\text{Aut}(G, S)$, where $\text{Aut}(G, S) = \{\alpha \in$
Aut(G) | S^α = S}). Note that Aut(X) is normal if and only if Aut(G, S) = Aut(X)_v for some v ∈ V(X).

It is well known that a vertex-transitive graph is a Cayley graph if and only if its automorphism group contains a subgroup acting regularly on its vertex set (see, for example, [32, Lemma 4]). There are vertex-transitive graphs which are not Cayley graphs and the smallest one is the well-known Petersen graph. Such a graph will be called a vertex-transitive non-Cayley graph, or a VN C-graph for short.

Many publications have been put into service of investigating the VN C-graphs from different perspectives. For example, in [19], Marušić asked for a determination of the set NC of non-Cayley numbers, that is, those numbers n for which there exists a VN C-graph of order n, and to settle this question, a lot of VN C-graphs were constructed in [12, 14, 15, 20, 21, 22, 23, 24, 25, 27, 30]. In [8], Feng considered the question to determine the smallest valency for VN C-graphs of any primes p of non-Cayley numbers, that is, those numbers N C different perspectives. For example, in [19], Marušić asked for a determination of the set of a graph and let G be a vertex-transitive subgroup of Aut(X). Let g be a vertex-transitive subgroup of Aut(X). By [31], the graph X is isomorphic to a coset graph Cos(A, H, D), where H = A_u is the stabilizer of u ∈ V(X) in
A and $D$ consists of all elements of $A$ which map $u$ to one of its neighbors. It is easy to show that $\text{Core}_A(H) = 1$ and that $D$ is a union of some double cosets of $H$ in $A$ satisfying $D = D^{-1}$. Recall that a graph $\Gamma = (V, E)$ is called an $(m, n)$-\textit{metacirculant}, where $m, n$ are positive integers, if $\Gamma$ is of order $|V| = mn$ and has two automorphisms $\rho, \sigma$ such that

(a) $\langle \rho \rangle$ is semiregular and has $m$ orbits on $V$,
(b) $\sigma$ cyclically permutes the $m$ orbits of $\langle \rho \rangle$ and normalizes $\langle \rho \rangle$, and
(c) $\sigma^m$ fixes at least one vertex of $\Gamma$.

A graph $\tilde{X}$ is called a \textit{covering} of a graph $X$ with projection $\varphi : \tilde{X} \rightarrow X$ if there is a surjection $\varphi : V(\tilde{X}) \rightarrow V(X)$ such that $\varphi|_{N_{\tilde{X}}(\tilde{v})} : N_{\tilde{X}}(\tilde{v}) \rightarrow N_X(v)$ is a bijection for any vertex $v \in V(X)$ and $\tilde{v} \in \varphi^{-1}(v)$. A covering $\tilde{X}$ of $X$ with a projection $\varphi$ is said to be \textit{regular} (or $K$-\textit{covering}) if there is a semiregular subgroup $K$ of the automorphism group $\text{Aut}(\tilde{X})$ such that the graph $X$ is isomorphic to the quotient graph $\tilde{X}/K$, say by $\mathcal{S}$, and the quotient map $\tilde{X} \rightarrow \tilde{X}/K$ is the composition $\varphi \mathcal{S}$ of $\varphi$ and $\mathcal{S}$ (for the purpose of this paper, all functions are composed from left to right). If $K$ is cyclic or elementary abelian then $\tilde{X}$ is called a \textit{cyclic} or an \textit{elementary abelian covering} of $X$, and if $\tilde{X}$ is connected $K$ becomes the covering transformation group. The \textit{fibre} of an edge or a vertex is its preimage under $\varphi$. An automorphism of $\tilde{X}$ is said to be \textit{fibre-preserving} if it maps a fibre to a fibre, while every covering transformation maps a fibre on to itself. All of fibre-preserving automorphisms form a group called the \textit{fibre-preserving group}.

Let $\tilde{X}$ be a $K$-covering of $X$ with a projection $\varphi$. If $\alpha \in \text{Aut}(X)$ and $\tilde{\alpha} \in \text{Aut}(\tilde{X})$ satisfy $\tilde{\alpha}\varphi = \varphi\alpha$, we call $\tilde{\alpha}$ a lift of $\alpha$, and $\alpha$ the projection of $\tilde{\alpha}$. Concepts such as a lift of a subgroup of $\text{Aut}(X)$ and the projection of a subgroup of $\text{Aut}(\tilde{X})$ are self-explanatory. The lifts and the projections of such subgroups are of course subgroups in $\text{Aut}(\tilde{X})$ and $\text{Aut}(X)$, respectively.

Let $m_1, m_2 > 1$ be two odd integers such that $(m_1, m_2) = 1$. Let $1 \leq t \leq m_2$, $(t, m_2) = 1$ be such that $t^2 \equiv -1$ (mod $m_2$). Let $H = \langle r \rangle \times \langle s \rangle \cong \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} = \mathbb{Z}_{m_1 m_2}$. Set $R = \{r, r^{-1}\}$, $L = \{r^t, r^{-t}\}$ and $S = \{1, s\}$ and $X_{m_1, m_2, t} = \text{BiCay}(H, R, L, S)$. Then we have the following result.

**Proposition 1.** [29, Theorem 4.3] Let $X$ be a connected tetravalent vertex-transitive bi-Cayley graph over a cyclic group of odd order $n$. Then $X$ is a $\mathcal{VNC}$-graph if and only if $X \cong X_{m_1, m_2, t}$ for some integers $m_1, m_2$ and $t$.

The following theorem is the main result of this paper.
Theorem 2. Let $X$ be a connected tetravalent vertex-transitive graph of order $6p$, where $p$ is a prime. Then $X$ is a $\mathcal{Y}N\mathcal{C}$-graph if and only if $X \cong X_{3,p,t}$, where $1 \leq t \leq p - 1$ and $t^2 \equiv -1 \pmod{p}$ or $X$ is one of the nine specified graphs.

2. Preliminaries

In this section, we introduce some notation and definitions as well as some preliminary results which will be used later in the paper.

For a regular graph $X$, use $d(X)$ to represent the valency of $X$, and for any subset $B$ of $V(X)$, the subgraph of $X$ induced by $B$ will be denoted by $X[B]$. Let $X$ be a connected vertex-transitive graph, and let $G \leq \text{Aut}(X)$ be vertex-transitive on $X$. For a $G$-invariant partition $\Omega$ of $V(X)$, the quotient graph $X_\Omega$ is defined as the graph with vertex set $\Omega$ such that, for any two vertices $B, C \in \Omega$, $B$ is adjacent to $C$ if and only if there exist $u \in B$ and $v \in C$ which are adjacent in $X$. Let $N$ be a normal subgroup of $G$. Then the set $\Omega$ of orbits of $N$ in $V(X)$ is a $G$-invariant partition of $V(X)$. In this case, the symbol $X_\Omega$ will be replaced by $X_N$.

For two groups $M$ and $N$, by $N \rtimes M$ we denote the semidirect product of $N$ by $M$. For a group $G$, the largest solvable normal subgroup of $G$ is called the solvable radical of $G$ and is denoted by $\text{Sol}(G)$. The outer automorphism group of the group $G$, is the quotient, $\text{Aut}(G)/\text{Inn}(G)$, where $\text{Aut}(G)$ is the automorphism group of $G$ and $\text{Inn}(G)$ is the subgroup consisting of inner automorphisms. The outer automorphism group is usually denoted by $\text{Out}(G)$. Also a lexicographic product of two graphs $X$ and $Y$ which is denoted by $X[Y]$ is defined as the graph with vertex set $V(X) \times V(Y)$ such that for any two vertices $u = (x_1, y_1)$ and $v = (x_2, y_2)$ in $X[Y]$, $u$ is adjacent to $v$ in $X[Y]$, whenever either $\{x_1, x_2\} \in E(X)$ or $x_1 = x_2$ and $\{y_1, y_2\} \in E(Y)$. Note that the lexicographic product of two Cayley graphs is a Cayley graph. For group and graph-theoretic terms not defined here we refer the reader to [4] and [35], respectively. Now we state the following well known result.

Proposition 3. Let $X$ be a $G$-vertex-transitive graph. Then $X$ is symmetric if and only if each vertex stabilizer $G_v$ acts transitively on the set of vertices adjacent to $v$, where $v \in V(X)$.

A finite simple group $G$ is said to be a $K_3$-group if its order has exactly three distinct prime divisors. By [11, Pages 12-14 ], $G$ is isomorphic to one of the following groups:

(1) $A_5, A_6, \text{PSL}_2(7), \text{PSL}_2(8), \text{PSL}_2(17), \text{PSL}_3(3), U_3(3), U_4(2)$.

The socle of a group $G$ is the subgroup generated by the set of all minimal normal subgroups of $G$, it is denoted by $\text{soc}(G)$. Also a group $G$ is said to be almost simple if
$T \leq G \leq \text{Aut}(T)$, where $T$ is a non-abelian simple group. It is well known that $G$ is an almost simple group if and only if $\text{soc}(G) = T$ for some non-abelian simple group $T$.

Now we prove the following lemma.

**Lemma 4.** Let $G$ be an almost simple group and $\text{soc}(G) = A$, where $A$ is a non-abelian simple $K_3$-group. Then

(i) If $A \cong A_5$ then $G \cong A_5$ or $S_5$.

(ii) If $A \cong A_6$ then $G \cong A_6$ or $S_6$ or $S_6 \rtimes \mathbb{Z}_2$.

(iii) If $A \cong \text{PSL}_2(p)$, where $p \in \{7, 17\}$, then $G \cong \text{PSL}_2(p)$ or $\text{PGL}_2(p)$.

(iv) If $A \cong \text{PSL}_2(8)$ then $G \cong \text{PSL}_2(8)$ or $G/\text{PSL}_2(8) \cong \mathbb{Z}_3$.

(v) If $A \cong \text{PSL}_3(3)$ then $G \cong \text{PSL}_3(3)$ or $G/\text{PSL}_3(3) \cong \mathbb{Z}_2$.

(vi) If $A \cong \text{U}_3(3)$ then $G \cong \text{U}_3(3)$ or $G/\text{U}_3(3) \cong \mathbb{Z}_2$.

(vii) If $A \cong \text{U}_4(2)$ then $G \cong \text{U}_4(2)$ or $G/\text{U}_4(2) \cong \mathbb{Z}_2$.

**Proof.** Since $G$ is an almost simple group and $\text{soc}(G) = A$, it implies that $A \leq G \leq \text{Aut}(A)$. If $A \cong A_5$, then $A_5 \leq G \leq \text{Aut}(A_5)$. Now since $|\text{Out}(A_5)| = 2$, it follows that $G \cong A_5$ or $S_5$ and (i) holds. Also if $A$ is isomorphic to one of $\text{PSL}_2(7)$, $\text{PSL}_2(17)$, $\text{PSL}_3(3)$, $\text{U}_3(3)$ or $\text{U}_4(2)$ then $|\text{Out}(A)| = 2$ and the assertions in (iii), (v), (vi) and (vii) hold. If $A \cong A_6$ then $|\text{Out}(A_6)| = 4$ and $\text{Aut}(A_6) \cong S_6 \rtimes \mathbb{Z}_2$. Thus $G \cong A_6$ or $S_6$ or $S_6 \rtimes \mathbb{Z}_2$ and (ii) holds. Finally if $A \cong \text{PSL}_2(8)$ then $|\text{Out}(\text{PSL}_2(8))| = 3$. Hence $G \cong \text{PSL}_2(8)$ or $G/\text{PSL}_2(8) \cong \mathbb{Z}_3$ and (iv) holds.

3. **Main Results**

In this section we classify all connected tetravalent VNC-graphs of order $6p$ where $p$ is a prime. To do this we prove the following propositions.

**Proposition 5.** Let $X$ be a graph, $A = \text{Aut}(X)$ and $N$ be the normal subgroup of $A$. Also let $\Omega$ be the set of orbits of $N$ on $V(X)$ and $X_N$ be a Cayley graph on subgroup of $A/K$, where $K$ is the kernel of action $N$ on $\Omega$. Then $X$ is a Cayley graph on some subgroup of $A$.

**Proof.** Suppose that $X_N = \text{Cay}(H, S)$, where $H \leq A/K$. Thus we may suppose that $H = T/K$, where $T \leq A$. Let $x, y$ be two arbitrary elements of $V(X)$. Set $X = x^N$ and $Y = y^N$. If $X = Y$ then there is $\alpha \in N$ so that $x^\alpha = y$. Thus we may suppose that $X \neq Y$. By our assumption there is an element $K\beta \in T/K$ such that $(x^N)^{K\beta} = (y^N)$ and so $(x^N)^\beta = (y^N)$. Now $x^{\beta n} = y^n$ for some $n \in N$. Now $x^{(\beta n^{-1})} = y$, where $\beta n^{-1} \in T$. Thus $T$ acts transitively on $V(X)$. Suppose that $x^t = x$, where $t \in T$. Now $(x^N)^{Kt} = (x^N)$, a contradiction. Thus $T$ acts regularly on $V(X)$ and $X$ is a Cayley graph on $T$. 


Proposition 6. Let \( X \) be a connected tetravalent vertex-transitive graph of order \( 3p \), where \( p \) is a prime. Then \( X \) is a \( \mathcal{VNC} \)-graph if and only if \( X \) is isomorphic to \( L(O_3) \), the line graph of the Petersen graph.

Proof. Suppose that \( X \) is a connected tetravalent \( \mathcal{VNC} \)-graph of order \( 3p \). By [26], all tetravalent vertex-transitive graphs of order 6, 9 and 21 are Cayley graphs. Also a tetravalent vertex-transitive graph of order 15 is not a Cayley graph and is isomorphic to \( L(O_3) \). Thus we may assume that \( p \geq 11 \). If \( X \) is symmetric then by [34, Page 215], is isomorphic to one of \( L_3(2)^4_{12} \) or \( G(3p, 2) \), which are Cayley graphs. Thus in the following we may assume that \( X \) is not symmetric. Let \( A = \text{Aut}(X) \) and \( v \in V(X) \). Since \( A_v \) is a \( \{2, 3\} \)-group we have \(|A| = 2^sp^t3^r\), for some non-negative integers \( s, t \). We claim that \( A \) is not solvable.

Suppose by contrary that \( A \) is solvable and let \( N \) be a minimal normal subgroup of \( A \). Then \( N \) is an elementary abelian \( r \)-group, where \( r \in \{2, 3, p\} \), \( N = T^k \), and \( T \) is a simple group. Let \( \Omega \) be the set of orbits of \( N \) on \( V(X) \) and \( K \) be the kernel of the action \( A \) on the set of orbits of \( N \). By considering the order of graph, we see that either \( r = 3 \) or \( r = p \). Assume first that \( N \) is an elementary abelian 3-group. Then \( X_N \) has order \( p \) and has valency 2 or 4. It is easy to see that \( K \) acts faithfully on each orbits and therefore \( K \leq S_3 \), which implies that \(|N| = 3 \). Let \( P \) be a Sylow \( p \)-subgroup of \( A \). Then \(|P| = p \) and since \( N \) is a normal subgroup of \( A \), it follows that the orbits of \( N \) form a complete block system of \( A \). Moreover \( P \times N \) acts regularly on \( V(X) \) which implies that \( X \) is a Cayley graph, a contradiction. Next assume that \( N \) is an elementary abelian \( p \)-group. Then \( X_N \) has order 3 and its valency is 2. Hence \( \text{Aut}(X_N) \cong D_6 \). Suppose that \( \Omega = \{\Delta_0, \Delta_1, \Delta_2\} \) and \( \Delta_i \sim \Delta_{i+1} \) where the subscripts are taken modulo 3. If \( \Delta_i \) has an edge, then \( X[\Delta_i] \cong C_p \) and it is easy to see that \(|K_v| \leq 2 \) where \( v \in V(X) \). Thus we have \(|K| \leq 2p \). In case \( \Delta_i \) has no edge, then since \(|V(X_N)| = 3 \) and \( p \geq 11 \) we conclude that \( X[\Delta_i \cup \Delta_{i+1}] \cong C_{2^p} \). Let \( K^* \) be the kernel of the action of \( K \) on \( X[\Delta_i \cup \Delta_{i+1}] \). Now the connectivity of \( X \) and transitivity of \( A/K \) on \( V(X_N) \) imply that \( K^* = 1 \) and consequently, \( K \leq \text{Aut}(X[\Delta_i \cup \Delta_{i+1}]) \cong D_{4p} \). Since \( K \) fixes \( \Delta_i \) it follows that \(|K| \leq 2p \). Since \( A/K \) is transitive on \( V(X_N) \) it follows that 3 divides \(|A/K| \) and \( A/K \) has an element of order 3, say \( K_\alpha \), where \( \alpha \in A \). Therefore \( \alpha^3 \in K \) and hence \( o(\alpha^3) = 1, 2, p \) or \( 2p \). If \( \alpha^3 = 1 \) then \( \langle \alpha \rangle \cdot P \) acts transitively and so regularly on \( V(X) \), a contradiction. Also if \( o(\alpha^3) = 2, p \) or \( 2p \) then \( \alpha^2, \alpha^p \) and \( \alpha^{2p} \) have order 3. In each case there is an element of order 3, say \( \beta \), in \( A \) such that \( \langle \beta \rangle \cdot P \) acts regularly on \( V(X) \), another contradiction. Hence \( A \) is not solvable.

Let \( L = \text{Sol}(A) \), \( \Sigma \) be the set of orbits of \( L \) on \( V(X) \) and \( K \) be the kernel of the action of \( A \) on the set of orbits of \( L \). Assume first that \( L \) is not trivial. Since \( K_v \) is a \( \{2, 3\} \)-group, it implies that \( K_v \) is solvable and by considering \( K = LK_v \) we see that \( K \) is solvable. By the
definition of solvable radical we conclude that $K = L$ and $A/L \leq \text{Aut}(X_L)$. If $\deg(X_L) = 0$ then $A/L$ is solvable which implies that $A$ is solvable, a contradiction. Since $|X| = 3p$, this implies that $\deg(X_L) \neq 1, 3$. In case $\deg(X_L) = 2$ we have $\text{Aut}(X_L) \cong D_{2n}$, where $|V(X_L)| = n$ and $n \in \{3, p\}$. Now $A/L$ is solvable and so $A$ is solvable, which is impossible. Thus we may assume that $\deg(X_L) = 4$ and $L \leq S_3$. Thus $X_L$ is a circulant graph and $X_L = \text{Cay}(G, S)$, where $G \cong \mathbb{Z}_p = \langle a \rangle$, $S = \{a^r, a^{-r}, a^s, a^{-s}\}$ where $(r, p) = (s, p) = 1$.

By [3, Theorem 1.2], $X_L$ is normal. Hence $\text{Aut}(G, S) = \text{Aut}(X_L)$. Since $\sigma : a \mapsto a^r$ is an automorphism of $G$ we have $S = \{a, a^{-1}, a^2, a^{-2}\}$ where $(i, p) = 1$.

Set $B = \text{Aut}(X_L)$ and $Y = X_L$. Let $B_1^*$ be the subgroup of $B_1$ which fixes $\{1\} \cup S$ and $Y_2(1)$ be the subgraph of $Y$ with vertex set

$$\{1\} \cup S \cup \{a^2, a^{-2}, a^{i+1}, a^{-i+1}, a^{2i}, a^{-2i}, a^{-i-1}, a^{i-1}\}.$$ 

Also

$$N_Y(a) = \{1, a^2, a^{i+1}, a^{-i+1}\}, N_Y(a^{-1}) = \{1, a^{-2}, a^{-i-1}, a^{i-1}\}, N_Y(a^i) = \{1, a^{i+1}, a^{2i}, a^{-i-1}\}$$

and $N_Y(a^{-i}) = \{1, a^{-i+1}, a^{-2i}, a^{i-1}\}$. By a tedious computation we find that $B_1^* = 1$ and $B_1$ have no any element of order 4. Thus $B_1$ acts faithfully on $S$ and so $B_1 \leq S_4$. Since $B_1$ has no any element of order 4, by considering the subgroups of $S_4$ we see that either $B_1 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $|B_1| \in \{1, 2, 6\}$. Since $|B| = \deg(Y) = |B_1||G|$ we see that $|B_1| \in \{p, 2p, 4p, 6p\}$. Moreover $\alpha : a \rightarrow a^{-1}$ is an element of $\text{Aut}(G, S)$ and $o(\alpha) = 2$. Hence $|B_1| = 2$ or 4 or 6. From the fact that $A/L$ acts transitively on $V(Y)$ we have $p \mid |A/L|$ and $|A/L| \in \{p, 2p, 4p, 6p\}$. By considering the order of $B$ we see that $G \leq A/L$. So $G = T/L$ where $T \leq A$ and by Proposition [5] $X$ is a Cayley graph, a contradiction.

Assume now that $L$ is trivial. Then $N$ is not solvable, where $N$ is the minimal normal subgroup of $A$. If $N$ is semiregular then $|N|$ divides $3p$ and $N$ is solvable, a contradiction. Therefore $N$ is not semiregular and the valency of quotient graph $X_N$ is not 4. By considering the order of graph we see that $X_N$ is not cubic. Thus $\deg(X_N) = 0$ or 2 and since $A/K \leq \text{Aut}(X_N)$ we conclude that $A/K$ is solvable. Moreover, since $K = K_v N$ and $K_v$ is a $\{2, 3\}$-group we conclude that $K/N$ is solvable. From the fact that $A/K \cong (A/N)/(K/N)$ we conclude that $A/N$ is solvable. We claim that $N$ is the unique minimal normal subgroup of $A$. Suppose by contrary that $M \neq N$ is another minimal normal subgroup of $A$. Since $MN/N \cong M/(M \cap N) \leq A/N$ we see that $M/(M \cap N)$ is solvable. But $M \cap N = 1$ and so $M$ is solvable, a contradiction. Hence $N$ is the only minimal normal subgroup of $A$. Also since $|A|$ is not divided by $p^2$ we conclude that $N$ is a simple group. Therefore $A$ is an almost simple group and $\text{soc}(A) = N$. From the fact that $A$ is a $\{2, 3, p\}$-group, we conclude that $N$ is a $K_3$-group. Also since $p \geq 11$ it follows that $N$ is isomorphic to either $\text{PSL}_2(17)$ or $\text{PSL}_3(3)$. 


First assume that \( N \cong \text{PSL}_2(17) \). Then by part \((iii)\) of Lemma 4, \( A \cong \text{PSL}_2(17) \) or \( \text{PGL}_2(17) \). If \( A \cong \text{PSL}_2(17) \) then \( |A : A_v| = 51 \). By [7], we see that \( A \) has not any subgroup of index 51, a contradiction. Also if \( A \cong \text{PGL}_2(17) \), then by using GAP [10], \( A \) has no any subgroup of index 51, another contradiction.

Assume now that \( N \cong \text{PSL}_3(3) \). In this case, by part \((v)\) of Lemma 4, \( A \cong \text{PSL}_3(3) \) or \( A = \text{Aut}(\text{PSL}_3(3)) \). In the first case, by using GAP [10], and the fact that \( A \) is a transitive permutation group of degree 39, we see that \( A \) has a regular subgroup which means that \( X \) is a Cayley graph, a contradiction. However the later case is impossible, because there is no transitive permutation representation of \( \text{Aut}(\text{PSL}_3(3)) \) of degree 39.

Let \( X \) be a connected tetravalent vertex-transitive graph of order \( 6p \). If \( p = 2 \) or 3, then by [26], we see that \( X \) is a Cayley graph. Also if \( p = 5 \) or \( p = 7 \) then by [13] and using [28] we see that \( X \) is isomorphic to one of the graphs in the following lemma.

**Lemma 7.** A connected tetravalent vertex-transitive non-Cayley graph of order at most \( 42 \) are listed below:

(i) The Coxeter graph;
(ii) the Desargues graph;
(iii) the dodecahedron graph;
(iv) one of the six graphs in Figure 1.

\[\text{Figure 1. Six graphs illustrated in Lemma 7}\]

**Proof of the Theorem.** Let \( X \) be a connected tetravalent vertex-transitive graph of order \( 6p \). If \( p \leq 7 \), then \( X \) is one of the nine specified graphs in Lemma [7]. So we may
assume that $p \geq 11$. Suppose that $X \cong X_{3,p,t}$, then by Proposition 11, $X$ is a $\mathcal{VNC}$-graph. Assume that $X$ is a $\mathcal{VNC}$-graph. Let $A = \text{Aut}(X)$ and $v \in V(X)$. Since $X$ is connected and tetravalent, $A_v$ is a $\{2,3\}$-group and so $|A| = 2^33^p$ for some integers $s, t \geq 1$. Assume that $N$ is a minimal normal subgroup of $A$. We know that $N = T^k$, where $T$ is a finite simple group. Let $\Omega$ be the set of orbits of $N$ on $V(X)$ and $K$ be the kernel of the action of $A$ on $\Omega$. Now we consider two cases:

**Case I.** $A$ is solvable.

In this case $N$ is an elementary abelian $r$-group, where $r \in \{2,3,p\}$. First suppose that $N$ is an elementary abelian 2-group. In this case $X_N$ has order $3p$ and valency 2 or 4. If $X_N$ has valency 2 then $X_N \cong C_{3p}$ and $\text{Aut}(X_N) \cong D_{6p}$. Suppose that $\Omega = \{\Delta_0, \Delta_1, \Delta_2, \ldots, \Delta_{3p-1}\}$, where the subscripts are taken modulo $3p$. Also suppose that $\Delta_i \sim \Delta_{i+1}$ in $X_N$. If for some $i$, the induced graph $X[\Delta_i]$ has an edge then by considering the valency of $X$, we see that $X[\Delta_{i-1} \cup \Delta_i] \cong C_4$ and $X[\Delta_i \cup \Delta_{i+1}] \cong K_4$ where $1 \leq i \leq 3p - 1$. Now by considering the induced graph $X[\Delta_0 \cup \Delta_{3p-1}]$, we have a contradiction. Thus we may assume that each $X[\Delta_i]$ has no any edge. Thus for each $i$, $X[\Delta_i \cup \Delta_{i+1}] \cong C_4$. It is easy to see that in this case $X \cong C_{3p}[2K_1]$, which is a Cayley graph, a contradiction.

Assume now that $X_N$ has valency 4. Then $K_v = 1$ for each vertex $v$. By Proposition 6, $X_N$ is a Cayley graph, unless the case where $X_N$ is isomorphic to the line graph of the Petersen graph. Since $p \geq 11$, $X_N$ is not isomorphic to the line graph of the Petersen graph. Hence $X_N$ is a Cayley graph and by [21, Proposition 1.2], is a $(3,p)$-metacirculant graph. Thus $X_N$ has an automorphism $\sigma$ of order $p$ such that $\langle \sigma \rangle$ is semiregular on the vertex set of $X_N$. Moreover $X_N$ has an automorphism say $\tau$ such that $\tau$ normalizing $\langle \sigma \rangle$ and cyclically permutes the 3 orbits of $\langle \sigma \rangle$ and has a cycle of size 3 in its cycle decomposition. Thus, for some positive integer $i$, the group $\langle \sigma, \tau^i \rangle$, has order $3p$ and acts regularly on $V(X_N)$. Therefore we may suppose that $X_N = \text{Cay}(H,S)$ where $H = \langle \sigma, \tau^i \rangle$. First suppose that $H \cong F_{3p}$. If $X_N$ is normal then $\text{Aut}(X_N)_1 = \text{Aut}(H,S)$. Since $X_N$ is connected and $\text{Aut}(H,S)$ acts faithfully on $S$ we conclude that $\text{Aut}(X_N)_1 \leq S_4$. Moreover $S = \{x, x^{-1}, y, y^{-1}\}$ where $x$ and $y$ have not the same order. It is easy to see that $\text{Aut}(H,S)$ has no any element of order 3 or 4. Thus by considering the subgroups of $S_4$ we see that $\text{Aut}(X_N)_1 = 1$ or $Z_2$ or $Z_2 \times Z_2$. Therefore $|\text{Aut}(X_N)| \leq 12p$ and so $|A/K|$ divides $12p$. By considering the order of $\text{Aut}(X_N)$ we see that $H \leq A/K$ and so $H = T/K$ for some $T \leq A$. Now $|T| = 6p$ and $T$ acts regularly on $V(X)$, a contradiction. Thus we may suppose that $X_N$ is not normal. If $X_N$ is not primitive then by [17, Theorem 3.1] we see that the valency of $X_N$ is not equal to 4. Also if $X_N$ is primitive then again by [34, Lemma 2.1] we have a contradiction. Therefore $X_N$ is a Cayley graph on $Z_{3p}$. 
Thus $X_N$ is a circulant graph and $X_N = \text{Cay}(H, S)$, where $S = \{a^i, a^{-i}, a^j, a^{-j}\}$ and either $(i, 3p) = 1$ or $(j, 3p) = 1$ or $(i - j, 3p) = 1$. Let $B = \text{Aut}(X_N)$ and $Y = X_N$. Let $B_1^∗$ be the subgroup of $B_1$ which fixes $\{1\} \cup S$ and $Y_2(1)$ be the subgraph of $Y$ with vertex set $\{1\} \cup S \cup (S^2 - (1 \cup S))$. We see that

$$S^2 - (1 \cup S) = \{a^{2i}, a^{i+j}, a^{-i-j}, a^{2i}, a^{-i+j}, a^{-i-j}, a^{2j}, a^{-2j}\}.$$ 

Also

$$N_Y(a^i) = \{1, a^{2i}, a^{i+j}, a^{-i-j}\}, N_Y(a^{-i}) = \{1, a^{-2i}, a^{-i+j}, a^{-i-j}\}, N_Y(a^j) = \{1, a^{i+j}, a^{-i+j}, a^{2j}\}$$

and $N_Y(a^{-j}) = \{1, a^{-i-j}, a^{-i-j}, a^{-2j}\}$. By a tedious computation we have $B_1^∗ = 1$ and $B_1$ has no any element of order 4. Thus $B_1$ acts faithfully on $S$ and so $B_1 \leq S_4$. Since $B_1$ has no any element of order 3, and the fact that $\alpha : a \rightarrow a^{-1}$ is an element of $\text{Aut}(H, S)$ and $o(\alpha) = 2$, it implies that $B_1 \cong \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$ and so $|\text{Aut}(X_N)| \in \{6p, 12p\}$. Since $A/N$ acts transitively on $V(Y)$ we have that $3p$ divides $|A/N|$. Thus $|A/N| \in \{3p, 6p, 12p\}$. By considering the order of $B$, we see that $H \leq A/N$ and so $H = T/N$ where $T \leq A$. Now by Proposition \[X\] $X$ is a Cayley graph, a contradiction.

Now suppose that $N$ is an elementary abelian 3-group and $\Omega = \{\Delta_0, \Delta_1, \Delta_2, \cdots, \Delta_{2p-1}\}$, where the subscripts are taken modulo $2p$. Then $X_N$ has order $2p$ and so $X_N$ has valency 2, 3 or 4. First suppose that $X_N$ has valency 2. Then $X_N \cong C_{2p}$ and $\text{Aut}(X_N) \cong D_{4p}$. Suppose that $\Delta_i \sim \Delta_{i+1}$ in $X_N$. If $\Delta_i$ has an edge then by considering the valency of $X$ we see that $N$ acts faithfully on $X[\Delta_i]$. Hence $|N| = 3$ and by considering the valency of $X$ we see that $X[\Delta_{i-1} \cup \Delta_i] \cong 3K_2$, $|K| \leq 2$ and $|K| \leq 6$. Since $A/K$ is transitive on $V(X_N)$ it follows that $2p$ divides $|A/K|$. Hence $A/K \cong \mathbb{Z}_{2p}$ or $A/K \cong D_{2p}$ or $A/K \cong D_{4p}$. If $A/K \cong \mathbb{Z}_{2p}$ or $A/K \cong D_{4p}$, then $A/K$ has an element of order $2p$, say $K\alpha$, where $\alpha \in A$. Therefore $\alpha^{2p} \in K$ and hence $o(\alpha^{2p}) = 1$ or 2, 3 or 6. If $\alpha^{2p} = 1$ then $\langle \alpha \rangle \times \langle \gamma \rangle$, where $\gamma$ is an element of order 3 in $K$, acts transitively and so regularly on $V(X)$, a contradiction. Also if $o(\alpha^{2p}) = 2, 3$ or 6 then $o^2, o^3$ and $o^6$ have order $2p$. Thus in each case there is an element of order $2p$, say $\beta$, in $A$ such that $\langle \beta \rangle \times \langle \gamma \rangle$ acts transitively and so regularly on $V(X)$, a contradiction. Thus we may suppose that $A/K \cong D_{2p}$. Thus $A/K$ has an element of order $p$, say $K\alpha$. Clearly, $K\alpha$ does not fix any element of $V(X_N)$ and so $\alpha$ does not. Clearly, $\alpha^p \in K$ and so $o(\alpha^p) = 1$ or 2 or 3 or 6. In each case we can find an element of order $p$, say $\beta$ such that $(\beta) \times (\gamma)$ acts semiregularly on $V(X)$. Set $H = (\beta) \times (\gamma)$, where $\gamma$ is an element of order 3 in $K$, acts semiregularly on $V(X)$ and has two orbits on $V(X)$, which means that $X$ is a bi-Cayley graph over $H \cong \mathbb{Z}_{3p}$. Then, by Proposition \[X\] $X \cong X_{3,p,t}$, where $1 \leq t \leq p - 1$, and $t^2 \equiv -1 \pmod{p}$.
Suppose that $\Delta_i$ has no edge. Then it is easy to see that either $X[\Delta_0 \cup \Delta_{2p-1}] = X[\Delta_{i-1} \cup \Delta_i] \cong C_6$ ($1 \leq i \leq 2p - 1$) or $X[\Delta_i \cup \Delta_{i+1}] \cong 3K_2$ ($1 \leq i \leq 2p - 1, i \text{ is odd}$) and $X[\Delta_i \cup \Delta_{i+1}] \cong K_{3,3}$ ($0 \leq i \leq 2p - 1, i \text{ is even}$).

First suppose that $X[\Delta_0 \cup \Delta_{2p-1}] = X[\Delta_{i-1} \cup \Delta_i] \cong C_6$ ($1 \leq i \leq 2p - 1$). Then we consider the action of $K$ in $X[\Delta_i]$ ($0 \leq i \leq 2p - 1$). Now it is easy to see that $K$ acts faithfully on $X[\Delta_i]$ and so $K \leq S_3$. Since $2p$ divides $|A/K|$ and $|A/K|$ divides $4p$ with a similar arguments as above we get a contradiction. Now suppose that $X[\Delta_i \cup \Delta_{i+1}] \cong 3K_2$ ($1 \leq i \leq 2p - 1, i \text{ is odd}$) and $X[\Delta_i \cup \Delta_{i+1}] \cong K_{3,3}$ ($0 \leq i \leq 2p - 2, i \text{ is even}$).

Let $V(\Delta_i) = \{a_i, b_i, c_i\}$ ($0 \leq i \leq 2p - 1$). Without loss of generality we may assume that $a_i \sim a_{i+1}, b_i \sim b_{i+1}$ and $c_i \sim c_{i+1}$. Now it is easy to see that $\alpha : a_i \mapsto b_i, b_i \mapsto c_i, c_i \mapsto a_i$ and $\beta : a_i \mapsto a_{i+2}, b_i \mapsto b_{i+2}, c_i \mapsto c_{i+2}$ are automorphisms of $X$ of orders 3 and $p$, respectively, where ($0 \leq i \leq 2p - 1$). Clearly $\langle \alpha \rangle \times \langle \beta \rangle \cong \mathbb{Z}_{3p}$ acts semiregularly and has two orbits on $V(X)$. Therefore $X$ is a bi-Cayley graph over the group $\mathbb{Z}_{3p}$ and again, by Proposition [1], $X = X_{3,p,t}$, where $1 \leq t \leq p - 1$ and $t^2 \equiv -1 \pmod{p}$. Suppose that $X_N$ has valency either 3 or 4. If $\text{Aut}(X_N)$ is primitive then by [18], Corollary 6.6], $p \geq 313$, $2p = m^2 + 1$ for some composite integer $m$. Also $\Gamma$ has valency either $\frac{m(m-1)}{2}$ or $\frac{m(m+1)}{2}$, a contradiction. Thus we may suppose that $\text{Aut}(X_N)$ is imprimitive. Now by [21], Theorem], $X_N$ is metacirculant. If $X_N$ has valency 3 then by [33], Main Theorem], $X_N$ is isomorphic to generalized Petersen graph $GP(p, k)$. By [9], Theorems 1, 2], we see that $|\text{Aut}(X_N)| = 4p$. If $\Delta_i$ has an edge then $X[\Delta_i] \cong C_3$ ($0 \leq i \leq 2p - 1$). We see that $X_N$ can not have valency 3, a contradiction. Thus we may suppose that $\Delta_i$ ($0 \leq i \leq 2p - 1$) has no edges. Let $V(\Delta_i) = \{a_i, b_i, c_i\}$ ($0 \leq i \leq 2p - 1$). Let $\Delta_{i+1} \sim \Delta_{i+2}$ and $\Delta_{i+3}$ are the vertices adjacent to $\Delta_i$. Since $X_N$ has valency 3, we may suppose that $X[\Delta_i \cup \Delta_{i+1}] \cong C_6$, $X[\Delta_i \cup \Delta_{i+2}] \cong 3K_2$ and $X[\Delta_i \cup \Delta_{i+3}] \cong 3K_2$. Now it is easy to see that $\alpha : a_i \mapsto b_i, b_i \mapsto c_i, c_i \mapsto a_i$ is an automorphism of $X$ of order 3. Also $K$ acts faithfully on $X[\Delta_i]$ and so $K = N \cong \mathbb{Z}_3$. By [1], Theorem 1.1], we see that the generalized Petersen graph is a normal bi-Cayley graph on $H = \mathbb{Z}_p$. Clearly, $H$ acts semiregularly on $V(X_N)$. If $H$ is not a subgroup of $A/K$ then $p^2 \mid |\text{Aut}(X_N)|$, a contradiction. Thus $H \leq A/K$ and $A/K$ has a semiregular element of order $p$, say $K\alpha$. Now $\alpha^p \in K$ and so $\text{o}(\alpha^p) = 1, 2$ or 3. In each case it is easy to find an element of order $p$, say $\beta$, which is a power of $\alpha$. If $\beta$ fixes an element then it is easy to see that $K\alpha$ or $K\alpha^2$ or $K\alpha^3$ fixes at least one $\Delta_i$, a contradiction. Now $\langle \beta \rangle \times \langle \gamma \rangle \cong \mathbb{Z}_{3p}$ acts semiregularly and has two orbits on $V(X)$ where $\gamma \in K$ and $O(\gamma) = 3$. Therefore $X$ is a bi-Cayley graph over group $\mathbb{Z}_{3p}$ and by Proposition [1], $X = X_{3,p,t}$, where $1 \leq t \leq p - 1$ and $t^2 \equiv -1 \pmod{p}$. Also if $X_N$ has valency 4 then $N = K$. Since $A_v$ is $\{2,3\}$-group we conclude that $A$ has a semiregular element of order $p$, say $\delta$. Since $N_A(N)/C_A(N)$ is a subgroup of $\text{Aut}(N) \cong \mathbb{Z}_2$ we conclude
that $\delta$ or $\delta^2 \in C_A(N)$. Also if $N = \langle \gamma \rangle$ then $\langle \delta, \gamma \rangle$ or $\langle \delta^2, \gamma \rangle$ is a semiregular subgroup of $\text{Aut}(X)$ which is isomorphic to $\mathbb{Z}_3 p$. Thus $X$ is a bi-Cayley graph over $H \cong \mathbb{Z}_3 p = \langle a \rangle$ and by Proposition $[1] X = X_{3,p,t}$, where $1 \leq t \leq p - 1$ and $t^2 \equiv -1 \pmod{p}$.

Finally, suppose that $N$ is an elementary abelian $p$-group. By considering the order of $A$ we see that $|N| = p$. Suppose that $N_v \neq 1$ for some $v \in V(X)$ and $w$ is another arbitrary vertex of $X$. Since $X$ is transitive, there exists $\alpha \in \text{Aut}(X)$ such that $w^\alpha = v$ and so $N_v = N_w^\alpha = \alpha^{-1} N_w \alpha = N_w$. Thus $N$ is trivial, a contradiction. Therefore $N$ acts semiregularly on $V(X)$ and so $X$ is an $N$-regular cover of $X_N$. By [26, Page 173] we know that $X_N$ is a Cayley graph. Thus $\text{Aut}(X_N)$ has a subgroup which acts regularly on $V(X)$, say $G$. Suppose that $\tilde{G}$ is a lift of $G$. Now it is easy to see that $\tilde{G}$ acts regularly on $V(X)$ and so $X$ is a Cayley graph, a contradiction.

Case II. $A$ is not solvable.

Let $L = \text{Sol}(A)$ be the solvable radical of $A$ and $\Omega$ be the set of orbits of $L$ on $V(X)$. First suppose that $L$ is not trivial. Let $K$ be the kernel of $A$ on $\Omega$. Since $K_v$ is a $\{2, 3\}$-group it implies that $K_v$ is solvable and so by considering $K = L K_v$ we see that $K$ is solvable. Now by the definition of $L$ we conclude that $K = L$ and $A/L \leq \text{Aut}(X_L)$. If $\text{deg}(X_L) = 0$ then $A/L$ is solvable and so $A$ is solvable, a contradiction. Also if $\text{deg}(X_L) = 2$ then $\text{Aut}(X_L) \cong D_{2n}$, where $|V(X_L)| = n$ and $n \in \{3, 6, p, 2p, 3p\}$. Now $A/L$ is solvable and so $A$ is solvable, a contradiction. Thus we may suppose that $\text{deg}(X_L) = 3$ or $4$. If $\text{deg}(X_L) = 3$ then $n \in \{6, 2p, 3p\}$. By [16, Theorem 1.1], we see that $\text{Aut}(X_L)$ is solvable and so $A/L$ is solvable. Now $A$ is solvable, a contradiction. Also if $\text{deg}(X_L) = 4$ then $L = K$ and $K$ acts faithfully on each orbits of $K$. Moreover, in this case $n \in \{6, p, 2p, 3p\}$. If $n = 6$, then $X_L$ is isomorphic to the octahedral graph and so $|\text{Aut}(X_L)| = 48$. Thus $\text{Aut}(X_L)$ is solvable which implies that $A$ is solvable, a contradiction. If $n = p$ then by $|V(X)| = |L| n$ we see that $|L| = 6$ and $L \cong S_3$ or $\mathbb{Z}_6$. Also since $A_v$ is a $\{2, 3\}$-group, it implies that $A$ has a semiregular element of order $p$, say $\alpha$. If $L$ as subgroup of $S_6$ is isomorphic to $S_3$ then since $L$ is semiregular we conclude that $S_3 \times \langle \alpha \rangle$ acts regularly on $V(X)$, a contradiction. Similarly if $L$ as subgroup of $S_6$ is isomorphic to $\mathbb{Z}_6 = \langle \beta \rangle$ then $\langle \alpha \rangle \times \langle \beta \rangle$ acts regularly on $V(X)$, a contradiction. Also if $|V(X_L)| = 2p$ then $|L| = 3$. Since $A_v$ is $\{2, 3\}$-group we conclude that $A$ has a semiregular element, say $\alpha$, of order $p$, . Since $N_A(L)/C_A(L)$ is a subgroup of $\text{Aut}(L)$ we conclude that $\alpha \in C_A(L)$. Also if $\beta \in L$ and $\beta$ has order $3$ then $\langle \alpha, \beta \rangle$ is a semiregular subgroup of $\text{Aut}(X)$ which is isomorphic to $\mathbb{Z}_3 p$. Thus $X$ is a bi-Cayley graph over $H$, where $H \cong \mathbb{Z}_3 p$. By Proposition $[1] X = X_{3,p,t}$, where $1 \leq t \leq p - 1$ and $t^2 \equiv -1 \pmod{p}$. Finally if $|V(X_L)| = 3p$ then $|L| = 2$. By Proposition $[6] X_N$ is a Cayley graph, unless the case where $X_N$ is isomorphic to the line graph of the Petersen graph. Since $p \geq 11$, it follows that $X_N$ is
not isomorphic to the line graph of the Petersen graph. Also if \( X_N \) is a Cayley graph then Aut\((X_N)\) has a subgroup say \( H \) which acts regularly on \( V(X_N) \). Thus \(|H| = 3p \) and by [21 Proposition 1.2], \( X_N \) is a \((3, p)\)-metacirculant graph. Thus \( X_N \) has an automorphism \( \sigma \) of order \( p \) such that \( \langle \sigma \rangle \) is semiregular on the vertex set of \( X_N \), and an automorphism \( \tau \) normalizing \( \langle \sigma \rangle \) and cyclically permuting the 3 orbits of \( \langle \sigma \rangle \) such that \( \tau \) has a cycle of size 3 in its cycle decomposition. Thus \( \langle \sigma, \tau^i \rangle \) for some positive integer \( i \), has order \( 3p \) and acts regularly on \( V(X_N) \). Therefore we may suppose that \( X_N = \text{Cay}(H, S) \). First suppose that \( H = \langle \sigma, \tau^i \rangle \cong F_{3p} \). If \( X_N \) is normal then Aut\((X_N)_1 = \text{Aut}(H, S) \). Since Aut\((H, S)\) acts faithfully on \( S \) we conclude that Aut\((X_N)_1 \leq S_4 \). Also since \( X_N \) is connected we conclude that \( S = \{x, x^{-1}, y, y^{-1}\} \) and \( x \) and \( y \) have not the same orders. Now it is easy to see that Aut\((H, S)\) has no any element of order 3 or 4. Thus by considering the subgroups of \( S_4 \) we see that Aut\((X_N)_1 = 1 \) or \( Z_2 \) or \( Z_2 \times Z_2 \). Therefore |Aut\((X_N)_1| \leq 12p \) and so |A/K| divides 12p. Now by considering the order of Aut\((X_N)\) we see that \( H \leq A/K \) and so \( H = T/K \) for some \( T \leq A \). Now \(|T| = 6p \) and \( T \) acts regularly on \( V(X) \), a contradiction. Thus we may suppose that \( X_N \) is not primitive then by [17 Theorem 3.1] we see that the valency of \( X_N \) is not equal to 4. Also if \( X_N \) is primitive then again by [34 Lemma 2.1] we have a contradiction. Therefore \( X_N \) is a Cayley graph on \( Z_{3p} \). Now similar to the case where \( A \) is solvable, we conclude that \( X \) is a Cayley graph, a contradiction.

Suppose that \( L \) is trivial. By considering the order of \( A \), we see that \( N \) is a \( K_3 \)-group. Since \( p \geq 11 \), it follows that \( N \cong \text{PSL}_2(17) \) or \( \text{PSL}_3(3) \). If \( N \) has more than two orbits then \( n \in \{3, 6, 2p, 3p\} \). First assume that \( n = 2p \) or \( 3p \). We know that \( A/K \) acts transitively on \( V(X_N) \) and \( K = K_vN \), where \( v \in V(X) \). Thus \( p^2 \) divides \(|A| \), a contradiction. Thus we may suppose that \( n = 3 \) or \( 6 \). By using GAP [10], it is easy to see that PSL\(_2(17)\) and PSL\(_3(3)\) does not have any orbit of order \( p \) or \( 2p \), where \( p \in \{13, 17\} \). Therefore \( N \) has at most two orbits. Now Aut\((X_N)\) is solvable and so \( A/K \) is solvable. Also since \( K_v \) is \{2, 3\}-group and \( K = K_vN \), it follows that \( K/N \) is solvable. Since \( A/K \cong (A/N)/(K/N) \) we conclude that \( A/N \) is also solvable. Suppose that \( M \) is another minimal normal subgroup of \( A \). Thus by the fact that \( MN/N \cong M/(M \cap N) \leq A/N \) we see that \( M/(M \cap N) \) is solvable. But \( M \cap N = 1 \) and so \( M \) is solvable, a contradiction. Hence \( N \) is the only minimal normal subgroup of \( A \). Since \(|A| \) is not divisible by \( p^2 \) we conclude that \( N \) is a simple group. Therefore \( A \) is an almost simple group and \( \text{soc}(A) = N \). Since \( A \) is \{2, 3, p\}-group, it follows that \( N \) is a \( K_3 \)-group. Also since \( p \geq 11 \) we conclude that \( N \cong \text{PSL}_2(17) \) or \( N \cong \text{PSL}_3(3) \).

First assume that \( N \cong \text{PSL}_2(17) \). Then \( A \cong \text{PSL}_2(17) \) or \( \text{PGL}_2(17) \). If \( A \cong \text{PSL}_2(17) \) then \(|A : A_v| = 102 \). By [7], we see that \( A \) has two subgroups of index 102 and both of them have an element of order 4. Also we see that \( A_v \) acts transitively on the set of
vertices adjacent to \( v \). By Proposition 3, \( X \) is a symmetric graph. If \( X \) is one-regular then \( |A| = 2^3 \cdot 3 \cdot 17 \), a contradiction. By [2, Theorem 1], we see that \( X \cong C_{3p}[2K_1] \) and so \( X \) is a Cayley graph, a contradiction. Also if \( A \cong \text{PGL}_2(17) \) then by using GAP [10], \( A \) has no any subgroup of index 102, a contradiction.

Assume that \( N \cong \text{PSL}_3(3) \). Then \( A \cong \text{PSL}_3(3) \) or \( A = \text{Aut}(\text{PSL}_3(3)) \). Let \( A = \text{Aut}(\text{PSL}_3(3)) \). Since \( A \leq S_{78} \) is a transitive permutation group of degree 78, there exists a core-free subgroup \( H \) of \( A \) of index 78 such that \( A \) is isomorphic to the permutation representation of the action of \( A \) on the right cosets of \( H \) in \( G \) by right multiplication. One can check, by using GAP [10], that in this case \( A \) has a regular subgroup of order 78 which means that \( X \) is a Cayley graph, a contradiction. Hence \( A = \text{PSL}_3(3) \) and \( X = \text{Cos}(\text{PSL}_3(3), H, D) \), is a coset graph, where \( H \cong (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes Q_8 \) and \( D \) is a union of double cosets of \( H \) in \( G \). Since \( X \) is 4-valent, we have \( |D|/|H| = 4 \). By using GAP [10], there are 15 possibilities for \( D \) and in each case \( \langle D \rangle \) is a proper subgroup of \( \text{PSL}_3(3) \) which means that \( X \) is disconnected, a contradiction. This complete the proof.

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